Integrability in String Theories

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Abstract

The solution term by term to the scattering of all consistent string theories is given. The moduli space of M-theory is derived and connects the various string theories. The solutions contain both the perturbative and non-perturbative sectors of the string. Modular forms found by differential equations on subspaces of the M-theory moduli space and transfinite algebras play an essential role in deriving the coefficients. Various results and identities in algebra are found from the explicit solution. Archetypes and models are presented in accord with phenomenology and cosmology.
I. Introduction

String theories in various dimensions have apparently been unified with an underlying structure, which arises from M-theory \[1\], \[2\]. The known five consistent string theories are Heterotic, IIB, IIA, I, and Spin(32), although there is no complete classification until now. The manifestation of the symmetries of M-theory, in conjunction with the basic principles of scattering, are used in this work to construct the full S-matrix solution to these five superstring theories in addition to an (unexpected) five more.

The full M-theory moduli space has not been presented in the literature. Neither has the full solution to the string scattering, including the ghost sector. The full solution means, as a power series expansion in $\alpha'$, every prefactor is a non-perturbative function of the string model specific couplings. In this work the moduli space and the non-perturbative scatterings are given, and with relatively simple differential equations the latter are defined and can be solved. The symmetries and the dualities of the various superstrings are both listed and manifested.

II. S-Matrices and Effective Actions

The S-matrix in perturbative string theory is defined formally in a couple of ways. One can use the path integral and its world-sheet action, and by gauge fixing all of the extraneous degrees of freedom sum over inequivalent random 2-d Riemann surfaces. The amplitudes are defined as the sum of these punctured surfaces with wavefunction overlaps provided by the set of vertex operators, which are in 1-to-1 correspondence with the physical states. The gauge fixing separates into several well known methods such as light-cone, but the integrals are very complicated to evaluate by hand.

In recent years an altered expansion has been developed, which is very compatible with the dualities in M-theory. The derivative expansion takes the form of the S-matrix and expands it as a power series in the number of derivatives and operators most of possess varying dimension. This is not a string coupling expansion; the field theory form of the expansion can be obtained in \[4\]-\[12\], and the string form can be found in \[13\]-\[19\], with some overlap between the two. However, every order in $\alpha'$ is explicitly non-perturbative in the string coupling. As $\alpha'$ does not transform under S-duality, the expansion is suitable to manifest the duality structures of the string(s). This expansion has been explicitly checked, to find out if there was anything wrong with S-duality without a trace. Supersymmetry and S-duality have been presumed independent, which is clearly not true in accord with this work and prior work. The
$\Box^3 R^4$ and lower orders have been generally investigated together with a few special series, and to genus two with the $\Box^2 R^4$ term [13], [20]-[27], [37]. Subtle issues in the genus expansion have been checked off as compatible with S-duality [28]-[36]. These calculations can show mechanisms for cancellations in the maximally supersymmetric IIB theory and were necessary to show the consistency of S-duality with the string and its low-energy limit. In the form of maximally extended supergravity defined in field theory there are implications for extended finiteness properties [13]-[14], [38]-[39].

Consider the graviton scattering in IIB superstring. Its expansion is, in Einstein frame,

$$S = \alpha'^{-3} \int \sqrt{g} \left[ R + \alpha'^3 f_{3/2} R^4 + \alpha'^5 f_{5/2} \Box^2 R^4 + \ldots \right],$$

which is power series without end in the number of derivatives. The gravitational portion is shown, but with supersymmetry and the higher order terms, including the tensor structure the functional is complete.

Formally,

$$S = \int \sqrt{g} \sum \alpha'^{-3+2j} f_j \mathcal{O}_j,$$

with $\mathcal{O}_j$ spanning the basis of operators, which depends specifically on the particular superstring theory and its compactification including fluxes. The derivative form is also useful in the background field expansion as exemplified in the holographic correspondence. Determination of the functions $f_j$ then 'solve' the string theory, apart from resummations which may be very relevant in certain energy regimes. Resummations also are expected to show the stringiness of the individual string fields such as the graviton $g_{\mu\nu}$, but from the target space-time point of view.

The expansion in (2) is perturbative, but in the string scale $\alpha'$. Variations of the action with respect to the fundamental string fields generate the on-shell S-matrix. Furthermore, ghost string fields may be included in the operator content. In this work, the coefficients $f_j$ are determined based on symmetries of trans finiteness and also the M-theory moduli space. These coefficients, which are functions in the couplings, are determined a specified and non-trivial differential equation, and the functions $f_j$ are related from one superstring theory to another.

III. Transfinite Algebra and Representations
The affine group transformations appear in the string in a variety of ways. First, they generate the modular subgroup of the Virasoro algebra on the world-sheet. Second, the target space conformal representations is expected to relate to the world-sheet Virasoro due to the conformal and Virasoro mapping. Third, there is reason to believe that the transfiniteness is related to the integrability in non-trivial curved backgrounds such as when black holes are present. For these reasons, transfinite groups and their actions are important to use and to clarify their role. Their use in this work involved representation content primarily, without the full group and dynamical implications. There are also powerful connections in mathematics, such as in algebra, differential forms and specifically equations, and number theory, with the transfinite algebra and its interwoven fabric in the full string scattering; this is not explored here for brevity.

The affine transfinite algebra $L_{a,b}^\alpha$ is characterized by the operations,

$$[L_a, L_b] = L_c + \alpha \delta_{a=b} L_a ,$$

with the Poisson-Lie commutator

$$\{L_a, L_b\} = L_a \times L_b = L_c ,$$

with $c = a \otimes b$. The affine extension of the Cartan algebra is labeled by the circle parameter $\alpha$. Two representations of $L_{a,b}$ at $\alpha = 0$ and at once are,

$$L_a = e^{i \frac{a \pi}{2N}} , \quad L_a = \frac{a}{p(a-p)} ,$$

where $p$ is defined as the largest prime factor of $a$; $a \otimes b$ are $a + b \mod 4N$ and $a \otimes b = p_a + p_2 \mod C_{p_1,p_2}^{p_1+p_2}$. A less trivial cyclotomic field representation is generated by the roots to the polynomial

$$P(x) = \sum_{\rho=0}^{N} x^\rho \mod p ,$$

with generator and root,

$$\alpha_a L_a \quad \alpha_a = \frac{S^{\alpha=1}}{f(N,p),g(N,p)} .$$
This representation is not used in this work but is important to string theory: it species branched covers of Riemann surfaces modulo five-form flux charge and conservation or flux conservation when a fiber disappears or is damaged. The cyclotomatic fields are generally used in so many areas of string theory that the representation is pervasive.

The commutation relations generate the invariant for the algebra $L_{a,b}^{\alpha=1}$,

$$[L_a, I] = I \quad I = \sum_{-\infty}^{\infty} \alpha_n L_n$$

in which the sum generating the $I$ ranges from minus infinity to infinity, which is a bit non-standard. There is a map that may be used to switch between the two labelings.

In the various string theories the components of the representations are the various operators found by taking various products of the fields. For example, the operators $O_1 = R^4$ which is a product of 8 Weyl tensors with a specific contraction, or $O_2 = \psi^4$ the product of four fermions such as gravitinos. The appropriate sum of representations span the full basis to the analytic part of the effective action after completely supersymmetrizing the elements. The commutator structure is realized as a direct product of the two operators, that is, when the operators are labeled at the same point. There are special representations that span the basis of the non-analytic operators, such as those required by unitarity; string theory constructs can be used to construct these.

The S-matrix is found by superimposing all of these terms, together with the supersymmetric extension, in the form

$$S = \int \sqrt{g} \sum \alpha_n L_{a_i, b_i}^{\alpha=1}.$$ (9)

The $L_{a_i, b_i}^{\alpha=1}$ is labeled by the set of integers from $i = 1, \ldots, N_R$, the number of representations. The representations are labeled by the indices $a_i$ and $b_i$. The transfinite representation that spans the ghost sector allows the action to be taken off-shell, with an unambiguous off-shell prescription.

**IV. Manifestation of Transfinite Symmetries**

The appearance of the transfiniteness of the string scattering is not apparent in perturbation theory.
Consider the following S-matrix as an element in the algebra, with generator,

\[
\prod \exp(t_j \int \mathcal{O}_j).
\]  

(10)

The generators correspond to conformal maps on the complex plane. The S-matrix then is a generator of conformal maps, including curves.

V. Moduli Space of the M-theory Web

The moduli space of the M-theory vacua is a fundamental space on and through which dualities act. These transformations through actions on the moduli space transform the various string theories into each other. For example, in the IIB sector there is S-duality which requires the group \( SL(2, Z) = SL(2, R)/\mathbb{Z} \) acting on a torus. This torus is larger in general, and the space follows essentially from the quantization of the various string theories.

Alternatively, determination of the moduli space a priori allows the various string theories to be interconnected, and solved for, in terms of the transfinite representations and modular solutions to a differential equation acting on a sector of the moduli space. The latter two quantities are the coefficients \( \alpha_n \) and \( Q_s \), which are an integer and a modular form of a particular weight. Both of which are found from the moduli space and multiply the operator content of the representations involved in the individual string scattering. Thus,

\[
S = \int \sqrt{g} \sum \alpha_n Q_s L^{\alpha=1}_{a_i, b_i}
\]

(11)

with \( n \) and \( s \) proportional depending on the theory and functions of the parameter labeling the elements in the representation \( L_{a_i, b_i}^{\alpha=1} \).

The basic elements of the M-theory moduli space are as follows:

1) a torus with mapping class group \( SL(2, R)/\mathbb{Z} \) (or \( SL(2, R)/\mathbb{Z}_p \))

2) a circle \( S^1 \) with group U(1) that models the eleventh dimension

3) two branched covers of the torus so that a pair of elliptic genus three surfaces are constructed with the moduli possibly fixed.

4) a branched cover of a unit \( S^3 \) with unramified points, i.e. a \( S^3 \) with one point removed or the space \( R^3 \); this is not part of the moduli space but rather used in
satisfying a unimodular condition with the points. The points may be resolved or blown up to construct the spaces in 1) to 3).

The spaces in 1) to 3) make up the the moduli space from which the couplings \( \tau = \tau_1 + i\tau_2 \) are found; the gauge coupling is roughly the square root of the gravitational coupling required by modular invariance or (anti-) holomorphicity. They transform under the modular group of the torus. In addition they are non-trivial fibers; for example the \( S^1 \) is a non-trivial fiber around the \( T^2 \) and the two elliptic surfaces are oriented in a diametrically opposite fashion and are fibered non-trivially over the \( S^3 \) in a disc like fashion involving only two coordinates. The disc-like fashion follows from one dimension oriented along the \( \tau_2 \) direction and another dimension along a non-intersecting path in \((S^1, \tau_1)\), both directions of which have periodicity. The path in \( S^3 \) is non-trivial because the \( S^1 \) has a non-trivial fiber over the entire \( T^2 \).

The construction of the moduli space begins with 12 points on a unit n-sphere embedded in 13 dimensional space, with signature specified below. The space has thirteen dimensions as there is split between 11 dimensions and a torus fiber. These points are used as unramified points placed on the n-sphere after compactifying the plane, so there are now 13 unramified points. The points are chosen in a symmetric fashion so that they are equidistant from each other, and also in a manner so that a polynomial of degree 5 specifies the unramified surface. The rotation of the configuration on the n-sphere allows one term in polynomial, the fifth, to be dropped through a Poincare reduction (fix one point and rotate).

The degree 4 polynomial

\[
P(x) = \alpha_1 x^4 + \alpha_2 x^3 + \alpha_3 x^2 + \alpha_4 x + \alpha_5,
\]

specifies the unramified surface described by an n-sphere with a symmetric ordering of 13 points. This polynomial also describes a torus, but with the parameters \( \alpha_1 = -1/\alpha_5 \) satisfying the antipodal constraint. This follows from specifying \( n = 5 \) together with the resolution of a degree 5 quintic in an ambient 13-dimensional space of arbitrary signature with the property that the toric is quintic together with its hyperkahler structure, which is Pennington’s theorem. Thus the polynomial describes a torus after a Poincare resolution. One coordinate is free but used to resolve the torus into a five-sphere. Due to the Poincare map, there is one ramified point on the surface, with all of the previous 13 unramified points taken off the surface into the ambient 13 dimensional spacetime. The torus will be utilized to describe the torus of the \( SL(2, Z) = SL(2, R)/Z \) and the remaining 13 points are used to resolve the
moduli space except four which are not necessary. These four points are used to close the $R^4$ into a ball in a fiber-wise action, which is still a 4-sphere.

It is necessary that the parameters labeling the 13 points satisfy a unimodularity constraint so that the Poincare reduction goes through. This is a determinant constraint on 13 scalar parameters describing the equidistant distance from one point to its neighbor using the round metric for example on the unit 5-sphere. These 13 parameters specify the configuration of the 13 points. The unimodularity constraint is that

$$det(\alpha_i\alpha_j) \neq n/p ,$$

which is an integer mod its largest prime number being an integer also. This is automatic if the following theorem due to is used: Consider a round ball with unit radius and determinant unity of its collection of n points. If n mod its reduction to p points is unity then the condition of unimodularity holds. In our case n= 13 and p= 5 generate unimodularity as they are coprime. Our condition holds, and the Poincare resolution is smooth.

The torus with one ramified point can accept a point, placing two punctures on it. The remaining 11 points are left off the torus. One of the unramified points on the torus is then bifurcated into a branch cut between two bifurcations, such as the location of two spin fields. Note that bifurcating the other point would make the torus a (hyperelliptic) genus two surface, although this is not done here; in this interpretation the independent $\alpha$ and $\beta$ cycles are correlated with one or two supersymmetries in the target space. The $S^1$ is then given a non-trivial fiber over the punctured and branch tori. This is trivial except for a gauge connection which gives monodromy around punctures of holomorphy 1, i.e. unramified poles or branch cuts. The fiber is specified by a pair of integers, or a complex number $n_R + in_R$. Alternatively the fiber can be seen as an $S^1$ over the hyperelliptic surface with the two independent homology cycles specifying the monopole gauge connections.

Two of the 11 points are resolved into two gauge independent circles of radius one. Radius one is invariant under T-duality, and there is no ambiguity. There is still an unramified point that is hidden because the circle closed on itself; imagine the point an $O(\epsilon)$ distance from the other end of the string and before closing add additional points. Place an additional three unramified points on each of the circles and branch them out after rotating one of the points so as to complete a torus with an unramified surface and a trivial fiber. The three unramified branch cuts can be extended to a genus three surface, which is elliptic by definition.
Figure 1: Elements of the moduli space. In the heterotic models an additional fiber would be added to represent the gauge degrees of freedom.

The previous describes the M-theory moduli space configuration, which accompanies the ten-dimensional component of spacetime. In the following the compactification of these dimensions is described.

The construction of the moduli space begins with 12 points on a unit $n$-sphere embedded in 13 dimensional space, with signature specified below. The space has thirteen dimensions as there is split between 11 dimensions and a torus fiber. These points are used as unramified points placed on the $n$-sphere after compactifying the plane, so there are now 13 unramified points. The points are chosen in a symmetric fashion so that they are equidistant from each other, and also in a manner so that a polynomial of degree 5 specifies the unramified surface. The rotation of the configuration on the $n$-sphere allows one term in polynomial, the fifth, to be dropped through a Poincare reduction (fix one point and rotate).

The degree 4 polynomial

\[ P(x) = \alpha_1 x^4 + \alpha_2 x^3 + \alpha_3 x^2 + \alpha_4 x + \alpha_5 , \tag{14} \]

specifies the unramified surface described by an $n$-sphere with a symmetric ordering of 13 points. This polynomial also describes a torus, but with the parameters $\alpha_1 = -1/\alpha_5$ satisfying the antipodal constraint. This follows from specifying $n = 5$
together with the resolution of a degree 5 quintic in an ambient 13-dimensional space of arbitrary signature with the property that the toric is quintic together with its hyperkahler structure, which is Pennington’s theorem. Thus the polynomial describes a torus after a Poincare resolution. One coordinate is free but used to resolve the torus into a five-sphere. Due to the Poincare map, there is one ramified point on the surface, with all of the previous 13 unramified points taken off the surface into the ambient 13 dimensional spacetime. The torus will be utilized to describe the torus of the $SL(2, Z) = SL(2, R)/Z$ and the remaining 13 points are used to resolve the moduli space except four which are not necessary. These four points are used to close the $R^4$ into a ball in a fiber-wise action, which is still a 4-sphere.

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There are five points remaining, and these are used to define the target spacetime theory. One is used to create an n-sphere by considering the n-sphere as an n+1-sphere and setting one coordinate to zero, or simply dropping it in a Poincare manner, in a recursive manner, so that the m-sphere is obtained. This is done for an $S^1$, an $S^2$, and an $S^3$ using three points. The fourth space is not as trivial through the sphere reduction.

The four-sphere is obtained by compactifying the four-dimensional flat geometry and adding one of the unramified points. A background background $F_5$ flux is added to the sphere to modify it to an anti-de Sitter signature. The point is added to
Figure 3: The picture shows the unramified extension of the elliptic curve into an elliptic Riemann surface of genus three. The pair are diametrically opposite as energy-coherence requires, for they are at rest in a stable minimum.

make the metric conformally flat instead of spatially flat; the latter is common in the literature. A point added changes the metric to conformally flat instead of spatially flat due to a hyperbolic resolvable singularity. This changes the curvature to round instead of flat for a given conformal class, and this results in the addition of five flux units thus changing the signature to round. The anti-de Sitter class is then changed to de Sitter, for the given amount of five-form flux. This breaks supersymmetry, but the scenario preserves modular invariance as the anomaly in the beta functions is still cancelled.

*VI. General String S-matrix and Effective Action*

The S-matrix of the general string is determined from three elements:

1) elements in the transfinite representation $L_{a,b}^\alpha=1$, with a possible Wick rotation

2) coefficients $\alpha_n$ multiplying the transfinite elements

3) modular functions containing the coupling dependence

The coefficients and modular functions in 2) and 3) multiply the operators that define the S-matrix. Only the analytic in derivative terms are considered, and their
form is

\[ S = \int \sqrt{g} \sum \alpha_n F_n(\tau, \bar{\tau}) O_n \]  \hspace{1cm} (16)

The non-analytic terms are also determined through both unitarity and special representations of the transfinite algebra. In fact, the computation of the string scattering to all genera and instanton number defines these special representations.

The representations to the string theories are listed in the table,

\[
\begin{pmatrix}
IIB & L_{1,1} & L'_{1,2} & L_{1,3} \\
IIA & L_{1,3} & L'_{1,4} & L_{1,2} \\
E_8 \times E_8 & L_{1,1} & L'_{1,1} & L_{1,4} \\
O(32)/Z_2 & L_{1,1} & L'_{1,4} & L_{4,1} \\
O_1(32)/Z_3 & L_{3,1} & L'_{3,3} & L'_{3,1} \\
O_2(32)/Z_4 & L_{3,3} & L_{3,1} & L'_{3,1} \\
O_3(32)/Z_8 & L_{3,1} & L'_{3,1} & L_{3,3}
\end{pmatrix}
\]  \hspace{1cm} (17)

or in a more natural notation,

\[
\begin{pmatrix}
IIB & L_{1,2} & L'_{1,3} & L_{1,4} \\
IIA & L_{3,1} & L'_{3,3} & L_{3,1} \\
E_8 \times E_8 & L_{1,1} & L'_{5,4} & L_{4,1} \\
O(32)/Z_2 & L_{1,1} & L'_{5,4} & L_{4,1} \\
O_1(32)/Z_3 & L_{3,1} & L'_{3,3} & L'_{3,1} \\
O_2(32)/Z_4 & L_{3,3} & L_{3,1} & L'_{3,1} \\
O_3(32)/Z_8 & L_{3,1} & L'_{3,1} & L_{3,3}
\end{pmatrix}
\]  \hspace{1cm} (18)

The representations and their particle content are further investigated in [42]. All of the representations are affine extensions \( L_{a,b}^{\alpha=1} \). The ghost sector, which contains one tachyon, is marked with a prime index. The left, right representations of \( L_{a,b} \) are a gauge multiplet and a set of supersymmetric partners. The elements in the representations must be supersymmetrized to fill out the entire string spectrum; after supersymmetrization the full set of operators is obtained, and completely describes the terms in the S-matrix. The last step requires summing over the tower of inequivalent, like Verma modules, field contents related to the primary ones listed above; the inequivalent ones span the numbers \( a \) which differ from those above by multiples of 3. \( b \) is a fixed parameter, which depends on the background and its fluctuations or spatial-temporal dependence.
The matter content of the theories must include the representations,

\[
E_8 \times E_8 \quad 3 \star 28 \star L_{3,1}^{\alpha' = 1/2} \\
O(32)/Z_2 \quad 3 \star 28 \star L_{3,1}^{\alpha' = 1/4} \\
O_1(32)/Z_3 \quad L_{3,1}^{\alpha' = 1/8} \\
O_2(32)/Z_4 \quad L_{3,1}^{\alpha' = 1/16} \\
O_3(32)/Z_8 \quad L_{3,1}^{\alpha' = 1/32},
\]

with coefficient \( \alpha_n = 1 \). The root lattices and the necessary fluxes are described in the following sections.

The corresponding roots associated to the elements in \( L_{3,1}^{\alpha' = 1} \) are,

\[
\begin{pmatrix}
IIB & 1, \sqrt{2}, \sqrt{2} & 1, 3, 3 \\
IIA & 1, \sqrt{2}, \sqrt{2} & 1, 2(4), 2(1/4) \\
E_8 \times E_8 & 1, 1, 1 & 2, 1(1), 1(5/2) \\
O(32)/Z_2 & 1, 1, 1 & 3, 1(1), 1(3) \\
\text{TypeI} & 3, 4, 6 & 3, 4(2) \\
O_1(32)/Z_2 & 1, 1, 1 & 3, 3(4), 5(1) \\
O_2(32)/Z_2 & 1, 1, 1 & 3, 5(1), 4(1) \\
O_3(32)/Z_2 & 1, 1, 1
\end{pmatrix}
\]

The \( \sqrt{2} \) numbers can be changed to 1 with the following modifications: \( E_8 \times E_8 \) has \((1,2,1)\), \( O(32)/Z_2 \) has \((1,2,1)\), typeI has \((1,2,1)\) and the rest \((1,2,1)\) and the set pertains to \( \alpha = 1/2 \) in the affine extension. These coefficients are stored in the roots \( \alpha_{n,b}^{a,b} \). The numbers in parenthesis are for spatially varying backgrounds, which could be a microscopic effect due to wormholes, or black holes, for example.

Apart from these coefficients the functions \( \mathcal{O}_s \) are required. These functions are determined in the string setting by an operator equation on the appropriate piece of the M-theory moduli space. In the case of the IIB superstring, for example, this operator is

\[
\Delta_{\tau,\bar{\tau}} f = \lambda_s f + \prod E_{s_i}(\tau, \bar{\tau}),
\]

for an index \( s \) partitioned into \( s = \sum s_i \), \( s_i \geq 3/2 \) with \( s_i \) an integer multiple of one half. The solution to this equation describes the functions \( \mathcal{O}_s \) in (11); the appropriate coefficient \( \alpha_{n,b}^{a,b} \) multiplies the function, which then multiplies the element in the
The different string theories have different coefficients, and the latter enter into the S-matrix through \( \Pi \).

The modular functions for the various string theories are found from the differential equations, or maps,

\[
\begin{align*}
\text{IIB} & \quad \Delta_{SL(2,R)} f_s = \lambda_s f_s + \sum_{\sigma} \prod_{E_s(\sigma)} f_{s\sigma(j)} \\
\text{IIA} & \quad \Delta_{SL(2,R)} f_s = \lambda_s f_s + \sum_{\sigma} \prod_{E_s(\sigma)} f_{s\sigma(j)} \\
E_8 \times E_8 & \\
\Delta_{SL(2,R)} f_s + \Delta_{\Gamma(E_8 \times E_8)} f_s = & \quad \left[ s(s-1) + s + \frac{3}{2} + \phi \sigma \right] f + \sum_{s\sigma} \prod_{E_s(\sigma)} f_{s\sigma(j)} - \delta_{1,s} \ln E_1 (24) \\
+ & \quad \sum_{s\sigma} \prod_{E_s(\sigma)} E_{s\sigma(\psi)} - \delta_{w,1} \ln^2 E_1^s - h(\tau_2) (25) \\
O(32)/Z_2 & \\
\Delta_{SL(2,R)} f_s + \Delta_{\Gamma(O(32))} f_s = & \quad \left[ s(s-1) + s + \frac{3}{2} + \phi \sigma \right] f_s + \sum_{s\sigma} \prod_{E_s(\sigma)} E_{s\sigma(j)} (26) \\
+ & \quad \delta_{0,1} \ln E_1(\tau, \bar{\tau}) + \sum_{s\sigma} \prod_{E_s(\sigma)} E_{s\sigma(j)}^{Spin(32)} - \delta_{w,1} \ln^2 E_1^{Spin(32)} - h(\tau_2) (27) \\
I & \quad \Delta f_s = \lambda^I f_s + \sum_{\sigma} \prod_{E_s(\sigma)} f_{s\sigma(j)} + \sum_{\pm} \tilde{\lambda}_I f_s f_{s\pm 1/2} . (28)
\end{align*}
\]

The \( \Delta \) Laplacian is invariant under \( \tau \to \tau + 1 \) and \( \tau \to -1/\tau \); those \( \Delta_{\Gamma(G)} \) are invariant under one the lattice groups \( E_8 \times E_8 \) and \( O(32) \). The eigenvalues \( \lambda_s \) depend on the model. The derivation of the differential equations are deduced in the following sections and require deforming the moduli space.

\textbf{VII. IIB S-matrix and Effective Action}

The S-matrix is found by summing all of the transfinite representations with the appropriate coefficients, that is the modular function and the root number,

\[
\sum \alpha_n f_n(\tau, \bar{\tau}) O_n . (29)
\]
\( \mathcal{O} \) is an element of a representation; \( f_n \) is the modular function, and the number \( \alpha_n \) is the root label of the transfinite representation. The content is

\[
L_{1,2}^\alpha, L_{1,3}'^\alpha, L_{1,4}^\alpha, 1, \sqrt{2}, \sqrt{2},
\]

(30)

together with the functions \( f_s \) which satisfy the differential equation in (22).

**VIII. IIA Superstring Theory**

The IIA theory is obtained from IIB by T-duality. Shrink the unit \( S^1 \) to zero radius, causing a conical point. T-dualize by using an S-duality transformation followed by a switch of the following transfinite representations in (18),

\[
L_{1,2} \leftrightarrow L_{1,3}^\prime, L_{1,4} \leftrightarrow L_{3,1}^\prime, L_{3,2} \leftrightarrow L_{3,4},
\]

(31)
in that order; the multiplets are the gravity, gauge, and matter multiplets. The point is blown back up to the unit \( S^1 \). If the circle is not blown down to a point then it is involuted to a circle with radius \( \alpha'/R \), i.e. the T-dual.

Then the previous modular functions \( \mathcal{E}_{s}^{IIB} \) are replaced to match the appropriate holomorphcity; the functions in IIA are the same as in IIB. The fact that these basis functions are identical can be seen from their definition, the differential equations (22)-(23).

The S-matrix is built from

\[
\sum \alpha_n f_n \mathcal{O}_n,
\]

(32)

from the representations and coefficients

\[
L_{1,1}^\alpha, L_{2,3}'^\alpha, L_{3,1}'^\alpha, 1, \sqrt{2}, \sqrt{2},
\]

(33)

and the functions \( f_s \) which satisfy the differential equation in (23). The inequivalent descendants of these representations are included in the sum, with their weights.

**IX. \( E_8 \times E_8 \) and \( SO(32) \) Heterotic Theories**
The quantum heterotic theory is obtained through the following steps. Blow down the $S^1$ and $T^2$, i.e. the $S^3$ Hopf fibration, to a double pole. The fiber structure holds together the configuration even when not a double pole from infinite expansion due to the presence of matter on the $S^3$. The origin of the matter is due primarily from real states in the ten dimensions made from the fiber bundle of n-spheres; the matter has a small but zero probability of quantum tunneling onto the $S^3$, and most likely the largest sphere with the most matter makes the largest contribution, but this depends on the relative position of the 3-sphere with the n-spheres and also with the two genus three hypersurfaces. (General comments about resolving singularities in the heterotic theory are found in [48, 58, 50, 59].)

The holomorphic (left) half of the IIB ten dimensions should be replaced with the holomorphic half of the bosonic string, with a 16-dimensional Narain lattice in the $E_8 \times E_8$ configuration. The $S^3$ still has a two unramified simple poles. Open the four n-spheres, from $n = 1, \ldots, 4$, by moving the four unramified points on the four n-spheres to the $S^3$, one at a time, using the unramified pole located at the origin of the $S^2$. There are five unramified poles on the $S^2$ and one on the fiber $S^1$. To complete the process, use the previous procedure to move the ramified pole at the origin of the now punctured $S^4$ to the $S^2$, making a total of seven poles in total in the $S^3$ fiber bundle.

Take the elliptic hypersurfaces, and by pinching the holes, change one into an $S^2$ with three unramified points (which were created into three branch cuts pinned by two poles of order a half). Then the $S^3$ can be extended with another fiber $S^2 \times S^2$ over $S^2$. The bundle is an $S^7$, which will admit an Poincare reduction to an $S^3$ by setting the $S^2$ coordinates to zero. There are now 12 unramified zeros which may be removed if the fiber is piecewise connected between one $S^2$ and the first $S^2$; it is, due to a homotopy condition of Ricci-flatness which is imposed and simple to implement. Consider traversing a closed path from the first 2-sphere to the second 2-sphere; this path is homotopy equivalent to a straight line, and as such, the same thing can be done in the inverse. Twisting these has no effect, but lifting them to the rest of the bundle could have an effect. The homotopy of any fiber bundle $S^3$ is zero, so that there is no obstruction to the assertion of trivial first homotopy.

Next, three points are removed and used to make an elliptic genus three hypersurface, as done in the moduli space construction. Note that there are 9 unramified points left in the 7-sphere bundle and one fixed point. Alternatively, the 12 poles are kept and to ensure that the $Z_7/7$ survives in the circle fiber over the 2-sphere when lifted over the $S^2 \otimes S^2$ the Donaldson-Yau invariant of the fiber bundle must remain as is (e.g. $s=400,t=40$).
In sum there are two moduli space configurations. The first is comprised of the 
$S^3$ and two elliptic genus three hypersurfaces. The second is from the elliptic surface 
and the $S^7$. Each elliptic surface is described by a curve,

$$P(x) = \sum_{i=8}^{i=8} a_i x^i.$$  

(34)

An involution to an 8-sphere is manifest; the round one has a $Z_8/8$ symmetry which 
becomes $Z_7/7$ when one of the coordinates is set to vanish. Generally there are only 
28 complex structures on the round 8-sphere \cite{K}, and the previous sphere specifies 
only 5 of them. Choose one that is compatible with duality as follows. Take the 
round 2-sphere and model it with a NUT and 2 parameters. This is because the 
elliptic surface really has an unramified point that has not become ramified yet; the 
initial curve is not exactly closed when those unramified points where placing onto 
the Riemann surface. However, before closing the torus, which is accomplished by 
pushing one of the poles onto the end of the string, a single Dehn twist is performed. 
This causes the blown up sphere, after unpinching the hole of the torus and placing 
the unramified point on the sphere, to have a NUT with charge one.

Having pinched the torus into the sphere, borrow an unramified point and put 
in the center and blow down to remake a torus; Only this torus has a NUT. Push 
the NUT into the origin where there are two poles on top of each other. The NUT 
is resolved into a 2-sphere with antipodal points identified, which are pushed away 
from the origin so that the solid angle subtended is less than $4\pi^2$. The fiber product 
$S^2 \otimes S^2$ is used to parameterize the complex structures of the round 7-sphere. There 
are four parameters and a radius in the fiber product.

The round 7-sphere is used as a guide in constructing the exact differential that 
defines the modular functions used in the S-matrix. The complex structure must 
be specified, and it relates to the $E_8 \times E_8$ complex moduli that enters into its fiber 
structure over the $S^3$ base. However, the complex moduli are not required for the 
scattering directly, but are required in order to specify which branch and the pole 
location of the differential operator that is used to determine the level $s$ functions, 
such as the $E_s$ ones in IIB superstring theory. Choose a direction in the bundle 
$S^2 \otimes S^2$ along with a radius and call it $v$; its norm is one due to it being defined 
on a direct product rather than on a bundle. Choose another direction in the fiber 
product and denote it by $w$; its norm is not one and is seemingly equivalent to $v$ despite 
an additional parameter in $w$, but the latter is projectively equivalent. This is clear 
from the topological index being the same, i.e. by Riemann-Roch evaluated at the
origin of the root lattice, the complex structure has a singularity everywhere but there
due to the vanishing of the radius. The vanishing signals that the complex moduli
jump, but they don't. The Riemann-Roch identity states that the moduli space will
jump one unit if there is a singularity or branch cut in the complex moduli plane,
but only if it cannot be removed to another location [11]. The space has a branch
from zero on out to infinity due to monodromy of a half, which is apparent in the
$S^2 \times S^2$ if the unramified point is split and mapped to a curve in the complex structure
moduli plane. Because the point is complex and also unramified, the loophole in the
Riemann-Roch identity is guaranteed to work. The topological index is the same as
the fixed radius to a unit $S^2 \otimes S^2$, and the number of free moduli is four, including
the fixed radii which are not allowed to vary.

The number of parameters in the differential equation require the specification of
the $SL(2, Z)$ coupling $\tau$ and $\bar{\tau}$, the holomorphic prepotential of the fiber and requires
its complex moduli to specify without ambiguity, and the lattice of the heterotic
model. Consider the following: Take a sphere and rotate it by one unit. Mark
the point and rotate back; then rotate again by another unit. Rotate back and
continue until the set of points fills out a lattice of points such as the one which
$E_8 \times E_8$ is described by, but in curvilinear planar coordinates. Two patches are
required to describe the set of data; however, only one patch is required to describe
the set of points as the ramified point at infinity does not have any data in its (small)
neighborhood. The patch with the lattice data can be mapped onto the 2-sphere fiber
in $S^2 \times S^2$, with the exception of a point that does not possess any data locally in its
neighborhood.

The lattice data describes the complex structure with a non-trivial mapping of
a unit disc onto the 2-sphere. Delete the origin point of the disc and map it onto
the unit 2-sphere; the anti-podal map defines complex conjugation on a section of a
$L^2$ line bundle on a twistor fibration. The twistor fibration is required to specify a
line bundle with the lattice data that can be used to construct a holomorphic field
with the complex data that is inserted into the differential map, in order to define
the modular functions. Consider the example of a 2-sphere with a line mapped to a
region; it maps out a line as it moves around, and hence fills a region. However it is
spinning and some of the time it is multiply covering previously covered areas in its
region; this is a problem if the area it sweeps has to be calculated. Consider the case
of uniform rotation and take a pullback to the disk of its rotation and covered data;
data is black and its covered region is blue, and the lattice points are green with a
white background. Take the covered region and map it to a second disk, together with
the lattice data; the twistor information is contained in the set of lattice points that
are mapped onto the secondary disc. The secondary disc with the lattice locations are then used in a twistor manner to find the time evolution of the spinning/nutating line segment.

In the heterotic case, the unit disc not only contains the lattice data but also potential toric information about the blow-up of a Calabi-Yau that the lattice could be a fiber over, in the ten-dimensional portion of the base space. This is useful, as the complex line is now a curve of cohomogeneity two in the base, but is a line in the fiber of $S^2 \times S^2$ using only one dimension of the base 2-sphere. The fiber $S^2 \times S^2$ over the base then describes the time evolution of the base manifold, including its complex structure which is what we want to find. The latter can be found as a projection of this multiply wound 2-sphere fiber on the base to the real line via a map called the Hurwitz construction, which this is partly based on. The Hurwitz construction is too simple when compared with the above; however, it has the same feature in determination of a cohomogenous curve from the fiber bundle as the Hurwitz map. The Hurwitz map is based on a twistor bundle and a line, and the one adopted here is based on an $S^3$ fiber, a curve of cohomogeneity one with respect to the $M_{\text{base}} \times S^3$ multiplying the line element and cohomogeniety with respect to the latter sphere cross line, together with a twistor space describing the time evolution of the cohomegenous one curve.

The twistor space allows us to create a function, given the appropriate twistor and lattice data, that reconstructs the fiber if the base manifold and its complex structure is known. Consider a flat ten-dimensional space, with trivial complex structure. The curve is not a three-manifold but rather a one-dimensional line element by Hodge duality. The Hodge property follows from extending the $S^1$ in the sphere product, picking up the curve in the complex plane where the data on the 2-sphere has been stored, Hodge-dualizing to a point and then resolving with the aid of the Hurwitz-like moment map; the $S^1$ then shrinks back to its former self. Else the $S^2$ can be extended as in the $S^1$ case, and this allows for a four-dimensional Hodge action to be performed on the 3-volume. The $S^2$ can now be shrunk to its former self, with one exception; this is that its conformal class has changed by one unit requiring another unit of five-form flux to be added. Also, the conformal class might induce scattering of five-form and gravity with the $E_8$ vectors in a non-trivial fashion because it requires instanton-like modes in the gravity sector. The instantons are in the ground ring of the $(0,0)$ model and their excitations are exponentially suppressed small $E_8$ instantons at strong coupling [51]-[56].

The Hurwitz-like map can be solved for using the linearization of the metric data of the map from the $S^2$ to the complex line. This map, when applied to a sample
propagating string, has only two excitations, longitudinal and transverse, and it obeys a field Monge-Ampere equation in the manner

$$\partial_{\tau} \partial_{\bar{\tau}} \sigma + \phi \sigma + \phi^* \sigma = 0.$$ (35)

The differential equation that the modular functions obey may be found directly by reading it off of the fiber bundle structure. The $S^1$ which is fibered over the $S^2$ is also fibered over each of the elliptic genus three hypersurfaces.

The $S^2$ is also fibered over each of the two hypersurfaces, and the fiber does not entangle with the $S^1$ fiber. This requires the following three conditions. The Chern class of the $S^2$ fiber is trivial, causing the $SL(2, \mathbb{Z})$ invariance to be trivially dependent on the auxiliary Riemann surface. The fiber should be called auxiliary as its sole function is to branch the $SL(2, \mathbb{Z})$ over the elliptic hypersurfaces; this causes the non-trivial fiber of the $E_8$ lattice over the elliptic genus three hypersurfaces to have non-trivial monodromy around the $SL(2, \mathbb{Z})$ and is not trivial. Last, the $S^2$ over the elliptic genus three hypersurfaces can be shrunk to a line, as the fiber is trivial, and also stable because only two unramified points are used in its overall construction, with one staying at the base of the fiber, and 4 are allowed in case points wander on it. The latter is important in the phenomenology as different scenarios require various placement of points and depending on the placements the elements in the fibered moduli space might be unstable to decay, with a 'pushing' and 'pulling' unramified points or the elements might explode.

Each elliptic hypersurface has an $E_8$ fiber on it. The same $E_8$ is also fibered over both the $S^1$, and for compact support to ensure the fibers don't 'break', it is fibered over $S_2$. The configuration is also relevant to mass generation of the fermions and is discussed in the section on phenomenology. The total fiber bundle, including the elements $S^1, S^2, M_{g=3}^{d=2}, E_{8}^{(1)},$ and $E_{8}^{(2)}$ are collected in a network of fiber structures; also, of the six in the list, the second through sixth items can be made holomorphic or anti-holomorphic with three 3 items in each class. The holomorphy is useful in that if the number of handles or genus is required to change, then it can be doubled from the lowest amount by doubling the curve into holomorphic and anti-holomorphic curves. The doubling can be used in mass generation and its use is explained in the phenomenology section.

The various fibers produce the five contributions to a differential equation whose solution represents the perturbative and non-perturbative contributions to the heterotic superstring. The $S^2$ fiber of the $SL(2, \mathbb{Z})$ has the contribution,
\[ \Delta_{SL(2,R)/\mathbb{Z}} f_s = \lambda_s f_s, \]  

(36)

with \( \lambda_s = s \) because there is no zero of the fiber; it routes also through the genus three hypersurface and back to the \( S^2 \). If the Chern class of the two fibers are the same then the zeros of the fiber can be chosen to coincide \cite{48, 40, 41} (these references are throughout in the following). The chern class of the first one is 4 and that of the second one is 8; a unit of flux on the first fiber raises it to 5 and lowers the second one to 7 because the unramified point moved. Add one more flux unit and the pair of fibers have flux 6 and 6; the flux arrangement is a bit unstable because the points want to move around. Ramification of all the points is not advised because particle dynamics in the ten dimensional space require some freedom in their placement and movement. The points can be ramified but not generally; in this case nail down the two unramified points on the \( S^1 \) and \( S^2 \) with the ramification process \cite{48} which is not simple as the \( S^1 \) fiber is not trivial. The ramification points can be brought into the \( S^2 \) via a hole that is made when an unramified point is pinched off the closed genus 0 surface. This point has to traverse the \( S^1 \) fiber to be able to ramify the unramified point there. This is possible if the fiber has a certain amount of flux to make it stable to ramification; this is possible only if the flux is configured even temporarily or in the early universe so that the ramification proceeds in a stable manner. The ramification point is then returned to the genus three hypersurface and is increased to genus three. The other two points are ramified and unramified with the hole now closed, e.g. the job of the former.\footnote{Recall that these two points tell us if there is an anomaly in spacetime via an integral of the stress current and supercurrent, and is in section 2.} The flux configuration of 1 unit and 3 units on the \( S^1 \) and the \( S^2 \) fibers; in the holomorphic factorization of the fiber, useful in phenomenology, 1 unit of 2 units on the \( P_3 \) and the \( P_2 \) should be consistent with duality if the ramifications are chosen to absorb the additional \( P \) fiber. This is possible if the ramifications are chosen to coincide with the fibers’ poles. Then the \( P_3 \) and \( P_2 \) collapse back to the former \( S^3 \) and \( S_2 \) with the former being a Hopf fibration on the \( S^2 \).

Much of the previous analysis is addressing the point of whether the eigenvalue is \( s \) or \( s(s - 1/2) \). The latter is more natural, which compares with previous modular forms and in previous studies of duality of the Heterotic superstring. There are two points on the \( S^2 \), one of which unramified and now is ramified due to the second point. The two ramified poles on the fiber, which open singularities in the base, coincide with the two ramified points there and this is reflected in \( \lambda_s \) being a polynomial degree two; the singularity is brought to Jordan normal form and resubstituted into...
conjugacy class 3 when it becomes a normal function. This causes the singularity to develop an ankle, or a branch, which stabilizes it further; furthermore the fiber develops an infinite class singularity when there is a cusp form [18]. The fiber is now related by duality, \( s \) to \( 1 - s \) and vice versa; on the keyhole region of \( SL(2, \mathbb{Z}) \), the duality acts this way, and the eigenvalue \( \lambda_s \) is \( s(1 - s) \) up to a constant which can be normalized to one by tree-level perturbation theory.

The \( S^1 \) fiber contributes the term to right hand side of (36),

\[
\sum \prod E_s - \delta_{1,s} \ln E_1 .
\] (37)

The delta function is to cancel the divergence at \( s = 1 \). This contribution is there to make the singularity at the top of the \( S^1 \) bundle go away, and is caused by an unramified point that has become ramified by a little more surgery as before. Then the same function as used in the IIB superstring case, which is explained now, contributes; there is a singularity at the ramified point at the top of the \( S^1 \) fiber which is nullified by the product of Eisenstein functions which is symmetrized due to the permutation of the modular ring involved. Also the singularity at \( s = 1 \) is removed by adding the appropriate cancellation term, which is an anomalous function in \( SL(2, R)/\mathbb{Z}_2 \),

\[
\sum \prod E_{s(j)} - \delta_{1,s} \ln E_1 .
\] (38)

There is a product with \( s_j \geq 3/2 \) because the fiber can be piecewise be broken up into its smallest and less smallest units labeled by \( s = n/2 \) of positive sign; the permutation group is applied because the various combinations of the pieces is totally symmetric.

The \( S^1 \) also a combination due to the chirality of the theory, that may be represented by a chiral boson somewhere in the fiber. It has one fixed point, fixed under the action of chirality, and so may be represented on the section (i.e. the modular function) by a function contiguous, homologous to its branched covering, to the unramified point at the base of the \( S^3 \). The function is taken to be

\[
s + \frac{3}{2} + \phi \sigma ,
\] (39)

where \( s \) represents its branched cover, which is a pole of order 3/2 (this is discussed in the section in phenomenology) and is chiral because the branch of order 3/2 has an orientation. The function \( \phi \) is a holomorphic function that contributes to the metric function as
and has a holomorphic differential. Here $\partial_z \phi_1$ is $\phi - 1$, and its abelian differential of the first kind pulls out the left term on the right hand side of the above equation except for the singularity at the branch cut of the form

$$g_{zz} = \partial_z \phi_1 + \partial_{\bar{z}} \phi_2$$

(40)

because the contour is anti-clockwise and passes over the singularity in $P_1$. This is a branch around the pole in $\phi_2$, the form of which is justified by the presence of a zero in the region bounded by the contour $P_1$; this means the contribution of the pole cancels because the integrand is now anti-holomorphic and the contributon of the branch and the zero integrate to zero mod 1 (the loop is the butterfly configuration surrounding the zero once and the branch twice to give it monodromy 0 mod 2). The net contribution is that in (39). The function $\sigma$ is genus three function on the genus two surface, the torus of duality, which has to be present due to the following reasons: it is holomorphic, it has two branches due to a double fiber on the $S^2$, it is holomorphic in $s$ because each piece of the dissected fiber is holomorphic, it has an anomaly at one point because the chiral boson can pick up a phase angle around the $S_1$ (the five form flux does not couple and there is a deficit of charge due to a fixed point at the top of the $S^1$ fiber), there is a holomorphic 2-form on the 2-sphere coming from the projection of the chiral boson onto the 2-sphere and can be used to cancel the anomaly from the ramified point of degree 3/2 with a branch extending to another point on the $S^1$. This occurs because the two unramified (one previously ramified) are in their most ramified form according to. There a Calabi-Yau is presented that, in accordance with S and T duality, encompasses the $S^1$ and is tangential to the ten-dimensional spacetime. This construction reflects this Calabi-Yau though the presence of 3 not 2 elliptic genus three hypersurfaces; the branch of degree 3 is resolved with three NUTS because of the order $3 = 3/2 + 3/2$ into a generic but specified genus three elliptic hypersurface which can be put together with the other two into a ramified but with circle fiber (the eleventh dimension) Calabi-Yau. It is stable because it has a non-trivial fiber with the $S^2$ and the heterotic gauge space without any singularities and is Ricci-flat. The last reason is that the removable singularity on the $S^1$ that is unramified can be brought to the surface of the Calabi-Yau and the five-form flux cancels its presence.
This results in a contribution of the form (36) with the function $\sigma$ of the form of a Weierstrauss P-function on a genus three hypersurface that is not hyperelliptic. This function has the form when evaluated at one, thus changing the gauge fiber to a ramified one with a branch at the origin and a pole at zero, encompassing the entire Calabi-Yau with a curve of cohomogeneity three,

$$\sigma = \sum_{k=0}^{\infty} \frac{1}{k!} k^n \sigma_k ,$$

and

$$\sigma_k = P(z_k) \quad z_k = \tan(z - 2\pi k - \frac{1}{2}).$$

The argument of the Weierstrass function is the evaluation at all points of the divisor of the reduced hypersurface to a hyperelliptic curve of degree 5. The moduli are the coupling $\tau$ and its anti-holomorphic counterpart $\bar{\tau}$ when treated as a holomorphic counterpart due to the only moduli present. The Calabi-Yau moduli are fixed by requiring that the gauge bundle is even and stable over all points. A fixed moduli can not be used in the Weierstrass function as there must be singularities present reflecting the presence of the points on the which there ia a fiber in the base of the $S^2$; there are two points sitting on top of eachother and the moduli chosen are $\tau$ and $\bar{\tau}$.

The final contribution arises from the gauge degrees of freedom fibered over the two genus three elliptic surfaces, which have merged into a non-trivial Calabi-Yau. There are three branched points from the holes of the two surfaces, for a total of six; three of which close with the other three with branches of order $1/2$ (degree 2). The contribution is of the same form as the $SL(2,\mathbb{Z})$ invariant function with the $\tau$ parameter holomorphic and the anti-holomorphic $\bar{\tau}$ treated as a holomorphic extension because both are needed to specify the anomaly which was cancelled (e.g. the cancellation and contribution of the Eisenstein contributions from the two genus three hypersurfaces and the $S^1$). There are no singularities on the fiber but there are three branches when represented on the complex plane. This suggests that the function contributing has the form,

$$\sum \prod E_{w}(j) E_{w}(j) - \delta_{w,1} \ln^2 E_{1}^8 - h(\tau_2) ,$$

with

25
\[ h = \sum_{i=-N}^{N} \tau_2^i/2. \]

The \( E_8 \) functions are defined by the root lattice, and the permutation is performed as before the fiber can be holomorphically split into sub-units of weight number \( (w_j, w_j) = (n/2, n/2) \) which are then permuted. The zero mode, or rather the anomalous contribution is then subtracted as before.

The function \( h(\tau_2) \) represents the contribution of a single degree of freedom which for sake of clarity we call a ghost mode. It exists simply because we require a polynomial to ramify over; the ramified point’s position and type is generated by the polynomial. In this case there are three branch points requiring six degrees of freedom per point on the Calabi-Yau for a total of 36. Two more specify the orientation of the three branches, as the third is oriented diametrically opposite to one of them so the homology makes sense in \( Sp(3 \times 2, \mathbb{Z}) \). The number \( N \) is then chosen to be \( N = 19, 2N = 38 \). The anomaly cancellation of the \( E_8 \times E_8 \) fiber requires 15 units of five-form flux to be placed on the 6 points evenly distributed over the points and branches. One more is place on the Calabi-Yau without structure of the points and branches so that the \( U(1) \) degree of freedom is cancelled there; this occurs when supersymmetry is broken. Without supersymmetry breaking the ghost mode cancels and there is an anomaly, so that a tachyon appears and is consistent with the theory. As a result, another transfinite representation is included to cancel the modes’ oscillation and the contribution from \( h \). The tachyon may or may not exist due to incomplete or complete cancellation of the function \( h \). The coefficients of the function \( h \) are taken to unity by moving the points and branches inside the Calabi-Yau before ramification. Notice that the contribution of the tachyon exists outside of perturbation theory as there are contributions with positive powers of \( \tau_2 \), and it is S-duality invariant. To complete the form of the heterotic gauge bundle, take \( w = s \).

There are several scenarios that change the flux organization; this is useful for compactifications. First, instead of flux put on the \( S^3 \), put it on the \( S^2 \) and arrange for the \( S^1 \) to be wrap a non-trivial 1-cycle rather be embedded in it through the ramified points. Second, instead of flux put on the \( S^1 \) fiber put it on the Calabi-Yau 3 fold around a non-trivial \( \pi_1 \) generator which could be ramified by one of the three oriented branch points; they are oriented because the resolution of a triple singularity requires that they have oriented directions. Third, the de Rham complex could be used to make the \( p \)-forms accept the flux; this is straightforward as the Calabi-Yau is made up of three fixed moduli Riemann surfaces with two non-trivial fibers. Four
fibers are redundant except for a branch cut due to having two fibers on the same surface which is resolved by adding more flux; more flux might destabilize the resolved the singular $R^4$ and make it unstable for cosmology which is the case and is discussed indirectly in the later section on phenomenology. The amount of flux is two units and only 4 more can be placed on the singular $R^4$; this would cause the singularity to collapse to a point and the big bang would occur far sooner than what we expect(ed). Some flux can be placed on the $E_8 \times E_8$ 16-dimensional holomorphic root space, but the total allowed strength can not exceed the first Chern class which means four units per $E_8$ branch. 8 units would compensate for the problem of the big bang occurring when it might occur(ed), but the extra degrees of freedom from 1 to 8 points might make gauge phenomenology difficult. Each flux unit produces a naive factor of a $U(1)$ which has to be broken by further modification of the compactification and the arrangement of flux, and generically 4 flux units on each branch are allowed. This scenario is not preferred from a gauge anomaly point of view, and also from a stability point of view as the closer to unstable the configuration is the more modification to a branched but non-singular $R^4$ becomes, and also the more the vacuum would produce particles such as gauge modes, ghosts, and/or tachyons.

The final issue is the modification of the index on the modular functions $E^{E_8 \times E_8}_w$. As observed in there is an obstruction to using $w = n/2$ as an index and the one $w = n/4$ or $w = n/5$ seems to be liked. There is an issue with embedding the Calabi-Yau in the manner done in which was not branched well. The paper has several branches emanating from the Calabi-Yau and based on five branched and unramified points at the top of the fiber. If they are ramified then the fiber would be too singular to accept compact support, or have functions exist as a result; only 2 or 3 ramifications are allowed as the Chern class is 7 for this fiber with an $S^2$ emanating from each branch or sub-fiber. A branched fiber requires three units, one each for the point, the branch (or a half) and the base point, and as a result only two or three of these branches can be held down by the ramifications. If they are not ramified, the fiber will presumably be unstable with the branches reverting back to their unbranched form. The Chern class contributes to the energy, and the minimal energy configuration is the previous.

There is a way to avoid the singularity. Branch the five points on a cylinder with two fiber points deleted or a cone with one fiber point deleted. The five points have base the cone and extend to the set of branch points on the Calabi-Yau. Two or one of the points have been resolved into spheres with the use of the fiber points, and they are ramified. Two more points are found diametrically opposite to the ramified points as the branch points have been allowed to pass through the holes and
ramified opposite to the holes; they’re not ramified until a base point together with its branch accepts one unit of flux, further flux of one unit is added each to the points opposite to the holes. The holes can now accept two more points, which make them immediately ramified as they are holes and no flux is necessary to make them stick. One more ramified point requires \( \frac{3}{2} \) or 2 units of flux to make it ramified anywhere on the cylinder or cone; the ramification of a point on a cylinder or a cone requires 1 less than than usual due to a gauge field present, which is turned on by the five-form wrapped around a non-trivial homology generator; the flux necessary is \( \frac{3}{2} \) for the ramified point in the Calabi-Yau and \( \frac{1}{2} \) unit for the base point at the top of the fiber, which is total not \( \frac{5}{2} \) or 3. The total flux is then 6 or 7, depending on whether a branched point requires \( \frac{1}{2} \) or 1 unit of flux to ramify\(^2\) Seven units of flux are not allowed on the fiber due to the Chern class of 3, which allows \( 2n + 1 \) for stability. Then this configuration is not allowed; however, if the other point of view of \( \frac{1}{2} \) per branch point is used, then the configuration is 6 and hence is meta-stable. The fiber is so close to unstable that it will probably cause particles to be emitted from the vacuum, such as gluons, \( E_8 \) fermions, and maybe tachyons. This would cause further damage to the fiber and it might decay.

The differential equation generating the modular functions is,

\[
\Delta_{SL(2,R)}f_s + \Delta_{\Gamma(E_8 \times E_8)}f_s = \left[s(s-1) + s + \frac{3}{2} + \phi \sigma \right]f + \sum \prod E_{s\sigma(j)} - \delta_{1,s} \ln E_1 \tag{46}
\]

\[
+ \sum \prod E_{w\sigma(j)} E_{s\sigma(j)} - \delta_{w,1} \ln^2 E_s^8 - h(\tau_2) \tag{47}
\]

with \( \sigma, \phi, \) and \( h \) defined earlier. \( E_8^w \) is an analog to the Eisenstein function on \( SL(2,R)/Z \). The contributions in (47) are from five (sub-)fibers.

The S-matrix is then generated by the sum,

\[
\sum \alpha_n f_n \mathcal{O}_n \tag{48}
\]

The representations and coefficients are

\[
L^{\alpha=1}_{4,1}, \quad L^{\alpha=1}_{4,4}, \quad L^{\alpha=1}_{4,4}, \quad 1, \quad 1, \quad 1, \quad , \tag{49}
\]

and the functions \( f_s \) which satisfy the differential equation in (47).

\(^2\)There is a discussion of this in [48]. The preference for 1 unit of flux is used in this work.
X. Type I and Type I’ Theories

In the type I theory the two unramified points on the $S^3$ are collected and placed at the origin of the $S^2$, thus producing a singularity of order two. The point at the origin of the $S^4$ in the compactification is pulled off and placed on the torus; the presence of the pole of order $(1, 1)$ signals a bolt $S^2/Z_2$ at the origin of $S^2$. Blowing up the singularity generates an additional $S^2$.

The $(1, 1)$ pole can be split into a pair of branch cuts with endpoints of the type, in pairs, $(\frac{1}{2}, \frac{1}{2})$. There are two orientations, $(1, 0)$ and $(0, 1)$, which require labeling the branch cuts with an orientation. The unramified double pole can be resolved into these branch cuts if the orientation is preserved. The branch cuts generate the double cover of the 2-sphere.

The fiber of the $S^1$ is specified by its behavior in the proximity of the two branch cuts on $S^2$, or rather, on $CP^1$. There are four contours generated the homology, of which three are independent. Call the line integrals of a 1-form around one of the branch cuts on the two sheets $\alpha_{\pm}$; the two sheets are labeled by $+$ and $–$. The line integral between two points of order $(0, 1/2)$ and $(0, 1/2)$ on different branch cuts is called $\beta_{\pm}$. Only three of the contour integrals are independent and $\beta_{–}$ is eliminated.

The integrals

$$\frac{1}{2} (\alpha_{+} + \alpha_{–})(\alpha_{+}, \alpha_{–}) \sim f_s(f_{s+1/2}, f_{s-1/2})$$

are in correspondence with the two functions $f_s f_{s\pm 1/2}$ with $s$ an integer multiple of a half. The gauge bundle is specified by integrals of the connection around the contours, and result in two integers $m, n$ given the $\alpha_{+}$ and $\beta_{+}$ contours. As long as these integers are non-vanishing the branch cuts can be moved around on the 2-sphere.

The differential operator specifying the modular functions is almost the same as in the IIB or IIA theory, but due to blowing up the bolt it is modified to,

$$\Delta_{SL(2, R)} f_s = \lambda_s f_s + \sum_{\sigma(j)} \prod E_{s_{\sigma}(j)} + \sum_{\pm} \tilde{\lambda}_s f_s f_{s \pm 1/2}.$$  

The sum $\pm$ represents the sum over the two independent contours used to label the fiber, the latter of which requires two half-integral numbers. The eigenvalues are $\tilde{\lambda}_s = 1/3$ and $\lambda_s = s(s - 1)$. Cusp forms with a residue at the tip of the keyhole region, at $\tau = 1/2 + i\sqrt{3}/2$, satisfy the differential equation $\Delta f_{cusp} = \lambda f_{cusp}$ (e.g. $s = 0$), and this normalizes the coefficient $\tilde{\lambda}$ to $1/3$. The eigenvalue $1/3$ is really a
normalization. For really large values of $s$ the first two terms appear to dominate due to the number of partitions of $s$ into a set of smaller numbers $s_i$; the fiber term $\sum f_s f_{s\pm 1/2}$ can be neglected; the $SL(2,R)/Z$ moduli space requires the eigenvalue $\lambda_s = s(s - 1)$.

The $S$-matrix is built from

$$\sum \alpha_n f_n O_n,$$

from the representations and coefficients

$$L^{a=1}, L^{a=1}, L^{a=1} 3, 4, 6,$$

and the functions $f_s$ which satisfy the differential equation in (51).

**XI. $SO(32)^{(n)}$ Heterotic Theories**

The $SO(32)$ superstring theory follows closely with one exception, the gauge bundle $E_8 \times E_8$ can be replaced with one of the four gauge groups: Spin(32), Spin(32)/$Z_4$, Spin(32)/$Z_8$ or Spin(32)/$Z_{16}$. The latter four can be exchanged for a lattice model once the points are identified on the fiber with a branch cut singularity extending from one point to another. The Spin(32)/$Z_8$ has two forms as the Cartan sub-algebra can be modded in several ways and two of them are inequivalent. They have rank 2, 4, 8, 16, and 32.

The inequivalent representation of the weight lattice is arranged to have the diametrically opposite Cartan generators in the Spin(32)/$Z_8$ algebra to be on a line or at an angle of 60 or 600 degrees; the latter is on the 6-fold copy of the realization of the weight lattice on the complex plane. The twisting of the gauge degrees of freedom require a specification of the five-form flux which is consistent with the remaining flux configuration and also modular invariance together with $S$- and $T$-duality. This is accomplished by turning on the five-form flux such that 2 units are applied to each resolved point on the gauge fiber and 2 units spread over the entire fiber. 2 units are applied to the resolved singularities so that the NUT that they contain can be converted into a $S^2$ sphere with a punctured hole; there is a branch cut emanating from the ramified point diametrically opposite representing the Cartan generator which passes through the hole to its modded partner Cartan generator. An additional
2 units of flux is all over the fiber so that it won't be unstable, as by Cheeger's theorem an equal amount of flux on ramified points together with some flux on the entire fiber can be stable only if the difference, i.e. the Cheeger quantity, is 6 or 12 on a branched two-dimensional surface; in this case there are 8 Cheeger numbers and 6 or 12 with a plus or minus sign are required for 16 dimensions. As there are 16 Cartan generators in the sub-algebra, the Cheeger number must be 6 or 12 mod \( n \), with \( n \) the number of ramified points. There are 32 ramified points and all are branched; thus a flux of 2 mod 32 is required. The first Chern class prefers 2 in order to be stable.

The embedding of the \( SO(32) \) algebra into \( Spin(32) \) allows for the possibility of a heterotic duality between the points and the algebra. There are sixteen resolved NUTs and one resolved bolt that each contain a \( S^2 \) of complex structure. One of which is preferred, which is the bolt. Take the bolt and cover it with a \( S^2 \) so that the hole is patched; this is accomplished with one unramified point used in the construction of the bolt resolution afterwards the bolt becomes ramified. The bolt can be rotated and its covered hole oriented to any other point in the fiber, using the Fubini-Study metric on the fiber base extended to act on the entire gauge bundle. Then there is an \( SO(30) \) action on the points due to a sequence of orderings together with a shift of the complex phase as directed for by the bolt; it is non-abelian due to the fact that the sphere is curved and the orderings of directings do not commute. This group fills out an \( E_8 \times E_8 \) model as the phase angles must fill out a representation of a maximally extended \( SU(2) \) which is either \( Spin(32) \) or \( E_8 \times E_8 \). The \( SU(2) \) is a 3-form which projects as a two-form on the \( S^2 \) but with the remaining component specifying one of the angles in the \( S^2 \); their is no redundancy as the radius has not been set to one. The actions of the rotations must match with the Cartan sub-algebra, which is non-trivial due to commutation relations; however, there is a subtlety in the construction as not all points exist on side of the bolt which could be anywhere. Fix it to be at the end of the gauge fiber. The spinning bolt can label the Cartan generators with a number and the remaining generators are resolved singularities with ramified points, which are all numbered.

The structure constants are defined to reflect the inherit curvature of the \( S^2 \) and the location of the generators, one of which is \( H_1 \) the preferred Cartan generator used to mod the Spin algebra by \( Z_p \). The \( Z_p \) modding reflects the generators by adding or subtracting \( 2\pi/p \times N \) to the rotation curve between one point and the other; \( N = 2 \) for 8 generators, \( N = 4 \) for 16 generators, \( N = 1 \) for 4 generators. Each curve sweeps out a trajectory in the \( S^2 \) with arclength equal to the phase angle with the modding \( \rho \rightarrow \rho \pm 2\pi/p \times N \), and \( N = 1 \) is special because it has no elliptic cover, i.e. a branch of the \( S^2 \) is not required to uniquely specify the path in case the curve
runs back on itself. The preferred generator $H_1$ is required not to do this. The $H_1$ requires a commutator with itself, and the line element could go around the sphere and come back to itself making a mod $p$ cover of the trivial traverse to a point and then traverse back. This construction is iterable, and the line elements form a group once the structure constants are defined. The line elements close an infinitesimal distance apart, proportional to the curvature of the 2-sphere.

The group is either $SO(32)/\mathbb{Z}_p$ or $E_8 \times E_8$ due to the following; the generators on the fiber are rank 16, and those of the line elements in the $S^2$ are 16 with the possibility that some duplicate or copy other elements’ actions. The group generators might roll in back of another group generator if the fiber is unstable, which appears not to be the case because the first Chern class is 0 mod $N$ on a manifold of dimension 16; this amount of charge is not normal and the bundle could unwrap [18]. The bundle may not only unwrap but rather it might dissipate if further electric charge enters, which means that it is not wanting of further charged particles like the electron, real or virtual. As the bundle is relatively stable, the gauge group is constructible and makes sense.

The gauge group must be $E_8 \times E_8$ and this is the Spin/$\mathbb{Z}_p$ theory. It is rank 16, is maximally extended from $SU(2)$, and it has a bilinear form of rank 2, inherited from the 3 form which can integrate on the $S^2$. The latter is degenerate over the patched hole on the $S^2$ as it doesn’t accept any more charge; it should be blown up into a surface that the 2-form could integrate over. The bundle is then an $S^4$ with two two-dimensional surfaces forming the direct product space without fiber; as such the product of the two-form can be used as an integration element and this cannot be a form as the infinitesimal differentials each appear twice but in an organized fashion. Each $SO(32)$ generator has a direction, if they don’t roll. First, if they roll, then the Cartan sub-algebra must be relabeled. Second, the set of 2-spheres might roll into an $E_8 \times E_8$ gauge group from the point of the bolt $S^4$, which still has a $S^2$ fiber; this is the basis for the duality. One can form an $SL(2, Z)$ invariant fiber shockwave that causes all the balls to move in particular directions to capture the $SO(32)/\mathbb{Z}_p$.

All of the balls are now well-ordered, but one of them, the Cartan generator has rolled to a new position behind another ball. This process occurs until all the balls are either behind or in front of their individual sets. The group depends on the configuration of the sets. In order for the gauge group to be maximally extended the balls must line up in an $E_8 \times E_8$ setting, which takes time without the specific shockwave applied at the base of the $S^2$ bolt; there is a temporary fiber from the $S^2$ of the $SL(2, Z)$ to the bolt and requires one unit of flux to pin down the ramified point at the top of the fiber. There is only one unit because the $S^2$ at the bottom of the
fiber requires only one unit and the gauge bundle is sufficiently rigid that no charge is necessary; here rigid means that it is close to unstable and naturally unramified points are broken and branched without need to ramify. The shockwave is instantaneous.

The configuration rolls into an $E_8 \times E_8$ configuration and T-duality allows one to unroll the balls for a moment to analyze their configuration. T-duality takes the complex structure of the bolt and exchanges it with the Kähler structure of the bolt, which is close to zero, and might even vanish. To keep it from vanishing a Taub-NUT singularity is added that makes the curvature far from zero; of course the singularity has to be added to cancel but for the moment it is present. To keep the singularity from almost vanishing, the bolt requires an almost complex extension, such as a Newijnhuis tensor without compact support. This makes the balls rotate backwards as time has no direction without support. If they rotate backwards then they will eventually roll back into their present position, and this can be facilitated by another quantum well such as that induced by a small and better focused shockwave. Then they are stuck in the heterotic position by applying five-form flux that rigidly holds the configuration. Thus T-duality has been allowed to map $SO(32)/\mathbb{Z}_p$ to $E_8 \times E_8$ by the flow of the balls from the exceptional heterotic to the orthogonal heterotic and then back by T-dualizing, and waiting for the balls to roll, to the $E_8 \times E_8$ theory. T-duality will work if the fact that the Kähler modulus and the complex structure modulus are interchanged is recognized; in this case, there are four allowed configurations: $Spin(32)/\mathbb{Z}_p$ with $\mathbb{Z}_p = 1, 2, 4, 8$. Because there is one hole and one singularity on the ball now, the ball sees two objects simultaneously one behind the other and is required to do so. That limits the group cases to the ones mentioned with the balls in each row related to each other by the quotient.

Now that the group action is identified, the balls must rotate into their respective position to fill out the vector or adjoint representation. In $E_8$ there is no adjoint but the vector takes its role, so that the adjoint may not be so trivial. The representation follows from symmetrizing the balls in their current position in each of the sets and calling the first ball their respective Cartan generator; the remaining balls in the set are descendents.

The fiber structure is almost the same between the heterotic orthogonal case and the exceptional heterotic theory, with one exception. Because there are now five theories, the lead candidate must be chosen to supervise the remaining candidate groups. This means that total supervision of the tower of heterotic orthogonal groups is under the control of the heterotic orthogonal $\mathbb{Z}_2$ group, and that once in this phase a simple transformation can alter the theory to any of its descendents.
The flux configuration is nearly identical in all of these theories with one exception. Instead of 2 fluxes on the gauge bundle there are 2 mod $N$ with $N$ being related to the Cheeger sum. That is, the sum of all the flux units on the gauge bundle must be equal to 2 modulus the sum of all the flux units on the unramified points to make them stick so that their combination mod $N$, the number of flux units on the points, is equal to zero. See the section above for further details. This is the primary distinction in the model(s) from the exceptional heterotic case. Another distinction is the amount of flux on the fibered Calabi-Yau; in this case, as there is fewer flux on the gauge bundle the flux can be increased in two ways on the 3-fold. One attach it to points on the manifold with branched covers, which have been glued and resolved. Second, attach it to the overall manifold without increasing the sum beyond the Chern class. Further alternatives are not mentioned. Third, a single point can still be added to the $S^2$ in the $S^3$ fiber bundle; this breaks S-duality as the quotient is now $SL(2, R)/Z_2$ which is not a group but rather a point-like object. The singularity can now accept four to eight more units of five-form flux without destroying the black hole singularity at the origin of the $R^4$ space. This could be useful for model building.

XII. Compactification and Supersymmetry Breaking

The compactifications considered here go against lore in one aspect; primarily anti-de Sitter spacetimes are considered, but locally de Sitter ones are presented that presumably model our neighborhood and local galaxies. Cosmological data cannot distinguish a redshift with positive or negative cosmological constant if the blueshift is treated as the microwave background anisotropy. This scenario fits with rings of compression separated by hundreds of millions of lightyears and as a shockwave of sorts in the early universe, near the time of big bang nucleosynthesis. The latter assertion would explain why there is little signature of the blueshifted matter and would be found primarily in the CMB dust. Also, our galactic halo could be considered redshifted if the compression wave entered our galaxy sooner than expected, for example in a few hundred million years with an additional compression wave explaining some of the blue-shift that data has observed in the halo; alternatively a compression wave may have entered our galaxy later such as in the billions of light-years. It is difficult to use current data to indicate either scenario, the given above or the one with a matter dominated universe with uniform expansion rate and constant cosmological. The expansion rate is consistent with the Hubble’s constant if the compression waves are placed apart and have the right movement outward from our point of view, and if Moore’s law is obeyed.
XIII. IIB Phenomenology

Remove the point from the $S^2$, after moving to the $S^1$. The point then becomes ramified at the origin of the $S^1$ due to the non-trivial fiber. This point can be unramified by taking in a point at infinity and then sending it back. The unramified point on the $S^1$ is used to open the $S^1$, then moved to the a four-dimensional subspace of the ten coordinates. Alternatively, a point on one of the elliptic surfaces can be unramified by pinching it off of one of the closed holes and used to unramify the one on the circle fiber. The fiber trivializes as the genus drops from genus 3 to genus 2; if not trivial then the 3-form can be used to make the fiber almost trivial, and giving it a vacuum expectation value. The vacuum expectation is going to modified anyway and it is assumed to be not the right value for phenomenology.

The point is taken off the end of the line segment and placed on a four-dimensional subspace in the ten dimensions. Because it placed by hand it is ramified, and can be blown up as a NUT of degree 1 into an anti-de Sitter space.

The 3-form is taken off the $S^1$ fiber where it was used to trivialize the fiber and moved onto the ten-dimensional space. The resolution of the NUT singularity requires five units of flux, which gives the appropriate number to cancel the target space gravitational beta function and restore modular invariance.

There are various scenarios in the following. Close the remaining five coordinates so that a $S^3$, $S^2$, and $S^1$ are made each with a point removed. First construct an $S^3$ fiber over the $S^2$, with the two singularities moved on top of each other; $\pi_3(S^2)$ is the set of integers and conformal class of one is required for modular invariance due to the absence of non-integer spins. One point is ramified due to the non-trivial fiber and one is unramified, both of which are located at the base of $S^2$. Then treat the two $S^1$ circles equally which means fiber both of them over the $S^3 \times S^2$ bundle; $\pi_1(S^3 \times S^2)$ is the set of integers by Kunneth's formula, including the resolution of the double pole at the origin. The unramified hole is on top of the ramified hole and both are resolved into a crosscap with a double charge located at its antipodal points. $\pi_2$ of the crosscap is the set of even integers and contributes not to the Kunneth formula except for a double charge divided by two. The crosscap is then a part of the $S^3$ fiber and can be closed by the addition of another two blown up points.

There are two $S_1$ fibers over the $S_3 \times S^2$ bundle. The fibers are each specified by an integer as the $S^3$ fiber is specified by a complex number, one for the Chern class and one for the conformal class evaluated on the tangent bundle for the latter. The flux is chosen to cancel that on the anti-de Sitter space. Two five forms can be
taken together as $F_5 \wedge F_5$ and given the vacuum expectation values. Indeed, if a triple product is taken, and the singularity on one of the genus 3 elliptic hypersurfaces is blown up appropriately, then the third form can be used to compensate for the flux in all directions so that they add to zero; this occurs if these two hypersurfaces, the $S^1$ fiber in the eleventh direction, and the first ten dimensions are taken as a product, this gives a manifold interpretation to the presence of the flux and its 15-form. The fluxes are chosen to agree with phenomenology.

In a different scenario, the amount of flux used to alter the bolt at the origin of $R^4$ can be altered by adding and subtracting no more than two units as the bolt can withstand only these values before requiring the additional resolution of a new singularity; the time scale of the instability is presumed shorter than the age of the universe. The remaining flux is spread among the 6 compactified dimensions with flux: one unit on the circle, two on the genus three elliptic hypersurface, and three on the $S^3 \times S^2$. This requires four units on the bolt located at the origin of the $R^4$ which is now resolved in a de Sitter space.

Another scenario is 4 on the elliptic hypersurface, 2 on the circle, and 4 on the bolt. Another is 4 on the elliptic hypersurface, 2 on the circle, and 4 on the bolt except one unit is stolen from the $S^1$ fiber with a puncture so as to make a total of 5 on the bolt; this is anti-de Sitter but in order to flip the sign of the 5 ten units are given to the $S^1$ branch singularity which it now becomes due to the large number of flux units and its instability (its instability is localized to a branch singularity which is the most general form of a point-like singularity of degree one).

(One more noteworthy solution is the ramification of the singularity caused by the ten units of flux and its placement on the 2-sphere which fills its hole, else it can be used to partially compensate for the pinched hole on the elliptic hypersurfaces. The presence of the point on the hypersurface might cause the entire Riemann surface to become unstable after blowing up the pinch to genus three. If it becomes unstable then it might pinch the $S^2$ fiber above the ten dimensional space to a point (which is zero coupling) and then also collapse the remaining genus 3 elliptic hypersurface. This continues until only the four-dimensional space survives, which will provide the solution is anti-de Sitter, and causes it become a de Sitter space with 4 or 5 units of flux. This scenario has a large order pole at the origin of spacetime.)

The presence of holes in the $S^1$ and and a required bolt filling-in in the $S^3 \times S^2$ fiber together with the application of the bolt resolution requires three points to be brought in from infinity and one returned. Depending on the accuracy of the phenomenology this scenario could be physical. Two points could also be effectively
removed from the genus three elliptic hypersurfaces. The stability of the configuration might be drastically altered, and is not preferred, as discussed in the section addressing this matter and its signature in cosmological data.

XIV. Heterotic Phenomenology

The configuration of the IIB superstring allowed for two supersymmetry breakings, either to (0, 0) or (0, 1). The heterotic breaking is identical to the IIB superstring, with the exception that two points (one ramified and one unramified) are required for a pole of order 6 and only one ramified point is required for a pole of order 8 [48]; these are located at the origin of the $R^4$. A higher degree singularity with unramified points are less singular than expected because the resolution is facilitated by their movability. The number of allowed flux units varies, from 1 to 4 in the former and 1 to 9 in the latter, leading to a wider class of supersymmetry breakings. M-theory suggests that the flux configuration would remain close to the same in the Heterotic phase as in the IIB superstring phase. A computation of the groundstate energy of the vacuum is required to determine the rate and possibility of a different string phase is one corner of the universe.

The mass and the couplings can be found by analyzing the singularity at the origin of the $R^4$. Examine the resolution of the heterotic pole at the origin. A NUT of type 1 is blown up into a $S^2$ with two points removed, suggesting a cylinder between them, i.e. $S^1 \times S^1$. The full blow up requires three points to be consistent with duality in a holomorphic setting; this results in $S^1$ times $S^1 \times S^1$ with a complex coordinate spanning the latter, a fiber of Chern class 3 as there are two unresolved blow up points and a ramified point at the origin of the former $S_1$. The ramification may not be removed as it as located at the origin and the fiber is non-trivial; more points could be brought in from infinity but this is not available as in the superstring case only one point from infinity was required and for consistency this condition is unaltered.

In [46] the masses of the known particles were found to high accuracy, satisfying the mass formula [47],

$$\Lambda^\frac{n}{16}\left[1 + 2i5^j + \left(\frac{\Lambda}{m_{pl}}\right)^{-3} + \ldots\right],$$

which is accurate to 1 fits the pattern with another suppression of $(\Lambda/m_{pl})$. For some
reason there are 27 combinations of numbers $2^i 5^j$ between 1 and 1000 with the scales 1, 10, 100, 1000.

The masses can be obtained in accord with the mass formula in (54). Take a bilinear fermion pair and move it onto a holomorphic $S^1$ fiber after passing the initial $S^1$; it completes a loop via the anti-holomorphic $S^1$ fiber. A phase could be added to the fermion bilinear, as described in [4], that acts as $\bar{\psi}\psi \rightarrow e^{-n/16}\bar{\psi}\psi$; the phase could also be interpreted as that of the dihedral group in $SO(4,2)$ which suggests that another four points are added to the $R^4$ and resolved. This action was identified in [4] as an orbifold of a $S^4$ by $\Gamma_{4,2}$, but an orbifold of an $S^5$ with group action from $\Gamma_{4,3}$; $\Gamma_{4,2}$ is $D_4$ and $\Gamma_{4,3}$ is $D_{11} \wedge D(\Gamma_{5,1})$ and they have dimension 8 and 19 [48]. Three of the latter are not close to the origin and are not used; the branch point which is unstable due to nine points on top of each other sits in between the origins of the two group actions.

The $D_{11}$ group is good for conformal points when all the points sit on top of each other, but resolved. The remainder acts when the points are separated and also resolved. Each action leaves the bilinear unchanged except one if the centralizer $H_1$ is chosen to be in the corresponding Lie algebra. The phase angle is $n/16$. There is no ambiguity as the axis, the Cartan generator $H_1$, is preferred. However, the other group actions are complicated but the leave the phase unchanged.

Next, the $\Gamma(5,1)$ is treated as a conformal extension of the $D_{11}$ known as Kummer’s algebra and permutes the six dihedral vertices amongst each other. This action changes the phase on each vertex as the wavefunction passes due to the action of the Cartan generator $H_1$ which permutes the phase angles at the vertices. The actions are chosen in an oriented pathwise fashion from the following steps: 1) choose which vertex the wavefunction is entering, 2) permute its phase angles by $H_1$, 3) choose another vertex to hop to, 4) permute its phase angle, 5) repeat until the wavefunction exits though the point it entered after hopping each vertex-vertex only once, which is unique. The phases allowed from hopping are: $e^{n/5}$, $e^{n/6}$, $e^{n/7}$, $e^{n/8}$, $e^{n/9}$, $e^{n/10}$ except for one which is resubstituted with $m \rightarrow m + 1$ due to angle deficit. The remaining four generators possess angle deficits from 30 degrees to 90 degrees in increments of 10 degrees, with a $\pm 10$.

Due to the uniqueness of the path through every vertex, the phase angles must add to $p/16$ with $p = 1, 2, 3, 4, 5, 6$ [7]. The quarks and leptons fall into this category. When a neutrino enters the chamber point split the first point in the finite lattice to include a branch and give it a phase angle of $n/4$ with the other vertices unchanged. The sum of these terms is $n/16 - 5$; the factor of 5 occurs due to the Cartan generator

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$H_1$ acting on too many vertices at the same time. In this case $H_1$ has an action on
the entire lattice each time a node is reached.

With the phase angle attached to the fermion bilinear, the pair exits the region by
traversing outward from the resolution of the points. Imagine a black hole, in the
early universe, and the fermion pair traverses through the horizon from the inside
outward (or because there is no casually disconnected region because the fermion
could exit through a path in the higher dimensions). It acquires a mass proportional
to the phase angle, with a scale set by $\Lambda = 1$ TeV and $m_{pl}$ so that the first term in
is found. This is alluded to partly in [46].

XV. M-theory in higher dimensions

The natural number of dimensions for M-theory is thirteen; the moduli space
presented contain a fibered set of three dimensions with a ten dimensional spacetime.
There are ten in the usual superstring description, and three more in the moduli
space; there is a circle fiber over a 2-sphere. The strong coupling limit of IIA is
obtained by an S-duality transformation followed by the limit $\tau_2 \to \infty$. The 2-sphere
in the IIA configuration is a 2-torus with complex structure $\tau$, and its volume goes to
infinity. The two torus is the same as a sphere with a puncture at one point and an
unramified point at the antipodal point; this is due to pinching off and unramifying
the point in the middle of the torus as the modulus hits the cusp point at $(1, \sqrt{3}/2)$
thus generating a 2-sphere with a hole at one end and an unramified point at the
other.

One limit is as follows. Recall that the volume $V_S$ of the circle satisfies $V_S = \sqrt{V_T}$,
the volume of the torus. The strong-coupling limit is via $g_s \to \infty$ which shrinks the
torus to a point on which the circle is fibered over. The unramified Rpoint is moved
onto the circle and splits into a line segment. The line segment can be extended to
fill the entire real line, and together with the ten dimensions, is eleven dimensional
spacetime.

A second limit is as follows. The unramified point exists at a location away from
the fiber on the circle, which is then equivalent to a line segment after it splits open
using its ramification the circle into two pieces as it has to be branched. The singular
points are absorbed by the branches and the two pieces are glued on top of each other,
which removes the branches. The volume to infinity of the punctured 2-sphere is the
complex plane and the fiber to the circle becomes trivial in this limit. This results in
a thirteen dimensional spacetime.
A conformal field description can be given to the eleventh dimension which is compatible with eleven dimensional supergravity. Put a spin-1, spin-2, spin-3/2, and spin-1/2 field with $2^{d/2}$ components on the circle which also contains the singularity of order two. The degenerate limit completes the eleven dimensional supergravity. A five form is added with one component fibered over the eleventh branch cut, due to disingularizing the two unramified points into an eleventh dimensional branch with a spin field at $x_{11} = \pm 1$ and another one in the tenth coordinate at $x_{11} = \pm 1$ which is oriented in the opposite direction. Desingularizing the non-trivial fiber of the five-form on both branch cuts requires fixing the fourth and fifth components on top of the branch, losing its field content. The three form is then used to fill out the topological degrees of freedom in the supergravity lagrangian. The five-form is relevant to carry over its and other degrees of freedom via supersymmetry to the IIB theory.

The two theories in 11 and 13 dimensions can be given consistent conformal field descriptions. In the first case there are ten flat spacetime dimensions and a compact circle of infinite radius fibered over the eleventh coordinate, with an unramified point added to the circle. The point at infinity is unramified and stolen; together with the unramified point at the origin, a singularity of order two is sitting at the origin of the eleventh dimension.

Rather than this description, the point is ramified to the origin of the eleventh dimension as the circle fiber is non-trivial. As the circle fiber is taken to have infinite radius, the point at infinity is stolen, unramified, and moved on top of the ramified point at the origin of the tenth dimension. Together the two points resolve a Taub-NUT class A singularity with a bolt and a nut, thus unramifying the formerly ramified point. The two unramified points are moved back onto the circle and placed at the origin. As the fiber becomes non-trivial the coordinate completes the flat space eleventh dimensional spacetime. Both this limit and the one in the previous paragraph lead to the same result.

In the thirteenth dimensional example a conformal field model can be presented based on the black hole work [47] together with that in [41]. Consider a black hole wound around the $S^1$ with a $U(1)$ (charge) fiber extending outward, in the other non-compact direction. T-duality changes these the $U(1)$ fiber with the other non-compact direction, which can be thought of as a wormhole due to a singularity at the origin of the $S^1$; this is due to the non-trivial $\pi_1$ and the winding of the soliton solution and its fiber. This blackhole and its wormhole relative have a description in terms of a holographic massive topological conformal field theory. Consider a WZW model at level 1 and its central extension; the former is described
\[ S = \int d^3 x \, \varepsilon^{\mu \nu \rho} W_\mu W_\nu W_\rho + \int d^2 x \sqrt{g} \, g^{\mu \nu} \partial_\mu W_\nu, \]  

(55)

with \( W_\mu \) a spin field transforming in the 1/2 representation. The central extension is given by the usual mass term in three dimensions,

\[ S = \int d^3 x \, \varepsilon^{\mu \nu \rho} W_\mu W_\nu W_\rho + \int d^2 x \sqrt{g} \, g^{\mu \nu} W_\mu W_\nu m^2. \]  

(56)

Both terms are included to preserve a transcendental symmetry so that upon \( m^2 = -k^2 \) the action of the combined terms vanishes. This configuration describes a self-dual wormhole that vanishes at the radius by momentum conservation.

For consistency, the compatibility with eleven dimensional supergravity is explained. Throw away the topological term; then dimensionally reduce the spin field \( W_\mu \) (recall that in 2-d, spin 1/2 is equivalent to spin 0). The mass term has \( m^2 = 2\pi n \). Take \( n = 1 \), which labels this sector as the \( d = 11 \) supergravity limit one. Absorb the \( 2\pi \) by a field redefinition; this is possible in another sector, but not all sectors simultaneously. The spin reduction generates the field content \((2, 3/2, 1)\) with the counting \((1, 2, 1)\) of the degrees of freedom. Take the reduced spin field and branch it over the two Riemann surfaces with the spin \( \pm 1 \) over the individual components. As the two surfaces are equal except for being diametrically opposite, project one to the other and multiply the components by a factor of two. There are eight homology generators on the elliptic genus three hypersurface. This means that in light-cone gauge the gravitational multiplet in \( d = 10 \) should extend to \( d = 11 \), and by supersymmetry the Noether current is subtended by a factor of three for dimensions, which become flat at large radius except for small corrections. This explains the origin of the field theory limit in from the \( d = 13 \) corner. The three-form arises by multiplying the individual components to make a three-form or by Hodge duality an 8-form by wedging all the components together; this shows that the topological term is not fundamental but rather a consequence of duality involving the three-form on the \( S^3 \), which is homotopically equivalent to a three-sphere. The duality extends holomorphically to the product of the three-sphere together with the \( d = 2 \) Riemann surfaces, which becomes a 7-sphere. The eight components of the spin field span the individual components subject to a constraint that light-cone tells us is spin independence, i.e. two independent spin structures generate the same spin content. This condition is reflected in the topological aspect of the three-form, as the former does not require a metric to distinguish the spin dynamics on the \( S^7 \) in a strong sense.
The fibered $S^3$ has a maximum invariance without the topological sector of $E_4$. The topological sector includes two copies of the graded Lie algebra $E_1$, which possesses a transfinite representation; the $Z_2$ breaks this down to $E_1$, which is interesting as the $E_4$ has two Cartan generators, enough to specify the indices $a$ and $b$ in the representation content $L_{a,b}$. Upon extending the $S^3$ fiber bundle to encompass one non-compact dimension such as the tenth, the $E_4$ becomes an $E_6$ broken down to $E_5$ because there is no point at infinity. The $E_5$ when taken as a semi-direct product with the single exceptional Lie algebra $E_1$, can be maximally extended into the gauge symmetry algebra of $E_{11,11}$ and is non-compact and isomorphic to the 3-ball. This is the purported maximum symmetry of M-theory, however, it is theoretically possible to extend further by including the remaining $E_1$ which was projected out by a discrete homomorphism. In doing so, the symmetry algebra becomes $E_{13,13}$, which has not been constructed yet, if possible; the $Z_2$ becomes one of the generators in the extension and in the surface parameterization, thus restricting its action to a hypersurface. This should be the maximum symmetry, as pertinent to the work here.

Reshuffling the indices on the representation labels, the eigenvalues of the two Casimirs of $E_4$, can lead to a group theoretic understanding of the representation numbers of $L_{a,b}$. However, once this is done there is a complication in the transfinite math that leads to the labeling in the first place analogous to reshuffling the spin representations of the Lorentz group which is simpler when the Poincare group is used in conjunction with the Pauli-Lubanski operator. Here $(a,b)$ are two Casimir eigenvalues of the $E_4$ and take on values which are integer multiples of a number in the string theory application. The numbers are ordered into $a > b$ by a similarity transformation which has action on the pairwise path integral and coherence path integral, together with the infinite tower of representations following the triples described earlier. This matrix is the Fubini-Study matrix of the transfinite extension of the $S^2$ without altering its complex structure; transfinite here means that several points were added to the $S^2$, without altering its metric or complex form (after deleting a point) \[45\]. This means that the matrix can be represented on the plane without boundary as a complex number; this also means that there is a non-linear field redefinition of string theory so that the representations enter Jordan normal form, or that the matrix is diagonal. The Jordan normal form can be thought of as a reduction of the classification of the representations $L_{a,b}$ if one thinks of the transfinite operations as conformal transformations in the complex plane, as discussed in the first section. Having only diagonal elements indicates that there is no central charge and thus no Weyl anomaly in the critical superstring, but rather from the target space-time point of view; this is also in the presence of three extra dimensions, which would naively
signal a $c = 3$ anomaly except in the $E_8 \times E_8$ and $O(n)(32)$ models, which would have an $c = 6$ anomaly. The Jordan normal form could also simplify the dynamics even in non-trivial backgrounds.

The $S^1$ fiber must contain a chiral boson \[57\] in order for the transfinite symmetry to have a central extension apparently. This chiral boson transforms like a scalar under the action of Poincare translations and as a Lorentz multiplet of spin $(1/2, 0)$ under Lorentz transformations in $d = 11$. The scalar seems to be required as the $d = 2$ sphere is holomorphically equivalent to the plane with the point removed and placed on the 1-sphere as an unramified point; the topology suggests that the radius is the affine parameter $\alpha$ and the chiral bosons partition function when interpreted as a conformal field theory enhances to an $SU(2)$ thus generating a conformal block with three primary towers. These towers generate the indices of the $L_{a,b}$ in an $SL(2, Z)$ invariant manner without any obstruction due to the chiral nature of the partition function; there is no anomaly if the point is placed at the origin, and the partition function is a scalar with a factor of $1/2$ multiplying the modes. This is T-dual to the coefficient of a 4 in the mode expansion.  

The factor of 4 is useful in transfinite number theory as it is often associated with the last non-transcendently solvable polynomial system, i.e. a quartic. It is conjectured that the expansion of the partition function into a series, as in [16], will generate all string related transfinite representations. The factor of 4 signifies that there are three contributions, the gauge and gravity, the ghost, and the matter components; also the Hardy-Littlewood result stating that one can replace 4 with 2 prevents the numbers in the series expansion to grow without bound and has practical application in the computations (even replacing 4 with 1 appears reasonable if the computation is such that the terms in the series contributing to a specific $x^N$ are collected appropriately).

XVI. Summary and Outlook

The superstring and its moduli space have been described with the goal of elucidating the dynamics even in non-trivial backgrounds. In type IIB a similar partition appears with a factor of 2 and generates the full contribution to the four-point function, with similar partition functions describing higher-point amplitude contributions. However, a result due to Hardy-Littlewood states that these partition functions are equal if the mode oscillators are normalized with that of the partition function of the one with a factor of 2 or 1; this appears trivial.

Apparently, if the tangent space of the $S^1$ plus a point has its tangent space replaced with the tangent space of an $S^1$ so there is no holomorphy then the IIB partition function in [16] is obtained, and this signals the transfiniteness of the IIB superstring without the circle fiber, which has been reduced to a point so that string modes cannot propagate on it.
cidating its M-theory relation as well as its exact solvability. In all cases, the exact S-matrix is constructed in terms of transfinite representations and certain moduli forms that satisfy a differential equation on a corner of moduli space. This work has simplified the approach to scattering so much that now it is possible to simply write down the S-matrix; this is advantageous for practical computing.

Phenomenology of the superstring, primarily in the IIB and exceptional heterotic, is described. The required five-form flux has to be distributed in the compactified spacetime and moduli space; this places constraints on the scenarios, some of which are described. The higher dimensional spacetime of $S^1 \times S^2 \times S^3$ appears to be one of the simplest realistic configurations, including possible quantization of the radii due to the quantized but ordered amount of flux on the circles. This scenario is improved by the resolution of a singularity at the origin of the $R^4$, and this has implications for cosmology. In this stable compactification the realistic mass value generation of the fundamental fields such as fermions are given.

It is possible to interpret the moduli space as a higher dimensional inclusion to the ten dimensional spacetime. Thirteen dimensions are required as the moduli space is three beyond the critical superstring dimension. There are a few bubbles of empty space that exist in this three-dimensional world, and they are partly populated with strings that quantum tunnel back and forth between the ten-dimensional and the three-dimensional space. The stability of this configuration, which requires the quantum coherence effect, is excellent and can be used as a further guide in finding the exact orientation of the three-dimensional configuration in various phases of string theory; vacuum selection is eliminated by including the coherence part of the energy. WMAP data appears to indicate that the above picture is correct [60].

The exact form of the scattering in any non-trivial theory usually has impact in other fields, including mathematics. The exact solution of all of the superstring theories, in uncompactified spacetime and where its clear, in compactifications such as tori or Calabi-Yau manifolds with toric moduli spaces, should have similar impact. Indeed, there are dozens of results following from solving the superstring theories.
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