CENTRAL PRODUCTS AND THE CHERMAK–DELMAGO LATTICE

WILLIAM COCKE AND RYAN MCCULLOCH

Abstract. The Chermak–Delgado lattice of a finite group is a modular, self-dual sublattice of the lattice of subgroups. We prove that the Chermak–Delgado lattice of a central product contains the product of the Chermak–Delgado lattices of the relevant central factors. Furthermore, we obtain information about heights of elements in the Chermak–Delgado lattice relative to their heights in the Chermak–Delgado lattices of central factors. We also explore how the central product can be used as a tool in investigating Chermak–Delgado lattices.

1. Introduction.

The Chermak–Delgado lattice consists of subgroups of a finite group that have maximal Chermak–Delgado measure. Due to the many unique properties of the Chermak–Delgado lattice, it has attracted attention from researchers interested in lattice theory, general finite group theory, and centralizers of groups.

Originally defined by Chermak and Delgado [8], the Chermak–Delgado lattice for a finite group $G$ is defined using the so-called Chermak–Delgado measure $m$ which takes subgroups of $G$ to positive integers via the formula

$$m_G(H) = |H| \cdot |C_G(H)|.$$  

It is quite interesting, and perhaps counter-intuitive that the subgroups with maximal value of $m$ in a group form a sublattice of the subgroup lattice of $G$. Recall that the subgroup lattice of $G$ is the poset of subgroups of $G$ with the operations of meet and join defined by subgroup intersection and subgroup generated by respectively, i.e., $H \wedge K = H \cap K$ and $H \vee K = \langle H, K \rangle$. Surprisingly, when $H$ and $K$ are in the Chermak–Delgado lattice, we have that $\langle H, K \rangle = HK$, i.e., the set theoretic product is actually a subgroup. Furthermore if $H$ is in the Chermak–Delgado lattice then $C_G(H)$ is in the Chermak–Delgado lattice. In addition, we know that for all $H$ in the Chermak–Delgado lattice of $G$, the subgroup $H$ must contain $Z(G)$, the center of $G$. The many properties we presented in this paragraph can be found in Isaacs [12, Section 1.G]. Because of these properties as well as current research about the Chermak–Delgado lattice for various groups, questions about the Chermak–Delgado lattice make for good research projects that can be accessible to students [16, 7].

Recent work on the Chermak–Delgado lattice can be broadly classified as coming in two broad themes. While not exhaustive, we list some references below.

1. Showing that certain types of structures occur as Chermak–Delgado lattices for various group: chains [5], antichains [6], diamonds [3], all subgroups containing $Z(G)$ in a group [15]; or how the Chermak–Delgado lattice relates to certain families of groups [11, 14, 2, 13].

2. Showing properties of the Chermak–Delgado lattice in general: the Chermak–Delgado lattice of a direct product is the direct product of the Chermak–Delgado lattices of the factors [4]. All subgroups of the Chermak–Delgado lattice of a finite group are subnormal in that group with subnormal depth bounded by their relevant position within the lattice [14, 3]. Most recently, the question was asked about how many groups are not contained in the Chermak–Delgado lattice [10].

Related to the direct product is the central product. Recall that a group $G$ is a central product of two of its subgroups $A$ and $B$ if $G = AB$ and $[A,B] = 1$, i.e., $ab = ba$ for all $a \in A$ and $b \in B$.

We prove that the Chermak–Delgado lattice of a central product contains the product of the Chermak–Delgado lattices of the relevant central factors.
Theorem A. Let $G$ be a finite group and $A, B \leq G$ such that $G$ is a central product of $A$ and $B$. Then $\text{CD}(A) \cdot \text{CD}(B) \subseteq \text{CD}(G)$. Furthermore, the top (resp. bottom) of $\text{CD}(G)$ is equal to the product of the tops (resp. bottoms) of $\text{CD}(A)$ and $\text{CD}(B)$.

Hence we have the following corollary.

Corollary 1. If $G$ is a central product of $A$ and $B$ then $m^*(G) = \frac{m^*(A) \cdot m^*(B)}{|A \cap B|^2}$.

In addition, we have the following structural information about the Cheramk–Delgado lattices of central products regarding the heights and depths of elements in the product.

Recall that the depth of $H \in \text{CD}(G)$ is the length of a maximal chain
\[ H = G_0 < G_1 < \cdots < G_n = T_G \]
of elements of $\text{CD}(G)$ where $T_G$ is the top element; the height of $H \in \text{CD}(G)$ is the length of a maximal chain
\[ B_G = G_0 < G_1 < \cdots < G_n = H \]
of elements of $\text{CD}(G)$ where $B_G$ is the bottom element. The height of $\text{CD}(G)$ is the height of the top element (or, equivalently, the depth of the bottom element). These quantities are well-defined since $\text{CD}(G)$ is a modular lattice, and so all of the maximal chains between two fixed elements are of the same length.

Theorem B. If $G$ is a central product of $A$ and $B$, then the height of the Cheramk–Delgado lattice of $G$ is equal to the sum of the heights of the Cheramk–Delgado lattices of $A$ and $B$ respectively. Moreover, an element $HK \in \text{CD}(G)$ with $A \cap B \leq H \leq A$ and $A \cap B \leq K \leq B$ has height (resp. depth) equal to the sum of the heights (resp. depths) of $H \in \text{CD}(A)$ and $K \in \text{CD}(B)$.

In Section 3 we show how the central product can be used to prove results about the Cheramk–Delgado lattice.

2. CENTRAL PRODUCTS AND THE CHERAMK–DELAGDO LATTICE.

A group $G$ is a central product of $A$ and $B$, which are subgroups of $G$, if $G = AB$ and $[A, B] = 1$. Note for every group $G$ we have the central product $G = GZ(G)$. Also note that if $G = AB$ is a central product, then both $A$ and $B$ are normal subgroups of $G$ and that $A \cap B \subseteq Z(G)$. Also for any $X_1 \leq A$ and for any $X_2 \leq B$, we have that $C_G(X_1X_2) = C_A(X_1)C_B(X_2)$. This last observation is almost enough to prove Theorem A.

Proposition 1. Suppose a finite group $G$ is a central product of $A$ and $B$. If $HK \in \text{CD}(G)$ for some $H \leq A$ and some $K \leq B$, then $m^*(G) = \frac{m^*(A) \cdot m^*(B)}{|A \cap B|^2}$, $\text{CD}(A) \cdot \text{CD}(B) \subseteq \text{CD}(G)$, and $H(A \cap B) \in \text{CD}(A)$ and $K(A \cap B) \in \text{CD}(B)$.

Proof. Suppose $HK \in \text{CD}(G)$, and so $HK = H(A \cap B)K(A \cap B)$. Also, $A \cap B \subseteq C_A(H)$ and $A \cap B \subseteq C_B(K)$.

Then
\[ m^*(G) = m_G(HK) = |HK| \cdot |C_H(K)| = |H(A \cap B)K(A \cap B)| \cdot |C_A(H)C_B(K)| = \frac{|H(A \cap B)| \cdot |K(A \cap B)|}{|A \cap B|} \cdot \frac{|C_A(H)| \cdot |C_B(K)|}{|A \cap B|} = \frac{m_A(H(A \cap B)) \cdot m_B(K(A \cap B))}{|A \cap B|^2} \]
Let $X \in \text{CD}(A)$ and $Y \in \text{CD}(B)$. Then $A \cap B \subseteq Z(A) \leq X$, $A \cap B \subseteq Z(B) \leq Y$, $A \cap B \subseteq C_A(X)$, and $A \cap B \subseteq C_B(Y)$. And so
\[ m_G(XY) = |XY| \cdot |C_G(XY)| = |XY| \cdot |C_A(X)C_B(Y)| = \frac{|X| \cdot |Y| \cdot |C_A(X)| \cdot |C_B(Y)|}{|A \cap B|} = \frac{m_A(X) \cdot m_B(Y)}{|A \cap B|^2} \]
Thus
\[ m_G(XY) = \frac{m^*(A) \cdot m^*(B)}{|A \cap B|^2} = m^*(G), \text{ and we have that } \text{CD}(A) \cdot \text{CD}(B) \subseteq \text{CD}(G), \text{ and also } H(A \cap B) \in \text{CD}(A) \text{ and } K(A \cap B) \in \text{CD}(B). \]

\[ \square \]
Lemma 2. Suppose that $C \leq G$. Let $Y$ be some subgroup of $G$, then $m_G(H) \cdot m_G(K) \leq m_G((H, K)) \cdot m_G(H \cap K)$.

Moreover, equality holds if and only if $(H, K) = HK$ and $C_G(H \cap K) = C_G(H)C_G(K)$.

Proof. $m_G(H) \cdot m_G(K) = \left| H \right| \cdot \left| C_G(H) \right| \cdot \left| K \right| \cdot \left| C_G(K) \right|$

$= \left| HK \right| \cdot \left| Y \cap K \right| \cdot \left| C_G(H)C_G(K) \right| \cdot \left| C_G(H) \cap C_G(K) \right|$

$\leq \left| (H, K) \right| \cdot \left| C_G(H) \cap C_G(K) \right| \cdot \left| Y \cap K \right| \cdot \left| C_G(H) \cap C_G(K) \right|$

$= m_G((H, K)) \cdot m_G(H \cap K)$.

□

The following lemma generalizes a result of An [1, Lemma 3.2].

Lemma 3. Suppose that $G$ is a finite group. If $K \leq X \leq H \leq G$, then $m_H(K) \cdot m_G(K) \leq m_H(X) \cdot m_G(X)$.

Moreover, equality holds if and only if $C_G(X) \subseteq HC_G(X)$.

Proof. Note that $m_H(K) \cdot m_G(K) = \frac{\left| K \right| \cdot \left| C_H(K) \right|}{\left| K \right| \cdot \left| C_G(K) \right|} = \frac{\left| C_H(K) \right|}{\left| C_G(K) \right|} = \frac{\left| H \cap C_G(K) \right|}{\left| C_G(K) \right|} = \frac{\left| H \right|}{\left| HC_G(K) \right|}$

and similarly $m_H(X) \cdot m_G(X) = \frac{\left| X \right| \cdot \left| C_H(X) \right|}{\left| X \right| \cdot \left| C_G(X) \right|} = \frac{\left| C_H(X) \right|}{\left| C_G(X) \right|} = \frac{\left| H \cap C_G(X) \right|}{\left| C_G(X) \right|} = \frac{\left| H \right|}{\left| HC_G(X) \right|}$.

Since $K \leq X$, $C_G(X) \subseteq C_G(K)$. So $HC_G(X) \subseteq HC_G(K)$ and we have $\left| HC_G(X) \right| \leq \left| HC_G(K) \right|$, where equality holds if and only if $C_G(K) \subseteq HC_G(X)$. Thus

$\frac{m_H(K)}{m_G(K)} = \frac{m_H(X) \cdot \left| HC_G(X) \right|}{m_G(K) \cdot \left| HC_G(K) \right|} \leq \frac{m_H(X)}{m_G(X)}$, where equality holds if and only if $C_G(K) \subseteq HC_G(X)$.

□

The last lemma we will use to prove Theorem 4 is one step below that of a central product. Here the group $G$ will be equal to $HC_G(X)$ where $X \leq H \leq G$. Note that a group $G$ is a central product if $G = HC_G(H)$ for some subgroup $H$ of $G$. This lemma generalizes a result of An [1, Lemma 3.3].

Lemma 4. Suppose that $G$ is a finite group and $X \leq H \leq G$ such that $G = HC_G(X)$. If $X \in CD(H)$, then for every $Y \in CD(G)$, we have $(X, Y) \in CD(G)$ and $X \cap Y \in CD(H)$. Furthermore, $(X, Y) = XY$ and $C_G(X \cap Y) = C_G(X)C_G(Y)$.

Proof. Let $Y \in CD(G)$. Since $C_G(X \cap Y) \leq G = HC_G(X)$, by Lemma 2

$\frac{m_H(X \cap Y)}{m_G(X \cap Y)} = \frac{m_H(X)}{m_G(X)}$.

Since $X \in CD(H)$, $m_H(X \cap Y) \leq m_H(X)$. It follows that $m_G(X \cap Y) \leq m_G(X)$. By Lemma 1
\[ m_G(X) \cdot m_G(Y) \leq m_G((X, Y)) \cdot m_G(X \cap Y). \]

It follows that \( m_G((X, Y)) \geq m_G(Y) = m^*(G) \). Hence \( m_G((X, Y)) = m_G(Y) \) and \((X, Y) \in \mathcal{CD}(G)\). Hence \( m_G(X) \leq m_G(X \cap Y) \), and so \( m_G(X) = m_G(X \cap Y) \).

And so

\[ m_G(X) \cdot m_G(Y) = m_G((X, Y)) \cdot m_G(X \cap Y), \]

and by Lemma 4 we have that \((X, Y) = XY \) and \( C_G(X \cap Y) = C_G(X)C_G(Y) \).

Finally, since

\[ \frac{m_H(X \cap Y)}{m_G(X \cap Y)} = \frac{m_H(X)}{m_G(X)}, \]

we conclude that \( m_H(X \cap Y) = m_H(X) = m^*(H) \). Hence \( X \cap Y \in \mathcal{CD}(H) \). \( \square \)

We can now prove Theorem A which states that for a finite group \( G = AB \) a central product, we have \( \mathcal{CD}(A) \cdot \mathcal{CD}(B) \subseteq \mathcal{CD}(G) \), and furthermore we have that the top (resp. bottom) of \( \mathcal{CD}(G) \) is equal to the product of the tops (resp. bottoms) of \( \mathcal{CD}(A) \) and \( \mathcal{CD}(B) \).

**Proof of Theorem A.** We write \( T_A, T_B, T_G \) for the top elements of the Chernak–Delgado lattices of \( A, B \) and \( G \) respectively. Similarly \( B_A, B_B, B_G \) refer to the bottom elements of these lattices.

For any \( X_1 \leq A, G = A C_G(X_1) \), and for any \( X_2 \leq B, G = B C_G(X_2) \), and so Lemma 3 applies for any \( X_1 \in \mathcal{CD}(A) \) and any \( X_2 \in \mathcal{CD}(B) \), with any \( Y \in \mathcal{CD}(G) \).

By Lemma 3, \( T_A T_G \in \mathcal{CD}(G) \), and so \( T_A \leq T_G \). Similarly \( T_B \leq T_G \). So \( T_A T_B \leq T_G \).

By Lemma 3, \( B_A \cap B_G \in \mathcal{CD}(A) \), and so \( B_A = B_A \cap B_G \). By Lemma 3, \( C_G(B_A) = C_G(B_A \cap B_G) = C_G(B_A)C_G(B_G) = C_G(B_A)T_G \). So \( T_G \leq C_G(B_A) = T_B \). Similarly \( T_G \leq A T_B \). We see that \( T_G \leq AT_B \cap T_A B = T_A T_B (A \cap B) \). And \( A \cap B \leq Z(A) \leq T_A \), and so \( T_G \leq T_A T_B \).

Thus, \( T_A T_B = T_G \), and so \( B_G = C_G(T_G) = C_A(T_A)C_B(T_B) = B_A B_B \). We apply Proposition 1 to complete the proof. \( \square \)

We provide two examples of equality in Theorem A.

**Proposition 2.** Suppose that a finite group \( G = A \times B \) is a direct product. Then \( \mathcal{CD}(G) = \mathcal{CD}(A) \cdot \mathcal{CD}(B) \).

**Proposition 3.** Suppose that a finite group \( G = AB \) with \( B \leq Z(G) \). Then \( \mathcal{CD}(G) = \mathcal{CD}(A) \cdot \{B\} \).

Proposition 2 appears in 4. We shall prove Proposition 3 in a moment, but first a few facts regarding central products and group products in general.

**Lemma 4.** Suppose that a group \( G = AB \). Suppose \( A \cap B \leq H_1 \leq A, A \cap B \leq H_2 \leq A, A \cap B \leq K_1 \leq B, \) and \( A \cap B \leq K_2 \leq B \). If \( H_1 K_1 \leq H_2 K_2 \), then \( H_1 \leq H_2 \) and \( K_1 \leq K_2 \); and if equality holds then \( H_1 = H_2 \) and \( K_1 = K_2 \).

**Proof.** Suppose \( H_1 K_1 \leq H_2 K_2 \), and let \( h_1 \in H_1 \) and \( k_1 \in K_1 \) be arbitrary. Then \( h_1 k_1 = h_2 k_2 \) for some \( h_2 \in H_2 \) and some \( k_2 \in K_2 \). And so \( x = h_2^{-1} h_1 = k_2 k_1^{-1} \in H_1 \cap K_1 \leq A \cap B \). And since \( A \cap B \leq H_2 \) and \( A \cap B \leq K_2 \), we have that \( h_1 = h_2 x \in H_2 \) and \( k_1 = x^{-1} k_2 \in K_2 \). Hence \( h_1 \leq H_2 \) and \( K_1 \leq K_2 \).

If \( H_1 K_1 \) and \( H_2 K_2 \) coincide, then \( H_1 K_1 \leq H_2 K_2 \) implies that \( H_1 \leq H_2 \) and \( K_1 \leq K_2 \), and similarly \( H_2 K_2 \leq H_1 K_1 \) implies that \( H_2 \leq H_1 \) and \( K_2 \leq K_1 \). \( \square \)

Note that Lemma 4 applies for any finite central product \( G = AB \) with \( H_1, H_2 \in \mathcal{CD}(A) \) and \( K_1, K_2 \in \mathcal{CD}(B) \), since \( A \cap B \leq Z(A) \) and \( A \cap B \leq Z(B) \) and all subgroups in the Chernak–Delgado lattice contain the center. It follows that for any finite central product \( G = AB \), we have that the sum of the heights of \( \mathcal{CD}(A) \) and \( \mathcal{CD}(B) \) cannot exceed the height of \( \mathcal{CD}(G) \). For if \( \mathcal{CD}(A) \) has height \( n \), and say \( B_A = G_0 < G_1 < \cdots < G_n = T_A \) is a chain in \( \mathcal{CD}(A) \), and if \( \mathcal{CD}(B) \) has height \( m \), and say \( B_B = H_0 < H_1 < \cdots < H_m = T_B \) is a chain in \( \mathcal{CD}(B) \), then we have by Lemma 3 that \( G_0 H_0 < G_1 H_0 < \cdots < G_n H_0 < G_1 H_1 < \cdots < G_n H_n \), and by Theorem A this chain of length \( n + m \) lives in \( \mathcal{CD}(G) \). Of course, our Theorem A gives the precise relationship between heights in \( \mathcal{CD}(A), \mathcal{CD}(B), \) and \( \mathcal{CD}(G) \).

If a group \( G = AB \) is a central product, then for any \( U \leq G \), we define \( \pi_A(U) = \{a | g = ab \text{ with } g \in U \text{ and } a \in A \} \) and \( \pi_B(U) = \{b | g = ab \text{ with } g \in U \text{ and } b \in B \} \).
Lemma 5. Suppose a group $G = AB$ is a central product, and $U \leq G$. Then $\pi_A(U)$ is a subgroup of $A$ and, furthermore, $A \cap B \leq \pi_A(U) \leq A$. A similar result is true for $\pi_B(U)$.

Proof. We prove the assertion for $\pi_A(U)$, and the assertion for $\pi_B(U)$ is similar.

Clearly $\pi_A(U) \subseteq A$. We show that $\pi_A(U)$ is a subgroup of $A$. Note that $\pi_A(U)$ is nonempty as $1 \in \pi_A(U)$. Let $x, y \in \pi_A(U)$. We show that $xy^{-1} \in \pi_A(U)$.

So there exists $u \in U$ and $v \in U$ so that $u = xb_1$ and $v = yb_2$ with $b_1, b_2 \in B$.

And so $v^{-1} = y^{-1}b_2^{-1}$, and we have $uw^{-1} = xy^{-1}b_1b_2^{-1}$ with $xy^{-1} \in A$ and $b_1b_2^{-1} \in B$. And since $uw^{-1} \in U$, it follows that $xy^{-1} \in \pi_A(U)$.

Finally, we show that $A \cap B \leq \pi_A(U)$. Let $z \in A \cap B$. Then $z = b$ for some $b \in B$. So $z^{-1} = b^{-1}$. Let $h \in \pi_A(U)$. Then there is $u \in U$ so that $u = hb'$ for some $b' \in B$. And so $u = ubz^{-1} = h'b^{-1}b^{-1} = hzb'b^{-1}$ and we have $hz \in A$ and $b'b^{-1} \in B$. Hence $hz \in \pi_A(U)$. And since $\pi_A(U)$ is a subgroup, $h^{-1}hz = z \in \pi_A(U)$. □

Lemma 6. Suppose a group $G = AB$ is a central product. Suppose $K \leq B$ and suppose $A \cap B \leq U \leq G$. If $K \leq U \leq AK$, then $U = \pi_A(U)K$.

Proof. If $K \leq U \leq AK$, then, of course, $K \leq U \leq \pi_A(U)K$. And so to prove that $U = \pi_A(U)K$, it suffices to prove that $\pi_A(U) \leq U$.

Let $ab \in U$ with $a \in A$ and $b \in B$. Since $U \leq AK$, $ab = a'k$ for some $a' \in A$ and some $k \in K$. So $a' = az$ for some $z \in A \cap B$. Since $K \leq U$, $a'k = a = az \in U$. Since $A \cap B \leq U$, $azz^{-1} = a \in U$. Hence $\pi_A(U) \leq U$. □

We now prove Proposition 3 which states that for a finite group $G = AB$ with $B \leq Z(G)$, we have that $CD(G) = CD(A) \cdot \{B\}$.

Proof of Proposition 3. We have that $G = AB$ is a central product, and since $B$ is abelian, $CD(B) = \{B\}$. It follows from Theorem A that $CD(A) \cdot \{B\} \leq CD(G)$.

Let $U \in CD(G)$. Then $Z(G) \leq U$, and so $B \leq U$, and so by Lemma 6, $U = \pi_A(U)B$. By Proposition 1, we have that $\pi_A(U)(A \cap B) = \pi_A(U) \in CD(A)$ and $B \in CD(B) = \{B\}$, and so $U = \pi_A(U)B \in CD(A) \cdot \{B\}$. □

Corollary 2. Suppose a finite group $G = AB$ is a central product. Then $G \in CD(G)$ if and only if $A \in CD(A)$ and $B \in CD(B)$.

Proof. If $A \in CD(A)$ and $B \in CD(B)$, then by Theorem A, $AB = G \in CD(G)$.

Conversely, suppose $G \in CD(G)$. Then by Theorem A, we have $T_AT_B = G \in CD(G)$, where $T_A$ is the top element of $CD(A)$ and $T_B$ is the top element of $CD(B)$. Now $A \cap B \leq Z(A) \leq T_A$, and $A \cap B \leq Z(B) \leq T_B$. Since $T_AT_B = AB$, it follows from Lemma 3 that $T_A = A$ and $T_B = B$.

We say that a central product $G = AB$ is proper if $Z(G) < A \subset G$ and $Z(G) < B \subset G$, and in such case we say that the group $G$ admits a proper central product.

Note that if a group $G = AB$ is a central product with one of $A$ or $B$ abelian, say $B$, then $B \leq Z(G)$, and we are in the situation of Proposition 3.

Given a finite group $G$ which admits a proper central product, one wonders if $CD(G)$ is equal to the subgroup collection of $G$ that is generated by all $CD(X) \cdot CD(Y)$ where $G = XY$ a proper central product. The answer is “no” in general.

Proposition 4. There exists a finite group $G$ with $G \in CD(G)$ so that $CD(G)$ has height 2 and $CD(G)$ possesses both abelian and nonabelian subgroups of height 1.

See 10 Proposition 10 for an explicit construction.

Proposition 5. There exists a finite group $G$ that admits a proper central product, so that for $C$ the subgroup collection of $G$ generated by all $CD(X) \cdot CD(Y)$ where $G = XY$ a proper central product, we have that $C$ is not equal to $CD(G)$.

Proof. Let $G$ be a group as in Proposition 4. Then there exists $H \in CD(G)$ nonabelian of height 1, and so $C_G(H) \in CD(G)$ is nonabelian of height 1, and so the central product $G = HC_G(H)$ is proper.

If $G = AB$ is an arbitrary proper central product, then by Corollary 2, we have $A \in CD(A)$ and $B \in CD(B)$. By Theorem A, we have that $CD(A) \cdot CD(B) \leq CD(G)$, and since $CD(G)$ has height 2, it follows that $CD(A) = \langle A, Z(A) \rangle$ and $CD(B) = \langle B, Z(B) \rangle$ both have height 1. Thus the only abelian subgroup in $CD(A) \cdot CD(B)$ is $Z(A)Z(B) = Z(G)$. And so if $C$ is the subgroup collection of $G$ generated by all $CD(X) \cdot CD(Y)$ where $G = XY$ a proper central product, then the only abelian subgroup in $C$ is $Z(G)$. Thus $C$ is not equal to $CD(G)$. □
We continue with an application of Theorem [A].

**Proposition 6.** Suppose that $G$ is a finite group and suppose that $\mathcal{CD}(G) = \{\mathbb{Z}(G), G\}$. Then $G$ admits no proper central product.

**Proof.** Note that if $G = \mathbb{Z}(G)$, then $G$ is abelian and the result is true. Suppose $G = AB$ is an arbitrary central product. We will show that one of $A$ or $B$ is abelian, and thus $G$ admits no proper central product.

By Corollary [2] we have $A \in \mathcal{CD}(A)$ and $B \in \mathcal{CD}(B)$. By Theorem [A] we have that $\mathcal{CD}(A) \cdot \mathcal{CD}(B) \subseteq \mathcal{CD}(G)$, and since $\mathcal{CD}(G)$ has height 1, it follows that one of $\mathcal{CD}(A)$ or $\mathcal{CD}(B)$ has height 0, say $\mathcal{CD}(A)$. So $\mathcal{CD}(A) = \{A\}$, and thus $A$ is abelian. □

We say that a subgroup $A \in \mathcal{CD}(G)$ is an atom if the height of $A \in \mathcal{CD}(G)$ is 1, and we say that a subgroup $B \in \mathcal{CD}(G)$ is a coatom if the depth of $B \in \mathcal{CD}(G)$ is 1.

**Lemma 7.** Suppose that $G$ is a finite group with $G \in \mathcal{CD}(G)$ and suppose that $\mathcal{CD}(G)$ has height greater than 1. If $A$ is a nonabelian atom in $\mathcal{CD}(G)$, then $AC_G(A)$ is a proper central product.

**Proof.** We note that $C_G(A)$ is a coatom of $\mathcal{CD}(G)$. Since $A$ is nonabelian it is not a subgroup of $C_G(A)$. Since $C_G(A)$ is a coatom, the only element of $\mathcal{CD}(G)$ properly containing it is $G$. Thus $AC_G(A)$ is all of $G$. Moreover, $C_G(A)$ is not abelian, else it would be contained in $C_G(C_G(A)) = A$, which would imply it is $A$ since $A$ is a nonabelian atom, but this implies that $A = C_G(A)$ which implies that $A$ is abelian. So the central product $G = AC_G(A)$ is proper. □

**Lemma 8.** Suppose that $G$ is a finite group with $G \in \mathcal{CD}(G)$ and suppose that $\mathcal{CD}(G)$ has height greater than 1. If $G$ admits no proper central product, then every atom in $\mathcal{CD}(G)$ is abelian.

**Proof.** If some atom $A \in \mathcal{CD}(G)$ is nonabelian, then $G = AC_G(A)$ is a proper central product. □

**Lemma 9.** Suppose that $G$ is a finite group with $G \in \mathcal{CD}(G)$ and suppose that $\mathcal{CD}(G)$ has height greater than 1. If $C$ is a coatom in $\mathcal{CD}(G)$, then either $C_G(C) = \mathbb{Z}(C)$, or $CC_G(C)$ is a proper central product.

We now prove Theorem [3] which states that for a finite group $G = AB$ a central product, the height of the Cheramk–Delgado lattice of $G$ is equal to the sum of the heights of the Cheramk–Delgado lattices of $A$ and $B$ respectively. Moreover, an element $HK \in \mathcal{CD}(G)$ with $A \cap B \leq H \leq A$ and $A \cap B \leq K \leq B$ has height (resp. depth) equal to the sum of the heights (resp. depths) of $H \in \mathcal{CD}(A)$ and $K \in \mathcal{CD}(B)$.

**Proof of Theorem [3].** We first prove the assertion in regards to height. It follows from Proposition [1] that given $HK \in \mathcal{CD}(G)$ with $A \cap B \leq H \leq A$ and $A \cap B \leq K \leq B$, we have $H \in \mathcal{CD}(A)$ and $K \in \mathcal{CD}(B)$, and it follows from Theorem [A] that such $HK \in \mathcal{CD}(G)$ exists, namely the product of the top element of $\mathcal{CD}(A)$ and the top element of $\mathcal{CD}(B)$.

Suppose $HK \in \mathcal{CD}(G)$ with $A \cap B \leq H \leq A$ and $A \cap B \leq K \leq B$, and suppose $H \in \mathcal{CD}(A)$ has height $i$ and $K \in \mathcal{CD}(B)$ has height $j$. We proceed by induction on $i + j$. If $i + j = 0$, the result is true by Theorem [A] and we have $H = B_A$, $K = B_B$, and $HK = B_A B_B = B_G$. Suppose $i + j > 0$ and suppose without loss of generality that $i > 0$. Let $H_0 \in \mathcal{CD}(A)$ of height $i - 1$ with $H_0 < H$. Then by induction $H_0 K \in \mathcal{CD}(G)$ is of height $i + j - 1$. Suppose $U \in \mathcal{CD}(G)$ with $H_0 K < U \leq HK$ is arbitrary. By Lemma [6], $U = \pi_A(U)K$. By Proposition [1], $\pi_A(U)(A \cap B) = \pi_A(U) \in \mathcal{CD}(A)$. And since $H_0, \pi_A(U)$, and $H$ all contain $A \cap B$, by Lemma [4] we have that $H_0 \leq \pi_A(U) \leq H$. Since $H_0 \in \mathcal{CD}(A)$ is of height $i - 1$ and $H \in \mathcal{CD}(A)$ is of height $i$, we have that $\pi_A(U) = H$, and so $U = HK$. Since $U$ was arbitrary, it follows that $HK \in \mathcal{CD}(G)$ is of height $i + j$.

We now establish the assertion in regards to depth. Let $n_A$, $n_B$, and $n_G$ be the heights of $\mathcal{CD}(A)$, $\mathcal{CD}(B)$, and $\mathcal{CD}(G)$, respectively. Suppose $H \in \mathcal{CD}(A)$ has height $i$ and $K \in \mathcal{CD}(B)$ has height $j$. Then we have just shown that $HK \in \mathcal{CD}(G)$ has height $i + j$. And so $H \in \mathcal{CD}(A)$ has depth $n_A - i$ and $K \in \mathcal{CD}(B)$ has depth $n_B - j$, and $HK \in \mathcal{CD}(G)$ has depth $n_G - (i + j)$. Now we have established that $n_A + n_B = n_G$, by applying the result to the heights of the top elements of $\mathcal{CD}(A)$, $\mathcal{CD}(B)$, and $\mathcal{CD}(G)$. Observe that $n_A - i + n_B - j = n_A + n_B - (i + j) = n_G - (i + j)$, and we have the result in regards to depth. □

Our last application of this section relies on both Theorem [A] and Theorem [B].

**Proposition 7.** Suppose that $G$ is a finite group with $G \in \mathcal{CD}(G)$ and suppose that $\mathcal{CD}(G)$ has height 2 or 3. Then $G$ admits no proper central product if and only if every atom in $\mathcal{CD}(G)$ is abelian.
Proof. One direction is true by Lemma 8.
Suppose that every atom in $\text{CD}(G)$ is abelian, and let $G = AB$ be an arbitrary central product. Since $G \in \text{CD}(G)$, by Corollary 2 we have that $A \in \text{CD}(A)$ and $B \in \text{CD}(B)$.

Now if one of $\text{CD}(A)$ or $\text{CD}(B)$ has height 0, say $\text{CD}(A)$, then $\text{CD}(A) = \{A\}$ and so $A$ is abelian, and the result follows.

Otherwise, since $\text{CD}(G)$ has height 2 or 3, we have that at least one of $\text{CD}(A)$ or $\text{CD}(B)$ has height 1, say $\text{CD}(A)$. So $\text{CD}(A) = \{A, Z(A)\}$. By Theorem 1 we have that $AZ(B)$ has height 1 in $\text{CD}(G)$, and thus is a nonabelian atom in $\text{CD}(G)$, a contradiction. □

The fun stops after height 3. Construct the central product $G = Q_8 \ast Q_8$, which is of course proper. Note that $\text{CD}(G)$ has height 4, and all of the atoms have order 4 so they are all abelian.

3. Investigating the Chermak–Delgado Lattice via the Central Product.

In this section we show how using the central product and the results from Section 2 we can obtain new proofs of some results about Chermak–Delgado lattice.

Let $\mathcal{L}(G)$ denote the lattice of all subgroups of a group $G$. Given a sublattice $\mathcal{C}$ of $\mathcal{L}(G)$, and given $H \leq K \leq G$, we denote the interval in $\mathcal{C}$ between $H$ and $K$ by $[H : K]_{\mathcal{C}} = \{X \in \mathcal{C} \mid H \leq X \leq K\}$. For example, when $G = Q_8$ the quaternion group of order 8, then $[\langle -1 \rangle : G]_{\mathcal{L}(G)}$ contains 5 groups: $\langle -1 \rangle, \langle i \rangle, \langle j \rangle, \langle k \rangle$, and $G$.

The next lemma originates from [4, Proposition 1.5].

Lemma 10. Let $G$ be a finite group and let $H \in \text{CD}(G)$. Then $\text{CD}(H \text{C}_G(H)) = [Z(H) : H \text{C}_G(H)]_{\text{CD}(G)}$.

Proof. First note that $H \text{C}_G(H)$ is a central product and so $Z(H \text{C}_G(H)) = Z(H)Z(\text{C}_G(H))$.

Now since $H \in \text{CD}(G)$, we have that $Z(\text{C}_G(H)) = \text{C}_G(\text{C}_G(H)) \cap \text{C}_G(H) = H \cap \text{C}_G(H) = Z(H)$. And so $Z(H \text{C}_G(H)) = Z(H)$.

We establish that for any $X$ with $Z(H) \leq X \leq H \text{C}_G(H)$, we have that $\text{C}_G(X) = H \text{C}_G(H)\text{C}_G(X)$. Then $\text{C}_G(H \text{C}_G(H)) \leq \text{C}_G(X) \leq \text{C}_G(H \cap \text{C}_G(H))$, and since $H \in \text{CD}(G)$, we have that $H \cap \text{C}_G(H) \leq \text{C}_G(X) \leq H \text{C}_G(H)$, and hence $\text{C}_G(X) = H \text{C}_G(H)\text{C}_G(X)$.

It follows that $X \in \text{CD}(H \text{C}_G(H))$ if and only if $X \in [Z(H) : H \text{C}_G(H)]_{\text{CD}(G)}$. □

The following corollary appears in An [1, Theorem 3.4] and [2, Theorem 4.4]. However, as stated at the start of the section, our goal is to show the power of the central product when investigating the Chermak–Delgado lattice. Of note, our proof utilizes Lemma 10 and Proposition 11 together with some standard results about the Chermak–Delgado lattice.

Corollary 3. Let $G$ be a finite group and let $H \in \text{CD}(G)$. Then $\text{CD}(H) = [Z(H) : H]_{\text{CD}(G)}$.

Proof. By Lemma 10 $\text{CD}(H \text{C}_G(H)) = [Z(H) : H \text{C}_G(H)]_{\text{CD}(G)}$.

So for any $X \in [Z(H) : H]_{\text{CD}(G)}$, $X \in \text{CD}(H \text{C}_G(H))$. Applying Proposition 11 with the central product $H \text{C}_G(H)$, we have that $X(H \cap \text{C}_G(H)) = XZ(H) = X \in \text{CD}(H)$. Thus $[Z(H) : H]_{\text{CD}(G)} \subseteq \text{CD}(H)$.

To see the reverse containment, note that every subgroup of $\text{CD}(H) \subseteq \text{CD}(H \text{C}_G(H))$, and every subgroup of $\text{CD}(\text{C}_G(H))$ contains $Z(\text{C}_G(H)) = \text{C}_G(\text{C}_G(H)) \cap \text{C}_G(H) = H \cap \text{C}_G(H) = Z(H)$. By Proposition 11, $\text{CD}(H) \cdot \text{CD}(\text{C}_G(H)) \subseteq \text{CD}(H \text{C}_G(H))$, and since $Z(H) \in \text{CD}(H \text{C}_G(H))$, it follows that $Z(H) \in \text{CD}(H)$ and $Z(H) \in \text{CD}(\text{C}_G(H))$. And so $\text{CD}(H) \subseteq \text{CD}(H \text{C}_G(H))$. And since $\text{CD}(H \text{C}_G(H)) = [Z(H) : H \text{C}_G(H)]_{\text{CD}(G)}$, we have that $\text{CD}(H) \subseteq [Z(H) : H]_{\text{CD}(G)}$. □

Of note, Corollary 3 can be used to argue that certain groups never appear in a Chermak–Delgado lattice for any group $G$. For example, the alternating group $A_4$ can never be in a Chermak–Delgado lattice since $A_4 \notin \text{CD}(A_4)$.

Taking the central product approach also allows us to prove a result of Tărnăuceanu [15, Corollary 4]. Tărnăuceanu’s result occurs as a corollary to their classification of groups $G$ satisfying $\text{CD}(G) = [Z(G) : G]_{\mathcal{L}(G)}$.

Once again, we are able to present a central product based proof.

Corollary 4. If $G$ is a finite group satisfying $\text{CD}(G) = [Z(G) : G]_{\mathcal{L}(G)}$ and $H$ is a subgroup of $G$, then $\text{CD}(H) = [Z(H) : H]_{\mathcal{L}(G)}$. 

Proof. By definition $CD(H) \subseteq [Z(H) : H]_{L(G)}$. For the reverse containment, suppose $X \in [Z(H) : H]_{L(G)}$. Since $Z(G) \leq C_G(H)$, we have that $HC_G(H) \in CD(G)$. It follows from Corollary 9 that $CD(HC_G(H)) = [Z(HC_G(H)) : HC_G(H)]_{L(G)} = [Z(H)[Z(C_G(H)) : HC_G(H)]_{L(G)}$.

And so $XC_G(H) \in CD(HC_G(H))$, and it follows by Proposition 1 that $X(C_G(H)) = XZ(H) = X \in CD(H)$. □

References

[1] Lijian An. Groups whose Chermak-Delgado lattice is a subgroup lattice of an elementary abelian $p$-group. *Communications in Algebra*, 50(7):2846–2853, 2022.
[2] Lijian An. Twisted centrally large subgroups of finite groups. *Journal of Algebra*, 604:87–106, 2022.
[3] Lijian An, Joseph Phillip Brennan, Haipeng Qu, and Elizabeth Wilcox. Chermak–Delgado lattice extension theorems. *Communications in Algebra*, 43(5):2201–2213, 2015.
[4] Ben Brewster and Elizabeth Wilcox. Some groups with computable Chermak–Delgado lattices. *Bulletin of the Australian Mathematical Society*, 86(1):29–40, 2012.
[5] Ben Brewster, Peter Hauck, and Elizabeth Wilcox. Groups whose Chermak–Delgado lattice is a chain. *Journal of Group Theory*, 17(2):253–279, 2014.
[6] Ben Brewster, Peter Hauck, and Elizabeth Wilcox. Quasi-antichain Chermak–Delgado lattices of finite groups. *Archiv der Mathematik*, 103(4):301–311, 2014.
[7] Erin Brush, Jill Dietz, Kendra Johnson-Tesch, and Brianne Power. On the Chermak–Delgado lattice of split metacyclic $p$-groups. *Involve*, 9(5):765–782, 2016.
[8] Andrew Chermak and Alberto Delgado. A measuring argument for finite groups. *Proceedings of the American Mathematical Society*, 107(4):907–914, 1989.
[9] William Cocke. Subnormality and the Chermak–Delgado lattice. *Journal of Algebra and its Applications*, 19(8):2050141, 2020.
[10] Georgiana Fasol˘ a and Marius T˘ arn˘ auceanu. Finite groups with large Chermak–Delgado lattices. *Bulletin of the Australian Mathematical Society*, pages 1–5, 2022. doi: 10.1017/S0004972722000806. URL http://dx.doi.org/10.1017/S0004972722000806.
[11] George Glauberman. Centrally large subgroups of finite $p$-groups. *Journal of Algebra*, 300:480–508, 2006.
[12] I. Martin Isaacs. *Finite Group Theory*, volume 92. American Mathematical Soc., 2008.
[13] Ryan McCulloch. Finite groups with a trivial Chermak–Delgado subgroup. *Journal of Group Theory*, 21(3):449–461, 2018.
[14] Alessandro Morresi Zuccari, Valentina Russo, and Carlo Maria Scoppola. The Chermak–Delgado measure in finite $p$-groups. *Journal of Algebra*, 502:262–276, 2018.
[15] Marius T˘ arn˘ auceanu. A note on the Chermak–Delgado lattice of a finite group. *Communications in Algebra*, 46(1):201–204, 2018.
[16] Elizabeth Wilcox. Exploring the Chermak–Delgado lattice. *Mathematics Magazine*, 89(1):38–44, 2016.