CATEGORICAL AND TOPOLOGICAL ENTROPIES ON SYMPLECTIC MANIFOLDS

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ABSTRACT. In this paper, we prove that if a symplectic automorphism is of a certain type, that we call "Penner type", then its categorical entropy bounds its topological entropy from below. Along the way, we show that its categorical entropy on the compact Fukaya category agrees with that on the wrapped Fukaya category. Also, we prove that for any compactly supported symplectic automorphism, its categorical entropy on the wrapped Fukaya category also agrees with that on any partially wrapped Fukaya category. This is a corollary of our result for general triangulated categories, which gives inequalities between the categorical entropy of functors on categories and their localisations.

1. INTRODUCTION

1.1. Introduction. In [DHKK14], the notion of categorical entropy is defined on a triangulated category. It is a generalization of the notion of topological entropy in category theory. With the notions, it would be natural to ask the following question: Let us assume that a topological space $X$ and an automorphism $\phi$ defined on $X$ are given so that

- $X$ induces a triangulated category $\mathcal{C}$, and
- $\phi$ induces an auto-functor on $\mathcal{C}$.

Then, the question is “what is the relation between the topological entropy of $\phi$ and the categorical entropy of the induced functor?”.

One can find examples of such pair $(X, \phi : X \to X)$ in algebraic geometry and symplectic topology. More precisely, in algebraic geometry, if $X$ is a smooth proper variety and $\phi$ is an endomorphism on $X$, then one can consider the bounded derived category of coherent sheaves on $X$, and there is a well-defined

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auto-functor induced from $\phi$. In symplectic topology, if $X$ is a symplectic manifold and $\phi$ is a symplectic automorphism, then one can consider the (triangulated closure of) Fukaya category of $X$, and there is a well-defined auto-functor induced from $\phi$.

The above question has been studied in algebraic geometry. For example, see [KT19] or [Mat21]. Especially, in [KT19], the authors proved that the categorical entropy of a surjective endomorphism of a smooth projective variety is equal to its topological entropy.

In this paper, we study the connection between the categorical entropy and the topological entropy on symplectic setting. More precisely, let $W$ be a Weinstein manifold equipped with a symplectic automorphism $\phi: W \simeq W$ of a certain type. Then, we prove that the categorical entropy of $\phi$ bounds the topological entropy of $\phi$ from below, i.e., we prove Theorem 8.3.

Also, we answer a purely categorical question. The question asks the relation between categorical entropies on a category $\mathcal{C}$, a full subcategory $\mathcal{D} \subset \mathcal{C}$, and the quotient (localisation) $\mathcal{C}/\mathcal{D}$. One motivation for this question can be found in symplectic topology. When one considers the wrapped Fukaya category and partially wrapped Fukaya categories of a Weinstein manifold, the above question arises naturally. Another motivation comes from the topological entropy. For the motivations, see the beginning of Section 3.

More detailed results will be given in Section 1.2.

1.2. Results. Let $W$ be a Weinstein manifold equipped with a symplectic automorphism $\phi: W \simeq W$. The problem we are interested in is the comparison of two entropy, one is the topological entropy of $\phi$, and the other is the categorical entropy of the induced functor on the Fukaya category.

However, there is an ambiguity. The ambiguity happens because there are various Fukaya categories of $W$, for example, the compact Fukaya category, the wrapped Fukaya category, or partially wrapped Fukaya categories. Thus, in order to define a categorical entropy of $\phi$, we should fix which Fukaya category we will consider.

In order to resolve the ambiguity, we compare the categorical entropies on the compact Fukaya category and the wrapped Fukaya category. We note that the comparison with partially wrapped Fukaya categories will appear in the later of this subsection.

In order to compare the categorical entropies on the compact and the wrapped Fukaya categories, we restrict our interests on the Weinstein manifolds such that the compact and the wrapped Fukaya categories satisfy the Koszul duality. More precisely, in this paper, we consider the plumbing spaces of $T^*S^n$ where $n \geq 3$ along a tree $T$. We note that the plumbing spaces admit the Koszul duality by [EL17]. Especially, there is a split-generator $S$ (resp. $L$) of the compact (resp. wrapped) Fukaya category such that $S$ and $L$ are dual to each other in the sense of Lemma 5.7.

The next task for comparing two categorical entropies is to fix a symplectic automorphism on a plumbing space. Fortunately, the plumbing spaces admit a large class of symplectic automorphisms. More precisely, the plumbing spaces of $T^*S^n$ have a collection of Lagrangian spheres, or the zero sections of $T^*S^n$. Then, each of Lagrangian spheres admits a Dehn twist along it, and by composing the positive or negative powers of Dehn twists, one can generate a lot of symplectic automorphisms.

Remark 1.1. We note that, if the plumbing space is a Milnor fiber of ADE-type, then [MW18] proved the split generation of any symplectic automorphism by Dehn twists.

For those symplectic automorphisms, we prove Theorem 1.2.

Theorem 1.2 (Technical statements are Theorems 6.6 and 6.13). Let $W$ be a plumbing of $T^*S^n$ with $n \geq 3$ along a tree $T$. Let $\phi$ be a product of positive or negative powers of Dehn twists along the zero sections. If
\(\Phi_C\) and \(\Phi_W\) denote the induced functors on the compact Fukaya category and the wrapped Fukaya category respectively, the following hold:

1. The categorical entropies of \(\Phi_C\) and \(\Phi_W\) agree with each other. Thus, the categorical entropy of \(\phi\) is well-defined.
2. The categorical entropy of \(\phi\) is computed by
   \[
   \lim_{m \to \infty} \frac{1}{m} \log \dim \text{Hom}^*(S, \phi^m(L)),
   \]
   where \(S\) (resp. \(L\)) is the split-generator of the compact (resp. wrapped) Fukaya category, and \(\text{Hom}\) denotes the morphism space of the wrapped Fukaya category.

Being motivated from Theorem 1.2, we would like to work with \(\phi\) such that one can easily keep track of
   \[
   \lim_{m \to \infty} \frac{1}{m} \log \dim \text{Hom}^*(S, \phi^m(L)).
   \]

A similar problem appears in surface theory. Especially, on a surface \(S\), if a surface automorphism \(\psi : S \to S\) is pseudo-Anosov, then there is a combinatorial tool, called train tracks, helping us to keep track of the isotopy class of \(\psi^m(C)\) for any closed curve \(C \subset S\) as \(m \to \infty\).

One of the well-known constructions of pseudo-Anosov surface automorphisms is the construction of Penner [Pen88]. We generalize the construction on the plumbing spaces. See Definition 7.6. Then, we have Theorem 1.3.

**Theorem 1.3** (Technical statement is Theorem 7.14. Also, see Section 7.2). If a symplectic automorphism \(\phi\), defined on a plumbing space, is of the certain type given in Definition 7.6, then, there is a combinatorial tool of tracking \(\phi^m(L)\) as \(m \to \infty\). Moreover, the categorical entropy of \(\phi\) is given by
   \[
   \lim_{m \to \infty} \frac{1}{m} \log |S \cap \phi^m(L)|.
   \]

Theorem 1.3 gives a topological way of computing the categorical entropy, which is counting the number of intersection points. Motivated from this, we prove one of the main theorems of this paper.

**Theorem 1.4** (Technical statement is Theorem 8.3). If a symplectic automorphism \(\phi\) satisfies the conditions of Theorem 1.3, then the categorical entropy of \(\phi\) is smaller than or equal to the topological entropy of \(\phi\).

Now, we discuss a question the relation between the categorical entropies on the wrapped Fukaya category and partially wrapped Fukaya categories, as mentioned in the second paragraph of the current subsection. Since the wrapped Fukaya category can be seen as a localisation of any partially wrapped Fukaya category, the question induces a more general, but purely categorical question: Let \(\mathcal{C}\) be a triangulated (dg) category, and let \(\mathcal{D}\) be a triangulated full subcategory of \(\mathcal{C}\). If a functor \(\Phi_\mathcal{C}\) preserves \(\mathcal{D}\), thus, there is an induced functor
   \[
   \Phi_{\mathcal{C}/\mathcal{D}} : \mathcal{C}/\mathcal{D} \to \mathcal{C}/\mathcal{D},
   \]
then what is the relation between the categorical entropies of \(\Phi_\mathcal{C}\) and \(\Phi_{\mathcal{C}/\mathcal{D}}\)?

For the question, we prove Theorem 1.5.

**Theorem 1.5** (Technical statement is Theorem 3.8). Let us assume that a triangulated category \(\mathcal{C}\), a full subcategory \(\mathcal{D}\) of \(\mathcal{C}\), and a functor \(\Phi_\mathcal{C} : \mathcal{C} \to \mathcal{C}\) satisfy the above conditions. When \(h_t(\Phi)\) means the categorical entropy of a functor \(\Phi\), the following inequalities hold:
   \[
   h_t(\Phi_{\mathcal{C}/\mathcal{D}}) \leq h_t(\Phi_\mathcal{C}) \leq \max\{h_t(\Phi_{\mathcal{C}/\mathcal{D}}), h_t(\Phi_\mathcal{D})\}.
   \]
Since the wrapped Fukaya category is a localisation of a partially wrapped Fukaya category, one obtains Theorem 1.6 as a corollary of Theorem 1.5.

**Theorem 1.6** (Technical statement is Theorem 4.2). If \( \phi : W \to W \) is a compactly supported symplectic automorphism on a Weinstein manifold \( W \). Then, the categorical entropy of \( \phi \) on the wrapped Fukaya category agrees with that of \( \Phi \) on any partially wrapped Fukaya category.

**Remark 1.7.** We would like to point out that Theorem 1.5, together with [DHKK14, Theorem 2.6], can give a specific way of computing the categorical entropies. For example, let \( W \) be a Weinstein manifold and let \( \varphi : W \to W \) be a compactly supported symplectomorphism. Then, there exists a stop such that the partially wrapped Fukaya category becomes smooth and proper. For example, one can find a stop such that the corresponding partially wrapped Fukaya category is a Fukaya-Seidel category. If \( \Phi_1 \) (resp. \( \Phi_2 \)) is the induced functor from \( \phi \) on the wrapped Fukaya category (resp. partially wrapped Fukaya category), then, by Theorem 1.5, the categorical entropies of \( \Phi_1 \) and \( \Phi_2 \) are the same. Moreover, by [DHKK14, Theorem 2.6], the categorical entropy of \( \Phi_2 \) can be computed from the dimensions of morphisms spaces in the partially wrapped Fukaya category, without using the original definition of categorical entropy. In other words, the categorical entropy of \( \Phi_2 \) is computed by the following formula

\[
\lim_{n \to \infty} \frac{1}{n} \log \dim \text{Hom}^*(G, \Phi_2^n(G)),
\]

where \( G \) is a split-generator of the wrapped Fukaya category. We note that this argument can be applied to any pair \((\mathcal{C}, \mathcal{C}/\mathcal{D})\) where \( \mathcal{C} \) is smooth and proper.

### 1.3. Structure of the paper

This paper consists of three parts, except the introduction section (Section 1).

Sections 2–4 are the first part. In the first part, we review preliminaries from category theory, for example, the notion of twisted complex, and the definition of categorical entropy. Then, we compare the categorical entropy of functors on a category \( \mathcal{C} \) and that of induced functors on the localisation of \( \mathcal{C} \). In other words, we prove Theorem 1.5. In Section 4 we apply the result to compare the categorical entropies on the wrapped and partially wrapped Fukaya categories, i.e., we prove Theorem 1.6.

Sections 5–7 are the second part. In the second part, we compare the categorical entropies of functors on the compact and the wrapped Fukaya categories, which are induced from the same symplectic automorphism. More precisely, we set the Weinstein manifolds and symplectic automorphisms in Section 5. Then, we prove Theorem 1.2 (resp. 1.3) in Section 6 (resp. 7).

The main theorem of this paper, Theorem 1.4, appears in the third part, i.e., Section 8. Then, we end this paper with an example in Section 9.

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2. Twisted complex formulation of the categorical entropy

In Section 2 we define the categorical entropy. The notion of categorical entropy is originally defined in [DHKK14]. We note that the original definition is defined by using a split-generator. In the current paper, we use a (split-)generating set instead of a split-generator. By using a generating set rather than
In order to introduce our approach using a generating set, we start this section by defining twisted complexes.

First, we set the notation. Let \( C \) be a dg category with the differential \( d \). Let \( k \) be the coefficient field. Let \( \text{hom}(A, B) \) be the morphism complex of the objects \( A, B \in C \), and \( \text{Hom}(A, B) \) be the cohomology of \( \text{hom}(A, B) \). Let \( A[n] \) be a shift of an object \( A \in C \), which can be thought as \( k[n] \otimes A \). We will ignore signs in this chapter. The reader can refer to [Sei08] for more details.

Now, we define the notion of twisted complexes.

**Definition 2.1.**

1. A twisted complex \((K, f)\) in \( C \) is given by an ordered decomposition
   
   \[ K = K_1 \oplus K_2 \oplus \ldots \oplus K_n, \]
   
   with \( K_i \in C \) (possibly with shift), and a strictly lower triangular degree 1 morphism \( f = (f_{ij}) : K \to K \) satisfying \( df + f f = 0 \). In other words, \( f \) consists of degree 1 morphisms \( f_{ij} : K_j \to K_i \) for \( i > j \), such that
   
   \[ df_{ij} + \sum_{i > l > j} f_{il} f_{lj} = 0. \]

2. We call \((K, f)\) a twisted complex with the components \( G_1, \ldots, G_m \), if
   
   \[ K = K_1 \oplus K_2 \oplus \ldots \oplus K_m \]
   
   such that \( K_i = G_{r_i} [n_i] \) for some \( r_i \) and \( n_i \).

3. A (homogeneous) morphism \( \tilde{\alpha} : (K, f) \to (L, g) \) of twisted complexes is a morphism \( \alpha : K \to L \) with the grading \( |\tilde{\alpha}| := |\alpha| \) and the differential
   
   \[ d\tilde{\alpha} := d\alpha + f\alpha + \alpha f. \]

4. A shift of \((K, f)\) is \((K[n], f)\) for some \( n \in \mathbb{Z} \).

5. The cone of \( \tilde{\alpha} \), denoted by \( \text{Cone}(\tilde{\alpha}) \), is the twisted complex
   
   \[ \left( K[1] \oplus L, \begin{pmatrix} f & 0 \\ \alpha & g \end{pmatrix} \right). \]

6. A homogeneous morphism \( \tilde{\alpha} \) is a homotopy equivalence if \( \text{Cone}(\tilde{\alpha}) \cong 0 \).

7. We write \( \text{Tw}(C) \) for the dg category of twisted complexes in \( C \). \( \text{Tw}(C) \) is also the pretriangulated closure of \( C \).

8. For any \( K \in \text{Tw}(C) \), we call \((K = K_1 \oplus \ldots \oplus K_n, f = (f_{ij})\) a twisted complex for \( K \) if they are homotopy equivalent in \( \text{Tw}(C) \).

We note that any object of \( C \) is trivially a twisted complex in \( C \). We allow \( K_1, \ldots, K_n \) to be in \( \text{Tw}(C) \) from now on.

**Remark 2.2.** Another notation for the twisted complex

\[ (K = K_1 \oplus \ldots \oplus K_n, f = (f_{ij})) \]
is

\[
\begin{array}{cccccc}
& K_1 & \rightarrow & \cdots & \rightarrow & K_{n-2} & \rightarrow & K_{n-1} & \rightarrow & K_n \\
\end{array}
\]

Also, a twisted complex \((K, f)\) can be seen as an iterated cone in \(\text{Tw}(\mathcal{C})\). Order of taking cone can be chosen freely after properly shifting \(K_i\)’s. One example of iterated cones corresponding to \((K, f)\) is

\[
\text{Cone}(K_1[-1] \rightarrow \cdots \rightarrow \text{Cone}(K_{n-2}[-1] \rightarrow (f_{n-1|n-2}, f_{n(n-2)}) \rightarrow \text{Cone}(K_{n-1}[-1] \rightarrow f_{n(n-1)} K_n))).
\]

Since the categorical entropy is defined for an auto-equivalence of a triangulated category, we need the following proposition later.

**Proposition 2.3.** If \(\tilde{K} \in \text{Tw}(\mathcal{C})\) is the twisted complex

\[
\{K_1 \oplus \cdots \oplus K_n, f = (f_{ij})\},
\]

and if \(\Phi: \text{Tw}(\mathcal{C}) \to \text{Tw}(\mathcal{C}')\) is a dg functor, then there exists a twisted complex for \(\Phi(\tilde{K}) \in \text{Tw}(\mathcal{C}')\),

\[
\{\Phi(K_1) \oplus \cdots \oplus \Phi(K_n), f' = (f'_{ij})\},
\]

for some \(f'_{ij}\).

**Proof.** We can interpret

\[
\{K_1 \oplus \cdots \oplus K_n, f = (f_{ij})\},
\]
as an iterated cone

\[
\text{Cone}(K_1[-1] \rightarrow \cdots \rightarrow \text{Cone}(K_{n-2}[-1] \rightarrow (f_{n-1|n-2}, f_{n(n-2)}) \rightarrow \text{Cone}(K_{n-1}[-1] \rightarrow f_{n(n-1)} K_n))).
\]

Since \(\Phi\) is a dg functor between pretriangulated categories, it preserves cones. This implies that

\[
\Phi(\text{Cone}(K_{n-1}[-1] \rightarrow f_{n(n-1)} K_n)) = \text{Cone}(\Phi(K_{n-1})[-1] \rightarrow \Phi(f_{n(n-1)} K_n)),
\]

and

\[
\Phi(\text{Cone}(K_{n-2}[-1] \rightarrow (f_{n-1|n-2}, f_{n(n-2)}) \rightarrow \text{Cone}(K_{n-1}[-1] \rightarrow f_{n(n-1)} K_n)))
\]

\[
= \text{Cone}(\Phi(K_{n-2})[-1] \rightarrow (\Phi(f_{n-1|n-2}, f_{n(n-2)}) \rightarrow \text{Cone}(\Phi(K_{n-1})[-1] \rightarrow \Phi(f_{n(n-1)} K_n))),
\]

and so on. This proves the proposition. \(\square\)

**Remark 2.4.** Note that we do not necessarily have

\[
\Phi(f_{n-1|n-2}, f_{n(n-2)}) = \Phi(f_{(n-1)(n-2)}, \Phi(f_{n(n-2)})).
\]

Thus, in general, we do not have

\[
f'_{ij} = \Phi(f_{ij}).
\]

Now, we define the notion of (split-)generators, in order to define the categorical entropy in Definition 2.8

**Definition 2.5.**

1. We say \(G_1, \ldots, G_m \in \mathcal{C}\) generate \(\mathcal{C}\), if the full dg subcategory \(\mathcal{D}\) of \(\mathcal{C}\) consisting of \(G_1, \ldots, G_m\) satisfies

\[
\text{Tw}(\mathcal{D}) \cong \text{Tw}(\mathcal{C}).
\]
(2) Let $\text{Perf}(\mathcal{C})$ be the split-closed pretriangulated dg category obtained by splitting direct summands in $\text{Tw}(\mathcal{C})$.

(3) We say $G_1, \ldots, G_m \in \mathcal{C}$ split-generate $\mathcal{C}$, if the full dg subcategory $\mathcal{D}$ of $\mathcal{C}$ consisting of $G_1, \ldots, G_m$ satisfies

$$\text{Perf}(\mathcal{D}) = \text{Perf}(\mathcal{C}).$$

Let $G_1, \cdots, G_m$ generate $\mathcal{C}$. Then, for any object $K$ in $\text{Tw}(\mathcal{C})$, there is a twisted complex for $K$ such that $G_1, \cdots, G_m$ are components of the twisted complex. Thus, Definition 2.6 makes sense.

**Definition 2.6.** Assume that the generators $G_1, \ldots, G_m$ satisfy

- $G_i \neq 0$ for all $i$,
- $G_i \neq G_j[n]$ for any $n \in \mathbb{Z}$ when $i \neq j$.

For a given twisted complex $(K = K_1 \oplus K_2 \oplus \cdots \oplus K_n, f)$ with the components $G_1, \ldots, G_m$ such that $K_i = G_i[n_i]$, and for a given $t \in \mathbb{R}$, the length of $(K, f)$ (with respect to $G_1, \ldots, G_m$ at $t$) is defined as

$$\text{len}_t(K, f) := \sum_{i=1}^{n} e^{n_i t}.$$

If $t = 0$, then the length of $K$ is just $n$.

From now on, we will assume that the generators/split-generators satisfy the assumptions above.

**Definition 2.7.** Let $C \in \text{Perf}(\mathcal{C})$ and let $G_1, \ldots, G_m$ be split-generators of $\mathcal{C}$. The complexity of $C$ with respect to $G_1, \ldots, G_m$ is defined as

$$\delta_t(G_1, \ldots, G_m; C) := \inf \{\text{len}_t((K, f)) : (K, f) \text{ is a twisted complex with components } G_1, \ldots, G_m \text{ for } C \oplus C' \text{ for some } C' \in \text{Perf}(\mathcal{C})\}.$$

Note that the complexity is zero if and only if $C = 0$.

Definition 2.8 is the original definition of categorical entropy, given in [DHKK14].

**Definition 2.8 ([DHKK14]).** Let $G$ be a split-generator of $\mathcal{C}$, and $\Phi : \text{Perf}(\mathcal{C}) \to \text{Perf}(\mathcal{C})$ be a dg functor. Then, for a given $t \in \mathbb{R}$, the categorical entropy of $\Phi$ is defined as

$$h_t(\Phi) := \lim_{n \to \infty} \frac{1}{n} \log \delta_t(G; \Phi^n(G)) \in (-\infty) \cup \mathbb{R}.$$

We note that the categorical entropy of $\Phi$ is well-defined and independent of the choice of the split-generator $G$. We also note that $h_t(\Phi) = -\infty$ if $\Phi^n(G) \approx 0$ for some $n$. One can also define the categorical entropy for $\Phi : \text{Tw}(\mathcal{C}) \to \text{Tw}(\mathcal{C})$ by considering its induced functor $\Phi : \text{Perf}(\mathcal{C}) \to \text{Perf}(\mathcal{C})$.

**Remark 2.9.** Since the notion of categorical entropy, Definition 2.8 is defined on the triangulated category with a split-generator, we assume that every triangulated category admits a split-generator through out this paper.

Definition 2.8 defines the categorical entropy with the length with respect to a split-generator. The following proposition shows that one can also compute the categorical entropy from the length with respect to a generating set.

**Proposition 2.10.** Let $G_1, \ldots, G_m$ be split-generators for $\mathcal{C}$. Let $G := G_1 \oplus \cdots \oplus G_m$. Let $\Phi : \text{Perf}(\mathcal{C}) \to \text{Perf}(\mathcal{C})$ be a dg functor. Then, for a given $t \in \mathbb{R}$, the categorical entropy of $\Phi$ can be given by

$$h_t(\Phi) = \lim_{n \to \infty} \frac{1}{n} \log \delta_t(G_1, \ldots, G_m; \Phi^n(G)).$$
Proof. First, note that $G$ is a split-generator for $\mathcal{C}$. Let $(K, f)$ be an arbitrary twisted complex for $\Phi^n(G) \oplus G'$ with the components $G_1, \ldots, G_m$ for some $G'$. Then one can replace each $G_i$ with $G$ to get a twisted complex $(K', f')$ for $\Phi^n(G) \oplus G' \oplus G''$ with the component $G$ for some $G''$. Note that

$$\text{len}_t(K', f') \text{ (with respect to } G) = \text{len}_t(K, f) \text{ (with respect to } G_1, \ldots, G_m).$$

This shows

$$\delta_t(G; \Phi^n(G)) \leq \delta_t(G_1, \ldots, G_m; \Phi^n(G)).$$

On the other hand, let $(L, g)$ be an arbitrary twisted complex for $\Phi^n(G) \oplus G'$ with the component $G$ for some $G'$. Then if you replace each $G$ with $G_1 \oplus \ldots \oplus G_m$, we get a twisted complex $(L', g')$ for $\Phi^n(G) \oplus G'$ with the components $G_1, \ldots, G_m$. Note that

$$\text{len}_t(L', g') \text{ (with respect to } G_1, \ldots, G_m) = m \times \text{len}_t(L, g) \text{ (with respect to } G).$$

This shows

$$\delta_t(G_1, \ldots, G_m; \Phi^n(G)) \leq m \delta_t(G; \Phi^n(G)).$$

By applying $\lim_{n \to \infty} \frac{1}{n} \log$ to the inequalities

$$\delta_t(G_1, \ldots, G_m; \Phi^n(G)) \leq m \delta_t(G; \Phi^n(G)) \leq m \delta_t(G; \Phi^n(G)),$$

we get

$$h_t(\Phi) \leq \lim_{n \to \infty} \frac{1}{n} \log \delta_t(G_1, \ldots, G_m; \Phi^n(G)) \leq h_t(\Phi).$$

\[\Box\]

3. Entropy for localisations of categories

The main theorem of Section 3, Theorem 3.8, is a comparison of categorical entropies on a category, its subcategory, and their localisation category. The motivation of Theorem 3.8 comes from the fact that the wrapped Fukaya category is a localisation of any partially wrapped Fukaya category. One can find another motivation from some properties of topological entropy. See Section 3.3.

We will prove Theorem 3.8 in Section 3.2. In Section 3.1, we prepare for stating and for proving Theorem 3.8.

3.1. Localisation of a category. Let $\mathcal{C}$ be a dg category. First, we define the notion of localisations of categories.

Definition 3.1 ([Dr04]).

1. If $\mathcal{D}$ is a full dg subcategory of $\mathcal{C}$, then the dg quotient $\mathcal{C} / \mathcal{D}$ is obtained from $\mathcal{C}$ by adding a degree $-1$ morphism $\epsilon_D : D \to D$ for each $D \in \mathcal{D}$ freely (in algebra level), such that

$$d \epsilon_D = 1_D.$$

2. The localisation functor

$$l : \mathcal{C} \to \mathcal{C} / \mathcal{D}$$

is the functor sending everything to itself.

It is easy to check that $D \cong 0$ in $\mathcal{C} / \mathcal{D}$ for all $D \in \mathcal{D} \subset \mathcal{C}$. Moreover, the localisation functor $l$ can be extend to the category of twisted complexes, i.e., there exists an extension, denoted by $l$ again,

$$l : \text{Tw}(\mathcal{C}) \to \text{Tw}(\mathcal{C} / \mathcal{D}).$$

Proposition 3.2 is about the extended localisation functor.

Proposition 3.2. The induced localisation functor $l : \text{Tw}(\mathcal{C}) \to \text{Tw}(\mathcal{C} / \mathcal{D})$ is essentially surjective.
\textbf{Proof.} By [Dri04], the functor $\text{Tw}(\mathcal{C})/\text{Tw}(\mathcal{D}) \to \text{Tw}(\mathcal{C}/\mathcal{D})$ is a quasi-equivalence. Since $\text{Tw}(\mathcal{C})$ and $\text{Tw}(\mathcal{C})/\text{Tw}(\mathcal{D})$ have the same objects, we get the result. \hfill $\square$

In Section 3, we compare the categorical entropies of functors on $\mathcal{C}, \mathcal{D}$ and $\mathcal{C}/\mathcal{D}$, where those functors are described in Proposition 3.3.

\textbf{Proposition 3.3.} Let $\mathcal{D}$ be a full dg subcategory of $\mathcal{C}$. Let $\Phi_\mathcal{C}: \text{Tw}(\mathcal{C}) \to \text{Tw}(\mathcal{C})$ be a dg functor satisfying $\Phi_\mathcal{C}(\mathcal{D}) \subset \text{Tw}(\mathcal{D})$. Let $\Phi_\mathcal{D}$ be the restriction of $\Phi_\mathcal{C}$ on $\text{Tw}(\mathcal{D})$, i.e.,

$$\Phi_\mathcal{D} = \Phi_\mathcal{C}|_{\text{Tw}(\mathcal{D})}: \text{Tw}(\mathcal{D}) \to \text{Tw}(\mathcal{D}).$$

Then, there exists a unique up to natural transformation dg functor

$$\Phi_\mathcal{C}/\mathcal{D}: \text{Tw}(\mathcal{C}/\mathcal{D}) \to \text{Tw}(\mathcal{C}/\mathcal{D}),$$

satisfying

$$\Phi_\mathcal{C}/\mathcal{D} \circ \mathcal{I} = \mathcal{I} \circ \Phi_\mathcal{C},$$

where $\mathcal{I}: \text{Tw}(\mathcal{C}) \to \text{Tw}(\mathcal{C}/\mathcal{D})$ is the (induced) localisation functor.

\textbf{Proof.} To define $\Phi_\mathcal{C}/\mathcal{D}$, consider the functor $\Phi_\mathcal{C}/|_{\mathcal{C}}: \mathcal{C} \to \text{Tw}(\mathcal{C})$. Since $\Phi_\mathcal{C}/|_{\mathcal{C}}(\mathcal{D}) \subset \text{Tw}(\mathcal{D})$, we have

$$(\mathcal{I} \circ \Phi_\mathcal{C}/|_{\mathcal{C}})(\mathcal{D}) \simeq 0.$$

Consequently, there is the unique decomposition of $\mathcal{I} \circ \Phi_\mathcal{C}/|_{\mathcal{C}}$ through $\mathcal{C}/\mathcal{D}$ (up to natural transformation), i.e., there exists

$$\Phi_\mathcal{C}/|_{\mathcal{C}}: \mathcal{C}/\mathcal{D} \to \text{Tw}(\mathcal{C}/\mathcal{D}),$$

such that

$$\Phi_\mathcal{C}/|_{\mathcal{C}} \circ \mathcal{I} = \mathcal{I} \circ \Phi_\mathcal{C}/|_{\mathcal{C}}.$$

We note that the domain of the above compositions is $\mathcal{C}$.

Then, $\Phi_\mathcal{C}/|_{\mathcal{C}}$ induces a functor

$$\Phi_\mathcal{C}/\mathcal{D}: \text{Tw}(\mathcal{C}/\mathcal{D}) \to \text{Tw}(\mathcal{C}/\mathcal{D}),$$

satisfying

$$\Phi_\mathcal{C}/\mathcal{D} \circ \mathcal{I} = \mathcal{I} \circ \Phi_\mathcal{C}/|_{\mathcal{C}}.$$ 

$\square$

In the rest of Section 3, we assume that there are $G_1, \ldots, G_m, D_1, \ldots, D_r \in \mathcal{C}$ such that split-generate $\mathcal{C}$. Also, we let $\mathcal{D}$ be the full dg subcategory of $\mathcal{C}$ such that the objects $D_1, \ldots, D_r$ split-generate $\mathcal{D}$. Then obviously, $G_1, \ldots, G_m$ split-generate $\mathcal{C}/\mathcal{D}$. Moreover, if we set

$$G := G_1 \oplus \cdots \oplus G_m, D := D_1 \oplus \cdots \oplus D_r,$$

and $\overline{G} := G \oplus D$,

then $G$ split-generates $\mathcal{C}/\mathcal{D}$, $D$ split-generates $\mathcal{D}$, and $\overline{G}$ split-generates $\mathcal{C}$.

From here to the end of Section 3, the length of a twisted complex in $\mathcal{D}$ (resp. $\mathcal{C}/\mathcal{D}$) means the length with respect to $D$ (resp. $G$) in the sense of Definition 2.6. For a twisted complex in $\mathcal{C}$, Definition 3.4 is useful.

\textbf{Definition 3.4.} Let $(K = K_1 \oplus K_2 \oplus \cdots \oplus K_n, f)$ be a twisted complex in $\mathcal{C}$ with the components $G$ and $D$. In other words, for any $i$, either $K_i = G[n_i]$ or $K_i = D[n_i]$ for some $n_i \in \mathbb{Z}$.

(1) $G$-components of $(K, f)$ (resp. $D$-components of $(K, f)$) are $K_i$’s such that

$$K_i = G[n_i] \ (\text{resp. } D[n_i]) \ \text{for some } n_i \in \mathbb{Z}$$
(2) For a given \( t \in \mathbb{R} \), we define
\[
\text{len}_{t,G}(K, f) := \sum_{K_i = G[n_i]} e^{n_i t},
\]
and
\[
\text{len}_{t,D}(K, f) := \sum_{K_j = D[n_j]} e^{n_j t}.
\]

Then obviously, we have
\[
\text{len}_t(K, f) \text{ (with respect to } G, D) = \text{len}_{t,G}(K, f) + \text{len}_{t,D}(K, f).
\]

**Remark 3.5.** In this section, we consider the generating set \( \{G, D\} \). We note that Definition 3.4 can be generalized for any generating set. For example, if one considers the generating set
\[
\{G_1, \ldots, G_m, D_1, \ldots, D_r\},
\]
then one can define \( G_i \) (resp. \( D_j \))-components and \( \text{len}_{t,G}(K, f) \) (resp. \( \text{len}_{t,D}(K, f) \)). We will use this in Section 7.

The following lemma will be the main ingredient of Theorem 3.8.

**Lemma 3.6.** Let \( \tilde{K} \in \text{Tw}(\mathcal{E}) \), and let \( \tilde{L} := l(\tilde{K}) \in \text{Tw}(\mathcal{E}/\mathcal{D}) \). If \((L, g)\) is a twisted complex for \( \tilde{L} \) in \( \text{Tw}(\mathcal{E}/\mathcal{D}) \)
with the component \( G \), then there exists a twisted complex \((\tilde{K}, f)\) for \( \tilde{K} \) in \( \text{Tw}(\mathcal{E}) \) with \( G, D \)-components such that the \( G \)-components of \((K, f)\) are the same as the \( G \)-components of \((L, g)\), i.e., the numbers of \( G \)-components in two twisted complexes (resp. the shifts of each of \( G \)-components) are the same. In particular, \[
\text{len}_{t,G}(K, f) = \text{len}_t(L, g).
\]

**Proof.** Before we start the proof, note that for any \( X, Y \in \text{Tw}(\mathcal{E}) \), we have the isomorphisms
\[
\lim_{(Y \to Z) \in Q_Y} \text{Hom}^\ast_{\text{Tw}(\mathcal{E})}(X, Z) \xrightarrow{\sim} \text{Hom}^\ast_{\text{Tw}(\mathcal{E}/\mathcal{D})}(X, Y) \xrightarrow{\sim} \text{Hom}^\ast_{\text{Tw}(\mathcal{E}/\mathcal{D})}(l(X), l(Y)),
\]
where \( Q_Y \) is the (filtered) category of morphisms \( f : Y \to Z \) in \( \text{Tw}(\mathcal{E}) \) such that \( \text{Cone}(f) \) is isomorphic to an object of \( \text{Tw}(\mathcal{D}) \). The first isomorphism is from \([\text{Ver}77, \text{Ver}96]\) (which is restated in \([\text{Dri}04]\), and the second isomorphism is from the fact that \( \text{Tw}(\mathcal{E})/\text{Tw}(\mathcal{D}) \) and \( \text{Tw}(\mathcal{E}/\mathcal{D}) \) are quasi-equivalent by \([\text{Dri}04]\).

This shows, for any \( \beta \in \text{Hom}^\ast_{\text{Tw}(\mathcal{E}/\mathcal{D})}(l(X), l(Y)) \), there exists \( Z \in \text{Tw}(\mathcal{E}) \) such that

(i) \( Z = \text{Cone}(W \to Y) \) for some \( W \in \text{Tw}(\mathcal{D}) \),

(ii) there is a morphism \( \alpha \in \text{Hom}^\ast_{\text{Tw}(\mathcal{E})}(X, Z) \) so that \( l(\alpha) = \beta \).

With the above arguments, we can prove the lemma by induction on the number of \( G \)-components of \((L, g)\).

For the base step, let us assume that \((L, g)\) has no \( G \)-components. Since \( \mathcal{E}/\mathcal{D} \) is generated by \( G \), \( l(\tilde{K}) = \tilde{L} \approx 0 \), i.e., \( \tilde{K} \) is in the kernel of \( l \). By \([\text{Kra}10]\), it means that \( \tilde{K} \) is isomorphic to an object in \( \text{Tw}(\mathcal{D}) \). Hence there is a twisted complex \((K, f)\) for \( \tilde{K} \) with \( D \)-components and no \( G \)-components. Thus, \((K, f)\) and \((L, g)\) have the same \( G \)-components.

For the inductive step, let us assume the induction hypothesis, i.e., we assume that the lemma holds for any \( \tilde{K} \) such that \( l(\tilde{K}) \) has a twisted complex \((L, g)\) with \( n \) many \( G \)-components.

Now we consider \( \tilde{K} \) such that \( \tilde{L} := l(\tilde{K}) \) admits a twisted complex \((L, g)\) with \((n+1)\) many \( G \)-components in \( \text{Tw}(\mathcal{E}/\mathcal{D}) \). We note that, as mentioned in Remark 2.2, the twisted complex \((L, g)\) can be written as an iterated cone. Thus, one can find another twisted complex \((L_1, g_1)\) in \( \text{Tw}(\mathcal{E}/\mathcal{D}) \) such that
\[
\tilde{L} \cong (L, g) = \text{Cone}\left((L_1, g_1) \xrightarrow{\beta} G[m]\right),
\]
for some $m \in \mathbb{Z}$. Especially, the number of $G$-components in $(L_1, g_1)$ is $n$ at $t = 0$.

From the fact that the localisation functor $l$ is essentially surjective, one knows the existence of a twisted complex $(K_1, f_1) \in \text{Tw}(\mathcal{E})$ such that

$$l(K_1, f_1) = (L_1, g_1).$$

Moreover, by the induction hypothesis, one can assume that two twisted complexes $(K_1, f_1)$ and $(L_1, g_1)$ have the same $G$-components.

We note that $G[m]$ can be seen as an object in $\text{Tw}(\mathcal{E})$. Then, from the isomorphisms in (3.1), one can show that there is an object $Z_1 \in \text{Tw}(\mathcal{E})$ such that

(i) $\quad Z_1 = \text{Cone}(W_1 \to G[m])$ for some $W_1 \in \text{Tw}(\mathcal{D})$,

(ii) there is a morphism $\alpha \in \text{Hom}_{\text{Tw}(\mathcal{E})}((K_1, f_1), Z_1)$ so that $l(\alpha) = \beta$.

Now, we consider the cone of $\alpha : (K_1, f_1) \to Z_1$. Let

$$\tilde{K}_2 := \text{Cone} \left( (K_1, f_1) \xrightarrow{\alpha} Z_1 \right) \in \text{Tw}(\mathcal{E}).$$

Then, one can easily observe that $\tilde{K}_2$ is a lift of

$$\tilde{L} := \text{Cone} \left( (L_1, g_1) \xrightarrow{\beta} G[m] \right) = l(\tilde{K}) \in \text{Tw}(\mathcal{E} / \mathcal{D}),$$

with respect to the localisation functor $l$, i.e., $l(\tilde{K}_2) = \tilde{L}$. Hence,

$$l(\tilde{K}) = l(\tilde{K}_2).$$

This shows that there is a homotopy equivalence $\delta : l(\tilde{K}) \to l(\tilde{K}_2)$. Consequently, from (3.1), one obtains $Z_2 \in \text{Tw}(\mathcal{E})$ such that

(i) $\quad Z_2 = \text{Cone}(W_2 \to \tilde{K}_2)$ for some $W_2 \in \text{Tw}(\mathcal{D})$,

(ii) there is a morphism $\gamma \in \text{Hom}_{\text{Tw}(\mathcal{E})}(\tilde{K}, Z_2)$ so that $l(\gamma) = \delta$.

Let

$$W_0 := \text{Cone}(\tilde{K} \xrightarrow{\gamma} Z_2).$$

Then, since $l(\gamma) \cong \delta$,

$$l(W_0) = l \left( \text{Cone}(\gamma) \right) \cong \text{Cone}(\delta) = 0.$$

In other words, $W_0 \in \text{Tw}(\mathcal{D})$ by [Kra10].

Collecting all the above results, we get

$$\tilde{K} = \text{Cone} \left( Z_2[-1] \to W_0[-1] \right)$$

$$\cong \text{Cone} \left( \text{Cone}(W_2 \to \tilde{K}_2)[-1] \to W_0[-1] \right)$$

$$\cong \text{Cone} \left( \text{Cone} \left( W_2 \to \text{Cone}((K_1, f_1) \xrightarrow{\alpha} Z_1)[-1] \to W_0[-1] \right) \right)$$

$$\cong \text{Cone} \left( \text{Cone} \left( W_2 \to \text{Cone}((K_1, f_1) \xrightarrow{\alpha} \text{Cone}(W_1 \to G[m]))[-1] \to W_0[-1] \right) \right).$$

As mentioned in Remark 2.2, one can convert the iterated cone in (3.2) to a twisted complex for $\tilde{K}$. As a twisted complex, the object part of $\tilde{K}$ is

$$W_2[1] \oplus K_1[1] \oplus W_1[1] \oplus G[m] \oplus W_0[-1].$$

Note that $W_0, W_1, W_2$ are objects in $\text{Tw}(\mathcal{D})$. Moreover, $(K_1, f_1)$ is a twisted complex in $\text{Tw}(\mathcal{E})$ such that the $G$-components in $(K_1, g_1)$ is the same to that of $(L_1, g_1)$.

As a twisted complex, the object part of $\tilde{L}$ is

$$L_1[1] \oplus G[m].$$
Then, the $G$-components in the twisted complexes in $\overline{K}$ and $\overline{L}$ should be the same. This completes the proof.

3.2. Comparison of entropies of $\Phi_{\overline{e}}, \Phi_{\overline{d}},$ and $\Phi_{\overline{e} \overline{d}}$. Now we are ready to compare the categorical entropies on $\mathcal{C}, \mathcal{D}$ and $\mathcal{C} \mathcal{D}$.

With Definition 3.4, it is easy to prove Proposition 3.7, which is the first comparison of the categorical entropies on $\mathcal{C}$ and $\mathcal{C} \mathcal{D}$.

**Proposition 3.7.** Let $\Phi_{\overline{e}} : \text{Tw}(\mathcal{C}) \rightarrow \text{Tw}(\mathcal{C})$ be a dg functor satisfying $\Phi_{\overline{e}}(\mathcal{D}) \subset \text{Tw}(\mathcal{D})$. Let $\Phi_{\overline{e} \overline{d}} : \text{Tw}(\mathcal{C} \mathcal{D}) \rightarrow \text{Tw}(\mathcal{C} \mathcal{D})$ be the induced functor (in the sense of Proposition 3.3). Then, for any $t \in \mathbb{R}$, we have

$$h_t(\Phi_{\overline{e}}) \leq h_t(\Phi_{\overline{e} \overline{d}}).$$

**Proof.** Let $(K, f)$ be a twisted complex for $\Phi^n_{\overline{e}}(G) \oplus \overline{K}$ in $\mathcal{C}$ with the components $G$ and $D$ for some $\overline{K}$. Applying the localisation functor $l$ to $(K, f)$, we get a twisted complex $(L, g)$ for

$$l(\Phi^n_{\overline{e}}(G) \oplus \overline{K}) = \Phi^n_{\overline{e} \overline{d}}(l(G)) \oplus l(\overline{K})$$

in $\mathcal{C} \mathcal{D}$ with the component $G$. Moreover, the $G$-components in $(K, f)$ are preserved in the new twisted complex $(L, g)$. Thus,

$$\text{len}_t(L, g) = \text{len}_{\mathcal{C}}(K, f) \leq \text{len}_t(K, f).$$

This shows that

$$\delta_t(G; \Phi^n_{\overline{e} \overline{d}}(G)) \leq \delta_t(G, D; \Phi^n_{\overline{e}}(G)),$$

which implies that

$$h_t(\Phi_{\overline{e} \overline{d}}) \leq h_t(\Phi_{\overline{e}}).$$

Proposition 3.7 gives a lower bound for the entropy of $\Phi_{\overline{e}}$. Theorem 3.8 gives an upper bound.

**Theorem 3.8.** Let $\mathcal{D}$ be a full dg subcategory of $\mathcal{C}$. Let $\Phi_{\overline{e}} : \text{Tw}(\mathcal{C}) \rightarrow \text{Tw}(\mathcal{C})$ be a dg functor satisfying $\Phi_{\overline{e}}(\mathcal{D}) \subset \text{Tw}(\mathcal{D})$, and let $\Phi_{\overline{d}} : \text{Tw}(\mathcal{D}) \rightarrow \text{Tw}(\mathcal{D})$ and $\Phi_{\overline{e} \overline{d}} : \text{Tw}(\mathcal{C} \mathcal{D}) \rightarrow \text{Tw}(\mathcal{C} \mathcal{D})$ be the induced functors (in the sense of Proposition 3.3). Then, for any $t \in \mathbb{R}$, we have

$$h_t(\Phi_{\overline{e} \overline{d}}) \leq h_t(\Phi_{\overline{e}}) \leq \max(h_t(\Phi_{\overline{e} \overline{d}}), h_t(\Phi_{\overline{d}})).$$

The main idea of Theorem 3.8 is to consider two twisted complexes in $\mathcal{C}$, which are related to $\Phi^n_{\overline{e}}(G)$. To be more precise, we recall that $h_t(\Phi_{\overline{e}})$ is defined as

$$h_t(\Phi_{\overline{e}}) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \delta_t(G; \Phi^n_{\overline{e}}(G)).$$

Thus, for sufficiently large $n$, there is a twisted complex $(K_n, f_n)$ in $\mathcal{C}$ such that

- there is an object $K'_n$ so that $(K_n, f_n)$ is a twisted complex for $\Phi^n_{\overline{e}}(G) \oplus K'_n$, and
- the length of $(K_n, f_n)$ with respect to $G$ is asymptotically $e^{nh_t(\Phi_{\overline{e}})}$.

Then, the above $(K_n, f_n)$ will be the first twisted complex which we will consider.

Similarly, in $\mathcal{C} \mathcal{D}$ (resp. $\mathcal{D}$), there is a twisted complex $(L_n, g_n)$ (resp. $(M_n, h_n)$) such that

- there is an object $L'_n$ (resp. $M'_n$) so that $(L_n, g_n)$ (resp. $(M_n, h_n)$) is a twisted complex for $\Phi^n_{\overline{e} \overline{d}}(G) \oplus L'_n$ (resp. $\Phi^n_{\overline{d}}(D) \oplus M'_n$), and
- the length of $(L_n, g_n)$ (resp. $(M_n, h_n)$) with respect to $G$ (resp. $D$) is asymptotically $e^{nh_t(\Phi_{\overline{e} \overline{d}})}$ (resp. $e^{nh_t(\Phi_{\overline{d}})}$).
Then, there is a twisted complex \((E_n, e_n)\) for \(\Phi^n_\g(C)\) in \(\g\) whose length related to those of \((L_n, f_n), (M_n, h_n)\) by Lemma \ref{lem:comparison}.

We will compare \(\text{len}_t(K_n, f_n) =: a_n\) and \(\text{len}_t(E_n, e_n) =: a'_n\) with Lemma \ref{lem:comp}. We note that, roughly, the length of \((K_n, f_n)\) should be bigger than that of \((E_n, e_n)\) with a small correction term. Since the logarithmic growth rate of length of \((E_n, e_n)\) is determined by the maximum of the logarithmic growth rates of the lengths of \((L_n, g_n)\) or \((M_n, h_n)\), the comparison will prove Theorem \ref{thm:main}. The detailed proof of Theorem \ref{thm:main} will appear after Lemma \ref{lem:comp}.

**Lemma 3.9.** Let \((a_n)_{n=0}^\infty\) and \((a'_n)_{n=0}^\infty\) be sequences of positive real numbers, and \((b_n)_{n=0}^\infty\) be a sequence of real numbers satisfying

1. \(\frac{a'_{n+1}}{a'_n} \leq \frac{a_{n+1}}{a_n}\) for sufficiently large \(n\),
2. \(a_n \leq a'_n + b_n\) for sufficiently large \(n\), and
3. there is a real number \(R \in \mathbb{R}\) such that \(\frac{b_n}{a'_n} < R\) for sufficiently large \(n\).

Then there exists an increasing sequence of natural numbers \(\{n_i|i \in \mathbb{Z}_{\geq 0}\}\) such that

\[
\lim_{i \to \infty} \frac{a_{n_i+1}}{a_{n_i}} = 1.
\]

**Proof.** Without loss of generality, we assume that the three conditions hold for all \(n\).

Let us assume the contrary. Then, there exists a pair \((M \in \mathbb{R}_{>1}, N \in \mathbb{N})\) such that

\[
\frac{a_{n+1}}{a'_{n+1}} \geq M,
\]

for all \(n \geq N\).

Since \(a'_n > 0\), the inequality \(a_n \leq a'_n + b_n\) gives

\[
\frac{a_n}{a'_n} \leq 1 + \frac{b_n}{a'_n}.
\]

If \(n > N\),

\[
\prod_{i=0}^{N-1} \left( \frac{a_{i+1}/a_i}{a'_{i+1}/a'_i} \right) M^{n-N} < \prod_{i=0}^{n-1} \left( \frac{a_{i+1}/a_i}{a'_{i+1}/a'_i} \right) = \frac{a_n/a_0}{a'_n/a'_0} = \frac{a_n}{a'_{0}} \leq \frac{a'_0}{a'_0} (1 + \frac{b_n}{a'_n}) = \frac{a'_n}{a'_0} (1 + \frac{b_n}{a'_n})\]

Finally, we have

\[
\prod_{i=0}^{N-1} \left( \frac{a_{i+1}/a_i}{a'_{i+1}/a'_i} \right) M^{n-N} \leq \frac{a'_0}{a'_0} (1 + \frac{b_n}{a'_n}) = \frac{a'_n}{a'_0} (1 + \frac{b_n}{a'_n}) = \frac{a'_n}{a'_0} (1 + R),
\]

because of the third condition of Lemma \ref{lem:comp}. This gives a contradiction since

\[
\lim_{n \to \infty} M^{n-N} = \infty.
\]

**Proof of Theorem \ref{thm:main}** The first inequality is Proposition \ref{prop:main}, thus it is enough to prove the second inequality

\[
h_t(\Phi_\g) \leq \max\{h_t(\Phi_\g/\emptyset), h_t(\Phi_\emptyset)\}.
\]

If \(h_t(\Phi_\g) = -\infty\), then the above holds. Thus, let us assume that \(h_t(\Phi_\g) > -\infty\).

We note that, in the rest of the proof, we use the following notation

\[
\alpha := h_t(\Phi_\g), \beta := h_t(\Phi_\g/\emptyset), \gamma := h_t(\Phi_\emptyset),
\]
for convenience. Moreover, we remark that the twisted complexes \((K_n, f_n), (E_n, e_n)\) are related to \(C/\mathcal{D}\), and \((M_n, h_n)\) is in \(\mathcal{D}\), in the rest of the proof.

**Step 1:** The first step is to choose a twisted complex \((K_n, f_n)\) in \(\mathcal{C}\) whose length is related to \(\alpha\).

Let \(\epsilon\) be a fixed positive real number. Then, there is a natural number \(N_1\) such that for all \(n \geq N_1\), the following holds:

\[
\left| \frac{1}{n} \log \delta_1(G, D; \Phi^n_{\bar{\varepsilon}}(G)) - \alpha \right| < \epsilon.
\]

Thus, there exists a twisted complex \((K_n, f_n)\) in \(\mathcal{C}\) such that

- there is an object \(K'_n\) in \(\mathcal{C}\) so that \((K_n, f_n) \simeq \Phi^n_{\bar{\varepsilon}}(G) \oplus K'_n\),
- the components of \((K_n, f_n)\) are \(G\) and \(D\),
- \(\text{len}_t(K_n, f_n)\) is sufficiently close to \(\delta_1(G, D; \Phi^n_{\bar{\varepsilon}}(G))\), or more precisely, the following hold:

\[
\begin{align*}
0 \leq \text{len}_t(K_n, f_n) - \delta_1(G, D; \Phi^n_{\bar{\varepsilon}}(G)) & \leq e^{n(\alpha - \epsilon)}, \\
e^{n(\alpha - \epsilon)} & \leq \text{len}_t(K_n, f_n) < e^{n(\alpha + \epsilon)}.
\end{align*}
\]

The second step is to choose twisted complexes \((L_n, g_n)\) and \((M_n, h_n)\) in \(C/\mathcal{D}\) and \(\mathcal{D}\), whose lengths are related to \(\beta\) and \(\gamma\) respectively. For the choices, we consider three different cases, the first case is \(\beta > -\infty, \gamma > -\infty\), the second case is \(\beta = -\infty, \gamma = -\infty\), and the third case is that one of \(|\beta, \gamma|\) is \(-\infty\) and the other is not. We will consider the first case in steps 2–6, the second case in step 7, and the third case in step 8.

**Step 2:** As mentioned above, we assume that \(\beta > -\infty, \gamma > -\infty\). Then, for a fixed \(\epsilon\), there is a natural number \(N_2\) (resp. \(N_3\)) such that

\[
\left| \frac{1}{n} \log \delta_1(G; \Phi^n_{\bar{\varepsilon} / \mathcal{D}}(G)) - \beta \right| < \epsilon, \text{ for all } n \geq N_2,
\]

\[
\left| \frac{1}{n} \log \delta_1(D; \Phi^n_{\bar{\varepsilon}}(D)) - \gamma \right| < \epsilon, \text{ for all } n \geq N_3.
\]

Moreover, for any \(n \geq N_2\) (resp. \(N_3\)), there is a twisted complex \((L_n, g_n)\) (resp. \((M_n, h_n)\)) in \(\mathcal{C}/\mathcal{D}\) (resp. \(\mathcal{D}\)) such that

- there is an object \(L'_n\) (resp. \(M'_n\)) in \(\mathcal{C}/\mathcal{D}\) (resp. \(\mathcal{D}\)) so that \((L_n, g_n) \simeq \Phi^n_{\bar{\varepsilon} / \mathcal{D}}(G) \oplus L'_n\) (resp. \((M_n, h_n) \simeq \Phi^n_{\bar{\varepsilon}}(D) \oplus M'_n\)),
- the components of \((L_n, g_n)\) (resp. \((M_n, h_n)\)) are \(G\) (resp. \(D\)),
- the following hold:

\[
\begin{align*}
\text{len}_t(L_n, g_n) & < e^{n(\beta + \epsilon)} , \\
\text{len}_t(M_n, h_n) & < e^{n(\gamma + \epsilon)}.
\end{align*}
\]

**Step 3:** For the third step, we fix a sufficiently large integer \(N\) such that

\[N \geq \max\{N_1, N_2, N_3\}.
\]

Then, we choose a twisted complex \((E_i, e_i)\) in \(\mathcal{C}\), which will be compared to \((K_n, f_n)\) as described right above of Lemma 3.9 so that

- \(\text{there is an object } E'_i\) in \(\mathcal{C}\) so that \((E_i, e_i) \simeq \Phi^{2 |N}(G) \oplus E'_i\),
- \(\text{the components of } (E_i, e_i)\) are \(G\) and \(D\),
- \(\text{the following hold:}

\[
\begin{align*}
\text{len}_t(L_n, g_n) & < e^{n(\beta + \epsilon)} , \\
\text{len}_t(M_n, h_n) & < e^{n(\gamma + \epsilon)}.
\end{align*}
\]

\[N \geq \max\{N_1, N_2, N_3\}.
\]
(iii) the following equality holds:

\[
\text{len}_{t,G}(E_i, e_i) = \text{len}_t(L_N, g_N)^{2^i}.
\]

We choose \((E_i, e_i)\) inductively.

For the base case, i.e., the case of \(i = 0\), by Lemma 3.6 first note that

\[
(L_N, g_N) \cong \Phi^N_{\mathcal{E} / \mathcal{D}}(G) \oplus L'_N \cong \Phi^N_{\mathcal{E} / \mathcal{D}}(l(G)) \oplus l(E'_0) \cong l(\Phi^N_{\mathcal{E} / \mathcal{D}}(G)) \oplus l(E'_0) = l(\Phi^N_{\mathcal{E} / \mathcal{D}}(G) \oplus E'_0)
\]

for some \(E'_0 \in \text{Tw}(\mathcal{E})\) since \(L'_N \in \text{Tw}(\mathcal{E} / \mathcal{D})\) and \(l: \text{Tw}(\mathcal{E}) \to \text{Tw}(\mathcal{E} / \mathcal{D})\) is essentially surjective by Proposition 3.2. We have indeed \(L'_N \in \text{Tw}(\mathcal{E} / \mathcal{D})\) since \(L'_N = \text{ Cone}(\Phi^N_{\mathcal{E} / \mathcal{D}}(G) \to (L_N, g_N))\), and \(\Phi^N_{\mathcal{E} / \mathcal{D}}(G)\) and \((L_N, g_N)\) are in \(\text{Tw}(\mathcal{E} / \mathcal{D})\).

Then, by Lemma 3.6 there exists a twisted complex \((E_0, e_0)\) in \(\mathcal{E}\) such that

- \((E_0, e_0) = \Phi^N_{\mathcal{E} / \mathcal{D}}(G) \oplus E'_0\),
- the components of \((E_0, e_0)\) are \(G\) and \(D\),
- \(\text{len}_{t,G}(E_0, e_0) = \text{len}_t(L_N, g_N)\).

We note that the last item is (3.7) for \(i = 0\).

In order to choose \((E_i, e_i)\) for all \(i \in \mathbb{N} \cup \{0\}\) inductively, let us assume that there is \((E_i, e_i)\) in \(\mathcal{E}\) satisfying (i)–(iii). One can apply \(\Phi^{2^i}G\) to the twisted complex \((E_i, e_i)\). Then, one obtains a twisted complex \((\tilde{E}_i, \tilde{e}_i)\) such that

- \((\tilde{E}_i, \tilde{e}_i) = \Phi^{2^i}G(\Phi^{2^i}G(\oplus E'_i) = \Phi^{2^{i+1}}G(\oplus \Phi^{2^i}G(E'_i)),
- the components of \((\tilde{E}_i, \tilde{e}_i)\) are \(\Phi^{2^i}G(G)\) and \(\Phi^{2^i}G(D)\),
- the number of \(\Phi^{2^i}G(G)\) (resp. \(\Phi^{2^i}G(D)\)) in \((\tilde{E}_i, \tilde{e}_i)\) is \(\text{len}_{t,G}(E_i, e_i)\) (resp. \(\text{len}_{t,D}(E_i, e_i)\)).

We note that we would like to choose a twisted complex \((E_{i+1}, e_{i+1})\) whose components are \(G\) and \(D\). Since the components in \((\tilde{E}_i, \tilde{e}_i)\) are \(\Phi^{2^i}G(G)\) and \(\Phi^{2^i}G(D)\), we modify the twisted complex. To do that, we replace the components \(\Phi^{2^i}G(D)\) in \((\tilde{E}_i, \tilde{e}_i)\) with

\[
\Phi^{2^i}G(D) \oplus M'_{2^{2^i}} \cong \Phi^{2^i}G(D) \oplus M'_{2^{2^i}} \cong (M'_{2^{2^i}}, h_{2^{2^i}}).
\]

Similarly, we replace the components \(\Phi^{2^i}G(G)\) in \((\tilde{E}_i, \tilde{e}_i)\) with

\[
\Phi^{2^i}G(G) \oplus \Phi^{2^i}G(D) \oplus E'_i = \Phi^{2^i}G(G) \oplus E'_i = (E_i, e_i).
\]

We note that, since the replacements can be understood as taking direct sums \(\oplus\) with \(M'_{2^{2^i}}\) or \(\Phi^{2^i}G(D)\oplus E'_i\), one obtains a new twisted complex equivalent to \(\Phi^{2^{i+1}}G(\oplus E'_{i+1})\) for some \(E'_{i+1}\) after the replacements. We also note that since in \(\mathcal{E}\), the components of \((E_i, e_i)\) (resp. \((M'_{2^{2^i}}, h_{2^{2^i}})\)) are \(G, D\) (resp. \(D\)), one can assume that the new twisted complex consists of components \(G, D\).

To sum up, let \((E_{i+1}, e_{i+1})\) be the new twisted complex obtained after the replacements. Then, the following hold:

- there is an object \(E'_{i+1}\) so that \((E_{i+1}, e_{i+1})\) is a twisted complex for \(\Phi^{2^{i+1}}G(\oplus E'_{i+1})\),
- the components of \((E_{i+1}, e_{i+1})\) are \(G\) and \(D\),
- by the construction, \(\text{len}_{t,G}(E_{i+1}, e_{i+1})\) is the product of the number of \(\Phi^{2^i}G(G)\) in \((\tilde{E}_i, \tilde{e}_i)\) and \(\text{len}_{t,D}(E_i, e_i)\), and both of them are \(\text{len}_t(L_N, g_N)^{2^i}\), thus,

\[
\text{len}_{t,G}(E_{i+1}, e_{i+1}) = \text{len}_t(L_N, g_N)^{2^{i+1}}.
\]
Before going further, we point out that

\[
\text{len}_i(E_{i+1}) = \text{len}_{i,G}(E_{i+1}) + \text{len}_{i,D}(E_{i+1})
\]

(3.8)

\[
= \text{len}_{i,G}(E_i) \cdot \text{len}_{i,G}(E_i) + \text{len}_{i,D}(E_i) \cdot \text{len}_{i,G}(E_i) + \text{len}_{i,D}(E_i) \cdot \text{len}(M_{2^i/N})
\]

\[
= \text{len}_{i,G}(E_i) \cdot \text{len}(E_i) + \text{len}_{i,D}(E_i) \cdot \text{len}(M_{2^i/N}).
\]

We note that in (3.3) we omitted the morphisms \( e_i, e_{i+1}, h_{2^i/N} \) of each twisted complexes for convenience.

**Step 4:** We are proving Theorem 3.8 for the first case, i.e., the case of \( \beta > -\infty, \gamma > -\infty \). If we assume that \( \beta \geq \gamma \), then we would like to prove that

\[
h_t(\Phi \epsilon) = \alpha \leq \beta = \max\{h_t(\Phi \epsilon/\epsilon) = \beta, h_t(\Phi \epsilon) = \gamma\}.
\]

Based on this, we will find a contradiction under the assumptions that \( \beta \geq \gamma \) and \( \alpha > \beta \) in steps 4 and 5.

In step 4, we show that

\[
a_i := \text{len}_{i}(K_{2^i/N}, f_{2^i/N}), a'_i := \text{len}_{i}(E_i, e_i), b_i := e^{2^i N(\alpha - \epsilon)}
\]

satisfy the conditions of Lemma 3.9.

In order to show that \( \frac{a'_{i+1}}{a'_i} \leq \frac{a_{i+1}}{a_i} \), we note that

\[
e^{2^i N(\alpha - \epsilon)} < a_i < e^{2^i N(\alpha + \epsilon)}, \quad e^{2^{i+1} N(\alpha - \epsilon)} < a_{i+1} < e^{2^{i+1} N(\alpha + \epsilon)},
\]

from (3.4). This induces that

\[
e^{2^i N(\alpha - 3\epsilon)} < \frac{a_{i+1}}{a_i} < e^{2^i N(\alpha + 3\epsilon)}.
\]

We also note that for sufficiently large \( i \),

\[
\frac{a'_{i+1}}{a'_i} = \frac{\text{len}_{i,G}(E_i) + \text{len}_{i,D}(E_i) \cdot \text{len}(M_{2^i/N})}{\text{len}_{i,G}(E_i)}
\]

\[
\leq \text{len}_{i}(L_N, g_N) 2^i + \text{len}_{i}(M_{2^i/N})
\]

\[
< e^{2^i N(\beta + \epsilon)} + e^{2^i N(\gamma + \epsilon)}
\]

\[
< (1 + e^{2^i N(\gamma - \beta)}) e^{2^i N(\beta + \epsilon)}
\]

\[
< e \cdot e^{2^i N(\beta + \epsilon)} = e^{2^i N(\beta + \epsilon) + 1},
\]

from (3.3), (3.8) and the assumption that \( \beta \geq \gamma \).

Since we assume that \( \alpha > \beta \), by choosing a sufficiently small \( \epsilon > 0 \) and a sufficiently large \( N \), one can have

\[
2^i N(\beta + \epsilon) + 1 < 2^i N(\alpha - 3\epsilon)
\]

Then, (3.9) and (3.10) show that

\[
\frac{a'_{i+1}}{a'_i} \leq \frac{a_{i+1}}{a_i}.
\]

In order to show that \( a_i \leq a'_i + b_i \) for all \( i \), the following is enough which is given by (3.3):

\[
a_i \leq \delta_{i}(G, D; \Phi_{\epsilon}^{2^i N} \rho_{\epsilon}((G))) + e^{2^i N(\alpha - \epsilon)} \leq \text{len}_{i}(E_i) + e^{2^i N(\alpha - \epsilon)} = a'_i + b_i.
\]

Finally, in order to show the last condition of Lemma 3.9 we note that

\[
a'_i = \text{len}_{i}(E_i) \geq \delta_{i}^{2^i N} \rho_{\epsilon}((G)) > e^{2^i N(\alpha - \epsilon)}.
\]
Then,

\[ \frac{b_i}{a_i} < e^{2^i N(a-\varepsilon) - 2^i N(a-\varepsilon)} = 1. \]

Thus, the last condition of Lemma 3.9 holds.

**Step 5:** By step 4, one can apply Lemma 3.9 for the above \(a_i, a'_i, b_i\). Thus, there is an increasing sequence of natural numbers \(\{i_n\}_{n \in \mathbb{N}}\) such that

\[ \lim_{n \to \infty} \frac{a_{i_n+1}}{a'_{i_n}} = 1. \]

On the other hand, by (3.9) and (3.10),

\[ \frac{a_{i_n+1}}{a'_{i_n}} > e^{2^i N(a-3\varepsilon) - 2^i N(\beta+\varepsilon) - 1}. \]

By choosing a sufficiently large \(N\) and a sufficiently small \(\varepsilon > 0\), the right hand side of the above inequality diverges to \(\infty\) as \(n \to \infty\). This is a contradiction. Thus, \(\alpha \leq \beta\) if \(\beta \geq \gamma\).

**Step 6:** Now we assume that \(\gamma > \beta\), then we would like to show that \(\alpha \leq \gamma\). Under the assumption that \(\alpha > \gamma\), the arguments in steps 4 and 5 give a contradiction again.

To sum up, steps 2–6 prove Theorem 3.8 for the first case.

**Step 7:** In step 7, we will consider the second case, i.e., the case that

\[ \beta = -\infty, \gamma = -\infty. \]

Since \(\beta = h_t(\Phi_{\mathcal{C}/\mathcal{D}}) = -\infty\), for any \(R \in \mathbb{R}\), there is a \(N_2 \in \mathbb{N}\) such that for all \(n \geq N_2\),

\[ \frac{1}{n} \log \delta_t(G; \Phi^n_{\mathcal{C}/\mathcal{D}}(G)) < R. \]

Furthermore, there is a twisted complex \((L_n, g_n)\) such that

\[ \text{len}_t(L_n, g_n) \leq e^{n(R+\varepsilon)}. \]

Similarly, there is \(N_3 \in \mathbb{N}\) such that for all \(n \geq N_3\), there exists a twisted complex \((M_n, h_n)\) in \(\mathcal{D}\) so that

\[ \text{len}_t(M_n, h_n) \leq e^{n(R+\varepsilon)}. \]

We can choose \(R < \alpha\). Then, by repeating the arguments in steps 2–6 with slight modifications, one can prove Theorem 3.8 for the second case. The modifications are using (3.5) and (3.6), instead of (3.5) and (3.6), and are replacing both of \(\beta\) and \(\gamma\) with \(R\).

**Step 8:** In step 8, we will consider the third case, i.e., only one of \(\beta, \gamma\) is \(-\infty\). For convenience, let us assume that \(\beta = -\infty < \gamma\). Then, the arguments in steps 2–6 will work after slight modification, as we did in step 7. The slight modifications are using (3.5) instead of (3.5), and replacing \(\beta\) with a sufficiently small \(R\).

When \(\beta > -\infty = \gamma\), the same logic works. \(\square\)

We end this section by pointing out that under some assumption, the categorical entropies on \(\mathcal{C}\) and \(\mathcal{C}/\mathcal{D}\) agree with each other. See Corollary 3.10 for the result.

**Corollary 3.10.** Assume \(h_0(\Phi_{\mathcal{D}}) \leq 0\) and \(\Phi^n_{\mathcal{C}/\mathcal{D}}(G) \neq 0\) for any \(n > 0\), then

\[ h_0(\Phi_{\mathcal{C}/\mathcal{D}}) = h_0(\Phi_{\mathcal{C}}). \]
Proof. When $t = 0$, the entropy $h_{t=0}(\Phi_{\epsilon_i/\emptyset}) \geq 0$ since $\Phi_{\epsilon_i/\emptyset}(G) \neq 0$ for any $n > 0$. This implies that
$$\max\{h_0(\Phi_{\epsilon_i/\emptyset}), h_0(\Phi_{\emptyset})\} = h_0(\Phi_{\epsilon_i/\emptyset}).$$
Then Theorem 3.8 gives the result.

3.3. Comparison with the topological entropy. Before moving on to the symplectic topology side of this paper, we compare the properties of categorical entropy, described in Section 3, to those of topological entropy.

The following are properties of topological entropy, which we consider in this subsection. We note that some other properties of topological entropy will be given in Section 8. For more details, we refer the reader to [Gro87, Section 1.6].

(i) Take two continuous self mappings of compact space, say $f_i : X_i \to X_i$ for $i = 1, 2$, and let $F : X_1 \to X_2$ be a continuous mapping such that $F \circ f_1 = f_2 \circ F$. If $F$ is onto, then
the topological entropy of $f_1 \geq$ the topological entropy of $f_2$.

(ii) Let $f : X \to X$ be a continuous self mapping and let us assume that there are two subsets $X_1$ and $X_2$ of $X$ such that $X_1 \cup X_2 = X$, $f(X_i) \subset X_i$.

It is known that
the topological entropy of $f = \max_{i=1,2} \{\text{the topological entropy of } f|_{X_i}\}$.

(iii) Let $f : X \to X$ be a continuous self mapping and let us assume that there is a subset $Y \subset X$ such that the restriction of $f$ onto $Y$ is a self mapping on $Y$. Then,
the topological entropy of $f \geq$ the topological entropy of $f|_{Y}$.

One can easily observe that Proposition 3.7 or the first inequality of Theorem 3.8 is the counterpart of (i) for the categorical entropy. The localisation functor plays the role of $F$ in (i).

Similarly, the second inequality of Theorem 3.8 is a counterpart of (ii) for the categorical entropy, but that is not a full counterpart of (ii). More precisely, we have an inequality in Theorem 3.8 but there is an equality in (ii). Moreover, there is no counterpart of (iii) in Section 3.2.

We roughly explain the reason why we could not have counterparts of (ii) and (iii) in Section 3.2. First, we note that, for measuring the topological entropy of $f$ in (iii), one needs to fix a metric $g$ on $X$. One also use the metric $g|_Y$ that is the restriction of $g$ onto $Y$, in order to measure the topological entropy of $f|_{Y}$. Since one uses the same metric, (iii) holds obviously. See [Gro87, Section 1.6] for more details. However, for measuring the categorical entropy, one should choose a split-generator $D$ for $\emptyset$ and a split-generator $\overline{G}$ for $\emptyset$. Since the relation between $D$ and $\overline{G}$ is arbitrary, it seems that proving the counterpart of (iii) is not an easy task.

Similarly, in order to turn Theorem 3.8 to
$$h_t(\Phi_{\emptyset}) = \max\{h_t(\Phi_{\epsilon_i/\emptyset}), h_t(\Phi_{\emptyset})\},$$
i.e., the counterpart of (ii), we need to prove
$$h_t(\Phi_{\emptyset}) \leq h_t(\Phi_{\epsilon_i/\emptyset}).$$
This is the counterpart of (iii). Thus, we could not have the counterpart of (ii) by the same reason.

Remark 3.11. We remark that the property (iii) of topological entropy can be induced from (ii) by setting $X_1 = Y$ and $X_2 = X$. We write (ii) and (iii) separately in order to emphasize the difference between topological and categorical entropies.
From the above arguments, a natural question arises: *if there are split-generators $D$ and $\mathcal{G}$ of $\mathcal{D}$ and $\mathcal{C}$, which are related to each other in a certain way, then could we have the counterparts of (ii) and (iii)?* Being motivated by the question, we consider the case of right-admissible or left-admissible full (dg) subcategory $\mathcal{D}$. For the definition of admissible subcategory, see Definition 3.12. We note that $\mathcal{D}$ is a full (dg) subcategory of $\mathcal{C}$ and both $\mathcal{C}$ and $\mathcal{D}$ are triangulated for the rest of the section.

**Definition 3.12.**

1. The right (resp. left) orthogonal complement $\mathcal{D}^\perp$ (resp. $\perp \mathcal{D}$) of $\mathcal{D}$ is the triangulated full (dg) subcategory of $\mathcal{C}$, consisting of objects $K \in \mathcal{C}$ such that
   \[
   \text{Hom}^* (L, K) = 0, \forall L \in \mathcal{D} \quad \text{(resp. Hom}^* (K, L) = 0, \forall L \in \mathcal{D}).
   \]
2. A triangulated full (dg) subcategory $\mathcal{D}$ of a triangulated (dg) category $\mathcal{C}$ is said to be right-admissible (resp. left-admissible) if and only if for any $L \in \mathcal{C}$, there is a distinguished triangle
   \[
   L' \to L \to L'' \to L'[1]
   \]
   for some $L' \in \mathcal{D}$ and $L'' \in \mathcal{D}^\perp$ (resp. for some $L' \in \perp \mathcal{D}$ and $L'' \in \mathcal{D}$) and such a triangle is unique up to unique isomorphism.

It is an easy consequence of Definition 3.12 that, for any right-admissible (resp. left-admissible) subcategory $\mathcal{D}$ of $\mathcal{C}$, its right (resp. left) orthogonal complement $\mathcal{D}^\perp$ (resp. $\perp \mathcal{D}$) is left-admissible (resp. right-admissible) and the left (resp. right) orthogonal complement $\perp \mathcal{D}$ (resp. $(\mathcal{D}^\perp)^\perp$) of $\mathcal{D}^\perp$ (resp. $(\perp \mathcal{D})^\perp$) is $\mathcal{D}$.

Let us consider the composition of the inclusion $\mathcal{D}^\perp \hookrightarrow \mathcal{C}$ (resp. $\perp \mathcal{D} \hookrightarrow \mathcal{C}$) and the localization functor $l : \mathcal{C} \to \mathcal{C} / \mathcal{D}$, which we will denote by $l_{\mathcal{D}}$ (resp. $l_{\perp \mathcal{D}}$). It was remarked in [Dri04] that a subcategory $\mathcal{D}$ of $\mathcal{C}$ is right-admissible (resp. left-admissible) if and only if $\mathcal{D}$ is split-closed and the functor $l_{\mathcal{D}} : \mathcal{D}^\perp \to \mathcal{C} / \mathcal{D}$ (resp. $l_{\perp \mathcal{D}} : \perp \mathcal{D} \to \mathcal{C} / \mathcal{D}$) is an equivalence.

Now assume that a dg subcategory $\mathcal{D}$ of a dg category $\mathcal{C}$ is right-admissible and hence that $\mathcal{D}^\perp$ is a left-admissible subcategory of $\mathcal{C}$. Then the above observations imply that the functor
\[
l_{\mathcal{D}} : \mathcal{D} = \{ - \mathcal{D}^\perp \to \mathcal{C} / \mathcal{D} \}
\]
is an equivalence. Let us denote by
\[
p : \mathcal{C} \to \mathcal{D} \quad \text{(resp. } q : \mathcal{C} \to \mathcal{D}^\perp)\]
the right (resp. left) adjoint of the inclusion
\[
i : \mathcal{D} \to \mathcal{C} \quad \text{(resp. } j : \mathcal{D}^\perp \to \mathcal{C})\]
whose existence can be proven using the definition of admissibility. We note that
\[
p(L) \simeq L' \quad \text{(resp. } q(L) \simeq L''[-1]),
\]
where $L, L', L''$ are the same as in Definition 3.12.

Furthermore assume that $\Phi_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is a dg functor. Then we define its induced functor $\Phi_{\mathcal{D}} : \mathcal{D}^\perp \to \mathcal{D}$ by
\[
\Phi_{\mathcal{D}} = q \circ \Phi_{\mathcal{C}} \circ j.
\]
Then we are ready to prove the following corollary of Theorem 3.8.

**Corollary 3.13.** Let $\mathcal{D}$ be a right-admissible (resp. left-admissible) full triangulated dg subcategory of $\mathcal{C}$. Let $\Phi_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ be a dg functor satisfying $\Phi_{\mathcal{C}}(\mathcal{D}) \subset \mathcal{D}$. Then, for any $t \in \mathbb{R}$, we have
\[
h_t(\Phi_{\mathcal{D}}) = \max\{h_t(\Phi_{\mathcal{C}}), h_t(\Phi_{\mathcal{D}})\} \quad \text{(resp. } h_t(\Phi_{\mathcal{C}}) = \max\{h_t(\Phi_{\mathcal{D}}), h_t(\Phi_{\mathcal{D}})\}).
\]
**Proof.** Let us consider the case that $\mathcal{D}$ is right-admissible only since the proof for the other case is similar.

First we show that, for any $t \in \mathbb{R}$,

$$h_t(\Phi_\mathcal{E}) \geq h_t(\Phi_\mathcal{D}). \tag{3.12}$$

For that purpose, take a split-generator $D$ for $\mathcal{D}$ and a split-generator $E$ for $\mathcal{D}^\perp$. Then $C := D \oplus E$ is a split-generator for $\mathcal{E}$.

It is easy to see that for any positive integer $m$, we have

$$\delta_t\left( (p \circ \Phi_\mathcal{E}^m(C)) \right) \geq \delta_t\left( (p \circ \Phi_\mathcal{E}^m(D)) \right). \tag{3.13}$$

But, for the right hand side of (3.13), we have

$$p(C) \simeq D,$$

and

$$(p \circ \Phi_\mathcal{E}^m)(C) = p\left( (\Phi_\mathcal{E}^m)(D) \oplus (\Phi_\mathcal{E}^m)(E) \right) = \Phi_\mathcal{E}^m(D) \oplus (p \circ \Phi_\mathcal{E}^m)(E).$$

Hence, we have, for any $t \in \mathbb{R}$,

$$\delta_t\left( (p \circ \Phi_\mathcal{E}^m)(C) \right) \geq \delta_t\left( (D; \Phi_\mathcal{E}^m(D)) \right). \tag{3.14}$$

The inequalities (3.13) and (3.14) prove the assertion (3.12).

On the other hand, we have

$$l(L) \simeq l\left( \text{Cone}(L''[-1] \to L') \right) \simeq l(L''[-1]),$$

$$(l \circ j \circ q)(L) \simeq l(L''[-1]),$$

where $L, L'$ and $L''$ are the same as in Definition 3.12. Thus, $(l \circ j \circ q)(L) \simeq l(L)$.

Now we show that

$$h_t(\Phi_\mathcal{E}/\mathcal{D}) = h_t(\Phi_\mathcal{E}/\mathcal{D}), \forall t \in \mathbb{R}. \tag{3.15}$$

For a proof, observe that the following diagram commutes.

```
\begin{array}{ccc}
\mathcal{D}^\perp & \xrightarrow{l_{\mathcal{D}^\perp}} & \mathcal{E}/\mathcal{D} \\
\Phi_{\mathcal{D}^\perp} \downarrow & & \downarrow \Phi_{\mathcal{E}/\mathcal{D}} \\
\mathcal{D}^\perp & \xrightarrow{l_{\mathcal{D}^\perp}} & \mathcal{E}/\mathcal{D}.
\end{array}
```

Indeed, we have, for any $K \in \mathcal{D}^\perp$,

$$\begin{align*}
(\Phi_{\mathcal{E}/\mathcal{D}} \circ l_{\mathcal{D}^\perp})(K) &= (\Phi_{\mathcal{E}/\mathcal{D}} \circ l \circ j)(K) \\
&\simeq (l \circ \Phi_\mathcal{E} \circ j)(K) \\
&\simeq (l \circ j \circ q \circ \Phi_\mathcal{E} \circ j)(K) \\
&\simeq (l \circ j \circ \Phi_{\mathcal{D}^\perp})(K) \\
&\simeq (l_{\mathcal{D}^\perp} \circ \Phi_{\mathcal{D}^\perp})(K).
\end{align*}$$

Here we used (3.3) for the third equality.
Hence, two functors $\Phi_{D} : D \to D$ and $\Phi_{C/D} : C/D \to C/D$ are identified on object level via the equivalence

$$l_{D} : D \to C/D.$$ 

Since the categorical entropy depends on how a functor acts on a generator iteratively, this proves the assertion.

Now, Theorem 3.8 and (3.15) say that, for any $t \in \mathbb{R}$, we have

$$(3.16) \quad h_{t}(\Phi_{D}) \leq h_{t}(\Phi_{C/D}) \leq \max\{h_{t}(\Phi_{D}), h_{t}(\Phi_{D})\}.$$ 

The desired equality (3.11) follows from (3.12) and (3.16). $\square$

Remark 3.14. Corollary 3.13 is also proved in a recent work [Kim22, Proposition 2.8]. We note that Theorem 3.8 can be applied to any full subcategory, but Corollary 3.13 cannot.

4. Entropy on wrapped Fukaya categories via symplectic automorphisms

In Section 4, we apply the results of the previous section to the Fukaya categories. More precisely, we will consider the following situation: Let $\mathcal{W}(W)$ be the wrapped Fukaya category of a Weinstein manifold $W$ with the Lagrangian cocores $G_1, \ldots, G_m$, and let $\mathcal{W}(W, \Lambda)$ be the partially wrapped Fukaya category of $W$ with the Legendrian stop $\Lambda \subset \partial W$. Define

$$G := G_1 \oplus \cdots \oplus G_m.$$ 

Let $D_1, \ldots, D_r$ be linking disks corresponding to $\Lambda$, and let $\mathcal{D}$ be the full subcategory of $\mathcal{W}(W, \Lambda)$ with the objects $D_1, \ldots, D_r$. Define

$$D := D_1 \oplus \cdots \oplus D_r$$ 

and

$$\overline{G} := G \oplus D.$$ 

By [GPS20], we can write

$$\mathcal{W}(W) \simeq \mathcal{W}(W, \Lambda)/\mathcal{D}.$$ 

Then there is the localisation functor

$$l : \mathcal{W}(W, \Lambda) \to \mathcal{W}(W).$$ 

Note that $\overline{G}$ split-generates $\mathcal{W}(W, \Lambda)$, $G$ split-generates $\mathcal{W}(W)$, and $D$ split-generates $\mathcal{D}$.

If $\phi : W \to W$ is a symplectic automorphism, then there are induced functors

$$\Phi_{\mathcal{W}(W, \Lambda)} : \text{Tw}(\mathcal{W}(W, \Lambda)) \to \text{Tw}(\mathcal{W}(W, \Lambda))$$ 

and

$$\Phi_{\mathcal{W}(W)} : \text{Tw}(\mathcal{W}(W)) \to \text{Tw}(\mathcal{W}(W)).$$ 

We have

$$\Phi_{\mathcal{W}(W)} \circ l(L) = l \circ \Phi_{\mathcal{W}(W, \Lambda)}(L)$$ 

for any $L \in \text{Tw}(\mathcal{W}(W, \Lambda))$.

Proposition 4.1. Let $\phi : W \to W$ be a compactly supported symplectic automorphism. Then, the induced functor

$$\Phi_{\mathcal{D}} : \text{Tw}(\mathcal{D}) \to \text{Tw}(\mathcal{D})$$ 

satisfies

$$h_{t}(\Phi_{\mathcal{D}}) = 0.$$ 

Proof. Since $\phi(D_i) = D_i$ for any $i$, $\Phi_{\mathcal{D}}$ is the identity functor. $\square$
Theorem 4.2. Let \( \phi: W \to W \) be a compactly supported symplectic automorphism. Then we have

\[
h_0(\Phi^W(W)) = h_0(\Phi^W(W,\Lambda)).
\]

If also \( h_t(\Phi^W(W)) \geq 0 \) for a given \( t \in \mathbb{R} \), then

\[
h_t(\Phi^W(W)) = h_t(\Phi^W(W,\Lambda)).
\]

Proof. Let \( \mathcal{C} := \mathcal{W}(W,\Lambda) \) and let \( \mathcal{D} \) be the category defined above. Then, \( \mathcal{C}/\mathcal{D} = \mathcal{W}(W) \). Moreover, we can apply Theorem 3.8 and Corollary 3.10 to get the result since Proposition 4.1 holds. \( \square \)

Corollary 4.3. If \( \Lambda \) and \( \Lambda' \) are two stops for \( W \), and \( \phi: W \to W \) is a compactly supported symplectomorphism, then

\[
h_0(\Phi^W(W,\Lambda)) = h_0(\Phi^W(W,\Lambda')).
\]

If also \( h_t(\Phi^W(W)) \geq 0 \) for a given \( t \in \mathbb{R} \), then

\[
h_t(\Phi^W(W,\Lambda)) = h_t(\Phi^W(W,\Lambda')).
\]

Proof. By Theorem 4.2, we have

\[
h_0(\Phi^W(W,\Lambda)) = h_0(\Phi^W(W)) = h_0(\Phi^W(W,\Lambda'))
\]

and when \( h_t(\Phi^W(W)) \geq 0 \), we have

\[
h_t(\Phi^W(W,\Lambda)) = h_t(\Phi^W(W)) = h_t(\Phi^W(W,\Lambda')).
\]

\( \square \)

5. SYMPLECTIC TOPOLOGICAL SET UP

In the rest of the paper, we investigate the categorical entropy (at \( t = 0 \)) on Fukaya categories. Especially, we will focus on Weinstein manifolds which are obtained by plumbing multiple copies of \( T^* S^n \) along trees for \( n \geq 3 \). In Section 5, we set the notation for the later use, but before setting up, we will briefly explain the reason why we care those specific spaces.

As mentioned in Section 1, when a Weinstein manifold \( W \) and an exact symplectic automorphism \( \phi \) on \( W \) are given, the present paper is interested in the induced functors from \( \phi \) on the various of Fukaya categories. Thus, it would be great if there is a connection between different Fukaya categories of \( W \).

If \( W \) is a plumbing space of \( T^* S^n \) with \( n \geq 3 \) along a tree, then it is known that the wrapped Fukaya category of \( W \) and the compact Fukaya category of \( W \) admit the "Koszul duality" as mentioned in [EL17]. Thus, the plumbing spaces can be good starting points to study the connection between entropies on different Fukaya categories of the same Weinstein manifold.

Moreover, on the plumbing spaces, it would be easy to construct a specific exact symplectic automorphism because the plumbing spaces have a natural collection of Lagrangian spheres. Then, by considering the Dehn twists along the Lagrangian spheres, or by considering their inverses and products, one can obtain a large class of exact symplectic automorphisms. See Remark 1.1.

Notation: Let \( T \) be a tree and let \( V(T) \) denote the set of vertices of \( T \). First, we define Definition 5.1

Definition 5.1. For a tree \( T \), let \( P_n(T) \) denote the Weinstein manifold obtained by plumbing multiple copies of \( T^* S^n \) along a tree \( T \).

On a Weinstein manifold \( P_n(T) \), one can set the following notation:

Definition 5.2.
(1) By definition, for any vertex \( v \in V(T) \), there is a Lagrangian sphere corresponding to \( v \). Let \( S_v \) denote the Lagrangian sphere corresponding to \( v \).

(2) There is a Dehn twist along \( S_v \). Let \( \tau_v \) denote the Dehn twist.

(3) Let \( L_v \) denote a Lagrangian cocore of \( S_v \). In other words, \( L_v \) is a Lagrangian disk such that

\[ |L_v \cap S_w| = \begin{cases} 
1, & \text{if } v = w, \\
0, & \text{otherwise.}
\end{cases} \]

Remark 5.3.

(i) To be more precise, we would like to point out that the choice of "Lagrangian cocore" in Definition 5.2 is not unique. In fact, when \( W \) is a Weinstein manifold, for any smooth point \( p \) of a Lagrangian skeleton of \( W \), there is a Lagrangian disk which transversally intersects the Lagrangian skeleton at \( p \). This is because, after a proper modification of the Liouville structure, the new Liouville structure near the smooth point \( p \) is exactly the same to the standard Liouville structure of a cotangent bundle. We note that the zero section of the cotangent bundle is a small neighborhood of \( p \) in the Lagrangian skeleton. For more details, we refer the reader to [CRGG17, Section 9.1]. Also, see [GPS18, Sections 1.1 and 7.1].

(ii) We also would like to point out that for a tree \( T \), \( P_n(T) \) admits a Weinstein structure whose Lagrangian skeleton is the union of the zero sections, i.e.,

\[ \bigcup_{v \in V(T)} S_v. \]

Thus, based on (i), for any point \( p \in S_v \) except the plumbing points, there is a Lagrangian cocore \( L_p \). Moreover, if \( p_1, p_2 \in S_v \), then it is easy to check that \( L_{p_1} \) and \( L_{p_2} \) are Hamiltonian isotopic. Thus, the Lagrangian cocore \( L_v \) in Definition 5.2 is well defined up to Hamiltonian isotopy.

For a given tree \( T \), we define the compact and wrapped Fukaya categories of \( P_n(T) \) as follows for convenience.

Definition 5.4. Let \( \mathcal{F}(P_n(T)) \) and \( \mathcal{W}(P_n(T)) \) denote the split-closure of the triangulated envelope of compact and wrapped Fukaya categories of \( P_n(T) \), respectively.

Remark 5.5. To be consistent with Section 2 and Section 3, we will occasionally (e.g. Section 6.1) identify the Fukaya categories \( \mathcal{F}(P_n(T)) \) and \( \mathcal{W}(P_n(T)) \) with their images under the Yoneda embedding, so that they can be regarded as dg categories.

Definition 5.6 will be needed in Section 7.

Definition 5.6. For \( v, w \in V(T) \), \( v \sim w \) if and only if \( v \) and \( w \) are connected by an edge in \( T \).

Grading convention: As mentioned before, it was shown in [EL17] that \( \mathcal{W}(P_n(T)) \) and \( \mathcal{F}(P_n(T)) \) are Koszul dual to each other for \( n \geq 3 \). This implies, as proven in [AS12], that one may grade \( S_i \)'s and \( L_i \)'s so that the endomorphism algebra \( \bigoplus_{i,j} \text{Hom}^*(L_i, L_j) \) is non-positively graded, while the endomorphism algebra \( \bigoplus_{i,j} \text{Hom}^*(S_i, S_j) \) is non-negatively graded.

Moreover, [AS12, Section 4.2] shows the following Lemma.

Lemma 5.7. One can choose a proper grading for \( \mathcal{W}(P_n(T)) \) so that the following hold:

(1) For each \( v \in V(T) \),

\[ \text{hom}^*(L_v, S_v) = \begin{cases} 
k(\overline{p}_v) \equiv k & \text{if } * = 0, \\
0 & \text{otherwise,}
\end{cases} \]
or equivalently,
\[ \text{hom}^*(S_v, L_v) = \begin{cases} k\langle p_v \rangle \equiv k & \text{if } * = n, \\ 0 & \text{otherwise}, \end{cases} \]
where \( \{p_v\} = S_v \cap L_v \).

2. For any \( v, w \in V(T) \) such that \( v \neq w \),
\[ \text{hom}^*(S_v, L_w) = \text{hom}^*(L_v, S_w) = 0. \]

3. For any \( v \in V(T) \),
\[ \text{hom}^*(S_v, S_v) = \begin{cases} k & \text{if } * = 0, n, \\ 0 & \text{otherwise}. \end{cases} \]

4. For any \( v, w \in V(T) \) such that \( v \sim w \), there exists an integer \( s_{vw} \in \{1, \ldots, n-1\} \) such that
\[ \text{hom}^*(S_v, S_w) = \begin{cases} k & * = s_{vw}, \\ 0 & \text{otherwise}, \end{cases} \]
and
\[ s_{vw} + s_{wv} = n. \]

5. For any \( v, w \in V(T) \) such that \( v \neq w, v \sim w \),
\[ \text{hom}^*(S_v, S_w) = 0. \]

6. For any \( v, w \in V(T) \) such that \( v \sim w \),
the shortest Reeb chord from \( L_v \) to \( L_w \) has degree \( s_{vw} - n + 1 \),
where the shortest Reeb chord is defined in \([AS12, \text{Section 4.2}]\). One can easily check that the sum of degrees of two shortest Reeb chords, one from \( L_v \) to \( L_w \) and the other from \( L_w \) to \( L_v \), is
\[ s_{vw} - n + 1 + s_{wv} - n + 1 = (s_{vw} + s_{wv}) - 2n + 2 = 2 - n. \]

6. Entropies of Products of Dehn Twists

The main goal of Section 6 is to prove Theorem 1.2 or its technical statement, i.e., Theorem 6.13. More precisely, we prove that if \( \phi : P_n(T) \rightarrow P_n(T) \) is an arbitrary product of \( \{t_v, t_v^{-1} \mid v \in V(T)\} \), the induced functors from \( \phi \) on \( \mathcal{F}(P_n(T)) \) and on \( \mathcal{W}(P_n(T)) \), have the same categorical entropies.

In Section 6.1, we prove Theorem 6.6. By Theorem 6.6 one can compute the categorical entropy on the compact Fukaya category from dimensions of Lagrangian Floer (co)homologies. And this leads us to the proof of Theorem 6.13 in Section 6.2.

Remark 6.1. We note that in order to relate the categorical entropies with Lagrangian Floer (co)homology or (co)chain complex, we only consider the categorical entropies at \( t = 0 \) in the rest of this paper.

6.1. Entropies of Products of Dehn Twists. We first introduce the notations for auto-functors induced by a symplectic automorphism.

Definition 6.2. Let \( \phi \) be a compactly-supported exact symplectic automorphism on \( P_n(T) \). Then let \( \Phi_{\mathcal{F}(P_n(T))} \) and \( \Phi_{\mathcal{W}(P_n(T))} \) denote the auto-functor induced by \( \phi \) on the compact Fukaya category \( \mathcal{F}(P_n(T)) \) and the wrapped Fukaya category \( \mathcal{W}(P_n(T)) \), respectively.

The generation results for both \( \mathcal{F}(P_n(T)) \) and \( \mathcal{W}(P_n(T)) \) are well-known. For example, see \([AS12, \text{Theorem 1.1}]\). More precisely, it was shown that the compact Fukaya category \( \mathcal{F}(P_n(T)) \) is generated by the spheres \( \{S_v, v \in V(T)\} \). Hence we can take \( S = \bigoplus_{v \in V(T)} S_v \) as a split-generator of \( \mathcal{F}(P_n(T)) \). On the other hand, the wrapped Fukaya category \( \mathcal{W}(P_n(T)) \) is generated by the cocores \( \{L_v, v \in V(T)\} \), and so we can take \( L = \bigoplus_{v \in V(T)} L_v \) as a split-generator of \( \mathcal{W}(P_n(T)) \).
Moreover, the following theorem holds in our case as the endomorphism algebra \( \bigoplus_{v,w} \text{Hom}^*(L_v, L_w) \) is non-positively graded.

**Theorem 6.3 ([AS12]).** For any exact, closed Lagrangian \( C \) of \( P_n(T) \), there is a twisted complex for \( C \) with components \( \{S_v, v \in V(T)\} \), in which none of the arrows are multiples of the identity morphisms.

We remark that the twisted complex mentioned in Theorem 6.3 is not only an object of the (triangulated closure of) compact Fukaya category, but also an object of the (triangulated closure of) wrapped Fukaya category. We consider the twisted complex as an object in \( \mathcal{W}(P_n(T)) \) and the hom-spaces below are morphism spaces of \( \mathcal{W}(P_n(T)) \).

By considering that \( S_v \cap L_w \neq \emptyset \) if and only if \( v = w \), and \( |S_v \cap L_v| = 1 \), we have

**Lemma 6.4.** Let \( \mathcal{S} \) be a twisted complex with components \( \{S_v, v \in V(T)\} \). For each \( v \in V(T) \), the following equality holds.

\[
\text{len}_{S_v}\mathcal{S} = \dim \text{hom}^*(\mathcal{S}, L_v).
\]

Theorem 6.3 induces Lemma 6.3.

**Lemma 6.5.** Let \( \phi \) be a compactly-supported exact symplectic automorphism on \( P_n(T) \). For any \( m \in \mathbb{N} \), there is a twisted complex \( \mathcal{S}_m \) with components \( \{S_v, v \in V(T)\} \) such that

- \( \mathcal{S}_m \simeq \phi^m(S) \), and
- for any \( v \in V(T) \), the cochain complex \( \text{hom}^*(\mathcal{S}_m, L_v) \) has the zero differential map.

**Proof.** We note that for any \( m \in \mathbb{N} \), \( \phi^m(S) \) is a closed exact Lagrangian of \( P_n(T) \). Thus, there is a twisted complex \( \mathcal{S}_m \) satisfying the conditions in Theorem 6.3.

Since the components of \( \mathcal{S}_m \) that contribute to \( \text{hom}^*(\mathcal{S}_m, L_v) \) are of the form \( S_v[d] \), we will consider the summands of \( \mathcal{S}_m \), that are the form of \( S_v[d] \) for some \( d \in \mathbb{Z} \). By collecting all these summands we have the following twisted complex:

\[
(6.1) \quad \left( \bigoplus_{1 \leq i \leq \text{len}_{S_v}(\mathcal{S}_m)} S_v[d_i], (f_{i,j}) \right),
\]

where \( (f_{i,j}) \in \text{hom}^1(S_v[d_i], S_v[d_j]) \), \( i < j \) are the arrows in the twisted complex \( \mathcal{S}_m \). We note that for any \( i, j, f_{i,j} \) is not a multiple of the identity morphisms by Theorem 6.3. This means that each arrow \( f_{i,j} \) comes from a multiple of the morphism of degree \( n \) in \( \text{hom}^*(S_v, S_v) \). See the last item of Lemma 6.4. This further implies that whenever the arrow \( f_{i,j} \) is nonzero, the following holds:

\[
(6.2) \quad d_i = d_j + 1 - n.
\]

Now (6.1) says that \( \text{hom}^*(\mathcal{S}_m, L_v) \) is given by

\[
(6.3) \quad \bigoplus_i \text{hom}^*(S_v[d_i], L_v) = \bigoplus_i k\langle p_v(d_i) \rangle,
\]

where \( p_v(d_i) \) denotes the morphism of degree \( (d_i + n) \) in \( \text{hom}^*(S_v[d_i], L_v) \). We note that for each \( i \), \( \text{hom}^*(S_v[d_i], L_v) \) is of dimension 1.

The differential map of (6.3) from the summand \( \text{hom}^*(S_v[d_j], L_v) = k\langle p_v(d_j) \rangle \) to the summand \( \text{hom}^*(S_v[d_i], L_v) = k\langle p_v(d_i) \rangle \) is given by the sum of maps of the following form:

\[
(6.4) \quad p_v(d_j) \rightarrow \mu^k(p_v(d_j), f_{i_1,i_2}, \ldots, f_{i_{k},i_{k+1}}), i = i_1 < i_2 < \cdots < i_k = j.
\]

One can check that each of such higher products vanishes due to a degree reason. Thus, the differential map is the zero map, i.e., Lemma 6.5 holds.
More precisely, if the higher product in \((6.4)\) does not vanish, then all the arrows \(f_{i_1,i_1} \) involved above are nonzero. Thus, one obtains
\[
d_i = d_{i+1} + 1 - n,
\]
from \((6.2)\), for any possible \(i\).

This concludes that
\[
d_i = d_{i+1} = d_2 + 1 - n = \cdots = d_k + (k-1)(1-n) = d_j + (k-1)(1-n).
\]

On the other hand, the degree of \(\mu^k(p_v(d_j), f_{i_k,i_1}, \ldots, f_{i_1,i_1}) \in \text{hom}^*(S_v[d_i], L_v)\) is given by
\[
\deg p_v(d_j) + \sum_{l=1}^{k-1} \deg f_{i_l,i_{l+1}} + \deg \mu^k = \deg p_v(d_j) + k - 1 + 2 - k
\]
\[
= d_j + n + 1.
\]

Moreover, since \(\mu^k(p_v(d_j), f_{i_k,i_1}, \ldots, f_{i_1,i_1})\) is an element of the one dimensional vector space \(\text{hom}^*(S_v[d_i], L_v)\), the degree of the higher product should be
\[
\deg p_v(d_i) = d_i + n.
\]
In other words, one obtains
\[
d_i + n = d_j + n + 1.
\]

Then, \((6.6)\) and \((6.5)\) contradict to each other. This completes the proof. \(\square\)

Now we are ready to prove the following.

**Theorem 6.6.** Let \(\phi\) be any compactly-supported exact symplectic automorphism on \(P_n(T)\). Then the following holds:
\[
h_0(\Phi, \mathcal{F}(P_n(T))) = \lim_{m \to \infty} \frac{1}{m} \log \text{dim} \text{Hom}^*(\phi^m(S), L).
\]

**Proof.** We recall that the categorical entropy \(h_0(\phi, \mathcal{F}(P_n(T)))\) measures the growth rate of the logarithm of
\[
\sum_{v \in V(T)} \inf \left\{ \text{len}_{S_v} (\mathcal{T}) | \mathcal{T} \cong \phi^m(S) \oplus C' \text{ for some } C' \in \mathcal{F}(P_n(T)) \right\}.
\]

We note that the twisted complex \(\mathcal{T}\) should consist of \(S_v\)'s.

Let \(\mathcal{I}_m\) be a twisted complex for \(\phi^m(S)\) given by Lemma 6.5. Then, for any twisted complex \(\mathcal{T}\) quasi-equivalent to \(\phi^m(S) \oplus C\) for some \(C \in \mathcal{F}(P_n(T))\), we have
\[
\text{len}_{S_v}(\mathcal{T}) = \text{dim} \text{hom}^*(\mathcal{T}, L_v) \quad (\because \text{Lemma 6.4})
\]
\[
\geq \text{dim} \text{Hom}^*(\phi^m(S) \oplus C, L_v)
\]
\[
\geq \text{dim} \text{Hom}^*(\phi^m(S), L_v)
\]
\[
= \text{dim} \text{hom}^*(\mathcal{I}_m, L_v)
\]
\[
= \text{len}_{S_v}(\mathcal{I}_m) \quad (\because \text{Lemma 6.4}).
\]

This concludes that \(\mathcal{I}_m\) is a twisted complex of \(\phi^m(S)\) giving the infimum in \((6.3)\).

By Lemma 6.5, for every \(v \in V(T)\), we have
\[
\text{dim} \text{Hom}^*(\phi^m(S), L_v) = \text{dim} \text{Hom}^*(\mathcal{I}_m, L_v) = \text{dim} \text{hom}^*(\mathcal{I}_m, L_v).
\]
The above observations imply that the categorical entropy \( h_0(\Phi_{\mathcal{F}(P_n(T_n))}) \) is achieved by the growth rate of the logarithm of
\[
\sum_{\nu} \text{len}_{S_{\nu}}(\mathcal{S}_m) = \sum_{\nu} \dim \hom^*(\mathcal{S}_m, L_{\nu}) = \sum_{\nu} \dim \Hom^*(\phi^m(S), L_{\nu}) = \Hom^*(\phi^m(S), L),
\]
as asserted.

**Remark 6.7.** Both Lemma 6.9 and Theorem 6.13 hold even when the ambient Liouville domain is given by the plumbing of cotangent bundles along a tree, if the zero sections of cotangent bundles are simply-connected closed manifolds. This is because the proof of Lemma 6.5 works as well for such a case except that (6.2) is replaced by
\[
d_i = d_j + 1 - \deg f \leq d_j,
\]
for some (homogeneous) element \( f \in H^*(Q) \setminus H^0(Q) \), where \( Q \) denotes a zero section. Thus, (6.5) still contradicts from (6.6). Then, the proof of Theorem 6.6 works without any modification.

### 6.2. Equality between categorical entropies

In Section 6.2 we prove Theorem 6.13. In order to do that, we start by proving Lemma 6.9 that holds in a more general setting than Theorem 6.13.

**Remark 6.8.** In order to prove Lemma 6.9, we need Lemmas 6.10–6.12. However, if \( \Phi \) is a product of positive or negative powers of Dehn twists, one can prove Lemma 6.9 by using Theorem 3.8 or Corollary 3.10 without using Lemmas 6.10–6.12. To be more precise, let \( \mathcal{C}, \mathcal{D} \) in Theorem 3.8 be the wrapped Fukaya category and the compact Fukaya category, respectively. Then, it is easy to check that the induced functor \( \Phi_{\mathcal{C}/\mathcal{D}} \) on the localised category is the identity functor. Thus, \( h_0(\Phi_{\mathcal{C}/\mathcal{D}}) = 0 \). By applying Theorem 3.8 or Corollary 3.10, one obtains
\[
0 \leq h_0(\Phi_{\mathcal{C}}) \leq \max\{h_0(\Phi_{\mathcal{C}/\mathcal{D}}) = 0, h_0(\Phi_{\mathcal{D}})\}.
\]
We note that at \( t = 0 \), \( h_0(\Phi_{\mathcal{D}}) \geq 0 \). Thus, it proves Lemma 6.9 for that specific case.

To state Lemma 6.9 let \( \mathcal{D} \) be a full dg subcategory of a dg category \( \mathcal{C} \), both of which are assumed to be split-closed and triangulated. Assume that there is an \( A_\infty \)-auto functor \( \Psi = \Psi_{\mathcal{C}} \) on \( \mathcal{C} \) such that
\[
\Psi_{\mathcal{C}}(\mathcal{D}) \subset \mathcal{D},
\]
which allows us to think of the restriction to \( \mathcal{D} \) of \( \Psi_{\mathcal{C}} \). Let \( \Psi_{\mathcal{D}} \) denote the restriction, i.e.,
\[
\Psi_{\mathcal{D}} := \Psi_{\mathcal{C}/\mathcal{D}} : \mathcal{D} \to \mathcal{D}.
\]

Lemma 6.9 compares the entropies of \( \Psi_{\mathcal{C}} \) and \( \Psi_{\mathcal{D}} \) under some assumptions.

**Lemma 6.9.** Let us assume that, for every object \( X \) of \( \mathcal{C} \), there exist some objects \( Y, Z \in \mathcal{D} \), and \( f \in \hom^0(Y, X), g \in \hom^0(X, Z) \) such that
\[
(6.10) \quad \Psi_{\mathcal{C}}(X) \cong \Cone(\Cone(Y \xrightarrow{f} X)) \xrightarrow{g} Z, \quad \text{where} \ \cong \ \text{means quasi-equivalence. Then, the following inequality holds}
\]
\[
h_0(\Psi_{\mathcal{C}}) \leq h_0(\Psi_{\mathcal{D}}).
\]

For a proof of Lemma 6.9 we need Lemmas 6.10–6.12. First, by applying (6.10) iteratively, we have

**Lemma 6.10.** For every \( m \in \mathbb{Z}_{>0} \), there exist \( Y_m, Z_m \in \mathcal{D}, f_m \in \hom^0(Y_m, X), \) and \( g_m \in \hom^0(X, Z_m) \) such that
\[
\text{• } Y_m = (\#_{i=1}^m \Psi_{m+i}(Y), (u_{ij})), \quad \text{where the right hand side represents a twisted complex with components} \Psi_{m+i}(Y), \ i = 1, \ldots, m.
\]
\[
\text{• } Z_m = (\#_{i=1}^m \Psi_{m-i}(Z), (v_{ij})), \quad \text{where the right hand side represents a twisted complex with components} \Psi_{m-i}(Z), \ i = 1, \ldots, m.
\]
• There is a quasi-isomorphism
\begin{equation}
\Psi^m(C) \equiv \text{Cone}(\text{Cone}(Y_m \xrightarrow{f_m} C) \xrightarrow{g_m} Z_m).
\end{equation}

One can easily prove Lemma 6.11 from Definition 2.7.

**Lemma 6.11.** If there is a triangle $F' \to F \to F'' \to F'[1]$ in $\text{Tw} \mathcal{C}$, then
\begin{equation}
\delta_0(E, F) \leq \delta_0(E, F') + \delta_0(E, F'').
\end{equation}

Lemma 6.12 is also straightforward to check from Definition 2.7.

**Lemma 6.12.** If both $E$ and $E'$ split-generate $F$ and if $E \equiv E' \oplus E''$ for some $E''$, then
\begin{equation}
\delta_0(E, F) \leq \delta_0(E', F).
\end{equation}

**Proof of Lemma 6.9.** Let $D$ be a split-generator of $\mathcal{D}$. Then there exists $C \in \mathcal{C}$ such that $C \oplus D$ split-generates $\mathcal{C}$. Hence the entropy $h_0(\Psi_\mathcal{C})$ is given by
\begin{equation}
h_0(\Psi_\mathcal{C}) = \lim_{m \to \infty} \frac{1}{m} \log \delta_0(C \oplus D, \Psi^m(C) \oplus \Psi^m(D))
\end{equation}

It is easy to check that
\begin{equation}
\delta_0(C \oplus D, \Psi^m(C) \oplus \Psi^m(D))
\end{equation}
is bounded above by
\begin{equation}
\delta_0(C \oplus D, \Psi^m(C)) + \delta_0(C \oplus D, \Psi^m(D)).
\end{equation}

Let $\lambda = \exp(h_0(\Psi_\mathcal{D})) \geq 1$. We will show that, for arbitrary $\epsilon > 0$, there is a sufficiently large $M > 0$ such that (6.12) is again bounded above as follow.
\begin{equation}
\delta_0(C \oplus D, \Psi^m(C)) + \delta_0(C \oplus D, \Psi^m(D)) \leq 3(\lambda + \epsilon)^m, \forall m \geq M,
\end{equation}
which will complete the proof.

Let $\epsilon > 0$ be given. We first bound the component $\delta_0(C \oplus D, \Psi^m(C))$ of (6.12). We apply Lemma 6.11 twice to (6.11) in Lemma 6.10 to show that
\begin{equation}
\delta_0(C \oplus D, \Psi^m(C)) \leq \delta_0(C \oplus D, Y_m) + \delta_0(C \oplus D, C) + \delta_0(C \oplus D, Z_m).
\end{equation}

By definition, $\delta_0(C \oplus D, C) = 1$. In order to bound $\delta_0(C \oplus D, Y_m)$ and $\delta_0(C \oplus D, Z_m)$ from above, we recall that $Y_m$ and $Z_m$ admit the twisted complex representations given in Lemma 6.10. Since the twisted complexes consist of $\Psi^i(Y)$ and $\Psi^i(Z)$, we consider $\delta_0(C \oplus D, \Psi^i(Y))$ and $\delta_0(C \oplus D, \Psi^i(Z))$.

One can apply Lemma 6.12 to $\delta_0(C \oplus D, \Psi^i(Y))$ and $\delta_0(C \oplus D, \Psi^i(Z))$ so that one obtains the following inequalities for sufficiently large $M_1 > 0$,
\begin{align*}
\delta_0(C \oplus D, \Psi^i(Y)) &\leq \delta_0(D, \Psi^i(Y)) < (\lambda + \frac{\epsilon}{3})^j \\
\delta_0(C \oplus D, \Psi^i(Z)) &\leq \delta_0(D, \Psi^i(Z)) < (\lambda + \frac{\epsilon}{3})^j
\end{align*}
for all $j \geq M_1$. We note that both $Y$ and $Z$ are split-generated by $D$ by assumption.

Now, iterative applications of Lemma 6.11 to the description of $Y_m$ and $Z_m$ in Lemma 6.10 show that, for sufficiently large integer $M_2 \geq M_1$, the following hold for all $m \geq M_2$
\begin{align}
\delta_0(C \oplus D, Y_m) &\leq M_1 \max\{\delta_0(C \oplus D, \Psi^j(Y)) | j = 0, \ldots, M_1 - 1\} + \sum_{j = M_1}^{m} (\lambda + \frac{\epsilon}{3})^j < (\lambda + \frac{\epsilon}{2})^m \\
\delta_0(C \oplus D, Z_m) &\leq M_1 \max\{\delta_0(C \oplus D, \Psi^j(Z)) | j = 0, \ldots, M_1 - 1\} + \sum_{j = M_1}^{m} (\lambda + \frac{\epsilon}{3})^j < (\lambda + \frac{\epsilon}{2})^m
\end{align}
As a combination of (6.14) and (6.15), we have
\[
\delta_0(C \oplus D, \Psi^m(C)) \leq 2(\lambda + \frac{\epsilon}{2})^m + 1, \forall m \geq M_2.
\]
This gives us the upper bound of the first component \(\delta_0(C \oplus D, \Psi^m(C))\) of (6.12).

For the second component \(\delta_0(C \oplus D, \Psi^m(D))\) of (6.12), we apply Lemma 6.12 to show that
\[
\delta_0(C \oplus D, \Psi^m(D)) \leq \delta_0(D, \Psi^m(D)).
\]
By definition of categorical entropy, for sufficiently large \(M_3 > 0\), we have
\[
\delta_0(C \oplus D, \Psi^m(D)) \leq (\lambda + \frac{\epsilon}{2})^m, \forall m \geq M_3.
\]
Finally, (6.16) and (6.17) show that (6.13) holds for sufficiently large \(M > \max\{M_2, M_3\}\). This completes the proof.

**Theorem 6.13.** Let \(\phi\) be any product of positive or negative powers of Dehn twists \(\tau_v\) for \(v \in V(T)\). Then we have
\[
h_0(\Phi_{\mathcal{W}(P_n(T))}) = h_0(\Phi_{\mathcal{F}(P_n(T))}).
\]

**Proof.** Let \(\mathcal{F}_m\) be the twisted complex for \(\phi^m(S)\) given by Lemma 6.5.

We note that for each \(v \in V(T)\) and \(m \in \mathbb{N}\), we have
\[
dim \text{Hom}^* (\phi^m(S), L_v) = \dim \text{Hom}^* (S, \phi^{-m}(L_v)).
\]
In this proof, we will consider \(\phi^{-m}(L_v)\).

Almost the same argument as in (6.9) except the last two equalities show that for any twisted complex \(\mathcal{L}\) consisting of \(L_v\)'s, which is quasi-equivalent to \(\phi^{-m}(L) \oplus C'\) for some \(C' \in \mathcal{W}(P_n(T))\), the following holds:
\[
\text{len}_{\mathcal{L}}(\mathcal{L}) \geq \dim \text{Hom}^* (S_w, \phi^{-m}(L)).
\]
This implies that there is an inequality
\[
h_0(\Phi_{\mathcal{W}(P_n(T))}^{-1}) \geq \lim_{m \to \infty} \frac{1}{m} \log \sum_{w \in V(T)} \dim \text{Hom}^* (S_w, \phi^{-m}(L)) = \lim_{m \to \infty} \frac{1}{m} \log \dim \text{Hom}^* (S, \phi^{-m}(L)).
\]
Combining (6.7), (6.18) and (6.19), we get an inequality
\[
h_0(\Phi_{\mathcal{W}(P_n(T))}^{-1}) \geq h_0(\Phi_{\mathcal{F}(P_n(T))}).
\]

Let \(\mathcal{C} = \mathcal{W}(P_n(T)), \mathcal{D} = \mathcal{F}(P_n(T))\) and \(\Psi_{\mathcal{C}} = \Phi_{\mathcal{W}(P_n(T))}^{-1}\). Then, they satisfy the assumptions of Lemma 6.9. In particular, (6.10) holds for \(\Phi_{\mathcal{W}(P_n(T))}^{-1}\) since \(\Phi_{\mathcal{W}(P_n(T))}^{-1}\) is induced by a product of positive or negative powers of Dehn twists \(\tau_v\) for \(v \in V(T)\). Hence, by applying Lemma 6.9 one obtains
\[
h_0(\Phi_{\mathcal{F}(P_n(T))}^{-1}) \geq h_0(\Phi_{\mathcal{W}(P_n(T))}^{-1}).
\]

Then, by combining (6.20) and (6.21), we obtain
\[
h_0(\Phi_{\mathcal{F}(P_n(T))}^{-1}) \geq h_0(\Phi_{\mathcal{F}(P_n(T))}).
\]

Since \(\phi^{-1}\) is also a product of positive or negative powers of Dehn twists \(\tau_v\) for \(v \in V(T)\), the above inequality holds even when \(\phi\) is replaced by \(\phi^{-1}\). Thus, we have the inequality opposite to (6.22). This means that
\[
h_0(\Phi_{\mathcal{W}(P_n(T))}) = h_0(\Phi_{\mathcal{F}(P_n(T))}).
\]
From the proof of Theorem 6.13 one also can see that the following holds:

**Corollary 6.14.** Let \( \phi \) be any product of positive or negative powers of Dehn twists \( \tau_v \) for \( v \in V(T) \). Then the following hold:

1. \( h_0(\Phi_{H_p}(P_n(T))) = h_0(\Phi_{H_p}(P_n(T))) = h_0(\Phi_{H_p}^{-1}(P_n(T))) = h_0(\Phi_{H_p}^{-1}(P_n(T))) \).

2. \( h_0(\Phi_{H_p}(P_n(T))) = \lim_{m \to \infty} \frac{1}{m} \log \dim \text{Hom}^*(S, \phi^m(L)) \).

7. **Entropies of symplectic automorphisms of Penner type**

Let \( \phi \) be a product of positive or negative powers of Dehn twists \( \tau_v \) for \( v \in V(T) \). Then, since the induced functors on the compact and wrapped Fukaya categories have the same categorical entropies, we simply say that \( \phi \), not the induced functors, has a categorical entropy. More precisely,

**Definition 7.1.** If \( \phi \) is a product of positive or negative powers of Dehn twists \( \tau_v \) for \( v \in V(T) \), then the categorical entropy of \( \phi \) is

\[ h_{\text{cat}}(\phi) = h_0(\Phi_{H_p}(P_n(T))) = h_0(\Phi_{H_p}(P_n(T))) \]

**Remark 7.2.** We compared the categorical entropies on the compact and the wrapped Fukaya categories, and they agree with each other for the given \( \phi \). Thanks to Corollary 3.10, the categorical entropy on partially wrapped Fukaya categories also agree with them.

The result of Section 6 is that

\[ h_{\text{cat}}(\phi) = \lim_{m \to \infty} \frac{1}{m} \log \dim \text{Hom}^*(S, \phi^m(L)) \]

Even if we have the above result, \( h_{\text{cat}}(\phi) \) is not easy to compute because measuring the growth rate of \( \dim \text{Hom}^*(S, \phi^m(L)) \) is not an easy task in general. In Section 7 we prove that, if \( \phi \) is of a specific type on \( P_n(T) \), its categorical entropy is given by the growth rate of the logarithms of the intersection numbers of \( S_v \) and \( \phi^m(L_w) \), for \( v, w \in V(T) \). The specific type will be defined in Section 7.3. These results are more precisely stated in Theorem 7.14.

### 7.1. Construction of a twisted complex

In Section 6 we constructed a twist complex by using a generation result [AS12] in the compact Fukaya category. In Section 7 we construct a twisted complex by using another generation result [CRGG17] in the wrapped Fukaya category. Theorem 7.3 is the generation result that we will use in the current section.

**Theorem 7.3 (CRGG17).** Let \( C \) be an admissible Lagrangian in \( \mathcal{W}(P_n(T)) \). If \( C \cap (\bigcup_v S_v) = \{a_1, \ldots, a_i\} \) so that \( a_i \) is not a plumbing point, and if the intersections are transverse, then \( C \) is isomorphic to a twisted complex built from the Lagrangian cocore disks \( L_{a_1}, \ldots, L_{a_i} \) in the category of twisted complexes of \( \mathcal{W}(P_n(T)) \).

See Remark 5.3 for cocore disks \( L_{a_i} \).

From Theorem 7.3 one can construct a twisted complex isomorphic to an admissible Lagrangian \( L \) by observing the intersection \( L \cap (\bigcup_v S_v) \), if \( L \) intersects \( \bigcup_v S_v \) transversally, outside of the plumbing points. Now, for a Lagrangian \( L \) transversally intersecting \( \bigcup_v S_v \), we consider \( \tau_v(L) \). More precisely, we would like to find \( \tau_v(L) \cap (\bigcup_v S_v) \), in order to find a twisted complex isomorphic to \( \tau_v(L) \). We note that by modifying the Dehn twist \( \tau_v \), one can assume that \( \tau_v(L) \) transversally intersects \( \bigcup_v S_v \) outside of the plumbing points, without loss of generality.

Moreover, one can assume that

\[ L \cap (\bigcup_{w \neq v} S_w) \]
is not contained in the support of \( \tau_v \), by choosing a proper Dehn twist \( \tau_v \). Thus, 

\[
\tau_v(L) \cap (\cup_{w \neq v} S_w) \cap \left( \text{Supp}(\tau_v) \right)^c = L \cap (\cup_{w \neq v} S_w) \cap \left( \text{Supp}(\tau_v) \right)^c.
\]

The superscript \( c \) in the above equation means the complement of the support.

Inside of the support, one can observe that \( \tau_v(L) \), roughly speaking, seems like a surgery of 

\[
L \text{ and } \Phi_{p \in L \cap S_v} S_v
\]
at every \( p \in L \cap S_v \). Thus, one can say that every \( p \in L \cap S_v \) creates one intersection point of \( \tau_v(L) \) and \( S_w \) for any \( w \) such that \( v \sim w \) or \( v = w \). Moreover, by using Seidel’s long exact sequence, one can see how the degrees of created intersection points in \( \tau_v(L) \cap S_w \) are determined.

We note that in the above arguments, it is essential to choose a proper Dehn twist. The existence of a proper Dehn twist can be proven by using the construction of Dehn twist given in [MW18, Section 2.1]. More precisely, by choosing a proper \textit{spherical Dehn twist profile}, one can obtain a proper Dehn twist. We note that one can find the definition of spherical Dehn twist profile in [MW18]. For more details on the construction of Dehn twists, and for the independence of choices, we refer the reader to [Sei99] and [Sei03].

Lemma 7.4 is the result of the above arguments when \( L \) is one of the cocores \( \{L_w | w \in V(T)\} \).

**Lemma 7.4.**

1. In the wrapped Fukaya category \( \mathcal{W}(P_n(T)) \), \( \tau_v(L_w) \) is isomorphic to the following twisted complex:

\[
\tau_v(L_w) \cong \begin{cases} L_w & v \neq w, \\ \{L_v[1-n] \oplus \bigoplus_{w \sim v} L_w[1-n+s_{vw}], \phi\} & v = w. \end{cases}
\]

2. In the wrapped Fukaya category \( \mathcal{W}(P_n(T)) \), \( \tau_v^{-1}(L_w) \) is isomorphic to the following twisted complex:

\[
\tau_v^{-1}(L_w) \cong \begin{cases} L_w & v \neq w, \\ \{L_v[n-1] \oplus \bigoplus_{w \sim v} L_w[s_{vw}-1], \phi\} & v = w. \end{cases}
\]

We note that \( \phi \) and \( g \) in (7.1) and (7.2) are some collections of morphisms, and we do not describe them above. We also note that in Lemma 7.4, we only used \( L_v \) for \( v \in V(T) \), and we did not use \( L_a \) which is used in Theorem 7.3. This is because if the intersection point \( a \) is in \( S_w \) for some \( w \in V(T) \), then \( L_a \) and \( L_w \) are isomorphic. See Remark 5.3.

In Lemma 7.4, we describe the twisted complexes corresponding to \( \tau_v(L_w) \) and \( \tau_v^{-1}(L_w) \). Now, we turn our attention to \( \phi(L_v) \) for \( \phi \) where \( \phi \) is a product of positive or negative powers of Dehn twists.

We note that there are two different ways of constructing a twisted complex for \( \phi^m(L_v) \). The first one is to use Theorem 7.3 and the second one is to apply Lemma 7.4 iteratively. To be more rigorous, one should show that \( \phi^m(L_v) \) transversally intersects \( \cup_v S_v \) in order to apply Theorem 7.3. It is easy to show that the condition can be achieved by choosing a proper representative of the mapping class \( [\phi] \). One can choose a proper representative by choosing a proper \textit{spherical Dehn twist profile}. Also, by using the freedom of choices, one can prove that the first and the second methods give the same twisted complex for \( \phi^m(L_v) \).

For the future convenience, let

\[
\mathcal{L}_{m,v}
\]
denote the twisted complex.
7.2. The notion of path. In the previous subsection, we discuss a construction of a twisted complex $L_{m,v}$ for $\phi^m(L_v)$. In this subsection, we introduce the notion of path, which we will use later for investigate $L_{m,v}$ more in detail.

**Definition 7.5.**

(1) A directed edge is an ordered sequence $[v_2, v_1]$ of two adjacent vertices $v_1, v_2$, i.e.,

$$v_1 \sim v_2 \in V(T).$$

(2) A constant loop at $v \in V(T)$ is the ordered sequence $[v, v]$.

(3) A path from $v_1$ to $v_k$ is an ordered sequence $[v_k, v_{k-1}, \ldots, v_1]$ of vertices $v_i \in V(T)$ such that $[v_{i+1}, v_i]$ be either a directed edge or a constant loop for $i = 1, \ldots, k-1$.

(4) The length of a path is the number of vertices in the path. For example, the length of a path $[v_k, \ldots, v_1]$ is $k$.

(5) A loop at $v \in V(T)$ is a path from $v$ to $v$ itself.

The notion of path can be used to describe the components of a twisted complex $L_{m,v}$. More precisely, one can associate a path from $v$ to $w$ to each component of the form $L_w[d]$ in the twisted complex isomorphic to $\phi(L_v)$. Let us explain it with the following example.

**Example.** Let $T$ be the Dynkin diagram of $A_3$ type. We label the set of vertices of $A_3$ by $V(A_3) = \{1, 2, 3\}$ in such a way that

$$1 \sim 2 \sim 3.$$ 

Let us assume that the cores $S_1, S_2$ and $S_3$ are graded so that

$$s_{12} = s_{23} = 1.$$ 

For a fixed $n \in \mathbb{N}$, let $\phi$ be the symplectic automorphism given by

$$\phi = \tau_3 \circ \tau_1 \circ \tau_2^{-1} \circ \tau_3 : P_n(A_3) \rightarrow P_n(A_3).$$

Now, let us describe the twisted complex isomorphic to $\phi(L_3)$, which obtained by applying Lemma 7.4.

First, for $\tau_3(L_3)$ we have

$$\tau_3(L_3) \cong \{L_2 \oplus L_3[1-n], f_1\}. \quad (7.4)$$

Here one can say that the component $L_2$ is created from $L_3$ by applying $\tau_3$. We associate the path $[2, 3]$ to this new component $L_2$ in order to indicate that this $L_2$ is a component of the twisted complex isomorphic to $\tau_3(L_3)$. Similarly, the $L_3[1-n]$ in $\tau_3(L_3)$ in (7.4) corresponds to a path $[3, 3]$.

Next, for $(\tau_2^{-1} \circ \tau_3)(L_3)$, we have

$$\tau_2^{-1} \circ \tau_3)(L_3) \cong \{(L_1[2-n] \oplus L_2[n-1] \oplus L_3) \oplus L_3[1-n], f_2\}. \quad (7.5)$$

Here the first three terms of the right hand side are parenthesized to emphasize that those terms are created from $L_2$ in (7.4) by applying $\tau_2^{-1}$. Then, one can say that the component $L_1[2-n]$ is created from $L_2$ by applying $\tau_2^{-1}$. Now we associate the path $[1, 2, 3]$ to the this component $L_1[n-2]$ to indicate that the component $L_1[n-2]$ is created in two steps. More precisely, $L_2$ is created by applying $\tau_3$ to $L_3$, then $L_1$ is created by applying $\tau_2$ to $L_2$ which is created in the first step. Similarly, the terms $L_2[2-n-1], L_3, L_3[1-n]$ in (7.5) correspond to the paths $[2, 2, 3], [3, 2, 3], [3, 3]$ respectively. We note that all three terms in the parenthesis in (7.5) correspond to paths ending with $[2, 3]$ since they are created to $L_2$ in (7.4), corresponding to the path $[2, 3]$.

For $(\tau_1 \circ \tau_2^{-1} \circ \tau_3)(L_3)$, we have

$$(\tau_1 \circ \tau_2^{-1} \circ \tau_3)(L_3) \cong \{(L_1[1-n] \oplus L_2) \oplus L_2[n-1] \oplus L_3 \oplus L_3[1-n], f_3\}. \quad (7.6)$$
Once again, here the first two terms of the right hand side are parenthesized to emphasize that those terms are created from \( L_1[n-2] \) in (7.3) by applying \( \tau_1 \). Then, one can say that the component \( L_1[-1] \) in (7.6) is created from \( L_1[n-2] \) in (7.3) by applying \( \tau_1 \). We associate the path \([1,1,2,3]\) to this component \( L_1[-1] \) to indicate that the component \( L_1[-1] \) is a component of the twisted complex isomorphic to \((\tau_1 \circ \tau_2^{-1} \circ \tau_3)(L_3)\). Similarly, the other terms \( L_2, L_2[n-1], L_3, L_3[1-n] \) correspond to paths \([2,1,2,3],[2,2,3],[3,2,3],[3,3,3],[2,3,3],[3,3,3]\).

Finally, \( \phi(L_3) = (\tau_3 \circ \tau_1 \circ \tau_2^{-1} \circ \tau_3)(L_3) \) is isomorphic to the following twisted complex:

\[
\phi(L_3) = (\tau_3 \circ \tau_1 \circ \tau_2^{-1} \circ \tau_3)(L_3) \cong (L_1[-1] \oplus L_2 \oplus L_2[n-1] \oplus L_2 \oplus L_3[1-n] \oplus L_2[1-n] \oplus L_3[2-2n], f_3).
\]

The seven terms in (7.8) correspond to the paths

\[
(7.8) \quad [1,1,2,3],[2,1,2,3],[2,2,3],[2,3,2,3],[3,3,2,3],[2,3,3],[3,3,3].
\]

**General rule.** From the above example, one can find a general rule explaining the above example. We explain the rule below.

Let \( \phi \) be a symplectic automorphism obtained by composing positive or negative powers of Dehn twists \( \tau_v \) for \( v \in V(T) \), i.e.,

\[
\phi = \tau_{v_1}^+ \circ \tau_{v_{i-1}}^+ \circ \cdots \circ \tau_{v_1}^+.
\]

Let \( J \) be an ordered sequence of vertices in \( V(T) \)

\[
[v_s, v_{s-1}, \ldots, v_1],
\]

which corresponds to \( \phi \).

By repeatedly applying Lemma 7.4, one can find a twisted complex isomorphic to \( \phi(L_v) \). Then, every term in the twisted complex, which is of the form \( L_w \) with a proper shift, corresponds to a path

\[
[w_k, w_{k-1}, \ldots, w_1],
\]

such that

(i) the path \([w_{k-1}, w_{k-2}, \ldots, w_1]\) is a subsequence of

\[
J = [v_s, v_{s-1}, \ldots, v_1],
\]

i.e., there is an increasing sequence \( j_1 \leq \cdots \leq j_{k-1} \) of natural numbers \( j_i \in \mathbb{N} \) so that

\[
w_{j_i} = v_{j_i},
\]

(ii) \( w_1 \) is the vertex \( v \) indicating the input \( L_v \) of \( \phi(L_v) \),

(iii) \( j_1 \) satisfies that \( j_1 = \min\{l|v_l = v\} \),

(iv) for any \( 1 \leq i \leq k-2 \), there is no \( t \) such that

\[
j_i < t < j_{i+1}, \text{ and } v_t = v_{j_i},
\]

(v) there is no \( j_{k-1} < i \leq s \) such that \( v_i = w_{k-1} \),

(vi) the last term \( w_k \) in the path is the vertex \( w \) indicating the term of the form \( L_w[d], d \in \mathbb{Z} \).

Moreover, it is easy to check that, if there is a path \([w_k, \ldots, w_1]\) satisfying the conditions (i)–(vi), then the twisted complex for \( \phi(L_v) \) should have a term of the form \( L_w[d] \) corresponding to the path.

We end this subsection by reviewing the above example. In the example, \( \phi = \tau_3 \circ \tau_1 \circ \tau_2^{-1} \circ \tau_3 \). Thus, the corresponding sequence should be

\[
J = [3,1,2,3].
\]

In (7.8), we have seven paths corresponding to the seven terms in (7.4). Then, the seven paths satisfy the conditions (i)–(v). Especially, for the condition (i), one can observe the following:

- For the path \([1,1,2,3]\), one can find a subsequence \([-1,1,2,3]\) of \( J \).
• For the path \([2, 1, 2, 3]\), one can find a subsequence \([-1, 2, 3]\) of \(J\).
• For the path \([2, 2, 3]\), one can find a subsequence \([-1, -2, 3]\) of \(J\).
• For the path \([2, 3, 2, 3]\), one can find a subsequence \([3, -1, 2, 3]\) of \(J\).
• For the path \([3, 3, 2, 3]\), one can find a subsequence \([3, -1, 2, 3]\) of \(J\).
• For the path \([2, 3, 3]\), one can find a subsequence \([-1, 3, -2, 3]\) of \(J\).
• For the path \([3, 3, 3]\), one can find a subsequence \([-1, 3, -2, 3]\) of \(J\).

7.3. Symplectic automorphisms of Penner type. Section 6 cares any product of positive or negative powers of Dehn twists, but as mentioned in the beginning of Section 7, we care symplectic automorphisms of a specific type. That specific type is called Penner type, and will be defined below.

Let \(T\) be a tree and let \(V(T)\) be the set of vertices. Then, by choosing a vertex \(v_0 \in V(T)\), one can define the following:

\[
\begin{align*}
V_+ (T) & := \{ v \in V(T) \mid v \text{ is connected to } v_0 \text{ by even number of edges.}\} \\
V_- (T) & := \{ v \in V(T) \mid v \text{ is connected to } v_0 \text{ by odd number of edges.}\}
\end{align*}
\]

The above \(V_+(T)\) and \(V_-(T)\) are disjoint decomposition of \(V(T)\).

Definition 7.6. A symplectic automorphism \(\phi : P_n(T) \rightarrow P_n(T)\) is of Penner type if either \(\phi\) or \(\phi^{-1}\) is a product of

- positive powers of \(\tau_v\) if \(v \in V_+(T)\),
- negative powers of \(\tau_w\) if \(w \in V_-(T)\).

Remark 7.7. We would like to point out that two sets \(V_+(T)\) and \(V_-(T)\) defined in Equations (7.9) and (7.10) are dependent on the choice of \(v_0\), but Definition 7.6 is independent from the choice. In the rest of this section, we assume an arbitrary chosen \(v_0\).

Before going further, we briefly discuss the motivation of considering symplectic automorphisms of Penner type. In [Pen88], Penner introduced a simple construction of pseudo-Anosov surface automorphisms. We note that every pseudo-Anosov surface automorphism admits a combinatorial tool helping us to keep track of the iteration of the surface automorphism. Since Definition 7.6 is a natural generalization of Penner construction on higher dimensional symplectic manifolds, we expect that there is a combinatorial tool. The notion of path explained in Section 7.2 is the expected tool.

Remark 7.8. We note that the original Penner construction [Pen88] in surface theory has one more condition than Definition 7.6. The higher dimensional generalization for the extra condition is the following Condition (P).

\((P)\) every \(v \in V(T)\) appears in the product.

We omit the condition (P) since without the condition (P), the properties in the rest of the paper hold. For more details on surface theory, we refer the eager reader to [FM12].

For the future use, we set the following notation.

Definition 7.9.

(1) For each \(v \in V(T)\), let \(\sigma_v\) be defined as follows:

\[
\sigma_v = \begin{cases} 
1 & \text{if } v \in V_+(T), \\
-1 & \text{if } v \in V_-(T).
\end{cases}
\]
(2) Let $J = [v_s, v_s-1, \ldots, v_1]$ be an ordered sequence of vertices $v_i \in V(T)$. Then, $\phi_J$ (resp. $\phi_{-J}$) is the symplectic automorphism of Penner type defined as follows:

$$\phi_J = r_{v_s} \circ r_{v_{s-1}} \circ \cdots \circ r_{v_1}, \quad \phi_{-J} = r_{v_s}^{-1} \circ r_{v_{s-1}}^{-1} \circ \cdots \circ r_{v_1}^{-1}.$$ 

We note that by Definition 7.6 for any $\phi$ of Penner type, there is an ordered sequence $J$ such that $\phi = \phi_J$ or $\phi = \phi_{-J}$.

### 7.4. Entropies of symplectic automorphisms of Penner type

In this subsection, we prove the main theorem of Section 7, i.e., Theorem 7.4. Theorem 7.4 says that if a symplectic automorphism $\phi$ is of Penner type, then the categorical entropy $h_{cat}(\phi)$ can be computed by counting the intersection points of two Lagrangians.

We roughly describe the proof of Theorem 7.4. In order to prove that, we show that $\mathcal{L}_{m,v}$ defined in (7.3) satisfies that

- for any $v, w \in V(T), m \in \mathbb{Z}$, $\text{hom}^*(S_w, \mathcal{L}_{m,v})$ has the zero differential map.

If it holds, then one can show that the (logarithmic growth rate of) length of the twisted complex $\mathcal{L}_{m,v}$ gives the categorical entropy of $\phi$. On the other hand, as mentioned at the end of Section 7.1, the length of $\mathcal{L}_{m,v}$ is given by number of intersection points $\phi^n(L_v) \cap (L_w S_w)$.

In order to show the above, we recall that each term in $\mathcal{L}_{m,v}$ is of the form $L_w[d]$ for some $w \in V(T)$ and $d \in \mathbb{Z}$. We start Section 7.4 by investigating how the shift $d$ is determined. The main tool of the investigation is the notion of path given in Section 7.2.

We recall that if $L_w[d]$ is a term in $\mathcal{L}_{m,v}$, then there is a path $I = [w = v_s, \ldots, v_1 = v]$ corresponding to $L_w[d]$ as mentioned right after Definition 7.5. Then, one can determine $d \in \mathbb{Z}$ by using the path $I$ and Lemma 7.4.

We note that since $\phi$ is of Penner type, there is a $J$ such that $\phi = \phi_J$ or $\phi = \phi_{-J}$. Then, $[v_{s-1}, v_{s-2}, \ldots, v_1]$ is a subsequence of $J[J = J\ldots J$.

We recall that by Lemma 7.4, $r_{v_s}^{\pm \sigma_{v_1}}(L_v)$ contains a term $L_{v_2}[d_1]$ for some integer $d_1$, which is determined by Lemma 7.4 and $s_{v_0 v_2}$. Similarly, $(r_{v_1}^{\pm \sigma_{v_1}}(r_{v_2}^{\pm \sigma_{v_1}}(L_{v_2}[d_1])))$ contains a term $L_{v_3}[d_1 + d_2]$ for some integer $d_2$, which is determined by Lemma 7.4 and $s_{v_2 v_3}$.

We repeat the same logic inductively, then one can observe that

$$r_{v_s}^{\pm \sigma_{v_1}} \circ r_{v_1}^{\pm \sigma_{v_1}} \circ \cdots \circ r_{v_1}^{\pm \sigma_{v_1}}(L_{v_1} = v)$$

contains $L_{v_{i+1}}[d_1 + d_2 + \cdots + d_i]$. The shifts $d_j$ are determined by Lemma 7.4 and $s_{v_i v_{i+1}}$ for $v, w \in V(T)$.

If the path $I$ is a loop, then one can observe that Lemma 7.10 holds.

**Lemma 7.10.** With the notation given above, if $I = [v = v_s, \ldots, v_1 = v]$ is a loop at a vertex $v \in V(T)$, then the shift $d$ of $L_v[d]$ satisfies

$$d = l(1-n), \text{ for some } l \in \mathbb{Z}.$$ 

**Proof.** We note that with the notation used above,

$$d = d_1 + \cdots + d_{s-1} = \sum_{\{i | v_i \neq v_{i+1}\}} d_i + \sum_{\{i | v_i = v_{i+1}\}} d_i.$$ 

First, we focus on the first term $\sum_{\{i | v_i \neq v_{i+1}\}} d_i$. For some $i_0 \in \{1, \ldots, s-1\}$, if $v_{i_0} \neq v_{i_0+1}$, then one can observe that there is $j \in \{1, \ldots, s-1\}$ such that

$$v_{i_0} = v_{j+1}, v_{i_0+1} = v_j,$$
since the path $I$ is a loop. For such $i_0$ and $j$, one can observe that
\[ d_{i_0} + d_j = 0. \]
(See Lemma 7.7 (6))

We note that since $v_{i_0} \sim v_{i_0 + 1}$, either $v_{i_0} \in V_+ (T)$, $v_{i_0 + 1} \in V_- (T)$ or $v_{i_0} \in V_- (T)$, $v_{i_0 + 1} \in V_+ (T)$. This is crucial to show that $d_{i_0} + d_j = 0$.

Moreover, one can observe that
\[ ||[i][v_{i_0}, v_{i_0 + 1}] = [v_i, v_{i+1}]|| = ||[j][v_{i_0 + 1}, v_{i_0}] = [v_j, v_{j+1}]||. \]
This concludes that
\[ \sum_{[i]v_i \neq v_i+1} d_i = 0. \]

For the second term $\sum_{[i]v_i = v_i+1} d_i$, one can observe that $d_i$ is either $1 - n$ or $n - 1$ from Lemma 7.4. This completes the proof.

Now, we consider any path $I$, not just loops.

**Lemma 7.11.** Let use the notation used above. For any vertex $v, w \in V(T)$, there is an integer $c_{vw}$ satisfying the following: If $L_w[d]$ is a term in $L_{m,v}$, then
\[ d = c_{vw} + l(1 - n), \]
for some $l \in \mathbb{Z}$.

**Proof.** Since $L_w[d]$ is a term in $L_{m,v}$, there is a path $I = [w = v_s, \ldots, v_1 = v]$ corresponding to $L_w[d]$ such that $I$ satisfies the general rule in Section 7.2. As explained above, the shift $d$ can be computed by using the path $I$ and Lemma 7.4.

Since $T$ is a tree, one can observe that the path $I$ can be decomposed as
\[ [v_s = v_{j_1}, \ldots, v_{i_k}], [v_{i_k}, \ldots, v_{j_{k-1}}], [v_{j_{k-1}}, \ldots, v_{i_{k-1}}], \ldots, [v_{j_1}, \ldots, v_i], \]
so that

(i) $[v_{j_1}, \ldots, v_i]$ is a path such that $v_{i_t}, v_{i_{t+1}}, \ldots, v_{j_t}$ are pairwise different vertices for all $t \in \{1, \ldots, k\}$, and

(ii) $[v_{i_{t+1}}, \ldots, v_{j_t}]$ is a loop, i.e., $v_{i_t} = v_{j_t}$ for all $t \in \{1, \ldots, k - 1\}$.

We note that the decomposition is not unique, but there exists at least a decomposition satisfying the conditions.

By Lemma 7.10 one can observe that the shift that is induced from the part of $I$ satisfying (ii) is an integer multiple of $(1 - n)$.

For the part satisfying (i), one can assume the concatenation
\[ [v_s = v_{j_1}, \ldots, v_{i_k} = v_{j_{k-1}}, v_{j_{k-1}} = v_{i_{k-2}}, \ldots, v_{i_2} = v_{j_1}, \ldots, v_i]. \]

Moreover, one can assume that all vertices in (7.12) are pairwise different. If not, in other words, there are two vertices in (7.12) such that they are the same, one can choose a different decomposition of $I$ so that the two vertices are two end points of a loop.

Finally, one can observe that the concatenation in (7.12) corresponds to the unique geometric path, i.e., the shortest path connecting two vertices $v$ and $w$ on the tree $T$, not a path in Definition 7.5. Thus, the shift induced from the part of $I$ satisfying (i) is determined by the unique geometric path. Let $c_{vw}$ be the shift determined by the geometric path. Then, it completes the proof.

In order to state and to prove Theorem 7.14, we need Lemma 7.12 and Definition 7.13.
Lemma 7.12 (cf. Lemma 6.4). Let $L$ be a twisted complex with components $L_v$, $v \in V(T)$. For each $w \in V(T)$, the following equality holds.

$$\text{len}_{L_w}(L) = \dim hom^*(S_w, L).$$

Proof. The proof of Lemma 6.4 also works for Lemma 7.12. \qed

Definition 7.13. Let $F : V \to V$ be a linear operator on a finite dimensional $\mathbb{R}$-vector space $V$. The spectral radius $\text{Rad}(F)$ of $F$ is defined by the maximum absolute value of the generalized eigenvalue of $F$.

We state and prove the main theorem of Section 7.

Theorem 7.14. Let $\phi$ be a symplectic automorphism of Penner type on $P_n(T)$. For any $v \in V(T), m \in \mathbb{N}$, let $L_{m,v}$ be the twisted complex defined in (7.3) for the triple $(\phi, v, m)$. Then,

1. The categorical entropy $h_{\text{cat}}(\phi)$ is given by

$$h_{\text{cat}}(\phi) = \lim_{m \to \infty} \frac{1}{m} \log \sum_{v,w} |S_w \cap \phi^m(L_v)|.$$

2. If $n$ is odd, then the categorical entropy $h_{\text{cat}}(\phi)$ equals

$$\log \text{Rad}(\phi : H_n(P_n(T), \partial P_n(T)) \to H_n(P_n(T), \partial P_n(T))).$$

Proof.

Proof of (1): Recall from Definition 2.8 that the categorical entropy $h_{\text{cat}}(\phi)$ is given by

$$h_{\text{cat}}(\phi) = \lim_{m \to \infty} \frac{1}{m} \log \sum_{v,w} \inf \{\text{len}_{L_w}(L) \mid L \cong \phi^m(L_v) \oplus C' \text{ for some } C' \in W(P_n(T))\}.$$  

The above equation induces that

$$h_{\text{cat}}(\phi) \leq \lim_{m \to \infty} \log \sum_{v,w} \text{len}_{L_w}(L_{m,v}).$$

Moreover, since

$$\dim hom^*(S_w, L_{m,v}) = \text{len}_{L_w}(L_{m,v}) = |S_w \cap \phi^m(L_v)|$$

by Lemma 7.12 and Theorem 7.3, one obtains

$$h_{\text{cat}}(\phi) \leq \lim_{m \to \infty} \log \sum_{v,w} \dim hom^*(S_w, L_{m,v}).$$

On the other hand, from Lemma 7.12 and an argument similar to (6.9), one can show that, for each $L$ in the right hand side of (7.13), the length $\text{len}_{L_w}L$ satisfies

$$\text{len}_{L_w}L \geq \dim \text{Hom}^*(S_w, \phi^m(L_v)) = \dim \text{Hom}^*(S_w, L_{m,v}).$$

Therefore, we obtain a lower bound for the categorical entropy $h_{\text{cat}}(\phi)$:

$$h_{\text{cat}}(\phi) \geq \lim_{m \to \infty} \frac{1}{m} \log \sum_{v,w} \dim \text{Hom}^*(S_w, L_{m,v}).$$

Then, inequalities (7.13) and (7.16) give us

$$\lim_{m \to \infty} \frac{1}{m} \log \sum_{v,w} \dim \text{Hom}^*(S_w, L_{m,v}) \leq h_{\text{cat}}(\phi) \leq \lim_{m \to \infty} \log \sum_{v,w} \dim hom^*(S_w, L_{m,v}).$$

Thus, by (7.14) and (7.17), it is enough to show that the differential map of $\text{Hom}^*(S_w, L_{m,v})$ is the zero map in order to complete the proof.
We point out that $\text{hom}^*(S_w, L_m, v)$ is determined by the terms in $L_m, v$ of the form $L_w[d]$. In other words, the cochain complex $\text{hom}^*(S_w, L_m, v)$ is identified with

$$\text{hom}^*(S_w, L_m, v) = \bigoplus_i \text{hom}^*(S_w, L_w[d_i]) = \bigoplus_i k[d_i - n],$$

as graded vector spaces.

By Lemma 7.11, for any $i$, $d_i = c_w w + l_i (1 - n)$ for some $l_i \in \mathbb{Z}$. This means that the differential map is zero for $n \geq 3$.

**Proof of (2):** We note that $B = \{[L_v]\}_{v \in V(T)}$ is a basis for the degree $n$ relative homology of $P_n(T)$ with the real coefficients, i.e.,

$$H_n(P_n(T), \partial P_n(T)) \otimes \mathbb{R}.$$

Moreover, $B$ admits a dual basis of $H_n(P_n(T))$ with respect to the intersection pairing. The dual basis is $\{[S_v]\}_{v \in V(T)}$.

We also note that by (7.11), if $n$ is odd, then every geometric intersection points of $S_w$ and $\phi^m(L_v)$ has the same sign as an algebraic intersection point. Thus, for any $v \in V(T)$, one obtains

$$(7.18) \quad [\phi^m(L_v)] = \sum_{w \in V(T)} |S_w \cap \phi^m(L_v)|[L_w] \in H_n(P_n(T), \partial P_n(T)) \otimes \mathbb{R}.$$

Let $B$ denote the matrix representation of a linear map $\phi^*: H_n(P_n(T), \partial P_n(T)) \otimes \mathbb{R} \rightarrow H_n(P_n(T), \partial P_n(T)) \otimes \mathbb{R}$, with respect to the basis $B$. Then, from (7.18), one can say that

$$B^m := ([S_w \cap \phi^m(L_v)]_{v, w \in V(T)}.$$

By (1), $h_{\text{cat}}(\phi)$ is the logarithmic growth rate of

$$\|B^m\|_1$$

as $m \rightarrow \infty$, where $\|\cdot\|_1$ means the standard $L_1$ norm.

Since the logarithmic growth rate of $\|B^m\|_1$ is the same as the logarithm of the spectral radius of $B$, (2) holds. $\square$

### 8. A COMPARISON OF THE CATEGORICAL AND TOPOLOGICAL ENTROPIES

In Section 7, we discussed how to compute the categorical entropies of symplectic automorphisms of Penner type. The result of the discussion, i.e., Theorem 7.14, shows that the categorical entropy has a strong connection with some topological invariants, for example, the geometric intersection number or the spectral radius.

In the current section, we investigate a connection between the categorical entropy and another topological invariant. More precisely, we prove that the categorical entropy of a symplectic automorphism of Penner type bounds the topological entropy of it from below, i.e., we prove Theorem 8.3.

In Section 8.1 we briefly recall definitions and properties related to the topological entropy. With the introduced definitions, we can state Theorem 8.3. For odd $n$, Theorem 8.3 is a straightforward corollary of Theorem 7.14. However, in order to prove the case of even $n$, we need a preparation that will be given in Section 8.2. Then, we prove Theorem 8.3 for even $n$ in Section 8.3.
8.1. **Preliminaries on topological entropy.** We briefly review the notion of topological entropy and some of its properties. For more details, we refer the reader to [Gro03], [Gro87], or [Yom87].

First, we introduce Definition 8.1.

**Definition 8.1** (See Section 1 of [Gro87]). Let $X$ be a topological manifold of dimension $n$ equipped with a metric $g$. Let $f : X \rightarrow X$ be an endomorphism.

1. The graph $\Gamma_f$ of $f$ is a subset of $X \times X$ such that
   \[ \Gamma_f := \{(x, f(x)) | x \in X\} \cdot \]

2. For a graph $\Gamma_f$, we denote $\Gamma_{f,k}$ the set of strings
   \[ (x, f(x), \ldots, f^k(x)) \in X^k := X \times \cdots \times X \text{ (k factors)} \cdot \]

3. An $\epsilon$-cubes in $X^k$ is a product of balls in $X$ of radius $\epsilon$.

4. For a subset $Y \subset X^k$, $\text{Cap}_\epsilon Y$ is the minimal number of $\epsilon$-cubes needed to cover $Y$.

5. The topological entropy of $f$, denoted by $h_{\text{top}}(f)$, is given by
   \[ h_{\text{top}}(f) := \lim_{\epsilon \to 0} \limsup_{k \to \infty} \frac{1}{k} \log \text{Cap}_\epsilon \Gamma_{f,k} \cdot \]

6. We denote by $\text{Vol}(\Gamma_{f,k})$ the $n$-dimensional volume of $\Gamma_{f,k} \subset X^k$ with respect to the product metric in $X^k$.

7. We set the log volume of $f$, $\text{lov}(f)$ as follows:
   \[ \text{lov}(f) := \limsup_{k \to \infty} \frac{1}{k} \log \text{Vol}(\Gamma_{f,k}) \cdot \]

We note that if $X$ is compact, then the definition of $h_{\text{top}}(f)$ does not depend on the choice of the metric [Bow71]. Thus, for the well definedness of topological entropy, we consider the plumbing spaces $P_n(T)$ as Weinstein domains, not Weinstein manifolds.

For the topological entropy of $f : X \rightarrow X$, Proposition 8.2 is well-known. See [Gro03 Sections 1 and 2].

**Proposition 8.2.**

1. If $Y$ is a subset of $X$ such that $f(Y) \subset Y$, then
   \[ h_{\text{top}}(f|_Y) \leq h_{\text{top}}(f) \cdot \]

2. Let $f_*$ be the induced linear map on $H_\ast(X)$. Then,
   \[ \log \text{Rad}(f_*) \leq \text{lov}(f) \cdot \]

3. The log volume of $f$ bounds the topological entropy of $f$ from below, i.e.,
   \[ \text{lov}(f) \leq h_{\text{top}}(f) \cdot \]

Together with the notion of topological entropy, we can state Theorem 8.3 saying a relation between categorical and topological entropies of symplectic automorphisms of Penner type.

**Theorem 8.3.** Let $T$ be a given tree. If $\phi : P_n(T) \xrightarrow{\approx} P_n(T)$ is a symplectic automorphism of Penner type, then the following inequality holds:

\[ h_{\text{cat}}(\phi) \leq h_{\text{top}}(\phi) \cdot \]
Proof for odd \( n \). First, we recall that \( \phi: P_n(T) \rightarrow P_n(T) \) induces two maps

\[
\phi_*: H_*(P_n(T)) \rightarrow H_*(P_n(T)), \\
\phi'_*: H_n(P_n(T), \partial P_n(T)) \rightarrow H_n(P_n(T), \partial P_n(T)).
\]

One can easily check that two induced maps \( \phi_*, \phi'_* \) have the same spectral radii. First, one can see that

\[
H_*(P_n(T)) = H_0(P_n(T)) \oplus H_n(P_n(T)).
\]

Since \( \phi_* \) acts as an identity map on \( H_0(P_n(T)) \), the relative Poincareé duality shows that \( \phi_* \) and \( \phi'_* \) have the same spectral radii.

We recall that if \( n \) is odd, then

\[
h_{cat}(\phi) = \log \text{Rad} \phi_*
\]

by the third item of Theorem 7.14. Then, Proposition 8.2 together with the above arguments, completes the proof. \( \square \)

8.2. Geometric preparation. Let \( k, n \) be natural numbers such that \( k \leq n \). In this subsection, we see that \( P_k(T) \) can be embedded in \( P_n(T) \).

It is easy to see that \( S^k \subset S^n \). To be more precise, let \( S^k \) (resp. \( S^n \)) be the unit sphere in \( \mathbb{R}^{k+1} \) (resp. \( \mathbb{R}^{n+1} \)) with respect to the standard Euclidean metric. Then, the embedding of \( \mathbb{R}^{k+1} \) to \( \mathbb{R}^{n+1} \)

\[
\mathbb{R}^{k+1} \hookrightarrow \mathbb{R}^{n+1}, (x_0, \ldots, x_k) \mapsto (x_0, \ldots, x_k, 0, \ldots, 0)
\]

induces an (isometric) embedding of \( S^k \) into \( S^n \). We note that the embedding not only is an isometric embedding, but also sends every geodesic in \( S^k \) to a geodesic in \( S^n \).

We note that if a smooth manifold admits a metric, one can identify the tangent and cotangent bundles of the manifold by using the metric. Also, we note that if \( N \) is a submanifold of \( M \), then the tangent bundle \( TN \) is a submanifold of \( TM \). Thus, one obtains the following:

\[
T^* S^k \cong TS^k \hookrightarrow TS^n \cong T^* S^n.
\]

Thus, one obtains an embedding of \( T^* S^k \) into \( T^* S^n \). Moreover, the embedding is a symplectic embedding when the cotangent bundles are equipped with the standard symplectic structures.

Let \( \tau_k \) (resp. \( \tau_n \)) be a Dehn twist defined on \( T^* S^k \) (resp. \( T^* S^n \)). Since the embedding of \( S^k \) into \( S^n \) sends geodesics to geodesics, one can say that

\[
\tau_n|_{T^* S^k} = \tau_k,
\]

where \( \tau_n|_{T^* S^k} \) means the restriction of \( \tau_n \) onto the embedded \( T^* S^k \).

The above discussion is given on cotangent bundles of spheres. We generalize it to plumbing spaces of them. In order to do that, we recall the plumbing procedure.

Let \( S^1_1, S^1_2 \) be \( n \)-dimensional spheres. If one plumbs \( T^* S^1_1 \) and \( T^* S^1_2 \) at \( p_i \in S^1_i \), one should choose a small neighborhood \( U_i \) of \( p_i \) in \( S^1_i \) such that \( U_i \cong \mathbb{D}^n \). Then, one identifies the disk cotangent bundles of \( U_1 \) and \( U_2 \), or simply identifies \( D^* U_1, D^* U_2 \) into \( T^* S^n \) so that the base of \( D^* U_1 \) is identified with the fiber of \( D^* U_2 \) at \( p_2 \).

For any natural number \( k \leq n \), one can assume that \( p_i \in S^k_i \subset S^n_i \) for \( i = 1, 2 \), where \( S^k_i \) is the embedded \( k \)-dimensional sphere \( S^k \) in \( S^n_i \). Moreover, one can assume that there is a Darboux chart of \( D^* U_1 \) such that the Darboux chart identifies \( U_1 \) (resp. \( U_1 \cap S^k \subset S^k_1 \)) with \( \mathbb{D}^k \) (resp. \( \mathbb{D}^k \subset \mathbb{D}^n \)). We note that \( \mathbb{D}^k \) is the unit disk in \( \mathbb{R}^k \), centered at the origin. Thus, the embedding of \( \mathbb{R}^k \) into \( \mathbb{R}^n \) induces an embedding of \( \mathbb{D}^k \) to \( \mathbb{D}^n \). Then, under the identification of \( D^* U_1 \) and \( D^* U_2 \), one can check that the \( T^* S^k_1 \cap D^* U_1 \cong D^* \mathbb{D}^k \)
is identified with \( T^*S_1^k \cap D^*U_2 \simeq D^*D^k \). In order words, the plumbing of \( T^*S_1^k \) and \( T^*S_2^k \) is embedded in the plumbing of \( T^*S_1^n \) and \( T^*S_2^n \).

We would like to point out that the plumbing procedure is a local procedure. Thus, for each of any plumbing points of \( P_n(T) \), we can apply the same argument. Then, this concludes that \( P_k(T) \subset P_n(T) \).

Based on the above arguments, Definitions 5.2 and 7.9 can be generalized as follows:

**Definition 8.4.** Let \( (k, n) \) be a pair of natural numbers such that \( k \leq n \) and let \( T \) be a tree.

1. Let \( S_v^n \) (resp. \( S_v^k \)) be the Lagrangian sphere in \( P_v(T) \) (resp. \( P_k(T) \)) corresponding to \( v \in V(T) \). Then, the embedding of \( P_k(T) \) into \( P_v(T) \) embeds \( S_v^k \) into \( S_v^n \).
2. Let \( \tau_{v,n} \) (resp. \( \tau_{v,k} \)) be a Dehn twist along \( S_v^n \) (resp. \( S_v^k \)). We note that the restriction of \( \tau_{v,n} \) on \( P_k(T) \) is \( \tau_{v,k} \).
3. Let \( L_v^n \) (resp. \( L_v^k \)) be the Lagrangian cocore of \( S_v^n \) (resp. \( S_v^k \)) in \( P_v(T) \). Then, the embedding of \( P_k(T) \) into \( P_v(T) \) embeds \( L_v^k \) into \( L_v^n \).
4. Let \( J = [v_s, \ldots, v]\) be an ordered sequence of vertices \( v_i \in V(T) \). Then, \( \phi_{\pm J,n} \) (resp. \( \phi_{\pm J,k} \)) be the symplectic automorphism of Penner type on \( P_v(T) \) (resp. \( P_k(T) \)), defined as follows:

\[
\phi_{J,n} = \tau_{v_1,n} \circ \tau_{v_1,v_{n-1},n} \circ \cdots \circ \tau_{v_1,v}, \quad \phi_{-J,n} = \tau_{v_1,n} \circ \tau_{v_1,v_{n-1},n} \circ \cdots \circ \tau_{v_1,v},
\]

\[
\phi_{J,k} = \tau_{v_1,k} \circ \tau_{v_1,v_{n-1},k} \circ \cdots \circ \tau_{v_1,v}, \quad \phi_{-J,k} = \tau_{v_1,k} \circ \tau_{v_1,v_{n-1},k} \circ \cdots \circ \tau_{v_1,v}.
\]

We note that the restriction of \( \phi_{\pm J,n} \) onto \( P_k(T) \) is \( \phi_{\pm J,k} \).

### 8.3. Theorem 8.3 for even \( n \).

In order to prove Theorem 8.3 for even \( n \), we need Lemma 8.5.

**Lemma 8.5.** Let \( T \) be a tree and let \( J = [v_1, \ldots, v] \) be an ordered sequence of vertices \( v_i \in V(T) \). Then, for any \( n, k \) such that \( 3 \leq k \leq n \),

\[
h_{cat}(\phi_{\pm J,n}) = h_{cat}(\phi_{\pm J,k}).
\]

**Proof.** Let us assume that \( 3 \leq k \leq n \). Then, from the arguments in Section 8.2 one can check that

\[
S_v^n \cap \phi_{\pm J,n}^m(L_w^n) \cap P_k(T) = S_v^k \cap \phi_{\pm J,k}^m(L_w^k),
\]

for any \( m \in \mathbb{Z} \), and for any \( v, w \in V(T) \). We note that \( P_k(T) \subset P_n(T) \).

Thus, for any \( m \in \mathbb{Z} \),

\[
|S_v^n \cap \phi_{\pm J,n}^m(L_w^n)| \geq |S_v^k \cap \phi_{\pm J,k}^m(L_w^k)|.
\]

Then, by the first item of Theorem 7.14

\[
h_{cat}(\phi_{\pm J,n}) \geq h_{cat}(\phi_{\pm J,k}).
\]

Especially, we have the following inequalities for any \( l \in \mathbb{N}_{\geq 2} \).

\[
h_{cat}(\phi_{\pm J,2l+1}) \geq h_{cat}(\phi_{\pm J,2l}) \geq h_{cat}(\phi_{\pm J,2l-1}).
\]

From the second item of Theorem 7.14 one can check that

\[
h_{cat}(\phi_{\pm J,2l+1}) = h_{cat}(\phi_{\pm J,2l-1}),
\]

for any \( l \in \mathbb{N}_{\geq 2} \). This is because they are given by the spectral radius of the same matrix. This completes the proof.

We prove Theorem 8.3 for even \( n \).
Proof of Theorem 8.3 for even $n$. By Definition 7.6, there is an ordered sequence $J = [v_k, \ldots, v_1]$ of vertices $v_j \in V(T)$ such that either $\phi = \phi_{J,n}$ or $\phi = \phi_{\bar{J},n}$.

The first item of Proposition 8.2 together with Section 8.2 implies that

$$h_{\text{top}}(\phi_{J,3}) \leq h_{\text{top}}(\phi_{J,4}) \leq \cdots \leq h_{\text{top}}(\phi_{\bar{J},n}) \leq \cdots.$$

On the other hand, by Lemma 8.5

$$h_{\text{cat}}(\phi_{J,3}) = h_{\text{cat}}(\phi_{J,4}) = \cdots = h_{\text{cat}}(\phi_{\bar{J},n}) = \cdots.$$

Since Theorem 8.3 for $n = 3$ is already proven above, we have

$$h_{\text{cat}}(\phi_{J,n}) = h_{\text{cat}}(\phi_{J,3}) \leq h_{\text{top}}(\phi_{J,3}) \leq h_{\text{top}}(\phi_{\bar{J},n}),$$

for any $n \geq 3$. This completes the proof. 

\[\Box\]

9. An Example

In this section, we give an example of computing a categorical entropy by using the previous sections.

We will consider the following specific example. Let the given tree $T$ be the Dynkin diagram of $A_3$ type. Then, we can label the vertices of $T = A_3$ as $V(A_3) = \{1, 2, 3\}$ so that the vertex $1$ (resp. $3$) is connected to the vertex $2$.

For a fixed $n \in \mathbb{N}_{\geq 3}$, let $\phi$ be the symplectic automorphism given as

$$\phi = \tau_1 \circ \tau_2^{-1} \circ \tau_3 : P_n(A_3) \to P_n(A_3).$$

We note that we are using the notation defined in Definition 5.2, not Definition 8.4.

It is easy to check that $L_1, L_2$ and $L_3$ generate the wrapped Fukaya category of $P_n(A_3)$. Moreover, $[L_1], [L_2], [L_3]$ generate the relative homology of $P_n(A_3)$ at degree $n$, i.e.,

$$H_n(P_n(T), \partial P_n(T)).$$

Similarly, it is easy to check that the Grothendieck group is also generated by a similar basis $\{L_1, L_2, L_3\}$. Also, it is easy to check that the effect of $\phi$ on $H_n(P_n(T), \partial P_n(T))$ is the same to that on the Grothendieck group. Thus, we will describe the effect of $\phi$ on the Grothendieck group, rather than $H_n(P_n(T), \partial P_n(T))$.

By using Seidel’s long exact sequences for Dehn twist $\tau_i$, one can have a linear map on the Grothendieck group, which is induced from $\tau_i$. The matrix representations for $\tau_1, \tau_2^{-1}, \tau_3$ are

$$B_1^+ := \begin{bmatrix} (-1)^{1-n} & 0 & 0 \\ 0 & (-1)^{1-n} & 0 \\ 0 & 0 & (-1)^{1-n} \end{bmatrix}, \quad B_2^- := \begin{bmatrix} 1 & (-1)^{n-2} & 0 \\ 0 & (-1)^{n-1} & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad B_3^+ := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & (-1)^{1-n} \end{bmatrix}.$$

We note that the matrix representations can vary by shifting $L_1, L_2, L_3$. Thus, to be more clear, we should have specified the grading information, but we omit that for convenience.

From the above computations, one can obtain a matrix representation of the linear map induced from $\phi$ on the Grothendieck group. The resulting matrix is

$$B_\phi = B_1^+ \circ B_2^- \circ B_3^+ = \begin{bmatrix} (-1)^{1-n} & -1 & -1 \\ (-1)^{2-n} & 1 + (-1)^{n-1} & 1 + (-1)^{n-1} \\ 0 & 1 & 1 + (-1)^{1-n} \end{bmatrix}.$$
If $n$ is an odd number, then,

$$B_\phi = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 2 \\ 0 & 1 & 2 \end{bmatrix}. $$

The eigenvalues of $B_\phi$ are $2 \pm \sqrt{3}$ and 1. Thus, the spectral radius is $2 + \sqrt{3}$. Then, Lemma 8.5 and Theorem 7.14 conclude that

$$h_{cat}(\phi) = 2 + \sqrt{3}, $$

for any $n$.

**Remark 9.1.**

(1) If $n$ is even, then we have

$$B_\phi = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. $$

Since the spectral radius of $B_\phi$ is 1, one can observe that the spectral radius is not equal to the categorical entropy of $\phi$. In other words, $\phi$ can be an example of showing that the categorical generalization of Gromov-Yomdin theorem [Yom87, Gro03, Gro87] does not hold in symplectic setting. See KO20 Section 1.2 for more details on the categorical generalization of Gromove-Yomdin theorem.

(2) Moreover, by Theorem 8.3 when $n$ is even, the topological entropy of the above $\phi$ is different from the spectral radius. In other words, $\phi$ should "not" satisfy the condition of Gromov-Yomdin theorem. This means that there is no complex structure on $P_n(A_3)$ making $\phi$ holomorphic.

(3) We also note that FFH+21 Section 3.1 explains a way of computing the categorical entropy of auto-functors for a specific case. The specific case is the following: The auto-functor is induced from a symplectic automorphism of Penner type $\phi: P_n(A_2) \to P_n(A_2)$, where $A_2$ is the Dynkin diagram of $A_2$ type, and where $n \geq 3$ is an odd integer. Also, the triangulated category which the induced functors are defined on is the compact Fukaya category of $P_n(A_2)$. FFH+21 computed the categorical entropy by using the spectral radius as we did in Section 9.

**References**

[AS12] Mohammed Abouzaid and Ivan Smith, Exact Lagrangians in plumbings, Geom. Funct. Anal. 22 (2012), no. 4, 785–831. MR 2984118

[Bow71] Rufus Bowen, Entropy for group endomorphisms and homogeneous spaces, Trans. Amer. Math. Soc. 153 (1971), 401–414. MR 274707

[CRGG17] Baptiste Chantraine, Georgios Dimitroglou Rizell, Paolo Ghiggini, and Roman Golovko, Geometric generation of the wrapped Fukaya category of Weinstein manifolds and sectors, arXiv preprint arXiv:1712.09126 (2017).

[DHKK14] G. Dimitrov, F. Haiden, L. Katzarkov, and M. Kontsevich, Dynamical systems and categories, The influence of Solomon Lefschetz in geometry and topology, Contemp. Math., vol. 621, Amer. Math. Soc., Providence, RI, 2014, pp. 133–170. MR 3289326

[Dri04] Vladimir Drinfeld, DG quotients of DG categories, J. Algebra 272 (2004), no. 2, 643–691. MR 2028075

[EL17] Tobias Ekholm and Yanki Lekili, Duality between lagrangian and legendrian invariants, arXiv preprint arXiv:1701.01284 (2017).

[FFH+21] Yu-Wei Fan, Simion Filip, Fabian Haiden, Ludmil Katzarkov, and Yijia Liu, On pseudo-Anosov autoequivalences, Adv. Math. 384 (2021), Paper No. 107732, 37. MR 4253251

[FM12] Benson Farb and Dan Margalit, A primer on mapping class groups, Princeton Mathematical Series, vol. 49, Princeton University Press, Princeton, NJ, 2012. MR 2850125

[GPS18] Sheel Ganatra, John Pardon, and Vivek Shende, Microlocal morse theory of wrapped Fukaya categories, arXiv preprint arXiv:1809.08807 (2018).

[GPS20] _____, Covariantly functorial wrapped Floer theory on Liouville sectors, Publ. Math. Inst. Hautes Études Sci. 131 (2020), 73–200. MR 4106794
