Symplectic maps of complex domains
into complex space forms

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Abstract

Let $M \subset \mathbb{C}^n$ be a complex domain of $\mathbb{C}^n$ endowed with a rotation
invariant Kähler form $\omega_\Phi = \frac{i}{2} \partial \bar{\partial} \Phi$. In this paper we describe sufficient
conditions on the Kähler potential $\Phi$ for $(M, \omega_\Phi)$ to admit a symplectic
embedding (explicitly described in terms of $\Phi$) into a complex space
form of the same dimension of $M$. In particular we also provide con-
ditions on $\Phi$ for $(M, \omega_\Phi)$ to admit global symplectic coordinates. As
an application of our results we prove that each of the Ricci flat (but
not flat) Kähler forms on $\mathbb{C}^2$ constructed by LeBrun in [15] admits
explicitly computable global symplectic coordinates.

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symplectic coordinates; Darboux theorem.

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1 Introduction and statements of the main results

Let $(M, \omega)$ and $(S, \Omega)$ be two symplectic manifolds of dimension $2n$ and $2N$, $n \leq N$, respectively. Then, one has the following natural and fundamental
question.

Question 1. Under which conditions there exists a symplectic embedding
$\Psi : (M, \omega) \to (S, \Omega)$, namely a smooth embedding $\Psi : M \to S$ satisfying
$\Psi^*(\Omega) = \omega$?

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Theorems A, B, C and D below give a topological answer to the previous question when $\Omega$ is the Kähler form of a $N$-dimensional complex space form $S$, namely $(S, \Omega)$ is either the complex Euclidean space $(\mathbb{C}^N, \omega_0)$, the complex hyperbolic space $(\mathbb{C}H^N, \omega_{hyp})$ or the complex projective space $(\mathbb{C}P^N, \omega_{FS})$ (see below for the definition of the symplectic (Kähler) forms $\omega_0$, $\omega_{hyp}$ and $\omega_{FS}$). Indeed these theorems are consequences of Gromov’s $h$-principle [12] (see also Chapter 12 in [9] for a beautiful description of Gromov’s work).

**Theorem A** (Gromov [12], see also [10]) Let $(M, \omega)$ be a contractible symplectic manifold. Then there exist a non-negative integer $N$ and a symplectic embedding $\Psi : (M, \omega) \rightarrow (\mathbb{C}^N, \omega_0)$, where $\omega_0 = \sum_{j=1}^{N} dx_j \wedge dy_j$ denotes the standard symplectic form on $\mathbb{C}^N = \mathbb{R}^{2N}$.

This was further generalized by Popov as follows.

**Theorem B** (Popov [20]) Let $(M, \omega)$ be a symplectic manifold. Assume $\omega$ is exact, namely $\omega = d\alpha$, for a 1-form $\alpha$. Then there exist a non-negative integer $N$ and a symplectic embedding $\Psi : (M, \omega) \rightarrow (\mathbb{C}^N, \omega_0)$.

Observe that the complex hyperbolic space $(\mathbb{C}H^N, \omega_{hyp})$, namely the unit ball $\mathbb{C}H^N = \{z = (z_1, \ldots, z_N) \in \mathbb{C}^N | \sum_{j=1}^{N} |z_j|^2 < 1\}$ in $\mathbb{C}^N$ endowed with the hyperbolic form $\omega_{hyp} = -\frac{i}{2} \partial \bar{\partial} \log(1 - \sum_{j=1}^{N} |z_j|^2)$ is globally symplectomorphic to $(\mathbb{C}^N, \omega_0)$ (see (20) in Lemma 2.2 below) hence Theorem B immediately implies

**Theorem C** Let $(M, \omega)$ be a symplectic manifold. Assume $\omega$ is exact. Then there exist a non-negative integer $N$ and a symplectic embedding $\Psi : (M, \omega) \rightarrow (\mathbb{C}H^N, \omega_{hyp})$.

The following theorem, further generalized by Popov [20] to the non-compact case, deals with the complex projective $\mathbb{C}P^N$, equipped with the Fubini–Study form $\omega_{FS}$. Recall that if $Z_0, \ldots, Z_N$ denote the homogeneous coordinates on $\mathbb{C}P^N$, then, in the affine chart $Z_0 \neq 0$ endowed with coordinates $z_j = \frac{Z_j}{Z_0}, j = 1, \ldots, N$, the Fubini-Study form reads as

$$\omega_{FS} = \frac{i}{2} \partial \bar{\partial} \log(1 + \sum_{j=1}^{N} |z_j|^2).$$
Theorem D (Gromov [10], see also Tischler [22]) Let \((M, \omega)\) be a compact symplectic manifold such that \(\omega\) is integral. Then there exist a non-negative integer \(N\) and a symplectic embedding \(\Psi : (M, \omega) \rightarrow (\mathbb{C}P^N, \omega_{FS})\).

At this point a natural problem is that to find the smallest dimension of the complex space form where a given symplectic manifold \((M, \omega)\) can be symplectically embedded. In particular one can study the case of equidimensional symplectic maps, as expressed by the following interesting question.

**Question 2.** Given a 2\(n\)-dimensional symplectic manifold \((M, \omega)\) under which conditions there exists a symplectic embedding \(\Psi\) of \((M, \omega)\) into \((\mathbb{C}^n, \omega_0)\) or \((\mathbb{C}P^n, \omega_{FS})\)?

Notice that locally there are not obstructions to the existence of such \(\Psi\). Indeed, by a well-known theorem of Darboux for every point \(p \in M\) there exist a neighbourhood \(U\) of \(p\) and an embedding \(\Psi : U \rightarrow \mathbb{R}^{2n} = \mathbb{C}^n\) such that \(\Psi^*(\omega_0) = \omega\). In order to get a local embedding into \((\mathbb{C}P^n, \omega_{FS})\) we can assume (by shrinking \(U\) if necessary) that \(\Psi(U) \subset CH^n\). Therefore \(f \circ \Psi : U \rightarrow (\mathbb{C}^n, \omega_{FS}) \subset (\mathbb{C}P^n, \omega_{FS})\), with \(f\) given by Lemma 2.2 below, is the desired embedding satisfying \((f \circ \Psi)^*(\omega_{FS}) = \Psi^*(\omega_0) = \omega\). Observe also that Darboux’s theorem is a special case of the following

**Theorem E (Gromov [13])** A 2\(n\)-dimensional symplectic manifold \((M, \omega)\) admits a symplectic immersion into \((\mathbb{C}^n, \omega_0)\) if and only if the following three conditions are satisfied: a) \(M\) is open, b) the form \(\omega\) is exact, c) the tangent bundle \((TM, \omega)\) is a trivial \(Sp(2n)\)-bundle. (Observe that a), b), c) are satisfied if \(M\) is contractible).

It is worth pointing out that the previous theorem is not of any help in order to attack Question 2 due to the existence of exotic symplectic structures on \(\mathbb{R}^{2n}\) (cfr. [11]). (We refer the reader to [1] for an explicit construction of a 4-dimensional symplectic manifold diffeomorphic to \(\mathbb{R}^4\) which cannot be symplectically embedded in \((\mathbb{R}^4, \omega_0)\)).

In the case when our symplectic manifold \((M, \omega)\) is a Kähler manifold, with associated Kähler metric \(g\), one can try to impose Riemannian or holomorphic conditions to answer the previous question. From the Riemannian point of view the only complete and known result (to the authors’ knowledge) is the following global version of Darboux’s theorem.

**Theorem F (McDuff [19])** Let \((M, g)\) be a simply-connected and complete \(n\)-dimensional Kähler manifold of non-positive sectional curvature. Then
there exists a diffeomorphism \( \Psi : M \to \mathbb{R}^{2n} \) such that \( \Psi^*(\omega_0) = \omega \).

(See also [4], [5], [6] and [8] for further properties of McDuff’s symplectomorphism).

The aim of this paper is to give an answer to Question 2 in terms of the Kähler potential of the Kähler metric of complex domains (open and connected) \( M \subset \mathbb{C}^n \) equipped with a Kähler form \( \omega \) which admits a rotation invariant Kähler potential. More precisely, throughout this paper we assume that there exists a Kähler potential for \( \omega \), namely a smooth function \( \Phi : M \to \mathbb{R} \) such that \( \omega = \frac{i}{2} \partial \overline{\partial} \Phi \), depending only on \( |z_1|^2, \ldots, |z_n|^2 \), where \( z_1, \ldots, z_n \) are the standard complex coordinates on \( \mathbb{C}^n \). Therefore, there exists a smooth function \( \tilde{\Phi} : \tilde{M} \to \mathbb{R} \), defined on the open subset \( \tilde{M} \subset \mathbb{R}^n \) given by

\[
\tilde{M} = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_j = |z_j|^2, z = (z_1, \ldots, z_n) \in M \} \tag{1}
\]

such that

\[
\Phi(z_1, \ldots, z_n) = \tilde{\Phi}(x_1, \ldots, x_n), \ x_j = |z_j|^2, \ j = 1, \ldots, n.
\]

We set \( \omega := \omega_{\Phi} \) and call \( \omega_{\Phi} \) a rotation invariant symplectic (Kähler) form with associated function \( \tilde{\Phi} \). It is worth pointing out that many interesting examples of Kähler forms on complex domains are rotation invariant (even radial, namely depending only on \( r = |z_1|^2 + \cdots + |z_n|^2 \)), since they often arise from solutions of ordinary differential equations on the variable \( r \) (see Example 3.3 below and also [3] in the case of extremal metrics).

Our first result is Theorem 1.1 below where we describe explicit conditions in terms of the potential \( \Phi \) for the existence of an explicit symplectic embedding of a rotation invariant domain \( (M, \omega_{\Phi}) \) into a given complex space form \( (S, \omega_\Xi) \) of the same dimension. In particular we find conditions on \( \Phi \) for the existence of global symplectic coordinates of \( (M, \omega_{\Phi}) \).

**Theorem 1.1** Let \( M \subseteq \mathbb{C}^n \) be a complex domain such that condition

\[
M \cap \{ z_j = 0 \} \neq \emptyset, \ j = 1, \ldots, n \tag{2}
\]

is satisfied \(^2\) and let \( \omega_{\Phi} = \frac{i}{2} \partial \overline{\partial} \Phi \) be a rotation invariant Kähler form on \( M \) with associated function \( \tilde{\Phi} : \tilde{M} \to \mathbb{R} \). Then

\(^2\)Obviously (2) is satisfied if \( 0 \in M \), but there are other interesting cases, see Examples 3.2 and 3.3 below, where this condition is fulfilled.
(i) there exists a uniquely determined special \(^3\) symplectic immersion
\[
\Psi_0 : (M, \omega_\Phi) \to (\mathbb{C}^n, \omega_0)
\]
(resp. \(\Psi_{hyp} : (M, \omega_\Phi) \to (\mathbb{C}H^n, \omega_{hyp})\) if and only if,
\[
\frac{\partial \Phi}{\partial x_k} \geq 0, \quad k = 1, \ldots, n. \tag{3}
\]

(ii) there exists a uniquely determined special symplectic immersion
\[
\Psi_{FS} : (M, \omega_\Phi) \to (\mathbb{C}^n, \omega_{FS}),
\]
if and only if
\[
\frac{\partial \Phi}{\partial x_k} \geq 0, \quad k = 1, \ldots, n \quad \text{and} \quad \sum_{j=1}^{n} \frac{\partial \Phi}{\partial x_j} x_j < 1, \tag{4}
\]
where we are looking at \(\mathbb{C}^n \ni \mathbb{C}P^n\) as the affine chart \(Z_0 \neq 0\) in \(\mathbb{C}P^n\) endowed with the restriction of the Fubini–Study form \(\omega_{FS}\).

Moreover, assume that \(0 \in M\). If (3) (resp. (4)) is satisfied then \(\Psi_0\) (resp. \(\Psi_{FS}\)) is a global symplectomorphism (and hence \(i \circ \Psi_{FS} : M \to \mathbb{C}P^n\) is a symplectic embedding) if and only if
\[
\frac{\partial \Phi}{\partial x_k} > 0, \quad k = 1, \ldots, n \tag{5}
\]
and
\[
\lim_{x \to \partial M} \sum_{j=1}^{n} \frac{\partial \Phi}{\partial x_j} x_j = +\infty \quad \text{(resp.} \lim_{x \to \partial M} \sum_{j=1}^{n} \frac{\partial \Phi}{\partial x_j} x_j = 1). \tag{6}
\]

\(^3\)See (7) in the next section for the definition of special map between complex domains.

\(^4\)For a rotation invariant continuous map \(F : M \to \mathbb{R}\) we write
\[
\lim_{x \to \partial M} \tilde{F}(x) = l \in \mathbb{R} \cup \{\infty\}, \quad x = (x_1, \ldots, x_n),
\]
if, for \(\|x\| \to +\infty\) or \(z \to z_0 \in \partial M\), we have \(\|\tilde{F}(x)\| \to l\), where \(\partial M\) denotes the boundary of \(M \subset \mathbb{C}^n\) and \(\tilde{F} : M \to \mathbb{R}, M\) given by (1), is the continuous map such that
\[
F(z_1, \ldots, z_n) = \tilde{F}(x_1, \ldots, x_n), \quad x_j = |z_j|^2.
\]
Observe that the maps $\Psi_0$, $\Psi_{hyp}$ and $\Psi_{FS}$ can be described explicitly (see (23), (24) and (25) below). This is a rare phenomenon. In fact the proofs of Theorems A, B, C, D and E above are existential and the explicit form of the symplectic embedding or symplectomorphism into a given complex space form is, in general, very hard to find.

Theorem 1.1 is an extension and a generalization of the results obtained by the first author and Fabrizio Cuccu in [7] for complete Reinhardt domains in $\mathbb{C}^2$. Actually, all the results obtained there become a straightforward corollary of our Theorem 1.1 (see Example 3.1 in Section 3).

Our second result is Theorem 1.2 below where we describe geometric conditions on $\Phi$, related to Calabi’s work on Kähler immersions, which implies the existence of a special symplectic immersion of $(M, \omega_{\Phi})$ in $(\mathbb{R}^{2n}, \omega_0)$, $n = \dim_{\mathbb{C}} M$ (and in particular the existence of global symplectic coordinates of $(M, \omega_{\Phi})$).

**Theorem 1.2** Let $M \subseteq \mathbb{C}^n$ be a complex domain such that $0 \in M$ endowed with a rotation invariant Kähler form $\omega_{\Phi}$. Assume that there exists a Kähler (i.e. a holomorphic and isometric) immersion of $(M, g_{\Phi})$ into some finite or infinite dimensional complex space form, where $g_{\Phi}$ is the metric associated to $\omega_{\Phi}$. Then, (5) is satisfied and hence there exists a special symplectic immersion $\Psi_0$ of $(M, \omega_{\Phi})$ into $(\mathbb{C}^n, \omega_0)$, which is a global symplectomorphism if and only if $\lim_{x \to \partial M} \sum_{j=1}^{n} \frac{\partial \Phi}{\partial x_j} x_j = +\infty$. If $\sum_{j=1}^{n} \frac{\partial \Phi}{\partial x_j} x_j < 1$ then there exists a symplectic immersion $\Psi_{FS}$ of $(M, \omega_{\Phi})$ into $(\mathbb{C}P^n, \omega_{FS})$ which is an embedding if and only if $\lim_{x \to \partial M} \sum_{j=1}^{n} \frac{\partial \Phi}{\partial x_j} x_j = 1$.

The paper is organized as follows. In the next section we prove Theorem 1.1 and Theorem 1.2. The later will follow by an application of Calabi’s results, which will be briefly recalled in that section. Finally, in Section 3 we apply Theorem 1.1 to some important cases. In particular we recover the results proved in [7] and we prove that each of the Ricci flat (but not flat) Kähler forms on $\mathbb{C}^2$ constructed by LeBrun in [15] admits explicitly computable global symplectic coordinates. Observe that this last result cannot be obtained by Theorem F above (see Remark 3.6 below).
2 Proof of the main results

The following general lemma, used in the proof of our main results Theorem 1.1 and Theorem 1.2, describes the structure of a special symplectic immersion between two complex domains \( M \subset \mathbb{C}^n \) and \( S \subset \mathbb{C}^n \) endowed with rotation invariant Kähler forms \( \omega_\Phi \) and \( \omega_\Xi \) respectively. In all the paper we consider smooth maps from \( M \) into \( S \) of the form

\[
\Psi : M \to S, \ z \mapsto (\Psi_1(z) = \tilde{\psi}_1(x)z_1, \ldots, \Psi_n(z) = \tilde{\psi}_n(x)z_n),
\]

\( z = (z_1, \ldots, z_n), \ x = (x_1, \ldots, x_n), \ x_j = |z_j|^2 \) for some real functions \( \tilde{\psi}_j : \tilde{M} \to \mathbb{R}, \ j = 1, \ldots, n, \) where \( \tilde{M} \subset \mathbb{R}^n \) is given by (1). A smooth map like (7) will be called a special map.

**Lemma 2.1** Let \( M \subseteq \mathbb{C}^n \) and \( S \subseteq \mathbb{C}^n \) be complex domains as above. A special map \( \Psi : M \to S, \ z \mapsto (\Psi_1(z), \ldots, \Psi_n(z)) \), is symplectic, namely \( \Psi^*(\omega_\Xi) = \omega_\Phi \), if and only if there exist constants \( c_k \in \mathbb{R} \) such that the following equalities hold on \( \tilde{M} \):

\[
\tilde{\psi}_2^k \frac{\partial \tilde{\Xi}}{\partial x_k}(\Psi) = \frac{\partial \tilde{\Phi}}{\partial x_k} + \frac{c_k}{x_k}, \ k = 1, \ldots, n,
\]

where \( \tilde{\Phi} \) (resp. \( \tilde{\Xi} \)) is the function associated to \( \omega_\Phi \) (resp. \( \omega_\Xi \)), and

\[
\frac{\partial \tilde{\Xi}}{\partial x_k}(\Psi) = \frac{\partial \tilde{\Xi}}{\partial x_k}(\tilde{\psi}_1^2x_1, \ldots, \tilde{\psi}_n^2x_n), \ k = 1, \ldots, n.
\]

**Proof:** From

\[
\omega_\Xi = \frac{i}{2} \sum_{i,j=1}^n \left( \frac{\partial^2 \tilde{\Xi}}{\partial x_i \partial x_j} \tilde{z}_j \tilde{z}_i + \frac{\partial \tilde{\Xi}}{\partial x_i} \delta_{ij} \right)_{x_1 = |z_1|^2, \ldots, x_n = |z_n|^2} dz_j \wedge d\bar{z}_i
\]

one gets

\[
\Psi^*(\omega_\Xi) = \frac{i}{2} \sum_{i,j=1}^n \left( \frac{\partial^2 \tilde{\Xi}}{\partial x_i \partial x_j}(\Psi) \tilde{\psi}_j \tilde{\psi}_i + \frac{\partial \tilde{\Xi}}{\partial x_j}(\Psi) \delta_{ij} \right)_{x_1 = |z_1|^2, \ldots, x_n = |z_n|^2} d\psi_j \wedge d\psi_i,
\]

where

\[
\frac{\partial^2 \tilde{\Xi}}{\partial x_i \partial x_j}(\Psi) = \frac{\partial^2 \tilde{\Xi}}{\partial x_i \partial x_j}(\tilde{\psi}_1^2x_1, \ldots, \tilde{\psi}_n^2x_n).
\]
If one denotes by
\[ \Psi^*(\omega \Xi) = \Psi^*(\omega \Xi)_{(2,0)} + \Psi^*(\omega \Xi)_{(1,1)} + \Psi^*(\omega \Xi)_{(0,2)} \]
the decomposition of \( \Psi^*(\omega \Xi) \) into addenda of type \((2, 0)\), \((1, 1)\) and \((0, 2)\) one has:

\[ \Psi^*(\omega \Xi)_{(2,0)} = \frac{i}{2} \sum_{i,j,k,l=1}^{n} \left( \frac{\partial^2 \tilde{\Xi}}{\partial x_i \partial x_j}(\Psi) \Psi_i \bar{\Psi}_j + \frac{\partial^2 \tilde{\Xi}}{\partial x_j}(\Psi) \delta_{ij} \right) \frac{\partial \Psi_j}{\partial z_k} \frac{\partial \bar{\Psi}_i}{\partial z_l} \, dz_k \wedge dz_l \]

(9)

\[ \Psi^*(\omega \Xi)_{(1,1)} = \frac{i}{2} \sum_{i,j,k,l=1}^{n} \left( \frac{\partial^2 \tilde{\Xi}}{\partial x_i \partial x_j}(\Psi) \Psi_i \bar{\Psi}_j + \frac{\partial^2 \tilde{\Xi}}{\partial x_j}(\Psi) \delta_{ij} \right) \left( \frac{\partial \Psi_j}{\partial z_k} \frac{\partial \bar{\Psi}_i}{\partial z_l} - \frac{\partial \Psi_j}{\partial z_l} \frac{\partial \bar{\Psi}_i}{\partial z_k} \right) \, dz_k \wedge d\bar{z}_l \]

(10)

\[ \Psi^*(\omega \Xi)_{(0,2)} = \frac{i}{2} \sum_{i,j,k,l=1}^{n} \left( \frac{\partial^2 \tilde{\Xi}}{\partial x_i \partial x_j}(\Psi) \Psi_i \bar{\Psi}_j + \frac{\partial^2 \tilde{\Xi}}{\partial x_j}(\Psi) \delta_{ij} \right) \frac{\partial \Psi_j}{\partial \bar{z}_k} \frac{\partial \bar{\Psi}_i}{\partial \bar{z}_l} \, d\bar{z}_k \wedge d\bar{z}_l. \]

(11)

(Here and below, with a slight abuse of notation, we are omitting the fact that all the previous expressions have to be evaluated at \( x_1 = |z_1|^2, \ldots, x_n = |z_n|^2. \)) Since \( \Psi_j(z) = \tilde{\psi}_j(|z_1|^2, \ldots, |z_n|^2) \), one has:

\[ \frac{\partial \Psi_i}{\partial z_k} = \frac{\partial \tilde{\psi}_i}{\partial x_k} z_i \bar{z}_k + \tilde{\psi}_i \delta_{ik}, \quad \frac{\partial \bar{\Psi}_i}{\partial z_k} = \frac{\partial \tilde{\psi}_i}{\partial x_k} \bar{z}_k z_i \]

(12)

and

\[ \frac{\partial \bar{\Psi}_i}{\partial \bar{z}_k} = \frac{\partial \tilde{\psi}_i}{\partial x_k} \bar{z}_i \bar{z}_k + \tilde{\psi}_i \delta_{ik}, \quad \frac{\partial \bar{\Psi}_i}{\partial \bar{z}_k} = \frac{\partial \tilde{\psi}_i}{\partial x_k} \bar{z}_k \bar{z}_i. \]

(13)

By inserting (12) and (13) into (9) and (10) after a long, but straightforward computation, one obtains:

\[ \Psi^*(\omega \Xi)_{(2,0)} = \frac{i}{2} \sum_{k,l=1}^{n} A_{kl} \bar{z}_k \bar{z}_l \, d\bar{z}_k \wedge d\bar{z}_l \]

(14)

and
\[ \Psi^*(\omega \Xi)_{(1,1)} = \frac{i}{2} \sum_{k,l=1}^{n} \left[ \left( A_{kl} + A_{lk} \right) + \frac{\partial^2 \tilde{\Xi}}{\partial x_k \partial x_l}(\Psi) \tilde{\psi}_k^2 \tilde{\psi}_l^2 \tilde{z}_k \tilde{z}_l + \frac{\partial \tilde{\Xi}}{\partial x_k}(\Psi) \delta_{kl} \tilde{\psi}_k^2 \right] dz_k \wedge d\bar{z}_l, \]  

where

\[ A_{kl} = \frac{\partial \tilde{\Xi}}{\partial x_k}(\Psi) \frac{\partial \tilde{\psi}_k^2}{\partial x_l} + \tilde{\psi}_k^2 \sum_{j=1}^{n} \frac{\partial^2 \tilde{\Xi}}{\partial x_j \partial x_k}(\Psi) \frac{\partial \tilde{\psi}_j^2}{\partial x_l} \left| z_j \right|^2. \]  

Now, we assume that

\[ \Psi^*(\omega \Xi) = \omega \Phi = \frac{i}{2} \sum_{k,l=1}^{n} \left( \frac{\partial^2 \tilde{\Phi}}{\partial x_k \partial x_l} \tilde{z}_k z_l + \frac{\partial \tilde{\Phi}}{\partial x_l} \delta_{lk} \right) \left| z_1 \right|^2 \ldots \left| z_n \right|^2 dz_k \wedge d\bar{z}_l. \]

Then the terms \( \Psi^*(\omega \Xi)_{(2,0)} \) and \( \Psi^*(\omega \Xi)_{(0,2)} \) are equal to zero. This is equivalent to the fact that (16) is symmetric in \( k,l \).

Hence, by setting \( \Gamma_l = \tilde{\psi}_l^2 \frac{\partial \tilde{\Xi}}{\partial x_l}(\Psi), \) \( l = 1, \ldots, n \) equation (15) becomes

\[ \Psi^*(\omega \Xi)_{(1,1)} = \frac{i}{2} \sum_{k,l=1}^{n} \left[ \left( A_{kl} + \frac{\partial^2 \tilde{\Xi}}{\partial x_k \partial x_l}(\Psi) \tilde{\psi}_k^2 \tilde{\psi}_l^2 \tilde{z}_k \tilde{z}_l + \frac{\partial \tilde{\Xi}}{\partial x_k}(\Psi) \delta_{kl} \tilde{\psi}_k^2 \right) \right] dz_k \wedge d\bar{z}_l = \]

\[ = \frac{i}{2} \sum_{k,l=1}^{n} \left( \frac{\partial \Gamma_l}{\partial x_k} \tilde{z}_k z_l + \Gamma_k \delta_{kl} \right) dz_k \wedge d\bar{z}_l. \]  

So, \( \Psi^*(\omega \Xi) = \omega \Phi \) implies

\[ \frac{i}{2} \sum_{k,l=1}^{n} \left( \frac{\partial \Gamma_l}{\partial x_k} \tilde{z}_k z_l + \Gamma_k \delta_{kl} \right) dz_k \wedge d\bar{z}_l = \frac{i}{2} \sum_{k,l=1}^{n} \left( \frac{\partial^2 \tilde{\Phi}}{\partial x_k \partial x_l} \tilde{z}_k z_l + \frac{\partial \tilde{\Phi}}{\partial x_l} \delta_{kl} \right) dz_k \wedge d\bar{z}_l. \]

In this equality, we distinguish the cases \( l \neq k \) and \( l = k \) and get respectively

\[ \frac{\partial \Gamma_l}{\partial x_k} = \frac{\partial^2 \tilde{\Phi}}{\partial x_k \partial x_l} \quad (k \neq l) \]

and
\[
\frac{\partial \Gamma_k}{\partial x_k} x_k + \Gamma_k = \frac{\partial^2 \tilde{\Phi}}{\partial x_k^2} x_k + \frac{\partial \tilde{\Phi}}{\partial x_k}.
\]

By defining \(A_k = \Gamma_k - \frac{\partial \tilde{\Phi}}{\partial x_k}\), these equations become respectively
\[
\frac{\partial A_k}{\partial x_l} = 0 \quad (l \neq k)
\]
and
\[
\frac{\partial A_k}{\partial x_k} x_k = -A_k.
\]
The first equation implies that \(A_k\) does not depend on \(x_l\) and so by the second one we have
\[
A_k = \Gamma_k - \frac{\partial \tilde{\Phi}}{\partial x_k} = \frac{c_k}{x_k},
\]
for some constant \(c_k \in \mathbb{R}\), i.e.
\[
\Gamma_k = \tilde{\psi}_k^2 \frac{\partial \tilde{\Xi}}{\partial x_k} (\Psi) = \frac{\partial \tilde{\Phi}}{\partial x_k} + \frac{c_k}{x_k}, \quad k = 1, \ldots, n,
\]
namely (8).

In order to prove the converse of Lemma 2.1, notice that by differentiating (8) with respect to \(l\) one gets:
\[
\frac{\partial^2 \tilde{\Phi}}{\partial x_k \partial x_l} - \frac{c_k}{x_k} \delta_{kl} = A_{kl} + \frac{\partial^2 \tilde{\Xi}}{\partial x_k \partial x_l} \tilde{\psi}_k \tilde{\omega}_l
\]
with \(A_{kl}\) given by (16). By \(\frac{\partial^2 \tilde{\Phi}}{\partial x_k \partial x_l} = \frac{\partial^2 \tilde{\Phi}}{\partial x_l \partial x_k}\) and \(\frac{\partial^2 \tilde{\Xi}}{\partial x_k \partial x_l} \tilde{\psi}_k \tilde{\omega}_l = \frac{\partial^2 \tilde{\Xi}}{\partial x_l \partial x_k} \tilde{\omega}_l \tilde{\psi}_k\) one gets \(A_{kl} = A_{lk}\). Then, by (14), the addenda of type \((2,0)\) (and \((0,2)\)) in \(\Psi^*(\omega_{\Xi})\) vanish. Moreover, by (16) and (17), it follows that \(\Psi^*(\omega_{\Xi}) = \omega_{\Phi}\).

In the proof of Theorem 1.1 we also need the following lemma whose proof follows by Lemma 2.1, or by a direct computation.

**Lemma 2.2** The map \(f : \mathbb{C}H^n \to \mathbb{C}^n\) given by
\[
(z_1, \ldots, z_n) \mapsto \left(\frac{z_1}{\sqrt{1 - \sum_{i=1}^n |z_i|^2}}, \ldots, \frac{z_n}{\sqrt{1 - \sum_{i=1}^n |z_i|^2}}\right)
\]
(19)
is a special global diffeomorphism satisfying

\[ f^*(\omega_0) = \omega_{hyp} \]  

(20)

and

\[ f^*(\omega_{FS}) = \omega_0, \]  

(21)

where, in the second equation, we are looking at \( \mathbb{C}^n \rightarrow \mathbb{C}P^n \) as the affine chart \( Z_0 \neq 0 \) in \( \mathbb{C}P^n \) endowed with the restriction of the Fubini-Study form \( \omega_{FS} \) and where \( \omega_0 \) denotes the restriction of the flat form of \( \mathbb{C}^n \) to \( \mathbb{C}H^n \subset \mathbb{C}^n \).

We are now in the position to prove our first result.

**Proof of Theorem 1.1** First of all observe that under assumption (2), the \( c_k \)'s appearing in the statement of Lemma 2.1 are forced to be zero. So, the existence of a special symplectic immersion \( \Psi : M \rightarrow S \) is equivalent to

\[ \tilde{\psi}_k \frac{\partial \tilde{\Xi}}{\partial x_k} (\Psi) = \frac{\partial \tilde{\Phi}}{\partial x_k}, \quad k = 1, \ldots, n. \]  

(22)

If we further assume \((S = \mathbb{C}^n, \omega_\Xi = \omega_0)\), namely \( \tilde{\Xi} = \sum_{j=1}^n x_j \), then condition (8) reduces to

\[ \tilde{\psi}_k = \frac{\partial \tilde{\Phi}}{\partial x_k}, \quad k = 1, \ldots, n, \]

and hence (3) follows by Lemma 2.1. Further \( \Psi_0 \) is given by:

\[ \Psi_0(z) = \left( \sqrt{\frac{\partial \tilde{\Phi}}{\partial x_1}} z_1, \ldots, \sqrt{\frac{\partial \tilde{\Phi}}{\partial x_n}} z_n \right)_{x_i = |z_i|^2} \]  

(23)

In order to prove (i) when \((S = \mathbb{C}H^n, \omega_\Xi = \omega_{hyp})\) observe that since the composition of two special maps is a special map it follows by (20) that the existence of a special symplectic map \( \Psi : (M, \omega_\Phi) \rightarrow (\mathbb{C}H^n, \omega_{hyp}) \) gives rise to a special symplectic map \( f \circ \Psi : (M, \omega_\Phi) \rightarrow (\mathbb{C}^n, \omega_0) \). The later is uniquely determined by the previous case, i.e. \( \Psi_0 = f \circ \Psi \). So \( \Psi_{hyp} := \Psi = f^{-1} \circ \Psi_0 \) and since the inverse of \( f \) is given by

\[ f^{-1} : \mathbb{C}^n \rightarrow \mathbb{C}H^n, \quad z \mapsto \left( \frac{z_1}{\sqrt{1 + \sum_{i=1}^n |z_i|^2}}, \ldots, \frac{z_n}{\sqrt{1 + \sum_{i=1}^n |z_i|^2}} \right), \]
one obtains:

$$\Psi_{hyp}(z) = \left( \sqrt{\frac{\partial \Phi}{\partial x_1}} \frac{z_1}{1 + \sum_{k=1}^{n} \frac{\partial \Phi}{\partial x_k} x_k}, \ldots, \sqrt{\frac{\partial \Phi}{\partial x_n}} \frac{z_n}{1 + \sum_{k=1}^{n} \frac{\partial \Phi}{\partial x_k} x_k} \right).$$

(24)

In order to prove (ii), notice that by (21) a special symplectic map $\Psi : (M, \omega_\Phi) \to (\mathbb{C}^n, \omega_{FS})$ is uniquely determined by the special symplectic map $\Psi_0 = f^{-1} \circ \Psi : (M, \omega_\Phi) \to (CH^n, \omega_0) \subset (\mathbb{C}^n, \omega_0)$ and therefore (4) is a straightforward consequence of the previous case (i). Furthermore $\Psi_{FS}$ is given by

$$\Psi_{FS}(z) = \left( \sqrt{1 - \sum_{k=1}^{n} \frac{\partial \Phi}{\partial x_k} x_k} \frac{z_1}{1 - \sum_{k=1}^{n} \frac{\partial \Phi}{\partial x_k} x_k}, \ldots, \sqrt{1 - \sum_{k=1}^{n} \frac{\partial \Phi}{\partial x_k} x_k} \frac{z_n}{1 - \sum_{k=1}^{n} \frac{\partial \Phi}{\partial x_k} x_k} \right)_{x_i = |z_i|^2}.$$  

(25)

Finally, notice that conditions (5) and (6) for the special map (23) (resp. (25)) are equivalent to $\Psi_0^{-1}(\{0\}) = \{0\}$ (resp. $\Psi_{FS}^{-1}(\{0\}) = \{0\}$) and to the properness of $\Psi_0$ (resp. $\Psi_{FS}$). Hence, the fact that under these conditions $\Psi_0$ (resp. $\Psi_{FS}$) is a global diffeomorphism follows by standard topological arguments.

**Remark 2.3** Observe that, by Theorem 1.1, if $(M, \omega_\Phi)$ admits a special symplectic immersion into $(\mathbb{C}^n, \omega_{FS})$, then it admits a special symplectic immersion in $(\mathbb{C}^n, \omega_0)$ (or $(CH^n, \omega_{hyp})$). The converse is false even if one restricts to an arbitrary small open set $U \subseteq M$ endowed with the restriction of $\omega_\Phi$ (see Remark 3.4 below).

In order to prove our second result (Theorem 1.2) we briefly recall Calabi’s work on Kähler immersions and his fundamental Theorem 2.9. We refer the reader to [2] for details and further results (see also [17] and [18]).

**Calabi’s work** In his seminal paper Calabi [2] gave a complete answer to the problem of the existence and uniqueness of Kähler immersions of a Kähler manifold $(M, g)$ into a finite or infinite dimensional complex space form. Calabi’s first observation was that if such Kähler immersion exists then the metric $g$ is forced to be real analytic being the pull-back via a holomorphic map of the real analytic metric of a complex space form. Then in a neighborhood of every point $p \in M$, one can introduce a very special Kähler potential $D_p^g$ for the metric $g$, which Calabi christened *diastasis*. 

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The construction goes as follows. Take a a real-analytic Kähler potential $\Phi$ around the point $p$ (it exists since $g$ is real analytic). By duplicating the variables $z$ and $\bar{z}$ $\Phi$ can be complex analytically continued to a function $\hat{\Phi}$ defined in a neighborhood $U$ of the diagonal containing $(p, \bar{p}) \in M \times M$ (here $\bar{M}$ denotes the manifold conjugated to $M$). The diastasis function is the Kähler potential $D_g^p$ around $p$ defined by

$$
D_g^p(q) = \hat{\Phi}(q, \bar{q}) + \hat{\Phi}(p, \bar{p}) - \hat{\Phi}(p, \bar{q}) - \hat{\Phi}(q, \bar{p}).
$$

**Example 2.4** Let $g_0$ be the Euclidean metric on $\mathbb{C}^N$, $N \leq \infty$, namely the metric whose associated Kähler form is given by $\omega_0 = \frac{i}{2} \sum_{j=1}^N dz_j \wedge d\bar{z}_j$. Here $\mathbb{C}^\infty$ is the complex Hilbert space $l^2(\mathbb{C})$ consisting of sequences $(z_j)_{j \geq 1}$, $z_j \in \mathbb{C}$ such that $\sum_{j=1}^{+\infty} |z_j|^2 < +\infty$. The diastasis function $D_{g_0}^0 : \mathbb{C}^N \to \mathbb{R}$ around the origin $0 \in \mathbb{C}^N$ is given by

$$
D_{g_0}^0(z) = \sum_{j=1}^N |z_j|^2.
$$

**Example 2.5** Let $(Z_0, Z_1, \ldots, Z_N)$ be the homogeneous coordinates in the complex projective space in $\mathbb{C}P^N$, $N \leq \infty$, endowed with the Fubini–Study metric $g_{FS}$. Let $p = [1,0,\ldots,0]$. In the affine chart $U_0 = \{Z_0 \neq 0\}$ endowed with coordinates $(z_1, \ldots, z_N)$, $z_j = \frac{Z_j}{Z_0}$ the diastasis around $p$ reads as:

$$
D_{p}^{g_{FS}}(z) = \log(1 + \sum_{j=1}^N |z_j|^2).
$$

**Example 2.6** Let $\mathbb{C}H^N = \{z \in \mathbb{C}^N \mid \sum_{j=1}^N |z_j|^2 < 1\} \subset \mathbb{C}^N, N \leq \infty$ be the complex hyperbolic space endowed with the hyperbolic metric $g_{hyp}$. Then the diastasis around the origin is given by:

$$
D_{0}^{g_{hyp}}(z) = -\log(1 - \sum_{j=1}^N |z_j|^2).
$$

A very useful characterization of the diastasis (see below) can be obtained as follows. Let $(z)$ be a system of complex coordinates in a neighbourhood of $p$ where $D_p^g$ is defined. Consider its power series development:

$$
D_p^g(z) = \sum_{j,k \geq 0} a_{jk}(g) z^m_j z^m_k.
$$
where we are using the following convention: we arrange every \( n \)-tuple of nonnegative integers as the sequence
\[
m_j = (m_{1,j}, m_{2,j}, \ldots, m_{n,j})_{j=0,1,\ldots}
\]
such that \( m_0 = (0, \ldots, 0) \), \( |m_j| \leq |m_{j+1}| \), with \( |m_j| = \sum_{\alpha=1}^{n} m_{\alpha,j} \) and \( z^{m_j} = \prod_{\alpha=1}^{n} (z_{\alpha})^{m_{\alpha,j}} \). Further, we order all the \( m_j \)'s with the same \( |m_j| \) using the lexicographic order.

Characterization of the diastasis: Among all the potentials the diastasis is characterized by the fact that in every coordinates system \((z)\) centered in \( p \) the coefficients \( a_{jk}(g) \) of the expansion (29) satisfy \( a_{j0}(g) = a_{0j}(g) = 0 \) for every nonnegative integer \( j \).

Definition 2.7 A Kähler immersion \( \varphi \) of \((M, g)\) into a complex space form \((S, G)\) is said to be full if \( \varphi(M) \) is not contained in a proper complex totally geodesic submanifold of \((S, G)\).

Definition 2.8 Let \( g \) be a real analytic Kähler metric on a complex manifold \( M \). The metric \( g \) is said to be resolvable of rank \( N \) if the \( \infty \times \infty \) matrix \( a_{jk}(g) \) given by (29) is positive semidefinite and of rank \( N \). Consider the function \( e^{D_p^g} - 1 \) (resp. \( 1 - e^{-D_p^g} \)) and its power series development:
\[
e^{D_p^g} - 1 = \sum_{j,k \geq 0} b_{jk}(g) z^{m_j} \bar{z}^{m_k}.
\]
(resp.
\[
1 - e^{-D_p^g} = \sum_{j,k \geq 0} c_{jk}(g) z^{m_j} \bar{z}^{m_k}.
\]

The metric \( g \) is said to be \( 1 \)-resolvable (resp. \(-1 \)-resolvable) of rank \( N \) at \( p \) if the \( \infty \times \infty \) matrix \( b_{jk}(g) \) (resp. \( c_{jk}(g) \)) is positive semidefinite and of rank \( N \).

We are now in the position to state Calabi’s fundamental theorem and to prove our Theorem 1.2.

Theorem 2.9 (Calabi) Let \( M \) be a complex manifold endowed with a real analytic Kähler metric \( g \). A neighbourhood of a point \( p \) admits a (full) Kähler immersion into \((\mathbb{C}^N, g_0)\) if and only if \( g \) is resolvable of rank at most (exactly) \( N \) at \( p \). A neighbourhood of a point \( p \) admits a (full) Kähler immersion into \((\mathbb{C}P^N, g_{FS})\) (resp. \((\mathbb{C}H^N, g_{hyp})\)) if and only if \( g \) is \( 1 \)-resolvable (resp. \(-1 \)-resolvable) of rank at most (exactly) \( N \) at \( p \).
Proof of Theorem 1.2 Without loss of generality we can assume $\Phi(0) = 0$. Then, it follows by the characterization of the diastasis function that $\Phi$ is indeed the (globally defined) diastasis function for the Kähler metric $g_\Phi$ (associated to $\omega_\Phi$) around the origin, namely $\Phi = D_{g_\Phi}^\omega$. Since $\Phi = D_{g_\Phi}^\omega$ is rotation invariant, namely it depends only on $|z_1|^2, \ldots, |z_n|^2$, the matrices $a_{jk}(g), b_{jk}(g)$ and $c_{jk}(g)$ above are diagonal, i.e.

$$a_{jk}(g) = a_j \delta_{jk}, \quad b_{jk}(g) = b_j \delta_{jk}, \quad c_{jk}(g) = c_j \delta_{jk}, \quad a_j, b_j, c_j \in \mathbb{R}.$$  (32)

Therefore, by Calabi’s Theorem 2.9 if $(M, g_\Phi)$ admits a (full) Kähler immersion into $(\mathbb{C}^N, g_0)$ (resp. $(\mathbb{CP}^N, g_{FS})$ or $(\mathbb{CH}^N, g_{hyp})$) then all the $a_j's$ (resp. the $b_j's$ or the $c_j's$) are greater or equal than 0 and at most (exactly) $N$ of them are positive. Moreover, it follows by the fact that the metric $g_\Phi$ is positive definite (at $0 \in M$) that the coefficients $a_k$ (resp. $b_k$ or $c_k$), $k = 1, \ldots, n$, are strictly greater than zero. Hence, by using (29) (resp. (30) or (31)) with $p = 0$ and $g = g_\Phi$ we get $\frac{\partial \Phi}{\partial x_k} (x) = a_k + P_0(x)$ (resp. $\frac{\partial \Phi}{\partial x_k} (x) = \frac{b_k + P_+(x)}{1 + \sum_j b_j x^m_j}$ or $\frac{\partial \Phi}{\partial x_k} (x) = \frac{c_k + P_-(x)}{1 - \sum_j c_j x^m_j}$) where $P_0$ (resp. $P_+$ or $P_-$) is a polynomial with non-negative coefficients in the variables $x = (x_1, \ldots, x_n), x_j = |z_j|^2$. Hence condition (3) above is satisfied. The last two assertions of Theorem 1.2 are immediately consequences of Theorem 1.1. \hfill \Box

3 Applications and further results

Example 3.1 (cfr. [7]) Let $x_0 \in \mathbb{R}^+ \cup \{+\infty\}$ and let $F : [0, x_0) \to (0, +\infty)$ be a non-increasing smooth function. Consider the domain

$$D_F = \{ (z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 < x_0, \ |z_2|^2 < F(|z_1|^2) \}$$

dowed with the 2-form $\omega_F = i \frac{\partial}{\partial \bar{\bar{z}}} \log \frac{1}{F(|z_1|^2) - |z_2|^2}$. If the function $A(x) = -\frac{x F'(x)}{F(x)}$ satisfies $A'(x) > 0$ for every $x \in [0, x_0)$, then $\omega_F$ is a Kähler form on $D_F$ and $(D_F, \omega_F)$ is called the complete Reinhardt domain associated with $F$. Notice that $\omega_F$ is rotation invariant with associated real function $\tilde{F}(x_1, x_2) = \log \frac{1}{F(x_1) - x_2}$. We now apply Theorem 1.1 to $(D_F, \omega_F)$. We have

$$\frac{\partial \tilde{F}}{\partial x_1} = -\frac{F'(x_1)}{F(x_1) - x_2} > 0, \quad \frac{\partial \tilde{F}}{\partial x_2} = \frac{1}{F(x_1) - x_2} > 0, \quad x_j = |z_j|^2, \quad j = 1, 2.$$
So, by Theorem 1.1, \((D_F, \omega_F)\) admits a special symplectic immersion in \((\mathbb{C}^2, \omega_0)\) (and in \((\mathbb{C}H^2, \omega_{hyp})\)). Moreover, this immersion is a global symplectomorphism only when

\[
\frac{\partial \bar{F}}{\partial x_1} x_1 + \frac{\partial \bar{F}}{\partial x_2} x_2 = x_2 - F'(x_1)x_1
\]

tends to infinity on the boundary of \(D_F\). For example, let \(F : [0, +\infty) \to \mathbb{R}^+\) given by \(F(x) = \frac{c}{1+x}, c > 0\) (resp. \(F(x) = \frac{1}{(1+x)^p}, p \in \mathbb{N}^+\)). Then \(\sum_{i=1}^{2} \frac{\partial \bar{F}}{\partial x_i} x_i = \frac{x_2(c+x_1)^2+c x_2}{c(c+x_1)x_2(1-x)}\) (resp. \(\sum_{i=1}^{2} \frac{\partial \bar{F}}{\partial x_i} x_i = \frac{x_2+p x_1(x_1)^{-p-1}}{(1+x_1)^p}\)) does not tend to infinity, for \(t \to \infty\), along the curve \(x_1 = t, x_2 = \frac{c}{c+t}\), for any \(\varepsilon \in (0,1)\) (resp. does not tend to infinity, for \(t \to \infty\), along the curve \(x_1 = t, x_2 = \varepsilon (1+t)^{-p}\), for any \(\varepsilon \in (0,1)\)). On the other hand, one verifies by a straight calculation that, if \(F : [0, +\infty) \to \mathbb{R}^+\) is given by \(F(x) = e^{-x}\) (resp. \(F : [0, 1) \to \mathbb{R}^+, F(x) = (1-x)^p, p > 0\)), then \(\sum_{i=1}^{2} \frac{\partial \bar{F}}{\partial x_i} x_i = \frac{x_1 x_2^p}{1-x_1 x_2}\) (resp. \(\sum_{i=1}^{2} \frac{\partial \bar{F}}{\partial x_i} x_i = \frac{x_2+x_p x_1(1-x_1)^{p-1}}{e^{x_1-x_1 x_2}}\)) tends to infinity on the boundary of \(D_F\). We then recover the conclusions of Examples 3.3, 3.4, 3.5, 3.6 in [7].

**Example 3.2** Let us endow \(\mathbb{C}^2 \setminus \{0\}\) with the rotation invariant Kähler form \(\omega_{\Phi} = \frac{i}{2} \partial \bar{\partial} \Phi\) with associated real function

\[
\bar{\Phi}(x_1, x_2) = a \log(x_1 + x_2) + b(x_1 + x_2) + c, \quad a, b, c > 0.
\]

The metric \(g_{\Phi}\) associated to \(\omega_{\Phi}\) is used in [21] (see also [16]) for the construction of Kähler metrics of constant scalar curvature on bundles on \(\mathbb{C}P^{n-1}\).

Since \(\frac{\partial \bar{\Phi}}{\partial x_i} = b + \frac{a}{x_1 + x_2} > 0\), by Theorem 1.1 there exists a special symplectic immersion of \((\mathbb{C}^2 \setminus \{0\}, \omega_{\Phi})\) in \((\mathbb{C}^2, \omega_0)\) (or in \((\mathbb{C}H^2, \omega_{hyp})\)).

**Example 3.3** Let us endow \(\mathbb{C}^2 \setminus \{0\}\) with the metric \(\omega_{\Phi} = \frac{i}{2} \partial \bar{\partial} \Phi\), where

\[
\Phi = \sqrt{r^4 + 1} + 2 \log r - \log(\sqrt{r^4 + 1} + 1), \quad r = \sqrt{|z_1|^2 + |z_2|^2}.
\]

The metric \(g_{\Phi}\) is used in [14] for the construction of the Eguchi–Hanson metric. A straight calculation shows that

\[
\frac{\partial \bar{\Phi}}{\partial x_i} = \frac{\partial \bar{\Phi}}{\partial r} \frac{\partial r}{\partial x_i} = \left[ \frac{4r^3}{2\sqrt{r^4 + 1}} \left( 1 - \frac{1}{\sqrt{r^4 + 1}} + \frac{1}{r} \right) \right] \frac{1}{2r} > 0,
\]

so by Theorem 1.1 there exists a special symplectic immersion of \((\mathbb{C}^2 \setminus \{0\}, \omega_{\Phi})\) in \((\mathbb{C}^2, \omega_0)\) (or in \((\mathbb{C}H^2, \omega_{hyp})\)).
Remark 3.4 Notice that in the previous Example 3.3 one has
\[ \frac{\partial \tilde{\Phi}}{\partial x_1} x_1 + \frac{\partial \tilde{\Phi}}{\partial x_2} x_2 = \frac{r^4}{\sqrt{r^4 + 1}} \left( 1 - \frac{1}{\sqrt{r^4 + 1}} \right) + 1 > 1, \]
so again by Theorem 1.1 it does not exist a special symplectic immersion of \((\mathbb{C}^2 \setminus \{0\}, \omega_\Phi)\) in \((\mathbb{C}^2, \omega_{FS})\). Moreover, such an immersion does not exist for any arbitrarily small \(U \subseteq \mathbb{C}^2 \setminus \{0\}\) endowed with the restriction of \(\omega_\Phi\) (cfr. Remark 2.3 above).

Example 3.5 In [15] Claude LeBrun constructed the following family of Kähler forms on \(\mathbb{C}^2\) defined by
\[ \omega_m = i \frac{2}{2} \partial \bar{\partial} \Phi_m, \]
where
\[ \Phi_m(u, v) = u^2 + v^2 + m(u^4 + v^4), \quad m \geq 0 \]
and \(u\) and \(v\) are implicitly defined by
\[ |z_1| = e^{m(u^2 - v^2)} u, \quad |z_2| = e^{m(v^2 - u^2)} v. \]
For \(m = 0\) one gets the flat metric, while for \(m > 0\) each of the metrics of this family represents the first example of complete Ricci flat (non-flat) metric on \(\mathbb{C}^2\) having the same volume form of the flat metric \(\omega_0\), namely \(\omega_m \wedge \omega_m = \omega_0 \wedge \omega_0\). Moreover, for \(m > 0\), these metrics are isometric (up to dilation and rescaling) to the Taub-NUT metric.

Now, with the aid of Theorem 1.1, we prove that for every \(m\) the Kähler manifold \((\mathbb{C}^2, \omega_m)\) admits global symplectic coordinates. Set \(u^2 = U, \quad v^2 = V\). Then \(\Phi_m\) (the function associated to \(\Phi_m\)) satisfies:
\[ \frac{\partial \Phi_m}{\partial x_1} = \frac{\partial \Phi_m}{\partial U} \frac{\partial U}{\partial x_1} + \frac{\partial \Phi_m}{\partial V} \frac{\partial V}{\partial x_1}, \]
\[ \frac{\partial \Phi_m}{\partial x_2} = \frac{\partial \Phi_m}{\partial U} \frac{\partial U}{\partial x_2} + \frac{\partial \Phi_m}{\partial V} \frac{\partial V}{\partial x_2}, \]
where \(x_j = |z_j|^2, j = 1, 2\). In order to calculate \(\frac{\partial U}{\partial x_j}\) and \(\frac{\partial V}{\partial x_j}, j = 1, 2\), let us consider the map
\[ G : \mathbb{R}^2 \to \mathbb{R}^2, \quad (U, V) \mapsto (x_1 = e^{2m(U-V)} U, \quad x_2 = e^{2m(V-U)} V) \]
and its Jacobian matrix
\[ J_G = \begin{pmatrix} (1 + 2mU) e^{2m(U-V)} & -2mU e^{2m(U-V)} \\ -2mV e^{2m(V-U)} & (1 + 2mV) e^{2m(V-U)} \end{pmatrix}. \]
We have $\det J_G = 1 + 2m(U + V) \neq 0$, so

$$J_G^{-1} = J_{G^{-1}} = \frac{1}{1 + 2m(U + V)} \begin{pmatrix} (1 + 2mV)e^{2m(V-U)} & 2mUe^{2m(U-V)} \\ 2mVe^{2m(V-U)} & (1 + 2mU)e^{2m(U-V)} \end{pmatrix}. $$

Since $J_{G^{-1}} = \left( \begin{array}{cc} \frac{\partial U}{\partial x_1} & \frac{\partial U}{\partial x_2} \\ \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{array} \right)$, by a straightforward calculation we get

$$\frac{\partial \tilde{\Phi}_m}{\partial x_1} = (1 + 2mV)e^{2m(V-U)} > 0, \quad \frac{\partial \tilde{\Phi}_m}{\partial x_2} = (1 + 2mU)e^{2m(U-V)} > 0,$$

and

$$\lim_{\|x\| \to +\infty} \left( \frac{\partial \tilde{\Phi}_m}{\partial x_1} x_1 + \frac{\partial \tilde{\Phi}_m}{\partial x_2} x_2 \right) = \lim_{\|x\| \to +\infty} (U + V + 4mUV) = +\infty,$$

namely (5) and (6) above respectively. Hence, by Theorem 1.1, the map

$$\Psi_0 : \mathbb{C}^2 \to \mathbb{C}^2, (z_1, z_2) \mapsto \left( (1 + 2mV)^{\frac{1}{2}} e^{m(V-U)} z_1, (1 + 2mU)^{\frac{1}{2}} e^{m(U-V)} z_2 \right)$$

is a special global symplectomorphism from $(\mathbb{C}^2, \omega_m)$ into $(\mathbb{C}^2, \omega_0)$.

Remark 3.6 Notice that for $m > 0$ we cannot apply McDuff’s Theorem F in the Introduction in order to get the existence of global symplectic coordinates on $(\mathbb{C}^2, \omega_m)$. Indeed, the sectional curvature of $(\mathbb{C}^2, g_m)$ (where $g_m$ is the Kähler metric associated to $\omega_m$) is positive at some point since $g_m$ is Ricci-flat but not flat.

Remark 3.7 In a forthcoming paper, among other properties, we prove that $(\mathbb{C}^2, g_m)$ cannot admit a Kähler immersion into any complex space form, for all $m > \frac{1}{2}$ (this is achieved by using Calabi’s diastasis function). Therefore, the previous example shows that the assumption in Theorem 1.2 that $(M, \omega_\Phi)$ admits a Kähler immersion into some complex space form is a sufficient but not a necessary condition for the existence of a special symplectic immersion into $(\mathbb{C}^n, \omega_0)$.

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