Proof of the Jacobi Property for the guiding-center Vlasov-Maxwell Bracket

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The proof of the Jacobi property of the guiding-center Vlasov-Maxwell bracket underlying the Hamiltonian structure of the guiding-center Vlasov-Maxwell equations is presented.

I. INTRODUCTION

The variational formulations of the guiding-center Vlasov-Maxwell equations were presented by Brizard and Tronci [1], whose work also included a derivation of exact conservation laws for energy-momentum and angular momentum through the Noether method. In a companion paper [2], the guiding-center Vlasov-Maxwell equations are expressed in Hamiltonian form

\[
\frac{\partial F_{gc}}{\partial t} = - \nabla \cdot \left( F_{gc} \frac{dX}{dt} \right) - \frac{\partial}{\partial p_\parallel} \left( F_{gc} \frac{dp_\parallel}{dt} \right) \equiv - \frac{\partial}{\partial Z^\alpha} \left( F_{gc} \frac{dZ^\alpha}{dt} \right),
\]

\[
\frac{\partial E}{\partial t} = 4\pi c \nabla \times \frac{\delta H_{gc}}{\delta B} - 4\pi q \int_p F_{gc} \frac{dX}{dt} - \nabla \times \left( 4\pi q \int_p P_\parallel \cdot \frac{dX}{dt} \right),
\]

\[
\frac{\partial B}{\partial t} = - 4\pi c \nabla \times \frac{\delta H_{gc}}{\delta E},
\]

where

\[
\frac{dZ^\alpha}{dt} = \left\{ Z^\alpha, \frac{\delta H_{gc}}{\delta F_{gc}} \right\}_{gc} + 4\pi q \left( \frac{\delta H_{gc}}{\delta E} + P_\parallel \cdot \nabla \times \frac{\delta H_{gc}}{\delta E} \right) \cdot \left\{ X, Z^\alpha \right\}_{gc}.
\]

Here, \( q \) denotes the charge of a guiding-center particle, with guiding-center position \( X \) and guiding-center parallel (kinetic) momentum \( p_\parallel \), and summation over charge species is implied wherever an integral of the guiding-center phase-space density \( F_{gc} \) appears (with \( \int_p \) denoting an integral over \( p_\parallel \) and the magnetic moment \( \mu \)). In addition, we introduced the symmetric dyadic tensor

\[
P_\parallel \equiv \frac{cp_\parallel}{qB} \left( I - \hat{b} \hat{b} \right),
\]

and the guiding-center Poisson bracket is

\[
\left\{ f, g \right\}_{gc} = \frac{B^*}{B^*_\parallel} \cdot \left( \nabla f \frac{\partial g}{\partial p_\parallel} - \frac{\partial f}{\partial p_\parallel} \nabla g \right) - \frac{c\hat{b}}{qB^*_\parallel} \cdot \nabla \times \nabla g
\]

\[
= \frac{1}{B^*_\parallel} \nabla \cdot \left( B^*_\parallel f \left\{ X, g \right\}_{gc} \right) + \frac{1}{B^*_\parallel} \frac{\partial}{\partial p_\parallel} \left( B^*_\parallel f \left\{ p_\parallel, g \right\}_{gc} \right),
\]

which can also be expressed in divergence form, where we have omitted the ignorable gyromotion pair \((\mu, \theta)\), and

\[
B^* \equiv B + \left( p_\parallel c/q \right) \nabla \times \hat{b}
\]

\[
B^*_\parallel \equiv \hat{b} \cdot B^* = B + \left( p_\parallel c/q \right) \hat{b} \cdot \nabla \times \hat{b}.
\]

We note that the guiding-center Poisson bracket \( \left\{ f, g \right\}_{gc} \) satisfies the Jacobi property

\[
\left\{ \left\{ f, g \right\}_{gc}, h \right\}_{gc} + \left\{ \left\{ g, h \right\}_{gc}, f \right\}_{gc} + \left\{ \left\{ h, f \right\}_{gc}, g \right\}_{gc} = 0,
\]

which holds for arbitrary functions \((f, g, h)\), subject to the condition

\[
\nabla \cdot B^* = \nabla \cdot B = 0,
\]

which is satisfied by the definition \( \left\{ f, g, h \right\}_{gc} \).
Lastly, the guiding-center Hamiltonian functional in Eqs. (11)-(3) is
\[ \mathcal{H}_{gc} \equiv \int_Z F_{gc} K_{gc} + \int_X \frac{1}{8\pi} (|E|^2 + |B|^2), \]
so that the Hamiltonian functional derivatives in the guiding-center Vlasov-Maxwell equations (11)-(3) are
\[ \left( \frac{\delta \mathcal{H}_{gc}}{\delta F_{gc}}, \frac{\delta \mathcal{H}_{gc}}{\delta E}, \frac{\delta \mathcal{H}_{gc}}{\delta B} \right) = \left( \frac{K_{gc}}{E/4\pi} \frac{B/4\pi + \int_p F_{gc} \mu_b}{}, \right), \]
where the guiding-center kinetic energy \( K_{gc} = p^2_{\parallel}/2m + \mu B \) yields the functional derivative \( \delta K_{gc}/\delta B = \mu \hat{b} \).

II. GUIDING-CENTER VLASOV-MAXWELL BRACKET

The guiding-center Vlasov-Maxwell bracket is initially constructed from the guiding-center Vlasov-Maxwell equations (11)-(3) and the Hamiltonian functional (10) as the functional identity
\[ \frac{\partial F}{\partial t} = \left[ F, \mathcal{H}_{gc} \right]_{gc} = \int_Z \frac{\partial F_{gc}}{\partial t} \frac{\delta F}{\delta F_{gc}} + \frac{\delta F}{\delta E} + \frac{\delta F}{\delta B} \bigg|_{\partial t} \bigg\rangle = \left\langle \frac{\delta F}{\partial \Psi^a} \bigg| \frac{\partial \Psi^a}{\partial t} \right\rangle, \]
where the guiding-center Vlasov-Maxwell bracket for two arbitrary functionals \((F, G)\) of the fields \( \Psi = (F_{gc}, E, B) \) is defined in terms of the Poisson structure:
\[ \left[ F, G \right]_{gc} = \left\langle \left( \frac{\delta F}{\delta \Psi^a} \left( \Psi \right) \right) ; J_{ab}^{gc}(\Psi) \left( \frac{\delta G}{\delta \Psi^b} \right) \right\rangle. \]
Here, the antisymmetric Poisson operator \( J_{ab}^{gc}(\Psi) \) guarantees the antisymmetry property: \( \left[ F, G \right]_{gc} = - \left[ G, F \right]_{gc}; \) and the bilinearity of Eq. (13) guarantees the Leibnitz property: \( \left[ F, G, K \right]_{gc} = \left[ F, G \right]_{gc} K + \left[ K, F \right]_{gc} \). The Jacobi property:
\[ J_{abc} \left[ F, G, K \right]_{gc} \equiv \left[ \left[ F, G \right]_{gc}, K \right]_{gc} + \left[ \left[ G, K \right]_{gc}, F \right]_{gc} + \left[ \left[ K, F \right]_{gc}, G \right]_{gc} = 0, \]
which holds for arbitrary functionals \((F, G, K)\), involves constraints on the Poisson operator \( J_{abc}^{gc}(\Psi) \). The purpose of the present notes is to provide an explicit proof that the guiding-center Vlasov-Maxwell bracket defined in Eq. (12) satisfies the Jacobi property (13) exactly.

From Eq. (12), we can now extract the guiding-center Vlasov-Maxwell bracket expressed in terms of two arbitrary guiding-center functionals \((F, G)\) as
\[ \left[ F, G \right]_{gc} = \int_Z F_{gc} \left( \frac{\delta F}{\delta F_{gc}}, \frac{\delta G}{\delta F_{gc}} \right) + 4\pi q \left( \frac{\delta^* F}{\delta E}, \delta \cdot \nabla \delta X \right)_{gc} + 4\pi c \left( \frac{\delta^* F}{\delta B}, \delta \cdot \nabla \delta X \right)_{gc}, \]
where we introduced the definition
\[ \frac{\delta^* F}{\delta E} = \frac{\delta F}{\delta E} + \frac{p_{\parallel}}{} \cdot \nabla \times \frac{\delta F}{\delta E}, \]
and
\[ \left( \frac{\delta^* F}{\delta F_{gc}}, \frac{\delta^* G}{\delta F_{gc}} \right)_{gc} = \left( \frac{\delta F}{\delta F_{gc}}, \frac{\delta G}{\delta F_{gc}} \right)_{gc}, \]
\[ \frac{\delta^* G}{\delta E} \left( \delta \cdot \nabla \delta X \right)_{gc} = \frac{c}{q} \left( \frac{\delta^* F}{\delta B}, \delta \cdot \nabla \delta X \right)_{gc}, \]
\[ \frac{\delta^* F}{\delta E} \left( \delta \cdot \nabla \delta X \right)_{gc} = - \frac{c}{q} \left( \frac{\delta^* G}{\delta B}, \delta \cdot \nabla \delta X \right)_{gc}. \]
with the notation \((f, g, k) \equiv (\delta F/\delta F_{gc}, \delta G/\delta F_{gc}, \delta K/\delta F_{gc})\) and \((F^*, G^*, K^*) \equiv (\delta^* F/\delta E, \delta^* G/\delta E, \delta^* K/\delta E)\). It is clear that the bracket \((15)\) is antisymmetric and, since it is bilinear in functional derivatives, it satisfies the Leibnitz property.

We now need to verify that the guiding-center bracket \((15)\) satisfies the Jacobi property \((14)\). According to the Bracket theorem \([3]\), the proof of the Jacobi property involves only the explicit dependence of the Poisson operator \(\partial_{gc}(F_{gc}, B)\), where we note that the dependence on the magnetic field \(B\) enters through the guiding-center Poisson bracket \((11)\), while the electric field \(E\) is explicitly absent. Hence, we can therefore write the double-bracket involving three arbitrary guiding-center functionals \((F, G, K)\):

\[
\left[ [F, G]_{gc}, K \right]_{gc}^F = \int_Z F_{gc} \left\{ \frac{\delta^P [F, G]_{gc}}{\delta F_{gc}}, \frac{\delta K}{\delta F_{gc}} \right\}_{gc} + 4\pi q \int_Z F_{gc} \frac{\delta^* K}{\delta E} \cdot \left\{ X, \frac{\delta^P [F, G]_{gc}}{\delta F_{gc}} \right\}_{gc} \\
- 4\pi c \int_X \frac{\delta P [F, G]_{gc}}{\delta B} \cdot \nabla \frac{\delta K}{\delta E},
\]

where the Poisson functional derivative \(\delta^P [F, G]/\delta F_{gc}\) only involves variations of the Poisson operator \(\partial_{gc}(F_{gc}, B)/\delta F_{gc}\) in the Vlasov and Interaction sub-brackets, while \(\delta^P [F, G]_{gc}/\delta B\) in the Maxwell sub-bracket involves the explicit dependence of \(\partial_{gc}(F_{gc}, B)\) on the magnetic field \(B\) appearing through \((b, B^*, B_0^*)\) in the guiding-center Poisson bracket \((11)\) and the dyadic tensor \(b\), where

\[
\delta B^* = \delta B + \nabla \times (P_\parallel \cdot \delta B) \\
(c/q) \delta \hat{b} = \delta B \cdot \partial P_\parallel / \partial p_\parallel,
\]

with \(P_\parallel\) defined in Eq. \((13)\).

### A. Vlasov and Interaction sub-brackets

From the guiding-center Vlasov-Maxwell bracket \((11)\), we find the Poisson functional derivative

\[
\frac{\delta^P [F, G]_{gc}}{\delta F_{gc}} = \{f, g\}_{gc} - 4\pi q \left( F^* \cdot \{X, g\}_{gc} - G^* \cdot \{X, f\}_{gc} \right) + (4\pi q)^2 \left( F^* \cdot \{X, X\}_{gc} \cdot G^* \right),
\]

where the functional derivatives \((f, g; F^*, G^*)\) are left intact according to the Bracket Theorem. Hence, the Vlasov sub-bracket in Eq. \((20)\) includes the terms

\[
\left\{ \frac{\delta^P [F, G]_{gc}}{\delta F_{gc}}, \frac{\delta K}{\delta F_{gc}} \right\}_{gc} = \left\{ \{f, g\}_{gc}, k \right\}_{gc} - 4\pi q \left\{ \left( F^* \cdot \{X, g\}_{gc} - G^* \cdot \{X, f\}_{gc} \right), k \right\}_{gc}
\]

\[+ (4\pi q)^2 \left\{ \left( F^* \cdot \{X, X\}_{gc} \cdot G^* \right), k \right\}_{gc},\]

while the Interaction sub-bracket in Eq. \((20)\) includes the terms

\[
4\pi q \frac{\delta^* K}{\delta E} \cdot \left\{ X, \frac{\delta^P [F, G]_{gc}}{\delta F_{gc}} \right\}_{gc} = 4\pi q K^* \cdot \left\{ X, \{f, g\}_{gc} \right\}_{gc} - (4\pi q)^2 K^* \cdot \left\{ X, \left( F^* \cdot \{X, g\}_{gc} - G^* \cdot \{X, f\}_{gc} \right) \right\}_{gc}
\]

\[+ (4\pi q)^3 K^* \cdot \left\{ X, \left( F^* \cdot \{X, X\}_{gc} \cdot G^* \right) \right\}_{gc}.
\]

We note, here, that the ordering \((4\pi q)^n\), with \(0 \leq n \leq 3\), will be useful in verifying the Jacobi property of the guiding-center Vlasov-Maxwell bracket \((15)\), i.e., the Jacobi property must hold separately for each power \(n\).

At the zeroth order in \(4\pi q\) in Eqs. \((23)-(24)\), the Vlasov-Maxwell contributions to the guiding-center Jacobi property \((14)\) are

\[
\int_Z F_{gc} \left\{ \{f, g\}_{gc}, k \right\}_{gc} + \left\{ \{g, k\}_{gc}, f \right\}_{gc} + \left\{ \{k, f\}_{gc}, g \right\}_{gc} \equiv \int_Z F_{gc} \text{Jac}_{V-M}^{(0)}[f, g, k].
\]

Next, at the first order in \(4\pi q\) in Eqs. \((23)-(24)\), the Vlasov-Interaction (VI) contributions to the guiding-center Jacobi property \((14)\) are

\[
4\pi q \int_Z F_{gc} \left( \text{Jac}_{V}^{(1)}[f, g; K^*] + \text{Jac}_{V}^{(1)}[g, k; F^*] + \text{Jac}_{V}^{(1)}[k, f; G^*] \right),
\]
where

\[
\text{Jac}_{V}^{(1)}[f, g; K^*] = K^* \cdot \left\{ X, \{f, g\}_{gc} \right\}_{gc} + \left\{ K^* \cdot \{X, g\}_{gc}, f \right\}_{gc} - \left\{ K^* \cdot \{X, f\}_{gc}, g \right\}_{gc}
\]

\[
= \text{Jac}_{V}^{(1)}[f, g; K^*] + \{K^*, f\}_{ge} \cdot \{X, g\}_{ge} - \{K^*, g\}_{ge} \cdot \{X, f\}_{ge},
\]

which is obtained after using the Leibnitz formula:

\[
\left\{ K^* \cdot \{X, g\}_{gc}, f \right\}_{gc} = \{K^*, f\}_{gc} \cdot \{X, g\}_{gc} + K^* \cdot \left\{ \{X, g\}_{gc}, f \right\}_{gc},
\]

and the first-order Vlasov-Maxwell (VM) contribution is defined as

\[
\text{Jac}_{V,M}^{(1)}[f, g; K^*] \equiv K_i^* \cdot \left( \left\{ X^i, \{f, g\}_{gc} \right\}_{gc} + \left\{ f, \{X, X\}_{gc} \cdot K^* \cdot \{X, f\}_{gc} \right\}_{gc}
\]

\[
= \text{Jac}_{V,M}^{(1)}[f, g; K^*] + \{G^*, f\}_{ge} \cdot \{X, X\}_{ge} \cdot K^* + \{G^* \cdot \{X, X\}_{ge} \cdot \{K^*, f\}_{ge}
\]

\[+ \left( K^* \cdot \{X, G^*\}_{ge} \cdot G^* \cdot \{X, K^*\}_{ge} \right) \cdot \{X, f\}_{ge},
\]

with the second-order Vlasov-Maxwell contribution defined as

\[
\text{Jac}_{V,M}^{(2)}[f; G^*, K^*] \equiv G_i^* K_j^* \left( \left\{ X^j, \{f, X^j\}_{gc} \right\}_{gc} + \left\{ f, \{X^j, X^i\}_{gc} \right\}_{gc}
\]

\[+ \left\{ f, \{X^j, X^i\}_{gc} \right\}_{gc}
\]

\[
= \text{Jac}_{V,M}^{(2)}[f; G^*, K^*] + \{F^* \cdot \{X, G^*\}_{gc} \cdot K^* \cdot \{X, X\}_{gc} \cdot \{K^*, f\}_{ge}
\]

\[+ \left( G^* \cdot \{X, K^*\}_{ge} \cdot K^* \cdot \{X, G^*\}_{gc} \right) \cdot \{X, X\}_{ge} \cdot F^*
\]

\[+ \left( K^* \cdot \{X, F^*\}_{ge} \cdot \{X, K^*\}_{gc} \right) \cdot \{X, X\}_{gc} \cdot G^*,
\]

Lastly, at the third order in \(4\pi q\) in Eqs. (23)-(24), the Vlasov-Interaction contributions to the guiding-center Jacobi property [14] are

\[
(4\pi q)^3 \int Z F_{ge} \left( \text{Jac}_{V}^{(2)}[f; G^*, K^*] + \text{Jac}_{V}^{(2)}[g; K^*, F^*] + \text{Jac}_{V}^{(2)}[k; F^*, G^*] \right),
\]

where

\[
\text{Jac}_{V}^{(2)}[f; G^*, K^*] = K^* \cdot \left\{ X, G^* \cdot \{X, f\}_{gc} \right\}_{gc} - G^* \cdot \left\{ X, K^* \cdot \{X, f\}_{gc} \right\}_{gc} + \left\{ G^* \cdot \{X, X\}_{gc} \cdot K^* \cdot f \right\}_{gc}
\]

\[= \text{Jac}_{V,M}^{(2)}[f; G^*, K^*] + \{G^*, f\}_{ge} \cdot \{X, X\}_{ge} \cdot K^* + \{G^* \cdot \{X, X\}_{ge} \cdot \{K^*, f\}_{ge}
\]

\[+ \left( K^* \cdot \{X, G^*\}_{ge} \cdot G^* \cdot \{X, K^*\}_{ge} \right) \cdot \{X, f\}_{ge},
\]

with the third-order Vlasov-Maxwell contribution defined as

\[
\text{Jac}_{V,M}^{(3)}[f, g; K^*] \equiv F_i^* G_j^* K_k^* \left( \left\{ X^i, \{X^j, X^k\}_{gc} \right\}_{gc} + \left\{ X^j, \{X^i, X^k\}_{gc} \right\}_{gc}
\]

\[+ \left\{ X^k, \{X^i, X^j\}_{gc} \right\}_{gc} \right).
\]
where the Jacobian variations (36)-(38) are excluded in Eq. (39). The Jacobian contributions to the Jacobi property (14) are given by Eq. (B4):

\[
4\pi q \int Z F_{gc} \left( \text{Jac}^{(1)}_j[f, g; K^*] + \right) + (4\pi q)^2 \int Z F_{gc} \left( \text{Jac}^{(2)}_j[f; G^*, K^*] + \right) + (4\pi q)^3 \int Z F_{gc} \text{Jac}^{(3)}_j[F^*, G^*, K^*],
\]

where \( \odot \) denotes cyclic permutations of the functionals \( (F, G, K) \) and the Jacobian contributions are

\[
\text{Jac}^{(1)}_j[f, g; K^*] = \{ X^i, K^*_i \}_gc \{ f, g \}_gc,
\]
\[
\text{Jac}^{(2)}_j[f; G^*, K^*] = \{ X^i, K^*_i \}_gc G^* \cdot \{ X, f \}_gc - \{ X^i, G^*_i \}_gc K^* \cdot \{ X, f \}_gc,
\]
\[
\text{Jac}^{(3)}_j[F^*, G^*, K^*] = \{ X^i, K^*_i \}_gc F^* \cdot \{ X, X \}_gc \cdot G^* + \{ X^i, F^*_i \}_gc G^* \cdot \{ X, X \}_gc \cdot K^*
\]
\[
\quad + \{ X^i, G^*_i \}_gc K^* \cdot \{ X, X \}_gc \cdot F^*,
\]

with \( \{ X^i, K^*_i \}_gc \) defined in Eq. (A2).

C. Maxwell sub-bracket

The calculation of the Poisson functional derivative \( \delta P[F, G]_gc/\delta B \) in Eq. (20) requires an extensive series of steps. Here, we derive the Poisson functional derivative of the guiding-center Vlasov-Maxwell bracket (15):

\[
\delta P[F, G]_gc = \int Z \frac{F_{gc}}{B^*_\parallel} \delta P \left( B^*_\parallel \left\{ \frac{\delta F}{\delta F_{gc}}, \frac{\delta G}{\delta F_{gc}} \right\}_gc \right)
\]
\[
+ 4\pi q \int Z \frac{F_{gc}}{B^*_\parallel} \delta P \left( B^*_\parallel \left\{ \frac{\delta F}{\delta E}, \{ X, \frac{\delta F}{\delta E} \}_gc \right\}_gc \right)
\]
\[
+ (4\pi q)^2 \int Z \frac{F_{gc}}{B^*_\parallel} \delta P \left( B^*_\parallel \left\{ \frac{\delta F}{\delta E} \cdot \{ X, X \}_gc \cdot \frac{\delta G}{\delta E} \right\}_gc \right),
\]

where the Jacobian variations (36)-(38) are excluded in Eq. (39).

1. Zeroth-order term

For the zeroth-order term in Eq. (39), we begin with Eq. (B7):

\[
\int X \nabla \times K^* \cdot \left( \int Z F_{gc} \{ f, g \}_gc \right) = \int Z F \left[ \nabla \times K^* \cdot \left( \nabla f \frac{\partial g}{\partial p_\parallel} - \frac{\partial f}{\partial p_\parallel} \nabla g \right) - \frac{\partial K^*}{\partial p_\parallel} \cdot \nabla f \times \nabla g \right].
\]

We now use the identity (A5), so that we may write

\[
F \nabla \times K^* \cdot \left( \nabla f \frac{\partial g}{\partial p_\parallel} - \frac{\partial f}{\partial p_\parallel} \nabla g \right) = \frac{4}{c} F_{gc} \nabla \times K^* \cdot \left( \{ X, f \}_gc \times \{ X, g \}_gc \right) + F_{B^*} \nabla \times K^* \cdot \{ f, g \}_gc,
\]

where the first term on the right side can be written as

\[
\nabla \times K^* \cdot \left( \{ X, f \}_gc \times \{ X, g \}_gc \right) = \{ X, f \}_gc \cdot \nabla K^* \cdot \{ X, g \}_gc - \{ X, g \}_gc \cdot \nabla K^* \cdot \{ X, f \}_gc
\]
\[
\quad + \frac{\partial K^*}{\partial p_\parallel} \cdot \left( \{ X, g \}_gc \frac{B^*}{B^*_\parallel} \cdot \nabla f - \{ X, f \}_gc \frac{B^*}{B^*_\parallel} \cdot \nabla g \right).
\]

B. Jacobian term

The next set of terms in Eq. (20) come from contributions due to the magnetic variations of the guiding-center Jacobian, where \( \delta B_\parallel^* \) is given by Eq. (B2). The Jacobian contributions to the Jacobi property (14) are given by Eq. (B4):

\[
4\pi q \int Z F_{gc} \left( \text{Jac}^{(1)}_j[f, g; K^*] + \right) + (4\pi q)^2 \int Z F_{gc} \left( \text{Jac}^{(2)}_j[f; G^*, K^*] + \right) + (4\pi q)^3 \int Z F_{gc} \text{Jac}^{(3)}_j[F^*, G^*, K^*],
\]
Next, we use the identity \( (A3) \) to write
\[
F \hat{b} \cdot \nabla \times K^* = \frac{q}{c} F_{gc} \left( \{ X^i, K^* \}_{gc} - \frac{B^*}{B^\parallel} \cdot \frac{\partial K^*}{\partial p^\parallel} \right),
\] (43)
and, after combining these expressions in Eq. (41), we obtain
\[
-4\pi c \int_X \nabla \times K^* \frac{\delta p}{\delta B} \left( \int_Z F_{gc} \{ f, g \}_{gc} \right) = 4\pi q \int_Z F_{gc} \text{Jac}_{M}^{(1)} \{ f, g; K^* \}. \tag{44}
\]
Here, all terms associated with \( \partial K^*/\partial p^\parallel \) have canceled out:
\[
\int_Z F \frac{\partial K^*}{\partial p^\parallel} \cdot \left[ \left( \frac{c}{q} \right) \nabla f \times \nabla g + B^* \{ f, g \}_{gc} + \{ X, f \}_{gc} (B^* \cdot \nabla g) - \{ X, g \}_{gc} (B^* \cdot \nabla f) \right] = 0,
\]
and the first-order Maxwell sub-bracket contribution is
\[
\text{Jac}_{M}^{(1)} \{ f, g; K^* \} = - \left\{ K^*, f \right\}_{gc} \cdot \{ X, g \}_{gc} + \left\{ K^*, g \right\}_{gc} \cdot \{ X, f \}_{gc} - \{ f, g \}_{gc} \left\{ X^i, K^* \right\}_{gc}.
\] (45)
Hence, by combining Eqs. (31), (36), and (45), we finally obtain
\[
\text{Jac}_{V}^{(1)} \{ f, g; K^* \} + \text{Jac}_{M}^{(1)} \{ f, g; K^* \} + \text{Jac}_{M}^{(1)} \{ f, g; K^* \} = \text{Jac}_{M}^{(1)} \{ f, g; K^* \},
\] (46)
where the first-order Vlasov-Maxwell term \( \text{Jac}_{V}^{(1)} \{ f, g; K^* \} \) is defined in Eq. (28).

2. First-order term

For the first-order term in \( 4\pi q \) in Eq. (59), we begin with Eq. (B11):
\[
\int_X \nabla \times K^* \cdot \left[ \left( \frac{c}{q} \right) \nabla f \times \nabla g + B^* \{ f, g \}_{gc} + \{ X, f \}_{gc} (B^* \cdot \nabla g) - \{ X, g \}_{gc} (B^* \cdot \nabla f) \right] = \int_Z F \left( \nabla \times K^* \cdot \left( G^* \cdot \frac{\partial f}{\partial p^\parallel} - F^* \cdot \frac{\partial g}{\partial p^\parallel} \right) - \frac{\partial K^*}{\partial p^\parallel} \cdot \left( G^* \times \nabla f - F^* \times \nabla g \right) \right). \tag{47}
\]
First, using the identity \( \partial f/\partial p^\parallel = \hat{b} \cdot \{ X, f \}_{gc} \), we can write
\[
G^* \cdot \nabla \times K^* \left( \hat{b} \cdot \{ X, f \}_{gc} \right) = \left( G^* \times \hat{b} \right) \cdot \left( \nabla \times K^* \times \{ X, f \}_{gc} \right),
\]
so that
\[
\nabla \times K^* \cdot \left( G^* \cdot \frac{\partial f}{\partial p^\parallel} - F^* \cdot \frac{\partial g}{\partial p^\parallel} \right) = \nabla \times K^* \cdot \left( \{ X, f \}_{gc} \times \left( G^* \times \hat{b} \right) - \{ X, g \}_{gc} \times \left( F^* \times \hat{b} \right) \right)
+ \hat{b} \cdot \nabla \times K^* \left( G^* \cdot \{ X, f \}_{gc} - F^* \cdot \{ X, g \}_{gc} \right).
\] (48)
Here, the last term on the right side of Eq. (48) can be written as
\[
\hat{b} \cdot \nabla \times K^* \left( G^* \cdot \{ X, f \}_{gc} - F^* \cdot \{ X, g \}_{gc} \right) = \frac{qB^*}{c} \left\{ X^i, K^* \right\}_{gc} \left( G^* \cdot \{ X, f \}_{gc} - F^* \cdot \{ X, g \}_{gc} \right) - \frac{q}{c} \hat{b} \cdot \frac{\partial K^*}{\partial p^\parallel} \left( G^* \cdot \{ X, f \}_{gc} - F^* \cdot \{ X, g \}_{gc} \right).
\] (49)
Next, we write
\[
\nabla \times K^* \cdot \left( \{ X, f \}_{gc} \times \left( G^* \times \hat{b} \right) \right) = \left( \{ K^*, f \}_{gc} + \frac{B^*}{B^\parallel} \cdot \nabla f \cdot \frac{\partial K^*}{\partial p^\parallel} \right) \cdot \left( G^* \times \hat{b} \right) - \left( G^* \times \hat{b} \right) \cdot \nabla K^* \cdot \{ X, f \}_{gc}
= - \frac{qB^*}{c} \left( \left\{ K^*, f \right\}_{gc} \cdot \{ X, X \}_{gc} \cdot G^* + \left\{ X, K^* \right\}_{gc} \cdot \{ X, f \}_{gc} \right)
+ \frac{\partial K^*}{\partial p^\parallel} \cdot \left( G^* \times \hat{b} \right) \frac{B^*}{B^\parallel} \cdot \nabla f + \left( \frac{q}{c} \hat{b} \cdot G^* \right) \{ X, f \}_{gc},
\] (50)
so that Eq. (48) becomes

\[ F \cdot \nabla K^* \cdot \left( \frac{G^*}{\partial f} \frac{\partial f}{\partial p_i} - \frac{F^*}{\partial g} \frac{\partial g}{\partial p_i} \right) = \frac{q}{c} F \cdot \left[ \left\{ X^i, K^*_i \right\} \frac{\partial}{\partial p_i} \left( G^* \cdot \left\{ X, f \right\} \right) - F^* \cdot \left\{ X, g \right\} \right] \]

\[ + \frac{q}{c} F \cdot \left( \left\{ K^*_i, g \right\} \cdot \left\{ X, X \right\} + F^* \cdot \left\{ X, K^*_i \right\} \cdot \left\{ X, g \right\} \right) \]

\[ - \frac{q}{c} F \cdot \left( \left\{ K^*_i, f \right\} \cdot \left\{ X, X \right\} + G^* \cdot \left\{ X, K^*_i \right\} \cdot \left\{ X, f \right\} \right) \]

\[ + F \cdot \frac{\partial K^*_i}{\partial p_i} \cdot \left[ \left( G^* \times \hat{b} \right) \frac{B^*}{B_{\parallel}} \cdot \nabla f - \left( F^* \times \hat{b} \right) \frac{B^*}{B_{\parallel}} \cdot \nabla g \right. \]

\[ + G^* \times \left( \frac{q}{c} \left\{ X, f \right\} \times B^* \right) - F^* \times \left( \frac{q}{c} \left\{ X, g \right\} \times B^* \right) \].

(51)

When we combine these expressions into Eq. (47), we obtain

\[ \int_X \nabla \times \mathbf{K} \cdot \frac{\delta P}{\delta \mathbf{B}} \left[ \int_Z F \cdot \left( G^* \cdot \left\{ X, f \right\} - F^* \cdot \left\{ X, g \right\} \right) \right] \]

\[ = \frac{q}{c} \int_Z F \cdot \left[ \left\{ X^i, K^*_i \right\} \frac{\partial}{\partial p_i} \left( G^* \cdot \left\{ X, f \right\} - F^* \cdot \left\{ X, g \right\} \right) \right] \]

\[ + \frac{q}{c} \int_Z F \cdot \left( \left\{ K^*_i, g \right\} \cdot \left\{ X, X \right\} + F^* \cdot \left\{ X, K^*_i \right\} \cdot \left\{ X, g \right\} \right) \]

\[ - \frac{q}{c} \int_Z F \cdot \left( \left\{ K^*_i, f \right\} \cdot \left\{ X, X \right\} + G^* \cdot \left\{ X, K^*_i \right\} \cdot \left\{ X, f \right\} \right) \]

\[ + \int_Z F \cdot \frac{\partial K^*_i}{\partial p_i} \cdot \left[ \left( G^* \times \hat{b} \right) \frac{B^*}{B_{\parallel}} \cdot \nabla f - \left( F^* \times \hat{b} \right) \frac{B^*}{B_{\parallel}} \cdot \nabla g \right. \]

\[ + G^* \times \left( \frac{q}{c} \left\{ X, f \right\} \times B^* \right) - F^* \times \left( \frac{q}{c} \left\{ X, g \right\} \times B^* \right) \].

(52)

We now note that the terms associated with \( \partial K^*/\partial p_i \) cancel out exactly, while the expression for the Jacobi property involves cyclic permutations of the functionals \( \left( F, G, K \right) \), with the combination \( \left\{ f, G^*, K^* \right\} \):

\[ -4\pi c \int_X \nabla \times \mathbf{K} \cdot \frac{\delta P}{\delta \mathbf{B}} \left[ \int_Z F \cdot \left( G^* \cdot \left\{ X, f \right\} \right) \right] - \nabla \times \mathbf{G} \cdot \frac{\delta P}{\delta \mathbf{B}} \left[ \int_Z \left( \frac{\delta P}{\delta \mathbf{B}} \int_Z \left( F \cdot \left\{ X, f \right\} \right) \right) \right] \]

\[ = 4\pi q \int_Z F \cdot \text{Jac}^{(2)}_M [f; G^*, K^*], \]

(53)

where the second-order Maxwell sub-bracket contribution is

\[ \text{Jac}^{(2)}_M [f; G^*, K^*] = \left\{ K^*, f \right\} \cdot \left\{ X, X \right\} \cdot G^* - \left\{ G^*, f \right\} \cdot \left\{ X, X \right\} \cdot K^* \]

\[ + G^* \cdot \left\{ X, K^*_i \right\} \cdot \left\{ X, f \right\} - K^* \cdot \left\{ X, G^*_i \right\} \cdot \left\{ X, f \right\} \]

\[ + \left\{ X^i, G^*_i \right\} \cdot K^* \cdot \left\{ X, f \right\} - \left\{ X^i, K^*_i \right\} \cdot G^* \cdot \left\{ X, f \right\}. \]

(54)

Hence, by combining Eqs. (30), (37), and (54), we finally obtain

\[ \text{Jac}^{(2)}_{V,M} [f; G^*, K^*] + \text{Jac}^{(2)}_{M} [f; G^*, K^*] + \text{Jac}^{(2)}_M [f; G^*, K^*] = \text{Jac}^{(2)}_{V,M} [f; G^*, K^*], \]

(55)

where the second-order Vlasov-Maxwell term \( \text{Jac}^{(2)}_{V,M} [f; G^*, K^*] \) is defined in Eq. (51).

3. Second-order term

For the second-order term in \( 4\pi q \) in Eq. (50), we begin with Eq. (313):

\[ -4\pi c \int_X \nabla \times \mathbf{K} \cdot \frac{\delta P}{\delta \mathbf{B}} \left[ \int_Z F \cdot F^* \cdot \left\{ X, X \right\} \cdot G^* \right] = 4\pi q \int_Z F \cdot \text{Jac}^{(3)}_M [F^*, G^*, K^*]. \]

(56)
where the third-order Maxwell sub-bracket contribution is

\[ \text{Jac}^{(3)}_{M} [F^*, G^*, K^*] = \frac{c}{q B_{||}} \left[ \frac{\partial F^*}{\partial p_{||}} \cdot (G^* \times K^*) + \frac{\partial G^*}{\partial p_{||}} \cdot (K^* \times F^*) + \frac{\partial K^*}{\partial p_{||}} \cdot (F^* \times G^*) \right]. \] (57)

We now use the identity

\[ \frac{c}{q B_{||}} \frac{\partial K^*}{\partial p_{||}} \cdot (F^* \times G^*) = \left( \frac{\partial K^*}{\partial p_{||}} \times \frac{c \hat{b}}{q B_{||}} \right) \cdot \left( \frac{F^* \times G^*}{B_{||}} \times \frac{B^*}{B_{||}} \right) + \frac{c \hat{b}}{q B_{||}} \cdot \left( \frac{F^* \times G^*}{B_{||}} \right) \cdot \frac{\partial K^*}{\partial p_{||}} \]
\[ = \left( \frac{F^* \cdot B^*}{B_{||}} \right) \frac{\partial K^*}{\partial p_{||}} \cdot \{X, X\}_{gc} \cdot G^* - \left( \frac{G^* \cdot B^*}{B_{||}} \right) \frac{\partial K^*}{\partial p_{||}} \cdot \{X, X\}_{gc} \cdot F^* \]
\[ + \frac{c \hat{b}}{q B_{||}} \cdot \left( \frac{F^* \times G^*}{B_{||}} \right) \cdot \frac{\partial K^*}{\partial p_{||}} \] (58)

where the last term on the right side is

\[ \frac{c \hat{b}}{q B_{||}} \cdot \left( \frac{F^* \times G^*}{B_{||}} \right) \cdot \frac{\partial K^*}{\partial p_{||}} = - \left\{ X^i, K^*_i \right\}_{gc} F^* \cdot \{X, X\}_{gc} \cdot G^* + \left( \frac{c \hat{b}}{q B_{||}} \cdot \nabla \times K^* \right) F^* \cdot \{X, X\}_{gc} \cdot G^*. \]

Next, we use the identity

\[ \left( K^* \cdot \{X, F^*\}_{gc} - F^* \cdot \{X, K^*\}_{gc} \right) \cdot \{X, X\}_{gc} \cdot G^* = \left[ \left( F^* \cdot \frac{B^*}{B_{||}} \right) \frac{\partial K^*}{\partial p_{||}} - \left( K^* \cdot \frac{B^*}{B_{||}} \right) \frac{\partial F^*}{\partial p_{||}} \right] \cdot G^* \times \frac{c \hat{b}}{q B_{||}} \]
\[ + \left( F^* \times \frac{c \hat{b}}{q B_{||}} \cdot \nabla K^* - K^* \times \frac{c \hat{b}}{q B_{||}} \cdot \nabla F^* \right) \cdot G^* \times \frac{c \hat{b}}{q B_{||}}, \] (59)

so that Eq. (57) yields the final expression for the third-order Maxwell sub-bracket contribution

\[ \text{Jac}^{(3)}_{M} [F^*, G^*, K^*] = - \text{Jac}^{(3)}_{j} [F^*, G^*, K^*] - \left[ \left( K^* \cdot \{X, F^*\}_{gc} - F^* \cdot \{X, K^*\}_{gc} \right) \cdot \{X, X\}_{gc} \cdot G^* + \circ \right], \] (60)

where \( \text{Jac}^{(3)}_{j} [F^*, G^*, K^*] \) is given by Eq. (38) and we used the identity

\[ F^* \times \frac{c \hat{b}}{q B_{||}} \cdot \nabla K^* \cdot G^* \times \frac{c \hat{b}}{q B_{||}} - G^* \times \frac{c \hat{b}}{q B_{||}} \cdot \nabla K^* \cdot F^* \times \frac{c \hat{b}}{q B_{||}} = \left( \frac{c \hat{b}}{q B_{||}} \cdot \nabla \times K^* \right) F^* \cdot \{X, X\}_{gc} \cdot G^*. \]

Hence, by combining Eqs. (38), (38), and (60), we finally obtain

\[ \text{Jac}^{(3)}_{V}[F^*, G^*, K^*] + \text{Jac}^{(3)}_{j}[F^*, G^*, K^*] + \text{Jac}^{(3)}_{M}[F^*, G^*, K^*] = \text{Jac}^{(3)}_{V,M}[f; G^*, K^*], \] (61)

where the third-order Vlasov-Maxwell term \( \text{Jac}^{(3)}_{V,M}[f; G^*, K^*] \) is defined in Eq. (34).
III. PROOF OF THE JACOBI PROPERTY

By combining the Vlasov-Maxwell contributions \( \langle 25, 28, 31, \rangle \), and \( 34 \), the Jacobi property \( 14 \) of the guiding-center Vlasov-Maxwell bracket \( 15 \) can now be expressed as a series in powers of \( 4\pi q \) up to third order:

\[
\mathcal{J}ac[F, G, K] = \int_{\mathcal{Z}} F_{gc} \text{Jac}^{(0)}_{VM}[f, g, k] + 4\pi q \int_{\mathcal{Z}} F_{gc} \left( \text{Jac}^{(1)}_{VM}[f, g; K^*] + \circ \right) \\
+ (4\pi q)^2 \int_{\mathcal{Z}} F_{gc} \left( \text{Jac}^{(2)}_{VM}[f; G^*, K^*] + \circ \right) + (4\pi q)^3 \int_{\mathcal{Z}} F_{gc} \text{Jac}^{(3)}_{VM}[F^*, G^*, K^*]
\]

\[= \int_{\mathcal{Z}} F_{gc} \left( \left\{ \left\{ f, g \right\}_{gc}, k \right\}_{gc} + \left\{ \left\{ g, k \right\}_{gc}, f \right\}_{gc} + \left\{ \left\{ k, f \right\}_{gc}, g \right\}_{gc} \right) + (4\pi q)^2 \int_{\mathcal{Z}} F_{gc} \left[ F_i^* \left( \left\{ \left\{ X^i, g \right\}_{gc}, k \right\}_{gc} + \left\{ \left\{ g, k \right\}_{gc}, X^i \right\}_{gc} + \left\{ \left\{ k, X^i \right\}_{gc}, g \right\}_{gc} \right) + \circ \right) \\
+ (4\pi q)^3 \int_{\mathcal{Z}} F_{gc} \left[ F_i^* G_j^* \left( \left\{ \left\{ X^i, X^j \right\}_{gc}, k \right\}_{gc} + \left\{ \left\{ X^j, k \right\}_{gc}, X^i \right\}_{gc} + \left\{ \left\{ k, X^i \right\}_{gc}, X^j \right\}_{gc} \right) + \circ \right) \\
+ (4\pi q)^3 \int_{\mathcal{Z}} F_{gc} \left[ F_i^* G_j^* K^*_\ell \left( \left\{ \left\{ X^i, X^j \right\}_{gc}, X^\ell \right\}_{gc} + \left\{ \left\{ X^\ell, X^i \right\}_{gc}, X^j \right\}_{gc} + \left\{ \left\{ X^j, X^i \right\}_{gc}, X^\ell \right\}_{gc} \right) \right),
\]

where all additional terms have cancelled out exactly as shown in Sec. III. In Eq. \( 62 \), it is clear that the Jacobi property of the guiding-center Vlasov-Maxwell bracket \( 15 \) is inherited from the Jacobi property \( 8 \) of the guiding-center Poisson bracket \( 9 \), since each term in Eq. \( 62 \) vanishes identically because of this latter property. Hence, the Jacobi property for the guiding-center Vlasov-Maxwell bracket \( 15 \) holds under the condition \( 9 \), which holds according to Eq. \( 7 \).

IV. SUMMARY

The proof of the Jacobi property of a functional bracket that forms the basis for the Hamiltonian structure of reduced Vlasov-Maxwell equations is often a challenging task (see, for example, Ref. \[4\]). While we can also appeal to rigorous theoretical grounds for the validity of the Jacobi property of the guiding-center Vlasov-Maxwell bracket \( 15 \), an explicit proof of Eq. \( 14 \) was presented in these notes. As expected, the Jacobi property is inherited from the Jacobi property \( 8 \) of the guiding-center Poisson bracket \( 9 \).

Appendix A: Poisson-bracket Identities

In this Appendix, we derive several identities associated with the guiding-center Poisson bracket \( 6 \). First, we have the integral identity

\[
\int_{\mathcal{Z}} B^\parallel \left\{ f, g \right\}_{gc} = \int_{\mathcal{Z}} \frac{\partial}{\partial Z^\alpha} \left( B^\parallel f \left\{ Z^\alpha, g \right\}_{gc} \right) = 0,
\]

(A1)

which holds for any functions \( (f, g) \). This identity yields the formula \( \int_{\mathcal{Z}} B^\parallel \left\{ f, g \right\}_{gc} k = \int_{\mathcal{Z}} B^\parallel f \left\{ g, k \right\}_{gc} \) after integration by parts is performed.

Next, for any arbitrary vector-valued function \( \mathbf{R} \), we find the divergence identity

\[
\left\{ X^i, R_j \right\}_{gc} = \frac{\hat{e}B}{qB^\parallel} \cdot \nabla \times \mathbf{R} + \frac{B^*}{B^\parallel} \cdot \frac{\partial}{\partial p^\parallel} \cdot \mathbf{R},
\]

(A2)

where summation over repeated indices on the left side is implied. In addition, for an arbitrary function \( f \), we find

\[
\left\{ \mathbf{R}, f \right\}_{gc} = \left\{ \mathbf{X}, f \right\}_{gc} \cdot \nabla \mathbf{R} - \left( \frac{B^*}{B^\parallel} \cdot \nabla f \right) \frac{\partial \mathbf{R}}{\partial p^\parallel},
\]

(A3)

where \( \left\{ \mathbf{X}, f \right\}_{gc} = (B^*/B^\parallel) \partial f / \partial p^\parallel + (\hat{e}B/qB^\parallel) \times \nabla f \).
Lastly, for two arbitrary functions \((f, g)\), we find the Poisson-bracket identity
\[
\left(\frac{c}{q}\right) \nabla f \times \nabla g = \{X, g\}_\text{gc} \ B^* \cdot \nabla f - \{X, f\}_\text{gc} \ B^* \cdot \nabla g - \{f, g\}_\text{gc}, \tag{A4}
\]
which is derived from the identity
\[
\hat{b} \times [B^* \times (\nabla f \times \nabla g)] = \begin{cases} 
\hat{b} \cdot (\nabla f \times \nabla g) B^* - B^* \nabla f \times \nabla g \\
\hat{b} \times \nabla f (B^* \cdot \nabla g) - \hat{b} \times \nabla g (B^* \cdot \nabla f)
\end{cases}
\]
Next, we use the identity
\[
\{X, f\}_\text{gc} \times \{X, g\}_\text{gc} = \left(\frac{B^*}{B^*_\parallel} \frac{\partial f}{\partial p_\parallel} + \frac{\hat{b}}{qB^*_\parallel} \times \nabla f\right) \times \left(\frac{B^*}{B^*_\parallel} \frac{\partial g}{\partial p_\parallel} + \frac{\hat{b}}{qB^*_\parallel} \times \nabla g\right)
\]
\[
= \frac{B^*}{B^*_\parallel} \times \left[\frac{\partial f}{\partial p_\parallel} \left(\frac{\hat{b}}{qB^*_\parallel} \times \nabla g\right) - \frac{\partial g}{\partial p_\parallel} \left(\frac{\hat{b}}{qB^*_\parallel} \times \nabla f\right)\right] + \left(\frac{\hat{b}}{qB^*_\parallel} \times \nabla f\right) \times \left(\frac{\hat{b}}{qB^*_\parallel} \times \nabla g\right)
\]
\[
= \frac{c}{qB^*_\parallel} \left[\nabla f \frac{\partial g}{\partial p_\parallel} - \frac{\hat{b}}{qB^*_\parallel} \times \nabla g\right] - \hat{b} \{f, g\}_\text{gc}, \tag{A5}
\]
to obtain
\[
\nabla \times \mathbf{R} \cdot \left(\{X, f\}_\text{gc} \times \{X, g\}_\text{gc}\right) = \frac{c}{qB^*_\parallel} \left[\nabla \times \mathbf{R} \cdot \left(\nabla f \frac{\partial g}{\partial p_\parallel} - \frac{\hat{b}}{qB^*_\parallel} \times \nabla g\right) - \hat{b} \cdot \nabla \mathbf{R} \{f, g\}_\text{gc}\right]
\]
\[
= \{\mathbf{R}, f\}_\text{gc} \cdot \{X, g\}_\text{gc} - \{\mathbf{R}, g\}_\text{gc} \cdot \{X, f\}_\text{gc}
\]
\[
+ \frac{\partial \mathbf{R}}{\partial p_\parallel} \cdot \left[\{X, g\}_\text{gc} \left(\frac{B^*}{B^*_\parallel} \cdot \nabla f\right) - \{X, f\}_\text{gc} \left(\frac{B^*}{B^*_\parallel} \cdot \nabla g\right)\right]. \tag{A6}
\]

**Appendix B: Magnetic Functional Derivatives in the Maxwell Sub-Bracket**

In this Appendix, we calculate the magnetic functional derivatives in the Maxwell sub-bracket that appear in Eq. (20). These functional derivatives are separated into Jacobian and \(n\)-th-order \((n = 0, 1, 2)\) derivatives.

### 1. Jacobian contributions

Each term in the Poisson functional derivative (22) includes a guiding-center Poisson bracket, which contains the Jacobian \(B^*_\parallel\) as a denominator. Hence, the double-bracket expression (20) includes the Jacobian contribution
\[
\delta^P \left[\mathcal{F}, \mathcal{G}\right]_\text{gc} \equiv - \int Z \frac{F_\text{gc}}{B^*_\parallel} \left[\mathcal{F}, \mathcal{G}\right]_\text{Fgc} \delta B^*_\parallel, \tag{B1}
\]
where \([\mathcal{F}, \mathcal{G}]_\text{Fgc} \equiv \delta^P [\mathcal{F}, \mathcal{G}]_\text{gc} / \delta F_\text{gc}\) is given by Eq. (22), and
\[
\delta B^*_\parallel = \hat{b} \cdot \mathbf{B}^* + \hat{b} \cdot \delta \mathbf{B}^* = \left(\frac{\partial \mathbf{B}^*}{\partial p_\parallel} \cdot \mathbf{p}^*_\parallel\right) \cdot \mathbf{B}^* + \hat{b} \cdot \left[\delta \mathbf{B} + \nabla \times (\mathbf{p}^*_\parallel \cdot \delta \mathbf{B})\right]
\]
\[
= \hat{b} \cdot \delta \mathbf{B} + \nabla \cdot \left[(\mathbf{p}^*_\parallel \cdot \delta \mathbf{B}) \times \hat{b}\right] + \frac{\partial}{\partial p_\parallel} \left[\delta \mathbf{B} \cdot \left(\frac{\hat{b}}{c} \mathbf{p}^*_\parallel \cdot \mathbf{B}^*\right)\right], \tag{B2}
\]
so that, for an arbitrary guiding-center function \(w\), we find (after integrating by parts)
\[
\int Z w \delta B^*_\parallel = \int X \delta \mathbf{B} \cdot \int P \left[w \hat{b} - \frac{q}{c} \mathbf{p}^*_\parallel \cdot \left(\mathbf{B}^* \frac{\partial w}{\partial p_\parallel} + \frac{\hat{b}}{q} \times \nabla w\right)\right] = \int X \delta \mathbf{B} \cdot \int P \left(w \hat{b} - \frac{qB^*_\parallel}{c} \mathbf{p}^*_\parallel \cdot \{\mathbf{X}, w\}_\text{gc}\right).
\]
For an arbitrary vector field $\mathbf{K}$, we obtain

$$4\pi c \int_X \nabla \times \mathbf{K} \cdot \int_P \frac{w \delta B^*}{\delta \mathbf{B}} = 4\pi q \int_Z \left[ w \frac{\delta \mathbf{B}}{q} \cdot \nabla \times \mathbf{K} - B^* \mathbf{P} \cdot \nabla \times \mathbf{K} \cdot \{ \mathbf{X}, w \}_{gc} \right]$$

where we used the identities (A1), (A2) and the definition $\mathbf{K}^* = \mathbf{K} + \mathbf{P} \cdot \nabla \times \mathbf{K}$. Integrating by parts, using the Poisson-bracket Leibnitz formula $\mathbf{K}^* \cdot \{ \mathbf{X}, w \}_{gc} = \{ \mathbf{X}^i, w K_i \}_{gc} - \{ \mathbf{X}^i, K_i^* \}_{gc} w$, we finally obtain the series expansion in powers of $4\pi q$:

$$4\pi c \int_X \nabla \times \mathbf{K} \cdot \int_P w \frac{\delta B^*}{\delta \mathbf{B}} = \int_Z F_{gc} \left\{ \mathbf{X}^i, K_i^* \right\}_{gc} \left[ 4\pi q \{ f, g \}_{gc} - (4\pi q)^2 \left( \mathbf{F}^* \cdot \{ \mathbf{X}, g \}_{gc} - \mathbf{G}^* \cdot \{ \mathbf{X}, f \}_{gc} \right) \\
+ (4\pi q)^3 \left( \mathbf{F}^* \cdot \{ \mathbf{X}, \mathbf{X} \}_{gc} \cdot \mathbf{G}^* \right) \right\}. \quad \text{(B4)}$$

after substituting $w \equiv \left| \mathbf{F}, \mathbf{G} \right|_{F_{gc}} F_{gc} / B^*_{gc}$.

2. Zeroth-order Maxwell sub-bracket

In the Maxwell sub-bracket appearing in Eq. (39), we begin with the magnetic variation of the zeroth-order term

$$\int_Z F \delta^P \left( B^* \{ f, g \}_{gc} \right) = \int_Z F \left[ \delta \mathbf{B}^* \cdot \left( \nabla f \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \right) - \frac{\mathbf{c}}{q} \delta \mathbf{b} \cdot \nabla f \times \nabla g \right]$$

$$= \int_Z \delta \mathbf{B} \cdot \left[ F \left( \nabla f \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \right) + \mathbf{P} \cdot \nabla \times \left( \nabla f \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \right) + \frac{\partial F}{\partial p} \mathbf{P} \cdot \left( \nabla f \times \nabla g \right) \right],$$

where $\delta \mathbf{B}^*$ and $\delta \mathbf{b}$ are given in Eq. (21), and integration by parts were carried out in order to release $\delta \mathbf{B}$. Hence, we obtain the magnetic functional derivative

$$\frac{\delta^P}{\delta \mathbf{B}} \left( \int_Z F_{gc} \{ f, g \}_{gc} \right) = \int_F \left( \nabla f \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \right) + \mathbf{P} \cdot \left[ \frac{\partial F}{\partial p} \left( \nabla f \times \nabla g \right) - \left( \nabla f \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \right) \times \nabla F \right]. \quad \text{(B5)}$$

We now insert this functional derivative in Eq. (20) to obtain

$$\int_X \nabla \times \mathbf{K} \cdot \frac{\delta^P}{\delta \mathbf{B}} \left( \int_Z F_{gc} \{ f, g \}_{gc} \right) = \int_Z F \nabla \times \mathbf{K} \cdot \left( \nabla f \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \right)$$

$$+ \int_Z \left( \mathbf{K}^* - \mathbf{K} \right) \cdot \left[ \frac{\partial F}{\partial p} \left( \nabla f \times \nabla g \right) - \left( \nabla f \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \right) \times \nabla F \right], \quad \text{(B6)}$$

where we used $\mathbf{P} \cdot \nabla \times \mathbf{X} = \mathbf{K}^* - \mathbf{K}$. Next, we integrate by parts the terms

$$\int_Z \left( \mathbf{K}^* - \mathbf{K} \right) \cdot \frac{\partial F}{\partial p} \left( \nabla f \times \nabla g \right) = - \int_Z F \left[ \frac{\partial K^*}{\partial p} \left( \nabla f \times \nabla g \right) + \left( \mathbf{K}^* - \mathbf{K} \right) \cdot \frac{\partial}{\partial p} \left( \nabla f \times \nabla g \right) \right],$$

$$\int_Z \nabla F \cdot \left( \nabla f \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \right) \times \left( \mathbf{K}^* - \mathbf{K} \right) = \int_Z F \nabla \times \left( \mathbf{K}^* - \mathbf{K} \right) \cdot \left( \nabla f \frac{\partial g}{\partial p} \frac{\partial f}{\partial p} \nabla g \right)$$

$$+ \int_Z F \left( \mathbf{K}^* - \mathbf{K} \right) \cdot \frac{\partial}{\partial p} \left( \nabla f \times \nabla g \right),$$

so that, after cancellations, we obtain the final expression to be used in Eq. (40):

$$\int_X \nabla \times \mathbf{K} \cdot \frac{\delta^P}{\delta \mathbf{B}} \left( \int_Z F_{gc} \{ f, g \}_{gc} \right) = \int_Z F \left[ \nabla \times \mathbf{K}^* \cdot \left( \nabla f \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \right) \nabla g \right] - \frac{\partial K^*}{\partial p} \cdot \left( \nabla f \times \nabla g \right). \quad \text{(B7)}$$
where we have used Eq. (B8) to obtain the identity
\[
\delta P G^* = \delta \mathbb{P}_\parallel \cdot \nabla \times G = -\frac{\delta B}{B} \cdot \left[ \mathbb{P}_\parallel (\hat{b} \cdot \nabla \times \mathbb{G}) + \hat{b} (\mathbb{P}_\parallel \cdot \nabla \times \mathbb{G}) + (\mathbb{P}_\parallel \cdot \nabla \times \mathbb{G}) \hat{b} \right] \equiv \delta B \cdot \frac{\delta P G^*}{\delta B}. \tag{B8}
\]

Hence, we obtain the magnetic functional derivative
\[
\frac{\delta P}{\delta B} \left( \int_Z F_{gc} G^* \cdot \{X, f\}_{gc} \right) = \int_Z F_{gc} \nabla \times K \cdot \frac{\delta P G^*}{\delta B} \cdot \{X, f\}_{gc} + F_{gc} \frac{\delta f}{\delta p} \nabla \times K \cdot (G^* + \mathbb{P}_\parallel \cdot \nabla \times G^*)
\]
\[
+ \int_Z (K^\star - K) \cdot \left[ \left( \frac{\delta f}{\delta p} \nabla F - \frac{\delta F}{\delta p} \nabla f \right) \times G^* - F \left( \nabla f \times \frac{\delta G^*}{\delta p} \right) \right]. \tag{B10}
\]

Next, we integrate by parts the terms
\[
\int_Z (K^\star - K) \cdot \left( \frac{\delta f}{\delta p} \nabla F - \frac{\delta F}{\delta p} \nabla f \right) \times G^* = \int_Z F \frac{\delta f}{\delta p} \left[ G^* \cdot \nabla \times K^\star - \nabla \times K \cdot (G^* + \mathbb{P}_\parallel \cdot \nabla \times G^*) \right]
\]
\[
+ \int_Z F \nabla f \cdot \left( \frac{\partial G^*}{\partial p_\parallel} \times (K^\star - K) + G^* \times \frac{\partial K^\star}{\partial p_\parallel} \right)
\]

so that, after cancellations, we obtain
\[
\int_X \nabla \times K \frac{\delta P}{\delta B} \left( \int_Z F_{gc} G^* \cdot \{X, f\}_{gc} \right) = \int_Z F_{gc} \nabla \times K \cdot \frac{\delta P G^*}{\delta B} \cdot \{X, f\}_{gc}
\]
\[
+ \int_Z F G^* \cdot \left( \frac{\delta f}{\delta p_\parallel} \nabla \times K^\star - \nabla f \times \frac{\partial K^\star}{\partial p_\parallel} \right),
\]

which yields the final expression to be used in Eq. (47):
\[
\int_X \nabla \times K \frac{\delta P}{\delta B} \left[ \int_Z F_{gc} \left( G^* \cdot \{X, f\}_{gc} - F^* \cdot \{X, g\}_{gc} \right) \right] = \int_Z F \nabla \times K^\star \cdot \left( G^* \frac{\partial f}{\partial p_\parallel} - F^* \frac{\partial g}{\partial p_\parallel} \right)
\]
\[
- \int_Z F \frac{\partial K^\star}{\partial p_\parallel} \cdot \left( G^* \nabla f - F^* \nabla g \right), \tag{B11}
\]

where we have used Eq. (B8) to obtain the identity
\[
B \left( \nabla \times K \frac{\delta P G^*}{\delta B} - \nabla \times G \cdot \frac{\delta P K^\star}{\delta B} \right) = \nabla \times G \cdot \left[ \mathbb{P}_\parallel (\hat{b} \cdot \nabla \times \mathbb{K}) + \hat{b} (\mathbb{P}_\parallel \cdot \nabla \times \mathbb{K}) + (\mathbb{P}_\parallel \cdot \nabla \times \mathbb{K}) \hat{b} \right] \tag{B12}
\]
\[
- \nabla \times K \cdot \left[ \mathbb{P}_\parallel (\hat{b} \cdot \nabla \times \mathbb{G}) + \hat{b} (\mathbb{P}_\parallel \cdot \nabla \times \mathbb{G}) + (\mathbb{P}_\parallel \cdot \nabla \times \mathbb{G}) \hat{b} \right] = 0.
\]

Hence, the magnetic functional derivatives \( (\delta F^*/\delta B, \ldots) \) do not contribute to the Maxwell sub-bracket and the Jacobi property \( (14) \), which may be viewed as an extension of the Bracket theorem.
4. Second-order Maxwell sub-bracket

In the Maxwell sub-bracket appearing in Eq. (39), we begin with the magnetic variation of the second-order term

\[
\int Z F_{\delta P} \left( B_{\parallel} F^* \cdot \{X, X\}_{gc} \cdot G^* \right) = - \int F \left[ \frac{\mathcal{E} \cdot \mathbf{F}^* \times \mathbf{G}^*}{q} + \delta P F^* \cdot \left( G^* \times \frac{\mathcal{E} \cdot \mathbf{b}}{q} \right) - \delta P G^* \cdot \left( F^* \times \frac{\mathcal{E} \cdot \mathbf{b}}{q} \right) \right]
\]

\[
= - \int X \mathbf{B} \cdot \int P F \left[ \frac{\partial P_{\parallel}}{\partial p_{\parallel}} \cdot F^* \times G^* + \delta P F^* \cdot \left( G^* \times \frac{\mathcal{E} \cdot \mathbf{b}}{q} \right) - \delta P G^* \cdot \left( F^* \times \frac{\mathcal{E} \cdot \mathbf{b}}{q} \right) \right]
\]

which yields the final expression to be used in Eq. (56):

\[
\int X \left[ \nabla \times K^* \cdot \frac{\delta P}{\delta \mathbf{B}} \left( \int Z F_{gc} F^* \cdot \{X, X\}_{gc} \cdot G^* \right) + \mathring{\circ} \right] = - \int Z F \left[ \frac{\partial F^*}{\partial p_{\parallel}} \cdot \left( G^* \times K^* \right) + \frac{\partial G^*}{\partial p_{\parallel}} \cdot \left( K^* \times F^* \right) + \frac{\partial K^*}{\partial p_{\parallel}} \cdot \left( F^* \times G^* \right) \right],
\]

where \( \mathring{\circ} \) denotes a cyclic permutation on the left side, we have used the cancellation identity (B12), and \( \frac{\partial K^*}{\partial p_{\parallel}} = \left( \frac{\partial P_{\parallel}}{\partial p_{\parallel}} \right) \cdot \nabla \times K \).

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[1] A.J. Brizard and C. Tronci, Phys. Plasmas 23, 062107 (2016).
[2] A.J. Brizard, *Hamiltonian structure of the guiding-center Vlasov-Maxwell equations*, in preparation (2021).
[3] P.J. Morrison, Phys. Plasmas 20, 012104 (2013).
[4] A.J. Brizard, P.J. Morrison, J.W. Burby, L. de Guillebon, and M. Vittot, J. Plasma Phys. 82, 905820608 (2016).