Operator-valued multipliers in vector-valued weighted Besov spaces and applications

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ABSTRACT

The operator-valued multiplier theorems in $E$-valued Besov spaces are studied, where $E_0$, $E$ are two Banach spaces and $E_0 \subset E$. These results permit us to show embedding theorems in $E_0$-valued weighted Besov-Lions type spaces $B^{l,s}_{p,q,\gamma}(\Omega; E_0, E)$. The most regular class of interpolation space $E_\alpha$, between $E_0$ and $E$ are found such that the mixed differential operator $D^\alpha$ is bounded from $B^{l,s}_{p,q,\gamma}(\Omega; E_0, E)$ to $B^s_{p,q,\gamma}(\Omega; E_\alpha)$ and Ehrling-Nirenberg-Gagliardo type sharp estimates are established. By using these results the separability properties of degenerate differential operators are studied. Especially, we prove that the associated differential operators are positive and also are generators of analytic semigroups. Moreover, maximal $B^s_{p,q,\gamma}$-regularity properties for abstract elliptic equation, Cauchy problem for degenerate abstract parabolic equation and the infinite systems of degenerate parabolic equations are studied.

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1. Introduction

Fourier multipliers in vector-valued function spaces has been studied e.g. in [17], [28], [32]. Operator-valued Fourier multipliers in weighted spaces have been investigated in [1], [9 – 12], [13], [30]. Mikhlin type Fourier multipliers in scalar weighted spaces have been studied e.g. in [14] and [30]. Moreover, operator-valued Fourier multipliers in weighted abstract $L_p$ spaces were investigated e.g. in [2], [7], [13], [16]. Regularity properties of differential-operator equations (DOEs) have been studied e.g. in [1], [3], [9], [21 – 26], [30 – 31]. A
comprehensive introduction to DOEs and historical references may be found in [1] and [31].

In this paper, operator-valued multiplier theorems in $E$-valued weighted Lebesgue and Besov spaces are shown. Then we consider the $E$-valued anisotropic Besov spaces $B^{s}_{p,q,\gamma}(\Omega; E_{0}, E)$, here $E_{0}$, $E$ are two Banach spaces, $E_{0}$ is continuously and densely embedded into $E$, and $\gamma = \gamma(x)$ is weighted function from $A_{p}, p \in (1, \infty)$ class. We prove boundedness and compactness of embedding operators in these spaces. This result generalized and improved the results [4, § 9, 27, § 1.7] for scalar Sobolev space, the result [15] for one dimensional Sobolev-Lions spaces and the results [22 – 23] for Hilbert-space valued class.

Finally, we consider differential-operator equation

$$Lu = \sum_{|\alpha|=2l} a_{\alpha}D^{\alpha}u + Au + \sum_{|\alpha|<2l} A_{\alpha}D^{\alpha}u = f$$  \hspace{1cm} (1.1)$$

where $a_{\alpha}$ are complex numbers, $A$ and $A_{\alpha}(x)$ are linear operators in a Banach space $E$, $\alpha = (\alpha_{1}, \alpha_{2}, ..., \alpha_{n})$.

We say that the problem $(1.1)$ is $B^{s}_{p,q,\gamma}(R^{n}; E)$-separable, if there exists a unique solution

$$u \in B^{2l,s}_{p,q,\gamma}(\Omega; E(A), E)$$

of $(1.1)$ for all $f \in B^{s}_{p,q,\gamma}(R^{n}; E)$ and there exists a positive constant $C$ independent of $f$ such that the coercive estimate holds

$$\sum_{|\alpha|=2l} \|D^{\alpha}u\|_{B^{s}_{p,q,\gamma}(R^{n}; E)} + \|Au\|_{B^{s}_{p,q,\gamma}(R^{n}; E)} \leq C \|f\|_{B^{s}_{p,q,\gamma}(R^{n}; E)}$$  \hspace{1cm} (1.2)$$

The estimate $(1.2)$ implies that if $f \in B^{s}_{p,q,\gamma}(R^{n}; E)$ and $u$ is the solution of the problem $(1.1)$ then all terms of the equation $(1.1)$ belong to $B^{s}_{p,q,\gamma}(R^{n}; E)$ (i.e. all terms are separable in $B^{s}_{p,q,\gamma}(R^{n}; E)$).

The above estimate implies that the inverse of the differential operator generated by $(1.1)$ is bounded from $B^{s}_{p,q,\gamma}(R^{n}; E)$ to $B^{2l,s}_{p,q,\gamma}(\Omega; E(A), E)$.

By using the separability properties of $(1.1)$ we show the maximal regularity properties of the following abstract parabolic Cauchy problem

$$\partial_{t}u + \sum_{|\alpha|=2l} a_{\alpha}D^{\alpha}u + Au = f(t,x),$$  \hspace{1cm} (1.3)$$

$$u(0,x) = 0$$
in weighted Besov spaces.

The paper is organized as follows. In Section 2 the necessary tools from Banach space theory and some background materials are given. In Sections 3-5 the multiplier theorems in vector-valued weighted Lebesgue and Besov spaces are proved. In Sections 6-8 by using these multiplier theorems, embedding
theorems in $E$-valued weighted Besov type spaces are shown. Finally, in Sections 9-14 the separability properties of problems (1.1), (1.3) and their applications are established.

2. Notations and background

Let $E$ be a Banach space and $\gamma = \gamma (x)$, $x = (x_1, x_2, ..., x_n)$ be a positive measurable function on the measurable subset $\Omega \subset \mathbb{R}^n$. Let $L_{p, \gamma} (\Omega; E)$ denote the space of strongly measurable $E$-valued functions that are defined on $\Omega$ with the norm

$$
\| f \|_{L_{p, \gamma} (\Omega; E)} = \left( \int \| f(x) \|_E^p \gamma (x) \, dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,
$$

$$
\| f \|_{L_{\infty, \gamma} (\Omega; E)} = \text{ess sup}_{x \in \Omega} \| f(x) \|_E \gamma (x), \quad p = \infty.
$$

For $\gamma (x) \equiv 1$, the space $L_{p, \gamma} (\Omega; E)$ will be denoted by $L_p (\Omega; E)$.

The weight $\gamma$ is said to satisfy an $A_p$ condition [18], i.e., $\gamma \in A_p$, $1 < p < \infty$ if there is a positive constant $C$ such that

$$
\left( \frac{1}{|Q|} \int_Q \gamma (x) \, dx \right) \left( \frac{1}{|Q|} \int_Q \frac{1}{\gamma^{-\frac{1}{p-1}} (x)} \, dx \right)^{p-1} \leq C,
$$

for all cubes $Q \subset \mathbb{R}^n$.

The Banach space $E$ is called a UMD-space and written as $E \in \text{UMD}$ if only if the Hilbert operator

$$
(H f) (x) = \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} \frac{f(y)}{x-y} \, dy
$$

is bounded in the space $L_p (\mathbb{R}, E)$, $p \in (1, \infty)$ (see e.g. [6]). UMD spaces include e.g. $L_p$, $l_p$ spaces and Lorentz spaces $L_{pq}$, $p, q \in (1, \infty)$.

Let $\mathbb{C}$ be a set of complex numbers and

$$
S_\varphi = \{ \xi; \; \xi \in \mathbb{C}, \; |\arg \xi| \leq \varphi \} \cup \{ 0 \}, \; 0 \leq \varphi < \pi.
$$

Let $E_1$ and $E_2$ be two Banach spaces. $B (E_1, E_2)$ denotes the space of bounded linear operators from $E_1$ to $E_2$. For $E_1 = E_2 = E$ it will denote by $B (E)$.

A linear operator $A$ is said to be positive in a Banach space $E$, with bound $M$ if $D (A)$ is dense on $E$ and

$$
\left\| (A + \xi I)^{-1} \right\|_{B(E)} \leq M \left( 1 + |\xi| \right)^{-1}
$$

with $\xi \in S_\varphi, \varphi \in [0, \pi)$, where $M$ is a positive constant and $I$ is an identity operator in $E$. Sometimes instead of $A + \xi I$ will be written $A + \xi$ and denoted
by $A$. It is known [29, §1.15.1] there exist fractional powers $A^\alpha$ of the positive operator $A$.

**Definition 2.1.** A positive operator $A$ is said to be $R$–positive in the Banach space $E$ if there exists $\varphi \in [0, \pi)$ such that the set

$$\{(\xi) (A + \xi I)^{-1} : \xi \in S_\varphi\}$$

is $R$-bounded (see e.g. [30]).

$\sigma_\infty (E)$ will denote the space of compact operators in $E$.

Let $E (A^\theta)$ denote the space $D (A^\theta)$ with graphical norm defined as

$$\|u\|_{E (A^\theta)} = \left( \|u\|^p + \|A^\theta u\|^p \right)^{\frac{1}{p}}, 1 \leq p < \infty, -\infty < \theta < \infty.$$ 

By $(E_1, E_2)_{\theta, p}$ will be denoted an interpolation space obtained from $\{E_1, E_2\}$ by the $K$–method [29, §1.3.1], where $\theta \in (0, 1), p \in [0, 1]$. We denote by $D (R^n; E)$ the space of $E$–valued $C^\infty$–function with compact support, equipped with the usual inductive limit topology and $S (E) = S (R^n; E)$ denote the $E$–valued Schwartz space of rapidly decreasing, smooth functions. For $E = \mathbb{C}$ we simply write $D (R^n)$ and $S = S (R^n)$, respectively. $D' (R^n; E) = L (D (R^n), E)$ denote the space of $E$–valued distributions and $S' (E) = S' (R^n; E)$ is a space of linear continued mapping from $S (R^n)$ into $E$. The Fourier transform for \( u \in S' (R^n; E) \) is defined by

$$F (u) (\varphi) = u (F (\varphi)), \varphi \in S (R^n).$$

Let $\gamma$ be such that $S (R^n; E_1)$ is dense in $L_{p, \gamma} (R^n; E_1)$. A function

$$\Psi \in C^{(i)} (R^n; B (E_1, E_2))$$

is called a multiplier from $L_{p, \gamma} (R^n; E_1)$ to $L_{q, \gamma} (R^n; E_2)$ if there exists a positive constant $C$ such that

$$\|F^{-1} \Psi (\xi) Fu\|_{L_{q, \gamma} (R^n; E_2)} \leq C \|u\|_{L_{p, \gamma} (R^n; E_1)}$$

for all $u \in S (R^n; E_1)$.

In a similar we can define the multiplier from $B_{p, q, \gamma}^* (R^n; E_1)$ to $B_{p, q, \gamma}^* (R^n; E_2)$.

We denote the set of all multipliers from $L_{p, \gamma} (R^n; E_1)$ to $L_{q, \gamma} (R^n; E_2)$ by $M_{p, q, \gamma}^* (E_1, E_2)$. For $E_1 = E_2 = E$ we denote the $M_{p, q, \gamma}^* (E_1, E_2)$ by $M_{p, q, \gamma}^* (E)$.

**Definition 2.2.** Let $\gamma$ be a positive measurable function on $R^n$. Assume $E$ is a Banach space and $p \in [1, 2]$. Suppose there exists a positive constant $C_0 = C_0 (p, \gamma, E)$ so that

$$\|Fu\|_{L_{p', \gamma} (R^n; E)} \leq C_0 \|Fu\|_{L_{p, \gamma} (R^n; E)}$$

(2.1)

for $\frac{1}{p} + \frac{1}{p'} = 1$ and each $u \in S (R^n; E)$. Then $E$ is called weighted Fourier type $\gamma$ and $p$. It is called Fourier type $p \in [1, 2]$ if $\gamma (x) \equiv 1$. 


Remark 2.1. The estimate \((2.1)\) shows that each Banach space \(E\) has weighted Fourier type \(\gamma\) and 1. By Bourgain [6] has shown that each \(B\)–convex Banach space (thus, in particular, each uniformly convex Banach space) has some non-trivial Fourier type \(p \in [1, 2]\), i.e. \(UMD\) spaces are Fourier type for some \(p \in [1, 2]\).

In order to define abstract Besov spaces we consider the dyadic-like subsets \(\{J_k\}_{k=0}^{\infty}\), \(\{I_k\}_{k=0}^{\infty}\) of \(R^n\) and partition of unity \(\{\varphi_k\}_{k=0}^{\infty}\) defined e.g. in [19].

Remark 2.2. Note the following useful properties are satisfied:

\[
supp \varphi_k \subset \tilde{I}_k \text{ for each } k \in N_0; \sum_{k=0}^{\infty} \varphi_k (s) = 1 \text{ for each } s \in R^n; I_m \cap supp \varphi_k = \emptyset \text{ if } |m - k| > 1; \varphi_{k-1} (s) + \varphi_k (s) + \varphi_{k+1} (s) = 1 \text{ for each } s \in supp \varphi_k \text{ and } k \in N_0.
\]

Among the many equivalent descriptions of Besov spaces, the most useful one for us is given in terms of the so-called Littlewood-Paley decomposition. This means that we consider \(f \in S' (E)\) as a distributional sum \(f = \sum_k f_k\) analytic functions \(f_k\) whose Fourier transforms have support in dyadic-like \(I_k\) and then define the Besov norm in terms of the \(f_k\)'s.

Definition 2.3. Let \(\gamma \in A_q\), \(1 \leq r, q \leq \infty\) and \(s \in \mathbb{R}\). The Besov space \(B^{s}_{q,r,\gamma} (R^n; E)\) is the space of all \(f \in S' (R^n; E)\) for which

\[
\|f\|_{B^{s}_{q,r,\gamma} (R^n; E)} = \left\{ \begin{array}{ll}
\left[ \sum_{k=0}^{\infty} 2^{ksr} \|\hat{\varphi}_k \ast f\|^r_{L^q \gamma (R^n; E)} \right]^{\frac{1}{r}}, & \text{if } 1 \leq r < \infty \\
\sup_{k \in N_0} \left[ \sum_{k=0}^{\infty} 2^{ks} \|\hat{\varphi}_k \ast f\|^r_{L^q \gamma (R^n; E)} \right], & \text{if } r = \infty
\end{array} \right.
\]

is finite. \(B^{s}_{q,r,\gamma} (R^n; E)\)–together with the norm in \((2.1)\), is a Banach space. \(B^{s}_{q,r,\gamma} (R^n; E)\) is the closure of \(S (R^n; E)\) in \(B^{s}_{q,r,\gamma} (R^n; E)\) with the induced norm. In a similar way as in [19, Lemma 3.2] it can be shown that different choices of \(\{\varphi_k\}\) lead to equivalent norms on \(B^{s}_{q,r,\gamma} (R^n; E)\).

Let \(\Omega\) be a domain in \(R^n\); \(B^{s}_{q,r,\gamma} (\Omega; E)\) denotes the space of restrictions to \(\Omega\) of all functions in \(B^{s}_{q,r,\gamma} (R^n; E)\) with the norm given by

\[
\|u\|_{B^{s}_{q,r,\gamma} (\Omega; E)} = \inf_{g \in B^{s}_{q,r,\gamma} (R^n; E)} \|g\|_{B^{s}_{q,r,\gamma} (R^n; E)},
\]

Let \(l = (l_1, l_2, ..., l_n)\), \(s \in \mathbb{R}\) and \(1 \leq q, r \leq \infty\). Here, \(B^{1,s}_{q,r,\gamma} (\Omega; E)\) denote a \(E\)-valued Sobolev-Besov weighted space of functions \(u \in B^{s}_{q,r,\gamma} (\Omega; E)\) that have generalized derivatives \(D^{l_k} u = \frac{\partial^{l_k}}{\partial x_k^{l_k}} u \in B^{s}_{q,r,\gamma} (\Omega; E)\), \(k = 1, 2, ..., n\) with
Let \( E_0 \) is continuously and densely belongs to \( E \). \( B_{q,\theta,\gamma}^l (\Omega; E) \) denotes the space \( B_{s,q,\theta,\gamma}^l (\Omega; E_0) \cap B_{q,\theta,\gamma}^l (\Omega; E) \) with the norm

\[
\|u\|_{B_{q,\theta,\gamma}^l (\Omega; E)} = \|u\|_{B_{q,\theta,\gamma}^l (\Omega; E_0)} + \sum_{k=1}^n \left\| D_k^l u \right\|_{B_{q,\theta,\gamma}^l (\Omega; E)} < \infty.
\]

Let \( (E(X); E^*(X^*)) \) be one of the pairs

\[
(L_{q,\gamma} (X), L_{q',\gamma'} (X^*)), \quad \left( B_{q,r,\gamma}^s (X), B_{q',r',\gamma'}^{-s} (X^*) \right),
\]

when \( 1 \leq q, r \leq \infty \), where

\[
\gamma' (.) = \frac{1}{\gamma - 1} (.)
\]

There is an embedding of \( E^*(X^*) \subset [E(X)]^* \) as a norming subspace for \( E(X) \). This embedding is given by the duality map

\[
\langle \cdot, \cdot \rangle_{E(X)} : E^*(X^*) \times E(X) \to \mathbb{C},
\]

where

\[
\langle g, f \rangle_{L_{q,\gamma} (X)} = \int_{\mathbb{R}^n} \langle g (t), f (t) \rangle_X dt = \int_{\mathbb{R}^n} g (t) f (t) dt
\]

in weighted Lebesgue space setting with \( E = L_{q,\gamma} \) and

\[
\langle g, f \rangle_{B_{q,r,\gamma}^s (X)} = \sum_{n,m \in \mathbb{N}_0} \langle \hat{\varphi}_n \ast g, \hat{\varphi}_m \ast f \rangle_{L_{q,\gamma} (X)}
\]

in Besov space setting with \( E = B_{q,r,\gamma}^s (X) \). One can check that this definition of duality is independent of the choice of the \( \{\varphi_k\}_{k=0}^\infty \).

### 3. The Fourier transform in weighted Besov spaces

By applying the Hausdor-Young inequality we get the following estimates for the Fourier transform on Besov spaces

**Theorem 3.1.** Assume \( \gamma \in A_\nu \) for \( \nu \in [1, \infty] \). Let \( E \) be a Banach space with weighted Fourier type \( \gamma \) and \( p \in [1, 2] \). Let \( 1 \leq q \leq p' \) and \( s \geq n \left( \frac{1}{q} - \frac{1}{p'} \right) \) and \( 1 \leq r \leq \infty \). Then there exists constant \( C \), depending only on \( C_0 (p, \gamma, E) \) so that if \( f \in B_{q,r,\gamma}^s (\mathbb{R}^n; E) \) then

\[
\left\| \left\{ \hat{f} \chi_{J_m} \right\}_{m=0}^\infty \right\|_{l_r (L_{q,\gamma}(\mathbb{R}^n; E))} \leq C \|f\|_{B_{q,r,\gamma}^s (\mathbb{R}^n; E)},
\]

(3.1)
where $C_0(p, \gamma, E)$ is a positive constant defined in the Definition 2.1.

An immediate corollary of Theorem 3.1 follows by choosing for $q = r = 1$ and $r = q = p'$ we obtain respectively

**Corollary 3.1.** Assume $\gamma \in A_q$ for $q \in [1, \infty]$. Let $E$ be a Banach space with Fourier type $p \in [1, 2]$. Then the Fourier transform $F$ defines the following bounded operators

\[
F : B_{p,1,\gamma}^\infty \left( \mathbb{R}^n; E \right) \to L_{1,\gamma} \left( \mathbb{R}^n; E \right) \tag{3.2}
\]

\[
F : B_{p,p',\gamma}^0 \left( \mathbb{R}^n; E \right) \to L_{p',\gamma} \left( \mathbb{R}^n; E \right). \tag{3.3}
\]

The norms of the above maps $F$ are bounded above by a constant depending only on $C_0(n,E)$.

Theorem 3.1 and Corollary 3.2 remain valid if $F$ is replaced with $F^{-1}$.

**Proof of Theorem 3.1.** Let $f \in B_{q,r,\gamma}^s \left( \mathbb{R}^n; E \right)$. Then, for each $k \in \mathbb{N}_0$, since $\hat{\varphi}_k * f \in L_{p,\gamma} \left( \mathbb{R}^n; E \right)$ and $E$ has weighted Fourier type $\gamma$ and $p$, we get

$$
\varphi_k \hat{f} = F (\hat{\varphi}_k * f) \in L_{p',\gamma} \left( \mathbb{R}^n; E \right).
$$

Thus by Remark 2.2,

$$
\hat{f} \chi_{J_m} = \left( \sum_{k=m-1}^{m+1} \varphi_k \hat{f} \right) \chi_{J_m} \in L_{q,\gamma} \left( \mathbb{R}^n; E \right) \text{ for each } m \in \mathbb{N}_0.
$$

Moreover, by Definition 2.2 we get

$$
\left\| \varphi_k \hat{f} \right\|_{L_{p',\gamma} \left( \mathbb{R}^n; E \right)} = \left\| F (\hat{\varphi}_k * f) \right\|_{L_{p',\gamma} \left( \mathbb{R}^n; E \right)} \leq C_0 \left\| \hat{\varphi}_k \hat{f} \right\|_{L_{p,\gamma} \left( \mathbb{R}^n; E \right)},
$$

i.e.

$$
\sum_{k=m-1}^{m+1} 2^{ks} \left\| \varphi_k \hat{f} \right\|_{L_{p',\gamma} \left( \mathbb{R}^n; E \right)} \leq C_0 \sum_{k=m-1}^{m+1} 2^{ks} \left\| \hat{\varphi}_k \hat{f} \right\|_{L_{p,\gamma} \left( \mathbb{R}^n; E \right)} \leq C_0 \left\| \hat{f} \right\|_{B_{q,r,\gamma}^s \left( \mathbb{R}^n; E \right)} \tag{3.4}
$$

\[CC_0 \left( p, \gamma, E \right) \left\| f \right\|_{B_{q,r,\gamma}^s \left( \mathbb{R}^n; E \right)}.
\]

In view of (3.4), it suffices to show that there exists the pozitive constant $C_1$ so that the following holds

$$
\left\| \hat{f} \chi_{J_m} \right\|_{L_{q,\gamma} \left( \mathbb{R}^n; E \right)} \leq C_1 \sum_{k=m-1}^{m+1} 2^{ks} \left\| \varphi_k \hat{f} \right\|_{L_{p',\gamma} \left( \mathbb{R}^n; E \right)} \tag{3.5}
$$
Firstly, consider the case where $q \neq p'$. Choose $1 \leq \sigma < p$ that $\frac{1}{q} = \frac{1}{p'} + \frac{1}{\sigma}$; so, $\frac{1}{\sigma} \leq s$. By the generalized Hölder’s inequality for each $m \in \mathbb{N}_0$,

$$
\left\| \hat{f} \chi_{J_m} \right\|_{L_{q, \gamma}(\mathbb{R}^n; E)} \leq \sum_{k=m-1}^{m+1} \left\| \varphi_k \hat{f} \chi_{J_m} \right\|_{L_{q, \gamma}(J_m; E)} \leq \left( \sum_{k=m-1}^{m+1} \left\| \varphi_k \right\|_{L_{q'}(J_m)} \right) \left\| \hat{f} \chi_{J_m} \right\|_{L_{q, \gamma}(J_m; E)} \leq C_2 \sum_{k=m-1}^{m+1} 2^{ks} \left\| \hat{f} \varphi_k \right\|_{L_{q', \gamma}(J_m; E)},
$$

where $C_2$ is a positive constant defined by

$$
C_2 = \left\| \hat{f} \chi_{J_m} \right\|_{L_{q, \gamma}(J_m; E)} \leq \left( \sum_{k=m-1}^{m+1} \left\| \varphi_k \right\|_{L_{q'}(J_m)} \right) \left\| \hat{f} \chi_{J_m} \right\|_{L_{q, \gamma}(J_m; E)} \leq C_2 \sum_{k=m-1}^{m+1} 2^{ks} \left\| \hat{f} \varphi_k \right\|_{L_{q', \gamma}(J_m; E)},
$$

(3.6)

$$
\sup_{m \in \mathbb{N}_0} \left\| \hat{f} \chi_{J_m} \right\|_{L_{q, \gamma}(\mathbb{R}^n; E)} \leq \left( \sum_{k=m-1}^{m+1} \left\| \varphi_k \right\|_{L_{q'}(J_m)} \right) \left\| \hat{f} \chi_{J_m} \right\|_{L_{q, \gamma}(J_m; E)} \leq C_2 \sum_{k=m-1}^{m+1} 2^{ks} \left\| \hat{f} \varphi_k \right\|_{L_{q', \gamma}(J_m; E)}.
$$

(3.7)

Since $\gamma \in A_\nu$ we have

$$
\sup_{m \in \mathbb{N}_0} \left\| \hat{f} \chi_{J_m} \right\|_{L_{q, \gamma}(\mathbb{R}^n; E)} \leq \left( \sum_{k=m-1}^{m+1} \left\| \varphi_k \right\|_{L_{q'}(J_m)} \right) \left\| \hat{f} \chi_{J_m} \right\|_{L_{q, \gamma}(J_m; E)} \leq C_2 \sum_{k=m-1}^{m+1} 2^{ks} \left\| \hat{f} \varphi_k \right\|_{L_{q', \gamma}(\mathbb{R}^n; E)}.
$$

So, from (3.6)-(3.8) we obtain (3.5).

**Remark 3.1.** By using the embedding $W_{p, \gamma}^j(\mathbb{R}^n; E) \subset B_{q, r, \gamma}^s(\mathbb{R}^n; E)$ for $s < j \in \mathbb{N}$ we get that the statement of Theorem 3.1 remains valid if $B_{q, r, \gamma}^s(\mathbb{R}^n; E)$ is replaced by $W_{p, \gamma}^j(\mathbb{R}^n; E)$.

Also, it follows from Corollary 3.2 that if $E$ has weighted Fourier type for $\gamma \in A_\nu$, $p \in [1; 2]$ and $j > \frac{q}{p}$ then the Fourier transform $F$ defines bounded operator:

$$
W_{p, \gamma}^j(\mathbb{R}^n; E) \rightarrow L_{1, \gamma}(\mathbb{R}^n; E).
$$
Furthermore, if $E$ has weighted Fourier type for $\gamma \in A_\nu$, $p \in [1, 2]$ and $j > \frac{n}{p}$ then there is a constant $C$ so that

$$\|f^\wedge\|_{L^1(R^n;E)} \leq C \|f\|_{L^p(R^n;E)} \left(\sum_{|\alpha| = j} \|D^\alpha f\|_{L^p(R^n;E)}\right)^{\frac{n}{jp}}$$

(3.9)

for each $f \in W^j_{p,\gamma}(R^n;E)$.

### 4. Fourier multipliers on weighted Lebesque spaces

Consider the bounded measurable function $m: R^n \rightarrow B(E_1, E_2)$. In this section, we identify conditions on $m$, generalizing the classical Mihlin condition so that the multiplication operator induced by $m$, i.e. the operator: $u \mapsto T_m = F^{-1} mFu$ is bounded from $L^q,\gamma(R^n; E_1)$ to $L^q,\gamma(R^n; E_2)$. We will rst give rather general criteria for Fourier multipliers in terms of the weighted Besov norm of the multiplier function; later we derive from these results analogues of the classical Mihlin and Hörmander conditions. To simplify the statements of our results, we let

$$M_{p,\gamma}(m) = \inf_{a > 0} \left\{ \|m(a,\cdot)\|_{L^p(R^n;B(E_1;E_2))}^{\frac{1}{p}} \right\}.$$  

Let

$$X_k = L^q,\gamma(E_k) = L^q,\gamma(R^n; E_k), \quad k = 1, 2, \quad Y = B_{p,1,\gamma}^\#(R^n; B(E_1, E_2)).$$

First we give a multiplier result from $X_1$ to $X_2$ in the spirit of Steklin’s theorem.

**Theorem 4.1.** Assume $\gamma \in A_\nu$ for $\nu \in [1, \infty]$. Let $E_1, E_2$ be a Banach spaces with weighted Fourier type $\gamma$ and $p \in [1, 2]$. Then there is a constant $C$, depending only on $C_{01}(p, \gamma, E_1)$ and $C_{02}(p, \gamma, E_2)$, so that if $m \in Y$, then $m$ is a Fourier multiplier from $X_1$ to $X_2$ and

$$\|T_m\|_{B(X_1, X_2)} \leq CM_{p,\gamma}(m)$$

for each $q \in [1, \infty]$.

Let $E^*$ denotes the dual space of $E$ and $A^*$—denotes the conjugate of the operator $A$.

The proof of Theorem 4.3 uses the following lemma.

**Lemma 4.1.** Assume $\gamma \in A_q$ for $q \in [1, \infty]$ and $k \in L^1(R^n; B(E_1, E_2))$. Suppose that there exists constants $C_i$ so that for each $x \in E_1$ and $x^* \in E_2^*$

$$\int_{R^n} \|k(s) x\|_{E_2} ds \leq M_0 \|x\|_{E_1}, \quad \int_{R^n} \|k^*(s) x^*\|_{E_2^*} ds \leq M_1 \|x^*\|_{E_2^*}.$$  

(4.1)
Then the convolution operator $K : X_1 \to X_2$ defined by

$$(Kf)(t) = \int_{\mathbb{R}^n} k(t-s) f(s) \, ds \text{ for } t \in \mathbb{R}^n \quad (4.2)$$

satisfies that

$$\|K\|_{B(X_1, X_2)} \leq M_0^{\frac{1}{n}} M_1^{1 - \frac{1}{n}}.$$

**Proof.** Since $k \in L_1(\mathbb{R}^n; B(E_1, E_2))$ it is well-known that (4.2) defines a bounded operator on $X_1$. Indeed, for $f \in X_1 \cap L_\infty(\mathbb{R}^n; E_1)$ we have

$$\int_{\mathbb{R}^n} \|k(t-s) f(s)\|_{E_2} \, ds = \int_{\mathbb{R}^n} \|k(s) f_s(t)\|_{E_2} \, ds \leq \|k\|_{L_1(\mathbb{R}^n; B(E_1, E_2))} \|f\|_{L_\infty(\mathbb{R}^n; E_1)}$$

for each $t \in \mathbb{R}^n$ and $f_s(t) = f(t-s)$. From (4.3) by applying the Minkowski’s inequality for integral with weight [20, § A.1] we get

$$\|Kf(.)\|_{X_2} \leq \int_{\mathbb{R}^n} \|k(s) f_s(t)\|_{X_2} \, ds \leq \int_{\mathbb{R}^n} \|k(s)\|_{B(E_1, E_2)} \|f_s\|_{X_1} \, ds =

\|k\|_{L_1(\mathbb{R}^n; B(E_1, E_2))} \|f_s\|_{X_1}.$$

Now, for $q = 1$ we have from (4.1)

$$\|Kf\|_{L_1, \gamma(\mathbb{R}^n; E_1)} \leq \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \|k(s) f_s(t)\|_{E_1} \, ds \right) \gamma(t) \, dt \leq

M_0 \int_{\mathbb{R}^n} \|f(t)\|_{E_1} \gamma(t) \, dt = M_0 \|f\|_{L_1, \gamma(\mathbb{R}^n; E_1)}.$$

Hence,

$$\|K\|_{B(L_1, \gamma(\mathbb{R}^n; E_1))} \leq M_0. \quad (4.4)$$

If $q = \infty$, then for each $L_\infty, \gamma(\mathbb{R}^n; E)$, $x^* \in E_2^*$ and $t \in \mathbb{R}^n$ by using (4.1) we get

$$\| (x^*, (Kf)(t))_{E_2} \| \leq \int_{\mathbb{R}^n} \| k^* (t-s) x^* f(s) \|_{E_1} \gamma(s) \, ds \leq

\int_{\mathbb{R}^n} \|k^*(t-s) x^*\|_{E_1^*} \|f(s)\|_{E_1} \gamma(s) \, ds \leq M_1 \|x^*\|_{E_1^*} \|f\|_{L_\infty, \gamma(\mathbb{R}^n; E)}.$$

Thus,

$$\|K\|_{B(L_\infty, \gamma(\mathbb{R}^n; E_1))} \leq M_1. \quad (4.5)$$
Let $L_{\infty, \gamma} (R^n; E_1)$ denotes the closure in $L_{\infty, \gamma} (R^n; E_1)$ norm of the simple functions $\sum_{k=1}^{m} x_k \chi_{A_k}$, where $x_k \in E_1$, $\text{vol} \ A_k < \infty$ and $m \in \mathbb{N}$. Then one can check that $K$ maps $L_{\infty, \gamma} (R^n; E_1)$ into $L_{\infty, \gamma} (R^n; E_2)$. Indeed, for $f = \chi_A$, we have

$$Kf(t) = \int k(s) x ds \to 0 \text{ for } t \to \infty$$

and $Kf$ is a continuous function from $R^n$ to $E_2$. Now, the Riesz-Thorin theorem (cf. [5, Thm 5.1.2]) yields the claim for $1 < p < \infty$.

**Proof of Theorem 4.1.** First assume in addition that $m \in S (B (E_1, E_2))$. Hence, $\hat{m} \in S (B (E_1, E_2))$. Fix $x \in E_1$. For an appropriate choice of $a > 0$, we can apply Corollary 3.1 to the function $t \to m(\alpha t)$ in $B^\frac{n}{p+1, \gamma} (R^n; E_2)$ and use that

$$F^{-1} [m(\alpha t)] (s) = a^{-n} \hat{m} \left(\frac{s}{\alpha}\right) x$$

to get

$$\|\hat{m} (\cdot) x\|_{L_{1, \gamma} (R^n; E_1)} = \|F^{-1} m(\alpha t) x\|_{L_{1, \gamma} (R^n; E_1)} \leq C_1 \|m(\alpha t) x\|_{B^\frac{n}{p+1, \gamma} (R^n; B(E_1, E_2))} \|x\|_{E_1} \leq 2C_1 M_{p, \gamma} \|x\|_{E_1},$$

for some constant $C_1$ which depends on $C_0 (p, \gamma, E_2)$.

By the additional assumption on $m$ we get

$$m^*(\cdot) \in S (B (E_2^*, E_1^*)^\circ), \text{ and } F^{-1} m^*(\cdot) = [\hat{m} (\cdot)]^* \in S (B (E_2^*, E_1^*)^\circ).$$

Let $x^* \in E_2^*$. Similarly, by applying Corollary 3.1 to an appropriate function

$$t \to [m(\alpha t)]^* x^* \text{ in } B^\frac{n}{p+1, \gamma} (R^n; E_1^*)$$

and using the fact that $M_{p, \gamma} (m) = M_{p, \gamma} (m^*)$, one has

$$\| [\hat{m} (\cdot)]^* x^* \|_{L_{1, \gamma} (R^n; E_1^*)} \leq 2C_2 M_{p, \gamma} (m) \|x^*\|_{E_2^*}$$

for some constant $C_2$ which depends $C_0 (p, \gamma, E_1^*)$. By Lemma 4.1, the convolution operator

$$(T_m f)(t) = \int \hat{m} (t-s) f(s) ds$$

satisfies

$$\|T_m\|_{B(X_1, X_2^*)} \leq CM_{p, \gamma} (m),$$

where $C = 2 \max \{C_1, C_2\}$. Furthermore, since $m \in L_1 (R^n; B(E_1, E_2))$, then $T_m$ satisfies the following

$$T_m f = F^{-1} m (\cdot) f (\cdot) \text{ for all } f \in S (R^n; E_1^*), \quad (4.6)$$

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also
\[ T_m \in C(\sigma(X_1, X_1^*), \sigma(X_2, X_2^*)) , \]
where \( \sigma(X_k, X_k^*) \) denote the interpolation spaces of \( X_k, X_k^* \).

For the general case, let \( m \in Y \). It is known that \( S(R^n; B(E_1, E_2)) \) is dense in \( Y \) when \( \gamma \in A_\nu, \nu \in [1, \infty] \). Now, let we choose a sequence \( \{m_n\}_n \subset S(R^n; B(E_1, E_2)) \) that converges to \( m \) in the \( Y \)-norm and obtain operators \( T_{m_n} \in B(X_1, X_2) \), where
\[ T_{m_n}f = F^{-1}m_n(\cdot) f(\cdot), \ f \in X_1. \]

It is clear to see that, the properties (4.6) and (4.7) pass from \( T_{m_n} \) to \( T_m \). One also has that
\[ \|T_m\|_{B(X_1, X_2)} \leq C \|m\|_{Y}. \]

Fix \( a > 0 \) such that \( m(a.) \in Y \). Then \( I_{E_2} \circ T_{m(a.)} = T_m \circ I_{E_1} \), where
\[ I_{Z} : L_{q,\gamma}(R^n; Z) \rightarrow L_{q,\gamma}(R^n; Z) \]
is the isometry
\[ T(f)(t) = a^{\frac{n}{\gamma}}f(at). \]
Thus,
\[ \|T_m\|_{B(X_1, X_2)} = \|T_{m(a.)}\|_{B(X_1, X_2)} \leq C \|m\|_{Y}, \]
i.e.
\[ \|T_m\|_{B(X_1, X_2)} \leq CM_{p,\gamma}(m). \]

The following remark collects some basic facts about the Fourier multiplier operators \( T_m \) given in Theorem 4.1 that will be used in the proof of Theorem 4.2.

**Remark 4.1.** Let \( f \in X_1 \) and \( \Omega \) be a closed subset of \( R^n \). Then the following are valid:
(a) Viewing \( f \) and \( T_m f \) as distributions, if \( \text{supp} \ \hat{f} \subset \Omega \) then \( \text{supp} \ F(T_m f) \subset \Omega \);
(b) \( T_{m_1+m_2} = T_{m_1} + T_{m_2} \). If \( \varphi \in S \), then \( \varphi * T_m f = T_m (\varphi * f) = T_{\varphi_m}(f) \);
(c) If \( \varphi \in S \) is 1 on \( \text{supp} \ \hat{f} \), then \( T_{\varphi_m}(f) = T_m(f) \);
(d) \( T_{m}^* \) restricted to \( L_{q,\gamma}(R^n; E_2^*) \) is \( T_{m^*(\cdot)} \).

5. Fourier multipliers on weighted Besov spaces

Consider the bounded measurable function \( m : R^n \rightarrow B(E_1, E_2) \). In this section we identify conditions on \( m \), generalizing the classical Mikhlin condition so that the multiplication operator induced by \( m \), i.e. the operator: \( u \rightarrow T_m = F^{-1}mFu \) is bounded from \( B^s_{p,q,\gamma}(R^n; E_1) \) to \( B^s_{p,q,\gamma}(R^n; E_2) \).

By applying this Theorem 4.1 to the blocks of the Littlewood Paley decomposition of Besov spaces we will now get the main result of this section. Let
\[ Y_i = B^s_{q,r,\gamma}(R^n; E_i), \ i = 1, 2. \]
Theorem 5.1. Assume $\gamma \in A_\nu$ for $\nu \in [1, \infty]$. Let $E_1, E_2$ be a Banach spaces with weighted Fourier type $\gamma$ and $p \in [1, 2]$. Then there is a constant $C$ depending only on $C_{01}(p, \gamma, E_1)$ and $C_{02}(p, \gamma, E_2)$, so that if

$$\varphi_k m \in Y \text{ and } M_{p, \gamma} (\varphi_k m) \leq A \text{ for each } k \in \mathbb{N}_0 \quad (5.1)$$

then $m$ is a Fourier multiplier from $Y_1$ to $Y_2$ and

$$\|T_m\|_{B(Y_1, Y_2)} \leq CA$$

for each $s \in \mathbb{R}$ and $q, r \in [1, \infty]$.

Proof. By definition partition of unity $\{\varphi_k\}_{k=0}^\infty$ we have

$$T_m f = F^{-1} m \hat{f} = \sum_{k \in \mathbb{N}_0} F^{-1} \left[ (\varphi_{k-1} + \varphi_k + \varphi_{k+1}) m F \left[ (\varphi_k \ast f) \right] \right] =$$

$$\sum_{k \in \mathbb{N}_0} T_{(\varphi_{k-1} + \varphi_k + \varphi_{k+1})} m F \left[ (\varphi_k \ast f) \right] , \quad (5.2)$$

where $T_m$ is the Fourier multiplier operator on $X_1$ given by Theorem 4.1. Theorem 4.1 gives that $m \varphi_k$ induces a Fourier multiplier operator $T_{m \varphi_k}$ with

$$\|T_{m \varphi_k}\|_{B(X_1, X_2)} \leq CM_{p, \gamma} (\varphi_k m) \leq CA$$

for some constant $C$ depending only on $C_{0,1}(p, \gamma, E_1)$ and $C_{0,2}(p, \gamma, E_2)$. Let

$$\psi_k = \varphi_{k-1} + \varphi_k + \varphi_{k+1}.$$

Note that $\psi_k(s) \equiv 1$ when $s \in \text{supp } \varphi_k$. Then $m \psi_k$ induces the Fourier multiplier operator $T_{m \psi_k}$ with

$$T_{m \psi_k} = T_{m \varphi_{k-1}} + T_{m \varphi_k} + T_{m \varphi_{k+1}} \in B(X_1, X_2)$$

and

$$\|T_{m \psi_k}\|_{B(X_1, X_2)} \leq 3CA.$$

Define $T_0 : S(E_1) \to S'(E_1)$ by

$$T_0 f = F^{-1} m(.) F f(.) .$$

If $f \in S(E_1)$, then

$$\hat{\psi}_k \ast T_0 f = T_{m \psi_k} (\hat{\varphi}_k \ast f)$$

for each $k \in \mathbb{N}_0$ since

$$F \left[ T_{m \psi_k} (\hat{\varphi}_k \ast f) \right] (.) = m(.) \psi_k(.) F \left[ (\varphi_k \ast f) (.) \right] =$$

$$\varphi_k(.) m(.) \hat{f}(.) = \varphi_k(.) F (T_0 f) = F \left[ (\hat{\varphi}_k \ast T_0 f) (.) \right] .$$

So, by the definition of the Besov norm

$$\|T_0 f\|_{Y_2} \leq 3CA \|T_0 f\|_{Y_1} .$$
Thus $T_0$ extends to a bounded linear operator from $\hat{B}_{q,r,\gamma}^s (\mathbb{R}^n; E_1)$ to $\hat{B}_{q,r,\gamma}^s (\mathbb{R}^n; E_2)$.

If $q, r < \infty$ then $\hat{B}_{q,r,\gamma}^s (\mathbb{R}^n; E) = B_{q,r,\gamma}^s (\mathbb{R}^n; E)$ and so all that would remain is to verify the weak continuity condition (4.7). However, we continue with the proof in order to also cover the case $q = \infty$ or $r = \infty$. We shall show that the operator $T_m : Y_1 \to Y_2$ defined by

$$T_m f = \sum_{k=1}^{\infty} f_k, \quad f_k = T_{m\psi_k} (\hat{\varphi}_k \ast f) \in X_2$$

is indeed a (norm) continuous operator. Fix $f \in Y_1$. First we show that the formal series (5.3) defines an element in $S'(E_2)$. Towards this, fix $\varphi \in S$. Remark 4.1 gives that supp $f_k \subset \bar{I}_k$. Thus

$$f_k (\varphi) = \hat{f}_k (\hat{\varphi}) = \hat{f}_k (\psi_k (-.) \hat{\varphi}) = f_k (\psi_k \ast \varphi)$$

and so by using Hölder inequality with weight $\gamma \in A_q$ as in (3.7) we get

$$\sum_{k=1}^{\infty} \| f_k (\varphi) \|_{E_2} \leq \sum_{k=1}^{\infty} \| f_k \|_{X_2} \left\| \gamma^{-\frac{1}{q}} (\psi_k \ast \varphi) \right\|_{L_{q'} (\mathbb{C})} \leq M \sum_{k=1}^{\infty} 2^{k s} \| \hat{\varphi}_k \ast f \|_{X_2} \| 2^{-k s} \psi_k \ast \varphi \|_{L_{q'} (\mathbb{C})} \leq M 2^{s} \| f \|_{Y_2} \| \varphi \|_{\hat{B}_{q',r',\sigma}^n (\mathbb{C})},$$

where

$$\sigma (.) = \gamma^{1-q} (.)$$

Thus $(T_m f) (\varphi)$ for $\varphi \in S$ defines a linear map from $S$ into $E_2$ which is continuous by well known inclusion

$$S (E_2) \subset Y_2 \subset S' (E_2).$$

By Remark 4.1, for each $j, k \in \mathbb{N}_0$

$$\hat{\varphi}_j \ast T_{m\psi_k} (\hat{\varphi}_k \ast f) = T_{m\psi_k} (\hat{\varphi}_j \ast \hat{\varphi}_k \ast f) = \hat{\varphi}_k \ast T_{m\psi_k} (\hat{\varphi}_j \ast f).$$

Thus, since the support of $\varphi_k$ intersects the support of $\varphi_j$ only for $|k - j| \leq 1$, applying Remark 4.1 further gives

$$\hat{\varphi}_k \ast T_m f = \sum_{j=k-1}^{k+1} \hat{\varphi}_j \ast T_{m\psi_j} (\hat{\varphi}_j \ast f) = \sum_{j=k-1}^{k+1} \hat{\varphi}_j \ast T_{m\psi_j} (\hat{\varphi}_k \ast f) = (5.4)$$

$$\sum_{j=k-1}^{k+1} T_{m\varphi_j \psi_j} (\hat{\varphi}_k \ast f) = T_{m\psi_k} (\hat{\varphi}_k \ast f).$$
Hence, $\hat{\varphi}_k * T_m f \in X_2$ and
\[
\| \hat{\varphi}_k * T_m f \|_{X_2} \leq 3CA \| \hat{\varphi}_k * f \|_{X_1},
\]
from which and in view of (5.2) it follows that range of $T_m$ is contained in $Y_1$ and that norm of $T_m$ as an operator from $Y_1$ to $Y_2$ is bounded by a constant depending on the items claimed. Furthermore, $T_m$ extends $T_0$; indeed, if $f \in S(E_1)$ then
\[
F(T_m f) = \sum_{k=1}^{\infty} F[T_m \psi_k (\hat{\varphi}_k * f)] = \sum_{k=1}^{\infty} m\psi_k \hat{\varphi}_k \hat{f} = \sum_{k=1}^{\infty} m\psi_k \hat{f} = F(T_0 f).
\]

It remains to show only that $T_m$ satisfies (4.7). Since $[m(-)]^* : R^n \to B(E_2^*; E_2^*)$ also satisfies condition (5.1), the Fourier multiplier operator $T_m^*(-)$, defined by (4.6), extends to $T_m^*(-) \in B(E^*(E_2), E^*(E_1))$, for $E = B_{q,r,\gamma}$. It suffices to show that $T_m^*$ restricted to $E^*(E_2)$ is $T_m^*(-)$. Hence, fix $g \in E^*(E_2)$, $f \in B_{q,r,\gamma}^*(E_1)$ and by using (5.4) and (2.3) we have
\[
\langle T_m g, f \rangle_{Y_1} = \sum_{n,k \in N_0} \langle \hat{\varphi}_n * g, T_m f \rangle_{L_{q,\gamma}(E_2)} = \sum_{n,k \in N_0} \langle \hat{\varphi}_n * g, T_m \psi_k (\hat{\varphi}_k * f) \rangle_{L_{q,\gamma}(E_2)}. \tag{5.5}
\]
and
\[
\langle T_m^*(-) g, f \rangle_{Y_1} = \sum_{n,k \in N_0} \langle \hat{\varphi}_n * T_m^*(-) g, \hat{\varphi}_k * f \rangle_{L_{q,\gamma}(E_1)} = \sum_{n,k \in N_0} \langle T_m^*(-) \psi_{\alpha,n} (\hat{\varphi}_n * g), \hat{\varphi}_k * f \rangle_{L_{q,\gamma}(E_1)}. \tag{5.6}
\]

Fix $K_0 \subset N_0$ and choose a radial $\psi \in S$ with compact support such that $\psi$ is 1 on $\cup_{k=1}^{K_0+1} \text{supp } \varphi_k$. If $n, k \in \{0,1,\ldots,K_0\}$, then by Remark 4.1 we get
\[
T_m \psi_k (\hat{\varphi}_k * f) = T_m \psi \hat{\varphi}_k (\hat{\varphi}_k * f) = T_m \psi (\hat{\varphi}_k * f) \tag{5.7}
\]
and
\[
T_m^*(-) \psi_{\alpha,n} (\hat{\varphi}_n * f) = T_m^*(-) \psi_{\alpha,n} (\hat{\varphi}_n * f) = T_m^*(-) \psi_{\alpha,n} (\hat{\varphi}_n * f). \tag{5.8}
\]

since $m\psi$ and $m^*(-) \psi_{\alpha,n}(\cdot)$ satisfy the assumptions of Theorem 4.1. Hence, by (5.5) – (5.8) and by Remark 4.1 we have
\[
\langle T_m g, f \rangle = \langle T_m^*(-) g, f \rangle.
\]

The next lemma gives a convenient way to verify the assumption of Theorem 4.8 in terms of derivatives.
By reasoning as Lemma 4.10 and Corollary 4.11 in [11] we obtain

**Lemma 5.1.** Let $\frac{2}{p} < l \in \mathbb{N}$ and $\sigma \in [p, \infty]$. If $m \in C^l(R^n; B(E_1, E_2))$ and there exists a positive constant $A$ so that

$$\|D^\alpha m\|_{L_\sigma(R^n; B(E_1, E_2))} \leq A$$

(5.9)

for each $k \in \mathbb{N}$, $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \leq l - 1$. Then $m$ satisfies condition (5.1) of Theorem 5.1.

**Corollary 5.1.** Let $q, r \in [1, \infty]$ and $s \in \mathbb{R}$. If $m \in C^l(R^n; B(E_1, E_2))$ and there exists a positive constant $A$ so that

$$\sup_{t \in \mathbb{R}^n} (1 + |t|)^{[\alpha]} \|D^\alpha m\|_{L_\sigma(R^n; B(E_1, E_2))} \leq A$$

(5.10)

for each $k \in \mathbb{N}$, $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \leq l$ and $m_k(.) = m(2^{k-1}.)$. Then $m$ is a Fourier multiplier from $Y_1$ to $Y_2$ provided one of the following conditions hold:

(a) $E_1$ and $E_2$ are arbitrary Banach spaces and $l = n + 1$;
(b) $E_1$ and $E_2$ are uniformly convex Banach spaces and $l = n$;
(c) $E_1$ and $E_2$ have Fourier type $p$ and $l = \left[\frac{n}{p}\right] + 1$.

### 6. Embedding theorems in Besov-Lions type spaces

From [23] we have

**Lemma 6.1.** Let $A$ be a positive operator on a Banach space $E$, $b$ be a nonnegative real number and $r = (r_1, r_2, ..., r_n)$ where $r_k \in \{0, b\}$. Let $t = (t_1, t_2, ..., t_n)$, $0 < t_k \leq T < \infty$, $k = 1, 2, ..., n$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $l = (l_1, l_2, ..., l_n)$, where $l_k$ are positive and $\alpha_k$ are nonnegative integers such that $\kappa = |(\alpha + r) : l| \leq 1$. For $0 < h \leq h_0 < \infty$ and $0 \leq \mu \leq 1 - \kappa$ the operator-function

$$\Psi_t(\xi) = \Psi_{t,h,\mu}(\xi) = \prod_{k=1}^n t_k^{-\frac{\alpha_k}{r_k}} \xi^r (i\xi)^\alpha A^{1-\kappa-\mu} h^{-\mu} [A + \eta(t, \xi)]^{-1}$$

is bounded operator in $E$ uniformly with respect to $\xi \in \mathbb{R}^n$, $h > 0$ and $t$, i.e there is a constant $C_\mu$ such that

$$\|\Psi_{t,h,\mu}(\xi)\|_{L(E)} \leq C_\mu$$

for all $\xi \in \mathbb{R}^n$ and $h > 0$, where,

$$\eta(t, \xi) = \sum_{k=1}^n t_k |\xi_k|^{l_k} + h^{-1}.$$
Let \( l = (l_1, l_2, ..., l_n) \), where \( l_k \) are positive integers. Let
\[
\nu(l) = \max_{k,j \in \{1, 2, ..., n\}} \left\{ \frac{1}{l_k} - \frac{1}{l_j} \right\}, \quad \eta(t) = \prod_{k=1}^{n} t_{k}^{\frac{\nu(l)}{l_k}}, \quad Y = B_{p,\theta,\gamma}^{l,s}(R^n; E(A), E).
\]

**Theorem 6.1.** Suppose the following conditions hold:
1. \( \gamma \in A_\nu \) for \( \nu \in [1, \infty] \). \( E \) is a Banach spaces with weighted Fourier type \( \gamma \) and \( \sigma \in [1, 2] \);
2. \( t = (t_1, t_2, ..., t_n) \), \( 0 < t_k \leq T < \infty \), \( k = 1, 2, ..., n \), \( 1 < p \leq q < \infty \), \( \theta \in [1, \infty] \);
3. \( l_k \) are positive and \( \alpha_k \) are nonnegative integers such that \( 0 < \varkappa + \nu(l) \leq 1 \), and let \( 0 \leq \mu \leq 1 - \varkappa - \nu(l) \);
4. \( A \) is a \( \varphi \)-positive operator in \( E \).

Then an embedding
\[
D^\alpha B_{p,\theta,\gamma}^{l,s}(R^n; E(A), E) \subset B_{p,\theta,\gamma}^{s}(R^n; E(A^{1-\varkappa-\mu}))
\]
is continuous and there exists a constant \( C_\mu > 0 \), depending only on \( \mu \), such that
\[
\| \eta(t) D^\alpha u \|_{B_{p,\theta,\gamma}^{s}(R^n; E(A^{1-\varkappa-\mu}))} \leq C_\mu [h^\mu \| u \|_{B_{p,\theta,\gamma}^{l,s}(R^n; E(A), E)} + h^{-(1-\mu)} \| u \|_{B_{p,\theta,\gamma}^{s}(R^n; E)}]
\]
for all \( u \in B_{p,\theta,\gamma}^{l,s}(R^n; E(A), E) \) and \( 0 < h \leq h_0 < \infty \).

**Proof.** We have
\[
\| D^\alpha u \|_{B_{p,\theta,\gamma}^{s}(R^n; E(A^{1-\varkappa-\mu}))} = \| A^{1-\varkappa-\mu} D^\alpha u \|_{B_{p,\theta,\gamma}^{s}(R^n; E)}
\]
for all \( u \) such that
\[
\| D^\alpha u \|_{B_{p,\theta,\gamma}^{s}(R^n; E(A^{1-\varkappa-\mu}))} < \infty.
\]

On the other hand by using the relation (6.2) we have
\[
A^{1-\alpha-\mu} D^\alpha u = F^{-1} FA^{1-\varkappa-\mu} D^\alpha u = F^{-1} A^{1-\varkappa-\mu} F D^\alpha u = F^{-1} A^{1-\varkappa-\mu} (i\xi)^\alpha F u = F^{-1} (i\xi)^\alpha A^{1-\varkappa-\mu} F u.
\]
Hence denoting \( \hat{F} u \) by \( \hat{u} \), we get from the relations (6.2) and (6.3)
\[
\| D^\alpha u \|_{B_{p,\theta,\gamma}^{s}(R^n; E(A^{1-\varkappa-\mu}))} \sim \| F^{-1} (i\xi)^\alpha A^{1-\varkappa-\mu} \hat{u} \|_{B_{p,\theta,\gamma}^{s}(R^n; E)}.
\]
Similarly, from definition of \( Y \) we have
\[
\| u \|_Y = \| u \|_{B_{p,\theta,\gamma}^{s}(E(A))} + \sum_{k=1}^{n} \| t_k D^k u \|_{B_{p,\theta,\gamma}^{s}(R^n; E)} = \]

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for all $u \in Y$. Thus proving the inequality (6.1) for some constants $C_\mu$ is equivalent to proving

$$
\eta \left\| F^{-1} (i \xi)^\alpha A^{1-\kappa-\mu} \hat{u} \right\|_{B^\mu_{p,q;\gamma} (R^n;E)} \leq C_\mu \left[ h^\mu \left( \left\| F^{-1} A \hat{u} \right\|_{B^\mu_{p,q;\gamma} (R^n;E)} + \sum_{k=1}^n \left\| \int k F^{-1} \left[ (i \xi_k)^{l_k} \hat{u} \right] \right\|_{B^\mu_{p,q;\gamma} (R^n;E)} \right) + h^{-(1-\mu)} \left\| F^{-1} \hat{u} \right\|_{B^\mu_{p,q;\gamma} (R^n;E)} \right].
$$

Thus the inequality (6.1) will be followed if we prove the following inequality

$$
\eta \left\| F^{-1} (i \xi)^\alpha A^{1-\kappa-\mu} \hat{u} \right\|_{B^\mu_{p,q;\gamma} (R^n;E)} \leq C_\mu \left\| F^{-1} [h^\mu (A + \psi(t,\xi)) \hat{u}] \right\|_{B^\mu_{p,q;\gamma} (R^n;E)}
$$

for a suitable $C_\mu > 0$ and for all $u \in Y$, where

$$
\psi = \psi(t,\xi) = \sum_{k=1}^n t_k |\xi_k|^{l_k} + h^{-1}.
$$

Let us express the left hand side of (6.3) as follows

$$
\frac{\eta}{C_\mu} \left\| F^{-1} (i \xi)^\alpha A^{1-\kappa-\mu} \hat{u} \right\|_{B^\mu_{p,q;\gamma} (R^n;E)} = \frac{\eta}{C_\mu} \left\| F^{-1} (i \xi)^\alpha A^{1-\kappa-\mu} [h^\mu (A + \psi)]^{-1} \left| h^\mu (A + \psi) \right| \right\|_{B^\mu_{p,q;\gamma} (R^n;E)}.
$$

(Since $A$ is a positive operator in $E$ and $-\psi(t,\xi) \in S(\varphi)$ so it is possible). It is clear that the inequality (6.4) will be followed immediately from (6.5) if we can prove that the operator-function

$$
\Psi_t = \Psi_{t,h,\mu} = \frac{\eta}{C_\mu} (i \xi)^\alpha A^{1-\kappa-\mu} \left[ h^\mu (A + \psi) \right]^{-1}
$$

is a multiplier in $M_{p,q,\gamma}^\mu (E)$, which is uniformly with respect to $h$ and $t$. In order to prove that $\Psi_t \in M_{p,q,\gamma}^\mu (E)$ it suffices to show that there exists a constant $M_\mu > 0$ with

$$
|\xi|^k \left\| D^\beta \Psi_t (\xi) \right\|_{L(E)} \leq C, \ k = 0, 1, ..., |\beta|
$$

for all

$$
\beta = (\beta_1, \beta_2, ..., \beta_n), \beta_k \in \{0, 1\}, \xi_k \neq 0.
$$
To see this, we apply Lemma 6.1 and get a constant $M_\mu > 0$ depending only on $\mu$ such that
\[
\|\Psi_t (\xi)\|_{L(E)} \leq M_\mu
\]
for all $\xi \in R^n$. This shows that the inequality (7.6) is satisfied for $\beta = (0, ..., 0)$.
We next consider (6.6) for $\beta = (\beta_1, ..., \beta_n)$ where $\beta_k = 1$ and $\beta_j = 0$ for $j \neq k$.
By using the condition $\kappa + \nu (l) \leq 1$ and well known inequality
\[
y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_n^{\alpha_n} \leq C \left[ 1 + \sum_{k=1}^{n} y_k^l \right], \quad y_k \geq 0,
\]
we have
\[
|\xi| |\xi_k| \|D_k \Psi_t (\xi)\|_{L(E)} \leq M_\mu, \quad k = 1, 2, ..., n.
\]
Repeating the above process we obtain the estimate (7.6).
Thus the operator-function $\Psi_{t, h, \mu}(\xi)$ is a uniform collection of multiplier with respect to $h$ and $t$ i.e
\[
\Psi_{t, h, \mu} \in \Phi_h \subset M_{p, \theta, \gamma}^\infty (E).
\]
This completes the proof of the Theorem 6.1. It is possible to state Theorem 6.1 in a more general setting. For this, we use the conception of extension operator.

**Condition 6.1.** Let $\gamma \in A_\nu$ for $\nu \in [1, \infty]$. Assume $E$ is a Banach spaces with weighted Fourier type $\gamma$ and $\sigma \in [1, 2]$. Suppose $A$ is a $\varphi$-positive operator in Banach spaces $E$. Let a region $\Omega \subset R^n$ be such that there exists a bounded linear extension operator $B$ from $B_{p, \theta, \gamma}^s (\Omega; E (A) , E)$ to $B_{p, \theta, \gamma}^s (R^n; E (A) , E)$, for $p, \theta \in [1, \infty]$.

**Remark 7.1.** If $\Omega \subset R^n$ is a region satisfying a strong l-horn condition (see [4], §18) $E = R, A = I$, then there exists a bounded linear extension operator from $B_{p, \theta}^s (\Omega) = B_{p, \theta}^s (\Omega; C, C)$ to $B_{p, \theta}^s (R^n; C, C)$.

Let
\[
Y = B_{p, \theta, \gamma}^s (R^n; E), \quad Y_0 = B_{p, \theta, \gamma}^{l,s} (\Omega; E (A), E)
\]

**Theorem 6.2.** Suppose all conditions of the Theorem 6.1 and the Condition 6.1 are hold. Then the embedding
\[
D^{\alpha} b_{p, \theta, \gamma}^{l,s} (\Omega; E (A) , E) \subset B_{q, \theta, \gamma}^s (\Omega; E (A^{1-\sigma-\nu}))
\]
is continuous and there exists a constant $C_{\mu}$ depending only on $\mu$ such that
\[
\eta \|D^{\alpha} u\|_{B_{q, \theta, \gamma}^s (\Omega; E (A^{1-\sigma-\nu}))} \leq (6.7)
\]
\[
C_{\mu} \left[ h^\mu \|u\|_{B_{p, \theta, \gamma}^{l,s} (\Omega; E (A), E)} + h^{-1-\mu} \|u\|_{B_{p, \theta, \gamma}^{l,s} (\Omega; E)} \right]
\]
for all $u \in Y_0$ and $0 < h \leq h_0 < \infty$.

**Proof.** It suffices to prove the estimate (7.7). Let $P$ be a bounded linear extension operator from $B_{q, \theta, \gamma}^s (\Omega; E)$ to $B_{q, \theta, \gamma}^s (R^n; E)$ and also from $Y_0$ to
Let $P_{Ω}$ be a restriction operator from $R^n$ to $Ω$. Then for any $u ∈ Y$ we have

\[ \| D^α u \|_{B^{s,θ,γ}_q(Ω; E(A))} = \| D^α P_Ω u \|_{B^{s,θ,γ}_q(Ω; E(A))} \]

\[ \leq C_μ \left[ h^μ \| Pu \|_{B^{1,1}_p,γ(E(A))} + h^{−(1−μ)} \| Pu \|_{B^{s,θ,γ}_q(R^n; E(A))} \right] \]

\[ \leq C_μ \left[ h^μ \| u \|_{B^{1,θ,γ}_p(E(A))} + h^{−(1−μ)} \| u \|_{B^{s,θ,γ}_q(E(A))} \right]. \]

**Result 6.1.** Let all conditions of Theorem 6.2 hold. Then for all $u ∈ Y_0$ we have the following multiplicative estimate

\[ \| D^α u \|_{B^{s,θ,γ}_q(Ω; E(A))} \leq C_μ \| u \|_{B^{1,θ,γ}_p(E(A))} \cdot \| u \|_{B^{s,θ,γ}_q(Ω; E(A))}. \] (6.8)

Indeed setting

\[ h = \| u \|_{B^{s,θ,γ}_q(Ω; E(A))} \cdot \| u \|_{B^{s,θ,γ}_q(Ω; E(A))}^{-1} \]

in (6.7) we obtain (6.8).

**Result 6.2.** If $l_1 = l_2 = \ldots = l_n = m$ then we obtain the continuity of embedding operators in the isotropic class

\[ B^{m,s}_{p,θ,γ}(Ω; E(A)) \]

For $E = C$, $A = I$ we obtain the embedding of weighted Besov type spaces

\[ D^α B^{1,s}_{p,θ,γ}(Ω) \subset B^{s}_{q,θ,γ}(Ω). \]

### 7. Application to vector-valued functions

Let $s > 0$ and consider the space [29, §1.18.2]

\[ l_q^σ = \{ u; \ u = \{ u_i \}, \ i = 1, 2, ..., ∞, \ u_i ∈ C \} \]

with the norm

\[ \| u \|_{l_q^σ} = \left( \sum_{i=1}^{∞} 2^{iqσ} |u_i|^q \right)^{1/q} < ∞. \]

Note that $l_0^0 = l_q$. Let $A$ is an infinite matrix defined in the space $l_q$ such that

\[ D(A) = l_q^σ, \ A = [δ_{ij}2^{σi}] \]

where $δ_{ij} = 0$, when $i ≠ j$, $δ_{ij} = 1$, when $i = j$, $i, j = 1, 2, ..., ∞$.

It is clear to see that this operator $A$ is positive in the space $l_q$. Then by Theorem 7.2 we obtain the continuous embedding

\[ D^α B^{1,s}_{p1,θ,γ}(Ω; l_q^σ, l_q) \subset B^{s}_{q,θ,γ}(Ω; l_q^{1−κ−μ}) \]

where $κ = \sum_{k=1}^{n} \frac{α_k + \frac{1}{m} − \frac{1}{n}}{l_k}$. 

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and the associate estimate \((6.7)\), where \(0 \leq \mu + \nu (l) \leq 1 - \varsigma\).

It should be noted that the above embedding haven’t been obtained with classical method up to this time.

8. B-separable DOE in \(R^n\)

Let us consider the differential-operator equation \((1.1)\).

**Condition 8.1.** Let

(a) \(K (\xi) = \sum_{|\alpha|=2l} a_\alpha (i\xi_1)^{\alpha_1} (i\xi_2)^{\alpha_2} \ldots (i\xi_n)^{\alpha_n} \in S (\varphi)\);

(b) There exists the positive constant \(M_0\) so that

\(|K (\xi)| \geq M_0 \sum_{k=1}^n \xi_k^{2l}\) for all \(\xi \in R^n, \xi \neq 0\).

**Definition 8.1.** The problem \((1.1)\) is said to be weighted \(B\)-separable (or weighted \(B_{s,\theta,\gamma}(R^n; E)\)-separable) if the problem \((1.1)\) has a unique solution \(u \in B_{s,\theta,\gamma}(R^n; E)\) for all \(f \in B_{s,\theta,\gamma}(R^n; E)\) and

\[\|Au\|_{B_{s,\theta,\gamma}(\Omega; E)} + \sum_{|\alpha|=2l} \|D^\alpha u\|_{B_{s,\theta,\gamma}(\Omega; E)} \leq C \|f\|_{B_{s,\theta,\gamma}(\Omega; E)}\cdot\]

Consider the following degenerate DOE

\[Lu = \sum_{|\alpha|=2l} a_{\alpha} D[^{\alpha}] u + Au + \sum_{|\alpha|<2l} A_{\alpha} D[^{\alpha}] u = f \quad (8.1)\]

where \(A (x), A_{\alpha} (x)\) are possible unbounded operators in a Banach space \(E\), \(a_k\) are complex-valued functions and

\[D[^{\alpha}] = \left(\gamma (x_k) \frac{\partial}{\partial x_k}\right)^{\alpha_k}, \quad D[^{\alpha}] = D_1[^{\alpha_1}] D_2[^{\alpha_2}] \ldots D_n[^{\alpha_n}]\]

**Remark 8.1.** Under the substitution

\[\tau_k = \int_0^{x_k} \gamma^{-1} (y) dy \quad (8.2)\]

spaces \(B_{s,\theta,\gamma}(R^n; E)\), \(B_{p,\theta,\gamma}[^{i}] (R^n; E)\) are mapped isomorphically onto the weighted spaces \(B_{s,\theta,\tilde{\gamma}}(R^n; E)\), \(B_{p,\theta,\tilde{\gamma}}[^{i}] (R^n; E)\), respectively, where

\[\gamma = \prod_{k=1}^n \gamma (x_k), \quad \tilde{\gamma} (\tau) = \prod_{k=1}^n \gamma (x_k (\tau_k))\]
Moreover, under the substitution (8.2) the degenerate problem (8.1) is mapped to the undegenerate problem (1.1) considered in the weighted space $B^s_{p,q,\theta,\bar{\gamma}}(\mathbb{R}^n; E)$.

Let

$$ Y = B^s_{q,\theta,\gamma}(\mathbb{R}^n; E), \ Y_0 = B^{2l,s}_{q,\theta,\gamma}(\mathbb{R}^n; E(A), E). $$

**Theorem 8.1.** Suppose the following conditions hold:
1. Condition 9.1 is hold;
2. $s > 0, 1 \leq q, \theta \leq \infty, k = 1, 2, ..., n$;
3. $\gamma \in A_{\nu}$ for $\nu \in [1, \infty]$. $E$ is a Banach spaces with weighted Fourier type $\gamma$ and $p \in [1, 2]$;
4. $A$ is a $\varphi$-positive operator in $E$ and

$$ A_{\alpha}(x) A^{-(1-|\alpha|-\mu)} \in L_{\infty}(\mathbb{R}^n; L(E)), \ 0 < \mu < 1 - |\alpha| 2l. $$

Then for all $f \in Y$ and for sufficiently large $|\lambda|, \lambda \in S(\varphi)$ equation (1.1) has a unique solution $u(x) \in Y_0$ and

$$ \sum_{|\alpha|=2l} \| D^\alpha u \|_Y + \| Au \|_Y \leq C \| f \|_Y. \quad (8.3) $$

**Proof.** Firstly, we will consider leading part of the equation (1.1) i.e. the differential-operator equation

$$ (L_0 + \lambda) u = \sum_{|\alpha|=2l} D^\alpha u + Au + \lambda u = f. \quad (8.4) $$

Then we apply the Fourier transform to equation (8.4) with respect to $x = (x_1, ..., x_n)$ and obtain

$$ \sum_{|\alpha|=2l} a_{\alpha} \xi^\alpha \hat{u}(\xi) + A\lambda \hat{u}(\xi) = \hat{f}(\xi). \quad (8.5) $$

Since $\sum_{|\alpha|=2l} a_{\alpha} \xi^\alpha \geq 0$ for all $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n$ therefore, $\omega = \omega(\lambda, \xi) = \lambda + \sum_{|\alpha|=2l} a_{\alpha} \xi^\alpha \in S(\varphi)$ for all $\xi \in \mathbb{R}^n$, i.e. operator $A + \omega$ is invertible in $E$.

Hence (8.5) implies that the solution of equation (8.4) can be represented in the form

$$ u(x) = F^{-1} (A + \omega)^{-1} f. \quad (8.6) $$

It is clear to see that the operator-function $\varphi_{\lambda}(\xi) = [A + \omega]^{-1}$ is a multiplier in $B^s_{p,\theta,\bar{\gamma}}(\mathbb{R}^n; E)$ uniformly with respect to $\lambda$. Actually, by definition of the positive operator, for all $\xi \in \mathbb{R}^n$ and $\lambda \geq 0$ we get

$$ \| \varphi_{\lambda}(\xi) \|_{L(E)} = \left\| (A + \omega)^{-1} \right\| \leq M (1 + |\omega|)^{-1} \leq M_0. $$
Moreover, since $D_k \varphi_\lambda (\xi) = \alpha_k a_\alpha \xi^\alpha (A + \omega)^{-2} \xi_k^{-1}$ then by using the resolvent properties of positive operator $A$ we have

$$\|\xi D_k \varphi_\lambda\|_{L(E)} \leq |\alpha_k a_\alpha| \|\xi^\alpha\|(A + \omega I)^{-2} \leq M. \quad (8.7)$$

Using the estimate (8.7) we show uniform estimate

$$|\xi|^\beta \|D_\xi^\beta \varphi_\lambda (\xi)\|_{B(E)} \leq C \quad (8.8)$$

for

$$\beta = (\beta_1, \ldots, \beta_n), \beta_i \in \{0, 1\}, \xi = (\xi_1, \ldots, \xi_n), \xi_i \neq 0.$$ 

In a similar way we prove that the operator-functions $\varphi_{\alpha\lambda} (\xi) = \xi^\alpha \varphi_{\lambda,t}, k = 1, 2, \ldots, n$ and $\varphi_{0\lambda} = A\varphi_{\lambda}$ satisfy the estimates

$$(1 + |\xi|)^{|\beta|} \|D_\xi^\beta \varphi_{\alpha\lambda} (\xi)\|_{B(E)} \leq C, \quad (1 + |\xi|)^{|\beta|} \|D_\xi^\beta \varphi_{0\lambda} (\xi)\|_{B(E)} \leq C. \quad (8.9)$$

Then in view of estimates (8.8) and (8.9) we obtain that operator-functions $\varphi_\lambda$, $\varphi_{\alpha\lambda}$, $\varphi_{0\lambda}$ are multipliers in $Y$. By (8.9) and in view of

$$\|D^\alpha u\|_Y = \|F^{-1} \xi^\alpha \hat{u}\|_Y = \|F^{-1} \xi^\alpha (A + \omega)^{-1} f\|_Y,$$

$$\|Au\|_Y = \|F^{-1} A \hat{u}\|_Y = \|F^{-1} \left[A (A + \omega)^{-1}\right] f\|_Y.$$  

we obtain that there exists a unique solution of equation (8.4) for all $f \in Y$ and the uniform estimate holds

$$\sum_{|\alpha| = 2l} \|D^\alpha u\|_Y + \|Au\|_Y \leq C \|f\|_Y. \quad (8.10)$$

Consider the differential operator $G_0$ generated by problem (8.4), that is

$$D (G_0) = B_{q, \delta, \gamma} (R^n; E (A), E), \quad G_0 u = \sum_{|\alpha| = 2l} D^\alpha u + Au.$$ 

The estimate (8.10) implies that the operator $G_0 + \lambda$ for all $\lambda \geq 0$ has a bounded inverse from $Y$ into $Y_0$. Let $G$ denote the differential operator in $Y$ generated by problem (1.1). Namely,

$$D (G) = Y_0, \quad Gu = G_0 u + L_1 u, \quad L_1 u = \sum_{|\alpha| < 2l} A_\alpha (x) D^\alpha u. \quad (8.11)$$

In view of (4) condition, by virtue of Theorem 6.1, for all $u \in Y$ we have

$$\|L_1 u\|_Y \leq \sum_{|\alpha| < 2l} \|A_\alpha (x) D^\alpha u\|_Y \leq \sum_{|\alpha| < 2l} \left\|A^{-1} f - \mu D^\alpha u\right\|_Y \leq \quad (8.12)$$
Let \( u \) be a unique solution to (8.1) and Remark 8.1 imply that the differential operator \( G \) is invertible from \( Y \) into \( Y_0 \). Then by choosing \( h \) and \( \lambda \) such that \( C h^\mu < 1, C_1 |\lambda|^{-1} h^{-1-\mu} < 1 \) from (9.15) we obtain the uniform estimate
\[
\| L_1 (G_0 + \lambda)^{-1} \|_{B(E)} < 1. 
\] (8.16)

Using the relation (8.11), estimates (8.10) and (8.16) and the perturbation theory of linear operators we obtain that the differential operator \( G + \lambda \) is invertible from \( Y \) into \( Y_0 \). This implies the estimate (8.3).

**Result 8.1.** The Theorem 8.1 implies that the differential operator \( G \) has a resolvent operator \( (G + \lambda)^{-1} \) for \( |\arg \lambda| \leq \phi \), and the following uniform estimate holds
\[
\sum_{|\alpha| \leq 2^n} |\lambda|^{-1-|\alpha|} \left\| D(\lambda + G)^{-1} \right\|_{B(Y)} + \left\| A(\lambda + G)^{-1} \right\|_{B(Y)} \leq C. 
\]

Let \( Q \) denote the operator in \( B_{\beta,\theta}^s (R^n, E) \) generated by problem (8.1). Theorem 8.1 and Remark 8.1 imply

**Result 8.2.** Let all conditions of Theorem 8.1 hold. Then for all \( f \in B_{\beta,\theta}^s (R^n, E) \), \( \lambda \in S(\varphi) \) and for sufficiently large \( |\lambda| \), the equation (8.1) has a unique solution \( u \in B_{\beta,\theta}^{[2\nu]} (R^n; E) \) and the coercive uniform estimate holds
\[
\sum_{|\alpha| \leq 2^n} |\lambda|^{-1-|\alpha|} \left\| D(\lambda + Q)^{-1} \right\|_{B(B_{\beta,\theta}^s (R^n, E))} \]
Remark 8.1. The Result 8.2 implies that operator $G$ is positive operator in $B_{q,\theta,\gamma}^s (\mathbb{R}^n; E)$. Then by virtue of [29, §1.14.5] the operator $G$ for $\varphi \in \left(\frac{\pi}{2}, \pi\right)$ is a generator of an analytic semigroup in $B_{q,\theta,\gamma}^s (\mathbb{R}^n; E)$.

9. The Cauchy problem for degenerate parabolic DOE

Consider the Cauchy problem for the degenerate parabolic CDOE

$$\partial_t u + \sum_{|\alpha|=2l} a_{\alpha} D_\alpha^u u + A u = f (t, x), \quad (9.1)$$

$$u(0, x) = 0, \quad x \in \mathbb{R}^n$$

in $B_{q, r, \gamma}^s (\mathbb{R}^{n+1}; E)$, where $A$ is a linear operator in a Banach space in $E$. Let

$$F = B_{q_1, r_1}^s (R_+^{n+1}; F), \quad F_0 = B_{q_1, \theta_1, \gamma}^s (\mathbb{R}^n; E),$$

$$F_1 = B_{q_1, \theta_1, \gamma}^{2l_1, s} (R_+^{n+1}; E (A), E) = B_{q_1, r_1}^{1, s} (R_+; D (G), F).$$

Theorem 9.1. Assume all conditions of Theorem 8.1 hold for $\varphi \in \left(\frac{\pi}{2}, \pi\right)$ and $s > 0$. Then for $f \in F$ the problem (9.1) has a unique solution $u \in F_1$ satisfying

$$\| D_t u \|_F + \sum_{|\alpha|=2l} \| D_\alpha^u u \|_F + \| A u \|_F \leq C \| f \|_F. \quad (9.2)$$

Proof. So, the problem (9.1) can be express as

$$\frac{du}{dt} + Gu (t) = f (t), \quad u (0) = 0, \quad t \in (0, \infty). \quad (9.3)$$

The Result 9.1 implies the positivity of $G$ for $\varphi \in \left(\frac{\pi}{2}, \pi\right)$. Then by virtue of [1, Proposition 8.10] we obtain that, for $f \in F$ the Cauchy problem (9.3) has a unique solution $u \in F_1$ satisfying

$$\| D_t u \|_{B_{q_1, r_1}^{s} (R_+; F_0)} + \| Gu \|_{B_{q_1, s_1}^{s} (R_+; F_0)} \leq C \| f \|_{B_{q_1, s_1}^{s} (R_+; F_0)}. \quad (9.4)$$

In view of Result 8.1 the operator $G$ is separable in $F_0$, therefore, the estimate (9.4) implies (9.2).

Consider now, the Cauchy problem for the degenerate parabolic CDOE

$$\partial_t u + \sum_{|\alpha|=2l} a_{\alpha} D_\alpha^{[\alpha]} u + A u = f (t, x), \quad (9.5)$$

$$u(0, x) = 0, \quad x \in \mathbb{R}^n.$$
Here, $B_{q,\theta,\gamma}^{[m],s} (\Omega; E)$ denote a $E$-valued Sobolev-Besov weighted space of functions $u \in B_{q,\theta,\gamma}^s (\Omega; E)$ that have generalized derivatives $D_k^{[m]} u \in B_{q,\theta,\gamma}^s (\Omega; E)$ with the norm

$$
\| u \|_{B_{q,\theta,\gamma}^{[m],s}(\Omega; E)} = \| u \|_{B_{q,\theta,\gamma}^s(\Omega; E)} + \sum_{k=1}^n \| D_k^{[m]} u \|_{B_{q,\theta,\gamma}^s(\Omega; E)} < \infty.
$$

Assume $E_0$ is continuously and densely belongs to $E$. Here, $B_{q,\theta,\gamma}^{[m],s} (\Omega; E_0, E)$ denotes the space $B_{q,\theta,\gamma}^s (\Omega; E_0) \cap B_{q,\theta,\gamma}^{[m],s} (\Omega; E)$ with the norm

$$
\| u \|_{B_{q,\theta,\gamma}^{[m],s}(\Omega; E_0, E)} = \| u \|_{B_{q,\theta,\gamma}^s(\Omega; E_0)} + \sum_{k=1}^n \| D_k^{[m]} u \|_{B_{q,\theta,\gamma}^s(\Omega; E)} < \infty.
$$

Let

$$
\Phi = B_{q,\theta,\gamma}^s (R^n+; E) = B_{q_1,r_1}^s (R^n+; F), \quad \Phi_0 = B_{q,\theta,\gamma}^s (R^n; E),
$$

$$
\Phi_1 = B_{q,\theta,\gamma}^{[m],s} (R^n+; E(A), E) = B_{q_1,r_1}^{[m],s} (R^n+; D(Q), F).
$$

From Theorem 8.1, Result 8.2 and Remark 8.1 we obtain the following

**Result 9.1.** Assume all conditions of Theorem 8.1 hold for $\varphi \in (\mathcal{F}, \pi)$ and $s > 0$. Then for $f \in F$ the equation (9.5) has a unique solution $u \in \Phi_1$ satisfying

$$
\| \partial_t u \|_\Phi + \sum_{|\alpha|=2l} \| D^{[\alpha]} u \|_\Phi + \| Au \|_\Phi \leq C \| f \|_\Phi. \quad (9.6)
$$

**Remark 9.1.** There are a lot of positive operators in concrete Banach spaces. Therefore, putting concrete Banach spaces instead of $E$ and concrete positive differential, pseudodifferential operators, or finite, infinite matrices, etc. instead of operator $A$ on DOE (8.1) and (9.5) by virtue of Results 8.2 and 9.1 we can obtain the maximal $B_{q,\theta,\gamma}^s$-regularity properties of different class of degenerate PDEs or system of other type equations.

10. Infinite systems of anisotropic elliptic equations

Consider the following infinity systems

$$
(L + \lambda) u_m = \sum_{|\alpha|=2l} a_\alpha D^\alpha u_m (x) + |d_m (x) + \lambda| u_m (x) + \sum_{|\alpha|<2l} d_{\alpha m} (x) D^\alpha u_m = f_m (x), \quad x \in \mathbb{R}^n, \ m = 1, 2, ..., \infty.
$$

Let

$$
Q (x) = \{ d_m (x) \}, \quad d_m > 0, \ u = \{ u_m \}, \ Qu = \{ d_m u_m \}, \ m = 1, 2, ... \infty,
$$
It is clear to see that this operator \( A \) we obtain (10.1). Let

\[
B = B \left( B_{\theta,\gamma}^\ast \left( R^n; l_p \right) \right).
\]

**Condition 10.1.** Assume \( \gamma \in A_\nu \) for \( \nu \in [1, \infty] \) and (b) assumption of Condition 8.1. is hold. There are positive constants \( C_1 \) and \( C_2 \) so that for \( \{d_j (x)\}_1^\infty \in l_q \) for all \( x \in R^n \) and some \( x_0 \in R^n \),

\[
C_1 |d_j (x_0)| \leq |d_j (x)| \leq C_2 |d_j (x_0)|.
\]

**Theorem 10.1.** Suppose the Condition 10.1 holds. Let \( a_m \in C_b (R^n) \), \( d_m \in C_b (R^n) \), \( d_{am} \in L_\infty (R^n) \) such that

\[
\max \sup_{m} \sum_{k=1}^{\infty} d_{am} (x) d_k^{-\left(1 - \frac{\omega}{2m} - \mu \right)} < M,
\]

for all \( x \in R^n \) and \( 0 < \mu < 1 - \frac{\omega}{2m} \).

Then:

(a) for all \( f (x) = \{f_m (x)\}_1^\infty \in B_{\theta,\gamma}^\ast (R^n; l_p) \), for \( |\arg \lambda| \leq \varphi \) and for sufficiently large \( |\lambda| \) problem (10.1) has a unique solution \( u = \{u_m (x)\}_1^\infty \) that belongs to space \( B_{\theta,\gamma}^{2\iota + \nu} (R^n, l_p (Q), l_p) \) and the uniform coercive estimate holds

\[
\sum_{|\alpha| \leq 2l} \|D^\alpha u\|_{B_{\theta,\gamma}^{\nu} (R^n; l_q)} + \|Qu\|_{B_{\theta,\gamma}^{\nu} (R^n; l_q)} \leq C \|f\|_{B_{\theta,\gamma}^{\nu} (R^n; l_q)}.
\]

(b) For \( |\arg \lambda| \leq \varphi \) and sufficiently large \( |\lambda| \) there exists a resolvent \( (O + \lambda)^{-1} \) of operator \( O \) and

\[
\sum_{|\alpha| \leq 2l} |\lambda|^{-\frac{\omega}{3m}} \left\| D^\alpha (O + \lambda)^{-1} \right\|_B + \left\| Q(O + \lambda)^{-1} \right\|_B \leq M.
\]

**Proof.** Really, let \( E = l_q \), \( A (x) \) and \( A_n (x) \) be infinite matrices, such that

\[
A = [d_m (x) \delta_{km}], \quad A_n (x) = [d_{amn} (x)], \quad k, m = 1, 2, \ldots, \infty.
\]

It is clear to see that this operator \( A \) is positive in \( l_p \). Therefore, by virtue of Theorem 9.1 we obtain that the problem (10.1) for all \( f \in B_{\theta,\gamma}^\ast (R^n; l_q) \), for \( |\arg \lambda| \leq \varphi \) and sufficiently large \( |\lambda| \) has a unique solution \( u \) that belongs to space \( B_{\theta,\gamma}^{2\iota + \nu} (R^n, l_p (Q), l_p) \) and the estimate (10.2) hold. From estimate (10.2) we obtain (10.3).
11. Cauchy problem for infinite systems of parabolic equations

Consider the following infinity systems of parabolic Cauchy problem

\[ \partial_t u_m + \sum_{|\alpha|=2l} a_\alpha D^\alpha u_m (t, x) + d_m (x) u_m (t, x) + \]

\[ + \sum_{|\alpha|<2l} d_{\alpha m} (x) D^\alpha u_m (y, x) = f_m (t, x), \quad t \in R_+, \quad x \in R^n, \]

\[ u_m (0, x) = 0, \quad m = 1, 2, \ldots, \infty, \]

\[ F = \tilde{B}^{s}_{q, \theta, \gamma} (R^{n+1}; l_p) = B^{s}_{q_1, r_1} (R_+; l_p), \quad F_0 = B^{1s}_{q, \theta, \gamma} (R^n; l_p), \]

\[ F_1 = \tilde{B}^{2l,1,s}_{q_1, r_1} (R^{n+1}; D (O), l_p) = B^{1l,s}_{q_1, r_1} (R_+; D (O), l_p), \]

where \( O \) is the operator in \( l_p \) generated by problem (10.1) for \( \lambda = 0 \).

In this section we show the following

**Theorem 11.1.** Let all conditions of Theorem 10.1 are hold. Then for \( f \in F \) the Cauchy problem (11.1) has a unique solution \( u \in F \) satisfying

\[ \| D_t u \|_F + \sum_{|\alpha|=2l} \| D^\alpha u \|_F + \| A u \|_F \leq C \| f \|_F. \]

**Proof.** Really, let \( E = l_q \), \( A \) and \( A_k (x) \) be infinite matrices, such that

\[ A = [d_m (x) \delta_{km}], \quad A_\alpha (x) = [d_{\alpha km} (x)], \quad k, m = 1, 2, \ldots, \infty. \]

Then the problem (11.1) can be express in a form (9.3) where \( G = O \) and

\[ A = [d_m (x) \delta_{km}], \quad A_\alpha (x) = [d_{\alpha km} (x)], \quad k, m = 1, 2, \ldots, \infty. \]

Then by virtue of Theorem 9.1 we obtain the assertion.

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