On the Nilpotency Class of a Generalized 3-Abelian Group

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Abstract. A group $G$ is called 3-abelian if the map $x \mapsto x^3$ is an endomorphism of $G$ and it is called generalized 3-abelian, if there exist elements $c_1, c_2, c_3 \in G$ such that the map $\varphi : x \mapsto x^{c_1} x^{c_2} x^{c_3}$ is an endomorphism of $G$. Abdollahi, Daoud and Endimioni have proved that a generalized 3-abelian group $G$ is nilpotent of class at most 10. Here, we improve the bound to 3 and we show that the exponent of its derived subgroup is finite and divides 9. We also prove that $G$ is 3-Levi, 9-central, 9-abelian and 3-nilpotent of class at most 2.

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1. Introduction and Results

Let $n \geq 2$ be an integer. A group $G$ is called $n$-abelian whenever $(xy)^n = x^n y^n$ for all $x, y \in G$; or equivalently the map $x \mapsto x^n$ is an endomorphism of $G$. Levi [7] has proved that a group $G$ is 3-abelian if and only if it is 2-Engel and the exponent of its derived subgroup $[G, G]$ divides 3. Trotter [10] has proved that a 3-abelian group $G$ is abelian whenever the map $x \mapsto x^3$ is an automorphism of $G$. A group $G$ is called generalized $n$-abelian whenever there exist elements $c_1, \ldots, c_n \in G$ such that the map $x \mapsto x^{c_1} \ldots x^{c_n}$ is an endomorphism of $G$. The class of generalized $n$-abelian groups is closed under the formation images, and finite direct products. Obviously, every $n$-abelian group is generalized $n$-abelian and it is easy to see that every generalized 2-abelian group is abelian. It is clear that by conjugating we may assume one of $c_1, \ldots, c_n$ to be the trivial element.

Abdollahi, Daoud and Endimioni [1, Theorem 3.1] have proved that a generalized 3-abelian group $G$ is nilpotent of class at most 10, and abelian

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whenever the map $x \mapsto x^{e_1} x^{e_2} x^{e_3}$ is a monomorphism of $G$ or the center $\zeta(G)$ has no element of order 3. Here we improve the previous results as following.

**Theorem 1.1.** Let $G$ be a generalized 3-abelian group. Then

i) $G$ is nilpotent of class at most 3.

ii) The exponent of $[G, G]$ divides 9 and the exponent of $\gamma_3(G)$ divides 3.

The free 4-generator group of exponent 3 is clearly a 3-abelian group and it is well known that it is nilpotent of class 3. Hence the bound 3 for the class of generalized 3-abelian groups is sharp.

Now for any non-zero integer $n$, a group $G$ is called $n$--Levi if $[x, y^n] = [x, y]$ for all $x, y \in G$. It is called $n$-central if $n \geq 1$ and $[x, y^n] = 1$ for all $x, y \in G$. We show that there is a relation between generalized 3-abelian groups $G$ and $n$--Levi or $n$-central groups.

**Theorem 1.2.** Let $G$ be a generalized 3-abelian group admitting an endomorphism of the form

$$\phi : x \mapsto x^a x^b$$

for some $a, b \in G$. Then

i) $G$ is 3-Levi, 9-central and 9-abelian.

ii) The subgroup $\text{Im} \phi$ is abelian. In particular, if $\phi$ is injective or surjective, $G$ is abelian.

Let $m \neq 0$ be an integer. Baer [2] introduced the $m$-center of a group $G$ as follows:

$$Z(G, m) = \{ a \in G \mid (ax)^m = a^m x^m \text{ and } (xa)^m = x^m a^m \text{ for all } x \in G \}. $$

The set $Z(G, m)$ is a characteristic subgroup of $G$ for any non-zero integer $m$. L.-C. Kappe and M. L. Newel [5] proved that

$$(ax)^m = a^m x^m \text{ for all } x \in G \iff (xa)^m = x^m a^m \text{ for all } x \in G.$$ 

Thus only one of the $m$-commutativity conditions suffices to define the $m$-center $Z(G, m)$. If $m$ is a positive integer, the upper $m$-central series $Z_i(G, m)$ is defined inductively as the following: $Z_0(G, m) = 1$, $Z_1(G, m) = Z(G, m)$ and $Z_{i+1}(G, m) / Z_i(G, m) = Z(G / Z_i(G, m), m)$ for $i \geq 1$.

We then get an ascending series.

$$1 = Z_0(G, m) \leq Z_1(G, m) \leq \cdots \leq Z_i(G, m) \leq Z_{i+1}(G, m) \leq \cdots .$$

A group $G$ is said to be $m$-nilpotent of class at most $k$, if $Z_k(G, m) = G$. Karasev [6] introduced the $m$-derived subgroup of a group $G$ as follows: For a given integer $m \geq 2$ and two elements $x, y$ of $G$, the $m$-commutator of $x$ and $y$ is defined by $[x, y]_m = (xy)^m y^{-m} x^{-m}$. The $m$-derived subgroup of $G$ is then the subgroup generated by the set of the $m$-commutators of $G$. It is a fully invariant subgroup of $G$ and is denoted by $[G, G]_m$. If $H$ is a normal subgroup of $G$, then $G/H$ is $m$-abelian if and only if $[G, G]_m \leq H$. $G/[G, G]$ being abelian, $[G, G]_m \leq [G, G]$. We have the following theorem:

**Theorem 1.3.** Every generalized 3-abelian group is 3-nilpotent of class at most 2. Moreover the exponent of its 3-derived subgroup divides 3.
2. Proofs

Notations used in this paper are standard. For a group $G$ and the elements $x_1, x_2, ..., x_n, x, y \in G$, the commutators $[x_1, x_2, ..., x_n]$ and $[x_n, y]$ are defined inductively by the rules:

$$[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2, \quad [x_1, y] = [x, y] = x^{-1}yx$$

and for $n \geq 2$ $[x_1, x_2, ..., x_{n+1}] = [[x_1, x_2, ..., x_n], x_{n+1}]$, $[x_n, y] = [[x_{n-1}, y], y]$.

For a given integer $i \geq 1$, we denote by $[G, G], \zeta_i(G)$ and $\gamma_3(G)$ respectively the derived subgroup, the $i$th-center and the third term of the lower central series of $G$. We denote by $H \leq G$ if $H$ is a subgroup of $G$. To prove Theorems 1.1, 1.2 and 1.3 we need the following lemmas.

Lemma 2.1. Let $G$ be a generalized $n$-abelian group admitting an endomorphism of the form $\psi : x \mapsto x^{c_1} \cdots x^{c_n}$ where $c_1, \ldots, c_n \in G$. Then, $G$ is $n$-abelian whenever $c_1, \ldots, c_n \in \zeta_2(G)$.

Proof. Let $x \in G$. As $c_1, \ldots, c_n \in \zeta_2(G)$ we have that $x, [x, c_1], \ldots, [x, c_n]$ commute. Thus

$$x^\psi = \prod_{1 \leq i \leq n} x [x, c_i] = x^n \prod_{1 \leq i \leq n} [x, c_i].$$

Therefore

$$(xy)^\psi = (xy)^n \prod_{1 \leq i \leq n} [xy, c_i] = (xy)^n \prod_{1 \leq i \leq n} [x, c_i][y, c_i]$$

$$x^\psi y^\psi = x^n \prod_{1 \leq i \leq n} [x, c_i]y^n \prod_{1 \leq i \leq n} [y, c_i] = x^n y^n \prod_{1 \leq i \leq n} [x, c_i][y, c_i]$$

whence $(xy)^n = x^n y^n$. \hfill \Box

Lemma 2.2. Let $G$ be a metabelian group, $x, y \in G$ and $u, v \in [G, G]$. Then, for any integer $n \geq 1$, the following assertions hold.

(a) $[uv, x] = [u, x][v, x]$. 

(b) $[x, y^n] = \prod_{1 \leq i \leq n} [x, y_i]^n$.

(c) $(xy^{-1})^n = x^n \prod_{0 < i+j < n} [x, y_i, x_{i+j+1}] y^{-n}$.

Proof. (a) is easy to prove as $[G, G]$ is abelian. For the proofs of (b) and (c) see [4]. \hfill \Box

Proof of Theorem 1.1. i) Let $G$ be a generalized $3$-abelian group with the given endomorphism $\phi$ defined by $x^\phi = x^a xx^b$ for all $x \in G$, where $a, b \in G$ are fixed. To prove that $G$ is nilpotent of class three, it is enough to show that every 4-generated subgroup $\langle g_1, g_2, g_3, g_4 \rangle$ of $G$ is nilpotent of class at most $3$. Now $H = \langle g_1, g_2, g_3, g_4, a, b \rangle$ is clearly invariant under the action of $\phi$ and is thus a generalized $3$-abelian group as well. We can thus replace $G$ by $H$ and it suffices then to show that $H$ is nilpotent of class at most $3$. By [1, Théorème 3.1], $H$ is nilpotent. Now one can use \textsc{np} package of Werner Nickel [8] implemented in \textsc{GAP} [9] and \textsc{MAGMA} [3] to find the nilpotency class.
of $H$. The package $\text{nq}$ has the capability of computing the largest nilpotent quotient (if it exists) of a finitely generated group with finitely many identical relations and finitely many relations. For example, if we want to construct the largest nilpotent quotient of a group $G$ with the following presentation

$$\langle x_1, \ldots, x_n \mid r_1(x_1, \ldots, x_n) = \cdots = r_m(x_1, \ldots, x_n) = 1, w(x_1, \ldots, x_n, y_1, \ldots, y_k) = 1 \rangle,$$

where $r_1, \ldots, r_m$ are relations on $x_1, \ldots, x_n$ and $w(x_1, \ldots, x_n, y_1, \ldots, y_k) = 1$ is an identical relation in the group $\langle x_1, \ldots, x_n \rangle$, one may apply the following code to use the package $\text{nq}$ in GAP:

```gap
LoadPackage("nq"); #nq package of Werner Nickel
F:=FreeGroup(n+k);
L:=F/[r1(F.1,...,F.n),...,rm(F.1,...,F.n),
w(F.1,...,F.n,F.(n+1),...,F.(n+k))];
H:=NilpotentQuotient(L,[F.(n+1),...,F.(n+k)]);
```

Note that we need to construct the free group of rank $n+k$ because as well as the $n$ generators for $G$ we also have an identical relation with $k$ free variables. Note that the function $\text{NilpotentQuotient}(L)$ attempts to compute the largest nilpotent quotient of $L$ and it will terminate only if $L$ has a largest nilpotent quotient. Note that our identical relation is $(xy)^\phi = x^\phi y^\phi$ for all $x, y \in G$, which can be written as follows:

$$(xy)^a (xy) (xy)^b = (x^a x^b) (y^a y^b)$$

or

$$y^a (xy) x^b (xx^b y^a y)^{-1} = 1.$$

So for our problem we need the following code:

```gap
LoadPackage("nq"); #nq package of Werner Nickel
F:=FreeGroup(8); x:=F.1; y:=F.2, a:=F.3; b:=F.4;
L:=F/[(F.2^F.3)*(F.1*F.2)*(F.1^F.4)*(F.1*(F.1^F.4)*(F.2^F.1)*F.2)^-1];
H:=NilpotentQuotient(L,[F.1,F.2]);
NilpotencyClassOfGroup(H);
```

This revealed that the class of $H$ is 3.

ii) $\phi$ induces on the quotient group $G/\gamma_3(G)$ the endomorphism

$$\bar{x} \mapsto \bar{x}^\bar{a} \bar{x} \bar{x}^\bar{b}.$$

$G/\gamma_3(G)$ is nilpotent of class at most 2; by lemma 2.1, it is 3-abelian. By Levi’s Theorem, $\exp\left(\left[\frac{G}{\gamma_3(G)}\right]^{\gamma_3(G)}\right)$, which is equal to $\exp\left(\left[\frac{[G,G]}{\gamma_3(G)}\right]_{\gamma_3(G)}\right)$, divides 3. Then $[G,G]^3 \leq \gamma_3(G)$ (*). It follows that $[G,G]^3 \leq \zeta(G)$ since $\gamma_3(G) \leq \zeta(G)$. Now let $x, y, z \in G$.

$$[x, y, z]^3 = [[x, y]^3, z] = 1$$
and so \( \gamma_3(G)^3 = \{1\} \). From the relation (*), we get \([G, G]^9 \leq \gamma_3(G)^3\). Hence \([G, G]^9 = \{1\}\).

This completes the proof.

**Proof of Theorem 1.2.** \(i\) Let \( x, y \in G \).

\[
[x, y^3] = \prod_{1 \leq i \leq 3} [x, y]^{(i)} = [x, y]^3[x, y, y]^3[y, y, y] = [x, y]^3.
\]

\[
[x, y^9] = \prod_{1 \leq i \leq 9} [x, y]^{(i)} = [x, y]^9[x, y, y]^9[3^6 \prod_{3 \leq i \leq 9} [x, y]^{(i)} = 1.
\]

\[
(xy^{-1})^9 = x^9 \prod_{1 \leq i+j \leq 8} [x, i+j x]^{(i+j+1)} y^{-9}
\]

\[
= x^9[x, y]^{3^6}[x, y, y]^{84}[x, y, y]^{84} \prod_{3 \leq i+j \leq 8} [x, i+j x]^{(i+j+1)} y^{-9} = x^9 y^{-9}.
\]

\(\square\)

\(ii\) Let \( x, y \in G \). From part (i) and Theorem 1.1, we know that \( G \) is nilpotent of class at most 3 and \( \gamma_3(G)^3 = \{1\} \). Therefore

\[
x^9 = x[x, a]x^2[x, b] = x^3[x, a][x, b][x, a, x]^2.
\]

Using furthermore the facts from part (i) and Theorem 1.2 that \([G, G]^9 = \{1\}\) and that \( G \) is 3-Levi we get

\[
[x^9, y^9] = [x^3[x, a][x, b], y^3[y, a][y, b]] = [x^3, y^3] = [x^3, y]^3 = [x, y]^9 = 1.
\]

This completes the proof.

**Proof of Theorem 1.3.** We have seen that the quotient group \( G/\gamma_3(G) \) is 3-abelian. So \([G, G]_3 \leq \gamma_3(G)^{(**)}) \) and \( \frac{G}{\mathcal{Z}(G, 3)} \) (which is isomorphic to \( \frac{\mathcal{Z}(G, 3)}{\gamma_3(G)^{(**)})} \)) also is 3-abelian. Therefore the series

\[
1 \leq \mathcal{Z}(G, 3) \leq G
\]

is 3-central and \( G \) is 3-nilpotent of class at most 2. The relation (***) yields \( ([G, G]_3)^3 \leq \gamma_3(G)^3 = 1 \). This completes the proof. \(\square\)

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