Polyakov Model in ’t Hooft flux background: A quantum mechanical reduction with memory

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Abstract: We construct a compactification of Polyakov model on $T^2 \times \mathbb{R}$ down to quantum mechanics which remembers non-perturbative aspects of field theory even at an arbitrarily small area. Standard compactification on small $T^2 \times \mathbb{R}$ possesses a unique perturbative vacuum (zero magnetic flux state), separated parametrically from higher flux states, and the instanton effects do not survive in the Born-Oppenheimer approximation. By turning on a background magnetic GNO flux in co-weight lattice corresponding to a non-zero ’t Hooft flux, we show that $N$-degenerate vacua appear at small torus, and there are $N - 1$ types of flux changing instantons between them. We construct QM instantons starting with QFT instantons using the method of replicas. For example, $SU(2)$ gauge theory with flux reduces to the double-well potential where each well is a fractional flux state. Despite the absence of a mixed anomaly, the vacuum structure of QFT and the one of QM are continuously connected. We also compare the quantum mechanical reduction of the Polyakov model with the deformed Yang-Mills, by coupling both theories to TQFTs. In particular, we compare the mass spectrum for dual photons and energy spectrum in the QM limit. We give a detailed description of critical points at infinity in the semi-classical expansion, and their role in resurgence structure.
1 Introduction

Polyakov model is a well-known non-abelian gauge theory on \( \mathbb{R}^3 \) with an algebra valued Higgs scalar in the adjoint representation. The model, studied by Polyakov in mid-70s, is the first example of calculable confinement and mass gap in a \( d \geq 3 \) gauge field theory [1]. The non-perturbative phenomena can be described by using semiclassical methods, in terms of proliferation of monopole configurations (instantons in 3d, not particles), which are leading order saddles in semi-classics in Euclidean path integral formulation. Even...
though this story is fairly well-known, and a textbook material by now [2–4], some important aspects of semi-classical analysis remain to be unsatisfactory. Especially the role of critical points at infinity (and relatedly correlated instanton events) are usually pushed under the rug. For example, third-order in semi-classics produce an imaginary ambiguous contribution to the mass gap $\Delta m_g^2 = c_1 e^{-3S_0} \pm i \pi e^{-3S_0}$ that is not even mentioned in old literature, and would be disastrous for Polyakov’s analysis as it stands [5].

Another interesting question in the Polyakov model that is partially addressed in textbooks (see Banks’ book or Tong’s lecture notes [3, 4]) is following. What classical configurations the monopoles are tunneling in between? Compactifying the theory on $T^2 \times \mathbb{R}$, the flux through $T^2$, $\Phi = \frac{2\pi}{g} n$, $n \in \mathbb{Z}$ is quantized, (these are magnetic flux of monopoles valued in co-root lattice of $SU(2)$) and the configurations with non-zero flux have higher energy than the ground state:

$$\Delta E_{n,0} = \frac{1}{2} \int_{T^2} B^2 = \frac{1}{2A_{T^2}} \left( \frac{2\pi}{g} n \right)^2 > 0$$

where $A_{T^2}$ is the area of torus. See Fig. 1, left. Therefore, zero flux vacuum and higher flux configurations are non-degenerate for finite values of $A_{T^2}$. Refs.[3, 4] point out that these flux states become degenerate in the large $T^2$ limit and the monopoles are the instantons which tunnel between these states. A consequence of the analysis of Refs.[3, 4] is that in the quantum mechanical small $T^2$ limit, the ground state is unique in perturbation theory, and within the Born-Oppenheimer approximation, the low energy limit is described by a single-well potential with no memory of instantons, corresponding to central well in Fig. 1.

In this paper, we pose the following question:

Is there a compactification of the $SU(N)$ Polyakov model in which $N - 1$ types of instantons on $\mathbb{R}^3$ and corresponding perturbatively degenerate vacua survive at finite and small $T^2 \times \mathbb{R}$ in a quantum mechanical limit?

Remarkably, the answer to this question turns out to be positive. And the construction goes through another time-honored method from mid 70s, the use of background ’t Hooft fluxes [6–8]. This method is applied successfully to understand Yang-Mills dynamics in lattice gauge theory to exhibit fractional instantons [9–11]. It also plays an important role in large-$N$ volume independence via TEK models [12], which, in retrospect, provides a non-perturbative lattice formulation of non-commutative field theory [13]. ’t Hooft fluxes also enter to recent literature as a particular realization of coupling a topological quantum field theory (TQFT) to QFT [14]. Up to our knowledge, and surprisingly, Polyakov model is not examined in ’t Hooft flux background since their more or less simultaneous inceptions about four decades ago. In this work, we explore benefits of merging these two ideas.

The way we implement ’t Hooft flux is slightly different from ’t Hooft original construction and other works. Our construction takes advantage of the dynamical abelianization of the Polyakov model [15]. In particular, we turn on a background magnetic GNO flux [16], e.g. for $SU(2)$ $\Phi_{bg} = \frac{1}{2}$ through $T^2$, which lives in co-weight lattice. Since co-weight module co-root $\Gamma^\vee_w/\Gamma^\vee_r \cong \mathbb{Z}_N$, for $SU(2)$ this is equivalent to the insertion of one unit of discrete
Figure 1: Compactification of SU(2) Polyakov model to QM for $\Phi_{bg} = 0$ and $\Phi_{bg} = \frac{1}{2}$ background magnetic GNO flux. $\Phi_{bg} = \frac{1}{2}$ GNO flux corresponds to one-unit of 't Hooft flux. In the former, the vacuum is unique perturbatively at finite $A_{T^2}$, and degeneracy only emerges in the $A_{T^2} \to \infty$ limit. In the latter case, there is a two-fold degenerate vacua perturbatively, and instantons persists even at small $A_{T^2}$. Generalizing this structure to SU(N) theory, we can achieve an $N$-fold perturbative degeneracy in the QM limit, and $N-1$ types of instantons.

't Hooft flux. And indeed, something interesting happens. The energy of configurations are modified into

$$E_n = \frac{1}{2A_{T^2}} \left( \frac{2\pi}{g} \left( n + \frac{1}{2} \right) \right)^2, \quad \Delta E_{n=-1,0} = 0. \quad (1.2)$$

The $n = -1$ and $n = 0$ states become degenerate classically. In particular, in the Born-Oppenheimer approximation, the system is described by a double-well potential, and monopole instantons survive in the compactification with 't Hooft flux even at small $A_{T^2}$ limit. See Fig. 1, right. Insertion of 't Hooft flux is useful for the remembrance of the things that exist in QFT in the quantum mechanical compactification or reduction.

One importance of our quantum mechanical reduction (which remembers the instantons and vacuum structure of the infinite volume theory) is in connection to resurgence properties [17]. In particular, the correlated $[M,\overline{M}]_{\pm}$ configuration (which is part of the thimble integration of the critical point at infinity as we make precise) is two-fold ambiguous [18–21] in QFT. It is difficult to demonstrate the cancellation of this ambiguity in full QFT because the large-order perturbative data and its ambiguity upon Borel resummation are not available. However, by building a QM reduction (which possess the same action instanton configurations), we can show all orders resurgent cancellations in multiple ways, most efficiently by using the exact WKB method [22, 23]. Even though this does not solve the problem in QFT, it makes it natural to conjecture that the same resurgent cancellation is also present in full QFT. Based on this conjecture, we provide estimates of large orders of perturbation theory in QFT.

Mass gap and string tension in the Polyakov model are induced by proliferation of the $M$ and $\overline{M}$ events, e.g. $m_g^2 \sim e^{-S_0}$. However, there are certain critical points at infinity, and their thimbles include correlated $[M,\overline{M}]_{\pm}$ and $[\overline{M}M,\overline{M}]_{\pm}$ effects, which carry the same
Figure 2: Dimensional reduction of $SU(N)$ Polyakov and dYM with the insertion of 't Hooft flux down to quantum mechanics on $T^2 \times \mathbb{R}$ and $T^2 \times \mathbb{R} \times S^1$, respectively. $N = 5$ above. Both theories have $N$-perturbative vacua, $N - 1$ ($N$) monopoles are associated with the simple (affine) root systems. In the absence of 't Hooft flux, there is a unique perturbative vacuum and monopoles of the QFT do not survive in QM limit in Born-Oppenheimer approximation.

magnetic charge as $\mathcal{M}$ and $\overline{\mathcal{M}}$, respectively. One may think that these are $e^{-3S_0}$ effects and can be neglected as argued in Coleman’s lectures [24]. However, the non-trivial issue is that these configurations give a qualitatively new effect; they are multi-fold ambiguous, and their contribution must be included. Their contribution to mass gap is ambiguous, of the form $\Delta m^2 = c_1 e^{-3S_0} \pm i\pi e^{-3S_0}$. As it stands, this would be an important problem concerning the mass gap in the Polyakov model. We are supposed to obtain a real mass gap, but we obtain corrections that have imaginary ambiguous parts! This is not something that one can be dismissive about. In our QM reduction, we show the resurgent cancellation of such ambiguities, leading to meaningful results.

These properties generalize to $SU(N)$ gauge theory Higgsed to $U(1)^{N-1}$. In this case, with the insertion of a non-trivial background magnetic flux in co-weight lattice $\Gamma_w$, $N$ distinct perturbative flux vacua remain degenerate in perturbation theory, and there are $N - 1$ instantons connecting them as shown in Fig. 2. Therefore, 't Hooft flux allows the instantons in QFT to survive in the QM limit even on small $T^2$.

This idea is similar to the role that global symmetry twisted boundary conditions play in sigma models on $\mathbb{R} \times S^1$ [26–35] and the role that center-symmetric holonomy plays on $\mathbb{R}^3 \times S^1$, see e.g. [17, 36].

This work can be viewed as an extension of the recent study in $SU(N)$ Yang-Mills theories in $\mathbb{Z}_N$ TQFT background formulated in $\mathbb{R} \times S^1 \times T^2$ [15] with center-stabilizing

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1 As a matter of fact, the generic confinement mechanism in gauge theories on $\mathbb{R}^3 \times S^1$ is due to second order ($e^{-2S_0}$) effects in semi-classics, called magnetic bions. Monopole-instantons, which appears at first order ($e^{-S_0}$), do not typically induce confinement due to their fermion zero mode structure. See [25].

2 In large volume, turning on a discrete 't Hooft flux background does not alter local dynamics. But on small space, turning on an appropriate 't Hooft flux changes dynamics in a better way relative to its absence! It helps the topological excitations of the large-volume theory to survive in a small volume limit. In this sense, perhaps, it is better to perform compactifications with 't Hooft flux insertion in general. This is also the perspective of old lattice works [9–11].
double-trace deformation in the $S^1$ circle. See also [37] for related ideas that connect YM to sigma model with global symmetry twisted boundary conditions [26]. The deformation on $S^1$ circle ensures adiabatic continuity between small $\mathbb{R}^3 \times S^1$ and $\mathbb{R}^4$ [38]. This is already checked on the lattice and it works, see e.g. [39]. In both dYM, and Polyakov, the long-distance dynamics abelianize to $U(1)^{N-1}$. In this sense, we take advantage of two aspects:

1) Dynamical abelianization and semi-classical calculability;

2) Background magnetic flux valued in the co-weight lattice $\Gamma^\vee_w$ and corresponding discrete ’t Hooft flux, which are related because $\Gamma^\vee_w/\Gamma^\vee_r \cong \mathbb{Z}_N$.

There are multiple differences between the present work and Ref. [15]. In the latter, the Higgs scalar is group valued (instead of being algebra valued), and there is also a topological $\theta$ angle. As a result, within Born-Oppenheimer approximation, the $SU(2)$ Polyakov model reduces to double-well potential on $\mathbb{R}$, and deformed YM on $\mathbb{R} \times S^1 \times T^2$ reduces to the particle on a circle $S^1$ with two-minima on the circle. The relation between these two constructions is discussed in Section. 6 and depicted in Fig. 2.

2 Quick summary of $SU(N)$ Polyakov Model

We first briefly review the $SU(N)$ Polyakov model [1], and set the notation for the rest of the paper. Consider a $SU(N)$ non-abelian gauge theory coupled to an adjoint scalar field with Euclidean action:

$$S = \int \frac{1}{2g_3^2} \left( \text{tr} (F \wedge \star F) + \text{tr} (D\phi \wedge \star D\phi) + V(\phi) \right)$$

(2.1)

where 

$$F = DA \quad , \quad D = d + iA.$$ 

We choose the vacuum structure of $V(\phi)$ such that the vacuum expectation value of the adjoint valued Higgs scalar is given by

$$\phi_v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix} = \begin{bmatrix} -\frac{N-1}{2}v \\ -\frac{N-3}{2}v \\ \vdots \\ +\frac{N-1}{2}v \end{bmatrix}$$

(2.2)

and leads to the breaking of $SU(N)$ symmetry to $U(1)^{N-1}$. For example, for $SU(2)$ theory, we take $V(\phi) = \lambda (\text{tr}(\phi^2) - \frac{1}{4} v^2)^2$ where $\lambda$ is dimensionless. We chose the distance between the eigenvalues to be $v$, but this can be relaxed through the discussion as long as $v_i$ are non-degenerate. For convenience, the classical mass dimensions of fields and parameters in the theory are:

$$\text{mass dimensions} : \quad [g_3^2] = [A] = [\phi] = +1, \quad [\lambda] = 0$$

(2.3)
As a result of the adjoint Higgsing, the perturbative spectrum contains \(N(N-1)\) massive gauge bosons, or \(W\)-bosons, and \(N-1\) massless gauge bosons corresponding to the photons of the unbroken \(U(1)^{N-1}\) gauge structure. For simplicity of the discussion, and without loss of generality, we will also take \(|v_i - v_{i+1}| = v\). As a result, the masses of the \(N-1\) lightest \(W\)-boson masses are equal to each other and given by

\[
m_W = |v^i - v^{i+1}| = v
\]  

There are also \(N^2 - 1\) massive scalars. These scalars also acquire masses due to adjoint Higgsing. The lightest \((N-1)\) scalar masses are given by

\[
m_\phi \sim \sqrt{\lambda}v \quad \text{(lightest scalar mass)}
\]

The lightest scalars may be heavier or lighter than the lightest \(W\)-bosons depending on the value of \(\lambda\).

The theory, at low energies can be described as a \(U(1)^{N-1}\) abelian gauge theory to all orders in perturbation theory. However, non-perturbatively, the gaplessness is destabilized by the monopole-instantons, the leading non-perturbative saddles in the problem. Constructing a dilute gas of these instantons and anti-instantons, Polyakov [1] showed that the dual photons acquire a non-perturbative exponentially small mass. Below, we briefly describe this phenomenon, and then, we will move to a more subtle subject of critical points at infinity.

In the vicinity of BPS limit, where \(\lambda \ll 1\), [40] monopole-instantons are solution to the self-duality equations:

\[
F = \pm \star_3 D\phi.
\]

The minimal action monopoles are associated with the simple root system. They carry a magnetic charge

\[
Q_{M_i} = \frac{2\pi}{g_3} \alpha_i
\]

and the classical actions of the \(\alpha_i\) monopole takes the values

\[
S_0^{(i)} = \frac{4\pi v}{g_3^2} = \frac{s_0}{g_2^2}
\]

where we defined the dimensionless coupling \(g_2^2 = g_3^2 v\) and the constant \(s_0 = 4\pi\). For later purposes, describing monopole interactions, it is also useful to define dimensionless position \(\tilde{r} = vr\).

Away from the BPS limit, where \(\lambda\) can take any value, the coupled second order Euclidean equations of motions, \(\frac{\delta S}{\delta A_\mu} = 0, \frac{\delta S}{\delta \phi} = 0\) also admit solutions associated with magnetic charge (2.7). In that case, the action of the monopoles are given by

\[
S_0(g^2, \lambda) = \frac{4\pi}{g^2} f(\lambda), \quad \text{where} \quad f(0) = 1, \quad f(\infty) = 1.787
\]
where \( f(\lambda) = 1 + \frac{\lambda^2}{2} + \ldots \) \([41]\) at small \( \lambda \). Even at arbitrarily large-\( \lambda \), the action of the elementary monopole saddles does not exceed \( 1.787S_0 \).

The monopole core size is the inverse \( W \)-boson mass, \( r_m \sim m_W^{-1} = v^{-1} \). At distances larger than \( r_m \), the dynamics should be described by an abelian gauge theory. In 3d, one can use abelian duality to express the \( N-1 \) photons in terms of dual scalar \( \sigma \). The reason for using dual fields is that the long distance EFT and the monopole operators has a natural representation in terms of dual photons. The abelian duality relation is \( \star_3 F = \frac{g^2}{8\pi^2} d\sigma \) where \( F \) denotes Cartan components of non-abelian gauge field. In the \( SU(N) \) theory, the \( \sigma \) field has a periodicity determined by weight lattice, \( \sigma \sim \sigma + 2\pi\mu_i, \ \mu_i \in \Gamma_w \), and

\[
\sigma \in \frac{\mathbb{R}^{N-1}}{2\pi\Gamma_w}
\]  

is the fundamental domain of \( \sigma \).

There are \( N-1 \) types of the fundamental monopole instantons that appears as leading order saddles, and the monopole operators are given by\(^3\)

\[
\mathcal{M}_{\alpha_i}(x) = Ke^{-S_0}e^{i\alpha_i \cdot \sigma(x)} \equiv \xi e^{i\alpha_i \cdot \sigma(x)} \quad i = 1, \ldots, N-1.
\]  

(2.11)

Since monopoles are finite action, they proliferate in the Euclidean vacuum. The density of the monopoles of any type is given by \( \xi \sim e^{-S_0} \). The grand canonical ensemble of the monopole gives the partition function:

\[
Z = \int D\sigma \exp \left[-\left( \frac{g^2}{8\pi^2} |d\sigma|^2 - 2 \sum_{i=1}^{N-1} \xi \cos (\alpha_i \cdot \sigma) \right) \right]
\]  

(2.12)

This partition function should be viewed with a UV cut-off which is of the order of inverse monopole core size,

\[
\text{UV cutoff in monopole EFT :} \quad \Lambda_{\text{UV}} \sim r_m^{-1} \sim m_W.
\]  

(2.13)

The EFT (2.12) based on monopoles generates mass gap for \( N-1 \) dual photons. The masses of \( N-1 \) types of dual photon can be obtained by diagonalizing quadratic part of the potential and is given by

\[
m_k^2 = m_\gamma^2 \sin^2 \left( \frac{\pi}{2N} k \right), \quad k = 1, 2, \ldots, N-1
\]  

(2.14)

where \( m_\gamma^2 \sim \xi/g^2 \). The fact that the mass gap is sourced by the fugacity of the monopole-instantons is the well-known result of Polyakov.

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\(^3\)If the diagonal of the adjoint scalar remains light or gapless for some reason (e.g. as in \( N = 2 \) SYM theory on \( \mathbb{R}^4 \) \([42]\)), it should also be included in EFT and the monopole operator should be modified into \( e^{-\frac{g^2}{8\pi^2} \phi(x) \sigma} + i \alpha_i \cdot \sigma \cdot \phi(x) \). In the present case, it is consistent to set the fluctuations \( \phi(x) \) to zero once we consider the physics at long distances.
3 Critical points at infinity and cluster expansion

In the Polyakov model, the instantons with a magnetic charge, as described above, are solutions to either BPS equations (as $\lambda \to 0$) or full second-order coupled equations of motions. Their proliferation leads to a dilute gas of monopoles described in (2.12), which is first order in $e^{-S_0}$. However, as in cluster expansion in statistical mechanics [43], some effects arise due to correlated events or molecules of instantons. These are higher-order effects in semi-classics. We briefly review the cluster expansion in Appendix B.

Even though the mass gap is of order $m_0^2 \sim e^{-S_0}$ at leading order in semi-classics, there are qualitatively new effects at higher order. For example, at third-order semi-classics, there are contributions to the mass gap that are of the form $\Delta m_0^2 \sim e^{-3S_0} \pm i\pi e^{-3S_0}$, where the imaginary part is two-fold ambiguous. These phenomena usually manifest themselves in a careful treatment of critical points at infinity.

In this section, we will describe some features of critical points at infinity and correlated $k$-events. In particular, we will assert that some of these ambiguous correlated events are related to the resurgence properties of QFT and cancel in the full resurgent semi-classical analysis. However, it is hard to prove this statement in full QFT. Yet in many quantum mechanical examples, our understanding of all order resurgent cancellations is much better. Therefore, it is desirable to reduce QFT to QM while keeping its non-perturbative aspects as intact as possible.

Up to our knowledge, no such compactification of the Polyakov model is known so far. The regular compactification with periodic boundary conditions on small $T^2 \times \mathbb{R}$ has a single perturbative vacuum and no memory of the instanton events in the Born-Oppenheimer approximation. In the next section, to circumvent this problem, we will construct the Polyakov model in 't Hooft flux background on small $T^2 \times \mathbb{R}$ that remembers the instanton events of full field theory. In that context, we can prove for example all order resurgent properties by using exact-WKB formalism. This leads us to a conjecture that the same resurgent cancellations also take place in the context of full QFT. Assuming this is the case, we will provide predictions of the large-order growth around the perturbative vacuum and monopole-instanton saddles.

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4The definition of the critical point at infinity is as follows: Consider a two instanton configuration with a (repulsive or attractive) interaction in between. At any finite separation, the combination is not a solution, because equations are non-linear and superposition of the solutions is not a solution, i.e., $\frac{dV_{\text{int}}(r)}{dr} \neq 0$ at any finite separation. But it becomes a genuine saddle at infinite separation because $\frac{dV_{\text{int}}(r)}{dr}|_{r=\infty} = 0$. If we map $r \in \mathbb{C}$ complex domain to Riemann sphere by using one-point compactification, the critical point is the North pole and is on the same footing with any other regular critical point in a certain sense. The feature that makes this saddle special is its non-Gaussian nature. Because of this property, one needs to integrate over the whole steepest descent cycle to obtain qualitatively correct non-perturbative contributions. Almost all saddles in QFT and QM are critical points at infinity. Yet, they are the least appreciated and understood ones. See Ref. [21] in the context of QM.

5However, if we consider flux $|\Phi = \frac{n \pi}{2}|n\rangle$, $n = \pm 1$ states in QM limit, these are degenerate at finite box too. The tunneling between them is two-monopole events with $\Delta n = 2$, see Fig. 1. But these states do not survive in BO approximation, and in the description of ground-state properties of QM.
Correlated and uncorrelated 2 instanton events: How to describe the critical point at infinity and correlated instanton events? Let us start with the possible combinations at the two instanton order, e.g., $M_i M_j$, $M_i M_j$ or $M_i, M_j$ combinations. The classical interaction between these events and their BPS ($\lambda \to 0$) limit are:

\[ V_{\text{int}}(r) = \begin{cases} \frac{Q_i Q_j}{4\pi r^2} (1 - e^{-m_\phi r}) \to 0 & \text{for } (M_i, M_j) \\ \frac{Q_i Q_j}{4\pi r^2} (1 + e^{-m_\phi r}) \to \frac{2\pi}{g_2^2} \alpha_i (\alpha_j) & \text{for } (M_i, \overline{M_j}) \end{cases} \] (3.1)

where $r$ is the distance between the two monopole-instantons, $1/r$ is induced by dual photon exchange and $e^{-m_\phi r}/r$ is due to scalar exchange. Close to BPS limit, where $\lambda \to 0$, the interaction between $M_i, M_j$ is zero, while the interaction between $M_i, \overline{M_j}$ doubles compared to just photon induced interactions, i.e, $(1 + e^{-m_\phi r}) \approx 2$ for any finite separation $r$. From hereon, we assume $M_i, M_j$ pairs are non-interacting, while $M_i, \overline{M_j}$ is interacting. This interaction can be repulsive, attractive or zero depending on charges because

\[ \alpha_i \cdot \alpha_j = 2\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1}, \quad i, j = 1, \ldots, N - 1 \] (3.2)

If the interaction is zero, the corresponding duo is a critical point at any separation (i.e. collection if critical point becomes a critical line), and there are no correlated 2-events. If the interaction is non-zero, this leads to a critical point at infinity and correlated 2-event amplitudes.

Due to the non-zero interaction between monopole-instantons, the partition function at 2 instanton order is expressed as

\[ Z_2 = \frac{[C_1]^2}{2!} \int d^3 r_1 d^3 r_2 e^{-V_{\text{int}}(|r_1 - r_2|)}, \] (3.3)

where

\[ [C_1] = \xi = Ke^{-S_0} = Av^3 S_0^2 e^{-S_0} \] (3.4)

is the monopole fugacity. $S_0^2$ arises from four zero-modes of monopoles, three positions, and 1 angular, each contributing $S_0^2/2$, and $v^3 = v^{-1}$ arises from the measure and Jacobian associated with the angular moduli, respectively, see e.g. [44, 45]. We will set $v = 1$ unless it is useful to see its existence explicitly. Therefore, all lengths are measured in units of monopole radius $r_m = v^{-1}$.

Defining $R = \frac{1}{2}(r_1 + r_2)$, $r = r_1 - r_2$, we can express (3.3) as an integral over the center coordinate and an integral over the relative coordinate. Let us denote $\int d^3 R = \mathcal{V}$ where $\mathcal{V}$ is the volume of the space-time manifold. Then, $Z_2$ takes the form

\[ Z_2 = \frac{1}{2}[C_1]^2 \mathcal{V} \int d^3 r e^{-V_{\text{int}}(r)} \] (3.5)

The integral, $\int d^3 r e^{-V_{\text{int}}(r)}$ as we will show by using various different regularization, has two parts. An infinite part (which can be regularized to the volume $\mathcal{V}$ of space-time manifold) and a universal finite part, a function of $g^2$. We will show that (3.5) can be written as

\[ Z_2 \sim \xi^2 (\mathcal{V}^2 + \mathcal{V}(g^2)) \] (3.6)
This form can be interpreted as a term in the cluster expansion (B.5). In our physical construction, in QFT, the relative position \( r \in \Gamma_{QZM} \) is a quasi-zero mode coordinate between two instanton event. However, in semi-classics, the cluster expansion needs to be suitably generalized to take into account steepest descent cycles over which one needs to integrate over.

**Repulsive interaction:** Let us assume first that the interaction is repulsive and we can write

\[
V_{\text{int}} = \frac{2\pi |\alpha_i \cdot \alpha_j|}{g^2 r}, \quad g^2 \equiv g_3^2 \frac{3}{v}, \quad \text{and} \quad \tilde{r} = vr. \tag{3.7}
\]

The integrand \( r^2 e^{-V_{\text{int}}} \) approaches 0 as \( r \to 0 \). For \( r \gg (g_3^2)^{-1} \), since \( e^{-V_{\text{int}}} \sim 1 \), the integral is divergent and takes the form \( \int d^3 r \sim V \).

To handle the divergence, we first use a hard cut-off in \( r \) integration, given by \( R \gg r_m = v^{-1} \). Then, changing the integration variable as

\[
z = \frac{2\pi |\alpha_i \cdot \alpha_j|}{g^2 \tilde{r}} \tag{3.8}
\]

the radial integral is written as

\[
J_2(g^2) = v^{-3} \left( \frac{2\pi |\alpha_i \cdot \alpha_j|}{g^2} \right)^3 4\pi \int_\delta^\infty dz z^{-4} e^{-z}. \tag{3.9}
\]

where \( \delta = \frac{2\pi |\alpha_i \cdot \alpha_j|}{Rg^2} \). The resulting integral is a representation of the incomplete gamma function:

\[
J_2(g^2) = 4\pi v^{-3} \lim_{\delta \to 0} \left\{ \left( \frac{2\pi |\alpha_i \cdot \alpha_j|}{g^2} \right)^3 \Gamma(-3, \delta) \right\}. \tag{3.10}
\]

Note that \( \Gamma(-3, \delta) \) has a branch cut along \( \delta = (-\infty, 0) \) [46, Sec. 8.2]. Its expansion around \( \delta = 0 \), which corresponds to large but finite separation of the monopole instantons, is

\[
\Gamma(-3, \delta) \simeq \frac{1}{3\delta^3} - \frac{1}{2\delta^2} + \frac{1}{2\delta} + O(\delta) + \frac{\ln(\delta) + \gamma}{6} - \frac{11}{36} \tag{3.11}
\]

Using \( \delta = \frac{2\pi |\alpha_i \cdot \alpha_j|}{Rg^2} \), we can write \( J_2(g^2) \) as

\[
J_2(g^2) \simeq \lim_{R \to \infty} \frac{4\pi}{3} R^3 \left( 1 + O(R^{-1}) \right) + 4\pi v^{-3} \left( \frac{2\pi |\alpha_i \cdot \alpha_j|}{g^2} \right)^3 \frac{1}{6} \left( \ln \left( \frac{2\pi |\alpha_i \cdot \alpha_j|}{g^2} \right) + \gamma - \frac{11}{6} \right) \tag{3.12}
\]

as a result of which

\[
J_2(g^2) = V + I(g^2) \tag{3.13}
\]

and we get the form (3.6) for \( Z_2 \).

**Remark:** In (3.12), the leading divergent part \( \left( \frac{4\pi}{3} R^3 \right) \) and the finite part \( I(g^2) \) are independent of the regularizations of integration. We used two other regularization as well to
demonstrate this point. In both, we obtain a divergent part (identified as volume) and a finite part denoted as $I(g^2)$ which agrees with (3.12). These two other regularizations are described in Appendix C.

Let us digest the result we have. Equation (3.6) is one aspect of critical point at infinity. For example, in the set of action $2S_0$ configurations, let us consider an $\mathcal{M}_{\alpha_i}$ and $\mathcal{M}_{\alpha_{i+1}}$ event. The interaction between them is repulsive so, $\frac{dV}{dr} = 0$ at $r = \infty$. The integral over QZM direction has two parts, a part divergent with volume $V$ and a finite part $I(g^2)$. There is also an overall factor of $V$ that arise from zero mode integration, producing (3.6).

As a result, we obtain $e^{-2S_0(\mathcal{V}^2 + \mathcal{V}I(g^2))}$ from critical point at infinity. The maximally extensive part in volume provides uncorrelated contribution of two instantons, square of the 1-cluster events $[C_1]^2$, while the sub-extensive part provides correlated 2-instanton contributions, from the set of 2-cluster events $[C_2]$. (e.g. $[M_{\alpha_i}M_{\alpha_{i+1}}] \sim e^{-2S_0 I(g^2)}$).

In this sense, the simplest critical point at infinity contains both uncorrelated two 1-instanton events and correlated 2-instanton events built in. Then, following the guideline in Appendix B, the equation (3.6) gets the following general form:

$$Z_2 = \mathcal{V}[C_1]^2 + \mathcal{V}[C_2]. \quad (3.14)$$

**Attractive Interactions:** Now, we turn our attention to the more interesting case, i.e. when the sign of $V_{\text{int}}$ becomes negative, and the interaction potential becomes attractive. The counterpart of the (3.3) integral is

$$J_2(g^2) = 4\pi \int dr r^2 e^{\pm \frac{2\pi}{g^2} |\alpha_i \cdot \alpha_j|} \quad (3.15)$$

and it can be handled in two related ways: One is analytic continuation and the other is judicious choice of the integration cycle\(^6\). Here we will use the analytical continuation method, while explain the integration cycles in Appendix D.

The analytical continuation idea, which was introduced in the context of QM in [18], is the following. Taking $g^2 \rightarrow -g^2$ in (3.15) maps the problem to the one studied above, which yields (3.11), $J_2(g^2)$. Taking $g^2 \rightarrow -g^2$ in $J_2(g^2)$, therefore, should bring us back

\(^6\)In the past, attractive interaction potentials caused too much confusion and hardship even in the simpler case of quasi-zero mode integral accounting for instanton anti-instanton interactions in quantum mechanics. Naive (and incorrect) perspective on such an integral was following. If we consider $e^{-\frac{\pi}{g^2}}$, it vanishes as $r \rightarrow 0$ very fast, while $\frac{e^{\frac{2\pi}{g^2} r^2}}{r^2}$ blows up as $r \rightarrow 0$. Both of the integrands tends to 1 as $r \rightarrow \infty$, which is responsible for the $\mathcal{V}$ factor that emanates from the integrals $\int drr^2 e^{\pm \frac{\pi}{g^2}}$. But clearly, the attractive interaction seem to pose a puzzle as $r \rightarrow 0$, as the integrand blows up as $r \rightarrow 0$. However, this perspective is incorrect in properly done semi-classics. In semi-classical formulation, the integration cycle is the steepest descent cycle. For the critical point at infinity, the descent cycles are

$$r \in [0, \infty], \quad \text{repulsive},$$

$$r \in [0, -\infty] \cup C^\infty_\infty, \quad \text{or} \quad r \in [0, -\infty] \cup C^\infty_\infty, \quad \text{attractive}, \quad (3.16)$$

So, for the attractive case, integration over $[0, -\infty]$ yields not a pathological integral, but same integral as in the repulsive case, and the integral over $C^\infty_\infty$ and $C^\infty_\infty$ picks the pole at $r = 0$, depending the orientation producing $\pm i$ in (3.17).
Figure 3: \(\Gamma(-3, \delta)\) has a branch cut along the negative real axis. It is possible to compute it at a point \(-\delta_0 < 0\) on the cut by analytically continue its value from \(\delta_0 > 0\). However, paths 1 and 2 lead complex conjugate results so that the function \(\Gamma(-3, \delta)\). This is the source of the ambiguous result of (3.17).

to the integral we would like to study, \(\tilde{J}_2(g^2)\). However, as we mentioned above, \(\Gamma(-3, \delta)\) has a branch cut along the negative real axis. Since \(g^2 \to -g^2\) corresponds to \(\delta \to -\delta\) in (3.10), there are two possible directions, clockwise and counter-clockwise for the analytical continuation, see Fig. 3. This freedom of choice leads to two complex conjugate results:

\[
\tilde{J}_2(g^2) = J_2(g^2 e^{\pm i\pi}) = \mathcal{V} - I(g^2) \pm \frac{2\pi^2}{3} \left( \frac{2|\alpha_i \cdot \alpha_j|}{g^2} \right)^3.
\]  

(3.17)

Now, as in the repulsive case, let us take an example in the set of instanton and anti-instanton configurations. If we consider an \(\mathcal{M}_{\alpha_i}\) and \(\overline{\mathcal{M}}_{\alpha_i}\) event, the interaction between them is attractive. The critical point is again at infinity. Now, the integral over the QZM direction has three ingredients, a part divergent with volume \(\mathcal{V}\) and a finite real part, and a finite imaginary two-fold ambiguous part. As a result, the contribution of critical point at infinity to partition function is of the form

\[
\xi^2 \int d^3r_1 d^3r_2 e^{-V_{\text{int}}(r_1-r_2)} = \xi^2 \left( \mathcal{V}^2 - \mathcal{V} I_{\pm}(g^2) \right)
\]

(3.18)

and we recover the form of (3.6) and (3.14). Note also that the imaginary part belongs to 2-cluster \([C_2]\) configuration \([\mathcal{M}_{\alpha_i}, \overline{\mathcal{M}}_{\alpha_i}]\).
is a characteristic feature of critical point at infinity, and the absence/presence of ambiguity is a feature of the corresponding steepest descent cycle. As in QFT, in QM, the exponentiation of the cluster expansion yields a contribution to vacuum energy of the form
\[ E_{0,\text{np}} \sim -e^{-S_0} - e^{-2S_0} I(g^2) - e^{-2S_0} I_\pm(g^2) + \ldots \]

3.1 Resurgence structure and predictions for large order behavior

**Vacuum Energy**: As it is defined in (B.1), the vacuum energy is related to the terms in the cluster expansion via their exponentiation
\[ E \sim -([C_1] + [C_2] + [C_3] + [C_4]) \] where \([C_k]\) are clusters of \(k\)-instanton events. For example,
\[ [C_2] = \left\{ [M_\alpha_iM_{\alpha_{i+1}}], [M_\alpha_i\overline{M}_{\alpha_{i}}], i \in [1, N-2], [M_\alpha_i,\overline{M}_{\alpha_{i}}]_\pm, i \in [1, N-1] \right\} \]

We view the Euclidean vacuum as a grand canonical example of all such configurations, not only as a gas of monopoles which takes into account the monopole-instanton events in \([C_1]\). The events in \([C_2]\) are called magnetic bions and neutral bions. The non-perturbative contribution to the ground state energy up to two instanton level comes from \([C_1]\) and \([C_2]\) and includes monopole terms of order \(e^{-S_0}\) and bion terms of order \(e^{-2S_0}\) effects. Most important for us is the fact that neutral bion contributions are two-fold ambiguous due to the \(I_\pm\) factor. The non-perturbative imaginary ambiguous part is given by:
\[ \text{Im}E_{\text{np}} = \text{Im}[M_\alpha_i\overline{M}_{\alpha_{i}}]_\pm = \pm \frac{2\pi^2}{3} \left( \frac{2\pi |\alpha_i \cdot \alpha_i|}{g^2} \right)^3 [M_\alpha_i][\overline{M}_{\alpha_{i}}] \]
\[ \sim \pm \left( \frac{s_0}{g^2} \right)^7 e^{-2s_0/g^2}. \]

where we used vacuum values of the monopole operators (3.4)
\[ M_\alpha_i = \overline{M}_{\alpha_i} = \left( \frac{s_0}{g^2} \right)^2 e^{-\frac{s_0}{g^2}}. \]

Note that in (3.20), \(7 = 2 + 2 + 3 = 2N_{\text{f}} + N_{\text{qg}}\) where factors of 2 are determined by number of zero modes of instantons divided by two and factor of 3 comes from the quasi-zero mode integration, and this will have a crucial effect in large order growth of perturbation theory.

In the Higgsed vacuum (2.2), perturbative contributions to the vacuum energy density \(E_0\) is expected to be an asymptotic divergent series, whose Borel resummation \(S_\pm E_0\) is also two-fold ambiguous. By resurgence, it is expected that the ambiguity in the \(S_\pm E_0\) to cancel against the ambiguity of the \([M_\alpha_i,\overline{M}_{\alpha_{i}}]_\pm\), i.e.,
\[ \text{Im}[S_\pm E_0 + [M_\alpha_i,\overline{M}_{\alpha_{i}}]_\pm] = 0 \]

Although one may think naively that they are small contributions to leading order monopole effects, in theories with fermions they lead to qualitatively new effects compared to monopoles. In QCD(adj) on \(\mathbb{R}^3 \times S^1\), there are \(N\) types of monopoles. Each acquires 2\(N_f\)-fermi zero modes and does not lead to a mass gap for gauge fluctuation. There are \(N\)-types of magnetic bions and they gap out the dual photons [47].
Similarly, perturbation theory around the monopole-instanton $M_{\alpha i} = (s_0/g^2)^2 e^{-s_0/g^2} P_{M}(g^2)$ is also expected to be divergent asymptotic expansion, whose Borel resummation must have ambiguity at three instanton order. Indeed, the QZM integration of the three instanton event has an ambiguity as well $[M \overline{M} M]_\pm$. For semi-classical expansion to be free of pathologies, the ambiguous imaginary parts (at three instanton level) must again cancel each other, and one should have

$$\text{Im} \left[[M] S_\pm P_{M} (g^2) + [M \overline{M} M]_\pm \right] = 0. \quad (3.23)$$

However, it is very difficult to prove (3.22) and (3.23) in full QFT, as we do not have immediate access to large orders of perturbative expansion. On the other hand, it is possible to prove such exact cancellations in quantum mechanics explicitly, e.g. in double-well potential [18–20, 48].

One may hope that by a dimensional reduction of the Polyakov model, one can prove the counterparts of (3.22) (3.23) at least in quantum mechanical context. However, it turns out that a naive reduction on $T^2 \times \mathbb{R}$ with periodic boundary conditions on $T^2$, there is a unique perturbative vacuum. Within Born-Oppenheimer approximation, instantons do not play a role in vacuum properties. This will be explained in detail in Section 5.1.

On the other hand, by turning on a non-trivial background GNO magnetic flux (for which discrete ’t Hooft flux is non-zero), we can reduce the Polyakov model to QM in such a way that the $N$ perturbative vacuum and $N - 1$ types of instantons of the QFT survive within Born-Oppenheimer approximation. This is the merit of turning a classical flux background in the Polyakov model. In the QM context, the resurgent cancellations such as (3.22) and (3.23) are already proven by either using the exact WKB method or by explicit computation. Below, we make a natural assumption that such resurgent cancellations survive in QFT as well and make non-trivial predictions for large-order growth of perturbation theory around the perturbative vacuum and monopole-instanton saddle.

**Prediction for large-order perturbative expansion:** The cancellations in (3.22) and (3.23) imply that the asymptotic behaviour of the perturbative expansions both around the perturbative vacuum saddle $E_0 = \sum_k b_k^{(0)} g^{2k}$ and instanton $P_{M}(g^2) = \sum_k b_k^{(1)} g^{2k}$ can be determined by using dispersion relations. For example,

$$b_k^{(0)} = \frac{1}{\pi} \int_0^\infty d(g^2) \text{Im} \mathcal{E}_{np}(g^2) \left(\frac{\text{Im} \mathcal{E}_{np}(g^2)}{(g^2)^{k+1}}\right), \quad (3.24)$$

where $b_k^{(0)}$ is perturbative coefficients around the vacuum. Then, using the results for $\text{Im}[M \overline{M}]$ in (3.20), we determine the large order behaviour of the perturbative expansions around the vacuum as

$$b_k^{(0)} \sim \frac{\Gamma(k + 7)}{(2s_0)^k}, \quad s_0 = 4\pi \quad (3.25)$$

---

8In general, the perturbative expansion is expected to be a double expansion in $g^2$ and $\lambda$. Indeed, the monopole-action (2.9) depends on both parameters, but the dependence on $\lambda$ becomes very weak in the BPS limit, $\lambda \to 0$. We work very close to this limit to avoid such subtleties, and study resurgence properties in terms of coupling $g^2$ only.
Similarly, the ambiguity in the three instanton sector is of the form (E.23)

\[ \text{Im}[\mathcal{M}\overline{\mathcal{M}}]_\pm \sim \pm \left( \frac{s_0}{g^2} \right)^{12} \ln \left( \frac{s_0}{g^2} \right) e^{-\frac{3s_0}{g^2}} \]  

(3.26)

Here, \( 12 = 2 + 2 + 2 + 3 + 3 = 3N_{zm} + 2N_{qzm} \) where factors of 2 are determined by number of zero modes of instantons divided by two and factors of threes are the consequence of integration over the two quasi-zero mode (QZM) coordinates. The log factor \( \ln \left( \frac{s_0}{g^2} \right) \) is also result of the QZM integrations and starts to appear at third order in semi-classics, and has an interesting effect on large-order growth. As one proceeds to order \( p \) in semi-classics, polynomials of order \( p - 2 \) in this logarithmic factor will arise. Given that instantons has prefactors that depend on \( g^2 \) as \( \mathcal{M}_\alpha = (s_0/g^2)^2 e^{-\frac{2s_0}{g^2}} \), we can determine the large order behaviour of perturbation theory around an instanton as:

\[ b_k^{(1)} \sim \frac{\Gamma(k + 10) \ln k}{(2s_0)^k} \]  

(3.27)

Again, the factor of ten in Gamma function is just \( 3N_{zm} + 2N_{qzm} - N_{zm} \) where the subtraction takes into account the prefactor of instanton.

Both in (3.25) and (3.27), the enhancement relative to \( k! \) behaviour is a consequence of the combination of zero and quasi-zero mode contributions. Furthermore, one can check that the extra \( \ln k \) enhancement in (3.27) is a consequence of \( \ln \left( \frac{s_0}{g^2} \right) \) in (3.26). Such \( \ln k \) enhancement of the large-order behaviour was first shown and tested in the context of quantum mechanics [49]. Up to our knowledge, its appearance in a QFT setting is new. These predictions can be tested by using stochastic perturbation theory and lattice techniques, see e.g. [50, 51].

**Remark on mass gap:** The observation of the cancellation between ambiguous parts is especially important in the light of Polyakov’s derivation of the mass gap. Indeed, if we consider the theory on \( \mathbb{R}^3 \), the mass gap would be sourced by the proliferation of the monopoles \( \mathcal{M}_\alpha \) at leading order, leading to a well-known result, \( m_g^2 \sim e^{-S_0} \). There are also contribution at second order in semi-classics induced by magnetic bions and there are unambiguous \( [\mathcal{M}_\alpha, \overline{\mathcal{M}}_{\alpha|\pm}] \) \( \sim e^{-2S_0} \). However, at third order in semi-classics, there are correction to mass gap sourced by the proliferation of 3-events, e.g. \( [\mathcal{M}_\alpha, \overline{\mathcal{M}}_\alpha, \mathcal{M}_\alpha] \) which possess two-fold ambiguous imaginary parts. Including both types of effects, the result would be \( m_g^2 \sim e^{-S_0} + e^{-2S_0} + e^{-3S_0} \pm e^{-3S_0} \), which would be quite undesirable. The implication of (3.23) is that this feature of semi-classics would be cured by a similar feature, sourced by non-Borel summability of the perturbation theory around the instanton.

The resurgent cancellations render the combination of semi-classical analysis and perturbation theory well-defined, even though each part is ambiguous in its own right. As stated earlier, we cannot currently prove these relations in the context of full QFT on \( \mathbb{R}^3 \). However, we can construct a compactification of the theory on \( T^2 \times \mathbb{R} \) in which the instantons of the model of \( \mathbb{R}^3 \) survive by using magnetic GNO (and ’t Hooft) flux background. Furthermore, the resurgent cancellation (5.16) holds in the quantum mechanical limit. This naturally leads us to conjecture that the relations such as (3.22) and (3.23) are valid in the QFT limit on \( \mathbb{R}^3 \).
4 Turning on ’t Hooft flux background

The original $SU(N)$ Polyakov model (2.1) has an exact $\mathbb{Z}^{[1]}_N$ 1-form symmetry, since the dynamical matter field in the theory is in the adjoint representation. We can consider turning on a classical background field for the one-form symmetry, called ’t Hooft flux background \[6–8\]. On $T^3$, we can insert $\mathbb{Z}^{[1]}_N$ fluxes $\ell_{ij}$ through $ij$ faces of the 3-torus, so we have the freedom to have $N^3$ fluxes. Mathematically, this corresponds to the classification of bundle topologies by 2nd Stiefel-Whitney classes $w_2 \in H^2(T^3, \mathbb{Z}_N)$. We will explicitly compactify $\mathbb{R}^3$ on $T^2 \times \mathbb{R}$, and consider flux $\ell_{12} \equiv \ell$ through $T^2$.

Let $\Omega_\mu$ denote the transition functions that we use in implementing ’t Hooft’s twisted boundary conditions connecting adjacent tori. $\Omega_1(x_2)$ is the transition function between $(x_1 + L_1, x_2) \sim (x_1, x_2)$, and is independent of $x_1$, but depends on $x_2$ coordinate, and vice versa for $\Omega_2(x_1)$. We have to impose these boundary conditions both on gauge field and adjoint scalar, $(A_\mu, \phi)$. For simplicity, let us write the explicit construction for $\phi$. Impose

$$
\phi(x_1 + L_1, x_2) = \Omega_1(x_2)\phi(x_1, x_2)\Omega_1^{-1}(x_2),
$$
$$
\phi(x_1, x_2 + L_2) = \Omega_2(x_1)\phi(x_1, x_2)\Omega_2^{-1}(x_1)
$$

(4.1)

We can connect the fields at the corners, $\phi(x_1 + L_1, x_2 + L_2)$ with $\phi(x_1, x_2)$, via two different paths: $(x_1, x_2) \rightarrow (x_1 + L_1, x_2) \rightarrow (x_1 + L_1, x_2 + L_2)$ or replacing middle point with $(x_1, x_2 + L_2)$. Consistency of the field at $\phi(x_1 + L_1, x_2 + L_2)$ demands the transition matrices $\Omega_\mu$ to obey

$$
\Omega_1(L_2)\Omega_2(0) = \Omega_2(L_1)\Omega_1(0)e^{2\pi \ell/N}
$$

(4.2)

where $\ell = 0, 1, \ldots, N-1$ is the ’t Hooft flux through 12. Under local gauge transformations, obviously, $\phi(x_1, x_2) \rightarrow g(x_1, x_2)\phi(x_1, x_2)g(x_1, x_2)^\dagger$. Therefore, the transition function must also transform for the gauge covariance of (4.1). The gauge invariant information in transition matrices (4.2) is just $\ell$, the discrete flux modulo $N$.

It is possible to undo twisted boundary conditions on $T^2 \times \mathbb{R}$, and replace them with a background $B^{(2)}$ fields associated with 1-form center symmetry $\mathbb{Z}^{[1]}_N$ given by

$$
\frac{N}{2\pi} \int_{12} B^{(2)} = \ell_{12} = \ell \mod N
$$

(4.3)

In the abelianized theories, it is also easy to relate the discrete ’t Hooft flux and magnetic GNO fluxes [15]. The magnetic GNO flux through the 12-surface can be written as

$$
\int_{12} B = \frac{2\pi}{g} \mu_\ell, \quad \mu_\ell \in \Gamma_w^\vee
$$

(4.4)

where $\ell$ is a non-zero magnetic $N$-ality, valued in $\mathbb{Z}_N \cong \Gamma_w^\vee/\Gamma_r^\vee$. Note that the dynamical magnetic monopole charges in the theory are associated with the co-root lattice $\alpha \in \Gamma_r^\vee$ and
do not change magnetic $N$-ality. Therefore, the GNO-flux configurations $\mu_1, \mu_1 - \alpha_1, \mu_1 - \alpha_1 - \alpha_2, \ldots$ are all associated with $\ell = 1$ discrete flux.\footnote{Recall that charges with non-zero $N$-ality are not present in the $SU(N)$ theory, but we can insert a GNO flux through $T^2$ to probe the dynamics of the theory. Even if we gauge $Z_N$ completely and move to pure $PSU(N)$ theory (without magnetic matter), charges with non-zero magnetic $N$-ality are still not present dynamically, but can be introduced as probes. In more general $PSU(N)$ theories, they can be introduced as dynamical magnetic matter fields.}

And finally, we may think the $\ell$-units 't Hooft flux as a non-dynamical center-vortex [52]. The holonomy of the gauge field around the 't Hooft flux insertion is a center element, $e^{i \oint a} = e^{i \frac{2\pi \ell}{N}}$.

4.1 Formal construction of coupling to $Z_N$ TQFT

We continue our discussion by coupling the $SU(N)$ Polyakov model to a $Z_N$ topological gauge theory. The resulting theory is $SU(N)/Z_N = PSU(N)$ gauge theory. Recall that there is no discrete theta angle in 3d gauge theory, hence, we only discuss one type of $PSU(N)$ theory. Coupling to $Z_N$ TQFT does not alter the local dynamics in the Polyakov model, but the spectrum of line operators and relatedly, the periodicity of dual photon field changes. In particular, 't Hooft operators with magnetic charges belonging to the co-weight lattice are useful operators in $PSU(N)$ and Wilson loops sourced by representation belonging to weight lattice are no longer genuine line operators.

To introduce a background gauge field for the $Z_N^{[1]}$ 1-form symmetry, one introduces a pair of $U(1)$ 2-form and 1-form gauge fields $(B^{(2)}, B^{(1)})$ satisfying [14, 53]

$$NB^{(2)} = dB^{(1)}, \quad N \int B^{(2)} = \int dB^{(1)} = 2\pi \mathbb{Z}.$$ (4.5)

In the standard 't Hooft construction, this amounts to saying that

$$\frac{N}{2\pi} \int_{\Sigma_{ij}} B^{(2)} = \ell_{ij} \mod N,$$ (4.6)

where $\ell_{ij}$ is 't Hooft flux through the 2-cycle $\Sigma_{ij}$ of the three torus $T^3$.

It is convenient to express the action of the $Z_N$ TQFT as

$$Z_{\text{top}} = \int DB^{(2)} DB^{(1)} DC^{(1)} e^{i \int C^{(1)} \wedge (NB^{(2)} - dB^{(1)})},$$ (4.7)

where $C^{(1)}$ is Lagrange multiplier. Equation (4.7) is invariant under the 1-form gauge transformation, $B^{(2)} \mapsto B^{(2)} + d\Lambda^{(1)}, \quad B^{(1)} \mapsto B^{(1)} + N\Lambda^{(1)}$.

To couple the 3d gauge theory to the background $B^{(2)}$ field, we first promote the $SU(N)$ gauge field $A$ into a $U(N)$ gauge field $\tilde{A}$ given by $\tilde{A} = A + \frac{1}{N} B^{(1)}$. As a result, under 1-form gauge transformation, $\tilde{A} \mapsto \tilde{A} + \Lambda^{(1)}, \quad F \mapsto F + d\Lambda^{(1)}$. i.e., the field strength is not gauge invariant under 1-form gauge transformations. The invariant combinations are...
\[ \tilde{G} = \tilde{F} - B^{(2)}, \text{ and } \tilde{D} = d - \tilde{A} - \frac{1}{N} B^{(1)}. \] Then, Polyakov model (2.1) in the \((B^{(2)}, B^{(1)})\) background field becomes:

\[
S[(B^{(2)}, B^{(1)})] = \frac{1}{2g^2} \int \left( \text{tr} \, \tilde{G} \wedge \ast \tilde{G} + \text{tr} \, \tilde{D} \phi \wedge \ast \tilde{D} \phi + V(\phi) \right),
\]

which is invariant under both 1-form and 0-form gauge transformations. To obtain the partition function of the \(SU(N)/\mathbb{Z}_N\) gauge theory, we also need to integrate over the 2-form gauge fields:

\[
Z_{\text{PSU}(N)} = \int \mathcal{D}B^{(2)} \mathcal{D}B^{(1)} \mathcal{D}C^{(1)} \mathcal{D}\tilde{A} e^{\int C^{(1)} \wedge (NB^{(2)} - dB^{(1)})} e^{-S[B^{(2)}, B^{(1)}, \tilde{A}]} \tag{4.9}
\]

This concretely relates the partition functions of \(PSU(N)\) theory and \(SU(N)\) theories as:

\[
Z_{\text{PSU}(N)} = \sum_{\ell_{ij} \in \mathbb{Z}_N^3} Z_{SU(N)}(\ell_{ij}) \tag{4.10}
\]

where \(Z_{SU(N)}(\ell_{ij})\) is the partition function of the \(SU(N)\) theory in the \((\ell_{12}, \ell_{23}, \ell_{31}) \in \mathbb{Z}_N^3\) discrete flux background.

5 QM reduction with instantons in Born-Oppenheimer limit

In this section, we focus on the properties of the quantum mechanical system on small \(T^2 \times \mathbb{R}\) without and with ’t Hooft flux. In the large volume limit where \(T^3\) is much larger than the correlation length \(m_g^{-1}\), there is no distinction in the local dynamics examined via \(Z_{SU(N)}, Z_{SU(N)}(\ell_{ij})\) and \(Z_{\text{PSU}(N)}\). However, in small \(T^2 \times \mathbb{R}\), a quantum mechanical limit, there is much more benefit in studying the dynamics of the quantum theory via \(Z_{SU(N)}(\ell_{ij})\) then \(Z_{SU(N)}\). In particular, in the Born-Oppenheimer approximation, \(Z_{SU(N)}\) does not remember the instantons of the theory on \(\mathbb{R}^3\) in the small \(T^2 \times \mathbb{R}\) limit, while the instanton configurations and degenerate harmonic vacua survive in \(Z_{SU(N)}(\ell_{12} \neq 0)\). We explain this structure below.

5.1 Discrete ’t Hooft and magnetic GNO flux, and Flux states

There are two fluxes that one can turn on for gauge theory formulated on \(T^2 \times \mathbb{R}\). These can be identified as:

- Magnetic GNO flux, \(\mu_{\ell} \in \Gamma_w^\vee\)

- Discrete ’t Hooft flux, \(\ell \in \mathbb{Z}_N \cong \Gamma_w^\vee/\Gamma_\nu^\vee\)

These two fluxes are not independent. The ’t Hooft flux \(\ell_{12}\) can be viewed as being classified by \(\mathbb{Z}_N\) that describes the quotient \(\Gamma_w^\vee/\Gamma_\nu^\vee\). On the other hand, the GNO flux takes possible values in \(\Gamma_w^\vee\).

Considering the system on \(T^2 \times \mathbb{R}\), let us express the magnetic GNO flux passing through \(T^2\) as

\[
\Phi = \int_{T^2} B = BA_{T^2} = \frac{2\pi}{g} \mu_{\ell}, \tag{5.1}
\]
Figure 4: Compactification of $SU(3)$ Polyakov model to QM for $\Phi_{bg} = 0$ and $\Phi_{bg} = \mu_1$ background magnetic flux. For $\Phi_{bg} = 0$, the vacuum is unique perturbatively and flux states have higher energy. For $\Phi_{bg} = \mu_1$, the vacua is three-fold degenerate perturbatively, and instanton persist in the quantum mechanical small $A(T^2)$ limit.

where $A_{T^2}$ is the area of $T^2$. The energy of the flux configuration is given by

$$ E = \frac{1}{2} \int_{T^2} B^2 = \frac{1}{2 A_{T^2}} \Phi^2. \quad (5.2) $$

The configuration with minimal energy does not have any flux, $\Phi = 0$. In the Polyakov model, the dynamical instantons changes the flux by $\Delta \Phi = \frac{2\pi}{g} \alpha$, and the energy of such configuration is $E = \frac{1}{2 A_{T^2}} \frac{8\pi^2}{g^2}$. This is much higher than perturbative vacuum with $E_0 = 0$. Therefore, in the QM limit of the Polyakov model, perturbative vacuum is unique. Only in the limit $A_{T^2} \to \infty$, different flux sectors become degenerate [3, 4]. See Fig. 4, left for $SU(3)$. Perturbative vacuum is unique and first excited state is six fold degenerate.

Now, let us check what happens once we turn on a classical background flux. In that case, we can write the flux of a configuration as $(\Phi_{bg} + \Phi)$, a background plus a dynamical part. This makes things more interesting. The energy of such configurations is:

$$ E = \frac{1}{2 A_{T^2}} (\Phi_{bg} + \Phi)^2 \quad (5.3) $$

The point is, even when $\Phi = 0$, energy associated with $\Phi_{bg}$ is positive. As such, changing flux by $\alpha \in \Gamma^\vee_r$ has a standing chance to be degenerate with $\Phi_{bg}$ configuration. And indeed, this is the case.

Assume $\Phi_{bg} = \frac{2\pi}{g} \mu_1$. Then, it is easy to see that the following $N$ magnetic flux configurations

$$ \nu_1 \equiv \mu_1, \quad \nu_2 \equiv \mu_1 - \alpha_1, \quad \nu_3 \equiv \mu_1 - \alpha_1 - \alpha_2, \quad \ldots \quad \nu_N \equiv \mu_1 - \sum_{a=1}^{N-1} \alpha_a, \quad (5.4) $$

are degenerate classically. The energy of these flux states are given by

$$ E_i = \frac{1}{2} \int_{T^2} B^2 = \frac{1}{2 A_{T^2}} \left( \frac{2\pi \nu_i}{g} \right)^2 = \frac{2\pi^2}{g^2 A_{T^2}} \left( 1 - \frac{1}{N} \right), \quad (5.5) $$
where we used the fact that weights of defining representation $\nu_i$ have constant length $\nu_i^2 = (1 - \frac{1}{N})$. Note that the other flux states which differ from the set (5.4) by roots necessarily have higher energies by factors of $\frac{1}{g^2 A_T^2}$ and are ignored within Born-Oppenheimer approximation. As shown in Fig. 4, right for $SU(3)$, perturbative vacua is now three-fold degenerate.

It is also important to note that since magnetic $N$-ality is defined modulo roots, these tunneling can never change the discrete flux of the background, which is given by $\ell = 1$. To construct higher discrete flux states, one needs to start with a background flux $\mu_\ell$ corresponding to a higher weight representation.

Let us denote the harmonic states in quantum mechanics by their flux configurations

$$|\nu_j\rangle = |\mu_1 - \sum_{a=1}^{j-1} \alpha_a\rangle$$

(5.6)

The simplest tunnelings are between the nearest neighbor states $|\nu_j\rangle \rightarrow |\nu_{j+1}\rangle$. We identify the changes in magnetic flux and discrete flux as:

$$\Delta \int_{T^2} B = \frac{2\pi}{g} (\nu_j - \nu_{j+1}) = -\frac{2\pi}{g} \alpha_a, \quad a = 1, \ldots, N - 1.$$  

$$\Delta \left( \frac{N}{2\pi} \int_{T^2} B^{(2)} \right) = 0$$

(5.7)

5.2 Tunneling between flux states and leading instanton effects

In $\ell = 1$ unit of discrete flux backgrounds, $N$-states given in (5.4) are degenerate and separated from higher states by a large perturbative gap. In Born-Oppenheimer approximation, we can focus on these lowest $N$ states split from each other only by tiny non-perturbative factors. The degeneracies between these $N$ states can lift via the tunneling effects. We can consider the Hamiltonian in tight-binding approximation, where only hopping between nearest-neighbor configurations are taken into account. Let us write

$$H = E_{bg} 1_N + \tilde{H},$$

(5.8)

where $\tilde{H}$ accounts for the leading instanton effects. Let us write first few cases explicitly:

$N=2$: The QM reduction of the $SU(2)$ Polyakov model in the BO approximation is same as the double-well potential shown in Fig.1, right. To describe the low end of the spectrum, we can use the tunneling Hamiltonian:

$$\tilde{H} = - \begin{bmatrix} 0 & \xi \\ \xi & 0 \end{bmatrix}$$

(5.9)

where $\xi = Ke^{-S_0}$ is the instanton fugacity. The ground state and first excited state are symmetric/anti-symmetric combinations of fractional flux states $| \pm \frac{1}{2} \rangle$:

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}} \left( |\frac{1}{2}\rangle + |\frac{-1}{2}\rangle \right) ,$$

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}} \left( |\frac{1}{2}\rangle - |\frac{-1}{2}\rangle \right) ,$$

(5.10)
with energy eigenvalues $\tilde{E}_0 = -\xi$ and $\tilde{E}_1 = +\xi$. The gap is given by an instanton factor, $\Delta E = 2\xi$ at leading order in semi-classics.

**N=3:** The QM reduction of the $SU(3)$ theory in the BO approximation is the triple-well potential on $\mathbb{R}^2$, shown in Fig.4, right. Low energy spectrum can be described by the tunneling Hamiltonian:

$$\tilde{H} = -\begin{bmatrix} 0 & \xi & 0 \\ \xi & 0 & \xi \\ 0 & \xi & 0 \end{bmatrix}$$

(5.11)

Note that the transition amplitudes $\langle \nu_2 | e^{-\beta H} | \nu_1 \rangle = \langle \nu_3 | e^{-\beta H} | \nu_2 \rangle = \xi$ is an instanton effect, but $\langle \nu_3 | e^{-\beta H} | \nu_1 \rangle \sim \xi^2$ is higher order in semi-classics, hence not included at leading order analysis.\(^{10}\) The energy eigenvalues are $\tilde{E}_0 = -\sqrt{2}\xi$, $\tilde{E}_1 = 0$, $\tilde{E}_2 = \sqrt{2}\xi$ at leading order in semiclassical expansion. The corresponding eigenstates are

$$|\Psi_0\rangle = \frac{1}{2} (|\nu_1\rangle + \sqrt{2} |\nu_2\rangle + |\nu_3\rangle),$$

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}} (|\nu_1\rangle - |\nu_3\rangle),$$

$$|\Psi_2\rangle = \frac{1}{2} (|\nu_1\rangle - \sqrt{2} |\nu_2\rangle + |\nu_3\rangle)$$

(5.12)

The degeneracy is lifted at 1-instanton order and the energy gap is an instanton factor.

**N=4:** The $SU(4)$ gauge theory reduces to a quantum mechanical system on $\mathbb{R}^3$ with four degenerate minima. The tunneling Hamiltonian takes the form:

$$\tilde{H} = -\begin{bmatrix} 0 & \xi & 0 & 0 \\ \xi & 0 & \xi & 0 \\ 0 & \xi & 0 & \xi \\ 0 & 0 & \xi & 0 \end{bmatrix}$$

(5.13)

with eigenfunctions

$$|\Psi_0\rangle = \mathcal{N} \left( +|\nu_1\rangle + \frac{1 + \sqrt{5}}{2} |\nu_2\rangle + \frac{1 + \sqrt{5}}{2} |\nu_3\rangle + |\nu_4\rangle \right),$$

$$|\Psi_1\rangle = \mathcal{N} \left( -|\nu_1\rangle + \frac{1 - \sqrt{5}}{2} |\nu_2\rangle - \frac{1 - \sqrt{5}}{2} |\nu_3\rangle + |\nu_4\rangle \right),$$

$$|\Psi_2\rangle = \mathcal{N} \left( +|\nu_1\rangle + \frac{1 - \sqrt{5}}{2} |\nu_2\rangle + \frac{1 - \sqrt{5}}{2} |\nu_3\rangle + |\nu_4\rangle \right),$$

$$|\Psi_3\rangle = \mathcal{N} \left( -|\nu_1\rangle + \frac{1 + \sqrt{5}}{2} |\nu_2\rangle - \frac{1 + \sqrt{5}}{2} |\nu_3\rangle + |\nu_4\rangle \right)$$

(5.14)

where $\mathcal{N}$ is the normalization constant. The energy eigenvalues are $\tilde{E}_0 = -\frac{1 + \sqrt{5}}{2} \xi$, $\tilde{E}_1 = \frac{1 - \sqrt{5}}{2} \xi$, $\tilde{E}_2 = -\frac{1 + \sqrt{5}}{2} \xi$ and $\tilde{E}_3 = \frac{1 + \sqrt{5}}{2} \xi$. The gap is again at 1-instanton order.

\(^{10}\) This is one of the main distinctions between deformed Yang-Mills [38] and Polyakov model [1], (or group vs. algebra valued adjoint scalar fields). In the former, $\langle \nu_3 | e^{-\beta H} | \nu_1 \rangle = \xi$ as well, and the affine monopole with magnetic charge $a_N$ is on the same footing with the rest of monopoles associated with the simple root system $a_1, \ldots, a_{N-1}$.
Note that in all cases, the eigenstates are a linear combination of the fractional flux states. Because of the background GNO flux insertion, all flux states are in co-weight lattice, but the important point is that the separation between the flux states is in the co-root lattice, corresponding to dynamical monopole-instantons in the theory.

5.3 Resurgence in QM reduction of Polyakov model with ’t Hooft flux

The $SU(2)$ gauge theory with $\ell_{12} = 1$ unit of ’t Hooft flux reduces to double-well quantum mechanics, as shown in Fig. 1. The working of resurgence in the double-well potential is well understood by now. Since this knowledge is present in literature [19, 20] we state here the main implication for our analysis.

- Perturbation theory around the degenerate harmonic flux vacua $|\pm \frac{1}{2}\rangle$ is an divergent asymptotic expansion, 
  \[ E_0 = \sum_k E_k(0) g^{2k}. \]
  This series is non-Borel summable. The Borel resummation $S_{\pm} E_0$ is two-fold ambiguous. Similarly, the instanton-antiinstanton configuration is also two-fold ambiguous. The two ambiguities cancel exactly.

  \[ \text{Im} [S_{\pm} E_0 + [II]_{\pm}] = 0 \]

  where $I$ is the magnetic flux changing instanton event (that extrapolates from $\Phi = -\frac{1}{2}$ to $\Phi = +\frac{1}{2}$. This is the dimensional reduction of the monopole $M$ in the original Polyakov model obtained by using method of replica. The mapping is formulated in Appendix A. By the appropriate scale separations in the geometric construction, the action of the instanton in QM must be the same as the action of the monopole in QFT, $S_I = S_M$. In fact, in dYM, on $\mathbb{R} \times T^2 \times S^1$, we can prove this statement rigorously, that $S_I = S_M = \frac{1}{N} \frac{8\pi^2}{g^2}$ due to the fact that we can turn on two types of discrete fluxes which guarantees that topological charge is $Q = 1/N$. Since monopole is a solution to BPS equation with the appropriate boundary condition, this show that these configuration in QM limit of dYM have fractional action.

- Exact WKB and exact quantization methods prove implicitly that this type of cancellations occurs to all non-perturbative orders, repeating itself around all non-perturbative saddles. See [22, 54]. In particular, all orders perturbative fluctuations around the instanton $E^{(1)} \sim e^{-S_0} P_I (g^2)$ can be determined by using P/NP relation [49, 55] and $P_I (g^2)$ is also a divergent asymptotic expansion. The exact quantization conditions implicitly prove that the ambiguous imaginary parts (at three instanton level) again cancel each other:

  \[ \text{Im} [[I]S_{\pm} P_I (g^2) + [ITT]_{\pm}] = 0. \]

The cancellations in (5.15) and (5.16) are proven in quantum mechanics. Since the instantons in quantum mechanics are mappings of monopoles in QFT compactified on $T^2 \times \mathbb{R}$
with the insertion of ‘t Hooft flux (which guarantees that instanton survives in QM limit), this leads us to conjecture that the counterpart of these resurgent relations, (3.22) and (3.23), hold in QFT. This leads to some predictions, (3.25) and (3.27) in QFT, that can be tested by lattice simulations.

6 Polyakov vs. deformed YM with ‘t Hooft flux

Polyakov model on $\mathbb{R}^3$ and deformed YM (dYM) on small $\mathbb{R}^3 \times S^1$ are intimately related, but different theories. Here, we would like to provide a succinct comparison of the properties of these two theories. But first, let us briefly remind basic aspects of dYM for completeness.

Deformed YM theory is the center-stabilizing double-trace deformation of YM theory on small $\mathbb{R}^3 \times S^1$ [38]. The gauge holonomy $U$ around $S^1$ plays the same role as the Higgs field in Polyakov model. However, in Polyakov model, $\phi$ is algebra valued, while in dYM, $U$ is group valued. In the vacuum, $U$ acquires a vev $U = \text{Diag}(e^{iv_1}, e^{iv_2}, \ldots, e^{iv_N})$, $|v_{i+1} - v_i| = \frac{2\pi}{N}$ and the theory undergoes dynamical abelianization $SU(N) \to U(1)^{N-1}$. Due to the compactness of the gauge holonomy in dYM, there exists an affine monopole associated with the affine root $\alpha_N$ which has the same action as the other monopoles [56, 57]. Even though there are fundamental differences between these two theories that start with their symmetries or even parameters that one can write in their action, as well as their global structures, their local dynamics are extremely similar.

6.1 Symmetries and ’t Hooft fluxes

First, let us start with formal differences. For Polyakov model, the center symmetry is $(\mathbb{Z}_N^1)_{E}$ while for dYM on $\mathbb{R}^3 \times S^1$, the center-symmetry is $(\mathbb{Z}_N^1)_{E} \times (\mathbb{Z}_N^0)_{E}$ which descends from $(\mathbb{Z}_N^1)_{E}$ on $\mathbb{R}^4$:

\[
\begin{align*}
SU(N) & : (\mathbb{Z}_N^1)_{E} \\
SU(N) & : (\mathbb{Z}_N^1)_{E} \times (\mathbb{Z}_N^0)_{E}
\end{align*}
\]  (6.1)

The charged operators under 1-form symmetry are Wilson line operators and under the 0-form symmetry, it is Polyakov loops.

In Polyakov model defined on $T^2 \times S^1$, we can turn on $N^3$ discrete fluxes valued in $H^2(T^2, \mathbb{Z}_N)$ and discrete theta angle is absent. In dYM on $T^3 \times S^1$, one can turn on $N^6$ discrete fluxes valued in $H^2(T^4, \mathbb{Z}_N)$ and there is also a discrete theta angle. Finally, gauging the center symmetry, i.e., summing over all possible fluxes gives us $PSU(N)$ theories. The partition function for the $PSU(N)$ theories can be written as

\[
\begin{align*}
P : & \quad Z_{PSU(N)}(\ell) = \sum_{\ell \in \mathbb{Z}_N^3} Z_{SU(N)}(\ell) \\
dYM : & \quad Z_{PSU(N)}(\ell, m) = \sum_{W \in \mathbb{Z}_N^6} e^{i \varphi_{\ell, m}^W} e^{i \theta (\ell W + |m|_N)} Z_{W}(\ell, m)
\end{align*}
\]  (6.2)
There is no theta angle in the Polyakov model, but since deformed YM is a locally 4d theory, there is a topological theta angle, and an associated topological charge quantized in integer units in $SU(N)$ theory. In $PSU(N)$, topological charge is quantized in units of $1/N$. The implication of this for the $SU(N)$ dynamics is discussed in [15].

Gauging the $(\mathbb{Z}_{N}[1])_{E}$ 1-form symmetry in the confining $SU(N)$ Polyakov model, we obtain $(\mathbb{Z}_{N}[0])_{M}$ form symmetry in the $PSU(N)$ theory. This process extends the periodicity of the dual photon field from the weight lattice to root lattice, $\sigma \sim \sigma + 2\pi \alpha_i, \ alpha_i \in \Gamma_r$. The action of the $(\mathbb{Z}_{N}[0])_{M}$ is to shift $\sigma \rightarrow \sigma + 2\pi \mu_i, \ mu_i \in \Gamma_w$, and the order parameter for this symmetry is monopole operators with magnetic charges in the co-weight lattice, $e^{i\mu_i \cdot \sigma(x)}$. In the infrared, we obtain a spontaneously broken $\mathbb{Z}_{N}[0]$ 0-form symmetry. As a result, there exists $N$ vacua in the thermodynamic limit of the $PSU(N)$ theory distinguished by the vacuum expectation values of the monopole operators

$$\langle e^{i\mu_i \cdot \sigma(x)} \rangle = e^{\frac{2\pi k}{N}}, \ k = 0, 1, \ldots$$ (6.3)

and the infrared limit of $PSU(N)$ Polyakov model is a $\mathbb{Z}_N$ TQFT. For dYM, the global symmetry becomes $(\mathbb{Z}_{N}[0])_{M} \times (\mathbb{Z}_{N}[1])_{M}$. The realization of this symmetry, say for $p = 0$ and varying $\theta \in [0, 2\pi N)$ subtly depends on the range of theta angle and we will not discuss it here.

6.2 Non-perturbative mass spectra for Polyakov vs. dYM

In both Polyakov model and dYM, monopole instantons induce a non-perturbative potential that gaps out all dual photons non-perturbatively. The potential is

$$P : \quad L_{m,1} = -2\xi \sum_{i=1}^{N-1} \cos (\alpha_i \cdot \sigma) \quad \text{or} \quad \alpha_i \in \Delta^0$$

$$dYM : \quad L_{m,2} = -2\xi \sum_{i=1}^{N} \cos (\alpha_i \cdot \sigma + \frac{\theta}{N}) \quad \text{or} \quad \alpha_i \in \hat{\Delta}^0$$ (6.4)

In Polyakov model, the leading saddles are monopoles in simple root system $\Delta^0 = \{\alpha_1, \ldots, \alpha_{N-1}\}$, while in dYM, monopoles in extended (affine) simple root system $\hat{\Delta}^0 = \Delta^0 \cup \{\alpha_N\}$ are contributing at leading order in semi-classics. This follows from the fact that the latter theory is locally four dimensional, and the affine root is in the same footing with the monopoles in $\Delta^0$.

To find the dual photon masses, we can expand non-perturbatively induced potential (6.4) to quadratic order as $\xi \sigma^T Q \sigma$. In writing the quadratic potential for $\sigma$, we use a basis of $N$-component vectors $(\sigma_1, \ldots, \sigma_N)$, where one component corresponding to $\frac{1}{\sqrt{N}}(\sigma_1 + \ldots + \sigma_N)$ is redundant, and decouples. In both cases, we have $N - 1$ physical dual photons.
only.

\[
P : \quad \mathcal{L}_{m,1} = \xi \sum_{j=1}^{N-1} (\sigma_j - \sigma_{j+1})^2
\]

\[
dYM : \quad \mathcal{L}_{m,2} = \xi \sum_{j=1}^{N} (\sigma_j - \sigma_{j+1})^2, \quad \sigma_{N+1} \equiv \sigma_1
\]

For example, for \( N = 7 \) theories, the \( Q \)-matrix that needs to be diagonalized is given by:

\[
Q_P = \begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 1
\end{pmatrix}
\]

\[
Q_{dYM} = \begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & -1 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 \\
-1 & 0 & 0 & 0 & 0 & -1 & 2
\end{pmatrix}
\]

Diagonalizing the mass matrix, we find the non-perturbative mass spectrum in both theories. Indeed, in both cases, there is a zero eigenvalue corresponding to unphysical mode and is removed from the spectrum. Masses of \( N - 1 \) dual photons in Polyakov model are:

\[
P : \quad m_k^2 = m_\gamma^2 \sin^2 \left( \frac{\pi k}{2N} \right), \quad k = 1, \ldots, N - 1
\]

while the ones in dYM are given by

\[
dYM : \quad m_{k,q}^2(\theta) = m_\gamma^2 \sin^2 \left( \frac{\pi k}{N} \right) \cos \left( \frac{\theta + 2\pi q}{N} \right), \quad k = 1, \ldots, N - 1, \quad q = 0, \ldots, N - 1 \mod N
\]

and \( q = 0, \ldots, N - 1 \mod N \) is the branch label. Setting \( \theta = 0, q = 0 \) for proper comparison, the difference by a factor of two in the argument of the sine function is indeed there. This difference can be understood in simple terms, as the difference of the normal modes and spectrum of \( N \) coupled springs and ball systems. The dYM corresponds to a ring of \( N \) coupled oscillators with periodic identifications at the end and the Polyakov model corresponds to \( N \) coupled oscillators with open boundary conditions and balls at the ends. The normal modes and frequencies of coupled oscillator systems are the same as photon eigenstates and masses.\(^{11}\)

In dYM, the mass spectrum is multi-branched as a function of \( \theta \) angle. It is composed of \( N \)-branches. Note that not all of the \( N \) branches are simultaneously stable. This is consistent with what is argued in [59]. For a given value of \( \theta \), approximately half of the branches \( \sim N/2 \) are stable. Which half of it is stable depends on the value of \( \theta \). The mass gap of the system is given by \( k = 1, N - 1 \), and for a given value of \( \theta \), it is equal to \( m_{\text{gap}}^2 = \text{Max}_q m_{k,q}^2(\theta) \). See [60] for further details.

\(^{11}\)Note that the distance between min and max of the dual photon mass square is an instanton factor, \( \xi \). In a natural abelian large \( N \) limit, this implies that \( N \to \infty \) states must fit into a continuous band, and the gap in the \( SU(N) \) model vanishes as \( \frac{1}{N} \) in the large-\( N \) limit. This construction admits an interpretation as an emergent dimension [58].
Figure 5: $\Delta E_k$ is the spectral gaps in reduced QM. $m_k^2$ are mass square for photons in full QFT, Polyakov model on $\mathbb{R}^3$ (red points) and deformed YM on $\mathbb{R}^3 \times S^1$ (black points). Both have a spectral band width of order $\xi \sim e^{-S_0}$. See text.

6.3 QM reductions and energy spectra in Born-Oppenheimer approximation

Compactifying the Polyakov model on $\mathbb{R} \times T^2$ and dYM theory on $\mathbb{R} \times T^2 \times S^1$, and inserting $\ell_{12} = 1$ unit of background 't Hooft flux through $T_2$ renders the flux configurations $|\nu_j\rangle$, $j = 1, \ldots N$ degenerate classically. However, QM description of these two systems are different. dYM has an extra $\mathbb{Z}_N$ form symmetry which cyclically permutes the magnetic flux vacua. The tunnelings in these two systems connect the vacua as

\[
P: \quad |\nu_1\rangle \xrightarrow{-\alpha_1} |\nu_2\rangle \xrightarrow{-\alpha_2} \cdots \xrightarrow{-\alpha_{N-1}} |\nu_N\rangle
\]
\[
dYM: \quad -\alpha_1 |\nu_1\rangle \xrightarrow{-\alpha_2} |\nu_2\rangle \xrightarrow{-\alpha_3} \cdots \xrightarrow{-\alpha_N} |\nu_1\rangle
\]

at leading order in semi-classics. See Fig. 2 for a depiction of states and global structure.

The tight-binding Hamiltonians capturing the tunnelings between the flux vacua $|\nu_j\rangle$ are given by:

\[
P: \quad \tilde{H}_1 = -\sum_{j=1}^{N-1} \xi |\nu_{j+1}\rangle\langle\nu_j| + \text{h.c.}
\]
\[
dYM: \quad \tilde{H}_2 = -\sum_{j=1}^{N} \xi e^{i\frac{2\pi j}{N}} |\nu_{j+1}\rangle\langle\nu_j| + \text{h.c.}
\]

The eigenstates of the Hamiltonian $\tilde{H}_1$ for $N = 2, 3, 4$ are given in Section 5.2. The eigenstates of the $\tilde{H}_2$ Hamiltonian are much easier to write thanks to the $\mathbb{Z}_N$ cyclic shift symmetry of the corresponding QM. They are given by the discrete Fourier transform of the magnetic flux states:

\[
|\Psi_k\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{i\frac{2\pi j k}{N}} |\nu_j\rangle, \quad k = 0, \ldots, N - 1
\]
The energy eigen-spectra of the low-lying $N$ states are given by

$$P: \quad E_k = -2\xi \cos \left( \frac{\pi}{N+1} (k+1) \right), \quad k = 0, \ldots, N-1$$

$$dYM: \quad E_k = -2\xi \cos \left( \frac{\theta + 2\pi k}{N} \right), \quad k = 0, \ldots, N-1 \quad (6.12)$$

The $N$-fold degeneracy is lifted non-perturbatively by instanton effects in Polyakov model and there is no state in the spectrum that remains degenerate. For dYM, at generic $\theta$, states are non-degenerate. The degeneracies occur in a certain pattern for special values $\theta = 0, \pi$ as

$$
\begin{align*}
N - \text{even} & \quad \theta = 0 \quad \text{degeneracies: } 12 \ldots 21 \\
N - \text{even} & \quad \theta = \pi \quad \text{degeneracies: } 22 \ldots 2 \\
N - \text{odd} & \quad \theta = 0 \quad \text{degeneracies: } 12 \ldots 2 \\
N - \text{odd} & \quad \theta = \pi \quad \text{degeneracies: } 2 \ldots 21 
\end{align*}
$$

The exact degeneracy for the $N$ even case at $\theta = \pi$ is the realization of the mixed anomaly between $Z_N^1$ form symmetry and time-reversal $T$ in QM. The pattern at odd $N$ is a consequence of global inconsistency, see [61]. Polyakov model does not possess a mixed-anomaly related to its $Z_N^1$ 1-form symmetry and the lowest-lying $N$ states are non-degenerate.

The gap between the ground state and higher states in the energy eigen-spectra of the low-lying $N$ states in quantum mechanics are given by

$$P: \quad \Delta E_k = 2\xi \left( \cos \left( \frac{\pi}{N+1} \right) - \cos \left( \frac{\pi(k+1)}{N+1} \right) \right), \quad k = 1, \ldots, N-1$$

$$dYM: \quad \Delta E_k(\theta = 0) = 4\xi \sin^2 \left( \frac{\pi k}{N} \right), \quad k = 1, \ldots, N-1 \quad (6.14)$$

$\Delta E_k(\theta = 0)$ in dYM quantum mechanics on tiny $T^2 \times S^1 \times \mathbb{R}$ is equal to mass square of the photons $m_k^2(\theta = 0)$ (6.8) in full QFT on $\mathbb{R}^3 \times S^1$. For Polyakov model, the same relation also holds, but approximately $\Delta E_k \approx m_k^2$, see (6.7). We find this correspondence quite striking. In both QFT and QM limits, the width of the non-perturbative band is controlled by monopole-instanton amplitude, and $N$ states fill the interval with the patterns shown in Fig. 5.

Note that the monopole instanton amplitude is controlled by the action $S_0 = \frac{4\pi v}{g_3}$ in Polyakov model and its QM reduction with ’t Hooft flux, and for dYM, since $v = \frac{2\pi}{LN}$, $S_0 = \frac{1}{N} \frac{8\pi^2}{g_4^2}$. Despite the fact that the theory is reduced to a tiny $T^2 \times S^1 \times \mathbb{R}$ in dYM, the ground state properties is controlled by fractional instanton action, $S_0 = \frac{1}{N} S_{4d}$.

In dYM, the fractional action is protected by the TQFT coupling. Since we know the resurgence techniques in the QM limit, and since the action of the fractional instantons remain invariant as the $T^3 \times \mathbb{R}$ size becomes larger, this construction has direct implications in QFT. Completeness of this semi-classical basis and the implications of this QM limit to the IR-renormalon problems are discussed in [62].

Note that the max and min of the band are separated by the monopole instanton factor which is at most $\sim 4\xi$. Since $\xi$ does not depend on $N$, $N$ states must fit into this interval.
in the large-$N$ limit. This implies that in the large-$N$ limit, the states in the QM system form a continuous band similar to the full QFT mass spectrum [58].

7 Prospects

Cluster expansion vs. Steepest descent: An intriguing aspect of our construction is following. Polyakov model, in the semi-classical approximation, maps to a grand canonical ensemble of Coulomb charges. In general, Coulomb gas can be studied via cluster expansion. Cluster expansion involves integrals such as (assume for simplicity $q = \pm 1$)

$$Z[q_1, q_2, \ldots, q_N] = \int \prod_{i=1}^{N} d^3 r_i \exp \left[ -\sum_{i<j} \frac{\pi q_i q_j}{g^2 |r_i - r_j|} \right]$$ (7.1)

and normally, (if we check textbooks on statistical physics), one would interpret the integral over the configuration space. However, semi-classical analysis instructs us to perform the integrations over the steepest descent cycles of critical points. This does not change the space of integration for repulsive charges, see Fig. 7, but for attractive charges, the steepest descent cycle lives in the complex domain, see Fig. 8. For a collection of arbitrary charges, semi-classics treatment requires complexification of configuration space. As a result, for example for $N = 2$, we obtain:

$$Z[+, +] = Z[-, -] = \mathcal{V}(\mathcal{V} + I(g^2))$$
$$Z[+, -] = Z[-, +] = \mathcal{V}(\mathcal{V} - I_{\pm}(g^2))$$ (7.2)

In the sense of involving complex and two-fold ambiguous results such as $I_{\pm}(g^2)$, our construction differs from standard cluster expansion in classical statistical mechanics. Perhaps, a better way to treat Coulomb gas in cluster expansion must go through this process of complexification. It is desirable to understand this aspect at a deeper level.

Even the minus in front of the second term in (7.2), which arises naturally from the steepest descent path is physical and important. For example, if we consider $\mathcal{N} = 2$ SYM theory in 3d with $SU(2)$ gauge group, first-order monopole terms in semi-classics do not contribute to bosonic potential due to index theorem. At second order, there are $[\mathcal{M}, \overline{\mathcal{M}}]$ configurations. The fact that this configurations contributes to the bosonic potential as $V(\tilde{\phi}) = +\xi^2 e^{-2\tilde{\phi}}$ arises from the minus sign in (7.2). (This leads to the physically correct consequence of positive potential and run-away vacua at infinity [42]. If it was not for the minus sign in (7.2), we would obtain $V(\tilde{\phi}) = -\xi^2 e^{-2\tilde{\phi}} < 0$ which is negative. This would be in contradiction with supersymmetry algebra which demands that vacuum energy must be positive semi-definite.

Relation to renormalization group: Another aspect is following. Sub-extensive factors in (7.2) correspond to contribution of 2-clusters to vacuum energy density. The 2-clusters are associated with $+2$ charges, $-2$ charges and dipoles with charges $0 = 1 - 1$. Naturally, there is a sense in which cluster expansion is tied with the renormalization group, since we are integrating over the separations between instantons. If the 2-cluster is supported
at separations $r \lesssim r_b$ for some length scale $r_b$, then it is meaningful to coarse-grain the system up to length scale $r_b$, and view the 2-cluster operators as independent operators in an EFT valid at length scales larger than $r_b$. The 2-cluster operators obtained in this way are called bion operators.

If we start with microscopic interactions $\mathcal{L}_{r_m} \supset -\xi(e^{i\sigma} + e^{-i\sigma})$ where $r_m$ is the short-distance cut-off, the 2-cluster effects would be of the form

$$
-\xi^2 I(g^2)e^{2i\sigma(r)},
-\xi^2 I(g^2)e^{-2i\sigma(r)},
+\xi^2 I_\pm(g^2)\left(e^{i\sigma(r)+\epsilon}e^{-i\sigma(r)} + c.c\right) \sim \xi^2 I_\pm(g^2)\cos(\epsilon \cdot \nabla \sigma)
$$

(7.3)

where $\epsilon$ is the direction of the dipole, and $|\epsilon| \lesssim r_b$. Since dipoles have a directionality, and they can come in any direction, we must average over the directionality to obtain:

$$
I_\pm(g^2)\xi^2\left(1 - \frac{1}{6}(\nabla \sigma)^2 + \ldots\right)
$$

(7.4)

In the leading term $\text{Re}[I_\pm(g^2)\xi^2 \times 1$ clearly contributes to vacuum energy density (proportional to second virial coefficient) and $\text{Re}[I_\pm(g^2)\xi^2 \times \frac{1}{6}(\nabla \sigma)^2$ corresponds to renormalization of the kinetic term of the dual photon at second order in semi-classics, and higher-order terms are suppressed by higher powers of derivative. Both terms are quite physical. The latter term reminds us of the physical fact that if we were to consider just a gas of dipoles, it would not generate a mass gap, and dipoles would only modify the dielectric constant of the medium. In fact, by using this line of reasoning, we would immediately prove that gas of dipoles do not induce a mass gap or screen [63]. So far, it all makes sense.

But semi-classics also tells us that there is an imaginary ambiguous part in second-order contribution, $\text{Im}[I_\pm(g^2)]\xi^2$. As emphasized in the paper, this is not a bug, but a feature. Indeed, according to resurgence, this ambiguity must cancel the ambiguity that arises from the left/right Borel resummation of perturbation theory, as in (5.15) and (5.16). In fact, in the quantum mechanical limit, this is already proven by resurgence methods. But here is a subtle issue. If we were to consider coarse-graining in the renormalization group rather than integration over the steepest descent cycle in the semi-classics, naively, we would not obtain the imaginary ambiguous parts in $[I_\pm(g^2)]\xi^2$, but just its real unambiguous part.

A natural question is, where do the imaginary ambiguous parts enter the story in the renormalization group procedure?

**Other abelianizing theories:** Several interesting QFTs undergo dynamical abelianization at long distances and that possess a 1-form center symmetry. The 4d examples are $\mathcal{N} = 1$ SYM and QCD(adj) on small $\mathbb{R}^3 \times S^1$ with periodic boundary conditions on fermions on $S^1$, $\mathcal{N} = 2$ and $\mathcal{N} = 4$ SYM on $\mathbb{R}^4$, and $SU(N) \times SU(N)$ QCD with bifundamental fermions and double-trace deformations on small $\mathbb{R}^3 \times S^1$, and its generalization to chiral quiver theories $SU(N) \times \ldots \times SU(N)$. The 3d examples are various generalizations of Polyakov models with adjoint fermions on $\mathbb{R}^3$, as well as variants as in 4d. To understand different dynamical consequences, the physical set-up we employ, either on $T^2 \times S^1 \times \mathbb{R}$
or $T^2 \times \mathbb{R}$ with magnetic GNO flux on $T^2$ (corresponding to a non-trivial ’t Hooft flux, but less abstract thanks to abelianization) can be used to understand different dynamical behaviors in these theories. In all these systems, the monopole tunneling events must be operative, but they must lead to vastly different dynamical consequences due to the interplay of fermionic zero modes and the existence of extra scalars in extended supersymmetric theories. In some sense, the idea is to bring long-distance semi-classical calculability with the ’t Hooft flux to understand detailed dynamics. We expect this to be a fertile playground to study a diverse set of theories.

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A Mapping QFT instantons to QM instantons by the method of replicas

In this section, we describe how the monopole in QFT with magnetic field $B = \frac{Q}{4\pi r^2} \hat{r}$ on $\mathbb{R}^3$ maps to instantons in QM on small $T^2 \times \mathbb{R}$. In QM, we describe the instantons as flux changing events through $T^2$. Defining $\Phi(\tau) = \int B$, the tunneling events correspond to

$$\Delta \Phi = \Phi(\tau = \infty) - \Phi(\tau = -\infty) = \frac{2\pi}{g} \alpha, \quad \alpha \in \Gamma_r^\vee. \quad (A.1)$$

Below, we show that indeed $B$ becomes independent of $(x,y)$ coordinates rather quickly and $\Phi(\tau)$ acquires a standard quantum mechanical instanton profile. We identify $\tau$ in QM with $z$ coordinate in QFT.

Assume $T^2$ is symmetric, with sizes $L_1 = L_2 = L$. We take $L$ much larger than the monopole core size $r_m$ and much smaller than characteristic monopoles separation $d_{mm}$ on $\mathbb{R}^3$:

$$r_m \ll L \ll d_{mm}. \quad (A.2)$$

In this way, we guarantee that generically there exists a single monopole per $T^2$, and the change in the local monopole profile relative to $\mathbb{R}^3$ is negligible. Furthermore, in the reduced QM system, the monopoles form a dilute gas in the Euclidean time direction $\mathbb{R}$.

Let us consider a single monopole on $T^2$, located at $(0,0,0)$. We would like to determine the magnetic field at $(x,y,z) \in T^2 \times \mathbb{R}$. We identify $z$ with the Euclidean time direction. The easiest way to proceed is to use the method of replicas. Because of the periodic boundary conditions, we can extend the charges periodically and form a charge lattice where charges are located at $(m,n)L \in \mathbb{Z} \times \mathbb{Z}$. (See Fig. 6).

Since the magnetic field can be written as $B(x,y,z) = -\nabla V(x,y,z)$ and it is easier to determine $V(x,y,z)$ “the magnetic potential”, let us do so due given charge distribution.
Figure 6: Each square of size $L \times L$ represents a torus $T^2$ in the compactified $x, y$ dimensions, with a monopole at the center. Due to periodic boundary conditions, the torus is replicated to form a two dimensional infinite array with monopoles located at $(m, n)L \in \mathbb{Z} \times \mathbb{Z}$. The field of the array of monopoles within small $T^2 \times \mathbb{R}$ is the one of a quantum mechanical instanton.

Our goal is to show that the potential (and magnetic field) becomes independent of the $(x, y)$ coordinate to very good accuracy for $z \gtrsim L$. To show this, let us write the potential at a point $(x, y, z)$ as a sum over the potential induced by individual charges:

$$V(x, y, z) = \sum_{m,n} V_{m,n}(x, y, z) = Q \frac{4\pi}{4\pi} \sum_{m,n} \frac{1}{\left[(x - mL)^2 + (y - nL)^2 + z^2\right]^{1/2}}. \quad (A.3)$$

Because of the periodic boundary conditions, the potential is a periodic function in $x, y$.

$$V(x + L, y, z) = V(x, y + L, z) = V(x, y, z) \quad (A.4)$$

Hence, we can expand it to a Fourier series

$$V(x, y, z) = \sum_{k_x, k_y} V_{k_x, k_y}(z) e^{\frac{2\pi i}{z}(k_x x + k_y y)}. \quad (A.5)$$
where \((k_x, k_y) \in \mathbb{Z} \times \mathbb{Z}\) and the Fourier coefficients are only functions of \(z\).

\[
V_{k_x, k_y}(z) = \frac{1}{L^2} \int_0^L \int_0^L dx \, dy \, V(x, y, z) \, e^{\frac{2\pi i}{L}(k_x x + k_y y)}
\]

\[
= \frac{1}{L^2} \sum_{m,n} \int_{Lm}^{L(m+1)} \int_{Ln}^{L(n+1)} dx \, dy \, V_{0,0}(x, y, z) \, e^{\frac{2\pi i}{L}(k_x x + k_y y)}
\]

\[
= \frac{Q}{4\pi L^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx \, dy \, \frac{1}{\sqrt{x^2 + y^2 + z^2}} e^{\frac{2\pi i}{L}(k_x x + k_y y)} \quad (A.6)
\]

It is convenient to express the last expression in the cylindrical coordinates. Defining \(\rho = (x, y)\) and \(k = (k_x, k_y)\), we re-write the Fourier coefficients as

\[
V_k(z) = \frac{Q}{4\pi L^2} \int_0^\infty d\rho \, \rho \int_0^{2\pi} d\theta \, e^{\frac{2\pi i}{L}|k| \rho \cos \theta} = \frac{Q}{2L^2} \int_0^\infty d\rho \rho \frac{J_0 \left( \frac{2\pi |k| \rho}{L} \right)}{\sqrt{\rho^2 + z^2}}. \quad (A.7)
\]

For \(k \neq 0\), we get

\[
V_{k \neq 0}(z) = \frac{Q}{2L^2} \left( \frac{L}{2\pi |k|} \right) e^{-\frac{2\pi |k||z|}{L}} \quad (A.8)
\]

Then, the associated magnetic field is

\[
B_z^{k \neq 0} = -\frac{dV_{k \neq 0}}{dz} = \frac{Q}{2L^2} \text{sign}(z) e^{-\frac{2\pi |k||z|}{L}}, \quad |k| \neq 0 \quad (A.9)
\]

where \(\text{sign}(z)\) indicates the magnetic field is always in outward direction from the plane of charges, and exponentially decaying rather rapidly. For the zero mode component, \(k = 0\), we find:

\[
B_z^{k=0}(z) = -\frac{dV_0}{dz} = \frac{Q}{2L^2} \int d\rho \rho \frac{1}{(\rho^2 + z^2)^{3/2}} = \frac{Q}{2L^2} \text{sign}(z). \quad (A.10)
\]

which does not decay at all as expected.

For \(k \neq 0\), all the Fourier coefficients decay exponentially fast, as in (A.9). Even for \(|k| = 1\) and at a distance \(z = L\), the decay of the mode is \(e^{-2\pi} \sim 10^{-3}\), and higher modes decay even faster. Therefore, the only mode that matters at distances \(|z| \gtrsim L\) is the zero mode of the Fourier decomposition in (A.10).

This is nothing but the magnetic field due to uniformly charged infinite plane, with magnetic charge density \(Q/L^2\). The magnetic flux passing through \(T^2\) and the change in the magnetic flux between \(z \equiv \tau = \pm L\) are given by

\[
\Phi(\tau) = \begin{cases} +\frac{Q}{2} & \tau > L \\ -\frac{Q}{2} & \tau < -L \end{cases} \quad \Delta \Phi = \frac{2\pi}{g} \alpha. \quad (A.11)
\]

These are the asymptotics of an instanton in quantum mechanics.

If we make this construction for the monopole in deformed YM on \(T^2 \times S^1 \times S^1\), the action of the instanton in reduced QM is \(S_0 = \frac{1}{N} S_{4d}\), and this is topologically protected because we can turn on two discrete fluxes in \(H^2(T^4, \mathbb{Z}_N)\) such that \(Q = \frac{(\ell m)}{N} = \frac{1}{N}\).
Since monopole solutions satisfy is BPS and satisfy appropriate boundary conditions, this shows that \( S_0 = \frac{1}{2} S_{1d} \). For Polyakov, there is no such topological reason for the action of the monopole to remain the same as we pass from QFT on \( \mathbb{R}^3 \) to QM on \( \mathbb{R} \times T^2 \). But the geometric scale separation (A.2) suggests that the action of monopole must remain approximately the same as the one on \( \mathbb{R}^3 \).

The role of 't Hooft flux in elementary description: Now, we can translate the crucial role of 't Hooft flux plays to an elementary language, in terms of parallel plate “magnetic capacitors.” Assume we do not insert a 't Hooft flux but consider an instanton-anti-instanton event. In the above language, this is a simple parallel plate (magnetic) capacitor. The energy difference between the system with and without the capacitor (whose plates are separated by \( d \)) is \( \Delta E = \frac{Q^2}{2} d \), stored in between the capacitors. Since \( \Delta E > 0 \) and \( \Delta E \to \infty \) as \( d \to \infty \), the state in between the parallel plates is not the vacuum state. This implies that instantons do not survive in the description of vacuum properties in the Polyakov model without discrete flux.

Now, let us insert a judiciously chosen background magnetic flux, say \( \Phi = \mu_1 \). This is similar to inserting parallel plates at infinity with charge \( \pm \mu_1 \) at two ends, and it causes a finite zero-point energy per unit length. Now, consider the anti-instanton and instanton pair, a magnetic capacitor. The flux in between the capacitors is \( \Phi = \mu_1 - \alpha_1 \). Now, the energy difference between the system with and without the capacitor is \( \Delta E = 0 \). Since \( \Delta E = 0 \) and is independent of the separation, the state in between the parallel plates has the same energy as the state outside, they are classically degenerate. Therefore, we can make all the tunneling events in QFT survive in a quantum mechanical limit in the Born-Oppenheimer approximation by using appropriate magnetic flux background.

B Cluster expansion, semi-classics and sub-extensiveness

For a gapped system, the partition function in the \( \beta \to \infty \) limit is dominated by the ground state. Therefore, we can write the partition function in the \( \beta \to \infty \) limit as \( Z(\beta) \sim e^{-VE_0} \) where \( E_0 \) is the ground state energy density and \( V \equiv V_{d+1} = \beta V_d \) is the Euclidean space-time volume. Let us write

\[
e^{-VE_0} \sim e^{V([C_1]+[C_2]+[C_3]+[C_4]+\ldots)}
\]

where \([\cdot]\) denotes class of saddle contributions organized according to their action \( S \sim nS_0 \) or fugacities \( \xi \sim e^{-nS_0} \). For example,

\[
[C_1] = \{M_{\alpha_i}, \overline{M}_{\alpha_i}, \ i \in [1, N - 1]\}
\]

\[
[C_2] = \{[M_{\alpha_i}, \overline{M}_{\alpha_{i+1}}], [\overline{M}_{\alpha_i}, M_{\alpha_{i+1}}], \ i \in [1, N - 2], \ [M_{\alpha_i}, \overline{M}_{\alpha_i}]_\pm, \ i \in [1, N - 1]\}
\]

\([C_1]\) denotes first order effects in semi-classics. These are regular saddles. \([C_2]\) are second order effects in semi-classics. Their relation to saddles in \([C_1]\) is following. Two configuration in \([C_1]\), if they interact, are not genuine saddles at any finite separation, but they
become saddles at infinite separation. The integral over the steepest descent cycle produce two-effects: one can be interpreted as $\frac{1}{2} V^2 |C_1|^2$, non-interacting two-instanton contribution and the other is $V|C_2|$, a 2-cluster contribution. The proliferation of these two type contributions exponentiate to produce the first two terms in (B.1). Magnetic and neutral bions are the elements of 2-cluster configurations.\footnote{In the BPS limit $\lambda \to 0$, $\mathcal{M}_{\alpha_0}$ does not interact with $\mathcal{M}_{\alpha_{1,1}}$ as the interaction due to dual photon exchange is cancelled exactly by the scalar exchange. But there are genuine BPS solutions with magnetic charge $\alpha_0 + \alpha_{1,1}$ with action $2S_0$ that needs to be included. These are $\mathcal{M}_{\alpha_0 + \alpha_{1,1}}$ that generate only quantitative differences, and to simplify the discussion, we will not include them below. But it is necessary to include the second order effects in $|C_2|$ because they generate qualitative new effects, such as ambiguities, and they naturally arise as part of the critical point at infinity of configurations in $|C_1|$, while $\mathcal{M}_{\alpha_0 + \alpha_{1,1}}$ do not arise in such manner.}

The semi-classical expansion is based on class $[C_1]$ and $k$-clusters $[C_k]$ (which arise naturally from critical points at infinity.)

Let us write the partition function as

$$e^{-\mathcal{V} \xi_0} \sim e^{\mathcal{V} ([C_1] + [C_2] + [C_3] + \ldots)}$$

$$= \prod_{\mathcal{F}} \sum_{n_{\mathcal{T}} = 0}^{\infty} \frac{1}{n_{\mathcal{T}}!} [\mathcal{V} \mathcal{F}]^{n_{\mathcal{T}}}$$

$$= \sum_{n_1 = 0}^{\infty} \frac{1}{n_{1,1}!} (\mathcal{V}[C_1])^{n_1} \sum_{n_2 = 0}^{\infty} \frac{1}{n_{2,2}!} (\mathcal{V}[C_2])^{n_2} \sum_{n_3 = 0}^{\infty} \frac{1}{n_{3,3}!} (\mathcal{V}[C_3])^{n_3} \ldots$$

$$= \sum_{n_1, n_2, n_3, \ldots} \frac{1}{n_{1,1}! n_{2,2}! n_{3,3}! \ldots} \mathcal{V}^{n_1 + n_2 + \ldots} ([C_1])^{n_1} ([C_2])^{n_2} ([C_3])^{n_3} \ldots \tag{B.3}$$

where $[C_k]$ is the sum over the configurations that are in the $k$-cluster, examples of which are shown in (B.2). For example, $[C_1] = \sum_{i=1}^{N-1} (\mathcal{M}_{\alpha_i} + \mathcal{M}_{\alpha_{i+1}})$ etc. So, the cluster expansion gets crowded rather quickly.

To highlight our main point, we will consider a very simple system whose merits carry over to the current problem without cluttering the expression more than necessary. Consider for example, a simple system like particle on a circle in the presence of a potential $V(q) = -\cos(q)$, $q \in [0, 2\pi]$. There is a unique minimum on the circle and there is an instanton and anti-instanton. To clarify the structure of the cluster expansion as much as possible, let us only keep track of instanton in the sum. (Anti-instantons can be incorporated easily.) Let us denote the $k$-clusters as

$$[C_k] = a_k(g^2)\xi^k, \quad a_1(g^2) = 1, \quad a_k(g^2) = P_{k-1}(\log(1/g^2)) \tag{B.4}$$

and use $\mathcal{V} = \beta$. Now, we can reorganize the partition function in the $\mathcal{V} \to \infty$ limit as an expansion in $\xi$ rather than an expansion in $\mathcal{V}$. This will illuminate some aspects of the
This expansion require some comments:

- Consider order $\xi^n$. The maximally extensive part in $\mathcal{V}$ is a non-interacting gas of single instantons.

- Let us first describe the meaning of sub-extensive part for $n = 2$. In (B.5) second line, the overall factor of $\mathcal{V}$ is sourced by integration over the center of action of the two instantons. The integral over one QZM direction in (B.5) yields $V_1 \frac{1}{2!} + \frac{(a_2)}{1!}$, a part extensive with volume, and a finite part. The combination of zero mode and quasi-zero mode integration yields $\xi^2$ term in (B.5). Sub-extensive pieces include effects of correlated events, in this case a 2-cluster. $a_2(g)$ is a first order polynomial in $\log(1/g^2)$. In general, $a_n(g^2)$ is an $(n-1)^{th}$ order polynomial in $\log(1/g^2)$. In general, $a_n(g^2)$ is an $(n-1)^{th}$ order polynomial in $\log(1/g^2)$.

- At order $\xi^n$, the terms are of the form $\mathcal{V}^k (\log(1/g^2))^{n-k}$. $\mathcal{V}^n$ part is the non-interacting gas. The last term $\mathcal{V}^1 (\log(1/g^2))^{n-1}$ is the contribution of $(n-1)$-cluster, and overall $\mathcal{V}$ is the integral over the center position of the cluster.

- Finally, and of course quite importantly, quasi-zero mode integration yields $\log(1/g^2)$ type terms for repulsive interactions and two-fold ambiguous $\log(1/g^2) \pm i \pi$ for the attractive interactions. As a result, the (B.5) will also have two-fold ambiguous terms once anti-instantons are included in the expansion.

If we apply the same strategy to the Polyakov model, the resulting expressions at
second order in semi-classics takes the form

$$\frac{\mathcal{V}^2}{2!}[\mathcal{C}_1]^2 + \frac{\mathcal{V}}{1!}[\mathcal{C}_2]$$

$$= \frac{\mathcal{V}^2}{2!} \left( \sum_{i=1}^{N-1} ([M_{\alpha_i}] + [\overline{M}_{\alpha_i}])^2 \right) + \frac{\mathcal{V}}{1!} \sum_{i=1}^{N-2} ([M_{\alpha_i} \overline{M}_{\alpha_{i+1}}] + [\overline{M}_{\alpha_i} M_{\alpha_{i+1}}])$$

$$+ \frac{\mathcal{V}}{1!} \sum_{i=1}^{N-1} ([M_{\alpha_i} \overline{M}_{\alpha_i}]_{\pm} + [\overline{M}_{\alpha_i} M_{\alpha_i}]_{\pm})$$

(B.6)

which have terms with three different characteristics.

- Terms which do not have a 2-cluster contribution, eg. \(\frac{\mathcal{V}^2}{2!}[M_{\alpha_i}]^2\) because \(M_{\alpha_i}\) does not interact with \(M_{\alpha_i}\). As such, there is no \([M_{\alpha_i} \overline{M}_{\alpha_i}]\) in \([\mathcal{C}_2]\).

- Terms which do have an unambiguous 2-cluster contribution, e.g.

$$\mathcal{V}^2[M_{\alpha_i}]\overline{M}_{\alpha_{i+1}} + \mathcal{V}[M_{\alpha_i} \overline{M}_{\alpha_{i+1}}] = \xi^2 (\mathcal{V}^2 + \mathcal{V}\alpha_2(g^2))$$

(B.7)

because some instanton and anti-instantons in \([\mathcal{C}_1]\) do interact repulsively with each other, and these are contribution of critical points at infinity of such configurations, integrated over the steepest descent cycle \(J_1\).

- Terms which do have a two-fold ambiguous 2-cluster contribution, e.g.

$$\mathcal{V}^2[M_{\alpha_i}]\overline{M}_{\alpha_i} + \mathcal{V}[M_{\alpha_i} \overline{M}_{\alpha_i}]_{\pm} = \xi^2 (\mathcal{V}^2 + \mathcal{V}\alpha_2(g^2\epsilon^{\pm i\pi}))$$

(B.8)

because some instanton and anti-instantons in \([\mathcal{C}_1]\) do interact attractively with each other, and these are contribution of critical points at infinity of such configurations, integrated over the steepest descent cycles \(J_2^\pm\). These cycles are inevitably two-fold ambiguous as described below.

### C QZM integral by integration by parts

**Repulsive interaction:** Another way to get the quasi-zero mode integral is to first introduce a hard cut-off \(R\) and then, using the integration by parts technique successively. Define

$$J^2(g^2) = 4\pi v^{-3} \lim_{R \to \infty} \int_0^R dr \frac{r^2 e^{- \frac{\pi |\alpha_i \cdot \alpha_j|}{g^2 r}}}{r^2}$$

$$= 4\pi v^{-3} \left( \frac{2\pi |\alpha_i \cdot \alpha_j|}{g^2} \right)^3 \int_0^{\tilde{R}} d\tilde{r} \tilde{r}^2 e^{-1/\tilde{r}}, \quad \tilde{R} = \frac{g^2}{2\pi |\alpha_i \cdot \alpha_j|} R$$

(C.1)

The radial integral is well-behaved at \(\tilde{r} = 0\) but diverges as \(\tilde{r} \to \infty\). We introduce a hard cutoff at \(\tilde{r} = \tilde{R}\). Applying integration by parts repeatedly, we give a precise meaning to divergent parts and finite parts. The divergent part is the volume of the system, and the finite part will be identified with the second virial expansion parameter, see eg. [43].
We can proceed with the repeated application of integration by parts, a procedure which eventually ends up with a finite integration:

\[
\left(\frac{3}{\pi} \frac{\pi |\alpha_i \cdot \alpha_j|}{g^2}\right)^{-3} J(g^2) = \left(\frac{\tilde{R}^3}{3} e^{-1/\tilde{R}} - \frac{1}{3} \int_0^{\tilde{R}} d\tilde{r} \tilde{r} e^{-1/\tilde{r}}\right)
= \left[\left(\frac{\tilde{R}^3}{3} - \frac{\tilde{R}^2}{6}\right) e^{-1/\tilde{R}} + \frac{1}{6} \int_0^{\tilde{R}} d\tilde{r} e^{-1/\tilde{r}}\right]
= \left[\left(\frac{\tilde{R}^3}{3} - \frac{\tilde{R}^2}{6} + \frac{\tilde{R}}{6}\right) e^{-1/\tilde{R}} - \frac{1}{6} \int_0^{\tilde{R}} d\tilde{r} \frac{1}{\tilde{r}} e^{-1/\tilde{r}}\right]
= \left[\left(\frac{\tilde{R}^3}{3} - \frac{\tilde{R}^2}{6} + \frac{\tilde{R}}{6} \ln \tilde{R}\right) e^{-1/\tilde{R}} + \frac{1}{6} \int_0^{\tilde{R}} d\tilde{r} \frac{\log \tilde{r}}{\tilde{r}^2} e^{-1/\tilde{r}}\right]
\]

(C.2)

Now, we carefully look at the behaviors of these expressions as \(\tilde{R} \to \infty\). In this limit, the last integral is finite, and it is just the Euler–Mascheroni constant \(\gamma\). The extensive term can be handled by expanding it around \(\tilde{R} = \infty\):

\[
\left(\frac{\tilde{R}^3}{3} - \frac{\tilde{R}^2}{6} + \frac{\tilde{R}}{6} \ln \tilde{R}\right) e^{-1/\tilde{R}} \approx \frac{\tilde{R}^3}{3} - \frac{\tilde{R}^2}{6} + \frac{\tilde{R}}{36} - \frac{\ln \tilde{R}}{6} + O\left(\frac{1}{\tilde{R}}\right)
\]

(C.3)

Identifying \(\tilde{R} = 1/\delta\), we observe that this produces the expansion of incomplete \(\Gamma\) function given in (3.11). As a result, we obtain:

\[
J_2(g^2) = V + I(g^2).
\]

in agreement with (3.13). We have few other comments on this result:

- In cluster expansion in statistical mechanics, there is a standard trick, subtracting and adding unity to the integrand:

\[
\int d^3r e^{-V_{\text{int}}(r)} = \int d^3r (e^{-V_{\text{int}}(r)} - 1 + 1) = V + \int d^3r (e^{-V_{\text{int}}(r)} - 1)
\]

(C.5)

where \((e^{-V_{\text{int}}(r)} - 1)\) is called Mayer-\(f\) function. But (C.5) is still formal for Coulomb gas, since the integral is still extensive with volume as \(V^{2/3}\). We believe the procedure described in (C.2) and the limit taken carefully in (C.3) and also around (3.10) are better ways to give meaning to this integral. \(I(g^2)\) can be identified with the second virial expansion parameter.

- One of the reasons for writing all steps of the integration (C.2) is to point out an interesting connection between QFT and QM. The integral in the third line of (C.2) \(\int d\tilde{r} \frac{1}{\tilde{r}} e^{-1/\tilde{r}}\) is actually the standard instanton-instanton interaction term in quantum mechanics. To see this, first, let us restore the coupling by writing \(\tilde{r} = \frac{g^2}{2\pi |\alpha_i \cdot \alpha_j|} \equiv \frac{g^2}{\alpha} r\) and use change of variables \(r = e^\tau\). Then, the integral becomes

\[
\int dr \frac{1}{r} e^{-\frac{\Delta}{g^2}} = \int d\tau e^{-\frac{\Delta}{g^2} e^{-\tau}}
\]

(C.6)
where $\tau$ is the quasi-zero mode parameter and $V_{\text{int}}(\tau) = \frac{A^2}{g^2} e^{-\tau}$ is the classical interaction between instantons. The interaction is repulsive for $A > 0$ and attractive for $A < 0$. This type of integrals appears naturally in QM and treated by using the critical point at infinity and Lefschetz thimble integration in [21].

- In QM, the quasi-zero mode integral is also divergent, with an extensive part ($\beta^1$) and a semi-finite part ($\beta^0$). If we compactify Euclidean time $\mathbb{R}$ to $S^1$, at second order in the semi-classical expansion, we obtain

$$
\int d\tau e^{-\frac{A^2}{g^2} e^{-\tau}} \rightarrow \begin{cases}
(\beta + I_{\text{qm}}(g^2)) = \beta + \log\left(\frac{A}{g^2}\right) + \gamma & A > 0 \\
(\beta + I_{\text{qm}}(g^2 e^{\pm i\pi})) = \beta + \log\left(\frac{A}{g^2}\right) + \gamma \pm i\pi & A < 0
\end{cases}
$$

The QZM integrals in QFT and QM are related in a certain way:

$$
J_{2,\text{qft}}(g^2) = V + 4\pi \left(\frac{2\pi |\alpha_i \cdot \alpha_j|}{g^2}\right)^\frac{3}{2} \left(I_{\text{qm}}(g^2) - \frac{11}{6}\right)
$$

where $I_{\text{qm}}(g^2)$ is given in (C.7).

Note that (C.8) connects a 3d calculation to a 1d calculation. It indicates that cluster expansion in the QFT and QM limit are not independent. Moreover, if there is an ambiguity in $I_{\text{qm}}(g^2)$, it indicates that there will be an ambiguity in $J_{2,\text{qft}}(g^2)$.

**Attractive interaction:** The case where the interaction is attractive can be obtained from (C.4) by using analytic continuation. The result is

$$
J(g^2 e^{\pm i\pi}) = V - I(g^2) \pm i \frac{2\pi}{3} \left(\frac{2\pi |\alpha_i \cdot \alpha_j|}{g^2}\right)^\frac{3}{2}.
$$

Clearly, there is a two-fold ambiguity in the attractive case, corresponding to $[\mathcal{M}_{\alpha_i, \overline{\mathcal{M}}_{\alpha_i}]_{\pm}$ amplitudes. Even after the ambiguity cancels via resurgence relations, note that there is an over-all phase difference between $\text{Re}[\mathcal{M}_{\alpha_i, \overline{\mathcal{M}}_{\alpha_i}]_{\pm}$ configuration and $[\mathcal{M}_{\alpha_i, \overline{\mathcal{M}}_{\alpha_i+1]}$. i.e,

$$
\text{Arg}[\text{Re}[\mathcal{M}_{\alpha_i, \overline{\mathcal{M}}_{\alpha_i}]_{\pm}] = \text{Arg}[\mathcal{M}_{\alpha_i, \overline{\mathcal{M}}_{\alpha_i+1]} + \pi
$$

This subtle phenomenon (the fact that these two configurations contribute oppositely to vacuum energy density) appeared earlier in the context of $\mathcal{N} = 1$ SYM and is called “hidden topological angle” [64]. It is a phase that arises from the difference of the quasi-zero mode integration cycles (thimbles) for the repulsive vs. attractive interactions.

**D Lefschetz thimbles for Coulomb interaction**

Consider an exponential integral of the form $\int_\Gamma dz e^{-\frac{1}{2} f(z) h(z)}$ where $|g^2| \ll 1$. In steepest descent formulation, we first determine the critical points of the action, by setting $\frac{df}{dz} = 0$,
which is satisfied at $z = \{z_1, \ldots, z_N\}$ where $N$ is the number of critical points. Critical points may be either at finite values of $z$ or $z = \infty$. The latter is called critical point at infinity and its discussion is slightly more subtle.

Associated with each critical point, one can determine a unique steepest descent cycle $J_i$ (called Lefschetz thimble in higher dimensions) on which $e^{-\frac{1}{g^2}f(z)}$ is ever decreasing and $e^{-\frac{1}{g^2}f(z)} \to 0$ on certain wedges (that we can be called good wedges) in $\mathbb{C}$. If $f(z)$ is a polynomial, the cycle $J_i$ starts in some good wedge and ends in some other good wedge as $|z| \to \infty$ and it passes through $z_i$. (A lucid explanation of this can be found in Witten’s work [65].) The steepest ascent cycle $K_i$ is the cycle on which $e^{-\frac{1}{g^2}f(z)}$ is ever-increasing, and it starts and ends at complementary wedges (bad domains), where exponential blows up. If $f(z)$ has a pole, the pole can also serve as an ending point of the thimble. For example, if $f(z)$ is a doubly periodic function on $\mathbb{C}$, the only place that steepest descent cycle $J_i$ can start and end are the poles as discussed in [66].

In our current example, $f(z) = \frac{1}{g^2} \frac{1}{z}$. $\text{Arg}(g^2) = 0$ is for repulsive interactions and $\text{Arg}(g^2) = \pi \pm \epsilon$ is for attractive interactions. In both cases, the critical point is at infinity, and there is a pole at $z = 0$. By using various regularizations, we can move the critical point to some $R_\star$, and let it go to infinity as the regulator is removed. Even with the hard-cutoff imposed in (C.2), although $f'(z)$ does not vanish anywhere, it can be made arbitrarily small at the boundary $z = \tilde{R}$, and the boundary of the integral behaves as a pseudo-critical point, which becomes genuine one as $\tilde{R}$ tends to infinity.

**Figure 7:** For repulsive interaction, $J_1$ is the steepest descent cycle. The critical point moves to infinity as the the regulator is removed. $J_1$ extends from the pole to $+\infty$.

**Repulsive interaction:** For the case $\text{Arg}(g^2) = 0$, the $J_1$ cycle is $[0, +\infty)$ as shown in Fig.(7). In this case,

$$\int_{J_1(0)} dz e^{-\frac{1}{g^2}f(z)} h(z) = V + I(g^2) \quad (D.1)$$

Note that $J_1$ leaves the critical point at infinity along the real direction, because it is the direction at which $e^{-\frac{1}{g^2}f(z)}$ drops fastest. It also enters to the pole in the $\text{Arg}(z) = 0$ direction, where the approach to zero is fastest. The steepest descent cycle changes smoothly as we change $\text{Arg}(g^2)$, so long as the change in $\text{Arg}(g^2)$ does not lead to crossing
of a Stokes line. In the current problem, Stokes line is at $\theta = \text{Arg}(g^2) = \pi$ which, as we know from the analytical continuation discussion in Section.3, corresponds to the attractive interaction.

**Attractive Interaction:** First of all, notice that we can move from the repulsive interactions to attractive interactions by dialing $\theta = \text{Arg}(g^2)$, and in fact, $\mathcal{J}_1(\theta = \pi \pm \epsilon) = \mathcal{J}_2(\theta = 0 \pm \epsilon)$ where $\mathcal{J}_2$ are the steepest descent cycles for the attractive interaction. Recall that at $\theta = 0$, the $\mathcal{J}_1(\theta) = 0$ cycles leaves the origin in $\text{arg}(z) = 0$ direction and enters to infinity at $\text{arg}(z) = 0$ direction. This guarantees that the integrand is well-behaved on the thimble in the vicinity of $z = 0$. For $\theta = \pi$, the $\mathcal{J}_1(\theta = \pi \pm \epsilon)$ leaves the pole at $z = 0$ in the $\text{arg}(z) = \pi \pm \epsilon$ (for convergence of the integral as $|z| \to 0$), however, eventually makes a clock-wise or counter-clockwise circle and then tends to $z = -\infty$ as shown in Fig. (8).

This guarantees that the integral along the thimble is well-behaved.

**Figure 8:** For attractive interactions, $\mathcal{J}_2^{\pm}$ are the two types of steepest descent cycles. $\mathcal{J}_2^{-}$ and $\mathcal{J}_2^{+}$ only differ by the direction of the circle. This generates the ambiguity in the $[M,M]_{\pm}$ event.

For the $\mathcal{J}_2^{\pm}$ cycles, the integral is not independent of the $\mathcal{J}_1(0)$ cycle. Integration over the cycles shown in Fig. (8) yields

$$\int_{\mathcal{J}_2^{\pm}} dz^2 e^{\frac{1}{g^2}f(z)} = V - I_{\pm}(g^2)$$

(D.2)

where the two-fold ambiguity arise from the orientation of the circle segment in $\mathcal{J}_2^{\pm}$. If we denote circle segment as $C_{\pm}$ depending on its orientation, the pole contribution

$$\oint_{C_{\pm}} dz^2 e^{\frac{1}{g^2}z^2} = \pm 2\pi i \frac{1}{6} \left( \frac{1}{g^2} \right)^3$$

(D.3)

The flip of orientation of this segment in $\mathcal{J}_2^{\pm}$ is the reason for the ambiguity in the $[M_\alpha,M_\alpha]_{\pm}$ amplitude in (3.17).

The fact that $\mathcal{J}_2(\theta)$ changes abruptly as $\theta$ crosses 0 means that for the attractive interaction, the Stokes line is at $\text{Arg}(g^2) = 0$. This simple fact means that we are trying to define the theory on a Stokes line. The phenomena we encounter (two-fold ambiguity that
arises from non-Borel summability of perturbation theory, and two-fold ambiguities in the \( \mathcal{M}_\alpha \mathcal{M}_\beta \) amplitudes are natural consequences of an effort to try to define the QFT on a Stokes line where it is most subtle. Yet, it is still possible to give it an unambiguous meaning by left/right Borel-Ecalle resummation.

### E 3 Instanton Order

In this section, we compute the partition function at the 3 instanton level and specifically, terms in the 3-cluster \([C_3]\). The partition function is written as

\[
Z_3 = \frac{[C_1]^3}{3!} \int \text{d}r_i \text{d}r_j \text{d}r_k e^{-V_{\text{int}}} , \quad [C_1] = a_1(g)e^{-S_0} = a_1(g)\xi \tag{E.1}
\]

where \( V_{\text{int}} \) is the sum of Coulomb interaction of the monopoles located at \( r_i, r_j \) and \( r_k \).

**Repulsive Interactions** We first assume that all the interactions between monopoles are repulsive. Then, the interaction potential is written as

\[
V_{\text{int}} = \frac{\pi}{g_3^2} \left[ \frac{|\alpha_i \alpha_j|}{|r_i - r_j|} + \frac{|\alpha_j \alpha_k|}{|r_j - r_k|} + \frac{|\alpha_i \alpha_k|}{|r_i - r_k|} \right]. \tag{E.2}
\]

To simplify the problem, we adapt the Jacobi coordinates as

\[
q_0 = \frac{1}{3}(r_i + r_j + r_k), \quad q_1 = r_j - r_i, \quad q_2 = r_k - \frac{1}{2}(r_j + r_i). \tag{E.3}
\]

Then, the spatial integral becomes

\[
J_3(g_3^2) = \mathcal{V} \int dq_1 dq_2 \exp \left\{ -\frac{\pi}{g_3^2} \left[ \frac{|\alpha_i \alpha_j|}{|q_1|} + \frac{|\alpha_j \alpha_k|}{|q_2 + \frac{1}{2}q_1|} + \frac{|\alpha_i \alpha_k|}{|q_2 - \frac{1}{2}q_1|} \right] \right\} \tag{E.4}
\]

The terms \(|q_2 + \frac{1}{2}q_1| = |r_k - r_j|\) and \(|q_2 - \frac{1}{2}q_1| = |r_k - r_i|\) can be expressed in a more convenient way using the geometry in Fig 9. Using the cosine rule, we write

\[
|q_2 + \frac{1}{2}q_1| = |r_k - r_j| = q_2^2 + \left(\frac{q_1}{2}\right)^2 + q_1q_2 \cos \theta \tag{E.5}
\]

\[
|q_2 - \frac{1}{2}q_1| = |r_k - r_i| = q_2^2 + \left(\frac{q_1}{2}\right)^2 - q_1q_2 \cos \theta \tag{E.6}
\]

where we set \( q_{1,2} = |q_{1,2}| \). Then, last two terms in the exponent in (E.4) becomes

\[
\frac{|\alpha_j \alpha_k|}{|q_2 + \frac{1}{2}q_1|} + \frac{|\alpha_i \alpha_k|}{|q_2 - \frac{1}{2}q_1|} = \frac{1}{q_2} \left( |\alpha_j \alpha_k| F_+(q_1, q_2) + |\alpha_i \alpha_k| F_-(q_1, q_2) \right) \tag{E.7}
\]

where

\[
F_\pm(q_1, q_2) = \left( 1 \pm \frac{q_1}{q_2} \cos \theta + \frac{q_1^2}{4q_2^2} \right)^{-1/2}. \tag{E.8}
\]

Since the large separation region dominates the integrals, both \( q_1 \) and \( q_2 \) are large and have approximately same size. Then, expanding \( F_\pm \) around \( q_1 = q_2 \) we get

\[
F_\pm(q_1, q_2) = f_\pm(\theta) + O\left(\frac{q_1}{q_2}\right) \tag{E.9}
\]
and at the leading order, up to a $\theta$ dependent pre-factor, the spatial integral reduces to

$$J_3(g_3^2) \simeq \mathcal{V} \int_0^\infty dq_1 q_1^2 e^{-\frac{2\pi|\alpha_i \cdot \alpha_j|}{g^2 q_1}} \int_0^\infty dq_2 q_2^2 e^{-\frac{2\pi(|\alpha_j \cdot \alpha_k| + |\alpha_i \cdot \alpha_k|)}{g^2 q_2}}$$  \hspace{1cm} (E.10)

These are in the same form with the integrals we get at 2 instanton level in Section 3. Thus, we can immediately deduce the partition function from (3.13) as

$$Z_3 = \frac{\xi^3}{3!} \mathcal{V} (\mathcal{V} + I_1(g^2)) (\mathcal{V} + I_2(g^2))$$  \hspace{1cm} (E.11)

where $I_1$ and $I_2$ are

$$I_1(g^2) = \frac{4\pi v^3}{6} \left( \frac{2\pi|\alpha_i \cdot \alpha_j|}{g^2} \right)^3 \left( \ln \left( \frac{2\pi|\alpha_i \cdot \alpha_j|}{g^2} \right) + \gamma - \frac{11}{6} \right),$$  \hspace{1cm} (E.12)

$$I_2(g^2) = \frac{4\pi v^3}{6} \left( \frac{2\pi}{g^2} (|\alpha_j \alpha_k| + |\alpha_i \alpha_k|) \right)^3 \left( \ln \left[ \frac{2\pi}{g^2} (|\alpha_j \alpha_k| + |\alpha_i \alpha_k|) \right] + \gamma - \frac{11}{6} \right).$$  \hspace{1cm} (E.13)

**Attractive Interactions:** Now, we turn our attention to the cases with attractive Coulomb potentials. An example suitable to the our choice of Jacobi coordinates in (E.3) is the case which there are 2 monopoles located at $r_i$ and $r_j$ and 1 anti-monopole at $r_k$. (See Fig. 9.) The associated interaction potential is given as

$$V_{\text{int}} = \frac{\pi}{g^2} \left[ \frac{|\alpha_i \alpha_j|}{|r_i - r_j|} - \frac{|\alpha_j \alpha_k|}{|r_j - r_k|} - \frac{|\alpha_i \alpha_k|}{|r_k - r_i|} \right]$$  \hspace{1cm} (E.14)

The partition function in this case can be probed by an analytical continuation of the $q_2$ integral in (E.10) as we did for attractive two instanton cases. This leads to a two fold ambiguous result in the form of (3.17) and we get

$$Z_3 = \frac{\xi^3}{3!} \mathcal{V} (\mathcal{V} + I_1(g^2)) \left( \mathcal{V} + I_2(g^2) \pm i\pi \left( \frac{\pi}{g^2} (|\alpha_j \alpha_k| + |\alpha_i \alpha_k|) \right)^3 \right)$$  \hspace{1cm} (E.15)
Remark: There are other possible sign combinations for the interaction potential in (E.14). In general, for any 3 instanton anti-instanton event, either all the signs are (+) as in (E.2) or there are 2 (−) sign along with 1 (+) sign as in (E.14). In the latter case, the Jacobi coordinates should be chosen appropriately so that the signs in the exponent of the $q_2$ integral should be the same. Otherwise, it is possible that they could cancel out and higher-order terms arising from (E.9) could lead to $\frac{1}{q_2^2}$ terms, i.e. dipole interaction. In these cases, the logarithmic behavior arising from the incomplete gamma function would be obscured. With appropriate coordinate choices, all of them lead to the same form with (E.15) up to modifications in charge-dependent terms.

Relation to the Cluster Expansion: Now let us understand our result in terms of the cluster expansion. First note that if we consider them separately, both $q_1$ and $q_2$ integrals in (E.10) can be interpreted as 2 body interactions. In this sense, the $q_1$ integral, along with the volume prefactor $\mathcal{V}$, corresponds to the interaction between the monopoles at $r_i$ and $r_j$. Its contribution is given as $\xi^2 \mathcal{V}(\mathcal{V} + I_1(g^2))$, which is in parallel with the 2 instanton events in Section 3. On the other hand, if we took the $q_2$ integral independently, it would correspond to the radial integration of an interaction between the monopole at $r_k$ and a monopole with a total charge $Q_{ij} = \frac{2\pi}{g^2} (\alpha_i + \alpha_j)$. In (E.10), this means that the monopole at $r_k$ coalesce into the 2 body interaction of the monopoles at $r_i$ and $r_j$.

This is indeed how we should relate (E.10) to the cluster expansion. Specifically, the contribution of the $q_1$ integral consists of 2-cluster connected and disconnected parts, i.e. $\xi^2 \mathcal{V}(\mathcal{V} + I_1(g^2)) \sim \mathcal{V}^2[C_1]^2 + \mathcal{V}[C_2]$, as in 2 instanton case. The contribution of the $q_2$, i.e. $\xi(\mathcal{V} + I_2(g^2))$, on the other hand, should be interpreted in terms of the connectedness of the monopole at $r_k$ to the two body interaction represented by the $q_1$ integral. Then,

- Volume contribution of $q_2$ integral refers that the monopole at $r_k$ is not connected to the neither of the other monopoles. Thus, it contributes as an independent 1-cluster, i.e. $\mathcal{V}[C_1]$, onto the expansion of the $q_2$ integral, i.e. $\mathcal{V}^2[C_1]^2 + \mathcal{V}[C_2]$.

- $I_2(g^2)$ part refers to the cases that the monopole at $r_k$ is attached to one or both of the other monopoles. Therefore, when $\xi I_2(g^2)$ combines with the two 1-cluster part $\xi^2 \mathcal{V}^2$, it results in a product of 1-cluster and 2-cluster, i.e. $\mathcal{V}^2[C_1]^2 \to \mathcal{V}^2[C_1][C_2]$. Similarly, the combination of $\xi I_2(g^2)$ with $\xi^2 \mathcal{V} I_1(g^2)$ turns the 2-cluster part of the $q_1$ integral into a 3-cluster, i.e. $\mathcal{V}[C_2] \to \mathcal{V}[C_3]$.

This indicates that the 3 instanton partition function in (E.11) indeed takes the form of $\xi^3$ order terms in the cluster expansion (B.5):

$$Z_3 = \frac{\mathcal{V}^3}{3!}[C_1]^3 + \frac{\mathcal{V}^2}{2!}[C_1][C_2] + \frac{\mathcal{V}}{1!}[C_3] \quad (E.16)$$

with the following identifications:

$$\xi^3 (I_1(g^2) + I_2(g^2)) \in [C_1][C_2], \quad (E.17)$$

$$\xi^3 I_1(g^2) I_2(g^2) \in [C_3]. \quad (E.18)$$
It is also possible to express (E.16) as a collection of the monopole events. First note that as stated in the footnote in page 34, the effective interaction between $M_{\alpha_i}$ and $M_{\alpha_j}$ vanishes. Therefore, they do not form 2 clusters which are restricted as in (B.6). However, contrary to the 2-cluster case, $M_{\alpha_i}$ and $M_{\alpha_j}$ can participate in the same 3-cluster term. Then, we get the following characteristics:

- Terms arising from the combination of 1-cluster events, e.g. $\frac{\nu^3}{\nu^3}[M_{\alpha_i}]^3$

- Unambiguous 2-cluster terms with an additional 1-cluster term, e.g

$$\nu^3[M_{\alpha_i}][M_{\alpha_{i+1}}]M_{\alpha_i}] + \nu^2[M_{\alpha_j}][M_{\alpha_i}][M_{\alpha_i}]_j, \quad j \neq i \pm 1 \quad (E.19)$$

which is a direct extension of the second order term.

- Ambiguous 2-cluster terms with an additional 1-cluster term, e.g

$$\nu^3[M_{\alpha_i}][M_{\alpha_i}][M_{\alpha_i}]M_{\alpha_i}] + \nu^2[M_{\alpha_j}][M_{\alpha_i}][M_{\alpha_i}]_j \pm \nu[M_{\alpha_i}][M_{\alpha_i}]_i \pm \nu[M_{\alpha_i}][M_{\alpha_i}]_i \pm (E.20)$$

- Unambiguous 3-cluster terms, e.g.

$$\nu^3[M_{\alpha_{i+1}}][M_{\alpha_{i+1}}][M_{\alpha_i}]M_{\alpha_i}] + \nu^2[M_{\alpha_{i+1}}][M_{\alpha_{i+1}}][M_{\alpha_i}] \pm \nu[M_{\alpha_{i+1}}][M_{\alpha_i}][M_{\alpha_i}] \mp \nu[M_{\alpha_i}][M_{\alpha_i}][M_{\alpha_i}] \pm (E.21)$$

- Ambiguous 3-cluster terms, e.g.

$$\nu^3[M_{\alpha_i}][M_{\alpha_i}][M_{\alpha_i}]M_{\alpha_i}] + \nu^2[M_{\alpha_i}][M_{\alpha_i}][M_{\alpha_i}] \pm \nu[M_{\alpha_i}][M_{\alpha_i}][M_{\alpha_i}] \pm (E.22)$$

Since only 3-cluster terms $C_3$ contribute to the spectrum at 3 instanton order, the ambiguous imaginary part of the spectrum arises from $\xi^3I_1(g^2)I_2(g^2)$ part of the partition function. Then, considering events with the ambiguity, we get the imaginary contribution to spectrum as

$$\text{Im}\mathcal{E}^{(3)} = \text{Im}[M\overline{M}M]_\pm \simeq \pm \left(\frac{s_0}{g^2}\right)^{12} \ln\left(\frac{1}{g^2}\right) e^{-\frac{12\pi}{g^2}}. \quad (E.23)$$

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