Quasi-projection operators
in the weighted $L_p$ spaces *

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Abstract

Approximation properties of multivariate quasi-projection operators are studied in the paper. Wide classes of such operators are considered, including the sampling and the Kantorovich-Kotelnikov type operators generated by different band-limited functions. The rate of convergence in the weighted $L_p$-spaces for these operators is investigated. The results allow to estimate the error for reconstruction of signals (approximated functions) whose decay is not enough to be in $L_p$.

Keywords Quasi-projection operators, band-limited functions, approximation order, modulus of smoothness, matrix dilation, weighted $L_p$ spaces.

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1 Introduction

The classical Kotelnikov formula (sampling expansion) provides exact reconstruction of band-limited signals based on the sampled values. Up to now, an overwhelming diversity of digital signal processing applications and devices are based on it and more than successfully use it. However, the class of band-limited signals is very small. To deal with sampling expansions for essentially wider classes of functions, one studies the convergence and error analysis of sampling expansions as the dilation factor goes to infinity (see, e.g., books [30], [36], and survey [34]). In the recent years, many works are dedicated to the study of approximation properties of sampling expansions and their generalizations in $L_p$-norm (see [3, 6, 7, 11, 19, 16, 27, 28, 29]).

The classical sampling expansion $\sum_{k \in \mathbb{Z}} f(M^{-j}k) \operatorname{sinc}(M^j x - k)$ is a special case of the quasi-projection operators (or scaling expansions)

$$Q_j(f; \tilde{\varphi}, \varphi) = \sum_{k \in \mathbb{Z}} M^j(f, \tilde{\varphi}(M^j \cdot - k)) \varphi(M^j \cdot - k)$$

with the Dirac delta-function as $\tilde{\varphi}$ and the sinc-function as $\varphi$.

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The operators $Q_j(f; \tilde{\varphi}, \varphi)$ are actively studied for different classes of functions $\varphi$ and functions/distributions $\tilde{\varphi}$. In particular, these operators with some special functions $\varphi$ and $\tilde{\varphi}$ play an important role in the wavelet theory, especially in the case of compactly supported $\varphi$ and $\tilde{\varphi}$. Approximation properties of $Q_j(f; \tilde{\varphi}, \varphi)$ with compactly supported $\varphi$ and $\tilde{\varphi}$ in the general situation were investigated in [13], [14], [18]. Another important special case is the case where $\varphi$ is a band-limited function and $\tilde{\varphi}$ is locally summable. Note that this class of quasi-projection operators includes the classical Kantorovich–Kotelnikov operators, where $\tilde{\varphi}$ is the characteristic function of $[0, 1]$. In this case $M^j_j(f, \tilde{\varphi}(M^j \cdot -k))$ is the averages value of $f$ near the node $M^{-j}k$. It is worth mentioning that using the averaging value instead of the exact value $f(M^{-j}k)$ in the sampling expansion allows to deal with discontinues signals and reduce the so-called time-jitter errors, which is very useful in the Signal and Image Processing. Moreover, the Kantorovich–Kotelnikov type operators are bounded in $L_p(\mathbb{R})$ and, therefore, provide better approximation order than the sampling operators. During the last years, the Kantorovich–Kotelnikov type operators as well as their different generalizations and refinements have been especially actively studied (see, e.g., [2, 12, 14, 16, 18, 19, 22, 26, 35]).

In the multidimensional case, the quasi-projection operator $Q_j(f; \tilde{\varphi}, \varphi)$ can be defined using a matrix $M$ as dilation as follows

$$Q_j(f; \tilde{\varphi}, \varphi) = \sum_{k \in \mathbb{Z}^d} \det M^j \langle f, \tilde{\varphi}(M^j \cdot -k) \rangle \varphi(M^j \cdot -k).$$

Let us mention some error estimates for $Q_j(f; \tilde{\varphi}, \varphi)$, which are closely related to our current investigation. For wide classes of functions $\tilde{\varphi}$ and band-limited functions $\varphi$, the multivariate quasi-projection operators with a matrix dilation were studied in [17]. Under the assumption that the Fourier transform of both the functions is $n + d + 1$ times continuously differentiable in a neighborhood of the origin (but not necessary continuous on $\mathbb{R}^d$), and all derivatives up to order $n - 1$ of the function $1 - \tilde{\varphi}^2 \varphi$ vanish at the origin, the following estimate was derived:

$$\|f - Q_j(f; \tilde{\varphi}, \varphi)\|_p \leq C \omega_n \left( \|f\|_{M^{-j}} \right)_p, \quad f \in L_p(\mathbb{R}^d), \quad 1 \leq p \leq \infty,$$

where $\omega_n$ is the modulus of smoothness of order $n$ and $C$ does not depend on $f$, $j$, and $M$.

Now let us consider the case of sampling expansions. If $Q_j(f; \tilde{\varphi}, \varphi)$ is the classical sampling expansion (i.e. $\varphi$ is the sinc-function and $\tilde{\varphi}$ is the Dirac delta-function), then well-known Brown’s inequality [5]

$$\|f - Q_j(f; \tilde{\varphi}, \varphi)\|_{\infty} \leq C \int_{|\xi| > M^{j/2}} |\hat{f}(\xi)| d\xi$$

holds for every $j \in \mathbb{Z}_+$ and a function $f : \mathbb{R} \to \mathbb{C}$ whose Fourier transform is summable on $\mathbb{R}$. This Brown’s result was strengthened in [16] in several directions. Namely, a similar inequality was established in $L_p$-norm, $2 \leq p \leq \infty$, and the multivariate operators $Q_j(f; \tilde{\varphi}, \varphi)$ with a matrix dilation were considered for a wide class of tempered distribution $\tilde{\varphi}$ and a wide class of band-limited functions $\varphi$ instead of the sinc-function. In particular, in the case of the Dirac delta-function as $\tilde{\varphi}$, under the assumption that the Fourier transform of $\varphi$ is identical 1 on a $\delta$-neighborhood of the origin, for any $\gamma > d/p$ and every function $f$ such that $\hat{f} \in L_q(\mathbb{R}^d)$, $\hat{f}(\xi) = O(|\xi|^{-\gamma - d/q})$, $1/p + 1/q = 1$, the following inequality was derived:

$$\|f - Q_j(f; \tilde{\varphi}, \varphi)\|_p \leq C \|M^{-j}\| \left( \int_{|M^{-j}||\xi| \geq \delta} |\xi|^\gamma |\hat{f}(\xi)|^q d\xi \right)^{1/q},$$

where $C$ does not depend on $f$, $j$, and $M$.

Error estimates (1) and (2) are aimed at the recovery of signals $f$, but they are not applicable to non-decaying signals and even for signals whose decay is not enough to belong to the space
$L_p(\mathbb{R}^d)$. The goal of the present paper is to obtain counterparts of these estimates in norms of some weighted $L_p$-spaces. An idea to extend sampling theory in this regard was recently suggested by H. Q. Nguyen and M. Unser [25]. In particular, they extended a number of basic facts of the theory of shift-invariant spaces in $L_p(\mathbb{R}^d)$ to the weighted spaces $L_{p,1/w}(\mathbb{R}^d)$, where the function $w$ belongs to the class of the so-called submultiplicative weights. Since this class contains weights with polynomial growth, signals which grow not faster than a polynomial are in an appropriate space $L_{p,1/w}(\mathbb{R}^d)$.

Using the technique developed in [17] and [19] as well as several basic facts obtained in [25] for the same class of submultiplicative weights, we derive an analog of estimate (1) given in terms of the best approximation and moduli of smoothness in the weighted $L_p$ spaces (see Theorems 11 and 13). However, this technique does not allow to work with slowly decaying functions $\varphi$, in particular, with the sinc-function. Thus, in contrast to the non-weighted case, the corresponding results for $L_{p,1/w}$ are proved under additional assumption on smoothness of the Fourier transform of $\varphi$. To fix this drawback, we apply and extend the technique developed in [29] and [16]. This is done in Sections 4.2 and 4.3 but only for weights $w$ satisfying some additional properties. Namely, the boundedness of the maximal function of the Hilbert transform in $L_{p,1/w}$ is required. For this purpose, we consider a certain subclass of Muckenhoupt type weights $A_p(\mathbb{R}^d)$, which consists of all admissible weights $w$ that belong to the classical Muckenhoupt class $A_p(\mathbb{R}^1)$ in each variable uniformly with respect to other variables. Note that because of this we have to restrict our consideration to diagonal dilation matrices $M$.

Approximation properties of the sampling expansions in $L_{p,1/w}$-norm for the submultiplicative Muckenhoupt weights $w$ are studied in Section 4.3. In particular, under the additional assumption that $w$ is band-limited and under certain assumptions on $\varphi$, we obtain an analog of estimate (2) (see Theorem 28). Without this additional assumption on $w$ and for a larger class of band-limited functions $\varphi$, an error estimate for the sampling expansions is derived in Theorem 31. Namely, the Fourier transform of $\varphi$ should be smooth enough near the origin and all derivatives up to order $n-1$ of the function $1-\tilde{\varphi}$ should vanish at the origin (instead of identical 1 in a neighborhood of the origin, as we suppose in Theorem 28). In this case, the approximation order represents a combination of estimates given in (1) and (2). As a consequence, we obtain the following approximation order:

$$
\left\| f - \sum_{k \in \mathbb{Z}^d} f(M^{-j}k)\varphi(M^{-j}k) \right\|_{p,1/w} = \begin{cases} 
\mathcal{O}(n^{-j(d/p+a)}) & \text{if } n > d/p + a, \\
\mathcal{O}(n^{-j(a+1/2)}) & \text{if } n = d/p + a, \\
\mathcal{O}(n^{-j}) & \text{if } n < d/p + a
\end{cases}
$$

whenever the Fourier transform of the function $f/w$ belongs to $L_q(\mathbb{R}^d)$, $1/p + 1/q = 1$, and has decay of order $d + a$, $a > 0$. Note that this estimate as well as Theorem 31 are new even in the non-weighted case.

The paper is organized as follows: in Section 2, we introduce notation and give some basic facts. In Section 3, we state and prove several auxiliary results. Section 4 is dedicated to the main results. In Subsection 4.1, we study approximation properties of the quasi-projection operator $Q_j(f;\tilde{\varphi},\varphi)$ in the case of $\varphi$ and $\tilde{\varphi}$ belonging to certain classes of functions whose Fourier transform has some smoothness. Only integrable functions belong to this class. In Subsection 4.2, we solve similar problems for the case of $\varphi$ belonging to a class of band-limited functions that includes non-integrable functions. Subsection 4.3 is dedicated to studying sampling expansions (i.e., the case of the Dirac delta-function as $\tilde{\varphi}$) for a wide class of band-limited functions $\varphi$.

## 2 Notation and basic facts

$\mathbb{N}$ is the set of positive integers, $\mathbb{R}$ is the set of real numbers, $\mathbb{C}$ is the set of complex numbers. $\mathbb{R}^d$ is the $d$-dimensional Euclidean space, $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ are its elements (vectors),
\( (x, y) = x_1 y_1 + \ldots + x_d y_d, \) \( |x| = \sqrt{\langle x, x \rangle}, \) \( 0 = (0, \ldots, 0) \in \mathbb{R}^d; \) \( B_r = \{ x \in \mathbb{R}^d : |x| \leq r \}, \) \( \mathbb{T}^d = \left[ -\frac{1}{2}, \frac{1}{2} \right]^d; \) \( \mathbb{Z}^d \) is the integer lattice in \( \mathbb{R}^d, \) \( \mathbb{Z}_+^d := \{ x \in \mathbb{Z}^d : x \geq 0 \}. \) If \( \alpha, \beta \in \mathbb{Z}_+^d, \) \( a, b \in \mathbb{R}^d, \) we set \( [\alpha] = \sum_{j=1}^d \alpha_j, \) \( \alpha! = \prod_{j=1}^d (\alpha_j!) \),

\[
\begin{align*}
\left( \frac{\beta}{\alpha} \right) &= \frac{\beta!}{\alpha! (\beta - \alpha)!}, \quad D^\alpha f = \frac{\partial^{[\alpha]} f}{\partial x^{\alpha}} = \frac{\partial^{[\alpha]} f}{\partial x_1 \ldots \partial x_d},
\end{align*}
\]

A \( d \times d \) real matrix \( M \) whose eigenvalues are bigger than 1 in modulus is called a dilation matrix. Since the spectrum of the operator \( M^{-1} \) is located in \( B_r \), where \( r = r(M^{-1}) := \lim_{j \to +\infty} \| M^{-j} \|^{1/j} \) is the spectral radius of \( M^{-1} \), and there exists at least one point of the spectrum on the boundary of \( B_r \), we have

\[
\| M^{-j} \| \leq C_{M, \vartheta} \vartheta^{-j}, \quad j \in \mathbb{Z}_+,
\]

for every positive number \( \vartheta \) which is smaller in modulus than any eigenvalue of \( M \). In particular, we can take \( \vartheta > 1 \), then

\[
\lim_{j \to +\infty} \| M^{-j} \| = 0.
\]

In what follows, the class of matrix dilations is denoted by \( \mathfrak{M}, \) \( m = |\det M|, \) \( M^* \) denotes the conjugate matrix to \( M, \) and the \( d \times d \) identity matrix is denoted by \( I_d, \) note also that \( M^0 := I_d. \)

We will say that a weight \( w : \mathbb{R}^d \to [1, \infty) \) belongs to the class \( \mathcal{W}^\alpha \) for some \( \alpha > 0 \) if the following conditions hold:

1) \( w \) is continuous and even;
2) there exist an even function \( w^* : \mathbb{R}^d \to \mathbb{R}_+ \) and a constant \( c_w > 0 \) such that

\[
w(x + y) \leq w^*(x) w(y), \quad w^*(x) \leq c_w (1 + |x|^2)^{\alpha/2}
\]

for all \( x, y \in \mathbb{R}^d; \)
3) there exists a constant \( C' > 0 \) such that

\[
w^*(M^{-j} x) \leq C' w^*(x)
\]

for all \( M \in \mathfrak{M}, \) \( x \in \mathbb{R}^d, \) and \( j \in \mathbb{Z}_+. \)

A model example of such a weight is

\[
w_\alpha(x) = (1 + |x|^2)^{\alpha/2}, \quad \alpha > 0.
\]

Note that if \( w \in \mathcal{W}^\alpha, \) then condition 2) is satisfied also for \( 1/w. \) Indeed,

\[
\frac{1}{w(x + y)} = \frac{w(x)}{w(x + y)w(x)} \leq \frac{w(y)}{w(x)}.
\]

Below, \( L_{p,w} \) denotes the weighted space \( L_p(\mathbb{R}^d, w), \) \( 1 \leq p \leq \infty, \) with the norm \( \| f \|_{p,w} = \| f w \|_{L_p(\mathbb{R}^d)}. \) In the unweighted case, we denote \( L_p = L_p(\mathbb{R}^d). \) Obviously, if \( w \in \mathcal{W}^\alpha, \) then \( L_{p,w} \subset L_p. \)

In what follows, we will often use the following simple inequality for \( f \in L_{p,1/w} \) and \( w \in \mathcal{W}^\alpha: \)

\[
\| f(\cdot + h) \|_{p,1/w} \leq w^*(h) \| f \|_{p,1/w}.
\]

In particular, if \( |h| \leq 1, \) then

\[
\| f(\cdot + h) \|_{p,1/w} \leq c_w 2^{\alpha/2} \| f \|_{p,1/w}.
\]
Similarly, \( \ell_{p,w} \) denotes the set of sequences \( c = \{ c_k \}_{k \in \mathbb{Z}^d} \) with the norm
\[
\|c\|_{\ell_{p,w}} = \left( \sum_{k \in \mathbb{Z}^d} |c_k w(k)|^p \right)^{1/p}.
\]

We use \( W^{n}_{p,w} \), \( 1 \leq p \leq \infty \), \( n \in \mathbb{N} \), to denote the Sobolev space on \( \mathbb{R}^d \), i.e. the set of functions whose partial derivatives up to order \( n \) are in \( L_{p,w} \), with the usual Sobolev semi-norm given by
\[
\|f\|_{W^{n}_{p,w}} = \sum_{|\nu| = n} \|D^\nu f\|_{p,w}.
\]

If \( f, g \) are functions defined on \( \mathbb{R}^d \) and \( \hat{f}, \hat{g} \in L_1 \), then \( \langle f, g \rangle := \int_{\mathbb{R}^d} \hat{f} \hat{g} \). If \( f \in L_1 \), then its Fourier transform is \( \mathcal{F} f (\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} \, dx \).

If \( \varphi \) is a function defined on \( \mathbb{R}^d \) and \( M \in \mathbb{N} \), we set
\[
\varphi_{jk}(x) := m^{i/2} \varphi(M^j x + k), \quad j \in \mathbb{Z}, \; k \in \mathbb{R}^d.
\]

Denote by \( \mathcal{S} \) the Schwartz class of functions defined on \( \mathbb{R}^d \). The dual space of \( \mathcal{S} \) is \( \mathcal{S}' \), i.e. \( \mathcal{S}' \) is the space of tempered distributions. The basic facts from distribution theory can be found, e.g., in [33]. Suppose \( f \in \mathcal{S}, \varphi \in \mathcal{S}' \), then \( \langle f, \varphi \rangle := \mathcal{F}(\hat{f})(\hat{\varphi}) \). If \( \varphi \in \mathcal{S}' \), then \( \hat{\varphi} = \mathcal{F}\varphi \) denotes its Fourier transform defined by \( \langle \hat{f}, \hat{\varphi} \rangle = \langle f, \varphi \rangle \), \( f \in \mathcal{S} \). If \( \varphi \in \mathcal{S}', j \in \mathbb{Z}, k \in \mathbb{R}^d \), then we define \( \varphi_{jk} \) by \( \langle \hat{f}, \hat{\varphi}_{jk} \rangle = \langle f, \varphi \rangle \). If \( \varphi \) is a function defined on \( \mathbb{R}^d \) and \( M \in \mathbb{N} \), we set
\[
\varphi_{jk}(x) := m^{i/2} \varphi(M^j x + k), \quad j \in \mathbb{Z}, \; k \in \mathbb{R}^d.
\]

Let \( 1 \leq p \leq \infty \). Denote by \( L_p \) the set
\[
L_p := \left\{ \varphi \in L_p : \|\varphi\|_{L_p} := \left( \sum_{k \in \mathbb{Z}^d} |\varphi(\cdot + k)|^p \right)^{1/p} < \infty \right\}.
\]

With the norm \( \|\cdot\|_{L_p} \), \( L_p \) is a Banach space. The simple properties are: \( L_1 = L_1 \), \( \|\varphi\|_p \leq \|\varphi\|_{L_p} \), \( \|\varphi\|_{L_q} \leq \|\varphi\|_{L_p} \) for \( 1 \leq q \leq p \leq \infty \). Therefore, \( L_p \subset L_q \) and \( L_p \subset L_q \) for \( 1 \leq q \leq p \leq \infty \). If \( \varphi \in L_p \) and compactly supported, then \( \varphi \in L_p \) for any \( p \geq 1 \). If \( \varphi \) decays fast enough, i.e. there exist constants \( C > 0 \) and \( \varepsilon > 0 \) such that \( |\varphi(x)| \leq C(1 + |x|)^{-d-\varepsilon} \) for all \( x \in \mathbb{R}^d \), then \( \varphi \in \mathcal{S}_\infty \).

The corresponding weighted \( L_{p,w} \) space with respect to a weight function \( w \) is defined according the following weighted norm
\[
\|\varphi\|_{L_{p,w}} := \|\varphi w\|_{L_p}.
\]

The modulus of smoothness \( \omega_n(f, \cdot)_{p,w} \) of order \( n \in \mathbb{N} \) for a function \( f \in L_{p,w} \) is defined by
\[
\omega_n(f, h)_{p,w} = \sup_{|\delta| < h, \delta \in \mathbb{R}^d} \|\Delta^n \delta f\|_{p,w},
\]
where
\[
\Delta^n \delta f(x) = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} f(x + \delta \nu).
\]
The modulus $\omega_n(f,h)_{p,w}$ is the classical modulus of smoothness. Together with the modulus (9), we will also use the following so-called anisotropic modulus of smoothness, in which the step $h$ is replaced by a $d \times d$ real matrix $M$:

$$\Omega_n(f,M^{-1})_{p,w} = \sup_{|M\delta| \leq 1, \delta \in \mathbb{R}^d} \| \Delta^n_{M}\delta} f \|_{p,w}.$$ 

As usual, in the unweighted case, i.e., $w_0(x) \equiv 1$, we use the following notation: $\omega_n(f,h)_{p} = \omega_n(f,h)_{p,w_0}$ and $\Omega_n(f,M^{-1})_{p} = \Omega_n(f,M^{-1})_{p,w_0}$.

It is obvious that for any $M \in \mathcal{M}$ and $f \in L_{p,w}$, one has

$$\Omega_n(f,M^{-1})_{p,w} \leq \omega_n(f,\|M^{-1}\|)_{p,w}.$$ 

Note also that in the case of the diagonal matrix $M$ with $\lambda_1, \ldots, \lambda_d$ on the diagonal, the modulus $\Omega_n(f,M)_{p}$ can be calculated by the following formula (see [31]):

$$\Omega_n(f,M^{-1})_{p} \simeq \sum_{j=1}^{d} \omega_n^{(j)}(f,\lambda_j)_{p}, \quad f \in L_{p}, \quad 1 < p < \infty,$$

where $\omega_n^{(j)}(f,h)_{p} = \sup_{|\delta_j| \leq h, \delta \in \mathbb{R}} \| \Delta^n_{\delta_j} f \|_{p}$ and $\{e_j\}_{j=1}^{d}$ denotes the standard basis in $\mathbb{R}^d$.

Let $A$ be a bounded measurable subset of $\mathbb{R}^d$. In what follows, the error of the best approximation of a function $f \in L_{p,w}$ is defined by

$$E_A(f)_{p,w} = \inf \{ \| f - g \|_{p,w} : g \in L_{p,w} \cap L_2, \text{ supp } \hat{g} \subset A \}.$$

## 3 Auxiliary results

The following auxiliary statements will be useful for us.

**Proposition 1** Let $1 \leq p \leq \infty$ and $w \in \mathcal{W}^\alpha$ for some $\alpha > 0$. If $\varphi \in \mathcal{L}_{p,w^*}$ and $c = \{c_k\}_{k \in \mathbb{Z}^d} \in \ell_{p,w}$, then

$$\left\| \sum_{k \in \mathbb{Z}^d} c_k \varphi_0 k \right\|_{p,w} \leq \| \varphi \|_{\mathcal{L}_{p,w^*}} \| c \|_{\ell_{p,w}}.$$

**Proposition 2** Let $1 \leq p \leq \infty$, $w \in \mathcal{W}^\alpha$ for some $\alpha > 0$, $\varphi \in \mathcal{L}_{p,w^*}$, and $c = \{c_k\}_{k \in \mathbb{Z}^d} \in \ell_{p,1/w}$. Then

$$\left\| \sum_{k \in \mathbb{Z}^d} c_k \varphi_0 k \right\|_{p,1/w} \leq \| \varphi \|_{\mathcal{L}_{p,w^*}} \| c \|_{\ell_{p,1/w}}.$$

**Proposition 3** Let $1 \leq p \leq \infty$, $w \in \mathcal{W}^\alpha$ for some $\alpha > 0$, $f \in L_{p,1/w}$, and $\varphi \in \mathcal{L}_{q,w^*}$, $1/p + 1/q = 1$. Then

$$\left( \sum_{k \in \mathbb{Z}^d} \left| \frac{\langle f, \varphi_0 k \rangle}{w(k)} \right|^p \right)^{1/p} \leq \| \varphi \|_{\mathcal{L}_{q,w^*}} \| f \|_{p,1/w}.$$ 

The above three propositions can be proved repeating step-by-step the proofs of Propositions 2, 4 and 5 in [25] respectively. ♦

**Corollary 4** Let $1 \leq p \leq \infty$ and $w \in \mathcal{W}^\alpha$ for some $\alpha > 0$. If $\varphi \in \mathcal{L}_{p,w^*}$ and $\tilde{\varphi} \in \mathcal{L}_{q,w^*}$, $1/p + 1/q = 1$, then for any $f \in L_{p,1/w}$

$$\left\| \sum_{k \in \mathbb{Z}^d} \langle f, \tilde{\varphi}_0 k \rangle \varphi_0 k \right\|_{p,1/w} \leq \| \varphi \|_{\mathcal{L}_{p,w^*}} \| \tilde{\varphi} \|_{\mathcal{L}_{q,w^*}} \| f \|_{p,1/w}. \quad (10)$$
Proof. The proof of (10) directly follows from Propositions 2 and 3. ◇

An order of approximation by the quasi-projection operators essentially depends on the compatibility of a function $\varphi$ and a distribution/function $\tilde{\varphi}$. Assuming that the Fourier transform of $\varphi$ and $\tilde{\varphi}$ is sufficiently smooth in a neighbourhood of the origin, we consider the following two types of compatibility.

Definition 5 A tempered distribution $\tilde{\varphi}$ and a function $\varphi$ are said to be strictly compatible if there exists $\delta > 0$ such that $\tilde{\varphi}(\xi)\varphi(\xi) = 1$ a.e. on $\{|\xi| < \delta\}$.

Definition 6 A tempered distribution $\tilde{\varphi}$ and a function $\varphi$ are said to be weakly compatible of order $n \in \mathbb{N}$ if $D^\beta(1 - \tilde{\varphi}\varphi)(0) = 0$ for all $|\beta| < n$, $\beta \in \mathbb{Z}_+^d$.

Remark 7 The condition $D^\beta(1 - \tilde{\varphi}\varphi)(0) = 0$, $|\beta| < n$, is a natural requirement for providing approximation order $n$ of quasi-projection operators generated by $\varphi$ and $\tilde{\varphi}$. This assumption often appears (especially in wavelet theory) in other terms, in particular, in terms of polynomial reproducing property (see [13, Lemma 3.2]). It is clear that to provide an infinitely large approximation order, these conditions should be satisfied for any $n$. Obviously, the latter holds for strictly compatible functions $\varphi$ and $\tilde{\varphi}$ while the weak compatibility of $\varphi$ and $\tilde{\varphi}$ provides approximation order $n$.

Proposition 8 Let $N \in \mathbb{Z}_+$, $\tilde{\varphi} \in \mathcal{S}_N^t$, $\varphi \in L_2$, $\sum_{k \in \mathbb{Z}^d} |\tilde{\varphi}(\cdot + k)|^2 \in L_\infty$, there exist $\delta \in (0, 1/2)$ such that $\tilde{\varphi}(\xi) = 0$ a.e. on $\{|\xi| < \delta\}$ for all $l \in \mathbb{Z}^d \setminus \{0\}$, and $\tilde{\varphi}$ and $\varphi$ be strictly compatible with respect to the parameter $\delta$. If a function $f \in L_2$ is such that its Fourier transform is supported in $\{|\xi| < \delta\}$, then

$$
f = \sum_{k \in \mathbb{Z}^d} \langle \hat{f}, \varphi_0k \rangle \varphi_0k \text{ a.e.} \quad (11)
$$

Proof. First of all, we check that the right hand side of (11) belongs to $L_2$. Set

$$G(\xi) = \sum_{l \in \mathbb{Z}^d} \hat{f}(\xi + l)\tilde{\varphi}(\xi + l).$$

By [19, Lemma 1], we have that $G \in L_2$ and $\hat{G}(k) := \langle \hat{f}, \varphi_0k \rangle$ is its $k$-th Fourier coefficient. Hence, the series $\sum_{k \in \mathbb{Z}^d} |\hat{G}(k)|^2$ is convergent. On the other hand, since the system $\{\varphi_0k\}_{k \in \mathbb{Z}^d}$ is Bessel (see, e.g., [20, Remarks 1.1.3 and 1.1.7]), we derive that

$$\sum_{k \in \mathbb{Z}^d} |\langle g, \varphi_0k \rangle|^2 \leq B\|g\|_2^2 \quad \text{for all} \quad g \in L_2, \quad (12)$$

where $B = \| \sum_{k \in \mathbb{Z}^d} |\tilde{\varphi}(\cdot + k)|^2 \|_\infty$.

If now $\Omega$ is a finite subset of $\mathbb{Z}^d$, by the Riesz representation theorem, there exists $g \in L_2$ such that $\|g\|_2 \leq 1$ and

$$\left\| \sum_{k \in \Omega} \langle \hat{f}, \varphi_0k \rangle \varphi_0k \right\|_2 = \left\| \sum_{k \in \Omega} \hat{G}(k)\varphi_0k \right\|_2 = \left| \sum_{k \in \Omega} \hat{G}(k)\langle \varphi_0k, g \rangle \right|.$$  

Next, using the Cauchy inequality and (12), we get

$$\left| \sum_{k \in \Omega} \hat{G}(k)\langle \varphi_0k, g \rangle \right| \leq \sqrt{B}\|g\|_2 \left( \sum_{k \in \Omega} |\langle \hat{f}, \varphi_0k \rangle|^2 \right)^{\frac{1}{2}} \leq \sqrt{B} \left( \sum_{k \in \Omega} |\langle \hat{f}, \varphi_0k \rangle|^2 \right)^{\frac{1}{2}}.$$  

7
This implies that the series \( \sum_{k \in \mathbb{Z}^d} \langle \hat{f}, \hat{\varphi_{0k}} \rangle \varphi_{0k} \) converges as the limit in \( L_2 \) of the cubic partial sums and hence its sum belongs to \( L_2 \). Using Carleson’s theorem, we have

\[
\mathcal{F} \left( \sum_{k \in \mathbb{Z}^d} \langle \hat{f}, \hat{\varphi_{0k}} \rangle \varphi_{0k} \right)(\xi) = \sum_{k \in \mathbb{Z}^d} \hat{G}(k) e^{2\pi i (k,\xi)} \hat{\varphi}(\xi) = G(\xi) \hat{\varphi}(\xi) = \sum_{l \in \mathbb{Z}^d} \hat{f}(\xi + l) \hat{\varphi}(\xi + l) \hat{\varphi}(\xi) \quad \text{a.e.}
\]

The sets \( \{ |\xi - l| < \delta \}, \ l \in \mathbb{Z}^d \), are mutually disjoint and their union contains the supports of the functions \( \hat{f}(\cdot + l) \). If \( |\xi - l| < \delta, \ l \neq 0 \), then \( \hat{\varphi}(\xi) = 0 \) and if \( |\xi| < \delta \), then \( \hat{\varphi}(\xi) \hat{\varphi}(\xi) = 1 \). Hence,

\[
\mathcal{F} \left( \sum_{k \in \mathbb{Z}^d} \langle \hat{f}, \hat{\varphi_{0k}} \rangle \varphi_{0k} \right)(\xi) = \hat{f}(\xi),
\]

which yields (11) \( \diamondsuit \).

Next we need several basic properties of the modulus of smoothness, which can be proved by standard way using also the inequalities (6) and (7) (see, e.g., [23, Ch. 4] and [4, Ch. 4]).

**Lemma 9** Let \( 1 \leq p \leq \infty, \ w \in \mathcal{W}^\alpha \) for some \( \alpha > 0 \), and \( n \in \mathbb{N} \). Then for any \( f, g \in L_{p,1/w} \) and \( \delta \in (0,1) \), we have

(i) \( \omega_n(f + g, \delta)_{p,1/w} \leq \omega_n(f, \delta)_{p,1/w} + \omega_n(g, \delta)_{p,1/w} \);

(ii) \( \omega_n(f, \delta)_{p,1/w} \leq C \|f\|_{p,1/w} \);

(iii) \( \omega_n(f, \lambda \delta)_{p,1/w} \leq C(1 + \lambda)^n \omega_n(f, \delta)_{p,1/w}, \ \lambda > 0, \)

where the constant \( C \) does not depend on \( f, \delta, \) and \( \lambda > 0, \)

**Proposition 10** Let \( 1 \leq p \leq \infty, \ w \in \mathcal{W}^\alpha \) for some \( \alpha > 0 \), and \( n \in \mathbb{N} \). Then for any \( f \in L_{p,1/w} \) there exists \( g \in W_{p,1/w}^n \) such that

\[
\|f - g\|_{p,1/w} \leq C \omega_n(f, 1)_{p,1/w} \quad \text{and} \quad \|g\|_{W_{p,1/w}^n} \leq C \omega_n(f, 1)_{p,1/w},
\]

where the constant \( C \) does not depend on \( f \) and \( w \).

**Proof.** In the unweighted case, inequalities (13) and (14) were proved in [4, Theorem 4.12, see also eq. (4.42)] by using the following function \( g \):

\[
g(x) = \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} \int_{[0,1]^d} \cdots \int_{[0,1]^d} f(x + k(u_1 + \cdots + u_n)) du_1 \cdots du_n.
\]

In the weighted case, inequalities (13) and (14) can be proved similarly using the same function \( g \) and inequalities (6) and (7) as well as Lemma 9 instead of its unweighted counterpart. \( \diamondsuit \)
4 Main results

4.1 The case of \( \varphi \in \mathcal{L}_{p,w^*} \) and \( \tilde{\varphi} \in \mathcal{L}_{q,w^*} \)

First, we consider the case of strictly compatible functions \( \varphi \) and \( \tilde{\varphi} \) and give the error estimate in terms of the best approximation.

**Theorem 11** Let \( 1 \leq p \leq \infty, w \in \mathcal{W}^\alpha \) for some \( \alpha > 0 \), and \( M \in \mathcal{M} \). Suppose 
1) \( \varphi \in \mathcal{L}_{p,w^*} \cap L_2 \) and \( \tilde{\varphi} \in \mathcal{L}_{q,w^*}, \) \( 1/p + 1/q = 1; \)
2) \( \varphi \) and \( \tilde{\varphi} \) are strictly compatible with respect to the parameter \( \delta > 0; \)
3) \( \text{supp} \tilde{\varphi} \subset (−1,1)^d. \)

Then for any \( f \in L_{p,1/w} \) and \( j \in \mathbb{Z}_+ \)

\[
\left\| f - \sum_{k \in \mathbb{Z}^d} \langle f, \varphi_{jk} \rangle \varphi_{jk} \right\|_{p,1/w} \leq CE_{\{∥M^{*-j\xi}<\delta\}}(f)_{p,1/w},
\]

where the constant \( C \) does not depend on \( f, M, \) and \( j. \)

**Proof.** Without loss of generality, we can assume that \( \delta \) is sufficiently small such that \( \tilde{\varphi}(\xi) = 0 \)
a.e. on \( [\xi - l] < \delta \) for all \( l \in \mathbb{Z}^d \setminus \{0\}. \)

If \( g \in L_{p,1/w} \cap L_2 \) and \( \text{supp } g \subset \{||\xi|| < \delta\}, \) then, due to Proposition 8, we have

\[
g = \sum_{k \in \mathbb{Z}^d} \langle g, \tilde{\varphi}_{0k} \rangle \tilde{\varphi}_{0k}.
\]

Corollary 4 and (17) imply that

\[
\left\| f - \sum_{k \in \mathbb{Z}^d} \langle f, \tilde{\varphi}_{0k} \rangle \tilde{\varphi}_{0k} \right\|_{p,1/w} \leq \left\| f - g \right\|_{p,1/w} + \left\| \sum_{k \in \mathbb{Z}^d} \langle g - f, \tilde{\varphi}_{0k} \rangle \tilde{\varphi}_{0k} \right\|_{p,1/w}
\]

\[
\leq (1 + \|\varphi\|_{\mathcal{L}_{p,w^*}} \|\tilde{\varphi}\|_{\mathcal{L}_{q,w^*},}) \|f - g\|_{p,1/w}.
\]

Let now \( j \in \mathbb{Z}_+ \) be fixed, \( G \) be a function in \( L_{p,1/w} \cap L_2 \) such that \( \text{supp } \tilde{\varphi} \subset \{||M^{*-j\xi}| < \delta\} \) and

\[
\|f - G\|_{p,1/w} \leq 2E_{\{∥M^{*-j\xi}| < \delta\}}(f)_{p,1/w}.
\]

Set \( g(x) = G(M^{-j}x). \) Obviously, \( g \in L_{p,1/w(M^{-j})} \cap L_2 \) and \( \text{supp } \tilde{\varphi} \subset \{||\xi|| < \delta\}. \) Thus, after the change of variable, using (18) with \( f(M^{-j}) \) instead of \( f \) and \( w(M^{-j}) \) instead of \( w, \) we have

\[
\left\| f - \sum_{k \in \mathbb{Z}^d} \langle f, \tilde{\varphi}_{0k} \rangle \tilde{\varphi}_{0k} \right\|_{p,1/w} = m^{-j/p} \left\| f - \sum_{k \in \mathbb{Z}^d} \langle f(M^{-j}) \tilde{\varphi}_{0k} \rangle \tilde{\varphi}_{0k} \right\|_{p,1/w(M^{-j})}
\]

\[
\leq (1 + \|\varphi\|_{\mathcal{L}_{p,w^*(M^{-j})}} \|\tilde{\varphi}\|_{\mathcal{L}_{q,w^*(M^{-j})}}) m^{-j/p} \|f(M^{-j}) - g\|_{p,1/w(M^{-j})}.
\]

To prove (16), it remains to note that

\[
m^{-j/p} \|f(M^{-j}) - g\|_{p,1/w(M^{-j})} = \|f - G\|_{p,1/w},
\]

apply (19), and take into account that

\[
\|\varphi\|_{\mathcal{L}_{p,w^*(M^{-j})}} \leq C'\|\varphi\|_{\mathcal{L}_{p,w^*}} \quad \text{and} \quad \|\tilde{\varphi}\|_{\mathcal{L}_{q,w^*(M^{-j})}} \leq C'\|\tilde{\varphi}\|_{\mathcal{L}_{q,w^*}},
\]

which proves the theorem. \( \diamond \)
**Corollary 12** Under the conditions of Theorem 11, we have
\[
\left\| f - \sum_{k \in \mathbb{Z}^d} \langle f, \tilde{\varphi}_{jk} \rangle \varphi_{jk} \right\|_{p,1/w} \leq C_1 \omega_n \left( f, \sqrt{d} \delta^{-1} M^{-j} \right)_{p,1/w} \leq C_2 \omega_n \left( f, \| M^{-j} \| \right)_{p,1/w},
\]
where \( C_1 \) and \( C_2 \) do not depend on \( f, M, \) and \( j. \)

**Proof.** The corollary follows from Theorem 11, the Jakson-type inequality given in Theorem 17, and Lemma 9 (iii). ◇

**Theorem 13** Let \( 1 \leq p \leq \infty, \omega \in \mathcal{W}_\alpha \) for some \( \alpha > 0, n \in \mathbb{N}, \) and \( M \in \mathfrak{M}. \) Suppose
1) \( \varphi \in \mathcal{L}_{p,w} \cap L_2 \) and \( \tilde{\varphi} \in \mathcal{L}_{q,w}, \ 1/p + 1/q = 1; \)
2) \( \varphi, \tilde{\varphi} \in C^\gamma(B_r) \) for some integer \( \gamma > n + d + p \alpha \) and \( \varepsilon > 0; \)
3) \( \varphi \) and \( \varphi \) are weakly compatible of order \( n; \)
4) \( \text{supp} \varphi \subset (-1,1)^d. \)

Then for any \( f \in L_{p,1/w} \) and \( j \in \mathbb{Z}_+ \)
\[
\left\| f - \sum_{k \in \mathbb{Z}^d} \langle f, \tilde{\varphi}_{jk} \rangle \varphi_{jk} \right\|_{p,1/w} \leq C_1 \omega_n \left( f, M^{-j} \right)_{p,1/w} \leq C_2 \omega_n \left( f, \| M^{-j} \| \right)_{p,1/w},
\]
where \( C \) does not depend on \( f, M, \) and \( j. \)

**Remark 14** To determine the approximation order in (20), one can use a natural approach replacing \( \omega_n \left( f, M^{-j} \right)_{p,1/w} \) by \( \omega_n \left( f, \| M^{-j} \| \right)_{p,1/w}. \) However this is not good because there exist matrices \( M \in \mathfrak{M} \) such that \( \| M^{-1} \| \geq 1. \) A better way is to use (3), which yields
\[
\left\| f - \sum_{k \in \mathbb{Z}^d} \langle f, \tilde{\varphi}_{jk} \rangle \varphi_{jk} \right\|_{p,1/w} \leq C_1 \omega_n \left( f, \vartheta^{-j} \right)_{p,1/w},
\]
where \( \vartheta \) is a positive number smaller (in absolute value) than any eigenvalue of \( M. \) If \( M \) is an isotropic matrix and \( \lambda \) is one of its eigenvalue (e.g., \( M = \lambda I_d \)), then one can take \( \vartheta = |\lambda|. \) If \( M \) is a diagonal matrix and \( \lambda \) is the smallest (in absolute value) its diagonal element, then one can take \( \vartheta = |\lambda|. \)

To prove Theorem 13 we need the following lemmas.

**Lemma 15** Let \( 1 \leq p \leq \infty, \omega \in \mathcal{W}_\alpha \) for some \( \alpha > 0, n \in \mathbb{N}. \) Suppose
1) \( \psi \in \mathcal{L}_{p,w} \) and \( \tilde{\psi} \in L_2; \)
2) \( \tilde{\psi} \in C^\gamma(\mathbb{R}^d) \) for some \( \gamma > n + d + p \alpha; \)
3) \( D^\beta \tilde{\psi}(0) = 0 \) for all \( |\beta| < n, \beta \in \mathbb{Z}_+^d. \)

Then for any \( f \in W^p_{p,1/w} \)
\[
\left\| \sum_{k \in \mathbb{Z}^d} \langle f, \tilde{\psi}_{0k} \rangle \psi_{0k} \right\|_{p,1/w} \leq C_1 \omega_n \left( f, \| \mathcal{L}_{p,w} \| f \| W^{-p}_{p,1/w} \right),
\]
where \( C \) does not depend on \( f \) and \( j. \)
**Proof.** Let $k \in \mathbb{Z}^d$ and $y \in [-1/2,1/2]^d - k$. Since $D^\beta \hat{\psi}(0) = 0$ whenever $[\beta] < n$, $\beta \in \mathbb{Z}_+^d$, we have

$$\int_{\mathbb{R}^d} x^\beta \hat{\psi}_{0k}(x) \, dx = 0, \quad [\beta] < n, \quad \beta \in \mathbb{Z}_+^d.$$ 

Hence, due to Taylor’s formula with the integral remainder,

$$|\langle f, \hat{\psi}_{0k} \rangle| = \left| \int_{\mathbb{R}^d} f(x) \hat{\psi}_{0k}(x) \, dx \right|$$

$$= \left| \int_{\mathbb{R}^d} \hat{\psi}_{0k}(x) \left( \sum_{\nu=0}^{n-1} \frac{1}{\nu!} ((x_1 - y_1) \partial_1 + \cdots + (x_d - y_d) \partial_d)^\nu f(y) + \frac{1}{(n-1)!} \left( \sum_{\nu=0}^{n-1} \frac{(1-t)^{n-1}}{(n-1)!} ((x_1 - y_1) \partial_1 + \cdots + (x_d - y_d) \partial_d)^n f(y + t(x - y)) \, dt \right) \right) \, dx \right|$$

$$\leq \int_{\mathbb{R}^d} dx |x - y|^n |\hat{\psi}_{0k}(x)| \int_0^1 \sum_{|\beta|=n} |D^\beta f(y + t(x - y))| \, dt.$$

From this, using Hölder’s inequality and taking into account that

$$|\hat{\psi}_{0k}(x)| \leq \frac{C_1}{(1 + |x + k|)^\gamma} \leq \frac{C_2}{(1 + |x - y|)^\gamma},$$

we obtain

$$|\langle f, \hat{\psi}_{0k} \rangle| \leq C_2 \int_{\mathbb{R}^d} dx \frac{|x - y|^n}{(1 + |x - y|)^\gamma} \int_0^1 \sum_{|\beta|=n} |D^\beta f(y + t(x - y))| \, dt$$

$$\leq C_2 \left( \int_{\mathbb{R}^d} \frac{|x - y|^n}{(1 + |x - y|)^\gamma} \, dx \right)^{1/q}$$

$$\times \left( \int_{\mathbb{R}^d} \frac{|x - y|^n}{(1 + |x - y|)^\gamma} \left( \int_0^1 \sum_{|\beta|=n} |D^\beta f(y + t(x - y))| \, dt \right)^p \, dx \right)^{1/p}$$

$$= C_3 \left( \int_{\mathbb{R}^d} \frac{|u|^n}{(1 + |u|)^\gamma} \int_0^1 \sum_{|\beta|=n} |D^\beta f(y + tu)|^p \, dt \right)^{1/p}. \tag{21}$$

Next, it follows from (21), properties of $w$, and Proposition 2 that
Lemma 16 Let $w$, $n$, $\gamma$, $\varphi$, and $\bar{\varphi}$ be as in Theorem 13. Then for any $f \in W_{n,1/w} \cap L_2$, we have

$$\left\| f - \sum_{k \in \mathbb{Z}^d} \langle f, \bar{\varphi}_{0k} \rangle \varphi_{0k} \right\|_{p,1/w} \leq \Upsilon(w^*) \| f \|_{W_{n,1/w}},$$

(22)

where the functional $\Upsilon$ is independent of $f$ and $\Upsilon(w^*(M^{-j})) \leq C \Upsilon(w^*)$ for all $j \in \mathbb{Z}_+$ and the constant $C$ depends only on $d$, $p$, $\varphi$, $\bar{\varphi}$, $\alpha$, $C'$, and $c_w$.

**Proof.** Choose $0 < \delta' < \delta'' < 1/2$ such that $\bar{\varphi}(\xi) \neq 0$ on $\{ |\xi| \leq \delta' \}$. Set

$$F(\xi) = \begin{cases} 
1 - \bar{\varphi}(\xi) \bar{\varphi}(\xi) & \text{if } |\xi| \leq \delta', \\
\bar{\varphi}(\xi) & \text{if } |\xi| \geq \delta''
\end{cases}$$

and extend this function such that $F \in C^\gamma(\mathbb{R}^d)$. Define $\bar{\psi}$ by $\bar{\psi} = F$. Obviously, the function $\bar{\psi}$ is continuous and $\bar{\psi}(x) = O(|x|^{-\gamma})$ as $|x| \to \infty$, where $\gamma > \alpha + d$, which yields that $\bar{\psi} \in L_{\infty,w^*}$, a fortiori $\bar{\psi} \in L_{q,w^*}$. On the other hand, $\bar{\psi}$ is obviously in $L_2$, and all assumptions of Lemma 15 with $\psi = \varphi \in L_{p,w^*}$ are satisfied. Hence,

$$\left\| \sum_{k \in \mathbb{Z}^d} \langle f, \bar{\psi}_{0k} \rangle \varphi_{0k} \right\|_{p,1/w} \leq C_0 c_w^2 \| \varphi \|_{Z_{p,w^*}} \| f \|_{W_{n,1/w}},$$

(23)

where $C_0$ depends on $d$, $p$, $\bar{\psi}$, and $\alpha$. Using (23), we derive

$$\left\| f - \sum_{k \in \mathbb{Z}^d} \langle f, \bar{\varphi}_{0k} \rangle \varphi_{0k} \right\|_{p,1/w} \leq C_0 c_w^2 \| \varphi \|_{Z_{p,w^*}} \| f \|_{W_{n,1/w}} + \left\| f - \sum_{k \in \mathbb{Z}^d} \langle f, \bar{\varphi}_{0k} + \bar{\psi}_{0k} \rangle \varphi_{0k} \right\|_{p,1/w}. \quad (24)$$

This proves the theorem. \( \diamond \)
It remains to estimate the second summand in the right-hand side of (24). For this, we will use the Meyer wavelets. Let \( \theta \) be the Meyer scaling function (see, e.g. [24, Sec. 1.4]). This function is band-limited, its Fourier transform is infinitely differentiable, supported in \([-2/3,2/3]\), and equals 1 on the interval \([-1/3,1/3]\); the integer translates of \( \theta \) form an orthonormal system. Set

\[
\Phi(x) = \prod_{i=1}^{d} \theta(x_i), \quad x \in \mathbb{R}^d.
\]

It is well known (see, e.g. [24, Sec. 2.1]) that \( \Phi \) generates a separable MRA in \( L_2 \) with respect to the matrix \( 2I_d \) and the corresponding wavelet functions \( \Psi^{(\nu)} \), \( \nu = 1, \ldots, 2^d - 1 \), such that for every \( j \in \mathbb{Z} \) the functions \( \Phi_{jk} = 2^{jd/2} \Phi(2^j \cdot + k) \) and \( \Psi^{(\nu)}_{jk} = 2^{jd/2} \Psi^{(\nu)}(2^j \cdot + k), k \in \mathbb{Z}^d, i \geq j, \nu = 1, \ldots, 2^j - 1 \), form an orthonormal basis for \( L_2 \). It follows that

\[
f = \sum_{j \in \mathbb{Z}^d} \langle f, \Phi_{jk} \rangle \Phi_{jk} + \sum_{\nu=1}^{2^d-1} \sum_{i=j}^{\infty} \sum_{k \in \mathbb{Z}^d} \langle f, \Psi^{(\nu)}_{ik} \rangle \Psi^{(\nu)}_{ik}.
\]

(25)

On the other hand, the functions \( \hat{\Psi}^{(\nu)} \) are infinitely differentiable and compactly supported. It follows that (see [24, Theorem 1.7.7 and Sec. 1.4])

\[
\int_{\mathbb{R}^d} y^\beta \hat{\Psi}^{(\nu)}(y) \, dy = 0 \quad \text{for all } \beta \in \mathbb{Z}_+^d.
\]

(26)

Thus, all assumptions of Lemma 15 with \( \psi = \tilde{\psi} = \Psi^{(\nu)} \) are satisfied. Hence

\[
\left\| \sum_{k \in \mathbb{Z}^d} \langle f, \Psi^{(\nu)}_{ik} \rangle \Psi^{(\nu)}_{ik} \right\|_{p,1/w} \leq C_1 c_w \|\Psi\|_{L_{p,w}} \|f\|_{W^{n}_{p,1/w}}, \quad \nu = 1, \ldots, 2^d - 1,
\]

where \( C_1 \) depends on \( d, p, \Psi \) and \( \alpha \). For every \( i \in \mathbb{Z} \) and \( \nu = 1, \ldots, 2^d - 1 \), after the change of variable, taking into account that \( w(2^{-i}) \in \mathcal{W}^\alpha \) and

\[
w^*(2^{-i}x) \leq c_w \max\{1, 2^{-i\alpha}\}(1 + |x|^2)^{\alpha/2},
\]

we have

\[
\left\| \sum_{k \in \mathbb{Z}^d} \langle f, \Psi^{(\nu)}_{ik} \rangle \Psi^{(\nu)}_{ik} \right\|_{p,1/w} \leq C_1 c_w \max\{1, 2^{-2\alpha}\} \|\Psi\|_{L_{p,w^*}(2^{-i})} 2^{-id/p} \|f(2^{-i})\|_{W^n_{p,1/w}(2^{-i})}.
\]

(27)

Choose \( j \in \mathbb{Z} \) such that \( 2^j < \delta' / \sqrt{d} \) and set \( G = \sum_{k \in \mathbb{Z}^d} \langle f, \Phi_{jk} \rangle \Phi_{jk} \). Note that \( j = j(\varphi, \tilde{\varphi}) \). Since \( \mathrm{supp} \; \tilde{\phi} \subset [-1,1]^d \), we have

\[
\mathrm{supp} \; G \subset 2^j[-1,1]^d \subset \delta' B_1.
\]

(28)

By construction, \( \overline{\varphi(\xi)(\tilde{\varphi}(\xi) + \tilde{\psi}(\xi))} = 1 \) whenever \( |\xi| \leq \delta' \). It follows from (28) and Proposition 8 that

\[
G = \sum_{k \in \mathbb{Z}^d} \langle G, \tilde{\varphi}_{0k} + \tilde{\psi}_{0k} \rangle \varphi_{0k}.
\]

(29)

Since \( \varphi, \tilde{\varphi} \in L_{q,m^*}, \varphi \in L_{p,w^*}, \) and \( f, G \in L_{p,1/w} \), due to Corollary 4, we derive

\[
\left\| \sum_{k \in \mathbb{Z}^d} \langle f - G, \tilde{\varphi}_{0k} + \tilde{\psi}_{0k} \rangle \varphi_{0k} \right\|_{p,1/w} \leq \|\varphi\|_{L_{p,w^*}} \|\tilde{\varphi} + \tilde{\psi}\|_{L_{q,w^*}} \|f - G\|_{p,1/w}.
\]

(30)
Combining (30), (25), and (29), we obtain
\[
\left\| f - \sum_{k \in \mathbb{Z}^d} \langle f, \check{\varphi}_{0k} \rangle \check{\varphi}_{0k} \right\|_{p,1/w} \leq \left\| f - G - \sum_{k \in \mathbb{Z}^d} \langle f - G, \check{\varphi}_{0k} \rangle \check{\varphi}_{0k} \right\|_{p,1/w}
\]
\[
= \left( 1 + \|\varphi\|_p,1/w, \|\check{\varphi} + \check{\psi}\|_{\mathcal{Q}_{p,w^*}} \right) \left\| f - G \right\|_{p,1/w}
\]
\[
= \left( 1 + \|\varphi\|_p,1/w, \|\check{\varphi} + \check{\psi}\|_{\mathcal{Q}_{p,w^*}} \right) \sum_{\nu=1}^{2^d-1} \sum_{i=j}^{\infty} \sum_{k \in \mathbb{Z}^d} \langle f, \Psi^{(\nu)}_{ik} \rangle \Psi^{(\nu)}_{ik} \right\|_{p,1/w} .
\]
It follows from (27) that
\[
\left\| \sum_{\nu=1}^{2^d-1} \sum_{i=j}^{\infty} \sum_{k \in \mathbb{Z}^d} \langle f, \Psi^{(\nu)}_{ik} \rangle \Psi^{(\nu)}_{ik} \right\|_{p,1/w} \leq C_1 e_w^2 \max\{1, 2^{-2j\alpha} \} \sup_{i \geq j} \|\Psi\|_{\mathcal{Q}_{p,w^*(2^{-i})}} \sum_{\nu=1}^{2^d-1} \sum_{i=j}^{\infty} 2^{-id/p} \left\| f(2^{-i}) \right\|_{W_{p,1/w}^{n}(2^{-i})} \]
\[
= C_1 e_w^2 \max\{1, 2^{-2j\alpha} \} \sup_{i \geq j} \|\Psi\|_{\mathcal{Q}_{p,w^*(2^{-i})}} \left( \sum_{\nu=1}^{2^d-1} \sum_{i=j}^{\infty} 2^{-ni} \right) \left\| f \right\|_{W_{p,1/w}^{n}} .
\]
Since \( w^* (x/2) \leq c_w (1 + |x|^2)^{\alpha/2} \) and \( \hat{\Psi} \) is infinitely differentiable, we have
\[
\sup_{i \geq j} \|\Psi\|_{\mathcal{Q}_{p,w^*(2^{-i})}} \leq C_2.
\]
Combining this with (31), (32), and (24), we complete the proof of (22) with
\[
\mathcal{Y}(w^*) = C_3 \max\left\{ 1, \|\varphi\|_{\mathcal{Q}_{p,w^*}}, \|\check{\varphi} + \check{\psi}\|_{\mathcal{Q}_{p,w^*}}, \|\varphi\|_{\mathcal{Q}_{p,w^*}} \right\},
\]
where \( C_3 \) depends only on \( d, p, \varphi, \check{\varphi}, \alpha, \) and \( c_w \).

**Proof of Theorem 13.** Let \( g \in W_{p,1/w}^n \) be defined by (15). Using Corollary 4, Lemma 16, and Proposition 10, we derive
\[
\left\| f - \sum_{k \in \mathbb{Z}^d} \langle f, \check{\varphi}_{0k} \rangle \check{\varphi}_{0k} \right\|_{p,1/w} \leq \left\| f - g \right\|_{p,1/w} + \left\| g - \sum_{k \in \mathbb{Z}^d} \langle g, \check{\varphi}_{0k} \rangle \check{\varphi}_{0k} \right\|_{p,1/w} + \left\| \sum_{k \in \mathbb{Z}^d} \langle f - g, \check{\varphi}_{0k} \rangle \check{\varphi}_{0k} \right\|_{p,1/w}
\]
\[
\leq C_1 \mathcal{Y}(w^*) \left\| f - g \right\|_{p,1/w} + \left\| g \right\|_{W_{p,1/w}^{n}} \leq C_2 \mathcal{Y}(w^*) \omega_n \left( f, 1 \right)_{p,1/w} ,
\]
where \( C_2 \) depends only on \( d, p, \varphi, \check{\varphi}, \alpha, \) and \( c_w \). This yields (20) for \( j = 0 \).

Consider now an arbitrary \( j \in \mathbb{Z}_+ \). After the change of variables, we have
\[
\left\| f - \sum_{k \in \mathbb{Z}^d} \langle f, \check{\varphi}_{jk} \rangle \check{\varphi}_{jk} \right\|_{p,1/w} = m^{-j} \left\| f(M^{-j} \cdot) - \sum_{k \in \mathbb{Z}^d} \langle f(M^{-j} \cdot), \check{\varphi}_{0k} \rangle \check{\varphi}_{0k} \right\|_{p,1/w(M^{-j} \cdot)} .
\]
Since \( w(M^{-j} \cdot) \in \mathcal{W}^\alpha \) and \( \mathcal{Y}(w^* (M^{-j} \cdot)) \leq C_3 \mathcal{Y}(w^*) \), it follows from (33) that
\[
\left\| f - \sum_{k \in \mathbb{Z}^d} \langle f, \check{\varphi}_{jk} \rangle \check{\varphi}_{jk} \right\|_{p,1/w} \leq C_4 \mathcal{Y}(w^*) m^{-j} \omega_n \left( f(M^{-j} \cdot), 1 \right)_{p,1/w(M^{-j} \cdot)}
\]
\[
\leq C_4 \mathcal{Y}(w^*) \Omega_n \left( f, M^{-j} \right)_{p,1/w} \leq C_5 \omega_n \left( f, \|M^{-j}\| \right)_{p,1/w} ,
\]
\[
\leq C_6 \omega_n \left( f, \|M^{-j}\| \right)_{p,1/w} ,
\]
\[
\leq C_7 \omega_n \left( f, \|M^{-j}\| \right)_{p,1/w} ,
\]
\[
\leq C_8 \omega_n \left( f, \|M^{-j}\| \right)_{p,1/w} .
\]
which proves the theorem. ◇

As a corollary from Theorem 13, we obtain the following Jackson-type theorem in the weighted spaces $L_{p,1/w}$.

**Theorem 17** (Jackson inequality) Let $1 \leq p \leq \infty$, $w \in \mathcal{W}^\alpha$ for some $\alpha > 0$, $n \in \mathbb{N}$, and $M \in \mathcal{M}$. Then for any $f \in L_{p,1/w}$ and $j \in \mathbb{Z}_+$

$$E_{\{|M^{-j}\xi|<\sqrt{Z}\}}(f)_{p,1/w} \leq C \Omega_n \left( f, M^{-j} \right)_{p,1/w} \leq C \omega_n \left( f, \|M^{-j}\| \right)_{p,1/w},$$

where $C$ does not depend on $f$, $M$, and $j$.

**Proof.** Choose a function $g \in \mathcal{S}$ such that

$$\|f - g\|_{p,1/w} \leq \Omega_n \left( f, M^{-j} \right)_{p,1/w} \quad (34)$$

and set

$$P = \sum_{k \in \mathbb{Z}^d} \langle g, \Phi_{jk} \rangle \Phi_{jk},$$

where $\Phi$ is the scaling function of the separable Meyer MRA (see the proof of Lemma 16). It is obvious that $\text{supp} \hat{P} \subset \{|M^{-j}\xi|<\sqrt{Z}\}$ and all conditions of Theorem 13 with $\varphi = \varphi = \Phi$ are satisfied. Thus, taking into account that $P \in L_{p,1/w} \cap L_2$, we obtain by Theorem 13 that

$$\|g - P\|_{p,1/w} \leq C_1 \Omega_n \left( f, M^{-j} \right)_{p,1/w}.$$

It remains to combine this with (34). ◇

### 4.2 The case of $\varphi \in \mathcal{B}$ and $\varphi \in \mathcal{L}_{q,w^*}$

Let $\mathcal{B}$ be a collection of all bounded sets in $\mathbb{R}^d$ and let $w$ be a nonnegative, locally integrable function. We say that $w$ belongs to $A_p(\mathbb{R}^d, \mathcal{B})$ for some $1 < p < \infty$ if there is a constant $c$ such that

$$\left( \frac{1}{\text{mes} I} \int_I w(x) \, dx \right) \left( \frac{1}{\text{mes} I} \int_I w(x)^{-1/(p-1)} \, dx \right)^{p-1} \leq c$$

for any $I \in \mathcal{B}$.

Now let $\mathcal{Q}_d$ and $\mathcal{R}_d$ denote the collection of all $d$-dimensional cubes and all $d$-dimensional rectangles with sides parallel to the coordinate axes, correspondingly. Then $A_p(\mathbb{R}^d, \mathcal{Q}_d)$ is the classical Muckenhoupt class $A_p(\mathbb{R}^d)$. If $d = 1$, then $A_p(\mathbb{R}^1) = A_p(\mathbb{R}^1, \mathcal{B}_1)$. At the same time, for $d > 1$, we have $A_p(\mathbb{R}^d, \mathcal{Q}_d) \subseteq A_p(\mathbb{R}^d)$. Recall also the fact that $|x|^{-\alpha} \in A_p(\mathbb{R}^d)$ for $-d < \alpha < d(p-1)$ while $|x|^\alpha \in A_p(\mathbb{R}^d, \mathcal{Q}_d)$ for $-1 < \alpha < p - 1$.

In what follows, for simplicity we denote $\mathcal{A}_p = \mathcal{A}_p(\mathbb{R}^d) = A_p(\mathbb{R}^d, \mathcal{Q}_d)$.

In the results formulated in the previous sections, we suppose that the weight $w$ belongs to the class $\mathcal{W}^\alpha$. In the next results, we will in addition suppose that $w^{-p} \in \mathcal{A}_p$. A model example of such a weight is

$$w_\alpha(x) = (1 + |x|^2)^{\alpha/2}, \quad 0 < \alpha < 1/p, \quad 1 < p < \infty.$$

**Lemma 18** (see [15, p. 453–454] and [21]) Let $1 < p < \infty$ and $d \geq 2$. The following assertions are equivalent:

1) $w \in \mathcal{A}_p(\mathbb{R}^d)$;
2) There exists a constant $C$ such that for almost every fixed vector $(x_1, \ldots, x_j, x_{j+1}, \ldots, x_d) \in \mathbb{R}^{d-1}$ and any interval $I \subset \mathbb{R}^1$ one has

$$\left( \frac{1}{\text{mes } I} \int_I w(x_1, \ldots, x_j, \ldots, x_d) dx_j \right) \left( \frac{1}{\text{mes } I} \int_I w^{-\frac{1}{p'}} (x_1, \ldots, x_j, \ldots, x_d) dx_j \right)^{p-1} \leq C.$$ 

In other words, belonging to $\mathcal{A}_p(\mathbb{R}^d)$ implies belonging to $A_p(\mathbb{R}^1)$ in each variable uniformly with respect to other variables.

3) For any $\delta = (\delta_1, \ldots, \delta_d) \in \mathbb{R}_+^d$, $\delta_j > 0$, $j = 1, \ldots, d$, one has $w(\delta_1 x_1, \ldots, \delta_d x_d) \in \mathcal{A}_p(\mathbb{R}^d)$.

**Proposition 19** Let $1 < q < \infty$, $\varphi \in \mathcal{B}$, $w \in \mathcal{W}^\alpha$ for some $\alpha \in (0,1)$, and $w^q \in \mathcal{A}_q$. Then for any $f \in L_{q,w}$

$$\left( \sum_{k \in \mathbb{Z}^d} |(f, \varphi_{0k}) w(k)|^q \right)^{1/q} \leq C \|f\|_{q,w}, \quad (35)$$

where $C$ depends only on $d$, $q$, $\alpha$, $c_w$ and $\varphi$.

Before the proof of Proposition 19, we introduce additional notation and prove one lemma. We set

$$U_k^0 = \{ t \in \mathbb{R} : |t - k| < 1 \} \quad \text{and} \quad U_k^1 = \mathbb{R} \setminus U_k^0, \quad k \in \mathbb{Z};$$

if $k \in \mathbb{Z}^d$ and $\chi = (\chi_1, \ldots, \chi_d) \in \{0,1\}^d$, then $U_k^\chi$ is defined by

$$U_k^\chi = U_{k_1}^{\chi_1} \times \cdots \times U_{k_d}^{\chi_d}.$$

**Lemma 20** Let $1 < q < \infty$, $w \in \mathcal{W}^\alpha(\mathbb{R})$ for some $\alpha \in (0,1)$, and $w^q \in A_q(\mathbb{R})$. Then for any $f \in L_{q,w}(\mathbb{R})$ and $u \in \mathbb{R}$

$$\left( \sum_{k \in \mathbb{Z}} \left| w(k) \int_{U_k^1} f(t) \frac{e^{2\pi i u(t - k)}}{t - k} dt \right|^q \right)^{1/q} \leq C \|f\|_{L_{q,w}(\mathbb{R})},$$

where $C$ depends only on $q$, $\alpha$, and $c_w$.

**Proof.** Using properties of $w$, we have

$$\sum_{k \in \mathbb{Z}} \left| w(k) \int_{U_k^1} f(t) \frac{e^{2\pi i u(t - k)}}{t - k} dt \right|^q \leq \sum_{k \in \mathbb{Z}} \int_{k-1/2}^{k+1/2} dx \left| w(x) w^*(x - k) \right| \int_{U_k^1} f(t) \frac{e^{2\pi i u t}}{t - k} dt \right|^q \leq 2^{\alpha q / 2} c_w^q \sum_{k \in \mathbb{Z}} \int_{k-1/2}^{k+1/2} dx \left| w(x) \int_{U_k^1} f(t) \frac{e^{2\pi i u t}}{t - k} dt \right|^q. \quad (36)$$

Taking into account that by Minkowski’s inequality

$$\sum_{k \in \mathbb{Z}} \int_{k-1/2}^{k+1/2} dx \left( w(x) \int_{U_k^1} |f(t)| \frac{1}{t - x} - \frac{1}{t - k} | dt \right)^q \leq C_1 \sum_{k \in \mathbb{Z}} \int_{k-1/2}^{k+1/2} dx \left( w(x) \int_{|t - x| \geq 1/2} \frac{|f(t)|}{(t - x)^q} dt \right)^q \leq C_1 \int dx \left( \int_{|t| \geq 1/2} \frac{c_w w^*(t) |w(t + x)f(t + x)|}{t^2} dt \right)^q \leq C_2 \|f\|_{q,w} \left( \int \int_{|t| \geq 1/2} \frac{|t|^\alpha}{t^2} dt \right)^q = C_3 \|f\|_{q,w}^q,$$
one can replace $t - k$ in the denominators of the integrand in (36) by $t - x$.

Next, we have

$$
\int_{\mathbb{R}} \left| \int_{|t-x| \geq 1} f(t) e^{2\pi i ut} \frac{q}{t-x} dt \right|^q w^q(x) dx \leq \int_{\mathbb{R}} \sup_{x > 0} \left| \int_{|t-x| \geq \epsilon} f(t) e^{2\pi i ut} \frac{q}{t-x} dt \right|^q w^q(x) dx
$$

where $f_u(t) = f(t)e^{2\pi i ut}$ and $M(g)$ is the maximal function of the Hilbert transform of a function $g$.

It remains to note that the operator $M$ is bounded in $L_{q,w}$ under our assumption $w^q \in A_q(\mathbb{R})$ (see, e.g., [10, Corollary 7.13]). 

**Proof of Proposition 19.** For convenience, we introduce the following notation. If $t \in \mathbb{R}^d$, then $\tilde{t} := (t_1, \ldots, t_{d-1}) \in \mathbb{R}^{d-1}$. For a function $g$ of $d$ variables $t_1, \ldots, t_{d-1}, s$, we set

$$
g_a(\tilde{t}) = g_{\tilde{t}}(s) := g(t_1, \ldots, t_{d-1}, s).
$$

Let also $\tilde{S} = [a_1, b_1] \times \cdots \times [a_{d-1}, b_{d-1}]$.

We have

$$
\left( \sum_{k \in \mathbb{Z}^d} |\langle f, \varphi_{0k} \rangle w(k)|^q \right)^{1/q} = \left( \sum_{k \in \mathbb{Z}^d} \left| w(k) \int_{\mathbb{R}^d} f(t) \varphi(t-k) dt \right|^q \right)^{1/q} \leq \sum_{\chi \in \{0,1\}^d} (I^x)^{1/q},
$$

where

$$
I^x = \sum_{k \in \mathbb{Z}^d} \left| w(k) \int_{U_{\chi}^x} f(t) \varphi(t-k) dt \right|^q.
$$

Thus, to prove (35) it suffices to show that for every $\chi \in \{0,1\}^d$

$$
I^x \leq C_0 \|\tilde{\theta}\|_{W_{\infty}^d(\tilde{S})}^q \|f\|_{L_{q,w}}^q,
$$

where $C_0$ depends on $d$, $q$, $\alpha$, $S$, and $c_w$.

We prove (37) by induction on $d$. We will verify the inductive step $d - 1 \to d$ and the base for $d = 1$ simultaneously using the same arguments.

To prove the inductive step $d - 1 \to d$, we assume that for any weight $\tilde{w}$ such that $\tilde{w}^q \in \mathcal{W}^{\alpha q} \cap \mathcal{A}^q(\mathbb{R}^{d-1})$, $g \in L_{q,w}(\mathbb{R}^{d-1})$, and $\varphi \in \mathcal{R}(\mathbb{R}^{d-1})$ (more precisely $\varphi = \mathcal{F}^{-1}\tilde{\theta}$, where $\tilde{\theta}$ is the same as in (8) with $\tilde{S}$ in place of $S$), we have

$$
\sum_{k \in \mathbb{Z}^{d-1}} \left| \tilde{w}(\tilde{k}) \int_{U_{\chi}^{d-1}} g(\tilde{t}) \varphi(\tilde{t} - \tilde{k}) d\tilde{t} \right|^q \leq C_1 \|\tilde{\theta}\|_{W_{\infty}^{d-1}(\tilde{S})}^q \|g\|_{L_{q,w}(\mathbb{R}^{d-1})}^q,
$$

where $C_1$ depends on $d$, $p$, $\alpha$, $\tilde{S}$, and $C_{\tilde{w}}$.

In what follows, we will use the fact that under our assumptions, we have $w^q_s \in \mathcal{W}^{\alpha q} \cap \mathcal{A}^q_s(\mathbb{R}^{d-1})$ for every $s \in \mathbb{R}$ and $w^q_{\tilde{t}} \in \mathcal{W}^{\alpha q} \cap A_q(\mathbb{R})$ for every $\tilde{t} \in \mathbb{R}^{d-1}$. This follows from Lemma 18 and basic properties of the weights in $\mathcal{W}^{\alpha q}$.

For any $\chi \in \{0,1\}^d$, we can write $\chi = (\chi_d, \chi_{d-1})$ and

$$
I^x = \sum_{\chi_d \in \{0,1\}} I(\hat{x}, \chi_d)
$$

Let us estimate $I(\hat{x}, \chi_d)$ for $\chi_d = 0$ and $\chi_d = 1$. 

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1) First let $\chi_d = 1$. In the case $d > 1$, we set
\[ \psi(x, \eta) = \int_S \theta_\eta(\xi)e^{2\pi i (\xi, x)}d\xi. \] (39)

Using the above notation and integrating by parts, we have
\[ \varphi(x) = \mathcal{F}^{-1}\theta(x) = \int \psi_\eta(\eta)e^{2\pi i x \eta}d\eta \]
\[ = \frac{\psi_\eta(b_d)e^{2\pi ibdx_d} - \psi_\eta(a_d)e^{2\pi ia_dx_d}}{2\pi i x_d} - \frac{1}{2\pi i x_d}\mathcal{F}^{-1}\psi_\eta(x_d). \] (40)

It follows from (40) that
\[ I^{(\xi, 1)} = \sum_{k \in \mathbb{Z}^{d-1}} \sum_{l \in \mathbb{Z}} \left| \int_{U_k^\xi} w_l(\tilde{k}) \int_{U_k^\xi} f_s(\tilde{l}) \varphi(\tilde{l} - k, s - \tilde{l}) d\tilde{l} d\tilde{s}\right|^q \leq (2\pi)^{-q}(I_1 + I_2), \] (41)

where
\[ I_1 = \sum_{k \in \mathbb{Z}^{d-1}} \sum_{l \in \mathbb{Z}} \left| \int_{U_k^\xi} w_l(\tilde{k}) \int_{U_k^\xi} f_s(\tilde{l}) \varphi(\tilde{l} - k, s - \tilde{l}) d\tilde{l} d\tilde{s}\right|^q \]
and
\[ I_2 = \sum_{k \in \mathbb{Z}^{d-1}} \sum_{l \in \mathbb{Z}} \left| \int_{U_k^\xi} w_l(\tilde{k}) \int_{U_k^\xi} f_s(\tilde{l}) \mathcal{F}^{-1}\psi_{l-k}(s - l) d\tilde{l} d\tilde{s}\right|^q. \] (42)

Let us consider $I_1$. Denoting
\[ F_{k,u}(s) = \int_{U_k^\xi} f_s(\tilde{l})\psi_{l-k}(u)d\tilde{l} \]
and using Lemma 20, we obtain
\[ I_1 \leq \sum_{u \in \{a_d, b_d\}} \sum_{k \in \mathbb{Z}^{d-1}} \sum_{l \in \mathbb{Z}} \left| \int_{U_k^\xi} w_l(\tilde{k}) F_{k,u}(s)e^{2\pi i u(s - l)}d\tilde{s}\right|^q \]
\[ \leq C_2 \sum_{u \in \{a_d, b_d\}} \sum_{k \in \mathbb{Z}^{d-1}} \|F_{k,u}\|_{L_2, w_k(R)}^q, \] (43)

where $C_2$ is the same constant as in Lemma 20.

Now, using the induction hypothesis (38), we derive
\[ \sum_{k \in \mathbb{Z}^{d-1}} \|F_{k,u}\|_{L_2, w_k(R)}^q = \int_{R} \sum_{k \in \mathbb{Z}^{d-1}} \left| \int_{U_k^\xi} w_s(\tilde{k}) \int_{U_k^\xi} f_s(\tilde{l})\mathcal{F}^{-1}\theta_u(\tilde{l} - \tilde{k})d\tilde{l} d\tilde{s}\right|^q d\tilde{s} \]
\[ \leq C_1\|\theta_u\|_{W_2^{d-1}(\bar{S})} \int_{R} \|f_s\|_{L_2, w_s(R^{d-1})}^q d\tilde{s}. \] (44)
Thus, combining (42) and (44), we get

\[ I_1 \leq C_1 C_2 \sum_{u \in \{a_d, b_d\}} \| \theta_u \|^q_{W^{-1}_{\infty}(S)} \| f \|^q_{L^q(S, w)} \leq 2C_1 C_2 \| \theta \|^q_{W^{-1}_{\infty}(S)} \| f \|^q_{L^q(S, w)}. \tag{45} \]

Let us consider \( I_2 \). Setting

\[ F_{k, \eta}^*(s) = \int \int_{U^k} \mathcal{F}^{-1} \psi_{\tilde{t} - \hat{k}}(s - l) \, d\tilde{t} \]

and using Hölder’s inequality, we obtain

\[
\left| \int \int_{U^k} \mathcal{F}^{-1} \psi_{\tilde{t} - \hat{k}}(s - l) \, d\tilde{t} \right|^q \leq \left( \int \int_{U^k} \left| \int \int_{U^k} \mathcal{F}^{-1} \psi_{\tilde{t} - \hat{k}}(s - l) \, d\tilde{t} \right|^q \right)^{1/2} \left( \int \int_{U^k} \left| \int \int_{U^k} \mathcal{F}^{-1} \psi_{\tilde{t} - \hat{k}}(s - l) \, d\tilde{t} \right|^q \right)^{1/2}
\]

\[ \leq (b_d - a_d)^{q-1} \int \int_{U^k} \left| \int \int_{U^k} F_{k, \eta}^*(s) \, d\eta \right|^q \, d\eta. \tag{46} \]

Thus, combining (42) and (46), using Lemma 20, and the induction hypothesis (38), we derive

\[
I_2 \leq (b_d - a_d)^{q-1} \sum_{k \in Z^{d-1}} \left( \sum_{l \in Z} w_l^q(k) \right)^{1/2} \left( \int \left| \int \int F_{k, \eta}^*(s) \, d\eta \right|^q \, d\eta \right)
\]

\[ = (b_d - a_d)^{q-1} \int \int_{U^k} \left| \int \int F_{k, \eta}^*(s) \, d\eta \right|^q \, d\eta \]

\[ \leq C_2 (b_d - a_d)^{q-1} \int \int_{U^k} \left| \int \int F_{k, \eta}^*(s) \, d\eta \right|^q \, d\eta \]

\[ = C_2 (b_d - a_d)^{q-1} \int \int_{U^k} \left| \int \int F_{k, \eta}^*(s) \, d\eta \right|^q \, d\eta \]

\[ \leq C_1 C_2 (b_d - a_d)^{q-1} \int \left| \int \int F_{k, \eta}^*(s) \, d\eta \right|^q \, d\eta \]

\[ \leq C_1 C_2 (b_d - a_d)^{q-1} \int \left| \int \int F_{k, \eta}^*(s) \, d\eta \right|^q \, d\eta \]

The above arguments are valid also for \( d = 1 \). In this case, the function \( \psi \) should be replaced by \( \theta \) while \( F_{k, u}(s) \) should be replaced by \( f(s) \theta(u) \) in (43). Similarly, \( F_{k, \eta}^*(s) \) should be replaced by \( f(s) \theta'(\eta) \) in (46). The sum over \( \tilde{k} \) and the integral over \( U^k \) are absent in this case.

Thus, combining (41), (45), and (47), we get (37) for any \( d \geq 1 \) and \( \chi = (\chi_1, \ldots, \chi_{d-1}, 1) \in \{0, 1\}^d \).

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2) Let now $\chi_d = 0$. In the case $d > 1$, we have

$$I(\tilde{\chi}, 0) = \sum_{l \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^{d-1}} \left| \int_{U^l_k} \tilde{w}_l(k) \int_{U^l_a} f_s(i) \varphi(i - k, s - t) \, dt \, ds \right|^q,$$

where $\psi$ is defined by (39).

Using Hölder’s inequality and induction hypothesis (38), we obtain

$$I(\tilde{\chi}, 0) \leq (b_d - a_d)^{q-1} \sum_{l \in \mathbb{Z}} \int_{|s - l| \leq 1} \sum_{a_d} \int_{|s - l| \leq 1} \sum_{k \in \mathbb{Z}^{d-1}} \left| \tilde{w}_l(k) \int_{U^l_a} f_s(i) \psi(i - k) e^{2\pi i \eta(s - t)} \, dt \right|^q,$$

Since

$$w_l(\tilde{t}) = w(t_1, \ldots, t_{d-1}, l) \leq w^*(0, \ldots, 0, l - s)w(t_1, \ldots, t_{d-1}, s) \leq (1 + |l - s|^2)^{\alpha/2}w(\tilde{t}),$$

it follows that

$$I(\tilde{\chi}, 0) \leq C_1 (b_d - a_d)^{q-1} \sum_{l \in \mathbb{Z}} \int_{|s - l| \leq 1} \sum_{a_d} \int_{|s - l| \leq 1} \sum_{k \in \mathbb{Z}^{d-1}} \left| \tilde{w}_l(k) \int_{U^l_a} f_s(i) \psi(i - k) e^{2\pi i \eta(s - t)} \, dt \right|^q,$$

which yields (37) for any $\chi = (\chi_1, \ldots, \chi_{d-1}, 0) \in \{0, 1\}^d$.

If $d = 1$, then due to properties of $w$ and Hölder’s inequality, we have

$$\sum_{k \in \mathbb{Z}} \left| w(k) \int_{U^l_k} f(t) \varphi(t - k) \, dt \right|^q \leq \left\| \varphi \right\|_{L^q_{2d}(R)}^q \sum_{k \in \mathbb{Z}} \left( \int_{|t - k| \leq 1} \left| w^*(t - k) \right| \, dt \right)^q \leq C_2^q \left( b_1 - a_1 \right)^q \left\| \varphi \right\|_{L^q_{2d}(R)}^q \sum_{k \in \mathbb{Z}} \left( \int_{|t - k| \leq 1} \left| f(t) \right| \, dt \right)^q.$$

Thus, inequality (37) holds for every $\chi \in \{0, 1\}^d$ and $d \geq 1$, which proves the proposition.

**Proposition 21** Let $1 < p < \infty$, $\varphi \in \mathcal{B}$, $w \in \mathcal{W}^\alpha$ for some $\alpha \in (0, 1)$, and $w^{-p} \in \mathcal{A}_p$. Then for any $a = \{a_k\}_{k \in \mathbb{Z}^d} \in \ell_{p, 1/w}$

$$\left\| \sum_{k \in \mathbb{Z}^d} a_k \varphi_{0k} \right\|_{\ell_{p, 1/w}} \leq C\|a\|_{\ell_{p, 1/w}},$$

where $C$ depends on $d$, $p$, $\alpha$, $c_w$, and $\varphi$. 

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Proof. It follows from Riesz's theorem that there exists \( g \in L_{q,w} \), \( 1/p + 1/q = 1 \), such that 
\[
\|g\|_{q,w} \leq 1 \text{ and } \left\| \sum_{k \in \mathbb{Z}^d} a_k \varphi_{0k} \right\|_{p,1/w} = \left\| \sum_{k \in \mathbb{Z}^d} a_k \langle g, \varphi_{0k} \rangle \right\| = \left\| \sum_{k \in \mathbb{Z}^d} \frac{a_k}{w(k)} \langle g, \varphi_{0k} \rangle w(k) \right\|.
\]
It is not difficult to check that \( w^g \in \mathcal{A}_p \). Thus, to prove (48) it remains to apply Hölder’s inequality and use Proposition 19 with \( f = g \).

Proposition 22 Let \( 1 < p < \infty, \varphi \in \mathcal{B}, w \in \mathcal{W}^\alpha \) for some \( \alpha \in (0,1) \), and \( w^{-p} \in \mathcal{A}_p \). Then for any \( f \in L_{p,1/w} \)
\[
\left( \sum_{k \in \mathbb{Z}^d} \left| \frac{(f, \varphi_{0k})}{w(k)} \right|^p \right)^{1/p} \leq C \|f\|_{p,1/w},
\]
where \( C \) depends on \( d, p, \alpha, c_w \) and \( \varphi \).

Proof. The proposition can be proved by following step by step the proof of Proposition 19 and using inequality (5) instead of (4).

Corollary 23 Let \( 1 < p < \infty, w \in \mathcal{W}^\alpha \) for some \( \alpha \in (0,1) \), and \( w^{-p} \in \mathcal{A}_p \). If \( \varphi, \tilde{\varphi} \in \mathcal{B} \), then for any \( f \in L_{p,1/w} \)
\[
\left\| \sum_{k \in \mathbb{Z}^d} (f, \varphi_{0k}) \varphi_{0k} \right\|_{p,1/w} \leq C \|f\|_{p,1/w},
\]
where \( C \) depends on \( d, p, \alpha, c_w, \varphi, \) and \( \tilde{\varphi} \).

Proof. The proof follows immediately from Propositions 21 and 22.

Corollary 24 Let \( 1 < p < \infty, w \in \mathcal{W}^\alpha \) for some \( \alpha \in (0,1) \), and \( w^{-p} \in \mathcal{A}_p \). If \( \varphi \in \mathcal{B} \) and \( \tilde{\varphi} \in L_{q,w^*} \), \( 1/p + 1/q = 1 \), then for any \( f \in L_{p,1/w} \)
\[
\left\| \sum_{k \in \mathbb{Z}^d} (f, \varphi_{0k}) \varphi_{0k} \right\|_{p,1/w} \leq C \|\tilde{\varphi}\|_{L_{q,w^*}} \|f\|_{p,1/w},
\]
where \( C \) depends on \( d, p, \alpha, c_w, \) and \( \varphi \).

Proof. The proof follows immediately from Propositions 3 and 21.

Lemma 25 Let \( 1 < p < \infty, w \in \mathcal{W}^\alpha \) for some \( \alpha \in (0,1) \), \( w^{-p} \in \mathcal{A}_p \), and \( n \in \mathbb{N} \). Suppose

1) \( \psi \in \mathcal{B} \) and \( \tilde{\psi} \in L_2 \);

2) \( \tilde{\psi} \in C^\gamma(\mathbb{R}^d) \) for some \( \gamma > n + d + p\alpha \);

3) \( D^\beta \tilde{\psi}(\mathbf{0}) = 0 \) for all \( |\beta| < n, \beta \in \mathbb{Z}_+^d \).

Then for any \( f \in W_{p,1/w}^n \)
\[
\left\| \sum_{k \in \mathbb{Z}^d} (f, \psi_{0k}) \psi_{0k} \right\|_{p,1/w} \leq C \|f\|_{W_{p,1/w}^n},
\]
where \( C \) depends only on \( d, p, n, \psi, \tilde{\psi}, c_w, \) and \( \alpha \).
Proof. The lemma can be proved repeating step by step the proof of Lemma 15. One needs only to use Proposition 21 instead of Proposition 2. ◇

**Theorem 26** Let $1 < p < \infty$, $w \in \mathcal{W}^\alpha$ for some $\alpha \in (0, 1)$, $w^{-p} \in \mathcal{A}_p$, and let $M \in \mathcal{M}$ be a diagonal matrix. Suppose

1) $\varphi \in \mathcal{B}$ and $\tilde{\varphi} \in \mathcal{L}_{q,w^*}$, $1/p + 1/q = 1$;

2) $\varphi$ and $\tilde{\varphi}$ are strictly compatible with respect to the parameter $\delta > 0$;

3) $\text{supp} \tilde{\varphi} \subset (-1,1)^d$.

Then for any $f \in L_{p,1/w}$ and $j \in \mathbb{Z}_+$

$$\left\| f - \sum_{k \in \mathbb{Z}^d} \langle f, \tilde{\varphi}_{jk} \rangle \varphi_{jk} \right\|_{p,1/w} \leq CE\{[|M|^{-j}|\leq \delta]\}(f)_{p,1/w},$$

where the constant $C$ does not depend on $f$, $M$, and $j$.

**Proof.** The proof is similar to the proof of Theorem 11. One only needs to use Corollary 24 instead of Corollary 4 in (30). Next we repeat all steps of the proof of Theorem 13, all arguments of the proof of Lemma 16, using Lemma 25 instead of Lemma 15 in (23) and using the constant $C$ where the functional $\Upsilon$ is independent of $f$.

In the case of weakly compatible functions $\varphi$ and $\tilde{\varphi}$, we have the following result given in terms of the moduli of smoothness.

**Theorem 27** Let $1 < p < \infty$, $w \in \mathcal{W}^\alpha$ for some $\alpha \in (0, 1)$, $w^{-p} \in \mathcal{A}_p$, $n \in \mathbb{N}$, and let $M \in \mathcal{M}$ be a diagonal matrix. Suppose

1) $\varphi \in \mathcal{B}$ and $\tilde{\varphi} \in \mathcal{L}_{q,w^*}$, $1/p + 1/q = 1$;

2) $\tilde{\varphi}, \hat{\varphi} \in C^\gamma(B_\varepsilon)$ for some integer $\gamma > n + d + pa$ and $\varepsilon > 0$;

3) $\tilde{\varphi}$ and $\varphi$ are weakly compatible of order $n$;

4) $\text{supp} \tilde{\varphi} \subset (-1,1)^d$.

Then for any $f \in L_{p,1/w}$ and $j \in \mathbb{Z}_+$, we have

$$\left\| f - \sum_{k \in \mathbb{Z}^d} \langle f, \tilde{\varphi}_{jk} \rangle \varphi_{jk} \right\|_{p,1/w} \leq C\Omega_n(f, M^{-j})_{p,1/w} \leq C\omega_n(f, |\lambda|^{-j})_{p,1/w},$$

where $\lambda$ is the smallest (in absolute value) diagonal element of $M$ and $C$ does not depend on $f$, $M$, and $j$.

**Proof.** First we assume that $f \in W^n_{p,1/w} \cap L_2$ and prove that

$$\left\| f - \sum_{k \in \mathbb{Z}^d} \langle f, \tilde{\varphi}_{0k} \rangle \varphi_{0k} \right\|_{p,1/w} \leq \Upsilon(w^*)\|f\|_{W^n_{p,1/w}}, \quad (50)$$

where the functional $\Upsilon$ is independent of $f$ and $\Upsilon(w^*(M^{-j} \cdot)) \leq C\Upsilon(w^*)$ for all $j \in \mathbb{Z}_+$, where the constant $C$ depends only on $d$, $p$, $\varphi$, $\tilde{\varphi}$, $\alpha$, $c_w$. To prove (50) we repeat step by step all arguments of the proof of Lemma 16, using Lemma 25 instead of Lemma 15 in (23) and using Corollary 24 instead of Corollary 4 in (30). Next we repeat all steps of the proof of Theorem 13, using (50) instead of Lemma 16 and taking into account that $(w(M^{-j} \cdot))^{-p} \in \mathcal{A}_p$ by Lemma 18 and $\|M^{-j}\| = |\lambda|^{-j}$. ◇
4.3 The case of sampling expansions ($\varphi \in \mathcal{B}$ and $\hat{\varphi}$ is the Dirac delta-function)

In this section, we study expansions $\sum_{k \in \mathbb{Z}^d} (f, \hat{\varphi}_{jk}) \varphi_{jk}$ where $\hat{\varphi}$ is the Dirac delta-function, i.e.,

$$\sum_{k \in \mathbb{Z}^d} (f, \hat{\varphi}_{jk}) \varphi_{jk} = \sum_{k \in \mathbb{Z}^d} (\hat{f}, \delta_k) \varphi_{jk} = m^{-j/2} \sum_{k \in \mathbb{Z}^d} f(-M^{-j}k) \varphi_{jk}.$$ 

In the theorems below, we will suppose that the function $\varphi$ and the Dirac delta-function are strictly compatible (weakly compatible of order $n \in \mathbb{N}$), which implies that $\hat{\varphi}(\xi) \equiv 1$ on the ball $B_{\delta}(D^\beta(1-\hat{\varphi})(0) = 0$ for all $|\beta| < n, \beta \in \mathbb{Z}_+^d$).

First, we consider band-limited weights and, as in the case of Theorem 27, we suppose that $w \in \mathcal{W}^\alpha$ and $w^{-p} \in \mathcal{A}_p$.

**Theorem 28** Let $2 \leq p < \infty$, $w \in \mathcal{W}^\alpha$ for some $\alpha \in (0, 1)$, $w^{-p} \in \mathcal{A}_p$, supp $\hat{w} \subset B_{4/2}$ for some $\delta \in (0, 1/2)$, and let $M \in \mathbb{M}$ be a diagonal matrix. Suppose

1) $\varphi \in \mathcal{B}$;
2) $\varphi$ is strictly compatible with the Dirac delta-function with respect to the parameter $\delta$;
3) supp $\hat{\varphi} \subset (-1 + \delta/2, 1 - \delta/2)^d$.

If a function $f \in L_{p,1/w}$ is such that $\mathcal{F}(w^{-1}f) \in L_{q,1/p+1/q = 1}$, and $\mathcal{F}(w^{-1}f)(\xi) = o(|\xi|^{-d-\alpha})$ as $|\xi| \to \infty$, $a > 0$, then for any $j \in \mathbb{Z}_+$ and $\gamma > d/p$

$$\left\| f - m^{-j/2} \sum_{k \in \mathbb{Z}^d} f(-M^{-j}k) \varphi_{jk} \right\|_{p,1/w}^q \leq C \|M^{-j}\|^{q\gamma} \int_{|M^{-j}\xi| \geq \delta/2} |\xi|^\gamma |\mathcal{F}(w^{-1}f)(\xi)|^q d\xi, \quad (51)$$

where $C$ does not depend on $M$, $j$, and $f$.

**Proof.** First, we consider the case $j = 0$ and prove that

$$\left\| f - \sum_{k \in \mathbb{Z}^d} f(-k) \varphi_{0k} \right\|_{p,1/w}^q \leq C_1 \int_{|\xi| \geq \delta/2} |\xi|^\gamma |\mathcal{F}(w^{-1}f)(\xi)|^q d\xi,$$

where $C_1$ depends on $p$, $\alpha$, $c_w$, and $\varphi$.

We set $g = w^{-1}f$ and $G(\xi) = \sum_{l \in \mathbb{Z}^d} \hat{g}(\xi + l)$.

Using Lemma 1 from [19] and the Hausdorff-Young inequality, we have

$$\langle \hat{g}, \delta_{0k} \rangle = \hat{G}(k)$$

and

$$\left( \sum_{k \in \mathbb{Z}^d} |(\hat{g}, \delta_{0k})|^p \right)^{1/p} = \left( \sum_{k \in \mathbb{Z}^d} |\hat{G}(k)|^p \right)^{1/p} \leq \|G\|_{L_q(\mathbb{T}^d)} < \infty, \quad (52)$$

Since

$$\langle \hat{g}, \delta_{0k} \rangle = \langle \mathcal{F}(w^{-1}f), \delta_{0k} \rangle = (w^{-1}f)(-k),$$

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it follows from Proposition 21 and inequality (52) that

\[
\left\| \sum_{x \in \mathbb{Z}^d} f(-k) \varphi_{0k} \right\|_{p,1/w} \leq C_2 \left( \sum_{x \in \mathbb{Z}^d} \left| \frac{f(-k)}{|w(-k)|} \right|^p \right)^{1/p} \leq C_3 \|G\|_{L_q(\mathbb{T}^d)}. \tag{53}
\]

Again using Lemma 1 from [19], we have

\[
\|G\|_{L_q(\mathbb{T}^d)} \leq C_4 \left( \int_{|\xi| \geq \delta} |\xi|^q |\hat{g}(\xi)|^q d\xi + \|\hat{g}\|_q \right)^{1/q}. \tag{54}
\]

Due to the du Bois-Reymond lemma, the function \(g(-x)\) coincides with \(\hat{g}(x)\) almost everywhere. It follows from the Hausdorff-Young inequality that \(\|g\|_p \leq \|\hat{g}\|_q\), which together with (53) and (54) yields

\[
\left\| f - \sum_{x \in \mathbb{Z}^d} f(-k) \varphi_{0k} \right\|_{p,1/w} \leq \|g\|_p + \left\| \sum_{x \in \mathbb{Z}^d} f(-k) \varphi_{0k} \right\|_{p,1/w} \leq C_5 \left( \int_{|\xi| \geq \delta} |\xi|^q |\hat{g}(\xi)|^q d\xi + \int_{|\xi| < \delta} |\hat{g}(\xi)|^q d\xi \right)^{1/q}, \tag{55}
\]

Denote

\[
F(x) = w(x)H(x), \quad H(x) = \mathcal{F}^{-1}h(x),
\]

where \(h \in C^\infty(\mathbb{R}^d)\) and \(\text{supp } h \subset B_{\delta/2}\). We can choose the function \(h\) such that

\[
\int_{|\xi| < \delta/2} |\mathcal{F}(w^{-1}f)(\xi)|^q d\xi < \int_{|\xi| > \delta} |\xi|^q |\mathcal{F}(w^{-1}f)(\xi)|^q d\xi. \tag{56}
\]

Note that \(F \in L_2\) since \(|F(x)| \leq w(x)|H(x)|\) and \(|H(x)| \leq c_N (1 + |x|)^{-N}\) for any \(N \in \mathbb{N}\). Moreover, by the Paley-Wiener-Schwartz theorem, \(\text{supp } \mathcal{F} \subset B_\delta\). Thus, by Proposition 8, (55), and (56), we obtain

\[
\left\| f - \sum_{x \in \mathbb{Z}^d} f(-k) \varphi_{0k} \right\|_{p,1/w}^q = \left\| f - \sum_{x \in \mathbb{Z}^d} (f(-k) - F(-k)) \varphi_{0k} \right\|_{p,1/w}^q \leq C_5^q \left( \int_{|\xi| \geq \delta} |\xi|^q |\mathcal{F}(w^{-1}(f-F))(\xi)|^q d\xi + \int_{|\xi| < \delta} |\mathcal{F}(w^{-1}(f-F))(\xi)|^q d\xi \right) \leq C_5^q \int_{|\xi| \geq \delta/2} |\xi|^q |\mathcal{F}(w^{-1}f)(\xi)|^q d\xi.
\]

This yields (51) for \(j = 0\). To get the required inequality for any \(j \in \mathbb{N}\), we replace \(f\) by \(f(M^{-j}\cdot)\) and \(w\) by \(w(M^{-j}\cdot)\), take into account that \((w(M^{-j}\cdot))^{-p} \in \mathcal{A}_p\) by Lemma 18, and change variables in the integrals. ◇
Corollary 29  Under the assumptions of Theorem 28, we have
\[
\left\| f - m^{-j/2} \sum_{k \in \mathbb{Z}^d} f(-M^{-j}k) \varphi_{jk} \right\|_{p,1/w}^q = \mathcal{O} \left( |\lambda|^{-j(a+d/p)} \right),
\]
where \(\lambda\) is the smallest (in absolute value) diagonal element of \(M\).

Remark 30  In Theorem 28 and Corollary 29, we suppose that \(\text{supp} \, \tilde{w} \subset B_{\delta/2}\), which is quite exotic in such type of problems. Nevertheless, such weights can be easily constructed by the following way. For some weight \(v\) such that \(v \in \mathcal{W}^\alpha\) and \(v^{-p} \in \mathcal{A}_p\), we set
\[
w(x) = v \ast V(x),
\]
where \(V\) is such that \(\text{supp} \, \tilde{V}\) is compact and \(V(x) \geq 0\) for all \(x \in \mathbb{R}^d\). It is not difficult to see that if \(v\) is positive, symmetric, and \(v(x+y) \leq v(x)v(y)\), then for all \(x \in \mathbb{R}^d\) one has the following two-sided inequality:
\[
v(x)\|V\|_{L_1,1/v} \leq w(x) \leq v(x)\|V\|_{L_1,\ast}.
\]
Thus, for an appropriate function \(V\), we have that \(w \in \mathcal{W}^\alpha\) with \(w^* = v^*\), \(w^{-p} \in \mathcal{A}_p\), and \(\text{supp} \, \tilde{w}\) is compact.

Theorem 31  Let \(2 \leq p < \infty\), \(n \in \mathbb{N}\), \(w \in \mathcal{W}^\alpha\) for some \(\alpha \in (0,1)\), \(w^{-p} \in \mathcal{A}_p\), \(w \in C^\infty(\mathbb{R}^d)\), \(|D^\beta w(x)| \leq c_{w,\beta}w(x)\) for every \(\beta \in \mathbb{Z}_+^d\), \(x \in \mathbb{R}^d\), and let \(M \in \mathfrak{M}\) be a diagonal matrix. Suppose
1) \(\varphi \in \mathcal{B}\);
2) \(\varphi \in C^r(B_c)\) for some integer \(r > n + d + pa\) and \(\varepsilon > 0\);
3) \(\varphi\) is weakly compatible of order \(n\) with the Dirac delta-function;
4) \(\text{supp} \, \varphi \subset (-1,1)^d\).
If a function \(f \in L_{p,1/w}\) is such that \(\mathcal{F}(w^{-1}f) \in L_q\), \(1/p + 1/q = 1\), and \(\mathcal{F}(w^{-1}f)(\xi) = \mathcal{O}(|\xi|^{-d-a})\) as \(|\xi| \to \infty\), \(a > 0\), then for any \(j \in \mathbb{Z}_+\) and any \(\gamma > d/p\)
\[
\left\| f - m^{-j/2} \sum_{k \in \mathbb{Z}^d} f(-M^{-j}k) \varphi_{jk} \right\|_{p,1/w}^q \leq C_1 \sum_{\nu=0}^n \|M^{-j}\|^{\gamma \omega_n - \nu}(f,\|M^{-j}\|_{p,1/w}^q
\]
\[
+ C_2 \|M^{-j}\|^{\gamma} \int_{|M^{-j}\xi| \geq 1/2} |\xi|^\nu |\mathcal{F} \left( \frac{f}{w} \right)(\xi)\|_q d\xi, \tag{57}
\]
where \(C_1\) and \(C_2\) do not depend on \(M, j,\) and \(f\).

Remark 32  Theorem 31 is new also in the unweighted case, i.e. for \(w(x) \equiv 1\). In this case, one can show that inequality (57) holds for any \(M \in \mathfrak{M}\) and has the following form:
\[
\left\| f - m^{-j/2} \sum_{k \in \mathbb{Z}^d} f(-M^{-j}k) \varphi_{jk} \right\|_p^q \leq C_1 \omega_n(f,\|M^{-j}\|_p^q)
\]
\[
+ C_2 \|M^{-j}\|^{\gamma} \int_{|M^{-j}\xi| \geq 1/2} |\xi|^\nu |\mathcal{F} f(\xi)|^q d\xi, \nonumber
\]
where \(C_1\) and \(C_2\) do not depend on \(M, j,\) and \(f\).
Remark 33 Similarly to Theorem 27 and Theorem 13, it is possible to obtain a shaper version of inequality (57) by replacing the modulus of smoothness $\omega_{n-\nu}(f,\|M^{-1}\|)_{p,1/w}$ by the anisotropic modulus of smoothness $\Omega_{n-\nu}(f,M^{-1})_{p,1/w}$.

Proof of Theorem 31. Let $\tilde{\varphi}$ be a function such that $\tilde{\varphi}$ is infinitely differentiable, supp $\tilde{\varphi} \subset [-1,1]^d$, and $\tilde{\varphi}(\xi) \equiv 1$ on $\mathbb{T}^d$. Since

$$f(-k) = w(-k)\langle \mathcal{F}\left(\frac{f}{w}\right), \tilde{\varphi}_0 \rangle,$$

we have

$$\left\| f - \sum_{k \in \mathbb{Z}^d} f(-k)\varphi_0 \right\|_{p,1/w} \leq \left\| \sum_{k \in \mathbb{Z}^d} w(-k)\langle \mathcal{F}\left(\frac{f}{w}\right), \tilde{\varphi}_0 \rangle \varphi_0 \right\|_{p,1/w} + \left\| f - \sum_{k \in \mathbb{Z}^d} w(-k)\langle \mathcal{F}\left(\frac{f}{w}\right), \tilde{\varphi}_0 \rangle \varphi_0 \right\|_{p,1/w} \leq: I_1(f, w) + I_2(f, w).$$

Using Proposition 21, the Hausdorff-Young inequality, Lemma 1 in [19], and taking into account that $1 - \tilde{\varphi}(\xi) = 0$ if $\xi \in \mathbb{T}^d$, we obtain

$$I_1(f, w) \leq C_1 \left( \sum_{k \in \mathbb{Z}^d} \left\| \langle \mathcal{F}\left(\frac{f}{w}\right), \tilde{\varphi}_0 \rangle \right\|_{p,1/w} \right)^{1/p} \leq C_2 \left( \sum_{k \in \mathbb{Z}^d} \left\| \mathcal{F}\left(\frac{f}{w}\right) \right\|_{L_1(\mathbb{T}^d)} \right)^{1/p} \leq C_3 \left( \int_{|\xi| \geq 1/2} \left| \mathcal{F}\left(\frac{f}{w}\right)(\xi) \right|^q \xi \right)^{1/q}.$$

The functions $\varphi, \tilde{\varphi}$ satisfy all assumptions of Theorem 27, hence,

$$I_2(f, w) \leq \left\| f - \sum_{k \in \mathbb{Z}^d} \langle f, \tilde{\varphi}_0 \rangle \varphi_0 \right\|_{p,1/w} \leq C_4 \omega_n(f,1)_{p,1/w} + I_3(f, w),$$

where

$$I_3(f, w) = \left\| \sum_{k \in \mathbb{Z}^d} \int w(-k) - w(t) f(t) \varphi_0(t) dt \varphi_0 \right\|_{p,1/w}.$$

Using Taylor’s formula, we have

$$w(-k) - w(t) = \sum_{0<|\beta| \leq n} \frac{(-1)^{|\beta|} D^\beta w(t)}{\beta!} (t + k)^\beta + \sum_{|\beta| = n+1} \frac{(-1)^{|\beta|} D^\beta w(t + \eta(t + k))}{\beta!} (t + k)^\beta$$

for some $\eta \in (-1,0)$, which gives

$$I_3(f, w) \leq \sum_{0<|\beta| \leq n} I_{3,\beta}(f, w) + \sum_{|\beta| = n+1} I_{4,\beta}(f, w),$$
where

\[
I_{3,\beta}(f, w) = \left\| \sum_{k \in \mathbb{Z}^d} \left\langle \frac{D^{\beta} w}{w} f, \psi_{ok} \right\rangle \phi_{ok} \right\|_{p,1/w}, \\
I_{4,\beta}(f, w) = \left\| \sum_{k \in \mathbb{Z}^d} \left\langle \frac{D^{\beta} w (\cdot + \eta (\cdot + k)}{w} f, \psi_{ok} \right\rangle \phi_{ok} \right\|_{p,1/w},
\]

and \( \psi(t) = t^{\beta} \bar{\psi}(t) \).

Fix \( \beta \in \mathbb{Z}_+^d \) with \( 0 < |\beta| \leq n \). Let a function \( g \) satisfy (13) and (14) with \( \frac{D^{\beta} w}{w} f \) instead of \( f \).

Using Corollary 24 and Lemma 25, taking into account that \( \hat{\psi}(\xi) = (-2\pi i)^{-|\beta|} D^{\beta} \hat{\bar{\psi}}(\xi) \), and hence \( D^\beta \hat{\psi}(0) = 0 \) for any \( \beta \in \mathbb{Z}_+^d \setminus \{0\} \), we obtain

\[
I_{3,\beta}(f, w) \leq \left\| \sum_{k \in \mathbb{Z}^d} \left\langle \frac{D^{\beta} w}{w} f - g, \psi_{ok} \right\rangle \phi_{ok} \right\|_{p,1/w} + \left\| \sum_{k \in \mathbb{Z}^d} (g, \psi_{ok}) \phi_{ok} \right\|_{p,1/w} \leq C_5 \left( \left\| \frac{D^{\beta} w}{w} f - g \right\|_{p,1/w} + \left\| g \right\|_{p,1/w} \right) \leq C_6 \omega_n \left( \frac{D^{\beta} w}{w} f, 1 \right)_{p,1/w}.
\]

Combining (58), (59), (60), and (61), we derive

\[
\left\| f - \sum_{k \in \mathbb{Z}^d} f(-k) \phi_{ok} \right\|_{p,1/w} \leq C_7 \left( \int_{|\xi| \geq 1/2} |\xi|^{-|\beta|} \left| \mathcal{F} \left( \frac{f}{w} \right)(\xi) \right|^q d\xi \right)^{1/q} + C_8 \omega_n (f, 1)_{p,1/w} + C_9 \sum_{|\beta| \leq n} \omega_n \left( \frac{D^{\beta} w}{w} f, 1 \right)_{p,1/w} + C_{10} \sum_{|\beta| = n+1} I_{4,\beta}(f, w).
\]

For an arbitrary \( j \in \mathbb{Z}_+ \), taking into account that \( (w(M^{-j} \cdot))^{-p} \in \mathcal{S}_p \), by Lemma 18, we can replace \( f \) by \( f(M^{-j} \cdot) \) and \( w \) by \( w(M^{-j} \cdot) \), and after change of variables in all integrals, we have

\[
\left\| f - m^{-j/2} \sum_{k \in \mathbb{Z}^d} f(-M^{-j} k) \phi_{jk} \right\|_{p,1/w} = m^{-j/p} \left\| f(M^{-j} \cdot) - \sum_{k \in \mathbb{Z}^d} f(M^{-j} k) \phi_{ok} \right\|_{p,1/w(M^{-j} \cdot)} \leq C_7 \left\| M^{-j} \right\|^\gamma \left( \int_{|\xi| \geq 1/2} |\xi|^{-|\beta|} \left| \mathcal{F} (w^{-1} f)(\xi) \right|^q d\xi \right)^{1/q} + C_8 \omega_n \left( f, \left\| M^{-j} \right\| \right)_{p,1/w} + C_9 \sum_{0 < |\beta| \leq n} \omega_n \left( \frac{D^{\beta} w}{w} f, \left\| M^{-j} \right\| \right)_{p,1/w} + C_{10} \sum_{|\beta| = n+1} m^{-j/p} I_{4,\beta}(f_j, w_j),
\]

where \( f_j(t) = f(M^{-j} t) \), \( w_j(t) = w(M^{-j} t) \).

To estimate the modulus \( \omega_n (\frac{D^{\beta} w}{w} f, \left\| M^{-j} \right\|)_{p,1/w} \), we use the following well-known relations:

\[
\Delta_h^n (f_1 f_2) = \sum_{\nu=0}^n \binom{n}{\nu} \Delta_h^{-\nu} (f_1) \Delta_h^{n-\nu} (f_2),
\]

\[
\omega_{\nu} (f_1, h) \leq C(\nu) h^{\nu} \left\| f_1 \right\|_{W_\infty^\nu}, \quad f_1 \in W_\infty^\nu,
\]

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which imply that
\[
\omega_n \left( \frac{D^β w}{w} f, \| M^{-j} \| \right)_{p,1/w} \leq \sum_{ν=0}^{n} \left( \frac{n}{ν} \right) \omega_ν \left( \frac{D^β w}{w}, \| M^{-j} \| \right) \omega_{n-ν} \left( f, \| M^{-j} \| \right)_{p,1/w} \leq C_{11} \sum_{ν=0}^{n} \| M^{-j} \| ^ν \omega_{n-ν} \left( f, \| M^{-j} \| \right)_{p,1/w}.
\]

(64)

To estimate \( I_{4,β}(f_j, w_j) \), \( |β| = n + 1 \), we note that
\[
|D^β w_j(t + η(t + k))| \leq \| M^{-j} \| ^{n+1} |D^β w( M^{-j} t + η M^{-j} (t + k))| \\
\leq c_{w,β} \| M^{-j} \| ^{n+1} w(M^{-j} t + η M^{-j} (t + k)) \\
\leq c_{w,β} c_w \| M^{-j} \| ^{n+1} w_j(t)(1 + |t + k|^α).
\]

Thus, using Proposition 21 and Proposition 3, taking into account that the function \((1 + |t|^α)|t^β \varphi(t)|\) belongs to \( L_{q,w^*} \), we obtain
\[
I_{4,β}(f_j, w_j) \leq c_{w,β} c_w \| M^{-j} \| ^{n+1} \left\| \sum_{k \in Z^d} \left( \frac{1}{w_j(k)} \int_{\mathbb{R}^d} |f_j(t)|(1 + |t + k|^α)(t + k)^β \varphi_0(t) |dt| \varphi_0 \right) \right\|_{p,1/w_j} \\
\leq C_{12} \| M^{-j} \| ^{n+1} \left( \sum_{k \in Z^d} \left( \frac{1}{w_j(k)} \int_{\mathbb{R}^d} |f_j(t)|(1 + |t + k|^α)(t + k)^β \varphi(t) |dt| \varphi \right) \right)^{1/p} \\
\leq C_{13} \| M^{-j} \| ^{n+1} \| f_j \|_{p,1/w_j} = C_{13} \| M^{-j} \| ^{n+1} \| f \|_{p,1/w}.
\]

Combining this with (62) and (64), we get (57).

État 34 Analysing the proof of Theorem 31, it is clear that inequality (57) remains valid for any \( 2 ≤ p ≤ ∞ \) under the assumption \( \varphi ∈ L_{p,w^*} \cap L_2 \) (instead of \( \varphi ∈ B \)). In this case, the condition \( w^{-ν} ∈ A_ν \) can be dropped and it suffices to assume that \( α > 0 \).

État 35 The first term in the right hand side of (57) can be replaced by
\[
C_1 \sum_{ν=0}^{n} \| M^{-j} \| ^ν \omega_n - ν \left( \frac{f}{w}, \| M^{-j} \| \right) ^q _p.
\]

Indeed, it follows from (63) that for any \( µ = 1, \ldots, n \) and \( δ = \| M^{-j} \| \)
\[
\omega_µ \left( f, δ \right) _{p,1/w} = \sup_{|t| ≤ δ} \| \frac{Δ^µ f}{w} \| _p ≤ \sum_{ν=0}^{µ} \left( \frac{µ}{ν} \right) \sup_{|t| ≤ δ} \| \frac{Δ^ν f}{w} \| _∞ \omega_{µ-ν} \left( \frac{f}{w}, δ \right) _p.
\]

Using Taylor’s formula with Lagrange’s remainder, we have
\[
\left| \frac{Δ^ν f(x)}{w(x)} \right| ≤ C(ν) \sum_{l=1}^{ν} \sum_{|β|=ν} w^{-1}(x) D^β w(x + θ_l t) |t|^ν,
\]
where \( |θ_l| ≤ ν \). It remains to note that, due to properties of \( w \),
\[
\left| \frac{D^β w(x + θ_l t)}{w(x)} \right| ≤ w^*(θ_l t) \left| \frac{D^β w(x + θ_l t)}{w(x + θ_l)} \right| ≤ C(w, n) < ∞.
\]

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Corollary 36  Under the assumptions of Theorem 31, the following estimate holds for $j \rightarrow \infty$

\[
\left\| f - m^{-j/2} \sum_{k \in \mathbb{Z}^d} f(-M^{-j}k) \varphi_{jk} \right\|_{p,1/w} = \begin{cases} O(|\lambda|^{-j(d/p+a)}) & \text{if } n > d/p + a, \\ O(|\lambda|^{-jn}) & \text{if } n = d/p + a, \\ O(|\lambda|^{-jn^{1/2}}) & \text{if } n < d/p + a, \end{cases}
\]

where $\lambda$ is the smallest (in absolute value) diagonal element of $M$.

Proof. Obviously,

\[
\|M^{-j}\|_{p,q} \int_{|M^{-j} \xi| \geq \delta} |\xi|^q \left| \mathcal{F} \left( \frac{f}{w} \right)(\xi) \right|^q d\xi = O \left( |\lambda|^{-jn(a+d/p)} \right), \quad j \rightarrow \infty.
\]

It remains to estimate the first term in the right hand side of (57). Due to Remark 35, it suffices to estimate the sum $\sum_{\nu=0}^n \delta^{\nu q} \omega_{n-\nu} (w^{-1} f, \delta)^q p$,.

Set

\[
V_\sigma f(x) = \mathcal{F}^{-1} \left( v \left( (\sigma^{-1} |\xi|) \right) \tilde{f} \right)(x),
\]

where $v \in C^\infty(\mathbb{R})$, $v(\xi) \leq 1$, $v(\xi) = 1$ for $|\xi| \leq 1/2$ and $v(\xi) = 0$ for $|\xi| \geq 1$. Using Pitt’s inequality (see, e.g., [8, inequality (1.1) for $s = 0$ and $p = q$]), we have

\[
\|\tilde{g}\|_{L_p(\mathbb{R}^d)} \leq C(p) \left( \int_{\mathbb{R}^d} |x|^d |g(x)|^p dx \right)^{1/p}, \quad 2 < p < \infty.
\]

It follows that

\[
E_{B_\sigma}(w^{-1} f)_p \leq \|w^{-1} f - V_\sigma (w^{-1} f)\|_p
\]

\[
\leq C(p) \left( \int_{|\xi| \leq \sigma/2} |\xi|^d (1 - v(\sigma^{-1} |\xi|)) \mathcal{F}(w^{-1} f)(\xi)|^p d\xi \right)^{1/p} = O \left( \sigma^{-(d/p+a)} \right), \quad \sigma \rightarrow \infty.
\]

If $n \geq d/p + a$, using the following Marchaud inequality (see [9]):

\[
\omega_\nu(w^{-1} f, 2^{-N})_p \leq C 2^{-\nu N} \left( \sum_{k=0}^N 2^{2\nu k} E_{B_{2^k}}(f) \right)^{1/2},
\]

we obtain

\[
\left( \sum_{\nu=0}^n \delta^{\nu q} \omega_{n-\nu} (w^{-1} f, \delta)^q p \right)^{1/2} = \begin{cases} O(\delta^{a+d/p}), & n > d/p + a, \\ O(\delta^n \log^{1/2}(\delta^{-1})), & n = d/p + a. \end{cases}
\]

If now $n < a + d/p$, then $d + a > n + d/q$, and hence the function $|\xi|^n \mathcal{F}(w^{-1} f)(\xi)$ belongs to $L_q$. It follows from the Hausdorff-Young inequality that $w^{-1} f \in W^p_q$, which yields that

\[
\omega_\nu(w^{-1} f, \delta)_p = O(\delta^\nu), \quad \delta \rightarrow 0,
\]

for any $1 \leq \nu \leq n$, which proves the corollary.  \(\diamond\)
Corollary 37 Let $p$, $w$, $n$, $M$, and $\varphi$ be as in Theorem 31. If $w^{-1} f \in L_p \cap L_1$ and $\omega_n(w^{-1} f, \delta)_1 = O(\delta^{d+a})$ as $\delta \to 0$, where $a \in (0, n-d)$, then
\[
\left\| f - m^{-j/2} \sum_{k \in \mathbb{Z}^d} f(-M^{-j} k) \varphi_{jk} \right\|_{p,1/w} = O \left( |\lambda|^{-j(a+d/p)} \right), \quad j \to \infty,
\]
where $\lambda$ is the smallest (in absolute value) diagonal element of $M$.

Proof. Applying the estimate
\[
|\mathcal{F} (w^{-1} f) (\xi)| \leq C \omega_n (w^{-1} f, |\xi|^{-1})_1 = O \left( |\xi|^{-d(a+d/p)} \right),
\]
which can be found, e.g., in [32], we see that all assumptions of Theorem 31 are satisfied. It is also obvious that the required estimate holds for the second term in the right hand side of (57). To estimate the first term, we can use the well-known embedding for the Besov spaces (see, e.g., [1, 6.5.1 and 6.2.5]):
\[
B^{\alpha+d}_{1,\infty}(\mathbb{R}^d) \subset B^{\alpha+d/p}_{p,\infty}(\mathbb{R}^d),
\]
which implies that $\omega_\nu(w^{-1} f, \delta)_p = O(\delta^{\alpha+d/p})$ for all integer $\nu \in (a+d/p, n]$. To estimate the moduli of smoothness of order $\nu \in [1, a+d/p]$, we can use the Marchaud inequality (65) and the Jackson inequality given in Theorem 17. This provides the following estimate
\[
\sum_{\nu=0}^{n} \delta^{\nu q} \omega_n - \nu (w^{-1} f, \delta)_p^q = O \left( \delta^{q(a+d/p)} \right),
\]
which together with Remark 35 proves the corollary. ☐

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