Transcendental Continued $\beta$-Fraction with Quadratic Pisot Basis over $F_q((x^{-1}))$

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Abstract. Let $F_q$ be a finite field and $F_q((x^{-1}))$ is the field of formal power series with coefficients in $F_q$. Let $\beta \in F_q((x^{-1}))$ be a quadratic Pisot series with $deg(\beta) = 2$. We establish a transcendence criterion depending on the continued $\beta$-fraction of one element of $F_q((x^{-1}))$.

1. Introduction

Let $F_q$ be a finite field of characteristic $q \geq 0$, $F_q[x]$ is the ring of polynomials with coefficients in $F_q$ and $F_q((x^{-1}))$ is the field of formal power series of the form:

$$f = \sum_{k \geq l} f_k x^{-k}, \quad f_k \in F_q, l \in \mathbb{Z},$$

where $l = deg(f)$ and by convention $deg(0) = -\infty$. We define the absolute value $|f| = q^{deg(f)}$ if $f \neq 0$, $|f| = 0$ otherwise. This absolute value is not archimedean over $F_q((x^{-1}))$.

Let $\beta \in F_q((x^{-1}))$ with $|\beta| > 1$. The continued $\beta$-fraction is a generalization of classic continued fraction with formal power series basis. Similar to the real case, the $\beta$-expansion of a formal power series $f$ is a unique representation $f = \sum d_i \beta^{-i}$ where $n \in \mathbb{Z}$ and $(d_i)_{i \geq n}$ is a polynomial sequence such that $deg(d_i) < deg(\beta)$ for all $i \geq n$ (see [4]). Thus, $f$ has a unique decomposition as follows $f = \sum_{i=0}^{N} d_i \beta^i + \sum_{i \in \mathbb{N}^*} d_{-i} \beta^{-i}$. Note that

$$\sum_{i=0}^{N} d_i \beta^i = [f]_{\beta}$$ is called $\beta$-polynomial part of $f$ and $\sum_{i \in \mathbb{N}^*} d_{-i} \beta^{-i} = \{f\}_{\beta}$ is called $\beta$-fractional part of $f$. If $\{f\}_{\beta} = 0$, then $f$ is $\beta$-polynomial. The $\beta$-polynomial’s set is denoted $F_q[x]_{\beta}$.

Let $f \in F_q((x^{-1}))$. Consider the transformation $T_{\beta}$ defined over $F_q((x^{-1}))$ by $T_{\beta}(f) = \frac{1}{f} - \lfloor \frac{1}{f} \rfloor_{\beta}$. The continued

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In this paper, we give a transcendence criterion of continued $\beta$-sequence $(f)$. Similarly, let $f$ be the $\beta$-fraction of $f$ has the following form: Set $f = f_0$,

$$f = [f_0]_\beta + T_\beta \left( \frac{1}{f_0} \right)$$

$$= [f_0]_\beta + \frac{1}{f_1}$$

$$= [f_0]_\beta + \frac{1}{\left[ f_1 \right]_\beta + T_\beta \left( \frac{1}{f_1} \right)}$$

$$= \left[ f_0 \right]_\beta + \frac{1}{\left[ f_1 \right]_\beta + \frac{1}{\left[ f_2 \right]_\beta + \frac{1}{\ldots + \frac{1}{[\ldots + \frac{1}{f_{n+1} + T_\beta \left( \frac{1}{f_{n+1}} \right)]} \ldots}}} \right]_\beta$$

where for all $i \geq 0$ the equality $\frac{1}{f_{i+1}} = T_\beta (\frac{1}{f_i})$ is satisfied if $f_i \notin F_\beta [x]$. Otherwise, the algorithm ends, and the sequence $(f_i)$ is finite.

In this paper, we give a transcendence criterion of continued $\beta$-sequence with quadratic Pisot series basis $\beta$ over $F_\beta((x^{-1}))$. This result has been developed on several researchers works [1, 7, 8], which contributes to prove Khintchine conjecture [5].

In [1] Baker proved that if $[a_0, a_1, a_2, \ldots ]$ is classic continued fraction of real number $x$ such that $a_n = a_{n+1} = \ldots = a_{n+\alpha(n)-1}$, where $\alpha(n)$ is a sequence of integers satisfying certain increasing properties, then $x$ is transcendental. In 2004, Mkaouar [7] built an other transcendence criterion of the classic continued fraction over $F_\beta((x^{-1}))$.

This paper is organized as follows. In section 2, basic arithmetics properties of continued $\beta$-fraction with Pisot series basis are introduced in $F_\beta((x^{-1}))$. In section 3, the main results are proved.

2. Continued $\beta$-fraction with Pisot series basis over $F_\beta((x^{-1}))$

Let $\beta \in F_\beta((x^{-1}))$ with $|\beta| > 1$ and $f \in F_\beta((x^{-1}))$, we have $f = [f]_\beta + \{f\}_\beta$.

If $\lambda_0 = [f]_\beta$. Then, we get

$$f = \lambda_0 + \frac{1}{\lambda_1 + \frac{1}{\lambda_2 + \ldots}}$$

where $\lambda_i = \left[ \frac{1}{f_{i+1}} \right]_\beta$ for all $i \geq 1$ and the map $T_\beta : f \rightarrow \{f\}_\beta$. The previous continued $\beta$-fraction of $f$ is denoted by $[\lambda_0, \lambda_1, \lambda_2, \ldots]_\beta$. The sequence $(\lambda_i)_{i \geq 0}$ is called the partial $\beta$-quotients of $f$ and the expansion $[\lambda_0, \lambda_1, \lambda_2, \ldots]_\beta$ is called the $n^{th}$-complete $\beta$-quotients of $f$, denoted by $f_n$. Similarly, let $f = [\lambda_0, \lambda_1, \ldots, \lambda_n, \ldots]_\beta$, we define two sequences $(P_n)_{n \geq 0}$ and $(Q_n)_{n \geq 0} \in F_\beta [x, \beta^{-1}]$ as follows:

$$\begin{align*}
P_0 &= \lambda_0, & P_1 &= \lambda_0 \lambda_1 + 1, & \text{and} & \{ P_n = \lambda_n P_{n-1} + P_{n-2} \\
Q_0 &= 1, & Q_1 &= \lambda_1, & Q_n &= \lambda_n Q_{n-1} + Q_{n-2} & \forall n \geq 2.
\end{align*}$$

Recall that $F_\beta [x]_\beta$ is not stable under usual multiplication. Let

$L_\phi = \{ n \in \mathbb{N} | \forall P_1, P_2 \in F_\beta [x]_\beta; d_\phi (P_1, P_2) \text{ is finite } \Rightarrow \beta^n P_1 P_2 \in F_\beta [x]_\beta \}$.

In [3], the value of $L_\phi$ is already calculated for some Pisot series over $F_\beta((x^{-1}))$.

**Corollary 2.1.** [3] Let $\beta$ be a quadratic Pisot series with $\deg(\beta) \geq 2$. Then, $L_\phi = 1$. 

3. Results

Let $\beta$ be a quadratic Pisot series with $\deg(\beta) = 2$. For $P = a_0 + a_1 \beta + \ldots + a_n \in \mathbb{F}_q[x]_{\beta}$. Define $\gamma(P)$ as the $\beta$-degree of $P$ as follows:

$$\text{deg}(\beta) \text{deg}(P) + \text{deg}(a_n) = 2s + \text{deg}(a_s).$$

In this case, we have $|P| = q^{|P|}$. 

**Theorem 3.1.** Let $\beta$ be a quadratic Pisot series with $\deg(\beta) = 2$ and $f$ is a formal power series such that its continued $\beta$-fraction is $[\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots]$. If

$$\log(\sum_{i=1}^{n} \gamma(\lambda_i))$$

$$\limsup_{n \to +\infty} \frac{\gamma(\lambda_i)}{n} = +\infty,$$

then $f$ is transcendental.

First, we use the following proposition and lemma.

**Proposition 3.2.** Let $\beta$ be a quadratic Pisot series with $\deg(\beta) = 2$. If $f$ is a formal power series such that its $n^{th}$ $\beta$-convergent is $\frac{P_n}{Q_n}$, then

$$(\beta^2 P_n) \in \mathbb{F}_q[x]_{\beta}, \quad (\beta^2 Q_n) \in \mathbb{F}_q[x]_{\beta}$$

and

$$\beta^{2s+1} (P_{n+1} Q_n - P_n Q_{n+1}) \in \mathbb{F}_q[x]_{\beta}.$$ 

**Proof.** According to Corollary 2.1, if $A_1, \ldots, A_n \in \mathbb{F}_q[x]_{\beta}$, then $\beta^2 (A_1 A_2 \ldots A_n) \in \mathbb{F}_q[x]_{\beta}$.

**Lemma 3.3.** Let $\beta$ be a quadratic Pisot series with $\deg(\beta) = 2$ and $f$ is an algebraic series of algebraic degree $d$. Then, there exists $c = c(f) > 0$ such that, for all $P, Q \in \mathbb{F}_q[x]_{\beta}$,

$$|\frac{P}{Q} - f| > \frac{c}{|\beta|^{\frac{d+1}{2}} |Q|^d}.$$ 

**Proof.** Similar to Liouville’s inequality [6]. Let $K(y) = A_d y^d + \ldots + A_0 \in \mathbb{F}_q[y]$ be an irreducible polynomial of $f$ such that $K(f) = 0$. According to Proposition 3.2, for all $P, Q \in \mathbb{F}_q[x]_{\beta}$ we get

$$\beta^{2s} Q^{d} K(\frac{P}{Q}) \in \mathbb{F}_q[x]_{\beta}.$$ 

So

$$|K(\frac{P}{Q})| \geq \frac{1}{|\beta|^{\frac{d+1}{2}} |Q|^d}.$$ 

As $K(f) = 0$, then

$$|K(\frac{P}{Q})| = |K(\frac{P}{Q}) - K(f)| \leq |\frac{P}{Q} - f| \max_{1 \leq i \leq d} |A_i f^{i-1}|.$$ 

Let $c_1 = \max_{1 \leq i \leq d} |A_i f^{i-1}|$ and $c = \min(1, \frac{1}{c_1})$. We obtain

$$|\frac{P}{Q} - f| \geq \frac{1}{c_1 |\beta|^{\frac{d+1}{2}} |Q|^d} > \frac{c}{|\beta|^{\frac{d+1}{2}} |Q|^d}.$$ 

\□
Proof. of Theorem 3.1
Let \((P_n/Q_n)_{n\in\mathbb{N}}\) be a \(\beta\)-convergent sequence of \(f\). Similar to the classical case,

\[
|f - P_n/Q_n| = \left| P_{n+1}Q_n - P_nQ_{n+1} \right|/Q_{n+1}Q_n.
\]

By Proposition 3.2, we obtain

\[
|\beta^{2n} (f - P_n/Q_n)| < \left| \beta \right|^{2n}/|Q_{n+1}|/|Q_n|.
\]

If \(f\) is an algebraic series of algebraic degree \(d > 1\). Then, according to Lemma 3.3, there exists \(c > 0\) such that

\[
|Q_{n+1}| < \frac{1}{c} |\beta|^{nd+2n+2} |Q_n|^{d-1}.
\]

Let \(c_1 = \frac{1}{c}\). Then,

\[
|Q_{n+1}| < c_1^{(d-1)n} |\beta|^{nd+2n+2} |Q_1|^{(d-1)n}.
\]

This implies,

\[
|Q_n| < c_1^{(d-1)n} |\beta|^{nd+2n+2} |Q_1|^{(d-1)n}.
\]

Moreover,

\[
\sum_{i=1}^{n} \gamma(\lambda_i) < (d-1)n \left[ \log(c_1) + \log(|\beta|^{nd+2n+2}) + \log(|Q_1|) \right].
\]

Hence,

\[
\log(\sum_{i=1}^{n} \gamma(\lambda_i)) < n \log(d-1) + \log(\log(c_1) + \left( \frac{n(d+2) + 2}{2} \right) \log(|\beta|) + \log(|Q_1|)),
\]

which is a contradiction with the fact that

\[
\limsup_{n \to +\infty} \frac{\log(\sum_{i=1}^{n} \gamma(\lambda_i))}{n} = +\infty.
\]

\(\square\)

**Theorem 3.4.** Let \(\beta\) be a quadratic Pisot series with \(\text{deg}(\beta) = 2\) and \(f\) is a formal power series such that its continued \(\beta\)-fraction \([B_1, B_2, B_3, \ldots]_{\beta}\) is not ultimately periodic, where \(B_i\) are finite blocks beginning with the repetition \(n_i\)-times of same partial \(\beta\)-quotient \(\lambda\) and \(\gamma(\lambda) > 1\). We denote by \(d_i\) the sum of \(\beta\)-degrees of \(B_i\)'s terms. If

\[
\liminf_{i \to +\infty} \frac{\sum_{j=1}^{i-1} d_j}{n_i} = 0,
\]

then \(f\) is transcendental.

The proof of Theorem 3.4 requires the following lemmas indeed.

Let \(f\) be an algebraic formal power series of minimal polynomial \(P(Y) = A_nY^n + \ldots + A_0\) where \(A_i \in F_q[x]\). Set \(H(f) = \max |A_i|\) and \(\sigma(f) = A_m\).
Lemma 3.5. Let \( \beta \) be a quadratic Pisot series with \( \text{deg}(\beta) = 2 \). If \( f \) is an algebraic series of algebraic degree \( d \) such that

\[
f = \lambda_1 + \frac{1}{\lambda_2 + \frac{1}{\ddots + \frac{1}{\lambda_1 + \frac{1}{R}}}}.
\]

where \( \lambda_i \in \mathbb{F}_q[x], h \in \mathbb{F}_q((x^{-1})) \) and \( |h| > 1 \). Then \( h \) is an algebraic series of algebraic degree \( d \) and

\[
H(h) \leq H(f)|\beta|^{\frac{d}{2}} \prod_{i=1}^{d} |\lambda_i|^d.
\]

Proof. Assume that \( f \) is an algebraic series of algebraic degree \( d \). Then,

\[
A_d f^d + A_{d-1} f^{d-1} + \ldots + A_0 = 0,
\]

where \( A_i \in \mathbb{F}_q[x] \).

If \( f = \lambda_1 + \frac{1}{R} \) such that \( \lambda_1 \in \mathbb{F}_q[x], h \in \mathbb{F}_q((x^{-1})) \) and \( |h| > 1 \). Thus, according to Proposition 3.2, we obtain

\[
\beta^d h^d (A_d(\lambda_1 + \frac{1}{R})^d + A_{d-1}(\lambda_1 + \frac{1}{R})^{d-1} + \ldots + A_0) = 0.
\]

This implies,

\[
B_0 h^d + B_{d-1} h^{d-1} + \ldots + B_0 = 0 \quad (\star)
\]

where

\[
B_{d-k} = \beta^\frac{d}{2} \sum_{j=0}^{d} \binom{d}{j} A_j \lambda_1^{j-k} \in \mathbb{F}_q[x].
\]

Let \( P = a_0 \beta^d + \ldots + a_1 \beta + a_0 \in \mathbb{F}_q[x], \) we denote by \( TC_\beta(P) = a_0 \). Then, by \( (\star) \) we have

\[
TC_\beta(B_d)h^d + TC_\beta(B_{d-1})h^{d-1} + \ldots + TC_\beta(B_0) = 0.
\]

Thus, we get

\[
|TC_\beta(B_d)| = |TC_\beta(\beta^d \sum_{j=0}^{d-1} \binom{d}{j} A_j \lambda_1^j + \binom{d}{0} A_d \beta^\frac{d}{2} \lambda_1^d)|
\]

\[
= |TC_\beta(\binom{d}{0} A_d \beta^\frac{d}{2} \lambda_1^d)|
\]

\[
\leq |A_d| |\beta^\frac{d}{2} \lambda_1^d|
\]

\[
\leq H(f)|\beta|^\frac{d}{2} |\lambda_1|^d
\]

and for \( k \geq 1 \),

\[
|TC_\beta(B_{d-k})| \leq |TC_\beta(B_d)| \leq H(f)|\beta|^\frac{d}{2} |\lambda_1|^d.
\]

Hence,

\[
H(h) \leq \sup_{0 \leq k \leq d} |TC_\beta(B_k)| \leq H(f)|\beta|^\frac{d}{2} |\lambda_1|^d.
\]

Consequently, if \( f = [\lambda_1, \lambda_2, \ldots, \lambda_s, h] \) where \( \lambda_i \in \mathbb{F}_q[x], h \in \mathbb{F}_q((x^{-1})) \) and \( |h| > 1 \), and set \( f_t = [\lambda_t, \lambda_{t+1}, \ldots, \lambda_s, h] \).

Then, we get by iterating the last case

\[
H(h) \leq H(f)|\beta|^\frac{d}{2} \prod_{t=1}^{s} |\lambda_t|^d. \quad \square
\]
Lemma 3.6. Let \( f \) be an algebraic series of algebraic degree \( d \) and \( g \) is a formal power series such that its continued \( \beta \)-fraction is purely periodic with its period is \( \text{Per}(g) = 1 \). We denote by \( \sigma(g) = Q_1 \) such that \( \frac{P_1}{Q_1} \) is the first \( \beta \)-convergent of \( g \). If \( f \neq g \), then
\[
|f - g| \geq \frac{1}{|\sigma(g)|^{(2d-1)H(f)^2}}.
\]

Proof. Combine Lemma (2) in [7] and Proposition 3.2, we get \( |f - g| \geq \frac{1}{|\sigma(g)|^{(2d-1)H(f)^2}}. \)

Proof. of Theorem 3.4
Assume that \( f \) is an algebraic series of algebraic degree \( d \) such that its continued \( \beta \)-fraction \( [B_1, B_2, B_3, \ldots]_\beta \) is not ultimately periodic where, for all \( i \geq 1 \), \( B_i \) is finite block which begins with \( n_i \)-times of \( \lambda \in \mathbb{F}_q[x] \) and \( \gamma(\lambda) > 1 \). We denote by \( g = [\lambda, \lambda, \lambda, \ldots]_\beta \) and we set \( f_i = [B_i, B_{i+1}, B_{i+2}, \ldots]_\beta \) such that its \( n_i \)-th \( \beta \)-convergent is \( \frac{P_n}{Q_n} \). As \( f_i \) and \( g \) have same first \( n_i \)-terms of their continued \( \beta \)-fraction, then according to Proposition 3.2, we have
\[
|\beta|^{-\frac{n_i}{2}} |f_i - g| \leq \sup\{|\beta|^{-\frac{n_i}{2}} |f_i - \frac{P_n}{Q_n}|, |\beta|^{-\frac{n_i}{2}} |g - \frac{P_n}{Q_n}|\}
\]
\[
\leq \frac{|\beta|^{-\frac{n_i}{2}}}{|Q_n|^2}.
\]
By Lemma 3.6, we obtain
\[
\frac{1}{|\beta|^{(d-2)|\sigma(g)|^{(2d-1)H(f_i)^2}}} \leq |\beta|^{-\frac{n_i}{2}} |f_i - g| \leq \frac{|\beta|^{-\frac{n_i}{2}}}{|Q_n|^2}.
\]
Thus,
\[
|Q_n|^2 \leq |\beta|^{-\frac{n_i}{2}} |\beta|^{(d-2)|\sigma(g)|^{(2d-1)H(f_i)^2}}.
\]
As \( \text{deg}(Q_n) = n_i \gamma(\lambda) - \text{deg}(g) \), then
\[
2n_i \gamma(\lambda) \leq \left( \frac{2n_i + 5}{2} \right) \text{deg}(\beta) + 2 \text{deg}(H(f_i)) + d \text{ deg}(g) + (2d - 1) \text{ deg}(\sigma(g)).
\]

Let \( \alpha = \sum_{j=1}^{i-1} \alpha_j \) with \( \alpha_j \) is the number of \( B_j \)'s terms. Then, by Lemma 3.5, we get
\[
2n_i \gamma(\lambda) \leq \text{deg}(H(f)) + \left( \frac{2n_i + 5}{2} \right) \text{deg}(\beta) + 2d \sum_{j=1}^{i-1} d_j + d \text{ deg}(g) + (2d - 1) \text{ deg}(\sigma(g)).
\]
Hence
\[
n_i (\gamma(\lambda) - 1) \leq \text{deg}(H(f)) + \left( \frac{5}{2} \right) d + \sum_{j=1}^{i-1} d_j + d \text{ deg}(g) + d \text{ deg}(\sigma(g)).
\]
Therefore, we get
\[
\liminf_{i \to +\infty} \frac{\sum_{j=1}^{i-1} d_j}{n_i} > 0.
\]
\( \square \)
Example 3.7. Let $\beta$ be a quadratic Pisot series with $\text{deg}(\beta) = 2$. Consider $f = [B_1, B_2, B_3, \ldots]_\beta$ where

$$B_i = [h\beta, \ldots, h\beta, (h + 1)\beta]_\beta,$$

with $h \in \mathbb{F}_q[X]$ and $\text{deg}(h) = 1$. We get:

$$\lim_{i \to +\infty} \frac{\sum_{j=1}^{i-1} j^i + 1}{i^i} = 0.$$

Then, by Theorem 3.4, $f$ is transcendental.

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