Test of the 4-th quark generation from the Cabibbo-Kobayashi-Maskawa matrix

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The structure of the mixing matrix, in the electroweak quark sector with four generations of quarks is investigated. We conclude that the area of the unitarity quadrangle is not a good choice as a possible measure of the CP violation. In search of new physics we analyze how the existence of the 4-th quark family may influence on the values of the Cabibbo-Kobayashi-Maskawa matrix and we show that one can test for the existence of the 4-th generation using the Jarlskog invariants of the known quarks only. The analysis based on the measured unitary triangle exhibits some tension with the assumption of three quark generations. The measurement of the unitarity triangle obtained from the scalar product of the second row/column of the CKM matrix by the complex conjugate of third row/column can provide information about the existence of the fourth generation of quarks.

Keywords: CP-conservation; Cabibbo-Kobayashi Maskawa matrix; Standard Model; fourth generation of quarks.

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1. Introduction

Some new and old experimental evidences in particle physics, represent a challenge to be studied and explained, such as the origin of the masses of the elementary particles and their hierarchy, the neutrino oscillations, the asymmetry between matter and antimatter, the presence of dark matter in the universe, etc. These issues that have to be faced, require consideration and explanations. Enlarging the Standard Model (SM) is one way to do it. The Standard Model (SM) of Elementary Particles, based on the gauge invariant field theory with the gauge group $U(1) \times SU(2)_L \times SU(3)_C$, has had an impressive phenomenological success, leading to the discovery of all the particles that were predicted by the model, and

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describing the spectra and interactions of elementary particles with great accuracy. Despite this success the SM has the former drawbacks and besides has some “anomalies” - in the experimental results. So, it has to be complemented with New Physics to explain with more precision the experimental reality. For these reasons the SM and its extensions are extensively studied in order to remove inconsistencies and to eliminate deficiencies in its description of elementary particles.

Motivated by the hypothesis, that is related with the fourth neutrino, arises the interest to study an extension of the SM, by increasing the number of families or generations and consider a fourth generation of quarks and leptons instead of the three of the SM, and investigate in this context, the new characteristics and consequences associated with the CP symmetry which is related with the asymmetry between matter and antimatter in the universe. For a recent review of the structure of the quark and lepton sector in the extended SM see Ref. 12 and a very complete set of references therein.

The problem that is related with the neutrino oscillations between the different “flavors” of them, that was observed by MiniBooNE in Fermilab in 2018, encouraged the idea of extending the number of families or flavors of the neutrinos. To explain the oscillations, the idea (which is not discarded conclusively) of a new neutrino, a “sterile” one, has been proposed, which is a possible dark matter candidate, that besides could be part of a fourth family of basic entities that compose the matter and antimatter. It has been also proposed that the heavy quarks of the fourth generation are contained in the dark matter as has been also explored.

The extension of the Standard Model by adding an additional family has been studied before (for a broad review, see Ref. 24 and a more recent update). Various phenomenological analysis examined the possibility of existence of the fourth quark family and its compatibility with experimental data. A very detailed study of the phenomenological consequences of the existence of the fourth generation is contained in Ref. 28. We use information from that paper to study a possibility of the detection of the fourth generation from the elements of the CKM matrix of the known quarks.

The structure of our paper is the following. The first two sections of the paper contain introductory material and define the notation, the next section is devoted to the description of the CKM matrix for the extended model. Next, we discuss the unitarity of the CKM matrix and the unitarity quadrangle. Finally we obtain the predictions for the matrix of the Jarlskog invariants for the known quarks in the extended model and find out how it deviates from the Standard Model. The paper is concluded by summarizing our results and the Appendices, where we include some details of the calculations.

2. The Mixing Matrix in the Standard Model

As we know, the SM has been notably successful (though the values of the mixing parameters are not predicted by the model, neither the masses of the elementary
quarks), on the basis that there are no experiments that are incompatible with the model. The generation of masses in the fermionic sector of the SM (quarks and leptons) arises through the Yukawa interactions with the Higgs doublet, using the mechanism of spontaneous symmetry breaking. The masses, are obtained then, by a diagonalization of the $3 \times 3$ matrix in the SM-Lagrangian’s Yukawa term, and are related with the experimental results for the quark masses.

This diagonalization, due to the unitary transformation of the original hypothetical massless quark fields into the massive ones, has important consequences which reverberate, in the charged interactions of them, by the appearance of the Cabibbo-Kobayashi-Maskawa Mixing Matrix (CKM).

A good understanding of this sector is crucial for the progress in the theory of elementary particles and its extensions.

In the SM the Yukawa interactions are described by the three complex $3 \times 3$ matrices for the up and down quarks and charged leptons. The charged lepton matrix is diagonal with matrix elements proportional to the lepton masses. The matrices of the quark Yukawa couplings are not diagonal and out of 18 complex matrix elements only 10 parameters have phenomenological significance: 6 quark masses and 4 parameters of the Cabibbo-Kobayashi-Maskawa matrix, three angles and a phase which is non vanishing and implies that the CP symmetry is broken. One of the reasons for such a reduction of the number of parameters is the freedom of choice of the phases of the quark fields (rephasing freedom). This is the reason that all the observables related with the Cabibbo-Kobayashi-Maskawa matrix must be rephasing invariant.

The SM is defined by its Lagrangian. The part of the Lagrangian that corresponds to the quark Yukawa interactions has the following form

$$y_u \bar{u}_R (\phi^+ u_L) + y_d \bar{d}_R (\phi d_L) + h.c. \quad (1)$$

Here $y_u$ and $y_d$ are the Yukawa couplings for the up and down quarks, $\phi$ is the Higgs field and $u_{L,R}$, $d_{L,R}$ are the quark fields. The $y_u$ and $y_d$ are $3 \times 3$ complex matrices and they are not observable in the SM. The $y_u$ and $y_d$ can be diagonalized by biunitary transformations ($Y_{i}^{u,d}$ being their eigenvalues)

$$\text{diag}(Y_1^{u}, Y_2^{u}, Y_3^{u}) = U_R^{u} y_u U_L^{u \dagger}, \quad \text{diag}(Y_1^{d}, Y_2^{d}, Y_3^{d}) = U_R^{d} y_d U_L^{d \dagger} \quad (2)$$

and the quark fields are transformed by the unitary transformations $U_{L,R}^{u,d}$. Upon this transformation and after the spontaneous symmetry breaking the terms of the Yukawa Lagrangian are transformed into the quark mass terms with the quark masses equal to $m_{i}^{u,d} = Y_{i}^{u,d} v / \sqrt{2}$ ($v$ is the Higgs field vacuum expectation value) and the charged current ceases to be diagonal and instead it is described by the
Cabibbo-Kobayashi-Maskawa (CKM) $V_{\text{CKM}}$

\[ \mathcal{L}_{\text{QW}} = -\frac{g}{\sqrt{2}} (\bar{u}_L, \bar{c}_L, \bar{t}_L) \gamma^\mu W^\mu_{\mu} V_{\text{CKM}} \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix} + \text{h.c.} \]
\[ V_{\text{CKM}} = U_U^u (U_d^l)^\dagger = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}. \]

The matrices $U_U^{u,d}_{L,R}$ in Eq. (2) are not equal for the up and down quarks and there is a rephasing freedom for the quark fields, which is the reason of the rephasing freedom of the CKM matrix. The quark masses (or $Y_{u,d}^i$) and the elements of the CKM matrix are measured observables in the SM.

The elements $V_{ij}$ of the $V_{\text{CKM}}$ matrix do represent the probability of transformation between quarks or interactions between them, which is allowed by the weak interaction.

The most frequently used representations of the mixing matrix are the following:

1. In terms of the angles $\theta_{12}, \theta_{13}, \theta_{23}$, and the phase $\delta$ the matrix $V_{\text{CKM}}$ can be represented as the product of the following three rotations, where $c_{ij} \equiv \cos \theta_{ij}, s_{ij} \equiv \sin \theta_{ij}$ are tagged by $i,j = 1,2,3$; and $\delta$ is the phase.

\[ R_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}, \quad R_{13} = \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix}, \quad R_{12} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]

as

\[ V_{\text{CKM}} = R_{23}R_{13}R_{12} \]
\[ = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}. \]

2. The Wolfenstein parametrization\textsuperscript{30} with the corrections by Buras, Lautenbacher and Ostermaier\textsuperscript{31} and the CKMfitter Group\textsuperscript{32} in terms of the parameters $A$, $\lambda$, $\rho$ and $\eta$, where

\[ s_{12} \equiv \lambda, \quad s_{23} \equiv A\lambda^2, \quad s_{13}e^{-i\delta} \equiv A\lambda^3 (\rho - i\eta), \]

is

\[ V_{\text{CKM}} \approx \begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda & A\lambda^3 (\rho - i\eta) \\ -\lambda & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3 (1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix}. \]
3. Extended Model

The subject of our analysis is the quark mixing matrix in the four families extension of the SM. The forth quark family consists of the quark doublet \((B,T)\) and the Mixing Matrix for four generations \(V_{4\text{Fam}}\) is a unitary \(4 \times 4\) matrix. After taking into account the rephasing freedom it depends on 3 phases \(\delta = \delta_{13}, \delta_{14}, \delta_{24}\) and 6 rotation angles \(\theta_{12}, \theta_{13}, \theta_{14}, \theta_{23}, \theta_{24}, \theta_{34}\). It can be represented by the product of the 6 rotation-like transformations \(R_{12}, R_{13}, R_{23}, R_{14}, R_{24}, R_{34}\) (see Ref. 33)

\[
R_{12} = \begin{pmatrix}
c_{12} & s_{12} & 0 & 0 \\
-s_{12} & c_{12} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}, \quad R_{13} = \begin{pmatrix}
c_{13} & 0 & s_{13}e^{-i\delta} & 0 \\
0 & 1 & 0 & 0 \\
-s_{13}e^{i\delta} & 0 & c_{13} & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}, \\
R_{23} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & c_{23} & s_{23} & 0 \\
0 & -s_{23} & c_{23} & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}, \quad R_{14} = \begin{pmatrix}
c_{14} & 0 & 0 & s_{14}e^{-i\delta_{14}} \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-s_{14}e^{i\delta_{14}} & 0 & 0 & c_{14} \\
\end{pmatrix}, \\
R_{24} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & c_{24} & 0 & s_{24}e^{-i\delta_{24}} \\
0 & 0 & 1 & 0 \\
0 & -s_{24}e^{i\delta_{24}} & 0 & c_{24} \\
\end{pmatrix}, \quad R_{34} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & c_{34} & s_{34} \\
0 & 0 & -s_{34} & c_{34} \\
\end{pmatrix},
\]

in the following way:

\[
V_{4\text{Fam}} = R_{34}R_{24}R_{14}R_{23}R_{13}R_{12} = \begin{pmatrix}
V_{ud} & V_{us} & V_{ub} & V_{uB} \\
V_{cd} & V_{cs} & V_{cb} & V_{cB} \\
V_{td} & V_{ts} & V_{tb} & V_{tB} \\
V_{Td} & V_{Ts} & V_{Tb} & V_{TB} \\
\end{pmatrix}.
\]

Here the product of \(R_{23}R_{13}R_{12}\) that contains only one phase is equivalent to the one obtained for three families of the SM. In order to simplify the further calculations, it is described in the following way

\[
V_{\text{CKM}} = R_{23}R_{13}R_{12} = \begin{pmatrix}
M_{11} & M_{12} & M_{13} & 0 \\
M_{21} & M_{22} & M_{23} & 0 \\
M_{31} & M_{32} & M_{33} & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix},
\]
where,
\[
\begin{align*}
M_{11} &= c_{12}c_{13}, \\
M_{12} &= s_{12}c_{13}, \\
M_{13} &= s_{13}e^{-i\delta}, \\
M_{21} &= -c_{23}s_{12} - s_{23}s_{13}e^{i\delta}c_{12}, \\
M_{22} &= c_{23}s_{12} - s_{23}s_{13}e^{i\delta}s_{12}, \\
M_{23} &= s_{23}c_{13}, \\
M_{31} &= s_{23}s_{12} - s_{13}e^{i\delta}c_{23}, \\
M_{32} &= -s_{23}c_{12} - c_{23}s_{13}e^{i\delta}s_{12}, \\
M_{33} &= c_{23}c_{13}.
\end{align*}
\]

In the case of \( R_{34}R_{24}R_{14} \) we obtain
\[
R_{34}R_{24}R_{14} = \begin{pmatrix}
    c_{14} & 0 & 0 & s_{14}e^{-i\delta_{14}} \\
    -s_{14}e^{i\delta_{14}}s_{24}e^{-i\delta_{24}} & c_{24} & 0 & s_{24}e^{-i\delta_{24}}c_{14} \\
    -s_{14}e^{i\delta_{14}}s_{34}c_{24} & -s_{34}s_{24}e^{i\delta_{24}} & c_{34} & s_{34}c_{24}c_{14} \\
    -s_{14}e^{i\delta_{14}}c_{34}c_{24} & -c_{34}s_{24}e^{i\delta_{24}} & -s_{34} & c_{34}c_{24}c_{14}
\end{pmatrix}.
\]

The following step is to evaluate the product \( V_{4\text{Fam}} = R_{34}R_{24}R_{14}V_{\text{CKM3}} \):
\[
V_{4\text{Fam}} = R_{34}R_{24}R_{14}V_{\text{CKM3}},
\]

which gives
\[
V_{4\text{Fam}} = \begin{pmatrix}
    c_{14}M_{11} & c_{14}M_{12} & c_{14}M_{13} & s_{14}e^{-i\delta_{14}} \\
    -s_{14}e^{i\delta_{14}}s_{24}e^{-i\delta_{24}}M_{11} & +c_{24}M_{21} & +c_{24}M_{22} & s_{24}e^{-i\delta_{24}}c_{14} \\
    -s_{14}e^{i\delta_{14}}s_{34}c_{24}M_{12} & -c_{34}s_{24}e^{i\delta_{24}}M_{21} & +c_{34}M_{31} & c_{34}c_{24}c_{14} \\
    -s_{14}e^{i\delta_{14}}c_{34}c_{24}M_{13} & -c_{34}s_{24}e^{i\delta_{24}}M_{22} & -s_{34}M_{32} & c_{34}c_{24}c_{14}
\end{pmatrix}.
\]

4. Explicit unitarity for \( n = 4 \)

To simplify notation, from now on, we consider that \( V = V_{4\text{Fam}} \).

The unitarity relations are described by the following 2 equations:
\[
\sum_{l=u,c,t,T} V_{lk}V_{lj}^* = \sum_{l=d,s,b,B} V_{kl}V_{jl}^* = \delta_{kj},
\]

which give 6 relations for \( k \neq j \)
\[
\sum_{j=d,s,b,B} V_{uj}V_{cj}^* = 0, \quad \sum_{j=d,s,b,B} V_{uj}V_{lj}^* = 0, \quad \sum_{j=d,s,b,B} V_{uj}V_{Tj}^* = 0, \quad \sum_{j=d,s,b,B} V_{cj}V_{lj}^* = 0, \quad \sum_{j=d,s,b,B} V_{cj}V_{Tj}^* = 0.
\]
In particular, we consider $k = u, j = c$

$$V_{ud}^* V_{cd} + V_{us}^* V_{cs} + V_{ub}^* V_{cb} + V_{uT}^* V_{cT} = 0. \quad (12)$$

As a consequence we find that

$$V_{ki} \left( \sum_{l=d,s,b,B} V_{kl} V_{jl}^* \right) V_{ji}^* = \delta_{kj} V_{ki}^* V_{ji}^*$$

$$\Rightarrow \text{Im} V_{ki} \left( \sum_{l \neq i} V_{kl} V_{jl}^* \right) V_{ji}^* = 0, \quad j \neq k. \quad (13)$$

As an example for $i = d, j = c$ and $k = u$ we have

$$V_{ud} (V_{ud}^* V_{cd} + V_{us}^* V_{cs} + V_{ub}^* V_{cb} + V_{uT}^* V_{cT}) V_{cd}^* = 0, \quad (14)$$

and

$$\text{Im} (V_{ud} V_{cs} V_{us}^* V_{cd}^* + V_{ud} V_{cb} V_{ub}^* V_{cd}^* + V_{ud} V_{cT} V_{uT}^* V_{cd}^*) = 0. \quad (15)$$

Note that each term in Eq. (15) is of the type $V_{\alpha j} V_{\beta k} V_{\alpha k}^* V_{\beta j}^*$ and by construction is invariant under the rephasing of the quark fields. Then, using the results from the Appendix B we have obtained that the imaginary part of the sum of three phase invariant elements are zero, which demonstrates that the parametrization that we use is compatible with unitarity.

5. Unitarity quadrangle

As an example, let us consider the representation of the following unitarity relation:

$$V_{ud} V_{cd}^* + V_{us} V_{cs}^* + V_{ub} V_{cb}^* + V_{uB} V_{cB}^* = 0. \quad (16)$$

which is represented in the complex plane as a quadrangle. The elements of the CKM matrix $V$ in Eq. (16) are labeled by the quark indices: $u, c, t, T$ for the $up$ quarks and $d, s, b, B$ for the $down$ quarks.
The angles $\phi_i$ in Fig. 1 are given by

$$
\phi_1 = \arg \left( \frac{V_{ud} V_{cd}^*}{V_{us} V_{cs}^*} \right),
\phi_2 = \arg \left( \frac{V_{us} V_{cs}^*}{V_{ub} V_{cb}^*} \right),
\phi_3 = \arg \left( \frac{V_{ub} V_{cb}^*}{V_{uB} V_{cB}^*} \right),
\phi_4 = \arg \left( \frac{V_{uB} V_{cB}^*}{V_{ud} V_{cd}^*} \right).
$$

The area of the quadrangle (see Appendix A) is evaluated in terms of the addition of the areas of two triangles 1, and 2 in which we decompose the quadrangle:

**Triangle 1** with sides $V_{us} V_{cs}^*$ and $V_{ud} V_{cd}^*$ with the angle $\phi_1$,

**Triangle 2** with sides $V_{ub} V_{cb}^*$ and $V_{uB} V_{cB}^*$ with the angle $\phi_3$.

The Areas of Triangles 1 and 2 are:

$$
{\text{Area (1)}} = \frac{1}{2} |V_{ud} V_{cd}^*| |V_{us} V_{cs}^*| \sin \phi_1 = \frac{1}{2} \text{Im} (V_{ud} V_{cs} V_{us}^* V_{cd}^*),
$$

$$
{\text{Area (2)}} = \frac{1}{2} \text{Im} (V_{ub} V_{uB} V_{cb}^* V_{cB}^*),
$$

and

$$
{\text{Area}_{\text{Tot}}} = \frac{1}{2} (|\text{Im} (V_{ud} V_{cs} V_{us}^* V_{cd}^*)| + |\text{Im} (V_{uB} V_{cb} V_{uB} V_{cB}^*)|).
$$

In the Appendix A we derive

$$
\text{Im} (V_{ud} V_{cs} V_{us}^* V_{cd}^*) = C_1 \sin \delta = J_{\text{Jarlskog}},
C_1 = c_{12} c_{13} s_{12} s_{13} s_{23}
$$

$$
\text{Im} V_{uB} V_{cb} V_{uB} V_{cB}^* = C_2 \text{Im} \left[ e^{-i(\delta_{14} - \delta_{24})} e^{i\delta} \right] = C_2 \sin (\delta + \delta_{24} - \delta_{14})
$$

where

$$
C_2 = a_1 a_2 a_4 a_5 s_{13} s_{23} c_{13},
$$

so

$$
{\text{Area}_{\text{Tot}}} = \frac{1}{2} (|C_1 \sin \delta| + |C_2 \sin (\delta + \delta_{24} - \delta_{14})|).
$$

Eq. (23) is compatible with the unitarity triangle area in the SM with three families, $V_{uB} = V_{cB} = 0$,

$$
\text{Area}_{\text{Tot}}^{\text{triang}} = \frac{1}{2} \text{Im} (V_{ud} V_{cs} V_{us}^* V_{cd}^*) = \frac{1}{2} J.
$$

Here $J$ is the Jarlskog invariant, which indicates, that the CP-symmetry is broken, if it is non vanishing

$$
J = c_{12} c_{13}^2 c_{23} s_{12} s_{13} s_{23} \sin \delta,
J \simeq A^2 \lambda^6 \eta.
$$

The area of the triangle (or the Jarlskog invariant) for the case of the three generations is the measure of the CP violation. For the case of four generations the situation is not so clear. If the area of the quadrangle is not zero this implies the presence of the CP violation in the CKM matrix, but the area of the quadrangle cannot be interpreted as the measure of the CP violation. This can be seen in Fig. 2.
where it is shown that the sides of the quadrangle can be ordered in six possible ways. Similar conclusion has been obtained in Ref. [37] for the four neutrino mixing. This results in three shapes of the quadrangle and the remaining three quadrangles are the mirror images. From Fig. 2 it is thus clear that the area of a unitarity triangle cannot be uniquely defined and thus it cannot serve as a measure of the CP violation.

![Fig. 2. The different shapes of a unitarity quadrangle. The numbers show how the sides are ordered.](image)

6. CKM matrix observable effects of the fourth generation

The CKM matrix is unitary by construction. This is true for three and four quark generations. The existence of the fourth generation is hypothetical and we will numerically investigate the influence of the fourth generation on the properties of the CKM matrix for three generations, which in such a case is a $3 \times 3$ submatrix of the $V_{4Fam}$ unitary matrix. If the fourth row and column of the $V_{4Fam}$ are not trivial (are not equal to (0,0,0,1)), then the $3 \times 3$ submatrix is not unitary and this has observable consequences. We will numerically analyze the $3 \times 3$ submatrix assuming that the parameters $\theta_{12}, \theta_{23}, \theta_{13}, \delta$ are those obtained from the best fit for the CKM matrix, which are (the units of angles are radians)

$$\theta_{12} = 0.2265, \quad \theta_{23} = 0.04216, \quad \theta_{13} = 0.00364, \quad \delta = 1.2405.$$  

(26)
In our numerical analysis of the \(4 \times 4\) CKM matrix we will consider the values of the hypothetical parameters \(\theta_{34}, \theta_{24}, \theta_{14}, \delta_{14}\) and \(\delta_{24}\) given in Table 1. The cases A, B, C and D were chosen in such a way that there is a hierarchy for the mixing angles \(\theta_{14}\) similar to that in the case of 3 generations and for the angle \(\theta_{34}\) we have chosen a very conservative value 0.01, which means smaller mixing that in the case of 3 generations. The values of the CP-violating phases \(\delta_{14}\) do not have to be small and we considered various combinations of their values in order to find out if it would be possible to determine which pattern of the CP violation is preferred. For the cases E, F and G the values of angles \(\theta_{14}\) and phases \(\delta_{14}\) were motivated by the results of Ref. 28.

Table 1. The values of the angles \(\theta_{34}, \theta_{24}, \theta_{14}\) and the phases \(\delta_{14}, \delta_{24}\) used for the numerical analysis of the observables of the CKM matrix for four generations. All values of the angles and phases are given in radians.

| Case   | \(\theta_{34}\) | \(\theta_{24}\) | \(\theta_{14}\) | \(\delta_{14}\) | \(\delta_{24}\) |
|--------|-----------------|-----------------|-----------------|-----------------|-----------------|
| A      | 0.01            | 0.001           | 0.0001          | 0               | 0               |
| B      | 0.01            | 0.001           | 0.0001          | 1               | 0               |
| C      | 0.01            | 0.001           | 0.0001          | 0               | 2               |
| D      | 0.25            | 0.05            | 0.02            | 1               | 1               |
| E      | 0.25            | 0.05            | 0.0125          | \(\frac{2}{3}\)\pi| \(\frac{2}{3}\)\pi|
| F      | 0.25            | 0.121           | 0.03            | 1               | 1               |
| G      | 0.25            | 0.121           | 0.03            | 1               | 1               |

The absolute values of the \(V_{3\text{Fam}}\) matrix are equal

\[
\begin{pmatrix}
0.974461 & 0.224529 & 0.00364299 \\
0.224379 & 0.97359 & 0.0421456 \\
0.00896436 & 0.041342 & 0.999105
\end{pmatrix}
\]

(27)

and let us compare this matrix with the matrices of the absolute values of the \(V_{4\text{Fam}}\) for seven cases

Case A:

\[
\begin{pmatrix}
0.974461 & 0.224529 & 0.00364299 & 0.0001 \\
0.224379 & 0.97359 & 0.0421456 & 0.001 \\
0.00896509 & 0.0413499 & 0.999054 & 0.0099983 \\
0.000552595 & 0.000582704 & 0.0100331 & 0.999949
\end{pmatrix}
\]

(28a)

Case B:

\[
\begin{pmatrix}
0.974461 & 0.224529 & 0.00364299 & 0.0001 \\
0.224379 & 0.97359 & 0.0421456 & 0.001 \\
0.00896581 & 0.0413498 & 0.999054 & 0.0099983 \\
0.000100905 & 0.000572439 & 0.0100334 & 0.999949
\end{pmatrix}
\]

(28b)

Case C:

\[
\begin{pmatrix}
0.974461 & 0.224529 & 0.00364299 & 0.0001 \\
0.224379 & 0.97359 & 0.0421456 & 0.001 \\
0.00896581 & 0.0413498 & 0.999054 & 0.0099983 \\
0.000100905 & 0.000572439 & 0.0100334 & 0.999949
\end{pmatrix}
\]

(28c)
Case D: 
\[
\begin{pmatrix}
0.974461 & 0.224529 & 0.00364299 & 0.0001 \\
0.224379 & 0.97359 & 0.0421456 & 0.001 \\
0.00896415 & 0.0413455 & 0.99055 & 0.00999983 \\
0.00141196 & 0.00839681 & 0.0100141 & 0.999949
\end{pmatrix}
\] (28d)

Case E: 
\[
\begin{pmatrix}
0.974266 & 0.224484 & 0.00364226 & 0.0199987 \\
0.225073 & 0.972149 & 0.0420917 & 0.0499692 \\
0.00854277 & 0.0486111 & 0.967746 & 0.247045 \\
0.00868513 & 0.0465875 & 0.24836 & 0.967508
\end{pmatrix}
\] (28e)

Case F: 
\[
\begin{pmatrix}
0.974385 & 0.224511 & 0.00364271 & 0.0124997 \\
0.224707 & 0.972233 & 0.0420922 & 0.0499753 \\
0.00860203 & 0.0418036 & 0.968056 & 0.247075 \\
0.00269613 & 0.051088 & 0.247149 & 0.967626
\end{pmatrix}
\] (28f)

Case G: 
\[
\begin{pmatrix}
0.974022 & 0.224427 & 0.00364135 & 0.0299955 \\
0.226267 & 0.965659 & 0.0418332 & 0.120651 \\
0.00860536 & 0.0625894 & 0.96734 & 0.245485 \\
0.00316871 & 0.11497 & 0.249982 & 0.961395
\end{pmatrix}
\] (28g)

From Eqs. (28) we see that significant differences appear only for the fourth row and column (we use 8 digits after decimal point to be able to spot the differences; the experimental precision for the measured matrix elements may be much smaller) and the \(3 \times 3\) submatrices (first three rows and columns) are compatible with Eq. (27).

One can conclude that at the level of the absolute values of the matrix elements, the fourth generation cannot be observed, due to the experimental precision of the measurements of the CKM matrix.

The analysis based on the absolute values of the CKM matrix elements is not very sensitive to phases of these matrix elements. To analyze the sensitivity of the CKM matrix to the values of the phases we will consider the Jarlskog invariants, which can be introduced for mixing in quark and lepton sectors. For 3 families the absolute values of all Jarlskog invariants are equal. This fact follows from the unitarity. For 4 families the \(3 \times 3\) CKM submatrix of observable quarks is not unitary and thus the Jarlskog invariants are not equal. For a \(3 \times 3\) matrix we can construct 9 Jarlskog invariants and they form a \(3 \times 3\) matrix. We will consider the absolute values of the Jarlskog invariants \(\Delta\) with matrix elements defined by

\[
\Delta = \begin{pmatrix}
|\text{Im}(V_{ts}V_{ub}^{*}V_{tb}^{*}V_{cs})| & |\text{Im}(V_{td}V_{ub}^{*}V_{tb}^{*}V_{cs})| & |\text{Im}(V_{td}V_{ts}V_{ub}^{*}V_{cs})| \\
|\text{Im}(V_{us}V_{tb}^{*}V_{ub}^{*}V_{ts})| & |\text{Im}(V_{ud}V_{tb}^{*}V_{ub}^{*}V_{ts})| & |\text{Im}(V_{ud}V_{ts}V_{ub}^{*}V_{cs})| \\
|\text{Im}(V_{us}V_{cd}V_{ub}^{*}V_{td})| & |\text{Im}(V_{ud}V_{cd}V_{ub}^{*}V_{td})| & |\text{Im}(V_{ud}V_{cd}V_{ts}V_{cs})|
\end{pmatrix}
\] (29)

As in the previous case we will consider seven sets of parameters defined in Table. [1]
The numerical values of the matrices $\Delta$ for these cases are equal

**Case A:**
\[
\begin{pmatrix}
3.25504 & 3.17461 & 3.17536 \\
3.17538 & 3.17504 & 3.17536 \\
3.17493 & 3.17493 & 3.17493
\end{pmatrix} \times 10^{-5}
\] (30a)

**Case B:**
\[
\begin{pmatrix}
3.25694 & 3.17539 & 3.17614 \\
3.17537 & 3.17528 & 3.17614 \\
3.17493 & 3.17493 & 3.17677
\end{pmatrix} \times 10^{-5}
\] (30b)

**Case C:**
\[
\begin{pmatrix}
3.25489 & 3.17282 & 3.1748 \\
3.17482 & 3.17447 & 3.1748 \\
3.17493 & 3.17493 & 3.17309
\end{pmatrix} \times 10^{-5}
\] (30c)

**Case D:**
\[
\begin{pmatrix}
3.25679 & 3.1736 & 3.17557 \\
3.17481 & 3.17472 & 3.17558 \\
3.17493 & 3.17493 & 3.17493
\end{pmatrix} \times 10^{-5}
\] (30d)

**Case E:**
\[
\begin{pmatrix}
5.27739 & 4.55467 & 6.92522 \\
3.22541 & 2.81132 & 6.91824 \\
3.16500 & 3.1795 & 3.16573
\end{pmatrix} \times 10^{-5}
\] (30e)

**Case F:**
\[
\begin{pmatrix}
0.57196 & 2.76940 & 0.53466 \\
3.30709 & 2.95447 & 0.53018 \\
3.16605 & 3.17511 & 3.16650
\end{pmatrix} \times 10^{-5}
\] (30f)

**Case G:**
\[
\begin{pmatrix}
8.26636 & 3.33730 & 9.05400 \\
3.55075 & 2.93561 & 9.05327 \\
3.12323 & 3.17538 & 3.12586
\end{pmatrix} \times 10^{-5}
\] (30g)

From Eqs. (30), we see that the value of the matrix element $\Delta_{11}$ is different than the remaining ones. This pattern is the same in all seven cases. The experimental value of the Jarlskog invariant $J$ quoted by the PDG is $J = (3.18 \pm 0.15) \times 10^{-5}$. The value of $\Delta_{11}$ exceeds the experimental value by more than $\sim 0.07 \times 10^{-5}$ in all the cases, but the case F, where it is much smaller. For the cases A, B, C and D all remaining matrix elements of the matrix $\Delta$ are compatible with the experimental value of $J$. For the cases E, F and G only the matrix elements in the third row are compatible with the experimental value of the Jarlskog invariant. It is worth to note that the value of $\Delta_{11}$ for the cases A, B, C and D is weakly dependent on the values of the phases $\delta_{14}$ and $\delta_{24}$. It means that a possible deviation of $\Delta_{11}$ from the remaining matrix elements does not give indication about the CP violation phases of the 4-th family.

The cases E, F and G differ significantly from those A-D. The values of the parameters $\theta_{34}$, $\theta_{24}$ and $\theta_{14}$ for the cases E, F and G, given in Table [4], were obtained
in Ref. [28] by analyzing the experimental data for the CKM matrix and they are tens or hundreds times bigger than those guessed from the hierarchy of the CKM matrix. Such a violation of the hierarchy only at the level of the fourth quark family would have a strong influence on the CKM matrix of the known quarks and there would be a significant violation of the unitarity of the CKM matrix for 3 generations.

The element $\Delta_{11}$ of the matrix $\Delta$ in Eq. (29) can be written in the following way

$$\Delta_{11} = \left| \text{Im}(V_{cs}V_{tb}V_{cb}^*V_{ts}^*) \right| = \left| (V_{cs}V_{tb}V_{cb}^*V_{ts}^*) \right| \times \left| \sin(\varphi_{cs} + \varphi_{tb} - \varphi_{cb} - \varphi_{ts}) \right|, \quad (31)$$

where $\varphi_{ij}$ are the phases of the $V_{ij}$ matrix elements of the CKM matrix. The absolute values of the matrix elements of the CKM matrix are well measured and for the comparison with the experimental data one must know the relative phases of the CKM matrix elements. The relative phases are determined from the angles of the unitarity triangles of the CKM matrix. The measured unitarity triangle follows from the scalar product of the first column of the CKM matrix by the complex conjugate of the third column

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* \quad (32)$$

and the angles of this triangle are equal

$$\phi_1 = \arg \left( \frac{V_{cd}V_{ub}^*}{V_{td}V_{tb}^*} \right) = \pi - \varphi_{cd} - \varphi_{tb} + \varphi_{cb} + \varphi_{td},$$

$$\phi_2 = \arg \left( \frac{V_{ed}V_{tb}^*}{V_{ud}V_{ub}^*} \right) = \pi - \varphi_{ed} - \varphi_{ub} + \varphi_{tb} + \varphi_{ud},$$

$$\phi_3 = \arg \left( \frac{V_{cd}V_{ub}^*}{V_{cd}V_{tb}^*} \right) = \pi - \varphi_{cd} - \varphi_{tb} + \varphi_{cb} + \varphi_{cd}. \quad (33)$$

From Eqs. (33) we see that the angles of the unitarity triangle can give the information required for the determination of the elements of the matrix $\Delta$ in Eq. (31). The unitarity triangles from which one can obtain the phases for the matrix element $\Delta_{11}$ are obtained by the scalar product of the second row of the CKM matrix by the complex conjugate of the third row (case (a)) or by the scalar product of the second column by the complex conjugate of the third column (case (b))

$$V_{cd}V_{td}^* + V_{cs}V_{ts}^* + V_{cb}V_{tb}^* \quad \text{case (a)},$$

$$V_{us}V_{ub}^* + V_{cs}V_{cb}^* + V_{ts}V_{tb}^* \quad \text{case (b)} \quad (34a)$$

and are shown in Fig. 3.

The angles of these triangles are equal

$$\psi_{1}^{(a)} = \arg \left( \frac{V_{cd}V_{ed}^*}{V_{cd}V_{ub}^*} \right) = \pi - \varphi_{cs} - \varphi_{tb} + \varphi_{ts} + \varphi_{cb},$$

$$\psi_{2}^{(a)} = \arg \left( \frac{V_{cb}V_{td}^*}{V_{cd}V_{td}^*} \right) = \pi - \varphi_{cb} - \varphi_{ed} + \varphi_{tb} + \varphi_{cd},$$

$$\psi_{3}^{(a)} = \arg \left( \frac{V_{cd}V_{ed}^*}{V_{cd}V_{ts}^*} \right) = \pi - \varphi_{cd} - \varphi_{ts} + \varphi_{td} + \varphi_{cs}. \quad (35a)$$
Fig. 3. Unitary triangles for the determination of the phase of the matrix element $\Delta_{11}$. The proportions of these triangles do not correspond to the experimental values of the CKM matrix elements; the triangles with sides, proportional to the experimental values would be very flat and the notation in the drawing might be confusing. Cases (a) and (b) are explained in text.

\[
\psi_1^{(b)} = \arg \left( -\frac{V_{cs}^* V_{cb}^*}{V_{ts}^* V_{tb}^*} \right) = \pi - \varphi_{cs} - \varphi_{tb} + \varphi_{cb} + \varphi_{ts}, \\
\psi_2^{(b)} = \arg \left( -\frac{V_{ts}^* V_{tb}^*}{V_{ub}^* V_{ub}^*} \right) = \pi - \varphi_{ts} - \varphi_{tb} + \varphi_{cb} + \varphi_{us}, \\
\psi_3^{(b)} = \arg \left( -\frac{V_{us}^* V_{ub}^*}{V_{cs}^* V_{cb}^*} \right) = \pi - \varphi_{us} - \varphi_{cb} + \varphi_{ub} + \varphi_{cs}.
\] (35b)

And we see that

\[
\Delta_{11} = |\text{Im}(V_{cs}^* V_{cb}^* V_{ts}^*)| = |(V_{cs}^* V_{cb}^* V_{ts}^*)| \times |\sin(\psi_1^{(a)})|, \\
\Delta_{11} = |\text{Im}(V_{us}^* V_{ub}^* V_{cb}^*)| = |(V_{us}^* V_{ub}^* V_{cb}^*)| \times |\sin(\psi_1^{(b)})|.
\] (36a)

\[
\Delta_{12} = |\text{Im}(V_{cd} V_{td}^* V_{tb}^*)| \times |\sin(\phi_1)| = (2.77 \pm 0.18) \times 10^{-5}, \\
\Delta_{22} = |\text{Im}(V_{ud} V_{ub}^* V_{td}^*)| \times |\sin(\phi_2)| = (3.00 \pm 0.24) \times 10^{-5}, \\
\Delta_{32} = |\text{Im}(V_{ud} V_{ub}^* V_{cd}^*)| \times |\sin(\phi_3)| = (3.21 \pm 0.25) \times 10^{-5}
\] (37)

and they are compatible within an error in the value of the Jarlskog invariant $J$ given in Ref. [1]

\[
J = (3.00^{+0.15}_{-0.09}) \times 10^{-5},
\] (38)

but the difference between $\Delta_{12}$ and $\Delta_{32}$ exceeds one standard deviation. The measurement of another unitary triangle may give a more conclusive result.
7. Results and Conclusions

We have analyzed the structure of the CKM matrix for the extension of the Standard Model by an additional generation of quarks. We analyzed the properties of the unitarity quadrangles and investigated if the area of the unitarity quadrangle might be a possible measure of the CP violation in the extended Standard Model. However the lack of the uniqueness of unitarity quadrangles does not permit to use its area as a good parameter of the CP violation.

Assuming, that the CKM mixture of the additional quark family follows the pattern in the $3 \times 3$ CKM matrix we analyze how this additional quark family changes the mixture matrix of the known quarks. We find that at least one of the Jarlskog invariant ($\Delta_{11}$) is significantly different than the remaining ones. This deviation may serve as a new test of the existence of the 4-th quark family. A possibility of such a deviation is hinted by the difference between the matrix elements $\Delta_{12}$ and $\Delta_{32}$ which exceeds one standard deviation.

Summarizing, we have analyzed what information about the fourth quark family and new physics can be obtained from the CKM matrix of the known quarks.

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Appendices

Appendix A.

Evaluation of the Quadrangle Area. Using the matrix elements of the Mixing Matrix:

$$V_{ud} = a_1 M_{11}, \quad V_{us} = a_1 M_{12}, \quad V_{ub} = a_1 M_{13},$$
$$V_{cB} = a_2 e^{-i\delta_{14}}, \quad V_{cd} = a_3 e^{(i\delta_{14} - i\delta_{24})} M_{11} + a_4 M_{21},$$
$$V_{cs} = a_3 e^{(i\delta_{14} - i\delta_{24})} M_{12} + a_4 M_{22},$$
$$V_{cb} = a_3 e^{(i\delta_{14} - i\delta_{24})} M_{13} + a_4 M_{23}, \quad V_{eB} = a_5 e^{-i\delta_{24}},$$
$$V_{td} = a_6 e^{i\delta_{14}} M_{11} + a_7 e^{i\delta_{24}} M_{21} + a_8 M_{31},$$
$$V_{ts} = a_6 e^{i\delta_{14}} M_{12} + a_7 e^{i\delta_{24}} M_{22} + a_8 M_{32},$$
$$V_{tb} = a_6 e^{i\delta_{14}} M_{13} + a_7 e^{i\delta_{24}} M_{23} + a_8 M_{33}, \quad V_{tB} = a_9,$$
$$V_{Td} = a_{10} e^{i\delta_{14}} M_{11} + a_{11} e^{i\delta_{24}} M_{21} + a_{12} M_{31},$$
$$V_{Ts} = a_{10} e^{i\delta_{14}} M_{12} + a_{11} e^{i\delta_{24}} M_{22} + a_{12} M_{32},$$
$$V_{Tb} = a_{10} e^{i\delta_{14}} M_{13} + a_{11} e^{i\delta_{24}} M_{23} + a_{12} M_{33}, \quad V_{TB} = a_{13}.$$
Here
\[ a_1 = c_{14}, \quad a_2 = s_{14} \quad a_3 = -s_{14}s_{24}, \quad a_4 = c_{24}, \quad a_5 = c_{14}s_{24}, \]
\[ a_6 = -s_{14}s_{34}c_{24}, \quad a_7 = -s_{24}s_{34}, \quad a_8 = c_{34}, \quad a_9 = c_{14}c_{24}s_{34}, \]
\[ a_{10} = -c_{24}s_{34}s_{14}, \quad a_{11} = -c_{34}s_{24}, \quad a_{12} = -s_{34}, \quad a_{13} = c_{14}c_{24}c_{34}. \]

and
\[ M_{11} = c_{12}c_{13}, \quad M_{12} = s_{12}c_{13}, \quad M_{13} = s_{13}e^{-i\delta}, \quad M_{14} = 0, \]
\[ M_{21} = -(c_{23}s_{12} + s_{23}s_{13}c_{12}e^{i\delta}), \quad M_{22} = c_{23}c_{12} - s_{12}s_{23}s_{13}e^{i\delta}, \]
\[ M_{23} = s_{23}c_{13}, \quad M_{14} = 0, \quad M_{31} = s_{23}s_{12} - s_{13}c_{12}c_{23}, \]
\[ M_{32} = -s_{23}c_{12} - c_{23}s_{13}e^{i\delta}s_{12}, \quad M_{33} = c_{23}c_{13}, \quad M_{34} = 0. \]

We obtain
\[
\text{Im} (V_{ud}V_{cs}V_{us}^{\ast}V_{cd}^{\ast}) = C_1 \sin \delta = J_{\text{Jarlskog}}, \quad C_1 = c_{12}c_{13}c_{23}s_{12}s_{13}s_{23}
\]
and
\[
V_{ub}V_{cb}V_{ub}^{\ast}V_{cb}^{\ast} = a_1a_2a_5e^{-i(\delta_{14} - \delta_{24})} \left( a_3e^{i(\delta_{14} + \delta_{24})} |M_{13}|^2 + a_4M_{13}^*M_{23} \right) = a_1a_2a_3a_5 |M_{13}|^2 + a_1a_2a_4a_5s_{13}s_{23}c_{13}e^{-i(\delta_{14} - \delta_{24} - \delta)},
\]
\[
\text{Im} [V_{ut}V_{cb}V_{ub}^{\ast}V_{cb}^{\ast}] = C_2 e^{-i(\delta_{14} - \delta_{24} - \delta)}, \quad C_2 = a_1a_2a_4a_5s_{13}s_{23}c_{13}
\]
\[
\text{Im} [V_{ut}V_{cb}V_{ub}^{\ast}V_{cb}^{\ast}] = C_2 \text{Im} e^{-i(\delta_{14} - \delta_{24})}e^{i\delta} = C_2 \sin (\delta + \delta_{24} - \delta_{14})
\]
\[
\text{Area}_{\text{Tot}} = \frac{1}{2} (|\text{Im} (V_{ud}V_{cs}V_{us}^{\ast}V_{cd}^{\ast})| + |\text{Im} (V_{ub}V_{cb}V_{ub}^{\ast}V_{cb}^{\ast})|)
\]
\[
\text{Area}_{\text{Tot}} = \frac{1}{2} (|C_1 \sin \delta| + |C_2 \sin (\delta + \delta_{24} - \delta_{14})|)
\]

Appendix B.

The explicit evaluation of the expressions 1,2,3, from our previous example, gives:
\[ V_{ud}^{\ast}V_{cd} + V_{us}^{\ast}V_{cs} + V_{ub}^{\ast}V_{cb} + V_{ub}^{\ast}V_{cb} = 0. \]

Where
\[ 1. - V_{ud}V_{cs}V_{us}^{\ast}V_{cd}^{\ast}, \quad 2. - V_{ud}V_{cb}V_{ub}^{\ast}V_{cd}^{\ast} \quad \text{and} \quad 3. - V_{ud}V_{cb}V_{ub}^{\ast}V_{cd}^{\ast}. \]

Case 1.- \( V_{ud}V_{cs}V_{us}^{\ast}V_{cd}^{\ast} \)
\[
V_{ud}V_{cs}^{\ast} = a_1 \left( a_2e^{-i(\delta_{14} - \delta_{24})} |M_{11}|^2 + a_4M_{11}^*M_{21}^* \right)
\]
\[
V_{cs}V_{us}^{\ast} = a_1 \left( a_3e^{i(\delta_{14} - \delta_{24})} |M_{12}|^2 + a_4M_{12}^*M_{22}^* \right)
\]
\[
V_{ud}V_{cs}V_{us}^{\ast}V_{cd}^{\ast} = a_1^2 (T_1 + T_2 + T_3 + T_4)
\]
\[
T_1 = a_2^2 |M_{11}|^2 |M_{12}|^2 , \quad T_2 = a_2a_4e^{-i(\delta_{14} - \delta_{24})} |M_{11}|^2 M_{22}M_{12}^*,
\]
\[
T_3 = a_3a_4e^{i(\delta_{14} - \delta_{24})} |M_{12}|^2 M_{11}M_{21}^*, \quad T_4 = a_2^2 M_{11}M_{22}M_{21}^*M_{12}^*.
\]
The results for the $T_i$ are

$$T_1 = (c_{12}c_{13}s_{12}c_{13})^2, \quad \text{Im} T_1 = 0.$$  

$$T_2 = a_3a_4 e^{-i(\delta_{14}-\delta_{24})} c_{12}^2 c_{13}^2 s_{12} c_{13} (c_{23} c_{12} - s_{12} s_{23} s_{13} e^{i\delta}).$$  

$$T_3 = a_3a_4 e^{i(\delta_{14}-\delta_{24})} s_{12}^2 c_{13}^2 [-c_{12} c_{13} (c_{23} s_{12} + s_{23} s_{13}) c_{12} e^{-i\delta}].$$  

$$T_4 = a_3^2 c_{12}^2 c_{13}^2 (b_1 + b_2 e^{i\delta}), \quad b_1 = -c_{23} s_{12} c_{12} - s_{23} s_{13} s_{12} c_{12} - 2c_{23} s_{23} s_{13}^2 e^{i\delta}.$$  

and

$$\text{Im} T_1 = 0,$$

$$\text{Im} T_2 = a_3a_4^2 c_{12}^2 s_{12} c_{13}^3 \text{Im} e^{-i(\delta_{14}-\delta_{24})} (c_{23} c_{12} - s_{12} s_{23} s_{13} e^{i\delta}),$$  

$$\text{Im} T_3 = a_3a_4 s_{12} c_{23}^2 s_{13}^2 c_{12} \text{Im} e^{i(\delta_{14}-\delta_{24})} (c_{23} s_{12} + s_{23} s_{13} c_{12} e^{-i\delta}),$$  

$$\text{Im} T_4 = a_3^2 c_{12}^2 c_{13}^2 (b_2 e^{i\delta}), \quad b_2 = c_{23} s_{23} s_{13}.$$  

Then

$$\text{Im} [V_{ud} V_{us}^* V_{cd}^* V_{cs}] / a_1^2 = -a_2 a_4 c_{12} c_{13}^2 s_{12} c_{13}^3 a_3^2 \text{Im} e^{-i\delta} - a_1 c_{12} s_{12} c_{23}^2 s_{23}^2 c_{13} a_3 a_4 \text{Im} e^{i(\delta_{14}-\delta_{24})}. \quad (B.1)$$

**Case 2.-** $V_{ud} V_{cb} V_{ub}^* V_{cd}^*$

$$V_{ud} V_{cd}^* = a_1 \left( a_3 e^{-i(\delta_{14}-\delta_{24})} |M_{11}|^2 + a_4 M_{11} M_{21}^* \right)$$

$$(V_{4Fam})_{cb} = a_3 e^{i(\delta_{14}-\delta_{24})} M_{13} + a_4 M_{23}, \quad (V_{4Fam})_{ub} = a_1 M_{13}$$

$$\text{Im} [V_{ud} V_{cb} V_{ub}^* V_{cd}^*] / a_1^2 = -a_2^2 s_{12} s_{13} s_{23} c_{12}^2 c_{13}^3 c_{23} \text{Im} e^{i\delta}$$

$$-a_3a_4 s_{13} c_{13} c_{12} \left[s_{13} c_{23} s_{12} \text{Im} e^{i(\delta_{14}-\delta_{24})} + s_{23} c_{12} \text{Im} (e^{i(\delta_{14}-\delta_{24})} e^{-i\delta}) \right]. \quad (B.2)$$

**Case 3.-** $V_{ud} V_{cb} V_{ub} V_{cd}^*$

$$V_{ud} V_{cb} V_{ub} V_{cd}^* = a_1 M_{11} a_5 e^{-i\delta_{24}} a_2 e^{-i\delta_{14}} \left(a_3 e^{i(\delta_{14}-\delta_{24})} M_{11} + a_4 M_{21}\right)^*$$

$$V_{ud} V_{cb} V_{ub} V_{cd}^*/a_1 a_2 a_5$$

$$= \left(a_3 (c_{12} c_{13})^2 + a_4 e^{i(\delta_{14}-\delta_{24})} c_{12} c_{13} (-c_{23} s_{12} - s_{23} s_{13} e^{-i\delta} c_{12}) \right)$$

$$\text{Im} [V_{ud} V_{cb} V_{ub} V_{cd}^*]$$

$$= -a_1 a_2 a_4 a_5 \text{Im} e^{i(\delta_{14}-\delta_{24})} c_{12} c_{13} (c_{23} s_{12} + s_{23} s_{13} c_{12} e^{-i\delta}). \quad (B.3)$$
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