Dynamical Behavior of an eco-epidemiological Model involving Disease in predator and stage structure in prey

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Abstract
An eco-epidemic model is proposed in this paper. It is assumed that there is a stage structure in prey and disease in predator. Existence, uniqueness and boundedness of the solution for the system are studied. The existence of each possible steady state points is discussed. The local condition for stability near each steady state point is investigated. Finally, global dynamics of the proposed model is studied numerically.

Keywords: Prey-predator model, stability, stage-structure, Disease.

1. Introduction
There are many factors that affect each of prey and predator, for example, pollution of the environment, and lack of food, predation, fishing and others. In addition to the factors heir important factor is the spread of infectious diseases between the prey alone, predator, or both. Therefore, the back of a great interest by researchers to study the effect of the spread of diseases, and this type for modeling is called eco-epidemiological, such as in 1986 Anderson and May [1] were the first who merged between it, ecology and epidemic systems, they created a prey-predator model with diseases. And there are researchers proposed and studied different prey-predator models with disease spread in prey population [2-5]. As well as, there are many papers about prey-predator model with disease for example, Bairagi et.al [6] studied prey predator model with harvest and disease. Chakraborty et al. [7] studied a ratio-dependent eco epidemic model with prey harvesting and they assumed that both the susceptible and infected prey are subjected to combine harvesting. Upadhyay and Roy [8] proposed an eco-epidemic model with simple law of mass action and modified functional response in [9]. In this work, we suggested idea eco-epidemic model describing prey-predator model with epidemic disease in the prey and involving top predator.
2. The mathematical Formulation:
In this section, the food web model consists of two compartments of predator (susceptible and infected) and a stage-structure prey in which the prey species growth logistically without of predation, while the predator decay exponentially in the absence of prey species. It is assumed that the prey population separate into two compartments: \( N_1(T) \) which represent the density of immature prey population at time \( T \), and \( N_2(T) \) that denotes to the density of mature prey population at time \( T \). Further the density of the susceptible predator at time \( T \) is denoted by \( N_4(T) \), while \( N_4(T) \) represents the density of infected predator population at time \( T \). Now in order to formulate the dynamics of such system, the following hypotheses are adopted:

1. The immature prey depends completely in its feeding on the mature prey that grows logistically with intrinsic growth rate \( r > 0 \) and carrying capacity \( k > 0 \). The immature prey individual grow up and become mature prey with growth up rate \( a_1 > 0 \). However the immature prey facing death with natural death rate \( d_1 > 0 \).
2. There is a kind of protection for the two stages of prey species from facing predation by the susceptible predator with refuge rate constant \( m_1, m_2 \in (0,1) \) respectively.
3. The susceptible predator consumed the immature prey individuals according to Holling type-II functional response with predation rate \( a_2 > 0 \), and half saturation constant \( b > 0 \). And consumed the mature prey individuals according to Holling type-II functional response with predation rate \( a_3 > 0 \) and contribute of portion of such food with conversion rate \( 0 < e_1 < 1 \). Moreover, the infected predator consumed the immature prey individuals according to lotka-volterra type of functional response with predation rate \( c > 0 \), \( c_1 \) represent the disease transmission from susceptible predator to infected predator and contributes a portion of such food with conversion rate \( 0 < e_2 < 1 \).
4. Finally, in the absence of food the susceptible predator. Facing death with natural death rate \( d_2 > 0 \) but the infected predator facing death due to disease and natural death rate \( d_4 > 0 \).

From above assumptions the system can be formulated mathematically with the following set of differential equations:

\[
\begin{align*}
\frac{dN_1}{dt} &= rN_2 \left( 1 - \frac{N_2}{k} \right) - a_1N_1 - \frac{a_2(1 - m_1)N_1N_3}{b + (1 - m_1)N_1 + (1 - m_2)N_2} \\
\frac{dN_2}{dt} &= a_1N_1 - \frac{a_3(1 - m_2)N_2N_3}{b + (1 - m_1)N_1 + (1 - m_2)N_2} \\
\frac{dN_3}{dt} &= \frac{e[a_2(1 - m_1)N_1 + a_3(1 - m_2)N_2]}{b + (1 - m_1)N_1 + (1 - m_2)N_2}N_3 - c_1N_3N_4 \\
\frac{dN_4}{dt} &= c_1N_3N_4 + e_1c(1 - m_1)N_1N_4 - d_3N_4
\end{align*}
\]

Now, by simplifying the model (1), the number of parameters is reduced by using the following dimensionless variables and parameters:

\[
\begin{align*}
t &= rT, & u_1 &= \frac{a_1}{r}, & u_2 &= s, & u_3 &= \frac{b}{k}, & u_4 &= \frac{a_3}{r}, \\
u_5 &= \frac{d_1}{r}, & u_6 &= \frac{c_1}{r}, & u_7 &= \frac{d_2}{r}, & u_8 &= \frac{c_1k}{N_3}, & u_9 &= \frac{eck}{r}, \\
u_{10} &= \frac{d_3}{r}, & x_1 &= \frac{k}{k}, & x_2 &= \frac{N_2}{k}, & x_3 &= \frac{N_3}{k}, & x_4 &= \frac{r}{r}
\end{align*}
\]

Accordingly, the dimensionless of system (1) becomes

\[
\frac{dx_1}{dt} = x_2(1 - x_2) - u_1x_1 - u_2(1 - m_1)x_3u_3 + (1 - m_1)x_1 + (1 - m_2)x_2 - (1 - m_1)x_1x_4
\]

\[
= f_1(x_1, x_2, x_3, x_4)
\]

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Clearly, the equations of system (2) are continuous and have continuous partial derivatives on the following positive 4th dim. space:

\[ R^4_+ = \{(x_1, x_2, x_3, x_4) \in R^4 : x_1 (0) \geq 0, x_2 (0) \geq 0, x_3 (0) \geq 0, x_4 (0) \geq 0\}. \]

Therefore, these equations are Lipschizian on \(R^4_+\), and hence the solution of system (2) exists and unique. Furthermore, each of the solutions of system (2) with positive initial condition is bounded as shown in the following.

**Theorem (1):** Each of the solutions of system (2) which are initiated in \(R^4_+\) are bounded.

**Proof:** Let \((x_1(t), x_2(t), x_3(t), x_4(t))\) be a solution of system (2) with positive initial condition \((x_1(0), x_2(0), x_3(0), x_4(0)) \in R^4_+\).

Now, define the function \(M(t) = x_1(t) + x_2(t) + x_3(t) + x_4(t)\) and then taken the time derivative of \(M(t)\) along the solution of system (2).

\[
\frac{dM}{dt} = (1 - u_6) x_2 - (1 - e) \left( \frac{u_2(1 - m_1)x_1 x_3}{u_3 + (1 - m_1)x_1 + (1 - m_2)x_2} - (1 - u_9)(1 - m_1)x_1 x_4 
\quad - (1 - e) \frac{u_4(1 - m_2)x_2 x_3}{u_3 + (1 - m_1)x_1 + (1 - m_2)x_2} - (u_6 - u_8)x_3 x_4 - u_7 x_3 - u_{10} x_4 \right)
\]

So, due to the fact that the conversion rate constant from prey population to predator population cannot exceeding the maximum predation rate constant from predator population to prey population, hence from the biological point of view, always \(u_5 < 1, e < 1, u_9 < 1, u_6 < u_8\) we get:

\[
\frac{dM}{dt} = x_1 + x_2 - (x_1 + x_2 + u_7 x_3
\quad + u_{10} x_4)
\]

\[
\frac{dM}{dt} \leq x_1 + x_2 - \mu(x_1 + x_2 + x_3
\quad + x_4)
\]

Where \(\mu = \min\{1, u_7, u_{10}\}\).

Since \(x_1 + x_2 = N\) represents prey specie which is growth logistically with carrying capacity(1), hence \(N \leq 1\).

So that,

\[
\frac{dM}{dt} \leq 1 - \mu M.
\]

Now, solve the differential equation with initial value \(M(0) = M_0\) we get:

\[
M(t) \leq \frac{1}{\mu} + \left(M_0 - \frac{1}{\mu}\right)e^{-\mu t},
\]

Then,

\[
\lim_{t \to \infty} M(t) \leq \left\{ \lim_{t \to \infty} \frac{1}{\mu} + \lim_{t \to \infty} \left(M_0 - \frac{1}{\mu}\right)e^{-\mu t} \right\}.
\]

So

\[
M(t) \leq \frac{1}{\mu}, \forall \ t > 0.
\]

Then each the solution of system (2) uniformly bounded.
3. Existence of the steady state points

In this part, the existence of all possible steady state points of system (2) is discussed. It is observed that, system (2) has only four steady state points, which are mentioned in the following:

- The steady state point $E_0 = (0,0,0,0)$, which is known as the varying point and is always exists.
- The two species steady state point $E_1 = (\bar{x}_1, \bar{x}_2, 0, 0)$ where:
  \[
  \bar{x}_1 = \left(\frac{1-u_5}{u_1}u_5 > 0 \right) \\
  \bar{x}_2 = (1 - u_5) > 0
  \]  
  (3a)

Exists under the following condition

- The steady state point $E_2 = (\bar{x}_1, \bar{x}_2, \bar{x}_3, 0)$

\[
\begin{align*}
\bar{x}_1 &= \frac{u_3u_7 + [u_7 - eu_5](1 - m_2)x_2}{[eu_2 - u_7](1 - m_1)} \\
(4a)
\end{align*}
\]

From the second equation of system (2) we have

\[
\begin{align*}
& u_1u_3x_1 + u_1(1 - m_1)x_1^2 + u_4(1 - m_2)x_1x_2 - u_4(1 - m_2)x_2x_3 - u_5u_3x_2 - u_5(1 - m_1)x_1x_2 \\
& - u_5(1 - m_2)x_2^2 = 0 \\
& (4b)
\end{align*}
\]

By substituting equation (4a) in equation (4b) we get

\[
\begin{align*}
& [u_1(u_7 - eu_5)/(eu_2 - u_7)(1 - m_1)(1 - m_2)]^2 + u_1(u_7 - eu_5)^2(1 - m_1)(1 - m_2)^2 \\
& - u_6(u_7 - eu_5)(eu_2 - u_7)(1 - m_1)(1 - m_2) \\
& - u_5(1 - m_2)((eu_2 - u_7)(1 - m_1)^2) x_2^2 \\
& + u_1u_3(u_7 - eu_5)(eu_2 - u_7)(1 - m_1)(1 - m_2) \\
& + 2u_1u_3u_7(u_7 - eu_5)(1 - m_1)(1 - m_2) + u_4u_3u_7(u_2 - u_7)(1 - m_1)(1 - m_2) \\
& - u_3u_7u_2((eu_2 - u_7)(1 - m_1)^2)x_2 - u_4(1 - m_2)((eu_2 - u_7)(1 - m_1)^2)x_3 \\
& + u_1u_3u_7u_2((eu_2 - u_7)(1 - m_1) + u_3u_7u_3u_7(1 - m_1) x_1x_3 \\
& - x_2 - x_2^2 - u_1x_1 - u_2(1 - m_1) - u_3 + (1 - m_1)x_1 + (1 - m_2)x_2 & = 0 \\
& (4c)
\end{align*}
\]

Also by substituting equation (4a) in equation(4c) we get:

\[
\begin{align*}
&\left(-(1 - m_2)((eu_2 - u_7)(1 - m_1)^2) - (1 - m_1)((eu_2 - u_7)(1 - m_2))((eu_2 - u_7)(1 - m_1)) \right)x_2^2 \\
& + \left(((1 - m_2) - u_3)((eu_2 - u_7)(1 - m_1)^2) \\
& + ((1 - m_1) - u_4(1 - m_2))((eu_2 - u_7)(1 - m_1))((eu_2 - u_7)(1 - m_2)) \\
& - u_3u_7(1 - m_1)((eu_2 - u_7)(1 - m_1)) - u_4(1 - m_2)((eu_2 - u_7)(1 - m_1)) \\
& + u_1u_3((eu_2 - u_7)(1 - m_1)^2) \\
& + u_3u_7((1 - m_1) - u_4(1 - m_2))((eu_2 - u_7)(1 - m_1)) \\
& - u_4u_3((eu_2 - u_7)(1 - m_1)) \\
& - 2u_1u_3u_7u_2(1 - m_1)((eu_2 - u_7)(1 - m_2))x_2 - u_1u_3u_7u_3u_7(1 - m_1) x_1x_3 \\
& - u_2(1 - m_1)((eu_2 - u_7)(1 - m_1))x_2 - u_2(1 - m_2)x_2x_3 & = 0 \\
& (4d)
\end{align*}
\]

Now, with some simplification we have:

\[
f_3(x_2, x_3) = a_1x_2^2 + a_2x_2 + a_3x_2x_3 + a_4 \\
= 0
\]
Now, in order to determine the values of $\hat{x}_2$ and $\hat{x}_3$, consider the two isoclines (4d) and (4e) as $x_3 \to 0$ which gives:

$$f_1(x_2) = a_4 x_2^2 + a_2 x_2 + a_4 = 0 \quad (i)$$

$$f_2(x_2) = b_4 x_2^3 + b_2 x_2^2 + b_3 x_2 + b_4 = 0 \quad (ii)$$

Obviously equation (i) is second degree polynomial equation, while equation (ii) is a third degree polynomial equation.

Consequently eq.(i) have a positive root say $x_{2a}$ provided that one of the following conditions hold:

$$a_1 > 0 \quad \text{and} \quad a_4 < 0\quad \text{or}$$

$$a_1 < 0 \quad \text{and} \quad a_4 > 0 \quad (4f)$$

However, equation (ii) has just one positive root, say $x_{2b}$, provided that one of the following conditions hold:

$$b_2 > 0 \quad \text{and} \quad b_1 > 0 \quad \text{or}$$

$$b_1 > 0 \quad \text{and} \quad b_3 < 0 \quad (4g)$$

Since we have $f(x_2, x_3) = 0$ then the derivative with respect to $x_2$ becomes:
\[ \frac{dx_3}{dx_2} = -\frac{a_3 x_2}{a_3 x_2} \]

Note that, \( \frac{dx_3}{dx_2} < 0 \) and hence the isoclines (4d) is Decreasing if the following condition hold:

\[ 2a_1 x_2 + a_2 + a_3 x_3 > 0 \quad (4h) \]

Similarly from equation (4e), we noted:

\[ \frac{dx_3}{dx_2} = -\frac{3b_1 x_2^2 + 2b_2 x_2 + b_3 + b_6 x_3}{b_5 + b_6 x_2} \]

Here \( \frac{dx_3}{dx_2} > 0 \) and hence the isoclines (4e) is increasing function iff the following condition hold:

\[ 3b_1 x_2^2 + 2b_2 x_2 + b_3 + b_6 x_3 < 0 \quad (4i) \]

Now, if \( x_{2b} < x_{2a} \), we get by the above analysis, it is noted that the two isoclines (4d) and (4e) intersect at unique point \( (x_2, x_3) \) iff the conditions (4f), (4g), (4h) and (4i) are satisfied, and hence the system (2) has only one positive steady state point if in addition to these conditions the following holds:

\[ \frac{e u_5(1-m_2)x_2}{u_3(1-m_2)x_2} > u_T > e u_5 \quad (4j) \]

\[ \bullet \quad \text{Lastly, the positive (coexistence) steady state point } E_3 = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4) \text{ exists if there is positive solution to the following set of equation:} \]

\[ x_2 - x_2^2 - u_1 x_1 - u_2(1 - m_1) \frac{x_1 x_3}{u_3 + (1 - m_1)x_1 + (1 - m_2)x_2} - (1 - m_1)x_1 x_4 = 0 \quad (5a) \]

\[ u_1 x_1 - u_4(1 - m_2) \frac{x_2 x_3}{u_3 + (1 - m_1)x_1 + (1 - m_2)x_2} - u_5 x_2 = 0 \]

\[ u_4 \left[ \frac{u_2(1 - m_1)x_1 + u_4(1 - m_2)x_2 - u_6 x_4 - u_7}{u_3 + (1 - m_1)x_1 + (1 - m_2)x_2} \right] = 0 \quad (5c) \]

\[ x_4 [u_6 x_3 + (1 - m_1)u_9 x_1 - u_10] = 0 \]

From equation (5d) we obtain:

\[ \bar{x}_3 = u_{10} - (1 - m_1)u_9 x_1 \quad (5e) \]

\[ From \ equation \ (5c) \ we \ obtain: \]

\[ \bar{x}_4 = \frac{(eu_2 - u_7)(1 - m_1)x_1 + (eu_4 - u_7)(1 - m_2) - u_3 u_7}{u_6 [u_3 + (1 - m_1)x_1 + (1 - m_2)x_2]} \quad (5f) \]

From equation (5b) we get:

\[ u_1 u_6(1 - m_1)x_2^2 + u_1 u_3 u_9 x_1 + [u_4 u_9(1 - m_2) + u_4 u_9(1 - m_1)(1 - m_2) - u_5 u_9(1 - m_1)]x_1 x_2 \]

\[ + [-u_4 u_{10}(1 - m_2) - u_3 u_5 u_9] x_1 - u_5 u_9(1 - m_2)x_2^2 = f(x_1, x_2) \]

Substituting equations (5e) and (5f) in equation (5a) we get:

\[ -u_6 u_8(1 - m_2)x_2^2 + [u_6 u_8(1 - m_2) - u_3 u_6 u_9] x_2^2 + u_6 u_8 u_3 x_2 - u_6 u_8(1 - m_1)x_1 x_2 \]

\[ + [u_6 u_8(1 - m_1) - u_4 u_6 u_8(1 - m_2) - u_6 (eu_4 - u_7)(1 - m_1)(1 - m_2)]x_1 x_2 \]

\[ + [u_2 u_4 u_9(1 - m_1)^2 - u_1 u_5 u_9(1 - m_1) - u_8 (eu_2 - u_7)(1 - m_1)^2] x_2^2 \]

\[ + [u_3 u_7 u_9(1 - m_1) - u_2 u_5 u_9] x_1 - u_5 u_9(1 - m_2) x_1 = 0 = g(x_1, x_2) \]

Now, with some simplification we have:

\[ f(x_1, x_2) = r_1 x_1^2 + r_3 x_1 + r_5 x_2 + r_4 x_2 + r_4 x_2^2 = 0 \quad (5g) \]

\[ g(x_1, x_2) = r_1 x_1^3 + r_2 x_2^3 + r_3 x_1 x_2 + r_4 x_1 x_2 + r_5 x_1 + r_6 x_2 + r_7 x_1 = 0 \quad (5h) \]

\[ \text{Where} \]

\[ r_1 = u_1 u_6(1 - m_1) > 0 \quad (5i) \]

\[ r_2 = u_1 u_3 u_9 > 0 \]

\[ r_3 = u_4 u_9(1 - m_2) + u_4 u_9(1 - m_1)(1 - m_2) - u_5 u_9(1 - m_1) \]

\[ r_4 = -u_4 u_{10}(1 - m_2) - u_4 u_9 u_8 < 0 \]

\[ r_5 = -u_5 u_9(1 - m_2) < 0 \]
Now, in order to determine the values of $\bar{x}_1$ and $\bar{x}_2$, consider the two isoclines i and ii as $x_1 \to 0$, which gives:

$$f(x_2) = r_4 x_2 + r_5 x_2^2 = x_2(r_4 + r_5 x_2) = 0 \quad (i)$$
$$g(x_2) = s_1 x_2^3 + s_2 x_2^2 + s_3 x_2 = 0 \quad (ii)$$

Obviously equation (i) is second degree polynomial equation, while equation (ii) is a third degree polynomial consequently due to Descartes’ rule equation (i) has two roots one of them, say $\bar{x}_{2m} = 0$ other of them say $x_{2m} = \frac{-r_4}{r_5} < 0$, However, equation (ii) has a unique positive root, say $x_{2n}$ and from equation (5g) it is easy to verify that

$$\frac{dx_1}{dx_2} = \frac{-r_3 x_1 + r_4 + 2r_5 x_2}{2r_3 x_1 + r_2 + r_3 x_2}$$

Hence $\frac{dx_1}{dx_2} > 0$ and hence the isoclines (5g) is increasing function if the following condition hold:

$$\begin{align*}
2r_1 x_1 + r_2 + r_3 x_2 > 0 \\
(5l)
\end{align*}$$

Similarly from equation (5h), we noted

$$\frac{dx_1}{dx_2} = \frac{-3s_1 x_2^2 + 2s_2 x_2 + s_3 + 2s_4 x_1 x_2 + s_5 x_2}{s_4 x_2^2 + s_5 x_2 + 2s_6 x_1 + s_7}$$

Note that $\frac{dx_1}{dx_2} < 0$ and hence the isoclines (5h) is decreasing iff the following condition hold:

$$\begin{align*}
3s_1 x_2^2 + 2s_2 x_2 + s_3 + 2s_4 x_1 x_2 + s_5 x_2 > 0 \\
(5j)
\end{align*}$$

Therefore the positive equilibrium point $E_3$ exists uniquely provided that in addition to the above conditions the following two conditions hold

$$\bar{x}_1 < \frac{u_0}{u_9(1-m_1)} \quad (5k)$$

$$\frac{(eu_2 - u_7)(1-m_1)x_1 + (eu_2 - u_7)(1-m_2)x_2}{u_3(1-m_1)x_1 + (1-m_2)x_2} > u_3 u_7 \quad (5l)$$

4. The Stability Conditions

In this part, the local conditions for stability near the steady state points of system (2) is investigated. It is to verify that the Jacobian matrix of system (2), at the general point $(x_1, x_2, x_3, x_4)$

$$f = (di g)_{4x4} \quad i, j = 1, 2, 3, 4.$$  

$$a_{11} = -[u_1 + (1-m_1)x_4] - \left[ \frac{u_2 u_3 (1-m_1)x_3 + u_2 (1-m_1)(1-m_2)x_2 x_3}{(u_3 + (1-m_1)x_1 + (1-m_2)x_2)^2} \right]$$
\[ a_{12} = 1 - 2x_2 + \left[ \frac{(u_2(1 - m_1)x_1x_3)(1 - m_2)}{(u_3 + (1 - m_1)x_1 + (1 - m_2)x_2)^2} \right] \]

\[ a_{13} = \frac{-u_2(1 - m_1)x_1}{u_3 + (1 - m_1)x_1 + (1 - m_2)x_2} \]
\[ a_{14} = -(1 - m_1)x_1 \]
\[ a_{21} = u_1 + \frac{u_4(1 - m_1)(1 - m_2)x_2x_3}{(u_3 + (1 - m_1)x_1 + (1 - m_2)x_2)^2} \]
\[ a_{22} = -u_5 - \frac{u_3u_4(1 - m_2)x_3 + u_4(1 - m_1)(1 - m_2)x_1x_3}{(u_3 + (1 - m_1)x_1 + (1 - m_2)x_2)^2} \]
\[ a_{23} = \frac{u_3 + (1 - m_1)x_1 + (1 - m_2)x_2}{u_3 + (1 - m_1)x_1 + (1 - m_2)x_2}\]
\[ a_{24} = 0 \]
\[ a_{31} = \frac{eu_2u_3(1 - m_1)x_1 + [u_2 - u_4]e(1 - m_1)(1 - m_2)x_2x_3}{(u_3 + (1 - m_1)x_1 + (1 - m_2)x_2)^2} \]
\[ a_{32} = \frac{eu_3u_4(1 - m_2)x_3 + [u_4 - u_2]e(1 - m_1)(1 - m_2)x_1x_3}{(u_3 + (1 - m_1)x_1 + (1 - m_2)x_2)^2} \]
\[ a_{33} = \frac{eu_2(1 - m_1)x_1 + eu_4(1 - m_2)x_2}{u_3 + (1 - m_1)x_1 + (1 - m_2)x_2} - (u_6x_4 + u_7) \]
\[ a_{34} = -u_6x_3 \]
\[ a_{42} = 0 \]
\[ a_{43} = u_8x_3 \]
\[ a_{44} = u_9x_3 + u_9(1 - m_1)x_1 - u_{10} \]

Therefore, the Jacobian matrix of system (2) at the vanishing steady state point \( E_0 \) is:

\[
J(E_0) = \begin{bmatrix}
-u_1 & 1 & 0 & 0 \\
u_1 & -u_5 & 0 & 0 \\
0 & 0 & -u_7 & 0 \\
0 & 0 & 0 & -u_{10}
\end{bmatrix}
\]

(7)

Thus, the eigenvalues of \( J(E_0) \) are:

\[
\lambda_{x_3} = -u_7 < 0 \quad \text{and} \quad \lambda_{x_4} = -u_{10} < 0 \quad \text{or} \quad \lambda^2 + B_1\lambda + B_2 = 0
\]

which gives two eigenvalues:

\[
\lambda_{x_1, x_2} = \frac{-B_1 \pm \sqrt{B_1^2 - 4B_2}}{2}
\]

where:

\[
B_1 = u_1 + u_5 > 0
\]
\[
B_2 = u_1(u_5 - 1) < 0
\]

Therefore, \( E_0 \) is a saddle point.

The Jacobian matrix of system (2) at \( E_1 \) is given by:

\[
J(E_1) = \begin{bmatrix}
-u_1 & 1 - 2\tilde{x}_2 & -u_2(1 - m_1)\tilde{x}_1 & -(1 - m_1)\tilde{x}_1 \\
u_1 & -u_5 & \frac{-u_2(1 - m_1)\tilde{x}_1}{u_3 + (1 - m_1)\tilde{x}_1 + (1 - m_2)\tilde{x}_2} & \frac{-(1 - m_1)\tilde{x}_1}{u_3 + (1 - m_1)\tilde{x}_1 + (1 - m_2)\tilde{x}_2} \\
0 & 0 & \frac{eu_2(1 - m_1)\tilde{x}_1 + eu_4(1 - m_2)\tilde{x}_2}{u_3 + (1 - m_1)\tilde{x}_1 + (1 - m_2)\tilde{x}_2} & u_7 \\
0 & 0 & 0 & u_9(1 - m_1)\tilde{x}_1 - u_{10}
\end{bmatrix}
\]

(8a)

Accordingly the characteristic equation of \( J(E_1) \) can be written as:
\[ \lambda - (u_9(1 - m_1)\bar{x}_1 - u_{10}) \left\{ \lambda - \left( \frac{e u_2(1 - m_1)\bar{x}_1 + e u_4(1 - m_2)\bar{x}_2}{u_3 + (1 - m_1)\bar{x}_1 + (1 - m_2)\bar{x}_2} - u_7 \right) \right\} \left[ \lambda^2 + B_1\lambda + B_2 \right] = 0 \quad \text{(Bb)} \]

Where
\[ B_1 = -[-u_1 - u_5] = u_1 + u_5 \]
\[ B_2 = u_4 u_5 - u_4 (1 - 2\bar{x}_2) \]

So either
\[ \lambda - (u_9(1 - m_1)\bar{x}_1 - u_{10}) \left\{ \lambda - \left( \frac{e u_2(1 - m_1)\bar{x}_1 + e u_4(1 - m_2)\bar{x}_2}{u_3 + (1 - m_1)\bar{x}_1 + (1 - m_2)\bar{x}_2} - u_7 \right) \right\} = 0 \]

We get the eigenvalues of \( J(E_1) \) in the \( x_3, x_4 \) direction respectively as:
\[ \lambda_{x_3} = \frac{e u_2(1 - m_1)\bar{x}_1 + e u_4(1 - m_2)\bar{x}_2}{u_3 + (1 - m_1)\bar{x}_1 + (1 - m_2)\bar{x}_2} - u_7 \]
\[ \lambda_{x_4} = u_9(1 - m_1)\bar{x}_1 - u_{10} \]
\[ \text{OR} \]
\[ \lambda^2 + B_1\lambda + B_2 = 0 \]

Hence, we get the other two eigenvalues of \( J(E_1) \) in the \( x_1, x_2 \) direction as:
\[ \lambda_{x_1}, \lambda_{x_2} = -\frac{B_1}{2} \pm \frac{1}{2} \sqrt{B_1^2 - 4B_2} \]

Then all the eigenvalues have negative real parts if the following conditions hold:
\[ u_9(1 - m_1)\bar{x}_1 < u_{10} \]
\[ e u_2(1 - m_1)\bar{x}_1 + e u_4(1 - m_2)\bar{x}_2 < u_7 \left( \frac{u_3 + (1 - m_1)\bar{x}_1}{+(1 - m_2)\bar{x}_2} \right) \]

So, \( E_1 \) is a local stable in the \( R^4 \). And it is unstable point on the other hand.

Thus Jacobian matrix of system (2) at \( E_2 \) is a given by:
\[ J(E_2) = \begin{bmatrix} c_{11} & c_{12} & c_{13} & -(1 - m_1)\bar{x}_1 \\ c_{21} & c_{22} & c_{23} & 0 \\ c_{31} & c_{32} & 0 & -u_6\bar{x}_3 \\ 0 & 0 & 0 & u_9\bar{x}_1 + u_9(1 - m_1)\bar{x}_1 - u_{10} \end{bmatrix} \quad \text{(9a)} \]

Where
\[ c_{11} = -u_1 - \left[ \frac{u_2 u_3(1 - m_1)\bar{x}_3 + u_2(1 - m_1)(1 - m_2)\bar{x}_2\bar{x}_3}{u_3 + (1 - m_1)\bar{x}_1 + (1 - m_2)\bar{x}_2} \right] \]
\[ c_{12} = 1 - 2\bar{x}_2 + \left[ \frac{u_2(1 - m_1)(1 - m_2)\bar{x}_1\bar{x}_3}{u_3 + (1 - m_1)\bar{x}_1 + (1 - m_2)\bar{x}_2} \right] \]
\[ c_{13} = \frac{-u_2(1 - m_1)\bar{x}_1}{u_3 + (1 - m_1)\bar{x}_1 + (1 - m_2)\bar{x}_2} \]
\[ c_{14} = -(1 - m_1)\bar{x}_1 \]
\[ c_{21} = u_1 + \frac{u_4(1 - m_1)(1 - m_2)\bar{x}_2\bar{x}_3}{\left( u_3 + (1 - m_1)\bar{x}_1 + (1 - m_2)\bar{x}_2 \right)^2} \]
\[ c_{22} = -u_5 - \left[ \frac{u_3 u_4(1 - m_2)\bar{x}_3 + u_4(1 - m_1)(1 - m_2)\bar{x}_1\bar{x}_3}{u_3 + (1 - m_1)\bar{x}_1 + (1 - m_2)\bar{x}_2} \right] \]
Then the eigenvalues of $J(F_2)$ are

$$\begin{align*}
\lambda - (u_8\dot{x}_3 + u_6(1-m_1)\dot{x}_1 - u_{10})[\lambda^3 + \dot{A}_1\lambda^2 + \dot{A}_2\lambda + \dot{A}_3] &= 0 \\
\dot{A}_1 &= -[c_{11} + c_{22} + c_{33}] \\
\dot{A}_2 &= c_{11}c_{22} - c_{12}c_{21} + c_{11}c_{33} - c_{13}c_{31} + c_{22}c_{33} - c_{23}c_{32} \\
\dot{A}_3 &= c_{11}c_{22}c_{33} - c_{12}c_{23}c_{31} - c_{13}c_{21}c_{32} + c_{13}c_{22}c_{31} + c_{11}c_{23}c_{32} + c_{12}c_{21}c_{33}
\end{align*}$$

Accordingly, either

$$\begin{align*}
[\lambda - (u_8\dot{x}_3 + u_6(1-m_1)\dot{x}_1 - u_{10})] &> 0 \\
\dot{A}_1 &= -[c_{11} + c_{22} + c_{33}] \\
\dot{A}_2 &= c_{11}c_{22} - c_{12}c_{21} + c_{11}c_{33} - c_{13}c_{31} + c_{22}c_{33} - c_{23}c_{32} \\
\dot{A}_3 &= c_{11}c_{22}c_{33} - c_{12}c_{23}c_{31} - c_{13}c_{21}c_{32} + c_{13}c_{22}c_{31} + c_{11}c_{23}c_{32} + c_{12}c_{21}c_{33}
\end{align*}$$


to ensure that all the eigenvalues have negative real parts so the steady state point $E_2 = (\dot{x}_1, \dot{x}_2, \dot{x}_3, 0)$ is local stable.

The Jacobian matrix of system (2) at steady state point $E_3 = (\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4)$:
\[ j(E_3) = \begin{bmatrix}
    d_{11} & d_{12} & d_{13} & -(1-m_1)\ddot{x}_1 \\
    d_{21} & d_{22} & d_{23} & 0 \\
    d_{31} & d_{32} & d_{33} & -u_6\ddot{x}_3 \\
    u_6(1-m_1)\ddot{x}_4 & 0 & u_6\ddot{x}_3 & u_6(1-m_1)\ddot{x}_1 - u_{10} \\
\end{bmatrix} \tag{10a} \]

Where

\[ d_{11} = [-u_4 + (1-m_1)\ddot{x}_4] - \frac{u_2u_3(1-m_1)\ddot{x}_3 + u_2(1-m_1)(1-m_2)\ddot{x}_2\ddot{x}_3}{(u_3 + (1-m_1)\ddot{x}_3 + (1-m_2)\ddot{x}_2)^2} \]

\[ d_{13} = \frac{u_2(1-m_1)(1-m_2)\ddot{x}_3}{u_2(1-m_1)\ddot{x}_1} \]

\[ d_{21} = u_1 + \frac{u_3(1-m_1)\ddot{x}_1 + (1-m_2)\ddot{x}_2}{u_4(1-m_1)\ddot{x}_2\ddot{x}_3} \]

\[ d_{22} = -u_5 - \frac{u_3u_4(1-m_2)\ddot{x}_3 + u_4(1-m_1)(1-m_2)\ddot{x}_1\ddot{x}_3}{(u_3 + (1-m_1)\ddot{x}_3 + (1-m_2)\ddot{x}_2)^2} \]

\[ d_{23} = \frac{u_3(1-m_1)\ddot{x}_3 + (1-m_2)\ddot{x}_2}{u_4(1-m_1)\ddot{x}_2\ddot{x}_3} \]

\[ d_{31} = \frac{u_3u_4(1-m_2)\ddot{x}_3 + u_4(1-m_1)(1-m_2)\ddot{x}_1\ddot{x}_3}{(u_3 + (1-m_1)\ddot{x}_3 + (1-m_2)\ddot{x}_2)^2} \]

\[ d_{32} = \frac{u_3u_4(1-m_2)\ddot{x}_3 + u_4(1-m_1)(1-m_2)\ddot{x}_1\ddot{x}_3}{(u_3 + (1-m_1)\ddot{x}_3 + (1-m_2)\ddot{x}_2)^2} \]

\[ d_{33} = \frac{u_3(1-m_1)\ddot{x}_3 + (1-m_2)\ddot{x}_2}{u_4(1-m_1)\ddot{x}_2\ddot{x}_3} - (u_6\ddot{x}_4 + u_7) = 0 \]

It is easy to verify that, the linearized system of system (2) can be written as:

\[ \frac{dR}{dt} = \frac{ds}{dt} = j(E_3)S \tag{10b} \]

Here

\[ R = (x_1, x_2, x_3, x_4)^t \text{ and } S = (s_1, s_2, s_3, s_4)^t \]

Where

\[ s_1 = x_1 - \ddot{x}_1, \quad s_2 = x_2 - \ddot{x}_2 \]

\[ s_3 = x_3 - \ddot{x}_3, \quad s_4 = x_4 - \ddot{x}_4 \]

Now, consider the following positive define function

\[ L_2 = \frac{a_1}{2}s_1^2 + \frac{a_2}{2}s_2^2 + \frac{a_3}{2}s_3^2 + \frac{a_4}{2}s_4^2 \tag{10c} \]

It is clearly that \( L_2 : \mathbb{R}_+^4 \rightarrow \mathbb{R} \) continuously differentiable function, So that \( L_2(\ddot{x}_1, \ddot{x}_2, \ddot{x}_3, \ddot{x}_4) = 0 \) when \( L_2(x_1, x_2, x_3, x_4) > 0 \) otherwise so by differentiating \( L_2 \) with respect to time \( t \), gives:

\[ \frac{dL_2}{dt} = a_1s_1\frac{ds_1}{dt} + a_2s_2\frac{ds_2}{dt} + a_3s_3\frac{ds_3}{dt} + a_4s_4\frac{ds_4}{dt} \]

We get:

\[ \frac{dL_2}{dt} = -\left[u_1 + (1-m_1)\ddot{x}_4\right] + \left[u_2u_3(1-m_1)\ddot{x}_3 + u_2(1-m_1)(1-m_2)\ddot{x}_2\ddot{x}_3\right]s_1^2 \]

\[ -\left[2\ddot{x}_2 - 1 - \frac{u_2(1-m_1)(1-m_2)\ddot{x}_1\ddot{x}_3}{(u_3 + (1-m_1)\ddot{x}_3 + (1-m_2)\ddot{x}_2)^2}\right]s_2^2 \]

\[ -\left[u_1 + \frac{u_4(1-m_1)(1-m_2)\ddot{x}_2\ddot{x}_3}{u_3 + (1-m_1)\ddot{x}_3 + (1-m_2)\ddot{x}_2}\right]s_3^2 \]

\[ -\left[u_5 + \frac{u_3u_4(1-m_2)\ddot{x}_3 + u_4(1-m_1)(1-m_2)\ddot{x}_1\ddot{x}_3}{(u_3 + (1-m_1)\ddot{x}_3 + (1-m_2)\ddot{x}_2)^2}\right]s_4^2 \]
\[
\frac{dL_2}{dt} = -[q_{11}s_1^2 + q_{12}s_1s_2 + q_{22}s_2^2]
\]

Where

\[
q_{11} = [u_1 + (1 - m_1)\bar{x}_1] + \left[\frac{u_2u_3(1 - m_1)\bar{x}_3 + u_4(1 - m_1)(1 - m_2)\bar{x}_2\bar{x}_3}{(u_3 + (1 - m_1)\bar{x}_1 + (1 - m_2)\bar{x}_2)^2}\right]
\]
\[
q_{12} = 2\bar{x}_2 - \left[1 + \frac{u_2(1 - m_1)(1 - m_2)\bar{x}_1\bar{x}_3}{(u_3 + (1 - m_1)\bar{x}_1 + (1 - m_2)\bar{x}_2)^2}\right] + u_1 + \frac{u_4(1 - m_1)(1 - m_2)\bar{x}_2\bar{x}_3}{(u_3 + (1 - m_1)\bar{x}_1 + (1 - m_2)\bar{x}_2)^2}
\]
\[
q_{22} = u_5 + \frac{u_3u_4(1 - m_2)\bar{x}_3 + u_4(1 - m_1)(1 - m_2)\bar{x}_4\bar{x}_3}{(u_3 + (1 - m_1)\bar{x}_1 + (1 - m_2)\bar{x}_2)^2}
\]

Now, it easy we have:

\[
\frac{dL_2}{dt} < -\left[\sqrt{q_{11}s_1} + \sqrt{q_{22}s_2}\right]^2 \text{ if and only if } \bar{x}_2
\]

\[
> \frac{1}{2}\left[1 + u_2\right] + \frac{(u_2\bar{x}_1 + u_4\bar{x}_2) + (1 - m_1)(1 - m_2)\bar{x}_2}{(u_3 + (1 - m_1)\bar{x}_1 + (1 - m_2)\bar{x}_2)^2}
\]

Therefore, \(\frac{dL_2}{dt}\) is negative definite and hence \(L_2\) is a Lyapunov function with respect to \(E_3\) in the sub region \(\Omega_2\). So \(E_3\) is asymptotically stable. Note that the function \(L_2\) is approaching to infant as any of its components to the same and its positive definite \(R^4\), however its derivative is negative definite on the sub region \(\Omega_2\) due to the given sufficient conditions. Therefore \(E_3\) is a globally asymptotically stable with in \(\Omega_2\).

**Theorem (2):** Assume that \(E_4\) is local stable in \(R^4\) if the following condition hold

\[
x_1 > \bar{x}_1 + e
\]

\[
x_2 > \bar{x}_2 + e
\]

\[
x_1 > \bar{x}_1 + u_9
\]

Then the steady state point \(E_4\) is global stable.

**Proof:** The following function

\[
v_1(x_1, x_2, x_3, x_4) = \frac{c_1}{2}(x_1 - \bar{x}_1)^2 + \frac{c_2}{2}(x_2 - \bar{x}_2)^2 + c_3x_3 + c_4x_4.
\]

It is easy to see that \(v_1(x_1, x_2, x_3, x_4) \in C'(R^4, R)\), in addition, \(v_1(\bar{x}_1, \bar{x}_2, 0,0) = 0\) while \(v_1(x_1, x_2, x_3, x_4) > 0 \forall (x_1, x_2, x_3, x_4) \in R^4\) and \((x_1, x_2, x_3, x_4) \neq (\bar{x}_1, \bar{x}_2, 0,0)\).

Furthermore, by the derivative with time and simplifying we get that:

\[
\frac{dv_1}{dt} = -c_1u_1(x_1 - \bar{x}_1)^2 - c_1(x_1 - \bar{x}_1)(x_2 - \bar{x}_2)(x_2 + \bar{x}_2) + c_1(x_1 - \bar{x}_1)(x_2 - \bar{x}_2)
\]

\[
+ c_2u_4(x_1 - \bar{x}_1)(x_2 - \bar{x}_2) + c_2u_5(x_2 - \bar{x}_2)^2
\]

\[
+ c_2u_4(1 - m_1)u_5(x_1 - \bar{x}_1)\bar{x}_2\bar{x}_3
\]

\[
- c_1(x_1 - \bar{x}_1)(1 - m_1)\bar{x}_1\bar{x}_4
\]

\[
- c_2u_4(1 - m_2)\bar{x}_3 - (1 - m_2)\bar{x}_2\bar{x}_3 + c_3u_4(1 - m_1)\bar{x}_1\bar{x}_3
\]

And then substituting \(c_1 = c_2 = c_3 = c_4 = 1\) in the above equation we get:
\[
\frac{dv_1}{dt} = -[u_1(x_1 - \tilde{x}_1)^2 - (x_2 + \tilde{x}_2 + 1 + u_1)(x_1 - \tilde{x}_1)(x_2 - \tilde{x}_2) + u_5(x_2 - \tilde{x}_2)^2] \\
- [x_1 - \tilde{x}_1 - e]\frac{u_2(1 - m_1)x_1x_3}{u_3 + (1 - m_1)x_1 + (1 - m_2)x_2} \\
- [x_2 - \tilde{x}_2 - e]\frac{u_3(1 - m_2)x_2x_3}{u_3 + (1 - m_1)x_1 + (1 - m_2)x_2} - [x_1 - \tilde{x}_1 - u_9](1 - m_1)x_4 \\
- (u_6 - u_8)x_3x_4 - u_7x_3 - u_10x_4.
\]

Obviously \(\frac{dv_1}{dt} < 0\) for every initial point and then \(v_1\) is a Lyapunov function provided that conditions (11a-11d) hold. Thus \(E_1\) is globally stable this completes the proof.

**Theorem (3):** Assume that \((\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, 0)\) is a locally in \(R_1^3\), then it is globally stable provided that the following conditions:

\[
\begin{align*}
1 & > \max\{(1 - m_1)\tilde{x}_1, (1 - m_2)\tilde{x}_2\} \\
u_4(1 - m_2)\tilde{x}_2 & > u_2(u_3 + (1 - m_2)\tilde{x}_2) \\
u_2(1 - m_1)\tilde{x}_1 & > u_4(u_3 + (1 - m_4)\tilde{x}_1) \\
(1 - m_1)x_1^2u_4 + u_6x_3x_4 & > (1 - m_1)\tilde{x}_1x_1x_4 + u_6x_4\tilde{x}_3
\end{align*}
\]

**Proof:** Consider the following function

\[
v_2(x_1, x_2, x_3, x_4) = \frac{1}{2}(x_1 - \tilde{x}_1)^2 + \frac{1}{2}(x_2 - \tilde{x}_2)^2 + \left(x_3 - \tilde{x}_3 - \tilde{x}_3 \ln \frac{x_3}{\tilde{x}_3}\right) + x_4
\]

It is easy to see that

\[
v_2(x_1, x_2, x_3, x_4) \in C(R_1^4, R), \quad \forall (x_1, x_2, x_3, x_4) \in R_1^4, \quad \text{and} \quad (x_1, x_2, x_3, x_4) \neq (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, 0)
\]

Furthermore by taking the derivative with time and simplifying we get that:

\[
\frac{dv_2}{dt} = (x_1 - \tilde{x}_1)(x_2 - \tilde{x}_2) - (x_2 - \tilde{x}_2)^2 - u_1(x_1 - \tilde{x}_1) \\
- u_2(1 - m_1)\frac{x_1x_3(u_3 + (1 - m_1)\tilde{x}_1 + (1 - m_2)\tilde{x}_2) - \tilde{x}_1\tilde{x}_3(u_3 + (1 - m_1)x_1 + (1 - m_2)x_2)}{(u_3 + (1 - m_1)x_1 + (1 - m_2)x_2)(u_3 + (1 - m_1)\tilde{x}_1 + (1 - m_2)\tilde{x}_2)} \\
+ (x_2 - \tilde{x}_2)u_1(x_1 - \tilde{x}_1) - u_5(x_2 - \tilde{x}_2) \\
- u_4(1 - m_2)\frac{x_2x_3(u_3 + (1 - m_1)\tilde{x}_1 + (1 - m_2)\tilde{x}_2) - \tilde{x}_2\tilde{x}_3(u_3 + (1 - m_1)x_1 + (1 - m_2)x_2)}{(u_3 + (1 - m_1)x_1 + (1 - m_2)x_2)(u_3 + (1 - m_1)\tilde{x}_1 + (1 - m_2)\tilde{x}_2)} \\
+(x_3 \tilde{x}_3)\left(\frac{e u_2(1 - m_1)(x_1(u_3 + (1 - m_1)\tilde{x}_1 + (1 - m_2)\tilde{x}_2) - \tilde{x}_1u_3 + (1 - m_1)x_1 + (1 - m_2)x_2)}{(u_3 + (1 - m_1)x_1 + (1 - m_2)x_2)(u_3 + (1 - m_1)\tilde{x}_1 + (1 - m_2)\tilde{x}_2)}\right) \\
- e u_4(1 - m_2)\left(\frac{x_2(u_3 + (1 - m_1)\tilde{x}_1 + (1 - m_2)\tilde{x}_2) - \tilde{x}_2(u_3 + (1 - m_1)x_1 + (1 - m_2)x_2)}{(u_3 + (1 - m_1)x_1 + (1 - m_2)x_2)(u_3 + (1 - m_1)\tilde{x}_1 + (1 - m_2)\tilde{x}_2)}\right) \\
- u_6x_4\right] + [u_8x_3x_4 + (1 - m_1)u_9x_1x_4 - u_10x_4]
Obviously, $\frac{dv_3}{dt} < 0$ for every initial point and then $v_3$ is a Lyap. function provided that conditions (11a-11f) hold. Thus $E_2$ is global stable this completes the proof.

**Theorem (4):** Assume that $E_3$ is local stable in $\mathbb{R}_+^4$. Then, it is a global stable in sub region of $\mathbb{R}_+^4$ that satisfies the following conditions

$$x_2 + x_2^* > 1 + u_1$$

$$k_1 + k_2 > u_2(1 - m_1)(x_1 - x_1^*)k_3$$

$$+ u_4(1 - m_2)(x_2 - x_2^*)k_4$$

$$[u_2 + u_4]e(1 - m_3)(1 - m_2)x_2^*x_2(x_3 - x_3^*)$$

$$> e(x_3 - x_3^*)[u_2(1 - m_1)k_5 + u_4(1 - m_2)k_6]$$

(13c)

Where

$$k_1 = u_2(1 - m_1)x_1x_2(x_1 - x_1^*)$$

$$k_2 = u_4(1 - m_2)x_2x_3(x_2 - x_2^*)$$

$$k_3 = (1 + (1 - m_1)(x_1 - x_1^*))x_1^*x_1^* + (1 - m_2)x_1^*x_2^*(x_2 - x_2^*)$$

$$k_4 = [1 + (1 - m_1)(x_1 - x_1^*) + (1 - m_2)(x_2 - x_2^*)]x_2^*x_2^*$$

$$k_5 = u_3(x_1 - x_1^*) + (1 - m_2)x_1x_2^*$$

$$k_6 = u_3(x_2 - x_2^*)$$

**Proof:** consider the following function

$$v_3(x_1, x_2, x_3, x_4) = \frac{1}{2}(x_1 - x_1^*)^2 + \frac{1}{2}(x_2 - x_2^*)^2 + \left(x_3 - x_3^* - x_3^* \ln \frac{x_4}{x_3^*} + \left(x_4 - x_4^* - x_4^* \ln \frac{x_4}{x_4^*}\right)\right)$$

It is easy to verify that $v_3(x_1, x_2, x_3, x_4) \in C'(\mathbb{R}^4_+, \mathbb{R})$ and $v_3(x_1, x_2, x_3, x_4) = 0$ while $v_3(x_1, x_2, x_3, x_4) > 0$ for all $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4_+$ and $(x_1, x_2, x_3, x_4) \neq (x_1^*, x_2^*, x_3^*, x_4^*)$ then by find the derivative with time, also simplifying it we get:

$$\frac{dv_3}{dt} = (x_1 - x_1^*) \left[ (x_2 - x_2^*) - (x_3^* - x_3^*) - u_4(x_1 - x_1^*) - (1 - m_1)(x_1x_4 - x_1^*x_4) \right]$$

$$- u_2(1 - m_1) \left[ x_1x_3(u_3 + (1 - m_1)x_1 + (1 - m_2)x_2)(u_3 + (1 - m_2)x_1 + (1 - m_2)x_2) \right]$$

$$+ (x_2 - x_2^*) \left[ u_4(x_1 - x_1^*) - u_5(x_2 - x_2^*) \right]$$

$$- u_4(1 - m_2) \left[ x_2x_3(u_3 + (1 - m_1)x_3 + (1 - m_2)x_3)(u_3 + (1 - m_1)x_3 + (1 - m_2)x_3) \right]$$

$$+ (x_3 - x_3^*) \left[ e(u_1 - m_3)(x_1 + (1 - m_1)x_1 + (1 - m_2)x_2) - (x_3 - x_3^*) \right]$$

$$- u_4(1 - m_2) \left[ x_2x_3(u_3 + (1 - m_1)x_3 + (1 - m_2)x_3)(u_3 + (1 - m_1)x_3 + (1 - m_2)x_3) \right]$$

Clearly, $\frac{dv_3}{dt} < 0$, and then $v_3$ is a Lyap. function provided that the given conditions (13a-13c) hold. Therefore, $E_3$ is global stable in the interior of a basin of attraction of $E_3$ and the proof is complete.

**5. Numerical illustrate**

In this section, the dynamical behavior of system (2) is studied numerically for different sets of initial values and different sets of parameters values.
It is observed that for the following set of hypothetical parameters system (2) has an asymptotical stable steady state point \( E_1 = (0.2, 0.99, 0.0) \) as shown in Figure-1

\[
\begin{align*}
&u_1 = 0.05, \; u_2 = 0.00001, \; u_3 = 0.25, \; u_4 = 0.00001, \; u_5 = 0.01 \\
&u_6 = 0.1, \; u_7 = 0.05, \; u_8 = 0.002, \; u_9 = 0.02, \; u_{10} = 0.1 \\
&m_1 = 0.01, \; m_2 = 0.02, \; e = 0.003
\end{align*}
\] (14)

Figure 1-Trajectory of system (2) that begin from different initial point, \((0.9, 0.7, 0.5, 0.3)\) and \((0.3, 0.9, 0.9, 0.8)\) for the data given by Eq. (14). (a) Trajectories of immature prey as a function of time (b) Trajectory of mature prey as a function of time. (c) Trajectory of susceptible predator as a function of a function of time.

from values of parameters that given in Eq. (14) with \(u_6 = 0.001, \; u_9 = 0.0002\), the solution of system (2) approaches to \( E_2 = (0.5, 0.9, 0.6, 0) \) as shown in Figure-2
Figure 2-Trajectory of solution of system (2) for above parameters from different set of initial points (0.9,0.7,0.5,0.3) and (0.3,0.3,0.9,0.6). (a) Trajectories of immature prey (b) Trajectory of mature prey (c) Trajectory of susceptible predator (d) Trajectory of infected predator.

It is observed that for the following set of hypothetical parameters that satisfies stable conditions of positive steady state point $E_3=(0.4,0.8,0.5,0.3)$ system (2) has asymptotic stable positive steady state point as shown in Figure-3

\begin{align*}
    u_1 &= 0.05, \quad u_2 = 0.00001 \quad u_3 = 0.25, \quad u_4 = 0.05, \quad u_5 = 0.01 \\
    u_6 &= 0.003, \quad u_7 = 0.05, \quad u_8 = 0.002, \quad u_9 = 0.02, \quad u_{10} = 0.01 \\
    m_1 &= 0.01, \quad m_2 = 0.02, \quad e = 0.003.
\end{align*} \tag{15}
Figure 3-Trajectory of system (2) that begin from different initial points (0.9, 0.7, 0.5, 0.3) and (0.1, 0.1, 0.9, 0.8). For the data given by Eq(15) (a) Trajectories of immature prey as a function of a time. (b) Trajectory of mature prey as a function of a time (c) Trajectory of susceptible predator as a function of a time (d) Trajectory of infected predator as a function of a time.

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