ON THE VANISHING OF HOMOLOGY FOR MODULES OF
FINITE COMPLETE INTERSECTION DIMENSION

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Abstract. We prove rigidity type results on the vanishing of stable Ext and Tor for modules of finite complete intersection dimension, results which generalize and improve upon known results. We also introduce a notion of pre-rigidity, which generalizes phenomena for modules of finite complete intersection dimension and complexity one. Using this concept, we prove results on length and vanishing of homology modules.

1. Introduction

The notion of rigidity of Tor was introduced by Auslander [Au] in order to study torsion in tensor products, and the zero divisor conjecture, for finitely generated modules over a commutative local ring. The general idea of rigidity of Tor for modules $M$ and $N$ over a ring $A$ is that the vanishing of $\text{Tor}_i^A(M, N)$ for some $i$ implies the vanishing of $\text{Tor}_j^A(M, N)$ for $j$’s different from $i$. Ever since its introduction by Auslander, rigidity of Tor has been a central topic in the theory of modules over commutative rings (see, for example, [PS], [Ho], and [He]).

Rigidity of Tor for finitely generated modules over unramified regular local rings was resolved by Auslander himself, and the ramified case was settled by Lichtenbaum [Li]. The next natural class of rings over which to study rigidity is that of complete intersections, and this was done in [HW1], [HW2], [Jo1], and more recently, [Da1] and [Da2]. Subsequent to the notion of complete intersection dimension, defined in [AGP], there has been a study of rigidity of Tor and Ext for modules of finite complete intersection dimension, for example [ArY], [Jo2], [AvB], and [Be2].

In this paper, we prove new rigidity results for Ext and Tor which generalize or improve upon many of the results in the above citations. We do so in the context of stable (co)homology. We show in Section 3 that the vanishing of $c$ ($c$ being the complexity of one of the modules) equally spaced stable Ext or Tor implies the vanishing of infinitely many of the remaining (co)homology modules. We also show that if $\dim R + 2$ consecutive stable Ext or Tor vanish infinitely often for negative or positive indices, respectively, then all the stable Ext or Tor must vanish.

In Section 4 we introduce a notion we call pre-rigidity, and show that it generalizes the vanishing phenomena of modules of finite complete intersection dimension and complexity one. We also show that it gives a formula for length which recovers known results for Betti numbers of modules over rings having an embedded deformation.

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In Section 2 we give preliminaries on complete intersection dimension, complexity, and stable (co)homology.

2. Finite complete intersection dimension

Throughout this section, we fix a local (meaning commutative Noetherian local) ring \((A, \mathfrak{m}, k)\), together with a finitely generated \(A\)-module \(M\). Given a minimal free resolution

\[
\cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0
\]

of \(M\), we denote the rank of the free module \(F_n\) by \(\beta_n(M)\). This integer, the \(n\)th Betti number of \(M\), is well-defined for all \(n\), since minimal free resolutions over local rings are unique up to isomorphisms. The complexity of \(M\), denoted \(\text{cx} M\), is defined as

\[
\text{cx} M \defeq \inf\{ t \in \mathbb{N} \cup \{0\} | \exists a \in \mathbb{R} \text{ such that } \beta_n(M) \leq an^{t-1} \text{ for all } n \gg 0 \}.
\]

The complexity of a finitely generated module over a local ring is not always finite; by a theorem of Gulliksen (cf. [Gul]), the local rings over which all finitely generated modules have finite complexity are precisely the complete intersections.

In [AGP], Avramov, Gasharov and Peeva defined and studied a class of modules behaving homologically as modules over complete intersections. Recall that a quasi-deformation of \(A\) is a diagram

\[
A \rightarrow R \leftarrow Q
\]

of local homomorphisms, in which \(A \rightarrow R\) is faithfully flat, and \(R \leftarrow Q\) is surjective with kernel generated by a regular sequence. The module \(M\) has finite complete intersection dimension if there exists such a quasi-deformation for which \(\text{pd}_Q(R \otimes_A M)\) is finite. The complete intersection dimension of \(M\), denoted \(\text{CI-dim} M\), is the infimum of all \(\text{pd}_Q(R \otimes_A M)\), the infimum taken over all quasi-deformations \(A \rightarrow R \leftarrow Q\) of \(A\). In the rest of the paper, we write “CI-dimension” instead of “complete intersection dimension”.

By [AGP, Theorem 5.3], every module of finite CI-dimension has finite complexity. Moreover, as we shall see in the next section, such a module also has reducible complexity in the sense of [Be1]. This reflects the fact that modules of finite CI-dimension behave homologically as modules over complete intersections. Since complete intersection rings are Gorenstein, modules of finite CI-dimension also behave, in some sense, as modules over Gorenstein rings. In order to make this precise, we recall the following, denoting the \(A\)-module \(\text{Hom}_A(M, A)\) by \(M^*\). We say that \(M\) is of Gorenstein dimension zero, denoted \(G \text{-dim} M = 0\), if it is reflexive (i.e. the canonical homomorphism \(M \rightarrow M^{**}\) is bijective) and \(\text{Ext}_A^n(M, A) = 0\) for \(n > 0\). The Gorenstein dimension of \(M\), denoted \(G \text{-dim} M\), is the infimum of the numbers \(n\), for which there exists an exact sequence

\[
0 \rightarrow G_n \rightarrow \cdots \rightarrow G_0 \rightarrow M \rightarrow 0
\]

in which \(G \text{-dim} G_i = 0\). By [AuB], a local ring is Gorenstein precisely when all its finitely generated modules have finite Gorenstein dimension.

If \(M\) has finite Gorenstein dimension \(d\), say, then by [AuB, Corollary 3.15], the module \(\Omega^d_A(M)\) has Gorenstein dimension zero. Choose a minimal free resolution \(S \rightarrow \Omega^d_A(M)^* \rightarrow 0\) of \(\Omega^d_A(M)^*\), and consider the dualized complex \(0 \rightarrow \Omega^d_A(M) \rightarrow S^*\). It follows directly from the defining properties of modules of Gorenstein dimension zero that this complex is exact. Splicing this complex with the minimal
free resolution of $\Omega^d_A(M)$, we obtain a doubly infinite minimal exact sequence

$$Q: \cdots \to Q_2 \xrightarrow{d_2} Q_1 \xrightarrow{d_1} Q_0 \xrightarrow{d_0} Q_{-1} \xrightarrow{d_{-1}} Q_{-2} \to \cdots$$

of free modules, in which $\text{Im} \ d_n = \Omega^d_A(M)$. Then $Q$ is a minimal complete resolution of $M$, and it is unique up to homotopy equivalence (cf. [Buc], [CoK]). Consequently, for every $n \in \mathbb{Z}$ and every $A$-module $N$, the stable homology and stable cohomology modules

$$\widehat{\text{Tor}}^A_n(M, N) \overset{\text{def}}{=} H_n(Q \otimes_A N)$$
$$\widehat{\text{Ext}}^A_n(M, N) \overset{\text{def}}{=} H_{-n}(\text{Hom}_A(Q, N))$$

are independent of the choice of complete resolution of $M$. By construction, there are isomorphisms $\widehat{\text{Tor}}^A_n(M, N) \cong \text{Tor}^A_n(M, N)$ and $\widehat{\text{Ext}}^A_n(M, N) \cong \text{Ext}^A_n(M, N)$ whenever $n > d$.

By [AGP, Theorem 1.4], if the CI-dimension of $M$ is finite, then $G\text{-dim } M = \text{CI-dim } M = \text{depth } A - \text{depth } M$.

Therefore $M$ admits a minimal complete resolution, and from the above we see that, for every $A$-module $N$, there are isomorphisms

$$\widehat{\text{Tor}}^A_n(M, N) \cong \text{Tor}^A_n(M, N)$$
$$\widehat{\text{Ext}}^A_n(M, N) \cong \text{Ext}^A_n(M, N)$$

for all $n > \text{depth } A - \text{depth } M$. Consequently, for a module of finite CI-dimension, vanishing patterns in stable (co)homology correspond to vanishing patterns in ordinary (co)homology beyond depth $A - \text{depth } M$. We shall therefore state the vanishing results in terms of stable (co)homology.

### 3. Vanishing of (co)homology

In this section, we establish our rigidity results for stable Ext and Tor for modules of finite CI-dimension. We start with the following lemma, which shows that a module of finite CI-dimension has reducible complexity.

**Lemma 3.1.** Let $A$ be a local ring, and $M$ a finitely generated $A$-module of finite CI-dimension and infinite projective dimension. Then, given any odd number $q$, there exists a faithfully flat extension $A \to R$ and an exact sequence

$$0 \to R \otimes_A M \to K \to \Omega^R_q(R \otimes_A M) \to 0$$

of $R$-modules, with $\text{cx}_R K = \text{cx}_A M - 1$. Moreover, the $R$-modules $R \otimes_A M$ and $K$ have finite CI-dimension, with $\text{CI-dim}_R(R \otimes_A M) = \text{CI-dim}_R K = \text{depth } A - \text{depth } M$.

**Proof.** By [Be2, Lemma 2.1], for any odd integer $q \geq 1$, there exists a quasi-deformation $A \to R \leftarrow Q$ and an exact sequence

$$0 \to R \otimes_A M \to K \to \Omega^R_{q-1}(R \otimes_A M) \to 0$$

of $R$-modules, with $\text{cx}_R K = \text{cx}_A M - 1$. Moreover, in the proof of [Be2, Lemma 2.1] it is shown that the CI-dimensions of both the $R$-modules $K$ and $R \otimes_A M$ are finite. Since the CI-dimension of $R \otimes_A M$ is finite, so is the CI-dimension of $\Omega^R_{q-1}(R \otimes_A M)$, and the result follows.
and by [AGP] Lemma 1.9 the inequality \( \text{depth}_R (R \otimes_A M) \leq \text{depth}_R \Omega^{2n-1}_R (R \otimes_A M) \) holds. But then \( \text{depth}_R K = \text{depth}_R (R \otimes_A M) \), and so

\[
\text{depth}_R R - \text{depth}_R K = \text{depth}_R R - \text{depth}_R (R \otimes_A M) = \text{depth}_A A - \text{depth}_A M,
\]

where the latter equality is due to faithful flatness. \( \square \)

Having established the necessary lemma, we now prove the first of the main results of this section.

**Theorem 3.2.** Let \( A \) be a local ring, and \( M \) a finitely generated \( A \)-module of finite CI-dimension and complexity \( c \). Furthermore, let \( N \) be a not necessarily finitely generated \( A \)-module. Suppose there is an integer \( n \in \mathbb{Z} \) and an odd number \( q \) such that

\[
\widetilde{\text{Tor}}_n^A(M, N) = \widetilde{\text{Tor}}_{n+q}^A(M, N) = \cdots = \widetilde{\text{Tor}}_{n+(c-1)q}^A(M, N) = 0.
\]

Then \( \widetilde{\text{Tor}}_{n-(i+1)q}(M, N) = \widetilde{\text{Tor}}_{n+(c-1)q+i(q+1)}(M, N) = 0 \) for all integers \( i \geq 1 \).

**Proof.** Denote depth \( A - \text{depth} M \) by \( d \). If \( c = 0 \), then there is nothing to prove since by the Auslander-Buchsbaum formula, the module \( \Omega^d_A(M) \) is free, and so \( \widetilde{\text{Tor}}_i^A(M, N) = 0 \) for all \( i \).

The proof proceeds by induction on the complexity \( c \) of \( M \). If \( c = 1 \), then by [AGP] Theorem 7.3] the module \( \Omega^d_A(M) \) is periodic of period at most two, hence so is the minimal complete resolution of \( M \). In particular, the modules \( \widetilde{\text{Tor}}_i^A(M, N) \) and \( \widetilde{\text{Tor}}_{i+2}^A(M, N) \) are isomorphic for all integers \( i \). Since \( q \) is an odd number, the case \( c = 1 \) follows.

Next, suppose that \( c \geq 2 \). Choose a faithfully flat extension \( A \rightarrow R \), together with an exact sequence

\[
0 \rightarrow R \otimes_A M \rightarrow K \rightarrow \Omega^d_R (R \otimes_A M) \rightarrow 0
\]

of \( R \)-modules, as in Lemma 3.1. Thus, the \( R \)-modules \( R \otimes_A M \) and \( K \) have finite CI-dimension, and the complexity of \( K \) is \( c-1 \). For every \( i \in \mathbb{Z} \) there is an isomorphism \( \widetilde{\text{Tor}}_i^R (R \otimes_A M, R \otimes_A N) \cong R \otimes_A \widetilde{\text{Tor}}_i^A (M, N) \), hence \( \widetilde{\text{Tor}}_i^A (M, N) \) vanishes if and only if \( \widetilde{\text{Tor}}_i^R (R \otimes_A M, R \otimes_A N) \) does. We may therefore, without loss of generality, assume that there exists an exact sequence

\[
0 \rightarrow M \rightarrow K \rightarrow \Omega^d_A(M) \rightarrow 0
\]

of \( A \)-modules, in which \( K \) has finite CI-dimension and complexity \( c-1 \). By the homology version of [AvM] Proposition 5.6], this short exact sequence induces a doubly infinite long exact sequence

\[
\cdots \rightarrow \widetilde{\text{Tor}}_{i+1}^A(K, N) \rightarrow \widetilde{\text{Tor}}_i^A(\Omega^d_A(M), N) \rightarrow \widetilde{\text{Tor}}_i^A(M, N) \rightarrow \widetilde{\text{Tor}}_{i+1}^A(K, N) \rightarrow \cdots
\]

of complete homology modules. Using [AvM] Proposition 5.6] once more, together with the fact that \( \widetilde{\text{Tor}}_i^A(F, N) = 0 \) for all \( i \) whenever \( F \) is free, we see that \( \widetilde{\text{Tor}}_i^A(\Omega^d_A(M), N) \) is isomorphic to \( \widetilde{\text{Tor}}_{i+q}^A(M, N) \) for all \( i \). Consequently, we obtain a long exact sequence

\[
\cdots \rightarrow \widetilde{\text{Tor}}_{i+1}^A(K, N) \rightarrow \widetilde{\text{Tor}}_{i+q+1}^A(M, N) \rightarrow \widetilde{\text{Tor}}_i^A(M, N) \rightarrow \widetilde{\text{Tor}}_{i+1}^A(K, N) \rightarrow \cdots
\]

of complete homology modules.
The vanishing assumption on $\text{Tor}^A_i(M, N)$ forces $\text{Tor}^A_i(K, N)$ to vanish for $i \in \{n, n + q, \ldots, n + (c - 2)q\}$. By induction, the modules $\text{Tor}_{n-i(q+1)}^A(K, N)$ and $\text{Tor}_{n+(c-2)q+i(q+1)}^A(K, N)$ vanish for all integers $i \geq 1$. Looking at the above long exact sequence again, we see that for all integers $i \geq 1$ the modules $\text{Tor}_{n-i(q+1)}^A(M, N)$ and $\text{Tor}_{n+(c-1)q+i(q+1)}^A(M, N)$ also must vanish. □

We include the cohomology version of Theorem 3.2 but omit the proof.

**Theorem 3.3.** Let $A$ be a local ring, and $M$ a finitely generated $A$-module of finite CI-dimension and complexity $c$. Furthermore, let $N$ be a not necessarily finitely generated $A$-module. Suppose there is an integer $n \in \mathbb{Z}$ and an odd number $q$ such that

$$\hat{\text{Ext}}_A^n(M, N) = \hat{\text{Ext}}_A^{n+q}(M, N) = \cdots = \hat{\text{Ext}}_A^{n+(c-1)q}(M, N) = 0.$$  

Then $\hat{\text{Ext}}_A^{n-i(q+1)}(M, N) = \hat{\text{Ext}}_A^{n+(c-1)q+i(q+1)}(M, N) = 0$ for all integers $i \geq 1$.

In the following corollaries, we record the special case $q = 1$ from the previous theorems.

**Corollary 3.4.** Let $A$ be a local ring, and $M$ a finitely generated $A$-module of finite CI-dimension and complexity $c$. Furthermore, let $N$ be a not necessarily finitely generated $A$-module. Suppose there is an integer $n \in \mathbb{Z}$ such that

$$\text{Tor}_n^A(M, N) = \text{Tor}_{n+1}^A(M, N) = \cdots = \text{Tor}_{n+c-1}^A(M, N) = 0.$$  

Then $\text{Tor}_{n-2i}^A(M, N) = \text{Tor}_{n+c-1+2i}^A(M, N) = 0$ for all integers $i \geq 1$.

**Corollary 3.5.** Let $A$ be a local ring, and $M$ a finitely generated $A$-module of finite CI-dimension and complexity $c$. Furthermore, let $N$ be a not necessarily finitely generated $A$-module. Suppose there is an integer $n \in \mathbb{Z}$ such that

$$\hat{\text{Ext}}_n^A(M, N) = \hat{\text{Ext}}_{n+1}^A(M, N) = \cdots = \hat{\text{Ext}}_{n+c-1}^A(M, N) = 0.$$  

Then $\hat{\text{Ext}}_{n-2i}^A(M, N) = \hat{\text{Ext}}_{n+c-1+2i}^A(M, N) = 0$ for all integers $i \geq 1$.

We note that Theorems 3.2 and 3.3 recover results of [Jo2] and [Be2] for the vanishing of $c_\lambda A M + 1$ consecutive $\text{Ext}$ and $\text{Tor}$ for modules of finite CI-dimension.

**Corollary 3.6.** Let $A$ be a local ring, and $M$ a finitely generated $A$-module of finite CI-dimension and complexity $c$. Furthermore, let $N$ be a not necessarily finitely generated $A$-module. Suppose there is an integer $n \geq \text{depth } A - \text{depth } M + 1$ such that

$$\text{Tor}_n^A(M, N) = \text{Tor}_{n+1}^A(M, N) = \cdots = \text{Tor}_{n+c}^A(M, N) = 0.$$  

Then $\text{Tor}_i^A(M, N) = 0$ for all integers $i \geq \text{depth } A - \text{depth } M + 1$.

**Corollary 3.7.** Let $A$ be a local ring, and $M$ a finitely generated $A$-module of finite CI-dimension and complexity $c$. Furthermore, let $N$ be a not necessarily finitely generated $A$-module. Suppose there is an integer $n \geq \text{depth } A - \text{depth } M + 1$ such that

$$\text{Ext}_n^A(M, N) = \text{Ext}_{n+1}^A(M, N) = \cdots = \text{Ext}_{n+c}^A(M, N) = 0.$$  

Then $\text{Ext}_i^A(M, N) = 0$ for all integers $i \geq \text{depth } A - \text{depth } M + 1$.  

\[\text{Vanishing of Homology}\]
We also generalize a result of [Jo1] for vanishing of \( \text{Tor} \) for modules over complete intersections.

**Theorem 3.8.** Let \( A \) be a local Cohen-Macaulay ring of dimension \( d \), and \( M \) and \( N \) finitely generated \( A \)-modules with \( M \) of finite CI-dimension. Then there exists an integer \( n_0 \) with the following property: if

\[
\hat{\text{Tor}}_i^A(M, N) = \hat{\text{Tor}}_{i+1}^A(M, N) = \cdots = \hat{\text{Tor}}_{i+d}^A(M, N) = 0
\]

for one even \( i \geq n_0 \), and

\[
\hat{\text{Tor}}_j^A(M, N) = \hat{\text{Tor}}_{j+1}^A(M, N) = \cdots = \hat{\text{Tor}}_{j+d}^A(M, N) = 0
\]

for one odd \( j \geq n_0 \), then \( \hat{\text{Tor}}_n^A(M, N) = 0 \) for all \( n \in \mathbb{Z} \).

**Remark.** Using the fact that for finitely generated \( A \)-modules \( M \) and \( N \) with \( M \) maximal Cohen-Macaulay,\n
\[
\hat{\text{Tor}}_i^R(M, N) \cong \hat{\text{Ext}}_{\hat{R}}^{-i-1}(M^*, N)
\]

for all \( i \in \mathbb{Z} \), one has a statement similar to that of [3.8] for vanishing of stable \( \text{Ext} \) with \( i \leq n_0 \) and \( j \leq n_0 \).

**Proof.** We prove this result in terms of vanishing of the ordinary homology modules \( \text{Tor}_n^A(M, N) \) for \( n > \text{depth } A - \text{depth } M \). For, by [AvB, Theorem 4.9], the complete homology modules \( \hat{\text{Tor}}_n^A(M, N) \) vanish for all \( n \in \mathbb{Z} \) if and only if \( \text{Tor}_n^A(M, N) = 0 \) for \( n > \text{depth } A - \text{depth } M \).

The proof is by induction on \( d \), the case \( d = 0 \) being covered by [Jo1, Theorem 3.1] (strictly speaking, the result [Jo1, Theorem 3.1] is formulated for modules over complete intersections, but the proof carries over verbatim to modules of finite CI-dimension). Suppose therefore that \( d \) is positive. We may assume that both \( M \) and \( N \) are of positive depth; if not, then we replace them by their first syzygies \( \Omega^1_A(M) \) and \( \Omega^1_A(N) \). By [AGP, Lemma 1.9], the module \( \Omega^1_A(M) \) also has finite CI-dimension.

Choose an element \( x \in A \) which is regular on \( M, N \) and \( A \), and consider the exact sequence

\[
0 \rightarrow M/xM \rightarrow M \rightarrow M/xM \rightarrow 0.
\]

This sequence induces a long exact sequence

\[
\cdots \rightarrow \text{Tor}^A_i(M, N) \xrightarrow{x} \text{Tor}^A_i(M, N) \rightarrow \text{Tor}^A_i(M/xM, N) \rightarrow \text{Tor}^A_{i-1}(M, N) \rightarrow \cdots
\]

in homology. Now denote the ring \( A/(x) \) by \( \hat{A} \), and the \( \hat{A} \)-modules \( M/xM \) and \( N/xN \) by \( \hat{M} \) and \( \hat{N} \), respectively. Note that, by [AGP, Proposition 1.12], \( \hat{M} \) has finite CI-dimension. Thus, since the dimension of \( \hat{A} \) is \( d - 1 \), by induction there exists an integer \( n_0 \) with the following property: if

\[
\hat{\text{Tor}}_i^\hat{A}(\hat{M}, \hat{N}) = \hat{\text{Tor}}_{i+1}^\hat{A}(\hat{M}, \hat{N}) = \cdots = \hat{\text{Tor}}_{i+d-1}^\hat{A}(\hat{M}, \hat{N}) = 0
\]

for one even \( i \geq n_0 \), and

\[
\hat{\text{Tor}}_j^\hat{A}(\hat{M}, \hat{N}) = \hat{\text{Tor}}_{j+1}^\hat{A}(\hat{M}, \hat{N}) = \cdots = \hat{\text{Tor}}_{j+d-1}^\hat{A}(\hat{M}, \hat{N}) = 0
\]

for one odd \( j \geq n_0 \), then \( \hat{\text{Tor}}_n^\hat{A}(\hat{M}, \hat{N}) = 0 \) for all \( n > \dim \hat{A} - \text{depth } \hat{M} \). Note that \( \dim \hat{A} - \text{depth } \hat{M} = d - \text{depth } M \).
Suppose
\[
\text{Tor}_i^A(M, N) = \text{Tor}_{i+1}^A(M, N) = \cdots = \text{Tor}_{i+d}^A(M, N) = 0
\]
for one even \(i \geq n_0 - 1\), and
\[
\text{Tor}_j^A(M, N) = \text{Tor}_{j+1}^A(M, N) = \cdots = \text{Tor}_{j+d}^A(M, N) = 0
\]
for one odd \(j \geq n_0 - 1\). Then the above long exact homology sequence implies that \(\text{Tor}_n^A(M, N) = 0\) for \(i + 1 \leq n \leq i + d\) and \(j + 1 \leq n \leq j + d\). By [Mat] Lemma 18.2(iii), there is an isomorphism \(\text{Tor}_n^A(M, N) \cong \text{Tor}_n^R(M, N)\) for every \(n > 0\), and so from above we see that \(\text{Tor}_n^A(M, N)\) vanishes for all \(n > d - \text{depth} M\). The long exact homology sequence then shows that \(\text{Tor}_n^A(M, N) = x \text{Tor}_n^A(M, N)\) for all \(n > d - \text{depth} M\), and by Nakayama’s Lemma we conclude that \(\text{Tor}_n^A(M, N) = 0\) for all \(n > d - \text{depth} M\). \(\square\)

Corollary 3.9. Let \(A\) be a local Cohen-Macaulay ring of dimension \(d\), and \(M\) and \(N\) finitely generated \(A\)-modules with \(M\) of finite CI-dimension. If for all positive integers \(n\) there exists an \(i \geq n\) such that
\[
\text{Tor}_i^A(M, N) = \text{Tor}_{i+1}^A(M, N) = \cdots = \text{Tor}_{i+d}^A(M, N) = 0
\]
then \(\text{Tor}_n^A(M, N) = 0\) for all \(n \in \mathbb{Z}\).

We remark that the examples of [Jo] 4.1 illustrate the sharpness of Theorems 3.2 and 3.3 in the \(q = 1\) case, in the sense that more vanishing cannot be concluded from the hypothesis. We recall these examples, in the context of stable (co)homology, and prove that certain homology modules remain nonzero.

Example 3.10. Let \(n\) be a positive integer and
\[
R = k[[X_1, \ldots, X_n, Y_1, \ldots, Y_n]]/(X_1Y_1, \ldots, X_nY_n),
\]
where \(k\) is a field and the \(X_i\) and \(Y_i\) are analytic indeterminates. Then \(R\) is a complete intersection of dimension \(n\) and codimension \(n\). Let \(M = R/(x_1, \ldots, x_n),\) \(N = R/(y_1, \ldots, y_n)\). Then, as is shown in [Jo], \(M\) and \(N\) are maximal Cohen-Macaulay \(R\)-modules of complexity \(n\) with \(\text{Ext}_{-i}^R(M, N) = 0\) for \(0 \leq i \leq n - 1\), and \(\text{Ext}_R^{-i}(M, N) \neq 0 \neq \text{Ext}_R^{-2i}(M, N)\). Theorem 3.3 shows that \(\text{Ext}_R^{-i}(M, N) = 0\) for all \(i \geq 1\). We moreover claim that \(\text{Ext}_R^{-i}(M, N) \neq 0\) for all \(i \geq 1\).

Indeed, note that \(M \cong M^*\). Consider the ring \(S = k[[X_1, Y_1]]/(X_1Y_1)\), and \(S\)-module \(M' = S/(x_1)\). One can construct a chain map \(f\) between the minimal resolution \(\cdots \rightarrow S \xrightarrow{x_1} S \xrightarrow{y_1} S \xrightarrow{x_1} S \rightarrow M' \rightarrow 0\) of \(M'\) over \(S\) and one \(F\) of \(M\) over \(R\) (and consequently one of \(M^*\) as well)

\[
\cdots \rightarrow S \xrightarrow{x_1} S \xrightarrow{y_1} S \xrightarrow{x_1} S \rightarrow M' \rightarrow 0
\]

\[
\cdots \rightarrow F_3 \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} R \rightarrow M \rightarrow 0
\]
such that \( f_i(1) \) is a basis element of \( F_i \) for all \( i \geq 0 \). Tensoring the top row with \( N' = S/(y_1) \) and the bottom row with \( N \), we get an induced commutative diagram

\[
\cdots \longrightarrow N' \xrightarrow{\bar{x}_1} N' \xrightarrow{0} N' \xrightarrow{\bar{x}_1} N' \longrightarrow 0
\]

\[
\cdots \longrightarrow F_3 \otimes_R N \xrightarrow{\partial_3 \otimes N} F_2 \otimes_R N \xrightarrow{\partial_2 \otimes N} F_1 \otimes_R N \xrightarrow{\partial_1 \otimes N} N \longrightarrow 0
\]

in which \( \bar{f}_i(1) \) is a minimal generator of \( F_i \otimes_R N \) for all \( i \geq 0 \). It follows that \( \text{Tor}^{2i}(M, N) \neq 0 \) for all \( i \geq 0 \). Finally we note that a complete resolution of \( M \cong M^* \) is given by

\[
\cdots \longrightarrow F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{[y_1 \ldots y_2]} F_0 \xrightarrow{\partial'_1} F_1 \xrightarrow{\partial'_2} F_2 \longrightarrow \cdots
\]

and so \( \text{Tor}_i^R(M, N) = \text{Ext}_{-i}^1(M, N) \) for all \( i \in \mathbb{Z} \). Thus \( \text{Ext}_{-2i}^1(M, N) \neq 0 \) for all \( i \geq 1 \), and this is what we claimed.

### 4. Pre-rigidity of Modules

Throughout this section, unless otherwise specified we let \((Q, n, k)\) be a local ring, \( x \) a non-zero divisor contained in the maximal ideal of \( Q \), and \( R = Q/(x) \). Let \( M \) be a finitely generated non-zero \( R \)-module, and \( F \) a \( Q \)-free resolution of \( M \). Assume that \( \{\sigma_i\}_{i \geq 0} \) is a system of higher homotopies on \( F \). That is, for all \( i \geq 0 \) each \( \sigma_i \) is a degree \( 2i - 1 \) endomorphism of \( F \) as a graded module with \( \sigma_0 = 0 \), \( \sigma_0 \sigma_1 + \sigma_1 \sigma_0 = x \text{Id}_F \) and \( \sum_{i+j=n} \sigma_i \sigma_j = 0 \) for \( n > 1 \). (Shamash shows in [Sha] that such a system always exists.)

**Definition 4.1.** We say that an \( R \)-module \( N \) is pre-rigid of degree \( r \) with respect to \( M \) and \( Q \) if there exists a \( Q \)-free resolution \( F \) of \( M \) and a system of higher homotopies \( \{\sigma_i\}_{i \geq 0} \) on \( F \) such that the induced maps

\[
(\sigma_i)_j \otimes_Q N : F_j \otimes_Q N \to F_{j+2i-1} \otimes_Q N
\]

are zero for \( j > r - (2i - 1) \), and all \( i \geq 1 \).

**Example 4.2.** If \( \text{pd}_Q M = r < \infty \), then every \( R \)-module \( N \) is pre-rigid of degree \( r \) with respect to \( M \) and \( Q \).

**Example 4.3.** Suppose that \( \sigma_i(F) \subseteq nF \) for all \( i \geq 1 \). Then \( k \) is pre-rigid of degree \( 0 \) with respect to \( M \) and \( Q \).

The following is the main result of this section. It motivates the choice of terminology.

**Theorem 4.4.** Let \( M \) be a finitely generated \( R \)-module, and assume that \( N \) is an \( R \)-module which is pre-rigid of degree \( r \) with respect to \( M \) and \( Q \). If \( \text{Tor}_{n}^{R}(M, N) = 0 \) for some \( n > r \), then \( \text{Tor}_{n-2i}^{Q}(M, N) = 0 \) for \( n \geq n - 2i > r \). If \( r = 0 \), then \( \text{Tor}_{n-2i}^{Q}(M, N) = 0 \) for all \( i \geq 0 \).

In preparation for the proof of Theorem 4.4 we want to describe a free resolution of \( M \) over \( R \) using one of \( M \) over \( Q \), following [Sha] (see also [AvB 3.1.3]).
Let $D$ be the complex of $R$-modules with trivial differential having $D_i = 0$ for $i < 0$, $D_{2i-1} = 0$ for $i \geq 1$, and $D_2$ the free $R$-module $Re_1$ on the singleton basis $e_i$ for $i \geq 0$. Let $F$ be a free resolution of $M$ over $Q$, and $\{\sigma_i\}_{i\geq 0}$ a system of higher homotopies on $F$ (recall that $\sigma_0$ is the differential of $F$). We equip the complex $D \otimes_Q F$ with the differential $\partial = \sum_j u_j \otimes \sigma_j$ where $u_j$ is defined by $u(e_i) = e_{i-j}$, so that $\partial(e_i \otimes f) = \sum_j e_{i-j} \otimes \sigma_j(f)$. Then $(D \otimes_Q F, \partial)$ is a free resolution of $M$ over $R$. [Sha].

**Proof.** We may compute $\text{Tor}^R_i(M, N)$ from the complex

$$F = (D \otimes_Q F) \otimes_R N \cong D \otimes_Q F \otimes_Q N.$$ Filtering this complex by $F_p = \sum_{i \leq p} D_{2i} \otimes_Q F \otimes_Q N$ one gets an upper semi-first-quadrant convergent spectral sequence whose $E^0$-page is

![Spectral Sequence Diagram](attachment:image.png)

with the convention that $E^0_{i,j} = D_{2i} \otimes_Q F_{j-i}$. Since $D_{2i} \cong R$ for all $i \geq 0$, the $E^1$-page of this spectral sequence is

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\text{Tor}^Q_3(M, N) \xleftarrow{d^1_{3,3}} \text{Tor}^Q_2(M, N) \xleftarrow{d^2_{2,3}} \text{Tor}^Q_1(M, N) \xleftarrow{d^3_{1,3}} \text{Tor}^Q_0(M, N)$$

$$\quad \text{Tor}^Q_2(M, N) \xleftarrow{d^1_{2,2}} \text{Tor}^Q_1(M, N) \xleftarrow{d^2_{1,2}} \text{Tor}^Q_0(M, N)$$

$$\quad \text{Tor}^Q_1(M, N) \xleftarrow{d^1_{1,1}} \text{Tor}^Q_0(M, N)$$

where the maps $d^i_{j,1}$ are induced by the maps

$$t \otimes (\sigma_1)_{i-1} \otimes N : D_2 \otimes F_{i-1} \otimes N \rightarrow D_0 \otimes F_i \otimes N$$
for $i \geq 1$. Note that $d_{i,j}^1 = d_{i+1,j+1}^1$ for all $i, j \geq 1$.

Now assume that $N$ is $r$-rigid with respect to $M$ and $Q$. Then it is clear that the maps $d_{i,j}^1 = 0$ for all $j \geq r$, and thus $d_{i,j}^1 = 0$ for all $j \geq i + r$.

It follows that $E_{i,j}^2 = E_{i,j}^1 = \text{Tor}_i^Q(M, N)$ for all $j \geq i + r + 1$. In general, the hypothesis that $N$ is $r$-rigid implies that the maps $d_{i,j}^n$ on the $E^{r^*}$-page of the spectral sequence are zero for all $j \geq i + r - (2s - 1) + 1$, and all $s \geq 1$, and thus the limit terms of the spectral sequence are given by

$$E_{i,j}^\infty = E_{i,j}^1 = \text{Tor}_i^Q(M, N)$$

for all $j \geq i + r + 1$.

Now taking the associated filtration $\Phi$ of the total homology $H$ of $F$ (see, for example, [Roj 11.13]), we have isomorphisms $\text{Tor}_j^Q(M, N) \cong \Phi^i H_{i+j}/\Phi^{i-1} H_{i+j}$ for $j \geq i + r + 1$. Since $H_n = \text{Tor}_n^A(M, N)$ for all $n$, the first statement of Theorem 4.4 follows easily.

When $r = 0$ we actually get that $E_{i,j}^1 = E_{i,j}^2 = \text{Tor}_i^Q(M, N)$ for all $j \geq i$, and so the second statement of the theorem holds.

The following main corollary of 4.4 shows that the notion of $r$-rigidity generalizes in a sense the behavior of modules of finite CI-dimension and complexity one.

**Corollary 4.5.** Let $A$ be a local ring, and assume that $M$ is a finitely $A$-module with finite CI-dimension. Let $A \to R \to Q$ be a codimension $c$ quasi-deformation with $R \cong Q/(x_1, \ldots, x_c)$ such that $\text{pd}_Q M \otimes_A R < \infty$. Assume that $N$ is an $A$-module such that $N \otimes_A R$ is $r$-rigid of degree $r$ with respect to $M \otimes_A R$ and $Q/(x_1, \ldots, x_c)$. Set $b = \max\{r, \text{depth}_A M - \text{depth}_A M + 1\} + c$. If $\text{Tor}_n^A(M, N) = 0$ for one even value of $n \geq b$ and one odd value of $n \geq b$, then $\text{Tor}_n^A(M, N) = 0$ for all $n \geq \text{depth}_A M - \text{depth}_M + 1$.

**Proof.** Suppose that $\text{Tor}_n^A(M, N) = 0$ for an even $n_e \geq b$ and $\text{Tor}_n^A(M, N) = 0$ for an odd $n_o \geq b$. By flatness we have $\text{Tor}_n^R(M', N') = \text{Tor}_n^R(M', N') = 0$, where $M' = R \otimes_A M$ and $N' = R \otimes_A N$. Let $Q' = Q/(x_2, \ldots, x_c)$. By assumption $N'$ is $r$-rigid of degree $r$ with respect to $M'$ and $Q'$. Since $b > r$, Theorem 4.3 applies to give $\text{Tor}_n^Q(M', N') = 0$ for $n \geq n - j > r$, where $n = \min\{n_o, n_e\}$. Since $n - r \geq b - r \geq c$, and $n - (\text{depth}_A M - \text{depth}_A M + 1) \geq b - (\text{depth}_A M + 1) \geq c$, we have at least $c$ consecutive vanishing $\text{Tor}_n^Q(M', N') = 0$ beyond $\text{depth}_A M - \text{depth}_A M + 1 = \text{depth}_A M - \text{depth}_A M'$. The complexity of $M'$ as a $Q'$-module is at most $c - 1$. Thus by [Jo2 2.2] we have $\text{Tor}_j^Q(M', N') = 0$ for all $j \geq \text{depth}_A Q' - \text{depth}_A Q' M' + 1$. A standard argument (see, for example, [Jo1 0.1]) now shows that $\text{Tor}_j^R(M', N') = \text{Tor}_j^R(M', N')$ for all $j \geq \text{depth}_R M' - \text{depth}_R M' + 1$. Finally, since $\text{Tor}_n^R(M', N') = \text{Tor}_n^R(M', N') = 0$ it follows that $\text{Tor}_j^R(M', N') = 0$ for all $j \geq \text{depth}_R M' - \text{depth}_R M' + 1$. Thus $\text{Tor}_n^A(M, N) = 0$ for all $j \geq \text{depth}_A M - \text{depth}_A M + 1$, which was the claim.

The next corollary is an immediate consequence of Theorem 4.4.

**Corollary 4.6.** Let $M$ be a finitely generated non-zero $R$-module. Suppose that $N$ is an $R$-module which is $r$-rigid of degree $0$ with respect to $M$ and $Q$. Then $\text{Tor}_n^R(M, N) = 0$ for some even $n \geq 0$ if and only if $N = 0$. 

The next theorem shows that the pre-rigidity condition gives a formula for relative lengths of Tor.

**Theorem 4.7.** Let $M$ be a finitely generated $R$-module. Suppose that $N$ is an $R$-module which is pre-rigid of degree $0$ with respect to $M$ and $Q$. If $\text{Tor}_n^R(M, N)$ has finite length for some $n \geq 0$, then $\text{Tor}_{n-2i}^Q(M, N)$ has finite length for all $i \geq 0$, and

$$\text{length} \text{Tor}_n^R(M, N) = \sum_{i \geq 0} \text{length} \text{Tor}_{n-2i}^Q(M, N)$$

**Proof.** Consider the spectral sequence in the proof of Theorem 4.4. The associated filtration $\Phi$ of the total homology $H_{\Phi}$ of $F$ is

$$0 = \Phi^{-1}H_n \subseteq \Phi^0H_n \subseteq \cdots \subseteq \Phi^nH_n = H_n$$

for all $n$, and we have $E_{i,j}^\infty \cong \Phi^iH_n/\Phi^{i+1}H_n$ for $i + j = n$, and all $n$. If $N$ is pre-rigid of degree $0$ with respect to $M$ and $Q$, then as we saw in the proof of Theorem 4.4 $E_{i,j}^\infty \cong \text{Tor}_j^Q(M, N)$ for all $i, j$. Recalling that $H_n \cong \text{Tor}_n^R(M, N)$, the claim is now clear. \hfill \Box

We single out a particular case of interest, which follows directly from Theorem 4.7. (See, for example, the definition of $\beta_{ij}^R(M, N)$ in [Da1] and [Da2].)

**Corollary 4.8.** Let $M$ be a finitely generated $R$-module. Suppose that $N$ is an $R$-module which is pre-rigid of degree $0$ with respect to $M$ and $Q$. Then $\text{Tor}_i^R(M, N)$ has finite length for some even $i \geq 0$ if and only if $M \otimes_R N$ has finite length.

**Remark.** Suppose that a free resolution $F$ of $M$ over $Q$ admits a system of higher homotopies $\{\sigma_i\}_{i \geq 0}$ such that $\sigma_i(F) \subseteq \ker F$ for all $i \geq 0$. Then the $R$-free resolution $D \otimes F$ of $M$ in the proof of Theorem 4.4 will be minimal. In this case we see that $k$ is pre-rigid of degree $0$ with respect to $M$ and $F$. Theorem 4.7 then gives a statement about Betti numbers: $\beta_{ij}^R(M) = \sum_{i \geq 0} \beta_{n-2i}^Q(M)$, which in terms of Poincaré series translates to $P_M^R(t) = P_M^Q(t)/(1 - t^2)$. Indeed, this was the main goal of [Sha]. It is shown in loc. cit. that the condition on the minimality of $D \otimes F$ is obtained, for example, when $x \in \ker \text{Ann}_Q M$.

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**References**

[Au] M. Auslander, *Modules over unramified regular local rings*, Illinois J. Math. 5 (1961), 631–647.

[AuB] M. Auslander, M. Bridger, *Stable module theory*, Mem. Amer. Math. Soc. 94 (1969).

[AvB] L. Avramov, *Infinite Free Resolutions* Six lectures on commutative algebra, (J. Elias, J.M. Giral, R.M. Miró-Roig, S. Zarzuela, eds.) Progress in Mathematics; Vol. 166, Birkhäuser, 1998

[AvB] L. Avramov, R.-O. Buchweitz, *Support varieties and cohomology over complete intersections*, Invent. Math. 142 (2000), 285-318.

[AGP] L. Avramov, V. Gasharov, I. Peeva, *Complete intersection dimension*, Inst. Hautes Études Sci. Publ. Math. No. 86 (1997), 67-114 (1998).
[ArY] T. Araya and Y. Yoshino, Remarks on a depth formula, a grade inequality and a conjecture of Auslander, Comm. Algebra 26 (1998), 3793-3806.

[AvM] L. Avramov, A. Martsinkovsky, Absolute, relative and Tate cohomology of modules of finite Gorenstein dimension, Proc. London Math. Soc. (3) 85 (2002), 393-440.

[Be1] P.A. Bergh, Modules with reducible complexity, J. Algebra 310 (2007), 132-147.

[Be2] P.A. Bergh, On the vanishing of (co)homology over local rings, J. Pure Appl. Algebra 212 (2008), no. 1, 262-270.

[Buc] R.-O. Buchweitz, Maximal Cohen-Macaulay modules and Tate cohomology over Gorenstein rings, preprint, University of Hannover, 1986.

[CoK] J. Cornick, P. H. Kropholler, On complete resolutions, Topology Appl. 78 (1997), 235-250.

[Da1] H. Dao, Decency and rigidity over hypersurfaces, arXive math.AC/0611568 preprint.

[Da2] H. Dao, Some observations on local and projective hypersurfaces, Math. Res. Lett. 15(2-3) (2008), 207–220.

[Gul] T.H. Gulliksen, On the deviations of a local ring, Math. Scand. 47 (1980), no. 1, 5-20.

[He] R. Heitmann, A counterexample to the rigidity conjecture for rings, Bull. Amer. Math. Soc. 29(1) (1993), 94–97.

[Ho] M. Hochster, Topics in the homological theory of modules over commutative rings, Expository lectures from the CBMS Regional Conference held at the University of Nebraska, Lincoln, Neb., June 24–28, 1974. Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 24, 1975.

[HW1] C. Huneke and R. Wiegand, Tensor products of modules and the rigidity of Tor, Math. Ann. 299 (1994), 449–476.

[HW2] C. Huneke and R. Wiegand, Tensor products of modules, rigidity and local cohomology, Math. Scand. 81 (1997), 161–183.

[Jo1] D. A. Jorgensen, Complexity and Tor on a complete intersection, J. Algebra 211 (1999), 578-598.

[Jo2] D. A. Jorgensen, Vanishing of (co)homology over commutative rings, Comm. Algebra 29(5) (2001), 1883-1898.

[Li] S. Lichtenbaum, On the vanishing of Tor in regular local rings, Illinois J. Math. 10 (1966), 220–226.

[Mat] H. Matsumura, Commutative ring theory, second edition, Cambridge Studies in Advanced Mathematics, 8, Cambridge University Press, Cambridge, 1989, xiv+320 pp.

[PS] C. Peskine and L. Szpiro, Dimension projective finie et cohomologie locale, Publ. Math. I.H.E.S. 42 (1972), 47–119.

[Ro] J. Rotman, An introduction to homological algebra, Academic Press, New York, 1979.

[Sha] J. Shamash, The Poincaré series of a local ring, J. Algebra 12 (1969), 453-470.

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