Volume of Seifert representations for graph manifolds and their finite covers

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Funding information
NSFC, Grant/Award Number: 11925101;
National Key R&D Program of China, Grant/Award Number: 2020YFA0712800

Abstract
For any closed orientable 3-manifold, there is a volume function defined on the space of all Seifert representations of the fundamental group. The maximum absolute value of this function agrees with the Seifert volume of the manifold due to Brooks and Goldman. For any Seifert representation of a graph manifold, the authors establish an effective formula for computing its volume, and obtain restrictions to the representation as analogous to the Milnor–Wood inequality (about transversely projective foliations on Seifert fiber spaces). It is shown that the Seifert volume of any graph manifold is a rational multiple of $\pi^2$. Among all finite covers of a given nongeometric graph manifold, the supremum ratio of the Seifert volume over the covering degree can be a positive number, and can be infinite. Examples of both possibilities are discovered, and confirmed with the explicit values determined for the finite ones.

MSC 2020
57M50, 53C20 (primary)

1 | INTRODUCTION

The Seifert volume $\text{SV}(M) \in [0, +\infty)$ of any orientable closed 3-manifold $M$ is introduced by Brooks and Goldman [3], as a generalization of Gromov’s simplicial volume [11]. This invariant can be obtained as the maximum absolute value of the volume function defined on the set of all $\tilde{\text{SL}}(2, \mathbb{R}) \times_\mathbb{R} \mathbb{R}$-representations of $\pi_1(M)$ (see Section 3.1). For any continuous map $f : M' \to M$
between 3-manifolds, the Seifert volume satisfies a similar inequality as with the simplicial volume:

\[ SV(M') \geq |\deg(f)| \times SV(M), \]

however, it does not have to achieve the equality for covering projections. Moreover, the Seifert volume is very intricately related with the geometric decomposition of the prime factors. The goal of the present paper is to unravel the relationship in the case of graph manifolds, and to understand the behavior of this invariant in finite covers.

We recall some recent advances to put our study into a context. First suppose that \( M \) is geometric in the sense of Thurston [23]. Then, \( SV(M) \) equals zero unless \( M \) supports the Seifert geometry \( \widetilde{SL}(2, \mathbb{R}) \) or the hyperbolic geometry \( \mathbb{H}^3 \). In the Seifert case, \( SV(M) \) actually equals the \( \widetilde{SL}(2, \mathbb{R}) \)-geometric volume. For any finite cover \( M' \) of \( M \), it follows that \( SV(M') \) is nonzero and is proportional to the covering degree \([M' : M]\). In the hyperbolic case, \( SV(M) \) may happen to be zero, but there are always finite covers \( M' \) of \( M \) with nonzero \( SV(M') \). Moreover, there are always some tower of finite covers \( \cdots \to M'_{n} \to \cdots \to M'_2 \to M'_1 \) of \( M \), such that the ratio \( SV(M'_{n})/[M'_n : M] \) is unbounded as \( n \) increases. In other words, virtual Seifert volume grows superlinearly fast (in some finite-covering towers), for hyperbolic 3-manifolds [6]. More generally, it is shown in [6] that virtual Seifert volume grows superlinearly fast for any orientable closed 3-manifold of nonzero simplicial volume. Since orientable closed 3-manifolds of zero simplicial volume are just connected sum of graph manifolds, essentially it remains interesting to ask, in the case of nongeometric graph manifolds, how fast virtual Seifert volume could possibly grow.

The covering Seifert volume as introduced in [6, Section 6] captures the supremum linear growth rate of virtual Seifert volume. For any orientable closed 3-manifold \( M \), it is defined as follows with value in \([0, +\infty] \):

\[
CSV(M) = \sup \left\{ \frac{SV(M')}{[M' : M]} : M' \text{ a finite cover of } M \right\}.
\]

(1.1)

It satisfies the covering property

\[ CSV(M') = [M' : M] \times CSV(M) \]

for finite covering projections \( M' \to M \). For instance, the above-mentioned result from [6] simply says that nonzero simplicial volume implies infinite covering Seifert volume. Besides, it is proved in [5] that any nongeometric graph manifold must have nonzero covering Seifert volume.

As the motivational application of our main results, we determine the covering Seifert volume for a sampling series of nongeometric graph manifolds. There we see examples of both possibilities of finite or of infinite covering Seifert volume. The infinite ones are somewhat more unexpected. Their existence brings about the new question of understanding the topological distinction behind the different growth behaviors.

### 1.1 Summary of results

Below we illustrate our main results with a family of very simply designed graph manifolds. Meanwhile, we point out the general theorems as actually proved in this paper.
Example 1.1 (Twisted doubling). Let $\Sigma_{g,1}$ be an orientable compact surface of genus $g$ and with a single boundary component. Denote by $\Sigma_+$ and $\Sigma_-$ a pair of oppositely oriented copies of $\Sigma_{g,1}$, and by $J_{\pm} = \Sigma_{\pm} \times S^1$ the oriented product manifolds of $\Sigma_{\pm}$ with the oriented circle $S^1$. For any entries $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$, we obtain an orientation-reversing homeomorphism $\varphi : \partial J_- \to \partial J_+$, such that the induced isomorphism $\varphi_* : H_1(\partial J_-) \to H_1(\partial J_+)$ operates on the standard bases as

$$\varphi_*([\partial \Sigma_-][S^1]) = ([\partial \Sigma_+][S^1]) \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$ 

We obtain a closed oriented 3-manifold

$$M \left( g; \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = J_- \cup_{\varphi} J_+$$

from $J_-$ and $J_+$ by identifying their boundaries using $\varphi$, or $M(\Sigma; a, b, c, d)$ for brevity.

Suppose $g > 0$ and $b \neq 0$. Then $M(\Sigma; a, b, c, d)$ is a graph manifold with two JSJ pieces $J_-$ and $J_+$ and a single JSJ torus $T = \partial J_+ = \varphi(\partial J_-)$. There is an obvious orientation-reversing homeomorphism $M(\Sigma; a, b, c, d) \cong M(\Sigma; d, -b, -c, a)$, as obtained by switching $\Sigma_{\pm}$ and inverting $\varphi$.

For any oriented graph manifold $M = M(\Sigma; a, b, c, d)$ as above and any representation $\rho : \pi_1(M) \to \widetilde{\text{SL}(2, \mathbb{R})} \times_{\mathbb{Z}} \mathbb{R}$, we obtain data $\xi_{\pm} = \xi_{\pm}(\rho) \in \mathbb{R}$ as follows. Fix an auxiliary base point on $T$, and obtain the van Kampen splitting $\pi_1(M) = \pi_1(J_-) *_{\pi_1(T)} \pi_1(J_+)$. Denote by $f_{\pm} \in \pi_1(M)$ the central element of the subgroup $\pi_1(J_{\pm}) \cong \pi_1(\Sigma_{\pm}) \times \pi_1(S^1)$ represented by the oriented fiber $S^1$ of $J_{\pm}$. If $\rho(f_{\pm})$ lies in the center $\mathbb{R}$ of $\widetilde{\text{SL}(2, \mathbb{R})} \times_{\mathbb{Z}} \mathbb{R}$, one may simply take $\xi_{\pm}$ as the value of $\rho(f_{\pm})$ in $\mathbb{R}$. In general, there is a continuous, real-valued conjugacy-class invariant for the elements of $\widetilde{\text{SL}(2, \mathbb{R})} \times_{\mathbb{Z}} \mathbb{R}$, which we call the essential winding number and introduce in Definition 2.3. Then we define $\xi_{\pm}$ as the essential winding number of $\rho(f_{\pm})$. Hence, we note that $\xi_{\pm}$ does not depend on the particular choice of the base point.

For any representation $\rho : \pi_1(M) \to \widetilde{\text{SL}(2, \mathbb{R})} \times_{\mathbb{Z}} \mathbb{R}$, the volume of $\rho$ is defined with value in $\mathbb{R}$, denoted as $\text{vol}_{\widetilde{\text{SL}(2, \mathbb{R})} \times_{\mathbb{Z}} \mathbb{R}}(M, \rho)$. This quantity depends only on the conjugacy class of $\rho$, and stays constant under continuous deformation of $\rho$. For $\rho$ ranging over all possible representations of $\pi_1(M)$ as above, it is known that the value set of $\text{vol}_{\widetilde{\text{SL}(2, \mathbb{R})} \times_{\mathbb{Z}} \mathbb{R}}(M, \rho)$ is always finite, so the maximum absolute value of those values is by definition $SV(M)$. See Section 3.1 for more details.

Specialized to twisted doublings and their representations, our main results can be summarized as follows.

- **Volume formula.** Setting $k_+ = d/b$ and $k_- = -a/b$,

$$\text{vol}_{\widetilde{\text{SL}(2, \mathbb{R})} \times_{\mathbb{Z}} \mathbb{R}}(M, \rho) = 4\pi^2 \cdot (k_+ \cdot \xi_+^2 - 2b^{-1} \cdot \xi_+ \xi_- + k_- \cdot \xi_-^2).$$

- **Generalized Milnor–Wood inequalities.**

$$|k_+ \cdot \xi_+ - b^{-1} \cdot \xi_-| \leq 2g - 1; \quad |b^{-1} \cdot \xi_+ + k_- \cdot \xi_-| \leq 2g - 1.$$
• **Rationality.**

\[
\frac{1}{\pi^2} \cdot \text{vol}_{\text{SL}(2,\mathbb{R}) \times \mathbb{Z}}(M, \rho) \in \mathbb{Q}.
\]

It follows by definition that \(SV(M)\) is also a rational multiple of \(\pi^2\). We notice that the volume formula is a homogeneous quadratic function of \(\xi_{\pm}(\rho)\) with coefficients determined by the topology of \(M\). Moreover, for any \(\eta_{\pm} \in \mathbb{Q}\) satisfying the strict inequality \(|k_{\pm} \cdot \eta_{\pm} - b_{-1} \cdot \eta_{\pm}| < 2g - 1\), we are able to construct some finite cover \(M'\) of \(M\) and some \(\text{SL}(2,\mathbb{R}) \times \mathbb{Z}\)–representation \(\rho'\) of \(\pi_1(M')\), such that the volume of \((M', \rho')\) equals \(4\pi^2 \cdot (k_+ \eta_+^2 - 2b_{-1} \cdot \eta_+ \cdot \eta_- + k_- \eta_-^2)\).

In this paper, we establish similar results for general graph manifolds. We consider any oriented closed graph manifold \(M\) with a simplicial JSJ graph \((V, E)\) and with oriented Seifert fibrations in the JSJ pieces, or what we call a *formatted graph manifold* (Definition 4.1). We show that the volume for any representation \(\rho : \pi_1(M) \to \text{SL}(2,\mathbb{R}) \times \mathbb{Z}\) is a homogeneous quadratic function of the essential winding numbers of \(\rho(f_v)\), where \(f_v\) corresponds to the oriented fiber of the JSJ piece \(J_v\) for each vertex \(v\). The quadratic function can be written down explicitly, involving the charges \(k_v\) associated to the JSJ pieces \(J_v\), and the fiber intersection numbers \(b_{v,w}\) associated to the JSJ tori \(T_{v,w}\). The symmetric matrix \(e_M \in \text{End}_{\mathbb{R}}(\mathbb{R}^V)\) that defines the quadratic function is called the Euler operator. Indeed, we find it playing a similar role in the general volume formula as the Euler number in the oriented Seifert fibration case (like quantization of classical observables). We obtain generalized Milnor–Wood inequalities, one with each vertex. We also prove the commensurability of volume values with \(\pi^2\). These results constitute Theorem 5.2. Moreover, we establish the generic virtual existence of \(\text{SL}(2,\mathbb{R}) \times \mathbb{Z}\)–volume values for formatted graph manifolds. This is Theorem 11.1.

**Example 1.2.** Suppose \(g > 0\).

\[
\text{CSV} \left( M \left( g; \begin{bmatrix} m & 1 \\ m^2 - 1 & m \end{bmatrix} \right) \right) = \begin{cases} +\infty & m = 0, \pm 1 \\ 8\pi^2 \cdot (2g - 1)^2/(|m| - 1) & m = \pm 2, \pm 3, \pm 4, ... \end{cases}
\]

Hence, we see that the covering Seifert volume can be finite and can be infinite for nongeometric graph manifolds. Example 1.2 is a consequence of Theorem 12.1, where the covering Seifert volume is determined for any graph manifold with constant charges and constant fiber intersection numbers, and with a cyclic JSJ graph (see Remark 12.2). The infinite case immediately leads to lots of other graph manifolds of infinite covering Seifert volume, once we require those manifolds to project \(M(g; 0,1,-1,0)\) or \(M(g; \pm 1,1,0,\pm 1)\) of nonzero degree. On the other hand, for a large class of what we call *strictly diagonally dominant* graph manifolds, we are able to confirm finiteness of their covering Seifert volume: Among twisted doublings, Example 1.3 below is the special case of Theorem 10.1. In certain occasions such as Example 1.2, the upper bounds are indeed sharp, and the conditions also happen to be necessary.

**Example 1.3.** Suppose \(g > 0\). If \(|a| > 1\) and \(|d| > 1\), and hence \(b \neq 0\),

\[
\text{CSV} \left( M \left( g; \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \right) \leq 4\pi^2 \cdot (2g - 1)^2 \cdot \left( |\frac{b}{|a| - 1} + |\frac{b}{|d| - 1}| \right).
\]
None of the twisted doublings in Example 1.3 admit finite covers fibering over a circle. The manifold $M(g; 1, 1, 0, 1)$ does fiber over a circle, as does its orientation-reversal $M(g; -1, 1, 0, -1)$. In fact, it is easy to identify $M(g; 1, 1, 0, 1)$ with the mapping torus of a Dehn twist, which acts on the doubling of $\Sigma_{g,1}$ twisting along the doubling curve. The manifold $M(g; 0, 1, -1, 0)$ is virtually fibered; it actually admits a Riemannian metric of nonpositive sectional curvature, because it is chargeless (meaning $k_+ = k_- = 0$). These topological descriptions all follow from Buyalo and Svetlov’s characterization on virtual properties of graph manifolds [4], plus simple observation.

Therefore, our examples seem to suggest that the difference between finite and infinite covering Seifert volume lies somewhere between the strictly diagonally dominant class and the virtually fibered class, for nongeometric graph manifolds in general.

For orientable closed 3-manifolds of nonzero simplicial volume, it remains unclear how to compute volume of their Seifert representations effectively. In particular, the authors do not know if any of those manifolds have Seifert volume incommensurable with $\pi^2$. It might be interesting to compare our phenomenon here with the rationality results in [18, 21] about $\text{PSL}(2, \mathbb{C})$-Chern–Simons invariants.

### 1.2 Ingredients of proofs

Below we explain some key ideas that are developed in this paper. There is another more technical outline in Section 5.2, where we are about to prove Theorem 5.2. As a general strategy, we wish to work out a fundamental case by direct means, and reduce any other case to that one, but we must also have a suitable formulation to proceed.

Suppose that $M$ is a graph manifold and $\rho$ is a $\tilde{\text{SL}}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$-representation of $\pi_1(M)$. If $\rho$ sends the center of any vertex group $\pi_1(J_v)$ to the central subgroup $\mathbb{Q}$, we can apply the additivity principle from [5] to compute volume. The volume turns out to depend only on the numbers $\rho(f_v) \in \mathbb{Q}$ and the structural data $k_v$ and $b_{v, w}$ about $M$. Inspired by the well-known volume formula and the Milnor–Wood inequality for Seifert geometric manifolds, we are able to formulate our analogous results in this case.

To deal with general representations, there are two prominent issues in front: One is to convert $\rho(f_v)$ somehow into a number; the other is to deform $\rho$ somehow into a representation as above. Both of these issues are related with the topological group structure of $\tilde{\text{SL}}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$. By introducing the essential winding number (Definition 2.3), we classify elements of $\tilde{\text{SL}}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$ up to conjugacy based on the classification of conjugacy classes in $\text{PSL}(2, \mathbb{R})$. This is done in Section 2.2, and resolves the first issue. Our solution to the second issue relies heavily on commutator factorization of paths in $\tilde{\text{SL}}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$. Generally speaking, for any topological group $G$, the commutator map $G \times G \to G : (a, b) \mapsto aba^{-1}b^{-1}$ does not satisfy the path lifting property. However, we are able to construct lifts for certain nice paths in $\tilde{\text{SL}}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$. The relevant techniques are developed in Section 7.

When studying existence of transverse foliations in Seifert fiber spaces, Eisenbud, Hirsch, and Neumann [8] considered the topological group $\mathcal{D}^+$ that consists of all the diffeomorphisms $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(r + 1) = \varphi(r) + 1$ for all $r \in \mathbb{R}$. They made use of the maximum $m(\varphi)$ and the minimum $\overline{m}(\varphi)$ of the function $\varphi(r) - r$, and analyzed the problem of factorizing any element $\varphi$ into a given number of commutators. Restricted to the subgroup $\tilde{\text{SL}}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$ of $\mathcal{D}^+$, our techniques can be regarded as refinement of theirs: Our essential winding number of $\varphi$ is a precise value to replace the former interval $[m(\varphi), \overline{m}(\varphi)]$. Our commutator path lifting lemmas are enhanced versions of former factorization results.
To deform a general representation $\rho$ continuously as desired, we actually have to work with the pull-back $\rho'$ of $\rho$ to another manifold $M'$. Here, $M'$ is some nicely constructed graph manifold that pinches nicely onto $M$. In particular, the volume of $(M',\rho')$ is the same as that of $(M,\rho)$. Our deformation path for $\rho'$ is constructed by compatibly assembling deformation paths of the restricted representations to the vertex groups of $\pi_1(M')$. We analyze the local types of $\rho'$ at the vertices, and employ our commutator path liftings to construct the local deformations case by case. The local types are explained in Section 5.1. The constructions of $M'$ and the deformation path for $\rho'$ are done in Section 8 (see also Section 5.2 for more explanation on the idea). With these constructions, we are able to complete the proof of Theorem 5.2. We also make use of similar constructions on some finite covers of $M$ to prove Theorem 11.1.

### 1.3 Organization

This paper can be divided into three parts: The first part is mostly preliminary and expository; the second part is about Theorem 5.2 and its application pertaining to Example 1.3; the third part is about Theorem 11.1 and its application pertaining to Example 1.2. These parts are organized as follows.

The first part consists of Sections 2–4. In Section 2, we study the structure of $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{Z}$, and in particular, we introduce the essential winding number (Definition 2.3). In Section 3, we review the Seifert volume from the approach of volume of representations. In Section 4, we review the topological theory about Seifert fiber spaces and graph manifolds.

The second part consists of Sections 5–10. In Section 5, we state our main result (Theorem 5.2) and outline its proof. In Sections 6–9, we develop necessary techniques and prove Theorem 5.2 (see Section 5.2 for details about their contents). In Section 10, we apply Theorem 5.2 to what we call strictly diagonally dominant graph manifolds, and bound their covering Seifert volume (Theorem 10.1).

The third part consists of Sections 11 and 12. In Section 11, we obtain the part converse of our main result (Theorem 11.1). In Section 12, we apply Theorems 5.2 and 11.1 to what we call constant cyclic graph manifolds, and determine their covering Seifert volume (Theorem 12.1).

There is an Appendix, where we clarify the normalization of the Seifert volume as mentioned in Section 3.1.

### 2 THE MOTION GROUP OF SEIFERT GEOMETRY

Seifert geometry is one of the eight 3-dimensional geometries as classified by Thurston [23]. In this section, we investigate this geometry from a transformation group perspective. We take the simply connected Lie group $\widetilde{SL}(2,\mathbb{R})$ as the model of its space. The identity component of its isomorphism group, which we refer to as the motion group of Seifert geometry, can be identified with $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{Z}$. We introduce a useful invariant for Seifert-geometric motions called the essential winding number. This is a continuous, conjugation-invariant function $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{Z} \to \mathbb{R}$ (see Definition 2.3). It plays a crucial role in the volume formula for Seifert-geometric motion representations of graph manifolds.
2.1 Construction via central extensions

Let $\widetilde{\operatorname{SL}}(2, \mathbb{R})$ be the universal covering Lie group of $\operatorname{SL}(2, \mathbb{R})$. (Throughout this paper, we think of Lie groups topologically as pointed spaces based at the identity. Universal covering Lie groups may be constructed canonically using relative homotopy classes of paths.) The expression

$$k(r) = \begin{bmatrix} \cos(\pi r) & -\sin(\pi r) \\ \sin(\pi r) & \cos(\pi r) \end{bmatrix}$$

for all $r \in \mathbb{R}$ defines a differentiable homomorphism $k : \mathbb{R} \to \operatorname{SL}(2, \mathbb{R})$, which lifts to be a differentiable homomorphism

$$\widetilde{k} : \mathbb{R} \to \widetilde{\operatorname{SL}}(2, \mathbb{R}).$$

The closed subgroup $\widetilde{\operatorname{SO}}(2)$ of $\widetilde{\operatorname{SL}}(2, \mathbb{R})$ that projects $\operatorname{SO}(2)$ is parametrized by $\mathbb{R}$ via $\widetilde{k}$. The center $Z(\widetilde{\operatorname{SL}}(2, \mathbb{R}))$ of $\widetilde{\operatorname{SL}}(2, \mathbb{R})$ can be recognized as the image of $\mathbb{Z}$ under $\widetilde{k}$.

Let $\widetilde{\operatorname{SL}}(2, \mathbb{R}) \times_\mathbb{Z} \mathbb{R}$ be the Lie group that is constructed as the quotient of $\widetilde{\operatorname{SL}}(2, \mathbb{R}) \times \mathbb{R}$ by the infinite cyclic, discrete, central subgroup $\{(\widetilde{k}(n), -n) : n \in \mathbb{Z}\}$. We denote elements of $\widetilde{\operatorname{SL}}(2, \mathbb{R}) \times_\mathbb{Z} \mathbb{R}$ as $g[s]$ for $g \in \widetilde{\operatorname{SL}}(2, \mathbb{R})$ and $s \in \mathbb{R}$, so the multiplication rule reads

$$g[s]g'[s'] = (gg')[s + s'],$$

and the relation

$$g\widetilde{k}(n)[s] = g[s + n]$$

holds for all $n \in \mathbb{Z}$. There are natural embeddings of $\widetilde{\operatorname{SL}}(2, \mathbb{R})$ and $\mathbb{R}$ into $\widetilde{\operatorname{SL}}(2, \mathbb{R}) \times_\mathbb{Z} \mathbb{R}$, namely, $g \mapsto g[0]$ and $s \mapsto \text{id}[s]$. We obtain the following short exact sequence of Lie groups homomorphisms:

$$\{0\} \longrightarrow \mathbb{R} \overset{\text{incl}}{\longrightarrow} \widetilde{\operatorname{SL}}(2, \mathbb{R}) \times_\mathbb{Z} \mathbb{R} \longrightarrow \operatorname{PSL}(2, \mathbb{R}) \longrightarrow \{1\}. \quad (2.1)$$

2.2 Classification of conjugacy classes

We introduce the essential winding number and use it to classify the conjugacy classes of $\widetilde{\operatorname{SL}}(2, \mathbb{R}) \times_\mathbb{Z} \mathbb{R}$.

Lemma 2.1. The expression

$$\overline{\omega} \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{cases} \text{sgn}(c-b) \cdot \arccos \left( \frac{a+d}{2} \right) \mod \pi \mathbb{Z} & \text{if } |a+d| < 2 \\ 0 \mod \pi \mathbb{Z} & \text{if } |a+d| \geq 2 \end{cases}$$

defines a continuous map $\overline{\omega} : \operatorname{PSL}(2, \mathbb{R}) \to \mathbb{R}/\pi \mathbb{Z}$, which is invariant under conjugation of $\operatorname{PSL}(2, \mathbb{R})$. Moreover, there exists a unique continuous, conjugation-invariant function
Let \( \tilde{\omega} : \tilde{\text{SL}}(2, \mathbb{R}) \to \mathbb{R} \) that lifts \( \tilde{\omega} \) and that satisfies
\[
\tilde{\omega}(\tilde{k}(r)) = \pi r
\]
for all \( r \in \mathbb{R} \).

**Proof.** We observe \( \text{sgn}(b - c) \cdot \arccos(-\frac{a + d}{2}) = \text{sgn}(c - b) \cdot \arccos((a + d)/2) - \pi \), so \( \tilde{\omega} \) is well defined. As \( |a + d| \) approaches 2 from below, the value of \( \tilde{\omega} \) approaches 0 mod \( \pi \mathbb{Z} \), so \( \tilde{\omega} \) is clearly continuous. For \( |a + d| < 2 \), the condition \( ad - bc = 1 \) implies either \( b < 0 < c \) or \( c < 0 < b \), so \( \text{sgn}(c - b) \) is either +1 or −1. Since \( \text{SL}(2, \mathbb{R}) \) is connected, and since \( (a + d)/2 \) is conjugation invariant, \( \tilde{\omega} \) must be constant on every conjugacy class of \( \text{PSL}(2, \mathbb{R}) \). Note that \( \tilde{\omega}(\pm k(r)) \) equals \( \pi r \) mod \( \pi \mathbb{Z} \), for all \( r \in \mathbb{R} \). It follows that \( \tilde{\omega} \) induces a group isomorphism \( \pi_1(\text{PSL}(2, \mathbb{R})) \to \pi_1(\mathbb{R}/\pi \mathbb{Z}) \). Therefore, there exists a unique continuous function \( \tilde{\omega} : \tilde{\text{SL}}(2, \mathbb{R}) \to \mathbb{R} \) that lifts \( \tilde{\omega} \) and that satisfies \( \tilde{\omega}(\tilde{k}(r)) = \pi r \), for all \( r \in \mathbb{R} \). For any \( g \in \tilde{\text{SL}}(2, \mathbb{R}) \), the conjugation map \( \tilde{\text{SL}}(2, \mathbb{R}) \to \tilde{\text{SL}}(2, \mathbb{R}) : u \mapsto ugu^{-1} \) is constant on the center \( \mathbb{Z} \), so it descends to a map \( \text{PSL}(2, \mathbb{R}) \to \tilde{\text{SL}}(2, \mathbb{R}) \). The latter is the lift of the conjugation map \( \text{PSL}(2, \mathbb{R}) \to \text{PSL}(2, \mathbb{R}) : \tilde{u} \mapsto \tilde{u} \tilde{g} \tilde{u}^{-1} \). Then \( \tilde{\omega} \) is invariant under conjugation of \( \tilde{\text{SL}}(2, \mathbb{R}) \), by the conjugation-invariance of \( \tilde{\omega} \) and the construction of \( \tilde{\omega} \). \( \square \)

**Remark 2.2.** The map \( \tilde{\omega} \) can be visualized as follows (compare [15, Section 2]). Denote by
\[
D = \{ (x, y, t) \in \mathbb{R}^3 : t^2 - x^2 - y^2 \leq 1 \}
\]
the region in 3-space between the two sheets of the hyperboloid \( t^2 - x^2 - y^2 = 1 \). Under the continuous map
\[
D \to \text{PSL}(2, \mathbb{R}) : (x, y, t) \mapsto \pm \begin{pmatrix}
\sqrt{1 + x^2 + y^2 - t^2 + y} & x + t \\
x - t & \sqrt{1 + x^2 + y^2 - t^2 - y}
\end{pmatrix},
\]
the interior of \( D \) projects homeomorphically onto the open subset of \( \text{PSL}(2, \mathbb{R}) \) consisting of the elements of nonzero trace. Each boundary component of \( D \) projects homeomorphically onto the closed subset of traceless elements, such that the points \( (x, y, t), (-x, -y, -t) \in \partial D \) project the same element. Therefore, \( \text{PSL}(2, \mathbb{R}) \) is homeomorphic to the quotient space of \( D \) identifying every antipodal pair of points on the boundary. The quotient space is, of course, an open solid torus. Under the homeomorphism, every conjugacy class of a hyperbolic element in \( \text{PSL}(2, \mathbb{R}) \) is a one-sheet hyperboloid \( t^2 - x^2 - y^2 = c \), for some \( c < 0 \); every conjugacy class of an elliptic element is a sheet of a hyperboloid \( t^2 - x^2 - y^2 = c \), for some \( 0 < c \leq 1 \); there are another two conjugacy classes of parabolic elements, corresponding to the components of the cone \( t^2 - x^2 - y^2 = 0 \) without the origin. One may easily check that the map \( \tilde{\omega} : \text{PSL}(2, \mathbb{R}) \to \mathbb{R}/\pi \mathbb{Z} \) collapses all the nonelliptic conjugacy classes to \( 0 \) mod \( \pi \mathbb{Z} \), and collapses every elliptic conjugacy class to a distinct point.

**Definition 2.3.** For any \( g[s] \in \tilde{\text{SL}}(2, \mathbb{R}) \times _{\mathbb{Z}} \mathbb{R} \), the **essential winding number** of \( g[s] \) is defined as
\[
\text{wind}(g[s]) = \frac{\tilde{\omega}(g)}{\pi} + s,
\]
which is a well-defined value in \( \mathbb{R} \). Here \( \tilde{\omega} : \tilde{\text{SL}}(2, \mathbb{R}) \to \mathbb{R} \) is as provided by Lemma 2.1.
We obtain a characterization of conjugate elements in \( \widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R} \) in terms of the essential winding number, as follows.

**Proposition 2.4.** Let \( g[s], g'[s'] \) be a pair of elements in \( \widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R} \). Denote by \( \tilde{g}, \tilde{g}' \) their images in \( PSL(2, \mathbb{R}) \) under the natural quotient homomorphism. Then the following statements are equivalent.

1. The elements \( g[s] \) and \( g'[s'] \) are conjugate in \( \widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R} \).
2. The elements \( g[s] \) and \( g'[s'] \) have equal essential winding number, and the elements \( \tilde{g} \) and \( \tilde{g}' \) are conjugate in \( PSL(2, \mathbb{R}) \).

**Proof.** Suppose that \( g[s] \) is conjugate to \( g'[s'] \) in \( \widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R} \). Then \( \tilde{g} \) and \( \tilde{g}' \) are obviously conjugate in \( PSL(2, \mathbb{R}) \). Below we argue \( \text{wind}(g[s]) = \text{wind}(g'[s']) \).

Note that there is a natural quotient homomorphism \( \widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}/\mathbb{Z} \), which sends any \( g[s] \) to \( s \mod \mathbb{Z} \). This implies that \( s \) must be \( s' + n \) for some \( n \in \mathbb{Z} \), due to the assumed conjugacy. We obtain \( g[s] = g[s' + n] = g\tilde{k}(n)[s'] \). Moreover, \( g\tilde{k}(n) \) must be conjugate to \( g' \) in \( \widetilde{SL}(2, \mathbb{R}) \). In fact, \( g[s] = u[t]g'[s'](u[t])^{-1} \) is equivalent to \( g\tilde{k}(n)[s'] = u g'u^{-1} [s'] \), and also equivalent to \( g\tilde{k}(n) = u g'u^{-1} \). By Lemma 2.1, we see \( \text{wind}(g[s]) = \text{wind}(g\tilde{k}(n)[s']) = \text{wind}(g'[s']) \).

Conversely, suppose \( \tilde{g} = \tilde{u}\tilde{g}'\tilde{u}^{-1} \) for some \( \tilde{u} \in PSL(2, \mathbb{R}) \), and also suppose \( \text{wind}(g[s]) = \text{wind}(g'[s']) \). Then, for any lift \( u \in \widetilde{SL}(2, \mathbb{R}) \) of \( \tilde{u} \), the element \( u g'u^{-1} g^{-1} \) is central, so there exists \( n \in \mathbb{Z} \) such that \( u g'u^{-1} = g\tilde{k}(n) \). Using Lemma 2.1, we obtain \( \text{wind}(g[s]) = \text{wind}(g\tilde{k}(n)[s - n]) = \text{wind}(g'[s' - n]) = \text{wind}(g'[s']) - n \). This means \( n = 0 \), and the conjugation relation \( u[0]g'[s'](u[0])^{-1} = g[s] \) follows, as desired. \( \square \)

### 2.3 Construction of the Seifert geometry

There is a natural action of the Lie group \( \widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R} \) on \( \widetilde{SL}(2, \mathbb{R}) \) by diffeomorphisms, namely,

\[ g[r].h = gh\tilde{k}(r) \]

for all \( g[r] \in \widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R} \) and \( h \in \widetilde{SL}(2, \mathbb{R}) \). The action is differentiable, transitive, and proper. In other words, it turns \( \widetilde{SL}(2, \mathbb{R}) \) into a 3-dimensional geometry in the sense of W. P. Thurston [23]. In fact, \( \widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R} \) is the identity component of the transformation group \( \text{Iso}(\widetilde{SL}(2, \mathbb{R})) \) of the Seifert geometry \( \widetilde{SL}(2, \mathbb{R}) \).

One may actually describe the maximal transformation group \( \text{Iso}(\widetilde{SL}(2, \mathbb{R})) \) of the Seifert geometry in terms of the central extension construction, as follows. Observe that there is a canonical involutive Lie group automorphism of \( SL(2, \mathbb{R}) \), defined as

\[ \nu : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}. \]

It lifts to a canonical involutive Lie group automorphism

\[ \nu : \widetilde{SL}(2, \mathbb{R}) \to \widetilde{SL}(2, \mathbb{R}). \]

As it turns out, \( \text{Iso}(\widetilde{SL}(2, \mathbb{R})) \) is the subgroup of \( \text{Diffeo}(\widetilde{SL}(2, \mathbb{R})) \) generated by \( \widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R} \) and \( \nu \). For any \( g[s] \in \widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R} \), the relation \( \nu \circ g[s] = \nu(g)[-s] \circ \nu \) holds. In other words,
Iso($\widetilde{SL}(2, \mathbb{R})$) factorizes as a semidirect product

$$\text{Iso}(\widetilde{SL}(2, \mathbb{R})) = (\widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}) \rtimes \{\text{id}, \widetilde{\nu}\}.$$  

**Remark 2.5.** A transverse foliation to the canonical fibration of $\widetilde{SL}(2, \mathbb{R})$ can be visualized as follows. Let $\mathbb{H}^2$ be the hyperbolic plane. We adopt the upper half-space model $\mathbb{H}^2 = \{z \in \mathbb{C} : \Im (z) > 0\}$, so $\text{PSL}(2, \mathbb{R})$, and hence $\widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$, acts isometrically on $\mathbb{H}^2$ by fractional linear transformations. For any point $p \in \mathbb{H}^2$, define

$$\tilde{k}_p : \mathbb{R} \to \widetilde{SL}(2, \mathbb{R})$$

as $\tilde{k}_p(r) = h \tilde{k}(r) h^{-1}$, for all $r \in \mathbb{R}$, where $h \in \widetilde{SL}(2, \mathbb{R})$ is any element that takes the imaginary unit $i$ to $p$. Note that $k_p$ depends only on the left coset $h \widetilde{SO}(2)$, and hence only on $p$. We adopt the ideal boundary $\partial_{\infty} \mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$ as the model of the real projective line. Denote by $\widetilde{\mathbb{R}P}^1$ be the universal covering space of $\mathbb{R}P^1$ with a distinguished base point $\tilde{0}$ that lifts 0. Then there is a canonical homeomorphism

$$\widetilde{SL}(2, \mathbb{R}) \cong \mathbb{H}^2 \times \widetilde{\mathbb{R}P}^1,$$

such that $g \in \widetilde{SL}(2, \mathbb{R})$ corresponds to $(g.i, g.\tilde{0})$. The action of $\widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$ on $\widetilde{SL}(2, \mathbb{R})$ is conjugate to the action

$$[g, r].(p, s) = (g.p, g\tilde{k}_p(r).s).$$

The projection of $\mathbb{H}^2 \times \widetilde{\mathbb{R}P}^1$ onto the $\mathbb{H}^2$ factor is a fibration that is invariant under the action of the center $\mathbb{R}$; the foliation with leaves parallel to $\mathbb{H}^2$ is transverse to the above fibration, with structure group $\widetilde{SL}(2, \mathbb{R})$.

### 3 | VOLUME OF MOTION REPRESENTATIONS IN SEIFERT GEOMETRY

In this section, we recall volume of representations, concentrating on the special case of Seifert geometry and its motion group. The corresponding representation volume recovers the Seifert volume for 3-manifolds as originally introduced by Brooks and Goldman [3]. The reader is referred to [7, Section 2] for complete details in a general setting.

For Seifert-geometric circle bundles over closed surfaces, the space of Seifert motion representations and the volume function are very well understood. We determine the volume function as a basic example of the theory (Theorem 3.2). In subsequent sections, this preliminary result is used to prove the graph-manifold case, and its statement also serves as a prototype of the general formula (Theorem 5.2).

#### 3.1 | Volume of representations

For any finitely generated group $\pi$, the set of homomorphisms $\pi \to \text{PSL}(2, \mathbb{R})$ naturally forms a real algebraic set, known as the $\text{PSL}(2, \mathbb{R})$-representation variety of $\pi$. For any Lie group $G$, we still
call any homomorphism $\pi \to G$ a representation of $\pi$ in $G$. The set of representations is no longer an algebraic variety in general. However, the set can be naturally topologized, so that a sequence of representations converges if and only if it converges in $G$ evaluated at any element of $\pi$. We denote by $R(\pi, G)$ the set of $G$-representations of $\pi$ with this topology, and refer to it the space of $G$-representations for $\pi$.

Given any representation $\rho : \pi \to \widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$, we can twist $\rho$ with any homomorphism $\alpha : \pi \to \mathbb{R}$, and obtain a new representation $\rho[\alpha] : \pi \to \widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$, such that

$$(\rho[\alpha])(\sigma) = \rho(\sigma) \cdot \text{id}[\alpha(\sigma)].$$

This gives rise to a free, continuous action of the Lie group $H^1(\pi; \mathbb{R})$ on $R(\pi, \widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R})$. A similar construction with integral coefficients yields free, discontinuous action of the discrete group $H^1(\pi; \mathbb{Z})$ on $R(\pi, \widetilde{SL}(2, \mathbb{R}))$. Given any representation $\tilde{\rho} : \pi \to \text{PSL}(2, \mathbb{R})$, we obtain an associated oriented circle bundle $B\pi \times_{\tilde{\rho}} \mathbb{R}P^1$ over the classifying space $B\pi \cong K(\pi, 1)$, using the canonical action of PSL(2, $\mathbb{R}$) on the real projective line $\mathbb{R}P^1$. This gives rise to an Euler class of $e(\tilde{\rho}) \in H^2(\pi; \mathbb{Z})$.

The structure of $R(\pi, \widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R})$ is described through the following theorem.

**Theorem 3.1** [7, Proposition 5.1]. For any finitely generated group $\pi$, the following statements hold true.

1. The space of representations $R(\pi, \text{PSL}(2, \mathbb{R}))$ has only finitely many path-connected components. The Euler class $e : R(\pi, \text{PSL}(2, \mathbb{R})) \to H^2(\pi; \mathbb{Z})$ is constant on each component.
2. The space of representations $R(\pi, \widetilde{SL}(2, \mathbb{R}))$ naturally projects the path-connected components of $R(\pi, \text{PSL}(2, \mathbb{R}))$ where the Euler class is trivial, and becomes a principal $H^1(\pi; \mathbb{Z})$-bundle over those components.
3. The space of representations $R(\pi, \widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R})$ naturally projects the path-connected components of $R(\pi, \text{PSL}(2, \mathbb{R}))$ where the Euler class is torsion, and becomes a principal $H^1(\pi; \mathbb{R})$-bundle over those components.

Let $M$ be an oriented connected closed 3-manifold. By denoting $\pi_1(M)$, we always assume that a universal covering space of $M$ is implicitly fixed, and $\pi_1(M)$ refers to the deck transformation group. There is a well-defined continuous function

$$R(\pi_1(M), \widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}) \to \mathbb{R} : \rho \mapsto \text{vol}_{\widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}}(M, \rho),$$

called the volume of $\widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$-representations for $M$. Moreover, the function $\text{vol}_{\widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}}(M, \cdot)$ is constant on every path-connected component of $R(\pi_1(M), \widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R})$. It follows from Theorem 3.1 that there are only finitely many path-connected components. The Seifert volume of $M$ is defined as

$$SV(M) = \sup \left\{ \left| \text{vol}_{\widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}}(M, \rho) \right| : \rho \in R(\pi_1(M), \widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}) \right\},$$

which values in $[0, +\infty)$.

Strictly speaking, the volume function depends on a chosen left-invariant volume form on $\widetilde{SL}(2, \mathbb{R})$, while other choices change the function by a nonzero scalar factor (see [7, Section 2]).
However, there is a preferred choice such that the Seifert volume agrees with Brooks and Goldman’s original definition. Indeed, this is the normalization that applies to Theorem 3.2 below. See the Appendix for an elaboration about the normalization.

### 3.2 Seifert geometric circle bundles

Let $N$ be an oriented circle bundle of Euler number $e \in \mathbb{Z}$ over an oriented closed surface $\Sigma$ of genus $g$. If $N$ supports the Seifert geometry, $g$ is at least 2 and $e$ is nonzero.

The space of representations $R(\pi_1(N), \tilde{SL}(2,\mathbb{R}) \times_{\mathbb{Z}} \mathbb{R})$ can be completely described as follows. The abelianization $\pi_1(N) \to H_1(N; \mathbb{Z})$ and the bundle projection $\pi_1(N) \to \pi_1(\Sigma)$ induce the embeddings of $R(H_1(N; \mathbb{Z}), PSL(2,\mathbb{R}))$ and $R(\pi_1(\Sigma), PSL(2,\mathbb{R}))$ into $R(\pi_1(N), PSL(2,\mathbb{R}))$, as real algebraic sets. The intersection of their images can be identified as $R(H_1(\Sigma; \mathbb{Z}), PSL(2,\mathbb{R}))$ embedded in $R(\pi_1(N), PSL(2,\mathbb{R}))$. Observe $H_1(N; \mathbb{Z}) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}/e\mathbb{Z}$. There are $|e|$ path-connected components of $R(H_1(N; \mathbb{Z}), PSL(2,\mathbb{R}))$, indexed by the $\mathbb{Z}/e\mathbb{Z}$. The component indexed by $j \in \mathbb{Z}/e\mathbb{Z}$ can be characterized by the rule that a fixed generator of the homological torsion subgroup goes to a rotation on $H^2$ of angle $2\pi j/e$. In particular, each of these components contains a representation with finite cyclic image, and hence has torsion Euler class. There are $(4g - 3)$ path-connected components of $R(\pi_1(\Sigma), PSL(2,\mathbb{R}))$, indexed by $0, \pm 1, \pm 2, \ldots, \pm (2g - 2)$, which indicate the Euler number $e(\psi) \in \mathbb{Z}$ of the associated oriented circle bundle $\Sigma \times_{\psi} \mathbb{R}P^1 \to \Sigma$ for any representation $\psi : \pi_1(\Sigma) \to PSL(2,\mathbb{R})$ in them [10]. The representation variety $R(H_1(\Sigma; \mathbb{Z}), PSL(2,\mathbb{R}))$ is path-connected. Note that any representation $\pi_1(N) \to PSL(2,\mathbb{R})$ either factors through the abelianization of $\pi_1(N)$, or trivializes the center of $\pi_1(N)$. Therefore, the representation variety $R(\pi_1(N), PSL(2,\mathbb{R}))$ has $(4g - 4 + |e|)$ path-connected components in total: Apart from one that contains the trivial representation, there are $(|e| - 1)$ path-connected components that consist only of abelian representations, and another $(4g - 4)$ that consist only of nonabelian ones. By Theorem 3.1 and the fact that $H^2(\Sigma; \mathbb{Z}) \to H^2(N; \mathbb{Z})$ has finite-cyclic image of order $|e|$, and the fact that the abelian components all have torsion Euler classes, the space of representations $R(\pi_1(N), \tilde{SL}(2,\mathbb{R}) \times_{\mathbb{Z}} \mathbb{R})$ projects onto $R(\pi_1(N), PSL(2,\mathbb{R}))$ as a principal $H^1(N; \mathbb{R})$-bundle.

We say that a representation $\phi \in R(\pi_1(N), \tilde{SL}(2,\mathbb{R}) \times_{\mathbb{Z}} \mathbb{R})$ is of central type if $\phi$ sends the center of $\pi_1(N)$ to the center of $\tilde{SL}(2,\mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$. For any central type representation $\phi$, we obtain the following a commutative diagram of group homomorphisms:

$$
\begin{array}{cccccc}
0 & \to & \mathbb{Z} & \to & \pi_1(N) & \to & \pi_1(\Sigma) & \to & 1 \\
\downarrow{\phi_{fib}} & & \downarrow{\phi} & & \downarrow{\phi_{base}} & & \\
0 & \to & \mathbb{R} & \to & \tilde{SL}(2,\mathbb{R}) \times_{\mathbb{Z}} \mathbb{R} & \to & PSL(2,\mathbb{R}) & \to & 1 \\
\end{array}
$$

(3.3)

We identify $\phi_{fib} : \mathbb{Z} \to \mathbb{R}$ as a real scalar, and the Euler class $e(\phi_{base}) \in H^2(\Sigma; \mathbb{Z})$ as an integer.

**Theorem 3.2.** Let $N$ be an oriented circle bundle over an oriented closed surface $\Sigma$. Suppose that the genus of $\Sigma$ is $g > 0$ and the Euler number of $N \to \Sigma$ is $e \neq 0$. Suppose that $\phi : \pi_1(N) \to \tilde{SL}(2,\mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$ is a representation of central type. Then the formulas hold:

$$
e(\phi_{base}) = e \times \phi_{fib},$$

Where $e(\phi_{base})$ is the Euler class of the representation $\phi_{base}$ and $\phi_{fib}$ is the fibration component.
and

\[ \text{vol}_{\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}}(N, \phi) = 4\pi^2 \phi_{\text{fib}} \times e(\phi_{\text{base}}). \]

In particular, \( \phi_{\text{fib}} \) is constant on the path-connected component of \( R(\pi_1(N), \widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}) \) that contains \( \phi \), and the possible values are precisely 0, \( \pm 1/e, \ldots, \pm (2g - 2)/e \).

**Proof.** If \( e(\phi_{\text{base}}) = 0 \), the representation \( \phi \) lies in the path-connected component of the trivial representation, by the above description of \( R(\pi_1(N), \widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}) \). As the trivial representation has volume 0, we obtain \( \text{vol}(N, \phi) = 0 \). The path-connected component of the trivial representation can be identified as the image of the pull-back embedding \( R(\pi_1(\Sigma), \widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}) \to R(\pi_1(N), \widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}) \), by Theorem 3.1, so we obtain \( \phi_{\text{fib}} = 0 \). This establishes the formulas for \( e(\phi_{\text{base}}) = 0 \).

It remains to prove the formulas for \( e(\phi_{\text{base}}) \neq 0 \). To this end, we consider the associated circle bundle \( \Sigma \times_{\phi_{\text{base}}} \mathbb{RP}^1 \) over \( \Sigma \), whose Euler number equals \( e(\phi_{\text{base}}) \). There is a natural transversely projective foliation \( \mathcal{F} \) on \( \Sigma \times_{\phi_{\text{base}}} \mathbb{RP}^1 \), which gives rise to a canonical holonomy representation

\[ \psi : \pi_1(\Sigma \times_{\phi_{\text{base}}} \mathbb{RP}^1) \to \widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}, \]

which actually factors through \( \tilde{SL}(2,\mathbb{R}) \). To be precise, we treat \( \pi_1(\Sigma \times_{\phi_{\text{base}}} \mathbb{RP}^1) \) as the deck transformation group of the universal covering space \( \tilde{\Sigma} \times \mathbb{RP}^1 \) of \( \Sigma \times_{\phi_{\text{base}}} \mathbb{RP}^1 \), where \( \tilde{\Sigma} \) and \( \mathbb{RP}^1 \) are fixed universal spaces of \( \Sigma \) and \( \mathbb{RP}^1 \), respectively. The covering projection is the composition of the regular covering projections \( \tilde{\Sigma} \times \mathbb{RP}^1 \to \Sigma \times \mathbb{RP}^1 \) and \( \Sigma \times \mathbb{RP}^1 \to \Sigma \times_{\phi_{\text{base}}} \mathbb{RP}^1 \), with deck transformation groups \( \pi_1(\mathbb{RP}^1) \) and \( \pi_1(\Sigma) \), respectively. The transversely projective foliation \( \mathcal{F} \) on \( \Sigma \times_{\phi_{\text{base}}} \mathbb{RP}^1 \) is therefore the quotient of the foliation on \( \tilde{\Sigma} \times \mathbb{RP}^1 \) with leaves parallel to \( \tilde{\Sigma} \). The holonomy representation \( \psi \) is the action of \( \pi_1(\mathbb{RP}^1) \) on \( \mathbb{RP}^1 \) factor. Note that \( \pi_1(\mathbb{RP}^1) \) is canonically isomorphic to \( \mathbb{Z} \), and acts on \( \mathbb{RP}^1 \) the same way as the center \( \mathbb{Z} \) of \( \tilde{SL}(2,\mathbb{R}) \). This means

\[ \psi_{\text{fib}} = +1. \]

Note

\[ \psi_{\text{base}} = \phi_{\text{base}}. \]

The volume of \( (\Sigma \times_{\phi_{\text{base}}} \mathbb{RP}^1, \psi) \) can be computed through Godbillon–Vey class of \( \mathcal{F} \):

\[ \text{vol}_{\tilde{SL}(2,\mathbb{R}) \times \mathbb{R}}(\Sigma \times_{\phi_{\text{base}}} \mathbb{RP}^1, \psi) = \int_{\Sigma} \int_{\mathbb{RP}^1} \text{gv}(\mathcal{F}) = 4\pi^2 e(\phi_{\text{base}}), \quad (3.4) \]

see [2, Proposition 2] (and also Section A).

As oriented circle bundles over \( \Sigma \) are classified by \( H^2(\Sigma; \mathbb{Z}) \cong \mathbb{Z} \), denote by \( \Sigma \times_m S^1 \) the oriented circle bundle over \( \Sigma \) of Euler number \( m \in \mathbb{Z} \). If \( m \) is nonzero, there is a fiber-preserving map \( \Sigma \times_1 S^1 \to \Sigma \times_m S^1 \) of degree \( m \), which is an cyclic covering of signed degree \( m \) restricted to every fiber. In particular, there are such maps \( \Sigma \times_1 S^1 \to N \) of degree \( e \), and \( \Sigma \times_1 S^1 \to \Sigma \times_{\phi_{\text{base}}} \mathbb{RP}^1 \) of degree \( e(\phi_{\text{base}}) \neq 0 \). Denote by \( \phi : \pi_1(N) \to \text{PSL}(2,\mathbb{R}) \) and \( \psi : \pi_1(\Sigma \times_{\phi_{\text{base}}} \mathbb{RP}^1) \to \text{PSL}(2,\mathbb{R}) \)
the pull-backs of $\phi_{\text{base}}$. The $\widetilde{SL}(2,\mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$-representation $\phi$ lifts $\tilde{\phi}$. Take some $\widetilde{SL}(2,\mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$-representation $\psi$ that lifts $\tilde{\psi}$. Denote by

$$\phi' : \pi_1(\Sigma \times S^1) \longrightarrow \pi_1(N) \longrightarrow \widetilde{SL}(2,\mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$$

and

$$\psi' : \pi_1(\Sigma \times S^1) \longrightarrow \pi_1(\Sigma \times_{\phi_{\text{base}}} \mathbb{R}P^1) \longrightarrow \widetilde{SL}(2,\mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$$

the pull-back representations of $\pi_1(\Sigma \times S^1)$. Note that both $\phi$ and $\psi$ factors through $\phi_{\text{base}} : \pi_1(\Sigma) \rightarrow \text{PSL}(2,\mathbb{R})$. It follows that $\phi'$ and $\psi'$ lies in the same component of $\mathcal{R}(\pi_1(\Sigma \times S^1), \widetilde{SL}(2,\mathbb{R}) \times_{\mathbb{Z}} \mathbb{R})$. We obtain

$$e \times \text{vol}(N, \phi) = \text{vol}(\Sigma \times S^1, \phi') = \text{vol}(\Sigma \times S^1, \psi') = e(\phi_{\text{base}}) \times \text{vol}(\Sigma \times_{\phi_{\text{base}}} \mathbb{R}P^1, \psi),$$

using (3.4). This implies

$$e \times \text{vol}(N, \phi) = 4\pi^2 e(\phi_{\text{base}})^2. \quad (3.5)$$

We show that $\phi_{\text{fib}}'$ and $\psi_{\text{fib}}'$ are equal, by arguing that the fiber image is constant on any path-connected component of $\mathcal{R}(\pi_1(\Sigma \times S^1), \widetilde{SL}(2,\mathbb{R}) \times_{\mathbb{Z}} \mathbb{R})$. In fact, because $H^2(\Sigma \times S^1, \mathbb{Z}) \cong H^1(\Sigma, \mathbb{Z})$ is torsion-free, $\mathcal{R}(\pi_1(\Sigma \times S^1), \widetilde{SL}(2,\mathbb{R}))$ intersects every path-connected component of $\mathcal{R}(\pi_1(\Sigma \times S^1), \widetilde{SL}(2,\mathbb{R}) \times_{\mathbb{Z}} \mathbb{R})$ (Theorem 3.1). The fiber image is constant on any path-connected component of $\mathcal{R}(\pi_1(\Sigma \times S^1), \widetilde{SL}(2,\mathbb{R}))$, as it takes discrete values in $\mathbb{Z}$. The fiber image is also constant for under the action of $H^1(\Sigma \times S^1; \mathbb{R})$, as the fiber is null-homologous in $\Sigma \times S^1$ (see (3.1)). Therefore, the fiber image is constant on any path-connected component of $\mathcal{R}(\pi_1(\Sigma \times S^1), \widetilde{SL}(2,\mathbb{R}) \times_{\mathbb{Z}} \mathbb{R})$, by Theorem 3.1. We obtain

$$e \times \phi_{\text{fib}}' = \phi_{\text{fib}}' = \psi_{\text{fib}}' = e(\phi_{\text{base}}) \times 1 = e(\phi_{\text{base}}). \quad (3.6)$$

Then asserted formulas follow from (3.5) and (3.6) for $e(\phi_{\text{base}}) \neq 0$. The possible values for $\phi_{\text{fib}}'$ are determined, since $e(\phi_{\text{base}})$ can take precisely the values $0, \pm 1, \ldots, \pm (2g - 2)$.

\[\Box\]

Remark 3.3.

1. The volume function on $\mathcal{R}(\pi_1(N), \widetilde{SL}(2,\mathbb{R}) \times_{\mathbb{Z}} \mathbb{R})$ is completely determined by Theorem 3.2. This is because any representation that lies over an abelian component of the $\text{PSL}(2,\mathbb{R})$-representation variety must have volume 0. In fact, any such representation is path connected with one that lies over a $\text{PSL}(2,\mathbb{R})$-representation with finite cyclic image.

2. Theorem 3.2 refines [5, Proposition 6.3] in the case of Seifert-geometric circle bundles. The point is that here we have enumerated all the actual volume values, thanks to Goldman’s result on $\text{PSL}(2,\mathbb{R})$-representation varieties of surface groups [10]. As the $\text{PSL}(2,\mathbb{R})$-representation varieties of Fuchsian groups have been described in Jankins–Neumann [14], it seems possible to obtain analogous refinement for all closed Seifert fiber spaces, with only extra notations and reductions.
4 | TOPOLOGY OF GRAPH MANIFOLDS

In this section, we review Seifert fiber spaces and graph manifolds. We recall a collection of numerical data that largely describes the topology of a graph manifold, including the intersection number on any JSJ torus between the adjacent fibers, and the Euler number of the JSJ pieces relative to the framing given by adjacent fibers. We organize the data and study behavior of the data under finite covering. For standard facts in 3-manifold topology, we refer to [1, 12].

In the literature, different authors refer to wider or narrower classes of 3-manifolds when they use the term graph manifold. We restrict ourselves to a reasonably general class, where the Seifert fibrations of the JSJ pieces are all orientable over orientable bases. We actually remember the orientations as auxiliary extra data, and assume the JSJ graph to be simplicial, so as to write down our volume formula in a convenient way (Theorem 5.2). This leads to what we call a formatted graph manifold, as introduced in Definition 4.1.

4.1 | Seifert fiber spaces

Recall that Seifert fiber space is a closed 3-manifold together with a foliation by circles. Every leaf admits a foliated tubular neighborhood that is topologically modeled on a foliated open solid torus $U(1) \times \mathbb{C}$. The leaves are the orbits of the $U(1)$-action $u.(w, z) = (u^\beta w, u^\alpha z)$, for all $u \in U(1)$ and $(w, z) \in U(1) \times \mathbb{C}$, where $\beta$ is some nonzero integer and $\alpha$ is some integer coprime to $\beta$. There are at most finitely many leaves that are locally modeled with $|\beta| > 1$, which are called the exceptional fibers. Any other leaf is called an ordinary fiber.

Circle bundles over closed surfaces are Seifert fiber spaces without exceptional fibers. In general, it is often instructive to think of a Seifert fiber space $\mathcal{N}$ analogously as a circle bundle over a closed 2-orbifold $\mathcal{O}$, where $\mathcal{O}$ can be concretely constructed as the leaf space of the foliation. The orbifold Euler characteristic $\chi(\mathcal{O}) \in \mathbb{Q}$ plays the same role as the usual Euler characteristic of the base for genuine circle bundles. When the Seifert fibration and the base 2-orbifold are both oriented, the Euler number $e(\mathcal{N} \to \mathcal{O}) \in \mathbb{Q}$ can be defined as an analog of the usual Euler number for oriented circle bundles over oriented closed surfaces.

The isomorphism type of any closed orientable 2-orbifold can be uniquely described with a symbol $(g; \beta_1, \ldots, \beta_k)$, which means genus $g \in \{0, 1, 2, \ldots\}$ with $k$ cone points of orders $\beta_i > 1$, respectively. The orbifold Euler characteristic can be computed by the formula

$$\chi(\mathcal{O}) = 2 - 2g - k + \sum_{i=1}^{k} \frac{1}{\beta_i}. \quad (4.1)$$

For any oriented closed 2-orbifold $\mathcal{O}$ of symbol $(g; \beta_1, \ldots, \beta_k)$, the oriented isomorphism type of any oriented Seifert fibration $\mathcal{N} \to \mathcal{O}$ can be uniquely described with a normalized symbol $(g; (1, \alpha_0), (\beta_1, \alpha_1), \ldots, (\beta_k, \alpha_k))$, where $\alpha_0$ is an integer, and where $\alpha_i \in \{1, 2, \ldots, \beta_i - 1\}$ are coprime to $\beta_i$ for $i = 1, \ldots, k$. The Euler number of the Seifert fibration can be computed by the formula

$$e(\mathcal{N} \to \mathcal{O}) = \alpha_0 + \sum_{i=1}^{k} \frac{\alpha_i}{\beta_i}. \quad (4.2)$$
**4.2 | Graph manifolds**

An orientable prime closed 3-manifold is said to be a *graph manifold* if there exists a possibly empty finite collection of mutually disjoint incompressible tori, and if the complement of their union admits a foliation by circles.

According to the Jaco–Shalen–Johanson (JSJ) decomposition theory in 3-manifold topology, every orientable prime closed 3-manifold contains an isotopically unique, minimal finite collection of mutually disjoint incompressible tori, such that the following dichotomy holds for every component of the complement of their union: Either the component admits a foliation by circles, or the component does not contain any incompressible tori that does not bound any toral end. (The two possibilities overlap when a component is a so-called small Seifert fiber space.) The tori are called the *JSJ tori*, and the complementary components are called the *JSJ pieces*. Graph manifolds are precisely those orientable prime 3-manifolds whose JSJ decomposition has only Seifert-fibered pieces. The *JSJ graph* is the dual graph of the JSJ decomposition: As a cell 1-complex, the vertices (0-cells) and the edges (1-cells) correspond the JSJ pieces and the JSJ tori, respectively, and the attaching maps are indicated by the adjacency relation between the JSJ objects.

**Definition 4.1.** We say that a closed orientable prime 3-manifold $M$ is a *formattable graph manifold*, if the JSJ graph of $M$ is a simplicial graph, and if the JSJ pieces of $M$ all admit orientable Seifert fibration over orientable 2-orbifolds (possibly with punctures). A *formatted graph manifold* is defined as an oriented formattable graph manifold $M$ enriched with an oriented Seifert fibration structure for every JSJ piece of $M$.

**Notation 4.2.** For any formatted graph manifold $M$ with a simplicial JSJ graph $(V, E)$, we introduce the following notations.

- For every vertex $v \in V$, denote by $J_v \to \mathcal{O}_v$ the oriented Seifert fibration of the corresponding JSJ piece. Denote by
  
  $$\chi_v = \chi(\mathcal{O}_v) \in \mathbb{Q}$$

  the orbifold Euler characteristic of $\mathcal{O}_v$.

- For every edge $\{v, w\} \in E$, denote by $T_{v, w}$ the corresponding JSJ torus with the outward orientation with respect to the orientation of $J_v$. Denote by
  
  $$b_{v, w} = I_{v, w}([f_v], [f_w]) \in \mathbb{Z}$$

  the algebraic intersection number between the oriented slopes $f_v$ and $f_w$ on $T_{v, w}$, that are parallel to the ordinary fibers of $J_v$ and $J_w$, respectively. Note that $T_{w, v}$ refers to the same JSJ torus with the reversed orientation, so $b_{w, v} = b_{v, w}$ depends only on $\{v, w\} \in E$.

- For every vertex $v \in V$, denote by $\hat{J}_v$ the oriented closed 3-manifold obtained from the Dehn filling of $J_v$ along the slopes $f_w$ on $T_{v, w}$, for all $w$ incident to $v$. Note that $J_v \to \mathcal{O}_v$ extends to a unique oriented Seifert fibration, over the closed oriented 2-orbifold $\hat{\mathcal{O}}_v$ obtained from $\mathcal{O}_v$ by filling all punctures. Denote by
  
  $$k_v = e(\hat{J}_v \to \hat{\mathcal{O}}_v) \in \mathbb{Q}$$

  the Euler number of the oriented Seifert fibration $\hat{J}_v \to \hat{\mathcal{O}}_v$. 
Remark 4.3. The quantity $k_v$ is sometimes called the charge of $M$ at the vertex $v$. The quantities as summarized in Notation 4.2 have been used in Buyalo–Svetlov [4] to characterize a list of significant properties, such as the existence of a nonpositively curved Riemannian metric, and the existence of a virtual fibering over the circle, and so on.

For any formatted graph manifold $M$ with the simplicial graph $(V, E)$, a Waldhausen basis can be chosen for the outward oriented JSJ tori $T_{v,w}$ that are adjacent to a JSJ piece $J_v$. To be precise, this is a basis $H_1(T_{v,w}; \mathbb{Q})$ of the form

$$\left([f_v], [s_{v,w}]\right)$$

as follows: The homology class $[f_v] \in H_1(T_{v,w}; \mathbb{Q})$ is represented by any oriented ordinary fiber of $J_v$; the homology classes $[s_{v,w}] \in H_1(T_{v,w}; \mathbb{Q})$ satisfy the relation $\sum_{\{v,w\} \in E} [s_{v,w}] = 0$ in $H_1(J_v; \mathbb{Q})$, under the natural inclusions $H_1(T_{v,w}; \mathbb{Q}) \to H_1(J_v; \mathbb{Q})$; and the algebraic intersection number on $H_1(T_{v,w}; \mathbb{Q})$ between $[f_v]$ and $[s_{v,w}]$ satisfies $I_{v,w}([f_v], [s_{v,w}]) = +1$, for all $\{v, w\} \in E$. With respect to given Waldhausen bases $([f_v], [s_{v,w}])$ for all $T_{v,w}$, the charge at vertices can be expressed explicitly as

$$k_v = \sum_{\{v,w\} \in E} \frac{a_{v,w}}{b_{v,w}}$$

where $a_{v,w} \in \mathbb{Q}$ are coefficients determined uniquely by the linear combination relation

$$[f_w] = a_{v,w}[f_v] + b_{v,w}[s_{v,w}]$$

in $H_1(T_{v,w}; \mathbb{Q})$. The existence of Waldhausen bases follows easily from the homology of orientable Seifert fibrations over orientable 2-orbifolds with punctures. If $J_v \to \mathcal{O}_v$ is a trivial circle bundle over an oriented surface, one may choose oriented slopes $s_{v,w}$ on $T_{v,w}$ so that they bound some horizontal lift of the base surface. In general, we do not require $[s_{v,w}]$ to have slope representatives.

### 4.3 Covering graph manifolds

**Definition 4.4.** A formatted covering projection is defined as a covering projection $M' \to M$ between formatted graph manifolds which preserves the orientations of the manifolds, and which respects the oriented Seifert fibration structures on the JSJ pieces, mapping fibers onto fibers preserving their orientations.

**Proposition 4.5.** Suppose that $M' \to M$ is a formatted covering projection between formatted graph manifolds.

1. For any covering pair of JSJ pieces $J'_{v'} \to J_v$,

$$\chi'_{v'} = \frac{[J'_{v'} : J_v]}{[f'_{v'} : f_v]} \times \chi_v.$$
(2) For any covering pair of JSJ pieces $J'_{v'} \rightarrow J_v$,

$$k_{v'} = \frac{[J'_{v'} : J_v]}{[f'_{v'} : f_v]} \times k_v.$$ 

(3) For any covering pair of JSJ tori $T'_{v',w'} \rightarrow T_{v,w}$,

$$b_{v',w'} = \frac{[f'_{v'} : f_v] \times [f'_{w'} : f_w]}{[T'_{v',w'} : T_{v,w}]} \times b_{v,w}.$$ 

(4) For any covering pairs of JSJ pieces $J'_{v'} \rightarrow J_v$ and any JSJ torus $T_{v,w}$,

$$\sum_{\{v',w'\} \in E'} \frac{[f'_{w'} : f_w]}{b_{v',w'}} = \frac{[J'_{v'} : J_v]}{[f'_{v'} : f_v]} \times \frac{1}{b_{v,w}}.$$ 

Here, the notation $[- : -]$ stands for the unsigned covering degree.

**Proof.** For any covering pair of JSJ pieces $J'_{v'} \rightarrow J_v$, there is an orbifold covering projection between the base 2-orbifolds $O'_{v'} \rightarrow O_v$. Denote by $J^*_{v'} \rightarrow O'_{v'}$, the pull-back of the Seifert fibration $J_v \rightarrow O_v$ using $O'_{v'} \rightarrow O_v$. Then $J'_{v'} \rightarrow J_v$ factorizes through the intermediate cover $J^*_{v'}$ of $J_v$, and $J'_{v'} \rightarrow J^*_{v'}$ is obtained from a fiber-preserving free action of a finite cyclic group, whose order equals $[f'_{v'} : f_v]$. We compute covering degree of $J'_{v'} \rightarrow J_v$ using the intermediate cover $J^*_{v'}$, obtaining the relation

$$[J'_{v'} : J_v] = [O'_{v'} : O_v] \times [f'_{v'} : f_v] = \frac{\chi_{v'}}{\chi_v} \times [f'_{v'} : f_v]. \tag{4.5}$$

We can compute the covering degree of $T'_{v',w'} \rightarrow T_{v,w}$ in two ways. Using the slope pairs $(f'_{v'}, f'_{w'})$ and $(f_v, f_w)$, we obtain the relation

$$[T'_{v',w'} : T_{v,w}] = [f'_{v'} : f_v] \times [f'_{w'} : f_w] \times \frac{b_{v,w}}{b_{v',w'}}. \tag{4.6}$$

On the other hand, we fix Waldhausen bases $([f_v], [s_{v,w}])$ on all $H_1(T_{v,w}; \mathbb{Q})$. There are uniquely induced Waldhausen bases $([f'_{v'}], [s'_{v',w'}])$ on all $H_1(T'_{v',w'}; \mathbb{Q})$, such that each $[s'_{v',w'}]$ becomes a positive rational multiple of $[s_{v,w}]$, under the induced homomorphism with respect to $T'_{v',w'} \rightarrow T_{v,w}$. Denote by the positive rational scalar as $[s'_{v',w'} : s_{v,w}] \in \mathbb{Q}$. Then, we obtain the relation

$$[T'_{v',w'} : T_{v,w}] = [f'_{v'} : f_v] \times [s'_{v',w'} : s_{v,w}]. \tag{4.7}$$

Using the formula (4.4), we can derive

$$\frac{a_{v',w'}}{b_{v',w'}} = \frac{[s'_{v',w'} : s_{v,w}]}{[f'_{v'} : f_v]} \times \frac{a_{v,w}}{b_{v,w}}, \tag{4.8}$$

where $[f'_{v'}] = a_{v',w'}[f'_{v'}] + b_{v',w'}[s'_{v',w'}]$ and $[f_w] = a_{v,w}[f_v] + b_{v,w}[s_{v,w}]$. 


By considering the covering degree over $J_v$ and any $T_{v,w}$, we obtain one more relation

$$[J'_{v'} : J_v] = \sum_{[v',w'] \in E'} [T'_{v',w'} : T_{v,w}].$$

(4.9)

The asserted formula regarding $\chi_{v'}$ is equivalent to (4.5). The asserted formula regarding $k_{v'}$ follows directly from (4.7), (4.8), (4.9), and (4.3). The asserted formula regarding $b_{v',w'}$ is equivalent to (4.6). The last asserted formula follows directly from (4.6) and (4.9).

□

5 | SEIFERT REPRESENTATIONS FOR GRAPH MANIFOLDS: THE STATEMENT

In this section, we give a precise statement of our main result (Theorem 5.2). The statement contains a volume formula for $\widetilde{\text{SL}}(2,\mathbb{R}) \times \mathbb{R}$-representations of formatted graph manifolds, and an estimate analogous to the Milnor–Wood inequality, and an assertion regarding rationality of the volume value. After that, we include some preliminary discussions to justify the formulation of Theorem 5.2, and to explain the structure of its proof. The entire proof of Theorem 5.2 occupies Sections 6–9 in the sequel.

**Notation 5.1.** For any formatted graph manifold $M$ with a simplicial JSJ graph $(V, E)$, and for any representation $\rho : \pi_1(M) \to \widetilde{\text{SL}}(2,\mathbb{R}) \times \mathbb{R}$, we introduce the following notations.

1. Denote by $\mathbb{R}^V$ the linear space of real functions on $V$. We think of $\mathbb{R}^V$ as furnished with the standard unordered basis $\{v^* : v \in V\}$, where $v^* \in \mathbb{R}^V$ stands for the characteristic function $v^*(v) = 1$ and $v^*(u) = 0$ for all $u \neq v$.
2. Denote by $\chi_M \in \mathbb{R}^V$ the rational vector such that $\chi_M(v) = \chi_v$.

Denote by $e_M \in \text{End}_{\mathbb{R}}(\mathbb{R}^V)$ the linear operator such that

$$e_M v^* = k_v v^* - \sum_{[v,w] \in E} \frac{w^*}{b_{v,w}}.$$

Note that $e_M$ is symmetric and rational with respect to the standard unordered basis. (See Notation 4.2.)

3. Denote by $\xi_\rho \in \mathbb{R}^V$ the vector such that

$$\xi_\rho(v) = \text{wind}(\rho(f_v)).$$

Here, $f_v$ is treated as an element of $\pi_1(M)$, and $\xi_\rho(v)$ depends only on its conjugacy class. (See Definition 2.3 and Proposition 2.4.)

4. For any edge $\{v, w\} \in E$, denote by

$$\tau_\rho(v, w) \in \mathbb{N} \cup \{\infty\}$$

...
the order of the image of $\pi_1(T_{v,w})$ under the induced representation $\tilde{\rho}: \pi_1(M) \to \text{PSL}(2, \mathbb{R})$. Here, $\pi_1(T_{v,w})$ is treated as a subgroup of $\pi_1(M)$, and $\tau_{\rho}(v, w)$ depends only on its conjugacy class.

**Theorem 5.2.** Let $M$ be a formatted graph manifold with a simplicial JSJ graph $(V, E)$ and $\rho: \pi_1(M) \to \tilde{\text{SL}}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$ be a representation. Adopt Notation 5.1. Then the following statements all hold true.

- **(Volume formula).**
  \[
  \text{vol}_{\tilde{\text{SL}}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}}(M, \rho) = 4\pi^2 \cdot \sum_{v \in V} \xi_{\rho}(v) \times (e_M \xi_{\rho})(v).
  \]

- **(Generalized Milnor–Wood Inequalities).** For all vertices $v \in V$,
  \[
  |(e_M \xi_{\rho})(v)| \leq \max \left\{ 0, -\chi_M(v) - \sum_{\{v, w\} \in E} \frac{1}{\tau_{\rho}(v, w)} \right\}.
  \]

- **(Rationality).** There is a solution $X \in \mathbb{Q}^V$ to the equation
  \[
  \text{vol}_{\tilde{\text{SL}}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}}(M, \rho) = 4\pi^2 \cdot \sum_{v \in V} X(v) \times (e_M X)(v).
  \]

**Remark 5.3.**

1. Denote by $(\cdot, \cdot)$ stands for the standard inner product on $\mathbb{R}^V$. Then the volume formula in Theorem 5.2 reads:
   \[
   \text{vol}_{\tilde{\text{SL}}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}}(M, \rho) = 4\pi^2 \cdot (\xi_{\rho}, e_M \xi_{\rho}),
   \]
   which closely looks like the formula in Theorem 3.2.

2. Let $N$ be an oriented circle bundle of Euler number $e \neq 0$ over an oriented surface of genus $g > 1$. Suppose that $N$ admits a transversely projective codimension-1 foliation with holonomy $\tilde{\phi}: \pi_1(N) \to \tilde{\text{SL}}(2, \mathbb{R})$. Then the estimate in Theorem 5.2 (or Theorem 3.2) applies to any lift $\phi: \pi_1(N) \to \tilde{\text{SL}}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$ of $\tilde{\phi}$. The resulting inequality
   \[
   |\xi_{\phi}| \leq |(2g - 2)/e|
   \]
   agrees with the Milnor–Wood inequality [17, 24], as observed in [2, Theorem 5 and Proof].

Theorem 5.2 essentially allows us to compute volume of Seifert representations for any orientable closed graph manifold. This is because any nongeometric graph manifold admits a finite cover that is formattable. In fact, one may first construct a finite cover where all the JSJ pieces are product (see [16, Proposition 4.4]); and then, construct a further finite cover that is induced by a finite cover of the JSJ graph, such that the covering JSJ graph has no loops with two or fewer edges (using residual finiteness of finitely generated free groups). The resulting cover is a formattable graph manifold. By contrast, for orientable closed 3-manifolds of nonzero simplicial volume, no
precise values of the Seifert volume have been determined, as far as the authors know. See [15, Section 7] for some examples with numerical computations.

We obtain the following consequence from the rationality part in Theorem 5.2 (see also Section 3.1).

**Corollary 5.4.** For any orientable closed graph manifold $M$, the Seifert volume $SV(M)$ is a rational multiple of $\pi^2$.

As we mentioned before, the volume of $(M, \rho)$ depends only on the induced representation $\bar{\rho} : \pi_1(M) \to PSL(2, \mathbb{R})$. Indeed, our volume formula also carries this feature in an evident way.

**Proposition 5.5.** In Theorem 5.2, the vector $e_M \xi_\rho$ in $\mathbb{R}^V$ depends only on the induced representation $\bar{\rho} : \pi_1(M) \to PSL(2, \mathbb{R})$.

**Proof.** Any representation $\pi_1(M) \to \widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$ that induces the same representation $\bar{\rho} : \pi_1(M) \to PSL(2, \mathbb{R})$ as that of $\rho$ is a twisted representation $\rho[\alpha]$ for some cohomology class $\alpha \in H^1(M; \mathbb{R})$, see (3.1) and Theorem 3.1. We observe $\xi_{\rho[\alpha]}(v) = \xi_{\rho}(v) + \hat{\alpha}(v)$, where $\hat{\alpha}(v) = \alpha([f_v])$ is the value of $\alpha$ at $[f_v] \in H_1(M; \mathbb{R})$. We may write $\hat{\alpha} = \sum_{v \in V} \alpha([f_v]) v^*$. With respect to any auxiliary Waldhausen bases $\{(f_{v,w}), [s_{v,w}]\}$, we use (4.3) to compute, for any vertex $v \in V$:

$$
(e_M \hat{\alpha})(v) = k_v \cdot \alpha([f_v]) - \sum_{[v,w] \in E} \frac{1}{b_{v,w}} \cdot \alpha([f_w])
$$

$$
= \sum_{[v,w] \in E} \left( \frac{a_{v,w}}{b_{v,w}} \cdot \alpha([f_v]) - \frac{1}{b_{v,w}} \cdot \alpha(a_{v,w}[f_v] + b_{v,w}[s_{v,w}]) \right)
$$

$$
= -\alpha \left( \sum_{[v,w] \in E} [s_{v,w}] \right)
$$

$$
= 0.
$$

The last step uses the property $\sum_{[v,w] \in E} [s_{v,w}] = 0$ in $H_1(J_v; \mathbb{Q})$ of Waldhausen bases. Therefore, $(e_M \xi_{\rho[\alpha]})(v)$ is constant as $\alpha$ ranges over $H^1(M; \mathbb{R})$. In other words, $e_M \xi_\rho$ depends only on $\bar{\rho}$. □

In the rest of this section, we take a closer look at the local structure of Seifert representations for graph manifolds, and give an outline of the proof of Theorem 5.2, in Section 5.2.

It turns out that edges with $\tau_{\rho}(v, w) = \infty$ are the most troublesome. We wish to get rid of them eventually by certain means of modifying $\rho$. In order to do so, we need to introduce some quantity $\delta_{\rho}(v)$ for measuring the local badness regarding such edges, and a more auxiliary relative version $\delta_{\rho}(v; C)$. These are introduced in Notation 5.6, after some preliminary discussion regarding local types of $\rho$, in Section 5.1.

## 5.1 Local types of Seifert representations

We analyze types of Seifert representations at vertices of the JSJ graph. Let $M$ be a formatted graph manifold and $\rho : \pi_1(M) \to \widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$ be a representation. Adopt Notation 5.1.
For any vertex \( v \in (V, E) \), if the image of any conjugate of \( f_v \) under the induced representation \( \bar{\rho} : \pi_1(M) \to \text{PSL}(2, \mathbb{R}) \) is nontrivial, the image of any conjugate of \( \pi_1(J_v) \) under \( \bar{\rho} \) must be abelian, centralizing some conjugate of \( \rho(f_v) \). In this case, \( \bar{\rho} (\pi_1(J_v)) \) can be conjugated into a 1-dimensional closed subgroup of \( \text{PSL}(2, \mathbb{R}) \), which may be elliptic, hyperbolic, or parabolic, depending on the type of \( \bar{\rho}(f_v) \).

We say that \( v \in V \) is a vertex of central type with respect to \( \rho \), if \( \bar{\rho}(f_v) \) is trivial in \( \text{PSL}(2, \mathbb{R}) \). Similarly, we call \( v \) a vertex of elliptic, hyperbolic, or parabolic type, if \( \bar{\rho}(f_v) \) is nontrivial of the corresponding type, which depends only on the conjugacy class of \( \bar{\rho}(f_v) \). We say that \( \{v, w\} \in E \) is an edge of central type with respect to \( \rho \), if both \( v \) and \( w \) are central. Similarly, we speak of edges of elliptic, hyperbolic, or parabolic types. Any other edge must connect a central type vertex with another of one of the three noncentral types, which we call the transiting type.

For any simplicial subgraph \( C = (V_C, E_C) \) of the simplicial JSJ graph \( (V, E) \), we denote by \( M_C \) the union of the JSJ pieces and the JSJ tori that occur in \( C \), and call \( M_C \) the JSJ block that corresponds to \( C \). Note that \( M_C \) is a 3-manifold possibly with toral ends.

A maximal noncentral component \( C \) refers to a maximal connected subgraph of \( (V, E) \) without central vertices. In other words, \( C \) is any connected component of the subgraph \( (V_{nc}, E_{nc}) \) obtained from \( (V, E) \) by removing all the central type vertices and all the central or transiting type edges. Thereby we call \( M_C \) a maximal noncentral block. The structure of \( \rho \) restricted to \( \pi_1(M_C) \) is described with more detail in Lemma 5.8 below.

**Notation 5.6.** Let \( M \) be a formatted graph manifold \( M \) with a simplicial JSJ graph \( (V, E) \), and \( \rho : \pi_1(M) \to \tilde{\text{SL}}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R} \) be a representation. Adopting Notation 5.1.

1. Let \( C = (V_C, E_C) \) be any maximal noncentral component with respect to \( \rho \). For any vertex \( v \in V \), denote by

\[
\delta_{\rho}(v; C) \in \mathbb{N} \cup \{0\}
\]

the number as follows: If \( v \in V \setminus V_C \), \( \delta_{\rho}(v; C) \) is the number of vertices \( w \in V_C \) incident to \( v \) with the property \( \tau_{\rho}(v, w) = \infty \). Otherwise, set \( \delta_{\rho}(v; C) = 0 \).

2. For any vertex \( v \), denote by

\[
\delta_{\rho}(v) \in \mathbb{N} \cup \{0\}
\]

the sum of all \( \delta_{\rho}(v; C) \), where \( C \) ranges over all the maximal noncentral components with respect to \( \rho \).

We can characterize \( \delta_{\rho}(v) \) alternatively as follows.

**Lemma 5.7.** Let \( M \) be a formatted graph manifold with a simplicial JSJ graph \( (V, E) \), and \( \rho : \pi_1(M) \to \tilde{\text{SL}}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R} \) be a representation.

1. If \( v \in V \) is of central type with respect to \( \rho \), then \( \delta_{\rho}(v) \) is the number of edges \( \{v, w\} \in E \) with \( \tau_{\rho}(v, w) = \infty \).
2. If \( v \in V \) is of noncentral type with respect to \( \rho \), then \( \delta_{\rho}(v) \) is zero.
Proof. For any \( \{v, w\} \in E \) with both \( v \) and \( w \) of central type, \( \tau_\rho(v, w) \) is necessary finite. So, for any \( v \in V \) of central type, \( \delta_\rho(v) = \sum_C \delta_\rho(v; C) \) counts all the edges \( \{v, w\} \in E \) with \( \tau_\rho(v, w) = \infty \). On the other hand, there are no edges \( \{v, w\} \) where \( v \) and \( w \) lie in distinct maximal noncentral components. So, for any \( v \in V \) of noncentral type, \( \delta_\rho(v; C) = 0 \) no matter \( C \) contains \( v \) or not, implying \( \delta_\rho(v) = 0 \). \( \square \)

**Lemma 5.8.** Let \( M \) be a formatted graph manifold and \( \rho : \pi_1(M) \to \widetilde{SL}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R} \) be a representation. Suppose that \( C = (V_C, E_C) \) is a subgraph of the JSJ graph \( (V, E) \), and is a maximal noncentral component with respect to \( \rho \).

Then, \( \rho \) is abelian over any spanning tree \( Y \) of \( C \), namely, \( \rho(\pi_1(M_Y)) \) is abelian. Moreover, \( \rho(\pi_1(M_C)) \) is contained in the normalizer of the centralizer of \( \rho(f_v) \) in \( \widetilde{SL}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R} \) for any \( v \in V_C \).

Proof. If \( e \) is an edge adjacent to a vertex \( v \in C \), the centralizer of \( \rho(\pi_1(T_e)) \) in \( \widetilde{SL}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R} \) is the same as the centralizer of \( \rho(\pi_1(J_v)) \), both equal to the centralizer of the noncentral element \( \rho(f_v) \). Moreover, if an element \( g \in \widetilde{SL}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R} \) conjugates \( \rho(\pi_1(T_e)) \) into the same centralizer, then \( g \) lies in the normalizer of the centralizer, since \( g \rho(f_v) g^{-1} \) commutes with \( \rho(f_v) \) (and by simple facts about elementary subgroups in \( PSL(2, \mathbb{R}) \)).

Joining other vertices with edges in \( C \) one by one, it follows that \( \rho(\pi_1(M_Y)) \) is contained in the unchanged centralizer, by induction and free amalgamation. In particular, \( \rho(\pi_1(M_Y)) \) is abelian. Joining the rest edges of \( C \) one by one, it follows that \( \rho(\pi_1(M_C)) \) is contained in the normalizer of the unchanged centralizer, by induction and HNN extension. See [22] for standard terminology about graph-of-group decompositions. \( \square \)

In particular, we can speak of elliptic, hyperbolic, or parabolic maximal noncentral components, according to the type of any of its vertex, and this will also be the type of any of its edge. Note that the normalizer of the centralizer of \( \pi_1(f_v) \) in \( \widetilde{SL}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R} \) quotient by the centralizer of \( \pi_1(f_v) \) is trivial if \( v \) is elliptic, or isomorphic to \( \mathbb{Z}/2\mathbb{Z} \) if \( v \) is hyperbolic, or isomorphic to \( \mathbb{R} \) if \( v \) is parabolic.

## 5.2 Outline of the proof

We prove Theorem 5.2 first in the virtually central case, where \( \rho \) is virtually central restricted to any vertex group \( \pi_1(J_v) \). This is equivalent to saying that \( \tau_\rho(v, w) \) are all finite. If \( \tau_\rho(v, w) \) are all 1, \( \rho \) is actually central restricted to any vertex group \( \pi_1(J_v) \). The volume in the central case can be suitably decomposed into contribution from each of the vertices, thanks to the additivity principle (see Theorem 6.3). Moreover, the individual contribution can be computed using Theorem 3.2. Then, the volume formula for the virtually central case can be derived by passing to a suitable finite cover of \( M \), and by approximating using virtually central representations with \( \xi \)-vectors in \( \mathbb{Q}^V \). The virtually central case is done in Section 6.

The proof of Theorem 5.2 would be complete if we were able to deform any representation to a virtually central one, through a continuous path in \( R(\pi_1(M), \widetilde{SL}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R}) \) along which the \( \xi \)-vector stays constant. Unfortunately, such a deformation appears unlikely to exist in general. However, we prove that there exist a formatted graph manifold \( M' \) that maps nicely onto \( M \) of degree 1, such that the pull-back representation \( \rho' : \pi_1(M') \to \widetilde{SL}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R} \)
deforms continuously to a virtually central representation. Since the volume of \((M', \rho')\) is equal to the volume of \((M, \rho)\), the construction allows us to reduce to the proof of the volume formula to the virtually central case on the level of \((M', \rho')\). We are also able to obtain a slightly weaker estimate for \(|e_{M', \rho}(v)|\) using \((M', \rho')\). To complete the proof of Theorem 5.2, we apply the covering trick again, and promote the weak estimate to the desired one.

The promotion appeals to certain efficient control on the complexity of the above \(M'\). To be more precise, the degree-one map \(M' \to M\) that we construct induces an isomorphism between the JSJ graphs, such that \(e_{M'} = e_M\) holds as we identify their JSJ graphs. Meanwhile, \(\chi_{M'}(v) \leq \chi_M(v)\) holds for any vertex \(v\). In effect, therefore, passing from \(M\) to \(M'\) increases the genera of the base 2-orbifolds of the JSJ pieces. We need sufficient increment of the genera to make enough room for the desired deformation, but we also want the increment at any vertex to be relatively small compared to the original genus. It turns out that \(\chi_{M'}(v) = \chi_M(v) - 2 \cdot \delta_{\rho}(v)\) suffices for our purpose. As one may infer from Notation 5.6, our deformation is obtained by assembling deformations restricted to the vertex subgroups (or JSJ-block subgroups). In Section 7, we develop techniques to factorize certain (continuous or sequential) families of elements in \(\tilde{\text{SL}}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R}\) nicely as families of commutators. In Section 8, we employ those techniques to construct \((M', \rho')\) for the desired reduction. In Section 9, we put all the partial results together, and conclude the proof of Theorem 5.2.

6 | THE VIRTUALLY CENTRAL CASE

In this section, we prove a special case of Theorem 5.2, as the following Theorem 6.1. Note that the additional assumption implies that \(\rho\) is virtually central restricted to the vertex groups, or equivalently, that \(\rho\) pulls back to a representation that is central restricted to the vertex groups, for some finite cover of \(M\).

**Theorem 6.1.** The statements of Theorem 5.2 hold true if \(\tau_{\rho}(v, w)\) is finite for all \(\{v, w\} \in E\).

The proof of Theorem 6.1 occupies the rest of this section.

6.1 | The fundamental computation

**Lemma 6.2.** The statements of Theorem 5.2 hold true under the following additional conditions:

- \(\xi_{\rho}(v) \in \mathbb{Q}\), for all vertices \(v \in V\), and
- \(\tau_{\rho}(v, w) = 1\), for all edges \(\{v, w\} \in E\).

The rest of this subsection is devoted to the proof of Lemma 6.2. We employ the following additivity principle for volume computation:

**Theorem 6.3** [5, Theorem 3.5]. Let \(M\) be an oriented closed irreducible 3-manifold with the JSJ tori \(T_1, ..., T_r\) and the JSJ pieces \(J_1, ..., J_k\). Let \(h_1, ..., h_r\) be slopes on \(T_1, ..., T_r\), respectively. Denote by \(N_i\) the oriented closed 3-manifold obtained from the Dehn filling of \(J_i\) along all the slopes \(h_j\) that occur on its boundary, for all \(i \in \{1, ..., k\}\).
Suppose that \( \rho : \pi_1(M) \to \tilde{\text{SL}}(2,\mathbb{R}) \times \mathbb{R} \) is a representation that is trivial on all the slopes \( h_j \).
Denote by \( \phi_i : \pi_1(N_i) \to \tilde{\text{SL}}(2,\mathbb{R}) \times \mathbb{R} \) the representation induced by \( \rho_i \), for all \( i \in \{1, \ldots, k\} \). Then the following formula holds:

\[
\text{vol}_{\tilde{\text{SL}}(2,\mathbb{R}) \times \mathbb{R}}(M, \rho) = \text{vol}_{\tilde{\text{SL}}(2,\mathbb{R}) \times \mathbb{R}}(N_1, \phi_1) + \cdots + \text{vol}_{\tilde{\text{SL}}(2,\mathbb{R}) \times \mathbb{R}}(N_k, \phi_k).
\]

Let us first unwrap the additional conditions of Lemma 6.2. The additional condition \( \tau_{\rho}(v, w) = 1 \) means that the representation \( \rho \) sends any JSJ torus subgroup \( \pi_1(T_{v, w}) \) to the central subgroup \( \mathbb{R} \). In particular, the image \( \rho(f_v) \) of the regular fiber of any JSJ piece \( J_v \) can be identified as a real value, which equals the essential winding number of \( \rho(f_v) \), by definition. Therefore, the additional condition \( \xi_{\rho}(f_v) \in \mathbb{Q} \) means \( \rho(f_v) \in \mathbb{Q} \). It follows that \( \rho(\pi_1(T_{v, w})) \) is also contained in the central subgroup \( \mathbb{Q} \) of \( \tilde{\text{SL}}(2,\mathbb{R}) \times \mathbb{R} \). In particular, there must exist a slope \( h_{v, w} \) on \( T_{v, w} \) that lies in the kernel of \( \rho \).

Choose a slope \( h_{v, w} \) for every edge \( \{v, w\} \in E \) as above, such that \( h_{w, v} \) is identified with \( h_{v, w} \) under the identification \( T_{v, w} \cong T_{w, v} \). For any JSJ piece \( J_v \) of \( M \), denote by \( N_v \) the Dehn filling of \( J_v \) along all the slopes \( h_{v, w} \) on \( \partial J_v \). Denote by \( \phi_v : \pi_1(N_v) \to \text{PSL}(2,\mathbb{R}) \) the representation induced by \( \rho \).

Theorem 6.3 allows us to compute the volume of \((M, \rho)\) from the volumes of \((N_v, \phi_v)\):

\[
\text{vol}_{\tilde{\text{SL}}(2,\mathbb{R}) \times \mathbb{R}}(M, \rho) = \sum_{v \in V} \text{vol}_{\tilde{\text{SL}}(2,\mathbb{R}) \times \mathbb{R}}(N_v, \phi_v). \quad (6.1)
\]

Note that if some \( h_{v, w} \) is parallel to the fiber \( f_v \) (in the unoriented sense), then \( \xi_{\rho}(v) \) equals 0. In this case, \( \text{vol}_{\tilde{\text{SL}}(2,\mathbb{R}) \times \mathbb{R}}(N_v, \phi_v) \) also equals 0, since \( \phi_v \) factors through the fundamental group of the base 2-orbifold of \( J_v \) (whose group homology is torsion on the dimension 3).

If \( J_v \) is any JSJ piece with \( \xi_{\rho}(v) \neq 0 \), there are no \( h_{v, w} \) parallel to the fiber \( f_v \). In this case, the Seifert fibration \( J_v \to \emptyset_v \) naturally extends to a Seifert fibration \( N_v \to \mathcal{R}_v \), where \( \mathcal{R}_v \) is the 2-orbifold obtained from \( \emptyset_v \) by filling the punctures with cone points. The orders of the cone points are the geometric intersection numbers between \( f_v \) and \( h_{v, w} \) on the tori \( T_{v, w} \) adjacent to \( J_v \).

**Lemma 6.4.** If \( \xi_{\rho}(v) \neq 0 \), then

\[
(e_M \xi_{\rho})(v) = e(N_v \to \mathcal{R}_v) \times \xi_{\rho}(v).
\]

**Proof.** Take auxiliary Waldhausen bases \( ([f_v], [s_{v, w}]) \) for all \( T_{v, w} \). Take an orientation for each \( h_{v, w} \). We write

\[
[f_w] = a_{v, w}[f_v] + b_{v, w}[s_{v, w}],
\]

\[
[h_{v, w}] = p_{v, w}[f_v] + q_{v, w}[s_{v, w}].
\]

Note that \( b_{v, w} \) and \( q_{v, w} \) are nonzero. We compute

\[
\frac{\xi_{\rho}(w)}{\xi_{\rho}(v)} = \frac{I_{v, w}([f_w], [h_{v, w}])}{I_{v, w}([f_v], [h_{v, w}])} = \frac{a_{v, w}q_{v, w} - b_{v, w}p_{v, w}}{q_{v, w}},
\]
where $I_{v,w} : H_1(T_{v,w};\mathbb{Q}) \times H_1(T_{v,w};\mathbb{Q}) \to \mathbb{Q}$ stands for the algebraic intersection pairing on the oriented JSJ torus $T_{v,w}$. This can be rearranged into

$$\xi_\rho(v) \times \left( \frac{a_{v,w}}{b_{v,w}} - \frac{p_{v,w}}{q_{v,w}} \right) = \xi_\rho(w) \times \frac{1}{b_{v,w}}.$$ 

Sum up over all the vertices $w$ incident to $v$. Then, we obtain

$$\xi_\rho(v) \times (k_v - e(N_v \to R_v)) = \sum_{\{v, w\} \in E} \xi_\rho(w) \times \frac{1}{b_{v,w}},$$

or equivalently,

$$(e_M \xi_\rho)(v) = e(N_v \to R_v) \times \xi_\rho(v).$$

as asserted.

Lemma 6.5. If $\xi_\rho(v) \neq 0$, then

$$\text{vol}_{\mathcal{SL}(2,\mathbb{R}) \times \mathbb{ZR}}(N_v, \phi_v) = 4\pi^2 \cdot e(N_v \to R_v) \times \xi_\rho(v)^2,$$

and moreover,

$$\left| e(N_v \to R_v) \times \xi_\rho(v) \right| \leq \max \{0, -\chi_M(v) - \text{valence}_{(V,E)}(v)\}.$$

Proof. For simplicity, we rewrite $N \to R$ for the Seifert fibration $N_v \to R_v$, and $f$ for the ordinary fiber $f_v$, and $\phi$ for the representation $\phi_v : \pi_1(N_v) \to \mathcal{SL}(2,\mathbb{R}) \times \mathbb{ZR}$. We rewrite $e_N$ for the Euler number $e(N_v \to R_v)$. We remember $\xi_\rho(v) = \phi(f)$.

The assertions are trivial if $e_N$ equals 0. If $e_N$ is not 0, and if $R$ has orbifold Euler number $\geq 0$, $\pi_1(R)$ is either virtually nonabelian nilpotent, or finite. Note that virtually nilpotent subgroups of $\mathcal{SL}(2,\mathbb{R}) \times \mathbb{ZR}$ are all abelian. In this case, we must have $\phi(f) = 0$, so the assertions are again trivial. It remains to prove for $e_N \neq 0$ and $\chi(R) < 0$.

Take covering projection $R' \to R$ of $R$ by a closed surface $R'$ (which always exists provided $\chi(R) < 0$). Denote by $N' \to R'$ the oriented circle bundle obtained from $N \to R$ by pull-back. For the pull-back representation $\phi' : \pi_1(N') \to \mathcal{SL}(2,\mathbb{R}) \times \mathbb{ZR}$, we observe that $\phi'$ is of central type, with $\phi'_{f \bot b} = \phi(f)$ and $e_{N'} = [R' : R] \times e_N$. As $R'$ has negative Euler characteristic and $N' \to R'$ has nonzero Euler number, we apply Theorem 3.2 to obtain the asserted volume formula:

$$\text{vol}_{\mathcal{SL}(2,\mathbb{R}) \times \mathbb{ZR}}(N, \phi) = \frac{\text{vol}_{\mathcal{SL}(2,\mathbb{R}) \times \mathbb{ZR}}(N', \phi')}{[R' : R]} = \frac{4\pi^2 \phi_{f \bot b} \times (e_{N'} \times \phi'_{f \bot b})}{[R' : R]} = 4\pi^2 e_N \times \xi_\rho(v)^2.$$

Let $P$ be the 2-orbifold obtained from $\mathcal{O}_v$ by filling the punctures with ordinary points. There is a natural orbifold map $R \to P$, which erases the filling cone points from $R$. The orbifold Euler characteristic of $P$ can be computed as

$$\chi(P) = \chi_M(v) + \text{valence}_{(V,E)}(v).$$
Suppose \( \chi(P) > 0 \). In this case, the 2-orbifold \( P \) is either spherical or bad. The induced representation \( \phi_{\text{base}} : \pi_1(R) \to \text{PSL}(2, \mathbb{R}) \) has necessarily finite image. The above volume formula implies \( \xi_P(v) = 0 \), so the asserted inequality holds for \( \chi(P) > 0 \), and indeed,

\[
|e_N \times \xi_P(v)| = 0.
\]

Suppose \( \chi(P) \leq 0 \). In this case, the 2-orbifold \( P \) admits a universal cover homeomorphic to an open disk. Then there exists a finite cover \( P'' \to P \) of \( P \) by a surface \( P'' \). Pull back \( P'' \to P \) along the composite map \( R' \to R \to P \). We obtain a finite covering projection \( R'' \to R' \), whose degree equals \( [P'' : P] \). The construction comes with a branched covering map \( R'' \to P'' \), whose mapping degree equals \( [R' : R] \). Denote by \( N'' \to P'' \) the pull-back of \( N' \to P' \) along \( R'' \to R' \). We obtain representations \( \phi'' : \pi_1(N'') \to \widetilde{\text{SL}}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R} \) and \( \phi' : \pi_1(N') \to \widetilde{\text{SL}}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R} \), which are naturally induced from \( \phi \) and of central type. Theorem 3.2 implies

\[
e(\phi''_{\text{base}}) = [P'' : P] \times e(\phi'_{\text{base}}) = [P'' : P] \times e_N \times \phi'_{\text{fib}} = [P'' : P] \times [R' : R] \times e_N \times \xi_P(v).
\]

On the other hand, \( \phi''_{\text{base}} : \pi_1(R'') \to \text{PSL}(2, \mathbb{R}) \) factors through a representation of \( \pi_1(P'') \), because of the extra assumption \( \tau_P(v, w) = 1 \) for all \{v, w\} \in E \) (from the assumptions of Lemma 6.2). The possible values for the Euler number of any representation \( \pi_1(P'') \to \text{PSL}(2, \mathbb{R}) \) are precisely \( 0, \pm 1, \pm 2, \ldots, \pm |\chi(P'')| \). By construction, we obtain

\[
|e(\phi''_{\text{base}})| \leq [R' : R] \times |\chi(P'')| = [R' : R] \times [P'' : P] \times |\chi(P)|.
\]

Therefore, we obtain

\[
|e_N \times \xi_P(v)| \leq |\chi(P)| = -\chi_M(v) - \text{valence}_{(V, E)}(v).
\]

so the asserted inequality also holds for \( \chi(P) \leq 0 \). \( \square \)

From Lemmas 6.4 and 6.5, we obtain

\[
\text{vol}_{\widetilde{\text{SL}}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R}}(N_v, \phi_v) = 4\pi^2 \cdot \xi_P(v) \times (e_M \xi_P)(v), \tag{6.2}
\]

and

\[
|e(N_v \to R_v) \times \xi_P(v)| \leq \max \{0, -\chi_M(v) - \text{valence}_{(V, E)}(v)\}. \tag{6.3}
\]

Both (6.2) and (6.3) hold for \( \xi_P(v) \neq 0 \) as well as for \( \xi_P(v) = 0 \). The inequality of Theorem 5.2 is exactly (6.3) under the additional condition \( \tau_P(v, w) = 1 \) for all \{v, w\} \in E \). The rationality of the inner product \( (\xi_P, e_M \xi_P) \) follows from Lemma 6.4 and the additional condition \( \xi_P(v) \in \mathbb{Q} \) for all \( v \in Q \). The volume formula of Theorem 5.2 follows from (6.1) and (6.2), under the additional conditions of Lemma 6.2.

This completes the proof of Lemma 6.2.
6.2 The covering trick

Lemma 6.6. The statements of Theorem 5.2 hold true under the following additional conditions:

- $\xi_\rho(v) \in \mathbb{Q}$, for all vertices $v \in V$, and
- $\tau_\rho(v, w) < \infty$, for all edges $\{v, w\} \in E$.

The rest of this subsection is devoted to the proof of Lemma 6.6. We reduce the problem to Lemma 6.2 by passing to a suitable finite cover of $M$. This invokes the following well-known fact, often referred to as Selberg’s lemma:

**Theorem 6.7 [20, Chapter 7, §7.6, Corollary 4].** Every finitely generated subgroup of $GL(n, \mathbb{C})$ contains a torsion-free normal subgroup of finite index.

Under the additional condition $\tau_\rho(v, w) < \infty$, the image of $\pi_1(T_{v, w})$ under $\bar{\rho} : \pi_1(M) \to PSL(2, \mathbb{R})$ is finite (and cyclic), for all edges $\{v, w\} \in E$. Note that $PSL(2, \mathbb{R}) \cong SO^+(2, 1)$ is linear, and that $\bar{\rho}(\pi_1(M))$ is a finitely generated group of $PSL(2, \mathbb{R})$. Then there exists some torsion-free finite-index normal subgroup of $\bar{\rho}(\pi_1(M))$, by Selberg’s lemma (Theorem 6.7). The preimage of that subgroup corresponds to a regular finite cover $M^*$ of $M$, where the JSJ tori subgroups all have trivial image under the pull-back representation $\bar{\rho}^* : \pi_1(M') \to PSL(2, \mathbb{R})$. For every covering pair of JSJ pieces $J^* \to J$, $J^*$ is furnished with an oriented Seifert fibration over an oriented base, as it covers such a Seifert fibration of $J$. Take a characteristic finite cover of the JSJ graph of $M^*$ that is a simplicial graph. Then, it induces a characteristic finite cover $M'$ over $M^*$, which is a formatted graph manifold.

From the above construction, we obtain a regular finite cover $M'$ over $M$, which is a formatted graph manifold. Denote by $\rho' : \pi_1(M') \to \tilde{SL}(2, \mathbb{R}) \times \mathbb{R}$ the pull-back representation of $\rho$, and by $(V', E')$ the JSJ graph of $M'$. The construction also guarantees

$$\tau_\rho'(v', w') = \frac{1}{[T'_{v', w'} : T_{v, w}]} \times \tau_\rho(v, w) = 1 \quad (6.4)$$

for any covering pair of JSJ tori $T'_{v', w'} \to T_{v, w}$. We observe

$$\xi_\rho'(v') = [f'_{v'} : f_{v}] \times \xi_\rho(v) \quad (6.5)$$

for all covering pair of JSJ pieces $J'_{v'} \to J_v$. In particular, Lemma 6.2 applies to $(M', \rho')$.

For any JSJ piece $J_v$ of $M$, the number of JSJ pieces $J'_{v'}$ of $M'$ that cover $J_v$ is precisely $[M' : M] / [J'_{v'} : J_v]$. For any boundary component $T_{v, w}$ of $J_v$, the number of boundary components $T'_{v', w'}$ of $J'_{v'}$ that cover $T_{v, w}$ is precisely $[J'_{v'} : J_v] / [T'_{v', w'} : T_{v, w}]$. The number $b_{v', w'}$ depends only on $\{v, w\}$ because of regular covering. By the formulas of Proposition 4.5 and (6.5), we compute:

$$(e_{M'} \xi_{\rho'})(v') = k_{v'} \xi_{\rho'}(v') - \sum_{\{v', w\} \in E'} \frac{\xi_{\rho'}(w')}{b_{v', w'}}$$

$$= \frac{[J'_{v'} : J_v]}{[f'_{v'} : f_{v}]} \times k_v \xi_\rho(v) - \sum_{\{v, w\} \in E} \frac{[J'_{v'} : J_v]}{[T'_{v', w'} : T_{v, w}]} \times \frac{[T'_{v', w'} : T_{v, w}]}{[f'_{v'} : f_{v}]} \times \frac{\xi_\rho(w)}{b_{v, w}}$$
\[
\frac{[J'_{v'} : J_v]}{[f'_{v'} : f_v]} \times \left( k_v \xi_{\rho}(v) - \sum_{\{v, w\} \in E} \frac{\xi_{\rho}(w)}{b_{v, w}} \right)
\]
\[
= [J'_{v'} : J_v] \times \left( e_M \xi_{\rho}(v) \right).
\]

By the volume formula in Theorem 5.2 (Lemma 6.2) for \((M', \rho')\) and (6.5), we compute:

\[
\text{vol}_{\mathbb{SL}(2, \mathbb{R}) \times \mathbb{R}}(M, \rho) = 1 \times \text{vol}_{\mathbb{SL}(2, \mathbb{R}) \times \mathbb{R}}(M', \rho')
\]
\[
= \frac{1}{[M' : M]} \times 4\pi^2 \cdot \left( \xi'_{\rho'}, e_M \xi'_{\rho'} \right)
\]
\[
= \frac{4\pi^2}{[M' : M]} \times \sum_{v' \in V'} \xi_{\rho'}(v') \times \left( e_M \xi_{\rho'} \right)(v')
\]
\[
= \frac{4\pi^2}{[M' : M]} \times \sum_{v \in V} \frac{[M' : M]}{[J'_{v'} : J_v]} \times [J'_{v'} : J_v] \times \xi_{\rho}(v) \times \left( e_M \xi_{\rho} \right)(v)
\]
\[
= 4\pi^2 \cdot \left( \xi_{\rho}, e_M \xi_{\rho} \right).
\]

By Proposition 4.5 and (6.4), we estimate, for any covering pair of JSJ pieces \(J'_{v'} \to J_v\):

\[
-\chi_{v'} - \sum_{\{v', w'\} \in E'} \frac{1}{\tau_{\rho'}(v', w')} = -\frac{[J'_{v'} : J_v]}{[f'_{v'} : f_v]} \times \chi_{v} - \sum_{\{v, w\} \in E} \frac{[J'_{v'} : J_v]}{[T'_{v', w'} : T_{v, w}]} \times \frac{T'_{v', w'} : T_{v, w}}{\tau_{\rho}(v, w)} \leq \frac{[J'_{v'} : J_v]}{[f'_{v'} : f_v]} \times \left( -\chi_{v} - \sum_{\{v, w\} \in E} \frac{1}{\tau_{\rho}(v, w)} \right).
\]

By the inequality in Theorem 5.2 (Lemma 6.2) for \((M', \rho')\), we obtain

\[
|\left(e_M \xi_{\rho}\right)(v)| = \frac{[f'_{v'} : f_v]}{[J'_{v'} : J_v]} \times \left|\left(e_M \xi_{\rho'}\right)(v')\right|
\]
\[
\leq \frac{[f'_{v'} : f_v]}{[J'_{v'} : J_v]} \times \max \left\{ 0, -\chi_{v'} - \sum_{\{v', w'\} \in E'} \frac{1}{\tau_{\rho'}(v', w')} \right\}
\]
\[
\leq \max \left\{ 0, -\chi_{v} - \sum_{\{v, w\} \in E} \frac{1}{\tau_{\rho}(v, w)} \right\}.
\]

The assertion about rationality is again obvious provided \(\xi_{\rho} \in \mathbb{Q}^V\).
This completes the proof of Lemma 6.6.
6.3 The rational approximation

**Lemma 6.8.** For any finitely generated group $\pi$ and any representation $\rho : \pi \to \tilde{\text{SL}}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$, there is a dense subset of cohomology classes $\alpha \in H^1(\pi; \mathbb{R})$, such that the image of the twisted representations $\rho[\alpha]$ of $\pi$ is contained in $\tilde{\text{SL}}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{Q}$.

To recall the notation $\rho[\alpha]$, see (3.1) and Section 2.1.

**Proof.** Denote by $\tilde{\rho} : \pi \to \text{PSL}(2, \mathbb{R})$ the induced representation. Let $u_1, \ldots, u_r$ be a generating set of $\pi$. For each generator $u_i$ of $\pi$, take a lift $g_i \in \tilde{\text{SL}}(2, \mathbb{R})$ of $\tilde{\rho}(u_i)$. Then, for any relator $R(u_1, \ldots, u_r)$ of $\pi$, the element $R(g_1, \ldots, g_r) \in \tilde{\text{SL}}(2, \mathbb{R})$ is central, so it can be identified as an integer $c_R \in \mathbb{Z}$. Suppose $s_1, \ldots, s_r \in \mathbb{R}$. Then the assignment $\rho'(u_i) = g_i[s_i]$ for all $i \in \{1, \ldots, r\}$ determines a representation $\rho' : \pi \to \tilde{\text{SL}}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$ if and only if the equation $R(g_1[s_1], \ldots, g_r[s_r]) = \text{id}[0]$ holds in $\tilde{\text{SL}}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$ for all relators $R$ of $\pi$. This gives rise to a linear system of equations with $r$ unknowns. Because of $\rho$, there is a real solution, so there is also a rational solution $\rho'$. By Theorem 3.1, we see $\rho' = \rho[\alpha']$ for some $\alpha' \in H^1(\pi; \mathbb{R})$. Then, the asserted subset of cohomology classes can be taken as the coset $\alpha' + H^1(\pi; \mathbb{Q})$. \[\square\]

With the above preparation, we finish the proof of Theorem 6.1 as follows.

By Lemma 6.8, we obtain a sequence of twisted representations $\rho[\alpha_n] : \pi_1(M) \to \tilde{\text{SL}}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{Q}$ such that $\alpha_n \to 0$ in $H^1(M; \mathbb{R})$. Note that $\tau_{\rho[\alpha_n]}$ are all equal to $\tau_{\rho'}$, so they are all finite by assumption. We also see $\xi_{\rho[\alpha_n]} \in Q^V$, because every $\pi_1(T,v,\omega)$ has virtually central and rational image in $\tilde{\text{SL}}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$ under any $\rho[\alpha_n]$. Apply Lemma 6.6 to all $\rho[\alpha_n]$ and take the limit. Then we obtain the volume formula and the generalized Milnor–Wood inequalities in Theorem 6.1. Note that $\rho[\alpha_n]$ all lie on the same component of $R(\pi_1(M), \tilde{\text{SL}}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R})$ as that of $\rho$, so they have the same volume of $\tilde{\text{SL}}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$-representations (see Section 3.1). We obtain the assertion on rationality, since any $\xi_{\rho[\alpha_n]}$ serves as an asserted solution $X \in Q^V$.

This completes the proof of Theorem 6.1.

7 PERTURBATION, DEFORMATION, AND CONJUGATION

In this section, we develop techniques for modifying certain $\tilde{\text{SL}}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$-representations of finitely generated abelian groups to obtain certain canonical families. The modification families are either sequential or continuous, and converge to the original representation. In particular, the modified representations all lie on the path-connected component of the original representation, in the space of $\tilde{\text{SL}}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$-representations. We also study commutator factorizations in $\tilde{\text{SL}}(2, \mathbb{R})$ (the commutator subgroup of $\tilde{\text{SL}}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$) of elements occurring in such modification. For given formatted graph manifolds, roughly speaking, these techniques allow us to modify $\tilde{\text{SL}}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$-representations restricted to the edge groups, and in certain circumstances, to extend the modification over the adjacent vertex groups. For both the modification and the commutator factorization, we develop parallel versions for elliptic, hyperbolic, and parabolic representations separately.
7.1 Noncentral representations of abelian groups

Lemma 7.1. Let $H$ be a finitely generated abelian group. If $\eta : H \to \widetilde{SL}(2,\mathbb{R}) \times_{\mathbb{R}} \mathbb{R}$ is a representation of elliptic type, then there exists a sequence of elliptic representations $\eta_n : H \to \widetilde{SL}(2,\mathbb{R}) \times_{\mathbb{R}} \mathbb{R}$, indexed by $n \in \mathbb{N}$, such that the following properties are all satisfied, for all $n \in \mathbb{N}$.

- For all $h \in H$, $\eta_n(h)$ lies in the coset $\eta(h)\widetilde{SL}(2,\mathbb{R})$.
- For all $h \in H$, $\eta_n(h)$ converges to $\eta(h)$ as $n$ tends to $\infty$.
- If $\eta(h)$ lies in the center $\mathbb{R}$, then $\eta_n(h)$ equals $\eta(h)$.
- The induced representation $\overline{\eta}_n : H \to PSL(2,\mathbb{R})$ has finite cyclic image.

In fact, $\eta_n$ can be constructed with image in the centralizer of $\eta(H)$ in $\widetilde{SL}(2,\mathbb{R}) \times_{\mathbb{R}} \mathbb{R}$.

Proof. Since $\eta$ is of elliptic type, the centralizer of $\eta(H)$ is a conjugate of the closed abelian subgroup $\widetilde{SO}(2) \times_{\mathbb{R}} \mathbb{R}$. We decompose the finitely generated abelian group $H$ as a Cartesian product of cyclic factors. Then, it suffices to construct $\eta_n$ for each of the factors, and this will determine a sequence of representations $\eta_n$ for $H$ with the asserted properties. So, the problem reduces to the case when $H$ is cyclic.

If $H$ is finite cyclic, or if $H$ is infinite cyclic but $\overline{\eta}(H)$ finite cyclic, we simply take $\eta_n$ to be $\eta$ for all $n \in \mathbb{N}$.

If $H$ is infinite cyclic and $\overline{\eta}(H)$ also infinite cyclic, we construct as follows. Identify $H$ with the additive group of integers $\mathbb{Z}$. Denote by $g[s] \in \widetilde{SL}(2,\mathbb{R}) \times_{\mathbb{R}} \mathbb{R}$ the image $\eta(1)$, for some $g \in \widetilde{SL}(2,\mathbb{R})$ and $s \in \mathbb{R}$. Since $\overline{\eta}(g) \in PSL(2,\mathbb{R})$ is elliptic, there is a sequence of finite-order elements $\tilde{g}_n \in PSL(2,\mathbb{R})$ which commute with $\overline{\eta}(g)$ and which converge to $\tilde{g} \in PSL(2,\mathbb{R})$ as $n$ tends to $\infty$. There is also a sequence of elements $g_n \in \widetilde{SL}(2,\mathbb{R})$ which lift $\tilde{g}_n \in PSL(2,\mathbb{R})$ and which converge to $g$ as $n$ tends to $\infty$. We construct $\eta_n$ by setting $\eta_n(1) = g_n[s]$, for all $n \in \mathbb{N}$. Note that $0 \in \mathbb{Z}$ is the only element with image in the center $\mathbb{R}$, since $\tilde{g}$ is faithful. Then the asserted properties are all obviously satisfied.\hfill $\square$

Lemma 7.2. Let $H$ be any abelian group. If $\eta : H \to \widetilde{SL}(2,\mathbb{R}) \times_{\mathbb{R}} \mathbb{R}$ is a representation of hyperbolic type, then there exists a continuous family of hyperbolic representations $\eta_t : H \to \widetilde{SL}(2,\mathbb{R}) \times_{\mathbb{R}} \mathbb{R}$, parametrized by $t \in (0,1]$, such that the following properties are all satisfied, for all $t \in (0,1]$.

- For all $h \in H$, $\eta_t(h)$ lies in the coset $\eta(h)\widetilde{SL}(2,\mathbb{R})$.
- For all $h \in H$, $\eta_t(h)$ equals $\eta(h)$.
- For all $h \in H$, $\eta_t(h)$ converges to the central element $\text{wind}(\eta(h)) \in \mathbb{R}$ as $t$ tends to $0$.
- If $\eta(h)$ lies in the center $\mathbb{R}$, then $\eta_t(h)$ equals $\eta(h)$ for all $t$.

In fact, $\eta_t$ can be constructed with image in the centralizer of $\eta(H)$ in $\widetilde{SL}(2,\mathbb{R}) \times_{\mathbb{R}} \mathbb{R}$.

Proof. Since $\eta$ is of hyperbolic type, the centralizer of $\eta(H)$ is the preimage in $\widetilde{SL}(2,\mathbb{R}) \times_{\mathbb{R}} \mathbb{R}$ of a real 1-dimensional subgroup of hyperbolic type in $PSL(2,\mathbb{R})$. It suffices to argue with $H$ replaced by that centralizer, and $\eta$ replaced by its inclusion into $\widetilde{SL}(2,\mathbb{R}) \times_{\mathbb{R}} \mathbb{R}$. Once we obtain a deformation family $\eta_t$ for the latter, satisfying the same properties as asserted, a desired deformation family of the original pair follows by composition. With this replacement, the construction as we explain below is sometimes called the canonical deformation.
Possibly after conjugation, we may assume \( \mathcal{H} = \mathbb{R} \times \mathbb{R} \) parametrized as \( h = (\lambda, s) \), and \( \eta : \mathcal{H} \to \widetilde{SL}(2, \mathbb{R}) \times_\mathbb{Z} \mathbb{R} \) expressed (with the notation in Section 2.1) as
\[
\eta(\lambda, s) = y^\lambda[s],
\]
where \( y^\lambda \in \widetilde{SL}(2, \mathbb{R}) \) denotes the unique lift (at \( y^0 = \text{id} \)) of the continuous homomorphism
\[
\mathbb{R} \to PSL(2, \mathbb{R}) : \lambda \mapsto \left\{ \pm \begin{bmatrix} e^\lambda & 0 \\ 0 & e^{-\lambda} \end{bmatrix} \right\}.
\]
For any \( t \in (0, 1] \) and \( (\lambda, s) \in \mathcal{H} \), we define a continuous family of representations \( \eta_t : \mathcal{H} \to \widetilde{SL}(2, \mathbb{R}) \times_\mathbb{Z} \mathbb{R} \) as
\[
\eta_t(\lambda, s) = y^\lambda t[s].
\]
By definition, we observe \( \text{wind}(y^\lambda t[s]) = s \) in the center \( \mathbb{R} \) for all \( t \). Therefore, the family \( \eta_t \) clearly satisfies all the asserted properties.

\[ \boxempty \]

**Lemma 7.3.** Let \( \mathcal{H} \) be any abelian group. If \( \eta : \mathcal{H} \to \widetilde{SL}(2, \mathbb{R}) \times_\mathbb{Z} \mathbb{R} \) is a representation of parabolic type, then there exists a continuous family of elements \( g_t \in \widetilde{SL}(2, \mathbb{R}) \), parametrized by \( t \in (0, 1] \), such that the following properties are all satisfied.

- At \( t = 1 \), \( g_t \) equals the identity.
- For all \( h \in \mathcal{H} \), \( g_t \eta(h) g_t^{-1} \) converges to the central element \( \text{wind}(\eta(h)) \in \mathbb{R} \) as \( t \) tends to 0.

In fact, \( g_t \) can be constructed to normalize the centralizer of \( \eta(\mathcal{H}) \).

**Proof.** For similar reasons as in the proof of Lemma 7.2, we may replace \((\mathcal{H}, \eta)\) with the centralizer of the image and its inclusion into \( \widetilde{SL}(2, \mathbb{R}) \times_\mathbb{Z} \mathbb{R} \). Possibly after conjugation, we assume \( \mathcal{H} = \mathbb{R} \times \mathbb{R} \) parametrized as \( h = (\beta, s) \), and \( \eta : \mathcal{H} \to \widetilde{SL}(2, \mathbb{R}) \times_\mathbb{Z} \mathbb{R} \) expressed as
\[
\eta(\lambda, s) = y^\beta[s],
\]
where \( y^\beta \in \widetilde{SL}(2, \mathbb{R}) \) denotes the unique lift (at \( y^0 = \text{id} \)) of the continuous homomorphism
\[
\mathbb{R} \to PSL(2, \mathbb{R}) : \lambda \mapsto \left\{ \pm \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix} \right\}.
\]
The expression
\[
\tilde{g}_t = \left\{ \pm \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} \right\}
\]
defines a path in \( PSL(2, \mathbb{R}) \) parametrized by \( t \in (0, 1] \), lifting to a unique path \( g_t \in \widetilde{SL}(2, \mathbb{R}) \times_\mathbb{Z} \mathbb{R} \) parametrized by \((0,1]\), such that \( g_1 \) equals the identity. Observe that the projection of the path...
$g_t \eta(\beta, s) g^{-1}_t$ takes the form

$$(0, 1) \to \text{PSL}(2, \mathbb{R}) : t \mapsto \left\{ \pm \begin{bmatrix} 1 & \beta t^2 \\ 0 & 1 \end{bmatrix} \right\}.$$ 

It follows that $g_t \eta(\beta, s) g^{-1}_t$ converges to the central element $\text{id}[s] = s$, as $t$ tends to 0, and $\text{wind}(g_t \eta(\beta, s) g^{-1}_t)$ stays constant along the way. The family $g_t$ satisfies the asserted properties, as desired. \hfill \square

### 7.2 Commutator factorization

**Lemma 7.4.** Suppose that $(\gamma_n)_{n \in \mathbb{N}}$ is a sequence of elliptic elements in $\tilde{\text{SL}}(2, \mathbb{R})$ that converge to the identity element as $n$ tends to $\infty$. Then there exist sequences $(\alpha_n)_{n \in \mathbb{N}}$ and $(\beta_n)_{n \in \mathbb{N}}$ in $\tilde{\text{SL}}(2, \mathbb{R})$, both converging to the identity element as $n$ tends to $\infty$, such that the relation

$$\gamma_n = \alpha_n \beta_n \alpha^{-1}_n \beta^{-1}_n$$

holds for all but finitely many $n \in \mathbb{N}$.

**Proof.** For any $\lambda \in (0, +\infty)$ and $\theta \in (0, \pi)$, we first factorize the matrix

$$C(\lambda, \theta) = \begin{bmatrix} \cos(\theta) & \lambda \sin(\theta) \\ -\lambda^{-1} \sin(\theta) & \cos(\theta) \end{bmatrix} \in \text{SL}(2, \mathbb{R})$$

as a commutator of two particularly constructed matrices in $\text{SL}(2, \mathbb{R})$ that depend continuously on $(\lambda, \theta)$.

To this end, we construct the following matrices $A, B \in \text{SL}(2, \mathbb{R})$, which depend continuously on the parameters $x, y \in [0, +\infty)$ and $\mu \in (0, +\infty)$:

$$A = \begin{bmatrix} \sqrt{1 + y^2} + y \sqrt{1 + x^2} & \mu xy \\ -\mu^{-1} xy & \sqrt{1 + y^2} - y \sqrt{1 + x^2} \end{bmatrix}, \quad B = \begin{bmatrix} \sqrt{1 + x^2} & \mu x \\ \mu^{-1} x & \sqrt{1 + x^2} \end{bmatrix}. \quad (7.1)$$

Then, the inverse matrices are easy to obtain, by switching diagonal entries and negating the off-diagonal ones:

$$A^{-1} = \begin{bmatrix} \sqrt{1 + y^2} - y \sqrt{1 + x^2} & -\mu xy \\ \mu^{-1} xy & \sqrt{1 + y^2} + y \sqrt{1 + x^2} \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} \sqrt{1 + x^2} & -\mu x \\ -\mu^{-1} x & \sqrt{1 + x^2} \end{bmatrix}.$$  

Direct computation shows

$$BA^{-1}B^{-1} = \begin{bmatrix} \sqrt{1 + y^2} - y \sqrt{1 + x^2} & \mu xy \\ -\mu^{-1} xy & \sqrt{1 + y^2} + y \sqrt{1 + x^2} \end{bmatrix}.$$
and
\[
ABA^{-1}B^{-1} = \begin{bmatrix}
1 - 2x^2y^2 & 2\mu xy\left(\sqrt{1 + y^2 + y\sqrt{1 + x^2}}\right) \\
-2\mu^{-1}xy\left(\sqrt{1 + y^2 - y\sqrt{1 + x^2}}\right) & 1 - 2x^2y^2
\end{bmatrix}.
\]

Therefore, given any parameters \((\lambda, \theta) \in (0, +\infty) \times (0, \pi)\), it is straightforward to verify the relation

\[
C(\lambda, \theta) = ABA^{-1}B^{-1}
\]

for any \(x, y \in [0, +\infty)\) and \(\mu \in (0, +\infty)\) that satisfy the following conditions:

\[
xy = \sin(\theta/2), \quad \mu = \lambda \times \sqrt{\frac{\sqrt{1+y^2+y\sqrt{1+x^2}}}{\sqrt{1+y^2+y\sqrt{1+x^2}}}}. \quad (7.2)
\]

We make the following particular assignments

\[
x = \sqrt{2 \tan(\theta/2)/(\lambda + \lambda^{-1})}, \quad y = \sqrt{\frac{2}{\lambda + \lambda^{-1}} \sin(\theta)/4}. \quad (7.3)
\]

Then, we obtain matrices \(A, B \in \text{SL}(2, \mathbb{R})\) that depend continuously on the parameters \((\lambda, \theta) \in (0, +\infty) \times (0, \pi)\), by putting (7.1), (7.2), and (7.3) altogether.

Our particular factorization \(C(\lambda, \theta) = ABA^{-1}B^{-1}\) satisfies \(A \approx 1\) and \(B \approx 1\) in \(\text{SL}(2, \mathbb{R})\) if \(C(\lambda, \theta) \approx 1\). In fact, if \(C(\lambda, \theta)\) lies in a small neighborhood of the identity matrix \(1 \in \text{SL}(2, \mathbb{R})\), we have equivalently \(\cos(\theta) \approx 1\) and \(\lambda^{\pm 1} \sin(\theta) \approx 0\), with small error. Then, we have \(x \approx 0\) and \(y \approx 0\), and therefore, \(\mu / \lambda \approx 1\). This implies the following key point of our construction:

\[
\mu x = (\mu / \lambda) \times \sqrt{\frac{2\lambda \sin(\theta)}{(1 + \lambda^{-2})(1 + \cos(\theta))}} \approx 0,
\]

and

\[
\mu^{-1}x = (\mu / \lambda)^{-1} \times \sqrt{\frac{2\lambda^{-1} \sin(\theta)}{(1 + \lambda^2)(1 + \cos(\theta))}} \approx 0.
\]

We see \(A \approx 1\) and \(B \approx 1\) in \(\text{SL}(2, \mathbb{R})\) from (7.1), (7.2), and (7.3).

In general, any elliptic element in \(\text{PSL}(2, \mathbb{R})\) can be conjugated to \(\{\pm C(\lambda, \theta)\}\) by some element in \(\text{SO}(2) / \{\pm 1\}\), for some parameters \((\lambda, \theta) \in (0, +\infty) \times (0, \pi)\). To see this, one may identify \(\text{PSL}(2, \mathbb{R})\) with the fractional linear transformation group acting on the upper-half complex plane. In terms of hyperbolic geometry, \(\{\pm C(\lambda, \theta)\}\) represents the rotation of angle \(2\theta\) counterclockwise about the point \(\lambda i\). The subgroup \(\text{SO}(2) / \{\pm 1\}\) of \(\text{PSL}(2, \mathbb{R})\) consists of all the rotations about \(i\). Therefore, any elliptic element in \(\text{PSL}(2, \mathbb{R})\) either fixes \(i\) already, or can be conjugated by some rotation about \(i\) to fix a point on the imaginary axis.

Suppose that \((\gamma_n)_{n \in \mathbb{N}}\) is a sequence of elliptic elements in \(\tilde{\text{SL}}(2, \mathbb{R})\) that converge to the identity element as \(n\) tends to \(\infty\). Denote by \(\tilde{\gamma}_n \in \text{PSL}(2, \mathbb{R})\) the image of \(\gamma_n\) under the canonical covering projection \(\tilde{\text{SL}}(2, \mathbb{R}) \to \text{PSL}(2, \mathbb{R})\). Then, we can find \(S_n \in \text{SO}(2)\) and \((\lambda_n, \theta_n) \in [1, +\infty) \times (0, \pi)\)
such that
\[ \bar{\gamma}_n = \{ \pm S_n C(\lambda_n, \vartheta_n) S_n^{-1} \}, \]

for all \( n \in \mathbb{N} \). Let \( A_n, B_n \in SL(2, \mathbb{R}) \) be the matrices as above, such that \( C(\lambda_n, \vartheta_n) = A_n B_n A_n^{-1} B_n^{-1} \). Then we obtain a commutator factorization
\[ \bar{\gamma}_n = \bar{\alpha}_n \bar{\beta}_n \bar{\alpha}_n^{-1} \bar{\beta}_n^{-1}, \]

where \( \bar{\alpha}_n = \{ \pm S_n A_n S_n^{-1} \} \) and \( \bar{\beta}_n = \{ \pm S_n B_n S_n^{-1} \} \) are elements of \( PSL(2, \mathbb{R}) \). Because \( SO(2) \) is compact, the convergence of \( (\gamma_n)_{n \in \mathbb{N}} \) implies \( C(\lambda_n, \vartheta_n) \to 1 \), and hence, \( A_n \to 1 \) and \( B_n \to 1 \), as \( n \to \infty \). It follows that for all but finitely many \( n \in \mathbb{N} \), \( \bar{\alpha}_n, \bar{\beta}_n \), and \( \bar{\gamma}_n \) lie in some small neighborhood of \( \{ \pm 1 \} \in PSL(2, \mathbb{R}) \) that lifts homeomorphically to a small neighborhood of the identity element in \( \widetilde{SL}(2, \mathbb{R}) \). Then, we obtain unique lifts \( \alpha_n, \beta_n \in \widetilde{SL}(2, \mathbb{R}) \) of \( \bar{\alpha}_n, \bar{\beta}_n \in PSL(2, \mathbb{R}) \), which converge to the identity as \( n \) tends to \( \infty \), and which satisfy the relation
\[ \gamma_n = \alpha_n \beta_n \alpha_n^{-1} \beta_n^{-1} \]
forall but finitely many \( n \in \mathbb{N} \), as asserted. \( \square \)

**Lemma 7.5.** Suppose that \( (\gamma_t)_{t \in [0, 1)} \) is a continuous family of hyperbolic elements in \( \widetilde{SL}(2, \mathbb{R}) \) that converge to the identity element as \( t \) tends to \( 1 \). Then there exist continuous families \( (\alpha_t)_{t \in [0, 1)} \) and \( (\beta_t)_{t \in [0, 1)} \) in \( \widetilde{SL}(2, \mathbb{R}) \), both converging to the identity element as \( t \) tends to \( 1 \), such that the relation
\[ \gamma_t = \alpha_t \beta_t \alpha_t^{-1} \beta_t^{-1} \]
holds for all \( t \in [0, 1) \).

**Proof.** For any \( \lambda \in (0, +\infty) \) and \( \tau \in (0, +\infty) \), we first factorize the matrix
\[ C(\lambda, \tau) = \begin{bmatrix} \cosh(\tau) & \lambda \sinh(\tau) \\ \lambda^{-1} \sinh(\tau) & \cosh(\tau) \end{bmatrix} \in SL(2, \mathbb{R}) \]
as a commutator of two particularly constructed matrices in \( SL(2, \mathbb{R}) \) which depends continuously on \( (\lambda, \tau) \).

We construct the following matrices \( A, B \in SL(2, \mathbb{R}) \), which depend continuously on the parameters \( x \in [0, 1], y \in [0, +\infty), \) and \( \mu \in (0, +\infty) \):
\[ A = \begin{bmatrix} \sqrt{1+y^2} + y\sqrt{1-x^2} & \mu xy \\ \mu^{-1} xy & \sqrt{1+y^2} - y\sqrt{1-x^2} \end{bmatrix}, \quad B = \begin{bmatrix} \sqrt{1-x^2} & \mu x \\ -\mu^{-1} x & \sqrt{1-x^2} \end{bmatrix}. \quad (7.4) \]

Then,
\[ ABA^{-1}B^{-1} = \begin{bmatrix} 1 + 2x^2y^2 & 2\mu xy(\sqrt{1+y^2} + y\sqrt{1-x^2}) \\ 2\mu^{-1} xy(\sqrt{1+y^2} - y\sqrt{1-x^2}) & 1 + 2x^2y^2 \end{bmatrix}. \]
Given any parameters \((\lambda, \tau) \in (0, +\infty) \times (0, +\infty)\), the relation

\[ C(\lambda, \tau) = ABA^{-1}B^{-1} \]

holds for any \(x \in (0, 1]\) and \(y \in (0, +\infty)\) that satisfy the following conditions:

\[ xy = \sinh(\tau/2), \quad \mu = \lambda \times \sqrt{\frac{\sqrt{1+y^2}-y\sqrt{1-x^2}}{\sqrt{1+y^2}+y\sqrt{1-x^2}}} \quad \text{(7.5)} \]

We make the following particular assignments:

\[ x = \sqrt{2\tanh(\tau/2)/(\lambda + \lambda^{-1})}, \quad y = \sqrt{(\lambda + \lambda^{-1})\sinh(\tau)/4} \quad \text{(7.6)} \]

Then, we obtain matrices \(A, B \in \text{SL}(2, \mathbb{R})\) that depend continuously on the parameters \((\lambda, \tau) \in (0, +\infty) \times (0, +\infty)\), by putting (7.4), (7.5), and (7.6) altogether. A similar argument as before shows \(A \approx 1\) and \(B \approx 1\) in \(\text{SL}(2, \mathbb{R})\), if \(C(\lambda, \tau) \approx 1\).

In general, any hyperbolic element in \(\text{PSL}(2, \mathbb{R})\) can be conjugated to \(\{\pm C(\lambda, \tau)\}\) by some unique element in \(\text{SO}(2)/\{\pm 1\}\), for some unique parameters \((\lambda, \theta) \in (0, +\infty) \times (0, +\infty)\). In fact, the conjugation element and the parameters vary continuously as the hyperbolic element varies in \(\text{PSL}(2, \mathbb{R})\). To see this, one may again identify \(\text{PSL}(2, \mathbb{R})\) with the fractional linear transformation group acting on the upper-half complex plane. In terms of hyperbolic geometry, \(\{\pm C(\lambda, \theta)\}\) represents the hyperbolic translation along the geodesic between the ideal endpoints \(\{\pm \lambda\}\) and of distance \(2\tau\) toward \(+\lambda\). The subgroup \(\text{SO}(2)/\{\pm 1\}\) of \(\text{PSL}(2, \mathbb{R})\) consists of all the rotations about \(\textbf{i}\). Therefore, any hyperbolic element in \(\text{PSL}(2, \mathbb{R})\) can be conjugated by some unique rotation about \(\textbf{i}\) to preserve a geodesic perpendicular to the positive imaginary axis, such that the ideal point 0 is moved rightward.

Suppose that \((\gamma_t)_{t \in [0, 1]}\) is a continuous family of hyperbolic elements in \(\tilde{\text{SL}}(2, \mathbb{R})\) that converge to the identity element as \(t\) tends to 1. Denote by \(\tilde{\gamma}_t \in \text{PSL}(2, \mathbb{R})\) the image of \(\gamma_t\) under the canonical covering projection \(\tilde{\text{SL}}(2, \mathbb{R}) \to \text{PSL}(2, \mathbb{R})\). Then we can find continuous families \(\{\pm S_t\} \in \text{SO}(2)/\{\pm 1\}\) and \((\lambda_t, \tau_t) \in (0, +\infty) \times (0, \pi)\) such that

\[ \tilde{\gamma}_t = \{\pm S_t C(\lambda_t, \tau_t)S_t^{-1}\}, \]

for all \(t \in [0, 1]\). (One may actually require \(S_t \in \text{SO}(2)\) to depend continuously on \(t\), by path lifting.) Let \(A_t, B_t \in \text{SL}(2, \mathbb{R})\) be the matrices as above, such that \(C(\lambda_t, \tau_t) = A_t B_t A_t^{-1} B_t^{-1}\). Then we obtain a commutator factorization

\[ \tilde{\gamma}_t = \tilde{\alpha}_t \tilde{\beta}_t \tilde{\alpha}_t^{-1} \tilde{\beta}_t^{-1}, \]

where \(\tilde{\alpha}_t = \{\pm S_t A_t S_t^{-1}\}\) and \(\tilde{\beta}_t = \{\pm S_t B_t S_t^{-1}\}\) are elements of \(\text{PSL}(2, \mathbb{R})\). Because \(\text{SO}(2)\) is compact, the convergence of \((\gamma_t)_{t \in [0, 1]}\) implies that \(\tilde{\alpha}_t, \tilde{\beta}_t\) converges to \(\{\pm 1\} \in \text{PSL}(2, \mathbb{R})\) as \(t\) tends to 1. Then, we obtain unique lifts \(\alpha_t, \beta_t \in \tilde{\text{SL}}(2, \mathbb{R})\) of \(\tilde{\alpha}_t, \tilde{\beta}_t \in \text{PSL}(2, \mathbb{R})\), which converge to the identity as \(t\) tends to 1, and which satisfy the relation

\[ \gamma_t = \alpha_t \beta_t \alpha_t^{-1} \beta_t^{-1} \quad \text{for all } t \in [0, 1], \]

as asserted.

\[ \square \]

**Lemma 7.6.** Suppose that \((\gamma_t)_{t \in [0, 1]}\) is a continuous family of parabolic elements in \(\tilde{\text{SL}}(2, \mathbb{R})\) that converge to the identity element as \(t\) tends to 1. Then there exist continuous families \((\alpha_t)_{t \in [0, 1]}\)
and \((\beta_t)_{t \in [0,1]}\) in \(\widetilde{SL}(2, \mathbb{R})\), both converging to the identity element as \(t\) tends to 1, such that the relation

\[ \gamma_t = \alpha_t \beta_t \alpha_t^{-1} \beta_t^{-1} \]

holds for all \(t \in [0,1]\).

**Proof.** For any \(u \in (0, +\infty)\), we factorize the matrices

\[ C(u) = \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} \in SL(2, \mathbb{R}) \]

as a commutator

\[ C(u) = ABA^{-1}B^{-1}, \]

where

\[ A = \begin{bmatrix} 1 & \sqrt{u} + u \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{\sqrt{1+\sqrt{u}}} & 0 \\ 0 & \sqrt{1+\sqrt{u}} \end{bmatrix}. \tag{7.7} \]

The matrices \(A, B \in SL(2, \mathbb{R})\) depend continuously on \(u \in (0, +\infty)\), and converge to the identity matrix as \(u\) tends to 0. Observe that \(C(u)^{-1}\) factorizes as a commutator \(BAB^{-1}A^{-1}\). In general, any parabolic element in \(PSL(2, \mathbb{R})\) can be conjugated to \(\{\pm C(u)\}\) or \(\{\pm C(u)^{-1}\}\) by some unique element in \(SO(2)\)/\{\(\pm 1\}\), for some unique parameter \(u \in (0, +\infty)\). In fact, the conjugation element and the parameter vary continuously as the parabolic element varies in \(PSL(2, \mathbb{R})\). On the upper-half complex plane, the conjugation and the parameter are determined by the requirement that the ideal fixed point of the parabolic element is conjugated to \(\infty\).

Suppose that \((\gamma_t)_{t \in [0,1]}\) is a continuous family of parabolic elements in \(\widetilde{SL}(2, \mathbb{R})\) that converge to the identity element as \(t\) tends to 1. Denote by \(\tilde{\gamma}_t \in PSL(2, \mathbb{R})\) the image of \(\gamma_t\) under the canonical covering projection \(\widetilde{SL}(2, \mathbb{R}) \to PSL(2, \mathbb{R})\). Then we can find continuous families \(\{\pm S_t\} \in SO(2)/\{\pm 1\}\) and \(u_t \in (0, +\infty) \times (0, \pi)\) such that

\[ \tilde{\gamma}_t = \{\pm S_t C(u_t)S_t^{-1}\} \text{ or } \{\pm S_t C(u_t)^{-1}S_t^{-1}\} \]

for all \(t \in [0,1]\). Let \(A_t, B_t \in SL(2, \mathbb{R})\) be the matrices as above, such that \(C(u_t) = A_t B_t A_t^{-1} B_t^{-1}\). Possibly after switching \(A_t\) and \(B_t\), we obtain a commutator factorization

\[ \tilde{\gamma}_t = \tilde{\alpha}_t \tilde{\beta}_t \tilde{\alpha}_t^{-1} \tilde{\beta}_t^{-1}, \]

where \(\tilde{\alpha}_t = \{\pm S_t A_t S_t^{-1}\}\) and \(\tilde{\beta}_t = \{\pm S_t B_t S_t^{-1}\}\) are elements of \(PSL(2, \mathbb{R})\). Then, we obtain unique lifts \(\alpha_t, \beta_t \in SL(2, \mathbb{R})\) of \(\tilde{\alpha}_t, \tilde{\beta}_t \in PSL(2, \mathbb{R})\), which converge to the identity as \(t\) tends to 1, and which satisfy the relation \(\gamma_t = \alpha_t \beta_t \alpha_t^{-1} \beta_t^{-1}\) for all \(t \in [0,1]\), as asserted. \(\square\)
8 | REDUCTION TO THE VIRTUALLY CENTRAL CASE WITH EXTRA GENERA

Let $M$ be a formatted graph manifold. For any ordinary fiber $f$ in a JSJ piece of $M$ with a fibered collar neighborhood parametrized as $f \times D^2$, we can construct a new formatted graph manifold:

$$M \#_f(f \times T^2) = (M \setminus \text{int}(f \times D^2)) \cup_{f \times \partial D^2} (f \times (D^2 \# T^2)),$$

where $D^2 \# T^2$ stands for (a fixed model of) a connected sum of a compact disk $D^2$ with a torus $T^2$. Namely, $M \#_f(f \times T^2)$ is obtained from $M$ by replacing the interior of $f \times D^2$ with the interior of $f \times (D^2 \# T^2)$, retaining the topology near $f \times \partial D^2$. The construction comes naturally with a degree-one map

$$M \#_f(f \times T^2) \to M,$$  \quad (8.1)

which is the identity map restricted to $M \setminus \text{int}(f \times D^2)$, and which is the product of the identity map $f \to f$ with the pinching map $D^2 \# T^2 \to D^2$ restricted to $f \times (D^2 \# T^2)$. We refer to $M \#_f(f \times T^2)$ as the fiber connected sum of $M$ and $f \times T^2$, and the map (8.1) as the horizontally pinching map.

The JSJ graph of $M \#_f(f \times T^2)$ can be identified with the simplicial JSJ graph $(V, E)$ of $M$, and the horizontally pinching map induces the identity map between the JSJ graphs. If $f$ is an ordinary fiber $f_v$ of the JSJ piece corresponding to a vertex $v \in V$, we observe the simple relations

$$e_{M \#_f(f \times T^2)} = e_M, \quad \chi_{M \#_f(f \times T^2)} = \chi_M - 2v^*$$  \quad (8.2)

(see Notation 5.1).

**Definition 8.1.** A map between formatted graph manifolds is called a formatted pinching map if it admits a factorization as the composition of finitely many maps between formatted graph manifolds, such that each of the factor maps is (formatted homeomorphically) conjugate to a horizontally pinching map of the form (8.1).

**Remark 8.2.**

1. Let $M$ be a formatted graph manifold with a simplicial JSJ graph $(V, E)$. For any union of finitely many mutually disjoint of ordinary fibers $F$ in JSJ pieces of $M$, we can construct the simultaneous fiber connected sum $M \#_F(F \times T^2)$, namely, $(M \setminus \text{int}(F \times D^2)) \cup_{F \times \partial D^2} (F \times (D^2 \# T^2))$. So, there is a simultaneous horizontally pinching map $M \#_F(F \times T^2) \to M$.

2. In general, every formatted pinching map $M' \to M$ can be written as a composite map $M' \to M \#_F(F \times T^2) \to M$, where the first factor is a formatted homeomorphism and the second factor is a simultaneous horizontally pinching map. It is instructive to look at the composition of two horizontally pinching maps, and proceed by induction.

3. One might also be interested in the fiber connected sum $M \#_f(f \times T^2)$ where $f$ is an exceptional fiber in a JSJ piece. It will give rise to a degree-1 map $M \#_f(f \times T^2) \to M$ defined...
similarly. In that case, however, the JSJ decomposition of $M \# f(f \times T^2)$ will have a new JSJ torus $f \times \partial D^2$ and a new JSJ piece $f \times \text{int}(D^2 \# T^2)$.

8.1 Local reduction

**Theorem 8.3.** Let $M' \to M$ be a formatted pinching map between formatted graph manifolds with identified simplicial JSJ graphs $(V, E)$. Let $\rho : \pi_1(M) \to \widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$ be a representation. Suppose that $C = (V_C, E_C)$ is a maximal noncentral component with respect to $\rho$ (see Section 5.1). Adopt Notations 5.1 and 5.6. Suppose that the inequality

$$\chi_{M'}(v) \leq \chi_M(v) - 2 \cdot \delta_{\rho}(v; C)$$

holds for all $v \in V$.

Then, for any constant $\varepsilon > 0$, there exists a representation $\rho' : \pi_1(M') \to \widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$ that satisfies all the following properties.

- In the space of representations $\mathcal{R}(\pi_1(M'), \widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R})$, $\rho'$ is path-connected with the pull-back representation of $\rho$.
- For any $v \in V$,

$$\left| \xi_{\rho'}(v) - \xi_{\rho}(v) \right| < \varepsilon.$$

- For any $\{v, w\} \in E$, $\tau_{\rho'}(v, w)$ equals $\tau_{\rho}(v, w)$ if $\tau_{\rho}(v, w)$ is finite.
- For any $v \in V_C$, $\delta_{\rho'}(v)$ equals 0.

In fact, one may require, furthermore, $\xi_{\rho'}(v) = \xi_{\rho}(v)$ if $v$ is not a vertex of $C$, and $\tau_{\rho'}(v, w) = \tau_{\rho}(v, w)$ if neither $v$ nor $w$ are vertices of $C$.

The rest of this subsection is devoted to the proof of Theorem 8.3.

8.1.1 The standard local model

Let $M$ be any formatted graph manifold with a simplicial JSJ graph $(V, E)$, and $\rho : \pi_1(M) \to \widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$ any representation. Let $C = (V_C, E_C)$ be a maximal noncentral component with respect to $\rho$. Denote by $M_C$ the block submanifold of $M$ over $C$, namely,

$$M_C = \bigcup_{w \in V_C} J_w \cup \bigcup_{\{w, u\} \in E_C} T_{w, u}.$$

Denote by $\Delta_{\rho}(C)$ the set of directed edges that depart from $V_C$ and arrive into $V \setminus V_C$, namely,

$$\Delta_{\rho}(C) = \{(w, v) \in V_C \times (V \setminus V_C) : \{v, w\} \in E \text{ and } \tau_{\rho}(v, w) = \infty\}.$$

These directed edges correspond to a subset of outward oriented JSJ tori on the boundary of the closure of $M_C$. We construct a particular formatted graph manifold $M^\dagger = M^\dagger(\Delta_{\rho}(C))$, together
with a formatted pinching map

\[ M^\dagger \to M, \]  

such that the equality

\[ \chi_{M^\dagger}(v) = \chi_M(v) - 2 \cdot \delta_\rho(v; C) \]  

holds for all \( v \in V \). The construction is as follows.

For any \((w, v) \in \Delta_\rho(C)\), take a collar neighborhood of \( T_{v,w} \) in \( \text{clos}(J_v) \), parametrized as \( T_{v,w} \times [0,1] \), where \( T_{v,w} \times \{0\} \) is the JSJ torus \( T_{v,w} \). We require that these \( T_{v,w} \times [0,1] \) are mutually disjoint in \( M \). Take an ordinary fiber \( f_v \) of \( J_v \) in \( T_{v,w} \times (0,1) \). We obtain a compact submanifold of \( M \) with boundary:

\[ W_C = \text{clos}(M_C) \cup \bigcup_{(w,v) \in \Delta_\rho(C)} T_{v,w} \times [0,1]. \]

Construct a simultaneous fiber connected sum of \( W_C \) with trivialized circle bundles over tori:

\[ W^\dagger_C = \text{clos}(M_C) \cup \bigcup_{(w,v) \in \Delta_\rho(C)} (T_{v,w} \times [0,1]) \# f_v (f_v \times T^2). \]

The formatted graph manifold \( M^\dagger \) is constructed as

\[ M^\dagger = (M \setminus \text{int}(W_C)) \cup \partial W_C W^\dagger_C. \]

The formatted pinching map \( M^\dagger \to M \) is defined as the identity map on \( M \setminus \text{int}(W_C) \) together with the simultaneous horizontal pinching map \( W^\dagger_C \to W_C \).

We introduce some additional notations to describe the structure of \( M^\dagger \). The collection of tori \( \partial W^\dagger_C \) decomposes \( M^\dagger \) into the interior of \( W^\dagger_C \), together with the connected components \( X_q \) of \( M \setminus W_C \), indexed by \( q \in \pi_0(M \setminus W_C) \). We denote by \( X_{q(v)} \) the component that contains \( J_v \). The collection of tori \( T_{v,w} \times \{0\} \) decomposes the interior of \( W^\dagger_C \) into \( M_C \) together with the interiors of the fiber-connected sums

\[ W^\dagger_{w,v} = (T_{v,w} \times [0,1]) \# f_v (f_v \times T^2). \]

For each \((w, v) \in \Delta_\rho(C)\), fix an oriented slope \( c_{v,w} \) on \( T_{v,w} \) such that \( T_{v,w} \) is parametrized as the product torus \( f_v \times c_{v,w} \). Then \( W^\dagger_{w,v} \) can be parametrized as a product

\[ W^\dagger_{w,v} = f_v \times \Omega_{w,v}, \]

where \( \Omega_{w,v} \) denotes the connected sum of the annulus \( c_{v,w} \times [0,1] \) with the torus \( T^2 \).

The following commutative diagram summarizes our above construction about any \((v, w) \in \Delta_\rho(C)\). Each arrow indicates an inclusion, and each equality symbol indicates a
parametrization:

\[
\begin{align*}
&f_u \times c_{v,w} \times \{0\} \xrightarrow{} T_{v,w} \times \{0\} \xrightarrow{} \text{clos}(M_C) \\
&f_u \times \Omega_{w,v} \xrightarrow{} W^+_{v,w} \xrightarrow{} W^+_C \xrightarrow{} M^+ \\
&f_u \times c_{v,w} \times \{1\} \xrightarrow{} T_{v,w} \times \{1\} \xrightarrow{} \text{clos}(X_{q(v)})
\end{align*}
\]

(8.5)

8.1.2 Associated graph-of-groups decompositions

The structure of \( \pi_1(M^+) \) can be described with graph-of-groups decompositions. We briefly recall the terminology in group theory for the reader’s reference, see [22]. Then we point out necessary choices and elaborate the decomposition.

Recall that a graph of groups \( \mathcal{G} \) refers to a collection of data as follows: A connected graph \( \Lambda \), regarded as a connected finite cell 1-complex; a group \( G_v \) for every vertex (0-cell) \( v \) of \( \Lambda \); a group \( G_e \) for every edge (1-cell) \( e \) of \( \Lambda \); and an injective homomorphism \( i_\delta : G_e \to G_v \) for every end \( \delta \) of an edge \( e \) attached to a vertex \( v \). If \( \Lambda \) contains a unique vertex \( v \) and no edges, the fundamental group \( \pi_1(\mathcal{G}) \) is defined as \( G_v \). If \( \Lambda \) contains at least one edge, \( \pi_1(\mathcal{G}) \) is defined recursively using free amalgamations and HNN extensions. To be precise, choose an orientation of \( e \) to distinguish its positive and negative ends \( e^\pm \). When \( e \) is separating in \( \Lambda \), \( \pi_1(\mathcal{G}) \) is defined as the free amalgamation

\[
\pi_1(\mathcal{G}) = \left( \pi_1(\mathcal{G}_{(\Lambda \setminus e)^-}) \ast \pi_1(\mathcal{G}_{(\Lambda \setminus e)^+}) \right) / \left\langle \left\langle i_e^-(g)^{-1}i_e^+(g) : g \in G_e \right\rangle \right\rangle,
\]

where \( \mathcal{G}_{(\Lambda \setminus e)^\pm} \) stands for the subgraph of groups over the connected component \( (\Lambda \setminus e)^\pm \) of \( \Lambda \setminus e \) attached on the end \( e^\pm \). When \( e \) is nonseparating in \( \Lambda \), \( \pi_1(\mathcal{G}) \) is defined as the HNN extension

\[
\pi_1(\mathcal{G}) = \left( \pi_1(\mathcal{G}_{\Lambda \setminus e}) \ast \langle t_e \rangle \right) / \left\langle \left\langle i_e^-(g)^{-1}t_ei_e^+(g)t_e^{-1} : g \in G_e \right\rangle \right\rangle,
\]

where \( \mathcal{G}_{\Lambda \setminus e} \) stands for the subgraph of groups over the connected subgraph \( \Lambda \setminus e \), and where \( t_e \) is a distinguished free letter called the stable letter. Hence, the group \( \pi_1(\mathcal{G}) \) admits a presentation whose generators are generators of the vertex groups together with \( b_1(\Lambda) \) stable letters. The relators of the presentation are either relators of the vertex groups or the relators arising from the free amalgamations and the HNN extensions, which correspond to generators of the edge groups. For different choices of the edge \( e \) and its orientation, the resulting fundamental group \( \pi_1(\mathcal{G}) \) from the reduction procedure is actually unique up to canonical isomorphisms. A graph-of-groups decomposition of a group \( G \) refers to a graph of groups \( \mathcal{G} \) together with an isomorphism \( G \cong \pi_1(\mathcal{G}) \).

Suppose that we have chosen basepoints for all the spaces in the diagram (8.5), and have chosen paths connecting from the included basepoints to the basepoints of the target spaces for all the arrows thereof. Then the arrows induce injective homomorphisms between the fundamental groups of the pointed subspaces. One may adjust the homomorphisms with conjugations in the target groups by adjusting of the paths relative to the basepoints. After fixing such choices, we
will obtain a graph-of-groups decomposition of $\pi_1(M^\dagger)$, whose vertex groups are $\pi_1(W_C^\dagger)$ and the subgroups $\pi_1(X_q)$, and whose edge groups are the subgroups $\pi_1(T)$ corresponding to the boundary components $T$ of $W_C^\dagger$. We will also obtain a graph-of-groups decomposition of $\pi_1(W_C^\dagger)$, whose vertex groups are $\pi_1(M_C)$ and the subgroups $\pi_1(W_{v,w}^\dagger)$, and whose edge groups are the subgroups $\pi_1(T_{v,w} \times \{0\})$.

We choose basepoints and paths as above for objects in the diagram (8.5), and then identify $\pi_1(M^\dagger)$ and $\pi_1(W_C^\dagger)$ with their graph-of-groups decompositions. We actually adjust the paths to make the homomorphisms $\pi_1(T_{v,w} \times \{0\}) \to \pi_1(W_{v,w}^\dagger)$ and $\pi_1(T_{v,w} \times \{1\}) \to \pi_1(W_{v,w}^\dagger)$ look nicer, as follows. Fix a presentation of $\pi_1(\Omega_{w,v})$ with four generators $x_{v,w}, y_{v,w}, c_{0,v,w}, c_{1,v,w}$ and one relation

$$x_{v,w}y_{v,w}x_{v,w}^{-1}y_{v,w}^{-1} = c_{1,v,w}c_{0,v,w}^{-1}.$$  

(8.6)

We require that the conjugacy classes of $c_{0,v,w}$ and $c_{1,v,w}$ in $\pi_1(\Omega_{w,v})$ represent the free homotopy classes of the loops $c_{v,w} \times \{0\}$ and $c_{v,w} \times \{1\}$, in $\Omega_{w,v}$, respectively. (The presentation follows immediately from the connected sum decomposition of $\Omega_{w,v}$ and the van Kampen theorem.) This yields a presentation of $\pi_1(W_{w,v}^\dagger)$ with five generators $f_v, x_{v,w}, y_{v,w}, c_{0,v,w}, c_{1,v,w}$, and the five relations, namely, the relation (8.6) and another four relations saying that $f_v$ commutes with any other generators. We also require that the conjugacy class of $f_v$ in $\pi_1(W_{w,v}^\dagger)$ represents the free homotopy classes of the ordinary fiber $f_v$. After suitable adjustment of the chosen paths, we assume that the image of $\pi_1(T_{v,w} \times \{0\}) \to \pi_1(W_{w,v}^\dagger)$ is the free abelian subgroup generated by $c_{0,v,w}$ and $f_v$. We also assume that the image of $\pi_1(T_{v,w} \times \{1\}) \to \pi_1(W_{w,v}^\dagger)$ is the free abelian subgroup generated by $c_{1,v,w}$ and $f_v$.

8.1.3 Reductions for the standard local model

**Lemma 8.4.** The statement of Theorem 8.3 holds true if $M'$ is the standard local model $M^\dagger$ and if $\rho$ is of elliptic type over $C$.

**Proof.** Let $M, (V, E), \rho,$ and $C = (V_C, E_C)$ be as assumed in Theorem 8.3. Suppose that $\rho$ is of elliptic type over $C$. Given any constant $\varepsilon > 0$, we show that a representation $\rho'$ of $\pi_1(M^\dagger)$ as asserted can be obtained by modifying the pull-back representation $\rho_{pb}^\dagger$ of $\rho$. Our modification is supported at the vertex subgroup $\pi_1(W_C^\dagger)$ of $\pi_1(M^\dagger)$, in the sense of the following description: On any vertex group $\pi_1(X_q)$ or any stable letter, $\rho'$ is defined as the restriction of $\rho_{pb}^\dagger$. Meanwhile, $\rho'$ is defined on the vertex group $\pi_1(W_C^\dagger)$, and equals $\rho_{pb}^\dagger$ on any incoming edge group, namely, the image of any $\pi_1(T_{v,w} \times \{1\}) \to \pi_1(W_C^\dagger)$. In fact, we construct a sequence of supported modifications $\{\rho_{n}^\dagger\}_{n \in \mathbb{N}}$, which converges to $\rho_{pb}^\dagger$ as $n$ tends to $\infty$. For all sufficiently large $n$, we show that $\rho'$ can be taken as $\rho_{n}^\dagger$.

The restriction of $\rho_{n}^\dagger$ to $\pi_1(\text{clos}(M_C))$ equals $\rho$. The image of $\rho(\pi_1(\text{clos}(M_C)))$ is contained in the normalizer of the centralizer of $\pi_1(f_v)$ for any $v \in V_C$ (Lemma 5.8), which coincides with the centralizer as $C$ is elliptic by assumption. Therefore, $\rho_{n}^\dagger$ factors through an elliptic-type representation $\eta : H \to \text{SL}(2, \mathbb{R}) \ltimes_{\mathbb{Z}} \mathbb{R}$, where $H$ denotes the abelianization of $\pi_1(\text{clos}(M_C))$. We apply Lemma 7.1, and obtain a sequence of perturbed elliptic-type representations $\eta_n : H \to$
\(\text{SL}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R},\) indexed by \(n \in \mathbb{N}.\) Denote by \(\rho^\dagger_n : \pi_1(\text{clos}(M_C)) \to \text{SL}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R}\) the pull-back representations of \(\eta_n,\) for all \(n \in \mathbb{N}.\) For any \((u, v) \in \Delta_p(C),\) the maximality of \(C\) implies that \(\rho\) is of central type at \(v,\) so \(\rho^\dagger_{ pb}(f_{nu})\) is by definition the central element \(\xi^\rho(u) \in \mathbb{R} \text{ of } \text{SL}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R}\). Then the asserted properties of \(\eta_n\) imply that \(\rho^\dagger_{ pb}(f_{nu}) = \xi^\rho(u)\) is central and constant for all \(n \in \mathbb{N}.\) We also observe \(\rho^\dagger_{ pb}(c_{1,v,u}) = \rho^\dagger_{ pb}(c_{0,v,u}).\) Then, the asserted properties of \(\eta_n\) imply \(\rho^\dagger_{ pb}(c_{1,v,u})\rho^\dagger_n(c_{0,v,u})^{-1} \in \text{SL}(2, \mathbb{R})\) for all \(n \in \mathbb{N},\) and \(\lim_{n \to \infty} \rho^\dagger_n(c_{0,v,u}) = \rho^\dagger_{ pb}(c_{0,v,u}).\) We apply Lemma 7.4 to \(\gamma_n = \rho^\dagger_{ pb}(c_{1,v,u})\rho^\dagger_n(c_{0,v,u})^{-1},\) and obtain some elements \(\alpha_n, \beta_n \in \text{SL}(2, \mathbb{R}),\) for all but finitely many \(n \in \mathbb{N}.\) Remember \(\gamma_n = \alpha_n \beta_n \gamma_{n-1}^{-1}\) and \(\lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \beta_n = \text{id.}\) Then we can extend the restriction of \(\rho^\dagger_n\) to \(\pi_1(T_{u,v} \times \{0\})\) to be a representation of \(\pi_1(W^\dagger_{ u,v}),\) defining

\[
\rho^\dagger_n(x_{u,v}) = \alpha_n, \quad \rho^\dagger_n(y_{u,v}) = \beta_n, \quad \rho^\dagger_n(c_{1,v,u}) = \rho^\dagger_{ pb}(c_{1,v,u}),
\]

for all but finitely many \(n \in \mathbb{N}.\) Note that the relation (8.6) is preserved under \(\rho^\dagger_n,\) namely,

\[
\rho^\dagger_n(x_{u,v})\rho^\dagger_n(y_{u,v})\rho^\dagger_n(x_{u,v})^{-1} = \alpha_n \beta_n \alpha_n^{-1} \beta_n^{-1} = \gamma = \rho^\dagger_n(c_{1,v,u})\rho^\dagger_n(c_{0,v,u})^{-1} ;
\]

the commutativity relations of \(f_{nu}\) with \(x_{u,v}, y_{u,v}, c_{0,v,u},\) and \(c_{1,v,u}\) are also preserved under \(\rho^\dagger_n,\) since \(\rho^\dagger_n(f_{nu}) = \xi^\rho(u)\) is central. Perform such extension for all \((u, v) \in \Delta_p(C).\) Then, for all but finitely many \(n \in \mathbb{N},\) we obtain representations \(\rho^\dagger_n : \pi_1(W^\dagger_{ u,v}) \to \text{SL}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R},\) which all coincide with \(\rho^\dagger_{ pb}\) on the incoming edge groups \(\pi_1(T_{u,v} \times \{1\}).\) Therefore, they extend uniquely to be representations \(\rho^\dagger_n : \pi_1(M') \to \text{SL}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R},\) which all coincide with \(\rho^\dagger_{ pb}\) on any vertex groups \(\pi_1(X_q)\) and any stable letter. Our construction guarantees \(\lim_{n \to \infty} \rho^\dagger_n = \rho^\dagger_{ pb}\) in \(\mathcal{R}(\pi_1(M'), \text{SL}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R}).\) By Theorem 3.1, \(\mathcal{R}(\pi_1(M'), \text{SL}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R})\) is locally path-connected. Then \(\rho^\dagger_n\) lies in the path-connected component of \(\rho^\dagger_{ pb}\) when \(n\) is sufficiently large. For any \(u \in V\), if \(u \notin V_C,\) the supported modification implies \(\xi^\rho_n(u) = \xi^\rho(u)\); otherwise, the continuity of the essential winding number implies \(\lim_{n \to \infty} \xi^\rho_n(u) = \xi^\rho(u).\) For any \((u, w) \in E,\) if \([u, w] \cap V_C = \emptyset,\) the supported modification implies \(\tau_{\rho_n}(u, w) = \tau_{\rho}(u, w);\) otherwise, the asserted properties of \(\eta_n\) imply that \(\tau_{\rho_n}(u, w) = \tau_{\rho_n}(u, w),\) and \(\lim_{n \to \infty} \tau_{\rho_n}(u, w) = \tau_{\rho}(u, w),\) so for any sufficiently large \(n, \tau_{\rho_n}(u, w) = \tau_{\rho}(u, w)\) if \(\tau_{\rho}(u, w)\) is finite. Note also that by construction (Lemma 7.1), every vertex \(u \in V_C\) has finite cyclic image in \(\text{PSL}(2, \mathbb{R})\) under the projectivization of \(\rho^\dagger_n,\) implying \(\delta^\dagger(u) = 0\) for all \(u \in V_C.\) Therefore, for the given constant \(\varepsilon > 0,\) we take \(\rho'\) to be \(\rho^\dagger_n\) for some sufficiently large \(n,\) and \(\rho'\) is as desired. \(\Box\)

**Lemma 8.5.** The statement of Theorem 8.3 holds true if \(M'\) is the standard local model \(M^\dagger\) and if \(\rho\) is of hyperbolic type over \(C.\)

**Proof.** Suppose that \(\rho\) is of hyperbolic type over \(C.\) We follow a similar procedure as in the elliptic case (Lemma 8.4), but take \(H\) as the abelianization of \(\pi_1(\text{clos}(M_Y)),\) where \(Y\) is a maximal spanning tree in \(C,\) and use a path of supported modifications of \(\rho^\dagger_{ pb}\) instead of a sequence.

To be precise, we apply Lemma 7.2 (and Lemma 5.8) to obtain a path of hyperbolic-type representations \(\eta \gamma : H \to \text{SL}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R},\) parametrized by \(t \in (0, 1].\) Extend the path continuously to \(t = 0\) by defining \(\eta_0(h) = \text{wind}(\eta(h))\) for all \(h \in H.\) Denote by \(\rho^\dagger_t : \pi_1(\text{clos}(M_Y)) \to \text{SL}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R}\) the pull-back representations of \(\eta_0,\) for all \(t \in [0, 1].\)

The family \(\rho^\dagger_t\) on \(\pi_1(\text{clos}(M_Y))\) automatically extends over \(\pi_1(\text{clos}(M_C))\) as representations \(\rho^\dagger_t : \pi_1(\text{clos}(M_C)) \to \text{SL}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R},\) such that the image of any stable letter (associated to any
directed edge in $C$ outside $Y$) stays the same as its image under $\rho$. This is plainly the following claim.

- If $u\eta(h)u^{-1} = \eta(h')$ holds for some $h, h' \in H$ and some $u \in \widetilde{\SL}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R}$, then $u\eta_i(h)u^{-1} = \eta_i(h')$ also holds for all $t$.

The claim is obvious if $\eta(h)$ is central, as $\eta(h')$ must also be central, and $\eta_i$ stays constant on such elements, by construction (Lemma 7.2). Otherwise $\eta(h)$ is hyperbolic. Since $\eta(h)$ commutes with $u\eta(h)u^{-1}$, we observe that $u$ normalizes the centralizer of $\eta(h)$ in $\widetilde{\SL}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R}$, acting by conjugation as either the identity or the inverse. It follows that $u\eta_i(h)u^{-1} = \eta_i(h)\pm 1 = \eta_i(h')$, the last equality because it holds at $t = 1$ and hence constant for all $t \in [0, 1]$. Therefore, the claim holds true.

At $t = 1$, the representation $\rho^+_t : \pi_1(\text{clos}(M_C)) \to \widetilde{\SL}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R}$ is the restriction of $\rho^+_\text{pb}$ to $\pi_1(\text{clos}(M_C))$. For any $(w, v) \in \Delta_\rho(C)$, we apply Lemma 7.5 to $\gamma_t = \rho^+_\text{pb}(c_{1,v,w})\rho_t(c_{0,v,w})^{-1}$, and obtain some elements $\alpha_t, \beta_t \in \widetilde{\SL}(2, \mathbb{R})$, for $t \in [0, 1)$. Remember $\gamma_t = \alpha_t \beta_t \alpha_t^{-1} \beta_t^{-1}$ and $\lim_{t \to -1} \alpha_t = \lim_{t \to 1} \beta_t = \text{id}$, so we define $\alpha_0 = \beta_0 = \text{id}$ at $t = 0$. Then, we can extend the restriction of $\rho_t^+$ to $\pi_1(T_{v,w} \times \{0\})$ to be a representation of $\pi_1(W_{v,w}^\dag)$, defining

$$
\rho^+_t(x_{v,w}) = \alpha_t, \quad \rho^+_t(y_{v,w}) = \beta_t, \quad \rho_t^+(c_{1,v,w}) = \rho^+_{\text{pb}}(c_{1,v,w}),
$$

for all $[0,1]$. Extend this way for all $\pi_1(W_{w,v}^\dag)$, and extend further by $\rho^+_{\text{pb}}$ on all $\pi_1(X_q)$ and stable letters. Then we obtain representations $\rho_t^+ : \pi_1(M_C^\dag) \to \widetilde{\SL}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R}$, which varies continuously for all $t \in [0, 1]$, and which equals $\rho_{\text{pb}}$ at $t = 1$. For any $v \in V$, $\xi_{\rho^+_t}(v) = \xi_{\rho_t}(v)$ holds for all $v \in V$ at $t = 0$. (The constructions in Lemmas 7.2 and 7.5 actually make $\xi_{\rho^+_t}(v) = \xi_{\rho_t}(v)$ for all $t \in [0, 1]$.) For any $(v, w) \in E$, if $(v, w) \cap V_C = \emptyset$, $\tau_{\rho^+_t}(v, w) = \tau_{\rho_t}(v, w)$ holds for all $t \in [0, 1]$; otherwise, $\tau_{\rho^+_t}(v, w) = 1$ holds at $t = 0$, while $\tau_{\rho_t}(v, w)$ is either 1 or $\infty$. Note also that by construction (Lemma 7.2), every vertex $v \in V_C$ becomes central with respect to $\rho_0^+$. Together with the fact $\tau_{\rho_0^+}(v, w) = 1$ for all $(v, w) \in \Delta_\rho(C)$, this implies $\delta_{\rho_0^+}(v) = 0$ for all $v \in V_C$. We take $\rho'$ to be $\rho_0^+$, as desired.

**Lemma 8.6.** The statement of Theorem 8.3 holds true if $M'$ is the standard local model $M^\dag$ and if $\rho$ is of parabolic type over $C$.

**Proof.** Suppose that $\rho$ is of parabolic type over $C$. This case is almost completely the same as the hyperbolic case (Lemma 8.5). Note that if we set $\eta_i = g_i \eta g_i^{-1}$ in the conclusion of Lemma 7.3, then $\eta_i$ satisfies exactly the same properties as listed in Lemma 7.2. Therefore, we basically repeat the argument of the hyperbolic case, applying Lemmas 7.3 and 7.6 instead of Lemmas 7.2 and 7.5. The same claim as in the proof of Lemma 8.5 still holds, except the argument needs to be replaced, as follows.

If $\eta(h)$ is central, again, $\eta(h')$ must be central, and $u\eta_i(h)u^{-1} = \eta_i(h')$ holds for all $t$, both sides being constant. If $\eta(h)$ is parabolic, $\eta(h')$ is a parabolic that commutes with $\eta(h)$, so $u$ lies in the normalizer of the centralizer of $\eta(h)$. By construction, $g_i$ also lies in the normalizer of the centralizer of $\eta(h)$ (Lemma 7.3). Then the commutator $[u, g_i]$ must commute with $\eta(h)$, implying $u\eta_i(h)u^{-1} = u g_i \eta(h) g_i^{-1} u^{-1} = g_i u \eta(h) u^{-1} g_i^{-1} = g_i \eta(h') g_i^{-1} = \eta_i(h')$ for all $t$. Therefore, the same claim remains true in the parabolic case.
Proceeding as before, we obtain representations \( \rho_t^+ : \pi_1(M^t) \to \widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R} \) for all \( t \in [0, 1] \), equal to \( \rho_{t_0} \) at \( t = 1 \), and central over \( C \) at \( t = 0 \). Again, \( \rho' \) can be taken as \( \rho_0^+ \) as desired. \( \square \)

8.1.4 Local reduction in general

We finish the proof Theorem 8.3 as follows. Let \( M \) be any formatted graph manifold with a simplicial JSJ graph \( (V, E) \), and \( \rho : \pi_1(M) \to \widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R} \) any representation. Let \( C = (V_C, E_C) \) be a maximal noncentral component with respect to \( \rho \).

Suppose that \( M' \to M \) is a formatted pinching map of a formatted graph manifold \( M' \), such that the inequality

\[
\chi_{M'}(v) \leq \chi_M(v) - 2 \cdot \delta_{\rho}(v; C)
\]

holds for all \( v \in V \). It follows from (8.4) that \( \chi_{M'}(v) \leq \chi_{M^t}(v) \) holds for all \( v \in V \). Therefore, \( M' \to M \) admits a factorization into a composition of formatted pinching maps \( M' \to M^t \to M \). Given any constant \( \varepsilon > 0 \), there is some representation \( \rho^+ \) of \( \pi_1(M^t) \) that satisfies the asserted properties for \( M^t \to M \) (Lemmas 8.4, 8.5, and 8.6). We take \( \rho' \) to be the pull-back of \( \rho^+ \) to \( \pi_1(M') \). Then \( \rho' \) also satisfies the properties, as desired.

This completes the proof of Theorem 8.3.

8.2 Global reduction

**Theorem 8.7.** Let \( M' \to M \) be a formatted pinching map between formatted graph manifolds with identified simplicial JSJ graphs \( (V, E) \). Let \( \rho : \pi_1(M) \to \widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R} \) be a representation. Adopt Notations 5.1 and 5.6. Suppose that

\[
\chi_{M'}(v) \leq \chi_M(v) - 2 \cdot \delta_{\rho}(v)
\]

holds for all vertices \( v \in V \).

Then, for any constant \( \varepsilon > 0 \), there exists a representation \( \rho' : \pi_1(M') \to \widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R} \), such that the following conditions are all satisfied.

- In the space of representations \( \mathcal{R}(\pi_1(M'), \widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}) \), \( \rho' \) is path-connected with the pull-back representation of \( \rho \).
- For all \( v \in V \),

\[
\left| \xi_{\rho'}(v) - \xi_{\rho}(v) \right| < \varepsilon.
\]

- For all \( \{v, w\} \in E \), \( \tau_{\rho'}(v, w) \) equals \( \tau_{\rho}(v, w) \) if \( \tau_{\rho}(v, w) \) is finite.
- For all \( v \in V \), \( \delta_{\rho'}(v) \) equals 0, and for all \( \{v, w\} \in E \), \( \tau_{\rho'}(v, w) \) is finite.

**Proof.** Set \( (M'_0, \rho'_0) = (M, \rho) \). Given any constant \( \varepsilon > 0 \), we construct \( (M'_n, \rho'_n) \) and \( M'_n \to M'_{n-1} \) inductively, for all \( n \in \mathbb{N} \) applicable, as follows. If there are \( m \) maximal noncentral components in \( (V, E) \) with respect to \( \rho \), the induction will be done within \( k \) steps with \( k \leq m \).
Suppose that \((M'_{n-1}, \rho'_{n-1})\) has been constructed, where \(M'_{n-1}\) is a formatted graph manifold with a simplicial JSJ graph identified with \((V, E)\), and where \(\rho'_{n-1} : \pi_1(M'_{n-1}) \to \widetilde{SL}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R}\) is a representation with \(\tau_{\rho'_{n-1}}(v_{n-1}, w_{n-1}) = \infty\) for some \(\{v_{n-1}, w_{n-1}\} \in E\). Without loss of generality, we may assume that \(w_{n-1}\) is noncentral with respect to \(\rho'_{n-1}\). Take the maximal noncentral component \(C_{n-1} = (V_{n-1}, E_{n-1})\) of \((V, E)\) with respect to \(\rho'_{n-1}\) such that \(w_{n-1}\) is a vertex of \(C_{n-1}\). (The vertex \(v_{n-1}\) may or may not be a vertex of \(C_{n-1}\).) Construct a formatted graph manifold \(M'_n\) and a formatted pinching map \(M'_n \to M'_{n-1}\), such that

\[\chi_{M'_n}(v) = \chi_{M'_{n-1}}(v) - 2 \cdot \delta_{\rho'}(v; C_{n-1}),\]

for all \(v \in V\). Indeed, \(M'_n\) can be obtained as a suitable simultaneous fiber-connected sum of \(M'_{n-1}\) (Remark 8.2). Apply Theorem 8.3 to construct a representation \(\rho'_n\) of \(\pi_1(M'_n)\) with \(\tau_{\rho'_n}(v_{n-1}, w_{n-1}) < \infty\). Moreover, \(\rho'_n\) lies in the path-connected component of the pull-back of \(\rho'_{n-1}\). The construction guarantees \(|\xi_{\rho'_n}(v) - \xi_{\rho'_{n-1}}(v)| < 2^{-n} \epsilon\) for all \(v \in V\). It also guarantees \(\tau_{\rho'_n}(v, w) = \tau_{\rho'_{n-1}}(v, w)\) for all \(\{v, w\} \in E\) unless \(\{v, w\} \cap V_{n-1} \neq \emptyset\) and \(\tau_{\rho}(v, w) = \infty\). In particular, this implies \(\delta_{\rho'_n}(v; C_j) = 0\) for all \(j = 1, 2, \ldots, n - 1\) and all \(v \in V\), by induction. Hence, \(C_1, \ldots, C_{n-1}\) must be distinct maximal noncentral components with respect to \(\rho\), at each step \(n\) (see Section 5.1).

The above construction terminates at some step \(n\) with \(\tau_{\rho'_n}(v, w)\) finite for all \(\{v, w\} \in E\). For this \(M'_n\), we obtain a formatted pinching map \(M'_n \to M\) as the composite map \(M'_n \to M'_{n-1} \to \cdots \to M_0\). Observe

\[\chi_{M'_n}(v) = \chi_M(v) - 2 \cdot \delta_{\rho}(v)\]

for all \(v \in V\), which follows immediately from the characterization of \(\delta_{\rho}(v)\) (Lemma 5.7) and the construction of \(M'_n\), by examining individually the effect at the vertices. We estimate

\[|\xi_{\rho'_n}(v) - \xi_{\rho}(v)| \leq \sum_{l=1}^{n} |\xi_{\rho'_l}(v) - \xi_{\rho'_{l-1}}(v)| < \sum_{l=1}^{n} 2^{-l} \epsilon < \epsilon.\]

Moreover, any finite \(\tau_{\rho}(v, w)\) remains unchanged throughout the construction, so \(\tau_{\rho'_n}(v, w)\) equals \(\tau_{\rho}(v, w)\) if \(\tau_{\rho}(v, w)\) is finite.

By the assumption about \(M'\), we have \(\chi_{M'(v)} \leq \chi_{M'_n}(v)\) for all \(v \in V\), so the formatted pinching map \(M' \to M\) factorizes as a composition of formatted pinching maps \(M' \to M'_n \to \cdots \to M_0\). Take \(\rho'\) to be the pull-back of \(\rho'_n\) to \(\pi_1(M')\). It follows that \(\rho'\) satisfies the properties as asserted. 

9 SEIFERT REPRESENTATIONS FOR GRAPH MANIFOLDS: THE PROOF

In this section, we prove Theorem 5.2. Let \(M\) be a formatted graph manifold with a simplicial JSJ graph \((V, E)\), and \(\rho : \pi_1(M) \to \widetilde{SL}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R}\) be a representation.

For each vertex \(v \in V\), we take \(\delta_v(\rho)\) mutually disjoint ordinary fibers in the corresponding JSJ piece \(J_v\) (see Notation 5.6). Construct a formatted graph manifold \(M'\) as the simultaneous fiber connected sum of \(M\) with trivial circle bundles over tori, with respect to the union of all these fibers (see Remark 8.2). Denote by \(M'\) the resulting formatted graph manifold and \(M' \to M\) the
associated formatted pinching map, namely, the simultaneous horizontally pinching of $M'$ onto $M$. We obtain the relation

$$\chi_{M'}(v) = \chi_M(v) - 2 \cdot \delta_{\rho}(v)$$

for all $v \in V$. Moreover, we obtain the relation

$$e_{M'} = e_M$$

in $\text{End}(\mathbb{R}^V)$. These obvious relations are the simultaneous version of (8.2).

We apply Theorem 8.7 to obtain a sequence representation $\rho_n'$ of $\pi_1(M')$, index by $n \in \mathbb{N}$, which are all path-connected to the pull-back representation of $\rho$, in the space of representations $\mathcal{R}(\pi_1(M'), \widetilde{\text{SL}}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R})$. We may require

$$\lim_{n \to \infty} \xi_{\rho_n'}(v) = \xi_{\rho}(v)$$

for all $v \in V$ and

$$\tau_{\rho_n'}(v, w) = \tau_{\rho}(v, w)$$

for all $\{v, w\} \in E$ with $\tau_{\rho}(v, w)$ finite. Moreover, we may require $\tau_{\rho_n'}(v, w)$ to be finite for all $\{v, w\} \in E$ and all $n \in \mathbb{N}$.

Since $\rho_n'$ are all path-connected with the pull-back of $\rho$, it follows that

$$\text{vol}_{\widetilde{\text{SL}}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R}}(M, \rho) = \text{vol}_{\widetilde{\text{SL}}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R}}(M', \rho_n').$$

By Theorem 6.1, we obtain

$$\text{vol}_{\widetilde{\text{SL}}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R}}(M', \rho_n') = 4\pi^2 \cdot \left( \xi_{\rho_n'}, e_{M'} \xi_{\rho_n'} \right).$$

Passing to the limit as $n$ tends to $\infty$, we obtain the volume formula

$$\text{vol}_{\widetilde{\text{SL}}(2, \mathbb{R}) \times \mathbb{Z} \mathbb{R}}(M, \rho) = 4\pi^2 \cdot \left( \xi_{\rho}, e_M \xi_{\rho} \right),$$

as asserted in Theorem 5.2.

Moreover, the asserted rationality in Theorem 5.2 about $(M, \rho)$ follows from the rationality about $(M', \rho_n')$ for sufficiently large $n$. After all, neither the Euler operator nor the volume has changed.

It remains to establish the asserted upper bound of $|(e_M \xi_{\rho}(v))|$ for all $v \in V$. At this point, we notice that the estimate for $|(e_M' \xi_{\rho_n'})(v)|$ from Theorem 6.1 only implies a weaker upper bound:

$$|(e_M \xi_{\rho}(v))| \leq \max \left\{ 0, -\chi_M(v) + 2 \cdot \delta_{\rho}(v) - \sum_{\{v, w\} \in E} \frac{1}{\tau_{\rho}(v, w)} \right\}.$$  (9.1)
This is already the desired bound for any $v$ that is noncentral with respect to $\rho$, since $\delta_\rho(v) = 0$ holds in that case (Lemma 5.7). To get rid of the unwanted term with $\delta_\rho(v)$ at any $v$ that is central with respect to $\rho$, we again appeal to the covering trick, as we did in Subsection 6.2. This is done as follows.

Denote by $\bar{\rho} : \pi_1(M) \to \text{PSL}(2,\mathbb{R})$ the induced representation of $\rho$. Since $\bar{\rho}(\pi_1(M))$ is a finitely generated subgroup of a linear group, it is residually finite. Recall that $\tau_\rho(v, w)$ is the order of $\bar{\rho}(\pi_1(T_{v,w}))$. Therefore, for any positive integer $D \in \mathbb{N}$, we can construct some quotient homomorphism $\bar{\rho}(\pi_1(M)) \to \Gamma$ onto a finite group $\Gamma$ (depending on $D$), with the following properties: For any $\{v, w\} \in E$, if $\tau_\rho(v, w)$ is finite, then $\bar{\rho}(\pi_1(T_{v,w}))$ injects $\Gamma$; otherwise, $\bar{\rho}(\pi_1(T_{v,w}))$ surjects a subgroup of $\Gamma$ of order at least $D$. Take $M^*$ to be the regular finite cover of $M$ that corresponds to the kernel of the composite homomorphism $\pi_1(M) \to \bar{\rho}(\pi_1(M)) \to \Gamma$. Take a characteristic finite cover of the JSJ graph of $M^*$ that is a simplicial graph. Take $M''$ to be the pull-back cover of $M^*$. Then $M''$ is a characteristic finite cover of $M^*$, and therefore, a regular finite cover of $M$. If $M''$ is not formatted with the lifted orientations of $M$ and its Seifert fibrations of the JSJ pieces, we replace $M''$ with a further finite cover that is regular over $M$ and that is formatted (Definition 4.1).

The construction makes sure that $M'' \to M$ is a formatted covering projection between formatted graph manifolds. Denote by $(V'', E'')$ the JSJ graph of $M''$. Denote by $\rho'' : \pi_1(M'') \to \text{PSL}(2,\mathbb{R}) \times \mathbb{Z}$ the pull-back representation of $\rho$. For any covering pair of JSJ pieces $J''_{v'} \to J_v$, the same computations as in Subsection 6.2 work, yielding the formula

$$\left| (e_{M''} \xi_{\rho''})(v'') \right| = \frac{[J''_{v'} : J_v]}{[f''_{v'} : f_v]} \times \left| (e_M \xi_\rho)(v) \right|, \quad (9.2)$$

and the comparison

$$-\chi_{M''}(v'') - \sum_{\{v'', w''\} \in E''} \frac{1}{\tau_{\rho''}(v'', w'')} \leq \frac{[J''_{v'} : J_v]}{[f''_{v'} : f_v]} \times \left( -\chi_M(v) - \sum_{\{v, w\} \in E} \frac{1}{\tau_\rho(v, w)} \right). \quad (9.3)$$

Note that terms with $\tau_{\rho''}(v'', w'') = \tau_\rho(v, w) = \infty$ have no contribution on both sides of the inequality.

Suppose that $v$ is a vertex at which $\rho$ is central. Then, for any JSJ torus $T_{v,w}$ of $M$ with $\tau_\rho(v, w) = \infty$, the number of JSJ tori $T''_{v'', w''}$ in $M''$ that cover $T_{v,w}$ is at most $[J''_{v'} : J_v]/[f''_{v'} : f_v]$ divided by $D$. In fact, the precise number should be $[J''_{v'} : J_v]/[T''_{v'', w''} : T_{v,w}]$, but we observe $[T''_{v'', w''} : T_{v,w}] \geq D$ from the construction, and observe $[f''_{v'} : f_v] = 1$, since $\bar{\rho}(f_v)$ has to be trivial in the central case. So, we reach the following comparison:

$$\delta_{\rho''}(v'') \leq \frac{[J''_{v'} : J_v]}{[f''_{v'} : f_v]} \times \frac{1}{D} \times \delta_\rho(v). \quad (9.4)$$

We apply the weaker upper bound (9.1) to $(M'', \rho'')$. Then:

$$\left| (e_{M''} \xi_{\rho''})(v'') \right| \leq \max \left\{ 0, -\chi_{M''}(v'') + 2 \cdot \delta_{\rho''}(v'') - \sum_{\{v'', w''\} \in E''} \frac{1}{\tau_{\rho''}(v'', w'')} \right\}. \quad (9.5)$$
Together with (9.2), (9.3), and (9.4), this yields:

\[
|e_M^\xi \rho)(v)| \leq \max \left\{ 0, -\chi_M(v) + \frac{2 \cdot \delta_\rho(v)}{D} - \sum_{[v, w] \in E} \frac{1}{\tau_\rho(v, w)} \right\}.
\]

Note that \(D\) is an arbitrary positive integer, and \(\delta_\rho(v)\) is at most the number of edges in \((V, E)\).

Therefore, if \(\rho\) is central at \(v\), we obtain the refined upper bound

\[
|e_M^\xi \rho)(v)| \leq \max \left\{ 0, -\chi_M(v) - \sum_{[v, w] \in E} \frac{1}{\tau_\rho(v, w)} \right\},
\]

as desired, by the covering trick argument as above. If \(\rho\) is noncentral at \(v\), the same bound follows trivially from (9.1) and \(\delta_\rho(v) = 0\) (Lemma 5.7).

This completes the proof of Theorem 5.2.

10 APPLICATION TO STRICLY DIAGONALLY DOMINANT GRAPH MANIFOLDS

As our first application of Theorem 5.2, we exhibit a class of graph manifolds whose covering Seifert volume (see (1.1)) can be effectively bounded. Example 1.3 follows immediately as a special case.

Theorem 10.1. Let \(M\) be a formatted graph manifold with a simplicial JSJ graph \((V, E)\). Adopt Notation 4.2. Suppose that the Euler operator \(e_M\) is strictly diagonally dominant, namely, the inequality

\[
|k_v| > \sum_{[v, w] \in E} \left| \frac{1}{b_v, w} \right|
\]

holds for all vertices \(v \in V\). Then

\[
CSV(M) \leq \sum_{v \in V} \left| k_v \right| - \sum_{[v, w] \in E} \left| \frac{4\pi^2 \chi_v^2}{\left| b_v, w \right|} \right|.
\]

To prove Theorem 10.1, we make use of the well-known Gershgorin circle theorem in matrix analysis:

Theorem 10.2 [13, Chapter 6, Theorem 6.1.1]. Let \(A = (a_{i, j})_{n \times n}\) be a square matrix of size \(n\) with entries in \(C\). Then the eigenvalues of \(A\) all lie in the union of disks \(D_1 \cup D_2 \cup \cdots \cup D_n\), where

\[
D_i = \left\{ z \in C : |z - a_{i, i}| \leq \sum_{j \neq i} |a_{i, j}| \right\},
\]

for \(i = 1, 2, \ldots, n\).
Lemma 10.3. Under the assumptions of Theorem 10.1, the Seifert volume of $M$ satisfies the inequality

$$SV(M) \leq \frac{4\pi^2 \cdot \sum_{v \in V} \chi^2_v}{\min \left\{ |k_v| - \sum_{\{v,w\} \in E} \frac{1}{b_{v,w}} : v \in V \right\}}.$$ 

Proof. The operator $e_M$ has a symmetric square matrix over the standard basis of $\mathbb{R}^V$. Theorem 10.2 implies that its eigenvalues $\lambda$ are all nonzero, and indeed,

$$|\lambda| \geq \min \left\{ |k_v| - \sum_{\{v,w\} \in E} \frac{1}{b_{v,w}} : v \in V \right\}.$$ 

In particular, $e_M$ is invertible. Note that the operator norm of $e_M^{-1}$ on the inner product vector space $\mathbb{R}^V$ is the largest $|1/\lambda|$. For any vector $\xi \in \mathbb{R}^V$ with $|(e_M\xi)(v)| \leq -\chi_v$ for all $v \in V$, we estimate

$$(\xi, e_M\xi) \leq \|e_M^{-1}\|_{\text{op}} \times \|e_M\xi\|_{\ell^2}^2 \leq \frac{\sum_{v \in V} \chi^2_v}{\min \left\{ |k_v| - \sum_{\{v,w\} \in E} \frac{1}{b_{v,w}} : v \in V \right\}}.$$ 

Then by Theorem 5.2, we obtain the asserted inequality. \qed

The following construction regarding finite covers of graph manifolds is somewhat well known. We supply a proof for the reader’s reference.

Lemma 10.4. Let $M$ be a formatted graph manifold with a simplicial JSJ graph $(V, E)$. Adopt Notation 5.1. Suppose $\chi_M(v) < 0$ holds for all $v \in V$. Then, for any formatted covering projection $M^* \to M$, there exist positive integers $m^*$ and $r^*$ with the following property:

For any positive integer $m$ divisible by $m^*$, and for any positive integers $r_v$ divisible by $r^*$, indexed by $v \in V$, there exist a formatted graph manifold $M'$, and a formatted covering projection $M' \to M$, such that the following conditions are all satisfied.

- Every JSJ torus $T'_{v',w'}$ of $M'$ projects a JSJ torus $T_{v,w}$ of $M$ as a characteristic cover of degree $m^2$.
- Every JSJ piece $J'_{v'}$ of $M'$ projects a JSJ piece $J_v$ of $M$ as a cover of degree $m^2r_v$.
- There are no exceptional fibers in any JSJ pieces of $M'$.
- The projection $M' \to M$ factors through $M^*$.

Proof. Since every JSJ piece $J_v$ of $M$ fibers over a 2-orbifold of negative Euler characteristic, we may replace $M^*$ with some finite cover, and assume that every JSJ piece of $M^*$ is homeomorphic to the product of a circle and a surface of positive genus and with an even number of punctures. For example, this can be done immediately by [5, Proposition 4.2] (applied to $M^*$ and a collection of product covers of the JSJ pieces as described). Moreover, we may replace $M^*$ with some finite cover, and assume that $M^*$ is regular over $M$, and that the JSJ torii of $M^*$ are all characteristic over JSJ torii of $M$ of the equal degree. For example, this is a special case of [5, Corollary 4.5] (applied to $M^* \to M$ as $N \to M$ thereof).

With these simplification assumptions, we take $m^*$ and $r^*$ according to the following conditions: The product $m^* \times m^*$ equals the covering degree $[T^*_{v^*,w^*} : T_{v^*,w^*}]$, for some (hence any)
covering pair of JSJ tori in $M^*$ and $M$; and the product $m^* \times m^* r^*$ equals the least common multiple of $[J^*_{v^*} : J_v]$, ranging over all the covering pairs of JSJ pieces in $M^*$ and $M$.

For any positive integers $s$ and $l$, we observe the following simple constructions: Given any orientable compact surface $\Sigma$ of positive genus with an even number of boundary components, we can always construct a finite cover $\Sigma'$ of $\Sigma'$ such that every component of $\partial \Sigma'$ covers a component $\partial \Sigma$ of degree $l$, and $\Sigma'$ covers $\Sigma$ of degree $l^s$. For example, one may first take an $l$-cyclic cover of $\Sigma$ dual to a union of disjoint simple arcs that intersects every component of $\partial \Sigma$ in exactly one point, and then take an $s$-cyclic cover of that cover dual to a nonseparating simple closed curve. It follows that the product of a circle with $\Sigma$ always admits a finite cover of degree $l^2s$, such that the boundary components projects as characteristic covers of degree $l^2$.

Suppose that $m$ and $r_v$, indexed by $v \in V$, are any given positive integers. For each JSJ piece $J^*_v$ of $M^*$, the above construction yields a finite cover $\text{clos}(J'_v) \to \text{clos}(J^*_v)$, which has degree $m^2 r_v /[J^*_{v^*} : J_v]$, and which is characteristic of degree $(m/m^*)^2$ restricted to each boundary component. Take $D$ to be a common multiple for all $[J^*_{v^*} : J_v]$ $. Take D/[J^*_{v^*} : J_v]$ copies of each $\text{clos}(J'_v)$. We may obtain a finite cover $M'$ over $M^*$ by gluing up these copies, identifying the common boundary components via covering transformations over the JSJ tori of $M^*$. Possibly after discarding extra connected components, we may require that $M'$ is connected. Possibly after passing to a pull-back cover induced by a cover of the JSJ graph, we may require that $M'$ has a simplicial JSJ graph. It is straightforward to verify that the composite covering projection $M' \to M^* \to M$ is as desired.

**Proof of Theorem 10.1.** Suppose that $M^* \to M$ is any given finite cover of $M$. Then, Lemma 10.4 applies and yields some positive integers $m^*$ and $r^*$. Take $m$ to be $m^*$, and for any vertex $v \in V$, take $r_v$ to be some positive integral multiple of $r^*$, such that the product

$$C = r_v \times \left( |k_v| - \sum_{[u,w] \in E} \left| \frac{1}{b_{u,w}} \right| \right)$$

becomes constant independent of $v$. Let $M'$ be a formatted cover of $M$ as guaranteed in Lemma 10.4, with respect to $m$ and these $r_v$. We observe

$$|k_{v'}| - \sum_{[v',w'] \in E'} \left| \frac{1}{b_{v',w'}} \right| = r_v \times \left( |k_v| - \sum_{[u,w] \in E} \left| \frac{1}{b_{u,w}} \right| \right) = C,$$

for any covering pair of JSJ pieces $J'_{v'} \to J_v$, by Proposition 4.5. Lemma 10.3 applies to $M'$, yielding

$$\text{SV}(M') \leq \frac{4\pi^2}{C} \sum_{v' \in V'} \chi^2_{v'}$$

$$= \sum_{v \in V} \left[ M' : M \right] \times \frac{m^2 r_v \times 4\pi^2 \chi^2_v}{r_v \times \left( |k_v| - \sum_{[u,w] \in E} \left| \frac{1}{b_{u,w}} \right| \right)}$$

$$= \left[ M' : M \right] \times \sum_{v \in V} \frac{4\pi^2 \chi^2_v}{|k_v| - \sum_{[u,w] \in E} \left| \frac{1}{b_{u,w}} \right|}.$$
Since $M' \to M$ factors through the given cover $M^*$, we obtain
\[
\frac{SV(M^*)}{[M^* : M]} \leq \frac{SV(M')}{[M' : M]} \leq \sum_{v \in V} \frac{4\pi^2 \chi_v^2}{|k_v| - \sum_{\{v,w\} \in E} |1/b_{v,w}|}.
\]
Since the finite cover $M^* \to M$ is arbitrary, we obtain the estimate for $CSV(M)$ as asserted in Theorem 10.1. \hfill \Box

11 | VIRTUAL EXISTENCE OF GENERIC VOLUME VALUES

Provided Theorem 5.2, it becomes suitable to ask which values in $[0, +\infty)$ actually arise as volume of $\widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$-representations for a given graph manifold. In this section, we offer a virtual and generic answer to this question, as Theorem 11.1. It is virtual because of the finite cover in the conclusion. It is generic because of the strict inequality in the hypothesis.

**Theorem 11.1.** Let $M$ be a formatted graph manifold with a simplicial JSJ graph $(V, E)$. Adopt Notation 5.1. Let $\xi \in \mathbb{Q}^V$ be any vector such that the following inequality holds for all $v \in V$:
\[
|\langle \mathbf{e}_M \xi \rangle(v)| < -\chi_M(v).
\]
Then, there exist a formatted finite cover $M' \to M$ and a representation $\rho' : \pi_1(M') \to \widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$ with the property
\[
\frac{\text{vol}_{\widetilde{SL}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}}(M', \rho')}{[M' : M]} = 4\pi^2 \cdot (\xi, \mathbf{e}_M \xi).
\]

The rest of this section is devoted to the proof of Theorem 11.1.

**Lemma 11.2.** Let $M$ be a formatted graph manifold with a simplicial JSJ graph $(V, E)$. Adopt Notation 5.1. Suppose that there are no exceptional fibers in any JSJ piece of $M$. Suppose that $\xi \in \mathbb{Z}^V$ is a vector, such that $b_{v,w}$ divides $\xi(v)$ for all $v \in V$ and $\{v, w\} \in E$, and such that
\[
|\langle \mathbf{e}_M \xi \rangle(v)| \leq -\chi_M(v) - \text{valence}_{(V,E)}(v)
\]
holds for all $v \in V$. Then, there exists a representation $\rho : \pi_1(M) \to \widetilde{SL}(2, \mathbb{R})$, such that $\rho(f_v)$ lies in the center $\mathbb{Z}$ and equals $\xi(v)$, for all $v \in V$.

**Proof.** We choose auxiliary basepoints of the JSJ pieces and the JSJ tori, and choose auxiliary paths to connect them, so the fundamental group $\pi_1(M)$ decomposes into a graph of groups accordingly. We construct representations $\pi_1(J_v) \to \widetilde{SL}(2, \mathbb{R})$ for all $v \in V$ such that the restricted representations to $\pi_1(T_{v,w})$ have image in the center $\mathbb{Z}$. We make sure that the restricted representations $\pi_1(T_{v,w}) \to \mathbb{Z}$ and $\pi_1(T_{w,v}) \to \mathbb{Z}$ are equal for each $\{v, w\} \in E$. Then these representations extend to a representation of $\pi_1(M)$, by any arbitrary assignments for the stable letters.

Since there are no exceptional fibers in any JSJ piece $J_v$ of $M$, we can identify clos$(J_v)$ as a product $f_v \times \Sigma_v$, where $\Sigma_v$ is a compact oriented surface. The components of $\partial \Sigma_v$ naturally
correspond to the edges \{v, w\} incident to v. With the induced orientations, they give rise to oriented slopes \( s_{v,w} \) on the corresponding JSJ tori \( T_{v,w} \). Therefore, we obtain a Waldhausen basis \([f_v], [s_{v,w}]\) for \( H_1(T_{v,w}; \mathbb{R}) \), which actually generates the integral lattice \( H_1(T_{v,w}; \mathbb{Z}) \). The equation \([f_w] = a_{v,w} [f_v] + b_{v,w} [s_{v,w}]\) on \( H_1(T_{v,w}; \mathbb{Z}) \) determines a unique integer \( a_{v,w} \in \mathbb{Z} \). We obtain a unique homomorphism \( \eta_{v,w} : H_1(T_{v,w}; \mathbb{Z}) \to \mathbb{Z} \) which sends \([f_v]\) to \( \zeta(v) \) and \([f_w]\) to \( \zeta(w) \). In fact, \([s_{v,w}]\) must go to \((\zeta(w) - a_{v,w} \zeta(v)) / b_{v,w}\), which lies in \( \mathbb{Z} \) because \( b_{v,w} \) divides both \( \zeta(v) \) and \( \zeta(w) \). We observe the relation

\[
(e_M \xi)(v) = k_v \xi(v) - \sum_{\{v,w\} \in E} \frac{1}{b_{v,w}} \times \xi(w) = (-1) \cdot \sum_{\{v,w\} \in E} \eta_{v,w}( [s_{v,w}] ).
\]

To simplify notations, we focus on a vertex \( v \), and suppose that \( \Sigma_v \) is of genus \( g \) with \( n \) boundary components. We enumerate the vertices incident to \( v \) as \( w_1, w_2, \ldots, w_n \). The fundamental group \( \pi_1(\Sigma_v) \) admits a well-known presentation of \( 2g + n \) generators \( x_1, y_1, x_2, y_2, \ldots, x_g, y_g, s_1, \ldots, s_n \), with a single relation

\[
[x_1, y_1][x_2, y_2] \cdots [x_g, y_g] = s_1 s_2 \cdots s_n,
\]

where \([x_i, y_i]\) stands for the commutator \( x_i y_i x_i^{-1} y_i^{-1} \). Then \( \pi_1(J_v) \) is generated as a direct product of \( \pi_1(\Sigma_v) \) and the infinite cyclic group \( \langle f_v \rangle \). We assume that the conjugacy class of \( s_j \) corresponds to the free homotopy class of the oriented slope \( s_{v,w_j} \). To define a homomorphism \( \rho_v : \pi_1(J_v) \to \widetilde{SL}(2, \mathbb{R}) \), we first assign \( \rho_v(f_v) \) to be the central element \( \xi(v) \in \mathbb{Z} \), and \( \rho_v(s_j) \) to be \( \eta_{v,w_j}( [s_{v,w_j}] ) \in \mathbb{Z} \), for \( j = 1, 2, \ldots, n \). Therefore, the assumption just says

\[
|\rho_v(s_1 s_2 \cdots s_n)| \leq 2g - 2.
\]

In this case, \( \rho_v(s_1 s_2 \cdots s_n) \) can be written as a product of \( g \) commutators \([\alpha_1, \beta_1][\alpha_2, \beta_2] \cdots [\alpha_g, \beta_g] \), where \( \alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_g, \beta_g \) are some elements of \( \widetilde{SL}(2, \mathbb{R}) \). In fact, this is a special case of [8, Theorems 2.3 and 4.1]. Alternatively, one may apply Theorem 3.2 to an oriented circle bundles \( N \) over closed oriented surfaces of genus \( g \) with Euler number 1: Since \( \pi_1(N) \) can be generated with \( 2g + 1 \) generators \( X_1, Y_1, X_2, Y_2, \ldots, X_g, Y_g, Z \), where \( Z = [X_1, Y_1][X_2, Y_2] \cdots [X_g, Y_g] \) is central, \( H^2(N; \mathbb{Z}) \) has no torsion, and there is a representation \( \phi : \pi_1(N) \to \widetilde{SL}(2, \mathbb{R}) \) that sends \( f \) to our \( \rho_v(s_1 s_2 \cdots s_n) \in \mathbb{Z} \). Hence, \( \alpha_i \) and \( \beta_i \) can be taken as \( \phi(X_i) \) and \( \phi(Y_i) \), respectively. (Yet, another manual proof can be derived from the explicit constructions in Lemmas 7.4, 7.5, and 7.6.) Anyways, taking a commutator factorization of \( \rho(s_1 s_2 \cdots s_n) \) of length \( g \) as above, we assign \( \rho_v(x_i) = \alpha_i \) and \( \rho_v(y_i) = \beta_i \). This defines a homomorphism \( \rho_v \) of \( \pi_1(J_v) \), whose restriction to the subgroup \( \pi_1(T_{v,w_j}) = \langle s_j, f_v \rangle \) is the pull-back of \( \eta_{v,w_j} \) via the abelianization isomorphism \( \pi_1(T_{v,w_j}) \cong H_1(T_{v,w_j}; \mathbb{Z}) \).

Perform the above construction for all \( v \in V \), and obtain representations \( \rho_v : \pi_1(J_v) \to \widetilde{SL}(2, \mathbb{R}) \). Then the restricted representations of \( \rho_v \) and \( \rho_w \) to \( \pi_1(T_{v,w}) = \pi_1(T_{w,v}) \) are obviously the same for any \( \{v, w\} \in E \). As explained, we can assemble these \( \rho_v \) and obtain a representation \( \rho : \pi_1(M) \to \widetilde{SL}(2, \mathbb{R}) \) as desired.

We are ready to prove Theorem 11.1 using Lemmas 11.2 and 10.4.

Let \( M \) be a formatted graph manifold with a simplicial graph \( (V, E) \). Note that the assumption in Theorem 11.1 about \( \xi \in \mathbb{Q}^V \) implies \( \chi_M(v) < 0 \) for all \( v \in V \). Let \( m^* \) and \( r^* \) be the positive integers...
as provided by Lemma 10.4, with respect to the trivial cover \( M^* = M \). For all \( v \in V \), we simply take \( r_v \) to be \( r^* \). On the other hand, we take \( m \) to be a sufficiently large positive integral multiple of \( m^* \). To be precise, we require

\[
m \cdot \xi(v)/b_{v,w} \in \mathbb{Z}
\]

for all \( v \in V \) and \( \{v, w\} \in E \), and moreover, we require the inequality

\[
|\langle e_M \xi(v) \rangle| \leq -\chi_M(v) - \frac{1}{m} \times \text{valence}_{(V,E)}(v)
\]

for all \( v \in V \). By Lemma 10.4, there is a finite cover \( M' \to M \) satisfying the listed conditions thereof.

In particular, we have \([J'_v : J_v] = m^2 r_* \) and \([f'_v : f_v] = m \) for any covering pair of JSJ pieces \( J'_v \to J_v \), and \([T'_{v,u} : T_{v,u}] = m^2 \) for any covering pairs of JSJ tori \( T'_{v,u} \to T_{v,u} \). So, we obtain

\[
\chi_{M'}(v') = mr^* \times \chi_M(v), \quad k_{v'} = r^* \times k_v, \quad \text{valence}_{(V',E')}(v') = r^* \times \text{valence}_{(V,E)}(v),
\]

and

\[
b_{v',w'} = b_{v,w}
\]

(see Proposition 4.5).

Let \( \xi' \in \mathbb{Z}^{V'} \) be the vector defined as

\[
\xi'(v') = m \times \xi(v)
\]

for any \( v' \in V' \), lying over a vertex \( v \in V \). Hence,

\[
k_{v'} \xi'(v') - \sum_{\{v',w'\} \in E'} \frac{\xi'(w')}{b_{v',w'}} = mr^* \times k_v \xi(v) - \sum_{\{v,w\} \in E} r^* \times \frac{m \times \xi(w)}{b_{v,w}},
\]

or equivalently,

\[
\langle e_{M'} \xi'(v') \rangle = mr^* \times \langle e_M \xi(v) \rangle.
\]

It follows that

\[
|\langle e_{M'} \xi'(v') \rangle| \leq -\chi_{M'}(v') - \text{valence}_{(V',E')}(v')
\]

holds for all \( v' \in V' \). Therefore, Lemma 11.2 applies to \( M' \) and \( \xi' \). We obtain some representation \( \rho' : \pi_1(M') \to \text{SL}(2, \mathbb{R}) \), such that

\[
\text{vol}_{\text{SL}(2,\mathbb{R}) \times \mathbb{R}}(M', \rho') = 4 \pi^2 \cdot (\xi', e_{M'} \xi') = 4 \pi^2 \cdot (\xi, e_M \xi) \times [M' : M].
\]

In other words, \((M', \rho')\) is as desired.
This completes the proof of Theorem 11.1.

12 APPLICATION TO CONSTANT CYCLIC GRAPH MANIFOLDS

We apply Theorems 5.2 and 11.1 to a family of graph manifolds, and determine their covering Seifert volume (see (1.1)). As a quick consequence, we also obtain Example 1.2 (see Remark 12.2).

**Theorem 12.1.** Let $M$ be a formatted graph manifold with a cyclic JSJ graph of $n$ vertices, $n \geq 3$. Adopt Notation 4.2. Enumerate the vertices as $j$ and the edges as $\{j, j+1\}$, where $j$ ranges over $\mathbb{Z}/n\mathbb{Z}$. Suppose that there are rational numbers $\chi < 0$, $b \neq 0$, and $k$, such that

$$
\chi_j = \chi, \quad b_{j,j+1} = b, \quad k_j = k,
$$

for all $j \in \mathbb{Z}/n\mathbb{Z}$. Then

$$
\text{CSV}(M) = \begin{cases} 
4n\pi^2\chi^2/(|k| - |2/b|) & \text{if } |k| > |2/b| \\
+\infty & \text{if } |k| \leq |2/b|.
\end{cases}
$$

**Remark 12.2.** We derive Example 1.2 as follows. Since $g > 0$, any $M(g; m, 1, m^2 - 1, m)$ admits a 2-fold cyclic cover $M'$ that restricts to a connected 2-fold cover to each of the JSJ pieces $J_x$. Take $M''$ to be the 2-fold cover of $M'$ induced by the 2-fold cover of its JSJ graph. Then $M''$ will satisfy the hypothesis of Theorem 12.1, with $\chi = 2(1 - 2g)$, $b = 1$, $k = 2m$, and $n = 4$. One may compute $\text{CSV}(M(g; m, 1, m^2 - 1, m)) = \text{CSV}(M'')/4$.

Below we divide into two cases: The finite case is done in Lemma 12.3, and the infinite case in Lemma 12.4. Together they make a complete proof of Theorem 12.1.

**Lemma 12.3.** Adopt the notations of Theorem 12.1. If $|k| > |2/b|$, then $\text{CSV}(M) = 4n\pi^2\chi^2/(|k| - |2/b|)$.

**Proof.** Without loss of generality, we assume the number of vertices $n$ to be even. In fact, for any odd $n$, we may work with the 2-fold cyclic cover $M'$ of $M$ induced by that of the JSJ graph, since $\text{CSV}(M') = 2 \cdot \text{CSV}(M)$ and $n' = 2n$. We actually make use of the parity assumption only when $k, b$ have opposite signs.

Under the assumption $|k| > |2/b|$, the Euler operator $e_M$ is strongly diagonally dominant. So, we apply Theorem 10.1 to obtain $\text{CSV}(M) \leq 4n\pi^2\chi^2/(|k| - |2/b|)$. On the other hand, suppose first that $k$ and $b$ have the same sign. Note that the Euler operator $e_M \in \text{End}_\mathbb{R}(\mathbb{R}^{\mathbb{Z}/n\mathbb{Z}})$ can be explicitly written down, as

$$
(e_M\eta)(j) = k \cdot \eta(j) - \frac{\eta(j - 1) + \eta(j + 1)}{b},
$$

for all $\eta \in \mathbb{R}^{\mathbb{Z}/n\mathbb{Z}}$ and $j \in \mathbb{Z}/n\mathbb{Z}$. For any $y \in \mathbb{Q}$ and $|y| < |\chi|/|k - 2/b|$, we construct $\eta \in \mathbb{Q}^{\mathbb{Z}/n\mathbb{Z}}$ such that $\eta(j) = y$ for all $j \in \mathbb{Z}/n\mathbb{Z}$. Then Theorem 11.1 applies to this case, so we obtain $\text{CSV}(M) \geq 4n\pi^2y^2 \cdot |k - 2/b|$. The right-hand side converges to $4n\pi^2\chi^2/(|k| - |2/b|)$ as $y$ tends
to $|\chi|/|k - 2/b|$. Therefore, we obtain
\[ \text{CSV}(M) = 4n\pi^2 \chi^2 / (|k| - |2/b|) \]
as asserted when $k$ and $b$ have the same sign.

When $k$ and $b$ have different signs, for any $y \in \mathbb{Q}$ and $|y| < |\chi|/|k + 2/b|$, we construct instead $\eta(j) = (-1)^j y$. This is well defined on $\mathbb{Z}/n\mathbb{Z}$ since $n$ is even. Apply Theorem 11.1 again, and let $y$ tend to $|\chi|/|k + 2/b|$, then we obtain
\[ \text{CSV}(M) = 4n\pi^2 \chi^2 / (|k| - |2/b|) \]
as asserted when $k$ and $b$ have different signs.

\[ \square \]

**Lemma 12.4.** Adopt the notations of Theorem 12.1. If $|k| \leq |2/b|$, then $\text{CSV}(M) = +\infty$.

**Proof.** Again, we assume without loss of generality that the number of vertices $n$ is even. Let $M$ be a formatted graph manifold in Theorem 12.1. For any $d \in \mathbb{N}$, we take the $d$-cyclic cover $M_d^*$ of $M$ as induced by the $d$-cyclic cover of its JSJ graph of degree $d$. Since $\text{CSV}(M_d^*) = d \cdot \text{CSV}(M)$, it suffices to show that $\text{CSV}(M_d^*)/d$ is unbounded as $d$ increases.

To this end, enumerate the vertices of the JSJ graph of $M_d^*$ as $j \in \mathbb{Z}/nd\mathbb{Z}$, and the edges as $\{j, j+1\}$. So, the covering projection between the JSJ graphs of $M_d^*$ and $M$ takes $j \in \mathbb{Z}/nd\mathbb{Z}$ to $jm o d n \in \mathbb{Z}/n\mathbb{Z}$, and takes $\{j, j+1\}$ to $\{j \mod n, j+1 \mod n\}$. Denote by $\mathbf{e}_d \in \text{End}_{\mathbb{R}}(\mathbb{R}\mathbb{Z}/nd\mathbb{Z})$ the Euler operator of $M_d^*$. For any $\eta \in \mathbb{R}\mathbb{Z}/nd\mathbb{Z}$ and $j \in \mathbb{Z}/nd\mathbb{Z}$, we obtain
\[ (\mathbf{e}_d \eta)(j) = k \cdot \eta(j) - \frac{\eta(j - 1) + \eta(j + 1)}{b}. \]

The eigenvalues and eigenvectors of $\mathbf{e}_d$ can be explicitly determined. The eigenvalues of $\mathbf{e}_d$ are precisely
\[ \lambda_m = k - \frac{2}{b} \cdot \cos \left( \frac{2m\pi}{nd} \right) = \left( k - \frac{2}{b} \right) + \frac{4}{b} \cdot \sin^2 \left( \frac{m\pi}{nd} \right), \]
where $m$ ranges over $\{0, 1, 2, ..., nd/2\}$. The bottom eigenvalue $\lambda_0$ and the top eigenvalue $\lambda_{nd/2}$ both have multiplicity 1, and there are corresponding normalized eigenvectors
\[ \Phi_0(j) = \sqrt{1/nd}, \quad \Phi_{nd/2}(j) = (-1)^j \sqrt{1/nd}. \]
Any other eigenvalue $\lambda_m$ has multiplicity 2, with an orthonormal pair of eigenvectors
\[ \Phi_m(j) = \sqrt{2/nd} \cdot \cos(2mj\pi/nd), \quad \Psi_m(j) = \sqrt{2/nd} \cdot \sin(2mj\pi/nd). \]

These eigenvalues and eigenvectors can be checked by direct computation. We also remark that when $nd$ is odd, the spectrum of $\mathbf{e}_d$ will be the same as above except dropping the top eigenvalue, and the corresponding eigenvector will be gone. In fact, when $k = 2$ and $b = -1$, one may identify $\mathbf{e}_d$ with the graph-theoretic Laplacian $L_{nd}$ of the cyclic graph of order $nd$ (see [19, Chapter 7]). In general, $\mathbf{e}_d$ is a scalar multiple of $L_{nd}$ plus a constant.

Under the assumption $|k| \leq |2/b|$, there exists some $\lambda_m$ (other than the top or the bottom eigenvalues) such that
\[ 0 < |\lambda_m| < C_1 \cdot d^{-1}, \]
where $C_1 = |8\pi/bn|$ is independent of $d$. This is because the eigenvalues are monotonically ordered between $\lambda_0 = k - 2/b$ and $\lambda_{nd/2} = k + 2/b$, and the difference between any consecutive pair of eigenvalues is at most $|8\pi/bnd|$, by elementary estimation. Take any such $\lambda_m$, and take $\eta_d = \sqrt{nd/2 \cdot \chi \cdot \lambda_m^{-1} \cdot \Phi_m}$ in $\mathbb{R}^{Z/ndZ}$. We estimate

$$|\langle e_d \eta_d(j) \rangle| = |\lambda_m \cdot \eta_d(j)| \leqslant |\chi|,$$

for all $j \in \mathbb{Z}/nd\mathbb{Z}$.

For any $0 < \epsilon < 1$, we can approximate $(1 - \epsilon) \cdot \eta_d \in \mathbb{R}^{Z/ndZ}$ by elements in $\mathcal{Q}^{Z/ndZ}$. Therefore, Theorem 11.1 applies to the approximating elements, and yields the estimation

$$\text{CSV}(M^*_d) \geqslant 4\pi^2 \cdot \langle \eta_d, e_d \eta_d \rangle = 2nd\pi^2 \chi^2 \cdot |\lambda_m|^{-1} > C_2 \cdot d^2,$$

where $C_2 = 2n\pi^2 \chi^2 \cdot C_1^{-1}$ is independent of $d$. This shows that $\text{CSV}(M^*_d)/d$ is unbounded as $d$ increases, as desired, so the proof is complete.

**APPENDIX: NORMALIZATION OF THE VOLUME IN SEIFERT GEOMETRY**

Brooks and Goldman defined the Seifert volume in terms of the Godbillon–Vey invariant [9] of certain transversely projective codimension-1 foliation [3]. On the other hand, the representation volume associated to $\widetilde{\text{SL}}(2, \mathbb{R}) \times \mathbb{Z}$ is determined only after fixing a left-invariant volume form on the homogeneous space $\widetilde{\text{SL}}(2, \mathbb{R})$ (see [7, Section 2]). In this section of appendix, we figure out the normalization explicitly.

Recall the definition of Brooks and Goldman as follows. Suppose that $M$ is a closed oriented smooth 3-manifold and $\bar{\rho} : \pi_1(M) \to \text{PSL}(2, \mathbb{R})$ is a representation. Then, we obtain a fiber bundle $M \times_{\bar{\rho}} \mathbb{P}^1$ over $M$, whose fiber is modeled on $\mathbb{P}^1$ with structure group $\text{PSL}(2, \mathbb{R})$, so the bundle space $M \times_{\bar{\rho}} \mathbb{P}^1$ is equipped with a codimension-1 foliation $\mathcal{F}_{\bar{\rho}}$ transverse to the fibers. The Godbillon–Vey class $gv(\mathcal{F}_{\bar{\rho}})$ lives in $H^3(M \times_{\bar{\rho}} \mathbb{P}^1; \mathbb{R})$. When the Euler class of $\bar{\rho}$ is torsion (see Theorem 3.1), one may naturally identify $H_3(M; \mathbb{R})$ with a direct summand of $H_3(M \times_{\bar{\rho}} \mathbb{P}^1; \mathbb{R}) \cong H_3(M; \mathbb{R}) \oplus H_2(M; \mathbb{R})$, which corresponds to the $(0,3)$-summand of the Serre spectral decomposition. Denote by $[M'] \in H_3(M \times_{\bar{\rho}} \mathbb{P}^1; \mathbb{R})$ the image of the fundamental class of $M$. Then, the Seifert volume $\text{SV}(M)$ of $M$ in the sense of Brooks and Goldman is defined as the maximum of $|\langle gv(\mathcal{F}_{\bar{\rho}}), [M'] \rangle|$ where $\bar{\rho}$ ranges over all the $\text{PSL}(2, \mathbb{R})$-representations of $\pi_1(M)$ whose Euler class is torsion (see [3, §3]).

We identify the Lie algebra of $\widetilde{\text{SL}}(2, \mathbb{R})$ as $\mathfrak{sl}(2, \mathbb{R})$, consisting of all traceless real $2 \times 2$-matrices. Denote by

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

the usual basis of $\mathfrak{sl}(2, \mathbb{R})$. They satisfy the relations $[H, E] = 2E$, $[H, F] = -2F$, and $[E, F] = H$. Denote by $\hat{E}, \hat{F}, \hat{H}$ the dual basis of $\mathfrak{sl}(2, \mathbb{R})^\vee = \text{Hom}_\mathbb{R}(\mathfrak{sl}(2, \mathbb{R}), \mathbb{R})$. The exterior form $\hat{H} \wedge \hat{E} \wedge \hat{F}$ can be regarded as an alternating trilinear function on $\mathfrak{sl}(2, \mathbb{R})$ such that $\hat{H} \wedge \hat{E} \wedge \hat{F}(H, E, F) = 1$, and it can also be naturally regarded as a left-invariant differential 3-form on $\widetilde{\text{SL}}(2, \mathbb{R})$. 
**Proposition A.1.** The Seifert volume in the sense of Brooks and Goldman coincides with the representation volume with respect to the triple \((G, X, \omega_X)\) as defined in [7], where

\[ G = \widetilde{SL}(2, \mathbb{R}) \times \mathbb{R}, \quad X = \widetilde{SL}(2, \mathbb{R}), \quad \omega_X = \pm 4H \wedge \hat{E} \wedge \hat{F}. \]

**Proof.** For any oriented closed hyperbolic surface \(\Sigma\), the unit vector bundle \(UT(\Sigma)\) is an oriented circle bundle over \(\Sigma\) of Euler number \(\chi(\Sigma) < 0\). The Seifert volume of \(UT(\Sigma)\) equals \(-4\pi^2 \chi(\Sigma)\), according to the definition of Brook and Goldman; moreover, it is realized by any discrete faithful representation \(\bar{\rho} : \pi_1(UT(\Sigma)) \to \widetilde{SL}(2, \mathbb{R})\), which must be central and induces a holonomy representation \(\pi_1 \Sigma \to \text{PSL}(2, \mathbb{R})\). (See [2, Proposition 2] and [3, Theorem 3]; or Theorem 3.2). As the discrete faithful representation realizes \(UT(\Sigma)\) as a \(\widetilde{SL}(2, \mathbb{R})\)-geometric manifold, it suffices to check that the asserted invariant volume form \(4H \wedge \hat{E} \wedge \hat{F}\) on \(\widetilde{SL}(2, \mathbb{R})\) results in the same volume value \(-4\pi^2 \chi(\Sigma)\) for \(UT(\Sigma)\).

To this end, we observe

\[
e^{tH} = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}, \quad e^{t(E+F)} = \begin{bmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{bmatrix}, \quad e^{t(E-F)} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}.
\]

Using the upper-half complex plane model of the hyperbolic plane \(\mathbb{H}^2\), it follows that the tangent map of \(PSL(2, \mathbb{R}) \to \mathbb{H}^2 : g \mapsto g\cdot i\) takes \(H\) and \(E + F\) to the tangent vectors \(2i\) and \(2\) at \(i\), respectively. (Here, we canonically identify the tangent space of \(\mathbb{C}\) at \(i\) with \(\mathbb{C}\).) The parametrized subgroup \(e^{t(E-F)}\) rotates any unit vector at \(i\) counterclockwise at constant angular speed 2. Therefore, if \(\mathfrak{sl}(2, \mathbb{R})\) is endowed with an inner product such that \(H/2, (E + F)/2, (E - F)/2\) form an orthonormal basis, then the induced left-invariant Riemannian metric on \(PSL(2, \mathbb{R})\) will have constant total length \(2\pi\) for all left-cosets of compact subgroup \(K = \text{SO}(2)/\{\pm 1\}\), and the homogeneous space \(PSL(2, \mathbb{R})/K\) with the induced left-invariant metric will be isometric to \(\mathbb{H}^2\).

This means that the desired normalized volume form \(\omega_{\widetilde{SL}(2, \mathbb{R})}\) must make \(|\omega_{\widetilde{SL}(2, \mathbb{R})}(H/2, (E + F)/2, (E - F)/2)| = 1. Since \(H \wedge \hat{E} \wedge \hat{F}(H/2, (E + F)/2, (E - F)/2) = -1/4\), we obtain \(\omega_{\widetilde{SL}(2, \mathbb{R})} = \pm 4H \wedge \hat{E} \wedge \hat{F}\), as asserted. \(\square\)

The sign ambiguity is inessential. It arises only because the Seifert volume is insensitive of orientation. However, it can be resolved if we orient the unit vector bundle \(M = UT(\Sigma)\) of any closed oriented hyperbolic surface \(\Sigma\) by a fixed orientation of \(PSL(2, \mathbb{R}) \cong UT(\mathbb{H}^2)\), and if we require \(\text{vol}_{G,X,\omega_X}(M) = \langle \text{gv}(\bar{\rho}_\beta), [M] \rangle\), where \(\bar{\rho} : \pi_1(M) \to \text{PSL}(2, \mathbb{R})\) is induced by any holonomy representation \(\rho : \pi_1(\Sigma) \to \widetilde{SL}(2, \mathbb{R}) \times \mathbb{R}\). This way \(\omega_X\) must agree with the fixed orientation of \(PSL(2, \mathbb{R})\), so the positive sign must apply if we orient \(\mathfrak{sl}(2, \mathbb{R})\) with the ordered basis \(H, E, F\).

In conclusion, we may (and do) fix the invariant volume form \(\omega_X\) to be \(4H \wedge \hat{E} \wedge \hat{F}\).

**Acknowledgments**

The authors thank Yao Fan, Hongbin Sun, and Ran Tao for helpful comments on preliminary versions of this paper. The authors thank the anonymous referee for many valuable comments, especially the correction of a formula in Proposition 4.5, and the suggestion regarding an incorrect formulation of Notation 5.6 in a draft version.

The author Y.L. is partially supported by NSFC Grant 11925101, and National Key R&D Program of China 2020YFA0712800.
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