A GENERALIZATION OF RICKART MODULES

BURCU UNGOR, SAIT HALICIÖGLU, AND ABDULLAH HARMANCI

Abstract. Let \( R \) be an arbitrary ring with identity and \( M \) a right \( R \)-module with \( S = \text{End}_R(M) \). In this paper we introduce \( \pi \)-Rickart modules as a generalization of generalized right principally projective rings as well as that of Rickart modules. The module \( M \) is called \( \pi \)-Rickart if for any \( f \in S \), there exist \( e^2 = e \in S \) and a positive integer \( n \) such that \( r_M(f^n) = eM \). We prove that several results of Rickart modules can be extended to \( \pi \)-Rickart modules for this general setting, and investigate relations between a \( \pi \)-Rickart module and its endomorphism ring.

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1. Introduction

Throughout this paper \( R \) denotes an associative ring with identity, and modules are unitary right \( R \)-modules. For a module \( M \), \( S = \text{End}_R(M) \) is the ring of all right \( R \)-module endomorphisms of \( M \). In this work, for the \((S, R)\)-bimodule \( M \), \( l_S(.) \) and \( r_M(.) \) are the left annihilator of a subset of \( M \) in \( S \) and the right annihilator of a subset of \( S \) in \( M \), respectively. A ring is called reduced if it has no nonzero nilpotent elements. By considering the right \( R \)-module \( M \) as an \((S, R)\)-bimodule the reduced ring concept was considered for modules in [1]. The module \( M \) is called reduced if for any \( f \in S \) and \( m \in M \), \( fm = 0 \) implies \( fM \cap Sm = 0 \). In [9] Baer rings are introduced as rings in which the right (left) annihilator of every nonempty subset is generated by an idempotent. Principally projective rings were introduced by Hattori [4] to study the torsion theory, that is, a ring is called left (right) principally projective if every principal left (right) ideal is projective. The concept of left (right) principally projective rings (or left (right) Rickart rings) has
been comprehensively studied in the literature. Regarding a generalization of Baer rings as well as principally projective rings, recall that a ring $R$ is called \emph{generalized left (right) principally projective} if for any $x \in R$, the left (right) annihilator of $x^n$ is generated by an idempotent for some positive integer $n$. A number of papers have been written on generalized principally projective rings (see [5] and [8]). According to Rizvi and Roman, an $R$-module $M$ is called \emph{Baer} if for any $R$-submodule $N$ of $M$, $l_S(N) = Se$ with $e^2 = e \in S$, while the module $M$ is said to be \emph{Rickart} if for any $f \in S$, $r_M(f) = eM$ for some $e^2 = e \in S$. Recently, Rickart modules are studied extensively by different authors (see [1] and [12]).

In what follows, we denote by $\mathbb{Z}$ and $\mathbb{Z}_n$ integers and the ring of integers modulo $n$, respectively, and $J(R)$ denotes the Jacobson radical of a ring $R$.

2. $\pi$-Rickart Modules

In this section, we introduce the concept of $\pi$-Rickart modules. We supply an example to show that all $\pi$-Rickart modules need not be Rickart. Although every direct summand of a $\pi$-Rickart module is $\pi$-Rickart, we present an example to show that a direct sum of $\pi$-Rickart modules is not $\pi$-Rickart. It is shown that the class of some abelian $\pi$-Rickart modules is closed under direct sums. We begin with our main definition.

**Definition 2.1.** Let $M$ be an $R$-module with $S = \text{End}_R(M)$. The module $M$ is called \emph{$\pi$-Rickart} if for any $f \in S$, there exist $e^2 = e \in S$ and a positive integer $n$ such that $r_M(f^n) = eM$.

For the sake of brevity, in the sequel, $S$ will stand for the endomorphism ring of the module $M$ considered.

**Remark 2.2.** $R$ is a $\pi$-Rickart $R$-module if and only if $R$ is a generalized right principally projective ring.

Every module of finite length, every semisimple, every nonsingular injective (or extending) and every Baer module is a $\pi$-Rickart module. Also every quasi-projective strongly co-Hopfian module, every quasi-injective strongly Hopfian module, every Artinian and Noetherian module is $\pi$-Rickart (see Corollary 2.29). Every finitely generated module over a right Artinian ring
is $\pi$-Rickart (see Proposition 2.31), every free module which its endomorphism ring is generalized right principally projective is $\pi$-Rickart (see Corollary 3.5), every finitely generated projective regular module is $\pi$-Rickart (see Corollary 3.7) and every finitely generated projective module over a commutative $\pi$-regular ring is $\pi$-Rickart (see Proposition 3.11).

One may suspect that every $\pi$-Rickart module is Rickart. But the following example illustrates that this is not the case.

**Example 2.3.** Consider $M = \mathbb{Z} \oplus \mathbb{Z}_2$ as a $\mathbb{Z}$-module. It can be easily determined that $S = \text{End}_\mathbb{Z}(M)$ is \[
\begin{bmatrix}
\mathbb{Z} & 0 \\
\mathbb{Z}_2 & \mathbb{Z}_2
\end{bmatrix}.
\] For any $f = \begin{bmatrix} a & 0 \\
b & c \end{bmatrix} \in S$, consider the following cases.

Case 1. Assume that $a = 0$, $b = \overline{0}$, $c = \overline{1}$ or $a = 0$, $b = c = \overline{1}$. In both cases $f$ is an idempotent, and so $r_M(f) = (1 - f)M$.

Case 2. If $a \neq 0$, $b = \overline{0}$, $c = \overline{1}$ or $a \neq 0$, $b = c = \overline{1}$, then $r_M(f) = 0$.

Case 3. If $a \neq 0$, $b = c = \overline{0}$ or $a \neq 0$, $b = \overline{1}$, $c = \overline{0}$, then $r_M(f) = 0 \oplus \mathbb{Z}_2$.

Case 4. If $a = 0$, $b = \overline{1}$, $c = \overline{0}$, then $f^2 = 0$. Hence $r_M(f^2) = M$.

Therefore $M$ is a $\pi$-Rickart module, but it is not Rickart by [12].

Our next endeavor is to find conditions under which a $\pi$-Rickart module is Rickart. We show that reduced rings play an important role in this direction.

**Proposition 2.4.** If $M$ is a Rickart module, then it is $\pi$-Rickart. The converse holds if $S$ is a reduced ring.

**Proof.** The first assertion is clear. For the second, let $f \in S$. Since $M$ is $\pi$-Rickart, $r_M(f^n) = eM$ for some positive integer $n$ and $e^2 = e \in S$. If $n = 1$, then there is nothing to do. Assume that $n > 1$. Since $S$ is a reduced ring, $e$ is central and so $(fe)^n = 0$. It follows that $fe = 0$. Hence $eM \leq r_M(f)$. On the other hand, always $r_M(f) \leq r_M(f^n) = eM$. Therefore $M$ is Rickart. □

Reduced modules are studied in [1] and it is shown that if $M$ is a reduced module, then $S$ is a reduced ring. Hence we have the following.

**Corollary 2.5.** If $M$ is a reduced module, then it is Rickart if and only if it is $\pi$-Rickart.

We obtain the following well known result (see [8 Lemma 1]).
Corollary 2.6. Let $R$ be a reduced ring. Then $R$ is a right Rickart ring if and only if $R$ is a generalized right principally projective ring.

Lemma 2.7. If $M$ is a $\pi$-Rickart module, then every non-nil left annihilator in $S$ contains a nonzero idempotent.

Proof. Let $I = l_S(N)$ be a non-nil left annihilator where $\emptyset \neq N \subseteq M$ and choose $f \in I$ be a non-nilpotent element. As $M$ is $\pi$-Rickart, $r_M(f^n) = eM$ for some idempotent $e \in S$ and a positive integer $n$. In addition $e \neq 1$. Due to $r_M(I) \subseteq r_M(f^n)$, we have $(1 - e)r_M(I) = 0$. It follows that $1 - e \in l_S(r_M(I)) = l_S(r_M(l_S(N))) = l_S(N) = I$. This completes the proof. □

We now give a relation among $\pi$-Rickart modules, Rickart modules and Baer modules by using Lemma 2.7.

Theorem 2.8. Let $M$ be a module. If $S$ has no infinite set of nonzero orthogonal idempotents and $J(S) = 0$, then the following are equivalent.

1. $M$ is a $\pi$-Rickart module.
2. $M$ is a Rickart module.
3. $M$ is a Baer module.

Proof. It is enough to show that (1) implies (3). Consider any left annihilator $I = l_S(N)$ where $\emptyset \neq N \subseteq M$. If $I$ is nil, then $I \subseteq J(S)$, and so $I = 0$. Thus we may assume that $I$ is not nil. By \[10\], Proposition 6.59, $S$ satisfies DCC on left direct summands, and so among all nonzero idempotents in $I$, choose $e \in I$ such that $S(1-e) = l_S(eM)$ is minimal. We claim that $I \cap l_S(eM) = 0$. Note that $I \cap l_S(eM) = l_S(N \cup eM)$. If $I \cap l_S(eM)$ is nil, then there is nothing to do. Now we assume that $I \cap l_S(eM)$ is not nil. If $I \cap l_S(eM) \neq 0$, then there exists $0 \neq f = f^2 \in I \cap l_S(eM)$ by Lemma 2.4. Since $fe = 0$, $e + (1 - e)f \in I$ is an idempotent, say $g = e + (1 - e)f$. Then $ge = e$, and so $g \neq 0$. Also $fg = f$. This implies that $l_S(gM) \subseteq l_S(eM)$. This contradicts to the choice of $e$. Hence $I \cap l_S(eM) = 0$. Due to $\varphi(1 - e) \in I \cap l_S(eM)$ for any $\varphi \in I$, we have $\varphi = \varphi e$. Thus $I \subseteq Se$, and clearly $Se = I = l_S(N)$. Therefore $M$ is Baer. □

Corollary 2.9. Let $R$ be a ring. If $R$ has no infinite set of nonzero orthogonal idempotents and $J(R) = 0$, then the following are equivalent.

1. $R$ is a generalized right principally projective ring.
(2) $R$ is a right Rickart ring.
(3) $R$ is a Baer ring.

**Corollary 2.10.** Let $M$ be a module. If $S$ is a semisimple ring, then the following are equivalent.

1. $M$ is a $\pi$-Rickart module.
2. $M$ is a Rickart module.
3. $M$ is a Baer module.

**Proof.** Since $S$ is semisimple, we have $J(S) = 0$ and $S$ is left Artinian. Then $S$ has no infinite set of nonzero orthogonal idempotents by [10, Proposition 6.59]. Hence Theorem 2.8 completes the proof. □

**Corollary 2.11.** If $M$ is Noetherian (Artinian) and $J(S) = 0$, then the following are equivalent.

1. $M$ is a $\pi$-Rickart module.
2. $M$ is a Rickart module.
3. $M$ is a Baer module.

**Proof.** $S$ has no infinite set of nonzero orthogonal idempotents in case $M$ is either Noetherian or Artinian. The rest is clear from Theorem 2.8. □

Modules which contain $\pi$-Rickart modules need not be $\pi$-Rickart, as the following example shows.

**Example 2.12.** Let $R$ denote the ring \[ \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix} \] and $M$ the right $R$-module \[ \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix} \]. Let $f \in S$ be defined by $f \begin{bmatrix} x & y \\ r & s \end{bmatrix} = \begin{bmatrix} 2x + 3r & 2y + 3s \\ 0 & 0 \end{bmatrix}$, where $\begin{bmatrix} x & y \\ r & s \end{bmatrix} \in M$. Then $r_M(f) = \left\{ \begin{bmatrix} 3k & 3z \\ -2k & -2z \end{bmatrix} : k, z \in \mathbb{Z} \right\}$. Since $r_M(f)$ is not a direct summand of $M$ and $r_M(f^n) = r_M(f^n)$ for any integer $n \geq 2$, $M$ is not a $\pi$-Rickart module. On the other hand, consider the submodule $N = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{bmatrix}$ of $M$. Then $\text{End}_R(N) = \begin{bmatrix} \mathbb{Z} & 0 \\ 0 & 0 \end{bmatrix}$. It is easy to show that $N$ is a Rickart module and so it is $\pi$-Rickart.

In [1] Proposition 2.4, it is shown that every direct summand of a Rickart module is Rickart. We now prove that every direct summand of a $\pi$-Rickart module inherits this property.
Proposition 2.13. Every direct summand of a \( \pi \)-Rickart module is \( \pi \)-Rickart.

Proof. Let \( M = N \oplus P \) be an \( R \)-module with \( S = \text{End}_R(M) \) and \( S_N = \text{End}_R(N) \). For any \( f \in S_N \), define \( g = f \oplus 0 \mid_P \) and so \( g \in S \). By hypothesis, there exist a positive integer \( n \) and \( e^2 = e \in S \) such that \( r_M(g^n) = eM \) and \( g^n = f^n \oplus 0 \mid_P \). Let \( M = eM \oplus Q \). Since \( P \subseteq eM \), there exists \( L \leq eM \) such that \( eM = P \oplus L \). So we have \( M = eM \oplus Q = P \oplus L \oplus Q \).

Let \( \pi_N : M \to N \) be the projection of \( M \) onto \( N \). Then \( \pi_N \mid_{Q \oplus L} : Q \oplus L \to N \) is an isomorphism. Hence \( N = \pi_N(Q) \oplus \pi_N(L) \). We claim that \( r_N(f^n) = \pi_N(L) \). We get \( g^n(L) = 0 \) since \( g^n(P \oplus L) = 0 \). But for all \( l \in L \), \( l = \pi_N(l) + \pi_P(l) \). Since \( g^n(l) = g^n\pi_N(l) + g^n\pi_P(l) \) and \( g^n(l) = 0 \) and \( g^n\pi_P(l) = 0 \) and \( g^n\pi_N(l) = f^n\pi_N(l) \), we have \( f^n\pi_N(L) = 0 \) and so \( \pi_N(L) \subseteq r_N(f^n) \).

For the reverse inclusion, let \( n \in r_N(f^n) \). Assume that \( n \notin \pi_N(L) \) and we reach a contradiction. Then \( n = n_1 + n_2 \) for some \( n_1 \in \pi_N(L) \) and some \( 0 \neq n_2 \in \pi_N(Q) \) and so there exists a \( q \in Q \) such that \( \pi_N(q) = n_2 \). Since \( Q \cap r_M(g^n) = 0 \), we have \( g^n(q) = (f^n \oplus 0 \mid_P)(q) = 0 \). But \( q = \pi_N(q) + \pi_P(q) \) and \( g^n\pi_P(q) = (f^n \oplus 0 \mid_P)\pi_P(q) = 0 \), we get \( f^n(q) = g^n(q) = f^n\pi_N(q) = 0 \). This implies \( n \notin r_N(f^n) \) which is the required contradiction. Hence \( r_N(f^n) \leq \pi_N(L) \). Therefore \( r_N(f^n) = \pi_N(L) \). \( \Box \)

Corollary 2.14. Let \( R \) be a generalized right principally projective ring with any idempotent \( e \) of \( R \). Then \( eR \) is a \( \pi \)-Rickart module.

Corollary 2.15. Let \( R = R_1 \oplus R_2 \) be a generalized right principally projective ring with direct sum of the rings \( R_1 \) and \( R_2 \). Then the rings \( R_1 \) and \( R_2 \) are generalized right principally projective.

We now characterize generalized right principally projective rings in terms of \( \pi \)-Rickart modules.

Theorem 2.16. Let \( R \) be a ring. Then \( R \) is generalized right principally projective if and only if every cyclic projective \( R \)-module is \( \pi \)-Rickart.

Proof. The sufficiency is clear. For the necessity, let \( M \) be a cyclic projective \( R \)-module. Then \( M \cong I \) for some direct summand right ideal \( I \) of \( R \). By Remark 2.2, \( R \) is \( \pi \)-Rickart as an \( R \)-module and by Proposition 2.13, \( I \) is \( \pi \)-Rickart, and so is \( M \). \( \Box \)
Theorem 2.17. Let $R$ be a ring and consider the following conditions.

1. Every free $R$-module is $\pi$-Rickart.
2. Every projective $R$-module is $\pi$-Rickart.
3. Every flat $R$-module is $\pi$-Rickart.

Then (3) $\Rightarrow$ (2) $\Leftrightarrow$ (1). Also (2) $\Rightarrow$ (3) holds for finitely presented modules.

Proof. (3) $\Rightarrow$ (2) $\Rightarrow$ (1) Clear. (1) $\Rightarrow$ (2) Let $M$ be a projective $R$-module. Then $M$ is a direct summand of a free $R$-module $F$. By (1), $F$ is $\pi$-Rickart, and so is $M$ due to Proposition 2.13.

(2) $\Rightarrow$ (3) is clear from the fact that finitely presented flat modules are projective. $\square$

Lemma 2.18. Let $M$ be a module and $f \in S$. If $r_M(f^n) = eM$ for some central idempotent $e \in S$ and a positive integer $n$, then $r_M(f^{n+1}) = eM$.

Proof. It is clear that $r_M(f^n) \leq r_M(f^{n+1})$. For the reverse inclusion, let $m \in r_M(f^{n+1})$. Then $fm \in r_M(f^n) = eM$, and so $fm = efm$. Since $e$ is central, $f^nm = f^{n-1}fm = f^{n-1}efm = f^{n-1}fem = f^nem = 0$. Hence $m \in r_M(f^n)$ and so $r_M(f^{n+1}) \leq r_M(f^n)$. $\square$

The next example reveals that a direct sum of $\pi$-Rickart modules need not be $\pi$-Rickart.

Example 2.19. Let $R$ denote the ring $\left[ \begin{array}{cc} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{array} \right]$ and $M$ the right $R$-module $\left[ \begin{array}{ccc} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{array} \right]$. Consider the submodules $N = \left[ \begin{array}{cc} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{array} \right]$ and $K = \left[ \begin{array}{cc} 0 & 0 \\ \mathbb{Z} & \mathbb{Z} \end{array} \right]$ of $M$. It is easy to check that every nonzero endomorphism of $N$ and $K$ is a monomorphism. Therefore $N$ and $K$ are $\pi$-Rickart modules but, as was claimed in Example 2.12, $M = N \oplus K$ is not $\pi$-Rickart.

A ring $R$ is called abelian if every idempotent is central, that is, $ae = ea$ for any $a, e^2 = e \in R$. A module $M$ is called abelian [17] if $fem = efm$ for any $f \in S$, $e^2 = e \in S$, $m \in M$. Note that $M$ is an abelian module if and only if $S$ is an abelian ring. In [8, Proposition 7], it is shown that the class of abelian generalized right principally projective rings is closed under direct sums. We extend this result as follows.
Proposition 2.20. Let $M_1$ and $M_2$ be $\pi$-Rickart $R$-modules. If $M_1$ and $M_2$ are abelian and $\text{Hom}_R(M_i, M_j) = 0$ for $i \neq j$, then $M = M_1 \oplus M_2$ is a $\pi$-Rickart module.

Proof. Let $S_i = \text{End}_R(M_i)$ for $i = 1, 2$ and $S = \text{End}_R(M)$. We may write $S$ as $\begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}$. Let $\begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix} \in S$ with $f_1 \in S_1$ and $f_2 \in S_2$. Then there exist positive integers $n, m$ and $e_1^2 = e_1 \in S_1$, $e_2^2 = e_2 \in S_2$ with $r_{M_1}(f_1^n) = e_1 M_1$ and $r_{M_2}(f_2^m) = e_2 M_2$. Consider the following cases:

(i) If $n = m$, then obviously $r_M(\begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}^m) = \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} M$.

(ii) If $n < m$, then by Lemma 2.18 we have $r_{M_1}(f_1^n) = r_{M_1}(f_1^m) = e_1 M_1$.

Thus $\begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}^m \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} = 0$, and so $\begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} M \leq r_M(\begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}^m)$.

Now let $\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \in r_M(\begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix})$. Then $m_1 \in r_{M_1}(f_1^m) = e_1 M_1$ and $m_2 \in r_{M_2}(f_2^m) = e_2 M_2$. Hence $\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$. Thus $\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \in e_1 M$. Therefore $r_M(\begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}^m) \leq \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} M$.

(iii) If $m < n$, then the proof is similar to case (ii), since $M_2$ is abelian. $\square$

Recall that a module $M$ is called duo if every submodule of $M$ is fully invariant, i.e., for a submodule $N$ of $M$, $f(N) \leq N$ for each $f \in S$. Our next aim is to find some conditions under which a fully invariant submodule of a $\pi$-Rickart module is also $\pi$-Rickart.

Lemma 2.21. Let $M$ be a module and $N$ a fully invariant submodule of $M$. If $M$ is $\pi$-Rickart and every endomorphism of $N$ can be extended to an endomorphism of $M$, then $N$ is $\pi$-Rickart.

Proof. Let $S = \text{End}_R(M)$ and $f \in \text{End}_R(N)$. By hypothesis, there exists $g \in S$ such that $g|_N = f$ and being $M$ $\pi$-Rickart, there exist a positive integer $n$ and an idempotent $e$ of $S$ such that $r_M(g^n) = e M$. Then $r_N(f^n) = N \cap r_M(g^n)$. Since $N$ is fully invariant, we have $r_N(f^n) = e N$, and so $r_N(f^n)$ is a direct summand of $N$. Therefore $N$ is $\pi$-Rickart. $\square$

The following result is an immediate consequence of Lemma 2.21.
Proposition 2.22. Let $M$ be a quasi-injective module and $E(M)$ denote the injective hull of $M$. If $E(M)$ is $\pi$-Rickart, then so is $M$.

Theorem 2.23. Let $M$ be a quasi-injective duo module. If $M$ is $\pi$-Rickart, then every submodule of $M$ is $\pi$-Rickart.

Proof. Let $M$ be a $\pi$-Rickart module and $N$ a submodule of $M$ and $f \in \text{End}_R(N)$. By quasi-injectivity of $M$, $f$ extends to an endomorphism $g$ of $M$. Then $r_M(g^n) = eM$ for some positive integer $n$ and $e^2 = e \in S$. Since $N$ is fully invariant under $g$, the proof follows from Lemma 2.21. □

Rizvi and Roman [15] introduced that the module $M$ is $K$-nonsingular if for any $f \in S$, $r_M(f)$ is essential in $M$ implies $f = 0$. They proved that every Rickart module is $K$-nonsingular. In order to obtain a similar result for $\pi$-Rickart modules, we now give a generalization of the notion of $K$-nonsingularity. The module $M$ is called generalized $K$-nonsingular, if $r_M(f)$ is essential in $M$ for any $f \in S$, then $f$ is nilpotent. It is clear that every $K$-nonsingular module is generalized $K$-nonsingular. The converse holds if the module is rigid. A ring $R$ is called $\pi$-regular if for each $a \in R$ there exist a positive integer $n$ and an element $x$ in $R$ such that $a^n = a^n xa^n$.

Lemma 2.24. Let $M$ be a module. If $S$ is a $\pi$-regular ring, then $M$ is generalized $K$-nonsingular.

Proof. Let $f \in S$ with $r_M(f)$ essential in $M$. By hypothesis, there exist a positive integer $n$ and $g \in S$ such that $f^n = f^n gf^n$. Then $gf^n$ is an idempotent of $S$ and so $r_M(f^n)$ is a direct summand of $M$. Since $r_M(f)$ is essential in $M$, $r_M(f^n)$ is also essential in $M$. Hence $r_M(gf^n) = M$ and so $gf^n = 0$. Therefore $f^n gf^n = f^n = 0$. □

Proposition 2.25. Every $\pi$-Rickart module is generalized $K$-nonsingular.

Proof. Let $M$ be a $\pi$-Rickart module and $f \in S$ with $r_M(f)$ essential in $M$. Then $r_M(f^n) = eM$ for some $e^2 = e \in S$ and a positive integer $n$. Hence $r_M(f^n)$ is essential in $M$. Thus $r_M(f^n) = M$ and so $f^n = 0$. □

Corollary 2.26. If $R$ is a generalized right principally projective ring, then $R$ is generalized $K$-nonsingular as an $R$-module.
Our next purpose is to find out the conditions when a \( \pi \)-Rickart module \( M \) is torsion-free as an \( S \)-module. So we consider the set \( T(\_S M) = \{ m \in M \mid fm = 0 \text{ for some nonzero } f \in S \} \) of all torsion elements of a module \( M \) with respect to \( S \). The subset \( T(\_S M) \) of \( M \) need not be a submodule of the modules \( S M \) and \( M_R \) in general. If \( S \) is a commutative domain, then \( T(\_S M) \) is an \( (S, R) \)-submodule of \( M \).

**Proposition 2.27.** Let \( M \) be a module with a commutative domain \( S \). If \( M \) is \( \pi \)-Rickart, then \( T(\_S M) = 0 \) and every nonzero element of \( S \) is a monomorphism.

**Proof.** Let \( 0 \neq f \in S \). Then there exist a positive integer \( n \) and \( e^2 = e \in S \) such that \( r_M(f^n) = eM \). Hence \( f^n e = 0 \). Since \( S \) is a domain, we have \( e = 0 \) and so \( r_M(f^n) = 0 \). This implies that \( \text{Ker} f = 0 \). Thus \( f \) is a monomorphism. On the other hand, if \( m \in T(\_S M) \) there exists \( 0 \neq f \in S \) such that \( fm = 0 \). Being \( f \) a monomorphism, we have \( m = 0 \), and so \( T(\_S M) = 0 \). \( \square \)

We close this section with the relations among strongly Hopfian modules, Fitting modules and \( \pi \)-Rickart modules. Recall that a module \( M \) is called **Hopfian** if every surjective endomorphism of \( M \) is an automorphism, while \( M \) is called **strongly Hopfian** \([6]\) if for any endomorphism \( f \) of \( M \) the ascending chain \( \text{Ker} f \subseteq \text{Ker} f^2 \subseteq \cdots \subseteq \text{Ker} f^n \subseteq \cdots \) stabilizes. We now give a relation between abelian and strongly Hopfian modules by using \( \pi \)-Rickart modules.

**Corollary 2.28.** Every abelian \( \pi \)-Rickart module is strongly Hopfian.

**Proof.** It follows from Lemma \[2.18\] and \[6\] Proposition 2.5]. \( \square \)

A module \( M \) is said to be a **Fitting module** \([6]\) if for any \( f \in S \), there exists an integer \( n \geq 1 \) such that \( M = \text{Ker} f^n \oplus \text{Im} f^n \). A ring \( R \) is called **strongly \( \pi \)-regular** if for every element \( a \) of \( R \) there exist a positive integer \( n \) (depending on \( a \)) and an element \( x \) of \( R \) such that \( a^n = a^{n+1}x \), equivalently, an element \( y \) of \( R \) such that \( a^n = ya^{n+1} \). Due to Armendariz, Fisher and Snider \([2]\), the module \( M \) is a Fitting module if and only if \( S \) is a strongly \( \pi \)-regular ring. In this direction we have the following result.

**Corollary 2.29.** Every Fitting module is a \( \pi \)-Rickart module.

**Corollary 2.30.** Let \( R \) be a ring and let \( n \) be a positive integer. If the matrix ring \( M_n(R) \) is strongly \( \pi \)-regular, then \( R^n \) is a \( \pi \)-Rickart \( R \)-module.
Proof. Let $M_n(R)$ be a strongly $\pi$-regular ring. Then by [6, Corollary 3.6], $R^n$ is a Fitting $R$-module and so it is $\pi$-Rickart. □

The following provides another source of examples for $\pi$-Rickart modules.

**Proposition 2.31.** Every finitely generated module over a right Artinian ring is $\pi$-Rickart.

**Proof.** Let $R$ be a right Artinian ring and $M$ a finitely generated $R$-module. Then $M$ is an Artinian and Noetherian module. Hence $M$ is a Fitting module. Thus Corollary 2.29 completes the proof. □

3. THE ENDOМОРPHISM RING OF A $\pi$-RICKАRT MODULE

In this section we study some relations between $\pi$-Rickart modules and their endomorphism rings. We prove that endomorphism ring of a $\pi$-Rickart module is always generalized right principally projective, the converse holds either the module is flat over its endomorphism ring or it is 1-epiretractable. Also modules whose endomorphism rings are $\pi$-regular are characterized.

**Lemma 3.1.** If $M$ is a $\pi$-Rickart module, then $S$ is a generalized right principally projective ring.

**Proof.** If $f \in S$, then $r_M(f^n) = eM$ for some $e^2 = e \in S$ and positive integer $n$. If $g \in r_S(f^n)$, then $gM \leq r_M(f^n) = eM$. This implies that $g = eg \in eS$, and so $r_S(f^n) \leq eS$. Let $h \in S$. Due to $f^n ehM \leq f^n eM = 0$, we have $f^n eh = 0$. Hence $eS \leq r_S(f^n)$. Therefore $r_S(f^n) = eS$. □

**Corollary 3.2.** Let $M$ be a module and $f \in S$. If $r_M(f^n)$ is a direct summand of $M$ for some positive integer $n$, then $f^n S$ is a projective right $S$-module.

The next result (see [8, Proposition 9]) is a consequence of Theorem 2.13 and Lemma 3.1.

**Corollary 3.3.** If $R$ is a generalized right principally projective ring, then so is $eRe$ for any $e^2 = e \in R$.

A module $M$ is called $n$-epiretractable [3] if every $n$-generated submodule of $M$ is a homomorphic image of $M$. We now show that 1-epiretractable modules allow us to get the converse of Lemma 3.1.
Proposition 3.4. Let $M$ be a 1-epiretractable module. Then $M$ is $\pi$-Rickart if and only if $S$ is a generalized right principally projective ring.

Proof. The necessity holds from Lemma 3.1. For the sufficiency, let $f \in S$. Since $S$ is generalized right principally projective, there exist a positive integer $n$ and $e^2 = e \in S$ such that $r_S(f^n) = eS$. Then $f^n e = 0$, and so $eM \leq r_M(f^n)$. In order to show the reverse inclusion, let $0 \neq m \in r_M(f^n)$. Being $M$ 1-epiretractable, there exists $0 \neq g \in S$ with $gM = mR$, and so $m = gm_1$ for some $m_1 \in M$. On the other hand, $f^n gM = f^n mR = 0$, and so $f^n g = 0$. Thus $g \in r_S(f^n) = eS$. It follows that $g = eg$. Hence we have $m = gm_1 = egm_1 = em \in eM$. Therefore $r_M(f^n) = eM$. □

Corollary 3.5. A free module is $\pi$-Rickart if and only if its endomorphism ring is generalized right principally projective.

A module $M$ is called regular (in the sense of Zelmanowitz [21]) if for any $m \in M$ there exists a right $R$-homomorphism $M \xrightarrow{\phi} R$ such that $m = m\phi(m)$. Every cyclic submodule of a regular module is a direct summand, and so it is 1-epiretractable. Then we have the following result.

Corollary 3.6. Let $M$ be a regular $R$-module. Then $S$ is generalized right principally projective if and only if $M$ is $\pi$-Rickart.

Corollary 3.7. Every finitely generated projective regular module is $\pi$-Rickart.

Proof. Let $M$ be a finitely generated projective regular module. By [19, Theorem 3.6], the endomorphism ring of $M$ is a generalized right principally projective ring. Hence, by Corollary 3.6 $M$ is $\pi$-Rickart. □

Let $\mathcal{U}$ be a nonempty set of $R$-modules. Recall that for an $R$-module $L$, the submodule $\text{Tr}(\mathcal{U}, L) = \sum\{\text{Im}h| h \in \text{Hom}(U, L), U \in \mathcal{U}\}$ is called the trace of $\mathcal{U}$ in $L$. If $\mathcal{U}$ consists of a single module $U$ we simply write $\text{Tr}(U, L)$.

The following result shows that the converse of Lemma 3.1 is also true for flat modules over their endomorphism rings. On the other hand, Theorem 3.8 generalizes the result [20, 39.10].

Theorem 3.8. Let $M$ be an $R$-module and $f \in S$. Then we have the following.
(1) If $f^nS$ is a projective right $S$-module for some positive integer $n$, then $\text{Tr}(M,r_M(f^n))$ is a direct summand of $M$.

(2) If $M$ is a flat left $S$-module and $S$ is a generalized right principally projective ring, then $M$ is $\pi$-Rickart as an $R$-module.

Proof. (1) Assume that $f^nS$ is a projective right $S$-module for some positive integer $n$. Then there exists $e^2 = e \in S$ with $r_S(f^n) = eS$. We show $\text{Tr}(M,r_M(f^n)) = eM$. Since $f^n eM = 0$, $eM \leq \text{Tr}(M,r_M(f^n))$. Let $g \in \text{Hom}(M,r_M(f^n))$. Hence $gM \leq r_M(f^n)$ or $f^n gM = 0$ or $f^n g = 0$. Thus $g \in r_S(f^n) = eS$ and so $eg = g$. It follows that $gM \leq egM \leq eM$ or $\text{Hom}(M,r_M(f^n))M \leq eM$.

(2) Assume that $M$ is a flat left $S$-module and $S$ is a generalized right principally projective ring. If $f \in S$, then $f^nS$ is a projective right $S$-module, since $r_S(f^n) = eS$ for some positive integer $n$ and $e^2 = e \in S$. As in the proof of (1), we have $\text{Tr}(M,r_M(f^n)) = eM$. Since $M$ is a flat left $S$-module and $f^n \in S$, $r_M(f^n)$ is $M$-generated by \cite[15.9]{20}. Again by \cite[13.5(2)]{20}, $\text{Tr}(M,r_M(f^n)) = r_M(f^n)$. Thus $r_M(f^n) = eM$. \hfill \Box

Recall that a ring $R$ is said to be von Neumann regular if for any $a \in R$ there exists $b \in R$ with $a = aba$. For a module $M$, it is shown that if $S$ is a von Neumann regular ring, then $M$ is a Rickart module (see \cite[Theorem 3.17]{12}). We obtain a similar result for $\pi$-Rickart modules.

**Lemma 3.9.** Let $M$ be a module. If $S$ is a $\pi$-regular ring, then $M$ is a $\pi$-Rickart module.

Proof. Let $f \in S$. Since $S$ is $\pi$-regular, there exist a positive integer $n$ and an element $g \in S$ such that $f^n = f^n gf^n$. Then $gf^n$ is an idempotent of $S$. Now we show that $r_M(f^n) = (1 - gf^n)M$. For $m \in M$, we have $f^n(1 - gf^n)m = (f^n - f^n gf^n)m = (f^n - f^n)m = 0$. Hence $(1 - gf^n)M \leq r_M(f^n)$. For the other side, if $m \in r_M(f^n)$, then $gf^n m = 0$. This implies that $m = (1 - gf^n)m \in (1 - gf^n)M$. Therefore $r_M(f^n) = (1 - gf^n)M$. \hfill \Box

Now we recall some known facts that will be needed about $\pi$-regular rings.

**Lemma 3.10.** Let $R$ be a ring. Then

(1) If $R$ is $\pi$-regular, then $eRe$ is also $\pi$-regular for any $e^2 = e \in R$.
(2) If $M_n(R)$ is $\pi$-regular for any positive integer $n$, then so is $R$. 
If $R$ is a commutative ring, then $R$ is $\pi$-regular if and only if $M_n(R)$ is $\pi$-regular for any positive integer $n$.

Proof. (1) Let $R$ be a $\pi$-regular ring, $e^2 = e \in R$ and $a \in eRe$. Then $a^n = a^nre^n$ for some positive integer $n$ and $r \in R$. Since $a^n = a^n e = ea^n$, we have $a^n = a^n(e re)a^n$. Therefore $eRe$ is $\pi$-regular.

(2) is clear from (1).

(3) Let $R$ be a commutative $\pi$-regular ring. By [11, Ex.4.15], every prime ideal of $R$ is maximal, and so every finitely generated $R$-module is co-Hopfian from [18]. Then for any positive integer $n$, $M_n(R)$ is $\pi$-regular by [2, Theorem 1.1]. The rest is known from (2). □

Proposition 3.11. Let $R$ be a commutative $\pi$-regular ring. Then every finitely generated projective $R$-module is $\pi$-Rickart.

Proof. Let $M$ be a finitely generated projective $R$-module. So the endomorphism ring of $M$ is $eM_n(R)e$ with some positive integer $n$ and an idempotent $e$ in $M_n(R)$. Since $R$ is commutative $\pi$-regular, $M_n(R)$ is also $\pi$-regular, and so is $eM_n(R)e$ by Lemma 3.10. Hence $M$ is $\pi$-Rickart by Lemma 3.9 □

The converse of Lemma 3.9 may not be true in general, as the following example shows.

Example 3.12. Consider $\mathbb{Z}$ as a $\mathbb{Z}$-module. Then it can be easily shown that $\mathbb{Z}$ is a $\pi$-Rickart module, but its endomorphism ring is not $\pi$-regular.

A module $M$ has $C_2$ condition if any submodule $N$ of $M$ which is isomorphic to a direct summand of $M$ is a direct summand. In [12, Theorem 3.17], it is proven that the module $M$ is Rickart with $C_2$ condition if and only if $S$ is von Neumann regular. The $C_2$ condition allows us to show the converse of Lemma 3.9.

Theorem 3.13. Let $M$ be a module with $C_2$ condition. Then $M$ is $\pi$-Rickart if and only if $S$ is a $\pi$-regular ring.

Proof. The sufficiency holds from Lemma 3.9. For the necessity, let $0 \neq f \in S$. Since $M$ is $\pi$-Rickart, $\text{Ker}f^n$ is a direct summand of $M$ for some positive integer $n$. Let $M = \text{Ker}f^n \oplus N$ for some $N \leq M$. It is clear that $f^n|_N$ is a monomorphism. By the $C_2$ condition, $f^nN$ is a direct summand of $M$. 


On the other hand, there exists $0 \neq g \in S$ such that $gf^n|_N = 1_N$. Hence $\left( f^n - f^ngf^n \right)M = (f^n - f^ngf^n)(Ker f^n \oplus N) = (f^n - f^ngf^n)N = 0$. Thus $f^n = f^ngf^n$, and so $S$ is a $\pi$-regular ring.

The following is a consequence of Proposition 3.11 and Theorem 3.13.

**Corollary 3.14.** Let $R$ be a commutative ring and satisfy $C_2$ condition. Then the following are equivalent.

1. $R$ is a $\pi$-regular ring.
2. Every finitely generated projective $R$-module is $\pi$-Rickart.

As every quasi-injective module has $C_2$ condition, we have the following.

**Corollary 3.15.** Let $M$ be a quasi-injective module. Then $M$ is $\pi$-Rickart if and only if $S$ is a $\pi$-regular ring.

**Corollary 3.16.** Every right self-injective ring is generalized right principally projective if and only if it is $\pi$-regular.

**Theorem 3.17.** Let $R$ be a right self-injective ring. Then the following are equivalent.

1. $M_n(R)$ is $\pi$-regular for every positive integer $n$.
2. Every finitely generated projective $R$-module is $\pi$-Rickart.

**Proof.** (1) $\Rightarrow$ (2) Let $M$ be a finitely generated projective $R$-module. Then $M \cong eR^n$ for some positive integer $n$ and $e^2 = e \in M_n(R)$. Hence $S$ is isomorphic to $eM_n(R)e$. By (1), $S$ is $\pi$-regular. Thus $M$ is $\pi$-Rickart due to Lemma 3.9.

(2) $\Rightarrow$ (1) $M_n(R)$ can be viewed as the endomorphism ring of a projective $R$-module $R^n$ for any positive integer $n$. By (2), $R^n$ is $\pi$-Rickart, and by hypothesis, it is quasi-injective. Then $M_n(R)$ is $\pi$-regular by Corollary 3.15.

The proof of Lemma 3.18 may be in the context. We include it as an easy reference.

**Lemma 3.18.** Let $M$ be a module. Then $S$ is a $\pi$-regular ring if and only if for each $f \in S$, there exists a positive integer $n$ such that $Ker f^n$ and $Im f^n$ are direct summands of $M$. 
Theorem 3.19. Let $M$ be a $\pi$-Rickart module. Then the right singular ideal $Z_r(S)$ of $S$ is nil and $Z_r(S) \subseteq J(S)$.

Proof. Let $f \in Z_r(S)$. Since $M$ is $\pi$-Rickart, $r_M(f^n) = eM$ for some positive integer $n$ and $e = e^2 \in S$. By Lemma [3.1], $r_S(f^n) = eS$. Since $r_S(f^n)$ is essential in $S$ as a right ideal, $r_S(f^n) = S$. This implies that $f^n = 0$, and so $Z_r(S)$ is nil. On the other hand, for any $g \in S$ and $f \in Z_r(S)$, according to previous discussion, $(fg)^n = 0$ for some positive integer $n$. Hence $1 - fg$ is invertible. Thus $f \in J(S)$. Therefore $Z_r(S) \subseteq J(S)$. □

Proposition 3.20. The following are equivalent for a module $M$.

1. Each element of $S$ is either a monomorphism or nilpotent.
2. $M$ is an indecomposable $\pi$-Rickart module.

Proof. (1) $\Rightarrow$ (2) Let $e = e^2 \in S$. If $e$ is nilpotent, then $e = 0$. If $e$ is a monomorphism, then $e(m - em) = 0$ implies $em = m$ for any $m \in M$. Hence $e = 1$, and so $M$ is indecomposable. Also for any $f \in S$, $r_M(f) = 0$ or $r_M(f^n) = M$ for some positive integer $n$. Therefore $M$ is $\pi$-Rickart.

(2) $\Rightarrow$ (1) Let $f \in S$. Then $r_M(f^n)$ is a direct summand of $M$ for some positive integer $n$. As $M$ is indecomposable, we see that $r_M(f^n) = 0$ or $r_M(f^n) = M$. This implies that $f$ is a monomorphism or nilpotent. □

Theorem 3.21. Consider the following conditions for a module $M$.

1. $S$ is a local ring with nil Jacobson radical.
2. $M$ is an indecomposable $\pi$-Rickart module.

Then (1) $\Rightarrow$ (2). If $M$ is a morphic module, then (2) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2) Clearly, each element of $S$ is either a monomorphism or nilpotent. Then $M$ is indecomposable $\pi$-Rickart due to Proposition [3.20]

(2) $\Rightarrow$ (1) Let $f \in S$. Then $r_M(f^n) = eM$ for some positive integer $n$ and an idempotent $e$ in $S$. If $e = 0$, then $f$ is a monomorphism. Since $M$ is morphic, $f$ is invertible by [14, Corollary 2]. If $e = 1$, then $f^n = 0$. Hence $1 - f$ is invertible. This implies that $S$ is a local ring. Now let $0 \neq f \in J(S)$. Since $f$ is not invertible, there exists a positive integer $n$ such that $r_M(f^n) = M$. Therefore $J(S)$ is nil. □

The next result can be obtained from Theorem [3.21] and [7, Lemma 2.11].
Corollary 3.22. Let $M$ be an indecomposable $\pi$-Rickart module. If $M$ is morphic, then $S$ is a left and right $\pi$-morphic ring.

In [13], a module $M$ is called dual $\text{Rickart}$ if for any $f \in S$, $\text{Im} f = eM$ for some $e^2 = e \in S$. In a subsequent paper the present authors continue studying some generalizations of dual Rickart modules. The module $M$ is called dual $\pi$-Rickart if for any $f \in S$, there exist $e^2 = e \in S$ and a positive integer $n$ such that $\text{Im} f^n = eM$. We end this paper to demonstrate the relations between $\pi$-Rickart and dual $\pi$-Rickart modules.

Proposition 3.23. Let $M$ be a module with $C_2$ condition. If $M$ is a $\pi$-Rickart module, then it is dual $\pi$-Rickart.

Proof. It follows from Theorem 3.13 and Lemma 3.18.

Recall that a module $M$ is said to have $D_2$ condition if any submodule $N$ of $M$ with $M/N$ isomorphic to a direct summand of $M$, then $N$ is a direct summand of $M$.

Proposition 3.24. Let $M$ be a module with $D_2$ condition. If $M$ is a dual $\pi$-Rickart, then it is $\pi$-Rickart.

Proof. Since $M/r_M(f^n) \cong \text{Im} f^n$ for any $f \in S$, $D_2$ condition completes the proof.

Proposition 3.25. Let $M$ be a projective morphic module. Then $M$ is $\pi$-Rickart if and only if it is dual $\pi$-Rickart.

Proof. Clear.

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