ELLIPITC THEORY OF DIFFERENTIAL EDGE OPERATORS, II: BOUNDARY VALUE PROBLEMS

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Abstract. This is a continuation of the first author’s development of the theory of elliptic differential operators with edge degeneracies. That first paper treated basic mapping theory, focusing on semi-Fredholm properties on weighted Sobolev and Hölder spaces and regularity in the form of asymptotic expansions of solutions. The present paper builds on this through the formulation of boundary conditions and the construction of parametrices for the associated boundary problems. As in [17], the emphasis is on the geometric microlocal structure of the Schwartz kernels of parametrices and generalized inverses.

1. Introduction

Degenerate elliptic operators on manifolds with boundary or corners arise naturally in many different problems in partial differential equations, geometric analysis, mathematical physics and elsewhere. Over the past several decades, many types of such equations have been studied, often by ad hoc methods but sometimes through the development of a more systematic theory to handle various classes of operators. One particularly fruitful direction concerns the elliptic operators associated to (complete or incomplete) iterated edge metrics on smoothly stratified spaces. The simplest examples of such operators include the Laplace operators for spaces with isolated conic singularities or with asymptotically cylindrical ends. Other important special cases include nondegenerate elliptic operators on manifolds with boundaries or the Laplacians on asymptotically hyperbolic (conformally compact) manifolds. The class of elliptic operators on spaces with simple edge singularities includes both of these sets of examples. A final example is the Laplacian (or any other elliptic operator) on a smooth manifold written in Fermi coordinates around a smooth embedded submanifold.

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To be more specific, let $M$ be a compact manifold with boundary, and suppose that $\partial M$ is the total space of a fibration with base $B$ and fibre $F$. Choose coordinates $(x, y, z)$ near the boundary so that $x = 0$ defines $\partial M$, $y$ is a set of coordinates on $B$ lifted to $\partial M$ and then extended into $M$ and $z$ are independent functions which restrict to a coordinate system on each fibre $F_y$. A differential operator of order $L$ is called an edge operator of order $m$ if it has the form

$$L = \sum_{j+|\alpha|+|\beta| \leq m} a_{j\alpha\beta}(x, y, z)(x\partial_x)^j(x\partial_y)^\alpha z^\beta.$$ 

We assume that the coefficients $a_{j\alpha\beta}$ are all $C^\infty$ on the closed manifold $M$; these can either be scalar or (if $L$ acts between sections of vector bundles) matrix-valued. We say that $L$ is edge elliptic if it is an elliptic combination of the constituent vector fields $x\partial_x$, $x\partial_y$, and $\partial_z$; an invariant definition is indicated in $\S$2. The examples above all fall into this class, or else are of the form $x^{-m}L$ where $L$ has this form; operators of this latter sort are called ‘incomplete’ edge operators since they include Laplacians of metrics with incomplete edge singularities.

The present paper is the continuation of a now rather old paper by the first author [17] which develops a framework for the analysis of elliptic differential edge operators based on the methods of geometric microlocal analysis. That paper establishes many fundamental results concerning the mapping properties of these operators and the regularity properties of solutions, and those results have had very many applications, both in analysis and geometry, in the intervening years. We review this theory in $\S$2. The mapping properties considered there were for an elliptic edge operator acting between weighted Sobolev and Hölder spaces. This left open, however, any development of a more general theory of “elliptic edge boundary value problems”. The present paper finally addresses this aspect of elliptic edge theory.

An important generalization involves the study of elliptic operators with iterated edge singularities; examples include Laplacians on $C^\infty$ polyhedra or conifolds, see [21], as well as on general smoothly stratified spaces [1], [2]. As of yet, there is no complete elliptic theory in this general setting, although many special cases and specific results have been obtained by a variety of authors. Certainly the closest to what we do here is the work of Gil, Krainer and Mendoza [7] and the recent and ongoing work of Krainer-Mendoza [14], [15]. In various respects, these last papers go much farther than what we do here. We mention also the theory developed by Schulze, [23], [24], [25]. The emphasis in those papers and monographs
is the development of a hierarchy of algebras of pseudodifferential operators, structured as in [4], with emphasis on the operator-symbol quantization associated to spaces with both simple and iterated edge singularities. Other notable contributions include the work of Maz’ya and his collaborators, see [16], as well as Nistor [22], Ammann-Lauter-Nistor [3] and Brüning-Seeley [5]. As noted above, Krainer and Mendoza [14], [15] also treat edge boundary problems, and in fact do so for more general operators with variable indicial roots, but their methods are somewhat different from the ones here. These last authors have very generously shared some important ideas from their work before the appearance of [14], described here in §3, which form a necessary part of our analysis.

There are many reasons for developing a theory of more general boundary conditions for elliptic edge operators, in particular from the point of view developed here. Perhaps most significant is the importance of mixed or global boundary conditions, either of local (Robin) or Atiyah-Patodi-Singer type, in the study of index theory for generalized Dirac operators on spaces with simple edge singularities, all of which appear in many natural problems. Similarly, the study of the eta invariant and analytic torsion for Dirac-type operators on spaces with various boundary conditions on spaces with isolated conic singularities has proved to be quite interesting. All of these directions fall within the scope of one or more of the other approaches cited above.

The geometric microlocal methods used here have a distinct advantage over other (e.g. more directly Fourier analytic) approaches: our primary focus is on Schwartz kernels rather than abstract mapping properties or methods too closely tied to more standard pseudodifferential theory, and because of this it is equally easy to obtain results adapted to any standard types of function spaces that one might wish to use, e.g. weighted Hölder or $L^p$ spaces. This transition between mapping properties on different types of spaces seems more difficult using those other approaches, although having such properties available is quite important when studying nonlinear geometric problems on spaces with edge singularities, cf. [11] for a recent example.

One limitation of the current development, however, is that we do not treat the delicate regularity issues associated with the possibility of smoothly varying indicial roots. As in [17], we make a standing assumption that all operators considered here have constant indicial roots, at least in the critical weight-range $(\delta, \delta)$. We refer to §2 for a description of all of this.

Because their precise description requires a number of preliminary definitions, we defer to §3 a careful description of our main results; however, we now state them briefly and somewhat informally. The starting point is the
basic statement that if $L$ is an elliptic edge operator, as above, then under appropriate hypotheses on the indicial roots and assuming the unique continuation property for the reduced Bessel operator $B(L)$, see §2 for these, the mapping

$$L : x^\delta H^m_e(M) \longrightarrow x^\delta L^2(M)$$

is essentially surjective, i.e. has closed range with finite dimensional cokernel, provided $\delta \leq \delta_0$ and $\delta$ is nonindicial, and is essentially injective, i.e. has closed range with finite dimensional nullspace, if $\delta \geq \delta_0$. Suppose that $u \in x^\delta H^m_e$ and $Lu = f \in x^\delta$. Then it is proved in [17] that

$$u \sim \sum_{j=1}^{N} \sum_{\ell, p \in \mathbb{N}_0} u_{j, p}(y, z)x^{\gamma_j + \ell}(\log x)^p + \tilde{u},$$

where the sum is over all indicial roots (see §2) of $L$ and indices $p, \ell$ such that $\gamma_j + \ell \in (\delta, \delta_0)$ and $p$ is no greater than some integer $N_{j, \ell}$, and where $\tilde{u} \in x^\delta H^m_e$. The subcollection of leading coefficients $\{u_{j, N_{j, 0}, 0}\}$ is called the Cauchy data of $u$ and denoted $\mathcal{C}(u)$. A boundary condition for this edge problem consists of a finite collection $Q = \{Q_{kj}\}$ of pseudodifferential operators acting on these leading coefficients. (Since the precise formulation is somewhat intricate, we defer this for now.) We then study the mappings:

$$Lu = f \in x^\delta L^2$$
$$Q(\mathcal{C}(u)) = \phi.$$

The main results here give conditions for when this mapping is Fredholm or semi-Fredholm acting on appropriate weighted Sobolev spaces. We follow the methods due originally to Calderon, described particularly well in the monograph of Chazarain and Piriou [3], and later extended significantly by Boutet de Monvel and others; however, we do not define the full Boutet de Monvel calculus in this edge setting. We also give the precise structure of the Schwartz kernel of the generalized inverse of this mapping, and consequently can study this problem on other function spaces. We do not treat any application of these results in this paper, but must rely on the reader’s knowledge of the centrality of elliptic boundary problems in the standard setting, and on his or her faith that this extension of that theory will also have broad applicability.

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2. A review of the edge calculus

We begin by recalling in more detail the geometric and analytic framework necessary to discuss the theory of differential and pseudodifferential edge operators, and then review the main theorems from [17] concerning the semi-Fredholm theory and asymptotics of solutions. This section is meant as a brief review, and is not meant to be self-contained. We refer the reader to [17] for elaboration and proofs of all the definitions and facts presented here.

**Edge structures** As in the introduction, let $M$ be compact manifold with boundary, and suppose that $\partial M$ is the total space of a fibration $\phi : \partial M \to B$ with fibre $F$. We set $b = \dim B$ and $f = \dim F$.

The fundamental object in this theory is the space $\mathcal{V}_e$ of all smooth vector fields on $M$ which are unconstrained in the interior and which are tangent to the fibres of $\phi$ at $\partial M$; clearly $\mathcal{V}_e$ is closed under Lie bracket. We shall routinely use local coordinate systems near the boundary of the following form: $x$ is a defining function for the boundary (i.e. $\partial M = \{x = 0\}$), $y_1, \ldots, y_b$ is a set of local coordinates on $B$ lifted to $\partial M$ and then extended into $M$, and $z_1, \ldots, z_f$ is a set of independent functions which restricts to a coordinate system on each fibre $F_y$. In terms of these,

\begin{equation}
\mathcal{V}_e = \text{Span}_{C^\infty} \{x\partial_x, x\partial_{y_1}, \ldots, x\partial_{y_b}, \partial_{z_1}, \ldots, \partial_{z_f}\}.
\end{equation}

In other words, any $V \in \mathcal{V}_e$ can be expressed locally as

\[ V = ax \partial_x + \sum b_i x \partial_{y_i} + \sum c_j \partial_{z_j}, \quad \text{where} \ a, b_i, c_j \in C^\infty(M). \]

Any differential operator can be expressed locally as the sum of products of vector fields, and so we can define interesting subclasses of operators by restricting the vector fields allowed in these decompositions. In particular, define $\text{Diff}^*_e(M)$ to consist of all differential operators which are locally finite sums of products of elements in $\mathcal{V}_e$. With the subscript corresponding to the
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usual order filtration, we have, in local coordinates,

\[(2.2) \quad \text{Diff}_m(M) \ni L = \sum_{j+|\alpha|+|\beta| \leq m} a_{j\alpha\beta}(x, y, z)(x \partial_x)^j(x \partial_y)^\alpha(z \partial_z)^\beta, \]

with all \(a_{j\alpha\beta} \in C^\infty\). Here and later we use standard multi-index notation to describe (differential) monomials. If \(L\) acts between sections of two bundles \(E\) and \(F\), then taking local trivializations of these bundles, the coefficients here are matrix-valued.

There is a natural edge tangent bundle \(eTM\) defined by the property that \(V_e\) coincides with its full space of \(C^\infty\) sections; its dual is the edge cotangent bundle \(eT^*M\), which has a local \(C^\infty\) basis of sections consisting of the 1-forms

\[\frac{dx}{x}, \frac{dy_1}{x}, \ldots, \frac{dy_b}{x}, dz_1, \ldots, dz_f.\]

Any \(L \in \text{Diff}_m(M)\) has symbol

\[e\sigma_m(L)(x, y, z, \xi, \eta, \zeta) = \sum_{j+|\alpha|+|\beta| = m} a_{j\alpha\beta}(x, y, z)\xi^j\eta^\alpha\zeta^\beta,\]

which is well-defined as a smooth function on \(eT^*M\) which is a homogenous polynomial of degree \(m\) on each fibre. If \(L\) acts between sections of two vector bundles \(E\) and \(F\), then \(e\sigma_m(L)\) takes values in \(\text{End}(\pi^*E, \pi^*F)\), where \(\pi: eT^*M \to M\). The operator \(L\) is said to be elliptic (in the edge sense) if this symbol is invertible when \((\xi, \eta, \zeta) \neq (0, 0, 0)\).

A (complete) edge metric is a smooth positive definite section of \(\text{Sym}^2(eT^*M)\). It is not hard to check that if \(g\) is any metric of this type, then its scalar Laplacian, Hodge Laplacian, and all other natural elliptic geometric operators (e.g. the rough Laplacian, the Lichnerowicz Laplacian, twisted Dirac operators, etc.) are all elliptic edge operators (N.B.; some of these operators are of this type only if expressed in terms of an appropriate basis of sections of the bundles on which they act). Similarly, an incomplete edge metric \(g\) is one of the form \(x^2\tilde{g}\), where \(\tilde{g}\) is a complete edge metric. Its Laplacian is of the form \(x^{-2}L\) where \(L \in \text{Diff}_2(M)\), and there are analogous assertions for the other elliptic operators mentioned above. In practice one often restricts to a smaller class of metrics (for example, requiring that \(g\) does not contain the term \(x^{-1}dxdy\), though even even more rigid hypotheses arise naturally), see [19] for more on this.

Model operators Let \(L\) be an elliptic edge operator of order \(m\), expressed as in (2.2). The analysis of the mapping properties of \(L\) relies on a variety of associated model operators.
First, the principal edge symbol $\sigma_m(L)$ is a purely algebraic model for $L$ at any point; the microlocal inversion of $L$, uniformly up to the boundary, relies on the invertibility of this object.

Next, associated to every point $y_0 \in B$ (taken as the origin in the $y$ coordinate system) is the normal operator
\begin{equation}
N_{y_0}(L) = \sum_{j+|\alpha|+|\beta| \leq m} a_{j\alpha\beta}(0,0,z) (s\partial_s)^j (s\partial_u)^\alpha \partial_z^\beta;
\end{equation}
this acts on functions on the model space $\mathbb{R}^+ \times \mathbb{R}^b \times F_{y_0}$, where $s$ and $u$ are global linear variables on a half-space which can be regarded as being the part of the tangent bundle which is the inward normal to $F_{y_0}$. This model space is naturally identified with the tangent cone with respect to the family of dilations $(x,y,z) \mapsto (\lambda x, \lambda y, z)$ as $\lambda \to \infty$.

Another operator which models the behaviour of $L$ near $F_{y_0}$ is the Bessel operator
\begin{equation}
B_{y_0, \hat{\eta}}(L) = \sum_{j+|\alpha|+|\beta| \leq m} a_{j\alpha\beta}(0,0,z) (t\partial_t)^j (-it\hat{\eta})^\alpha \partial_z^\beta.
\end{equation}
Here $\hat{\eta} = \eta/|\eta|$, where $\eta \in T^*_y B$ (i.e. $\hat{\eta}$ lies in the spherical conormal bundle $S^*_y B$). This is obtained from $N(L)$ by first passing to the Fourier transform in $u$ (which transforms $s\partial_u$ to $-is\eta$) and then rescaling by setting $t = s|\eta|$. These operations are reversible so the family $B_{y_0, \hat{\eta}}(L)$ is completely equivalent to $N_{y_0}(L)$ even though it appears to be simpler. Note that the structure of $B(L)$ as $t \to \infty$ captures the behaviour as $|\eta| \to \infty$, and hence corresponds to local behaviour for $N(L)$.

Finally, the indicial operator, which is an elliptic $b$-operator in the sense of [20], [17], on $\mathbb{R}^+ \times F$, is defined by
\begin{equation}
I_{y_0}(L) = \sum_{j+|\beta| \leq m} a_{j0\beta}(0,0,z) (t\partial_t)^j \partial_z^\beta.
\end{equation}
This is obtained from the Bessel-normal operator by dropping the terms which are lower order in the $b$-theory. Thus $B_{y_0, \hat{\eta}}(L) = I_{y_0}(L) + E$, where $E$ is truly lower order on any finite interval $0 < t < t_0$. However, as remarked above, the large $t$ behaviour of $B_{y_0, \hat{\eta}}(L)$ contains important information missing in the indicial operator.

**Indicial roots and the trace bundle** The indicial operator can be conjugated, via the Mellin transform, to the operator pencil,
\begin{equation}
I_{y_0}(L)(\zeta) := \sum_{j+|\beta| \leq m} a_{j0\beta}(0,0,z) (-i\zeta)^j \partial_z^\beta,
\end{equation}
which depends smoothly on \(y_0 \in B\); this is often called the indicial family of \(L\). (An operator pencil, an important generalization of a resolvent family, is simply a polynomial family with operator coefficients.) Because the coefficient of \(ζ^j\) has order \(m - j\), and the coefficient of \(ζ^0\) is an elliptic operator on \(F_{y_0}\) of order \(m\). Hence the analytic Fredholm theorem may be applied, and this implies that this family is either never invertible, for any \(ζ \in \mathbb{C}\), or else that this inverse is meromorphic, and the Laurent coefficients at each pole are operators of finite rank. It is certainly necessary to assume that \(I_{y_0}(L)(0) = \sum_{|β|=m} a_{00β}(z)∂_z^β\) has index zero, otherwise we are necessarily in the first, nowhere invertible, case. A standard condition to ensure that the inverse exists at one point, and hence away from a discrete set, is that the resolvent of the (ordinary) symbol, \((σ_m(I_{y_0}(L)(0)) - λ)^{-1}\), satisfy standard elliptic symbol-with-parameter estimates in some open conic sector. In any case, we shall assume that we are in some setting which allows us to conclude that the indicial family has meromorphic inverse.

**Definition 2.1.** The boundary spectrum of \(L\), \(\text{Spec}_b(L)\), is the set of locations of the poles of the meromorphic family \(I_{y_0}(L)^{-1}\) (this is also called the spectrum of the operator pencil); elements of \(\text{Spec}_b(L)\) are called the indicial roots of \(L\) (at \(y_0\)).

We have tacitly suppressed the fact that these indicial roots may vary with \(y_0\). The analysis of edge operators with indicial roots depending nontrivially on \(y_0\) is an interesting and difficult topic, and is discussed in detail in the forthcoming work of Krainer and Mendoza. However, partly because this behaviour often does not occur for the natural examples of edge operators, we choose to make the basic

**Assumption 2.2 (Constancy of indicial roots).** The spectrum of the indicial family is discrete and the location of the poles of \(I_{y_0}(L)(ζ)^{-1}\) does not depend on \(y_0 \in B\).

It is necessary to make one further hypothesis:

**Assumption 2.3.** For each \((y_0, η)\), the Bessel operator \(B_{y_0, δ}(L)\) is injective on \(t^δ L^2(dt dz)\) for \(δ \gg 0\), and is surjective when \(δ \ll 0\).

This holds in many interesting situations, see [18].

We henceforth always work with operators satisfying both of these assumptions. In the following, we fix two nonindicial values \(δ\) and \(δ\) such that \(B(L)\) is injective on \(t^δ L^2\) and surjective on \(t^{δ} L^2\). We then write \(\mathcal{G}(L)\) for the set of all indicial roots in this critical strip, omitting their multiplicity:

\[
\mathcal{G}(L) := \{ζ_j \in \mathbb{C} : \bar{δ} - 1/2 < \Im ζ_0 < \bar{δ} - 1/2 \text{ and } \exists p \in \mathbb{N} \text{ s.t. } (ζ_j, p) \in \text{Spec}_b(L)\}.
\]
The shift by $1/2$ appears because we are using the measure $dtdz$.

**Proposition 2.4.** Let $\omega \in t^{2}L^{2}(dtdz)$ and $\int_{\gamma_{0}}(L)\omega = f \in t^{5}L^{2}(dtdz)$. Then
\[
\omega = \sum_{\zeta_{j} \in \mathcal{S}(L)} \sum_{p=0}^{p_{j}} \omega_{j,p}(z)t^{-i\zeta + \ell}(\log t)^{p} + \tilde{\omega}, \quad \omega_{j,p} \in C^{\infty}(F), \; \tilde{\omega} \in t^{3}L^{2}(dtdz).
\]

Here $p_{j} + 1$ is equal to the order of the pole of $\int_{\gamma_{0}}(L)\omega^{-1}$ at $\zeta_{j}$.

Dropping the subscript $y_{0}$, we pass to the Mellin transform of this equation (see [17]), which is $I_{\mathcal{L}}(\zeta)\omega_{M}(\zeta, z) = f_{M}(\zeta, z)$. The Mellin transforms $\omega_{M}(\zeta, z)$ and $f_{M}(\zeta, z)$ are holomorphic in $\Im \zeta < \delta - 1/2$ and $\Im \zeta < \tilde{\delta} - 1/2$, respectively, both with values in $L^{2}(F; dz)$. Thus $\omega_{M} = -I_{\mathcal{L}}(\zeta)^{-1}f_{M}$ extends meromorphically to this larger half-plane. Taking the inverse Mellin transform by integrating along the line $\Im \zeta = \delta < \tilde{\delta} - 1/2$ and then shifting the contour across the poles in this horizontal strip produces the expansion.

Using this result, we can define the trace of $\omega$ to consist of the set of functions $\omega_{j,p}$ over all $\zeta_{j} \in \mathcal{S}(L)$ and $p \leq p_{j}$. There is a subtlety here in that although we require that each $\zeta_{j}$ is independent of $y_{0}$, the same may not be true of the $p_{j}$, so we need to explain carefully the sense in which this expansion depends smoothly on $y_{0}$.

To describe this we need make a small detour. First note that the algebraic multiplicity of each pole is well-defined. As in the special case of the resolvent family of a non self-adjoint operator, this algebraic multiplicity is a positive integer which measures the dimension of the space of generalized eigenvectors associated to that indicial root. Since there are several different-looking (but equivalent) definitions of this quantity, we provide a slightly longer description than strictly necessary, for the reader’s convenience. For simplicity of notation, omit the dependence on $y_{0}$ for the moment, and suppose that $\zeta_{0}$ is the indicial root in question. The geometric eigenspace of $I_{\mathcal{L}}(\zeta_{0})\phi_{0} = 0$, and its dimension is called the geometric multiplicity of the indicial root. Suppose now that there exist additional functions $\phi_{j} \in C^{\infty}(F), j = 1, \ldots, k - 1$ such that
\[
(2.7) \quad \sum_{j=0}^{\ell} \frac{1}{j!}(\partial_{\zeta}^{j}I_{\mathcal{L}})(\zeta_{0})\phi_{j} = 0, \; \ell = 1, \ldots, k - 1;
\]
this sequence of equations is equivalent to the single condition
\[
I_{\mathcal{L}}(\zeta)\phi_{0} + (\zeta - \zeta_{0})\phi_{1} + \ldots (\zeta - \zeta_{0})^{k-1}\phi_{k-1} = O(|\zeta - \zeta_{0}|^{k}).
\]
The ordered $k$-tuple $(\phi_{0}, \ldots, \phi_{k-1})$ is called a generalized eigenvector, and the maximal length of all such chains beginning with $\phi_{0}$ is called the
multiplicity of $\phi_0$ and denoted $m(\phi_0)$. In other words, $m(\phi_0)$ measures the order to which $\phi_0$ can be extended as a formal series solution of $I(L)(\zeta) \sum (\zeta - \zeta_0)^j \phi_j \equiv 0$. We refer to $\phi_0(\zeta) = \sum (\zeta - \zeta_0)^j \phi_j$ as the root function associated to the eigenvector $\phi_0$. Following [16 §1.1], choose a basis $\{\phi_{0,1}, \ldots, \phi_{0,N}\}$ for the geometric eigenspace of $I(L)(\zeta_0)$ so that $m(\phi_{0,1}) \leq m(\phi_{0,2}) \leq \cdots \leq m(\phi_{0,N})$, and then define the algebraic multiplicity of the pole to be the number

$$m(\zeta_0) = \sum_{j=1}^{N} m(\phi_{0,j}).$$

There is an alternate description, following [8 Ch. XI], which gives a slightly different intuition for this number. The first step is to write the holomorphic family $I(L)(\zeta)$ as the product $E(\zeta)D(\zeta)F(\zeta)$, locally near $\zeta_0$. Here $E(\zeta)$ and $F(\zeta)$ are holomorphic and invertible for $|\zeta - \zeta_0| < \epsilon$ and

$$D(\zeta) = P_0 + (\zeta - \zeta_0)^{\kappa_1} P_1 + \cdots + (\zeta - \zeta_0)^{\kappa_r} P_r,$$

where the $P_j$ are mutually disjoint projectors (i.e. $P_i P_j = 0$ if $i \neq j$), $P_0$ has infinite rank, rank $P_j = 1, j > 0$, and $P_0 + \cdots + P_r = \text{Id}$. Clearly $\kappa_r$ is the order of the pole of $I(L)^{-1}(\zeta_0)$ at $\zeta_0$, and a straightforward calculation shows that the algebraic multiplicity $m(\zeta_0)$ is equal to $\kappa_1 + \cdots + \kappa_r$.

In any case, by an extension of the theorem of Keldyš [12, 13], see Gohberg-Sigal [9] and Menniken-Möller [21], the generalized eigenvectors characterize the singular part of the Laurent expansion of $I_{y_0}(L)(\zeta)^{-1}$ at $\zeta_0$, as follows. These sources prove that there exists a set of polynomials in $\zeta$, $(\psi_1(\zeta), \ldots, \psi_N(\zeta))$, taking values in $\mathcal{D}'(F)$ (distributions on $F$), and an operator-valued family $H(\zeta)$ which is holomorphic near $\zeta_0$, such that

$$I_{y_0}(L)(\zeta)^{-1} = \sum_{j=1}^{N} (\zeta - \zeta_0)^{-k_j} \phi_{0,j}(\zeta) \otimes \psi_j(\zeta) + H(\zeta).$$

(2.8)

Here, for each $j$, $\phi_{0,j}(\zeta)$ is the root function corresponding to the element $\phi_{0,j}$ in the geometric eigenspace and $k_j = m(\phi_{0,j})$. This implies that for any holomorphic function $u(\zeta)$ taking values in $C^{\infty}(F)$, each singular Laurent coefficient of $I_{y_0}(L)(\zeta)^{-1} u(\zeta)$ at $\zeta_0$ is a linear combinations of the coefficients $\phi_{0,j}$ of the root functions $\phi_{0,j}(\zeta) = \sum \phi_{\ell,j}(\zeta - \zeta_0)^{\ell}$.

Going back to Proposition 2.2.4 and using the notation there, it is clear that each of the singular Laurent coefficients of $\omega_M$ at any pole $\zeta_0$ in the horizontal strip is linear combination of coefficients of the root functions for $I(L)(\zeta_0)$, hence the coefficients of the terms $t^{-i\zeta_0}(\log t)^p$ in the partial expansion for $\omega$ are constituted by these same functions.
We now finally come to the issue of smooth dependence on \( y_0 \). The key fact is that the algebraic multiplicity \( m(\zeta_0) \) of each pole is invariant under small perturbations, and hence is locally independent of \( y_0 \). This follows from an operator-valued version of Rouché’s theorem; we refer to [16, §1.1.2] for a more careful description, and to [8, Theorem 9.2] for a proof.

In fact, slightly more is true: the direct sum of the coefficients of the root functions \( \{\phi_{0,j}(\zeta), j \leq N\} \) form a vector space \( \mathcal{E}_{y_0}(\zeta_0) \) of dimension \( m(\zeta_0) \), and as \( y_0 \) varies, these vector spaces fit together as a smooth vector bundle \( \mathcal{E}(\zeta_0) = \mathcal{E}(L; \zeta_0) \) over each connected component of \( B \).

To define this bundle, write \( M(\zeta_0) \) and \( \mathcal{H}(\zeta_0) \) for the spaces of germs of meromorphic and holomorphic functions, respectively, at \( \zeta_0 \). Following the definitions above, if \( u(\zeta) \) is a holomorphic \( \mathcal{C}^\infty(F) \)-valued function defined near \( \zeta_0 \), then the Laurent coefficients of \( I_{y_0}(L)(\zeta)^{-1}u(\zeta) \) at \( \zeta_0 \) lie in \( \mathcal{E}_{y_0}(\zeta_0) \). We take this as our primary definition and hence let

\[
\mathcal{E}_{y_0}(\zeta_0) = \{ [u] \in M(\zeta_0)/\mathcal{H}(\zeta_0) : [I_{y_0}(L)(\zeta)^{-1}u] = 0 \};
\]
equivalently, \( \mathcal{E}_{y_0}(\zeta_0) \) is identified with the kernel of \( I_{y_0}(L) \) on the space of all finite combinations \( \sum_q a_{j,q}(z)t^{-i\phi_0}(\log t)^q \) with \( a_{j,q} \in \mathcal{C}^\infty(F) \).

**Proposition 2.5** (Krainer and Mendoza [14]).

\[
\mathcal{E}(L; \zeta_0) := \prod_{y_0 \in B} \mathcal{E}_{y_0}(L; \zeta_0) \xrightarrow{\pi} B
\]
is a smooth vector bundle of rank \( m(\zeta_0) \).

We sketch some elements of the proof (recalling however that those authors work in the more general setting where the \( \zeta_j \) may also vary with \( y \)). For each \( y_0 \in B \), and indicial root \( \zeta_0 \in \mathcal{S}(L) \), Krainer and Mendoza construct an independent set of smooth functions \( \{\phi_{y_0,j}\}_{j=1}^{m_0}, \ m_0 = m(\zeta_0), \) in a neighbourhood \( U \) of \( y_0 \), which form a basis of \( \mathcal{E}_y(\zeta_0) \) for each \( y \in U \).

They then show that if \( \phi(t,y,z) \in t^2L^2 \) depends smoothly on \( y \in U \subset B \) and \( z \in F \), and \( I_y(L)\phi \equiv 0 \), then there exist smooth functions \( f_j : U \to \mathbb{C} \), \( j = 1, \ldots, m_0 \) such that

\[
\phi(t,y,z) = \sum_{j=1}^{m_0} f_j(y)\phi_{y,j}(t,z).
\]

It follows from this that \( \mathcal{E}(L; \zeta_0) \) is a smooth vector bundle over \( B \). A nonobvious consequence is that the ranges of the various singular Laurent coefficients of \( I_y(L)(\zeta)^{-1} \) remain independent of one another as \( y \) varies.

To conclude, let us remark that the full strength of Assumption 2.2 is not needed: it is only necessary that every indicial root with real part in the critical interval \([\delta_1, \delta_2]\) for any \( y_0 \) is independent of \( y_0 \), so in particular,
there are no indicial roots with imaginary parts crossing the levels $\delta$, $\bar{\delta}$. We shall phrase most results as if all indicial roots are constant, but remark at various points how results change in this slightly more general setting.

**Mapping properties** Each of the model operators described above plays an important role in determining the refined mapping properties of $L$. The basic result, stated more carefully below, is that if both $e^\sigma_m(L)$ and $N(L)$ are invertible (as a bundle map and as an operator between weighted Sobolev spaces, respectively); we encompass this pair of properties by saying that $L$ is *fully elliptic* -- then $L$ itself is Fredholm between the analogous weighted Sobolev spaces. For this reason, the pair $(e^\sigma_m(L), N(L))$ should be regarded as the full symbol of $L$. This is the simplest nontrivial case of a symbol hierarchy for iterated edge structures (as in Schulze’s work).

We shall let $L$ act on weighted Sobolev and Hölder spaces. Fix a reference measure $dV = dx dy dz$ (more precisely, $dV$ is a smooth, strictly positive multiple of Lebesgue measure). For any $k \in \mathbb{N}_0$, define

$$H^k_e(M) = \{ u : V_1 \ldots V_\ell u \in L^2(dV) \ \forall \ V_j \in \mathcal{V}_e \ \ell \leq k \}.$$  

Using interpolation and duality (or using edge pseudodifferential operators) one also defines $H^s_e(M)$ for any $s \in \mathbb{R}$. We also define their weighted versions

$$x^\delta H^s_e(M) = \{ u = x^\delta v : v \in H^s_e(M) \}.$$  

Note that these are the Sobolev spaces associated to any complete edge metric $g$ (though the measure $dV$ is equal to $x^{b+1}$ times the Riemannian density for such a metric).

Similarly, we define the Hölder seminorm

$$[u]_{e,0,\alpha} = \sup_{(x,y,z) \neq (x',y',z')} \frac{|u(x, y, z) - u(x', y', z')|(x + x')^{\alpha}}{|x - x'|^{\alpha} + |y - y'|^{\alpha} + (x + x')^{\alpha}|z - z'|^{\alpha}}.$$  

This is simply the standard Hölder seminorm associated to the Riemannian distance associated to the complete metric $g$. The edge Hölder space $\Lambda^0,\alpha_e(M)$ consists of functions $u$ such that $\sup |u| + [u]_{e,0,\alpha} < \infty$. We also define the weighted edge Hölder spaces

$$x^\delta \Lambda^0,\alpha_e(M) = \{ u = x^\delta v : V_1 \ldots V_\ell v \in \Lambda^0,\alpha_e(M) \ \ell \leq k \text{ and } V_j \in \mathcal{V}_e \}.$$  

It is clear from the definitions that if $L \in \text{Diff}_e^m(M)$, then

(2.9) \hspace{1cm} L : x^\delta H^s_e(M) \longrightarrow x^\delta H^{s-m}_e(M)  

(2.10) \hspace{1cm} L : x^\delta \Lambda^{k+m,\alpha}_e(M) \longrightarrow x^\delta \Lambda^{k,\alpha}_e(M)  

are bounded mappings for every $\delta, s \in \mathbb{R}$ and $k \in \mathbb{N}_0$. This is [17, Cor. 3.23].

The basic and most important mapping property for elliptic edge operators is the following.
Proposition 2.6. (17, Thm. 6.1) Suppose that $L \in \Diff^m_e(M)$ is elliptic satisfying the Assumption 2.2, and that $\delta \notin \Spec_b(L)$. Suppose finally that $B_{y_0,\hat{\eta}}(L) : t^\delta H^m_b(\mathbb{R}^+ \times F; t^{-1}dtdz) \rightarrow t^\delta L^2(\mathbb{R}^+ \times F; t^{-1}dtdz)$ is invertible for every $(y_0, \hat{\eta})$. Then both (2.9) and (2.10) are Fredholm mappings. If we only know that $B(L)$ is injective for all $(y_0, \hat{\eta})$, then (2.9) and (2.10) are semi-Fredholm and essentially injective; if $B(L)$ is surjective for every $(y_0, \hat{\eta})$, then (2.9) and (2.10) are semi-Fredholm and essentially surjective.

Normalizations and conventions. We first rewrite Assumption 2.3 in the following form:

Assumption 2.7. There exists values $\delta < \bar{\delta}, \underline{\delta}, \bar{\delta} \notin \Spec_b(L)$, such that, for every $(y_0, \hat{\eta})$,

$$B_{y_0,\hat{\eta}}(L) : t^{\underline{\delta}} H^m_b \rightarrow t^{\underline{\delta}} L^2$$

is surjective, and

$$B_{y_0,\hat{\eta}}(L) : t^{\bar{\delta}} H^m_b \rightarrow t^{\bar{\delta}} L^2$$

is injective.

Remark 2.8. It is enough to assume that an ‘injectivity weight’ $\bar{\delta}$ exists for both $B(L)$ and its adjoint $B(L)^*$ (taken with respect to any fixed measure of the form $t^\gamma dtdz$). This holds simply because injectivity of $B(L)^*$ on some $t^{\delta^*} L^2$ is equivalent to surjectivity of $B(L)$ on another space $t^{\delta^*} L^2$, where $\delta^*$ is determined by $\delta$ and $\gamma$.

Based on this, we see that Assumption 2.7 will hold if both $B(L)$ and $B(L)^*$ satisfy the more basic

Assumption 2.9 (Unique continuation property). Any solution $u$ to $B(L)u = 0$ which vanishes to infinite order at $t = 0$ and which has subexponential growth as $t \rightarrow \infty$ is the trivial solution $u \equiv 0$.

That this should always be true is quite believable, but has not been proved in general. It is known to hold in the special case where $L$ is second order with diagonal principal part and $\dim F = 0$, see [18].

Remark 2.10. Another observation which simplifies notation below is that the precise choice of measure $t^\gamma dtdz$ for $B(L)$, or $x^\delta dx dy dz$ for $L$, (or other measures which differ from these by a smooth function $J$ which is uniformly bounded above and away from 0) is irrelevant for these various mapping and regularity properties. Obviously, the values of $\gamma$ and $J$ enter into the precise computations of adjoints, normalization of weight parameters, etc., but do not in any way effect the nature of the any of the results below.
Thus we always assume that we are working with respect to the measure \( dtdz \), or \( dxdydz \). We also fix the two values \( \delta \) and \( \overline{\delta} \) (and this choice of fixed measures) henceforth for the rest of the paper.

One final remark: as noted earlier, we really only need to assume constancy of indicial roots with real part in the interval \( [\delta, \overline{\delta}] \), though in that more general case, one has slightly weaker regularity statements (conormality rather than complete polyhomogeneity).

**Generalized inverses** Assume that \( L \in \text{Diff}_e^m(M) \) is elliptic and satisfies Assumptions 2.2 and 2.9, and that \( \delta \) and \( \overline{\delta} \) have been chosen as above. By Proposition 2.6, the mapping (2.9) is semi-Fredholm whenever \( \delta \geq \overline{\delta} \) or \( \delta \leq \delta \), and in either case, \( \delta \notin \text{Spec}_b(L) \); in other words, this mapping has closed range, and either finite dimensional nullspace or finite dimensional cokernel, respectively. General functional analysis then gives, for each \( s \in \mathbb{R} \), the existence of a generalized inverse \( G \) for (2.9), which is to say, there exists a bounded map

\[
(2.11) \quad G : x^\delta H^s_e(M) \to x^\delta H^{s+m}_e(M)
\]

which satisfies \( GL = I - P_1 \) and \( LG = I - P_2 \) where \( P_1 \) and \( P_2 \) are the orthogonal projectors onto the nullspace of \( L \) and orthogonal complement of the range of \( L \), respectively. By the simplest form of elliptic regularity in the edge setting, we obtain that \( P_1 \) and \( P_2 \) are both smoothing in the sense that \( P_1 : x^\delta H^{s+m}_e \to x^\delta H^{s+m}_e \) and \( P_2 : x^\delta H^s_e \to x^\delta H^s_e \) are bounded for any \( t > s \).

Much more is true, and one of the strengths of the pseudodifferential edge theory is that it allows one to give a fairly explicit description of the Schwartz kernels of these operators. Fix \( \delta \) as above, and set \( s = 0 \) (to normalize the choice of projectors). Then Theorem 6.1 in [17] asserts that \( G, P_1 \) and \( P_2 \) are all pseudodifferential edge operators. When \( \delta > \overline{\delta} \), then \( P_1 \) has finite rank and maps into the space of polyhomogeneous functions, while when \( \delta < \delta \), then \( P_2 \) has finite rank and maps into the space of polyhomogeneous functions. We shall recall the definitions of these spaces of pseudodifferential operators in §4, but for now point out that this description of their Schwartz kernels has a number of important ramifications. For example, once one establishes a general boundedness theorem for pseudodifferential edge operators on weighted edge Hölder spaces, then it is an immediate consequence of this Sobolev semi-Fredholmness that one can then deduce that for this same value of \( \delta \), the mapping (2.10) is also semi-Fredholm, and that \( P_1 \) and \( P_2 \) are the appropriate projectors in that case too. Indeed, the equations \( GL = I - P_1 \) and \( LG = I - P_2 \) still hold, and all operators are bounded on the appropriate spaces. Note in particular that if \( \delta < \delta \), for example, then the nullspace of (2.10) is infinite dimensional and in this case
it does not follow from general theory that this nullspace is complemented in $x^\delta \Lambda_{m+k}$. Nonetheless, since the infinite rank projectors $P_1$ and $I - P_1$ are bounded, we see that this nullspace has a complement, as claimed.

3. Outline and statement of the main result

We are now in a position to provide a more careful statement of our main results and to sketch the arguments to prove them.

There are several closely related conceptual frameworks for studying elliptic boundary problems; the one we follow here is very close to the one developed by Boutet de Monvel [4], and used in many other places since, including by Schulze [23,24] for edge operators. This theory is centered around the idea of extending the use of ‘interior’ pseudodifferential edge operators by introducing the associated spaces of trace and Poisson operators, as well as boundary operators along the edge $B$.

What distinguishes our approach here is the focus on the geometric structures of the Schwartz kernels of these various types of operators. As in [17], any one of these operators has a Schwartz kernel which is a polyhomogeneous distribution on a certain blown up space. Section 4 describes all of this more carefully. Amongst the tasks we must face is to show that the composition of an interior edge operator and a trace operator (interior to boundary) is again a trace operator, and similarly the composition of a Poisson operator (boundary to interior) with an interior edge operator is again of Poisson type. These composition formulæ are perhaps the most technically demanding part of this presentation.

The operators which arise in these elliptic boundary problems are of a somewhat more special type, which we call representable. This is described in §5, where we introduce these subclasses of interior, trace and Poisson edge operators and examine their normal operators.

Following these more general ‘structural’ definitions and results, we turn to the analysis specific to elliptic differential edge operators. In the steps below, we first define each object at the level of Bessel operators, where the issues are typically finite dimensional. We then rescale and take inverse Fourier transforms and obtain the corresponding objects at the level of normal operators. Although everything becomes infinite dimensional, it is still completely equivalent to the finite dimensional problem. The last step is to extend each object from the normal operator level to that of the actual operators, and this is where the special class of representable operators becomes important.
The starting point is to identify the spaces on which the boundary trace map is well defined. We define

\[
\mathcal{H}_B^{(L)} = \{ u \in t^\delta H^m(\mathbb{R}^+ \times F_{y_0}; dt \, dz) \mid B_{y_0, \tilde{\eta}}(L)u \in t^\delta L^2 \}
\]

(3.1)

\[
\mathcal{H}_N^{(L)} = \{ u \in s^\delta H^m(\mathbb{R}^b \times F_{y_0}; ds \, dY \, dz) \mid N_{y_0}(L)u \in s^\delta L^2 \}
\]

\[
\mathcal{H}_L^{(L)} = \{ u \in x^\delta H^m(M; dx \, dy \, dz) \mid Lu \in x^\delta L^2 \},
\]

where by implication the second inclusion is supposed to hold for all \(y_0, \tilde{\eta}\).

For simplicity we often denote these simply as \(\mathcal{H}_B, \mathcal{H}_N\) and \(\mathcal{H}\), omitting the subscript \(\delta, \delta\). These are Hilbert spaces with respect to the norms

\[
\|u\|_{\mathcal{H}_B} = \|u\|_{L^2} + \|B(L)u\|_{L^2}
\]

\[
\|u\|_{\mathcal{H}_N} = \|u\|_{L^2} + \|N(L)u\|_{L^2}
\]

\[
\|u\|_{\mathcal{H}_L} = \|u\|_{L^2} + \|Lu\|_{L^2}.
\]

(3.2)

In §6 we construct successively the trace and Poisson operators associated to an elliptic edge operator \(L\) by first constructing the corresponding operators for \(B(L)\) and \(N(L)\). Since \(B(L)\) is Fredholm at all nonindicial weights, most of the considerations for it are finite dimensional and we may formulate the analogue of the Calderon, or Lopatinski-Schapiro conditions directly. The starting point is that \(\mathcal{H}_B\) is the natural domain for the boundary trace map for \(B(L)\), and in fact for each \(y_0 \in B\),

\[
\text{Tr}_{B(L)} : \mathcal{H}_B \longrightarrow \mathcal{E}_{y_0} := \bigoplus \mathcal{E}_{y_0}(L, \zeta_j),
\]

where \(\mathcal{E}_{y_0}(L, \zeta_j)\) is the fibre of the trace bundle \((2,5)\) associated to the indicial root \(\zeta_j\) at \(y_0\) and the direct sum is over all indicial roots with imaginary part in the interval \((\delta - 1/2, \delta - 1/2)\). The corresponding trace map for the normal operator \(N(L)\) is obtained by rescaling and taking the inverse Fourier transform, and

\[
\text{Tr}_{N(L)} : \mathcal{H}_N \longrightarrow \bigoplus H^{-(3(\zeta_j) - \delta + 1/2)}(\mathbb{R}^b, \mathcal{E}_{y_0}).
\]

The trace map for \(L\) itself is bounded as a map

\[
\text{Tr}_L : \mathcal{H}_L \longrightarrow \bigoplus H^{-(3(\zeta_j) - \delta + 1/2)}(B; \mathcal{E}).
\]

In a similar way, we construct the Poisson edge operators \(P_{B(L)}, P_{N(L)}\) and \(P_L\) as

\[
P_L : \bigoplus H^{-(3(\zeta_j) - \delta + 1/2)}(B, \mathcal{E}(\zeta_j)) \longrightarrow \ker L \cap x^\delta H^\infty_e(M).
\]
By construction, $P_L \circ \text{Tr}_L$ is the identity on $\ker L \cap \mathcal{H}$. The Calderon subspaces

$$
\mathcal{C}_{B(L)} = \text{Tr}_{B(L)}(\ker B(L) \cap \mathcal{H}^B), \quad \mathcal{C}_{N(L)} = \text{Tr}_{N(L)}(\ker N(L) \cap \mathcal{H}^N), \quad \text{and}
$$

$$
\mathcal{C}_L = \text{Tr}_L(\ker L \cap \mathcal{H})
$$

are of fundamental importance. For $B(L)$ this subspace depends smoothly on $(y_0, \hat{\eta})$, and for $N(L)$ it depends smoothly on $y_0$.

Let us now explain how to formulate a boundary problem for the edge operator $L$. Fix a vector bundle $W$ over $B$ and a pseudodifferential operator $Q : C^\infty(B, \mathcal{E}) \to C^\infty(B, W)$.

For many operators of interest, $W$ splits as a finite direct sum $\bigoplus W_k$, and of course $\mathcal{E}$ also splits into the summands corresponding to each indicial root, so $Q$ has a matrix form $(Q_{jk})$ where the different components may have different orders.

**Definition 3.1.** With all notation as above, an edge boundary value problem $(L, Q)$ is a system

$$
Lu = f \in x^5L^2(M), \quad u \in \mathcal{H}_{s, \delta} \subset x^4H^m_e(M),
$$

$$
Q(\text{Tr}_Lu) = \phi \in \bigoplus_{k=1}^{M} H^{4-d_k-1/2}(B, W_k).
$$

As in the classical theory on a manifold with boundary, the determina nt of whether this problem is Fredholm is formulated using the (left or right) invertibility of the principal symbol of the boundary conditions restricted to the Calderon subspace:

**Definition 3.2.** The boundary conditions $Q$ of an edge boundary value problem $(L, Q)$ are

(i) right-elliptic if $\sigma(Q)(y_0, \hat{\eta}) | \mathcal{C}_{B(L)(y_0, \hat{\eta})} : \mathcal{C}_{B(L)(y_0, \hat{\eta})} \to \pi^*W_{y_0}$ is surjective,

(ii) left-elliptic if $\sigma(Q)(y_0, \hat{\eta}) | \mathcal{C}_{B(L)(y_0, \hat{\eta})} : \mathcal{C}_{B(L)(y_0, \hat{\eta})} \to \pi^*W_{y_0}$ is injective, and

(iii) elliptic if $\sigma(Q)(y_0, \hat{\eta}) | \mathcal{C}_{B(L)(y_0, \hat{\eta})} : \mathcal{C}_{B(L)(y_0, \hat{\eta})} \to \pi^*W_{y_0}$ is an isomorphism

for all $(y_0, \hat{\eta}) \in S^*B$, where $\pi : S^*B \to B$ is the standard projection.

The final section, §7, assembles the various types of operators considered earlier to construct parametrices in each of these three cases. Our main result is the
Theorem 3.3. Let \((L, Q)\) be right-elliptic. Let \(G\) be the generalized inverse for \(L\) on \(x^\delta L^2\). Then
\[
(L, Q) : (\mathcal{H}, \| \cdot \|_\mathcal{H}) \to x^\delta L^2(M) \oplus \left( \bigoplus_{k=1}^M H^{\frac{d_k-1}{2}}(B, W_k) \right),
\]
is semi-Fredholm with right parametrix
\[
G(f, \phi) = Gf + P_L[K(\phi - Q(\text{Tr}_L Gf))].
\]
In particular, \((L, Q)\) has closed range of finite codimension.

Theorem 3.4. Let \((L, Q)\) be left-elliptic. Then
\[
(L, Q) : (\mathcal{H}, \| \cdot \|_\mathcal{H}) \to x^\delta L^2(M) \oplus \left( \bigoplus_{k=1}^M H^{\frac{d_k-1}{2}}(B, W_k) \right),
\]
is semi-Fredholm with left parametrix
\[
G(f, \phi) = Gf + P_L[K(\phi - Q(\text{Tr}_L Gf))].
\]
In particular, \((L, Q)\) has a finite-dimensional kernel.

These results together prove that an elliptic edge boundary problem gives a Fredholm mapping.

4. INTERIOR, TRACE AND POISSON EDGE OPERATORS

In this section we recall the space of pseudodifferential edge operators and introduce the corresponding spaces of trace and Poisson operators. As explained earlier, our focus is on the Schwartz kernels of these operators, in particular their structure as polyhomogeneous distributions. We keep the notation of the preceding sections.

The definitions below are phrased in the language of manifolds with corners and various spaces of conormal or polyhomogeneous functions on them, so we review some of this now. A manifold with corners is a space locally diffeomorphically modelled on neighbourhoods in the standard orthant \((\mathbb{R}^+)^\ell \times \mathbb{R}^{n-\ell}\). A standing assumption is that every boundary face of a manifold with corners is embedded. This implies, in particular, that if \(H\) is a boundary hypersurface, then there is a globally defined boundary defining function \(\rho_H\) which vanishes precisely on \(H\) and is strictly positive everywhere else, and is such that \(d\rho_H \neq 0\) at \(H\).

The most useful and natural classes of ‘smooth’ functions on a manifold with corners \(\mathcal{M}\) are the conormal and polyhomogeneous distributions. Let \(\{(H_i, \rho_i)\}_{i=1}^N\) enumerate the boundary hypersurfaces and corresponding defining functions of \(\mathcal{M}\). For any multi-index \(b = (b_1, \ldots, b_N) \in \mathbb{C}^N\)
set \( \rho^b = \rho_1^{b_1} \cdots \rho_N^{b_N} \). Similarly, for \( p = (p_1, \ldots, p_N) \in \mathbb{N}_0^N \), we write \((\log \rho)^p = (\log \rho_1)^{p_1} \cdots (\log \rho_N)^{p_N}\). Finally, let \( \mathcal{V}_b(W) \) be the space of all smooth vector fields on \( W \) which are unconstrained in the interior but which lie tangent to all boundary faces.

**Definition 4.1.** A distribution \( u \) on \( W \) is said to be conormal of order \( b \) at the faces of \( W \), written \( u \in A^b(W) \), if \( u \in \rho^b L^\infty(W) \) for some \( b \in \mathbb{C}^N \) and \( V_1 \cdots V_{\ell} u \in \rho^b L^\infty(W) \) for all \( V_j \in \mathcal{V}_b(W) \) and for every \( \ell \geq 0 \).

An index set \( E \) is a collection of pairs \( \{(\gamma, p)\} \subset \mathbb{C} \times \mathbb{N}_0 \) satisfying the following hypotheses:

(i) \( \Re \gamma \) accumulates only at plus infinity, while the second index \( p \) for a given \( \gamma \) is bounded above by a constant depending on \( \gamma \), i.e. \( p \leq P_\gamma < \infty \);

(ii) If \( (\gamma, p) \in E \), then \( (\gamma + j, p') \in E \) for all \( j \in \mathbb{N} \) and \( 0 \leq p' \leq p \).

An index family \( E = (E_1, \ldots, E_N) \) is an \( N \)-tuple of index sets associated to each of the boundary hypersurfaces of \( W \). In the rest of this paper, we typically let \( k \) stand for the simple index set \( \{(k + \ell, 0) : \ell \in \mathbb{N}_0\} \).

A conormal distribution \( u \) on \( W \) is said to be polyhomogeneous with index family \( E \), \( u \in A^E_p \), and if in addition, near each \( H_i \),

\[
    u \sim \sum_{(\gamma, p) \in E_i} a_{\gamma, p} \rho_i^2 (\log \rho_i)^p, \quad \text{as} \quad \rho_i \to 0,
\]

with coefficients \( a_{\gamma, p} \) conormal on \( H_i \), polyhomogeneous with index \( E_j \) at any \( H_i \cap H_j \). We also require that \( u \) have product type expansions at all corners of \( W \).

A \( p \)-submanifold in a manifold with corners \( W \) is an embedded submanifold with the property that if \( p \in S \), then it is possible to choose coordinates \( (x, y) \in (\mathbb{R}^+)^k \times \mathbb{R}^{n-k} \) for \( W \) with \( p = (0, 0) \), and such that \( S = \{(x, y) : x'' = 0, y'' = 0\} \), where \( x = (x', x'') \) and \( y = (y', y'') \) are some subdivisions of these sets of coordinates. In other words, \( W \) has a product structure near \( S \). We may then define the new manifold with corners \([W, S]\) by blowing up \( W \) around \( S \). This consists of taking the disjoint union \( W \setminus S \) and the inward-pointing normal bundle of \( S \), and endowing this set with the structure of a smooth manifold with corners, with the unique minimal differential structure so that smooth functions on \( W \) and polar coordinates around \( S \) all lift to be smooth. This blown up space has a ‘front face’, which is a new boundary hypersurface which projects down to \( S \) in the ‘blowdown’; it is the total space of a fibration over \( S \) with fibre some spherical orthant.
4.1. **Pseudodifferential edge operators.** Let $M^2_e$ denote the double edge space, which is obtained by blowing up the fibre diagonal of $(\partial M)^2$ in the product $M^2$, $M^2_e = [M^2; \text{fdiag}]$. In standard adapted local coordinates $(x, y, z)$ on $M$ near $\partial M$, with $(\tilde{x}, \tilde{y}, \tilde{z})$ a copy of these coordinates on the other factor of $M$ in $M^2$, the fibre diagonal $\text{fdiag}$ is the submanifold \{x = \tilde{x} = 0, y = \tilde{y}\}; it is the total space of a fibration over $\text{diag}(B \times B)$ with fibre $S^n_+ \times F \times F$. The space $M^2_e$ is a manifold with corners up to codimension three; there are three boundary hypersurfaces, denoted $\text{ff}$ (the front face), $\text{lf}$ (the left face) and $\text{rf}$ (the right face). The front face is the one created by the blowup; it is the total space of a fibration over $\text{fdiag}$ with each fibre a copy of the quarter-sphere $\{\omega = (\omega_0, \omega', \omega_n) \in S^n : \omega_0, \omega_n \geq 0\}$.

It is often more convenient to use projective coordinates rather than polar coordinates. Thus away from $\text{rf}$, we use

\[(4.1) \quad s = \frac{x}{\tilde{x}}, \quad Y = \frac{y - \tilde{y}}{\tilde{x}}, \quad z, \tilde{x}, \tilde{y}, \tilde{z},\]

where $\tilde{x}$ and $s$ are defining functions of $\text{ff}$ and $\text{rf}$, respectively. Note that in these coordinates, $\text{ff}$ is the face where $\tilde{x} = 0$. There are analogous coordinates valid away from $\text{rf}$, obtained by interchanging the roles of $x$ and $\tilde{x}$.

Figure 1 illustrates $M^2_e$

![Figure 1. The edge double space $M^2_e$.](image)

This space has a distinguished submanifold, the edge diagonal $\text{diag}_e$, which is the lift of the diagonal to $M^2_e$. (Strictly speaking, it is the closure of the lift of the interior of the diagonal.)

A linear operator $A$ on $M$ is called a pseudodifferential edge operator of order $m$ and with index family $\mathcal{E}$, $A \in \Psi_m^e(M)$, if the lift of its Schwarz kernel $K_A$ to $M^2_e$ is polyhomogeneous distribution on this space, where the index sets $\mathcal{E} = (E_{\text{ff}}, E_{\text{lf}}, E_{\text{rf}})$ describe the expansions at the three faces. The superscript $-\infty$ indicates the pseudodifferential order, hence the lifted Schwartz kernel is smooth along $\text{diag}_e$. The full space of pseudodifferential edge operators, $\Psi_{-\infty}^e(M)$, consists of the space of sums $A + B$ where $A$ is an
operator of order $-\infty$ as above, and where the lift of the Schwartz kernel of $B$ to $M_2^\partial$ is supported near diag$_e$, has a classical conormal singularity along that submanifold, and is smoothly extendible (after factoring out a certain singular density) across ff. To understand the singular density here, note that the identity operator has Schwartz kernel which lifts as

$$\delta(x - \tilde{x})\delta(y - \tilde{y})\delta(z - \tilde{z})d\tilde{x}d\tilde{y}d\tilde{z} = \delta(s - 1)\delta(Y)\delta(z - \tilde{z})\tilde{x}^{-b-1}d\tilde{x}d\tilde{y}d\tilde{z}.$$ 

This is smoothly extendible across the front face, which in these projective coordinates is where $\tilde{x} = 0$, provided we factor out the final singular measure. In the language above, $\text{Id} \in \Psi^{b,0}_e(M)$.

There is a distinguished subalgebra $\Psi^*_e(M)$, called the small calculus, which consists of operators which vanish to infinite order at the left and right faces, $E_{\text{lf}} = E_{\text{rf}} = \emptyset$, and with $E_{\text{ff}} = 0$. The residual calculus $\Psi^{-\infty,0}_e,E_{\text{lf}},E_{\text{rf}}(M)$ consists of operators with no singularity along the lifted diagonal and with standard index set 0 at the front face.

Many details have been suppressed here, and we refer to [17] where all of this is described more carefully.

4.2. Edge trace operators. Whereas the edge operators introduced in the previous subsection map functions on $M$ to functions on $M$, the other two classes of operators we consider map functions on $M$ to functions on $\partial M$ (these are the edge trace operators) or functions on $\partial M$ to functions on $M$ (these are the edge Poisson operators). We now describe the former of these.

An edge trace operator $T$ is again described in terms of the lifting properties of its Schwartz kernel. Initially this Schwartz kernel is a distribution on $\partial M \times M$; this space has the same distinguished submanifold as before, namely the fibre diagonal of $(\partial M)^2$, $\text{fdiag} = \{\tilde{x} = 0, y = \tilde{y}\}$. We define the edge trace double space

$$T^2_e = [\partial M \times M; \text{fdiag}]$$

note that this is nothing other than the right face rf of $M_2^\partial$. It has two boundary hypersurfaces, the new front face of which, still denoted here by ff, is simply one boundary face of the front face of $M_2^\partial$, and hence a bundle of hemispheres $S^{n-1}_+\tilde{e}$ over fdiag. The lift of the original face here is denoted of, and still called the original face.

We can use the same projective coordinates as before, namely $(Y, z, \tilde{x}, \tilde{y}, \tilde{z})$ with $Y = (y - \tilde{y})/\tilde{x}$. Figure 2 illustrates this space.

**Definition 4.2.** The space $\Psi^{k,F_{\text{lit}}}_e(M)$ of trace operators of order $k \in \mathbb{N}_0$ is the space of all operators $T$ with Schwartz kernels $K_T$ which are pushforwards from polyhomogeneous conormal distributions $\kappa_T$ on the trace blowup space
Figure 2. The trace blowup $T^2_e$.

$T^2_e$ which have index set $F_{of}$ at the original face, and index set $F_{ff} = (-1 - b + k) + \mathbb{N}_0$ at the front face.

4.3. **Edge Poisson operators.** The last class of operators we define are those which act from functions on $\partial M$ to functions on $M$. The ones amongst these in which we are particularly interested are analogues of the classical Poisson operators, and hence take functions on the boundary to functions in the interior which are solutions of an elliptic edge operator $L$. However, it is advantageous to consider the full class of all operators with the relevant structure.

The Schwartz kernel of an edge Poisson operator $P$ is a distribution on $M \times \partial M$, and as usual, we consider distributions which lift to be polyhomogeneous on the edge Poisson double space $P^2_e$, obtained from $M \times \partial M$ by blowing up the same fibre diagonal $\text{fdiag}$. The space $P^2_e$ is naturally identified with the left face $lf$ of $M^2_e$; it has two boundary hypersurfaces, the front face $ff$, which is ‘the other’ boundary hypersurface of the front face of $M^2_e$, and the original face of. We often use projective coordinates $(x, z, Y, \tilde{y}, \tilde{z})$ with $Y = (y - \tilde{y})/x$. It is illustrated in Figure 3 (which is just the ‘transpose’ of Figure 2).

Figure 3. The Poisson double space $P_e$. 

Definition 4.3. The space $\Psi_k^{k,J} (P^2_e)$ of edge Poisson operators of order $k \in \mathbb{N}_0$ is the space of all operators $P$ with Schwartz kernels $K_P$ which are pushforwards from polyhomogeneous conormal distributions $\kappa_P$ on the edge Poisson double space $P^2_e$, with index sets $J_{of}$ at the original face, and $J_{ff} = (-1 - b + k) + \mathbb{N}_0$ at the front face.

Comparing with the Boutet de Monvel calculus, one expects that we should also include operators mapping functions on $\partial M$ to functions on $\partial M$. Indeed, a complete analogue of that calculus (as in the work of Schulze) would indeed include these, but this is not necessary for our purposes here. Note that the operators of this type we would need are not of any particularly standard type; their Schwartz kernels on $(\partial M)^2$ should be conormal at the fibre diagonal $f_{diag}$, rather than the diagonal of the boundary. These are, in some sense, lifts of pseudodifferential operators from $B^2$ to $(\partial M)^2$.

4.4. Composition formulæ. The key fact which makes the definitions above useful is that these classes of operators are closed under composition. This statement must be qualified to account for two issues. The first is the trivial observation that one can only compose operators of the appropriate types, e.g. $T \circ A$ is defined if $A$ is an interior edge operator and $T$ is an edge trace operator, and similarly, $A \circ P$ is defined if $P$ is an edge Poisson operator and $A$ an interior edge operator, but of course $P \circ T$ is not defined, etc. More seriously, however, even when composing two interior edge operators, the composition may not be defined because of integrability issues. Thus if $A \in \Psi^k_{e,e}$ and $A' \in \Psi^{k}_{e',e'}$, then $A \circ A'$ is defined only if $E_{ef} + E'_{ef} > -1$ (this lower bound depends on the choice of reference measure). The full composition theorem for interior edge operators is proved in [17], and we prove here the analogous results for compositions involving edge Poisson and trace operators. The main point in all of this is the more subtle fact that if two operators have Schwartz kernels which lift to be polyhomogeneous on the appropriate blown-up space, then the same is true for the composition. This can be verified ‘by hand’, breaking up the regions of integration into different neighbourhoods and using projective coordinate systems to check the polyhomogeneity of these localized integrals. There is a much more elegant and conceptual way, due to Melrose, and employed in [17] (and many other places), using the ‘pushforward theorem’. This states that under appropriate conditions on a map $f : X \to X'$ between two manifolds with corners, the pushforward of a polyhomogeneous distribution is polyhomogeneous. We review this result now and apply it to state the composition formulæ.
First introduce some terminology. Let $X$ and $X'$ be two compact manifolds with corners, and $f : X \to X'$ a smooth map. Let $\{H_i\}$ and $\{H'_j\}$ be enumerations of the codimension one boundary faces of $X$ and $X'$, respectively, and let $\rho_i$, $\rho'_j$ be global defining functions for $H_i$, resp. $H'_j$. We say that the map $f$ is a $b$-map if

$$f^*\rho'_j = A_{ij} \prod_i e^{e(i,j)}, \quad 0 < A_{ij} \in C^\infty(X), \quad e(i,j) \in \mathbb{N} \cup \{0\};$$

in other words, $f^*\rho'_j$ vanishes to constant order along each boundary face of $X$. In particular, this means that if $f(H_i) \cap H'_j \neq \emptyset$, then $f(H_i) \subset H'_j$, and the order of vanishing of $f$ in the direction normal to $H_i$ is constant along the entire face.

Next, $f$ is called a $b$-submersion if $f_*$ induces a surjective map between the $b$-tangent bundles of $X$ and $X'$. (The $b$-tangent space at a point $p$ of $\partial X$ on a codimension $k$ corner is spanned locally by the sections $x_1 \partial x_1, \ldots, x_k \partial x_k, \partial y_j$, where $x_1, \ldots, x_k$ are the defining functions for the faces meeting at $p$ and the $y_j$ are local coordinates on the corner through $p$.) Finally, if we require that $f$ is not only a $b$-submersion, but that in addition, for each $j$ there is at most one $i$ such that $e(i,j) \neq 0$ (this condition simply means that each hypersurface face $H_i$ in $X$ gets mapped into at most one $H'_j$ in $X'$, or in other words, no hypersurface in $X$ gets mapped to a corner in $X'$), then $f$ is called a $b$-fibration.

Let $\nu_0$ be any smooth density on $X$ which is everywhere nonvanishing and smooth up to all boundary faces of $X$. A smooth $b$-density $\nu_b$ is, by definition, any density of the form $\nu_b = \nu_0(\Pi \rho_i)^{-1}$. Fix smooth nonvanishing $b$-densities $\nu_b$ on $X$ and $\nu'_b$ on $X'$.

**Proposition 4.4** (The Pushforward Theorem (Melrose)). Let $u$ be a polyhomogeneous function on $X$ with index set $E_j$ at the face $H_j$, for all $j$. Suppose that if $e(i,j) = 0$ for all $i$, i.e. $H_j$ is mapped to the interior of $X'$, then $\text{Re} z > 0$ for all $(z,p) \in E_j$. In this case, the pushforward $f_*(u\nu_b)$ is well-defined and equals $h\nu'_b$ where $h$ is polyhomogeneous on $X'$ and has an index family $f_b(\mathcal{E})$ given by an explicit formula in terms of the index family $\mathcal{E}$ for $X$.

We do not state the formula for the index set of the pushforward in generality, but give an informal description sufficient for the present situation. If $H_{j_1}$ and $H_{j_2}$ are both mapped to a face $H'_i$, and if $H_{j_1} \cap H_{j_2} = \emptyset$, then the pushforward has index set $E_{j_1} + E_{j_2}$ at $H'_i$. If they do intersect, then the contribution is the extended union $H_{j_1} \cup H_{j_2}$. For any two index sets $E, E'$
their extended union $E \overline{\cup} E'$ is defined by

$$E \overline{\cup} E' = E \cup E' \cup \{(z, p + q + 1) : \exists (z, p) \in E, \text{ and } (z, q) \in E'\}. \tag{4.2}$$

After these generalities, we can now state the composition results between interior and Poisson operators and between Poisson and trace operators. Note that the composition formula between trace and interior operators is the adjoint of the interior-Poisson composition, so we do not state it separately.

4.4.1. Interior $\circ$ Poisson. Let $G$ be an interior edge operator and $P$ a Poisson edge operator and consider the (only possible) composition $A = G \circ P$. To show that this is again a Poisson edge operator, we must verify that the Schwartz kernel of this composition lifts to be polyhomogeneous on $P_e$ and has the stated index sets. This is accomplished by constructing the interior-Poisson triple space $M^3_{i-p}$, obtained by a sequence of blowups from $M \times M \times \partial M$. Recall the fibre diagonal $f_{diag}$ which is blown up in the definitions of the interior edge and Poisson operators. Here, using local coordinates $(x, y, z)$, $(x', y', z')$ and $(y'', z'')$ in the three factors, $f_{diag} := \{x = x' = 0, y = y'\}$ is the fibre diagonal that needs to blown for polyhomogeneity of $G$, $f_{diag}_p := \{x' = 0, y' = y''\}$ is the fibre diagonal that needs to blown for polyhomogeneity of $P$, and finally $f_{diag}_a := \{x = 0, y = y''\}$ is the fibre diagonal that needs to blown for polyhomogeneity of $A$. All the three sub-manifolds intersect at $f_{diag}_0 := \{x = x' = x'' = 0, y = y' = y''\}$. We define the triple space by

$$M^3_{i-p} := [[M \times M \times \partial M; f_{diag}_0]; f_{diag}_g, f_{diag}_p, f_{diag}_a].$$

Then there exist natural projections

$$\pi_p : M^3_{i-p} \to P^2_e \times M_{(x,y,z)} \to P^2_e,$$
$$\pi_a : M^3_{i-p} \to P^2_e \times M_{(x',y',z')} \to P^2_e,$$
$$\pi_g : M^3_{i-p} \to M^2_e \times \partial M_{(y'',z'')} \to M^2_e.$$ 

The maps $\pi_*$ are $b$-fibrations by construction of the triple space. The triple space is also equipped with the natural blowdown map $\beta_3 : M^3_{i-p} \to M \times M \times \partial M$. We also consider the natural blowdown maps $\beta_2 : M^2_e \to M^2$ and $\beta_1 : P^2_e \to M \times \partial M$.

The Schwartz kernel $K_G$ of an interior edge operator $G \in \Psi^{-\infty,\gamma;E_t_0,E_t_0}(M^2_e)$ lifts to a polyhomogeneous conormal distribution $\kappa_G = \beta^*_2 K_G$ on $M^2_e$ of leading order $(-1 - b + \gamma)$ at the front face. The Schwartz kernel $K_P$ of an edge Poisson operator $P \in \Psi^{-\infty,\rho;E_0,E_t}(P^2_e)$ lifts to a polyhomogeneous conormal distribution $\kappa_P = \beta^*_1 K_P$ on $P^2_e$ of leading order $(-1 - b + \rho)$ at the
front face. The kernel of the composition $A = G \circ P$ can be expressed using pullbacks and pushforwards as

$$\kappa_A := \beta_1^*(K_A) = (\pi_a)_* [\pi_g^* \kappa_G \cdot \pi_p^* \kappa_P].$$

Applying the pushforward theorem we obtain the

**Theorem 4.5.** If $\Re E_{tt} + \Re J_{tt} > -1$ then

$$\Psi_e^{-\infty, \gamma, E_{tt}, E_{tt}}(M^2_e) \circ \Psi_e^{-\infty, \rho, J_{tt}}(P^2_e) \subset \Psi_e^{-\infty, \gamma + \rho, E_{tt}}(P^2_e).$$

**Proof.** In view of the pushforward theorem, it remains to identify the leading order behaviour at the various boundary faces. Denote the boundary defining functions of the boundary faces introduced by blowing up $\text{diag}_{a}$ by $\rho_*$, where $* \in \{0, g, p, a\}$. We write $\rho_{t}$ for the front face defining functions in the double spaces $M^2_e$ and $P^2_e$. The defining functions of the boundary faces $\{x = 0\}$ and $\{x' = 0\}$ in $M^2_{e-p}$ are denoted by $\rho_r$ and $\rho_l$, respectively. The defining functions of the boundary faces $\{x = 0\}$ and $\{x' = 0\}$ in either $M^2_e$ or $P^2_e$ are denoted by $\rho_{tt}$ and $\rho_{tt}$, respectively. We then obtain

$$\pi_g^*(\rho_{tt}) = \rho_0 \rho_g, \quad \pi_g^*(\rho_{tt}) = \rho_{r,l},$$

$$\pi_p^*(\rho_{tt}) = \rho_0 \rho_p, \quad \pi_p^*(\rho_{tt}) = \rho_l,$$

$$\pi_a^*(\rho_{tt}) = \rho_0 \rho_a, \quad \pi_a^*(\rho_{tt}) = \rho_r.$$

We denote by $\nu_3$ a $b$-volume on $M^3_{e-p}$ and by $\nu_1$ a $b$-volume on $P^2_e$. We compute

$$\beta_3^* (dx \, dy \, dz \, dx' \, dy' \, dz' \, dy'' \, dz'') = \rho_0^{2+2b}(\rho_g \rho_p \rho_r \rho_l)^{1+b} \nu_3,$$

$$\beta_1^* (dx \, dy \, dz \, dy'' \, dz'') = \rho_{tt}^{1+b} \nu_1.$$

The leading order behaviour of $\kappa_A$ at the various boundary faces of $P^2_e$ follows now from the following computation

$$\kappa_A := \beta_1^* (dx \, dy \, dz \, dy'' \, dz'') = \beta_1^* (K_A \, dx \, dy \, dz \, dy'' \, dz''),$$

$$= (\pi_a)_*[\pi_g^* \kappa_G \cdot \pi_p^* \kappa_P \cdot \beta_3^* (dx \, dy \, dz \, dx' \, dy' \, dz' \, dy'' \, dz'')]$$

$$= (\pi_a)_*[\rho_0^{1+b} \rho_g \rho_p \rho_r \rho_l^{1+b} \rho_{tt}^{1+b} \rho_{tt}^{1+b} \nu_3]$$

$$= \rho_{tt}^{-1+b+\gamma + \rho \cdot E_{tt} \cdot \beta_1^*(dx \, dy \, dz \, dy'' \, dz'').$$

This proves the statement. \qed
4.4.2. Poisson \circ trace. Let \( P \) be a Poisson and \( T \) a trace edge operator, and consider the (only possible) composition \( G = P \circ T \). To show that this is again an interior edge operator, we must verify that the Schwartz kernel of this composition lifts to be polyhomogeneous conormal on \( M^2_e \) and has the stated index sets. This is accomplished by constructing the Poisson-trace triple space \( M^3_{p-t} \), obtained by a sequence of blowups from \( M \times \partial M \times M \).

Recall the fibre diagonal \( \text{fdiag} \) which is blown up in the definitions of the Poisson and the trace edge operators. Here, using local coordinates \((x, y, z)\), \((y', z')\) and \((x'', y'', z'')\) in the three factors, \( \text{fdiag}_p := \{x = 0, y = y'\} \) is the fibre diagonal that needs to blown for polyhomogeneity of \( P \), \( \text{fdiag}_t := \{x'' = 0, y'' = y''\} \) is the fibre diagonal that needs to blown for polyhomogeneity of \( T \), and finally \( \text{fdiag}_g := \{x = x'' = 0, y = y''\} \) is the fibre diagonal that needs to blown for polyhomogeneity of \( G \). All the three submanifolds intersect at \( \text{fdiag}_0 := \{x = x'' = 0, y = y' = y''\} \). We define the triple space by

\[
M^3_{p-t} := \left[ [M \times \partial M \times M] ; \text{fdiag}_0 \right] ; \text{fdiag}_g, \text{fdiag}_p, \text{fdiag}_t.
\]

Then there exist natural projections

- \( \pi_p : M^3_{i-p} \to P^2_e \times M_{(x'', y'', z'')} \to P^2_e \),
- \( \pi_t : M^3_{i-p} \to T^2_e \times M_{(x, y, z)} \to T^2_e \),
- \( \pi_g : M^3_{i-p} \to M^2_e \times \partial M_{(y', z')} \to M^2_e \).

The maps \( \pi_* \) are \( b \)-fibrations by construction of the triple space. The triple space is also equipped with the natural blowdown map \( \beta_3 : M^3_{p-t} \to M \times \partial M \times M \). We also consider the natural blowdown maps \( \beta_g : M^2_e \to M^2 \), \( \beta_p : P^2_e \to M \times \partial M \) and \( \beta_t : T^2_e \to \partial M \times M \).

The Schwartz kernel \( K_P \) of an edge Poisson operator \( P \in \Psi_{e^{-\infty, \rho, J,t}}(P^2_e) \) lifts to a polyhomogeneous conormal distribution \( \kappa_P = \beta^*_p K_P \) on \( P^2_e \) of leading order \((-1 - b + \rho)\) at the front face. The Schwartz kernel \( K_T \) of an edge trace operator \( T \in \Psi_{e^{-\infty, \tau, F,it}}(T^2_e) \) lifts to a polyhomogeneous conormal distribution \( \kappa_T = \beta^*_t K_T \) on \( T^2_e \) of leading order \((-1 - b + \tau)\) at the front face. The kernel of the composition \( G = P \circ T \) can be expressed using pullbacks and pushforwards as

\[
\kappa_G := \beta^*_2(K_G) = (\pi_g)_* [\pi^*_p \kappa_P \cdot \pi^*_t \kappa_T].
\]

Applying the pushforward theorem we obtain the

**Theorem 4.6.**

\[
\Psi_{e^{-\infty, \rho, J,t}}(P^2_e) \circ \Psi_{e^{-\infty, \tau, F,it}}(T^2_e) \subset \Psi_{e^{-\infty, \rho + \tau, J,it}}(M^2_e).
\]

**Proof.** In view of the pushforward theorem, it remains to identify the leading order behaviour at the various boundary faces. Denote the boundary
defining functions of the boundary faces introduced by blowing up \( f_{\text{diag}} \), by \( \rho_* \), where \( * \in \{0, g, p, t\} \). We write \( \rho_{\Pi} \) for the front face defining functions in the double spaces \( M_e^2 \) and \( P_e^2, T_e^2 \). The defining functions of the boundary faces \( \{x = 0\} \) and \( \{x' = 0\} \) in \( M_{i-p}^3 \) are denoted by \( \rho_r \) and \( \rho_t \), respectively. The defining functions of the boundary faces \( \{x = 0\} \) and \( \{x' = 0\} \) in either \( M_e^2 \) or \( P_e^2, T_e^2 \) are denoted by \( \rho_{\Pi t} \) and \( \rho_{\Pi f} \), respectively. We then obtain

\[
\begin{align*}
\pi^p_\rho(\rho_{\Pi}) &= \rho_0 \rho_p, \quad \pi^p_\rho(\rho_{\Pi t}) = \rho_r, \\
\pi^p_\rho(\rho_{\Pi}) &= \rho_0 \rho_p, \quad \pi^p_\rho(\rho_{\Pi t}) = \rho_t, \\
\pi^p_\rho(\rho_{\Pi}) &= \rho_0 \rho_g, \quad \pi^p_\rho(\rho_{\Pi t, \Pi}) = \rho_{r, t}.
\end{align*}
\]

We denote by \( \nu_3 \) a \( b \)-volume on \( M_{i-p}^3 \) and by \( \nu_2 \) a \( b \)-volume on \( M_e^2 \). We compute

\[
\begin{align*}
\beta^3_g(dx \, dy \, dz \, dx' \, dy' \, dz') &= \rho_0^{2+2b}(\rho_p \rho_t \rho_{r, t})^{1+b} \nu_3, \\
\beta^2_g(dx \, dy \, dz \, dx'' \, dy'' \, dz'') &= \rho_{\Pi}^{1+b} \rho_{\Pi t}^{1+b} \rho_{\Pi t}^{1+b} \nu_2.
\end{align*}
\]

The leading order behaviour of \( \kappa_G \) at the various boundary faces of \( M_e^2 \) follows now from the following computation

\[
\begin{align*}
\kappa_G \cdot \beta^g_g(dx \, dy \, dz \, dx'' \, dy'' \, dz'') &= \beta^g_g(K_G dx \, dy \, dz \, dx'' \, dy'' \, dz'') \\
&= (\pi_g)_*[\pi^p_G \cdot \pi^p_T \cdot \pi^p_{\Pi} \cdot \beta^g_g(dx \, dy \, dz \, dy' \, dz') \cdot \beta^g_g(dx'' \, dy'' \, dz'')] \\
&= (\pi_g)_*[\rho_0^{\rho + \tau} \rho_{\Pi t}^{1+b+J_{\Pi}} \rho_{\Pi t}^{1+b+F_{\Pi}} \nu_3] \\
&= \rho_{\Pi f}^{\rho + \tau} \rho_{\Pi t}^{1+b+J_{\Pi}} \rho_{\Pi t}^{1+b+F_{\Pi}} \nu_2 = \rho_{\Pi f}^{1-b+\rho + \tau} \rho_{\Pi t}^{J_{\Pi}} \beta^g_g(dx \, dy \, dz \, dx'' \, dy'' \, dz'').
\end{align*}
\]

This proves the statement. \( \square \)

5. Representable subclass of edge, trace and Poisson operators

Within the more general classes of residual edge, Poisson and trace operators there are subclasses of operators for which the restriction to the front face has a particular representation formula. We call these the subclasses of representable operators. We introduce these now, and then show how the composition formulae specialize in this setting, proving in particular that the composition of representable operators is again representable.

5.1. Representable residual edge operators. We may consider \( \mathbb{R}^+ \times F \) as a manifold with boundary with a trivial edge structure, where the base \( B \) reduces to a single point. The corresponding edge double-space thus corresponds to the somewhat simpler \( b \)-double space, from [20], [17], and is denoted \( (\mathbb{R}^+ \times F)^2_b \). This is a manifold with corners, with three boundary
Definition 5.1. Let $G_0(t, z, \tilde{t}, \tilde{z}; \tilde{y}, \tilde{\eta}) \in \mathcal{A}_{\text{phg}}^{\mathcal{E}'}((\mathbb{R}^+ \times F)^2)$, where $\mathcal{E}' = (E_{\text{ff}} = N_0, \mathcal{E} = (E_{\text{ff}}, E_{\text{ff}}'))$, and the lf, rf index sets are constant (at least in the critical range) when varying in smooth parameters $(\tilde{y}, \tilde{\eta}) \in S^*B$.

Following [17, (5.18)], if $G_0$ is a edge Bessel operator, $G_0 \in \Psi^{-\infty,\mathcal{E}'}_b((\mathbb{R}^+ \times F)^2)$, if it satisfies the following two conditions:

(i) $G_0(t, z, \tilde{t}, \tilde{z}; \tilde{y}, \tilde{\eta})$ decreases rapidly as $t \to \infty$, locally uniformly in $(z, \tilde{t}, \tilde{z})$, and as $\tilde{t} \to \infty$, locally uniformly in $(t, z, \tilde{z})$;

(ii) $G_0(t, z, \tilde{t}, \tilde{z}; \tilde{y}, \tilde{\eta})$ admits a polyhomogeneous expansion as $t \to 0$, where the coefficient functions decrease rapidly as $\tilde{t} \to \infty$, uniformly in the other coordinates, and vice versa.

Following [17, (5.18)], if $G_0$ is a edge Bessel operator and $k \in \mathbb{N}_0$, set

\begin{equation}
N_k(G_0) = \int_{\mathbb{R}^b} e^{i\gamma \eta} G_0(s|\eta|, z, |\eta|, \tilde{z}; \tilde{y}, \tilde{\eta})|\eta|^{-k+1}d\eta.
\end{equation}

The proof of [17, Prop. 5.19] shows that $N_k(G_0)$ is polyhomogeneous on the front face ff of the edge double space $(\mathbb{R}^+ \times \mathbb{R}^b \times F)^2$.

It will be convenient below to use the homogeneity rescaling

\begin{equation}
\kappa_\lambda u(x, \cdot) := u(\lambda x, \cdot), \quad x \in \mathbb{R}^+.
\end{equation}

Consider a residual edge operator $\text{Op}(G_0) \in \Psi^{-\infty,k,\mathcal{E}}_e(M^2_e)$. By definition this acts on test functions $u$ supported near $\partial M$ by

\begin{equation}
[\text{Op}(G_0)u](x, y, z) = \int e^{i(y-\tilde{y})\eta} \kappa_{|\eta|} \circ G_0(x, z, \tilde{x}, \tilde{z}; \tilde{y}, \tilde{\eta}) \circ \kappa_{|\eta|}^{-1} u(\tilde{x}, \tilde{y}, \tilde{z})|\eta|^{-k+1}d\eta d\tilde{x} d\tilde{y} d\tilde{z}.
\end{equation}

The Schwartz kernel is thus

\begin{equation}
K_{\text{Op}(G_0)}(x, y, z, \tilde{x}, \tilde{y}, \tilde{z}) = \int_{\mathbb{R}^b} e^{i(y-\tilde{y})\eta} G_0(s|\eta|, z, |\eta|, \tilde{z}; \tilde{y}, \tilde{\eta})|\eta|^{-k+1}d\eta
\end{equation}

\begin{align*}
&= \tilde{x}^{-1-b+k} \int_{\mathbb{R}^b} e^{i\gamma \eta} G_0(s|\eta|, z, |\eta|, \tilde{z}; \tilde{y} + \tilde{x}Y, \tilde{\eta})|\eta|^{-k+1}d\eta \\
&= \tilde{x}^{-1-b+k} N_k(G_0) + O(\tilde{x}^{-b+k}).
\end{align*}

The representable subcalculus of residual edge operators consists of those operators $G \in \Psi^{-\infty,k,\mathcal{E}}_e(M^2_e)$, whose normal operator $N(G)$, defined as the restriction of $\rho^{1+b-k}_{\text{ff}} \kappa_G$ to ff, is given by $N_k(G_0)$ for some $G_0 \in \Psi^{-\infty,\mathcal{E}'}_b(\mathbb{R}^+ \times F)$. 
Recall that if \( L \) is an elliptic edge operator, then for any nonindicial weight \( \delta \in (\delta, \delta) \), there is a generalized inverse \( G \) and projectors \( P_1 \) and \( P_2 \) onto the nullspace and cokernel. By [17] (4.22), the lift of the Schwartz kernel of \( P_1 \to M^2_e \) is polyhomogeneous with index set

\[
E_{\ell f} = \{ (\zeta, p) \in \text{Spec}_e(L) \mid 3 \zeta > \delta - 1/2 \},
\]

\[
E_{\ell f} = \{ (\zeta, p) \in \mathbb{C} \times \mathbb{N}_0 \mid (\zeta + 2\delta, p) \in E_{\ell f} \}, \quad E_{\ell f} = \mathbb{N}_0.
\]

Furthermore, its normal operator \( N(P_1) \) equals \( N_0(P_{01}) \) where \( P_{01} \in \Psi^{-\infty, F}_b((\mathbb{R}^+ \times F)^2) \) is the projector onto the nullspace for the Bessel operator \( B(L) \). Similarly, the lift of the Schwartz kernel of \( P_2 \to M^2_e \) is polyhomogeneous with index set

\[
F_{\ell f} = \{ (\zeta, p) \in \mathbb{C} \times \mathbb{N}_0 \mid (-\zeta - 2\delta - 1, p) \in \text{Spec}_e(L), 3 \zeta > -\delta - 1/2 \},
\]

\[
F_{\ell f} = \{ (\zeta, p) \in \mathbb{C} \times \mathbb{N}_0 \mid (\zeta - 2\delta, p) \in F_{\ell f} \}, \quad F_{\ell f} = \mathbb{N}_0,
\]

and has normal operator \( N(P_2) = N_0(P_{02}) \), where \( P_{02} \in \Psi^{-\infty, \mathcal{F}}_b((\mathbb{R}^+ \times F)^2), \quad \mathcal{F} = (F_{\ell f}, F_{\ell f}, F_{\ell f}) \). Note that if \( \delta > \delta \) then \( P_{01} = 0 \) while if \( \delta < \delta \) then \( P_{02} = 0 \). Finally, the lift of the Schwartz kernel of \( G \) is polyhomogeneous on \( M^2_e \) with index set

\[
H_{\ell f} = E_{\ell f} \overline{\mathcal{F}}_{\ell f}, \quad H_{\ell f} = E_{\ell f} \overline{\mathcal{F}}_{\ell f}, \quad H_{\ell f} = \mathbb{N}_0,
\]

This has normal operator \( N(G) = N_0(G_0) \) for \( G_0 \in \Psi^{-\infty, \mathcal{H}}_b((\mathbb{R}^+ \times F)^2), \quad \mathcal{H} = (H_{\ell f}, H_{\ell f}, H_{\ell f}) \).

5.2. **Representable trace operators.** We next introduce the Bessel trace kernels.

**Definition 5.2.** Let \( T_0(\tilde{t}, z, \tilde{z}; \tilde{y}, \tilde{\eta}) \) be polyhomogeneous on \( F^2 \times \mathbb{R}^+ \), smooth in the interior, and varying smoothly in \( (\tilde{y}, \tilde{\eta}) \in S^* B \), with index sets \( \mathcal{F} = (F_{\ell f}, F_{\ell f}) \) constant (at least in the critical range) when varying in \( (\tilde{y}, \tilde{\eta}) \). Then \( T_0 \) is called a trace Bessel kernel, \( T_0 \in \Psi^{-\infty, \mathcal{F}}_b(F^2 \times \mathbb{R}^+) \), if it satisfies:

(i) \( T_0(\tilde{t}, z, \tilde{z}; \tilde{y}, \tilde{\eta}) \) is rapidly decreasing as \( \tilde{t} \to \infty \), locally uniformly in \( (z, \tilde{z}) \);

(ii) \( T_0(\tilde{t}, z, \tilde{z}; \tilde{y}, \tilde{\eta}) \) admits a polyhomogeneous expansion as \( \tilde{t} \to 0 \), uniformly in the other variables.

The class of representable trace operators consists of those operators \( T \in \Psi^{-\infty, k, F_{\ell f}}(T_e) \), whose normal operator \( N(T) \), defined as the restriction of \( \rho_{\ell f}^{1+b-k} \kappa_T \) to \( \ell f \), is given by

\[
N_k(T_0) := \int_{\mathbb{R}^b} e^{i\mathcal{Y} \eta} T_0(\eta, z, \tilde{z}; \tilde{y}, \tilde{\eta})|\eta|^{-k+1} d\eta,
\]
for some $T_0 \in \Psi^{-\infty,F}_b(F^2 \times \mathbb{R}^+)$. An example is a trace operator $\text{Op}(T_0) \in \Psi^{-\infty,k,F}_e(T_e)$, defined on test functions $u$ supported near $\partial M$ by

$$\text{Op}(T_0)u (y, z) := \int e^{i(y-\tilde{y})\eta} T_0(\tilde{x}, z; \tilde{y}, \tilde{\eta}) \circ \kappa^{-1}_{|\eta} u(\tilde{x}, \tilde{y}, \tilde{z}) |\eta|^{-k} d\eta d\tilde{x} d\tilde{y} d\tilde{z}.$$ 

and extended trivially away from the singular neighborhood. The corresponding operator kernel is given in local coordinates by

$$K_{\text{Op}(T_0)}(y, z, x, \tilde{y}, \tilde{z}) = \int_{\mathbb{R}^b} e^{i(y-\tilde{y})\eta} T_0(\tilde{x}|\eta|, z, \tilde{z}; y, \tilde{\eta}) |\eta|^{-k+1} d\eta$$

$$= \tilde{x}^{-1-b+k} \int_{\mathbb{R}^b} e^{iY\eta} T_0(|\eta|, z, \tilde{z}; y + \tilde{x} Y, \tilde{\eta}) |\eta|^{-k+1} d\eta$$

$$= \tilde{x}^{-1-b+k} N_k(T_0) + O(\tilde{x}^{-b+k}).$$

**5.3. Representable Poisson operators.** Finally, we introduce the Bessel Poisson kernels.

**Definition 5.3.** Let $P_0(t, z, \tilde{z}; \tilde{y}, \tilde{\eta})$ be polyhomogeneous on $\mathbb{R}^+ \times F^2$ with index set $J = (J_{lf}, J_{hf})$, parametrized and varying smoothly in $(\tilde{y}, \tilde{\eta}) \in S^*B$. Then $P_0$ is called a Bessel Poisson operator, $P_0 \in \Psi^{-\infty,J}_b(\mathbb{R}^+ \times F^2)$, if:

(i) $P_0(t, z, \tilde{z}; \tilde{y}, \tilde{\eta})$ is rapidly decreasing as $t \to \infty$, locally uniformly in $(z, \tilde{z})$;

(ii) $P_0(t, z, \tilde{z}; \tilde{y}, \tilde{\eta})$ admits a polyhomogeneous expansion as $t \to 0$, uniformly in the other variables.

The representable Poisson operators are operators $P \in \Psi^{-\infty,k,J_{lf}}_e(P_e)$ with leading coefficient at the front face, the normal operator $N(P)$, given by

$$N_k(P_0) := \int_{\mathbb{R}^b} e^{iY\eta} P_0(|\eta|, z, \tilde{z}; \tilde{y}, \tilde{\eta}) |\eta|^{-k+1} d\eta$$

for some $P_0 \in \Psi^{-\infty,J_{lf},J'_{lf}}_b(\mathbb{R}^+ \times F)$. If $\text{Op}(P_0) \in \Psi^{-\infty,k,J_{lf}}_e(P_e)$ is defined near $\partial M$ by

$$[\text{Op}(P_0)u] (x, y, z) := \int e^{i(y-\tilde{y})\eta} \kappa_{|\eta} P_0(x, z, \tilde{z}; y, \tilde{\eta}) u(\tilde{y}, \tilde{z}) |\eta|^{-k+1} d\eta d\tilde{y} d\tilde{z},$$

$$= \tilde{x}^{-1-b+k} \int e^{iY\eta} P_0(|\eta|, z, \tilde{z}; y + \tilde{x} Y, \tilde{\eta}) |\eta|^{-k+1} d\eta$$

$$= \tilde{x}^{-1-b+k} N_k(T_0) + O(\tilde{x}^{-b+k}).$$

**5.3. Representable Poisson operators.** Finally, we introduce the Bessel Poisson kernels.
then the Schwartz kernel is given locally by
\begin{equation}
K_{\text{Op}(P_0)}(x, y, z, \tilde{y}, \tilde{z}) = \int_{\mathbb{R}^b} e^{i(y-\tilde{y})\eta} P_0(x|\eta|, z, \tilde{z}; y, \tilde{\eta})|\eta|^{-k+1}d\eta
= x^{-1-b+k} \int_{\mathbb{R}^b} e^{iY\eta} P_0(|\eta|, z, \tilde{z}; \tilde{y} + xY, \tilde{\eta})|\eta|^{-k+1}d\eta,
= x^{-1-b+k}N_k(P_0) + O(x^{-b+k}).
\end{equation}

5.4. Composition of representable operators. We conclude this section by proving that the property of being representable is closed under composition.

**Residual \circ Poisson:** Let \(\text{Op}(G_0)\) and \(\text{Op}(P_0)\) be a residual edge and an edge Poisson operator associated to the Bessel operator \(G_0\) Bessel Poisson operator \(P_0\), respectively. Using (5.3) and (5.9), the composition \(\text{Op}(G_0) \circ \text{Op}(P_0)\) is given by
\begin{align}
K_{\text{Op}(G_0) \circ \text{Op}(P_0)}(x, y, z, \tilde{y}, \tilde{z}) &= \int \int e^{i(y-y')\eta} G_0(x|\eta|, z, \tilde{z}; y, \tilde{\eta})|\eta|^{-g+1}d\eta \\
&= \int \int e^{i(y-y')\eta'} P_0(x|\eta'|, z, \tilde{z}; y', \tilde{\eta}')|\eta'|^{-p+1}d\eta' d\tilde{x} dy' dz'
\end{align}
where we have substituted
\begin{equation}
Y = \frac{y - \tilde{y}}{x}, \quad Y' = \frac{y' - \tilde{y}}{x}, \quad t = \tilde{x}.
\end{equation}
Replacing \(Y'\) by \((-Y')\) we find for the leading \(x^{-1-b+(p+g)}\) coefficient
\begin{equation}
N(\text{Op}(G_0) \circ \text{Op}(P_0)) = \int e^{iY\eta} G_0(|\eta|, z, t|\eta|, z; \tilde{y}, \tilde{\eta})|\eta|^{-g+1}
\times \int e^{i(Y'-Y)\eta'} P_0(t|\eta'|, z; \tilde{y}, \tilde{\eta}')|\eta'|^{-p+1}d\eta' dY' dt dz'
= \int e^{iY\eta} (G_0 \circ P_0)(|\eta|, z, \tilde{z}; \tilde{y}, \tilde{\eta})|\eta|^{-p-g+1}d\eta = N_g(G_0) \circ N_p(P_0).
\end{equation}
This proves that the normal operator of this composition is representable and has the form (5.8).
**Poisson \(\circ\) trace:** Now consider a Poisson operator \(\text{Op}(P_0)\) associated to the Bessel Poisson kernel \(P_0\) and a trace operator \(\text{Op}(T_0)\) associated to the Bessel trace kernel \(T_0\). The composition in (5.9) and (5.7) takes the form

\[
K_{\text{Op}(P_0)\circ\text{Op}(T_0)}(x, y, z, \tilde{x}, \tilde{y}, \tilde{z}) = \int \int e^{i(y-y')\eta} P_0(x|\eta|, z, z'; y, \tilde{\eta})|\eta|^{-p+1} d\eta
\]

\[
e^{i(y-y')\eta'} T_0(\tilde{x}|\eta'|, z', \tilde{z}; y', \tilde{\eta'})|\eta'|^{-\tau+1} d\eta' dy' dz' = \int \int e^{i(Y-Y')\eta} P_0(s|\eta|, z, z'; \tilde{y} + \tilde{x}Y, \tilde{\eta})x^{-1-b+p}|\eta|^{-p+1} d\eta
\]

\[
e^{Y'\eta'} T_0(|\eta'|, z', \tilde{z}; \tilde{y} + \tilde{x}Y', \tilde{\eta'})x^{\tau}|\eta'|^{-\tau+1} d\eta' dY' d\tilde{z}',
\]

where

\[
Y = \frac{y - \tilde{y}}{x}, \quad Y' = \frac{y' - \tilde{y}}{x}, \quad s = \frac{x}{\tilde{x}}.
\]

As before, substituting \(Y'\) by \((-Y')\), we obtain

\[
N(\text{Op}(P_0) \circ \text{Op}(T_0)) = \int e^{Y\eta} P_0(s|\eta|, z, z'; \tilde{y}, \tilde{\eta})|\eta|^{-p+1}
\]

\[
\times \int e^{i(\eta-\eta')Y'} T_0(|\eta'|, z', \tilde{z}; \tilde{y}, \tilde{\eta}')|\eta'|^{-\tau+1} d\eta' dY' d\eta d\tilde{z}' = \int e^{Y\eta} (P_0 \circ T_0)(s|\eta|, z, |\eta|, \tilde{z}; \tilde{y}, \tilde{\eta})|\eta|^{-p+1} d\eta = N_p(P_0) \circ N_r(T_0),
\]

so this composition is again representable.

**Trace \(\circ\) Poisson:** Finally, if \(\text{Op}(T_0)\) is a trace operator associated to the Bessel trace kernel \(T_0\) and \(\text{Op}(P_0)\) is a Poisson operator associated to the Bessel Poisson kernel \(P_0\), then (5.7) and (5.9) becomes

\[
K_{\text{Op}(T_0)\circ\text{Op}(P_0)}(y, z, \tilde{y}, \tilde{z}) = \int \int e^{i(y-y')\eta} T_0(x|\eta|, z, z'; y, \tilde{\eta})|\eta|^{-\tau+1} d\eta
\]

\[
e^{i(y-y')\eta'} P_0(x|\eta'|, z', \tilde{z}; y', \tilde{\eta'})|\eta'|^{-p+1} d\eta' dx dy' dz' = \int \int e^{i(Y-Y')\eta} T_0(t|\eta|, z, z'; \tilde{y} + rY, \tilde{\eta})r^{-1-b+p}|\eta|^{-\tau+1} d\eta
\]

\[
e^{Y'\eta'} P_0(t|\eta'|, z', \tilde{z}; \tilde{y} + rY', \tilde{\eta'})r^{p}|\eta'|^{-p+1} d\eta' dt dY' d\tilde{z}',
\]

where

\[
Y = \frac{y - \tilde{y}}{r}, \quad Y' = \frac{y' - \tilde{y}}{r}, \quad t = \frac{x}{r}.
\]
Substituting $Y'$ by $(-Y')$, and taking the limit $r \to 0$, we obtain the principal symbol of a pseudodifferential operator on the closed manifold $B$ acting on sections of the trace bundle:

$$N(\Op(T_0) \circ \Op(P_0)) = \int e^{iY\eta} T_0(t|\eta|, z, z'; \tilde{y}, \hat{\eta})|\eta|^{-r+1}$$

$$(5.18)$$

$$\times \int e^{i(\eta-\eta')Y'} P_0(t|\eta'|, z'; \tilde{z}; \tilde{y}, \hat{\eta}')|\eta'|^{-p+1} \, d\eta' \, d\eta' \, dx \, dz'$$

$$= \int e^{iY\eta} (T_0 \circ P_0)(z, \tilde{z}; \tilde{y}, \hat{\eta})|\eta|^{-p-r+1} \, d\eta = N_r(T_0) \circ N_p(P_0).$$

6. Trace and Poisson operators of an elliptic edge operator

Let $L \in \Diff^m_e(M)$ be an elliptic differential edge operator. We use all the same notation as above, and assume, in particular, that $B(L)$ is injective on $t^2L^2$ and surjective on $t^2L^2$.

Define

$$(6.1) \quad H_{\Delta,\delta}(L) = \{ u \in x^\delta L^2 : Lu \in x^\delta L^2 \}.$$

We often refer to this as $H_{\Delta,\delta}$, or even just $H$. This is a Hilbert space with respect the graph norm

$$||u||_H = ||u||_{x^\delta L^2} + ||Lu||_{x^\delta L^2}.$$

In this section we define and study the trace map, which assigns to any $u \in H_{\Delta,\delta}$ the set of leading coefficients in its expansion with exponents between $\delta$ and $\delta$. We also construct the Poisson operator for $L$, which assigns to an appropriate set of leading coefficients an element of $\ker L \cap H_{\Delta,\delta}$.

A subtlety in these definitions is that leading coefficients are sections of the trace bundle

$$\mathcal{E}(L) = \bigoplus_{j=0}^N \mathcal{E}(L; \zeta_j)$$

introduced in §2. A standing assumption in this paper is that the $\zeta_j$ are independent of $y \in B$, and because of this, the different subbundles $\mathcal{E}(L; \zeta_j)$ do not interact with one another. Thus, to simplify the notation in this section, we suppose that there is only a single indicial root $\zeta_0 \in \mathcal{G}(L)$, and we $\mathcal{E}(L) = \mathcal{E}(L; \zeta_0)$.

6.1. The trace map for the model Bessel operator. The model Bessel operator corresponding to $L$ is

$$B_{\tilde{y},\hat{\eta}}(L) = \sum_{j+|\alpha|+|\beta| \leq m} a_{j,\alpha,\beta}(0, \tilde{y}, z)(t \partial_t)^j (it\hat{\eta})^\alpha \partial_z^\beta,$$
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which acts (as an unbounded operator) on $t^\delta H^m_{loc}(\mathbb{R}^+ \times F; dt dz)$. Just as for $L$, however, we are primarily interested in its restriction to

$$
\mathcal{H}^B_{\delta, \delta} = \{ \omega \in t^\delta L^2(\mathbb{R}^+ \times F; dt dz) : B(L)\omega \in t^\delta L^2 \}.
$$

If $\omega \in \mathcal{H}^B_{\delta, \delta}$, then we can follow the same strategy as in the proof of Proposition 2.4 to obtain the (strong) expansion

$$
\omega \sim \sum_{\ell \geq 0} \sum_{p=0}^{p_0} t^{-i\zeta_0 + \ell}(\log t)^p \omega_{\ell,p}(\tilde{y}, z) + \tilde{\omega}, \quad \tilde{\omega} \in \bigcap_{\epsilon > 0} t^{\delta-\epsilon} L^2.
$$

Indeed, writing $B(L) = I(L) + E$, where $E$ contains all terms with ‘extra’ powers of $t$, then $B(L)\omega = f$ becomes $I(L)\omega = f - E\omega$. The term $E\omega$ creates new ‘higher order’ terms $t^{-i\zeta_0 + \ell}$ with $\ell > 0$, but discarding these we obtain

$$
(6.2) \quad I_{\tilde{y}}(L) \left( \sum_{p=0}^{p_0} t^{-i\zeta_0}(\log t)^p \omega_{0,p}(\tilde{y}, z) \right) = 0.
$$

By definition of the fibres of the trace bundle, this expression in parentheses lies in $\mathcal{E}_{\tilde{y}}(L; \zeta_0)$ for each $\tilde{y}$.

Now consider how this expansion varies as a function of $\tilde{y}$. Even if $f$ depends smoothly on $\tilde{y}$, the individual coefficients $\omega_{0,p}$ may fail to be smooth (or even continuous) in $\tilde{y}$ because the order $p_0$ of the indicial root may vary. This is where the properties of the trace bundle from [14], discussed above in §2, become crucial. As explained there, on any neighbourhood $U \subset B$ over which $E$ is trivialized, there exist smooth functions $\phi_{\tilde{y},k}(t, z)$, $k \leq m_0 = m(\zeta_0)$, such that

$$
\sum_{p=0}^{p_0} t^{-i\zeta_0}(\log t)^p \omega_{0,p}(\tilde{y}, z) = \sum_{k=1}^{m_0} f_k(\tilde{y}) \phi_{\tilde{y},k}(t, z),
$$

where, somewhat remarkably, $f_k \in C^\infty(U)$ even though the number of terms in the sum on the left may be discontinuous.

Using all of this, we can now state the

**Definition 6.1.** The Bessel trace map $\text{Tr}_{B(L)}$ is the operator which assigns to each $\omega \in C^\infty(S^*B; H^B_{2,2})$ a section of $\mathcal{E}(L; \zeta_0)$ which is represented in a neighbourhood $U \subset B$ in which $\mathcal{E}(L; \zeta_0)$ is trivialized by the smooth basis of sections $\phi_{\tilde{y},k}$ by the $m_0$-tuple $\{f_1, \ldots, f_{m_0}\}$.

Note that if $\omega(\tilde{y}, \tilde{\eta}) \in t^\delta L^2$ for each $(\tilde{y}, \tilde{\eta})$, then $\text{Tr}_{B(L)}\omega = 0$.

**Proposition 6.2.** The operator $\text{Tr}_{B(L)}$ is a representable Bessel trace kernel in the sense of Definition 5.2.
Proof. Recall the definition of the trace bundle in Proposition 2.5. Then for a solution $\omega$, the singular part of its Mellin transform is a section of $\mathcal{E}(L)$. Consider, following [14], the Hilbert space adjoint $I_{\tilde{y}}(L)(\zeta)^*$ of the indicial operator pencil and set $I_{\tilde{y}}(L)^*(\zeta) := I_{\tilde{y}}(L)(\tilde{\zeta})^*$. This depends smoothly on $y$ and is a holomorphic family of Fredholm operators in $\zeta \in \mathbb{C}$. Its indicial roots are the complex conjugates of elements of $\text{Spec}_{\mathcal{B}}(L)$. We denote its trace bundle by $\mathcal{E}^*(L)$. This suggestive notation is vindicated by a central result in [14, Theorem 5.3], which asserts the nondegeneracy of the pairing

$$\mathcal{E}^*(L)(\zeta_0) \times \mathcal{E}^*(L)(\bar{\zeta_0}) \to \mathbb{C},$$

(6.3)

for any sufficiently small $\epsilon > 0$. Identifying $\mathcal{E}^*(L, \zeta_0)$ with the kernel of $I_{\tilde{y}}(L)$ on the space of finite combinations $\sum a_q(z)t^{-i\zeta_0} \log^d(t), a_j \in C^\infty(F)$, we may assign to each basis element $\phi_{\tilde{y},j}$ its dual, $\phi^*_{\tilde{y},j}$, with respect to this pairing.

If $\chi \in C^\infty_0(\mathbb{R})$ is a cutoff function which equals one near 0, then the integral kernel of the Bessel trace map is

$$\text{Tr}_{B(L)}(t, \tilde{z}; \tilde{y}) = \frac{1}{2\pi} \bigoplus_{j=1}^{m_0} \int_{B_\epsilon(\zeta_0)} t^{i\zeta-1}\chi(t)I_{\tilde{y}}(L)^*(\tilde{\zeta})\phi^*_{\tilde{y},j}(\tilde{\zeta}, \tilde{z}) d\zeta$$

(6.4)

This satisfies the conditions of Definition 5.2 and hence $\text{Tr}_{B(L)}$ is a representable Bessel trace kernel.

The absence of the variable $z$ in this formula is a result of the identification of the asymptotic coefficients of $\omega$ with local sections of the trace bundle, since this bundle is trivialized by the smooth basis $\{\phi_{\tilde{y},j}\}$, which has coefficients $\{f_1, ..., f_{m_0}\}$ depending only on $\tilde{y}$. 

6.2. Trace of solutions to the normal operator. The next step is to carry out a similar analysis of the trace operator for the normal operator $N(L)$. Recall that $N(L)$ is identified with the restriction of the lift $\beta^*L$ to the front face in $M^2_e$ with respect to the blowdown map $\beta : M^2_e \to M^2$, and in the projective coordinates $(s, Y, z)$ from (4.1) this takes the form (2.3) (with $Y$ replacing the variable $u$ there). The normal operator is equivalent to the Bessel operator (2.4) through Fourier transform (in $Y$) and rescaling (setting $s = t/|\eta|$):

$$\mathcal{F} \circ N_{\tilde{y}}(L) \circ \mathcal{F}^{-1} \big|_{s=t/|\eta|} = B_{\tilde{y},\tilde{\eta}}(L).$$
Thus if \( \omega \in s^\delta \mathcal{L}^2(ds \ dY \ dz) \) is such that \( N(L)\omega \in s^\delta \mathcal{L}^2 \), then its Fourier transform \( \hat{\omega} \) evaluated at \( s = t/|\eta| \) is an element of \( \mathcal{H}^B_{\delta, \delta} \). As such, it can be written locally as

\[
\hat{\omega}(s, \eta, z) = \sum_{k=1}^{m_0} a_k(\tilde{y}, \eta) \phi_{\tilde{y}, k}(s|\eta|, z).
\]

We define the trace map for \( N_{\tilde{y}}(L) \) as

\[
\text{Tr}_{N(L)} \omega := \bigoplus_{j=1}^{m_0} \int e^{i(Y - \tilde{Y}) \eta} \Phi_j(s|\eta|, \tilde{y}, \tilde{z}) \omega(s, \tilde{Y}, \tilde{z}) |\eta|^{-i\zeta_0 + 1} \, ds \, \eta \, d\tilde{Y} \, d\tilde{z}
\]

\[
= \bigoplus_{j=1}^{m_0} \int_{\mathbb{R}^b} e^{iY \eta} a_j(\tilde{y}, \eta) |\eta|^{-i\zeta_0 + 1} \, d\eta \in H^{-(3\zeta_0 - \delta - 1/2)}(\mathbb{R}^b, dY) \otimes \mathcal{E}_{\tilde{y}}(L; \zeta_0),
\]

where we used the regularity result \cite[Thm. 7.3]{17}. From (5.6) and since \( \text{Tr}_{B(L)} \) is a Bessel trace kernel, we infer that

\[
(6.5) \quad \text{Tr}_{N(L)}(Y, \tilde{y}, \tilde{z}) = \int_{\mathbb{R}^b} e^{iY \eta} \text{Tr}_{B(L)}(|\eta|, \tilde{y}, \tilde{z}) |\eta|^{-i\zeta_0 + 1} \, d\eta,
\]

is smooth on the front face of \( T_e \) and polyhomogeneous at the boundaries of this face.

6.3. **The trace map of \( L \).** The construction above determines a Schwartz kernel representation for a trace map of the operator \( L \) itself. Indeed, following (6.5), define in local coordinates of the corner neighborhood in \( M^2 \)

\[
\text{Tr}_L(x, y, \tilde{y}, z) := \int_{\mathbb{R}^b} e^{i\eta(y - \tilde{y})} \text{Tr}_{B(L)}(x|\eta|, \tilde{y}, \tilde{z}) |\eta|^{-i\zeta_0 + 1} \, d\eta,
\]

and extend smoothly to the interior. From the work above,

\[
\text{Tr}_L : \mathcal{H}_{\delta, \delta} \rightarrow H^{-(3\zeta_0 - \delta - 1/2)}(B, \mathcal{E}(L; \zeta_0))
\]

is a bounded mapping, a representable trace operator. Note that this operator \( \text{Tr}_L \) is by no means unique.

6.4. **The edge Poisson operator.** We define the Bessel Poisson operator

\[
P_0 : \mathcal{E}(L, \zeta_0) \to \mathcal{H}^B_{\delta, \delta}, \quad (f_1, \ldots, f_{m_0}) \mapsto \sum_{j=1}^{m_0} f_j \phi_{\tilde{y}, j},
\]

with the integral kernel (as before \( \tilde{z} \) is absent)

\[
P_0(t, z; \tilde{y}) = \bigoplus_{j=1}^{m_0} \phi_{\tilde{y}, j}(t, z).
\]
In particular, \( P_0 \) is representable in the sense of Definition 5.3 with the index set \( J_0 \) determined by the asymptotic expansion of each \( \phi_{\bar{g},j} \). The associated normal operator is given by

\[
N_{-i\zeta_0 + 1}(P_0) = \int_{\mathbb{R}^b} e^{i\sigma_\eta} P_0(|\eta|, z; \bar{y})|\eta|^{i\zeta_0} d\eta,
\]

which we extend off the front face to define an edge Poisson operator

\[
\text{Op}(P_0) : H^{-((\zeta_0) - \ell/2)}(B, \mathcal{E}(\zeta_0)) \rightarrow \mathcal{E}_c^\infty(M, g).
\]

Consider the orthogonal projector (cf. [17])

\[
(6.6) \quad P_1 : \mathcal{E}_c^\infty(M, g) \rightarrow \ker L \cap \mathcal{E}_c^\infty(M, g),
\]

which is a residual edge operator discussed in [4.4] with the corresponding edge Bessel kernel \( P_{01} \), which is the Schwartz kernel of the orthogonal projection of \( t^\ell L^2(dt dz) \) onto \( \ker B(L)(\bar{y}, \bar{\eta}) \cap t^\ell L^2(dt dz) \). We define

\[
(6.7) \quad P_L = P_1 \circ \text{Op}(P_0).
\]

By the composition rule (5.12) we find

\[
N(P_L) = N_0(P_1) \circ N_1(P_0) = \int e^{i\sigma_\eta}(P_0 \circ P_0)(|\eta|, z)d\eta.
\]

The restriction of Bessel trace map \( \text{Tr}_{B(L)} \) to \( \ker B(L) \cap t^\ell L^2 \) is injective, since \( B(L) \) is injective on \( t^\ell L^2 \). Hence \( \text{Tr}^{-1}_{B(L)} \) admits a left-inverse \( \text{Tr}^{-1}_{B(L)} \), mapping \( \mathcal{E}_\gamma(L, \zeta_0) \) to \( \ker B(L) \cap t^\ell L^2 \), which is a true inverse when restricted to \( \text{im} \text{Tr}_{B(L)}(\ker B(L) \cap t^\ell L^2) \).

**Lemma 6.3.** \( \text{Tr}^{-1}_{B(L)} = P_0 \circ P_0 | \text{im} \text{Tr}_{B(L)}(\ker B(L) \cap t^\ell L^2) \).

**Proof.** Note that by [17] (5.8) there exists a generalized inverse \( G_0 \) such that \( G_0 B(L) = I - P_{01} \). Consequently, for any \( \omega \in \mathcal{H}^B_{L^2} \) we find \( \omega - P_{01}\omega = G_0 B(L)\omega \in t^\ell L^2 \). Hence, \( \text{Tr}_{B(L)}\omega = \text{Tr}_{B(L)} P_0 \omega \). Thus

\[
\text{Tr}_{B(L)} P_0 \circ P_0(\text{Tr}_{B(L)}\omega) = \text{Tr}_{B(L)} P_0(\text{Tr}_{B(L)}\omega) = \text{Tr}_{B(L)}\omega.
\]

If \( \omega \in \ker B(L) \cap t^\ell L^2 \), then \( \omega = P_0 \circ P_0(\text{Tr}_{B(L)}\omega) \) since \( B(L) \) is injective on \( t^\ell L^2 \).

**Proposition 6.4.** \( (N(P_L) \circ \text{Tr}_{N(L)})\omega = \omega \), for \( \omega \in N(L) \cap s^\ell L^2 \).

**Proof.** We compute according to (5.15)

\[
N(P_L) \circ \text{Tr}_{N(L)} = \int e^{i\sigma_\eta}(P_0 \circ P_0 \circ \text{Tr}_{B(L)(\bar{y}, \bar{\eta})})(|s|, \bar{s}|\eta|, z; \bar{z})d\eta.
\]
which is the normal operator of a residual edge operator. Consider
\( \omega(s, Y, z) \in \ker N(L) \cap s^2L^2 \). As before, \( \tilde{\omega}(t/|\eta|, \eta, z) \in \ker B(L)(\tilde{y}, \tilde{\eta}) \cap t^2L^2 \).
Thus we compute by Lemma 6.3

\[
N(P_L) \circ \text{Tr}_N(L) \omega = \int e^{i(Y - \tilde{Y}) \eta}(P_0 \circ P_0 \circ \text{Tr}_B(L)(\tilde{y}, \tilde{\eta}))(s|\eta|, s|\eta|, z, \tilde{z}) \omega(s, \tilde{Y}, \tilde{\eta}) d\eta d\tilde{s} d\tilde{Y} d\tilde{\eta}
\]

\[
= \int e^{iY \eta}(P_0 \circ P_0 \circ \text{Tr}_B(L)(\tilde{y}, \tilde{\eta}))(s|\eta|, s|\eta|, z, \tilde{z}) \tilde{\omega}(s, \eta, \tilde{z}) d\eta d\tilde{s} d\tilde{Y} d\tilde{\eta}
\]

\[
= \int e^{iY \eta} \tilde{\omega}(s, \eta, z) d\eta = \omega(s, Y, z).
\]

\[\square\]

7. Fredholm theory of elliptic edge boundary value problems

We return now to the general situation, where \( \mathcal{S}(L) = \{\zeta_0, ..., \zeta_N\} \). Fix a collection \( E_1, ..., E_M \) of finite rank vector bundles over \( B \) and set \( E = \bigoplus_{k=1}^M E_k \). Now consider the collection of classical pseudodifferential operators

\[
Q_{kj} \in \Psi^{d_k - 3(\zeta_j)}(B; \mathcal{E}(L; \zeta_j), E_k), \quad j = 1, ..., N, \quad k = 1, ..., M,
\]

\[
Q_{kj} : H^s(B, \mathcal{E}(L, \zeta_j)) \rightarrow H^{s - d_k + 3(\zeta_j)}(B, E_k), \quad s \in \mathbb{R}.
\]

Define the homogeneity rescalings

\[
\eta(L) : \mathcal{E} \rightarrow \mathcal{E}, \quad (u_1, ..., u_N) \mapsto (|\eta|^{3\zeta_j} u_1, ..., |\eta|^{3\zeta_N} u_N),
\]

\[
\eta(Q) : E \rightarrow E, \quad (e_1, ..., e_M) \mapsto (|\eta|^{d_1} e_1, ..., |\eta|^{d_M} e_M).
\]

The matrix \( (Q_{kj}) \) defines the pseudodifferential system \( Q \) where

\[
\sigma_0(Q)(\tilde{y}, \eta) = \eta(Q) \circ \sigma_0(Q)(\tilde{y}, \tilde{\eta}) \circ \eta(L)^{-1}.
\]

(Note that \( \eta \) appears on the left and \( \tilde{\eta} = \eta/|\eta| \) on the right.)

We now recall the form of the general edge boundary value problem:

**Definition 7.1.** Let \( L \in \text{Diff}^m_e(M) \) be edge elliptic, and suppose that \( Q = (Q_{kj}) \) is as above. Then the edge boundary value problem \( (L, Q) \) is the set of equations

\[
Lu = f \in x^L^2(M),
\]

\[
Q(\text{Tr}_L u) = \phi \in \bigoplus_{k=1}^M H^{\frac{d_k}{2} - \frac{1}{2}}(B, E_k).
\]

for \( u \in x^L H^m_e(M) \).
We have already stated, in Definition 3.2, the definitions of right-, left- and full ellipticity of the boundary problem \((L, Q)\).

Clearly
\[(L, Q) : \mathcal{H} \rightarrow \check{x}^{\hat{\delta}}L^2(M, g) \oplus \left( \bigoplus_{k=1}^{M} H^{\delta-d_k-1/2}(B, E_k) \right),\]
\[u \mapsto (Lu, Q(\text{Tr}_L u)).\]

is continuous. Our goal is to show that it is semi-Fredholm if \((L, Q)\) satisfies conditions i) or ii) of Definition 3.2, and Fredholm if \((L, Q)\) satisfies condition iii). This is proved by a parametrix construction.

7.1. **The right-elliptic case.** Consider a right-elliptic system \((L, Q)\). Since \(\sigma(Q) \upharpoonright \text{im } \text{Tr}_B(L) : \text{im } \text{Tr}_B(L) \rightarrow E\) is surjective, there exists a right parametrix \(K : H^{\delta-d_k-1/2}(B, E_k) \rightarrow H^{\delta-3(\zeta_j)-1/2}(B, \mathcal{E}(L; \zeta_j))\) for \(Q\); this has principal symbol
\[\sigma(K)(\tilde{y}, \eta) = \eta(L) \circ \sigma(Q)^{-1}(\tilde{y}, \tilde{\eta}) \circ \eta(Q)^{-1},\]
\[\sigma(Q)^{-1}(\tilde{y}, \tilde{\eta}) : E_{\tilde{y}} \rightarrow \text{im } \text{Tr}_B(L(\tilde{y}, \tilde{\eta})),\]
where \(\sigma(Q)^{-1}(\tilde{y}, \tilde{\eta})\) is some choice of right-inverse for \(\sigma(Q)(\tilde{y}, \tilde{\eta}) \upharpoonright \text{im } \text{Tr}_B(L)\) which varies smoothly in \((\tilde{y}, \tilde{\eta})\).

**Theorem 7.2.** If \((L, Q)\) is right-elliptic, then (7.1) is semi-Fredholm, with closed range of finite codimension. A right parametrix for it is given by
\[G(f, \phi) = Gf + P_L[K(\phi - Q(\text{Tr}_L Gf))],\]
where \(G\) is the generalized inverse for \(L\) on \(x^{\hat{\delta}}H^m_e(M)\).

**Proof.** By definition, \(LG = \text{Id} - P_2\), where \(P_2\) is the orthogonal projection onto the finite-dimensional space \(\ker L \cap x^{\hat{\delta}}L^2\). Thus if \(f \in x^{\hat{\delta}}L^2\), then
\[\|Gf\|_\mathcal{H} = \|Gf\|_{x^{\hat{\delta}}H^m_e} + \|LGf\|_{x^{\hat{\delta}}L^2} \leq \|Gf\|_{x^{\hat{\delta}}H^m_e} + \|f\|_{x^{\hat{\delta}}L^2} + \|P_2f\|_{x^{\hat{\delta}}L^2}.\]

Since \(G\) is bounded on \(x^{\hat{\delta}}L^2\) and \(\|f\|_{x^{\hat{\delta}}L^2} \leq \|f\|_{x^{\hat{\delta}}L^2}\), we have \(\|Gf\|_\mathcal{H} \leq C(\|f\|_{x^{\hat{\delta}}L^2} + \|P_2f\|_{x^{\hat{\delta}}L^2})\). Hence \(G : \ker P_2 \cap x^{\hat{\delta}}L^2 \rightarrow \mathcal{H}\) is bounded. For simplicity below, we assume that \(P_2 \equiv 0\); if this projector is nontrivial, it only changes things by a finite dimensional amount, which does not affect any of the Fredholmness statements below.
Next, both
\[
P_L : \bigoplus_{j=0}^N H^{-(3(\zeta_j)-\frac{d+2}{2})}(B, \mathcal{E}(L, \zeta_j)) \to \ker L \cap x^\delta H^\infty_e(M) \subset \mathcal{H}
\]
and
\[
\begin{equation}
\mathbf{Tr}_L : \mathcal{H} \to \bigoplus_{j=0}^N H^{-(3(\zeta_j)-\frac{d+2}{2})}(B, \mathcal{E}(L, \zeta_j)),
\end{equation}
\]
are continuous, the latter by the discussion in \(\S 6.3\). All of this, together with continuity of the pseudodifferential operators \(Q\) and \(K\) between appropriate Sobolev spaces over \(B\), shows that the parametrix \(G\) is a bounded mapping.

We now compute the error term \(\mathbf{Tr}_L G(f, \phi)\). Since \(LP_L = 0\), and we are assuming that the cokernel of \(L\) is trivial, we have \(LG(f, \phi) = f\). Next,
\[
Q \mathbf{Tr}_L G(f, \phi) = Q[\mathbf{Tr}_L G f + \mathbf{Tr}_L P_L (K(\phi - Q(\mathbf{Tr}_L G f)))]
\]
\[
= Q\mathbf{Tr}_L G f + (Q \circ \mathbf{Tr}_L \circ P_L \circ K)(\phi - Q(\mathbf{Tr}_L G f))
\]
\[
= \phi + (Q \circ \mathbf{Tr}_L \circ P_L \circ K - I)(\phi - Q(\mathbf{Tr}_L G f)).
\]

It thus remains to prove that
\[
(Q \circ \mathbf{Tr}_L \circ P_L \circ K - I) : \bigoplus_{k=1}^M H^{d-\delta_{d_k} - \frac{d+2}{2}}(B, E_k) \to \bigoplus_{k=1}^M H^{d-\delta_{d_k} - \frac{d+2}{2}}(B, E_k)
\]
is compact. This is however simply a pseudo-differential operator over the closed manifold \(B\), so it suffices to check that its principal symbol vanishes. We compute, using \(5.18\), that
\[
\sigma_0(Q \circ \mathbf{Tr}_L \circ P_L \circ K - I)(\tilde{y}, \eta)
\]
\[
= \sigma(Q)(\tilde{y}, \eta) \circ (\mathbf{Tr}_{B(L)} \circ P_{01} \circ P_0) \circ \sigma(K)(\tilde{y}, \eta) - I.
\]
By definition, \(\sigma(K)\) maps into \(\mathcal{C}_{B(L)}\), so all terms cancel and this principal symbol vanishes. This completes the proof. \(\square\)

### 7.2. Left-elliptic edge boundary value problem

Now consider a set of boundary operators \(Q\) which satisfy the left-elliptic conditions. Since \(\sigma_Q(\tilde{y}, \tilde{\eta}) \upharpoonright \mathcal{C}_{B(L)}\) is injective, there exists a matrix of pseudodifferential operators
\[
K : \bigoplus_{k=1}^M H^{d-\delta_{d_k} - \frac{d+2}{2}}(B, E_k) \to \bigoplus_{j=1}^N H^{d-3(\zeta_j) - \frac{d+2}{2}}(B, \mathcal{E}(L; \zeta_j)),
\]
with principal symbol
\[ \sigma(K)(\tilde{y}, \eta) = \eta(L) \circ \sigma(Q)^{-1}(\tilde{y}, \hat{\eta}) \circ \eta(Q)^{-1} \]
where
\[ \sigma(Q)^{-1}(\tilde{y}, \hat{\eta}) : E_{\tilde{y}} \to C_B(L), \]
is a left-inverse to \( \sigma(Q)(\tilde{y}, \hat{\eta}) \upharpoonright C_B(L) \). Note that \( K \) is not necessarily a left-parametrix for \( Q \), since \( \sigma(Q)^{-1}(\tilde{y}, \hat{\eta}) \) does not invert the full symbol \( \sigma(Q)(\tilde{y}, \hat{\eta}) \), but this is not required for our argument.

**Theorem 7.3.** If \( (L, Q) \) is left-elliptic, then
\[ (L, Q) : \mathcal{H} \to x^\gamma L^2(M) \oplus \left( \bigoplus_{k=1}^M H^{\delta-d_k-1/2}(B, E_k) \right), \]
is semi-Fredholm with left parametrix
\[ \mathcal{G}(f, \phi) = Gf + P_L[K(\phi - Q(\text{Tr}_L Gf))]. \]

**Proof.** As before, \( \mathcal{G} \) is a bounded operator and we compute for any \( u \in \mathcal{H} \),
\[ \mathcal{G}(L, Q)u = GLu + P_L[K(Q \text{Tr}_L u - Q \text{Tr}_L GLu)] \]
\[ = GLu + (P_L \circ K \circ Q \circ \text{Tr}_L)(u - GLu) \]
\[ = u + (P_L \circ K \circ Q \circ \text{Tr}_L - I)P_1u, \]
where \( P_1 \) is the orthogonal projection onto the nullspace of \( L \) in \( x^\delta L^2 \). Hence we must show that \( (P_L \circ K \circ Q \circ \text{Tr}_L - I) \circ P_1 \) is compact on \( \mathcal{H} \).

By the form of \( || \cdot ||_\mathcal{H} \) and since \( LP_L = 0 \) and \( LP_1 = 0 \), we need only check compactness of
\[ (P_L \circ K \circ Q \circ \text{Tr}_L - I) \circ P_1 : \mathcal{H} \to \ker L \cap \mathcal{H}. \]
By the composition results in §4.4, \( (P_L \circ K \circ Q \circ \text{Tr}_L) \) is an edge operator of order \(-\infty\), i.e. has no diagonal singularity, and has normal operator
\[ N(P_L \circ K \circ Q \circ \text{Tr}_L) = \]
\[ \int e^{\gamma \eta}(P_{01} \circ P_0 \circ \sigma(K) \circ \sigma(Q) \circ \text{Tr}_{B(L)})(s|\eta|, |\eta|, z, \tilde{z}; \tilde{y}, \hat{\eta})d\eta, \]
whence
\[ N((P_L \circ K \circ Q \circ \text{Tr}_L - I) \circ P_1) = \]
\[ \int e^{\gamma \eta}(P_{01} \circ P_0 \circ \sigma(K) \circ \sigma(Q) \circ \text{Tr}_{B(L)} \circ P_0 - P_{01})(s|\eta|, |\eta|, z, \tilde{z}; \tilde{y}, \hat{\eta})|\eta|d\eta. \]
In this combination, $\sigma(Q)(\tilde{y}, \tilde{\eta})$ acts on $C_B(L)$, so that $\sigma(Q)(\tilde{y}, \tilde{\eta})$ and $\sigma(Q)^{-1}(\tilde{y}, \tilde{\eta})$ cancel. After further obvious cancellations, this normal operator reduces to

$$\int e^{iy\eta}(P_{01} - P_{01})(s|\eta|, |\eta|, z, \tilde{z}; \tilde{y}, \tilde{\eta}) |\eta| d\eta = 0.$$  

Finally, using the boundedness properties of $P_1$, $P_L$ and $\text{Tr}_L$, we see that

$$R := (P_L \circ K \circ Q \circ \text{Tr}_L - I) \circ P_1 : \mathcal{H} \to \ker L \cap x^{\delta}H^\infty_e(M) \hookrightarrow x^{\delta}L^2$$

is bounded as well. From the composition results in \cite{11}, $R \in \Psi^{-\infty,0,E_{it},E_{it}}(M^2)$ and $N(R) = 0$, so in fact $R \in \Psi^{-\infty,1,E_{it},E_{it}}(M^2)$, with index sets

$$(7.5) \quad E_{\text{lf}} = \{ (\zeta, p) \in \text{Spec}_b(L) \mid \Im \zeta > \delta - 1/2 \},$$

$$(7.5) \quad E_{\text{rf}} = \{ (\zeta, p) \in \mathbb{C} \times \mathbb{N}_0 \mid (\zeta + 2\delta, p) \in E_{\text{lf}} \},$$

see \cite{5.4}. Its compactness is now a consequence of \cite{17}, Prop. 3.29. \hfill \Box

From Theorems \ref{thm:7.2} and \ref{thm:7.3} we now conclude the

**Corollary 7.4.** Let $(L, Q)$ be elliptic. Then

$$(L, Q) : \mathcal{H}_{\Delta} \to x^{\delta}L^2(M) \oplus \left( \bigoplus_{k=1}^M H^{\delta-d_k-1/2}(B, E_k) \right),$$

is Fredholm, with parametrix

$$\mathcal{G}(f, \phi) = Gf + P_L[K(\phi - Q(\text{Tr}_L Gf))].$$

We conclude by presenting one simple application of this machinery.

**Proposition 7.5.** Let $u \in x^{\delta}L^2(M)$ and suppose that $Lu = 0$ and $\text{Tr}_L u = 0$. Then $u \in x^{\delta}H^\infty_e(M)$.

**Proof.** Choose any left elliptic boundary value problem $(L, Q)$, and let $\mathcal{G}$ be its left parametrix, as constructed above, so that $\mathcal{G} \circ (L, Q) = \text{Id} - \mathcal{R}$. Then $\text{Tr}_L u = 0$, so $u = \mathcal{R}u$. Since $N(\mathcal{R}) = 0$, \cite{17}, Thm. 3.25] gives that

$$\mathcal{R} : x^{\delta}H^s_e(M) \to x^{\delta+\epsilon}H^\infty_e(M), \ s \geq 0$$

is bounded for some $\epsilon > 0$ which depends only on $\mathcal{R}$. Iterating this statement gives the result. \hfill \Box
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