Stochastic Gradient Methods with Compressed Communication for Decentralized Saddle Point Problems

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Abstract

We propose two stochastic gradient algorithms to solve a class of saddle-point problems in a decentralized setting (without a central server). The proposed algorithms are the first to achieve sub-linear/linear computation and communication complexities using respectively stochastic gradient/stochastic variance reduced gradient oracles with compressed information exchange to solve non-smooth strongly-convex strongly-concave saddle-point problems in decentralized setting. Our first algorithm is a Restart-based Decentralized Proximal Stochastic Gradient method with Compression (C-RDPSG) for general stochastic settings. We provide rigorous theoretical guarantees of C-RDPSG with gradient computation complexity and communication complexity of order $O((1 + \delta)^4 \frac{1}{L^2} \kappa_f^2 \kappa_g^2 \frac{1}{\epsilon})$, to achieve an $\epsilon$-accurate saddle-point solution, where $\delta$ denotes the compression factor, $\kappa_f$ and $\kappa_g$ denote respectively the condition numbers of objective function and communication graph, and $L$ denotes the smoothness parameter of the smooth part of the objective function. Next, we present a Decentralized Proximal Stochastic Variance Reduced Gradient algorithm with Compression (C-DPSVRG) for finite sum setting which exhibits gradient computation complexity and communication complexity of order $O((1 + \delta)^2 \kappa_f^2 \kappa_g \log(\frac{1}{\epsilon}))$. Extensive numerical experiments show competitive performance of the proposed algorithms and provide support to the theoretical results obtained.

1 Introduction

Decentralized environments are useful for large-scale machine learning settings where privacy and other constraints on data sharing (e.g. legal, geographical), prevent the availability of entire data set in a single computing machine (or node). In this work, we consider a decentralized environment where the computing nodes possess similar processing and storage capabilities; further, we do not assume a central server. The (static) topology of the decentralized environment is represented using an undirected, connected, simple graph $G = (V, E)$, where $V = \{1, 2, \ldots, m\} := [m]$ denotes the set of $m$ computing nodes and an edge $e_{ij} \in E$ denotes the fact that nodes $i, j \in V$ are connected. Also, we assume that the communication is synchronous and at every synchronization step, node $i$ communicates only with its neighbors $\mathcal{N}(i) = \{j \in V : e_{ij} \in E\}$.

We are interested in developing first-order stochastic algorithms that are efficient in terms of local gradient computation complexity as well as overall inter-node communication complexity, for solving the following saddle point (or mini-max) problem in the aforementioned decentralized environment:

$$\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} \Psi(x, y) := \frac{1}{m} \sum_{i=1}^{m} (f_i(x, y) + g(x) - r(y)),$$

(SPP)
where $\mathcal{X} \subset \mathbb{R}^{d_x}$, $\mathcal{Y} \subset \mathbb{R}^{d_y}$ are convex and compact sets, $f_i : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$ for every node $i \in [m]$ is smooth, strongly-convex strongly-concave in $(x, y)$, and $g : \mathcal{X} \rightarrow \mathbb{R}$ and $r : \mathcal{Y} \rightarrow \mathbb{R}$ are proper, continuous, convex functions which might be non-smooth. This class of saddle point problems finds its use in robust classification and regression applications and AUC maximization problems [29, 40].

It is well-known that using compressed information exchange (or compression) in decentralized optimization results in efficient communication and resource utilization [1] [34] [2] [19]. Notably, recent works (e.g. [13] [23] [18]) achieve sublinear/convergence rates with compression and low per iteration complexity, in solving decentralized minimization problems, similar to the rates available for non-compressed communication settings. However, to our knowledge, it is not known if it is possible to achieve fast convergence rates with compression and low per iteration complexity, when solving problems of the form (SPP). In this work, we design algorithms which can achieve sublinear/convergence rates to solve problem (SPP), by using stochastic oracles for efficient gradient computation, and compressed information exchange for efficient communication. In particular, we focus on the following two settings of problem (SPP): (i) general stochastic setting (ii) finite sum setting. In general stochastic setting, we assume the following form for the local function: $f_i(x, y) = E_{x \sim D_i}[f_i(x; y; \xi_i)]$, where $D_i$ is the local sample distribution in node $i$, allowing for general heterogeneous data distributions across different nodes. In finite sum setting, we assume that each $f_i(x, y)$ is represented as the average over local samples, hence $f_i(x, y) = \frac{1}{n} \sum_{j=1}^{\n} f_{ij}(x, y)$, where $f_{ij}(x, y)$ represents the loss function at $j$th batch of samples at node $i$.

**Contributions of our work:** Inspired by algorithms developed for decentralized minimization problems [23] [18], we design two algorithms namely Restart-based Decentralized Proximal Stochastic Gradient method with Compression (C-RDPSG) and Decentralized Proximal Stochastic Variance Reduced Gradient algorithm with Compression (C-DPSVRG), the first algorithms promoting the effective combination of stochastic gradient estimator oracles and information exchange with compression for solving non-smooth saddle point problems of form (SPP) in a decentralized environment (henceforth referred to as DENCS). Our rigorous analysis establishes C-RDPSG and C-DPSVRG as the first compression based algorithms to achieve respectively, sublinear and linear convergence rates for solving (SPP) in DENCS (similar rates are known so far only for decentralized smooth saddle point problems with no compression and no stochastic variance reduction oracles [25] [28] [4] [5]). For general stochastic settings, C-RDPSG achieves gradient computation and communication complexity of order $O\left((1 + \delta)^2 \frac{1}{\epsilon^2} \frac{\kappa_f^2 \kappa_g}{\epsilon} \frac{1}{\epsilon} \right)$, to obtain an $\epsilon$-accurate saddle-point solution, where $\delta$ denotes the compression factor, $\kappa_f$ and $\kappa_g$ denote respectively the condition numbers of objective function $f$ and communication graph $G$, and $L$ denotes the smoothness parameter of smooth part $\frac{1}{m} \sum_{i=1}^{m} f_i$. In finite sum setting common in machine learning, C-DPSVRG is shown to have gradient computation complexity and communication complexity of order $O\left((1 + \delta) \frac{\kappa_f^2 \kappa_g}{\epsilon} \log \left( \frac{1}{\delta} \right) \right)$. Using extensive experiments, we showcase empirical evidence supporting the theoretical guarantees obtained in our analysis.

**Notations:** The following notations will be used in the paper. The $k$-tuple $(u^1, u^2, \ldots, u^k)$ of $k$ vectors $u^1, u^2, \ldots, u^k$, each of size $d \times 1$ denotes the vector $\left( (u^1)^\top, (u^2)^\top, \ldots, (u^k)^\top \right)^\top$ of size $kd \times 1$. The notation $z = (x, y) \in \mathbb{R}^{d_x+d_y}$ denotes the pair of primal variable $x$ and dual variable $y$, and $z^* = (x^*, y^*)$ denotes a saddle point of problem (SPP). The communication link between a pair of nodes $(i, j) \in V \times V$ is assumed to have associated weight $W_{ij} \in [0, 1]$, which are collected into a weight matrix $W$ of size $m \times m$. $I_p$ denotes a $p \times p$ identity matrix, $1$ denotes a $m \times 1$ column vector of ones and $J = \frac{1}{m} 11^\top$ is an $m \times m$ matrix of uniform weights equal to $\frac{1}{m}$. We define $f(z) := f(x, y) := \sum_{i=1}^{m} f_i(x, y)$. The notation $\langle u, v \rangle$ denotes the inner product between two $d \times 1$ vectors $u$ and $v$. For a $d \times 1$ vector $u$ and for some $d \times d$ symmetric positive semi-definite (p.s.d) matrix $A$, we define $\|u\|_A^2 = u^\top Au$. $A \otimes B$ denotes the Kronecker product of two matrices $A$ and $B$.

**Paper Organization:** A short survey of state-of-the-art algorithms for decentralized minimization and saddle point problems is provided in Section 2. We then provide a consensus perspective of problem (SPP) in Section 3 which leads to the development of primal and dual variable updates using an inexact primal dual hybrid algorithm. We discuss the technical assumptions useful for our algorithm development in Section 4. C-RDPSG algorithm for general stochastic setting is illustrated in Section 5 followed by a discussion of C-DPSVRG algorithm for finite sum setting in Section 6. Experiments are provided in Section 7. We conclude the paper in Section 8 with pointers to future work.
2 Related Work

Decentralized minimization: Several existing works address decentralized convex minimization: e.g. distributed gradient methods [31], gradient tracking methods [33, 30], stochastic gradient oracle (SGO) based methods [27], SGO+compression based methods [14, 23, 18]; see [18] for detailed survey. Recent works focus on non-convex optimization in decentralized settings [11].

Decentralized saddle-point problems: A distributed saddle-point algorithm with Laplacian averaging (DSPAwLA) in [25], based on gradient descent ascent updates to solve non-smooth convex-concave saddle point problems achieves $O(1/\epsilon^2)$ convergence rate. An extragradient method with gradient tracking (GT-EG) proposed in [28] is shown to have linear convergence rates for solving decentralized strongly convex strongly concave problems, under a positive lower bound assumption on the gradient difference norm. The dependence of rates in [28] on condition number $\kappa_g$ is unknown. Recently, multiple works [22, 56, 4] have proposed using minibatch gradients for solving decentralized saddle point problems. Decentralized extra step (DES) [4] shows linear communication complexity with dependence on the graph condition number as $\sqrt{\kappa_g}$, obtained at the cost of incorporating multiple rounds of communication of primal and dual updates. A near optimal distributed Min-Max data similarity (MMDS) algorithm is proposed in [5] for saddle point problems with a suitable data similarity assumption. Decentralized parallel optimistic stochastic gradient method (DPOS) was proposed in [22] for nonconvex-nonconcave saddle point problems. A gradient tracking based algorithm called DMHSGD for solving nonconvex-strongly concave saddle point problems is proposed in [36]. A detailed discussion on related work is provided in Appendix A. Table 1 positions our work in the context of existing methods.

Table 1: Comparison of proposed optimization algorithms for decentralized saddle point problems with state-of-the-art algorithms. SGO and VR denote respectively Stochastic Gradient Oracle and Variance Reduction.

| Algorithm      | Compression | SGO | VR | Non-smooth case | Gossip step | Computation complexity | Communication complexity |
|----------------|-------------|-----|----|-----------------|-------------|------------------------|--------------------------|
| MMDs [5]       | X           | X   | X  | X               | ✓           | $O(\kappa^2\epsilon^2)$ | $O(\kappa_g^2\epsilon^2)$ |
| DMHSGD [36]    | X           | ✓   | X  | ✓               | ✓           | $O(\kappa^2\epsilon^2)$ | $O(\kappa_g^2\epsilon^2)$ |
| DES [4]        | ✓           | ✓   | X  | ✓               | ✓           | $O(\kappa^2\epsilon^2)$ | $O(\kappa_g^2\epsilon^2)$ |
| DPOS [22]      | ✓           | ✓   | ✓  | ✓               | ✓           | $O(\kappa^2\epsilon^2)$ | $O(\kappa_g^2\epsilon^2)$ |
| GT-EG [28]     | X           | X   | X  | ✓               | ✓           | $O(\kappa^2\epsilon^2)$ | $O(\kappa_g^2\epsilon^2)$ |
| DSPAwLA [25]   | ✓           | ✓   | ✓  | ✓               | ✓           | $O(\kappa^2\epsilon^2)$ | $O(\kappa_g^2\epsilon^2)$ |
| C-RDPSG (this work) | ✓       | ✓   | ✓  | ✓               | ✓           | $O(\kappa^2\epsilon^2)$ | $O(\kappa_g^2\epsilon^2)$ |
| C-DPSVRG (this work) | ✓       | ✓   | ✓  | ✓               | ✓           | $O(\kappa^2\epsilon^2)$ | $O(\kappa_g^2\epsilon^2)$ |

3 A Consensus Viewpoint of Problem (SPP)

Assuming the local copy of $(x, y)$ in $i$-th node as $(x^i, y^i)$, we collect local primal and dual variables into $x = (x^1, x^2, \ldots, x^m) \in \mathbb{R}^{md_x}$ and $y = (y^1, y^2, \ldots, y^m) \in \mathbb{R}^{md_y}$. Using this notation, we first formulate problem (SPP) as the following optimization problem with consensus constraints on the variables $x$ and $y$:

$$\min_{x \in \mathbb{R}^{md_x}} \max_{y \in \mathbb{R}^{md_y}} \sum_{i=1}^{m} f_i(x^i, y^i), \ G(x) = \sum_{i=1}^{m} (g(x^i) + \delta x(x^i)), \ R(y) = \sum_{i=1}^{m} (r(y^i) + \delta y(y^i))$$

and $U = \sqrt{I_m} - \bar{W}$. The assumptions on $W$ (to be made later) would imply $I_m - W$ to be symmetric p.s.d. and hence the existence of $\sqrt{I_m} - \bar{W}$. Further, considering $\sqrt{I_m} - \bar{W}$ will be convenient for our computations. $\delta_{C\{u\}}$ denotes the indicator function of set $C$ which is 0 when $u \in C$ and $+\infty$ otherwise. Formulating an unconstrained decentralized optimization problem as an equivalent problem with a consensus constraint on the local variables is well-known [25].

We have the following Lagrangian function of problem (1):

$$\mathcal{L}(x, y; S^x, S^y) = F(x, y) + G(x) - R(y) + \langle S^x, (U \otimes I_{d_x}) x \rangle + \langle S^y, (U \otimes I_{d_y}) y \rangle,$$

2

where $S^x \in \mathbb{R}^{md_x}$ and $S^y \in \mathbb{R}^{md_y}$ denote the Lagrange multipliers associated with consensus constraints on variables $x$ and $y$ respectively. Solving constrained problem (1) is equivalent to
Towards this end, we follow [26, 23] to compress a related difference vector instead of directly solving the following problem (see Theorem C.1 in Appendix C):

$$\min_{x \in \mathbb{R}^{md_x}, y \in \mathbb{R}^{md_y}} \max_{S^x, S^y} \mathcal{L}(x, y; S^x, S^y).$$

(3)

To solve problem (3), we propose inexact Primal Dual Hybrid Gradient (PDHG) updates alternatively for the primal-dual variable pair \(x, S^x\) and dual-primal pair \(y, S^y\), illustrated in eqs. (P1) and (D1).

**Updates to primal-dual pair \(x, S^x\):**

$$\nu^x_{t+1} = \arg \min_{x \in \mathbb{R}^{md_x}} \left\{ \langle x, \nabla_x F(x_t, y_t) \rangle + \frac{\gamma}{2} \| x - x_t \|^2 \right\}$$

$$S^x_{t+1} = S^x_t - \frac{\gamma}{2s} (U \otimes I_d_x) \nu^x_{t+1}$$

$$x_{t+1} = \text{prox}_{\gamma G}(\hat{x}_{t+1}).$$

$$\nu^y_{t+1} = \arg \max_{y \in \mathbb{R}^{md_y}} \left\{ \langle y, \nabla_y F(x_t, y_t) \rangle - \frac{\gamma}{2} \| y - y_t \|^2 \right\}$$

$$S^y_{t+1} = S^y_t - \frac{\gamma}{2s} (U \otimes I_d_y) \nu^y_{t+1}$$

$$y_{t+1} = \text{prox}_{\gamma F}(\hat{y}_{t+1}).$$

(P1)

(P2)

Note that in eqs. (P1), \(\nu^x_{t+1}\) is found using a prox-linear step involving linearization of \(\mathcal{L}\) with respect to \(x\) and a penalized cost-to-move term \(\frac{\gamma}{2s} \| x - x_t \|^2\), followed by an ascent step to update the Lagrange dual variables \(S^x\). Then \(\hat{x}_{t+1}\) is found using prox-linear step similar to the first step but using the recent \(S^x_{t+1}\). Finally \(x_{t+1}\) is found by a prox step where \(\text{prox}_{\gamma G}(x) = \min_{u \in \mathbb{R}^{md_x}} G(u) + \frac{\gamma}{2} \| u - x \|^2\). Letting \(D^x = (U \otimes I_d_x) S^x_t\), and pre-multiplying by \(U \otimes I_{d_y}\) in the update step of \(S^x\) equations (P1) reduce to those in equations (P2). Similarly, the updates to \(y, S^y\) can be done using appropriate gradient ascent-descent steps which lead to corresponding equations (D1) and (D2).

Inexact PDHG type updates are known for solving a Lagrangian function of an underlying convex minimization problem in single machine setting (e.g. proximal alternating predictor-corrector (PAPC) algorithm [7, 24], primal-dual fixed point (PDFP) algorithm [8]), and in decentralized setting [18]. In contrast to PAPC, PDFP, and the algorithm in [18], the type of inexact PDHG updates proposed in our work address a Lagrangian function corresponding to an underlying saddle point problem (SPP). To our knowledge, our work is the first to extend inexact PDHG type updates to solve Lagrangian of saddle point problem of form (SPP). For other perspectives and generalizations of inexact PHDG, see [6, 11, 9, 55, 38, 18].

Observe that the term \((I_m - W) \otimes I_{d_y}) \nu^y_{t+1}\) in eq. (P2) and \((I_m - W) \otimes I_{d_x}) \nu^x_{t+1}\) in eq. (D2) denote the communication of \(\nu^x_{t+1}\) and \(\nu^y_{t+1}\) across the nodes. Further note that \(\nu^x_{t+1}\) and \(\nu^y_{t+1}\) need to be communicated only once for updating \(D^x_{t+1}, \hat{x}_{t+1}\) and \(D^y_{t+1}, \hat{y}_{t+1}\) in the inexact PDHG updates for \(x\) and \(y\) respectively. This results in cheaper communication in every iteration when compared to the multiple communications that happen in a single iteration of the algorithms in [5, 36, 22]. To improve the communication efficiency further, we propose to compress \(\nu^x_{t+1}\) and \(\nu^y_{t+1}\). We recall that compression has not yet been used in existing algorithms for decentralized saddle point problems.

Towards this end, we follow [26, 23] to compress a related difference vector instead of directly compressing \(\nu^x_{t+1}\) and \(\nu^y_{t+1}\). Each node \(i\) is assumed to maintain a local vector \(H^{i,x}\) and a stochastic compression operator \(Q\) is applied on the difference vector \(\nu^i_{t+1} - H^{i,x}\). The concise form of these updates is illustrated in Algorithm [4] (COMM procedure) in Appendix [13]. Algorithm 1 illustrates the proposed inexact PDHG updates with compression. Thus the inexact PDHG update steps to obtain \(\nu^x_{t+1}\) and \(\nu^y_{t+1}\) involve computing the gradients \(\nabla_x F(x_t, y_t) = (\nabla_x f_1(x^1, y^1), \ldots, \nabla_x f_m(x^m, y^m))\) and
Algorithm 1 Inexact primal dual hybrid algorithm updates using gradient computation oracle $G$ (IPDHG)

1: **INPUT:** $x, y, D^x, D^y, H^x, H^y, H^{w,x}, H^{w,y}, s, \gamma_x, \gamma_y, \alpha_x, \alpha_y, G$
2: Compute gradients $G^x$ and $G^y$ at $(x, y)$ via oracle $G$
3: $\nu^x = x - sG^x - sD^x$
4: $\hat{\nu}^x, \hat{\nu}^{w,x}, H^x_{new}, H^x_{new} = \text{COMMIT}(\nu^x, H^x_{new}, H^{w,x}_{new}, \alpha_x)$
5: $D^x_{new} = D^x + \frac{\gamma}{2\nu^x} (\hat{\nu}^x - \hat{\nu}^{w,x})$
6: $\tilde{x} = \nu^y - \frac{\gamma}{2\nu^x} (\hat{\nu}^x - \hat{\nu}^{w,x})$
7: $x_{new} = \text{prox}_{\delta G} (\tilde{x})$
8: $\nu^y = y + sG^y - sD^y$
9: $\hat{\nu}^y, \hat{\nu}^{w,y}, H^y_{new}, H^{y}_{new} = \text{COMMIT}(\nu^y, H^y_{new}, H^{w,y}_{new}, \alpha_y)$
10: $D^y_{new} = D^y + \frac{\gamma}{2\nu^y} (\hat{\nu}^y - \hat{\nu}^{w,y})$
11: $\hat{y} = \nu^y - \frac{\gamma}{2\nu^y} (\hat{\nu}^y - \hat{\nu}^{w,y})$
12: $y_{new} = \text{prox}_{\delta H} (\hat{y})$
13: **RETURN:** $x_{new}, y_{new}, D^x_{new}, D^y_{new}, H^x_{new}, H^y_{new}, H^{w,x}_{new}, H^{w,y}_{new}$.

∇$_y F(x_i, y_i) = (\nabla_y f_1(x^1, y^1), \ldots, \nabla_y f_m(x^m, y^m))$ using a gradient computation oracle $G$. We now discuss methods based on two different stochastic gradient oracles to compute the gradients. Before discussing the methods, we state technical assumptions common to both the methods.

4 Assumptions

We list below the assumptions to be used throughout this work.

**Assumption 4.1.** Each $f_i(\cdot, y)$ is $\mu_x$-strongly convex for every $y \in \mathcal{Y}$; hence for any $x_1, x_2 \in \mathcal{X}$ and fixed $y \in \mathcal{Y}$, it holds: $f_i(x_1, y) \geq f_i(x_2, y) + \langle \nabla_{x_i} f_i(x_2, y), x_1 - x_2 \rangle + \frac{\mu_x}{2} \|x_1 - x_2\|^2$.

**Assumption 4.2.** Each $f_i(x, \cdot)$ is $\mu_y$-strongly concave for every $x \in \mathcal{X}$; hence for any $y_1, y_2 \in \mathcal{Y}$ and fixed $x \in \mathcal{X}$, it holds: $f_i(x, y_1) \leq f_i(x, y_2) + \langle \nabla_{y_i} f_i(x, y_2), y_1 - y_2 \rangle - \frac{\mu_y}{2} \|y_1 - y_2\|^2$.

**Assumption 4.3.** $g(x)$ and $r(y)$ are proper, convex, continuous and possibly non-smooth functions.

**Assumption 4.4.** The compression operator $Q$ satisfies the following for every $u \in \mathbb{R}^d$: $Q(u)$ is an unbiased estimate of $u$, thus $E[Q(u)] = u$ and $Q(u)$ has bounded variance: $E[\|Q(x) - x\|^2] \leq \delta \|x\|^2$, where the constant $0 \leq \delta \leq 1$ denotes the amount of compression induced by operator $Q$ and is called a compression factor. When $\delta = 0$, $Q$ achieves no compression.

**Assumption 4.5.** The weight matrix $W$ satisfies the following conditions: $W$ is symmetric and row stochastic, $W_{ij} > 0$ if and only if $(i, j) \in E$ and $W_{ii} > 0$ for all $i \in [m]$. The eigenvalues of $W$ denoted by $\lambda_1, \ldots, \lambda_m$ satisfy: $-1 < \lambda_m \leq \lambda_{m-1} \leq \ldots \leq \lambda_2 < 1$.

Graph condition number $\kappa_g$ is defined as the ratio of largest eigenvalue and second smallest eigenvalue of $I - W$ and the condition number $\kappa_f$ is defined as $L/\mu$, where $\mu = \min(\mu_x, \mu_y)$. Note that Assumptions 4.1, 4.3 and Assumption 4.5 are standard in the study of saddle point problems (e.g. [4, 5, 28, 22]). Assumptions on bounded variance and unbiasedness of compression operator $Q$ are also standard in existing works (e.g. [11, 18, 23]).

5 General Stochastic Setting

In the general stochastic setting, we allow for availability of heterogeneous data distributions in each node and assume that the gradients $\nabla_x f_i(x, y)$ and $\nabla_y f_i(x, y)$ are computed using an oracle $G$ described below.

**General Stochastic Gradient Oracle (GSGO):**

1. Sample a mini-batch of samples $\xi^i \sim D_i$ in each node $i$, where $D_i$ is the data distribution local to node $i$.
2. Compute stochastic gradients: $G_{i,x}^i = \nabla_x f_i(x^i, y^i; \xi^i)$ and $G_{i,y}^i = \nabla_y f_i(x^i, y^i; \xi^i)$.

Note that algorithms based on GSGO converge only to a neighborhood of the optimal solution using constant step size and require diminishing step size to converge to an exact solution [23]. However diminishing step sizes result in slow convergence of the algorithms [23]. Restart based GSGO schemes in single machine setting [39, 40] have shown convergence to exact solution without step size decay at every iterate. Inspired by these schemes, we design in this work, a restart based
stochastic gradient method illustrated in Algorithm 2 which invokes in every iteration \( k \), a sequence of \( t_k \) primal and dual variable updates using inexact PDHG with compression (Algorithm 1). In Algorithm 2, step length \( s_k \) is chosen to decrease geometrically only at every restart step \( k \), resulting in better convergence. The input parameter \( \rho \) of Algorithm 2 is defined in Appendix E.3. The step lengths \( \gamma_k^x, \gamma_k^y \) used for updating \( x \) and \( y \) using Algorithm 1 are computed in Step 4 of Algorithm 2 using \( s_k \), the strong convexity parameters \( \mu_x, \mu_y \), the Lipschitz constants \( L_{xy}, L_{yx} \), the largest eigenvalue \( \lambda_{\text{max}} \) of \( I_m - W \) matrix, and compression factor \( \delta \). The parameters \( \alpha_{x,k}, \alpha_{y,k} \) useful in COMM procedure and constants \( M_{x,k}, M_{y,k}, M_k \) are then computed in Step 6 which are further useful to determine the number of inner iterations \( t_k \). However, the restart scheme proposed in Algorithm 2 is simpler than that in [40], where the restart scheme in single machine setting requires the computation of Fenchel conjugate at every restart incurring additional \( O(d_y) \) operations. Our scheme is also different from other restart based schemes studied for saddle point problems in single machine setting [42][43][44].

**Algorithm 2** Restart-based Decentralized Proximal Stochastic Gradient method with Compression (C-RDPSG)

1. **INPUT:** \( x_{0,0} = x_0, y_{0,0} = y_0, s_0 = \frac{1}{\lambda L_k} \), Number of iterations \( K, \rho, \mathcal{G} \) obtained using GSGO.
2. for \( k = 0 \) to \( K - 1 \) do
3. \( s_k = \frac{1}{2\sqrt{\rho}}, \bar{b}_{x,k} = \mu_x s_k - 4s_k^2 L_{xy}, \bar{b}_{y,k} = \mu_y s_k - 4s_k^2 L_{yx}, \)
4. \( \gamma_k^x = \frac{\bar{b}_{x,k}}{\lambda_{\text{max}}(I_m - W)}, \gamma_k^y = \frac{\bar{b}_{y,k}}{\lambda_{\text{max}}(I_m - W)} \)
5. \( \alpha_{x,k} = \frac{\bar{b}_{x,k}}{\lambda_{\text{max}}}, \alpha_{y,k} = \frac{\bar{b}_{y,k}}{\lambda_{\text{max}}} \)
6. \( M_{x,k} = 1 - \frac{\sqrt{\lambda_{\text{max}}(I - W)}}{1 - 2\sqrt{\lambda_{\text{max}}}}, M_{y,k} = 1 - \frac{\sqrt{\lambda_{\text{max}}(I - W)}}{1 - 2\sqrt{\lambda_{\text{max}}}} \)
7. \( M_k = \min(M_{x,k}, M_{y,k}) \)
8. Set \( t_k = \frac{1}{-\log(1 - \frac{1}{2\sqrt{\rho}})} \max\{\log\left(\frac{3M_k + 6\sqrt{\rho}}{M_k}\right), \log\left(\frac{3M_k + 6\sqrt{\rho}}{M_k}\right), \log\left(\frac{1}{3\sqrt{\rho}}\right)\} \)
9. \( D_{x,k}^0 = D_{y,k}^0 = 0, H_{x,k}^0 = x_k, H_{y,k}^0 = y_k, H_{w,x,k}^0 = (W \otimes I_d)H_{x,k}^0, H_{w,y,k}^0 = (W \otimes I_d)H_{y,k}^0 \)
10. for \( t = 0 \) to \( t_k - 1 \) do
11. \( x_{k,t+1}, y_{k,t+1}, D_{x,k,t+1}, D_{y,k,t+1}, H_{x,k,t+1}, H_{y,k,t+1}, H_{w,x,k}^t, H_{w,y,k}^t = IPDHG(x_{k,t}, y_{k,t}, D_{x,k,t}, D_{y,k,t}, H_{x,k,t}, H_{y,k,t}, H_{w,x,k}^t, H_{w,y,k}^t, s_k, \gamma_k^x, \gamma_k^y, \alpha_{x,k}, \alpha_{y,k}, \mathcal{G}) \)
12. end for
13. \( x_{K,1} = x_{K-1}, y_{K,1} = y_{K-1} \)
14. end for
15. RETURN: \( x_K, y_K \).

Under appropriate assumptions on unbiasedness and smoothness of the stochastic gradients (see Appendix E), we have the following convergence result of Algorithm 2.

**Theorem 5.1.** Suppose \( \{x_{k,0}\}_k \) and \( \{y_{k,0}\}_k \) are the sequences generated by Algorithm 2. Then with at most

\[
K(\epsilon) = \max\left\{ \log_2 \left( \frac{\|z_0 - 1x^*\|^2}{\epsilon} \right), 2\log_2 \left( \frac{(1 + \delta)^2}{L^2 \sqrt{\delta}} + \frac{m\sigma^2}{\kappa_f^2 \kappa_g (1 + \delta)^2 L^2 \epsilon} \right) \right\},
\]

iterations, \( E[\|x_{K,0}\|] - 1x^* \|^2 \leq \epsilon \). Moreover the total gradient computation complexity and communication complexity to achieve \( \epsilon \)-accurate saddle point solution in expectation are

\[
T_{\text{grad}}(\epsilon) = \mathcal{O}(\max(\frac{(1 + \delta)^2\|z_0 - 1x^*\|^2}{\epsilon}, \frac{(1 + \delta)^4\kappa_f^2 \kappa_g^2}{L^2 \epsilon} + \frac{(1 + \delta)^4 \sigma^2 \kappa_f^2 \kappa_g^2}{L^2 \epsilon})),
\]

and \( T_{\text{comm}}(\epsilon) = T_{\text{grad}}(\epsilon) + K(\epsilon) \) respectively.

**Proof sketch of Theorem 5.1** We first define appropriate quantities \( D_{x,k}^0, D_{y,k}^0, H_{w,x,k}^0, H_{w,y,k}^0, V_{(x_1, x_2)} \) and \( V_{(y_1, y_2)} \) (see Appendix E.3) to capture the interactions between inexact PDHG updates done on primal-dual pair \( x, S^x \) and dual-primal pair \( y, S^y \). Then we derive a recursive relation on the sequence \( E[\Phi_{k,0}] \) (defined in Appendix E.3) depending on the primal and dual iterates (see Lemma E.5 in Appendix E). Further, we represent \( E[\Phi_{k,0}] \) in terms of restart quantity \( E[\Phi_{k,0}] \) by setting the algorithm parameters carefully (see Lemma E.6 in Appendix E). We show the feasibility of Algorithm 2 parameters; and derive suitable lower and upper bounds depending on the condition
numbers $\kappa_f$ and $\kappa_g$ in Appendix G. The proof of Theorem 5.1 then follows by obtaining a recursive relation of restart iterates $x_{k,0}, y_{k,0}$ and is given in Appendix F.

We note that our analysis techniques are different from the analysis in [39, 40] due to different structure of primal-dual updates and inclusion of communication compression. Further the smooth objective function $F(x, y)$ and consensus constraints in (1) depend on primal and dual iterates and thus the analysis of Algorithm 2 is more complicated than that of the algorithms in [18].

6 Finite Sum Setting

In the finite sum setting, we assume that each local function $f_i(x, y)$ is of the form $\frac{1}{n} \sum_{j=1}^{n} f_{ij}(x, y)$. For simplicity, we assume that each node $i$ has same number of batches $n$. However, our analysis easily extends to different number of batches $n_i$. Let $P_i = \{p_{il} : l \in \{1, 2, \ldots, n\}\}$ be the probability distribution with which index $l$ is sampled. Note that GSGO in Algorithm 2 shows sublinear convergence for solving (SPP). Stochastic variance reduction techniques ([13, 16, 18, 37]) are known to accelerate convergence of GSGO methods for minimization problems. Inspired by this success, we propose a Stochastic Variance Reduced Gradient (SVRG) oracle comprising the following steps.

**Stochastic Variance Reduced Gradient Oracle (SVRGO):**

1. **Index sampling:** Sample $l \in \{1, 2, \ldots, n\} \sim P_i$ for every node $i$.
2. **Stochastic gradient computation with variance reduction:** For a reference point $z^t = (\tilde{x}^t, \tilde{y}^t)$, compute stochastic gradients at $z^t = (x^t, y^t)$ with respect to $x$ and $y$ as follows:
   
   $$G_{i,x}^t = \frac{1}{np_{il}} \left[ \nabla_x f_{i,l}(z^t) - \nabla_x f_{i,l}(\tilde{z}^t) \right] + \nabla_x f_i(z^t),$$
   
   $$G_{i,y}^t = \frac{1}{np_{il}} \left[ \nabla_y f_{i,l}(z^t) - \nabla_y f_{i,l}(\tilde{z}^t) \right] + \nabla_y f_i(z^t).$$

3. **Reference point update:** Sample $\omega \in \{0, 1\} \sim \text{Bernoulli}(\hat{\rho})$ and update the reference points as follows:
   
   $$\tilde{x}_{i+1} = \omega x^t_i + (1 - \omega) \tilde{x}^t_i,$$
   
   $$\tilde{y}_{i+1} = \omega y^t_i + (1 - \omega) \tilde{y}^t_i.$$ 

The SVRGO setup above requires computation of full-batch gradient at the reference point periodically, but is memory-efficient compared to other variance reduction schemes (e.g. SAGA [10]). Note also that the SVRGO based algorithm in [29] is for a differently structured problem than (SPP). The C-DPSVRG methodology using SVRGO is illustrated in Algorithm 3 with step sizes defined in Appendix G.

**Discussion on Complexity Results:** Theorem 5.1 and 6.1 demonstrate the sublinear and linear convergence of C-RDPSG and C-DPSVRG respectively with explicit dependence on $\kappa_f$ and $\kappa_g$. Computation complexity of Algorithm 3 depends on $1/p, p$ being the reference probability. Smaller
We distribute the samples across 20 nodes and create 20 mini batches of local samples for all the nodes. We evaluate the performance of proposed algorithms on robust logistic regression problem with three noncompression baseline algorithms: (1) Distributed Min-Max Data similarity algorithm, (2) Decentralized Parallel Optimistic Stochastic Gradient (DPOSG) algorithm [22] and, (3) Decentralized Minimix Hybrid Stochastic Gradient Descent (DM-HSGD) algorithm [35]. The parameters are chosen based on the theory developed in the respective papers. More details are provided in Appendix I.

Algorithm 3 Decentralized Proximal Stochastic Variance Reduction method with Compression (C-DPSVRG)

1: INPUT: $x_0, y_0, D^0_0 = D^N_0 = 0, H^0_0 = x_0, H^y_0 = y_0, H^{w,x}_0 = (W \otimes I_d) x_0, H^{w,y}_0 = (W \otimes I_d) y_0, s = \frac{a_{DPbin}}{2M}, \alpha_x, \alpha_y, \gamma_x, \gamma_y, \sigma$, defined using SVRGO.
2: for $t = 0$ to $T - 1$ in parallel for all nodes $i$ do
3: $x_{t+1}, y_{t+1}, D^x_{t+1}, D^y_{t+1}, H^x_{t+1}, H^y_{t+1}, H^{w,x}_{t+1}, H^{w,y}_{t+1} =$ IPDHG($x_t, y_t, D^x_t, D^y_t, H^x_t, H^y_t, H^{w,x}_t, H^{w,y}_t, s, \gamma_x, \gamma_y, \alpha_x, \alpha_y, \sigma$)
4: end for
5: RETURN: $x_T, y_T$.

$p$ yields rare computation of full batch gradients and hence needs more calls of SVRGO; it is noticed that small value of $p$ gives faster convergence in terms of gradient computations (see results in Appendix I). Complexity of C-RDPSG shows dependence on graph condition number as $O(\kappa^2_g)$; whereas C-DPSVRG shows such dependence as $O(\kappa_g)$. This makes C-RDPSG more sensitive to sparse topology as demonstrated in Figure 9 (see Appendix I). We notice that our C-RDPSG method converges to saddle point solution with linear rates in the deterministic setting as well ($\sigma = 0$). This can be obtained by removing restart update and using a constant step size in the convergence analysis.

7 Numerical Experiments

We evaluate the performance of proposed algorithms on robust logistic regression problem

$$\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} \Psi(x, y) = C \sum_{i=1}^{N} \log (1 + \exp \left(-b_i x^T (a_i + y)\right)) + \lambda \frac{1}{2} \|x\|_2^2 - \frac{\beta}{2} \|y\|_2^2$$

over a binary classification data set $D = \{(a_i, b_i)\}_{i=1}^{N}$ and $C = \frac{1}{\max N}$. We consider constraint sets $\mathcal{X}$ and $\mathcal{Y}$ as 12 ball of radius 100 and 1 respectively. We compute smoothness parameters $L_{xx}, L_{yy}, L_{xy}$ and $L_{y}x$ using Hessian information of the objective function and set strong convexity and strong concavity parameters to $\lambda$ and $\beta$ respectively. Unless stated otherwise, we set $\lambda = \beta = 10$, number of nodes to $m = 20$ and number of batches to $n = 20$ in all our experiments. We implement all the experiments in Python Programming Language.

Datasets: We rely on four binary classification datasets namely, a4a, phishing and ijcnn1 from https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/ and sido data from http://www.causality.inf.ethz.ch/data/SIDO.html. Dataset details are available in Appendix I. We distribute the samples across 20 nodes and create 20 mini batches of local samples for all the datasets.

Network Setting: We conduct the experiments for 2D torus topology and ring topology. For 2D torus, we generate the weight matrix $W$ with $W_{ij} = 1/5$ for all $(i, j) \in E \cup \{(i, i)\}$. For ring topology, weight matrix $W$ is constructed by setting $W_{ij} = 1/3$ for all $(i, j) \in E \cup \{(i, i)\}$.

Compression Operator: In all our experiments, we consider an unbiased $b$-bits quantization operator $Q_{\infty}(x) = (\|x\|_\infty 2^{-(b-1)} \text{sign}(x)) \cdot \frac{2^{b-1} |x|}{\|x\|_\infty} + u$, where $\cdot$ represents Hadamard product, $|x|$ denotes elementwise absolute value and $u$ is a random vector uniformly distributed in $[0, 1]^d$.

Baseline methods: We compare the performance of proposed algorithms C-RDPSG and C-DPSVRG with three noncompression baseline algorithms: (1) Distributed Min-Max Data similarity algorithm [5] (2) Decentralized Parallel Optimistic Stochastic Gradient (DPOSG) algorithm [22] and, (3) Decentralized Minimix Hybrid Stochastic Gradient Descent (DM-HSGD) algorithm [35]. The parameters are chosen based on the theory developed in the respective papers. More details are provided in Appendix I.

Benchmark Quantities: We run the centralized and uncompressed version of C-DPSVRG for 50,000 iterations to get saddle point solution $z^* = (x^*, y^*)$ of problem (8). The performance of all the methods is measured using $\sum_{i=1}^{m} \|z_i^* - z^*\|^2$.

Observations: C-RDPSG converges faster in the beginning and slows down after reaching around $10^{-8}$ accuracy as depicted in Figure 1. C-DPSVRG converges faster than other baseline methods. DPOSG and DM-HSGD converges only to a neighborhood of the saddle point solution and starts oscillating after sometime. The inclusion of restart scheme in C-RDPSG helps to mitigate the flat
and oscillatory behavior at the later iterations unlike DPOSG and DM-HSGD. The performance of C-RDPSG is competitive with DPOSG and DM-HSGD in the long run as demonstrated in Figure 1. We observe that C-DPSVRG and C-RDPSG are faster than Min-max similarity, DPOSG, DMHSGD for obtaining low accurate solutions in terms of bits transmission and communication cost. We observe similar performance of proposed methods for ring topology in Figure 2. More analysis of these and additional experiments are presented in Appendix I.

**Figure 1:** Convergence behavior of iterates to saddle point vs. Gradient computations (Column 1), Communications (Column 2), Number of bits transmitted (Column 3) for different algorithms in 2D torus topology. a4a, ijcnn, datasets are in Rows 1,2 respectively.

**Figure 2:** Convergence behavior of iterates to saddle point vs. Gradient computations (Column 1), Communications (Column 2), Number of bits transmitted (Column 3) for different algorithms in ring topology. a4a, ijcnn datasets are in Rows 1,2 respectively.

### 8 Conclusion

We have proposed two stochastic gradient algorithms for decentralized optimization for saddle point problems, with compression. Both the algorithms offer practical advantages and are shown to have rigorous theoretical guarantees. It would be interesting to adapt both C-RDPSG and C-DPSVRG to cases where some of the constants are unknown in the problem setup.
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A Related work

Decentralized saddle-point problems: A distributed saddle-point algorithm with Laplacian averaging (DSPAwLA) in [25], based on gradient descent updates to solve non-smooth convex-concave saddle point problems achieves $O(1/\epsilon^2)$ convergence rate. An extragradient method with gradient tracking (GT-EG) proposed in [28] is shown to have linear convergence rates for solving decentralized strongly convex strongly concave problems, under a positive lower bound assumption on the gradient difference norm. However such assumptions might not hold for problems without bilinear structure, and the dependence of rates in [28] on condition number $\kappa_g$ is unknown. Both [28] and [25] do not consider compression and are based on full batch gradient computations. Recently, multiple works [22, 36, 4] have proposed using minibatch gradients for solving decentralized saddle point problems. Decentralized extra step (DES) [4] shows linear communication complexity with dependence on the graph condition number as $\sqrt{\kappa_g}$, obtained at the cost of incorporating multiple rounds of communication of primal and dual updates. The dependence of computation cost of DES on $\kappa_g$ is however not made explicit in [4]. A near optimal distributed Min-Max data similarity (MMDs) algorithm is proposed in [5] for saddle point problems with a suitable data similarity assumption. MMDs is based on full batch gradient computations and requires solving an inner saddle point problem at every iteration. MMDs allows for communication efficiency by choosing only one node uniformly at random to update the iterates. However, every node computes the full batch gradient before heading to next gradient based updates. Moreover, this scheme employs accelerated gossip [20] multiple times to propagate the gradients and model updates to the entire network. Communication complexity of MMDs is shown to depend on eigengap of weight matrix $W$, while gradient computation complexity is not investigated. Decentralized parallel optimistic stochastic gradient method (DPOSG) was proposed in [22] for nonconvex-nonconcave saddle point problems. This method involves local model averaging step (multiple communication rounds) to reduce the effect of consensus error. A gradient tracking based algorithm called DMHSGD for solving nonconvex-strongly concave saddle point problems in [36], uses a large mini-batch at the first iteration and requires the nodes to communicate multiple times to share both model and gradient updates, to achieve better aggregates of quantities. Table 1 positions our work in the context of existing methods.

B Compression Algorithm of [26]

We follow [26, 23] to compress a related difference vector instead of directly compressing $\nu_{t+1}$ and $\nu_{t+1}^y$. We now describe the compression related updates for $\nu_{t+1}^x$ and $\nu_{t+1}^y$. Each node $i$ is assumed to maintain a local vector $H_t^i$ and a stochastic compression operator $Q$ is applied on the difference vector $\nu_{t+1}^i - H_{t+1}^i$. Hence the compressed estimate $\tilde{\nu}_{t+1}^i \nu_{t+1}^i$ is obtained by adding the local vector and the compressed difference vector using $\tilde{\nu}_{t+1}^i = H_{t+1}^i + Q(\nu_{t+1}^i - H_{t+1}^i)$. The local vector $H_{t+1}^i$ is then updated using a convex combination of the previous local vector information and the new estimate $\tilde{\nu}_{t+1}^i$ using $H_{t+1}^i = (1 - \alpha)H_{t+1}^i + \alpha\tilde{\nu}_{t+1}^i$ for a suitable $\alpha \in [0, 1]$. Collecting the quantities in individual nodes into $H_{t+1} = (H_{t+1}^1, \ldots, H_{t+1}^m)$ and $\nu_{t+1} = (\nu_{t+1}^1, \ldots, \nu_{t+1}^m)$, the update step can be written as $H_{t+1}^i = (1 - \alpha)H_{t+1}^i + \alpha\tilde{\nu}_{t+1}^i$. Pre-multiplying both sides of $H_{t+1}^i$ update by $W \otimes I$, and denoting $(W \otimes I)H_{t+1}^i$ by $H_{t+1}^w$, and $(W \otimes I)\tilde{\nu}_{t+1}^i$ by $\tilde{\nu}_{t+1}^w$ we have: $H_{t+1}^w = (1 - \alpha)H_{t+1}^w + \alpha\tilde{\nu}_{t+1}^w$. $\nu_{t+1}^w$ can be further simplified as $\tilde{\nu}_{t+1}^w = (W \otimes I)\tilde{\nu}_{t+1}^w = (W \otimes I)(H_{t+1}^w + Q(\nu_{t+1}^w - H_{t+1}^w)) = H_{t+1}^w + (W \otimes I)Q(\nu_{t+1}^w - H_{t+1}^w)$. A similar update scheme is used for compressing $\nu_{t+1}^y$. The entire procedure is illustrated in Algorithm 1 where we have used $\nu_{t+1} = (\nu_{t+1}^1, \nu_{t+1}^y)$, $H_t = (H_t^1, H_t^y)$, $H_{t+1}^w = (H_{t+1}^w, H_{t+1}^w)$, $\nu_{t+1}^w = (\nu_{t+1}^w, \nu_{t+1}^w)$. Further recall that $\nu_{t+1} = (\nu_{t+1}^1, \ldots, \nu_{t+1}^m)$ denotes the collection of the local variables at $m$ nodes. Similar is the case for the other variables $\nu_{t+1}^y$, $H_t^x$, $H_t^y$, $H_{t+1}^w$, $H_{t+1}^w$.

C Basic Results and Inequalities

Equivalence between problems (1) and (3).

Theorem C.1. Under assumptions of compactness of feasible sets, (sub)gradient boundedness and continuity of $f_1(x, y), g(x)$ and $r(y)$ over $\mathcal{X}$ and $\mathcal{Y}$, problem (1) is equivalent to problem (3) in the sense that for any solution $(x^*, y^*, \hat{S}^x, \hat{S}^y)$ of (3), the point $(x^*, y^*)$ is a solution to (1).
We assume that where the last equality follows from the previous equality due to separability of the objective function.

Algorithm 4: Compressed Communication Procedure (COMM) [18]

1: INPUT: \( \nu_{t+1}, H_t, H_t^w, \alpha \)
2: \( Q_t^i = Q(\nu_{t+1}^i - H_t^i) \) \{ (compression) \}
3: \( \nu_{t+1}^i = H_t^i + Q_t^i \)
4: \( H_t^{i+1} = (1 - \alpha)H_t^i + \alpha \nu_{t+1}^i \)
5: \( \nu_{t+1}^{i,w} = H_t^{i,w} + \sum_{j=1}^{m} W_{ij}Q_{t}^{j} \) \{ (communicating compressed vectors) \}
6: \( H_t^{i,w} = (1 - \alpha)H_t^{i,w} + \alpha \nu_{t+1}^{i,w} \)
7: RETURN: \( \nu_{t+1}^i, \nu_{t+1}^{i,w}, H_t^{i+1}, H_t^{i,w} \) for each node \( i \).

Proof. The proof is based on the analysis of a similar result in [32]. Let \( \tilde{\Psi}(x, y) = F(x, y) + \sum_{i=1}^{m} (g(x_i) - r(y_i)) \). For simplicity of notation, we assume \( U \odot I_{d_x} = U_x \) and \( U \odot I_{d_y} = U_y \). The objective function \( \tilde{\Psi}(x, y) \) is convex-concave and feasible sets \( X^m, Y^m \) are compact. Then, using Sion-Kakutani Theorem [3],

\[
\min_{x \in X^m} \max_{y \in Y^m} \tilde{\Psi}(x, y) = \max_{y \in Y^m} \min_{x \in X^m} \tilde{\Psi}(x, y).
\]  

Consider the following problem:

\[
\min_{x \in X^m} \tilde{\Psi}(x, y) \text{ such that } U_x x = 0.
\]

We assume that \((x^*, y^*)\) is the saddle point solution of problem (1). Therefore, constraint qualification \((U_x x^* = 0)\) holds. Then using Lagrange strong duality,

\[
\min_{x \in X^m} \tilde{\Psi}(x, y) = \max_{x \in X^m} \min_{y \in Y^m} \tilde{\Psi}(x, y) + \langle S^x, U_x x \rangle.
\]

Since \( X \) and \( Y \) are compact sets, the gradients and subgradients respectively of \( f, g \) and \( r \) with respect to \( x \) will be bounded by a suitable constant \( B_1 \). Therefore using Theorem 2 in [17], there exists an optimal dual multiplier \( S^x \) for r.h.s in (1) such that \( \| S^x \|_2 \leq \frac{\sqrt{\| B_1 \|_{\lambda_{\min}(U_x)}}}{X} = : R_1 \), where \( \lambda_{\min}(U_x) \) denotes the smallest nonzero eigenvalue of \( U_x \). Therefore,

\[
\max_{x \in X^m} \min_{y \in Y^m} \tilde{\Psi}(x, y) + \langle S^x, U_x x \rangle = \max_{\| S^x \|_2 \leq R_1} \min_{x \in X^m} \tilde{\Psi}(x, y) + \langle S^x, U_x x \rangle.
\]

This implies that (11) can be rewritten as

\[
\min_{x \in X^m} \tilde{\Psi}(x, y) = \max_{\| S^x \|_2 \leq R_1} \min_{x \in X^m} \tilde{\Psi}(x, y) + \langle S^x, U_x x \rangle.
\]

Plugging above equality into (9) and by repeated application of Sion-Kakutani theorem [3], we have:

\[
\min_{x \in X^m} \max_{y \in Y^m} \tilde{\Psi}(x, y) = \max_{y \in Y^m} \min_{x \in X^m} \tilde{\Psi}(x, y) + \langle S^x, U_x x \rangle
\]

\[
= \max_{y \in Y^m} \min_{x \in X^m} \left[ \max_{\| S^x \|_2 \leq R_1} \tilde{\Psi}(x, y) + \langle S^x, U_x x \rangle \right]
\]

\[
= \min_{x \in X^m} \max_{y \in Y^m} \tilde{\Psi}(x, y) + \langle S^x, U_x x \rangle
\]

where the last equality follows from the previous equality due to separability of the objective function in (14) in \( y \) and \( S^x \). Next, we consider a maximization problem associated with the consensus constraint \( U_y y = 0 \) to write (14) in the form of (3). Towards this end, consider

\[
\max_{y \in Y^m} \tilde{\Psi}(x, y) + \langle S^x, U_x x \rangle \text{ such that } U_y y = 0.
\]

The dual formulation of (15) is given by

\[
\min_{S^y \in Y^m} \max_{y \in Y^m} \tilde{\Psi}(x, y) + \langle S^x, U_x x \rangle + \langle S^y, U_y y \rangle.
\]

Again using Lagrange strong duality, we have

\[
\max_{y \in Y^m} \tilde{\Psi}(x, y) + \langle S^x, U_x x \rangle = \min_{S^y \in Y^m} \max_{y \in Y^m} \tilde{\Psi}(x, y) + \langle S^x, U_x x \rangle + \langle S^y, U_y y \rangle.
\]
By following arguments similar to the primal problem with constraint \( U_y \), we can write
\[
\max_{y \in Y^m} \tilde{\Psi}(x, y) + \langle S^*, U_x x \rangle = \min_{\|S^*\|_2 \leq R_1} \left[ \min_{y \in Y^m} \max_{x \in X^m} \tilde{\Psi}(x, y) + \langle S^*, U_x x \rangle + \langle S^*, U_y y \rangle \right],
\] (16)
where \( R_2 := \sqrt{\frac{m R_2}{\lambda_{\text{min}}(U_y)}} \). By substituting (16) in (14), we obtain
\[
\min_{x \in X^m} \max_{y \in Y^m} \tilde{\Psi}(x, y) = \min_{x \in X^m} \max_{\|S^*\|_2 \leq R_1} \left[ \min_{y \in Y^m} \max_{x \in X^m} \tilde{\Psi}(x, y) + \langle S^*, U_x x \rangle + \langle S^*, U_y y \rangle \right]
= \min_{x \in X^m} \max_{\|S^*\|_2 \leq R_1} \left[ \min_{y \in Y^m} \tilde{\Psi}(x, y) + \langle S^*, U_x x \rangle + \langle S^*, U_y y \rangle \right]
= \min_{x \in X^m} \min_{\|S^*\|_2 \leq R_1} \tilde{\Psi}(x, y) + \langle S^*, U_x x \rangle + \langle S^*, U_y y \rangle.
\] (17)

Since optimal Lagrange dual variables \( \tilde{S}^* \) and \( \tilde{S}^*_y \) always lie in \( \ell_2 \) balls of radius \( R_1 \) and \( R_2 \) respectively, equation (17) can be equivalently written as
\[
\min_{x \in X^m} \max_{y \in Y^m} \tilde{\Psi}(x, y) = \min_{x \in X^m, S^*} \max_{y \in Y^m} \tilde{\Psi}(x, y) + \langle S^*, U_x x \rangle + \langle S^*, U_y y \rangle
= \min_{x \in X^m, S^*} \max_{y \in Y^m} F(x, y) + G(x) - R(y) + \langle S^*, U_x x \rangle + \langle S^*, U_y y \rangle.
\]

This completes the proof of Theorem C.1 \( \square \)

We now provide the optimality conditions for the optimization problem (SPP).

**Optimality Conditions of problem (SPP).** Let \( f(x, y) := \sum_{i=1}^m f_i(x, y) \). Since \((x^*, y^*)\) is the saddle point solution to (SPP), we have \( x^* = \arg \min_{x \in X} \Psi(x, y^*) \) and \( y^* = \arg \max_{y \in Y} \Psi(x^*, y) \).

\[
0 \in \frac{1}{m} \nabla_x f(x^*, y^*) + \partial g(x^*) + \partial I_X(x^*)
= \partial g(x^*) + \partial I_X(x^*) + \frac{1}{s} \left( x^* - \left( x^* - \frac{s}{m} \nabla_x f(x^*, y^*) \right) \right)
= \partial \left( g(x) + I_X(x) + \frac{1}{2s} \left\| x - \left( x^* - \frac{s}{m} \nabla_x f(x^*, y^*) \right) \right\| ^2 \right)_{x=x^*}.
\] (18)

This implies that
\[
x^* = \arg \min_{x \in X} g(x) + I_X(x) + \frac{1}{2s} \left\| x - \left( x^* - \frac{s}{m} \nabla_x f(x^*, y^*) \right) \right\| ^2
= \arg \min_{x \in X} g(x) + \frac{1}{2s} \left\| x - \left( x^* - \frac{s}{m} \nabla_x f(x^*, y^*) \right) \right\| ^2
= \text{prox}_{s g} \left( x^* - \frac{s}{m} \nabla_x f(x^*, y^*) \right).
\] (19)

We also have \( y^* = \arg \min_{y \in Y} \Psi(x^*, y) = \arg \min_{y \in Y^m} (-\Psi(x^*, y) + I_Y(y)) \). Therefore,
\[
0 \in \partial_y (-\Psi(x^*, y^*)) + \partial I_Y(y^*)
= -\frac{1}{m} \nabla_y f(x^*, y^*) + \partial r(y^*) + \partial I_Y(y^*)
= \partial r(y^*) + \partial I_Y(y^*) + \frac{1}{s} \left( y^* - \left( y^* + \frac{s}{m} \nabla_y f(x^*, y^*) \right) \right)
= \partial \left( r(y) + I_Y + \frac{1}{2s} \left\| y - \left( y^* + \frac{s}{m} \nabla_y f(x^*, y^*) \right) \right\| ^2 \right)_{y=y^*}.
\] (20)

Therefore, \( y^* = \text{prox}_{s r} \left( y^* + \frac{s}{m} \nabla_y f(x^*, y^*) \right). \)
Notations useful for further analysis:

In the subsequent analysis, we assume $d_x = d_y = 1$ for simplicity of representation. Our analysis still holds for $d_x > 1$ and $d_y > 1$ by incorporating Kronecker product. We define Bregman distance with respect to each function $f_i(\cdot, y)$ and $f_i(\cdot, z)$ as

$$V_{f_i, y}(x_1, x_2) = f_i(x_1, y; \xi) - f_i(x_2, y; \xi) - \langle \nabla_x f_i(x_2, y), x_1 - x_2 \rangle$$

$$V_{f_i, z}(y_1, y_2) = f_i(x, y_1; \xi) + f_i(x, y_2; \xi) - \langle -\nabla_y f_i(x, y_2), y_1 - y_2 \rangle,$$

respectively. Let $L = \max \{L_{xx}, L_{yy}, L_{xy}, L_{yx} \}$ and $\mu = \min \{\mu_x, \mu_y \}$. Suppose $\lambda_{\text{max}}(I - W)$, $\lambda_{m-1}(I - W)$ and $(I - W)^\dagger$ denote the largest eigenvalue, second smallest eigenvalue and pseudo inverse of $I - W$ respectively. Let $\kappa_f = L/\mu$ and $\kappa_y = \lambda_{\text{max}}(I - W)/\lambda_{m-1}(I - W)$ denote the condition number of function $f$ and graph $G$ respectively. We further define $D^*_x := -(I - J)\nabla_x F(1z^*)$, $D^*_y := (I - J)\nabla_y F(1z^*)$, $H^*_x := 1(x^* - \frac{\mu}{\nu} \nabla_x f(z^*))$ and $H^*_y := 1(y^* + \frac{\mu}{\nu} \nabla_y f(z^*))$.

We now state a few preliminary results that will be used in later sections. These results may be of independent interest as well.

**Proposition C.2.** Let $W$ satisfy Assumption 4.5 and let $D^*_x$ and $D^*_y$ be as defined in above paragraph. Then, $D^*_x \in \text{Range}(I - W)$ and $D^*_y \in \text{Range}(I - W)$.

**Proof.** To prove this result, we first show that $\text{Range}(I - W) = (\text{Range}(1^\dagger))^\perp$. Then we prove that both $D^*_x$ and $D^*_y$ lie in $(\text{Range}(1^\dagger))^\perp$.

$$\text{Range}(I - W) = \{z^\top(I - W) : z \in \mathbb{R}^m\}, \quad \text{Range}(1^\dagger) = \{\alpha 1^\top : \alpha \in \mathbb{R}\},$$

$$\text{Null}(I - W) = \{ z^\top : z^\top(I - W) = 0 \},$$

$$(\text{Range}(1^\dagger))^\perp = \{ x : \langle x, y \rangle = 0 \text{ for all } y \in \text{Range}(1^\dagger) \}.$$  

$(I - W)$ is a symmetric matrix, therefore, $z^\top(I - W)$ represents the linear combination of the columns of $I - W$. We first show that $\text{Range}(I - W) \subseteq (\text{Range}(1^\dagger))^\perp$. Towards that end, let $y^\top \in \text{Range}(I - W)$. This implies that there exists a $z \in \mathbb{R}^m$ such that $z^\top(I - W) = y^\top$. Therefore,

$$(1^\dagger y) = \alpha(1^\top) = \alpha(1^\top(I - W)z) = 0.$$  

The last step follows from $1^\top W = 1^\top$. This implies that $y \in (\text{Range}(1^\dagger))^\perp$. Therefore, $\text{Range}(I - W) \subseteq (\text{Range}(1^\dagger))^\perp$. Now we show that $\text{dim}(\text{Range}(I - W)) = \text{dim}(\text{Range}(1^\dagger))^\perp$. We know that $\text{Null}(I - W) = \text{Range}(1^\dagger)$. This means that $\text{dim}(\text{Null}(I - W)) = 1$. Using rank-nullity Theorem, we get $\text{dim}(\text{Range}(I - W)) = m - 1$. Moreover, $\text{Range}(1^\dagger)$ is a subspace of $\mathbb{R}^m$, and $\text{dim}(\text{Range}(1^\dagger))^\perp = m - 1$. Therefore, $\text{dim}(\text{Range}(I - W)) = \text{dim}(\text{Range}(1^\dagger))^\perp$, and

$$\text{Range}(I - W) = (\text{Range}(1^\dagger))^\perp.$$  

This completes the first part of the proof. Recall

$$D^*_x = -(I - J)\nabla_x F(1z^*)$$

$$D^*_y = (I - J)\nabla_y F(1z^*).$$

Therefore, $1^\top D^*_x = -(I - J)\nabla_x F(1z^*) = 0$ because $1^\top J = 1^\top$. Similarly, $1^\top D^*_y = (I - J)\nabla_y F(1z^*) = 0$. Hence, $D^*_x \in (\text{Range}(1^\dagger))^\perp = \text{Range}(I - W)$ and $D^*_y \in (\text{Range}(1^\dagger))^\perp = \text{Range}(I - W)$.

**Proposition C.3.** *(Smoothness in $x$)* Assume that $f(\cdot, y)$ is convex and $L_{xx}$-smooth in $x$ for any fixed $y$. Then

$$\frac{1}{2L_{xx}} \| \nabla_x f(x_1, y) - \nabla_x f(x_2, y) \|^2 \leq V_{f, y}(x_1, x_2) \leq \frac{L_{xx}}{2} \| x_1 - x_2 \|^2 \text{ for all } x_1, x_2.$$  

**Proof.** Using the smoothness of $f(\cdot, y)$, we have

$$f(x_1, y) \leq f(x_2, y) + \langle \nabla_x f(x_2, y), x_1 - x_2 \rangle + \frac{L_{xx}}{2} \| x_1 - x_2 \|^2$$

$$f(x_1, y) - f(x_2, y) - \langle \nabla_x f(x_2, y), x_1 - x_2 \rangle \leq \frac{L_{xx}}{2} \| x_1 - x_2 \|^2$$

$$V_{f, y}(x_1, x_2) \leq \frac{L_{xx}}{2} \| x_1 - x_2 \|^2.$$  

(36)
This completes the proof of second inequality. Let \( h(x_1) := V_{f,y}(x_1, x_2) \) for a given \( y \) and \( x_2 \). Notice that \( h(x_1) = 0 \) at \( x_1 = x_2 \). Using convexity of \( f(x,y) \) in \( x \), \( h(x_1) \geq 0 \). Therefore, \( h(x_1) \) achieves its minimum value at \( x_2 \) and the minimum value is 0.

\[
\|\nabla_x h(x_1) - \nabla_x h(x'_1)\| = \|\nabla_x f(x_1, y) - \nabla_x f(x'_1, y)\| \\
\leq L_{xx} \|x_1 - x'_1\|.
\]

This implies that \( h(x_1) \) is also \( L_{xx} \)-smooth.

\[
h(\bar{x}) \leq h(x_1) + \langle \nabla_x h(x_1), \bar{x} - x_1 \rangle + \frac{L_{xx}}{2} \|\bar{x} - x_1\|^2
\]

Take minimization over \( \bar{x} \) on both sides.

\[
\min h(\bar{x}) \leq h(x_1) + \min_{\bar{x}} \left( \langle \nabla_x h(x_1), \bar{x} - x_1 \rangle + \frac{L_{xx}}{2} \|\bar{x} - x_1\|^2 \right).
\]

Let \( \delta(\bar{x}) = \langle \nabla_x h(x_1), \bar{x} - x_1 \rangle + \frac{L_{xx}}{2} \|\bar{x} - x_1\|^2 \).

\[
\nabla \delta(\bar{x}) = \nabla_x h(x_1) + L_{xx} (\bar{x} - x_1), \quad \nabla^2 \delta(\bar{x}) = L_{xx} I \succ 0.
\]

Therefore,

\[
\min_{\bar{x}} \delta(\bar{x}) = \left( \nabla_x h(x_1), -\frac{\nabla_x h(x_1)}{L_{xx}} + x_1 - x_1 \right) + \frac{L_{xx}}{2} \left\| \frac{\nabla_x h(x_1)}{L_{xx}} + x_1 - x_1 \right\|^2
\]

\[
= -\frac{1}{L_{xx}} \|\nabla_x h(x_1)\|^2 + \frac{\|\nabla_x h(x_1)\|^2}{2L_{xx}}
\]

\[
= -\frac{\|\nabla_x h(x_1)\|^2}{2L_{xx}}.
\]

Plug in above minimum value into \( (43) \).

\[
\min h(\bar{x}) \leq h(x_1) - \frac{\|\nabla_x h(x_1)\|^2}{2L_{xx}}
\]

\[
0 \leq h(x_1) - \frac{\|\nabla_x h(x_1)\|^2}{2L_{xx}}
\]

\[
= V_{f,y}(x_1, x_2) - \frac{\|\nabla_x f(x_1, y) - \nabla_x f(x_2, y)\|^2}{2L_{xx}}.
\]

This gives

\[
\frac{\|\nabla_x f(x_1, y) - \nabla_x f(x_2, y)\|^2}{2L_{xx}} \leq V_{f,y}(x_1, x_2).
\]

Proposition C.4. (Smoothness in \( y \)) Assume that \( -f(x, y) \) is convex and \( L_{yy} \)-smooth in \( y \) for any fixed \( x \). Then

\[
\frac{1}{2L_{yy}} \| -\nabla_y f(x, y_1) + \nabla_y f(x, y_2) \|^2 \leq V_{-f,x}(y_1, y_2) \leq \frac{L_{yy}}{2} \|y_1 - y_2\|^2.
\]

The proof of Proposition C.4 is similar to that of Proposition C.3 and is omitted.

D  A recursion relationship useful for further analysis

In the following discussion, we use the shorthand notation \( 1u \) to represent \((1 \otimes I_d)u \), for any \( u \in \mathbb{R}^d \), and the notation \( A^t \) denotes the pseudo-inverse of a square matrix \( A \).

Lemma D.1. Let \( x_{k,t+1}, y_{k,t+1}, D_{k,t+1}^x, D_{k,t+1}^y, H_{k,t+1}^x, H_{k,t+1}^y, H_{k,t+1}^{w,x}, H_{k,t+1}^{w,y} \) be obtained from Algorithm \( \boxed{} \) using IPDHG\((x_{k,t}, y_{k,t}, D_{k,t}^x, D_{k,t}^y, H_{k,t}^x, H_{k,t}^y, H_{k,t}^{w,x}, H_{k,t}^{w,y}, s, \gamma_x, \gamma_y, \alpha_x, \alpha_y, \mathcal{G}) \). Let Assumption 4.4 and Assumption 4.5 hold. Suppose \( \alpha_y \in (0, (1 + \delta)^{-1}) \) and

\[
\gamma_y \in \left(0, \min \left\{ \frac{2 - 2\sqrt{\delta} \alpha_y}{\lambda_{\max}(I - W)}, \frac{\alpha_y - (1 + \delta) \alpha_y^2}{\sqrt{\delta} \lambda_{\max}(I - W)} \right\} \right).
\]
Then the following holds for all \( t \geq 0 \):
\[
M_y E \left\| y_{k,t+1} - 1 y^* \right\|^2 + \frac{2s^2}{\gamma_y} E \left\| D^y_{k,t+1} - D^y \right\|_{(I-W)^t}^2 + \sqrt{\delta} E \left\| H^y_{k,t+1} - H^y \right\|^2 \\
\leq \left\| y_{k,t} - 1 y^* + sG^y_{k,t} - s \nabla_y f(1x^*, 1y^*) \right\|^2 + \frac{2s^2}{\gamma_y} \left( 1 - \frac{\gamma_y}{2} \lambda_{m-1}(I - W) \right) \left\| D^y_{k,t} - D^y \right\|_{(I-W)^t}^2 \\
+ \sqrt{\delta} (1 - \alpha_y) \left\| H^y_{k,t} - H^y \right\|^2,
\] (54)
where \( E \) denotes the conditional expectation on stochastic compression at \( t \)-th update step and
\[
M_y = 1 - \frac{\sqrt{\delta} \alpha_y}{2 \lambda_{\text{max}}(I-W)}
\] and \( H^y = 1(x^* - \frac{m}{\alpha} \nabla_x f(x^*, y^*)) \) and \( H^y = 1(y^* + \frac{m}{\alpha} \nabla_y f(x^*, y^*)) \).

**Proof of Lemma D.1**

We follow [13] to prove Lemma D.1. We have \( H^y = 1(x^* - \frac{m}{\alpha} \nabla_x f(x^*, y^*)) \) and \( H^y = 1(y^* + \frac{m}{\alpha} \nabla_y f(x^*, y^*)) \). First, we bound the terms appearing on the l.h.s. of (54) as
\[
M_y E \left\| y_{k,t+1} - 1 y^* \right\|^2 \leq \left\| y_{k,t+1} - H^y \right\|^2_{(1 - \gamma_y (I-W) - \alpha_y \sqrt{\delta} t)} + \frac{\gamma_y^2 M_y}{4} E \left\| \nu^y_{k,t+1} - \nu^y_{k,t+1} \right\|^2_{(I-W)^t},
\] (55)
and
\[
\left( \frac{2s^2}{\gamma_y} E \left\| D^y_{k,t+1} - D^y \right\|_{(I-W)^t}^2 + \sqrt{\delta} E \left\| H^y_{k,t+1} - H^y \right\|^2 \right) + \left\| \nu^y_{k,t+1} - H^y \right\|^2_{(1 - \gamma_y (I-W) - \alpha_y \sqrt{\delta} t)}
\] \[= \frac{s^2}{\gamma_y} \left\| D^y_{k,t} - D^y \right\|^2_{(1 - \gamma_y (I-W) + \alpha_y \sqrt{\delta} t)} + \frac{1}{2} E \left\| \nu^y_{k,t+1} - \nu^y_{k,t+1} \right\|^2_{(1 - \gamma_y (I-W) + \alpha_y \sqrt{\delta} t)}
\] \[+ \left\| \nu^y_{k,t+1} - H^y \right\|^2_{(1 - \gamma_y (I-W) + \alpha_y \sqrt{\delta} t)}.
\] (56)

Proofs of (55) and (56) are provided in Sections D.1 and D.2 respectively.

On adding (55) and (56), we obtain
\[
M_y E \left\| y_{k,t+1} - 1 y^* \right\|^2 + \frac{2s^2}{\gamma_y} E \left\| D^y_{k,t+1} - D^y \right\|_{(I-W)^t}^2 + \sqrt{\delta} E \left\| H^y_{k,t+1} - H^y \right\|^2
\] \[\leq \left\| y_{k,t} - 1 y^* + sG^y_{k,t} - s \nabla_y f(1z^*) \right\|^2 + \frac{s^2}{\gamma_y} \left\| D^y_{k,t} - D^y \right\|^2_{(1 - \gamma_y (I-W) + \alpha_y \sqrt{\delta} t)}
\] \[+ \sqrt{\delta} (1 - \alpha_y) \left\| H^y_{k,t} - H^y \right\|^2 - \sqrt{\delta} \alpha_y (1 - \alpha_y) \left\| \nu^y_{k,t+1} - H^y \right\|^2
\] \[+ \frac{1}{2} E \left\| \nu^y_{k,t+1} - \nu^y_{k,t+1} \right\|^2_{(1 - \gamma_y (I-W) + \alpha_y \sqrt{\delta} t)} + \frac{\gamma_y^2 M_y}{4} E \left\| \nu^y_{k,t+1} - \nu^y_{k,t+1} \right\|^2_{(I-W)^t}.
\] (57)

We now bound the terms on the r.h.s. of (57). First, observe that
\[
\frac{1}{2} E \left\| \nu^y_{k,t+1} - \nu^y_{k,t+1} \right\|^2_{(1 - \gamma_y (I-W) + \alpha_y \sqrt{\delta} t)} + \frac{\gamma_y^2 M_y}{4} E \left\| \nu^y_{k,t+1} - \nu^y_{k,t+1} \right\|^2_{(I-W)^t}
\] \[= \frac{1}{2} E \left\| \sqrt{\gamma_y (I-W) + 2 \sqrt{\delta} \alpha_y ^2} (\nu^y_{k,t+1} - \nu^y_{k,t+1}) \right\|^2 + \frac{\gamma_y^2 M_y}{4} E \left\| (I-W) (\nu^y_{k,t+1} - \nu^y_{k,t+1}) \right\|^2
\] \[\leq \frac{1}{2} \left( \gamma_y \lambda_{\text{max}} (I-W) + 2 \sqrt{\delta} \alpha_y ^2 \right) E \left\| \nu^y_{k,t+1} - \nu^y_{k,t+1} \right\|^2 + \frac{\gamma_y^2 M_y}{4} \left\| (I-W) \right\|^2 E \left\| \nu^y_{k,t+1} - \nu^y_{k,t+1} \right\|^2
\] \[\leq \left( \frac{1}{2} \gamma_y \lambda_{\text{max}} (I-W) + 2 \sqrt{\delta} \alpha_y ^2 + \frac{\gamma_y^2 M_y \lambda_{\text{max}} (I-W) }{4} \right) E \left\| \nu^y_{k,t+1} - \nu^y_{k,t+1} \right\|^2.
\] (58)

We also have
\[
\nu^y_{k,t+1} - \nu^y_{k,t+1} = Q (\nu^y_{k,t+1} - H^y_{k,t}) - (\nu^y_{k,t+1} - H^y_{k,t})
\] (59)
\[
E \left\| \nu^y_{k,t+1} - \nu^y_{k,t+1} \right\|^2 = E \left\| Q (\nu^y_{k,t+1} - H^y_{k,t}) - (\nu^y_{k,t+1} - H^y_{k,t}) \right\|^2
\] \[\leq \delta \left\| \nu^y_{k,t+1} - H^y_{k,t} \right\|^2.
\] (60)
Substituting this inequality in (58), we bound the last two terms in the r.h.s of (57) as
\[
\frac{1}{2} E \left\| \frac{\gamma_y}{4} \left( \frac{\gamma_y}{2} M_y E \right)^2 + \frac{\gamma_y}{4} \delta \alpha_y^2 \left( I - W \right) \right\|_2^2 \leq \left( \frac{1}{2} \left( \gamma_y \lambda_{\text{max}} (I - W) + 2 \sqrt{\delta} \alpha_y^2 \right) + \frac{\gamma_y}{4} \delta \alpha_y^2 \right) \frac{1}{2} \left( \gamma_y \lambda_{\text{max}} (I - W) + \frac{\gamma_y}{4} \delta \alpha_y^2 \right).
\]

We will now bound the term \( s y \left\| D_y \right\|^2 \) as
\[
\frac{s^2 y}{\gamma_y} \left\| D_y \right\|^2 = \frac{s^2 y}{\gamma_y} \left\| D_y \right\|^2 - \frac{s^2 y}{\gamma_y} \left\| D_y \right\|^2 + \frac{s^2 y}{\gamma_y} \left\| D_y \right\|^2.
\]

By substituting (61) and (62) in (57), we get
\[
M_y E \left\| y_{k+1} - 1 y^* \right\|^2 + \frac{2s^2 y}{\gamma_y} \left\| D_y \right\|^2 + \frac{s^2 y}{\gamma_y} \left\| D_y \right\|^2 \leq \left\| y_{k+1} - 1 y^* + s G_y \nabla \Phi (1 z^*) \right\|^2 + \frac{2s^2 y}{\gamma_y} \left\| D_y \right\|^2 + \left( \frac{\gamma_y}{2} \lambda_{\text{max}} (I - W) + \frac{\gamma_y}{2} \lambda_{\text{max}} (I - W) + \sqrt{\delta} \alpha_y^2 - \sqrt{\delta} \alpha_y (1 - \alpha_y) \right) \frac{1}{2} \left( \gamma_y \lambda_{\text{max}} (I - W) + \frac{\gamma_y}{4} \delta \alpha_y^2 \right) \frac{1}{2} \left( \gamma_y \lambda_{\text{max}} (I - W) + \frac{\gamma_y}{4} \delta \alpha_y^2 \right),
\]

The coefficient of \( \left\| y_{k+1} - 1 y^* + s G_y \nabla \Phi (1 z^*) \right\|^2 \) in (63) is
\[
\frac{\delta^2 \gamma_y^2}{2} \lambda_{\text{max}} (I - W) = \frac{\gamma_y^2}{2} \lambda_{\text{max}} (I - W) + \sqrt{\delta} \alpha_y^2 - \sqrt{\delta} \alpha_y (1 - \alpha_y)
\]

\[
\frac{\delta^2 \gamma_y^2}{2} \lambda_{\text{max}} (I - W) < \frac{\gamma_y^2}{2} \lambda_{\text{max}} (I - W) + \sqrt{\delta} \alpha_y^2 - \sqrt{\delta} \alpha_y (1 - \alpha_y)
\]

\[
\frac{\gamma_y^2}{2} \lambda_{\text{max}} (I - W) + \frac{\gamma_y^2}{2} \lambda_{\text{max}} (I - W) - \sqrt{\delta} \alpha_y (1 + \delta) \alpha_y^2
\]

\[
\frac{\gamma_y^2}{2} \lambda_{\text{max}} (I - W) - \sqrt{\delta} \alpha_y (1 - \delta) \alpha_y^2
\]

\[
\leq 0, \quad \text{as} \quad \gamma_y \leq \frac{\alpha_y - (1 + \delta) \alpha_y^2}{\sqrt{\delta} \lambda_{\text{max}} (I - W)}.
\]
In deriving (64), the first inequality follows from $M_y < 1$ (to be proved later in Section E.3) and the second inequality follows from $\gamma_y \lambda_{\max}^\gamma (I - W) < 1$, since from assumption (53), it holds that $0 < \gamma_y < \frac{2 - 2\gamma_y \lambda_{\max}^\gamma (I - W)}{\lambda_{\max}^\gamma (I - W)}$. As a result, (63) can be simplified as

$$
\begin{align*}
M_y E \left\| y_{k,t+1} - 1 y^* \right\|^2 &\leq \left\| y_{k,t} - 1 y^* + s Q_y^y (-s \nabla_y F(1 z^*)) \right\|^2 + 2 \frac{s^2}{\gamma_y} \left( 1 - \frac{\gamma_y}{\lambda_{m-1}^\gamma (I - W)} \right) \left\| D_y^y - D_y^* \right\|^2_{(I - W)} + 2 \frac{s^2}{\gamma_y} \left( 1 - \frac{\gamma_y}{\lambda_{m-1}^\gamma (I - W)} \right) \left\| D_y^y - D_y^* \right\|^2_{(I - W)} + 2 \frac{s^2}{\gamma_y} \left( 1 - \frac{\gamma_y}{\lambda_{m-1}^\gamma (I - W)} \right)^2 + \sqrt{\delta} \left\| H_y^y - H_y^* \right\|^2,
\end{align*}
$$

proving Lemma D.1. The remainder of this section is devoted to prove (55) and (56).

### D.1 Proof of (55)

First, observe that

$$
\begin{align*}
\left\| y_{k,t+1} - 1 y^* \right\|^2 &= \sum_{i=1}^m \left\| y_{k,t+1}^i - y^* \right\|^2 \\
&= \sum_{i=1}^m \left\| \text{prox}_{s \gamma_y} (\hat{y}_{k,t+1}^i) - \text{prox}_{s \gamma_y} (y^* + \frac{s}{m} \nabla_y f(x^*, y^*)) \right\|^2 \\
&= \sum_{i=1}^m \left\| \text{prox}_{s \gamma_y} (\hat{y}_{k,t+1}^i) - \text{prox}_{s \gamma_y} (H^*_y) \right\|^2 \\
&\leq \sum_{i=1}^m \left\| \hat{y}_{k,t+1}^i - H^*_y \right\|^2 \quad \text{(from non-expansivity of prox)} \\
&= \left\| \hat{y}_{k,t+1} - H^*_y \right\|^2 \\
&= \left\| \nu_{k,t+1}^y - \frac{\gamma_y}{2} (I - W) \nu_{k,t+1}^y \nu_{k,t+1}^y - H^*_y \right\|^2 \\
&= \left\| (I - \frac{\gamma_y}{2} (I - W)) (\nu_{k,t+1}^y - H^*_y) \right\|^2 + \frac{\gamma_y^2}{4} \left\| (I - W) (\nu_{k,t+1}^y - \nu_{k,t+1}^y) \right\|^2 \\
&\quad + 2 \left( (I - \frac{\gamma_y}{2} (I - W)) (\nu_{k,t+1}^y - H^*_y), (I - W) E (\nu_{k,t+1}^y - \nu_{k,t+1}^y) \right).
\end{align*}
$$

By taking conditional expectation over stochastic compression at $t$-th iterate, we obtain

$$
\begin{align*}
E \left\| y_{k,t+1} - 1 y^* \right\|^2 &\leq \left\| (I - \frac{\gamma_y}{2} (I - W)) (\nu_{k,t+1}^y - H^*_y) \right\|^2 + \frac{\gamma_y^2}{4} \left\| (I - W) (\nu_{k,t+1}^y - \nu_{k,t+1}^y) \right\|^2 \\
&\quad - \gamma_y \left( (I - \frac{\gamma_y}{2} (I - W)) (\nu_{k,t+1}^y - H^*_y), (I - W) E (\nu_{k,t+1}^y - \nu_{k,t+1}^y) \right).
\end{align*}
$$

We have

$$
\begin{align*}
\nu_{k,t+1}^y - \nu_{k,t+1}^y &= H_{k,t}^y + Q (\nu_{k,t+1} - H_{k,t}^y) - \nu_{k,t+1}^y \\
&= Q (\nu_{k,t+1} - H_{k,t}^y) - \left( \nu_{k,t+1} - H_{k,t}^y \right) \\
E (\nu_{k,t+1} - \nu_{k,t+1}^y) &= E \left( Q (\nu_{k,t+1} - H_{k,t}^y) - \nu_{k,t+1}^y \right) = 0.
\end{align*}
$$

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The last equality follows from Assumption \ref{assumption-4.4}. By substituting the above equation into \eqref{eq:67}, we obtain
\begin{equation}
E \|y_{k,t+1} - 1y^*\|^2 \leq \left\| \left( I - \frac{\gamma_y}{2} (I - W) \right) \left( v_{y,k,t+1}^* - H_y^* \right) \right\|^2 + \frac{\gamma_y^2}{4} E \left\| (I - W)(\hat{v}_{y,k,t+1}^* - v_{y,k,t+1}^*) \right\|^2.
\end{equation}

We will now convert the square norm terms of \eqref{eq:69} into matrix-norm based terms. Towards that end, observe that
\begin{equation}
\begin{aligned}
\left( I - \frac{\gamma_y}{2} (I - W) \right)^2 &= I + \frac{\gamma_y^2}{4} (I - W)^2 - \gamma_y (I - W) \\
&= I - \frac{\gamma_y}{2} (I - W) + \frac{\gamma_y^2}{4} (I - W)^2 - \gamma_y (I - W) \\
&= I - \frac{\gamma_y}{2} (I - W) - \frac{\gamma_y}{2} (I - W) + \frac{\gamma_y^2}{4} (I - W)^2 \\
&= I - \frac{\gamma_y}{2} (I - W) + \frac{\gamma_y}{2} (I - W) \left( -I + \frac{\gamma_y}{2} (I - W) \right) \\
&= I - \frac{\gamma_y}{2} (I - W) + \frac{\gamma_y}{2} (I - W)^{1/2} \left( -I + \frac{\gamma_y}{2} (I - W) \right) (I - W)^{1/2}.
\end{aligned}
\end{equation}

Note that \(-I + \frac{\gamma_y}{2} (I - W)\) is a negative semidefinite matrix because \(0 < \frac{\gamma_y}{2} \lambda_{\text{max}}(I - W) < 1\) from the choice of \(\gamma_y\). Using this fact in equation \eqref{eq:70}, we get \(\forall x \in \mathbb{R}^m\),
\begin{equation}
x^T \left( I - \frac{\gamma_y}{2} (I - W) \right)^2 x \leq x^T (I - \frac{\gamma_y}{2} (I - W)) x.
\end{equation}

Consider
\begin{equation}
\left\| \left( I - \frac{\gamma_y}{2} (I - W) \right) \left( v_{y,k,t+1}^* - H_y^* \right) \right\|^2 = \left( v_{y,k,t+1}^* - H_y^* \right)^T \left( I - \frac{\gamma_y}{2} (I - W) \right)^2 \left( v_{y,k,t+1}^* - H_y^* \right)
\end{equation}
\begin{equation}
\leq \left\| v_{y,k,t+1}^* - H_y^* \right\|^2 \left( I - \frac{\gamma_y}{2} (I - W) \right)^2 \left( v_{y,k,t+1}^* - H_y^* \right).
\end{equation}

Moreover,
\begin{equation}
\left\| (I - W)(\hat{v}_{y,k,t+1}^* - v_{y,k,t+1}^*) \right\|^2 = \left( \hat{v}_{y,k,t+1}^* - v_{y,k,t+1}^* \right)^T (I - W)^2 \left( \hat{v}_{y,k,t+1}^* - v_{y,k,t+1}^* \right).
\end{equation}

On substituting above two equalities \eqref{eq:72} and \eqref{eq:73} in \eqref{eq:69}, we obtain
\begin{equation}
E \|y_{k,t+1} - 1y^*\|^2 \leq \left\| v_{y,k,t+1}^* - H_y^* \right\|^2 \left( I - \frac{\gamma_y}{2} (I - W) \right)^2 + \frac{\gamma_y^2}{4} E \left\| \hat{v}_{y,k,t+1}^* - v_{y,k,t+1}^* \right\|^2 (I - W)^2.
\end{equation}

To complete the proof of \eqref{eq:55}, we first show that the first term on the r.h.s. of \eqref{eq:74} is at most
\begin{equation}
M_y^{-1} \left\| v_{y,k,t+1}^* - H_y^* \right\|^2 \left( I - \frac{\gamma_y}{2} (I - W) \right)^2 \alpha_y \beta_I^2,
\end{equation}
where \(M_y = 1 - \frac{\sqrt{\alpha_y}}{1 - \frac{\gamma_y}{2} \lambda_{\text{max}}(I - W)}\). To this end, consider
\begin{equation}
\begin{aligned}
\sqrt{\alpha_y} I - \frac{\sqrt{\alpha_y}}{1 - \frac{\gamma_y}{2} \lambda_{\text{max}}(I - W)} \left( I - \frac{\gamma_y}{2} (I - W) \right)
&= \sqrt{\alpha_y} \left( 1 - \frac{\gamma_y}{2} \lambda_{\text{max}}(I - W) \right) I - \sqrt{\alpha_y} \left( I - \frac{\gamma_y}{2} (I - W) \right)
&= \frac{1}{1 - \frac{\gamma_y}{2} \lambda_{\text{max}}(I - W)} \left( -\sqrt{\alpha_y} \gamma_y \lambda_{\text{max}}(I - W) I + \frac{\sqrt{\alpha_y} \gamma_y}{2} \lambda_{\text{max}}(I - W) \right).
\end{aligned}
\end{equation}

The largest eigenvalue of \(\sqrt{\alpha_y} I - \frac{\sqrt{\alpha_y}}{1 - \frac{\gamma_y}{2} \lambda_{\text{max}}(I - W)} \left( I - \frac{\gamma_y}{2} (I - W) \right)\) is
\begin{equation}
\frac{1}{1 - \frac{\gamma_y}{2} \lambda_{\text{max}}(I - W)} \left( -\sqrt{\alpha_y} \gamma_y \lambda_{\text{max}}(I - W) I + \frac{\sqrt{\alpha_y} \gamma_y}{2} \lambda_{\text{max}}(I - W) \right) = 0.
\end{equation}

Therefore, \(\sqrt{\alpha_y} I - \frac{\sqrt{\alpha_y}}{1 - \frac{\gamma_y}{2} \lambda_{\text{max}}(I - W)} \left( I - \frac{\gamma_y}{2} (I - W) \right)\) is negative semidefinite, and
D.2 Proof of (56)

Multiplying throughout by \( M \)

Observe that

\[
\text{On pre-multiplying both sides by } \sqrt{\delta \alpha_y}.
\]

In Algorithm 3, for every \( k \)

Substituting \( \nu_{k,t}^y \) into the above inequality and using the definition of \( \|x\|^2_A \), we obtain

\[
\left\| \nu_{k,t+1}^y - H_y^* \right\|_{I - \gamma_y^2 (I - W)}^2 \leq M_y^{-1} \left\| \nu_{k,t+1}^y - H_y^* \right\|^2_{I - \gamma_y^2 (I - W) - \delta \sqrt{t}}.
\] (77)

From (74) and (78), we see that

\[
E \left\| y_{k,t+1}^y - 1 \right\|^2 \leq M_y^{-1} \left\| \nu_{k,t+1}^y - H_y^* \right\|^2_{I - \gamma_y^2 (I - W) - \delta \sqrt{t}} + \frac{\gamma_y^2}{4} E \left\| \nu_{k,t+1}^y - \nu_{k,t+1}^y \right\|^2_{(I - W)^2}.
\]

Multiplying throughout by \( M_y \), the proof of (55) is complete.

D.2 Proof of (56)

For every \( k \), let \( G_x^y \) and \( G_y^y \) denote the stochastic gradient oracles at iterate \( t \). For Algorithm 2, \( G_x^y = \nabla_x F(z_{k,t}; \xi_{k,t}) \) and \( G_y^y = \nabla_y F(z_{k,t}; \xi_{k,t}) \) are obtained using general stochastic gradient oracle. In Algorithm 3, \( G_x^x \) and \( G_y^y \) are obtained from the SVRG oracle.

Step 1: Computing \( E \left\| D_{k,t+1}^y - D_y^* \right\|^2_{(I - W)^t} \)

Observe that

\[
D_{k,t+1}^y - D_y^* = D_{k,t}^y + \frac{\gamma_y}{2s} (I - W) \nu_{k,t+1}^y - D_y^* \\
= D_{k,t}^y + \frac{\gamma_y}{2s} (I - W) \nu_{k,t+1}^y - D_y^* + \frac{\gamma_y}{2s} (I - W) H_y^* \\
= D_{k,t}^y - D_y^* + \frac{\gamma_y}{2s} (I - W) \nu_{k,t+1}^y - H_y^* \\
= D_{k,t}^y - D_y^* + \frac{\gamma_y}{2s} (I - W) (\nu_{k,t+1}^y - H_y^*) + \frac{\gamma_y}{2s} (I - W) (\nu_{k,t+1}^y - H_y^*) \\
\] (79)

On pre-multiplying both sides by \( \sqrt{(I - W)^t} \) and taking square norm on the resulting equality, we obtain

\[
\left\| \sqrt{(I - W)^t} (D_{k,t+1}^y - D_y^*) \right\|^2 \\
= \left\| \sqrt{(I - W)^t} (D_{k,t}^y - D_y^*) + \frac{\gamma_y}{2s} \sqrt{(I - W)^t} (I - W) (\nu_{k,t+1}^y - H_y^*) \right\|^2 \\
+ \frac{\gamma_y^2}{4s^2} \left\| \sqrt{(I - W)^t} (I - W) (\nu_{k,t+1}^y - \nu_{k,t+1}^y) \right\|^2 \\
+ 2 \left( \sqrt{(I - W)^t} (D_{k,t}^y - D_y^*) + \frac{\gamma_y}{2s} \sqrt{(I - W)^t} (I - W) (\nu_{k,t+1}^y - H_y^*) \right. \\
\left. + \frac{\gamma_y}{2s} \sqrt{(I - W)^t} (I - W) (\nu_{k,t+1}^y - H_y^*) \right). \\
\] (80)
By taking conditional expectation over compression at $t$-th iterate and using the result
\[ E \left( \nu^y_{k,t+1} - \nu^y_{k,t+1} \right) = 0, \]
we obtain
\[
E \left\| (I - W)^\dagger (D_{k,t+1}^y - D_y^*) \right\|^2 = \left\| (I - W)^\dagger (D_{k,t}^y - D_y^*) + \frac{\gamma^y}{4s^2} \sqrt{(I - W)^\dagger (I - W)(\nu^y_{k,t+1} - H_y^*)} \right\|^2
+ \frac{\gamma^y}{4s^2} E \left\| (I - W)^\dagger (I - W)(\nu^y_{k,t+1} - H_y^*) \right\|^2
+ 2 \left\langle (I - W)^\dagger (D_{k,t}^y - D_y^*), \frac{\gamma^y}{2s} \sqrt{(I - W)^\dagger (I - W)(\nu^y_{k,t+1} - H_y^*)} \right\rangle
+ \frac{\gamma^y}{4s^2} E \left\| (I - W)^\dagger (I - W)(\nu^y_{k,t+1} - H_y^*) \right\|^2. \tag{81}
\]

We have
\[
\left( (I - W)^\dagger (I - W) \right)^\top \left( (I - W)^\dagger (I - W) \right) = (I - W)^\dagger (I - W)^\dagger (I - W)
= (I - W)^\dagger (I - W)
= I - W; \tag{82}
\]
where the last equality follows from the definition of pseudoinverse. Consider
\[
\left\| (I - W)^\dagger (I - W)(\nu^y_{k,t+1} - H_y^*) \right\|^2
= (\nu^y_{k,t+1} - H_y^*)^\top \left( (I - W)^\dagger (I - W) \right)^\top (I - W)(\nu^y_{k,t+1} - H_y^*)
= \left\| (\nu^y_{k,t+1} - H_y^*) \right\|_{I - W}^2. \tag{83}
\]

Similarly, we see that
\[
E \left\| D_{k,t+1}^y - D_y^* \right\|^2_{(I - W)^\dagger}
= \left\| D_{k,t}^y - D_y^* \right\|^2_{(I - W)^\dagger} + \frac{\gamma^y}{4s^2} \left\| \nu^y_{k,t+1} - H_y^* \right\|_{I - W}^2 + \frac{\gamma^y}{4s^2} E \left\| \nu^y_{k,t+1} - \nu^y_{k,t+1} \right\|_{I - W}^2
+ \frac{\gamma^y}{8} \left\langle \sqrt{(I - W)^\dagger (D_{k,t}^y - D_y^*)}, \sqrt{(I - W)^\dagger (I - W)(\nu^y_{k,t+1} - H_y^*)} \right\rangle. \tag{84}
\]

Now, we will simplify the last term of (84). From the property of adjoints,
\[
\left\langle \sqrt{(I - W)^\dagger (D_{k,t}^y - D_y^*)}, \sqrt{(I - W)^\dagger (I - W)(\nu^y_{k,t+1} - H_y^*)} \right\rangle
= \left\langle (I - W) \sqrt{(I - W)^\dagger (I - W)^\dagger (D_{k,t}^y - D_y^*)}, \nu^y_{k,t+1} - H_y^* \right\rangle
= \left\langle (I - W)(I - W)^\dagger (D_{k,t}^y - D_y^*), \nu^y_{k,t+1} - H_y^* \right\rangle. \tag{85}
\]
Note that $D_y^* \in \text{Range}(I - W)$ using Proposition C.2. Further note that $D_{k,t}^y \in \text{Range}(I - W)$ because of update process (Step 10 in Algorithm 1). Therefore, there exists $D_{k,t}^y$ and $\tilde{D}_y$ such that
Taking square norm on both sides of (89), we obtain
\[
\begin{align*}
D_{k,t}^y &= (I - W) \tilde{D}_{k,t}^y, \quad \text{and} \quad D_y^* = (I - W) \tilde{D}_y. \\
(I - W)(I - W)^\dagger (D_{k,t}^y - D_y^*) &= (I - W)(I - W)^\dagger (I - W)(I - W) (\tilde{D}_{k,t}^y - (I - W) \tilde{D}_y) \\
&= (I - W) (I - W)^\dagger (I - W) (\tilde{D}_{k,t}^y - \tilde{D}_y) \\
&= (I - W) \tilde{D}_{k,t}^y - (I - W) \tilde{D}_y \\
&= D_{k,t}^y - D_y^*.
\end{align*}
\] (86)

This gives
\[
\begin{align*}
\left( \sqrt{(I - W)^\dagger (D_{k,t}^y - D_y^*)}, \sqrt{(I - W)^\dagger (I - W)(\nu_{k,t+1} - H_y^*)} \right) &= \left( D_{k,t}^y - D_y^*, \nu_{k,t+1} - H_y^* \right).
\end{align*}
\] (87)

On substituting above equality in (84), we obtain
\[
\begin{align*}
E \left\| D_{k,t+1}^y - D_y^* \right\|^2_{(I - W)^\dagger} &= \left\| D_{k,t}^y - D_y^* \right\|^2_{(I - W)^\dagger} + \frac{\gamma_y^2}{4s^2} \left\| \nu_{k,t+1}^y - H_y^* \right\|^2_{I - W} + \frac{\gamma_y^2}{4s^2} E \left\| \nu_{k,t+1}^y - \nu_{k,t+1}^y \right\|^2_{I - W} \\
&+ \frac{\gamma_y}{s} \left( D_{k,t}^y - D_y^*, \nu_{k,t+1} - H_y^* \right).
\end{align*}
\] (88)

Note that \( \frac{1}{m} \nabla_y f(z^*) = \frac{1}{m} \sum_{i=1}^{m} \nabla_y f_i(z^*) = J \nabla_y F(z^*) \). Then we can write \( H_y^* = 1y^* + sJ \nabla_y F(z^*) \). We have
\[
\begin{align*}
\nu_{k,t+1}^y - H_y^* &= \nu_{k,t+1}^y - (1y^* + sJ \nabla_y F(z^*)) \\
&= y_{k,t} + sG_{k,t}^y - sD_{k,t}^y + sD^y - 1y^* - sJ \nabla_y F(1z^*) \\
&= y_{k,t} - 1y^* + s \left( G_{k,t}^y - \nabla_y F(1z^*) \right) - s \left( D_{k,t}^y - D_y^* \right).
\end{align*}
\] (89)

In deriving equation (89), the second equality follows from the definition of \( D_y^* = (I - J) \nabla_y F(1z^*) \), and the third equality follows from the update step of \( \nu_{k,t+1}^y \) (Step 9 in Algorithm 1).

Substituting (89) in (88),
\[
\begin{align*}
E \left\| D_{k,t+1}^y - D_y^* \right\|^2_{(I - W)^\dagger} &= \left\| D_{k,t}^y - D_y^* \right\|^2_{(I - W)^\dagger} + \frac{\gamma_y^2}{4s^2} \left\| \nu_{k,t+1}^y - H_y^* \right\|^2_{I - W} + \frac{\gamma_y^2}{4s^2} E \left\| \nu_{k,t+1}^y - \nu_{k,t+1}^y \right\|^2_{I - W} \\
&+ \frac{\gamma_y}{s} \left( D_{k,t}^y - D_y^*, y_{k,t} - 1y^* + s \left( G_{k,t}^y - \nabla_y F(1z^*) \right) \right) - \gamma_y \left\| D_{k,t}^y - D_y^* \right\|^2.
\end{align*}
\] (90)

**Step 2: Computing**
\[
\begin{align*}
2s^2 \gamma_y E \left\| D_{k,t+1}^y - D_y^* \right\|^2_{(I - W)^\dagger} + \left\| \nu_{k,t+1}^y - H_y^* \right\|^2_{I - 2\nu_y^2 (I - W)}
\end{align*}
\]

Taking square norm on both sides of (89), we obtain
\[
\begin{align*}
\left\| \nu_{k,t+1}^y - H_y^* \right\|^2 &= \left\| y_{k,t} - 1y^* + s \left( G_{k,t}^y - \nabla_y F(1z^*) \right) \right\|^2 + s^2 \left\| D_{k,t}^y - D_y^* \right\|^2 \\
&- 2s \left( D_{k,t}^y - D_y^*, y_{k,t} - 1y^* + sG_{k,t}^y - s \nabla_y F(1z^*) \right).
\end{align*}
\] (91)

Multiply (90) by \( \frac{2s^2}{\gamma_y} \) and adding the resulting inequality with (91),
\[
\begin{align*}
\frac{2s^2}{\gamma_y} E \left\| D_{k,t+1}^y - D_y^* \right\|^2_{(I - W)^\dagger} + \left\| \nu_{k,t+1}^y - H_y^* \right\|^2
\end{align*}
\]
We have
\[
\left\| \nu_{k,t+1}^y - H_y^* \right\|_I^2 - \frac{\gamma_y}{2} \left\| \nu_{k,t+1}^y - H_y^* \right\|_I^2 \] 
\[
= (\nu_{k,t+1}^y - H_y^*)^T (I - W) (\nu_{k,t+1}^y - H_y^*) + \frac{\gamma_y}{2} (I - W) (\nu_{k,t+1}^y - H_y^*) 
\]
\[
= (\nu_{k,t+1}^y - H_y^*)^T \left( I - \frac{\gamma_y}{2} (I - W) \right) (\nu_{k,t+1}^y - H_y^*) 
\]
\[
= \left\| \nu_{k,t+1}^y - H_y^* \right\|_I^2 \cdot \frac{\gamma_y}{2} \left( I - \frac{\gamma_y}{2} (I - W) \right) .
\]

Therefore,
\[
\left\| \nu_{k,t+1}^y - H_y^* \right\|_I^2 = \left\| \nu_{k,t+1}^y - H_y^* \right\|_I^2 - \frac{\gamma_y}{2} \left\| \nu_{k,t+1}^y - H_y^* \right\|_I^2 \cdot \frac{\gamma_y}{2} (I - W) .
\]

By substituting above equality in (92), we obtain
\[
\frac{2s^2}{\gamma_y} E \left\| D_{k,t+1}^y - D_y^* \right\|_I^4 = \frac{2s^2}{\gamma_y} \left\| D_{k,t}^y - D_y^* \right\|_I^4 + \left\| \nu_{k,t+1}^y - H_y^* \right\|_I^2 
\]
\[
= \frac{2s^2}{\gamma_y} \left\| D_{k,t}^y - D_y^* \right\|_I^4 + \left\| \nu_{k,t,k+1}^y - H_y^* \right\|_I^2 \cdot \frac{\gamma_y}{2} (I - W) .
\]

Hence we get
\[
\frac{2s^2}{\gamma_y} E \left\| D_{k,t+1}^y - D_y^* \right\|_I^4 = \left\| D_{k,t}^y - D_y^* \right\|_I^4 \cdot \frac{\gamma_y}{2} (I - W) + \left\| \nu_{k,t,k+1}^y - H_y^* \right\|_I^2 \cdot \frac{\gamma_y}{2} (I - W) .
\]

Now we write \( \frac{2s^2}{\gamma_y} \left\| D_{k,t}^y - D_y^* \right\|_I^4 \) in terms of
\[
\frac{s^2}{\gamma_y} \left\| D_{k,t}^y - D_y^* \right\|^2_2 (I - W)^T \gamma_y I \cdot 
\frac{2s^2}{\gamma_y} \left\| D_{k,t}^y - D_y^* \right\|^2_2 (I - W)^T \gamma_y I \cdot 
\frac{2s^2}{\gamma_y} \left\| D_{k,t}^y - D_y^* \right\|^2_2 (I - W)^T \gamma_y I \cdot 
\frac{2s^2}{\gamma_y} \left\| D_{k,t}^y - D_y^* \right\|^2_2 (I - W)^T \gamma_y I .
\]

By substituting this equality in (94), we obtain
\[
\frac{2s^2}{\gamma_y} E \left\| D_{k,t+1}^y - D_y^* \right\|_I^4 = \left\| D_{k,t}^y - D_y^* \right\|^2_2 (I - W)^T \gamma_y I \cdot 
\frac{s^2}{\gamma_y} \left\| D_{k,t}^y - D_y^* \right\|^2_2 (I - W)^T \gamma_y I .
\]

Step 3: Computing \( \sqrt{\gamma^2} E \left\| H_{k,t+1}^y - H_y^* \right\|^2 \) and finishing the proof

Observe from Step 4 in Algorithm 3 that \( H_{k,t+1}^y = (1 - \alpha_y) H_{k,t+1}^y + \alpha_y \nu_{k,t+1}^y \), and as a result,
\[
H_{k,t+1}^y - H_y^* = (1 - \alpha_y) (H_{k,t}^y - H_y^*) + \alpha_y (\nu_{k,t+1}^y - \nu_{k,t+1}^y) + \alpha_y (\nu_{k,t+1}^y - H_y^*) ,
\]
and
\[
\left\| H_{k,t+1}^y - H_y^* \right\|^2 = \left\| (1 - \alpha_y) (H_{k,t}^y - H_y^*) + \alpha_y (\nu_{k,t+1}^y - \nu_{k,t+1}^y) \right\|^2 + \alpha_y (\nu_{k,t+1}^y - H_y^*) \right\|^2 + 2 \left\langle (1 - \alpha_y) (H_{k,t}^y - H_y^*) + \alpha_y (\nu_{k,t+1}^y - \nu_{k,t+1}^y) , \alpha_y (\nu_{k,t+1}^y - \nu_{k,t+1}^y) \right\rangle .
\]
Taking conditional expectation over compression at $t$-th iterate on both sides and substituting $E\left(\hat{\nu}_{k,t+1} - \nu_{k,t+1}\right) = 0$, we see that
\[
E\left[H_{k,t+1} - H_y\right] = \left\| (1 - \alpha_y)(H_{k,t} - H_y^\star) + \alpha_y(\nu_{k,t+1} - H_y^\star) \right\|^2 + \alpha_y^2 E\left[\hat{\nu}_{k,t+1} - \nu_{k,t+1}\right]^2
\]
\[
= (1 - \alpha_y)\left\|H_{k,t} - H_y\right\|^2 + \alpha_y\left\|\nu_{k,t+1} - H_y^\star\right\|^2 - \alpha_y(1 - \alpha_y, k)\left\|\nu_{k,t+1} - H_y^\star\right\|^2
\]
\[+ \alpha_y^2 E\left[\hat{\nu}_{k,t+1} - \nu_{k,t+1}\right]^2.
\]
(98)
The last equality follows from the identity $\|x + \alpha y\|^2 = (1 - \alpha)\|x\|^2 + \alpha\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2$. On multiplying both sides of (98) by $\sqrt{\delta}$, we obtain
\[
\sqrt{\delta} E\left[H_{k,t+1} - H_y^\star\right] = \sqrt{\delta}(1 - \alpha_y)\left(H_{k,t} - H_y^\star\right) + \sqrt{\delta}(\nu_{k,t+1} - H_y^\star) + \sqrt{\delta}(\nu_{k,t+1} - H_y^\star)\left\|\nu_{k,t+1} - H_y^\star\right\|^2
\]
\[\cdot (1 - \alpha_y)\left\|\nu_{k,t+1} - H_y^\star\right\|^2.
\]
(99)
We know that
\[
\left\|\nu_{k,t+1} - H_y^\star\right\|^2 = \left\|\nu_{k,t+1} - H_y^\star\right\|^2 - \left\|\nu_{k,t+1} - H_y^\star\right\|^2.
\]
(100)
Therefore,
\[
\frac{2s^2}{\gamma_y} E\left[D_{k,t+1}^y - D_y^\star\right] = \frac{2s^2}{\gamma_y} E\left[D_{k,t+1}^y - D_y^\star\right]^2 + \frac{s^2}{\gamma_y} E\left[\nu_{k,t+1} - H_y^\star\right]^2
\]
\[\cdot \left\|\nu_{k,t+1} - H_y^\star\right\|^2.
\]
(101)
where the last equality follows from (98). Now we add above equality with (99) and obtain the following expression:
\[
\frac{2s^2}{\gamma_y} E\left[D_{k,t+1}^y - D_y^\star\right] = \frac{2s^2}{\gamma_y} E\left[D_{k,t+1}^y - D_y^\star\right]^2 + \frac{s^2}{\gamma_y} E\left[\nu_{k,t+1} - H_y^\star\right]^2
\]
\[\cdot \left\|\nu_{k,t+1} - H_y^\star\right\|^2.
\]
(102)
Rearranging the r.h.s. of the above equation, we obtain (56).

We have a similar recursion result in terms of $x$.

**Lemma D.2.** Let $x_{k,t+1}, y_{k,t+1}, D_{k,t+1}^x, D_{k,t+1}^y, H_{k,t+1}^x, H_{k,t+1}^y, H_w^x, H_w^y, H_{w,x}, H_{w,y}, H_{w,x}^w, H_{w,y}^w$ be obtained from Algorithm 1 using IPDHG$(x_{k,t}, y_{k,t}, D_{k,t}^x, D_{k,t}^y, H_{k,t}^x, H_{k,t}^y, H_{w,x}, H_{w,y}, s, \gamma, \alpha, \gamma, \alpha, \gamma, \alpha, \gamma, \alpha, \gamma, \alpha, \gamma)$ Let Assumption 4.4 and Assumption 4.5 hold. Suppose $\alpha_x \in \left((0, 1 - \frac{1}{\lambda})\right)$ and
\[
\gamma_x \in \left(0, \min \left\{ \frac{2 - 2\sqrt{\delta}\alpha_x - \alpha_x (1 - \frac{1}{\lambda})\alpha_x^2}{\lambda_{\max}(I - W), \sqrt{\lambda_{\max}(I - W)}} \right\} \right).
\]
(103)
Then the following holds for all $t \geq 0$:
\[
M_z E\left|x_{k,t+1} - 1x^*\right|^2 + \frac{2s^2}{\gamma_x} E\left[D_{k,t+1}^x - D_x^\star\right] = \frac{2s^2}{\gamma_x} E\left[D_{k,t+1}^x - D_x^\star\right]^2 + \sqrt{\delta} E\left[H_{k,t+1}^x - H^x_x\right]
\]
\[\leq \left\|x_{k,t} - 1x^* - sG_{k,t}^x + s\nabla x F(1x^*, 1y^*) \right\|^2 + \frac{2s^2}{\gamma_x} \left(1 - \frac{\gamma_x}{2}\lambda_{m-1}(I - W)\right) \left\|D_{k,t+1}^x - D_x^\star\right\|^2.
\]
(104)
Assumption E.2, the following hold for all $t$:

We define a quantity $\Phi$ where

$$\Phi = \frac{\sqrt{3}\alpha_x}{1 - \frac{\sqrt{3}}{4}\lambda_{\max}(I-W)}.$$ 

We omit the proof of Lemma [D.2] as it is similar to the proof of Lemma [D.1].

E  Proofs for General Stochastic Setting

In this section, we state and present proofs of the results for general stochastic setting discussed in Section 5.

We now make the following assumptions on the stochastic gradients.

**Assumption E.1. (Unbiasedness and local bounded variance of stochastic gradients)** Stochastic gradients $\nabla_x f_i(x, y; \xi^i) \text{ and } \nabla_y f_i(x, y; \xi^i)$ for each node $i$ are unbiased estimates of the respective true gradients: $E[\nabla_x f_i(x, y; \xi^i)] = \nabla_x f_i(x, y), E[\nabla_y f_i(x, y; \xi^i)] = \nabla_y f_i(x, y)$ and have bounded variance:

$$E \left[ \| \nabla_x f_i(x^*, y^*; \xi^i) - \nabla_x f_i(x^*, y^*) \|^2 \right] \leq \sigma_x^2,$$

$$E \left[ \| \nabla_y f_i(x^*, y^*; \xi^i) - \nabla_y f_i(x^*, y^*) \|^2 \right] \leq \sigma_y^2,$$

where $(x^*, y^*)$ is the saddle point of (SPP).

**Assumption E.2. (Smoothness)**

1. Each $f_i(\cdot, y; \xi^i)$ is $L_{xx}$ smooth in expectation; for all $x_1, x_2 \in \mathbb{R}^{d_x},$

$$E \left[ \| \nabla_x f_i(x_1, y; \xi^i) - \nabla_x f_i(x_2, y; \xi^i) \|^2 \right] \leq 2L_{xx} (f_i(x_1, y) - f_i(x_2, y) - \langle \nabla_x f_i(x_2, y), x_1 - x_2 \rangle).$$

2. Each $-f_i(\cdot, \xi^i)$ is $L_{yy}$ smooth in expectation; for all $y_1, y_2 \in \mathbb{R}^{d_y},$

$$E \left[ \| -\nabla_y f_i(x_1, y; \xi^i) - \nabla_y f_i(x_2, y; \xi^i) \|^2 \right] \leq 2L_{yy} (-f_i(x_1, y) + f_i(x_2, y) - \langle -\nabla_y f_i(x_2, y), y_1 - y_2 \rangle).$$

3. For every $x \in \mathbb{R}^{d_x},$ the following holds: $E \left[ \| \nabla_x f_i(x_1; \xi^i) - \nabla_x f_i(x_2; \xi^i) \|^2 \right] \leq L_{xx}^2 \| y_1 - y_2 \|^2$ for all $y_1, y_2 \in \mathbb{R}^{d_y}.$

4. For every $y \in \mathbb{R}^{d_y},$ the following holds: $E \left[ \| \nabla_y f_i(x_1; \xi^i) - \nabla_y f_i(x_2; \xi^i) \|^2 \right] \leq L_{yy}^2 \| x_1 - x_2 \|^2$ for all $x_1, x_2 \in \mathbb{R}^{d_x}.$

We define a quantity $\Phi_{k,t}$ consisting of primal and dual updates which is instrumental in deriving the convergence rate of Algorithm 2:

$$\Phi_{k,t} = M_{x,k} \| x_{k,t} - 1x^* \|^2 + 2\frac{\gamma_x}{\gamma_{x,k}} \left\| D_{x,t}^e - D_x^* \right\|^2_{(I-W)^t} + \sqrt{\delta} \| H_{x,t}^e - H_{x,t}^* \|^2$$

$$+ M_{y,k} \| y_{k,t} - 1y^* \|^2 + 2\frac{\gamma_y}{\gamma_{y,k}} \left\| D_{y,t}^e - D_y^* \right\|^2_{(I-W)^t} + \sqrt{\delta} \| H_{y,t}^e - H_{y,t}^* \|^2$$

for all $t \geq 0$ and $k \geq 0.$

**Lemma E.3.** Suppose $\{x_{k,t}\}_t$ and $\{y_{k,t}\}_t$ are the sequences generated by Algorithm 2 with $G = [\nabla_x F(\cdot; \xi); -\nabla_y F(\cdot; \xi)].$ Then, under Assumption 4.1 Assumption 4.2 Assumption 4.7 and Assumption E.2 the following hold for all $t \geq 0:

$$E \left[ \| x_{k,t} - 1x^* - s_k G_{x,t}^e + s_k \nabla_x F(1x^*, 1y^*) \|^2 \right]$$

$$\leq (1 - \mu_x s_k) \| x_{k,t} - 1x^* \|^2 + 4s_k^2 L_{xx}^2 \| y_{k,t} - 1y^* \|^2 - (2s_k - 8s_k^2 L_{xx}) \sum_{i=1}^m V_{f_i,y_k,t}(x^*, x_{k,t})$$

$$+ 2s_k \left( F(x_{k,t}, 1y^*) - F(1x^*, 1y^*) + F(1x^*, y_{k,t}) - F(z_{k,t}) \right) + 2m \sigma_x^2 s_k^2.$$ 

$$E \left[ \| y_{k,t} - 1y^* - s_k G_{y,t}^e - s_k \nabla_y F(1x^*, 1y^*) \|^2 \right]$$

$$\leq (1 - \mu_y s_k) \| y_{k,t} - 1y^* \|^2 + 4s_k^2 L_{yy}^2 \| x_{k,t} - 1x^* \|^2 - (2s_k - 8s_k^2 L_{yy}) \sum_{i=1}^m V_{f_j, y_{k,t}^i}(y^*, y_{k,t})$$

$$+ 2s_k \left( -F(x_{k,t}, 1y^*) + F(1x^*, 1y^*) - F(1x^*, y_{k,t}) + F(z_{k,t}) \right) + 2m \sigma_y^2 s_k^2,$$

where $E$ denotes the conditional expectation on stochastic gradient at $t$-th update step.
E.1 Proof of Lemma [E.3]

We will derive inequality (107) here. The proof of inequality (108) is similar and is omitted.

In the general stochastic setting, we have $G_{k,t}^{i,x} = \nabla_x f_i(z_{k,t}^i; x_{k,t}^i)$, $G_{k,t}^{i,y} = \nabla_y f_i(z_{k,t}^i; \xi_{k,t}^i)$ and step size is $s_k$. We now have

\[
E \|x_{k,t} - 1x^* - s_k G_{k,t}^{x} + s_k \nabla_x F(1x^*, 1y^*)\|^2 = \|x_{k,t} - 1x^*\|^2 + s_k^2 E \|\nabla_x F(1x^*, 1y^*) - \nabla_x F(z_{k,t}; \xi_{k,t})\|^2 \\
+ 2s_k E \langle x_{k,t} - 1x^*, \nabla_x F(1x^*, 1y^*) - \nabla_x F(z_{k,t}; \xi_{k,t}) \rangle = \|x_{k,t} - 1x^*\|^2 + s_k^2 E \|\nabla_x F(1x^*, 1y^*) - \nabla_x F(z_{k,t}; \xi_{k,t})\|^2 \\
+ 2s_k \langle x_{k,t} - 1x^*, \nabla_x F(1x^*, 1y^*) - \nabla_x F(z_{k,t}) \rangle \\
\leq \|x_{k,t} - 1x^*\|^2 + 2s_k^2 E \|\nabla_x F(1x^*, 1y^*) - \nabla_x F(z_{k,t}; \xi_{k,t})\|^2 \\
+ 2s_k \langle x_{k,t} - 1x^*, \nabla_x F(1x^*, 1y^*) - \nabla_x F(z_{k,t}) \rangle \\
\leq \|x_{k,t} - 1x^*\|^2 + 2s_k^2 E \|\nabla_x F(1x^*, 1y^*; \xi_{k,t}) - \nabla_x F(z_{k,t}; \xi_{k,t})\|^2 \\
+ 2s_k \langle x_{k,t} - 1x^*, \nabla_x F(1x^*, 1y^*) - \nabla_x F(z_{k,t}; \xi_{k,t}) \rangle + 2m^2 s_k^2 \sigma_2.
\]

Also,

\[
E \|\nabla_x F(1x^*, 1y^*; \xi_{k,t}) - \nabla_x F(z_{k,t}; \xi_{k,t})\|^2 \\
= \sum_{i=1}^m E \|\nabla_x f_i(z^*; \xi_{k,t}^i) - \nabla_x f_i(z_{k,t}^i; \xi_{k,t}^i)\|^2 \\
= \sum_{i=1}^m E \|\nabla_x f_i(z^*; \xi_{k,t}^i) - \nabla_x f_i(x^*, y_{k,t}^i; \xi_{k,t}^i) + \nabla_x f_i(x^*, y_{k,t}^i; \xi_{k,t}^i) - \nabla_x f_i(z_{k,t}^i; \xi_{k,t}^i)\|^2 \\
\leq \sum_{i=1}^m \left( 2E \|\nabla_x f_i(z^*; \xi_{k,t}^i) - \nabla_x f_i(x^*, y_{k,t}^i; \xi_{k,t}^i)\|^2 + 2E \|\nabla_x f_i(x^*, y_{k,t}^i; \xi_{k,t}^i) - \nabla_x f_i(z_{k,t}^i; \xi_{k,t}^i)\|^2 \right) \\
\leq \sum_{i=1}^m \left( 2L_{xy}^2 \|y_{k,t}^i - y^*\|^2 + 4L_{xx} V_{f_i,y_{k,t}^i}(x^*, x_{k,t}^i) \right) \quad \text{(from Assumption E.2 and equation (28))} \\
= 2L_{xy}^2 \|y_{k,t} - 1y^*\|^2 + 4L_{xx} \sum_{i=1}^m V_{f_i,y_{k,t}^i}(x^*, x_{k,t}^i).
\]

We now simplify the inner product $\langle x_{k,t} - 1x^*, \nabla_x F(1x^*, 1y^*) - \nabla_x F(z_{k,t}; \xi_{k,t}) \rangle$ term present in the r.h.s of (109). Recall the definition of Bregman distance $V_{f_i,y}(x_1, x_2)$:

\[
V_{f_i,y}(x_1, x_2) = f_i(x_1, y) - f_i(x_2, y) - \langle \nabla_x f_i(x_1, y), x_1 - x_2 \rangle
\]

\[
\langle \nabla_x f_i(z_{k,t}^i, -x^* + x_{k,t}^i) \rangle = -f_i(x^*, y_{k,t}^i) + f_i(z_{k,t}^i) + V_{f_i,y_{k,t}^i}(x^*, x_{k,t}^i).
\]

Using $\mu_x$ strong convexity of $f_i(\cdot, y)$, we have

\[
f_i(x_{k,t}^i, y^*) \geq f_i(z^*) + \langle \nabla_x f_i(z^*), x_{k,t}^i - x^* \rangle + \frac{\mu_x}{2} \|x_{k,t}^i - x^*\|^2
\]

\[
\langle \nabla_x f_i(z^*), x_{k,t}^i - x^* \rangle \leq f_i(x_{k,t}^i, y^*) - f_i(z^*) - \frac{\mu_x}{2} \|x_{k,t}^i - x^*\|^2.
\]

We now compute

\[
\langle x_{k,t} - 1x^*, \nabla_x F(1x^*, 1y^*) - \nabla_x F(z_{k,t}; \xi_{k,t}) \rangle = \sum_{i=1}^m \langle x_{k,t}^i - x^*, \nabla_x f_i(z^*) - \nabla_x f_i(z_{k,t}^i) \rangle \\
\leq \sum_{i=1}^m \left( f_i(x_{k,t}^i, y^*) - f_i(z_{k,t}^i) - \frac{\mu_x}{2} \|x_{k,t}^i - x^*\|^2 + f_i(x^*, y_{k,t}^i) - f_i(z_{k,t}^i) - V_{f_i,y_{k,t}^i}(x^*, x_{k,t}^i) \right) \\
= F(x_{k,t}^i, y^*) - F(1x^*, 1y^*) + F(1x^*, y_{k,t}^i) - F(z_{k,t}^i) - \frac{\mu_x}{2} \|x_{k,t}^i - 1x^*\|^2
\]

\[
- \sum_{i=1}^m V_{f_i,y_{k,t}^i}(x^*, x_{k,t}^i),
\]

(116)
We have the following corollary. We now show that

\[ s \]

where the second last step follows from (112) and (114). On substituting (110) and (116) in (109), we obtain

\[ E \| x_{k,t} - 1x^* - s_k G_{k,t}^x + s_k \nabla_x F(x_*^*, y_*^*) \| ^2 \]

\[ \leq \| x_{k,t} - 1x^* \|^2 + 2ms_k^2 \sigma_x^2 + 2s_k^2 \left( 2L_{2y}^2 \| y_{k,t} - 1y^* \|^2 + 4L_{xy} \sum_{i=1}^m V_{f_i,y_{k,t}}(x_i^*, x_{k,t}^i) \right) \]

\[ + 2s_k \left( F(x_{k,t}, y_*^*) - F(1x^*, y_*^*) + F(x_*^*, y_{k,t}) - F(z_{k,t}) - \frac{\mu_x}{2} \| x_{k,t} - 1x^* \|^2 - \sum_{i=1}^m V_{f_i,y_{k,t}}(x_i^*, x_{k,t}^i) \right) \]

\[ = (1 - \mu_x) \| x_{k,t} - 1x^* \|^2 + 4s_k^2 L_{xy}^2 \| y_{k,t} - 1y^* \|^2 - (2s_k - 8s_k^2 L_{xx}) \sum_{i=1}^m V_{f_i,y_{k,t}}(x_i^*, x_{k,t}^i) \]

\[ + 2s_k \left( F(x_{k,t}, y_*^*) - F(1x^*, y_*^*) + F(x_*^*, y_{k,t}) - F(z_{k,t}) + 2ms_k^2 \sigma_x^2, \right) \]

(107)

completing the proof.

We have the following corollary.

**Corollary E.4.** Let \( s_k = \frac{1}{4L_f \kappa_{f2k/2}} \) for every \( k \geq 0 \). Then, under the setting of Lemma E.3,

\[ E \| x_{k,t} - 1x^* - s_k G_{k,t}^x + s_k \nabla_x F(x_*^*, y_*^*) \| ^2 \]

\[ \leq (1 - b_{x,k}) \| x_{k,t} - 1x^* \|^2 + (1 - b_{y,k}) \| y_{k,t} - 1y^* \|^2 + 2ms_k^2(\sigma_x^2 + \sigma_y^2), \]

for all \( t \geq 1 \), where \( b_{x,k} = \mu_x s_k - 4s_k^2 L_{xy}^2, b_{y,k} = \mu_y s_k - 4s_k^2 L_{yy}^2 \).

**E.2 Proof of Corollary E.4**

As the step size \( s_k = \frac{1}{4L_f \kappa_{f2k/2}} \), we have \( s_k \leq \frac{1}{4L} \). We now show that the terms \( 2s_k - 8s_k^2 L_{xx} \) and \( 2s_k - 8s_k^2 L_{yy} \) appearing in (107) and (108) are non-negative:

\[ 2s_k - 8s_k^2 L_{xx} = \frac{2}{4L_f \kappa_{f2k/2}} - 8L_{xx} \frac{1}{16L_f^2 \kappa_{f2k/2}} \]

\[ \geq \frac{1}{2L_f \kappa_{f2k/2}} - 8L \frac{1}{16L_f^2 \kappa_{f2k/2}} = \frac{1}{2L_f \kappa_{f2k/2}} - \frac{1}{2L_f \kappa_{f2k/2}} \]

\[ = \kappa_{f2k/2} - \frac{1}{2L_f \kappa_{f2k/2}} \geq 0. \]

(118)

Similarly, we get \( 2s_k - 8s_k^2 L_{yy} \geq 0 \). Recall

\[ b_{x,k} = \mu_x s_k - 4s_k^2 L_{xy}^2 \]

\[ = \frac{\mu_x}{4L_f \kappa_{f2k/2}} - \frac{4L_{xy}^2}{16L_f^2 \kappa_{f2k/2}} \]

\[ \geq \frac{\mu_x}{4L_f \kappa_{f2k/2}} - \frac{4L}{16L_f^2 \kappa_{f2k/2}} \]

\[ = \frac{1}{4L_f^2 \kappa_{f2k/2}} - \frac{1}{4L_f^2 \kappa_{f2k/2}} \]

\[ \geq \frac{1}{4L_f^2 \kappa_{f2k/2}} - \frac{1}{4L_f^2 \kappa_{f2k/2}} \]

\[ = \left( 1 - \frac{1}{\sqrt{2}} \right) \frac{1}{4L_f^2 \kappa_{f2k/2}}. \]

(119)

We now show that \( b_{x,k} < 1 \).

\[ b_{x,k} < \mu_x s_k = \frac{\mu_x}{4L_f \kappa_{f2k/2}} \leq \frac{\mu_x}{4L_{xx} \kappa_{f2k/2}} = \frac{1}{4L_{xx} \kappa_{f2k/2}} < 1. \]

(120)
Therefore, \( b_{x,k} \in (0, 1) \) for every \( k \geq 0 \). In a similar fashion, we obtain \( b_{y,k} \in (0, 1) \) for every \( k \geq 0 \).

On adding (107) and (108), we obtain

\[
E \left[ x_{k,t} - 1x^* - s_k G^x_{k,t} + s_k \nabla_x F(1z^*) \right]^2 + E \left[ y_{k,t} - 1y^* + s_k G^y_{k,t} - s_k \nabla_y F(1z^*) \right]^2 \\
\leq (1 - \mu_x s_k + 4s_k^2 \mathbb{L}_{x,t}) \left\| x_{k,t} - 1x^* \right\|^2 \leq \left\| x_{k,t} - 1x^* \right\|^2 \\
+ (1 - \mu_y s_k + 4s_k^2 \mathbb{L}_{y,t}) \left\| y_{k,t} - 1y^* \right\|^2 \\
- (2s_k - 8s_k^2 \mathbb{L}_{x,t}) \sum_{i=1}^m V_{f,i,k}(x^*, x^*_i, k, t) - (2s_k - 8s_k^2 \mathbb{L}_{y,t}) \sum_{i=1}^m V_{f,i,k}(y^*, y^*_i, k, t) + 2ms_k^2 (\sigma_x^2 + \sigma_y^2) \\
\leq (1 - b_{x,k}) \left\| x_{k,t} - 1x^* \right\|^2 + (1 - b_{y,k}) \left\| y_{k,t} - 1y^* \right\|^2 + 2ms_k^2 (\sigma_x^2 + \sigma_y^2). \\
\] (121)

The last inequality follows from non-negativity of \( V_{f,i,k}(x^*, x^*_i, k, t), V_{f,i,k}(y^*, y^*_i, k, t), 2s_k - 8s_k^2 \mathbb{L}_{x,t} \) and \( 2s_k - 8s_k^2 \mathbb{L}_{y,t} \).

We now establish a recursion for \( E[\Phi_{k,t}] \).

**Lemma E.5.** Suppose \( \{x_k,t\} \) and \( \{y_k,t\} \) are the sequences generated by algorithm 2 with \( G = [\nabla_x F(\xi), -\nabla_y F(\xi)] \). Suppose Assumptions 4.1, 4.5 and Assumptions E.1, E.2 hold. Let step size \( s_k \) is chosen according to Corollary E.4. Then, for every \( k \geq 0 \), the following holds in total expectation:

\[
E[\Phi_{k,t+1}] \leq \rho_k E[\Phi_{k,t}] + 2ms_k^2 (\sigma_x^2 + \sigma_y^2), \\
\] (122)

for all \( t \geq 0 \), where \( \rho_k \) is defined in equation (126).

### E.3 Proof of Lemma E.5

On adding (54) and (104), we have

\[
M_{x,k} E \left[ x_{k,t+1} - 1x^* \right]^2 + \frac{2s_k^2}{\gamma_{x,k}} E \left\| D^x_{k,t+1} - D^x_{k,t} \right\|^2 \left( I - W \right)^t + \sqrt{\delta} E \left\| H^x_{k,t+1} - H^x_{k,t} \right\|^2 \\
+ M_{y,k} E \left[ y_{k,t+1} - 1y^* \right]^2 + \frac{2s_k^2}{\gamma_{y,k}} E \left\| D^y_{k,t+1} - D^y_{k,t} \right\|^2 \left( I - W \right)^t + \sqrt{\delta} E \left\| H^y_{k,t+1} - H^y_{k,t} \right\|^2 \\
\leq \left\| x_{k,t} - 1x^* - s_k G^x_{k,t} + s_k \nabla_x F(1x^*) \right\|^2 + \frac{2s_k^2}{\gamma_{x,k}} \left( 1 - \frac{\gamma_{x,k}}{2} \lambda_{m-1}(I - W) \right) \left\| D^x_{k,t} - D^x_{k} \right\|^2 \left( I - W \right)^t \\
+ \frac{2s_k^2}{\gamma_{y,k}} \left( 1 - \frac{\gamma_{y,k}}{2} \lambda_{m-1}(I - W) \right) \left\| D^y_{k,t} - D^y_{k} \right\|^2 \left( I - W \right)^t \\
+ \sqrt{\delta} (1 - \alpha_{x,k}) \left\| H^x_{k,t} - H^x_{k,t} \right\|^2 + \left\| y_{k,t} - 1y^* + s_k G^y_{k,t} - s_k \nabla_y F(1z^*) \right\|^2 \\
+ \sqrt{\delta} (1 - \alpha_{y,k}) \left\| H^y_{k,t} - H^y_{k,t} \right\|^2. \\
\] (123)

By taking conditional expectation on stochastic gradient at \( t \)-th step on both sides of above inequality and applying Tower property, we obtain

\[
M_{x,k} E \left[ x_{k,t+1} - 1x^* \right]^2 + \frac{2s_k^2}{\gamma_{x,k}} E \left\| D^x_{k,t+1} - D^x_{k,t} \right\|^2 \left( I - W \right)^t + \sqrt{\delta} E \left\| H^x_{k,t+1} - H^x_{k,t} \right\|^2 \\
+ M_{y,k} E \left[ y_{k,t+1} - 1y^* \right]^2 + \frac{2s_k^2}{\gamma_{y,k}} E \left\| D^y_{k,t+1} - D^y_{k,t} \right\|^2 \left( I - W \right)^t + \sqrt{\delta} E \left\| H^y_{k,t+1} - H^y_{k,t} \right\|^2 \\
\leq E \left[ x_{k,t} - 1x^* - s_k G^x_{k,t} + s_k \nabla_x F(1x^*) \right]^2 + \frac{2s_k^2}{\gamma_{x,k}} \left( 1 - \frac{\gamma_{x,k}}{2} \lambda_{m-1}(I - W) \right) \left\| D^x_{k,t} - D^x_{k} \right\|^2 \left( I - W \right)^t \\
+ \frac{2s_k^2}{\gamma_{y,k}} \left( 1 - \frac{\gamma_{y,k}}{2} \lambda_{m-1}(I - W) \right) \left\| D^y_{k,t} - D^y_{k} \right\|^2 \left( I - W \right)^t \\
+ \sqrt{\delta} (1 - \alpha_{x,k}) \left\| H^x_{k,t} - H^x_{k,t} \right\|^2 + E \left[ y_{k,t} - 1y^* + s_k G^y_{k,t} - s_k \nabla_y F(1x^*, 1y^*) \right]^2 \\
+ \sqrt{\delta} (1 - \alpha_{y,k}) \left\| H^y_{k,t} - H^y_{k,t} \right\|^2 \\
\leq (1 - b_{x,k}) \left\| x_{k,t} - 1x^* \right\|^2 + (1 - b_{y,k}) \left\| y_{k,t} - 1y^* \right\|^2 \\
+ \frac{2s_k^2}{\gamma_{x,k}} \left( 1 - \frac{\gamma_{x,k}}{2} \lambda_{m-1}(I - W) \right) \left\| D^x_{k,t} - D^x_{k} \right\|^2 \left( I - W \right)^t + \frac{2s_k^2}{\gamma_{y,k}} \left( 1 - \frac{\gamma_{y,k}}{2} \lambda_{m-1}(I - W) \right) \left\| D^y_{k,t} - D^y_{k} \right\|^2 \left( I - W \right)^t \\
+ \sqrt{\delta} (1 - \alpha_{x,k}) \left\| H^x_{k,t} - H^x_{k,t} \right\|^2 + \sqrt{\delta} (1 - \alpha_{y,k}) \left\| H^y_{k,t} - H^y_{k,t} \right\|^2 + 2ms_k^2 (\sigma_x^2 + \sigma_y^2) \\
\] (124)

where the last inequality follows from inequality (121).
By taking total expectation on both sides of above inequality and using the definition of \( \Phi_{k,t} \), we obtain

\[
E[\Phi_{k,t+1}]
\leq (1 - b_{x,k})E \| x_{k,t} - 1x^* \|^2 + (1 - b_{y,k})E \| y_{k,t} - 1y^* \|^2
+ \frac{2s_k^2}{\gamma_{x,k}} \left( 1 - \frac{\gamma_{x,k}}{2} \lambda_{m-1}(I - W) \right) E \left\| D_{k,t}^x - D_{k,t}^y \right\|^2_{(I-W)^t} + \frac{2s_k^2}{\gamma_{y,k}} \left( 1 - \frac{\gamma_{y,k}}{2} \lambda_{m-1}(I - W) \right) E \left\| D_{k,t}^y - D_{k,t}^y \right\|^2_{(I-W)^t}
+ \sqrt{\delta}(1 - \alpha_{x,k})E \left\| H_{k,t}^x - H_{x,k}^* \right\|^2 + \sqrt{\delta}(1 - \alpha_{y,k})E \left\| H_{k,t}^y - H_{y,k}^* \right\|^2 + 2ms_k^2(\sigma_x^2 + \sigma_y^2)
\]

\[
= \frac{(1 - b_{x,k})}{M_{x,k}} E \| x_{k,t} - 1x^* \|^2 + \frac{(1 - b_{y,k})}{M_{y,k}} E \| y_{k,t} - 1y^* \|^2
+ \frac{2s_k^2}{\gamma_{x,k}} \left( 1 - \frac{\gamma_{x,k}}{2} \lambda_{m-1}(I - W) \right) E \left\| D_{k,t}^x - D_{k,t}^y \right\|^2_{(I-W)^t} + \frac{2s_k^2}{\gamma_{y,k}} \left( 1 - \frac{\gamma_{y,k}}{2} \lambda_{m-1}(I - W) \right) E \left\| D_{k,t}^y - D_{k,t}^y \right\|^2_{(I-W)^t}
+ \sqrt{\delta}(1 - \alpha_{x,k})E \left\| H_{k,t}^x - H_{x,k}^* \right\|^2 + \sqrt{\delta}(1 - \alpha_{y,k})E \left\| H_{k,t}^y - H_{y,k}^* \right\|^2 + 2ms_k^2(\sigma_x^2 + \sigma_y^2)
\leq \max \left\{ \frac{1 - b_{x,k}}{M_{x,k}}, \frac{1 - b_{y,k}}{M_{y,k}}, 1 - \frac{\gamma_{x,k}}{2} \lambda_{m-1}(I - W), 1 - \frac{\gamma_{y,k}}{2} \lambda_{m-1}(I - W), 1 - \alpha_{x,k}, 1 - \alpha_{y,k} \right\} \times E[\Phi_{k,t}]
+ 2ms_k^2(\sigma_x^2 + \sigma_y^2),
\]

where \( \rho_k := \max \left\{ \frac{1 - b_{x,k}}{M_{x,k}}, \frac{1 - b_{y,k}}{M_{y,k}}, 1 - \frac{\gamma_{x,k}}{2} \lambda_{m-1}(I - W), 1 - \frac{\gamma_{y,k}}{2} \lambda_{m-1}(I - W), 1 - \alpha_{x,k}, 1 - \alpha_{y,k} \right\} \). \hspace{1cm} (125)

\[
E.4 \quad \text{Feasibility of Parameters for Algorithm 2}
\]

**Parameters setting:** From Corollary E.4, the step size used in Algorithm 2 is \( s_k = \frac{1}{4\kappa_f L^2\delta^2} \)
for every \( k \geq 0 \). We choose the parameters involved in \textsc{Comm} procedure and other parameters \( \gamma_{x,k}, \gamma_{y,k} \) as follows:

\[
\alpha_{x,k} = \frac{b_{x,k}}{1 + \delta}, \quad \alpha_{y,k} = \frac{b_{y,k}}{1 + \delta}, \hspace{1cm} (127)
\gamma_{x,k} = \frac{b_{x,k}}{2(1 + \delta)^2 \lambda_{\max}(I - W)}, \quad \gamma_{y,k} = \frac{b_{y,k}}{2(1 + \delta)^2 \lambda_{\max}(I - W)}, \hspace{1cm} (128)
M_{x,k} = 1 - \frac{\sqrt{\delta} \alpha_{x,k}}{1 - \frac{2\gamma_{x,k}}{\lambda_{\max}(I - W)}}, \quad M_{y,k} = 1 - \frac{\sqrt{\delta} \alpha_{y,k}}{1 - \frac{2\gamma_{y,k}}{\lambda_{\max}(I - W)}}, \hspace{1cm} (129)
\]

**Feasibility of parameters:** Above choice of parameters should satisfy the following conditions:

\[
\alpha_{x,k} < \min \left\{ \frac{b_{x,k}}{\sqrt{\delta}}, \frac{1}{1 + \delta} \right\}, \quad \alpha_{y,k} < \min \left\{ \frac{b_{y,k}}{\sqrt{\delta}}, \frac{1}{1 + \delta} \right\}, \hspace{1cm} (130)
\gamma_{x,k} \in \left( 0, \min \left\{ 2 - 2\sqrt{\delta} \alpha_{x,k}, \frac{\alpha_{x,k} - (1 + \delta)\alpha_{x,k}^2}{\lambda_{\max}(I - W)} \sqrt{\delta} \lambda_{\max}(I - W) \right\} \right), \hspace{1cm} (131)
\gamma_{y,k} \in \left( 0, \min \left\{ 2 - 2\sqrt{\delta} \alpha_{y,k}, \frac{\alpha_{y,k} - (1 + \delta)\alpha_{y,k}^2}{\lambda_{\max}(I - W)} \sqrt{\delta} \lambda_{\max}(I - W) \right\} \right), \hspace{1cm} (132)
\frac{\gamma_{x,k}}{2} \lambda_{m-1}(I - W) \in (0,1), \quad \frac{\gamma_{y,k}}{2} \lambda_{m-1}(I - W) \in (0,1), \hspace{1cm} (133)
M_{x,k} \in (0,1), \quad M_{y,k} \in (0,1), \hspace{1cm} (134)
1 - b_{x,k} \in (0,1), \quad 1 - b_{y,k} \in (0,1), \hspace{1cm} (135)
\]

In this section, we show that all parameters specified in (128) satisfy all requirements of (130)-(135).
Feasibility of $\alpha_{x,k}$ and $\alpha_{y,k}$. From [119] and [120], we have $0 < b_{x,k} < 1$. Therefore, $\alpha_{x,k} < \frac{1}{1+\delta}$. Moreover, $\frac{\sqrt{3}}{1+\delta} \leq 1/2$ as $\delta \in [0,1]$. Therefore, $\alpha_{x,k} \leq \frac{b_{x,k}}{2\sqrt{3}} < b_{x,k}/\sqrt{3}$. Hence, $\alpha_{x,k} < \min\left\{\frac{b_{x,k}}{\sqrt{3}}, \frac{1}{1+\delta}\right\}$. Similarly, $\alpha_{y,k} < \min\left\{\frac{b_{y,k}}{\sqrt{3}}, \frac{1}{1+\delta}\right\}$ because $b_{y,k} \in (0,1)$.

Feasibility of $\gamma_{x,k}$ and $\gamma_{y,k}$. Consider

$$\frac{\alpha_{x,k} - (1 + \delta)\alpha_{x,k}^2}{\sqrt{\delta}\lambda_{\max}(I - W)} = \frac{b_{x,k} - b_{x,k}^2}{\sqrt{\delta}(1 + \delta)\lambda_{\max}(I - W)} \geq \frac{2(b_{x,k} - b_{x,k}^2)}{(1 + \delta)(1 + \delta)\lambda_{\max}(I - W)}.$$  \hspace{1cm} (136)

The last inequality uses the relation $\frac{\sqrt{3}}{1+\delta} \leq \frac{1}{2}$. Using [120], we have $b_{x,k} \leq \frac{1}{4\kappa_2 n/2^{k/2}} < 0.25$. This allows us to use the inequality $2x - 2x^2 \geq x/2$ for all $0 \leq x \leq 0.75$. Therefore,

$$\frac{\alpha_{x,k} - (1 + \delta)\alpha_{x,k}^2}{\sqrt{\delta}\lambda_{\max}(I - W)} > \frac{2(1 + \delta)^2\lambda_{\max}(I - W)}{b_{x,k}} = \gamma_{x,k}.$$ \hspace{1cm} (137)

Consider

$$2 - 2\sqrt{\delta}\alpha_{x,k} = \left(2 - \frac{2\sqrt{\delta}b_{x,k}}{1 + \delta}\right)\frac{1}{\lambda_{\max}(I - W)} \geq \left(2 - \frac{2\sqrt{\delta}}{1 + \delta}\right)\frac{1}{\lambda_{\max}(I - W)} \geq \frac{1}{\lambda_{\max}(I - W)} \geq \frac{b_{x,k}}{2(1 + \delta)^2\lambda_{\max}(I - W)} = \gamma_{x,k},$$ \hspace{1cm} (138)

where the third inequality uses $\frac{\sqrt{3}}{1+\delta} \leq \frac{1}{2}$ and fourth inequality uses $\frac{b_{x,k}}{2(1+\delta)^2} < 1$. We know that $b_{y,k} \in (0,1)$. Therefore, by following similar steps, the chosen $\gamma_{y,k}$ is also feasible. As $\gamma_{x,k} < \frac{2 - 2\sqrt{\delta}\alpha_{x,k}}{\lambda_{\max}(I - W)} < \frac{2}{\lambda_{\max}(I - W)}$. Notice that $\lambda_{m+1}(I - W) < \lambda_{\max}(I - W)$ Therefore,

$$\frac{\gamma_{x,k}}{\lambda_{m+1}(I - W)} < \frac{\gamma_{x,k}}{2\lambda_{\max}(I - W)} < 1.$$ \hspace{1cm} (139)

Similarly, $\frac{\gamma_{y,k}}{2} - \frac{\gamma_{y,k}}{2\lambda_{\max}(I - W)} < 1$.

Feasibility of $M_{x,k}$ and $M_{y,k}$. Recall $M_{x,k} = 1 - \frac{\sqrt{\delta}\alpha_{x,k}}{1 - \frac{\sqrt{\delta}\alpha_{x,k}}{\lambda_{\max}(I - W)}}$ and $M_{y,k} = 1 - \frac{\sqrt{\delta}\alpha_{y,k}}{1 - \frac{\sqrt{\delta}\alpha_{y,k}}{\lambda_{\max}(I - W)}}$. We have

$$\gamma_{x,k} < \frac{2 - 2\sqrt{\delta}\alpha_{x,k}}{\lambda_{\max}(I - W)} \Rightarrow \frac{\gamma_{x,k}\lambda_{\max}(I - W)}{2} < 1 - \sqrt{\delta}\alpha_{x,k}$$

$$\Rightarrow 1 - \frac{\gamma_{x,k}\lambda_{\max}(I - W)}{2} > \sqrt{\delta}\alpha_{x,k}$$

$$\Rightarrow \frac{\sqrt{3}\alpha_{x,k}}{1 - \frac{\gamma_{x,k}\lambda_{\max}(I - W)}{2}} < 1.$$ \hspace{1cm} (140)

Moreover, $\frac{1}{\gamma_{x,k}\lambda_{\max}(I - W)} > 0$. Therefore, $M_{x,k} \in (0,1)$. The feasibility of $M_{y,k}$ can be proved similarly.
Feasibility of \(\frac{1-b_{x,k}}{M_{x,k}}\) and \(\frac{1-b_{y,k}}{M_{y,k}}\). Observe that
\[
1 - \frac{b_{x,k}}{M_{x,k}} = \frac{1 - b_{x,k}}{1 - \sqrt{\delta} b_{x,k}} = \frac{1 - b_{x,k}}{1 - \sqrt{b_{x,k}} \lambda_{\max}(I - W)} = (1 - b_{x,k}) \left(1 - \frac{\sqrt{\delta} b_{x,k}}{2} \lambda_{\max}(I - W)\right) \\
= (1 - b_{x,k})(1 - \frac{b_{x,k}}{1 + \delta}) - \frac{b_{x,k}}{1 + \delta} \lambda_{\max}(I - W) - \sqrt{\delta} b_{x,k} \\
= (1 - b_{x,k}) \left(1 - \frac{b_{x,k}}{1 + \delta}\right) - \frac{b_{x,k}}{1 + \delta} \lambda_{\max}(I - W) - \sqrt{\delta} b_{x,k}
\]
(141).

Let \(\theta = 1 - b_{x,k}\), \(\sigma = 1 - \frac{b_{x,k}}{1 + \delta}\) and \(\chi = \frac{\sqrt{\delta} b_{x,k}}{1 + \delta}\). We now show that \(\chi \leq b - \alpha\).

\[
\chi = \frac{\sqrt{\delta} b_{x,k}}{1 + \delta} \\
\leq \frac{b_{x,k}}{2} \\
= b_{x,k} \left(1 - \frac{1}{4(1 + \delta)}\right) - b_{x,k} \left(1 - \frac{1}{4(1 + \delta)}\right) \\
= b_{x,k} \left(1 - \frac{1}{4(1 + \delta)}\right) + b_{x,k} \left(\frac{1}{2} - b_{x,k} + \frac{b_{x,k}}{4(1 + \delta)}\right) \\
\leq b_{x,k} \left(1 - \frac{1}{4(1 + \delta)}\right) + b_{x,k} \left(\frac{1}{2} - b_{x,k} + \frac{b_{x,k}}{2}\right) \\
= b_{x,k} \left(1 - \frac{1}{4(1 + \delta)}\right)
\]
(142)

Notice that \(0 \leq \chi \leq \sigma - \theta\), and \(\sigma \theta \leq (\theta + \chi)(\sigma - \chi)\), i.e.,
\[
(1 - b_{x,k}) \left(1 - \frac{b_{x,k}}{4(1 + \delta)}\right) \leq \left(1 - b_{x,k} + \frac{\sqrt{\delta} b_{x,k}}{1 + \delta}\right) \left(1 - \frac{b_{x,k}}{4(1 + \delta)} - \frac{\sqrt{\delta} b_{x,k}}{1 + \delta}\right)
\]
(143).

Substituting the above inequality in (141), we obtain
\[
1 - \frac{b_{x,k}}{M_{x,k}} \leq \left(1 - b_{x,k} + \frac{\sqrt{\delta} b_{x,k}}{1 + \delta}\right) \left(1 - \frac{b_{x,k}}{4(1 + \delta)} - \frac{\sqrt{\delta} b_{x,k}}{1 + \delta}\right) \\
= 1 - b_{x,k} + \frac{\sqrt{\delta} b_{x,k}}{1 + \delta} \\
= 1 - \left(1 - \frac{\sqrt{\delta}}{1 + \delta}\right) b_{x,k} \\
\leq 1 - \frac{b_{x,k}}{2} \\
\leq 1 - \left(1 - \frac{1}{\sqrt{2}}\right) \frac{1}{8k^2} \frac{1}{2^{k/2}},
\]
(144)

where the last step follows from (119). Notice that \(1 - \frac{1}{\sqrt{2}}\) \(\frac{1}{8k^2} \frac{1}{2^{k/2}} \in (0, 1)\) and hence \(\frac{1 - b_{x,k}}{M_{x,k}} \in (0, 1)\). Similarly, we obtain
\[
1 - \frac{b_{y,k}}{M_{y,k}} \leq 1 - \left(1 - \frac{1}{\sqrt{2}}\right) \frac{1}{8k^2} \frac{1}{2^{k/2}}.
\]
(145)

We now prove another intermediate result that shows the convergence behavior of \(E[\Phi_{k,t+1}]\).

**Lemma E.6.** Let \(\Phi_{k,t+1}\) and \(p\) be as defined in (152). Under the settings of Lemma E.3
\[
E[\Phi_{k,t+1}] \leq \left(1 - \frac{p}{2^{k/2}}\right)^{t+1} E[\Phi_{k,0}] + \frac{m(\sigma^2 + \sigma'^2)}{8\rho L^2} \frac{1}{2^{k/2}},
\]
(146)

for all \(t \geq 0\).
E.5 Proof of Lemma E.6

First, we show that $\rho_k \leq 1 - \frac{\rho}{2k^2}$, where these quantities are defined in (152). To prove this relation, we first simplify the terms $1 - \frac{\gamma_k}{\gamma_{k-1}} \lambda_m - (I - W)$ and $1 - \alpha_{x,k}$ appearing in the definition of $\rho_k$:

$$1 - \frac{\gamma_{x,k}}{2} \lambda_{m-1}(I - W) = 1 - \frac{b_{x,k}}{4(1 + \delta_0)^2 \lambda_{\max}(I - W)}$$

$$\leq 1 - \frac{1}{4(1 + \delta_0)^2 \lambda_{\max}(I - W)} \left( 1 - \frac{1}{\sqrt{2}} \right) \frac{1}{4\kappa_\gamma^2 2^{k/2}}$$

$$= 1 - \left( 1 - \frac{1}{\sqrt{2}} \right) \frac{1}{16(1 + \delta_0)^2 \kappa_\gamma^2 2^{k/2}}. \quad (147)$$

Similarly, we obtain

$$1 - \frac{\gamma_{y,k}}{2} \lambda_{m-1}(I - W) \leq 1 - \left( 1 - \frac{1}{\sqrt{2}} \right) \frac{1}{16(1 + \delta_0)^2 \kappa_\gamma^2 2^{k/2}}. \quad (148)$$

Consider

$$1 - \alpha_{x,k} = 1 - \frac{b_{x,k}}{1 + \delta} \leq 1 - \frac{1}{1 + \delta} \left( 1 - \frac{1}{\sqrt{2}} \right) \frac{1}{4\kappa_\gamma^2 2^{k/2}}$$

$$= 1 - \left( 1 - \frac{1}{\sqrt{2}} \right) \frac{1}{4\kappa_\gamma^2 2^{k/2}}. \quad (149)$$

$$1 - \alpha_{y,k} = 1 - \frac{b_{y,k}}{1 + \delta} \leq 1 - \frac{1}{1 + \delta} \left( 1 - \frac{1}{\sqrt{2}} \right) \frac{1}{4\kappa_\gamma^2 2^{k/2}}.$$  \hspace{1cm} (150)

Let us recall $\rho_k$:

$$\rho_k = \max \left\{ 1 - \frac{b_{x,k}}{M_x \kappa_\gamma}, 1 - \frac{b_{y,k}}{M_y \kappa_\gamma}, 1 - \frac{\gamma_{x,k}}{2} \lambda_{m-1}(I - W), 1 - \frac{\gamma_{y,k}}{2} \lambda_{m-1}(I - W), 1 - \alpha_{x,k}, 1 - \alpha_{y,k} \right\}$$

$$\leq \max \left\{ 1 - \left( 1 - \frac{1}{\sqrt{2}} \right) \frac{1}{16(1 + \delta_0)^2 \kappa_\gamma^2 2^{k/2}}, 1 - \left( \frac{1}{\sqrt{2}} \right) \frac{1}{1 + \delta} \left( 1 - \frac{1}{\sqrt{2}} \right) \frac{1}{4\kappa_\gamma^2 2^{k/2}} \right\}$$

$$= 1 - \min \left\{ 1 - \left( 1 - \frac{1}{\sqrt{2}} \right) \frac{1}{16(1 + \delta_0)^2 \kappa_\gamma^2 2^{k/2}}, 1 - \left( \frac{1}{\sqrt{2}} \right) \frac{1}{1 + \delta} \left( 1 - \frac{1}{\sqrt{2}} \right) \frac{1}{4\kappa_\gamma^2 2^{k/2}} \right\}$$

$$= 1 - \rho \frac{\kappa_\gamma^2}{2k^2}, \quad (151)$$

where $\rho$ is defined as

$$\rho = \min \left\{ \left( 1 - \frac{1}{\sqrt{2}} \right) \frac{1}{16(1 + \delta_0)^2 \kappa_\gamma^2 2^{k/2}}, 1 - \left( \frac{1}{\sqrt{2}} \right) \frac{1}{1 + \delta} \left( 1 - \frac{1}{\sqrt{2}} \right) \frac{1}{4\kappa_\gamma^2 2^{k/2}} \right\}. \quad (152)$$

Using Lemma E.5 we have

$$E[\Phi_{k,t+1}] \leq \rho_k E[\Phi_{k,t}] + 2m s_k^2 (\sigma_x^2 + \sigma_y^2) \leq \left( 1 - \frac{\rho}{2k^2} \right) E[\Phi_{k,t}] + 2m s_k^2 (\sigma_x^2 + \sigma_y^2)$$

$$\Rightarrow a_k E[\Phi_{k,t}] + C_1 s_k^2,$$  \hspace{1cm} (153)

where $a_k = 1 - \frac{\rho}{2k^2}$ and $C_1 = 2m(\sigma_x^2 + \sigma_y^2)$. We now unroll the above recursion to obtain

$$E[\Phi_{k,t+1}] \leq a_k^{t+1} E[\Phi_{k,0}] + \sum_{l=0}^t a_k^{t-l} C_1 s_k^2 = a_k^{t+1} E[\Phi_{k,0}] + C_1 s_k^2 a_k^t \sum_{l=0}^t a_k^l$$

$$= a_k^{t+1} E[\Phi_{k,0}] + C_1 s_k^2 a_k^t \frac{a_k^{t+1} - 1}{a_k - 1}$$

$$\leq a_k^{t+1} E[\Phi_{k,0}] + C_1 s_k^2 a_k^t \frac{a_k - 1}{a_k - 1} = a_k^{t+1} E[\Phi_{k,0}] + C_1 s_k^2 \frac{a_k - 1}{a_k - 1}$$

$$= a_k^{t+1} E[\Phi_{k,0}] + C_1 s_k^2 \frac{1}{1 - a_k} = \left( 1 - \frac{\rho}{2k^2} \right)^{t+1} E[\Phi_{k,0}] + C_1 s_k^2 2^{k/2} \frac{1}{\rho}$$

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Using $C_1 = 2m(\sigma_x^2 + \sigma_y^2)$ we have
\[
E [\Phi_{k,t+1}] \leq \left(1 - \frac{\rho}{2^{k/2}}\right)^{t+1} E [\Phi_{k,0}] + 2m(\sigma_x^2 + \sigma_y^2) \frac{1}{16L^2\kappa_f^2} 2^{k/2} \frac{1}{\rho} 
= \left(1 - \frac{\rho}{2^{k/2}}\right)^{t+1} E [\Phi_{k,0}] + \frac{m(\sigma_x^2 + \sigma_y^2)}{8\rho L^2\kappa_f^2} \frac{1}{2^{k/2}}.
\] (154)

\section{Proof of Theorem 5.1}

This proof is based on several intermediate results proved in Appendices C-E. Hence it would be useful to refer to those results in order to appreciate the proof of Theorem 5.1.

We divide the proof of Theorem 5.1 into two parts. We first find the total number of outer iterations required by Algorithm 2 to achieve target accuracy $\epsilon$. Then we derive the total gradient computation complexity of Algorithm 2.

\subsection*{F.0.1 Total Number of Outer Iterations}

We have the following initializations at every $k + 1$ outer iterate:
\[
x_{k+1,0} = x_{k,t_k}, \; y_{k+1,0} = y_{k,t_k}.
\]
Therefore,
\[
\|x_{k+1,0} - 1x^*\|^2 + \|y_{k+1,0} - 1y^*\|^2 = \|x_{k,t_k} - 1x^*\|^2 + \|y_{k,t_k} - 1y^*\|^2
\leq \frac{1}{\min\{M_x,k, M_y,k\}} \left(M_x,k \|x_{k,t_k} - 1x^*\|^2 + M_y,k \|y_{k,t_k} - 1y^*\|^2\right)
\leq \frac{1}{M_k} \Phi_{k,t_k}.
\] (156)

By taking total expectation on both sides and using Lemma E.6 we obtain
\[
E \|x_{k+1,0} - 1x^*\|^2 + E \|y_{k+1,0} - 1y^*\|^2 \leq \frac{1}{M_k} \left(1 - \frac{\rho}{2^{k/2}}\right)^{t_k} E [\Phi_{k,0}] + \frac{m(\sigma_x^2 + \sigma_y^2)}{8\rho L^2\kappa_f^2} \frac{1}{2^{k/2}}
\leq E [\Phi_{k,0}] \left(1 - \frac{\rho}{2^{k/2}}\right)^{t_k} + \frac{m(\sigma_x^2 + \sigma_y^2)}{8\rho L^2\kappa_f^2} \frac{1}{2^{k/2}}.
\] (157)

We now focus on bounding $\Phi_{k,0}$ in terms of $\|x_{k,0} - 1x^*\|^2$ and $\|y_{k,0} - 1y^*\|^2$. Recall from (152) that
\[
\Phi_{k,0} = M_x,k \|x_{k,0} - 1x^*\|^2 + \frac{2\delta^2}{\gamma_{x,k}} \left\| D_{k,0}^x - D_x^* \right\|_{(I-W)}^2 + \sqrt{\delta} \left\| H_{k,0}^x - H_{x,k}^* \right\|^2 + M_y,k \|y_{k,0} - 1y^*\|^2 + \frac{2\delta^2}{\gamma_{y,k}} \left\| D_{k,0}^y - D_y^* \right\|_{(I-W)}^2 + \sqrt{\delta} \left\| H_{k,0}^y - H_{y,k}^* \right\|^2.
\] (158)

We now bound the various terms appearing on the r.h.s. of the above equation. First,
\[
\left\| H_{k,0}^x - H_{x,k}^* \right\|^2 = \left\| x_{k,0} - 1(x^* - \frac{s_k}{m} \nabla_x f(z^*)) \right\|^2 
\leq 2 \left\| x_{k,0} - 1x^* \right\|^2 + \frac{2\delta^2}{m} \left\| \nabla_x f(z^*) \right\|^2
\] (160)

Moreover,
\[
\left\| H_{k,0}^y - H_{y,k}^* \right\|^2 = \left\| y_{k,0} - 1(y^* + \frac{s_k}{m} \nabla_y f(z^*)) \right\|^2 
\leq 2 \left\| y_{k,0} - 1y^* \right\|^2 + \frac{2\delta^2}{m} \left\| \nabla_y f(z^*) \right\|^2.
\] (161)

We now have
\[
\left\| D_{k,0}^x - D_x^* \right\|_{(I-W)}^2 = \| (I - J) \nabla_x F(1z^*) \|_{(I-W)}^2 = C_2.
\] (162)
\[
\left\| D_{t,0}^y - D_t^y \right\|^2_{(I-W)^t} = \left\| (I - J) \nabla_y F(1z^*) \right\|^2_{(I-W)^t}
\]
(163)

Therefore,
\[
\frac{2s^2}{\gamma_{x,k}} \left\| D_{t,0}^y - D_t^y \right\|^2_{(I-W)^t} = \frac{4(1 + \delta)^2 \lambda_{\max}(I - W)s^2 b_{x,k}}{b_{x,k} C_2} 
\]
(164)

Substituting the above inequality in (157), we obtain
\[
E \leq \frac{(1 + \delta)^2 \lambda_{\max}(I - W)}{b_{x,k} 16L^2 \kappa_j^2 2^k} C_2
\]
\[
= \frac{(1 + \delta)^2 \lambda_{\max}(I - W)}{b_{x,k} 4L^2 \kappa_j^2 2^k} C_2
\]
\[
\leq \frac{(1 + \delta)^2 4k_j^2 2^k}{4L^2 \kappa_j^2 2^k C_2} \left(1 - \frac{1}{\sqrt{2}}\right)
\]
\[
= \frac{2(1 + \delta)^2 C_2}{L^2 2^k \left(1 - \frac{1}{\sqrt{2}}\right)}. \tag{165}
\]

Similarly, we have
\[
\frac{2s^2}{\gamma_{y,k}} \left\| D_{t,0}^y - D_t^y \right\|^2_{(I-W)^t} \leq \frac{2(1 + \delta)^2 C_3}{L^2 2^k \left(1 - \frac{1}{\sqrt{2}}\right)}. \tag{166}
\]

On substituting the above inequality in (157), we obtain
\[
E \| z_{t+1,0} - 1z^* \|^2 \leq \frac{1}{M_k} \left(1 - \frac{\rho}{2k^2}\right)^t_k \left( (M_{x,k} + 2\sqrt{\delta}) E \| x_{t,0} - 1x^* \|^2 + (M_{y,k} + 2\sqrt{\delta}) E \| y_{t,0} - 1y^* \|^2 \right)
\]
\[
+ \frac{1}{M_k} \left(1 - \frac{\rho}{2k^2}\right)^t_k \left( \frac{\sqrt{\delta}}{8ML^2 \kappa_j^2 2^k} \left( \| \nabla_x f(z^*) \|^2 + \| \nabla_y f(z^*) \|^2 \right) + \frac{2(1 + \delta)^2 (C_2 + C_3)}{L^2 2^k \left(1 - \frac{1}{\sqrt{2}}\right)} \right)
\]
\[
+ \frac{m(\sigma_x^2 + \sigma_y^2)}{8M_k \rho L^2 \kappa_j^2 2^k} \frac{1}{2k^2}. \tag{167}
\]

We have
\[
t_k = \frac{1}{-\log \left(1 - \frac{\rho}{2k^2}\right)} \max \left\{ \log \left( \frac{3M_{x,k} + 6\sqrt{\delta}}{M_k} \right), \log \left( \frac{3M_{y,k} + 6\sqrt{\delta}}{M_k} \right), \log \left( \frac{3}{M_k} \right) \right\}. \tag{168}
\]

Therefore,
\[
t_k \geq \frac{1}{-\log \left(1 - \frac{\rho}{2k^2}\right)} \log \left( \frac{3M_{x,k} + 6\sqrt{\delta}}{M_k} \right)
\]
\[
\Rightarrow -t_k \log \left(1 - \frac{\rho}{2k^2}\right) \geq \log \left( \frac{3M_{x,k} + 6\sqrt{\delta}}{M_k} \right) = -\log \left( \frac{M_k}{3M_{x,k} + 6\sqrt{\delta}} \right)
\]
\[
\Rightarrow t_k \log \left(1 - \frac{\rho}{2k^2}\right) \leq \log \left( \frac{M_k}{3M_{x,k} + 6\sqrt{\delta}} \right)
\]
\[
\Rightarrow \left(1 - \frac{\rho}{2k^2}\right)^{t_k} \leq \frac{M_k}{3(M_{x,k} + 2\sqrt{\delta})}. \tag{170}
\]
Moreover,

\[ t_k \geq - \log \left( 1 - \frac{\rho}{2^{k/2}} \right) \log \left( \frac{3}{M_k} \right) \]

\[ \Rightarrow -t_k \log \left( 1 - \frac{\rho}{2^{k/2}} \right) \geq \log \left( \frac{3}{M_k} \right) \]

\[ \Rightarrow t_k \log \left( 1 - \frac{\rho}{2^{k/2}} \right) \leq \log \left( \frac{M_k}{3} \right) \]

\[ \Rightarrow \left( 1 - \frac{\rho}{2^{k/2}} \right)^{t_k} \leq \frac{M_k}{3}. \quad (171) \]

By using above inequalities into (168), we obtain,

\[ E \left\| z_{k+1} - 1z^* \right\|^2 \]

\[ \leq \frac{1}{3} E \| x_{k,0} - 1x^* \|^2 + \frac{1}{3} E \| y_{k,0} - 1y^* \|^2 \]

\[ + \frac{1}{3} \left( \frac{\sqrt{\delta}}{8mL^2r_2^{12}k} \left( \| \nabla_x f(z^*) \|^2 + \| \nabla_y f(z^*) \|^2 \right) + \frac{2(1+\delta)^2(C_2+C_3)}{L2^k/2 \left( 1 - \frac{1}{r_2} \right)} \right) + \frac{m(\sigma_x^2 + \sigma_y^2)}{8M_k\rho L^2r_2^2} \frac{1}{2^k/2}, \quad (172) \]

where \( A_1 = \frac{\sqrt{\delta}}{24mL^2r_2^2} \left( \| \nabla_x f(z^*) \|^2 + \| \nabla_y f(z^*) \|^2 \right), \ A_2 = \frac{m(1+\delta)^2(C_2+C_3)}{3L^2(1-\frac{1}{12})}. \)

To proceed further, we derive lower bounds on \( M_{x,k} \) and \( M_{y,k} \).

**Lower bound on \( M_{x,k} \).** Using (120), we have

\[ b_{x,k} \leq \frac{1}{4K_xr_f} \]

\[ \Rightarrow \frac{b_{x,k}}{4(1+\delta)^2} \leq \frac{1}{16(1+\delta)^2K_xr_f} \]

\[ \Rightarrow 1 - \frac{b_{x,k}}{4(1+\delta)^2} \geq 1 - \frac{1}{16(1+\delta)^2K_xr_f} \]

\[ \Rightarrow \frac{1}{1 - \frac{b_{x,k}}{4(1+\delta)^2}} \leq \frac{1}{1 - \frac{1}{16(1+\delta)^2K_xr_f}} \quad \quad \quad \quad \quad \quad \quad (174) \]

We also have

\[ \frac{\alpha_{x,k}\sqrt{\delta}}{1 - \gamma_{x,k}r_{\max}(I-W)} \leq \frac{\alpha_{x,k}\sqrt{\delta}}{1 - \frac{1}{16(1+\delta)^2K_xr_f}} \]

\[ = \frac{b_{x,k}\sqrt{\delta}}{1 - \frac{1}{16(1+\delta)^2K_xr_f}} = \sqrt{\delta}b_{x,k}16(1+\delta)^2K_xr_f \]

\[ = \frac{\sqrt{\delta}b_{x,k}16(1+\delta)^2K_xr_f}{(1+\delta)(16(1+\delta)^2K_xr_f - 1)} \quad \quad \quad \quad \quad \quad \quad (175) \]

\[ = \frac{15(1+\delta)^2K_xr_f + (1+\delta)^2K_xr_f - 1}{15(1+\delta)^2K_xr_f} \]

\[ \leq \frac{4\sqrt{\delta}(1+\delta)}{15(1+\delta)K_xr_f}. \quad \quad \quad \quad \quad \quad \quad (176) \]

Therefore, \( M_{x,k} \) is lower bounded by

\[ M_{x,k} = 1 - \frac{\alpha_{x,k}\sqrt{\delta}}{1 - \gamma_{x,k}r_{\max}(I-W)} \geq 1 - \frac{4\sqrt{\delta}}{15(1+\delta)K_xr_f} =: \tilde{M}. \quad \quad \quad \quad \quad \quad \quad (177) \]
Consider
\[
\frac{m(\sigma^2 + \sigma_0^2)}{8M^2 \rho L^2 \kappa F^2} \frac{1}{2k/2} \leq \frac{m(\sigma^2 + \sigma_0^2)}{8M^2 \rho L^2 \kappa F^2} \frac{1}{2k/2} =: A_3 \quad \text{(178)}
\]
where \( A_3 = \frac{m(\sigma^2 + \sigma_0^2)}{8M^2 \rho L^2 \kappa F^2} \). On substituting these bounds in (168), we obtain
\[
E \| z_{k+1,0} - z^* \|^2 \leq E \| z_{k,0} - z^* \|^2 + A_1 \frac{1}{2^k} + \frac{(A_2 + A_3)}{2k/2}
\]
Using (179) recursively yields
\[
E \| z_{k+1,0} - z^* \|^2 \leq \frac{1}{3k+1} E \| z_0 - z^* \|^2 + \sum_{l=0}^{k} \frac{1}{3k+1} E \| z_0 - z^* \|^2 + \sum_{l=0}^{k} \frac{1}{3k+1} E \| z_0 - z^* \|^2 + (A_2 + A_3) \frac{1}{2^l}
\]
where \( \epsilon_k = \frac{A_1}{2^k} + \frac{(A_2 + A_3)}{2k/2} \).

By choosing \( k = K(\epsilon) = \max\{ \log_2 \left( \frac{2E \| z_0 - z^* \|^2 + A_1}{\epsilon} \right), 2 \log_2 \left( \frac{6(A_2 + A_3)}{(3\sqrt{2})^m} \right) \} \), we obtain
\[
E \| z_{K(\epsilon)+1,0} - z^* \|^2 \leq \epsilon. \quad \text{(181)}
\]
F.0.2 Gradient Computation Complexity

The gradient computation complexity $T_{\text{grad}}(\epsilon)$ is bounded by the following computation

$$T_{\text{grad}}(\epsilon) = \sum_{k=0}^{K(\epsilon)-1} t_k$$

$$= \sum_{k=0}^{K(\epsilon)-1} \frac{1}{-\log \left( 1 - \frac{\rho}{2^{|k|}} \right)} \max \left\{ \log \left( \frac{3M_{x,k} + 6\sqrt{\delta}}{M_k} \right), \log \left( \frac{3M_{v,k} + 6\sqrt{\delta}}{M_k} \right), \log \left( \frac{3}{M_k} \right) \right\}$$

$$\leq \sum_{k=0}^{K(\epsilon)-1} \frac{1}{-\log \left( 1 - \frac{\rho}{2^{|k|}} \right)} \max \left\{ \log \left( \frac{3 + 6\sqrt{\delta}}{M_k} \right), \log \left( \frac{3 + 6\sqrt{\delta}}{M_k} \right), \log \left( \frac{3}{M_k} \right) \right\}$$

$$= \log \left( \frac{3 + 6\sqrt{\delta}}{M} \right) \sum_{k=0}^{K(\epsilon)-1} \frac{1}{-\log \left( 1 - \frac{\rho}{2^{|k|}} \right)}$$

$$\leq \log \left( \frac{3 + 6\sqrt{\delta}}{M} \right) \sum_{k=0}^{K(\epsilon)-1} \frac{2^{K(\epsilon)/2} - 1}{\rho}$$

$$= 5 \log \left( \frac{3 + 6\sqrt{\delta}}{M} \right) \frac{2^{K(\epsilon)/2} - 1}{\rho(\sqrt{2} - 1)}$$

$$= 5 \log \left( \frac{3 + 6\sqrt{\delta}}{M} \right) \frac{2^{K(\epsilon)/2}}{\rho(\sqrt{2} - 1)}$$

$$= 5 \log \left( \frac{3 + 6\sqrt{\delta}}{M} \right) \frac{1}{\rho(\sqrt{2} - 1)}$$

$$= 5 \log \left( \frac{3 + 6\sqrt{\delta}}{M} \right) \frac{1}{\sqrt{2} - 1}$$

$$= 5 \log \left( \frac{3 + 6\sqrt{\delta}}{M} \right) \frac{1}{\sqrt{2} - 1}$$

$$= \frac{5}{\rho(\sqrt{2} - 1)} \log \left( \frac{3 + 6\sqrt{\delta}}{M} \right)$$

$$= \frac{5}{\rho(\sqrt{2} - 1)} \log \left( \frac{3 + 6\sqrt{\delta}}{M} \right) \sqrt{2} \max \left\{ \log_2 \left( \frac{2\|z_0 - 1\| + 6A_1}{\epsilon} \right), \log_2 \left( \frac{6(A_2 + A_3)}{M(3-\sqrt{2})\epsilon} \right) \right\}$$

$$= \frac{10}{\rho(\sqrt{2} - 1)} \log \left( \frac{3 + 6\sqrt{\delta}}{M} \right) \max \left\{ \sqrt{2} \frac{\|z_0 - 1\| + 6A_1}{\epsilon}, \frac{6(A_2 + A_3)}{M(3-\sqrt{2})\epsilon} \right\}$$

$$= \frac{5}{\sqrt{2} - 1} \log \left( \frac{3 + 6\sqrt{\delta}}{M} \right) \sqrt{2} \max \left\{ \log_2 \left( \frac{2\|z_0 - 1\| + 6A_1}{\epsilon} \right), \log_2 \left( \frac{6(A_2 + A_3)}{M(3-\sqrt{2})\epsilon} \right) \right\}$$

$$= \frac{5}{\sqrt{2} - 1} \log \left( \frac{3 + 6\sqrt{\delta}}{M} \right) \sqrt{2} \max \left\{ \log_2 \left( \frac{2\|z_0 - 1\| + 6A_1}{\epsilon} \right), \log_2 \left( \frac{6(A_2 + A_3)}{M(3-\sqrt{2})\epsilon} \right) \right\}$$

Notice that above complexity does not contain $1/\hat{M}$ term because $\hat{M} \in \left[ \frac{11}{15}, 1 \right]$.  

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We begin with few intermediate results which will help us in getting the final convergence result.

Assumption G.3.

Assumption G.2.

Assume that each $\rho$ where

For theoretical analysis of Algorithm 3, we make the following smoothness assumptions on $f$

G Proofs for the Finite Sum Setting

F.1 Algorithm 2 behavior in deterministic setting

In this section, we briefly discuss that Algorithm 2 converges to the saddle point solution with linear rates when $\sigma_x = \sigma_y = 0$, where $\sigma_x$ and $\sigma_y$ are the bounds on the stochastic gradients of GSGO (see Assumption G.1). Recall the recursive relation in Lemma E.5

On substituting $k = 0$ and $\sigma_x = \sigma_y = 0$ in above inequality, we obtain

where $\rho_0$ is defined in (126). By unrolling above recursion in $t$, we get

Note that $M_{x,0} \|x_{0,t} - 1x^{*}\|^2 + M_{y,0} \|y_{0,t} - 1y^{*}\|^2 \leq \Phi_{0,t}$ from (106). Therefore, $E \|x_{0,t} - 1x^{*}\|^2 + E \|y_{0,t} - 1y^{*}\|^2 \leq \frac{E[\Phi_{0,t}]}{\min(M_{x,0}, M_{y,0})}$. Under the above settings, Algorithm 2 needs $T_{grad}(\epsilon) = \frac{1}{\log(1/\rho_0)} \log \left( \frac{\Phi_{0,0}}{\epsilon \min\{M_{x,0}, M_{y,0}\}} \right)$ gradient computations and communications to achieve $E \|x_{0,T_{grad}(\epsilon)} - 1x^{*}\|^2 + E \|y_{0,T_{grad}(\epsilon)} - 1y^{*}\|^2 \leq \epsilon$. We now write $T_{grad}(\epsilon)$ in terms of $\kappa_f, \kappa_g$ and $\delta$. Using (131), we have $\rho_0 \leq 1 - \rho$, where $\rho$ is defined in (132). Notice that

Therefore,

\[
T_{grad}(\epsilon) = 5 \left( \min \left\{ \left(1 - \frac{1}{\sqrt{2}}\right) \frac{1}{16(1 + \delta)^2 \kappa_f \kappa_g}, \frac{1}{1 + \delta} \left(1 - \frac{1}{\sqrt{2}}\right) \frac{1}{4\kappa_f^2} \right\} \right)^{-1} 
\]

In this section, we prove all results related to convergence analysis of Algorithm 3 in Section 6

For theoretical analysis of Algorithm 3 we make the following smoothness assumptions on $f_{ij}$ particular to the finite sum setting.

**Assumption G.1.** Assume that each $f_{ij}(x, y)$ is $L_{xx}$ smooth in $x$; for every fixed $y$, $\|\nabla_x f_{ij}(x, y) - \nabla_x f_{ij}(x', y)\| \leq L_{xx} \|x - x'\|$, $\forall x, x' \in \mathbb{R}^d_x$

**Assumption G.2.** Assume that each $f_{ij}(x, y)$ is $L_{yy}$ smooth in $y$ i.e., for every fixed $x$, $\|\nabla_y f_{ij}(x, y) + \nabla_y f_{ij}(x, y')\| \leq L_{yy} \|y - y'\|$, $\forall y, y' \in \mathbb{R}^d_y$

**Assumption G.3.** Assume that each $\nabla_x f_{ij}(x, y)$ is $L_{xy}$ Lipschitz in $y$ i.e., for every fixed $x$, $\|\nabla_x f_{ij}(x, y) - \nabla_x f_{ij}(x, y')\| \leq L_{xy} \|y - y'\|$, $\forall y, y' \in \mathbb{R}^d_y$

**Assumption G.4.** Assume that each $\nabla_y f_{ij}(x, y)$ is $L_{yx}$ Lipschitz in $x$ i.e., for every fixed $y$, $\|\nabla_y f_{ij}(x, y) - \nabla_y f_{ij}(x', y)\| \leq L_{yx} \|x - x'\|$, $\forall x, x' \in \mathbb{R}^d_x$

We begin with few intermediate results which will help us in getting the final convergence result.
Lemma G.5. Let \( \{x_t\}, \{y_t\} \) be the sequences generated by Algorithm 3 with \( G^t_i \) and \( G^y_i \) obtained from SVRG. Then, under Assumptions 4.2 and Assumptions 8.2, the following holds for all \( t \geq 1 \):

\[
\begin{align*}
E & \left[ ||x_t - 1x^* - sG^t_i + s\nabla_x F(1x^*, 1y^*)||^2 + E ||y_t - 1y^* + sG^y_i - s\nabla_y F(1x^*, 1y^*)||^2 \right] \\
& \leq \left( 1 - \mu_x s \right) ||x_t - 1x^*||^2 + \left( 1 - s\mu_y \right) ||y_t - 1y^*||^2 \\
& - 2s \sum_{i=1}^m V_{f_i,y_t}(x^*, x_t) - 2s F(1x^*, y_t) - 2s F(x_t, 1y^*) - 2s F(1x^*, 1y^*) \\
& + \frac{2s^2}{n^2p_{\min}} \sum_{i=1}^m \sum_{j=1}^n \left| \left| \nabla_x f_{i,j}(z^*_i) - \nabla_x f_{i,j}(z^*) \right| \right|^2 + \frac{2s^2}{n^2p_{\min}} \sum_{i=1}^m \sum_{j=1}^n \left| \left| \nabla_y f_{i,j}(z^*_i) - \nabla_y f_{i,j}(z^*) \right| \right|^2.
\end{align*}
\]

where \( p_{\min} \) is defined as \( \min_{i,j} \{ p_{ij} \} \).

G.1 Proof of Lemma G.5.

We begin the proof by bounding the primal (\( x \)) and dual (\( y \)) updates on the l.h.s. of (186) separately. In particular, we show that

\[
\begin{align*}
E & \left[ ||x_t - 1x^* - sG^t_i + s\nabla_x F(1x^*, 1y^*)||^2 \right] \\
& \leq \left( 1 - \mu_x s \right) ||x_t - 1x^*||^2 \\
& + \frac{2s^2}{n^2p_{\min}} \sum_{i=1}^m \sum_{j=1}^n \left| \left| \nabla_x f_{i,j}(z^*_i) - \nabla_x f_{i,j}(z^*) \right| \right|^2 + \frac{2s^2}{n^2p_{\min}} \sum_{i=1}^m \sum_{j=1}^n \left| \left| \nabla_y f_{i,j}(z^*_i) - \nabla_y f_{i,j}(z^*) \right| \right|^2 \\
& - 2s \sum_{i=1}^m V_{f_i,y_t}(x^*, x_t)
\end{align*}
\]

and

\[
\begin{align*}
E & \left[ ||y_t - 1y^* + sG^y_i - s\nabla_y F(1x^*, 1y^*)||^2 \right] \\
& \leq \left( 1 - s\mu_y \right) ||y_t - 1y^*||^2 + 2s \left( -F(x_t, 1y^*) - F(z_t) + F(x_t, 1y^*) - F(1x^*, 1y^*) \right) \\
& + \frac{2s^2}{n^2p_{\min}} \sum_{i=1}^m \sum_{j=1}^n \left| \left| \nabla_y f_{i,j}(z^*_i) - \nabla_y f_{i,j}(z^*) \right| \right|^2.
\end{align*}
\]

Observe that (187) and (188) are similar, and we only prove (187) in Section G.1.1 below. Adding (187) and (188), we obtain

\[
\begin{align*}
E & \left[ ||x_t - 1x^* - sG^t_i + s\nabla_x F(1x^*, 1y^*)||^2 + E ||y_t - 1y^* + sG^y_i - s\nabla_y F(1x^*, 1y^*)||^2 \right] \\
& \leq \left( 1 - \mu_x s \right) ||x_t - 1x^*||^2 + \left( 1 - s\mu_y \right) ||y_t - 1y^*||^2 \\
& - 2s \sum_{i=1}^m V_{f,i,y_t}(x^*, x_t) - 2s \sum_{i=1}^m V_{f,i,x_t}(y^*, y_t) \\
& + \frac{2s^2}{n^2p_{\min}} \sum_{i=1}^m \sum_{j=1}^n \left| \left| \nabla_x f_{i,j}(z^*_i) - \nabla_x f_{i,j}(z^*) \right| \right|^2 + \frac{2s^2}{n^2p_{\min}} \sum_{i=1}^m \sum_{j=1}^n \left| \left| \nabla_y f_{i,j}(z^*_i) - \nabla_y f_{i,j}(z^*) \right| \right|^2.
\end{align*}
\]

To finish the proof of Lemma G.5, we bound the last two terms of (190) as shown in Section G.1.2.

G.1.1 Proof of (187)

First, consider the primal update term
\[ E \| x_t - 1x^* - sG_t^2 + s\nabla_x F(1x^*, 1y^*) \|^2 \]
\[ = \sum_{i=1}^{m} E \left\| x_t^i - x^* - sG_t^{i,x} + s\nabla_x f_i(x^*, y^*) \right\|^2 \]
\[ = \sum_{i=1}^{m} \| x_t^i - x^* \|^2 + s^2 \sum_{i=1}^{m} E \left\| G_t^{i,x} - \nabla_x f_i(x^*, y^*) \right\|^2 \]
\[ - 2s \sum_{i=1}^{m} E \left( x_t^i - x^*, G_t^{i,x} - \nabla_x f_i(x^*, y^*) \right) \]
\[ = \sum_{i=1}^{m} \| x_t^i - x^* \|^2 + s^2 \sum_{i=1}^{m} E \left\| \frac{1}{np_{i,t}} \left( \nabla_x f_i(\bar{z}_t^i) - \nabla_x f_i(\bar{z}_t^i) \right) + \nabla_x f_i(\bar{z}_t^i) - \nabla_x f_i(x^*, y^*) \right\|^2 \]
\[ - 2s \sum_{i=1}^{m} E \left( x_t^i - x^*, \frac{1}{np_{i,t}} \left( \nabla_x f_i(\bar{z}_t^i) - \nabla_x f_i(\bar{z}_t^i) \right) + \nabla_x f_i(\bar{z}_t^i) - \nabla_x f_i(x^*, y^*) \right) \]. (191)

Observe that
\[ E \left[ \frac{1}{np_{i,t}} \left( \nabla_x f_i(\bar{z}_t^i) - \nabla_x f_i(\bar{z}_t^i) \right) + \nabla_x f_i(\bar{z}_t^i) - \nabla_x f_i(x^*, y^*) \right] \]
\[ = \frac{1}{n} \sum_{i=1}^{n} \nabla_x f_i(\bar{z}_t^i) - \frac{1}{n} \sum_{i=1}^{n} \nabla_x f_i(\bar{z}_t^i) + \nabla_x f_i(\bar{z}_t^i) - \nabla_x f_i(z^* ) \]
\[ = \nabla_x f_i(\bar{z}_t^i) - \nabla_x f_i(\bar{z}_t^i) + \nabla_x f_i(\bar{z}_t^i) - \nabla_x f_i(z^* ) \]
\[ = \nabla_x f_i(z^* ) - \nabla_x f_i(z^* ), \] (192)

where the first equality and second last equality follows respectively from step (1) of SVRGO and definition of \( f_i(x, y) \). Substituting the above in the last term of (191), we see that
\[ E \| x_t - 1x^* - sG_t^2 + s\nabla_x F(1x^*, 1y^*) \|^2 \]
\[ \leq \sum_{i=1}^{m} \| x_t^i - x^* \|^2 + s^2 \sum_{i=1}^{m} E \left\| \frac{1}{np_{i,t}} \left( \nabla_x f_i(\bar{z}_t^i) - \nabla_x f_i(\bar{z}_t^i) \right) + \nabla_x f_i(\bar{z}_t^i) - \nabla_x f_i(x^*, y^*) \right\|^2 \]
\[ - 2s \sum_{i=1}^{m} \left( x_t^i - x^*, \nabla_x f_i(\bar{z}_t^i) - \nabla_x f_i(z^* ) \right) . \] (193)

Substituting (112) (i.e., Bregman distance) and (114) (i.e., strong convexity of \( f \)) in (193), we obtain
\[ E \| x_t - 1x^* - sG_t^2 + s\nabla_x F(1x^*, 1y^*) \|^2 \]
\[ \leq \sum_{i=1}^{m} \| x_t^i - x^* \|^2 + s^2 \sum_{i=1}^{m} E \left\| \frac{1}{np_{i,t}} \left( \nabla_x f_i(\bar{z}_t^i) - \nabla_x f_i(\bar{z}_t^i) \right) + \nabla_x f_i(\bar{z}_t^i) - \nabla_x f_i(x^*, y^*) \right\|^2 \]
\[ - 2s \sum_{i=1}^{m} \left( - f_i(x^*, y_t^i) + f_i(z_t^i) + V_{f_i,y_t^i}(x^*, x_t^i) \right) + 2s \sum_{i=1}^{m} \left( f_i(x_t^i, y^*) - f_i(z^*) - \frac{\mu_x}{2} \| x_t^i - x^* \|^2 \right) \]
\[ = \sum_{i=1}^{m} \| x_t^i - x^* \|^2 + s^2 \sum_{i=1}^{m} E \left\| \frac{1}{np_{i,t}} \left( \nabla_x f_i(\bar{z}_t^i) - \nabla_x f_i(\bar{z}_t^i) \right) + \nabla_x f_i(\bar{z}_t^i) - \nabla_x f_i(x^*, y^*) \right\|^2 \]
\[ + 2s \left( F(1x^*, y_t^i) - F(z_t^i) \right) - 2s \sum_{i=1}^{m} V_{f_i,y_t^i}(x^*, x_t^i) + 2s \left( F(x_t^i, 1y^*) - F(1z^*) \right) - s\mu_x \| x_t - 1x^* \|^2 \]
\[ = (1 - \mu_x s) \| x_t - 1x^* \|^2 + s^2 \sum_{i=1}^{m} E \left\| \frac{1}{np_{i,t}} \left( \nabla_x f_i(\bar{z}_t^i) - \nabla_x f_i(\bar{z}_t^i) \right) + \nabla_x f_i(\bar{z}_t^i) - \nabla_x f_i(x^*, y^*) \right\|^2 \]
\[ - 2s \sum_{i=1}^{m} V_{f_i,y_t^i}(x^*, x_t^i) + 2s \left( F(1x^*, y_t^i) - F(z_t^i) + F(x_t^i, 1y^*) - F(1z^*) \right) . \] (194)
Now we bound the second term on the r.h.s. of \( \|u_t - 1x^*\|^2 \) and \( \|y_t - 1y^*\|^2 \) as follows:

\[
\begin{align*}
\sum_{i=1}^{m} E \left[ \frac{1}{np_{ij}} \left\| \nabla_x f_i(z_i^*) - \nabla_x f_i(z_i) \right\| \right] \\
= \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} \left[ \frac{1}{np_{ij}} \left\| \nabla_x f_i(z_i^*) - \nabla_x f_i(z_i) \right\| \right] \\
\leq \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} \left\| \nabla_x f_i(z_i^*) - \nabla_x f_i(z_i) \right\|^2 \\
+ \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} \left[ \frac{1}{np_{ij}} \left\| \nabla_x f_i(z_i^*) - \nabla_x f_i(z_i) \right\| \right] \\
\leq \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} \left\| \nabla_x f_i(z_i^*) - \nabla_x f_i(z_i) \right\|^2 \\
+ \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} \left[ \frac{1}{np_{ij}} \left\| \nabla_x f_i(z_i^*) - \nabla_x f_i(z_i) \right\| \right] \\
\leq \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} \left\| \nabla_x f_i(z_i^*) - \nabla_x f_i(z_i) \right\|^2 \\
\leq \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} \left\| \nabla_x f_i(z_i^*) - \nabla_x f_i(z_i) \right\|^2 ,
\end{align*}
\]

where \( p_{\min} = \min_{i,j} \{ p_{ij} \} \). Let \( u_i = \left\{ \frac{v}_{x f_i(z_i)} - \nabla_x f_i(z_i^*) \; : \; l \in \{1,2,\ldots,n\} \right\} \) be a random variable with probability distribution \( P_i = \{ p_{il} : l \in \{1,2,\ldots,n\} \} \).

\[
\begin{align*}
E[u_i] &= E \left[ \frac{\nabla_x f_i(z_i^*) - \nabla_x f_i(z_i)}{np_{il}} \right] \\
&= \frac{1}{n} \sum_{l=1}^{n} \nabla_x f_i(z_i^*) - \frac{1}{n} \sum_{l=1}^{n} \nabla_x f_i(z_i) \\
&= \nabla_x f_i(z_i^*) - \nabla_x f_i(z_i) .
\end{align*}
\]

We know that \( E \|u_i - E[u_i]\|^2 \leq E \|u_i\|^2 \). Therefore,

\[
\begin{align*}
E \left[ \left\| \frac{\nabla_x f_i(z_i^*) - \nabla_x f_i(z_i^*)}{np_{ij}} - \nabla_x f_i(z_i) \right\|^2 \right] \\
\leq E \left[ \left\| \frac{\nabla_x f_i(z_i^*) - \nabla_x f_i(z_i)}{np_{ij}} \right\|^2 \right] \\
= \frac{1}{n^2} \sum_{j=1}^{n} \left\| \frac{\nabla_x f_i(z_i^*) - \nabla_x f_i(z_i)}{p_{ij}} \right\|^2 \\
= \frac{1}{n^2} \sum_{j=1}^{n} \frac{1}{p_{ij}} \left\| \nabla_x f_i(z_i^*) - \nabla_x f_i(z_i) \right\|^2 \\
\leq \frac{1}{n^2 p_{\min}} \sum_{j=1}^{n} \left\| \nabla_x f_i(z_i^*) - \nabla_x f_i(z_i) \right\|^2 .
\end{align*}
\]
By substituting the above inequality in (195), we obtain
\[
\sum_{i=1}^{m} E \left[ \frac{1}{np_{il}} \left( \nabla x f_i(z_i^*) - \nabla x f_i(z^*) \right) + \nabla x f_i(z_i^*) - \nabla x f_i(z^*) \right] \leq \frac{2s^2}{n^2 \rho_{\text{min}}} \sum_{i=1}^{m} \sum_{j=1}^{n} \left\| \nabla x f_{ij}(z_i^*) - \nabla x f_{ij}(z^*) \right\|^2 + \frac{2s^2}{n^2 \rho_{\text{min}}} \sum_{i=1}^{m} \sum_{j=1}^{n} \left\| \nabla x f_{ij}(z_i^*) - \nabla x f_{ij}(z^*) \right\|^2.
\]
(198)

Substituting this inequality in (194), we obtain (187).

G.1.2 Finishing the Proof of Lemma G.5

We now compute upper bounds on the last two terms present in (190) using smoothness assumptions. First, observe that
\[
\|\nabla x f_{ij}(z_i^*) - \nabla x f_{ij}(z^*)\|^2 + \|\nabla y f_{ij}(z_i^*) - \nabla y f_{ij}(z^*)\|^2
\]
\[
= \|\nabla x f_{ij}(z_i^*) - \nabla x f_{ij}(z^*)\|^2 + \|\nabla y f_{ij}(z_i^*) - \nabla y f_{ij}(z^*)\|^2
\]
\[
+ \|\nabla y f_{ij}(z_i^*) - \nabla y f_{ij}(z^*)\|^2 + \|\nabla y f_{ij}(z_i^*) - \nabla y f_{ij}(z^*)\|^2
\]
\[
\leq 2 \left( \|\nabla x f_{ij}(z_i^*) - \nabla x f_{ij}(z^*)\|^2 + \|\nabla x f_{ij}(z_i^*) - \nabla x f_{ij}(z^*)\|^2 \right)
\]
\[
+ 2 \left( \|\nabla y f_{ij}(z_i^*) - \nabla y f_{ij}(z^*)\|^2 + \|\nabla y f_{ij}(z_i^*) - \nabla y f_{ij}(z^*)\|^2 \right)
\]
\[
\leq 2L_{xy} V_{f_{ij}}(x_i^*, x_i^*) + 2L_{y}^2 \left\| y_i^* - y^* \right\|^2 + 4L_{yy} V_{f_{ij}, x_i^*}(y_i^*, y_i^*) + 2L_{yx}^2 \left\| x_i^* - x^* \right\|^2,
\]
(199)
where the last inequality follows from Proposition C.3, Proposition C.4 and Assumptions G.3-G.4.

Adding up the above inequality for \( j = 1 \) to \( n \) and using (28)-(29), we obtain
\[
\sum_{j=1}^{n} \left( \|\nabla x f_{ij}(z_i^*) - \nabla x f_{ij}(z^*)\|^2 + \|\nabla y f_{ij}(z_i^*) - \nabla y f_{ij}(z^*)\|^2 \right)
\]
\[
\leq 2L_{xy} \sum_{j=1}^{n} V_{f_{ij}}(x_i^*, x_i^*) + 4L_{yy} \sum_{j=1}^{n} V_{f_{ij}, x_i^*}(y_i^*, y_i^*) + 2nL_{xy} \left\| y_i^* - y^* \right\|^2 + 2nL_{yx}^2 \left\| x_i^* - x^* \right\|^2
\]
\[
= 2L_{xy} \sum_{j=1}^{n} (f_{ij}(x_i^*, y_i^*) - f_{ij}(x_i^*, y_i^*) - \langle \nabla x f_{ij}(x_i^*, y_i^*), x_i^* - x^* \rangle)
\]
\[
+ 4L_{yy} \sum_{j=1}^{n} (-f_{ij}(x_i^*, y_i^*) + f_{ij}(x_i^*, y_i^*) - \langle -\nabla y f_{ij}(x_i^*, y_i^*), y_i^* - y^* \rangle)
\]
\[
+ 2nL_{xy}^2 \left\| y_i^* - y^* \right\|^2 + 2nL_{yx}^2 \left\| x_i^* - x^* \right\|^2
\]
\[
= 4L_{xy} \left( \left( \frac{1}{n} \sum_{j=1}^{n} f_{ij}(x, y) \right) - \frac{1}{n} \sum_{j=1}^{n} f_{ij}(x, y) - \langle \nabla x f_{ij}(x, y), x - x^* \rangle \right)
\]
\[
+ 4L_{yy} \left( \left( \frac{1}{n} \sum_{j=1}^{n} f_{ij}(x, y) \right) - \frac{1}{n} \sum_{j=1}^{n} f_{ij}(x, y) - \langle -\nabla y f_{ij}(x, y), y - y^* \rangle \right) + 2nL_{xy}^2 \left\| y_i^* - y^* \right\|^2 + 2nL_{yx}^2 \left\| x_i^* - x^* \right\|^2
\]
\[
= 4nL_{xy} V_{f_i, y}(x^*, x_i^*) + 4nL_{yy} V_{f_i, x}(y^*, y_i^*) + 2nL_{xy} \left\| y_i^* - y^* \right\|^2 + 2nL_{yx}^2 \left\| x_i^* - x^* \right\|^2,
\]
(200)
where the second last step follows from the structure of \( f_i(x, y) = \frac{1}{n} \sum_{j=1}^{n} f_{ij}(x, y) \). Therefore,
\[
\frac{2s^2}{n^2 \rho_{\text{min}}} \sum_{i=1}^{m} \sum_{j=1}^{n} \left( \|\nabla x f_{ij}(z_i^*) - \nabla x f_{ij}(z^*)\|^2 + \|\nabla y f_{ij}(z_i^*) - \nabla y f_{ij}(z^*)\|^2 \right)
\]
\[
\leq \frac{8s^2}{n^2 \rho_{\text{min}}} \sum_{i=1}^{m} V_{f_i, y}(x^*, x_i^*) + \frac{8s^2}{n^2 \rho_{\text{min}}} \sum_{i=1}^{m} V_{f_i, x}(y^*, y_i^*) + \frac{4s^2}{n^2 \rho_{\text{min}}} \left\| y_i^* - y^* \right\|^2 + \frac{4s^2}{n^2 \rho_{\text{min}}} \left\| x_i^* - x^* \right\|^2.
\]
(201)
Similarly, we bound the last term of (190) as
\[
\frac{2s^2}{n^2p_{\min}} \sum_{i=1}^{m} \sum_{j=1}^{n} \left( \| \nabla_x f_{ij}(z_i^t) - \nabla_x f_{ij}(z^*) \|^2 + \| \nabla_y f_{ij}(z_i^t) - \nabla_y f_{ij}(z^*) \|^2 \right)
\leq \frac{4s^2(L_{xx}^2 + L_{yx}^2)}{np_{\min}} \| \tilde{x}_t - 1x^* \|^2 + \frac{4s^2(L_{yy}^2 + L_{xy}^2)}{np_{\min}} \| \tilde{y}_t - 1y^* \|^2.
\] (202)

On substituting (201) and (202) in (190), we obtain
\[
E \| x_t - 1x^* - sG_t^x + s\nabla_x F(1x^*, 1y^*) \|^2 + E \| y_t - 1y^* + sG_t^y - s\nabla_y F(1x^*, 1y^*) \|^2
\leq \left( 1 - \mu_x s + \frac{4s^2L_{xx}^2}{np_{\min}} \right) \| x_t - 1x^* \|^2 + \left( 1 - \mu_y s + \frac{4s^2L_{xy}^2}{np_{\min}} \right) \| y_t - 1y^* \|^2
\leq \left( 1 - \mu_x s + \frac{4s^2L_{xx}^2}{np_{\min}} \right) \| x_t - 1x^* \|^2 + \left( 1 - \mu_y s + \frac{4s^2L_{xy}^2}{np_{\min}} \right) \| y_t - 1y^* \|^2
\leq \frac{4s^2(L_{xx}^2 + L_{yx}^2)}{np_{\min}} \| \tilde{x}_t - 1x^* \|^2 + \frac{4s^2(L_{yy}^2 + L_{xy}^2)}{np_{\min}} \| \tilde{y}_t - 1y^* \|^2.
\] (203)

proving Lemma G.5.

We now have the following corollary.

**Corollary G.6.** Let \( s = \frac{\mu_{p_{\min}}}{24L} \). Then under the settings of Lemma G.5
\[
E \| x_t - 1x^* - sG_t^x + s\nabla_x F(1x^*, 1y^*) \|^2 + E \| y_t - 1y^* + sG_t^y - s\nabla_y F(1x^*, 1y^*) \|^2
\leq \left( 1 - \mu_x s + \frac{4s^2L_{xx}^2}{np_{\min}} \right) \| x_t - 1x^* \|^2 + \left( 1 - \mu_y s + \frac{4s^2L_{xy}^2}{np_{\min}} \right) \| y_t - 1y^* \|^2
\leq \frac{4s^2(L_{xx}^2 + L_{yx}^2)}{np_{\min}} \| \tilde{x}_t - 1x^* \|^2 + \frac{4s^2(L_{yy}^2 + L_{xy}^2)}{np_{\min}} \| \tilde{y}_t - 1y^* \|^2.
\] (204)

G.2 Proof of Corollary G.6

From the statement of the corollary, we have \( s = \frac{\mu_{p_{\min}}}{24L} \leq \frac{np_{\min}}{24L} < \frac{np_{\min}}{4L} \leq \frac{np_{\min}}{4L_{xx}} \). This implies that
\[
\frac{4sL_{xx}}{np_{\min}} \leq 1 \text{ i.e., } \frac{8s^2L_{xx}}{np_{\min}} \leq 2s.
\] (205)

Notice that \( V_{f_{t},y_{t}}(x^*, x_t^t) \geq 0 \). Therefore, \( 2s - \frac{8s^2L_{xx}}{np_{\min}} \sum_{i=1}^{m} V_{f_{t},y_{t}}(x^*, x_t^t) \geq 0 \). We also have \( s \leq \frac{np_{\min}}{8s^2L_{xx}} \) because \( L = \max \{ L_{xx}, L_{yy}, L_{xy}, L_{yx} \} \). Therefore, \( \frac{4s^2L_{xx}}{np_{\min}} \leq 2s \). Due to the concavity of \( f_t(x, y) \) in \( y, V_{f_{t},x_{t}}(y^*, y_t^t) \) is nonnegative. Therefore, \( 2s - \frac{8s^2L_{xx}}{np_{\min}} \sum_{i=1}^{m} V_{f_{t},x_{t}}(y^*, y_t^t) \geq 0 \). By substituting these lower bounds in (186), we get the desired result.

**Parameters setup**

Let \( p_{\min} = \min_{i,j} \{ p_{ij} \} \). We define the following quantities which are instrumental in simplifying the bounds and in Algorithm 3 implementation.
\[
\hat{c}_x := \frac{8s^2(L_{xx}^2 + L_{yx}^2)}{np_{\min}p}, \quad \hat{c}_y := \frac{8s^2(L_{yy}^2 + L_{xy}^2)}{np_{\min}p},
\] (207)
\[
b_x := s\mu_x - \frac{4s^2L_{xx}^2}{np_{\min}} - \hat{c}_xp, \quad b_y := s\mu_y - \frac{4s^2L_{xy}^2}{np_{\min}} - \hat{c}_yp,
\] (208)
\[
\alpha_x := \frac{b_x}{(1 + \delta)}, \quad \alpha_y := \frac{b_y}{(1 + \delta)}.
\] (209)
\[
\gamma_x := \min \left\{ \frac{b_x}{4\sqrt{\delta}(1 + \delta)\lambda_{\max}(I - W)}, \frac{1}{4(1 + \delta)\lambda_{\max}(I - W)} \right\},
\] (210)
\[
\gamma_y := \min \left\{ \frac{b_y}{4\sqrt{\delta}(1 + \delta)\lambda_{\max}(I - W)}, \frac{1}{4(1 + \delta)\lambda_{\max}(I - W)} \right\},
\] (211)
\[ \Phi_t := M_x \| x_t - 1x^* \|^2 + \frac{2s^2}{\gamma_x} \| D_t^x - D_{x^*}^x \|^2_{(I-W)t} + \sqrt{\delta} \| H_t^x - H_x^* \|^2 \] 

\[ + M_y \| y_t - 1y^* \|^2 + \frac{2s^2}{\gamma_y} \| D_t^y - D_{y^*}^y \|^2_{(I-W)y} + \sqrt{\delta} \| H_t^y - H_y^* \|^2, \]

\[ \rho = \max \left\{ \frac{1 - b_x}{M_x}, \frac{1 - b_y}{M_y}, 1 - \frac{\gamma_x}{2} \lambda_{m-1}(I - W), 1 - \frac{\gamma_y}{2} \lambda_{m-1}(I - W), 1 - \alpha_x, 1 - \alpha_y, 1 - \frac{p}{2} \right\} \]

\[ \hat{\Phi}_t = \Phi_t + \tilde{c}_x \| \tilde{x}_t - 1x^* \|^2 + \tilde{c}_y \| \tilde{y}_t - 1y^* \|^2. \]

**Lemma G.7. Parameters Feasibility** The parameters defined in (208), (209), (210) and (211) satisfy the followings:

\[ b_x \in (0, 1), \quad b_y \in (0, 1), \] (217)

\[ \alpha_x < \min \left\{ \frac{b_x}{\sqrt{\delta}}, \frac{1}{1 + \delta} \right\}, \quad \alpha_y < \min \left\{ \frac{b_y}{\sqrt{\delta}}, \frac{1}{1 + \delta} \right\} \] (218)

\[ \gamma_x \in \left( 0, \min \left\{ \frac{2 - 2\sqrt{\delta} \alpha_x}{\lambda_{\max}(I - W)} \frac{\alpha_x - (1 + \delta) \alpha_x^2}{\lambda_{\max}(I - W)} \right\} \right), \] (219)

\[ \gamma_y \in \left( 0, \min \left\{ \frac{2 - 2\sqrt{\delta} \alpha_y}{\lambda_{\max}(I - W)} \frac{\alpha_y - (1 + \delta) \alpha_y^2}{\lambda_{\max}(I - W)} \right\} \right), \] (220)

\[ \frac{\gamma_x}{2} \lambda_{m-1}(I - W) \in (0, 1), \quad \frac{\gamma_y}{2} \lambda_{m-1}(I - W) \in (0, 1), \] (221)

\[ M_x \in (0, 1), \quad M_y \in (0, 1), \] (222)

\[ 1 - \frac{b_x}{M_x} \in (0, 1), \quad 1 - \frac{b_y}{M_y} \in (0, 1). \] (223)

Moreover,

\[ M_x \geq 1 - \frac{8b_x \sqrt{\delta}}{7(1 + \delta)} \geq 1 - \frac{4b_x}{7}, \] (224)

\[ M_y \geq 1 - \frac{8b_y \sqrt{\delta}}{7(1 + \delta)} \geq 1 - \frac{4b_y}{7}, \] (225)

\[ 1 - \frac{b_x}{M_x} < 1 - \frac{3b_x}{7}, \quad 1 - \frac{b_y}{M_y} < 1 - \frac{3b_y}{7}, \] (226)

\[ 1 - \frac{\gamma_x}{2} \lambda_{m-1}(I - W) = \begin{cases} 1 - \frac{b_x}{8\sqrt{\delta}(1 + \delta) \alpha_x} & \text{if } b_x \leq \sqrt{\delta} \\ 1 - \frac{1}{8\sqrt{\delta}(1 + \delta) \alpha_x} & \text{if } b_x > \sqrt{\delta} \end{cases}. \] (227)

### G.3 Proof of Lemma G.7

In this section, we show that the chosen parameters \( \alpha_x, \alpha_y, M_x, M_y, \gamma_x, \gamma_y \) satisfy the following conditions given in Lemma G.7:

\[ \alpha_x < \min \left\{ \frac{b_x}{\sqrt{\delta}}, \frac{1}{1 + \delta} \right\}, \quad \alpha_y < \min \left\{ \frac{b_y}{\sqrt{\delta}}, \frac{1}{1 + \delta} \right\} \]

\[ \gamma_x \in \left( 0, \min \left\{ \frac{2 - 2\sqrt{\delta} \alpha_x}{\lambda_{\max}(I - W)} \frac{\alpha_x - (1 + \delta) \alpha_x^2}{\lambda_{\max}(I - W)} \right\} \right), \gamma_y \in \left( 0, \min \left\{ \frac{2 - 2\sqrt{\delta} \alpha_y}{\lambda_{\max}(I - W)} \frac{\alpha_y - (1 + \delta) \alpha_y^2}{\lambda_{\max}(I - W)} \right\} \right), \]

\[ \frac{\gamma_x}{2} \lambda_{m-1}(I - W) \in (0, 1), \quad \frac{\gamma_y}{2} \lambda_{m-1}(I - W) \in (0, 1), \quad M_x \in (0, 1), \quad M_y \in (0, 1), \]

\[ 1 - \frac{b_x}{M_x} \in (0, 1), \quad 1 - \frac{b_y}{M_y} \in (0, 1). \] (228)

We first show that \( b_x \in (0, 1) \) and \( b_y \in (0, 1) \). From definition,

\[ b_x = s\mu_x - \frac{4s^2 \mu_x^2}{n\mu_{\min}^2} - \tilde{c}_xp < s\mu_x = \frac{\mu_{\min} \mu_x}{24L^2} \leq \frac{n\mu_{\min} \mu_x}{24L^2} = \frac{n\mu_{\min}^2 \mu_x^2}{24L^2} \leq \frac{1}{24} < 1. \] (229)
Similarly,
\[ b_y = s\mu_y - \frac{4s^2L_{yy}^2}{n\mu_{\min}^2} - \tilde{c}_yp < s\mu_y = \frac{\mu n\mu_{\min}^2}{24L^2} \leq \frac{n\mu_{\min}^2}{24L_{yy}^2} = \frac{n\mu_{\min}^2}{24\kappa_y^2} \leq \frac{1}{24} < 1. \] (230)

We now focus on the lower bound on \( b_x \) and \( b_y \).
\[ b_x = s\mu_x - \frac{4s^2L_{yx}^2}{n\mu_{\min}^2} - \tilde{c}_xp \]
\[ = s\mu_x - \frac{4s^2L_{yx}^2}{n\mu_{\min}} - \frac{8s^2(L_{xx}^2 + L_{yx}^2)}{n\mu_{\min}} \]
\[ = s\mu_x - \frac{12s^2L_{yx}^2}{n\mu_{\min}} - \frac{8s^2L_{xx}^2}{n\mu_{\min}} \]
\[ \geq s\mu - \frac{12s^2L_{yx}^2}{n\mu_{\min}} - \frac{8s^2L_{xx}^2}{n\mu_{\min}} \]
\[ = s\mu - \frac{20s^2L_{yx}^2}{n\mu_{\min}} \]
\[ = \frac{\mu^2 n\mu_{\min}}{24L^2} - \frac{\mu^2 n^2\mu_{\min}^2}{576L^4} \frac{20L^2}{n\mu_{\min}} \]
\[ = \frac{n\mu_{\min}}{24\kappa_f^2} - \frac{n\mu_{\min}}{576\kappa_f^2} \geq \frac{n\mu_{\min}}{144\kappa_f^2} > 0. \] (231)

In a similar fashion, we get \( b_y < 1 \) and
\[ b_y \geq \frac{n\mu_{\min}}{144\kappa_f^2} > 0. \] (232)

**Feasibility of \( \alpha_x \) and \( \alpha_y \).**

We have, \( 0 < b_x < 1 \). Therefore, \( \alpha_x < \frac{1}{1+\delta} \). Moreover, \( \frac{\sqrt{\delta}}{1+\delta} \leq \frac{1}{2} \) as \( \delta \in [0,1] \). Therefore, \( \alpha_x \leq \frac{b_x}{2\sqrt{\delta}} < \frac{b_x}{\sqrt{\delta}} \). Hence, \( \alpha_x < \min \left\{ \frac{b_x}{\sqrt{\delta}}, \frac{1}{1+\delta} \right\} \). Similarly, \( \alpha_y < \min \left\{ \frac{b_y}{\sqrt{\delta}}, \frac{1}{1+\delta} \right\} \) because \( b_y \in (0,1) \).

**Feasibility of \( \gamma_{x,k} \) and \( \gamma_{y,k} \).** We consider two cases to verify the feasibility of \( \gamma_x \) and \( \gamma_y \).

**Case 1: \( b_x \leq \sqrt{\delta} \).**

This gives \( \gamma_x = \frac{b_x}{4\sqrt{\delta}(1+\delta)\lambda_{\max}(I-W)} \). Consider
\[ \frac{\alpha_x - (1+\delta)\alpha_x}{\sqrt{\delta}\lambda_{\max}(I-W)} = \frac{b_x - b_x^2}{\sqrt{\delta}(1+\delta)\lambda_{\max}(I-W)}. \] (234)

The last inequality uses the relation \( \frac{\sqrt{\delta}}{1+\delta} \leq \frac{1}{2} \). Using (229), we have \( b_x \leq \frac{1}{24\kappa_f^2} < 0.75 \). This allows us to use the inequality \( 2x - 2x^2 \geq x/2 \) for all \( 0 \leq x \leq 0.75 \). Therefore,
\[ \frac{\alpha_x - (1+\delta)\alpha_x^2}{\sqrt{\delta}\lambda_{\max}(I-W)} > \frac{b_x}{4\sqrt{\delta}(1+\delta)\lambda_{\max}(I-W)} \]
\[ = \gamma_{x,k}. \] (235)

We also have
\[ \frac{2 - 2\sqrt{\delta}\alpha_x}{\lambda_{\max}(I-W)} = \left( 2 - \frac{2\sqrt{\delta}b_x}{1+\delta} \right) \frac{1}{\lambda_{\max}(I-W)} \geq \left( 2 - \frac{2\sqrt{\delta}}{1+\delta} \right) \frac{1}{\lambda_{\max}(I-W)} \]
\[ \geq \frac{1}{\lambda_{\max}(I-W)} > \frac{1}{4(1+\delta)\lambda_{\max}(I-W)} \]
\[ > \frac{1}{4\sqrt{\delta}(1+\delta)\lambda_{\max}(I-W)} \]
\[ = \gamma_{x,k}. \] (236)
where the second last inequality uses $b_x \leq \sqrt{\delta}$. We know that $b_y \in (0, 1)$. Therefore, by following similar steps, the chosen $\gamma_y$ is also feasible.

**Case II: $b_x > \sqrt{\delta}$**

This gives $\gamma_x = \frac{1}{4(1+\delta)\lambda_{\max}(I-W)}$.

\[
\frac{\alpha_x - (1+\delta)\alpha^2_x}{\sqrt{\delta}\lambda_{\max}(I-W)} = \frac{b_x - b^2_x}{\sqrt{\delta}(1+\delta)\lambda_{\max}(I-W)} \geq \frac{b_x}{4\sqrt{(1+\delta)\lambda_{\max}(I-W)}} > \frac{1}{4(1+\delta)\lambda_{\max}(I-W)} = \gamma_x.
\]

Consider

\[
\frac{2 - 2\sqrt{\delta}\alpha_x}{\lambda_{\max}(I-W)} = \left(2 - \frac{2\sqrt{\delta}b_x}{1+\delta}\right) \frac{1}{\lambda_{\max}(I-W)} \geq \left(2 - \frac{2\sqrt{\delta}}{1+\delta}\right) \frac{1}{\lambda_{\max}(I-W)} \geq \frac{1}{\lambda_{\max}(I-W)} > \frac{1}{4(1+\delta)\lambda_{\max}(I-W)} = \gamma_x.
\]

Therefore, $\gamma_x < \min\left\{ \frac{\alpha_x - (1+\delta)\alpha^2_x}{\sqrt{\delta}\lambda_{\max}(I-W)}, \frac{2 - 2\sqrt{\delta}\alpha_x}{\lambda_{\max}(I-W)} \right\}$.

As $\gamma_x < \frac{2 - 2\sqrt{\delta}/\alpha_x}{\lambda_{\max}(I-W)} < \frac{2}{\lambda_{\max}(I-W)}$, notice that $\lambda_{m-1}(I-W) < \lambda_{\max}(I-W)$ Therefore,

\[
\frac{\gamma_x}{2} \lambda_{m-1}(I-W) < \frac{\gamma_x}{2} \lambda_{\max}(I-W) < 1.
\]

Similarly, $\frac{\gamma_y}{2} \lambda_{m-1}(I-W) < 1$.

**Feasibility of $M_x$ and $M_y$.**

Recall $M_x = 1 - \frac{\sqrt{\delta}b_x}{\lambda_{\max}(I-W)}$ and $M_y = 1 - \frac{\sqrt{\delta}b_y}{\lambda_{\max}(I-W)}$. We have

\[
\frac{\gamma_x}{2} \lambda_{\max}(I-W) < 1 - \sqrt{\delta}\alpha_x
\]

\[
1 - \frac{\gamma_x}{2} \lambda_{\max}(I-W) > \sqrt{\delta}\alpha_x
\]

\[
\frac{\sqrt{\delta}\alpha_x}{1 - \frac{\gamma_x}{2} \lambda_{\max}(I-W)} > 1.
\]

Moreover, $\frac{\sqrt{\delta}b_x}{\lambda_{\max}(I-W)} > 0$. Therefore, $M_x \in (0, 1)$. Similar steps follow to prove the feasibility of $M_y$.

**Feasibility of $\frac{1-b_x}{M_x}$ and $\frac{1-b_y}{M_y}$.**

We derive upper bounds on $\frac{1-b_x}{M_x}$ and $\frac{1-b_y}{M_y}$ to verify the feasibility. We divide the derivation into two cases.

**Case I: $b_x \leq \sqrt{\delta}$**
This implies that
\[
\gamma_x = \frac{b_x}{4\sqrt{\delta}(1 + \delta)\lambda_{\max}(I - W)}
\]  
(241)
\[
\frac{\gamma_x}{2\lambda_{\max}(I - W)} = \frac{b_x}{8\sqrt{\delta}(1 + \delta)}.
\]  
(242)

Recall \(M_x\):
\[
M_x = 1 - \frac{\sqrt{\delta}\alpha_x}{1 - \frac{2\delta}{\sqrt{\lambda_{\max}(I - W)}}}
\]
\[
= 1 - \frac{\sqrt{\delta}b_x}{8\sqrt{\delta}(1 + \delta)}
\]
\[
= 1 - \frac{\sqrt{\delta}b_x \times 8\sqrt{\delta}(1 + \delta)}{(1 + \delta) \left(8\sqrt{\delta}(1 + \delta) - b_x\right)}
\]
\[
= 1 - \frac{8\delta b_x}{(8\sqrt{\delta}(1 + \delta) - b_x)}
\]
\[
= 1 - \frac{8\delta}{b_x}.
\]  
(243)
We know that \(\frac{\sqrt{\delta}}{b_x} \geq 1\). Therefore, \(\frac{\sqrt{\delta}(1 + \delta)}{b_x} > 1\) which in turn implies that
\[
\frac{8\sqrt{\delta}(1 + \delta)}{b_x} - 1 > \frac{8\sqrt{\delta}(1 + \delta)}{b_x} - \frac{\sqrt{\delta}(1 + \delta)}{b_x}
\]
\[
= \frac{7\sqrt{\delta}(1 + \delta)}{b_x}
\]
\[
\frac{1}{\frac{8\sqrt{\delta}(1 + \delta)}{b_x} - 1} < \frac{b_x}{7\sqrt{\delta}(1 + \delta)}.
\]  
(244)

By using above relation in (243), we obtain
\[
M_x \geq 1 - \frac{8\delta b_x}{7\sqrt{\delta}(1 + \delta)} = 1 - \frac{8b_x\sqrt{\delta}}{7(1 + \delta)}
\]  
(245)
\[
\geq 1 - \frac{8b_x}{7} \times \frac{1}{2} = 1 - \frac{4b_x}{7},
\]  
(246)
where the second last inequality uses \(\frac{\sqrt{\delta}}{b_x} \leq \frac{1}{2}\).
\[
\frac{1 - b_x}{M_x} = 1 + \frac{1 - b_x}{M_x} - 1 \leq 1 + \frac{1 - b_x}{1 - \frac{8b_x\sqrt{\delta}}{7(1 + \delta)}} - 1
\]
\[
= 1 + \frac{1 - b_x - 1 + \frac{8b_x\sqrt{\delta}}{7(1 + \delta)}}{1 - \frac{8b_x\sqrt{\delta}}{7(1 + \delta)}} = 1 - \frac{b_x - \frac{8b_x\sqrt{\delta}}{7(1 + \delta)}}{1 - \frac{8b_x\sqrt{\delta}}{7(1 + \delta)}}
\]
\[
= 1 - \frac{7b_x(1 + \delta) - 8b_x\sqrt{\delta}}{7(1 + \delta) - 8b_x\sqrt{\delta}} = 1 - \frac{7(1 + \delta) - 8\sqrt{\delta}}{b_x - 8\sqrt{\delta}}
\]
\[
\leq 1 - \frac{7(1 + \delta) - \frac{8(1 + \delta)}{2}}{b_x - 8\sqrt{\delta}} = 1 - \frac{3(1 + \delta)}{b_x - 8\sqrt{\delta}}
\]
\[
< 1 - \frac{3(1 + \delta)}{b_x - 8\sqrt{\delta}} < 1.
\]  
(247)
Similarly, we obtain

\[ M_y \geq 1 - \frac{8by\sqrt{\delta}}{t(1 + \delta)} \geq 1 - \frac{4by}{t} \]  \hspace{1cm} (248)

\[ 1 - b_y \frac{M_y}{1 - 3b_y} < \frac{1}{7}. \]  \hspace{1cm} (249)

**Case II**: \( b_x > \sqrt{\delta} \).

\[ \gamma_x = \frac{1}{4(1 + \delta)\lambda_{\text{max}}(I - W)}. \]  \hspace{1cm} (250)

We have

\begin{align*}
M_x &= 1 - \frac{\sqrt{\delta}\alpha_x}{1 - \frac{\gamma_x}{2} \lambda_{\text{max}}(I - W)} \\
&= 1 - \frac{\sqrt{\delta}b_x}{1 - \frac{\gamma_x}{2} \lambda_{\text{max}}(I - W)} \\
&= 1 - \frac{\sqrt{\delta}b_x \times 8(1 + \delta)}{(1 + \delta)(8(1 + \delta) - 1)} \\
&= 1 - \frac{8\sqrt{\delta}b_x}{8(1 + \delta) - 1}.
\end{align*}  \hspace{1cm} (251)

As \( 8(1 + \delta) - 1 > 8(1 + \delta) - 1 - \delta = 7(1 + \delta) \). Therefore,

\[ M_x \geq 1 - \frac{8\sqrt{\delta}b_x}{7(1 + \delta)}. \]  \hspace{1cm} (252)

Notice that above lower bound matches with lower bound in (245). Therefore, by following steps similar to Case I, we obtain

\begin{align*}
1 - b_x \frac{M_x}{1 - 3b_x} &< 1 - \frac{3b_x}{7}, \quad \text{and} \\
1 - b_y \frac{M_y}{1 - 3b_y} &< 1 - \frac{3b_y}{7}. \quad \text{(253) and (254)}
\end{align*}

**Feasibility of** \( 1 - b_x \lambda_{m-1}(I - W) \) and \( 1 - \frac{\gamma_x}{2} \lambda_{m-1}(I - W) \).

If \( b_x \leq \sqrt{\delta} \), then \( \gamma_x = \frac{b_x}{4\sqrt{\delta}(1 + \delta)\lambda_{\text{max}}(I - W)} \).

\begin{align*}
1 - \frac{\gamma_x}{2} \lambda_{m-1}(I - W) &= 1 - \frac{b_x}{8\sqrt{\delta}(1 + \delta)\lambda_{\text{max}}(I - W)} \lambda_{m-1}(I - W) \\
&= 1 - \frac{b_x}{8\sqrt{\delta}(1 + \delta)\kappa_g}.
\end{align*}  \hspace{1cm} (255)

If \( b_x > \sqrt{\delta} \), then \( \gamma_x = \frac{b_x}{4(1 + \delta)\lambda_{\text{max}}(I - W)} \).

\begin{align*}
1 - \frac{\gamma_x}{2} \lambda_{m-1}(I - W) &= 1 - \frac{b_x}{8(1 + \delta)\lambda_{\text{max}}(I - W)} \lambda_{m-1}(I - W) \\
&= 1 - \frac{b_x}{8(1 + \delta)\kappa_g}.
\end{align*}  \hspace{1cm} (256)

We now have a result on recursive relationship on \( E[\tilde{\Phi}_t] \).

**Lemma G.8.** Let \( \{x_t\}_t, \{y_t\}_t \) be the sequences generated by Algorithm 3. Suppose Assumptions 4.1-4.5 and Assumptions G.1-G.4 hold. Then for every \( t \geq 1 \):

\[ E[\tilde{\Phi}_t] \leq \rho^t \tilde{\Phi}_0, \]  \hspace{1cm} (258)

where \( \rho \) is defined in equation (215).
G.4 Proof of Lemma G.8

Iterates \( \{x_t, y_t\} \) of Algorithm 3 are obtained by calling Algorithm 1 at iterate \( t - 1 \). Therefore, Lemma D.1 and Lemma D.2 also hold for Algorithm 3. Adding inequalities (54) and (103) (Lemma D.1 and Lemma D.2), we have

\[
M_x E\left\| x_{t+1} - 1x^* \right\|^2 + \frac{2s^2}{\gamma_x} E\left\| D_t^x - D^*_x \right\|_{(I-W)^t}^2 + \sqrt{\delta} E\left\| H_{t+1}^x - H^*_x \right\|^2 + \\
M_y E\left\| y_{t+1} - 1y^* \right\|^2 + \frac{2s^2}{\gamma_y} E\left\| D_t^y - D^*_y \right\|_{(I-W)^t}^2 + \sqrt{\delta} E\left\| H_{t+1}^y - H^*_y \right\|^2
\]

\[
\leq \left\| x_t - 1x^* - sG_t^x + s\nabla_x F(1z^*) \right\|^2 + \frac{2s^2}{\gamma_x} \left( 1 - \frac{\gamma_x}{2} \lambda_{m-1} (I - W) \right) \left\| D_t^x - D^*_x \right\|^2_{(I-W)^t} + \\
\sqrt{\delta} (1 - \alpha_x) \left\| H_t^x - H^*_x \right\|^2 + \\
\left\| y_t - 1y^* + sG_t^y - s\nabla_y F(1z^*) \right\|^2 + \frac{2s^2}{\gamma_y} \left( 1 - \frac{\gamma_y}{2} \lambda_{m-1} (I - W) \right) \left\| D_t^y - D^*_y \right\|^2_{(I-W)^t} + \\
\sqrt{\delta} (1 - \alpha_y) \left\| H_t^y - H^*_y \right\|^2.
\]

By the definition of \( \Phi_t \), the above inequality can be rewritten as

\[
E[\Phi_{t+1}]
\]

\[
\leq \left\| x_t - 1x^* - sG_t^x + s\nabla_x F(1z^*) \right\|^2 + \frac{2s^2}{\gamma_x} \left( 1 - \frac{\gamma_x}{2} \lambda_{m-1} (I - W) \right) \left\| D_t^x - D^*_x \right\|^2_{(I-W)^t} + \\
\sqrt{\delta} (1 - \alpha_x) \left\| H_t^x - H^*_x \right\|^2 + \\
E\left\| y_t - 1y^* + sG_t^y - s\nabla_y F(1z^*) \right\|^2 + \frac{2s^2}{\gamma_y} \left( 1 - \frac{\gamma_y}{2} \lambda_{m-1} (I - W) \right) \left\| D_t^y - D^*_y \right\|^2_{(I-W)^t} + \\
\sqrt{\delta} (1 - \alpha_y) \left\| H_t^y - H^*_y \right\|^2 + \frac{2s^2}{\gamma_x} \left( 1 - \frac{\gamma_x}{2} \lambda_{m-1} (I - W) \right) \left\| D_t^x - D^*_x \right\|^2_{(I-W)^t} + \\
\sqrt{\delta} (1 - \alpha_y) \left\| H_t^y - H^*_y \right\|^2.
\]

Taking conditional expectation on stochastic gradient at \( t \)-th step on both sides of above inequality and applying Tower property, we obtain

\[
E[\Phi_{t+1}]
\]

\[
\leq \left( 1 - \mu_x s + \frac{4s^2L^2_{yx}}{n_{p\min}} \right) \left\| x_t - 1x^* \right\| + \left( 1 - s\mu_y + \frac{4s^2L^2_{yy}}{n_{p\min}} \right) \left\| y_t - 1y^* \right\| + \\
\frac{4s^2(L^2_{yx} + L^2_{yy})}{n_{p\min}} \left\| x_t - 1x^* \right\|^2 + \frac{4s^2(L^2_{yx} + L^2_{yy})}{n_{p\min}} \left\| y_t - 1y^* \right\|^2 + \\
\frac{2s^2}{\gamma_x} \left( 1 - \frac{\gamma_x}{2} \lambda_{m-1} (I - W) \right) \left\| D_t^x - D^*_x \right\|^2_{(I-W)^t} + \\
\frac{2s^2}{\gamma_y} \left( 1 - \frac{\gamma_y}{2} \lambda_{m-1} (I - W) \right) \left\| D_t^y - D^*_y \right\|^2_{(I-W)^t} + \\
\sqrt{\delta} (1 - \alpha_x) \left\| H_t^x - H^*_x \right\|^2 + \sqrt{\delta} (1 - \alpha_y) \left\| H_t^y - H^*_y \right\|^2.
\]

The last step holds due to Corollary G.6. From SVRG oracle, we have

\[
\left\| \tilde{x}_{t+1} - x^* \right\|^2 = \frac{\left\| x_t - 1x^* \right\|^2}{\gamma_x} \text{ with probability } p, \quad \text{ and } \\
\left\| \tilde{y}_{t+1} - y^* \right\|^2 = \frac{\left\| y_t - 1y^* \right\|^2}{\gamma_y} \text{ with probability } p
\]

\[
\left\| \tilde{x}_{t+1} - x^* \right\|^2 = \frac{\left\| x_t - 1x^* \right\|^2}{\gamma_x} \text{ with probability } 1 - p, \quad \text{ and } \\
\left\| \tilde{y}_{t+1} - y^* \right\|^2 = \frac{\left\| y_t - 1y^* \right\|^2}{\gamma_y} \text{ with probability } 1 - p.
\]

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Therefore,
\[ E \| \tilde{x}_{t+1} - 1x^* \|^2 + E \| \tilde{y}_{t+1} - 1y^* \|^2 \]
\[ = \sum_{i=1}^{m} E \| \tilde{x}_{i,t+1} - x^* \|^2 + \sum_{i=1}^{m} E \| \tilde{y}_{i,t+1} - y^* \|^2 \]
\[ = \sum_{i=1}^{m} \left( p \| x_{i,t} - x^* \|^2 + (1 - p) \| x_{i,t} - x^* \|^2 \right) + \sum_{i=1}^{m} \left( p \| y_{i,t} - y^* \|^2 + (1 - p) \| y_{i,t} - y^* \|^2 \right) \]
\[ = p \| x_t - 1x^* \|^2 + (1 - p) \| \tilde{x}_t - 1x^* \|^2 + p \| y_t - 1y^* \|^2 + (1 - p) \| \tilde{y}_t - 1y^* \|^2 . \]

Using above equality and (261), we obtain
\[ E [\Phi_{t+1}] + \tilde{c}_x E \| \tilde{x}_{t+1} - 1x^* \|^2 + \tilde{c}_y E \| \tilde{y}_{t+1} - 1y^* \|^2 \]
\[ \leq \left( 1 - \mu_x s + \frac{4s^2 L_{xy}}{\eta_{\text{min}}} + \tilde{c}_x p \right) \| x_t - 1x^* \|^2 + \left( 1 - s \mu_y + \frac{4s^2 L_{xy}}{\eta_{\text{min}}} + \tilde{c}_y p \right) \| y_t - 1y^* \|^2 \]
\[ + \frac{\tilde{c}_x (1 - p)}{n_{\text{min}}} \left( \frac{4s^2 (L_{xx}^2 + L_{xy}^2)}{n_{\text{min}}} \right) \| \tilde{x}_t - 1x^* \|^2 + \frac{\tilde{c}_y (1 - p)}{n_{\text{min}}} \left( \frac{4s^2 (L_{yy}^2 + L_{xy}^2)}{n_{\text{min}}} \right) \| \tilde{y}_t - 1y^* \|^2 \]
\[ + \frac{2s^2}{\gamma_x} \left( 1 - \frac{\gamma_x}{2} \lambda_{m-1}(I - W) \right) \| D_x^t - D_x^* \|^2_{(I - W)^t} + \frac{2s^2}{\gamma_y} \left( 1 - \frac{\gamma_y}{2} \lambda_{m-1}(I - W) \right) \| D_y^t - D_y^* \|^2_{(I - W)^t} \]
\[ + \sqrt{\delta} (1 - \alpha_x) \| H_x^t - H_x^* \|^2 + \sqrt{\delta} (1 - \alpha_y) \| H_y^t - H_y^* \|^2 . \]

We have \( \tilde{c}_x = \frac{8s^2(L_{xx}^2 + L_{xy}^2)}{n_{\text{min}} p} \) in (263) is
\[ \tilde{c}_x (1 - p) + \frac{4s^2 (L_{xx}^2 + L_{xy}^2)}{n_{\text{min}}} = \tilde{c}_x \left( 1 - p + \frac{4s^2 (L_{xx}^2 + L_{xy}^2)}{n_{\text{min}} p} \right) \]
\[ = \tilde{c}_x \left( 1 - p + \frac{4s^2 (L_{xx}^2 + L_{xy}^2)}{8s^2 (L_{xx}^2 + L_{xy}^2)} \right) \]
\[ = \tilde{c}_x \left( 1 - p + \frac{p}{2} \right) \]
\[ = \tilde{c}_x \left( 1 - p + \frac{p}{2} \right) , \]
and the coefficient of \( \| \tilde{y}_t - 1y^* \|^2 \) in (263) is
\[ \tilde{c}_y (1 - p) + \frac{4s^2 (L_{yy}^2 + L_{xy}^2)}{n_{\text{min}}} = \tilde{c}_y \left( 1 - p + \frac{4s^2 (L_{yy}^2 + L_{xy}^2)}{n_{\text{min}} p} \right) \]
\[ = \tilde{c}_y \left( 1 - p + \frac{4s^2 (L_{yy}^2 + L_{xy}^2)}{8s^2 (L_{yy}^2 + L_{xy}^2)} \right) \]
\[ = \tilde{c}_y \left( 1 - p + \frac{p}{2} \right) \]
\[ = \tilde{c}_y \left( 1 - p + \frac{p}{2} \right) . \]

Substituting the above simplified coefficients into (261), we see that
\[ E [\Phi_{t+1}] + \tilde{c}_x E \| \tilde{x}_{t+1} - 1x^* \|^2 + \tilde{c}_y E \| \tilde{y}_{t+1} - 1y^* \|^2 \]
\[ \leq \left( 1 - \mu_x s + \frac{4s^2 L_{xy}}{\eta_{\text{min}}} + \tilde{c}_x p \right) \| x_t - 1x^* \|^2 + \left( 1 - s \mu_y + \frac{4s^2 L_{xy}}{\eta_{\text{min}}} + \tilde{c}_y p \right) \| y_t - 1y^* \|^2 \]
\[ + \tilde{c}_x \left( 1 - p + \frac{p}{2} \right) \| \tilde{x}_t - 1x^* \|^2 + \tilde{c}_y \left( 1 - p + \frac{p}{2} \right) \| \tilde{y}_t - 1y^* \|^2 \]
\[ + \frac{2s^2}{\gamma_x} \left( 1 - \frac{\gamma_x}{2} \lambda_{m-1}(I - W) \right) \| D_x^t - D_x^* \|^2_{(I - W)^t} + \frac{2s^2}{\gamma_y} \left( 1 - \frac{\gamma_y}{2} \lambda_{m-1}(I - W) \right) \| D_y^t - D_y^* \|^2_{(I - W)^t} \]
\[ + \sqrt{\delta} (1 - \alpha_x) \| H_x^t - H_x^* \|^2 + \sqrt{\delta} (1 - \alpha_y) \| H_y^t - H_y^* \|^2 . \]
By taking total expectation on both sides and using the definition of $b_x$ and $b_y$, we obtain
\[
E[\Phi_{t+1}] + \tilde{c}_x E[\tilde{x}_{t+1} - 1x^*]^2 + \tilde{c}_y E[\tilde{y}_{t+1} - 1y^*]^2
\leq (1 - b_x) E[xx_t - 1x^*]^2 + (1 - b_y) E[y_t - 1y^*]^2
+ \frac{2s^2}{\gamma_x} \left(1 - \frac{\gamma_x}{2} \lambda_{m-1}(I - W)\right) E[|D_x^r - D_x^s|_{t-W}]^2 + \frac{2s^2}{\gamma_y} \left(1 - \frac{\gamma_y}{2} \lambda_{m-1}(I - W)\right) E[|D_y^r - D_y^s|_{t-W}]^2
+ \sqrt{\delta} (1 - \alpha_x) E[|H_x^r - H_x^s|^2] + \sqrt{\delta} (1 - \alpha_y) E[|H_y^r - H_y^s|^2]^2 = \left(\frac{1-b_x}{M_x}\right) M_x E[xx_t - 1x^*]^2 + \left(\frac{1-b_y}{M_y}\right) M_y E[y_t - 1y^*]^2
+ \tilde{c}_x \left(1 - \frac{p}{2}\right) E[\tilde{x}_t - 1x^*]^2 + \tilde{c}_y \left(1 - \frac{p}{2}\right) E[\tilde{y}_t - 1y^*]^2
+ \frac{2s^2}{\gamma_x} \left(1 - \frac{\gamma_x}{2} \lambda_{m-1}(I - W)\right) E[|D_x^r - D_x^s|_{t-W}]^2 + \frac{2s^2}{\gamma_y} \left(1 - \frac{\gamma_y}{2} \lambda_{m-1}(I - W)\right) E[|D_y^r - D_y^s|_{t-W}]^2
+ \sqrt{\delta} (1 - \alpha_x) E[|H_x^r - H_x^s|^2] + \sqrt{\delta} (1 - \alpha_y) E[|H_y^r - H_y^s|^2]^2
= \max \left\{ \frac{1-b_x}{M_x}, \frac{1-b_y}{M_y}, 1 - \frac{\gamma_x}{2} \lambda_{m-1}(I - W), 1 - \frac{\gamma_y}{2} \lambda_{m-1}(I - W), 1 - \alpha_x, 1 - \alpha_y, 1 - \frac{p}{2} \right\}
\times \left( E[\Phi_{t}] + \tilde{c}_x E[\tilde{x}_t - 1x^*] + \tilde{c}_y E[\tilde{y}_t - 1y^*] \right)
= \rho \left( E[\Phi_{t}] + \tilde{c}_x E[\tilde{x}_t - 1x^*] + \tilde{c}_y E[\tilde{y}_t - 1y^*] \right),
\]
where
\[
\rho = \max \left\{ \frac{1-b_x}{M_x}, \frac{1-b_y}{M_y}, 1 - \frac{\gamma_x}{2} \lambda_{m-1}(I - W), 1 - \frac{\gamma_y}{2} \lambda_{m-1}(I - W), 1 - \alpha_x, 1 - \alpha_y, 1 - \frac{p}{2} \right\}.
\]
By the definition of $\tilde{\Phi}_t = \Phi_{t} + \tilde{c}_x \|x_t - 1x^*\|^2 + \tilde{c}_y \|y_t - 1y^*\|^2$, (268) reduces to
\[
E[\tilde{\Phi}_{t+1}] \leq \rho E[\tilde{\Phi}_t].
\]
Therefore, $E[\tilde{\Phi}_{t+1}] \leq \rho^{t+1} \tilde{\Phi}_0$.

H Proof of Theorem 6.1

This proof is based on several intermediate results proved in Appendices C, E. Hence it would be useful to refer to those results in order to appreciate the proof of Theorem 6.1.

Observe that
\[
E[xx_{t+1} - 1x^*]^2 + E[y_{t+1} - 1y^*]^2 \leq \frac{1}{\min\{M_x, M_y\}} \left( M_x E[xx_{t+1} - 1x^*]^2 + M_y E[y_{t+1} - 1y^*]^2 \right)
\leq \frac{1}{\min\{M_x, M_y\}} E[\Phi_{t+1}]
\leq \frac{1}{M} \rho^{t+1} \Phi_0,
\]
where last inequality follows from Lemma 6.8 and $M := \min\{M_x, M_y\}$. Hence, (271) reduces to
\[
E[x_{T(\epsilon)} - 1x^*]^2 + E[y_{T(\epsilon)} - 1y^*]^2 \leq \epsilon,
\]
for $T(\epsilon) = \frac{1}{-\log \rho} \log \left( \frac{\tilde{\Phi}_0}{M\epsilon} \right)$.

H.0.1 Gradient Computation Complexity

Recall
\[
\rho = \max \left\{ \frac{1-b_x}{M_x}, \frac{1-b_y}{M_y}, 1 - \frac{\gamma_x}{2} \lambda_{m-1}(I - W), 1 - \frac{\gamma_y}{2} \lambda_{m-1}(I - W), 1 - \alpha_x, 1 - \alpha_y, 1 - \frac{p}{2} \right\}.
\]
Using Lemma \ref{lemma:rho-bound}, \( \rho \) can be upper bounded as
\[
\rho \leq \max \left\{ 1 - \frac{3b_x}{7}, 1 - \frac{3b_y}{7}, 1 - \frac{b_x}{8\sqrt{(1 + \delta)\kappa_g}}, 1 - \frac{1}{8(1 + \delta)\kappa_g}, 1 - \frac{b_y}{8\sqrt{(1 + \delta)\kappa_g}} \right\}, \tag{275}
\]
\[
1 - \frac{b_x}{1 + \delta}, 1 - \frac{b_y}{1 + \delta}, 1 - \frac{p}{2} \right\}. \tag{276}
\]
Using (232) and (233), we have
\[
1 - \frac{3b_x}{7} \leq 1 - \frac{3n_{p_{\min}}}{144\kappa_f^2}, 1 - \frac{3b_y}{7} \leq 1 - \frac{3n_{p_{\min}}}{144\kappa_f^2}, 1 - \frac{b_x}{1 + \delta} \leq 1 - \frac{n_{p_{\min}}}{144(1 + \delta)\kappa_f^2}, 1 - \frac{b_y}{1 + \delta} \leq 1 - \frac{n_{p_{\min}}}{144(1 + \delta)\kappa_f^2}
\]
\[
1 - \frac{b_x}{8\sqrt{(1 + \delta)\kappa_g}} \leq 1 - \frac{n_{p_{\min}}}{1152\sqrt{(1 + \delta)\kappa_g\kappa_f^2}}, 1 - \frac{b_y}{8\sqrt{(1 + \delta)\kappa_g}} \leq 1 - \frac{n_{p_{\min}}}{1152\sqrt{(1 + \delta)\kappa_g\kappa_f^2}}. \tag{277}
\]
Therefore,
\[
\rho \leq \max \left\{ 1 - \frac{n_{p_{\min}}}{336\kappa_f^2}, 1 - \frac{n_{p_{\min}}}{1152\sqrt{(1 + \delta)\kappa_g\kappa_f^2}}, 1 - \frac{1}{8(1 + \delta)\kappa_g}, 1 - \frac{n_{p_{\min}}}{144(1 + \delta)\kappa_f^2}, 1 - \frac{p}{2} \right\} \tag{278}
\]
\[
\leq 1 - \min \left\{ \frac{n_{p_{\min}}}{336\kappa_f^2}, \frac{n_{p_{\min}}}{1152\sqrt{(1 + \delta)\kappa_g\kappa_f^2}}, \frac{1}{8(1 + \delta)\kappa_g}, \frac{n_{p_{\min}}}{144(1 + \delta)\kappa_f^2}, \frac{p}{2} \right\} \tag{279}
\]
\[
= 1 - \tilde{C}. \tag{280}
\]
By taking log on both sides, we obtain
\[
\log \rho \leq \log(1 - \tilde{C})
\]
\[
- \log \rho \geq - \log(1 - \tilde{C})
\]
\[
\frac{1}{- \log \rho} \leq \frac{1}{- \log(1 - \tilde{C})}
\]
\[
\leq \frac{5}{\tilde{C}}
\]
\[
= 5 \left( \min \left\{ \frac{n_{p_{\min}}}{336\kappa_f^2}, \frac{n_{p_{\min}}}{1152\sqrt{(1 + \delta)\kappa_g\kappa_f^2}}, \frac{1}{8(1 + \delta)\kappa_g}, \frac{n_{p_{\min}}}{144(1 + \delta)\kappa_f^2}, \frac{p}{2} \right\} \right)^{-1}
\]
\[
= 5 \max \left\{ \frac{336\kappa_f^2}{n_{p_{\min}}}, \frac{1152\sqrt{(1 + \delta)\kappa_g\kappa_f^2}}{n_{p_{\min}}}, \frac{8(1 + \delta)\kappa_g}{n_{p_{\min}}}, \frac{144(1 + \delta)\kappa_f^2}{n_{p_{\min}}}, \frac{2}{p} \right\}, \tag{282}
\]
where the fourth inequality uses the fact that \((1/ - \log(1 - x)) \leq 5/x\) for all \(0 < x < 1\). Using Lemma \ref{lemma:rho-bound} we have \(M_x \geq 1 - \frac{3b_x}{7} \). Therefore, \(M_x \geq 1 - \frac{b_x}{7} \) because \(0 < b_x < 1\). Moreover, \(M_y > \frac{3}{7} \) as \(0 < b_y < 1\). Therefore, log \((\phi_y/M_x) \leq \log \left( \frac{2\phi_y}{3x_e} \right) \). Hence,
\[
T(\epsilon) = O \left( \max \left\{ \frac{\kappa_f^2}{n_{p_{\min}}}, \frac{\sqrt{(1 + \delta)\kappa_g\kappa_f^2}}{n_{p_{\min}}}, \frac{1}{(1 + \delta)\kappa_g}, \frac{(1 + \delta)\kappa_f^2}{n_{p_{\min}}}, \frac{2}{p} \right\} \log_2 \left( \frac{\phi_y}{\epsilon} \right) \right). \tag{283}
\]

H.1 Discussion on the analysis techniques

In this section, we discuss and compare the analysis techniques of our work with those in existing works. In \cite{18} a convex composite minimization problem is studied and inexact PDHG method is applied to its saddle point formulation. In this work, we study a different problem \cite{1} where a smooth function depends jointly on primal and dual variables. We prove that it is equivalent to study unconstrained saddle point problem \cite{3} to get the solution of \cite{1}. However, \cite{18} uses a well known equivalence between a convex minimization problem and its Lagrangian formulation \cite{17}. We
define additional quantities $D^*_{y}$, $H^*_{y}$ and Bregman distance functions $V_{f,y}(x_1, x_2)$, $V_{-f,x}(y_1, y_2)$ in Appendix C to obtain appropriate bounds.

**Algorithm 2 analysis:** In contrast to [18], we get complicated upper bounds depending on primal and dual iterates in Lemma E.3 which yields different set of parameters. We prove the feasibility of these parameters and derive useful lower and upper bounds in Appendix E.4. [18] uses diminishing step size and an induction approach to prove the convergence to exact solution. However, we use a different method summarized below to derive the convergence rate of Algorithm (2). We first derive a relation which connects iterate $t$ information with the restart iterates (see Lemma E.6). Then by appropriately choosing the restart iterates $x_{k,0}, y_{k,0}, D^*_{k,0}, D^y_{k,0}, H^*_{k,0}$ and inner iterates $t_k$, we get the complexity result in Appendix E. Then it is worth noting that proof techniques of Lemma E.6 and Theorem 5.1 are different from those in [18] due to different algorithm structure involving a restart scheme, complicated bounds, and different sets of parameters.

**Algorithm 3 analysis:** Using smoothness, strong convexity strong concavity assumptions, and definitions of $V_{f,y}(x_1, x_2)$ and $V_{-f,x}(y_1, y_2)$, we upper bound $E \|x_t - 1x^* - sG^*_f + s\nabla_x F(1x^*, 1y^*)\|^2 + E \|y_t - 1y^* + sG^*_y - s\nabla_y F(1x^*, 1y^*)\|^2$ in terms of $x_t, y_t, \bar{x}_t, \bar{y}_t, V_{f,y}(x_1, x_2)$ and $V_{-f,x}(y_1, y_2)$ in Lemma G.5. Note that the upper bound in Lemma G.5 is complicated and different from that of [18] because we have additional terms contributed by dual variable $y$ with different coefficients and terms containing square norms dependent on the reference points. This intermediate result generates different bounds and sets of parameters in the subsequent analysis. We carefully set the step size and choose algorithm parameters with proven feasibility in Lemma G.7. We rigorously compute lower and upper bounds on chosen parameters in terms of $\kappa_f, \kappa_y$ and $\delta$ in Lemma G.7 and Appendix H. In our work, these derivations are more involved in comparison to [18].

Analysis methods of [36] and [22] are based on averaging quantities; for example average of iterates with proven feasibility in Lemma G.7. We rigorously compute lower and upper bounds depending on the smoothness of saddle point problem. In contrast to [36] and [22], our analysis does not demand any separate bound on consensus error and gradient estimation errors and depends in addition on the smoothness of saddle point problem. In Lemma G.5 is complicated and different from that of [18] because we have additional terms dependent on the reference points.

### I Numerical Experiments

We evaluate the effectiveness of proposed algorithms on robust logistic regression problem

$$
\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} \Psi(x, y) = \frac{1}{N} \sum_{i=1}^{N} \log \left(1 + \exp \left(-b_i x^\top (a_i + y)\right)\right) + \frac{\lambda}{2} \|x\|^2_2 - \frac{\beta}{2} \|y\|^2_2
$$

over a binary classification data set $\mathcal{D} = \{(a_i, b_i)\}_{i=1}^{N}$. We take $\mathcal{X}$ and $\mathcal{Y}$ respectively as $\ell_2$ balls of radius 100 and 1. We compute smoothness parameters $L_{xx}$, $L_{yy}$, $L_{xy}$ and $L_{yx}$ using Hessian information of the objective function (see Section 5.5). Unless stated otherwise, we set strong convexity parameter as $\lambda = 10$ and strong concavity parameter as $\beta = 10$, and number of nodes and number of batches to 20 in all experiments. We use Python Programming Language for implementing all methods. The experiments were run on a Linux machine with 32 2.10GHz Intel Xeon CPUs.

**I.1 Experimental Setup:**

**Datasets:** We rely on four binary classification datasets namely, a4a, phishing and ijcn1 from https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/ and sido data from http://www.causality.inf.ethz.ch/data/SID0.html. The characteristics of these datasets are reported in Table 2. We distribute the samples across 20 nodes and create 20 mini batches of local samples for all datasets.

**Network Setting:** We conduct the experiments for 2D torus topology and ring topology. We generate weight matrix $W$ with $W_{ij} = 1/5$ and $W_{ij} = 1/3$ for all $(i, j) \in \mathcal{E} \cup \{(i, i)\}$ for 2D torus topology and ring topology respectively.
Table 2: Data Sets used for experiments. \( N \) and \( d \) denote respectively the number of samples and number of features.

| Data set | \( N \)  | \( d \) |
|----------|---------|---------|
| a4a      | 4781    | 122     |
| phishing | 11,055  | 68      |
| ijcnn1   | 49,990  | 22      |
| sido     | 2536    | 4932    |

**Compression Operator:** We consider an unbiased \( b \)-bits quantization operator \( Q(x) \) throughout our empirical study.

\[
Q(x) = \left( \|x\|_\infty 2^{-b-1} \text{sign}(x) \right) \cdot \left( \frac{2^{b-1} |x|}{\|x\|_\infty} + u \right),
\]

where \( \cdot \) represents Hadamard product, \( |x| \) denotes elementwise absolute value and \( u \) is a random vector uniformly distributed in \([0, 1]^d\).

### I.2 Baseline methods

We compare the performance of proposed algorithms C-RDPSG and C-DPSVRG with three non-compression based baseline algorithms: (1) Distributed Min-Max Data similarity algorithm [5] (2) Decentralized Parallel Optimistic Stochastic Gradient (DPOSG) algorithm [22] and, (3) Decentralized Minimax Hybrid Stochastic Gradient Descent (DM-HSGD) algorithm [36].

**Distributed Min-Max data similarity:** This algorithm is based on accelerated gossip scheme employed on model updates and gradient vectors [5]. The number of iterates in accelerated gossip scheme and the step size are computed according to the theoretical details provided in [5]. This algorithm requires approximate solution of an inner saddle point problem at every iterate. We run extragradient method [15] to solve the inner saddle point problem with a desired precision accuracy provided in [5]. Throughout this section, we use the shorthand notation for Distributed Min-Max data similarity algorithm as Min-Max similarity.

**Decentralized Parallel Optimistic Stochastic Gradient (DPOSG):** DPOSG [22] is a two step algorithm with local model averaging designed for solving unconstrained saddle-point problems in a decentralized fashion. We include the projection steps to both update sequences of DPOSG as we are solving constrained problem (284). The step size and the number of local model averaging steps are tuned according to Theorem 1 in [22].

**Decentralized Minimax Hybrid Stochastic Gradient Descent (DM-HSGD):** DM-HSGD [36] is a gradient tracking based algorithm designed for solving saddle point problems with a constraint set on dual variable. We incorporate projection step to the model update of primal variable. We use grid search to find the best step sizes for primal and dual variable updates. Other parameters like initial large batch size and parameters involved in gradient tracking update sequence are chosen according to the experimental setting in [36].

### I.3 Benchmark Quantities

We run the centralized and uncompressed version of C-DPSVRG for 50,000 iterations to find saddle point solution \( z^* = (x^*, y^*) \) of problem (284). The performance of all the methods is measured using \( \sum_{t=1}^m \|z_t - z^*\|^2 \).

**Number of gradient computations and communications:** We calculate the total number of gradient computations according to the number of samples used in the gradient computation at a given iterate \( t \). The number of communications per iterate are computed as the number of times a node exchanges information with its neighbors.

**Number of bits transmitted:** We set number of bits \( b = 4 \) in compression operator \( Q(x) \) for C-RDPSG and C-DPSVRG. Similar to [14], we assume that on an average 5 bits (1 bit for sign and 4 bits for quantization level) are transmitted at every iterate for C-RDPSG and C-DPSVRG. We assume that on an average 32 bits are transmitted per communication for DPOSG, DM-HSGD and Min-Max similarity algorithm.
1.4 Observations

Comparison to baselines: C-RDPSG converges faster in the beginning and slows down after reaching approximately $10^{-8}$ accuracy as depicted in Figure 3 and Figure 4. C-DPSVRG converges faster than other baseline methods. DPOSG and DM-HSGD converge only to a neighbourhood of the saddle point solution and start oscillating after some time. The restart scheme’s inclusion in C-RDPSG helps mitigate the flat and oscillatory behaviour at the later iterations, unlike DPOSG and DM-HSGD. C-RDPSG is faster than C-DPSVRG, DPOSG, DM-HSGD and Min-Max similarity in terms of gradient computations, communications and bits transmitted to achieve saddle points of moderate accuracy. The performance of C-RDPSG is competitive with DPOSG and DM-HSGD in the long run as demonstrated in Figure 3 and Figure 4.

Compression effect: Plots in Figure 3 depict that C-DPSVRG is 1000 times faster than Min-Max similarity, DPOSG algorithm and 10 times faster than DM-HSGD in terms of transmitted bits. We observe that C-RDPSG is also 1000 times faster than Min-max similarity and DPOSG for obtaining saddle point solutions of moderate accuracy, in terms of transmitted bits.

Communication efficiency: The one-time communication at every iterate in C-DPSVRG speeds up communication and makes C-DPSVRG to be 100 times faster than Min-Max similarity and DPOSG methods as shown in Figure 5. C-RDPSG is 10 times faster than DM-HSGD and 100 times faster than DPOSG and Min-Max similarity in terms of communications at the initial stages of the algorithm.

Different number of bits: We also conduct experiments to observe the behaviour of C-RDPSG and C-DPSVRG with different numbers of bits transmitted. Results are depicted in Figure 5 and Figure 6. Figure 5 demonstrates that C-DPSVRG with 16 bits and 32 bits require same number of gradient computations, but the number of bits transmitted is much lesser in 16 bits than in 32 bits. From Figure 5 the number of gradient computations in C-DPSVRG with 8 bits, 16 bits and 32 bits are almost same to achieve $10^{-8}$ accuracy, but C-DPSVRG with 8 bits transmits less number of bits and hence makes the communication faster. C-RDPSG performance does not change drastically in the long run in terms of gradient computations against the number of bits as shown in Figure 6.

Different choices of reference probability parameters: The full batch gradient computations in C-DPSVRG depends on the reference probability parameter $p$. Motivated from [16], we run C-DPSVRG with five different reference probabilities as $1/n, 1/(\kappa n^3)^{1/4}, 1/(\kappa n)^{1/2}, 1/(\kappa n^5)^{1/4}$ and $1/\kappa$. From Figure 7 we observe that setting $p = 1/n$ requires the least number of gradient computations as it corresponds to the less frequent computation of full batch gradients.

Impact of topology: From Figure 4 we observe that the convergence behaviour of all methods is similar to 2D torus topology. We also note that ring topology requires large number of communications compared to 2D torus due to its sparse connectivity.

Compression error: We plot compression error $\|Q(\nu^x) - \nu^x\|^2 + \|Q(\nu^y) - \nu^y\|^2$ against number of transmitted bits for C-RDPSG and C-DPSVRG as shown in Figure 8. We observe that C-DPSVRG with 2 bits achieves compression error $10^{-25}$ in less than 20,000 transmitted bits. It shows a clear advantage of using smaller number of bits in C-DPSVRG while maintaining low compression error. In C-RDPSG, larger the number of bits used in the quantization operator, smaller the compression error. There are sharp jumps in the decay of compression error during a restart of C-RDPSG as depicted in Figure 8.

Impact of number of nodes: As the number of nodes increases, C-RDPSG requires fewer gradient computations in the initial phase. However, smaller number of nodes gives faster convergence at the later iterations for 2D torus and ring topology, as depicted in Figure 11 and Figure 13. C-DPSVRG also requires small number of gradient computations in 2D torus with larger number of nodes because it assigns smaller batch sizes to every node. As depicted in Figure 10, C-DPSVRG performance does not change much in terms of the number of communications and bits transmitted for 2D torus. The sparsity level of ring topology is higher than that of 2D torus and increases with number of nodes. It leads C-DPSVRG to achieve fast convergence eventually against gradient computations with a smaller number of nodes, as demonstrated in Figure 12. In contrast to the performance of C-DPSVRG against communications in 2D torus (Figure 10), C-DPSVRG requires more communications for large number of nodes in a ring topology, as shown in Figure 12. DM-HSGD and C-RDPSG are competitive in terms of gradient computations and communications with larger number of nodes as demonstrated in Figure 5 and 14. However it is to be noted that DM-HSGD exhibits an oscillatory behavior in the saddle point solutions, after reaching a moderate solution accuracy.
1.5 Estimating Lipschitz parameters

In this section, we estimate Lipschitz parameters $L_{xx}, L_{yy}, L_{xy}, L_{yx}$ of robust logistic regression problem (284). Assume that each node $i$ has $N_i$ number of local samples such that $\sum_{i=1}^{m} N_i = N$. Recall objective function $\Psi(x, y)$ in equation (284):

$$\Psi(x, y) = \frac{1}{N} \sum_{i=1}^{N} \log \left(1 + \exp \left(-b_i x^\top (a_i + y)\right)\right) + \frac{\lambda}{2} \|x\|_2^2 - \frac{\beta}{2} \|y\|_2^2$$

$$= \frac{1}{N} \sum_{i=1}^{m} \sum_{l=1}^{N_i} \log \left(1 + \exp \left(-b_{il} x^\top (a_{il} + y)\right)\right) + \frac{\lambda}{2} \|x\|_2^2 - \frac{\beta}{2} \|y\|_2^2$$

$$= \sum_{i=1}^{m} \left( \frac{1}{N_i} \sum_{l=1}^{N_i} \log \left(1 + \exp \left(-b_{il} x^\top (a_{il} + y)\right)\right) + \frac{\lambda}{2m} \|x\|_2^2 - \frac{\beta}{2m} \|y\|_2^2 \right)$$

$$= \sum_{i=1}^{m} f_i(x, y),$$

where $f_i(x, y) = \frac{1}{N_i} \sum_{l=1}^{N_i} \log \left(1 + \exp \left(-b_{il} x^\top (a_{il} + y)\right)\right) + \frac{\lambda}{2m} \|x\|_2^2 - \frac{\beta}{2m} \|y\|_2^2$. Gradients of $f_i(x, y)$ with respect to $x$ and $y$ are given by

$$\nabla_x f_i(x, y) = \frac{1}{N_i} \sum_{l=1}^{N_i} \frac{-b_{il} (a_{il} + y)}{1 + \exp(b_{il} x^\top (a_{il} + y))} + \frac{\lambda}{m} x$$

$$\nabla_y f_i(x, y) = \frac{1}{N_i} \sum_{l=1}^{N_i} \frac{b_{il} x}{1 + \exp(b_{il} x^\top (a_{il} + y))} - \frac{\beta}{m} y.$$

We create $n$ batches $\{N_{i_1}, \ldots, N_{i_n}\}$ of local samples $N_i$ and write $f_i(x, y)$ in the form of $\frac{1}{n} f_{ij}(x, y)$.

$$f_i(x, y) = \frac{1}{N} \sum_{l=1}^{N_i} \log \left(1 + \exp \left(-b_{il} x^\top (a_{il} + y)\right)\right) + \frac{\lambda}{2m} \|x\|_2^2 - \frac{\beta}{2m} \|y\|_2^2$$

$$= \frac{1}{n} \sum_{j=1}^{n} \sum_{l=1}^{N_i} \log \left(1 + \exp \left(-b_{il} x^\top (a_{il} + y)\right)\right) + \frac{\lambda}{2mn} \|x\|_2^2 - \frac{\beta}{2mn} \|y\|_2^2$$

$$= \frac{1}{n} \sum_{j=1}^{n} \left( \frac{1}{N_i} \sum_{l=1}^{N_i} \log \left(1 + \exp \left(-b_{il} x^\top (a_{il} + y)\right)\right) + \frac{\lambda}{2mn} \|x\|_2^2 - \frac{\beta}{2mn} \|y\|_2^2 \right)$$

$$= \frac{1}{n} \sum_{j=1}^{n} f_{ij}(x, y),$$

where $f_{ij}(x, y) = \frac{1}{N_i} \sum_{l=1}^{N_i} \log \left(1 + \exp \left(-b_{il} x^\top (a_{il} + y)\right)\right) + \frac{\lambda}{2mn} \|x\|_2^2 - \frac{\beta}{2mn} \|y\|_2^2$. We are now ready to find required Lipschitz parameters.

**Computing $L_{xx}^{ij}$**:  

$$\nabla^2_{xx} f_{ij}(x, y) = \frac{n}{N} \sum_{l=1}^{N_i} \frac{(a_{il}^2 + y)(a_{il} + y)^\top \exp \left(b_{il} x^\top (a_{il} + y)\right)}{1 + \exp \left(b_{il} x^\top (a_{il} + y)\right)^2} + \frac{\lambda}{m} f_{ij}$$

$$\implies \|\nabla^2_{xx} f_{ij}(x, y)\|_2 \leq \frac{n}{4N} \sum_{l=1}^{N_i} (2\|a_{il}\|_2^2 + 2R^2_y) + \frac{\lambda}{m}$$

$$= \frac{n}{2N} \sum_{l=1}^{N_i} \|a_{il}\|_2^2 + \frac{nN_i R_y^2}{2N} + \frac{\lambda}{m} =: L_{xx}^{ij}.$$
Computing $L_{yy}^{ij}$:
\[
\nabla^{2}_{yy}f_{ij}(x, y) = \frac{n}{N} \sum_{l=1}^{N_{ij}} \exp \left( b_{il}' x^\top (a_{il}' + y) \right) (b_{il}')^2 x x^\top - \frac{\beta}{m} I
\]
\[
\Rightarrow \left\| \nabla^{2}_{xx}f_{ij}(x, y) \right\|_2 \leq \frac{n}{N} \sum_{l=1}^{N_{ij}} \left\| xx^\top \right\|_2 + \frac{\beta}{m} m =: L_{yy}^{ij}.
\]

Computing $L_{xy}^{ij}$:
\[
\nabla_y (\nabla_x f_{ij}(x, y)) = \frac{n}{N} \sum_{l=1}^{N_{ij}} \left( -b_{il}' I \right) + (b_{il}')^2 (a_{il}' + y) x^\top \frac{\exp \left( b_{il}' x^\top (a_{il}' + y) \right)}{(1 + \exp(b_{il}' x^\top (a_{il}' + y))^2)
\]
Hence we have
\[
\left\| \nabla^{2}_{xy}f_{ij}(x, y) \right\|_2 \leq \frac{n}{N} \sum_{l=1}^{N_{ij}} \left( 1 + \frac{1}{4} \left\| (a_{il}' + y) x^\top \right\|_2 \right)
\]
\[
\leq \frac{n}{N} \sum_{l=1}^{N_{ij}} \left( 1 + \frac{R_x}{4} \left\| (a_{il}')_2 + R_y \right\| \right)
\]
\[
\leq \frac{n}{N} \sum_{l=1}^{N_{ij}} \left( 1 + \frac{R_x}{4} \sum_{l=1}^{N_{ij}} \left\| (a_{il}')_2 \right\| \right) =: L_{xy}^{ij}.
\]
We set $L_{xx} = \max_{i,j} \{ L_{xx}^{ij} \}$, $L_{yy} = \max_{i,j} \{ L_{yy}^{ij} \}$ and $L_{xy} = L_{yx} = \max_{i,j} \{ L_{xy}^{ij} \}$. The strong convexity and strong concavity parameters are respectively set to $\lambda$ and $\beta$.

Figure 3: Convergence behavior of iterates to saddle point vs. Gradient computations (Row 1), Communications (Row 2), Number of bits transmitted (Row 3) for different algorithms in 2D torus topology. a4a, phishing, ijcnn, sido datasets are in Columns 1,2,3,4 respectively.
Figure 4: Convergence behavior of iterates to saddle point vs. Gradient computations (Row 1), Communications (Row 2), Number of bits transmitted (Row 3) for different algorithms in ring topology. a4a, phishing, ijcnn, sido datasets are in Columns 1,2,3,4 respectively.

Figure 5: Convergence behavior of iterates to saddle point vs. Gradient computations (Row 1), Number of bits transmitted (Row 2) for C-DPSVRG behavior with different number of bits in 2D torus topology. a4a, phishing, ijcnn, sido datasets are in Columns 1,2,3,4 respectively.

Figure 6: Convergence of iterates to saddle point vs. Gradient computations (Row 1), Number of bits transmitted (Row 2) for C-RDPSG behavior with different number of bits in 2D torus topology. a4a, phishing, ijcnn, sido datasets are in Columns 1,2,3,4 respectively.
Figure 7: Performance of C-DPSVRG with different reference probabilities in 2D torus. a4a, phishing, ijcnn, sido datasets are in Columns 1,2,3,4 respectively.

Figure 8: Compression error with different number of bits. Row 1: C-DPSVRG, Row 2: C-RDPSG. a4a, phishing, ijcnn, sido datasets are in Columns 1,2,3,4 respectively.

Figure 9: Comparison with baselines with different number of nodes on a 2D torus topology with ijcnn data. Column 1: 56 nodes, Column 2: 110 nodes, Column 3: 210 nodes.
Figure 10: Performance of C-DPSVRG with different number of nodes on 2D torus topology with ijcnn data.

Figure 11: Performance of C-RDPSG with different number of nodes on 2D torus topology with ijcnn data.

Figure 12: Performance of C-DPSVRG with different number of nodes on ring topology with ijcnn data.

Figure 13: Performance of C-RDPSG with different number of nodes on ring topology with ijcnn data.
Figure 14: Comparison with baselines with different number of nodes on a ring topology with ijcnn data. Column 1: 56 nodes, Column 2: 110 nodes, Column 3: 210 nodes.