Combinatorics in tensor-integral reduction

June-Haak Ee, Dong-Won Jung, U-Rae Kim and Jungil Lee

Department of Physics, Korea University, Seoul 02841, Korea
E-mail: jungil@korea.ac.kr

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Abstract
We illustrate a rigorous approach to express the totally symmetric isotropic tensors of arbitrary rank in the \( n \)-dimensional Euclidean space as a linear combination of products of Kronecker deltas. By making full use of the symmetries, one can greatly reduce the efforts to compute cumbersome angular integrals into straightforward combinatoric counts. This method is generalised into the cases in which such symmetries are present in subspaces. We further demonstrate the mechanism of the tensor-integral reduction that is widely used in various physics problems such as perturbative calculations of the gauge-field theory in which divergent integrals are regularised in \( d = 4 - 2\epsilon \) space–time dimensions. The main derivation is given in the \( n \)-dimensional Euclidean space. The generalisation of the result to the Minkowski space is also discussed in order to provide graduate students and researchers with techniques of tensor-integral reduction for particle physics problems.

Keywords: combinatorics, tensor angular integral, tensor-integral reduction, isotropic tensor, Feynman integral

1. Introduction

As geometric objects generalised from scalars and vectors, tensors act an important role to represent various physical quantities. The fundamental classification to distinguish scalars, vectors, and tensors is based on their transformation properties under rotation [1]. A scalar is a single invariant quantity under rotation. In the \( n \)-dimensional Euclidean space, a vector \( V = (V^1, \ldots, V^n) \) has \( n \) Cartesian components and a vector index \( i \) is used to identify the \( i \)th component \( V_i \). Under a rotation, the corresponding linear transformation matrices of any vectors are identical to a single rotation matrix \( R^{ij} \) that transforms a radial vector \( \rho = (x^1, \ldots, x^n) \) into \( \rho' = (x'^1, \ldots, x'^n) \) like \( x'^i = \sum_j R^{ij} x^j \) keeping the magnitude \( |\rho| = \sqrt{\sum_j x^j x^j} \) invariant. This relation also holds for any polar vector \( V \).

Then, the direct product of \( k \) polar vectors, \( V^{h_1 \ldots h_k} \equiv V^{h_1} \cdots V^{h_k} \) must transform like
$V^{h_{i}...i_{k}} = \sum_{k_{1}=1}^{n}...\sum_{k_{h_{k}}=1}^{n} \partial^{h_{1}}...\partial^{h_{k}} V^{h_{i}...i_{k}}$. An object with $k$ vector indices that transforms like $V^{h_{i}...i_{k}}$ is called a rank-$k$ Cartesian tensor $T^{h_{i}...i_{k}}_{(k)}$. If a tensor is invariant under rotation, then we call it an isotropic tensor. A tensor is called symmetric (antisymmetric) if it is invariant (flips the sign) under exchange of two given vector indices. Elementary examples of symmetric rank-2 tensors are the tensor of polarisability, tensor of inertia, and tensor of stress. A cross product of two polar vectors such as the angular momentum or torque is an antisymmetric rank-2 tensor [2]. If a tensor is invariant (flips the sign) under exchange of any two vector indices, then we call it totally symmetric (totally antisymmetric).

Among various tensors, the totally symmetric isotropic tensors $\tilde{T}^{h_{i}...i_{k}}_{(k)}$ are particularly important because of their invariance properties: $\tilde{T}^{h_{i}...i_{k}}_{(k)}$ remains the same under rotation and/or exchange of any two vector indices. The most elementary example is the Kronecker delta $\delta^{ij}$ which is of rank 2. Because of the rotational symmetry of $\delta^{ij}$, it is trivial to verify that the scalar product $A \cdot B$ of any two vectors $A$ and $B$ is invariant under rotation. If $\delta^{ij}$ is multiplied to a rank-2 tensor $T^{ij}_{(2)}$, and the vector indices are contracted, then one obtains the trace of the tensor which is invariant under rotation.

A considerable amount of research on isotropic tensors to study various physics problems has been made such as in crystallography or rheology describing material properties [3–6], fluid dynamics to study turbulence effects [7–9], and molecular dynamics involving multi-photon emission/absorption processes [10–13]. All of the approaches listed above are restricted to three dimensions. In fact, dimensional regularisation in the gauge-field theory of particle physics requires information of $\tilde{T}^{h_{i}...i_{k}}_{(k)}$ in $d = 4 - 2\epsilon$ space–time dimensions, where $\epsilon$ is an infinitesimally small number. In this approach a divergent loop integral appearing in perturbative expansions is regularised into powers of $1/\epsilon$. Eventually, after a proper renormalisation, the resultant physical quantities become finite as $\epsilon \rightarrow 0$ and the theory retains the predictive power [14–18]. In employing dimensional regularisation, physical observables are first calculated in $n = d - 1$ spatial dimensions assuming that $n$ is a countable number. Then, any functions of $n$, such as the gamma function, appearing in physical variables are analytically continued to $n = 3 - 2\epsilon$. The loop integrals are integrated over a loop momentum $p = (p^{0}, p^{1}, ..., p^{d-1})$ whose every component runs from $-\infty$ to $\infty$. Here, $p$ is a d-vector, which is the $d$-dimensional analogue of the four-vector $p = (p^{0}, p^{1}, p^{2}, p^{3})$. After evaluating the residue by integrating over the energy component $p^{0}$, the resultant integrals over $p^{0}$’s involve angle averages. This integral is in general a linear combination of tensor integrals that are to be simplified into a linear combination of constant tensors with the coefficients proportional to scalar integrals. This procedure is consistent with the standard approach called the Passarino–Veltman reduction [19].

In this paper, we illustrate a systematic approach to find the explicit form of the totally symmetric isotropic tensor $\tilde{T}^{h_{i}...i_{k}}_{(k)}$ as a linear combination of Kronecker deltas. In principle, the expression can be found by taking the average of $\hat{r}^{h_{1}}...\hat{r}^{h_{k}}$ over the angle of a unit radial vector $\hat{r}$. For example, $\tilde{T}^{ij}_{(2)} = \delta^{ij}/3$ in three dimensions [20]. While a direct evaluation of the average over $n = 2$ polar angles and an azimuthal angle in $n$ dimensions requires great efforts to deal with quite a few beta functions, our rigorous derivation based on only the abstract algebraic structure makes extensive use of the symmetries leading to a great simplification of the evaluation steps into simple counts of combinatorics. Our methods are generalised into the cases in which such symmetries are present in subspaces. As applications, we demonstrate how to make use of $\tilde{T}^{h_{i}...i_{k}}_{(k)}$ in evaluating angular integrals and in tensor-integral reductions.

This paper is organised as follows. In section 2, we list definitions of fundamental terminologies of tensor analyses that are frequently used in the remainder of this paper.
Section 3 provides the derivation of the explicit form of the totally symmetric isotropic tensor $T_{(k)}^{(n)}$. Applications of $T_{(k)}^{(n)}$ to the computation of angular integrals and the tensor-integral reduction are given in section 4 which is followed by a summary in section 5. Appendices provide technical formulas: a standard parametrisation of the spherical polar coordinates in the $n$-dimensional Euclidean space is given in appendix A. Explicit evaluations of angle averages in $n$ dimensions are listed in appendix B. The extension of our results to problems in the $d$-dimensional Minkowski space is summarised in appendix C.

2. Definitions

In this section, we list definitions of terminologies involving the vector and tensor analyses presented in this paper. We work in the $n$-dimensional Euclidean space $\mathbb{R}^n$.

2.1. Vector

A vector $V \in \mathbb{R}^n$ can be expressed as a linear combination

$$V = V^i \hat{e}_i,$$

where $\hat{e}_i$ and $V_i$ are the unit basis vector along the $i$th Cartesian axis and the $i$th component, respectively. The $n$-tuple $(V^1, \cdots, V^n)$ is also used to denote $V$. Every Cartesian axis of $\mathbb{R}^n$ is homogeneous and isotropic so that the unit basis vectors satisfy the orthonormal conditions:

$$\hat{e}_i \cdot \hat{e}_j = \delta^{ij},$$

where the Kronecker delta $\delta^{ij}$ is the $ij$ element of the $n \times n$ identity matrix $I$. Thus the scalar product of two vectors $V$ and $W$ can be expressed as

$$V \cdot W = V^i W^i.$$

Here and in the remainder of this paper we use the Einstein’s convention for summation of repeated vector indices; $V^i W^i$ represents $\sum_{i=1}^{n} V^i W^i$. The square and the magnitude of a vector $V$ are defined by $V^2 \equiv V \cdot V$ and $|V| \equiv \sqrt{V^2}$, respectively, and $V^{2k} \equiv (V^2)^k$ for any positive integer $k$. The unit vector $\hat{V}$ along the direction of $V$ is defined by $\hat{V} \equiv V/|V|$. The trace of the identity matrix is the dimension of $\mathbb{R}^n$:

$$\delta^{ii} = n.$$

Under rotation, a vector $V$ transforms into $V'$ as

$$V'^i = \mathcal{R}^i_j V^j,$$

where $\mathcal{R}$ is the rotation matrix. Because $V'^2 = V^2$, $\mathcal{R}$ is an orthogonal matrix:

$$\mathcal{R}^T \mathcal{R} = \mathcal{R} \mathcal{R}^T = I.$$

2.2. Gram–Schmidt orthogonalisation

A convenient way of constructing orthonormal basis vectors is Gram–Schmidt orthogonalisation [21]. If $a_1, \cdots, a_m$ are linearly independent vectors in $\mathbb{R}^n$ with $m \leq n$, then one can construct unit basis vectors as
\[
\hat{p} = \frac{1}{\sqrt{D(p-1)D(p)}} \left| \begin{array}{ccc}
 a_1 & a_2 & \cdots & a_p \\
 a_1 \cdot a_1 & a_2 \cdot a_1 & \cdots & a_p \cdot a_1 \\
 \vdots & \vdots & \ddots & \vdots \\
 a_1 \cdot a_{p-1} & a_2 \cdot a_{p-1} & \cdots & a_p \cdot a_{p-1} \\
\end{array} \right|, \quad (7)
\]

where \( p = 1, \cdots, m, \) \( D(p) = \det[A_p(p)] \) is called the Gram determinant of a \( p \times p \) square matrix \( A_p(p) \) with elements \( A_p^{ij} = a_i \cdot a_j \), and \( D(0) = 1 \).

2.3. Tensor

The symbol \( T_{(k)}^{i_1 \cdots i_k} \) denotes the Cartesian tensor of rank \( k \), where \( k \) is the number of vector indices. Under rotation, \( T_{(k)}^{i_1 \cdots i_k} \) transforms like

\[
T_{(k)}^{i_1 \cdots i_k} = R^{i_1 i_h} \cdots R^{i_k i_h} T_{(k)}^{h_1 \cdots h_k}. \quad (8)
\]

2.4. Permutation

In the remainder of this paper, we use the symbol \( \sigma \) to represent one of the \( k! \) permutations \( [\sigma(i_1), \ldots, \sigma(i_k)] \) of the ordered list of vector indices \( (i_1, \ldots, i_k) \). For example, the permutation \( [\sigma(i_1), \sigma(i_2), \sigma(i_3)] \) of the ordered list \( (i_1, i_2, i_3) \) is one of the following cases \( (i_1, i_2, i_3), (i_1, i_3, i_2), (i_2, i_1, i_3), (i_2, i_3, i_1), (i_3, i_1, i_2), \) or \( (i_3, i_2, i_1) \). Thus, the summation over \( \sigma \) is defined, for example, by

\[
\sum_{\sigma} T_{(3)}^{i_{\sigma(i_1)} i_{\sigma(i_2)} i_{\sigma(i_3)}} = T_{(3)}^{i_1 i_2 i_3} + T_{(3)}^{i_1 i_3 i_2} + T_{(3)}^{i_2 i_1 i_3} + T_{(3)}^{i_2 i_3 i_1} + T_{(3)}^{i_3 i_1 i_2} + T_{(3)}^{i_3 i_2 i_1}. \quad (9)
\]

Summation over \( \sigma \) can also be used for tensor products like

\[
\sum_{\sigma} A_{(1)}^{\sigma(i_1)} B_{(2)}^{\sigma(i_2)} C_{(2)}^{\sigma(i_3)} = [A_{(1)}^h B_{(1)}^{h_i} + A_{(1)}^i B_{(1)}^{h_j}] [C_{(2)}^{h_i} + C_{(2)}^{h_j}]
+ [A_{(1)}^h B_{(1)}^{i_k} + A_{(1)}^i B_{(1)}^{h_j}] [C_{(2)}^{i_k} + C_{(2)}^{h_j}]
+ [A_{(1)}^h B_{(1)}^{j_l} + A_{(1)}^i B_{(1)}^{i_k}] [C_{(2)}^{j_l} + C_{(2)}^{i_k}]
+ [A_{(1)}^h B_{(1)}^{l_m} + A_{(1)}^i B_{(1)}^{j_l}] [C_{(2)}^{l_m} + C_{(2)}^{j_l}]
+ [A_{(1)}^h B_{(1)}^{m_p} + A_{(1)}^i B_{(1)}^{l_m}] [C_{(2)}^{m_p} + C_{(2)}^{l_m}]. \quad (10)
\]

2.5. Rotationally invariant (isotropic) tensor

If \( T_{(k)}^{i_1 \cdots i_k} \) is invariant under any rotation,

\[
T_{(k)}^{i_1 \cdots i_k} = R^{i_1 i_h} \cdots R^{i_k i_h} T_{(k)}^{h_1 \cdots h_k}, \quad (11)
\]

then it is called a rotationally invariant (isotropic) tensor. Because \( R \) is orthogonal, \( \delta^{ij} \) is the isotropic tensor of rank 2: \( R^{ij} R^{k j} \delta_{hk} = [R^j R^i]^{k} = \delta^{ij} \). In a similar manner, a product of Kronecker deltas is always isotropic: \( R^{i_1 i_2} \cdots R^{i_{2k-1} i_{2k}} \delta_{h_1 h_2} \cdots \delta_{h_{2k-1} h_{2k}} = [R^h R^j]^{i_1 i_2} \cdots [R^{i_{2k-1}} R^{i_{2k}}]^{i_{2k-1} i_{2k}} \).

In general, a tensor is isotropic if and only if it is expressible as a linear combination of products of Kronecker deltas and possibly one Levi-Civita tensor \([22, 23]\):
where the constants $c_{nk}$ and $d_{nk}$ depend on the permutation $\sigma$ defined in section 2.4, the number of dimensions $n$, and the rank $k$. The first term in the brackets survives only if $k$ is even. The second term in the brackets survives only if $k \geq n$ and $k - n$ is even.

2.6. Totally symmetric isotropic tensor

A tensor is called totally symmetric if it is invariant under exchange of any two vector indices. We denote $\mathcal{T}^{h_{1} \cdots h_{k}}_{(k)}$ by the totally symmetric isotropic tensor of rank $k$. Thus $\mathcal{T}^{h_{1} \cdots h_{k}}_{(k)}$ can be constructed by symmetrising the indices of terms in equation (12) with the coefficient $c_{nk}$ only. Then non-vanishing entries are of rank even only:

$$
\mathcal{T}^{h_{1} \cdots h_{k}}_{(2k)} = \frac{c_{nk}}{(2k)!!} \sum_{\sigma} \delta^{\sigma(1) \cdots \sigma(n)} \cdots \delta^{\sigma((2k-1) \cdots (2k)},
$$

where $c_{nk}$ is independent of $\sigma$ and depends only on $n$ and $k$. The additional factor $(2k)!! = (2!)^{k}k!$ is divided to cancel the over-counts of the summation over $\sigma$ in equation (13). Here, the factor $k!$ appears because of the commutativity of the multiplication of $k$ Kronecker deltas and the factor $(2!)^{k}$ appears because each of $k$ Kronecker deltas is symmetric. As a result, the constant $c_{nk}$ is the normalisation factor for a single distinct product of Kronecker deltas. We choose the normalisation

$$
\mathcal{T}^{h_{1} \cdots h_{k}}_{(2k)} = 1,
$$

which determines the constant $c_{nk}$ uniquely.

2.7. Decomposition

Any vector $V \in \mathcal{V}_{(m)}$ is expressed as the sum of the longitudinal vector $V_{\parallel} \in \mathcal{V}_{(m)}$ and the transverse vector $V_{\perp} \in \mathcal{V}_{(n-m)}$. Here, $\mathcal{V}_{(m)}$ and $\mathcal{V}_{(n-m)}$ are the vector spaces spanned by $\{ \hat{\varepsilon}_{p} \}_{p=1, \cdots, m}$ defined in equation (7) and $\{ \hat{\varepsilon}_{q} \}_{q=1, \cdots, m}$, respectively. The projections onto the spaces $\mathcal{V}_{(m)}$ and $\mathcal{V}_{(n-m)}$ can be made by multiplying the projection operators $\delta_{\parallel}^{ij}$ and $\delta_{\perp}^{ij}$ as

$$
V^{i}_{\parallel} = \delta_{\parallel}^{ij} V^{j},
V^{i}_{\perp} = \delta_{\perp}^{ij} V^{j},
$$

where the longitudinal and transverse projection operators are defined, respectively, by

$$
\delta_{\parallel}^{ij} = \sum_{p=1}^{m} \hat{\varepsilon}_{p}^{i} \hat{\varepsilon}_{p}^{j},
\delta_{\perp}^{ij} = \sum_{q=m+1}^{n} \hat{\varepsilon}_{q}^{i} \hat{\varepsilon}_{q}^{j},
$$

and $\delta_{\parallel}^{ij} = \delta_{\parallel}^{ij} + \delta_{\perp}^{ij}$. In a similar manner, any rank-$k$ tensor $T^{h_{1} \cdots h_{k}}_{(k)}$ can be decomposed as

$$
T^{h_{1} \cdots h_{k}}_{(k)} = (\delta_{\parallel}^{h_{1}} + \delta_{\perp}^{h_{1}}) \cdots (\delta_{\parallel}^{h_{k}} + \delta_{\perp}^{h_{k}}) T^{h_{1} \cdots h_{k}}_{(k)}.
$$
For example, a rank-$k$ Cartesian tensor $q^i \cdots q^k$ can be decomposed as
\[
q^i \cdots q^k = \sum_{r=0}^{k} \frac{1}{r!(k-r)!} \sum_{\sigma=1}^{\sigma(k)} q^{\sigma(i_i)} q^{\sigma(i_{i+1})} \cdots q^{\sigma(i_k)},
\]
where the factor $r!(k-r)!$ is divided for a given $r$ to cancel the over-counted permutations of $r$ $q^i$'s and $k-r$ $q_j$'s that are identical, respectively. For the rank-3 case $q^i q^j q^k$, the explicit expansion is given by
\[
q^i q^j q^k = q_i^2 q_j^2 q_k^2 + (q_i^1 q_j^2 q_k^2 + q_i^2 q_j^1 q_k^2 + q_i^2 q_j^2 q_k^1) + (q_i^1 q_j^1 q_k^2 + q_i^2 q_j^2 q_k^1 + q_i^2 q_j^1 q_k^2) + q_i^1 q_j^1 q_k^1.
\]

3. Totally symmetric isotropic tensor $\mathcal{I}_{(2k)}^{(2k)}$

In this section, we carry out a rigorous derivation of the totally symmetric isotropic tensors $\mathcal{I}_{(2k)}^{(2k)}$ of arbitrary ranks in $\mathfrak{S}(n)$ based on only the symmetries of the abstract algebraic structure.

3.1. Recurrence relation

According to equation (13), $\mathcal{I}_{(2k)}^{(2k)}$ must be a linear combination of products of Kronecker deltas. If we factor out $\delta^{ij}$ for $j = i_2, \cdots, i_{2k}$, then its coefficient must be a totally symmetric isotropic tensor of rank $2k-2$ so that
\[
\mathcal{I}_{(2k)}^{(2k)} = \lambda_{2k} [\delta^{ij} \mathcal{I}_{(2k-2)}^{(2k-2)} + \delta^{ij} \mathcal{I}_{(2k-2)}^{(2k-2)} + \cdots + \delta^{ij} \mathcal{I}_{(2k-2)}^{(2k-2)}],
\]
where $\lambda_{2k}$ is a constant and there are $2k-1$ terms in the brackets. If we multiply $\delta^{i_1 i_2} \delta^{i_3 i_4} \cdots \delta^{i_{2k-1} i_{2k}}$ and impose the normalisation condition in equation (14), then the term in the brackets proportional to $\delta^{i_1 i_2}$ gives $n$ and each of the remaining $2k-2$ terms gives unity. As a result, we determine $\lambda_{2k} = 1/(n+2k-2)$ and the recurrence relation as
\[
\mathcal{I}_{(2k)}^{(2k)} = \frac{1}{n+2k-2} [\delta^{ij} \mathcal{I}_{(2k-2)}^{(2k-2)} + \delta^{ij} \mathcal{I}_{(2k-2)}^{(2k-2)} + \cdots + \delta^{ij} \mathcal{I}_{(2k-2)}^{(2k-2)}].
\]

3.2. Complete reduction into Kronecker deltas

We are ready to find the explicit form of $\mathcal{I}_{(2k)}^{(2k)}$ by making recursive use of equation (21). Substituting $k = 1$ into equation (21), we determine $\mathcal{I}_{(2)}^{(2)}$ as
\[
\mathcal{I}_{(2)}^{(2)} = \frac{\delta^{ij}}{n},
\]
where we have set $\mathcal{I}_{(0)} = 1$ to be consistent with the normalisation in equation (14). In this manner, we can find $\mathcal{I}_{(2k+2)}^{(2k+2)}$ once $\mathcal{I}_{(2k)}^{(2k)}$ is known. The next two entries are given by
Each term in the brackets of the second equalities in equation (23) represents a single distinct product of Kronecker deltas. In general, the normalisation factor $c_{nk}$ in equation (13) is determined as

$$c_{nk} = \frac{1}{n(n+2) \cdots (n+2k-2)}. \quad (24)$$

Our final result for the explicit form of $\widetilde{T}^{ij_1\cdots ij_{2k-2},ij_2}_{(2k)}$ is

$$\widetilde{T}^{ij_1\cdots ij_{2k-2},ij_2}_{(2k)} = \frac{1}{n(n+2) \cdots (n+2k-2)} \frac{1}{(2k)!!} \sum_\sigma \delta^{\sigma(i_1)\sigma(i_2)} \cdots \delta^{\sigma(i_{2k-2})\sigma(i_{2k})}. \quad (25)$$

While the summation in equation (25) is over the $(2k)!!$ permutations of $(i_1,\cdots,i_{2k})$, the number of distinct terms in $\widetilde{T}^{ij_1\cdots ij_{2k-2},ij_2}_{(2k)}$ is $N(2k) \equiv (2k)!/(2k)!! = (2k-1)!!$.

### 3.3. Projection operator

The normalisation condition $\widetilde{T}^{ij_1\cdots ij_{2k-2},ij_2}_{(2k)} = 1$ in equation (14) requires that

$$\widetilde{T}^{ij_1\cdots ij_{2k-2},ij_2}_{(2k)} = \mathcal{N}(2k)c_{nk} \delta^{ij_1} \cdots \delta^{ij_{2k-2}} \widetilde{T}^{ij_1\cdots ij_{2k-2},ij_2}_{(2k)}$$

$$= \mathcal{N}(2k)c_{nk}$$

$$= \frac{(2k-1)!!}{n(n+2) \cdots (n+2k-2)}, \quad (26)$$

where we have made use of the symmetric property of $\widetilde{T}^{ij_1\cdots ij_{2k-2},ij_2}_{(2k)}$: every one of $\mathcal{N}(2k)$ distinct terms in a factor $\widetilde{T}^{ij_1\cdots ij_{2k-2},ij_2}_{(2k)}$ on the left side has the identical contribution $c_{nk} \delta^{ij_1} \cdots \delta^{ij_{2k-2}}$ to the product. It is straightforward to project out the totally symmetric isotropic part $\mathcal{T}^{ij_1\cdots ij_{2k-2}}_{(2k)}$ of a tensor $T^{ij_1\cdots ij_{2k-2}}_{(2k)}$ as

$$\mathcal{T}^{ij_1\cdots ij_{2k-2}}_{(2k)} = \Pi^{h_1\cdots h_{2k-2}}_{(2k)} \mathcal{T}^{ij_1\cdots ij_{2k-2}}_{(2k)}, \quad (27a)$$

where the projection operator $\Pi^{h_1\cdots h_{2k-2}}_{(2k)}$ is defined by

$$\Pi^{h_1\cdots h_{2k-2}}_{(2k)} \equiv \frac{n(n+2) \cdots (n+2k-2)}{(2k-1)!!} \mathcal{T}^{ij_1\cdots ij_{2k-2}}_{(2k)} \widetilde{T}^{ij_1\cdots ij_{2k-2},ij_2}_{(2k)}, \quad (27b)$$
4. Application to tensor integrals

In this section, we apply the result in equation (25) for the totally symmetric isotropic tensor of arbitrary rank in \( n \) dimensions to compute various tensor integrals involving angle averages. This leads to a great simplification of the evaluation steps into simple counts of combinatorics.

4.1. Angle average

Let us consider the tensor integral

\[
\langle \hat{\mathbf{r}}^i \cdots \hat{\mathbf{r}}^k \rangle_{\hat{\mathbf{r}}} = \frac{1}{\Omega^{(n)}} \int d\Omega^{(n)} \hat{\mathbf{r}}^i \cdots \hat{\mathbf{r}}^k,
\]

where \( \hat{\mathbf{r}} \equiv (\hat{r}^1, \hat{r}^2, \ldots, \hat{r}^n) \) is the unit radial vector \( (\hat{r}^2 = 1) \) and the tensor is independent of any specific vectors. The analytic expressions for the \( n \)-dimensional solid angle \( \Omega^{(n)} \) and its differential element \( d\Omega^{(n)} \) expressed in terms of \( n - 2 \) polar angles and an azimuthal angle are listed in appendix A.

It is manifest that \( \langle \hat{\mathbf{r}}^i \cdots \hat{\mathbf{r}}^k \rangle_{\hat{\mathbf{r}}} \) is isotropic and totally symmetric and, therefore, it must be proportional to \( \Omega_{(k)}^{(n)} \). Because \( \Omega_{(k)}^{(n)} = 0 \) for any \( k \) odd, the only non-vanishing components are \( \langle \hat{\mathbf{r}}^i \cdots \hat{\mathbf{r}}^k \rangle_{\hat{\mathbf{r}}} \). By multiplying \( \delta_{i_1 i_2} \delta_{i_3 i_4} \cdots \delta_{i_{2k-1} i_{2k}} \), summing over the indices, and substituting \( \hat{r}^2 = 1 \), we find that \( \langle \hat{\mathbf{r}}^i \cdots \hat{\mathbf{r}}^k \rangle_{\hat{\mathbf{r}}} \) satisfies the normalisation condition in equation (14). As a result,

\[
\langle \hat{\mathbf{r}}^i \cdots \hat{\mathbf{r}}^k \rangle_{\hat{\mathbf{r}}} = \Omega_{(2k)}^{(n)}.
\]

This can be applied to derive the angle-average formulas \( \langle (\mathbf{a} \cdot \hat{\mathbf{r}})^{2k-1} \rangle_{\hat{\mathbf{r}}} = 0 \) and

\[
\langle (\mathbf{a} \cdot \hat{\mathbf{r}})^{2k} \rangle_{\hat{\mathbf{r}}} = a^i a^{i_2} \cdots a^{i_{2k}} \Omega_{(2k)}^{(n)} = \frac{(2k - 1)!!}{n(n + 2) \cdots (n + 2k - 2)} a^{2k},
\]

where \( \mathbf{a} \) is a constant vector and \( k \) is a positive integer. This result agrees with equation (B6) that is obtained by direct evaluations of angular integrals. In general, for any constant vectors \( \mathbf{a}_1, \ldots, \mathbf{a}_m \), we obtain

\[
\langle (\mathbf{a}_1 \cdot \hat{\mathbf{r}})(\mathbf{a}_2 \cdot \hat{\mathbf{r}}) \cdots (\mathbf{a}_m \cdot \hat{\mathbf{r}}) \rangle_{\hat{\mathbf{r}}} = a_1^i a_2^{i_2} \cdots a_m^{i_{2k}} \Omega_{(m)}^{(n)}.
\]

The expression vanishes for all \( m \) odd and \( \mathbf{a}_i \)'s do not have to be distinct. As is discussed in appendix B, an explicit integration over polar and azimuthal angles is extremely tedious even in two or three dimensions and it is non-trivial to obtain the general form in equation (31) directly by integration. Therefore, our strategy to make use of \( \Omega_{(m)}^{(n)} \) is a quite efficient way to evaluate the angular integrals.

4.2. Tensor-integral reduction

In general, the integrand of a tensor angular integral may have a scalar factor that depends on constant vectors as well as the integral variable \( q \), while the angle average in equation (28) is independent of any specific vectors. The simplest case is that the scalar factor depends only on \( q \):

\[
A_{(k)}^{(n)} = \int q^i \cdots q^n f(q),
\]

where \( f(q) \) is an arbitrary function.
where $q = (q^1, \cdots, q^n)$, $f(q)$ is a scalar, and the symbol $\int_q$ is defined by
\[
\int_q \equiv \int_{-\infty}^{\infty} dq^1 \cdots \int_{-\infty}^{\infty} dq^n = \int_0^\infty \, \|q\|^{n-1} \, d\Omega^{(n)}_q. \tag{33}
\]
Here, $d\Omega^{(n)}_q$ is the differential solid-angle element of $\hat{q}$. It is manifest that $A_{(k)}^{i_1 \cdots i_k}$ in equation (32) is totally symmetric and isotropic so that only the even-rank case survives: $A_{(2k)}^{i_1 \cdots i_{2k}} = \Pi^{i_1 \cdots i_{2k} j_1 \cdots j_{2k}} A_{(2k)}^{j_1 \cdots j_{2k}}$ according to equation (27). Therefore, the non-vanishing elements are completely determined as
\[
A_{(2k)}^{i_1 \cdots i_{2k}} = \int_q q^{i_1} f(q), \tag{34}
\]
where we have used $\int_q q^i j^j q^{j_{2k-1}} q^{j_{2k-1}} = 0$ and $\int_q q^i j^j q^{j_{2k}} = \frac{(2k-1)!}{(n+2) - (n+2k-2)} q^{2k}$. As a result, one has to compute only a single scalar integral $\int_q q^{2k} f(q)$ without evaluating all of the $n^{2k}$ components of $A_{(2k)}^{i_1 \cdots i_{2k}}$.

A more complicated situation is that $B_{(k)}^{i_1 \cdots i_k}$ is a scalar that depends on $a$ and a constant vector $\hat{a}$ and because $\int_q$ is totally symmetric and isotropic so that only the even-rank case survives: $B_{(2k)}^{i_1 \cdots i_{2k}} = \Pi^{i_1 \cdots i_{2k} j_1 \cdots j_{2k}} B_{(2k)}^{j_1 \cdots j_{2k}}$ according to equation (27). Therefore, the non-vanishing elements are completely determined as
\[
B_{(2k)}^{i_1 \cdots i_{2k}} = \int_q q^{i_1} \cdots q^{i_{2k}} g(q, a), \tag{35}
\]
where $g(q, a)$ is a scalar that depends on $q$ and a constant vector $a$. It is manifest that $B_{(2k)}^{i_1 \cdots i_k}$ is symmetric under exchange of any two vector indices and it must depend only on $a$ because $q$ is integrated out. We call $\Omega^{(n)(1)}$ the one-dimensional Euclidean space spanned by $a$ and $\Omega^{(n)(n-1)}$ the space perpendicular to $a$. According to equation (15), $q' = q + a$, with $q_i = (q - a) \hat{a} \in \Omega^{(n)(1)}$ and $q = q_i \in \Omega^{(n)(n-1)}$. By making use of equation (18), we decompose $B_{(2k)}^{i_1 \cdots i_k}$ as
\[
B_{(2k)}^{i_1 \cdots i_k} = \sum_{r=0}^{k} \frac{1}{r! (k-r)!} \sum_{\sigma} \delta^{(i_1)}(\xi) \cdots \delta^{(i_k)}(\xi) \int_q q^{(i_{r+1})} \cdots q^{(i_{k})} g(q, a), \tag{36}
\]
where $g(q, a) \equiv (q \cdot \hat{a}) g(q, a)$. The first summation is over the number of longitudinal components, $r$. The constant tensor $\delta^{(i_1)}(\xi) \cdots \delta^{(i_k)}(\xi)$ of rank $r$ and the remaining tensor integral $\int_q q^{(i_{r+1})} \cdots q^{(i_{k})} g(q, a)$ of rank $k-r$ are defined in $\Omega^{(n)(1)}$ and $\Omega^{(n)(n-1)}$, respectively. The tensor integral in $\Omega^{(n)(n-1)}$ is totally symmetric and isotropic. Thus we find that
\[
\int_q q^{(i_{r+1})} \cdots q^{(i_{k})} g_r(q, a) = \int_{\Omega^{(n)(n-1)}} q^{(i_{r+1})} \cdots q^{(i_{k})} g_r(q, a), \tag{37}
\]
where $\int_{\Omega^{(n)(n-1)}}$ is the $\Omega^{(n)(n-1)}$ analogue of $\int_{\Omega^{(n)}}$ defined in $\Omega^{(n)}$, the explicit form of $\int_{\Omega^{(n)}}$ can be obtained by replacing every Kronecker delta with the corresponding one in $\Omega^{(n)(n-1)}$ that is given in equation (16) and replacing the dimension $n$ with $n-1$ as
\[
\int_{\Omega^{(n)(n-1)}} = \int_{\Omega^{(n)}} |\hat{a}|^{n-1}. \tag{38}
\]
As a result, $B_{(2k)}^{i_1 \cdots i_k}$ can be expressed as the following linear combination:
\[
B_{(2k)}^{i_1 \cdots i_k} = \sum_{r=0}^{k} E_{(r)[k]}^{i_1 \cdots i_k} \int_q |q|^{k-r} g_r(q, a), \tag{39}
\]
where the constant tensor $E_{(r)[k]}^{i_1 \cdots i_k}$ of rank $k$ with $r$ longitudinal indices is defined by
\[
E_{(r)[k]}^{i_1 \cdots i_k} = \begin{cases} \frac{1}{r! (k-r)!} \sum_{\sigma} \delta^{(i_1)}(\xi) \cdots \delta^{(i_k)}(\xi) \int_{\Omega^{(n)(n-1)}} q^{(i_{r+1})} \cdots q^{(i_{k})} g_r(q, a), & \text{if } k-r \text{ is even}, \\ 0, & \text{if } k-r \text{ is odd}. \end{cases} \tag{40}
\]
Here, $\mathcal{E}^{(i)k}_{i(k)}$ is totally symmetric. Although this tensor is not completely isotropic, there is a partial isotropy among transverse components only. For example, we list the reduction formulas for the first four entries of $\mathcal{E}^{(i)k}_{i(k)}$:

\[
\int q q^i g(q, a) = \mathcal{E}^{(1,1)}_{(1,1)} \int q g_1(q, a),
\]
\[
\int q^2 q^i g(q, a) = \mathcal{E}^{(0,2)}_{(0,2)} \int q g_2(q, a) + \mathcal{E}^{(1,2)}_{(1,2)} \int q g_2(q, a),
\]
\[
\int q^3 q^i q^j g(q, a) = \mathcal{E}^{(1,3)}_{(1,3)} \int q g_3(q, a) + \mathcal{E}^{(1,2)}_{(2,2)} \int q g_2(q, a) + \mathcal{E}^{(1,2)}_{(3,3)} \int q g_2(q, a),
\]
\[
\int q^4 q^i q^j q^k g(q, a) = \mathcal{E}^{(2,4)}_{(2,4)} \int q^4 g_4(q, a) + \mathcal{E}^{(2,4)}_{(3,4)} \int q^4 g_4(q, a) + \mathcal{E}^{(2,4)}_{(4,4)} \int q^4 g_4(q, a),
\]

where

\[
\mathcal{E}^{(1,1)}_{(1,1)} = \alpha^i,
\]
\[
\mathcal{E}^{(0,2)}_{(0,2)} = \delta^{ij},
\]
\[
\mathcal{E}^{(1,2)}_{(1,2)} = \alpha^i \alpha^j,
\]
\[
\mathcal{E}^{(1,3)}_{(1,3)} = \frac{1}{n-1} (\alpha^i \delta^{jk} + \alpha^j \delta^{ki} + \alpha^k \delta^{ij}),
\]
\[
\mathcal{E}^{(2,4)}_{(2,4)} = \frac{1}{(n-1)(n+1)} (\delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk}),
\]
\[
\mathcal{E}^{(2,4)}_{(3,4)} = \frac{1}{n-1} (\alpha^i \alpha^j \delta^{kl} + \alpha^i \alpha^k \delta^{jl} + \alpha^i \alpha^l \delta^{kj} + \alpha^j \alpha^k \delta^{il} + \alpha^j \alpha^l \delta^{ki} + \alpha^k \alpha^l \delta^{ij}),
\]
\[
\mathcal{E}^{(2,4)}_{(4,4)} = \alpha^i \alpha^j \delta^{kl}.
\]

The most general case is that the integrand depends on $m$ linearly independent constant vectors $a_1, \ldots, a_m$:

\[
\mathcal{C}^{(i)h}_{(k)} = \int q q^i \cdots q^h h(q, a_1, \ldots, a_m),
\]

where $h(q, a_1, \ldots, a_m)$ is a scalar. The tensor $\mathcal{C}^{(i)h}_{(k)}$ is symmetric under exchange of any two vector indices. We call $\Omega_{(m)}$ the $m$-dimensional Euclidean space spanned by $a_1, \ldots, a_m$, and $\Omega_{(n-m)}$ the space perpendicular to those constant vectors. We choose the unit basis vectors $\hat{e}_1, \ldots, \hat{e}_m$ in equation (7) to span $\Omega_{(m)}$. Then we can decompose $q$ as $q = q_\parallel + q_\perp$, where $q_\parallel = \sum_{p=1}^m \hat{e}_p (\hat{e}_p \cdot q) \in \Omega_{(m)}$ and $q_\perp = q - q_\parallel \in \Omega_{(n-m)}$. By making use of the identity in equation (18), we can express $\mathcal{C}^{(i)h}_{(k)}$ as

\[
\mathcal{C}^{(i)h}_{(k)} = \sum_{r=0}^k \frac{1}{r!(k-r)!} \sum_{\alpha} \int q_\parallel q_\perp \cdots q_\parallel q_\perp q_\parallel \cdots q_\parallel h(q, a_1, \ldots, a_m),
\]

where the first summation is over $r$, the number of longitudinal components. Because every $\hat{e}_p^i$ is independent of $q$, we find that
where \( p_1, \ldots, p_r \) are indices of Cartesian axes of \( \mathcal{V}_{(m)} \) and the scalar \( h_{p_1, \ldots, p_r} \) is defined by
\[
h_{p_1, \ldots, p_r}(q, a_1, \ldots, a_m) \equiv (q \cdot \hat{e}_{p_1}) \cdots (q \cdot \hat{e}_{p_r}) h(q, a_1, \ldots, a_m),
\]
In equation (45), we have made a replacement \( \eta^{\sigma_i(1)} \cdots \eta^{\sigma_i(1)} \rightarrow \eta_{(k-r)}^{\sigma_i(1)} \cdots \eta_{(k-r)}^{\sigma_i(1)} \) by taking into account the isotropy of \( \epsilon^{\sigma_i(1) \cdots \sigma_i(1)} \) in the space \( \mathcal{V}_{(n)} \). Here, \( \eta_{(k-r)}^{\sigma_i(1) \cdots \sigma_i(1)} \) is the analogue of \( \epsilon^{\sigma_i(1) \cdots \sigma_i(1)} \) that is defined in \( \mathcal{V}_{(n)} \): the explicit form of \( \eta_{(k-r)}^{\sigma_i(1) \cdots \sigma_i(1)} \) can be obtained by replacing every Kronecker delta with the corresponding one in \( \mathcal{V}_{(k-r)} \) that is given in equation (16) and replacing the dimension \( n \) with \( n - m \) as
\[
\eta_{(k-r)}^{\sigma_i(1) \cdots \sigma_i(1)} = \eta^{\sigma_i(1) \cdots \sigma_i(1)}_{(n-m)},
\]
As an example, we list the first four entries of the tensor-integral reduction formulas for \( m = 2 \):
\[
\int q^i h(q, a_1, a_2) = \sum_{a=1}^{2} \epsilon^{i}_{a} \int q \ h_{a}(q, a_1, a_2),
\]
\[
\int q^i q^j h(q, a_1, a_2) = \sum_{a,b=1}^{2} \epsilon^{i}_{a} \epsilon^{j}_{b} \int q \ h_{a,b}(q, a_1, a_2) + \frac{\delta^{ij}}{n-2} \int q_{k}^{2} h(q, a_1, a_2),
\]
\[
\int q^i q^j q^k h(q, a_1, a_2) = \sum_{a,b,c=1}^{2} \epsilon^{i}_{a} \epsilon^{j}_{b} \epsilon^{k}_{c} \int q \ h_{a,b,c}(q, a_1, a_2)
\]
\[
+ \frac{2}{n-2} \int q_{i}^{2} h_{a}(q, a_1, a_2)
\]
\[
\int q^i q^j q^k q^l h(q, a_1, a_2) = \sum_{a,b,c,d=1}^{2} \epsilon^{i}_{a} \epsilon^{j}_{b} \epsilon^{k}_{c} \epsilon^{l}_{d} \int q \ h_{a,b,c,d}(q, a_1, a_2)
\]
\[
+ \frac{2}{n-2} \int q_{i}^{2} h_{a,b}(q, a_1, a_2)
\]
\[
+ \frac{4}{(n-2)n} \int q_{i}^{4} h(q, a_1, a_2).
\]

As an example, we list the first four entries of the tensor-integral reduction formulas for \( m = 2 \):
is the generalised version of the Kronecker delta of rank-2 tensor appears in various applications of particle physics, molecular dynamics, fluid dynamics, material sciences, and so forth. Thus, it is essential to know exact formulas of those tensors to calculate angle averages of an isotropic system and the corresponding value for a system that has partial isotropies in subspaces.

We have illustrated a rigorous approach to derive the totally symmetric isotropic tensor $\mathcal{T}_{ii}^{(2k)}$ of arbitrary rank $k$ in the $n$-dimensional Euclidean space. The derivation is based on only the abstract algebraic structure and symmetries. All of the tensors of rank odd vanish and $\mathcal{T}_{ii}^{(2k)}$ is expressed as a linear combination of products of Kronecker deltas as shown in equation (25). The approach has been generalised to analyse a physical system that has totally symmetric isotropic components in a subspace. As an immediate application, we have demonstrated that angle averages can be evaluated without carrying out cumbersome angular integration. Instead, the symmetric properties of the tensor $\mathcal{T}_{ii}^{(2k)}$ enable us to determine the averages by only counting combinatoric multiplicity factors.

Loop integrals appearing in perturbative calculations of the gauge-field theory may have divergences. To carry out a standard renormalisation procedure, one first regularises such an integral using dimensional regularisation. After imposing dimensional regularisation, there are numerous tensor integrals in $n = 3 - 2\varepsilon$ spatial dimensions. We have demonstrated a systematic procedure to reduce those tensor integrals as a linear combination of constant tensors whose coefficients are scalar integrals. This straightforward demonstration of the tensor-integral reduction in the $n$-dimensional Euclidean space is equally applicable to the Minkowski space problems such as the Passarino–Veltman reduction without losing generality.

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Appendix A. Spherical polar coordinates in $n$ dimensions

In this appendix, we illustrate the standard parametrisation of the spherical polar coordinates and the corresponding solid-angle element in $n$ dimensions.

A.1. Polar and azimuthal angles

The spherical polar coordinates consist of the radius $r$, $n - 2$ polar angles $\theta_i$, and a single azimuthal angle $\phi$, where $i = 1, \cdots, n - 2$. The construction of this system can be easily achieved by applying the Pythagoras theorem recursively.

The radial vector $\mathbf{r}$ can be expressed in terms of the Cartesian coordinates as

$$ \mathbf{r} = (x^1, \cdots, x^n). \tag{A1} $$

Its magnitude is the radius, $r = \sqrt{x^2} = \sqrt{x^1 x^n}$, and the unit radial vector $\hat{r} \equiv \mathbf{r}/r$ is a function of polar and azimuthal angles. If $n = 2$, then the constraint $\hat{r}^2 = (\hat{r}^1)^2 + (\hat{r}^2)^2 = 1$
allows us to parametrise the coordinates as \( \hat{r}^1 = \sin \phi \) and \( \hat{r}^2 = \cos \phi \). Similarly, one can parametrise the last two coordinates for \( n \geq 3 \) as \( \hat{r}^n = \sqrt{1 - \sum_{k=1}^{n-2} (\hat{r}^k)^2} \sin \phi \) and \( \hat{r}^{n-1} = \sqrt{1 - \sum_{k=1}^{n-3} (\hat{r}^k)^2} \sin 1 \hat{n} \). Thus we require a single azimuthal angle \( \phi \) for all \( n \geq 2 \) with the allowed range \( 0 \leq \phi \leq 2\pi \). The first \( n - 2 \) coordinates \( \hat{r}^1, \ldots, \hat{r}^{n-2} \) for \( n \geq 3 \) are parametrised by only polar angles as follows: we set \( \gamma_k \equiv \hat{r}_k \). We can always decompose \( \gamma_k \) into \( \gamma_k = \alpha_k + \beta_1 \), where \( \alpha_k \equiv \left( \gamma_k^1, 0, \ldots, 0 \right) \) and \( \beta_1 \equiv \left( 0, \gamma_k^2, \ldots, \gamma_k^n \right) \) are along and perpendicular to the \( x^1 \) axis, respectively. Because \( \gamma_k^1 = \alpha_k^1 + \beta_1 \), we can introduce a polar angle \( \theta_1 \) such that \(-1 \leq \hat{r}^1 = \gamma_k^1 = \cos \theta_1 \leq 1 \) and \( 0 \leq |\beta_1| = \sqrt{\sum_{i=2}^{n} (\gamma_k^i)^2} = \sin \theta_1 \leq 1 \). The allowed range of the polar angle \( \theta_1 \) is \( 0 \leq \theta_1 \leq \pi \). We can define the unit vector \( \gamma_k^1 = \beta_{k-1} / \sin \theta_{k-1} \) recursively for \( k = 2, \ldots, n - 2 \). In a similar manner, we can decompose \( \gamma_k \) into \( \gamma_k = \alpha_k + \beta_k \) with \( \alpha_k \equiv \left( \gamma_k^1, 0, \ldots, 0 \right) \) and \( \beta_k \equiv \left( 0, \gamma_k^{k+1}, \ldots, \gamma_k^n \right) \), where we have neglected the \( x^1, \ldots, x^{k-1} \) components that are vanishing. Because \( \gamma_k^1 = \alpha_k^1 + \beta_k \), we can introduce a polar angle \( \theta_k \) such that \(-1 \leq \gamma_k^1 = \cos \theta_k \leq 1 \) and \( 0 \leq |\beta_k| = \sqrt{\sum_{i=k+1}^{n} (\gamma_k^i)^2} = \sin \theta_k \leq 1 \), where \( 0 \leq \theta_k \leq \pi \).

In summary, the Cartesian coordinates for a unit radial vector for \( n \geq 2 \) is parametrised with \( n - 2 \) polar angles and an azimuthal angle as

\[
\begin{align*}
\hat{r}^1 &= \cos \theta_1, \\
\hat{r}^2 &= \sin \theta_1 \cos \theta_2, \\
\hat{r}^3 &= \sin \theta_1 \sin \theta_2 \cos \theta_3, \\
&\vdots \\
\hat{r}^{n-2} &= \sin \theta_1 \sin \theta_2 \sin \theta_3 \cdots \sin \theta_{n-3} \cos \theta_{n-2}, \\
\hat{r}^{n-1} &= \sin \theta_1 \sin \theta_2 \sin \theta_3 \cdots \sin \theta_{n-3} \sin \theta_{n-2} \sin \phi, \\
\hat{r}^n &= \sin \theta_1 \sin \theta_2 \sin \theta_3 \cdots \sin \theta_{n-3} \sin \theta_{n-2} \cos \phi.
\end{align*}
\]

For \( n = 2 \), only an azimuthal angle is required to express \( \hat{r}^1 = \sin \phi \) and \( \hat{r}^2 = \cos \phi \).

**A.2. Solid angle**

**A.2.1. Gaussian-integral method.** The computation of the solid angle \( \Omega^{(n)} \) in the \( n \)-dimensional Euclidean space can be carried out by making use of a Gaussian integral:

\[
I^{(1)} \equiv \int_{-\infty}^{\infty} dx \; e^{-x^2} = \sqrt{\pi}.
\]

(A3)

The \( n \)-dimensional Gaussian integral,

\[
I^{(n)} \equiv \int_{-\infty}^{\infty} dx^1 \int_{-\infty}^{\infty} dx^2 \cdots \int_{-\infty}^{\infty} dx^n \; e^{-\left(\sum_{i=1}^{n} x_i^2\right)},
\]

(A4)

can be evaluated in the spherical polar coordinate system in which the integrand is independent of the direction of the Euclidean vector \( r = (x^1, \ldots, x^n) \) whose radius is defined by \( r = \sqrt{x^1^2 + \cdots + x^n^2} \). The integrand of (A4) depends only on \( r \) and it is independent of the direction of \( r \). Then the radial integral for \( I^{(n)} \) is evaluated as

\[
I^{(n)} = \Omega^{(n)} \int_{0}^{\infty} dr \; r^{n-1} e^{-r^2} = \Omega^{(n)} \frac{\Gamma\left(\frac{n}{2}\right)}{2},
\]

(A5)
where the gamma function is defined by
\[ \Gamma(x) \equiv \int_0^\infty t^{x-1}e^{-t}dt. \]  
(A6)

Because \( I^{(n)} \) is the \( n \)th power of \( I^{(1)} \), we have \( I^{(n)} = \pi^{n/2} \). This determines the solid angle \( \Omega^{(n)} \) in \( n \) dimensions as
\[ \Omega^{(n)} = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{1}{2}n\right)}. \]  
(A7)

### A.2.2. Angular-integral derivation.

The parametrisation (A2) can be used to express an \( n \)-dimensional volume integral for \( \Omega^{(n)} \) as a product of the radial integral and the angular one as:
\[ \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx^2 \cdots \int_{-\infty}^{\infty} dx^n = \int_0^\infty dr \int_0^{2\pi} d\phi \int_0^\pi d\theta_1 \int_0^{\pi} d\theta_2 \cdots \int_0^{\pi} d\theta_{n-2} J \]
\[ = \int_0^\infty dr \int_0^{2\pi} d\phi \Omega^{(n)}, \]  
(A8)

where \( d\Omega^{(n)}_r \) and \( J \) are the solid-angle element of the unit radial vector \( \hat{r} \) and the Jacobian, respectively, and they are defined by
\[ d\Omega^{(n)}_r = d\phi \, d\theta_1 \cdots d\theta_{n-2} J, \]
\[ J = \sin^{n-2} \theta_1 \sin^{n-3} \theta_2 \cdots \sin^2 \theta_{n-3} \sin \theta_{n-2}. \]  
(A9)

For \( n = 2 \), there is no polar angle and \( J = 1 \). Changing the integration variables for polar angles from \( \theta_i \) to \( z_i \equiv \cos \theta_i \), we find that
\[ d\Omega^{(n)}_r = d\phi \, dz_1 \cdots dz_{n-2} J, \]
\[ J = (1 - z_1^2) + (1 - z_2^2)^2 + \cdots (1 - z_{n-3}^2)^2 \cdot 1. \]  
(A10)

Note that \( \sin \theta_i = \sqrt{1 - z_i^2} \geq 0 \) because \( 0 \leq \theta_i \leq \pi \). By integrating over \( \phi \) and \( z_i \)'s, we can reproduce the solid-angle formula (A7):
\[ \Omega^{(n)} = \int d\Omega^{(n)}_r \]
\[ = \int_0^{2\pi} d\phi \prod_{j=1}^{n-2} \int_1^{\infty} dz_j (1 - z_j^2)^{\frac{\nu - 1}{2}} \]
\[ = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{1}{2}n\right)} \frac{\Gamma\left(\frac{1}{2}n - \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}n - \frac{1}{2}\right)} \cdots \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}. \]  
(A11)

where we have used \( \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \) and the integral table
\[ \int_{-1}^{1} (1 - x^2)^{\nu}dx = \frac{\Gamma(n + 1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n + \frac{3}{2}\right)}. \]  
(A12)
Appendix B. Direct evaluation of angle average

By making use of the definition for the tensor angular integral in equation (28), we can express the average of \((\mathbf{a} \cdot \hat{r})^k\) over the direction of the unit radial vector \(\hat{r}\) as

\[
\langle (\mathbf{a} \cdot \hat{r})^k \rangle_{\hat{r}} = \frac{1}{\Omega^{(n)}} \int d\Omega^{(n)}_{\hat{r}} (\mathbf{a} \cdot \hat{r})^k,
\]

where \(k\) is a non-negative integer and \(\mathbf{a}\) is a constant vector. The solid-angle element \(d\Omega^{(n)}_{\hat{r}}\) and the \(n\)-dimensional solid angle \(\Omega^{(n)}\) are defined in equations (A10) and (A11), respectively.

Because the integrand \((\mathbf{a} \cdot \hat{r})^k\) is a scalar, the average is invariant under rotation. Thus, there exists a rotational transformation to make \(x^1\) Cartesian axis parallel to \(\mathbf{a}\) so that \(\mathbf{a} = (|\mathbf{a}|, 0, \cdots, 0)\). In that case, the average is simplified as

\[
\langle (\mathbf{a} \cdot \hat{r})^k \rangle = |\mathbf{a}|^k \langle (\hat{r})^k \rangle.
\]

By making use of the parametrisation for \(d\Omega^{(n)}_{\hat{r}}\) in equation (A10) and integrating over \(\phi, z_2, \cdots, z_{n-2}\), we find that

\[
\langle (\hat{r})^k \rangle = \int_0^1 dz_1 z_1^k (1 - z_1^2)^{\frac{n-3}{2}}.
\]

For all \(k\) odd, the integrand of the numerator is odd to make the integral vanish. By making use of the integral table,

\[
\int_0^1 (1 - x^2)^n x^k dx = \frac{1 + (-1)^k}{2} \frac{\Gamma(1 + a) \Gamma\left(\frac{1}{2} + \frac{1}{2} b \right)}{\Gamma\left(a + \frac{1}{2} b + \frac{3}{2}\right)},
\]

and the identities

\[
\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{(k - 1)!!}{\sqrt{2^k}},
\]

\[
\frac{\sqrt{2^k} \Gamma\left(\frac{n+k}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} = n(n + 2) \cdots (n + k - 4)(n + k - 2),
\]

we find that

\[
\langle (\mathbf{a} \cdot \hat{r})^k \rangle = \begin{cases} 0, & k \text{ odd}, \\ |\mathbf{a}|^k \frac{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{n+k}{2}\right)} = \frac{|\mathbf{a}|^k (k - 1)!!}{n(n + 2)(n + 4) \cdots (n + k - 2)}, & k \text{ even}. \end{cases}
\]

This result for the angular integral agrees with equation (30) that is obtained by taking into account the symmetries only.

The most general form of the angle average is

\[
\langle (\mathbf{a}_1 \cdot \hat{r})(\mathbf{a}_2 \cdot \hat{r}) \cdots (\mathbf{a}_m \cdot \hat{r}) \rangle_{\hat{r}} = \frac{1}{\Omega^{(n)}} \int d\Omega^{(n)}_{\hat{r}} (\mathbf{a}_1 \cdot \hat{r})(\mathbf{a}_2 \cdot \hat{r}) \cdots (\mathbf{a}_m \cdot \hat{r}),
\]

where all of the constant vectors \(\mathbf{a}_i\) do not have to be distinct and \(m\) is a positive integer which is not bounded above. The integral can be expressed as a linear combination of
where \( k_i \)'s are non-negative integers satisfying \( \sum_{i=1}^{n} k_i = m \). In principle, the evaluation of the integral is straightforward. However, the corresponding calculations consist of many steps and one should take extreme care to avoid mistakes during such a tedious computation. Instead, by making use of the totally symmetric isotropic tensor given in equation \((31)\), one can greatly reduce the efforts and obtain the result only by counting combinatoric multiplicity factors.

**Appendix C. Reduction in the \( d \)-dimensional Minkowski space**

In evaluating divergent loop integrals coming from perturbative calculations of the gauge-field theory, one regularises those integrals by analytic continuation of the space–time dimensions from \( 4 \) to \( d = 4 - 2\epsilon \) to express the integral measure as

\[
\int_{q} \equiv \int d^{d}q = \int_{-\infty}^{\infty} dq^{0} \int_{-\infty}^{\infty} dq^{1} \cdots \int_{-\infty}^{\infty} dq^{d-1},
\]

where the contravariant \( d \)-vector \( q^{\mu} = (q^{0}, q^{1}, \cdots, q^{d-1}) \) is the loop momentum. Here, \( q^{0} \) and \( q^{i} \) correspond to the time and spatial components, respectively. Note that a Greek letter is used for a \( d \)-vector index in a Minkowski space while an italic index is used for the Euclidean space.

One can integrate out \( q^{0} \) by closing the contour on the complex \( q^{0} \) plane to find the residue originated from the relevant propagator factors. Then the integral reduces into a form defined in the \((d-1)\)-dimensional Euclidean space that can be always evaluated by making use of formulas presented in the text. Alternatively, without carrying out the \( q^{0} \) integral first, one can directly evaluate the \( d \)-dimensional integral as follows.

The scalar product of two \( d \)-vectors \( a \) and \( b \), which is invariant under Lorentz transformation, is defined by

\[
a \cdot b = a^{\mu}b^{\nu}g_{\mu\nu} = a^{0}b^{0} - a^{1}b^{1} - \cdots - a^{d-1}b^{d-1},
\]

where \( g_{\mu\nu} = g^{\mu\nu} = \text{diag}[1, -1, -1, \cdots, -1] \) is the metric tensor for the \( d \)-dimensional Minkowski space that corresponds to the Kronecker delta \( \delta^{\mu} \) in the Euclidean space. The covariant \( d \)-vector \( a_{\mu} \) corresponding to the contravariant \( d \)-vector \( a^{\mu} = (a^{0}, a^{1}, a^{2}, \cdots, a^{d-1}) \) is defined by \( a_{\mu} = g_{\mu\nu}a^{\nu} = (a^{0}, -a^{1}, -a^{2}, \cdots, -a^{d-1}) \) and \( a^{\mu} = g^{\mu\nu}a_{\nu} \).

By generalising equation \((34)\) to the \( d \)-dimensional Minkowski space, we can reduce the tensor loop integral that depends only on the loop momentum \( q \) into the following form

\[
\int_{q} q^{\mu_{1}} \cdots q^{\mu_{2k}} f(q) = \int_{(2k)} \int_{q} (q \cdot q)^{k} f(q),
\]

where \( f(q) \) is a Lorentz scalar which is invariant under Lorentz transformation and we have neglected the vanishing contributions of rank odd. The totally symmetric isotropic tensor of rank \( 2k \),
The generalised version into the d-dimensional Minkowski space: η and δij in equation (25) are replaced with d and gμμ, respectively.

In a similar manner, we can reduce the tensor loop integral that depends on the loop momentum q and an external momentum a for a massive particle (a · a > 0) as

\[ \int q^{\mu_1} \cdots q^{\mu_k} f(q, a) = \frac{1}{d(d + 1) \cdots (d + 2k - 2)(2k)!!} \sum_\sigma g^{\sigma(\mu_1) \cdots \sigma(\mu_k)} g_{\perp(\mu_{k-1})}^{\sigma(\mu_k)} \times \int \frac{(a \cdot a') (q \cdot a')^{(k-r)/2}}{(a \cdot a')^{(k-r)/2}} f(q, a), \]

where the summation over r is for even k − r only. The totally symmetric isotropic tensor of rank even is given by

\[ \tilde{T}^{(2k)}_{\perp} = \frac{1}{(d - 1)(d + 1) \cdots (d + 2k - 3)(2k)!!} \sum_\sigma g^{\sigma(\mu_1) \cdots \sigma(\mu_k)} g_{\perp(\mu_{k-1})}^{\sigma(\mu_k)}. \]

Here, qμ = gμν qν and gμν is defined by

\[ g_{\perp}^{\mu\nu} = \bar{g}^{\mu\nu} - \frac{a^\mu a^\nu}{a \cdot a}. \]

If the scalar function f(q, a) is replaced with f(q, a₁, ..., a₀), where aᵢ is the d-momentum of the i-th external massive particle, then one can generalise the method to obtain equation (43) in a similar manner that we have employed to derive the relativistic version in equation (C5) in the presence of a single external particle.

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