ON THE LIFESPAN OF THREE-DIMENSIONAL GRAVITY WATER WAVES WITH VORTICITY

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ABSTRACT. We prove a long-term regularity result for three-dimensional gravity water waves with small initial data but nonzero initial vorticity. We consider solutions whose vorticity vanishes on the free boundary and use this to derive a system for the evolution of the free boundary which reduces to the Zakharov/Craig-Sulem formulation in the irrotational case. We are able to continue the solution until a time determined by the size of the initial vorticity in such a way that if the vorticity is zero, one recovers a lifespan $T \sim \varepsilon^{-N}$ where $N$ can be taken arbitrarily large if the initial data is taken to be arbitrarily smooth.

1. Introduction

The motion of an inviscid incompressible fluid occupying a region $D = \cup_{0 \leq t \leq T} \{t\} \times D_t, D_t \subset \mathbb{R}^3$ is described by the fluid velocity $v = (v_1, v_2, v_3)$ and a non-negative function $p$ known as the pressure. If the fluid body is subject to the force of gravity, then the equations of motion are given by Euler’s equations:

$$\left( \partial_t + v^k \partial_k \right) v_i = -\partial_i p - e_3 \quad \text{in } D_t,$$

where $\partial_i = \frac{\partial}{\partial x_i}$, and conservation of mass:

$$\text{div } v = \partial_i v^i = 0 \quad \text{in } D_t.$$  

Here, we are using the Einstein summation convention and summing over repeated upper and lower indices and writing $v^i = \delta_{ij} v_j$. We have also chosen units so that the acceleration due to gravity is one and are writing $e_3 = (0, 0, 1)$. Fluid particles on the boundary move with the velocity of the fluid, so that:

$$v \cdot n = \kappa,$$

where $\kappa$ is the normal velocity of $\partial D_t$ and $n$ is the unit normal to $\partial D_t$. We assume that $D_t$ is given by $D_t = \{(x_1, x_2, y) : x_1, x_2 \in \mathbb{R}^2, y \leq h(t, x_1, x_2)\}$ for some function $h$, in which case (1.3) can be re-written as:

$$\partial_t h + v^1 \partial_1 h + v^2 \partial_2 h = v^3 \quad \text{on } \partial D_t.$$

If the fluid body moves in vacuum and there is no surface tension on the boundary then the pressure satisfies:

$$p = 0 \quad \text{on } \partial D_t.$$  

Given $h_0 : \mathbb{R}^2 \rightarrow \mathbb{R}$, set $D_0 = \{(x_1, x_2, y) | y \leq h_0(x_1, x_2)\}$. If $v_0 : D_0 \rightarrow \mathbb{R}^3$ is a vector field satisfying the constraint $\text{div } v_0 = 0$, we want to find a function $h$ and a vector field $v$ so that with $D_t = \{(x_1, x_2, y) | y \leq h(t, x_1, x_2)\}$, $v$ satisfies (1.1)-(1.2) and the initial conditions:

$$h(0, x_1, x_2) = h_0(x_1, x_2), \quad v = v_0 \quad \text{on } \{0\} \times D_0.$$  

This problem is ill-posed unless the following “Taylor sign condition” holds (see (1)):

$$-\nabla_n p(x, t) \geq \delta_0 > 0 \quad \text{on } \partial D_t,$$

where $\nabla_n = n^i \nabla_i$. 

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where $n$ denotes the unit normal to $\partial D_t$. This condition ensures that the pressure is positive in the interior of the fluid and prevents the Rayleigh-Taylor instability from occurring.

In the irrotational case ($\omega \equiv \text{curl} v = 0$), the velocity $v$ is given by $v = \nabla \psi$ for a harmonic function $\psi : D_t \to \mathbb{R}$, and the motion of the fluid is determined entirely by $h$ and $\varphi = \psi|_{\partial D_t}$. This, and related problems have been studied extensively by several authors in the case that the fluid domain $D_t$ is diffeomorphic to the half-space. See for example [2], [3], [4], [5], as well as [6] for a recent overview of these problems. Let us single out the works [7], [8], in which the authors independently proved that in the irrotational case, (1.1)-(1.5) is globally well-posed for sufficiently small and well-localized initial data.

In the case that $\omega \neq 0$, Lindblad-Christodoulou [9] used the Taylor sign condition (1.7) to prove energy estimates for the system (1.1)-(1.5) in the case that $D_t$ is a bounded domain, and later Lindblad [10] proved that this problem is locally well-posed in Sobolev spaces using a Nash-Moser iteration. The same result was later shown by Coutand-Shkoller [11] using a tangential smoothing operator as well as by [12] who used a more geometric approach which also applies on an unbounded domain.

Relatively little is known about the long-term behavior of solutions to the problem (1.1)-(1.5) with nonzero vorticity. We recall that in the case without free boundary and without gravity:

\[
\partial_t + v^k \partial_k v^i + \partial_i p = 0 \quad \text{in } \mathbb{R}^3, \\
\text{div } v = 0 \quad \text{in } \mathbb{R}^3,
\]

non-trivial vorticity is the obstacle to obtaining a global-in-time solution. By [13], if there are constants $M_0, T^*$ so that if $T < T^*$ and $v \in C([0, T]; H^s(\mathbb{R}^3)) \cap C^1([0, T]; H^{s-1}(\mathbb{R}^3))$ solves (1.8)-(1.9) and the a priori estimate:

\[
\int_0^T ||\omega(s)||_{L^\infty(\mathbb{R}^3)} \, ds \leq M_0,
\]

holds, then the solution can be extended to $v \in C([0, T^*]; H^s(\mathbb{R}^3)) \cap C^1([0, T^*]; H^{s-1}(\mathbb{R}^3))$. It then follows from the fact that:

\[
(\partial_t + v^k \partial_k)\omega = \omega \cdot \partial v
\]

and this result that if $\omega = 0$ at $t = 0$, sufficiently regular solutions to (1.8)-(1.9) can be extended to $T = \infty$. See also [14] for an extension to the case of a fixed domain with Neumann boundary condition and [15] for an extension to the free-boundary problem on a bounded domain.

In [16], the authors consider the Euler-Maxwell one-fluid system with nontrivial vorticity but without free boundary in three dimensions. They prove that there is a norm $|| \cdot ||$ so that if $||\text{curl} v(0, \cdot)|| \leq \delta$ for sufficiently small $\delta$, then one can continue the solution up to $T \sim \delta^{-1}$. In particular, this provides a proof of global existence when $\text{curl} v(0, \cdot) = 0$ for the Euler-Maxwell system.

Returning to the free boundary problem, to the best of our knowledge, the only papers that address the issue of the long-time behavior of solutions in the prescence of nontrivial vorticity are [17], [18] and [19]. In [17] Ifrim-Tataru prove that in two space dimensions (with one-dimensional boundary), solutions with constant vorticity can be continued up to $T \sim \varepsilon^{-2}$ if the initial data is of size $\varepsilon$. This is in constrast to the lifespan $T \sim \varepsilon^{-1}$ which is guaranteed by the local well-posedness theory. See also [18] in which Bieri-Miao-Shahshahani-Wu prove a similar result for a self-gravitating liquid occupying a bounded region. In [19], the authors consider the problem in arbitrary dimension and prove that the solution can be continued so long as the mean curvature of the boundary and $||\nabla v||_{L^\infty(D_t)}$ are bounded.
For our result, we will measure the regularity of $\omega$ in the norm:

$$
||\omega(t)||_{H^1_r(D_t)}^2 = \sum_{k \leq r} \int_{D_t} (1 + |x|^2 + y^2)^2 |\partial_x^k \omega(t, x, y)|^2 \, dxdy,
$$

(1.12)

and we will be considering solutions of Euler’s equation with $\omega \cdot n|_{\partial D_t} = 0$. Our main theorem is an analog of the result in [16]:

**Theorem 1.1.** Fix $N_1 \geq 6$ and $N \gg 1$. Define $N_0 = 2NN_1$. There are constants $0 < \varepsilon_1^* \ll \varepsilon_0^* \ll 1$ satisfying the following property. Suppose that $v_0, h_0$ satisfy:

$$
||v_0||_{H^{N_0}(D_0)} + ||h_0||_{H^{N_0}(\mathbb{R}^3)} \leq \varepsilon_0 \leq \varepsilon_0^*,
$$

(1.13)

and that the Taylor sign condition (1.7) holds at $t = 0$. Suppose in addition that $\omega_0 = \text{curl} v_0$ satisfies the bound:

$$
||\omega_0||_{H^{N_1}(D_0)} \leq \varepsilon_1 \leq \varepsilon_1^*.
$$

(1.14)

Let $(v, h)$ be the solution to (1.10)-(1.15) with initial data $v_0, h_0$. Let $T_\omega$ be the largest time so that $(\omega \cdot n)|_{\partial D_t} = 0$ for $0 \leq t \leq T_\omega$. Then the problem (1.10)-(1.13) has a unique solution $(v, h)$ with initial data $(v_0, h_0)$ with $v(t) \in H^{N_0}(D_t)$, $h(t) \in H^{N_0}(\mathbb{R}^3)$ for $0 \leq t \leq T'_\varepsilon, \varepsilon_1$, where:

$$
T'_{\varepsilon, \varepsilon_1} = C_N \min \left( \frac{\varepsilon_0}{\varepsilon_1^{1/3}}, \frac{1}{\varepsilon_0}, T_\omega \right),
$$

(1.15)

for a constant $C_N$ depending only on $N$ and $||(-\nabla n p_0)^{-1}||_{L_\infty(\partial D_0)}$.

Here, $p_0$ is determined from $v_0, h_0$ by solving:

$$
\Delta p_0 = -(\partial_t v_0^j)(\partial_j v_0^i), \quad \text{in } D_0,
$$

(1.16)

$$
p_0 = 0, \quad \text{on } \partial D_0.
$$

(1.17)

One simple way to ensure that the condition $(\omega \cdot n)|_{\partial D_t} = 0$ holds for all time is to assume that $\omega_0|_{\partial D_0} = 0$, since by the transport equation (1.11) it then follows that $\omega|_{\partial D_t} = 0$ for $t > 0$ as well (see Lemma 5,79). We therefore have the following corollary:

**Corollary 1.1.** With the same hypotheses as Theorem 1.1, suppose in addition that $\omega_0|_{\partial D_0} = 0$. Then the solution $(v, h)$ can be continued until:

$$
T_{\varepsilon, \varepsilon_1} = C_N \min \left( \frac{1}{\varepsilon_1^{1/3}}, \frac{1}{\varepsilon_0} \right).
$$

(1.18)

In particular, if $\omega_0 = 0$, this gives a proof that the solution can be continued until $T \sim \varepsilon_0^{-N}$. See also [20] for a similar lifespan bound for irrotational water waves on a periodic domain. Let us make a few remarks. The assumption that $(\omega \cdot n)|_{\partial D_t}$ for $t \geq 0$ is crucial here; as we will see in Section 3 this allows us to derive an equation for the evolution of the variables on the boundary which we will need in order to prove dispersive estimates.

Next, by the results [8], [7], comparing to the result in [16], one would expect to be able to take $T_{\varepsilon_0, \varepsilon_1} \sim \frac{\varepsilon_0}{\varepsilon_1}$ which would in turn give a new proof of global existence in the irrotational case. The difference between that work and this one is that solutions to the linearization of the system (1.1)-(1.2) with zero vorticity decay at a rate $1/t$, while in [16], solutions to the linearized system decay at a rate $1/t^{1+\beta}$ for small $\beta$.

We also remark that at the heuristic level the vorticity satisfies an equation of the form $W' = W^2$ which has lifespan $\sim 1/W_0$ and not $\sim 1/W_0^{1/3}$. We hope to address both of these issues in future work.
1.1. Outline of the proof. As in other works on the global behavior of solutions to dispersive equations, the result follows from a bootstrap argument, consisting of energy estimates to control the $L^2$-based norms and dispersive estimates to control the $L^\infty$-based norms. We start with the energy estimates.

The system (1.1)-(1.3) has the following conserved quantity:

$$E_0(t) = \int_{\mathcal{D}_t} |v(t)|^2 \, dx \, dy + \int_{\mathbb{R}^2} |h(t)|^2 \, dS.$$  \hfill (1.19)

Here, we are writing $\mathcal{D}_t = \{(x, y)|x \in \mathbb{R}^2, y \leq h(t, x)\}$. In the case $\omega = 0$, one can use that the system (1.1)-(1.3) reduces to a Hamiltonian system on the boundary (see (3.1)-(3.2)) and this leads to higher-order energy estimates. Since we are considering the case $\omega \neq 0$, we prove energy estimates for the system (1.1)-(1.3) directly. These energy estimates are based on the estimates in [9], and we extend their approach to the case of an unbounded domain. (See also [21] where similar estimates were proved for the compressible Euler equations with free boundary in an unbounded domain).

The energies are of the form:

$$\mathcal{E}^r(t) = \int_{\mathcal{D}_t} Q(D^r v, D^r v) \, dx \, dy + \int_{\partial \mathcal{D}_t} |\overline{D}^{r-2}\theta|^2 (-\nabla_n p)^{-1} \, dS + \int_{\mathcal{D}_t} |D^{r-1}\omega|^2 \, dx \, dy$$  \hfill (1.20)

where $D$ is the covariant derivative in $\mathcal{D}_t$, $\overline{D}$ is the covariant derivative on $\partial \mathcal{D}_t$ and $\theta$ is the second fundamental form of $\partial \mathcal{D}_t$; writing $n$ for the unit normal to $\partial \mathcal{D}_t$ and $\Pi_{ij} = \delta_{ij} - n_i n_j$ for the projection to the tangent space at the boundary, it is given by:

$$\theta_{ij} = \Pi_{ij} D_h n_t.$$  \hfill (1.21)

Here $Q$ is a quadratic form which is the usual norm $Q(\beta, \beta) = |\beta|^2$ away from the boundary and which is the norm of the projection to the tangent space at the boundary when restricted to the boundary, $Q(\beta, \beta) = |\Pi \beta|^2$. See Section 5 for a precise definition. These energies appear to lose control over normal derivatives of $v$ near the boundary, but it follows from the elliptic estimates in section 4 (see, in particular Lemma 5.2) that $\mathcal{E}^r$ controls $||v||^2_{H^r(\mathcal{D}_t)}$. In Theorem 5.1 we prove that:

$$\frac{d}{dt} \mathcal{E}^r(t) \leq A(t) \left( \mathcal{E}^r(t) + A(t) P(\mathcal{E}^{r-1}(t), ..., \mathcal{E}^0(t)) \right),$$  \hfill (1.22)

where $P$ is a homogeneous polynomial with positive coefficients and $A$ is given by:

$$A(t) = ||Dv(t)||_{L^\infty(\mathcal{D}_t)} + ||\theta(t)||_{W^2,\infty(\partial \mathcal{D}_t)} + ||Dp(t)||_{L^\infty(\mathcal{D}_t)} + ||D^2 p(t)||_{L^\infty(\partial \mathcal{D}_t)} + ||DD_t p||_{L^\infty(\partial \mathcal{D}_t)}.$$  \hfill (1.23)

We now turn to the more difficult task of proving dispersive estimates, and for this we will need to change variables. In $\mathcal{D}_t$, we write:

$$v = \nabla_{x,y} \psi + v_\omega, \quad \Delta_{x,y} \psi = 0,$$  \hfill (1.24)

with $\nabla_n \psi = v \cdot n$ on $\partial \mathcal{D}_t$ and where $\text{curl } v_\omega = \omega$. We also write $\varphi = \psi|_{\partial \mathcal{D}_t}$. It will be important that the energies $\mathcal{E}^r$ control norms of $\varphi, h$ and $v_\omega$. To see why this is the case, note that $\theta_{ij} = (1 + |\nabla h|^2)^{-1/2} \nabla_j \nabla_i h$ for $i, j = 1, 2$ and that $||h||^2_{L^2(\mathbb{R}^2)}$ is bounded by the conserved energy, from which it follows that $||h||^2_{H^r(\mathbb{R}^2)} \leq \mathcal{E}^r$. To control $\varphi$, we start with the observation that:

$$\int_{\partial \mathcal{D}_t} \varphi \mathcal{N} \varphi \, dS = ||\nabla_{x,y} \psi||^2_{L^2(\mathcal{D}_t)} \leq ||v||^2_{L^2(\mathcal{D}_t)},$$  \hfill (1.25)

where $\mathcal{N}$ is the Dirichlet-to-Neumann map. The left hand side controls $||\Lambda^{1/2} \varphi||_{L^2(\partial \mathcal{D}_t)}$ where $\Lambda = |\nabla|$. To control higher derivatives, we could repeat this argument with $\varphi$ replaced by $\nabla^r \varphi$ but...
this would require controlling the commutator $[\mathcal{N}, \nabla^r]$ which is nontrivial. Instead it will suffice for our purposes to use a slightly weaker version of the trace inequality \cite{[26]} and estimate:

$$
||\nabla_x \psi||^2_{H^{-1}(\partial D_t)} \lesssim ||\nabla_{x,y} \psi||^2_{H^{r}(D_t)} \lesssim ||v||^2_{H^{r}(D_t)} + ||v_w||^2_{H^{r}(D_t)} \lesssim \mathcal{E}',
$$

(1.26)

where the estimate for $||v_w||_{H^{r}(D_t)}$ follows from the elliptic estimates in Section 4 since \text{curl} \ v_w = \omega and \text{curl} \ v_w \cdot n|_{\partial D_t} = 0. See Proposition 5.2 To highest order, the left-hand side here controls $||\nabla_x \phi||_{H^{-1}(\mathbb{R}^3)}$.

With the $L^2$ estimates out of the way, we now want to prove $L^\infty$ estimates for $\phi, h$. In section 3 we derive a system satisfied by $\phi$ and $h$. This system is well-known in the case that $\omega = 0$ (see e.g. \cite{[22]}) but the formulation that we use appears to be new in the case $\omega \neq 0$. To motivate this formulation, we recall the basic idea behind the “good unknown” introduced in \cite{[23]}. We write $V = v^i|_{\partial D_t}$, $i = 1, 2$ and $B = v^3|_{\partial D_t}$ as well as $U = V + \nabla h B$. After restricting Euler’s equation (1.1) to $\partial D_t$ and using the boundary condition (1.5), $V$ and $B$ satisfy the following equations:

$$
\hat{D}_t V = -a \nabla h,
$$

(1.27)

$$
\hat{D}_t B = a - 1,
$$

(1.28)

where $a = (\partial_y p)|_{\partial D_t}$ and $\hat{D}_t = \partial_t + V^1 \partial_1 + V^2 \partial_2$. In particular, we have:

$$
\hat{D}_t (V + \nabla h B) = -\nabla h - \hat{D}_t \nabla h.
$$

(1.29)

In the case $\omega = 0$, $V = \partial_x \psi|_{\partial D_t}$ and $B = \partial_y \psi|_{\partial D_t}$, so by the chain rule, we have:

$$
\nabla_x \phi(x) = (\nabla_x \psi)(x, h(x)) + \nabla_x h(x)(\nabla_y \psi)(x, h(x)) = V + \nabla h B,
$$

(1.30)

with $\phi(x) = \psi(x, h(x))$. Plugging this into (1.29) gives an evolution equation for $\nabla \phi$. It turns out that in the irrotational case, after making this substitution (1.29), is of the form:

$$
\partial_t \nabla \phi = \nabla F(\phi, h),
$$

(1.31)

for a nonlinearity $F(\phi, h)$ which also depends on the derivatives of $\phi, h$. This leads to an equation for $\partial_t \phi$.

When $\omega \neq 0$, we write $v = \nabla_{x,y} \psi + v_w$ in $D_t$ and let $V^i_w = v^i_w|_{\partial D_t}$ for $i = 1, 2$ and $B^3_w = v^3_w|_{\partial D_t}$. Repeating the above calculation leads to an equation of the form:

$$
\partial_t \nabla \phi + \partial_t (V_\omega + \nabla h B_\omega) = \nabla F(\phi, h) + G(\phi, h, V_\omega, B_\omega),
$$

(1.32)

with the same $F$ as above. Writing $U_\omega = V_\omega + \nabla h B_\omega$, the crucial observation is that:

$$
\text{curl}_2 U_\omega = \partial_1 U^2_\omega - \partial_2 U^1_\omega = \omega \cdot n \quad \text{on } \mathbb{R}^2.
$$

(1.33)

See Theorem 3.1 In particular, if $\omega \cdot n = 0$ it follows that $U_\omega = \nabla a_\omega$ for a function $a_\omega$. Making this substitution in (1.32), it turns out that $G$ is a gradient, $G = \nabla H(\phi, h, V_\omega, B_\omega)$ for some other nonlinearity $H$, and the system becomes:

$$
\partial_t (\nabla \phi + \nabla a_\omega) = \nabla (F(\phi, h) + H(\phi, h, V_\omega, B_\omega)),
$$

(1.34)

which gives an evolution equation for $\phi_\omega = \phi + a_\omega$. Setting $u = h + i\Lambda^{1/2} \phi_\omega$ and writing $w = (V_\omega, B_\omega)$ (1.34) and (1.1) lead to an equation of the form:

$$
(\partial_t + i\Lambda)u = N(u) + L(w) + N_1(u, w) + N_2(w, w),
$$

(1.35)

where $N, N_1, N_2$ are a nonlinear operators and $L$ is linear. See Proposition 3.1 for the precise form of the right-hand side.

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1The good unknown used in \cite{[23]} is actually given by $U = V + T_{\phi} h B$ where $T$ is Bony’s paraproduct but it will suffice to use this simpler definition for our purposes.
The nonlinearity $N$ is the same one that occurs in \cite{S}, and can be handled using simple modifications of the arguments there. Specifically, we start with the Duhamel representation of system (1.35):

\[
e^{it\Lambda^{1/2}} u(t) = u_0 + \int_0^t e^{is\Lambda^{1/2}} N(u) \, ds + \int_0^t e^{is\Lambda^{1/2}} \left( L(w) + N_1(u, w) + N_2(w, w) \right) \, ds \tag{1.36}
\]

\[
\equiv u_0 + f_1(u) + f_2(u, w). \tag{1.37}
\]

We follow \cite{S} and define:

\[
||u||_X = (1 + t)||u||_{W^{4, \infty}(\mathbb{R}^2)} + (1 + t)^{-\delta} ||u||_{H^{N_0}(\mathbb{R}^2)} + ||\Lambda^t x e^{it\Lambda^{1/2}} u||_{L^2(\mathbb{R}^2)}), \tag{1.38}
\]

where here $\epsilon, \delta$ are sufficiently small constants.

Minor changes to the arguments in \cite{S} (which we outline in Section 7) show that:

\[
(1 + t)||e^{-it\Lambda^{1/2}} f_1(u)||_{W^{4, \infty}(\mathbb{R}^2)} \lesssim ||u(t)||^2_X + (1 + t)^{2+\delta} ||\omega(t)||_{H^{N_1}_w(D_0)}, \tag{1.39}
\]

\[
(1 + t)^{-\delta} ||\Lambda^t x f_1(u)||_{L^2(\mathbb{R}^2)} \lesssim ||u(t)||^2_X + (1 + t)||\omega(t)||_{H^{N_1}_w(D_0)} \tag{1.40}
\]

Next, in Section 6 we establish bounds of the above form for $f_2$:

\[
(1 + t)||e^{-it\Lambda^{1/2}} f_2(u, w)||_{W^{4, \infty}(\mathbb{R}^2)} \lesssim ||u(t)||^2_X + (1 + t)^{2+\delta} ||\omega(t)||_{H^{N_1}_w(D_0)}, \tag{1.41}
\]

\[
(1 + t)^{-\delta} ||\Lambda^t x f_2(u, w)||_{L^2(\mathbb{R}^2)} \lesssim ||u(t)||^2_X + (1 + t)||\omega(t)||_{H^{N_1}_w(D_0)} \tag{1.42}
\]

The proof of (1.41)-(1.42) requires bounding norms of $w = (V, B_w)$ on the boundary in terms of $\omega$ in the interior, for which we use the elliptic estimates in Section 4. These estimates combined with the above the above energy estimates and a continuity argument show that the solution can be continued until $T \sim T_{\varepsilon_0, \varepsilon_1}$.

2. PROOF OF THE MAIN THEOREM

We begin by decomposing our initial velocity $v_0$ into its irrotational and rotational parts. Given $h_0 : \mathbb{R} \to \mathbb{R}$, set $D_0 = \{ (x_1, x_2, y) | y \leq h_0(x_1, x_2) \}$. We now write $v_0 = \nabla_{x,y} \psi_0 + v_{w_0}$, where $\Lambda \psi_0 = 0$ in $D_0$, $\nabla_n \psi_0 = v_0 \cdot n$ on $\partial D_0$, and where $\text{curl } v_{w_0} = \omega_0 \equiv \text{curl } v_0$. We also write $V^i_w = v^i_w |_{\partial D_0}$, $i = 1, 2$ and $B_{w_0} = v^3_w |_{\partial D_0}$. In Section 6 we prove that if $\omega_0 |_{\partial D_0} = 0$, then $V_w + \nabla h_0 B_{w_0} = \nabla a_{w_0}$ for a function $a_{w_0}$. We then write $\varphi_w = \psi_0 |_{\partial D_0}$ as well as $\varphi_{w_0} = \varphi_w + a_{w_0}$ and $u_0 = h_0 + i\Lambda^{1/2} \varphi_{w_0}$, where $\Lambda = |\nabla|$.

We now fix $N_1 \geq 6$, $N \gg 1$ and set $N_0 = 2NN_1$. With the above notation and with $|| \cdot ||_{H^{N_1}_w}$ defined by (1.12), we suppose that $v_0, \omega_0, h_0$ satisfy:

\[
\omega_0 \cdot n_0 |_{\partial D_0} = 0, \tag{2.1}
\]

\[
||v_0||_{L^\infty(D_0)} + ||v_0||_{H^{N_0}(D_0)} + ||h_0||_{H^{N_0}(D_0)} + ||u_0||_{W^{4, \infty}(\mathbb{R}^2)} + ||\Lambda^t x u_0||_{L^2(\mathbb{R}^2)} \leq \frac{1}{2} \varepsilon_0, \tag{2.2}
\]

\[
||\omega_0||_{H^{N_1}_w(D_0)} \leq \frac{1}{2} \varepsilon_1 \ll \varepsilon_0, \tag{2.3}
\]

\[
||\omega_0||_{H^{N_1}_w(D_0)} \leq \frac{1}{2} \varepsilon_1 \ll \varepsilon_0, \tag{2.4}
\]

for sufficiently small $\varepsilon_0$ and $\varepsilon_1$, where $n_0$ is the unit normal to $\partial D_0$.

We now define $p_0 : D_0 \to \mathbb{R}$ by:

\[
\Delta p_0 = -(\partial_i v^i_0)(\partial_j v^j_0) \quad \text{in } D_0, \tag{2.5}
\]

\[
p_0 = 0 \quad \text{on } \partial D_0. \tag{2.6}
\]

In order for the initial value problem (1.1)-(1.6) to be well-posed, we need to ensure that $(-\nabla n_0 p_0) \geq \delta_0 > 0$ for some $\delta_0$. In the irrotational case, this condition holds automatically, essentially because then $\Delta p_0 = -(\nabla v)^2 \leq 0$ (see \cite{24}). When $\text{curl } v_0 \neq 0$ we instead have the following result:
Lemma 2.1. Suppose that \( \| \omega_0 \|_{L^\infty(D_0)} \leq \frac{1}{2} \| v_0 \|_{L^\infty(D_0)} \). Then, with \( p_0 \) defined by (2.6), there is a constant \( c_0 > 0 \) so that:

\[
( - \nabla n_0 p_0 ) \geq 2c_0 > 0 \text{ on } \partial D_0.
\]

**Proof.** We follow the argument in [24]. We fix a function \( f : \partial D_0 \to \mathbb{R} \) and let \( F \) denote its harmonic extension to \( D_0 \). By Green’s identity:

\[
\int_{\partial D_0} f \nabla n_0 (p_0 + y) - (p_0 + y) \nabla n_0 f = \int_{D_0} \Delta (p_0 + y) F.
\]

We now note that \( \Delta p_0 = -\left( \partial_t v_0^0 \right) \right) \partial_0 v_0^0 ) = -\left( \partial_t v_0^0 \right) \partial_0 v_0^0 ) + \left( \partial_t v_0^0 \right) \delta^t (\text{curl } v_0) j_t \). By assumption we have that \( \| \nabla v_0 - \text{curl } \omega \|_{L^\infty(D_1)} \geq \frac{1}{2} \| \nabla v_0 \|_{L^\infty(D_1)} \) and so in particular we have that \( \Delta p_0 < 0 \). Therefore by (2.8) and the fact that \( p_0 = 0 \) on \( \partial D_0 \), we have:

\[
\int_{\partial D_0} f D_n (p_0 + y) - y D_n f > 0.
\]

The rest of the proof of Lemma 4.1 from [24] now goes through without change. \( \square \)

We will use the following local well-posedness result, which follows from Theorem B in [12] and the above lemma:

**Proposition 2.1.** Let \( h_0 \in H^{N_0} (\mathbb{R}^2) \), \( D_0 = \{(x_1, x_2, y) | y \leq h_0 (x_1, x_2) \} \) \( v_0 \in H^{N_0} (D_0) \). Suppose that \( \| \omega_0 \|_{L^\infty(D_0)} \leq \frac{1}{2} \| v_0 \|_{L^\infty(D_0)} \). Then there is a \( T = T(v_0, h_0) > 0 \), a function \( h : [0, T] \times \mathbb{R}^2 \to \mathbb{R} \) and a vector field \( v = v(t) \) defined on \( D_t = \{(x_1, y) | y \leq h(t, x) \} \) for \( 0 \leq t \leq T \), so that \( v \big|_{t=0} = v_0, D_t(t=0) = D_0 \), \( (v, D_t) \) satisfy (1.1)-(1.5) and \( v(t, \cdot) \in H^{N_0} (D_t), h(t, \cdot) \in H^{N_0} (\mathbb{R}^2) \) for \( t \leq T \).

We now want to extend the time \( T \) in this theorem to \( T'_{\varepsilon_0, \varepsilon_1} \) defined in (1.15), provided that the vorticity \( \omega \) vanishes on \( \partial D_t \). We suppose that \( u, v \) satisfy the following bootstrap assumptions for \( t \geq 0 \):

\[
\| u(t) \|_{W^{4, \infty}(\mathbb{R}^2)} \leq \frac{\varepsilon_0}{1 + t} \tag{2.10}
\]

\[
\| v(t) \|_{H^{N_0}(D_t)} + \| h(t) \|_{H^{N_0}(\mathbb{R}^2)} + \| \Lambda^{1/2} x e^{it \Lambda^{1/2}} u(t) \|_{L^2(\mathbb{R}^2)} \leq \varepsilon_0 (1 + t)^\delta, \tag{2.11}
\]

\[
\| \omega(t) \|_{H^1(D_t)} \leq \varepsilon_1 (1 + t)^\delta, \tag{2.12}
\]

and that \( \omega \cdot n \big|_{\partial D_t} = 0 \), where here \( \varepsilon_1 > 0 \) is a small constant. In Section 5.1 we show that:

\[
\| \Lambda^{1/2} \varphi \omega \|_{H^{N_0-1}(\mathbb{R}^2)} \lesssim \| v(t) \|_{H^{N_0}(D_t)} + \| h(t) \|_{H^{N_0}(\partial D_t)} + O(\varepsilon_0^2), \tag{2.13}
\]

if (2.10)-(2.11) hold. In particular, the assumption (2.11) implies an estimate for \( \| u \|_{H^{N_0-1}(\mathbb{R}^2)} \), a fact which is used several times in the proofs of the following theorems.

Recalling the definitions in (1.37), and that we are writing \( w = (V_\omega, B_\omega) \), we have:

**Proposition 2.2.** If the bootstrap assumptions (2.10)-(2.12) hold for sufficiently small \( \varepsilon_0, \varepsilon_1 \) and \( \omega \cdot n \big|_{\partial D_t} = 0 \) for \( 0 \leq t \leq T \), then:

\[
\| e^{-it \Lambda^{1/2}} f_1(u) \|_{W^{4, \infty}(\mathbb{R}^2)} + \| e^{-it \Lambda^{1/2}} f_2(u, w) \|_{W^{4, \infty}(\mathbb{R}^2)} \lesssim \frac{\varepsilon_0^2}{1 + t} + \varepsilon_1 (1 + t)^{1+2\delta}, \tag{2.14}
\]

\[
\| \Lambda^{1/2} (x f_1(u)) \|_{L^2(\mathbb{R}^2)} + \| \Lambda^{1/2} (x f_2(u, w)) \|_{L^2(\mathbb{R}^2)} \lesssim \varepsilon_0^2 (1 + t)^\delta + \varepsilon_1 (1 + t)^{2+2\delta}, \tag{2.15}
\]

for \( 0 \leq t \leq T \).

The term \( f_1 \) can be estimated by simple modifications of the estimates in and we outline this approach in Section 7. The estimates for \( f_2 \) can be found in Sections 6.1-6.3.

We now need some estimates to control the size of \( \omega \), which we prove in Section 5.5.
Proposition 2.3. If the assumptions (2.10)-(2.12) hold, then there is a constant $C_N$ so that:

$$
||\omega(t)||^2_{H^{N_1}(D_t)} \leq ||\omega(0)||^2_{H^{N_1}(D_0)} + C_N(\epsilon_0(1+t)^{1/N})(\epsilon_1(1+t)^{1+\delta})\epsilon_1^2(1+t)^{2\delta}.
$$

(2.16)

In particular, if:

$$2C_N t \leq T_1 \equiv \min\left(\frac{1}{\epsilon_1^{1/3}}, \frac{1}{\epsilon_0^N}\right),$$

(2.17)

this implies that:

$$||\omega(t)||^2_{H^{N_1}(D_t)} \leq \frac{3}{4} \epsilon_1^2(1+t)^{2\delta}.
$$

(2.18)

The last ingredient we need is an energy estimate for the entire system, which we prove in Section 5.

Proposition 2.4. If $||v_0||_{H^{N_0}(D_0)} + ||\omega_0||_{H^{N_0-1}(D_0)} \leq \epsilon_0/2$ and the bootstrap assumptions (2.10)-(2.12) hold, then with $c_0$ as in Lemma 2.1, there is a constant $C_{N_0}^E = C_{N_0}^E(c_0)$ so that:

$$
||v(t)||^2_{H^{N_0}(D_t)} + ||h(t)||^2_{H^{N_0}(R^2)} \leq \frac{\epsilon_0^2}{4} + C_{N_0}^E(\epsilon_0 + \epsilon_1(1+t)^{2+\delta})\epsilon_1^2(1+t)^{2\delta}.
$$

(2.19)

In particular, if $t$ is such that:

$$2C_{N_0}^E t \leq T_2 \equiv \frac{1}{\epsilon_1^{1/3}},$$

(2.20)

this implies that:

$$||v(t)||^2_{H^{N_0}(D_t)} + ||h(t)||^2_{H^{N_0}(D_t)} \leq \frac{\epsilon_0^2}{4} + \epsilon_1^3(1+t)^{2\delta}.
$$

(2.21)

Setting $T_{\epsilon_0, \epsilon_1} = \min(T_1, T_2)$, a standard continuity argument then gives Theorem 1.1.

3. DERIVATION OF THE EQUATIONS ON THE BOUNDARY

We will use the equations (1.1)-(1.3) directly to prove energy estimates. However, to prove the dispersive estimates in Proposition 2.2, we will need to use equations for $h$ and $v|_{\partial D_t}$. In the irrotational case, $v_i = \partial_i \psi$ for a harmonic function $\psi$ satisfying $\nabla_n \psi = v \cdot n$ on $\partial D_t$. Letting $\varphi = \psi|_{\partial D_t}$, one can show that $h, \varphi$ satisfy the system:

$$
\partial_1 h = G(h)\varphi,
$$

(3.1)

$$
\partial_2 \varphi = -h - \frac{1}{2} |\nabla \varphi|^2 + \frac{(G(h)\varphi + \nabla h \cdot \nabla \varphi)^2}{2(1 + |\nabla h|^2)},
$$

(3.2)

where $G(h)$ is the rescaled Dirichlet-to-Neumann map (see (3.7)) and we are writing $\nabla = (\partial_1, \partial_2)$. This system is derived from the fact that when $\omega = 0$, Euler’s equations become:

$$
\partial_1 (\partial_1 \psi + |\partial_1 \psi|^2 + p + y) = 0, \quad \text{in } D_t.
$$

(3.3)

See e.g. [22] or [25] for a derivation.

This no longer works when $\omega \neq 0$ and so another approach is needed. Our derivation of the equations on the boundary is partially based on the ideas in [23] (see in particular Section 4.1 there). We define:

$$
V^i = v^i|_{\partial D_t} \text{ for } i = 1, 2, \quad B = v^3|_{\partial D_t}.
$$

(3.4)

In what follows we will write $\partial_i = \frac{\partial}{\partial x^i}$ for $i = 0, 1, 2, 3$ with the convention that $x^0 = t$. We will also occasionally write $\partial_y = \partial_3$. We will also write $\nabla$ for the derivative of quantities defined on $R^2 \sim \partial D_t$ and $\partial$ when differentiating quantities defined on $D_t$. We now collect a few well-known
and elementary identities. Given $f : \mathcal{D}_t \to \mathbb{R}$, write $F(x) = f(x, h(x)) = f|_{\partial\mathcal{D}_t}(x)$. Then, by the chain rule:

$$\partial_i F = (\partial_i f)|_{\partial\mathcal{D}_t} + \nabla_i (\partial_y f)|_{\partial\mathcal{D}_t}, \quad i = 0, 1, 2 \quad (3.5)$$

If $f$ is harmonic on $\mathcal{D}_t$ then additionally:

$$\partial_i F = \frac{1}{1 + |\nabla h|^2} \left( G(h)F + \nabla h \cdot \nabla F \right), \quad (3.6)$$

where $G(h)$ is the rescaled Dirichlet-to-Neumann operator:

$$G(h)F = \sqrt{1 + |\nabla h|^2} \nabla n f|_{\partial\mathcal{D}_t}. \quad (3.7)$$

We also recall that the boundary condition (1.3) can be written:

$$\partial_t h + V^1 \nabla_1 h + V^2 \nabla_2 h = B. \quad (3.8)$$

As a consequence, writing $\hat{\mathcal{D}}_t = \partial_t + V^1 \partial_1 + V^2 \partial_2$, we have:

$$\hat{\mathcal{D}}_t F = (\mathcal{D}_t f)|_{\partial\mathcal{D}_t}. \quad (3.9)$$

Next, in $\mathcal{D}_t$ we define $\psi \in L^6(\mathcal{D}_t) \cap \dot{H}^1(\mathcal{D}_t)$ to be the harmonic extension of $v \cdot n$ to $\mathcal{D}_t$, that is, $\psi$ satisfies:

$$\Delta \psi = 0 \quad \text{in} \quad \mathcal{D}_t, \quad \nabla n \psi = n \cdot v \quad \text{on} \quad \partial\mathcal{D}_t. \quad (3.10)$$

The function $\psi$ is unique since $\text{div} v = 0$ in $\mathcal{D}_t$. It then follows that $v_\omega \equiv v - \partial \psi$ satisfies:

$$\text{curl} v_\omega = \omega, \quad \text{div} v_\omega = 0, \text{ on } \mathcal{D}_t, \quad n \cdot v_\omega = 0 \text{ on } \partial\mathcal{D}_t. \quad (3.11)$$

We will write:

$$\varphi = \psi|_{\partial\mathcal{D}_t}, \quad V^i_\omega = v^i_\omega|_{\partial\mathcal{D}_t}, \quad i = 1, 2, \quad B_\omega = v^3_\omega|_{\partial\mathcal{D}_t}. \quad (3.12)$$

The following derivation is inspired by the approach of [23] and [19]. The main result of this section is the following:

**Theorem 3.1.** With the above notation:

1. Writing $U_\omega = V_\omega + \nabla h B_\omega$, we have $\nabla U^2_\omega - \nabla U^1_\omega = \omega|_{\partial\mathcal{D}_t} \cdot n$, In particular, $U_\omega = \nabla a_\omega$ for a function $a_\omega : \mathbb{R}^2 \to \mathbb{R}$.
2. The variables $\varphi, h, V_\omega$ and $B_\omega$ satisfy the system:

$$\partial_t h = G(h)\varphi, \quad (3.13)$$

$$\partial_t (\varphi + a_\omega) = -h - \frac{1}{2} |\nabla \varphi|^2 + \frac{1}{2} \frac{(G(h)\varphi + \nabla h \cdot \nabla \varphi)^2}{1 + |\nabla h|^2} + R_\omega, \quad (3.14)$$

where:

$$R_\omega = -\frac{1}{2} |V_\omega|^2 - \nabla \varphi \cdot V_\omega - \frac{1}{2} (V_\omega \cdot \nabla h)^2 + (G(h)\varphi)V_\omega \cdot \nabla h \quad (3.15)$$
Proof. By \(3.5\):

\[
\nabla_1 U_\omega^2 - \nabla_2 U_\omega^1 = \nabla_1 V_\omega^2 - \nabla_2 V_\omega^1 + (\nabla h) \nabla_1 B_\omega - (\nabla h) \nabla_2 B_\omega
\]

\(3.16\)

\[
= (\partial_1 v_\omega^2)|_{\partial D_t} - (\partial_2 v_\omega^1)|_{\partial D_t} + (\nabla h)(\partial_1 v_\omega^2)|_{\partial D_t} - (\nabla h)(\partial_2 v_\omega^1)|_{\partial D_t}
\]

\(3.17\)

\[
+ (\nabla h)(\partial_1 v_\omega^3)|_{\partial D_t} - (\nabla h)(\partial_2 v_\omega^3)|_{\partial D_t} + (\nabla h)(\partial_3 v_\omega^3)|_{\partial D_t} - (\nabla h)(\nabla h)(\partial_3 v_\omega^3)|_{\partial D_t}
\]

\(3.18\)

\[
= (\partial_1 v_\omega^2 - \partial_2 v_\omega^1)|_{\partial D_t} - \left((\nabla h)(\partial_2 v_\omega^3 - \partial_3 v_\omega^3)|_{\partial D_t} + (\nabla h)(\partial_3 v_\omega^3 - \partial_1 v_\omega^3)|_{\partial D_t}\right)
\]

\(3.19\)

\[
= \omega^3|_{\partial D_t} - (\nabla h)|_{\partial D_t}
\]

\(3.20\)

which gives the first result.

We now derive \(3.13\) - \(3.14\). Differentiating \(3.8\) and using the fact that \([\hat{D}_t, \nabla] = -\nabla V^k\nabla_k\) gives:

\[
\hat{D}_t \partial_t h = \nabla_1 B - \nabla_1 V^k\nabla_k h
\]

\(3.21\)

Writing \(a = (\partial_y p)|_{\partial D_t}\), restricting Euler’s equations \(1.1\) to the boundary and using that \(p = 0\) on \(\partial D_t\) gives:

\[
\hat{D}_t V_i = -a\nabla_i h, \quad i = 1, 2 \\
\hat{D}_t B = a - 1.
\]

\(3.22\)

\(3.23\)

Therefore:

\[
\hat{D}_t(\nabla \varphi + U_\omega) = \hat{D}_t(V + \nabla h B) = -\nabla h + (\hat{D}_t \nabla h) B
\]

\(3.24\)

\[
= -\nabla h + (\nabla B - \nabla V^k\nabla_k h) B
\]

\(3.25\)

\[
= -\nabla h + \frac{1}{2} |B|^2 - \nabla V^k\nabla_k h B.
\]

\(3.26\)

We will write \(f_b = f|_{\partial D_t}\) for the restriction to the boundary. Expanding out the definition of \(\hat{D}_t\) and recalling by convention, sums over repeated upper and lower indices run over only the first two indices gives that:

\[
\hat{D}_t(\nabla \varphi + U_\omega) = \partial_t(\nabla \varphi + U_\omega) + (\partial_k \psi)_b \nabla^k \nabla \varphi + (\partial_k \psi)_b \nabla^k U_\omega + V^k \nabla_k \nabla \varphi + V^k \nabla_k U_\omega
\]

\(3.27\)

Combining this with \(3.26\) and expanding \((\partial_k \psi)_b = \nabla_k \varphi - \nabla^k h(\partial_y \psi)_b\), we have:

\[
\partial_t(\nabla \varphi + U_\omega) = -\nabla h + \frac{1}{2} |B|^2 - \nabla V^k\nabla_k h B - (\nabla^k \varphi - (\partial_y \psi)_b \nabla^k h) \nabla_k \nabla \varphi
\]

\(3.28\)

\[
- \nabla^k \varphi \nabla_k U_\omega + (\partial_y \psi)_b \nabla^k h \nabla_k U_\omega - V^k \nabla_k \nabla \varphi - V^k \nabla_k U_\omega.
\]

\(3.29\)

Expanding \(V, B\) in terms of \(\psi\) and \(v_\omega\) and using \(3.5\):

\[
\nabla V^k \nabla_k h B = \nabla(\partial_k \psi)_b \nabla^k h(\partial_y \psi)_b + \nabla V^k \nabla_k h B_\omega + \nabla(\partial_k \psi)_b \nabla^k h B_\omega + \nabla V^k \nabla_k h(\partial_y \psi)_b
\]

\(3.30\)

\[
= (\nabla \nabla^k \psi)_b \nabla^k h(\partial_y \psi)_b + \nabla V^k \nabla_k h B_\omega + \nabla(\nabla^k \varphi) \nabla_k h B_\omega
\]

\(3.31\)

\[
- \nabla(\nabla^k \varphi) \nabla_k h B_\omega - \nabla(\nabla_k h(\partial_y \psi)_b) \nabla^k h(\partial_y \psi)_b + \nabla V^k \nabla_k h(\partial_y \psi)_b
\]

\(3.32\)
We insert this expression into the previous one to get:
\[
\partial_t(\nabla \varphi + U_\omega) = A(\varphi, h) + \frac{1}{2} \nabla |B_\omega|^2 + \nabla((\partial_y \psi)_b B_\omega) \\
- \nabla V_\omega^k \nabla_k h B_\omega - V_\omega^k \nabla_k U_\omega \\
- \nabla \nabla^k \varphi (\nabla_k h B_\omega) - \nabla \nabla^k \varphi (\nabla_k U_\omega) - V_\omega^k \nabla_k \nabla \varphi \\
+ (\partial_y \psi)_b \nabla^k h \nabla_k U_\omega + \nabla (\nabla^k h (\partial_y \psi)_b) \nabla_k h B_\omega - \nabla V_\omega^k \nabla_k h (\partial_y \psi)_b, \\
\]
where \( A \) is given by:
\[
A = -\nabla h + \frac{1}{2} \nabla (\partial_y \psi)_b^2 - \frac{1}{2} \nabla |\nabla \varphi|^2 + (\partial_y \psi)_b^2 \nabla^k h \nabla_k \nabla \varphi \\
- \left( \nabla (\nabla^k \varphi) \nabla_k h + \nabla (\nabla^k h (\partial_y \psi)_b) \right) (\partial_y \psi)_b \\
= -\nabla h + \frac{1}{2} \nabla (\partial_y \psi)_b^2 - \frac{1}{2} \nabla |\nabla \varphi|^2 - \nabla \nabla^k h (\partial_y \psi)_b (\partial_y \psi)_b, \\
\]
using (3.35) in the last step. Applying (3.36) shows that:
\[
A = \nabla \left( -h - \frac{1}{2} |\nabla \varphi|^2 + \frac{(G(h) \varphi + \nabla h \cdot \nabla \varphi)^2}{2(1 + |\nabla h|^2)} \right) \\
\]
We now want to show that all of the other terms in (3.36) are also gradients.
To handle the terms on the second row of (3.36), we note that by the definition of \( U_\omega \):
\[
\nabla V_\omega^k \nabla_k h B_\omega + V_\omega^k \nabla_k U_\omega = \nabla V_\omega^k (\delta_{kl} U_\omega^l) + V_\omega^k \nabla_k U_\omega - \nabla V_\omega^k (\delta_{kl} V_\omega^l) \\
= \nabla (\delta_{kl} V_\omega^k U_\omega^l) - \frac{1}{2} \nabla |V_\omega|^2 \\
\]
where we used the fact that curl \( U_\omega = 0 \) in the last step.
To deal with the terms on the third row of (3.36) we note that:
\[
\nabla_k \varphi \nabla^k U_\omega + V_\omega^k \nabla_k \varphi + \nabla \nabla_k \varphi \nabla^k h B_\omega = \nabla_k \varphi \nabla^k U_\omega + \nabla_k \varphi (V_\omega^k + \nabla^k h B_\omega) = \nabla (\nabla_k \varphi U_\omega^k). \\
\]
Finally, to handle the terms on the last line of (3.36), we again use that curl \( U_\omega = 0 \) and expand out \( U_\omega = \nabla \omega + \nabla h B_\omega \) and write the result as:
\[
(\partial_y \psi)_b \left( \nabla_k h \nabla_i U_\omega^k + \nabla_i \nabla^k h \nabla_k h B_\omega - \nabla_i V_\omega^k \nabla_k h \right) + |\nabla h|^2 B_\omega \nabla_i (\partial_y \psi)_b \\
= (\partial_y \psi)_b \left( \nabla_k h \nabla_i V_\omega^k + \nabla_k h \nabla_i (\nabla_k h B_\omega) + \nabla_i \nabla_k h \nabla^k h B_\omega - \nabla_i V_\omega^k \nabla_k h \right) + |\nabla h|^2 B_\omega \nabla_i (\partial_y \psi)_b \\
= \nabla_i (|\nabla h|^2 (\partial_y \psi)_b B_\omega). \\
\]
Combining the results of (3.31)-(3.45), we see that (3.36) becomes:
\[
\partial_t (\nabla \varphi + \nabla a_\omega) = A(\varphi, h) \\
+ \frac{1}{2} \nabla |B_\omega|^2 + \frac{1}{2} \nabla |V_\omega|^2 - \nabla (V_\omega \cdot U_\omega) + \nabla ((1 + |\nabla h|^2)(\partial_y \psi)_b B_\omega) - \nabla (\nabla \varphi \cdot U_\omega) \\
\]
Now we note that since \( v_\omega \cdot n = 0 \), we have \( B_\omega = V_\omega^k \nabla_k h \) which further implies \( U_\omega = V_\omega + \nabla h (V_\omega \cdot \nabla h) \). The second line of (3.46) then becomes the gradient of:
\[
- \frac{1}{2} (\nabla h \cdot V_\omega)^2 - \frac{1}{2} (V_\omega)^2 + \left( (1 + |\nabla h|^2)(\partial_y \psi)_b - \nabla \varphi \cdot \nabla h \right) (\nabla h \cdot V_\omega) - \nabla \varphi \cdot V_\omega \\
= - \frac{1}{2} (\nabla h \cdot V_\omega)^2 - \frac{1}{2} (V_\omega)^2 + (G(h) \varphi)(V_\omega \cdot \nabla h) - \nabla \varphi \cdot V_\omega \\
\]
Next, we note that the vorticity does not enter into $h$ when we write $v$ in terms of $\psi, v_\omega$. Indeed, recalling that $v_\omega \cdot n = 0$ on $\partial D_t$ and using (3.5) gives:

$$\partial_t h = - (\partial \psi)_b \cdot \nabla h - V_\omega \cdot \nabla h + (\partial \psi)_b + B_\omega$$

$$= \sqrt{1 + |\nabla h|^2} ((\partial \psi)_b \cdot n + (v_\omega)_b \cdot n)$$

$$= G(h) \varphi,$$  \hspace{1cm} (3.48)

where in the last step we used that $D \psi = 0$ in $D_t$.

Combining the result of the above calculation with (3.50) completes the proof.

It is a little awkward to work in terms of $a_\omega$, since it depends on the vorticity in the interior in a complicated way, and moreover we only control $\nabla a_\omega$, not $a_\omega$ itself. For this reason we set:

$$\varphi_\omega = \varphi + a_\omega.$$  \hspace{1cm} (3.51)

The above system becomes:

$$\partial_t h = G(h) \varphi_\omega - G(h) a_\omega,$$  \hspace{1cm} (3.52)

$$\partial_t \varphi_\omega = - h - |\nabla \varphi_\omega|^2 + \frac{(G(h) \varphi_\omega + \nabla h \cdot \nabla \varphi_\omega)^2}{1 + |\nabla h|^2} + \tilde{R}_\omega,$$  \hspace{1cm} (3.53)

where:

$$\tilde{R}_\omega = \int \nabla a_\omega)^2 + \nabla \varphi_\omega \cdot \nabla a_\omega + (\nabla h)^2 \frac{1}{2} (G(h) a_\omega + \nabla h \cdot \nabla a_\omega)(G(h) \varphi_\omega + \nabla h \cdot \nabla \varphi_\omega)$$

$$- \frac{1}{2} |V_\omega|^2 - \nabla \varphi_\omega \cdot V_\omega - \frac{1}{2} (V_\omega \cdot \nabla h)^2 + (G(h) \varphi_\omega) V_\omega \cdot \nabla h + \nabla a_\omega \cdot V_\omega - (G(h) a_\omega) V_\omega \cdot \nabla h.$$  \hspace{1cm} (3.54)

We now recall that $\nabla a_\omega = V_\omega + \nabla h B_\omega$ and that $B_\omega = \nabla h \cdot V_\omega$. Writing $G(h) a_\omega = G(h) \Lambda^{-1} R \cdot \nabla a_\omega$, we note that $V_\omega$ enters linearly into these equations, since:

$$\partial_t h = G(h) \varphi_\omega - G(h) (\Lambda^{-1} R \cdot V_\omega) - G(h) (\Lambda^{-1} R \cdot (\nabla h \cdot V_\omega)).$$  \hspace{1cm} (3.55)

We also note that $V_\omega, B_\omega$ enter no more than quadratically into the remaining terms.

Using these identities, can further re-write:

$$\tilde{R}_\omega = - |V_\omega \cdot \nabla h|^2 + (\nabla \varphi_\omega \cdot \nabla h)(V_\omega \cdot \nabla h) + G(h) \left[ \varphi_\omega - \Lambda^{-1} R \cdot V_\omega - \Lambda^{-1} R \cdot (\nabla h B_\omega) \right](V_\omega \cdot \nabla h)$$

$$+ (\nabla h)^2 \frac{1}{2} \left( G(h) [\Lambda^{-1} R \cdot V_\omega] + \nabla h \cdot V_\omega \right)^2 - (G(h) \Lambda^{-1} R \cdot V_\omega + \nabla h \cdot V_\omega)(G(h) \varphi_\omega + \nabla h \cdot \nabla \varphi_\omega)$$

$$+ \text{more nonlinear terms}.$$  \hspace{1cm} (3.56)

We now expand out $G(h)$ in powers of $h$. We recall the following expansion of $G(h)$ in powers of $h$:

$$G(h) = \Lambda + G_2(h) + G_3(h) + G_4(h),$$  \hspace{1cm} (3.57)

with:

$$G_2(h) = - \nabla \cdot (h \nabla) + \Lambda (h \Lambda),$$  \hspace{1cm} (3.58)

$$G_3(h) = \Lambda (h^2 \Lambda^2) + \Lambda^2 (h^2 \Lambda) - 2(h \Lambda (h \Lambda)),$$  \hspace{1cm} (3.59)

and where $G_4(h) \equiv (h - \Lambda - G_1(h) - G_2(h)$ vanishes to order 3 when $h = 0$. See [26] for a formal derivation of this expansion, and e.g. Appendix F of [S] for rigorous estimates for $G_4$. Here, we are using the notation:

$$\Lambda^s f = F^{-1}(|\xi|^s F f), \quad s \in \mathbb{R},$$  \hspace{1cm} (3.60)

where $F$ is the Fourier transform on $\mathbb{R}^2$. 
In particular, keeping track of just the terms which are linear or quadratic, the above equations become:

\[
\begin{align*}
\partial_t h &= \Lambda \varphi_\omega - R \cdot V_\omega - \nabla \cdot (h \nabla \varphi_\omega) - \Lambda (h \Lambda \varphi_\omega) + \Lambda (h R \cdot V_\omega) + \nabla \cdot (h V_\omega) + \ldots \\
\partial_t \varphi_\omega &= -h - |\nabla \varphi_\omega|^2 + (\Lambda \varphi_\omega)^2 + |R \cdot V_\omega|^2 + (R \cdot V_\omega) \Lambda \varphi_\omega + \ldots
\end{align*}
\] (3.61) (3.62)

We now set:

\[ u = h + i \Lambda^{1/2} \varphi_\omega \] (3.63)

With this definition, we can recover \( h, \varphi_\omega \) from \( u \):

\[ h = \text{Re} \, u, \quad \varphi_\omega = \Lambda^{-1/2} \text{Im} \, u. \] (3.64)

In what follows, we will write \( u_R = \text{Re} \, u \) and \( u_I = \text{Im} \, u \). We will also write \( R_i \) for the Riesz transform:

\[ \mathcal{F}(R_i f) (\xi) = \frac{\xi_i}{|\xi|} (\mathcal{F} f) (\xi), \quad i = 1, 2. \] (3.65)

**Proposition 3.1.** With the above definitions, we have:

\[ (\partial_t + i \Lambda^{1/2}) u = N(u) + L(V_\omega) + N_1(u, V_\omega) + N_2(u, V_\omega) + N_3(u, V_\omega), \] (3.66)

where \( N(u) = B(u) + T(u) + R(u) \) and:

\[ B(u) = \Lambda u_R (\Lambda^{1/2} u_I) + \nabla \cdot (u_R (\Lambda^{-1/2} \nabla u_I)) + i \Lambda^{1/2} \left( |\Lambda^{-1/2} \nabla u_I|^2 + |\Lambda^{1/2} u_I|^2 \right) \] (3.67)

\[ T(u) = -\frac{1}{2} \Lambda (u_R^2 \Lambda^{3/2} u_I) + \Lambda^2 (u_R \Lambda^{1/2} u_I) - 2 \Lambda (u_R \Lambda (u_R \Lambda^{1/2} u_I)) + i \Lambda^{1/2} \left( \Lambda^{1/2} u_I (u_R \Lambda^{3/2} u_I - \Lambda (u_R \Lambda^{1/2} u_I)) \right) \] (3.68)

\[ L(V_\omega) = -R \cdot V_\omega, \] (3.69)

\[ N_1(u, V_\omega) = \Lambda^{1/2} (R \cdot V_\omega \Lambda^{1/2} u_I) - \nabla \cdot (u_R V_\omega) + \Lambda (u_R R \cdot V_\omega) \] (3.70)

\[ N_2(V_\omega, V_\omega) = \Lambda^{1/2} (R \cdot V_\omega)^2, \] (3.71)

and where \( R(u) \) (resp. \( N_3(u, V_\omega) \)) vanish to order 4 (resp. 3) when \( h = 0 \), and where \( N_3(u, V_\omega) \) is quadratic in \( V_\omega \) and its derivatives.

For later use, we record the Duhamel form of these equations:

\[ e^{i t \Lambda^{1/2}} u(t) - u_0 = g_1(t) + g_2(t) + g_3(t) + g_4(t) + g_5(t), \] (3.72)

where:

\[ g_1(t) = \int_0^t e^{is \Lambda^{1/2}} N(u) \, ds, \] (3.73)

\[ g_2(t) = \int_0^t e^{is \Lambda^{1/2}} L(w) \, ds, \] (3.74)

\[ g_3(t) = \int_0^t e^{is \Lambda^{1/2}} N_1(u, w) \, ds, \] (3.75)

\[ g_4(t) = \int_0^t e^{is \Lambda^{1/2}} N_2(w, w) \, ds, \]

\[ g_5(t) = \int_0^t e^{is \Lambda^{1/2}} N_3(u, w) \, ds, \]

4. **Elliptic Estimates and the Regularity of the Free Boundary**

Much of the material in the following sections is based heavily on the estimates and ideas in [9]. In [9], the authors consider the free boundary problem for a bounded fluid region, but extending their approach to the case of an unbounded domain is straightforward.

It is convenient to work in terms of Lagrangian coordinates, which we now define. We let \( \Omega \) denote the lower half-plane in \( \mathbb{R}^3 \). In this section, we will use the convention that points in \( D_\ell \) are
denoted by $x$ and points in $\Omega$ are denoted by $y$. The Lagrangian coordinates $x(t) : \Omega \to D_t$ are then defined by:

$$\frac{d}{dt} x_i(t, y) = v_i(t, x(t, y)) \quad y \in \Omega, \quad (4.1)$$

$$x(0, y) = y. \quad (4.2)$$

In these coordinates, the material derivative $D_t = \partial_t + v^k \partial_k$ becomes the usual time derivative:

$$D_t = \left. \frac{\partial}{\partial t} \right|_{y = \text{const.}} + v^k \frac{\partial}{\partial x_k}. \quad (4.3)$$

The Lagrangian coordinates $x$ induce a time dependent (co)metric $g$ on $\Omega$:

$$g_{ab} = \delta_{ij} \frac{dx^i}{dy^a} \frac{dx^j}{dy^b}, \quad g^{ab} = \delta_{ij} \frac{dy^a}{dx^i} \frac{dy^b}{dx^j}. \quad (4.4)$$

We use the convention that indices $a, b, c, ...$ denote quantities expressed in Lagrangian coordinates and indices $i, j, k, ...$ denote quantities expressed in the $x$ coordinates. We let $D$ denote the covariant derivative on $\Omega$ with respect to the metric $g$. We write $\Gamma^a_{bc}$ for the Christoffel symbols:

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd} \left( \frac{\partial}{\partial y^a} g_{bd} + \frac{\partial}{\partial y^b} g_{ad} - \frac{\partial}{\partial y^d} g_{ab} \right), \quad (4.5)$$

and the covariant derivative of a $(0, r)$ tensor $\beta$ is then:

$$D_a \beta_{a_1 \ldots a_r} = \partial_y \beta_{a_1 \ldots a_r} - \Gamma^d_{aa_1} \beta_{da_2 \ldots a_r} - \cdots - \Gamma_{aa_1 \ldots a_r} \beta_{a_1 \ldots a_r}. \quad (4.6)$$

We let $d = d(t, p) = \text{dist}_g(p, \partial \Omega)$ denote the geodesic distance with respect to the metric $g$ from $p \in \Omega$ to $\partial \Omega$, and we define the unit normal to $\partial \Omega$ by:

$$n_a = \partial_a d, \quad n^a = g^{ab} n_b. \quad (4.7)$$

We will also write $n_i$ for the normal expressed in Eulerian coordinates:

$$n_i = \frac{\partial y^a}{\partial x^i} n_a, \quad n^i = \delta^{ij} n_j. \quad (4.8)$$

We let $\iota_0 = \iota_0(t)$ denote the injectivity radius of $\partial D_t$. By definition, this is the largest number $\iota_0$ so that the map:

$$(x, \iota) \to x + \iota n(x), \quad x \in \partial D_t \quad (4.9)$$

is injective from $\partial D_t \times (-\iota_0, \iota_0) \to \{ x \in D_t : d(t, p) < \iota_0 \}$.

The (co)metric on $\partial \Omega$ is given by:

$$\gamma_{ab} = g_{ab} - n_a n_b, \quad \gamma^b_a = \delta^b_a - n_a n^b, \quad (4.10)$$

and the second fundamental form of $\partial \Omega$ is:

$$\theta_{ab} = \gamma^{c}_{a} \gamma^{d}_{b} \nabla_c n_d. \quad (4.11)$$

We note that on $\partial \Omega$, if $D$ denotes the covariant derivative on $\partial \Omega$ with respect to the metric $\gamma$, then:

$$D_a \beta_{a_1 \ldots a_r} = \gamma^b_a \partial_{a_1}^b \cdots \partial_{a_r}^b D_{b} \beta_{b_1 \ldots b_r}. \quad (4.12)$$

In particular this implies that if $q$ is a function on $\Omega$ with $q = 0$ on $\partial \Omega$ then $\gamma^b_a D_b q = 0$ on $\partial \Omega$. 
4.1. The extension of the normal to the interior. Since $d$ is the geodesic distance, we have $D_DaDa = 0$ and so $DDa = \theta$, where $\theta$ is the second fundamental form for the surfaces $\{d = \text{const}\}$. We will also write $\theta$ for the second fundamental form of $\partial \Omega$; if $n_a$ is the unit normal vector to $\partial \Omega$, then:

$$\theta_{ab} = (\delta^c_a - n_a n^c)(\delta^d - n_d n^d) D_c n_d.$$  \hspace{1cm} (4.13)

We now define an extension of the normal to a neighborhood of the boundary. We fix $d_0$ with $\eta_0/16 \leq d_0 \leq \eta_0/2$ and let $\eta \in C^\infty(\mathbb{R})$ be a function with $\eta(s) = 1$ when $|s| \leq 1/2$, $\eta(s) = 0$ when $|s| \geq 3/4$, $0 \leq \eta(s) \leq 1$ and $|\eta'| \leq 4$. We then define:

$$\tilde{n}_a(p) = \eta \left( \frac{d(p)}{d_0} \right) D_a d(x, y).$$ \hspace{1cm} (4.14)

Close to the boundary, we have $\tilde{n}_a = D_a d$ and away from the boundary, $\tilde{n}_a = 0$. We will not need the following lemma explicitly but it is useful to note that we can control the regularity of $\tilde{n}$. See Lemma 3.10 in [9] for the proof.

**Lemma 4.1.** With the above definitions, for each $y \in \partial \Omega$, if $d \leq \eta_0/2$:

$$|D\tilde{n}(q, d)| \leq 2|\theta(q)|, \hspace{1cm} |D\tilde{n}(q, d)| \leq 6||h||L^\infty(\Omega),$$  \hspace{1cm} (4.15)

where $h_{ab} = \frac{1}{2} D_t g_{ab}$.

We now extend $\gamma$ to the interior $\Omega$. Abusing notation, we will write:

$$\gamma_{ab} = g_{ab} - \tilde{n}_a \tilde{n}_b, \hspace{1cm} \gamma^b_a = \delta^{ac} \gamma_{bc}, \hspace{1cm} \gamma^{ab} = g^{ac} g^{bd} \gamma_{cd}.$$  \hspace{1cm} (4.16)

On $\partial \Omega$, $\gamma_{ab}$ (resp. $\gamma^{ab}$) is just the metric (resp. cometric) on $\partial \Omega$ induced by $g$, and $\gamma^a_a$ is the projection to $T(\partial \Omega)$. Away from $\partial \Omega$, $\gamma_{ab} = g_{ab}$ and $\gamma^a_a$ is the identity map. The estimates in Lemma 4.1 then imply (see Lemma 3.11 in [9]):

**Lemma 4.2.** With the above definitions, we have:

$$|D \gamma| \leq C \left( ||\theta|| L^\infty(\Omega) + \frac{1}{\eta_0} \right), \hspace{1cm} |D_t \gamma| \leq C ||h|| L^\infty(\partial \Omega).$$  \hspace{1cm} (4.17)

4.2. Elliptic estimates. For notational convenience, in this section we write $x_3 = y$. We will use multi-index notation and write $I = (i_1, \ldots, i_r)$. We will write $D^r$ for the operator which has components:

$$D^r_I = D_{i_1} \cdots D_{i_r},$$  \hspace{1cm} (4.18)

If $i_j = 1, 2$ for each $j = 1, \ldots, r$, we will also write $\nabla^r$ for the operator:

$$\nabla^r_I = \nabla_{i_1} \cdots \nabla_{i_r}.\hspace{1cm} (4.19)$$

We will also write:

$$\gamma^I_J = \gamma^{i_1}_{j_1} \cdots \gamma^{i_r}_{j_r}.$$  \hspace{1cm} (4.20)

Let $\beta$ be a $(0, r + 1)$ tensor with $\beta_{i_1, \ldots, i_r} = D^r_I \alpha_i$ for some $(0,1)$-tensor $\alpha$. We write:

$$(\text{div} \beta)_I = \delta^j_i D_j \beta_I = D^r_I (\delta^j_i D_j \alpha_i), \hspace{1cm} (4.21)$$

We will also write:

$$(\nabla \beta)_J = D_I \beta_{i_j} - D_j \beta_{Ii} = D^r_I (D_I \alpha_{i_j} - D_j \alpha_i).$$  \hspace{1cm} (4.22)

We will rely heavily on the following pointwise estimate in $D_t$, which is originally from [9]:

**Lemma 4.3.** If $\beta$ is as above, then:

$$|D \beta|^2 \leq C (\delta^{ij} \gamma^{kl} \gamma^{IJ} (D^k \beta_{II})(D^l \beta_{JJ}) + |\text{div} \beta|^2 + |\text{curl} \beta|^2),$$  \hspace{1cm} (4.25)
We will also use the following $L^2$ estimates:

**Lemma 4.4.** With the above notation, if $|\theta| + \frac{1}{\alpha} \leq K$ then:

\[
\begin{align*}
||\beta||^p_{L^p(\partial \Omega)} &\leq C\left(||D\beta||_{L^p(\Omega)} + K||\beta||_{L^p(\Omega)}\right), \quad 1 < p < \infty, \\
||\beta||^2_{L^2(\partial \Omega)} &\leq C||\Pi \beta||^2_{L^2(\partial \Omega)} + C\left(||\text{div } \beta||_{L^2(\Omega)} + ||\text{curl } \beta||_{L^2(\Omega)} + K||\beta||_{L^2(\Omega)}\right)||\beta||_{L^2(\Omega)}, \\
||\beta||^2_{L^2(\partial \Omega)} &\leq C||n \cdot \beta||^2_{L^2(\partial \Omega)} + C\left(||\text{div } \beta||_{L^2(\Omega)} + ||\text{curl } \beta||_{L^2(\Omega)} + K||\beta||_{L^2(\Omega)}\right)||\beta||_{L^2(\Omega)},
\end{align*}
\]

and

\[
\begin{align*}
||D\beta||^2_{L^2(\Omega)} &\leq C||D\beta||_{L^2(\partial \Omega)}^2 + C\left(||\text{div } \beta||_{L^2(\Omega)} + ||\text{curl } \beta||_{L^2(\Omega)}\right)^2, \\
||D\beta||^2_{L^2(\Omega)} &\leq C||\Pi D\beta||_{L^2(\partial \Omega)}||\Pi n \cdot \beta||_{L^2(\Omega)} + C\left(||\text{div } \beta||_{L^2(\Omega)} + ||\text{curl } \beta||_{L^2(\Omega)} + K||\beta||_{L^2(\Omega)}\right)^2, \\
||D\beta||^2_{L^2(\Omega)} &\leq C||\Pi n \cdot \nabla \beta||_{L^2(\partial \Omega)}||\Pi \beta||_{L^2(\partial \Omega)} + C\left(||\text{div } \beta||_{L^2(\Omega)} + ||\text{curl } \beta||_{L^2(\Omega)} + K||\beta||_{L^2(\Omega)}\right)^2.
\end{align*}
\]

**Proof.** Other than (4.26) for $p \neq 2$, all of the above inequalities are in Lemma 5.6 in [9]. To prove (4.26) for $p \neq 2$ we can argue in essentially the same way as the $p = 2$ case; by Stokes’ theorem:

\[
||\beta||^p_{L^p(\partial \Omega)} = \int_{\partial \Omega} \tilde{n}_i \tilde{n}^i ||\beta||^p dS = \int_{\Omega} (D_i \tilde{n}^i) ||\beta||^p + p \nabla \beta \cdot \beta ||\beta||^{p-2}.
\]

By Lemma 4.4, the first term is bounded by $K||\beta||^p_{L^p(\Omega)}$. To bound the second term, we just note that by Hölder’s inequality and Young’s inequality, it is bounded by $||\nabla \beta||_{L^p(\Omega)} ||\beta||^p_{L^p(\Omega)} \leq ||\nabla \beta||_{L^p(\Omega)} + ||\beta||^p_{L^p(\Omega)}$.

The estimates (4.25) will be used to show that the energy (defined in (5.8)) controls all derivatives of $v$. The estimates in (4.4) will be used to show that the energies control $v$ on the boundary, and we will also use them with $\alpha = \nabla q$ for a function $q$ to control solutions of the Dirichlet problem. We will assume in many of the following estimates that $K \leq 1$. This is only for notational convenience and is not essential to the arguments; many of the estimates will involve constants which can be bounded in terms of $1 + K$ and so this assumption allows us to ignore the unimportant dependence on $K$. We will make it clear when this assumption is used. Versions of these estimates with more explicit dependence on $K$ can be found in [9].

First, we show that derivatives of $q$ can be controlled by projected derivatives of $q$ on the boundary and derivatives of $\Delta q$:

**Proposition 4.1.** If $K \leq 1$ then for $r \geq 1$:

\[
||D^r q||_{L^2(\partial \Omega)} + ||D^r q||_{L^2(\Omega)} \leq C\left(||\Pi D^r q||_{L^2(\partial \Omega)} + \sum_{s \leq r-1} ||D^s \Delta q||_{L^2(\partial \Omega)} + ||D q||_{L^2(\Omega)}\right),
\]

and for any $\delta > 0$:

\[
||D^r q||_{L^2(\partial \Omega)} + ||D^{r-1} q||_{L^2(\partial \Omega)} \leq \delta ||\Pi D^r q||_{L^2(\partial \Omega)} + C(1/\delta) \sum_{s \leq r-2} ||D^s \Delta q||_{L^2(\partial \Omega)} + ||D q||_{L^2(\Omega)}.
\]

**Proof.** By (4.27) with $\beta = D^r q$:

\[
||D^r q||^2_{L^2(\partial \Omega)} \leq ||\Pi D^r q||^2_{L^2(\partial \Omega)} + C\left(||D^{r-1} \Delta q||_{L^2(\partial \Omega)} + K||D^r q||_{L^2(\Omega)}\right)||D^r q||_{L^2(\partial \Omega)},
\]

and by (4.30) with $\beta = D^{r-1} q$:

\[
||D^r q||^2_{L^2(\partial \Omega)} \leq C||\Pi D^r q||_{L^2(\partial \Omega)}||D^{r-1} q||_{L^2(\partial \Omega)} + C\left(||D^{r-1} \Delta q||_{L^2(\partial \Omega)} + K||D^{r-1} q||_{L^2(\partial \Omega)}\right)^2.
\]

Combining these inequalities and using induction gives (4.33) and (4.34).
We will use this proposition in two ways. First, in our energy estimates we will directly control \( \|\Pi D^r p\|_{L^2(\Omega)} \) if the Taylor sign condition \((4.7)\) holds and since \( \Delta p = -(\partial_i v^j)(\partial_j v^i) \), we control this as well. We will also use this estimate to control derivatives \( D_i p \) on \( \partial \mathcal{D}_t \), and we will rely on the observation that \( \Pi D^r q \) is lower order if \( q = 0 \) on \( \partial \Omega \). This is clear when \( r = 0, 1 \), and for \( r = 2 \) we have:

\[
\Pi_l^i \Pi_k^j D_j D_l q = \Pi_l^i D_j (\Pi_k^j D_l q) - \Pi_l^i D_j (\Pi_k^j D_l q),
\]

and when \( q = 0 \) on \( \partial \Omega \), the first term is zero and the second term is \(- (\Pi_l^i D_j n^j) n^i \nabla q \), so that \( \Pi \nabla^2 q = \theta D_n q \). We also record the \( r = 3 \) case for later use:

\[
\Pi D^3 q = \bar{\nabla}^3 q - 2 \theta \otimes (\theta \cdot \hat{\nabla} q) + (\bar{\nabla} \theta) D_N q + 3 \theta \otimes (\bar{\nabla} D_N q).
\]

It will not be important in our argument exactly which indices appear where.

One can use the following heuristic argument from [9] to see what the higher-order version of the formula is. If \( d(x) = \text{dist}(x, \partial \Omega) \) then \( q/d \) is smooth up to the boundary, and:

\[
\Pi D^r q = \Pi D^r \left( \frac{d q}{d} \right) = \sum_{s=0}^{r} \Pi(D^s d) \otimes D^{r-s} \left( \frac{q}{d} \right).
\]

Restricting this formula to the boundary, we see that the \( s = 0, 1 \) terms drop out and that \( q/d \sim \nabla_n q \). Further, if we knew that all the derivatives falling on \( d \) were purely tangential, then arguing as above we could replace \( D^s d \) with \( D^s - \theta \). We therefore write \( D_i = (\Pi_i^j + n_i n^j) D_j \) and further note that \( n_i n^j D_j D_k d = 0 \) because \( d \) is the geodesic distance. Each time we make this substitution, some derivatives will fall onto the factors of \( N \) we have introduced and this generates more factors of \( \theta \), but at the same time less derivatives land on the function \( q \). This suggests that we should expect:

\[
\Pi D^r q \sim \sum_{s=0}^{r-2} \bar{\nabla}^s \theta \otimes D^{r-s} D_n q
\]

Also note that the \( s = r - 2 \) term of the expansion \((4.40)\) is \((\bar{\nabla}^{r-2} \theta) D_n q \) and so if the lower order terms and \( |Dq|^{-1} \) are bounded, this gives an estimate for \( \theta \) in terms of \( q \).

The rigorous version of these observations is:

**Proposition 4.2.** Let \( q : \mathcal{D}_t \to \mathbb{R} \) be a function. If \( ||\theta||_{L^\infty(\partial \Omega)} \leq 1 \), then for \( m = 0, 1 \):

\[
||\Pi D^r q||_{L^2(\partial \Omega)}^2 \leq ||\bar{\nabla}^r q||_{L^2(\partial \Omega)} + 2||\bar{\nabla}^{r-2} \theta||_{L^2(\partial \Omega)} ||D_n q||_{L^\infty(\partial \Omega)}
\]

\[
+ C\left(||\theta||_{L^\infty(\partial \Omega)} + \sum_{k \leq r-2} ||\bar{\nabla}^k \theta||_{L^2(\partial \Omega)} \right) \sum_{k \leq r-2+m} ||D^k q||_{L^2(\partial \Omega)} + C \sum_{k=1}^{r-1} ||D^{r-k} q||_{L^2(\partial \Omega)}
\]

and if \( |D_n q| > \delta_0 > 0 \):

\[
||\bar{\nabla}^{r-2} \theta||_{L^2(\partial \Omega)} \leq C\delta_0^{-1} \left(||\Pi D^r q||_{L^2(\partial \Omega)} + \sum_{k=1}^{r-1} ||D^{r-k} q||_{L^2(\partial \Omega)} \right)
\]

\[
+ C\delta_0^{-1} \left(||\theta||_{L^\infty(\partial \Omega)} + \sum_{k \leq r-3} ||\bar{\nabla}^{r-3} \theta||_{L^2(\partial \Omega)} \right) \sum_{k \leq r-1} ||D^k q||_{L^2(\partial \Omega)}
\]

Combining these two propositions, we have:
Corollary 4.1. If $K \leq 1$ and $q : D_t \rightarrow \mathbb{R}$ is a function with $q = 0$ on $\partial\Omega$, then for $r \geq 3$:
\[
||D^{-1}q||_{L^2(\partial\Omega)} \leq \beta C \left(||D^{-3}\theta||_{L^2(\partial\Omega)}||D_n q||_{L^\infty(\partial\Omega)} + ||D^{-2}\Delta q||_{L^2(\Omega)} \right) \\
+ C(||\theta||_{L^2(\partial\Omega)}, ..., ||D^{-4}q||_{L^2(\partial\Omega)}) \left(||D_n q||_{L^\infty(\partial\Omega)} + \sum_{s \leq r-3} ||D^s \Delta q||_{L^2(\Omega)} + ||Dq||_{L^2(\Omega)} \right),
\]
and for $r > 3$:
\[
||D^{-1}q||_{L^2(\partial\Omega)} + ||Dq||_{L^\infty(\partial\Omega)} \leq C ||D^{-2}\Delta q||_{L^2(\Omega)} + C(||\theta||_{L^2(\partial\Omega)}, ..., ||D^{-3}\theta||_{L^2(\partial\Omega)}) \sum_{s \leq r-3} ||D^s \Delta q||_{L^2(\Omega)}. 
\]

(4.44)

(4.45)

4.3. Estimates for $v_\omega$. Unlike the previous section, in this section we will work on $D_t$. We will therefore write:
\[
\gamma_{ij} = \frac{\partial y^a}{\partial x^i} \frac{\partial y^b}{\partial x^j} \gamma_{ab},
\]
and similarly for $\gamma_i^j$, $D_t$, etc. In Section 11 we use some of the ideas from [27] to show that $v_\omega = \text{curl } \beta$, where $\beta$ satisfies:
\[
\Delta \beta = \omega \quad \text{in } D_t, 
\]
\[
\gamma_i^j \beta_j = 0, \quad j = 1, 2, 3, \text{ on } \partial D_t, 
\]
\[
D_n (\beta \cdot n) + H \beta \cdot n = 0 \quad \text{on } \partial D_t. 
\]
(4.47)
(4.48)
(4.49)
Taking the divergence of (4.47) and noting that $D \cdot \beta|_{\partial D_t} = \gamma D \cdot (\gamma \cdot \beta) + D_n (\beta \cdot n) + H \beta \cdot n = 0$, it follows that $\text{div } \beta = 0$ in $D_t$ if $\beta$. We have the basic elliptic estimate:

**Lemma 4.5.** With $\beta$ as defined above:
\[
||\beta||_{L^6(D_t)} + ||D\beta||_{L^2(D_t)} + ||\text{curl } \beta||_{L^2(D_t)} \lesssim ||\omega||_{L^{6/5}(D_t)}. 
\]
(4.50)

**Proof.** First, by the Sobolev inequality (A.2), $||\beta||_{L^6(D_t)} \lesssim ||D\beta||_{L^2(D_t)}$. We next show that $||D\beta||_{L^2(D_t)} \lesssim ||\text{curl } \beta||_{L^2(D_t)}$. Note that:
\[
\int_{D_t} \delta^j \delta^k (D_i \beta_k D_j \beta_i) = \int_{D_t} \delta^j \delta^k (D_i \beta_k D_j \beta_i) + \int_{D_t} \delta^j \delta^k (D_i \beta_k) \text{curl } \beta_j. 
\]
(4.51)
Integrating by parts, the first term is:
\[
\int_{\partial D_t} \delta^j \delta^k n^k D_i \beta_k \beta_j - \int_{D_t} \delta^j \delta^k (D_i \beta_k) \beta_j 
\]
(4.52)
The interior term vanishes since $\text{div } \beta = 0$. To handle the boundary term, we note that since $\gamma \cdot \beta = 0$ on $\partial D_t$:
\[
n^k \beta^i D_i \beta_k = n^k \beta^i D_i n^\ell \beta_k = n^i (\beta^\ell n_\ell) D_i n^k \beta_k - n^i (\beta^\ell n_\ell) H n^k \beta_k = (\text{div } \beta - \gamma_i^k D_i (\gamma^\ell_k \beta^\ell)) (\beta^\ell n_\ell) = 0, 
\]
(4.53)
where we have used that $\text{div } \beta = 0$. Returning to (4.51), we have:
\[
||D\beta||_{L^2(D_t)}^2 \lesssim ||D\beta||_{L^2(D_t)} ||\text{curl } \beta||_{L^2(D_t)}, 
\]
(4.54)
which implies the bound for $||D\beta||_{L^2(D_t)}$.

Finally we show that $||\text{curl } \beta||_{L^2(D_t)} \lesssim ||\omega||_{L^{6/5}(D_t)}$. Integrating by parts:
\[
\int_{D_t} |\text{curl } \beta|^2 = \int_{\partial D_t} (n \times \beta) \text{curl } \beta - \int_{D_t} \beta \text{curl}^2 \beta. 
\]
(4.55)
Since the tangential components of $\beta$ vanish on $\partial D_t$, it follows that $n \times \beta = 0$. The interior term is bounded by $||\beta||_{L^6(D_t)} ||\omega||_{L^{6/5}(D_t)}$, which completes the proof. \qed
The above estimates combined with the elliptic estimates in the previous section will allow us to bound \( \|v_\omega\|_{W^{k,1}(\mathbb{R}^2)} \). In the proof of the dispersive estimates, we will also need to bound \( \|V_\omega\|_{L^p(\partial D_t)} \) for \( 1 < p < 2 \). Recall that in the interior, we have \( V_\omega = \text{curl} \beta \) with \( \Delta \beta = \omega \). In the flat case \( (h = 0) \), a simple calculation using the Newtonian potential shows that for any \( z \in \{(z_1, z_2, z_3) | z_3 \leq 0\} \), we have \( |v_\omega(z)| = |\text{curl} \beta(z)| \leq \frac{1}{1 + |z|^2} |(1 + |z|^2)\omega|_{L^1(D)} \). Restricting this to \( z = (x, 0) \in \partial D_t \) gives that \( V_\omega \in L^p(\partial D_t) \) for \( p > 1 \). To handle the case with \( h \neq 0 \), in Proposition D.1, we follow the approach of [28] to construct a Green’s function for \( D_t \) which satisfies the same estimates as the Newton potential, and this can be used to prove estimates for \( \|V_\omega\|_{L^p(\partial D_t)} \) for \( 1 < p \).

**Proposition 4.3.** If \( \|h\|_{W^{4,\infty}(\mathbb{R}^2)} + \|h\|_{H^{N_1}(\mathbb{R}^2)} \leq 1 \), then for \( 2 \leq p < \infty \), \( 0 \leq r \leq N_1 - 2 \):

\[
\|\nabla^r V_\omega\|_{L^p(\mathbb{R}^2)} + \|D^r v_\omega\|_{L^2(D_t)} \lesssim \|\omega\|_{H^{N_1}(D_t)},
\]

and for \( 1 < p \leq 2 \):

\[
\|V_\omega\|_{L^p(\mathbb{R}^2)} \lesssim \|\omega\|_{H^{N_1}(D_t)}.
\]

The assumption on the size of \( h \) is for notational convenience and can be avoided. We remark that by the interpolation inequality \( \text{(A.3)} \), if the bootstrap assumptions \( 2.10, \text{ and } 2.12 \) hold then we have:

\[
\|h\|_{H^{N_1}(\mathbb{R}^2)} \lesssim \varepsilon_0(1 + t^\sigma) + \varepsilon_1(1 + t)^\delta,
\]

where \( \sigma = \frac{N_1 - 1}{N_0} (1 + \delta) \) and so this quantity is less than one until \( t \sim T_{\sigma_0, \varepsilon_1} \). We will be forced to take \( t \lesssim T_{\sigma_0, \varepsilon_1} \) at other points anyways, so this is not a serious restriction.

**Proof.** First, by \( \text{(3.3)} \), \( \nabla^r V_\omega = (D^r v_\omega)_{|\partial D_t} - \nabla^r h(D_y v_\omega)_{|\partial D_t} + (\nabla h)^r(D_y^r v_\omega)_{|\partial D_t} + \ldots \), up to similar terms. We show how to prove the estimates for the first term, as the other terms can be handled similarly. We consider the cases \( 1 < p < 2 \) and \( p \geq 2 \) separately.

When \( p \geq 2 \), by Holder’s, Young’s and Sobolev’s inequalities, it suffices to control \( \|D^k V_\omega\|_{L^2(D_t)} \) for \( 0 \leq k \leq r + 2 \). Since \( v_\omega \cdot n = 0 \) on \( \partial D_t \), repeatedly applying the trace inequality \( \text{(4.27)} \) gives:

\[
\|D^k v_\omega\|_{L^2(\partial D_t)} \leq C \left( \|D^k \omega\|_{L^2(D_t)} + (1 + K) \|v_\omega\|_{L^2(D_t)} \right).
\]

The constant here depends on bounds for \( \|\theta\|_{L^\infty(\partial D_t)} \) as well as \( \|\theta\|_{H^{k-2}(\partial D_t)} \) and by assumption these are both bounded. By the estimate \( \text{(4.58)} \), we have \( \|v_\omega\|_{L^2(D_t)} \lesssim \|\omega\|_{H^{N_1}(D_t)} \) and by Holder’s inequality, we have \( \|\omega\|_{L^6(\partial D_t)} \lesssim \|\omega\|_{H^{N_1}(D_t)} \). Since \( k \leq r + 2 \leq N_1 \) we bound the first term here as well.

The estimate \( \text{(4.57)} \) follows from \( \text{(D.12)} \).

\[ \square \]

5. **Energy Estimates**

The system \( \text{(1.1), (1.3)} \) has a conserved energy:

\[
E_0(t) = \frac{1}{2} \int_{D_t} |v(t, x, y)|^2 dx \, dy + \frac{1}{2} \int_{\mathbb{R}^2} h(t, x)^2 \, dx = \frac{1}{2} \int_{\mathbb{R}^2} \int_{-\infty}^{h(t, x)} |v(t, x, y)|^2 \, dx \, dy + \frac{1}{2} \int_{\mathbb{R}^2} |h(t, x)|^2 \, dx.
\]

We have:

\[
\frac{d}{dt} E_0(t) = \int_{D_t} v^i \partial_i v_i \, dx \, dy + \frac{1}{2} \int_{\mathbb{R}^2} \partial_h |v|^2 \, dx + \int_{\mathbb{R}^2} h \partial_t h \, dx
\]

\[
= - \int_{D_t} v^i (v^k \partial_k v_i + \partial_i (p + y)) \, dx \, dy + \frac{1}{2} \int_{\mathbb{R}^2} \partial_h |v|^2 \, dx + \int_{\mathbb{R}^2} h \partial_t h \, dx
\]

\[
= - \frac{1}{2} \int_{\partial D_t} n_k v^k |v|^2 \, dS - \int_{\partial D_t} n_k v^k h \, dS + \frac{1}{2} \int_{\mathbb{R}^2} \partial_h |v|^2 \, dx + \int_{\mathbb{R}^2} h \partial_t h \, dx,
\]
where we used that $\text{div} \, v = 0$ in $D_t$ and that $p = 0$ on $\partial D_t$. Using (1.4) the first and third, and second and fourth terms here cancel.

To get higher-order energies, in the irrotational case ($\omega = 0$) one can use the system (3.1) - (5.2) directly to prove energy estimates. See [8] or [23] for this approach. In the case $\omega \neq 0$, the corresponding system (3.13) - (3.14) is more complicated to work with and we instead choose to model our approach on [9] and prove energy estimates for Euler’s equation (1.1) - (1.3) directly. The advantage is that the estimates can be proved using elementary techniques, relying only on integration by parts and simple geometric facts (such as (4.40), (4.27)).

We define the projection $\gamma$ as in (4.16). We will write:

$$\gamma^{ij} = \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b} \gamma^{ab},$$

(5.5)

for $\gamma$ expressed in the $x$-coordinates. We also write:

$$\gamma^{\bar{i} \cdot \bar{i} \cdot \bar{j} \cdot j \cdot r} = \gamma^{1 \cdot 1 \cdot 2 \cdot 2 \cdot 3} \gamma^{3 \cdot 3 \cdot 2} \gamma^{2 \cdot 2 \cdot 1} \gamma^{1 \cdot 1 \cdot 2} \gamma^{3 \cdot 3 \cdot 2},$$

(5.6)

For $(0, r)$-tensors $\alpha, \beta$, we define:

$$Q(\alpha, \beta) = \gamma^{\bar{i} \cdot \bar{i} \cdot \bar{j} \cdot j} \alpha_{\bar{i} \cdot \bar{i} \cdot \bar{j} \cdot j} \beta_{\bar{j} \cdot \bar{j} \cdot \bar{i} \cdot i}.$$

(5.7)

The energies are:

$$E^r(t) = \int_{D_t} \delta^{ij} Q(D^r v_i, D^r v_j) \, dV + \int_{\partial D_t} Q(D^r p, D^r p) |D p| \, dS + \int_{D_t} |D^{r-1} \omega|^2 \, dV. (5.8)$$

We will see that since $p = 0$ on $\partial D_t$, $Q(D^r p, D^r p) = Q(\overline{D}^{r-2} \theta, \overline{D}^{r-2} \theta) |D_n p|^2$ to highest order (see the discussion after (4.37) and the estimate (5.39)). In particular since $\theta \sim \nabla^2 h$, bounds for $E^r$ imply bounds for $h$. Moreover, in Theorem 5.2 we will see that $||u||_{H^N_0}^2 \leq E^0$ to highest order (recall that $u$ is defined in 3.63).

We will prove the energy estimates in the following sections assuming the following a priori bounds:

$$|\theta(t)| + \frac{1}{\theta_0(t)} \leq K \quad \text{on } \partial D_t, \quad (5.9)$$

$$-\nabla_n p(t) \geq \delta_0 > 0 \quad \text{on } \partial D_t, \quad (5.10)$$

$$|D^2 p(t)| + |D_n D_i p(t)| \leq L \quad \text{on } \partial D_t, \quad (5.11)$$

$$|D_v(t)| + |D^2_v p(t)| \leq M \quad \text{on } \partial D_t. \quad (5.12)$$

Recall that we are writing $\theta_0(t)$ for the injectivity radius of $\partial D_t$. We will assume in the estimates that $K \leq 1$. This is only for notational convenience and is not essential to the arguments; many of the estimates will involve coefficients that can be bounded in terms of $1 + K$ and this allows us to ignore the unimportant dependence on $K$. We also remark that $\frac{1}{\theta_0} \leq ||\theta||_{L^\infty(\partial D_t)}$ and so the definition of $K$ is somewhat overcomplicated. We choose to keep track of both terms because it turns out that if one is interested in proving energy estimates which depend on as few derivatives of $v$ as possible in $L^\infty$, it is difficult to control the time evolution of $\theta_0$. For this reason, in [9], the authors introduce another radius which they denote $\theta_1$ (see Definition 3.5 there) which can be used to control $\theta_0$. For our purposes this distinction will not be important, because we will eventually need to assume bounds for more derivatives of $v$ in any case, but if one is interested in studying this problem with less regular data it is useful to keep track of both terms.

The main result of this section is the following energy estimate:

**Proposition 5.1.** Suppose that the a priori assumptions (5.9) - (5.12) hold. There are continuous functions $C_r = C_r(\delta_0^{-1})$ and homogeneous polynomials $P_r$ with positive coefficients so that for $r \geq 0$:

$$\left| \frac{d}{dt} E^r(t) \right| \leq C_r(\delta_0^{-1})(K + L + M) \left( E^r(t) + (K + L + M) P_r(\mathcal{E}^r_{n-1}(t), K, L, M) \right),$$

(5.13)
with \( \mathcal{E}_{r-1}^s = \sum_{s\leq r-1} \mathcal{E}_s \).

We prove this in the next two subsections. Next, we relate the energy \( \mathcal{E}_r \) and the a priori assumptions (5.9)-(5.12) to the dispersive variable \( u \) and the vorticity.

**Lemma 5.1.** If the bootstrap assumptions (2.10)-(2.12) hold, then with:

\[
\mathcal{A}(t) = \|\theta(t)\|_{L^\infty(\partial D_1)} + \frac{1}{t_0(t)} + ||D^2p||_{L^\infty(\partial D_1)} + ||DD_t p||_{L^\infty(\partial D_1)} + ||Dv(t)||_{L^\infty(D_1)} + ||Dp(t)||_{L^\infty(D_1)},
\]

and

\[
\mathcal{B}(t) = \|h(t)\|_{W^{4,\infty}(\mathbb{R}^2)} + \|\varphi(t)\|_{W^{4,\infty}(\mathbb{R}^2)} + \|\omega(t)\|_{H^s_{w_1}(D_1)},
\]

we have:

\[
\mathcal{A}(t) \lesssim \mathcal{B}(t) \left(1 + \mathcal{B}(t) \sqrt{\mathcal{E}_3^s(t)}\right),
\]

where \( \mathcal{E}_3^s = \sum_{s\leq 3} \mathcal{E}_s \). Furthermore, if \( 0 \leq t \leq T_{\varepsilon_0, \varepsilon_1} \) with \( T_{\varepsilon_0, \varepsilon_1} \) defined by (1.15), then:

\[
-\nabla_v p(t) \geq \frac{1}{2} (-\nabla_v p(0)) \quad \text{on} \quad \partial D_1.
\]

We remark that one could replace \( ||\omega||_{H^s_{w_1}(D_1)} \) in (5.16) with an \( L^\infty \)-based norm with fewer derivatives by using a Schauder estimate, but this will suffice for our purposes. We also note that the fact that \( \sqrt{\mathcal{E}_3} \) shows up on the right-hand side of (5.16) is because we need to control \( ||DD_t p||_{L^\infty(\partial D_1)} \). We bound this by Sobolev embedding and then the elliptic estimates in Section 4. Since \( \Delta D_t p \) is cubic (see (5.31)), this can be bounded by \( B^2 \sqrt{\mathcal{E}_3^s} \).

Recall that \( \varphi = \psi|_{\partial D_1} \) where \( \nabla_v \psi = n \cdot v \) on \( \partial D_1 \). Since by Lemma 5.2, the energies control derivatives of \( v \) on \( \partial D_1 \) as well as derivatives of \( \theta \), we have the following estimate, which is proved in Section 5.4.

**Proposition 5.2.** With \( \varphi_\omega \) defined by (3.31), if \( ||h||_{W^{4,\infty}(\mathbb{R}^2)} \ll 1 \), then for any \( r \geq 1 \):

\[
||h||_{H^{r,\infty}(\mathbb{R}^2)} + ||A^{1/2} \varphi_\omega||_{L^2(\mathbb{R}^2)} + \|\nabla \varphi_\omega\|_{H^{r-1}(\mathbb{R}^2)} \leq C \mathcal{E}_s^r + AP(\mathcal{E}_s^{r-1}, A),
\]

where \( \mathcal{E}_s^r = \sum_{s\leq r} \mathcal{E}_s \) and \( A \) defined by (5.14).

We will then see that the energy estimates (5.13) and this lemma imply:

**Proposition 5.3.** If the bootstrap assumptions (2.10)-(2.12) hold, then:

\[
||\omega(t)||_{H^{s_1}_{w_1}(D_1)} \leq ||\omega_0||_{H^{s_1}_{w_1}(D_0)} + C_N \int_0^t \left(||u(s)||_{W^{s_1+2,\infty}(\mathbb{R}^2)} + ||\omega(s)||_{H^{s_1}_{w_1}(D_s)}\right)||\omega(s)||_{H^{s_1}_{w_1}(D_s)} ds.
\]

We will need to take \( N_1 \geq 6 \) to prove the dispersive estimates and since we only control \( ||u||_{W^{4,\infty}} \), the result is that \( ||u||_{W^{s_1+2,\infty}(\mathbb{R}^2)} \) decays slightly slower than the critical rate of \( 1/t \), and this is why we are only able to follow the solution until \( T \sim \varepsilon_0^{-N} \).

Assuming these results for the moment, we can now provide the proofs of Theorem 2.4 and 2.3.

**Proof of Theorem 2.4.** By Grönwall’s inequality and induction, if \( \delta \) is sufficiently small then the estimate (5.13) combined with (5.16) implies that there is a constant \( C_{N_0}^s \) such that:

\[
\mathcal{E}_{N_0}(t) \leq \mathcal{E}_{N_0}(0) + C_{N_0}^s \int_0^t \left(\frac{\varepsilon_0}{1+s} + \varepsilon_1(1+s)^3\varepsilon_0^2(1+s)^{2\delta} \right) ds.
\]

\[
\leq \mathcal{E}_{N_0}(0) + C_{N_0}^s (\varepsilon_0^3(1+t)^2 + \varepsilon_1^2(1+t)^{1+3\delta}).
\]

Using (5.39) and the fact that \( \theta \sim \nabla \cdot h \) completes the proof.
Proof of Theorem 2.3. By the interpolation inequality (A.3) combined with the estimate (5.18) for \( \|u\|_{H^{N_0}} \),
\[
\|u\|_{W^{N_1+2,\infty}(\mathbb{R}^2)} \lesssim \frac{\varepsilon_0}{(1 + t)^{1-\sigma}},
\]
with \( \sigma' = \frac{N_0 + 2}{N_0 - 1}(1 + \delta) \), provided the assumptions (2.10)-(2.12) hold. Recalling that \( N_0 = 2N_1 \), this implies:
\[
\|u\|_{W^{N_1,\infty}(\mathbb{R}^2)} \lesssim \frac{\varepsilon_0}{(1 + t)^{1-1/N}}.
\]
Combining this with (5.19) and using the assumptions (2.10)-(2.12), we have:
\[
\|\omega(t)\|^2_{H^{N_1}_w(D_t)} \leq \|\omega(0)\|^2_{H^{N_1}_w(D_0)} + C_{N_1} \int_0^t \left( \varepsilon_0 (1 + s)^{-1+\sigma} + \varepsilon_1 (1 + s)^{1+\delta} \right) \varepsilon_1^2 (1 + s)^{2\delta} \, ds \leq \frac{1}{4} \varepsilon^2 + C_{N_1} (\varepsilon_0 (1 + t)^{\sigma}) (\varepsilon_1 (1 + t)^{\delta}) \varepsilon_1^2 (1 + t)^{2\delta},
\]
as required.

As in [9], before proving the energy estimates (5.13), it is convenient to first prove that \( \mathcal{E} \) controls norms of \( v, p \) and the second fundamental form \( \theta \).

5.1. Quantities controlled by \( \mathcal{E} \). We start with the equations for \( \omega \) and \( p \). Taking the curl of (1.1) shows that \( \omega \) satisfies:
\[
D_t \omega_{ij} = \omega_{ik} D^k v_j.
\]
Taking the divergence of (1.1) and using (1.2) gives that \( p \) satisfies:
\[
\Delta p = - (D_t v^j) (D_j v^i) = - D_t (v^j D_j v^i),
\]
where we used that \( \text{div} \, v = 0 \). We will also need to use the equation for \( D_t p \). We apply \( D_t \) to both sides of (5.27), and the right-hand side is:
\[
-D_t (D_t (v^j D_j v^i)) = - D_t (D_t (v^j D_j v^i)) + D_k (v^k D_i (v^j D_j v^i)),
\]
while:
\[
D_t \Delta p = D_t D_t D^j p - D_t v^k D_k D^j p = \Delta D_t p - D_t (v^k D_k D^j p) - D_k (D_t v^k D^j p).
\]
In particular, rearranging the indices this shows that:
\[
\Delta D_t p = D_t \left( v^k D_k D^j p - D_k v^j D^k p - D_t (v^j D_j v^i) + v^j D_k (v^j D_j v^k) \right).
\]
We shall need that the right-hand side of (5.30) is the divergence of a vector field, but for most of our applications it is more useful to use (1.2) and re-write this in the following slightly more attractive way:
\[
\Delta D_t p = 4 \text{tr} \left( (Dv) \cdot D^2 p \right) + 2 \text{tr} \left( (Dv)^3 \right) - (\Delta v) \cdot Dp,
\]
where we are writing \( (Dv) \cdot D^2 p \) and \( (Dv)^3 \). The next lemma follows from these observations, the interpolation inequalities (A.6)-(A.7), and the fact that \( [D_t, \partial_t] = - (\partial_t v^j) \partial_j \).

Lemma 5.2. If \( K \leq 1 \) then there are constants \( C_\tau > 0 \) so that:
\[
\|D_t D^r v + D^{r+1} p\|_{L^2(D_t)} + \|D_t D^{r-1} \omega\|_{L^2(D_t)} + \|\Delta D^{r-1} p\|_{L^2(D_t)} \leq C_\tau \|Dv\|_{L^\infty(D_t)} \sum_{k=0}^r \|D^k v\|_{L^2(D_t)},
\]
(5.32)
\[
\|\Pi(D_t D^r p + (D^r v) \cdot Dp - D^r D_t p)\|_{L^2(\partial D_t)} \leq C_\tau \sum_{s=1}^{r-2} \|\Pi((D^{1+s} v) \cdot (D^{r-s} p))\|_{L^2(\partial D_t)}
\]
(5.33)
and
\[ ||D^{r-2}\Delta D_t p - (D^{r-2}\Delta v) \cdot Dp||_{L^2(\mathcal{D}_t)} \leq C_r \left( ||Dv||_{L^\infty(\mathcal{D}_t)} + ||Dp||_{L^\infty(\mathcal{D}_t)} \right) \]
\[ + C_r||Dv||_{L^\infty(\mathcal{D}_t)} \sum_{s=1}^{r-1} ||D^s v||_{L^2(\mathcal{D}_t)} + ||D^s p||_{L^2(\mathcal{D}_t)} \]
\[ + C_r||Dv||_{L^\infty(\mathcal{D}_t)} \sum_{s=1}^{r-1} ||D^s v||_{L^2(\mathcal{D}_t)} (5.34) \]

The elliptic estimates in Section 4 give us the following coercive estimates. These are essentially from \cite{9}; the only difference here is that these estimates hold when \(Vol \mathcal{D}_t = \infty\).

**Lemma 5.3.** Suppose that \(K \leq 1\). Then there are constants \(C_r\) with:
\[ ||D^r v||_{L^2(\mathcal{D}_t)} \leq C_r \mathcal{E}_r, \quad (5.35) \]
\[ ||\Pi D^r p||_{L^2(\partial \mathcal{D}_t)} \leq ||Dp||_{L^\infty(\partial \mathcal{D}_t)} \mathcal{E}_r, \quad (5.36) \]
In addition, for \(r \geq 1\):
\[ ||D^r p||_{L^2(\mathcal{D}_t)} + ||D^r p||_{L^2(\partial \mathcal{D}_t)} \leq C_r \left( ||Dp||_{L^\infty(\partial \mathcal{D}_t)} + ||Dv||_{L^\infty(\mathcal{D}_t)} \right)^2 \mathcal{E}_r^*, \quad (5.37) \]
with \(\mathcal{E}_r^* = \sum_{k \leq r} \mathcal{E}_k\), and:
\[ ||\Pi D^r D_t p||_{L^2(\partial \mathcal{D}_t)} + ||\Pi D^r D_t p||_{L^2(\partial \mathcal{D}_t)} \leq C_r \left( ||Dp||_{L^\infty(\mathcal{D}_t)} + ||Dv||_{L^\infty(\mathcal{D}_t)} \right)^2 \mathcal{E}_r(t) \]
\[ + P \left( \mathcal{E}_r^*, ||Dp||_{L^\infty(\mathcal{D}_t)}, ||Dv||_{L^\infty(\mathcal{D}_t)}, ||D^2 p||_{L^\infty(\partial \mathcal{D}_t)} \right). \quad (5.38) \]
Furthermore, if \(-\nabla_n p \geq \delta_0 > 0\), then:
\[ ||\Pi D^{r-2}\theta||_{L^2(\partial \mathcal{D}_t)} \leq ||(D_n p)^{-1}||_{L^\infty(\partial \mathcal{D}_t)} \left( \mathcal{E}_r + P \left( \mathcal{E}_r^*, ||Dv||_{L^\infty(\mathcal{D}_t)}, ||Dp||_{L^\infty(\mathcal{D}_t)}, ||D^2 p||_{L^\infty(\partial \mathcal{D}_t)} \right) \right) \]
\[ (5.39) \]
where \(P\) is a homogeneous polynomial with positive coefficients.

**Proof.** The estimate (5.35) follows from (4.25) and (5.36) follows from the definition of the boundary term in the energy. To prove (5.37), we apply (4.33), (5.32) and (5.36), which gives (5.37) with an extra term \(||Dp||_{L^2(\mathcal{D}_t)}\) on the right-hand side. To control this, we integrate by parts twice and use (5.27):
\[ \int_{\mathcal{D}_t} |Dp|^2 = - \int_{\mathcal{D}_t} p\Delta p = \int_{\mathcal{D}_t} D_t p(D_i v^j D_j v^i) = \int_{\mathcal{D}_t} D_t p(v^j D_j v^i). \quad (5.40) \]
Bounding the right hand side by \(||Dv||_{L^\infty(\mathcal{D}_t)}||Dp||_{L^2(\mathcal{D}_t)}||v||_{L^2(\mathcal{D}_t)}\) and dividing both sides by \(||Dp||_{L^2(\mathcal{D}_t)}\) gives the result.

Similarly, applying (5.34), (4.33) and (4.42) gives (5.38) with an extra term \(||D^2 D_t p||_{L^2(\mathcal{D}_t)}\) on the right-hand side. This can be handled by using the fact that \(D_t p = 0\) on \(\partial \mathcal{D}_t\), the equation (5.30) and integrating by parts twice:
\[ \int_{\mathcal{D}_t} |D D_t p|^2 = - \int_{\mathcal{D}_t} D_t p(D_i X^i) = \int_{\mathcal{D}_t} (D_t D_t p) X^i, \quad (5.41) \]
where \(X_i = v^k D_k D_i p - D_i v^j D_k v^j - D_i (v^j D_j v^i) + v_i D_k (v^j D_j v^k)\). The result now follows after using (5.37) and (5.35) to control \(||X||_{L^2(\mathcal{D}_t)}\).

The estimate (5.39) follows from (4.43) and the estimates we have just proved. \(\square\)
5.2. Proof of Theorem 5.1. We start by applying Proposition 5.11 from [9] with \( \alpha = -D^r p, \beta = D^{r-1} v \) and \( \nu = |Dp|^{-1} \), which gives:

\[
\frac{d}{dt} \mathcal{E}_r \leq C \sqrt{\mathcal{E}_r} \left( |\Pi(D_t D^r p + (D_k p) D^r u^k)|_{L^2(\partial D_1)} + |D_t D^r u + D^{r+1} p|_{L^2(D_1)} \right) + CK \mathcal{E}_r + C \left( \| \text{curl} D^r v \|_{L^2(D_1)} + \| \Delta D^{r-1} p \|_{L^2(D_1)} \right) + K \left( \| D^{r-1} v \|_{L^2(D_1)} + \| D^r p \|_{L^2(D_1)} \right)^2. \tag{5.42}
\]

By Lemma 5.3, every term except the first one above is bounded by the right-hand side of (5.13). By (5.33) and (5.38), it suffices to prove the following bound:

\[
\sum_{s=1}^{r-2} \left| \Pi((D^{1+s} u)(D^{r-s} p)) \right|_{L^2(\partial D_1)} \leq C(K + L + M) \left( \mathcal{E}_r + (K + L + M)P(\mathcal{E}_0, ..., \mathcal{E}_{r-1}, K, L, M) \right), \tag{5.45}
\]

for a polynomial \( P \). We write \( (\Pi^{r-s} D^s p)_I = \gamma_I^r D^r p \) and \( (\Pi^{s+1} D^s v)_I = \gamma_I^s D^s v \). Then:

\[
\left| \Pi((D^{s+1} v)(D^{r-s} p)) \right|_{L^2(\partial D_1)} \leq \left| \Pi^{s+1} D^s v \right| \left| \Pi^{r-s} D^s p \right|_{L^2(\partial D_1)} + \left| \Pi^s N^k D^s v_k \right| \left| \Pi^{r-s} N^k D^{r-s-1} D p \right|_{L^2(\partial D_1)}. \tag{5.46}
\]

We now apply the interpolation inequality (A.6) which shows that see that each of these terms is bounded by the right-hand side of (5.13).

5.3. Proof of Lemma 5.1. To control \( \| \theta \|_{L^\infty(\partial D_1)} + \frac{1}{10} \) we start by noting that \( \frac{1}{10} \leq C \| \theta \|_{L^\infty(\partial D_1)} \) and that by the elementary formula \( \theta_j = (1 + |\nabla h|^2)^{-1/2} \nabla \nabla_j h \), we have \( \| \theta \|_{L^\infty(\partial D_1)} \leq C (\nu h)_{C^2(\mathbb{R}^2)} \). We note that \( \Delta |D^3 \psi|^2 = |D^3 \psi|^2 \geq 0 \), so writing \( v = D^3 \psi + v_\omega \), applying the maximum principle to control \( \| D^2 \psi \|_{L^\infty(D_1)} \leq \| D^2 \psi \|_{L^\infty(\partial D_1)} \) and the estimate (4.50), we have:

\[
\| D^2 v \|_{L^\infty(D_1)} \leq \| D^2 \psi \|_{L^\infty(D_1)} + \| D^2 v_\omega \|_{L^\infty(D_1)} \leq \| D^2 \psi \|_{L^\infty(\partial D_1)} + \| v_\omega \|_{H^2_{\partial} D_1}. \tag{5.49}
\]

To control \( D^2 \psi \) on \( \partial D_1 \), we can either use (3.5) and (3.6) or just use the pointwise inequality (4.25) on \( \partial D_1 \) which shows that \( |D^2 \psi| \leq |\Delta \psi| + |\Pi D^2 \psi| \). By the projection formula (4.37) we have \( |\Pi D^2 \psi| \leq |\Delta^2 \psi| + |\theta(\sum D^r \psi) \| \leq |\Delta^2 \varphi| + |\theta(\sum D^r \varphi) \| \) where \( \Delta D \) denotes the covariant derivative on \( \partial D_1 \). By the estimate for the Dirichlet-to-Neumann map (C.3), this proves the bound for \( \| D^3 \psi \|_{L^\infty(\partial D_1)} \).

The estimates for \( \| D^2 p \|_{L^\infty(\partial D_1)} \) follow from the pointwise estimate (4.25), the fact that \( \Delta p = -(Dv) \cdot (Dv) \) and the bounds we just proved. To bound \( \| DD_p p \|_{L^\infty(\partial D_1)} \), we apply Sobolev embedding (A.4) on \( \partial D_1 \) and the elliptic estimate (4.31) it suffices to bound:

\[
\| \Pi D^3 D_p p \|_{L^2(\partial D_1)} + \| \Pi D^2 D_p p \|_{L^2(\partial D_1)} + \sum_{s \leq 2} \| D^s \Delta D_p p \|_{L^2(D_1)} + \| DD_p p \|_{L^2(\partial D_1)}. \tag{5.50}
\]

Using the identity (5.41) gives:

\[
\| DD_p p \|_{L^2(D_1)} \leq C \left( \| D^2 p \|_{L^\infty(D_1)} \| v \|_{L^2(D_1)} + \| Dv \|_{L^\infty(D_1)} \| Dp \|_{L^2(D_1)} + \| Dv \|_{L^\infty(D_1)} \| Dv \|_{L^2(D_1)} \right). \tag{5.51}
\]
and using the estimates we have just proved and Lemma 5.2 gives that \(|DD_t p|_{L^2(D_t)}\) is bounded by the right-hand side of (5.16). To control \(|ID^3 D_t p|_{L^2(D_t)} + |ID^2 D_t p|_{L^2(D_t)}\), we use the formulas (4.37), (4.38) and the estimates we have just proved.

To get a lower bound for \(\nabla N P\) on \(\partial D_t\), we start by noting that since \(p = 0\) on \(\partial D_t\) and \((D_t N_i) N_i = 0\), so that \(D_t D_t N_t = D_N D_t \rho\) on \(\partial D_t\). Since \(p = 0\) on \(\partial D_t\) and \((D_t N^i) N_i = 0\) it follows that \(D_t \partial N_t = (D_t N^i) \partial_i \rho + D_t D_t \rho = D_N D_t \rho\). Applying Sobolev embedding on \(\partial D_t\), the estimate (5.38), and the bootstrap assumptions (2.10)-(2.12), we have:

\[
|\nabla N P(t)| \geq |\nabla N P(0)| - \int_0^t |\nabla N D_t \rho(s)| \, ds \geq |\nabla N P(0)| - C \int_0^t \frac{\epsilon_0^3}{1 + s} (1 + s)^{\delta + \epsilon_0^3} \, ds. \tag{5.52}
\]

The second term is bounded by \(\frac{1}{2} |\nabla N P(0)|\) so long as \(t \leq C(|\nabla N P(0)|^{-1}) \epsilon_0^{-1/3}\) and \(\epsilon_0\) is taken sufficiently small.

4.4 Proof of Proposition 5.2. We now show how the energies in the previous section control Sobolev norms of \(\varphi, h\). Recall that \(u = h + iA^{1/2} \omega\), where \(\omega = \varphi + a\omega\) and \(\nabla a\omega = V_\omega + \nabla h B_\omega\).

We begin by noting that by the definition of \(\omega = \varphi + a\omega\), the fact that \(\nabla a\omega = V_\omega + \nabla h B_\omega\), and the fact that \(B_\omega = -\nabla h \cdot V_\omega\) (since \(v_\omega \cdot N = 0\)) it suffices to prove the following estimate:

\[
||h||_{H^2(\mathbb{R}^2)} + ||A^{1/2} \varphi||_{L^2(\mathbb{R}^2)} + ||\nabla \varphi||_{H^{-1}(\mathbb{R}^2)} + ||A^{1/2} a\omega||_{L^2(\mathbb{R}^2)} + ||V_\omega||_{L^2(\mathbb{R}^2)} \lesssim \mathcal{E} + AP(\mathcal{E}^{-1}). \tag{5.53}
\]

We start with bounds for \(h\). By the elementary formula:

\[
\theta_{i j} = \frac{1}{\sqrt{1 + |\nabla h|^2}} \nabla_i \nabla_j h
\]

we have \(\nabla^i h \sim \nabla h \cdot \theta + O(\nabla^2 \theta, ..., \nabla h)\). We can therefore bound \(||h||_{H^N(\mathbb{R}^2)}\) by the right-hand side of (5.18) provided we also control \(||\nabla h||_{L^2(\mathbb{R}^2)} + ||h||_{L^2(\mathbb{R}^2)}\). Note that \(||h||_{L^2(\mathbb{R}^2)} \leq E_0\) where \(E_0\) is the conserved energy (defined in (5.11), and a bound for \(||\nabla h||_{L^2(\mathbb{R}^2)}\) follows from this and the bound for \(||\nabla^2 h||_{L^2(\mathbb{R}^2)}\).

Next we bound \(\varphi\). First, we have:

\[
- \int_{\partial D_t} \varphi N \varphi = - \int_{\partial D_t} |\rho D_t \varphi| = - \int_{\partial D_t} \psi D_t \varphi = - \int_{D_t} \psi D_t \varphi - \int_{D_t} |\psi|^2 \delta + \int_{D_t} |\nabla \psi|^2 \leq ||\psi||_{L^2(D_t)}^2 + ||\nabla \psi||_{L^2(D_t)}^2. \tag{5.55}
\]

The left-hand side is:

\[
||A^{1/2} \varphi||_{L^2(\mathbb{R}^2)} \sim ||A^{1/2} \varphi||_{L^2(\mathbb{R}^2)}^2. \tag{5.56}
\]

which follows from the remarks after Proposition 2.2 in [8].

To control \(A^{1/2} a\omega\), we note that by the fractional integration estimate (3.11), \(||A^{1/2} a\omega||_{L^2(\mathbb{R}^2)} = ||A^{-1/2} A a\omega||_{L^2(\mathbb{R}^2)} \lesssim ||A a\omega||_{L^{4/3}(\mathbb{R}^2)}\) and by the fact that the Riesz transform is bounded on \(L^{4/3}\) it follows that \(||A^{1/2} a\omega||_{L^2(\mathbb{R}^2)} \lesssim ||\nabla a\omega||_{L^{4/3}(\mathbb{R}^2)}\). Since \(\nabla a\omega = V_\omega + \nabla h B_\omega\), we have

\[
||A^{1/2} a\omega||_{L^2(\mathbb{R}^2)} \lesssim ||(1 + |x|^2)^{1/2} V_\omega||_{L^2(\mathbb{R}^2)} + ||\nabla h||_{L^\infty(\mathbb{R}^2)} ||(1 + |x|^2)^{1/2} B_\omega||_{L^2(\mathbb{R}^2)} \tag{5.57}
\]

and by (4.56), this is bounded by the right-hand side of (5.18).

To control the higher norms of \(\varphi\) and \(V_\omega\), we use the following:

Lemma 5.4. Under the hypotheses of Proposition 5.2 we have:

\[
||\nabla \varphi||_{L^2(\mathbb{R}^2)}^2 + ||\nabla^{-1} V_\omega||_{L^2(\mathbb{R}^2)}^2 \lesssim \mathcal{E} + AP(\mathcal{E}^{-1}), \tag{5.58}
\]

where \(\mathcal{E}^{-1} = \sum_{s \leq -1} \mathcal{E}^r\) and \(A\) is defined by (5.14).
Proof. The estimates for $V_\omega$ follow from (4.56). To bound $\nabla^r \varphi$, we start with the fact that:

$$||D\psi||_{L^2(\partial D_i)} \lesssim ||v||_{L^2(D_i)} + ||v_\omega||_{L^2(D_i)},$$

(5.59)

By the chain rule, we have:

$$||\nabla \varphi||_{L^2(\mathbb{R}^2)} \leq ||D\psi||_{L^2(\partial D_i)} + ||\nabla h D_y \psi||_{L^2(\partial D_i)}.$$

(5.60)

Bounds for the second term will follow in a similar way to the bounds for the first term so we just show how to bound the first term. By the inequality (4.27):

$$||D\psi||_{L^2(\partial D_i)} \leq ||D_N \psi||_{L^2(D_i)} + ||\Delta \psi||_{L^2(D_i)} + K||D\psi||_{L^2(D_i)}$$

(5.61)

and so the trace inequality (4.26) and the estimate (5.59) imply:

$$||D\varphi||_{L^2(\mathbb{R}^2)} \lesssim ||Dv||_{L^2(D_i)} + ||v||_{L^2(D_i)} + ||v_\omega||_{L^2(D_i)},$$

(5.63)

where the implicit constant depends only on $K$. The first two terms are bounded by $\mathcal{E}^1 + \mathcal{E}^0$ and the last term can be bounded by $||\omega||_{H^2_\omega(D_i)}$ by (4.3). To explain the strategy for higher-order derivatives we first consider what happens when $r = 2$. Using (4.27) again:

$$||D^2\psi||_{L^2(\partial D_i)} \lesssim ||D_N D\psi||_{L^2(\partial D_i)} + K||D\psi||_{L^2(D_i)}.$$

(5.64)

By the estimate (4.27):

$$||D_N D\psi||_{L^2(\partial D_i)} \lesssim ||\Pi D_N D\psi||_{L^2(\partial D_i)} + ||\text{div} D_N D\psi||_{L^2(D_i)} + ||\text{curl} D_N D\psi||_{L^2(D_i)} + K||D_N \psi||_{L^2(D_i)}.$$

(5.65)

Note that:

$$\Pi_j D_N D_i \psi = \Pi_j D_i D_N \psi - (\Pi_j D_i N_k) D_k \psi.$$

(5.66)

The first term is $\overline{D}_i (v \cdot N)$ and the second term is $-\theta_k D_k \psi$. Also both $\text{div} D_N D\psi$ and $\text{curl} D_N \psi$ are lower order. The first is because to highest order it is $D_N \Delta \psi = 0$ and the second because $\text{curl} D \psi = 0$.

Therefore we have:

$$||D^2\psi||_{L^2(\partial D_i)} \lesssim ||\overline{D}_i (v \cdot N)||_{L^2(\partial D_i)} + K||D\psi||_{L^2(\partial D_i)} + ||D\psi||_{L^2(D_i)}.$$

(5.67)

Using the trace inequality to bound the first term and the above argument to bound the lower-order norms of $\psi$ gives that:

$$||D^2\psi||_{L^2(\partial D_i)} \lesssim ||\overline{D}_i (v \cdot N)||_{L^2(\partial D_i)} + ||v||_{L^2(D_i)} + ||v_\omega||_{L^2(D_i)},$$

(5.68)

where the implicit constant depends only on $K$.

We now prove a higher-order version of this. Repeatedly applying the chain rule (5.5), to highest order we have:

$$\nabla^r \varphi \sim \nabla^r \psi + \nabla^r h (D_y \psi) + ...$$

(5.69)

where the missing terms are all bounded pointwise by $\sum_{k \leq r-1} |D_k^r \psi|$ times a polynomial in $\sum_{k \leq r-1} |\nabla^k h|$. We now want to replace $\nabla^r \psi$ with $\nabla^{r-1} \nabla N \psi \sim \overline{D}^{r-1} (v \cdot N)$ and lower order terms. By the inequality (4.27):

$$||\nabla^r \psi||_{L^2(\partial D_i)} \lesssim ||\nabla^r \nabla^{-1} \psi||_{L^2(\partial D_i)} + ||\nabla^{r-1} \psi||_{L^2(D_i)},$$

(5.70)

with implicit constant depending on $K$. Next, with $\beta = \nabla^{r-1} \psi$, we apply the estimate (4.27) and have:

$$||\nabla \beta ||_{L^2(\partial D_i)}$$

$$\lesssim ||\Pi \nabla^r \beta ||_{L^2(\partial D_i)} + ||\text{div} \nabla^r \beta ||_{L^2(D_i)} + ||\text{curl} \nabla^r \beta ||_{L^2(D_i)} + ||\nabla^r \beta ||_{L^2(D_i)}.$$

(5.71)
The interior terms are all lower order by the same observation as above, and so we just need to deal with the boundary term. We note that:

$$\Pi_j^I \nabla_n \nabla_{r-1}^I \psi = \Pi_j^I \nabla_{r-1}^I \nabla_n \psi - \sum_{K,L} \Pi_j^I (\nabla_{K \cdot n}) (\nabla_{L \cdot n} \psi)$$  \hspace{1cm} (5.72)

where the sum is over all multi-indices $K, L$ with $K + L = I$ and $|K| \leq |I| - 1$.

Since $\nabla_n \psi = n \cdot v$ on $\partial D_t$, using (4.2) to replace $\Pi_j^I \nabla_{r-1}^I \nabla_n \psi$ with $D^{r-1} \nabla_n \psi$ and applying Lemma 5.2 to control $D^{r-1} (n \cdot v)$ by the energy shows that the first term in (5.72) is controlled by the energy. The worst term appearing in the sum in (5.72) from the point of view of the regularity of $\theta$ is the case $K = I$. This involves $r - 1$ projected derivatives of $n$ and by Proposition 4.11 of [9] and the definition $\theta = \Pi \nabla N$, this can be bounded by $||D^{-r/2} \theta||_{L^2(\partial D_t)}$ to highest order. We can now use induction and interpolation (A.6) to deal with the lower-order terms.

Having now bounded $\varphi$, let us see how to control $V_\omega$ and $B_\omega$. First, since $v_\omega \cdot n = 0$ on $\partial D_t$, we have $B_\omega = V_\omega \cdot \nabla h$ and so it is enough to bound $V_\omega$. Since $V_\omega = v_\omega |_{\partial D_t} = (v - \nabla \psi) |_{\partial D_t}$, estimates for $V_\omega$ follow from the above estimates for $\psi$ and the estimates in Lemma 5.3.

\[ 5.5. \text{ Proof of Proposition 5.3.} \] A short calculation using the fact that $[D_t, D] = -D^k D_k$, $D_t (1 + |z|^2)^2 = 4|z|^2 z \cdot v$ and the equation for the vorticity (5.26) shows that:

$$D_t D^m ((1 + |z|^2)^2 \omega) = (1 + |z|^2)^2 (D^{m+1} v \cdot \omega + D v \cdot D^m \omega) + R,$$  \hspace{1cm} (5.73)

where $R$ is a sum of terms which can be bounded pointwise by $(1 + |z|^2)^2 \sum_{a=1}^m |D^a v(z)||D^{m-a} \omega(z)|$.

We next write $v = D \psi + v_\omega$ and the result as:

$$D_t ((1 + |z|^2)^2 D^m \omega) = (1 + |z|^2)^2 (D^{m+2} \psi \cdot \omega + D^2 \psi \cdot D^m \omega + D^{m+1} v_\omega \cdot \omega + D v_\omega \cdot D^m \omega) + R.$$  \hspace{1cm} (5.74)

Taking $m \leq N_0$ By the Reynolds transport theorem, the above calculation and Sobolev embedding, we have:

$$\frac{d}{dt} ||D^m \omega(t)||^2_{L^2_w} \leq \int_{D_t} ((1 + |z|^2)^2 (||D^{m+2} \psi|| \omega + |D^2 \psi||D^m \omega| + |D^{m+1} v_\omega|| \omega | + |D v_\omega||D^m \omega| + R) |D^m \omega| dz$$  \hspace{1cm} (5.75)

$$\lesssim \left( ||D^2 \psi||_{W^{m, \infty}(D_t)} + ||D^{m+1} v_\omega||_{L^2(D_t)} + ||v_\omega||_{L^\infty(D_t)} \right) ||\omega||^{2}_{W^{1,1}_w(D_t)}.$$  \hspace{1cm} (5.76)

To control the first term, we use the maximum principle as in the proof of Lemma 5.1 which gives that $||D^2 \psi||_{W^{m, \infty}(D_t)} \leq ||D^2 \psi||_{W^{m, \infty}(\partial D_t)}$. Using (3.5) and (3.6) repeatedly shows that $||D^2 \psi||_{W^{m, \infty}(\partial D_t)} \lesssim ||D^2 \varphi||_{W^{m+1, \infty}(R^3)} < \infty$, up to lower order terms.

To control the other two terms from (5.77), we use (1.56):

$$||v_\omega||_{L^\infty(D_t)} + ||D^{m+1} v_\omega||_{L^2(D_t)} \lesssim ||\omega||_{W^{1,1}_w(D_t)},$$  \hspace{1cm} (5.78)

which proves (5.19).

We also note the following, which is used in the proof of Corollary 1.1.

**Lemma 5.5.** If $\omega_0|_{\partial D_0} = 0$ and $\int_0^T ||\partial v||_{L^{\infty}(\partial D_t)} < \infty$, for some $T > 0$, then $\omega|_{\partial D_t} = 0$ for $t \leq T$. 

Proof. Changing to Lagrangian coordinates and letting \(\mu_\gamma\) denote the volume element on \(\partial \Omega\) with respect to the metric \(\gamma\), we have:
\[
\frac{d}{dt} \int_{\partial \Omega} |\omega(t)|^2 d\mu = \frac{d}{dt} \int_{\Omega} |\omega(t)|^2 d\mu_\gamma = \int_{\Omega} D_t \omega(t) \cdot \omega(t) + |\omega(t)|^2 D_t d\mu_\gamma.
\]  
(5.79)
By Lemma 3.9 in [9], we have
\[
\frac{d}{dt} ||\omega(t)||^2_{L^2(\partial \Omega)} \leq C||\partial \nu||_{L^\infty(\partial \Omega)} ||\omega(t)||^2_{L^2(\partial \Omega)}.
\]  
(5.80)
Multiplying both sides by the integrating factor \(e^{-C \int_0^t ||\partial \nu(s)||_{L^\infty(\partial \Omega)}} ds\) and integrating gives that:
\[
||\omega(t)||^2_{L^2(\partial \Omega)} \leq C \exp \left( \int_0^t ||\partial \nu(s)||_{L^\infty(\partial \Omega)} ds \right) ||\omega(0)||^2_{L^2(\partial \Omega)},
\]  
(5.81)
from which the result follows.

6. Dispersive estimates for terms involving the vorticity

We now prove the estimates for the terms \(g_2, ..., g_5\) from (3.72). We recall that \(R_j\) denotes the Riesz transform and \(\Lambda^s\) denotes fractional differentiation on \(\mathbb{R}^2\). We will also ignore the difference between \(\text{Reu}, \text{Imu}\) and just write \(u\). Then the terms we want to estimate are:
\[
g_2(t) = \int_0^t e^{is\Lambda^{1/2}} R \cdot V\omega(s) \, ds,
\]  
(6.1)
\[
g_3(t) = \int_0^t e^{is\Lambda^{1/2}} \Lambda^{1/2}((R \cdot V\omega)(\Lambda^{1/2} u)) - \nabla \cdot (u V\omega) + \Lambda(u R \cdot V\omega),
\]  
(6.2)
\[
g_4(t) = \int_0^t e^{is\Lambda^{1/2}} (R \cdot V\omega)^2 \, ds.
\]  
(6.3)

In the next three sections, we prove:

Proposition 6.1. If (2.10) holds with \(\varepsilon_0 \ll 1\), then:
\[
||\nabla^k e^{-it\Lambda^{1/2}} g(t) ||_{L^\infty(\mathbb{R}^2)} \lesssim \int_0^t \left( 1 + ||\omega(s)||_{H_{w_1}^{N_1}(D_a)} + ||u(s)||_{W^{k+3}}(\mathbb{R}^2) \right) ||\omega(s)||_{H_{w_1}^{N_1}(D_a)} \, ds,
\]  
(6.4)
for \(k \leq N_1 + 4\), and
\[
||\Lambda^I xg(t) ||_{L^2(\mathbb{R}^2)} \lesssim \int_0^t (1 + s) \left( 1 + ||\omega(s)||_{H_{w_1}^{N_1}(D_a)} + ||u(s)||_{W^{4}}(\mathbb{R}^2) + ||u(s)||_{H_0^0} ||\omega(s)||_{H_{w_1}^{N_1}(D_a)} \right) \, ds,
\]  
(6.5)
for \(I = 2, 3, 4\).

Assuming this holds for the moment, we show how it implies the estimates for \(f_2\) in (2.14)-(2.15). If the assumptions (2.10)-(2.12) hold, then using Lemma A.3 (6.4) implies:
\[
||\nabla^k e^{-it\Lambda^{1/2}} g(t) ||_{L^\infty(\mathbb{R}^2)} \lesssim (1 + t)^{1+\delta} \varepsilon_1 + (1 + t)^{-1+\sigma} \varepsilon_0^2 \varepsilon_1 + (1 + t)^{2\delta} \varepsilon_1^2
\]  
(6.6)
where \(\sigma \leq \frac{1}{4}\). Since \(\varepsilon_1 \ll \varepsilon_0\), this implies the second inequality in (2.14).

Similarly, we have:
\[
||\Lambda^I xg(t) ||_{L^2(\mathbb{R}^2)} \lesssim (1 + t)^{2+\delta} \varepsilon_1 + (1 + t)^{2+2\delta} \varepsilon_0^2 \varepsilon_1 + (1 + t)^{2+2\delta} \varepsilon_1^2
\]  
(6.7)
which implies the second inequality in (2.15).
6.1. Estimates for $g_2$.

**Lemma 6.1.** If $v$ satisfies (2.10), (2.11) with $\varepsilon_0 \ll 1$, then:

$$\|\nabla^k e^{-it\Lambda^{1/2}} g_2(t)\|_{L^\infty(\mathbb{R}^2)} \lesssim \int_0^t \|\omega(s)\|_{H_{w_0}^{N_1}(D_\alpha)} ds, \quad k \leq N_1 - 2$$  \hspace{1cm} (6.8)

$$\|\Lambda^t x g_2(t)\|_{L^2(\mathbb{R}^2)} \lesssim \int_0^t (1 + s)\|\omega(s)\|_{H_{w_0}^{N_1}(D_\alpha)} ds.$$  \hspace{1cm} (6.9)

**Proof.** By Sobolev embedding, we have:

$$\|\nabla^k e^{it\Lambda^{1/2}} \Lambda^t R \cdot V_\omega(s)\|_{L^\infty(\mathbb{R}^2)} \lesssim \|R \cdot V_\omega(s)\|_{H^{3/2+k}(\mathbb{R}^2)} \lesssim \|V_\omega(s)\|_{H^{2+k}(\mathbb{R}^2)}.$$  \hspace{1cm} (6.10)

By (1.3), $\|V_\omega(s)\|_{H^{2+k}(\mathbb{R}^2)} \lesssim \|\omega(t)\|_{H_{w_0}^{N_1}(D_\alpha)^*}$, which implies (6.8). Note that this estimate loses more than half a derivative, but we are avoiding the use of fractional derivatives in $D_t$.

By Plancherel’s theorem, $\|\Lambda^t e^{it\Lambda^{1/2}} g_2\|_{L^2} = \|\xi|^t \partial_\xi (e^{it|\xi|^{1/2}} \hat{g}_2)\|_{L^2}$. We have:

$$\partial_\xi e^{it|\xi|^{1/2}} \hat{g}_2 = \partial_\xi \left( \int_0^t e^{is|\xi|^{1/2}} \frac{\xi}{|\xi|} \cdot \hat{V}_\omega(s, \xi) \, ds \right)$$  \hspace{1cm} (6.11)

$$= \int_0^t s \frac{\xi}{|\xi|^{3/2}} e^{is|\xi|^{1/2}} \hat{V}_\omega(s, \xi) \, ds + \int_0^t e^{is|\xi|^{1/2}} \partial_\xi \left( \frac{\xi}{|\xi|} \right) \hat{V}_\omega(s, \xi) \, ds + \int_0^t e^{is|\xi|^{1/2}} \partial_\xi \hat{V}_\omega(s, \xi) \, ds$$  \hspace{1cm} (6.12)

$$\equiv \int_0^t s \hat{g}_2^1(s, \xi) \, ds + \int_0^t \hat{g}_2^2(s, \xi) \, ds + \int_0^t \hat{g}_2^3(s, \xi) \, ds.$$  \hspace{1cm} (6.13)

By the fractional integration lemma (C.1), we have:

$$\|\Lambda^t g_2^1\|_{L^2} \lesssim \|\Lambda^{-1/2+1} V_\omega\|_{L^2} \lesssim \|V_\omega\|_{L^p_1},$$  \hspace{1cm} (6.14)

where $p_1 = 2(2 - \iota)/3$.

Similarly, bounding $\|\xi|^t \partial_\xi(|\xi|^{-1} \xi)\| \lesssim |\xi|^{-1+1}$ and taking $p_2 = 2/(2 - \iota) > 1$, we have that:

$$\|\Lambda^t g_2^2\|_{L^2} \lesssim \int_0^t \|\Lambda^{-1+1} V_\omega(s)\|_{L^2} \, ds \lesssim \int_0^t \|V_\omega(s)\|_{L^{p_2}} ds \lesssim \int_0^t \|\omega(s)\|_{H_{w_0}^{N_1}(D_\alpha)} ds,$$  \hspace{1cm} (6.15)

by (1.57).

To control $\Lambda^t g_2^3$, we write $\Lambda^t = \Lambda^{-1} \Lambda = -\Lambda^{-1} R \cdot \nabla$. Using fractional integration again, we have:

$$\|\Lambda^t (x V_\omega)\|_{L^2} \lesssim \|\Lambda^{-1+1} \nabla (x V_\omega)\|_{L^2} \lesssim \|\nabla (x V_\omega)\|_{L^{p_2}},$$  \hspace{1cm} (6.16)

Combining the above estimates and using Proposition (1.3) gives (6.9). \hspace{1cm} $\square$

6.2. **Estimates for $g_3$.** We now bound the term involving both $u$ and $v_\omega$. This is:

$$g_3(t) = \int_0^t e^{is\Lambda^{1/2}} N_1(u, w) \, ds = \sum_{n=1,2,3} \int_0^t e^{is\Lambda^{1/2}} g_3^n(s) \, ds,$$  \hspace{1cm} (6.17)

with:

$$g_3^1 = \Lambda^{1/2}((R \cdot V_\omega)(\Lambda^{1/2} u)),$$  \hspace{1cm} (6.18)

$$g_3^2 = -\nabla \cdot (u V_\omega),$$  \hspace{1cm} (6.19)

$$g_3^3 = \Lambda (u R \cdot V_\omega).$$  \hspace{1cm} (6.20)
Lemma 6.2. If (2.10) - (2.12) hold with $\varepsilon_0 \ll 1$, then:

$$
\|\nabla^k e^{-it\Lambda^{1/2}} g^j(t)\|_{L^\infty(\mathbb{R}^2)} \lesssim \int_0^t \|\omega(s)\|_{H^k_w(\mathbb{R}^2)} \|u(s)\|_{W^{k+3,\infty}(\mathbb{R}^2)} \, ds
$$

(6.21)

$$
\|\Lambda^i(xg_3(t))\|_{L^2} \lesssim \int_0^t (1 + s) \|\omega(s)\|_{H^k_w(\mathbb{R}^2)} (\|u(s)\|_{W^4,\infty} + \|u(s)\|_{H^0}) \, ds
$$

(6.22)

Proof. The estimates for each of the terms $g^j_1, I = 1, 2, 3$ are similar, so we just show how to bound $g^j_3$. Applying Sobolev embedding and using the fact that the Riesz transform maps $L^2 \to L^2$, we have:

$$
\|\nabla^k \Lambda^{1/2} (R \cdot V_\omega)(\Lambda^{1/2} u)\|_{L^\infty} \lesssim \|(R \cdot V_\omega)(\Lambda^{1/2} u)\|_{H^{k+2}} \lesssim \|V_\omega\|_{H^{k+2}} \|u\|_{W^{k+3,\infty}},
$$

(6.23)

say. By (6.3) this is bounded by the right-hand side of (6.21).

To prove the bound for $xg_3$, we write:

$$
\partial_\xi g^j_3 = \int_{\mathbb{R}^2} \partial_\xi \left( e^{is|\xi|^{1/2}} |\xi|^{1/2} (\xi - \eta) |\eta|\right)^2 \hat{\omega}(\xi - \eta) \, d\eta
$$

(6.24)

$$
= \int_{\mathbb{R}^2} e^{is|\xi|^{1/2}} |\eta|^{1/2} \left( i s - \frac{(\xi - \eta) \xi}{|\xi|} + \frac{|\xi|}{|\xi|} m_0(\xi - \eta) + m_1(\xi) \frac{(|\xi| - \eta)}{|\xi|} \right) \hat{\omega}(\xi - \eta) \, d\eta
$$

(6.25)

$$
+ \int_{\mathbb{R}^2} e^{is|\xi|^{1/2}} |\xi|^{1/2} (\xi - \eta) |\eta|^{1/2} \partial_\xi \hat{\omega}(\xi - \eta) \, d\eta,
$$

(6.26)

with $m_0(\xi, \eta) = \partial_\xi \left( \frac{\xi - \eta}{|R - \eta|} \right)$ and $m_1(\xi) = \partial_\xi |\xi|^{1/2}$.

In physical space, after applying $\Lambda^i$ the first term in (6.25) is:

$$
e^{is\Lambda^{1/2}} \Lambda^i \left( (R \cdot V_\omega)(\Lambda^{1/2} u) \right),
$$

(6.27)

and using the fractional product rule (C.2), this is bounded by the right-hand side of (6.22).

The second term in (6.25) contributes:

$$
e^{is\Lambda^{1/2}} \Lambda^{1/2+i} \left( (m_0(\nabla) V_\omega)(\Lambda^{1/2} u) \right).
$$

(6.28)

Since $|m_0(\xi - \eta)| \lesssim |\xi - \eta|$, we bound the result in $L^2$ by:

$$
\|\Lambda^{-1} V_\omega\|_{L^{p_1}} \|\Lambda^{i+1} e^{is\Lambda^{1/2}} u\|_{L^{p_2}} + \|\Lambda^{-1/2+i} V_\omega\|_{L^4} \|\Lambda^{1/2} e^{is\Lambda^{1/2}} u\|_{L^{4}}
$$

(6.29)

where $1/p_1 + 1/p_2 = 1/2$. We then take $p_1$ so large that $\|\Lambda^{-1} V_\omega\|_{L^{p_1}} \lesssim \|V_\omega\|_{L^{4/3}} (see (C.1)), say, and since $p_2 > 2$ we bound $\|\Lambda^{i+1} u\|_{L^{p_2}} \lesssim \|u\|_{H^0} + \|u\|_{W^4,\infty}$. Again using (C.1), the second term can be bounded by $\|V_\omega\|_{L^2} (\|u\|_{W^4,\infty} + \|u\|_{H^0})$. Using (4.3) to control the factors of $V_\omega$, the result is bounded by the right-hand side of (6.22).

The third term in (6.25) is:

$$
e^{is\Lambda^{1/2}} m_1(\nabla) \left( (R \cdot V_\omega)(\Lambda^{1/2} u) \right),
$$

(6.30)

and recall $|m_1(\xi)| \lesssim |\xi|^{-1/2}$. With $1 < p < 2$ so that $\|\Lambda^{-1/2+i} F\|_{L^2} \lesssim \|F\|_{L^p}$, we therefore have:

$$
\|\Lambda^{-1/2+i} \left( (R \cdot V_\omega)(\Lambda^{1/2} u) \right)\|_{L^2} \lesssim \|(R \cdot V_\omega)(\Lambda^{1/2} u)\|_{L^p} \lesssim \|V_\omega\|_{L^p} \|u\|_{W^4,\infty}.
$$

(6.31)

The estimate for (6.26) can be performed similarly to how we controlled $g^j_2$ in the previous section. □
6.3. Estimates for \( g_4 \). We now bound the term which is quadratic in the vorticity:

\[
g_4(t) = \int_0^t e^{is\Lambda^{1/2}} N(w, w) \, ds = \int_0^t e^{is\Lambda^{1/2}} \Lambda^{1/2+i} (R \cdot V_\omega(s))^2 \, ds \tag{6.32}
\]

We prove:

**Lemma 6.3.** If \( \nu \) satisfies (2.10)-(2.12) with \( \varepsilon_0 \ll 1 \), then:

\[
\| \nabla^k e^{-it\Lambda^{1/2}} g_4(t) \|_{L^\infty(\mathbb{R}^2)} \lesssim \int_0^t \|\omega(s)\|^2_{H^N_1(D_+)} \, ds,
\]

\[
\| \Lambda^s xg_4(t) \|_{L^2(\mathbb{R}^2)} \lesssim \int_0^t (1 + s) \|\omega(s)\|^2_{H^N_1(D_+)} \, ds
\]

which implies (6.33).

A calculation similar to the one in the proof of (6.22) shows that in order to bound \( xg_2 \), we need to control the time integral of:

\[
s \| \Lambda^s (R \cdot V_\omega)^2 \|_{L^2} + \| \Lambda^{-1/2} (R \cdot V_\omega)^2 \|_{L^2} + \| \Lambda^{1/2+i} ((\Lambda^{-1} V_\omega) R \cdot V_\omega) \|_{L^2} + \| \Lambda^{1/2+i} ((R \cdot (x V_\omega)(R \cdot V_\omega)) \|_{L^2}
\]

Using (C.2) and (C.1) as in the proof of the previous lemma, it is straightforward to bound each of these terms by \( (1 + s) \|(1 + |x|^2)^{1/2} V_\omega \|_{H^3(\mathbb{R}^2)} \).

6.4. Estimates for \( g_5 \). Recall that \( g_5 \) contains all terms of order three or higher which involve \( V_\omega \). There are two such types of terms: the terms coming from the first line of (3.56), and the terms of degree 2 and higher from expanding the rescaled Dirichlet-to-Neumann map \( G(h) \) in powers of \( h \) and inserting this into (3.56). In either case, the vorticity enters at most quadratically. We illustrate how to handle the term corresponding to the first term on the right-hand side of (3.56), which is:

\[
R_1 = -\int_0^t e^{is \Lambda^{1/2}} |V_\omega \cdot \Lambda^{-1/2} \nabla u|^2.
\]

Using Sobolev embedding and e.g. the Hörmander-Mikhlin multiplier theorem, it is straightforward to estimate:

\[
\| \nabla^k R_1 \|_{L^\infty(\mathbb{R}^2)} \lesssim \| R_1 \|_{H^{k+2}(\mathbb{R}^2)} \lesssim \int_0^t \| V_\omega(s) \|_{W^{1+\epsilon,k+2}(\mathbb{R}^2)}^2 \| u(s) \|_{W^{1+\epsilon,k+2}(\mathbb{R}^2)}^2 \,
\]

for arbitrary \( \epsilon > 0 \). Using the interpolation inequality (A.3) and Young’s inequality \( |ab| \lesssim |a|^p + |b|^q \) for \( 1/p + 1/q = 1 \), this shows that:

\[
\| \nabla^k R_1 \|_{L^\infty(\mathbb{R}^2)} \lesssim \left( \frac{\varepsilon_0^2}{(1 + t)^{\sigma'}} \right) (\varepsilon_1^2 (1 + t)^{2+2\delta}) \lesssim \varepsilon_0^2 \frac{1}{1 + t} + \varepsilon_1^2 (1 + t)^{3+3\delta}
\]

where \( \sigma' \ll 1 \). To estimate the terms coming from the expansion of \( G(h) \), one can argue as above, but using additionally the estimates from Appendix F of [8].

The estimates for \( \Lambda^s xg_5 \) are similar to the above and the estimates we have already proved.

7. Estimates for the dispersive terms

In this section we bound the term \( g_1 \) defined in (3.72). We can actually proceed nearly exactly as in [8] to handle these terms. The only differences here are that (1) after performing the normal forms transformation (integration by parts in time), there are additional terms involving the vorticity that need to be bounded and (2) we want to control \( \| \Lambda^s xg_1 \|_{L^2(\mathbb{R}^2)} \) instead of \( \| xg_1 \|_{L^2(\mathbb{R}^2)} \).
Applying the bootstrap assumptions (2.10)-(2.11) and the interpolation inequality (A.3), we get:

\[ \int_{\mathbb{R}^2} \mu(\xi, \eta)e^{it\varphi_{\alpha \beta}(\xi, \eta)} f_{-\alpha}(t, \xi - \eta) f_-(t, \eta) \, d\eta - \int_{0}^{t} \int_{\mathbb{R}^2} \mu(\xi, \eta)e^{is\varphi_{\alpha \beta}(\xi, \eta)} \partial_s \left( f_{-\alpha}(s, \xi - \eta) f_-(s, \eta) \right) \, d\eta = A_1 + A_2 \quad (7.1) \]

where \( \alpha, \beta \in \{+, -\} \), \( \varphi_{\pm \pm} = |\xi|^{1/2} \pm |\xi - \eta|^{1/2} \pm |\eta|^{1/2} \), \( f_+ = f, f_- = \overline{f} \) with \( f = e^{itA^{1/2}} u \), and \( \mu \) is a bilinear multiplier which is in the class \( B_1 \), defined in Appendix C of [8]. The first term here can be estimated exactly as in Section 5 of [8] which gives:

\[ ||\nabla^k e^{-itA^{1/2}} F^{-1} A_1||_{L^\infty(\mathbb{R}^2)} \lesssim \frac{\varepsilon_0^2}{1 + t} \quad (7.2) \]

To control \( A_2 \), it is enough to consider the case that \( \partial_s \) falls on the second factor. Using the equation (3.72), this generates two types of terms: those involving just \( \hat{f} \) and those involving \( V_\omega \). The first type of term can be dealt with just as in [8]. There are a large number of terms involving \( V_\omega \) however they can all be dealt with similarly to the estimates from the previous section. This is because none of the above estimates involve any special cancellations are are just performed by applying Sobolev embedding, Holder’s inequality and various simple facts from Harmonic analysis.

We just need to use Theorem C.1 from [8] in place of Holder’s inequality. For example, the term coming from \( g_2 \) in (3.72) is:

\[ A_3 = \int_{0}^{t} \int_{\mathbb{R}^2} e^{is\varphi_{\alpha \beta}(\xi, \eta)} \mu(\xi, \eta) \frac{\eta}{|\eta|} \hat{f}_{-\alpha}(s, \xi - \eta) e^{-\beta|s| |\eta|^{1/2}} \overline{V_\omega}(s, \eta) \, d\eta ds \quad (7.3) \]

Applying Sobolev embedding \( ||q||_{L^\infty(\mathbb{R}^2)} \lesssim ||q||_{W^{1, p}(\mathbb{R}^2)} \) for \( p > 10 \), say, Theorem C.1 from [8] gives:

\[ ||\nabla^k e^{-itA^{1/2}} F^{-1} A_3||_{L^\infty(\mathbb{R}^2)} \lesssim \int_{0}^{t} ||\nabla^k e^{-itA^{1/2}} F^{-1} B_\mu(u, V_\omega)||_{W^{1, p}(\mathbb{R}^2)} \]

\[ \lesssim \int_{0}^{t} ||u(s)||_{W^{k+1, 2p}(\mathbb{R}^2)} ||V_\omega(s)||_{W^{k+1, 2p}(\mathbb{R}^2)} \quad (7.4) \]

Applying the bootstrap assumptions (2.10)-(2.11) and the interpolation inequality (A.3), we get:

\[ ||\nabla^k e^{-itA^{1/2}} F^{-1} A_3||_{L^\infty(\mathbb{R}^2)} \lesssim \int_{0}^{t} \frac{\varepsilon_0}{(1 + s)^{1 - \sigma}} \varepsilon_1(1 + s)^\delta \, ds \lesssim \varepsilon_0 \varepsilon_1(1 + t)^{1 + \delta} \quad (7.5) \]

The estimates for \( ||A^t x g_1||_{L^2(\mathbb{R}^2)} \) can be proven in a similar manner as above by following the outline in [8]. The only difference is that one needs to use the assumption \( ||A^t(x e^{itA^{1/2}} u)||_{L^2} \leq (1 + t)^\delta \) in place of the assumption \( ||e^{itA^{1/2}} u||_{L^2} \leq (1 + t)^\delta \) in [8]. Summing up and noting that \( V_\omega \) enters no more than quadratically into any of the above terms, we get:

**Proposition 7.1.** If (2.10)-(2.12) hold for \( \varepsilon_0 \ll 1 \) and \( \varepsilon_1 \ll \varepsilon_0 \), then:

\[ ||\nabla^k e^{-itA^{1/2}} g_1||_{L^\infty(\mathbb{R}^2)} \lesssim \frac{\varepsilon_0^2}{1 + t} + \varepsilon_1(1 + t)^{1 + 2\delta} \quad (7.6) \]

for \( k \leq N_1 + 4 \), and

\[ ||A^t(x g_1)||_{L^2(\mathbb{R}^2)} \lesssim \varepsilon_0^2(1 + t)^\delta + \varepsilon_0 \varepsilon_1(1 + t)^{1 + 2\delta} \quad (7.7) \]

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ON THE LIFESPAN OF THREE-DIMENSIONAL GRAVITY WATER WAVES WITH VORTICITY

APPENDIX A. INTERPOLATION AND SOBOLEV INEQUALITIES

In this section we will assume that \( \partial D_t \) is given by the graph of a function, \( \partial D_t = \{(x, h(t, x)), x \in \mathbb{R}^2\} \), and further that we have a bound for the second fundamental form and injectivity radius of \( \partial D_t \), as well as a bound for \(|\nabla h|\):

\[ |\theta| + \frac{1}{t_0} + |\nabla h| \leq K. \]  

(A.1)

Note that \( \theta \sim \nabla^2 h \). We then have the following Sobolev inequalities:

Lemma A.1. If \( \|u\|_{L^6(D_t)} + \|Du\|_{L^2(D_t)} < \infty \), then:

\[ \|u\|_{L^6(D_t)} \leq C(K)\|Du\|_{L^2(D_t)}. \]  

(A.2)

If \( u \in W^{k,p}(D_t) \) then for \( k > \frac{3}{p} \):

\[ \|u\|_{L^\infty(D_t)} \leq C(K)\|u\|_{W^{k,p}(D_t)}, \]  

(A.3)

and if \( u \in W^{k,p}(\partial D_t) \), then for \( k > \frac{2}{p} \):

\[ \|u\|_{L^\infty(\partial D_t)} \leq C(K)\|u\|_{W^{k,p}(\partial D_t)}, \]  

(A.4)

These estimates all follow from the estimates in the appendix of [9]. The estimates there are all stated for the case of a bounded domain but it is clear that the proof goes through for an unbounded domain.

We will also need interpolation estimates on \( \partial D_t \) and \( D_t \):

Lemma A.2. Let \( 2 \leq p \leq s \leq \infty \) and \( 0 \leq k \leq m \). Suppose that:

\[ \frac{m}{s} = \frac{k}{p} + \frac{m-k}{q} \]  

(A.5)

If \( \alpha \) is a \((0,r)\) tensor then with \( a = \frac{k}{m} \),

\[ \|\nabla^k \alpha\|_{L^s(\partial D_t)} \leq C\|\alpha\|_{L^q(\partial D_t)}^{1-a} \|\nabla^m \alpha\|_{L^p(\partial D_t)}^a \]  

(A.6)

and if \( |\theta| + \frac{1}{ta} \leq K \), then:

\[ \sum_{j=0}^k \|D^j \alpha\|_{L^s(D_t)} \leq C(K)\|\alpha\|_{L^q(D_t)}^{1-a} \left( \sum_{j=0}^m \|D^j \alpha\|_{L^p(D_t)} \right)^a. \]  

(A.7)

Finally, we will use the following interpolation inequality which is Lemma 5.1 in [8]:

Lemma A.3. If \( 2 \leq p \leq \infty, k \leq N_0 + \frac{2}{p} - 1 \), then:

\[ \|\nabla^k u\|_{L^p(\mathbb{R}^2)} \leq (1+t)^{-1+\frac{k}{p}+\sigma} ((1+t)\|u(t)\|_{W^{4,\infty}(\mathbb{R}^2)} + (1+t)^{-\delta}\|u(t)\|_{H^{N_0}(\mathbb{R}^2)}), \]  

(A.8)

where \( \sigma = \sigma(k,p,N_0,\delta) = \frac{k}{N_0 + \frac{2}{p} - 1} (\delta - \frac{2}{p} + 1) \).

APPENDIX B. SCHAUDER ESTIMATES

The following result is well-known (see e.g. Theorem 7.3 in [30]):

Proposition B.1. If \( f = F \) on \( \partial D_t \) and \( \partial D_t \) is given by the graph of \( h : \mathbb{R}^2 \to \mathbb{R} \), then for \( k \geq 2 \):

\[ \|f\|_{C^{k,\alpha}(D_t)} \leq C\|h\|_{C^{k,\alpha}(\mathbb{R}^2)} \left( \|f\|_{C^{k-2,\alpha}(D_t)} + \|\Delta f\|_{C^{k-2,\alpha}(\partial D_t)} + \|f\|_{L^\infty(D_t)} \right). \]  

(B.1)

We will also need standard \( L^p \) estimates (see e.g. Theorem 15.2 in [30]):

Proposition B.2. If \( f \in W^{2,p}(D_t), f = F \) on \( \partial D_t \) and \( \partial D_t \) is given by the graph of \( h : \mathbb{R}^2 \to \mathbb{R} \), then:

\[ \|f\|_{W^{k,p}(D_t)} \leq C\|h\|_{C^{k}(\mathbb{R}^2)} \left( \|\Delta f\|_{W^{k-2,p}(D_t)} + \|f\|_{W^{k-1,p}(\partial D_t)} + \|f\|_{L^p(D_t)} \right). \]  

(B.2)
APPENDIX C. ESTIMATES FROM HARMONIC ANALYSIS

We collect a few results that we will use frequently.

Lemma C.1.  
- If $1 < p \leq q < \infty$ and $\alpha = \frac{2}{p} - \frac{2}{q}$ then:
  \[
  ||\Lambda^{-\alpha} f||_{L^q} \lesssim ||f||_{L^p} \tag{C.1}
  \]
- If $1 < p < \infty$ and $s \geq 0$, then for any $1 < p_1, p_2, q_1, q_2 < \infty$ with $1/p_1 + 1/p_2 = 1/q_1 + 1/q_2 = 1/p$,
  \[
  ||\Lambda^s (fg)||_p \lesssim ||\Lambda^s f||_{p_1} ||g||_{p_2} + ||f||_{q_1} ||\Lambda^s g||_{q_2}. \tag{C.2}
  \]

The estimate (C.1) is known as the Hardy-Littlewood fractional integration lemma; for a proof, see [31]. For a proof of (C.2), see [32].

We will also use the following estimate for the Dirichlet-to-Neumann map, which is Proposition 2.2 from [8]. As mentioned there, this is not optimal (both in terms of the regularity assumed of $h$ and the number of derivatives of $\varphi$ on the right-hand side) but this will suffice for our purposes.

Proposition C.1.  If $\varphi : \partial \Omega \to \mathbb{R}$ where $\partial \Omega$ is the graph of a function $h$ with $h \in W^{4,\infty}(\mathbb{R}^2)$ then:
  \[
  ||N \varphi||_{W^{2,\infty}(\mathbb{R}^2)} \lesssim ||\nabla \varphi||_{W^{3,\infty}(\mathbb{R}^2)} + ||\Lambda^{1/2} \varphi||_{L^{\infty}(\mathbb{R}^2)}, \tag{C.3}
  \]
with implicit constant depending on $||h||_{W^{4,\infty}(\mathbb{R}^2)}$.

APPENDIX D. ELLIPTIC SYSTEMS

We follow the approach of [27] and [33]. First, we define the space $Y$ to be closure of $C^\infty(\mathcal{D}_t)$ with respect to the norm:
  \[
  ||u||_Y \equiv ||u||_{L^6(\mathcal{D}_t)} + ||Du||_{L^2(\mathcal{D}_t)}. \tag{D.1}
  \]
We note that by the Sobolev inequality (A.2), $Y$ is actually a Hilbert space with inner product:
  \[
  (u, v)_Y \equiv \int_{\mathcal{D}_t} Du \cdot Dv. \tag{D.2}
  \]

The goal of this section is to construct a solution $\beta$ to the system:
  \[
  \begin{align*}
  \text{div } \beta &= 0, & \text{in } \mathcal{D}_t, \\
  \text{curl } \beta &= \alpha, & \text{in } \mathcal{D}_t, \\
  \beta \cdot N &= 0 & \text{on } \partial \mathcal{D}_t,
  \end{align*}
  \tag{D.3-5}
  \]

where $\alpha \in L^{6/5}(\mathcal{D}_t)$. Suppose for the moment that the following system has a unique weak solution $\beta'$:
  \[
  \begin{align*}
  \Delta \beta' &= \alpha & \text{in } \mathcal{D}_t, \\
  \gamma_j^i \beta'_i &= 0 & \text{on } \partial \mathcal{D}_t, \\
  D_N \beta'_N &= -H \beta'_N & \text{on } \partial \mathcal{D}_t,
  \end{align*}
  \tag{D.6-8}
  \]

where $D_N = N^j D_j$, $H$ is the mean curvature of $\partial \mathcal{D}_t$, $H = \text{tr } \theta$ and $\beta'_N = N^i \beta'_i$. We now recall that by the definition of the second fundamental form we have $\text{div } \beta'|_{\partial \mathcal{D}_t} = \text{tr } D\beta'|_{\partial \mathcal{D}_t} = \text{div}_{\partial \mathcal{D}_t} (\Pi \beta') + H \beta'_N$. Taking the divergence of (D.6) and applying this formula shows that $\beta'$ satisfies:
  \[
  \begin{align*}
  \Delta \text{div } \beta' &= 0 & \text{in } \mathcal{D}_t, \\
  \text{div } \beta' &= 0 & \text{on } \partial \mathcal{D}_t
  \end{align*}
  \tag{D.9-10}
  \]
so that $\text{div } \beta' = 0$ in $\mathcal{D}_t$. In particular this implies that $\Delta \beta' = \text{curl}^2 \beta'$. If we then set $\beta = \text{curl } \beta'$, it follows that $\beta$ satisfies (D.3) and (D.4). To see that $\beta$ satisfies (D.5), we just note that $N \cdot \text{curl } \beta$ only involves tangential derivatives of $\gamma \cdot \beta'$ and thus this vanishes if (D.7) holds. We also remark
that this choice of $\beta$ is actually unique; if $\beta_1, \beta_2$ satisfy (D.3)-(D.5) it follows that $\beta_1 - \beta_2 = D\phi$ for some harmonic function $\phi$ which satisfies a Neumann problem with zero boundary data and is thus a constant.

We now prove that (D.6)-(D.8) has a unique weak solution:

**Proposition D.1.** Let $\alpha \in L^{6/5}(D_t)$ and suppose that $H$, the mean curvature, satisfies:

$$||H||_{L^3(\partial D_t)} + ||\nabla H||_{L^{3/2}(\partial D_t)} \ll 1.$$  \hfill (D.11)

Then the problem (D.6)-(D.8) has a unique solution $\beta' \in H^1(D_t)$. Furthermore, with $\beta = \text{curl} \beta'$, under the above hypotheses we have:

$$|\beta(z)| \lesssim \frac{1}{(1 + |z|)^2} \int_{D_t} (1 + |z'|)|\alpha(z')| dz'$$  \hfill (D.12)

**Proof.** We let $C^\infty_{\text{tan}}(D_t)$ denote the collection of smooth one-forms $\alpha$ on $D_t$ so that $\gamma^i_j \alpha_j$ is compactly supported in $D_t$, and we let $Y_0$ denote the closure of $C^\infty_{\text{tan}}(D_t)$ with respect to the norm $||u||_Y = ||u||_{L^6(D_t)} + ||Du||_{L^2(D_t)}$. We define the bilinear form:

$$B[u, \varphi] = \int_{D_t} \delta^i_k \delta^j_l (D_i u_j)(D_k \varphi_l) + \int_{\partial D_t} H u_N \varphi_N,$$  \hfill (D.13)

for $u, \varphi \in Y_0$ and with $w_N = N^i u_i$. We want to find $u \in Y_0$ so that:

$$B[u, \varphi] = \int_{D_t} \delta^i_j \alpha_i \varphi_j,$$  \hfill (D.14)

for all $\varphi \in Y_0$. The map $\varphi \mapsto \int_{D_t} \alpha \cdot \varphi$ is a continuous linear map on $Y$ since $\alpha \in L^{6/5}(D_t)$, and so by the Lax-Milgram theorem it suffices to prove that $B$ is bounded and coercive.

Fix a smooth cutoff function $\chi = \chi(x,y)$ so that $\chi \equiv 1$ when $|y - h(x)| \leq \rho$ and $\chi \equiv 0$ when $|y - h(x)| \geq 2\rho$ for some fixed $\rho > 0$. Let $\tilde{H} = \chi H$, and note that by Stokes’ theorem we have:

$$||\sqrt{\tilde{H}} w||_{L^2(\partial D_t)}^2 = \int_{D_t} D_k \left( N^k \tilde{H} \right) |w|^2 dxdy = \int_{D_t} D_k \left( N^k \tilde{H} \right) |w|^2 dxdy + 2 \int_{D_t} \tilde{H} N^k D_k w \cdot w dxdy \lesssim ||\nabla \tilde{H}||_{L^{3/2}(D_t)} ||w||_{L^6(D_t)}^2 ||w||_{L^6(D_t)} \lesssim \epsilon^* ||w||_Y^2,$$  \hfill (D.15)

where we used that $||\nabla \tilde{H}||_{L^{3/2}(D_t)} \lesssim ||\nabla H||_{L^{3/2}(D_t)}$ and $||\tilde{H}||_{L^3(D_t)} \lesssim ||H||_{L^3(D_t)}$. In particular this shows that the bilinear form $B$ is bounded on $Y$ and also, for sufficiently small $\epsilon^*$, that it is coercive on $Y$.

We now prove the decay estimate (D.12). For this, we will construct a Green’s function $G$ for the problem (D.6)-(D.8), following the approach of [28] and [33]. We fix $\rho > 0$ and let $G_\rho = G_\rho(z, z')$ denote the weak solution to the problem (D.6)-(D.8) with $\alpha_i = \frac{1}{|z'|^3} \chi_{B_\rho(z')}(z)$, $i = 1, 2, 3$, where $B_\rho(z')$ denotes the ball of radius $\rho$ centered at $z'$ and $\chi$ is the cutoff function supported on this ball. Following the argument in section 4 of [28], one can prove that:

$$||G_\rho(z, \cdot)||_{W^{1,p}(B_{d_z}(z))} \leq C(d_y),$$  \hfill (D.16)

for some $p$ with $p \in (1, 3/2)$ and where $d_z$ denotes the distance from $z$ to $\partial D_t$. The constant here depends on $||h||_{C^1(B_2)}$. Taking a diagonal subsequence, for each $z$ we get a function $G(z, \cdot) \in W^{1,p}(B_{d_z}(z))$ with $G_\rho(z, \cdot) \to G(z, \cdot)$ weakly in $W^{1,p}(B_{d_z}(z))$. We would like to conclude the following two estimates:

$$|G(z, z')| \leq C |z - z'|^{-1}, \quad |D_z G(z, z')| \leq C |z - z'|^{-2},$$  \hfill (D.17)

where $C = C(||h||_{W^{1,\infty}(B_2)})$. These estimates follows as in Section 5 of [33] and Theorem 3.13 in [34], provided that the system (D.6)-(D.8) satisfies the condition “(LH)” in [34]. However this follows from Corollary 4.9 there provided that the system (D.6)-(D.8) is sufficiently close to a
diagonal system. Since we are assuming that \( \|h\|_{W^{4,\infty}(\mathbb{R}^2)} \) is small, this follows after straightening the boundary. We can now prove (D.12). We can assume \( |z| \geq 1 \). If \( |z'| \leq \frac{1}{2}|z| \), then \( |z-z'| \geq \frac{1}{2}|z| \) so that:

\[
|\int_{D_1} D_z \mathcal{G}(z, z') \alpha(z') \, dz'| \lesssim \int_{D_1} \frac{1}{|z-z'|^2} |\alpha(z')| \, dz' \lesssim \frac{1}{|z|^2} \|\alpha\|_{L^1(D_1)}.
\]

(D.18)

When \( |z'| \geq \frac{1}{2}|z| \), we instead estimate:

\[
|\int_{D_1} D_z \mathcal{G}(z, z') \alpha(z') \, dz'| \lesssim \int_{D_1} \frac{1}{|z'|^2} |\alpha(z-z')| \, dz' \lesssim \int_{D_1} \frac{1}{|z|^2} |\alpha(z-z')| \, dz' \lesssim \frac{1}{|z|^2} \|\alpha\|_{L^1(D_1)},
\]

(D.19)

as required.

\( \square \)

**Remark.** For the applications we have in mind, the assumption \( \|H\|_{L^3(\partial D_1)} \ll 1 \) is not a serious restriction because we will actually have \( \|H\|_{L^3(\partial D_1)} \leq \varepsilon_0 \). On the other hand, the assumption \( \|\nabla H\|_{L^{3/2}(\partial D_1)} \ll 1 \) only holds until \( t \sim \varepsilon_0^{-1/8} \sim \varepsilon_0^{-N} \). This condition is already forced on us by essentially the result of Proposition 5.3, and so this is not a serious restriction for our purposes. We note that it may be possible to remove this assumption by arguing as in section 5.2 of [27], but the arguments there are somewhat involved.

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