Spectral Zeta Functions for a Cylinder and a Circle

V. V. Nesterenko, I. G. Pirozhenko

Bogoliubov Laboratory of Theoretical Physics
Joint Institute for Nuclear Research, Dubna, 141980, Russia
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Abstract

Spectral zeta functions $\zeta(s)$ for the massless scalar fields obeying the Dirichlet and Neumann boundary conditions on a surface of an infinite cylinder are constructed. These functions are defined explicitly in a finite domain of the complex plane $s$ containing the closed interval of real axis $-1 \leq \Re s \leq 0$. Proceeding from this the spectral zeta functions for the boundary conditions given on a circle (boundary value problem on a plane) are obtained without any additional calculations. The Casimir energy for the relevant field configurations is deduced.

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I. INTRODUCTION

The zeta function technique is widely used in quantum field theory, specifically, for calculation of the vacuum energy (the Casimir energy) of quantized fields in compactified configuration space. For the boundaries with spherical geometry this method has been essentially worked out in papers. The Casimir energy is determined by the value of the corresponding zeta function $\zeta(s)$ at a certain point (usually at $s = -1$). Therefore, when it regards the vacuum energy calculation, the zeta function is investigated only at a separate point or at least in its infinitesimal neighborhood.

However, in some problems it proves to be useful to construct explicitly the spectral zeta function for a finite range of its argument $s$. For example, proceeding from the zeta function for an infinite cylinder $\zeta_{\text{cyl}}(s)$ defined in the domain $-1 \leq \text{Re} \ s \leq 0$ one can easily express both the Casimir energy of a cylinder via $\zeta_{\text{cyl}}(-1)$ and the Casimir energy of a circle in terms of $\zeta_{\text{cyl}}(0)$. Such an approach has obvious advantage not only for shortening the calculations (a unique zeta function is applicable to two problems) but also when treating the divergences. For instance, the zeta function technique applied to the boundary conditions given on the surface of an infinite cylinder at once gives the finite value of the vacuum energy (the renormalization is carried out simultaneously with the regularization). As for the boundary conditions defined on a circle, the zeta function regularization is unable to remove all the divergences. The answers obtained by different authors for the finite part of the corresponding Casimir energy do not coincide. In view of this it would be interesting to express the spectral zeta function $\zeta_{\text{cir}}(s)$ in terms of $\zeta_{\text{cyl}}(s)$ which is already ”normalized” (being free of the divergences) at the point $s = -1$. It is this that we are going to do in the present note.

The layout of the paper is as follows. In Sec. II we derive the relation between the zeta functions for the boundary conditions given on the surface of an infinite cylinder, $\zeta_{\text{cyl}}(s)$, and on the circle, $\zeta_{\text{cir}}(s)$. In Sec. III the spectral zeta functions for the massless scalar fields obeying the Dirichlet and Neumann boundary conditions on the lateral area of an infinite cylinder are constructed explicitly. As in Ref. the central part is played here by the uniform asymptotic expansion of the Bessel functions. Explicit formulas defining these zeta functions in a finite region $\Omega$ of the complex plane $s$ containing the closed interval of real axis $-1 \leq \text{Re} \ s \leq 0$ are derived. In Sec. IV using these formulas we obtain the spectral zeta functions for the relevant plane problem with the Dirichlet or Neumann boundary conditions given on a circle. The Casimir energies for the field configurations in hand are calculated. In Conclusion (Sec. V) the obtained results are shortly discussed and compared with those of other authors.

II. RELATION BETWEEN THE SPECTRAL ZETA FUNCTIONS FOR A CYLINDER AND A CIRCLE

Let us remind briefly how to find out the relation between the zeta function for a three dimensional problem with boundary conditions given on the lateral area of an infinite cylinder of radius $a$ and the zeta function for a two dimensional problem with boundary conditions given on a circle of the same radius. For simplicity in both cases the massless scalar field obeying the Dirichlet or Neumann boundary conditions is considered.

In the case of an infinite cylinder the eigenfunctions are proportional to $\exp(-i\omega t + ik_z z + in\theta)$ where $\{r, \theta, z\}$ are cylindrical coordinates. The eigenfrequencies $\omega$ inside the cylinder...
for the Dirichlet and Neumann boundary conditions are given respectively by the equations
\[ J_l(\lambda r)|_{r=a} = 0, \quad J'_l(\lambda r)|_{r=a} = 0. \] (2.1)
For the outside region we have
\[ H^{(1)}_l(\lambda r)|_{r=a} = 0, \quad H^{(1)'}_l(\lambda r)|_{r=a} = 0. \] (2.2)
Here the notation \( \lambda^2 = \omega^2 - k^2 \) is introduced.
The eigenfrequency equations for the scalar field with boundary conditions defined on a circle are obtained by putting \( k^2 = 0 \) in (2.1) and (2.2).

In order to construct the spectral zeta function for an infinite cylinder or a circle one can employ the standard definition
\[ \zeta(s) = \sum_{\{p\}} (\omega_p^{-s} - \bar{\omega}_p^{-s}). \] (2.3)
Here \( \omega_p \) are the eigenfrequencies of the scalar field under certain boundary conditions, \( \bar{\omega}_p \) are the same frequencies when the boundaries are removed. The summation (or integration) should be done over all the quantum numbers \( \{p\} \) specifying the spectrum. To make the sum convergent the parameter \( s \) should belong to a region of the complex plane \( s \) where \( \Re s \) is large enough. However for the massless fields considered here there also exists a restriction for the maximal values of \( \Re s \) in order to ensure the convergent integration at the origin (see below).

For a cylinder and a circle the general formula (2.3) looks as follows
\[ \zeta_{cyl}(s) = \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \sum_{l,n} \left[ (\lambda^2_{ln}(a) + k_z^2)^{-s/2} - (\lambda^2_{ln}(\infty) + k_z^2)^{-s/2} \right], \] (2.4)
\[ \zeta_{cir}(s) = \sum_{l,n} \left[ \lambda^s_{ln}(a) - \lambda^s_{ln}(\infty) \right], \] (2.5)
with \( \lambda_{ln} \) being defined by (2.1) and (2.2) for the both zeta functions. Integration over \( k_z \) in (2.4) can be accomplished by making use of the formula
\[ \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} (k_z^2 + b^2)^{-s/2} = \frac{b^{1-s}}{2\pi} B \left( \frac{1}{2}, \frac{s-1}{2} \right), \quad \Re s > 1, \]
where \( B(x, y) \) is the Euler beta function
\[ B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y). \]
Comparing the result of the integration with (2.5) one arrives at the relation between \( \zeta_{cyl}(s) \) and \( \zeta_{cir}(s) \)
\[ \zeta_{cyl}(s) = \frac{1}{2\pi} B \left( \frac{1}{2}, \frac{s-1}{2} \right) \zeta_{cir}(s-1). \] (2.6)
When calculating the Casimir energy by making use of the zeta function technique usually one puts
\[ E_C = \frac{1}{2} \zeta(s = -1). \] (2.7)
To use this formula, for example, in the case of a circle one should find the analytic continuation of \( \zeta_{cir}(s) \) into the point \( s = -1 \). On the other hand, in accordance with (2.6) the zeta function \( \zeta_{cir}(s) \) at the point \( s = -1 \) can be expressed through \( \zeta_{cyl}(s = 0) \). Thus the analytic continuation of the zeta function \( \zeta_{cyl}(s) \) into the region \( -1 \leq \Re s \leq 0 \) provides the opportunity to calculate the Casimir energy both for an infinite cylinder and for a circle.
III. SPECTRAL ZETA FUNCTIONS FOR A CYLINDER WITH THE DIRICHLET AND NEUMANN BOUNDARY CONDITIONS

In Ref. 8 a consistent procedure has been developed for constructing the spectral zeta functions for the boundary conditions given on a sphere and on the lateral area of an infinite cylinder. Here we follow the same approach and start with consideration of the spectral zeta function $\zeta_{cyl}(s)$ for the massless scalar field obeying the Dirichlet boundary conditions on an infinite cylinder.

Taking into account the contributions of the field oscillations inside (Eq. (2.1)) and outside (Eq. (2.2)) the cylinder and representing the sum over $l$ in (2.4) in terms of contour integral one obtains

$$\zeta_{cyl}^D(s) = \frac{a^{s-1}}{2\sqrt{\pi}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{3-s}{2}\right)} \sum_{n=-\infty}^{\infty} \int_{0}^{\infty} dy y^{1-s} \frac{d}{dy} \ln\left[2nyI_n(y)K_n(y)\right].$$

(3.1)

The contour $C$ consists of the imaginary axis ($-i\infty, i\infty$) and a semi-circle of an infinite radius in the right half plane of a complex variable $\lambda$. Keeping in mind the behavior of the integrand at all the segments of the contour $C$ and integrating over $k_x$ Eq. (3.1) becomes

$$\zeta_{cyl}^D(s) = \frac{a^{s-1}}{2\sqrt{\pi}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{3-s}{2}\right)} \sum_{n=-\infty}^{\infty} \int_{0}^{\infty} dy y^{1-s} \frac{d}{dy} \ln\left[2nyI_n(y)K_n(y)\right].$$

(3.2)

Then, in order to accomplish the analytic continuation of (3.2) into the region $-1 \leq \text{Re} s < 0$ we express $\zeta_{cyl}(s)$ in terms of the Riemann zeta function with the well-known analytic continuation. After changing the integration variable $y \rightarrow ny$ in (3.2) we employ the uniform asymptotic expansion (UAE) of the Bessel functions[10] up to the order $n^{-4}$

$$\ln\left(2nyI_n(ny)K_n(ny)\right) = \ln(yt) + \frac{t^2(1 - 6t^2 + 5t^4)}{8n^2} + \frac{t^4(13 - 284t^2 + 1062t^4 - 1356t^6 + 565t^8)}{64n^4} + O(n^{-6}),$$

(3.3)

where $t = 1/\sqrt{1+y^2}$. Substituting (3.3) in all the terms of the series (3.2), where $n \neq 0$, we obtain

$$\zeta_{cyl}^D(s) = C(s)\left(Z_0(s) + Z_1(s) + Z_2(s) + Z_3(s)\right),$$

(3.4)

$$Z_0(s) = \int_{0}^{\infty} dy y^{1-s} \frac{d}{dy} \left\{ \ln(2nyI_0(y)K_0(y)) - \frac{t^2}{8}(1 - 6t^2 + 5t^4) \right\},$$

(3.5)

$$Z_1(s) = \sum_{n=1}^{\infty} n^{-1-s} \int_{0}^{\infty} dy y^{1-s} \frac{d}{dy} \ln\left(\frac{y^2}{1+y^2}\right),$$

(3.6)

$$Z_2(s) = \frac{1}{4} \sum_{n=1}^{\infty} n^{-1-s} + \frac{1}{2} \int_{0}^{\infty} dy y^{1-s} \frac{d}{dy} \left[ t^2(1 - 6t^2 + 5t^4) \right],$$

(3.7)

$$Z_3(s) = \frac{1}{32} \sum_{n=1}^{\infty} n^{-3-s} \int_{0}^{\infty} dy y^{1-s} \frac{d}{dy} \left[ t^4(13 - 284t^2 + 1062t^4 - 1356t^6 + 565t^8) \right],$$

(3.8)

where

$$C(s) = \frac{a^{s-1}}{2\sqrt{\pi}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{3-s}{2}\right)}.$$
Here the notation $Z_0(s)$ is introduced for the difference between the term with $n = 0$ in (3.2) and the integral

$$A(s) = \frac{1}{8} \int_0^\infty dy \, y^{1-s} \frac{d}{dy} \left[ t^2(1 - 6 t^2 + 5 t^4) \right], \quad -1 < \text{Re} \, s < 3. \quad (3.10)$$

The function $Z_1(s)$ corresponds to the first term in the UAE (3.3). $Z_2(s)$ involves the contribution of the $1/n^2$-order term of the latter together with the integral $A(s)$, $Z_3(s)$ is generated by the terms of order $1/n^4$ in the expansion (3.3).

Taking into account the asymptotics

$$2yI_0(y)K_0(y) = -2y \ln y + (2 \ln 2 - 2\gamma)y + O(y^3), \quad y \to 0,$$

$$2yI_0(y)K_0(y) = 1 + \frac{1}{8y^2} + \frac{27}{128y^4} + O(y^{-6}), \quad y \to \infty, \quad (3.11)$$

where $\gamma$ is the Euler constant, one can ascertain the domain of variation of the complex variable $s$ so that the integrals in (3.5)–(3.8) exist. The ultraviolet behavior ($y \to \infty$) of the integrands in (3.5)–(3.8) determines the lower bound for Re $s$ and the infrared one ($y \to 0$) is responsible for the upper bound. Formula (3.5) defines $Z_0(s)$ as an analytic function of $s$ if $-3 < \text{Re} \, s < 1$. When this condition holds one can perform in (3.5) the integration by parts

$$Z_0(s) = -(1 - s) \int_0^\infty dy \, y^{-s} \left[ \ln(2yI_0(y)K_0(y)) - \frac{t^2}{8} (1 - 6t^2 + 5t^4) \right]. \quad (3.12)$$

The integral defining $Z_1(s)$ in (3.6) exists if $-1 < \text{Re} \, s < 1$. The sum over $n$ in this formula is finite for $\text{Re} \, s > 2$. As these two regions do not overlap, the introduction of the parameter $s$ in the original formula (2.3) does not regularize completely the divergences in $Z_1(s)$ on this stage of our consideration. An additional infrared regularization should be used here, for example, by introducing the photon ”mass” $\mu$. As a result the integration in Eq. (3.6) will be restricted from below by $\mu$, and the constraint $\text{Re} \, s < 1$ will be removed. The function $Z_1(s)$, regularized in this way, can be used for required analytic continuation (see below).

The integral in Eq. (3.7) defining $Z_2(s)$ exists when $-1 < \text{Re} \, s < 3$. The sum over $l$ in (3.7) is finite when $\text{Re} \, s > 0$. The integral in (3.8) converges if $-3 < \text{Re} \, s < 3$. The sum over $n$ in this formula is finite for $\text{Re} \, s > -2$. Thus the regions, where the integrals and the sums exist, overlap and these formulas can be used for constructing the analytic continuation needed.

As it was stressed in the preceding section, we are interested in the evaluation of $\zeta_{cyl}^D(s)$ in the region $\Omega$ of the complex plane $s$ containing the closed interval of the real axis $-1 \leq \text{Re} \, s \leq 0$. For the zeta function $\zeta_{cyl}^D(s)$ defined by the sum (3.4) we shall construct the analytic continuation into this region in the following way.

It has been already noticed that the functions $Z_0(s)$ in (3.5) and $Z_3(s)$ in (3.8) are analytic in the region under consideration. In order to obtain the required analytic continuation of the function $Z_2(s)$ it as sufficient to express the sum over $n$ in (3.7) in terms of the Riemann zeta function

$$\sum_{n=1}^\infty \frac{1}{n^s} = \zeta(z), \quad (3.13)$$

and the integral in (3.7) in terms of the Euler gamma function using the equality

$$\int_0^\infty dy \, y^{1-s} \frac{d}{dy} t^{2(\rho-1)} = (1 - \rho) \frac{\Gamma \left( \frac{3-s}{2} \right) \Gamma \left( \rho - \frac{3-s}{2} \right)}{\Gamma(\rho)}, \quad 3 - 2 \text{Re} \, \rho < \text{Re} \, s < 3. \quad (3.14)$$
It gives
\[ Z_2(s) = \frac{1}{4} \left[ \zeta(s + 1) + \frac{1}{2} \right] \Gamma\left(\frac{3-s}{2}\right) \Gamma\left(\frac{1+s}{2}\right) \left[ -1 + 3(1+s) - \frac{5}{8}(3+s)(1+s) \right]. \] (3.15)

Making use of Eq. (3.14) for integrating in (3.8) we get
\[ Z_3(s) = \frac{1}{32} \zeta(s + 3) \Gamma\left(\frac{3-s}{2}\right) \left[ -13 \Gamma\left(\frac{3+s}{2}\right) + 142 \Gamma\left(\frac{5+s}{2}\right) - 177 \Gamma\left(\frac{7+s}{2}\right) \right. \\
\left. + \frac{113}{2} \Gamma\left(\frac{9+s}{2}\right) - \frac{113}{24} \Gamma\left(\frac{11+s}{2}\right) \right]. \] (3.16)

Keeping in mind that we have introduced the photon mass \( \mu \) into Eq. (3.6) we can substitute here the sum in terms of the Riemann zeta function according to Eq. (3.13). After that the photon mass can be put to zero. In order to obtain the required analytic continuation of the integral in (3.6) we first expand the logarithm
\[ \ln\frac{y^2}{1+y^2} = \ln\left(1 - \frac{1}{1+y^2}\right) = - \sum_{m=1}^{\infty} \frac{1}{m(1+y^2)^m}. \] (3.17)

After that one can carry out the integration in (3.6) with the result
\[ Z_1(s) = \frac{1-s}{2} \zeta(s - 1) \Gamma\left(\frac{1-s}{2}\right) \sum_{m=1}^{\infty} \frac{1}{m} \frac{\Gamma\left(m - \frac{1-s}{2}\right)}{\Gamma(m)}. \] (3.18)

In order for the domain of the convergence of the series in (3.18) to be determined it is convenient to use the formula 8.328.2 from Ref. 11
\[ \frac{\Gamma(m+z)}{\Gamma(m)} \bigg|_{m \to \infty} \to \frac{1}{m^{(1-z)/2}}. \] (3.19)

From Eq. (3.19) it follows that the series (3.18) converges when Re \( s < 1 \). But it is not dangerous now because we have replaced the sum in Eq. (3.6) in terms of the Riemann zeta function \( \zeta(s - 1) \) which is defined everywhere except for the point \( s = 2 \).

Finally Eqs. (3.4), (3.9), (3.12), (3.15), (3.16), and (3.18) define the spectral zeta function \( \zeta_{\text{cyl}}(s) \) as an analytic function of the complex variable \( s \) in the region \( \Omega \). Here it is worth noting that the analytic continuation does not ensure that in the region \( \Omega \), we are interested in, the spectral zeta function \( \zeta_{\text{cyl}}(s) \) is free of singularities. Really Eqs. (3.13) and (3.14) used for analytic continuation contain the Riemann zeta function \( \zeta(s) \) and the Euler gamma function \( \Gamma(s) \) having singularities (poles) at certain isolated points. Therefore, \( \zeta_{\text{cyl}}(s) \) considered in \( \Omega \) may also possess the singularities of the same type. As the analytic continuation is unique the removal of these divergences is obviously impossible. This manifests the inability of the present approach to remove the divergences in all the cases. If the considered quantity is expressed via the value of the spectral zeta function at its singularity point then the zeta function technique does not give a finite answer.

In order to obtain the Casimir energy of the massless scalar field obeying the Dirichlet boundary conditions on an infinite cylinder of radius \( a \) let us calculate, according to (2.7), the spectral zeta function \( \zeta_{\text{cyl}}(s) \) at the point \( s = -1 \). Numerical integration in (3.12) yields
\[ Z_0(-1) = -0.021926 + \frac{3}{4} - \frac{5}{16} = 0.415574. \] (3.20)
Now we turn to Eq. (3.18) with \( s \) tending to \(-1\)

\[
Z_1(-1) = \lim_{s \to -1} \zeta(s - 1) \left[ \Gamma\left(\frac{1+s}{2}\right) + \sum_{m=2}^{\infty} \frac{1}{m(m-1)} \right].
\]

Keeping in mind the relations

\[
\Gamma(x) = \frac{1}{x} - \gamma + O(x),
\]

\[
\sum_{m=2}^{\infty} \frac{1}{m(m-1)} = 1, \quad \zeta(-2) = 0,
\]

where \( \gamma \) is the Euler constant, \( \gamma = 0.577215 \ldots \), one derives

\[
Z_1(-1) = 2 \lim_{s \to -1} \frac{\zeta(s - 1)}{1 + s} + \zeta(-2)(1 - \gamma) = 2 \zeta'(-2) = -0.060897.
\]

Now we evaluate the value of \( Z_2(-1) \) using Eq. (3.15) and taking into account that \( \Gamma((1 + s)/2) \) has a pole at the point \( s = -1 \) (see Ref. 5)

\[
Z_2(-1) = \frac{1}{4} \lim_{s \to -1} \left[ \zeta(s + 1) + \frac{1}{2} \right] \Gamma\left(\frac{1+s}{2}\right)
= \frac{1}{4} \lim_{s \to -1} \left[ \zeta(0) + \zeta'(0)(s + 1) + O\left((s + 1)^2\right) + \frac{1}{2}\right] \cdot \left[ \frac{2}{1 + s} - \gamma + O\left(\frac{1+s}{2}\right) \right]
= -\frac{1}{4} 2 \zeta'(0) = -\frac{1}{4} \ln(2\pi).
\]

Here we have used the values of the Riemann zeta function and its derivative at the origin

\[
\zeta(0) = -\frac{1}{2}, \quad \zeta'(0) = -\frac{1}{2} \ln(2\pi).
\]

The formula (3.16) gives the following value for \( Z_3(-1) \)

\[
Z_3(-1) = \frac{1}{32} \zeta(2) = \frac{\pi^2}{192} = 0.051404.
\]

One can make this result more precise. The point is that, from the very beginning, a complete expression

\[
\tilde{Z}_3(s) = 2 \sum_{n=1}^{\infty} n^{1-s} \int_{0}^{\infty} dy y^{1-s} \frac{d}{dy} \left[ \ln(2yn L_n(y)K_n(ny)) - \ln \frac{y}{\sqrt{1+y^2}} - \frac{t^2(1 - 6t^2 + 5t^4)}{8n^2} \right]
\]

(3.27)

can be considered instead of the function \( Z_3(s) \) defined by (3.8). The formula (3.27) reduces to (3.8) after substituting the logarithm by its uniform asymptotic expansion (3.3). It is easy to show that \( \tilde{Z}_3(s) \) is an analytic function of \( s \) when \(-3 < \text{Re} s < 1\). It means that \( \tilde{Z}_3(s) \), as well as \( Z_3(s) \), does not need analytic continuation. In practice Eq. (3.27) is used for several first values of \( n, n \leq n_0 \), and for \( n > n_0 \) one applies (3.8). The reason is that the uniform asymptotic expansion does not provide sufficient accuracy when \( n < n_0 = 6 \div 10 \).

Evaluating \( Z_3(-1) \) according to this algorithm with \( n_0 = 6 \) we obtain an improved value (compare with (3.26))

\[
\tilde{Z}_3(-1) = 0.045611.
\]
For the remaining coefficient $C(s = -1)$ in (3.4) one finds

$$C(-1) = -\frac{1}{4\pi a^2}. \quad (3.29)$$

Finally, summing up Eqs. (3.20), (3.23), (3.24), (3.28), and (3.29) we get for $\zeta_{\text{cyl}}^D(-1)$

$$\zeta_{\text{cyl}}^D(-1) = -\frac{1}{4\pi a} \left[ 0.415574 - 0.060897 - \frac{1}{4} \ln(2\pi) + 0.045611 \right] = \frac{0.001213}{a^2}. \quad (3.30)$$

It gives the following value for the Casimir energy of massless scalar field obeying the Dirichlet boundary conditions on the lateral area of an infinite cylinder of radius $a$

$$E_{\text{cyl}}^D = \frac{1}{2} \zeta_{\text{cyl}}^D(-1) = \frac{0.00606}{a^2}. \quad (3.31)$$

It is not necessary to calculate the spectral zeta function for the Neumann boundary conditions $\zeta_{\text{cyl}}^N(s)$. The point is that in Ref. the spectral zeta function for the electromagnetic field with boundary conditions defined on an infinitely thin perfectly conducting cylindrical shell was constructed. This zeta function is the sum of two spectral zeta functions for scalar fields obeying the Dirichlet and Neumann boundary conditions on a cylinder. Therefore

$$\zeta_{\text{cyl}}^N(s) = \zeta_{\text{cyl}}^\text{shell}(s) - \zeta_{\text{cyl}}^D(s). \quad (3.32)$$

We shall not quote here the expression for $\zeta_{\text{cyl}}^\text{shell}(s)$ found in Ref. (see the next section). Taking into account Eq. (3.32) at $s = -1$ we obtain the Casimir energy of massless scalar field with Neumann boundary conditions on an infinite cylinder

$$E_{\text{cyl}}^N = E_{\text{cyl}}^\text{EM} - E_{\text{cyl}}^D = -\frac{0.01356}{a^2} - \frac{0.00061}{a^2} = -\frac{0.01417}{a^2}. \quad (3.33)$$

Here we have borrowed the value of the Casimir energy $E_{\text{cyl}}^\text{shell}$ from Ref. where its consistent derivation is presented. Recently the values of the zeta functions $\zeta_{\text{cyl}}^D(-1)$ and $\zeta_{\text{cyl}}^N(-1)$ were evaluated with higher accuracy (see the Conclusion).

**IV. SPECTRAL ZETA FUNCTIONS FOR A CIRCLE**

Having defined the zeta functions for an infinite cylinder $\zeta_{\text{cyl}}^D(s)$ and $\zeta_{\text{cyl}}^N(s)$ in the region $\Omega$ of the complex plane $s$, containing the segment of the real axis $-1 \leq \text{Re } s \leq 0$, we can at once obtain the zeta functions for a circle, $\zeta_{\text{cir}}^D(s)$ and $\zeta_{\text{cir}}^N(s)$, making use of Eq. (2.6)

$$\zeta_{\text{cir}}^{D,N}(s) = 2\sqrt{\pi} \frac{\Gamma\left(\frac{s + 1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \zeta_{\text{cyl}}^{D,N}(s + 1). \quad (4.1)$$

It is important to note that here there is no need in additional calculations or analytic continuation because the values of $\zeta_{\text{cir}}^{D,N}(-1)$, defining the relevant Casimir energies, are expressed, according to (4.1), in terms of $\zeta_{\text{cyl}}^{D,N}(0)$.

Let us first derive $\zeta_{\text{cir}}^D(-1)$ substituting Eqs. (3.4) and (3.9) into (4.1)

$$\zeta_{\text{cir}}^D(-1) = -\frac{1}{\pi} \lim_{s \to 0} \sum_{i=0}^{3} Z_i(s). \quad (4.2)$$
Numerical calculation and integration according to (3.14) in (3.12) with \( s = 0 \) gives

\[
Z_0(0) = - \int_0^\infty dy \left[ \ln(2yI_0(y)K_0(y)) - \frac{I^2}{8}(1 - 6t^2 + 5t^4) \right] = \pi \left( 0.02815 - \frac{1}{128} \right). \tag{4.3}
\]

In Eq. (3.6) defining \( Z_1(s) \) we put \( s = 0 \) and integrate by parts

\[
Z_1(0) = -2 \zeta(-1) \int_0^\infty dy \ln \frac{y}{\sqrt{1 + y^2}} = -2 \left( -\frac{1}{12} \right) \left( -\frac{\pi}{2} \right) = -\frac{\pi}{12}. \tag{4.4}
\]

Without pretending to high accuracy the value of \( Z_3(0) \) can be evaluated by making use of Eq. (3.16)

\[
Z_3(0) = \frac{1}{32} \zeta(3) \Gamma \left( \frac{3}{2} \right) \left[ -13 \Gamma \left( \frac{3}{2} \right) + 142 \Gamma \left( \frac{5}{2} \right) - \frac{1062}{6} \Gamma \left( \frac{7}{2} \right) + \frac{1356}{24} \Gamma \left( \frac{9}{2} \right) \right] - \frac{565}{720} \Gamma \left( \frac{11}{2} \right) = \frac{\pi}{64} (-0.136719) \zeta(3). \tag{4.5}
\]

The function \( Z_2(s) \) determined in (3.15) has a pole at the point \( s = 0 \) because

\[
\zeta(1 + s) \simeq \frac{1}{s} + \gamma + \ldots, \quad s \to 0. \tag{4.6}
\]

Therefore, we can only extract the finite and divergent parts in \( Z_2(0) \)

\[
Z_2(0) = \frac{1}{4} \Gamma \left( \frac{3}{2} \right) \Gamma \left( \frac{1}{2} \right) \frac{1}{8} \left( \lim_{s \to 0} \zeta(1 + s) + \frac{1}{2} \right) = \frac{\pi}{64} \left( \frac{1}{s} \bigg|_{s \to 0} + \gamma \right) + \frac{\pi}{128}. \tag{4.7}
\]

Finally, substituting Eqs. (4.3), (4.4), (4.5), and (4.7) into Eq. (4.2) we get

\[
\zeta^D(\text{circ})(-1) = -\frac{1}{\pi} \left[ 0.028156 \pi + \zeta(-1) \pi - \frac{\pi}{64} 0.136719 \zeta(3) + \frac{\pi}{64} \left( \frac{1}{s} \bigg|_{s \to 0} + \gamma \right) \right] = \frac{1}{a} \left( 0.047189 - \frac{1}{64} \frac{1}{s} \bigg|_{s \to 0} \right). \tag{4.8}
\]

It gives the following result for the Casimir energy of the scalar massless field obeying the Dirichlet boundary conditions on a circle

\[
E^D_{\text{circ}} = \frac{1}{2} \zeta^D(\text{circ})(-1) = \frac{1}{a} \left( 0.0023595 - \frac{1}{128} \frac{1}{s} \bigg|_{s \to 0} \right). \tag{4.9}
\]

As in the case of an infinite cylinder, it is convenient first to construct the sum of two zeta functions for the Dirichlet and Neumann boundary conditions \( \zeta^D+N_{\text{circ}}(s) \) and then find \( \zeta^N_{\text{circ}}(s) \) as a difference \( \zeta^D+N_{\text{circ}}(s) - \zeta^D_{\text{circ}}(s) \). The zeta function \( \zeta^D+N_{\text{circ}}(s) \) is again expressed through the corresponding zeta function of a cylinder

\[
\zeta^D+N_{\text{circ}}(s) = 2\sqrt{\pi} \frac{\Gamma \left( \frac{s}{2} + \frac{1}{2} \right)}{\Gamma \left( \frac{s}{2} \right)} \zeta^\text{shell}(s+1), \tag{4.10}
\]
where \( \zeta_{\text{shell}}(s) \) is the zeta function of electromagnetic field with boundary conditions defined on a surface of a perfectly conducting cylindrical shell, \( \zeta_{\text{shell}}(s) = \zeta_{D}(s) + \zeta_{N}(s) \). The spectral zeta function \( \zeta_{\text{shell}}(s) \) has been explicitly constructed in Ref. 5. The relevant formulas read

\[
\zeta_{\text{shell}}(s) = \tilde{Z}_{1}(s) + \tilde{Z}_{2}(s) + \tilde{Z}_{3}(s),
\]

\[
\tilde{Z}_{1}(s) = \frac{(s - 1)a^{s-1}}{2\sqrt{\pi}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{3-s}{2}\right)} \int_{0}^{\infty} dy y^{-s} \left\{ \ln[1 - \mu_{0}^{2}(y)] + \frac{y^{4}t_{6}(y)}{4} \right\},
\]

\[
\tilde{Z}_{2}(s) = \frac{(1-s)(3-s)a^{s-1}}{64\sqrt{\pi}} [2\zeta(s+1) + 1] \frac{\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)},
\]

\[
\tilde{Z}_{3}(s) = \frac{(1-s)(3-s)(71s^{2} - 52s - 235)a^{s-1}}{61440\sqrt{\pi}} \zeta(s+3) \frac{\Gamma\left(\frac{3+s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)},
\]

where \( \mu_{0}(y) = y(I_{0}(y)K_{0}(y))' \).

We are again interested in the value of \( \zeta_{\text{cir}}^{D+N}(s) \) at the point \( s = -1 \) which is expressed through \( \zeta_{\text{shell}}(0) \) according to (4.10). The functions \( \tilde{Z}_{1}(s) \) and \( \tilde{Z}_{3}(s) \) give the finite contributions to \( \zeta_{\text{cir}}^{D+N}(-1) \)

\[
- \frac{1}{a} 0.531627 \quad \text{and} \quad \frac{1}{a} 0.006896,
\]

respectively. However, as in the case of the Dirichlet boundary conditions, \( Z_{2}(s) \) gives a pole-like contribution to (4.10)

\[
- \frac{3}{32a} \left( \frac{1}{s} \right)_{s \to 0} + \gamma + \frac{1}{2}.
\]

Hence, for the Casimir energy in question one obtains

\[
E_{\text{cir}}^{D+N} = \frac{1}{2} \zeta^{D+N}(-1) = - \frac{3}{64a} \frac{1}{s} \bigg|_{s \to 0} - \frac{1}{a} 0.2895.
\]

Finding the difference between (4.17) and (4.9) we arrive at the Casimir energy of scalar massless field obeying the Neumann boundary conditions on a circle

\[
E_{\text{cir}}^{N} = \frac{1}{a} \left( -0.3131 - \frac{5}{128} \frac{1}{s} \bigg|_{s \to 0} \right).
\]

The zeta function technique does not lead to a finite answer for the Casimir energy in the plane problem considered here (two space-like dimensions), as well as in all the cases of arbitrary even space dimensions. As usual the coefficients in front of the pole-like contributions, calculated by different methods coincide, but the finite parts of the answers differ. In this respect our consideration has a certain advantage, because when calculating the Casimir energy of the fields on a plane we employ Eq. (4.1). Thereby we in fact make use of the spectral zeta function \( \zeta_{\text{cyl}}(s) \) for a cylinder which has already been "normalized" by a finite answer for the Casimir energy of an infinite cylinder.

Of course the problem of the Casimir energy calculation in the even dimensional spaces is far from being completely solved. To obtain an acceptable finite answer for this energy one should invoke some additional physical arguments providing the removal of the pole-like contributions from Eqs. (4.9), (4.17), and (4.18) or use new mathematical methods which will not result in such divergent terms.
V. CONCLUSION

In the present paper the explicit expressions are derived which define the spectral zeta functions for an infinite cylinder \( \zeta_{\text{cyl}}(s) \) in a finite range of complex variable \( s \) containing the segment of real axis \(-1 \leq \text{Re} \ s \leq 0\). It enables one to find the spectral zeta function for a circle making use of the zeta function for an infinite cylinder according to the relation (2.6). In Ref. 8 this relation was applied directly, i.e., for constructing \( \zeta_{\text{cyl}}(s) \) from \( \zeta_{\text{cir}}(s) \). However, to obtain the value of \( \zeta_{\text{cyl}}(s) \) at the point \( s = -1 \), the authors of this paper have to make additional analytic continuation of the function \( \zeta_{\text{cir}}(s) \) from the neighborhood of the point \( s = -1 \) to the point \( s = -2 \).

In our consideration, as well as in treatment of the analogous problems by other authors, the central part was played by the uniform asymptotic expansion for the product of the Bessel functions. The lack of such expansions for the eigenfunctions in problems with other geometry of the boundaries (for example, with boundary conditions defined on the surface of a spheroid) does not permit to expand directly this approach beyond the systems with spherical symmetry.

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