Abstract

The training of artificial neural networks (ANNs) is nowadays a highly relevant algorithmic procedure with many applications in science and industry. Roughly speaking, ANNs can be regarded as iterated compositions between affine linear functions and certain fixed nonlinear functions, which are usually multidimensional versions of a one-dimensional so-called activation function. The most popular choice of such a one-dimensional activation function is the rectified linear unit (ReLU) activation function which maps a real number to its positive part \( \mathbb{R} \ni x \mapsto \max\{x, 0\} \in \mathbb{R} \).

In this article we propose and analyze a modified variant of the standard training procedure of such ReLU ANNs in the sense that we propose to restrict the negative gradient flow dynamics to a large submanifold of the ANN parameter space, which is a strict \( C^{\infty} \)-submanifold of the entire ANN parameter space that seems to enjoy better regularity properties than the entire ANN parameter space but which is also sufficiently large and sufficiently high dimensional so that it can represent all ANN realization functions that can be represented through the entire ANN parameter space. In the special situation of shallow ANNs with just one-dimensional ANN layers we also prove for every Lipschitz continuous target function that every gradient flow trajectory on this large submanifold of the ANN parameter space is globally bounded. For the standard gradient flow on the entire ANN parameter space with Lipschitz continuous target functions it remains an open problem of research to prove or disprove the global boundedness of gradient flow trajectories even in the situation of shallow ANNs with just one-dimensional ANN layers.
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1 Introduction

The training of deep artificial neural networks (ANNs) is nowadays a highly relevant technical procedure with many applications in science and industry. In the most simple form we can think of a deep ANN as a tuple of real numbers describing a function, the so-called realization function of the ANN, which consists of multiple compositions of affine linear functions and certain fixed nonlinear functions. To be more specific, the realization function of such an ANN with \( L \in \mathbb{N} \cap (1, \infty) = \{2, 3, 4, \ldots\} \) affine linear transformations and layer dimensions \( \ell_0, \ell_1, \ldots, \ell_L \in \mathbb{N} = \{1, 2, 3, \ldots\} \) is given through an affine linear function from \( \mathbb{R}^{\ell_0} \) to \( \mathbb{R}^{\ell_1} \) (1st affine linear transformation), then a fixed nonlinear function from \( \mathbb{R}^{\ell_1} \) to \( \mathbb{R}^{\ell_1} \), then again an affine linear function from \( \mathbb{R}^{\ell_1} \) to \( \mathbb{R}^{\ell_2} \) (2nd affine linear transformation), then again a fixed nonlinear function from \( \mathbb{R}^{\ell_2} \) to \( \mathbb{R}^{\ell_2} \), \ldots and, finally, an affine linear function from \( \mathbb{R}^{\ell_{L-1}} \) to \( \mathbb{R}^{\ell_L} \) (\( L \)-th affine linear transformation). There are thus \( \ell_0 \ell_1 + \ell_1 \) real numbers to describe the 1st affine linear transformation in the ANN (affine linear transformation from \( \mathbb{R}^{\ell_0} \) to \( \mathbb{R}^{\ell_1} \)), there are thus \( \ell_1 \ell_2 + \ell_2 \) real numbers to describe the 2nd affine linear transformation in the ANN (affine linear transformation from \( \mathbb{R}^{\ell_1} \) to \( \mathbb{R}^{\ell_2} \)), \ldots and there are thus \( \ell_{L-1} \ell_L + \ell_L \) real numbers to describe the \( L \)-th affine linear transformation (affine linear transformation from \( \mathbb{R}^{\ell_{L-1}} \) to \( \mathbb{R}^{\ell_L} \)). The overall number \( d \in \mathbb{N} \) of real ANN parameters thus satisfies

\[
d = \sum_{k=1}^{L} (\ell_{k-1} \ell_k + \ell_k) = \sum_{k=1}^{L} \ell_k(\ell_{k-1} + 1). \tag{1.1}
\]

We also refer to Figure 1 for a graphical illustration of the architecture of such an ANN.
Figure 1: Graphical illustration for the architecture of an ANN with $L \in \mathbb{N} \cap (1, \infty)$ affine linear transformations, with $\ell_0 \in \mathbb{N}$ neurons on the input layer, with $\ell_1$ neurons on the 1st hidden layer, with $\ell_2$ neurons on the 2nd hidden layer, ..., with $\ell_{L-1}$ neurons on the $(L-1)^{th}$ hidden layer, and with $\ell_L$ neurons on the output layer.

The nonlinear functions in between the affine linear transformation are usually multi-dimensional versions of a fixed one-dimensional function $a : \mathbb{R} \to \mathbb{R}$ in the sense that the nonlinear function after the $k$-th affine linear transformation with $k \in \{1, 2, \ldots, L-1\}$ is the function from $\mathbb{R}^{\ell_k}$ to $\mathbb{R}^{\ell_k}$ given by

$$\mathbb{R}^{\ell_k} \ni (x_1, \ldots, x_{\ell_k}) \mapsto (a(x_1), \ldots, a(x_{\ell_k})) \in \mathbb{R}^{\ell_k}$$

and the one-dimensional function $a : \mathbb{R} \to \mathbb{R}$ is then referred to as activation function of the considered ANN. In numerical simulations maybe the most popular choice for the activation function $a : \mathbb{R} \to \mathbb{R}$ in (1.2) is the ReLU activation function which is given by

$$\mathbb{R} \ni x \mapsto \max\{x, 0\} \in \mathbb{R}.$$

There are also very good analytical reasons why the ReLU activation function in (1.3) seems to be so popular in numerical simulations. More formally, in the case of the ReLU activation in (1.3) it has been proven (see [10]) for Lipschitz continuous target functions that there exist global minimum points in the risk landscape in the training of ANNs in the shallow situation $(L, \ell_0, \ell_1, \ell_2) \in \{2\} \times \{1\} \times \mathbb{N} \times \{1\}$ while for other smooth activation functions such as the standard logistic activation function $\mathbb{R} \ni x \mapsto (1 + \exp(-x))^{-1} \in \mathbb{R}$ the existence of global minimum points has been disproven (see [5, 12]) and the existence of global minimum points in the risk landscape, in turn, seems to be closely related to the boundedness of gradient descent (GD) trajectories; see [5].

Despite the common usage of the ReLU activation function in deep ANNs, it remains an open problem of research to rigorously prove (or disprove) the convergence of GD trajectories. This lack of theoretical understanding applies to the ReLU activation function but also to other activation function. While for other smooth activation function, the boundedness of GD trajectories is often not even expected (see [5]), for the
ReLU activation function it remains an open problem to prove (or disprove) the boundedness of GD trajectories in the training of ReLU ANNs. Another key difficulty in the mathematical analysis of the training process of ReLU ANNs is the fact that the ReLU activation in (1.3) fails to be differentiable at 0 and this lack of differentiability of the activation function transfers to the risk function, which, in turn, makes it difficult to analyze time-discrete GD processes as analyses of such methods rely on local Lipschitz continuity properties of the gradient of the risk function (see, e.g., [1, 10, 11]).

In this article we propose and analyze a modified variant of the standard training process of ReLU ANNs. More formally, in this work we modify the usual gradient flow dynamics in a way so that the gradient flow remains the entire training process on a large submanifold of the ANN parameter space \( \mathbb{R}^d \). Specifically, in this work we consider a suitable \((d - \sum_{k=1}^{L-1} \ell_k)\)-dimensional \( C^\infty \)-submanifold of the \( d \)-dimensional ANN parameter space \( \mathbb{R}^d \) and modify the gradient flow dynamics in a way so that the modified gradient flow remains on this submanifold.

The advantages of this gradient descent dynamics on this \((d - \sum_{k=1}^{L-1} \ell_k)\)-dimensional \( C^\infty \)-submanifold of the ANN parameter space \( \mathbb{R}^d \) are

(i) that the risk function seems to have better differentiability properties than on the whole ANN parameter space \( \mathbb{R}^d \) and

(ii) that nearly all parameters on the submanifold are bounded and it thus seems to be easier to verify the boundedness of gradient flow trajectories on this submanifold.

In particular, in the special shallow ANN situation \((L, \ell_0, \ell_1, \ell_2) = (2, 1, 1, 1)\) we rigorously prove for every Lipschitz continuous target function \( f \) the global boundedness of every gradient flow trajectory; see Theorem 3.17 in Section 3 below. For the standard gradient flow on the entire ANN parameter space \( \mathbb{R}^d \) with Lipschitz continuous target functions it remains an open problem of research to prove or disprove the global boundedness of gradient flow trajectories even in the special shallow ANN situation shallow ANN situation \((L, \ell_0, \ell_1, \ell_2) = (2, 1, 1, 1)\).

Let us also add a few references which are more or less related to the approach proposed in this article. In a very vague sense the approach in this article is related to the famous batch normalization procedure (see Ioffe & Szegedy [8]) in the sense that in the batch normalization approach the data processed through the different ANN layers are normalized in a certain sense while in this article not the data processed through the ANN layers but the ANN parameters itself are normalized in a suitable sense.

As mentioned above, the risk function along the modified gradient flow trajectory appears to have better smoothness properties than on the entire ANN parameter space. This is due to the fact that, roughly speaking, the input parameters of each hidden neuron have constant non-zero norm along the entire trajectory. The fact that in the case of shallow ANNs with one hidden layer certain differentiability properties can be ensured if one assumes that the inner ANN parameters are bounded away from zero in a suitable sense has previously been observed in, e.g., Chizat & Bach [3], Wojtowytsch [14], and [9, Proposition 2.11].

The remainder of this article is organized as follows. In Section 2 we describe the modified gradient flow optimization dynamics in the situation of general deep ANNs with an arbitrary large number \( L \in \mathbb{N} \cap (1, \infty) \) of affine linear transformations and arbitrary layer dimensions \( \ell_0, \ell_1, \ldots, \ell_L \in \mathbb{N} \). In Section 3 we consider the special situation of shallow ANNs with one-dimensional layer dimensions in the sense that \( L = 2 \) and
\(\ell_0 = \ell_1 = \ell_2 = 1\) and prove in Theorem 3.17 in this special situation for every Lipschitz continuous target function that every GF trajectory is globally bounded.

## 2 Normalized gradient flow optimization in the training of deep ReLU artificial neural networks (ANNs)

In this section we describe and study the modified gradient flow optimization dynamics in the situation of general deep ANNs with an arbitrary large number \(L \in \mathbb{N} \cap (1, \infty)\) of affine linear transformations and arbitrary layer dimensions \(\ell_0, \ell_1, \ldots, \ell_L \in \mathbb{N}\).

### 2.1 Gradient flow optimization on submanifolds on the ANN parameter space

In the following abstract result, Lemma 2.1, we introduce a modification of a standard gradient flow \((\Theta_t)_{t \in [0, \tau]}\) with \(\frac{d}{dt} \Theta_t = \mathcal{G}(\Theta_t)\) (see (2.2)) with the property that certain quantities \(\psi_k(\Theta_t) \in \mathbb{R}, k \in \{1, 2, \ldots, K\}\), are time-invariant. Roughly speaking, at each time \(t \in [0, \tau)\) the derivative vector \(\mathcal{G}(\Theta_t) \in \mathbb{R}^\theta\) is projected onto the tangent space to a certain submanifold of \(\mathbb{R}^\theta\) on which all \(\psi_k\) are constant. Intuitively, this causes the gradient flow to move only tangentially to the manifold and therefore the quantities \(\psi_k(\Theta_t)\) remain invariant.

**Lemma 2.1** (Gradient flow dynamics on submanifolds). Let \(\mathcal{G} : \mathbb{R}^\theta \to \mathbb{R}^\theta\) be measurable, let \(K \in \mathbb{N}\), for every \(k \in \{1, 2, \ldots, K\}\) let \(\psi_k : \mathbb{R}^\theta \to \mathbb{R}\) be continuously differentiable, assume for all \(k, l \in \{1, 2, \ldots, K\}, \theta \in \mathbb{R}^\theta\) with \(\min_{m \in \{1, 2, \ldots, K\}} \|\nabla \psi_m(\theta)\| > 0\) and \(k \neq l\) that

\[
\langle (\nabla \psi_k)(\theta), (\nabla \psi_l)(\theta) \rangle = 0,
\]

let \(\tau \in (0, \infty]\), and let \(\Theta \in C([0, \tau), \mathbb{R}^\theta)\) satisfy for all \(t \in [0, \tau)\) that \(\inf_{s \in [0, t]} \min_{k \in \{1, 2, \ldots, K\}} \|\nabla \psi_k(\Theta_s)\| > 0\), \(\int_0^t \|\mathcal{G}(\Theta_s)\| \, ds < \infty\), and

\[
\Theta_t = \Theta_0 + \int_0^t \left(\mathcal{G}(\Theta_s) - \sum_{k=1}^K \|\nabla \psi_k(\Theta_s)\|^{-2} \langle \mathcal{G}(\Theta_s), (\nabla \psi_k)(\Theta_s) \rangle (\nabla \psi_k)(\Theta_s)\right) \, ds. \tag{2.2}
\]

Then it holds for all \(k \in \{1, 2, \ldots, K\}\), \(t \in [0, \tau)\) that \(\psi_k(\Theta_t) = \psi_k(\Theta_0)\).

**Proof of Lemma 2.1.** Observe that (2.1) ensures for all \(k \in \{1, 2, \ldots, K\}, \theta \in \mathbb{R}^\theta\) with \(\min_{l \in \{1, 2, \ldots, K\}} \|\nabla \psi_l(\theta)\| > 0\) that

\[
\left\langle (\nabla \psi_k)(\theta), \mathcal{G}(\theta) - \sum_{l=1}^K \|\nabla \psi_l(\theta)\|^{-2} \langle \mathcal{G}(\theta), (\nabla \psi_l)(\theta) \rangle (\nabla \psi_l)(\theta) \right\rangle = 0.
\]

\[
\left\langle (\nabla \psi_k)(\theta), \mathcal{G}(\theta) \right\rangle = \sum_{l=1}^K \left\langle (\nabla \psi_k)(\theta), \|\nabla \psi_l(\theta)\|^{-2} \langle \mathcal{G}(\theta), (\nabla \psi_l)(\theta) \rangle (\nabla \psi_l)(\theta) \right\rangle
\]

\[
\left\langle (\nabla \psi_k)(\theta), \mathcal{G}(\theta) \right\rangle - \sum_{l=1}^K \left\|\nabla \psi_l(\theta)\|^{-2} \langle \mathcal{G}(\theta), (\nabla \psi_l)(\theta) \rangle \right\| (\nabla \psi_k)(\theta), (\nabla \psi_l)(\theta) \right\rangle
\]

\[
\left\langle (\nabla \psi_k)(\theta), \mathcal{G}(\theta) \right\rangle - \left\|\nabla \psi_k(\theta)\|^{-2} \langle \mathcal{G}(\theta), (\nabla \psi_k)(\theta) \rangle \right\| (\nabla \psi_k)(\theta), (\nabla \psi_k)(\theta) \right\rangle
\]

\[
\left\langle (\nabla \psi_k)(\theta), \mathcal{G}(\theta) \right\rangle - (\mathcal{G}(\theta), (\nabla \psi_k)(\theta)) = 0.
\]
The generalized chain rule and (2.2) hence imply that for all \( k \in \{1, 2, \ldots, K\} \), \( t \in [0, \tau) \) it holds that
\[
\psi_k(\Theta_t) = \psi_k(\Theta_0). \tag{2.4}
\]
The proof of Lemma 2.1 is thus complete.

### 2.2 Descent property for modified gradient flows

In this subsection we show in an abstract setting that the considered modified gradient flow still has a descent property in the sense that the value of the objective function \( \mathcal{L}(\Theta_t), t \in [0, \infty) \), is monotonically non-increasing in time. Notice that we do not assume that the objective function \( \mathcal{L} : U \to \mathbb{R} \) is continuously differentiable. Instead, we only assume that \( \mathcal{L} \) can be approximated by differentiable functions \( \mathcal{L}_r \in C^1(U, \mathbb{R}) \), \( r \in \mathbb{N} \), in a suitable sense (see below (2.5) in Proposition 2.2). This will be important when applying our results to the risk functions occurring in the training of ANNs with the non-differentiable ReLU activation. For the proof of Proposition 2.2 we will apply the generalized chain rule from Cheridito et al. [2, Lemma 3.3].

**Proposition 2.2** (Energy dynamics for modified gradient flows). Let \( \mathcal{D}, K \in \mathbb{N} \), for every \( k \in \{1, 2, \ldots, K\} \) let \( \phi_k : \mathbb{R}^d \to \mathbb{R} \) and \( \psi_k : \mathbb{R}^d \to \mathbb{R}^d \) be locally bounded and measurable, let \( U \subseteq \mathbb{R}^d \) be open, let \( \gamma \in C([0, \infty), [0, \infty)) \), \( \mathcal{L} \in C(U, \mathbb{R}) \), let \( \mathcal{G} : U \to \mathbb{R}^d \) be locally bounded and measurable, let \( \Theta \in C([0, \infty), U) \) satisfy for all \( t \in [0, \infty) \) that
\[
\Theta_t = \Theta_0 - \int_0^t \phi_k(\Theta_s) \psi_k(\Theta_s) \psi_k(\Theta_s) ds, \tag{2.5}
\]
and assume that there exist \( \mathcal{L}_r \in C^1(U, \mathbb{R}^d), r \in \mathbb{N} \), which satisfy for all compact \( K \subseteq U \) that \( \sup_{r \in \mathbb{N}} \sup_{\theta \in K} \| \nabla \mathcal{L}_r(\theta) \| < \infty \) and which satisfy for all \( \theta \in U \) that \( \lim_{r \to \infty} \mathcal{L}_r(\theta) = \mathcal{L}(\theta) \) and \( \lim_{r \to \infty} \nabla \mathcal{L}_r(\theta) = \mathcal{G}(\theta) \). Then it holds for all \( t \in [0, \infty) \) that
\[
\mathcal{L}(\Theta_t) = \mathcal{L}(\Theta_0) - \int_0^t \gamma(s) \left( \| \mathcal{G}(\Theta_s) \|^2 + K \sum_{k=1}^{K} \phi_k(\Theta_s) \psi_k(\Theta_s) \psi_k(\Theta_s) \right) ds. \tag{2.6}
\]

**Proof of Proposition 2.2.** Note that the assumption that \( \phi_k, k \in \{1, 2, \ldots, K\} \), \( \psi_k, k \in \{1, 2, \ldots, K\} \), and \( \mathcal{G} \) are locally bounded and measurable and the fact that \( \Theta \) is continuous show for all \( t \in [0, \infty) \) that \( [0, t] \ni s \mapsto \gamma(s)(\mathcal{G}(\Theta_s) + \sum_{k=1}^{K} \phi_k(\Theta_s) \psi_k(\Theta_s) \psi_k(\Theta_s)) \in \mathbb{R}^d \) is bounded and measurable. Combining this with (2.5) and the generalized chain rule (cf., e.g., Cheridito et al. [2, Lemma 3.3]) proves for all \( t \in [0, \infty) \), \( r \in \mathbb{N} \) that
\[
\mathcal{L}_r(\Theta_t) = \mathcal{L}_r(\Theta_0) - \int_0^t \gamma(s) \left( \langle \mathcal{G}(\Theta_s), \nabla \mathcal{L}_r(\Theta_s) \rangle + \sum_{k=1}^{K} \phi_k(\Theta_s) \psi_k(\Theta_s) \psi_k(\Theta_s) \right) ds \tag{2.7}
\]
In addition, observe that the fact that \( \Theta \) is continuous demonstrates for every \( t \in [0, \infty) \) that \( \{\Theta_s : s \in [0, t]\} \subseteq U \) is compact. Combining this with the assumption that for all compact \( K \subseteq U \) it holds that \( \sup_{r \in \mathbb{N}} \sup_{\theta \in K} \| \nabla \mathcal{L}_r(\theta) \| < \infty \), the assumption that for all \( \theta \in K \) it holds that \( \lim_{r \to \infty} \mathcal{L}_r(\theta) = \mathcal{L}(\theta) \) and \( \lim_{r \to \infty} \nabla \mathcal{L}_r(\theta) = \mathcal{G}(\theta) \), and the dominated convergence theorem establishes (2.6). The proof of Proposition 2.2 is thus complete. \( \square \)
Corollary 2.3 (Energy dynamics for modified gradient flows). Let $\mathfrak{d}, K \in \mathbb{N}$, for every $k \in \{1, 2, \ldots, K\}$ let $\psi_k : \mathbb{R}^{\mathfrak{d}} \to \mathbb{R}$ be locally bounded and measurable, let $U \subseteq \mathbb{R}^{\mathfrak{d}}$ be open, let $\gamma \in C([0, \infty), [0, \infty))$, $\mathcal{L} \in C(U, \mathbb{R})$, let $\mathcal{G} : U \to \mathbb{R}^{\mathfrak{d}}$ be locally bounded and measurable, let $\Theta \in C([0, \infty), U)$ satisfy for all $t \in [0, \infty)$ that

$$
\Theta_t = \Theta_0 - \int_0^t \gamma(s) \left( \mathcal{G}(\Theta_s) + \sum_{k=1}^{K} \langle \psi_k(\Theta_s), \mathcal{G}(\Theta_s) \psi_k(\Theta_s) \rangle \right) ds,
$$

and assume that there exist $\mathcal{L}_r \in C^1(U, \mathbb{R}^r)$, $r \in \mathbb{N}$, which satisfy for all compact $K \subseteq U$ that $\sup_{r \in \mathbb{N}} \sup_{\theta \in K} \| \nabla \mathcal{L}_r(\theta) \| < \infty$ and which satisfy for all $\theta \in U$ that $\lim_{r \to \infty} \mathcal{L}_r(\theta) = \mathcal{L}(\theta)$ and $\lim_{r \to \infty} \nabla \mathcal{L}_r(\theta) = \mathcal{G}(\theta)$. Then it holds for all $t \in [0, \infty)$ that

$$
\mathcal{L}(\Theta_t) = \mathcal{L}(\Theta_0) - \int_0^t \gamma(s) \left( \| \mathcal{G}(\Theta_s) \|^2 + \sum_{k=1}^{K} \| \langle \psi_k(\Theta_s), \mathcal{G}(\Theta_s) \psi_k(\Theta_s) \rangle \|^2 \right) ds.
$$

Proof of Corollary 2.3. Note that Proposition 2.2 establishes (2.9). The proof of Corollary 2.3 is thus complete. \qed

In the next result we apply the more general Proposition 2.2 to the modified gradient flow from Lemma 2.1. Using Parseval’s identity for the orthogonal gradient vectors $\nabla \psi_k(\Theta_t) \in \mathbb{R}^{\mathfrak{d}}$, $k \in \{1, 2, \ldots, K\}$, we establish that the value $\mathcal{L}(\Theta_t)$ is non-increasing in the time variable $t$.

Corollary 2.4 (Gradient flow dynamics on submanifolds). Let $\mathfrak{d}, K \in \mathbb{N}$, for every $k \in \{1, 2, \ldots, K\}$ let $\psi_k : \mathbb{R}^{\mathfrak{d}} \to \mathbb{R}$ be continuously differentiable, let $U \subseteq \mathbb{R}^{\mathfrak{d}}$ satisfy

$$
U = \{ \theta \in \mathbb{R}^{\mathfrak{d}} : \min_{k \in \{1, 2, \ldots, K\}} \| (\nabla \psi_k)(\theta) \| > 0 \},
$$

let $\mathcal{L} \in C(U, \mathbb{R})$, let $\mathcal{G} : U \to \mathbb{R}^{\mathfrak{d}}$ be locally bounded and measurable, assume for all $\theta \in U$, $k, l \in \{1, 2, \ldots, K\}$ with $k \neq l$ that

$$
\langle (\nabla \psi_k)(\theta), (\nabla \psi_l)(\theta) \rangle = 0,
$$

let $\gamma \in C([0, \infty), [0, \infty))$, $\Theta \in C([0, \infty), U)$ satisfy for all $t \in [0, \infty)$ that

$$
\Theta_t = \Theta_0 - \int_0^t \gamma(s) \left( \mathcal{G}(\Theta_s) - \sum_{k=1}^{K} \| (\nabla \psi_k)(\Theta_s) \|^{-2} \langle \mathcal{G}(\Theta_s), (\nabla \psi_k)(\Theta_s) \rangle (\nabla \psi_k)(\Theta_s) \right) ds,
$$

and assume that there exist $\mathcal{L}_r \in C^1(U, \mathbb{R}^r)$, $r \in \mathbb{N}$, which satisfy for all compact $K \subseteq U$ that $\sup_{r \in \mathbb{N}} \sup_{\theta \in K} \| \nabla \mathcal{L}_r(\theta) \| < \infty$ and which satisfy for all $\theta \in U$ that $\lim_{r \to \infty} \mathcal{L}_r(\theta) = \mathcal{L}(\theta)$ and $\lim_{r \to \infty} \nabla \mathcal{L}_r(\theta) = \mathcal{G}(\theta)$. Then

(i) it holds for all $k \in \{1, 2, \ldots, K\}$, $t \in [0, \infty)$ that $\psi_k(\Theta_t) = \psi_k(\Theta_0)$ and

(ii) it holds for all $s, t \in [0, \infty)$ with $s \leq t$ that

$$
\mathcal{L}(\Theta_t) = \mathcal{L}(\Theta_s) - \int_s^t \gamma(u) \left( \| \mathcal{G}(\Theta_u) \|^2 - \sum_{k=1}^{K} \| (\nabla \psi_k)(\Theta_u) \|^{-2} \| \langle \mathcal{G}(\Theta_u), (\nabla \psi_k)(\Theta_u) \rangle \| \right) du
$$

$$
= \mathcal{L}(\Theta_s) - \int_s^t \gamma(u) \| \mathcal{G}(\Theta_u) \| - \sum_{k=1}^{K} \| (\nabla \psi_k)(\Theta_u) \|^{-2} \| \langle \mathcal{G}(\Theta_s), (\nabla \psi_k)(\Theta_u) \rangle (\nabla \psi_k)(\Theta_u) \| \|^2 du
$$

$$
\leq \mathcal{L}(\Theta_s).
$$

(2.13)
Proof of Corollary 2.4. Throughout this proof for every \( t \in [0, \infty) \) let \( P_t : \mathbb{R}^p \to \mathbb{R}^p \) satisfy for all \( v \in \mathbb{R}^p \) that

\[
P_t(v) = \sum_{k=1}^{K} \frac{\| (\nabla \psi_k)(\Theta_t) \|^2 - 2 \langle (\nabla \psi_k)(\Theta_t), v \rangle (\nabla \psi_k)(\Theta_t)}{\| (\nabla \psi_k)(\Theta_t) \|^2 - 2 \langle (\nabla \psi_k)(\Theta_t), (\nabla \psi_k)(\Theta_t) \rangle}. \tag{2.14}
\]

Observe that Parseval’s identity ensures that for all \( t \in [0, \infty) \) it holds that

\[
\| P_t(v) \|^2 = \sum_{k=1}^{K} \left\| \frac{(\nabla \psi_k)(\Theta_t)}{\| (\nabla \psi_k)(\Theta_t) \|^2 - 2 \langle (\nabla \psi_k)(\Theta_t), v \rangle (\nabla \psi_k)(\Theta_t)} \right\|^2 \leq \| v \|^2. \tag{2.15}
\]

In addition, Lemma 2.1 and Proposition 2.2 imply for all \( s, t \in [0, \infty) \) with \( s \leq t \) that

\[
\mathcal{L}(\Theta_t) = \mathcal{L}(\Theta_s) - \int_s^t \gamma(u) \left( \| \mathcal{G}(\Theta_u) \|^2 - \sum_{k=1}^{K} \| (\nabla \psi_k)(\Theta_u) \|^2 - 2 \langle (\nabla \psi_k)(\Theta_u), (\nabla \psi_k)(\Theta_u) \rangle \right) du
\]

\[
= \mathcal{L}(\Theta_s) - \int_s^t \gamma(u) \left( \| \mathcal{G}(\Theta_u) \|^2 - \| P_u(\Theta_u) \|^2 \right) du \tag{2.16}
\]

The proof of Corollary 2.4 is thus complete. \( \square \)

Next, in Corollary 2.5 we specialize the above results to the case of a continuously differentiable objective function \( \mathcal{L} \in C^1(U, \mathbb{R}) \).

**Corollary 2.5** (Gradient flow dynamics on submanifolds, differentiable case). Let \( \vartheta, K \in \mathbb{N} \), for every \( k \in \{1, 2, \ldots, K\} \) let \( \psi_k : \mathbb{R}^p \to \mathbb{R} \) be continuously differentiable, let \( U \subseteq \mathbb{R}^p \) satisfy

\[
U = \{ \theta \in \mathbb{R}^p : \min_{k \in \{1, 2, \ldots, K\}} \| (\nabla \psi_k)(\theta) \| > 0 \}, \tag{2.17}
\]

let \( \mathcal{L} \in C^1(U, \mathbb{R}) \), assume for all \( \theta \in U, k, l \in \{1, 2, \ldots, K\} \) with \( k \neq l \) that

\[
\langle (\nabla \psi_k)(\theta), (\nabla \psi_l)(\theta) \rangle = 0, \tag{2.18}
\]

and let \( \gamma \in C([0, \infty), [0, \infty)) \), \( \Theta \in C([0, \infty), U) \) satisfy for all \( t \in [0, \infty) \) that

\[
\Theta_t = \Theta_0 - \int_0^t \gamma(s) \left( (\nabla \mathcal{L})(\Theta_s) - \sum_{k=1}^{K} \| (\nabla \psi_k)(\Theta_s) \|^2 - 2 \langle (\nabla \psi_k)(\Theta_s), (\nabla \psi_k)(\Theta_s) \rangle \right) ds. \tag{2.19}
\]

Then

(i) it holds for all \( k \in \{1, 2, \ldots, K\} \), \( t \in [0, \infty) \) that \( \psi_k(\Theta_t) = \psi_k(\Theta_0) \) and

(ii) it holds for all \( s, t \in [0, \infty) \) with \( s \leq t \) that

\[
\mathcal{L}(\Theta_t) = \mathcal{L}(\Theta_s) - \int_s^t \gamma(u) \left( \| (\nabla \mathcal{L})(\Theta_u) \|^2 - \sum_{k=1}^{K} \| (\nabla \psi_k)(\Theta_u) \|^2 - 2 \langle (\nabla \psi_k)(\Theta_u), (\nabla \psi_k)(\Theta_u) \rangle \right) du
\]

\[
= \mathcal{L}(\Theta_s) - \int_s^t \gamma(u) \left( \| (\nabla \mathcal{L})(\Theta_s) \|^2 - \sum_{k=1}^{K} \| (\nabla \psi_k)(\Theta_s) \|^2 - 2 \langle (\nabla \psi_k)(\Theta_s), (\nabla \psi_k)(\Theta_s) \rangle \right) \| (\nabla \psi_k)(\Theta_u) \|^2 du \leq \mathcal{L}(\Theta_s). \tag{2.20}
\]
Proof of Corollary 2.5. This is a special case of Corollary 2.4 (applied with \( L \cap L, G \cap \nabla L, (\mathcal{L}_r)_{r \in \mathbb{N}} \cap (\mathcal{L})_{r \in \mathbb{N}} \) in the notation of Corollary 2.4). The proof of Corollary 2.5 is thus complete.

In the final result of this subsection, Corollary 2.6, we show a modified version of Corollary 2.5. More specifically, we prove that the time-dependent factors \( \gamma(t) \in [0, \infty), \ t \in [0, \infty) \), can be chosen in such a way that the value \( L(\Theta_t), t \in [0, \infty) \), decreases at the same rate as for the standard gradient flow; see item (ii) below.

**Corollary 2.6** (Gradient flow dynamics on submanifolds). Let \( \mathfrak{d}, K \in \mathbb{N} \), for every \( k \in \{1, 2, \ldots, K\} \) let \( \psi_k : \mathbb{R}^q \to \mathbb{R} \) be continuously differentiable, let \( U \subseteq \mathbb{R}^q \) satisfy

\[
U = \{ \theta \in \mathbb{R}^q : \min_{k \in \{1, 2, \ldots, K\}} \| (\nabla \psi_k)(\theta) \| > 0 \}, \tag{2.21}
\]

assume for all \( \theta \in U, k, l \in \{1, 2, \ldots, K\} \) with \( k \neq l \) that

\[
\langle (\nabla \psi_k)(\theta), (\nabla \psi_l)(\theta) \rangle = 0, \tag{2.22}
\]

and let \( \gamma \in C([0, \infty), [0, \infty)), L \in C^1(U, \mathbb{R}), \Theta \in C([0, \infty), U) \) satisfy for all \( t \in [0, \infty) \)

\[
\gamma(t) \| (\nabla L)(\Theta_t) - \sum_{k=1}^{K} (\nabla \psi_k)(\Theta_t) \| \leq \| (\nabla L)(\Theta_t) \| \tag{2.23}
\]

and

\[
\Theta_t = \Theta_0 - \int_0^t \gamma(s) \left( (\nabla L)(\Theta_s) - \sum_{k=1}^{K} (\nabla \psi_k)(\Theta_s) \right) ds. \tag{2.24}
\]

Then

(i) it holds for all \( k \in \{1, 2, \ldots, K\}, t \in [0, \infty) \) that \( \psi_k(\Theta_t) = \psi_k(\Theta_0) \) and

(ii) it holds for all \( s, t \in [0, \infty) \) with \( s \leq t \) that

\[
L(\Theta_t) = L(\Theta_s) - \int_s^t \| (\nabla L)(\Theta_u) \|^2 du. \tag{2.25}
\]

**Proof of Corollary 2.6.** This is a special case of Corollary 2.5. The proof of Corollary 2.6 is thus complete.

**2.3 Normalized gradient descent in the training of deep ReLU ANNs**

In the following we introduce our notation for deep ANNs with ReLU activation. Setting 2.7 below is inspired by Hutzenreuter et al. [6, Setting 2.1].

In Setting 2.7 we first introduce the depth \( L \in \mathbb{N} \cap (1, \infty) \) of the considered ANN, the layer dimensions \( \ell_0, \ell_1, \ldots, \ell_L \in \mathbb{N} \), the continuously differentiable approximations \( \mathfrak{R}_r \in C^1(\mathbb{R}, \mathbb{R}), r \in \mathbb{N} \), for the ReLU activation function \( \mathfrak{R}_\infty(x) = \max\{x, 0\} \), the unnormalized probability distribution \( \mu : \mathcal{B}([a, b]^\ell_0) \to [0, \infty] \) of the input data, and the measurable target function \( f : [a, b]^\ell_0 \to \mathbb{R}^{\ell_L} \). Note that in the definition of the ANN realization functions in (2.30) we subtract in the last layer (the case \( k + 1 = L \)) the average value of the output of the previous layer with respect to the input distribution.
\( \mu \). In (2.31) we introduce the risk functions \( \mathcal{L}_r : \mathbb{R}^q \to \mathbb{R} \), \( r \in \mathbb{N} \cup \{\infty\} \), and we define the generalized gradient \( \mathcal{G} : \mathbb{R}^q \to \mathbb{R}^q \) as the pointwise limit of the approximate gradients \( \nabla \mathcal{L}_r : \mathbb{R}^q \to \mathbb{R}^q \) as \( r \to \infty \).

In (2.36) we inductively define the layer-wise rescaling operations \( \Psi_k : \mathbb{R}^q \to \mathbb{R}^q \), \( k \in \{0, 1, \ldots, L - 1\} \), which have the property that certain sub-vectors of the parameter vector \( \Psi_{L-1}(\theta) \) are modified in order to have norm 1 without changing the realization function; see Proposition 2.8 below for details. Finally, we define the modified gradient flow process \( \Theta : [0, \infty) \times \Omega \to \mathbb{R}^q \) with random initialization \( \xi : \Omega \to \mathbb{R}^q \) and the modified gradient descent process \( \Theta : \mathbb{N}_0 \times \Omega \to \mathbb{R}^q \).

**Setting 2.7.** Let \( a \in \mathbb{R}, b \in (a, \infty), (\ell_k)_{k \in \mathbb{N}_0} \subseteq \mathbb{N}, L \in \mathbb{N}\{1\} \) satisfy \( \Theta = \sum_{k=1}^{L} \ell_k(\ell_k - 1 + 1) \), for every \( \theta = (\theta_1, \ldots, \theta_L) \in \mathbb{R}^q \) let \( w^{k, \theta} = (w_{i,j}^{k, \theta})_{(i,j) \in \ell_k \times \ell_{k-1}} \in \mathbb{R}^{\ell_k \times \ell_{k-1}}, k \in \mathbb{N}, \) and \( b^{k, \theta} = (b_{i}^{k, \theta}, \ldots, b_{\ell_k}^{k, \theta}) \in \mathbb{R}^{\ell_k}, k \in \mathbb{N}, \) satisfy for all \( k \in \{1, \ldots, L\}, i \in \{1, \ldots, \ell_k\}, j \in \{1, \ldots, \ell_{k-1}\} \) that

\[
\begin{align*}
  w_{i,j}^{k, \theta} &= \theta_{(i-1)\ell_{k-1} + j + \sum_{h=1}^{k-1} \ell_h(\ell_h - 1 + 1)} \quad \text{and} \quad b_{i}^{k, \theta} &= \theta_{\ell_k \ell_{k-1} + i + \sum_{h=1}^{k-1} \ell_h(\ell_h - 1 + 1)},
\end{align*}
\]

(2.26)

for every \( k \in \mathbb{N}, \theta \in \mathbb{R}^q \) let \( A^\theta_k = (A^\theta_{k,1}, \ldots, A^\theta_{k,\ell_k}) : \mathbb{R}^{\ell_k - 1} \to \mathbb{R}^{\ell_k} \) satisfy for all \( x \in \mathbb{R}^{\ell_k-1} \) that

\[
A^\theta_k(x) = b^{k, \theta} + w^{k, \theta}x,
\]

(2.27)

let \( \mathcal{R}_r : \mathbb{R} \to \mathbb{R}, r \in \mathbb{N} \cup \{\infty\} \), satisfy for all \( x \in \mathbb{R} \) that \( (\bigcup_{r \in \mathbb{N}} \{\mathcal{R}_r\}) \subseteq C^1(\mathbb{R}, \mathbb{R}), \)

\[
\sup_{r \in \mathbb{N}} \sup_{y \in [-|x|,|x|]} \left| (\mathcal{R}_r)'(y) \right| < \infty, \mathcal{R}_\infty(x) = \max\{x, 0\}, \text{ and }
\]

\[
\lim_{r \to \infty} \sup_{x \in \mathbb{R}} \left[ \sum_{r=0}^{\infty} |(\mathcal{R}_r(x) - \mathcal{R}_\infty(x))| + |(\mathcal{R}_r)'(x) - 1(0,\infty)(x)| \right] = 0,
\]

(2.28)

for every \( r \in \mathbb{N} \cup \{\infty\}, k \in \mathbb{N} \) let \( \mathcal{M}_r^k : \mathbb{R}^q \to \mathbb{R}^q \) satisfy for all \( x = (x_1, \ldots, x_{\ell_k}) \in \mathbb{R}^{\ell_k} \) that

\[
\mathcal{M}_r^k(x) = (\mathcal{R}_r(x_1), \ldots, \mathcal{R}_r(x_{\ell_k})),
\]

(2.29)

let \( \mu : \mathcal{B}([a, b]^{\ell_0}) \to [0, \infty] \) be a finite measure, for every \( \theta \in \mathbb{R}^q, r \in \mathbb{N} \cup \{\infty\} \) let \( \mathcal{N}_r^{k, \theta} = (\mathcal{N}_r^{k,1}, \ldots, \mathcal{N}_r^{k,\ell_k}) : \mathbb{R}^q \to \mathbb{R}^{\ell_k}, k \in \mathbb{N}, \) satisfy for all \( k \in \mathbb{N}, i \in \{1, \ldots, \ell_k\}, x \in \mathbb{R}^q \) that \( \mathcal{N}_r^{k,1} = \mathcal{A}_1^\theta(x) \) and

\[
\mathcal{N}_{r+1,1}^{k, \theta}(x) = \mathcal{A}^\theta_{k+1}((\mathcal{M}_r^k \circ \mathcal{N}_r^{k, \theta})(x) - 1(1_{\ell_k}) \sum_{i=0}^{\ell_k} (\mathcal{M}_r^k \circ \mathcal{N}_r^{k, \theta})(y) \mu(dy)),
\]

(2.30)

let \( f = (f_1, \ldots, f_{\ell_k}) : [a, b]^{\ell_0} \to \mathbb{R}^q \) be measurable, for every \( r \in \mathbb{N} \cup \{\infty\} \) let \( \mathcal{L}_r : \mathbb{R}^q \to \mathbb{R} \) satisfy for all \( \theta \in \mathbb{R}^q \) that

\[
\mathcal{L}_r(\theta) = \int_{[a, b]^{\ell_0}} \|\mathcal{N}_r^{k, \theta}(x) - f(x)\|^2 \mu(dx),
\]

(2.31)

let \( \mathcal{G} = (G_1, \ldots, G_0) : \mathbb{R}^q \to \mathbb{R}^q \) satisfy for all \( \theta \in \{\theta \in \mathbb{R}^q : ((\nabla \mathcal{L}_r(\theta))_{r \in \mathbb{R}} \text{ is convergent}) \) that \( \mathcal{G}(\theta) = \lim_{r \to \infty} \nabla \mathcal{L}_r(\theta) \), for every \( k \in \mathbb{N}, \theta \in \mathbb{R}^q \), \( i \in \{1, \ldots, \ell_k\} \) let \( V^{k, \theta}_i = (V^{k, \theta}_{i,1}, \ldots, V^{k, \theta}_{i,\ell_k-1}) \in \mathbb{R}^{\ell_k - 1} \) satisfy

\[
V^{k, \theta}_i = (w^{k, \theta}_i, \ldots, w^{k, \theta}_{i,\ell_k}, b^{k, \theta}_i),
\]

(2.32)

for every \( k \in \mathbb{N}, i \in \{1, \ldots, \ell_k\} \) let \( \psi^k_i : \mathbb{R}^q \to \mathbb{R} \) satisfy for all \( \theta \in \mathbb{R}^q \) that \( \psi^{k, \theta}_i(\theta) = \|V^{k, \theta}_i\|^2,\) let \( \Lambda \subseteq \mathbb{N}^2 \) satisfy \( \Lambda = \bigcup_{k=1}^{L-1} (\{k\} \times \{1, 2, \ldots, \ell_k\}) \), let \( \rho : (\bigcup_{n \in \mathbb{N}} \mathbb{R}^n) \to (\bigcup_{n \in \mathbb{N}} \mathbb{R}^n) \) satisfy for all \( n \in \mathbb{N}, x \in \mathbb{R}^n \) that

\[
\rho(x) = \left[\|x\| + 1(0)(\|x\|)\right]^{-1} x,
\]

(2.33)
let $G : \mathbb{R}^p \to \mathbb{R}^p$ satisfy for all $\theta \in \mathbb{R}^p$ that
\[
G(\theta) = G(\theta) - \sum_{(k,i) \in \Lambda} \langle \rho((\nabla \psi^k_i)(\theta)), G(\theta) \rangle \rho((\nabla \psi^k_i)(\theta)),
\] (2.34)

let $\phi : \mathbb{R}^p \to \mathbb{R}^p$ satisfy for all $k \in \{1, \ldots, L\}, \theta \in \mathbb{R}^p, i \in \{1, \ldots, \ell_k\}$ that
\[
V^{k,\phi(\theta)}_i = \begin{cases} \rho(V^{k,\theta}_i) & : k < L \\ V^{k,\theta}_i & : k = L, \end{cases}
\] (2.35)

let $\Psi_k : \mathbb{R}^p \to \mathbb{R}^p, k \in \mathbb{N}_0$, satisfy for all $k, K \in \mathbb{N}, i \in \{1, \ldots, \ell_K\}, \theta \in \mathbb{R}^p$ that $\Psi_0(\theta) = \theta$ and
\[
V^{K,\Psi_k}_i(\theta) = \begin{cases} \rho(V^{K,\Psi_{k-1}}_i(\theta)) & : K = k \\ \text{diag}([\|V^{K,\Psi_{k-1}}_i(\theta)\|, \ldots, \|V^{K,\Psi_{k-1}}_i(\theta)\|, 1]) V^{K,\Psi_{k-1}}_i(\theta) & : K = k + 1 \\ V^{K,\Psi_{k-1}}_i(\theta) & : K \notin \{k, k + 1\}, \end{cases}
\] (2.36)

let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\xi : \Omega \to \mathbb{R}^p$ be a random variable, let $\Theta : [0, \infty) \times \Omega \to \mathbb{R}^p$ satisfy for all $t \in [0, \infty), \omega \in \Omega$ that $\int_0^t \|G(\Theta_s(\omega))\| \, ds < \infty$ and
\[
\Theta_t(\omega) = \Psi_{L-1}(\xi(\omega)) - \int_0^t G(\Theta_s(\omega)) \, ds,
\] (2.37)

let $(\gamma_n)_{n \in \mathbb{N}_0} \subseteq \mathbb{R}$, and let $\Theta_n : \Omega \to \mathbb{R}^p, n \in \mathbb{N}_0$, satisfy for all $n \in \mathbb{N}_0, \omega \in \Omega$ that $\Theta_0(\omega) = \Psi_{L-1}(\xi(\omega))$ and
\[
\Theta_{n+1}(\omega) = \phi(\Theta_n(\omega) - \gamma_n G(\Theta_n(\omega))).
\] (2.38)

Next, in Proposition 2.8 we verify some basic properties of the rescaling operation $\Psi_{L-1} : \mathbb{R}^p \to \mathbb{R}^p$. In particular, we show for every parameter vector $\theta \in \{\vartheta \in \mathbb{R}^p : \min_{(k,i) \in \Lambda} \|V^{k,\theta}_i\| > 0\}$ that the rescaled vector $\Psi_{L-1} \in \mathbb{R}^p$ is an element of a suitable $C^\infty$-submanifold of the parameter space $\mathbb{R}^p$; see items (i) and (iii). In addition, we demonstrate that the rescaling map $\Psi_{L-1}$ does not change the output of the considered ANN with ReLU activation; see item (ii).

**Proposition 2.8** (Properties of ANNs with normalized parameter vectors). **Assume Setting 2.7.** Then

(i) it holds that
\[
\{\theta \in \mathbb{R}^p : (\forall (k,i) \in \Lambda : \|V^{k,\theta}_i\| = 1)\}
\] (2.39)
is a $(\vartheta - (\sum_{k=1}^{L-1} \ell_k))$-dimensional $C^\infty$-submanifold of the $\mathbb{R}^p$,

(ii) it holds for all $\theta \in \mathbb{R}^p$ that
\[
\mathcal{N}^{L,\theta}_\infty = \mathcal{N}^{L,\Psi_{L-1}(\theta)}_\infty,
\] (2.40)
and

(iii) it holds for all $\theta \in \{\vartheta \in \mathbb{R}^p : \min_{(k,i) \in \Lambda} \|V^{k,\theta}_i\| > 0\}$, $(k,i) \in \Lambda$ that $\|V^{k,\Psi_{L-1}(\theta)}_i\| = 1$. 


Proof of Proposition 2.8. First, to show item (i) let $\mathcal{V}: \mathbb{R}^\theta \to \mathbb{R}^{#\Lambda}$ satisfy for all $\theta \in \mathbb{R}^\theta$ that $\mathcal{V}(\theta) = (\psi^k(\theta))_{(k,i) \in \Lambda}$. Note that $\mathcal{V} \in C^\infty(\mathbb{R}^\theta, \mathbb{R}^{#\Lambda})$. In addition, observe that for all $\theta \in \{ \theta \in \mathbb{R}^\theta : (\forall (k,i) \in \Lambda: \psi^k(\theta) \neq 0) \}$ it holds that $\operatorname{rank}(\mathcal{V}(\theta)) = #\Lambda = \sum_{k=1}^{L-1} \ell_k$. Combining this with the preimage theorem (cf., e.g., Tu [13, Theorem 9.9]) proves that
\begin{equation}
\{ \theta \in \mathbb{R}^\theta : (\forall (k,i) \in \Lambda: \| V_i^{k,\theta} \| = 1) \} = \{ \theta \in \mathbb{R}^\theta : \mathcal{V}(\theta) = (1,1,\ldots,1) \}
\end{equation}
is a $(\sigma - (\sum_{k=1}^{L-1} \ell_k))$-dimensional $C^\infty$-submanifold of the $\mathbb{R}^\theta$. This establishes item (i).

To prove item (ii) let $K \in \{1,2,\ldots,L-1\}$, $\theta \in \mathbb{R}^\theta$, $y \in \mathbb{R}^{\ell_0}$ be fixed and for every $\theta \in \mathbb{R}^\theta$ let $\mathcal{N}^{\theta_0}_\infty: \mathbb{R}^{\ell_0} \to \mathbb{R}^{\ell_0}$ satisfy for all $x \in \mathbb{R}^{\ell_0}$ that $\mathcal{N}^{\theta_0}_\infty(x) = x$. Note that the fact that for all $k \in \{1,2,\ldots,K-1\}$, $i \in \{1,2,\ldots,\ell_K\}$ it holds that $V_i^{k,\Psi_{K-1}(\theta)} = V_i^{k,\Psi_{K-1}(\theta)}$ demonstrates that $\mathcal{N}^{K-1,\Psi_{K}(\theta)}_\infty(y) = \mathcal{N}^{K-1,\Psi_{K-1}(\theta)}_\infty(y)$. Moreover, observe that (2.36) ensures for all $i \in \{1,2,\ldots,\ell_K\}$, $j \in \{1,2,\ldots,\ell_{K-1}\}$ that
\begin{equation}
\begin{align*}
\mathbf{w}_{i,j}^{K,\Psi_{K}(\theta)} &= \mathbf{w}_{i,j}^{K,\Psi_{K-1}(\theta)} (\| V_i^{K,\Psi_{K-1}(\theta)} \| + 1_{\{0\}}(\| V_i^{K,\Psi_{K-1}(\theta)} \|))^{-1}, \\
\mathbf{b}_{i}^{K,\Psi_{K}(\theta)} &= \mathbf{b}_{i}^{K,\Psi_{K-1}(\theta)} (\| V_i^{K,\Psi_{K-1}(\theta)} \| + 1_{\{0\}}(\| V_i^{K,\Psi_{K-1}(\theta)} \|))^{-1}.
\end{align*}
\end{equation}

Therefore, we get for all $i \in \{1,2,\ldots,\ell_K\}$ that
\begin{equation}
\mathcal{N}^{K,\Psi_{K}(\theta)}_\infty(y) = \mathbf{b}_{i}^{K,\Psi_{K}(\theta)} + \sum_{j=1}^{\ell_{K-1}} \mathbf{w}_{i,j}^{K,\Psi_{K}(\theta)} \mathcal{R}_\infty(\mathcal{N}^{K-1,\Psi_{K}(\theta)}_\infty(y))
\end{equation}
\begin{equation}
= \left( \mathbf{b}_{i}^{K,\Psi_{K-1}(\theta)} + \sum_{j=1}^{\ell_{K-1}} \mathbf{w}_{i,j}^{K,\Psi_{K-1}(\theta)} \mathcal{R}_\infty(\mathcal{N}^{K-1,\Psi_{K-1}(\theta)}_\infty(y)) \right) \\
\times (\| V_i^{K,\Psi_{K-1}(\theta)} \| + 1_{\{0\}}(\| V_i^{K,\Psi_{K-1}(\theta)} \|))^{-1}
\end{equation}
\begin{equation}
= (\| V_i^{K,\Psi_{K-1}(\theta)} \| + 1_{\{0\}}(\| V_i^{K,\Psi_{K-1}(\theta)} \|))^{-1} \mathcal{R}_\infty(\mathcal{N}^{K,\Psi_{K-1}(\theta)}_\infty(y)).
\end{equation}

This and the fact that $\forall u \in \mathbb{R}, \eta \in [0,\infty): \mathcal{R}_\infty(\eta u) = \eta \mathcal{R}_\infty(u)$ imply for all $i \in \{1,2,\ldots,\ell_K\}$ that
\begin{equation}
\mathcal{R}_\infty(\mathcal{N}^{K,\Psi_{K}(\theta)}_\infty(y)) = (\| V_i^{K,\Psi_{K-1}(\theta)} \| + 1_{\{0\}}(\| V_i^{K,\Psi_{K-1}(\theta)} \|))^{-1} \mathcal{R}_\infty(\mathcal{N}^{K,\Psi_{K-1}(\theta)}_\infty(y)).
\end{equation}

In addition, note that (2.36) shows for all $i \in \{1,2,\ldots,\ell_{K+1}\}$, $j \in \{1,2,\ldots,\ell_K\}$ that
\begin{equation}
\mathbf{w}_{i,j}^{K+1,\Psi_{K}(\theta)} = \mathbf{w}_{i,j}^{K+1,\Psi_{K-1}(\theta)} \| V_j^{K+1,\Psi_{K-1}(\theta)} \| \quad \text{and} \quad \mathbf{b}_{i}^{K+1,\Psi_{K}(\theta)} = \mathbf{b}_{i}^{K+1,\Psi_{K-1}(\theta)}.
\end{equation}

Combining this with the fact that for all $j \in \{1,2,\ldots,\ell_K\}$ with $\| V_j^{K,\Psi_{K-1}(\theta)} \| = 0$ it holds that $\mathcal{N}^{K,\Psi_{K-1}(\theta)}_\infty(y) = \mathcal{N}^{K,\Psi_{K}(\theta)}_\infty(y) = 0$ establishes that for all $i \in \{1,2,\ldots,\ell_{K+1}\}$ we have that
\begin{equation}
\begin{align*}
\mathbf{b}_{i}^{K+1,\Psi_{K-1}(\theta)} + \sum_{j=1}^{\ell_{K+1}} \mathbf{w}_{i,j}^{K+1,\Psi_{K}(\theta)} \mathcal{R}_\infty(\mathcal{N}^{K,\Psi_{K}(\theta)}_\infty(y))
&= \mathbf{b}_{i}^{K+1,\Psi_{K-1}(\theta)} + \sum_{j=1}^{\ell_{K}} \frac{\| V_j^{K,\Psi_{K-1}(\theta)} \|}{\| V_j^{K,\Psi_{K-1}(\theta)} \| + 1_{\{0\}}(\| V_j^{K,\Psi_{K-1}(\theta)} \|)} \mathbf{w}_{i,j}^{K+1,\Psi_{K-1}(\theta)} \mathcal{R}_\infty(\mathcal{N}^{K,\Psi_{K-1}(\theta)}_\infty(y)) \\
&= \mathbf{b}_{i}^{K+1,\Psi_{K-1}(\theta)} + \sum_{j=1}^{\ell_{K}} \mathbf{w}_{i,j}^{K+1,\Psi_{K-1}(\theta)} \mathcal{R}_\infty(\mathcal{N}^{K,\Psi_{K-1}(\theta)}_\infty(y)).
\end{align*}
\end{equation}
This and (2.30) prove that $\mathcal{N}_{\infty}^{r,K,\Psi_{K}(\theta)}(y) = \mathcal{N}_{\infty}^{r-1,K-1,\Psi_{K-1}(\theta)}(y)$. Hence, we obtain that $\mathcal{N}_{\infty}^{L,r,K,\Psi_{K}(\theta)} = \mathcal{N}_{\infty}^{L,r-1,K-1,\Psi_{K-1}(\theta)}$. Induction therefore establishes item (ii).

Next observe that (2.36), (2.33), and induction demonstrate for all $\theta \in \mathbb{R}^{n}$, $j \in \{1, 2, \ldots, L - 1\}$, $(k, i) \in \Lambda$ with $\|V_{i}^{k,\Psi_{j}(\theta)}\| > 0$ and $k \leq j$ that $\|V_{i}^{k,\Psi_{j}(\theta)}\| = 1$. This establishes item (iii). The proof of Proposition 2.8 is thus complete.

In the following result, Proposition 2.9, we establish some invariance properties of the considered modified GF and GD processes in Setting 2.7. In particular, we show for every $\omega \in \Omega$ for which the initial value $\xi(\omega) \in \mathbb{R}^{n}$ is non-degenerate in a suitable sense that the corresponding GF trajectory $(\Theta_{t}(\omega))_{t \in [0, \infty)}$ stays on the considered $C^{\infty}$-submanifold of the parameter space (see item (i)) and has non-increasing risk value $\mathcal{L}_{\infty}(\Theta_{t}(\omega))$, $t \in [0, \infty)$ (see item (ii)). For the proof we employ Lemma 2.1 and Corollary 2.4.

**Proposition 2.9** (Properties of modified GF and GD processes). Assume Setting 2.7. Then

(i) it holds for all $(k, i) \in \Lambda$, $t \in [0, \infty)$, $\omega \in \Omega$ with $\min_{(k,i)\in \Lambda} \|V_{i}^{k,\xi(\omega)}\| > 0$ that

$$
\psi_{i}^{k}(\Theta_{t}(\omega)) = 1,
$$

(2.47)

(ii) it holds for all $s \in [0, \infty)$, $t \in [s, \infty)$, $\omega \in \Omega$ with $\min_{(k,i)\in \Lambda} \|V_{i}^{k,\xi(\omega)}\| > 0$ that $\mathcal{L}_{\infty}(\Theta_{t}(\omega)) \leq \mathcal{L}_{\infty}(\Theta_{s}(\omega))$, and

(iii) it holds for all $(k, i) \in \Lambda$, $n \in \mathbb{N}_{0}$, $\omega \in \Omega$ with $\min_{(k,i)\in \Lambda} \|V_{i}^{k,\Theta_{n}(\omega)-\gamma_{n}G(\Theta_{n}(\omega))}\| > 0$ that

$$
\psi_{i}^{k}(\Theta_{n+1}(\omega)) = 1.
$$

(2.48)

**Proof of Proposition 2.9.** First, to prove item (i) let $\omega \in \Omega$ satisfy $\min_{(k,i)\in \Lambda} \|V_{i}^{k,\xi(\omega)}\| > 0$ and denote

$$
\tau = \inf\{t \in [0, \infty) : \min_{(k,i)\in \Lambda} \|V_{i}^{k,\Theta_{t}(\omega)}\| = 0\} \cup \{\infty\} \in [0, \infty].
$$

(2.49)

Note that item (iii) in Proposition 2.9 and (2.37) ensure for all $(k, i) \in \Lambda$ that $\|V_{i}^{k,\Theta_{0}(\omega)}\| = \|V_{i}^{k,\Psi_{L-1}(\xi(\omega))}\| = 1$. Hence, we obtain for all $(k, i) \in \Lambda$ that $\psi_{i}^{k}(\Theta_{0}(\omega)) = 1$. Furthermore, the fact that $\Theta$ is continuous implies that $\tau > 0$. In addition, observe that for all $t \in [0, \tau)$ we have that

$$
G(\Theta_{t}(\omega)) = G(\Theta_{0}(\omega)) - \sum_{(k,i)\in \Lambda} \rho(\nabla \psi_{i}^{k}(\Theta_{0}(\omega)), G(\Theta_{0}(\omega))) \nabla \psi_{i}^{k}(\Theta_{t}(\omega)).
$$

(2.50)

Combining this with (2.37), the fact that for all $(k, i) \in \Lambda$ it holds that $\psi_{i}^{k} \in C^{1}(\mathbb{R}^{n}, \mathbb{R})$, the fact that for all $t \in [0, \tau)$ it holds that $\inf_{s \in [0,t]} \min_{(k,i)\in \Lambda} \|\nabla \psi_{i}^{k}(\Theta_{s}(\omega))\| > 0$, and Lemma 2.1 (applied with $K \cap \emptyset \Lambda$, $(\psi_{i})_{i \in \{1,2,\ldots,K\}} \cap (\psi_{i})_{(k,i)\in \Lambda}$ in the notation of Lemma 2.1) shows for all $t \in [0, \tau)$ that $\psi^{k}_{i}(\Theta_{t}(\omega)) = \|V_{i}^{k,\Theta_{t}(\omega)}\| = 1$. This, (2.49), and the fact that $\Theta \in C([0, \infty), \mathbb{R}^{n})$ prove that $\tau = \infty$, which establishes item (i).

Next note that Hutzenhailer et al. [6, Theorem 2.9] demonstrates for all $\theta \in \mathbb{R}^{n}$ that $\bigcup_{\gamma \in \mathbb{N}} \{\mathcal{L}_{r}\} \subseteq C^{1}(\mathbb{R}^{n}, \mathbb{R})$, $\lim_{r \to \infty} \mathcal{L}_{r}(\theta) = \mathcal{L}_{\infty}(\theta)$, and $\lim_{r \to \infty} \nabla \mathcal{L}_{r}(\theta) = G(\theta)$. Furthermore, [6, Lemma 3.6] ensures for all compact $K \subseteq \mathbb{R}^{n}$ that $\sup_{\theta \in K} \sup_{r \in \mathbb{N}} \|\nabla \mathcal{L}_{r}(\theta)\| < \infty$. Combining this with Corollary 2.4 establishes item (ii).

Finally, to prove item (iii) let $n \in \mathbb{N}_{0}$, $\omega \in \Omega$ satisfy $\min_{(k,i)\in \Lambda} \|V_{i}^{k,\Theta_{n}(\omega)-\gamma_{n}G(\Theta_{n}(\omega))}\| > 0$. Observe that (2.33) implies for all $(k, i) \in \Lambda$ that

$$
\psi_{i}^{k}(\Theta_{n+1}(\omega)) = \|V_{i}^{k,\Theta_{n+1}(\omega)}\| = \|\rho(\Theta_{i}^{k,\Theta_{n}(\omega)-\gamma_{n}G(\Theta_{n}(\omega)))\| = 1.
$$

(2.51)

This establishes item (iii). The proof of Proposition 2.9 is thus complete. □
3 Global boundedness of normalized gradient flows in the training of shallow ReLU ANNs with one hidden neuron

In this section we prove that the modified gradient flow considered in Section 2 is uniformly bounded in the case of shallow ANNs with one-dimensional input, one neuron on the hidden layer, one-dimensional output, and uniformly distributed input data; see Theorem 3.17 below. For convenience we first introduce the simplified notation we will employ throughout this section.

3.1 Notation

Let $\mu: \mathcal{B}(\mathbb{R}) \to [0, \infty]$ be the Lebesgue measure on $\mathbb{R}$. Let $f \in C([0, 1], \mathbb{R})$ be the target function and let $\mathcal{I} = \int_0^1 f(x) \, dx$. Let $m: \mathbb{R}^3 \to \mathbb{R}$ satisfy for every $\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$ that $m(\theta) = \int_0^1 \max\{\theta_1 s + \theta_2, 0\} \, ds$. We consider the risk function $\mathcal{L}: \mathbb{R}^3 \to \mathbb{R}$ which satisfies for all $\theta \in \mathbb{R}^3$

$$\mathcal{L}(\theta) = \int_0^1 \left( \theta_3 (\max\{\theta_1 s + \theta_2, 0\} - m(\theta)) + \mathcal{I} - f(s) \right)^2 \, ds. \quad (3.1)$$

Let $g: \mathbb{R}^3 \to \mathbb{R}$ satisfy for all $\theta \in \mathbb{R}^3$ that $g(\theta) = |\theta_1|^2 + |\theta_2|^2$ and consider the two-dimensional $C^\infty$-manifold $\mathcal{M} = g^{-1}(1) \subseteq \mathbb{R}^3$. We want the gradient flow to stay on this manifold.

Note that Ibragimov et al. [7, Corollary 2.3] ensures for all $\theta \in \mathbb{R}^3$ with $|\theta_1| + |\theta_2| > 0$ that $m$ is differentiable at $\theta$. Using [7, Corollary 2.3] again ensures for all $\theta \in \mathbb{R}^3$ with $|\theta_1| + |\theta_2| > 0$ that all partial derivatives of $\mathcal{L}$ at $\theta$ exist. It is also not hard to see that these derivatives are continuous around $\theta$. Consider a modified gradient $\mathcal{G}: \mathbb{R}^3 \to \mathbb{R}^3$ which is locally bounded and measurable and satisfies for all $\theta \in \mathbb{R}^3$ with $|\theta_1| + |\theta_2| > 0$ that

$$\mathcal{G}(\theta) = \nabla \mathcal{L}(\theta) - \|\nabla g(\theta)\|^2 \langle \nabla \mathcal{L}(\theta), \nabla g(\theta) \rangle \nabla g(\theta). \quad (3.2)$$

From [9, Proposition 2.11] we know for all $\theta \in \mathbb{R}^3$ with $|\theta_1| + |\theta_2| > 0$ that $\nabla \mathcal{L}(\theta)$ agrees with the generalized gradient introduced in Setting 2.7. For every $\theta \in \mathbb{R}^3$ let $I^\theta = \{s \in [0, 1]: \theta_1 s + \theta_2 > 0\}$ and let

$$q^\theta = \begin{cases} \frac{-\theta_2}{\theta_1} & : \theta_1 \neq 0 \\ \infty & : \text{else.} \end{cases} \quad (3.3)$$

In the following we consider a gradient flow (GF) trajectory $\Theta = (\Theta_1, \Theta_2, \Theta_3): [0, \infty) \to \mathbb{R}^3$ which satisfies $\Theta(0) \in \mathcal{M}$ and $\forall t \in [0, \infty): \Theta(t) = \Theta(0) - \int_0^t \mathcal{G}(\Theta(u)) \, du$.

3.2 Basic properties of the gradient and the GF trajectory

Lemma 3.1. Consider the notation in Subsection 3.1. Then

1For simplicity we assume that the outer bias has the fixed value $\mathcal{I}$. So the risk function only depends on three parameters.
\begin{equation}
\langle \nabla L(\theta), \nabla g(\theta) \rangle = 4\theta_3 \int_0^1 (\theta_3 (\max \{\theta_1 s + \theta_2, 0\} - m(\theta)) + \overline{T} - f(s)) \times (\theta_1 s \mathbb{1}_{I^e(s)} + \theta_2 \mathbb{1}_{I^e(s)}) \, ds,
\end{equation}

and

\begin{equation}
\langle \nabla^2 L(\theta), \nabla g(\theta) \rangle = 4\theta_3 \int_0^1 (\theta_3 (\max \{\theta_1 s + \theta_2, 0\} - m(\theta)) + \overline{T} - f(s)) \times (\theta_1 s \mathbb{1}_{I^e(s)} + \theta_2 \mathbb{1}_{I^e(s)}) \, ds.
\end{equation}
Therefore, we obtain for all $\theta \in \mathcal{M}$ that

\[
\mathbf{G}_1(\theta) = 2\theta_3 \int_0^1 \left(\theta_3(\max\{\theta_1s + \theta_2, 0\} - m(\theta)) + \overline{\tau} - f(s)\right)(s - \theta_1(\theta_1s + \theta_2)) \mathbf{1}_{I^\theta}(s) \, ds
\]

\[
= 2\theta_3 \int_0^1 \left(\theta_3(\max\{\theta_1s + \theta_2, 0\} - m(\theta)) + \overline{\tau} - f(s)\right)(\theta_2^2 s - \theta_1\theta_2) \mathbf{1}_{I^\theta}(s) \, ds,
\]

\[
\mathbf{G}_2(\theta) = 2\theta_3 \int_0^1 \left(\theta_3(\max\{\theta_1s + \theta_2, 0\} - m(\theta)) + \overline{\tau} - f(s)\right)(1 - \theta_2(\theta_1s + \theta_2)) \mathbf{1}_{I^\theta}(s) \, ds
\]

\[
= 2\theta_3 \int_0^1 \left(\theta_3(\max\{\theta_1s + \theta_2, 0\} - m(\theta)) + \overline{\tau} - f(s)\right)(\theta_1^2 - \theta_1\theta_2 s) \mathbf{1}_{I^\theta}(s) \, ds,
\]

and $\mathbf{G}_3(\theta) = \frac{\partial}{\partial \theta_3} \mathcal{L}(\theta)$. Combining this with (3.5) establishes item (i). Item (ii) follows from [4, Corollary 2.6]. The proof of Lemma 3.1 is thus complete.

As a consequence of item (ii), Lemma 2.1, and Corollary 2.4 we obtain:

**Lemma 3.2.** Consider the notation in Subsection 3.1. Then

(i) it holds that $\Theta \in C^1([0, \infty), \mathbb{R}^3)$,

(ii) it holds for all $t \in [0, \infty)$ that $\Theta(t) \in \mathcal{M}$, and

(iii) it holds that $[0, \infty) \ni t \mapsto \mathcal{L}(\Theta(t)) \in \mathbb{R}$ is non-increasing.

**Remark 3.3.** It is not hard to see that for all $\theta \in \mathcal{M}$ the following properties hold:

- If $\theta_1 > 0$ then $I^\theta = (q^\theta, \infty) \cap [0, 1]$, and if $\theta_1 < 0$ then $I^\theta = (-\infty, q^\theta) \cap [0, 1]$.
- It holds that $\mu(I^\theta) \in (0, 1)$ if and only if $q^\theta \in (0, 1)$.
- It holds that $\mu(I^\theta) = 0$ if and only if $I^\theta = \emptyset$.
- It holds that $\mu(I^\theta) = 1$ if and only if $I^\theta \supseteq (0, 1)$.

This and the fact that $\Theta \in C([0, \infty), \mathbb{R}^3)$ easily imply that $[0, \infty) \ni t \mapsto \mu(I^{\Theta(t)}) \in \mathbb{R}$ is continuous.

### 3.3 Proof of the boundedness in simple cases

We first show the following:

**Lemma 3.4.** Consider the notation in Subsection 3.1. Then for every $\varepsilon > 0$ it holds that

\[
\sup_{t \in [0, \infty)} \left[\|\Theta(t)\| \mathbf{1}_{[\varepsilon, 1]}(\mu(I^{\Theta(t)}))\right] < \infty.
\]  

(3.8)

Notice that, due to Lemma 3.2, it suffices to bound $\Theta_3(t)$. For this we use the following elementary lemma, which is proved, e.g., in [9, Corollary 5.2].

**Lemma 3.5.** Let $\alpha, \beta \in \mathbb{R}$ and let $I \subseteq \mathbb{R}$ be a bounded interval. Then $\int_I (\alpha x + \beta)^2 \, dx \geq \frac{\alpha^2}{3\pi^2} (\mu(I))^3$.
Proof of Lemma 3.4. Throughout this proof let \( \varepsilon > 0 \), let \( \mathcal{C} = \| f - \overline{T} \|_{L^2([0,1])} \), and let \( T \subseteq [0, \infty) \) satisfy \( T = \{ t \in [0, \infty) : \mu(I^{\Theta(t)}) \in [\varepsilon, 1] \} \). Note that for all \( t \in T \) it holds that \( q^{\Theta(t)} \in (0,1) \) and \( |\Theta_2(t)| \leq |\Theta_1(t)| \). Hence, we obtain for all \( t \in T \) that \( |\Theta_1(t)| \geq 2^{-1/2} \). Furthermore, the triangle inequality proves for all \( \theta \in \mathbb{R}^3 \) that

\[
\sqrt{\mathcal{L}(\theta)} \geq \left( \int_0^1 (\theta_3(\max\{\theta_1 s + \theta_2, 0\} - m(\theta)))^2 ds \right)^{1/2} - \mathcal{C}
\]

\[
\geq \left( |\theta_3|^2 \int_s^1 (\theta_1 s + \theta_2 - m(\theta))^2 ds \right)^{1/2} - \mathcal{C}.
\]

Combining this with Lemma 3.5, item (iii) in Lemma 3.2, and the fact that \( \forall t \in T : |\Theta_1(t)| \geq 2^{-1/2} \) implies for all \( t \in T \) that

\[
\sqrt{\mathcal{L}(\Theta(0))} \geq \sqrt{\mathcal{L}(\Theta(t))} \geq |\Theta_3(t)| \left( \int_{I^{\Theta(t)}} (\Theta_1(t)s + \Theta_2(t) - m(\Theta(t)))^2 ds \right)^{1/2} - \mathcal{C}
\]

\[
\geq |\Theta_3(t)||\Theta_1(t)| (\frac{1}{17} \mu(I^{\Theta(t)}))^{1/2} - \mathcal{C} \geq 24^{-1/2}|\Theta_3(t)|^{3/2} - \mathcal{C}.
\]

This establishes that \( \sup_{t \in T} |\Theta_3(t)| < \infty \). The proof of Lemma 3.4 is thus complete. \( \Box \)

From Lemma 3.4 we obtain the boundedness if \( I^{\Theta(t)} \) is not the entire interval \((0,1)\), but has a positive measure bounded away from zero.

Lemma 3.6. Consider the notation in Subsection 3.1. Then for every \( \varepsilon > 0 \) it holds that

\[
\sup_{t \in [0, \infty)} [\| \Theta(t) \|_{1_\varepsilon}} \cdot \| I^{\Theta_1(t)} \| \cdot \mu(I^{\Theta(t)})] < \infty \quad (3.11)
\]

Proof of Lemma 3.6. Throughout this proof let \( \varepsilon > 0 \), let \( \mathcal{C} = \| f - \overline{T} \|_{L^2([0,1])} \), and let \( T \subseteq [0, \infty) \) satisfy \( T = \{ t \in [0, \infty) : |\Theta_1(t)| \geq \varepsilon, \mu(I^{\Theta(t)}) = 1 \} \). Using the same arguments as in the proof of Lemma 3.6 yields for all \( t \in T \) that

\[
\sqrt{\mathcal{L}(\Theta(0))} \geq \sqrt{\mathcal{L}(\Theta(t))} \geq |\Theta_3(t)| \left( \int_0^1 (\Theta_1(t)s + \Theta_2(t) - m(\Theta(t)))^2 ds \right)^{1/2} - \mathcal{C}
\]

\[
\geq \left( \frac{1}{17} |\Theta_3(t)||\Theta_1(t)| (\frac{1}{17} \mu(I^{\Theta(t)}))^{1/2} - \mathcal{C} \geq \frac{1}{4} |\Theta_3(t)|^2 - \mathcal{C}.
\]

This shows that \( \sup_{t \in T} |\Theta_3(t)| < \infty \). The proof of Lemma 3.6 is thus complete. \( \Box \)

Proposition 3.7. Consider the notation in Subsection 3.1. Then it holds for all \( t \in [0, \infty) \) with \( q^{\Theta(t)} \notin (0,1) \) and \( |\Theta_1(t)| < 1 \) that

\[
\frac{d}{dt}(|\Theta_3(t)|^2 + \ln(1 - |\Theta_1(t)|^2)) = 0.
\]

Proof of Proposition 3.7. Observe that for all \( t \in [0, \infty) \) with \( |\Theta_1(t)| < 1 \) we have that

\[
\frac{d}{dt}(|\Theta_3(t)|^2 + \ln(1 - |\Theta_1(t)|^2)) = 2\Theta_3(t)\Theta_3(\Theta(t)) - 2\frac{\Theta_1(t)}{1 - |\Theta_1(t)|^2}\Theta_1(\Theta(t))
\]

\[
= 2\Theta_3(t)\Theta_3(\Theta(t)) - 2\frac{\Theta_1(t)}{|\Theta_1(t)|^2}\Theta_1(\Theta(t)).
\]

Furthermore, if \( q^{\Theta(t)} \notin (0,1) \) we either have \( I^{\Theta(t)} = \emptyset \) or \( I^{\Theta(t)} \supseteq (0,1) \). In the first case Lemma 3.1 demonstrates that \( \Theta_1(\Theta(t)) = \Theta_3(\Theta(t)) = 0 \). (3.14) therefore ensures (3.13).
In the second case we obtain from Lemma 3.1 that
\[
\mathcal{E}_1(\Theta(t)) = 2(\Theta_2(t))^2 \Theta_3(t) \int_0^1 (\Theta_3(t)(\max\{\Theta_1(t)s + \Theta_2(t), 0\} - m(\Theta(t))) + \mathcal{F} - f(s))s \, ds
\]
and
\[
\mathcal{E}_3(\Theta(t)) = 2\Theta_1(t)\Theta_3(t) \int_0^1 (\Theta_3(t)(\max\{\Theta_1(t)s + \Theta_2(t), 0\} - m(\Theta(t))) + \mathcal{F} - f(s))s \, ds
\]
Combining this with (3.14) establishes (3.13). The proof of Proposition 3.7 is thus complete. □

Remark 3.8. An analogous statement to Proposition 3.7 can be proved for any number \( H \in \mathbb{N} \) of neurons on the hidden layer, using similar identities for the gradient components.

Using the last two results, we get boundedness in the case \( q^\Theta(t) \notin (0, 1) \). Indeed, if \( I^\Theta(t) = \emptyset \) then \( \mathcal{E}(\Theta(t)) = 0 \), so it cannot diverge. If \( I^\Theta(t) \supseteq (0, 1) \) and \(|\Theta_1(t)|\) is bounded away from zero the boundedness follows from Lemma 3.6. If \( I^\Theta(t) \supseteq (0, 1) \) and \(|\Theta_1(t)|\) is bounded away from 1 the boundedness follows from Proposition 3.7.

The remaining and more difficult cases occur when \( I^\Theta(t) \) has small positive measure. This is the content of the next two subsubsections.

### 3.4 The case that the breakpoint is close to 1

In this subsection we will deal with the case that the activity interval \( I^\Theta(t) \) is non-empty and contained in some interval \([1 - \varepsilon, 1]\) for a small \( \varepsilon > 0 \), which is not covered by the previous results. Note that \( I^\Theta(t) \) can only be of the considered form if \( q^\Theta(t) \in (0, 1) \) and \( \Theta_2(t) < 0 < \Theta_1(t) \). Furthermore, we have \(|\Theta_2(t)|^2 < \frac{1}{2} < |\Theta_1(t)|^2\). This will be used throughout this section.

**Lemma 3.9.** Consider the notation in Subsection 3.1 and let \( \theta \in \mathcal{M} \) satisfy \( q^\theta \in (0, 1) \) and \( \theta_1 > 0 \). Then

(i) it holds that \( m(\theta) = \frac{\theta_1}{2}(1 - q^\theta)^2 \),

(ii) it holds that
\[
\int_{\mathcal{F}} (\max\{\theta_1s + \theta_2, 0\} - m(\theta)) \, ds = \frac{\theta_1}{2}(1 - q^\theta)^2q^\theta,
\]

(iii) it holds that
\[
\int_0^1 (\max\{\theta_1s + \theta_2, 0\} - m(\theta))^2 \, ds = \theta_1^2(1 - q^\theta)^3\left(\frac{1}{12} + \frac{q^\theta}{4}\right),
\]

and

(iv) it holds that
\[
\mathcal{E}_1(\theta) = 2\theta_3\left(\frac{\theta_1\theta_2^2\theta_3}{12}(1 - q^\theta)^2(7 + 2q^\theta + 3(q^\theta)^2) - \int_{q^\theta}^1 (\mathcal{F} - f(s))(\theta_2^3s - \theta_1\theta_2) \, ds\right),
\]
\[
\mathcal{E}_3(\theta) = 2\theta_1^2\theta_3(1 - q^\theta)^3\left(\frac{1}{12} + \frac{q^\theta}{4}\right) + 2\int_{q^\theta}^1 (\mathcal{F} - f(s)) \max\{\theta_1s + \theta_2, 0\} \, ds.
\]
Proof of Lemma 3.9. First, we have

\[ m(\theta) = \int_0^1 \max\{\theta_1 s + \theta_2, 0\} \, ds = \theta_1 \int_{\theta^2}^1 (s - q^\theta) \, ds = \frac{\theta_1}{2} (1 - q^\theta)^2. \]  

(3.20)

This establishes item (i). Next, item (i) implies that

\[ \int_{\theta^2}^1 (\max\{\theta_1 s + \theta_2, 0\} - m(\theta)) \, ds = m(\theta) - \int_{\theta^2}^1 m(\theta) \, ds = m(\theta) - (1 - q^\theta)m(\theta) \]

(3.21)

\[ = q^\theta m(\theta) = \frac{\theta_1}{2} (1 - q^\theta)^2 q^\theta. \]

This establishes item (ii). Moreover, observe that

\[ \int_0^1 (\max\{\theta_1 s + \theta_2, 0\} - m(\theta))^2 \, ds \]

\[ = \int_0^1 (\max\{\theta_1 s + \theta_2, 0\})^2 \, ds - 2m(\theta) \int_0^1 \max\{\theta_1 s + \theta_2, 0\} \, ds + \int_0^1 m(\theta)^2 \, ds \]

\[ = \theta_1^2 \int_{\theta^2}^1 (s - q^\theta)^2 \, ds - 2m(\theta)^2 + m(\theta)^2 = \theta_1^2 \int_{\theta^2}^1 s^2 \, ds - m(\theta)^2 \]

\[ = \frac{\theta_1^2}{3} (1 - q^\theta)^3 - \frac{\theta_1^2}{4} (1 - q^\theta)^4 = \theta_1^2 (1 - q^\theta)^3 \left( \frac{1}{3} - \frac{1 - q^\theta}{4} \right) = \theta_1^2 (1 - q^\theta)^3 \left( \frac{1}{12} + \frac{q^\theta}{4} \right). \]

This establishes item (iii). In addition, (3.4) assures that

\[ \mathfrak{S}_3(\theta) = 2\theta_3 \int_0^1 (\max\{\theta_1 s + \theta_2, 0\} - m(\theta))^2 \, ds \]

\[ + 2 \int_0^1 (\overline{f} - f(s))(\max\{\theta_1 s + \theta_2, 0\} - m(\theta)) \, ds \]

\[ = 2\theta_1^2 \theta_3 (1 - q^\theta)^3 \left( \frac{1}{12} + \frac{q^\theta}{4} \right) + 2 \int_{\theta^2}^1 (\overline{f} - f(s)) \max\{\theta_1 s + \theta_2, 0\} \, ds. \]

(3.23)

Furthermore, note that

\[ \int_{\theta^2}^1 (\max\{\theta_1 s + \theta_2, 0\} - m(\theta))(\theta_2^2 s - \theta_1 \theta_2) \, ds \]

\[ = -\theta_1 \theta_2 \int_{\theta^2}^1 (\max\{\theta_1 s + \theta_2, 0\} - m(\theta)) \, ds + \theta_2^2 \int_{\theta^2}^1 s(\theta_1 s + \theta_2) \, ds - \theta_2^2 m(\theta) \int_{\theta^2}^1 s \, ds \]

\[ = -\frac{\theta_1 \theta_2}{2} (1 - q^\theta)^2 q^\theta + \theta_2^2 \theta_1 \left( \frac{1 - (q^\theta)^3}{3} - q^\theta (1 - (q^\theta)^2) \right) - \frac{\theta_2^2 \theta_1}{2} (1 - q^\theta)^2 \left( \frac{1 - (q^\theta)^2}{2} \right) \]

\[ = (1 - q^\theta) \theta_1 \theta_2^2 \left( \frac{1 - q^\theta}{2} + 1 + q^\theta (q^\theta)^2 - q^\theta (1 + q^\theta) - \frac{1 - (q^\theta)^2}{4} (1 - q^\theta)^2 \right) \]

\[ = (1 - q^\theta)^2 \frac{\theta_1 \theta_2^2}{12} (7 + 2q^\theta + 3(q^\theta)^2). \]

(3.24)

Combining this with (3.4) establishes item (iv). The proof of Lemma 3.9 is thus complete.

Next, by symmetry we may assume wlog that \( \overline{f} \leq f(1) \). (Otherwise replace \( f \mapsto -f, \theta_3 \mapsto -\theta_3 \).)
Lemma 3.10. Consider the notation in Subsection 3.1, assume \( \overline{f} = f(1) \), and assume that \( f \) is Lipschitz continuous. Then there exists \( c \in \mathbb{R} \) which satisfy for all \( t \in [0, \infty) \) with \( I^{\Theta(t)} \subseteq [\frac{1}{2}, 1] \) and \( |\Theta_3(t)| \geq c \) that
\[
\frac{d}{dt}|\Theta_3(t)|^2 \leq 0. \tag{3.25}
\]

Proof of Lemma 3.10. First, the assumption that \( f \) is Lipschitz continuous ensures that there exists \( L \in (0, \infty) \) which satisfies for all \( s \in [0, 1] \) that \( |f(s) - f(1)| = |f(s) - \overline{f}| \leq L(1 - s) \). Combining this with Lemma 3.9 demonstrates for all \( \theta \in \mathcal{M} \) with \( \Theta \neq I^{\theta} \subseteq [\frac{1}{2}, 1] \) that
\[
\left| \int_{q^\theta}^{1} (\overline{f} - f(s)) \max\{\theta_1 s + \theta_2, 0\} \, ds \right| \leq L \theta_1 \int_{q^\theta}^{1} (1 - s)(s - q^\theta) \, ds = \frac{L \theta_1}{6}(1 - q^\theta)^3. \tag{3.26}
\]
This, the chain rule, and the fact that for all \( t \in [0, \infty) \) with \( \Theta \neq I^{\Theta(t)} \subseteq [\frac{1}{2}, 1] \) it holds that \( 2^{-1/2} \leq |\Theta_1(t)| \leq 1 \) show that for all \( t \in [0, \infty) \) with \( \Theta \neq I^{\Theta(t)} \subseteq [\frac{1}{2}, 1] \) we have that
\[
\frac{d}{dt}|\Theta_3(t)|^2 = -2\Theta_3(t) \Theta_3(\Theta(t)) \leq -2|\Theta_1(t)\Theta_3(t)|^2(1 - q^{\Theta(t)})^3 \left( \frac{1}{12} + \frac{q^{\Theta(t)}}{3} \right) + L \frac{|\Theta_1(t)\Theta_3(t)|}{3}(1 - q^{\Theta(t)})^3 \tag{3.27}
\]
\[
\leq (1 - q^{\Theta(t)})^3 \left( -\frac{|\Theta_3(t)|^2}{12} + \frac{L |\Theta_3(t)|}{3} \right) = (1 - q^{\Theta(t)})^3 |\Theta_3(t)|^2 \left( 4L - |\Theta_3(t)| \right). \]
Hence, we obtain for all \( t \in [0, \infty) \) with \( I^{\Theta(t)} \subseteq [\frac{1}{2}, 1] \) and \( |\Theta_3(t)| \geq 4L \) that \( \frac{d}{dt}|\Theta_3(t)|^2 \leq 0 \). The proof of Lemma 3.10 is thus complete. \( \square \)

Lemma 3.11. Consider the notation in Subsection 3.1 and assume \( \overline{f} < f(1) \). Then there exists \( \varepsilon \in (0, 1/2) \) which satisfies for all \( t \in [0, \infty) \) with \( \Theta \neq I^{\Theta(t)} \subseteq [1 - \varepsilon, 1] \) and \( \Theta_3(t) \leq 0 \) that
\[
\frac{d}{dt}|\Theta_3(t)|^2 \leq 0. \tag{3.28}
\]

Proof of Lemma 3.11. Observe that the fact that \( f \) is continuous assures that there exists \( \varepsilon \in (0, 1/2) \) which satisfies for all \( s \in [1 - \varepsilon, 1] \) that \( f(s) > \overline{f} \). This implies for all \( t \in [0, \infty) \) with \( \Theta \neq I^{\Theta(t)} \subseteq [1 - \varepsilon, 1] \) that
\[
\int_{q^\theta}^{1} (\overline{f} - f(s)) \max\{\theta_1 s + \theta_2, 0\} \, ds \leq 0.
\]
Combining this with Lemma 3.9 demonstrates for all \( t \in [0, \infty) \) with \( \Theta \neq I^{\Theta(t)} \subseteq [1 - \varepsilon, 1] \) and \( \Theta_3(t) \leq 0 \) that \( \frac{d}{dt}|\Theta_3(t)|^2 = -2\Theta_3(t) \Theta_3(\Theta(t)) \leq 0 \). The proof of Lemma 3.11 is thus complete. \( \square \)

Lemma 3.12. Consider the notation in Subsection 3.1 and assume \( \overline{f} < f(1) \). Then there exists \( \varepsilon \in (0, 1/2) \) which satisfies for all \( t \in [0, \infty) \) with \( \Theta \neq I^{\Theta(t)} \subseteq [1 - \varepsilon, 1] \) and \( \Theta_3(t) > 0 \) that
\[
\frac{d}{dt} \left( |\Theta_3(t)|^2 - \frac{3}{8}|\Theta_1(t) - 2^{-1/2}|^2 \right) \leq 0. \tag{3.29}
\]

Proof of Lemma 3.12. First, the fact that \( f \) is continuous ensures that there exist \( \beta \in (0, \infty) \), \( \varepsilon \in (0, 1/2) \) which satisfy for all \( s \in [1 - \varepsilon, 1] \) that \( \overline{f} + \beta < f(s) < \overline{f} + \frac{10\beta}{9} \). This implies for all \( \theta \in \mathcal{M} \) with \( \Theta \neq I^{\theta} \subseteq [1 - \varepsilon, 1] \) that
\[
\int_{q^\theta}^{1} (f(s) - \overline{f}) \max\{\theta_1 s + \theta_2, 0\} \, ds \leq \frac{10\beta}{9} \int_{q^\theta}^{1} \max\{\theta_1 s + \theta_2, 0\} \, ds = \frac{5\beta}{9} \theta_1 (1 - q^\theta)^2 \tag{3.30}
\]
and
\[
\int_{\theta} (f(s) - \mathcal{T})(\theta_2^2s - \theta_1s_{\theta_2}) \, ds \geq \frac{\beta}{2} \theta_1^2 q^0 (1 - q^0)(2 + q^0 + (q^0)^2). \tag{3.31}
\]

Combining this with Lemma 3.9 demonstrates for all \( \theta \in \mathcal{M} \) with \( \emptyset \neq I^0 \subseteq [1 - \varepsilon, 1] \) and \( \theta_3 \geq 0 \) that
\[
\mathcal{G}_3(\theta) \geq \theta_1 (1 - q^0)^2 \left( 2\theta_1 \theta_3(1 - q^0) \left( \frac{1}{16} + \frac{q^0}{4} \right) - \frac{10\beta}{9} \right) \tag{3.32}
\]
and
\[
\mathcal{G}_1(\theta) \leq 2\theta_3 \theta_2^2 \theta_3(1 - q^0)^2 - \frac{\beta}{2} \theta_1 q^0 (1 - q^0)(2 + q^0 + (q^0)^2)) \leq \theta_1 \theta_3(1 - q^0)(\theta_3(1 - q^0) - \beta \theta q^0(2 + q^0 + (q^0)^2)). \tag{3.33}
\]

In addition, note that for all \( \theta \in \mathcal{M} \) with \( \emptyset \neq I^0 \subseteq [1 - \varepsilon, 1] \) it holds that
\[
\theta_1 - 2^{-1/2} = \frac{\theta_1}{\sqrt{2}} \left( \sqrt{2} - \theta_1^{-1} \sqrt{\theta_1^2 + \theta_2^2} \right) = \frac{\theta_1}{\sqrt{2}} \left( \sqrt{2} - \sqrt{1 + (q^0)^2} \right) \tag{3.34}
\]

This and the chain rule show for all \( t \in [0, \infty) \) with \( \emptyset \neq I^\Theta(t) \subseteq [1 - \varepsilon, 1] \) and \( \Theta_3(t) > 0 \) that
\[
\begin{align*}
\frac{d}{dt} \left( (\Theta_3(t))^2 - \frac{5}{8} \Theta_1(t) - 2^{-1/2} \right)^2 \\
= -2\Theta_3(t) \mathcal{G}_3(\Theta(t)) + \frac{5}{4} \mathcal{G}_1(\Theta(t))(\Theta(t) - 2^{-1/2}) \\
\leq 2\Theta_1(t) \Theta_3(t)(1 - q^{\Theta(t)})^2 \left( \frac{10\beta}{9} - \Theta_1(t) \Theta_3(t)(1 - q^{\Theta(t)}) \left( \frac{1}{6} + \frac{q^{\Theta(t)}}{2} \right) \\
- \frac{5\Theta_1(t)(1 + q^{\Theta(t)})}{8 + 4\sqrt{2}(1 + (q^{\Theta(t)})^2)} (\beta \Theta_1(t) q^{\Theta(t)}(2 + q^{\Theta(t)} + (q^{\Theta(t)})^2) - \Theta_3(t)(1 - q^{\Theta(t)})) \right).
\end{align*}
\tag{3.35}
\]

Next observe that the fact that for all \( q \in [0, 1] \): \( \frac{1 + q}{2 + \sqrt{2}(1 + q^2)} \leq \frac{1}{2} \) ensures that there exists \( \eta \in (0, \varepsilon) \) which satisfies for all \( t \in [0, \infty) \) with \( \emptyset \neq I^\Theta(t) \subseteq [1 - \eta, 1] \) that
\[
\frac{5(\Theta_1(t))^2(1 + q^{\Theta(t)})}{8 + 4\sqrt{2}(1 + (q^{\Theta(t)})^2)} q^{\Theta(t)}(2 + q^{\Theta(t)} + (q^{\Theta(t)})^2) \geq \frac{10}{9},
\]
\[
\frac{5(1 + q^{\Theta(t)})}{8 + 4\sqrt{2}(1 + (q^{\Theta(t)})^2)} \leq \frac{5}{8}, \quad \text{and} \quad \frac{1}{6} + \frac{q^{\Theta(t)}}{2} > \frac{5}{8}. \tag{3.36}
\]

Therefore, we obtain for all \( t \in [0, \infty) \) with \( \emptyset \neq I^\Theta(t) \subseteq [1 - \eta, 1] \) and \( \Theta_3(t) > 0 \) that
\[
\begin{align*}
\frac{d}{dt} \left( (\Theta_3(t))^2 - \frac{5}{8} \Theta_1(t) - 2^{-1/2} \right)^2 \\
\leq 2\Theta_1(t) \Theta_3(t)(1 - q^{\Theta(t)})^2 \left( \frac{10\beta}{9} - \Theta_1(t) \Theta_3(t)(1 - q^{\Theta(t)}) \left( \frac{1}{6} + \frac{q^{\Theta(t)}}{2} \right) \\
- \frac{10\beta}{9} + \frac{5}{8} \Theta_1(t) \Theta_3(t)(1 - q^{\Theta(t)}) \right) \leq 0.
\end{align*}
\tag{3.37}
\]

The proof of Lemma 3.12 is thus complete. \( \square \)
3.5 The case that the breakpoint is close to 0

Finally, we consider the case where the activity interval \( I^\Theta(t) \) is non-empty and contained in some interval \([0, \varepsilon]\) with \( \varepsilon > 0 \) small. The arguments are essentially analogous to the previous case. Note that this time we must have \( q^\Theta(t) \in (0, 1) \) and \( \Theta_2(t) > 0 > \Theta_1(t) \). Furthermore, for small \( \varepsilon > 0 \) we have that \( \Theta_1(t) \) is close to \(-1\) and \( \Theta_2(t) \) is close to \(0).

**Lemma 3.13.** Consider the notation in Subsection 3.1 and let \( \theta \in \mathcal{M} \) satisfy \( q^\theta \in (0, 1) \) and \( \theta_1 < 0 \). Then

(i) it holds that \( m(\theta) = \frac{\theta_1}{2}(q^\theta)^2 \),

(ii) it holds that

\[
\int_{t^\theta} \left( \max\{\theta_1 s + \theta_2, 0\} - m(\theta) \right) ds = \frac{\theta_1}{2}(1 - q^\theta)(q^\theta)^2, \tag{3.38}
\]

(iii) it holds that

\[
\int_{0}^{1} \left( \max\{\theta_1 s + \theta_2, 0\} - m(\theta) \right)^2 ds = \theta_1^2(q^\theta)^3 \left( \frac{1}{3} - \frac{q^\theta}{4} \right), \tag{3.39}
\]

and

(iv) it holds that

\[
\Theta_1(\theta) = 2\theta_3 \left( -\frac{\theta_1\theta_3}{12} (q^\theta)^3 (6 + 6q^\theta + 2(q^\theta)^2 + 3(q^\theta)^3) + \int_{0}^{q^\theta} (f(s) - f(s)) (\theta_2 s - \theta_1 \theta_2) ds \right),
\]

\[
\Theta_3(\theta) = 2\theta_1^2\theta_3(q^\theta)^3 \left( \frac{1}{3} - \frac{q^\theta}{4} \right) + 2 \int_{0}^{q^\theta} (f(s) - f(s)) \max\{\theta_1 s + \theta_2, 0\} ds. \tag{3.40}
\]

In the following consider the case \( \overline{f} \leq f(0) \), the case \( \overline{f} > f(0) \) being analogous.

**Lemma 3.14.** Consider the notation in Subsection 3.1, assume \( \overline{f} = f(0) \), and assume that \( f \) is Lipschitz continuous. Then there exists \( c \in \mathbb{R} \) which satisfy for all \( t \in [0, \infty) \) with \( I^\Theta(t) \subseteq [0, \frac{1}{2}] \) and \( |\Theta_3(t)| \geq c \) that

\[
\frac{d}{dt} |\Theta_3(t)|^2 \leq 0. \tag{3.41}
\]

**Proof of Lemma 3.14.** First, the assumption that \( f \) is Lipschitz continuous ensures that there exists \( L \in (0, \infty) \) which satisfies for all \( s \in [0, 1] \) that

\[
|f(s) - f(0)| = |f(s) - \overline{f}| \leq L s.
\]

Combining this with Lemma 3.13 demonstrates for all \( \theta \in \mathcal{M} \) with \( \emptyset \neq I^\theta \subseteq [0, \frac{1}{2}] \) that

\[
\left| \int_{0}^{q^\theta} (f(s) - f(s)) \max\{\theta_1 s + \theta_2, 0\} ds \right| \leq L|\theta_1| \int_{0}^{q^\theta} s(q^\theta - s) ds = \frac{Ld}{6}(q^\theta)^3. \tag{3.42}
\]

This and the chain rule show for all \( t \in [0, \infty) \) with \( \emptyset \neq I^\Theta(t) \subseteq [0, \frac{1}{2}] \) that

\[
\frac{d}{dt} |\Theta_3(t)|^2 = -2\Theta_3(t) \Theta_3(\Theta(t)) \leq -2|\Theta_1(t)\Theta_3(t)|^2(q^\Theta(t))^3 \left( \frac{1}{3} - \frac{q^\Theta(t)}{4} \right) + L|\Theta_1(t)|\Theta_3(t)(q^\Theta(t))^3 \leq (q^\Theta(t))^3 \left( \frac{|\Theta_3(t)|^2}{12} + \frac{L|\Theta_3(t)|}{3} \right) = (q^\Theta(t))^3 \frac{|\Theta_3(t)|^2}{12} (4L - |\Theta_3(t)|). \tag{3.43}
\]
Hence, we obtain for all $t \in [0, \infty)$ with $I^{\theta(t)} \subseteq [0, \frac{1}{2}]$ and $|\Theta_3(t)| \geq 4L$ that $\frac{d}{dt}|\Theta_3(t)|^2 \leq 0$. The proof of Lemma 3.14 is thus complete. □

**Lemma 3.15.** Consider the notation in Subsection 3.1 and assume $\tilde{f} < f(0)$. Then there exists $\varepsilon \in (0, \frac{1}{2})$ which satisfies for all $t \in [0, \infty)$ with $\emptyset \neq I^{\theta(t)} \subseteq [0, \varepsilon]$ and $\Theta_3(t) \leq 0$ that

$$
\frac{d}{dt}|\Theta_3(t)|^2 \leq 0. \quad (3.44)
$$

**Proof of Lemma 3.15.** Note that the fact that $f$ is continuous assures that there exists $\varepsilon \in (0, \frac{1}{2})$ which satisfies for all $s \in [0, \varepsilon]$ that $f(s) > \tilde{f}$. This implies for all $t \in [0, \infty)$ with $\emptyset \neq I^{\theta(t)} \subseteq [0, \varepsilon]$ that $\int_0^{\theta(t)} (\tilde{f} - f(s)) \max\{\Theta_1(t)_s + \Theta_2(t), 0\} \, ds \leq 0$. Combining this with Lemma 3.13 demonstrates for all $t \in [0, \infty)$ with $\emptyset \neq I^{\theta(t)} \subseteq [0, \varepsilon]$ and $\Theta_3(t) \leq 0$ that $\frac{d}{dt}|\Theta_3(t)|^2 = -2\Theta_3(t)|\Theta_3(\Theta(t))| \leq 0$. The proof of Lemma 3.15 is thus complete. □

**Lemma 3.16.** Consider the notation in Subsection 3.1 and assume $\tilde{f} < f(0)$. Then there exists $\varepsilon \in (0, \frac{1}{2})$ which satisfies for all $t \in [0, \infty)$ with $\emptyset \neq I^{\theta(t)} \subseteq [0, \varepsilon]$ and $\Theta_3(t) > 0$ that

$$
\frac{d}{dt}(|\Theta_3(t)|^2 + \frac{5}{8}|\Theta_1(t)|^2) \leq 0. \quad (3.45)
$$

**Proof of Lemma 3.16.** First, the fact that $f$ is continuous ensures that there exist $\beta \in (0, \infty), \varepsilon \in (0, \frac{1}{2})$ which satisfy for all $s \in [0, \varepsilon]$ that $\tilde{f} + \beta < f(s) < \tilde{f} + \frac{10\beta}{9}$. This implies for all $t \in \mathcal{M}$ with $\emptyset \neq I^\theta \subseteq [0, \varepsilon]$ that

$$
\int_\rho^\theta (f(s) - \tilde{f}) \max\{\theta_1 s + \theta_2, 0\} \, ds \leq \frac{10\beta}{9} \int_0^{\rho^\theta} \max\{\theta_1 s + \theta_2, 0\} \, ds = -\frac{5\beta}{9} \theta_1 (q^\theta)^2 \quad (3.46)
$$

and

$$
\int_\rho^\theta (f(s) - \tilde{f})(\theta_2^2 s - \theta_1 \theta_2) \, ds \geq \frac{\beta}{2} \theta_1^2 (q^\theta)^2 (2 + (q^\theta)^2). \quad (3.47)
$$

Combining this with Lemma 3.13 demonstrates for all $\theta \in \mathcal{M}$ with $\emptyset \neq I^\theta \subseteq [0, \varepsilon]$ and $\Theta_3(t) \geq 0$ that

$$
|\Theta_3(\theta)| \geq \theta_1 (q^\theta)^2 \left(2 \theta_1 \theta_3 q^\theta \left(\frac{1}{3} - \frac{q^\theta}{4}\right) + \frac{10\beta}{9}\right) \quad (3.48)
$$

and

$$
|\Theta_1(\theta)| \leq 2 \theta_3 (-\theta_1^2 \theta_3 (q^\theta)^3 \left(\frac{1}{2} + q^\theta\right) - \frac{\beta}{2} \theta_1^2 (q^\theta)^2 (2 + (q^\theta)^2)) = \theta_1^2 \theta_3 (q^\theta)^2 \left(-\theta_1 \theta_3 q^\theta (1 + 2q^\theta) - \beta (2 + (q^\theta)^2)\right). \quad (3.49)
$$

This and the chain rule show for all $t \in [0, \infty)$ with $\emptyset \neq I^{\theta(t)} \subseteq [0, \varepsilon]$ and $\Theta_3(t) > 0$ that

$$
\frac{d}{dt}(||\Theta_3(t)||^2 + \frac{5}{8}|\Theta_1(t)|^2) = -2\Theta_3(t)|\Theta_3(\Theta(t)) + \frac{5}{4}\Theta_1(t)|\Theta_1(\Theta(t))
\leq \Theta_1(t)\Theta_3(t)(q^{\Theta(t)})^2 \left(\frac{1}{4} - \frac{q^{\Theta(t)}}{4}\right)
\leq \frac{5}{8}|\Theta_1(t)|^2 \left(2 + (q^{\Theta(t)})^2\right) + \frac{5}{4}(\Theta_1(t))^3 \Theta_3(t)(q^{\Theta(t)})(1 + 2q^{\Theta(t)}) \quad (3.50)
$$

Next observe that there exists $\eta \in (0, \varepsilon)$ which satisfies for all $t \in [0, \infty)$ with $\emptyset \neq I^{\theta(t)} \subseteq [0, \eta]$ that

$$
\frac{5}{4}|\Theta_1(t)|^2 (2 + (q^{\Theta(t)})^2) > \frac{20}{9} \quad \text{and} \quad -\frac{1}{3} + q^{\Theta(t)} + \frac{5}{4}(\Theta_1(t))^3 (1 + 2q^{\Theta(t)}) < 0. \quad (3.51)
$$
Therefore, we obtain for all \( t \in [0, \infty) \) with \( \emptyset \neq I^{\Theta(t)} \subseteq [0, \eta] \) and \( \Theta_3(t) > 0 \) that
\[
\frac{d}{dt} \left( |\Theta_3(t)|^2 + \frac{3}{8}|\Theta_1(t)|^2 \right) \\
\leq -\Theta_1(t)\Theta_3(t)(q^{\Theta(t)})^2\beta \left( \frac{2q}{3} - \frac{3}{4}(\Theta_1(t))^2 \right)(2 + (q^{\Theta(t)})) \\
+ |\Theta_1(t)\Theta_3(t)|^2(q^{\Theta(t)})^2\left( -\frac{1}{3} + q^{\Theta(t)} + \frac{3q}{4}(\Theta_1(t))^2 \right)(1 + 2q^{\Theta(t)}) \leq 0.
\]
(3.52)
The proof of Lemma 3.16 is thus complete.

3.6 Proof of the main boundedness result

We now combine the results for the different cases to establish the conjecture that the entire trajectory remains bounded; see Theorem 3.17 below. The main difficulty in the proof is that the gradient flow may change between the different regimes.

Theorem 3.17. Consider the notation in Subsection 3.1 and assume that \( f \) is Lipschitz continuous. Then \( \sup_{t \in [0, \infty]} \|\Theta(t)\| < \infty \).

Remark 3.18. The assumption that \( f \) is Lipschitz is only needed in the special cases \( \overline{\mathcal{G}} = f(0) \) (see Lemma 3.14) and \( \overline{\mathcal{G}} = f(1) \) (see Lemma 3.10). If one assumes \( f(0) \neq \overline{\mathcal{G}} \neq f(1) \) it is sufficient if \( f \) is merely continuous.

Proof of Theorem 3.17. First note that if there exists \( t \in [0, \infty) \) with \( \mu(I^{\Theta(t)}) = 0 \) then \( \Phi(\Theta(t)) = 0 \). By uniqueness of solutions (since \( \Phi \) is locally Lipschitz on \( \mathcal{M} \)), we obtain for all \( u \in [0, \infty) \) that \( \Phi(\Theta(u)) = 0 \) and, hence, \( \Theta(u) = \Theta(0) \). In this case the statement clearly holds.

From now on we assume \( \forall t \in [0, \infty) : \mu(I^{\Theta(t)}) > 0 \). We consider the case \( \overline{\mathcal{G}} < \min \{ f(0), f(1) \} \). The remaining cases are analogous, using Lemmas 3.10 and 3.14. Observe that Lemmas 3.11, 3.12, 3.15, and 3.16 assure that there exists \( \epsilon \in (0, 1/2) \) which satisfies the following properties:

(I) It holds for all \( t \in [0, \infty) \) with \( \emptyset \neq I^{\Theta(t)} \subseteq [1 - \epsilon, 1] \) and \( \Theta_3(t) \leq 0 \) that
\[
\frac{d}{dt} \left( |\Theta_3(t)|^2 \right) \leq 0,
\]
(II) it holds for all \( t \in [0, \infty) \) with \( \emptyset \neq I^{\Theta(t)} \subseteq [0, \epsilon] \) and \( \Theta_3(t) \leq 0 \) that \( \frac{d}{dt} \left( |\Theta_3(t)|^2 \right) \leq 0 \),
(III) it holds for all \( t \in [0, \infty) \) with \( \emptyset \neq I^{\Theta(t)} \subseteq [1 - \epsilon, 1] \) and \( \Theta_3(t) > 0 \) that
\[
\frac{d}{dt} \left( |\Theta_3(t)|^2 - \frac{3}{8}|\Theta_1(t)|^2 \right) \leq 0,
\]
(IV) it holds for all \( t \in [0, \infty) \) with \( \emptyset \neq I^{\Theta(t)} \subseteq [0, \epsilon] \) and \( \Theta_3(t) > 0 \) that
\[
\frac{d}{dt} \left( |\Theta_3(t)|^2 \right) \leq 0.
\]
Next let \( \mathcal{T} \subseteq [0, \infty) \) satisfy
\[
\mathcal{T} = \{ t \in [0, \infty) : \mu(I^{\Theta(t)}) \in [\epsilon, 1] \} \cup \{ t \in [0, \infty) : |\Theta_1(t)| \geq \epsilon, \mu(I^{\Theta(t)}) = 1 \}.
\]
(3.53)
Note that Lemmas 3.4 and 3.6 imply that \( \mathcal{C} = |\Theta_3(0)|^2 + \sup_{t \in \mathcal{T}} |\Theta_3(t)|^2 < \infty \). Now let \( \tau \in [0, \infty) \) be arbitrary, we will show that \( |\Theta_3(\tau)|^2 < \mathcal{C} + 3 \). Define
\[
u = \sup \{ t \in [0, \tau) : |\Theta_3(t)|^2 \leq \mathcal{C} \}
\]
(3.54)
and assume without loss of generality that \( \nu < \tau \) and \( \tau \notin \mathcal{T} \). Observe that this implies that \( \mu(I^{\Theta(\tau)}) < \epsilon \) or \( \mu(I^{\Theta(\tau)}) = 1 \). We now consider four cases.
Case 1. Assume $\mu(I^{\Theta(\tau)}) = 1$. In this case, we necessarily have $\forall t \in (u, \tau): \mu(I^{\Theta(t)}) = 1$. Indeed, otherwise by continuity of $t \mapsto \mu(I^{\Theta(t)})$ there would exist $t \in (u, \tau)$ with $\mu(I^{\Theta(t)}) \in (\varepsilon, 1)$. Hence $t \in T$ and $(\Theta_3(t))^2 \leq \mathcal{C}$, which contradicts (3.54). Furthermore, from (3.54) we obtain for all $t \in (u, \tau)$ that $|\Theta_1(t)| < \varepsilon$. In addition, Proposition 3.7 ensures for all $t \in (u, \tau)$ that $\frac{d}{dt}((|\Theta_3(t)|^2 + \ln(1 - |\Theta_1(t)|^2)) = 0$. Hence, we obtain that
\[
|\Theta_3(\tau)|^2 \leq |\Theta_3(u)|^2 + \ln(1 - |\Theta_1(\tau)|^2) - \ln(1 - |\Theta_1(u)|^2)] 
\leq |\Theta_3(u)|^2 + 2\ln(1 - \varepsilon^2)] \leq \mathcal{C} + 2|\Theta_1(u)|^2 < \mathcal{C} + 3. \tag{3.55}
\]

Case 2. Assume $\mu(I^{\Theta(\tau)}) < \varepsilon$ and $\Theta_3(\tau) < 0$. Since $[0, \infty) \ni t \mapsto \Theta_3(t) \in \mathbb{R}$ and $[0, \infty) \ni t \mapsto \mu(I^{\Theta(t)}) \in \mathbb{R}$ are continuous, (3.54) shows for all $t \in (u, \tau)$ that $\Theta_3(t) < -\sqrt{\mathcal{C}} < 0$ and $(I^{\Theta(t)} \subseteq [0, \varepsilon]) \vee (I^{\Theta(t)} \subseteq [1 - \varepsilon, 1])$. (I) and (II) therefore imply for all $t \in (u, \tau)$ that $\frac{d}{dt}(|\Theta_3(t)|^2) \leq 0$. Hence, we obtain that $|\Theta_3(\tau)|^2 \leq |\Theta_3(u)|^2 < \mathcal{C}$.

Case 3. Assume $\mu(I^{\Theta(\tau)}) < \varepsilon$, $\Theta_3(\tau) > 0$, and $I^{\Theta(t)} \subseteq [1 - \varepsilon, 1]$. By continuity of $[0, \infty) \ni t \mapsto \Theta_3(t) \in \mathbb{R}$ and (3.54) we obtain for all $t \in (u, \tau)$ that $\Theta_3(t) > \sqrt{\mathcal{C}} > 0$ and $I^{\Theta(t)} \subseteq [1 - \varepsilon, 1]$. (III) therefore demonstrates for all $t \in (u, \tau)$ that $\frac{d}{dt}((|\Theta_3(t)|^2 - \frac{5}{3}|\Theta_1(t) - 2^{-1/2}|^2) \leq 0$. This yields that
\[
|\Theta_3(\tau)|^2 \leq |\Theta_3(u)|^2 + \frac{5}{3}|\Theta_1(u)| - 2^{-1/2}|^2 - |\Theta_1(\tau) - 2^{-1/2}|^2] 
\leq \mathcal{C} + \frac{5}{2} < \mathcal{C} + 3. \tag{3.56}
\]

Case 4. Assume $\mu(I^{\Theta(\tau)}) < \varepsilon$, $\Theta_3(\tau) > 0$, and $I^{\Theta(t)} \subseteq [0, \varepsilon]$. By continuity of $[0, \infty) \ni t \mapsto \Theta_3(t) \in \mathbb{R}$ and (3.54) we obtain for all $t \in (u, \tau)$ that $\Theta_3(t) > \sqrt{\mathcal{C}} > 0$ and $I^{\Theta(t)} \subseteq [0, \varepsilon]$. (IV) therefore proves for all $t \in (u, \tau)$ that $\frac{d}{dt}((|\Theta_3(t)|^2 + \frac{1}{2}|\Theta_1(t)|^2) \leq 0$. This implies that
\[
|\Theta_3(\tau)|^2 \leq |\Theta_3(u)|^2 + \frac{5}{2}|\Theta_1(u)|^2 - |\Theta_1(\tau)|^2] \leq \mathcal{C} + \frac{5}{2} < \mathcal{C} + 3. \tag{3.57}
\]

The proof of Theorem 3.17 is thus complete. \hspace{1cm} \square

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