A SERIES OF SPECTRAL GAPS FOR THE ALMOST MATHIEU OPERATOR WITH A SMALL COUPLING CONSTANT

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Abstract. For the almost Mathieu operator with a small coupling constant, for a series of spectral gaps, we describe the asymptotic locations of the gaps and get lower bounds for their lengths. The results are obtained by analysing a monodromy matrix.

1. Introduction

We consider the almost Mathieu operator, acting in $l^2(\mathbb{Z})$ by the formula

$$ (H_\theta f)_k = f_{k+1} + f_{k-1} + 2\lambda \cos(2\pi(\theta + hk))f_k, \quad k \in \mathbb{Z}, $$

where $\lambda > 0$, $0 \leq \theta < 1$, and $0 < h < 1$ are parameters. The parameter $\lambda$ is called coupling constant. The operator (1.1) arises when studying an electron in a crystal submitted to a constant magnetic field when the field is weak, when it is strong, in semiclassical regime etc, see, e.g., [18] and references therein. This operator attracts attention of mathematicians as well as physicists thanks to its rich and unusual properties. One of the most difficult and interesting problems is the problem of describing the geometry of the spectrum of $H_\theta$. During three decades, efforts of many mathematicians have been aimed at proving that for irrational $h$ the spectrum is a Cantor set. Among them are A. Avila, J. Bellissard, B. Helffer, S. Zhitomirskaya, R. Krikorian, Y. Last, J. Puig, B. Simon, J. Sjöstrand and many others, see [1], where the proof was completed, and [20], which is one of the last articles devoted to analysis of the geometry of the spectrum of (1.1).

Among the papers of physicists explaining the cantorian structure of the spectrum, we single out the paper [24] where M. Wilkinson containing heuristic analysis clear for mathematicians. In the semiclassical approximation, he has successively described sequences of shorter and shorter spectral gaps, i.e., obtained a constructive description of the spectrum as a Cantor set. The spectrum located on certain intervals of the real line, being “put under the microscope”, looks like the spectrum of the almost Mathieu operator with new parameters. That is why the approach described by Wilkinson was called a renormalization method. Using methods of the pseudodifferential operator theory, B. Helffer and J. Sjöstrand developed a rigorous asymptotic renormalization method, and turned the heuristic results into mathematical theorems.

Let us note that the asymptotic renormalization methods can be used when the parameter $h$ can be represented by a continued fraction with sufficiently large elements.

Later, V. Buslaev and A. Fedotov suggested the monodromization method, one more renormalization approach that arose when trying to use the Bloch-Floquet theory ideas to study the geometry of the spectrum of difference operators in $L^2(\mathbb{R})$.

Key words and phrases. Almost Mathieu operator, small coupling, monodromy matrix, spectral gaps, asymptotics.

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The method was further developed by A. Fedotov and F. Klopp when studying adiabatic quasiperiodic operators. More details can be found in review [13]. The monodromization method can be used for one-dimensional two-frequency difference and differential quasiperiodic operators independently of any assumptions on the continued fraction. If such an equation contains an asymptotic parameter, one can effectively describe the asymptotic geometry of the spectrum. In [13] we described how to apply the monodromization method to get a constructive asymptotic description of the spectrum as a Cantor set in the case studied by B. Helffer and J. Sjöstrand. This paper is the first step to a similar constructive asymptotic description of the spectrum of the Harper equation with a small coupling constant. Here, we make only the first renormalization. This allows to get lower bounds for the gap lengths for a series of the longest spectral gaps.

The results we prove in this paper were announced in the short note [16]. Below \( C \) denotes positive constants independent of any parameters of the problem, variables and indices. When writing \( a = O(b) \), we mean that \( |a| \leq C|b| \), and when writing \( a = O_H(b) \), we mean that \( |a| \leq e^{C/h}|b| \).

Furthermore, for \( z \in \mathbb{C} \), we often use the notations \( x = \text{Re}\ z \) and \( y = \text{Im}\ z \).

2. Main results

It is well known, see, for example, [19], that for the irrational \( h \), as a set, the spectrum of the almost Mathieu operator is independent of the parameter \( \theta \) and coincides with the spectrum of the operator acting in \( L^2(\mathbb{R}) \) by the formula \( H\psi(x) = \psi(x+h) + \psi(x-h) + 2\lambda \cos(2\pi x)\psi(x) \). This operator is called the Harper operator. It is a difference Schrödinger operator with a 1-periodic potential. Below we discuss only this operator.

As for the one-dimensional periodic differential operators, for the operator \( H \), one can define a monodromy matrix. Here, we describe asymptotics of a monodromy matrix and spectral results obtained by means of these asymptotics.

2.1. Monodromy matrix.

2.1.1. Let us consider the Harper equation

\[
\psi(x + h) + \psi(x - h) + 2\lambda \cos(2\pi x)\psi(x) = E\psi(x), \quad x \in \mathbb{R},
\]

where \( E \in \mathbb{C} \) is a spectral parameter. Its solution space is invariant with respect to the operator of translation by one. Let us fix a basis in the solution space. The corresponding monodromy matrix represents the restriction of the translation operator to the solution space. Equation (2.1) being a second-order difference equation on \( \mathbb{R} \), and its solution space is a two-dimensional module over the ring of \( h \)-periodic functions, see, e.g. [8]. Thus, a monodromy matrix is a \( 2 \times 2 \) matrix \( h \)-periodic in \( x \). The reader finds the formal definition in section 3.1.3.

The following two theorems describe the functional structure of one of the monodromy matrices.

**Theorem 2.1.** In the solution space of (2.1), there a basis such that the corresponding monodromy matrix is of the form

\[
M(x) = \begin{pmatrix}
a - 2\lambda_1 \cos(2\pi x) & s + t e^{-2\pi i x} \\
-s - t e^{2\pi i x} & st/\lambda_1
\end{pmatrix}, \quad a = \lambda_1 \frac{1 - s^2 - t^2}{st}, \quad \lambda_1 = \lambda_1^k,
\]

where the coefficients \( s \) and \( t \) are independent of \( x \) and meromorphic in \( E \).
This theorem is a part of Theorem 7.2 from [8]. The basis solutions are minimal entire solutions, i.e., solutions entire functions of $x$ and growing the most slowly both as $\text{Im} \ x \to -\infty$ and as $\text{Im} \ x \to +\infty$. These solutions are meromorphic in $E$.

**Remark 2.1.** It follows from the proof of this theorem (see [8]) that if, for given $h = h_0 \in (0, 1)$ and $\lambda = \lambda_0 \in (0, \infty)$, the coefficients $s$ and $t$ are analytic at $E = E_0$, then they are continuous in $(h, \lambda, E)$ in a neighborhood of $(h_0, \lambda_0, E_0)$.

In Section 6, we check

**Theorem 2.2.** For $E \in \mathbb{R}$,

$$
(2.3) \quad t \in i\mathbb{R}, \quad |s| = \lambda_1 \sqrt{1 + |t|^2 / \lambda_1^2 + |t|^2},
$$

and the zeroth Fourier coefficient of the trace of the monodromy matrix equals

$$
(2.4) \quad L = \frac{2i}{t} \sqrt{(1 + |t|^2)(\lambda_1^2 + |t|^2) \cos(\arg(is))}.
$$

Relations (2.3) and (2.4) reflect the self-adjointness of the Harper operator.

2.1.2. Pick $a \in (0, \pi)$. The asymptotics of the coefficients $s$ and $t$ as $\lambda \to 0$ are described in terms of a meromorphic function $\sigma_a$ satisfying the equation

$$
(2.5) \quad \sigma_a(z + a) = (1 + e^{-iz})\sigma_a(z - a), \quad z \in \mathbb{C}.
$$

Let $S = \{z \in \mathbb{C} : |\text{Re} \ z| < \pi + a\}$. The function $\sigma_a$ is uniquely characterized by the following properties. In the strip $S$, it is analytic, does not vanish, tends to one as $y \to -\infty$ and has the minimal possible growth as $y \to +\infty$. This function and functions related to it arose in different areas of mathematical physics, see, e.g., [8, 2, 3, 10, 15, 21, 23]. We discuss $\sigma_a$ in section 8. Let

$$
(2.6) \quad F_0(p) = \sigma_{\pi h}(4\pi p - \pi + \pi h) \sigma_{\pi h}(4\pi p - \pi + \pi h).
$$

Below, we also use the parameter $p$ related to $E$ by the equation

$$
(2.7) \quad E = 2 \cos(2\pi p)
$$

The most of this paper is devoted to obtaining the asymptotics of $t$ and $s$ as $\lambda \to 0$. Let $\xi = \frac{h}{2\pi} \ln \lambda$. One has

**Theorem 2.3.** Pick $\beta$, $0 < \beta < 1/2$. Let $|\text{Im} \ p| \leq h$ and $h/4 < \text{Re} \ p < 1/2 - h/4$. As $\lambda \to 0$

$$
(2.8) \quad t = \frac{ie^{4\pi(1/2 - p)/\hbar}}{2 \sin(2\pi p)} F_0(p) (1 + O_H(\lambda^\beta)), \quad s = \frac{-2i e^{4\pi(1/2 + p)/\hbar} \sin(2\pi p)}{F_0(p)} (1 + O_H(\lambda^\beta)).
$$

The asymptotics of $s$ and $t$ near the points $p = 0$ and $p = 1/2$ are more complicated. They are described by Theorem 7.1.

To get the asymptotics of $s$ and $t$, we obtain asymptotics of minimal entire solutions to equation (2.1) as $\lambda \to 0$. For small $\lambda$, they appear to be close to solutions to the equations $\psi(x + h) + \psi(x - h) + \lambda e^{\pm 2\pi i x/a} \psi(x) = E \psi(x)$ in the half-planes $\mathbb{C}_\pm$ respectively. Having constructed analytic in $\mathbb{C}_\pm$ solutions to the Harper equation, we made of them the minimal entire solutions with the help of a Riemann-Hilbert problem.

Our asymptotic method works if $\lambda << e^{-C/h}$. If $h$ is so small that $\lambda >> e^{-C/h}$, then the asymptotics of the solutions to the Harper equation can be obtained by semi-classical methods, see, e.g., [4].
2.2. Spectral gaps. First renormalization of the monodromization method consists in replacing equation (2.1) with the first monodromy equation
\[
\Psi(z + h_1) = M_1(z)\Psi(z), \quad z \in \mathbb{C}, \quad h_1 = \{1/h\},
\]
where $M_1$ is a monodromy matrix, and $\{x\}$ is the fractional part of $x \in \mathbb{R}$. Equations (2.1) and (2.9) simultaneously have pairs of linearly independent solutions, one solution of a pair decaying exponentially as $x \to +\infty$, and the other decaying as $x \to -\infty$, see [13]. This allows to find gaps in the spectrum of the Harper equation by studying solutions to the monodromy equation. In section 3.2 we prove

**Theorem 2.4.** Let $I \subset \mathbb{R}$ be an open interval. There exists such a constant $C$ independent of $h$ and $E$ that if
\[
(L/2)^2 \geq (1 + C\lambda_1)(1 + |t|^2), \quad \lambda_1 \leq |t| \leq 1, \quad \forall E \in I,
\]
then $I$ is in a gap of the Harper operator.

It is useful to compare this theorem with a well-known theorem from the theory of the one-dimensional periodic differential Schrödinger operators. It says that the spectrum of a periodic operator is located on the intervals where the absolute value of trace of a monodromy matrix is less than or equal to two.

Using Theorem 2.4, formula (2.4) and the asymptotics $s$ and $t$ described in Theorems 2.3 and 7.1, one can describe a sequence of the longest gaps in the spectrum of the Harper operator. As its spectrum is symmetric with respect to zero, and as the spectra for the frequencies $h$ and $1-h$ coincide, we consider only the spectrum located on the positive semi-axis $\mathbb{R}_+$ in the case where $0 < h < 1/2$. Let $[x]$ denotes the integer part of $x \in \mathbb{R}$. One has

**Theorem 2.5.** Let $0 < h < 1/2$ and $\beta \in (0,1/2)$. Let us assume that $\lambda \leq e^{-c/h}$ with a sufficiently large $c$. If $[1/h]$ is even, we also require that $\lambda^{h_1} \leq e^{-c/h}$. Then, in the spectrum of the Harper equation on $\mathbb{R}_+$, there is a sequence of gaps $\{g_k\}_{k=1}^K$, $K = [1/2h]$. The $k$-th gap contains the point
\[
E_k = 2\cos(\pi h k + O_h(\lambda^p)),
\]
and its length $|g_k|$ satisfies the estimate
\[
|g_k| \geq 4h \left( \frac{\lambda}{4} \right)^k \frac{(1 + O_h(\lambda^p))}{\sin^2(2\pi h)\ldots\sin^2(\pi h(k - 1))}. \tag{2.11}
\]

In these formulas, $p = \beta$ if $1 \leq k \leq K - 1$ or if $k = K$ and $[1/h]$ is odd, and $p = \min\{h_1, \beta\}$ if $k = K$ and $[1/h]$ is even. Furthermore, for $k = 1$, the product of the sines in (2.11) has to be replaced with one.

In the case of even $[1/h]$, the $K$th interval containing the spectrum is the most difficult to describe. This reflects the fact that, for small $hh_1$, it is located near zero that is a very special point. If $h = p/q$ where $p$ and $q$ are coprime integers, and $q$ is even, there is a gap containing zero, and if $q$ is odd, zero is inside the spectrum. The complexity of the spectrum as a set near zero for $h \notin \mathbb{Q}$ is well-known. One can find a series of new results in [20].

We also note that, under the condition $\lambda \leq e^{-c/h}$, the number of gaps, $K$, can be of the order of $\ln \frac{1}{h}$, and thus, can be large.

The results described in this theorem agree with numerical results from [18]. In the next paper, we will give a proof that the expression in the right hand side of (2.11) is the leading term of the asymptotics of the length of the $k$th gap. It is interesting to compare our results with the results obtained in the case of small $h$ and $\lambda = 1$, see [13]. In this case, there is a statement similar to Theorem 2.4. However, the asymptotics of the coefficients $t$ and $s$ turn out to be quite different,
on the most of the interval $[-4, 4]$ containing the spectrum, $L(E)$ oscillates with an exponentially large with respect to $h$ amplitude, whereas in our case, for most $E \in (0, 2)$, $L \approx 2 \cos(p/h)$. Thus, in the case of small $h$, the spectrum is located on a series of exponentially small intervals, and in the case of small $\lambda$, we observe small gaps in the spectrum.

2.3. Other gaps. Since the matrix $M_1$ is 1-periodic, for equation (2.9), we can also define a monodromy matrix $M_2$ and consider the second monodromy equation that can be obtained from the first one by replacing $M_1$ with $M_2$ and $h_1$ with $h_2 = \{1/h_1\}$. Continuing, we can construct an infinite sequence of difference equations. In the next paper, studying consequently the equations of this sequence, we will describe consequently series of shorter and shorter gaps. We also obtain upper bounds for the gaps lengths. For this, we compute the increments of the integrated density of states between the gap we study and two gaps close to its ends. In the framework of the monodromization method, such computations are quite natural, see [11].

2.4. The plan of the paper. In section 3, we give the definition of a monodromy matrix, and prove Theorems 2.4 and 2.5. To prove the last one, we use Theorem 2.3 on the asymptotics of the monodromy matrix described in Theorem 2.1. The most of the remaining part of the paper is devoted to the proof of Theorem 2.3. In section 4, we construct and analyze analytic solutions to the model equation (4.1). In the upper half-plane, they appear to be close to solutions to the Harper equation. In section 5, in terms of the solutions to the model equation, we construct solutions to the Harper equation analytic in the upper half-plane. Recall that the monodromy matrix described in Theorem 2.1 corresponds to a basis of two minimal entire solutions to the Harper equation. In section 6, we recall the definition of minimal entire solutions and prove Theorem 2.2. In section 7, for sufficiently small $\lambda$, using the analytic solutions to the Harper equation constructed in section 5, we construct and study the minimal entire solutions, and prove Theorem 2.3. In Section 8, we recall properties of the function $\sigma_a$ that are used in this paper.

3. Monodromy matrices, monodromy equation and spectral results

Here we remind the definition of a monodromy matrix, describe relations between solutions of a difference equation with periodic coefficients and solutions of a corresponding monodromy equation, and prove Theorems 2.4 and 2.5.

3.1. Monodromy matrices and monodromy equation.

3.1.1. Definition and elementary properties of a monodromy matrix. Here, following [13] we discuss the difference equations of the form

\begin{equation}
\Psi(x + h) = M(x)\Psi(x),
\end{equation}

where $x$ is a real variable, $M : \mathbb{R} \to SL(2, \mathbb{C})$ is a given 1-periodic function, and $h \in (0, 1)$ is a fixed number. Obviously, for any solution $\Psi$ to (3.1), we have $\det \Psi(x + h) = \det \Psi(x)$, $x \in \mathbb{R}$. We say that a solution $\Psi : \mathbb{R} \to M_2(\mathbb{C})$ is fundamental, if its determinant is a nonzero constant.

Note that, to construct a fundamental solution, it suffices to define it arbitrarily on the interval $0 < x < h$, and then, to define its values outside of this interval directly with the help of equation (3.1).
It can be shown that a matrix function $\tilde{\Psi} \in M_2(\mathbb{C})$ is a matrix solution to (3.1) if and only if it can be represented in the form
\[(3.2) \quad \tilde{\Psi}(x) = \Psi(x) p(x), \quad x \in \mathbb{R},\]
where $p : \mathbb{R} \to M_2(\mathbb{C})$ is an $h$-periodic function, and $\Psi$ is a fundamental solution.

Note that this representation implies that the space of solutions to (3.1) is a module over the ring of $h$-periodic functions.

Let $\psi_1, \psi_2 : \mathbb{R} \to \mathbb{C}^2$ be two vector solutions to (3.1). We say that they are linearly independent if $\det(\psi_1, \psi_2)$ does not vanish. In this case, a function $\psi : \mathbb{R} \to \mathbb{C}^2$ is a solution to (3.1) if and only if it is a linear combination of $\psi_1$ and $\psi_2$ with $h$-periodic coefficients.

Let $\Psi$ be a fundamental solution. As $M$ is 1-periodic, the function $x \mapsto \Psi(x + 1)$ is also a solution to (3.1), and we can write
\[(3.3) \quad \Psi(x + 1) = \Psi(x) p(x), \quad x \in \mathbb{R}.\]

The matrix $M_1(x) = p'(hx)$, where $'$ denotes transposition, is called the monodromy matrix corresponding to the fundamental solution $\Psi$.

Note that, by construction, the monodromy matrix is 1-periodic, and that its definition and the definition of a fundamental solution imply that the monodromy matrix is unimodular.

In early papers, the $h$-periodic matrix $p(x)$ was called the monodromy matrix. It turns out to be more natural to consider 1-periodic monodromy matrices.

### 3.1.2. Monodromy equation

Let $M_1$ be a monodromy matrix corresponding to a fundamental solution $\Psi$ to (3.1). Let us consider the first monodromy equation (2.9).

It appears that the behavior of the solutions to (3.1) at infinity “copies” the one for (2.9). Let us formulate the precise statement.

Let $M$ be a $SL(2, \mathbb{C})$-valued function of real variable, and $h > 0$. Let $k \in \mathbb{Z}$ and $x \in \mathbb{R}$. We put
\[P_k(M, x, h) = M(x + h(k - 1)) \cdots M(x + h)M(x), \quad k \geq 0,
\]
and
\[P_k(M, x, h) = M^{-1}(x + 2h) \cdots M^{-1}(x - 2h)M^{-1}(x - h), \quad k < 0.
\]
Clearly, if $\psi : \mathbb{R} \to \mathbb{C}^2$ satisfies (3.1), then
\[(3.4) \quad \psi(x + hk) = P_k(M, x, h)\psi(x).
\]
One has

**Theorem 3.1.** [14] Let $\Psi$ be a fundamental solution to (3.1), and let $M_1$ be the corresponding monodromy matrix. Then, for all $N \in \mathbb{Z}$,
\[(3.5) \quad P_N(M, h, x) = \Psi([x + Nh])\sigma_2 P_N(M_1, h_1, x_1)\sigma_2^{-1}(x),
\]
\[(3.6) \quad N_1 = -[\theta + N\theta], \quad h_1 = \{1/h\}, \quad x_1 = \{x/h\},
\]
where $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ is the Pauli matrix.

In this paper we use

**Corollary 3.1.** Let in the case of Theorem 3.1 $\Psi \in L_{\text{loc}}^\infty(\mathbb{R}, SL(2, \mathbb{C}))$, and let there exist two vector solutions $\psi_\pm^{(1)}$ to the monodromy equation satisfying the estimates
\[(3.7) \quad \|\psi_\pm^{(1)}(\pm x)\|_{C^2} \leq C_0 e^{\mp \kappa x}, \quad x \geq 0,
\]
with some positive constants \( C_0 \) and \( \kappa \). Then there two vector solutions \( \psi^{(0)}_\pm \) to equation (3.1) such that

\[
\det \left( \psi^{(0)}_-(x), \psi^{(0)}_+(x) \right) = \det \left( \psi^{(1)}_-(\{x/h\}), \psi^{(1)}_+(\{x/h\}) \right), \quad \forall x \in \mathbb{R},
\]

\[
\|\psi^{(0)}_\pm(\pm x)\|^2 \leq C_1 e^{\mp \kappa x/H} x, \quad \forall x \geq 0,
\]

with a positive constant \( C_1 \).

Proof. As \( P_N(M, h, x)\Psi(x) = \Psi(x + N h) \), formula (3.5) implies that

\[
\Psi(x + N h) \sigma_2 = \Psi(\{x + N h\}) \sigma_2 P_N(M, h_1, x_1).
\]

Let us define the solutions \( \psi^{(0)}_\pm : \mathbb{R} \to \mathbb{C}^2 \) to (3.1) by the formulas

\[
\psi^{(0)}_\pm(x) = \Psi(x) \sigma_2 \psi^{(1)}_\pm(\{x/h\}).
\]

As \( \det \Psi \equiv 1 \), relation (3.8) is obvious. Furthermore, as \( x_1 = \{x/h\} \), and \( N_1 = -[x + N h] \), formulas (3.11) and (3.10) lead to the relation

\[
\psi^{(0)}_\pm(x + N h) = \Psi(\{x + N h\}) \sigma_2 \psi^{(1)}_\pm(x_1 - [x + N h] h_1), \quad N \in \mathbb{Z},
\]

that can be rewritten in the form

\[
\psi^{(0)}_\pm(x) = \Psi(\{x\}) \sigma_2 \psi^{(1)}_\pm(x_1 - [x] h_1), \quad x \in \mathbb{R}.
\]

This formula and estimates (3.7) imply (3.9). The proof is complete. \( \square \)

3.1.3. Monodromy matrices for difference Schrödinger equations. Let \( h > 0 \) and \( v : \mathbb{R} \to \mathbb{C} \). The difference Schrödinger equation

\[
\psi(x + h) + \psi(x - h) + v(x) \psi(x) = E \psi(x), \quad x \in \mathbb{R},
\]

is equivalent to (3.1) with

\[
M(z) = \begin{pmatrix} E - v(x) & -1 \\ 1 & 0 \end{pmatrix}.
\]

More precisely, a function \( \Psi : \mathbb{R} \to \mathbb{C}^2 \) satisfies (3.1) with this matrix if and only if \( \Psi(x) = \begin{pmatrix} \psi(x) \\ \psi(x - h) \end{pmatrix} \), and \( \psi \) is a solution to the Harper equation. This allows to transform all observations made for (3.1) into observations for (3.12) .

Let \( \psi_1 \) and \( \psi_2 \) be two solutions to (2.1). The expression

\[
\{\psi_1(x), \psi_2(x)\} = \psi_1(x + h) \psi_2(x) - \psi_1(x) \psi_2(x + h),
\]

their Wronskian, is \( h \)-periodic in \( x \).

Assume that the Wronskian is constant and nonzero. Then \( \psi_{1,2} \) form a basis in the space of solutions, and a function \( \psi \) satisfies (3.12) if and only if

\[
\psi(x) = a(x) \psi_1(x) + b(x) \psi_2(x),
\]

where \( a \) and \( b \) are \( h \)-periodic coefficients.

One easily proves that

\[
a(x) = \frac{\{\psi_1(x), \psi_2(x)\}}{\{\psi_1(x), \psi_2(x)\}}, \quad b(x) = \frac{\{\psi_1(x), \psi_2(x)\}}{\{\psi_1(x), \psi_2(x)\}}.
\]

If \( v \) is 1-periodic, the functions \( x \to \psi_1(x + 1) \) and \( x \to \psi_2(x + 1) \) are also solutions to (3.12), and one can write

\[
\Psi(x + 1) = M_1(x/h) \Psi(x), \quad \Psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}, \quad x \in \mathbb{R},
\]

where \( M_1 \) is a 1-periodic two-by-two matrix. It is the matrix monodromy corresponding to the basis \( \psi_1 \) and \( \psi_2 \). It coincides with a monodromy matrix for (3.1) with the matrix (3.13).
Lemma 3.1. In the case of Theorem 2.4, for any $E \in \mathbb{I}$, there exist two vector solutions $\psi^{(1)}_\pm$ to (2.9) such that $\det(\psi^{(1)}_+(x), \psi^{(1)}_-(x))$ is a nonzero constant, and, for $x \geq 0$, \[ \|\psi^{(1)}_\pm(x)\|_{C^2} \leq C e^{tx} \ln \frac{1}{|\lambda|}, \] $L$ being the zeroth Fourier coefficient of trace of the monodromy matrix.

This proposition implies

Lemma 3.1. In the case of Theorem 2.4, for any $E \in \mathbb{I}$, there are no polynomially bounded nontrivial solutions to the Harper equation (2.1).

First, using the last proposition and lemma, we prove Theorem 2.4. Assume that, for some $\theta \in \mathbb{R}$, the almost Mathieu operator (1.1) has some spectrum on $I$. Then by the Avron-Simon theorem from section 2.4 in [9], on $I$ almost everywhere with respect to the spectral measure, there is a nontrivial polynomially bounded solution $f$ to the almost Mathieu equation

\[
(3.18) \quad f_{k+1} + f_{k-1} + 2\lambda \cos(2\pi(\theta + kh))f_k = Ef_k, \quad k \in \mathbb{Z}.
\]

One defines a solution $\psi$ to the Harper equation so that $\psi(x) = f_k(\theta)$ if $x = \theta + kh$ with $k \in \mathbb{Z}$, and $\psi(x) = 0$ otherwise. This $\psi$ is a non-trivial polynomially bounded solution to the Harper equation. But it can not exist in view of Lemma 3.1.

Thus, for any $\theta \in \mathbb{R}$, on $I$ there is no spectrum of the almost Mathieu operator. This implies that on $I$ there is no spectrum of the Harper operator as the latter is a direct integral of the almost Mathieu operators (with respect to $\theta$).

Now, to prove Theorem 2.4, we have to check Lemma 3.1 and Proposition 3.1.

Proof of Lemma 3.1. Let us assume that, for an $E \in \mathbb{I}$, there is a nontrivial polynomially bounded solution $\psi$. For this $E$ we construct the solutions to the monodromy equation described in Proposition 3.1. Then, in terms of these solutions, we construct the solutions $\psi^{(0)}_\pm$ to equation (3.1) with matrix (3.13) as described in Corollary 3.1 (this is possible as the fundamental solution used to define the monodromy matrix is entire in $x$).

Let $\psi_1$ be the first entry of $\psi^{(0)}_+$, and $\psi_2$ be the one of $\psi^{(0)}_-$. One has

\[
\{\psi_1(x), \psi_2(x)\} = \det(\psi^{(0)}_+(x), \psi^{(0)}_-(x)) = \det(\psi^{(1)}_+(x), \psi^{(1)}_-(x)),
\]

where we used Corollary 3.1. Thus, by Proposition 3.1, $\{\psi_1(x), \psi_2(x)\}$ is a nonzero constant. Therefore, one has (3.15)–(3.16). Now, it suffices to show that the Wronskians $\{\psi(x), \psi_j(x)\}$, $j = 1, 2$, equal zero. But this is obvious, as these Wronskians are periodic and, on the other hand, $\{\psi(x), \psi_\pm(x)\} \to 0$ as $x \to \pm \infty$ since $\psi$ is polynomially bounded, and $\psi_\pm$ are exponentially decreasing as $x \to \pm \infty$. The proof of Lemma 3.1 is complete.

Now, let us prove Proposition 3.1.

Proof. Below we assume that $E \in \mathbb{I}$. In view of Theorem 2.1, for $x \in \mathbb{R}$, we can represent the monodromy matrix in the form

\[
(3.19) \quad M(x) = M_0 + \tilde{M}(x), \quad \tilde{M}(x) = O(t), \quad M_0 = \begin{pmatrix} \lambda s/\alpha & (1 - s^2 - t^2) \alpha \\ -s & -s \end{pmatrix}.
\]

Assuming that $M_0$ has two distinct real eigenvalues, we transform the monodromy equation to the form

\[
(3.20) \quad \phi(x + h) = p(D + \Delta(x)) \phi(x), \quad D = \begin{pmatrix} 1/U & 0 \\ 0 & U \end{pmatrix}, \quad x \in \mathbb{R},
\]
where \( p \geq 1 \), \( U \geq 1 \) and \( \Delta(x) \) is a matrix “sufficiently small” in the case of Theorem 2.5. Then, we construct two solutions to (3.20) by means of

**Proposition 3.2.** Let us consider equation (3.20) with given constants \( h > 0 \), \( p \geq 1 \) and \( U > 1 \), and \( p(D + \Delta) \in L^\infty(\mathbb{R}, SL(2, \mathbb{C})) \). Let

\[
(3.21) \quad U - U^{-1} > 4m, \quad m = \max_{1 \leq i,j \leq 2} \sup_{x \in \mathbb{R}} |\Delta_{ij}(x)|.
\]

There exist \( \phi_{\pm} \in L^\infty_{\text{loc}}(\mathbb{R}, \mathbb{C}^2) \), vector solutions to (3.20), such that

\[
(3.22) \quad \|\phi_{\pm}(x)\|_{C^2} \leq C e^{\frac{\pi}{8} \ln \frac{p(U^{-1} + U)}{4}} \forall x \geq 0, \quad \sup_{x \in \mathbb{R}} |\det(\phi_{\pm}(x), \phi_{\mp}(x))| < 1.
\]

This proposition is a generalized version of Proposition 4.1 from [12]: there we assumed that \( p = 1 \). Mutatis mutandis, the proof of Proposition 3.2 repeats the old one.

The \( \det(\phi_{\pm}, \phi_{\mp}) \) being \( h \)-periodic, the function \( x \mapsto \phi_{\pm}(x)/\det(\phi_{\pm}(x), \phi_{\mp}(x)) \) satisfies (3.20). Now we denote this new function by \( \phi_{\pm} \). This new function belongs to \( L^\infty_{\text{loc}}(\mathbb{R}, \mathbb{C}^2) \), satisfies the old estimate (3.22), and, for this new function, we have \( \det(\phi_{\pm}(x), \phi_{\mp}(x)) = 1 \). So, we can and do assume that this equality is valid for \( \phi_{\pm} \) from Proposition 3.2.

Let us transform the monodromy equation to the form (3.20). Therefore, we compute the eigenvalues and eigenvectors of \( M_0 \). In view of Theorem 2.2, one has

\[
(3.23) \quad t = i\tau, \quad \tau \in \mathbb{R}, \quad s = -i\lambda_1 \sqrt{1 + \tau^2 \lambda_1^2} e^{i\alpha}, \quad \alpha \in \mathbb{R},
\]

Let

\[
(3.24) \quad p = \sqrt{1 + \tau^2}, \quad q = \frac{1}{\tau} \sqrt{\tau^2 + \lambda_1^2}, \quad Q = \sqrt{q^2 \cos^2 \alpha - 1}.
\]

Then

\[
(3.25) \quad \text{tr} \, M_0 = 2pq \cos \alpha, \quad \det M_0 = p^2.
\]

The eigenvalues \( \nu_{\pm} \) and the corresponding eigenvectors \( v_{\pm} \) are given by the formulæ

\[
(3.26) \quad \nu_{\pm} = p(q \cos \alpha \pm Q), \quad v_{\pm} = \left( 1, \frac{1}{q} \left( q \cos \alpha - \frac{1}{q} e^{i\alpha} \mp Q \right) \right).
\]

Let \( V = (v_+, v_-) \) be the matrix with the columns \( v_{\pm} \). We represent a vector solution to (2.9) in the form

\[
(3.27) \quad \psi(x) = V \phi(x).
\]

Then \( \phi \) satisfies equation (3.20) with

\[
(3.28) \quad U = q \cos \alpha + Q \quad \text{and} \quad \Delta(x) = \frac{1}{p} V^{-1} M(x) V.
\]

Let us determine the conditions under which \( U \) and \( \Delta \) from (3.28) satisfy the assumptions of Proposition 3.2. Let

\[
(3.29) \quad q^2 \cos^2 \alpha > 1.
\]

We can and do assume that \( q \cos \alpha > 1 \), \( Q > 0 \). Then \( U > 1 \).

Now, let us estimate the entries of \( \Delta \). One has

\[
|\sin \alpha| = \sqrt{1 - \cos^2 \alpha} \leq \sqrt{1 - 1/q^2} \leq \frac{|\lambda_1|}{\tau},
\]

\[
0 < Q = \sqrt{ \left( 1 + \frac{\lambda_1^2}{\tau^2} \right) \cos^2 \alpha - 1} \leq \frac{|\lambda_1|}{\tau}.
\]
Therefore and as $\lambda_1 \leq |\tau|$, the second entries of $v_\pm$ are uniformly bounded:

$$
\left| \frac{p}{s} \left( q \cos \alpha - \frac{1}{q} e^{i\alpha} + Q \right) \right| = \left| \frac{p}{qs} \right| \frac{\lambda_1^2}{\tau^2} \cos \alpha - i \sin \alpha \mp qQ \leq C \frac{p \lambda_1}{|q s \tau|} = C.
$$

As $\det V = -2pQ/s$, this estimate implies that, for all $i, j \in \{1, 2\}$,

$$
(3.30) \quad \max_{x \in \mathbb{R}} |\Delta_{ij}(x)| \leq C \left( \frac{s}{p^2 Q} \right) = C \left| \frac{\lambda_1}{pqQ} \right| < C \frac{\lambda_1}{Q}.
$$

Therefore, the second condition from (3.21) is satisfied if

$$
(3.31) \quad Q = \frac{U - U^{-1}}{2} \geq C \frac{\lambda_1}{Q} \iff q^2 \cos^2 \alpha \geq 1 + C \lambda_1.
$$

Clearly, this condition implies (3.29) and is equivalent to (2.10).

Let assume that (3.31) is satisfied. Then, using Proposition 3.2 and formula (3.27), we construct two vector solutions $\psi_\pm = V \phi_\pm$ to the monodromy equation. As $p(U^{-1} + U)/2 = pq \cos \alpha = L/2$, estimates (3.22) imply the estimates for $\psi_\pm$ from Proposition 3.1. Furthermore, one has

$$
\det(\psi_+(x), \psi_-(x)) = \det V \det(\phi_+(x), \phi_-(x)) = \det V \neq 0.
$$

The proof is complete. \(\square\)

### 3.3. Gaps in the spectrum of the Harper equation: proof of Theorem 2.5.

Theorem 2.5 describing a sequence of gaps in the spectrum of the Harper operator follows from Theorem 2.4, a sufficient condition for $E$ to be in a gap written in terms of the coefficients of a monodromy matrix, and Theorem 2.3 describing the asymptotics of the monodromy matrix as $\lambda \to 0$.

Below we assume that

$$
(3.32) \quad p \in I_p = [h/4, 1/4].
$$

where $p$ is the parameter related to $E$ by (2.7). In view of (2.4), condition (2.10) can be written in the form

$$
(3.33) \quad (1 + X^2) \cos^2 \alpha \geq 1 + C_0 \lambda_1, \quad X = \lambda_1/\tau = i \lambda_1/\ell,
$$

where $\alpha = \arg is$, $s$ and $\ell$ being the coefficients from Theorem 2.1, $\lambda_1 = \lambda_1/\tau$, and $C_0$ is a certain positive constant.

Below, when analyzing (3.33), in most of the statements, we make

**Hypothesis 1.** For a $\beta \in (0, 1/2)$, one has $\lambda \leq c^{-c/h}$, where $c$ is a sufficiently large positive constant.

#### 3.3.1. Locations of gaps.

Inequality (3.33) can be rewritten as $X^2 - C_0 \lambda_1 \geq \sin^2 \alpha$, and assuming that $\lambda_1/X^2$ and $X^2$ are sufficiently small, we transform it to the form

$$
(3.34) \quad |X| (1 + O(\lambda_1/X^2) + O(X^2)) \geq |\sin \alpha|,
$$

and see that there are gaps located near the points $E_k$ defined by the relations

$$
(3.35) \quad E_k = 2 \cos(2\pi p_k), \quad \alpha(p_k) = \pi k, \quad k \in \mathbb{Z}.
$$

Later, we shall see that all $p_k \in I_p$ are really located in gaps.

To continue, we need the following two lemmas:

**Lemma 3.2.** Under hypothesis 1, for $p \in I_p$, one has

$$
(3.36) \quad \alpha = \frac{2\pi p}{h} + O_H(\lambda^3).
$$

If $p > Ch > h/4$, the error term is analytic in $p$ and satisfies the estimate

$$
(3.37) \quad \left( O_H(\lambda^3) \right)'_p = O_H(\lambda^3).
$$
Proof. Pick \( c \) sufficiently small. Assume that \( p \) is in \( V_{hA} \), the \((ch)\)-neighborhood of the interval defined in (3.32). Then formula (3.36) follows from Theorem 2.3.

The description of the zeros and poles of the function \( \sigma_a \) from section 8.1.2 implies that the expression \( \sin(2\pi p)/F_0(p) \) is analytic and has no zeros in \( V_{ch} \). As \( s \) is a meromorphic function of \( E \), this observation and the second formula in (2.8) imply that (1) \( s \) is bounded and, thus, analytic in \( p \in V_{ch} \), and thus that (2) the error term in this formula is also analytic in \( V_{ch} \).

The estimate (3.37) follows from the analyticity of the error term and the Cauchy representation for the derivatives of analytic functions. \( \square \)

**Lemma 3.3.** Under hypothesis 1, one has

\[
(3.38) \quad X = X_0(p)\left(1 + O_h(\lambda^3)\right), \quad X_0(p) = e^{2p\ln\lambda/h} \frac{2\sin(2\pi p)}{F_0(p)},
\]

\[
(3.39) \quad \ln X_0(p) = \frac{2p\ln\lambda}{h} + O\left(\frac{1}{h}\right), \quad \left(O\left(\frac{1}{h}\right)\right)'_p = O\left(\frac{1}{h}\right).
\]

Proof. Formulas (3.38) follow from Theorem 2.3.

Under the condition (3.32), we get

\[
(3.40) \quad \ln X_0 = \frac{2p\ln\lambda}{h} - \ln F_0(p) + \ln p + O(1), \quad \frac{dO(1)}{dp} = O(1).
\]

Recall that \( F_0 \) is given by (2.6). So, we have to study the function \( p \mapsto f(p) = \ln \sigma_{ch}(4\pi p - \pi + \pi h) \). For this we fix \( \delta \in (0, \pi) \).

By Corollary 8.1 we get

\[
(3.41) \quad f(p) = O(1/h) \quad \text{if} \quad |4\pi p + \pi h| \geq \delta, \quad -h/4 \leq Re p \leq 1 \quad \text{and} \quad |Im p| \leq 1.
\]

Estimate (3.41) and the Cauchy representations for the derivatives of analytic functions imply that

\[
(3.42) \quad f'(p) = O(1/h) \quad \text{if} \quad \delta/4\pi - h/4 \leq p \leq 1/4.
\]

It looks like that here one has to replace \( \delta \) with a greater number, but, \( \delta \) being chosen quite arbitrarily, (3.42) is valid for any fixed \( \delta \in (0, \pi) \).

If \( \max\{\delta/4\pi - h/4, h/4\} \leq p \leq 1/4 \), then estimates (3.41), (3.42), and (3.40) imply (3.39).

If \( |4\pi p + \pi h| \leq \delta \), then Theorem 8.2 implies that

\[
(3.43) \quad f(p) = \frac{2p\ln h}{h} + \ln \Gamma\left(\frac{2p}{h} + 1\right) + O\left(\frac{1}{h}\right), \quad \left(O\left(\frac{1}{h}\right)\right)'_p = O\left(\frac{1}{h}\right).
\]

When deriving this formula, we estimated the derivative of the error term from formula (8.10) arguing as when proving (3.42).

To estimate \( \ln \Gamma \), we use the Stirling formula \( (\ln \Gamma)(x + 1) = x(\ln x - 1) + O(\ln x) \). Note that the error term satisfies the estimate \( (O(\ln x))' = O(1/x) \). We get

\[
\begin{align*}
\frac{2p\ln p}{h} + O\left(\frac{1}{h}\right) &= O\left(\frac{1}{h}\right), \quad \left(O\left(\frac{1}{h}\right)\right)'_p = O\left(\frac{1}{h}\right), \quad \frac{h}{4} \leq p \leq \frac{\delta}{4\pi} - \frac{h}{4},
\end{align*}
\]

and thus, in view of (3.40), estimate (3.39). The proof is complete. \( \square \)

Lemma 3.2 immediately implies

**Corollary 3.2.** In the case of Lemma 3.2, for \( p_k \in I_p \), one has

\[
(3.44) \quad p_k = \frac{hk}{2} + O_H(\lambda^3)
\]

where the error is uniform uniform in \( k \).
One can easily see that $hk/2 \in I_p$ if and only if

$$(3.45) \quad 1 \leq k \leq K, \quad K = \left\lfloor \frac{1}{2h} \right\rfloor,$$

where $[x]$ denotes the integer part of $x \in \mathbb{R}$. We note also that

$$\frac{1}{4} - pK = \frac{h}{4} \left\{ \begin{array}{ll} h_1 & \text{if } [1/h] \text{ is even}, \\ 1 + h_1 & \text{if } [1/h] \text{ is odd}. \end{array} \right.$$ 

For the points $p_k$ one has

**Corollary 3.3.** In the case of Lemma 3.2, we fix a positive constant $C_1$ so that the error term in (3.36) be bounded by $\delta_0 = \lambda^3 e^{C_1/h}$. Assume that $\delta_0/\pi \leq \min\{1/3, h_1/2\}$. If $k$ satisfies (3.45), then there is $p_k \in I_p$.

**Proof.** By Lemma (3.2), the function $\alpha$ is monotonous on the interval $[h/3, 1/4]$, and $[2\pi/3 + \delta_0, \pi/(2h) - \delta_0] \subset \alpha([h/3, 1/4])$. Thus, for any $k$ satisfying

$$\frac{2}{3} + \frac{\delta_0}{\pi} \leq k \leq \frac{1}{2h} - \frac{\delta_0}{\pi},$$

the equation $\alpha(p_k) = \pi k$ has a unique solution in $[h/3, 1/4]$. As $\delta_0/\pi \leq 1/3$, the minimal possible value of $k$ equals 1. Recall that $1/h = [1/h] + h_1, \ h_1 \in (0, 1)$. Thus, one has

$$\frac{1}{2h} = \left\{ \begin{array}{ll} \left[ \frac{1}{\pi} \right] + \frac{h_1}{2}, & \text{if } [1/h] \text{ is even}, \\ \left[ \frac{1}{\pi} \right] + \frac{1+h_1}{2}, & \text{if } [1/h] \text{ is odd}, \end{array} \right.$$ 

and as $\delta_0/\pi \leq h_1/2$, the maximal value of $k$ equals $\left[ \frac{1}{\pi} \right]$. The proof is complete. $\square$

**Remark 3.1.** As seen from the proof, either if $\left[ \frac{1}{\pi} \right]$ is odd, or if $k < \left[ \frac{1}{\pi} \right]$, then the condition on $\delta_0$ can be weakened and replaced with $|\delta_0| \leq 1/3$.

Let $K = \left[ \frac{1}{\pi} \right]$. One has

**Lemma 3.4.** There is a $c > 0$ such that if $\lambda < e^{-\frac{\pi}{2}}$, then following holds.

For any $k = 1, 2, \ldots, K - 1$, the point $E_k = 2\cos(2\pi p_k)$ is located in a gap. The point $E_K$ is in a gap if either $\left[ \frac{1}{\pi} \right]$ is odd, or if $\left[ \frac{1}{\pi} \right]$ is even, and $\lambda^{h_1} < e^{-\frac{\pi}{2}}$.

**Proof.** In view of (3.33) and the definition of $p_k$, see (3.35), it suffices to prove that $X(p_k) \geq C_0 \lambda_1$, where $\lambda_1 = \lambda^{h_1}$, and $C_0 > 0$ is certain constant. Using Lemma 3.3 and (3.44), for sufficiently large $c$, we get

$$(3.46) \quad X(p_k) = e^{\frac{2p_k \ln \lambda}{\pi} + O\left( \frac{1}{\lambda} \right)} = e^{k \ln \lambda + O(h^2 \ln \lambda) + O\left( \frac{1}{\lambda} \right)} = \lambda^k O\left( \frac{1}{\lambda} \right),$$

Therefore, $\frac{\lambda}{X(p_k)} \leq \lambda^{h_1 - 2k + \frac{c}{\pi}}$, and we get

$$(3.47) \quad \frac{\lambda}{X(p_k)} \leq e^{\frac{c}{\pi}} \left\{ \begin{array}{ll} \lambda^2 & \text{if } 1 \leq k \leq K - 1, \\ \lambda^{1 + h_1} & \text{if } k = K, \text{ and } \left[ \frac{1}{\pi} \right] \text{ is odd,} \\ \lambda^{h_1} & \text{if } k = K, \text{ and } \left[ \frac{1}{\pi} \right] \text{ is even.} \end{array} \right.$$ 

This implies the needed. $\square$ 

Below, we denote by $g_k$ the gap containing $E_k$. We have proved the statement of Theorem 2.5 on the location of $g_k, \ k = 1, 2, \ldots, K$. 

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3.3.2. The lengths of gaps. Here we prove (2.11). Let us fix \( 1 \leq k \leq K \). To get a lower bound for \(|g_k|\), the length of \( g_k \), we first assume that 
\[
|p - p_k| \leq C h X(p_k).
\]
To check (3.33), we prove

**Lemma 3.5.** Under hypothesis 1, one has
\[
X(p) = X_0(p_k)(1 + O_H(\lambda^\beta)), \quad \sin \alpha(p) = \frac{2\pi(p - p_k)}{h} (1 + O_H(\lambda^\beta)).
\]

**Proof.** Below, we assume that \( \lambda \) satisfies hypothesis 1.

As when proving Lemma 3.4, we see that, for sufficiently large \( c \), \( X(p_k) = \lambda^k e^{O(1/h)} \).

Therefore,
\[
|p - p_k| \leq h \lambda^k e^{C/h}.
\]

Lemma 3.3 and (3.50) imply that one has
\[
X(p) = X_0(p)(1 + O_H(\lambda^\beta)) = X_0(p_k)e^{\ln \lambda \lambda^k e^{C/h}} (1 + O_H(\lambda^\beta)).
\]

This implies the first formula in (3.49).

Lemma 3.2 implies that \( \alpha(p) = \frac{2\pi(p - p_k)}{h} (1 + O_H(\lambda^\beta)) \). Thus, in view of (3.50), we get
\[
\sin \alpha = \frac{2\pi(p - p_k)}{h} \left( 1 + O\left( \frac{(p - p_k)^2}{h^2} \right) \right) (1 + O_H(\lambda^\beta)) = \frac{2\pi(p - p_k)}{h} (1 + O_H(\lambda^\beta)).
\]

We proved the second formula in (3.49).

Now, we can readily prove

**Proposition 3.3.** Let hypothesis 1 be satisfied, and let \( \lambda^{h_c} e^{c/h} \) be sufficiently small if \( k = K \) and \([1/h] \) is even. Then (3.34) can be transformed to the form
\[
|p - p_k| \leq \frac{h}{2\pi} X_0(p_k)(1 + O_H(\lambda^\beta))
\]
where \( p = \beta \) if \( 1 \leq k \leq K - 1 \) or \([1/h] \) is odd, and \( p = \min\{h_1, \beta\} \) if \( k = K \) and \([1/h] \) is even.

**Proof.** The statement follows from Lemma 3.5 and formulas (3.46) and (3.47).

Now, to complete the proof of (2.11), we need to compute \( X_0(p_k) \) and to write (3.51) in terms of \( E \). We begin with

**Lemma 3.6.** Under Hypothesis 1, one has
\[
X_0(p_k) = \left( \frac{\lambda}{4} \right)^k \frac{\sin(\pi h k)}{\sin^2(\pi h) \ldots \sin^2(\pi h)} (1 + O_H(\lambda^\beta)).
\]

**Proof.** By the second estimate in (3.39), for sufficiently large \( c \), we have
\[
X_0(p_k) = X_0 \left( \frac{h_k}{2} \right) (1 + O_H(\lambda^\beta))
\]
as \( \beta \) in Lemma 3.3 can be chosen greater then in this lemma.

Let us recall that \( X_0 \) is defined in (3.38) with \( F_0 \) given by (2.6). Using equation (2.5), we get
\[
F_0 \left( \frac{h_k}{2} \right) = \prod_{k=1}^k \left( 1 + e^{-(2\pi h_k - \pi)^2} |\sigma_{\pi h}(\pi h)|^2 \right). \]

Now, let us compute \( \sigma_{\pi h}(\pi h) \). Equation (2.5) implies that \( |\sigma_{\pi h}(\pi h)| = 2 |\sigma_{\pi h}(-\pi h)| \).

On the other hand, in view of (8.4), we get \( |\sigma_{\pi h}(\pi h)\sigma_{\pi h}(-\pi h)| = 1 \). These two relations implies that \( |\sigma_{\pi h}(\pi h)|^2 = 2 \). Thus,
\[
F_0 \left( \frac{h_k}{2} \right) = 2 \cdot 4^k \prod_{k=1}^k \left| \sin(\pi h_k) \right|^2.
\]
This formula and formulas (3.38) and (2.6) imply (3.52).

Finally, we rewrite (3.51) in terms of \( E \). We denote the right hand side in (3.51) by \( \Delta_k \). As \( E = 2 \cos(2\pi p) \), we get

\[
|g_k| \geq 2 \cos(2\pi(p_k - \Delta_k)) - \cos(2\pi(p_k + \Delta_k)) = 4 \sin(2\pi p_k) \sin(2\pi \Delta_k).
\]

As \( X_0(k) = O_H(\lambda^k) \), one also has \(|\text{Delta}_k| = O_H(\lambda^k)\), and

\[
|g_k| \geq 8\pi \Delta_k \sin(2\pi p_k)(1 + O_H(\lambda^{2k})�)
\]

By means of (4.34), we get finally

\[
|g_k| \geq 8\pi \Delta_k \sin(\pi \mu(1 + O_H(\lambda^\beta))�)
\]

Substituting into this estimate instead of \( \Delta_k \) the right hand side from (3.51) and using formula (3.52), we come to (2.11).

This completes the proof Theorem 2.5.

4. Model equation

Entire solutions to equation

\[
\mu(z + h) + \mu(z - h) + e^{-2\pi iz} \mu(z) = 2 \cos(2\pi p) \mu(z), \quad z \in \mathbb{C}.
\]

were constructed in [17]. Here, we briefly recall construction of these solutions, and get for them estimates uniform in \( h \).

4.1. Construction of solutions.

4.1.1. Integral representation. We construct solutions in the form:

\[
\mu(z) = \frac{1}{\sqrt{h}} \int_{\gamma} e^{\frac{2\pi iz}{h}} v(k) \, dk,
\]

where \( \gamma \) is a curve in the complex plane that we describe later, and \( v \) is a function analytic in a sufficiently large neighborhood of \( \gamma \). The function \( \mu \) satisfies (4.1) if

\[
v(k + h) = 2(\cos(2\pi p) - \cos(2\pi k)) v(k).
\]

We note that

\[
2(\cos(2\pi p) - \cos(2\pi k)) = -e^{2\pi ik}(1 - e^{-2\pi i(k-p)})(1 - e^{-2\pi i(k+p)}).
\]

Therefore, one can choose

\[
v(k) = e^{\frac{i\pi k}{2} - \frac{\pi}{2} \sigma_H} \sigma_H(2\pi(k - p - \frac{h}{2} - \frac{1}{2})) \sigma_H(2\pi(k + p - \frac{h}{2} - \frac{1}{2}))
\]

where \( \sigma_n \) is a solution to (2.5). In this paper, we use the meromorphic solution described in Section 8.

4.1.2. Properties of the function \( v \). As \( \sigma_n \) the function \( v \) is meromorphic. In Section 8.1.2 we list all the poles of \( \sigma_n \). This description implies that the poles of \( v \) are located at the points

\[
k = \pm p - lh - m, \quad l, m = 0, 1, 2, 3, \ldots.
\]

Let \( l_{\pm} = \pm p - (-\infty, 0] \). The rays \( l_{\pm} \) contain all the poles of \( v \).

Let \( |\text{Im}k| > C |\text{Re}k| \). Corollary 8.1 implies the asymptotic representations

\[
v(k) \approx v_- e^{\frac{i\pi}{k} - \frac{\pi}{2} \sigma_H} + o(1), \quad v_- = -ie^{-\frac{\pi}{k} - \frac{\pi}{2} \sigma_H}, \quad k \to -i\infty,
\]

\[
v(k) \approx v_+ e^{-i\pi k \frac{1}{2} - \frac{\pi}{2} \sigma_H} + o(1), \quad v_+ = -ie^{-\frac{2\pi i k^2}{h} - \frac{\pi}{2} \sigma_H}, \quad k \to +i\infty.
\]
4.1.3. Integration path. We choose the integration path in (4.2) so that it does not intersect \( l_\pm \), from infinity from bottom to top along the line \( e^{i\pi/4} \mathbb{R} \) and goes to infinity upward along the line \( e^{i\pi/4} \mathbb{R} \). This completes the construction of \( \mu \).

4.1.4. Notations. Fix \( X > 0 \). We study \( \mu \) in the strip \( |x| \leq X \) assuming that \( p \in P \),

\[
P = \{ p \in \mathbb{C} : 0 \leq \text{Re} \, p \leq 1/2, | \text{Im} \, p | \leq h \}.
\]

4.2. Estimates in the upper half-plane. Let

\[
\xi_\pm(z) = e^{\pm \frac{\pi z^2}{h^2} + \frac{\pi i z}{\sqrt{h}}}.
\]

One has

**Proposition 4.1.** Let us pick \( X,Y > 0 \). Assume that \( p \in P \). One has

\[
\mu = \nu_+ \xi_+ + \nu_- \xi_-,
\]

and for \( |x| \leq X \) the functions \( F_\pm \) satisfy the uniform in \( x \) estimates:

\[
F_\pm = O(H) \quad \text{if} \quad y \geq -Y, \quad \text{and} \quad F_\pm = 1 + o(1) \quad \text{as} \quad \xi \rightarrow +\infty.
\]

**Proof.** Below, for \( k \in \mathbb{C} \), we write \( r = \text{Re} \, k \) and \( q = \text{Im} \, k \). In view of (4.6)–(4.7), as \( \xi \rightarrow \pm \infty \) the behavior of the integrand in (4.2) is described by the exponentials

\[
e^{-\frac{2\pi i}{h}k \cdot e^{-\frac{i\pi}{2}k^2}} = \xi_\pm(z) e^{-\frac{\pi i (k \cdot z)}{h}}, \quad \text{where} \quad k_\pm(z) = \pm z + \frac{h}{2} + \frac{1}{2}.
\]

Let us consider the straight lines

\[
L_\pm(z) = k_\pm(z) + e^{\mp \pi i/4} \mathbb{R}.
\]

The lines \( L_\pm(z) \) are lines of steepest descent for the functions \( k \mapsto e^{\mp \pi i (k \cdot z)^2} \).

They intersect one another at \( k_* = y + ix + h/2 + 1/2 \).

Let us pick \( d_0 > 0 \). There is an \( Y > 0 \) such that if \( y > Y \), then for all \( (h,p,x) \in (0,1) \times P \times [-X,X] \), the distance from \( L_\pm \) to \( l_\pm \) is greater than \( d_0 \).

First, we assume that \( y > Y \). In this case, we choose the integration path \( \gamma \) in (4.2) so that it go upwards first along \( L_\pm \) from infinity to \( k_* \) and then along \( L_\pm \) from \( k_* \) to infinity. We denote by \( \gamma_- (\gamma_+ \) the part of \( \gamma \) below (resp., above) \( k_* \). We define two functions \( \mu_\pm \) by same the formula as \( \mu \), i.e. by (4.2), but with the integration path \( \gamma \) replaced with \( \gamma_\pm \). It suffices to show that

\[
\mu_\pm = \mp e^{\mp \pi i/4} v_\pm \xi_\pm F_\pm,
\]

with \( F_\pm \) satisfying the estimates (4.11). We prove (4.14) only for \( \mu_- ; \mu_+ \) is estimated similarly.

Substituting (4.4) into (4.2) and using (4.12), we get

\[
\mu_- = \xi_- (z) v_- \nu, \quad \nu = \frac{1}{\sqrt{h}} \int_{\gamma_-} e^{\frac{\pi i (k \cdot \gamma_- (z))^2}{h}} F(k,p,h) \, dk,
\]

where \( F(k,p,h) = \sigma_{\text{eh}}(2\pi(k - p - b/2 - 1/2)) \sigma_{\text{eh}}(k + p - b/2 + 1/2)) \).

We remind that \( k = r + iq, \ r,q \in \mathbb{R} \). Let \( 0 < k < 1 \). By Corollary 8.1, for \( k \in \gamma_- \)

\[
F(k,p,h) = e^{O(h^{-1} e^{-2\pi \kappa |q|} (1 + |r|))}.
\]

As along \( \gamma_- \) one has

\[
k - k_-(z) = \sqrt{2} e^{\pi/4} \text{Im} (k - k_-(z)), \quad \text{Im} (k - k_-(z)) = q + y, \quad |r| \leq C + |q + y|,
\]

\[
k_-(z) = \frac{1}{\sqrt{2} h} e^{\frac{\pi z^2}{h^2} + \frac{\pi i z}{\sqrt{h}}} \mathbb{R},
\]

\[
F_0 = O(1)
\]

\[
e^{\frac{\pi i z}{\sqrt{h}}} = e^{\mp \frac{2\pi i}{h}k \cdot e^{-\frac{i\pi}{2}k^2}} = \xi_\pm(z) e^{-\frac{\pi i (k \cdot z)}{h}},
\]

\[
\xi_\pm(z) = e^{\pm \frac{\pi z^2}{h^2} + \frac{\pi i z}{\sqrt{h}}}.
\]

One has

\[
F = O(H) \quad \text{if} \quad y \geq -Y, \quad \text{and} \quad F = 1 + o(1) \quad \text{as} \quad \xi \rightarrow +\infty.
\]
So, representation (4.17) implies that as $y \to 0$, and $\beta < \alpha < 1$, we have

$$
\nu = e^{\frac{\pi}{4}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} e^{-\frac{2\pi y^2}{n}} + O\left(\frac{e^{-2\pi \sqrt{y}}}{\sqrt{\pi \hbar}}\right) \, dq
$$

$$
(4.17)
$$

Therefore,

$$
|\nu| \leq C \sqrt{\frac{2\pi}{\hbar}} \int_{-\infty}^{\infty} e^{-\frac{2\pi y^2}{n} + C(1+|t|)} \, dt \leq H \int_{-\infty}^{\infty} \left(e^{-\frac{4\pi y}{\hbar}} + e^{-\frac{4\pi (1+|t|)}{\hbar}}\right) \, dt \leq H.
$$

This proves the first estimate in (4.11) for $y > Y$. To prove the second one, we note that for sufficiently large $y$

if $\frac{Y}{2} \leq t < y$, then $e^{-2\pi \kappa |t-y|} (1 + |t|) \leq C_2 e^{-\pi \kappa (y-t)} (1 + |t|) \leq C e^{-\pi y} y$.

So, representation (4.17) implies that as $y \to \infty$

$$
\nu = e^{\frac{\pi}{4}} \sqrt{\frac{2}{\pi}} \left(\int_{-\infty}^{\frac{Y}{2}} e^{-\frac{2\pi y^2}{n} + O\left(\frac{1}{\sqrt{\hbar}}\right)} \, dt + \int_{\frac{Y}{2}}^{\infty} e^{-\frac{2\pi y^2}{n} + O\left(\frac{1}{\hbar}\right)} \, dt\right) = e^{\pi / 4} + O(1).
$$

This proves the second estimate in (4.11).

To complete the proof, it suffices to check that if $|x| \leq X$ and $|y| \leq Y$, then $\mu \leq H$. In this case, we pick $\delta, \sigma > 0$ and choose the integration path $\gamma$ in (4.2) that goes along $L_-(z)$ upwards from infinity to the circle $c_r$ with radius $r$ and center at $k_0$, then along $c_r$ in the anticlockwise direction to the upper point of intersection of $L_+(z)$ and $c_r$, and finally along $L_+(z)$ upwards from this point to infinity. We assume that $r$ is sufficiently large so that the distance between $\gamma$ and the rays $l_{\pm}$ is greater than $\delta$. In view of Corollary 8.1, on $\gamma \cap c_r$, the integrand in (4.2) is bounded by $H$. On $\gamma \setminus c_r$ it is estimated as when proving the first estimate in (4.11).

4.3. Estimates in the lower half-plane. Set

$$
(4.18)
$$

$$
a(p) = e^{-\frac{\pi p^2}{16}} e^{\frac{\pi p^2}{16} - \frac{\pi p}{2} + \frac{\pi}{8} - \frac{\pi}{4} p} \sigma_{\pi h}(2\pi (2p - \frac{3}{2} - \frac{1}{2})).
$$

We note that $a$ is meromorphic in $p$, and its poles in $(-\infty, 0)$.

One has

**Proposition 4.2.** Pick positive $X$. Let $p \in P$. There is an $Y > 0$ independent of $p$ and $h$ and such that, for $y \leq -Y$ and $|x| \leq X$, one has the following results

- **Fix $\alpha$, $\beta$ so that $0 < \beta < \alpha < 1$. Then,
  $$(4.19) \quad \mu(z) = a(p) + O_H\left(e^{-2\pi \beta |y|}\right), \quad \text{if } \Re p \geq \alpha h / 2,$$

- **Fix $\alpha$ and $\beta'$ so that $0 < \alpha < 1$ and $0 < \beta' < 1$. Then
  $$(4.20) \quad \mu(z) = a(p) e^{\frac{2\pi \beta x}{h}} + a(-p) e^{-\frac{2\pi \beta x}{h}} + O_H\left(e^{\frac{2\pi \beta x}{h} - 2\pi \beta' |y|}\right), \quad \text{if } 0 \leq \Re p \leq \alpha h / 2.$$**

**Proof.** Let us begin with justifying (4.19). Remind that $\mu$ has the integral representation (4.2). For $y < 0$, the behavior of $\mu$ appears to be determined by the rightmost poles of $v$.

The poles of $v$ are at the points listed in (4.5). As $p \in P$, they are inside the strip $|\Im k| \leq h$. As $\Re p \geq \alpha h / 2$, $0 < \alpha \leq 1$, we see that, to the right of the line $\Re k = \Re p - \alpha h$, the function $v$ has only one simple pole; it is situated at $k = p$. 

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We deform $\gamma$, the integration contour in (4.2), to a curve that goes from $-e^{\frac{\pi i}{4}}\infty$ to $+e^{\frac{\pi i}{4}}\infty$, has the same asymptotes as $\gamma$, but, instead of staying to the right of all the poles of $v$, it goes between the pole at $k = p$ and the other ones (they stay to the left of this curve). We keep for the new integration curve the old notation $\gamma$. The function $\mu$ can be represented in the form

$$\mu = A + B,$$

(4.21)

$$A = \frac{2\pi i}{\sqrt{h}} \text{res}_{k=p} I(k), \quad B = \frac{1}{\sqrt{h}} \int_{\gamma} I(k) \, dk, \quad I = e^{\frac{2\pi i k}{h}} v(k).$$

Using the representation (4.4), the information on the poles of the function $\sigma(z)$ from section 8.1.2, and formula (8.6), we get

$$A = a(p) e^{\frac{2\pi i p}{h}}$$

with $a(p)$ given by (4.18).

Now, to complete the proof of the proposition, we need only to estimate the term $B$ in (4.21). Let $\gamma_+$ ($\gamma_-$) be the part of $\gamma$ situated above (resp., below) the line $\text{Im} \, k = \text{Im} \, p$. First, we choose $\gamma_{\pm}$, and then, we prove that

$$\left| \int_{\gamma_{\pm}} I(k) \, dk \right| \leq H \left| e^{\frac{2\pi i p}{h}} \right| e^{-2\pi |\beta| |y|}.$$  

(4.23)

We begin with estimating the integral along $\gamma_-$. We remind that the exponential $e^{\pi i (k - k_-^2)}$ governs the behavior of $I(k)$, the integrand in (4.2), as $\text{Im} \, k \to -\infty$, see the beginning of the proof of Proposition 4.1. We assume that $y < -Y$ with a positive $Y$. Therefore, $\text{Im} \, k_- > Y$.

For $\xi \in \mathbb{C}$, we denote by $H_r(\xi)$ the smooth curve described by an equation of the form $\text{Im} \, (k - k_-) = c$, $c \in \mathbb{R}$, and containing $\xi$. If $c = 0$ this curve is one of the straight lines $k_- + \mathbb{R}$ and $k_- + i\mathbb{R}$, otherwise it is a hyperbola located in one of the sectors bounded by these lines. Its asymptotes are two half lines of these straight lines. Let $H_r(\xi)$ be the smooth curve described by an equation of the form $\text{Re} \, (k - k_-) = c$, $c \in \mathbb{R}$, and containing $\xi$. If $c = 0$ this curve is one of the straight lines $k_- + \mathbb{R}$ and $e^{\frac{\pi i}{4}} \mathbb{R}$, otherwise it is a hyperbola located in one of the sectors bounded by these lines. Two half lines of these straight lines are its asymptotes.

Set $k_1 = p - \beta h$. As $0 < \beta < \alpha < 1$, the point $k_1$ is to the right of the line $\text{Re} \, k = p - \alpha h$ and to the left of $p$. As $p \in P$, one has $|\text{Im} \, k_1| \leq h < 1$.

Let $Y$ be sufficiently large, and $y < -Y$. Then the hyperbola $H_r(k_1)$ stays in the half plane $\text{Im} \, k < \text{Im} \, k_- (z)$ and intersects the line $\text{Im} \, k = -2$ at a point $k_2$. We denote by $\gamma_1$ its segment of $H_r(k_1)$ between $k_1$ and $k_2$. Furthermore, if $Y$ is sufficiently large, $H_r(k_2)$ is a hyperbola located below $k_- (z)$. We denote by $\gamma_2$ its segment between $k_2$ and $\infty$ along which $\text{Re} \, k \to -\infty$ as $k \to \infty$. The curve $\gamma_-$ is the union of $\gamma_1$ and $\gamma_2$, see Fig. 1.

If $Y$ is sufficiently big, then (1) the curve $\gamma_-$ does stays between $p$ and all the other poles of $v$; (2) its segment $\gamma_2$ is located below the poles of the integrand at a distance greater than 1.

Let us estimate $\int_{\gamma_2} I(p) \, dp$. We note that, by the definition of $H_r$ and by (4.12), the expression $\text{Im} \, (2kz + k^2 - k - kh)$ is constant on $H_r(k_1)$. Therefore, by (4.4)
we get

\begin{equation}
\frac{1}{\sqrt{H}} \left| \int_{I_1} e^{2\pi i k} v(k) dk \right| \leq \frac{C \left| e^{-\frac{C(Y)}{2}} \right|}{\sqrt{H}} \times \\
\times \int_{I_1} |\sigma_{\pi h}(2\pi (k + p - \frac{h}{2} - \frac{1}{2})) \sigma_{\pi h}(2\pi (k - p - \frac{h}{2} - \frac{1}{2}))| dk \]
\end{equation}

where

\[ C(H) = \pi \Im \langle 2kz + k^2 - k - kh \rangle_{H,(k_1)}. \]

Computing \( C(H) \) at the point \( k_1 \), we get

\begin{equation}
\left| e^{-\frac{C(Y)}{k_1}} \right| \leq C \left| e^{\frac{\pi h}{k_1}} \right| e^{-2\pi \beta |y|}. 
\end{equation}

Let us estimate the integrand in the right hand side of (4.24). Using (2.5), we get

\[ \sigma_{\pi h}(2\pi (k - p - \frac{h}{2} - \frac{1}{2})) = \frac{\sigma_{\pi h}(2\pi (k - p + \frac{h}{2} - \frac{1}{2}))}{1 - e^{-2\pi i (k-p)}}. \]

For \( k \in I_1 \), one has

\[ \Re (k + p - h/2 - 1/2) \geq -1/2 - h/2 + 2\Re p - \beta h - C(Y)|\Im (k - p)| \]
\[ \geq -1/2 - h/2 + (\alpha - \beta)h - C(Y)|\Im (k - p)|. \]

where \( C(Y) > 0 \) tends to zero as \( Y \to \infty \). These observations and Corollaries 8.1–8.2 imply that, for sufficiently large \( Y \) and \( k \in I_1 \),

\[ |\sigma_{\pi h}(2\pi (k + p - \frac{h}{2} - \frac{1}{2})) \sigma_{\pi h}(2\pi (k - p - \frac{h}{2} - \frac{1}{2}))| \leq H. \]

This estimate and (4.25) imply estimate (4.23) with \( I_1 \) instead of \( \gamma_{\pm} \).

Consider the integral \( \int_{\gamma_1} I(k) dk \) at \( Y \) stays below the poles of \( I \), at a distance greater than 1, by means of Corollary 8.1, one immediately obtains

\[ |\sigma_{\pi h}(2\pi (k + p - \frac{h}{2} - \frac{1}{2})) \sigma_{\pi h}(2\pi (k - p - \frac{h}{2} - \frac{1}{2}))| \leq H, \quad k \in I_1, \]

and

\[ \left| \int_{\gamma_1} I dk \right| \leq H \left| e^{rac{\pi h}{k_1}} \frac{(2kz + k^2 - k - kh)}{z = k_2} \right| \left| \int_{\gamma_2} e^{\frac{\pi h}{k}} ((k-k_-)(z)^2 - (k-k_-)(z)^2) dk \right|. \]

Clearly,

\[ \left| e^{rac{\pi h}{k_1}} \frac{(2kz + k^2 - k - kh)}{z = k_2} \right| = e^{-\frac{C(Y)}{k_1}} \leq C \left| e^{\frac{\pi h}{k_1}} \right| e^{-2\pi \beta |y|}. \]
We remind that curve $\gamma_2 \subset H_r(k_2)$ goes to infinity approaching the asymptote $e^{i\pi/4}(-\infty, 0]$. Integrating by parts, we get

$$\int_{\gamma_2} e^{\mp i((k-k_-(z))^2-(k_2-k_-(z))^2)} \, dk \leq C h/k_2 - k_-. $$

These estimates imply that $\int_{\gamma_2} I \, dk$ satisfies an estimate of the form (4.23). This implies (4.23) with $\gamma_-$. The estimates of the integral along $\gamma_+$, the part of $\gamma$ above the line $\text{Im} \, k = \text{Im} \, p$ are similar. We omit the details and mention only that now the role of $e^{\mp i((k-k_+(z))^2)}$ is played by the exponential $e^{-i((k-k_+(z))^2)}$. The obtained estimates and the formula for $A$ imply (4.19). This completes the proof of (4.19). Representation (4.20) is obtained similarly.

\section*{4.4. Rough estimates.} We shall need

Lemma 4.1. Pick $X > 0$. Let $p \in P$ and $|x| \leq X$. One has

(4.26) $|\mu(z)| \leq Hw(x,y)(1+y_-)$, \hspace{1em} $|\mu'(z)| \leq Hw(x,y)(1+y_+)$, \hspace{1em} $y_\pm = (|y| \pm y)/2$,

(4.27) $w(x,y) = \begin{cases} e^{\frac{2\pi i (|x|+1/2)}{2\pi} - \pi y}, & y \geq 0, \\ e^{\frac{2\pi i p(x,y)}{2\pi}}, & y \leq 0. \end{cases}$

When proving this lemma we use

Lemma 4.2. Pick $\alpha \in (0,1)$. For $p \in P$ one has

(4.28) $|a(p)| \leq H$ \hspace{1em} if \hspace{1em} $|p| \geq \alpha h/2$,

(4.29) $|a(p) + a(-p)| \leq H$, \hspace{1em} $|pa(p)| \leq H$ \hspace{1em} if \hspace{1em} $|p| \leq \alpha h/2$.

Proof. Let $|p| \geq \alpha h/2$ and $p \in P$. Corollaries 8.1 and 8.2 imply that $|\sigma_{\pi h}(2\pi(2p - \frac{h}{2} - \frac{1}{2}))| \leq H$. This and (4.18) lead to (4.28).

Assume that $|p| \leq \alpha h/2$ ($p$ is not necessarily in $P$). In view of (4.18), it suffices to check that

$$|\sigma_{\pi h}(2\pi(2p - \frac{h}{2} - \frac{1}{2})) + \sigma_{\pi h}(2\pi(-2p - \frac{h}{2} - \frac{1}{2}))| \leq \frac{C}{h},$$

$$|p\sigma_{\pi h}(2\pi(2p - \frac{h}{2} - \frac{1}{2}))| \leq C.$$ Both the functions $p \mapsto \sigma_{\pi h}(2\pi(2p - \frac{h}{2} - \frac{1}{2})) + \sigma_{\pi h}(2\pi(-2p - \frac{h}{2} - \frac{1}{2}))$ and $p \mapsto p\sigma_{\pi h}(2\pi(2p - \frac{h}{2} - \frac{1}{2}))$ are analytic in $p$ in the $\frac{\alpha h}{2}$-neighborhood of zero (see section 8.1.2). By Theorem 8.2 $\sigma_{\pi h}(2\pi(2p - \frac{h}{2} - \frac{1}{2}))$ is bounded by $C/h$ at the boundary of this neighborhood. This and the maximum principle imply the needed estimates for the $\sigma$-function.

Now we can check Lemma 4.1.

Proof. Let $p \in P$. Pick $Y$ sufficiently large.

For $y \leq -Y$, the estimate for $\mu$ follows directly from Proposition 4.1.

For $|y| \leq Y$, the estimate for $\mu'$ follows from the estimate for $\mu$ and the Cauchy estimates for the derivatives of analytic functions (as $X$ and $Y$ were chosen rather arbitrarily).

Let $y \geq Y$ and $|x| \leq X$. The first estimate in (4.26) implies that, in the $(h/y)$-neighborhood of $z$, $\mu$ is bounded by $Hw(x,y)$ (we again use the fact that $X$ and $Y$ were chosen rather arbitrarily). This and the Cauchy estimates for the derivatives of analytic functions lead then to the estimate $|\mu'(z)| \leq yHw(x,y).$ This completes the proof of (4.26) for $y \geq -Y$. 

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Let us prove (4.26) for \( y \leq -Y \). Pick \( 0 < \alpha < 1 \). Let \( |x| \leq X, \ y \leq -Y \) and \( \Re p \geq \alpha h/2 \). Estimate (4.28) and Proposition 4.2 imply that

\[
|\mu(z)e^{-\frac{2\pi i z}{h}}| \leq H. 
\]

By means of the Cauchy estimates for the derivatives of analytic functions we get the estimate

\[
\left| \frac{d}{dz} \left( \mu(z)e^{-\frac{2\pi i z}{h}} \right) \right| \leq H. 
\]

Estimates (4.30)–(4.31) lead to (4.26) for \( |x| \leq X, \ y \leq -Y \) and \( \Re p \geq \alpha h/2 \). If \( 0 \leq \Re p \leq \alpha h/2 \), the estimates for \( \mu \) and its derivative are deduced from (4.20) and (4.29) similarly. We omit further details. \( \square \)

4.5. **One more solution to the model equation.** Let

\[
\tilde{\mu}(z, p) = e^{-i\pi z/h} \mu(z + 1/2, 1/2 - p),
\]

were we indicated the dependence of \( \mu \) on \( p \) explicitly. Together with \( \mu, \ \tilde{\mu} \) is a solution to (4.1). It is entire in \( z \) and \( p \). We use it to construct entire solutions to the Harper equation. Here, we compute the Wronskian \( \{ \mu, \tilde{\mu} \} = \mu(z + h)\tilde{\mu}(z) - \mu(z)\tilde{\mu}(z + h) \).

**Lemma 4.3.** For all \( p \in \mathbb{C} \),

\[
\{ \mu, \tilde{\mu} \} = i e^{\frac{2\pi i z^2}{h^2} - \frac{i \pi z}{h} - \frac{\alpha h}{2}}.
\]

**Proof.** Pick \( X > 0 \) and \( 0 < \alpha < \frac{1}{2} \). Assume that \( \alpha h/2 \leq p \leq 1/2 - \alpha h/2 \). Then, by means of (4.32), (4.19) and (4.28), we check that, uniformly in \( x \in [-X, X] \), as \( y \to +\infty \) the Wronskian \( \{ \mu, \tilde{\mu} \} \) tends to

\[
2i e^{\frac{i \pi}{h} - \frac{i \pi z}{h}} a(1/2 - p)a(p) \sin(2\pi p).
\]

On the other hand, by means of (4.4), we check that, as \( y \to +\infty \) uniformly in \( x \in [-h, 0] \),

\[
|\{ \mu(z), \tilde{\mu}(z) \}| \leq H \left( e^{\frac{\pi z}{h} \left( x - \frac{1}{2} + |x|/2 \right)} + 1 \right) \leq H.
\]

The Wronskian being entire (as \( \mu \) and \( \tilde{\mu} \)) and \( h \)-periodic in \( z \) (as the Wronskian of any two solutions of a one-dimensional difference Schrödinger equation, see section 3.1.3), we conclude that it is independent of \( z \) and equals the expression in (4.34). As the Wronskian is entire in \( p \), this statement is valid for all \( p \). Finally, using the definition of \( a \), equation (2.5) and formula (8.4), we check that

\[
2i a(1/2 - p)a(p) \sin(2\pi p) = i e^{\frac{-2\pi i z^2}{h^2} + \frac{i \pi z}{h} - \frac{\alpha h}{2} - \frac{\alpha h}{2}}.
\]

This leads to the statement of the lemma. \( \square \)

5. **Analytic solution to Harper equation**

5.1. **Preliminaries.** For \( Y \in \mathbb{R} \), we set \( \mathbb{C}_+(Y) = \{ y \geq Y \} \). Here, we pick \( Y > 0 \) and for sufficiently small \( \lambda > 0 \) construct a solution to (2.1) analytic in \( \mathbb{C}_+(Y) \).

Below, we represent the spectral parameter \( E \) in the form \( E = 2 \cos(2\pi p) \) and consider solutions to (2.1) as functions of the parameter \( p \).

As \( \lambda \) is small, then, when constructing solutions to (2.1) in \( \mathbb{C}_+(Y) \), it is natural to rewrite this equation in the form

\[
\psi(z + h) + \psi(z - h) + \lambda e^{-2\pi i z} \psi(z) - 2\cos(2\pi p) \psi(z) = \lambda e^{2\pi i z} \psi(z),
\]

so that the term in the right hand side could be considered as a perturbation. Let \( \xi = \frac{1}{2\pi} \ln \lambda \). Then \( \mu(z + i\xi) \) is a solution to the unperturbed equation

\[
\psi(z + h) + \psi(z - h) + \lambda e^{-2\pi i z} \psi(z) - 2\cos(2\pi p) \psi(z) = 0.
\]
We construct $\psi$, an analytic solution to equation (5.1) close to $\mu(z + i\xi)$. We prove

**Theorem 5.1.** Pick positive $Y$. There exists a positive constant $C$ such that if $\lambda \leq e^{-\frac{\pi i}{2}}$, then:
- There is $\psi_0$, a solution to (2.1) analytic in $(z,p) \in \mathbb{C}_+(-Y) \times P$;
- Pick positive $X$. As $y \to +\infty$, uniformly in $x \in [-X,X]$

\[
(5.3) \quad \psi_0(z) = e^{\frac{i\pi}{2} \xi_-(z + i\xi)} v_-(1 + z_0 + o(1)) - e^{-\frac{i\pi}{2} \xi_+(z + i\xi)} v_+(1 + o(1)),
\]

where $z_0$ is a constant satisfying the estimate

\[
|z_0| \leq H \lambda^{1+\nu(p)} (1 + |\xi|)^3, \quad \nu(p) = \min \left\{ 1, \frac{1-2\text{Re}p}{h} \right\}.
\]

- Assuming that $\lambda$ is so small that $2h < -\xi$, we pick $Y \in (2h, -\xi)$ and $X > 0$. For $|x| \leq X$ and $y \leq Y/2$, one has

\[
(5.4) \quad |\psi_0(z) - \mu(z + i\xi)| \leq H \lambda (1 + |\xi|)^3 e^{2\text{Re}p|\xi|/h}.
\]

The rest of the section is devoted to the proof of Theorem 5.1.

Let us explain the idea of the proof of Theorem 5.1. Let $\gamma = \{z \in i\mathbb{R} : y \geq -Y\}$. We study the integral equation

\[
(5.5) \quad \psi(z) = \mu(z + i\xi) - (K\psi)(z), \quad K\psi(z) = \int_\gamma \kappa(z, z') \psi(z') \, dz', \quad z \in \gamma.
\]

The kernel $\kappa$ is constructed in terms of $\mu(\cdot + i\xi)$ and $\tilde{\mu}(\cdot + i\xi)$, two linearly independent solutions of the unperturbed equation (5.2),

\[
(5.6) \quad \kappa(z, z') = \frac{\lambda}{2ih} \Theta(z, z') \left[ \mu(z + i\xi) \tilde{\mu}(z' + i\xi) - \mu(z' + i\xi) \tilde{\mu}(z + i\xi) \right] e^{2\pi i z'},
\]

were

\[
(5.7) \quad \Theta(z, z') = \cot \frac{\pi(z' - z)}{h} - i.
\]

Similar integral operators have appeared in [7]. The kernel $\kappa$ can be considered as a difference analog of the resolvent kernel arising in the theory of differential equations. First, we construct a solution to the integral equation (5.5), and then, we check that it is analytic in $z$ and satisfies the difference equation (2.1). Finally, we obtain the asymptotics of this solution for $y \to +\infty$ and for $y \sim 0$.

**5.2. Integral equation.** Here, we prove the existence of a solution (continuous in $z$ and analytic in $p$) to the integral equation.

Below,

\[
(5.8) \quad q(y) = (1 + |y|) \begin{cases} e^{-\frac{\pi y}{2h} - \frac{\pi y}{h}}, & y \geq 0, \\ e^{\frac{\pi y}{2h} - \frac{\pi y}{h}}, & y \leq 0 \end{cases}, \quad \bar{q}(y) = (1 + |y|) \begin{cases} e^{\frac{\pi y}{2h} - \frac{\pi y}{h}}, & y \geq 0, \\ e^{-\frac{\pi y}{2h} + \frac{\pi y}{h}}, & y \leq 0 \end{cases}.
\]

**5.2.1. Estimates of $\mu$ and $\tilde{\mu}$.** To estimate the norm of the integral operator, we use

**Corollary 5.1.** On the curve $\gamma$, the functions $\mu$ and $\tilde{\mu}$ satisfy the estimates

\[
(5.9) \quad |\mu(z)|, |\mu'(z)| \leq Hq(y), \quad |\tilde{\mu}(z)|, |\tilde{\mu}'(z)| \leq H\bar{q}(y),
\]

This Corollary follows directly from Lemma 4.1.
5.2.2. A solution of the integral equation. One has

**Proposition 5.1.** Fix positive $\alpha < 1$. There is a positive constant $C$ such that if $\lambda \leq e^{-\frac{\pi}{2}}$, then the integral equation (5.5) has a solution $\psi_0$ continuous in $z \in \gamma$, analytic in $p \in P$ and satisfying the estimate

\begin{equation}
|\psi_0(z) - \mu(z + i\xi)| \leq \lambda^\alpha H q(y + \xi), \quad z \in \gamma.
\end{equation}

Proof. Let $C(\gamma, q)$ be the space of functions defined and continuous on $\gamma$ and having the finite norm $\|f\| = \sup_{z \in \gamma} |q^{-1}(y + \xi) f(z)|$. The proof of the proposition is based on

**Lemma 5.1.** For $z, z' \in \gamma$, one has

\begin{equation}
|q^{-1}(y + \xi) \kappa(z, z') q(y' + \xi)| \leq \lambda H e^{-2\pi y'} (1 + \xi^2).
\end{equation}

First, we prove the proposition, and then, we check estimate (5.11). This estimate implies that the norm of $K$ as an operator acting in $C(\gamma, q)$ is bounded by $\lambda H (1 + \xi^2)$. By Corollary 5.1, $\mu(\cdot, + \xi) \in C(\gamma, q)$. So, there is a positive constant $C$ such that, if $\lambda(1 + \xi^2) < e^{-C/h}$, then there is $\psi_0$, a solution to (5.5) from $C(\gamma, q)$.

The estimate of the norm of the integral operator implies that

\[ |\psi_0(z) - \mu(z)| \leq \lambda(1 + \xi^2) H q(y + \xi), \quad z \in \gamma. \]

This implies (5.10) for any fixed positive $\alpha < 1$. The analyst of $\psi_0$ in $p$ follows from the analyticity of $\mu$ and the uniformity of the estimates. This completes the proof.

Let us prove Lemma 5.1. Below, $z, z' \in \gamma$. We note that by (4.33), for $p \in P$ one has $C^{-1} \leq \{\mu(z), \mu(z')\} \leq C$.

First, we consider the case where $|y - y'| \geq h$. In view of Corollary 5.1, we get

\[ |q^{-1}(y + \xi) \kappa(z, z') q(y' + \xi)| \leq \lambda H e^{-2\pi y'} (E_1 + E_2), \]

\[ E_1 = |\Theta(z, z')| \bar{q}(y' + \xi) q(y' + \xi), \quad E_2 = |\Theta(z, z')| \bar{q}(y + \xi) q(y + \xi) q^2(y' + \xi). \]

To justify (5.11), it suffices to check that $E_{1,2} \leq C (1 + |\xi|)^2$. Note that

\begin{equation}
|\Theta(z, z')| \leq C \begin{cases} e^{-\frac{2\pi |y-y'|}{h}} & \text{if } y - y' \geq h, \\ 1 & \text{if } y' - y \geq h. \end{cases}
\end{equation}

Clearly, $E_1 \leq C \bar{q}(y' + \xi) q(y' + \xi)$. For $y' \geq -Y$, we have

\begin{equation}
\bar{q}(y' + \xi) q(y' + \xi) \leq (1 + |y' + \xi|)^2 e^{-2\pi(y' + \xi)} \leq C, \quad \text{if } y' + \xi \geq 0, \\
\bar{q}(y' + \xi) q(y' + \xi) \leq (1 + |y' + \xi|)^2 \leq C (1 + |\xi|)^2 \quad \text{otherwise}. \end{equation}

This implies that $E_1 \leq C (1 + |\xi|)^2$. To estimate $E_2$, we have to consider four cases. If $y + \xi, y' + \xi \geq 0$, we have

\[ E_2 \leq |\Theta(z, z')| (1 + |y' + \xi|)^2 e^{-\frac{2\pi |y-y'|}{h}} - 2\pi(y' + \xi) \leq C |\Theta(z, z')| e^{-\frac{2\pi |y-y'|}{h}} \leq C. \]

If $y + \xi \geq 0 \geq y' + \xi$, then

\[ E_2 \leq (1 + |y' + \xi|)^2 |\Theta(z, z')| e^{-\frac{2\pi |y-y'|}{h}} - \frac{4\pi \text{Re}(\frac{y' + \xi}{h})}{h} \leq C (1 + \xi^2) e^{-\frac{2\pi |y-y'|}{h}} \leq C (1 + \xi^2). \]

If $y' + \xi \geq 0 \geq y + \xi$, then

\[ E_2 \leq (1 + |y' + \xi|)^2 |\Theta(z, z')| e^{-\frac{4\pi \text{Re}(\frac{y' + \xi}{h})}{h}} - \frac{2\pi |y' + \xi|}{h} \leq C |\Theta(z, z')| e^{-\frac{2\pi |y-y'|}{h}} \leq C |\Theta(z, z')| \leq C. \]
Finally, if \(y + \xi, y' + \xi \leq 0\), we get
\[
E_2 \leq (1 + |y' + \xi|)^2 |\Theta(z, z')| e^{\frac{1}{2} \text{Re} \mu(z - z')} \leq C (1 + |\xi|)^2.
\]

These estimates imply that \(E_2 \leq C (1 + |\xi|)^2\). This completes the proof in the case where \(|y - y'| \geq h\).

Let us consider the case where \(|y - y'| \leq h\). Let \(\eta = \text{Im} \zeta\). Using (4.26) we get
\[
|\Theta(z, z') (\mu(z + i\xi) \tilde{\mu}(z' + i\xi) - \mu(z' + i\xi) \tilde{\mu}(z + i\xi))| \\
\leq Ch \max_{\zeta \in \gamma, |\eta - \nu| \leq h} |\mu(z + i\xi) \mu'(\zeta + i\xi) - \mu'(\zeta + i\xi) \tilde{\mu}(z + i\xi)| \\
\leq H (q(y + \xi) \max_{|\eta - \nu| \leq h} \tilde{q}(\eta + \xi) + \tilde{q}(y + \xi) \max_{|\eta - \nu| \leq h} q(\eta + \xi)) \\
\leq Hq(y + \xi)\tilde{q}(y + \xi),
\]
and, using (5.13), we again come to (5.11). This completes the proof. \(\square\)

Note that Corollary 5.1 and Proposition 5.1 imply

**Corollary 5.2.** In the case of Proposition 5.1, there is a constant \(C\) such that, if \(\lambda \leq e^{-\frac{2\pi i}{2h}}\), then
\[
|\psi_0(z)| \leq Hq(y + \xi), \quad z \in \gamma.
\]

3. Analytic continuation of the solution of the integral equation. Here, we prove the first point of Theorem 5.1. One has

**Lemma 5.2.** The solution \(\psi_0\) can be analytically continued in \(\mathbb{C}_+(-Y)\).

Similar statements were checked in [7] and [8], we outline the proof only for the convenience of the reader.

**Proof.** For \(z' \in \gamma\), the kernel \(\kappa(z, z')\) is analytic in \(z \in S_h = \{|\text{Re} z| < h, y > -Y\}\), and the function \(K\psi_0\) can be analytically continued in \(S_h\); \(\psi_0\), being a solution to (5.5), can be also analytically continued in \(S_h\).

Having proved that \(\psi_0\) is analytic in \(S_h\), one can deform the integration contour in the formula for \(K\psi_0\) inside \(S_h\), and check that, in fact, \(\psi_0\) can be analytically continued in \(S_{2h} = \{|\text{Re} z| < 2h, y > -Y\}\). Continuing in this way, one comes to the statement of the Lemma. \(\square\)

Below, we denote by \(\psi_0\) also the analytic continuation of the old \(\psi_0\).

**5.3.1. Function \(\psi_0\) and the difference equation.** To check that \(\psi_0\) satisfies (5.1), we again borrow an argument from [7] and [8].

We call a curve in \(\mathbb{C}\) vertical if along it \(x\) is a smooth function of \(y\). For \(z \in \mathbb{C}_+(-Y)\) we denote by \(\gamma(z)\) a vertical curve that begins at \(-iY\), goes upward to \(z\), then comes back to the imaginary axis and goes along it to \(+\infty\). One can compute \(K\psi_0(z)\) by the formula in (5.5) with the integration path \(\gamma\) replaced with \(\gamma(z)\).

Set \((H_0 f)(z) = f(z + h) + f(z - h) + \lambda e^{-2\pi i z} f(z) - 2 \cos(2\pi p) f(z)\). Then, \(H_0\psi_0 = -H_0 K\psi_0\). Using (5.5) with \(\gamma\) replaced with \(\gamma(z), z \in \mathbb{C}_+(-Y)\), we easily get
\[
(H_0 K\psi_0)(z) = 2\pi i \text{res}_{z' = z} \kappa(z + h, z') \psi_0(z') = \lambda e^{2\pi i z} \psi_0(z).
\]

Thus, we come to

**Lemma 5.3.** The solution \(\psi_0\) satisfies equation (2.1) in \(\mathbb{C}_+(-Y)\).

The last two lemmas imply the first point of the Theorem 5.1.
5.4. Asymptotics in the upper half-plane. We get asymptotics (5.3) using the integral equation for \( \psi_0 \). First, we pick \( \delta \in (0, 1) \) and sufficiently large \( Y > 0 \), assume that \( |x| \leq \delta h \) and \( y > Y \), and represent \( (K\psi_0)(z) \) in the form
\[
(K\psi_0)(z) = \frac{\lambda}{2\rho h(x, y)} (\mu (z + i\xi)(I(\mu) + J(\mu)) - \tilde{\mu} (z + i\xi)(I(\mu) + J(\mu))) ,
\]
where
\[
I(f) = \int_{\gamma, |y - y'| \geq h} \Theta(z, z') f(z' + i\xi) e^{2\pi i z'} \psi_0(z') dz',
\]
\[
J(f) = \int_{\gamma, |y - y'| \leq h} \Theta(z, z') (f(z' + i\xi) - f(z + i\xi)) e^{2\pi i z'} \psi_0(z') dz' .
\]
Let us estimate \( I(\mu) \), \( I(\tilde{\mu}) \), \( J(\mu) \) and \( J(\tilde{\mu}) \). The first two are defined \( \forall x \). One has

**Lemma 5.4.** Let \( p \in P \). We pick \( X > 0 \). As \( y \to +\infty \), uniformly in \( x \in [-X, X] \)
\[
I(\tilde{\mu})(z) = o(1), \quad I(\mu)(z) = e^{2\pi i (\xi + \frac{1}{h})} (a + o(1)) ,
\]
where
\[
a = 2i \int e^{2\pi i (\xi + \frac{1}{h})} \mu (z' + i\xi) e^{2\pi i z'} \psi_0(z') dz'.
\]
One has
\[
a = O \left( (1 + |\xi|)^2 \lambda^{\nu(p)} H \right) , \quad \nu(p) = \min \left\{ 1, \frac{1 - 2\text{Re} \rho}{h} \right\} .
\]
Note that the integral in (5.19) converges in view of (5.9) and (5.14).

**Proof.** Let us estimate \( I(\tilde{\mu}) \). We assume that \( y + \xi > 0 \). Using (5.12), (5.9) and (5.14), we get
\[
|I(\tilde{\mu})(z)| \leq H \left( \int_y^\infty (1 + |y' + \xi|)^2 e^{-2\pi (y' + \xi)} e^{-2\pi y'} dy' + \int_{-\xi}^y e^{-2\pi (y' - \xi)} (1 + |y' + \xi|)^2 e^{-2\pi y'} e^{-2\pi y'} dy' \right).
\]
The expression in the brackets is bounded by
\[
\int_{-\xi}^\infty e^{-2\pi y'} dy' + \int_{-\xi}^y e^{-2\pi (y' - \xi)} e^{-2\pi y'} dy' \left( 1 + |Y - \xi|^2 \right) \int_{-Y}^{-\xi} e^{2\pi (y' - \xi)} e^{-2\pi y'} dy'.
\]
As \( h < 1 \), this implies (5.18).
Let us turn to \( I(\mu) \). Arguing as when estimating \( I(\tilde{\mu}) \), we check that
\[
\left| \int_{\gamma, y > y' + h} \Theta(z, z') \mu (z' + i\xi) e^{2\pi i z'} \psi_0(z') dz' \right| \leq H \int_{y + h}^\infty (1 + |y' + \xi|)^2 e^{-2\pi (y' - \xi)} e^{-2\pi y'} dy' = o(e^{-2\pi x})
\]
uniformly in \( x \) as \( y \to +\infty \). Then, using the estimate
\[
\left| \Theta(z, z') - 2ie^{-2\pi i (\xi + \xi')^2} \right| \leq C e^{-2\pi (y' - \xi)} , \quad y - y' \geq h ,
\]
and again arguing as before, we prove that as \( y \to \infty \), uniformly in \( x \)
\[
\int_{\gamma, y > y' + h} \Theta(z, z') \mu (z' + i\xi) e^{2\pi i z'} \psi_0(z') dz' = e^{2\pi i (\xi + \xi')^2} a + o \left( e^{-2\pi y} \right)
\]
with $a$ from (5.19). Estimates (5.21)–(5.22) imply the second estimate in (5.18).

Let us prove (5.20). Using (5.19) and estimates (5.9) and (5.14), we get

$$|a| \leq H(1 + |\xi|)^2 \int_{-\xi}^{\xi} e^{2\pi(1 - 2\nu p)(z+i\xi)} - 2\pi y' + H \int_{-\xi}^{+\infty} e^{-2\pi(2y' + \xi)}(1 + |y' + \xi|)^2 dy'.$$

One has $\max_{0 \leq y' \leq -\xi} e^{2\pi(1 - 2\nu p)(z+i\xi)} - 2\pi y' = e^{2\pi(\nu(p)\xi)} = \lambda^\nu(p)$ with $\nu(p)$ defined by the second formula in (5.20). Furthermore, the second integral in the last estimate for $a$ equals $\lambda^{+\infty} e^{-4\pi t(1 + t)^2 dt}$. These observations lead to (5.20). This completes the proof of the lemma. □

Let us turn to the terms $J(\mu)$ and $J(\tilde{\mu})$. One has

**Lemma 5.5.** Let $p \in P$. We pick $\delta \in (0, 1)$. As $y \to +\infty$, uniformly in $x \in [-\delta h, \delta h]$,

$$(5.23) \quad J(\tilde{\mu})(z) = o(1), \quad J(\mu)(z) = o(e^{-2\pi y}).$$

**Proof.** Assume that $y + \xi \geq h$. Using estimates (4.26) for $d\mu/dz$ and estimate (5.14), we get

$$|J(\mu)| \leq H(1 + |y + \xi|)^2 e^{2\pi(1 - 2\nu p)(y+i\xi)} - 2\pi(1 + t)^2 dt. \leq H e^{-2\pi(y+i\xi)} - 2\pi y, $$

where we used the inequalities $|x| \leq \delta h$ and $0 < \delta < 1$. This implies the first estimate in (5.23). Similarly one proves the second one. □

Now we are ready to prove (5.3). We do it in three steps. Below we assume that $p \in P$. All the $o(1)$ are uniform in $x$.

1. First, we pick $\delta \in (0, 1)$ and assume that $|x| \leq \delta h$. Estimates (5.18) and (5.23) imply that uniformly in $x$ as $y \to \infty$

$$(5.24) \quad \psi_0 = \mu(z + i\xi) - (K\psi_0)(z) = \mu(z + i\xi)(1 + o(1)) + e^{2\pi i(\nu + i\xi)} - 2\pi(1 + t)^2 dt \leq H e^{-2\pi(y+i\xi)} - 2\pi y,$$

where we denoted $\tilde{\mu} = (i\lambda \delta)/\mu(\mu, \tilde{\mu})$. Representation (5.24), and formulas (4.32) and (4.10) imply (5.3) with $x_0 = ie^{2\pi i\delta}$. This completes the proof of (5.3) for $x \in [-\delta h, \delta h]$. We note that (5.3) implies that, for sufficiently large $y$,

$$(5.25) \quad |\psi_0(z)| \leq C(1 + |x_0|) e^{2\pi i(\nu + i\xi)} - 2\pi(1 + t)^2 dt.$$ 

2. Now, we pick $\delta \in (1/2, 1)$ and justify (5.3) assuming that $h \delta \leq x \leq h(1 + \delta)$. Let $\epsilon \in (0, 1 - \delta)$. We denote by $\gamma(z)$ the curve that goes first along a straight line from $i(y - h)$ to the point $z = h + \epsilon h$ and next along a straight line from this point to $i(y + h)$. One obtains representation (5.15) with $J(f)$ defined by the new formula:

$$(5.26) \quad J(f)(z) = \int_{\gamma(z)} \Theta(z', z') f(z' + i\xi) e^{2\pi i z'} \psi_0(z') dz'.$$

We note that on $\gamma(z)$, one has $|\Theta(z, z')| \leq C$. As

$$-\delta h \leq (\delta - 1)h < x - h + \epsilon h \leq (\delta + \epsilon)h,$$

and as $\delta + \epsilon < 1$, we can and do assume that on $\gamma(z)$ for sufficiently large $y$ solution $\psi_0$ satisfies (5.25). Using (5.25) and (4.26), we check that as $y \to +\infty$

$$|J(\mu)(z)| \leq H(1 + |x_0|) e^{2\pi i(\nu + i\xi)} - 2\pi(1 + |y + \xi|) = o(e^{-2\pi(y+i\xi)}).$$

Reasoning analogously, we also prove that $J(\tilde{\mu})(z) = o(1)$ as $y \to +\infty$. These two estimates and (5.18) lead again to (5.3). This completes the proof of (5.3) for $0 \leq x \leq (1 + \delta)h$. The case of $(1 + \delta)h \leq x \leq 0$ is analysed similarly.

3. To justify (5.3) for larger $|x|$, one uses equation (5.1). We discuss only the
case of $x \geq 0$ and omit further details. Pick $\delta \in (0, 1)$ and $X > 0$. In the case of Theorem 5.3, we can assume that $|\kappa_0| \leq 1/2$. Then, $1 + \kappa_0 \neq 0$ and for $x \in [\delta h, X]$ (5.3) actually means that
\[
\psi_0(z) = A\kappa(z + i\xi)(1 + o(1)), \quad A = e^{\frac{\pi}{2}}(1 + \kappa_0)v_0. 
\]
By (5.1), one can write
\[
\psi_0(z) = -\lambda e^{-2\pi i(z-h)}\psi_0(z-h) \left(1 + o(1) + e^{2\pi i(z+i\xi-h)}\psi_0(z-2h)/\psi_0(z-h)\right). 
\]
Let $(1 + \delta)h \leq x \leq (2 + \delta)h$. Then $\psi_0(z-h) = A\kappa_0(z+i\xi-h)(1 + o(1))$. This and (5.25) (that is valid on any given compact subinterval of $(-h,h)$) imply that $e^{2\pi i\xi}\psi_0(z-2h)/\psi_0(z-h) = o(1)$. Therefore,
\[
\psi_0(z) = -\lambda e^{-2\pi i(z-h)}\psi_0(z-h) (1 + o(1)) = A\kappa(z+i\xi)(1 + o(1)).
\]
So, we proved (5.3) for $x \in [0, (2 + \delta)h]$. Continuing in this way, one proves (5.3) for all $x$ such that $0 \leq x \leq X$. This completes the proof of (5.3).

5.5. **Asymptotics in the upper half-plane.** Here, we prove the third statement of Theorem 5.1, i.e., the asymptotic representation (5.4). We pick $X > 0$ and a sufficiently large $Y > 0$ and assume that $|x| \leq X$ and $|y| \leq Y/2$. We also assume that $\lambda$ is so small (or $\lambda \leq e^{C/h}$ with $C$ so large) that $Y < -\xi$.

The proof follows the same plan and uses the same estimates for $\mu$, $\bar{\mu}$ and $\psi_0$ as for studying $\psi_0$ as $y \to \infty$. So, we omit elementary details.

1. First, we represent $K\psi_0$ in the form (5.15) with $I$ and $J$ given by (5.16)–(5.17). Then, using (5.9) and (5.14) and the rough estimate $|\Theta(z, z')| \leq C$ for $|y - y'| \geq h$, we get
\[
(I(\bar{\mu})(z)) \leq H(1 + |\xi|)^2, \quad |J(\mu)(z)| \leq H(1 + |\xi|)^2 e^{4\pi Re p|\xi|/h}. 
\]

2. Let us turn to the terms $J(\mu)$ and $J(\bar{\mu})$. We first fix a positive $\delta < 1$, and consider the case where $|x| \leq \delta h$. By means of Lemma 4.1 and (5.14), we get
\[
|J(\bar{\mu})(z)| \leq H(1 + |\xi|)^2, \quad |J(\mu)(z)| \leq H(1 + |\xi|)^2 e^{4\pi Re p|\xi|/h}. 
\]

3. We recall that $\psi_0(z) - \mu(z + i\xi) = K\psi_0(z)$. Estimating the right hand side by means of (5.27)–(5.28), the estimate from Lemma 4.1 for $\mu$ and (4.32), the definition of $\bar{\mu}$, we get
\[
|\psi_0(z) - \mu(z + i\xi)| \leq \lambda H(1 + |\xi|)^3 e^{2\pi Re p|\xi|/h}, 
\]
i.e., representation (5.4) for $|x| \leq \delta h$

4. Now, we pick $\delta \in (1/2, 1)$ and justify (5.29) for $|x| \leq (1 + \delta)h$. As $|y| < Y/2$, we can assume that the point $z - h$ is above the lower end of $\gamma$. This allows to choose the curve $\gamma(z)$ as in section 5.4, and redefine $J$ by (5.26).

We can and do assume that estimate (5.29) is proved on $\gamma(z)$. This estimate and (4.26) imply that on $\gamma(z)$ one has
\[
|\psi_0(z)| \leq H(1 + |\xi|) e^{2\pi Re p|\xi|/h}. 
\]
Using this and (4.26) we obtain for the new $J(\mu)$ and $J(\bar{\mu})$ the old estimates (5.28), and, therefore, we come to (5.29) for $\delta \leq x \leq (1 + \delta)h/2$. The case of negative $x$ is treated similarly.

5. Let us prove (5.29) for all $|x| \leq X$. Therefore, we use a difference analog of the Grönwall’s inequality. We discuss only the case where $x > 0$. The case of $x < 0$ is treated similarly.

Let $\delta(z) = \psi_0(z) - \mu(z + i\xi)$. Equations (2.1) and (4.1) for $\psi_0$ and $\mu$ imply that
\[
\delta(z + h) + \delta(z - h) + 2(\lambda \cos(2\pi z) - \cos(2\pi p))\delta(z) = -\lambda e^{2\pi i\xi}\mu(z + i\xi)
\]
Therefore,
\[ \Delta(z + h) = M(z)\Delta(z) - \lambda e^{2\pi i z} \mu(z + i\xi) e_1, \]
where
\[ \Delta(z) = \begin{pmatrix} \delta(z) \\ \delta(z - h) \end{pmatrix}, \quad M(z) = \begin{pmatrix} 2(\cos(2\pi p) - \lambda \cos(2\pi z)) & -1 \\ \lambda \cos(2\pi z) & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \]
Let \( n(z) = \|\Delta(z)\|_{C^2} \) and \( A = \max_{|\mu| \leq Y} \|M(z)\|, \) where \( \|\cdot\| \) is the Hilbert-Schmidt norm. In view of (4.26) we obtain
\[ n(z) \leq B + A n(z - h), \quad B = \lambda(1 + |\xi|)H e^{2\pi \text{Re} p/|h|.} \]
Therefore, for \( N \in \mathbb{N}, \) we have
\[ n(z) \leq B \left( 1 + A + \ldots A^{N-1} + A^N \right) n(z - Nh) = B \frac{A^N - 1}{A - 1} + A^N n(z - Nh). \]
Assume that \( \frac{Nh}{2} \leq x \leq X \) and choose \( N \) so that \( \frac{Nh}{2} \leq x - Nh \leq \frac{3Nh}{2}. \) Then,
- by (5.29), \( n(z - Nh) \leq \lambda H(1 + |\xi|)^3 e^{2\pi \text{Re} p/|h|.}; \)
- \( 1 \leq N \leq C/\lambda, \) and \( A^N \leq H. \)
Using these observations and the estimate for \( B \) from (5.30), we deduce from (5.31) estimate (5.29) for \( 0 \leq x \leq X. \) This completes the proof of Theorem 5.1.

6. Monodromy matrix for Harper equation

Here, following [8], we recall the definition of the minimal entire solutions to Harper equation and then describe the monodromy matrix corresponding to a basis of two minimal entire solutions. This is the matrix described in Theorem 2.1. Then, we prove Theorem 2.2.

In [8] the authors considered equation (3.1) with an \( SL(2, \mathbb{C}) \)-valued \( 2\pi \)-periodic entire function \( M. \) Using the equivalence described in section 3.1.3, we describe such results for the Harper equation.

For a function \( f \) of \( z \) and \( E, \) we let \( f^*(z, E) = \overline{f(z, E)}. \)
Clearly, \( f \) and \( f^* \) satisfy (2.1) simultaneously. Using this symmetry, we prove Theorem 2.2 in the end of this section.

6.1. Minimal entire solutions and monodromy matrices.

6.1.1. Solutions with the simplest behavior as \( y \to \pm \infty. \) To characterize the behavior of a minimal entire solution as \( y \to \pm \infty, \) we express it in terms of solutions having the simplest asymptotic behavior as \( y \to \pm \infty. \) Let us describe these solutions. The next theorem follows from Theorem 1.1a from [8].

**Theorem 6.1.** If \( Y_1 > 0 \) is sufficiently large, there exist two solutions \( u_{\pm} \) of (2.1) that are analytic in the half-plane \( \mathbb{C}_+(Y_1) = \{ z \in \mathbb{C} : y \geq Y_1 \} \) and admit the representations
\[ u_{\pm}(z) = e^{\pm \frac{i\pi}{N}} (z - \frac{1}{2} + i\xi)^2 + i\pi z + o(1), \quad y \to +\infty. \]

One has
\[ \{ u_{+}(z), u_{-}(z) \} = \lambda. \]

Moreover, \( u_{\pm} \) are Bloch solutions in the sense of [8], i.e., the ratios \( u_{\pm}(z + h)/u_{\pm}(z) \) are \( h \)-periodic in \( z. \)

**Remark 6.1.** The expressions \( u_{\pm}^0(z) = e^{\pm \frac{i\pi}{N}} (z - \frac{1}{2} + i\xi)^2 + i\pi z, \) the leading terms in (6.1), satisfy the equations \( u_{\pm}^0(z + h) + \lambda e^{-2\pi i z} u_{\pm}^0(z) = 0 \) (compare it with Harper equation!).

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We construct two solutions with the simplest asymptotic behavior as \( y \to -\infty \) by the formulas
\[
d_{\pm}(z) = -u_{\pm}(1 - z).
\]
We use

**Lemma 6.1.** One has
\[
d_{\pm}'(z) = \alpha_{\pm}(z)u_\mp(z), \quad y \geq Y_1.
\]
\( \alpha_{\pm} \) being analytic and \( h \)-periodic, and \( \alpha(z) = 1 + o(1) \) as \( y \to \infty \) uniformly in \( x \).

*Proof.* We check (6.4) for \( d_+ \). Mutatis mutandis, for \( d_- \), the analysis is the same.

As the solution space is a two-dimensional modul over the ring of \( h \)-periodic functions, see section 3.1.3,
\[
d_{\pm}'(z) = \alpha(z)u_-(z) + \beta(z)u_+(z), \quad \beta(z) = \frac{\{d_{\pm}'(z), u_-(z)\}}{\{u_+(z), u_-(z)\}}.
\]
where \( \alpha \) and \( \beta \) are \( h \)-periodic coefficients analytic in \( \mathbb{C}_+(Y_1) \).

We recall that \( u_- \) is a Bloch solution. It means that the ratio \( r(z) = u_-(z+1)/u_-(z) \) is \( h \)-periodic. Using (6.1), we get the asymptotic representation
\[
r(z) = -\lambda^{1/h} e^{-2\pi iz/h + o(1)}, \quad y \to \infty,
\]
the error estimate being uniform in \( x \). This implies that, for sufficiently large \( y \), the solution \( u_- \) tends to zero as \( x \to -\infty \). Similarly one proves that \( d_{\pm}' \) does the same. Therefore, being periodic, the Wronskian \( \{d_{\pm}'(z), u_-(z)\} \) equals zero. In view of (6.2), this implies that \( \beta = 0 \).

Using (6.1), we check that \( \alpha(z) = 1 + o(1) \) as \( y \to \infty \) uniformly in \( x \). \( \square \)

**6.1.2. The minimal solutions.** Let \( \psi \) be an entire solution, and let \( Y_1 \) be as in Theorem 6.1. Then, \( \psi \) admits the representations:
\[
\psi(z) = A(z)u_+(z) + B(z)u_-(z), \quad y \geq Y_1,
\]
\[
\psi(z) = C(z)d_+(z) + D(z)d_-(z), \quad y \leq Y_1,
\]
where \( A, B, C \) and \( D \) are analytic and \( h \)-periodic in \( z \). The solution \( \psi \) is called *minimal* if \( A, B, C \) and \( D \) are bounded and one of them tends to zero as \( |y| \) tends to infinity.

Let \( \psi \) be a minimal solution such that \( \lim_{y \to -\infty} D(z) = 0 \) and \( C(-i\infty) = \lim_{y \to -\infty} C(z) \neq 0 \). We set \( \psi_D(z) = \psi(z)/C(-i\infty) \).

In section 7, for sufficiently small \( \lambda \), we construct \( \psi_D \) in terms of the solution \( \psi_0 \) from section 5.

Let \( A, B, C \) and \( D \) be the coefficients defined for \( \psi = \psi_D \) by (6.6) and (6.7). The limits
\[
A_D = \lim_{y \to -\infty} A(z), \quad B_D = \lim_{y \to -\infty} B(z), \quad C_D = \lim_{y \to -\infty} C(z), \quad D_D = \lim_{y \to -\infty} e^{2\pi iz/h} D(z)
\]
are called the *asymptotic coefficients* of \( \psi_D \). By definition of \( \psi_D \) one has \( C_D = 1 \).

**6.1.3. The monodromy matrix.** In terms of \( \psi_D \), we define one more solution to Harper equation (2.1) by the formula
\[
\psi_B(z) = \psi_D(1 - z).
\]
Clearly, \( \psi_B \) is one more minimal entire solution.

Theorem 7.2 from [8] can be formulated as follows:
Theorem 6.2. The minimal entire solutions $\psi_D$ and $\psi_B$ exist. They, their asymptotic coefficients and their Wronskian are nontrivial meromorphic functions of $E$. The Wronskian is independent of $z$. The monodromy matrix corresponding to $\psi_D$ and $\psi_B$ is of the form (2.2), and

$$s = -\lambda_1 \frac{D_D}{B_D}, \quad t = -\lambda_1 A_D, \quad \lambda_1 = \lambda_1^\dagger = e^{2\pi i/\pi}.$$  

(6.9)

The analysis of the poles of $s$ and $t$ was done in [7] in the case of $\lambda = 1$. Mutatis mutandis, it can be done similarly in the general case. One can see that $s$ and $t$ are analytic on the interval $I = [-2 - 2\lambda, 2 + 2\lambda]$. In section 7, for small $\lambda$ we compute the asymptotics of $s$ and $t$ as $\lambda \to 0$ on the interval $I I = [-2 - 2\lambda, 2 + 2\lambda]$, and see again that they are analytic on $I$.

6.2. Real analytic symmetry and the monodromy matrix coefficients.

Here we prove Theorem 2.2

6.2.1. A relation for the monodromy matrix. Let us consider $E$ such that $\psi_D$ and $\psi_B$ form a basis in the space of solutions of Harper equation. The monodromy matrix corresponding to this basis is defined by (3.17) with $\psi_1 = \psi_D$ and $\psi_2 = \psi_B$. As $\psi_D^*$ and $\psi_B^*$ also are solutions of Harper equation, one can write

$$\Psi^*(z) = S(z/h)\Psi(z), \quad \Psi^*(z) = \left(\begin{array}{c} \psi_D^*(z) \\ \psi_B^*(z) \end{array}\right),$$

where $S$ is a $2 \times 2$ matrix with 1-periodic coefficients.

The matrix $S$ is entire in $z$ and meromorphic in $E$ as the basis solutions do. One has

Lemma 6.2. The matrices $M$ and $S$ satisfy the relation

$$S(z + h_1) M(z) = M^*(z) S(z), \quad h_1 = \{1/h\},$$

where $M^*$ is obtained from $M$ by applying the operation $^*$ to each of its entries.

Proof. The definition of the monodromy matrix and (6.10) imply that

$$S ((z + 1)/h) M(z/h) \Psi(z) = M^*(z/h) S(z/h) \Psi(z).$$

Let $(\Psi(z + h), \Psi(z))$ be the matrix made of the column vectors $\Psi(z + h)$ and $\Psi(z)$. As the functions $S$ and $M$ are 1-periodic, we get

$$S ((z + 1)/h) M(z/h) (\Psi(z + h), \Psi(z)) = M^*(z/h) S(z/h) (\Psi(z + h), \Psi(z)).$$

As

$$\det(\Psi(z + h), \Psi(z)) = \{\psi_D(z), \psi_B(z)\},$$

the determinant of $(\Psi(z + h), \Psi(z))$ is nontrivial, and we come to the relation $S(z/h + 1/h) M(z/h) = M^*(z/h) S(z/h)$. As $S$ is 1-periodic, it implies (6.11). \hfill $\square$

6.3. The matrix $S$. Let us study properties of the matrix $S$. We shall need the following elementary observation.

Lemma 6.3. One has

$$\sigma_1 S(h_1 - z) \sigma_1 = S(z), \quad \sigma_1 = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right).$$

(6.12)

Proof. In view of (6.8), we have $\Psi(1 - z) = \sigma_1 \Psi(z)$. This and (6.10) imply that $\Psi^*(z) = \sigma_1 S(h_1 - z/h) \sigma_1 \Psi(z)$. Using (6.10) once more, we obtain the relation $S(z/h) \Psi(z) = \sigma_1 S(h_1 - z/h) \sigma_1 \Psi(z)$. Now arguing as in the end of the proof of Lemma 6.2, we deduce (6.12) from this relation. This completes the proof of the lemma. \hfill $\square$

Now, we prove
Proposition 6.1. The coefficients of $S$ are independent of $z$. One has

\begin{align}
(6.13) & \quad S_{11} = S_{22} = \frac{1}{B_D}, \quad S_{12} = S_{21} = \frac{A_D}{B_D}, \\
(6.14) & \quad A'_D = A_D, \quad B_D B'_D - A_D A'_D = 1.
\end{align}

Proof. We prove formulas \((6.13)\) for $S_{11}$ and $S_{12}$. These formulas and relation \((6.12)\) imply the formulas for the other entries of $M$.

According to \((3.15)\)– \((3.16)\), relation \((6.10)\) implies that

\begin{equation}
(6.15) \quad S_{11}(z/h) = \frac{\{\psi_D(z), \psi_B(z)\}}{\psi_D(z), \psi_B(z)}, \quad S_{12}(z/h) = \frac{\{\psi_D(z), \psi_D(z)\}}{\psi_D(z), \psi_B(z)}.
\end{equation}

Below, we compute the Wronskians in \((6.15)\) in terms of the asymptotic coefficients of the solution $\Psi_D$.

Let us begin with \(\{\psi_D(z), \psi_B(z)\}\). We recall that, for sufficiently large $Y_1$, solution $\psi = \psi_D$ admits representations \((6.6)\) and \((6.7)\) with $h$-periodic coefficients $A$, $B$, $C$ and $D$, and $D = e^{-2\pi iz/h} D_1(z)$, where $D_1$ is bounded in the half-plane $y < -Y_1$. By means of \((6.8)\) and \((6.3)\) we get for $y \geq Y_1$

\[ \{\psi_D(z), \psi_B(z)\} = \left\{ A(z)u_+(z) + B(z)u_-(z), -C(1 - z)u_+(z) - e^{-2\pi i(1 - z) h} D_1(1 - z) u_-(z) \right\} \]

\[ = \left\{ -e^{-2\pi i(1 - z) h} A(z) D_1(1 - z) + D(z) C(1 - z) \right\} \{u_+(z), u_-(z)\}. \]

Using this representation, \((6.2)\) and the definitions of the asymptotic coefficients $\psi_D$, see \((6.1.2)\), we check that

\[ \{\psi_D(z), \psi_B(z)\} \to \lambda B_D C_D = \lambda B_D \quad \text{as} \quad y \to \infty. \]

Similarly, we prove that

\[ \{\psi_D(z), \psi_B(z)\} = \left\{ C(z) d_+(z) + e^{-2\pi i z/h} D_1(z) d_-(z), -A(1 - z) d_+(z) - B(1 - z) d_-(z) \right\} \]

\[ \to \lambda B_D \quad \text{as} \quad y \to -\infty. \]

As $\{\psi_D(z), \psi_B(z)\}$ is an $h$-periodic entire function, these observations imply that

\begin{equation}
(6.16) \quad \{\psi_D(z), \psi_B(z)\} = \lambda B_D.
\end{equation}

Arguing similarly one computes $\{\psi_D^*(z), \psi_B(z)\}$ and $\{\psi_D(z), \psi_D^*(z)\}$, and obtains the formulas

\[ \{\psi_D^*(z), \psi_B(z)\} = \lambda + o(1) \quad \text{as} \quad y \to +\infty, \]

\[ \{\psi_D^*(z), \psi_B(z)\} = \lambda B_D B'_D - A_D A'_D + o(1) \quad \text{as} \quad y \to -\infty, \]

and

\[ \{\psi_D(z), \psi_B^*(z)\} = \lambda A_D + o(1) \quad \text{as} \quad y \to +\infty, \]

\[ \{\psi_D(z), \psi_B^*(z)\} = -\lambda A_D + o(1) \quad \text{as} \quad y \to -\infty. \]

Omitting elementary details we note that, to get these formulas, one uses \((6.1)\).

The last four formulas imply that

\[ \{\psi_D^*(z), \psi_B(z)\} = \lambda, \quad \{\psi_D(z), \psi_B^*(z)\} = \lambda A_D. \]

This, \((6.16)\) and \((6.15)\) imply \((6.13)\).
6.3.1. Proof of Theorem 2.2. Formulas (6.11) and (6.13) imply the relation
\[ \tilde{S} M(z) = M^*(z) \tilde{S}, \quad \forall z \in \mathbb{C}, \quad \tilde{S} = \begin{pmatrix} 1 & A_D \\ A_D & 1 \end{pmatrix}. \]
This relation implies that
\[ M_{12}(z) + A_D M_{22}(z) = M_{11}^*(z) A_D + M_{12}(z), \quad \forall z \in \mathbb{C}. \]
Substituting into this formula the expressions for the monodromy matrix coefficients from (2.2), we get
\[ s + t e^{-2\pi i z} + A_D \frac{st}{\lambda_1} = A_D \left( a^* - 2\lambda_1 \cos(2\pi z) \right) + s^* + t^* e^{2\pi i z}, \quad \forall z \in \mathbb{C}. \]
This equality of two trigonometric polynomials leads to the relations
\[ t = -\lambda_1 A_D, \quad s + A_D \frac{st}{\lambda_1} = a^* A_D + s^*. \]
The first of these two relations and the first formula in (6.14) imply that \( t = -t^* \), and substituting in the second one the formula \( A_D = -t/\lambda_1 \) and the formula for \( a \) from (2.2), one easily checks that
\[ ss^* = \lambda_1^2 \frac{1 - t^2}{\lambda_1^2 - t^2}. \]
These two observations imply (2.3).

Let \( E \in \mathbb{R} \). One has \( t = i \tau \) and \( s = -i \lambda_1 \sqrt{1 + t^2} e^{i \alpha} \) with real \( \tau \) and \( \alpha \). Using these representations, we get the following formula for the zeroth Fourier coefficient of the trace of the monodromy matrix described in Theorem 2.1:
\[ (\text{Tr} M)_0 = \frac{\lambda_1}{st} \left( 1 - s^2 - t^2 \right) + \frac{st}{\lambda_1} = \frac{2}{\tau} \sqrt{(1 + \tau^2)(\lambda_1 + \tau^2)} \cos \alpha. \]
This leads to (2.4). The proof of Theorem 2.2 is complete.

7. Asymptotics of the monodromy matrix coefficients

7.1. Formulation of the Riemann problem. To construct the solution \( \psi_D \), we paste it of solutions analytic in \( \mathbb{C}_+ = \{ z \in \mathbb{C} : y \geq 0 \} \) and \( \mathbb{C}_- = \{ z \in \mathbb{C} : y \leq 0 \} \) by means of a Riemann problem. Here, we formulate this problem.

7.1.1. Relations between entire solutions and solutions analytic in \( \mathbb{C}_\pm \). Let \( S_\pm \) be the set of solutions to Harper equation that are analytic in \( \mathbb{C}_\pm \), and let \( K_\pm \) be the set of the complex valued functions that are analytic and \( b \)-periodic in in \( \mathbb{C}_\pm \). Assume that \( \psi_\pm \) and \( \phi_\pm \) belong \( S_\pm \). Let \( w_\pm(z) = \{ \psi_\pm(z), \phi_\pm(z) \} \). Clearly, \( w_\pm \in K_\pm \). We assume that \( w_\pm \) does not vanish in \( \mathbb{C}_\pm \). Then, the pair \( \psi_\pm, \phi_\pm \) is a basis in \( S_\pm \). Any entire solution \( \psi \) to (3.1) admits the representations
\[ \psi(z) = a_+(z) \psi_+(z) + b_+(z) \phi_+(z), \quad z \in \mathbb{C}_+, \quad a_+, b_+ \in K_+, \]
\[ \psi(z) = a_-(z) \psi_-(z) + b_-(z) \phi_-(z), \quad z \in \mathbb{C}_-, \quad a_-, b_- \in K_-, \]
with
\[ a_\pm(z) = \frac{1}{w_\pm(z)} \{ \psi(z), \phi_\pm(z) \}, \quad b_\pm(z) = \frac{1}{w_\pm(z)} \{ \psi_\pm(z), \psi(z) \}. \]
Both representations (7.1) and (7.2) are valid on the real line. Therefore,
\[ a_+(z) \psi_+(z) + b_+(z) \phi_+(z) = a_-(z) \psi_-(z) + b_-(z) \phi_-(z), \quad z \in \mathbb{R}. \]
This implies that
\[ V_+ = GV_-, \quad z \in \mathbb{R}, \quad V_+ = \begin{pmatrix} a_+ \\ b_+ \end{pmatrix}, \quad V_- = \begin{pmatrix} a_- \\ b_- \end{pmatrix}, \]
(7.5)
where $G$ is the matrix described by the formulas

\begin{align}
G_{11}(z) &= \frac{w_-(z)}{w_+(z)} \{\psi_-(z), \phi_+(z)\}, \\
G_{12}(z) &= \frac{1}{w_+(z)} \{\phi_-(z), \phi_+(z)\}, \\
G_{21}(z) &= \frac{1}{w_+(z)} \{\psi_+(z), \psi_-(z)\}, \\
G_{22}(z) &= \frac{1}{w_+(z)} \{\psi_+(z), \phi_-(z)\}.
\end{align}

**Remark 7.1.** One has

\begin{align}
\det G(z) &= \frac{1}{w_-(z)} w_+(z), \quad z \in \mathbb{R}.
\end{align}

Indeed, (7.4) implies also the relation

\[ V_- = \frac{1}{w_-(z)} \begin{pmatrix}
\{\psi_+(z), \phi_-(z)\} & \{\phi_+(z), \phi_-(z)\} \\
\{\psi_-(z), \psi_+(z)\} & \{\psi_-(z), \phi_+(z)\}
\end{pmatrix} V_+.
\]

One also can express $V_-$ via $V_+$ by inverting the matrix $G$ in (7.5). Comparing the results, one comes to (7.7).

We have checked

**Lemma 7.1.** Any entire solution of (3.1) can be represented by (7.1) – (7.2) with $a_+,$ $b_+ \in \mathbb{K}_+$ and $a_-, b_- \in \mathbb{K}_-$ and these coefficients satisfy the relation (7.5) with the matrix $G$ given by (7.6).

One can easily prove also

**Lemma 7.2.** If $a_+$ and $b_+$ belong to $\mathbb{K}_+,$ $a_-$ and $b_-$ belong to $\mathbb{K}_-$, and if these four functions satisfy relation (7.5) with the matrix $G$ given by (7.6), then formulae (7.1) – (7.2) describe an entire solution of (3.1).

### 7.1.2. Change of variable. Let $\zeta = e^{2\pi i z/h}$ and

\[ T = \{\zeta \in \mathbb{C} : |\zeta| = 1\}, \quad B_o = \{\zeta \in \mathbb{C} : |\zeta| \leq 1\}, \quad B_\infty = \{\zeta \in \mathbb{C} : |\zeta| \geq 1\} \cup \{\infty\}.
\]

The functions $V_{\pm}$ being $h$-periodic, we can consider them as functions of $\zeta.$ Then, $V_+$ appears to be analytic in $B_o \setminus \{0\}$ and $V_-$ is analytic in $B_\infty \setminus \{\infty\}.

### 7.1.3. Basis solutions for constructing $\psi_D.$ Let $\psi_0(z, p)$ be the solution to (2.1) described in Proposition 5.1, and let

\begin{align}
\psi_+(z, p) &= \psi_0(z, p), \quad \phi_+(z, p) = e^{-\frac{\pi i(z + \xi)}{h}} \psi_0(z - 1/2, 1/2 - p),
\end{align}

and

\begin{align}
\psi_-(z, p) &= \psi_0^*(z, p), \quad \phi_-(z, p) = \phi_0^*(z, p).
\end{align}

Clearly, $\psi_+, \phi_+ \in \mathbb{S}_+,$ and $\psi_-, \phi_- \in \mathbb{S}_-.$

To work with $\psi_\pm$ and $\phi_\pm$, we need to describe their behavior for large $y$ and for $y \sim 0.$ By means of (5.3), (6.1) and (6.3), we get

**Corollary 7.1.** For sufficiently large $Y_1$ and for all $z \in \mathbb{C}_+(Y_1)$

\begin{align}
\psi_+(z, p) &= A_\psi(z) e^{\frac{2\pi i(z + \xi)}{h}} u_+(z) + B_\psi(z) u_-(z), \\
\phi_+(z, p) &= A_\psi(z) u_+(z) + B_\psi(z) u_-(z).
\end{align}

For $z \in \mathbb{C}_-(-Y_1) = \{z \in \mathbb{C} : y \leq -Y_1\}$

\begin{align}
\psi_-(z, p) &= \frac{B_\psi(z)}{A_\psi(z)} d_+(z) + \frac{A_\psi(z)}{B_\psi(z)} e^{-\frac{2\pi i(z - \xi)}{h}} d_-(z), \\
\phi_-(z, p) &= \frac{B_\psi(z)}{A_\psi(z)} d_+(z) + \frac{A_\psi(z)}{B_\psi(z)} d_-(z).
\end{align}
where $\alpha_{\pm}, A_{\psi}, B_{\psi}, A_{\phi}$ and $B_{\phi}$ are $h$-periodic and analytic in $z$; $\alpha_{\pm}$ are described in (6.1), and one has

\begin{align}
A_{\psi}(z) &= A_{\psi,0}(1 + o(1)), \quad B_{\psi}(z) = B_{\psi,0}(1 + o(1)), \quad y \to +\infty, \\
A_{\psi,0} &= e^{\frac{\beta}{h} - \frac{2\pi i p}{h} - \frac{\pi}{h} + \frac{\beta}{h} - \pi} e^{-\beta}, \quad B_{\psi,0} = e^{-\frac{2\pi i p}{h} - \frac{\pi}{h} - \pi} (1 + o_{\pm}), \\
A_{\phi}(z) &= A_{\phi,0}(1 + o(1)), \quad B_{\phi}(z) = B_{\phi,0}(1 + o(1)), \quad y \to +\infty, \\
A_{\phi,0}(p) &= -iA_{\phi,0}(1/2 - p) e^{-\frac{2\pi i p}{h}} - B_{\phi,0}(p) = -iB_{\phi,0}(1/2 - p) e^{-\frac{2\pi i p}{h}}.
\end{align}

These asymptotics are uniform in $z$.

Furthermore, the third point of Theorem 5.1 and Proposition 4.2 imply

**Corollary 7.2.** Pick $\alpha \in (0, 1)$ and $X > 0$. For $p \in P$, $|y| \leq \frac{Y}{2}$ and $|x| \leq X$ the following holds.

- Pick $\beta \in (0, \alpha)$. If $\text{Re} p \geq \alpha h/2$

\begin{align}
\psi_+(z) &= e^{\frac{2\pi i p(z + i)}{h}} \left( a(p) + O(\lambda \beta H) \right), \\
\phi_+(z) &= e^{-\frac{2\pi i p(z + i)}{h}} \left( \frac{e^{-\frac{\pi i}{2}(1/2 - p)}}{a(1/2 - p) + O(\lambda \beta H)} \right).
\end{align}

- Pick $\beta \in (0, 1)$. If $\text{Re} p \leq \alpha h/2$,

\begin{align}
\psi_+(z) &= a(p) e^{\frac{2\pi i p(z + i)}{h}} + a(-p) e^{-\frac{2\pi i p(z + i)}{h}} + O \left( \lambda \beta H e^{\frac{2\pi p}{h}} \right), \\
\phi_+(z) &= e^{-\frac{2\pi i p(z + i)}{h}} e^{-\frac{\pi i}{2}(1/2 - p)} a(1/2 - p) + O \left( \lambda \beta H e^{\frac{2\pi p}{h}} \right).
\end{align}

We complete this section by computing the Wronskians $w_+ = w_+(z)$. As

\begin{equation}
w_-(z) = w_+(z),
\end{equation}

we need to compute only $w_+$. Pick $Y$ as in Theorem 5.1. One has

**Lemma 7.3.** Pick $0 < \beta < 1$. For $p \in P$ and $z \in \mathbb{C}_+(-Y/2)$, one has

\begin{equation}
w_+ = w_0 + O(\lambda \beta H), \quad w_0 = ie^{-\frac{2\pi i p}{h} + \frac{2\pi i p}{h}} - \frac{\pi i}{h} - \frac{\beta}{h}.
\end{equation}

**Remark 7.2.** Lemma 7.3 implies that $\psi_+$ and $\phi_+$ are linearly independent if the quantity $\lambda \beta H$ is sufficiently small.

**Proof.** First, we compute $w_+$ as $y \to +\infty$. Using (7.10) we get

\begin{equation}
\{\psi_+, \phi_+\} = \left\{ A_{\psi} e^{\frac{2\pi i p(z + i)}{h}} u_+ + B_{\psi} u_- + A_{\phi} u_+ + B_{\phi} u_- \right\} = \left\{ A_{\psi} B_{\phi} e^{\frac{2\pi i p(z + i)}{h}} - B_{\psi} A_{\phi} \right\} \{u_+, u_-\} = -\lambda A_{\phi,0} B_{\phi,0} + o(1).
\end{equation}

So, the Wronskian is bounded as $y \to +\infty$.

Let us pick $\beta \in (0, 1)$ and check that

\begin{equation}
w_+ = 2i \sin(2\pi p) a(p) a(1/2 - p) e^{-\frac{\pi i}{2}(1/2 - p)} + O(\lambda \beta H), \quad |y| \leq Y/2.
\end{equation}

Note that in view of (4.35) this already implies representation (7.19) for $|y| \leq Y/2$.

To prove (7.20), we pick $\alpha \in (\beta, 1)$ and consequently consider four cases. In the case where $\alpha_{\pm} \leq \text{Re} p \leq 1/2 - \alpha_{\pm}$ formula (7.20) follows from representations (7.14),
As we can assume that \( \beta \) in the last formula is larger than in (7.20), we again obtain (7.20). The case where \( \Re p \geq \max\{a_b, \frac{1}{2} - \frac{a_b}{2}\} \) is treated similarly (by means of (7.14) and (7.17)). Finally, if \( \frac{1}{2} - \frac{a_b}{2} \leq \Re p \leq \frac{a_b}{2} \), we get

\[
w_+ = 2i \sin(2\pi p) a(p) e^{-\frac{i\pi(1/2-p)}{2}} + O(\xi^3 H).
\]

As we can assume that \( \beta \) in the last formula is larger than in (7.20), we again obtain (7.20). The case where \( \Re p \geq \max\{a_b, \frac{1}{2} - \frac{a_b}{2}\} \) is treated similarly (by means of (7.14) and (7.17)). Finally, if \( \frac{1}{2} - \frac{a_b}{2} \leq \Re p \leq \frac{a_b}{2} \), we get

\[
w_+ = 2i \sin(2\pi p) a(p) e^{-\frac{i\pi(1/2-p)}{2}} - 2i \sin(2\pi p) a(-p) e^{-\frac{i\pi(1/2+p)}{2}} e^{-\frac{\pi(1)}{4}} + O(\xi^3 H).
\]

Let us estimate the second term in the right hand side of (7.21). Denote it by \( T \). According to (4.18), for \( y \rightarrow \infty \), we have

\[
|T| \leq C\lambda^{1/2} |\sin(2\pi p)\sigma_{\pi h}(-4\pi p - \pi h - \pi)\sigma_{\pi h}(4\pi(p - 1/2) - \pi h - \pi)|.
\]

By means of (2.5) and (8.3), we get

\[
(7.22) \quad |T| \leq C\lambda^{1/2} \frac{|\sigma_{\pi h}(4\pi(h/2 - p) - \pi h - \pi)\sigma_{\pi h}(4\pi(p - \pi h - \pi)|}{1 - e^{-4\pi(p-1/2)}}.
\]

As \( \frac{1}{2} - \frac{a_b}{2} \leq \Re p \leq \frac{a_b}{2} \), one has \( h \geq \frac{1}{2a} > \frac{1}{2} \), and also

\[
4\pi(h/2 - \Re p) \geq 2\pi h(1 - \alpha) > \pi(1 - \alpha), \quad 4\pi\Re p \geq 2\pi(1 - \alpha h) \geq 2\pi(1 - \alpha),
\]

\[
(1-\alpha)/2 \leq \Re p \leq \alpha/2.
\]

The first two inequalities and Corollary 8.1 imply that \( |\sigma_{\pi h}(4\pi(h/2 - p) - \pi h - \pi)\sigma_{\pi h}(4\pi(p - \pi h - \pi)| \leq C \). The third inequality implies that \( |1 - e^{-4\pi(p-1/2)}| \geq C \). These observations prove that \( |T| \leq C\lambda^{1/2} \). As \( 1/2 < h < 1 \), the term \( T \) can be included in error term. This completes the proof of (7.19) for \( |y| \leq Y/2 \).

The expression \( w_+ - w_0 \) is \( h \)-periodic and analytic in \( z \). Since it is bounded as \( y \rightarrow +\infty \), representation (7.19) justified for \( |y| \leq Y/2 \) and the maximum principle imply that \( w_+ - w_0 = O(H\lambda^3) \) for all \( z \in \mathbb{C}_+(Y/2) \) uniformly in \( p \in P \). This completes the proof. \( \square \)

7.1.4. Riemann problem for constructing \( \psi_D \). Let \( \psi_+ \) and \( \phi_+ \) be the bases chosen in section 7.1.3. The minimal solution \( \psi = \psi_D \) admits the representations (7.1)-(7.2). The coefficients \( a_+, b_+ \in K_\pm \) satisfy the equation (7.5) with the matrix \( G \) defined by (7.6). To formulate the Riemann problem for these coefficients, we need to study their behavior at \( \pm i\infty \).

The coefficients \( a_+ \) and \( b_+ \) being \( h \)-periodic, we shall regard them as functions of the variable \( \zeta = e^{2\pi i\zeta}/h \) and use the notations introduced in the section 7.1.2.

Substituting (7.10) into (7.1), we see that if \( a_+ \) and \( b_+ \) are bounded as \( \zeta \rightarrow 0 \) \((y \rightarrow +\infty)\), then \( \psi_D \) admits representation (6.6) with \( A \) and \( B \) staying bounded as \( y \rightarrow +\infty \), and one has

\[
(7.23) \quad A_D = A_{\psi, 0} a_+(0), \quad B_D = B_{\psi, 0} a_+(0) + B_{\psi, 0} b_+(0).
\]

Substituting (7.11) into (7.2) and taking into account Lemma 6.1, we see that if, as \( \zeta \rightarrow \infty \), the coefficient \( a_- \) is bounded and \( b_-(\zeta) \rightarrow 0 \), then \( \psi_D \) admits representation (6.7) with \( C \) staying bounded and \( D \) vanishing as \( \zeta \rightarrow \infty \), and one has

\[
(7.24) \quad 1 = C_D = B^*_{\psi, 0} a_-(\infty), \quad D_D = e^{-2\pi \xi} A^*_{\psi, 0} a_-(\infty) + A_{\psi, 0} b_-^{(1)}, \quad b_-^{(1)} = \lim_{\zeta \rightarrow \infty} (\zeta b_-(\zeta)).
\]
Let us collect the obtained information on the coefficients $a_{\pm}$ and $b_{\pm}$. One has
\begin{equation}
V_+(\zeta) = G(\zeta) V_-(\zeta), \quad \zeta \in T,
\end{equation}
\begin{equation}
V_+ \text{ is analytic in } B_0, \quad V_- \text{ is analytic in } B_\infty, \quad V_-(\infty) = \frac{1}{|\nu_0|} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\end{equation}
Equation (7.25) and conditions (7.26) form a Riemann problem. We shall see that, for sufficiently small $\lambda$, this problem has a unique solution. Having solved this problem, we shall reconstruct the coefficients of the minimal solution $\psi_D$ by the formulae (7.23) and (7.24).

7.2. Matrix $G$. In this section, we study the matrix $G$.

7.2.1. Functional relations. The properties of the matrix $G$ we discuss here immediately follow from its definition (7.6) and from the choice of the solutions $\psi_{\pm}$ and $\phi_{\pm}$, see (7.8)–(7.9). When describing these properties, we use the variable $z$, assume that $\lambda$ is sufficiently small, and that $p \in P$.

As $w_+$ is bounded away from zero in the domain $(z, p) \in \mathbb{C}_+(-Y/2) \times P$, the matrix $G$ is analytic there.

Let
\begin{equation}
g_{ij}(z, p) = w_+(z, p) G_{ij}(z, p), \quad i, j \in \{1, 2\}.
\end{equation}

As $\psi_+ = \psi_+^* + \phi_+^*$, (7.6) implies that
\begin{equation}
g_{22}(z, p) = g_{11}^+(z, p), \quad g_{12}(z, p) = -g_{12}^+(z, p), \quad g_{21}(z, p) = -g_{21}^+(z, p).
\end{equation}

Furthermore, relation (7.18) and the formula (7.7) imply that
\begin{equation}
\det G^* = \det G^{-1}, \quad g_{11} g_{22} - g_{12} g_{21} = w_+^* w_+.
\end{equation}

Finally, as $\phi_+(z, p) = e^{-\frac{i\pi(z+P)}{2}} \psi_+(z - 1/2, 1/2 - p)$, we get
\begin{equation}
g_{12}(z, p) = e^{\frac{2\pi i}{\lambda}} g_{21}(z - 1/2, 1/2 - p).
\end{equation}

7.2.2. The asymptotics of $G$ for $p$ bounded away from 0 and $\pi$. As $\lambda \to 0$, $G$ appears to be close to a constant matrix. One has the following two statements:

**Proposition 7.1.** Pick $0 < \beta < 1/2$. There is a positive constant $C_1$ such that if $\lambda < e^{-C_1/h}$, then for $p \in P$ such that $h/4 \leq \text{Re} p \leq 1/2 - h/4$ and for $|y| \leq \frac{Y}{2}$
\begin{equation}
G = \frac{1}{w_0} \begin{pmatrix} \delta & \delta e^{4\pi i y} (2i F(p) \sin(2\pi p) + \delta) \\ e^{-4\pi i y} (2i F(p) \sin(2\pi p) + \delta) & \delta \end{pmatrix},
\end{equation}
where $w_0$ is defined in (7.19), $\delta$ denotes $O(\lambda^\beta H)$, and $F$ is a meromorphic function such that
\begin{equation}
F(p) = |\sigma_{\text{nh}}(4\pi p - \pi - \pi h)|^2, \quad p \in \mathbb{R}.
\end{equation}
Moreover, one has $|F(p)| \leq H$ for $p \in P$ such that $h/4 \leq \text{Re} p$.

**Lemma 7.4.** One has
\begin{equation}
F(p + 1/2) = 4 \sin^2 \frac{2\pi p}{h} F(p),
\end{equation}
\begin{equation}
4 \sin^2(2\pi p) F(1/2 - p) F(p) = 1.
\end{equation}

**Proof.** Formula (7.32) follows from (8.3), and (7.33) follows from (8.4) and (2.5). \[\square\]
Proof of Proposition 7.1. Below, we assume that all the hypotheses of the proposition are satisfied. First, we estimate the Wronskian $g_{11} = \{\psi_-, \phi_+\}$. Using (7.14)–(7.15) and the definition $\psi_-$ from (7.9), we get

$$g_{11} = \left\{ e^{-\frac{2\pi p (z-\xi)}{h}}(\alpha^*(p) + \delta), e^{-\frac{2\pi p (z-\xi)}{h}}(\alpha^*(1/2 - p) + \delta) \right\}.$$  

Obviously, the leading term equals zero, and using estimate (4.28) we prove that $g_{11} = O(\lambda^3 H)$. By means of the first relation from (7.28) we also see that $g_{22} = O(\lambda^3 H)$. As $G_{1j} = (w_+)^{-1} Q_{ij}$, $j = 1, 2$, and in view of Lemma 7.3, we get the announced estimate for the diagonal elements of the matrix $G$.

Now consider $g_{12} = \{\phi_-(z), \phi_+(z)\}$. Using (7.15), (7.9) and (4.28), we get

$$g_{12} = \left\{ e^{-\frac{2\pi p (z-\xi)}{h}}(\alpha^*(1/2 - p) + \delta), e^{-\frac{2\pi p (z-\xi)}{h}}(\alpha(1/2 - p) + \delta) \right\} =
\left\{ e^{-\frac{2\pi p (z-\xi)}{h}}(2i \sin(2\pi p) |a(1/2 - p)|^2 + O(\lambda^3 H)) \right\}.
$$

Let us note that, to get this formula, instead of $h/4 \leq p \leq 1/2 - h/4$, we have only to assume that $Re p \leq 1/2 - h/4$. The definition of $a$, formula (4.18), implies that

$$F(p) = |a(p)|^2.$$

Therefore,

$$g_{12} = e^{-\frac{4\pi p (z-\xi)}{h}} (2i F(1/2 - p) \sin(2\pi p) + \delta),$$

and also, in view of (4.28) $|F(p)| \leq H$ for $p \in P$ such that $Re p \geq h/4$. Representations (7.35), (7.19) and the last estimate imply the formula for $G_{12}$ announced in the proposition. We note that it is valid for all $p \in P$ such that $Re p \leq \pi - h/4$.

Formula (7.35) and relation (7.30) imply the formula for $G_{21}$ announced in the proposition. It is valid for all $p \in P$ such that $h/4 \leq Re p$.

We have checked all the statements of the proposition. □

7.2.3. The case of $0 \leq p \leq h/4$.

Proposition 7.2. Pick $0 < \beta < 1$. There is a positive constant $C_2$ such that if $\lambda < e^{-C_2/h}$, then for $p \in P$, $0 \leq Re p \leq h/4$ and $|y| \leq Y/2$ one has

$$G = \frac{1}{w_0} \left( e^{\frac{4\pi p}{h}} F_d(p) + \delta \right) e^{-\frac{4\pi p}{h}} (2i F(1/2 - p) \sin(2\pi p) + \delta),$$

where $\delta$ denotes $O(\lambda^3 H)$,

(7.36) $F_d(p) = -4ie^{\frac{2\pi p}{h}} \frac{2\pi p}{h} \sin(2\pi p) F(-p),$

(7.37) $F_a(p) = 2i \sin(2\pi p) F(p) - e^{-\frac{4\pi p}{h}} F(-p),$

and $F$, $F_d$ and $F_a$ satisfy the estimates

(7.38) $p^2 F(p) = O(H)$, $F_d(p) = O(H)$, $F_a(p) = O(\lambda H)$.

Proof. Below we assume that $p$ and $z$ satisfy the hypotheses of the proposition. Let us compute $g_{11}$. Formulas (7.16), (7.15) and (7.9) imply that

(7.39) $g_{11} = \{a^*(p) e^{\frac{2\pi p (z-\xi)}{h}} + a^*(1/2 - p) e^{\frac{2\pi p (z-\xi)}{h}} + O(e^{-\frac{4\pi p}{h}} \lambda^3 H), e^{-\frac{2\pi p (z-\xi)}{h}}(e^{\frac{4\pi p}{h}} a(1/2 - p) + \delta) \}.$

Using estimates (4.28) and (4.29) we get the formula

$$g_{11} = e^{-\frac{4\pi p}{h}} F_d(p) + O((1 + |\xi|) \lambda^3 H).$$
with \( \tilde{F}_d(p) = 2i\sin(2\pi p) e^{-\frac{ie(1/2-p)}{\pi}} a^*(-p)a(1/2 - p) \). Let us show that \( \tilde{F}_d = F_d \).

In view of (4.18),
\[
\frac{i e^{-\frac{ie(1/2-p)}{\pi}}} a^*(-p) a(1/2 - p) = e^{-\frac{4\pi p}{\pi}} \sigma_{\pi h}(-4\pi p - \pi - \pi h) \sigma_{\pi h}(-4\pi p + \pi - \pi h).
\]

By (8.3)
\[
\sigma_{\pi h}(-4\pi p + \pi - \pi h) = -2ie^{\frac{4\pi p}{h}} \sin\frac{2\pi p}{h} \sigma_{\pi h}(-4\pi p - \pi - \pi h).
\]

The last two relations imply that \( \tilde{F}_d = F_d \). So, we have

\[
g_{11} = e^{\frac{4\pi p}{\pi}} F_d(p) + O(H\lambda^3(1 + |\xi|)).
\]

The definition of \( F_d \) and estimates (4.28) and (4.29) imply that \( |F_d(p)| \leq H \).

As we can assume that \( \beta \) in (7.40) is larger than in Proposition 7.2, this formula, (7.19) and the estimate \( |F_d(p)| \leq H \) imply the representation for \( G_{11} \) that we have to prove. The representation for \( G_{22} \) follows from (7.40) and the first relation from (7.28).

Now let us turn to the anti-diagonal coefficients of \( G \). The coefficient \( G_{12} \) was already computed in the proof of Proposition 7.1: when computing \( G_{12} \), we had only to assume that \( 0 \leq \Re p \leq 1/2 - h/4 \). So, \( G_{21} \) is the only element of the matrix \( G \) that we still have to compute. Using (7.16), (7.9) and estimates (4.28) and (4.29), we get

\[
g_{21} \equiv \{\psi_+ (z), \psi_-(z)\} = \{a(p)e^\frac{2\pi p(1+i)}{\pi} + a(-p)e^{-\frac{2\pi p(1+i)}{\pi}} + O(\lambda^\beta H e^{-\frac{2\pi p}{\pi}}),
\]
\[
a^*(p)e^{-\frac{2\pi p(1+i)}{\pi}} + a^*(-p)e^\frac{2\pi p(1+i)}{\pi} + O(\lambda^\beta H e^{-\frac{2\pi p}{\pi}})\}\}
\]
\[
e^{-\frac{4\pi p}{\pi}} \left(2i\sin(2\pi p)(|a(p)|^2 - e^{\frac{2\pi p}{\pi}}|a(-p)|^2) + O(\lambda^\beta(1 + |\xi|)H)\right)
\]
\[
e^{-\frac{4\pi p}{\pi}} \left(F_a(p) + O(\lambda^\beta(1 + |\xi|)H)\right),
\]

where, at the last step, we used the definition of \( F_a \) and relation (7.34).

Now, let us discuss the estimates for \( F \) and \( F_a \). Estimate for \( p^2 F(p) \) immediately follows from (7.34) and (4.29). The estimate \( F_a(p) = O((1 + |\xi|)H) \) follows from the estimate for \( F \), and the observations that the function \( p \mapsto p(F(p) - F(-p)) \) is analytic in a \( Ch \)-neighborhood of zero, estimate (4.28) and the maximum modulus principle.

As we can assume that, in the obtained formula for \( g_{21} \), the number \( \beta \) is larger than \( \beta \) in statement of Proposition 7.2, the formula for \( g_{12} \) and the estimate for \( F_a \) lead to the needed formula for \( G_{21} \).

\[\square\]

7.3. **Proof of Theorem 2.3.** Here, we compute the coefficients \( s \) and \( t \) of the monodromy matrix in the case where \( p \in P \) is bounded away from 0 and \( \pi \). Therefore, first, we solve the Riemann problem (7.25)–(7.26) to find the asymptotics of \( a_+(0), b_+(0), a_- (\infty) \) and \( \lim_{\xi \to \infty} f (-\xi) \). Then, by means of formulae (7.23) and (7.24), we compute the coefficients \( A_D, B_D \) and \( D_D \) of the minimal entire solution \( \psi_D \). Finally, using formulae (6.9), we compute \( s \) and \( t \).

7.3.1. **Solving the Riemann problem.** The leading term of the asymptotics of the matrix \( G \) being independent of \( z \), the analysis of the Riemann problem is elementary. Below we assume that \( z \) and \( p \) satisfy assumptions of Proposition 7.1. All the matrices we consider are two-by-two matrices with complex entries.

Let
\[
G_0 = \frac{2i\sin(2\pi p)}{w_0} \begin{pmatrix} 0 & F(1/2 - p) \\ F(p) & 0 \end{pmatrix}, \quad T = \begin{pmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{pmatrix}, \quad \tau = e^{\frac{2\pi p}{\pi}}.
\]

Relation (7.33) implies that \( \det G_0 = 1/w_0^2 \). In view of Proposition 7.1,
\[
G = T G_0 (I + \Delta)^{-1}, \quad \Delta = O(\lambda^3 H).
\]
The term \( \Delta \) is analytic in \((z, p) \in \{(y) \leq \frac{p}{T}\} \times \{p \in P : h/4 \leq \Re p \leq 1/2 - h/4\}\).

Now, we pass to the variable \( \zeta = e^{2\pi i z/h} \). Let \( \|\| \) be a matrix norm. Pick \( \alpha \in (0, 1) \). For matrix functions on \( T = \{ |\zeta| = 1 \} \) denote by \( \|\|_\alpha \) the standard Hölder norm defined in terms of \( \|\| \). One has

**Lemma 7.5.** Let \( \Delta \) be a matrix valued function on \( T \). Pick \( \alpha \in (0, 1) \). If \( \|\Delta\|_\alpha \) is sufficiently small, then there exist unique matrix functions \( W_\pm \) such that
\[
W_+ \text{ is analytic in } \mathbb{B}_\alpha, \quad W_- \text{ is analytic in } \mathbb{B}_\infty, \quad W_-(\infty) = I,
\]
\[
W_+ (\zeta) = (I + \Delta(\zeta)) W_-(\zeta), \quad \zeta \in T.
\]

These functions satisfy the estimates:
\[
\|W_+(\zeta) - I\| \leq C\|\Delta\|_\alpha, \quad |\zeta| \leq 1; \quad \|W_-(\zeta) - I\| \leq \frac{C\|\Delta\|_\alpha}{|\zeta|}, \quad |\zeta| \geq 1.
\]

**Proof.** The Lemma follows from the standard results of the theory of singular integral operators, see, e.g., [22]. So, we describe the proof omitting standard details. First, in \( H_\alpha(T) \), the space of Hölder functions on \( T \), one constructs a solution to the equation
\[
W_- = I + S(\Delta W_-),
\]
where, for \( f \in H_\alpha(T) \),
\[
S(f)(\zeta) = -\frac{1}{2} f(\zeta) + \frac{1}{2\pi i} \text{v.p.} \int_T \frac{f(\zeta')}{|\zeta' - \zeta|} d\zeta', \quad \zeta \in T,
\]
where the orientation of \( T \) is positive. As \( S \) is a bounded operator in \( H_\alpha(T) \), and as for \( f, g \in H_\alpha(T) \) one has \( \|fg\|_\alpha \leq \|f\|_\alpha \|g\|_\alpha \), equation (7.46) has a unique solution provided \( \|\Delta\|_\alpha \) is sufficiently small.

One defines \( W_+(\zeta) \) for \( \zeta \in \mathbb{B}_\alpha \) and \( W_-(\zeta) \) for \( \zeta \in \mathbb{B}_\infty \) by the formula
\[
W_\pm (\zeta) = I + \frac{1}{2\pi i} \int_T \frac{\Delta(\zeta')W_-(\zeta')}{|\zeta' - \zeta|} d\zeta',
\]
and checks that these two function have all the properties described in Lemma 7.5. We omit further details.

In our case, \( \Delta \) is analytic in a ring \( e^{-\pi Y/h} \leq |\zeta| \leq e^{\pi Y/h} \), and in this ring \( \|\Delta(\zeta)\| \leq H\lambda^3 \). Therefore, for any fixed \( \beta \in (0, 1/2) \), \( \|\Delta\|_\alpha \leq C(\beta) H\lambda^3 \). So, there is a positive constant \( C \) such that if \( \lambda \leq e^{-C/h} \), then \( \Delta \) satisfies the condition of Lemma 7.5. Below, we assume that this is the case.

By means of Lemma 7.5, one immediately constructs a solution to the Riemann problem (7.26)–(7.25):
\[
V_+(\zeta) = T G_0 W_+(\zeta) T^{-1} e, \quad V_-(\zeta) = T G_0 W_-(\zeta) T^{-1} e, \quad e = \frac{1}{B_{\gamma,0}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

Indeed, these vector functions \( V_\pm \) have the analytic properties required in (7.26), and \( V_-(\infty) = T G_0(I + \Delta(\zeta)) T^{-1} e = e \). Moreover, by (7.41) and (7.44), for \( |\zeta| = 1 \),
\[
V_+(\zeta) = T G_0 W_+(\zeta) T^{-1} e = T G_0(I + \Delta(\zeta)) W_-(\zeta) T^{-1} e =
\]
\[
= T G_0(I + \Delta(\zeta)) T^{-1} V_-(\zeta) = G(\zeta) V_-(\zeta).
\]

So, \( V_\pm \) satisfy also (7.25).

To compute the coefficients of the monodromy matrix, we need to compute \( a_+(0) \) and \( b_+(0) \), the first and the second components of the vector \( V_+(0) \), \( a_-(\infty) \), the
first component of the vector $V_-(\infty)$, and $b_{-1}$ defined in (7.24) in terms of $b_-$, the second component of the vector $V_-$. From (7.47) we easily deduce that
\begin{equation}
(7.48) \quad a_+(0) = O(e^{\pi \xi \lambda^2 H}), \quad b_+(0) = \frac{e^{-\frac{4\pi \xi}{u_0}B^*_{v,0}}}{u_0 B^*_{v,0}} (2i \sin(2\pi p) F(p) + O(\lambda^3 H)),
\end{equation}
To get the estimate for $a_+$ we used the estimate $\zeta_0 = O(H \lambda)$ (see the third point of Proposition 5.1). Furthermore,
\begin{equation}
(7.49) \quad a_-(\infty) = \frac{1}{B^*_{v,0}}, \quad b_{-1} = \zeta b_-(\zeta) \bigg|_{\zeta=\infty} = O(\lambda^3 e^{-4\pi \xi / h p + \pi \xi H}).
\end{equation}
To get the last estimate, we used Lemma 7.5.

7.3.2. Asymptotics of the coefficients $s$ and $t$. Using (6.9), (7.23), (7.24), we get
\begin{equation}
(7.50) \quad t = -\lambda_1 A_{\phi,0} b_+(0), \quad s = -\frac{A^*_{\phi,0} a_-(\infty) + \lambda_1 A^*_{\phi,0} b_{-1}}{B^*_{v,0} a_+(0) + B_{\phi,0} b_+(0)}.
\end{equation}
Using (7.12) and (7.13), formulas for $A_{\phi,0}$, $A_{\phi,0}$, $B_{\phi,0}$ and $B_{\phi,0}$, (7.48)–(7.49), estimates for $a_+$, $b_+$ and $b_{-1}$, and the estimate $\zeta_0 = O(H \lambda)$, we obtain
\begin{equation}
t = 2ie^{\frac{4n_1}{4(1/2-p)}} (F(p) \sin(2\pi p) + O(\lambda^3 H)) (1 + O(\lambda H)),
\end{equation}
\begin{equation}
s = \frac{1}{2i} \frac{e^{\frac{4n_1}{4(1/2-p)}} e^{\frac{2n_1}{4(1/2-p)}} (1 + e^{\frac{4n_1}{4(1/2-p)}} O(\lambda^3 H))}{e^{\frac{2n_1}{4(1/2-p)}} O(\lambda^3 H) + (F(p) \sin(2\pi p) + O(\lambda^3 H)) (1 + O(\lambda H))}.
\end{equation}
Let us simplify these formulae. For $h/4 \leq \text{Re} p \leq 1/2 - h/4$, one has $|F(p)| \leq H$ (see Proposition 7.1). By this estimate and (7.33), one also has
\begin{equation}
|2F(p) \sin(2\pi p)|^{-1} = |2\sin(2\pi p) F(1/2 - p)| \leq H.
\end{equation}
Using these observations, we get
\begin{equation}
t = 2ie^{\frac{4n_1}{4(1/2-p)}} F(p) \sin(2\pi p) (1 + O(\lambda^3 H)), \quad s = \frac{e^{\frac{4n_1}{4(1/2-p)}} e^{\frac{2n_1}{4(1/2-p)}} (1 + O(\lambda^3 H))}{2i F(p) \sin(2\pi p)}.
\end{equation}
Finally, by means of (2.5), we check that, for $p \in \mathbb{R}$,
\begin{equation}
2i \sin(2\pi p) F(p) = 2i \sin(2\pi p) |\sigma_{\pi h}(4\pi p - \pi + \pi h)|^2 = - \frac{\sigma(4\pi p - \pi + \pi h)^2}{2i \sin(2\pi p)}.
\end{equation}
This relation and (7.51) imply the statement of Theorem 2.3.\hfill \Box

7.4. Asymptotics of $s$ and $t$ for $p$ close to zero. Here, we prove

Theorem 7.1. Pick $\beta \in (0, 1)$. Let $p \in P$ and $0 \leq \text{Re} p \leq h/4$. As $\lambda \to 0$
\begin{equation}
t = ie^{\frac{4n_1}{4(1/2-p)}} F_0(p) \left[ 1 - \frac{e^{\frac{4n_1}{4(1/2-p)}} \rho(p)}{2 \sin(2\pi p)} + \delta \right], \quad s = \frac{-2ie^{\frac{4n_1}{4(1/2-p)}} + 2i \sin(2\pi p)}{F_0(p) \left[ 1 - \frac{e^{\frac{4n_1}{4(1/2-p)}} \rho(p)}{2 \sin(2\pi p)} + \delta \right]},
\end{equation}
where for $p \in \mathbb{R}$, one has $\rho(p) = \left| \frac{\sigma(-4\pi p - \pi + \pi h)}{\pi(4\pi p - \pi - \pi h)} \right|^2$, and $\delta$ again denotes $O(\lambda^3 H)$.

The plan of the proof is similar to one of Theorem 2.3.
7.4.1. Coefficients $a_+, b_+$ and $b_-$. Let

$$G_0 = \frac{1}{w_0} \begin{pmatrix} e^{\frac{2\pi i}{h}} F_d(p) & 2i F(1/2 - p) \sin(2\pi p) \\ F_a(p) & e^{\frac{4\pi i}{h}} F_d^*(p) \end{pmatrix}.$$  

(7.53)

One has

Lemma 7.6. $\det(w_0 G_0) = 1$.

Proof. For $p \in \mathbb{R}$,

$$\det(w_0 G_0) = e^{\frac{4\pi i}{h}} |F_d(p)|^2 - 2i \sin(2\pi p) F_a(p) F(1/2 - p) =$$

$$= 16 e^{\frac{4\pi i}{h}} \sin^2(2\pi p/h) \sin^2(2\pi p) F^2(-p) + 4 \sin^2(2\pi p) (F(p) - e^{\frac{4\pi i}{h}} F(-p)) F(1/2 - p).$$

(7.54)

Now, we note that (7.32) and (7.31) imply that

$$16 \sin^2(2\pi p/h) \sin^2(2\pi p) F(-p) = 1/F(p), \quad \text{and} \quad 4 \sin^2 p F(1/2 - p) = 1/F(p).$$

Therefore, (7.54) implies the statement of the lemma. \hfill \Box

The above lemma implies that $\det G_0 \neq 0$. Let us check that $G$ admits representation (7.41) with the new $G_0$. Using the formula for $G$ from Proposition 7.2, the definition of $T$ and Lemma 7.6, we get $G_0^{-1} T^{-1} G_T = I + \Delta$ with the $a$ matrix $\Delta$ such that

$$\Delta = \begin{pmatrix} O \left( \left| e^{\frac{4\pi i}{h}} F_d \right| + \left| F \right| \right) \lambda^\beta H & O \left( \left| e^{\frac{4\pi i}{h}} F_d^* \right| + \left| F \right| \right) \lambda^\beta H \\ O \left( \left| F_a \right| + \left| e^{\frac{4\pi i}{h}} F_d \right| \right) \lambda^\beta H & O \left( \left| F_a \right| + \left| e^{\frac{4\pi i}{h}} F_d^* \right| \right) \lambda^\beta H \end{pmatrix}$$

with $F_a = F_a(p)$, $F_d = F_d(p)$, $F = F(\pi - p)$.

In the case we consider $1/2 \geq 1/2 - \Re p \geq 1/2 - h/4 \geq h/4$ as $0 \leq h \leq 1$. So, by Proposition 7.1, $F(1/2 - p) = O(H)$. Using this estimate, the estimates for $F_d$ and $F_a$ from Proposition 7.2 and the inequality $|\Re \xi/h| \leq 1$, we get that $\Delta = O(\lambda^\beta H)$ with a smaller $\beta$, which is not important as we could take the old $\beta$ larger. So, we get representation (7.41). Note that the matrix $\Delta$ is analytic under the hypotheses of Proposition 7.2.

Having obtained (7.41), one proceeds precisely as in the case where $h/4 \leq \Re p$. In result, one again obtains (7.47) with new matrices $W_\pm$ having the representations $W_+ = I + O(H \lambda^\beta)$ and $W_- = I + O(H \lambda^\beta)/|\xi|$ in $B_0$ and $B_\infty$ respectively. Now, instead of (7.48), one obtains

$$a_+(0) = \frac{e^{\frac{4\pi i}{h}} F_d(p) + O(\lambda^\beta H)}{w_0 B^*_\psi,0}, \quad b_+(0) = \frac{e^{-\frac{4\pi i}{h}} (F_a(p) + O(\lambda^\beta H))}{w_0 B^*_\psi,0},$$

(7.55) $a_-(\infty) = \frac{1}{B^*_\psi,0}, \quad b_-(\infty) = O(e^{-\frac{4\pi i}{h} + \pi \lambda^\beta H}).$

(7.56)

7.4.2. Asymptotics of $s$ and $t$. One computes the coefficients $s$ and $t$ of the monodromy matrix by means of formulas (7.50), asymptotic representations (7.55) and (7.56), and the estimates for $F$, $F_a$ and $F_d$ from Proposition 7.2. Computations similar to ones carried out in the previous section lead to the formulas

$$t = e^{\frac{4\pi (1/2 - p) i}{h}} (F_a(p) + O(\lambda^\beta H)),$$

$$s = \frac{e^{\frac{4\pi i}{h} + \frac{4\pi i}{h}} F_a(p) + i e^{\frac{4\pi i}{h} + \frac{4\pi i}{h}} F_d(p) + O(\lambda^\beta H)}{F_a(p) + ise^{\frac{4\pi i}{h} + \frac{4\pi i}{h}} F_d(p) + O(\lambda^\beta H)}.$$

Let us prove (7.52) for $t$. In view of (2.5) and (7.31), for $p \in \mathbb{R}$ one has

$$\rho(p) = \frac{\sigma(-4\pi p - \pi + \pi h)}{\sigma(4\pi p - \pi + \pi h)} \geq \frac{\sigma(-4\pi p - \pi + \pi h)^2}{\sigma(4\pi p - \pi + \pi h)^2} = \frac{F(-p)}{F(p)}.$$
Therefore,

\begin{equation}
F_a(p) = -4 \sin^2(2\pi p) F(p) \frac{1 - e^{\frac{\pi p}{\pi}} \rho(p)}{2i \sin(2\pi p)}.
\end{equation}

Furthermore, one has $1/2 - \text{Re } p \geq 1/2 - h/4 \geq h/4$. This and the estimate for $F$ from Proposition 7.1 imply that $F(1/2 - p) = O(H)$. So, by (7.33) we get $(4 \sin^2(2\pi p) F(p))^{-1} = O(H)$. This estimate and (7.57) imply that

\begin{equation}
t = -4 \sin^2(2\pi p) F(p) e^{\frac{4}{\pi} \rho(p)} \left( 1 - \frac{e^{\frac{\pi p}{\pi}} \rho(p)}{2i \sin(2\pi p)} + O(\lambda^3 H) \right).
\end{equation}

In view of (2.5), we also see that $4 \sin^2(2\pi p) F(p) = |\sigma^2(4\pi p - \pi + \pi h)|^2 = F_0(p)$ for $p \in \mathbb{R}$. This and the last formula for $t$ imply the formula for $t$ from (7.52).

To prove the formula for $s$ from (7.52), one checks that

\begin{equation}
F_a(p) + ie^{\frac{\pi p}{\pi} + \frac{3\pi p}{2}} F_d(p) = 2i \sin(2\pi p) F(p) \left( 1 - e^{\frac{\pi p}{\pi} (\xi + i/2) \frac{F(-p)}{F(p)}} \right)
\end{equation}

\begin{equation}
= -4 \sin^2(2\pi p) F(p) \frac{1 - e^{\frac{\pi p}{\pi} (\xi + i/2) \rho(p)}}{2i \sin(2\pi p)}.
\end{equation}

This and the last formula for $s$ imply the formula for $s$ from (7.52). The proof of Theorem 7.1 is complete. □

8. A trigonometric analog of the Euler Gamma-function

Here, following mostly [8, 15], we discuss a meromorphic solution to equation (2.5).

8.1. Definition and elementary properties.

8.1.1. There exists a unique meromorphic solution $\sigma_a$ to equation (2.5) that is analytic and does not vanish in the strip $S_0 = \{|\text{Re } z| < \pi + \alpha\}$, and admits in $S_0$ the asymptotic representations

\begin{equation}
\sigma_a(z) = 1 + o(e^{-\alpha|y|}), \quad y \to -\infty;
\end{equation}

and

\begin{equation}
\sigma_a(z) = e^{-i\pi \frac{z}{\alpha} + i \pi \frac{z}{2\alpha^2} + i \frac{\pi}{12} + o(e^{-\alpha|y|})}, \quad y \to +\infty,
\end{equation}

where $\alpha$ is any number fixed so that $0 < \alpha < 1$.

The function $\sigma_a$ is continuous in $a > 0$; the asymptotic representations for $\sigma_a$ are uniform in $a$ only for $a$ bounded away from zero.

8.1.2. Using equation (2.5), one can analytically continue $\sigma_a$ to a meromorphic function. Its poles are located at the points $-\pi - a - 2\pi l - 2ak$, $l, k = 0, 1, 2, \ldots$, and its zeros are at the points $\pi + a + 2\pi l + 2ak$, $l, k = 0, 1, 2, \ldots$.

Its zero at $\pi + a$ and its pole at $-\pi - a$ are simple. The other zeros and poles are simple if $a/\pi \not\in \mathbb{Q}$. 

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Theorem 8.2. We assume that Estimate (8.12) follows from (8.11) and (8.5). Let us prove (8.11). Below
Proof.

\begin{align}
(8.12) \quad \sigma_a(z) &= (1 + e^{-\frac{i\pi}{a}z}) \sigma_a(z - \pi), \\
(8.14) \quad \sigma_a(-z) &= e^{-\frac{1}{4\pi} z^2 + i\pi^2 + \frac{1}{12} - \frac{1}{\sigma_a(z)}}, \\
(8.16) \quad \overline{\sigma_a(z)} &= e^{\frac{1}{4\pi} z^2 - \frac{1}{\overline{\sigma_a(z)}}} - \frac{i\pi}{12} \sigma_a(z).
\end{align}

8.2. Quasiclassical asymptotics. Here, we discuss \( \sigma_a \) for small \( a > 0 \).

8.2.1. Below we use the branch of the function \( z \mapsto \ln (1 + e^{-i\pi} z) \) analytic in \( \mathbb{C}_- \),

the lower halfplane, and satisfying the condition

\begin{align}
(8.21) \quad \ln (1 + e^{-i\pi} z) &\rightarrow 0, \quad z \rightarrow -i\infty.
\end{align}

We set

\begin{align}
(8.22) \quad L(z) &= \int_{-i\infty}^{z} \ln (1 + e^{-i\pi} z') dz',
\end{align}

where we integrate in \( \mathbb{C}_- \), say, along the line \( \Re z' = \Re z \). One has

**Theorem 8.1.** [15] Pick \( 0 < \delta < \pi \). In \( \mathbb{C}_- \cup \mathbb{R} \) outside the \( \delta \)-neighborhood of the

half-lines \( z \geq \pi \) and \( z \leq -\pi \), for sufficiently small \( a \), \( \sigma_a \) admits the representation

\begin{align}
(8.23) \quad \sigma_a(z) &= \exp \left( \frac{1}{2a} L(z) + O \left( a (1 + |x|) e^{-|y|} \right) \right).
\end{align}

8.2.2. Here, we discuss \( \sigma_a \) in a neighborhood of the point \( -\pi \). The behavior of \( \sigma_a \)

for \( z \sim \pi \) can be described by means of (8.4). One has

**Theorem 8.2.** [15] Let \( 0 < \delta < 2\pi \). For \( t \) in the \( \delta \)-neighborhood of zero,

\begin{align}
(8.24) \quad \sigma_a(-\pi + t) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} T \left( \frac{t + a}{2a} \right) e^{\frac{i\pi}{2a} \tilde{l}(\zeta) \zeta - \frac{1}{4a} + O(a)} \tilde{l}(\zeta) = \ln \frac{1 - e^{-i\zeta}}{\zeta},
\end{align}

where one uses the analytic branch \( \tilde{l} \) such that \( \tilde{l}(0) = i\pi/2 \). The error term in (8.24)

is analytic in \( t \).

8.3. Corollaries.

8.3.1. Estimates outside a neighborhood of \( -\pi \). Pick \( 0 < \delta < \pi \) and \( 0 < \kappa < 1 \). One has

**Corollary 8.1.** Outside the \( \delta \)-neighborhood of the ray \( z < -\pi \),

\begin{align}
(8.25) \quad \sigma_a(z) &= e^{O(a^{-1} e^{-\kappa|y|} (1 + |x|))}, \quad y \leq 0, \\
(8.26) \quad \sigma_a(z) &= e^{-\frac{1}{4\pi} z^2 + i\pi^2 + \frac{1}{12} + O(a^{-1} e^{-\kappa|y|} (1 + |x|))}, \quad y \geq 0.
\end{align}

**Proof.** Estimate (8.26) follows from (8.25) and (8.5). Let us prove (8.25). Below

we assume that \( y \leq 0 \).

First, we note that (8.25) is valid for \( |x| \leq a \). Indeed, let \( a_0 > 0 \) be so small that

(8.27) holds for all \( 0 < a < a_0 \). For these \( a \), representation (8.25) follows
directly from (8.5). For \( a_0 \leq a \leq 2\pi \), representation (8.25) immediately follows
from (8.1) that is valid and uniform in \( x \) if \( |x| \leq a \).
Now, we assume that \( z \) is outside the \( \delta \)-neighborhood of the ray \( z < -\pi \). We choose \( n \in \mathbb{Z} \) so that \( |x - 2an| \leq a \). By (2.5)
\[
\sigma_a(z) = \prod_{j=1}^{[n]} \left( 1 + e^{-i(z + (2j - 1)a)} \right)^{\pm 1} \sigma_a(z - 2na) \quad \text{if} \quad \pm n \geq 1.
\]
For \( z \) we consider, one has \( |n| \leq \frac{|x|}{2a} + \frac{1}{2} \), and
\[
\prod_{j=1}^{[n]} \left( 1 + e^{-i(z + (2j - 1)a)} \right)^{\pm 1} = e^{O(|n|e^{-|u|})} = e^{O(xe^{-|u|}/a)}.
\]
Representation (8.11) valid for \( |x| \leq a \) and the last estimate imply (8.11) for all \( z \) we consider.

8.3.2. Estimates for \( z \) in a neighborhood of \(-\pi\). Fix positive \( c_1, c_2 \) and \( \delta < 2\pi \). One has

**Corollary 8.2.** Let \( |z + \pi| \leq \delta \), and let \( x \geq -\pi - a + c_1a - c_2|y| \). Then, \( |\sigma_a(z)| \leq Ce^{C/a} \).

**Proof.** 1) Let \( u = \frac{z + \pi + a}{2a} \). Then, under the hypothesis of the corollary,
\[
\text{Re } u \geq \frac{c_1}{2} - c_2|\text{Im } u|.
\]
Let \( D \) be the domain defined by this inequality in the complex plane of \( u \).

2) For \( u \in D \), we set
\[
Y(u) = \sqrt{\frac{n}{2\pi}} e^{-u(ln u - 1)} \Gamma(u),
\]
where the branches of \( \ln \) and \( \sqrt{\cdot} \) are analytic in \( D \) and such that \( \ln 1 = 0 \) and \( \sqrt{1} = 1 \). The function \( Y \) is bounded in \( D \).

3. By (8.10) and the previous steps, under the hypothesis of the Corollary,
\[
|\sigma_a(z)| \leq C \left| \exp \left( u\ln u - 1 \right) - \frac{1}{2} \ln u + \frac{\ln(2a)}{2a} (z + \pi) + \frac{1}{2a} \int_{0}^{z+\pi} \tilde{I}(\zeta) d\zeta \right|
\leq Ce^{C/a} \left| \exp \left( u\ln u + \frac{\ln(2a)}{2a} (z + \pi) \right) \right| \leq Ce^{C/a} |e^{u\ln(2au)}|.
\]

as \( \tilde{I} \) is analytic in the \( \delta \)-neighborhood of zero and as \( |u| \geq C \) in \( D \).

4. As \( |2au| = |z + \pi + a| \leq C \), then one also has \( |2au \ln(2au)| \leq C \), and (8.15) implies that \( |\sigma_a(z)| \leq Ce^{C/a} \).

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