Symmetry Breaking and Enhanced Condensate Fraction in a Matter-Wave Bright Soliton

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An exact diagonalization study reveals that a matter-wave bright soliton and the Goldstone mode are simultaneously created in a quasi-one-dimensional attractive Bose-Einstein condensate by superpositions of quasi-degenerate low-lying many-body states. Upon formation of the soliton the maximum eigenvalue of the single-particle density matrix increases dramatically, indicating that a fragmented condensate converts into a single condensate as a consequence of the breaking of translation symmetry.

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Fragmentation of a Bose-Einstein condensate (BEC), which occurs as a consequence of a certain exact symmetry of the system, has recently been discussed in a number of articles [1, 2, 3]. In contrast to the conventional BEC, characterized by a unique macroscopic eigenvalue in the single-particle density matrix [4], the fragmented BEC is characterized by more than one macroscopic eigenvalue [5]. If the system has an exact symmetry and if the many-body theory predicts fragmentation of the ground state, the Gross-Pitaevskii (GP) mean-field theory does not predict a fragmented condensate but approximates it with a single condensate whose symmetry is spontaneously broken. For example, a quasi-one-dimensional (1D) BEC with attractive interaction forms bright solitons [6], which are well described by the GP theory [7]. Efforts to elucidate how such symmetry-broken states emerge from exact many-body states have been made in diverse systems [6].

In this Letter, we show that the formation of a broken-symmetry soliton and an enhancement of the condensate fraction are caused by superpositions of the low-lying states of the symmetry-preserving many-body Hamiltonian. We find that the many-body spectrum exhibits a number of quasi-degenerate states in the regime where the exact ground state is a fragmented condensate. Superposition of these quasi-degenerate levels simultaneously generates the broken-symmetry bright soliton and the Goldstone mode, accompanied by a significant increase in the condensate fraction. By introducing a small symmetry-breaking perturbation or by considering the action of a quantum measurement, we explicitly show that the fragmented condensate is very fragile against the soliton formation. Also elucidated in the language of the many-body theory is the mechanism underlying a partial breaking of the quantized circulation in the presence of a rotating drive.

We consider a system of N attractive bosons with mass m on a 1D ring with circumference 2πR. Length and energy are measured in units of R and ℏ²/(2mR²), respectively. The Hamiltonian for our system is given by

\[ \hat{H} = \int_0^{2\pi} d\theta \left[ -\hat{\psi}^{\dagger}(\theta) \frac{\partial^2}{\partial \theta^2} \hat{\psi}(\theta) - \frac{\pi g}{2} \hat{\psi}^{\dagger}(\theta) \hat{\psi}(\theta)^2 \right], \] (1)

where \( \hat{\psi}(\theta) \) is the field operator, which annihilates an atom at position \( \theta \), and \( g (> 0) \) denotes the strength of attractive interaction. According to the GP mean-field approximation for the Hamiltonian \( \hat{\psi} \), the ground state is either a uniform condensate or a broken-symmetry bright soliton, depending on whether the parameter \( gN \) is below or above the critical value, \( gN = 1 \). In contrast, all eigenstates of the original Hamiltonian are translation invariant, and many-body theory predicts that the ground state is either a single \( (gN < 1) \) or fragmented \( (gN \gtrsim 1) \) condensate [8].

Figure 1 (a) shows the low-lying spectrum obtained by exact diagonalization of the Hamiltonian \( \hat{\psi} \). The dramatic change in the landscape of the energy spectrum around \( gN \approx 1 \) is a consequence of the quantum phase transition between a single condensate and a fragmented one. Figure 1 (b) presents the Bogoliubov spectrum obtained from the Bogoliubov-de Gennes equations. By comparing Figs. 1(a) and (b), we find that the Bogoliubov spectrum has a one-to-one correspondence with the many-body spectrum for \( gN \lesssim 1 \). For \( gN \gtrsim 1 \), however, the many-body spectrum becomes much more intricate than the Bogoliubov one. In the Bogoliubov spectrum for \( gN \gtrsim 1 \), there appears a Goldstone mode \( A' \) (the translation mode of the soliton) associated with the symmetry breaking of the ground state, the breathing mode \( B' \), and the second harmonic of the breathing mode \( C' \). In Fig. 1 (a) for \( gN \gtrsim 1 \), in contrast, a number of quasi-degenerate levels appear with the density of states peaking around the Bogoliubov levels; we denote the corresponding groups as \( A, B, \) and \( C \), respectively. The basis states for the diagonalization are restricted to the angular-momentum states \( l = 0, \pm 1 \) \( (l_c = 1) \) unless otherwise stated, and the field operator is given by
where the degeneracy is almost maintained. As $gN$ approaches 1, each branch begins to ramify, and the energy landscape for $gN > 1$ is characterized by the index $\sigma$ and $\mathcal{L}$ as $E_{0,\sigma} \lesssim E_{\pm 1,\sigma} \lesssim E_{\pm 2,\sigma} \lesssim \cdots$. There is no Goldstone mode because the ground state possesses the translation symmetry, and the lowest excited states $|\pm 1\rangle_A$ have a finite energy gap $\Delta E = E_{\pm 1,\sigma} - E_{0,\sigma}$, since the system is finite. However, the density of states above the ground state becomes higher for larger $N$, and the gap $\Delta E$ collapses as $1/N$ [Fig. 4(c)]. The ground state is therefore unstable against excitations of the quasi-degenerate low-lying states.

We construct the many-body counterparts of the bright soliton $|\Psi_\theta\rangle$ and the Goldstone mode $|\Phi_\theta\rangle$, such that $\langle \Psi_\theta | \Phi_\theta \rangle = 0$ by superpositions of the ground and quasi-degenerate states:

$$|\Psi_\theta\rangle = e^{-i\hat{L}_\theta} \left[ \beta_0|0\rangle_A + \sum_{\mathcal{L} \neq 0} \beta_{\mathcal{L}} (|\mathcal{L}\rangle_A - | - \mathcal{L}\rangle_A) \right],$$

(2)

$$|\Phi_\theta\rangle = \frac{d}{d\theta} |\Psi_\theta\rangle = -ie^{-i\hat{L}_\theta} \sum_{\mathcal{L} > 0} \mathcal{L} \beta_{\mathcal{L}} (|\mathcal{L}\rangle_A - | - \mathcal{L}\rangle_A),$$

(3)

where $\hat{L} \equiv \int d\theta \hat{\psi}^\dagger \hat{\psi} (\theta) ( -i \partial_\theta \hat{\psi} (\theta) )$ is the angular momentum operator, and $\beta_{\mathcal{L}}$’s satisfy $\sum_{\mathcal{L}} |\beta_{\mathcal{L}}|^2 = 1$. The energy cost associated with the superposition (2), $E_{\Psi_\theta} - E_{\Phi_\theta} = \sum_{\mathcal{L}} |\beta_{\mathcal{L}}|^2 (E_{|\mathcal{L}\rangle_A} - E_{|0\rangle_A})$, is on the order of $1/N$. This indicates that the symmetry breaking from the exact ground state $|0\rangle_A$ to the bright soliton $|\Psi_\theta\rangle$ costs little energy, and the superposition thus occurs by an “infinitesimal” perturbation on the order of $1/N$.

The emergence of quasi-degenerate levels also plays a crucial role in breaking the quantized circulation. In the presence of a rotating drive with angular frequency $2\Omega$, the many-body ground state is either a single or fragmented condensate, depending on whether $f(g, N, \Omega) = (1 - gN)^2 - 2(\Omega - 1/2)^2$ is positive or negative [10], where $[\Omega + 1/2]$ denotes the maximum integer that does not exceed $\Omega + 1/2$. Figure 5 shows the total angular momentum of the ground state $\mathcal{L}_g$ and low-lying eigenvalues of the Hamiltonian $\hat{H} - 2\hat{\Omega} \hat{L} + \Omega^2$ in the rotating frame. There clearly appear two distinct regimes where the density of states of excitations is low ($f > 0$) and high ($f < 0$) in the spectrum, and the circulation $\hbar \mathcal{L}_g / m$ is quantized only when $f > 0$. When the density of states above the ground state is sparse, the ground state cannot make a transition to states with higher angular momenta even if $\Omega$ is increased, since the term $-2\Omega \hat{L}$ is not large enough to make up for the excitation energy due to a large energy gap. The total angular momentum $\mathcal{L}_g$ therefore does not increase with $\Omega$, and is quantized at integral multiples of $N$ for $f > 0$. However, a number of branches with higher angular momenta decrease in energy as $f$ becomes negative; then, as $\Omega$ is increased, one branch after another takes the place of the ground level upon intersection [Fig. 5(b)]. The angular momentum
of the ground state $L_g$ thus increases stepwise each time the excitation energy collapses. The interval between collapses of the excitation energy, i.e., the width of the steps of $L_g$, becomes narrower as $N$ becomes larger, eventually resulting in the breaking of the quantized circulation, as indicated by the slopes in the thick line in Fig. 2(a). The emergence of quasi-degenerate levels also induces symmetry breaking in a manner similar to Eq. 2. In fact, the ground state in the GP theory is a localized soliton for $f < 0$.

We next investigate the effect of a symmetry-breaking perturbation by diagonalizing the Hamiltonian $\hat{H} \pm \varepsilon \hat{V}$ where $\hat{H}$ is given by Eq. 1 and $\hat{V} = \int d\theta \hat{\psi}^\dagger(\theta) \cos \theta \hat{\psi}(\theta)$. As the perturbation is switched on, the ground state becomes localized by superposing the quasi-degenerate states. The largest eigenvalue $\lambda^{(c)}_\text{M}$ of the single-particle density matrix is plotted as a function of $\varepsilon N^2$ in Fig. 3. The fragmented condensate approaches the single condensate as $\varepsilon N^2$ increases. We note that $\lambda^{(c)}_\text{M}$ depends only on $\varepsilon N^2$, which indicates that a single condensate is realized for $\varepsilon \gtrsim N^{-2}$. The fragmented condensate with $N \gg 1$ is hence very fragile against the symmetry-breaking perturbation.

The condensate fraction $\lambda^{(c)}_\text{M}$ and the distribution of $\beta_L$ in Eq. 2 can be derived analytically with the Bogoliubov approximation 11. Since the frequency of the translation mode of the broken-symmetry soliton is proportional to $\varepsilon^{1/2}$ and small, the number of excited atoms in this mode is much larger than in the other modes. We thus consider only the translation mode, and obtain the depletion as $(2N)^{-1} \varepsilon^{-1/2} [2F^{1/2} - \varepsilon^{1/2} + O(\varepsilon)] = 1 - \lambda^{(c)}_\text{B}$, where $F \equiv 7(3gN + 4)^{1/2}/[(5gN + 2)(2gN - 2)^{1/2}]$ is a decreasing function for $gN > 1$. The behavior of $\lambda^{(c)}_\text{B}$ is in excellent agreement with the numerical result as shown in Fig. 3. Assuming that $\beta_L |L\rangle_A$ in Eq. 2 corresponds to the projection of the Bogoliubov ground state on the angular momentum $L$, i.e., $\beta_L |L\rangle_A \approx \int d\theta \hat{\psi}(\theta) |L\rangle_B$, we obtain the distribution of $\beta_L$ as

$$|\beta_L|^2 \approx \frac{1}{\sqrt{2\pi d^2}} \exp \left[ -\frac{L^2}{2d^2} \right],$$

$$d^2 = gN \varepsilon^{1/2} \left( \frac{(2F)^{1/2} + 2\varepsilon^{1/2} + O(\varepsilon)}{7gN} \right),$$

which is in excellent agreement with the numerical results as shown in the inset of Fig. 3. The behavior of the width $d \propto N^{1/2} \varepsilon^{-1/4}$ indicates that the center-of-mass fluctuation is proportional to $N^{-1} \varepsilon^{-1/4}$.

Finally, we investigate what happens to the exact ground state under the action of a quantum measurement 12. The ground state $|0\rangle_A$ is prepared as an initial state with the number of atoms $N_{\text{init}}$, so that $gN_{\text{init}} > 1$. Suppose that one atom is detected at position $\theta_i$ in the $j$-th measurement. The postmeasurement state $|\Psi(j)\rangle$ is related to the premeasurement one $|\Psi(j-1)\rangle$ by $|\Psi(j)\rangle = \hat{\psi}(\theta_j)|\Psi(j-1)\rangle/\langle \Psi(j-1) | \hat{\psi}(\theta_j) | \Psi(j-1) \rangle$. The normalized single-particle density is given by $n^{(j)}(\theta) = \langle \Psi(j) | \hat{\psi}(\theta) | \Psi(j) \rangle / N_j$, and the position of the $(j+1)$-th measurement $\theta_{j+1}$ is probabilistically determined after the $j$-th measurement according to the probability distribution $n^{(j)}(\theta)$. Since each run is a stochastic process, we numerically perform sequential runs of independent sim-

FIG. 2: Low-lying spectrum (thin lines, left scale) and expectation value of the total angular momentum (thick line, right scale) of the ground state versus the angular frequency of the rotating drive for $g = 2 \times 10^{-3}$ and $N = 100$. (b) is an enlargement of (a).

FIG. 3: The largest eigenvalue of the reduced single-particle density matrix obtained by the exact diagonalization ($\lambda^{(c)}_\text{M}$) and Bogoliubov theory ($\lambda^{(c)}_\text{B}$) versus $\varepsilon N^2$ for $gN \equiv 1.4$. Inset: Distribution of $|\beta_L|^2 = |A(L|\Psi(\theta))|^2$ with $\varepsilon = 1 \times 10^{-4}$. The solid curves depict Eq. 4 with Eq. 5.
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\[ \lambda_{\text{M}}^{(j)} = \frac{1}{2} \left[ 1 - c + ce^{-2\alpha} \right] \]

where \( \alpha = 1/(8cj) \). These results are in excellent agreement with the numerical ones as shown in Fig. 4.

In conclusion, we employed the exact diagonalization method to investigate the simultaneous emergence of a bright soliton and the Goldstone mode in a 1D attractive BEC. We found that the existence of a number of quasi-degenerate states above a critical strength of attractive interaction makes the ground state fragile

cations and take the ensemble average for each \( j \). We find that the ensemble-averaged value of the condensate fraction \( \overline{\lambda}_{\text{M}}^{(j)} \) is independent of \( N_{\text{init}} \) for a fixed \( gN_{\text{init}} \), and that it monotonically increases as shown in Fig. 4. Therefore, when \( N_{\text{init}} \gg 1 \), \( \overline{\lambda}_{\text{M}}^{(j)} \) reaches the order of one for \( j/N_{\text{init}} \ll 1 \), indicating that the fragmented condensate rapidly becomes a single condensate by the action of quantum measurements. The single-particle density \( n^{(j)}(\theta) \) localizes and the state \( |\Psi^{(j)}(\theta)\rangle \) reaches the form of Eq. (8).

Suppose that the initial state is a uniform superposition of the soliton state \( \int d\theta \delta(\theta - \theta_0) \phi_{\text{sol}}^{G}\Psi(\theta) \), where \( \theta_0 \) is the center of mass and \( A(\theta) \) is a constant for \( j = 0 \). We can show \[ \int d\theta \lambda_{\text{M}}^{(j)}(\theta, \theta_0) \]

as \( \Delta \theta_0 \propto j^{-1/2} \) by the \( j \)-time measurements. The condensate fraction can also be derived analytically \[ \lambda_{\text{M}}^{(j)}(\theta) \]

with \( c = 2/(4gN_{\text{init}} - 1)/(7gN_{\text{init}}) \), and hence the distribution of \( 2cj \). The center-of-mass fluctuation thus reduces to \( \Delta \theta_0 \propto j^{-1/2} \) by the \( j \)-time measurements.

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