THE CURVED SYMMETRIC 2– AND 3–CENTER PROBLEM ON CONSTANT NEGATIVE SURFACES

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Abstract. We study the motion of the negative curved symmetric two and three center problem on the Poincaré upper semi plane model for a surface of constant negative curvature $\kappa$, which without loss of generality we assume $\kappa = -1$. Using this model, we first derive the equations of motion for the 2-and 3-center problems. We prove that for 2–center problem, there exists a unique equilibrium point and we study the dynamics around it. For the motion restricted to the invariant $y$–axis, we prove that it is a center, but for the general two center problem it is unstable. For the 3–center problem, we show the non-existence of equilibrium points. We study two particular integrable cases, first when the motion of the free particle is restricted to the $y$–axis, and second when all particles are along the same geodesic. We classify the singularities of the problem and introduce a local and a global regularization of all them. We show some numerical simulations for each situation.

1. Introduction. The classical $N$-body problem has a long history. Isaac Newton proposed it in 1687 in the first edition of *Principia* in the context of the Moon's motion. He assumed that universal gravitation acts among celestial bodies (reduced to point masses) in direct proportion with the product of the masses and in inverse proportion with the square of the distance.

The idea of extending the gravitational force among point masses to spaces of constant curvature occurred soon after the discovery of the hyperbolic geometry. In the 1830’s, independently of each other, Bolyai and Lobachevsky realized that there must be an intimate connection between the laws of physics and the geometry of these spaces. The second author has been partially supported by Conacyt-México project A1-S-10112 and Asociación Mexicana de Cultura A.C.

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of the universe, [4, 25, 23]. A few years earlier, Gauss had interpreted Newton’s gravitational law as stating that the attracting force among bodies is inversely proportional with the area of the sphere of radius equal to the distance between the point masses (i.e. proportional to $1/r^2$, where $r$ is the distance). Using this idea, Bolyai and Lobachevsky suggested that, the space must be hyperbolic and that, the attracting force among bodies must be inversely proportional to the hyperbolic area of the corresponding hyperbolic sphere (i.e. proportional to $1/\sinh(|\kappa|^{1/2}r)$, where $r$ is the distance and $\kappa < 0$ the curvature of the hyperbolic space). This is equivalent to saying that, in hyperbolic space, the potential that describes the gravitational force is proportional to $\coth(|\kappa|^{1/2}r)$.

The above analytic expression of the potential was first introduced by Schering, [29, 30], and then extended to elliptic space by Killing, [19, 20, 21]. But with no ways of checking the validity of this generalization of the gravitational force, it was unclear whether the cotangent potential had any physical meaning, the more so since Lipschitz had proposed a different extension of the law, which turned out to be short-lived, [24].

The curved $N$-body problem became somewhat neglected after the birth of general relativity, but was revived after the discretization of Einstein’s equation showed that a $N$-body problem in spaces of variable curvature is too complicated to be treated with analytical tools. In the 1990s, the Russian school of celestial mechanics did a deep analysis of the curved Kepler and the curved 2-body problem, that contrary to the Newtonian case, they are not equivalent, see [22, 31] and the references therein for an historical background of the problem. After understanding that, unlike in the Euclidean case, these problems are not equivalent, the latter failing to be integrable, [31], the 2-body case was intensively studied by several researchers of this school. More recently, the work of Diacu, Santoprete, and Pérez-Chavela consider the curved $N$-body problem for $N > 2$ in a unified way, leading to many interesting results, [1, 2, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 28]. Other researchers developed these ideas further, [1, 18, 26, 27, 32, 33, 34, 35, 36], and the problem is growing in popularity.

The problem of two (three) fixed centers concerns to determining the motion of a particle of positive mass which is moving under to the Newton attraction of two (three) gravitational fixed masses. Euler studied the two center problem, and he showed the integrability of this problem (1760). The classification of the motion for the plane case was further studied by Charlier, Tallqvist and Badalyan (1907-1927), [35]. In this paper we described the problem of two and three fixed centers in the The Poincaré upper semiplane model of constant negative curvature $\kappa$, which without loss of generality we assume $\kappa = -1$, it corresponds to a model of the hyperbolic geometry. Along this paper by short, we will denote it by $\mathbb{H}_2$, where

$$\mathbb{H}_2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}, \quad (1.1)$$

endowed with the metric $ds^2 = (dx^2 + dy^2)/y^2$.

For the classical three center problem in celestial mechanics, in order to prove the existence of chaotic behaviour in this problem, the people usually take a perturbation of the integrable two center problem. In this direction, we analyze a couple of integrable particular cases of the three center problem defined on a surface of constant negative curvature, for this we have to regularize the singularities due to collision on the problem, first in a local sense, and second in a global sense. We
believe that in order to study the integrability or not of the problem, it will be useful to have both kind of regularization, we will tackle it in a forthcoming paper.

After the introduction, in Section 2, we state the equations of motion. In Sections 3 and 4 we analyze the two and three center problem, in the last case we introduce a new local regularization of the singularities for the two integrable cases that we analyze along this paper. Finally, in Section 5, we obtain a global regularization for the collision of the free particle with the three fixed ones.

2. Equations of motion. The \(N\)-body problem in the Poincaré upper semi plane model \(\mathbb{H}^2\) can be defined as the simple mechanical system on the configuration space \(Q = (\mathbb{H}^2)^N \setminus \Delta\), where \(\Delta\) is the set of collision configurations, and whose Lagrangian \(L : TQ \to \mathbb{R}\) is given by

\[
L = \frac{1}{2} \left( \sum_i m_i v_i^2 \right) - \sum_{i<j} m_i m_j U(d_{ij}).
\]

In the above equation \(m_i\) and \(v_i^2\) are, respectively, the mass and the square of the hyperbolic norm of the velocity of the \(i^{th}\) particle. The positive number \(d_{ij}\) is the hyperbolic distance between the \(i^{th}\) and the \(j^{th}\) particle and the potential \(U : \mathbb{R}^+ \to \mathbb{R}\) is given by

\[
U(d) = -\coth(d), \quad d > 0. \tag{2.1}
\]

This choice of potential is a generalization of the classical \(N\)-body problem in the following way: The Newtonian potential \(U_N(d) = \frac{1}{d}\) is (proportional to) the fundamental solution of the Laplacian operator on \(\mathbb{R}^3\), while the proposed potential \(U(d)\) is (proportional to) the fundamental solution of the Laplace-Beltrami operator on \(\mathbb{H}^2\). For more information see [6, 22].

We now proceed to write explicit formulae for the equations of motion for the \(N\)-body problem in the \(\mathbb{H}^2\) model. The equations are written in terms of the global Cartesian coordinates.

Using that the Riemannian distance \(d\), between two points \((x_1, y_1), (x_2, y_2) \in \mathbb{H}^2\) satisfies

\[
cosh(d) = 1 + \frac{(x_1 - x_2)^2 + (y_1 - y_2)^2}{2y_1 y_2}. \tag{2.2}
\]

We obtain that

\[
d_{ij} = \cosh^{-1}\left(1 + \frac{(x_i - x_j)^2 + (y_i - y_j)^2}{2y_i y_j}\right) \tag{2.3}
\]

now, using the relationship

\[
\coth(\cosh^{-1}(z)) = \frac{z}{\sqrt{z^2 - 1}}
\]

that is valid for all \(z > 1\) we can write

\[
L = \frac{1}{2} \left( \sum_i m_i \frac{x_i^2 + y_i^2}{y_i^2} \right) + \sum_{i<j} m_i m_j \frac{(x_i - x_j)^2 + y_i^2 + y_j^2}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2)((x_i - x_j)^2 + (y_i + y_j)^2)}. \tag{2.4}
\]
The Euler-Lagrange equations for the above Lagrangian can be written in the form:

\[
\ddot{x}_i = \frac{2\ddot{y}_i}{y_i} + \sum_{j \neq i} \frac{m_j 8(x_j - x_i) y_i^2 y_j^2}{((x_i - x_j)^2 + (y_i - y_j)^2)((x_i - x_j)^2 + (y_i + y_j)^2)^{3/2}}, \\
\ddot{y}_i = \frac{\ddot{x}_i^2 - \ddot{z}_i^2}{y_i} + \sum_{j \neq i} \frac{m_j 4y_i^4 y_j^2 ((x_i - x_j)^2 + y_j^2 - y_i^2)}{((x_i - x_j)^2 + (y_i - y_j)^2)((x_i - x_j)^2 + (y_i + y_j)^2)^{3/2}},
\]

(2.5)

where \(i\) runs from 1 to \(N\). The first term appearing on the right hand side of the above equations is the inertia of the point mass \(m_i\) manifesting its tendency to move along a geodesic of \(\mathbb{H}^2\). The second term is the force due to the gravitational interaction with the other masses.

3. **The symmetrical two center problem on \(\mathbb{H}^2\)**. In order to get acquainted with the three center problem we start our analysis with the simplest, but illustrative case of the two center problem.

We consider two point particles with the same positive mass \(m_1\) (called primaries), located symmetrically on the same geodesic with respect to the middle point between them. In [18], the authors prove that any geodesic as above can be mapped into the unit semicircle centered at the origin, where the middle point goes to the point \((0, 1)\) and the two fixed center are now located symmetrically with respect to the \(y\)-axis, on the unit circle. Then without loss of generality, we assume that the geodesic is the unitary half circle centered at the origin, so the particles are fixed at positions \(q_1 = (x_1, y_1)\) and \(q_2 = (-x_1, y_1)\) with \(|q_1| = |q_2| = 1, y_1 > 0\). We consider a particle of positive mass \(m\) with position \(q = (x, y)\) moving under the influence of the primaries, and we fix \(m_1 = 1\). According to (2.5), the equations of motion for the last particle are

\[
\ddot{x} = \frac{2\ddot{y}}{y} + \frac{8(x_1 - x)y_1^2}{2((x - x_1)^2 + (y - y_1)^2)((x - x_1)^2 + (y + y_1)^2)^{3/2}}, \\
\ddot{y} = \frac{\ddot{x}^2 - \ddot{z}^2}{y} + \frac{4y_1^2 ((x - x_1)^2 + y_1^2 - y^2)}{2((x - x_1)^2 + (y - y_1)^2)((x - x_1)^2 + (y + y_1)^2)^{3/2}},
\]

(3.1)

It is easy to verify that the Poincaré upper half plane model described above, can be written in complex variables as \(\mathbb{H}^2 = \{w \in \mathbb{C} | \text{Im}(w) > 0\}\) with the Riemannian metric

\[-ds^2 = \frac{4}{(w - \bar{w})^2} dwd\bar{w}.
\]

The particle \(q\) has coordinates \((x, y)\), without loss of generality we consider the points \(r_1\) and \(r_2\) at \((x_1, y_1)\) and \((-x_1, y_1)\) respectively, with \(|r_1| = |r_2| = 1\).

In this space, the cotangent potential takes the form

\[U(q, \bar{q}) = -\sum_{i=1}^{2} \frac{(\bar{q} - q)(\bar{r}_i - r_i) - 2|q|^2 + 1}{T_i},\]

(3.2)
Proof. In order to prove this claim, we will use polar coordinates. Let be $(x, y, \theta, r)$ and

$$Dynamics around the equilibrium point. First that all, we prove that the system defined in equation (3.1) has exactly one equilibrium point.

**Theorem 3.1.** The symmetrical two center problem on $H^2$ with centers at $(x_1, y_1)$ and $(-x_1, y_1)$, where $x_1^2 + y_1^2 = 1$ has exactly one equilibrium point given by $(x, y, \dot{x}, \dot{y}) = (0, 1, 0, 0)$.

**Proof.** In order to prove this claim, we will use polar coordinates. Let be $x = r(t) \cos \theta(t)$, $y = r(t) \sin \theta(t)$. The kinetic and potential energy become

$$T(r, \theta, \dot{r}, \dot{\theta}) = \frac{\dot{r}^2 + \dot{\theta}^2}{2r^2 \sin^2 \theta},$$

$$U(r, \theta) = \frac{1}{2} \sum_{i=1}^{2} \frac{(x_i - r \cos \theta)^2 + y_i^2 + r^2 \sin^2 \theta}{(x_i - r \cos \theta)^2 + (y_i - r \sin \theta)^2}.$$  

Values $x_i$ and $y_i$ stand for $\cos \theta_i$ and $\sin \theta_i$, respectively.

The Euler-Lagrange equations are

$$\frac{\dot{\theta}}{r \sin^2 \theta} = \frac{-\dot{r}^2}{r^3 \sin^2 \theta} + \frac{2 \dot{r} (r \cos \theta) \dot{\theta} + (\sin \theta) \dot{r}}{r^3 \sin^2 \theta} + U_r,$$

$$\frac{\dot{\theta}}{\sin^2 \theta} = \frac{- \cos \theta (\dot{\theta}^2 / r^2 + \dot{\theta}^2)}{r^2 \sin^2 \theta} + \frac{2 \theta \cos \theta}{\sin^2 \theta} + U_\theta,$$

with

$$U_r = -\sum_{i=1}^{2} \frac{4(\sin^2 \theta) y_i^2 (r - 1)}{[(r^2 - 2(\cos \theta) r x_i + 1)^2 - (2(\sin \theta) r y_i)^2]^{3/2}},$$

$$U_\theta = \sum_{i=1}^{2} \frac{4r^2 \sin \theta y_i^2 ((r^2 + 1) \cos \theta - 2r x_i)}{[(r^2 - 2(\cos \theta) r x_i + 1)^2 - (2(\sin \theta) r y_i)^2]^{3/2}}.$$  

Equilibrium points are those satisfying $\dot{\theta} = \dot{r} = 0$, with this we have

$$U_r = 0, \quad and \quad U_\theta = 0.$$  

We observe that $U_r = 0$ holds if and only if $r = 1$, which means that the equilibrium points (in case they exist), must lie on the unit circle. Since $y_1 = y_2$, $x_1 = -x_2$ and
using \( r = 1 \), we obtain that \( U_\theta = 0 \) can be written as
\[
4y^2 \sin \theta \left( \frac{2 \cos \theta - 2x_1}{[(1 - 2x_1 \cos \theta + 1)^2 - (2y_1 \sin \theta)^2]^{3/2}} + \frac{2 \cos \theta + 2x_1}{[(1 + 2x_1 \cos \theta + 1)^2 - (2y_1 \sin \theta)^2]^{3/2}} \right) = 0.
\]
To simplify the above expression, consider the change of coordinate \( z = \cos \theta \) with \( \theta \in (-\pi/2,\pi/2) \) in order to stain in \( \mathbb{H}^2 \). Hence we have
\[
\sqrt{1 - z^2} y_1 \left( \frac{z - x_1}{[(1 - z x_1)^2 - (1 - z^2)y_1^2]^{3/2}} + \frac{z + x_1}{[(1 + z x_1)^2 - (1 - z^2)y_1^2]^{3/2}} \right) = 0,
\]
using that \( x_1^2 + y_1^2 = 1 \), after some straightforward computations we obtain
\[
\sqrt{1 - z^2} y_1 \left( \frac{-(z + x_1)^2 \text{sign}(x_1 - z) + (x_1 - z)^2}{(x_1^2 - z^2)^2} \right) = 0. \tag{3.4}
\]
If \( z = \pm 1 \) then \( \theta = \pm \pi/2 \) which is not allowed. Notice that if \( x_1 - z < 0 \), then above expression is not satisfied. Therefore we should have \( x_1 - z > 0 \). Using this inequality we have that Equation (3.4) holds if and only if
\[
-(z + x_1)^2 + (x_1 - z)^2 = -4zx_1 = 0,
\]
and this holds only for \( z = 0 \), or \( \cos \theta = 0 \). The point \( r = 1 \), \( \cos \theta = 0 \) corresponds to the point \( x = 0, y = 1 \). This ends the proof of Theorem 3.1.

Now we observe that the geodesic corresponding to the \( y \)-axis is invariant if \( \dot{x}(0) = 0 \). In the following we analyze the motion on this invariant set.

Using that \( x(t) \equiv 0 \), the equation corresponding to \( y \) in (3.1) takes the form
\[
\dot{y} = \frac{\dot{y}^2}{y} + \frac{8y^3y_1^2(1 - y^2)}{[(x_1^2 + (y - y_1)^2)(x_1^2 + (y + y_1)^2)]^{3/2}}. \tag{3.5}
\]
We have the following result.

**Proposition 1.** The equilibrium point \((x(t), y(t)) = (0, 1)\) restricted to the geodesic identify with the \( y \)-axis is a center, that is all orbits in a neighborhood of this point are periodic.

**Proof.** The system given by equation (3.5) can be written as
\[
\dot{y} = \nu, \quad \dot{\nu} = \frac{\nu^2}{y} + \frac{8y^3y_1^2(1 - y^2)}{[(x_1^2 + (y - y_1)^2)(x_1^2 + (y + y_1)^2)]^{3/2}}. \tag{3.6}
\]
The slope of the vector field is given by \( g(y, \nu) = \dot{\nu} / \nu \). Since \( g(y, -\nu) = -g(y, \nu) \), the flow is symmetric with respect to the \( y \)-axis.

The direction of the solution curves of system (3.6) is horizontal (the slope of the flow is zero) along the curves
\[
\nu = \pm \sqrt{-\frac{8y^3y_1^2(1 - y^2)}{[(x_1^2 + (y - y_1)^2)(x_1^2 + (y + y_1)^2)]^{3/2}}} =: h(y).
\]
We observe that the above function is not defined for \( y < 1 \), which means that for \( y < 1 \), the slope of the solution curves are never zero. The behavior of the slopes given by the function \( g \) can be seen by checking that (see Figure 1)
\[
\lim_{\nu \to 0} g(y, \nu) = \infty \quad \text{if} \quad y < 1, \quad \text{or} \quad -\infty \quad \text{if} \quad y > 1.
\]
If we focus on \( \nu > 0 \), the function \( g(y, \nu) \) is positive for any \( y < h(y) \) and negative for any \( y > h(y) \), from this observation and the symmetries of the flow we get the result.

3.2. Behavior of the global flow near \( (0, 1, 0, 0) \). Remembering that \((x_1, y_1)\) is in the first quadrant, we can take \((x_1, y_1) = (\cos \theta, \sin \theta)\) for some \( 0 < \theta < \pi/2 \), and splitting the two second-order differential equations of equation (3.1) into 4 first-order differential equations, we get

\[
\begin{align*}
\dot{x} &= u, \\
\dot{y} &= v, \\
\dot{u} &= \frac{2uv}{y} + \frac{E}{(AB)^{3/2}} - \frac{F}{(CD)^{3/2}}, \\
\dot{v} &= \frac{v^2 - u^2}{y} + \frac{G}{(AB)^{3/2}} + \frac{H}{(CD)^{3/2}},
\end{align*}
\]

(3.7)

where

\[
\begin{align*}
A &= (x - \cos \theta)^2 + (y - \sin \theta)^2, \\
B &= (x - \cos \theta)^2 + (y + \sin \theta)^2, \\
C &= (x + \cos \theta)^2 + (y - \sin \theta)^2, \\
D &= (x + \cos \theta)^2 + (y + \sin \theta)^2, \\
E &= 8(\cos \theta - x)y^4 \sin^2 \theta, \\
F &= 8(\cos \theta + x)y^4 \sin^2 \theta, \\
G &= 4y^3 \sin^2 \theta [(x - \cos \theta)^2 + \sin^2 \theta - y^2], \\
H &= 4y^3 \sin^2 \theta [(x + \cos \theta)^2 + \sin^2 \theta - y^2].
\end{align*}
\]

By straightforward computations, we obtain that the linear part of the flow given by (3.1) around the equilibrium point \((0, 1, 0, 0)\) is given by
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Since this equation has one positive root, the equilibrium point $(0, 1, 0, 0)$ is unstable. We have thus proved the following result.

**Proposition 2.** The equilibrium point $(0, 1, 0, 0)$ of the general 2–center problem is unstable.

In [35], the author proves that the two center problem in spaces of constant curvature, positive or negative, is integrable.

4. The symmetrical three center problem on $\mathbb{H}^2$. We consider three point particles on the same geodesic $l$, two of them with positive mass $m$, located symmetrically with respect to the third point with positive mass $\mu$; remembering that on $\mathbb{H}^2$ there are symmetries which are analogous to the usual rotations on the plane, using this fact and an similar argument for the tangent vector at the position of the particle with mass $\mu$, we can prove the existence of one isometry which maps $l$ onto the upper half of the unit circle, and the position of the mass $\mu$ onto the point $(0, 1)$ (in [18] you can find the explicit formula of this isometry). Then, without loss of generality we assume that the geodesic $l$ is the unitary half circle centered at the origin, so the three particles are fixed at positions $q_1 = (x_1, y_1)$, $q_2 = (-x_1, y_1)$ and $q_3 = (0, 1)$ with $|q_1| = |q_2| = |q_3| = 1$. Now we consider a fourth particle with position $q = (x, y), y > 0$ of positive mass $M$, which after normalization we can assume, without loss of generality, $M = 1$, this free particle moves under the influence of the three fixed masses. The phase space is $T^*(\mathbb{H}^2 \setminus \Delta)$ where $\Delta = \{(\pm x_1, y_1), (0, 1)\}$ is the collision set. We see that the potential is an analytic function in $\mathbb{H}^2 \setminus \Delta$.

According to (2.5), we express the equations of motion for the last particle in the following form

\[
\ddot{x} = \frac{2\dot{y}}{y} + \frac{8m(x_1 - x)y_1^2}{[((x - x_1)^2 + (y - y_1)^2)((x - x_1)^2 + (y + y_1)^2)]^{3/2}} - \frac{8\mu xy^4}{[x^2 + (y - 1)^2(x^2 + (y + 1)^2)]^{3/2}},
\]

\[
\ddot{y} = \frac{2\dot{x}}{y} - \frac{4my^3}{y_1^2}((x - x_1)^2 + y_1^2 - y^2)
+ \frac{4my^3}{y_1^2}((x + x_1)^2 + y_1^2 - y^2)
+ \frac{4\mu y^3}{y_1^2}((x - x_1)^2 + (y - y_1)^2)((x - x_1)^2 + (y + y_1)^2)]^{3/2}.
\]
Consider the three-center problem on \( \mathbb{H}^2 \), where the three fixed particles are located at \( q_1 = (x_1, y_1), q_2 = (-x_1, y_1) \) and \( q_3 = (0, 1) \), with \( x_1^2 + y_1^2 = 1 \); and masses \( M, m_1, m_2 \) and \( \mu \), respectively, with \( M = 1, m_2 = m1 = m \). This problem has no equilibrium points.

**Theorem 4.1.** Consider the three-center problem on \( \mathbb{H}^2 \), where the three fixed particles are located at \( q_1 = (x_1, y_1), q_2 = (-x_1, y_1) \) and \( q_3 = (0, 1) \), with \( x_1^2 + y_1^2 = 1 \); and masses \( M, m_1, m_2 \) and \( \mu \), respectively, with \( M = 1, m_2 = m1 = m \). This problem has no equilibrium points.

**Proof.** Consider, as in the proof of Theorem 3.1, polar coordinates. By symmetry it is enough to do the analysis with \( x \geq 0 \).

The kinetic and potential energy are

\[
T(r, \theta, \dot{r}, \dot{\theta}) = \frac{\dot{r}^2 + \dot{\theta}^2}{2r^2 \sin^2 \theta},
\]

\[
U(r, \theta) = \sum_{i=1}^{3} m_i \frac{(x_i - r \cos \theta)^2 + y_i^2 + r^2 \sin^2 \theta}{((x_i - r \cos \theta)^2 + (y_i - r \sin \theta)^2)^{1/2} ((x_i - r \cos \theta)^2 + (y_i + r \sin \theta)^2)^{1/2}}.
\]

The Euler-Lagrange equations are

\[
\frac{\dot{r}}{r^2 \sin^2 \theta} = \frac{-\dot{r}^2}{r^3 \sin^2 \theta} + \frac{2\dot{r} (r \cos \theta) \dot{\theta} + (\sin \theta) \ddot{r}}{r^3 \sin^3 \theta} + U_r,
\]

\[
\frac{\dot{\theta}}{\sin^2 \theta} = \frac{-\cos \theta (\ddot{r} r^2 + \dot{r}^2)}{r^2 \sin^3 \theta} + \frac{2\dot{\theta} \cos \theta}{\sin^3 \theta} + U_\theta,
\]

with

\[
U_r = -\sum_{i=1}^{3} m_i \frac{4(\sin^2 \theta)x_i^2 r (r - 1)}{[(r^2 - 2rx_i \cos \theta + 1)^2 - (2ry_i \sin \theta)^2]^{3/2}},
\]

\[
U_\theta = \sum_{i=1}^{3} m_i \frac{4r^2 \sin \theta y_i^2 ((r^2 + 1) \cos \theta - 2rx_i)}{[(r^2 - 2rx_i \cos \theta + 1)^2 - (2ry_i \sin \theta)^2]^{3/2}}
\]

\[
= \sum_{i=1}^{2} m_i \frac{4r^2 \sin \theta y_i^2 ((r^2 + 1) \cos \theta - 2rx_i)}{[(r^2 - 2rx_i \cos \theta + 1)^2 - (2ry_i \sin \theta)^2]^{3/2}} + \mu \frac{4(r^2 + 1)r^2 \cos \theta \sin \theta}{(4r^2 \cos^2 \theta + (r^2 - 1)^2)^{3/2}}.
\]

Equilibrium points are those satisfying \( \dot{\theta} = \dot{r} = 0 \), with this we have

\[
U_r = 0, \quad \text{and} \quad U_\theta = 0.
\]

First equation of above system holds if and only if \( r = 1 \), which means that if there are equilibrium points, then they should lie on the unit circle. Since \( y_1 = y_2 \),
\( x_1 = -x_2 \) and using \( r = 1 \), then \( U_\theta = 0 \) can be written as

\[
4m \sin \theta y_1 \left( \frac{2 \cos \theta - 2x_1}{(1 - 2 \cos \theta x_1 + 1)^2 - (2 \sin \theta y_1)^2} \right)^{3/2} + \frac{2 \cos \theta + 2x_1}{(1 + 2 \cos \theta x_1 + 1)^2 - (2 \sin \theta y_1)^2} + \mu \frac{\sin \theta}{\cos^2 \theta} = 0.
\]

(4.4)

To simplify the above expression, consider the change of coordinate \( z = \cos \theta \).

Hence, we have

\[
\sqrt{z^2 - 1} \left[ -z^2 m y_1 (z + x_1)^2 \text{sign}(x - z) + (z^2 m y_1 + \mu (z + x_1)^2)(x_1 - z)^2 \right] = 0.
\]

(4.5)

Which is zero if and only if

\[-z^2 m y_1 (z + x_1)^2 \text{sign}(x - z) + (z^2 m y_1 + \mu (z + x_1)^2)(x_1 - z)^2 = 0.\]

Notice that last equation is never satisfied if \( x - z < 0 \), hence we should have \( x - z > 0 \). Considering this inequality we need to look for solutions of

\[-z^2 m y_1 (z + x_1)^2 + (z^2 m y_1 + \mu (z + x_1)^2)(x_1 - z)^2 = 0,\]

rearranging the last equations we get,

\[
\mu z^4 - 4my_1x_1 z^3 - 2\mu x_1^2 z^2 + \mu x_1^4 = 0.
\]

(4.6)

This quartic polynomial in \( z \) has discriminant \( p(\mu, m, y_1, x_1) \) given by \( p(\mu, m, y_1, x_1) = -256M^2 x_1^2 m^2 y_1^2 (27m^2 y_1^2 + 16\mu^2) < 0. \)

Since the discriminant is negative, then polynomial (4.6) has only complex roots. It implies that do not exist any \( \theta \) that satisfies (4.4), and therefore no equilibrium points exist for the problem. This ends the proof of Theorem 4.1. 

\( \square \)
4.1. The vertical case. We restrict to the free particle with mass \( M = 1 \) to move on the \( y \)-axis. So let \( q_1 = (0, y_1), \ y > 0 \), then (4.1) becomes

\[
\ddot{y} = \frac{\dot{y}^2}{y} + \frac{8my^2(1 - y^2)}{[(x_1^2 + (y - y_1)^2)(x_1^2 + (y + y_1)^2)]^{3/2}} + \frac{4\mu y^3}{(y^2 - 1)^2}. \tag{4.7}
\]

On this invariant axis the equation of motion becomes singular at the collision \( q_3 = (0, 1) \). So, the motion of the free particle \( M \) will form a binary collision with the fixed particle \( \mu \).

**Theorem 4.2.** Consider the symmetrical three-center problem where the free particle with coordinates \( q = (0, y) \) and mass \( M \) moves under the influence of the primaries at positions \( q_1 = (x_1, y_1) \) \((|q_1| = 1)\), \( q_2 = (-x_2, y_2) \), \( q_3 = (0, 1) \) with masses \( m_1 \), \( m_2 \) and \( \mu \) respectively. Then, the binary collision of \( q \) and \( q_3 \) can be regularized via the coordinate transformation \( y = 1 + u^2 \), and a time transformation \( t = g(\tau) \) such that \( \frac{dt}{d\tau} = 4u^2 \).

**Proof.** Consider the Hamiltonian function \( H \) of the problem (remember that we have fixed \( M = 1 \))

\[
H(y, \dot{y}) = \frac{\dot{y}^2}{2y} - U,
\]

with

\[
U = m \left( \frac{x_1^2 + y^2 + y_1^2}{\sqrt{x_1^2 + (y - y_1)^2}} \frac{x_1^2 + (1 + u^2)^2 + y_1^2}{\sqrt{x_1^2 + (1 + u^2 - y_1)^2}} + m \frac{x_2^2 + y^2 + y_2^2}{\sqrt{x_2^2 + (y - y_2)^2}} \frac{x_2^2 + (1 + u^2)^2 + y_2^2}{\sqrt{x_2^2 + (1 + u^2 + y_2)^2}} + \mu \frac{y^2 + 1}{|y^2 - 1|} \right). \tag{4.8}
\]

Now let us take the change of coordinates \( y = f(u) = 1 + u^2 \), which implies \( \dot{y} = 2u\dot{u} \).

The Hamiltonian takes the form

\[
H(u, \dot{u}) = \frac{4u^2\dot{u}^2}{2(1 + u^2)} - m \left( \frac{x_1^2 + (1 + u^2)^2 + y_1^2}{\sqrt{x_1^2 + (1 + u^2 - y_1)^2}} \frac{x_2^2 + (1 + u^2)^2 + y_2^2}{\sqrt{x_2^2 + (1 + u^2 + y_2)^2}} - \mu \frac{(1 + u^2)^2 + 1}{u^2(2 + u^2)} \right). \tag{4.9}
\]

Consider the new time parametrization \( \frac{dt}{d\tau} = f'(u)^2 \) and the new Hamiltonian \( \dot{H} \) given by \( \dot{H} = f'(u)^2(H + \frac{C}{2}) \). It is well-known that the flow associated to \( H \) at level \( H = -\frac{C}{2} \) is the same to the flow generated by \( \dot{H} \) at level zero.

In this way, the Hamiltonian \( \dot{H} \) is

\[
\dot{H}(u, \dot{u}) = \frac{u^2}{2(1 + u^2)} - 4mu^2 \left( \frac{x_1^2 + (1 + u^2)^2 + y_1^2}{\sqrt{x_1^2 + (1 + u^2 - y_1)^2}} \frac{x_2^2 + (1 + u^2)^2 + y_2^2}{\sqrt{x_2^2 + (1 + u^2 + y_2)^2}} - \mu \frac{(1 + u^2)^2 + 1}{2 + u^2} \right) + 2u^2C, \tag{4.10}
\]

where (‘) stands for the derivative respect to \( \tau \).

The collision between \( M \) and \( \mu \) occurs at \( u = 0 \), and we can see in the above equation that the singularity given by such collision has been removed. The regularized
Hamiltonian in terms of $u$ and its generalized momentum $p_u$ is
\[
\tilde{H}(u, p_u) = \frac{(1 + u^2)p_u^2}{2} - 4mu^2 \frac{x_1^2 + (1 + u^2)^2 + y_1^2}{\sqrt{x_1^2 + (1 + u^2 - y_1)^2}} \frac{\sqrt{x_1^2 + (1 + u^2 + y_1)^2}}{x_2^2 + (1 + u^2)^2 + y_2^2} - 4mu^2 \frac{(1 + u^2)^2 + 1}{(2 + u^2)} + 2u^2C,
\]
with $p_u = \frac{u'}{1 + u^2}$.

**Numerical example.** Let us consider the masses $m_1 = m_2 = \mu = M = 1$ and positions $x_1 = 1/2 = -x_2$ with energy constant $C = 0$. The equations of motion with these values in non-regularized coordinates are:
\[
\begin{align*}
\dot{y} &= p_y y, \\
\dot{p}_y &= -1/2 py^2 + 16 \frac{y}{\sqrt{1 + 4 (y - 1/2 \sqrt{3})^2}} \frac{1}{\sqrt{1 + 4 (y + 1/2 \sqrt{3})^2}} \\
&\quad - 4 \frac{(y^2 + 1) (8y + 4 \sqrt{3})}{(y^2 + 1) (8y + 4 \sqrt{3})} \frac{1}{\sqrt{1 + 4 (y + 1/2 \sqrt{3})^2}} + 2 \frac{y}{\sqrt{(y - 1)^2 (y + 1)^2}} + 1/2 \frac{(2y - 2)}{(y - 1)^{3/2}} - \frac{y^2 + 1}{(y + 1)^{3/2}}. 
\end{align*}
\]

While the equations of motion in regularized coordinates are (in the new time $\tau$):
\[
\begin{align*}
P'_u &= -up_u^2 + 32 \frac{(u^2 + 1)^2 + 1}{\sqrt{1 + 4 (u^2 - 1/2 \sqrt{3} + 1)^2}} \frac{u^2 (u^2 + 1)}{\sqrt{1 + 4 (u^2 + 1/2 \sqrt{3} + 1)^2}} \\
&\quad + 128 \frac{u^2}{\sqrt{1 + 4 (u^2 - 1/2 \sqrt{3} + 1)^2}} \frac{u^2}{\sqrt{1 + 4 (u^2 + 1/2 \sqrt{3} + 1)^2}} \\
&\quad - 256 \frac{u^2 (u^2 + 1)^2 + 1}{\sqrt{1 + 4 (u^2 - 1/2 \sqrt{3} + 1)^2}} \frac{u^2 + 1/2 \sqrt{3} + 1}{(2 + u^2)} \frac{u^2 + 1/2 \sqrt{3} + 1}{(1 + 4 (u^2 + 1/2 \sqrt{3} + 1)^2)^{3/2}} \\
&\quad - 256 \frac{u^2 (u^2 + 1)^2 + 1}{(u^2 + 1)^2} \frac{u^2 + 1/2 \sqrt{3} + 1}{(1 + 4 (u^2 + 1/2 \sqrt{3} + 1)^2)^{3/2}} \\
&\quad + 16 \frac{(u^2 + 1) u}{u^2 + 2} - 8 \frac{(u^2 + 1)^2 + 1}{(u^2 + 2)^2}.
\end{align*}
\]
In order to see the behavior of the particle in the regularized and the non-regularized space, we have plotted both phase portraits. Let us consider an example with the following initial conditions: \( y(0) = 2, \ p_y(0) = -0.25 \) \((\dot{y} = -0.5)\), these conditions lead to \( u(0) = 1, \ p_u(0) = -1, \) see Figures (3a) and (3b).

4.2. **The geodesic case.** This is a particular “collinear” four-body problem with three fixed masses \( m_1 = m, \ m_2 = m, \ m_3 = \mu, \) and a free fourth body with mass \( m_4 = M = 1, \) all of them moving on the same geodesic. Let be

\[
q_1 = (\cos \theta_1, \sin \theta_1), \quad q_2 = (-\cos \theta_1, \sin \theta_1), \quad q_3 = (0, 1), \quad q_4 = (\cos \theta(t), \sin \theta(t),
\]

where \( \theta_1 \in (0, \pi/2) \) is a fixed angle; \( \theta(t) \) is an angle in \((0, \theta_1) \cup (\theta_1, \pi/2)\), the case \( \theta(t) \in (\pi/2, \pi) \) is getting by symmetry. We observe that the values \( \theta(t) = \theta_1 \) or \( \theta(t) = \pi/2 \) correspond to singularities due to binary collision (see Figure 4).
According to (2.5), we express the equations of motion for the last particle in the following form
\[ \ddot{x} = -2 \sin \theta + \frac{8m(\cos \theta_1 - \cos \theta) \sin^4 \theta \sin^2 \theta_1}{[(2 - 2(\cos \theta_1 \cos \theta + \sin \theta_1 \sin \theta))(2 - 2(\cos \theta_1 \cos \theta - \sin \theta_1 \sin \theta))]^{3/2}} \]
\[ + \frac{8m(- \cos \theta_1 - \cos \theta) \sin^4 \theta \sin^2 \theta_1}{[(2 + 2(\cos \theta_1 \cos \theta + \sin \theta_1 \sin \theta))(2 - 2(\sin \theta_1 \sin \theta - \cos \theta_1 \cos \theta))]^{3/2}} \]
\[ - \frac{8\mu \cos \theta \sin^4 \theta}{[(2 - 2 \sin \theta)(2 + 2 \sin \theta)]^{3/2}}. \]
\[ \ddot{y} = \frac{1}{\sin \theta} \]
\[ + \frac{4m \sin^3 \theta \sin^2 \theta_1((\cos \theta - \cos \theta_1)^2 + \sin^2 \theta_1 - \sin^2 \theta)}{[(2 - 2(\cos \theta_1 \cos \theta + \sin \theta_1 \sin \theta))(2 - 2(\cos \theta_1 \cos \theta - \sin \theta_1 \sin \theta))]^{3/2}} \]
\[ + \frac{4m \sin^3 \theta \sin^2 \theta_1((\cos \theta + \cos \theta_1)^2 + \sin^2 \theta_1 - \sin^2 \theta)}{[(2 + 2(\cos \theta_1 \cos \theta + \sin \theta_1 \sin \theta))(2 + 2(\cos \theta_1 \cos \theta + \sin \theta_1 \sin \theta))]^{3/2}} \]
\[ + \frac{8\mu \sin^3 \theta \cos^2 \theta}{[(2 - 2 \sin \theta)(2 + 2 \sin \theta)]^{3/2}}. \]

Notice that the common denominator in the equations of motion
\[ [(2 - 2(\cos \theta_1 \cos \theta + \sin \theta_1 \sin \theta))(2 - 2(\cos \theta_1 \cos \theta - \sin \theta_1 \sin \theta))]^{3/2}, \]
vanishes when \( \theta = \theta_1 \). This means the free particle \( M \) and \( m \) form a collision. Further, the denominator in the equation of motion
\[ [(2 - 2 \sin^2 \theta)(2 + 2 \sin^2 \theta)]^{3/2}, \]
vanishes when \( \theta = \pi/2 \). This means that the free particle \( M \) and \( \mu \) form a collision.

4.3. Binary collision singularities. A binary collision is caused when the distance between two masses becomes zero in finite time. During the collision, the potential function is infinite and consequently, the velocity becomes infinite. Thus, the potential function \( U(q,q) \) is singular on \( \Delta \).

4.3.1. Classifications of the binary collisions. In the geodesic three center problem, we consider two particular cases:

- **The first case.** The fourth particle moves on the same geodesic between \( m \) and \( \mu \). Note that in this case we have two binary collisions which happen when \( q_4 = q_1 \) or \( q_4 = q_3 \).
- **The second case.** The fourth particle moves on the same geodesic in which all the three fixed primaries lie. Let \( q_4 = (x_4, y_4), y_4 < y_1, x_4 > x_1 \) with \( |q_4| = 1 \). Here we will have the 4 collinear problem. Note that the binary collision happens when \( q_4 = q_1 \).

Consider the case where the particle \( q \) with mass \( M \) lies between \( q_3 \) and \( q_1 \) with masses \( \mu \) and \( m_1 = m \). It is possible to regularize both collisions using one coordinate transformation (global regularization). For this case it is worth to consider first the analogous problem, which results by rotating (anti-clockwise) the particles \( \pi/2 \) around the hyperbolic circle with center at \( m \).
The angle of rotation is the angle between one geodesic and its image under $T$. With the above, we explicitly gave the isometry which rotates the geodesic $z = i$ fixed, hence in this case Moebius transformations satisfy $T(i) = i$, or

$$i = \frac{ai + b}{ci + d},$$

which implies $a = d$, $b = -c$ and $a^2 + b^2 = 1$. From this fact, we can see that $T$ has the form

$$T(z) = \frac{\cos \theta z + \sin \theta}{-\sin \theta z + \cos \theta}.$$

The angle of rotation is the angle between one geodesic and its image under $T$. If we consider an infinitesimal change in the initial position $z$, then the derivative of $T$ is the corresponding change of $z$ under $T$.

In this way, the change in rotation can be observed from the derivative at the center of rotation

$$\frac{dT}{dz}(i) = \frac{-1}{(\cos \theta - i \sin \theta)^2} = e^{2\theta}.$$

So, $T$ is a hyperbolic rotation by an angle $\phi = 2\theta$.

On the other hand, let $z = x + iy$ be any point on the half circle, $|z|^2 = 1$, $z \in \mathbb{H}^2$. Since we want a rotation that maps $x + iy$ into the imaginary axis, then we should have $Re(T(x + iy)) = 0$, that is

$$\frac{-2 \cos^2 \theta x + x}{2 \cos \theta \sin \theta x} = 0,$$

which implies $\cos^2 \theta = \frac{1}{2}$, or in other words $\theta = \frac{(2k + 1)\pi}{4}$, $k \in \{0, 1, 2, \ldots\}$.

If we take the rotation corresponding to $k = 0$, i.e., $\theta = \frac{\pi}{4}$ then we have that the rotation is $\phi = 2\theta = \frac{\pi}{2}$.

Summarizing, by means of isometries we can always map one isometry to any other. With the above, we explicitly gave the isometry which rotates the geodesic “unit circle” onto the geodesic “vertical line” through the point $(0, 1)$.

After applying this transformation to the centers, then they will have coordinates $q_1 = (0, \tilde{y}_1)$, $q_2 = (0, \tilde{y}_2)$, $q_3 = (0, 1)$ and we can consider that the particle $q$ has coordinates $q = (0, y)$ with $y \in (1, \tilde{y}_1)$, see Figure 5.

The Hamiltonian takes the form

$$H(y, \dot{y}) = \frac{\dot{y}^2}{2y} - U(y),$$

with

$$U = m \frac{y^2 + \tilde{y}_1^2}{|y - \tilde{y}_1|(y + \tilde{y}_1)} + m \frac{y^2 + \tilde{y}_2^2}{|y - \tilde{y}_2|(y + \tilde{y}_2)} + \mu \frac{y^2 + 1}{|y^2 - 1|}.$$
Theorem 4.3. In the geodesic case of the three-center problem with primary masses described as above, the collision of $M$ with $\mu$, and the collision of $M$ with $m_1$ can both be regularized via the coordinate transformation

$$y = f(u) = \left(\frac{y_1 - 1}{2}(1 - \cos u) + 1\right),$$

and a time transformation $t = g(\tau)$ such that $\frac{dt}{d\tau} = \left(\frac{y_1 - 1}{2}\right)^2 \sin^2 u$.

Proof. Consider the Hamiltonian (4.14), and the transformation

$$y = f(u) = \left(\frac{y_1 - 1}{2}(1 - \cos u) + 1\right).$$

Now let us deal with the factors $y - \bar{y}_1$ and $y - 1$. We have

$$y - \bar{y}_1 = -\frac{y_1 - 1}{2}(1 + \cos u), \quad \text{and} \quad y - 1 = \frac{y_1 - 1}{2}(1 - \cos u).$$

From the above expressions we can easily see that

$$|y - \bar{y}_1||y - 1| = \left(\frac{y_1 - 1}{2}\right)^2 (1 - \cos^2 u) = \left(\frac{y_1 - 1}{2}\right)^2 \sin^2 u = f'(u)^2.$$

Consider the new time parametrization $\tau$ with $\frac{dt}{d\tau} = f''(u)^2$, and the new Hamiltonian $\tilde{H}$ given by $\tilde{H} = f''(u)^2(H + \frac{\mu}{2})$. It is well-known that the flow associated to $H$ at level $H = -\frac{\mu}{2}$ is the same to the flow generated by $\tilde{H}$ at level zero.

Notice that $\dot{y} = \frac{y_1 - 1}{2} \sin u \dot{u}$ and $u' = \dot{u}(\frac{y_1 - 1}{2} \sin u)^2$, where $(\cdot)'$ is for the derivative respect $u$.

In this way, the Hamiltonian $\tilde{H}$ is

$$\tilde{H}(u, u') = \frac{u'^2}{2 \left(\frac{y_1 - 1}{2}(1 - \cos u) + 1\right)^2} - \left( m \frac{y^2 + \bar{y}_1^2}{|y - \bar{y}_1|(y + \bar{y}_1)} + m \frac{y^2 + \bar{y}_2^2}{|y - \bar{y}_2|(y + \bar{y}_2)} \right) + \mu \frac{y^2 + 1}{|y^2 - 1|} \left(\frac{y_1 - 1}{2}\right)^2 \sin^2 u.$$
The collision corresponding to $y = 1$ becomes in the new coordinates $u = \pm 2n\pi$, $n \in 0, 1, 2, \cdots$; and the collision corresponding to $y = \bar{y}_1$ becomes $u = \pm (2n - 1)\pi$, $n \in 1, 2, 3, \cdots$. We can see in the above expression that the singularities that come from those collisions have been removed.

The Hamiltonian in terms of the generalized momentum is

$$H(u, p_u) = \frac{u'^2}{2\left(\frac{\bar{y}_1 - 1}{2} (1 - \cos u) + 1\right)^2} - m \frac{\bar{y}_1 - 1}{2} (1 - \cos u) + 1 + \bar{y}_1$$

$$- m \left( \left(\frac{\bar{y}_1 - 1}{2} (1 - \cos u) + 1\right)^2 + \bar{y}_1^2 \right) \left(\frac{\bar{y}_1 - 1}{2} (1 - \cos u) + 1 + \bar{y}_2\right)$$

$$- \mu \frac{\bar{y}_1 - 1}{2} (1 - \cos u) + 1 + \bar{y}_1$$

$$- \frac{C}{2} \left(\frac{\bar{y}_1 - 1}{2}\right)^2 \sin^2 u. \quad (4.16)$$

The collision corresponding to $y = 1$ becomes in the new coordinates $u = \pm 2n\pi$, $n \in 0, 1, 2, \cdots$; and the collision corresponding to $y = \bar{y}_1$ becomes $u = \pm (2n - 1)\pi$, $n \in 1, 2, 3, \cdots$. We can see in the above expression that the singularities that come from those collisions have been removed.

The Hamiltonian in terms of the generalized momentum is

$$H(u, p_u) = \frac{p_u^2 \left(\frac{\bar{y}_1 - 1}{2} (1 - \cos u) + 1\right)}{2} - m \left(\frac{\bar{y}_1 - 1}{2} (1 - \cos u) + 1\right)^2 + \bar{y}_1^2$$

$$- m \left(\frac{\bar{y}_1 - 1}{2} (1 - \cos u) + 1\right)^2 + \bar{y}_2^2 \left(\frac{\bar{y}_1 - 1}{2} (1 - \cos u) + 1 + \bar{y}_2\right)$$

$$- \mu \left(\frac{\bar{y}_1 - 1}{2} (1 - \cos u) + 1\right)^2 + \bar{y}_1$$

$$- \frac{C}{2} \left(\frac{\bar{y}_1 - 1}{2}\right)^2 \sin^2 u. \quad (4.17)$$

with $p_u = \frac{u'}{\frac{\bar{y}_1 - 1}{2} (1 - \cos u) + 1}$.

**Numerical example.**

Let us consider the masses $m_1 = m_2 = \mu = M = 1$ and positions $y_1 = 3/2$, $y_2 = 1/2$ with energy constant $C = 0$, and take the example with the following initial conditions: $y(0) = 5/4$, $p_y(0) = 2/25$, $(\dot{y}(0) = 0.1)$, these condition lead to
Theorem 4.4. In the geodesic three-center problem with primary masses at \( q_1 = (0, \bar{y}_1) \), \( q_2 = (0, \bar{y}_2) \), \( q_3 = (0, 1) \) and \( q = (0, y) \) with \( y > \bar{y}_1 \) and masses \( m_1 \), \( m_2 \), \( \mu \) and \( M \), respectively. The binary collision of \( M \) and \( m_1 \) can be regularized via the coordinate transformation \( y = \bar{y}_1 + u^2 \), and a time transformation \( t = g(\tau) \) such that \( \frac{dt}{d\tau} = 4u^2 \).

Proof. The proof is similar as in Theorem 4.2.

5. Global regularization of the three center problem. Consider the three center problem, with centers at \( q_1 = (0, \bar{y}_1) \), \( q_2 = (0, \bar{y}_2) \) and \( q_3 = (0, 1) \) with \( (\bar{y}_1 > 1) \). The potential energy is (remember that \( M = 1) \)

\[
U = m_1 \frac{x^2 + \bar{y}_1^2 + y^2}{\sqrt{x^2 + (y - \bar{y}_1)^2}} + \mu \frac{x^2 + y^2 + 1}{\sqrt{x^2 + (y - 1)^2}} + m_2 \frac{x^2 + \bar{y}_2^2 + y}{\sqrt{x^2 + (y - \bar{y}_2)^2}} + 1 + \bar{y}_2^2 + 1 \]

(5.1)

In order to regularize the collision with all centers, we will introduce a Thiele-Burrau type transformation.

Theorem 5.1. In the geodesic three-center problem with centers at \( q_1 = (0, \bar{y}_1) \), \( q_2 = (0, \bar{y}_2) \), \( q_3 = (0, 1) \) and masses \( m_1 \), \( m_2 \), \( \mu \) and \( M \), respectively. The binary collision of \( M \) with \( m_1 \), \( m_2 \) and \( \mu \) can be regularized via a suitable change of coordinates and a reparametrization of the time, that is, in this case we have a global regularization.

Proof. Consider the following change of coordinates

\[
x = \left( \frac{\bar{y}_1 - 1}{2} \right) \sin \xi \sinh \eta, \quad y = - \left( \frac{\bar{y}_1 - 1}{2} \right) \cos \xi \cosh \eta + \left( 1 + \bar{y}_1 \right). \]

(5.2)
implies $\xi > 0$ now, from the first equation we have that $\xi \eta$ Lagrangian in $\tau$ as a function of $q$ point in the regularized plane. Let us write the transformations as Regularized equations. 5.1. formations can be written as $U$ The following factors that appear in the potential $U$ with $f$ and $\bar{G}$ | $y$ $f = (\bar{y} - 1)^2 (\cos^2 \eta - \cos^2 \xi)$. On the other hand, if $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R} \times \mathbb{R}^+$ defined as $f(\xi, \eta) = (x, y)$ then $|\frac{\partial f}{\partial \xi}|^2 = (\bar{y} - 1)^2 (\cos^2 \eta - \cos^2 \xi)$. We will consider the new time reparametrization $\tau$ as a function of $t$ satisfying $\frac{dt}{\tau} = \frac{\partial f}{\partial \xi}^2$. It is well known that the flow given by the lagrangian action $\int L dt$ is equivalent to the flow given by $\int L d\tau$ with $L = L \frac{dt}{\tau}$ (See [2] for more details).

In this way, we can consider the Lagrangian given by $L$. The potential for this Lagrangian in $\xi \eta$-coordinates is

$$U(\xi, \eta) \left| \frac{\partial f}{\partial \xi} \right|^2 \quad (5.5)$$

$$= m_1 \left( \left( \frac{\bar{y}_1 - 1}{2} \right) \sin \xi \sinh \eta \right)^2 + \bar{y}_1^2 + \left( - \left( \frac{\bar{y}_1 - 1}{2} \right) \cos \xi \cosh \eta + \left( \frac{1 + \bar{y}_1}{2} \right) \right)^2 \right) f_1$$

$$+ m_2 \left( \left( \frac{\bar{y}_1 - 1}{2} \right) \sin \xi \sinh \eta \right)^2 + \left( - \left( \frac{\bar{y}_1 - 1}{2} \right) \cos \xi \cosh \eta + \left( \frac{1 + \bar{y}_1}{2} \right) \right)^2 \right) f_2$$

$$+ m_2 \left( \left( \frac{\bar{y}_1 - 1}{2} \right) \sin \xi \sinh \eta \right)^2 + \left( - \left( \frac{\bar{y}_1 - 1}{2} \right) \cos \xi \cosh \eta + \left( \frac{1 + \bar{y}_1}{2} \right) \right)^2 \right) f_3, \quad (5.3)$$

with

$$f_1 = \left( \frac{\bar{y}_1 - 1}{2} \right) \left( \cosh \eta + \sin \xi \right),$$

$$f_2 = \left( \frac{\bar{y}_1 - 1}{2} \right) \left( \cosh \eta - \sin \xi \right),$$

$$f_3 = \left( \frac{\bar{y}_1 - 1}{2} \right)^2 \left( \cosh^2 \eta - \sin^2 \xi \right),$$

$$G = \sqrt{\left( \left( \frac{\bar{y}_1 - 1}{2} \right) \sin \xi \sinh \eta \right)^2 + \left( - \left( \frac{\bar{y}_1 - 1}{2} \right) \cos \xi \cosh \eta + \left( \frac{1 + \bar{y}_1}{2} \right) \right)^2 + \bar{y}_2^2}. \quad (5.4)$$

Here we can see that the singularities given by the collision between $q$ and $q_1$ and between $q$ and $q_3$ have been removed.\[ Q.E.D. \]

5.1. Regularized equations. In this subsection we will write the initial conditions in the regularized plane. Let us write the transformations as $x = A \sin \xi \sinh \eta, \quad y = -A \cos \xi \cosh \eta + B.$ (5.6)

Notice that $y > 0$ implies, from second equation of (5.6) that $\cos \xi \cosh \eta < \frac{y}{A}$; now, from the first equation we have that $x > 0$ implies $\xi \in (0, \pi)$, and $x < 0$ implies $\xi \in (\pi, 2\pi)$, see Figure 7. On the other hand, it is easy to see that the point $q_1$ corresponds to $\eta = 0$, $\xi = \pm \pi, \pm 3\pi, \pm 5\pi, \cdots$ in the $\xi \eta$-plane; the point $q_3$
Figure 7. Shaped area corresponds to the points \((\xi, \eta)\) with 
\[ \cos \xi \cosh \eta < \frac{B}{A} \] 
with \(\bar{y}_1 = 2\). Region 1 is for \(x > 0, y > 0\); and region 2 for \(x < 0, y > 0\).

corresponds to the point \(\eta = 0, \xi = 0, \pm 2\pi, \pm 4\pi \cdots\); and the point \(q_2\) is mapped to 
the points \(\xi = 0, \pm 2\pi, \pm 4\pi, \cdots\), \(\eta = \arccosh(\frac{\bar{y}_2 - B}{A})\).

From system (5.6) we have 
\[
\begin{align*}
&\cos^2 \xi - \left(\frac{y - B}{A}\right)^2 + 1 + \left(\frac{\bar{y}_1}{A}\right)^2 = 0, \\
&\left(\frac{\bar{y}_2 - B}{A}\right)^2 + 1 + \left(\frac{\bar{y}_1}{A}\right)^2 = 0.
\end{align*}
\]

From here we get the coordinate \(\xi\) given \(x\) and \(y\). Using any of the equations in (5.6) we get \(\eta\).

Using equations (5.6) we have 
\[
\dot{x} = A[A_1 \dot{\eta} + A_2 \dot{\xi}], \quad \dot{y} = A[-A_2 \dot{\eta} + A_1 \dot{\xi}],
\]
with \(A_1 = \sin \xi \cosh \eta, A_2 = \sinh \eta \cos \xi\).

Solving last system we get \(\dot{\xi} = \frac{A_2 \dot{x} + A_1 \dot{y}}{A_2 + A_1}\) and \(\dot{\eta} = \frac{A_1 \dot{x} - A_2 \dot{y}}{A_1 + A_2}\). Considering that \(\dot{\xi} = \xi' \frac{dx}{dt}\) we have 
\[
\xi' = \frac{A_2 \dot{x} + A_1 \dot{y}}{A_2 + A_1} \frac{dt}{\frac{dx}{dr}}, \quad \eta' = \frac{A_1 \dot{x} - A_2 \dot{y}}{A_1 + A_2} \frac{dt}{\frac{dx}{dr}}.
\]

Numerical example.
Consider the following initial conditions in the \(xy\)-plane, \(x(0) = 0.5, y(0) = 1.5, \dot{x}(0) = 0.5, \dot{y}(0) = 0.5\) and masses \(m_1 = m_2 = M = \mu = 1\) and positions of the centers with \(\bar{y}_1 = 2\) and \(\bar{y}_2 = 1/4\), see Figure 8a. From the previous section, these conditions take the form (see Figure 8b) 
\[\xi(0) = \pi/2, \quad \eta(0) = 0.88137, \quad \xi'(0) = 0.1249997924, \quad \eta'(0) = 0.1249997924,\]

As a second example consider as initial conditions in the \(xy\)-plane, \(x(0) = -0.4, y(0) = 0.5, \dot{x}(0) = 0.5, \dot{y}(0) = 0.5\) and masses \(m_1 = 7, m_2 = 3, \mu = M = 1\) and positions of the centers with \(\bar{y}_1 = 3\) and \(\bar{y}_2 = 1/4\), see Figure 9a. From the previous section, these conditions take the form (see Figure 9b) 
\[\xi(0) = 0.970604, \quad \eta(0) = 0.962423650,\]
Figure 8. An orbit in the $xy$-plane and its corresponding in the regularized plane

\[
\xi'(0) = 0.784500196, \quad \eta'(0) = 2.418430032,
\]

Figure 9. An orbit in the $xy$-plane and its corresponding in the regularized plane

\[
\xi'(0) = 0.784500196, \quad \eta'(0) = 2.418430032.
\]

Figure 10. We plot the same constrained graphic as Figure 9b where we can see the positions of the centers plotted.

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