WAVE PROPAGATION AND SCATTERING FOR THE RS2 BRANE COSMOLOGY MODEL

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ABSTRACT. We study the wave equation for the gravitational fluctuations in the Randall-Sundrum brane cosmology model. We solve the global Cauchy problem and we establish that the solutions are the sum of a slowly decaying massless wave localized near the brane, and a superposition of massive dispersive waves. We compute the kernel of the truncated resolvent. We prove some $L^1 - L^\infty$, $L^2 - L^\infty$ decay estimates and global $L^p$ Strichartz type inequalities. We develop the complete scattering theory: existence and asymptotic completeness of the wave operators, computation of the scattering matrix, determination of the resonances on the logarithmic Riemann surface.

I. Introduction

The fundamental question in mathematical cosmology is the stability of the models of the universe. It consists in solving the global Cauchy problem for the Einstein equations. This is an extremely hard task because of the deep nonlinearity of these equations. We recall the impressive work by D. Christodoulou and S. Klainerman [11] on the global nonlinear stability of the Minkowski space-time which, however, is the simplest model. In this paper, our aim is much more modest. We adress the question of the linear stability of a famous model of the brane cosmology, i.e. we investigate the global properties of the gravitational fluctuations that are the solutions of a linear hyperbolic equation with a singular potential. We consider the so-called RS2 brane world, proposed by L. Randall and R. Sundrum in [30], in which our observable universe is a 4-D Minkowski brane embedded in a 5-D Anti-de Sitter space bulk. This model plays a considerable role in cosmology: it is a phenomenological realization of M theory ideas; it also provides a framework for exploring the holographical principle and the AdS/CFT correspondence. The seminal papers by L. Randall and R. Sundrum [29] and [30] have generated a huge litterature. Among important publications, we can cite the following reviews, [9], [19], [21], and the book by P.D. Mannheim [22].

The RS2 brane world, introduced by L. Randall and R. Sundrum in [30] is described by the 5-dimensional lorentzian manifold endowed with a warped metric:

$$\mathcal{M} = \mathbb{R}_T \times \mathbb{R}^3_X \times \mathbb{R}_y, \quad ds^2_{\mathcal{M}} = e^{-2k|y|} (dT^2 - dX^2) - dy^2.$$ 

The warp coefficient $k$ is a strictly positive real number. This metric is not smooth at $y = 0$ and we have to distinguish two submanifolds: the positive tension Minkowski brane that corresponds to our flat world

$$\mathcal{M} = \mathbb{R}_T \times \mathbb{R}^3_X \times \{y = 0\}, \quad ds^2_{\mathcal{M}} = dT^2 - dX^2,$$

and the bulk $\mathcal{B}$, associated with the extra transverse and non compact dimension $y$, in which the brane is imbedded:

$$\mathcal{B} = \mathcal{B}^+ \cup \mathcal{B}^-, \quad \mathcal{B}^\pm = \mathbb{R}_T \times \mathbb{R}^3_X \times \{\pm y > 0\}, \quad ds^2_{\mathcal{B}^\pm} = e^{\mp 2ky} (dT^2 - dX^2) - dy^2.$$ 

$\mathcal{B}^+$ and $\mathcal{B}^-$, that are isometric, are parts of the Anti-De Ditter universe $AdS$, with constant negative curvature $-k^2$. To see that, we introduce new coordinates

$$U^0 = \frac{1}{2} e^{-ky} \left( e^{2ky} + \frac{1}{k^2} + |X|^2 - T^2 \right), \quad U^i = \frac{1}{k} e^{-ky} X^i, \quad 1 \leq i \leq 3.$$
\[ U^4 = \frac{1}{2} e^{-ky} \left( e^{2ky} - \frac{1}{k^2} + |X|^2 - T^2 \right), \quad U^5 = \frac{T}{k} e^{-ky}, \]
and we check that \( \mathcal{B}^+ \) is included in the region \( U^0 > U^4 \) of \( \text{AdS} \) which is defined as the quadric
\[
\text{AdS} : \quad (U^0)^2 + (U^5)^2 - \sum_{i=1}^{4} (U^i)^2 = \frac{1}{k^2}
\]
embedded in the 6 dimensional flat space \( \mathbb{R}^6 \) with the metric
\[
ds^2 = (dU^0)^2 + (dU^5)^2 - \sum_{i=1}^{4} (dU^i)^2.
\]
To understand the causal structure of this universe, we can also construct \( \mathcal{M} \) by gluing together the time-like boundary of two copies of a piece \( \mathcal{B}^+ \) of \( \text{AdS} \), of which the boundary is isometric to the Minkowski manifold. This construction is depicted in Figure 1 by a Penrose diagram where the radial null geodesics travel at \( \pm 45^\circ \) angles. The dotted line is the time-like infinity, the dashed lines have to be identified. We see that unlike the whole Anti-De Sitter space-time, \( \mathcal{M} \) is globally hyperbolic and there is no problem of causality (see \cite{2} for a discussion on a global Cauchy problem in \( \text{AdS} \)).

\[ T = \pm \infty \quad y = \pm \infty \quad y = 0 \quad \text{BRANE glue together} \]

\( \mathcal{B}^- \)

\( \mathcal{B}^+ \)

\[ Figure 1. \text{ Conformal diagram of } \mathcal{M}. \]

We are interested in the gravitational fluctuations around this background. Using the linearized Einstein equations, Randall and Sundrum established that these gravity waves obey to the master equation:
\[(\text{I.1}) \quad \left[ e^{2k|y|} \left( \partial^2_x - \Delta x \right) - \partial^2_y - 4k \delta_0(y) + 4k^2 \right] \Psi = 0, \]
where the Dirac distribution denotes the simple layer on the brane. Our work is devoted to a complete analysis of this equation. The main assertions that are stated heuristically in the papers by the physicists (see e.g. \cite{22}, \cite{23}, \cite{30}, \cite{31}) are the following : (i) the general solution \( \Psi \) of the
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Klein tower decays as $t \sim \nu_0(T, \mathbf{X}, y)$ of the truncated resolvent, and we show the existence of its analytic continuation on the universal spectral analysis and the proof of the representation (I.2). We also compute explicitly the kernel we prove that the hamiltonian is a well defined self-adjoint operator. In section 3, we perform its theory for the master equation. In section 2, we construct the suitable functional framework, and

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In this paper we establish these results in a rigorous way, and we develop the complete scattering theory for the master equation. In section 2, we construct the suitable functional framework, and we prove that the hamiltonian is a well defined self-adjoint operator. In section 3, we perform its spectral analysis and the proof of the representation (I.2). We also compute explicitly the kernel of the truncated resolvent, and we show the existence of its analytic continuation on the universal covering $\mathbb{C}^*$ of $\mathbb{C}^{*}$, outside a lattice of half hyperbolas. In the next section, we prove that the Kaluza-Klein tower decays as $t^{-\frac{1}{12}}$ by establishing some $L^1 - L^\infty$ and $L^2 - L^\infty$ estimates in suitable weighted spaces. Moreover we get a Strichartz type estimate near the brane in $L^\infty([-R, R]_y; L^4(\mathbb{R}_T \times \mathbb{R}^3_y))$, and for the Kaluza-Klein tower in $L^\infty([R, R]_y, \mathbb{R}^3_y)$. Section 5 is devoted to the scattering theory: we prove the existence and asymptotic completeness of the wave operators describing the scattering of the Kaluza-Klein tower by the brane. In the last part, we calculate the scattering matrix, and we determine the set of resonances: it is a lattice of radial half straight lines on $\mathbb{C}^*$, of which the origins are the $z$-zeros of the Hankel functions $H^{(j)}_\nu(z)$, $\nu, j = 1, 2$. All our results of asymptotic behaviours, suggest that this brane cosmology model is linearly stable. The non-linear stability of the Minkowski brane is a huge open problem.

II. The Cauchy problem.

It is convenient to use the Poincaré coordinates $(t, \mathbf{x}, z)$ with

$$t = kT, \quad \mathbf{x} = k\mathbf{X}, \quad z = \frac{y}{|y|} \left( e^{k|y|} - 1 \right),$$

for which the Randall-Sundrum Universe is described by the conformally flat manifold

$$\mathcal{M} = \mathbb{R}_t \times \mathbb{R}^3_x \times \mathbb{R}_z, \quad ds^2 = \frac{1}{k^2} \left( \frac{1}{1 + |z|} \right)^2 \left( dt^2 - dx^2 - dz^2 \right).$$

We change the unknown field by putting

$$\Phi(t, \mathbf{x}, z) = e^{\frac{2}{3}\frac{1}{|y|}} \Psi(T, \mathbf{X}, y).$$

Since the simple layer $\delta_0$ is a homogeneous distribution of degree $-1$ and $\delta_0(y) = k\delta_0(z)$, we see that $\Psi$ is solution of (II.1) iff $\Phi$ is solution of:

$$(\Box_{t, \mathbf{x}, z} + \frac{15}{4} \left( \frac{1}{1 + |z|} \right)^2 - 3\delta_0(z)) \Phi = 0, \quad (t, \mathbf{x}, z) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}.$$ 

Therefore Equation (II.1) is equivalent to the D’Alembertian on the five-dimensional Minkowski space-time, $\Box_{t, \mathbf{x}, z} := \partial_t^2 - \Delta_\mathbf{x} - \partial_z^2$, perturbed by a singular potential, $\frac{15}{4} \left( \frac{1}{1 + |z|} \right)^2 - 3\delta_0(z)$, the
so-called volcano potential. In this section we investigate the Cauchy problem for the equation (II.1), with the initial data

\[(II.2) \quad \Phi(0, \mathbf{x}, z) = \Phi_0(\mathbf{x}, z), \quad \partial_t \Phi(0, \mathbf{x}, z) = \Phi_1(\mathbf{x}, z), \quad (\mathbf{x}, z) \in \mathbb{R}^3 \times \mathbb{R}.\]

We introduce the differential operator

\[(II.3) \quad P(\partial) := -\Delta_x - \partial_z^2 + \frac{15}{4} \left( \frac{1}{1 + |z|} \right)^2 - 3\delta_0(z),\]

considered as a densely defined operator \(H_0\) on \(L^2(\mathbb{R}^4)\), endowed with the domain:

\[(II.4) \quad \mathcal{D}(H_0) := \{ u \in H^1(\mathbb{R}^3_x \times \mathbb{R}_z) ; \quad P(\partial)u \in L^2(\mathbb{R}^3_x \times \mathbb{R}_z) \},\]

which makes sense since \(H^1(\mathbb{R}^4) \subset C^0(\mathbb{R}_z; H^{\frac{1}{2}}(\mathbb{R}^3_x))\) and so \(u(\mathbf{x}, 0)\delta_0(z)\) is a well defined distribution in \(H^{-1}(\mathbb{R}^4)\).

To analyse this operator and to be able to construct the suitable functional framework, we recall some definitions of spaces of distributions. Given an Hilbert space \(X\), the Beppo Levi space \(BL^1(\mathbb{R}^n; X)\) is defined as the completion of the space of the \(X\)-valued test functions on \(\mathbb{R}^n\), \(C_0^\infty(\mathbb{R}^n; X)\), for the Dirichlet norm \(\| \nabla u \|_{L^2(\mathbb{R}^n; X)}\) (see [13]). On the other hand, we introduce the weighted Sobolev space \(W^1_0(\mathbb{R}^3; X)\) that is the subspace of \(\mathcal{D}'(\mathbb{R}^n; X)\) satisfying:

\[(II.5) \quad \| f \|_{W^1_0(\mathbb{R}^3; X)}^2 := \int_{-\infty}^{\infty} | f'(z) |^2 + \frac{15}{4} \left( \frac{1}{1 + |z|} \right)^2 \| f(z) \|_X^2 \, dz < \infty.\]

We have \(W^1_0(\mathbb{R}^3; X) \subset C^0(\mathbb{R}^3; X)\), and thanks to the Hardy inequality,

\[(II.6) \quad \forall f \in L^2(\mathbb{R}^3; X), \quad \int_{-\infty}^{\infty} \frac{1}{z^2} \int_0^z f(\xi) d\xi \, dz \leq 4 \int_{-\infty}^{\infty} | f(z) |^2_X \, dz,\]

there exists \(C > 0\) such that

\[(II.7) \quad \forall f \in W^1_0(\mathbb{R}^3; X), \quad f(0) = 0 \Rightarrow \| f \|_{W^1_0(\mathbb{R}^3; X)} \leq C \| f' \|_{L^2(\mathbb{R}^3; X)}.\]

**Lemma II.1.** The operator \((H_0, \mathcal{D}(H_0))\) is a positive self-adjoint operator on \(L^2(\mathbb{R}^4)\) and 0 is not an eigenvalue. The domain of \(H_0^2\) is \(H^1(\mathbb{R}^4)\) and for any \(u \in H^1(\mathbb{R}^4)\) the three following quantities are equal to \(\| H_0^2 u \|_{L^2(\mathbb{R}^4)}^2\) :

\[(II.8) \quad \| \nabla u \|_{L^2(\mathbb{R}^3_x \times \mathbb{R}_z)}^2 + \| \partial_z u + \frac{3}{2} \frac{z}{|z|} \left( \frac{1}{1 + |z|} \right) u \|_{L^2(\mathbb{R}^3_x \times \mathbb{R}_z)},\]

\[(II.9) \quad \| u \|_{L^2(\mathbb{R}^3_x; BL^1(\mathbb{R}^3_z))}^2 + \| u \|_{W^1_0(\mathbb{R}^3_x; L^2(\mathbb{R}^3_z))}^2 - 3 \| u(\mathbf{x}, 0) \|_{L^2(\mathbb{R}^3_z)}^2,\]

\[(II.10) \quad \| u \|_{L^2(\mathbb{R}^3_x; BL^1(\mathbb{R}^3_z))}^2 + \| u - u(\mathbf{x}, 0)(1 + |z|)^{-\frac{3}{2}} \|_{W^1_0(\mathbb{R}^3_x; L^2(\mathbb{R}^3_z))}^2.\]

Moreover for any \(m > 0\), we have the elliptic estimate:

\[(II.11) \quad \min \left( \frac{m^2}{3 + m^2}, \frac{m^2}{4} \right) \| u \|_{H^1(\mathbb{R}^4)} \leq \| H_0^2 u \|_{L^2(\mathbb{R}^4)}^2 + m^2 \| u \|_{L^2(\mathbb{R}^4)}^2.\]

**Proof of Lemma II.1.** We know that when \(u \in H^1(\mathbb{R}^3_x \times ]0, \infty[)\) satisfies \(\Delta_{x,z} u \in L^2(\mathbb{R}^3_x \times ]0, \infty[)\), where \(\Delta_{x,z}\) is the euclidean Laplace operator on \(\mathbb{R}^3_x \times \mathbb{R}_z\), then \(\partial_z u \in C^0([0, \infty[; H^{-\frac{1}{2}}(\mathbb{R}^3_x))\) (see [20], Theorem 7.3, p. 201). We deduce that \(u \in \mathcal{D}(H_0)\) iff \(u \in H^1(\mathbb{R}^3_x \times \mathbb{R}_z)\) and

\[(II.12) \quad \Delta_{x,z} u|_{\mathbb{R}^3_x \times \mathbb{R}_z} \in L^2(\mathbb{R}^3_x \times \mathbb{R}_z).\]
Now for \( u, v \in \mathcal{D}(H_0) \), using the jumps formula and (II.13), we get by the Green formula:

\[
\langle H_0 u, v \rangle_{L^2} = Q(u, v),
\]

where we have introduced the quadratic form \( Q \) on \( H^1(\mathbb{R}^4) \) given by

\[
Q(u, v) := \int_{\mathbb{R}^4} \nabla_x u \nabla_x v + \partial_x u \partial_x v + \frac{15}{4} \left( \frac{1}{1 + |z|} \right)^2 u v dz \quad -3 \int_{\mathbb{R}^3} u(x, 0) v(x, 0) dx,
\]

and we see that \( H_0 \) is symmetric. To pursue the proof, it is useful to present some results on the quadratic forms.

Given a Hilbert space \( X \), we consider the quadratic form

\[
f \in W_0^1(\mathbb{R}; X), \quad q(f, f) := \int_{-\infty}^{\infty} |f'(z)|_x^2 + \frac{15}{4} \left( \frac{1}{1 + |z|} \right)^2 |f(z)|_x^2 dz - 3 |f(0)|_x^2.
\]

We immediately check with an integration by part that:

\[
q(f, f) = \int_{-\infty}^{\infty} |f'(z) + \frac{3}{2} |z| \left( \frac{1}{1 + |z|} \right) f(z)|_x^2 dz \geq 0.
\]

We simply use the inequalities

\[
|f'(z)|_x \leq |f'(z) + \frac{3}{2} |z| \left( \frac{1}{1 + |z|} \right) f(z)|_x + \frac{3}{2} |f(z)|_x,
\]

\[
|f'(z)|_x |f(z)|_x \leq \frac{1}{2\alpha} |f'(z)|_x^2 + \frac{\alpha}{2} |f(z)|_x^2
\]

for any \( \alpha > 0 \), to get an elliptic estimate: given \( m > 0 \), we write

\[
\int_{-\infty}^{\infty} |f'(z)|_x^2 \left( 1 - \frac{3}{2\alpha} \right) + |f(z)|_x^2 \left( \frac{9}{4} - \frac{3\alpha}{2} + m^2 \right) dz \leq q(f, f) + m^2 \int_{-\infty}^{\infty} |f(z)|_x^2 dz,
\]

and we choose \( \alpha = \frac{3 + m^2}{2} \) to obtain:

\[
\min \left( \frac{m^2}{3 + m^2}, \frac{m^2}{4} \right) \int_{-\infty}^{\infty} |f'(z)|_x^2 + |f(z)|_x^2 \ dz \leq q(f, f) + m^2 \int_{-\infty}^{\infty} |f(z)|_x^2 \ dz.
\]

(II.16) leads also to the following crucial result. If we introduce

\[
f_0(z) := (1 + |z|)^{-\frac{3}{2}},
\]

then we have for any \( x \in X, f \in W_0^1(\mathbb{R}; X) \):

\[
q(f_0 x, f) = 0,
\]

and thus in particular

\[
q(f, f) = \int_{-\infty}^{\infty} |f'(z) - f_0'(z) f(0)|_x^2 + \frac{15}{4} \left( \frac{1}{1 + |z|} \right)^2 |f(z) - f_0(z) f(0)|_x^2 dz.
\]

In the sequel we take \( X = L^2(\mathbb{R}^4) \). Then the equivalence between (II.18), (II.19) and (II.20) follows from (II.16) and (II.20), hence \( H_0 \) is positive and 0 is not an eigenvalue. Moreover (II.17) gives

\[
\min \left( \frac{m^2}{3 + m^2}, \frac{m^2}{4} \right) \int_{\mathbb{R}^4} |\nabla_x z u(x, z)|^2 + |u(x, z)|^2 dx dz \leq Q(u, u) + m^2 \int_{\mathbb{R}^4} |u(x, z)|^2 dx dz.
\]

Now to prove the self-adjointness, we consider \( g \in L^2(\mathbb{R}^4), \varepsilon = \pm 1 \), and we have to solve

\[
\varepsilon \int_{\mathbb{R}^4} u(x, z) g(x, z) dx dz = \varepsilon iu + P(\partial) u + \varepsilon iu = g.
\]
This problem is equivalent to find $u \in \mathcal{D} (\mathcal{H}_0)$ such that

$$\tag{II.23} A(u, \varphi) := Q(u, \varphi) + \varepsilon i \int_{\mathbb{R}^4} u(x, z) \varphi(x, z) \, dx \, dz = \int_{\mathbb{R}^4} g(x, z) \varphi(x, z) \, dx \, dz, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^4).$$

Since the sesquilinear form $A$ is continuous on $H^1(\mathbb{R}^4)$, (II.23) is equivalent to

$$\tag{II.24} A(u, v) := Q(u, v) + \varepsilon i \int_{\mathbb{R}^4} u(x, z) \overline{v(x, z)} \, dx \, dz = \int_{\mathbb{R}^4} g(x, z) \overline{v(x, z)} \, dx \, dz, \quad \forall v \in H^1(\mathbb{R}^4).$$

Since (II.21) assures that $A$ is coercive on $H^1(\mathbb{R}^4)$, the Lax-Milgram theorem implies that there exists $u \in H^1(\mathbb{R}^4)$ solution of (II.24). Taking $v$ an arbitrary test function in $\mathcal{D}(\mathbb{R}_x^3 \times \mathbb{R}_z^*)$ we deduce that

$$-\Delta_x u - \partial_z^2 u + \frac{15}{4} \left( \frac{1}{1+|z|} \right)^2 u + \varepsilon i u = g \quad \text{in} \quad \mathcal{D}'(\mathbb{R}_x^3 \times \mathbb{R}_z^*).$$

Thus (II.12) is satisfied and an integration by parts in (II.24) gives

$$\left\langle \partial_z u(x, 0^+) - \partial_z u(x, 0^-) + 3u(x, 0), \overline{v(x, 0)} \right\rangle_{H^{-\frac{1}{2}}(\mathbb{R}_z^3), H^{\frac{1}{2}}(\mathbb{R}_z^3)} = 0, \quad \forall v \in H^1(\mathbb{R}^4).$$

We conclude that $u$ belongs to the domain of $\mathcal{H}_0$, which is self-adjoint.

Finally (II.21) assures that the quadratic form $Q$ on $L^2(\mathbb{R}^4)$ with its natural form domain $\mathcal{D}(Q) = H^1(\mathbb{R}^4)$ is closed. The standard results of the spectral theory (see e.g. [27], section 2.7) states that the domain of $\mathcal{H}_0^1$ is $\mathcal{D}(Q)$ and (II.11) follows from (II.21).

Q.E.D.

This result allows to easily solve the Cauchy problem (II.1), (II.2), for $\Phi_0 \in H^1(\mathbb{R}^4)$, $\Phi_1 \in L^2(\mathbb{R}^4)$ by putting

$$\Phi(t) = \cos \left( t \mathcal{H}_0^1 \right) \Phi_0 + \mathcal{H}_0^1 \sin \left( t \mathcal{H}_0^1 \right) \Phi_1.$$

It is obvious that $\Phi \in C^0(\mathbb{R}; H^1(\mathbb{R}^4)) \cap C^1(\mathbb{R}; L^2(\mathbb{R}^4))$ and satisfies the conservation of the energy :

$$Q(\Phi(t), \Phi(t)) + \| \partial_z \Phi(t) \|^2_{L^2(\mathbb{R}^4)} = Q(\Phi_0, \Phi_0) + \| \Phi_1 \|^2_{L^2(\mathbb{R}^4)}.$$

Since this conserved quantity is positive, it is natural to consider the Cauchy problem in the larger functional space, associated with this energy. (II.8) shows that $\sqrt{Q(u, u)}$ is a norm on $H^1(\mathbb{R}^4)$, therefore we introduce the Hilbert space $\mathfrak{W}^1(\mathbb{R}^4)$ that is the closure of $H^1(\mathbb{R}^4)$ for the norm

$$\tag{II.25} \| u \|_{\mathfrak{W}^1} := \| \nabla_x u \|_{L^2(\mathbb{R}_x^3 \times \mathbb{R}_z)} + \| \partial_z u + \frac{3}{2} \frac{z}{|z|} \left( \frac{1}{1+|z|} \right) u \|_{L^2(\mathbb{R}_x^3 \times \mathbb{R}_z)}.$$

$\mathfrak{W}^1(\mathbb{R}^4)$ is a space of distributions on $\mathbb{R}^4$, and

$$\mathfrak{W}^1(\mathbb{R}^4) = \left\{ u \in L^2(\mathbb{R}_z; BL^1(\mathbb{R}^3_\mathfrak{X})); \quad \partial_z u + \frac{3}{2} \frac{z}{|z|} \left( \frac{1}{1+|z|} \right) u \in L^2(\mathbb{R}^4) \right\}.$$

We make some warnings on this space : By the classical embedding of the Beppo Levi space :

$$BL^1(\mathbb{R}^3_\mathfrak{X}) \subset L^2 \left( \mathbb{R}^3_\mathfrak{X}, \frac{1}{1+|x|^2} \, dx \right),$$

we can see that the functions $u \in \mathfrak{W}^1(\mathbb{R}^4)$ satisfy

$$u, \partial_z u \in L^2 \left( \mathbb{R}_z; L^2 \left( \mathbb{R}^3_\mathfrak{X}, \frac{1}{1+|x|^2} \, dx \right) \right),$$

hence

$$u \in C^0 \left( \mathbb{R}_z; L^2 \left( \mathbb{R}^3_\mathfrak{X}, \frac{1}{1+|x|^2} \, dx \right) \right).$$
and thus $u(x,0)\delta_0(z)$ is well defined in $H^{-1}_{loc}(\mathbb{R}^4)$. But we have to be careful by using expressions (II.9) or (II.10) to evaluate $Q(u,u)$ when $u \in \mathcal{W}^1(\mathbb{R}^4) \setminus H^1(\mathbb{R}^4)$, because there exists $u \in \mathcal{W}^1(\mathbb{R}^4)$ such that
\[
\partial_z u \notin L^2(\mathbb{R}^4), \quad u \notin L^2\left(\mathbb{R}^3 \times \mathbb{R}_z, \frac{1}{1+z^2}dxdz\right), \quad u(x,0) \notin L^2(\mathbb{R}^3_x).
\]
To see that, we take $\chi \in \mathcal{D}(\mathbb{R}^3)$ such that $\|\chi\|_{L^2} = \|\nabla \chi\|_{L^2} = 1$. For any $n \in \mathbb{N}$, we put $u_n(x,z) = n^{-\frac{2}{3}}(\chi(z))(1+|z|)^{-\frac{4}{3}}$. We compute $Q(u_n,u_n) = 1$, $\|\partial_z u_n\|_{L^2(\mathbb{R}^4)}^2 = \frac{n^2}{2}$, $\|u_n\|_{L^2(\mathbb{R}^4)}^2 = \frac{1}{2}n^2$, $\|u_n(.,0)\|_{L^2(\mathbb{R}^3)}^2 = n^2$.

We return to the Cauchy problem (II.1), (II.2) for $\Phi_0 \in \mathcal{W}^1(\mathbb{R}^4)$, $\Phi_1 \in L^2(\mathbb{R}^4)$. On $\mathcal{W}^1(\mathbb{R}^4) \times L^2(\mathbb{R}^4)$ we introduce the operator
\[
A_0 := \frac{1}{i}\begin{pmatrix} 0 & 1 \\ -\partial_t & 0 \end{pmatrix}, \quad \mathcal{D}(A_0) = \mathcal{D}(H_0) \times H^1(\mathbb{R}^4).
\]
From the properties of $H_0$ we deduce that $A_0$ is a densely defined, symmetric operator on $\mathcal{W}^1(\mathbb{R}^4) \times L^2(\mathbb{R}^4)$ and $H^1(\mathbb{R}^4) \times L^2(\mathbb{R}^4) \subset \text{Ran}(A_0 \pm i)$ where $\text{Ran}(f)$ denotes the range of any map $f$. Therefore $A_0$ is essentially self-adjoint and its self-adjoint closure $A$ generates a unitary group $e^{itA}$. Then
\[
\begin{pmatrix} \Phi(t) \\ \partial_t \Phi(t) \end{pmatrix} = e^{itA} \begin{pmatrix} \Phi_0 \\ \Phi_1 \end{pmatrix}
\]
is a solution of the Cauchy problem that satisfies
\[
\Phi \in \Xi := \{ \Phi \in C^0(\mathbb{R}_t; \mathcal{W}^1(\mathbb{R}^4)), \quad \partial_t \Phi \in C^0(\mathbb{R}_t; L^2(\mathbb{R}^4)) \},
\]
and the conservation of the energy:
\[
\forall t \in \mathbb{R}, \quad \|\Phi(t)\|_{2\mathcal{W}^1(\mathbb{R}^4)}^2 + \|\partial_t \Phi(t)\|_{L^2(\mathbb{R}^4)}^2 = \|\Phi_0\|_{2\mathcal{W}^1(\mathbb{R}^4)}^2 + \|\Phi_1\|_{L^2(\mathbb{R}^4)}^2.
\]
We prove the uniqueness by the usual way. If $\Phi$ is a solution of (II.1), (II.2), (II.26), we take a test function $\theta \in C_0^\infty(\mathbb{R})$, such that $\int \theta(t)dt = 1$ and for all integer $n > 0$, we put $\Phi_n(t) = \int \Phi(s)\theta(n(t-s))ds$. Since $\Phi_n \in C^\infty(\mathbb{R}_t; \mathcal{W}^1(\mathbb{R}^4))$, $\partial_t \Phi_n \in C^\infty(\mathbb{R}_t; L^2(\mathbb{R}^4))$ we can multiply equation (II.1) by $\partial_t \Phi_n$ and an integration in $(x,z)$ leads to the conservation
\[
\forall t \in \mathbb{R}, \quad \|\Phi_n(t)\|_{2\mathcal{W}^1(\mathbb{R}^4)}^2 + \|\partial_t \Phi_n(t)\|_{L^2(\mathbb{R}^4)}^2 = \|\Phi_n(0)\|_{2\mathcal{W}^1(\mathbb{R}^4)}^2 + \|\partial_t \Phi_n(0)\|_{L^2(\mathbb{R}^4)}^2.
\]
Since $\Phi_n$ and $\partial_t \Phi_n$ tend to $\Phi$ and $\partial_t \Phi$ as $n \to \infty$, respectively in $C^0(\mathbb{R}_t; \mathcal{W}^1(\mathbb{R}^4))$ and $C^0(\mathbb{R}_t; L^2(\mathbb{R}^4))$, we conclude that (II.27) holds and $\Phi = 0$ when $\Phi_0 = \Phi_1 = 0$.

We summarize our results:

**Theorem II.2.** Given $\Phi_0 \in \mathcal{W}^1(\mathbb{R}^4)$, $\Phi_1 \in L^2(\mathbb{R}^4)$, there exists a unique solution $\Phi$ of the Cauchy problem (II.1), (II.2), (II.26). Moreover this solution satisfies (II.27).

The solutions given by this theorem are called finite energy solutions.

**Remark II.3.** We could interpret this Cauchy problem for a wave equation with the singular potential $\frac{15}{4} \left(\frac{1}{1+z^2}\right)^2 - 3\delta_0(z)$, as two mixed problems with smooth coefficients on the half-space, and boundary condition on the brane $z = 0$. If we introduce
\[
\Box (\partial_t, x, z) \Phi_\pm(t, x, z) := \Phi(t, x, z) \pm \Phi(t, x, -z),
\]
$\Phi$ is solution of (II.7) iff $\Phi_\pm$ are solutions of the boundary problems in $\mathbb{R}_t \times \mathbb{R}^3_x \times ]0, \infty[_z$:
\[
\Box(t, x, z) \Phi_\pm + \frac{15}{4} \left(\frac{1}{1+z}\right)^2 \Phi_\pm = 0, \quad (t, x, z) \in \mathbb{R} \times \mathbb{R}^3_x \times ]0, \infty[.\]
We shall see that in this case, (II.33) is true for any $a, b \in R$. 

The homogeneous Dirichlet problem for $\Phi_-$ is trivial, but the Robin problem (with the “bad” sign) for $\Phi_+$, that is the part of the wave that is physically pertinent, needs a careful analysis, similar to the previous one, of the quadratic form:

$$\int_{R^3_x} \int_0^\infty |\nabla_x \Phi_+(x, z)|^2 + |\partial_z \Phi_+(x, z)|^2 + \frac{15}{4} \left( \frac{1}{1 + |z|} \right)^2 |\Phi_+(x, z)|^2 dz dx - \frac{3}{2} \int_{R^3} |\Phi_+(x, 0)|^2 dx.$$

This approach can be useful for the numerical experiments (e.g. [34]).

We now present some classes of solutions. First of all we note that the equation is invariant with respect to the transform $z \rightarrow -z$, therefore if $\Phi$ is solution, then $\Phi_+$ and $\Phi_-$ defined by (II.28) are also solutions, respectively called $z$-even wave and $z$-odd wave. In particular we have $\Phi_-(t, x, 0) = 0$, hence $\Phi_-$ is a wave, equal to zero on the brane, that is solution of

$$\Box_{t,x} \Phi_- + \frac{15}{4} \left( \frac{1}{1 + |z|} \right)^2 \Phi_- = 0, \quad (t, x, z) \in R \times R^3 \times R.$$

This equation is a smooth perturbation of the D’Alembertian with a non-decreasing potential with cartesian anisotropy (see [33]). Thanks to (II.9) and the Hardy inequality (II.6), the $W^1(R^4)$-norm is equivalent to the $BL^1(R^4)$-norm on the subspace of the $z$-odd functions, and Theorem II.2 assures that the Cauchy problem for (II.32) is well posed in $BL^1(R^4) \times L^2(R^4)$. We shall see that its solutions are asymptotically free.

Much more interesting is the even part $\Phi_+$ of $\Phi$. We can very simply perform a large family of waves that are confined in the vicinity of the brane. For any $\varphi_0 \in BL^1(R^3_x)$, $\varphi_1 \in L^2(R^3_x)$, we put

$$\Phi(t, x, z) = \phi_0(t, x) f_0(z),$$

with $f_0(z) := (1 + |z|)^{-\frac{4}{3}}$, and $\phi_0 \in C^0(R; BL^1(R^3_x))$, $\partial_t \phi_0 \in C^0(R; L^2(R^3_x))$ is solution of the wave equation on the Minkowski brane:

$$\partial^2_t \phi_0 - \Delta_x \phi_0 = 0, \quad \phi_0(0, x) = \varphi_0(x), \quad \partial_t \phi_0(0, x) = \varphi_1(x).$$

We call massless graviton wave any solution of (II.1), (II.2) with initial data

$$\Phi_0 \in BL^1(R^3_x) \otimes C f_0(z), \quad \Phi_1 \in L^2(R^3_x) \otimes \mathbb{C} f_0(z).$$

Its energy is obviously localized near the brane on which it is propagating: for all $a < 0$, $b > 0$, there exists $C(a, b) > 0$ such that

$$\forall t \in R, \quad \| \nabla_x \Phi(t) \|^2_{L^2(R^3_x \times [a, b])} + \| \partial_t \Phi(t) \|^2_{L^2(R^3_x \times [a, b])} = C(a, b) \left( \| \phi_0 \|^2_{BL^1(R^3)} + \| \phi_1 \|^2_{L^2(R^3)} \right).$$

In opposite, a Kaluza-Klein wave is a solution for which there exists $a < 0$, $b > 0$ such that

$$\lim_{|t| \to \infty} \| \nabla_x \Phi(t) \|^2_{L^2(R^3_x \times [a, b])} + \| \partial_t \Phi(t) \|^2_{L^2(R^3_x \times [a, b])} = 0.$$

We shall see that in this case, (II.33) is true for any $a, b$ and the Kaluza-Klein waves are asymptotic to a free wave of the Minkowski space time $R^{1+4}$. The main results of the next part state that any finite energy solution, is the sum of a massless graviton and a Kaluza-Klein wave. Since the unitary group $e^{itA}$ leaves invariant $(BL^1 \otimes C f_0) \times (L^2 \otimes C f_0)$, we have to study its action on the orthogonal of this subspace. We introduce:

$$\mathcal{A}^1(R^4) := (BL^1(R^3_x) \otimes C f_0(z))^{\perp_{L^2}}, \quad \mathcal{A}^0(R^4) := (L^2(R^3_x) \otimes C f_0(z))^{\perp_{L^2}}.$$
that
\[ (\text{II.36}) \quad \varphi(x), f_0(z), f >_{\mathbb{M}} 0. \]

Since we have
\[ < \varphi(x)f_0(z), f >_{\mathbb{M}} = - < \Delta x \int_{-\infty}^{\infty} f(., z)f_0(z)dz, \varphi >_{\mathcal{D}'(\mathbb{R}^d), \mathcal{D}(\mathbb{R}^d)}, \]
and \( f(., z)f_0(z)dz \) is an integral absolutely converging in \( BL^1(\mathbb{R}^3_x) \), we conclude that
\[ (\text{II.35}) \quad \forall f \in \mathbb{W}^1(\mathbb{R}^4), \ f \in \mathbb{R}^1(\mathbb{R}^4) \Leftrightarrow \int_{-\infty}^{\infty} f(., z)f_0(z)dz = 0. \]

It is obvious that the \( z \)-odd functions belongs to \( \mathbb{R}^0 \) but we are mainly interested by the \( z \)-even functions that are associated with the Kaluza-Klein waves.

Finally we can solve the inhomogeneous Cauchy problem describing the propagation of the gravitational perturbations due to a source \( S(t, x) \), localized on the brane.

**Theorem II.4.** Given \( \Phi_0 \in \mathbb{W}^1(\mathbb{R}^4), \ \Phi_1 \in L^2(\mathbb{R}^4), \ S \in C^0(\mathbb{R}^d; H^1(\mathbb{R}^3_x)) \cap C^1(\mathbb{R}^d; L^2(\mathbb{R}^3_x)) \) such that \( (\partial_t^2 - \Delta_x)S \in C^0(\mathbb{R}^d; L^2(\mathbb{R}^3_x)) \), there exists a unique solution \( \Phi \in \Xi \) of the Cauchy problem
\[ (\text{II.36}) \quad \left( \square_{t,x,z} + \frac{15}{4} \left( \frac{1}{1 + |z|} \right)^2 - 3\delta_0(z) \right) \Phi = S(t, x) \otimes \delta_0(z), \ (t, x, z) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}, \]
\[ (\text{II.37}) \quad \Phi(0, x, z) = \Phi_0(x, z), \ \partial_t \Phi(0, x, z) = \Phi_1(x, z), \ (x, z) \in \mathbb{R}^3 \times \mathbb{R}. \]

**Proof of Theorem II.4.** We take \( \theta \in C^\infty(\mathbb{R}_x) \) such that \( \theta(0) = -\frac{1}{4} \). Then \( \Phi \) is solution iff
\[ \Psi(t, x, z) := \Phi(t, x) - S(t, x) \otimes \theta(z) \]
is solution of
\[ \left( \square_{t,x,z} + \frac{15}{4} \left( \frac{1}{1 + |z|} \right)^2 - 3\delta_0(z) \right) \Psi = - (\partial_t^2 - \Delta_x)S \otimes \theta - S \otimes \left[ -\theta'' + \frac{15}{4} \left( \frac{1}{1 + |z|} \right)^2 \theta \right] := F, \]
\[ \Psi(0, x, z) = \Phi_0(x, z) - S(0, x)\theta(z) := \Psi_0(x, z), \ \partial_t \Psi(0, x, z) = \Phi_1(x, z) - \partial_t S(0, x)\theta(z) := \Psi_1(x, z). \]
Since \( \Psi_0 \in \mathbb{W}^1(\mathbb{R}^4), \ \Psi_1 \in L^2(\mathbb{R}^4) \) and \( F \in C^0(\mathbb{R}^d; L^2(\mathbb{R}^4)) \), we get the solution of this Cauchy problem easily with the Duhamel formula.

\[ Q.E.D. \]

### III. Spectral expansion.

In this part we prove that any finite energy solution is the sum of a massless graviton and a Kaluza-Klein tower. The tool is the spectral decomposition of \( H_0 \). We also compute the kernel of the truncated (in brane energy) resolvent.

**Theorem III.1.** There exists \( f_0^\pm(z) \in C^0([0, \infty[, \mathbb{R} \times \mathbb{R}_z) \) with \( f_0^\pm(z) = 0 \), such that for any \( \Phi_0 \in \mathbb{W}^1(\mathbb{R}^4), \ \Phi_1 \in L^2(\mathbb{R}^4) \), there exists \( \phi_0 \in C^0(\mathbb{R}^d; BL^1(\mathbb{R}^3_x)) \) with \( \partial_t \phi_0 \in C^0(\mathbb{R}^d; L^2(\mathbb{R}^3_x)) \), and for almost all \( m > 0 \), \( \phi_m \in C^0(\mathbb{R}^d; H^1(\mathbb{R}^3_x)) \cap C^1(\mathbb{R}^d; L^2(\mathbb{R}^3_x)) \) such that
\[ \partial_t^2 \phi_0 - \Delta_x \phi_0 = 0, \ \partial_t^2 \phi_m - \Delta_x \phi_m^\pm + m^2 \phi_m^\pm = 0, \]
\[ (\text{III.1}) \quad \int_0^\infty \| \nabla_{t,x} \phi_m^\pm(t) \|_{L^2(\mathbb{R}^3)}^2 + m^2 \| \phi_m^\pm(t) \|_{L^2(\mathbb{R}^3)}^2 dm < \infty, \]
and the solution \( \Phi \in \Xi \) of the Cauchy problem \( (\text{II.1}), (\text{II.2}) \), can be written as
\[ (\text{III.2}) \quad \Phi(t, x, z) = \phi_0(t, x)f_0(z) + \sum_{\pm} \lim_{M \to \infty} \int_0^M \phi_m^\pm(t, x)f_m^\pm(z)dm, \]
where the limit holds in $\Xi$. Moreover we have
\begin{equation}
(\text{III.3})
\| \Phi_0 \|_{L^2(\mathbb{R}^4)}^2 + \| \Phi_1 \|_{L^2(\mathbb{R}^4)}^2 = \| \nabla_{t,x} \phi_0(t) \|_{L^2(\mathbb{R}^3)}^2 + 2 \sum_{m \pm} \int_0^\infty \| \nabla_{t,x} \phi_m^\pm(t) \|_{L^2(\mathbb{R}^3)}^2 + m^2 \| \phi_m^\pm(t) \|_{L^2(\mathbb{R}^3)}^2 \, dm.
\end{equation}

$\phi_0(t,x)f_0(z)$ and $(\phi_m^\pm(t,x)f_m^\pm(z))_{0 < m}$ are solutions of equation (II.1). The first one is the massless graviton, its energy is finite, and the second one is called the tower of massive Kaluza-Klein modes. Since $f_m^\pm \notin L^2(\mathbb{R}^4)$, the energy of these modes is infinite.

**Proof of Theorem (III.1).** First we develop the spectral analysis of the one-dimensional operator
\begin{equation}
(\text{III.4})
\mathbf{h} := -\frac{d^2}{dz^2} + \frac{15}{4} \left( \frac{1}{1+|z|} \right)^2 - 3\delta_0(z), \quad \mathcal{D}(\mathbf{h}) := \{ u \in H^1(\mathbb{R}); \mathbf{h}u \in L^2(\mathbb{R}) \},
\end{equation}
in particular, we determine its pure point spectrum $\sigma_{pp}(\mathbf{h})$ and its absolutely continuous spectrum $\sigma_{ac}(\mathbf{h})$.

**Lemma III.2.** $(\mathbf{h}, \mathcal{D}(\mathbf{h}))$ is a self-adjoint positive operator on $L^2(\mathbb{R})$, and we have :
\begin{equation}
(\text{III.5})
\sigma_{pp}(\mathbf{h}) = \{ 0 \}, \quad \text{Ker}(\mathbf{h}) = \mathbb{C}f_0,
\end{equation}
\begin{equation}
(\text{III.6})
\sigma_{ac}(\mathbf{h}) = [0, \infty[.
\end{equation}

**Proof of Lemma (III.2).** We use the quadratic form (II.15) with $X = \mathbb{C}$. For $f, g \in \mathcal{D}(\mathbf{h})$, an integration by part and (II.16) give :
\begin{equation}
<\mathbf{h}f, g >_{L^2} = q(f,g), \quad <\mathbf{h}f, f >_{L^2} \geq 0,
\end{equation}
thus $\mathbf{h}$ is symmetric and positive. Now given $g \in L^2(\mathbb{R})$, $f_\pm \in \mathcal{D}(\mathbf{h})$ is solution of $\mathbf{h}f_\pm \pm if_\pm = g$ iff
\begin{equation}
(\text{II.17})
\forall v \in H^1(\mathbb{R}), \quad a_\pm(f_\pm, v) := q(f_\pm, v) \pm i < f_\pm, v >_{L^2} = < g, v >_{L^2}.
\end{equation}
(II.17) implies that $a_\pm$ is continuous and coercive on $H^1(\mathbb{R})$, hence this problem has a unique solution and $\mathbf{h}$ is self-adjoint. Moreover, since $q$ is a closed form, and its form domain is $H^1(\mathbb{R})$, a classical result assures that $\mathcal{D}(\mathbf{h}^{1/2}) = H^1(\mathbb{R})$. To investigate the spectrum, we solve for $m \geq 0$ the equation
\begin{equation}
\mathbf{h}u_m = m^2 u_m, \quad u_m \in \mathcal{D}'(\mathbb{R}).
\end{equation}
When $m = 0$, an explicit calculation with the jump formula gives
\begin{equation}
u_0(z) = \lambda(1+|z|)^{-\frac{3}{2}} + \mu \frac{z}{|z|} \left[ (1+|z|)^{\frac{3}{2}} - (1+|z|)^{-\frac{3}{2}} \right], \quad \lambda, \mu \in \mathbb{C}.
\end{equation}
For $0 < m$, we use the Bessel equation satisfied by the Hankel functions $H_2^{(1)}$, $H_2^{(2)}$ :
\begin{equation}
x^2 u'' + xu' + (x^2 - 4)u = 0, \quad x \in \mathbb{R},
\end{equation}
to check that
\begin{equation}
\pm z > 0 \Rightarrow u_m(z) = \lambda_+^{(1)} \sqrt{1+|z|} H_2^{(1)}(m(1+|z|)) + \lambda_+^{(2)} \sqrt{1+|z|} H_2^{(2)}(m(1+|z|)), \quad \lambda_+^{(j)} \in \mathbb{C}.
\end{equation}
Since we have the asymptotics (RS, 7.2)
\begin{equation}
(\text{III.7})
H_2^{(1)}(x) \sim -\sqrt{\frac{2}{\pi x}} e^{i(x-\frac{\pi}{4})}, \quad H_2^{(2)}(x) \sim -\sqrt{\frac{2}{\pi x}} e^{-i(x-\frac{\pi}{4})}, \quad x \to \infty,
\end{equation}
we conclude that 0 is the only eigenvalue and \( u_0 \in \text{Ker}(h) \) iff \( \mu = 0 \).

To investigate the continuous part of the spectrum of \( h \), it is convenient to split the functions into even and odd parts like in Remark II.3 by putting
\[
\sqrt{2} u_\pm(z) = u(z) \pm u(-z), \quad Pu := (u_+, u_-)
\]
and the map \( P : u \mapsto Pu \) is an isometry from \( L^2(\mathbb{R}) \) onto \( L^2(\mathbb{R}^+) \times L^2(\mathbb{R}^+) \). Since \( u \in \mathcal{D}(h) \) iff \( u \in H^1(\mathbb{R}), u'_{\mid \mathbb{R}^+} \in L^2(\mathbb{R}^+), u'(0^+) - u'(0^-) + 3u(0) = 0 \), we introduce the self-adjoint operators \((h_\pm, \mathcal{D}(h_\pm))\), on \( L^2(\mathbb{R}^+) \) defined by :
\[
h_\pm := -\frac{d^2}{dz^2} + \frac{15}{4} \left( \frac{1}{1 + z} \right)^2,
\]
\[
\mathcal{D}(h_+) := \left\{ u_+ \in H^2(\mathbb{R}^+); u'_+(0) + \frac{3}{2} u_+(0) = 0 \right\}, \quad \mathcal{D}(h_-) := \left\{ u_- \in H^2(\mathbb{R}^+); u_-(0) = 0 \right\}.
\]
We have
(III.8) \quad Phu = (h_+ u_+, h_- u_-),
due to the operator \((h, \mathcal{D}(h))\) is unitarily equivalent with the selfadjoint operator \((h_+, h_-)\) on \( L^2(\mathbb{R}^+) \times L^2(\mathbb{R}^+) \). As previous, we can easily show that
\[
\sigma_{pp}(h_+) = \{0\}, \quad \sigma_{pp}(h_-) = \emptyset,
\]
\[
\mathcal{D} \left( (h_+)^{\frac{1}{2}} \right) = H^1(\mathbb{R}^+), \quad \mathcal{D} \left( (h_-)^{\frac{1}{2}} \right) = H^1_0(\mathbb{R}^+).
\]
Since \( \sigma_{ac}(h) = \sigma_{ac}(h_+) \cap \sigma_{ac}(h_-) \), the proof of (III.6) is reduced to investigating the spectrum of \( h_\pm \). To establish that \( \sigma_{ac}(h_\pm) = [0, \infty[ \), we use a nice result on the Schrödinger operator (Theorem 7.3 of [27]) : it sufficient to prove that for any \( m > 0 \) and all solutions \( u_1, u_2 \neq 0 \) of
(III.9) \quad -u'' + \frac{15}{4} \left( \frac{1}{1 + z} \right)^2 u = m^2 u, \quad z > 0,
we have
\[
0 < \liminf_{b \to \infty} \int_0^b \frac{|u_1(z)|^2}{|u_2(z)|^2} dz.
\]
Since the solutions \( u \neq 0 \) of (III.9) have the form
(III.10) \quad u(z) = \lambda^{(1)}(u) \sqrt{1 + z} \mathcal{H}_2^{(1)}(m(1 + z)) + \lambda^{(2)}(u) \sqrt{1 + z} \mathcal{H}_2^{(2)}(m(1 + z)), \quad \lambda^{(j)} \in \mathbb{C},
with \( |\lambda^{(1)}| + |\lambda^{(2)}| \neq 0 \), we deduce from the asymptotics (III.7) that
\[
\int_0^b |u(z)|^2 dz \sim \frac{2k}{\pi m} \left( |\lambda^{(1)}(u)|^2 + |\lambda^{(2)}(u)|^2 \right) b, \quad b \to \infty,
\]
therefore
\[
\liminf_{b \to \infty} \int_0^b \frac{|u_1(z)|^2}{|u_2(z)|^2} dz = \frac{|\lambda^{(1)}(u_1)|^2 + |\lambda^{(2)}(u_1)|^2}{|\lambda^{(1)}(u_2)|^2 + |\lambda^{(2)}(u_2)|^2} > 0.
\]
Q.E.D.

We now construct the explicit spectral representations of \( h_\pm \). We denote \( L^2_{ac} \) the absolutely continuous sub-space of \( h_+ \),
\[
L^2_{ac} := \left\{ f \in L^2(\mathbb{R}^+), \int_0^\infty f(z)f_0(z)dz = 0 \right\}.
\]
We introduce the functions \( u_\pm(z, m) \) defined for \( 0 < m, 0 < z \) by:
moreover there exists $C > 0$ such that for all $z \geq 0$, $m \geq M > 0$

(III.16) \[ \sup_{0 \leq z < \infty} \left| u_+(z, m) + \sqrt{2 \pi} \cos(mz) \right| + \left| u_-(z, m) - \sqrt{2 \pi} \sin(mz) \right| \leq \frac{C}{m}, \]

(III.17) \[ \sup_{M \leq m < \infty} \left| u_+^{-1}(z, m) - \sqrt{2 \pi} \sin \left( mz + m - \frac{5\pi}{4} - \arg H_1^{(1)[2]}(m) \right) \right| \leq \frac{C}{Mz}, \]

Proof of Lemma III.3. Since the Hankel functions have no real zero, $u_\pm$ are well defined in $C^\infty([0, \infty]\times[0, \infty])$. To get the asymptotic behaviours at low energy, it is convenient to express $u_\pm$ in terms of Bessel and Neuman functions. We have:

$$u_+(z, m) = \sqrt{m(1+z)} J_1(m) N_2(m(1+z)) - N_1(m) J_2(m(1+z)) e^{i\theta_1(m)},$$

$$u_-(z, m) = \sqrt{m(1+z)} J_2(m) N_2(m(1+z)) - N_2(m) J_2(m(1+z)) e^{i\theta_2(m)}.$$

We recall that the Bessel functions are entiere functions (38, 3.1):

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+\nu)!} \left( \frac{x}{2} \right)^{\nu+2n},$$

and we get from the Neumann expansion of the Bessel functions of second kind (25, 3.1), that the Neuman functions are analytic functions defined on the Riemann surface of the logarithm with the following asymptotics near zero:

$$N_1(x) + \frac{2}{\pi x} - \frac{x}{\pi} \log(x) = O(x) \in C^2([0, \infty], x \to 0^+),$$

$$N_2(x) + \frac{4}{\pi x^2} + \frac{1}{\pi} - \frac{x^2}{4\pi} \log(x) = O(x^2) \in C^2([0, \infty], x \to 0^+).$$

Elementary calculations lead to:

(III.18) \[ u_+(z, m) + \sqrt{m(1+z)}^{-\frac{3}{2}} = O \left( m^\frac{5}{2}(1+z)^\frac{5}{2} \right) \in C^2([0, \infty]\times[0, \infty], m \to 0, \]
Replacing the Hankel functions by these asymptotics in (III.11) and (III.12), we obtain the appropriate boundary condition at $z$ that is a solution satisfying

$$u_+(z, m) = \sqrt{\frac{2}{\pi}} \cos(mz) + O\left(\frac{1}{m}\right), \quad m \to \infty,$$

$$u_-(z, m) = \sqrt{\frac{2}{\pi}} \sin(mz) + O\left(\frac{1}{m}\right), \quad m \to \infty,$$

(III.20)

$$\begin{cases}
  u_+(z, m) = \sqrt{\frac{2}{\pi}} \sin\left(mz + m - \frac{5\pi}{4} - \arg H_1^{(1)}(m) \right) + O\left(\frac{1}{m(1+z)}\right), \quad z \to \infty, \\
  u_-(z, m) = \sqrt{\frac{2}{\pi}} \sin\left(mz + m - \frac{5\pi}{4} - \arg H_2^{(1)}(m) \right) + O\left(\frac{1}{m(1+z)}\right), \quad z \to \infty.
\end{cases}$$

Q.E.D.

We are now ready to introduce the distorted Fourier transforms associated with $h_\pm$. For $f \in C^0_0([0, \infty], \tilde{f} \in C^0_0([0, \infty], m)$, we put

$$0 < m, \quad F_{\pm}(f)(m) := \int_0^\infty f(z) u_{\pm}(z, m) dz,$$

$$0 < z, \quad \tilde{F}_{\pm}(\tilde{f})(z) := \int_0^\infty \tilde{f}(m) u_{\pm}(z, m) dm.$$

**Lemma III.4.** $F_+$ (respectively $F_-$) can be extended into an isometry from $L^2_0$ (respectively from $L^2(\mathbb{R}_+^2)$) onto $L^2(\mathbb{R}_+^2)$ and $\tilde{F}_\pm = F_{\pm}^{-1}$. $h_\pm$ is implemented by the operator of multiplication by $m^2$

(III.21)

$$f \in \mathcal{D}(h_\pm), \quad F_{\pm} (h_\pm f)(m) = m^2 F_{\pm}(f)(m).$$

**Proof of Lemma III.4.** We construct the spectral representation by the usual way (see e.g. [27] Theorem 7.4). First, we have to find a normalized $\varphi(z, \lambda)$ upper solution of the Schrödinger equation

$$-\partial_z^2 \varphi + \frac{15}{4} \left(\frac{1}{1 + z}\right)^2 \varphi = \lambda \varphi, \quad z > 0,$$

that is a solution satisfying

for almost $\lambda > 0$, $\varphi(z, \lambda) = \lim_{\varepsilon \to 0^+} \varphi(z, \lambda + i\varepsilon)$ in $L^2_{loc}(\mathbb{R}_+^2)$,

with

$$\forall \varepsilon > 0, \quad \varphi(z, \lambda + i\varepsilon) \in L^2(\mathbb{R}_+^2),$$

and

$$\varphi \partial_z \varphi - \varphi \partial_z \varphi = -i.$$

From (III.7) and (III.10), we obtain

$$0 < \lambda, \quad 0 < z, \quad \varphi(z, \lambda) = \frac{1}{2} \sqrt{\pi} \sqrt{1 + z} H_2^{(1)}(\sqrt{\lambda}(1 + z)).$$

The next step consists in finding a real solution $v_{\pm}(z, \lambda)$ of the Schrödinger equation, satisfying the appropriate boundary condition at $z = 0$, i.e. $\partial_z v_+(0, \lambda) + \frac{i}{\lambda} v_+(0, \lambda) = 0, v_-(0, \lambda) = 0,$ and
normalized to have spectral amplitude $A := 2 \mid v_\pm \partial_x v_\pm - \varphi \partial_z v_\pm \mid = 1$. Up to the sign, this solution is unique, and a tedious computation based on (III.10) leads to

$$v_-(z, \lambda) := \frac{1}{4} \sqrt{\pi(1 + z)} \left[ \frac{H_2^{(2)}(\sqrt{\lambda})}{H_2^{(1)}(\sqrt{\lambda})} H_2^{(1)}(\sqrt{\lambda}(1 + z)) - H_2^{(2)}(\sqrt{\lambda}(1 + z)) \right] e^{i \theta_2(\sqrt{\lambda})},$$

$$v_+(z, \lambda) := \frac{1}{4} \sqrt{\pi(1 + z)} \left[ \frac{2H_2^{(2)}(\sqrt{\lambda})}{2H_2^{(1)}(\sqrt{\lambda})} + \frac{\sqrt{\lambda}(H_2^{(2)}')}{\sqrt{\lambda}(H_2^{(1)}')} \right] H_2^{(1)}(\sqrt{\lambda}(1 + z)) - H_2^{(2)}(\sqrt{\lambda}(1 + z)) \right] e^{i \theta_1(\sqrt{\lambda})},$$

where $\theta_j$ is given by (III.13). Finally we put $u_\pm(z, m) := 2 \sqrt{\frac{\pi}{\lambda}} v_\pm(z, m^2)$, and we get (III.12), and also (III.11) by using the recurrence relation

$$2H_2^{(j)}(x) + x \frac{d}{dx} H_2^{(j)}(x) = xH_1^{(j)}(x).$$

Then $F_\pm$ are unitary representations that satisfy (III.21).

Q.E.D.

Now we are ready to prove the theorem. We define

$$\varphi^M_j(x) := \int_{-\infty}^{\infty} \Phi_j(x, \zeta) f_0(\zeta) d\zeta, \quad \Phi^M_j(x, z) := \varphi^M_j(x)f_0(z), \quad \Phi^B_j := \Phi_j - \Phi^M_j.$$

The solution $\Phi$ of the Cauchy problem (III.1), (III.2), (III.26) can be written as

$$\Phi = \Phi^M + \Phi^B, \quad \Phi^M(t, x, z) = \phi_0(t, x)f_0(z),$$

where $\phi_0 \in C^0(\mathbb{R}_t; BL^1(\mathbb{R}^3_x))$ with $\partial_t \phi_0 \in C^0(\mathbb{R}_t; L^2(\mathbb{R}^3_x))$, is solution of

$$\partial^2_t \phi_0 - \Delta_x \phi_0 = 0, \quad \phi_0(0, x) = \varphi^M_0(x), \quad \partial_t \phi_0(0, x) = \varphi^M_1(x),$$

and $\Phi^B \in C^0(\mathbb{R}_t; \mathbb{R}^4)$ with $\partial_t \Phi^B \in C^0(\mathbb{R}_t; \mathbb{R}^0(\mathbb{R}^4))$, is solution of (III.11) with initial data

$$\Phi^B(0, x, z) = \Phi^B_0(x, z), \quad \partial_t \Phi^B(0, x, z) = \Phi^B_1(x, z).$$

If we put

$$2\Phi_\pm(t, x, z) := \Phi^B(t, x, z) \pm \Phi^B(t, x, -z),$$

we have

$$\Phi^B(t, x, z) = \Phi_+(t, x, |z|) + \frac{z}{|z|} \Phi_-(t, x, |z|).$$

We note that $\Phi_+(t, \cdot)$ and $\Phi_-(t, \cdot)$ are orthogonal in $\mathbb{R}^1(\mathbb{R}^4)$ and $L^2(\mathbb{R}^4)$, and

$$\| \Phi^B_0 \|^2_{\mathbb{R}^1(\mathbb{R}^4)} + \| \Phi^B_1 \|^2_{L^2(\mathbb{R}^4)} = 2 \sum_{\pm} \int_{-\infty}^{\infty} \int_{\mathbb{R}^3_x} |\nabla_x \Phi_\pm|^2 + \left| \partial_z \Phi_\pm + \frac{3}{2} \frac{z}{|z|} \frac{1}{1 + z^2} \Phi_\pm \right|^2 dz dx.$$ 

Given $t \in \mathbb{R}$, for almost $x \in \mathbb{R}^3_x$, the maps $z \mapsto \nabla_x \Phi_-(t, x, z)$ and $z \mapsto \nabla_x \Phi_+(t, x, z)$ respectively belong to $L^2(\mathbb{R}^3_z)$ and to $L^2_{ac}$. Thus we can apply Lemma (III.4) and for almost all $m > 0$ we introduce

$$\phi^\pm_\alpha(t, x, z) := \lim_{M \to \infty} \int_0^M \Phi_\pm(t, x, z) u_\pm(z, m) dz,$$

that is converging in $L^2(\mathbb{R}_t^+; BL^1(\mathbb{R}^3_x))$. Moreover, since $\Phi_\pm \in C^0(\mathbb{R}_t; \mathbb{R}^1(\mathbb{R}^4))$ with $\partial_t \Phi_\pm \in C^0(\mathbb{R}_t; \mathbb{R}^0(\mathbb{R}^4))$, then

$$(III.23) \quad \phi^\pm_\alpha \in C^0(\mathbb{R}_t; L^2(\mathbb{R}_m^+; BL^1(\mathbb{R}^3_x))), \quad \partial_t \phi^\pm_\alpha \in C^0(\mathbb{R}_t; L^2(\mathbb{R}_m^+; L^2(\mathbb{R}^3_x))).$$
and

\[(\text{III.24})\quad \phi^\pm_m(0, x) = \frac{1}{2} \lim_{M \to \infty} \int_0^M (\Phi^B_0(x, z) \pm \Phi^B_m(x, -z)) u_\pm(z, m) dz \in L^2(\mathbb{R}^m_+; BL^1(\mathbb{R}^3_\chi)), \]

\[(\text{III.25})\quad \partial_t \phi^\pm_m(0, x) = \frac{1}{2} \lim_{M \to \infty} \int_0^M (\Phi^B_1(x, z) \pm \Phi^B_1(x, -z)) u_\pm(z, m) dz \in L^2(\mathbb{R}^m_+; L^2(\mathbb{R}^3_\chi)). \]

Therefore, if we introduce

\[(\text{III.26})\quad f^+_m(z) := u_+(|z|, m), \quad f^-_m(z) := \frac{z}{|z|} u_-(|z|, m), \]

Lemma \textbf{III.21} assures that

\[\nabla_{t, x} \Phi^\pm(t, x, z) = \lim_{M \to \infty} \int_0^M \nabla_{t, x} \phi^\pm_m(t, x) f^\pm_m(z) dm \quad \text{in} \quad C^0(\mathbb{R}^t_t; L^2(\mathbb{R}^4_{x, z})), \]

\[\int_0^\infty \int_{\mathbb{R}^3_\chi} |\nabla_{t, x} \Phi^\pm(t, x, z)|^2 \, dz \, dx = \int_0^\infty \int_{\mathbb{R}^3_\chi} |\nabla_{t, x} \phi^\pm_m(t, x, m)|^2 \, dm \, dx. \]

We remark that for \(u \in \mathcal{D}(h^-)\) we have

\[\int_0^\infty \left| f'(z) + \frac{3}{2} \frac{1}{1+z} f(z) \right|^2 \, dz = <h \pm f, f >_{L^2(\mathbb{R}^+)} = \int_0^\infty |F_{\pm}(f)|^2 \, m^2 \, dm. \]

This equality can be extended by a density to a norm in \(\mathcal{D}(h^-)\). We deduce that \(m\phi^\pm_m \in C^0(\mathbb{R}^t_t; L^2(\mathbb{R}^m_+; L^2(\mathbb{R}^3_\chi)))\) hence with \textbf{III.23}

\[(\text{III.27})\quad \forall \alpha > 0, \quad \phi^\pm_m \in C^0(\mathbb{R}^t_t; L^2([a, \infty)_m; H^1(\mathbb{R}^3_\chi))), \]

and

\[\int_0^\infty \int_{\mathbb{R}^3_\chi} \left| \partial_x \Phi^\pm(t, x, z) + \frac{3}{2} \frac{1}{1+z} \Phi^\pm(t, x, z) \right|^2 \, dz \, dx = \int_0^\infty \int_{\mathbb{R}^3_\chi} m^2 \left| \phi^\pm_m(t, x, m) \right|^2 \, dm \, dx, \]

therefore \textbf{III.1}, \textbf{III.2}, \textbf{III.3} are established.

It remains to prove that \(\phi^\pm_m\) is a finite energy solution of the Klein-Gordon equation for almost all \(m \neq 0\). Thanks to \textbf{III.3}, we see that the map \((\Phi_0, \Phi_1) \mapsto (\phi^\pm_m, \partial_t \phi^\pm_m)\) is continuous from \(\mathfrak{M}_1(\mathbb{R}^4) \times L^2(\mathbb{R}^4)\) to \(C^0(\mathbb{R}^t_t; L^2(\mathbb{R}^m_+; BL^1(\mathbb{R}^3_\chi))) \times C^0(\mathbb{R}^t_t; L^2(\mathbb{R}^m_+; L^2(\mathbb{R}^3_\chi)))\), hence it is sufficient to prove that

\[(\text{III.28})\quad (\partial_t^2 - \Delta_x + m^2)\phi^\pm_m = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^t_t \times \mathbb{R}^3_\chi \times [0, \infty)_m). \]

for a dense set of initial data. We choose \(\Phi_1 \in C^0(\mathbb{R}^4), \) and \(\Phi_0 \in \mathcal{D}(H_0),\) compactly supported in \(\{(x, z); \quad |x|, |z| \leq R\},\) and such that \(\Phi_0 \in C^0(\mathbb{R}^3 z; C^0(\mathbb{R}^3_\chi)).\) It is easy to see that the set of such data is dense in \(\mathfrak{M}_1(\mathbb{R}^4)\) : given \(\Phi_0 \in \mathcal{D}(H_0),\) we take \(\chi \in C^0(\mathbb{R}^4)\) equal to 1 on a neighborhood of \(0 \) and \(\theta \in C^0(\mathbb{R}^3_\chi), 0 \leq \theta, \int \theta(x) dx = 1.\) Then \(\Phi_{n,p}(x, z) := p^3 \int \theta(p(x - x')) \chi(x, \frac{z}{n}) \Phi_0(x', z) dx'\) belongs to \(\mathcal{D}(H_0) \cap \mathcal{E}'(\mathbb{R}^4) \cap C^0(\mathbb{R}_{x}; C^0(\mathbb{R}^3_\chi)),\) and \(\Phi_{n,p}\) tends to \(\Phi_0\) in \(H^1(\mathbb{R}^4)\) as \(n, p \to \infty.\) The solution of the Cauchy problem satisfies

\[\Phi \in C^2(\mathbb{R}^t_t; L^2(\mathbb{R}^4)) \cap C^1(\mathbb{R}^t_t; H^1(\mathbb{R}^4)), \]

and since \(\Delta_x\) and \(P(\partial)\) defined by \textbf{III.3} are commuting, \(\Delta_x \Phi\) is also a finite energy solution, compactly supported in space at each time, hence

\[\Delta_x \Phi \in C^0(\mathbb{R}^t_t; L^2(\mathbb{R}^4)) \cap C^1(\mathbb{R}^t_t; H^1(\mathbb{R}^4)). \]

This implies that

\[\Phi, \partial_t \Phi \in C^0(\mathbb{R}^t_t \times \mathbb{R}^3 z; H^\frac{2}{2}(\mathbb{R}^3_\chi) \subset C^0(\mathbb{R}^5)). \]
We deduce that for all \( t \in \mathbb{R}, x \in \mathbb{R}^3 \), the map \( z \mapsto \Phi(t, x, z) \) belongs to \( \mathcal{D}(h_\pm) \), and for \( |t| \leq T \)

\[
\phi^\pm_m(t, x) = \frac{1}{2} \int_0^{\infty} (\Phi(t, x, z) \pm \Phi(t, x, -z)) u_\pm(z, m) \, dm \in C^1([-T, T]; H^1(\mathbb{R}_x^3)),
\]

\[
(\partial_t^2 - \Delta_x + m^2)\phi^\pm_m = \int_0^{\infty} (\partial_t^2 + P(\partial)) (\Phi(t, x, z) \pm \Phi(t, x, -z)) u_\pm(z, m) \, dm = 0.
\]

It remains to prove that in the general case where \((\Phi_0, \Phi_1) \in \mathcal{W}^1(\mathbb{R}^4) \times L^2(\mathbb{R}^4), \phi^\pm_m \) belongs to \( C^0(\mathbb{R}_t; H^1(\mathbb{R}_x^3)) \cap C^1(\mathbb{R}_t; L^2(\mathbb{R}_x^3)) \) for almost all \( m > 0 \). We have established that for \( 0 < a, \phi^\pm_m \in C^0(\mathbb{R}_t; L^2([a, \infty[, H^1(\mathbb{R}_x^3)]) \cap C^1(\mathbb{R}_t; L^2([a, \infty[, L^2(\mathbb{R}_x^3)]) \) is solution of \( \text{(III.24)}, \text{(III.25)} \) and \( \text{(III.28)} \). We remark that this Cauchy problem is well posed in this functional framework because of the conservation of the energy:

\[
\int_0^{\infty} \int_{\mathbb{R}_x^3} |\nabla_x \phi^\pm_m(t, x)|^2 + m^2 |\phi^\pm_m(t, x)|^2 \, dx \, dt = Cst.
\]

This equality can be easily proved by the usual way from the Klein-Gordon equation when \( \phi^\pm_m \in C^1(\mathbb{R}_t; L^2([a, \infty[, H^1(\mathbb{R}_x^3)]) \cap C^2(\mathbb{R}_t; L^2([a, \infty[, L^2(\mathbb{R}_x^3)]) \). In the general case, we use \( \phi_\varepsilon(t, x) := \varepsilon^{-1} \int_{t-\varepsilon}^{t+\varepsilon} \phi^\pm_m(s, x) \, ds \) that belongs to this space and tends to \( \phi^\pm_m \) in \( C^0(\mathbb{R}_t; L^2([a, \infty[, H^1(\mathbb{R}_x^3)]) \cap C^1(\mathbb{R}_t; L^2([a, \infty[, L^2(\mathbb{R}_x^3)]) \) as \( \varepsilon \to 0 \). Now \( \text{(III.25)} \) and \( \text{(III.27)} \) assure that for almost \( m > 0 \), \( \phi^\pm_m(0, x) \in H^1(\mathbb{R}_x^3) \), \( \partial_t \phi^\pm_m(0, x) \in L^2(\mathbb{R}_x^3) \). For such an \( m \) we consider \( \psi^\pm_m \in C^0(\mathbb{R}_t; H^1(\mathbb{R}_x^3)) \cap C^1(\mathbb{R}_t; L^2(\mathbb{R}_x^3)) \) solution of the Klein-Gordon equation with initial data \( \psi^\pm_m(0, x) = \phi^\pm_m(0, x), \partial_t \psi^\pm_m(0, x) = \partial_t \phi^\pm_m(0, x) \). The energy estimate implies that

\[
\psi^\pm_m \in L^2([a, \infty[, C^0(\mathbb{R}_t; H^1(\mathbb{R}_x^3)) \cap C^1(\mathbb{R}_t; L^2(\mathbb{R}_x^3))).
\]

Since this space is included in \( C^0(\mathbb{R}_t; L^2([a, \infty[, H^1(\mathbb{R}_x^3)]) \cap C^1(\mathbb{R}_t; L^2([a, \infty[, L^2(\mathbb{R}_x^3)]) \), the uniqueness implies \( \psi^\pm_m = \phi^\pm_m \), therefore we have proved that \( \phi^\pm_m \) belongs to \( C^0(\mathbb{R}_t; H^1(\mathbb{R}_x^3)) \cap C^1(\mathbb{R}_t; L^2(\mathbb{R}_x^3)) \) for almost all \( m > 0 \).

Q.E.D.

We achieve this part devoted to the spectral analysis of \( H_0 \), by the computation of the kernel of its resolvent. Since \( H_0 \) is a positive self-adjoint operator, \((H_0 - \lambda^2)^{-1}\) is well defined in \( L^2(\mathbb{R}_4) \) for all \( \lambda \in \mathbb{C}^* \) with \( 0 < \arg \lambda < \pi \). If we add a cut-off in energy relatively to the brane, i.e. we consider \((H_0 - \lambda^2)^{-1} 1_{[0, \pi]}(|\nabla_x|) \), we can express the kernel explicitly.

We recall that the Hankel functions are holomorphic on the whole Riemann surface of the logarithm \( \sqrt{\pi} \). We introduce

\[
K(m; z, z') := \frac{\pi}{4i} \sqrt{(1 + z)(1 + z')} \left[ H^{(2)}_2(m) H^{(1)}_2(m(1 + z \wedge z')) - H^{(1)}_2(m) H^{(2)}_2(m(1 + z \wedge z')) \right] H^{(1)}_2(m(1 + z \vee z'))
\]

(III.29)

where \( z \wedge z' := \min(z, z'), z \vee z' := \max(z, z') \). For \( x \in \mathbb{C}^* \), we denote \( \sqrt{\pi} \) the branch of the square root defined by \( 0 \leq \arg \sqrt{\pi} < \pi \) when \( 0 \leq \arg x < 2\pi \).

**Theorem III.5.** For any \( R > 0, \lambda \in \mathbb{C}^*, 0 < \arg \lambda < \pi, F \in L^1 \cap L^2(\mathbb{R}^4), \) we have

\[
(H_0 - \lambda^2)^{-1} 1_{[0, \pi]}(|\nabla_x|) F(x, z) = \int_{\mathbb{R}^4} K_R(\lambda; x, z; x', z') F(x', z') \, dx' \, dz',
\]
where this integral converges absolutely and the kernel of the truncated resolvent is given by

\[ K_R(\lambda; x, z; x', z') := \int_0^R \frac{\sin(r | x - x'|)}{4\pi^2 | x - x'|} \left[ \frac{zz'}{|zz'|} \frac{1}{H_2^{(1)}(\sqrt{\lambda^2 - r^2})} - \frac{1}{H_1^{(1)}(\sqrt{\lambda^2 - r^2})} \right] K\left(\sqrt{\lambda^2 - r^2}; | z |, | z' | \right) rdr. \]

Proof of Theorem III.3 Given \( m \in \mathbb{C} \), \( 0 < \arg m < \pi \), we compute the kernel \( K_\pm(m; z, z') \) of the resolvents \( (H_\pm - m^2)^{-1} \) by the usual way (see [27], p. 262). First we determine \( f_j(m; z) \) the solutions of equation (III.9) with \( f_1(m; 0) = 0, \partial_z f_1(m; 0) = 1, f_2(m; 0) = 1, \partial_z f_2(m; 0) = 0 \). Using (III.10), the wronskian relation \( H_\nu^{(1)}(x) \frac{d}{dx} H_\nu^{(2)}(x) - H_\nu^{(2)}(x) \frac{d}{dx} H_\nu^{(1)}(x) = -\frac{\nu}{x^2} \), and the identity \( x \frac{d}{dx} H^{(j)}_2(x) + 2H^{(j)}_2(x) = xH^{(j)}_1(x) \), we find after tedious calculations:

\[
\begin{align*}
 f_1(m; z) &= \frac{\pi}{4i} \sqrt{1 + z} \left[ H_2^{(2)}(m) H_1^{(1)}(m(1 + z)) - H_2^{(1)}(m) H_2^{(2)}(m(1 + z)) \right], \\
 f_2(m; z) &= -\frac{\pi}{4i} \sqrt{1 + z} \left[ \left( m H_2^{(1)}(m) - \frac{3}{2} H_2^{(2)}(m) \right) H_1^{(1)}(m(1 + z)) \right. \\
 & \quad \left. - \left( m H_1^{(1)}(m) - \frac{3}{2} H_1^{(2)}(m) \right) H_2^{(2)}(m(1 + z)) \right].
\end{align*}
\]

Secondly we have to determine \( C(m) \in \mathbb{C} \) such that \( f_2(m; z) + C(m) f_1(m; z) \in L^2(\mathbb{R}_+^2) \). The bounds for the Hankel functions (25), P. 267 assure that for any \( \delta > 0 \) there exists \( C_\delta > 0 \) such that for all \( x \in \mathbb{C}^* \), with \( 0 < \arg x < \pi \) we have

\[ (III.30) \quad \forall x \in \mathbb{C}^*, \ 0 < \arg x < \pi, \quad |H_\nu^{(1)}(x)| \leq C_\delta |e^{ix}|, \quad |H_\nu^{(2)}(x)| \leq C_\delta |e^{-ix}|. \]

Since \( 0 < \arg m < \pi \), the function \( H_2^{(1)}(m(1 + z)) \) is exponentially decreasing as \( z \to \infty \), and \( H_2^{(2)}(m(1 + z)) \) is exponentially increasing as \( z \to \infty \). Thus we get

\[ C(m) = \frac{m H_2^{(1)}(m) - \frac{3}{2} H_2^{(2)}(m)}{H_1^{(1)}(m)}, \]

\[ f_2(m; z) + C(m) f_1(m; z) = \sqrt{1 + z} \frac{H_2^{(1)}(m(1 + z))}{H_1^{(1)}(m)}. \]

Finally we know that

\[
\begin{align*}
 K_-(m; z, z') &= f_1(m, z \land z') \left[ f_2(m; z \lor z') + C(m) f_1(m; z \lor z') \right], \\
 K_+(m; z, z') &= -f_1(m, z \land z') \left[ \frac{3}{2} f_2(m; z \lor z') + C(m) f_1(m; z \lor z') \right],
\end{align*}
\]

and so we conclude that

\[ (III.31) \quad K_+(m; z, z') = -\frac{K(m; z, z')}{H_1^{(1)}(m)}, \quad K_-(m; z, z') = \frac{K(m; z, z')}{H_1^{(1)}(m)}. \]

Now given \( \lambda \in \mathbb{C} \), \( 0 < \arg \lambda < \pi \), and \( F \in L^2(\mathbb{R}_\lambda^3 \times \mathbb{R}_z) \) we introduce \( F_\pm(x, z) := \frac{1}{2}(F(x, z) \pm F(x, z)) \), and we solve \( (H_0 - \lambda^2) \Phi_\pm = 1_{[0,R]}(\varphi_x) F_\pm \). Then the solution \( \Phi \) of \( (H_0 - \lambda^2) \Phi = 1_{[0,R]}(\varphi_x) F \) is given by \( \Phi(x, z) = \Phi_+(x, |z|) + \frac{1}{|z|} \Phi_-(x, |z|) \). We use the partial Fourier transform \( F_x f(\xi, z) = \hat{f}(\xi, z) \) of \( f(\cdot, z) \) with respect to \( x \) given for \( f \in C_0^\infty(\mathbb{R}^4) \) by

\[ (III.32) \quad \hat{f}(\xi, z) := \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{-ix\xi} f(x, z) dx. \]
Then for almost all $\xi \in \mathbb{R}^3$, the function $r \in \mathbb{R}^+ \mapsto \hat{\Phi}_\pm(\xi,\cdot)$ belongs to $\mathcal{D}(h_\pm)$, and
\[
(h_\pm + |\xi|^2 - \lambda^2)\hat{\Phi}_\pm(\xi, z) = 1_{[0,R]}(|\xi|)\hat{F}_\pm(\xi, z), \quad 0 < z.
\]

In the sequel, we assume that $F \in L^1 \cap L^2(\mathbb{R}^3)$, hence all the following integrals make sense. Taking advantage of the Hankel transform, we have for $0 < z$
\[
\Phi_\pm(x, z) = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \int_{|\xi| \leq R} e^{i\xi \cdot x} \left(\int_0^{\infty} K_\pm \left(\sqrt{\lambda^2 - |\xi|^2}; z, z'\right) \hat{F}_\pm(\xi, z') dz'\right) d\xi
\]
\[
= \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \int_{\mathbb{R}^3} \int_0^{\infty} \left[ \int_{|\xi| \leq R} e^{i(x-x') \cdot \xi} K_\pm \left(\sqrt{\lambda^2 - |\xi|^2}; z, z'\right) d\xi \right] F_\pm(x', z') dz' dx'
\]
\[
= \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \int_{\mathbb{R}^3} \int_0^{\infty} \frac{1}{|x-x'|} \left[ \int_0^{R} J_\frac{3}{2}(|x-x'| r) K_\pm \left(\sqrt{\lambda^2 - r^2}; z, z'\right) r^2 dr \right] F_\pm(x', z') dz' dx'.
\]

We conclude that for all $x \in \mathbb{R}^3, z \in \mathbb{R}$, we have:
\[
\Phi(x, z) = \int_{\mathbb{R}^4} K_R(\lambda; x, z; x', z') F(x', z') dx' dz',
\]
with
\[
K_R(\lambda; x, z; x', z') := \int_0^{R} \frac{\sin(r |x-x'|)}{4\pi^2 |x-x'|^2} \left[ K_+ \left(\sqrt{\lambda^2 - r^2}; |z|, |z'|\right) + \frac{zz'}{|z'|} K_- \left(\sqrt{\lambda^2 - r^2}; |z|, |z'|\right) \right] r dr.
\]

Now the result follows from (III.31) and (III.30) that assure that for any compact $A \subset \{\lambda \in \mathbb{C}^*; \ 0 < \arg \lambda < \pi\}$, and all $R, R_1 \in [0, \infty[$, there exist $C, \gamma > 0$ such that
\[
(III.33) \quad \sup_{x,x' \in \mathbb{R}^4} \sup_{|z| \leq R_1} \sup_{\lambda \in A} |K_R(\lambda; x, z; x', z')| \leq Ce^{-\gamma |z'|}.
\]

Q.E.D.

Finally we investigate the domain of analyticity of the continuation of the truncated resolvents (see Figure 3, section VI).

**Corollary III.6.** For any $R_j > 0$, the truncated resolvent
\[
1_{[0,R_j]}(|x| + |z|) \left(\mathbf{H}_0 - \lambda^2\right)^{-1} 1_{[0,R_0]}(|\nabla x|) 1_{[0,R_2]}(|x| + |z|)
\]
considered as a $\mathcal{L}(L^2(\mathbb{R}^4_{x,z}))$-valued function of $\lambda$, has an analytic continuation on
\[
\mathcal{O}_{R_0} := \left\{ \lambda \in \mathbb{C}^*; \ H^{(1)}_{\nu_0} \left(\sqrt{\lambda^2 - r^2}\right) \neq 0 \ \forall r \in [0, R_0], \ \nu = 1, 2 \right\}.
\]

**Proof of Corollary III.6.** By the Dunford theorem (39, p. 128), it is sufficient to show that given $F \in L^2(\mathbb{R}^4)$, the map
\[
\lambda \mapsto 1_{[0,R_j]}(|x| + |z|) \int_{\mathbb{R}^4} K_{R_0}(\lambda; x, z; x', z') 1_{[0,R_2]}(|x'| + |z'|) F(x', z') dx' dz'
\]
is a $L^2(\mathbb{R}^4_{x,z})$-valued, holomorphic function on $\mathcal{O}_{R_0}$. This property is easily proved with (III.33) by remarking that the map
\[
\lambda \mapsto 1_{[0,R_j]}(|x| + |z|) K_{R_0}(\lambda; x, z; x', z') 1_{[0,R_2]}(|x'| + |z'|)
\]
is a $L^\infty(\mathbb{R}^4_{x,z} \times \mathbb{R}^4_{x',z'})$-valued, holomorphic function on $\mathcal{O}_{R_0}$.

Q.E.D.
IV. Decay near the Brane and Strichartz estimates

It is clear that the massless graviton decays uniformly as \( t^{-1} \) for regular data, and belongs to \( L^4(\mathbb{R}^5) \) according to [35]. In this part we investigate the decay of the Kaluza-Klein tower. We know that for initial data in \( C_0^\infty(\mathbb{R}^4) \), the solutions of the D'Alembertian on the Minkowski space-time \( \mathbb{R}^{1+4} \) decay uniformly in space as \( t^{-\frac{3}{2}} \) (Von Wahl estimate [37]). In this section we first prove that the same estimate holds for the Kaluza-Klein tower near the brane. These results can appear to be surprising since, although these waves are superpositions of Klein-Gordon field on \( \mathbb{R}^{1+3} \) for which this decay holds, there is a continuum of mass on \([0, \infty[\), without gap to separate the zero mass. The key of the phenomenon is the behaviour at zero of the spectral kernels \( f_m^\pm(z) = O(\sqrt{m}) \). Since the Von Wahl estimates involve the norms of the initial data in some Sobolev spaces based on \( L^1 \), we have to construct the functional framework adapted to our problem. We introduce for \( N = 0, 1 \) and \( \varepsilon > 0 \):

\[
\mathcal{X}_\varepsilon^N := \left\{ \Phi \in \mathbb{R}^0 \cap \mathcal{D} \left( H_0^{N+1} \right) : | \alpha | \leq 2 + N, 1 \leq l \leq 3 + N - | \alpha | \right\} (IV.1)
\]

where the derivatives with respect to \( z \) are taken in the sense of the distributions in \( \mathbb{R}_x^3 \times \mathbb{R}_z^3 \). This space is endowed with its natural norm \( \| \Phi \|_{L^2(\mathbb{R}^4)} + \| H_0^{N+1} \Phi \|_{L^2(\mathbb{R}^4)} + \| \| (1 + |z|)^{\varepsilon} \partial_x^\alpha \Phi \|_{L^1(\mathbb{R}^4)} \right\} + \sum \| \partial_x^\alpha \partial_z^\beta \Phi \|_{L^1(\mathbb{R}_x^3 \times \mathbb{R}_z^3)} \right\}.

Theorem IV.1. For any \( R > 0 \) there exists \( C_R > 0 \) such that for any \( \varepsilon \in [0, \frac{1}{2}] \), and for all \( \Phi_0 \in \mathcal{X}_\varepsilon^1 \), \( \Phi_1 \in \mathcal{X}_\varepsilon^0 \), the solution \( \Phi \) of the Cauchy problem (II.1), (II.2), (II.26) satisfies:

\[
\| \Phi(t,.) \|_{L^\infty(\mathbb{R}_x^3 \times [-R,R]^3)} \leq C_R \left[ \frac{1}{\varepsilon} \sum_{|\alpha|+j \leq 3} \left( 1 + |z| \right)^{\varepsilon} \partial_x^\alpha \Phi_j \right] .
\]

The first term of the right-hand side of this estimate, \( \frac{1}{\varepsilon} \sum_{|\alpha|+j \leq 3} \left( 1 + |z| \right)^{\varepsilon} \partial_x^\alpha \Phi_j \), controls the contribution of the light Kaluza-Klein modes with the mass \( m \in [0,1] \), while the second one, \( \sum_{|\alpha|+j \leq 3} \sum_{l=1}^{4-|\alpha|-j} \| \partial_x^\alpha \partial_z^j \Phi_j \|_{L^1(\mathbb{R}_x^3 \times \mathbb{R}_z^3)} \), is due to the heavy Kaluza-Klein modes with the mass \( m \geq 1 \).

Proof of Theorem IV.1. According to Theorem III.1 we write the solution as

\[
\Phi(t, x, z) = \lim_{M \to -\infty} \Phi_M(t, x, z), \quad \Phi_M(t, x, z) := \int_0^M \phi_m^\pm(t, x) f_m^\pm(z) \, dm,
\]

where for \( m > 0 \), \( \phi_m^\pm \) is solution of the Klein-Gordon equation:

\[
\partial_t^2 \phi_m^\pm - \Delta_x \phi_m^\pm + m^2 \phi_m^\pm = 0, \quad \phi_m^\pm(0, x) = \phi_0^m(0, x), \quad \partial_t \phi_m^\pm(0, x) = \phi_1^m(0, x).
\]

Then \( v(t, x) := \frac{1}{m} \phi_m^\pm(t/m, x/m) \) is solution of \( \partial_t^2 v - \Delta_x v + v = 0, \quad v(0, x) = v_0(x) := \phi_0^m(x/m), \quad \partial_t v(0, x) = v_1(x) := \frac{1}{m} \phi_1^m(x/m) \). The \( L^1 - L^\infty \) estimate due to Von Wahl (see [37]) assures that

\[
\| v(t, x) \|_{L^\infty(\mathbb{R}_x^3)} \leq C(1 + |t|)^{-\frac{3}{2}} \sum_{|\alpha|+j \leq 3} \| \partial_x^\alpha v_j(x) \|_{L^1(\mathbb{R}_x^3)} .
\]

We deduce that

\[
\| \phi_m^\pm(t, x) \|_{L^\infty(\mathbb{R}_x^3)} \leq C(1 + m |t|)^{-\frac{3}{2}} \sum_{|\alpha|+j \leq 3} m^{3-|\alpha|-j} \| \partial_x^\alpha \phi_m^\pm(x) \|_{L^1(\mathbb{R}_x^3)} .
\]
and \( \| \Phi_M(t, x, z) \|_{L^\infty(\mathbb{R}^3 \times [-R, R]_z)} \leq C_R \sum_{\pm} \sum_{|\alpha| + j \leq 3} \int_0^1 (1 + m | t |)^{-\frac{3}{2}} \sqrt{m} \| \partial_x^\alpha \phi_m^j \|_{L^1(\mathbb{R}^3)} dm \)

(IV.7)

To estimate the integral for \( m \in [0, 1] \), we write

\[
\partial_x^\alpha \phi_m^j(x) = \frac{1}{2} \int_0^1 \left( \partial_x^\alpha \Phi_j(x, z) \pm \partial_x^\alpha \Phi_j(x, -z) \right) u_\pm(z, m) dz
\]

(III.18) and (III.19) assure that for any \( \alpha \in [0, \frac{1}{2}] \), we have

\[
m(1 + z) \leq 1 \Rightarrow |u_+(z, m)| \leq \sqrt{m(1 + z)^{-\frac{3}{2}}} + Cm^2 (1 + | z |)^{\frac{3}{2}},
\]

\[
m(1 + z) \leq 1 \Rightarrow |u_-(z, m)| \leq \frac{1}{8} (m)^{\frac{3}{2}} (1 + z)^{\frac{3}{2}} + Cm^2,
\]

hence for any \( \varepsilon \in [0, \frac{1}{2}] \), we have

(IV.8)

\[
m(1 + z) \leq 1 \Rightarrow |u_\pm(z, m)| \leq C m^\varepsilon (1 + k | z |)^\varepsilon.
\]

On the other hand (III.16) implies that

(IV.9)

\[
\sup_{m(1 + z) \geq 1} |u_\pm(z, m)| \leq C < \infty.
\]

Therefore we get for all \( 0 < m \leq 1 \) and \( 0 < \varepsilon \leq \frac{1}{2} \):

\[
\| \partial_x^\alpha \phi_m^j \|_{L^1(\mathbb{R}^3)} \leq C m^{\varepsilon} \| (1 + | z |)^\varepsilon \partial_x^\alpha \Phi_j \|_{L^1(\mathbb{R}^{4+j})},
\]

hence

(IV.10)

\[
\int_0^1 (1 + m | t |)^{-\frac{3}{2}} \sqrt{m} \| \partial_x^\alpha \phi_m^j \|_{L^1(\mathbb{R}^3)} dm \leq C \varepsilon (1 + | z |)^{\frac{3}{2}} \| (1 + | z |)^\varepsilon \partial_x^\alpha \Phi_j \|_{L^1(\mathbb{R}^{4+j})}.
\]

We now estimate the integral for large \( m \). We need a regularization with respect to \( x \). Let \( (\theta_n)_n \subset C_0^\infty(\mathbb{R}^3) \) such that \( 0 \leq \theta_n \), \( \text{Supp} \theta_n \subset \{ |x| \leq 1/n \} \), \( \int \theta_n(x) dx = 1 \). We introduce \( \Phi_j^n(x, z) := \int \Phi_j(x - y, z) \theta_n(y) dy \). We can easily check that \( \Phi_j^n \) tends to \( \Phi_j \) in \( \mathcal{X}^{-1-j} \) as \( n \) tends to infinity. Moreover, for all \( N, \Delta_N^j \partial_x^\alpha \Phi_j^n \) belongs to \( \mathcal{D}(H_0^{2-j}) \cap \mathcal{K}^0 \). We conclude with a classical argument of density, that it is sufficient to establish the result when the data satisfy \( \Delta_N^j \partial_x^\alpha \Phi_j \in \mathcal{D}(H_0^{2-j}) \cap \mathcal{K}^0 \) for all \( N \). In this case, for almost all \( x \in \mathbb{R}^3 \), the map \( z \mapsto \partial_x^\alpha \Phi_j(x, z) \pm \partial_x^\alpha \Phi_j(x, -z) \) belongs to \( \mathcal{D}(h_2^{2-j}) \). Thanks to Lemma (III.4) we have

(IV.11)

\[
\partial_x^\alpha \phi_m^j(x) = m^{-2N} F_\pm \left( h_{2-j}^j \left[ \partial_x^\alpha \Phi_j(x, z) \pm \partial_x^\alpha \Phi_j(x, -z) \right] \right) (m),
\]
hence with (IV.9) we get for $m \geq 1$

$$||\partial_x^\alpha \phi_m^\pm||_{L^1(\mathbb{R}^4)} \leq m^{-2N} ||h_2^{-j} [\partial_x^\alpha \Phi_j(x,z) \pm \partial_x^\alpha \Phi_j(x,-z)]||_{L^1(\mathbb{R}^4 \times [0,\infty[)}.$$ 

Since

$$||h_2^N [\partial_x^\alpha \Phi_j(x,z) \pm \partial_x^\alpha \Phi_j(x,-z)]||_{L^1(\mathbb{R}^4 \times [0,\infty[)} \leq C \sum_{l=0}^{4-2j} ||\partial_z^l \partial_x^\alpha \Phi_j||_{L^1(\mathbb{R}^4 \times \mathbb{R}^4)},$$

where the $z$-derivatives are taken in the sense of the distributions in $\mathbb{R}^4 \setminus \{z = 0\}$, we conclude that

$$\int_1^M (1 + m |t|)^{-3/2} m^{3-|\alpha|-j} ||\partial_x^\alpha \phi_m^\pm||_{L^1(\mathbb{R}^4)} dm \leq C \sum_{l=0}^{4-2j} ||\partial_z^l \partial_x^\alpha \Phi_j||_{L^1(\mathbb{R}^4 \times \mathbb{R}^4)} \int_1^M (1 + m |t|)^{-3/2} m^{3-|\alpha|} dm.$$

Since

$$\int_1^\infty (1 + m |t|)^{-3/2} m^{3-|\alpha|-j} dm \leq C \quad |t|^{-3/2}, \quad |\alpha| + j \leq 3, \quad j = 0, 1,$$

we obtain (IV.12)

$$\sum_{|\alpha| + j \leq 3} \int_1^\infty (1 + m |t|)^{-3/2} m^{3-|\alpha|-j} ||\partial_x^\alpha \phi_m^\pm||_{L^1(\mathbb{R}^4)} dm \leq C \quad |t|^{-3/2} \sum_{l+|\alpha| + j \leq 4} ||\partial_z^l \partial_x^\alpha \Phi_j||_{L^1(\mathbb{R}^4 \times \mathbb{R}^4)}.$$

Now the theorem follows from (IV.7), (IV.10) and (IV.12).

Q.E.D.

The $L^2 - L^\infty$ estimates are very useful to the study of non-linear problems. In the case of the massive Klein-Gordon equation in the Minkowski space-time, they were initially proved by S. Klainerman [18] for compactly supported initial data, refined in [4] and [16] and extended to the non compact data by Georgiev [14]. We establish such estimates for the Kaluza-Klein solutions.

We introduce suitable weighted spaces:

(IV.13)

$$\mathcal{Y}_\epsilon^N : = \{ \Phi \in \mathcal{R}^0 \cap \mathfrak{D} \left( H_0^{N+1} \right); \sum_{|\alpha| \leq N+1} \sum_{0 \leq p \leq q} ||\chi_p(|l y|)\chi_q(|z|)(1 + |y|)|^{3/2}(1 + |z|)|^{3/2} \partial_y^\alpha \Phi_j||_{L^2(\mathbb{R}^4 \times \mathbb{R}^4)} + \sum_{|\alpha| = N+2} \sum_{0 \leq p \leq q} ||\chi_p(|l y|)\chi_q(|z|)(1 + |y|)|^{3/2}(1 + |z|)|^{3/2} \partial_y^\alpha \Phi_j||_{L^2(\mathbb{R}^4 \times \mathbb{R}^4)} + \sum_{l=1}^{N+3} \sum_{|\alpha| \leq N+3-l} \sum_{0 \leq p} ||\chi_p(|l y|)(1 + |y|)|^{3/2} \partial_z^l \partial_y^\alpha \Phi_j||_{L^2(\mathbb{R}^4 \times \mathbb{R}^4)} < \infty \}.$$

Here $\chi_p$ is the dyadic partition of the unity on $[0, \infty[$, defined by

$$\chi_0 = 1_{[0,1[}, \quad 1 \leq p \Rightarrow \chi_p = 1_{[2^{p-1},2^p[}.$$
Theorem IV.2. For any \( R > 0 \) there exists \( C_R > 0 \) such that for any \( \varepsilon \in [0, \frac{1}{2}] \), and for all \( \Phi_0 \in \mathcal{Y}^1_{\varepsilon}, \Phi_1 \in \mathcal{Y}^0_{\varepsilon} \), the solution \( \Phi \) of the Cauchy problem (II.1), (II.2), (II.26) satisfies:

\[
\| \Phi(t, x, \cdot) \|_{L^\infty([-R, R])} \leq C_R \left( \frac{1 + t}{m} + |x| \right)^{-\frac{5}{2}} \left[ \sum_{|\alpha| + |j| \leq 2} \| \partial_\alpha \Phi_0(x) \|_{L^2(R)} \right]
\]

\[
\left. + \frac{1}{\varepsilon} \sum_{|\alpha| + |j| = 3} \sum_{p,q} \| \chi_p(|y|) \chi_q(|z|) (1 + |y|)^{\frac{3}{2}} (1 + |z|)^{\frac{3}{2} + \varepsilon} \partial_\alpha^p \Phi_j \|_{L^2(R^{3})} \right]
\]

\[
\left. + \sum_{l=1}^4 \sum_{|\alpha| + j \leq 4 - l} \sum_{p} \| \chi_p(|y|) (1 + |y|)^{\frac{3}{2}} \partial_\alpha \partial_\alpha^p \Phi_j \|_{L^2(R^{3})} \right]
\]

Like in the previous theorem, the term with the weight \( (1 + |y|)^{\frac{3}{2} + \varepsilon} \) is used to control the contribution of the light Kaluza-Klein modes, while the \( z \)-derivatives are useful to estimate the heavy modes.

Proof of Theorem IV.2. We start with expression (IV.2). We have to control carefully the dependence of the decay estimates with respect to the mass, so we need the following:

**Lemma IV.3.** There exists \( C > 0 \) such that for any \( m > 0 \), the solution \( \phi_m \) of the Klein-Gordon equation \( \partial_\nu^2 \phi_m - \Delta_x \phi_m + m^2 \phi_m = 0 \), \( \phi_m(0, x) = \phi_0^m(x) \), \( \partial_i \phi_m(0, x) = \phi_1^m(x) \), satisfies for all \( t \in \mathbb{R}, x \in \mathbb{R}^3 \):

\[
| \phi_m(t, x) | \leq C \left( 1 + \frac{m}{1 + |x|} \right) (1 + |t|) \left[ \sum_{|\alpha| + j \leq 3} \| \partial_\nu^\alpha \phi_m(t, y) \|_{L^2(R^{3})} \right]
\]

\[
\left. + C(1 + m + |t| + m |x|)^{-\frac{3}{2}} \sum_{|\alpha| + j \leq 3} \sum_{p=0}^\infty m^{-|\alpha|} \| \chi_p(|y|) (1 + |y|)^{\frac{3}{2}} \partial_\nu^\alpha \phi_m(t, y) \|_{L^2(R^{3})} \right]
\]

provided the norms of the right-hand side are finite.

Proof of Lemma IV.3. We follow closely the method employed by Georgiev [14]. By the Sobolev inequality on \( \mathbb{R}^3 \), there exists \( C > 0 \) such that for any \( t \in \mathbb{R}, x \in \mathbb{R}^3 \) we have:

(IV.14) \[
| \phi_m(t, x) | \leq C \sum_{|\alpha| \leq 2} \| \partial_\nu^\alpha \phi_m(t, y) \|_{L^2(R^{3})}.
\]

We can control the \( L^2 \)-norm of the right-hand side by using the energy equality

\[
\int_{\mathbb{R}^3} | \partial_t \partial_\nu^\alpha \phi_m(t, y) |^2 + | \nabla_y \partial_\nu^\alpha \phi_m(t, y) |^2 + m^2 | \partial_\nu^\alpha \phi_m(t, y) |^2 \, dy
\]

\[
= \int_{\mathbb{R}^3} | \partial_\nu^\alpha \phi_m(t, y) |^2 + | \nabla_y \partial_\nu^\alpha \phi_m(t, y) |^2 + m^2 | \partial_\nu^\alpha \phi_m(t, y) |^2 \, dy,
\]

to get

\[
| \phi_m(t, x) | \leq C \left( 1 + \frac{m}{1 + |x|} \right) \sum_{|\alpha| + j \leq 3} \| \partial_\nu^\alpha \phi_m(t, y) \|_{L^2(R^{3})}.
\]

Now we choose some function \( \chi \in C_0^\infty(\mathbb{R}^3) \) such that \( \chi(y) = 0 \) when \( |y| \geq 2 \) and \( \chi(y) = 1 \) when \( |y| \leq 2 \). Given \( t \in \mathbb{R} \) and \( x \in \mathbb{R}^3 \) fixed, we consider the solution \( \psi(s, y) \) of \( \partial_\nu^2 \psi - \Delta_y \psi + m^2 \psi = 0, \)
ψ(0, y) = χ \left( \frac{y-x}{1+|t|} \right) φ_m^0(y), \quad \partial_y ψ(0, y) = χ \left( \frac{y-x}{1+|t|} \right) \overline{φ}_m(y). \quad \text{The finite dependence domain argument implies that} \quad φ_m(t, x) = ψ(t, x), \quad \text{thus by applying the previous estimate to} \quad ψ \quad \text{we obtain:}

|φ_m(t, x)| ≤ C \left(1 + \frac{1}{m} \right) \sum_{|α|+j \leq 3} \|\partial_φ^α φ_m(y)\|_{L^2(|y-x|\leq 2+2|t|)}.

If \quad |x| ≥ 4 \quad |t| + 4, \quad \text{and} \quad |y-x| ≤ 2 \quad |t|, \quad \text{we have} \quad |y| ≥ \frac{1}{2} \quad |x|, \quad \text{hence}

|φ_m(t, x)| ≤ C \left(1 + \frac{1}{m} \right) \sum_{|α|+j \leq 3} ||(1+ |y|)|\frac{3}{2} \partial_φ^α φ_m(y)||_{L^2(\mathbb{R}^3)}.

When \quad |x| ≤ 4 \quad |t| + 4 \quad \text{and} \quad |t| ≤ 1, \quad \text{the Sobolev inequality (IV.14) gives the result. Finally, when} \quad |x| ≤ 4 \quad |t| + 4 \quad \text{and} \quad |t| ≥ 1, \quad \text{we use the Von Wahl estimate (IV.4) and the inequality}

\|φ\|_{L^1(\mathbb{R}^3)} ≤ \sqrt{4π \log(2)} \sum_{k=0}^{∞} \|χ_k(|y|)(1+ |y|)|\frac{3}{2} φ(y)\|_{L^2(\mathbb{R}^3)}.

Q.E.D.

Now we use Lemma [IV.3] and the estimates [IV.8] and [IV.9], to get from (IV.2):

\|Φ(t, x, )\|_{L^∞([-R,R]_x)} ≤ C_R \sum_{|α|+j \leq 3} (1+ |x| + |t|)^{-\frac{3}{2}} \int_0^1 \frac{1}{\sqrt{m}} \|((1+ |y|)|\frac{3}{2} \partial_φ^α φ_m^+ ||_{L^2(\mathbb{R}^3)}dm

+ \sum_{p=0}^{∞} \int_0^1 (1+ m |x| + m |t|)^{-\frac{3}{2}} \sum_{|α|+j \leq 3} \|χ_{p,}(|y|)(1+ |y|)|\frac{3}{2} \partial_φ^α φ_m^+ ||_{L^2(\mathbb{R}^3)}dm

+ (1+ |x| + |t|)^{-\frac{3}{2}} \int_1^∞ \|((1+ |y|)|\frac{3}{2} \partial_φ^α φ_m^+ ||_{L^2(\mathbb{R}^3)}dm

+ \sum_{p=0}^{∞} \int_1^∞ (1+ m |x| + m |t|)^{-\frac{3}{2}} \sum_{|α|+j \leq 3} \|χ_{p,}(|y|)(1+ |y|)|\frac{3}{2} \partial_φ^α φ_m^+ ||_{L^2(\mathbb{R}^3)}dm.

To estimate the integral on the light mass, we use [IV.8] and [IV.9]. If P is a partial differential operator on \mathbb{R}^3_x, we obtain for any 0 \leq ε and 0 < m ≤ 1

\|P φ_m^+ ||_{L^2(\mathbb{R}^3)} ≤ C m^ε \|((1+k |z|)^ε P \Phi_j ||_{L^1(\mathbb{R}_z \times L^2(\mathbb{R}^3_x))} ≤ C m^ε \sum_{p=0}^{∞} \|χ_{p,}(|z|)(1+ |z|)|\frac{3}{2} + ε \|P \Phi_j ||_{L^2(\mathbb{R}^3_x)}.

For P = (1+ |y|)|\frac{3}{2} \partial_φ^α, we take ε = 0 and we get:

(IV.15) \int_0^1 \frac{1}{\sqrt{m}} \|\partial_φ^α φ_m^+ ||_{L^2(\mathbb{R}^3)}dm ≤ C \sum_{p=0}^{∞} \|χ_{p,}(|z|)(1+ |y|)|\frac{3}{2} \|P \Phi_j ||_{L^2(\mathbb{R}^3_x)}.

For P = χ_{p,}(y)(1+ |y|)|\frac{3}{2} \partial_φ^α with |α| + j ≤ 2, we take ε = 0 again and we obtain:

\int_0^1 (1+ m |x| + m |t|)^{-\frac{3}{2}} \sum_{|α|+j \leq 2} ||χ_{q,}(|z|)χ_{p,}(|y|)(1+ |z|)|\frac{3}{2} \|P \Phi_j ||_{L^2(\mathbb{R}^3_x)}.

C(|x| + |t|)^{-\frac{3}{2}} \sum_{q=0}^{∞} ||χ_{q,}(|z|)χ_{p,}(|y|)(1+ |z|)|\frac{3}{2} \|P \Phi_j ||_{L^2(\mathbb{R}^3_x)}.
Finally for \( P = \chi_p(y)(1+ |y|)^{\frac{3}{2}} \partial_y^n \) with \( |\alpha| + j = 3 \), we take \( \epsilon > 0 \) and we obtain:

\[
\int_1^\infty (1 + m |x| + m |t|) - \frac{3}{2} m^{\frac{3}{2} - |\alpha| - j} \| \chi_p(|y|) (1 + |y|)^{\frac{3}{2}} \partial_y^n \phi_m^\pm \|_{L^2(\mathbb{R}^3)} dm \\
\leq \frac{C}{\epsilon} \left( |x| + |t| \right) - \frac{3}{2} \sum_{q=0}^\infty \| \chi_q(|z|) \chi_p(|y|)(1 + |z|)^{\frac{3}{2} + \epsilon}(1 + |y|)^{\frac{3}{2}} \partial_y^n \Phi_j \|_{L^2(\mathbb{R}^3 \times \mathbb{R}^+)}.
\]

To estimate the integrals on the large mass, \( m \geq 1 \), we use the procedure of regularization employed in the proof of the previous theorem. Thanks to (IV.11) and Lemma III.4, we have

\[
(IV.17)
\]

\[
Q.E.D.
\]

We know that Strichartz has proved in [35] that for suitable initial data, the solutions of the D’Alembertian in the Minkowski space-time \( \mathbb{R}^{1+3} \) belong to \( L^4(\mathbb{R}^4) \), and the solutions of the massive Klein-Gordon equation belong to \( L^q(\mathbb{R}^4) \), \( \frac{4}{q} \leq q \leq 4 \). Moreover the massless fields in the Minkowski space-time \( \mathbb{R}^{1+4} \) belong to \( L^{10}(\mathbb{R}^{1+4}) \). Therefore, we expect that near the brane, the solution of the master equation are in \( L^\infty([-R,R_z];L^4(\mathbb{R}_t \times \mathbb{R}^3_{x}) \), and the Kaluza-Klein tower that is more dispersive because of the mass, is in \( L^{\frac{10}{3}}(\mathbb{R}_t \times \mathbb{R}^3_x \times [-R,R_z]) \). This is indeed the case.

**Theorem IV.4.** There exists \( C > 0 \) and for any \( R > 0 \), some \( C_R > 0 \), such that the solution \( \Phi \) of the Cauchy problem [11,12], [11,22] satisfies the following inequalities. When \( \Phi_0 \in \mathcal{D}(H^{1}) \), \( \Phi_1 \in \mathcal{D}(H^{\frac{1}{2}}) \) we have:

\[
(IV.16)
\]

When \( \Phi_j \in \mathcal{D}(H^{\frac{1}{2}(1-j)+\varepsilon}) \) for some \( \varepsilon > 0 \), we have:

\[
(IV.17)
\]
When $\Phi$ is a massless graviton, we have

$$(\text{IV.18}) \\
\| \Phi \|_{L^\infty(\mathbb{R}_t; L^4(\mathbb{R}_t \times \mathbb{R}^4_x))} \leq C \sum_{j=0,1} \| H_0^{\frac{1}{2},j} \Phi_j \|_{L^2(\mathbb{R}^4_t)}.$$  

**Proof of Theorem IV.4.** The global $L^9(\mathbb{R}_t \times \mathbb{R}^4_x)$ estimate due to Strichartz [35] assures that the solution $v$ of the Klein-Gordon equation with the mass equal to 1 in $\mathbb{R}_t \times \mathbb{R}^3_x$, solution of $\partial_t^2 v - \Delta_x v + v = 0$, $v(0,x) = v_0(x)$, $\partial_t v(0,x) = v_1(x)$, satisfies

$$\frac{10}{3} \leq q \leq 4, \quad \| v(t,x) \|_{L^9(\mathbb{R}_t \times \mathbb{R}^3_x)} \leq C \sum_{j=0,1} \| (1 - \Delta_x)^{\frac{1}{2}} v_j(x) \|_{L^2(\mathbb{R}^3_x)}.$$  

Therefore we deduce that the solution $\phi_m$ of the Klein-Gordon equation with mass $m > 0$, $\partial_t^2 \phi_m - \Delta_x \phi_m + m^2 \phi_m = 0$, $\phi_m(0,x) = \phi_0^0(x)$, $\partial_t \phi_m(0,x) = \phi_1^0(x)$, satisfies:

$$(\text{IV.19}) \quad \frac{10}{3} \leq q \leq 4, \quad \| \phi_m(t,x) \|_{L^9(\mathbb{R}_t \times \mathbb{R}^3_x)} \leq C^{m-\frac{4}{9}} \sum_{j=0,1} \| (m^2 - \Delta_x)^{\frac{1}{2}} \phi_j(x) \|_{L^2(\mathbb{R}^3_x)}.$$  

Concerning the massless free wave $\phi_0$, solution of $\partial_t^2 \phi_0 - \Delta_x \phi_0 = 0$, we also have:

$$(\text{IV.20}) \quad \| \phi_0(t,x) \|_{L^9(\mathbb{R}_t \times \mathbb{R}^3_x)} \leq C \sum_{j=0,1} \| (-\Delta_x)^{\frac{1}{2}} \phi_j(x) \|_{L^2(\mathbb{R}^3_x)}.$$  

We use the representation (IV.2), (IV.3). We deduce from (III.16), (III.26) and (IV.5) that for any $z \in [-R,R]$ we have for any $M > 1$:

$$\| \Phi_M(\cdot,\cdot,z) \|_{L^9(\mathbb{R}_t \times \mathbb{R}^3_x)} \leq C_R \left[ \sum_j \int_0^1 \sqrt{m} \| \phi_m^\pm \|_{L^9(\mathbb{R}_t \times \mathbb{R}^3_x)} dm + \int_1^M \phi_m^+ g^+(z,m) dm \right] \left[ \int_1^M \frac{1}{m} \| \phi_m^\pm \|_{L^9(\mathbb{R}_t \times \mathbb{R}^3_x)} dm \right],$$  

where $g^+(z,m) = \sqrt{\frac{2}{\pi}} \cos(mz)$, $g^-(z,m) = \sqrt{\frac{2}{\pi}} \sin(mz)$. We use (IV.19) with $q = \frac{10}{3}$ to get

$$\int_0^1 \sqrt{m} \| \phi_m^\pm \|_{L^{10}(\mathbb{R}_t \times \mathbb{R}^3_x)} dm + \int_1^M \frac{1}{m} \| \phi_m^\pm \|_{L^{10}(\mathbb{R}_t \times \mathbb{R}^3_x)} dm \leq$$

$$C \sum_{j=0,1} \left( \int_0^M \| (m^2 - \Delta_x)^{\frac{1}{2}} \phi_j^\pm(x) \|_{L^2(\mathbb{R}^3_x)} dm \right)^{\frac{1}{2}}.$$  

Since Lemma (III.21) assures that the distorted Fourier transforms are isometries, we have:

$$\sum_j \int_0^\infty \| (m^2 - \Delta_x)^{\frac{1}{2}} \phi_m^\pm(x) \|_{L^2(\mathbb{R}^3_x)} dm$$

$$= \frac{1}{4} \sum_j \int_0^\infty \| (n^2 - \Delta_x)^{\frac{1}{2}} [\Phi_j(x,z) \mp \Phi_j(x,-z)] \|_{L^2(\mathbb{R}^3_x)} dz$$

$$= \int_{-\infty}^\infty \| (n - \Delta_x)^{\frac{1}{2}} \Phi_j(x,z) \|_{L^2(\mathbb{R}^3_x)} dz = \frac{1}{2} \| H_0^{\frac{1}{2},j} \Phi_j \|_{L^2(\mathbb{R}^3_x)}.$$  

We conclude that

$$\sum_j \int_0^1 \sqrt{m} \| \phi_m^\pm \|_{L^{10}(\mathbb{R}_t \times \mathbb{R}^3_x)} dm + \int_1^\infty \frac{1}{m} \| \phi_m^\pm \|_{L^{10}(\mathbb{R}_t \times \mathbb{R}^3_x)} dm \leq C \sum_{j=0,1} \| H_0^{\frac{1}{2},j} \Phi_j \|_{L^2(\mathbb{R}^4_{t,x})}.$$
To estimate the third term, we remark that \( \int_1^M \phi_m^\pm(t, x) g^\pm(z, m) \, dm \) is solution of the free wave equation in the Minkowski space-time \( \mathbb{R}_t \times \mathbb{R}^4_{x, z} \), thus the Strichartz estimate assures that

\[
\left\| \int_1^M \phi_m^\pm(t, x) g^\pm(z, m) \, dm \right\|_{L^\infty(\mathbb{R}_t \times \mathbb{R}^4_{x, z})} \leq C \sum_{j=0,1} \left\| \left( -\Delta_x - \partial_x^2 \right)^{\frac{1}{2} - j} \int_1^M \phi_m^{j, \pm}(x) g^\pm(z, m) \, dm \right\|_{L^2(\mathbb{R}^4_{x, z})}.
\]

Taking account of the expression of \( g^\pm \), we have

\[
\left( -\Delta_x - \partial_x^2 \right)^{\frac{1}{2} - j} \int_1^M \phi_m^{j, \pm}(x) g^\pm(z, m) \, dm = \int_1^M \left( -\Delta_x + m^2 \right)^{\frac{1}{2} - j} \phi_m^{j, \pm}(x) g^\pm(z, m) \, dm,
\]

hence we get by Lemma 3.12 that for almost all \( x \) fixed in \( \mathbb{R}^3 \):

\[
\left\| \left( -\Delta_x - \partial_x^2 \right)^{\frac{1}{2} - j} \int_1^M \phi_m^{j, \pm}(x) g^\pm(z, m) \, dm \right\|_{L^2(\mathbb{R}^4_x)} = \left\| \left( -\Delta_x + m^2 \right)^{\frac{1}{2} - j} \phi_m^{j, \pm}(x) \right\|_{L^2([1, M]_m)} \\
\leq \left\| \left( -\Delta_x + m^2 \right)^{\frac{1}{2} - j} \phi_m^{j, \pm}(x) \right\|_{L^2(\mathbb{R}^4_x)} \\
\leq \left\| \left( -\Delta_x + h^\pm \right)^{\frac{1}{2} - j} [\Phi_j(x, z) \pm \Phi_j(x, -z)] \right\|_{L^2(\mathbb{R}^4_x)} \\
\leq C \left\| H_0^{j, \pm}(\frac{1}{2} - j) \Phi_j(x, z) \right\|_{L^2(\mathbb{R}^4_x)}.
\]

Finally we take the \( L^2(\mathbb{R}^4_x) \)-norm and we get

\[
\sup_{M>1} \left\| \int_1^M \phi_m^\pm(t, x) g^\pm(z, m) \, dm \right\|_{L^\infty(\mathbb{R}_t \times \mathbb{R}^4_{x, z})} \leq C \sum_{j=0,1} \left\| H_0^{j, \pm}(\frac{1}{2} - j) \Phi_j \right\|_{L^2(\mathbb{R}^4_{x, z})},
\]

and the proof of (IV.16) is complete.

The proof of the \( L^\infty(\mathbb{R}^4_x) \) estimate for the Kaluza-Klein tower is similar. When \( \Phi_j \in \mathbb{R}^0 \cap \mathfrak{D}(H_{1/2}^{1-j} + \varepsilon) \), we deduce from (IV.5), (IV.6) and (IV.19) that for any \( z \in [-R, R] \) we have for any \( M > 1 \) :

\[
\| \Phi_M(\cdot, \cdot, z) \|_{L^4(\mathbb{R}_t \times \mathbb{R}^4_x)} \leq C_R \sum_{j=0,1} \int_0^M \| \phi_m^\pm \|_{L^4(\mathbb{R}_t \times \mathbb{R}^4_x)} \, dm \\
\leq C_R \sum_{j=0,1} \int_0^1 \left( m^2 - \Delta_x \right)^{\frac{1}{2} - j} \phi_m^{j, \pm}(x) \|_{L^2(\mathbb{R}^4_x)} \, dm \\
+ \int_1^M \left( m^2 - \Delta_x \right)^{\frac{1}{2} - j + \varepsilon} \phi_m^{j, \pm}(x) \|_{L^2(\mathbb{R}^4_x)} \, dm \leq M^{-\frac{1}{2} - 2\varepsilon} \sum_{j=0,1} \left( \int_0^M \left( m^2 - \Delta_x \right)^{\frac{1}{2} - j + \varepsilon} \phi_m^{j, \pm}(x) \|_{L^2(\mathbb{R}^4_x)}^2 \, dm \right)^{1/2} \\
\leq C_R \sqrt{\varepsilon} \sum_{j=0,1} \left\| H_0^{j, \pm}(1-j + \varepsilon) \Phi_j \right\|_{L^2(\mathbb{R}^4_x)},
\]

hence (IV.17) is established. At last, estimate (IV.18) for a massless graviton with initial data \( \Phi_j(x, z) = \phi_j(x) \otimes f_0(z) \) is a direct consequence of (IV.20) since \( (\Delta_x)^{\frac{1}{2} - j} \phi_j \otimes f_0 = H_0^{\frac{1}{2} - j} \Phi_j \).

Q.E.D.
V. Brane-World Scattering

In this section we prove that the finite energy solutions of (I.1) with data in $\mathbb{R}^1 \times \mathbb{R}^0$, that are the Kaluza-Klein towers, are asymptotic to free waves $\Phi_{\infty}^\pm$ with finite energy on the five-dimensional Minkowski space-time:

(V.1) \[ (\partial_t^2 - \Delta_x - \partial_z^2) \Phi_{\infty}^\pm = 0, \quad (t, x, z) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}, \]

(V.2) \[ \Phi_{\infty}^\pm \in C^0 \left( \mathbb{R}_t; BL^1 \left( \mathbb{R}^4_{x,z} \right) \right), \quad \partial_t \Phi_{\infty}^\pm \in C^0 \left( \mathbb{R}_t; L^2 \left( \mathbb{R}^4_{x,z} \right) \right). \]

**Theorem V.1.** Any finite energy solution $\Phi$ with initial data $\Phi_0 \in \mathbb{R}^1$, $\Phi_1 \in \mathbb{R}^0$, satisfies the decay estimate [II.33] for any $a, b$, and there exist unique free waves $\Phi_{\infty}^\pm$ satisfying (V.1), (V.2) and

(V.3) \[ \lim_{t \to \pm \infty} \left\| \partial_t \Phi(t) - \partial_t \Phi_{\infty}^\pm(t) \right\|_{L^2(\mathbb{R}^4_{x,z})} = 0. \]

Moreover, these fields satisfy:

(V.4) \[ \lim_{t \to \pm \infty} \left\| \nabla_{t,x} \Phi(t) - \nabla_{t,x} \Phi_{\infty}^\pm(t) \right\|_{L^2(\mathbb{R}^4_{x,z})} = 0, \]

(V.5) \[ \lim_{t \to \pm \infty} \left\| \partial_z \Phi(t) + \frac{3}{2} \frac{z}{|z|} \left( \frac{1}{1+|z|} \right) \Phi(t) - \partial_z \Phi_{\infty}^\pm(t) \right\|_{L^2(\mathbb{R}^4_{x,z})}^2 = 0, \]

and when $\Phi$ is a $z$-odd wave, we also have:

(V.6) \[ \lim_{t \to \pm \infty} \left\| \partial_z \Phi(t) - \partial_z \Phi_{\infty}^\pm(t) \right\|_{L^2(\mathbb{R}^4_{x,z})} = 0. \]

The wave operators $W^\pm : (\Phi_0, \Phi_1) \mapsto \left( \Phi_{\infty}^\pm(0, \cdot), \partial_t \Phi_{\infty}^\pm(0, \cdot) \right)$, are isometries from $\mathbb{R}^1 \times \mathbb{R}^0$ onto $BL^1 \left( \mathbb{R}^4_{x,z} \right) \times L^2 \left( \mathbb{R}^4_{x,z} \right)$.

We can introduce the Scattering Operator

$$ S := W^+ (W^-)^{-1}, $$

that is unitary on $BL^1 \left( \mathbb{R}^4_{x,z} \right) \times L^2 \left( \mathbb{R}^4_{x,z} \right)$ and describes the scattering of the Kaluza-Klein waves by the Minkowski brane. We can interpret the existence of these operators in term of a Goursat problem on the manifolds $B^\pm$ described by Figure 1. The asymptotic completeness of the wave operators $W^+$ assures that the characteristic problem with the data specified on the conformal null infinity $T = -\infty$, $y = \pm \infty$, and the boundary condition on the brane, is well posed, and the existence of $W^+$ implies that the solution has a trace on the future characteristic boundary $T = +\infty$, $y = \pm \infty$. This approach of the Goursat problem in general relativity, was used in [5] for the Maxwell system on the Schwarzschild manifold for which the timelike infinity is singular, and in [24] for the Maxwell and Dirac equations in a large class of non-stationary vacuum space-times admitting a conformal compactification that is smooth at null and timelike infinity.

**Proof of Theorem V.1.** First we prove the uniqueness of the asymptotic fields. Given some finite energy solution $\Phi$ with initial data $\Phi_0 \in \mathbb{R}^1$, $\Phi_1 \in \mathbb{R}^0$, assume there exist two free waves $\Phi_{\infty}^1$, $\Phi_{\infty}^2$, solutions of (V.1) and (V.2), satisfying

$$ \lim_{t \to \infty} \left\| \partial_t \Phi(t) - \partial_t \Phi_{\infty}^j(t) \right\|_{L^2(\mathbb{R}^4_{x,z})} = 0. $$

Then

$$ \lim_{t \to \infty} \left\| \partial_t \left( \Phi_{\infty}^1 - \Phi_{\infty}^2 \right)(t) \right\|_{L^2(\mathbb{R}^4_{x,z})} = 0. $$

The well-known result of the equipartition of the energy (see e.g. [3]) assures that

$$ \lim_{t \to \infty} \left\| \partial_t \left( \Phi_{\infty}^1 - \Phi_{\infty}^2 \right)(t) \right\|_{L^2(\mathbb{R}^4_{x,z})}^2 = \frac{1}{2} \left\| \nabla_{t,x,z} \left( \Phi_{\infty}^1 - \Phi_{\infty}^2 \right)(0) \right\|_{L^2(\mathbb{R}^4_{x,z})}^2. $$
We conclude that $\Phi_{1}^{\infty} = \Phi_{2}^{\infty}$. Conversely, given some free finite energy solution $\Phi_{\infty}$ of (V.1) and (V.2), assume there exists two waves $\Phi^1$, $\Phi^2$, solutions of (III.1) with initial data in $W^{1}_{x} \times L^{2}$, satisfying
\[
\lim_{t \to \infty} \| \partial_{t} \Phi_{\infty}(t) - \partial_{t} \Phi^{j}(t) \|_{L^{2}(\mathbb{R}^{4}_{x,z})} = 0.
\]
Then
\[
\lim_{t \to \infty} \| \partial_{t} \left( \Phi^{1} - \Phi^{2} \right)(t) \|_{L^{2}(\mathbb{R}^{4}_{x,z})} = 0.
\]
Since 0 is not eigenvalue of $H_{0}$, an abstract result of [13] assures that
\[
\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \| \partial_{t} \left( \Phi^{1} - \Phi^{2} \right)(t) \|^{2}_{L^{2}(\mathbb{R}^{4}_{x,z})} dt = \frac{1}{2} \| \partial_{t} \left( \Phi^{1} - \Phi^{2} \right)(0) \|^{2}_{L^{2}(\mathbb{R}^{4}_{x,z})} + \frac{1}{2} \| \Phi^{1}(0) - \Phi^{2}(0) \|^{2}_{W},
\]
and we conclude that $\Phi^{1} = \Phi^{2}$.

A similar argument shows that the wave operators are isometric since
\[
\| \partial_{t} \Phi_{1} \|^{2}_{L^{2}(\mathbb{R}^{4}_{x,z})} + \| \Phi_{0} \|^{2}_{W} = \lim_{T \to \infty} \frac{2}{T} \int_{0}^{T} \| \partial_{t} \Phi(t) \|^{2}_{L^{2}(\mathbb{R}^{4}_{x,z})} dt
\]
\[
= \lim_{T \to \infty} \frac{2}{T} \int_{0}^{T} \| \partial_{t} \Phi_{\infty}(t) \|^{2}_{L^{2}(\mathbb{R}^{4}_{x,z})} dt
\]
\[
= \lim_{T \to \infty} 2 \| \partial_{t} \Phi_{\infty}(t) \|^{2}_{L^{2}(\mathbb{R}^{4}_{x,z})}
\]
\[
= \| \partial_{t} \Phi_{\infty}(0) \|^{2}_{L^{2}(\mathbb{R}^{4}_{x,z})} + \| \Phi_{\infty}(0) \|^{2}_{BL^{1}(\mathbb{R}^{4})}.
\]

Thanks to this property and the conservation of the energy, it is sufficient to construct $W^{\pm}$ on a dense subspace of $\mathcal{R}^{1} \times \mathcal{R}^{0}$, and $(W^{\pm})^{-1}$ on a dense subspace of $BL^{1}(\mathbb{R}^{4}) \times L^{2}(\mathbb{R}^{4})$. If $\hat{f}$ is the partial Fourier transform of $f$ with respect to $x$ defined by (III.32), $f \in W^{1}_{x}$ if $\hat{f} \in L^{2}(\mathbb{R}_{x}^{3} \times \mathbb{R}_{z}; |\xi|^{2} d\xi dz)$ and $\partial_{z} \hat{f} + \frac{3}{2} \frac{z}{|z|} \left( \frac{1}{1+|z|} \right) \hat{f} \in L^{2}(\mathbb{R}_{x,z}^{4})$. We deduce from (III.35) and the Fubini theorem that when $f \in \mathcal{R}^{1}$, (respect. $f \in \mathcal{R}^{0}$), we have for almost all $\xi \in \mathbb{R}^{3}$:
\[
\hat{f}(\xi, z) \in H^{1}(\mathbb{R}_{z}) \quad (\text{respect. } L^{2}(\mathbb{R}_{z})) , \quad \int_{-\infty}^{\infty} \hat{f}(\xi, z) f_{0}(z) dz = 0.
\]
We introduce a space of regular data
\[
\mathcal{D}_{0} := \left\{ f \in H^{1}(\mathbb{R}^{4}); \hat{f} \in C_{0}^{\infty} \left( \left( \mathbb{R}_{\xi}^{3} \setminus \{0\} \right) \times \mathbb{R}_{z} \right) \right\}.
\]
By cut-off and regularization, we can easily show that
\[
\mathcal{D}_{0} := \left\{ f \in \mathcal{D}_{0}, \int_{-\infty}^{\infty} f(x, z) f_{0}(z) dz = 0 \right\}
\]
is dense in $\mathcal{R}^{1} \times \mathcal{R}^{0}$.

In the sequel, we take $(\Phi_{0}, \Phi_{1}) \in \mathcal{D}_{0} \times \mathcal{D}_{0}$, and $\hat{\Phi}_{j}(\xi, z) = 0$ when $|\xi| \not\in [\alpha, \beta]$ for some $0 < \alpha < \beta < \infty$, and we consider the unique finite energy solution $\Phi$ with this initial data. For any $\xi \in \mathbb{R}^{3}$, Lemma [III.2] assures there exists a unique solution $u(t, z; \xi) \in C^{0}(\mathbb{R}_{t}; H^{1}(\mathbb{R}_{z})) \cap C^{1}(\mathbb{R}_{t}; L^{2}(\mathbb{R}_{z}))$ of the Klein-Gordon equation
\[
\partial_{t}^{2} u + hu + |\xi|^{2} u = 0, \quad t, z \in \mathbb{R},
\]
with Cauchy data
\[
u(0, z; \xi) = \hat{\Phi}_{0}(\xi, z), \quad \partial_{t} u(0, z; \xi) = \hat{\Phi}_{1}(\xi, z),
\]
and $u$ satisfies the constraint
\[
\forall t \in \mathbb{R}, \quad \int_{-\infty}^{\infty} u(t, z; \xi) f_{0}(z) dz = 0.
\]
Moreover the usual tools of the spectral theory (see e.g. [31], Theorem VIII.25, Theorem VIII.20) imply that the map $\xi \mapsto h^+ |\xi|^2$ is $C^\infty$ in the strong resolvent sense, hence

$$u \in C_0^\infty \left( \mathbb{R}_t^3; C^0(\mathbb{R}; H^1(\mathbb{R})) \cap C^1(\mathbb{R}; L^2(\mathbb{R})) \right)$$

and we conclude that

$$\Phi(t, x, z) = \frac{1}{(2\pi)^2} \int_{\alpha \leq |\xi| \leq \beta} e^{ix\xi} u(t, z; \xi) d\xi.$$ 

Furthermore the map $x \mapsto \Phi(t, x, \cdot) \in H^1(\mathbb{R}_z)$ is a function of rapid decrease, hence

$$u(t, z; \xi) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} e^{-ix\xi} \Phi(t, x, z) dx.$$ 

This elementary Fourier analysis allows to reduce the study of the asymptotic behaviour of $\Phi$ to

the scattering theory for a 1+1-dimensional Klein-Gordon equation.

Let $m$ be a strictly positive real number. We compare the solutions of

(V.8) \quad \partial_t^2 u + hu + m^2 u = 0,

with the solutions of the free Klein-Gordon equation

(V.9) \quad \partial_t^2 u - \partial_z^2 u + m^2 u = 0, \quad t, z \in \mathbb{R}.

We denote $H^1_m(\mathbb{R})$ the usual Sobolev space $H^1(\mathbb{R})$, endowed with the norm

$$\| u \|^2_{H^1_m} := \int_{-\infty}^{\infty} |u'(z)|^2 + m^2 |u(z)|^2 dz,$$

and we introduce the spaces

$$K^1 := \left\{ u \in H^1(\mathbb{R}), \int_{-\infty}^{\infty} u(z)f_0(z)dz = 0 \right\}, \quad K^0 := \left\{ u \in L^2(\mathbb{R}), \int_{-\infty}^{\infty} u(z)f_0(z)dz = 0 \right\},$$

and $K^1_m$ denotes $K^1$ equipped with the norm

(V.10) \quad \| u \|^2_{K^1_m} := \int_{-\infty}^{\infty} \left[ u'(z) + \frac{3}{2} \frac{z}{|z|} \left( \frac{1}{1 + |z|} \right) u(z) \right]^2 + m^2 |u(z)|^2 dz.

From (II.17), we see that $\| u \|_{K^1_m}$ and $\| u \|_{H^1_m}$ are two equivalent norms. We now develop the scattering theory for (V.8). In particular, the following result provides a rigorous justification of the numerical experiments by the physicists (see e.g. [34]).

**Lemma V.2.** For any $u_0 \in K^1$, $u_1 \in K^0$, there exist unique $u^\pm \in C^0(\mathbb{R}; H^1(\mathbb{R})) \cap C^1(\mathbb{R}; L^2(\mathbb{R}))$ solutions of (V.9) such that

$$\| u(t) - u^\pm(t) \|_{H^1_m} + \| \partial_t u(t) - \partial_t u^\pm(t) \|_{L^2(\mathbb{R})} \to 0, \quad t \to \pm \infty,$$

where $u \in C^0(\mathbb{R}; H^1(\mathbb{R})) \cap C^1(\mathbb{R}; L^2(\mathbb{R}))$ is the solution of (V.8) with the initial data $u(0, z) = u_0(z)$, $\partial_t u(0, z) = u_1(z)$. The maps

$$W_m^\pm : (u_0, u_1) \mapsto (u^\pm(0), \partial_t u^\pm(0))$$

are isometries from $K^1_m \times K^0$ onto $H^1_m(\mathbb{R}) \times L^2(\mathbb{R})$.

For any $u_0 \in H^1(\mathbb{R})$, $u_1 \in L^2(\mathbb{R})$, there exist unique $A, B \in \mathbb{C}$, $u^\pm, \varrho^\pm \in C^0(\mathbb{R}; H^1(\mathbb{R})) \cap C^1(\mathbb{R}; L^2(\mathbb{R}))$, $u^\pm$ solutions of (V.9), such that

$$u(t, z) = [A e^{imt} + B e^{-imt}] (1 + |z|)^{-\frac{3}{2}} + u^\pm(t, z) + \varrho^\pm(t, z),$$

$$\| \varrho^\pm(t) \|_{H^1_m} + \| \partial_t \varrho^\pm(t) \|_{L^2(\mathbb{R})} \to 0, \quad t \to \pm \infty.$$
\textbf{Proof of Lemma} V.2. We use the scattering theory with two Hilbert spaces by T. Kato \cite{17}, that deals with the comparison between two abstract wave equations
\begin{equation}
\frac{d^2}{dt^2}u + A_j u = 0, \quad j = 1, 2, \quad t \in \mathbb{R}.
\end{equation}

We consider some densely defined self-adjoint operators $A_j$ on a separable Hilbert space $(\mathfrak{h}, \| \cdot \|)$. We assume $A_j$ is strictly positive,
\begin{equation}
\forall u \in \mathcal{D}(A_j) \setminus \{0\}, \quad 0 < (A_j u, u),
\end{equation}
hence $B_j := A_j^{\frac{1}{2}}$ is well defined and the null space of $B_j$ is \{0\}. Furthermore, we suppose that
\begin{equation}
(A_2 + 1)^{-1} - (A_1 + 1)^{-1} \text{ is trace class}.
\end{equation}

We refer to \cite{31} for the trace class ideal. We know that with these assumptions, the principle of invariance (see e.g. \cite{32}, Theorem XI.11, Corollary 2, p. 30), assures that the wave operators
\begin{equation}
\Omega^\pm(A_2, A_1) := s - \lim_{t \to \pm \infty} e^{-itA_2}e^{itA_1}Q_i = s - \lim_{t \to \pm \infty} e^{-itB_2}e^{itB_1}Q_i, \quad i = 1, 2,
\end{equation}
where $Q_j$ is the projection for $\mathfrak{h}$ on the subspace of absolute continuity for $A_j$.

We introduce the Hilbert space $\mathfrak{h}_j := [\mathcal{D}(B_j)] \times \mathfrak{h}$, equipped with the norm
\begin{equation}
\| (u, v) \|^2_{\mathfrak{h}_j} := \| B_j u \|^2 + \| v \|^2,
\end{equation}
where $[\mathcal{D}(B_j)]$ is the closure of the domain of $B_j$, $\mathcal{D}(B_j)$, for the norm $\| B_j u \|$. We assume that
\begin{equation}
\mathcal{D}(B_1) = \mathcal{D}(B_2),
\end{equation}
and there exists $M > 0$ such that for any $u \in \mathcal{D}(B_j)$, we have
\begin{equation}
\| B_1 u \| \leq \| B_2 u \| \leq M \| B_1 u \|.
\end{equation}

As consequence, $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are the same linear space, endowed with two equivalent norms. The finite energy solutions of the wave equations are given by the unitary groups $U_j(t)$ on $\mathfrak{h}_j$, specified on $\mathcal{D}(B_j) \times \mathfrak{h}$ by
\begin{equation}
U_j(t) = \begin{pmatrix} \cos tB_j & B_j^{-1}\sin tB_j \\ -B_j^{-1}\sin tB_j & \cos tB_j \end{pmatrix}.
\end{equation}

$P_j$ denoting the projection on the subspace of absolute continuity for the selfadjoint operator that generates $U_j(t)$, we define the wave operators
\begin{equation}
W^\pm(A_2, A_1) := s - \lim_{t \to \pm \infty} U_2(-t)U_1(t)P_1.
\end{equation}

Theorem 10.3, Theorem 10.5 and Remark 10.6 of \cite{17} assure that assumptions (V.11), (V.13), (V.14) and (V.15) imply that $W^\pm(A_2, A_1)$ exist, are complete and partially isometric with initial projection $P_1$ and final projection $P_2$. Moreover $W^\pm(A_1, A_2)$ exist, are complete and equal to $[W^\pm(A_2, A_1)]^*$. Finally, using the principle of invariance and formula (9.10) of \cite{17}, we obtain a nice expression of the wave operators:
\begin{equation}
W^\pm(A_2, A_1) = \frac{1}{2} \begin{pmatrix} \Omega^+(A_2, A_1) + \Omega^-(A_2, A_1) & i\tilde{B}_2^{-1}[\Omega^-(A_2, A_1) - \Omega^+(A_2, A_1)] \\ -i\tilde{B}_2[\Omega^-(A_2, A_1) - \Omega^+(A_2, A_1)] & \Omega^+(A_2, A_1) + \Omega^-(A_2, A_1) \end{pmatrix},
\end{equation}
where $\tilde{B}_j$ is the unitary map from $[\mathcal{D}(B_j)]$ onto $\mathfrak{h}$ defined as the unique continuous extension of $B_j$.

To apply these abstract results, we take $\mathfrak{h} = L^2(\mathbb{R})$ and we consider operator $\mathfrak{h}$ defined by (III.4), and we introduce operator
\begin{equation}
\mathfrak{h}_1 := \mathfrak{h} + m^2, \quad \mathcal{D}(\mathfrak{h}_1) = \mathcal{D}(\mathfrak{h})
\end{equation}
which we want to compare with the free hamiltonian
\[ h_2 := -\frac{d^2}{dz^2} + m^2, \quad \mathcal{D}(h_2) = H^2(\mathbb{R}). \]

Because of (V.15), we need a third operator
\[ h_3 := -\frac{d^2}{dz^2} + m^2 + \frac{15}{4} \left( \frac{1}{1 + |z|} \right)^2, \quad \mathcal{D}(h_3) = H^2(\mathbb{R}). \]

First we choose \( A_1 = h_2, A_2 = h_3 \). It is obvious that \( h_2 \) and \( h_3 \) are strictly positive selfadjoint operators. Moreover \( h_3 \) is a short range perturbation of \( h_2 \), by potential \( V(z) := \frac{15}{4} \left( \frac{1}{1 + |z|} \right)^2 \) and putting \( g(\zeta) = \left( \zeta^2 + m^2 \right)^{-1} \), we write
\[ (h_3 + 1)^{-1} - (h_2 + 1)^{-1} = (h_3 + 1)^{-1} V(z)g \left( \frac{d}{dz} \right). \]

Since \((1 + z^2)^{\frac{3}{2}} V(z)\) and \((1 + \zeta^2)^{\frac{3}{2}} g(\zeta)\) are in \( L^2(\mathbb{R}) \), a well known result (see [32], Theorem XI.21) assures that \( V(z)g \left( \frac{d}{dz} \right) \) is trace class, so (V.12) is satisfied. Moreover \( \mathcal{D}(h_3^2) = \mathcal{D}(h_2^2) = H^1(\mathbb{R}) \), hence \( \mathcal{F}_1 = \mathcal{F}_2 = H^1(\mathbb{R}) \times L^2(\mathbb{R}) \), and we see that hypotheses (V.11), (V.13), (V.14) and (V.15) are satisfied. We conclude that (V.17)
\[ W^\pm(h_2, h_3) \text{ exist and are complete.} \]

Now we take \( A_1 = h_1, A_2 = h_3 \). According to Lemma III.2 \( h_1 \) is strictly positive. (III.17) assures that \( \mathcal{D}(h_1^\frac{1}{2}) = \mathcal{D}(h_3^\frac{1}{2}) = H^1(\mathbb{R}) \) and for \( u \in H^1(\mathbb{R}) \) we have
\[ \min \left( 1, \frac{2m^2}{11} \right) \| h_2^\frac{1}{2} u \|^2 \leq \| h_3^\frac{1}{2} u \|^2 \leq \left( 1 + \frac{15}{4m^2} \right) \| h_2^\frac{1}{2} u \|^2 \]

hence (V.11), (V.13), (V.14) and (V.15) are satisfied. At last, given \( f \in L^2(\mathbb{R}), u = (h_3 + 1)^{-1} f - (h_1 + 1)^{-1} f \) is solution of
\[ -u'' + \frac{15}{4} \left( \frac{1}{1 + |z|} \right)^2 u + m^2 u = 0, \quad z \neq 0, \]

thus \((h_3 + 1)^{-1} - (h_1 + 1)^{-1}\) is a finite rank operator and (V.12) is satisfied. We conclude that (V.18)
\[ W^\pm(h_3, h_1) \text{ exist and are complete.} \]

We deduce from (V.17), (V.18) and the Chain Rule Theorem, that \( W^\pm(h_2, h_1) \) exist, are complete and partially isometric with initial projection \( P_1 \) and final projection \( P_2 \). It is clear that \( P_2 = \text{Id} \) and \( \mathcal{F}_2 = H_0^1(\mathbb{R}) \times L^2(\mathbb{R}) \). Moreover, Lemma 8.1 in [17] assures that \( (u, v) \in P_1 \mathcal{F}_1 \) iff \( v \in Q_1 h \) and \( h_1^\frac{1}{2} u \in Q_1 h \). Lemma III.2 implies that \( Q_1 h = K^0 \), and since \((h_1^\frac{1}{2} u, f_0) = (u, f_0)\), we deduce that \( h_1^\frac{1}{2} u \in Q_1 h \) iff \( u \in K^1 \) and \( P_1 \mathcal{F}_1 = K^1_{m} \times K^0 \). We conclude that \( W^\pm_m = W^\pm(h_2, h_1) \) is an isometry from \( K^1_{m} \times K^0 \) onto \( H_0^1(\mathbb{R}) \times L^2(\mathbb{R}) \). Finally for general initial data \( u_0 \in H^1(\mathbb{R}), u_1 \in L^2(\mathbb{R}) \), we apply the previous result to the projections on \( K^1 \times K^0 \) and we compute:
\[ A = \frac{k}{2} \int (1 + |z|)^{-\frac{3}{2}} \left[ u_0(z) - im^{-1}u_1(z) \right] dz, \quad B = \frac{k}{2} \int (1 + |z|)^{-\frac{3}{2}} \left[ u_0(z) + im^{-1}u_1(z) \right] dz. \]
The uniqueness of \( A, B, u^\pm, g^\pm \) is obvious.

\[ Q.E.D. \]
We return to the asymptotic behaviour of $\Phi$. We introduce
\[
\Phi_\infty^\pm(t, x, z) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\alpha \leq |\xi| \leq \beta} e^{i|\xi|} u^\pm(t, z; \xi) d\xi,
\]
where $u^\pm(t, z; \xi)$ is the solution of the free Klein-Gordon equation (V.9) with $m = |\xi|$, given by the previous Lemma and satisfying
\[
\| u(t, :, \xi) - u^\pm(t, :, \xi) \|_{H^1_{|\xi|}} + \| \partial_t u(t, :, \xi) - \partial_t u^\pm(t, :, \xi) \|_{L^2(\mathbb{R})} \to 0, \ t \to \pm \infty.
\]
Since $u^\pm \in C^0_0(\mathbb{R}_t^3; C^0(\mathbb{R}_t^3; H^1(\mathbb{R}_z^2)) \cap C^1(\mathbb{R}_t^3; L^2(\mathbb{R}_z^2)))$, $\Phi_\infty^\pm$ is a free wave satisfying (V.1) and (V.2). The Parseval equality gives
\[
\| \nabla_{t, x, z} \Phi(t) - \nabla_{t, x, z} \Phi_\infty^\pm(t) \|_{L^2(\mathbb{R}_x^3)}^2 + \| \partial_t \Phi(t) - \partial_t \Phi_\infty^\pm(t) \|_{L^2(\mathbb{R}_x^3)}^2 = \int_{\alpha \leq |\xi| \leq \beta} \| u(t, :, \xi) - u^\pm(t, :, \xi) \|_{H^1_{|\xi|}}^2 + \| \partial_t u(t, :, \xi) - \partial_t u^\pm(t, :, \xi) \|_{L^2(\mathbb{R})}^2 d\xi.
\]
We evaluate the integrand:
\[
\| u(t, :, \xi) - u^\pm(t, :, \xi) \|_{H^1_{|\xi|}}^2 + \| \partial_t u(t, :, \xi) - \partial_t u^\pm(t, :, \xi) \|_{L^2(\mathbb{R})}^2 \leq C \left( \| u(t, :, \xi) \|_{L^2(\mathbb{R}_x^3)}^2 + \| u^\pm(t, :, \xi) \|_{H^1_{|\xi|}}^2 + \| \partial_t u(t, :, \xi) \|_{L^2(\mathbb{R})}^2 + \| \partial_t u^\pm(t, :, \xi) \|_{L^2(\mathbb{R})}^2 \right).
\]
We conclude by the dominated convergence theorem that
\[
\| \nabla_{t, x, z} \Phi(t) - \nabla_{t, x, z} \Phi_\infty^\pm(t) \|_{L^2(\mathbb{R}_x^3)}^2 + \| \partial_t \Phi(t) - \partial_t \Phi_\infty^\pm(t) \|_{L^2(\mathbb{R}_x^3)}^2 \to 0, \ t \to \pm \infty,
\]
and the Wave Operators $W^\pm$ are well defined on $\mathcal{D}_0 \times \mathcal{D}_0$. Since these operators are isometric, they can be extended by density on $\mathbb{R}^1 \times \mathbb{R}^0$ and (V.3) and (V.6) are met. To prove (V.5), it is sufficient to show that when $\Phi_0, \Phi_1 \in \mathcal{D}_0$, then
\[
\int_{\mathbb{R}^4} \left( \frac{1}{1 + |z|} \right)^2 | \Phi(t, x, z) |^2 dxdz \to 0, \ |t| \to \infty.
\]
To prove this result, we use the $L^\infty$ estimate for the solutions of the Klein-Gordon equation (V.9), $| u(t, z) | \leq C(m + \frac{1}{|t|}) | mt |^{-\frac{1}{2}} \sum_{i=0,1} \sum_{j \leq 2^{-i}} \| \partial_i \partial_j^2 u(0, :, ) \|_{L^1(\mathbb{R}_z)}$:
\[
\int_{\mathbb{R}^4} \left( \frac{1}{1 + |z|} \right)^2 | \Phi(t, x, z) |^2 dxdz = \int_{\alpha \leq |\xi| \leq \beta} \left( \int_{-\infty}^{\infty} | u(t, z; \xi) |^2 \left( \frac{1}{1 + |z|} \right)^2 dz \right) d\xi \leq C |t|^{-1} \sum_{i=0,1} \sum_{j \leq 2^{-i}} \| \partial_i \partial_j^2 u(0, :, ) \|_{L^1(\mathbb{R}_z)} d\xi \leq C' |t|^{-1} \sum_{i=0,1} \sum_{j \leq 2^{-i}} \int_{\alpha \leq |\xi| \leq \beta} \left( (1 + |z|) \| \partial_i \partial_j^2 u(0, :, ) \|_{L^2(\mathbb{R}_z)}^2 d\xi \right) \leq C' |t|^{-1} \sum_{i=0,1} \sum_{j \leq 2^{-i}} \| (1 + |z|) \partial_i \partial_j^2 \Phi(t, x, z) \|_{L^2(\mathbb{R}_z)}^2 d\xi.
\]
Conversely we take $\Phi_\infty^j \in \mathcal{D}_0$ with $\Phi_\infty^j(\xi, z) = 0$ when $|\xi| \notin [\alpha, \beta]$ for some $0 < \alpha < \beta < \infty$, and we consider $\Phi_\infty \in C^1(\mathbb{R}_t; \mathcal{D}_0)$ solution of (V.1) and (V.2) with $\Phi_\infty(0, x, z) = \Phi_0^0(\xi, x, z)$. 


∂tΦ∞(0, x, z) = Φ1∞(x, z). Then u∞(t, z; ξ):= Φ∞(t, ξ, z) ∈ C0((R3; C0(R; H1(R)) ∩ C1(R; L2(R)))

is solution of (V.9) with m = |ξ|. Lemma V.2 assures there exists u±(t, z; ξ) ∈ C0((R3; C0(R; K1) ∩ C1(R; K0))
solution of (V.8) with m = |ξ| satisfying

\[ \| u∞(t, :, ξ) - u±(t, :, ξ) \|_{H^1[|t|]} + \| ∂tu∞(t, :, ξ) - ∂tu±(t, :, ξ) \|_{L^2(\mathbb{R})} \rightarrow 0, \quad t \rightarrow \pm \infty. \]

We put

\[ \Phi±(t, x, z) := \frac{1}{(2\pi)^{3/2}} \int_{|\alpha| \leq |\beta|} e^{i\alpha \cdot x} u±(t, z; \xi) d\xi. \]

that is solution of (II.1) in C0(R; R) and ∂tΦ± ∈ C0(R; R). As above we have

\[ \| \nabla_{t,x,z} \Phi±(t) - \nabla_{t,x,z} \Phi∞(t) \|_{L^2(\mathbb{R}^4)} + \| ∂tΦ±(t) - ∂tΦ∞(t) \|_{L^2(\mathbb{R}^4)} \]

\[ = \int_{|\alpha| \leq |\beta|} \| u±(t, :, ξ) - u∞(t, :, ξ) \|_{H^1[|t|]} + \| ∂tu±(t, :, ξ) - ∂tu∞(t, :, ξ) \|_{L^2(\mathbb{R})} \rightarrow 0, \quad t \rightarrow \pm \infty. \]

We conclude that we have constructed (W±)−1 on D0 × D0 that is a dense subspace of BL1(\mathbb{R}^4) × L2(\mathbb{R}^4). Finally (V.6) follows from (V.4) since \| Φ \|_{BL1(\mathbb{R}^4)} = \| Φ \|_{2\mathbb{R}} for the z-odd functions.

Q.E.D.

VI. SCATTERING AMPLITUDE AND BRANE RESONANCES

In this section we compute the scattering matrix and we investigate its holomorphic continuation. We recall that the spectral representation R of the free wave equation in R × R is defined by

\[ R: (Φ0, Φ1) ∈ BL1(\mathbb{R}^4) \times L^2(\mathbb{R}^4) \rightarrow \frac{|σ|^{3/2}}{\sqrt{2}} [σF_{x,z}Φ0(σω)+F_{x,z}Φ1(σω)] \in L^2(\mathbb{R} × S^3_ω), \]

where S^3_ω is the unit sphere of the euclidean space \mathbb{R}^4, \sigmaω = (ξ, \zeta) ∈ \mathbb{R}^3 × \mathbb{R}, and we have used the Fourier transform on \mathbb{R}^3 × \mathbb{R} defined for Φ ∈ L1(\mathbb{R}^3 × \mathbb{R}) by

\[ F_{x,z}Φ(ξ, ζ) = \frac{1}{(2π)^{3/2}} \int_{\mathbb{R}^4} e^{-i(x ξ + zζ)} Φ(x, z) dx dz. \]

We know (see e.g. [28]) that R is an isometry from BL1(\mathbb{R}^4 × \mathbb{R}^4) onto L2(\mathbb{R} × S^3_ω) and if Φ is a finite energy solution of the wave equation ∂t^2Φ − Δx,zΦ = 0, we have

\[ R(Φ(t,.), ∂tΦ(t,.)) = e^{iσt}R(Φ(0,.), ∂tΦ(0,.)). \]

It will be useful to distinguish the odd and the even part of the scattering operator. Thus we introduce spaces E± associated with E = R^1 × R^3 × BL^1 × L^2 by

\[ E± := \{ Φ ∈ E: Φ(x, −z) = ±Φ(x, z) \}. \]

It is obvious that R^1 × R^3 and BL^1 × L^2 are left invariant, respectively by the perturbed and the free dynamics. Therefore S± defined as the restriction of S to BL^1 × L^2 is an isometry of this space. We also introduce

\[ L^2_±(\mathbb{R} × S^3_ω) := \{ f ∈ L^2(\mathbb{R} × S^3_ω), f(σ, ω_1, ω_2, ω_3, −ω_4) = ±f(σ, ω_1, ω_2, ω_3, ω_4) \}. \]

Then, R is an isometry from BL^1(\mathbb{R}^4 × \mathbb{R}^4) onto L^2_±(\mathbb{R}^4 × \mathbb{R}^4), and in this part we establish first that the Scattering Operator S = S+ ⊕ S− is implemented by an explicit Scattering Amplitude acting as a multiplication operator on L^2_+(\mathbb{R} × S^3_ω) ⊕ L^2_-(\mathbb{R} × S^3_ω).
From the proof of Theorem V.1, we have by using the partial Fourier transform (III.32):

\[ W \]

Moreover these Scattering Amplitudes are explicitly known:

\[ U \]

Proof of Theorem VI.1. We compute \( S := RSR^{-1} \). If the inverse Fourier transform on \( \mathbb{R}^4_\xi \) is denoted by \( \mathcal{F}_{\xi,\zeta}^{-1} \), we easily check that

\[ W \]

where \( U \) is the unitary map given by

\[ Uf(\xi,\zeta) = \frac{\sigma - \frac{3}{2}}{i\sqrt{2}} \left( \frac{\sigma - f(\sigma,\omega) - f(-\sigma,-\omega)}{if(\sigma,\omega) + if(-\sigma,-\omega)} \right) \]

From the proof of Theorem VI.1 we have by using the partial Fourier transform (III.32):

\[ W \]

where \( W_{[\xi]}^m \) is given by Lemma V.2 with \( m = |\xi| \), and with (V.16) we get

\[ W \]

with

\[ 0 \leq m, \ h_m := -\partial_x^2 + m^2, \ \mathcal{D}(h_m) = H^2(\mathbb{R}) \]

We deduce that the scattering operator has the form:

\[ S \]

where \( s \) is the usual scattering operator associated with the Schrödinger equation \( i\partial_t u - hu = 0 \):

\[ s := \Omega^+(h_0, h) \circ [\Omega^-(h_0, h)]^{-1} \]

We introduce the self-adjoint operators \( (h_{0,\pm}, \mathcal{D}(h_{0,\pm})) \), on \( L^2(\mathbb{R}^+) \) defined by:

\[ h_{0,\pm} := -\frac{d^2}{dz^2} \]

We easily see that \( (h_{0,\pm}, h_0) \) satisfies (V.12) and so the waves operators \( \Omega^\pm(h_{0,\pm}, h_\pm) \) exist and are complete on \( L^2(\mathbb{R}^+_z) \). From the intertwining relations (III.8) and \( (h_{0,\pm} \oplus h_{0,\pm}) \mathcal{P} = \mathcal{P}h_0 \), we deduce that:

\[ \Omega^\pm(h_0, h) = \mathcal{P}^{-1} [\Omega^\pm(h_{0,\pm}, h_\pm) \oplus \Omega^\pm(h_{0,\pm}, h_\pm)] \mathcal{P} \]
and if we introduce the isometries on \( L^2(\mathbb{R}^+) \) given by
\[
s_{\pm} := \Omega^\pm(h_{0,\pm}, h_{\pm}) \circ \left[ \Omega^{-}(h_{0,\pm}, h_{\pm}) \right]^{-1},
\]
we obtain the splitting of the scattering operator as the sum of an even part and an odd part :
\[
s = P^{-1}[s_{+} \oplus s_{-}] P.
\]
We now want to determine the form of \( \hat{s} := \mathcal{F}_z s \mathcal{F}_z^{-1} \) on \( L^2(\mathbb{R}_\zeta) \) where \( \mathcal{F}_z \) is the Fourier transform
\[
f \in L^1(\mathbb{R}_z), \quad \mathcal{F}_z f(\zeta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iz\zeta} f(z) dz.
\]
We need the Fourier sine and Fourier cosine transforms which are the unitary maps \( \mathcal{F}_s \) on \( L^2(\mathbb{R}^+) \) defined by
\[
f \in L^1(\mathbb{R}^+), \quad \mathcal{F}_s f(m) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \cos(mz)f(z) dz, \quad \mathcal{F}_c f(m) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \sin(mz)f(z) dz.
\]
We now can express the scattering matrix :
\[
\hat{s} = P^{-1}[\mathcal{F}_s s_+ \mathcal{F}_s^{-1} \oplus \mathcal{F}_s s_- \mathcal{F}_s^{-1}] P.
\]
In a first time, we admit that \( \mathcal{F}_s s_+ \mathcal{F}_s^{-1} \) is a multiplication operator \( \hat{s}_+(m) \) on \( L^2(\mathbb{R}_m^+) \). Given \( f \in L^2(\mathbb{R}_\sigma \times S^3_\omega) \) we write \( f = f_+ \oplus f_- \), \( f_{\pm} \in L^2(\mathbb{R}_\sigma \times S^3_\omega) \), and we compute :
\[
Sf_{\pm}(\sigma, \omega) = \frac{1}{2} \left( 1 + \frac{\sigma}{\sigma} \right) \hat{s}_{\pm}(\mu_m l)(m) f_{\pm}(\sigma, \omega) + \frac{1}{2} \left( 1 - \frac{\sigma}{\sigma} \right) \hat{s}_{\pm}(\mu)(m) f_{\pm}(\sigma, \omega)
\]
and we deduce that
\[
Sf(\sigma, \omega) = \left[ \hat{s}_+(\sigma, \omega) f_+(\sigma, \omega) + \hat{s}_-(\sigma, \omega) f_-(\sigma, \omega) \right] 1_{0,\infty}(|\sigma|) + \left[ \hat{s}_+(\sigma, \omega) f_+(\sigma, \omega) - \hat{s}_-(\sigma, \omega) f_-(\sigma, \omega) \right] 1_{-\infty,0}(|\sigma|).
\]
We now compute \( \hat{s}_\pm \). We know (see e.g. [27], Theorem 8.1), that when the free hamiltonian is \( h_{0,-} \), the scattering operator for the perturbed Schrödinger operator on the half line, is determined by the scattering phase shift \( \delta_-(m) \) defined by \( u_-(z, m) = \sqrt{\frac{2}{\pi}} \sin (mz + \delta_-(m) + o(1)) \) with \( \lim_{z \to \infty} o(1) = 0 \), and the formula
\[
\hat{s}_-(m) = e^{i2\delta_-(m)}.
\]
From (III.17) we get
\[
\hat{s}_-(m) = i e^{2im \left( \frac{H_2(2)}{H_2(1)} \right)(m)}.
\]
The same argument assures that if
\[
s_+ := \Omega^+(h_{0,-}, h_{+}) \circ \left[ \Omega^{-}(h_{0,-}, h_{+}) \right]^{-1},
\]
then for \( f \in L^2(\mathbb{R}_m^+) \) we have :
\[
\mathcal{F}_- \mathcal{F}_+^{-1} f(m) = i e^{2im \left( \frac{H_2(2)}{H_2(1)} \right)(m)} f(m).
\]
Since \( \cos(mz) = \sin(mz + \frac{\pi}{2}) \), the scattering phase shift for the pair \( (h_{0,-}, h_{0,+}) \) is \( \frac{\pi}{2} \), hence formula (8.2.6) in [27] assures that \( \Omega^{\pm}(h_{0,-}, h_{0,+}) = \pm i \mathcal{F}_-^{-1} \circ \mathcal{F}_+ \). Now from the chain rule, we have \( s_+ = (\Omega^+(h_{0,-}, h_{0,+}))^{-1} \circ s_+ \circ \Omega^{-}(h_{0,-}, h_{0,+}) \), then we conclude that
\[
\hat{s}_+(m) = -i e^{2im \left( \frac{H_1(2)}{H_1(1)} \right)(m)}.
\]
Thus (VI.1) and (VI.2) follow from (VI.3). Moreover, by the relation of conjugation

$$H^{(1)}_\nu(z) = \overline{H^{(2)}_{\nu}(z)},$$

and the asymptotic behaviour \(\lim_{x \to 0^+} \frac{H^{(2)}_{\nu}(x)}{H^{(1)}_\nu(x)} = -1\), we get \(S_\pm \in C^0(\mathbb{R}_\sigma^* \times S^3_\omega; S^1)\). Q.E.D.

We now study the analytic continuation of the Scattering Amplitudes. We denote \(Z^{(j)}_\nu\) the set of the zeros of \(H^{(j)}_\nu(z)\), \(\nu \in \mathbb{N}\), in the Riemann surface of the logarithm \(\tilde{\mathbb{C}}^*\). Thanks to the relation of conjugation, \(z \in Z^{(1)}_\nu \Leftrightarrow \overline{z} \in Z^{(2)}_\nu\). The zeros of \(H^{(1)}_\nu\) in the principal Riemann sheet \(\{z; -\pi < \arg z \leq \pi\}\) are well known ([1], p.373, [26], figure 14) ; they are located in the open lower half-plane \(\{z; -\pi < \arg z < 0\}\) and are of two types : (i) an infinite number of zeros \(z_n\) just below the negative real semiaxis with \(\Re z_n < -\nu\) and \(\lim_{n \to \infty} \Im z_n = -\frac{1}{2} \log 2\); (ii) a finite set of \(\nu\) zeros with \(|\Re z| < \nu\), which lie along the lower half of the boundary of an eye-shapes domain around the origin. By using the relation

$$H^{(1)}_{\nu}(z e^{-\im \pi}) = (-1)^{\nu m} \left( (m + 1)H^{(1)}_\nu(z) + mH^{(1)}_\nu(\overline{z}) \right),$$

Olver [26] has proved that the asymptotic repartition in the other Riemann sheets are analogous, and the projection of \(Z^{(j)}_\nu\) on \(\mathbb{C}^*\) is a lattice with asymptotes \(\Im z = \frac{1}{2} \log \left(\frac{|4m-1|+1}{|4m-1|-1}\right)\), \(\Im z = \frac{1}{2} \log \left(\frac{|4m-1|+3}{|4m-1|-3}\right)\). We note that the previous relation shows the symmetry of the zeros with respect to the imaginary axis:

$$H^{(1)}_{\nu}(z) = 0 \Leftrightarrow H^{(1)}_{\nu}(e^{\im \pi}z) = 0.$$  

We emphasize that on each sheet, there is no accumulation near the real axis, but there exists a sequence \(z_n \in Z^{(j)}_\nu\) such that \(\arg z_n \to \infty\), \(\Im z_n \to 0\), \(n \to \infty\). We recall the value of the firsts of them (see e.g. [12]).

| Zeros of \(H^{(1)}_1\) | Zeros of \(H^{(1)}_2\) |
|------------------------|------------------------|
| \(-\pi < \arg z \leq \pi\) | \(-\pi < \arg z \leq \pi\) |
| \(-3\pi < \arg z \leq -\pi\) | \(-3\pi < \arg z \leq -\pi\) |
| \(-0, 419 - 0.577 i\) | 0, 333 + 0, 413i |
| \(-3, 832 - 0.355 i\) | 3, 832 + 0, 208i |
| \(-7, 016 - 0, 349 i\) | 7, 016 + 0, 204i |
| \(-10, 173 - 0, 348 i\) | 10, 137 + 0, 203i |
| \(-13, 326 - 0, 348 i\) | 13, 326 + 0, 204i |

| \(-0, 429 - 1, 281 i\) | 0, 429 + 1, 281i |
| \(-1, 317 - 0, 836 i\) | 1, 146 + 0, 652i |
| \(-5, 138 - 0, 372 i\) | 5, 136 + 0, 281i |
| \(-8, 418 - 0, 356 i\) | 8, 417 + 0, 208i |
| \(-11, 620 - 0, 351 i\) | 11, 620 + 0, 206i |

We show that the singularities of the Scattering Amplitudes \(\sigma \mapsto S_\pm(\sigma) \in C^0(S^3_\omega)\), called Brane Resonances, form a lattice of radial half straight lines :

$$\Sigma^{(1)}_\nu := \left\{ z = \alpha z_\ast \in \tilde{\mathbb{C}}^*, \ 1 \leq \alpha, \ H^{(1)}_{\nu}(z_\ast) = 0 \right\}, \quad \Sigma^{(2)}_\nu := \left\{ z = \alpha z_\ast \in \tilde{\mathbb{C}}^*, \ 1 \leq \alpha, \ H^{(2)}_{\nu}(-z_\ast) = 0 \right\},$$

with \(\nu = 1, 2\). In the figure below, the first lines of resonances in the principal sheet \(-\pi < \arg z \leq \pi\), and the second sheet \(-3\pi < \arg z \leq -\pi\), are depicted.
Theorem VI.2. The Scattering Amplitude \( S_{+[-]}(\sigma, \omega) \) considered as a \( C^0(\Sigma^3_\omega) \)-valued function of \( \sigma \in [0, \infty[ \) (respectively \( \sigma \in ]-\infty, 0] \)) has an analytic continuation on \( \Sigma^\alpha \) (respectively \( \Sigma^\beta \)). For \( \sigma \in \Sigma^\alpha \), there exists \( C > 0 \) such that \( C \mid \sigma - \sigma_* \mid^{-1} \leq \| S_{+[-]}(\sigma) \|_{L^\infty(S^3_\omega)} \) as \( \sigma \to \sigma_* \), \( \sigma \in \Sigma^\alpha \setminus \Sigma^\beta \).

Proof of Theorem VI.2. First we fix \( \omega \in S^3 \) with \( \omega_4 \neq 0 \). We know that \( H_\nu^{(j)}(z) \) is holomorphic on \( \Sigma^\alpha \) and its \( z \)-zeros are non real and simple ( [25], p. 244). Moreover \( H_\nu^{(1)}(z) \) and \( H_\nu^{(2)}(z) \) have no common zero, because otherwise such a complex would be a non real zero of \( J_\nu \) that has only real zeros ( [25], p. 245). We conclude that \( \sigma \in [0, \infty[ \to S_{+[-]}(\sigma, \omega) \) has a meromorphic continuation on \( \Sigma^\alpha \), is holomorphic in \( D_{+[-]} := \Sigma^\alpha \setminus \Sigma^\beta \), and its poles are simple. Now for \( \sigma_0 \) fixed in \( \Sigma^\alpha \setminus \Sigma^\beta \), \( \sigma_0 \mid \omega_4 \mid \) belongs to \( D_{+[-]} \) for \( \omega_4 \neq 0 \). Moreover the Neumann expansions of the Bessel functions of first and second kind ( [25], p.243), imply that \( H_\nu^{(2)}(\sigma_0 \mid \omega_4 \mid)/H_\nu^{(1)}(\sigma_0 \mid \omega_4 \mid) \to -1 \) as \( \omega_4 \to 0 \). We deduce that \( S_{\pm}(\sigma_0) \in C^0(S^3_\omega) \). To prove that the Scattering Amplitudes are \( C^0(S^3_\omega) \)-valued holomorphic functions on \( D_{+[-]} \), we use the Dunford theorem ( [39], p. 128). We take a Radon measure \( \mu \) on \( S^3_\omega \). Then for any piecewise-\( C^1 \) loop \( \gamma \) in \( D_{+[-]} \), the Fubini theorem and the Cauchy theorem give

\[
\oint_{\gamma} \left( \int_{S^3} S_\pm(\sigma, \omega) d\mu(\omega) \right) d\sigma = \int_{S^3} \left( \oint_{\gamma} S_\pm(\sigma, \omega) d\sigma \right) d\mu(\omega) = 0,
\]

thus the Moreira theorem assures that \( \mu(\Sigma_{\pm}(\sigma, \cdot)) \) is holomorphic and we get the result by the Dunford theorem. To prove that the resonances are actually singularities, we pick \( \sigma_* \in \Sigma^\alpha \) and we choose \( \omega_* \in S^3 \) with \( \omega_4 = \alpha^{-1} \). Since \( z_* \) is a simple zero of \( H_\nu^{(1)}(z) \) and not a zero for \( H_\nu^{(2)}(z) \), there exists \( \varepsilon, \delta, C > 0 \) such that:

\[
|\sigma - \sigma_*| \leq \delta \Rightarrow \left| H_\nu^{(2)}(\sigma \mid \omega_4 \mid) \right| \geq \varepsilon, \quad \left| H_\nu^{(1)}(\sigma \mid \omega_4 \mid) \right| \leq C \mid \sigma - \sigma_* \mid.
\]
We conclude that \(| S_{+[-]}(\sigma,\omega) | \geq \delta | \sigma - \sigma_* |^{-1}\) when \(| \sigma - \sigma_* | \leq \delta\). The proof of the holomorphic continuation of \(\sigma \in] - \infty, 0]\) \(\mapsto S_{+[-]}(\sigma,\omega)\) on \(\mathbb{C}^* \setminus \Sigma_{1[2]}^{(2)}\) is similar.

\[Q.E.D.\]

We end this part by some remarks on the \textit{brane quasimodes}. In the physical litterature, this term means a solution of the master equation with an exponential dumping in time as \(t \to \infty\) and this solution is associated with some resonance. We can easily construct such solutions. For any \(\zeta^{(j)}_{\nu} \in \mathcal{Z}^{(j)}_{\nu}\) there functions

\[
u \in \mathcal{Z}^{(j)}_{\nu}\)

\[u^{(j)}_{\nu}(z) := \sqrt{1 + | z |} H^{(j)}_{2}(\zeta^{(j)}_{\nu}(1 + | z |))\]

\[u^{(j)}_{\nu}(z) := \frac{z}{1 + | z |} \sqrt{1 + | z |} H^{(j)}_{2}(\zeta^{(j)}_{2}(1 + | z |))\]

are solution of \(\nu u^{(j)}_{\nu} = (\zeta^{(j)}_{\nu})^2 u^{(j)}_{\nu}\). If we choose any solution \(\phi(\zeta^{(j)}_{\nu}; t, x)\) of the Klein-Gordon equation with complex mass \(k \zeta^{(j)}_{\nu}\),

\[(VI.4)\]

\[\partial^2_t \phi - \Delta_x \phi + (\zeta^{(j)}_{\nu})^2 \phi = 0,\]

then \(\Phi(t, x, z) = \phi(\zeta^{(j)}_{\nu}; t, x) u^{(j)}_{\nu}(z)\) is solution of the master equation \((III.1)\). In particular, for any singularity \(\sigma^{(j)}_{\nu} \in \Sigma^{(j)}_{\nu}\) of the Scattering Matrix, we can associate a quasimode \(\Phi(\sigma^{(j)}_{\nu}; t, x, z)\) such that

\[\Phi(\sigma^{(j)}_{\nu}; t, x, z) \sim C_{\pm} e^{i \zeta^{(j)}_{\nu} |x| |z| - t}, \quad z \to \pm \infty\]

where \((\omega', \omega_4) \in S^3\). Given \(\sigma^{(1)}_{\nu} = \alpha \zeta^{(1)}_{\nu} \in \Sigma^{(1)}_{\nu}\) or \(\sigma^{(2)}_{\nu} = -\alpha \zeta^{(2)}_{\nu} \in \Sigma^{(2)}_{\nu}, 1 \leq \alpha,\) we choose \(\omega \in S^3\) with \(| \omega_4 | = \alpha^{-1}\), and we put

\[\Phi(\sigma^{(j)}_{\nu}; t, x, z) := e^{i \zeta^{(j)}_{\nu} (x, \omega' - t)} u^{(j)}_{\nu}(z).\]

An elementary Fourier analysis shows that the finite energy solutions of \((VI.4)\) are superpositions of plane waves solutions, \(e^{i (\lambda_{\nu} t + x \cdot \xi)}\) with \(\xi \in \mathbb{R}^3\), and \(\lambda_{\nu}\) satisfies the relation of dispersion

\[(VI.5)\]

\[\lambda_{\nu}^2 = | \xi |^2 + \left( \zeta^{(j)}_{\nu} \right)^2.\]

The behaviour in time of these plane waves is exponential since \(\Im \lambda_{\nu} \neq 0\). In the same spirit, a second kind of quasimodes that play an important role in physics of the branes, are the solutions that propagate on the brane as an harmonic plane wave \(e^{i x \cdot \xi}\):

\[\Phi(\lambda_{\nu}; t, x, z) := e^{i (\lambda_{\nu} t + x \cdot \xi)} u^{(j)}_{\nu}(z).\]

These last modes can be associated with the singularities of the analytic continuation of the truncated resolvents (see Corollary \((III.6)\) of which the common domain of analyticity \(\cap_{R_0 > 0} \mathcal{O}_{R_0}\) is depicted on Figure 3 (the curves are defined by the relation of dispersion \((VI.5)\)). We remark that the usual definitions of the resonances, on the one hand as singularities of the scattering matrix, and on the other hand as singularities of the truncated resolvent, are not equivalent in the case of the brane scattering. The common singularities are the zeros of the Hankel functions, that are the ends of the half-hyperbolas and the half straight lines.
We let open the hard problem consisting to give a rigorous meaning to the following possible asymptotic representation of the solutions of (II.1):

\[
\Phi(t, x, z) \sim \sum_{\zeta^{(j)} \in \mathbb{Z}, -\pi < \arg \zeta^{(j)} < 0} \phi(\zeta^{(j)}; t, x) u^{(j)}(z), \quad t \to \infty.
\]

A similar Lax-Phillips formula has been established for another important cosmological model, the De Sitter-Schwarzschild manifold \[8\]. It allows to perform a numerical scheme of computation of the resonances and leads to a direct method to observe a black-hole from the detection of the gravitational waves \[6\].

In the case of the brane scattering that we have investigated in this paper, expansion (VI.6) is expected by the physicists (see e.g. \[34\]). It would be interpreted as a “dark radiation” on the brane, constituted of metastable massive gravitons with mass equal to $\Re \zeta^{(j)}$ and mean lifetime $\sim \Im \zeta^{(j)}^{-1}$ (\[34\], formula (12)). While the space dimension is even and the brane is a non-decaying perturbation, such a resonances expansion is not totally hopeless since there is no accumulation of singularities of the scattering matrix near the real axis on the first sheet, although it is the case for the singularities of the resolvent.

We refer to the deep works by Burq and Zworski \[10\] and Tang and Zworski \[36\] for similar results in the euclidean framework.

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