ON THE CLASSIFICATION OF KANTOR PAIRS AND STRUCTURABLE ALGEBRAS IN CHARACTERISTIC 5

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Abstract. We observe that any finite-dimensional central simple 5-graded Lie algebra over a field $k$ of characteristic $\neq 2,3$ is necessarily classical, i.e. a twisted form of a central quotient of a Chevalley Lie algebra. Consequently, the classification of central simple structurable algebras and Kantor pairs over fields of characteristic 5 derives from the classification of simple algebraic groups.

1. Introduction

In 2008 A. Premet and H. Strade [PS08b] completed the classification of finite-dimensional simple Lie algebras over an algebraically closed field of characteristic $\neq 2,3$. As a result, the Block–Wilson–Premet–Strade classification theorem [Str09] establishes that every such Lie algebra is one of the following:

- “classical” simple Lie algebras, classified by Dynkin diagrams $A_l$, $D_l$, $E_6$, $E_7$, $E_8$, $F_4$, $G_2$.
- Cartan type Lie algebras in characteristic $p > 3$.
- Melikian Lie algebras in characteristic $p = 5$.

We say that a finite-dimensional central simple Lie algebra $\mathcal{L}$ over a field $k$ is of Chevalley type, if it is the central quotient of the tangent Lie algebra of a simply connected simple algebraic group over $k$, or, equivalently, the derived Lie algebra of the Lie algebra of an adjoint simple algebraic group. In [BDMS1] we have established that if $\mathcal{L}$ is a central simple Lie algebra of Chevalley type with a non-trivial $\mathbb{Z}$-grading over a field $k$ of characteristic $\neq 2,3$, then there is a simple Kantor pair $\mathcal{K}$ over $k$ such that $\mathcal{L}$ is the 5-graded Lie algebra to $\mathcal{K}$. If, moreover, $\text{char } k \neq 5$, then we showed that the converse is also true, namely, any central simple 5-graded Lie algebra is of Chevalley type.

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Previously, the non-degeneracy of simple Kantor pairs over a ring with invertible 2, 3, 5 was established in [GGLN11]. Among the simple Lie algebras in the Block–Wilson–Premet–Strade classification, only the Lie algebras of Chevalley type are non-degenerate [Sel67, p. 124], while all simple Lie algebras of Cartan and Melikian type are degenerate (this observation goes back to A. I. Kostrikin and A. Premet; see e.g. [Wil76] and [Str09, Corollary 12.4.7]). Thus, the non-degeneracy of simple Kantor pairs implies that the associated 5-graded Lie algebras are of Chevalley type.

In the present paper we extend the above results to the characteristic 5 case. Using the Block–Wilson–Premet–Strade classification, we deduce that any central simple 5-graded Lie algebra over a field of characteristic $\neq 2, 3, 5$ is of Chevalley type. Consequently, for any central simple structurable algebra or Kantor pair over such a field the associated 5-graded Lie algebra is of Chevalley type, and hence their classification derives from the classification of isotropic simple algebraic groups. This allows to extend to $\text{char} = 5$ the classification of central simple structurable algebras over fields of characteristic $\neq 2, 3, 5$ due to Allison and Smirnov [All78, Smi92].

We also describe all 5-gradings on simple Lie algebras of Chevalley type over algebraically closed fields $k$ of characteristic $\neq 2, 3$ that make them into 5-graded Lie algebras associated with structurable algebras. The classification is combinatorial and independent of $k$. It derives from the classification of nilpotent classes in such algebras [LS12].

All commutative rings we consider are assumed to be associative and unital. All algebras are finite-dimensional over the respective scalars unless explicitly mentioned otherwise.

2. Graded Lie algebras

**Definition 2.1.** Let $R$ be a commutative ring, and let $\mathcal{L} = \bigoplus_{i \in \mathbb{Z}} \mathcal{L}_i$ be a $\mathbb{Z}$-graded Lie algebra over $R$. We say that $\mathcal{L}$ is $(2n+1)$-graded, if $\mathcal{L}_i = 0$ for all $i \in \mathbb{Z}$ such that $|i| > n$. The grading is non-trivial, if $\mathcal{L} \neq \mathcal{L}_0$.

**Construction 2.2.** For any $(2n+1)$-graded Lie algebra $\mathcal{L}$ over $R$ we define the $(2n+1)$-graded Lie algebra

$$\tilde{\mathcal{L}} = \bigoplus_{i \in \mathbb{Z}} \tilde{\mathcal{L}}_i$$

over $R$ as follows. For any $i \neq 0$, $\tilde{\mathcal{L}}_i = \mathcal{L}_i$. For $i = 0$, we define $\tilde{\mathcal{L}}_0$ to be the set of all $\phi = (\phi_i) \in \prod_{i \in \mathbb{Z} \setminus \{0\}} \text{End}_R(\mathcal{L}_i)$ satisfying the following conditions:

$$\phi_{i+j}([a, b]) = [\phi_i(a), b] + [a, \phi_j(b)]$$

for all $-n \leq i, j \leq n$, $i, j \neq 0$, $i \neq -j$; $a \in \mathcal{L}_i$, $b \in \mathcal{L}_j$;

$$\phi_j([a, b], c) = [[\phi_i(a), b], c] + [[a, \phi_i(b)], c] + [[a, b], \phi_j(c)]$$

for all $-n \leq i, j \leq n$, $i, j \neq 0$, $a \in \mathcal{L}_i$, $b \in \mathcal{L}_{-i}$, $c \in \mathcal{L}_j$.

In other words, $\phi \in \tilde{\mathcal{L}}_0$ behaves as a derivation of $\mathcal{L}$ that preserves grading, except that $\phi$ is not defined on $\mathcal{L}_0$. Clearly, $\tilde{\mathcal{L}}_0$ is an $R$-module.
We define the Lie bracket \([-,-]_\tilde{\mathcal{L}}\) on \(\tilde{\mathcal{L}}\) in terms of the Lie bracket \([-,-]\) on \(\mathcal{L}\) as follows. For any \(u \in \tilde{\mathcal{L}}_i, v \in \tilde{\mathcal{L}}_j, i, j \in \mathbb{Z}\), we let

\[
[u,v]_\tilde{\mathcal{L}} = \begin{cases} 
[u,v] & \text{if } i \neq j, \ i, j \neq 0; \\
(\text{ad}([u,v])|_{\mathcal{L}_i})_{i \in \mathbb{Z}\setminus\{0\}} & \text{if } i = -j, \ i \neq 0; \\
u(v) & \text{if } i = 0, \ j \neq 0; \\
-v(u) & \text{if } i \neq 0, \ j = 0; \\
uv - vu & \text{if } i = j = 0.
\end{cases}
\]

\[\text{(2)}\]

It is straightforward to check that \(\tilde{\mathcal{L}}\) is indeed a \((2n+1)\)-graded Lie algebra over \(R\). Since it is not likely to provoke confusion, in what follows we denote the Lie bracket on \(\tilde{\mathcal{L}}\) simply by \([-,-]\).

Note that the Lie algebra \(\tilde{\mathcal{L}}\) by construction contains an element \(\zeta \in \tilde{\mathcal{L}}_0\) which acts as a grading derivation:

\[\zeta, x] = ix \quad \text{for any } i \in \mathbb{Z} \text{ and } x \in \tilde{\mathcal{L}}_i.\]

It is also easy to see that there is a natural graded Lie algebra homomorphism \(\mathcal{L} \to \tilde{\mathcal{L}}\) that sends any element \(x \in \mathcal{L}_0\) to \((\text{ad}(x)|_{\mathcal{L}_i})_{i \in \mathbb{Z}\setminus\{0\}}\).

**Lemma 2.3.** [BDMSt, Lemma 4.1.4] Let \(\mathcal{L}\) be a \((2n+1)\)-graded finite-dimensional Lie algebra over a field \(k\). Then for any commutative \(k\)-algebra \(R\), there is a natural isomorphism of \((2n+1)\)-graded \(R\)-Lie algebras

\[\tilde{\mathcal{L}} \otimes_k R \cong (\mathcal{L} \otimes_k R)\tilde{\mathcal{L}}.\]

To simplify the notation, under the assumptions of Lemma 2.3 we will consider the Lie algebras \(\mathcal{L}\) and \(\tilde{\mathcal{L}}\) as functors on the category of \(k\)-algebras \(R\), so that, by definition,

\[\tilde{\mathcal{L}}(R) = \tilde{\mathcal{L}} \otimes_k R \cong (\mathcal{L} \otimes_k R)\tilde{\mathcal{L}}.\]

Similarly, \(\text{Aut}(\mathcal{L})\) and \(\text{Der}(\mathcal{L})\) will stand for the functors of Lie automorphisms and Lie derivations of \(\mathcal{L}\) respectively:

\[\text{Aut}(\mathcal{L})(R) = \text{Aut}_R(\mathcal{L} \otimes_k R), \quad \text{Der}(\mathcal{L})(R) = \text{Der}_R(\mathcal{L} \otimes_k R).\]

Note that \(\text{Aut}(\mathcal{L})\) and \(\text{Der}(\mathcal{L})\) are naturally represented by closed \(k\)-subschemas of the affine \(k\)-scheme of linear endomorphisms of \(\mathcal{L}\). In particular, Lemmas 2.3 and 2.4 imply that there is an isomorphism of functors

\[\tilde{\mathcal{L}} \cong \text{Der}(\mathcal{L}).\]

**Lemma 2.4.** [BDMSt, Lemma 4.1.5] Let \(\mathcal{L}\) be a central simple finite-dimensional \((2n+1)\)-graded Lie algebra over a field \(k\), and let \(K\) be a field extension of \(k\).

(i) Let \(I_i \subseteq \mathcal{L}(K)_i, i \in \mathbb{Z}\setminus\{0\},\) be \(K\)-subspaces that satisfy the following conditions: for all \(i, j \in \mathbb{Z}\setminus\{0\}\) such that \(i \neq -j\) we have

\[\mathcal{L}(K)_i, I_j] \subseteq I_{i+j};\]

\[\mathcal{L}(K)_i, I_{-i}] \mathcal{L}(K)_i] \subseteq I_i;\]

\[\mathcal{L}(K)_i, I_{-i}] \mathcal{L}(K)_{-i}] \subseteq I_{-i}.\]

Then either \(I_i = 0\) for all \(i \in \mathbb{Z}\setminus\{0\}\), or \(I_i = \mathcal{L}(K)_i\) for all \(i \in \mathbb{Z}\setminus\{0\}\).
(ii) If \( f \in \text{Aut}(\mathcal{L})(K) \) satisfies \( f|_{\mathcal{L}(K)} = \text{id}_{\mathcal{L}(K)} \) for all \( i \in \mathbb{Z} \setminus \{0\} \), then \( f = \text{id}_{\mathcal{L}(K)} \).

**Definition 2.5.** Let \( \mathcal{L} = \bigoplus_{i \in \mathbb{Z}} \mathcal{L}_i \) be a \( \mathbb{Z} \)-graded Lie algebra over a field \( k \). Consider the 1-dimensional split \( k \)-subtorus \( S \cong \mathbb{G}_m \) of \( \text{Aut}(\mathcal{L}) \) defined as follows: for any \( k \)-algebra \( R \), any \( t \in \mathbb{G}_m(R) \), and any \( i \in \mathbb{Z}, v \in \mathcal{L}_i \otimes_k R \), we set \( t \cdot v = t^i v \). We call \( S \) the grading torus of \( \mathcal{L} \).

**Notation 2.6.** Let \( \mathcal{L} \) be a finite-dimensional 5-graded Lie algebra over a field \( k, \text{char } k \neq 2, 3 \). For any commutative \( k \)-algebra \( R \) and any \( (x, s) \in (\mathcal{L} \otimes_k R)_\sigma \oplus (\mathcal{L} \otimes_k R)_{2\sigma} = (\mathcal{L}_\sigma \oplus \mathcal{L}_{2\sigma}) \otimes_k R \) we set

\[
e_{\sigma}(x, s) = \sum_{i=0}^{4} \frac{1}{i!} \text{ad}(x + s)^i \in \text{End}_R(\mathcal{L} \otimes_k R).
\]

**Definition 2.7.** Let \( \mathcal{L} \) be a finite-dimensional 5-graded Lie algebra over \( k \). The grading derivation on \( \mathcal{L} \) is the derivation \( \zeta \in \text{Der}_k(\mathcal{L}) \) such that for any \( -2 \leq i \leq 2 \) and any \( x \in \mathcal{L}_i \) one has

\[
\zeta(x) = i \cdot x \quad \text{for any } -2 \leq i \leq 2 \text{ and any } x \in \mathcal{L}_i.
\]

If \( \mathcal{L} \) contains an element \( \zeta \) such that \( \text{ad}_\zeta \) is the grading derivation, we call \( \zeta \) a grading derivation of \( \mathcal{L} \) by abuse of language.

**Definition 2.8.** Let \( \mathcal{L} \) be a finite-dimensional 5-graded Lie algebra over a commutative ring \( R \) such that \( 2, 3 \in R^\times \). We say that \( x \in \mathcal{L} \) is algebraic, if the endomorphism \( \sum_{i=0}^{4} \frac{1}{i!} \text{ad}(x)^i \) is an automorphism of \( \mathcal{L} \) as an \( R \)-Lie algebra. We say that \( \mathcal{L} \) is algebraic, if all elements \( (x, s) \in \mathcal{L}_\sigma \oplus \mathcal{L}_{2\sigma} \) are algebraic.

**Remark 2.9.** By [BDMS1] Lemma 3.1.7] any 3-graded Lie algebra, and hence any Jordan algebra over \( R \) with \( 2, 3 \in R^\times \) is algebraic; any 5-graded Lie algebra, and hence any structurable algebra over \( R \) with \( 2, 3, 5 \in R^\times \) is algebraic. (Despite that lemma is stated for algebras over a field, the proof is also valid over commutative rings.) By [BDMS1] Theorem 4.2.8] any central simple structurable division algebra over a field of characteristic \( \neq 2, 3 \) is algebraic.

**Lemma 2.10.** Let \( \mathcal{L} \) be a finite-dimensional 5-graded Lie algebra over a field \( k, \text{char } k \neq 2, 3 \).

(i) Let \( R \) be any commutative associative unital \( k \)-algebra. If \( (x, s) \in \mathcal{L}_\sigma \oplus \mathcal{L}_{2\sigma} \) is algebraic, then \((\lambda x, \mu s)\) is an algebraic element of \( \mathcal{L} \otimes_k R \) for any \( \lambda, \mu \in R \).

(ii) Let \( \mathcal{L}' \) be another finite-dimensional 5-graded Lie algebra over \( k \), and let \( f: \mathcal{L} \to \mathcal{L}' \) be a graded \( k \)-homomorphism of Lie algebras, such that \( f|_{\mathcal{L}_i}: \mathcal{L}_i \to \mathcal{L}'_i \) is a bijection for all \( i \in \{ \pm 1, \pm 2 \} \). If \( (x, s) \in \mathcal{L}_\sigma \oplus \mathcal{L}_{2\sigma} \) is algebraic in \( \mathcal{L} \), then \( f(x, s) \) is algebraic in \( \mathcal{L}' \).

**Proof.** The proof of (i) is the same as in [BDMS1] Lemma 3.1.6]. The proof of (ii) is the same as in [BDMS1] Lemma 3.1.8]. \( \square \)

The following results relate algebraicity to the classification of simple Lie algebras.

**Theorem 2.11.** [BDMS1] Theorem 4.1.8] Let \( \mathcal{L} \) be an algebraic central simple 5-graded Lie algebra over a field \( k \) of characteristic different from 2, 3, such that \( \mathcal{L} \neq \mathcal{L}_0 \). Then the algebraic \( k \)-group \( G = \text{Aut}(\mathcal{L})^9 \) is an adjoint absolutely simple group of \( k \)-rank \( \geq 1 \), satisfying \( \mathcal{L} = [\text{Lie}(G), \text{Lie}(G)] \) and \( \text{Lie}(G) \cong \mathcal{L} \).
Lemma 2.12. [BDMSt, Lemma 4.2.4 (i)] Let \( k \) be a field, \( \text{char} k \neq 2, 3 \). Let \( G \) be an adjoint simple algebraic group over \( k \). Let \( L = \text{Lie}(G) \) be its Lie algebra. Let \( L = \bigoplus_{i=-2}^{5} L_i \) be any 5-grading on \( L \) such that \( L_1 \oplus L_{-1} \neq 0 \). The 5-graded Lie algebra \( L \) is algebraic.

Theorem 2.13. Let \( L \) be a central simple 5-graded Lie algebra over a field \( k \) of characteristic different from 2, 3, such that \( L \neq L_0 \). Then \( L \) is of Chevalley type if and only if \( L \) is algebraic.

Proof. This follows from the two previous results and Lemma 2.10, since Lie algebras of adjoint simple algebraic groups are by definition the central simple Lie algebras of Chevalley type. \( \square \)

3. 5-Graded Simple Lie Algebras

Lemma 3.1. Let \( L \neq L_0 \) be a simple 5-graded Lie algebra over a field \( k \) of characteristic \( \neq 2, 3 \), and let \( S \) be the corresponding grading torus. Let \( L = \bigoplus_{i=-r}^{s} L_i \) be another \( \mathbb{Z} \)-grading on \( L \) such that \( \bigoplus_{i=-r}^{1} L_i \) is generated as a Lie algebra by \( L_{[-1]} \neq 0 \). If the second grading is preserved by \( S(k) \), then \( L_{[-1]} \not\subseteq L_0 \).

Proof. Since \( S \) preserves the second grading, we have

\[
L_{[i]} = \bigoplus_{j=-2}^{2} (L_{[j]} \cap L_{[i]})
\]

for all \(-r \leq i \leq s\). Assume that \( L_{[-1]} \subseteq L_0 \). Then \( L_{[-1]} \subseteq L_0 \) for all \( 1 \leq i \leq r \). For any \( x \in L_j \), \( -2 \leq j \leq 2 \), we have \( t \cdot x = t^j x \) for any \( t \in S(k) \). Since \( \text{char}(k) \neq 2, 3 \), there is \( t \in k \) such that \( t^{\pm 1} \neq 1 \) and \( t^{\pm 2} \neq 1 \). Then \( L_j \subseteq \bigoplus_{i \geq 0} L_{[i]} \) for all \( j \neq 0 \). Since \( L \) is simple, it is generated by \( L_j \), \( j \neq 0 \), hence \( L = \bigoplus_{i \geq 0} L_{[i]} \). However, this contradicts \( L_{[-1]} \neq 0 \). \( \square \)

Lemma 3.2. Let \( L \) be one of the simple graded Cartan type Lie algebras \( X(m, \frac{1}{2})^{(2)} \), \( X \in \{ W, S, H, K \} \), in the notation of [Str04], over an algebraically closed field \( k \) of characteristic 5. Then \( L \) does not have a non-trivial 5-grading.

Proof. Let \( L = \bigoplus_{i=-2}^{2} L_i \) a non-trivial 5-grading on \( L \), and let \( L = \bigoplus_{i=-r}^{s} L_{[i]} \) be the standard grading on \( L \). Note that \( r = 1 \) for \( X = W, S, H \), and \( r = 2 \), \( \dim(L_{[-2]}) = 1 \) for \( X = K \).

By [Str04, Theorem 7.4.1] we can assume that the grading torus \( S \) of the 5-grading preserves the standard grading. Then by Lemma 3.1 the 5-grading on \( L_{[-1]} \) is non-trivial. Since \( L_{[-1]} \) is contained in the restricted subalgebra \( X(m, \frac{1}{2})^{(2)} \) of \( L \), the induced 5-grading on the restricted subalgebra is non-trivial, and hence we can assume that \( L = X(m, \frac{1}{2})^{(2)} \) from the start. In particular, \( \text{ad}(L_{[-2]})^5 = 0 \).

Let \( x \in L_{[-1]} \) be an element of non-zero 5-grading. Then \( \text{ad}(x)^3(L) \) is contained in a nilpotent Lie subalgebra \( L_{\pm 1} \oplus L_{\pm 2} \) of \( L \). By [Wil75, Theorem 2] the space \( L_{[-1]} \) is an irreducible representation for the group of Lie algebra automorphisms of \( L \) preserving the standard grading. Thus, \( L_{[-1]} \) has a basis consisting of elements \( y \) such that \( \text{ad}(y)^3(L) \) is contained in a nilpotent Lie subalgebra of \( L \). Since \( L \) is restricted, we have \( \text{ad}(L_{[-2]})^5 = 0 \). Then the Jacobi identity readily implies that

\[
\text{ad}(L_{[-1]})^{2 \cdot \dim(L_{[-1]}) + 1 + 2 \cdot 4 \cdot \dim(L_{[-2]})}(L) \subseteq \text{ad}(y)^3(L)
\]

for an element \( y \) as above. On the other hand, we have \( L_{[0]} \subseteq \text{ad}(L_{[-1]})^s(L) \). Since \( L_{[0]} \) is not nilpotent, this implies that

\[
s < 2 \cdot \dim(L_{[-1]}) + 1 + 2 \cdot 4 \cdot \dim(L_{[-2]}).
\]
If $X = W, S, H$, then $\dim(L_{-1}) = m$. Hence (7) becomes $s < 2m + 1$. If $X = W$, then $m \geq 1$ and $s = 4m - 1$. If $X = S$, then $m \geq 2$ and $s = 4m - 2$. If $X = H$, then $s = 4m - 3$ and $m \geq 2$. This contradicts (7).

If $X = K$, then $\dim(L_{-1}) = m - 1$. Hence (7) becomes $s < 2(m - 1) + 9 = 2m + 7$. We have $s = 4m$, if $m + 3 \equiv 0 \pmod{5}$, and $s = 4m + 1$ otherwise. Since $m \geq 3$, both cases contradict (7).

**Theorem 3.3.** Let $L$ be a central simple $5$-graded Lie algebra over an algebraically closed field $k$ of characteristic different from $2, 3$, such that $L \neq L_0$. Then $L$ is a classical simple (Chevalley) Lie algebra.

**Proof.** If $\text{char} \ k \neq 5$, then $L$ is a Chevalley Lie algebra by Remark 2.13 combined with Theorem 2.13. Assume $\text{char} \ k = 5$. According to the Block–Wilson–Premet–Strade classification theorem [Str04], it is enough to check that $L$ is not of Cartan or Melikian type.

Assume first that $L$ is a simple Lie algebra of Cartan type [Str04, Definition 4.2.4]. Let $L = L_{(-r)} \supseteq \ldots \supseteq L_{(s)}$ be a standard filtration of $L$. By [Str04, Theorem 4.2.7 (3)] the standard filtration is invariant under all automorphisms of $L$. In particular, it is invariant under the grading torus $S$ of the 5-grading. Let

$$Gr(L) = \bigoplus_{i=-r}^{s} Gr(L)_{[i]}$$

be the associated graded Lie algebra. Hence $Gr(L)$ carries an induced 5-grading. The induced 5-grading is non-trivial, since there is $0 \neq x \in L_i$, $i \neq 0$, and $t \in F^* \subseteq S(k)$ such that $t \cdot x = t^ix \neq x$, and hence $t$ acts non-trivially on the image of $x$ in $Gr(L)$.

The derived series of $Gr(L)$ also inherits the 5-grading, hence $Gr(L)_{(\infty)}$ is 5-graded. We show that the induced 5-grading on $Gr(L)_{(\infty)}$ is also non-trivial. Indeed, if it were trivial, then $Gr(L)_{(\infty)} \subseteq Gr(L)_{(0)}$. Since $Gr(L)$ is also a Lie algebra of Cartan type in the sense of [Str04, Definition 4.2.4], $Gr(L)_{(\infty)}$ is simple by [Str04, Theorem 4.2.7 (1)]. Then we have

$$Gr(L)_{(\infty)} \subseteq (Gr(L)_{(0)})_{(\infty)} \subseteq Gr(L)_{(\infty)},$$

which implies $Gr(L)_{(\infty)} = (Gr(L)_{(0)})_{(\infty)}$. Then by [Str04, Lemma 4.2.5] we have $Gr(L) = Gr(L)_{(0)}$, which contradicts the non-triviality of the 5-grading on $Gr(L)$.

By [Str04, Theorem 4.2.7 (2)] we have $Gr(L)_{(\infty)} = X(m,n)^{(2)}$, $X = W, S, H, K$. By Lemma 3.2 these algebras do not have non-trivial 5-gradings.

Now let $L = M(2, n_1, n_2)$ be a simple Lie algebra of Melikian type. By [BKM15, Theorem 1.2] we can assume that the grading torus corresponding to the 5-grading under consideration preserves the standard grading $L = \bigoplus_{i=-3}^{s} L_{[i]}$ of $L$. Then the grading derivation $\zeta$ corresponding to the 5-grading is a homogenous derivation of $L$. Let $W(2, n_1, n_2)$ be the standard simple Witt subalgebra of $L$. Then by [Str04, Theorem 7.1.4] any homogeneous derivation of $L$ acts non-trivially on $W(2, n_1, n_2)$, and hence this subalgebra carries a non-trivial 5-grading induced by $\zeta$. However, this is not possible by Lemma 3.2. □

**Theorem 3.4.** Let $L$ be a central simple 5-graded Lie algebra over a field $k$ of characteristic different from $2, 3$, such that $L \neq L_0$. Then $L$ is of Chevalley type.

**Proof.** Let $\bar{k}$ be the algebraic closure of $k$. Then $L \otimes_k \bar{k}$ is a central simple Lie algebra over $\bar{k}$ with a non-trivial 5-grading. Then by Theorem 3.3 $L \otimes_k \bar{k}$ is a Lie algebra of Chevalley type.
Then it is algebraic by Theorem 2.13. Hence $L$ is algebraic, since $L$ embeds into $L \otimes_k \bar{k}$. Again by Theorem 2.13 we conclude that $L$ is of Chevalley type. \hfill \square

4. Structurable algebras and Kantor pairs associated to algebraic groups

Definition 4.1. A structurable algebra over a field $k$ of characteristic not 2 or 3 is a finite-dimensional, unital $k$-algebra with involution $(A, \bar{\cdot})$ such that

\begin{equation}
[V_{x,y}, V_{z,w}] = V_{(x,y,z),w} - V_{z,(y,x,w)}
\end{equation}

for $x, y, z, w \in A$, where the left hand side denotes the Lie bracket of the two operators, and where

$$V_{x,y}z := \{x \bar{y} z\} := (x \bar{y})z + (z \bar{y})x - (z \bar{x})y.$$  

For all $x, y, z \in A$, we write $U_{x,y}z := V_{x,z}y$ and $U_{x}y := U_{x,x}y$. The trilinear map $(x, y, z) \mapsto \{x \bar{y} z\}$ is called the triple product of the structurable algebra.

In [All78] and [All79], a structurable algebra is defined as an algebra with involution such that

\begin{equation}
[T_z, V_{x,y}] = V_{T_z x, y} - V_{x, T_z y}
\end{equation}

for all $x, y, z \in A$ with $T_x := V_{x,1}$. The equivalence of (8) and (9) follows from [All79, Corollary 5.(v)].

Definition 4.2. Let $(A, \bar{\cdot})$ be a structurable algebra; then $A = \mathcal{H} \oplus \mathcal{S}$ for

$$\mathcal{H} = \{h \in A \mid \bar{h} = h\} \quad \text{and} \quad \mathcal{S} = \{s \in A \mid s = -s\}.$$

The elements of $\mathcal{H}$ are called hermitian elements, the elements of $\mathcal{S}$ are called skew-hermitian elements or briefly skew elements. The dimension of $\mathcal{S}$ is called the skew-dimension of $A$.

As usual, the commutator and the associator are defined as

$$[x, y] = xy - yx, \quad [x, y, z] = (xy)z - x(yz),$$

for all $x, y, z \in A$. For each $s \in \mathcal{S}$, we define the operator $L_s : A \to A$ by

$$L_s x := sx.$$  

The following map is of crucial importance in the study of structurable algebras:

$$\psi : A \times A \to \mathcal{S} : (x, y) \mapsto x \bar{y} - y \bar{x}.$$  

Definition 4.3. An ideal of $A$ is a two-sided ideal stabilized by $\bar{\cdot}$. A structurable algebra $(A, \bar{\cdot})$ is simple if its only ideals are $\{0\}$ and $A$, and it is called central if its center $Z(A, \bar{\cdot}) = Z(A) \cap \mathcal{H}$ is equal to $k1$. The radical of $A$ is the largest solvable ideal of $A$. A structurable algebra is semisimple if its radical is zero.
If \( \text{char}(k) \neq 2, 3, 5 \), a semisimple structurable algebra is the direct sum of simple structurable algebras [Smi92, Section 2].

Recall the generalization of the Tits–Kantor–Koecher construction that associates to any structurable algebra \( \mathcal{A} \) a 5-graded Lie algebra \( K(\mathcal{A}) \).

Let \( \operatorname{End}(\mathcal{A}) \) be the ring of \( k \)-linear maps from \( \mathcal{A} \) to \( \mathcal{A} \). For each \( A \in \operatorname{End}(\mathcal{A}) \), we define new \( k \)-linear maps

\[
A^e = A - L_{A^{(1)+}}, \\
A^\delta = A + R_{A^{(1)}}.
\]

where \( L_x \) and \( R_x \) denote left and right multiplication by an element \( x \in \mathcal{A} \), respectively. Define the Lie subalgebra \( \text{Strl}(\mathcal{A}, -) \) of \( \operatorname{End}(\mathcal{A}) \) as

\[
\text{Strl}(\mathcal{A}, -) = \{ A \in \operatorname{End}(\mathcal{A}) \mid [A, V_{x,y}] = V_{Ax,y} + V_{x,Ay} \}.
\]

(This definition follows from [All78, Corollary 5].) It follows from the definition of structurable algebras that \( V_{x,y} \in \text{Strl}(\mathcal{A}, -) \), so we can define the Lie subalgebra

\[
\text{Inn}(\mathcal{A}, -) = \text{Span}\{V_{x,y} \mid x, y \in \mathcal{A}\},
\]

which is, in fact, an ideal of \( \text{Strl}(\mathcal{A}, -) \).

**Definition 4.4.** Consider two copies \( \mathcal{A}_+ \) and \( \mathcal{A}_- \) of \( \mathcal{A} \) with corresponding isomorphisms \( \mathcal{A} \to \mathcal{A}_+ : x \mapsto x_+ \) and \( \mathcal{A} \to \mathcal{A}_- : x \mapsto x_- \), and let \( \mathcal{S}_+ \subset \mathcal{A}_+ \) and \( \mathcal{S}_- \subset \mathcal{A}_- \) be the corresponding subspaces of skew elements. Let

\[
K(\mathcal{A}) = \mathcal{S}_- \oplus \mathcal{A}_- \oplus \text{Inn}(\mathcal{A}) \oplus \mathcal{A}_+ \oplus \mathcal{S}_+
\]

as a vector space; as in [All79, §3], we make \( K(\mathcal{A}) \) into a Lie algebra by extending the Lie algebra structure of \( \text{Inn}(\mathcal{A}) \) as follows:

- **[Inn, \( K(\mathcal{A}) \)]**

  \[
  [V_{a,b}, V_{a',b'}] = V_{(a,b,a'),(b,a',b)} - V_{a',(b,a,b')}, \quad [V_{a,b}, y_-] = V_{Ax,y} + V_{x,Ay}, \quad [V_{a,b}, y_-] = (-V_{b,a}y)_- \in \mathcal{A}_-, \n  \]

- **[\( \mathcal{S}_\pm, \mathcal{A}_\pm \)]**

  \[
  [s_+, x_+] := 0, \quad [t_-, y_-] := 0, \quad [s_+, y_-] := (sy)_+ \in \mathcal{A}_+, \quad [t_-, x_+] := (tx)_- \in \mathcal{A}_-, \n  \]

- **[\( \mathcal{A}_\pm, \mathcal{A}_\pm \)]**

  \[
  [x_+, y_-] := V_{x,y} \in \text{Inn}(\mathcal{A}), \quad [x_+, x_+] := \psi(x, x')_+ \in \mathcal{S}_+, \quad [y_-, y_-] := \psi(y, y')_- \in \mathcal{S}_-, \n  \]

- **[\( \mathcal{S}_\pm, \mathcal{S}_\pm \)]**

  \[
  [s_+, s'_+] := 0, \quad [t_-, t'_-] := 0, \n  \]


for all $x, x', y, y' \in \mathcal{A}$, all $s, s', t, t' \in \mathcal{S}$, and all $V_{a, b}, V_{a', b'} \in \text{Inn}(A)$.

From the definition of the Lie bracket we clearly see that the Lie algebra $K(A)$ has a 5-grading given by $K(A)_j = 0$ for all $|j| > 2$ and

$$K(A)_{-2} = \mathcal{S}_-, \quad K(A)_{-1} = \mathcal{A}_-, \quad K(A)_0 = \text{Inn}(A),$$

$$K(A)_1 = \mathcal{A}_+., \quad K(A)_2 = \mathcal{S}_+.$$  

In the case where $\mathcal{A}$ is a Jordan algebra, we have $\mathcal{S} = 0$, and thus the Lie algebra $K(A)$ has a 3-grading; in this case $K(A)$ is exactly the Tits–Kantor–Koecher construction of a Lie algebra from a Jordan algebra.

It is shown in [All79, §5] that the structurable algebra $\mathcal{A}$ is simple if and only if $K(\mathcal{A})$ is a simple Lie algebra, and that $\mathcal{A}$ is central if and only if $K(\mathcal{A})$ is central.

**Definition 4.5.** A structurable algebra $\mathcal{A}$ is called algebraic, if $K(\mathcal{A})$ is algebraic.

**Definition 4.6.** Let $\mathcal{A}$ be a structurable algebra over a field $k$ of characteristic $\neq 2, 3$. An element $x \in \mathcal{A}$ is called an absolute zero divisor if $Uxy = 0$ for any $y \in \mathcal{A}$. The algebra $\mathcal{A}$ is called non-degenerate if it has no non-trivial absolute zero divisors.

If an element $x \in K(\mathcal{A})_\sigma = \mathcal{A}_\sigma$ is an absolute zero divisor of $K(\mathcal{A})$, then it is represented by an absolute zero divisor of $\mathcal{A}$; this follows from the fact that by Definition 4.4,

$$[x_\sigma, [x_\sigma, y_\sigma]] = -V_{x,y}x \in \mathcal{A}_\sigma$$

for all $x, y \in \mathcal{A}$.

The following theorem strengthens [BDMS04, Theorem 4.1.1].

**Theorem 4.7.** Let $\mathcal{A}$ be a central simple structurable algebra over a field $k$ of characteristic different from 2, 3. Then the algebraic $k$-group $G = \text{Aut}(K(\mathcal{A}))^\circ$ is an adjoint absolutely simple group of $k$-rank $\geq 1$, and $K(\mathcal{A}) = [\text{Lie}(G), \text{Lie}(G)]$.

**Proof.** The same claim was established in [BDMS04, Theorem 4.1.1] under the additional assumption that $\mathcal{A}$ is algebraic. The Lie algebra $K(\mathcal{A})$ is a central simple Lie algebra with a non-trivial 5-grading, hence it is a Lie algebra of Chevalley type by Theorem 3.4. Hence it is algebraic by Theorem 2.13.

Given a non-trivial 5-grading on a Lie algebra of Chevalley type, there may not be a structurable algebra associated to it. However, we always obtain an associated Kantor pair by [BDMS04, Lemma 4.3.3].

**Definition 4.8** ([AF99]). A Kantor pair is a pair of finite-dimensional vector spaces $(K_+, K_-)$ over $k$ equipped with a trilinear product

$$\{\cdot, \cdot, \cdot\}: K_+ \times K_- \times K_+ \to K_\sigma, \quad \sigma \in \{-1, 1\},$$

satisfying the following two identities:

(KP1) $[V_{x,y}, V_{z,w}] = V_{\{x,y,z\}, w} = V_{z,\{y,x,w\}}$;

(KP2) $K_{a,b}V_{x,y} + V_{y,x}K_{a,b} = K_{K_{a,b}, x,y}$;

where $V_{x,y,z} := \{xyz\}$ and $K_{a,b,z} := \{azb\} - \{bza\}$. 
There is a tight connection between Kantor pairs and Lie triple systems. In [AF99], a \( \mathbb{Z} \)
graded Lie triple system \( \mathcal{T} = \bigoplus_{i \in \mathbb{Z}} \mathcal{T}_i \) is called sign-graded, if \( \mathcal{T}_i = 0 \) for all \( i \neq \pm 1 \). By [AF99] Theorem 7 for any Kantor pair on \( \mathcal{T} = K_+ \oplus K_- \) there is a natural structure of a sign-graded Lie triple system with two non-zero graded components \( \mathcal{T}_i = K_+ \) and \( \mathcal{T}_{-i} = K_- \).

Let \( \mathcal{T} = \mathcal{T}_i \oplus \mathcal{T}_{-i} \) be a sign-graded Lie triple system over \( k \). In [AF99, p. 532] the 5-graded Lie algebra \( g(\mathcal{T}) = \bigoplus_{i=-2}^2 g(\mathcal{T}_i) \) is defined, which is called the standard graded embedding of \( \mathcal{T} \).

If \( \mathcal{T} = K_+ \oplus K_- \) is the Lie triple system corresponding to a Kantor pair \( (K_+, K_-) \), the 5-graded Lie algebra \( g(K_+ \oplus K_-) \) is also called the standard graded embedding of this Kantor pair.

According to the following result, every 5-grading on a simple Lie algebra of Chevalley type over a field of characteristic \( \neq 2,3 \) arises from a Kantor pair. In particular, every isotropic adjoint simple algebraic group can be constructed from a simple Kantor pair.

**Lemma 4.9.** Let \( k \) be a field of characteristic different from 2,3. Let \( G \) be an adjoint simple algebraic group over \( k \). Let \( \mathcal{L} = \text{Lie}(G) \) be its Lie algebra, and let \( \mathcal{L} = \bigoplus_{i=-2}^2 \mathcal{L}_i \) be any 5-grading on \( \mathcal{L} \) such that \( \mathcal{L}_1 \oplus \mathcal{L}_{-1} \neq 0 \). Then \( (\mathcal{L}_1, \mathcal{L}_{-1}) \) is a central simple Kantor pair with respect to the triple product operation \( \mathcal{L}_\sigma \times \mathcal{L}_{-\sigma} \times \mathcal{L}_{\sigma} \to \mathcal{L}_{\sigma} \) given by

\[
\{x, y, z\} = -[[x, y], z],
\]

and its standard 5-graded Lie algebra \( g = g(\mathcal{L}_1 \oplus \mathcal{L}_{-1}) \) is canonically isomorphic to the graded subalgebra \([\mathcal{L}, \mathcal{L}] + k\zeta \) of \( \mathcal{L} \).

**Proof.** By [BDMS7] Lemma 4.3.3 \( (\mathcal{L}_1, \mathcal{L}_{-1}) \) is a Kantor pair, and \( g(\mathcal{L}_1 \oplus \mathcal{L}_{-1}) \) is as required. It remains to prove centrality and simplicity. By [BDMS8] Lemma 4.1.6 the \( k \)-Lie algebra \([\mathcal{L}, \mathcal{L}] \) is central simple, and differs from \( \mathcal{L} \) only in the grading 0 component. Let \( (\mathcal{L}_1, \mathcal{L}_{-1})' \) be the Kantor pair which differs from \( (\mathcal{L}_1, \mathcal{L}_{-1}) \) by the sign of the triple product, i.e. \( \{x, y, z\}' = [[x, y], z] \). Then \([\mathcal{L}, \mathcal{L}] \) envelopes the Kantor pair \( (\mathcal{L}_1, \mathcal{L}_{-1})' \) in the sense of [AFS17]. Then by [AFS17] Theorem 4.20 \( (\mathcal{L}_1, \mathcal{L}_{-1})' \) is central simple. Clearly, this is equivalent to \( (\mathcal{L}_1, \mathcal{L}_{-1}) \) being central simple. \( \square \)

Now we can establish the converse as well, namely, that any simple Kantor pair over a field of characteristic \( \neq 2,3 \) arises from an isotropic simple algebraic group.

**Theorem 4.10.** Let \( (K_+, K_-) \) be a central simple Kantor pair over a field \( k \) of characteristic \( \neq 2,3 \), and let \( g(K_+ \oplus K_-) \) be its standard 5-graded Lie algebra. Then the algebraic \( k \)-group \( G = \text{Aut}(g(K_+ \oplus K_-))^{\circ} \) is an adjoint absolutely simple group of \( k \)-rank \( \geq 1 \), and \( g(K_+ \oplus K_-) \cong [\text{Lie}(G), \text{Lie}(G)] + k\zeta \).

**Proof.** Set \( \mathcal{L} = g(K_+ \oplus K_-) \), so that \( \mathcal{L}_1 = K_+ \), \( \mathcal{L}_{-1} = K_- \). Let \( (\mathcal{L}_1, \mathcal{L}_{-1})' \) be the Kantor pair which differs from \( (\mathcal{L}_1, \mathcal{L}_{-1}) \) by the sign of the triple product, i.e. \( \{x, y, z\}' = -\{x, y, z\} = [[x, y], z] \). Since \( (\mathcal{L}_1, \mathcal{L}_{-1}) \) is central simple, \( (\mathcal{L}_1, \mathcal{L}_{-1})' \) is also central simple. Let \( \mathcal{K} = \mathfrak{K}(\mathcal{L}_1, \mathcal{L}_{-1})' \) be the 5-graded Lie algebra associated to \( (\mathcal{L}_1, \mathcal{L}_{-1})' \) in [AFS17, § 4.3]. By construction, \( g(K_+ \oplus K_-) = k\zeta + \mathcal{K} \). By [AFS17] Corollary 4.14 \( \mathcal{K} \) is central simple. Then by Theorem 3.3 \( \mathcal{K} \) is of Chevalley type. Then by Theorem 2.13 it is algebraic. Then by [BDMS7] Theorem 4.1.8 \( G = \text{Aut}(\mathcal{K})^{\circ} \) is an adjoint absolutely simple group of \( k \)-rank \( \geq 1 \), and \( \mathcal{K} \cong [\text{Lie}(G), \text{Lie}(G)] \). Then by [BDMS8] Lemma 4.1.6 we have \( \text{Aut}(g(K_+ \oplus K_-))^{\circ} \cong G \). \( \square \)
Corollary 4.11. Let \((K_+, K_-)\) be a simple Kantor pair over a commutative ring \(k\) such that \(2, 3 \in \ker(k)\). Then \((K_+, K_-)\) is non-degenerate in the sense of [GGLN11].

Proof. Since \((K_+, K_-)\) is simple, \(k\) does not have non-trivial ideals, and hence is a field. Extending \(k\), we can assume that \((K_+, K_-)\) is central simple. Then by Theorem 4.10 its standard 5-graded embedding \(L\) is a central simple Lie algebra of Chevalley type. Then \(L \otimes_k \bar{k}\) is non-degenerate. Then \(L\) is non-degenerate. Then by [GGLN11, Corollary 2.5] \((K_+, K_-)\) is non-degenerate.

Lemma 4.12. Let \(R\) be a commutative ring with \(2, 3 \in \ker(R)\). Let \(L = g(L_{-1} \oplus L_1)\) be the 5-graded Lie algebra over \(R\) associated with a Kantor pair \((\L_{-1}, \L_1)\) over \(R\). Then \((\L_{-1}, \L_1)\) is the Kantor pair associated with a structurable algebra \(A\) over \(R\) if and only if there exist \(u \in \L_1, v \in \L_{-1}\) such that \(\zeta = [u, v]\) is the grading derivation of \(L\).

Proof. Assume first that \(L\) is graded-isomorphic to \(K(A)\). Let \(1 \in \L_1, \hat{1} \in \L_{-1}\) denote the images of \(1 \in A\). Then \([1, \hat{1}] \in \L_0\) acts as the grading derivation \(\zeta\) on \(L \cong K(A)\).

Conversely, assume that \(\zeta = [u, v]\) for some \(u \in \L_1, v \in \L_{-1}\). By [AF99, Corollary 14] an element \((x, 0) \in \L_1 \oplus \L_2\) is 1-invertible if and only if there is \(\hat{x} \in \L_{-1}\) such that \(V_{\hat{x}, x} = \hat{2} \text{id}_{\L}, V_{\hat{x}, x} = 2 \text{id}_{\L_{-1}}\). Since \([u, v] = \zeta\), we have \(V_{u, v} = -\text{id}_{\L}, V_{v, u} = -\text{id}_{\L_{-1}}\). Then \(x = u\) is 1-invertible with \(\hat{x} = -2v\). Since \((\L_1, \L_{-1})\) contains a 1-invertible element of the form \((x, 0)\), by [AF99, Corollary 15] it is isomorphic to a Kantor pair associated with a structurable algebra.

Lemma 4.13. Let \(k\) be a field of characteristic different from \(2, 3\). Let \(G\) be an adjoint simple algebraic group over \(k\). Let \(L = \text{Lie}(G)\) be its Lie algebra, and let \(L = \bigoplus_{i=2}^2 L_i\) be any 5-grading on \(L\) such that \(L_1 \oplus L_{-1} \neq 0\), and let \(\zeta \in \L_0\) be the grading derivation. Then \(\zeta = [u, v]\) for some \(u \in \L_1, v \in \L_{-1}\) if and only if there is a structurable algebra \(A\) over \(k\) such that \([L, L]\) is graded-isomorphic to \(K(A)\).

Proof. Assume first that \(L\) is graded-isomorphic to \(K(A)\). Let \(1 \in \L_1, \hat{1} \in \L_{-1}\) denote the images of \(1 \in A\). Then \([1, \hat{1}] \in \L_0\) acts as the grading derivation \(\zeta\) on \([L, L]\). Since \(L \cong \text{Der}([L, L])\) by [BDMST, Lemma 4.1.6], \([1, \hat{1}] = \zeta\).

Conversely, assume that \(\zeta = [u, v]\) for some \(u \in \L_1, v \in \L_{-1}\). Consider the Kantor pair \((\L_{-1}, \L_1)\) with the triple product operation \(\L_\alpha \times \L_{-\alpha} \times \L_\alpha \to \L_\alpha\) given by

\[
\{x, y, z\} = -[[x, y], z].
\]

By Lemma 4.10 \(g(L_1 \oplus L_{-1})\) is canonically isomorphic to the graded subalgebra \([L, L] + k\zeta\) of \(L\). Then by Lemma 4.12 the pair \((\L_{-1}, \L_1)\) is associated to a structurable algebra \(A\). Then \(K(A)\) is graded-isomorphic to \([g(L_1 \oplus L_{-1}), g(L_1 \oplus L_{-1})] \cong [L, L]\). □

5. 5-Gradings that correspond to structurable algebras

Definition 5.1. Let \(G\) be an algebraic group over a field \(k\) and \(T \subseteq G\) be a split \(n\)-dimensional \(k\)-subtorus of \(G\). Let \(X^*(T) \cong \mathbb{Z}^n\) be the group of characters of \(T\), and let

\[\text{Lie}(G) = \bigoplus_{\alpha \in X^*(T)} \text{Lie}(G)_\alpha\]
be the $\mathbb{Z}^n$-grading on $\text{Lie}(G)$ induced by the adjoint action of $T$. We call
\[ \Phi(T, G) = \{ \alpha \in X^*(T) \mid \text{Lie}(G)_\alpha \neq 0 \} \]
the set of roots of $G$ with respect to $T$.

If $G$ is a reductive algebraic group over $k$ and $T$ is a maximal split $k$-subtorus of $G$, then \( \Phi(T, G) \setminus \{0\} \) is a root system in the sense of Bourbaki [BT65]. By abuse of language, we call $\Phi(T, G)$ a root system of $G$.

Let $\Phi$ be a root system and $\Pi \subseteq \Phi$ be a system of simple roots. For any $\alpha \in \Phi$ we write $\alpha = \sum_{\beta \in \Pi} m_\beta(\alpha) \beta$, where the coefficients $m_\beta(\alpha)$ are either all non-negative, or all non-positive. Once $\Pi$ is fixed, we denote the corresponding sets of positive and negative roots by $\Phi^+$. Recall that for any pair of opposite parabolic subgroups $P_\pm$ of $G$ with unipotent radicals $U_\pm$, there is a maximal split $k$-subtorus $T$ of $G$ such that
\[ T \subseteq P_+ \cap P_- \]
and a system of simple roots $\Pi \subseteq \Phi$ and a non-empty subset $J \subseteq \Pi$ such that
\begin{align*}
\Phi(T, P_\sigma) &= \Phi^\sigma \cup (\Phi \cap \mathbb{Z}(\Pi \setminus J)), \\
\Phi(T, U_\sigma) &= \Phi^\sigma \setminus \mathbb{Z}(\Pi \setminus J), \\
\Phi(T, P_+ \cap P_-) &= \Phi \cap \mathbb{Z}(\Pi \setminus J).
\end{align*}

The set $t(P_+) = \Pi \setminus J$ is called the type of $P_+$; it is a system of simple roots of the root system of the Levi subgroup $P_+ \cap P_-$ of $P_+$. In general, the type of a parabolic subgroup $P$ is the type of $P$ over an algebraic closure of the field of definition.

**Notation 5.2.** Let $G$ be a reductive algebraic group over a field $k$, and let $\lambda : \mathbb{G}_m \to G$ be a cocharacter of $G$ over $k$. Then $\lambda$ induces a $\mathbb{Z}$-grading on $\text{Lie}(G)$, and we denote by $\text{Lie}(G)(\lambda, i)$ the $i$-th component of this grading, i.e.
\[ \text{Lie}(G)(\lambda, i) = \{ v \in \text{Lie}(G) \mid \lambda(t) \cdot v = t^i v \text{ for any } t \in \mathbb{G}_m(k) = k^x \}. \]

By [DG70, Exp. XXVI, Prop. 6.1] there is a unique pair of (not necessarily proper) opposite parabolic subgroups $P(\lambda)$ and $P(-\lambda)$ in $G$ such that
\[ \text{Lie}(P(\lambda)) = \bigoplus_{i \geq 0} \text{Lie}(G)(\lambda, i) \quad \text{and} \quad \text{Lie}(P(-\lambda)) = \bigoplus_{i \geq 0} \text{Lie}(G)(\lambda, -i). \]

We denote by $C_G(\lambda)$ the centralizer of $\lambda(\mathbb{G}_m)$ in $G$; this is a Levi subgroup $P(\lambda) \cap P(-\lambda)$ of $P(\lambda)$.

In order to classify $5$-gradings that correspond to structurable algebras, we rely on the theory of nilpotent orbits in Lie algebras of simple algebraic groups over an algebraically closed field. This theory originates from the work of E. Dynkin (1952) [Dyn52], with further developments by B. Kostant, G. E. Wall, R. W. Richardson, T.A. Springer, R. Steinberg, G. B. Elkington, P. Bala and R. Carter, K. Pommerening, and many others. As a result, the particular statements we need are seriously scattered in the literature, and we cite them according to more recent sources where they are the most explicit.
We recall the essence of Dynkin’s classification of nilpotent elements in complex simple Lie algebras. Let \( \mathcal{L}_\mathbb{C} \) be a simple Lie algebra over \( \mathbb{C} \), and let \( e \in \mathcal{L}_\mathbb{C} \) be a nilpotent element. By the Jacobson–Morozov theorem, \( \mathcal{L}_\mathbb{C} \) contains an \( sl_2 \)-triple of the form \( \{e, h, f\} \). Let \( \mathcal{H} \leq \mathcal{L}_\mathbb{C} \) be a Cartan subalgebra of \( \mathcal{L}_\mathbb{C} \) containing \( h \). Let \( \Phi \) be the root system of \( \mathcal{L}_\mathbb{C} \), and let \( \alpha, \beta \in \Phi \), be the standard root vectors in a Chevalley basis of \( \mathcal{L}_\mathbb{C} \) with respect to \( \mathcal{H} \). There is a choice of a system of simple roots \( \Pi \subseteq \Phi \) such that for any \( \alpha \in \Pi \) one has \( \alpha(h) \geq 0 \). Dynkin established that, moreover, \( \alpha(h) \in \{0, 1, 2\} \). The Dynkin diagram of \( \Phi \) with the integers \( \alpha(h) \) associated to the nodes corresponding to roots \( \alpha \in \Pi \) is called a weighted Dynkin diagram of \( e \). It is uniquely determined by \( e \), and the weighted diagrams of two nilpotents \( e, e' \) coincide if and only if \( e \) and \( e' \) are conjugate by an inner automorphism of \( \mathcal{L}_\mathbb{C} \) [Dyn52, Theorems 8.1 and 8.3].

Let \( \mathcal{L}_\mathbb{Z} \) be the \( \mathbb{Z} \)-Lie subalgebra of \( \mathcal{L}_\mathbb{C} \) generated by all \( e_\alpha \) and \( h_\alpha \), \( \alpha \in \Phi \). Then \( \mathcal{L}_\mathbb{Z} \) is a \( \mathbb{Z} \)-form of \( \mathcal{L}_\mathbb{C} \), i.e. \( \mathcal{L}_\mathbb{C} = \mathcal{L}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{C} \). Let \( k \) be any algebraically closed field, and let \( G^{sc} \) be a simply connected simple algebraic group over \( k \) of the same type \( \Phi \) as \( \mathcal{L}_\mathbb{C} \), and let \( G^{ad} \) be the corresponding adjoint group. Then \( \text{Lie}(G^{sc}) \cong \mathcal{L}_\mathbb{Z} \otimes_\mathbb{Z} k \), see e.g. [Mil17, Theorem 23.72]. Define the cocharacter \( \lambda : \mathbb{G}_m \to G^{ad} \leq \text{Aut}(\text{Lie}(G^{sc})) \) in such a way that \( \lambda(t) \cdot e_\alpha = t^{\alpha(h)} e_\alpha \) and \( \lambda(t) \cdot h_\alpha = h_\alpha \) for any \( \alpha \in \Pi \cup (-\Pi) \) and \( t \in k^\times \). Thus, we associate to any weighted Dynkin diagram a \( k \)-cocharacter \( \lambda : \mathbb{G}_m \to G^{ad} \).

The classification of weighted Dynkin diagrams corresponding to nilpotents involves the notion of a distinguished parabolic subgroup. Note that the classification of types of distinguished parabolic subgroups, as defined below, is independent of the characteristic of the base field.

**Definition 5.3.** [LS12 § 2.6] Let \( G \) be a semisimple algebraic group over a field \( k \), and let \( \Phi = \Phi(T, G) \) be the root system of \( G \). A parabolic subgroup \( P \) of \( G \) is called distinguished if

\[
\dim L_P = \dim \bigoplus_{\alpha \in \Phi: \sum_{\beta \in \mathcal{J}} m_\beta(\alpha) = 1} \text{Lie}(G)_\alpha,
\]

where \( L_P \) is a Levi subgroup of \( P \), and \( \Pi \setminus J \) is the type of \( P \) in a system of simple roots \( \Pi \) of \( \Phi \).

**Theorem 5.4.** [Pre03 LS12] Let \( G \) be an adjoint simple algebraic group over an algebraically closed field \( k \) of type \( \Phi \) such that \( \text{char}(k) \neq 2 \) if \( \Phi = B_l, C_l \ (l \geq 2) \) or \( \Phi = D_l \ (l \geq 4) \), and \( \text{char}(k) \neq 2, 3 \) if \( \Phi = E_6, E_7, E_8, G_2, F_4 \).

1. Let \( \lambda : \mathbb{G}_m \to G \) be a \( k \)-cocharacter of \( G \) corresponding to a weighted Dynkin diagram of a nilpotent element in a complex simple Lie algebra of the same type as \( G \). Then
   (i) \( C_G(\lambda) \) has a unique dense open orbit \( V \) in \( \text{Lie}(G)(\lambda, 2) \).
   (ii) For any \( e \in V(k) \), \( C_G(e) \leq P(\lambda) \).
   (iii) For any \( e \in V(k) \), let \( C(\lambda, e) = C_G(\lambda) \cap C_G(e) \). Then \( C(\lambda, e)^0 \) is a reductive subgroup of \( G \), and for any maximal torus \( S \) of \( C(\lambda, e) \) the parabolic subgroup \( Q = P(\lambda) \cap H \) is a distinguished parabolic subgroup of the semisimple group \( H = [C_G(S), C_G(S)] \). Moreover, \( \lambda(\mathbb{G}_m) \leq H \), and \( e \) lies in the dense open orbit of \( C_H(\lambda) \) in \( \text{Lie}(H)(\lambda, 2) \).

2. Conversely, for any nilpotent element \( e \in \text{Lie}(G) \) there is a cocharacter \( \lambda : \mathbb{G}_m \to G \) as above, such that \( e \) belongs to the unique dense open orbit of \( C_G(\lambda) \) in \( \text{Lie}(G)(\lambda, 2) \).
Proof. (1) Set $\mathcal{L} = \text{Lie}(G)$, $P = P(\lambda)$, $L_P = C_G(\lambda)$, and $L_Q = C_H(\lambda)$ for short. Clearly, $L_P$ is a Levi subgroup of $P$ and $L_Q$ is a Levi subgroup of $Q$.

Assume first that $G$ is not of type $E_8$ if $\text{char}(k) = 5$. Then the characteristic of $k$ is good for $G$. There is reductive group $\tilde{G}$ over $k$ such that $[\tilde{G}, \tilde{G}]$ is the simply connected group isogenous to $G$, and $\text{Lie}(\tilde{G})$ admits a non-degenerate $\tilde{G}$-invariant trace form, see [Pre03, Theorem 2.3]. To prove our theorem, clearly, we can replace $G$ by $\tilde{G}$; this makes other results of Premet applicable.

By the discussion before [Pre03, Theorem 2.3] $L_P$ has a dense open orbit $V$ in $\mathcal{L}(\lambda, 2)$, and for any $e \in V(k)$, $C_G(e) \leq P$. By [Pre03, Theorem 2.3 (iii)] $C(\lambda, e)$ is a reductive subgroup of $G$. By [Pre03, Theorem 2.3 (ii)] $C_G(e) \leq P$. The remaining statements of (iii) follow from [Pre03, Proposition 2.5].

Assume that $G$ is of type $E_8$ and $\text{char}(k) \neq 2, 3$. Consider the table [LS12, Table 22.1.1]. By [LS12, Theorem 15.1] the weighted diagrams of complex nilpotent elements are exactly the ones in the 2nd column of this table, and, conversely, for any $\lambda$ corresponding to such a diagram, and any field $k$ as above, there is a nilpotent element $e \in \mathcal{L}(\lambda, 2)$ such that $e^P$ is dense in $\bigoplus_{i \geq 2} \mathcal{L}(\lambda, i)$ and $C_G(e) \leq P$. This implies that $e^{L_P} = V$ is a dense open orbit of $L_P$ in $\mathcal{L}(\lambda, 2)$. Since any two such orbits would intersect, this orbit is unique. Clearly, it is enough to establish (iii) for this particular element $e \in V(k)$.

By [LS12, Theorem 1 (c)] $C_G(e) = C(\lambda, e) R_\lambda(C_G(e))$, where $R_\lambda(C_G(e))$ is the unipotent radical of $C_G(e)$ and $C(\lambda, e)^\circ$ is a reductive group (note that, contrary to our conventions, in [LS12] reductive groups are not required to be connected). Let $S$ be a maximal torus of $C(\lambda, e)$, then $C_G(S)$ is a Levi subgroup of a parabolic subgroup of $G$ by [LS12, Lemma 2.2]. Set $H = [C_G(S), C_G(S)]$. Clearly, $e \in \mathcal{L}(\lambda, 2) \cap \text{Lie}(H)$, since $e$ is nilpotent and belongs to $\text{Lie}(C_G(S))$. By the proof of [LS12, Lemma 2.13] $e$ is a distinguished element of $H = [C_G(S), C_G(S)]$, i.e. $C_H(e)^\circ$ is a unipotent group. Since $S \leq C_G(e)$, this implies that $S = \text{Cent}(C_G(S))^\circ$. On the other hand, by the actual statement of [LS12, Lemma 2.13], $C_G(S)$ is conjugate to the Levi subgroup $\bar{L}$ of a parabolic subgroup of $G$ used in the original construction of the 1-dimensional torus $\lambda(\mathbb{G}_m) = T$ given in [LS12, Lemma 15.3 (i)]. Let $\bar{S} = \text{Cent}(\bar{L})^\circ$, then $\bar{S} \leq C_G(e)$. Since $S$ and $\bar{S}$ are conjugate in $G$, they have the same dimension, and hence they are both maximal tori in $C_G(e)$. Moreover, they are both contained in $C(\lambda, e)^\circ \leq C_G(e)^\circ$, hence they are conjugate in this group, i.e. by an element centralizing $T$. Then the remaining statements of our claim (iii) follow from the corresponding properties of $T$ with respect to $\bar{L}$ stated in [LS12, Lemma 15.3 (i)].

(2) Let $e \in \text{Lie}(G)$ be any nilpotent. The classification of nilpotent classes [LS12, Theorem 1 (c)] implies that there is a cocharacter $\lambda : \mathbb{G}_m \to G$, that corresponds to a weighted Dynkin diagram, and such that $e$ belongs to the dense open orbit of $C_G(\lambda)$ in $\text{Lie}(G)(\lambda, 2)$. Moreover, the explicit classification of occurring weighted Dynkin diagrams [LS12, Theorem 3.1; Tables 22.1.1-22.1.5] is independent of the ground field under the assumption $\text{char}(k) \neq 2, 3$, hence any such weighted Dynkin diagram is a diagram of a complex nilpotent.

Lemma 5.5. In the setting of Theorem 5.4 (1) (iii), assume moreover that $H$ is not of type $E_8$ if $\text{char}(k) = 5$. Then $[\text{Lie}(H)(\lambda, -2), e] = \text{Lie}(H)(\lambda, 0)$.

Proof. Let $\Psi$ be the root system of $H$, let $\Sigma$ be a system of simple roots of $\Psi$, and let $J \subset \Sigma$ be the set of simple roots corresponding to the parabolic subgroup $Q = P(\lambda) \cap H$ of $H$. Since $Q$ is distinguished, one has $\lambda(\alpha) = 2$ for all $\alpha \in J$, see [LS12, Lemma 10.3]. Hence one has

$$\dim(\text{Lie}(H)(\lambda, 2)) = \dim(\text{Lie}(H)(\lambda, -2)) = \dim(\text{Lie}(H)(\lambda, 0)).$$
Hence it remains to prove that \([u,e] \neq 0\) for any \(0 \neq u \in \text{Lie}(H)(\lambda, -2)\). If \((\Psi, \text{char}(k)) \neq (E_8, 5)\), then this follows from \([\text{Pre03}, \text{Theorem 2.3 (iv)}]\).

Assume that \(k\) is a field of characteristic \(\neq 2, 3\), and \(G\) is an adjoint simple algebraic group over \(k\). Denote by \(\bar{k}\) an algebraic closure of \(k\). Let \(L = \text{Lie}(G)\) be the Lie algebra of \(G\), and let \(\mathcal{L} = \bigoplus_{i \in \mathbb{Z}} \mathcal{L}_i\) be any \(\mathbb{Z}\)-grading on \(L\) such that \(\mathcal{L} \neq L_0\). By \([\text{BDMS01}, \text{Lemma 4.1.6}]\) one has \(G = \text{Aut}(\text{Lie}(G))^\circ\), hence there is a unique closed embedding of a 1-dimensional split \(k\)-torus \(\lambda : S \to G\), such that \(L_i = \text{Lie}(G)(\lambda, i)\) for all \(i \in \mathbb{Z}\).

**Definition 5.6.** Assume that the \(\mathbb{Z}\)-grading \(\text{Lie}(G) = \bigoplus_{i \in \mathbb{Z}} \text{Lie}(G)_i\) of the Lie algebra of an adjoint simple algebraic group \(G\) corresponds to the cocharacter \(\lambda : \mathbb{G}_m \to G\). We define the type of the grading to be the type \(t(P(\lambda))\) of the corresponding positive parabolic subgroup \(P(\lambda)\).

**Theorem 5.7.** Let \(\mathcal{L} = 2 \bigoplus_{i = -2}^{2} \mathcal{L}_i\) be a 5-grading on the Lie algebra \(L = \text{Lie}(G)\) of \(G\) such that \(\mathcal{L}_1 \oplus \mathcal{L}_{-1} \neq 0\), and let \(\lambda : \mathbb{G}_m \to G\) be the corresponding cocharacter of \(G\). Let \(\Delta\) be the weighted Dynkin diagram such that simple roots in \(t(P(\lambda))\) have weight 0, and roots in \(\Pi \setminus t(P(\lambda))\) have weight 2. Then \([\mathcal{L}, \mathcal{L}]\) is graded-isomorphic to \(K(A)\) for a structurable algebra \(A\) over \(k\) if and only if \(\Delta\) is a weighted Dynkin diagram of a nilpotent element in a complex simple Lie algebra of the same type as \(G\).

**Proof.** Consider the cocharacter \(2\lambda : \mathbb{G}_m \to G\), so that \(L_i = \text{Lie}(G)(2\lambda, 2i), i \in \mathbb{Z}\). Then \(\Delta\) is the weighted Dynkin diagram corresponding to \(2\lambda\) over \(\bar{k}\).

Assume the condition on \(\Delta\) is satisfied. We show that there are \(u \in \mathcal{L}_{-1}, e \in \mathcal{L}_1\) such that \([u, e] = \zeta\), the grading derivation of \(L\); then \([\mathcal{L}, \mathcal{L}]\) is graded-isomorphic to \(K(A)\) by \([\text{BDMS01}, \text{Lemma 4.1.3}]\). Let \(1 + e \in \mathbb{G}_m(k[e])\) be the unit element of \(\text{Lie}(\mathbb{G}_m)(k) \cong k\). Then \(2\zeta = 2\lambda(1 + e) \in \text{Lie}(G)\) is the grading derivation of \(\text{Lie}(G)\) with respect to \(2\lambda\). We have

\[2\zeta \in \text{Lie}(G)(2\lambda, 0) \cap \text{Lie}(2\lambda(\mathbb{G}_m)) \subseteq \text{Lie}(H)(2\lambda, 0),\]

where \(H\) is as in Theorem \(5.4\). By Lemma \(5.5\) there is a dense open orbit \(V \subseteq \mathcal{L}_1 = \text{Lie}(G)(2\lambda, 2)\) of \(L_P\), such that for any \(e \in V(k)\) one has \([\text{Lie}(H)(2\lambda, -2), e] = \text{Lie}(H)(2\lambda, 0)\). In particular, \(\zeta \in [\text{Lie}(G)(2\lambda, -2), e] \subseteq [\mathcal{L}_{-1}, e]\).

Now let \(k\) be not necessarily algebraically closed, and let \(\bar{k}\) be its algebraic closure. Note that \(P(\pm \lambda)\) and \(C_G(\lambda) = P(\lambda) \cap P(-\lambda)\) are defined over \(k\). Since \(C_G(\lambda) \times_k \bar{k}\) has a unique dense open orbit \(V \subseteq \mathcal{L}_1 \otimes_k \bar{k}\), the open subvariety \(V\) of the affine space \(\mathcal{L}_1\) is defined over \(k\) (although the action of \(C_G(k)\) on it does not have to be transitive). If \(k\) is infinite, then \(V(k) \neq \emptyset\) just because \(V\) is an open subvariety of an affine space. If \(k\) is finite, then \(V(k) \neq \emptyset\) by \([\text{SS70}, \text{2.7}]\), since \(V\) is a homogeneous space for \(C_G(\lambda)\). Thus, there is a \(k\)-point \(e \in V(k)\). Then \(\zeta \in [\mathcal{L}_{-1}, e]\), since the same holds over \(\bar{k}\).

Next, assume that \([\mathcal{L}, \mathcal{L}]\) is graded-isomorphic to \(K(A)\), and show that \(\Delta\) is a weighted Dynkin diagram of a nilpotent element in a complex simple Lie algebra of type \(\Phi\). We can assume that \(k\) is algebraically closed without loss of generality. Let \(e = 1_+ \in \mathcal{L}_1\) and \(f = 1_- \in \mathcal{L}_{-1}\) be the elements representing the unit of the structurable algebra \(A\). By the very definition of \(K(A)\), we conclude that \([e, f] = \zeta \in \mathcal{L}_0\). Furthermore, for any \(0 \neq x \in A\) one has \(V_{\lambda, x}(1) = 2\bar{x} - x \neq 0\), since \(2\bar{x} = x\) implies \(2x = \bar{x}\) and \(3\bar{x} = 0\), whence \(x = 0\).

By Theorem \(5.4\) there is a cocharacter \(\lambda : \mathbb{G}_m \to G\) that corresponds to a weighted Dynkin diagram of complex nilpotent, and such that \(e\) belongs to the dense open orbit of \(C_G(\lambda')\) in
Lemma 5.8. Let \( \zeta' \) be the grading derivation of \( \mathcal{L} \) corresponding to \( \lambda' \). Then \([\zeta', e] = 2e\). Assume for the moment that the subgroup \( H \) corresponding to \( \lambda' \) and \( e \) is not of type \( E_8 \) if \( \text{char} \, k = 5 \). Then by Lemma 5.3 there is \( f' \in \text{Lie}(G)(\lambda', -2) \) such that \([e, f'] = \zeta'\). We have \( \lambda(\mathbb{G}_m), \lambda(\mathbb{G}_m) \leq N_G(k \cdot e) \). Hence after conjugating \( \lambda(\mathbb{G}_m) \) by an element of \( C_G(e)(k) \leq N_G(k \cdot e)(k) \) we can assume that \( \lambda(\mathbb{G}_m) \) and \( \lambda(\mathbb{G}_m) \) lie in the same maximal torus of \( N_G(k \cdot e) \), and thus centralize each other. In particular, \( \zeta' \in \mathcal{L}_0 \), and hence without loss of generality \( f' \in \mathcal{L}_{-1} \). Then \( 2\zeta - \zeta' = [e, 2f - f'] \) and \([2\zeta - \zeta', e] = 0\). In other words, \([e, 2f - f'], e] = 0\). However, \([e, 2f - f'] \) acts on \( \mathcal{L}_1 \) as \( V_{1,2f-f'} \) acts on \( \mathcal{A} \), whence \( 2f - f' = 0 \) by the above computation. Thus \( 2\zeta = \zeta' \), and we are done.

Assume that \( H = G \) has type \( E_8 \); then \( P(\lambda') \) is a distinguished parabolic subgroup of \( G \). We claim that this case cannot occur in our setting. The types of distinguished parabolic subgroups are listed in [LS12, Table 13.2]. Denote by \( \mathcal{L} = \bigoplus_{i \in \mathbb{Z}} \mathcal{L}[i] \) the grading on \( \mathcal{L} \) induced by \( \lambda' \). In all cases, one readily sees that \( \mathcal{L}[2i] \neq 0 \) for all \( 1 \leq i \leq k \), where \( k \geq 5 \). Then by [LS12, proof of Propositions 13.4 and 13.5], under the assumption \( \text{char} \, k \neq 2,3 \) there is a nilpotent \( e' \in \mathcal{L}[2] \) such that \( e' \) lies in the dense orbit of \( C_G(\lambda') \) in \( \mathcal{L}[2] \), and one has \( (\text{ad}(e'))|_{\mathcal{L}[2]} \neq 0 \). Clearly, \( e \) and \( e' \) are \( C_G(\lambda') \)-conjugate. However, since \( e \in \mathcal{L}_1 \), and \( \mathcal{L} \) is \( 5 \)-graded, one has \( \text{ad}(e)^5 = 0 \), hence \( \text{ad}(e')^5 = 0 \) as well, a contradiction. \( \square \)

Lemma 5.8. Let \( \mathcal{L} = \bigoplus_{i=-2}^{2} \mathcal{L}_i \) be a \( 5 \)-grading on the Lie algebra \( \mathcal{L} = \text{Lie}(G) \) of \( G \) such that \( \mathcal{L}_1 \oplus \mathcal{L}_{-1} \neq 0 \), and let \( \lambda : \mathbb{G}_m \to G \) be the corresponding cocharacter of \( G \). Let \( \tilde{\alpha} \) be the root of maximal height in \( \Phi \) with respect to \( \Pi \). Let \( J = \Pi \setminus t(P(\lambda)) \), then \( J \) satisfies (a) \( J = \{\alpha_1\} \) or (b) \( J = \{\alpha_1, \alpha_2\} \). If (a) holds, then \( m_{\alpha_1}(\tilde{\alpha}) = 1 \) or \( 2 \). If (b) holds, then \( m_{\alpha_1}(\tilde{\alpha}) = m_{\alpha_2}(\tilde{\alpha}) = 1 \).

In both cases \( \lambda(t) \cdot e_{\alpha} = te_{\alpha} \) for any \( \alpha \in J \) and \( t \in k^\times \).

Proof. By [BDMSt, Lemma 4.2.2] the root system \( \Phi \cap \mathbb{Z}J \) is of type \( A_1, BC_1, A_2, \) or \( A_1 \times A_1 \), and for any \(-2 \leq i \leq 2\) we have

\[
\mathcal{L}_i = \bigoplus_{\alpha \in \Phi: \sum_{\beta \in J} m_\beta(\alpha) = i} \text{Lie}(G)_\alpha.
\]

Then \( \lambda(t)e_{\alpha} = te_{\alpha} \) for \( \alpha \in J \) by the definition of \( \lambda \). Since \( \sum_{\alpha \in J} m_\alpha(\tilde{\alpha}) \leq 2 \), the remaining claims are clear. \( \square \)

Theorem 5.9. Let \( G \) be an adjoint simple algebraic group over a field \( k \), \( \text{char} \, k \neq 2,3 \). Let \( \mathcal{L} = \bigoplus_{i=-2}^{2} \mathcal{L}_i \) be a \( 5 \)-grading on the Lie algebra \( \mathcal{L} = \text{Lie}(G) \) of \( G \) such that \( \mathcal{L}_1 \oplus \mathcal{L}_{-1} \neq 0 \). Then \( [\mathcal{L}, \mathcal{L}] \) is graded-isomorphic to \( K(\mathcal{A}) \) for a structurable \( k \)-algebra \( \mathcal{A} \) if and only if the type of grading is the complement of the set \( J \) of simple roots of the root system \( \Phi \) of \( G \) listed in the following table.
Proof. Let $\lambda : \mathbb{G}_m \to G$ be the cocharacter of $G$ corresponding to the grading. By Theorem 5.7 it remains to check if $2\lambda$ is a $k$-cocharacter corresponding to a weighted Dynkin diagram of a nilpotent element in the complex case. By Lemma 5.8 one has $2\lambda(t) \cdot e_\alpha = t^2 e_\alpha$ for all $\alpha \in J$ and $2\lambda(t) \cdot e_\alpha = t^0 e_\alpha$ for all $\alpha \in \Pi \setminus J$.

If $\Phi$ is of exceptional type, then one readily checks that in all cases $2\lambda$ is as required, since its weighted Dynkin diagram occurs in the Tables 22.1.1–22.1.5 of nilpotent conjugacy classes in \cite{LS12}.

Assume $\Phi$ is of classical type. We use the descriptions of weighted Dynkin diagrams of nilpotent orbits given in \cite{CM93}.

| Case | Description |
|------|-------------|
| $A_l, l \geq 1$ | $\{\alpha_i, \alpha_{i+1-i}\}, 1 \leq i \leq (l+1)/3$; $\{\alpha_{l+1/2}\}$ if $l$ is odd |
| $B_l, l \geq 2$ | $\{\alpha_i\}, 1 \leq i \leq (2l+1)/3$ |
| $C_l, l \geq 3$ | $\{\alpha_i\}, 1 \leq i \leq 2l/3$, $i$ is even; $\{\alpha_l\}$ |
| $D_l, l \geq 4$ | $\{\alpha_i\}, 1 \leq i \leq 2l/3$; $\{\alpha_{l-1}\}$ and $\{\alpha_l\}$ if $l$ is even |
| $E_6$ | $\{\alpha_1, \alpha_6\}; \{\alpha_2\}$ |
| $E_7$ | $\{\alpha_1\}; \{\alpha_2\}; \{\alpha_6\}; \{\alpha_7\}$ |
| $E_8$ | $\{\alpha_1\}; \{\alpha_8\}$ |
| $F_4$ | $\{\alpha_1\}; \{\alpha_4\}$ |
| $G_2$ | $\{\alpha_2\}$ |

The table above shows the classification of nilpotent orbits for each type of Dynkin diagram. The case $A_l$ involves a sequence of $l+1$ numbers, with the number of $1$s and $-1$s being $m$ and $n-m$, respectively. The case $B_l$ involves a sequence of $2l+1$ numbers, with the number of $1$s and $-1$s being $l$. The case $C_l$ involves a sequence of $2l$ numbers, with the number of $1$s and $-1$s being $l$. The case $D_l$ involves a sequence of $2l$ numbers, with the number of $1$s and $-1$s being $l$. The case $E_6$ involves a sequence of $6$ numbers, with the number of $1$s, $2$s, $6$s, and $7$s being $1$, $2$, $6$, and $7$. The case $E_7$ involves a sequence of $7$ numbers, with the number of $1$s, $2$s, $6$s, and $7$s being $1$, $2$, $6$, and $7$. The case $E_8$ involves a sequence of $8$ numbers, with the number of $1$s, $2$s, $8$s being $1$, $2$, and $8$. The case $F_4$ involves a sequence of $4$ numbers, with the number of $1$s, $2$s, $4$s being $1$, $2$, and $4$. The case $G_2$ involves a sequence of $2$ numbers, with the number of $1$s and $2$s being $1$ and $2$. In each case, the conditions on the number of $1$s and $-1$s ensure that the corresponding orbit is a nilpotent orbit in $G$. The proof uses the properties of the cocharacter $\lambda$ and the weighted Dynkin diagram to verify that the orbit is nilpotent. The table provides a complete classification of nilpotent orbits for each type of Dynkin diagram.
Case $\Phi = C_l$. In the notation of [CM93], we have $n = l$. Nilpotent orbits are classified by partitions $2n = \sum_{i=1}^{2n} d_i$ of $2n$ in which odd parts occur with even multiplicity [CM93, Theorem 5.1.3]. In [CM93, Lemma 5.3.1], $h_1 + h_n = 2$ is the sum of labels of the weighted Dynkin diagram. Since $2h_n$ is the label of $\alpha_n$, we have $h_n = 0$ or $h_n = 1$.

If $h_n = 1$, then $h_1 = 1$, and hence $h_i = 1$ for all $i$. Then $d_i = 2$ for all non-zero $d_i$, hence the partition is $2n = 2 + \ldots + 2$. The condition on odd parts is satisfied, hence $J = \{\alpha_i\}$ is valid.

If $h_n = 0$, then $h_1 = 2$ and $d_i \in \{1, 2, 3\}$ for $1 \leq i \leq 2n$. Since $h_i - h_{i+1} = 2$ only for one $i$ between 1 and $n - 1$, we conclude that $h_1 = \ldots = h_i = 2$ and $h_{i+1} = \ldots = h_{n-1} = 0$. Then $2n$ is partitioned into the sum of $i$ times number 3, and $2n - 3i \geq 0$ times number 1, where $i$ is even. Summing up, $J = \{\alpha_i\}$ with even $i = 2m$, $1 \leq m \leq l/3$, or $J = \{\alpha_i\}$.

Case $\Phi = D_l$. In our notation, $l = n$. Nilpotent orbits are classified by partitions $2n = \sum_{i=1}^{2n} d_i$ of $2n$ in which even parts occur with even multiplicity, except that "very even" partitions with only even parts (each having even multiplicity) correspond to two different orbits [CM93, Theorem 5.1.4]. Since all weights of the Dynkin diagram in our setting are even, all numbers $h_1, \ldots, h_n$ have the same parity. Assume first that the partition is not very even, i.e. the numbers $d_i$ are odd, and the numbers $h_i$ are even. By [CM93, Lemma 5.3.4] the sum of labels on the Dynkin diagram equals $h_1 + h_{n-1} \in \{2, 4\}$. Since $h_1 \geq h_{n-1}$, we have $h_1 = 2$, $h_{n-1} = 0$ or $h_1 = h_{n-1} = 2$. If $h_1 = h_{n-1} = 2$, then $h_1 = h_2 = \ldots = h_{n-1} = 2$ and $h_n = 0$, which is not possible (not enough zeroes). Hence $h_1 = 2$, $h_{n-1} = 0$. Then $h_n = 0$ as well. The partition consists of $i$ times 3, where $i \geq 1$, and $2n - 3i$ numbers 1. The multiplicity condition is void. Since numbers 3 produce triples $2, 0, -2$, one has $2n \geq 3i$. This case corresponds to $J = \{\alpha_i\}$, $1 \leq i \leq 2n/3$.

Assume the partition is very even. Then by [CM93, Lemma 5.3.5] the sum of labels of vertices $\alpha_n$ and $\alpha_{n-1}$ equals 2. Then either all other labels are 0, or there is label 2 at $\alpha_1$. In the first case we have $h_1 = h_2 = \ldots = h_{n-1}$, and since all $d_i$ are even, this implies that the partition is $n$ numbers 2. Since it is very even, $n$ is even. In the second case $h_1 - h_2 = 2$, $h_2 = \ldots = h_{n-1}$. Since $h_1 \geq h_2 \geq h_n$, then $d_1 = h_1 + 1$ can only have multiplicity one, which is wrong. Therefore, this case does not take place in our setting. Summing up, very even partitions occur with $J = \{\alpha_{n-1}\}$ and $J = \{\alpha_n\}$ for $n$ even. \qed

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