Strange attractors for Overbeck -Boussinesq model

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Abstract. In this paper, we consider dynamics defined by the Navier-Stokes equations in the Oberbeck-Boussinesq approximation in a two dimensional domain. This model of fluid dynamics involve fundamental physical effects: convection, and diffusion. The main result is as follows: local semiflows, induced by this problem, can generate all possible structurally stable dynamics defined by \( C^1 \) smooth vector fields on compact smooth manifolds (up to an orbital topological equivalency). To generate a prescribed dynamics, it is sufficient to adjust some parameters in the equations, namely, the viscosity coefficient, an external heat source, some parameters in boundary conditions and the small perturbation of the gravitational force.

1. Introduction

The hypothesis that turbulence can be generated by strange (chaotic) attractors was pioneered in [16, 17]. In this paper, an analytical proof of this hypothesis for the Oberbeck-Boussinesq (OB) approximation of Navier-Stokes equations is stated. The OB equations describes a model of fluid dynamics, which involves fundamental effects: convection and heat transfer. Boussinesq flows are common in nature (such as atmospheric fronts, oceanic circulation), and industry. It is known the OB approximation is very accurate for many flows important in applications [6].

The main result of this paper can be outlined as follows. We consider the initial boundary values problem (IBVP) defined by the Navier-Stokes equations in the OB approximation and standard boundary conditions on a rectangle \( \Omega \subset \mathbb{R}^2 \). Global semiflows, induced by that IBVP, can generate all possible hyperbolic dynamics defined by \( C^1 \)-smooth vector fields on finite dimensional compact smooth manifolds (up to an orbital topological equivalencies). The well known examples of hyperbolic dynamics with a "chaotic" behaviour are the Anosov flows, the Smale A-axiom systems and the Smale horseshoes [20, 12, 9]. To generate a prescribed hyperbolic dynamics, it is sufficient to adjust some parameters involved in the IBVP formulation, in particular, the viscosity coefficient \( \nu \), a spatially inhomogeneous heat source and a small spatially inhomogeneous perturbation of the gravitational force.

The main technical tool used in the proof is the method of realization of vector fields (RVF) proposed by P. Poláčik [18, 19, 4]. Let us outline the RVF method and some previously obtained results.

Let us consider an IBVP associated with a system of PDE’s and involving a parameter \( \mathcal{P} \). Assume that for each value of \( \mathcal{P} \) that IBVP generates a global semiflow \( S^t \). We obtain then a family \( \mathcal{F} \) of global semiflows \( S^t_{\mathcal{P}} \), where each semiflow
depends on the parameter $\mathcal{P}$. Suppose that for an integer $n > 0$ there is an appropriate value $P_n$ of the parameter $\mathcal{P}$ such that the corresponding global semiflow $S^t_{P_n}$ has an $n$-dimensional finite $C^1$-smooth locally invariant manifold $M_n$. The semiflow $S^t_{P_n}$ restricted to $M_n$ is defined by a vector field $Q$ on $M^n$. Then we say that the family $S^t_{P_n}$ realizes the vector field $Q$.

By these realizations it is shown that semiflows associated with some special quasilinear parabolic equations in two dimensional domains can generate complicated hyperbolic dynamics \[19, 4\]. For a large class of reaction-diffusion systems the RVF method allows us to prove existence of chaotic attractors \[23\]. One can show that, for each integer $n$, semiflows induced by these systems can realize a dense set in the space of all $C^1$-smooth vector fields on the unit ball $B^n \subset \mathbb{R}^n$ \[23\]. Therefore, such systems generate all structurally stable (persistent under sufficiently small $C^1$-perturbations) dynamics, up to orbital topological equivalence \[20, 12\]. The families of the semiflows enjoying such property of the dense realization can be called \textit{maximally complex}. If a family of semiflows is maximally complex, that family generates all hyperbolic dynamics on finite dimensional compact smooth manifolds. By this terminology, the main result of this paper is as follows. The family of semiflows, associated with the IBVP’s generated by the OB equations, is maximally complex (see Theorem 4.1).

Using the RVF method for the OB equations we encounter the following main difficulty: how to reduce the OB dynamics to a system of differential equations with quadratic nonlinearities

\[
\frac{dX}{dt} = K(X) + MX + f, \quad X \in \mathbb{R}^N,
\]

where $X(t)$ is a unknown function, $X = (X_1, ..., X_N) \in \mathbb{R}^N$, $K(X)$ is a quadratic term, $f \in \mathbb{R}^N$, the linear term $MX$ is defined by a $N \times N$ matrix $M$. This reduction is based on a construction of locally attracting invariant manifolds $M_N$ for the OB equations. To this end, we choose the IBVP parameters in a special way, since it is impossible to prove existence of locally attracting invariant or inertial manifolds for the general OB equations. We adjust the parameter $\mathcal{P}$ such that a linear operator $L$, that determines the linearization of the OB equations, has special spectral properties. Namely, this operator has $N$ zero eigenvalues and all the rest spectrum of $L$ lies in the negative half plane and it is separated by a positive barrier from the imaginary axis. To find the operator $L$ having such properties is not easy, and the construction of $L$ is a main technical part of the paper. The further application of the RVF method to (1.1) follows works \[24, 25, 23\] with small modifications.

Let us outline this reduction on $M_N$ in more detail. We are seeking for a solution $(\mathbf{v}, u)$, where $\mathbf{v}$ is the fluid velocity, $u$ is the temperature (or the impurity density), $x, y$ are horizontal and vertical coordinates, respectively, and the gravitational force is directed along $y$. Let us assume that the solutions are small perturbations of the flow $\mathbf{v} = 0, u = U(y)$, which have the form $\mathbf{v} = \gamma \tilde{v}, u(x, y, t) = U(y) + \gamma u_1(x, y) + \gamma w(x, y, t)$, where $\tilde{v}, w$ are new unknown functions, the terms $U$ and $u_1$ are adjusted in a special way and $\gamma > 0$ is a small parameter. The operator $L$ is defined via $U$ and $L$ does not depend on $\gamma$. The function $u_1$ defines the matrix $M$ in equations (1.1).

Since $U$ depends only on $y$, we can separate variables in the spectral problem for the operator $L$ (this method is well known, see \[5, 8\]). Eigenfunctions of $L$ have the
form \( e_k = (\Psi_k(y) \sin(kx), \Theta(y) \cos(kx))^{tr} \) with eigenvalues \( \lambda_k \), where \( k = 1, 2, \ldots \).

In the classical approach \( U(y) \) is a linear function of \( y \) \([5, 8]\). Then one obtains that there are possible bifurcations, where \( \lambda_k \) changes its sign at a value \( P_0 \) for a \( k = k_0 \).

For small \( Re \lambda(k_0) > 0 \) and \( \gamma > 0 \) the solution is \( X e_{k_0} \) (up to small corrections) and the magnitude \( X \) can be obtained from a simple nonlinear equation for \( X \).

In this paper, the main trick is as follows. We take \( U \) as a fast decreasing exponent perturbed by a small polynomial, \( U = C_U b^{1-s_1} \exp(-by) + \mu y P_N(y) \), where \( \mu = b^{-s_2}, s_1, s_2 \in (0, 1) \), \( b \) is a large parameter, independent of \( \gamma \) and \( C_U \) is a parameter of the order 1 (it does not depend on \( b \) for large \( b \)). The function \( P_N \) is a polynomial of the degree \( N \). For each \( k \) the spectral problem for \( L \) can be reduced to a nonlinear equation for \( \lambda_k \). In the limit \( b \to \infty, \nu \to +\infty \), and if \( P_N(y) = 0 \), the equation for \( \lambda_k \) has a simple limit form, which does not involve \( k \). Furthermore, we can use the small term \( \mu y P_N \) to control location of the roots \( \lambda_k \) of this equation. Namely, let us choose some \( k_1, k_2, \ldots, k_N < < b \). Under a special choice of \( P_N, C_U \) and some other parameters we have \( \lambda_k = 0, k = k_1, k_2, \ldots, k_N \), whereas all others \( \lambda_k \) satisfy \( Re \lambda_k < -\delta(b) \), where \( \delta > 0 \). When we vary \( C_U \), all \( \lambda_k \) with \( k = k_j \) pass through 0 simultaneously. In this case there is a bifurcation, which involves a number of the unstable modes \( e_k \) (this effect was found in \([24]\) for the Marangoni problem).

The special spectral properties of the operator \( L \) allow us to proceed the reduction of the OB equations to \([1, 1]\) by a quite routine procedure, which uses the well known results of invariant manifold theory \([10, 11, 2]\). This procedure shows that \( K \) and \( M \) depends on the problem parameters differently, namely, the coefficients involved in \( K \) depend on the eigenfunctions of \( L \) whereas \( M \) is a linear functional \( M = M[u_1] \) of \( u_1(x, y) \). We show that, as \( u_1 \) runs over the set of all smooth functions defined on \( \Omega \), the range of this functional is dense in the linear space of all \( N \times N \) matrices. This fact allows us to apply the results on quadratic systems \([1, 1]\) from Sect. \([10]\) and completes the proof.

Note that systems \([1, 1]\) have important applications, in particular, in chemistry, where they describe bimolecular chemical reactions \([26]\), and for population dynamics. Results on existence of complex dynamics for \([1, 1]\) were first obtained in \([13]\) (see also \([26]\)). In \([25]\) the RVF method is applied to investigate dynamics of systems \([1, 1]\). It is shown that systems \([1, 1]\) generate a maximally complex family of semiflows, where parameters are \( N, M \), coefficients of the bilinear form \( K \) and \( f \). In this paper, we are dealing with a more complicated situation, when the coefficients of the quadratic forms \( K(X, X) \) are fixed. This difficulty is not too hard and it can be overcome by the methods of the invariant manifold theory \([10, 2, 1]\) that allows us to reduce systems \([1, 1]\) of large dimension to analogous systems of smaller dimension, where the coefficients, which define \( K(X) \) can be considered as free parameters (for more detail, see Sect. \([10]\) and \([21]\)).

In physical words, one can say that the interaction of the corresponding slow modes associated with the eigenfunctions \( e_{k_j} \) can generate a complicated dynamics and spatio-temporal patterns. The relations, obtained in the paper, give an analytical description of spatio-temporal patterns induced by strange attractors. The patterns are similar to found in \([24]\) for Marangoni flows, they are quasiperiodic in \( x \), and have the boundary layer form, i.e., located at the top boundary \( y = 0 \).

The paper is organized as follows. In the next sections we formulate the problem and describe the RVF method. In Sect. \([4]\) we state the main result. In Sect. \([5]\) it
is shown that the IBVP is well posed and defines a global semiflow. In the next section we introduce the operator \( L \). In Sect. 7, which is a key technical part of the paper, we investigate that operator, and show that \( L \) has needed spectral properties. In Sect. 8, we prove existence of the finite dimensional invariant manifold. In Sect. 9, we check conditions, which is critically important for the RVF method. Here we show that, for each fixed \( N \), by a choice \( u_1(x, y) \), we can obtain any prescribed matrices \( M \). In Sect. 10, we consider quadratic systems (1.1). The remaining part of the proof is stated in Sect. 11.

Below we use the following standard convention: all positive constants, independent of the parameters \( b, \gamma \) and \( \nu \), are denoted by \( c_i, C_j \). To diminish a formidable number of indices \( i, j \), we shall use sometimes the same indices assuming that the constants can vary from a line to a line.

2. Statement of the problem

We consider the Oberbeck-Boussinesq of the Navier Stokes equations:

\[
\begin{align*}
v_t + (v \cdot \nabla)v &= \nu \Delta v - \nabla p + \kappa e (1 + \gamma g_1) (u - u_0), \\
\nabla \cdot v &= 0, \\
u_t + (v \cdot \nabla)u &= \Delta u + \eta, 
\end{align*}
\]

where \( v = (v_1(x, y, t), v_2(x, y, t))^T \), \( u = u(x, y, t) \), \( p = p(x, y, t) \) are unknown functions defined on \( \Omega \times \{ t \geq 0 \} \), the domain \( \Omega \) is a rectangle, \( \Omega = [0, \pi] \times [0, h] \subset \mathbb{R}^2 \). Here \( v \) is the fluid velocity, where \( v_1 \) and \( v_2 \) are the normal and tangent velocity components, \( \nu \) is the viscosity coefficient, \( p \) is the pressure, \( u \) is the temperature, \( \eta(x, y) \) is a function describing a distributed heat source, \( v \cdot \nabla \) denotes the advection operator \( v_1 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} \). The unit vector \( e \) is directed along the vertical \( y \)-axis: \( e = (0, 1)^T \), \( \kappa \) is the coefficient of thermal expansion and a constant \( u_0 \) is the reference temperature. The term \( \gamma g_1(x, y) \) is a space inhomogeneous perturbation of the gravitational force, where \( \gamma > 0 \) is a small parameter, and \( g_1(x, y) \) is a smooth function. We assume that the non-perturbed density \( \rho_0 = 1 \).

The initial conditions are

\[
\begin{align*}
v(x, y, 0) &= v^0(x, y), & p(x, y, 0) &= p^0(x, y), & u(x, y, 0) &= u^0(x, y). 
\end{align*}
\]

The function \( u \) satisfies the boundary conditions

\[
\begin{align*}
u_x(x, y, t)|_{x=0, \pi} &= 0, \\
u_y(x, y, t)|_{y=0} &= \beta u(x, 0, t), & u_y(x, y, t)|_{y=h} &= \beta_1 u(x, h, t). 
\end{align*}
\]

For the fluid velocity we set conditions of the free surface at the vertical boundaries \( x = 0, \pi \):

\[
\begin{align*}
v_1(x, y, t)|_{x=0, \pi} &= 0, & \frac{\partial v_2(x, y, t)}{\partial x}|_{x=0, \pi} &= 0 
\end{align*}
\]

and the no-flip condition at \( y = 0, y = h \):

\[
\begin{align*}
v(x, y, t)|_{y=0, h} &= 0. 
\end{align*}
\]
3. RVF method

Before to formulate the main theorem, let us describe the method of the realization of vector fields (RVF) invented by P. Poláčik (see \[18, 19\]). We change slightly the original version to adapt it for our goals.

Let us consider a family of local semiflows \( S_t^P \) in a fixed Banach space \( B \). Assume these semiflows depend on a parameter \( P \in B_1 \), where \( B_1 \) is another Banach space. Denote by \( B_n(\mathbb{R}) \) the ball \( \{ q : |q| \leq R \} \) in \( \mathbb{R}^n \), where \( q = (q_1, q_2, \ldots, q_n) \) and \(|q|^2 = q_1^2 + \cdots + q_n^2\). For \( R = 1 \) we will omit the radius \( R, B^n = B^n(1) \). Remind that a set \( M \) is said to be locally invariant in an open set \( W \subset B \) under a semiflow \( S_t \) in \( B \) if \( M \) is a subset of \( W \) and each trajectories of \( S_t \) leaving \( M \) simultaneously leaves \( W \). In this paper, all \( W \) are tubular neighborhoods of the balls \( B^n(R) \), which have small widths. Consider a system of differential equations defined on the ball \( B^n \):

\[
\frac{dq}{dt} = Q(q),
\]

where

\[
Q \in C^1(B^n), \quad \sup_{q \in B^n} |\nabla Q(q)| < 1.
\]

Assume the vector field \( Q \) is directed strictly inward at the boundary \( \partial B^n = \{ q : |q| = 1 \} \):

\[
Q(q) \cdot q < 0, \quad q \in \partial B^n.
\]

Then system (3.10) defines a global semiflow on \( B^n \). Let \( \epsilon \) be a positive number.

**Definition 3.1. (realization of vector fields)** We say that the family of local semiflows \( S_t^P \) realizes the vector field \( Q \) (dynamics (3.10)) with accuracy \( \epsilon \) (briefly, \( \epsilon \)-realizes), if there exists a parameter \( P = P(Q, \epsilon, n) \in B_1 \) such that

(i) semiflow \( S_t^P \) has a locally invariant in a open domain \( W \subset B \) and locally attracting manifold \( M_n \subset B \) diffeomorphic to the unit ball \( B^n \);

(ii) this manifold is embedded into \( B \) by a map

\[
z = Z(q), \quad q \in B^n, \quad z \in B, \quad Z \in C^{1+r}(B^n),
\]

where \( r > 0 \);

(iii) the restriction of the semiflow \( S_t^P \) to \( M_n \) is defined by the system of differential equations

\[
\frac{dq}{dt} = Q(q) + \tilde{Q}(q, P), \quad Q \in C^1(B^n),
\]

where

\[
|\tilde{Q}(\cdot, P)|_{C^1(B^n)} < \epsilon.
\]

**Definition 3.2.** Let \( \Phi \) be a family of vector fields \( Q \), where each \( Q \) is defined on a ball \( B^n \), positive integers \( n \) may be different. We say that the family \( F \) of local semiflows \( S_t^P \) realizes the family \( \Phi \) if for each \( \epsilon > 0 \) and each \( Q \in \Phi \) the filed \( Q \) can be \( \epsilon \)-realized by the family \( F \).

We say that the family \( F \) is maximally dynamically complex if that family realizes all \( C^1 \)-smooth finite dimensional fields defined on all unit balls \( B^n \).
4. Main results

The IBVP defined by (2.2) -(2.9) involves the coefficients $\nu, h, \gamma, \beta, \beta_1$ and the functions $\eta(x,y), g_1(x,y)$. We set $\mathcal{P} = \{h, \nu, \gamma, \beta, \beta_1, u_0, \eta(\cdot, \cdot), g_1(\cdot, \cdot)\}$. The main result is as follows:

**Theorem 4.1.** The family of the semiflows defined by IBVP (2.2) -(2.9) is maximally dynamically complex, that is, for each integer $n$, each $\epsilon > 0$ and each vector field $Q$ satisfying (3.11) and (3.12), there exists a value of the parameter $P(Q, \epsilon)$ such that IBVP (2.2) -(2.9) defines a semiflow $S^t_P$, which $\epsilon$-realizes the vector field $Q$.

Persistence of hyperbolic sets [12] and some additional arguments then implies the following corollary.

**Theorem 4.2.** The family of semiflows $S^t_P$ induced by IBVP (2.2) -(2.9) generates all (up to orbital topological equivalencies) hyperbolic dynamics on compact invariant hyperbolic sets defined by $C^1$-smooth vector fields on finite dimensional smooth compact manifolds.

In particular, we obtain that the OB dynamics can generate Smale axiom A flows, Ruelle-Takens attractors [16] [17] and the Anosov flows.

5. Existence and uniqueness

5.1. Function spaces and embeddings. We use the standard Hilbert spaces [10]. We denote by $H$ the closure in $L_2(\Omega, \mathbb{R}^2)$ of the set of $C^1$-smooth vector valued functions $v = (v_1, v_2)$ such that $\nabla \cdot v = 0$ and satisfying boundary conditions (2.8), (2.9).

The space $H$ is equipped by norms $|| ||$, where $||v||^2 = \langle v_1, v_1 \rangle + \langle v_2, v_2 \rangle$ and $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\Omega)$ defined by

$$\langle f, g \rangle = \int_0^h \int_0^\pi f(x,y)g(x,y)dxdy.$$ (5.16)

Let us denote by $H_\alpha$ the fractional spaces

$$H_\alpha = \{v \in H : ||v||_\alpha = ||(I - \Delta_D)^\alpha v|| < \infty\},$$ (5.17)

where $\Delta_D$ is the Laplace operator with the domain corresponding to boundary conditions (2.8), (2.9). Let $\tilde{H}_\alpha$ be another fractional space associated with $L_2(\Omega)$:

$$\tilde{H}_\alpha = \{u \in L_2(\Omega) : ||u||_\alpha = ||(I - \Delta_N)^\alpha u|| < \infty\},$$ (5.18)

where $\Delta_N$ is the Laplace operator with the domain corresponding to the boundary conditions (2.6), (2.7). Below we omit the indices $N, D$.

The Sobolev embeddings

$$H_\alpha \subset C^s(\Omega), \quad 0 \leq s < 2(\alpha - 1/2),$$ (5.19)

and

$$\tilde{H}_\alpha \subset L_q(\Omega), \quad 1/q > 1/2 - \alpha, \quad q \geq 2$$ (5.20)

will be used below, and analogous embeddings for $H_\alpha$. We consider IBVP (2.2)-(2.9) in the phase space $\mathcal{H} = H \times \tilde{H}$.

In coming subsection our aim is to prove that the IBVP (2.2)-(2.9) defines a global semiflow.
5.2. Evolution equations. To show local existence of solutions we use the standard semigroup methods. Let $P$ be the Leray projection (see [22]) and $v \in PH$. Then we can rewrite our IBVP as an evolution equation for the pair $z = (v, u)^{tr}$:

\begin{equation}
(5.21) \quad z_t = Az + F(z),
\end{equation}

where

\begin{align*}
A &= (\nu P \Delta_D, \Delta_N)^{tr}, \quad F = (F_1, F_2)^{tr}, \\
F_1 &= P\left( - (v \cdot \nabla)v + \kappa e(1 + \gamma g_1)(u - u_0) \right), \\
F_2 &= -(v \cdot \nabla)u + \eta.
\end{align*}

By (5.20) we observe that

\begin{equation}
(5.22) \quad ||(v \cdot \nabla)v|| \leq ||v||_\infty ||v||_{\alpha},
\end{equation}

where $\alpha \in (1/2, 1)$ and $|f|_\infty$ denotes the supremum norm:

$|f|_\infty = \sup_{x,y \in \Omega} |f(x,y)|$, $|v|_\infty = |v_1|_\infty + |v_2|_\infty$.

Estimate (5.22) and an analogous estimate for $||(v \cdot \nabla)u||$ show that for $\alpha > 1/2$ the map $F$ is a bounded $C^1$- map from a bounded domain in $H_\alpha = H_\alpha \times H_n$ to $H$. This fact implies a local existence and uniqueness of solutions of (5.21). So, eq. (5.21) defines a local semiflow in $H$.

**Proposition 5.1.** Let $\beta > 0$ and $\beta > \beta_1$. Then the IBVP defined by (2.2)-(2.9) generates a global semiflow in $H$.

**Proof.** Global existence and boundedness of solutions can be derived by the differential inequalities

\begin{equation}
(5.23) \quad \frac{1}{2} \frac{d}{dt} ||v||^2 \leq -\nu ||\nabla v||^2 + \kappa (1 + \gamma g_1) ||v_2|| ||u - u_0||,
\end{equation}

and

\begin{equation}
(5.24) \quad \frac{1}{2} \frac{d}{dt} ||u||^2 \leq -||\nabla u||^2 + ||\eta|| ||u|| + c_1 ||v_2|| ||u|| + I_\beta(u),
\end{equation}

where

\begin{equation}
(5.25) \quad I_\beta(u) = \beta_1 \int_0^\pi u^2(x, h) dx - \beta \int_0^\pi u^2(x, 0) dx.
\end{equation}

For $\beta > 0$ one has

\begin{equation}
(5.26) \quad I_\beta \leq \bar{\beta}_1 \int_0^\pi \left( u^2(x, h) - u^2(x, 0) \right) dx = \bar{\beta}_1 \int_0^h \int_0^\pi u_y u dx dy,
\end{equation}

where $\bar{\beta}_1 = \max\{\beta_1, 0\}$. Thus for each $a > 0$

\begin{equation}
(5.27) \quad I_\beta \leq \frac{\bar{\beta}_1}{2} ||u_y|| ||u|| \leq \frac{\bar{\beta}_1}{4} (a ||\nabla u||^2 + a^{-1} ||u||^2).
\end{equation}

Choosing an appropriate $a$ we see that inequalities (5.23), (5.24) and estimate (5.27) lead to by a priori estimate $||u(\cdot, t)|| + ||v(\cdot, t)|| < c_2 \exp(c_3 t)$ for all $t \geq 0$. Thus, we can conclude that eq. (5.21) defines a global semiflow in $H$. □
6. Linearization of the problem

First we follow the standard approach developed for the Rayleigh–Bénard convection \cite{5,8} but with small modifications.

Let $U(y)$ be a $C^\infty$-smooth function of $y \in [0, h]$ such that

\begin{align}
\frac{dU}{dy}|_{y=0} &= \beta U(0), \\
\frac{dU}{dy}|_{y=h} &= \beta_1 U(h).
\end{align}

For sufficiently small $\gamma$ and $u_0$ such that $u_0 > |U|_\infty$ we set

\begin{equation}
(6.29)
\frac{dU}{dy}|_{y=0} = \beta U(0),
\end{equation}

\begin{equation}
(6.29)
\frac{dU}{dy}|_{y=h} = \beta_1 U(h).
\end{equation}

We suppose that

\begin{equation}
(6.30)
\frac{\partial g_1(x,y)}{\partial x}|_{x=0, \pi} = 0 \quad \forall \ y \in [0, h],
\end{equation}

and

\begin{equation}
(6.31)
\frac{\partial g_1(x,y)}{\partial y}|_{y=0, h} = 0 \quad \forall \ x \in [0, \pi].
\end{equation}

Then

\begin{equation}
(6.32)
\frac{\partial u_1(x,y)}{\partial x}|_{x=0, \pi} = 0 \quad \forall \ y \in [0, h],
\end{equation}

and

\begin{equation}
(6.33)
\frac{\partial u_1(x,y)}{\partial y}|_{y=0} = \beta u_1(x,0) \quad \forall \ x \in [0, \pi],
\end{equation}

\begin{equation}
(6.34)
\frac{\partial u_1(x,y)}{\partial y}|_{y=h} = \beta_1 u_1(x, h) \quad \forall \ x \in [0, \pi].
\end{equation}

Assume that

\begin{equation}
\eta = \eta_0 + \gamma^2 \eta_1, \quad \eta_0 = -\Delta (U + \gamma u_1),
\end{equation}

where $\eta_1$ is a smooth function, which will be considered as a parameter. Let us represent $u$ and $v$ as

\begin{equation}
(6.36)
u = U + \gamma u_1 + \gamma w, \quad v = \gamma \tilde{v},
\end{equation}

where $w, \tilde{v}$ a new unknown functions. Taking into account that the Leary projection of a gradient field is zero, and using substitution \cite{6.30} we note that eq. \cite{6.21} can be rewritten as

\begin{equation}
(6.37)
\tilde{v} = P \left( \nu \Delta \tilde{v} - \gamma (\tilde{v} \cdot \nabla)v + \kappa \rho w (1 + \gamma g_1) \right),
\end{equation}

\begin{equation}
(6.38)
w_t = \Delta w - \tilde{v}_2 U_y - \gamma (\tilde{v} \cdot \nabla)(u_1 + w) + \gamma \eta_1.
\end{equation}

Due to \cite{6.28}, \cite{6.21}, \cite{6.33}, \cite{6.34}, and \cite{6.35} the new unknowns $\tilde{v}$ and $w$ satisfies the same homogeneous boundary conditions that $v$ and $u$.

Removing the terms of the order $\gamma$ in \cite{6.37}, \cite{6.38}, we obtain the linear operator

\begin{equation}
(6.39)
L \ddot{z} = (\ddot{L}_1 \dot{z}, \ddot{L}_2 \ddot{z})^t, \quad \ddot{z} = (\tilde{v}, w)^t
\end{equation}
where the operators $\tilde{L}_k$ are defined by

$$\tilde{L}_1 v = \mathbb{P}(\nu \Delta v + \kappa e w), \quad \tilde{L}_2 w = \Delta w - \tilde{v}_2 U_y,$$

The spectral problem for the operator $L$ has the form

$$\lambda v = \mathbb{P}(\nu \Delta v + \kappa e w),$$

$$\lambda w = \Delta w - v_2 U_y,$$

where $v$ satisfies boundary conditions \(2.8\) and \(2.9\), and $w$ satisfies the boundary conditions

$$w_x(x, y)|_{x=0,\pi} = 0, \quad \forall \ y \in [0, h],$$

$$w_y(x, y)|_{y=0} = \beta w(x, 0), \quad w_y(x, y)|_{y=h} = \beta_1 w(x, h), \quad \forall \ x \in [0, \pi].$$

This spectral problem is investigated in coming sections but first we consider some properties of the operator $L$.

### 6.1. Properties of $L$

In order to apply the standard technique \([10]\), first let us show that the operator $L$ is sectorial.

**Lemma 6.1.** $L$ is a sectorial operator.

**Proof.** We use the following result \([11, 10]\): if $L^{(0)}$ is a self adjoint operator in a Banach space $X$, $L^{(0)} : X \to X$ and $B$ is a linear operator, $B : X \to X$ such that $\text{Dom} \ L^{(0)} \subset \text{Dom} \ B$ and for all $\rho \in \text{Dom} \ L^{(0)}$

$$||B\rho|| \leq \sigma ||L^{(0)}\rho|| + K(\sigma)||\rho||$$

for $0 < \sigma < 1$ and a constant $K(\sigma) > 0$, then $L^{(0)} + B$ also is a sectorial operator.

Let us define the unperturbed operator $L^{(0)}$ by the relations

$$\tilde{L}_1^{(0)}(v, w)^{tr} = \nu \mathbb{P} \Delta v, \quad \tilde{L}_2^{(0)}(v, w)^{tr} = \Delta w,$$

where $\rho = (v, w)^{tr} \in \mathcal{H}$. The operator $L^{(0)}$ is self-adjoint in the space $\mathcal{H}$, its spectrum is discrete and lies in the interval $(-\infty, 0)$. Therefore, $-L^{(0)}$ is a sectorial.

The operator $B$ is given then by

$$B(v, w)^{tr} = (\kappa w, -v_2 U_y)^{tr}.$$

It is clear that estimate \([6.45]\) is satisfied. □

**Lemma 6.2.** For some $\lambda > 0$ and positive $\beta$ such that $\beta > \beta_1$ the resolvent $(L - \lambda)^{-1}$ is a compact operator from $\mathcal{H}$ to $\mathcal{H}$.

**Proof.** Consider equations

$$\lambda v = \mathbb{P}(\nu \Delta v + \kappa e w + g),$$

$$\lambda w = \Delta w - v_2 U_y + f,$$

where $g, f$ lie in $H$ and $\tilde{H}$, respectively. These equations imply the estimates

$$\lambda ||v||^2 + \nu ||\nabla v||^2 \leq \kappa ||w|| ||v|| + ||g|| ||v||,$$

$$\lambda ||w||^2 + ||\nabla w||^2 \leq \kappa_0 ||w|| ||v|| + ||f|| ||w|| + I_\beta(w),$$

where $\kappa_0 > 0$ is independent of $\lambda$ and $I_\beta(w)$ is defined by \([6.25]\). For sufficiently large positive $\lambda \in \mathbb{R}$ the above inequalities and \([5.27]\) imply

$$||\nabla w|| + \nu ||\nabla v|| \leq c_1 (||f|| + ||g||).$$
Consequently, $L - \lambda$ is invertible and $(L - \lambda)^{-1}$ is a compact operator.

According to (see [11], Ch. III, Theorem 6.29) the last lemma implies that the spectrum of $L$ is discrete (consists of isolated eigenvalues), each eigenvalue has a finite multiplicity $n(\lambda)$, and the resolvent $(L - \lambda)^{-1}$ is a compact operator for all $\lambda$, where $(L - \lambda)^{-1}$ is bounded. We investigate the spectrum of $L$ in the next section.

7. Spectrum of linear operator $L$

To study the spectral problem for $L$, we use the stream function-vorticity reformulation of the Navier Stokes equations [3]. The velocity $v$ can be expressed via the stream function $\psi(x, y)$ by the relations

\[ v_1 = \psi_y, \quad v_2 = -\psi_x. \]

Given a $v$, the function $\psi$ can be found by the relation

\[ (7.48) \quad \psi(x, y) = -\int_y^h v_1(x, s)ds. \]

As a result of the standard transformations, eqs. (6.41), (6.42) take the form

\[ \lambda \Delta \psi = \nu \Delta^2 \psi - \kappa w_x, \]
\[ (7.49) \]
\[ \lambda w = \Delta w + \psi_x U_y, \]
\[ (7.50) \]

We obtain the following boundary conditions for $\psi$:

\[ \psi(x, y, \lambda)|_{x=0,\pi} = \Delta \psi(x, y, \lambda)|_{x=0,\pi} = 0, \]
\[ (7.51) \]
\[ \psi_x(x, y, \lambda)|_{y=0,\pi} = \psi_y(x, y, \lambda)|_{y=0,\pi} = 0. \]
\[ (7.52) \]

7.1. Some preliminaries. Let us consider the spectral problem defined by (7.49), (7.50), (7.54), (7.58), (6.43) and (6.44). We seek eigenfunctions $e(x, y, \lambda) = (\psi, w)^{tr}$ with eigenvalues $\lambda \in \mathbb{C}_a$, where $\mathbb{C}_a$ denotes the half-plane

\[ (7.53) \quad \mathbb{C}_a = \{ \lambda \in \mathbb{C} : \text{Re } \lambda > -a \}. \]

In fact, we are interested in $\lambda \in \mathbb{C}_{1/2}$ because for small $\gamma$ only the eigenfunctions with the eigenvalues $\lambda \in \mathbb{C}_a$, where $a >> \gamma$, are involved in the construction of the locally invariant manifold $\mathcal{M}_N$.

Since $U = U(y)$ is independent of $x$, we seek the eigenfunctions in the form

\[ (7.54) \quad w(x, y, \lambda) = w_k(y, \lambda) \cos(kx), \]
\[ (7.55) \quad \psi(x, y, \lambda) = \psi_k(y, \lambda) \sin(kx), \]

where $k$ are positive integers, $k \in \mathbb{N} = \{1, 2, ..., \}$. Let us introduce the operator $L_k = D_y^2 - k^2$. Moreover, to simplify formulas, we set $\kappa = \nu$. Then for $\psi_k$ and $w_k$ one obtains the following boundary value problem on $[0, h]$:

\[ (7.56) \quad \lambda_k \nu^{-1} L_k \psi_k = L_k^2 \psi_k + k^2 w_k, \]
\[ (7.57) \]
\[ \lambda_k w_k = L_k w_k - U_y \psi_k, \]
\[ (7.58) \]
\[ \psi_k(0) = \psi_k(h) = \frac{d\psi_k}{dy}|_{y=0,h} = 0, \]
\[ (7.59) \]
\[ \frac{dw_k(y)}{dy}|_{y=0} = \beta w_k(0), \quad \frac{dw_k(y)}{dy}|_{y=h} = \beta_1 w_k(h). \]
Let us denote $\bar{k} = \sqrt{k^2 + \lambda_k}$. We can suppose, without loss of generality, that $Re \bar{k} > 0$ for $\lambda_k \in \mathbb{C}_{1/2}$, since $\bar{k}$ is involved in eq. (7.57) only via $\bar{k}^2$.

7.2. Choice of profile $U$ and parameters. To simplify the spectral problem, let us introduce a large parameter $b > 0$ and assume that

$$\nu > b^{10}, \quad h = -10 \log b.$$  

Moreover, we set

$$\beta = rb, \quad r = b^{-s_0},$$

where the value $s_0 \in (0, 1)$ will be precise below. The key trick is the following choice of $U$:

$$U(y) = \bar{C}_U + \int_0^y \left( C_U rb^4 \exp(-bs) + \mu s P_N(s) \right) ds,$$

where

$$\mu = b^{-s_2}, \quad s_2 \in (0, 1),$$

$P_N(y)$ is a polynomial of degree $N$ and $C_U, \bar{C}_U \neq 0$ are coefficients. We shall precise the value of $C_U$ in the end of this section, where it will be shown that $C_U$ does not depend on $b$ as $b \to \infty$. To satisfy condition (6.28), we set

$$\bar{C}_U = \beta^{-1} C_U rb^4 = C_U b^3.$$

We adjust $\beta_1$ from condition (6.29) and relation (7.62) and as a result, one obtains

$$\beta_1 = \left( C_U rb^4 \exp(-bh) + \mu h P_N(h) \right) B^{-1},$$

where

$$B = \left( \bar{C}_U + \int_0^h \left( C_U rb^4 \exp(-bs) + \mu s P_N(s) \right) ds \right).$$

Note that $\beta > 0$ and for large $b$ one has $\beta_1 < \beta$, therefore, $\beta$ and $\beta_1$ satisfy the conditions of Prop. 5.1 and Lemma 6.2.

7.3. Main result on spectrum of operator $L$. Let us formulate the assertion.

**Proposition 7.1.** Let (7.60)-(7.65) hold, $N$ be a positive integer and $K_N = \{k_1, \ldots, k_N\} \subset \mathbb{Z}_+$. Then there exists a polynomial $P_N(y)$ such that for sufficiently large $b$ the eigenvalues $\lambda_k$ of BVP (7.56)-(7.59) satisfy

$$\lambda_k = 0 \quad k \in K_N,$$

$$Re \lambda_k < -C_N b^{-cN} \quad k \notin K_N,$$

where positive $C_N, c_N$ are uniform in $b$ as $b \to +\infty$.

The plan of the proof is as follows. We show that the values $\lambda_k \in \mathbb{C}_{1/2}$ only under the condition $k < c_1 b^{s_1}$, where $s_1 \in (0, 1)$. That result allows us to find an asymptotics for the eigenfunctions, which is valid for $k/b \ll 1$. For $\lambda_k$ we obtain a nonlinear equation. The asymptotics of eigenfunctions and the property $k/b \ll 1$ allows us to simplify this equation for $\lambda_k$. By a variable rescaling we show that this equation is a small perturbation of a simple cubic one. As a result, for small $k/b$ that equation can be investigated by a perturbation technique.
First we prove a series of auxiliary assertions. Consider the Green function $\Gamma_k(y, y_0)$ of the operator $L_k$ defined by the equation

$$L_k \Gamma_k = \delta(y - y_0)$$

and the boundary conditions

$$\frac{d\Gamma_k}{dy}(y, y_0)|_{y=0} = \beta \Gamma_k(0, y_0),$$
$$\frac{d\Gamma_k}{dy}(y, y_0)|_{y=h} = \beta_1 \Gamma_k(h, y_0).$$

**Lemma 7.2.** Let $\text{Re} \ k > 0$. Then $\Gamma_k$ satisfies the estimate

$$|\Gamma_k(y, y_0) - \tilde{\Gamma}_k(y, y_0)| < C_0 \exp(-\text{Re} \ k |h - y| + |h - y_0|)),$$

where $C_0 > 0$ is a constant and $\tilde{\Gamma}_k$ is defined by

$$\tilde{\Gamma}_k(y, y_0) = \frac{\exp(-\bar{k}y_0)(\sinh(\bar{k}y) + \bar{k}\beta^{-1}\cosh(\bar{k}y))}{k(1 + k\beta^{-1})}, \quad y < y_0,$$

and

$$\tilde{\Gamma}_k(y, y_0) = \frac{\exp(-\bar{k}y)(\sinh(\bar{k}y_0) + \bar{k}\beta^{-1}\cosh(\bar{k}y_0))}{k(1 + k\beta^{-1})}, \quad y \geq y_0.$$

**Proof.** Let us represent $\Gamma_{k,h}$ as a sum $\Gamma_{k,h} = \hat{\Gamma}_k(y, y_0) + \tilde{\Gamma}_k(y, y_0)$, where $\hat{\Gamma}_k(y, y_0)$ is the Green function of the operator $L_k$ on $[0, +\infty)$ under boundary conditions

$$\frac{d\hat{\Gamma}_k}{dy}(y, y_0)|_{y=0} = \beta \hat{\Gamma}_k(0, y_0), \quad \lim_{y \to +\infty} \hat{\Gamma}_k(y, y_0) = 0.$$

Then $\hat{\Gamma}$ is defined by (7.71), (7.70) and $\tilde{\Gamma}_k$ is the solution of the following boundary value problem:

$$L_k \tilde{\Gamma}_k = 0,$$

$$\frac{d\tilde{\Gamma}_k(y)}{dy}|_{y=0} = \beta \tilde{\Gamma}_k(0, y_0),$$

$$\frac{d\tilde{\Gamma}_k(y)}{dy}|_{y=h} = \beta_1 \tilde{\Gamma}_k(h, y_0) + \beta_2; \quad \beta_2 = \beta \hat{\Gamma}_k(h, y_0) - \frac{d\hat{\Gamma}_k(y, y_0)}{dy}|_{y=h}.$$

Note that $|\hat{\Gamma}_k(h, y_0)| < c|\bar{k}|^{-1}|\exp(-\text{Re} \ k |h - y_0|))$. Therefore,

$$\beta_2 < c_1|\bar{k}|^{-1}|\exp(-\text{Re} \ k |h - y_0|)).$$

Resolving the BVP defined by (7.71), (7.72), (7.73) and taking into account the above estimate for $\beta_2$ we see that $|\tilde{\Gamma}_k| < C_0 \exp(-\text{Re} \ k |h - y| + |h - y_0|)).$ \hfill \Box

Roughly speaking Lemma 7.2 asserts that for large $h$ the Green function $\Gamma_k$ consists of two terms, the first one can be computed explicitly and the second one is a exponentially decreasing boundary layer term. Such a structure simplifies the analysis of the spectral problem. All terms induced by the boundary layers are negligible as $b \to +\infty$ due to our choice (7.60) of $h$. Note that for $\lambda \in \mathbb{C}_{1/2}$

$$\text{Re} \ k(\lambda) > \sqrt{k^2 - 1/2} > \frac{k}{2}.$$
Thus estimate (7.68) implies
\begin{equation}
|\Gamma_k(y, y_0) - \Gamma_{\bar{k}}(y, y_0)| < C_0 \exp(-k||h - y| + |h - y_0|/2).
\end{equation}

**Lemma 7.3.** For \( m = 0, 1, 2, 3 \) and \( \lambda \in \mathbb{C}_{1/2} \) the solution \( \psi_k \) of eq. (7.56) satisfies
\begin{equation}
|D^m_y \psi_k|_{\infty} \leq c_m k^{m-2} h|w_k|_{\infty}
\end{equation}
and for \( m = 0, 1, 2 \) the solution \( w_k \) of eq. (7.57) satisfies
\begin{equation}
|D^m w_k|_{\infty} \leq c_m k^{m-1} |\bar{k}|^{-1} |\psi_k|_{\infty} \sup_{y \in [0, h]} |U_y|.
\end{equation}

**Proof.** To prove (7.70) we use Lemma 7.2 and (7.74), which show that
\begin{equation}
\int_0^h |D^m_y \Gamma_k(y, y_0)|dy < C_m k^{m-1} |\bar{k}|^{-1}, \quad m = 0, 1.
\end{equation}
These estimates imply (7.70) for \( m = 0, 1 \). For \( m = 2 \) estimate (7.76) follows from eq. (7.56) and (7.74).
To prove (7.75), we use the relation
\begin{equation}
||D^2_y \psi_k||^2 + (2k^2 + \frac{Re \lambda}{\nu})||D_y \psi_k||^2 + (k^4 + \frac{k^2 Re \lambda}{\nu})||\psi_k||^2 = k^2 Re \langle \psi_k^*, w_k \rangle,
\end{equation}
which follows from (7.56) and where \( \langle f, g \rangle = \int_0^h f g dy \), \( \psi_k^* \) is complex conjugate to \( \psi_k \), and \( ||f||^2 = \langle f, f^* \rangle \).

For \( \lambda \in \mathbb{C}_{1/2} \) implies
\begin{equation}
||D^m_y \psi_k|| \leq c_m k^{m-2} ||w_k|| \leq c_m \sqrt{h} k^{m-2} |w_k|_{\infty}
\end{equation}
for \( m = 0, 1, 2 \). Using (7.79) and eq. (7.56) we extend (7.79) on the case \( m = 4 \) and by \( ||D^2_y \psi_k||^2 \geq ||D^4_y \psi_k||^2 ||D^2_y \psi_k|| \) we obtain (7.79) for \( m = 3 \).

Now the Sobolev embeddings
\begin{equation}
|D^m_y \psi_k|_{\infty} \leq c_m \sqrt{h} ||D^{m+1} \psi_k||, \quad m = 0, 1, 2, 3
\end{equation}
lead to (7.76). \( \square \)

**Lemma 7.4.** If \( Re \lambda_k > -1/2 \), and \( \mu, s_2, s_3 \) are defined by (7.63) and a non-trivial solution of BVP (7.56)-(7.59) exists. Then
\begin{equation}
|\bar{k}| < c_1 hrb = c_1 h b^{1-s_0},
\end{equation}
where \( c_1 > 0 \) is a constant independent of \( b, k \).

**Proof.** One has \( 2|\psi_k(y)| \leq y^2 |D^2_y \psi|_{\infty} \). That estimate and eq. (7.57) imply that
\begin{equation}
|w_k|_{\infty} \leq c_2 |D^2_y \psi_k|_{\infty} \sup_{y \in [0, h]} I_2(y),
\end{equation}
where
\begin{equation}
I_m(y) = \int_0^h |\Gamma_k(y, y_0)\left(C_U rb^4 \exp(-by_0) + \mu y_0 P_N(y_0)\right)|y_0^m dy_0.
\end{equation}
By Lemma 7.2 and (7.60) we find that for sufficiently large \( b \) one has
\begin{equation}
|I_2| < c_3 |\bar{k}|^{-1}(rb + b^{-s_3}),
\end{equation}
where
\( (7.83) \quad s_3 = s_2/2. \)

According to Lemma 7.3
\( (7.84) \quad |D^2_y \psi_k|_\infty \leq c_0 h |w_k|_\infty. \)

The above estimate of \(|I_2|\) and (7.84), (7.81) imply
\( (7.85) \quad |w_k|_\infty \leq c_4 h (rb + b^{-s_3}) |\bar{k}|^{-1} |w_k|_\infty \)
that entails (7.80). \(\Box\)

**Lemma 7.5.** If \( \text{Re} \lambda_k > -1/2 \), then for sufficiently large \( b > 0 \) a nontrivial solution of problem (7.56)-(7.59) exists only under condition
\( (7.86) \quad \rho_2 = \frac{1}{2} \left. \frac{d^2 \psi_k(y)}{dy^2} \right|_{y=0} \neq 0. \)

**Proof.** The proof is analogous to the previous one. If \( \rho_2 = 0 \), then \( 6|\psi_k(y)| \leq y^3 |D^3_y \psi|_\infty. \) This estimate and eq. (7.57) imply that
\( (7.87) \quad |w_k|_\infty \leq C_3 |D^3_y \psi_k|_\infty \sup_{y \in [0,k]} I_3(y), \)
where \( I_3 \) is defined by (7.82). As above one has
\( |I_3| < c_4 |\bar{k}|^{-1} (r + b^{-s_3}). \)

This estimate and the inequality
\( (7.88) \quad |D^3_y \psi_k|_\infty \leq c_1 kh |w_k|_\infty, \)
that follows from Lemma 7.3 imply
\( (7.89) \quad |w_k|_\infty \leq c_5 h (r + b^{-s_3}) |\bar{k}|^{-1} |w_k|_\infty. \)
For large \( b \) one has \( c_5 h (r + b^{-s_3}) < 1, \) thus \( |w_k|_\infty = 0. \) \(\Box\)

The next step is to find an asymptotics for \( \psi_k \) under condition (7.80).

**7.4. Asymptotics of eigenfunctions.** Let us find an asymptotic for \( \psi_k \) and \( w_k \) with respect to the parameter \( k/b \), which is small due to Lemma 7.4. Using the Taylor expansion for \( \psi_k \) at \( y = 0 \) we introduce the function \( \bar{W}_k \) as a solution of the equation
\( (7.90) \quad L \bar{W}_k - \lambda_k \bar{W}_k = C_U r b^4 \rho_2 y^2 \exp(-by), \)
where, according to Lemma 7.3 without any loss of generality one can set \( \rho_2 = 1. \) Solving eq. (7.90) one has
\( (7.91) \quad \bar{W}_k = C_U r b^4 D_b^2 \left( \frac{\exp(-by) - \xi_k \exp(-b\bar{y})}{b^2 - k^2} \right), \)
where
\[ \xi_k = \frac{1 + r}{r(1 + k/\beta)}, \quad D_b = \frac{\partial}{\partial b} \]
We represent \( \psi_k \) by
\( (7.92) \quad \psi_k = \bar{\Psi}_k + \tilde{\Psi}_k, \)
where
\[ \tilde{\Psi}_k = C_U r k^2 b^4 D_b^2 \left( \frac{(b^2 - k^2)}{(b^2 - \bar{k}^2)} = 1 (k_{\bar{b}}(b, y) - \xi_k \Phi_k(k_{\bar{b}}, y)) \right) \]
and
\begin{equation}
\Phi_k(p, y) = \frac{\exp(-py) - \exp(-ky) + y(p - k) \exp(-ky)}{(p^2 - k^2)(p^2 - k^2 - \lambda \nu^{-1})}.
\end{equation}

We see that
\begin{equation}
\bar{\Psi}_k(\bar{k}, y) = -6C_U k^2 \tilde{\xi}_k \Phi_k(\bar{k}, y) + Z_k(\bar{k}, y),
\end{equation}
where
\begin{equation}
\tilde{\xi}_k = \frac{1 + r}{1 + k\beta - 1},
\end{equation}
and $Z_k$ is defined by relations
\begin{equation}
Z_k(\bar{k}, y) = Z_k^{(1)}(\bar{k}, y) + Z_k^{(2)}(\bar{k}, y).
\end{equation}
Here
\begin{align*}
Z_k^{(1)}(\bar{k}, y) &= C_U r k^2 b^4 D_\nu^2 (\Phi_k(b, y)), \\
Z_k^{(2)}(\bar{k}, y) &= -C_U k^2 \tilde{\xi}_k (b^4 D_\nu^2 ((b^2 - \bar{k}^2)^{-1}) - 6) \Phi_k(\bar{k}, y).
\end{align*}
By these relations and Lemma 7.4 one obtains
\begin{equation}
\sup_{y \in [0, \delta]} |Z_k(\bar{k}, y)| < c_0 |C_U| \bar{k}^2 / b^2
\end{equation}
and thus the function $Z_k$ is a small correction to the main term $\bar{\Psi}_k$.

**Lemma 7.6.** In the domain $\lambda \in \mathbb{C}_{1/2}$ the function $\bar{\Psi}_k(\bar{k}, y)$ is an analytic in $\lambda$ and for sufficiently large $b > 0$ satisfies the estimate
\begin{equation}
|\bar{\Psi}_k(\bar{k}, y)| < C_0 |C_U|,
\end{equation}
where $C_0 > 0$ is a constant independent of $k$ and $b$.

**Proof.** Note that according to definition (7.61) of $\beta$ and $r$, one has $|\tilde{\xi}_k| < 2$ for $\lambda \in \mathbb{C}_{1/2}$. Consider the function $\Phi_k(\bar{k}, y)$. We make the substitution $\lambda = k^2 \tau$, where $\tau$ is a new complex variable defined in the domain $Re \tau > -1/2$. Then $k - \bar{k} = k(\sqrt{1 + \tau} - 1)$. Using the Taylor series one has
\begin{equation}
\Phi_k(\bar{k}, y) = \frac{y^2 (\sqrt{1 + \tau} - 1)^2}{2k^2 \tau^2 (1 - \nu^{-1})} T(ky, \tau) \exp(-ky),
\end{equation}
where
\begin{equation}
T(z, \tau) = 1 - 2z \frac{\sqrt{1 + \tau} - 1}{3!} + 2z^2 \frac{\sqrt{1 + \tau} - 1}{4!} - ...
\end{equation}
Note that the function $(\sqrt{1 + \tau} - 1)^2 \tau^{-2}$ is uniformly bounded for $Re \tau > -1/2$. Since $(ky)^m \exp(-ky) \leq m^m \exp(-m)$ the series for $T$ converges and uniformly bounded in the domain $|\sqrt{1 + \tau} - 1| < 1$. Consequently, in that domain one has
\begin{equation}
|\Phi_k(\bar{k}(\tau), y)| < C_0.
\end{equation}
On the other hand for $|\tau| > \delta$ we have
\begin{equation}
|\Phi_k(\bar{k}, y)| = \frac{|\exp(-\bar{k}y) - \exp(-ky) + y(\bar{k} - k) \exp(-ky)|}{k \bar{k}^2 \tau^2 (1 - \nu^{-1})} < c_0(\delta) k^{-2}.
\end{equation}
Using (7.101), (7.102), (7.93) and (7.94) one obtains (7.98). \qed
7.5. Perturbation theory. We represent \( \psi_k \) by (7.92). Then for \( \tilde{\Psi}_k \) one obtains the equation

\[
\lambda \nu^{-1} L_k \tilde{\Psi}_k = L_k^2 \tilde{\Psi}_k + k^2 \tilde{W}_k,
\]

where \( \tilde{W}_k \) satisfies

\[
\lambda_k \tilde{W}_k = L_k \tilde{W}_k + \mu y P_N(y) (\tilde{\Psi}_k + \tilde{\phi}_k) + S_k,
\]

and \( S_k \) admits the estimate

\[
|S_k(y)| < C_3 r b^3 y^3 \exp(-by)|D^3_y(\tilde{\Psi}_k + \tilde{\phi}_k)|_{\infty}.
\]

The functions \( \tilde{\Psi}_k \) and \( \tilde{W}_k \) satisfy the boundary conditions

\[
\frac{d\tilde{W}_k(y)}{dy} |_{y=0} = \beta \tilde{W}_k(0), \quad \frac{d\tilde{W}_k(y)}{dy} |_{y=h} = \beta_1 \tilde{W}_k(h),
\]

\[
\tilde{\Psi}_k(0) = 0, \quad \frac{d\tilde{\Psi}_k(y)}{dy} |_{y=0} = 0,
\]

\[
\tilde{\Psi}_k(h) = p_k, \quad \frac{d\tilde{\Psi}_k(y)}{dy} |_{y=h} = q_k,
\]

where \(|p_k|, |q_k| < c_1 \exp(-k h) < c_2 b^{-10} \).

We can resolve the BVP defined by (7.103), (7.104), (7.106), (7.107), and (7.108) by iterations that follows from the next lemma. That BVP defines \( \tilde{\Psi}_k \) via \( \tilde{\phi}_k \), i.e., \( \tilde{\Psi}_k = A(\tilde{\phi}_k) \), where \( A \) is a linear operator. We consider that operator on the space \( C^2[0,h] \) of functions \( f \) with the bounded norm

\[
|f|_3 = \sum_{m=0}^{3} \sup_{y \in [0,h]} |D_y^m f|.
\]

**Lemma 7.7.** For \( |\hat{k}| < c_0 b^{s_1} \), where \( s_1 \in (0,1) \), and sufficiently large \( b \) the operator \( A \) is a contraction on \( C^3[0,h] \). More precisely, the solutions of the BVP defined by (7.103), (7.104), (7.106), (7.107), and (7.108) satisfy

\[
|\tilde{\Psi}_k|_3 < c_1 b^{-s_4} |\tilde{\phi}_k|_3, \quad s_4 = \min\{s_3, 4s_0/5\} > 0.
\]

**Proof.** We have \( \tilde{W}_k(y) = J_P(y) + J_S(y) \), where

\[
J_P = \int_0^h \Gamma_k(y,y_0) y_0 P_N(y_0) (\tilde{\Psi}_k(y_0) + \tilde{\phi}_k(y_0)) dy_0,
\]

\[
J_S = \int_0^h \Gamma_k(y,y_0) S_k(y_0) dy_0.
\]

By (7.77) and using that for large \( b \) one has \( \mu \max_{y \in [0,h]} |y P_N(y)| < c_4 b^{-s_3} \), one finds

\[
|J_P| \leq c_2 b^{-s_3} |\hat{k}|^{-1} k^{-1} (|\tilde{\Psi}_k|_{\infty} + |\tilde{\phi}_k|_{\infty}).
\]

To estimate \( |J_S| \) we note that

\[
\int_0^h |\Gamma_k(y,y_0)| y_0^3 \exp(-by_0) dy_0 \leq c_3 b^{-4} |\hat{k}|^{-1}.
\]

Thus, by (7.105) one has

\[
|J_S| < c_4 r |\hat{k}|^{-1} (|D^3_y \tilde{\Psi}_k|_{\infty} + |D^3_y \tilde{\phi}_k|_{\infty})).
\]
Consequently

\[ |\tilde{W}_k|_\infty < c_5 \left( b^{-\alpha_3} \frac{1}{|k|} (|\tilde{\Psi}_k|_\infty + |\tilde{\Psi}_k|_\infty) + \frac{r}{|k|} (|D^3_y\tilde{\Psi}_k|_\infty + |D^3_y\tilde{\Psi}_k|_\infty) \right). \]  

From Lemma 7.3 it follows that

\[ |\tilde{\Psi}_k|_\infty \leq c_6 k^{-2} h |\tilde{W}_k|_\infty, \quad |D^3_y\tilde{\Psi}_k|_\infty \leq c_7 k h |\tilde{W}_k|_\infty. \]  

Substituting those inequalities into (7.114) and noticing that for large \( b \) one has \( rh < b^{-4\alpha_0/5} \), we find that

\[ |\tilde{W}_k|_\infty < c_7 \left( b^{-\alpha_3} (|k|)^{-1} |\tilde{\Psi}_k|_\infty + r |\tilde{k}|^{-1} |D^3_y\tilde{\Psi}_k|_\infty \right). \]

Again using (7.115), one has (7.109).

**7.6. Nonlinear equation for \( \lambda_k \).** Let us make the substitution \( \tilde{z} = k(\lambda)/k \), where \( \tilde{z} \) is a new complex unknown. Since \( \lambda \in \mathbb{C} \), for each fixed \( k \) the variable \( \tilde{z} \) lies in the domain

\[ \mathbb{D}_{k,b} = \{ z \in \mathbb{C} : \text{Re } z > \sqrt{k^2 - 1/2}, |z| < c_0 hr b/k \}. \]

According to (7.93), (7.94), (7.96), (7.99) and (7.101) one has

\[ \frac{d^2 \tilde{\Psi}_k(y, \lambda)}{dy^2}|_{y=0} = -3 \tilde{\xi}_k(z) C_U (1 - \nu^{-1})^{-1} g(z) + H_k(z, b), \]

where

\[
\begin{align*}
g(z) &= \frac{1}{(z+1)^2}, \quad \tilde{\xi}_k(z) = \frac{1 + r}{1 + k z/rb}, \\
H_k &= H_{k,0} + H_{k,1}, \\
H_{k,0} &= C_U r b^4 D^3_y \left( \frac{b-k}{(b^2 - k^2 z^2)(b+k)(b^2 - k^2(1 + (z^2 - 1)\nu^{-1}))} \right), \\
2H_{k,1} &= -C_U \tilde{\xi}_k(z) \left( b^4 (D^2_y z^2 - 6) g(z) \right).
\end{align*}
\]

Let us set

\[ 3C_U = -8 (1 - \nu^{-1})(1 + r)^{-1}. \]

We note that \(|1 + z| > 1\) and thus according to estimate (7.97) and Lemma 7.4 for large \( b \) one has

\[ \sup_{z \in \mathbb{D}_{k,b}} |H_k(z, b)| < c_1 b^{-2\alpha_0}. \]

Let us introduce

\[ \tilde{H}_k(z, b) := \frac{d^2 \tilde{\Psi}_k(y, \lambda(z))}{dy^2}|_{y=0}. \]

Due to Lemma 7.7 the term \( \tilde{H}_k \) is a smooth uniformly bounded function in the domain \( \mathbb{D}_{k,b} \):

\[ \sup_{z \in \mathbb{D}_{k,b}} |\tilde{H}_k(\tau, b)| < c_3 b^{-\alpha_4}. \]

Then we use that \( \psi_k = \tilde{\Psi}_k + \tilde{\Psi}_k \) and compute \( \rho_4 \) defined by (7.86). Without any loss of generality one can set \( \rho_2 = 1 \). As a result, we obtain the equation for \( z \):

\[ (z+1)^2 = 4(1 + az)^{-1} + Y_k(z, b), \]

where

\[ 2Y_k = (1 + z)^2 (\tilde{H}_k(z, b) + H_k(z, b)), \quad a = k/rb. \]
We note that $Y_k$ admits the estimate
\begin{equation}
|Y_k(z,b)| < C_1 b^{-s_4} |1 + z|^2, \quad z \in \mathbb{D}_{k,b}.
\end{equation}

\textbf{7.7. Investigation of equation (7.121).} As $b \to \infty$ we have $Y_k \to 0$ and the limit cubic equation, which arises from (7.121), has three roots: a real and two imaginary ones.

**Lemma 7.8.** (a) If the root $z_k$ of (7.121) lies in the domain
$$
E_{k,b,c_1} = \{ z \in \mathbb{D}_{k,b} : \text{Re } z > 1 - c_1 b^{-s_4} \},
$$
then $z_k \in I_b$, where
\begin{equation}
I(b) = \{ z \in \mathbb{C} : |z - 1| < c_4 b^{-s_4} \},
\end{equation}
where $c_4 > 0$ is a constant. The subdomain $I_b$ contains only a single root of (7.121); (b) Let
\begin{equation}
\text{Re } Y_k(z,b) < -c_1 b^{-s_4}, \quad z \in I(b).
\end{equation}
Then for sufficiently large $b$ the root $z_k$ of (7.121) satisfies
\begin{equation}
\text{Re } z_k < 1 - c_2 b^{-s_4}.
\end{equation}

\textbf{Proof.} (a) Since $\text{Re } z_k > 0$, we have $|1 + az_k| > 1$ and then (7.121) implies that
\begin{equation}
|1 + z_k| < 2 + c_1 b^{-s_4} < 3.
\end{equation}
Then (7.122) can rewritten as
\begin{equation}
|Y_k(z,b)| < C_2 b^{-s_4}, \quad z \in \mathbb{D}_{k,b}.
\end{equation}
We take the imaginary part of (7.121) and one has
$$
2(\text{Re } z_k + 1)\text{Im } z_k = \frac{4a\text{Im } z_k}{1 + a^2|z_k|^2} + \text{Im } Y_k.
$$
That relation and (7.121) give
\begin{equation}
|\text{Im } z_k| < c_2 b^{-s_4}.
\end{equation}
Now we take the real part of (7.121) that entails
\begin{equation}
(\text{Re } z_k + 1)^2 = (\text{Im } z_k)^2 + 4(1 - a\text{Re } z_k)(1 + a^2|z_k|^2)^{-1} + \text{Re } Y_k.
\end{equation}
By that relation, estimates (7.127), (7.128). $\text{Re } z_k > 0$ and $a > 0$ one obtains that
\begin{equation}
\text{Re } z_k < 1 + c_2 b^{-s_4}.
\end{equation}
That inequality and (7.128) entail the assertion (a).

Let us prove the uniqueness of the roots $z_k$ in the case (a). One has $z_k(a) \in I(b)$. For bounded $k$ the distance $d_{E,D} = \text{dist}(I_b, \partial \mathbb{D}_{k,b})$ between the boundary of the domain $\mathbb{D}_{k,b}$ and $I_b$ satisfies $d_{E,D} > 1/4$. Note that $H_k(z,b)$ is an analytical function of $z$ in the domain $\mathbb{D}_{k,b}$ (see the proof of Lemma 7.6). Therefore, estimate (7.119) entails
\begin{equation}
\frac{|dH_k|}{dz} < c_3 b^{-2s_0}, \quad z \in I_b.
\end{equation}
The perturbation \( \tilde{H}_k \) also is analytic in \( z \) in the domain \( \mathbb{D}_{k,b} \). Indeed, \( \tilde{H}_k \) is the derivative of \( \Psi_k \), which is a fixed point of of contraction mapping and that contraction analytically depends on the parameter \( z \) if \( z \in \mathbb{D}_{k,b} \). Therefore, (7.120) gives

\[
(7.132) \quad \left| \frac{d\tilde{H}_k}{dz} \right| < c_7 b^{-s_4}, \quad z \in I_b.
\]

Estimates (7.131) and (7.132) imply that

\[
(7.133) \quad \left| \frac{dY_k}{dz} \right| < c_8 b^{-s_5}, \quad z \in I(b), \quad s_5 = \min\{s_4, 2s_0\}
\]

for some \( c_8 > 0 \). Now the uniqueness of the root \( z_k = 0 \) for \( k = k_1, ..., k_N \) follows from (7.133) and the Implicit Function Theorem.

Consider assertion (b). If \( \text{Re} \ Y_k < -c_1 b^{-s_4} \), then relations (7.128) and (7.129) imply

\[
(7.134) \quad \text{Re} \ z_k + 1 < -c_2 b^{-s_4} + 4(1 + a^2|z_k|^2)^{-1},
\]

and (7.125) follows. \( \square \)

For large \( b \) and \( z \in I(b) \) the main term of the asymptotics of \( \tilde{H}_k \) can be found by the system of equations

\[
(7.135) \quad L_k^2 \Psi_k^{(0)} = k^2 W_k^{(0)},
\]

\[
(7.136) \quad L_k W_k^{(0)} = y^3 P_N(y) \exp(-ky) := V_k(y),
\]

for the unknown functions \( \Psi_k^{(0)}(y) \) and \( W_k^{(0)}(y) \), which satisfy no-flip boundary conditions analogous to ones for \( \tilde{Ψ}_0 \) and \( \tilde{W}_0 \). For \( z = 1 \) the perturbation \( \tilde{H}_k(z,b) \) can be expressed via \( \Psi_k^{(0)}(y) \) by the relation

\[
(7.137) \quad \tilde{H}_k(1,b) = \mu \frac{d^2 \Psi_k^{(0)}(y)}{dy^2} |_{y=0} + O(b^{-s_4}).
\]

By a variation of \( P_N(y) \) in (7.62), we can obtain a large class of perturbations \( \tilde{H}_k \) that follows from the next lemma.

**Lemma 7.9.** For any polynomial \( Z_N(p) = q_0 + q_1 p + \ldots + q_N p^N \) of variable \( p = k^{-1} \) there exists a polynomial \( \tilde{P}_N(y) = \sum_{n=0}^{N} \tilde{r}_n y^n \) such that

\[
(7.138) \quad D^2_y \Psi_k^{(0)}(y) |_{y=0} = k^{-6} (Z_N(k^{-1}) + \frac{k}{(\beta + k)} \tilde{Z}_N(k^{-1}) + O(b^{-10})), \quad b \to \infty
\]

where

\[
(7.139) \quad \tilde{Z}_N(k^{-1}) = a_0 q_0 + a_1 q_1 k^{-1} + \ldots + a_N q_N k^{-N},
\]

and \( a_n > 0, n = 1, ..., N \) are some coefficients independent of \( k,b \).

**Proof.** Consider equation (7.135) under no-flip conditions at \( y = 0,h \). We multiply the both hand sides of (7.135) by \( f_k(y) = (ky + 1) \exp(-ky) \). Integrating by parts on \([0,h]\) and using that \( df_k(y)/dy = 0, \Psi_k^{(0)}(y) = 0 \) and \( d\Psi_k^{(0)}/dy = 0 \) at \( y = 0 \), we obtain

\[
(7.140) \quad \frac{d^2 \Psi_k^{(0)}}{dy^2} |_{y=0} = \int_0^h W_k^{(0)}(y)(ky + 1) \exp(-ky)dy + O(\exp(-kh)).
\]
Let
\[ \rho_k(y) = -\left(\frac{3}{4k(\beta + k)} + \frac{3y^2}{4k} + \frac{y^4}{4}\right) \exp(-ky). \]

Note that \( L_k \rho_k = f_k(y) \). Suppose \( w(y) \) be a smooth function defined on \([0, h]\) and satisfying the condition
\[ \frac{dw(y)}{dy}\big|_{y=0} = \beta w(0). \]

The function \( \rho_k \) also satisfies the same boundary condition. Then, integrating by parts one obtains
\[ \int_0^h (L_k w(y)) \rho_k(y) dy = O(\exp(-kh)) + \int_0^h w(y)(ky + 1) \exp(-ky) dy. \]

We multiply the both hand sides of eq. (7.136) by \( \rho_k \). Then, again integrating by parts, and using the last equality, we find that
\[ \frac{d^2 \Psi_k^{(0)}}{dy^2}\big|_{y=0} = \int_0^h y^3 P(y) \rho_k(y) dy + O(b^{-10}). \]

We substitution \( P_N \) into that relation that gives
\[ \frac{d^2 \Psi_k^{(0)}}{dy^2}\big|_{y=0} = \sum_{n=0}^N \tilde{r}_n (2k)^{-n-6} \left( \frac{3(n+3)!}{\beta + k} + \frac{3(n+4)!}{2} + \frac{(n+5)!}{4} \right) + O(b^{-10}). \]

Then using relation (7.142) we can find coefficients \( \tilde{r}_n \) such that
\[ k^{-6} \sum_{l=0}^N q_l k^{-l} = \sum_{n=0}^N \tilde{r}_n (2k)^{-n-6} \left( \frac{3(n+4)!}{2} + \frac{(n+5)!}{4} \right) \]
for all \( k \neq 0 \). As a result, we obtain (7.138) and (7.139), where
\[ a_n = 3 \left( \frac{3(n+4)}{2} + \frac{(n+4)(n+5)}{4} \right)^{-1}, \]
that entails the assertion of the Lemma.

In order to use the last lemma we take \( Z \) of a special form:
\[ Z(p, d) = -\left( \prod_{j=1}^N (p - (k_j + d_j)^{-1}) \right)^2, \]
where \( k_j \) are defined by Prop. 7.1 and \( d_j \) are parameters. The polynomial \( \tilde{Z}_k \) is defined then by (7.139).

**LEMMA 7.10.** For sufficiently large \( b > 0 \) one can find coefficients \( d_j(b) \) such that the real roots \( z_k \) of eq. (7.121) satisfy
\[ z_k < 1 - c_0 b^{-c_1}, \quad k \notin K_N, \]
for some \( c_1 > 0 \) independent of \( b \) and
\[ z_k = 1, \quad k \in K_N. \]
Moreover, \( d_j(b) \to 0 \) as \( b \to +\infty \).
Let us prove that this system has a solution both negative. Thus we obtain that for \( k \neq 0 \) hand side of (7.146) with respect to the contribution \( Z \) use relation (7.146). Since (7.144).

\[ rb \]

\[ (7.148) \]

\[ Y_k(z, b, d) = \mu k^{-6} (Z(k^{-1}, d) + \frac{k}{\beta + k} \tilde{Z}(k^{-1}, d)) + O(\mu b^{-s_4}) + O(b^{-2s_0}). \]

Now we make a special choice of the exponents \( s_0 \) and \( s_2 \). Let us take \( s_2 \in (0, 1/10) \). Then we choose \( s_0 \in (1 - s_2/2, 1) \) close to 1 such that

\[ s_0 = (6 + 2N)(1 - s_0) > s_4. \]

Using Lemma (7.4) and definitions (7.143) and (7.139) of \( Z \) and \( \tilde{Z} \), we see then that for all positive \( k \) such that \( k < \min\{\beta^{1/2}, c_1 b^{1-s_0}\} \), and \( k \notin \mathcal{K}_N \) the term \( \mu k^{-6} Z(k^{-1}, d) \) is dominant in the right hand side of (7.146) and that term is negative. One has \( Z(k^{-1}, d) < -c_0 b^{-s_0} < 0 \).

For all positive \( k \) such that \( k < \beta^{1/2} \) but \( k < c_0 b^{1-s_0} \), the both terms \( \mu k^{-6} Z(k^{-1}, d) \) and \( \mu k^{-6} \frac{k}{\beta + k} \tilde{Z}(k^{-1}, d) \) are dominant (as \( b \to +\infty \)) in the right hand side of (7.146) with respect to the contribution \( O(b^{-s_4}) \). Those terms are both negative. Thus we obtain that for \( k \notin \mathcal{K}_N \) the roots of eq. (7.121) satisfy (7.144).

Consider (7.145). We define \( d_j \) by the system

\[ rb + k_j \]

\[ (7.148) \]

\[ \frac{\partial Y_k(z, b, d)}{\partial b(1 + r)} + \mu Y_k(z, b, d)|_{z=1} = 0, \quad \forall \ j = 1, ..., N. \]

Let us prove that this system has a solution \( d \) such that \(|d|\) is small enough. We use relation (7.146). Since \( Z(k_j^{-1}, 0) = 0 \) and the Jacobian matrix defined by

\[ J_{ij}(d) = \frac{\partial Z(k_j^{-1}, d)}{\partial d_i} \]

is invertible for small \(|d|\) (it follows from definition (7.143) of \( Z \)) we can apply to eq. (7.148) the Implicit Function Theorem and find a solution \( d \) of (7.148) such that \(|d| \to 0 \) as \( b \to 0 \). Now relation (7.145) follows from Lemma (7.8) \[ \square \]

**Proof.** of Prop. (7.1) Proposition 7.1 follows from the last lemma. \[ \square \]

### 7.8. Conjugate spectral problem

Standard computations show that the corresponding conjugate spectral problem is defined by the following equations and the boundary conditions:

\[ \lambda \Delta \phi = \nu \Delta^2 \phi - U_y \tilde{w}_x, \]

\[ \lambda \tilde{w} = \Delta \tilde{w} + \nu \phi_x, \]

\[ \tilde{w}_x(x, y)|_{x=0, \pi} = 0, \]

\[ \tilde{w}_y(x, y)|_{y=0} = \beta \tilde{w}(x, 0), \quad \tilde{w}_y(x, y)|_{y=\pi} = \beta_1 \tilde{w}(x, \pi), \]

\[ \phi(x, y)|_{y=0, \pi} = \Delta \phi(x, y)|_{x=0, \pi} = 0. \]

The Fourier decomposition of that problem is as follows:

\[ \lambda_k L_k \phi_k = \nu L_k^2 \phi_k - k^2 U_y \tilde{w}_k, \]
As it follows from (7.162) the norms of the functions \( \psi \) have the form
\begin{equation}
\psi \quad \text{where} \quad || \psi || = \| \psi \| = \beta \psi(0),
\end{equation}
\begin{equation}
\frac{d \psi_k}{dy}(y)|_{y=0} = \frac{d \psi_k}{dy}(y)|_{y=h} = \beta \psi_k(h).
\end{equation}

7.9. Eigenfunctions of \( L \) and \( L^* \) with zero eigenvalues. Let us consider the eigenfunctions \( e_k \) of \( L \) with the zero eigenvalues. By the stream function, they can be represented as have the form
\begin{equation}
e_j = (v_j, \theta_j)^{tr},
\end{equation}
where \( j = 1, ..., N, v_j = (D_y \psi_j, -D_x \psi_j)^{tr} \) and
\begin{equation}
\psi_j = \Psi(y) \sin(k_j x), \quad \theta_j(y) = \Theta_j \cos(k_j x).
\end{equation}

Relations (7.93) and (7.94) show that
\begin{equation}
\Theta_j(y) = \beta_j \left( \exp(-by) - \exp(-k_j y) + \tilde{\theta}_j(y) \right),
\end{equation}
where \( |\tilde{\theta}_j| < c_0 b^{-c_1} \) for some \( c_1 > 0 \). Moreover, by (7.91) one obtains
\begin{equation}
\theta_j = \phi_1 b \exp(-k_j y) + \tilde{\theta}_j(y),
\end{equation}
where \( \phi_1 < c_3 b^{-c_1}, c_1 > 0 \). Substituting that relation into (7.165), after some computations we obtain
\begin{equation}
\theta_j = \Theta_j(y) \cos(k_j x),
\end{equation}
(7.164)
\begin{equation}
\Theta_j = \tilde{\alpha}_j \left( (k_j y^2 + y) \exp(-k_j y) + \tilde{\theta}_j \right),
\end{equation}
where \( |\tilde{\theta}_j| < c_4 b^{-c_5} \), \( c_1 > 0 \), \( \tilde{\alpha}_j \) are constants, which should be chosen to satisfy biorthogonality relations
\begin{equation}
\langle e_j, e_i \rangle = \delta_{ij},
\end{equation}
where \( \delta_{ij} \) stands for the Kronecker symbol. For \( \psi_j^* \) one has \( v_j = (D_y \psi_j^*, -D_x \psi_j^*)^{tr} \).

As it follows from (7.166) the norms of the functions \( \psi_j^* \) are small:
\begin{equation}
||\psi_j^*|| < C\nu^{-1}.
\end{equation}

The next lemma concludes the investigation of spectral properties of the operator \( L \).

**Lemma 7.11.** The eigenvalue 0 of the operator \( L \) has no generalized eigenfunctions.
Proof. Let us check that generalized eigenfunctions are absent. Since 0 has a finite multiplicity, we can use the Jordan representation. Assume that there exists a generalized eigenfunction \( e_g \). Then there is a non-zero \( b \) such that \( L e_g = b = \sum_{j=1}^{N} b_j e_l \) for some \( b_l, l \in \{1,\ldots,N\} \). Then \( \langle b, e^*_k \rangle = 0 \) for all \( k \in \{1,\ldots,N\} \). Eigenfunctions \( e^*_k \) and \( e_l \) are orthogonal for \( k \neq l \), and thus all coefficients \( b_l = 0 \).

7.10. Estimates for semigroup \( \exp(Lt) \). The operator \( L \) is sectorial and, according to Prop. 7.1, satisfies the Spectral Gap Condition. Therefore \( \| \exp(Lt)v \| \leq M \exp(-\rho t) \| v \| \) for some \( \kappa > 0 \) and for all \( v = (\tilde{\omega}, w)^{tr} \) such that \( \langle v, \tilde{e}_j \rangle = 0, \quad j = 1,\ldots,N \), one has the following estimates

\[
\| \exp(Lt)v \| \leq M \exp(-\rho t) \| v \|, \quad (7.166)
\]

\[
\| \exp(Lt)v \|_\alpha \leq \bar{M} S_\alpha(t) \exp(-\rho t) \| v \|, \quad (7.167)
\]

Here \( \alpha > 1/2 \) and

\[ S_\alpha(t) = t^{-\alpha}, \quad 0 < t \leq \rho^{-1}, \]

and

\[ S_\alpha(t) = \rho^{-\alpha}, \quad t > \rho^{-1}. \]

The constants \( M, \bar{M} \) and \( \rho \) can depend on \( b \) but they are independent of \( \gamma \). Estimates (7.166) and (7.167) will be used in the proof of existence of the invariant manifold, see Appendix.

8. Finite dimensional invariant manifold

In this section, we reduce the Navier-Stokes dynamics to a system of ordinary differential equations following Sect. 4. Let \( E_N \) be the finite dimensional subspace \( E_N = \text{Span}\{e_1,\ldots,e_N\} \) of the phase space \( H \), where \( e_j = (v_j, \theta_j)^{tr} \) are the eigenfunctions of the operator \( L \) with the zero eigenvalues. Let \( e^*_j = (v^*_j, \theta^*_j)^{tr} \) be the corresponding eigenfunctions of the conjugate operator \( L^* \). We assume that \( e^*_j \) and \( e_j \) are biorthogonal. Let \( P_N, Q_N \) be the projection operators

\[
P_N z = \sum_{j=1}^{N} \langle z, e^*_j \rangle e_j, \quad Q_N = I - P_N. \quad (8.168)
\]

We denote by \( P_{v,N}, P_{w,N}, Q_{v,N} \) and \( Q_{w,N} \) the components of these operators. For \( P_{v,N} \) and \( P_{w,N} \) one has

\[
P_{v,N} z = \sum_{j=1}^{N} \langle z, e^*_j \rangle v_j, \quad P_{w,N} = \sum_{j=1}^{N} \langle z, e^*_j \rangle \theta_j. \quad (8.169)
\]

We transform equations (6.37),(6.38) into a system with "fast" and "slow" variables. Let us introduce the auxiliary functions \( R_v(X) \) and \( R_w(X) \) by

\[
R_v(X) = \sum_{j=1}^{N} X_j v_j, \quad R_w(X) = \sum_{j=1}^{N} X_j \theta_j,
\]

where \( X = (X_1,\ldots,X_N)^{tr} \). We represent \( \tilde{v} \) and \( w \) by

\[
\tilde{v} = R_v(X) + \tilde{v}, \quad w = R_w(X) + \tilde{w}. \quad (8.170)
\]
where
\[ P_N(\tilde{\psi}, \tilde{\nu}) = 0, \]
and \( \tilde{\psi}, \tilde{\nu} \) and \( X(t) \) are new unknown functions.

We substitute relations (8.170) in eqs. (6.37) and (6.38). As a result, one obtains the system
\[
\frac{dX_t}{dt} = \gamma F_i(X, \tilde{\psi}, \tilde{\nu}),
\]
(8.171)\[
\tilde{v}_t = P(\nu \Delta \tilde{\psi} + \kappa \tilde{e} \tilde{\nu} + \gamma Q_{1,N} F(X, \tilde{\psi}, \tilde{\nu})),
\]
(8.172)\[
\tilde{w}_t = \Delta \tilde{w} - \tilde{\nu}_2 U_v + \gamma Q_{2,N} G(X, \tilde{\psi}, \tilde{\nu}),
\]
(8.173)\[
F = \kappa e g_1(R_w(X) + w) - (R_v(X) + \psi) \cdot \nabla(R_w(X) + \tilde{\psi}),
\]
(8.174)\[
G = \eta_1 - (R_w(X) + \tilde{\psi}) \cdot \nabla(R_w(X) + \tilde{w} + u_1),
\]
(8.175)\[
F_i = (F, v_i^*) + (G, \theta_i^*).
\]
(8.176)

We consider equations (8.171), (8.172) and (8.173) in the domain
\[
W_{\gamma,R_0,C(1),\alpha} = \{(X, \tilde{\psi}, \tilde{\nu}) : |X| < R_0, ||\tilde{\psi}||_\alpha + ||\tilde{\nu}||_\alpha < C(1)\gamma\}.
\]
(8.177)

That domain is a tubular neighborhood of the ball \( B^N(R_0) \) and the parameter \( C(1) > 0 \) is independent of \( \gamma \) for small \( \gamma \). The width of that neighborhood is \( C(1)\gamma \).

**Lemma 8.1.** Let \( \delta \in (0,1) \) and \( \alpha \in (1/2,1) \). Assume \( \gamma > 0 \) is small enough: \( \gamma < \gamma_0(N,R_0,b,\alpha,\delta,C^{(1)}) \). Then the local semiflow \( S_t \), defined by equations (8.171), (8.172), and (8.173) has a locally invariant manifold \( M^{(N)}_\gamma \) of dimension \( N \). This manifold is defined by
\[
\tilde{v} = \tilde{v}_0(X,\gamma), \quad \tilde{w} = \tilde{w}_0(X,\gamma),
\]
(8.178)

where \( \tilde{v}_0(X,\gamma), \tilde{w}_0(X,\gamma) \) are maps from the ball \( B^N(R_0) \) to \( H_\alpha \) and \( \tilde{H}_\alpha \), respectively. They are bounded in \( C^{1+\delta} \)-norm:
\[
|\tilde{v}_0(X,\gamma)|_{C^{1+\delta}(B^N(R_0))} < C_1\gamma,
\]
(8.179)\[
|\tilde{w}_0(X,\gamma)|_{C^{1+\delta}(B^N(R_0))} < C_2\gamma,
\]
(8.180)

constants \( C_i > 0 \) are uniform in \( \gamma \). The restriction of the semiflow \( S_t \) on \( M_N \) is defined by the system of differential equations
\[
\frac{dX_t}{dt} = \gamma(V_i(X) + \tilde{V}_i(X,\gamma)),
\]
(8.181)

where
\[
V_i(X) = F_i(X,0,0)
\]
(8.182)

and the corrections \( \tilde{V}_i(X,\gamma) = F_i(X,\tilde{v}_0(X,\gamma),\tilde{w}_0(X,\gamma)) - F_i(X,0,0) \) satisfy the estimates
\[
|\tilde{V}_i|, |D_X \tilde{V}_i| < c_1\gamma^s, \quad s > 0.
\]
(8.183)
This assertion is proved in Appendix.

Due to Theorem on persistence of hyperbolic sets [12], for sufficiently small \( \gamma \) we can remove small corrections \( \hat{V} \) in the right hands of (8.181). Then, after a time rescaling, we obtain from (8.181) the following system of differential equations with quadratic nonlinearities:

\[
\frac{dX_i}{dt} = V_i(X) = K_i(X) + M_i(X) + f_i,
\]

which does not involve the small parameter \( \gamma \). Let us note that \( K_i \) and \( M_i \) can be represented as

\[
K_i(X) = \sum_{j=1}^{N} K_{ij}X_jX_i, \quad M_i(X) = \sum_{j=1}^{N} M_{ij}X_j.
\]

Using the stream- function representation of the eigenfunctions \( \psi_j \) we see that the coefficients \( K_{ij} \) and \( M_{ij} \) in (8.185) can be computed by the relations

\[
M_{ij}(u_1(x,y)) = \langle \{\psi_j, \theta_i^* \}, u_1 \rangle,
\]

and for large \( \nu \)

\[
|K_{ij} - \langle \{\psi_j, \theta_i^* \} \rangle| < c_0\nu^{-1}.
\]

Here we have used estimate (7.165), which implies that the scalar products, where the fluid components \( \psi_j^* \) of the conjugate eigenfunctions \( e_j^* \) are involved, have the order \( O(\nu^{-1}) \).

9. Control of linear terms in system (8.184)

In this section, we show that the coefficients \( M_{ij} \) involved in system (8.184) are completely controllable by the function \( u_1(x,y) \).

To calculate the entries of \( M_{ij} \) we use the relation

\[
M_{ij}(u_1(x,y)) = \langle \{\psi_j, \theta_i^* \}, u_1 \rangle.
\]

Using (7.159), (7.160), (7.161), (7.163) and (7.164) one obtains

\[
M_{ij}(u_1(x,y)) = 12 \sum_{j,l=1}^{N} (\bar{a}_i \bar{b}_j \bar{a}_k \bar{b}_l) \int_0^{2\pi} \int_0^{\gamma} \tilde{\zeta}_{ij}(y) \cos((k_i + k_j)x) + \tilde{\zeta}_{ij} \cos((k_i - k_j)x))dxdy,
\]

where

\[
\tilde{\zeta}_{ij} = k_j \Psi_j(y) - d\Theta^*_i(y) dy - k_i d\Psi_j(y) dy \Theta^*_i(y),
\]

\[
\tilde{\zeta}_{ij} = k_j \Psi_j(y) - d\Theta^*_i(y) dy - k_i d\Psi_j(y) dy \Theta^*_i(y).
\]

Using relations (7.160), (7.161) and (7.164), one obtains

\[
\tilde{\zeta}_{ij} = \bar{a}_i \bar{b}_j \eta_{ij}, \quad \zeta_{ij} = \bar{a}_i \bar{b}_j \eta_{ij},
\]

where \( \bar{a}_i, \bar{b}_j \) are some non-zero coefficients, and

\[
\eta_{ij} = y^2(k_j + 2k_i - 2k_r) y - 2(2k_i - 2k_r) y^2 \exp(-(k_i + k_j)y) + O(b^{-r_0}),
\]

\[
\tilde{\eta}_{ij} = y^2(k_j - 2k_i + 2k_i(k_i - k_j)y) \exp(-(k_i + k_j)y) + O(b^{-r_0})
\]

for some \( r_0 > 0 \).
Lemma 9.1. For each \(N \times N\) matrix \(T\) and each \(\delta > 0\) there exists a \(2\pi\)-periodic in \(x\) smooth function \(g_1(x, y)\) such that for sufficiently large \(b\) one has
\[
|M_{jl}(u_1(\cdot, \cdot)) - T_{jl}| < \delta, \quad \forall j, l = 1, \ldots, N,
\]
where \(u_1\) is defined via \(\tilde{u}_1\) by (6.30).

Proof. We are looking for \(u_1\) satisfying (9.193). Just \(u_1\) is found we use relation (6.30) and define \(g_1\) as
\[
g_1 = \frac{u_1}{(U - u_0)(1 + \gamma u_1)}
\]
choosing a sufficiently large \(u_0 > 0\) such that the function \(U - u_0\) has no roots.

We represent \(u_1(x, y)\) by a Fourier series:
\[
u_1(x, y) = \hat{u}_0(y) + \sum_{k=1}^{+\infty} \hat{u}_k(y) \cos(kx).
\]

Then relation (9.189) gives
\[
M_{ij}(u_1) = \frac{1}{2} \int_0^h \left( \tilde{u}_{ij}(y) \tilde{u}_{k_1 + k_2}(y) + \tilde{u}_{ij} \tilde{u}_{k_1 - k_2}(y) \right) dy.
\]

We introduce the auxiliary quantities \(V_{n,m}^{(p)}\) by
\[
V_{n,m}^{(p)} = \int_0^h y^{2+p} \exp(-my) \hat{u}_n(y) dy,
\]
where \(n > 0, m, p \geq 0\) are integer indices. Let us prove an auxiliary assertion.

Lemma 9.2. For any \(a_{m,p}\), where \(m = 1, \ldots, M\) and \(p = 0, 1, 2\), we can find a function \(W(y) \in C^2([0, h])\) such that
\[
\int_0^h y^p \exp(-my) W(y) dy = a_{m,p}, \quad \forall m = 1, \ldots, M, \quad p = 0, 1, 2.
\]

and
\[
W(0) = W(h) = W'(0) = W'(h) = 0.
\]

Proof. Consider the map \(W(\cdot) \rightarrow a_{m,p}\) defined by (9.196) on the space \(E_3\) of the functions \(\in C^2([0, h])\) and satisfying (9.197). The range of that map is a linear subspace of \(\mathbb{R}^{3M}\). If this assertion is not fulfilled, there is a vector orthogonal to the closure of that linear subspace and thus there exist numbers \(b_{m,p}\) such that
\[
\sum_{m=1}^{M} \sum_{p=0}^{2} |b_{m,p}| = 1
\]
and
\[
\sum_{m=1, \ldots, M} \sum_{p=0,1,2} \int_0^h b_{m,p} y^p \exp(-my) W(y) dy = 0
\]
for all \(W(y) \in C^2([0, h])\) such that (9.197) holds. This means that
\[
\sum_{m=1, \ldots, M} \sum_{p=0,1,2} b_{m,p} y^p \exp(-my) = 0, \quad \forall y \in (0, h),
\]
i.e., the functions \(y^p \exp(-my)\) are linearly dependent that is not the case. \(\square\)
Using that lemma we assume that all $V^{(p)}_{n,m} = 0$ for all $n = m = k_i + k_j$. Then by (9.194) and (9.196) estimate (9.193) can be rewritten as follows:

\[
\left|\begin{pmatrix} k_j + 2k_i \end{pmatrix} V^{(0)}_{|k_j - k_i|, k_j + k_i} + 2k_i^2 V^{(1)}_{|k_j - k_i|, k_j + k_i} - 2k_j k_i^2 V^{(2)}_{|k_j - k_i|, k_j + k_i} + \hat{B}_{ij} (b) - T_{ij} \right| < \delta,
\]

where $\hat{B}_{ij}$, $i, j = 1, ..., N$ satisfy $|\hat{B}_{ij}| < c_1 b^{-c_2}$, $c_1, c_2 > 0$ and $T_{ij} = T_{ij} \hat{a}_i \hat{b}_j$, where $\hat{a}_i, \hat{b}_j$ are coefficients.

Estimate (9.198) shows that, in order to prove the assertion, it suffices to find $X_{n,m} = V^{(0)}_{n,m}$ and $Y_{n,m} = V^{(1)}_{n,m}$ satisfying the system

\[
(k_j + 2k_i) X_{|k_j - k_i|, k_j + k_i} + 2k_i^2 Y_{|k_j - k_i|, k_j + k_i} = \bar{T}_{ij}, \quad i, j \in 1, ..., N.
\]

We decompose that system into the symmetric and antisymmetric parts. Then we obtain

\[
\begin{align*}
3 \left( k_j + 2k_i \right) X_{|k_j - k_i|, k_j + k_i} + & \left( k_i^2 + k_j^2 \right) Y_{|k_j - k_i|, k_j + k_i} = \bar{T}_{ij}^s, \quad i \geq j, \\
\frac{1}{2} \left( k_i - k_j \right) X_{|k_j - k_i|, k_j + k_i} + & \left( k_i^2 - k_j^2 \right) Y_{|k_j - k_i|, k_j + k_i} = \bar{T}_{ij}^a, \quad i > j,
\end{align*}
\]

for some $\bar{T}_{ij}^s$ and $\bar{T}_{ij}^a$, where $i, j \in 1, ..., N$. Consider the map $R_N : (i, j) \rightarrow |k_j - k_i|, k_j + k_i$ defined on the set of the pairs $(i, j)$ where $i = 1, ..., N$, $j = 1, ..., N$ and $i \geq j$. That map is an injection, therefore, the system of equations (9.200) and (9.201) can be represented as a set of independent systems of two linear equations for two unknowns. For each $(i, j)$ the corresponding linear $2 \times 2$ system is resolvable that can be checked by the determinant calculation.

\[\square\]

Let us formulate a lemma about control $f$ by $\eta_1$.

**Lemma 9.3.** Given a vector $f = (f_1, ..., f_N)$, there exists a smooth $2\pi$-periodic in $x$ function $\eta_1(x, y)$ such that

\[\langle \tilde{\theta}, \eta_1 \rangle = f_i, \quad i = 1, ..., N.\]

We omit an elementary proof.

In coming sections we investigate system (8.181) mainly following works [24] and [25, 23].

**10. Quadratic systems**

System (8.184) defines a local semiflow $S^t(f, M)$ in the ball $B^N(R_0) \subset \mathbb{R}^N$ of the radius $R_0$ centered at 0. We shall consider the vector $f$ and the matrix $M$ as parameters of this semiflow whereas the entries $K_{ijl}$ will be fixed.

Let us formulate an assumption on entries $K_{ijl}$. We represent $X$ as a pair $X = (Y, Z)$, where

\[Y_l = X_l, \quad l \in I_p, \quad Z_j = X_{j+p}, \quad j \in J_p,\]

where $I_p = \{1, ..., p\}$ and $J_p = \{1, ..., N-p\}$. Then system (8.184) can be rewritten as

\[\frac{dY}{dt} = K^{(1)}(Y) + K^{(2)}(Y, Z) + K^{(3)}(Z) + RY + PZ + f,\]
(10.203) \( \frac{dZ}{dt} = \dot{K}^{(1)}(Y) + \dot{K}^{(2)}(Y, Z) + \dot{K}^{(3)}(Z) + \ddot{R}Y + \ddot{P}Z + \ddot{f}, \)

where for \( i = 1, \ldots, p \)

(10.204) \( K_i^{(1)}(Y) = \sum_{j \in J_p} \sum_{l \in I_p} K_i^{(1)j} Y_{jl}, \quad K_i^{(3)}(Z) = \sum_{j \in J_p} \sum_{l \in I_p} K_i^{(3)j} Z_{jl}, \)

(10.205) \( K_i^{(2)}(Y, Z) = \sum_{j \in I_p} \sum_{l \in J_p} K_i^{(2)jl} Y_{jl}, \)

and for \( k = 1, \ldots, N - p \)

(10.206) \( \dot{K}_k^{(1)}(Y) = \sum_{j \in I_p} \sum_{l \in I_p} \dot{K}_k^{(1)jl} Y_{jl}, \quad \dot{K}_k^{(3)}(Z) = \sum_{j \in I_p} \sum_{l \in I_p} \dot{K}_k^{(3)jl} Z_{jl}, \)

(10.207) \( \dot{K}_k^{(2)}(Y, Z) = \sum_{j \in I_p} \sum_{l \in J_p} \dot{K}_k^{(2)jl} Y_{jl}. \)

Note that

(10.208) \( \dot{K}_k^{(1)} = K_{k+p,jl}, \quad k = 1, \ldots, p, \quad j, l = 1, \ldots, p. \)

The linear terms \( MX \) take the form

(10.209) \( (RY)_i = \sum_{j \in I_p} R_{ij} Y_j, \quad (\dot{R}Y)_k = \sum_{j \in I_p} \dot{R}_{kj} Y_j, \)

(10.210) \( (PZ)_i = \sum_{j \in J_p} P_{ij} Z_j, \quad (\dot{P}Z)_k = \sum_{j \in J_p} \dot{P}_{kj} Z_j, \)

and \( f = (f_1, \ldots, f_p), \quad \ddot{f} = (\ddot{f}_1, \ldots, \ddot{f}_{N-p}). \)

We denote by \( S^4(\mathcal{P}) \) the local semiflow defined by (10.202) and (10.203). Here \( \mathcal{P} \) is a semiflow parameter, \( \mathcal{P} = \{ f, \ddot{f}, P, \dot{P}, R, \dot{R} \} \). Let us formulate an assumption on quadratic terms \( K_i(X) \).

**p-Decomposition Condition** Suppose entries \( K_{ij} \) satisfy the following condition. For some \( p \) there exists a decomposition \( X = (Y, Z) \), where \( Y \in \mathbb{R}^p \) and \( Z \in \mathbb{R}^{N-p} \) such that for all \( b_{jl} \) the linear system

(10.211) \( \sum_{i \in J_p} \dot{K}_{i,jl} \chi_i = b_{jl}, \quad l, j \in I_p \)

has a solution \( \chi \).

Clearly that for \( N > p^2 + p \) and generic matrices \( K \) this condition is valid.

Let us formulate some conditions to the matrices \( R, \dot{R}, P \) and \( \dot{P} \). Let \( \xi > 0 \) be a parameter. We suppose that

(10.212) \( \ddot{P}_{ij} = -\xi^{-1} \delta_{i,j}, \quad i = 1, \ldots, N - p, \quad j = 1, \ldots, \)

(10.213) \( \ddot{R}_{ij} = 0, \quad \ddot{f}_i = 0, \quad i = 1, \ldots, N - p, \quad j = 1, \ldots, p, \)

(10.214) \( P_{ij} = \xi^{-1} T_{ij}, \quad |T_{ij}| < C_0, \quad i = 1, \ldots, p, \quad j = 1, \ldots, N - p, \)

(10.215) \( |R_{ij}| < C, \quad i = 1, \ldots, p, \quad j = 1, \ldots, p. \)

Let us define the domain in \( \mathbb{R}^N \):

(10.216) \( \mathcal{W}_{\dot{R},C^{(2)},\xi} = \{ X = (Y, Z) : \quad |Y| < \dot{R}, \quad |Z| < C^{(2)}\xi, \quad C^{(2)} > 0. \)
Note that \( W_{R,C} \) is a tubular neighborhood of the ball \( B^p(\tilde{R}) \) of the small width \( C(2) \).

**Lemma 10.1.** Assume \((10.214), (10.215), (10.216) \) and \((10.215) \) hold and \( \tilde{R} > 0, c_0 > 0 \) are constants. For sufficiently small positive \( \xi < \xi_0(R, \delta, M, k, p, N, c_0) \) and \( C(2) > c_0 \) the local semiflow \( S^t(\mathcal{P}) \) defined by system \((10.202), (10.203) \) has a locally invariant in the domain \( W_{R,C}\xi \) and locally attracting manifold \( \mathcal{M}_p^{(2)} \). This manifold is defined by equations

\[
Z = \xi(K^{(1)}(Y) + W(Y, \xi)), \quad Y \in B^p(\tilde{R})
\]

where \( W \) is a \( C^{1+\delta} \) smooth map defined on the ball \( B^p(\tilde{R}) \) to \( \mathbb{R}^{N-p} \) and such that

\[
|W(\cdot, \xi)|_{C^{1}(B^p(\tilde{R}))} < C_1 \xi^{s}, \quad s > 0.
\]

**Proof** can be found in the paper \[24\].

The semiflow \( S^t \) restricted to \( \mathcal{M}_p \) is defined by the equations

\[
dY = \xi S(Y, \xi),
\]

where

\[
S(Y, \xi) = K^{(1)}(Y) + \xi K^{(2)}(Y, \tilde{K}^{(1)}(Y) + W(Y, \xi)) + \\
+ \xi^2 K^{(3)}(\tilde{K}^{(1)}(Y) + W(Y, \xi)) + RY + T\tilde{K}^{(1)}(Y) + W(Y, \xi) + f.
\]

The estimates for \( W \) show that \( S \) can be presented as

\[
S(Y, \xi) = K^{(1)}(Y) + RY + T\tilde{K}^{(1)}(Y) + f + \tilde{S}(Y, \xi)
\]

where a small correction \( \tilde{S}(Y, \xi) \) satisfies

\[
|\tilde{S}(Y, \xi)|_{C^{1}(B^p(\tilde{R}))} < c_0 \xi^{1/2}.
\]

In \((10.220) \) \( R \) and \( f \) are free parameters. The quadratic form \( D(Y) = K^{(1)} + T\tilde{K}^{(1)} \) can be also considered as a free parameter according to \( p \)-Decomposition Condition. Therefore, we have proved the following assertion.

**Lemma 10.2.** Let

\[
W(Y) = D(Y) + RY + f
\]

be a quadratic vector field on the ball \( B^p(\tilde{R}) \), \( \tilde{R} > 0 \), where

\[
D_l(Y) = \sum_{i=1}^{p} \sum_{j=1}^{p} D_{ij} Y_i Y_j, \quad (RY)_l = \sum_{j=1}^{p} R_{ij} Y_j.
\]

Consider system \((10.202), (10.203) \). Let \( p \)- Decomposition Condition hold. Then for any \( \epsilon > 0 \) and \( R_0 > 0 \) the field \( W \) can be \( \epsilon \) - realized by local semiflow defined by system \((10.202), (10.203) \) on \( C^{1+\delta} \)-smooth locally invariant in a neighborhood \( W_{R,C}\xi \) of ball \( B^p(\tilde{R}) \) and locally attracting manifold \( \mathcal{M}_p^{(2)} \) of dimension \( p \). Here parameters \( \mathcal{P} \) are the matrices \( P, R, \tilde{R} \), the vectors \( f, \tilde{f} \).

Lemma 10.2 and results \[25\] imply the following assertion. Let us consider the families \( \Phi_{2,R_0} \) of quadratic fields \( V(X) \) defined on the ball \( B^N(R_0) \) by \((8.181) \) and depending on a parameter \( \mathcal{P} \) as follows. Each field in a family \( \Phi_{2,R_0} \) is defined by numbers \( p, N \), the matrix \( M \), the coefficients \( K_{ij,l} \), where \( i, j, l = 1, \ldots, N \), and the vector \( f \). We consider \( p, N \), the matrix \( M \) and the vector \( f \) as free parameters, i.e., the parameter \( \mathcal{P} \) of our family is a quadruple \( \{p, N, M, f\} \), where \( p, N \) runs over
the set of all positive integers $\mathbb{N}_+$, and $p^2 + p < N$. For each fixed $N$ the range of the parameter $M$ is the set of all square $N \times N$ matrices and the range of $f$ is $\mathbb{R}^N$. For each $P$ the corresponding coefficients $K_{ijl}$ are defined uniquely and satisfy $p$-Decomposition condition.

**Proposition 10.3.** Consider a family of the semiflows defined by systems (8.184), where $V \in \Phi_{2,R_0}$. Then that family enjoys the following property. For each integer $n$, each $\epsilon > 0$ and each vector field $Q$ satisfying (3.11) and (3.12), there exists a value of the parameter $P = P(\epsilon,n)$ such that the corresponding system (8.184) defines a semiflow, which $\epsilon$-realizes the vector field $Q$.

**10.1. Verification of $p$-Decomposition condition for system (8.181).** Let us fix a $p$. To verify $p$-Decomposition condition for system (8.184) we choose the set $K_N$ from Prop. 6.2 in a special way. Namely, we set $K_{p,N} = K_{p,N}$, where the set $K_{p,N}$ is defined below as follows.

Let us denote by $P_{2,p}$ the set of non-ordered pairs $(i,j)$, where $i,j \in \{1,\ldots,p\}$. The equality $(i,j) = (i',j')$ means that either $i = i'$, $j = j'$ or $i = j'$, $j = i'$. Let $S_p$ be the set consisting of all sums $k_i + k_j$, where $i,j \in \{1,\ldots,p\}$. Consider the map $S_p : P_{2,p} \rightarrow S_p$ from the set $P_{2,p}$ on the set $S_p$ defined by $S_p((i,j)) = k_i + k_j$. Let us prove an auxiliary lemma.

**Lemma 10.4.** For each $p > 1$ there exists a set $\bar{K}_p = \{\bar{k}_1,\ldots,\bar{k}_p\}$ of integers $\bar{k}_i > 0$ such that
\begin{equation}
\bar{k}_i \neq 5n, \quad \forall n \in \mathbb{N}
\end{equation}
and all the sums $\bar{k}_i + \bar{k}_j$ are mutually distinct, i.e.,
\begin{equation}
\bar{k}_i + \bar{k}_j = \bar{k}_i' + \bar{k}_j' \implies (i,j) = (i',j').
\end{equation}
In the other words, the map $S_p$ is injective.

**Proof.** The set $K_p = \{\bar{k}_1,\bar{k}_2,\ldots,\bar{k}_p\}$ can be found by an induction. For $p = 2$ we set $\bar{k}_1 = 1, \bar{k}_2 = 7$. Suppose $K_p$ is found for some $p$. Then we take an odd $k_{p+1}$ such that $k_{p+1} > k_{j_1} + k_{j_2}$ for all $j_1,j_2 \in \{1,\ldots,p\}$. Then the extended set $K_{p+1} = \{\bar{k}_1,\ldots,\bar{k}_p,\bar{k}_{p+1}\}$ satisfies conditions (10.224).

To show it, consider two sums from (10.221). If the pairs $(i,j)$ and $(i',j')$ do not include the index $p + 1$ then $(i,j) = (i',j')$ by the induction assumption. If one of $i,j$ equals $p + 1$ but the pair $(i',j')$ does not include $p + 1$, equality (10.224) is not fulfilled that follows from the construction of $k_{p+1}$. Therefore, the both pairs $(i,j)$ and $(i',j')$ contains $p + 1$. Then we can exclude this index, that gives either $i = i'$ or $i = j'$.

For all $p$ we define the sets $K_{p,N} = \{k_1,k_2,\ldots,k_N\}$, where $N = p(p+1)/2$, as follows. Let us take the set $K_p = \{k_1,\ldots,k_p\}$ satisfying the conclusion of lemma (10.4). For $i \leq p$ we take $k_i = \bar{k}_i$. The integers $k_i \in K_{p,N}$ with $i > p$ we define as different sums from $K_p$, i.e., $k_i = k_{i_1} + k_{i_2}$ for some $i_1,i_2 \in \{1,\ldots,p\}$.

Let us set $Y_l = X_l, l = 1,\ldots,p$. Respectively, all the rest variables $X_j$ with $j > p$ will be $Z_l$, where $l = j - p$. Then, due to relation (10.208), to verify $p$-Decomposition condition (10) it is sufficient to check that the linear system
\begin{equation}
\sum_{i=p+1}^{N} K_{ijl} \chi_i = b_{jl}, \quad j,l \in I_p = \{1,\ldots,p\}
\end{equation}
has a solution for any given $b_{jl}$.

To verify it, let us calculate the coefficients $K_{ijl}$ defined by (8.187). Integrating by parts one has

$$K_{ijl} = -\langle \{ \psi_j^*, \theta_i^* \}, \theta_l \rangle + O(\nu^{-1}).$$

Thus by definition (8.186)

$$K_{ijl} = -M_{ij}(\theta_l, \cdot) + O(\nu^{-1}).$$

Using that relation and (9.189) one obtains

$$K_{ijl} = \frac{1}{4} (\delta_{k_i, k_l} + k_j I_{ijl} + O(b^{-c_1})), \quad c_1 > 0,$$

and

$$I_{ijl} = \int_0^h \tilde{\zeta}_{ij}(y) \Theta_l(y) dy, \quad p + 1 \leq i \leq N, \quad j, l = 1, \ldots, p.$$ 

Relation (10.228) means that for each fixed pair $(j, l)$ the sum in the left-hand side of system (10.225) consists of a single term with the index $i$ defined by $k_i = k_j + k_l$. Due to Lemma 10.4 system (10.225) can be decomposed in independent linear equations, each of them involves only a single unknown $\chi_i$. Thus system (10.225) is resolvable under condition that all coefficients $I_{ijl}$ are not equal 0. To compute those coefficients we take into account relation (9.192) for $\tilde{\zeta}_{ij}$. For large $b$ we find that

$$I_{ijl} = \bar{a}_i \bar{b}_j \beta_l \left( J(\bar{k}_j, \bar{k}_l) + O(b^{-r_2}) \right) \delta_{k_i, k_l}, \quad r_2 > 0,$$

where $\bar{a}_i, \bar{b}_j, \beta_l \neq 0$ and

$$J(\bar{k}_j, \bar{k}_l) = 2(2(\bar{k}_l + \bar{k}_j))^{-3}(\bar{k}_j - 5\bar{k}_l).$$

We note (10.229) implies that $J(\bar{k}_j, \bar{k}_l) \neq 0$ for all integers $j, l = 1, \ldots, p$. Thus for each integer $p$ we can solve system (10.225) and the $p$-Decomposition condition is fulfilled.

11. Proof of Theorems

Proof. of Theorem 4.1. Let $\epsilon > 0$. We suppose that a vector field $Q$ defined on the ball $B^n$ satisfy (3.11) and (3.12). Our goal is to find parameters $\mathcal{P}$ of IBVP (2.2) - (2.9) such that the corresponding family of semiflows, generated by that IBVP, $\epsilon$-realizes $Q$.

Step 1. According to Proposition 10.3 for each $\epsilon_0 > 0$ we can $\epsilon_0$-realize the field $Q$ by a semiflow defined by a quadratic vector field $V(X)$ from a family $\Phi_{2,R_0}$. The field $V$ is defined on a ball $B^N \subset \mathbb{R}^N$.

Step 2. Consider the family $F_{OB}$ of global semiflows defined by by IBVP (2.2) - (2.9) with parameters $\mathcal{P} = \{ h, \nu, \gamma, \beta, I_1, \eta(\cdot, \cdot), g_1(\eta(\cdot, \cdot)) \}$. For any $\epsilon_1 > 0$ that family $\epsilon_1$-realizes (in the sense of definition 8.2) a family of semiflows $\Phi_{2,R_0}$ considered at the previous step. Therefore, if $\epsilon_0, \epsilon_1$ are small enough, the family $F_{OB}$ realizes $Q$ with accuracy $\epsilon$. 

Proof. of Theorem 4.2

Consider a global semiflow on finite dimensional smooth compact manifold defined by a $C^{1}$-smooth vector field and having a hyperbolic compact invariant set $\Gamma$. 

□
For an integer $n > 0$ we can find a smooth vector field $Q$ on a unit ball $B^n$, which generates a semiflow having a topologically equivalent hyperbolic compact invariant set $\Gamma'$ (and the corresponding restricted dynamics are orbitally topologically equivalent). Due to the Theorem on Persistence of Hyperbolic sets (see [20]) we find a sufficiently small $\epsilon(\Gamma', Q) > 0$ such that for all $C^1$ perturbations $\bar{Q}$ of $Q$ satisfying $|\bar{Q}|_{C^1(B^n)} < \epsilon$ the perturbed systems

$$dq/dt = Q(q) + \bar{Q}(q)$$

have hyperbolic compact invariant sets $\bar{\Gamma}$ topologically equivalent to $\Gamma$ (and the corresponding restricted dynamics are orbitally topologically equivalent). Then we $\epsilon$- realize this field by Theorem 4.1.

Now, to finish proof, it is sufficient to prove that trajectories defined by system (11.230) on $B^n$ do not leave the locally invariant manifold $M_n$. By definition of locally invariant manifolds (see Sect. 3), it suffices to prove that those trajectories do not leave the corresponding domain $W$, where that manifold is locally invariant. For sufficiently small $\epsilon > 0$ the trajectories $q$ are bounded. Indeed, the corresponding perturbed field $Q(q) + \bar{Q}(q)$, defined on $M_n$, directed inward the ball $B^n$ at the boundary $\partial B^n$ and thus the corresponding semitrajectories do not leave that ball. The corresponding trajectories $z(t) = (X(t), \tilde{v}, w(t))$ of semiflow $S^t$ generated by our IBVP on the manifold $M^{(1)}_n$ also are bounded. We choose radius $R_0$ and the width $C^{(1)}$ such that the domain $W_{R_0, C^{(1)}, \alpha}$ contains these trajectories $z(t)$. We make an analogous choice for all locally invariant manifolds $M^{(2)}_p, M^{(3)}$, ... involved in our realization, adjusting $R_d, \bar{R}$ and the corresponding width parameters $C^{(2)}$ and $C^{(3)}$. Then $z(t) = (X(t), \tilde{v}, w(t))$ do not leave $M_n$. It finishes the proof.

12. Conclusion

The idea that a complicated behaviour of dissipative dynamical systems, associated with fundamental models of physics, chemistry and biology can be generated by a strange (chaotic) attractor was pioneered in the seminal work of D. Ruelle and F. Takens [16]. In this paper, it is shown that classical system of hydrodynamics, which appears in many applications, can exhibit all kinds of structurally stable chaotic behaviour. The mathematical method admits a transparent physical interpretation: complicated large time dynamics can be produced by an exponentially decreasing at the no-flip boundary temperature profile and small space inhomogeneous perturbations of the gravity force and that profile. In this paper, the complete analytical description of complex turbulent patterns is given.

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I dedicate this paper to the memory of my friend Vladimir Shelkovich.

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14. Appendix

Proof of Lemma [8.1]. This assertion is a consequence of Theorem 6.1.7 [10]. In the variables $\hat{z} = (\hat{v}, \hat{w})^T$, $X$ system (8.171), (8.172), (8.173) can be rewritten
as

\[(14.231)\]

\[X_t = \gamma \hat{F}(X, \hat{z}),\]

\[(14.232)\]

\[\hat{z}_t = L\hat{z} + \hat{G}(X, \hat{z}).\]

Using the standard truncation trick we modify eq. (14.231) as follows:

\[(14.233)\]

\[X_t = \gamma \hat{F}(x, \tilde{w}) \chi_{R_0}(X),\]

where \(\chi_{R_0}\) is a smooth function such that \(\chi_{R_0}(X) = 1\) for \(|X| < R_0\) and \(\chi_{R_0}(X) = 0\) for \(|X| > 2R_0\). As a result of that modification, \(X\)-trajectories of (14.233) are defined for all \(t \in (-\infty, +\infty)\) (as in Theorem 6.1.7 [10]). Then an invariant manifold for the semiflow defined by system (14.233), (14.232) is a locally invariant one for the semiflow generated by (14.231), (14.232).

Let us consider the semigroup \(\exp(Lt)\). We have estimates (7.166), (7.167), where \(M, \bar{M}, \rho > 0\) do not depend on \(\gamma\). Moreover,

\[(14.234)\]

\[M_0 = \gamma \sup_{(X, \hat{z}) \in D_0, 2R_0, \alpha} ||\hat{F}\chi_{R_0}|| < c_2\gamma,\]

\[(14.235)\]

\[\lambda = \gamma \sup_{(X, \hat{z}) \in D_0, 2R_0, \alpha} ||D_X\hat{F}\chi_{R_0}|| + ||D_z\hat{F}\chi_{R_0}|| < c_3\gamma,\]

\[(14.236)\]

\[M_2 = \gamma \sup_{(X, \hat{w}) \in D_0, 2R_0, \alpha} ||D_z\hat{G}|| < c_4\gamma,\]

We set \(\mu_0 = \kappa/4\). Then for small \(\gamma\)

\[(14.237)\]

\[M_3 = \gamma \sup_{(X, \hat{z}) \in D_0, 2R_0, \alpha} ||D_X\hat{G}|| < c_5\gamma.\]

We set \(\delta_1 = 2\theta_1\), where

\[(14.238)\]

\[\theta_p = \lambda M_0 \int_0^{\infty} u^{-\alpha} \exp(-(\kappa - p\mu')u)du, \quad 1 \leq p \leq 1 + \delta,\]

and \(\mu' = \mu_0 + \delta_1 M_2\). For sufficiently small \(\gamma\) one has \(\mu' < \rho/2\), therefore, the integral in the right hand side of (14.238) converges and, according to (14.236), one obtains \(\theta < c_6\gamma\) (since \(M\) is independent of \(\gamma\)). We notice then that for sufficiently small \(\gamma\) the following estimates

\[(1 + \delta)\mu' < \rho/2,\]

\[\theta_1 < \delta_1 (1 + \delta_1)^{-1} < 1, \quad \theta_1 (1 + \delta_1)M_2\mu'^{-1} < 1,\]

and

\[\theta_p(1 + (1 + \delta_1)M_2 \mu'^{-1}) < 1\]

hold. Those estimates show that all conditions of Theorem 6.1.7 [10] are satisfied, and Lemma 8.1 is proved.
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