The kinematic component of the cosmological redshift

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ABSTRACT

It is widely believed that the cosmological redshift is not a Doppler shift. However, Bunn & Hogg have recently pointed out that to settle properly this problem, one has to transport parallelly the velocity four-vector of a distant galaxy to the observer’s position. Performing such a transport along the null geodesic of photons arriving from the galaxy, they found that the cosmological redshift is purely kinematic. Here we argue that one should rather transport the velocity four-vector along the geodesic connecting the points of intersection of the world-lines of the galaxy and the observer with the hypersurface of constant cosmic time. We find that the resulting relation between the transported velocity and the redshift of arriving photons is not given by a relativistic Doppler formula. Instead, for small redshifts it coincides with the well known non-relativistic decomposition of the redshift into a Doppler (kinematic) component and a gravitational one. We perform such a decomposition for arbitrary large redshifts and derive a formula for the kinematic component of the cosmological redshift, valid for any FLRW cosmology. In particular, in a universe with $\Omega_m = 0.24$ and $\Omega_\Lambda = 0.76$, a quasar at a redshift 6, at the time of emission of photons reaching us today had the recession velocity $v = 0.997c$. This can be contrasted with $v = 0.96c$, had the redshift been entirely kinematic. Thus, for recession velocities of such high-redshift sources, the effect of deceleration of the early Universe clearly prevails over the effect of its relatively recent acceleration. Last but not least, we show that the so-called proper recession velocities of galaxies, commonly used in cosmology, are in fact radial components of the galaxies’ four-velocity vectors. As such, they can indeed attain superluminal values, but should not be regarded as real velocities.

Key words: methods: analytical – cosmology: theory

1 INTRODUCTION

A standard interpretation of the cosmological redshift in the framework of the Friedman-Lemaître-Robertson-Walker (FLRW) models is that it is an effect of the expansion of the Universe. This interpretation is obviously correct since $1 + z = a(t_o)/a(t_e)$, where $z$ is the value of the redshift, $a(t)$ is the scale factor of the Universe and $t_e$ and $t_o$ are respectively the times of emission and observation of a sent photon. In semi-popular literature (e.g. Kaufmann & Freedman 1999; Franknoi, Morrison & Wolff 2004; Seeds 2007), but also in professional (e.g. Harrison 2000; Abramowicz et al. 2007), one can often find statements that distant galaxies are ‘really’ at rest and the observed redshift is caused by the ‘expansion of space’. According to other authors (e.g. Peacock 1999; Whiting 2004; Chodorowski 2007a,b; Bunn & Hogg 2009, hereafter BH9), such statements are misleading and cause misunderstandings about the cosmological expansion. A presentation of the ongoing debate in the literature on this issue is beyond the scope of the present work; a (possibly non-exhaustive) list of papers includes Davis & Lineweaver (2001), Davis & Lineweaver (2004), Barnes et al. (2006), Francis et al. (2007), Lewis et al. (2007), Lewis et al. (2008), Grøn & Elgarøy (2007), Peacock (2008), Abramowicz et al. (2009), Chodorowski (2008), Cook & Burns (2009).

On the other hand, there is broad agreement that the cosmological redshift is not a pure Doppler shift; the gravitational field must also generate a gravitational shift. Gravity can be neglected only locally, in the local inertial frame (LIF) of an observer. It turns out that for small redshifts, the cosmological redshift can be decomposed into a Doppler shift and a Newtonian gravitational one (Bondi 1947). The latter is a shift induced by the Newtonian gravitational potential. Can the cosmological redshift be decomposed into a Doppler shift and a gravitational shift (not necessarily Newtonian) for an arbitrary value of the redshift? This is the question which we want to deal with in this Paper. Formally, the answer is no. There is no invariant definition of the recession velocity of a distant galaxy in general relativity (GR). This velocity is a relative velocity of the galaxy and the observer, and in curved spacetime there is no unique way to compare vectors at widely separated points. A natural way to define the recession velocity is to transport parallelly the velocity four-vector of the distant galaxy to the observer, but the result will depend on the chosen path. (This is just the definition of curvature.) In practice, however, as a ‘preferred’ path one can choose a geodesic connecting the galaxy and the observer. Moreover, in FLRW models there is a natural foliation of spacetime, into space-like hypersurfaces of constant cosmic time. In our Paper, as the geodesic we will adopt the geodesic lying on such
a hypersurface, i.e. connecting the points of intersection of the worldlines of the galaxy and the observer with the hypersurface of constant cosmic time.

In a seminal study of the cosmological redshift, BH9, following Synge (1960) and Narlikar (1994), adopted another geodesic for the parallel transport: the null geodesic along which the photon is travelling from the source to the observer. This approach results in one ‘effective’ velocity, while we think it is important to make a distinction between the velocity at the time of emission and the velocity at the time of observation. These two velocities are obtained by transporting parallelly the velocity four-vector of the source respectively on the hypersurface of constant $t_o$ and $t_e$. Not surprisingly, our result differs from that obtained by BH9. However, unlike theirs, ours correctly reproduces the small-redshift decomposition of the cosmological redshift into a Doppler component and a gravitational component, mentioned above. Specifically, our decomposition coincides with that of Bondi (1947) for isotropic and homogeneous matter distribution.

This Paper is organized as follows. In Section 2 we transport parallelly the velocity four-vector of a distant galaxy to the observer. From the transported vector we calculate the recession velocity of the galaxy, which turns out to depend on the galaxy’s comoving distance and the assumed background cosmological model. In Section 3 we find specific relations between the cosmological redshift and its Dopplerian component for two particularly simple cosmological models: the empty model and the Einstein-de Sitter model. In Section 4, using only the Principle of Equivalence, we derive the small-redshift decomposition of the cosmological redshift and find it to be identical with that obtained in Section 3 using generally relativistic methods. Synge (1960) and Narlikar (1994), adopted another geodesic for the velocity four-vector, and its Dopplerian component for two particularly simple cosmological models: the empty model and the Einstein-de Sitter model. In Section 5 we find specific relations between the cosmological redshift and its Dopplerian component for two particularly simple cosmological models: the empty model and the Einstein-de Sitter model. In Section 6 the scale factor $a(t)$ relates fixed, or comoving, coordinates, to physical, or proper, coordinates, $r$: $r = ax$. The RW coordinates are $\{x^\alpha\} = \{ct, x, \theta, \phi\}$. The curve considered here is a radial geodesic on the hypersurface of constant cosmic time, hence $dx^\mu = \delta^\mu_0 dx$, where $\delta^\mu_0$ is the Kronecker delta. This gives

$$
dU^\alpha = -\Gamma^\alpha_{\beta \gamma} U^\beta.
$$

(5)

The vector $U$ is here the four-velocity of a distant galaxy, parallelly transported to the central observer (i.e. to the origin of the coordinate system). By definition, $U^\alpha = dx^\alpha/d\lambda$, where $x^\alpha(t)$ denotes the galaxy world-line as a function of the cosmic time. In the RW (co-moving) coordinates every galaxy has fixed $\theta$, $\phi$, and $x$, so $dx = cdt$ and $U^\alpha = dx^\alpha/d(ct)$. For these reasons, the initial value for $U^\alpha$ is $\delta^\alpha_0$. Specifically, the initial conditions for Equations (5) are

$$
U^\alpha(x_\ast) = \delta^\alpha_0.
$$

(6)

where $x_\ast$ is the comoving radial coordinate of the emitting galaxy.

We calculate $\Gamma^\alpha_{\beta \gamma}$ using equations (2)–(3) and obtain the following set of two linear equations for $U^0$ and $U^1$:

$$
\frac{dU^0}{dx} = -\frac{a^2(t)H(t)}{c} U^1,
$$

(7)

$$
\frac{dU^1}{dx} = -\frac{H(t)}{c} U^0.
$$

(8)
We remind that since $t$ is fixed, so are $a$ and $H$. The corresponding equations for $U^2$ and $U^3$ are simple to solve; with conditions (6), the result is $U^2(x) = U^3(x) = 0$. Differentiating Equation (7) with respect to $x$ and using (8) to eliminate $dU^1/dx$ yields

$$\frac{d^2U^0}{dx^2} = \frac{a^2H^2}{c^2} U^0.$$ (9)

The solution is

$$U^0(x) = C_1e^{aHx/c} + C_2e^{-aHx/c},$$ (10)

where $C_1$ and $C_2$ are the integration constants. Imposing initial conditions and noticing that $aH = \dot{a}$ we obtain

$$U^0(x) = \frac{1}{2} \left( a(e^{x-x_0}/c) + e^{-a(e^{x-x_0}/c)} \right),$$ (11)

$$U^1(x) = -\frac{1}{2} \left( a(e^{x-x_0}/c) + e^{-a(e^{x-x_0}/c)} \right),$$ (12)

hence

$$U^0(x = 0) = \cosh (\dot{a}x/c),$$ (13)

$$U^1(x = 0) = a^{-1} \sinh (\dot{a}x/c).$$ (14)

We see that $(g_{0\beta}U^0U^\beta)_{|x=0} = 1$, as it should.

In order to identify the transported velocity four-vector of the distant galaxy with its recession velocity, we now have to transform the vector from the RW coordinates to the coordinates of the Local Inertial Frame (LIF) of the central observer. A general coordinate transformation $x' = x'(x)$ transforms the (unprimed) components of the metric, $g_{\alpha\beta}$, to

$$(g_{\alpha\beta})' = \frac{\partial x'^{\alpha}}{\partial x^{\beta}}g_{\alpha\beta}.$$ (15)

The following transformation of the RW coordinates: $t' = t, x' = a(t)x, \theta' = \theta, \phi' = \phi$, yields $g_{\alpha\beta}'$, which in the limit $x' \rightarrow 0$ tend to $\eta_{\alpha\beta}'$, i.e. to the Minkowski metric. Thus, the primed coordinates are indeed the coordinates of the observer’s LIF. A transformation of vector components, $U' = (\partial x'/\partial x^\alpha)U^\alpha$, yields here $U' = U^0$ and $U^1 = a(t)U^1$, hence

$$U'^0(x' = 0) = \cosh (\dot{a}x/c),$$ (16)

$$U'^1(x' = 0) = \sinh (\dot{a}x/c).$$ (17)

Again, the components $U'^\alpha$ are properly normalized:

$$\eta_{\alpha\beta}'(U'^\alpha U'^\beta)|_{x'=0} = 1.$$ (18)

In the LIF of the observer, the radial component of the parallely-transported four-velocity of the distant galaxy is non-zero. We interpret this effect as a non-zero recession velocity of the galaxy, $v$. Quantitatively, $U'^0 = \gamma$ and $U'^1 = \beta\gamma$, where $\beta \equiv v/c$ and $\gamma \equiv (1 - \beta^2)^{-1/2}$. This yields immediately

$$\gamma = \cosh (\dot{a}x/c),$$ (19)

$$\beta\gamma = \sinh (\dot{a}x/c),$$ (20)

hence

$$\beta = \tanh (\dot{a}x/c).$$ (21)

Finally, since $\sinh y + \cosh y = \exp y$, we obtain

$$\exp \left( \frac{\dot{a}x}{c} \right) = \left( \frac{1 + \beta}{1 - \beta} \right)^{1/2} = 1 + z_D,$$ (21)

where $z_D$ is the Doppler component of the (total) cosmological redshift of the galaxy, $z$. In section 3 we will explain how to use this equation to relate the cosmological redshift of a distant galaxy to its Dopplerian component. Equation (21) is the central formula of the present paper.

### 3.1 Time of observation

At the time of observation $\dot{a}|_{t_0} = H_0$ (we use the normalization $a_0 = 1$, so that the comoving radial distance equals to the proper distance today), hence

$$\exp \left[ \frac{H_0 x(z)}{c} \right] = 1 + z_D.$$ (22)

Here, $z_D$ is a Doppler shift due to the present value of the recession velocity, $v_0$.

#### 3.1.1 Empty universe

In the empty model ($\Omega_m = \Omega_\Lambda = 0$), the comoving radial distance is $x(z) = cH_0^{-1} \ln(1 + z)$, hence $e^{\ln(1+z)} = 1 + z_D$, or $z = z_D$. (23)

In an empty universe the origin of the cosmological redshift is thus entirely Dopplerian. This result is expected, since spacetime of an empty universe is the Minkowski spacetime, where redshift is simply a Doppler shift.

Some cosmologists still believe that the cosmological redshift is not a Doppler shift even in the case of an empty universe. The origin of this erroneous belief is a wrong interpretation of the FLRW metric for an empty universe in the RW coordinates. In these coordinates, space (3-D hypersurface of constant cosmic time) does have non-zero curvature, but what matters here is the full 4-D curvature, which is zero. Indeed, a simple coordinate transformation transforms the metric to the standard Minkowskian form. What else could spacetime of an empty universe be? There is no matter so there is no gravity, and a complete relativistic description is provided by special relativity. Indeed, Chodorowski (2007a) proved that in an empty universe $z = z_D$, using only specially relativistic concepts. More specifically, he showed that $1 + z_D = a(t_0)/a(t_0)$, where $a(t) = t/t_0$ is the scale factor.

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1 To check this, instead of using eq. (18) it is much simpler to note that $x = x'/a(t')$ and to take its full differential.

2 The quantity $\dot{a}x/c$ plays here essentially a role of the so called ‘velocity-parameter’ of the Lorentz transformation. See e.g. Rindler (1977) and Schutz (1985).
factor of the empty model. Another derivation of the redshift in the empty model will be presented in Section 3.

There is some remaining subtlety here that deserves further explanation. In the specially-relativistic formula for a Doppler shift, \( z_D = \sqrt{1 + v/c}/(1 - v/c) \), the recession velocity \( v \) is an inertial velocity of a distant galaxy, call it \( v_{\text{distant}} \). The velocity that appears in Equation (21) and resulting (23) is a ‘local’ velocity, \( v_{\text{local}} \), transported parallelly from the galaxy to the central observer using the RW coordinates. Are these two velocities equal? Redshift is an observable, so the answer is ‘they must be’, but let us understand, why.

In Minkowski, i.e. flat, spacetime, components of a transported four-vector in inertial coordinates are constant (see Eq. 1), so \( U''_{\text{local}} \) (i.e., transported) is equal to \( U''_{\text{distant}} \). Here, double primes denote the global inertial coordinates of the central observer, in which \( v_{\text{distant}} \) is measured. This equality is no longer true for non-inertial coordinates (like RW), but parallel transport does not depend on the coordinate system used, so the transported vector is the same. We have transformed the RW components of the transported four-velocity, \( U''_{\text{local}} \), to the components in the coordinates of the LIF of the observer, \( U''_{\text{local}} \), and obtained \( v_{\text{local}} \). Since the global inertial frame of the central observer is an extension of its LI (possible only in an empty universe), \( U''_{\text{local}} \) and \( U''_{\text{local}} \) are the components of the same vector in the same coordinate frame, so they must be equal. We thus have \( U''_{\text{distant}} = U''_{\text{local}} = U''_{\text{local}} \), hence \( v_{\text{distant}} = v_{\text{local}} \).

Summing up, our definition of the recession velocity of a distant cosmological object correctly implies that in the case of an empty universe the origin of the cosmological redshift is purely Dopplerian.

3.2 Time of emission

At the time of emission, \( \ddot{a}_{\text{em}} = (\dot{a}/a)_{\text{em}} = H_0/(1 + z) \), where \( H_0 \) is the Hubble constant at the time of emission. Using this equality in Formula (21) we obtain

\[
\exp \left[ \frac{H(z)x(z)}{c(1+z)} \right] = 1 + z_D. \tag{26}
\]

Writing the above we have used the fact that there is a unique correspondence between \( t_e \) and \( z \), so \( H_0 \) can be expressed in terms of the latter. A general formula for the Hubble constant as a function of redshift is

\[
H^2(z) = H_0^2 \left[ \Omega_\Lambda + \Omega_m(1+z)^3 - (\Omega - 1)(1+z)^2 \right]. \tag{27}
\]

Here, \( \Omega = \Omega_m + \Omega_\Lambda \). In Equation (26), \( z_D \) is a Doppler shift due to the recession velocity at the time of emission, \( v_e \).

3.2.1 Empty universe

From Equation (27) we obtain \( H(z) = H_0(1 + z) \), hence \( H(z)x(z)/[c(1+z)] = \ln(1 + z) \). This yields again \( z = z_D \).

Equations (28) and (23) involve two different Doppler shifts: respectively at the time of emission, \( z_D \), and at the time of observation, \( z_o \). These equations are consistent because in the empty model the expansion of the universe is kinematic and recession velocities are constant in time. This is not true for any other cosmological model.

3.2.2 Einstein-de Sitter universe

In the EdS model \( H(z) = H_0(1 + z)^{3/2} \), hence \( z_D = \exp \left\{ \frac{2}{3} \left[ (1 + z)^{1/2} - 1 \right] \right\} - 1 \). \tag{29}

Infinite value of \( z \) corresponds to infinite value of \( z_D \), so at the time of the Big-Bang (corresponding to \( z = \infty \)), the horizon was receding from the observer with the velocity of light. Let us decompose the cosmological redshift into a Doppler shift and the ‘rest’, \( z = z_D + \text{‘rest’} \). From Equation (29) we see that for high redshifts, the Dopplerian component of the cosmological redshift is (much) bigger than the total value of the latter. Therefore, in the EdS model the ‘rest’ must in fact be negative, i.e., it is a blueshift. We will show later that for small redshifts, the ‘rest’ can be identified with a gravitational shift. It will be demonstrated that a gravitational shift is indeed a blueshift for all matter-only models. A homogeneous and isotropic distribution of ordinary matter creates a gravitational potential well with the observer at the center and the work performed by photons climbing down this well is negative.

3.2.3 Small redshifts

From Equation (27) we have \( H(z) = H_0[1 + (1 + q_0)z + O(z^2)] \), hence \( H(z)x(z)/[c(1+z)] \simeq z + (q_0 - 1)z^2/2 \). Inserted in Equation (26), this yields

\[
z = z_D + \frac{q_0}{2} z_D^2 + O(z_D^3). \tag{30}
\]

Again, in Section 3 we will see that the above result is identical to that obtained using the PoE.
4 PRINCIPLE OF EQUIVALENCE

In Section 4 we studied the cosmological redshift using generally-relativistic concepts. In this Section we will restrict our analysis to small values of the cosmological redshift; this will allow us to use the Principle of Equivalence (PoE). It is well known that the PoE enables one to study local effects of light propagation in a weak gravitational field. Specifically, one can replace a static reference frame in this field by a freely falling frame in empty space. From the PoE it follows that the latter is locally inertial, with the laws of light propagation being given by special relativity. The classical examples of phenomena studied in this way are the gravitational redshift and the bending of light rays. In some instances it is possible to calculate correctly non-local effects by summing up infinitesimal contributions from adjacent local inertial frames, freely falling along light ray trajectory. For example, the (small) gravitational redshift of a photon sent from the surface of a non-relativistic star (like, e.g., the Sun) to infinity can be calculated in this way. In some other instances it is not possible; a classical example is the bending of light rays. The calculation based on the PoE gives only half of the value of bending predicted by GR in the weak-field approximation (but see Will 2006 and references therein).

It is well known that the cosmological redshift can be interpreted as ‘an accumulation of the infinitesimal Doppler shifts caused by photons passing between fundamental observers separated by a small distance’ (Peacock 1999; see also Peebles 1993, Padmanabhan 1993). The derivation is very simple but let us recall it here. Consider two neighbouring comoving observers, located on the trajectory of a light ray, separated by an infinitesimal proper distance \( \delta l = a(t) \delta x \), where \( a(t) \) is the scale factor and \( \delta x \) is their comoving separation. The velocity of the second observer relative to the first is \( \delta v = (d/dt) \delta l = a \delta \dot{x} \). The equation for the radial light propagation in the RW metric is \( c \delta t = a \delta x \), where \( \delta t \) is the light travel time between the two observers, so \( \delta v = c \delta \dot{x} / a \). This small recession velocity of the second observer causes a non-relativistic Doppler shift of the passing ray, \( \delta \lambda / \lambda = \delta v / c = \delta a / a \). (Reference frames of the FOs are locally inertial.) Integrating the relation between \( \lambda \) and \( a \) we obtain \( 1 + z \equiv \lambda / \lambda_e = a / a_e \).

The above derivation resembles a derivation using the PoE, which will be presented below. There is, however, an important difference. In the above we have applied the generally relativistic RW metric and the law of light propagation in it. The RW coordinates are local inertial coordinates of (freely falling) Fundamental Observers and the cosmic time \( t \) is their local proper time. This has enabled us to integrate the effect (along the ray’s trajectory) trivially and exactly. Not surprisingly, then, the result is correct for an arbitrary value of the redshift. In an application of the PoE, the gravitational field of the system under consideration is described in terms of non-relativistic Newtonian dynamics, so the field must be weak. In the FLRW cosmological models the gravitational field is weak only at small distances from an observer, more specifically much smaller than the Hubble radius. Therefore, the PoE can give correct predictions only for small values of the cosmological redshift. Still, we think that calculations using the PoE are worth performing. The reasons are as follows.

First, the aim of our Paper is to decompose the cosmological redshift into a global Doppler shift (i.e., due to the relative velocity of the emitter and the observer) and the ‘rest’. In the above calculation of the redshift we summed up contributions from infinitesimal relative velocities of neighbouring FOs, but this did not help us to express the answer in terms of the relative velocity of the emitter and the observer. As stated already many times, for large cosmological distances the recession velocity a priori is not well defined. In this Paper we proposed its generally relativistic definition. This definition is however quite formal, while for small redshifts, velocity is a well defined pre-relativistic concept. Therefore, the identification of a global Doppler component of the small cosmological redshift will be simple and unambiguous. Second, in previous sections, using GR, we have shown that the cosmological redshift, of any value, can be indeed decomposed into a Doppler shift and the ‘rest’. In this section, using the PoE, we will similarly decompose the redshift into a global Doppler shift and the rest and will show that the ‘rest’ is the gravitational redshift in the Newtonian sense. In other words, we will prove that for small redshifts the rest is a classical Newtonian gravitational shift.

Among all the FLRW models there is a toy model which can be described using special relativity only. This is the empty, or Milne model (Milne 1933). At first sight, the relation of this model to the PoE is problematic, since in an empty universe there is no gravity. However, the relevance of the model will be seen later. Spacetime of an empty universe is the Minkowski spacetime. Let us denote the global inertial coordinates of an observer by \( T, X, Y, Z \). In the standard spherical coordinates, \( R, \Theta, \Phi \), the metric is

\[
 ds^2 = dT^2 - dR^2 - R^2 (d\Theta^2 + \sin^2 \Theta \, d\Phi^2).
\]

Like above, we will calculate the cosmological redshift summing up the infinitesimal Doppler shifts caused by photons passing between neighbouring Fundamental Observers, located along the photons’ path. We have seen that such a summation can be easily done in the RW coordinates, for any FLRW model. However, it turns out that in the empty model, it is possible to perform this summation using also the Minkowski coordinates \( T \) and \( R \).

In the Milne model the recession velocities of Fundamental Observers are constant; for example, in the rest-frame \( O \) of the central observer \( v(T, R) = R / T = H(T) R = \text{const} \). Let us consider a Fundamental Observer located at a distance \( R \) from the central observer at time \( T \). In his rest-frame \( O' \), he observes a small Doppler shift of a photon sent (in fact, transmitted) by a nearby observer located at \( R + \delta R \). Due to the Hubble expansion, the latter observer has a small recession velocity \( \delta v' = \delta R' / T' = H(T') \delta R' \) relative to the former. Like in the RW coordinates (locally coinciding with the coordinates \( T' \) and \( R' \)), the relative wavelength shift is \( \delta \lambda / \lambda = \delta v' / c \). However, in the empty model we can use the special-relativistic law of composition of velocities to relate the velocity \( \delta v' \) to the velocity \( \delta v \), measured in the observer’s frame \( O \). This law can be applied, because the relative velocity of the frames \( O \) and \( O' \), \( v(T, R) \), is constant. We have

\[
 \delta v' = \frac{v(T, R + \delta R) - v(T, R)}{1 - v(T, R + \delta R) v(T, R)/c^2},
\]

hence we obtain

\[
 \frac{\delta \lambda}{\lambda} = \frac{\delta v(T, R)/c}{1 - [v(T, R)/c]^2}.
\]

Let us denote the distance of the source at the time of emission by \( D_0 \). Our present task is to integrate this equation along the photon’s trajectory, \( c(T - T_0) = D_0 - R \). (The photon travels towards the central observer.) However, it can be easily checked that since the velocity of any FO is constant, \( \delta v(T, R) = H(T) \delta R \) equals to \( \delta v_e = H(T_e) \delta R_e \). Therefore, the integration simplifies to an integration over \( R_e \) at fixed \( T = T_e \).

\[
 \int_{R_e}^{\lambda_0} \frac{d\lambda}{\lambda} = - \int_{R_e}^{0} \frac{H_e(c)}{1 - (H_e R_e/c)^2} \, dR_e.
\]

(The minus sign accounts for the fact that for a photon travelling from...
the source to the observer \( R \) decreases, so \( \delta R \) is negative.) Defining 
\[ \beta \equiv H_0 R/c \text{ and } \beta_o \equiv H_0 D_o/c, \]
we obtain
\[ \int_{\lambda_o}^{\lambda} d\lambda = \int_0^{\beta_o} \frac{d\beta}{1 - \beta^2}. \]
(35)
The left-hand-side integral is equal to \( \ln(1 + z) \). The integration of the right-hand-side is elementary and yields \( \ln[(1 + \beta_o)/(1 - \beta_o)]/2 \). Hence,
\[ 1 + z = \left( \frac{1 + \beta_o}{1 - \beta_o} \right)^{1/2} = 1 + z_D. \]
(36)
In sum, we have accumulated infinitesimal Doppler shifts using the inertial coordinates of the central observer and expressed the total redshift in terms of the recession velocity of the source. The relationship between the cosmological redshift and the recession velocity turned out to be given by the relativistic Doppler formula, in agreement with generally-relativistic Equation (28). Thus, as already stated earlier, in an empty universe the cosmological redshift is entirely a global Doppler shift. A similar derivation of the redshift in the empty model using Minkowskian coordinates with an explicit integration along the photon’s trajectory was presented in Appendix A and Section 2 of Chodorowski (2007a).

Now we proceed to derive, using the PoE, the relation between the recession velocity of a nearby source and its redshift, for empty universes. Why was the exercise with the Milne model useful for this purpose? High redshifts correspond to relativistic velocities and in the empty model their effects manifest as the \( 1 - \beta^2 \) factor in the integrand on the RHS of Equation (35). Expansion of this integrand for small velocities and subsequent integration yields \( \beta_o + \mathcal{O}(\beta_o^2) \). Therefore, in our non-relativistic calculations we will be allowed to retain terms up to second order in \( \beta_o \). (We will see later that the leading-order contribution to the velocity–redshift relation coming from Newtonian gravity is already of the second order in \( \beta_o \).) Like previously, we denote Newtonian coordinates of the central observer by \( T \) and \( R \). They are now inertial only locally, because there is a gravitational field. We ‘transform this field away’ using local inertial frames of FOs. Contrary to the empty model, FOs do not move now with constant speeds. According to the Newtonian interpretation of the FL equations (Milne & Mc Crea 1934), the change of speed of any FO located on a sphere centered on another observer is induced by the gravitational force exerted on the FO by the mass encompassed within this sphere. (This is Newton’s theorem for an isotropic and homogeneous universe. The speed of the FO is, of course, measured relative to the central observer). The effects of the cosmological constant \( \Lambda \) can be incorporated easily because \( \Lambda \) results in a force proportional to distance, similarly to the classical gravitational force for a homogeneous mass distribution. The FO located at time \( T \) at a distance \( R \) from the central observer has velocity related to his velocity at the time of emission of a photon, \( v_o \), in the following way:
\[ v(T, R) = v_o + g(T_o, R_o) \Delta T + \mathcal{O}(\Delta T^2). \]
(37)
Here, \( T_o \) is the time of emission, \( R_o = R(T_o) \), \( \Delta T = T - T_o \) and \( g(T_o, R_o) \) is the initial gravitational acceleration of the FO. Writing the above formula we have approximated changing acceleration of the FO by its initial value. (The result of this approximation for \( v(T, R) \) is already of second order in \( \Delta T \).) It is straightforward to calculate \( g(T, R) \) from Newton’s theorem and to express it in terms of the cosmological parameters. However, there is a faster way of doing this: by differentiating the Hubble law in Newtonian coordinates, \( v = H(T)R \), with respect to time. The result is
\[ g(T, R) = -q(T)H^2(T)R, \]
(38)
where \( q \) is the deceleration parameter.

From Equation (35), \( v(T_o, R_o) = -q_o H_o^2 R_o \approx -q_o H_o^2 R_o \). Inserting these expressions for \( v_o \) and \( g \) in Equation (37), we obtain up to terms linear in \( \Delta T \)
\[ v(T, R) = H_0 R_o (1 - q_o H_o \Delta T) \]
(39)
\( R \approx R_o + H_0 R_o \Delta T \). The velocity of a FO is described here in terms of \( T \) and his initial position, \( R_o \). A small relative velocity of two neighbouring FOs at time \( T \), \( \delta v_T \), can be expressed as \( \langle \delta v(T)/\partial R \rangle T \delta R_o = H_0 (1 - q_o H_o \Delta T) \delta R_o \). This relative velocity causes a non-relativistic Doppler shift of the frequency of a passing photon. To obtain the total redshift we need to integrate these infinitesimal Doppler shifts along the photon’s path (i.e., null geodesic). Unlike in the empty model, here \( T \) and \( R \) are only locally inertial coordinates. However, their extent is quite limited, \( D_o \ll c H_0^{-1} \), so \( H_0 \Delta T \) equals to \( H_0 (D_o - R)/c \approx H_0 (D_o - R_o)/c \). Thus terms already of higher order in \( (D_o - R_o)/c \). Therefore, \( \int_{T_o}^{T} d\lambda/\lambda = \ln(1 + z) \) equals to
\[ \frac{1}{c} \int_{T_o}^{T} H_0 \left[ 1 - q_o (H_0 (D_o - R_o)/c) \right] dR_o = \beta_o - \frac{1}{2} q_o \beta_o^2. \]
(40)
where, as before, \( \beta_o \equiv H_0 D_o/c \). Hence we obtain finally
\[ z = \exp \left[ \beta_o - \frac{1}{2} q_o \beta_o^2 \right] - 1 = \beta_o + \frac{1}{2} \beta_o^2 - \frac{q_o}{2} \beta_o^2 + \mathcal{O}(\beta_o^3). \]
(41)
Expansion of the Doppler formula (40) to second order in \( \beta_o \) gives \( z_D \). Therefore, the first two terms on the RHS of Equation (41) can be identified with a (global) Doppler shift due to the recession velocity of the source at the time of emission. Since the third term is of second order in \( \beta_o \), we can write \( q_o \beta_o^2/2 = q_o z_D^2/2 + \mathcal{O}(\beta_o^3) \). In such a way we find out that Equation (41) is identical to Equation (29), the latter derived using a generally-relativistic approach. In Equation (28), the kinematic component of the redshift is related by the Doppler formula to the recession velocity of the source at the time of observation. We recover this equation by noting (with some small labour) that Equation (39) yields \( \beta_o = \beta_o + q_o \beta_o^2 \), and using this equality in Equation (41).

The term \( -q_o \beta_o^2/2 \) is a gravitational shift: the work, per unit-mass rest energy, performed by a photon travelling from the source to the observer. From Equation (38) we have \( g(T_o, R_o) = -q_o H_o^2 R_o \). The gravitational potential \( \Psi(R) \approx q_o H_o^2 R^2/2 \). We thus indeed obtain \( (\Psi_o - \Psi_c)/c^2 = -q_o \beta_o^2/2 \). Whether the observer is located in the centre of a potential well or on the top of a potential hill depends on the value of \( q_o \). For all matter-only cosmological models \( q_o \) is positive, so the observer is in the centre of a potential well. Then a gravitational shift is negative, so it is in fact a blueshift. This reflects the fact that the photon climbs down a potential well so the performed work is negative. For an accelerating universe (like, as we currently believe, ours) \( q_o < 0 \) and a gravitational shift is a redshift.

Peacock (1999) derived Equation (41) in a much simpler way. He stressed that in the definition of the redshift, the wavelength at the time of emission is the wavelength measured in the source’s rest-frame \( O' \). \( \lambda_o ' \). In the observer’s rest-frame \( O \), this wavelength is Doppler-shifted, \( \lambda_o /\lambda_o ' = 1 + z_D \). Next, on the path to the observer the photon undergoes a gravitational shift, \( \lambda_o /\lambda_o = 1 + z_c \). Hence, \( 1 + z \equiv \lambda_o /\lambda_o ' = (1 + z_D)(1 + z_c) \). This equality gives Formula (41) because, as we have verified above, \( z_D = -q_o \beta_o^2/2 \), and because \( z_D z_c \) is already of third order in \( \beta_o \). However, freely-falling observers employed in his calculation of \( z \) were not, strictly speaking, the freely-falling observers of the studied problem. In the RW models freely-falling observers are (by definition) FOs, partaking in
the Hubble expansion. Peacock employed the standard ‘Einstein-lift’
observers, starting to fall only when the photon passes the top of their
cabin. Accounting for non-zero initial velocities of freely-falling ob-
servers complicated our calculations, but enabled us to apply the PoE
‘from first principles’. We performed our calculations consistently in
the same way as we did using global inertial coordinates in the empty
model, and the RW coordinates in all FLRW models.

Peacock’s aim was to derive a formula for the angular diam-
eter distance up to the terms quadratic in redshift. In order to obtain
the correct result, he had to expand the relativistic Doppler formula
up to the term quadratic in \( \beta_c \). Strictly speaking, this term is a rela-
tivistic correction and in principle one is not allowed to account for
it in the calculations using the PoE, where velocities are assumed to
be non-relativistic. However, the justification for including this term
can be found analyzing our calculations in the Milne model. We ob-
tained \( \ln(1+z_D) = \beta_c + O(\beta_c^2) \), or \( 1+z_D = \exp[\beta_c + O(\beta_c^2)] \),
where \( O(\beta_c^2) \) was the lowest-order relativistic correction. Therefore,
solving for \( z_D \) we were allowed to expand the exponential function up
to a term quadratic in \( \beta_c \), still remaining in the non-relativistic
regime. Since the exponential function has the same second-order ex-
ansion as the Doppler formula, we correctly obtained the second-
order Doppler relation, also in our non-relativistic calculations for
non-empty models (see Eq. (41)).

We stress that we did not assume that \( z = z_0 + z_D \). In our
calculations both terms appeared naturally as a result of an applica-
tion of the PoE. The first person to note that the cosmological re-
shift, if small, can be decomposed into a Doppler shift and a grav-
itational shift was Bondi (1947). He performed such a decomposi-
tion in a spherically symmetric generalization of the RW models, the
Lemaître-Tolman-Bondi model. Our formula (41) is a special case of
his formula (52), in a sense that \( z_D = -q_0 \beta_c^2 / 2 \) only for a homo-
geogeneous distribution of cosmic matter and small distances.

In the approach presented in this section, the cosmological red-
shift of any value is generated by relative motions of FOs. For small
values of the redshift the application of the PoE allowed us to de-
compose the redshift into a Doppler component and a gravitational com-
ponent. We have found that a Doppler component of the red-
shift is obviously generated by a kinematic component of the motions,
while a gravitational component of the redshift is generated by their
varying-in-time component. Of course, this interpretation of a grav-
atational component is fully consistent with the standard interpreta-
tion of a gravitational shift, because this is a gravitational field that causes
FOs to accelerate.

5 RECESSION VELOCITIES

Calculations in Section 4, based on our definition of recession ve-
locity, led us to Equation (20). This equation is a general formula for
the recession speed of a galaxy, valid for an arbitrarily comoving po-

tion of the galaxy and an arbitrary instant of time. The hyperbolic
tangent on the RHS of this equation ensures that the recession veloc-
ity of a source with an arbitrarily high redshift is subluminal (\( \nu < c \)).
However, it is widely believed that recession velocities of sufficiently
distant galaxies can be, and indeed are, superluminal (e.g. Davis &
Lineweaver 2001; Davis & Lineweaver 2004; Grøn & Elgarøy 2007;
Lewis et al. 2007). In Subsection 5.1 we will find a relation between
our recession velocity and a commonly used estimator of the veloc-
ity: the proper recession velocity. Basing on this relation we will show
that the conception of the superluminal expansion of the Universe is
wrong. In Subsection 5.2 we will intercompare our recession veloc-
ities of galaxies for some specific cosmological models.

5.1 No superluminal velocities

Let us remind that in the comoving coordinates \( \{x^\mu\} = \{ct, x, \theta, \phi\} \),
the FLRW metric takes on the form given by Equation (3). For a parti-
cle (or a galaxy) participating in the homogeneous and isotropic ex-
ansion of the cosmic fluid, \( \theta = \text{const}, \phi = \text{const} \) and \( x = \text{const} \).
Hence, \( ds = cdt \) and

\[
U^\alpha = \frac{dx^\alpha}{cdt} = (1, 0, 0, 0),
\]

Let us change coordinates to \( \{x'^\alpha\} = \{ct, r, \theta, \phi\} \), where \( r = a(t)x \).
Spacetime interval \( ds \) is an invariant, so also in the primed coordi-

\[
U'^\alpha = \frac{dx'^\alpha}{cdt} = (1, \frac{Hr}{c}, 0, 0),
\]

where \( H(t) = \dot{a}/a \) is the Hubble constant.

It is a standard practice in the literature to use as an estimator of the recession velocity of a galaxy its so-called proper recession
velocity,

\[
v_{\text{prop}} \equiv \frac{dr}{dt} = Hr,
\]

that is the derivative of the proper distance between the galaxy and
the observer with respect to the proper cosmic time. Comparing equa-
tion (43) with (44) we see that

\[
v_{\text{prop}} = c U'^1,
\]

or that \( v_{\text{prop}}/c \) is nothing more than the radial component of the
galaxy’s four-velocity in the proper coordinates \( \{ct, r, \theta, \phi\} \). There-
fore, the proper velocity can indeed attain superluminal values\( \frac{c}{4} \) but
should not be regarded as a recession velocity. As described earlier,
to calculate the latter, one should

A. Transport parallely the galaxy’s velocity four-vector to the ob-
server’s position;

B. Transform the components of the transported vector to the com-
ponents in the locally inertial coordinates of the observer;

C. Read off the value of relative velocity using the fact that in these
coordinates, \( U'^1 = \beta \gamma (\nu) \).

We stress that \( U'' \) and \( U'^\alpha \) given in Equations (42) and (43) are just
different components of the same four-vector \( U(t, r) \) in respectively
comoving and proper coordinates. It has been noted earlier that in
the vicinity of the observer, the proper coordinates reduce to her/his
local inertial coordinates. Therefore, in this case the operation B.
should not be necessary. Indeed, a straightforward (though somewhat
more complicated) calculation of the parallel transport gives directly

\[
U^1(t, r = 0) = \sinh (Hr/c),
\]

in agreement with Equation (17).

Note that for sufficiently distant objects, the transported (in proper coordinates) radial component of the four-velocity is even more
superluminal than the initial one \( \sinh (Hr/c) \) compared to
4 For instance, in Fig. 4 the present value of the proper velocity of the source is 1.24c.

3 Another derivation of this decomposition was presented in Grøn & Elgarøy (2007).
However, using the equation given in point C. of the above list, we get

$$\beta_{\text{our}} = \tanh\left(\frac{v_{\text{prop}}}{c}\right), \quad (46)$$

where $v_{\text{prop}} = Hr$. Equation (46), which coincides with equation (20), relates the proper recession velocity of a galaxy to the velocity which follows from our definition. Whatever the value of $r$ (whence $v_{\text{prop}}$), $\beta_{\text{our}}$ remains smaller than $c$.

The aim of this Paper is to derive a decomposition of the cosmological redshift, so we have assumed that the distant galaxy lies on the past light cone of the observer. However, both Equations (7)–(8) of the parallel transport and their initial conditions (6) are completely independent of this assumption. Consequently, in the solutions of these equations and in the resulting Equation (46), the proper coordinate of a distant galaxy can be arbitrarily large; in particular, not limited by the radius of the particle horizon. As a corollary, not only the part of the cosmic substratum contained within the particle horizon, but entire cosmic substratum expands subluminally. Needless to say, the issue whether the Universe is open or closed does not change anything: even in an open universe, at any instant of time every particle has – though allowable arbitrarily large – finite proper coordinate.

The only exception for subluminality of the expansion of the Universe is the moment of initial singularity. As then $r \rightarrow 0$, it is better to write $Hr = \dot{a}x$. The early Universe is dominated by ultra-relativistic particles and radiation so its scale factor evolves as $a(t) \propto t^{1/2}$. Therefore, $\dot{a} \propto t^{-1/2}$, so it diverges at $t = 0$. The only point in space which in spite of this divergence is stationary at the Big-Bang is $x = 0$. This is obvious because $x = 0$ is the origin of the coordinate system and all velocities are measured relative to it. However, if the velocity of the central observer is measured by another observer, non-zero separation of the two observers and the fact that velocities are relative imply that the velocity of the central observer relative to all other observers is non-zero. (In order to prove it more formally, use the FLRW metric centered on any other point.) Thus, relative velocities of all FOs are non-zero and at the initial singularity all velocities attain the velocity of light.

Summing up, the expansion of the Universe is never superluminal. A common misconception is that the expansion is superluminal based on the wrong identification of recession velocity with the ‘proper’ recession velocity.

This subject is interesting in its own right and worth of further study. Here we have presented it only briefly, to convince the reader that a decomposition of the cosmological redshift into a gravitational shift and a kinematic one for an arbitrary redshift is indeed possible. This is so because for arbitrary redshifts, recession velocities of galaxies are subluminal. We plan to study in more detail the issue of (the lack of) superluminal expansion of the Universe on superhorizon scales in a follow-up work.

### 5.2 Recession velocities for specific models

Figure 1 shows parallel transport of the velocity four-vector of a distant galaxy, $U$, from the galaxy to the observer along the geodesic on the hypersurface of constant cosmic time. The transport is plotted for two values of time: the observation time, $t_o$ (dashed horizontal line), and the emission time, $t_e$ (solid horizontal line), both in units of $t_o$. The adopted cosmological model is a matter-only, open universe with $\Omega_m = 0.24$. The observer is at the origin of the coordinate system, $x = 0$, and the comoving coordinate of the emitting galaxy is $x_\nu$ (vertical line) in units of $cH_0^{-1}$. The adopted redshift of the source at $t_e$ (determining the value of $x_\nu$) is equal to 3. The transport is performed in the comoving coordinates; in these coordinates all spatial components of the galaxy’s velocity four-vector are initially zero. During the transport towards the observer, the radial component of the four-velocity gradually increases. In the plot this component is multiplied by $a(t)$, to enable a physical interpretation of the transported vector, $U(0)$, in the framework of special relativity. Namely, at $a = 0$ the scaled comoving components, $(U^0, aU^i)$, are equal to the (non-scaled) components of $U$ in the local inertial coordinates of the observer, $(U^0_{\nu}, U^i_{\nu})$. Slanted long-dashed lines show the direction of null vectors at the events $(t_o, 0)$ and $(t_e, 0)$ in the latter coordinates.

In Figure 1, $U^0_{\nu}(0) = aU^0(0)$ is greater than $t_e$ than for $t_o$. Thus, the recession velocity of the source at the time of emission was greater than it is at present. This conclusion is confirmed by the directions of the vectors: both vectors are timelike (as required), but the direction of the vector at $(t_e, 0)$ is closer to the null direction than the direction of the corresponding vector at $(t_o, 0)$. It is expected, since a matter-only model constantly decelerates, so its expansion must have been faster in the past.

In practice, it is more interesting to know the recession velocity of a source at the time it emitted the photons which are reaching the observer today. In other words, the velocities at the time of emission are more interesting than the velocities at the time of observation. (There is even no guarantee that the emitting galaxy exists today.)

Combining Equation (20) with (21) and (26), we obtain

$$\beta_e = \tanh \left[ \frac{H(z)x(z)}{c(1 + z)} \right]. \quad (47)$$

Again, Equation (47) assures that the recession velocity (at the time of emission) of a source with an arbitrarily high redshift is subluminal ($v_e < c$). In the empty (Milne) model, from (47) we obtain $\beta_e = \tanh[\ln(1 + z)]$, or

$$\beta_e = \frac{(1 + z)^2 - 1}{(1 + z)^2 + 1}, \quad (48)$$

in agreement with the Doppler formula. In the EdS model we obtain

$$\beta_e = \tanh \left\{ 2 \left[ (1 + z)^{1/2} - 1 \right] \right\}. \quad (49)$$

In the currently favoured, flat model with non-zero $\Lambda$, Equation (27) yields

$$H(z) = H_0 \sqrt{1 - \Omega_m - \Omega_\Lambda(1 + z)^2}. \quad (50)$$

The comoving distance is $x(z) = cH_0^{-1} \int_0^z [1 - \Omega_m - \Omega_\Lambda(1 + y)^3]^{-1/2} dy$.

Let us now consider a source located at a redshift, e.g., 6. For the Milne model, the corresponding recession velocity at the time of emission, $v_e^{(M)}$, is 0.96c. For the EdS model, this velocity, $v_e^{(E)}$, is about 0.997c. For the flat model with $\Lambda$, adopting $\Omega_m = 0.24$, the velocity, $v_e^{(A)}$, equals approximately to 0.991c. We interpret these values in the following way. A Milne universe is coasting, so $v_e^{(M)} = v_e^{(M)}$. An EdS universe decelerates, so it was expanding faster in the past. Therefore, it is logical that $v_e^{(E)} > v_e^{(E)}$. Similarly, one can also expect that $v_e^{(A)} > v_e^{(A)}$: in the flat $\Lambda$ model, the assumed value of $\Omega_m$.

5 In the model used to construct Fig. 1 the final value of the radial component transported on the hypersurface $t = t_o$ is 1.58 compared to its initial value 1.24.

6 An alternative explanation of this inequality follows from the results of subsection 3.2.2. In the EdS model there is a large gravitational blueshift, so for the total redshift to be equal to that in the Milne model, a Doppler shift must be much larger. In other words, $v_e^{(E)}$ must be much closer to the velocity of light than $v_e^{(E)}$. 

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is smaller than unity; moreover, there is a late-time phase of acceleration due to the cosmological constant. Interestingly, however, the value of \( v^e(\Lambda) \) is greater than \( v^o(\Lambda) \) and much closer to \( v^e(h) \). Apparently, for such a high redshift (and the adopted value \( \Omega_m = 0.24 \)), the effect of initial deceleration of the Universe on recession velocities strongly prevails over the opposing effect of its latter acceleration. In fact this is not so surprising, since for \( \Omega_m = 0.24 \) the flat model starts to accelerate only for the redshift as small as about 0.5.

On the other hand, for an arbitrarily high redshift there is always a range of sufficiently small values of \( \Omega_m \), for which \( v^e(\Lambda) < v^o(\Lambda) \). We have checked that for \( \Omega_m = 0.24 \), the critical redshift for which \( v^e(\Lambda) = v^o(\Lambda) \) is about 1.6.

6 DISCUSSION

Chodorowski (2007a) noted that in conformally flat coordinates in open, zero-\( \Lambda \) models, coordinate velocities of all galaxies are constant and subluminal. Lewis et al. (2007) pointed out that even in these coordinates, the proper velocity \( e \) of sufficiently distant galaxies are superluminal. Lewis’s et al. (2007) criticism was fully justified, because coordinate velocities are meaningless. However, in this Paper we have shown that the proper velocity is not a real velocity of a galaxy, so the fact that the former can be superluminal does not imply the same for the latter. Indeed, basing on the definition of relative velocity given in this Paper we have demonstrated that velocities of all particles of the cosmic substratum are in fact not superluminal.

In a recent paper, Faraoni (2010) studied the generation of the cosmological redshift by an instantaneous expansion of otherwise Minkowski spacetime. A source and an observer were assumed to be at rest relative to each other both at the time of emission and observation of photons. It was also assumed that the universe expands only at a single instant of time \( t_X \) and \( t_e < t_X < t_o \). Faraoni (2010) showed that the induced redshift is given by the canonical formula for all FLRW universes, \( 1 + z = a(t_o)/a(t_e) \). On the basis of this finding and the fact that the source and the observer were initially and finally at rest, he deduced that in the studied case the cosmological redshift is entirely gravitational. This is in agreement with our results, as will be explained below. Equations (20)–(21) are valid for any function \( a(t) \), not necessarily determined by the FL equations. The arguments made by Faraoni imply that \( a(t_e) = a(t_o) = 0 \) and from Equation (21) one can see that, both at the time of emission and observation, the Dopplerian component of the cosmological redshift is zero. [Alternatively, from Equation (20) it follows that the recession velocity vanishes.] However, Faraoni’s (2010) conclusion that the cosmological redshift is in general gravitational is incorrect. The quantity \( a \) vanishes only for closed FLRW models and only for a single instant of time. More importantly, our Universe is certainly expanding now, and whatever the exact values of cosmological parameters, its expansion has been continuous in the past. Therefore, the cosmological redshift must be partly Dopplerian.

BH9 transported the four-velocity of a distant galaxy along the null geodesic connecting the source and the observer. [The original idea dates back to Synge (1960) and Narlikar (1994)]. Instead, as the path of parallel transport we have chosen the geodesic connecting the source and the observer on the hypersurface of constant cosmic time. Both definitions are equally justified mathematically: both are intrinsic (i.e. they do not depend on any coordinate system) and reduce to the SR definition of relative velocity if the two observers are at the same point. The reason that as a path of transport we have chosen a geodesic on the hypersurface of constant cosmic time was to separate the effects of the curvature of the Universe from the effects of its evolution. We wanted to define the recession velocity at a well-specified instant of time. Transporting along a null geodesic provides \( v_{BH9} \), which is a sort of time-average (over the time interval between the times of emission and observation) of our, instantaneous recession velocity. Indeed, it can be easily verified that for small redshifts, hence also small time intervals, \( v_{BH9} = (v_e + v_o)/2 \). As a consequence of their definition of relative velocity, BH9 found that in any FLRW cosmology and for any value of the redshift the cosmological redshift is related to the transported velocity \( v_{BH9} \) by the relativistic Doppler formula. From this fact they deduced that the cosmological redshift is entirely Dopplerian. Our interpretation of the redshift is different: in a non-empty Universe there is gravitational field, inducing a gravitational shift. Therefore, with an exception of the empty model, the origin of the cosmological redshift must be partly gravitational. Using our definition of relative velocity, we have thoroughly proven this conjecture in the present Paper. Thus, the difference between the finding of BH9 and ours is yet another hint that \( v_{BH9} \) is not the recession velocity of a galaxy at a given instant of cosmic time. Using Equations (27) and (41) it can be shown that for small redshifts, \( z = (\beta_e + \beta_o)/2 + (\beta_e + \beta_o)/2^2/2 \). Comparing this equation with Equation (41) we see that the relation between \( z \) and the effective velocity \( (\beta_e + \beta_o)/2 \) is indeed Dopplerian. But this effective velocity is just \( v_{BH9} \).

Different definitions of relative velocity lead to different interpretations of the cosmological redshift. However, the definition of BH9 and ours imply the same very important conclusion: recession velocities of all galaxies inside our particle horizon are subluminal. This fact is perhaps more fundamental than the specific value of recession velocity for a given redshift.

7 SUMMARY

In this Paper we have decomposed the cosmological redshift into a Dopplerian shift and a ‘rest’, which we have interpreted as a gravitational shift. In order to extract the kinematic component from the cosmological redshift, we had to define properly the recession velocity of a distant galaxy. This velocity is a relative velocity of the galaxy and the observer, and in GR one cannot directly compare vectors at widely separated points. First, one has to transport parallelly the four-velocity of the distant galaxy to the observer. Next, the transported four-velocity is transformed to the local inertial coordinates of the observer and then the recession speed is typically extracted from the (transformed) radial component of the four-velocity. In general, parallel transport depends on the chosen path, but geodesics are special curves in GR and they seem to be a very natural choice for the problem under consideration. As the path, Synge (1960), Narlikar (1994) and BH9 adopted the null geodesic of photons emitted towards the observer. For reasons explained in Section 6, we have chosen a different path: the geodesic connecting the source with the observer on the hypersurface of constant cosmic time. BH9 found that in any FLRW cosmology and for any value of the redshift, the cosmological redshift is related to the transported velocity, as defined by them, \( v_{BH9} \), by the relativistic Doppler formula. From this fact they deduced that

\[ \frac{v}{c} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \]

7 In this case, the proper distance is measured on the hypersurface of constant conformal time.

\[ \frac{v}{c} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \]

8 Which means that, in particular, for small redshifts the formula of BH9 does not reduce to the non-relativistic limit of Bondi (1947)
the cosmological redshift is entirely Dopplerian. However, from the
Principle of Equivalence (PoE) it follows that the cosmological red-
shift must be partly gravitational. The explanation of this apparent
paradox is that $v_{\text{DH}}$ is not a recession velocity at a specific instant
of time, but rather a sort of an effective, time-averaged velocity.

In Section 3 we have presented analytical relations between the
cosmological redshift and its Doppler component for two particularly
simple cosmological models. In an empty universe, the cosmological
redshift is entirely kinematic. This is expected, because in the empty
model there is no gravitational field. In an Einstein-de Sitter universe,
at the time of emission of photons the Doppler shift is greater than the
total redshift, so a gravitational shift is in fact a blueshift. At the time of
observation, the situation is different: for large enough redshifts
the gravitational component is dominant; in particular, even the par-
ticle horizon is receding with a speed smaller than that of light. This
marked difference in the magnitude of kinematic components of the
total redshift at the time of emission and at the time of observation is
due to the fact that all matter-only models have expanded faster in the
past.

Using only the Principle of Equivalence, in Section 4 we have in-
dependently decomposed the (non-relativistic) cosmological redshift
into a Doppler shift and a ‘rest’, and have shown that in this regime
the ‘rest’ is induced by the Newtonian gravitational potential. The
obtained result is in agreement with a second-order expansion of the
exact formula derived here in the framework of GR (Sec. 3), as well as
with the classical formula of Bondi (1947) and equivalent findings
of Peacock (1999) and Grøn & Elgarøy (2007).

Last but not least, in Section 5 we have critically revised a
widespread opinion that the recession velocities of (sufficiently dis-
tant galaxies are superluminal. We have demonstrated that the com-
monly used proper velocity is not a recession velocity, but merely
a radial component of the galaxy’s velocity four-vector (when ex-
pressed in proper coordinates). As such, it can indeed attain super-
luminal values. On the contrary, the definition of the recession ve-
locity, presented in this Paper, naturally implies that the velocity is
subluminal for any value of the galaxy’s redshift. The definition of
BH9 implies exactly the same conclusion. Basing on our definition
we have also argued that the expansion of the Universe is subluminal
even on superhorizon scales. We plan to study this issue more com-
prehensively in a future work.

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REFERENCES

Abramowicz, M. A., Bajtlik, S., Lasota J.-P., Moudens, A. 2007, AcA, 57, 139
Abramowicz, M. A., Bajtlik, S., Lasota J.-P., Moudens, A. 2009, AcA, 59, 131
Barnes, L. A., Francis, M. J., James, J. B., Lewis, G. F. 2006, MNRAS, 373, 382
Bondi, H. 1947, MNRAS, 107, 410
Bunn, E. F., Hogg, D. W. 2009, AmJPh, 77, 688 (BH9)
Chodorowski, M. J. 2007a, Old New Concepts Phys., 4, 15
Chodorowski, M. J. 2007b, MNRAS, 378, 239
Chodorowski, M. J. 2008, preprint (arXiv:0812.3972)
Cook, R. J., Burns, M. S. 2009, AmJPh, 77, 59
Davis, T. M., Lineweaver, C. H. 2001, AIPC, 555, 348
Davis, T. M., Lineweaver, C. H. 2004, PASA, 21, 97
Faraoni, V. 2010, GReGr, 42, 851
Fraknoi, A., Morrison, D., Wolff, S. 2004, Voyages Through the Universe,
(Belmont, CA: Thomson Brooks/Cole), 3rd ed.
Francis, M. J., Barnes, L. A., James, J. B., Lewis, G. F. 2007, PASA, 24, 95
Grøn, Ø., Elgarøy, Ø. 2007, AmJPh, 75, 151
Harrison, E. 2000, Cosmology, (Cambridge: Cambridge University Press),
2nd ed.
Kauffmann, W. J., Freedman, R. 1999, Universe, (W. H. Freeman, New York),
5th ed.
Lewis, G. F., Francis, M. J., Barnes, L. A., James, J. B. 2007, MNRAS, 381,
L50
Lewis, G. F., Francis, M. J., Barnes, L. A., Kwan, J., James, J. B. 2008, MNRAS, 388, 960
Milne, E. A. 1933, Zeitschrift für Astrophysik, 6, 1
Milne, E. A., McCrea, W. H. 1934, QuartMath, 5, 73; reprinted in 2000 by
GRG, 32, 1949
Narlikar, J. V. 1994, AmPh, 62, 903
Padmanabhan, T. 1993, Structure Formation in the Universe, (Cambridge, UK: Cambridge University Press), p. 54
Peacock, J. A. 1999, Cosmological Physics, (Cambridge, UK: Cambridge University Press), p. 87
Peacock, J. A. 2008, preprint (arXiv:0809.4573)
Peebles, P. J. E. 1993, Principles of Physical Cosmology, (Princeton, NJ:
Princeton University Press), p. 94
Rindler, W. 1977, Essential Relativity, (New York, Heidelberg, Berlin: Springer-Verlag), revised 2nd ed.
Schutz, B. F. 1985, A First Course in General Relativity, (Cambridge, UK:
Cambridge University Press)
Seeds, M. A. 2007, Foundations of Astronomy, (Belmont, CA: Thomson Brooks/Cole), 10th ed.
Synge, J. L. 1960, Relativity: The General Theory, (Amsterdam: North-
Holland)
Whiting, A. B. 2004, The Observatory, 124, 174
Will, C. M. 2006, Living Reviews in Relativity, 9