NONDISCRETE $P$-GROUPS CAN BE REFLEXIVE

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Abstract. We present a series of examples of nondiscrete reflexive $P$-groups (i.e., groups in which all $G_{δ}$-sets are open) as well as noncompact reflexive $ω$-bounded groups (in which the closure of every countable set is compact). Our main result implies that every product of feathered (equivalently, almost metrizable) Abelian groups equipped with the $P$-modified topology is a reflexive group. In particular, every compact Abelian group with the $P$-modified topology is reflexive. This answers a question posed by S. Hernández and P. Nickolas and solves a problem raised by Ardanza-Trevijano, Chasco, Domínguez, and Tkachenko.

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1. Introduction

Extending Pontryagin’s duality to diverse classes of topological groups beyond locally compact ones has been the object of attention through the last 60 years. It has become patent in recent times that the duality properties of precompact groups and of projective limits of discrete groups, two otherwise well studied classes of topological groups, are poorly understood. We refer the reader to [7] for the case of precompact groups and to [14] and [15] where the need of an improved...
knowledge of the duality properties emerged in the study of the structure of projective limits of Lie groups.

The duality theory of $P$-groups involves immediately both classes of topological groups. A $P$-group is a topological group in which the intersection of any countable family of open sets is open. Some basic information about $P$-groups can be found in [3, Section 4.4].

It is easy to see that $P$-groups are projective limits of discrete groups that do not contain infinite compact subsets. Because of this latter property the character group of a $P$-group is always precompact. Even more, it has the far stronger property of being $\omega$-bounded (the closure of any countable subset is compact), see Lemma 5.1 below. Therefore $P$-groups occupy the region of pro-Lie groups that is farthest to locally compact groups whereas $\omega$-bounded groups are among the most compact-like classes of topological groups.

Despite a considerable interest to the subject, the only result in the literature concerning the duality of $P$-groups is Leptin’s example [17], later reproduced by Noble [18] and Banaszczyk [5, Example 7.11]. This example is the inverse limit of an uncountable family of discrete groups which turns out to be a nondiscrete $P$-group whose second dual is discrete. Hence Leptin’s group is not reflexive. This motivates the following question posed by S. Hernández and P. Nickolas at the International Workshop on Topological Groups and Dynamic Systems in Madrid, 2008:

**Question 1.** Must every reflexive Abelian $P$-group be discrete?

This problem is naturally linked with the question on whether a precompact, noncompact group can be reflexive (recall that the dual group of a $P$-group is $\omega$-bounded and hence precompact). The first examples of precompact, noncompact reflexive groups have been recently obtained in [1] and [11], but the examples presented there are precompact groups with no infinite compact subsets; so they are far from being $\omega$-bounded. Question 1 has therefore the natural accompanying question.

**Question 2.** How close to being compact can a reflexive precompact Abelian group be? In particular, can a noncompact $\omega$-bounded group be reflexive?

The main objective of this paper is to answer Question 1 (in the negative) and Question 2 (in the positive). We do this in Theorem 6.8 below by proving that every product of discrete Abelian groups with the $P$-modified topology is reflexive. This result is then extended to products of feathered (equivalently, almost metrizable) Abelian groups. In an attempt to trace the borders
between reflexive and nonreflexive \( P \)-groups we also give a number of new examples of reflexive and nonreflexive \( P \)-groups.

We also establish in Propositions 5.6 and 5.9 that the class of reflexive \( P \)-groups has unexpectedly good permanence properties—it contains quotient groups and if a reflexive \( P \)-group \( G \) is a dense subgroup of a topological group \( H \), then \( H \) is reflexive \( P \)-group as well.

2. Notation

All groups considered here are assumed to be Abelian if otherwise is not specified explicitly. The complex plane with its usual multiplication and topology is denoted by \( \mathbb{C} \). A character on a group \( G \) is a homomorphism of \( G \) to the circle group \( T = \{ z \in \mathbb{C} : |z| = 1 \} \). The set \( \{ e^{i\varphi} : -\pi/2 < \varphi < \pi/2 \} \) is denoted by \( T_+ \). Pontryagin’s duality theory is based on relating a topological group \( G \) with the group \( G^\wedge \) of continuous characters of \( G \). The group \( G^\wedge \) will be equipped with the topology of uniform convergence on the compact subsets of \( G \). This topology has as a neighbourhood basis at the identity the sets

\[
K^\wedge = \{ \chi \in G^\wedge : \chi(x) \in T_+ \text{ for all } x \in K \},
\]

where \( K \) runs over the family of compact subsets of \( G \). If a topological group \( G \) is well-represented by its continuous characters, one can recover \( G \) from \( G^\wedge \) by forming the bidual group \( G^{\wedge\wedge} = (G^\wedge)^\wedge \) and then considering the canonical evaluation homomorphism \( \alpha_G : G \to G^{\wedge\wedge} \) defined by \( \alpha_G(x)(\chi) = \chi(x), \) for all \( x \in G \) and \( \chi \in G^\wedge \). We say accordingly that \( G \) is reflexive if the homomorphism \( \alpha_G : G \to G^{\wedge\wedge} \) is a topological isomorphism.

We will use the concept of nuclear group as it appears in [4, 5]. It is worth mentioning that every closed subgroup \( H \) of a nuclear group \( G \) is dually embedded in \( G \), i.e., for every \( x \in G \setminus H \) there exists a continuous character \( \chi : G \to \mathbb{T} \) such that \( \chi(H) = \{1\} \) and \( \chi(x) \neq 1 \) (see [5, Corollary 8.6]). In particular, continuous characters of a nuclear group \( G \) separate points of \( G \) which in its turn implies that the evaluation homomorphism \( \alpha_G : G \to G^{\wedge\wedge} \) is injective.

Given a topological group \( H \), we denote by \( PH \) the \( P \)-modification of \( H \) which is the same underlying group \( H \) endowed with the finer topological group topology whose base is formed by the family of \( G_\delta \)-sets in the original group \( H \). The subgroup of \( H \) generated by a set \( A \subseteq H \) is \( \langle A \rangle \). Sometimes we use \( \langle A, B \rangle \) for \( A, B \subseteq H \) to denote the group \( \langle A \cup B \rangle \).
A group $G$ is boolean if every $x \in G$ satisfies $x + x = 0_G$. The group $\mathbb{Z}_2 = \{0, 1\}$ and all powers of $\mathbb{Z}_2$ are boolean groups.

Let $\{D_i : i \in I\}$ be a family of topological groups and $D = \prod_{i \in I} D_i$ the product group with the usual Tychonoff product topology. Elements of $D$ will be regarded as functions $x : I \to \bigcup_{i \in I} D_i$ such that $x(i) \in D_i$ for all $i \in I$. If $x \in D$, we define the support of $x$ as

$$\text{supp}(x) = \{i \in I : x(i) \neq 0_i\},$$

where $0_i$ is the neutral element of $D_i$. With these notations, the subgroup

$$\Sigma D = \{x \in D : |\text{supp}(x)| \leq \omega\}$$

of $D$ is called the $\Sigma$-product of the family $\{D_i : i \in I\}$. Similarly,

$$\sigma D = \{x \in D : |\text{supp}(x)| < \omega\}$$

is a subgroup of $D$ which is called the $\sigma$-product of the family $\{D_i : i \in I\}$.

Let $[I]^{\leq \omega}$ denote the family of all countable subsets of the index set $I$. For every $i \in I$, we denote by $S_i$ the family of all subgroups of type $G_\delta$ in $D_i$. It is clear that $S_i$ is a base of $PD_i$ at $0_i$. The collection of sets

$$U(J, f) = \{x \in D : x(i) \in f(i) \text{ for all } i \in J\},$$

where $J \in [I]^{\leq \omega}$ and $f$ a function with the domain $J$ such that $f(i) \in S_i$ for each $i \in J$, constitutes a base at the identity of $PD$, the $P$-modification of the product group $D$. Clearly, $\Sigma D$ is a dense subgroup of $PD$, while $\sigma D$ is a dense subgroup of $D$.

In what follows we will also use the sets

$$U(J) = \{x \in D : x(i) = 0_i \text{ for all } i \in J\},$$

with $J \subset I$. Notice that if the groups $D_i$'s are discrete, then the family $\{U(J) : J \in [I]^{\leq \omega}\}$ forms a local base at the identity of $PD$.

We will mainly be working with topological groups $G$ such that

$$\Sigma D \subseteq G \subseteq PD.$$ 

The subgroup $\Sigma D$ of $PD$ will always carry the topology inherited from $PD$, i.e., $\Sigma D$ is a $P$-group.

Let $G$ be a subgroup of $PD$ and $J \subset I$. We will say that a character $\chi : G \to \mathbb{T}$ depends on $J$ if there is $x \in G$ with $\text{supp}(x) \subset J$ such that $\chi(x) \neq 1$ (notice that $x \in G \cap U(I \setminus J)$).
It is easy to see (see Lemma 4.1 below) that a character $\chi: G \to T$ is continuous if and only if there are $J \in [I]^{\leq \omega}$ and a function $f$ such that $G \cap U(J, f) \subset \ker \chi$. Therefore, for every continuous character $\chi: G \to T$, there is $J \in [I]^{\leq \omega}$ such that $\chi$ does not depend on $I \setminus J$. We say in this case that $\chi$ depends (at most) on countably many coordinates. Similarly, we say that a set $K \subset G^\wedge$ depends (at most) on countably many coordinates if there is $J \in [I]^{\leq \omega}$ such that every $\chi \in K$ does not depend on $I \setminus J$.

3. The results

As we mentioned above, Leptin [17] gave an example of a nonreflexive $P$-group which was the subgroup of $P\mathbb{Z}_2^\omega$ consisting of elements with finite support (i.e. the $\sigma$-product of $\omega_1$ copies of the discrete two-element group $\mathbb{Z}_2$). Here we extend Leptin’s argument to deduce the following result (see Proposition 7.3):

**Theorem 1.** The $\Sigma$-product $\Sigma D \subset PD$ is not reflexive, where $D = \prod_{i \in I} D_i$ is the product of an uncountable family of nontrivial discrete Abelian groups.

We also extend Theorem 1 to certain subgroups between $\Sigma = \Sigma\mathbb{Z}_2^\tau$ and $P\mathbb{Z}_2^\tau$ as follows (see Theorem 7.8):

**Theorem 2.** If $L$ is a countable subgroup of $P\mathbb{Z}_2^\tau$, for an uncountable cardinal $\tau$, then the subgroup $G_L = \Sigma + L$ of $P\mathbb{Z}_2^\tau$ is not reflexive.

It may be worth to observe that while the proof of Theorem 1 uses an argument close to Leptin’s, this argument does not work for $G_L$ and a different one is needed for Theorem 2.

Somewhat surprisingly we also find a wealth of non-discrete reflexive $P$-groups (and hence reflexive noncompact $\omega$-bounded groups). In particular, we prove the following fact in Theorem 6.8 (and Corollary 6.9):

**Theorem 3.** Let $D = \prod_{i \in I} D_i$ be a product of nontrivial discrete Abelian groups, where $|I| > \omega$. Then the non-discrete $P$-group $\Pi = PD$ and the noncompact $\omega$-bounded group $\Pi^\wedge$ are reflexive.

The non-reflexive groups in Theorems 1 and 2 are evidently non-complete, while the reflexive groups presented in Theorem 3 are complete (apply [12, Theorem 8]). One can conjecture, therefore, that reflexive $P$-groups are complete. We show in Theorem 9 below that this is not the case.
Theorem 4. Let $D = \prod_{i \in I} D_i$ be a product of feathered Abelian groups. Then the group $PD$ is reflexive.

As a step towards the proof of Theorem 1, we show that every compact Abelian group with the $P$-modified topology is reflexive.

The following reflexion principle turns out to be quite useful when trying to extend the class of reflexive $P$-groups (see Theorem 6.14):

Theorem 5. Suppose that $D = \prod_{i \in I} D_i$ is a product of topological groups and $\Sigma D \subset G \subset PD$. Then the group $G$ is reflexive iff the subgroup $\pi_J(G)$ of $PD_J$ is reflexive, for every set $J \subset I$ satisfying $|J| \leq \aleph_1$ (here $\pi_J: D \rightarrow \prod_{i \in J} D_i$ is the projection).

Theorem 5 enables us to present examples of dense reflexive subgroups $G$ of the groups $PD$ (see Corollary 6.15). Clearly, $G$ is not complete provided that $G \neq PD$.

Theorem 6. Let $D = \prod_{i \in I} D_i$ be a product of feathered Abelian groups. Then the subgroup

$$\Sigma_{\aleph_1} D = \{ x \in PD : |\text{supp}(x)| \leq \aleph_1 \}$$

of $PD$ is reflexive and every subgroup $G$ of $PD$ containing $\Sigma_{\aleph_1} D$ is also reflexive.

Once we have established that the class of reflexive $P$-groups is quite wide, it is natural to clarify the permanence properties of this class. In the next result we present two of them (see Propositions 5.6 and 5.9):

Theorem 7. Let $G$ be a reflexive $P$-group.

(a) If $G$ is a dense subgroup of a topological group $H$, then $H$ is also a reflexive $P$-group.

(b) If $\pi: G \rightarrow K$ is a continuous open homomorphism of $G$ onto $K$, then $K$ is a reflexive $P$-group.

Let $\tau > \omega$ be a cardinal. To extend Theorem 3 to subgroups slightly smaller than $\Pi = P\mathbb{Z}_\tau^*$ we consider an arbitrary ultrafilter $\xi$ on $\tau$ containing all subsets of $\tau$ with countable complement.
Note that $A \in \xi$ implies $|A| > \omega$. We then define

$$G_\xi = \{ x \in \Pi : \text{supp}(x) \notin \xi \}.$$ 

It is straightforward to check that $G_\xi$ is a subgroup of $\Pi$. Also, if $x \in \Sigma$, then $|\text{supp}(x)| \leq \omega$. So, $\text{supp}(x) \notin \xi$ and therefore $\Sigma \Pi \subset G_\xi$. It is clear that $G_\xi \neq \Pi$ because the constant function 1 is not in $G_\xi$. Actually this is the “most important” element absent in $G_\xi$, as $\Pi = \langle G_\xi, 1 \rangle = G_\xi \oplus \langle 1 \rangle$.

We prove in Theorem 6.17 that this smaller subgroup $G_\xi$ of $\Pi$ is also reflexive:

**Theorem 8.** The nondiscrete $P$-group $G_\xi$ is reflexive, for every ultrafilter $\xi$ on $\tau$ containing the complements to countable sets.

### 4. Factorization of characters on subgroups of a product group

Here we collect several results of technical nature which will be used later. Throughout this section $D = \prod_{i \in I} D_i$ stands for the Tychonoff product of a family $\{D_i : i \in I\}$ of topological groups, not necessarily Abelian. For $J \subset I$, we denote by $D_J$ the corresponding subproduct $\prod_{i \in J} D_i$ and by $\pi_J : D \rightarrow D_J$ the projection.

**Lemma 4.1.** Let $G$ be a subgroup of $PD$ and $\chi : G \rightarrow \mathbb{T}$ be a character. The following assertions are then equivalent:

1. $\chi$ is continuous, i.e., $\chi \in G^\wedge$.
2. There are $J \in [I]^{\leq \omega}$ and a function $f$ such that $G \cap U(J, f) \subset \ker \chi$ (in particular, $\chi$ does not depend on $I \setminus J$).
3. There are a countable set $J \subset I$ and a continuous character $\chi_J$ on the subgroup $\pi_J(G)$ of $PD_J$ such that $\chi = \chi_J \circ \pi_J|_G$.

**Proof.** Suppose $\chi$ is continuous. Since $G$ is a $P$-group, the kernel of $\chi$ is an open subgroup of $G$. It follows from the definition of the topology of $PD$ that there exist a countable set $J \subset I$ and a function $f$ with the domain $J$ such that $f(i)$ is an open subgroup of $PD_i$ for each $i \in J$ and the basic open set $U(J, f) = \{ x \in D : x(i) \in f(i) \text{ for all } i \in J \}$ in $PD$ satisfies $G \cap U(J, f) \subset \ker \chi$. Clearly $\chi$ does not depend on $I \setminus J$. Hence (1) implies (2).

Suppose now that $G \cap U(J, f) \subset \ker \chi$ for a countable set $J \subset I$ and a corresponding function $f$. We define a character $\chi_J$ on $\pi_J(G)$ by $\chi_J(\pi_J(x)) = \chi(x)$ for any element $x \in G$. This definition is correct since the equality $\pi_J(x) = \pi_J(y)$ implies that $x^{-1}y \in G \cap U(J, f)$ and $\chi(x) = \chi(y)$. By the
definition of \( \chi_J \), we see that \( \chi = \chi_J \circ \pi_J|_G \). It also follows from the definition of \( \chi_J \) that its kernel contains the set \( \pi_J(G) \cap U_f \), where \( U_f = \prod_{i \in J} f(i) \) and \( f(i) \) is an open subgroup of \( PD_i \) for each \( i \in J \). Since \( U_f \) is an open subgroup of \( PD_J \), we conclude that \( \chi_J \) is a continuous character on the group \( \pi_J(G) \). So (2) implies (3).

Finally suppose that there is a countable set \( J \subset I \) and a continuous character \( \chi_J \) on \( \pi_J(G) \) such that \( \chi = \chi_J \circ \pi_J|_G \). Since the projection \( \pi_J: PD \to PD_J \) is continuous, we see that so is the character \( \chi \). Hence (3) implies (1).

The following result is close in the spirit to [8, Theorem 4.6], where the product space carries the usual Tychonoff product topology.

Lemma 4.2. Let \( G \) be a dense subgroup of \( PD \) and \( \chi \in G^\wedge \). Then \( \chi \) admits a continuous extension to a character \( \overline{\chi} \) on \( PD \) and, for every set \( J \subset I \), \( \overline{\chi} \) does not depend on \( I \setminus J \) if and only if there exists a continuous character \( \psi \) on \( \pi_J(G) \) such that \( \chi = \psi \circ \pi_J|_G \).

Proof. Since \( G \) is dense in \( PD \) and the circle group \( T \) is compact (hence complete), \( \chi \) extends to a continuous character \( \overline{\chi} \) on the group \( PD \). If \( \overline{\chi} \) does not depend on \( I \setminus J \), where \( J \subset I \), there exists a character \( \overline{\psi} \) on \( PD_J \) such that \( \overline{\chi} = \overline{\psi} \circ \pi_J \), where \( D_J = \prod_{i \in I} D_i \) and \( \pi_J: D \to D_J \) is the projection. Since \( \pi_J: PD \to PD_J \) is open, the character \( \overline{\psi} \) on \( PD_J \) is continuous. Then \( \chi = \psi \circ \pi_J|_G \), where \( \psi \) is the restriction of \( \overline{\psi} \) to the subgroup \( \pi_J(G) \) of \( PD_J \). Clearly, the character \( \psi \) is continuous.

Conversely, suppose that there exists a continuous character \( \psi \) on the subgroup \( \pi_J(G) \) of \( PD_J \) such that \( \chi = \psi \circ \pi_J|_G \). Since \( \pi_J(G) \) is dense in \( PD_J \), \( \psi \) extends to a continuous character \( \overline{\psi} \) on the group \( PD_J \). Clearly, the characters \( \overline{\chi} \) and \( \overline{\psi} \circ \pi_J \) coincide on the dense subgroup \( G \) of \( PD \). Since the group \( T \) is Hausdorff, we see that \( \overline{\chi} = \overline{\psi} \circ \pi_J \). It follows that \( \overline{\chi} \) does not depend on \( I \setminus J \). 

In general, the existence of a continuous character \( \psi \) on \( \pi_J(G) \) satisfying \( \chi = \psi \circ \pi_J|_G \) in Lemma 4.2 cannot be weakened to a simpler condition that \( \chi \) does not depend on \( I \setminus J \). However, this weakening is possible for special subgroups of \( PD \) as we will see in Corollary 4.4 below. First we need a lemma.

Lemma 4.3. If \( \Sigma D \subset G \subset PD \), then the restriction of the projection \( \pi_J: PD \to PD_J \) to \( G \) is an open homomorphism of \( G \) onto \( \pi_J(G) \), for every nonempty set \( J \subset I \).
Proof. Let $J$ be a nonempty subset of $I$. Since $G$ is a subgroup of $PD$, the restriction of $\pi_J$ to $G$ is a continuous homomorphism. Hence it suffices to verify that the image $\pi_J(V \cap G)$ is open in $\pi_J(G)$, for every basic open neighbourhood $V$ of the identity $e$ in $PD$. In fact, we will show that $\pi_J(V \cap G) = \pi_J(V) \cap \pi_J(G)$.

Given a basic open neighbourhood $V$ of $e$ in $PD$, one can find a countable set $C \subset I$ and open subgroups $V_i$ of $D_i$ for $i \in C$ such that

$$V = \{x \in D : x(i) \in V_i \text{ for each } i \in C\}.$$ 

Let $F = C \cap J$ and $E = C \setminus J$. It is clear that $F$ and $E$ are disjoint countable sets and $C = F \cup E$.

Take an arbitrary point $y \in \pi_J(V) \cap \pi_J(G)$. There exists an element $x \in G$ with $\pi_J(x) = y$. Clearly, $x(i) = y(i) \in V_i$ for each $i \in F$. Since $E$ is countable, we can find an element $x_0 \in \Sigma D$ such that $\text{supp}(x_0) \cap J = \emptyset$ and $x_0(i) = x(i)$ for each $i \in E$. Then the element $z = x \cdot x_0^{-1}$ of $D$ satisfies $z(i) \in V_i$ for each $i \in C$, so $z \in V$. Since $x_0 \in \Sigma D \subset G$, we see that $z \in G$. Hence $z \in V \cap G$ and $\pi_J(z) = \pi_J(x) \cdot (\pi_J(x_0))^{-1} = y$. This implies that $\pi_J(V \cap G) \subset \pi_J(V \cap G)$. The inverse inclusion is obvious, so the equality $\pi_J(V \cap G) = \pi_J(V) \cap \pi_J(G)$ is proved. Therefore the restriction of the homomorphism $\pi_J$ to $G$ is open when considered as a mapping onto its image. □

Corollary 4.4. Let $G$ be a group with $\Sigma D \subset G \subset PD$, and $\chi \in G^\wedge$. Then, for every set $J \subset I$, the continuous extension $\overline{\chi}$ of $\chi$ over PD does not depend on $I \setminus J$ if and only if $\chi$ does not depend on $I \setminus J$.

Proof. By Lemma 4.2, $\overline{\chi}$ is a continuous character on $PD$. Hence it suffices to verify that if $\chi$ does not depend on $I \setminus J$, neither does $\overline{\chi}$. Under this assumption, there exists a character $\psi$ on $\pi_J(G)$ such that $\chi = \psi \circ \pi_J|_G$. Since the restriction to $G$ of the projection $\pi_J$ is open by Lemma 4.3, the character $\psi$ is continuous. We apply Lemma 4.2 once again to conclude that $\overline{\chi}$ does not depend on $I \setminus J$. □

5. Reflexivity of $G$ and compact subsets of $G^\wedge$

Here we characterize the reflexivity of a $P$-group $G$ in terms of compact subsets of the dual group $G^\wedge$. First we present a simple but useful piece of information.

Lemma 5.1. If $G$ is a $P$-group, then the dual group $G^\wedge$ is $\omega$-bounded.
Proof. Since $G$ is a $P$-group, all compact subsets of $G$ are finite. Hence $G^\wedge$ is a topological subgroup of $\mathbb{T}^G$, where $\mathbb{T}$ is the circle group with its usual compact topology inherited from the complex plane. Let $C$ be a countable subset of $G^\wedge$. Since the circle group $\mathbb{T}$ and its power $\mathbb{T}^G$ are compact, the set $\overline{C}$ is also compact (the closure is taken in $\mathbb{T}^G$). We claim that $\overline{C} \subset G^\wedge$.

To verify this inclusion, take an arbitrary element $\varphi \in \overline{C}$. It is well known (and easy to see) that $\varphi$ is a homomorphism of $G$ to $\mathbb{T}$, so it suffices to prove the continuity of $\varphi$ at the neutral element $0_G$ of $G$. Since each $\chi \in C$ is continuous at $0_G$, there exists an open neighbourhood $U_\chi$ of $0_G$ in $G$ such that $\chi(U_\chi) = \{1\}$; here $1$ is the neutral element of $\mathbb{T}$. Then $V = \bigcap_{\chi \in C} U_\chi$ is an open neighbourhood of $0_G$, and we claim that $\varphi(V) = \{1\}$. Indeed, if $x \in V$, then $\chi(x) = 1$ for each $\chi \in C$, so it follows from $\varphi \in \overline{C}$ that $\varphi(x) = 1$. Thus, $\varphi$ is also continuous at $0_G$, i.e., $\varphi \in G^\wedge$.

Therefore the group $G^\wedge$ is $\omega$-bounded. \hfill \square

Let us isolate a property that a $P$-group must possess in order to be reflexive. In what follows we say that a set $K \subseteq G^\wedge$ is constant on a subgroup $H$ of $G$ if every $\chi \in K$ is constant on $H$.

**Lemma 5.2.** Let $G$ be a $P$-group. The evaluation mapping $\alpha_G : G \to G^{\wedge^\wedge}$ is continuous if and only if every compact set $K \subseteq G^\wedge$ is constant on an open subgroup of $G$.

Proof. *Necessity.* Let $K$ be a compact subset of $G^\wedge$. If $\alpha_G$ is continuous, there exists an open neighbourhood $U$ of the neutral element $e$ in $G$ such that $\alpha_G(U) \subset K^{\triangleright}$. Since $G$ is a $P$-group, it follows from [3] Lemma 4.4.1a) that there exists an open subgroup $V$ of $G$ such that $V \subset U$. Clearly the set $\mathbb{T}_+$ does not contain nontrivial subgroups, so $\chi(V) = \{1\}$ for each $\chi \in K$. Thus $K$ is constant on $V$.

* Sufficiency. It suffices to verify the continuity of the homomorphism $\alpha_G$ at the neutral element of $G$. Let $K^{\triangleright}$ be a basic open set in $G^{\wedge^\wedge}$, with $K$ a compact subset of $G^\wedge$. By hypothesis, there exists an open subgroup $V$ of $G$ such that every $\chi \in K$ is constant on $V$. Then $\alpha_G(V) \subset K^{\triangleright}$. Therefore $\alpha_G$ is continuous at $e$. \hfill \square

Since we are mainly concerned with (subgroups of) product groups, it is worth to reformulate the above lemma for this special case.

**Corollary 5.3.** Let $G$ be a subgroup of $PD$, where $D = \prod_{i \in I} D_i$ is the product of an arbitrary family of topological groups. The evaluation mapping $\alpha_G : G \to G^{\wedge^\wedge}$ is continuous if and only if
for every compact set $K \subseteq G^\wedge$, one can find a set $J \in [I]^{\leq \omega}$ and an open subgroup $U$ of $PD_J$ such that every $\chi \in K$ is constant on the set $G \cap \pi_J^{-1}(U)$.

**Proof.** By Lemma 5.2, the continuity of $\alpha_G$ means that every compact set $K \subseteq G^\wedge$ is constant on an open subgroup $V$ of $G$. Since the sets $G \cap \pi_J^{-1}(U)$, where $J \in [I]^{\leq \omega}$ and $U$ is an open subgroup of $PD_J$, form an open basis at the neutral element of $G$, the required conclusion is immediate. □

**Theorem 5.4.** A $P$-group $G$ is reflexive if and only if every compact set $K \subseteq G^\wedge$ is constant on an open subgroup of $G$.

**Proof.** By [3, Lemma 4.4.1 a)], every $P$-group has a base at the identity consisting of open subgroups. Hence $G$ is a topological subgroup of a product of discrete groups. Since the class of nuclear groups contains discrete Abelian groups and is closed under taking products and arbitrary subgroups (here we apply Propositions 7.3, 7.5, and 7.6 from [5]), it follows that the group $G$ is nuclear.

We already know that the evaluation homomorphism $\alpha_G : G \to G^\wedge$ is injective because $G$ is a nuclear group. By the same reason, the mapping $\alpha_G : G \to \alpha_G(G)$ is open, where $\alpha_G(G)$ carries the topology inherited from $G^\wedge$ (see Theorem 8.5 and Lemma 14.3 of [5]).

Since the $P$-group $G$ has no infinite compact subsets, $G^\wedge$ carries the topology of pointwise convergence on elements of $G$. It follows that $G^\wedge = \alpha_G(G)$ (see for instance [9, Theorem 1.3]). The $P$-group $G$ is therefore reflexive if and only if $\alpha_G$ is continuous. The theorem is then a direct consequence of Lemma 5.2. □

Here is a coordinatewise form of Theorem 5.4 which is immediate after Corollary 5.3.

**Theorem 5.5.** Let $D = \prod_{i \in I} D_i$ be a product of topological groups. A subgroup $G$ of $PD$ is reflexive if and only if for every compact set $K \subseteq G^\wedge$, there exist a set $J \in [I]^{\leq \omega}$ and an open subgroup $U$ of $PD_J$ such that $K$ is constant on $G \cap \pi_J^{-1}(U)$.

The following two somewhat unexpected facts fail to hold outside the class of $P$-groups. The first of them says that reflexivity in $P$-groups extends from a dense subgroup to the whole group:

**Proposition 5.6.** Let $G$ be a a dense subgroup of a topological group $H$. If $G$ is a reflexive $P$-group, so is $H$. 
Proof. Suppose that $G$ is a reflexive $P$-group. Since $G$ is dense in $H$, it follows from [3, Lemma 4.4.1 d)] that $H$ is a $P$-group. According to Theorem 5.4 it suffices to verify that every compact set $K \subset H^\wedge$ is constant on an open subgroup of $H$. Denote by $r$ the natural restriction homomorphism of $H^\wedge$ to $G^\wedge$ defined by $r(\chi) = \chi|_G$, for each $\chi \in H^\wedge$. Since $G \subset H$ and the dual groups $G^\wedge$ and $H^\wedge$ carry the topologies of pointwise convergence on elements of $G$ and $H$, respectively, $r$ is continuous. It also follows from the density of $G$ in $H$ that $r$ is one-to-one and onto. In other words, $r$ is a continuous isomorphism of $H^\wedge$ onto $G^\wedge$.

Since $G$ is reflexive, Theorem 5.4 implies that the compact set $r(K) \subset G^\wedge$ is constant on an open subgroup $V$ of $G$. Let $U$ be the closure of $V$ in $H$. Then $U$ is an open subgroup of $H$ and every $\chi \in K$ is constant on the dense subset $V$ of $U$. By a continuity argument, $\chi$ is constant on $U$. Therefore $K$ is constant on $U$, whence the reflexivity of $H$ follows. □

The second fact establishes that the class of reflexive $P$-groups is stable under taking quotients. Its proof makes use of dual homomorphisms. Since this tool will be used several times in the article, we give a lemma explaining basic properties of dual homomorphisms.

Let us recall that a surjective mapping $f : X \to Y$ is compact covering if for every compact set $K \subset Y$, there exists a compact set $C \subset X$ such that $f(C) = K$.

Lemma 5.7. Let $\pi : G \to H$ be a continuous homomorphism of topological Abelian groups. Let also $\pi^\wedge : H^\wedge \to G^\wedge$ be the dual homomorphism defined by $\pi^\wedge(\chi) = \chi \circ \pi$, for each $\chi \in H^\wedge$. Then:

(a) $\pi^\wedge$ is continuous.

(b) If $\pi$ is compact covering, then $\pi^\wedge$ is a topological isomorphism of $H^\wedge$ onto its image $\pi^\wedge(H^\wedge)$.

(c) If the homomorphism $\pi$ is open, then the image $\pi^\wedge(H^\wedge)$ is closed in $G^\wedge$.

Proof. (a) is well known. Indeed, let $C$ be a compact subset of $G$ and $U = C^\circ$ a basic open neighbourhood of the neutral element in $G^\wedge$. Then $K = \pi(C)$ is a compact subset of $H$ and $V = K^\circ$ is an open neighbourhood of the neutral element in $H^\wedge$. For any $\chi \in V$, we have $\pi^\wedge(\chi)(C) = (\chi \circ \pi)(C) = \chi(\pi(C)) = \chi(K) \subset \mathbb{T}_+$, that is, $\pi^\wedge(\chi) \in C^\circ = U$. This implies the continuity of $\pi^\wedge$ and proves (a).

(b) follows from [3, Lemma 5.17].
(c) Suppose that \( \pi \) is an open homomorphism of \( G \) to \( H \) and let \( K = \pi(G) \). Then \( K \) is an open subgroup of \( H \), so every continuous character on \( K \) extends to a continuous character on \( H \). Hence \( \pi^\wedge(K^\wedge) = \pi^\wedge(H^\wedge) \) and we can assume without loss of generality that \( \pi(G) = H \).

Our further argument is very close to the proof of item 2) of Corollary 0.4.8 from [2]. Indeed, let \( \psi \) be in the closure of \( \pi^\wedge(H^\wedge) \) in \( G^\wedge \). Since finite sets are compact, the topology of \( G^\wedge \) contains the topology of pointwise convergence on elements of \( G \). Applying this fact, one easily verifies that \( \psi \) is a homomorphism of \( G \) to \( T \) and that \( \psi \) is constant on each fiber \( \pi^{-1}(y) \), \( y \in H \). Hence \( \psi = \chi \circ \pi \) implies that \( \chi \) is continuous as well. Thus, \( \chi \in H^\wedge \) and \( \psi = \pi^\wedge(\chi) \in \pi^\wedge(H^\wedge) \).

**Corollary 5.8.** Let \( \pi: G \to H \) be a continuous onto homomorphism. If all compact subsets of \( H \) are finite, then \( \pi^\wedge \) is a topological isomorphism of \( H^\wedge \) onto the subgroup \( \pi^\wedge(H^\wedge) \) of \( G^\wedge \). In particular, this is the case when \( H \) is a \( P \)-group.

**Proof.** Since all compact subsets of \( H \) are finite, the homomorphism \( \pi \) is compact covering. The required conclusion now follows from (b) of Lemma 5.7.

**Proposition 5.9.** Let \( \pi: G \to H \) be a continuous open epimorphism of topological groups. If \( G \) is a reflexive \( P \)-group, so is \( H \).

**Proof.** The fact that the image \( H = \pi(G) \) is a \( P \)-group follows from [3, Lemma 4.4.1c)]. Let us show that \( H \) is reflexive. Take an arbitrary compact subset \( C \) of \( H^\wedge \). By (a) of Lemma 5.7, the dual homomorphism \( \pi^\wedge: H^\wedge \to G^\wedge \) is continuous, so \( K = \pi^\wedge(C) \) is a compact subset of \( G^\wedge \). Since the group \( G \) is reflexive, Theorem 5.4 implies that there exists an open subgroup \( U \) of \( G \) such that \( K \) is constant on \( U \). Then \( C \) is constant on the open subgroup \( V = \pi(U) \) of \( H \). Indeed, if \( \chi \in C \) and \( y \in V \), take \( x \in U \) with \( \pi(x) = y \). Then \( \chi(y) = \chi(\pi(x)) = \pi^\wedge(\chi)(x) = 1 \) since \( \pi^\wedge(\chi) \in K \). Applying Theorem 5.4 once again, we conclude that the group \( H \) is reflexive.

We finish this section with two special cases of Theorem 5.5.

**Corollary 5.10.** Let \( D = \prod_{i \in I} D_i \) be a product of topological groups. The group \( PD \) is reflexive if and only if the following hold:

(a) the group \( PD_J = P(\prod_{i \in J} D_i) \) is reflexive for each \( J \in [I]^{\leq \omega} \);
(b) every compact set \( K \subset (PD)^\wedge \) depends at most on countably many coordinates.

Proof. Necessity. It is easy to see that the projection \( \pi_J: PD \to PD_J \) is open for each \( J \subset I \).

If the group \( PD \) is reflexive, then the reflexivity of the groups \( PD_J \) follows from Proposition 5.9. Suppose that \( K \) is a compact subset of \( (PD)^\wedge \). Theorem 5.5 implies that there exist \( J \in [I]^{\leq \omega} \) and an open subgroup \( U \) of \( PD_J \) such that \( K \) is constant on \( \pi_J^{-1}(U) \). Then every \( \chi \in K \) does not depend on \( I \setminus J \), that is, \( K \) does not depend on \( I \setminus J \). Hence conditions (a) and (b) hold true.

Sufficiency. Let us deduce the reflexivity of \( PD \) from (a) and (b). Take a compact set \( K \subset (PD)^\wedge \). By (b), there exists a countable set \( J \subset I \) such that \( K \) does not depend on \( I \setminus J \).

Denote by \( \varphi: (PD_J)^\wedge \to (PD)^\wedge \) the homomorphism dual to the projection \( \pi_J: PD \to PD_J \). Since the projection \( \pi_J: PD \to PD_J \) is open and all compact subsets of \( PD_J \) are finite, it follows from Corollary 5.8 that \( \varphi \) is a topological isomorphism of \( (PD_J)^\wedge \) onto a subgroup of \( (PD)^\wedge \).

We claim that \( K \subset \varphi((PD_J)^\wedge) \). Indeed, take an arbitrary character \( \chi \in K \). Since \( \chi \) does not depend on \( I \setminus J \), there exists a character \( \zeta \) on \( PD_J \) such that \( \chi = \zeta \circ \pi_J \). The character \( \zeta \) is continuous since the projection \( \pi_J \) is open. Hence \( \zeta \in (PD_J)^\wedge \) and \( \chi = \varphi(\zeta) \in \varphi((PD_J)^\wedge) \).

Let \( C = \varphi^{-1}(K) \). Then \( C \) is a compact subset of the group \( (PD_J)^\wedge \). By (a), the group \( PD_J \) is reflexive. According to Theorem 5.4, \( PD_J \) contains an open subgroup \( U \) such that every character \( \zeta \in C \) is constant on \( U \). Since \( \varphi(C) = K \), we see that every \( \chi \in K \) is constant on the open subgroup \( \pi_J^{-1}(U) \) of \( PD \). The reflexivity of the group \( PD \) now follows from Theorem 5.5. \( \square \)

We shall see in Proposition 6.7 that one can drop item (b) in the above corollary. The next result is a slight modification of Corollary 5.10, so we omit its proof.

**Corollary 5.11.** Let \( D = \prod_{i \in I} D_i \) be a product of topological groups such that the group \( PD_J \) is reflexive for each \( J \in [I]^{\leq \omega} \). Suppose that \( G \) is a subgroup of \( PD \) satisfying \( \pi_J(G) = PD_J \) for each \( J \in [I]^{\leq \omega} \). Then \( G \) is reflexive if and only if every compact set \( K \subset G^\wedge \) depends at most on countably many coordinates.

### 6. Reflexive \( P \)-groups

We prepare here our way to show that some nondiscrete \( P \)-groups are reflexive. The lemma below is obvious and its proof is omitted.

**Lemma 6.1.** Let \( b, t \) be elements of \( T \) and \( t \neq 1 \). Then there is an integer \( k \) such that \( t^k \cdot b \notin T_+ \).
In the following proposition and in Lemmas 6.3, 6.5 \( D = \prod_{i \in I} D_i \) stands for the product of an arbitrary family of topological Abelian groups.

**Proposition 6.2.** Suppose that \( C = \{ \chi_{\eta} : \eta < \omega_1 \} \subset (PD)^\land \), \( J = \{ J_\eta : \eta < \omega_1 \} \subset [I]^{\leq \omega} \), and \( X = \{ x_\eta : \eta < \omega_1 \} \subset PD \) satisfy the following conditions for each \( \eta < \omega_1 \):

1. \( \chi_\eta \) does not depend on \( I \setminus J_\eta \);
2. supp \((x_\eta) \subset J_\eta \);
3. \( \chi_\eta(x_\eta) \in T \setminus T_+ \);
4. if \( \zeta < \eta \), then \( J_\zeta \cap \text{supp}(x_\eta) = \emptyset \).

Then every element of \( \bigcap_{\gamma < \omega_1} \{ \chi_\eta : \eta \geq \gamma \}^{PD} \) is discontinuous as a character on the group \( PD \).

**Proof.** For each \( \gamma < \omega_1 \), let \( C_\gamma = \{ \chi_\eta : \eta \geq \gamma \} \) and \( K_\gamma = \overline{C_\gamma}^{PD} \). Since the family \( \{ K_\gamma : \gamma < \omega_1 \} \) of compact sets is decreasing, we see that \( K = \bigcap_{\gamma < \omega_1} K_\gamma \) is nonempty. Take \( \rho \in K \). Suppose toward a contradiction that \( \rho \) is continuous, i.e., that \( \rho \in (PD)^\land \). By Lemma 4.1, there exists \( J \in [I]^{\leq \omega} \) such that \( \rho \) does not depend on \( I \setminus J \). It follows from (2) and (4) that the family \( \{ \text{supp}(x_\eta) : \eta < \omega_1 \} \) is pairwise disjoint. Take a countable ordinal \( \eta_0 \) such that \( J \cap \text{supp}(x_\eta) = \emptyset \) for all \( \eta \) satisfying \( \eta_0 \leq \eta < \omega_1 \). It then follows that \( \rho(x_\eta) = 1 \) for every countable ordinal \( \eta \geq \eta_0 \).

Given a family \( \{ g_\alpha : \alpha \in A \} \subset D \) such that \( \text{supp}(g_\alpha) \cap \text{supp}(g_\beta) = \emptyset \) for distinct \( \alpha, \beta \in A \), we can define an element \( g = \prod_{\alpha \in A} g_\alpha \in D \) by the requirements that \( \text{supp}(g) = \bigcup_{\alpha \in A} \text{supp}(g_\alpha) \) and for every \( \alpha \in A \), the elements \( g \) and \( g_\alpha \) coincide on \( \text{supp}(g_\alpha) \).

Now for every countable ordinal \( \eta \geq \eta_0 \), we define a point \( g_\eta \in D \) satisfying the following two conditions:

(a) \( g_\eta \) is either the neutral element \( 0 \) of \( D \) or \( k_\eta x_\eta \), for some \( k_\eta \in \mathbb{Z} \);
(b) \( \chi_\eta \left( \prod_{\eta_0 \leq \beta \leq \eta} g_\beta \right) \in T \setminus T_+ \).

To begin, we put \( g_{\eta_0} = x_{\eta_0} \). Then conditions (a) and (b) hold. Suppose now that \( \eta_0 < \sigma < \omega_1 \) and that \( g_\eta \) have been defined for all \( \eta \) with \( \eta_0 \leq \eta < \sigma \) such that (a) and (b) hold. Notice that the family \( \{ \text{supp}(g_\eta) : \eta_0 \leq \eta < \sigma \} \) is pairwise disjoint, by (a). If \( \chi_\sigma \left( \prod_{\eta_0 \leq \beta < \sigma} g_\beta \right) \notin T_+ \), put \( g_\sigma = 0 \). If \( z = \chi_\sigma \left( \prod_{\eta_0 \leq \beta < \sigma} g_\beta \right) \in T_+ \), we apply Lemma 6.4 to \( z \) and \( t = \chi_\sigma(x_\sigma) \) to find an integer \( k_\sigma \) such that \( z \cdot t^{k_\sigma} \notin T_+ \). We then put \( g_\sigma = k_\sigma x_\sigma \). Since \( \chi_\sigma \left( \prod_{\eta_0 \leq \beta \leq \sigma} g_\beta \right) = z \cdot t^{k_\sigma} \notin T_+ \), the element \( g_\sigma \) satisfies (a) and (b) at the stage \( \sigma \). The recursive definitions are complete.

By (a), the supports of \( g_\eta \)’s with \( \eta \geq \eta_0 \) are disjoint, so we can put \( h_0 = \prod_{\eta_0 \leq \eta < \omega_1} g_\eta \). Again by (a), \( \text{supp}(h_0) \subset \bigcup \{ \text{supp}(x_\eta) : \eta_0 \leq \eta < \omega_1 \} \) and, since \( J \cap \text{supp}(x_\eta) = \emptyset \) for every countable ordinal
\( \eta \geq \eta_0 \), we see that \( J \cap \text{supp}(h_0) = \emptyset \). Therefore \( \rho(h_0) = 1 \). Since \( \rho \in K_{\eta_0} \) and \( (T_+)_{h_0} \times T^D \setminus \{h_0\} \) is a basic open set in \( T^D \) containing \( \rho \), there exists \( \eta \geq \eta_0 \) such that \( \chi_\eta(h_0) \in T_+ \). However, it follows from (4) of the proposition that \( J_\eta \cap \text{supp}(x_\beta) = \emptyset \) if \( \eta < \beta < \omega_1 \), while condition (a) implies that \( \text{supp}(g_\beta) \subset \text{supp}(x_\beta) \). Therefore, the sets \( J_\eta \) and \( \text{supp} \left( \prod_{\eta < \beta < \omega_1} g_\beta \right) \) are disjoint and hence (1) of the proposition implies that \( \chi_\eta \left( \prod_{\eta < \beta < \omega_1} g_\beta \right) = 1 \). It now follows from (b) that

\[
\chi_\eta(h_0) = \chi_\eta \left( \prod_{\eta_0 \leq \beta \leq \eta} g_\beta \right) \cdot \chi_\eta \left( \prod_{\eta < \beta < \omega_1} g_\beta \right) = \chi_\eta \left( \prod_{\eta_0 \leq \beta \leq \eta} g_\beta \right) \cdot 1 \notin T_+.
\]

This contradiction shows that \( \rho \) is discontinuous. \( \square \)

**Lemma 6.3.** Suppose that \( \Sigma D \subset G \subset PD \), \( g \in G \), \( \chi \in G^\land \), and \( \chi(g) \neq 1 \). If \( \chi \) does not depend on \( I \setminus J \), for a countable set \( J \subset I \), then there exists a point \( x \in G \) with \( \text{supp}(x) \subset J \cap \text{supp}(g) \) such that \( \chi(x) \notin T_+ \).

**Proof.** We can find elements \( y, z \in PD \) such that \( g = y + z \), \( \text{supp}(y) = J \cap \text{supp}(g) \), and \( \text{supp}(z) \cap J = \emptyset \). Notice that \( y \in \Sigma D \subset G \), so \( z \in G \). It follows from our choice of \( z \) that \( \chi(z) = 1 \). Since \( \chi(y) = \chi(y) \cdot \chi(z) = \chi(y + z) = \chi(g) \neq 1 \), we see that \( t = \chi(y) \neq 1 \). Take an integer \( n \) such that \( t^n \notin T_+ \). The point \( x = ny \in G \) is as required. \( \square \)

**Lemma 6.4.** Suppose that \( \Sigma D \subset G \subset PD \) and \( K \subset G^\land \). If \( K \) depends on uncountably many coordinates, then \( K \) contains a subset \( C = \{ \chi_\eta : \eta < \omega_1 \} \) and \( G \) contains a subset \( X = \{ x_\eta : \eta < \omega_1 \} \) such that conditions (1)–(4) of Proposition 6.2 hold for \( C \), \( X \), and a suitable collection \( \mathcal{J} = \{ J_\eta : \eta < \omega_1 \} \) of countable subsets of \( I \).

**Proof.** To begin, we choose a family \( \{ J_\chi : \chi \in K \} \) of countable subsets of \( I \) such that \( \chi \) does not depend on \( I \setminus J_\chi \), for each \( \chi \in K \).

Since \( K \) depends on uncountably many coordinates, there is a nontrivial character \( \chi_0 \in K \). It follows from Lemma 6.3 that there is a point \( x_0 \in G \) with \( \text{supp}(x_0) \subset J_{\chi_0} \) such that \( \chi_0(x_0) \notin T_+ \). We put \( J_0 = J_{\chi_0} \). Then (1)–(4) of Proposition 6.2 hold for \( \eta = 0 \).

Suppose now that \( \sigma < \omega_1 \) and that \( \chi_\eta, x_\eta, \) and \( J_\eta \) have been defined to satisfy (1)–(4) of Proposition 6.2 for all \( \eta < \sigma \). Then the set \( T = \bigcup_{\eta < \sigma} J_\eta \) is countable. By assumptions of the lemma, there is a character \( \chi_\sigma \in K \) depending on \( I \setminus T \), i.e., there is a point \( g \in G \) such that \( g(i) \) is the neutral element of \( D_i \) for each \( i \in T \) and \( \chi_\sigma(g) \neq 1 \). It now follows from Lemma 6.3 that there exists a point \( x_\sigma \in G \) with \( \text{supp}(x_\sigma) \subset \text{supp}(g) \cap J_{\chi_\sigma} \) such that \( \chi_\sigma(x_\sigma) \notin T_+ \). Let \( J_\sigma = J_{\chi_\sigma} \). It is
clear that $\chi_\eta$, $x_\eta$, and $J_\eta$ satisfy (1)–(4) of Proposition 6.2 for all $\eta \leq \sigma$. The recursive definitions are complete which finishes the proof.

A slight modification in the above argument can be made in order to deduce the following lemma which will be applied in the proof of Proposition 6.16.

**Lemma 6.5.** Suppose that $\Sigma D \subset G \subset PD$ and that two sets $C = \{\chi_\eta : \eta < \omega_1\} \subset G^\land$ and $\mathcal{J} = \{J_\eta : \eta < \omega_1\} \subset [I]^{\leq \omega}$ satisfy the following conditions for all $\eta < \omega_1$:

(a) $\chi_\eta$ does not depend on $I \setminus J_\eta$;
(b) $J_\zeta \subset J_\eta$ if $\zeta < \eta$;
(c) $\chi_\eta$ depends on $I \setminus \bigcup_{\zeta < \eta} J_\zeta$.

Then there exists a set $X = \{x_\eta : \eta < \omega_1\} \subset G$ such that $C$, $\mathcal{J}$, and $X$ satisfy conditions (1)–(4) of Proposition 6.2.

**Proposition 6.6.** Let $D = \prod_{i \in I} D_i$ be a product of topological groups. Then every compact set $K \subset (PD)^\land$ depends at most on countably many coordinates.

**Proof.** Let $\Pi = PD$. Suppose to the contrary that a compact set $K \subset \Pi^\land$ depends on uncountably many coordinates. By Lemma 6.4, there exist $C \subset K$, $X \subset G$, and $\mathcal{J} \subset [I]^{\leq \omega}$ satisfying conditions (1)–(4) of Proposition 6.2. By the latter proposition, there is an element $\rho$ in the closure of $C$ in $T^\Pi$ such that $\rho \notin \Pi^\land$. But $K$, being compact, must be closed in $T^\Pi$, whence $\rho \in K \subset \Pi^\land$. This contradiction shows that there must exist $J \in [I]^{\leq \omega}$ such that $K$ does not depend on $I \setminus J$.  

The above proposition shows that item (b) in Corollary 5.10 can be omitted:

**Proposition 6.7.** Let $D = \prod_{i \in I} D_i$ be a product of topological groups. Then the product group $PD$ is reflexive if and only if $PD_J$ is reflexive for each $J \in [I]^{\leq \omega}$.

**Theorem 6.8.** Let $D = \prod_{i \in I} D_i$ be a product of discrete Abelian groups. Then the $P$-group $\Pi = PD$ and the $\omega$-bounded group $\Pi^\land$ are reflexive.

**Proof.** The group $PD_J$ is discrete and hence reflexive, for every $J \in [I]^{\leq \omega}$. Therefore the reflexivity of $\Pi$ follows from Proposition 6.7. Hence the dual group $\Pi^\land$ is reflexive as well.  

In the case when the product $D = \prod_{i \in I} D_i$ in the above theorem contains uncountably many nontrivial factors, we obtain the following result that answers a question posed by S. Hernández and P. Nickolas and solves a problem raised in a comment after Proposition 2.10 in [1].
Corollary 6.9. There exist non-discrete reflexive $P$-groups as well as $\omega$-bounded noncompact reflexive groups.

We can now establish the reflexivity of certain $P$-groups which are not necessarily $P$-modifications of products of discrete groups. A simple auxiliary lemma is in order:

**Lemma 6.10.** Suppose that $\pi: G \to H$ is a continuous onto homomorphism of compact groups. Then the homomorphism $\pi: PG \to PH$ is open, where $PG$ and $PH$ are $P$-modifications of the groups $G$ and $H$, respectively.

**Proof.** Let $e$ be the neutral element of $G$. It is clear that the sets of the form $V = \bigcap_{n \in \omega} U_n$, where $U_n$'s are open neighbourhoods of $e$ in $G$ and $\overline{U}_n+1 \subset U_n$ for each $n \in \omega$ (the closure is taken in $G$), constitute a base at $e$ in $PG$. Therefore, it suffices to verify that every image $\pi(V)$ is open in $PH$. Notice that the continuous epimorphism $\pi: G \to H$ is open since $G$ is compact. Using the compactness of $G$ once again we see that $\pi(V) = \bigcap_{n \in \omega} \pi(U_n)$, so $\pi(V)$ is a $G_\delta$-set in $H$. Hence $\pi(V)$ is open in $PH$. □

**Proposition 6.11.** Let $H$ be a compact Abelian group and $PH$ the $P$-modification of $H$. Then the group $PH$ is reflexive.

**Proof.** It is well known that one can find a compact Abelian group $G$ of the form $G = \prod_{i \in I} G_i$, with compact metrizable factors $G_i$, and a continuous homomorphism $\pi$ of $G$ onto $H$ (see [13, Lemma 1.6]). By Lemma 6.10 the homomorphism $\pi: PG \to PH$ is open. For every $i \in I$, let $D_i$ be the group $G_i$ with the discrete topology. Denote by $D$ the product group $\prod_{i \in I} D_i$. Since the factors $G_i$ are metrizable, the topological groups $PG$ and $PD$ coincide. It now follows from Theorem 6.8 that the group $PG$ is reflexive, while Proposition 5.9 implies the reflexivity of $PH$. □

According to [3, Section 4.3], a topological group $H$ is *feathered* if it contains a nonempty compact set with a countable neighbourhood base in $H$. Let us call a topological group $H$ *pseudo-feathered* if there exists a nonempty compact set of type $G_\delta$ in $H$. It is clear that every feathered group is pseudo-feathered and that $H$ is pseudo-feathered if and only if it contains a compact subgroup of type $G_\delta$. An Abelian group is pseudo-feathered iff it admits an open continuous homomorphism with compact kernel onto a group of countable pseudocharacter. In the following result we extend the conclusion of Proposition 6.11 to pseudo-feathered groups.
Proposition 6.12. Let $H$ be a pseudo-feathered Abelian group. Then the group $PH$ is reflexive.

Proof. Let $C$ be a compact subgroup of type $G_δ$ in $H$. Clearly, $PC$ is then an open subgroup of $PH$. Since a topological group admitting an open subgroup that is reflexive is itself reflexive (Proposition 2.2 of [6]) and the group $PC$ is reflexive by Proposition 6.11, we conclude that $PH$ is reflexive. □

The next result is a common generalization of Theorem 6.8 and Proposition 6.12:

Theorem 6.13. Let $H = \prod_{i \in I} H_i$ be the product of a family of pseudo-feathered Abelian groups. Then the group $PH$ is reflexive.

Proof. It is easy to verify that if $C_n$ is a compact set of type $G_δ$ in a space $X_n$, for each $n \in \omega$, then the compact set $C = \prod_{n \in \omega} C_n$ has type $G_δ$ in the product space $X = \prod_{n \in \omega} X_n$. This observation implies that the group $H_J = \prod_{i \in J} H_i$ is pseudo-feathered for each $J \subseteq I \leq \omega$. The reflexivity of $PH$ now follows from Proposition 6.7. □

In Theorem 6.14 below we characterize the reflexivity of certain subgroups $G$ of “big” products $PD = P\prod_{i \in I} D_i$ of topological groups in terms of projections $\pi_J(G)$ of $G$ to relatively small subproducts $PD_J = P\prod_{i \in J} D_i$.

Theorem 6.14. Suppose that $D = \prod_{i \in I} D_i$ is a product of topological groups and $\Sigma D \subset G \subset PD$. Then the group $G$ is reflexive iff the subgroup $\pi_J(G)$ of $PD_J$ is reflexive, for every set $J \subset I$ satisfying $|J| \leq \aleph_1$.

Proof. Necessity. Let $G$ be reflexive. Take any $J \subset I$ satisfying $|J| \leq \aleph_1$ and put $H = \pi_J(G)$. By Lemma 4.3 the restriction to $G$ of the projection $\pi_J: PD \to PD_J$ is an open homomorphism of $G$ onto $H$. Hence the reflexivity of $H$ follows from Proposition 5.9.

Sufficiency. Suppose that $\pi_J(G)$ is reflexive, for each $J \subset I$ with $|J| \leq \aleph_1$. Since $\Sigma D \subset G$, the equality $\pi_J(G) = D_J$ holds for each $J \in [I]^{\leq \omega}$. Therefore, according to Corollary 5.11 it suffices to show that every compact set $K \subset G^\wedge$ depends at most on countably many coordinates.

Suppose to the contrary that $G^\wedge$ contains a compact set $K$ which depends on uncountably many coordinates. Apply Lemma 6.3 to choose families $\{x_\eta : \eta < \omega_1\} \subset K$, $\{x_\eta : \eta < \omega_1\} \subset G$, and $\{J_\eta : \eta < \omega_1\} \subset [I]^{\leq \omega}$ satisfying conditions (1)–(4) of Proposition 6.2. Let $J = \bigcup_{\eta < \omega_1} J_\eta$. Then $J \subset I$ and $|J| \leq \aleph_1$. Hence the subgroup $H = \pi_J(G)$ of $PD_J$ is reflexive. Notice that by
Lemma 4.3] the restriction to $G$ of the homomorphism $π_J$ is open when considered as a mapping of $G$ onto $H$. Let $η < ω_1$. Since $χ_η$ does not depend on $I \setminus J_η$ and $J_η ⊂ J$, there exists a continuous character $ψ_η$ on $H$ such that $χ_η = ψ_η ∘ π_J|_G$. We put $Ψ = \{ψ_η : η < ω_1\}$.

Denote by $φ$ the continuous homomorphism $(π_J|_G)^\wedge$ of $H^\wedge$ to $G^\wedge$. Then $φ(ψ_η) = χ_η ∈ K$ for each $η < ω_1$, so $φ(Ψ) ⊂ K$. Since $H$ is a $P$-group, all compact subsets of $H$ are finite. Hence Corollary 5.8 implies that $φ$ is a topological isomorphism of $H^\wedge$ onto the subgroup $φ(H^\wedge)$ of $G^\wedge$.

Further, since the homomorphism $π_J|_G$ of $G$ onto $H$ is open, it follows from item (c) of Lemma 5.7 that $φ(H^\wedge)$ is a closed subgroup of $G^\wedge$. Therefore, $C = K \cap φ(H^\wedge)$ is a compact subset of $φ(H^\wedge)$ and $L = φ^\−1(C)$ is a compact subset of $H^\wedge$. It follows from $Ψ ⊂ H^\wedge$ and $φ(Ψ) ⊂ K$ that $Ψ ⊂ L$. The latter inclusion and the definition of the set $Ψ$ together imply that $L$ depends on uncountably many coordinates.

Indeed, suppose that for some countable set $A ⊂ J$, every element of $L$ does not depend on $J \setminus A$. In particular, $ψ_η$ does not depend on $J \setminus A$, for each $η < ω_1$. Since $χ_η = ψ_η ∘ π_J|_G$, we see that each $χ_η$ does not depend on $I \setminus A$. It follows from our choice of the families $\{χ_η : η < ω_1\}$, $\{x_η : η < ω_1\}$, and $\{J_η : η < ω_1\}$ (see conditions (2)–(4) of Proposition 6.2) that $\text{supp}(x_η) ⊂ J_η \setminus \bigcup_{ζ < η} J_ζ$ and $χ_η(x_η) ≠ 1$, for each $η < ω_1$. Since the sets $A_η = J_η \setminus \bigcup_{ζ < η} J_ζ$ are pairwise disjoint, there exists $η < ω_1$ such that $A \cap A_η = ∅$. Since $χ_η$ does not depend on $I \setminus A$, this implies that $χ_η(x_η) = 1$, which is a contradiction. We have thus proved that every compact subset $K$ of $G^\wedge$ depends at most on countably many coordinates and, therefore, $G$ is reflexive. □

Theorem 6.14 makes it possible to find many proper dense reflexive subgroups of big products of pseudo-feathered groups endowed with the $P$-modified topology:

**Corollary 6.15.** Suppose that $D = \prod_{i \in I} D_i$ is a product of pseudo-feathered Abelian groups and let

$$\Sigma_{N_1} D = \{x ∈ PD : |\text{supp}(x)| ≤ N_1\}.$$ 

Then every group $G$ with $Σ_{N_1} D ⊂ G ⊂ PD$ is reflexive.

**Proof.** According to Proposition 5.6 it suffices to show that $Σ_{N_1} D$ is reflexive. It is clear that $π_J(Σ_{N_1} D) = D_J = \prod_{i \in J} D_i$ for each $J ⊂ I$ with $|J| ≤ N_1$, where $π_J : D → D_J$ is the projection. By Theorem 6.14 the groups $PD_J$ are reflexive. One applies Theorem 6.14 to conclude that the group $Σ_{N_1} D$ is reflexive as well. □
In what follows we identify the additive group \( \mathbb{Z}_2 = \{0, 1\} \) with the multiplicative subgroup \( \{1, -1\} \) of \( \mathbb{T} \). Hence the dual group \( G^\wedge \) of every boolean \( P \)-group \( G \) is topologically isomorphic to a subgroup of \( \mathbb{Z}_2^G \). We will now show that the \( P \)-group \( II = P\mathbb{Z}_2^{\omega_1} \) contains proper dense reflexive subgroups of the form \( G_\xi \) defined after Theorem 2.

**Proposition 6.16.** Every compact set \( K \subset (G_\xi)^\wedge \) depends at most on countably many coordinates, where \( \xi \) is an ultrafilter on \( \omega_1 \) containing the complements to countable sets. Hence the group \( G_\xi \) is reflexive.

**Proof.** On the contrary, suppose that a compact set \( K \subset (G_\xi)^\wedge \) depends on uncountably many coordinates. We construct two sets \( \{\chi_\eta : \eta < \omega_1\} \subset K \) and \( \{J_\eta : \eta < \omega_1\} \subset [\omega_1]^{\leq \omega} \) satisfying the following conditions for all \( \eta < \omega_1 \):

(i) \( \chi_\eta \) does not depend on \( \omega_1 \setminus J_\eta \);

(ii) \( J_\zeta \subset J_\eta \) if \( \zeta < \eta \);

(iii) \( \eta \in J_\eta \);

(iv) \( \chi_\eta \) depends on the set \( \omega_1 \setminus (\{\eta\} \cup \bigcup_{\zeta < \eta} J_\zeta) \).

Let \( \chi_0 \in K \) be a nontrivial character. Take a countable set \( J_0 \subset \omega_1 \) such that \( 0 \in J_0 \) and \( \chi_0 \) does not depend on \( \omega_1 \setminus J_0 \). Suppose that for some \( \eta < \omega_1 \), the sequences \( \{\chi_\zeta : \zeta < \eta\} \subset K \) and \( \{J_\zeta : \zeta < \eta\} \subset [\omega_1]^{\leq \omega} \) have been defined to satisfy conditions (i)–(iv). Then we put \( T_\eta = \bigcup_{\zeta < \eta} J_\zeta \) and choose \( \chi_\eta \in K \) such that \( \chi_\eta \) depends on the set \( \omega_1 \setminus (T_\eta \cup \{\eta\}) \). Such a choice of \( \chi_\eta \) is possible since the set \( T_\eta \cup \{\eta\} \) is countable and \( K \) depends on uncountably many coordinates. Let \( J_\eta' \) be a countable subset of \( \omega_1 \) such that \( \chi_\eta \) does not depend on \( \omega_1 \setminus J_\eta' \). Then the set \( J_\eta = J_\eta' \cup T_\eta \cup \{\eta\} \) is countable and \( \chi_\eta \) does not depend on \( \omega_1 \setminus J_\eta \). Therefore, the sets \( \{\chi_\zeta : \zeta \leq \eta\} \) and \( \{J_\zeta : \zeta \leq \eta\} \) satisfy (i)–(iv) at the step \( \eta \).

For every \( A \in \xi \), we put \( F_A = \overline{\{\chi_\eta : \eta \in A\}}^{G_\xi} \) and \( \mathcal{C} = \{F_A : A \in \xi\} \). It follows from \( \chi_\eta \in K \) for all \( \eta < \omega_1 \) and the compactness of \( K \) that \( F_A \subset K \), for each \( A \in \xi \). Since \( \mathcal{C} \) is a family of closed subsets of the compact space \( \mathbb{T}^{G_\xi} \) with the finite intersection property, \( \mathcal{C} \) has non-empty intersection. Let \( \rho \) be a point in \( \bigcap \{F_A : A \in \xi\} \). Clearly, \( \rho \in K \), so \( \rho \) is continuous. Let \( J_\rho \) be a countable subset of \( \omega_1 \) such that \( \rho \) does not depend on \( \omega_1 \setminus J_\rho \).

Since \( \Sigma \) is a dense subgroup of both \( G_\xi \) and \( II = P\mathbb{Z}_2^{\omega_1} \), the characters \( \rho \) and \( \chi_\eta \) admit continuous extensions \( \overline{\rho} : II \to \mathbb{T} \) and \( \overline{\chi_\eta} : II \to \mathbb{T} \), for each \( \eta < \omega_1 \). Again, the density of \( G_\xi \) in \( II \) implies that \( \overline{\rho} \) does not depend on \( \omega_1 \setminus J_\rho \) and \( \overline{\chi_\eta} \) does not depend on \( \omega_1 \setminus J_\eta \).
Denote by 1 the element of $\Pi$ all of whose coordinates are equal to 1. For every $\eta < \omega_1$, let $H_\eta = \{ x \in \Pi : x(\eta) = 0 \}$ and take a character $\psi_\eta$ on $\Pi = \langle H_\eta, 1 \rangle$ defined by $\psi_\eta(x) = \overline{\chi}_\eta(x)$ and $\psi_\eta(x+1) = \overline{\chi}_\eta(x) + \overline{p}(1)$, for all $x \in H_\eta$. Since $\eta \in J_\eta$, we have $U(J_\eta) \subset H_\eta$. It then follows from the above definition that $\psi_\eta$ does not depend on $\omega_1 \setminus J_\eta$.

**Claim 1.** Put $T_\eta = \bigcup_{\zeta < \eta} J_\zeta$. For every $\eta < \omega_1$, the character $\psi_\eta$ depends on $\omega_1 \setminus T_\eta$.

**Proof of Claim 1.** Indeed, by (v) of the recursive construction, $\chi_\eta$ depends on $\omega_1 \setminus (\{ \eta \} \cup T_\eta)$. Hence there exists $x \in G_\xi \cap U(\{ \eta \} \cup T_\eta)$ such that $\chi_\eta(x) \neq 1$. Then $x \in H_\eta$ and $\psi_\eta(x) = \overline{\chi}_\eta(x) = \chi_\eta(x) \neq 1$, and we see that $\psi_\eta$ depends on $\omega_1 \setminus T_\eta$.

**Claim 2.** For all $\alpha < \omega_1$, $\overline{\psi} \in \{ \overline{\psi}_\eta : \alpha \leq \eta < \omega_1 \}^{\tau^\omega}$.

**Proof of Claim 2.** Fix $\alpha < \omega_1$ and take $\{g_1, \ldots, g_n\} \subset \Pi$. We can assume that there exists $m \leq n$ such that $\{g_1, \ldots, g_m\} \subset G_\xi$ and $\{g_{m+1}, \ldots, g_n\} \subset \Pi \setminus G_\xi$. Then $\{g_1, \ldots, g_{m}, g_{m+1}, \ldots, g_n\} \subset G_\xi$. Let $A = \omega_1 \setminus \bigcup_{i \leq m} \text{supp}(g_i)$, $B = \bigcap_{m < k \leq n} \text{supp}(g_k)$, and $C = A \cap B \cap \{ \alpha, \omega_1 \}$. It follows from our choice of $g_1, \ldots, g_n$ that $C \in \xi$. So, $\rho \in F_\xi$. Take $\eta \in C$ such that $\rho(g_i) = \chi_\eta(g_i)$ whenever $1 \leq i \leq m$ and $\rho(g_k + 1) = \chi_\eta(g_k + 1)$ whenever $m < k \leq n$. If $1 \leq i \leq m$, then $g_i(\eta) = 0$ because $\eta \in A$. So, $g_i \in H_\eta$ and $\psi_\eta(g_i) = \overline{\chi}_\eta(g_i) = \chi_\eta(g_i) = \rho(g_i) = \overline{\rho}(g_i)$. If $m < k \leq n$, then $(g_k + 1)(\eta) = 0$ because $\eta \in B$. So, $(g_k + 1) \in H_\eta$ and $\psi_\eta(g_k + 1) = \overline{\chi}_\eta(g_k + 1) = \chi_\eta(g_k + 1) = \rho(g_k + 1) = \overline{\rho}(g_k + 1)$. Since $\psi_\eta$ and $\overline{\rho}$ are homomorphisms and $\psi_\eta(1) = \overline{\rho}(1)$, we see that $\psi_\eta(g_k) = \overline{\rho}(g_k)$. Therefore, $\overline{\rho} \in \{ \psi_\eta : \alpha \leq \eta < \omega_1 \}^{\tau^\omega}$. This completes the proof of Claim 2.

Now, we have a character $\overline{\rho} \in \Pi^\omega$, a family of characters $\{ \psi_\eta : \eta < \omega_1 \} \subset \Pi^\omega$, and a family $\{J_\eta : \eta < \omega_1 \}$ of countable subsets of $\omega_1$. If follows from our definition of the characters $\psi_\eta$’s and the above conditions (i), (ii), and (v) that $\{ \psi_\eta : \eta < \omega_1 \}$ and $\{J_\eta : \eta < \omega_1 \}$ satisfy (a)–(c) of Lemma 6.7$^+$ (with $\psi_\eta$’s in place of $\chi_\eta$’s). Since $\overline{\rho} \in \bigcap_{\alpha < \omega_1} \{ \psi_\eta : \alpha \leq \eta < \omega_1 \}^{\tau^\omega}$, we are in position to use Proposition 6.2 to obtain a contradiction with the fact that $\overline{\rho}$ is continuous. This contradiction shows that the compact set $K$ depends at most on countably many coordinates. The reflexivity of $G_\xi$ now follows from Theorem 5.5.

To finish this section, we extend the conclusion of Proposition 6.16 to subgroups $G_\xi$ of the group $P\mathbb{Z}_2^\omega$, for any uncountable cardinal $\tau$.

**Theorem 6.17.** Let $\tau > \omega$ be a cardinal and $\xi$ an ultrafilter on $\tau$ containing the complements to countable sets. Then subgroup $G_\xi$ of $P\mathbb{Z}_2^\omega$ is reflexive.
Proof. According to Theorem 6.14 it suffices to verify that the subgroup \( \pi_J(G_{\xi}) \) of the group \( P\mathbb{Z}_2^J \) is reflexive, for every set \( J \subset \tau \) satisfying \( |J| \leq \aleph_1 \). Let us consider two possible cases.

Case 1. \( J \not\in \xi \). Then the definition of \( G_{\xi} \) implies that \( \pi_J(G_{\xi}) = \mathbb{Z}_2^J \), so the reflexivity of \( \pi_J(G_{\xi}) \) is immediate from Theorem 6.8.

Case 2. \( J \in \xi \). Put \( \eta = \{J \cap A : A \in \xi\} \). Then \( \eta \) is an ultrafilter on \( J \) containing the complements to countable sets. Further, the definition of \( G_{\xi} \) implies that the projection \( \pi_J(G_{\xi}) \) of \( G_{\xi} \) coincides with the subgroup \( G_{\eta} \) of \( P\mathbb{Z}_2^J \). Identifying \( J \) and \( \omega_1 \) and applying Proposition 6.16, we see that the group \( \pi_J(G_{\xi}) \) is again reflexive. \( \square \)

7. Non-reflexive \( P \)-groups

We would like to trace the border between reflexivity and non-reflexivity for \( P \)-groups \( G \) such that \( \Sigma D \subset G \subset PD \), where \( D = \prod_{i \in I} D_i \) is a product of discrete groups. Recall that by an old result of Leptin in [17] (see also [5, Example 17.11]), the subgroup 
\[
\sigma\mathbb{Z}_2^{\omega_1} = \{x \in \mathbb{Z}_2^{\omega_1} : \text{supp}(x) \text{ is finite}\}
\] of \( P\mathbb{Z}_2^{\omega_1} \) is not reflexive. We now extend this fact to some dense subgroups of the groups of the form \( PD \).

Let \( G \) be a subgroup of \( PD \) containing \( \Sigma D \). For each \( i \in I \), let \( \pi_i : G \to D_i \) be the projection, \( \pi_i(x) = x(i) \). For a set \( J \subset I \), we also put \( F_J = \{\varphi \circ \pi_i : i \in J, \varphi \in (D_i)^\wedge \} \cup \{1\} \), where \( 1 \) is the identity of \( G^\wedge \).

Lemma 7.1. For every \( J \in [I]^{\leq \omega} \), \( F_J \subset G^\wedge \).

Proof. Suppose that \( J \in [I]^{\leq \omega} \) and take any \( \rho \in F_J \). Then \( U(J) \) is an open set in \( PD \) containing \( 0_G \) such that \( \rho(U(J) \cap G) = \{1\} \). Therefore \( \rho \) is continuous. \( \square \)

Lemma 7.2. Let \( J \) be a nonempty subset of \( I \) and \( \psi \in F_J \), \( \psi \neq 1 \). Then \( \psi \) is continuous as a character on \( G \) if and only if there exists \( x \in \Sigma D \) such that \( \psi(x) \neq 1 \).

Proof. Suppose that \( \psi \) is continuous. Since \( \psi \neq 1 \) and \( \Sigma D \) is dense in \( G \), there is a point \( x \in \Sigma D \) such that \( \psi(x) \neq 1 \).

Conversely, take \( x \in \Sigma D \) such that \( \psi(x) \neq 1 \) and write \( F_J = F_J \cap \text{supp}(x) \cup F_J \setminus \text{supp}(x) \). Since \( (\chi \circ \pi_i)(x) = 1 \) for all \( i \in J \setminus \text{supp}(x) \) and all \( \chi \in (D_i)^\wedge \), we have that \( \psi \notin F_J \setminus \text{supp}(x) \). Then \( \psi \in F_J \cap \text{supp}(x) \). It now follows from Lemma 7.1 that \( \psi \) is continuous. \( \square \)
Proposition 7.3. Let \( D = \prod_{i \in I} D_i \) be a product of nontrivial discrete Abelian groups. Then the subgroup \( \Sigma = \Sigma D \) of \( PD \) is not reflexive provided that \( |I| > \omega \). Furthermore, the bidual group \( \Sigma^{\wedge \wedge} \) is discrete.

Proof. Given \( \psi \in F_I \), \( \psi \neq 1 \), there exists \( x \in \Sigma \) such that \( \psi(x) \neq 1 \). It now follows from Lemma 7.2 that \( F_I \subset \Sigma^\wedge \). The set \( F_I \) is compact as a closed subset of \( \mathbb{T}^\Sigma \). Since \( F_I \) does not depend on countably many coordinates (it actually depends on every index \( i \in I \)), the group \( \Sigma \) is not reflexive by Theorem 5.5. It is easy to see that \( F_I \) generates a dense subgroup of \( \Sigma^\wedge \), so \( (F_I)^{\perp} \) contains only the neutral element of \( G^{\wedge \wedge} \). Hence the bidual group \( \Sigma^{\wedge \wedge} \) is discrete. \( \square \)

Sets of the form \( F_J \), with \( J \neq I \), can also be used to show that some subgroups larger than \( \Sigma = \Sigma D \) are not reflexive. This is done in Lemma 7.3 for groups of the form \( G_L = \langle \Sigma, L \rangle \) with \( L \subset PD \) satisfying \( |I \setminus \bigcup_{x \in L} \text{supp}(x)| \geq \omega_1 \).

Lemma 7.4. Let \( I \) be a subset of \( PD \) and \( J = I \setminus \bigcup_{x \in L} \text{supp}(x) \). If \( |J| > \omega \), then \( F_J = \{ \varphi \circ \pi_i : i \in J, \varphi \in (D_i)^\wedge \}^{D_L} \cup \{1\} \) is a compact subset of \( (G_L)^\wedge \).

Proof. Take \( \psi \in F_J \), \( \psi \neq 1 \). Assume that \( \psi(z) = 1 \) for all \( z \in \Sigma \). Take \( g \in G \) such that \( \psi(g) \neq 1 \) and write \( g = z + n_1 x_1 + \ldots + n_k x_k \), where \( z \in \Sigma \), \( x_1, \ldots, x_k \in L \), and \( n_1, \ldots, n_k \in \mathbb{Z} \). Since \( \psi(g) \neq 1 \) and \( U = \{ y \in \mathbb{T}^G : y(g) \neq 1 \} \) is an open neighbourhood of \( \psi \), there are \( j \in J \) and \( \varphi \in (D_j)^\wedge \) such that \( (\varphi \circ \pi_j)(g) \in U \). Then \( (\varphi \circ \pi_j)(g) \neq 1 \) which is a contradiction because

\[
(\varphi \circ \pi_j)(g) = (\varphi \circ \pi_j)(n_1 x_1 + \ldots + n_k x_k) = \varphi(n_1 x_1(j) + \ldots + n_k x_k(j)) = \varphi(0_G) = 1.
\]

So, there is \( z \in \Sigma \) such that \( \psi(z) \neq 1 \). Now the continuity of \( \psi \) follows from Lemma 7.2. \( \square \)

Theorem 7.5. Let \( I \) be a subset of \( PD \) such that the set \( J = I \setminus \bigcup_{x \in L} \text{supp}(x) \) is uncountable. Then the subgroup \( G = \langle \Sigma, L \rangle \) of \( PD \) is not reflexive.

Proof. Put \( F_J = \{ \varphi \circ \pi_i : i \in J, \varphi \in (D_i)^\wedge \}^{D_L} \). It follows from Lemma 7.4 that \( F_J \subset G^\wedge \). Since \( F_J \) depends on every coordinate \( i \in J \), we conclude that \( G \) is not reflexive in view of Theorem 5.5. \( \square \)

The preceding results make use of sets of the form \( F_J \) to see that some subgroups \( G \) with \( \Sigma D \subset G \subset PD \) are not reflexive. Sets of this sort were already used by Leptin [17] to evidence the nonreflexivity of \( \sigma \mathbb{Z}_2^{\omega_1} \). We see next that \( F_J \) may not be contained in \( G^\wedge \), for some \( G = \langle \Sigma D, a \rangle \) and \( J \subset I \). This makes necessary a different approach to show, as we do in Theorem 7.8, that these groups are not reflexive.
Let $\Pi = PD$, $\Sigma = \Sigma D$, and $a$ be a point in $\Pi \setminus \Sigma$ such that $na \in \Sigma$ for some integer $n > 1$. We put $G = \langle \Sigma, a \rangle$, $J = \text{supp}(a)$, and $F_J = \{ \varphi \circ \pi_i : i \in J, \varphi \in (D_i)^\wedge \}^{\Sigma \Gamma}$. Note that $G = \{ g + ka : g \in \Sigma, 0 \leq k < n \}$.

**Lemma 7.6.** If $G = \langle \Sigma, a \rangle$, then $F_J$ is not contained in $G^\wedge$.

**Proof.** Since $na \in \Sigma$, there exists a countable set $C \subset I$ such that $na(i) = 0$, for each $i \in I \setminus C$. Therefore, we can find a divisor $m$ of $n$ with $m > 1$ and an uncountable set $A \subset J \setminus C$ such that the order of $a(i)$ equals $m$ for each $i \in A$.

Let us define $\psi$ in $\text{Hom}(G, \mathbb{T})$ as follows: $\psi(g + la) = e^{(2\pi l/m)i}$ for all $g \in \Sigma$ and $l \in \mathbb{Z}$. Since $\psi(g) = 1$ for each $g \in \Sigma$ and $\psi(l_1a) = \psi(l_2a)$ whenever $m$ divides $l_1 - l_2$, our definition of $\psi$ is correct. We now show that $\psi \in F_A$. For this, let $g_k$ be points in $\Sigma$, $l_k$ be integers, and $V_k$ open sets in $\mathbb{T}$ such that $\psi(g_k + l_k a) \in V_k$ for all $k \leq N$, where $N \in \mathbb{N}$. Since $| \bigcup_{0 \leq k \leq N} \text{supp}(g_k) | \leq \omega$ and $| A | \geq \omega_1$, we can choose $j \in A \setminus \bigcup_{0 \leq k \leq N} \text{supp}(g_k)$. Since the order of $a(j)$ equals $m$, there is a character $\rho \in (D_j)^\wedge$ such that $\rho(a(j)) = e^{(2\pi l/m)i}$. Now it is easy to see that $(\rho \circ \pi_j)(g_k + l_k a) = \psi(g_k + l_k a) \in V_k$ whenever $0 \leq k \leq N$. Therefore $\psi \in F_A \subset F_J$. The discontinuity of $\psi$ follows from Lemma 7.2 because $\psi \neq 1$ and $\psi(g) = 1$ for all $g \in \Sigma$. This proves that $\psi \in F_J \cap G^\wedge \neq \emptyset$. □

By Lemma 7.6 the argument used in Proposition 7.3 for $\Sigma D$ does not work for $\langle \Sigma, \bar{T} \rangle$, where $\Sigma = \Sigma \mathbb{Z}_{2^\omega}$ and $\bar{T}$ is the element of $\mathbb{Z}_{2^\omega}$ all of whose coordinates are equal to 1. However, we show in Theorem 7.3 that the group $\langle \Sigma, \bar{T} \rangle$ is not reflexive either.

**Lemma 7.7.** Let $\Pi = PD$ and $\Sigma = \Sigma \Pi$. Suppose that a subgroup $L$ of $\Pi$ has the property that for each $\alpha < \omega_1$, there exists an ordinal $\beta(\alpha)$ such that $\alpha < \beta(\alpha) < \omega_1$ and the restriction to $L$ of the projection $p_{J(\alpha)} : \Pi \to \mathbb{Z}_2^{J(\alpha)}$ is injective, where $J(\alpha) = \beta(\alpha) \setminus \alpha$. Then the subgroup $G = \Sigma + L$ of $\Pi$ is not reflexive.

**Proof.** For every $\alpha < \omega_1$, let $\gamma(\alpha) = \beta(\alpha) + 1$ and decompose $\mathbb{Z}_2^{\gamma(\alpha)} = H_\alpha \oplus L_\alpha$, with $L_\alpha = p_{\gamma(\alpha)}(L)$, where $p_{\gamma(\alpha)} : \Pi \to \mathbb{Z}_2^{\gamma(\alpha)}$ is the projection. Observe that we can (and so we will) choose $H_\alpha$ to satisfy the following condition:

$$\{ x \in \mathbb{Z}_2^{\gamma(\alpha)} : x(\eta) = 0 \text{ for all } \eta \in J(\alpha) \} \subset H_\alpha. \tag{1}$$

We also choose, for every $\alpha < \omega_1$, a homomorphism $\psi_\alpha : \mathbb{Z}_2^{\gamma(\alpha)} \to \mathbb{Z}_2$ such that $\psi_\alpha(L_\alpha) = \{ 0 \}$ and $\psi_\alpha|H_\alpha = \pi_{\beta(\alpha)}|H_\alpha$, where $\pi_{\beta(\alpha)}$ is the projection of $\mathbb{Z}_2^{\gamma(\alpha)}$ to the $\beta(\alpha)$th factor $(\mathbb{Z}_2)^{\beta(\alpha)}$. Let us put $\chi_\alpha = \psi_\alpha \circ p_{\gamma(\alpha)}$. Since both homomorphisms $\psi_\alpha$ and $p_{\gamma(\alpha)}$ are continuous, so is $\chi_\alpha$. 


For every \( \alpha \leq \omega_1 \), let \( E_\alpha = \{ \chi_\sigma : \sigma < \alpha \}^2 \). We claim that \( E_\omega \subset G^\wedge \). Indeed, let \( \rho \in E_\omega \). If \( \rho \neq 0 \), there is \( g \in G \) with \( \rho(g) = 1 \). Take \( g_1 \in \Sigma \) and \( g_2 \in L \) such that \( g = g_1 + g_2 \). If \( \rho(g_2) = 1 \), then there exists \( \alpha < \omega_1 \) such that \( \chi_\alpha(g_2) = 1 \) which is not possible because \( \chi_\alpha(g_2) = \psi_\alpha(p_{\gamma(\alpha)}(g_2)) = 0 \).

We thus have \( \rho(g_1) = 1 \). Now, mimicking the proof of Lemma 7.2, we may take \( \alpha < \omega_1 \) such that \( \supp(g_1) \subset \alpha \), and write

\[
E_\omega = E_\alpha \bigcup \{ \chi_\sigma : \sigma \geq \alpha \}^2 \subset G^\wedge.
\]

Take any \( \sigma \geq \alpha \). It follows from (1) that \( p_{\gamma(\sigma)}(g_1) \in H_\sigma \) and hence \( \chi_\sigma(g_1) = \psi_\sigma(p_{\gamma(\sigma)}(g_1)) = \pi_{\beta(\sigma)}(p_{\gamma(\sigma)}(g_1)) = 0 \). Since \( \sigma \geq \alpha \) is arbitrary, we conclude that \( \rho \notin \{ \chi_\sigma : \sigma \geq \alpha \}^2 \) and, therefore, \( \rho \in E_\alpha \). We have thus that \( E_\omega = \bigcup_{\alpha < \omega_1} E_\alpha \). Since all elements of \( E_\alpha \) are continuous (they all do not depend on \( \omega_1 \setminus \gamma(\alpha) \)), we see that \( E_\omega \) is a compact subset of \( G^\wedge \).

Given \( \alpha < \omega_1 \), we define \( b_\alpha \in \Sigma \) by \( b_\alpha(\gamma) = 0 \) if \( \gamma \neq \beta(\alpha) \) and \( b_\alpha(\beta(\alpha)) = 1 \). This implies that \( p_{\gamma(\alpha)}(b_\alpha) \in H_\alpha \) and, therefore, \( \chi_\alpha(b_\alpha) = \psi_\alpha(p_{\gamma(\alpha)}(b_\alpha)) = \pi_{\beta(\alpha)}(p_{\gamma(\alpha)}(b_\alpha)) = b_\alpha(\beta(\alpha)) = 1 \).

It follows that \( \chi_\alpha \) depends on the index \( \beta(\alpha) \) which in its turn implies that \( E_\omega \) depends on uncountably many coordinates. By Theorem 5.5, the group \( G \) is not reflexive.

**Theorem 7.8.** Let \( \tau \) be an uncountable cardinal, \( \Pi = \mathbb{PZ}_2^T \), and \( \Sigma = \Sigma \Pi \). Then, for every countable subgroup \( L \) of \( \Pi \), the group \( G = \Sigma + L < \Pi \) is not reflexive.

**Proof.** Let \( L_0 = L \cap \Sigma \). Since every subgroup of the boolean group \( L \) is a direct summand, there exists a subgroup \( L_1 \) of \( L \) such that \( L = L_0 \oplus L_1 \). Since \( L_0 \subset \Sigma \), we see that \( G = \Sigma + L = \Sigma + L_0 + L_1 = \Sigma + L_1 \). It follows from our definition of \( L_1 \) that the intersection \( \Sigma \cap L_1 \) is trivial, so the set \( \supp(x) \) is uncountable, for each \( x \in L_1 \) distinct from \( 0_G \).

Take a subset \( J \) of \( \tau \) such that \( |J| = \aleph_1 \) and \( |J \cap \supp(x)| = \aleph_1 \) for each \( x \in L_1 \), \( x \neq 0_G \).

By Theorem 6.14 it suffices to show that the subgroup \( \pi_J(G) \) of \( \mathbb{PZ}_2^J \) is not reflexive. Since \( \pi_J(G) = \pi_J(\Sigma) + \pi_J(L_1) \) and \( \pi_J(\Sigma) = \Sigma \mathbb{Z}_2^J \), we can assume without loss of generality that \( \tau = J = \omega_1 \). Hence \( G \) is a subgroup of \( \Pi = \mathbb{PZ}_2^\omega \).

In view of Lemma 7.1, it suffices to show that for every \( \alpha < \omega_1 \), there is a countable ordinal \( \beta(\alpha) > \alpha \) such that the restriction to \( L_1 \) of the projection \( p_{(\alpha, \beta(\alpha))} \) is injective. Given an ordinal \( \alpha < \omega_1 \), we take an element \( \beta(x) \in \supp(x) \setminus \alpha \), for each \( x \in L_1 \setminus \{0_G\} \), and put

\[
B = \{ \beta(x) : x \in L_1, \ x \neq 0_G \}.
\]
Then \( |B| \leq |L_1| \leq \omega \), so there exists a countable ordinal \( \beta(\alpha) > \alpha \) such that \( B \subseteq \beta(\alpha) \). Put \( J(\alpha) = \beta(\alpha) \setminus \alpha \). It is clear that the restriction to \( L_1 \) of the projection \( p_{J(\alpha)} \) is one-to-one since \( B \subset J(\alpha) \) and \( B \) intersects the set \( \text{supp}(x) \setminus \alpha \), for each \( x \in L_1 \) distinct from \( 0_G \). This finishes the proof of the theorem.

\[ \square \]

8. Problem section

Here we present several problems whose solutions can substantially improve our understanding of the duality theory for \( P \)-groups. The first of them arises in an attempt to extend Theorem 6.8 to arbitrary products of reflexive \( P \)-groups:

Problem 1. Let \( \Pi = \prod_{i \in I} G_i \) be the product of a family of reflexive \( P \)-groups. Is the group \( P\Pi \) reflexive?

According to Proposition 6.7 it suffices to consider the case when the index set \( I \) in the above problem is countable. One can try to prove (or refute) a more general form of the above problem inspired by Proposition 6.11:

Problem 2. Let \( G \) be a reflexive topological group. Is the group \( PG \) then reflexive?

A direct verification shows that every reflexive \( P \)-group constructed so far contains a discrete (hence closed) subgroup of cardinality \( 2^{\omega} \). This explains the origin of the following problem:

Problem 3. Does there exist a nondiscrete reflexive Lindelöf \( P \)-group?

In the next problem we pretend to generalize Theorem 6.17:

Problem 4. Let \( \tau \) be an uncountable cardinal and \( G \) be a subgroup of \( \Pi = P\mathbb{Z}_2^\tau \) such that \( |\Pi/G| < \omega \) (or \( |\Pi/G| \leq \omega \)). Is \( G \) reflexive? What if, additionally, \( G \) contains \( \Sigma \Pi \)?

We do not know whether Theorem 6.8 extends to bigger subgroups of \( P\mathbb{Z}_2^{\omega_1} \):

Problem 5. Is is true that the subgroup \( \Sigma + L \) of \( \Pi = P\mathbb{Z}_2^{\omega_1} \) fails to be reflexive, for any subgroup \( L \) of \( \Pi \) satisfying \( |L| \leq \aleph_1 \)?
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