The Implementation of the Trapezoidal Fuzzy Number toward the Solution of the A Fuzzy Inventory Model with Shortages

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Abstract—In this paper, fuzzy inventory model with shortages has been considered. Trapezoidal fuzzy number with Weibull distribution is used to fuzzify the data of an inventory model. All the parameters used in the inventory model is converted to trapezoidal fuzzy numbers using function principle. We find the estimate of the total inventory cost by defuzzifying the trapezoidal fuzzy numbers using the the graded mean integration. The optimum total cost for the inventory model is found. Keywords—Inventory · Trapezoidal fuzzy number · Defuzzification

1. Introduction
Inventory models are the models which is used to determine the optimum levels of the inventories which has to be maintained during the various stages of production, storing, ordering and managing the goods.
Inventory models deals with certainty and uncertainty of demand which occurs before and after the production of goods. The inventories during manufacturing can be classified as raw materials, work-in-progress and finished goods. After finishing the product, we have two costs that deals with holding the inventories. They are named as ordering costs and carrying costs.
There are two major models in the world of inventory. One is the deterministic model which deals with no uncertainty of demand and replenishment of inventories. The Probabilistic model which deal with uncertainty with demand pattern and lead time of the inventories.
Inventory models with and without Back orders are considered by Yao et al.[24] in fuzzy environment. Trapezoidal number is used to fuzzify the order quantity q. With the help of centroid strategy along with the extension principle, they found the membership functions of fuzzy total cost.

2. Preliminaries
“Definition 2.1: A Weibull distribution is defined as a random variable X is said to have a Weibull distribution with parameters α and β (α > 0; β > 0) if the probability density function of X is

\[ f(x; \alpha, \beta) = \begin{cases} \frac{\alpha}{\beta} x^{\alpha-1} e^{-(x/\beta)^\alpha} & x \geq 0 \\ 0 & x < 0 \end{cases} \]

where α is a shape parameter and β is a scale parameter.”
3. Methodology

Graded Mean Integration by Chen and Hsieh[6] is used to defuzzify the fuzzy number which is based on the mean integral value with h-level. The generalised fuzzy number is described as follows.

Suppose $\tilde{X}$ is the fuzzy number considered then $\tilde{X} \subseteq \mathbb{R}$. The following conditions are met by the membership function $\mu_{\tilde{X}}$ of $\tilde{X}$:

i) $\mu_{\tilde{X}}: \mathbb{R} \rightarrow [0,1]$,  
ii) $\mu_{\tilde{X}} = 0, -\infty < x \leq x_1$,  
iii) $\mu_{\tilde{X}} = L(x)$ is strictly increasing on $[x_1, x_2]$,  
v) $\mu_{\tilde{X}} = R(x)$ is strictly decreasing on $[x_3, x_4]$,  
vi) $\mu_{\tilde{X}} = 0, x_4 \leq x < \infty$,

Where $0 < w_X \leq 1$, and $x_1, x_2, x_3$, and $x_4$ are real numbers.

Also, the generalised fuzzy number is denoted by $\tilde{X} = (x_1, x_2, x_3, x_4; w_X)_{LR}$.

When $w_X = 1$, it can be simplified as $\tilde{X} = (x_1, x_2, x_3, x_4)_{LR}$.

Also, upon referring to and employing the Graded Mean Integration Representation approach, we have

$L^{-1}$ and $R^{-1}$ are as the inverted functions linked to L and R in that order. For the generalised fuzzy number, its associated graded mean h-level value of in

$$\tilde{X} = (x_1, x_2, x_3, x_4; w_X)_{LR},$$

Then the Graded Mean Integration Representation of $\tilde{X}$ is $P(\tilde{X})$ with graded $w_X$, where

$$P(\tilde{X}) = \int_0^{w_X} \frac{k(L^{-1}(k)+R^{-1}(k))}{2} x_3 R^{-1}(k) x_4$$

In the proposed fuzzy inventory model, we will now represent the fuzzified number as the parameter type. Let us assume that $\tilde{Y}$ as a trapezoidal fuzzy number which is represent as $\tilde{Y} = (y_1, y_2, y_3, y_4)$. Calculation of the Graded Mean Integration of $\tilde{Y}$ is given by the formula (3.1) as

$$P(\tilde{Y}) = \int_0^{k} \left(\frac{y_1+y_4+(y_2-y_1-y_4)y}{2}\right) dk / \int_0^{1} k dk$$
4. Function principle's arithmetical operations

Arithmetic operations on trapezoidal fuzzy number is done using Function. The following are the arithmetical operations on fuzzy numbers done under the function principle. Suppose \( \tilde{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \) and \( \tilde{\theta} = (\Theta_1, m_2, m_3, m_4) \) are the two trapezoidal fuzzy numbers. Then,

1. The addition of \( \tilde{\lambda} \) and \( \tilde{\theta} \) is \( \tilde{\lambda} \oplus \tilde{\theta} = (\lambda_1 + \Theta_1, \lambda_2 + \Theta_2, \lambda_3 + \Theta_3, \lambda_4 + \Theta_4) \), where \( \lambda_1, \Theta_1, \lambda_2, \Theta_2, \lambda_3, \Theta_3, \lambda_4 \) and \( \Theta_4 \) are any \( \mathbb{R} \).
2. The multiplication of \( \tilde{\lambda} \) and \( \tilde{\theta} \) is \( \tilde{\lambda} \otimes \tilde{\theta} = (\phi_1, \phi_2, \phi_3, \phi_4) \), where

\[
T = \{\lambda_1 \Theta_1, \lambda_2 \Theta_1, \lambda_3 \Theta_1, \lambda_4 \Theta_1\}, \quad T_1 = \{\lambda_2 \Theta_2, \lambda_3 \Theta_2, \lambda_4 \Theta_2\}, \quad \phi_1 = \text{min} T, \quad \phi_2 = \text{max} T_1, \quad \phi_3 = \text{max} T_1, \quad \phi_4 = \text{min} T_1.
\]

If \( \lambda_1, \Theta_1, \lambda_2, \Theta_2, \lambda_3, \Theta_3, \lambda_4 \) and \( \Theta_4 \) are \( \mathbb{R} \) which are \( \geq 0 \),
then \( \tilde{\lambda} \otimes \tilde{\theta} = (\lambda_1 \Theta_1, \lambda_2 \Theta_2, \lambda_3 \Theta_3, \lambda_4 \Theta_4) \).
3. \( -\tilde{\theta} = (-\Theta_4, -\Theta_3, -\Theta_2, -\Theta_1) \), then the subtraction of \( \tilde{\lambda} \) and \( \tilde{\theta} \) is \( \tilde{\lambda} \ominus \tilde{\theta} = (\lambda_1 - \Theta_4, \lambda_2 - \Theta_3, \lambda_3 - \Theta_2, \lambda_4 - \Theta_1) \), Where \( \lambda_1, \Theta_1, \lambda_2, \Theta_2, \lambda_3, \Theta_3, \lambda_4 \) and \( \Theta_4 \) are \( \mathbb{R} \).
4. \( \frac{\tilde{\lambda}}{\tilde{\theta}} = \frac{1}{\phi_4 \Theta_4} \frac{1}{\phi_3 \Theta_3} \frac{1}{\phi_2 \Theta_2} \frac{1}{\phi_1 \Theta_1} \) , where \( \phi_1, \phi_2, \phi_3, \phi_4, \Theta_1, \Theta_2, \Theta_3, \Theta_4 \) and \( \Theta_4 \) which are \( \geq 0 \), then the quotient of \( \tilde{\lambda} \) and \( \tilde{\theta} \) is \( \tilde{\lambda} \ominus \tilde{\theta} = (\frac{\lambda_1}{\Theta_4}, \frac{\lambda_2}{\Theta_3}, \frac{\lambda_3}{\Theta_2}, \frac{\lambda_4}{\Theta_1}) \).

5. Symbols used in the model

- \( \tilde{\rho} \) : Fuzzy demand rate
- \( \tilde{Q} \) : The fuzzy order quantity during a cycle of length \( T \)
- \( \tilde{Q}_1 \) : The positive inventory fuzzy level at time \( t \), \( 0 \leq t \leq t_1 \)
- \( \tilde{Q}_2 \) : The negative inventory fuzzy level at time \( t \), \( t_1 \leq t \leq T \)
- \( \tilde{S} \) : Fuzzy shortage cost / time unit
- \( \tilde{A} \) : Fuzzy ordering cost / order
- \( \tilde{T} \) : Fuzzy total cost / time unit
- \( \tilde{\lambda}, \tilde{\theta} \) : Fuzzy numbers

Trapezoidal Fuzzy Numbers:

\( \tilde{\rho} = (r_1, r_2, r_3, r_4) \) : Fuzzy Demand rate
\( \tilde{\ell} = (l_1, l_2, l_3, l_4) \) : Fuzzy number
\( \tilde{m} = (m_1, m_2, m_3, m_4) \) : Fuzzy number
\( \tilde{A} = (k_1, k_2, k_3, k_4) \) : Fuzzy ordering cost / order

The above are the representation of generalized fuzzy numbers in trapezoidal format.

Mathematical Formulation:

During the positive stock interval \([0, t_1]\) and shortage interval \([t_1, T]\), the rate of change of the inventory is given by the differential equations as follows.

\[
\frac{d\tilde{Q}_1(t)}{dt} + \tilde{\rho}(t) = -\tilde{\rho} \quad \text{(5.1)}
\]

\[
\frac{d\tilde{Q}_2(t)}{dt} = -\tilde{\rho} \quad \text{(5.2)}
\]
With the boundary conditions
\[ Q_1(t) = Q_2(t) = 0 \text{ at } t = t_1 \text{ and } Q_1(t) = IM \text{ and } Q_2(t) = IB \text{ at } t = T \] ------ (5.3) (5.4)

6. Mathematical solution for the model

Case I: Inventory Level Without Shortage:
The inventory diminishes due to demand in the interval \([0,t_1]\). Therefore, level of the inventory at any
time in the interval \([0,t_1]\) is described by differential equation

From equation (5.1)
\[ \frac{dQ_1(t)}{dt} + \varphi(t)\dot{Q}_1(t) = -\bar{R} \] ------ (6.1)

With the boundary conditions are \(Q_1(0) = IM, Q_2(t_1) = 0\)

From equation (5.3)
\[ \ddot{Q}_1(t) = \bar{R} \left[ t_1 - t + \alpha t^{\beta + 1} + e^{\alpha t^{\beta + 1}} \right] \] ------ (6.2)

Case II: Inventory Level with Shortage:
Differential equation is used to describe the state of inventory during \([t_1,T]\).

From equation (5.2)
\[ \frac{d\ddot{Q}_2(t)}{dt} = -\ddot{R} \]
\[ \frac{d\ddot{Q}_1(t)}{dt} = (r_4, r_3, r_2, r_1) \]

where \(t_1 < t < T\)

We have the boundary conditions as \(Q_2(t_1) = 0\) and \(Q_2(t) = IB\)

\[ \dot{Q}_2(t) = -\bar{R} (t - t_1) \]

The solution of differential equation, from equation (5.4) is

\[ Q_2(t) = \bar{R} (t_1 - t) \]
\[ Q_2(t) = (r_1, r_2, r_3, r_4)(t_1 - t) \]

Therefore, for one renewal cycle the total cost has the following elements,

Holding cost:
\[ \bar{H}C = \int_0^{t_1} H(t)\dot{Q}_1(t)dt \]

From the equation (6.2)
\[ \bar{H}C = IR \left[ \frac{t_1^2}{2} + \frac{a \beta t_1^{\beta + 2}}{(\beta + 1)(\beta + 2)} \right] + \bar{m}\bar{R} \left[ \frac{t_1^3}{6} + \frac{a \beta t_1^{\beta + 3}}{2(\beta + 2)(\beta + 3)} \right] \] ------ (6.3)

\[ \bar{H}C = \left\{ \begin{array}{l}
(\bar{r}_1 t_1^2, \bar{r}_2 t_1^2, \bar{r}_3 t_1^2, \bar{r}_4 t_1^2) \\
+ (\bar{m}_1 t_1^2, \bar{m}_2 t_1^2, \bar{m}_3 t_1^2, \bar{m}_4 t_1^2) \end{array} \right\} \] ------ (6.4)

Shortage Cost during \([t_1,T]\):
\[
\overline{SC} = \pi \int_{t_1}^{T} \tilde{R}(t_1 - t) dt
\]

\[
\overline{SC} = \frac{1}{2} \pi \tilde{R}(T - t_1)^2 \quad \text{---------(6.5)}
\]

\[
\overline{SC} = \frac{1}{2} \pi (r_1, r_2, r_3, r_4)(T - t_1)^2 \quad \text{---------(6.6)}
\]

Ordering Cost per order:
\[
\tilde{A} = (k_1, k_2, k_3, k_4) \quad \text{---------(6.7)}
\]

Total cost function per time unit:
\[
\overline{TC}(t_1, T) = \frac{1}{T} [\overline{HC} + \overline{SC} + \tilde{A}]
\]

From equations (6.3), (6.5) and (6.7)
\[
\overline{TC}(t_1, T) = \frac{1}{T} \left\{ \tilde{m} \tilde{R} \otimes \left[ \frac{t_1^2}{2} + \frac{\alpha \beta t_1^{\beta + 2}}{(\beta + 1)(\beta + 2)} \right] \oplus \tilde{R} \otimes \left[ \frac{t_1^3}{6} + \frac{\alpha \beta t_1^{\beta + 3}}{2(\beta + 2)(\beta + 3)} \right] \right\}
\]

From equations (6.4), (6.6) and (6.8)
\[
\overline{TC}(t_1, T) = \frac{1}{T} \left\{ \left( r_1 \alpha 1, r_2 \alpha 2, r_3 \alpha 3, r_4 \alpha 4 \right) \left[ \frac{t_1^2}{2} + \frac{\alpha \beta t_1^{\beta + 2}}{(\beta + 1)(\beta + 2)} \right] \right\} + \left( r_1 \theta 1, r_2 \theta 2, r_3 \theta 3, r_4 \theta 4 \right) \left[ \frac{t_1^3}{6} + \frac{\alpha \beta t_1^{\beta + 3}}{2(\beta + 2)(\beta + 3)} \right] \right\}
\]

The Necessary condition for the total cost per time unit to be minimized is
\[
\frac{\partial \overline{TC}}{\partial t_1} = 0 \text{ and } \frac{\partial \overline{TC}}{\partial T} = 0
\]

Given
\[
\left( \frac{\partial^2 \overline{TC}}{\partial t_1^2} \right) \left( \frac{\partial^2 \overline{TC}}{\partial T^2} \right) - \left( \frac{\partial^2 \overline{TC}}{\partial t_1 \partial T} \right)^2 > 0 \text{ and } \left( \frac{\partial^2 \overline{TC}}{\partial T^2} \right) > 0
\]

7. Numerical example and sensitivity study
In this paper, we consider an inventory problem with variables with correct units as follows:
R=2000, \( \lambda = 2.0 \), \( \theta = 1.5 \), \( C = 10 \), \( O = 50 \), \( \alpha = 0.15 \), \( \beta = 1.5 \), \( A = 50 \).

Let us convert the above-mentioned ordinary crispnumbers to trapezoidal fuzzy numbers using, “greater or less than Y” or “about Y”:
1. “greater or less than X” = (0.92Y, 0.94Y, 1.02Y, 1.11Y), and
2. “about X” = (0.92Y, Y, Y, 1.02Y)

Then by the above injunction, the fuzzy parameters is modified as follows:
i) Fuzzy Demand rate, \( R_0 \)
ii) Fuzzy ordering cost, \( A = \)
iii) Fuzzy numbers, $\lambda = \text{"greater or less than 2"} = (1.84, 1.88, 2.04, 2.2)$

$\theta = \text{"greater or less than 1.5"} = (1.38, 1.41, 1.53, 1.55)$

iv) $\alpha = \text{"greater or less than 0.15"} = (0.13, 0.14, 0.15, 0.16)$

$\beta = \text{"greater or less than 1.5"} = (1.3, 1.4, 1.5, 1.6)$

Substituting the above values in the equations

Table 1: Effect of changes in limitation of the inventory problem

| Parameters | %Change | R | TC |
|------------|---------|---|----|
| $+40\%$    | 4156    | 70.4151 | 7205 |
| $+20\%$    | 9571    | 70.3380 | 0196 |
| $-20\%$    | 1837    | 70.0471 | 0639 |
| $-40\%$    | 1837    | 70.0471 | 0639 |

| %Change | $\alpha$ | TC |
|---------|---------|----|
| $+40\%$ | 5009    | 70.3120 | 0099 |
| $+20\%$ | 6363    | 70.3072 | 0667 |
| $-20\%$ | 8646    | 70.2972 | 0613 |
| $-40\%$ | 1837    | 70.2972 | 0613 |

| %Change | $\beta$ | TC |
|---------|---------|----|
| $+40\%$ | 8828    | 70.2768 | 0736 |
| $+20\%$ | 6363    | 70.2825 | 0357 |
| $-20\%$ | 8646    | 70.3636 | 0558 |
| $-40\%$ | 1837    | 70.3636 | 0558 |

| %Change | $\lambda$ | TC |
|---------|-----------|----|
| $+40\%$ | 4479      | 70.4491 | 0165 |
| $+20\%$ | 5301      | 70.3754 | 0871 |
| $-20\%$ | 1973      | 70.2297 | 1249 |
| $-40\%$ | 5064      | 70.1575 | 4767 |

| %Change | $\theta$ | TC |
|---------|---------|----|
| $+40\%$ | 196      | 70.3029 | 0545 |
| $+20\%$ | 2987     | 70.3029 | 0533 |
| $-20\%$ | 5194     | 70.3029 | 0667 |
| $-40\%$ | 1837     | 70.3029 | 0667 |

Substituting the above values in the equations

$t_1 = (0.015286761, 0.016010255, 0.017695545, 0.01853819)$

$T = (1.2867336, 1.3582188, 1.5011892, 1.5726744)$

Substituting the values of $R, l, m, \alpha, \beta$ in equation $(6.8)$ the total cost is found as

$\bar{TC}(t_1,T) = (63.268925, 66.774421, 73.635413, 77.541909)$
From the formula given in equation (3.1) and (3.2) and using the graded mean integration the total cost can be defuzzified. The Defuzzified Total Cost is given by $T_C = 70.2716503$

Comparing the above defuzzified value of the total cost with the crisp value of the total cost is negligible.

![Fig.1: $(T_C \text{ vs } t_1 \text{ and } T)$](image)

![Figure 2. (TC Vs T at $t_1 = 0.01$)(Figure-3 (TC Vs T at $t_1 = 0.01$))](image)

8. Conclusion
The above example has been defuzzified using centroid method in Graded mean Integration and relative changes have been computed. We can see that the total cost maximizes or minimizes with mere change which is considered and within our expectation level.

An analytical solution of an inventory model with holding cost and shortages under Weibull distribution is obtained. Here all the variables are considered as fuzzy. We have used Function Principle to fuzzify the inventory model and Graded Mean Integration to defuzzify the trapezoidal fuzzify numbers and solved the model. Practical problem is illustrated to support the variability of the method given above. The fuzzy inventory model is more real world and accurate than the crisp inventory model.

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