Exotic $Z_2$ Symmetry Breaking Transitions in 2D Correlated Systems

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The Landau paradigm of phase transitions is one of the backbones in critical phenomena. With a $Z_2$ symmetry, it describes the Ising universality class whose central charge is one half ($c = 1/2$) in two spatial dimensions (2D). Recent experiments in strongly correlated systems, however, suggest intriguing possibilities beyond the Landau paradigm. We uncover an exotic universality class of a $Z_2$ symmetry breaking transition with $c = 1$. It is shown that fractionalization of discrete symmetry order parameters may realize the exotic class. In addition to novel critical exponents, we find that the onset of an order parameter may be super-linear in contrast to the sub-linear onset of the Ising class. We argue that a super-linear onset of a $Z_2$ order parameter without breaking a bigger symmetry than $Z_2$ is evidence of exotic phenomena, and our results are applied to recent experiments in phase transitions at pseudo-gap temperatures.

Fathoming exotic phenomena beyond the Landau paradigm is one of the most fundamental problems in strongly correlated systems. Prime examples include strange metals and pseudo-gap phenomena in high-temperature superconductors $^1$-$^2$, and recent advances in topological matter research expand territories of the exotic phenomena since topology further classifies phases of their same form. Namely, the cosine term with $\Phi$ describes a topological defect, $2\pi$ vortex, as in the Kosterlitz-Thouless (KT) transition $^3$-$^5$. Characteristic interplay between the self-dual cosine terms gives rise to the Ising class.

One obvious way for a non-Ising CFT is to give up the $Z_2$ symmetry adopting a bigger symmetry such as $Z_N$ latticel symmetry group, $C_{4v}$, whose basis functions are $(x^2 - y^2, xy)$. Under a reflection, for example $y \rightarrow -y$, $(B_1, B_2)$ acts differently. A $90^\circ$ rotation $(R_{\pi/2})$ acts as a $Z_2$ symmetry in the two representations because the four fold rotational symmetry is broken down to the two-fold one $(C_4 \rightarrow C_2)$, so-called Ising-nematicity. The Landau theory for $B_1$ becomes

$$S_{\text{Landau}} = \int_x \left( \frac{1}{2} \nabla^2 B_1(x) \right)^2 + \frac{r}{2} B_1(x)^2 + \frac{g}{24} B_1(x)^4 + \cdots,$$

with coupling constants, $(r, g)$ and a shortened notation $\int_x = \int d^D x$, and including $B_2$ is straightforward. In 2D, the self-duality of the Ising class is one of the main characteristics, which can be conveniently understood by the $Z_2$ clock model. Using the polar representation of the order parameters, $(B_1 = \rho \cos(\Phi), B_2 = \rho \sin(\Phi))$, the symmetry actions become

$$\Phi \rightarrow \Phi + \pi \ (R_{\pi/2}), \quad \Phi \rightarrow -\Phi \ (\text{reflection}).$$

Then, the $Z_2$ clock model is naturally introduced $S_{Z_2} = \int_x \frac{1}{2\pi K} (\nabla \Phi)^2 - u_2 \cos(2\Phi)$, with two parameters $(K, u_2)$ in a continuum form. Its dual theory is

$$S_{Z_2, \text{dual}} = \int_x \frac{(\nabla \Phi)^2}{2\pi K} - u_2 \cos(2\Phi) - u_2 \cos(2\Phi), \quad (1)$$

with a dual field $(\tilde{\Phi})$ $^6$-$^10$. The sign of $u_2$ chooses broken-symmetry states, bond nematicity ($B_1$) for $u_2 > 0$ and diagonal nematicity ($B_2$) for $u_2 < 0$. The self-duality is manifested by the same form of the two cosine terms, $\cos(2\Phi), \cos(2\tilde{\Phi})$. We stress that the two cosine terms describe different physical quantities in spite of their same form. Namely, the cosine term with $\Phi$ describes energy density of the order parameter, $\cos(2\Phi) \propto B_{1,2}^2$, while the cosine term with $\tilde{\Phi}$ describes a topological defect, $2\pi$ vortex, as in the Kosterlitz-Thouless (KT) transition $^3$-$^5$. Characteristic interplay between the self-dual cosine terms gives rise to the Ising class.

One obvious way for a non-Ising CFT is to give up the $Z_2$ symmetry adopting a bigger symmetry such as $Z_N$...
with \( N > 2 \). For example, vector-type order parameters such as valence bond solid or loop current order break the four-fold rotational symmetry completely \((C_4 \rightarrow C_1)\). The \( Z_2 \) clock model may describe their phase transitions, 
\[
S_{Z_2,dual} = \int_x \left( \frac{(\nabla \Phi)^2}{2\pi K} - u_{2\pi} \cos(2\Phi) - u_4 \cos(4\Phi) \right),
\]
where \( K \) is the plaquette index. The dual lattice with a plaquette index \( f \) with \( f \) even, \( f = \pm 1 \), \( \alpha = 1, 2 \), \( \mu = \pm 1 \), \( \nu = \pm 1 \), \( \phi (x) \), \( u_2, u_4 \), and \( u_8 \) are straightforward. For example, an inversion symmetry \((\phi (x)) \) is imposed and the symmetry actions are
\[
(\phi (x), \phi (x)) \rightarrow \left( \phi (x), \phi (x) + \frac{\pi}{2}, \phi (x) - \frac{\pi}{2} \right),
\]
\( (\phi (x), \phi (x)) \rightarrow \left( -\phi (x), -\phi (x) \right) \) (reflection).

Roughly speaking, one may interpret that the angle variable \( \Phi \) in the \( Z_2 \) clock model is split into the two angle variables \((\phi (x), \phi (x)) \) with the gauge constraint (see SI). We remark that generalizations to other \( Z_2 \) symmetries are straightforward. For example, an inversion symmetry with a two fold symmetry group \((C_{2h})\) allows two representations \((A_u, B_u)\) whose fractionalized representations can be similarly obtained.

Many body physics with the fractionalized discrete order parameters may be investigated by considering a generic lattice Hamiltonian,
\[
H = \sum_{i,j} t_{ij} \left( \psi_{\alpha}^\dagger (i) \psi_{\alpha} (j) e^{im_{ij}} + h.c. \right) + \sum_{i} \left( f_i^2 \right) \frac{2e^2}{4\Lambda} + \sum_{\alpha \beta} \left( \psi_{\alpha}^\dagger (i) \psi_{\beta} (j) + h.c. \right)^2 + \cdots
\]
with \( \alpha = 1, 2 \). The site index \((i, j)\) on a square lattice are introduced. The link-variable \( a_{ij} \) is for the \( U(1) \) gauge potential, and its field strength, \( f_i \), is defined on a dual lattice with a plaquette index \( \nu = \pm 1 \). Several parameters including a gauge charge \( e \) may access a variety of phases and their transitions. Microscopic information of the bosonic fields may be associated with doped spin liquid physics and naturally connected with fractionalization of valence-bond-solid orders as in the recent work \[13\] (See SI). In this work we focus on a phenomenological model, leaving microscopic discussion to future works. A \( Z_2 \) symmetry breaking transition may be realized by tuning \( u_2 \), and its low energy theory becomes
\[
S_{eff} = \int_x \left[ \frac{(\partial_i \phi)^2}{2}\phi + \frac{\rho}{2} (\partial_i \phi - 2a_\mu)^2 + \frac{r}{2}\phi^2 + \frac{q}{4}\rho^2 + \frac{f^2}{2\Lambda^2} + \cdots \right] + \int_x \left[ \frac{n_{\phi}^2}{2} (\partial_i \phi)^2 - 2\mu_2 \rho \cos(2\phi) \right],
\]
with \( \phi \pm = \phi_1 \pm \phi_2 \). For simplicity, we consider the case \(( \phi_1, \phi_2) \) giving \( \rho = \rho_1 \pm \rho_2 \neq 0 \) whose differences may be treated perturbatively. The second line of the action is similar to the \( Z_2 \) clock model with \( \phi \). Notice that \( \phi_1 \) is gauge neutral while \( \phi_2 \) has gauge charge two coupled to a \( U(1) \) gauge field. Accordingly, vortex configurations of \( \phi \) is free of gauge flux, but ones with \( \phi_+ \) is attached to gauge flux with the flux-quantization rule, \( \Phi_{flux} = \frac{1}{2 \pi} a_{\mu} d\mu = (2\pi n_i')/2 \).

We stress that vortex configurations of the two variables \((\phi_-, \phi_+)\) are not independent. The vortex numbers are defined by \( n_i' = n_i' = n_i' + n_i' \). For example, \( \mu_2 \) indicates that the parity of \( n_i' \) is the same as the one of \( n_i' \). The 2\( \pi \) vortex of \( \phi_- \) indicates an odd integer \( n_i' \). By controlling a Hamiltonian of \((\phi_+, f \)) sector, we may suppress the \( 2\pi \) vortex of \( \phi_- \). One obvious way for the suppression is to pay energy penalty to field strength by tuning the field strength energy terms, \( f^2 + f(s)^2 + \cdots \). For example, taking the limit of \( \Lambda \rightarrow 0 \), the energy penalty enforces \( n_+ \rightarrow 0 \) with an even number of \( n_- \). Thus, the \( 2\pi \) vortex is suppressed energetically.

Alternatively, one can use non-local interactions of the \((\phi_- + f \)) sector to suppress the \( 2\pi \) vortex of \( \phi_+ \). Let us consider a lattice model,
\[
S_{\delta} = -\frac{2\kappa}{\pi} \sum_{i,\mu,\alpha} \cos(\Delta_\mu \phi_\alpha - a_\mu) + \sum_{i} f(i) f^2 \frac{2e^2}{4\Lambda} + i\frac{\rho}{2\pi} f(i)
\]
, where the amplitude \( \rho \) in \( S_{eff} \) is treated to be fixed with a coupling constant \( \kappa \). The partition function \( Z_{\delta} = \int_{[\phi]} e^{-\sum_{\phi} S_{\delta}} \) may be analytically solved treating the \( u_2 \) term perturbatively. By using the standard Villain approximation, we find
\[
Z_{\delta} = \sum_{m_1, m_2} \int_{[\phi]} \exp \left[ -\sum_{r,\mu} \frac{\Delta_\mu (m_1)^2 + (\Delta_\mu (m_2)^2)}{4\kappa/\pi} \right] \times \exp \left[ -\sum_{r,\mu} \frac{(f^2)}{2e^2} + i f(m_1 + m_2 + \frac{\rho}{2\pi}) \right]
\]
where the integer variables \( m_1, m_2 \) are defined on a dual lattice. It is obvious that the role of the non-local term \((\rho = \pi)\) is to shift the integer variable \( m_1 + m_2 \) by a half-integer. Microscopically, this term is related to a non-local interaction term such as a background object associated with a half-flux (see SI). The Poisson summa-
The explicit form of $F$ and $\delta$ sharp contrasts to ones of the Ising class, $\eta$ is $\kappa$. The order parameter scaling dimension $\delta_{\text{dual}}$ obtained by integrating $Z$ is for the universal class of the inverted clock model.

The suppression of the $2\pi$ vortex gives the dual theory,

$$S_{\text{dual}}^{\text{ICM}} = \int \frac{(\partial_{\phi} \tilde{\phi})^2}{2\pi \kappa} - u_{4\epsilon} \cos(4\phi) - u_2 \cos(2\phi),$$

where we drop the subscript ($\cdot$) hereafter. The $4\pi$ vortex term is manifest with the cosine term, $\cos(4\phi)$. Interestingly, the form of the cosine terms is similar to the one of $S_{\text{dual}}^{\text{ICM}}$. It is easy to construct a mapping between ($\phi, \tilde{\phi}, \kappa$) and ($\Phi, \tilde{\Phi}, K$) indicating a critical theory of the ICM model has $c = 1$.

Striking properties of the universality class of ICM (ICM class) can be investigated by modifying analysis of the clock models [31, 33]. Scaling dimensions of the cosine terms are $[\cos(4\phi)] = 4\kappa$ and $[\cos(2\phi)] = 1/\kappa$ near the Gaussian fixed point. The critical value of $\kappa$ is $\kappa_c = 1/2$, where the two cosine terms are marginal. Away from $\kappa_c$, either one of the cosine terms becomes relevant. The scaling dimension of an order parameter is $[e^{i\phi}] = 1/2$, which gives the anomalous dimension and external field exponent, $\eta_{\text{ICM}} = 1$ and $\delta_{\text{ICM}} = 3$ in sharp contrasts to ones of the Ising class, $\eta_{\text{Ising}} = 1/4$ and $\delta_{\text{Ising}} = 15$. These are summarized in Table I.

The renormalization group (RG) equations are obtained by modifying the RG analysis of the $Z_4$ clock model [31, 33, 38, 39] (see SI also),

$$\frac{d}{dt}H_1 = H_2 H_3,$$
$$\frac{d}{dt}H_2 = H_1 H_3,$$
$$\frac{d}{dt}H_3 = H_1 H_2$$

upto the leading order with the renormalization parameter $l$ with $H_1 = 4(\kappa - \frac{1}{2}), H_2 = 2(u_{4\epsilon} + u_2)$, and $H_3 = 2(u_{4\epsilon} - u_2)$. There are critical lines ($H_i = H_j = 0$ for $i \neq j$) with the multi-critical point at the origin ($H_{1,2,3} = 0$). Remarkably, critical exponents are non-universal, so critical properties depend on how to approach the critical lines. Near the origin, three different behaviors of the correlation length are obtained by analysing the RG equation with the different limits;

1) $\xi_{K\text{T}} \sim \exp\left(\frac{-a}{\sqrt{|t|}}\right)$ in the limit of $H_i = H_j$ and $i \neq j$
2) $\xi_{SU(2)} \sim \exp\left(\frac{a}{|t|}\right)$ in the limit of $H_1 = H_2 = H_3$
3) $\xi_{\text{power}} \sim |t|^{-1/2}$ for $H_2 \neq 0$ and $H_3 \ll 1$.

The dimensionless temperature $t \equiv \frac{T}{T_c} - 1$ and a non-universal positive constant ($a$) are introduced. The first and second limits have the same RG equations of the KT transition and SU(2) spin chain. The third one is obtained by $\frac{d}{dt}(H_1 \pm H_3) = \pm H_2(H_1 \pm H_3)$ with $H_2 > 0$ giving $\nu = 1/2$. The scaling analysis further gives the order parameter onset behaviors, $\Delta \sim e^{-c/|t|}$ for $T < T_c$. Remarkably, the order parameter onset of the ICM class may be super-linear, which is distinctly different from the sub-linear onset of the Ising-class with $\beta = 1/8$.

Notice that the $2\pi$ vortex suppression endows $S_{\text{dual}}^{\text{ICM}}$ with the inverted structure of $S_{Z_4, \text{dual}}$. If $4\pi$ vortex configurations are further suppressed, then the ICM class may have an inverted structure of the $Z_2$ clock model. In analogy with the $Z_2$ clock model, the symmetry breaking transition is in the KT universality class with the correlation length, $\xi_{K\text{T}}$. One plausible mechanism of $4\pi$ vortex suppression is to incorporate a non-trivial quantum number inside vortex cores as shown in [41, 43] and impose corresponding symmetry. Doped quantum spin liquids, where fermionic excitations naturally appear, may naturally host mechanisms of the suppression.

We further discuss implication of a super-linear onset of an order parameter. In 2D, powerful structures of CFTs guarantee that the Landau theory with a $Z_2$ symmetry is in the Ising class with $\beta = 1/8$, and thus a super-linear onset is incompatible with a $Z_2$ symmetry under the Landau paradigm. In the ICM class, we resolve the incompatibility by going beyond the Landau paradigm. The suppression of the $2\pi$ vortex induces the CFT with $c = 1$. An alternative way is to enlarge a symmetry, say from $Z_2$ to $Z_4$, and then, a super-linear onset of an order parameter may be realized, the Ashkin-Teller CFT with $c = 1$, even under the Landau paradigm. Therefore, observation of a super-linear onset in experiments indicates that either a broken symmetry is bigger than $Z_2$ or exotic transition beyond the Landau paradigm appears.

Let us apply our results to recent experiments in cuprates which report super-linear onsets of a nematic transition formula allows to extract the $\vartheta$ contribution,

$$Z_\vartheta = \sum_{[n_1],[n_2]} \prod_{\nu} \left( \cos\left(\frac{\vartheta}{2}(n_1^\nu + n_2^\nu)\right) \right)^2 F[n_1^\nu, n_2^\nu].$$

TABLE I. Universality class with a $Z_2$ symmetry, and their critical exponents. The order parameter scaling dimension $[\Delta], \text{anomalous dimension } \eta, \text{and external field dependence exponent } \delta$ are presented. MFT is for mean field theory, and ICM is for the universality class of the inverted clock model.
Z_c = 1 transition. First, the transition is under the Landau paradigm and the black lines are to distinguish bond (d-density wave) or diagonal nematicity (u_2 > 0) and diagonal nematicity (u_2 < 0). (b) Order parameter onset of the ICM class, \( \Delta(T) = A e^{-\sqrt{2-u_2}} \) (the KT type onset). Data (red dots) fitting of the magneto torque experiment [25] is presented. The dashed line is a typical onset of the Ising class.

We provide three phenomenological scenarios for the c = 1 transition. First, the transition is under the Landau paradigm with a bigger symmetry group, say Z_4. An Ising-nematic order cannot describe the transition, but the order parameters such as loop-current [44] or d-density wave [45] orders may be plausible. However, the onset of such order parameters requires additional experimental signatures. Namely, time-reversal (translational) symmetry breaking must be broken by loop-current order, but signals of the broken symmetries are under debates in spite of manifest signatures of the broken rotational symmetry. Second, the transition is beyond the Landau paradigm and only Z_2 symmetry is broken as in the ICM class. The presence of the enigmatic strange metal and violation of the Luttinger theorem in under-doped cuprates [25] seems naturally connected to this scenario. This scenario may also be related to deconfinement of doped quantum spin-liquids and the recent work about topological orders [2] [3] [18] (see SI). The final one is that the transition is beyond the Landau paradigm and the broken symmetry is Z_4. The universality class may be described by another Sine-Gordon model with the self-dual cosine terms with \( \cos(4\phi) \) and \( \cos(4\phi') \). Here, the order parameter onset is also the KT type, and there is additional KT phase transition at high temperature. Note that all of symmetry breaking transitions of the three scenarios are associated with the CFT with c = 1.

In conclusion, we present mechanisms of Z_2 symmetry breaking beyond the Landau paradigm by employing fractionalization of discrete order parameters. Characteristic critical exponents and correlation length behaviors are obtained. We find that the onset of an order parameter may be super-linear in contrast to the sub-linear onset of the Ising class. Our results of the non sub-linear onset are applied to recent experiments in cuprates. Further theoretical studies including disorder effects and numerical analysis of microscopic lattice models in terms of valence bond solid or loop-current are highly desired.

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[1] S. A. Kivelson, I. P. Bindloss, E. Fradkin, V. Oganesyan, J. M. Tranquada, A. Kapitulnik, and C. Howald, How to detect fluctuating stripes in the high-temperature superconductors, Rev. Mod. Phys. 75, 1201 (2003).
[2] S. Sachdev, Order and quantum phase transitions in the cuprate superconductors, Rev. Mod. Phys. 75, 913 (2003).
[3] P. A. Lee, N. Nagaosa, and X. G. Wen, Doping a Mott insulator: Physics of high-temperature superconductivity, Rev. Mod. Phys. 78, 17 (2006).
[4] M. Vojta, Lattice symmetry breaking in cuprate superconductors: stripes, nematics, and superconductivity. Adv. Phys. 58, 699-820 (2009).
[5] E. Berg, E. Fradkin, S. A. Kivelson, and J. M. Tranquada, Striped superconductors: how spin, charge and superconducting orders intertwine in the cuprates New Journal of Physics 11, 115004 (2009).
[6] T. M. Rice, K. Y. Yang, and F. C. Zhang, A phenomenological theory of the anomalous pseudogap phase in underdoped cuprates, Rep. Prog. Phys. 75, 016502 (2012).
[7] L. Taillefer, Scattering and Pairing in Cuprate Superconductors, Ann. Rev. Con. Mat. Phys. 1, 51 (2010).
[8] B. Keimer, S. A. Kivelson, M. R. Norman, S. Uchida & J. Zaanen, From quantum matter to high-temperature superconductivity in copper oxides. Nature 518, 179-186 (2015).

[9] A. Kitaev, Anyons in an exactly solved model and beyond Ann. Phys. 321, 2-111 (2006).

[10] X. G. Wen, Zoo of quantum-topological phases of matter, Rev. Mod. Phys. 89, 041004 (2017).

[11] T. Senthil, Symmetry-Protected Topological Phases of Quantum Matter, Ann. Rev. Con. Mat. Phys. 6, 299 (2015).

[12] Y. Machida, S. Nakatsuji, S. Onoda, T. Tayama & T. Sakakibara, Time-reversal symmetry breaking and spontaneous Hall effect without magnetic dipole order, Nature, 463, 210 (2010).

[13] D. Pesin and L. Balents, Mott physics and band topology in materials with strong spinorbit interaction Nat. Phys. 6, 376 (2010).

[14] E.-G. Moon, C. Xu, Y. B. Kim, and L. Balents, Non-fermi-liquid and topological states with strong spin-orbit coupling Phys. Rev. Lett. 111, 200401 (2013).

[15] I. F. Herbut and L. Janssen, Topological Mott Insulator in Three-Dimensional Systems with Quadratic Band Touching Phys. Rev. Lett. 113, 106401 (2014).

[16] T. Kondo, M. Nakayama, R. Chen, J. J. Ishikawa, E.-G. Moon, T. Yamamoto, H. Kumaishi, K. Ono, H. Yamamoto, Y. Ota, W. Malaeb, H. Kanai, Y. Nakashima, Y. Ishida, R. Yoshida, H. Yamamoto, M. Matsumani, S. Kimura, N. Inami, K. Ono, H. Kunitsugu, S. Nakatsuji, L. Balents & S. Shin Nat. Comm. 6, 10042 (2015).

[17] L. Savary, E.-G. Moon, and L. Balents, New Type of Quantum Criticality in the Pyrochlore Iridates Phys. Rev. X 4, 041027 (2014).

[18] S. Chatterjee, S. Sachdev, and M. S. Scheurer, Intertwining topological order and broken symmetry in a theory of fluctuating spin density waves Phys. Rev. Lett. 119, 227002 (2017).

[19] T. Senthil, A. Vishwanath, L. Balents, S. Sachdev, M. P. A. Fisher, Deconfined quantum critical points. Science, 303 1490-1494, (2004).

[20] M. Oleie, A. Vishwanath, Emergent photons and new transitions in the O(3) sigma model with hedgehog suppression. Phys. Rev. B 70, 075104 (2004).

[21] J. Xia, E. Schemm, G. Deutscher, S. A. Kivelson, D. A. Bonn, W. N. Hardy, R. Liang, W. Siemens, G. Koster, M. M. Fejer, and A. Kapitulnik, Polar Kerr-Effect Measurements of the High-Temperature YBaCu3O6+δ Superconductor: Evidence for Broken Symmetry near the Pseudogap Temperature. Phys. Rev. Lett. 100, 127002 (2008).

[22] T. Wu, H. Mayaffre, S. Krmer, M. Horvati, C. Berthier, W. N. Hardy, R. Liang, D.A. Bonn & M. H. Julien, Incipient charge order observed by NMR in the normal state of YBa2Cu3O6. Nat. Commun. 6, 6438 (2015).

[23] S. Badoux, W. Tabis, F. Lalibert, G. Grissonnanche, B. Vignolle, D. Vignolles, J. Bard, D. A. Bonn, W. N. Hardy, R. Liang, N. Doiron-Leyraud, L. Taillefer & C. Proust, Change of carrier density at the pseudogap critical point of a cuprate superconductor. Nature, 531, 210 (2016).

[24] R. Daou, J. Chang, D. LeBoeuf, O. C. Choinire, F. Lalibert, N. D. Leyraud, B. J. Ramshaw, R. Liang, D. A. Bonn, W. N. Hardy & L. Taillefer Broken rotational symmetry in the pseudogap phase of a high-Tc superconductor. Nature 463, 519-522 (2010).

[25] Y. Sato, S. Kasahara, H. Murayama, Y. Kasahara, E.-G. Moon, N. H. Sung, J. D. Denlinger & B. J. Kim, Observation of a d-wave gap in electron-doped Sr2IrO4. Nat. Phys. 12, 37 (2016).

[26] M. J. Lawler, K. Fujita, J. Lee, A. R. Schmidt, Y. Kohsaka, C. K.Kim, H. Esaki, S. Uchida, J. C. Davis, J. P. Sethna & E. A. Kim, Intra-unit-cell electronic nematicity of the high-Tc copper-oxide pseudogap states. Nature 466, 347-351 (2010).

[27] T. P. Croft, E. Blackburn, J. Kudla, R. Liang, W. N. Hardy, and S. M. Hayden, No evidence for orbital loop currents in charge-ordered YBa2Cu3O6.456 from polarized neutron diffraction, Phys. Rev. B 96, 214504 (2017).

[28] Y. K. Kim, N. H. Sung, J. D. Denlinger & B. J. Kim, Observation of a d-wave gap in electron-doped Sr2IrO4. Nature, 545, 372-375 (2008).

[29] L. Zhao, C. A. Belvin, R. Liang, D. A. Bonn, W. N. Hardy, N. P. Armitage & D. Hsieh, A global inversion-symmetry-broken phase inside the pseudogap region of YBa2Cu3O6, Nat. Phys. 13, 250-255 (2016).

[30] T. P. Croft, E. Blackburn, J. Kudla, R. Liang, W. N. Hardy, and S. M. Hayden, No evidence for orbital loop currents in charge-ordered YBa2Cu3O6.456 from polarized neutron diffraction, Phys. Rev. B 96, 214504 (2017).

[31] Y. K. Kim, N. H. Sung, J. D. Denlinger & B. J. Kim, Observation of a d-wave gap in electron-doped Sr2IrO4. Nat. Phys. 12, 37 (2016).

[32] L. Zhao, C. A. Belvin, R. Liang, D. A. Bonn, W. N. Hardy, N. P. Armitage & D. Hsieh, A global inversion-symmetry-broken phase inside the pseudogap region of YBa2Cu3O6, Nat. Phys. 13, 250-255 (2016).

[33] M. Kosterlitz and D. J. Thouless, Ordering, metastability and phase transitions in two-dimensional planar models. Phys. Rev. B 16, 1217 (1977).

[34] L. Zhao, D. H. Torchinsky, H. Chu, V. Ivanov, R. Lifshitz, R. Flint, T. Qi, G. Cao & D. Hsieh, Evidence of an odd-parity hidden order in a spin-orbit coupled correlated iridate Nat. Phys., 12, 32-36 (2017).

[35] P. Francesco, P. Mathieu, and D. Senechal, Conformal Field Theory. (Springer, 1996).

[36] J. V. Jos, L. P. Kadanoff, S. Kirkpatrick, and D. R. Nelson, Renormalization, vortices, and symmetry-breaking perturbations in the two-dimensional planar model. Phys. Rev. B 20, 214504 (2017).

[37] J. P. Sethna & E. A. Kim, Intra-unit-cell electronic nematicity of the high-Tc copper-oxide pseudogap states. Nature 466, 347-351 (2010).

[38] M. J. Lawler, K. Fujita, J. Lee, A. R. Schmidt, Y. Kohsaka, C. K.Kim, H. Esaki, S. Uchida, J. C. Davis, J. P. Sethna & E. A. Kim, Intra-unit-cell electronic nematicity of the high-Tc copper-oxide pseudogap states. Nature 466, 347-351 (2010).

[39] T. Senthil, A. Vishwanath, L. Balents, and M. S. Scheurer, Intertwining topological order and broken symmetry in a theory of fluctuating spin density waves Phys. Rev. Lett. 119, 227002 (2017).

[40] T. Grover, and T. Senthil, Topological Spin Hall States, Nature 463, 519-522 (2010).
Charged Skyrmions, and Superconductivity in Two Dimensions. Phys. Rev. Lett. 100, 156804 (2008).

[43] E.-G. Moon, Skyrmions with quadratic band touching fermions: A way to achieve charge 4e superconductivity Phys. Rev. B 85, 245123 (2012).

[44] C. Varma, Non-Fermi-liquid states and pairing instability of a general model of copper oxide metals. Phys. Rev. B 55, 14554 (2009).

[45] S. Chakravarty, R. B. Laughlin, D. K. Morr, and C. Nayak, Hidden order in the cuprates. Phys. Rev. B 63, 094503 (2001).

Appendix A: Representations with fractionalized degrees of freedom

Let us consider representations with fractionalized degrees of freedom,

\[ \begin{align*}
B_1 &= \rho_1 \cos(2\phi_1 - 2\phi_s), \\
B_2 &= \rho_2 \sin(2\phi_2 - 2\phi_s),
\end{align*} \]

with the transformations of the \( \pi/2 \) rotation (\( R_{\pi/2} \)) and the reflection (\( \sigma_y : y \to -y \)),

\[ \begin{align*}
(\phi_1, \phi_2, \phi_{s1}, \phi_{s2}) &\to (\phi_1 + \pi/2, \phi_2 - \pi/2, \phi_{s1}, \phi_{s2}) \quad (R_{\pi/2}) \\
(\phi_1, \phi_2, \phi_{s1}, \phi_{s2}) &\to (-\phi_1, -\phi_2, -\phi_{s1}, -\phi_{s2}) \quad (\sigma_y).
\end{align*} \]

Then, the condensation of the spectator fields \( \phi_{s1,2} \) makes the representation conventional. Other than the conventional order parameters, one can have emergent order parameters,

\[ \begin{align*}
b_1 &= \sqrt{\rho_1 \rho_2} \cos(\phi_1 - \phi_2), \\
b_2 &= \sqrt{\rho_1 \rho_2} \sin(\phi_1 - \phi_2),
\end{align*} \]

which transform precisely as \( (B_1, B_2) \).

Appendix B: Lattice model analysis

In this section, we discuss how to realize the lattice model with the non-local interaction in the main-text,

\[ S_\theta = -\frac{2\kappa}{\pi} \sum_{i,\mu,k} \cos(\Delta_\mu \phi_k - a_\mu) + \sum_{i,y} f(i_p)^2 + i \frac{\vartheta}{2\pi} f(i_p). \]

The standard notation for lattice derivation is used and the gauge potential \( a_\mu \) is defined on links. The field strength \( f \) is defined on the dual lattice. The presence of the unusual \( \vartheta \) term may be understood by introducing high energy degrees of freedom with a half of the gauge charge of \( \phi_{1,2} \). Let us consider an action,

\[ S_{UV} = S_{\theta=0} - \frac{t}{\pi} \sum_{i,\mu} \cos(\Delta_\mu \Theta - \frac{1}{2} a_\mu). \]

The Villain approximation gives

\[ e^{t \cos(\Delta_\mu \Theta - \frac{1}{2} a_\mu)} \to \sum_{M_\mu} e^{-\frac{1}{4}(\Delta_\mu \Theta - \frac{1}{2} a_\mu - 2\pi M_\mu)^2} \]

and the Poisson formula gives

\[ \sum_{J_\mu} e^{-\frac{\pi}{4} J_\mu^2 - iJ_\mu \Delta_\mu \Theta + i\frac{1}{2} J_\mu a_\mu}. \]

The integer variables \( J_\mu, M_\mu \) on links are introduced. The angle variable integration endows the delta function, which may be replaced by \( J_\mu = \epsilon_{\mu\nu} \Delta_\nu N \) with an integer variable \( N \) on plaquette sites. The action becomes

\[ S_{eff} = S_{\theta=0} + \sum_{i,y} i\frac{N}{2} f(i_p) + \frac{\pi}{2t} (\Delta_\mu N)^2 \]
with a plaquette index $i_\mu$. It is obvious that in the limit of $t \to \infty$ the fluctuation term vanishes and $S_{\epsilon ff}$ with the $N=1$ background becomes equivalent to $S_\theta$. With the lattice model $S_\theta$, one can analytically calculate the partition function,

$$Z_\theta = \int_{[\phi]} e^{-S_\theta}. \quad \text{(B1)}$$

Again, using the Villain approximation, the Poisson summation formula, and the $\phi_{1,2}$ integrations, we find

$$Z_\theta = \int_{[\phi]} \sum_{l_\mu=-\infty}^{l_\mu=+\infty} \exp\left[ -\sum_{r,\mu} \left( \frac{l_\mu^2 + l_\mu}{4\kappa/\pi} + i(l_\mu + l_\mu a_{\mu}) \right) \right] \times \exp\left[ -\sum_{l,\mu} \left( \frac{f^2}{2\epsilon^2} + i \frac{\theta}{2\pi} f \right) \right] \prod_l \delta(\Delta l_{1\mu}) \delta(\Delta l_{2\mu})$$

The delta functions can be rewritten by introducing the number variables $n_1, n_2$ on the dual lattice,

$$l_{1\mu} = \epsilon_{\mu\alpha} \Delta n_{1\alpha}, \quad l_{2\mu} = \epsilon_{\mu\alpha} \Delta n_{2\alpha}.$$ 

The gauge field integration gives the partition function,

$$Z_\theta = \sum_{n_1, n_2} \exp\left[ -\sum_{r,\mu} \frac{(\Delta n_{1\mu})^2 + (\Delta n_{2\mu})^2}{4\kappa/\pi} \right] \times \exp\left[ -\sum_{l,\mu} \frac{e^2}{2} (n_1 + n_2 + \frac{\theta}{2\pi})^2 \right]$$

Using the Poisson formula again and rescaling the fields $\phi_{1,2} \to 2\phi_{1,2}/\pi$, we have

$$Z_\theta = \int_{[\phi]} \sum_{n_1, n_2} \exp\left[ -\sum_{r,\mu} \frac{(\Delta n_{1\mu})^2 + (\Delta n_{2\mu})^2}{\pi\kappa} \right] \times \exp\left[ i4(\hat{\phi}_1 m_1 + \hat{\phi}_2 m_2) + \frac{e^2}{2} \left( \frac{2(\hat{\phi}_1 + \hat{\phi}_2)}{\pi} + \frac{\theta}{2\pi} \right)^2 \right]$$

Introducing $\phi_+ = \phi_1 \pm \phi_2$ and shifting the variable, we have Shifting the variable $\phi_+ \to \phi_+ - \frac{\theta}{4}$, we obtain

$$Z_\theta = \sum_{m_1, m_2} e^{\Sigma_{r,-i\hat{\phi}(m_1 + m_2)} [F[m_1, m_2]]}$$

$$F[m_1, m_2] = \int_{[\phi]} \prod_{r,\mu} \exp\left[ -\frac{(\Delta n_{1\mu})^2 + (\Delta n_{2\mu})^2}{2\pi\kappa} \right] \times \exp\left[ i(2\hat{\phi}_+(m_1 + m_2) + 2\hat{\phi}_-(m_1 - m_2)) \right] \times \exp\left[ \frac{e^2}{2} \left( \frac{\hat{\phi}_+(m_1 + m_2)}{\pi} \right)^2 \right]$$

It is obvious to see that $F[m_1, m_2] = F[-m_1, -m_2]$, so we have the final form,

$$Z_\theta = \sum_{n^+_1, n^+_2} \prod_{i_\mu} \left( \cos\left( \frac{\theta}{2} (n^+_1 + n^+_2) \right) \right) F[n^+_1, n^+_2]$$

The destructive interference with $\theta = \pi$ for an odd number of $n^+_r$ is obvious.

The parity of $n^+_r$ is the same as one of $n^+_r$, and thus the minimal vortex of $\phi_-$ is not a $2\pi$ vortex but a $4\pi$ vortex. Moreover, the $\phi_+$ is short-ranged by the presence of $e^2$ term, so the low energy theory becomes

$$S_{\theta,dual} = \int_x \left( \frac{\partial \phi_+}{\pi} \right)^2 - u_4 \cos(4\tilde{\phi}_-). \quad \text{(B2)}$$

As usual in the derivation of the Sine-Gordon model, we put the fugacity term of the vortex, $u_4$, by hand. Notice that the $\theta$ term naturally appears in $s=1/2$ spin chains, and our discussion on the exotic $Z_2$ transition with the inverted clock model may be directly applied to spin $s=1/2$ chain.

**Appendix C: Properties of the inverted clock model**

In this section, we explicitly present differences between the $Z_4$ clock model and the inverted clock model. The $Z_4$ clock model is

$$S_{Z_4} = \int_x \frac{K}{2\pi} \left( \partial_\mu \tilde{\Phi} \right)^2 - u_4 \cos(4\Phi),$$

whose dual theory is

$$S_{Z_4,dual} = \int_x \frac{1}{2\pi K} \left( \partial_\mu \Phi \right)^2 - u_{2\nu} \cos(2\Phi) - u_4 \cos(4\Phi).$$

Around the Gaussian point, the scaling dimensions of the two cosine potential scaling are

$$[\cos(4\Phi)] = \frac{4}{K}, \quad [\cos(2\Phi)] = K,$$

and at $K = K_c = 2$, the two operators are marginal. Away from $K_c$, either one of the operators become relevant while the other one is irrelevant. Also, the scaling dimension of the $Z_4$ order parameter, $\Delta_{Z_4} = \langle \cos(\Phi) \rangle$, is determined by

$$[\cos(\Phi)] = \frac{1}{4K}.$$ 

At $K_c$, the order parameter scaling dimension is $[\Delta_{Z_4}] = 1/8$.

The inverted clock model with the $2\pi$ vortex suppression is

$$S_{dual}^{LCM} = \int_x \frac{1}{2\pi K} \left( \partial_\mu \tilde{\phi} \right)^2 - u_4 \cos(4\tilde{\phi}) - u_2 \cos(2\phi).$$

We drop the subscript $-$ for simplicity. Around the Gaussian point, the scaling dimensions of the two cosine potential scaling are

$$[\cos(4\tilde{\phi})] = 4K, \quad [\cos(2\phi)] = \frac{1}{K},$$

and at $K = K_c = 1/2$, the two operators are marginal. Also, the scaling dimension of the $Z_2$ order parameter of
the ICM class, $\Delta_{Z_2} = \langle \cos(\phi) \rangle$, is determined by

$$[\cos(\phi)] = \frac{1}{4\kappa}.$$  

At $\kappa_c$, the order parameter scaling dimension is $[\Delta_{Z_2}] = 1/2$.

The scaling dimension of the external field is determined by

$$\int_x h_4 \cos(\Phi) \to [h_4] = 2 - [\cos(\Phi)] = \frac{15}{8},$$

$$\int_x h_2 \cos(\phi) \to [h_2] = 2 - [\cos(\phi)] = \frac{3}{2}.$$  

The scaling analysis gives

$$\Delta \propto (\xi^{-1})^{[\Delta]} \propto h \frac{[\Delta]}{[\delta]} \to \delta = \frac{[h]}{[\Delta]},$$  

(C1)

which gives

$$\delta_{Z_2} = 15, \quad \delta_{Z_2,ICM} = 3.$$  

The susceptibility can be obtained in a similar way,

$$\chi = \frac{\partial \Delta}{\partial h} \propto h \frac{[\Delta]}{[\delta]} \propto (\xi^{-1})^{[\Delta]-[h]},$$  

(C2)

Note that if $4\pi$ vortex is further suppressed, then similar analysis to the $Z_N$ clock model with $N > 4$ can be straightforwardly done.

As mentioned in the main-text, further suppression of $4\pi$ vortex configurations makes the $Z_2$ symmetry breaking transition as in the KT class. There are three possible phases; a symmetry broken phase at low temperature, KT phase at intermediate phase, and disordered phase at high temperature. The two transitions are both described by the KT transitions $[34, 36]$, and at the symmetry breaking transition temperature, the scaling dimensions of physical operators are determined by the condition, $[\cos(2\phi)] = 2$, around the Gaussian point, giving

$$\Delta_{Z_2} = \frac{1}{2}, \quad [\delta] = 3.$$  

Finally, it is possible that the system has a $Z_4$ symmetry with $2\pi$ vortex suppression. Then, the critical theory becomes

$$\mathcal{S}_{Z_4,ICM}^{dual} = \int_x \left( \frac{\partial^2 \phi}{2\pi \kappa} - u_{4\nu} \cos(4\phi) - u_4 \cos(4\phi) \right).$$

The scaling dimensions of the cosine terms are $[\cos(4\phi)] = \frac{1}{4}$ and $[\cos(4\phi)] = 4\kappa$ around the Gaussian point. There are two critical points $\kappa_{c1} = 1/2$ and $\kappa_{c2} = 2$. At low temperature ($\kappa > \kappa_{c2}$), the $Z_4$ symmetry is broken and the criticality is described by the order parameter $\Delta_{Z_4} = \langle e^{i\phi} \rangle$ whose onset is the KT type with the scaling dimension, $[\Delta_{Z_4}] = 1/8$. In the intermediate region $\kappa_{c1} < \kappa < \kappa_{c2}$, the KT phase appears.

1. Renormalization group analysis

The universality class of the ICM class can be investigated by applying the RG analysis of the clock model. Near the Gaussian point ($u_2 = u_{4d} = 0$), it is easy to obtain

$$\frac{d}{dt} \kappa = -u_{4d}^2 + 4\kappa^2 u_2^2,$$

$$\frac{d}{dt} u_{4d} = (2 - 4\kappa)u_{4d},$$

$$\frac{d}{dt} u_2 = (2 - 4\kappa)u_2.$$  

Notice that the duality at the level of the partition functions ($\mathcal{Z}(\kappa, u_{4d}) = \mathcal{Z}_{dual}((4\kappa)^{-1}, u_{4d}, u_2)$) are well established.

Introducing new variables,

$$H_1 = 4(\kappa - \frac{1}{2}), \quad H_2 = 2(u_2 + u_{4d}), \quad H_3 = 2(u_2 - u_{4d}),$$  

the RG equations are simplified,

$$\frac{d}{dt} H_1 = H_2 H_3, \quad \frac{d}{dt} H_2 = H_1 H_3, \quad \frac{d}{dt} H_3 = H_1 H_2,$$

which is the same as the RG equations obtained by the work by Kadanoff $[38]$.

The critical lines are determined by the condition, $H_i = H_j = 0$ and $i \neq j$. Near a fixed point with $H_2 > 0$ and $H_2 \ll 1$, the RG equation becomes

$$\frac{d}{dt} (H_1 \pm H_3) = \pm H_2 (H_1 \pm H_3).$$

$H_1 - H_3$ is irrelevant, and $H_1 + H_3$ is relevant giving $\nu = 1/H_2$. Notice the scaling dimension can be very small in the limit, $H_2 \ll 1$. It is easy to observe that the fine-tuned limit $H_1 = H_2 = H_3 = H$ gives the RG equation,

$$\frac{dH}{dt} = H^2,$$

and the correlation length becomes

$$\xi \sim e^{\frac{1}{H^2}}.$$  

Appendix D: Relations with microscopic models

In this section, we discuss relations with microscopic models to realize the exotic $Z_2$ transitions with the ICM class. Especially, we show close relation to the previous work on topological order $[18]$. As mentioned in the main-text, one concrete way is to consider fractionalization of physical operators. Let us consider the Schwinger boson representation, which is useful to describe phases nearby conventional magnetic symmetry broken phases. We start with writing the anti-ferromagnetic component
of spin operators, \( \vec{S}(i) = (-1)^i \vec{n}(i) + \cdots \),
\[
\vec{n}(i) = \frac{1}{2} b^\dagger \alpha(i) (\vec{\sigma})_{\alpha\beta} b^\beta(i),
\]
The valence bond solid operator can be defined as
\[
V_x(i) \equiv \vec{S}(i) \cdot \vec{S}(i+\hat{x}) (-1)^{ix}, \quad i = (i_x, i_y)
\]
\[
V_y(i) \equiv \vec{S}(i) \cdot \vec{S}(i+\hat{y}) (-1)^{iy}.
\]
Rewriting the dimer operators in terms of the Schwinger bosons, one can show
\[
V_x \propto (-1)^{ix} Q^\dagger_x Q_x, \quad V_y \propto (-1)^{iy} Q^\dagger_y Q_y,
\]
with
\[
Q_x(i) = \varepsilon_{\alpha\beta} b_{i,\alpha} b_{i+x,\beta}, \quad Q_y(i) = \varepsilon_{\alpha\beta} b_{i,\alpha} b_{i+y,\beta}.
\]
The operators \( (Q_{x,y}) \) have twice bigger gauge charge than the Schwinger bosons. Then, one can interpret \( Q_{x,y} \) as the VBS fractionalized operators. With some modifications, one can find a mapping between \( Q_{x,y} \) and the fractionalized representation \((\psi_1,\psi_2)\) in the main-text. The presence of the emergent order parameter is manifest, for example \( Q^\dagger_x Q_y + Q^\dagger_y Q_x \), which has the same symmetry transformation as \( V_x V_y \). More detailed discussion about the relations will be presented in other places.

**Appendix E: Data fitting**

The non sub-linear onset in the ICM class can be applied to the recent experiments in cuprates. As an example, we apply our results to the recent torque experiments in cuprates around the pseudo-gap temperatures. In the experiments, the materials (YBCO) have a spin chain structure along one of the bond directions, which plays a role of an external field. To take account of the spin chain effects, we modify the formula,
\[
\eta(T) = \eta(T > T^*) + (\xi^{-1}(T))^{|\Delta|}.
\]
Note that \( T^* \) and \( \eta(T > T^*) \) are experimentally observed. The off-set value \( \eta(T > T^*) \) can be reasonably read off from the data, but the pseudo-gap temperature \( T^* \) is not easy to be determined accurately because of the super-linear onsets. We first fit the data with two fitting parameters for the three types of the correlation functions, \( \xi_{KT}, \xi_{SU(2)}, \) and \( \xi_{\text{power}} \) using the reported values of \( \eta(T > T^*) \) and \( T^* \). The \( R^2 \) in statistics is bigger than 0.99 for all cases, and we find that \( \xi_{KT} \) and \( \xi_{\text{power}} \) show better fittings than \( \xi_{SU(2)} \). Next, we also perform data fitting varying with \( T^* \) because of uncertainty of \( T^* \) in experiments, and we find that the fitting becomes better by setting the pseudo-gap temperatures higher than experimentally observed ones, and we present best results with the KT onsets in Fig. 2. We expect that systems without the spin-chain effects such as Hg compounds cuprates would be a better system to determine \( T^* \) better.
FIG. 2. Fitting results of the torque experiments of [28] with the KT onset, $\eta(T) = \eta(T > T^*) + Ae^{-\sqrt{B - T/T^*}}$. We read off the offset values $\eta(T > T^*)$ from the data, and the three parameters ($T^*$, $A$, $B$) are used. The statistical $R^2$ values are also presented.