Detecting scaling in phase transitions on the truncated Heisenberg algebra

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ABSTRACT: We construct and analyze the phase diagram of the self-interacting matrix field coupled to curvature of the non-commutative truncated Heisenberg space. The model reduces to renormalizable Grosse-Wulkenhaar model in the infinite matrix size limit and exhibits the purely non-commutative non-uniformly ordered phase. Particular attention is given to the scaling of the model’s parameters. We additionally provide the infinite matrix size limit for the disordered to ordered phase transition line.

KEYWORDS: matrix models, non-commutative geometry, phase transitions
1 Introduction

Non-commutativity (NC) of space-time was conjured in early days of quantum field theory in hopes of fighting arising infinities [1] but soon the magic of renormalization prevailed and it was forgotten. Since then it was seen to lurk in different corners of physics at different energies, from condensed matter physics to quantum gravity, either as an effective description of encountered phenomena [2, 3] or as a postulated fundamental property of nature. Realization that string theory hides NC at low energies [4] — they even appear to share much closer connection [5] — finally rekindled the interest for it after many years. But, as if in revenge for abandoning it decades ago, NC cast a severe curse upon field theories on NC spaces: the mixing of UV and IR divergences of non-planar diagrams that damages their renormalizability [6–8].

Grosse-Wulkenhaar (GW) model [9–13] is one of rare NC models immune to UV/IR mixing [14–16]. It describes a self-interacting real scalar field on the NC Moyal space confined in the external harmonic oscillator potential. The oscillator term, which shields its renormalizability, can be reinterpreted [17] as coupling with the curvature of the underlying NC space of the truncated Heisenberg algebra $\mathfrak{h}^{tr}$. All attempts at generalizing this construction to renormalizable NC gauge models have so far been unsuccessful.

A common feature of NC field theories is that simultaneously with UV/IR mixing, emerges the translation breaking striped phase in which field oscillates around different values at different points in space and where periodic non-uniform magnetisation patterns appear [18–20]. It is believed that this new order lies at the root of UV/IR mixing [21]. A while back, we examined a GW inspired gauge model on $\mathfrak{h}^{tr}$ space [22, 23] which in
addition to trivial vacuum possesses another position-dependent one — a possible hallmark of striped behaviour. Lengthy analytical treatment showed that divergent non-local derivative counterterms render this model non-renormalizable. We are, in light of this, interested whether the numerical exploration of phases and critical behaviour could indicate nonrenormalizability in advance and save time with future approaches. To that end, in this and the following papers we will numerically compare the behaviour of the matrix regularization of two-dimensional GW model — whose renormalizability was originally explored in matrix base — with and without the curvature term. It would be interesting to see if the way the curvature term is turned off affects the limiting phase diagram. This would correspond to the particular way the oscillator term of GW model needs to be turned off by cutoff parameter, in order to assure the two-dimensional NC $\phi^4$-model’s renormalizability [9].

Phase diagrams of matrix models on fuzzy spaces have been extensively studied both analytically [21, 24–30] and numerically [31–39]. Notable example is $\phi^4$-model on the fuzzy sphere, where we encounter three phases that meet at a triple point. In disordered phase field eigenvalues oscillate around zero, and in ordered phase around one of the two oppositely signed minima of the effective eigenvalue potential. Due to eigenvalue repulsion there is also the third, non-uniformly ordered phase where eigenvalues populate both of these minima. Since we can, in a way, view different eigenvalues as field at different points of space, this phase corresponds to the above-mentioned stripe phase. In fact, there might exist entire series of non-uniformly ordered phases [39].

In this paper we analyze in detail the detection of scaling of parameters of each term in the action; this turns out to be nontrivial due to slow convergence and the triple point drifting. We also present the phase diagram for matrices of size $N = 24$ and results for infinite matrix size limit of disordered to ordered phase transition line.

2 The model

The GW model [9]

$$S_{GW} = \int \frac{1}{2} (\partial \phi)^2 + \frac{\Omega^2}{2} ((\theta^{-1}x)\phi)^2 + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4,$$  

(2.1)

with NC embedded in the Moyal-Weyl star product

$$(f \star g)(x) = \exp\left(\frac{\imath \theta \mu \nu}{2} \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial z^\nu}\right) f(y)g(z) \bigg|_{x} \implies [x^\mu, x^{\nu}] = i \theta^{\mu \nu},$$  

(2.2)

is in [17] identified with that of a scalar field coupled with a NC curvature

$$S_R = \int \sqrt{g}\left(\frac{1}{2} (\partial \phi)^2 - \frac{\xi}{2} R \phi^2 + \frac{M^2}{2} \phi^2 + \frac{\Lambda}{4!} \phi^4\right).$$  

(2.3)

The underlying $h^{1\!+\!1}$ space satisfies

$$[\mu x, \mu y] = i \epsilon (1 - \mu z), \quad [x, z] = i \epsilon \{y, z\}, \quad [y, z] = -i \epsilon \{x, z\},$$  

(2.4)
where $\epsilon$ the strength and $\mu$ the mass scale of NC. For $\epsilon = 1$, $\mu x$ and $\mu y$ can be represented by finitely-truncated matrices of the Heisenberg algebra

$$X = \frac{1}{\sqrt{2}} \begin{bmatrix} +1 & & \\ +\sqrt{2} & +\sqrt{3} & \\ +\sqrt{3} & & \ddots \end{bmatrix}_{N \times N}, \quad Y = \frac{i}{\sqrt{2}} \begin{bmatrix} +1 & & \\ +\sqrt{2} & -\sqrt{3} & \\ +\sqrt{3} & & \ddots \end{bmatrix}_{N \times N}. \quad (2.5)$$

The model (2.3) was analysed in the frame formalism, with geometry defined by the choice of momenta $p_\mu$ as functions of elements of algebra

$$\epsilon_{p_1} = i\mu^2 y, \quad \epsilon_{p_2} = -i\mu^2 x, \quad \epsilon_{p_3} = i\mu \left( \mu z - \frac{1}{2} \right), \quad (2.6)$$

and derivatives realized as commutators $\partial_\mu f = [p_\mu, f]$ with these momenta.

We investigated a matrix regularization of (2.3)

$$S_N = \text{Tr} \left( c_k \Phi[P_\alpha, [P_\alpha, \Phi]] + c_r R\Phi^2 + c_2 \Phi^2 + c_4 \Phi^4 \right), \quad (2.7)$$

the field $\Phi$ being $N \times N$ hermitian matrix, $P_\alpha$ momenta and $R$ the curvature of 4th space projected onto $Z = 0$ section

$$P_1 = -Y, \quad P_2 = X, \quad R = R \pm 8 \left( X^2 + Y^2 \right). \quad (2.8)$$

All originally dimensionfull quantities are expressed in units of $\mu$. We used hybrid Monte Carlo, executed in $2^6$ parallel threads each with at least $2^{10}$ decorrelated steps, to measure thermodynamic observables:

- energy $E = \langle S \rangle$,
- heat capacity $C = \langle S^2 \rangle - \langle S \rangle^2$,
- magnetization $M = \langle |\text{Tr } \Phi| \rangle$,
- magnetic susceptibility $\chi = \langle |\text{Tr } \Phi|^2 \rangle - \langle |\text{Tr } \Phi| \rangle^2$,
- Binder cumulant $U = 1 - \langle |\text{Tr } \Phi|^4 \rangle / (3^2 \langle |\text{Tr } \Phi|^2 \rangle^2)$,

as well as the control Schwinger-Dyson identity

$$\langle \text{Tr} \left( 2c_k \Phi[P_\alpha, [P_\alpha, \Phi]] + 2c_r R\Phi^2 + 2c_2 \Phi^2 + 4c_4 \Phi^4 \right) \rangle = N^2. \quad (2.9)$$

We also kept an eye on the distribution of eigenvalues and traces of the field. Expectation value $\langle \mathcal{O} \rangle$ of the observable $\mathcal{O}$ is given by

$$\langle \mathcal{O} \rangle = \frac{\int d\Phi \mathcal{O} \exp(-S)}{\int d\Phi \exp(-S)}. \quad (2.10)$$
Figure 1. Thermodynamic observables for $N = 16$, $c_k = 1$, $c_r = 0$, $\tilde{c}_4 = c_4/N = 0.25$, as functions of rescaled mass parameter $\tilde{c}_2 = c_2/N$. Errorbars are mostly covered by data markers. $M_{\text{max}} = N\sqrt{-\tilde{c}_2/(2\tilde{c}_4)}$ is defined by maximal trace of the PP solution in (2.12). We see two transitions as two peaks in $C$ and matching (would-be-) peaks in $\chi$. We also clearly see two phases in plots of $M$ and $U$, and the indication of the third phase as dents in their slope profiles. Disordered phase is colored in orange and ordered phases in different shades of blue.

We computed standard uncertainties $\Delta O$ from decorrelated data at 68% confidence level. Phase transitions in finite system manifest as smeared finite peaks and edges in relevant quantities. We scanned through parameter space by varying mass parameter at fixed quartic coupling and searched for peaks in $C$ and $\chi$ (Figure 1). For finite $N$ they do not coincide perfectly, but they converge when matrix size increases. We modeled peaks with triangular distribution of width $w$ and then took $w/(2\sqrt{6})$ as a measure of uncertainty of their position, which gives 65% confidence interval. The edges of triangular distribution had to be at least $2 - 3$ standard errors below the best choice for the maximum, and there had to be at least 2 points in proper increasing/decreasing order on the each side of the maximum.

In the absence of kinetic and curvature terms, it is possible to simplify the integration over hermitian matrices in (2.10), leaving only computationally much cheaper integration over eigenvalues. Since in our case it is not possible to simultaneously diagonalize all four terms, this simplification could not be utilized and we had to settle with working with relatively small matrix sizes.

Already the analysis of the classical action provides a clue about the structure of the phase diagram. We assume $c_4 > 0$, to ensure that $S$ is bounded from below. The equation
of motion reads
\[ 2c_k [P_\alpha, [P_\alpha, \Phi]] + c_r \{ R, \Phi \} + \Phi (2c_2 + 4c_4\Phi^2) = 0, \tag{2.11} \]
and its kinetic, curvature and pure potential parts have respective solutions:
\[
\Phi = \frac{\text{Tr} \Phi}{N} \mathbb{1}, \quad \Phi = 0, \quad \Phi^2 = \begin{cases} \frac{c_2}{2c_4} \mathbb{1} \quad \text{for } c_2 \geq 0, \\
-\frac{c_2}{2c_4} \mathbb{1} \quad \text{for } c_2 < 0. \end{cases} \tag{2.12} \]

Obviously, competition is at work between three types of vacua characteristic of three phases discovered in the related matrix models [21]:

- **disordered phase**: dominant contributions come from oscillations around the trivial vacuum \( \langle \Phi \rangle_\uparrow = 0 \),

- **non-uniformly ordered phase** (striped phase, matrix phase): dominant contributions come from oscillations around \( \langle \Phi \rangle_{\uparrow \downarrow} \propto U \mathbb{1}_\pm U^\dagger \), \( U \) being a unitary matrix and \( \mathbb{1}_\pm \) non-trivial square roots of identity matrix,

- **uniformly ordered phase** (Ising phase): dominant contributions come from oscillations around \( \langle \Phi \rangle_{\uparrow \uparrow} \propto \mathbb{1} \).

We will denote them \( \uparrow, \uparrow \downarrow \) and \( \uparrow \uparrow \), respectively. The pure potential (PP) model, with only mass and quartic term, exhibits the \( \uparrow \) phase for \( c_2 > 0 \) and a phase transition between \( \uparrow \) and \( \uparrow \downarrow \) phases for \( c_2 < 0 \). When the kinetic term is turned on, the \( \uparrow \uparrow \) phase also appears.

The phases can also be characterised by field’s eigenvalue distribution: whether it has connected or disconnected support (one- or multi-cut) and whether it is symmetric or asymmetric. Symmetric connected deformed Wigner semicircle distribution corresponds to \( \uparrow \) phase, two-cut distribution to \( \uparrow \downarrow \) and asymmetric one-cut distribution to \( \uparrow \uparrow \) phase (Figure 2). Additionally, Binder cumulant changes sigmoidally with mass parameter, going from 0 in the \( \uparrow \) phase to 2/3 in the \( \uparrow \uparrow \) phase, deviating into a valley in the \( \uparrow \downarrow \) phase (Figure 1).

The possibility arises of the novel modification of ordered phases. In the limit of negligible kinetic term, a diagonal solution exists that combines the effects of the curvature and the potential
\[ \Phi^2 = -\frac{c_2}{2c_4} \mathbb{1} + c_r R, \tag{2.13} \]
provided that
\[ c_2 \leq \min_j (-c_r R_{jj}). \tag{2.14} \]

A preliminary analysis of positions of peaks of distribution of eigenvalues and traces seem to corroborate this. We here concentrate mostly on the model without curvature, while the detailed investigation of curvature effects is pending.

For the inspected part of parameter space, the \( \uparrow \rightarrow \uparrow \downarrow \) transition is visible for \( N \geq 16 \) and the transition to \( \uparrow \uparrow \) phase is hard to access (similarly to [31]) for values of \( c_4 \) that allow all 3 phases to occur. The anchoring of the phase diagram is done mostly on the \( \uparrow \rightarrow \uparrow \uparrow \) transition line.
Figure 2. Characteristic shapes of eigenvalue ($\lambda$) and action ($S$) distributions ($\rho$) in different phases for $N = 16$, $c_k = 1$, $c_r = 0$, $\tilde{c}_4 = 0.25$, and $\tilde{c}_2 = 0.5$ (top left), $\tilde{c}_2 = 1.0$ (top right), $\tilde{c}_2 = 1.4$ (bottom left), $\tilde{c}_2 = 2.0$ (bottom right). $\lambda$ is given in units of $M_{\text{max}}/N = \sqrt{-\tilde{c}_2/2\tilde{c}_4}$ and $S$ in units of $N^2\tilde{c}_2^2/(4\tilde{c}_4)$. Bottom left figure lives near the border of two ordered phases, and we see two competing energy levels each belonging to one of them. This is typical of 1st order transition.

3 Scaling

Phase diagram of family of models $S_N(c_k, c_r, c_2, c_4; \Phi)$ is expected to converge to a well defined non-trivial large $N$ limit only if we properly choose the scaling of the models’ parameters. We will denote scaling of a quantity $q$ with $\nu_q$, so that

$$q = \tilde{q}N^{\nu_q},$$

where $\nu_S = 2$ stands for the scaling of the action, $\nu_{\Phi}$ for the field/its eigenvalues, $\nu_P = 1/2$ for the momenta, $\nu_R = 1$ for the curvature and $\nu_k$, $\nu_r$, $\nu_2$, $\nu_4$ for the coefficients in front of the kinetic, curvature, mass and quatic term respectively.

Requiring each term in the action to behave as $\mathcal{O}(N^2)$ leads, by power counting, to system of equations (Tr increases power by 1)

$$\nu_S = \nu_k + 2\nu_P + 2\nu_{\Phi} + 1$$  
$$\nu_S = \nu_r + \nu_R + 2\nu_{\Phi} + 1$$  
$$\nu_S = \nu_2 + 2\nu_{\Phi} + 1$$  
$$\nu_S = \nu_4 + 4\nu_{\Phi} + 1$$
solved by
\[ \nu_4 = 2\nu_2 - 1, \quad \nu_r = \nu_2 - \nu_R, \quad \nu_k = \nu_2 - 2\nu_P \quad 2\nu_\Phi = 1 - \nu_2. \] (3.2)

For values of \( \nu_2 \) and \( \nu_4 \) used in the PP model and on the fuzzy sphere, this amounts to
\[ \nu_2 = 3/2, \quad \nu_4 = 2, \quad \nu_r = 1/2, \quad \nu_k = 1/2, \quad \nu_\Phi = -1/4. \] (3.3)

We wish to examine a simpler choice:
\[ \nu_2 = 1, \quad \nu_4 = 1, \quad \nu_r = 0, \quad \nu_k = 0, \quad \nu_\Phi = 0. \] (3.4)

We will also, without loss of generality, set \( \tilde{c}_k = 1 \).

The wrong choice of scaling would instead of large \( N \) stabilization cause the drifting of transition points either towards zero or infinite values in the parameter space. This can be used to identify the correct choice of scaling. It turns out, however, that discriminating between choices based on data is not trivial.

We will first look at the PP term and then see how the kinetic and the curvature terms behave against this well established background.

4 Pure potential term

The PP model is well studied both analytically and numerically so it can provide the basic calibration of the method. Let us denote the correct scaling as in the previous section and the scaling we wish to test with primes
\[ c'_i = \tilde{c}_i N^{\nu'_i} = \tilde{c}_i N^{\nu_i} N^{\nu'_i - \nu_i} = c_i N^{\Delta \nu_i}, \quad i = 2, 4. \] (4.1)

Since for the correct scaling and \( N \to \infty \) the phase transition happens at
\[ \tilde{c}_2 = -2\sqrt{\tilde{c}_4}, \] (4.2)

for the primed scaling this becomes
\[ \tilde{c}_2 N^{\Delta \nu_2} = -2\sqrt{\tilde{c}_4 N^{\Delta \nu_4}}. \] (4.3)

The slope of the logarithmic plot of the last equality
\[ \log |\tilde{c}_2| = \frac{\Delta \nu_4 - 2\Delta \nu_2}{2} \log N + \frac{\log 4\tilde{c}_4}{2} \] (4.4)
is therefore changed from zero (up to \( O(1/N) \) effects) to \( \Delta \nu_4/2 - \Delta \nu_2 \), and Figure 3 and Table 1 show how it is affected by different choices of scaling.

Both \( \nu_2 = 3/2, \nu_4 = 2 \) and \( \nu_2 = 1, \nu_4 = 1 \) lead to the correct zero slope and therefore to matrix size independent behaviour. The latter choice automatically collapses the peaks of \( \chi \), while for the former they scale approximately as \( \sqrt{1/N} \).

That both peaks of \( \chi \) and \( C \) converge to the same value is demonstrated for \( \tilde{c}_4 = 0.01 \), where the the large \( N \) limit of the transition \( \tilde{c}_2 \) gives respective values \(-0.201(8)\) and \(-0.215(7)\); the theoretical value is \(-0.2\).
There is a slight systematic difference (+0.04 on average) between measured and theoretical slopes in Table 1. It can be explained as a finite size effect, that disappears for large enough matrices. Namely, since the equation (4.2) is based on the infinite matrix limit, we could account for the finite matrix size by modifying it into

$$\tilde{c}_2 = -2\sqrt{\tilde{c}_4 \left(1 + \frac{\gamma}{\sqrt{N}} + \cdots\right)},$$  \hspace{1cm} (4.5)$$

which in turn modifies (4.4) into

$$\log |\tilde{c}_2| = \frac{\Delta \nu_4 - 2\Delta \nu_2}{2} \log N + \frac{\log 4\tilde{c}_4}{2} - \frac{\gamma}{2\sqrt{N}}.$$ \hspace{1cm} (4.6)$$

The modified plot is indiscernible from the linear one on the data points, but the intercept and the slope of $\log |\tilde{c}_2| + \gamma/(2\sqrt{N})$ is perfectly aligned with the theoretical value. The choice of $\sqrt{1/N}$- over $1/N$-series will be addressed later.

The results in this section justify the assumption that both conventional and tested choice of scaling are valid, and that there are in fact infinitely many possible ones.
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$\nu_4$ & intercept & slope &  \\
& expected & measured & expected & measured \\
\hline
0.0 & $-1.75(4)$ & $-0.50$ & $-0.47(2)$ &  \\
0.5 & $-1.77(4)$ & $-0.25$ & $-0.21(2)$ &  \\
1.0 & $-1.61$ & 0.00 & $+0.05(2)$ &  \\
1.5 & $-1.78(3)$ & $+0.25$ & $+0.30(2)$ &  \\
2.0 & $-1.76(4)$ & $+0.50$ & $+0.53(2)$ &  \\
\hline
\end{tabular}
\caption{log $|\tilde{c}_2|$ vs. log $N$ linear fits for $\chi$-transitions for $\tilde{c}_4 = 0.01$, $\nu_2 = 1$ and different choices of $\nu_4$.}
\end{table}

5 Curvature term

Let us briefly inspect the relevant case where $c_r < 0$. The NC curvature of the model is a negative diagonal matrix

$$R_{jj} = R + 8 - \begin{cases} 16j, & 1 < j < N, \\ 8N, & j = N, \end{cases} \quad (5.1)$$

where $R = 15/2$; in simulation we erroneously used $R = 15/4$ but that does not change the conclusions of this section because they depend on $O(N)$ part of curvature. Diagonality yields $\text{Tr} (R \Phi^2) = R_{jj} (\Phi^2)_{jj}$, bounding the curvature term in the action by

$$\text{Tr} \left( |c_r| \min_j |R_{jj}| \Phi^2 \right) \leq \text{Tr} \left( c_r R \Phi^2 \right) \leq \text{Tr} \left( |c_r| \max_j |R_{jj}| \Phi^2 \right), \quad (5.2)$$

which translates to

$$\text{Tr} \left( (8 - R) |\tilde{c}_r| \Phi^2 \right) \leq \text{Tr} \left( c_r R \Phi^2 \right) \leq \text{Tr} \left( (16N - (24 + R)) |\tilde{c}_r| \Phi^2 \right). \quad (5.3)$$

Treating this as a bounded contribution to the mass term, we could naively expect it to be reflected in a deformation of the transition line $\tilde{c}_2 \to \tilde{c}_{2,r}$

$$\tilde{c}_2 - \left( 16 - \frac{24 + R}{N} \right) |\tilde{c}_r| \leq \tilde{c}_{2,r} \leq \tilde{c}_2 - \frac{8 - R}{N} |\tilde{c}_r|. \quad (5.4)$$

The wrong choice of scaling would change this into

$$\tilde{c}_2 - \left( 16 - \frac{24 + R}{N} \right) |\tilde{c}_r| N^{\Delta\nu_r} \leq \tilde{c}_{2,r} \leq \tilde{c}_2 - \frac{8 - R}{N} |\tilde{c}_r| N^{\Delta\nu_r}. \quad (5.5)$$

This means that for $\Delta\nu_r < 0$ we would practically see the PP case and for $\Delta\nu_r > 0$ the $N^{\Delta\nu_r}$ runaway effect towards large negative values of the mass parameter.

This is exactly what we see in Figure 4. There are multiple peaks of $M$ for $\Delta\nu_r = 1$, the topmost coinciding with the peaks of $\chi$ which we use as the indicator of the phase transition. The equation of the line traversing them

$$1.01(3) \log N - 1.83(9) - \frac{2.0(2)}{N} \quad (5.6)$$
Figure 4. $\uparrow\rightarrow\uparrow\downarrow$ transition in the PP model with curvature for $\tilde{c}_4 = 0.01$, $\tilde{c}_r = -0.01$, $4 \leq N \leq 16$ and fixed $\nu_2 = \nu_4 = 1$, observed as peaks in $\chi$ and $M$. The green/center data represents the correct choice of scaling $\nu_r = 0$, the orange/top $\Delta\nu_r = +1$ and the red/bottom $\Delta\nu_r = -1$. For $\Delta\nu_r = +1$, $M$ peaks multiple times until $\chi$ reaches its maximum. The pale-red dashed line represents the PP model. Errorbars are mostly covered by data markers and pale coloured stripes represent the 68% confidence intervals.

fits very well with the expansion of the left-hand side of (5.5) (with (4.2) substituted)

$$\log N - 1.83 - \frac{2.98}{N},$$

and the slope 1.01(3) with $\Delta\nu_r = 1$.

Looking at the eigenvalue distribution as the mass parameter decreases towards larger negative values, one by one separate peaks break off the edge of the shrinking deformed Wigner semi-circle. Meanwhile the trace distribution stays centered at zero. Finally, when the Wigner semi-circle completely disappears, trace distribution starts to show off-zero peaks. We tentatively interpret this as curvature eigenvalues activating one by one in pairs of opposite signs, introducing more and more order into the disordered phase, until the susceptibility peaks and system transitions into a modified matrix phase.

The left-hand side of (5.5) also predicts the shift between the $\Delta\nu_r = 0$ and the PP-line to be less than $16|\tilde{c}_r| = 0.16$ and the actual difference at $N = 16$ is 0.15(4). As for the $\Delta\nu_r = -1$ line, it is practically indiscernible from the PP-line, as expected.

6 Kinetic term

As far as transitions go, the action with $(\tilde{c}_k N^{\Delta\nu_k}, \tilde{c}_2, \tilde{c}_4)$ is equivalent, via absorption of the coefficient into the field, to the one with $(\tilde{c}_k, \tilde{c}_2 N^{-\Delta\nu_k}, \tilde{c}_4 N^{-2\Delta\nu_k})$. Thus would the wrong choice of scaling force the transition points to drift towards zero or the infinity.
The analysis is now complicated by the fact that we lack the analytical prediction for the transition line with kinetic term turned on, so the exact rate of the above mentioned drift is unknown. Furthermore, discrimination of different scalings based on the data is not clear cut. For example, although Figure 5 shows convincing convergence, looking at the transition plots for $\nu_k = 0$ and $\nu_k = 0.5$ in Figure 6, it is not immediately clear which represents the correct choice. At first glance, the wrong choice $\nu_k = 0.5$ appears to converge to a non-trivial finite value instead of zero, and the correct choice $\nu_k = 0$ to ever increase, possibly towards infinity. One reason for this could be the convergence of the position of the triple point with increasing $N$ closer towards the origin — the effect demonstrated in [39] — causing the system with fixed $\tilde{c}_4$ to go from 2-phase to 3-phase regime as $N$ increases. The other explanation could be the anomalous negative scaling of the kinetic term, causing the shift towards infinity. Using our data it is not possible to rule out the second option and fix the scaling to precision less than $\pm0.5$, as this would require inspecting much larger matrices. However we can strengthen the case for the choice $\nu_k = 0$.

Firstly, Figure 6 (top) allows finite near-linear extrapolation for $1/N \to 0$ (in green and blue). Secondly, the change from 2-phase to 3-phase regime for smaller examined $\tilde{c}_4$ happens at larger $N$, which is consistent with triple point converging towards smaller $\tilde{c}_4$. Thirdly, as we will see, extrapolation of the data for $N < 16$ (in red and orange) converges to a value consistent with stable linear transition line passing through other smaller values of $\tilde{c}_4$: had the system not entered 3-phase regime with increasing $N$, the transition line would have passed through $\tilde{c}_4 = 0.01$ as well at this extrapolated value of $\tilde{c}_2$. 

Figure 5. $\downarrow\uparrow\uparrow\uparrow$ transition for $\tilde{c}_4 = 0.001$, $\nu_k = 0$ and $N \leq 40$, observed as peaks in $C$ (orange/top) and $\chi$ (red/bottom). Pale-coloured stripes represent the 68% confidence intervals. The large $N$ limit is zoomed-in.
Figure 6. (top) Transitions for $\tilde{c}_4 = 0.01$, $\nu_k = 0$, $N \leq 50$ with zoomed-in large $N$ limit. Top plots represent $C$ (red and green) and the bottom ones $\chi$ (orange and blue). $N < 16$ is the 2-phase regime (red and orange) and $N > 16$ is the 3-phase regime (blue and green). The $\downarrow\rightarrow\uparrow\downarrow$ transition peak fully separates from $\uparrow\downarrow\rightarrow\uparrow\uparrow$ peak for $N \geq 50$. Pale-coloured stripes represent the 68% confidence intervals. (bottom) Transitions for $\tilde{c}_4 = 0.01$, $\nu_k = 0.5$, $N \leq 32$ with two zoomed-in regions. The orange/top line represents the linear fit for $N \geq 8$, the red/bottom one is our model’s prediction. Pale-coloured stripes represent the 68% confidence intervals.
Figure 7. $\alpha(N)$ and $\beta(N)$ coefficients of the $\downarrow\rightarrow\uparrow\uparrow$ transition line constructed from peaks in $C$ (orange/larger errors) and $\chi$ (red/smaller errors) for $N \leq 40$. Pale-coloured stripes represent the 68% confidence intervals. The large $N$ limits are zoomed-in. As we can see, the $\alpha$-governed square root behaviour of the transition line completely disappears in the infinite matrix limit, leaving only the linear one.
The model on the fuzzy sphere [39] exhibits linear $\uparrow\rightarrow\uparrow\uparrow$ transition line in the large $N$ limit

$$|\tilde{c}_2| \propto \tilde{c}_4.$$  

(6.1)

In our model, transition for $\nu_k = 0$ and fixed $N$ appears to follow the empirical law

$$|\tilde{c}_2| = \alpha(N)\sqrt{\tilde{c}_4} + \beta(N)\tilde{c}_4,$$  

(6.2)

where $\alpha(N)$ decreases for larger matrices (Figure 7). The coefficients remain stable when higher power of $\tilde{c}_4$ is added, while the uncertainty makes the higher term indistinguishable from zero. We are hoping that RG approach [40–42] could replicate this form of the transition line; the work on this is currently on the way.

The wrong choice of scaling would transform (6.2) into

$$|\tilde{c}_2|N^{-\Delta\nu_k} = \alpha(N)\sqrt{\tilde{c}_4N^{-2\Delta\nu_k}} + \beta(N)\tilde{c}_4N^{-2\Delta\nu_k},$$  

(6.3)

giving

$$|\tilde{c}_2| = \alpha(N)\sqrt{\tilde{c}_4} + \beta(N)\tilde{c}_4\left(\frac{1}{N}\right)^{\Delta\nu_k}.$$  

(6.4)

We examined several models of perturbative expansion of $\alpha$ and $\beta$ as well as a few non-perturbative ones; we did not examine the more complicated possibility that they contain residual dependence on $\tilde{c}_4$. The series in $1/\sqrt{N}$ showed excellent agreement with the collected data:

$$\alpha(N) = \sum_{k=0}^{\infty} \frac{\alpha_i}{\sqrt{N^k}}, \quad \beta(N) = \sum_{k=0}^{\infty} \frac{\beta_i}{\sqrt{N^k}}. \quad \text{(6.5)}$$

In order to access the large $N$ convergence of the transition line and subsequently that of $\alpha$ and $\beta$, we compared two approaches:

- method I: for fixed $\tilde{c}_4$ and various fixed $\nu_k$, we varied $N$ and for each detected $\tilde{c}_2(N)$ at which transition occurs; we then fitted the $1/\sqrt{N}$-expansion of (6.4) to get the combinations of $\alpha_i$, $\beta_i$ and $\tilde{c}_4$ (Table 2);
- method II: for fixed $N$ and $\nu_k = 0$, we constructed the transition line for a range of $\tilde{c}_4$ and then extracted $\alpha(N)$ and $\beta(N)$ using (6.2); we then varied $N$ and fitted series (6.5) to get $\alpha_i$ and $\beta_i$ (Figure 7).

The comparison of these two approaches is given in Table 2: we see that the choice of $\nu_k = 0$ scaling of the kinetic term leads to consistent values for coefficients of the transition line. Also, with increasing matrix size $\Delta\nu_k > 0$ transition points collapse to zero in the predicted manner which is for $\Delta\nu_k \geq 1$ practically linear.

Applying method II to the $\chi$-data from Figure 7, we get the following expansions

$$\alpha(N) = 0.01(2) + \frac{0.07(7)}{\sqrt{N}} + \frac{2.06(9)}{N},$$  

(6.6a)

$$\beta(N) = 10.5(5) - \frac{31(4)}{\sqrt{N}} + \frac{43(9)}{N} - \frac{24(8)}{N\sqrt{N}}.$$  

(6.6b)
| $\mathcal{O}$ | $\Delta \nu_k$ | $\tilde{c}_4$ | expression | method I | method II |
|-----------|-------------|----------|------------|----------|----------|
| $N$       | −1.0       | $1 \cdot 10^{-5}$ | $\beta_0$ | 11.4(9)  | 10.5(5)  |
| $\sqrt{N}$ | −1.0       | $1 \cdot 10^{-5}$ | $\beta_1$ | −36(6)   | −31(4)   |
|           | −0.5       | $5 \cdot 10^{-3}$ | $\beta_0$ | 10.6(6)  | 10.5(5)  |
|           | −1.0       | $1 \cdot 10^{-5}$ | $\beta_2 + \alpha_0/\sqrt{\tilde{c}_4}$ | 55(9) | 46(9) |
|           | −0.5       | $5 \cdot 10^{-3}$ | $\beta_1 + \alpha_0/\sqrt{\tilde{c}_4}$ | −32(3) | −31(4) |
|           | 0.0        | $1 \cdot 10^{-3}$ | $\beta_0 + \alpha_0/\sqrt{\tilde{c}_4}$ | 10.7(5) | 10.8(5) |
|           | 2.0        | $1 \cdot 10^{-2}$ | $\alpha_0$ | −0.00(4) | 0.1(2) |
| $\frac{1}{\sqrt{N}}$ | −0.5       | $5 \cdot 10^{-3}$ | $\beta_2 + \alpha_1/\sqrt{\tilde{c}_4}$ | 50(3) | 44(9) |
|           | 0.0        | $1 \cdot 10^{-3}$ | $\beta_1 + \alpha_1/\sqrt{\tilde{c}_4}$ | −24(4) | −29(4) |
|           | 0.5        | $1 \cdot 10^{-2}$ | $\beta_0 + \alpha_1/\sqrt{\tilde{c}_4}$ | 12(1) | 11.2(9) |
|           | 2.0        |                        | $\alpha_0$ | −0.0(2) | 0.07(7) |
| $\frac{1}{N}$ | −1.0       | $1 \cdot 10^{-5}$ | $\alpha_2 + \beta_4\sqrt{\tilde{c}_4}$ | 1.99(7) | 2.0(2) |
|           | 0.0        | $1 \cdot 10^{-3}$ | $\beta_2 + \alpha_2/\sqrt{\tilde{c}_4}$ | 86(6) | 109(9) |
|           | 0.5        | $1 \cdot 10^{-2}$ | $\beta_1 + \alpha_2/\sqrt{\tilde{c}_4}$ | −17(8) | −11(4) |
|           | 1.0        | $1 \cdot 10^{-2}$ | $\alpha_2 + \beta_0\sqrt{\tilde{c}_4}$ | 3.3(3) | 3.1(1) |
|           | 2.0        |                        | $\alpha_2$ | 2.3(3) | 2.06(9) |

Table 2. Comparison of the estimates of $\alpha_i$ and $\beta_i$ using fits for different $\Delta \nu_k$ and fixed $\tilde{c}_4$ (method I, using (6.4)) to the estimates from $\nu_k = 0$ and variable $\tilde{c}_4$ and $N$ (method II, using (6.6)).

where we used the lowest order polynomial in $1/\sqrt{N}$ that fits well with the data. The higher terms turn out to be indiscernible form zero within their large uncertainties. The C-data have much less predictive power since the peaks of $C/N^2$ are wide, skewed, nearly flat and do not scale with $N$, unlike the peaks in $\chi$ which scale linearly.

We can now explain peculiar behaviour of $\nu_k = 0.5$ plot in Figure 6. Combining (6.4) and (6.6), we expect it to change as

$$\frac{\alpha_1\sqrt{\tilde{c}_4} + \beta_0\tilde{c}_4}{\sqrt{N}} + \frac{\alpha_2\sqrt{\tilde{c}_4} + \beta_1\tilde{c}_4}{N} + \frac{\alpha_3\sqrt{\tilde{c}_4} + \beta_2\tilde{c}_4}{N\sqrt{N}},$$

having near constant slope around

$$N = 3 \cdot \frac{\alpha_3\sqrt{\tilde{c}_4} + \beta_2\tilde{c}_4}{\alpha_1\sqrt{\tilde{c}_4} + \beta_0\tilde{c}_4} \approx 3 \cdot \frac{\beta_2}{\beta_0} = 12(3),$$

which falls right in the middle of observed flat region $8 \leq N \leq 32$ on $1/N$ axes, but would ultimately behave as $1/\sqrt{N}$ for large enough matrices.

7 Phase Diagram

Let us now look at Figure 8 and the structure of the phase diagram for $N = 24$ obtained from peaks in $C$. From $\tilde{c}_4 = 0$ to $\tilde{c}_4 \approx 0.015$, stretches the $\uparrow\rightarrow\uparrow\uparrow$ transition line that can
be approximated as
\[ L_1 : \quad |\tilde{c}_2| = 0.0015(4) + 8.8(1)\tilde{c}_4, \quad (7.1) \]
followed by the ↑↓→↑↑ transition line
\[ L_3 : \quad |\tilde{c}_2| = -0.007(3) + 9.4(1)\tilde{c}_4, \quad (7.2) \]
The slopes of these lines are very similar, making it difficult to determine which points belong to which line: this has to be determined from the \( \chi \)-data in Figure 8 that clearly shows the transition from \( L_1 \) to the \( L_3 \). Near \( \tilde{c}_4 \approx 0.05 \), \( C \)-diagram enters a 3-phase regime and ↓→↑↓ transition line appears, which is linear for smaller \( \tilde{c}_4 \)
\[ L_2 : \quad |\tilde{c}_2| = -0.12(3) + 3.5(3)\tilde{c}_4, \quad (7.3) \]
and for larger values of \( \tilde{c}_4 \) exhibits square root behaviour characteristic for the limiting PP model
\[ |\tilde{c}_2| = 2.62(5)\sqrt{\tilde{c}_4} - 0.48(5) + \frac{0.039(9)}{\sqrt{\tilde{c}_4}}. \quad (7.4) \]
This can also be seen on the fuzzy sphere [29], where it holds
\[ |\tilde{c}_2| = 2.5\sqrt{\tilde{c}_4} + \frac{0.5}{1 - \exp(1/\sqrt{\tilde{c}_4})} \approx 2\sqrt{\tilde{c}_4} + 0.25 - \frac{0.042}{\sqrt{\tilde{c}_4}}. \quad (7.5) \]
It would be interesting to compare these two once the large \( N \) extrapolation of the \( L_2 \) is obtained. A very crude linear extrapolation of \( N = 16, 20, 24 \) gives promising \( 2.0(4) \) for the square root coefficient.

The extrapolation of \( L_2 \) intersects \( L_{1/3} \) at \( \tilde{c}_4 \approx 0.02 \), which is in the vicinity of the meeting point of \( L_1 \) and \( L_3 \) at \( \tilde{c}_4 \approx 0.015 \), placing the would-be triple point nearby. The pale yellow triangle formed by the meeting point of \( L_1 \) and \( L_3 \) and the starting point of \( L_2 \) should collapse into a triple point when \( N \to \infty \). This effect is in fact demonstrated on the fuzzy sphere [39]. In this region the two transition peaks are still convoluted into a single one (like peaks of \( \chi \) in Figure 1).

Expression for \( L_3 \) should be taken with a grain of salt. This is where the ergodicity of algorithm starts to falter, contributing to an unknown systematic error.

Based on the analysis of \( \alpha(N) \) and \( \beta(N) \) from Figure 7, the ↓→↑↑ transition line in the large \( N \) limit extrapolates to
\[
C : \quad |\tilde{c}_2| = -0.03(7)\sqrt{\tilde{c}_4} + 13(3)\tilde{c}_4, \quad (7.6a)
\]
\[
\chi : \quad |\tilde{c}_2| = +0.01(2)\sqrt{\tilde{c}_4} + 10.5(5)\tilde{c}_4, \quad (7.6b)
\]
These two expressions agree, as they should, or we could otherwise conclude that the triple point is located at the origin, and that 3-phase regime exists throughout the parameter space. Apparently, the \( \sqrt{c_1} \) effect completely disappears.

Equation of the ↓→↑↑ line in Figure 9, obtained from from linear fit through large \( N \) limits at fixed \( \tilde{c}_4 \), reads
\[
\chi : \quad |\tilde{c}_2| = +0.0004(3) + 10.1(5)\tilde{c}_4. \quad (7.7)
\]
Figure 8. Phase diagram for $N = 24$. Pale-gray stripes represent 68% confidence intervals. Top diagram uses $C$-data and bottom one $\chi$-data.
Based on the extrapolation estimates of points that with increasing matrix size switch from 2-phase to 3-phase regime, there exists a possibility of systematic error from such still unidentified points, that could lower the true slope in (7.7). Namely, as triple point slides towards zero, it deforms the about-to-be-shortened end of the transition line close to it towards the less slanted $\downarrow\rightarrow\uparrow\downarrow$ transition line. Also, inclusion of the $\tilde{c}_4^{3/2}$ term into (6.2) gives somewhat higher estimates for the linear term, although consistent with the reported ones.

The smallest $\tilde{c}_4$ for which we detected change from 2-phase to 3-phase regime is $\tilde{c}_4 = 0.005$ at $N = 28$. For all $\tilde{c}_4 < 0.005$ and $N \leq 40$ we see only two phases. This implies that $\downarrow\rightarrow\uparrow\uparrow$ transition line ends in the triple point at $\tilde{c}_4(T) \leq 0.005$.

8 Conclusion

We detailedly tested several choices for scaling of terms in the action of our model and chose the convergent albeit non-standard one: $\nu_2 = 1$, $\nu_4 = 1$, $\nu_r = 0$, $\nu_k = 0$. The choice replicated the known results for the PP model. Varying scalings around this choice led to transition lines without stable non-trivial infinite matrix limit. We semi-empirically determined equation (6.2) of the $\downarrow\rightarrow\uparrow\uparrow$ transition line when the kinetic term is turned on and found that it contains a part that captures the finite size effects and which disappears for larger matrices. The careful inspection of various scalings using two different approaches allowed us to non-trivially extrapolate this line from relatively small matrix sizes to the large $N$ limit.

We mapped phases of the model with turned off curvature in mass parameter-quartic coupling plane. The resulting diagram for $N = 24$ is presented in Figure 8 and it consists, as expected, of three phases with different degree and kind of field eigenvalue activation. $\downarrow\rightarrow\uparrow\downarrow$ transition appears to be 3rd-order and $\uparrow\downarrow\rightarrow\uparrow\uparrow$ 1st-order. As for the $\downarrow\rightarrow\uparrow\uparrow$ transition,
specific heat for large matrices is practically constant compared to its large mass limit and fine details are buried under the data uncertainty. However, peaks in susceptibility scale linearly with matrix size, unlike in $\uparrow\downarrow\to\uparrow\downarrow$ transition where they stay roughly constant, which indicates that this transition could be 2nd-order.

In Figure 9, we extrapolated $\uparrow\downarrow\to\uparrow\uparrow\uparrow$ border using matrices of sizes $N \leq 40$ and observed a convincing convergence. The extrapolated line radiates from the origin with the slope 10.6(3) (the combined estimate). This could be the consequence of shortness of $\uparrow\downarrow\to\uparrow\uparrow\uparrow$ line, but clear disappearance of square root effects in Figure 7 indicates that the line is indeed linear. This surprisingly linear behaviour is observed also in the model on fuzzy sphere [39]. We also demonstrated phase diagram convergence on token points from $\uparrow\downarrow\to\uparrow\uparrow\uparrow$ and $\uparrow\downarrow\to\uparrow\downarrow\downarrow$ transition lines. This is the part of the ongoing work of finding their large $N$ limits.

The triple point of the model is estimated to lie at $\tilde{c}_4(T) \leq 0.005$. This is significantly smaller than in the model on fuzzy sphere [39] where the best estimate is $\tilde{c}_4(T) = 0.021(2)$, especially when larger matrices could pull it even closer to the origin. We still do not have enough data to extrapolate its drifting towards origin to its final position. The methods of Section 6 could rule out the possibility that triple point lies at the origin. Namely, if we manage to choose such scaling that stretches equally both $\tilde{c}_2$ and $\tilde{c}_4$ axes and reverses the direction of triple point drifting, that would mean that, with the original scaling, its coordinates have non-zero values. Once we find the limits of the remaining transition lines, we will be able to pinpoint it properly.

We also plan to compare these extrapolated lines with recent analytical results [29] for the fuzzy sphere in the regime where the two models could behave similarly, namely $\uparrow\downarrow\to\uparrow\uparrow\uparrow$ line for large $\tilde{c}_4$, where they should mimic the PP model, and $\uparrow\downarrow\to\uparrow\uparrow\uparrow$ line where kinetic terms grow smaller as the field, up to a prefactor, oscillates closer to identity matrix.

While inspecting the scaling of the curvature term, we confirmed that it alters both eigenvalue distribution and the border of $\uparrow\downarrow$ phase when added to the PP model. Based on a cross section of the diagram, it seems that $\downarrow\to\uparrow\downarrow$ line gets shifted proportionally to the curvature parameter $|\tilde{c}_r|$ and to the scaled maximal eigenvalue of curvature. The next important step is to see how it affects the full model in order to shed more light on its connection to renormalizability: the work on it is on the way.

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