Higgs-Boson Production and Decay Close to Thresholds

BERND A. KNIEHL, 1 CAESAR P. PALISOC, 2 ALBERTO SIRLIN 1, 3
1 II. Institut für Theoretische Physik, Universität Hamburg,
Luruper Chaussee 149, 22761 Hamburg, Germany
2 National Institute of Physics, University of the Philippines,
Diliman, Quezon City 1101, Philippines
3 Department of Physics, New York University,
4 Washington Place, New York, New York 10003, USA

Abstract

At one loop in the conventional on-mass-shell renormalization scheme, the production and decay rates of the Higgs boson $H$ exhibit singularities proportional to $(2M_V - M)^{-1/2}$ as the Higgs-boson mass $M$ approaches from below the pair-production threshold of a vector boson $V$ with mass $M_V$. This problem is of phenomenological interest because the values $2M_W$ and $2M_Z$, corresponding to the $W$- and $Z$-boson thresholds, lie within the $M$ range presently favoured by electroweak precision data. We demonstrate how these threshold singularities are eliminated when the definitions of mass and total decay width of the Higgs boson are based on the complex-valued pole of its propagator. We illustrate the phenomenological implications of this modification for the partial width of the $H \rightarrow W^+W^-$ decay.

PACS numbers: 11.15.Bt, 12.15.Lk, 14.80.Bn
1 Introduction

The conventional definitions of the mass $M$ and the total decay width $\Gamma$ of an unstable boson are given by

\begin{align}
M^2 &= M_0^2 + \text{Re} A(M^2), \\
M\Gamma &= -\frac{\text{Im} A(M^2)}{1 - \text{Re} A'(M^2)},
\end{align}

where $M_0$ is the bare mass and $A(s)$ is the self-energy in the case of scalar bosons and the transverse self-energy in the case of vector bosons. In fact, most calculations of the total decay rates are based on Eq. (2). $M$ and $\Gamma$ are conventionally referred to as the on-shell mass and width, respectively.

However, over the last decade it has been shown that, in the context of gauge theories, $M$ and $\Gamma$ become gauge dependent in $O(g^4)$ and $O(g^6)$, respectively, where $g$ is a generic gauge coupling \[1,2,3\]. A solution to this problem can be achieved by defining the mass and width in terms of the complex-valued position of the propagator’s pole:

\begin{equation}
\bar{s} = M_0^2 + A(\bar{s}),
\end{equation}

an idea that goes back to well-known postulates of scattering (S) matrix theory \[4\]. An important advantage of Eq. (3) is that $\bar{s}$ is gauge independent to all orders in perturbation theory \[1,2,3\]. A frequently employed parameterization is \[1,5\]

\begin{equation}
\bar{s} = m_2^2 - im_2 \Gamma_2,
\end{equation}

where we use the notation of Ref. \[1\]. Identifying $m_2$ and $\Gamma_2$ with the gauge-independent definitions of mass and width, we have

\begin{align}
m_2^2 &= M_0^2 + \text{Re} A(\bar{s}), \\
m_2 \Gamma_2 &= -\text{Im} A(\bar{s}).
\end{align}

Alternative, gauge-independent definitions of mass and width based on $\bar{s}$, with particular merits, have been discussed in the literature \[1,2,3,6\]. A phenomenologically relevant application of the S-matrix approach is to observables at the $Z$-boson resonance \[7\].

Over a period of two decades, the on-shell renormalization scheme \[8,9,10\] has provided a very convenient framework for the calculation of quantum corrections in electroweak perturbation theory. In fact, many important calculations have been performed in this scheme. One of its principal aims is the parameterization of $S$-matrix elements in terms of physical masses and coupling constants. For most calculations at the one-loop level, Eqs. (1) and (2) are satisfactory and, in fact, the original papers \[8,9\] employed such definitions. In higher orders, the gauge dependence of $M$ and $\Gamma$ precludes their identification with physical quantities. It is then natural to remedy this deficiency by replacing Eqs. (1) and (2) by Eqs. (5) and (6), respectively. In this way, the calculations are parameterized in terms of constants, such as $m_2$ and $\Gamma_2$, that can be identified with
physical observables to all orders in perturbation theory. In particular, we observe from Eq. (5) that the mass counterterm, a basic quantity in the renormalization procedure, is given by \( \text{Re} A(\bar{s}) \), rather than \( \text{Re} A(M^2) \). We shall refer to this improved formulation, based on Eqs. (5) and (6), as the pole scheme.

There is another significant pitfall of Eqs. (1) and (2), which has gone almost unnoticed so far. At the one-loop order, the production cross sections and total and partial decay widths of the Higgs boson \( H \) exhibit singularities proportional to \( (2M_V - M)^{-1/2} \) as the Higgs-boson mass \( M \) approaches from below the pair-production threshold of a vector boson \( V \) with mass \( M_V \). This problem is of phenomenological interest because the values \( 2M_W \) and \( 2M_Z \), corresponding to the \( W \)- and \( Z \)-boson thresholds, lie within the \( M \) range presently favoured by electroweak precision data [16,17]. On the other hand, there is no such singularity at the pair-production threshold of a fermion \( f \) [9,11,12,13,14,15]. This circumstance may be related, by the use of an appropriate dispersion relation, to the different threshold behaviours of the lowest-order partial widths of the decays \( H \rightarrow VV \) and \( H \rightarrow f\bar{f} \), which are proportional to \( (M - 2M_V)^{1/2} \) and \( (M - 2M_f)^{3/2} \), respectively [13]. In the case of a two-body threshold, this kind of singularity generally occurs if the two particles form an \( S \)-wave state and the sum of their masses is degenerate with that of the primary particle [13]. For example, it would also occur in \( Z \)-boson production and decay if the mass relation \( M_Z = 2M_t \), where \( t \) denotes the top quark, were satisfied [13], since in this case the \( t\bar{t} \) pair can be in an \( S \)-wave state. By the same token, it would occur for an extra neutral vector boson \( Z' \) with mass \( M_{Z'} = 2M_t \). For definiteness, in the following, we focus our attention on the Higgs boson \( H \) of the standard model (SM).

As explained in Refs. [13,14,15], the threshold singularity is an artifact of treating an unstable particle, such as the Higgs boson, as an asymptotic state of the \( S \) matrix. Detailed inspection [13,14,15] reveals that it originates from the wave-function renormalization constant in the on-shell scheme,

\[
Z = \frac{1}{1 - \text{Re} A'(M^2)}, \tag{7}
\]

which also appears in the definition of \( \Gamma \) in Eq. (2). One way to obtain Eq. (7) is to consider the Taylor expansion of the inverse propagator \( s - M_0^2 - A(s) \) about \( s = M^2 \). This procedure tacitly assumes that \( A(s) \) is analytic near \( s = M^2 \), so that the Taylor expansion can be performed. In most cases, this assumption is valid. However, \( A(s) \) possesses a branch point if \( s \) is at a threshold. If the threshold is due to a two-particle state with zero orbital angular momentum, then \( \text{Re} A'(s) \) diverges as \( 1/\beta \), where \( \beta \) is the relative velocity common to the two particles, as the threshold is approached from below [18]. Another case where the non-analyticity of \( A(s) \) in the neighbourhood of \( s = M^2 \) leads to serious problems in the on-shell formalism is the behaviour of the resonant amplitude when the unstable particle is coupled to massless quanta, such as photons and gluons [20].

The purpose of this paper is to show how the threshold singularities are rigorously removed by adopting the pole scheme. The salient point is that \( Z \) is redefined in such a way that the derivative term \( \text{Re} A'(M^2) \) is replaced by an appropriate ratio of differences,
where the increment of the argument is of order mass times width. In this way, the
treshold singularities are regularized by the very width of the primary particle. This
mechanism was illustrated for a toy model, consisting of a real scalar particle coupled to
two stable, complex scalar particles, through a numerical simulation in Ref. [18]. Here, we
analytically elaborate the underlying formalism in a general, model-independent way and
apply it to a case of phenomenological interest, namely the thresholds of the SM Higgs
boson at $M = 2M_V$, with $V = W, Z$.

If the threshold particles are unstable, an alternative way of eliminating the threshold
singularities in a physically meaningful way is to incorporate their widths in the one-loop
calculation [21]. Notice that the on-shell and pole formulations are equivalent through
the one-loop order, as may be seen, for example, by Taylor expanding Eq. (6) about $m^2$
[1,3]. Furthermore, as will become apparent later on, threshold singularities only appear
in connection with physical thresholds. Therefore, this regularization procedure does not
spoil the gauge independence of the physical predictions. If the widths of the threshold
particles are much larger than the one of the primary particle, then it appears plausible to
adopt this method. In general, however, the two regularizing effects, associated with the
widths of the primary and threshold particles, should be combined in a unified analysis.
We explain how this can be achieved.

This paper is organized as follows. In Section 2, we present the one-loop expressions for
the Higgs-boson self-energy in $R_\xi$ gauge [22] and in the pinch-technique (PT) framework
[23,24], explicitly exhibit the threshold singularities at $M = 2M_V$, and show that the
latter are gauge independent. In Section 3, starting from the definition of $\Gamma_2$ in Eq. (6),
we derive the counterpart of Eq. (7) in the pole scheme and show that it is devoid of
threshold singularities. We also present a simple substitution rule which allows us to
translate existing results for Higgs-boson observables from the on-shell scheme to the pole
scheme if the threshold particles are stable. In Section 4, we generalize this substitution
rule to the general case of unstable threshold particles and discuss in detail the situation
when the widths of the latter are much larger than the one of the primary particle. In
Section 5, we present a numerical analysis for the $H \rightarrow W^+W^-$ partial decay widths
of the Higgs boson, based on the results from Ref. [13]. In Section 6, we summarize our
conclusions.

2 On-shell formulation

In this section, we pin down the origin of the threshold singularities in Higgs-boson ob-
servables at $M = 2M_V$ by inspecting the corresponding expression of $Z$ in Eq. (7). The
relevant ingredient is the Higgs-boson self-energy $A(s)$. Since we wish to clarify if the
threshold singularities give rise to spurious gauge dependence, we consider the one-loop
expressions for $A(s)$ in $R_\xi$ gauge [22] and in the PT framework [23,24].

In $R_\xi$ gauge, we have to consider the Feynman diagrams depicted in Fig. 1. They
yield

\[ A(s) = \frac{G}{\pi} \left\{ - \left( \frac{s}{2} + 3M_W^2 \right) A_0 (M_W^2) + \frac{1}{2} (s - M^2) A_0 (\xi_W M_W^2) \right. \]

\[- \left( \frac{s^2}{4} - sM_W^2 + 3M_W^2 \right) B_0 (s, M_W^2, M_W^2) \]

\[ + \frac{1}{4} (s^2 - M^4) B_0 (s, \xi_W M_W^2, \xi_W M_W^2) + \frac{1}{2} (W \to Z) \]

\[- \frac{3}{4} M^2 A_0 (M^2) - \frac{9}{8} M^4 B_0 (s, M^2, M^2) \]

\[ + \sum_f N_f m_f^2 \left[ 2 A_0 \left( M_f^2 \right) - \left( \frac{s}{2} - 2M_f^2 \right) B_0 \left( s, M_f^2, M_f^2 \right) \right] \right\}, \tag{8} \]

where the sum is over fermion flavours \( f \), \( N_f = 1 \) (3) for leptons (quarks), \( G \) is related to Fermi’s constant \( G_\mu \) by \( G = G_\mu / (2\pi^2) \), \( \xi_W \) is the gauge parameter associated with the \( W \) boson, and the term \( (W \to Z) \) signifies the contribution involving the \( Z \) boson, which is obtained from the one involving the \( W \) boson by replacing \( M_W \) and \( \xi_W \) with \( M_Z \) and \( \xi_Z \), respectively. In dimensional regularization, the scalar one-loop one- and two-point integrals are defined as

\[ A_0 \left( m_0^2 \right) = - \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int \frac{d^D q}{q^2 - m_0^2 + i\varepsilon}, \]

\[ B_0 \left( p^2, m_0^2, m_1^2 \right) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int \frac{d^D q}{(q^2 - m_0^2 + i\varepsilon) [(q + p)^2 - m_1^2 + i\varepsilon]}, \tag{9} \]

where \( \mu \) is the 't Hooft mass scale and \( D = 4 - 2\varepsilon \) is the space-time dimensionality. Their solutions in the physical limit \( D \to 4 \) may, for example, be found in Ref. [15]. The absorptive part of Eq. (8) was already presented in Ref. [3]. In the 't Hooft-Feynman gauge, with \( \xi_W = \xi_Z = 1 \), Eq. (8) agrees with the corresponding result given in Eqs. (B.2) and (B.3) of Ref. [25]. Notice that \( A(s) \) is gauge independent at \( s = M^2 \). From Eqs. (1) and (2) it hence follows that the one-loop expression for \( M \) and the tree-level one for \( \Gamma \) are gauge independent, too, as expected. We note in passing that the gauge independence of \( \text{Re} A(M^2) \) requires the inclusion of the tadpole contribution in Eq. (8).

Next, we present the PT expression for \( A(s) \). We recall that the PT is a prescription that combines the conventional self-energies with so-called pinch parts from vertex and box diagrams in such a manner that the modified self-energies are gauge independent and exhibit desirable theoretical properties [23,24]. We calculate the pinch part \( \Delta A(s) \) in \( R_\xi \) gauge by means of the \( S \)-matrix PT framework elaborated in Ref. [24]. We choose the elastic scattering of two fermions via a Higgs boson in the \( s \) channel as the reference process. Our result is independent of this choice [23,24]. The relevant Feynman diagrams are depicted in Fig. 4. In the formulation of Ref. [24], the corresponding amplitudes reflecting the interactions of the vector bosons with the external fermions are described in terms of matrix elements of Fourier transforms of time-ordered products of current...
operators. Through successive current contractions with the longitudinal four-momenta found in the propagators and vertices of the massive vector bosons, Ward identities are triggered, after which the relevant pinch contributions are identified with amplitudes involving appropriate equal-time commutators of currents. Setting aside the details, the pinch contribution is found to be

$$\Delta A(s) = \frac{G}{\pi} (s - M^2) \left\{ \frac{1}{2} \left[ A_0 \left( M_W^2 \right) - A_0 \left( \xi_W M_W^2 \right) \right] + \left[ \frac{1}{4} (s + M^2) + M_W^2 \right] B_0 \left( s, M_W^2, M_W^2 \right) - \frac{1}{4} (s + M^2) B_0 \left( s, \xi_W M_W^2, \xi_W M_W^2 \right) + \frac{1}{2} (W \rightarrow Z) \right\}. \quad (10)$$

The second and third lines of Eq. (10) agree with Eq. (2.21) of Ref. [26], where the seagull and tadpole diagrams were omitted because they were not needed for the purpose of that paper. As expected, the PT self-energy of the Higgs boson,

$$a(s) = A(s) + \Delta A(s), \quad (11)$$

is independent of $\xi_W$ and $\xi_Z$ for all values of $s$. Furthermore, $\Delta A(s)$ vanishes at $s = M^2$, so that the one-loop expression for $M$ and the tree-level one for $\Gamma$ are not affected by the application of the PT.

The one-loop radiative corrections to physical observables characterizing the production or decay of a real Higgs boson involve its wave-function renormalization constant $Z$. The on-shell definition of the latter, given in Eq. (7), contains the term $\text{Re} A'(M^2)$, which is the source of the threshold singularities at $M = 2M_V$. To see that, let us consider the expression

$$\frac{\partial}{\partial s} B_0 \left( s, M_V^2, M_V^2 \right) \bigg|_{s=M^2} = \begin{cases} -\frac{1}{M^2} \left[ 1 + \frac{A}{\sqrt{1-A}} \text{arsinh} \sqrt{-\frac{1}{A}} \right], & A < 0, \\ -\frac{1}{M^2} \left[ 1 + \frac{A}{\sqrt{1-A}} \left( \text{arcosh} \sqrt{\frac{1}{A}} - i\frac{\pi}{2} \right) \right], & 0 < A < 1, \\ -\frac{1}{M^2} \left[ 1 - \frac{A}{\sqrt{1-A}} \text{arcsin} \sqrt{\frac{1}{A}} \right], & A > 1, \end{cases} \quad (12)$$

where $A = 4M_V^2/M^2$, which appears in $A'(M^2)$ with the prefactor $-(G/\pi) (M^4/4 - M^2M_V^2 + 3M_V^4)$. Equation (12) follows from [15]

$$B_0 \left( s, M_V^2, M_V^2 \right) = \frac{1}{\epsilon - \gamma_E + \ln \frac{4\pi \mu^2}{M_V^2} + 2 - 2\sqrt{1 - \frac{4 (M_V^2 - i\varepsilon)}{s}} \text{arsinh} \sqrt{-\frac{s}{4 (M_V^2 - i\varepsilon)}}} + O(\varepsilon), \quad (13)$$

\footnote{In Eq. (12), we also consider the case $A < 0$, which provides a convenient starting point for the analytic continuation to be performed in Section 4.}
where $\gamma_E$ is Euler’s constant. As $M$ approaches $2M_V$ from below, Eq. (12) develops a real singularity proportional to $(2M_V - M)^{-1/2}$. In a one-loop calculation, one expands $Z$ as $Z = 1 + \text{Re} A'(M^2) + O(g^4)$, which then exhibits the same threshold singularity. As $M$ surpasses $2M_V$, this threshold singularity is shifted from the real part to the imaginary one and, therefore, it does not affect $Z$. Since the prefactors of $B_0(s, \xi_V M_V^2, \xi_V M_V^2)$ in Eq. (8) vanish at $s = M^2$, there are no threshold singularities at the unphysical thresholds $M = 2\sqrt{\xi_V}M_V$ in $R_\xi$ gauge. On the other hand, in the PT framework, there are no gauge-dependent thresholds, and Eq. (12) appears in $a'(M^2)$ with the same prefactor as in $A'(M^2)$. In conclusion, the threshold singularities are gauge independent and only affect physical thresholds.

3 Pole formulation

We now explain how the threshold singularities are avoided in the pole scheme. Throughout this section, we assume that the threshold particles are stable. The general case of unstable threshold particles will be discussed in Section 4. As is well known [1,3], the on-shell and pole definitions of mass and widths, given in Eqs. (1), (2), (5), and (6), are equivalent through next-to-leading order, i.e. through $O(g^2)$ and $O(g^4)$, respectively. In particular, this implies that the perturbative expansion of $m^2 \Gamma_2$ through $O(g^4)$ resembles the one of $M \Gamma$,

$$M \Gamma = -\text{Im} A^{(1)}(M^2) \left[1 + \text{Re} A^{(1)\prime}(M^2)\right] - \text{Im} A^{(2)}(M^2) + O(g^6), \quad (14)$$

where the superscripts refer to the number of quantum loops, and thus also suffers from threshold singularities. In fact, Taylor expanding the right-hand side of Eq. (8) about $m_2$ and retaining only terms through $O(g^4)$, we obtain

$$m_2 \Gamma_2 = -\text{Im} A^{(1)} \left( m_2^2 \right) + m_2 \Gamma_2 \text{Re} A^{(1)\prime} \left( m_2^2 \right) - \text{Im} A^{(2)} \left( m_2^2 \right) + O(g^6)$$

$$= -\text{Im} A^{(1)} \left( m_2^2 \right) \left[1 + \text{Re} A^{(1)\prime} \left( m_2^2 \right)\right] - \text{Im} A^{(2)} \left( m_2^2 \right) + O(g^6), \quad (15)$$

which is equivalent to Eq. (14) through $O(g^4)$ and thus also prone to threshold singularities. In order to avoid the latter, we have to undo the Taylor expansion in Eq. (15), i.e. we have to substitute

$$\text{Re} A^{(1)\prime} \left( m_2^2 \right) = \frac{\text{Im} A^{(1)} \left( m_2^2 \right) - \text{Im} A^{(1)\prime} \left( \bar{s} \right)}{m_2 \Gamma_2} + O(g^4)$$

$$= -1 - \frac{\text{Im} A^{(1)} \left( m_2^2 - im_2 \Gamma_2^{(0)} \right)}{m_2 \Gamma_2^{(0)}} + O(g^4), \quad (16)$$

where, consistent with our approximation, we have replaced $\Gamma_2$ with the tree-level width $\Gamma_2^{(0)} = -\text{Im} A^{(1)} \left( m_2^2 \right) / m_2$. By the same token, the substitution rule of Eq. (14) allows us to eliminate, in the spirit of the pole scheme, the threshold singularities which have
been encountered in the on-shell scheme \cite{11,12,13,14,15,19}. To that end, we abandon Eq. (12) and instead substitute

\[ \text{Re} \frac{\partial}{\partial s} B_0 \left( s, m_{2,V}^2, m_{2,V}^2 \right) \bigg|_{s=m_2^2} = \frac{\text{Im} \ B_0 \left( m_2^2, m_{2,V}^2, m_{2,V}^2 \right) - \text{Im} \ B_0 \left( \bar{s}, m_{2,V}^2, m_{2,V}^2 \right)}{m_2 \Gamma_2} + O(g^2), \tag{17} \]

where \( m_{2,V} \) is the pole mass of the threshold particles, together with

\[ \text{Im} \ B_0 \left( m_2^2, m_{2,V}^2, m_{2,V}^2 \right) = \pi \sqrt{1-a \theta (1-a)}, \tag{18} \]

\[ \text{Im} \ B_0 \left( \bar{s}, m_{2,V}^2, m_{2,V}^2 \right) = f(a, \gamma) + 2 \pi \sqrt{1-a \theta (1-a) \theta (\gamma)}, \tag{19} \]

where \( a = 4 m_{2,V}^2/m_2^2 \) and \( \gamma = \Gamma_2/m_2 \). Here, we have used the auxiliary function

\[ f(a, \gamma) \equiv -2 \text{Im} \left( \sqrt{1-\bar{a} \text{arsinh} \sqrt{1/\bar{a}}} \right) = \text{sign}(\gamma) \sqrt{2} \left\{ \frac{1}{2} \sqrt{b(c-b) + a} \right. \\
\times \ln \left[ \frac{1}{a} \left( b + c + \sqrt{(b-1)(c+a-1) + (b+1)(c-a+1)} \right) \right] \\
- \sqrt{b(c-b) - a} \arctan \frac{\sqrt{b+1 + \sqrt{c-a+1}}}{\sqrt{b-1 + \sqrt{c+a-1}}} \right\}, \tag{20} \]

where \( \bar{a} = 4 m_{2,V}^2/\bar{s}, b = \sqrt{1+\gamma^2}, c = \sqrt{(a-1)^2 + \gamma^2}, \) and \( \text{sign}(x) = \theta (x) - \theta (-x) \). The term proportional to \( \theta (1-a) \) in Eq. (18) guarantees that Eqs. (18) and (19) refer to the same Riemann sheet, so that their difference appearing in Eq. (17) vanishes in the limit \( \Gamma_2 \to 0 \) and the derivative expression on the left-hand side of that equation is recovered if \( m_2 \neq 2 m_{2,V} \). The factors \( \theta (\gamma) \) and \( \text{sign}(\gamma) \) in Eqs. (19) and (20), respectively, make sure that the case \( \gamma < 0 \) is covered, too, a generalization that will be useful in Section 4. At threshold, where \( m_2 = 2 m_{2,V} \), Eq. (17) leads to

\[ \text{Re} \frac{\partial}{\partial s} B_0 \left( s, m_{2,V}^2, m_{2,V}^2 \right) \bigg|_{s=m_2^2} = \frac{1}{m_2^2} \left[ \frac{\pi}{\sqrt{2} \gamma} \left( 1 + \frac{\gamma}{2} \right) - 2 + O \left( \gamma^{3/2} \right) \right] + O(g^2), \tag{21} \]

i.e. the threshold singularity of the on-shell scheme is automatically regularized in the pole scheme by the width \( \Gamma_2 \) of the primary particle.

Finally, we explain how Eq. (16) is generalized to higher orders. For that purpose, we rewrite Eq. (1) as

\[ m_2 \Gamma_2 = - \frac{\text{Im} \ A \left( m_2^2 \right) - \text{Im} \ A \left( \bar{s} \right)}{1 - \left[ \text{Im} \ A \left( m_2^2 \right) - \text{Im} \ A \left( \bar{s} \right) \right]/(m_2 \Gamma_2)}. \tag{22} \]

Solving Eq. (22) for \( m_2 \Gamma_2 \), we recover Eq. (1), so that the two expressions are equivalent. The usefulness of Eq. (22) may be appreciated by observing that, in order to calculate \( \Gamma_2 \)
to $O(g^{2n+2})$, we only need to insert the $O(g^{2n})$ expression for $\Gamma_2$ on the right-hand side of that equation. Comparing Eq. (22) with Eq. (2), we see that

$$Z_2 = \frac{1}{1 - [\text{Im} A(m_2^2) - \text{Im} A(\bar{s})]/(m_2^2\Gamma_2)}$$

plays the role of the wave-function renormalization constant for unstable particles in the pole scheme. This is to be compared with its counterpart in the on-shell scheme, given in Eq. (7).

### 4 Unstable threshold particles

So far, we have assumed the threshold particles to be stable. We now allow for them to have a finite pole width $\Gamma_{2,V}$. This can be achieved by replacing in the loop amplitude $A(m_2^2)$ the square of their mass $m_{2,V}^2$ by the complex position $\bar{s}_V = m_{2,V}^2 - im_{2,V}\Gamma_{2,V}$ of the pole of their propagator. In this way, the substitution rule given in Eq. (17) becomes

$$\text{Re} \left. \frac{\partial}{\partial s} B_0(s, \bar{s}_V, \bar{s}_V) \right|_{s=m_2^2} = \frac{\text{Im} B_0(m_2^2, \bar{s}_V, \bar{s}_V) - \text{Im} B_0(s, \bar{s}_V, \bar{s}_V)}{m_2^2\Gamma_2} + O(g^2).$$

The expression for $\text{Im} B_0(s, \bar{s}_V, \bar{s}_V)$ has a structure analogous to Eq. (19). Comparing $4\bar{s}_V/s = [a(1 + \gamma_V^2)/(1 + \gamma_V\gamma_V)]/[1 - i(\gamma - \gamma_V)/(1 + \gamma_V\gamma_V)]$, where $\gamma_V = \Gamma_{2,V}/m_{2,V}$, with $\bar{a} = a/(1 - i\gamma)$, we see that in Eq. (20) $a$ and $\gamma$ are to be replaced by $a(1 + \gamma_V^2)/(1 + \gamma_V\gamma_V)$ and $(\gamma - \gamma_V)/(1 + \gamma_V\gamma_V)$, respectively. This leads to

$$\text{Im} B_0(s, \bar{s}_V, \bar{s}_V) = f \left( \frac{1 + \gamma_V^2}{1 + \gamma_V\gamma_V}, \frac{\gamma - \gamma_V}{1 + \gamma_V\gamma_V} \right) + 2\pi\sqrt{1 - a\theta(1 - a)\theta(\gamma - \gamma_V)}. \quad (25)$$

Setting $\gamma = 0$ in Eq. (25), we obtain

$$\text{Im} B_0 \left( m_2^2, \bar{s}_V, \bar{s}_V \right) = f \left( a(1 + \gamma_V^2), -\gamma_V \right). \quad (26)$$

The second term on the right-hand side of Eq. (24) ensures that Eqs. (23) and (24) refer to the same Riemann sheet, so that Eq. (24) reduces to Eq. (17) in the limit $\Gamma_{2,V} \to 0$.

In a situation when the threshold particles are much more unstable than the primary particle, i.e. if $\Gamma_2/m_2 \ll \Gamma_{2,V}/m_{2,V}$, we may take the limit $\Gamma_2 \to 0$ on the right-hand side of Eq. (24) and thus effectively return to the derivative expression on the left-hand side of that equation, which reads

$$\text{Re} \left. \frac{\partial}{\partial s} B_0(s, \bar{s}_V, \bar{s}_V) \right|_{s=m_2^2} = -\frac{1}{m_2^2} \text{Re} \left( 1 + \frac{\bar{a}_V}{\sqrt{1 - \bar{a}_V}} \arcsinh \sqrt{\frac{-1}{\bar{a}_V}} \right)$$

$$= -\frac{1}{m_2^2} \left\{ 1 + \frac{a}{\sqrt{2c_V}} \left( \frac{1}{2} \left( \sqrt{c_V - a + 1} - \gamma_V \sqrt{c_V + a - 1} \right) \right) \right\}.$$
\[
\times \ln \left\{ \frac{1}{ab_V^2} \left( b_V + b_V c_V + \sqrt{(b_V - 1) (b_V c_V + ab_V^2 - 1)} \right) \\
+ \sqrt{(b_V + 1) (b_V c_V - ab_V^2 + 1)} \right\} \\
- \left( \sqrt{c_V + a - 1 + \gamma_V \sqrt{c_V - a + 1}} \right)\right), \quad (27)
\]

where \( \bar{a}_V = 4\bar{s}_V/m_2^2, b_V = \sqrt{1 + \gamma_V^2} \), and \( c_V = (a - 1)^2 + a^2 \gamma_V^2 \). At threshold, we have

\[
\text{Re} \frac{\partial}{\partial s} B_0 (s, \bar{s}_V, \bar{s}_V)\bigg|_{s=m_2^2} = \frac{1}{m_2^2} \left[ \frac{\pi}{2\sqrt{2}\gamma_V} (1 + \gamma_V) - 2 + O(\gamma_V^2) \right] + O(g^2), \quad (28)
\]

i.e. the threshold singularity is now entirely regularized by the width \( \Gamma_{2,V} \) of the threshold particles.

### 5 Discussion

We are now in a position to investigate the phenomenological implications of our results. The SM Higgs boson with mass \( m_2 \) in the vicinity of \( 2m_2 \) dominantly decays to a pair of \( W \) bosons, with a branching fraction of about 90\% \cite{15}. Therefore, we choose the partial width of this decay as an example to illustrate the threshold singularity and its removal.

The complete one-loop radiative correction to this observable was obtained within the on-shell scheme in Refs. \cite{9,12,13}. Its structure is exhibited in Eq. (14) if we include in Im \( A^{(1)}(M^2) \) and Im \( A^{(2)}(M^2) \) only intermediate states containing a \( W^+W^- \) pair. Specifically, \( -\text{Im} A^{(1)}(M^2) \) represents the tree-level result, written with \( G_\mu, Z = 1+\text{Re} A^{(1)}(M^2) \) is the Higgs-boson wave-function renormalization constant of Eq. (7), and \( -\text{Im} A^{(2)}(M^2) \) comprises the proper \( HW^+W^- \) vertex correction, the \( HW^+W^- \) coupling and \( W \)-boson wave-function renormalization constants, and the real-photon bremsstrahlung correction.

As we have seen in Section 2, the Taylor expansion of Eq. (1), given in Eq. (13), is equivalent to Eq. (14). In the following, we work in the pole scheme, on the basis of Eq. (15). All the ingredients of Eq. (13) may be found, in analytic form, in Ref. [3].

As explained in the context of Eq. (12), the threshold singularities arise from the term \( \text{Re} \frac{\partial}{\partial s} B_0 (s, m_2^2, m_{2,V}^2) \bigg|_{s=m_2^2} \), which is contained in \( \text{Re} A^{(1)}(m_2^2) \). If the threshold particles are stable, these threshold singularities are regularized by \( \Gamma_2 \) according to the substitution rule of Eq. (17). In the case of unstable threshold particles, this substitution rule is generalized by including \( \Gamma_{2,V} \) as described in Eq. (24). In the limiting case \( \Gamma_2/m_2 \ll \Gamma_{2,V}/m_{2,V} \), we recover the derivative expression of Eq. (27), in which the threshold singularity is regularized by \( \Gamma_{2,V} \). In the case under consideration, we have \( V = Z \) and \( \Gamma_2/m_2 : (\Gamma_{2,V}/m_{2,V}) \approx 1 : 7 \), so that the second method of regularization, which incorporates both \( \Gamma_2 \) and \( \Gamma_{2,V} \), should be most appropriate, while the third one,
which is solely based on \( \Gamma_{2,V} \), should provide a good approximation. On the other hand, the first scheme, which only includes \( \Gamma_2 \), should be unrealistic in this particular case.

In our numerical analysis, we use \( m_{2,W} = 80.391 \text{ GeV} \), \( m_{2,Z} = 91.153 \text{ GeV} \), and \( \Gamma_{2,Z} = 2.493 \text{ GeV} \) \cite{16}, and adopt the residual input parameters from Ref. \cite{27}. We remind the reader that \( m_2^2 = m_1^2 / (1 + \Gamma_1^2/m_1^2) \) and \( \Gamma_2^2 = \Gamma_2^2 / (1 + \Gamma_2^2/m_1^2) \), where \( m_1 \) and \( \Gamma_1 \) can be identified with the measured values \cite{1}. We evaluate the Higgs-boson total decay width \( \Gamma_2 \), which enters Eqs. (17) and (24), in the Born approximation. In Fig. 3, we show the \( H \to W^+W^- \) partial decay width at one loop in the pole scheme as a function of \( m_2 \) in the vicinity of the threshold at \( m_2 = 2m_{2,Z} \). The evaluation from the Taylor-expanded expression given in Eq. (15) on the basis of Eq. (12) (dotted line), which exhibits a threshold singularity, is compared with the one where this threshold singularity is jointly regularized by \( \Gamma_2 \) and \( \Gamma_{2,Z} \) according to the substitution rule of Eq. (24) (solid line). For comparison, we also display the tree-level result (dashed line). We observe that the regularized result smoothly interpolates across the threshold region and merges with the unregularized result sufficiently far away from the threshold. Below (above) threshold, the regularization leads to an increase (decrease) of the result relative to the unregularized case. In the threshold region, the regularized correction increases the Born result by about 7%. In Fig. 4, we compare the regularized result shown in Fig. 3 (solid line) with the results based on the regularizations by \( \Gamma_2 \) according to Eq. (17) (dashed line) and by \( \Gamma_{2,Z} \) according to Eq. (27) (dot-dashed line). For reference, we also show the unregularized result (dotted line). Comparing the dashed and solid lines, we observe that \( \Gamma_{2,Z} \) plays a crucial role in the regularization of the threshold singularity, as anticipated above. On the other hand, we infer from the closeness of the dot-dashed and solid lines that the relative importance of \( \Gamma_2 \) in the combined regularization approach is minor, due to the smallness of \( \Gamma_2/m_2 \) as compared to \( \Gamma_{2,V}/m_{2,V} \).

6 Conclusions

As explained in the text, the threshold singularity, associated with the conventional on-shell wave-function renormalization constant, affects the production and decay rates of the unstable particle whenever its mass is degenerate with the sum of masses of an interacting pair of virtual particles that form an \( S \)-wave state. It is important to realize that, since the wave-function renormalization constant is a universal prefactor in all decay and production amplitudes, the associated singularity affects all production and decay processes of the unstable particle.

For definiteness, we focussed our analysis on a case of phenomenological interest, namely the \( H \to W^+W^- \) decay of the SM Higgs boson when its mass \( M \) is very close to \( 2M_Z \). By examining the one-loop Higgs-boson self-energy in \( R_\xi \) gauge, we showed that the threshold singularity is gauge independent and, for that reason, also affects the conventional wave-function renormalization constant in the PT framework. We then demonstrated how the one-loop threshold singularity is removed in the pole formulation. In particular, the conventional on shell wave-function renormalization constant given in
Eq. (7) is replaced by Eq. (23), which plays the rôle of the wave-function renormalization for the unstable particle in the pole scheme. When the virtual particles in the mass-degenerate pair are themselves unstable, their widths can also be used to tame the threshold singularity. We then showed how the two regularizing effects, associated with the widths of the primary and threshold particles, can be combined in a unified analysis. The various effects are illustrated for the $H \rightarrow W^+W^-$ decay width in Figs. 3 and 4. As a byproduct, we presented in Eqs. (8) and (11) the complete Higgs-boson self-energies at one loop in $R_\xi$ gauge and in the PT framework, respectively.

Acknowledgements

B.A.K. thanks Oleg Yakovlev for fruitful discussions concerning Ref. [21]. A.S. is grateful to the Theory Group of the 2nd Institute for Theoretical Physics for the hospitality extended to him during a visit when this manuscript was prepared. The work of B.A.K. was supported in part by the Deutsche Forschungsgemeinschaft through Grant No. KN 365/1-1, by the Bundesministerium für Bildung und Forschung through Grant No. 05 HT9GUA 3, and by the European Commission through the Research Training Network Quantum Chromodynamics and the Deep Structure of Elementary Particles under Contract No. ERBFMRX-CT98-0194. The work of C.P.P. was supported by the German Academic Exchange Service (DAAD) through Grant No. A/97/00746. The work of A.S. was supported in part by the Alexander von Humboldt Foundation through Research Award No. IV USA 1051120 USS, and by the National Science Foundation through Grant No. PHY-9722083.

References

[1] A. Sirlin, Phys. Rev. Lett. 67 (1991) 2127; Phys. Lett. B 267 (1991) 240.

[2] S. Willenbrock, G. Valencia, Phys. Lett. B 259 (1991) 373; R.G. Stuart, Phys. Lett. B 262 (1991) 113; Phys. Lett. B 272 (1991) 353; Phys. Rev. Lett. 70 (1993) 3193; H. Veltman, Z. Phys. C 62 (1994) 35; M. Passera, A. Sirlin, Phys. Rev. Lett. 77 (1996) 4146; P. Gambino, P.A. Grassi, Phys. Rev. D 62 (2000) 076002.

[3] B.A. Kniehl, A. Sirlin, Phys. Rev. Lett. 81 (1998) 1373; Phys. Lett. B 440 (1998) 136.

[4] R.E. Peierls, in The Proceedings of the 1954 Glasgow Conference on Nuclear and Meson Physics, edited by E.H. Bellamy and R.G. Moorhouse (Pergamon Press, London and New York, 1955), p. 296; M. Lévy, Nuovo Cimento XIII (1959) 115; R.J. Eden, P.V. Landshoff, D.I. Olive, J.C. Polkinghorne, The Analytic S-Matrix (Cambridge University Press, Cambridge, England, 1966), p. 247.
[5] M. Consoli, A. Sirlin, in *Physics at LEP*, CERN Yellow Report No. 86-02 (February 1986), Vol. 1, p. 63.

[6] A.R. Bohm, N.L. Harshman, Nucl. Phys. B 581 (2000) 91.

[7] A. Leike, T. Riemann, J. Rose, Phys. Lett. B 273 (1991) 513; T. Riemann, Phys. Lett. B 293 (1992) 451.

[8] A. Sirlin, Phys. Rev. D 22 (1980) 971; S. Sakakibara, Phys. Rev. D 24 (1981) 1149; K-I. Aoki, Z. Hioki, R. Kawabe, M. Konuma, T. Muta, Prog. Theor. Phys. Suppl. 73 (1982) 1; M. Böhm, H. Spiesberger, W. Hollik, Fortschr. Phys. 34 (1986) 687; W.F.L. Hollik, Fortschr. Phys. 38 (1990) 165; A. Denner, Fortschr. Phys. 41 (1993) 307.

[9] J. Fleischer, F. Jegerlehner, Phys. Rev. D 23 (1981) 2001.

[10] E. Kraus, Ann. Phys. (N.Y.) 262 (1998) 155; P.A. Grassi, Nucl. Phys. B 560 (1999) 499.

[11] D.Yu. Bardin, B.M. Vilenskiĭ, P.Kh. Khristova, Sov. J. Nucl. Phys. 53 (1991) 152 [Yad. Fiz. 53 (1991) 240].

[12] D.Yu. Bardin, B.M. Vilenskiĭ, P.Kh. Khristova, Sov. J. Nucl. Phys. 54 (1991) 833 [Yad. Fiz. 54 (1991) 1366].

[13] B.A. Kniehl, Nucl. Phys. B 357 (1991) 439.

[14] B.A. Kniehl, Nucl. Phys. B 376 (1992) 3; Z. Phys. C 55 (1992) 605.

[15] B.A. Kniehl, Phys. Rep. 240 (1994) 211.

[16] The LEP Collaborations ALEPH, DELPHI, L3, OPAL, the LEP Electroweak Working Group and the SLD Heavy Flavour and Electroweak Groups, D. Abbaneo et al., Report No. CERN-EP-2000-016, January 2000 (URL: [http://lepewwg.web.cern.ch/LEPEWWG](http://lepewwg.web.cern.ch/LEPEWWG)).

[17] B.A. Kniehl, A. Sirlin, Eur. Phys. J. C 16 (2000) 635.

[18] T. Bhattacharya, S. Willenbrock, Phys. Rev. D 47 (1993) 4022.

[19] J. Fleischer, F. Jegerlehner, Nucl. Phys. B 216 (1983) 469.

[20] M. Passera, A. Sirlin, Phys. Rev. D 58 (1998) 113010.

[21] K. Melnikov, M. Spira, O. Yakovlev, Z. Phys. C 64 (1994) 401.

[22] K. Fujikawa, B.W. Lee, A.I. Sanda, Phys. Rev. D 6 (1972) 2923.
[23] J.M. Cornwall, in *The 1981 French-American Seminar “Theoretical Aspects of Quantum Chromodynamics”*, Marseille, France, 1981, edited by J.W. Dash, Report No. CPT-81/P.1345, p. 95; Phys. Rev. D 26 (1982) 1453; J.M. Cornwall, J. Papavassiliou, Phys. Rev. D 40 (1989) 3474; J. Papavassiliou, Phys. Rev. D 41 (1990) 3179.

[24] G. Degrassi, A. Sirlin, Phys. Rev. D 46 (1992) 3104.

[25] B.A. Kniehl, Nucl. Phys. B 352 (1991) 1.

[26] J. Papavassiliou, A. Pilaftsis, Phys. Rev. D 58 (1998) 053002.

[27] Particle Data Group, D.E. Groom et al., Eur. Phys. J. C 15 (2000) 1.
Figure 1: Feynman diagrams pertinent to the conventional self-energy of the SM Higgs boson in $R_\xi$ gauge.
Figure 2: Feynman diagrams pertinent to the pinch parts of the self-energy of the SM Higgs boson in $R_\xi$ gauge.
Figure 3: $H \rightarrow W^+W^-$ partial decay width at one loop in the pole scheme as a function of the Higgs-boson pole mass $m_2$ in the vicinity of the threshold at $m_2 = 2m_{2,Z}$. The evaluation from the Taylor-expanded expression given in Eq. (13) on the basis of Eq. (12) (dotted line), which exhibits a threshold singularity, is compared with the one where this threshold singularity is jointly regularized by $\Gamma_2$ and $\Gamma_{2,Z}$ according to the substitution rule of Eq. (21) (solid line). For comparison, also the tree-level result is shown (dashed line).
Figure 4: $H \rightarrow W^+W^-$ partial decay width at one loop in the pole scheme as a function of the Higgs-boson pole mass $m_2$ in the vicinity of the threshold at $m_2 = 2m_{Z'}$. The threshold singularity is (a) not regularized (dotted line) or regularized (b) by $\Gamma_2$ according to Eq. (17) (dashed line), (c) by $\Gamma_{2,Z}$ according to Eq. (27) (dot-dashed line), or (d) by $\Gamma_2$ and $\Gamma_{2,Z}$ according to Eq. (24) (solid line).