UNIVERSAL ABELIAN VARIETY AND SIEGEL MODULAR FORMS

SHOUHEI MA

Abstract. We give a correspondence between Siegel modular forms and pluricanonical forms on the universal family of abelian varieties (or more generally the Kuga family) and its compactification, for every arithmetic group for a symplectic form of rank $2g > 2$. We first show that the graded ring of Siegel modular forms of weight divisible by $g + s + 1$ is isomorphic to the ring of pluricanonical forms on the $s$-fold Kuga family. Then we prove that for a certain class of compactification of the Kuga variety, this extends to an isomorphism with its log canonical ring. The same principle also leads to a bound of the Kodaira-Iitaka dimension of the compactification in terms of modular forms. In most cases, the Kuga variety has canonical singularities.

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1. Introduction

Our purpose in this article is to establish a correspondence between Siegel modular forms and pluricanonical forms on the universal family of abelian varieties and its compactification, which connects modular forms to the geometry of the universal family. We proceed in three steps. First we

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give the correspondence on the universal family before compactification. Next this is extended to the correspondence on a certain class of compactification. Finally, we derive a bound of the Kodaira-Iitaka dimension of such a compactification in terms of modular forms.

Let $\Lambda$ be a free $\mathbb{Z}$-module of rank $2g > 2$ equipped with a nondegenerate symplectic form $\Lambda \times \Lambda \to \mathbb{Z}$, and $\Gamma$ be a finite-index subgroup of the symplectic group $\text{Sp}(\Lambda)$ of $\Lambda$. Let $A(\Gamma) = \mathcal{D}/\Gamma$ be the Siegel modular variety defined by $\Gamma$, where $\mathcal{D}$ is the Hermitian symmetric domain attached to $\Lambda$. Over $A(\Gamma)$ we have the universal family of abelian (or Kummer) varieties. More generally, we have the $s$-fold Kuga family $X^s(\Gamma) \to A(\Gamma)$, whose general fibers are $s$-fold self products of the abelian varieties or their quotient by $-1$ according to whether $-1 \not\in \Gamma$ or $-1 \in \Gamma$. The space $X^s(\Gamma)$ is a normal quasi-projective variety of dimension $g(g + 2s + 1)/2$. Let $H^0(X^s(\Gamma), K_{X^s(\Gamma)}^{\otimes m})$ be the space of holomorphic $m$-canonical forms on (the regular locus of) $X^s(\Gamma)$. Let $M_k(\Gamma)$ be the space of Siegel modular forms of weight $k$ with respect to $\Gamma$. Our starting point is the following correspondence.

**Theorem 1.1.** We have a natural isomorphism

$$\bigoplus_{m \geq 0} H^0(X^s(\Gamma), K_{X^s(\Gamma)}^{\otimes m}) \cong \bigoplus_{m \geq 0} M_{(g+s+1)m}(\Gamma)$$

of graded rings.

We also show that $X^s(\Gamma)$ has canonical singularities in most cases (§10). Hence in that case, we have

$$\bigoplus_{m \geq 0} H^0(X, K_X^{\otimes m}) \cong \bigoplus_{m \geq 0} M_{(g+s+1)m}(\Gamma)$$

for every desingularization $X \to X^s(\Gamma)$ of $X^s(\Gamma)$.

A typical application of (1.1) would be to draw some information on the geometry of $X^s(\Gamma)$ from the knowledge about modular forms. For example, (1.1) tells us that the $\mathbb{C}$-algebra $\bigoplus_m H^0(K_{X^s(\Gamma)}^{\otimes m})$ is finitely generated with transcendental degree $g(g + 1)/2 + 1$, even though $X^s(\Gamma)$ is not compact. We also find that when $g, s, m$ are odd and $-1 \in \Gamma$, there is no nonzero $m$-canonical form on any smooth projective model of $X^s(\Gamma)$.

The proof of (1.1) is simple, based on natural isomorphisms between the relevant line bundles. We also give a higher analogue of (1.1) in the form of a Leray spectral sequence that relates vector-valued Siegel modular forms to the cohomology of $K_{X^s(\Gamma)}^{\otimes m}$.

Our main result is the extension of (1.1) to a certain class of compactification $\bar{X}$ of $X^s(\Gamma)$. Although our principal interest would be in compact $\bar{X}$, the result also applies to not fully compact $\bar{X}$ as well.
Theorem 1.2. Let $\tilde{X}$ be a complex analytic variety which contains $X^s(\Gamma)$ as a Zariski open set. Assume that

- the singular locus of $\tilde{X}$ has codimension $\geq 2$,
- $X^s(\Gamma) \to A(\Gamma)$ extends to a morphism $\tilde{X} \to A(\Gamma)^\Sigma$ to some toroidal compactification $A(\Gamma)^\Sigma$ of $A(\Gamma)$, and
- every irreducible component of the boundary divisor $\Delta_X = \tilde{X} - X^s(\Gamma)$ of $\tilde{X}$ dominates some irreducible component of the boundary divisor $\Delta_A = A(\Gamma)^\Sigma - A(\Gamma)$ of $A(\Gamma)^\Sigma$.

Then the isomorphism (1.1) extends to an isomorphism

$$
\bigoplus_{m \geq 0} H^0(\tilde{X}, K^{\text{sm}}_{\tilde{X}}(m\Delta_X)) \cong \bigoplus_{m \geq 0} M_{(g+s+1)m}(\Gamma).
$$

This maps the subspace $S_{(g+s+1)m}(\Gamma)$ of cusp forms into $H^0(K^{\text{sm}}_{\tilde{X}}((m-1)\Delta_X))$.

Here $H^0(K^{\text{sm}}_{\tilde{X}}(l\Delta_X))$ is the space of meromorphic $m$-canonical forms on the regular locus of $\tilde{X}$ which is holomorphic on $X^s(\Gamma)$ and has at most pole of order $l$ along every irreducible component of $\Delta_X$. We also show that the restricted map $S_{(g+s+1)m}(\Gamma) \hookrightarrow H^0(K^{\text{sm}}_{\tilde{X}}((m-1)\Delta_X))$ is surjective under certain conditions, mainly on the singularities of the pair $(\tilde{X}, \Delta_X)$.

The assumptions in Theorem 1.2 would be natural if one wants to view $\tilde{X}$ as an extension of the family $X^s(\Gamma) \to A(\Gamma)$. Namikawa [20] was the first to construct such an extension $\tilde{X}$ for $s = 1$ and $\Gamma$ the principal congruence subgroups of $\text{Sp}(2g, \mathbb{Z})$ of even level $\geq 4$, where $\tilde{X}$ is nonsingular and is a projective family over the 2nd Voronoi compactification of $A(\Gamma)$. Namikawa constructed his $\tilde{X}$ as a toroidal compactification of $X^1(\Gamma)$. General theory of toroidal compactification of $X^s(\Gamma)$ has been then developed in [21], [4], [13]. On the other hand, a feature of Theorem 1.2 is that the isomorphism (1.2) is obtained without knowing specific geometry of the boundary of $\tilde{X}$. As a consequence, we find that the log canonical ring $\bigoplus_m H^0(K^{\text{sm}}_{\tilde{X}}(m\Delta_X))$ is invariant for all compactifications $\tilde{X}$ satisfying those conditions.

Theorem 1.2 is proved by showing that every $m$-canonical form $\omega$ on $X^s(\Gamma)$ has at most pole of order $m$ along every component of $\Delta_X$ (a Koecher
type statement). We deduce this property by deriving an asymptotic estimate of the $L^{2/m}$ norm of $\omega$ around $\Delta_X$. As a key step, we use the isomorphism (1.1) to translate the $L^{2/m}$ norm of $\omega$ into the Petersson norm of the corresponding modular form. The problem is then reduced to the asymptotic estimate of the Petersson norm of modular forms around $\Delta_A$, which is derived by a standard calculation.

By a similar argument with local modular forms, we also derive the following bound of the Iitaka dimension $\kappa(K_{\tilde{X}})$ of the canonical divisor $K_{\tilde{X}}$ of $\tilde{X}$, without knowing specific geometry of the boundary nor having moduli interpretation of $\tilde{X}$.

**Theorem 1.3.** Let $\tilde{X} \supset X^s(\Gamma)$ be a normal compact complex analytic variety which satisfies the conditions in Theorem 1.2. Then we have

$$\kappa(A(\Gamma)^{\mathbb{Q}}, (g + s + 1)L - \Delta_A) \leq \kappa(K_{\tilde{X}}) \leq g(g + 1)/2,$$

where $L$ is the $\mathbb{Q}$-line bundle of modular forms of weight 1. In particular, $\kappa(K_{\tilde{X}})$ stabilizes to $g(g + 1)/2$ for large $s$.

Here the second inequality $\kappa(K_{\tilde{X}}) \leq g(g + 1)/2$ is just a consequence of Theorem 1.1 and the content of Theorem 1.3 is the first inequality.

Recall ([12], [19]) that the Iitaka dimension $\kappa(D) = \kappa(V, D)$ of a Weil divisor $D$ on a normal compact (analytic) variety $V$ is defined as the maximum of the dimension of the image of $\phi_{|mD|} : V \to \mathbb{P}^N$ as $m$ runs. When $\tilde{X}$ has canonical singularities, $\kappa(K_{\tilde{X}})$ equals to the Kodaira dimension $\kappa(X^s(\Gamma))$ of $X^s(\Gamma)$ (which by definition is $\kappa(K_{\tilde{X}})$ for a smooth projective model $\tilde{X}$ of $X^s(\Gamma)$). In general, we only know $\kappa(X^s(\Gamma)) \leq \kappa(K_{\tilde{X}})$ due to the obstruction coming from possible non-canonical singularities (in the boundary) of $\tilde{X}$. Note that the inequality $\kappa(X^s(\Gamma)) \leq g(g + 1)/2$ also follows from Iitaka’s addition formula [12]. We also have $\kappa(X^s(\Gamma)) \geq \kappa(A(\Gamma))$ by Iitaka’s subadditivity conjecture proved in this case ([30]). Hence $\kappa(X^s(\Gamma)) = g(g + 1)/2$ when $A(\Gamma)$ is of general type.

A similar mechanism of correspondence works also for polarized $K3$ surfaces, including those with rational double points. In that case, there arises a ramification divisor in the total space produced by the $(-2)$-reflection, which gives rise to a vanishing condition at the $(-2)$-Heegner divisor for modular forms. This will be discussed elsewhere.

I wish to thank Gavril Farkas for valuable comments which led me to study Theorem 1.3.

**Organization of the paper.** §2 is a summary of basic definitions, such as Siegel modular variety, Siegel modular forms and Kuga variety. In §3 we prove Theorem 1.1 §4 and §5 are recollection of Siegel domain realization and toroidal compactification. For our purpose, we give an explicit
description of the Siegel domain realization in a self-contained manner. In §6 we prepare an asymptotic estimate of the Petersson norm of local modular forms. In §7 we prepare a general $L^{2/m}$ criterion for log pluricanonical forms. In §8 we prove Theorem 1.2. In §9 we prove Theorem 1.3. In §10 which is rather independent of other sections, we prove that $X'(\Gamma)$ has canonical singularities in most cases. The logical relation between these sections is as follows.

2. Preliminaries

In this section we recall Siegel modular variety, Siegel modular forms, and Kuga family.

2.1. Period domain. Let $\Lambda$ be a free abelian group of rank $2g > 2$ endowed with a nondegenerate symplectic form $(\cdot, \cdot): \Lambda \times \Lambda \to \mathbb{Z}$. Let $G(g, \Lambda C)$ be the Grassmannian parametrizing $g$-dimensional subspaces of $\Lambda C$. Let $LG(\Lambda C) = LG(g, \Lambda C)$ be the Lagrangian Grassmannian parametrizing $g$-dimensional (= maximal) totally isotropic subspaces. The Hermitian symmetric domain attached to $\Lambda$ is the open subset of $LG(\Lambda C)$ defined by $D = \{ [V] \in LG(\Lambda C) | i(\cdot, \cdot)_V > 0 \}$. Here $i(\cdot, \cdot)_V > 0$ means the condition that the Hermitian form $i(\cdot, \cdot)_V$ on $V$ is positive definite. This ensures that $\Lambda C = V \oplus \bar{V}$ for $[V] \in D$.

Let $\Lambda C = D \times \Lambda C$ be the product vector bundle over $D$, $E \to D$ the tautological sub vector bundle of $\Lambda C$ whose fiber over $[V] \in D$ is $V \subset C_{\Lambda}$, and $F = \Lambda C/E$ the universal quotient bundle. By the symplectic pairing we have a canonical isomorphism $F \cong E'$. Via the canonical isomorphism

$TG(g, \Lambda C)_{\mathcal{D}} \cong Hom(E, F) \cong F \otimes F$

for the tangent bundle of $G(g, \Lambda C)$, the tangent bundle of $\mathcal{D}$ is canonically isomorphic to $Sym^2 F$.

2.2. Universal marked family. The local system $\underline{\Lambda} = D \times \Lambda$ inside $\Lambda C$ induces a local system of sections of $F \to D$. The universal family $f: \mathcal{X} \to D$ of abelian varieties over $\mathcal{D}$ is defined by

$\mathcal{X} = F/\underline{\Lambda} = \underline{\Lambda C}/(E + \underline{\Lambda}) \cong E'/\underline{\Lambda}$. 
Here $Λ$ gives a local system of sections of $E^\vee$ by the symplectic pairing. By construction, the fiber of $f : X \to D$ over $[V] \in D$ is the abelian variety

$A = Λ_C/(V + Λ) \cong V^\vee/Λ$

polarized by the symplectic form on $Λ \cong H_1(A, \mathbb{Z})$. We can naturally identify $H^0(Ω^1_A) = V$. Hence if $Ω^1_f$ is the relative cotangent bundle of $f$, we have a canonical isomorphism $f^*Ω^1_f ≃ E$.

Let $L = \det E$, which is restriction of the tautological line bundle $O(-1)$ over $P(Λ^g \Lambda_C)$ by the Plücker embedding

$D \subset LG(Λ_C) \subset G(g, Λ_C) \hookrightarrow P(g \bigwedge Λ_C)$.

The fiber of $L$ over $[V] \in D$ is $\det V = H^0(K_A)$. Hence if $K_f = \det Ω^1_f$ is the relative canonical bundle of $f$, we have a natural isomorphism $f^*K_f \cong L$. Since $K_f|_A ≃ O_A$ for every fiber $A$ of $f$, the natural homomorphism $f^*f^*K_f \to K_f$ is isomorphic. Therefore

$K_f \cong f^*L$.

On the other hand, taking determinant of $Ω^1_D \cong \text{Sym}^2E$, we also have an $\text{Sp}(Λ_R)$-equivariant isomorphism

$K_D \cong \det(\text{Sym}^2E) \cong L^{g+1}$.

For a natural number $s$ we write

$X^{(s)} = X \times_D \cdots \times_D X \cong F^{\otimes s}/Λ^{\otimes s}$

for the $s$-fold self fiber product of $X$ over $D$. Let $f^{(s)} : X^{(s)} \to D$ be the projection. The fiber of $f^{(s)}$ over $[V] \in D$ is $A \times \cdots \times A$ ($s$ times) where $A = V^\vee/Λ$. Since $X^{(s)}$ is pullback of $X \to D$ by $X^{(s-1)} \to D$, we see inductively that the relative canonical bundle of $f^{(s)}$ is isomorphic to

$K_{f^{(s)}} \cong (f^{(s)})^*L^{\otimes s}$.

2.3. Quotient by $Γ$. Let $Γ$ be a finite-index subgroup of the symplectic group $\text{Sp}(Λ)$ of $Λ$. The action of $Γ$ on $Λ_C$ induces the action of $Γ$ on $D$ which is properly discontinuous. The quotient space

$A(Γ) = D/Γ$

is the Siegel modular variety defined by $Γ$. By Baily-Borel [3], $A(Γ)$ has the structure of a normal quasi-projective variety of dimension $g(g+1)/2$.

The group $Γ$ acts on the vector bundle $Λ_C$ equivariantly. This preserves the sub bundle $E$ and the local system $Λ$, and thus $Γ$ acts on $X^{(s)}$. The quotient space

$X^{(s)}(Γ) = X^{(s)}/Γ \cong F^{\otimes s}/(Λ^{\otimes s} \rtimes Γ)$
is called the $s$-fold Kuga family. This is a normal quasi-projective variety of dimension $g(g + 2s + 1)/2$ fibered over $A(\Gamma)$. Here the quasi-projectivity reduces to the case $\Gamma = \text{Sp}(\Lambda)$ by Grothendieck’s Riemann existence theorem ([8] p. 442), and this case follows from Mumford’s GIT construction ([13] Chapter 7, §2 – §3). For $s = 1$ and some torsion-free $\Gamma$, Shimura ([24]) constructed a projective embedding of $X^1(\Gamma)$ using theta functions. In a special case in $g = 2$, its defining equation is determined in [7]. General members of the fibration $X^s(\Gamma) \to A(\Gamma)$ are abelian varieties when $-1 \in \text{element } \Gamma$, and Kummer varieties when $-1 \in \Gamma$. We do not exclude the Kummer case in this paper.

The group $\Gamma$ acts on the line bundle $L$ equivariantly. A $\Gamma$-invariant section of $L$ is called a Siegel modular form of weight $k$ with respect to $\Gamma$. We write $M_k(\Gamma)$ for the space of them. We do not need to impose cusp condition by the Koecher principle (see, e.g., [5]).

2.4. Petersson metric. We define an $\text{Sp}(\Lambda_\mathbb{R})$-invariant Hermitian metric on $L$ and an $\text{Sp}(\Lambda_\mathbb{R})$-invariant volume form on $D$, and explain their relationship with canonical forms.

We fix an isomorphism $\text{det} \Lambda \simeq \mathbb{Z}$. Let $[V] \in D$. For two vectors $\omega = v_1 \wedge \cdots \wedge v_g$, $\eta = w_1 \wedge \cdots \wedge w_g$ of $L_{[V]} = \text{det} V$, the wedge product

$$(2.1) \quad \omega \wedge \bar{\eta} = v_1 \wedge \cdots \wedge v_g \wedge \bar{w}_1 \wedge \cdots \wedge \bar{w}_g$$

is a vector of $\text{det} \Lambda_{\mathbb{C}}$. We define the inner product of $\omega$ and $\eta$ to be the image of $i^g \omega \wedge \bar{\eta}$ in $\text{det} \Lambda_{\mathbb{C}} \simeq \mathbb{C}$. This defines a Hermitian metric on the line bundle $L$, which is $\text{Sp}(\Lambda_\mathbb{R})$-invariant by construction. Its $k$-th power defines an $\text{Sp}(\Lambda_\mathbb{R})$-invariant Hermitian metric on $L^\otimes k$ which we denote by $(\cdot, \cdot)^k$.

Alternatively, one can also define $(\cdot, \cdot)_1$ as follows. For $[V] \in D$ we have the natural Hermitian inner product $i(\cdot, \gamma)|_V$ on $V$. This defines an $\text{Sp}(\Lambda_\mathbb{R})$-invariant Hermitian metric on the vector bundle $E$. The induced metric on $L = \text{det} E$ is also $\text{Sp}(\Lambda_\mathbb{R})$-invariant, so coincides with $(\cdot, \cdot)_1$ up to a constant.

Geometrically, $(\cdot, \cdot)_1$ is the Hodge metric for the abelian fibration $X \to D$.

Lemma 2.1. Let $A = V^g/\Lambda$ be the abelian variety over $[V] \in D$. We identify $\omega, \eta \in L_{[V]} = \text{det} V$ with canonical forms on $A$ via the natural isomorphism $\text{det} V \simeq H^0(K_A)$. Then we have

$$(\omega, \eta)_1 = i^g \int_A \omega \wedge \bar{\eta}.$$ 

Proof. The wedge product (2.1) corresponds to the $(g, g)$ form $\omega \wedge \bar{\eta}$ on $A$ via the isomorphism $\text{det} \Lambda_{\mathbb{C}} \simeq H^{2g}(A, \mathbb{C})$. The isomorphism $\text{det} \Lambda_{\mathbb{C}} \simeq \mathbb{C}$ coincides with the integration map $\int_A : H^{2g}(A, \mathbb{C}) \to \mathbb{C}$. □

We next define an invariant volume form $\text{vol}_D$ on $D$. Via the isomorphism $K_D \simeq L_{g+1}^{\otimes}$, the metric $(\cdot, \cdot)_{g+1}$ induces an invariant Hermitian metric
on $K_D$. For each $[V] \in D$, we choose a vector $\omega \neq 0$ in the fiber $(K_D)_{[V]}$ of $K_D$ and define

$$(\text{vol}_D)_{[V]} = i^{N^2} \frac{\omega \wedge \bar{\omega}}{(\omega, \omega)_{g+1}}$$

where $N = \dim D = g(g + 1)/2$. This does not depend on the choice of $\omega$ and defines a volume form on $D$, i.e., a real form of top degree which is everywhere positive with respect to the orientation. Since $(\ , \ )_{g+1}$ is $\text{Sp}(\Lambda_{\mathbb{R}})$-invariant, so is $\text{vol}_D$.

**Lemma 2.2.** Let $\omega, \eta$ be two local sections of $K_D$ over a subset of $D$. Then we have

$$(\omega, \eta)_{g+1} \text{vol}_D = i^{N^2} \omega \wedge \bar{\eta}.$$ 

**Proof.** It suffices to check this equality at each point. We may assume $\omega, \eta \neq 0$. Then $\eta = \alpha \omega$ for some $\alpha \in \mathbb{C}$, so we have

$$\frac{\omega \wedge \bar{\eta}}{(\omega, \eta)_{g+1}} = \frac{\bar{\alpha} \omega \wedge \bar{\omega}}{\bar{\alpha}(\omega, \omega)_{g+1}} = i^{-N^2} \text{vol}_D.$$

□

2.5. **Siegel upper half space.** The traditional style defining Siegel modular forms as functions on the Siegel upper half space can be realized if we pick up a 0-dimensional cusp of $D$, which corresponds to a maximal ($= \text{rank } g$ primitive) totally isotropic sublattice of $\Lambda$. In this subsection we recall this translation.

Choose a maximal totally isotropic sublattice $J$ of $\Lambda$. We also choose a maximal totally isotropic subspace $J'_Q$ of $\Lambda_Q$ such that $\Lambda_Q = J_Q \oplus J'_Q$. (The role of $J'_Q$ will be auxiliary.) We can identify $J'_Q \simeq J'_Q^\vee$ and $(J'_Q)^\vee \simeq J_Q$ by the symplectic pairing. The choice of $J_C$ determines the Zariski open set $\{ [V] | V \cap J_C = \{ 0 \} \}$ of the Grassmannian $G(g, \Lambda_C)$. Via the splitting $\Lambda_Q = J_Q \oplus J'_Q$, this open set is mapped isomorphically to the linear space

$$\text{Hom}(J'_C, J_C) \simeq J_C \otimes J_C$$

by associating to a linear map $J'_C \to J_C$ its graph. Taking intersection with $\text{LG}(\Lambda_C)$, we obtain the Zariski open set

$$H_J := \{ [V] \in \text{LG}(\Lambda_C) | V \cap J_C = \{ 0 \} \}$$

of $\text{LG}(\Lambda_C)$. Since totally isotropicity of the graph of $J'_C \to J_C$ is equivalent to the symmetricity of the corresponding vector of $J_C \otimes J_C$, we obtain

$$H_J \simeq \text{Sym}^2 J_C.$$ 

The domain $D$ is contained in $H_J$, and its image by $H_J \to \text{Sym}^2 J_C$ is

$$\delta_J := \{ \Omega \in \text{Sym}^2 J_C | \text{Im} \Omega > 0 \}.$$
This is a realization of \( \mathcal{D} \) as the Siegel upper half space. If we use another \( J'_Q \) in place of \( J_Q' \), the isomorphism \( H_J \to \text{Sym}^2 J_C \) is shifted by a translation.

We choose an orientation of \( J \). This determines a generator of \( \wedge^g J \approx \mathbb{Z} \) which we denote by \( \text{det} J \). Then we can define a nowhere vanishing section \( s_J \) of the line bundle \( L \) by the condition

\[
(s_J([V]), \text{det} J) = 1, \quad [V] \in \mathcal{D}.
\]

Here \( ( , ) \) is the paring between \( L[V] = \text{det} V \subset \wedge^g \Lambda_C \) and \( \text{det} J \in \wedge^g \Lambda_C \) induced from the symplectic form on \( \Lambda_C \). The factor of automorphy associated to the frame \( s_J \) is given by

\[
j(\gamma, [V]) = \frac{\gamma(s_J([V]))}{s_J(\gamma [V])} = (\gamma(s_J([V])), \text{det} J).
\]

Via the trivialization of \( L^\otimes k \) by \( s_J^\otimes k \), Siegel modular forms of weight \( k \) are identified with holomorphic functions \( F \) on \( \mathcal{D} \). The invariance under \( \Gamma \) is equivalent to the condition

\[
F([\gamma V]) = j(\gamma, [V])^k F([V]), \quad \gamma \in \Gamma, \quad [V] \in \mathcal{D}.
\]

We calculate \( j(\gamma, [V]) \) on the Siegel upper half space \( \mathfrak{H}_J \). Choose a basis \( l_1, \cdots, l_g \) of \( J \) of positive orientation, and let \( m_1, \cdots, m_g \in J_Q' \) be its dual basis, namely \( (m_i, l_j) = \delta_{ij} \). For \( [V] \in \mathcal{D} \) we can take the basis \( \omega_1, \cdots, \omega_g \) of \( V \) such that \( (\omega_i, l_j) = \delta_{ij} \) (normalized basis). If we write the matrix expression of \( (\omega_1, \cdots, \omega_g) \) with respect to \( (l_i), (m_j) \) in the form \( (\Omega I_g) \), then \( \Omega \) is the symmetric matrix representing the image of \( [V] \) in \( \text{Sym}^2 J_C \) with respect to \( (l_i) \). Let \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) be the matrix representation of \( \gamma \) with respect to \( (l_i), (m_j) \). Since \( s_J([V]) = \omega_1 \wedge \cdots \wedge \omega_g \), we have

\[
j(\gamma, [V]) = (\gamma \omega_1 \wedge \cdots \wedge \gamma \omega_g, l_1 \wedge \cdots \wedge l_g) = \text{det}(C\Omega + D).
\]

This is the classical form of factor of automorphy.

Next we calculate the Petersson metric on \( L \) over \( \mathfrak{H}_J \).

**Lemma 2.3.** Let \( \Omega \) be the matrix expression of the image of \( [V] \in \mathcal{D} \) in \( \mathfrak{H}_J \). Then we have

\[
(s_J([V]), s_J([V]))_1 = \text{det} \text{Im} \Omega
\]

up to a constant independent of \( V \).

**Proof.** We use the notation above. Since \( s_J([V]) = \omega_1 \wedge \cdots \wedge \omega_g \) and

\[
\omega_1 \wedge \cdots \wedge \omega_g \wedge \bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_g
\]

\[
= \text{det} \begin{pmatrix} \Omega & \bar{\Omega} \\ l_g & l_g \end{pmatrix} l_1 \wedge \cdots \wedge l_g \wedge m_1 \wedge \cdots \wedge m_g,
\]

\[
\text{det} \left( \begin{pmatrix} \Omega & \bar{\Omega} \\ l_g & l_g \end{pmatrix} l_1 \wedge \cdots \wedge l_g \wedge m_1 \wedge \cdots \wedge m_g, \right)
\]

\[
\text{det} (\Omega + \bar{\Omega}) \text{det} \begin{pmatrix} l_1 \wedge \cdots \wedge l_g \\ m_1 \wedge \cdots \wedge m_g \end{pmatrix}.
\]
then \((s_J([V]), s_J([V]))_1\) equals to a constant multiple of
\[
\det \begin{pmatrix} \Omega & \bar{\Omega} \\ I_g & I_g \end{pmatrix} = \det \begin{pmatrix} \Omega & \bar{\Omega} \\ O & I_g \end{pmatrix} = (2i)^g \det(\text{Im } \Omega).
\]

\[\square\]

**Corollary 2.4.** Let \(F\) and \(G\) be local sections of \(L^g\) over some subset of \(D\). We identify \(F, G\) with holomorphic functions \(F(\Omega), G(\bar{\Omega})\) on the corresponding subset of \(S_J\) via the frame \(s_J^g\) and the isomorphism \(D \to S_J\). Then we have
\[
(F([V]), G([V]))_g = F(\Omega) \cdot G(\bar{\Omega}) \cdot \det(\text{Im } \Omega)^g.
\]

Finally, we express the invariant volume form \(\text{vol}_D\) in terms of the flat volume form on \(\text{Sym}^2 J_C\).

**Lemma 2.5.** Let \(\text{vol}_J\) be a flat volume form on \(\text{Sym}^2 J_C\). Under the isomorphism \(D \cong S_J\) we have
\[
\text{vol}_D = \frac{1}{\det(\text{Im } \Omega)^{g+1}} \text{vol}_J
\]
up to a constant.

**Proof.** The canonical form \(\omega_J\) on \(D\) corresponding to the section \(s_J^g\) of \(L^g\) extends to a translation-invariant canonical form on \(\text{Sym}^2 J_C\). Hence \(\omega_J = dz_1 \wedge \cdots \wedge dz_N\) for some coordinate \(z_1, \cdots, z_N\) on \(\text{Sym}^2 J_C\). Then
\[
\text{vol}_D = \frac{1}{\text{det}(\text{Im } \Omega)^{g+1}} \text{vol}_J = \frac{\text{vol}_J}{(s_J, s_J)^{g+1}} = \frac{\text{vol}_J}{\det(\text{Im } \Omega)^{g+1}}.
\]

\[\square\]

We thus recover the classical form of Petersson inner product.

### 3. Proof of Theorem 1.1

In this section we prove Theorem 1.1 and discuss its examples and a higher analogue. We use the common notation \(f: \mathcal{X}^{(s)} \to D\) and \(f: X^s(\Gamma) \to A(\Gamma)\) for both projections.

#### 3.1. Proof of Theorem 1.1

The correspondence is simple. Recall from §2.2 that we have the isomorphism \(K_f \simeq f^* L^{g} \) over \(\mathcal{X}^{(s)}\) and \(K_D \simeq L^{g+1}\) over \(D\). Combining them, we obtain the isomorphism
\[
(3.1) \quad K_{\mathcal{X}^{(s)}} \simeq K_f \otimes f^* K_{\mathcal{D}}^{(s)} \simeq f^* L^{(g+1)m}
\]
over \(\mathcal{X}^{(s)}\), and \(f_s(K_{\mathcal{X}^{(s)}}) \simeq L^{(g+1)m}\) over \(D\).

Let \(\Gamma\) be a finite-index subgroup of \(\text{Sp}(\Lambda)\). We first consider the case \(\Gamma\) is torsion-free. The \(\Gamma\)-linearized line bundles \(K_{\mathcal{X}^{(s)}}, K_f, K_D\) on \(\mathcal{X}^{(s)}\) and \(D\) descend to the line bundles \(K_{\mathcal{X}^{(s)}(\Gamma)}, K_f, K_{\text{A}(\Gamma)}\) on \(X^s(\Gamma)\) and \(A(\Gamma)\). Also \(L\)
descends to a line bundle on $A(\Gamma)$ which we again denote by $L$. Since the isomorphism (3.1) is $\Gamma$-equivariant, it descends to the isomorphism

$$K^{\otimes m}_{X^s(\Gamma)} \simeq f^* L^{\otimes (g+s+1)m}$$

of line bundles over $X^s(\Gamma)$. Taking global sections over $X^s(\Gamma)$ gives

$$H^0(X^s(\Gamma), K^{\otimes m}_{X^s(\Gamma)}) \simeq H^0(X^s(\Gamma), f^* L^{\otimes (g+s+1)m}) \simeq H^0(A(\Gamma), L^{\otimes (g+s+1)m}) = M_{(g+s+1)m}(\Gamma).$$

Clearly this isomorphism is compatible with multiplication, and we obtain Theorem 1.1 in this case.

We next consider the general case $\Gamma$ is not necessarily torsion-free.

**Lemma 3.1.** The projection $X^{(s)} \to X^s(\Gamma)$ is unramified in codimension 1.

**Proof.** Let $\gamma \neq \text{id} \in \Gamma$ be an element of finite order. It suffices to show that the fixed locus of $\gamma$ on $X^{(s)}$ has codimension $\geq 2$. When $\gamma = -\text{id}$, the fixed locus is the sections of order $\leq 2$ points in the abelian fibration $X^{(s)} \to D$, which has codimension $gs \geq 2$. When $\gamma \neq \pm \text{id}$, $\gamma$ acts on $D$ nontrivially. If $[V] \in D$ is a fixed point of $\gamma$, the $\gamma$-action on the linear space $V$ is nontrivial, so $\gamma$ acts on the fiber $(V/\Lambda)^s$ nontrivially. Then the fixed locus of $\gamma$ on $X^{(s)}$ has codimension $\geq 1 + 1 = 2$. $\square$

Now we choose a torsion-free normal subgroup $\Gamma' \triangleleft \Gamma$ of finite index. Let $G = \Gamma/\Gamma'$. The finite group $G$ acts on $X^s(\Gamma') \to A(\Gamma')$ with quotient $X^s(\Gamma) \to A(\Gamma)$. By the previous step for $\Gamma'$, we have an isomorphism

$$H^0(X^s(\Gamma'), K^{\otimes m}_{X^s(\Gamma')}) \simeq M_{(g+s+1)m}(\Gamma')$$

which by construction is $G$-equivariant. We take the $G$-invariant part of this isomorphism. For the right side we have by definition

$$M_{(g+s+1)m}(\Gamma')^G = M_{(g+s+1)m}(\Gamma).$$

For the left side, since $X^s(\Gamma') \to X^s(\Gamma)$ is unramified in codimension 1 by Lemma 3.1 we have

$$H^0(X^s(\Gamma'), K^{\otimes m}_{X^s(\Gamma')})^G = H^0(X^s(\Gamma), K^{\otimes m}_{X^s(\Gamma)}).$$

This proves Theorem 1.1.

3.2. **Examples.** We discuss a few easy consequences of Theorem 1.1.

**Corollary 3.2.** Let $g, s, m$ be odd and assume $-1 \in \Gamma$. Then there is no nonzero $m$-canonical form on $X^s(\Gamma)$. In particular, there is no nonzero $m$-canonical form on any smooth projective model of $X^s(\Gamma)$. 
Proof. The first assertion follows from the standard fact that when \(-1 \in \Gamma\) and \(g\) is odd, \(M_k(\Gamma) = 0\) for \(k = (g + s + 1)m\) odd. Indeed, \(-1 \in \Gamma\) acts on \(L^g\) by multiplication by \((-1)^g = -1\), so there is no nonzero section of \(L^g\) invariant under \(-1 \in \Gamma\). The second assertion holds because we can take a projective compactification \(\bar{X}\) of \(X^r(\Gamma)\) and its desingularization \(\tilde{X} \to \bar{X}\) that is isomorphic over the regular locus of \(\bar{X}\), so we can restrict pluricanonical forms on \(\tilde{X}\) to the regular locus of \(X^r(\Gamma)\). □

Corollary 3.3. There always exists a nonzero bi-canonical form on \(X^r(\Gamma)\). When \(g + s\) is odd, there always exists a nonzero canonical form on \(X^r(\Gamma)\).

Proof. This holds because we have the Eisenstein series of even weight \(> g + 1\) (cf. [5]). □

The pluricanonical forms corresponding to the Eisenstein series, however, cannot extend holomorphically over the compactifications \(\bar{X}\) of \(X^r(\Gamma)\) as in Theorem 1.2 in general (Proposition 8.5).

In some special cases, the ring of Siegel modular forms is explicitly determined. This tells us the structure of the ring \(\otimes_m H^0(\mathcal{K}_{X^r(\Gamma)}^{\otimes m})\).

Example 3.4. Let \(\Gamma = \text{Sp}(4, \mathbb{Z})\). By Igusa [11], the ring of modular forms of even weight is free:
\[
\bigoplus_{k \in 2\mathbb{Z}} M_k(\text{Sp}(4, \mathbb{Z})) = \mathbb{C}[e_4, e_6, \chi_{10}, \chi_{12}],
\]
where \(e_k\) is the Eisenstein series of weight \(k\), and \(\chi_{10}, \chi_{12}\) are the unique cusp forms of weight 10, 12 respectively. This implies that \(\otimes_m H^0(\mathcal{K}_{X}^{\otimes m})\) for \(X = X^1(\text{Sp}(4, \mathbb{Z}))\) is isomorphic to
\[
\bigoplus_{k \in 4\mathbb{Z}} M_k(\text{Sp}(4, \mathbb{Z})) = \mathbb{C}[e_4, e_6^2, \chi_{12}, e_6 \chi_{10}, \chi_{10}^2]/(e_6^2 \cdot \chi_{10}^2 = (e_6 \chi_{10})^2).
\]
See, e.g., [11], [6], [29] and the references there for other cases where the ring structure is determined. In much more cases, explicit dimension formula for \(S_k(\Gamma)\) is known: see [28], [32] for recent general results, and the references in [32] for known cases in small \(g\). For the opposite direction, we refer to, e.g., [31] for the birational type of some \(X^1(\Gamma)\) in small \(g\).

3.3. Vector-valued Siegel modular forms. This subsection is a sort of appendix to §3.1. We combine a generalization of the argument in §3.1 with the Leray spectral sequence for \(X^r(\Gamma) \to A(\Gamma)\). This gives a spectral sequence that relates the cohomology of \(\mathcal{K}_{X^r(\Gamma)}^{\otimes m}\) with the cohomology of certain vector bundles on \(A(\Gamma)\) whose sections are vector-valued modular forms. The result of this subsection will not be used in other sections.

We assume that \(\Gamma\) is torsion-free. The universal quotient bundle \(F\) over \(\mathcal{D}\) descends to a vector bundle over \(A(\Gamma)\) which we again denote by \(F\).
Proposition 3.5. Assume that $\Gamma$ is torsion-free. For each $m \geq 0$ there exists a spectral sequence

\[
E^p_{2q} = H^p(A(\Gamma), \bigwedge^q (F^{\oplus s}) \otimes L^{\otimes (g+s+1)m}) \Rightarrow H^{p+q}(X^s(\Gamma), K^{\otimes m}_{X^s(\Gamma)}).
\]

Proof. We abbreviate $X = X^s(\Gamma)$. We shall rewrite the $E_2$ page of the Leray spectral sequence

\[
E^p_{2q} = H^p(A(\Gamma), \mathcal{R}^q f_*(K^{\otimes m}_X)) \Rightarrow H^{p+q}(X, K^{\otimes m}_X).
\]

Since $K_X \simeq f^* L^{\otimes g+s+1}$, we have

\[
\mathcal{R}^q f_*(K^{\otimes m}_X) \simeq \mathcal{R}^q f_* f^* L^{\otimes (g+s+1)m} \simeq \mathcal{R}^q f_* O_X \otimes L^{\otimes (g+s+1)m}.
\]

by the projection formula. We shall show that

\[
\mathcal{R}^q f_* O_X \simeq \bigwedge^q (F^{\oplus s}).
\]

By Grauert’s theorem ([8] III.12.9), $\mathcal{R}^q f_* O_X$ is locally free and its fiber over a point $[A] = [V^s/\Lambda]$ of $A(\Gamma)$ is identified with $H^q(\mathcal{O}_{A^s})$. We have the canonical isomorphisms

\[
H^q(\mathcal{O}_{A^s}) \simeq H^0(\Omega^q_{A^s}) \simeq \bigwedge^q H^0(\Omega^1_{A^s}) \simeq \bigwedge^q (H^0(\Omega^1_{A^s}))^{\oplus s} \simeq \bigwedge^q (V^s)^{\oplus s}
\]

Here the first isomorphism is induced from the Hodge pairing

\[
H^{0,0}(A^s) \times H^{0,q}(A^s) \to \mathbb{C}, \quad (\omega, \eta) \mapsto \int_{A^s} \omega \wedge \eta \wedge h^{g+s-q},
\]

where $h$ is the polarization on $A^s$ induced from the given symplectic form. The space $\bigwedge^q (V^s)^{\oplus s}$ is the fiber of $\bigwedge^q (E^\vee)^{\oplus s} \simeq \bigwedge^q F^{\oplus s}$ over $[A]$. \qed

Sections of $\bigwedge^q (F^{\oplus s}) \otimes L^{\otimes k}$ over $A(\Gamma)$ are identified with modular forms of weight $k$ for $\Gamma$ with values in $\bigwedge^q (\text{St}^{\oplus s})^\vee$ where St is the standard representation of $\text{GL}_g(\mathbb{C})$. Thus the edge morphism of (3.2) at the side $p = 0$ takes the form

\[
H^q(X^s(\Gamma), K^{\otimes m}_{X^s(\Gamma)}) \to M_{(g+s+1)m}(\Gamma, \bigwedge^q (\text{St}^{\oplus s})^\vee).
\]

The edge morphism at the other side $q = 0$ takes the form

\[
H^p(A(\Gamma), L^{\otimes (g+s+1)m}) \to H^p(X^s(\Gamma), K^{\otimes m}_{X^s(\Gamma)}).
\]
4. Siegel domain realization

In this section we recall the Siegel domain realization (of the third kind) of \( \mathcal{D} \) associated to each cusp. We give an explicit and self-contained description of the Siegel domain realization that does not depend on coordinate (so as to be suitable for dealing with general \( \Gamma \)). We follow the style of \([15]\). Throughout this section we fix a primitive totally isotropic sublattice \( I \) of \( \Lambda \), say of rank \( g' \). This corresponds to a cusp (rational boundary component) of \( \mathcal{D} \). We set \( g'' = g - g' \). We write \( \Lambda(I) = I^\perp / I \), which is a nondegenerate symplectic lattice of rank \( 2g'' \).

4.1. Structure of the stabilizer. Let \( \Gamma(I)_Q \) be the stabilizer of \( I_Q \) in \( \text{Sp}(\Lambda_Q) \). We describe the structure of \( \Gamma(I)_Q \) and \( \Gamma(I)_Z = \Gamma(I)_Q \cap \Gamma \).

4.1.1. Over \( \mathbb{Q} \). Let \( W(I)_Q < \Gamma(I)_Q \) be the kernel of the natural map \( \Gamma(I)_Q \to \text{Sp}(\Lambda(I)_Q) \times \text{GL}(I_Q) \). Since this is surjective, we have the canonical exact sequence

\[
1 \to W(I)_Q \to \Gamma(I)_Q \to \text{Sp}(\Lambda(I)_Q) \times \text{GL}(I_Q) \to 1.
\]

If we choose a lift \( \Lambda(I)_Q \hookrightarrow I_Q^+ \) of \( \Lambda(I)_Q \) and a totally isotropic subspace \( I_Q' \) of \( \Lambda_Q \) with \( \Lambda(I)_Q^+ = I_Q \oplus I_Q' \), we obtain a non-canonical splitting of this sequence, by letting \( \text{Sp}(\Lambda(I)_Q) \) act on the lifted \( \Lambda(I)_Q \) and \( \text{GL}(I_Q) \) on \( I_Q \oplus I_Q' \cong I_Q \oplus I_Q' \). This gives a non-canonical isomorphism

\[
\Gamma(I)_Q \cong (\text{Sp}(\Lambda(I)_Q) \times \text{GL}(I_Q)) \rtimes W(I)_Q.
\]

Let \( U(I)_Q < W(I)_Q \) be the kernel of the natural map \( \Gamma(I)_Q \to \text{GL}(I_Q^+) \). We put \( V(I)_Q = W(I)_Q / U(I)_Q \).

Elements of \( W(I)_Q \) can be described as follows. For \( m \in I_Q^+ \) and \( l \in I_Q \) we define \( T_{m,l} \in \text{Sp}(\Lambda_Q) \) by

\[
T_{m,l}(v) = v + (m, v)l + (l, v)m, \quad v \in \Lambda_Q.
\]

Since \( T_{m,l} \) acts on \( I_Q^+ \) by \( T_{m,l}(v) = v + (m, v)l \) for \( v \in I_Q^+ \), it acts trivially on \( I_Q \) and \( \Lambda(I)_Q \), so is an element of \( W(I)_Q \).

**Lemma 4.1.** The following relations hold.

1. \( T_{\alpha m, l} = T_{m, \alpha l} \) for \( \alpha \in \mathbb{Q} \).
2. \( T_{m,l} \circ T_{m', l'} = T_{m + m', l + l'} \).
3. \( T_{m,l} \circ T_{m', l'} = T_{m + m', l + l'} \) where \( \alpha = (m, m')/2 \).
4. \( T_{l,l'} = T_{l', l} \) if \( l, l' \in I_Q \).

**Proof.** This can be checked by a direct calculation. \( \square \)

**Proposition 4.2.** (1) The group \( W(I)_Q \) is generated by the elements \( T_{m,l} \).

More specifically, if we take a basis \( l_1, \ldots, l_{2g-g'} \) of \( I_Q^+ \) such that \( l_1, \ldots, l_{g'} \)
span \( I_Q \), elements of \( W(I)_Q \) can be written as compositions of \( T_{\alpha_i, j_i, l_i} \) for some \( \alpha_{i, j} \in \mathbb{Q} \) where \( 1 \leq i \leq 2g - g' \) and \( 1 \leq j \leq g' \).

(2) We have the canonical isomorphisms

\[
\text{Sym}^2 I_Q \cong U(I)_Q, \quad l \cdot l' \mapsto T_{l, l'},
\]

\[
\Lambda(I)_Q \otimes I_Q \cong V(I)_Q, \quad m \otimes l \mapsto [T_{m, l}],
\]

where \( m \in I_Q' \) is a lift of \( m \in \Lambda(I)_Q \). In particular, \( U(I)_Q \) and \( V(I)_Q \) are \( \mathbb{Q} \)-vector spaces.

**Proof.** (1) Let \( \gamma \in W(I)_Q \). It acts on \( I_Q^\perp \) by \( \gamma(v) = v + \varphi(v) \) for some linear map \( \varphi : I_Q^\perp \to I_Q \) such that \( \varphi(I_Q^\perp) = 0 \), namely \( \varphi : \Lambda(I)_Q \to I_Q \). By the nondegeneracy of \( \Lambda(I)_Q \), we can write \( \varphi(\cdot) = \sum \langle m_\alpha, \cdot \rangle l_\alpha \) for some \( \sum \alpha \cdot m_\alpha \otimes l_\alpha \in \Lambda(I)_Q \otimes I_Q \). Thus, composing \( \gamma \) with \( T_{m_\alpha, l_\alpha} \), we may assume that \( \gamma \) acts trivially on \( I_Q^\perp \), namely \( \gamma \in U(I)_Q \). Next we choose a totally isotropic subspace \( I_Q' \subset I_Q \) such that \( \Lambda_Q = I_Q \oplus I_Q' \) and put \( \Lambda_Q' = I_Q \oplus I_Q' \). Since \( \gamma \) acts trivially on \( \Lambda_Q' \), it preserves \( \Lambda_Q' \). Since \( \gamma \) acts trivially on \( \Lambda_Q' / I_Q = I_Q' \), it sends \( v \in I_Q' \to \gamma(v) = v + \psi(v) \) for some linear map \( \psi : I_Q' \to I_Q \). As before we can write \( \psi(\cdot) = \sum \alpha_i l_i \otimes l_i \) for some \( \sum \alpha_i l_i \otimes l_i \in I_Q \otimes I_Q \). Since \( \gamma(I_Q') \) is totally isotropic, then \( \alpha_i l_i = \alpha_i l_i' \). Hence \( \gamma \) equals to the composition of \( T_{\alpha_i l_i, l_i' l_i} \). Assertion (2) follows from the proof of (1) and Lemma 4.1.

Thus \( U(I)_Q \) is the center of \( W(I)_Q \), and we have the exact sequence

\[
0 \to \text{Sym}^2 I_Q \to W(I)_Q \to \Lambda(I)_Q \otimes I_Q \to 0.
\]

When \( g' = 1 \), this gives \( W(I)_Q \) the structure of a Heisenberg group.

Since \( U(I)_Q \) is a normal subgroup of \( \Gamma(I)_Q \), we have the adjoint action of \( \Gamma(I)_Q \) on \( U(I)_Q \). Since \( \gamma \circ T_{m, l} \circ \gamma^{-1} = T_{m, l} \gamma \) for \( \gamma \in \Gamma(I)_Q \), this coincides with the natural action of \( \Gamma(I)_Q \) on \( \text{Sym}^2 I_Q \).

We describe the action of \( U(I)_Q \) on \( \mathcal{D} \) in a special case. (The general case will be studied later.)

**Lemma 4.3.** Suppose that \( I \) is maximal, i.e., \( g' = g \). We take an isomorphism \( H_I \cong \text{Sym}^2 I_C \) as in \( \text{\S}2.5 \). Then the \( U(I)_Q \)-action on \( H_I \) coincides with the translation by \( \text{Sym}^2 I_Q \) on \( \text{Sym}^2 I_C \).

**Proof.** As in \( \text{\S}2.5 \) we choose a maximal totally isotropic subspace \( I_Q' \) of \( \Lambda_Q \) with \( \Lambda_Q = I_Q \oplus I_Q' \). Take a vector in \( \text{Sym}^2 I_C \) of the form \( v \cdot w = v \otimes w + w \otimes v \) where \( v, w \in I_C \). The corresponding linear map \( I_Q' \to I_C \) is

\[
\varphi_{v, w}(\cdot) = (v, \cdot)w + (w, \cdot)v.
\]

Then \( T_{l, l'} \in U(I)_Q \) sends the graph of \( \varphi_{v, w} \) to the graph of

\[
\varphi_{v, w + l, l'}(\cdot) = (v, \cdot)w + (w, \cdot)v + (l, \cdot)l' + (l', \cdot)l.
\]
This proves our claim. □

4.1.2. Over $\mathbb{Z}$. Now let $\Gamma$ be a finite-index subgroup of $\text{Sp}(\Lambda)$ and $\Gamma(I)_\mathbb{Z} = \Gamma(I)_\mathbb{Q} \cap \Gamma$ be the stabilizer of $I$ in $\Gamma$. We put

$$W(I)_\mathbb{Z} = W(I)_\mathbb{Q} \cap \Gamma, \quad U(I)_\mathbb{Z} = U(I)_\mathbb{Q} \cap \Gamma, \quad V(I)_\mathbb{Z} = W(I)_\mathbb{Z} / U(I)_\mathbb{Z}.$$  

Then $U(I)_\mathbb{Z}$ is a lattice in $U(I)_\mathbb{Q} \cong \text{Sym}^2 I_\mathbb{Q}$, and $V(I)_\mathbb{Z}$ is a lattice in $V(I)_\mathbb{Q} \cong \Lambda(I)_\mathbb{Q} \otimes I_\mathbb{Q}$. We also set

$$\Gamma(I)_\mathbb{Z} = \overline{\Gamma(I)}_\mathbb{Z}/U(I)_\mathbb{Z}, \quad \Gamma_I = \Gamma(I)_\mathbb{Z}/W(I)_\mathbb{Z}.$$  

Then $\Gamma_I$ is mapped injectively into $\text{Sp}(\Lambda(I)) \times \text{GL}(I)$. By definition we have the canonical exact sequences

$$(4.3) \quad 0 \to U(I)_\mathbb{Z} \to W(I)_\mathbb{Z} \to V(I)_\mathbb{Z} \to 0,$$

$$(4.4) \quad 0 \to W(I)_\mathbb{Z} \to \Gamma(I)_\mathbb{Z} \to \Gamma_I \to 0.$$  

4.2. **Siegel domain realization.** The choice of the totally isotropic subspace $I_\mathbb{C}$ determines the 2-step projection

$$(4.5) \quad \text{LG}(\Lambda_\mathbb{C}) \xrightarrow{\pi_1} \text{LG}(g'', I_\mathbb{C}^+) \xrightarrow{\pi_2} \text{LG}(g'', \Lambda(I)_\mathbb{C}),$$

where $\pi_1$ sends $V$ to $W = V \cap I_\mathbb{C}^+$ and $\pi_2$ sends $W$ to its image in $\Lambda(I)_\mathbb{C}$. We shall show that the restriction of $(4.5)$ to $D \subset \text{LG}(\Lambda_\mathbb{C})$ gives an embedded 2-step fibration

$$(4.6) \quad D \hookrightarrow D(I) \hookrightarrow \text{LG}(\mathcal{K}_I) \xrightarrow{\pi_1} \mathcal{V}_I \xrightarrow{\pi_2} D_{\Lambda(I)}$$

where

- $D_{\Lambda(I)}$ is the Hermitian symmetric domain attached to $\Lambda(I)$,
- $\mathcal{V}_I \to D_{\Lambda(I)}$ an affine space bundle for a vector bundle,
- $\text{LG}(\mathcal{K}_I) \to \mathcal{V}_I$ a relative Lagrangian Grassmannian,
- $D(I) \to \mathcal{V}_I$ a principal $\text{Sym}^2 I_\mathbb{C}$-bundle, and
- $D \to \mathcal{V}_I$ a Siegel upper half space bundle.

This is an explicit form of the Siegel domain realization of $D$ at the cusp for $I$. When $g' = g$, this is the Siegel upper half space model in §2.3.

4.2.1. **Linear algebra.** It will be useful to enlarge $D$ to the open set

$$D(I) = \{ [V] \in \text{LG}(\Lambda_\mathbb{C}) \mid i(\cdot, \cdot)|_{W \cap I_\mathbb{C}^+} > 0 \}$$

of $\text{LG}(\Lambda_\mathbb{C})$. We begin by clarifying the linear algebra construction

$$(4.7) \quad V \mapsto (W, V/W) \mapsto W \mapsto \text{Im}(W \to \Lambda(I)_\mathbb{C})$$

for $[V] \in D(I)$ where $W = V \cap I_\mathbb{C}^+$. 

Lemma 4.4. For $[V] \in \mathcal{D}(I)$ let $W = V \cap I_C^\perp$. Then $W$ is totally isotropic, $i\langle \cdot, \cdot \rangle_W > 0$, $\dim W = g''$ and $W \cap I_C = \{0\}$. In particular, the natural map $W \to \Lambda(I)_C$ is injective and its image is a point of $\mathcal{D}_{\Lambda(I)}$. \\
Proof. The first two assertions are obvious. Since $\langle \cdot, \cdot \rangle|_{I_C} \equiv 0$, we have $W \cap I_C = \{0\}$. Hence $W \to \Lambda(I)_C$ is injective. Since its image is a totally isotropic subspace of $\Lambda(I)_C$, we have $\dim W \leq g''$. We also have $\dim W \geq g''$ because $W$ is the kernel of the pairing map $V \to I_C'$. \hfill \Box \\
We next look at the quotient space $V/W$. We set $\Lambda(W) = (W^\perp \cap \Lambda_C)/W$, which is a nondegenerate symplectic space of dimension $2g'$ over $\mathbb{C}$. Then $V/W$ is a maximal totally isotropic subspace of $\Lambda(W)$, namely a point of $\text{LG}(\Lambda(W))$. We have another, distinguished point of $\text{LG}(\Lambda(W))$ as follows.

Lemma 4.5. Let $I_W = (I_C \oplus W)/W$ be the image of $I_C$ in $\Lambda(W)$. Then $I_W$ is a maximal totally isotropic subspace of $\Lambda(W)$, and we have $(V/W)\cap I_W = \{0\}$. \\
Proof. Since $I_C \subset W^\perp$, we have a natural map $I_C \to \Lambda(W)$. This is injective by $I_C \cap W = \{0\}$, hence $I_W$ has dimension $g'$. Since $V \cap I_C = \{0\}$ and $W \subset V$, we have $V \cap (W \oplus I_C) = W$ and hence $(V/W)\cap I_W = \{0\}$. \hfill \Box \\
Thus the point $V/W$ of $\text{LG}(\Lambda(W))$ is contained in the Zariski open set 
$$H_W = \{ [V^+] \in \text{LG}(\Lambda(W)) \mid V^+ \cap I_W = \{0\} \}$$

of $\text{LG}(\Lambda(W))$. If we choose another maximal totally isotropic subspace $I'_W$ of $\Lambda(W)$ such that $\Lambda(W) = I_W \oplus I'_W$, we obtain an isomorphism 

$$H_W \simeq \text{Sym}^2 I_W \simeq \text{Sym}^2 I_C$$

by the graph construction as in §2.5. Thus $H_W$ is an affine space for $\text{Sym}^2 I_C$.

Conversely, $H_W$ is contained in $\mathcal{D}(I)$ in the following sense.

Lemma 4.6. Let $[V'/W]$ be a point of $H_W$ where $V'$ is a $g$-dimensional subspace of $W^\perp$ containing $W$. Then $[V'] \in \mathcal{D}(I)$ and $V' \cap I_C^\perp = W$. \\
Proof. Clearly $V'$ is totally isotropic. We see that $V' \cap I_C^\perp = W$ from $W \subset V' \cap I_C^\perp$ and $\dim(V' \cap I_C^\perp) \leq g''$, where the latter holds because the projection $V' \cap I_C^\perp \to \Lambda(I)_C$ is injective by $V' \cap (W \oplus I_C) = W$ (and so $V' \cap I_C = \{0\}$). \hfill \Box \\

4.2.2. Structure of $\mathcal{V}_I \to \mathcal{D}_{\Lambda(I)}$. We now formulate (4.7) into (4.6). The linear subspaces $W$ of $I_C^\perp$ satisfying the conditions in Lemma 4.4 are parametrized by the space

$$\mathcal{V}_I = \{ [W] \in \text{LG}(g'', I_C^\perp) \mid i\langle \cdot, \cdot \rangle|_W > 0 \}.$$ 

Note that the property $W \cap I_C = \{0\}$ automatically follows from the positivity of $i\langle \cdot, \cdot \rangle|_W$. Restriction of (4.5) to $\mathcal{D}(I) \subset \text{LG}(\Lambda_C)$ gives

$$\mathcal{D}(I) \overset{\xi_1}{\to} \mathcal{V}_I \overset{\xi_2}{\to} \mathcal{D}_{\Lambda(I)}.$$
We first describe the structure of \( \pi_2 : \mathcal{V}_I \to \mathcal{D}_{\Lambda(I)} \). Let \( F_I \to \mathcal{D}_{\Lambda(I)} \) be the universal quotient bundle over \( \mathcal{D}_{\Lambda(I)} \) of rank \( g'' \).

**Lemma 4.7.** \( \mathcal{V}_I \) is an affine space bundle for the vector bundle \( F_I \otimes I_C \). A choice of a lift \( \Lambda(I)_C \hookrightarrow I_C \) of \( \Lambda(I)_C \) determines a section of \( \pi_2 \) and hence an isomorphism \( \mathcal{V}_I \cong F_I \otimes I_C \).

**Proof.** Let \([U]\) be a point of \( \mathcal{D}_{\Lambda(I)} \) where \( U \) is a subspace of \( \Lambda(I)_C \). Let \( U' \) be the inverse image of \( U \) in \( I_C^+ \). Then \( U' \) is totally isotropic of dimension \( g \), and \( i(\cdot, \cdot)|_{U'} \) is semi positive definite with kernel \( I_C \). This shows that

\[
\pi_2^{-1}([U]) = \{ W \subset U' \mid \dim W = g'', W \cap I_C = \{0\} \}.
\]

If we choose a lift \( U' \cong I_C \oplus U \) of \( U \), we obtain an isomorphism \( \pi_2^{-1}([U]) \cong \text{Hom}(U, I_C) \) by taking the graph of a linear map \( U \to I_C \). Thus \( \pi_2^{-1}([U]) \) is an affine space for \( U' \otimes I_C \). If we take a lift \( \Lambda(I)_C \hookrightarrow I_C \) of \( \Lambda(I)_C \), it determines a lift of every \( U \) and hence a section of \( \pi_2 \). \( \square \)

### 4.2.3. Structure of \( \mathcal{D}(I) \to \mathcal{V}_I \)

We next describe the structure of the map \( \pi_1 : \mathcal{D}(I) \to \mathcal{V}_I \). Let \( \mathcal{K}_I \to \mathcal{V}_I \) be the rank \( 2g' \) symplectic vector bundle over \( \mathcal{V}_I \) whose fiber over \( [W] \in \mathcal{V}_I \) is \( \Lambda(W) = W^\perp/W \). Let

\[
\text{LG}(\mathcal{K}_I) = \bigcup_{[W]\in \mathcal{V}_I} \text{LG}(\Lambda(W))
\]

be its relative Lagrangian Grassmannian. Then \( \mathcal{D}(I) \to \mathcal{V}_I \) factorizes through the embedding

\[
\mathcal{D}(I) \hookrightarrow \text{LG}(\mathcal{K}_I), \quad V \mapsto (V/W, W),
\]

where \( W = V \cap I_C^+ \). By Lemmas 4.5 and 4.6 this is an isomorphism to the Zariski open set \( \cup_{[W]} H_W \) of \( \text{LG}(\mathcal{K}_I) \).

Let \( U(I)_C = U(I)_Q \otimes Q_C \cong \text{Sym}^2 I_C \). By the same definition as (4.2), we have \( U(I)_C \subset \text{Sp}(\Lambda_C) \). Since \( U(I)_C \) acts trivially on \( I_C^+ \), it preserves \( \mathcal{D}(I) \) and acts trivially on \( \mathcal{V}_I \), hence acts on each fiber \( H_W \) of \( \mathcal{D}(I) \to \mathcal{V}_I \).

**Lemma 4.8.** \( \mathcal{D}(I) \) is a principal \( U(I)_C \)-bundle over \( \mathcal{V}_I \).

**Proof.** By the same calculation as Lemma 4.3, the action of \( U(I)_C \) on each \( H_W \) coincides with translation by \( \text{Sym}^2 I_C \cong \text{Sym}^2 I_W \). \( \square \)

### 4.2.4. Structure of \( \mathcal{D} \hookrightarrow \mathcal{D}(I) \)

Finally, we describe the structure of the embedding \( \mathcal{D} \hookrightarrow \mathcal{D}(I) \) relatively over \( \mathcal{V}_I \). We identify \( \mathcal{D}(I) = \cup_{[W]} H_W \) as above. Let \( \mathbb{H}_I \) be the Siegel upper half space in \( \text{Sym}^2 I_C \).

**Lemma 4.9.** For each \( [W] \in \mathcal{V}_I \), the intersection \( \mathcal{D} \cap H_W \subset H_W \) is a translation of \( \mathbb{H}_I \) in \( H_W \cong \text{Sym}^2 I_C \).
Proof. We only need to prove this for one particular $[W] \in \mathcal{V}_I$, as $\Gamma(I)_{\mathbb{Z}}$ acts on $\mathcal{V}_I$ transitively. We take a lift of $\Lambda(I)_{\mathbb{Q}}$ and choose $W$ from $\Lambda(I)_{\mathbb{C}} \subset I_\mathbb{C}^\perp$. Then $\Lambda(W) \simeq \Lambda(I)_{\mathbb{C}}^\perp$ naturally. We show that every $[V] \in \pi_1^{-1}([W])$ can be written as $V = W \oplus V'$ where $V' = V \cap \Lambda(I)_{\mathbb{C}}^\perp$. Indeed, it suffices to check that $\dim V' \geq g'$, which in turn follows from the observation that the image of the projection $V \to \Lambda(I)_{\mathbb{C}}$ from $\Lambda(I)_{\mathbb{C}}^\perp$ is contained in $W^\perp = W$. Now we have $i(\cdot, \cdot)|_{V'} > 0$ if and only if $i(\cdot, \cdot)|_{V} > 0$, namely $V' \in \mathcal{S}_I$. \hfill \Box

We have thus obtained a 2-step fibration as in (4.6).

4.3. Action of the stabilizer. We describe the action of the stabilizer $\Gamma(I)_{\mathbb{Z}}$ of $I$ on the Siegel domain realization. Note that the action of $\Gamma(I)_{\mathbb{Z}}$ on $\mathcal{D}$ extends to $\mathcal{D}(I)$. Following the filtration (4.3), (4.4) of $\Gamma(I)_{\mathbb{Z}}$, we proceed in three steps: first by $U(I)_{\mathbb{Z}}$, then by $V(I)_{\mathbb{Z}}$, and finally by $\Gamma_I$. The first step was done in Lemma 4.8, and we consider the remaining steps. Let $T_I = U(I)_{\mathbb{C}}/U(I)_{\mathbb{Z}}$ be the algebraic torus associated with the lattice $U(I)_{\mathbb{Z}}$. We have $\mathcal{S}_I/U(I)_{\mathbb{Z}} = \text{ord}^{-1}(C_I)$ inside $T_I$, where $C_I \subset \text{Sym}^2 I_{\mathbb{Z}}$ is the cone of positive definite forms and ord: $T_I \to U(I)_{\mathbb{R}}$ is the projection map as in [2] p.2.

Proposition 4.10. The following holds.

(1) The quotient $\mathcal{V}_I = \mathcal{D}(I)/U(I)_{\mathbb{Z}}$ is a principal $T_I$-bundle over $\mathcal{V}_I$, and the quotient $\mathcal{B}_I = \mathcal{D}/U(I)_{\mathbb{Z}}$ is a ord$^{-1}(C_I)$-bundle inside it.

(2) The group $V(I)_{\mathbb{Z}}$ acts on $\mathcal{D}_{\mathcal{A}(I)}$ trivially, and on the fibers of $\mathcal{V}_I = F_I \otimes I_{\mathbb{C}}$ by translation by the lattice $V(I)_{\mathbb{Z}}$ of $\Lambda(I)_{\mathbb{Q}} \otimes I_{\mathbb{Q}}$. Thus $\mathcal{V}_I/V(I)_{\mathbb{Z}}$ is a fibration of abelian varieties over $\mathcal{D}_{\mathcal{A}(I)}$.

(3) The group $\Gamma_I$ acts on $\mathcal{V}_I/V(I)_{\mathbb{Z}} \to \mathcal{D}_{\mathcal{A}(I)}$ by the equivariant action of $\text{Sp}(\Lambda(I)) \times \text{GL}(I)$ on $F_I \otimes I_{\mathbb{C}}$ plus translation on the fibers.

Proof. (1) follows from Lemmas 4.8 and 4.9.

(2) Since $V(I)_{\mathbb{Z}}$ acts on $\Lambda(I)$ trivially, it acts on $\mathcal{D}_{\mathcal{A}(I)}$ trivially. Take a point $[U] \in \mathcal{D}_{\mathcal{A}(I)}$ and choose $[W] \in \pi_1^{-1}([U])$. Recall that $\pi_2^{-1}([U])$ is identified with $\text{Hom}(W, I_C) \simeq \text{Hom}(U, I_C)$ by associating to $\varphi: W \to I_C$ its graph. Since $T_{m,l}$ acts on $I_C^\perp$ by $T_{m,l}(v) = v + (m, v)l$, it maps the graph of $\varphi$ to the graph of $\varphi + (m, \cdot)l$. This is translation on $U^\vee \otimes I_C$ by $m \otimes l \in V(I)_{\mathbb{Q}}$ via the pairing map $\Lambda(I)_{\mathbb{Q}} \to U^\vee$.

(3) We have $(\Gamma_I)_{\mathbb{Q}} = \text{Sp}(\Lambda(I)_{\mathbb{Q}}) \times \text{GL}(I_{\mathbb{Q}})$. Choose a lift $\Lambda(I)_{\mathbb{Q}} \leftarrow I^\perp_{\mathbb{Q}}$ of $\Lambda(I)_{\mathbb{Q}}$ and a totally isotropic subspace $I'_\mathbb{Q}$ of $\Lambda_{\mathbb{Q}}$ with $\Lambda_{\mathbb{Q}} = I^\perp_{\mathbb{Q}} \oplus I'_\mathbb{Q}$. As explained before, this induces a section $s: (\Gamma_I)_{\mathbb{Q}} \leftarrow \Gamma(I)_{\mathbb{Q}}$ of (4.1) and also a section of $\mathcal{V}_I \to \mathcal{D}_{\mathcal{A}(I)}$. Since $s((\Gamma_I)_{\mathbb{Q}})$ preserves the lifted $\Lambda(I)_{\mathbb{Q}} \subset \Lambda_{\mathbb{Q}}$, its element $s(\gamma_1, \gamma_2)$ maps $\pi_2^{-1}([U])$ to $\pi_2^{-1}([\gamma_1 U])$ by

$$
\text{Hom}(U, I_C) \to \text{Hom}(\gamma_1 U, I_C), \quad \varphi \mapsto \gamma_2 \circ \varphi \circ \gamma_1^{-1}.
$$
Thus \( s/(\Gamma I) \mathbb{Z} \) preserves the chosen section of \( \mathcal{V}_I \rightarrow \mathcal{D}_{\Lambda(I)} \) and acts on \( \mathcal{V}_I \) via the natural equivariant action on \( F_I \otimes I_C \).

Now let \( \gamma \) be an element of \( \Gamma_I \). If we take its lift \( \tilde{\gamma} \in \Gamma(I)\mathbb{Z} \), we can write \( \tilde{\gamma} = \alpha \cdot s(\gamma) \) for some \( \alpha \in W(I)_{\mathbb{Q}} \). The \( \gamma \)-action on \( \mathcal{V}_I/\mathcal{V}(I)_{\mathbb{Z}} \) is the composition of the above action of \( s(\gamma) \) (which preserves the chosen section) and translation by \([\alpha] \in V(I)_{\mathbb{Q}}/V(I)_{\mathbb{Z}} \).

If the exact sequence (4.4) splits, we can take \( \alpha = 0 \).

5. Toroidal compactification

In this section we recall the construction of toroidal compactification of \( A(I) \) following [2], [10]. We denote by \( T(N) = N_C/N \) the algebraic torus associated to a free \( \mathbb{Z} \)-module \( N \) of finite rank. We especially write \( T_I = T(U(I)_{\mathbb{Z}}) \).

5.1. Relative torus embedding. Let \( I \) be a primitive totally isotropic sub-lattice of \( \Lambda \). We equip \( U(I)_{\mathbb{R}} \cong \text{Sym}^2 I_{\mathbb{R}} \) with a \( \mathbb{Z} \)-structure by \( U(I)_{\mathbb{Z}} \). Let \( C_I \subset U(I)_{\mathbb{R}} \) be the cone of positive definite forms on \( I_{\mathbb{R}} \), and \( C_I^* \subset U(I)_{\mathbb{R}} \) the cone of semi positive definite forms whose kernel is defined over \( \mathbb{Q} \). In other words, \( C_I^* = \bigcup_{I'} C_{I'} \) where \( I' \) ranges over all primitive sublattices of \( I \) (including \( I' = \{0\} \)). Recall that \( \Gamma(I)_{\mathbb{R}} \) acts on \( U(I)_{\mathbb{R}} \) by the adjoint action which coincides with the natural action on \( \text{Sym}^2 I_{\mathbb{R}} \). Since \( \text{GL}(I) \) acts on \( C_I \) properly discontinuously, so does the image of \( \Gamma(I)_{\mathbb{Z}} \) in \( \text{GL}(I) \).

A fan \( \Sigma = (\sigma_a)_{a} \) in \( U(I)_{\mathbb{R}} \) is called \( \Gamma(I)_{\mathbb{Z}} \)-admissible if

1. the support of \( \Sigma \) is \( C_I^* \),
2. \( \Sigma \) is preserved by the action of \( \Gamma(I)_{\mathbb{Z}} \), and
3. \( \Sigma/\Gamma(I)_{\mathbb{Z}} \) consists of only finitely many cones.

Let \( T_I \hookrightarrow T_I^\Sigma \) be the torus embedding defined by the fan \( \Sigma \). A ray \( \mathbb{R}_{\geq 0} Q \) in \( \Sigma \) corresponds to a \( T_I \)-orbit of codimension 1 in the boundary of \( T_I^\Sigma \), say \( \Delta_Q \). We always assume that \( Q \) is chosen as a primitive vector of \( U(I)_{\mathbb{Z}} \). Then \( \Delta_Q \) is identified the quotient torus \( T_I/Q \cong T(U(I)_{\mathbb{Z}}/\mathbb{Z}Q) \) where \( T_Q = T(\mathbb{Z}Q) \).

To be more specific, let \( T_I^Q \) be the torus embedding of \( T_I \) defined by the ray \( \mathbb{R}_{\geq 0} Q \). We have \( T_I \subset T_I^Q \subset T_I^\Sigma \), and \( \Delta_Q \) is the unique boundary divisor of \( T_I^Q \). If \( \tilde{T}_Q \cong \mathbb{C} \) is the standard partial compactification of \( T_Q \cong \mathbb{C}^\times \), then \( T_I^Q \cong T_I \times_{T_Q} \tilde{T}_Q \). The embedding \( T_Q \hookrightarrow T_I \) extends to \( \tilde{T}_Q \hookrightarrow T_I^Q \) which gives a normal space of \( \Delta_Q \) at its base point.

A character \( e^x = \exp(2\pi i \chi(\cdot)) \) of \( T_I \), where \( \chi \in U(I)_{\mathbb{Z}}^\times \), extends holomorphically over \( \Delta_Q \) if and only if \( \chi(Q) \geq 0 \), and it is identically 0 at \( \Delta_Q \) if and only if \( \chi(Q) > 0 \). By restriction, the character group of \( \Delta_Q \) is identified with \( Q^\times \cap U(I)_{\mathbb{Z}}^\times \). If we choose \( \chi \in U(I)_{\mathbb{Z}}^\times \) such that \( \chi(Q) = 1 \) (this is possible because \( Q \) is primitive in \( U(I)_{\mathbb{Z}} \)), then \( \Delta_Q \) is defined by \( e^x = 0 \) and \( e^x \) gives a normal parameter around \( \Delta_Q \).
Now let $\mathcal{T}_I \to \mathcal{V}_I$ be the principal $T_I$-bundle constructed in Proposition 4.10. We can form the relative torus embedding

$$\mathcal{T}^\Sigma_I = \mathcal{T}_I \times_{T_I} \mathcal{T}^\Sigma_I = (\mathcal{T}_I \times T^\Sigma_I)/T_I.$$ Let $\mathcal{B}^\Sigma_I$ be the interior of the closure of $\mathcal{B}_I$ in $\mathcal{T}^\Sigma_I$. This is the partial compactification in the direction of $I$ defined by $\Sigma$.

Since $\Gamma(I)_\mathbb{Z}$ preserves $\Sigma$, the $\Gamma(I)_\mathbb{Z}$-action on $T_I$ extends to $\mathcal{T}^\Sigma_I$. The $\Gamma(I)_\mathbb{Z}$-action and the $T_I$-action on $\mathcal{B}_I$ are compatible in the sense that

$$\gamma(gx) = \text{Ad}_\gamma(g)(yx), \quad x \in \mathcal{T}_I, \ g \in T_I, \ \gamma \in \Gamma(I)_\mathbb{Z}.$$ Thus the $\Gamma(I)_\mathbb{Z}$-action on $\mathcal{T}_I$ extends to $\mathcal{T}^\Sigma_I$, and so the $\Gamma(I)_\mathbb{Z}$-action on $\mathcal{B}_I$ extends to $\mathcal{B}^\Sigma_I$. By \cite{[2]}, the $\Gamma(I)_\mathbb{Z}$-action on $\mathcal{B}^\Sigma_I$ is properly discontinuous.

### 5.2. Adjacent cusps.
Let $J$ be a primitive totally isotropic sublattice of $\Lambda$ that contains $I$. The cusp of $\mathcal{D}$ associated to $J$ is in the closure of the cusp associated to $I$. We shall describe the relationship between the relative torus embeddings for $I$ and for $J$. First note that $U(I)_\mathbb{R} \approx \text{Sym}^2 J_\mathbb{R}$ is contained in $U(J)_\mathbb{R} \approx \text{Sym}^2 J_\mathbb{R}$. Then $U(I)_\mathbb{Z}$ is a primitive sublattice of $U(J)_\mathbb{Z}$, so $T_I$ is a sub torus of $T_J$. The cone $C^*_I$ is an extremal sub cone of $C^*_J$. If we have a fan $\Sigma_J$ in $U(J)_\mathbb{R}$ with support $C^*_J$, its restriction $\Sigma_I = \Sigma_J|_I$ is compatible with the action of $U(I)_\mathbb{R}$ in $U(I)_\mathbb{R}$ with support $C^*_I$. Here $\Sigma_J|_I$ consists of cones $\sigma_\alpha$ in $\Sigma_J$ with $\sigma_\alpha \subset C^*_J$. The embedding $T_I \hookrightarrow T_J$ extends to $T^\Sigma_I \hookrightarrow T^\Sigma_J$.

We set $\overline{U(J)}_\mathbb{Z} = U(J)_\mathbb{Z}/U(I)_\mathbb{Z}$. We have the quotient map

$$\mathcal{B}_I = \mathcal{D}/U(I)_\mathbb{Z} \to \mathcal{B}_J = \mathcal{D}/U(J)_\mathbb{Z}$$

by $\overline{U(J)}_\mathbb{Z}$. Note that $U(J)_\mathbb{Z} \subset \Gamma(I)_\mathbb{Z}$ and so $\overline{U(J)}_\mathbb{Z} \subset \Gamma(I)_\mathbb{Z}$.

**Lemma 5.1.** For $I \subset J$ the quotient map $\mathcal{B}_I \to \mathcal{B}_J$ extends to an etale map $\mathcal{B}^\Sigma_I \to \mathcal{B}^\Sigma_J$. More specifically, it factorizes as

$$\mathcal{B}^\Sigma_I \to \mathcal{B}^\Sigma_I/\overline{U(J)}_\mathbb{Z} \hookrightarrow \mathcal{B}^\Sigma_J,$$

where the first map is a free quotient map, and the second is an open immersion whose image does not intersect with the boundary strata of $\mathcal{B}^\Sigma_J$ corresponding to the cones in $\Sigma_J - \Sigma_I$.

**Proof.** Since $I \subset J \subset J^\perp \subset I^\perp$, we have $\mathcal{D}(I) \subset \mathcal{D}(J)$. This is clearly compatible with the action of $U(I)_\mathbb{C} \subset U(J)_\mathbb{C}$. Dividing by $U(I)_\mathbb{Z} \subset U(J)_\mathbb{Z}$, we obtain a map $\mathcal{T}_I \to \mathcal{T}_J$ which is compatible with the action of $T_I \hookrightarrow T_J$. Hence $\mathcal{T}_I \to \mathcal{T}_J$ extends to $\mathcal{T}^\Sigma_I \to \mathcal{T}^\Sigma_J$ whose restriction gives $\mathcal{B}^\Sigma_I \to \mathcal{B}^\Sigma_J$.

We shall observe $\mathcal{T}^\Sigma_I \to \mathcal{T}^\Sigma_J$ more closely. We put

$$\mathcal{T}^\Sigma_J = \mathcal{T}_J \times_{T_J} T^\Sigma_J \simeq \mathcal{T}_J \times_{T_I} T^\Sigma_I.$$
Then $\mathcal{T}_I^{\Sigma_I} \to \mathcal{T}_J^{\Sigma_I}$ factorizes as

$$\mathcal{T}_I^{\Sigma_I} \to \mathcal{T}_J^{\Sigma_I} \to \mathcal{T}_J^{\Sigma_J}. $$

The second map $\mathcal{T}_I^{\Sigma_I} \to \mathcal{T}_J^{\Sigma_I}$ is an open embedding, whose complement consists of boundary strata corresponding to the cones in $\Sigma_J - \Sigma_I$. As for the first map $\mathcal{T}_I^{\Sigma_I} \to \mathcal{T}_J^{\Sigma_I}$, note that $\mathcal{T}_I^{\Sigma_I} \to \mathcal{T}_J$ is a $T_I$-equivariant map between the principal $T_I$-bundles $\mathcal{T}_I \to \mathcal{V}_I$ and $\mathcal{T}_J \to \mathcal{T}_J/T_I$. The map between the bases $\mathcal{V}_I \to \mathcal{V}_J/T_I$ factorizes as

$$\mathcal{V}_I = \mathcal{D}(I)/U(I)_C \to \mathcal{D}(I)/U(I)_C + U(J)_Z \hookrightarrow \mathcal{D}(J)/U(I)_C + U(J)_Z = \mathcal{T}_J/T_I. $$

The first map is a free quotient by $U(J)_Z$, and the second is an open embedding. Thus $\mathcal{T}_I^{\Sigma_I} \to \mathcal{T}_J^{\Sigma_I}$ is also a composition of a free quotient by $U(J)_Z$ and an open embedding. □

5.3. **Toroidal compactification.** A toroidal compactification of $A(\Gamma)$ is constructed from the following data.

**Definition 5.2 ([2], [10]).** An admissible collection of fans for $\Gamma$ is a collection $\Sigma = (\Sigma_I)_I$ of fans, one for each primitive totally isotropic sublattice $I$ of $\Lambda$, which satisfies the following conditions:

1. $\Sigma_I$ is a $\Gamma(I)_Z$-admissible fan in $U(I)_\mathbb{R}$,
2. $\gamma(\Sigma_I) = \Sigma_{\gamma I}$ for $\gamma \in \Gamma$,
3. when $I \subset J$, then $\Sigma_J|_I = \Sigma_I$.

We will abbreviate $\mathcal{B}_I^{\Sigma_I} = \mathcal{B}_I^{\Sigma}$ when no confusion is likely to occur. The toroidal compactification of $A(\Gamma)$ by $\Sigma$ is defined as ([2], [10])

$$A(\Gamma)^{\Sigma} = \left( \bigcup_I \mathcal{B}_I^{\Sigma_I} \right)/\sim,$$

where $I$ ranges over all primitive totally isotropic sublattices of $\Lambda$ (including $I = \{0\}$ where $\mathcal{B}_I = \mathcal{B}_I^{\Sigma_I} = \mathcal{D}$), and $\sim$ is the equivalence relation generated by the following relations:

1. the isomorphism $\gamma : \mathcal{B}_I^{\Sigma_I} \to \mathcal{B}_{\gamma I}^{\Sigma_I}$ by $\gamma \in \Gamma$, and
2. the etale map $\mathcal{B}_I^{\Sigma_I} \to \mathcal{B}_J^{\Sigma_J}$ for $I \subset J$ as in Lemma 5.1.

Let $\Sigma^o_I = \Sigma_I \setminus \bigcup_{K \subset I} \Sigma_K$ be the set of cones in $\Sigma_I$ whose relative interior is contained in $C_I$. We write $\Delta_{\sigma, I}$ for the boundary stratum of $\mathcal{B}_I^\Sigma$ corresponding to a cone $\sigma \in \Sigma_I$, and let

$$\Delta_I = \bigcup_{\sigma \in \Sigma^o_I} \Delta_{\sigma, I}$$

be the union of boundary strata that does not come from higher dimensional cusps adjacent to $I$. By Lemma 5.1 the natural map $\Delta_I/\Gamma(I)_Z \to A(\Gamma)^{\Sigma}$ is injective.
Theorem 5.3 \([\square]\). Let \(\Sigma\) be an admissible collection of fans for \(\Gamma\).

1. The space \(A(\Gamma)_{\Sigma}\) is a compact Moishezon space and contains \(A(\Gamma)\) as a Zariski open set.
2. For each primitive totally isotropic sublattice \(I\) of \(\Lambda\), the natural map
\[
\mathcal{B}_I^Z/\Gamma(I)_Z \to A(\Gamma)_{\Sigma}
\]
is isomorphic on an open neighborhood of \(\Delta_I/\Gamma(I)_Z\).
3. There is a surjective morphism from \(A(\Gamma)_{\Sigma}\) to the Satake compactification of \(A(\Gamma)\) which is identity on \(A(\Gamma)\). The image of \(\Delta_I\) is the boundary component associated to \(I\).

By property (2) (see \([\square]\) p. 175), the quotient space \(\mathcal{B}_I^Z/\Gamma(I)_Z\) gives a local model of \(A(\Gamma)_{\Sigma}\) around the boundary strata lying over the \(I\)-cusp.

5.4. Extension of the modular line bundle. There is a natural number \(k'\) such that for every \(x \in D\) and \(\gamma \in \Gamma\) with \(\gamma(x) = x\), \(\gamma\) acts trivially on \(L \otimes k'\). Then \(L \otimes k'\) descends to a line bundle over \(A(\Gamma)_{\Sigma}\). In this subsection we extend some multiple of this line bundle over \(A(\Gamma)_{\Sigma}\). This is an explicit form of Mumford's extension \([\text{17}]\). We proceed in two steps:

1. first extend \(L\) from \(\mathcal{B}_I\) to \(\mathcal{B}_I^Z\) for each \(I\), and then
2. for some \(k\), \(L \otimes k\) descends from \(\sqcup I \mathcal{B}_I^Z\) to \(A(\Gamma)_{\Sigma}\).

As the first step, let \(I\) be a primitive totally isotropic sublattice of \(\Lambda\). We choose a maximal totally isotropic sublattice \(J \subset \Lambda\) containing \(I\). Fix an orientation of \(I\) and \(J\). Let \(s_J\) be the distinguished frame of \(L\) over \(D\) associated to \(J\) (see §2.5). Since \(s_J\) is invariant under \(U(I)_Z \subset U(J)_Z\), it descends to a frame of \(L\) over \(\mathcal{B}_I = D/U(I)_Z\) which we again denote by \(s_J\). Then there exists a unique extension of \(L\) to a line bundle over \(\mathcal{B}_I^Z\) such that \(s_J\) extends to its frame. We again denote by \(L\) the extended line bundle over \(\mathcal{B}_I^Z\). By construction, a section \(s\) of \(L \otimes k\) over \(\mathcal{B}_I\) extends holomorphically over \(\mathcal{B}_I^Z\) if and only if the function \(s/s_J \otimes k\) over \(\mathcal{B}_I\) extends holomorphically over \(\mathcal{B}_I^Z\).

Proposition 5.4. The extension of \(L\) defined above is independent of the choice of \(J\) up to isomorphism. Moreover, the equivariant action of \(\Gamma(I)_Z\) on \(L\) over \(\mathcal{B}_I\) extends to an equivariant action over \(\mathcal{B}_I^Z\).

In order to prove this, we consider a decomposition of \(s_J\). Let
\[
D \hookrightarrow \text{LG}(K_I) \xrightarrow{\pi_1} \mathcal{V}_I \xrightarrow{\pi_2} \mathcal{D}_{\Lambda(I)}
\]
be the Siegel domain realization with respect to \(I\), and \(\pi = \pi_2 \circ \pi_1\). Let \(E_I \to \mathcal{D}_{\Lambda(I)}\) be the tautological bundle over \(\mathcal{D}_{\Lambda(I)}\), and \(L_I = \det E_I\) the modular line bundle over \(\mathcal{D}_{\Lambda(I)}\). We have the distinguished frame \(s_{J/I}\) of \(L_I\) associated to the oriented, maximal totally isotropic sublattice \(J/I\) of \(\Lambda(I)\). On the other
Lemma 5.5. We have a natural isomorphism \( L \cong L_{\eta_1} \otimes \pi^* L_I \). Under this isomorphism we have \( s_J = s_I \otimes \pi^* s_{J/I} \).

Proof. By varying the exact sequence of vector spaces 
\[
0 \to V \cap I_2^J \to V \xrightarrow{=} V/(V \cap I_2^J) \to 0
\]
over \( \mathcal{D} \), we obtain the exact sequence of vector bundles
\[
0 \to \pi^* E_I \to E \to E_{\eta_1} \to 0.
\]
This shows that \( L \cong L_{\eta_1} \otimes \pi^* L_I \). We have \( s_J = s_I \otimes \pi^* s_{J/I} \) because
\[
(s_I \otimes \pi^* s_{J/I}, \det J) = (s_I, \det I) \cdot (s_{J/I}, \det(J/I)) = 1.
\]

(Proof of Proposition 5.4). If \( J' \supset I \) is another maximal totally isotropic sublattice, then \( s_{J'}/s' = \pi^*(s_{J/I}/s_{J'/I}) \) is the pullback of a nowhere vanishing function on \( \mathcal{D}_{\Sigma(I)} \). Since the partial compactification \( \mathcal{B}_I \hookrightarrow \mathcal{B}^E_I \) is done relatively over \( \mathcal{V}_I \), \( s_{J'/I} \) extends to a nowhere vanishing function on \( \mathcal{B}^E_I \). This shows the independence of the extension from \( J \). If we consider \( J' = \gamma J \) for \( \gamma \in \Gamma(I)_Z \), this also implies the second assertion.

We consider the collection of these extended line bundles over the whole \( \sqcup_I \mathcal{B}^E_I \) and denote it again by \( L \). The \( \Gamma \)-action on \( L \) over \( \sqcup_I \mathcal{B}_I \) extends over \( \sqcup_I \mathcal{B}^E_I \) by Proposition 5.4. Furthermore, if \( p : \mathcal{B}^E_I \to \mathcal{B}^E_I \) is the etale map for \( I \subset J \) as in Lemma 5.1, the isomorphism \( p^*(L|_{\mathcal{B}^E_I}) \cong L|_{\mathcal{B}^E_I} \) over \( \mathcal{B}_I \) extends over \( \mathcal{B}^E_I \), because we can use a common frame \( s_K \) for the extension over both \( \mathcal{B}^E_I \) and \( \mathcal{B}^E_J \) where \( K \) is maximal with \( K \supset J \supset I \).

Lemma 5.6 (cf. [17]). A modular form \( F \) of weight \( k \), as a section of \( L^{0k} \) over \( \mathcal{B}_I \), extends holomorphically over \( \mathcal{B}^E_I \). \( F \) is a cusp form if and only if it vanishes at the boundary divisor of \( \mathcal{B}^E_I \) for all \( I \).

Proof. By the above gluing, we may assume that \( I \) is maximal. We identify \( F \) with a function on \( \mathcal{B}_I \) via the frame \( s^{0k}_I \), which has Fourier expansion
\[
F = \sum_{\chi \in U(I)} a_\chi e^\chi, \quad e^\chi = \exp(2\pi i \chi(\cdot)).
\]
By the Koecher principle, we have \( a_\chi \neq 0 \) only when \( \chi \) is semi positive definite. Then \( \chi(Q) \geq 0 \) for every ray \( \mathbb{R}_{\geq 0} Q \) in \( \Sigma_I \), so \( e^\chi \) extends holomorphically over \( \mathcal{B}^E_I \) for such \( \chi \). This proves the first claim.
By definition, \( F \) is a cusp form if and only if \( a_\chi \neq 0 \) only for positive definite \( \chi \) at all maximal \( I \). Let \( U(I)_z^+ \) (resp. \( U(I)_z^{+*} \)) be the set of semi positive definite \( Q \in U(I)_z \) (resp. \( \chi \in U(I)_z^{+*} \)). Since the set of strictly semi positive definite \( \chi \in U(I)_z^+ \) coincides with

\[
\bigcup_{Q \in U(I)_z^+, \, \text{rk}(Q) = 1} Q^+ \cap U(I)_z^{+*} = \bigcup_{Q \in U(I)_z^{+*}, \, \text{rk}(Q) \geq 1} Q^+ \cap U(I)_z^{+*},
\]

the cuspidal condition is equivalent to \( a_\chi = 0 \) for all \( \chi \in Q^+ \cap U(I)_z^{+*} \) for every ray \( \mathbb{R}_{\geq 0} Q \) in \( \Sigma_I \) at every maximal \( I \). Since \( Q^+ \cap U(I)_z^{+*} \) is the character group of the boundary torus associated to \( \mathbb{R}_{\geq 0} Q \), this is equivalent to the vanishing of \( F \) at the boundary of \( \mathcal{B}_I^\Sigma \) for every maximal \( I \).

We choose a natural number \( k \) such that for every \( I, x \in \mathcal{B}_I^\Sigma \), \( \gamma \in \Gamma(I)_z \) with \( \gamma(x) = x \), \( \gamma \) acts trivially on \( L^\Sigma \). Then the line bundle \( L^\Sigma \) over \( \bigcup_I \mathcal{B}_I^\Sigma \) descends to a line bundle over \( A(\Gamma)^\Sigma = (\bigcup_I \mathcal{B}_I^\Sigma)/\sim \). This will be denoted as \( L^\Sigma \) by abuse of notation (for \( L \) might not exist as a line bundle over \( A(\Gamma)^\Sigma \)). By Lemma 5.6, we have \( H^0(A(\Gamma)^\Sigma, L^\Sigma) \simeq M_k(\Gamma) \).

6. Asymptotic estimate of Petersson norm

Let \( A(\Gamma)^\Sigma \) be a toroidal compactification of \( A(\Gamma) \). Let \( J \) be a maximal totally isotropic sublattice of \( A \) and \( \mathbb{R}_{\geq 0} Q \) be a ray in \( \Sigma_J \). Let \( \Delta_Q \subseteq \Delta_{Q,J} \) be the corresponding stratum in the boundary of \( \mathcal{B}_J^\Sigma \). The image of \( \Delta_Q \) in \( A(\Gamma)^\Sigma \) is a Zariski open set of an irreducible component of the boundary divisor of \( A(\Gamma)^\Sigma \). In this section we prepare an asymptotic estimate of the Petersson norm of a local modular form as the period approaches \( \Delta_Q \).

We choose \( \chi \in U(J)_z^{+*} \) with \( \chi(Q) = 1 \). Recall that \( q = \exp(2\pi i\chi(\cdot)) \) gives a normal parameter around \( \Delta_Q \). We choose an arbitrary point \( x \) of \( \Delta_Q \) and take a small neighborhood \( \Delta_x \) of \( x \) in \( \Delta_Q \). Let \( T_r \subseteq \mathcal{B}_J^\Sigma \) be the tubular neighborhood of \( \Delta_x \) of radius \( r \), defined by \( |q| \leq r \). We fix a sufficiently small \( 0 < R \ll 1 \) and set \( W_\varepsilon = T_R - T_\varepsilon \) for \( 0 < \varepsilon < R \), which is the annulus bundle around \( \Delta_x \) of radius \( [\varepsilon, R] \). Let \( F \) be a local section of \( L^\Sigma \) defined on a neighborhood of \( x \) (local modular form). Let \( \beta > 0 \). We want to give an asymptotic estimate of

\[
\int_{W_\varepsilon} (F, F)^{\beta}_{\Sigma} \, \text{vol}_\varepsilon \quad (\varepsilon \to 0).
\]

We first compute the asymptotic behavior of the Petersson metric on \( L \).

**Lemma 6.1.** Let \( s_f \) be the distinguished frame of \( L \) associated to \( J \). Around each point \( x \) of \( \Delta_Q \), we have

\[
(s_f, s_f) \sim C_x \cdot (-\log |q|)^{\text{rk}(Q)} \quad (|q| \to 0)
\]

for some constant \( C_x > 0 \), where \( \text{rk}(Q) \) is the rank of \( Q \) as a quadratic form.
Proposition 6.2. Let $F$ be a local section of $L^\otimes k$ defined over a neighborhood of $x \in \Delta_Q \subset \mathcal{B}_J$. Let $\beta > 0$ be a positive real number. Then

$$
\int_{W_x} (F, F)_{\epsilon}^\beta \text{vol}_D = o(\epsilon^{-\alpha}) \quad (\epsilon \to 0)
$$

for every $\alpha > 0$. Moreover, when $k\beta \geq g+1$, $F$ vanishes at $\Delta_Q$ if and only if

$$
\int_{W_x} (F, F)_{\epsilon}^\beta \text{vol}_D = O(1) \quad (\epsilon \to 0).
$$

Proof. Via the frame $s_j^\otimes$ we identify $F$ with a holomorphic function $F(\Omega)$ defined around $x$. Let $\text{vol}_J$ be a flat volume form on $\text{Sym}^2 J_\mathbb{C}$. By Corollary 2.4 and Lemma 2.5, we have

$$
\int_{W_x} (F, F)_{\epsilon}^\beta \text{vol}_D = \int_{W_x} |F(\Omega)|^{2\beta} \cdot \text{det}(\text{Im} \Omega)^{k\beta-g-1} \text{vol}_J.
$$

Locally around $x$, we can write (up to constant)

$$
\text{vol}_J = dx \wedge d\bar{x} \wedge \text{vol}_{\Delta_x} = |q|^{-2} dq \wedge d\bar{q} \wedge \text{vol}_{\Delta_x} = r^{-1} dr \wedge d\theta \wedge \text{vol}_{\Delta_x}
$$

for some volume form $\text{vol}_{\Delta_x}$ on $\Delta_x \subset \Delta_Q$, where $q = re^{i\theta}$. Therefore

$$
\int_{W_x} (F, F)_{\epsilon}^\beta \text{vol}_D = C \cdot \int_{\epsilon}^{\infty} r^{-1} dr \int_0^{2\pi} d\theta \int_{\Delta_x} |F(\Omega)|^{2\beta} \cdot \text{det}(\text{Im} \Omega)^{k\beta-g-1} \text{vol}_{\Delta_x}.
$$
Since $F(\Omega) = O(1)$ as $r = |q| \to 0$, Lemma 6.1 implies that
\[
\int_{W_\varepsilon} (F, F)^{\beta}_k \vol_D \leq C \int_{\varepsilon}^R r^{-1} dr \int_0^{2\pi} \int_{D_{\varepsilon}} \log r |\varepsilon^{(k\beta-g-1)} \vol_\Delta|
\]
\[
= C \int_{\varepsilon}^R \log r |\varepsilon^{(k\beta-g-1)} r^{-1} dr
\]
where $C > 0$ are some constants independent of $\varepsilon$ and $g'$ is the rank of $Q$. We have $\log r = o(r^{-\alpha'})$ for any $\alpha' > 0$ as $r \to 0$. Hence
\[
\int_{\varepsilon}^R \log r |\varepsilon^{(k\beta-g-1)} r^{-1} dr = o(r^{-1-\alpha}) \quad (r \to 0)
\]
for any $\alpha > 0$. It follows that
\[
\int_{\varepsilon}^R \log r |\varepsilon^{(k\beta-g-1)} r^{-1} dr = o(\varepsilon^{-\alpha}) \quad (\varepsilon \to 0),
\]
which proves the first assertion.

When $F|_{\Delta_Q} \neq 0$, this calculation also shows that
\[
\int_{W_\varepsilon} (F, F)^{\beta}_k \vol_D \geq C' \int_{\varepsilon}^R \log r |\varepsilon^{(k\beta-g-1)} r^{-1} dr + (\text{const})
\]
for some $C' > 0$ independent of $\varepsilon \ll R$. When $k\beta \geq g + 1$, the right hand side diverges as $\varepsilon \to 0$. On the other hand, when $F|_{\Delta_Q} = 0$, we have $|F(\Omega)|^{2\beta} = O(r^{2\beta})$ and so
\[
\int_{W_\varepsilon} (F, F)^{\beta}_k \vol_D \leq C \int_{\varepsilon}^R \log r |\varepsilon^{(k\beta-g-1)} r^{-1+2\beta} dr \leq C \int_{\varepsilon}^R r^\delta dr
\]
for some $\delta > -1$. Therefore $\int_{W_\varepsilon} (F, F)^{\beta}_k \vol_D$ converges in this case. □

Note that the “only if” direction in the second assertion holds with no restriction on $k\beta$.

7. $L^{2/m}$ CRITERION

This section is independent of the previous sections. We prepare a general criterion for the pole order of a pluricanonical form in terms of the asymptotic behavior of its integral. This will be used in §8 and §9.

7.1. $L^{2/m}$ norm of $m$-canonical forms. Let $U$ be a complex manifold of dimension $n$, and $\omega$ a (holomorphic) $m$-canonical form on $U$. We define the $L^{2/m}$ norm of $\omega$ as follows. Let $\bar{\omega}$ be the complex conjugate of $\omega$. After a constant multiple, $\omega \wedge \bar{\omega}$ gives a real, nonnegative $C^\infty$ section of the real line bundle $(\bigwedge^{2n} \Omega_{U,\mathbb{R}})^{\otimes m}$. Here $\Omega_{U,\mathbb{R}}$ is the real cotangent bundle of $U$. 
To be more precise, if we locally write \( \omega = f(z)(dz_1 \wedge \cdots \wedge dz_n)^s \) with \( z_a = x_a + iy_a \), then
\[
\omega \wedge \bar{\omega} = |f(z)|^2(dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n)^s_m
= m(n-2m)|f(z)|^2(dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n)^s_m.
\]
Globally, if we choose a volume form \( \text{vol}_U \) on \( U \), we can write
\[
\omega \wedge \bar{\omega} = m(n-2m)\varphi(z) \text{vol}_U^s_m
\]
for some real, nonnegative \( C^\infty \) function \( \varphi(z) \) on \( U \). We put \( \|\omega\|^2 = \varphi(z)\text{vol}_U^s_m \) and define its \( m \)-th root by
\[
\|\omega\|^{2/m} = \sqrt[m]{\varphi(z)} \text{vol}_U.
\]
Then \( \|\omega\|^{2/m} \) is a real, nonnegative, continuous \((n,n)\) form on \( U \) which is \( C^\infty \) outside the zero divisor of \( \omega \). This definition does not depend on the choice of \( \text{vol}_U \). The integral \( \int_U \|\omega\|^{2/m} \) is the norm of \( \omega \) we want to use.

### 7.2. Criterion for pole order.

Now let \( X \) be a complex manifold and \( \Delta \subset X \) be a smooth irreducible divisor. We take a normal parameter of \( \Delta \) and denote by \( T_r \) the tubular neighborhood of \( \Delta \) of radius \( r \). We fix a sufficiently small \( 0 < R \ll 1 \). For \( 0 < \varepsilon < R \) we set \( U_\varepsilon = T_R - T_\varepsilon \), which is the annulus bundle of radius \( [\varepsilon, R] \) around \( \Delta \). Let \( \omega \) be an \( m \)-canonical form on \( X - \Delta \). Our purpose is to relate the pole order of \( \omega \) along \( \Delta \) to the asymptotic behavior of the integral
\[
\int_{U_\varepsilon} \|\omega\|^{2/m} \quad (\varepsilon \to 0).
\]

Since the problem is local, we shall assume that \( X \) is a polydisc in \( \mathbb{C}^n \), with coordinate \((z_1, \cdots, z_n)\), \( \Delta \) is defined by \( z_1 = 0 \), and \( T_r \) is given by \( |z_1| < r \). Then \( U_\varepsilon \) is defined by \( \varepsilon \leq |z_1| \leq R \). We can express \( \omega = f(z)(dz_1 \wedge \cdots \wedge dz_n)^s \) for a holomorphic function \( f(z) \) on \( X - \Delta \). Then
\[
\|\omega\|^{2/m} = |f(z)|^{2/m}dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n
= |f(z)|^{2/m}rdr \wedge d\theta \wedge dx_2 \wedge dy_2 \wedge \cdots \wedge dx_n \wedge dy_n,
\]
where \( z_a = x_a + iy_a \) and \( z_1 = re^{i\theta} \). So its integral over \( U_\varepsilon \) is expressed as
\[
\int_{U_\varepsilon} \|\omega\|^{2/m} = \int_\varepsilon^R r dr \int_0^{2\pi} d\theta \int_\Delta |f(z)|^{2/m}dx_2 \wedge \cdots \wedge dy_n.
\]

The function \( f(z) \) is meromorphic over \( X \) if and only if we can write \( f(z) = \frac{g(z)}{z_1^\nu} \) for some \( \nu \in \mathbb{Z} \) and a holomorphic function \( g(z) \) on \( X \) such that \( g|_\Delta \neq 0 \). This \( \nu \) is the pole order of \( f \) (and of \( \omega \)) along \( \Delta \).
Proposition 7.1. Let \( \nu \) be the pole order of \( \omega \) along \( \Delta \).

1. We have \( \nu \leq m \) if and only if \( \int_{U_\varepsilon} \|\omega\|^{2/m} = o(\varepsilon^{-2/m}) \).

2. We have \( \nu \leq m - 1 \) if and only if \( \int_{U_\varepsilon} \|\omega\|^{2/m} = O(1) \).

Proof. If \( f(z) \) is not meromorphic over \( X \), then for any \( N > 0 \), \( |f(z)|^2 \) diverges faster than \( |z_1|^{-N} \) along an open subset of \( \Delta \). Then \( \int_{U_\varepsilon} \|\omega\|^{2/m} \) diverges faster than \( \varepsilon^{-M} \) for any \( M > 0 \). So we may assume that \( f(z) \) is meromorphic over \( X \) and write \( f(z) = g(z)/z_1^\nu \) where \( g(z) \) is holomorphic over \( X \) with \( g|_\Delta \neq 0 \), and \( \nu \) is the pole order of \( \omega \) along \( \Delta \).

By (7.1), we have

\[
\int_{C} \|\omega\|^{2/m} = \int_{C} r^{-1-2\nu/m} dr \int_0^{2\pi} d\theta \int_\Delta |g(z)|^{2/m} dx_2 \wedge \cdots \wedge dy_n.
\]

As a function of \( r \), the integral

\[
\int_0^{2\pi} d\theta \int_\Delta |g(z)|^{2/m} dx_2 \wedge \cdots \wedge dy_n
\]

is continuous at \( 0 \leq r \leq R \), and has a nonzero value at \( r = 0 \) by \( g|_\Delta \neq 0 \).

Therefore

\[
\int_0^{2\pi} d\theta \int_\Delta |g(z)|^{2/m} dx_2 \wedge \cdots \wedge dy_n = C + o(1) \quad (r \to 0)
\]

for some constant \( C > 0 \). It follows that

\[
\int_{U_\varepsilon} \|\omega\|^{2/m} = \int_{C} r^{-1-2\nu/m}(C + o(1))dr.
\]

When \( \nu < m \), this shows that \( \int_{U_\varepsilon} \|\omega\|^{2/m} = O(1) \). When \( \nu \geq m \), we obtain

\[
C' \log \varepsilon + \text{(const)} \leq \int_{U_\varepsilon} \|\omega\|^{2/m} \leq C'' |\log \varepsilon| + \text{(const)} \quad \text{when } \nu = m,
\]

\[
C' \varepsilon^{2(1-\nu/m)} + \text{(const)} \leq \int_{U_\varepsilon} \|\omega\|^{2/m} \leq C'' \varepsilon^{2(1-\nu/m)} + \text{(const)} \quad \text{when } \nu > m,
\]

for some constants \( C', C'' > 0 \) independent of \( \varepsilon \ll 1 \). This first shows the equivalence in (2). The smallest \( \nu \) with \( \nu > m \) is \( \nu = m + 1 \), for which \( \varepsilon^{2(1-\nu/m)} = \varepsilon^{-2/m} \). This implies the equivalence in (1). \( \square \)

For our argument in §8 it is crucial in (1) to pass from the bound \( O(\log \varepsilon) \) to the (seemingly) weaker \( o(\varepsilon^{-2/m}) \), which creates a room for the estimate.

Remark 7.2. This criterion does not depend on the choice of the normal parameter \( z_1 \) for \( \Delta \). Indeed, if \( z'_1 \) is another normal parameter, there exist constants \( c, c' > 0 \) such that \( c|z_1| \leq |z_1| \leq |z'_1| \) around \( \Delta \). If \( T'_r = \{|z_1| \leq r\} \) is the tubular neighborhood of radius \( r \) with respect to \( z'_1 \), then \( T'_{c'r} \subset T_r \) and \( T_{cr} \subset T'_r \). Writing \( U'_r = T'_{c'r} - T'_r \), we have \( U_\varepsilon \subset U'_{c'r} \) and \( U'_r \subset
$U_{\varepsilon}$ up to a region independent of $0 < \varepsilon \ll 1$. Thus, if $\int_{U_{\varepsilon}} ||\omega||^{2/m} = o(\varepsilon^{-a})$ holds, then
\[ \int_{U_{\varepsilon}} ||\omega||^{2/m} \leq \int_{U_{\varepsilon}} ||\omega||^{2/m} + (\text{const}) = o(c^{-a} \varepsilon^{-a}) = o(\varepsilon^{-a}), \]
and vise versa.

We also want to have a simple normal crossing version of (2). Let $X$ be again a polydisc in $\mathbb{C}^m$ and let $\Delta$ now be defined by $z_1 \cdots z_k = 0$. We take a smaller closed polydisc $V \subset X$.

**Proposition 7.3.** An $m$-canonical form $\omega$ on $X - \Delta$ has at most pole of order $m - 1$ along every component of $\Delta$ if and only if $\int_{V - \Delta} ||\omega||^{2/m} < \infty$.

**Proof.** The “if” direction follows from Proposition 7.1 (2), so we only have to consider the “only if” direction. We can write
\[ \omega = f(z) \cdot (z_1 \cdots z_k)^{1-m} \cdot (dz_1 \wedge \cdots \wedge dz_n)^{\otimes m} \]
for some holomorphic function $f(z)$ on $X$. Writing $z_{\alpha} = r_{\alpha} e^{i\theta_{\alpha}}$, we have
\[ ||\omega||^{2/m} = C \cdot |f(z)|^{2/m} (r_1 \cdots r_k)^{2/m-2} d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n \]
\[ \leq C' \cdot (r_1 \cdots r_k)^{2/m-1} dr_1 \wedge d\theta_1 \wedge \cdots \wedge dr_k \wedge d\theta_k \wedge d\bar{z}_k+1 \wedge \cdots \wedge d\bar{z}_n. \]
Since $\int_0^1 r^\delta dr = O(1)$ if $\delta > -1$, this shows that $\int_{V - \Delta} ||\omega||^{2/m} < \infty$. \[\Box\]

8. **Proof of Theorem 1.2**

In this section we prove Theorem 1.2. Let us begin with recalling the setting. Let $\bar{X}$ be a complex analytic variety containing $X^s(\Gamma)$ as a Zariski open set. We are imposing the conditions that
- the singular locus of $\bar{X}$ has codimension $\geq 2$,
- $X^s(\Gamma) \to A(\Gamma)$ extends to a morphism $f : \bar{X} \to A(\Gamma)^\Sigma$ to some toroidal compactification of $A(\Gamma)$, and
- every irreducible component of $\Delta_X = \bar{X} - X^s(\Gamma)$ dominates some irreducible component of $\Delta_A = A(\Gamma)^\Sigma - A(\Gamma)$.

Note that by our second condition, $\Delta_X = f^{-1}(\Delta_A)$ is a divisor of $\bar{X}$.

We want to show that the isomorphism $H^0(K_{X^s(\Gamma)}^{\otimes m}) = M_{(g+s+1)m}(\Gamma)$ of §3.1 extends to an isomorphism
\[ H^0(\bar{X}, K_{\bar{X}}^{\otimes m}(m\Delta_X)) \cong M_{(g+s+1)m}(\Gamma). \]
Since restriction to $X^s(\Gamma) \subset \bar{X}$ gives an inclusion
\[ H^0(\bar{X}, K_{\bar{X}}^{\otimes m}(m\Delta_X)) \hookrightarrow H^0(X^s(\Gamma), K_{X^s(\Gamma)}^{\otimes m}), \]
it is sufficient to show that this is actually equality. In other words, we want to show that every $m$-canonical form on $X^s(\Gamma)$ has at most pole of order
4.10, we can show that for the ramification index of \( \Delta_X \), we will deduce this property by applying the \( L^{2/m} \) criterion of §7. In order to verify the estimate as in Proposition 7.1(1), we translate the \( L^{2/m} \) norm of \( m \)-canonical forms to the Petersson norm of the corresponding modular forms. The problem is then reduced to the asymptotic estimate of the Petersson norm of modular forms, which we have prepared in §6 in advance. The assertion for cusp forms is proved by a slight modification of this argument.

8.1. **Pullback to relative torus embedding.** In this subsection we translate the \( L^{2/m} \) criterion on \( \bar{X} \) to that on the family over the torus fibration associated to each cusp. Let \( I \) be a primitive totally isotropic sublattice of \( \Lambda \). Let \( \mathcal{X}^{(s)}_I = \mathcal{X}^{(s)}/U(I)_\mathbb{Z} \) be the \( s \)-fold Kuga family over \( \mathcal{B}_I = \mathcal{D}/U(I)_\mathbb{Z} \). The situation around the \( I \)-cusp is as follows:

\[
\begin{array}{ccc}
\mathcal{X}^{(s)}_I & \xrightarrow{\hat{p}_1} & \mathcal{X}^{(s)}/\Gamma(I)_\mathbb{Z} \\
\downarrow f & & \downarrow f \\
\mathcal{B}_I & \xrightarrow{p_1} & \mathcal{B}_{I/\Gamma(I)}_\mathbb{Z} \\
\end{array}
\]

We write \( p = p_2 \circ p_1 \) and \( \hat{p} = \hat{p}_2 \circ \hat{p}_1 \).

Let \( \mathbb{R}_{\geq 0} Q \subset U(I)_\mathbb{R} \) be a positive definite ray in \( \Sigma_I \), where \( Q \) is a primitive vector of \( U(I)_\mathbb{Z} \). Let \( \Delta_Q = \Delta_{Q,I} \) be the boundary stratum of \( \mathcal{B}_I^\Sigma \) of codimension 1 corresponding to \( \mathbb{R}_{\geq 0} Q \), and we put \( \Delta_Q' = p(\Delta_Q) \subset A(\Gamma)^\Sigma \). We write \( a \) for the ramification index of \( \mathcal{B}_I^\Sigma \to A(\Gamma)^\Sigma \) at \( \Delta_Q \). We shall localize the situation. We take a general point \( x \) of \( \Delta_Q \), its small neighborhood \( V \) in \( \mathcal{B}_I^\Sigma \), and its small neighborhood \( \Delta_x \) in \( \Delta_Q \) contained in \( V \). Then \( y = p(x) \) is a general point of \( \Delta_Q' \), \( V' = p(V) \) is a small neighborhood of \( y \) in \( A(\Gamma)^\Sigma \), \( \Delta_y' = p(\Delta_x) \) is a small neighborhood of \( y \) in \( \Delta_Q' \), and \( p: \Delta_x \to \Delta_y' \) is isomorphic. In some local coordinates around \( x \) and \( y \), \( p: V \to V' \) is expressed as

\[
(z, z_1, \cdots, z_n) \mapsto (z', z_1, \cdots, z_n),
\]

with \( \Delta_Q \) defined by \( z = 0 \) and \( \Delta_Q' \) defined by \( z' = 0 \).

**Remark 8.1.** By Theorem 5.3(2), \( a \) equals to the ramification index of \( \mathcal{B}_I^\Sigma \to \mathcal{B}_{I/\Gamma(I)}_\mathbb{Z} \) at \( \Delta_Q \). Let \( U(I)_\mathbb{Z}^* = U(I)_\mathbb{Q} \cap \langle \Gamma, -1 \rangle \). Using Proposition 4.10 we can show that

\[
a = \begin{cases} 
1 & Q/2 \not\in U(I)_\mathbb{Z}^*, \\
2 & Q/2 \in U(I)_\mathbb{Z}^*,
\end{cases}
\]
like the classification of regular/irregular cusps in the case $g = 1$. We do not need this precise information for the proof of Theorem 1.2.

We set $U' = f^{-1}(V') \subset \tilde{X}$, $V^o = V \setminus \Delta_Q \subset B_I$ and $U^o = f^{-1}(V^o) \subset X_I^{(s)}$.

The situation is

\[
\begin{array}{c}
U^o \xrightarrow{\hat{p}} U' \cap X^i(\Gamma) \\
\downarrow f \quad \downarrow f \\
V^o \xrightarrow{p} V' \setminus \Delta_Q \xrightarrow{f} U' \\
\downarrow f \\
V \xrightarrow{p} V'
\end{array}
\]

which is a localization of (8.1). Here $U^o \to V^o$ is an abelian fibration, $U' \cap X^i(\Gamma) \to V' \setminus \Delta_Q'$ is an abelian or Kummer fibration, $p$ has degree $a$, and $\hat{p}$ has degree $a$ or $2a$.

Let $T_r$ be the tubular neighborhood of $\Delta_x$ of radius $|z| = r$. Then $p(T_r)$ is the tubular neighborhood of $\Delta'_q$ of radius $|z'| = r^a$. We fix a sufficiently small $0 < R \ll 1$, and for $0 < \varepsilon < R$ we set

\[
V_\varepsilon = T_R - T_\varepsilon \subset V^o, \\
V_\varepsilon' = p(V_\varepsilon) \subset V' \setminus \Delta_Q.
\]

Then $V_\varepsilon$ is the annulus bundle of radius $\varepsilon \leq |z| \leq R$ around $\Delta_x$, and $V_\varepsilon'$ is the annulus bundle of radius $\varepsilon^a \leq |z'| \leq R^a$ around $\Delta'_q$. Let

\[
U_\varepsilon = f^{-1}(V_\varepsilon) \subset U^o, \\
U_\varepsilon' = f^{-1}(V_\varepsilon') \subset U' \cap X^i(\Gamma),
\]

be the families over these bases. We have as restriction of (8.2)

\[
\begin{array}{c}
U_\varepsilon \xrightarrow{\hat{p}} U_\varepsilon' \\
\downarrow f \quad \downarrow f \\
V_\varepsilon \xrightarrow{p} V_\varepsilon'
\end{array}
\]

Then $U_\varepsilon'$ gives an annulus bundle around general point of each component of $\Delta_x$ lying over $\Delta'_q$. To be more precise, let $f^{-1}(\Delta'_q) = \sum_i \Delta_i$ be the irreducible decomposition of the reduced divisor $f^{-1}(\Delta'_q) \subset \tilde{X}$. We can write $f^\ast \Delta'_q = \sum_i d_i \Delta_i$ for some natural numbers $d_i$. By our third assumption of Theorem 1.2 each $\Delta_i$ dominates $\Delta'_q$, so we can choose a general point $y_i$ of $\Delta_i$ such that $f(y_i) = y$ and the tangent map $T_{y_i} \Delta_i \to T_y \Delta'_q$ is surjective. In some local coordinate around $y_i$, the projection $f: \tilde{X} \to A(\Gamma)^2$ is expressed as

\[
(z'', z_1, \ldots, z_n, w_1, \ldots, w_{g_3}) \mapsto (z' = (z'')^{d_i}, z_1, \ldots, z_n),
\]
with \( \Delta_i \) defined by \( z'' = 0 \). We take a small neighborhood \( \Delta_{y_i} \) of \( y_i \) in \( \Delta_i \) and let \( U'_{\varepsilon,i} \) be the restriction of \( U'_e \) around \( \Delta_{y_i} \). Then \( U'_{\varepsilon,i} \) is the annulus bundle of radius \( \varepsilon^{a/d_i} \leq |z''| \leq R^{a/d_i} \) around \( \Delta_{y_i} \).

We can now state the version of the \( L^{2/m} \) criterion we will apply.

**Lemma 8.2.** Let \( \omega_X \) be an \( m \)-canonical form on \( U' \cap X^s(\Gamma) \) and \( \omega_I \) be the pullback of \( \omega_X \) to \( U' \).

1. Assume that for every positive number \( \alpha > 0 \) the asymptotic estimate

   \[
   \int_{U'_e} \|\omega_I\|^{2/m} = o(\varepsilon^{-\alpha}) \quad (\varepsilon \to 0)
   \]

   holds. Then \( \omega_X \) has at most pole of order \( m \) along \( U' \cap \Delta_i \) for every irreducible component \( \Delta_i \) of \( f^{-1}(D'_Q) \).

2. If \( \int_{U'_e} \|\omega_I\|^{2/m} = O(1) \), then \( \omega_X \) has at most pole of order \( m - 1 \) along \( U' \cap \Delta_i \) for every \( \Delta_i \).

**Proof.** (1) It is sufficient to look at the pole order of \( \omega_X \) around \( \Delta_{y_i} \). What has to be shown is that the asymptotic estimate (8.3) for \( \omega_I \) implies an asymptotic estimate for \( \omega_X \) around \( \Delta_{y_i} \) as in Proposition 7.1 (1). Since \( U'_{\varepsilon,i} \) is the annulus bundle of radius \( [\varepsilon^{a/d_i}, R^{a/d_i}] \) around \( \Delta_{y_i} \), the estimate in Proposition 7.1 (1) for \( \omega_X \) is equivalent to the estimate

   \[
   \int_{U'_{\varepsilon,i}} \|\omega_X\|^{2/m} = o((\varepsilon^{a/d_i})^{-2/m}) = o(e^{-2a/\alpha d_i}) \quad (\varepsilon \to 0).
   \]

Thus it suffices to show that (8.3) implies (8.4).

Since \( U'_{\varepsilon,i} \subset U'_e \) and \( \|\omega_X\|^{2/m} \) is a nonnegative multiple of a volume form, we have

\[
\int_{U'_{\varepsilon,i}} \|\omega_X\|^{2/m} \leq \int_{U'_e} \|\omega_X\|^{2/m}.
\]

Pulling back \( \omega_X \) to \( U_e \) by \( U_e \to U'_e \), we have

\[
\int_{U'_e} \|\omega_X\|^{2/m} \leq \int_{U_e} \|\omega_I\|^{2/m}.
\]

Then, if we substitute \( \alpha = 2a/\alpha d_i \) into (8.3), we obtain

\[
\int_{U_e} \|\omega_I\|^{2/m} = o(e^{-2a/\alpha d_i}).
\]

This gives (8.4).

The proof of (2) is similar, using Proposition 7.1 (2) in place of Proposition 7.1 (1). □
8.2. **Translation to modular forms.** As the next step, we translate the $L^{2/m}$ norm of $\omega_f$ over $U_1$ to the Petersson norm of the corresponding local modular form over $V_2$. This reduces the proof of Theorem 1.2 to the asymptotic estimate of the latter. The problem is local with respect to the base, so we work over an open set of the period domain $D$. Let $f : X^{(s)} \to D$ be the $s$-fold Kuga family over $D$.

**Proposition 8.3.** Let $B$ be an open set of $D$ and $X = f^{-1}(B) \subset X^{(s)}$. Let $F$ be a section of $L^{(g+s+1)m}$ over $B$ and $\omega$ be the $m$-canonical form on $X$ corresponding to $f^*F$ via the isomorphism $K^{\otimes m}_{X^{(s)}} \simeq f^*L^{(g+s+1)m}$. Then we have

$$\int_X \|\omega\|^{2/m} = \int_B (F, F)_{(g+s+1)m} \mathrm{vol}_D$$

up to a constant independent of $F$ and $B$.

**Proof.** Since the problem is local over $D$, we may assume that $B$ is sufficiently small. Since $K^{\otimes m}_{X^{(s)}} \simeq f^*K^{\otimes m}_D \otimes K^{\otimes m}_f$ and $f_*K_f$ is invertible, we can write $\omega$ as

$$\omega = f^*\varphi \cdot f^*\omega^\otimes B \otimes \omega^\otimes f,$$

where $\varphi$ is a holomorphic function on $B$, $\omega_B$ a nowhere vanishing canonical form on $B$, and $\omega_f$ a nowhere vanishing relative canonical form for $f$ on $X$. Then we have

$$\|\omega\|^{2/m} = C \cdot f^*|\varphi|^{2/m} \cdot f^*(\omega_B \wedge \bar{\omega}_B) \otimes (\omega_f \wedge \bar{\omega}_f),$$

and hence

$$\int_X \|\omega\|^{2/m} = C \cdot \int_B \left( \int_{X/B} \omega_f \wedge \bar{\omega}_f \right) |\varphi|^{2/m} \omega_B \wedge \bar{\omega}_B,$$

where $\int_{X/B}$ means fiber integral.

On the other hand, under the isomorphism $K_f \simeq f^*L^{\otimes g}$, we have $\omega_f = f^*F_1$ for some nowhere vanishing section $F_1$ of $L^{\otimes g}$ over $B$. Also, under the isomorphism $K_D \simeq L^{\otimes g+1}$, we have $\omega_B = F_2$ for some nowhere vanishing section $F_2$ of $L^{\otimes g+1}$ over $B$. By construction of the isomorphism $K^{\otimes (s)}_{X^{(s)}} \simeq f^*L^{(g+s+1)}$, the local modular form $F$ decomposes as $F = \varphi \cdot F_1^{\otimes m} \otimes F_2^{\otimes m}$. Therefore we have

$$(F, F)_{(g+s+1)m} = |\varphi|^2 \cdot (F_1, F_1)_{s}^m \cdot (F_2, F_2)_{s+1}^m.$$}

By iterated application of Lemma 2.1 we see that

$$(F_1, F_1)_s = \int_{X/B} \omega_f \wedge \bar{\omega}_f$$
up to a constant. Together with Lemma 2.2, this shows that

$$\int_B (F,F)_{(g+s+1)m}^{1/m} \cdot \omega_D = \int_B |\varphi|^{2/m} \cdot (F_1,F_1)_s \cdot (F_2,F_2)_{g+1} \cdot \omega_D$$

$$= \int_B \left( \int_{X/B} \omega_f \wedge \bar{\omega}_f \right) |\varphi|^{2/m} \omega_B \wedge \bar{\omega}_B$$

$$= \int_X |\omega|^{2/m}$$

up to a constant. This proves Proposition 8.3.

8.3. Completion of the proof. Combining the argument so far, we obtain the following.

**Proposition 8.4.** Let $\omega_X$ and $\omega_I$ be as in Lemma 8.2 and $k = (g+s+1)m$. Let $F$ be the section of $L^{g^k}$ over $V^s$ such that $\omega_I = f^* F$ under the isomorphism $K_{X_0}^m \cong f^* L^{g^k}$ of line bundles over $U^s$.

1. If $F$ extends to a holomorphic section of the extended line bundle $L^{g^k}$ over $V \subset \mathcal{B}^2_I$ (cf. 5.4), then $\omega_X$ has at most pole of order $m$ along $U' \cap \Delta_i$ for every irreducible component $\Delta_i$ of $f^{-1}(\Delta_Q)$.

2. If furthermore $F$ vanishes at $\Delta_Q$, then $\omega_X$ has at most pole of order $m-1$ along $U' \cap \Delta_i$ for every $\Delta_i$.

**Proof.** (1) We shall show that the estimate (8.3) holds, which by Proposition 8.3 is equivalent to the estimate

$$\int_{V_\alpha} (F,F)_k^{1/m} \cdot \omega_D = o(\varepsilon^{-\alpha}).$$

We reduce this to Proposition 6.2 by passing from the $I$-cusp to an adjacent 0-dimensional cusp. Choose a maximal totally isotropic sublattice $J$ of $\Lambda$ containing $I$. By Lemma 5.1, the projection $\mathcal{B}_I \to \mathcal{B}_J$ extends to an etale map $\pi: \mathcal{B}^2_I \to \mathcal{B}^2_J$. We choose $\chi \in U(J)_0$ such that $\chi(Q) = 1$ and put $q = \exp(2\pi i \chi(\cdot))$ on $\mathcal{B}_J \subset T_J$. Let $W_e \subset \mathcal{B}_J$ be the annulus bundle around $\pi(\Delta_e)$ of radius $\varepsilon \leq |q| \leq R$. As explained in Remark 7.2, the asymptotic behavior of the integral over $V_e = \pi(V_e)$ is equivalent to that over $W_e$. Now we have

$$\int_{W_e} (F,F)_k^{1/m} \cdot \omega_D = o(\varepsilon^{-\alpha})$$

by the first part of Proposition 6.2 with $\beta = 1/m$, which implies (8.5).

(2) Similarly, we are reduced to showing that

$$\int_{V_\alpha} (F,F)_k^{1/m} \cdot \omega_D = O(1),$$

which follows from the second part of Proposition 6.2. □
We can now complete the proof of Theorem 1.2. (Proof of Theorem 1.2.) Let $\omega_X$ be an $m$-canonical form on $X^s(\Gamma)$. Let $F$ be the corresponding (global) modular form. Then $\omega_f$ corresponds to the restriction of $F$ to $V^o$. By Lemma 5.6, $F$ extends holomorphically over $V$, so we can apply Proposition 8.4 (1). The assertion for cusp forms follows from the second part of Lemma 5.6 and Proposition 8.4 (2).

In the rest of this section we show that the restricted map

\[(8.6) \quad S_{(g+s+1)m}(\Gamma) \hookrightarrow H^0(K_X^{\otimes m}(m-1)\Delta_X))\]

for cusp forms is surjective under a certain condition on the singularities of the pair $(\tilde{X}, \Delta_X)$ and properness of $\tilde{X} \to A(\Gamma)^\Sigma$.

Let $X$ be a normal complex analytic variety and $\Delta$ be an effective $\mathbb{Q}$-Weil divisor such that $[\Delta] = 0$. Recall ([14], §2.3) that the pair $(X, \Delta)$ is called Kawamata log terminal if $K_X + \Delta$ is $\mathbb{Q}$-Cartier and for some (hence all) log resolution $\pi: Y \to X$ of $(X, \Delta)$, we have $K_Y = \pi^*(K_X + \Delta) + \sum a(E_i)E_i$ with $a(E_i) > -1$. If $\omega$ is a meromorphic $m$-canonical form on the regular locus of $X$ whose pole divisor satisfies $\leq m\Delta$, its pullback to $Y$ has at most pole of order $m - 1$ along every component of the exceptional divisor. (This is the only property where we need the klt condition.)

Our result is as follows. We write $\Delta_{U''} = \Delta_X \cap U'' = f^{-1}(\Delta_Q') \cap U'$.

**Proposition 8.5.** Assume that $f: \tilde{X} \to A(\Gamma)^\Sigma$ is proper and that the pair $(U', (1-m^{-1})\Delta_{U'})$ is klt for a general point of every irreducible component $\Delta_Q'$ of $\Delta_A$. Then the map (8.6) is surjective.

**Proof.** Let $\omega_X$ be an element of $H^0(K_X^{\otimes m}(m-1)\Delta_X))$ and $F$ be the corresponding modular form of weight $k = (g + s + 1)m$. In view of Lemma 5.6, we want to show that $F$ vanishes at $\Delta_Q^I$ for every $I$ and positive definite $\mathbb{R}_{>0}Q \in \Sigma_I$. If we choose a maximal $J \supset I$, this is equivalent to the vanishing of $F$ at $\Delta_Q^J$. By the second part of Proposition 6.2 with $\beta = 1/m$ (note that $km^{-1} \geq g + 1$), it suffices to show that $\int_{W_k} (F, F)^{1/m}_{\text{vol}_D} = O(1)$.

Going back by the etale gluing $B_I^{\Sigma_I} \to B_J^{\Sigma_J}$, we are reduced to showing that $\int_{U''} (F, F)^{1/m}_{\text{vol}_D} < \infty$. By Proposition 8.3, this is translated to $\int_{U''} ||\omega_f||^{2/m}_{\text{vol}_D} < \infty$, which in turn is equivalent to $\int_{U''-\Delta_{U''}} ||\omega_f||^{2/m} < \infty$.

We take a log resolution $(U'', \Delta_{U''})$ of $(U', \Delta_{U'})$ and let $E$ be its exceptional divisor. Then the above condition is rewritten as

\[(8.7) \quad \int_{U''-E-\Delta_{U''}} ||\omega_f||^{2/m} < \infty.\]

The divisor $E + \Delta_{U''}$ is simple normal crossing, and can be covered by finitely many local charts of $U''$ by the properness of $\tilde{X} \to A(\Gamma)^\Sigma$. Since the pole divisor of $\omega_X$ satisfies $\leq m(1-m^{-1})\Delta_{U''}$ by our assumption on $\omega_X$, we have that $\omega_X$ vanishes at $\Delta_{U''}$.
the klt condition for \((U', (1 - m^{-1})\Delta_{U'})\) implies that \(\omega_X\) has at most pole of order \(m - 1\) along every component of \(E + \Delta_{U'}\). Then the assertion \((8.7)\) follows from Proposition \(7.3\).

Example 8.6. (1) When \(m = 1\), the klt condition is just \(U'\) being log terminal (e.g., having only quotient singularities). Thus \(S_{g+s+1}(\Gamma) \approx H^0(K_{\tilde{X}})\) when \(\tilde{X}\) is log terminal. This extends the result of Hatada \([9]\) where the case \(\tilde{X}\) smooth is considered.

(2) When \(U'\) is smooth and \(\Delta_{U'}\) is simple normal crossing, the pair \((U', (1 - m^{-1})\Delta_{U'})\) is always klt. Hence

\[
S_{(g+s+1)m}(\Gamma) \approx H^0(K_{\tilde{X}}^\otimes ((m - 1)\Delta_X))
\]

for every \(m\) when \(\tilde{X}\) is smooth and \(\Delta_X\) is simple normal crossing over general points of \(\Delta_A\).

9. Proof of Theorem 1.3

In this section we prove Theorem 1.3. Let \(\tilde{X} \supset X^s(\Gamma)\) be a normal analytic variety satisfying the conditions of Theorem 1.2, and \(f: \tilde{X} \to A(\Gamma)^{\Sigma}\) the extended morphism. We write \(K_{\tilde{X}}\) for the canonical divisor of \(\tilde{X}\) (as a Weil divisor) or the corresponding rank 1 reflexive sheaf on \(\tilde{X}\).

We choose and fix a natural number \(m\) such that \(L_{\otimes m}\) can be defined as a line bundle over \(A(\Gamma)^{\Sigma}\) in the sense of the last paragraph of \(8.4\). Since \(K_{X^s(\Gamma)} = f^* L_{\otimes k}\) over \(X^s(\Gamma)\) where \(k = (g + s + 1)m\), then \(K_{X^s(\Gamma)}^{\otimes m}\) also descends to a line bundle over \(X^s(\Gamma)\) which we denote by \(K_{X^s(\Gamma)}^{\otimes m}\) (by abuse of notation). Since \(X^s(\Gamma) \to X^s(\Gamma)\) is unramified in codimension 1 by Lemma 3.1, \(K_{X^s(\Gamma)}^{\otimes m}\) to the regular locus of \(X^s(\Gamma)\) is indeed isomorphic to the \(m\)-power of its canonical bundle. The isomorphism \(K_{X^s(\Gamma)}^{\otimes m} = f^* L_{\otimes k}\) over \(X^s(\Gamma)\) descends to an isomorphism \(K_{X^s(\Gamma)}^{\otimes m} = f^* L_{\otimes k}|_{X^s(\Gamma)}\) of line bundles over \(X^s(\Gamma)\).

Proposition 9.1. Let \(\tilde{X}, m\) and \(k = (g + s + 1)m\) be as above. Then the isomorphism \(f^* L_{\otimes k}|_{X^s(\Gamma)} \approx K_{X^s(\Gamma)}^{\otimes m}\) of line bundles over \(X^s(\Gamma)\) extends to an injective homomorphism

\[
f^* L_{\otimes k} \hookrightarrow K_{X^s(\Gamma)}^{\otimes m}(m\Delta_X)
\]

of sheaves on \(\tilde{X}\). In particular, we have

\[mK_{\tilde{X}} \geq f^*(kL - m\Delta_A)\]

Proof. We keep the notation from \(8\) concerning the boundary. Let \(y\) be a general point of an irreducible component \(\Delta'_{U'}\) of \(\Delta_A\). Let \(F\) be a local frame of \(L_{\otimes k}\) on a small neighborhood \(V'\) of \(y\) in \(A(\Gamma)^{\Sigma}\). Then \(f^* F\) is a local frame of \(f^* L_{\otimes k}\) on the neighborhood \(U' = f^{-1}(V')\) of \(f^{-1}(y)\) in \(\tilde{X}\). Let \(\omega\) be the \(m\)-canonical form on \(U' \cap X^s(\Gamma)\) corresponding to the restriction of
\( f^*F \) to \( U' \cap X'(\Gamma) \) via \( K^{{\mathrm{gm}}}_{X'(\Gamma)} \cong f^*L^{{\mathrm{deg}}}_{X'(\Gamma)} \). By Proposition \([8,4](1)\), \( \omega \) has at most pole of order \( m \) along \( U' \cap \Delta_i \) for every irreducible component \( \Delta_i \) of \( f^{-1}(\Delta'_Q) \). This shows that the isomorphism \( f^*L^{{\mathrm{deg}}}_{X'(\Gamma)} \cong K^{{\mathrm{gm}}}_{X'(\Gamma)} \) at \( U' \cap X'(\Gamma) \) extends to a sheaf homomorphism \( f^*L^{{\mathrm{deg}}}_{U'} \hookrightarrow K^{{\mathrm{gm}}}_{U'}(m\Delta_U') \) over \( U' \). Since we obtain this for general points of every component of \( \Delta_A \), our condition on \( \bar{X} \to A(\Gamma)\Sigma \) ensures that we obtain a homomorphism \( f^*L^{{\mathrm{deg}}} \hookrightarrow K^{{\mathrm{gm}}}_{\bar{X}}(m\Delta_{\bar{X}}) \) outside a codimension \( \geq 2 \) locus in \( \bar{X} \). By the normality of \( \bar{X} \), this extends over the whole \( \bar{X} \).

As for the second assertion, we have \( mK_{\bar{X}} + m\Delta_X \geq f^*(kL) \) by the first assertion. By our condition on \( \bar{X} \to A(\Gamma)\Sigma \), we can pullback \( \Delta_A \) to a Weil divisor of \( \bar{X} \) whose support is \( \Delta_X \). (Pullback \( \Delta_A \to A(\Gamma)\Sigma \) as a Cartier divisor and take closure in \( \bar{X} \).) Then \( f^*\Delta_A \geq \Delta_X \) and so \( mK_{\bar{X}} + mf^*\Delta_A \geq f^*(kL) \). □

Theorem \([1.3]\) is deduced as follows.

(Proof of Theorem \([1.3]\).) By Proposition \([9.1]\), we have

\[
\kappa(\bar{X}, K_{\bar{X}}) \geq \kappa(\bar{X}, f^*((g + s + 1)L - \Delta_A)) \\
\geq \kappa(A(\Gamma)_\Sigma, (g + s + 1)L - \Delta_A).
\]

On the other hand, since

\[
H^0(\bar{X}, K^{{\mathrm{gm}}}_{\bar{X}}) \subset H^0(\bar{X}, K^{{\mathrm{gm}}}_{\bar{X}}(m\Delta_X)) \cong M_{m(g+s+1)}(\Gamma)
\]

for every \( m' \), we have

\[
\kappa(\bar{X}, K_{\bar{X}}) \leq \kappa(A(\Gamma)_\Sigma, (g + s + 1)L) = g(g + 1)/2.
\]

\( □ \)

10. Singularities

In this section we prove that the Kuga variety \( X'(\Gamma) \) has canonical singularities in most cases (Proposition \([10.3]\)). Below, by a representation of a finite group \( G \) over a field \( K \), we mean a finite dimensional \( K \)-linear space equipped with a linear action of \( G \). (\( K \) will be either \( \mathbb{Q} \) or \( \mathbb{C} \).) We write \( e(\alpha) = \exp(2\pi i\alpha) \) for \( \alpha \in \mathbb{Q}/\mathbb{Z} \).

Let \( W \) be a representation of a finite group \( G \) over \( \mathbb{C} \). The Reid–Shepherd-Barron–Tai criterion \([22], [27]\) tells whether \( W/G \) has canonical singularities in terms of the eigenvalues of elements of \( G \). Let \( \gamma \in G \) and \( e(\alpha_1), \ldots, e(\alpha_d) \) be the eigenvalues of \( \gamma \) on \( W \) where \( d = \dim W \). We choose \( \alpha_i \in \mathbb{Q} \) from \( 0 \leq \alpha_i < 1 \). The Reid-Tai sum of \( \gamma \) is defined by

\[
RT_\gamma(W) = \sum_{i=1}^{d} \alpha_i.
\]

The action of \( \gamma \) on \( W \) is called quasi-reflection (or pseudo-reflection) if its eigenvalues are \( 1, \ldots, 1, \lambda \) with \( \lambda \neq 1 \).
Theorem 10.1 ([23], [27]). Assume that $G$ contains no quasi-reflection on $W$. Then $W/G$ has canonical singularities if and only if $RT_\gamma(W) \geq 1$ for every element $\gamma \neq \text{id}$ of $G$.

We will apply this RST criterion for $W$ the tangent space $T_pX(s)$ of $X(s)$ at a point $p \in X(s)$ and $G$ the stabilizer of $p$ in $\Gamma$.

10.1. Distribution of eigenvalues. We first prepare a lemma on the distribution of eigenvalues. Let $G = \mathbb{Z}/n$ be the standard cyclic group of order $n$. For $k \in \mathbb{Z}/n$ we write $\chi_{k/n}$ for the 1-dimensional $\mathbb{C}$-representation of $G$ on which the standard generator $\bar{1} \in G$ acts by $e(k/n)$. Recall ([23] §13.1) that there is a unique faithful $\mathbb{Q}$-representation $V_n$ of $G$ that is irreducible over $\mathbb{Q}$. This can be defined as the kernel of $\Phi_n(A) : \mathbb{Q}G \to \mathbb{Q}G$ where $A : \mathbb{Q}G \to \mathbb{Q}G$ is the multiplication by $\bar{1} \in G$ and $\Phi_n(x)$ the $n$-th cyclotomic polynomial. The complexification of $V_n$ decomposes as $V_n := V_n \otimes_\mathbb{Q} \mathbb{C} \cong \bigoplus_{k \in (\mathbb{Z}/n)^*} \chi_{k/n}$.

For $d | n$, $V_d$ is a representation of $G$ via the reduction map $\mathbb{Z}/n \to \mathbb{Z}/d$. It is classical ([23] §13.1) that every $\mathbb{Q}$-representation of $G$ decomposes over $\mathbb{Q}$ into a direct sum of $V_{d_1}, \cdots, V_{d_N}$ for some $d_1, \cdots, d_N | n$. (It may happen that $d_i = d_j$ for $i \neq j$.)

Lemma 10.2. Let $\Lambda_\mathbb{Q}$ be a representation of $G$ over $\mathbb{Q}$ and

(10.1) $$\Lambda_\mathbb{Q} = \bigoplus_{i=1}^N V_{d_i}$$

be an irreducible decomposition of $\Lambda_\mathbb{Q}$ over $\mathbb{Q}$. Assume that $G$ preserves a weight 1 Hodge decomposition $\Lambda_\mathbb{C} = V \oplus \bar{V}$ of $\Lambda_\mathbb{C}$. Then the following holds.

(1) Let $d > 2$. If $V_{d|\mathbb{C}}$ appears in (10.1), there is a sub $G$-representation $W_d$ of $V$ such that $W_d \oplus \bar{W}_d \cong V_{d|\mathbb{C}}$ as representations of $G$ over $\mathbb{C}$.

(2) For $d = 1, 2$ the multiplicity of $V_d$ in (10.1) is even, say $2k$, and $V$ contains a sub $G$-representation $V'$ isomorphic to $V_{d|\mathbb{C}}$.

Proof. (1) Let $d > 2$. For a $\mathbb{C}$-representation $W$ of $G$ we write $\lambda(W)$ for the set of eigenvalues of $\bar{1} \in G$ counted with multiplicity. We choose eigendecompositions of $V$ and $\bar{V}$ with respect to $\bar{1} \in G$:

$$V = \bigoplus_{\lambda_\alpha \in \lambda(V)} \mathbb{C}v(\lambda_\alpha), \quad \bar{V} = \bigoplus_{\lambda_\beta' \in \bar{\lambda(V)}} \mathbb{C}w(\lambda_\beta'),$$

where $v(\lambda_\alpha) \in V$ is a $\lambda_\alpha$-eigenvector and $w(\lambda_\beta') \in \bar{V}$ a $\lambda_\beta'$-eigenvector. We also fix a decomposition $\lambda(\Lambda_\mathbb{C}) = \lambda(V) \cup \bar{\lambda(\bar{V})}$. Now, since $\Lambda_\mathbb{Q}$ contains $V_{d|\mathbb{C}}$, ...
there exists some embedding
\[ \theta : \lambda(V_{d}^{\text{reg}}) \hookrightarrow \lambda(\Lambda_{C}) = \lambda(V) \sqcup \lambda(\bar{V}). \]

(\mu \in \lambda(V_{d}^{\text{reg}}) and \theta(\mu) \in \lambda(\Lambda_{C}) are the same number.) We put the elements of \( \lambda(V_{d}^{\text{reg}}) \) by the order of their angle in \((0, 2\pi)\), say \( \lambda(V_{d}^{\text{reg}}) = \{ \mu_{1}, \cdots, \mu_{l} \} \). Then \( \mu_{i+1} = \mu_{i} \), and \( \mu_{i} \) has angle in \((0, \pi)\) if \( i \leq l/2 \). We put
\[
W_{d}^{+} := \bigoplus_{\{ i \leq l/2 \} \delta(\mu_{i}) \in \lambda(V)} \mathcal{O}_{V}(\theta(\mu_{i})), \\
W_{d}^{-} := \bigoplus_{\{ i \leq l/2 \} \theta(\mu_{i}) \in \lambda(V)} \mathcal{O}_{W}(\theta(\mu_{j})).
\]

Since \( v(\lambda_{s}) \in V \) and \( w(\Lambda_{p}') \in \bar{V} \), we have \( W_{d}^{+}, W_{d}^{-} \subset V \). We also have \( W_{d}^{+} \cap W_{d}^{-} = \{ 0 \} \) because elements of \( \lambda(W_{d}^{+}) \) have angle in \((0, \pi)\) while those of \( \lambda(W_{d}^{-}) \) in \((\pi, 2\pi)\). We then put
\[ W_{d} := W_{d}^{+} \oplus W_{d}^{-} \subset V. \]

Since \( \lambda(W_{d}^{+}) \sqcup \lambda(W_{d}^{-}) = \{ \mu_{1}, \cdots, \mu_{l/2} \} \) by construction, we have
\[ \lambda(W_{d}^{+} \oplus \bar{W}_{d}) = \lambda(W_{d}^{+}) \sqcup \lambda(W_{d}^{-}) \sqcup \lambda(\bar{W}_{d}^{+}) \sqcup \lambda(\bar{W}_{d}^{-}) = \{ \mu_{1}, \cdots, \mu_{l} \}. \]

Therefore \( W_{d} \oplus \bar{W}_{d} \simeq V_{d}^{\text{reg}} \) as abstract \( G \)-representations.

(2) Let \( d = 1 \) or 2. Let \( \mathcal{W} \subset \Lambda_{Q} \) be the direct sum of all components \( \mathcal{W}_{d} \) in (10.1) such that \( d_{i} = d \). Then \( W := \mathcal{W} \otimes_{Q} \mathbb{R} \) is the \((\pm 1)\)-eigenspace of \( \bar{I} \in G \) on \( \Lambda_{R} \). Let \( J : \Lambda_{R} \to \Lambda_{R} \) be the complex structure corresponding to the Hodge decomposition \( \Lambda_{C} = V \oplus \bar{V} \). Since the \( G \)-action commutes with \( J, J \) preserves \( W \) and hence gives a complex structure on \( W \). In particular, \( \mathcal{W} \) has even dimension. If \( W_{C} = V' \oplus \bar{V}' \) is the Hodge decomposition given by \( J_{w} \), then \( V' = V \cap W_{C} \) is the \((\pm 1)\)-eigenspace of \( \bar{I} \in G \) on \( V \).

10.2. **Singularities of** \( X'(\Gamma) \). As before, let \( \Lambda \) be a symplectic lattice of rank \( 2g > 2 \) and \( \Gamma \) a finite-index subgroup of \( \text{Sp}(\Lambda) \). Our main result of §10 is the following.

**Proposition 10.3.** The Kuga variety \( X'(\Gamma) \) has canonical singularities unless when \((g, s) = (2, 1), (3, 1), (2, 2) \) and \( \Gamma \) contains an element of order 6 whose eigenvalues on \( \Lambda_{C} \) are \( e(1/6), e(-1/6), 1, \cdots, 1 \).

**Proof.** Recall from §2.3 that \( X'(\Gamma) = X^{(s)}/\Gamma \). Let \( p = ([V], x_{1}, \cdots, x_{s}) \) be a point of \( X^{(s)} \) where \([V] \in D \) and \( x_{i} \in V'/\Lambda \). It suffices to show that \( T_{p}X^{(s)}/\Gamma_{p} \) has canonical singularities where \( \Gamma_{p} < \Gamma \) is the stabilizer of \( p \). By Lemma 8.1, \( \Gamma_{p} \) contains no quasi-reflection on \( T_{p}X^{(s)} \). Thus it suffices to show that \( RT_{\gamma}(T_{p}X^{(s)}) \geq 1 \) for every \( \gamma \neq \text{id} \in \Gamma_{p} \). The \( \Gamma_{p} \)-representation \( T_{p}X^{(s)} \) decomposes as
\[
T_{p}X^{(s)} \simeq T_{x_{1}}(V'/\Lambda) \oplus \cdots \oplus T_{x_{s}}(V'/\Lambda) \oplus T_{[V]}D
\]
\[ \simeq (V')^{\otimes s} \oplus \text{Sym}^{2}V'. \]
So the problem is reduced to the following calculation in linear algebra. (We rewrite $V'$ as $V$, and $\Lambda'_Q$ as $\Lambda_Q$.)

**Lemma 10.4.** Let $G = \langle \gamma \rangle$ be a finite cyclic group and $\Lambda_Q$ be a $G$-representation over $\mathbb{Q}$ of dimension $2g > 2$. Assume that $G$ preserves a Hodge decomposition $\Lambda_C = V \oplus \bar{V}$. Assume also that $(s, \Lambda_Q)$ is not one of the following:

- $s = 1, \Lambda_Q \simeq V_6 \oplus V_1^{g_2g-2}$ with $g = 2, 3$.
- $s = 2, \Lambda_Q \simeq V_6 \oplus V_1^{g_2}$ ($g = 2$).

Then $RT(\gamma (V^{\otimes s} \oplus \text{Sym}^2 V)) \geq 1$.

**Proof.** As $G$-representation, one of the following cases occur:

1. $\Lambda_Q \supset V_d, \varphi(d) > 2$;  
2. $\Lambda_Q \supset V_3$;  
3. $\Lambda_Q \supset V_4$;  
4. $\Lambda_Q \supset V_6$;  
5. $\Lambda_Q = V_1^{g_2k} \oplus V_2^{g_2l}$.

We shall estimate the Reid-Tai sum case-by-case according to this classification. As in the proof of Lemma [10.1], for a $G$-representation $W$ over $\mathbb{C}$, we write $\lambda(W)$ for the set of eigenvalues of $\gamma$ counted with multiplicity. By associating $\alpha \in [0, 1)$ to an eigenvalue $e(\alpha)$, we identify elements of $\lambda(W)$ with rational numbers in $[0, 1)$.

We first consider the case $s = 1$. We write $RT = RT_\gamma (V \oplus \text{Sym}^2 V)$. We shall show that $RT \geq 1$ unless $\Lambda_Q \simeq V_6 \oplus V_1^{g_2g-2}$ with $g = 2, 3$.

1. Let $W_d \subset V$ be the sub $G$-representation such that $W_d \oplus \bar{W}_d \simeq V_d$ as constructed in Lemma [10.2]. Firstly, if $\lambda(W_d)$ contains two elements $\lambda, \lambda'$ from $(1/2, 1)$, we have $RT > \lambda + \lambda' > 1$. Secondly, suppose that $\lambda(W_d)$ contains exactly one element $\lambda$ from $(1/2, 1)$. Since $W_d \oplus \bar{W}_d \simeq V_d$, every element of $\lambda(W_d) \cap (0, 1/2)$ except $1 - \lambda$ appears in $\lambda(W_d)$. Let $\lambda'$ be the maximal element of $\lambda(W_d) \cap (0, 1/2)$. When $\lambda' > 1 - \lambda$, we have $RT \geq \lambda + \lambda' > 1$. When $\lambda' < 1 - \lambda$, we have $\lambda + \lambda' \in \lambda(\text{Sym}^2 V)$ and $\lambda + \lambda' < 1$. Then $RT \geq \lambda + (\lambda + \lambda') > 2\lambda > 1$. Thirdly, if all elements of $\lambda(W_d)$ are contained in $(0, 1/2)$, we have $\lambda(W_d) = \lambda(V_d) \cap (0, 1/2) / V_d \simeq W_d \oplus \bar{W}_d$. Let $\lambda$ be the maximal element of $\lambda(W_d)$. Then $\lambda > 1/4$. Since $\lambda(\text{Sym}^2 V)$ contains $\lambda + \lambda' < 1$ for every $\lambda' \in \lambda(V)$, we have $RT > (2 + \varphi(d)/2) \lambda \geq 4\lambda > 1$.

2. By Lemma [10.2] we have either $\chi_{1/3} \subset V$ or $\chi_{2/3} \subset V$. In the first case, we have $1/3 \in \lambda(V)$ and $2/3 \in \lambda(\text{Sym}^2 V)$, and so $RT \geq 1/3 + 2/3 = 1$. In the second case, we have $2/3 \in \lambda(V)$ and $1/3 \in \lambda(\text{Sym}^2 V)$, so again $RT \geq 1$.

3. Since $g > 1$, we have $\Lambda_Q \neq V_4$. In view of the cases (1), (2), we only need to consider the case $V_4 \oplus V_d \subset \Lambda_Q$ with $d = 1, 2, 4, 6$. When $V_4 \subset \Lambda_Q$, $\lambda(V)$ contains two elements from $\{1/4, 1/4, 3/4, 3/4\}$ by Lemma [10.2], so
\[\lambda(\text{Sym}^2 V)\text{ contains two }1/2\text{ and hence }RT > 1.\] When \(\mathbb{V}_4 \oplus \mathbb{V}_6 \subset \Lambda_Q\), \(\lambda(V)\) contains \(\{1/4 \text{ or } 3/4, 1/6 \text{ or } 5/6\}\) by Lemma 10.2. Then \(\lambda(\text{Sym}^2 V)\) contains \(\{1/2, 1/3 \text{ or } 2/3\}\). Hence \(RT > 1\). Similarly, when \(\mathbb{V}_4 \oplus \mathbb{V}_2 \subset \Lambda_Q\), \(\lambda(V)\) contains \(\{1/2, 1/4 \text{ or } 3/4\}\), and so \(\lambda(\text{Sym}^2 V)\) contains \(3/4 \text{ or } 1/4, 1/2\), which implies \(RT > 1\). Finally, when \(\mathbb{V}_4 \oplus \mathbb{V}_1 \subset \Lambda_Q\), \(\lambda(V)\) contains \(\{0, 1/4 \text{ or } 3/4\}\), and so \(\lambda(\text{Sym}^2 V)\) contains \(\{1/2, 1/4 \text{ or } 3/4\}\). This proves \(RT \geq 1\).

(4) In view of the cases (1) – (3), we only need to cover the cases

\[\Lambda_Q \supset \mathbb{V}_6 \oplus \mathbb{V}_6, \quad \Lambda_Q \supset \mathbb{V}_6 \oplus \mathbb{V}_2, \quad \Lambda_Q = \mathbb{V}_6 \oplus \gamma^{g_{2g-2}} (g \geq 4).\]

When \(\mathbb{V}_6 \subset \Lambda_Q\), \(\lambda(V)\) contains two elements from \(\{1/6, 1/6, 5/6, 5/6\}\) by Lemma 10.2, so \(\lambda(\text{Sym}^2 V)\) contains two elements from \(\{1/3, 1/3, 2/3, 2/3\}\). It follows that \(RT \geq 1\). When \(\mathbb{V}_6 \oplus \mathbb{V}_2 \subset \Lambda_Q\), \(\lambda(V)\) contains \(\{1/2, 1/6 \text{ or } 5/6\}\), so \(\lambda(\text{Sym}^2 V)\) contains \(1/3 \text{ or } 2/3\). Thus \(RT \geq 1\). Finally, when \(\Lambda_Q = \mathbb{V}_6 \oplus \mathbb{V}_6 \oplus \mathbb{V}_6 \oplus \mathbb{V}_6 \oplus \mathbb{V}_6 \oplus \mathbb{V}_6 \oplus \mathbb{V}_6 \oplus \mathbb{V}_6 \oplus \mathbb{V}_6 \oplus \mathbb{V}_6 \), we have \(V \simeq \gamma^{1/6} \oplus \gamma^{1/6} \oplus \gamma^{1/6} \oplus \gamma^{1/6} \oplus \gamma^{1/6} \oplus \gamma^{1/6} \oplus \gamma^{1/6} \oplus \gamma^{1/6} \oplus \gamma^{1/6} \oplus \gamma^{1/6}\). If \(V \simeq \gamma^{1/6} \oplus \gamma^{1/6} \oplus \gamma^{1/6} \oplus \gamma^{1/6} \oplus \gamma^{1/6} \oplus \gamma^{1/6} \oplus \gamma^{1/6} \oplus \gamma^{1/6} \oplus \gamma^{1/6} \oplus \gamma^{1/6}\), then \(\lambda(V) = \{1/6, 0, 0, 0, 0, 0, \cdots, 0\}\) and \(\lambda(\text{Sym}^2 V) = \{1/3, 1/6, \cdots, 1/6, 0, 0, 0, \cdots, 0\}\) where \(1/6\) has multiplicity \(g - 1\). Hence \(RT = 1/3 + g/6 \geq 1\) by the assumption \(g \geq 4\).

(5) In this case we have \(V \simeq \gamma^{1/6} \oplus \gamma^{1/6} \) by Lemma 10.2. When \(l \geq 2\), we have \(RT \geq l/2 \geq 1\). When \(l = 1\), we have \(k \geq 1\) by \(g \geq 2\). Then \(1/2 \in \lambda(V)\) and \(1/2 \in \lambda(\text{Sym}^2 V)\), so \(RT \geq 1\). This finishes the proof for the case \(s = 1\).

Next let \(s \geq 2\). Since

\[RT_{\gamma}(V^{\oplus s} \oplus \text{Sym}^2 V) \geq RT_{\gamma}(V \oplus \text{Sym}^2 V),\]

we only need to consider the case \(\Lambda_Q = \mathbb{V}_6 \oplus \mathbb{V}_6 \oplus \mathbb{V}_6 \oplus \mathbb{V}_6 \oplus \mathbb{V}_6 \oplus \mathbb{V}_6 \oplus \mathbb{V}_6 \oplus \mathbb{V}_6 \oplus \mathbb{V}_6 \oplus \mathbb{V}_6 \) with \(g = 2, 3\) by the above proof for the case \(s = 1\). In this case the Reid-Tai sum is \(1/3 + (g + s - 1)/6\), which is smaller than \(1\) only when \((g, s) = (2, 2)\). This completes the proof of Lemma 10.4 and hence of Proposition 10.3.

\[\square\]

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DEPARTMENT OF MATHEMATICS, TOKYO INSTITUTE OF TECHNOLOGY, TOKYO 152-8551, JAPAN
E-mail address: ma@math.titech.ac.jp