VIRTUAL VECTOR BUNDLES AND GRADED THOM SPECTRA

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Abstract. We introduce a convenient framework for constructing and analyzing orthogonal Thom spectra arising from virtual vector bundles. This framework enables us to set up a theory of orientations and graded Thom isomorphisms with good multiplicative properties. The theory is applied to the analysis of logarithmic structures on commutative ring spectra.

1. Introduction

Classically, the starting point for the construction of a Thom spectrum is a sequence of spaces $X_0 \to X_1 \to X_2 \to \ldots$, together with a compatible family of maps $f_n : X_n \to BO(n)$. The Thom space $T(f_n)$ of the resulting vector bundle on $X_n$ then constitutes the $n$th space in the corresponding Thom spectrum. From a slightly different point of view, one may start with a space $X$ and a map $X \to BO$ to the classifying space for stable vector bundles, then choose a suitable filtration of this map, and proceed with this data as above. This point of view has been developed in detail by Lewis [7, Section IX]. It is also of interest to consider Thom spectra derived from virtual vector bundles in the sense of maps $f : X \to BO \times \mathbb{Z}$ with target the classifying space for $KO$-theory. The naive approach to this is to apply the above procedure over each of the components $BO \times \{n\}$ and then pass to the $n$-fold suspension of the resulting Thom spectrum. However, this construction does not reflect the $E_\infty$ structure of $BO \times \mathbb{Z}$ and some care is needed in order to define a graded Thom spectrum functor with good multiplicative properties.

One of the main objectives of the present paper is to introduce a convenient framework for the construction and analysis of graded Thom spectra. Let $\mathcal{O}$ be the topological category with objects the standard inner product spaces $\mathbb{R}^n$ for $n \geq 0$, and morphisms the linear isometric isomorphisms. This is a permutative topological category under direct sum. We define a topological category $\mathcal{W}$ by applying Quillen’s localization construction [3] to $\mathcal{O}$, that is, $\mathcal{W} = \mathcal{O}^{-1} \mathcal{O}$. This definition makes $\mathcal{W}$ a permutative topological category whose classifying space is a model of $BO \times \mathbb{Z}$. It follows that the objects in $\mathcal{W}$ have a canonical $\mathbb{Z}$-grading and we write $\mathcal{W}_{\{n\}}$ for the $n$th component of $\mathcal{W}$.

Writing $Top$ for the category of compactly generated weak Hausdorff spaces, we define a $\mathcal{W}$-space to be a continuous functor from $\mathcal{W}$ to $Top$. We shall be working with the functor category of $\mathcal{W}$-spaces which we denote by $Top^\mathcal{W}$. The permutative structure of $\mathcal{W}$ induces a symmetric monoidal convolution product on $Top^\mathcal{W}$, and we use the term commutative $\mathcal{W}$-space monoid for a commutative monoid in $Top^\mathcal{W}$. Such commutative monoids arise naturally, and in particular we show that the definition of the classical Stiefel manifolds can be upgraded to give a commutative $\mathcal{W}$-space monoid that we denote by $\tilde{V}$. There also is an oriented Stiefel $\mathcal{W}$-space $V$, which gives rise to a commutative $\mathcal{W}$-space monoid $\tilde{V}_{ev}$ when restricted to even degrees.

We shall prove that the Stiefel $\mathcal{W}$-space $V$ defines a graded frame bundle over a suitable graded version of the Grassmannians which we denote by $Gr$. In order to
explain this properly we need the permutative topological category $\mathcal{V}$ with objects $\mathbb{R}^n$ for $n \geq 0$, and morphisms the (not necessarily surjective) linear isometries. The graded Grassmannian $\text{Gr}$ is then realised as a commutative monoid in the corresponding symmetric monoidal category of $\mathcal{V}$-spaces $\text{Top}^\mathcal{V}$, and there is a chain of Quillen equivalences of topological model categories

\begin{equation}
\text{Top}/\text{Gr}_{h\mathcal{V}} \simeq \text{Top}^\mathcal{V}/\text{Gr} \simeq \text{Top}^W/\mathcal{V} \simeq \text{Top}^W
\end{equation}

where $\text{Gr}_{h\mathcal{V}}$ denotes the homotopy colimit of $\text{Gr}$ over $\mathcal{V}$. Since $\text{Gr}_{h\mathcal{V}}$ is a model of $BO \times \mathbb{Z}$, it follows that $W$-spaces represent virtual vector bundles via these equivalences.

The category $\text{Top}^W$ is closely related to the symmetric monoidal category of orthogonal spectra $\text{Sp}^D$ introduced in [10]: There is an adjunction

$$S^W: \text{Top}^W \rightleftarrows \Omega^W: W$$

whose left adjoint $S^W$ is strong symmetric monoidal with respect to the convolution product on $\text{Top}^W$ and the smash product on $\text{Sp}^D$. Applying $S^W$ to $V$ we get a commutative orthogonal ring spectrum that turns out to be a model of the periodic unoriented cobordism spectrum. Motivated by this we write $\text{MOP} = S^W[\mathcal{V}]$. The graded orthogonal Thom spectrum functor is now defined to be the composition of the functors

\begin{equation}
T: \text{Top}/\text{Gr}_{h\mathcal{V}} \to \text{Top}^\mathcal{V}/\text{Gr} \to \text{Top}^W/\mathcal{V} \xrightarrow{S^W} \text{Sp}^D/\text{MOP}.
\end{equation}

Here we implicitly follow Lewis [7, Section IX] by precomposing with a Hurewicz fibrant replacement functor to ensure that the composite functor takes weak homotopy equivalences over $\text{Gr}_{h\mathcal{V}}$ to stable equivalences over $\text{MOP}$. The next theorem (proved in Section 6.3) shows that this construction of the graded Thom spectrum functor has good multiplicative properties.

**Theorem 1.1.** Let $\mathcal{D}$ be an operad augmented over the Barratt-Eccles operad. Then the graded Thom spectrum functor in (1.2) induces a functor of $\mathcal{D}$-algebras

$$T: \text{Top}[\mathcal{D}]/\text{Gr}_{h\mathcal{V}} \to \text{Sp}^D[\mathcal{D}]/\text{MOP}.$$ 

Here we require $\mathcal{D}$ to be augmented over the Barratt-Eccles operad in order to ensure that $\text{Gr}_{h\mathcal{V}}$ inherits the structure of a $\mathcal{D}$-algebra. In the case where $\mathcal{D}$ is an $E_\infty$ operad, we see that the graded Thom spectrum functor takes $E_\infty$ spaces over $\text{Gr}_{h\mathcal{V}}$ to $E_\infty$ orthogonal ring spectra over $\text{MOP}$.

### 1.2. Orientations and the graded Thom isomorphism.

The theory of orientations also has an attractive formulation in the setting of $W$-spaces. For a commutative orthogonal ring spectrum $R$ there is an associated commutative $W$-space monoid $\text{GL}_1^W(R)$ of graded $W$-space units. This is the $W$-space analogue of the corresponding construction for symmetric spectra introduced by the authors in [13]. In the case of the Eilenberg-Mac Lane spectrum $HZ/2$, the $W$-space units $\text{GL}_1^W(HZ/2)$ is equivalent to the 0th part $V_{(0)}$ of the Stiefel $W$-space, whereas for $HZ$, the graded $W$-space units $\text{GL}_1^W(HZ)$ is equivalent to the 0th part $\tilde{V}_{(0)}$ of the oriented Stiefel $W$-space. For a general orthogonal ring spectrum $R$, an $R$-orientation of a $W$-space is simply encoded by a map to $\text{GL}_1^W(R)$. Restricted to the degree 0 part, this approach to orientations reduces to the orientation theory developed in [13] as follows from the fibration sequence

$$\text{GL}_1^W(R)_{hW_{(0)}} \to BO \to B\text{GL}_1(R),$$

where $\text{GL}_1(R)$ denotes the traditional (ungraded) grouplike monoid of units. For an $E_\infty$ ring spectrum, Ando et. al. [1] construct $A_\infty$ and $E_\infty$ $R$-algebra Thom spectra associated to spaces over $B\text{GL}_1(R)$. Using the graded units of ring spectra...
introduced by the authors, it is possible to give a unified treatment that includes both Thom spectra for spaces over $B\text{GL}_1(R)$ and the present approach to graded Thom spectra. We plan to return to this in a separate paper.

In the present paper we find it convenient to work with a very general notion of orientations relative to a grouplike commutative $W$-space monoid $W$. This then leads to the following formulation of the corresponding graded Thom isomorphism:

\begin{equation*}
\text{We consider a commutative orthogonal ring spectrum } R \text{ which is an algebra over the degree 0 part } S^W[W_0] \text{. In this situation there is an associated commutative periodic ring spectrum } RP \simeq \bigvee_{n \in \mathbb{Z}} \Sigma^n R, \text{ where } d \text{ is a generator for the image of } \pi_0(W_{hW}) \text{ in } \pi_0(BW) \cong \mathbb{Z}. \text{ The following theorem is then obtained by combining Theorems 7.5 and 7.9.}
\end{equation*}

**Theorem 1.3.** For a $W$-space space $X$ equipped with a $W$-orientation, there is a stable equivalence of $R$-module spectra $R \wedge S^W[X] \simeq RP \triangleright X_{hW}$. This is a stable equivalence of $E_\infty$-$R$-algebras provided that $X$ is a commutative $W$-space monoid equipped with a multiplicative orientation.

Here $RP \triangleright X_{hW}$ denotes an $R$-module (respectively an $E_\infty$-$R$-algebra) whose underlying orthogonal spectrum is stably equivalent to $\bigvee_{n \in \mathbb{Z}} \Sigma^n R \wedge (X_{hW(n)^+})$. More generally, we establish a multiplicative version of the graded Thom isomorphism for any operad augmented over the Barratt-Eccles operad (see Section 7.6).

**Corollary 1.4.** For a commutative $W$-space monoid $X$ equipped with a multiplicative $W$-orientation, there is an isomorphism of graded rings $R_*(S^W[X]) \cong \bigoplus_{n \in \mathbb{Z}} \Sigma^n R_*(X_{hW(n)})$.

As a special case of this, a multiplicative orientation of a commutative $W$-space monoid $X$ with respect to the evenly graded Stiefel $W$-space $\tilde{V}_{ev}$ gives rise to an evenly graded Thom isomorphism

\begin{equation*}
H_*(S^W[X], \mathbb{Z}) \cong \bigoplus_{n \in \mathbb{Z}} \Sigma^{2n} H_*(X_{hW(2n)}, \mathbb{Z}).
\end{equation*}

Having developed the orientation theory for $W$-spaces, we extend this to spaces over $Gr_{hW}$ via the Quillen equivalences in (1.1). Combining Theorems 1.1 and 1.5 we then get multiplicative Thom isomorphisms for graded Thom spectra (see Section 7.13 for details).

1.5. Applications to logarithmic structures on commutative ring spectra. The theory of $W$-spaces and graded Thom spectra developed in this paper is motivated in part by the applications to the analysis of logarithmic structures on commutative ring spectra and the corresponding logarithmic topological Hochschild homology introduced by the authors in joint work with John Rognes [15, 16]. We discuss these applications in Section 8 which also contains an introduction to this circle of ideas.

2. Categories of linear isometries

We first review some general material on topological categories before turning to the relevant categories of linear isometries.

2.1. Topological categories. Let $\text{Top}$ denote the category of compactly generated weak Hausdorff topological spaces. For us a topological category will mean a (not necessarily small) category enriched in $\text{Top}$. This means that the morphism sets are topologized so as to be objects in $\text{Top}$ and that composition is continuous. For a small topological category $K$, a $K$-space is by definition a continuous functor $X: K \to \text{Top}$, and we write $\text{Top}^K$ for the topological category of $K$-spaces.
In the following we briefly review some basic constructions on $K$-spaces, referring the reader to [5] for more details. Given a $K$-space $X$ and a $K^{op}$-space $Y$, the bar construction $B(Y; K, X)$ is defined as the realization of the simplicial space with $p$-simplices

$$B_p(Y; K, X) = \prod_{k_0, \ldots, k_p} Y(k_0) \times K(k_1, k_0) \times \cdots \times K(k_p, k_{p-1}) \times X(k_p)$$

where the coproduct is over all $(p+1)$-tuples of objects in $K$. The simplicial structure maps are of the usual bar construction type as detailed in [5, Section 3].

Let $F: K \to L$ be a continuous functor between small topological categories. Given a $K$-space $X$, the homotopy left Kan extension along $F$ is by definition the $L$-space $F^h(X)$ defined by

$$F^h(X)(l) = B(L(F(-), l), K, X)$$

for $l$ an object in $L$ and $L(F(-), l)$ the associated $K^{op}$-space. As a special case of this construction, the bar resolution $\overline{X}$ of a $K$-space $X$ is the homotopy left Kan extension along the identity functor on $K$,

$$\overline{X}(k) = B(K(−, k), K, X).$$

There is canonical “evaluation” map of $K$-spaces $\epsilon: \overline{X} \to X$ which is a level-wise equivalence, cf. [3 Proposition 3.1]. Another special case is the homotopy colimit of a $K$-space $X$ defined by

$$\text{hocolim}_K X = B(*, K, X)$$

where $*$ denotes the terminal $K$-space. In particular, the homotopy colimit of $*$ is the classifying space $BK = B(*, K, *)$. It follows from the definitions that there is a natural homeomorphism $\text{colim}_K \overline{X} \cong \text{hocolim}_K X$. For convenience we often write $X_hK$ instead of $\text{hocolim}_K X$.

2.2. The $\boxtimes$-product on $\text{Top}^K$. We say that a topological category $K$ is permutative if it has a continuous symmetric strict monoidal product $\boxplus$ with strict unit $0$. A permutative structure on $K$ induces a symmetric monoidal convolution product on $\text{Top}^K$: Given $K$-spaces $X$ and $Y$, we define $X \boxtimes Y$ to be the left Kan extension of the product diagram

$$X \times Y: K \times K \xrightarrow{X \times Y} \text{Top} \times \text{Top} \xrightarrow{h} \text{Top}$$

along $\boxplus: K \times K \to K$. Thus, with the $\otimes$-notation from [4], we have

$$(X \boxtimes Y)(k) = K(− \boxplus −, k) \otimes_{K \times K} (X \times Y),$$

also known as the coend of the evident $(K \times K)^{op} \times (K \times K)$-diagram. By the universal property of the left Kan extension, a map of $K$-spaces $X \boxtimes Y \to Z$ amounts to a natural transformation of $K \times K$-diagrams

$$X(h) \times Y(k) \to Z(h \boxplus k).$$

We write $UK = K(0, −)$ for the $K$-space defined by the monoidal unit for $K$. The following proposition is proved by arguments that are by now quite standard (see e.g. [2] or the discussion in [10] of the analogous situation for based topological index categories).

Proposition 2.3. The $\boxtimes$-product makes $\text{Top}^K$ a closed symmetric monoidal category with $UK$ as monoidal unit. \hfill $\Box$

We use the term $K$-space monoid for a monoid in $\text{Top}^K$ with respect to the $\boxtimes$-product. The next lemma is a formal consequence of the universal properties of the free $K$-spaces $F^h_K(K) = K(k, −) \times K$ and the $\boxtimes$-product.
Lemma 2.4. Given a pair of spaces $K$ and $L$, and a pair of objects $h$ and $k$ in $K$, there is a natural isomorphism
\[
F^K_h(K) \otimes F^K_k(L) \cong F^K_{h \otimes k}(K \times L).
\]

2.5. Categories of linear isometries. Let $V$ be the topological category with objects the standard inner product spaces $\mathbb{R}^n$ for $n \geq 0$ and morphisms the (not necessarily surjective) isometries. The canonical identification of $\mathbb{R}^n \oplus \mathbb{R}^n$ with $\mathbb{R}^{m+n}$ makes $V$ a permutative topological category with symmetry isomorphisms $\chi_{m,n} : \mathbb{R}^m \oplus \mathbb{R}^n \to \mathbb{R}^n \oplus \mathbb{R}^m$ defined by $\chi_{m,n}(u,v) = (v,u)$ for $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$. It is convenient to identify the objects of $V$ with the natural numbers $n \geq 0$. Thus, we shall write $V(n)$ instead of $V(\mathbb{R}^n, \mathbb{R}^n)$ and $m \oplus n$ instead of $\mathbb{R}^m \oplus \mathbb{R}^n$. With this notation, the orthogonal group $O(n)$ is the group of endomorphisms $V(n,n)$ of $n$. We write $O$ for the subcategory of isometric isomorphisms in $V$.

Definition 2.6. Let $W = O^{-1}O$ be Quillen’s localization construction applied to $O$. In detail, we specify that the objects of $W$ be pairs of natural numbers $(m_1, n_2)$. The morphism space $W((m_1, m_2), (n_1, n_2))$ is non-empty if and only if there exists a natural number $m$ such that $m_1 + m = n_1$ and $m_2 + m = n_2$. In this case
\[
W((m_1, m_2), (n_1, n_2)) = (O(m_1 \oplus m, n_1) \times O(m_2 \oplus m, n_2))/O(m)
\]
where the orthogonal group $O(m)$ acts diagonally from the right via the inclusions $O(m) \to O(m_i \oplus m)$ for $i = 1, 2$, extending an isometry of $\mathbb{R}^m$ by the identity on $\mathbb{R}^{m_i}$. The elements in this morphism space are written in the form $[m, \sigma_1, \sigma_2]$ for $\sigma_1$ in $O(m_1)$ and $\sigma_2$ in $O(m_2)$. With this notation composition is given by
\[
[m, \tau_1, \tau_2] \circ [m, \sigma_1, \sigma_2] = [m \oplus n, \tau_1 \circ (\sigma_1 \oplus id_{\mathbb{R}^n}), \tau_2 \circ (\sigma_2 \oplus id_{\mathbb{R}^n})].
\]

The next lemma gives a geometric interpretation of the morphisms spaces in $W$. Here we write $\mathbb{R}^n \oplus V$ for the orthogonal complement of a linear subspace $V$ in $\mathbb{R}^n$.

Lemma 2.7. Suppose that $m_1 + m = n_1$ and $m_2 + m = n_2$. Then the map
\[
W((m_1, m_2), (n_1, n_2)) \to V(m_1, n_1) \times V(m_2, n_2), \quad [m, \sigma_1, \sigma_2] \mapsto ([m, \sigma_1] \oplus [m_2, \sigma_2])|_{\mathbb{R}^{m_1} \oplus \alpha_1|_{\mathbb{R}^{m_1}}}, \quad [m, \sigma_1, \sigma_2] \mapsto ([m_2, \sigma_2] \oplus \alpha_2|_{\mathbb{R}^{m_2}})
\]
is a fiber bundle whose fiber over an element $(\alpha_1, \alpha_2)$ can be identified with the space of isometric isomorphisms between $\mathbb{R}^{m_1} \oplus \alpha_1(\mathbb{R}^{m_1})$ and $\mathbb{R}^{m_2} \oplus \alpha_2(\mathbb{R}^{m_2})$.

The permutative structure of $W$ is defined on objects by
\[
(m_1, m_2) \oplus (n_1, n_2) = (m_1 \oplus n_1, m_2 \oplus n_2).
\]

For a pair of morphisms
\[
[m, \sigma_1, \sigma_2] : (m_1, m_2) \to (m'_1, m'_2), \quad [n_1, n_2] \to (n'_1, n'_2),
\]
we define $[n_1, n_2, \sigma_1, \sigma_2] \oplus [m_1, m_2]$ to be the morphism
\[
[m \oplus n, (\sigma_1 \oplus \tau_1) \circ (id_{\mathbb{R}^{m_1}} \oplus \chi_{m_1, m} \oplus id_{\mathbb{R}^n}), (\sigma_2 \oplus \tau_2) \circ (id_{\mathbb{R}^{m_2}} \oplus \chi_{m_2, m} \oplus id_{\mathbb{R}^n})].
\]
The object $(0,0)$ is a strict unit for this product and we have the coordinate-wise symmetry isomorphism on $W$ defined by
\[
[\chi_{m_1, n_1}, \chi_{m_2, n_2}] : (m_1, m_2) \oplus (n_1, n_2) \to (n_1, n_2) \oplus (m_1, m_2).
\]

We define the degree of an object $(n_1, n_2)$ to be the integer $n_2 - n_1$ and write $W(d)$ for the full subcategory of $W$ with objects of degree $d$. Thus, $W$ decomposes as a coproduct of the categories $W(d)$ for $d \in \mathbb{Z}$ and the permutative structure restricts to functors $\oplus : W(d) \times W(e) \to W(d+e)$.
2.8. The Grothendieck construction. It will be convenient to have a description of the 0-component \( W_{(0)} \) in terms of a certain Grothendieck construction (in the sense of Thomason [24]). Let us write \( \text{TopCut} \) for the (topological) category of small topological categories. Consider the functor \( O : \mathcal{V} \to \text{TopCut} \) that takes \( \mathbb{R}^n \) to \( O(n) \), thought of as a topological category with a single object, and that takes an isometry \( \alpha : \mathbb{R}^m \to \mathbb{R}^n \) to the functor (that is, group homomorphism) \( O_{\alpha} : O(m) \to O(n) \) defined as follows: an element \( a \in O(m) \) is mapped to the element \( O_{\alpha}(a) \in O(n) \) determined by the commutativity of the diagram

\[
\begin{array}{ccc}
\mathbb{R}^n & \xrightarrow{O_{\alpha}(a)} & \mathbb{R}^n \\
{\{a, \text{inc}\}} & & {\{a, \text{inc}\}} \\
\mathbb{R}^m \oplus (\mathbb{R}^n \ominus \alpha(\mathbb{R}^m)) & \xrightarrow{\alpha \oplus \text{id}} & \mathbb{R}^m \oplus (\mathbb{R}^n \ominus \alpha(\mathbb{R}^m)).
\end{array}
\]

Here \( \mathbb{R}^n \ominus \alpha(\mathbb{R}^m) \) again denotes the orthogonal complement of \( \alpha(\mathbb{R}^m) \) and \( \text{inc} \) is the inclusion. The functor \( O \) gives rise to the topological category \( \mathcal{V} \mathcal{O} \), the Grothendieck construction, with objects the natural numbers \( n \geq 0 \) and morphisms \((a, \alpha) : m \to n\) given by a linear isometry \( \alpha : \mathbb{R}^m \to \mathbb{R}^n \) in \( \mathcal{V} \) and an element \( a \in O(n) \). Composition of morphisms is defined by

\[
(b, \beta) \circ (a, \alpha) = (b \cdot O_{\beta}(a), \beta \alpha).
\]

The direct sum on objects and morphisms inherited from \( \mathcal{V} \) makes \( \mathcal{V} \mathcal{O} \) a permutative category with symmetry isomorphism \((\text{id}_{\mathbb{R}^{m+n}}, \chi_{m,n}) : m \oplus n \to n \oplus m\) induced by the symmetry isomorphism \( \chi_{m,n} \) from \( \mathcal{V} \).

Lemma 2.9. The categories \( \mathcal{V} \mathcal{O} \) and \( W_{(0)} \) are isomorphic as permutative topological categories.

Proof. We appeal to the universal property of \( \mathcal{V} \mathcal{O} \) (see [24 Proposition 1.3.1]) for the construction of a functor \( \mathcal{V} \mathcal{O} \to W_{(0)} \). Identifying \( W((n, n), (n, n)) \) with \( O(n) \times O(n) \) we assign to each \( n \geq 0 \) the object \((n, n)\) in \( W_{(0)} \) with \( O(n) \)-action in the second variable. The action on morphisms is given by the diagonal inclusion

\[
\mathcal{V}(m, n) \cong O(m \oplus l, n)/O(l) \to (O(m \oplus l, n) \times O(m \oplus l, n))/O(l).
\]

It is easy to check that these data satisfy the requirements for defining a functor and that the induced maps of morphism spaces are homeomorphisms. The inverse functor takes a morphism \([l, \sigma_1, \sigma_2] : (m, m) \to (n, n)\) to \((\sigma_2 \sigma_1^{-1}, \sigma_1|_{\mathbb{R}^m})\). \( \square \)

Introducing the Grothendieck construction \( \mathcal{V} \mathcal{O} \) in this context allows us to invoke Thomason’s homotopy colimit theorem as in the next lemma. Let \( \mathcal{N} \) be the ordered set of natural numbers and define a functor \( \mathcal{N} \to \mathcal{V} \) by mapping \( n \) to \( \mathbb{R}^n \) and \( n \to n + 1 \) to the inclusion \( \mathbb{R}^n \oplus \mathbb{R}^0 \to \mathbb{R}^n \oplus \mathbb{R} \).

Proposition 2.10. For a \( \mathcal{V} \mathcal{O} \)-space \( X \) there are natural weak homotopy equivalences

\[
\text{hocolim}_{n \in \mathcal{N}} X(n)_{hO(n)} \xrightarrow{\sim} \text{hocolim}_{n \in \mathcal{V}} X(n)_{hO(n)} \xrightarrow{\sim} \text{hocolim}_{\mathcal{V} \mathcal{O}} X
\]

where \( X(n)_{hO(n)} \) denotes the homotopy orbits of the left \( O(n) \)-space \( X(n) \).

Proof. The first map is induced by the functor \( \mathcal{N} \to \mathcal{V} \) and is a weak homotopy equivalence by [3 Lemma 7.3]. The second map is defined in analogy with the map considered by Thomason [23 Lemma 1.2.1]. It is a weak equivalence by a slight generalisation of the argument used in the proof of [24 Theorem 1.2] (see also [5 Proposition 6.2] and the dual argument in [20 Theorem 2.3]). \( \square \)
Applied to the terminal $V/O$-space, the above proposition determines the homotopy type of $B(V/O)$ and hence of $BW(0)$,

$$\text{hocolim}_{n \in \mathbb{N}} BO(n) \cong \text{hocolim}_{n \in V} BO(n) \cong B(V/O) \cong BW(0).$$

It follows that $BW$ is a model of $BO \times \mathbb{Z}$, the classifying space for $KO$-theory, which was first observed by Thomason based on the analysis in [3].

3. The homotopy theory of $W$-spaces

In this section we set up the basic homotopy theory of $W$-spaces. This is analogous to the homotopy theory of $J$-spaces considered in [18]. The corresponding homotopy theory of $V$-spaces has been worked out by Lind [8]. Much of this material is quite standard and we only cover the minimum needed for the rest of the paper.

3.1. The $W$-model structure. We first consider the level model structure on $Top^W$ and say that a map of $W$-spaces $X \to Y$ is a level equivalence (respectively a level fibration) if $X(n_1, n_2) \to Y(n_1, n_2)$ is a weak homotopy equivalence (respectively a fibration) for all objects $(n_1, n_2)$ in $W$. We say that $X \to Y$ is a cofibration if it has the left lifting property with respect to maps of $W$-spaces that are both level equivalences and level fibrations. Consider for each object $(d_1, d_2)$ the pair of adjoint functors

$$F^W_{(d_1, d_2)}: Top \rightleftarrows Top^W: Ev_{(d_1, d_2)}$$

where the evaluation functor $Ev_{(d_1, d_2)}$ takes a $W$-space $X$ to $X(d_1, d_2)$ and the action of the left adjoint $F^W_{(d_1, d_2)}$ on a space $K$ is given by

$$F^W_{(d_1, d_2)}(K) = W((d_1, d_2), -) \times K.$$

Let $I$ be the standard set of generating cofibrations for $Top$ of the form $S^{n-1} \to D^n$ for $n \geq 0$, and let $J$ be the standard set of generating acyclic cofibrations of the form $D^n \to D^n \times I$ for $n \geq 0$. We let $F J$ (respectively $F I$) denote the set of maps in $Top^W$ of the form $F^W_{(d_1, d_2)}(i)$ for $i$ an element in $I$ (respectively $J$). The following result is standard (see for instance the analogous results for discrete index categories in [11 Theorem 11.6.1] and based topological index categories in [10 Theorem 6.5]).

Proposition 3.2. The level equivalences, level fibrations, and cofibrations specify a cofibrantly generated model structure on $Top^W$ with generating cofibrations $FI$ and generating acyclic cofibrations $FJ$.

We shall be more interested in a certain localization of the level model structure and say that a map of $W$-spaces $X \to Y$ is a

- $W$-equivalence if the induced map of homotopy colimits $XhW \to YhW$ is a weak homotopy equivalence,
- $W$-fibration if it is a level fibration and the diagram

$$\begin{array}{ccc}
X(m_1, m_2) & \longrightarrow & X(n_1, n_2) \\
\downarrow & & \downarrow \\
Y(m_1, m_2) & \longrightarrow & Y(n_1, n_2)
\end{array}$$

is homotopy cartesian for any morphism $(m_1, m_2) \to (n_1, n_2)$ in $W$,

- $W$-cofibration if it is a cofibration in the level model structure.

Proposition 3.3. The $W$-equivalences, $W$-fibrations, and $W$-cofibrations specify a cofibrantly generated proper model structure on $Top^W$. 

We shall refer to this as the $\mathcal{W}$-model structure on $\text{Top}^\mathcal{W}$. The $\mathcal{W}$-cofibrations will usually be referred to simply as cofibrations. The proof of the proposition will be based on the following lemma. It is a variant of [14, Proposition 4.4] that applies to continuous functors $\mathcal{W} \to \text{Top}$ as opposed to functors defined only on discrete indexing categories.

Lemma 3.4. Let $X \to Y$ be a map in $\text{Top}^\mathcal{W}$ such that the left hand square in

\[
\begin{array}{ccc}
X(m_1, m_2) & \to & Y(m_1, m_2) \\
\downarrow & & \downarrow \\
X(n_1, n_2) & \to & Y(n_1, n_2)
\end{array}
\]

is homotopy cartesian for every morphism $[m, \sigma_1, \sigma_2]: (m_1, m_2) \to (n_1, n_2)$ in $\mathcal{W}$. Then the right hand square is homotopy cartesian for every object $(m_1, m_2)$ in $\mathcal{W}$.

Proof. We may assume without loss of generality that $(m_1, m_2)$ has degree 0, since for any object $(m_1, m_2)$, restricting $X \to Y$ along the homotopy cofinal functor $(m_2, m_1) \oplus (-): \mathcal{W} \to \mathcal{W}$ reduces the statement to the degree 0 case. Next we use the identification of $\mathcal{W}_{[0]}$ with the Grothendieck construction $\mathcal{V}/\mathcal{O}$ from [2, 2.10] and view $X$ and $Y$ as $\mathcal{V}/\mathcal{O}$-diagrams. By Proposition 2.10 it now suffices to show that the outer square in the commutative diagram

\[
\begin{array}{ccc}
X(m, m) & \to & \text{hocolim}_{m \in N} X(m, m)_{h\mathcal{O}(m)} \\
\downarrow & & \downarrow \\
Y(m, m) & \to & \text{hocolim}_{m \in N} Y(m, m)_{h\mathcal{O}(m)}
\end{array}
\]

is homotopy cartesian for every $m \geq 0$. Here the left hand square is homotopy cartesian by [14, Theorem 7.6], whereas the right hand square is homotopy cartesian by the version of the lemma for discrete indexing categories, see [14, Proposition 4.4] and [13, Lemma 6.12 and Remark 6.13].

Proof of Proposition 3.3. This is essentially [13, Proposition 6.16] with $\mathcal{K} = \mathcal{W}$ and $\mathcal{A}$ being the subcategory of all identity morphisms in $\mathcal{W}$. The only change, forced by the topological enrichment of $\mathcal{W}$, is that Lemma 3.4 replaces [13, Lemma 6.12]. Properness follows as in [13, §11], where in the proof of [13, Proposition 11.3] we again have to replace [13, Lemma 6.12] by Lemma 3.4.

The following lemma justifies thinking of the homotopy colimit functor as a derived version of the colimit functor.

Lemma 3.5. For a cofibrant $\mathcal{W}$-space $X$, the canonical map $X_{h\mathcal{W}} \to \text{colim}_{\mathcal{W}} X$ is a weak equivalence.

Proof. With the notation from [3, §2], the map in question takes the form

\[
B(*)_{\mathcal{W}} \to \text{colim}_{\mathcal{W}} \to X.
\]

An adjunction argument as in the proof of [3, Theorem 18.4.1] shows that $\text{colim}_{\mathcal{W}}$ sends level equivalences between cofibrant objects to weak equivalences. The claim now follows by [3, 3.1 Proposition (5)], using Lemma 3.7 below.

The next lemma provides an important source for cofibrant $\mathcal{W}$-spaces.

Lemma 3.6. Let $\mathcal{K}$ be a small topological category and let $F: \mathcal{K} \to \mathcal{W}$ be a continuous functor. Assume that the morphism spaces in $\mathcal{K}$ are cofibrant and that the inclusions of the identity morphisms are cofibrations. Let $X$ be a $\mathcal{K}$-space such that $X(k)$ is cofibrant for all objects $k$ in $\mathcal{K}$. Then the homotopy left Kan extension $F_k^i(X)$ is a cofibrant $\mathcal{W}$-space.
Proof. Consider in general a simplicial $\mathcal{W}$-space $Z_*$ and observe that the geometric realization $Z = |Z_*|$ admits a filtration by $\mathcal{W}$-spaces $Z^{(n)}$ for $n \geq 0$, such that $Z^{(0)} = Z_0$, $\operatorname{colim}_n Z^{(n)} = Z$, and there are pushout diagrams

$$SZ_{n-1} \times \Delta^n \cup_{SZ_{n-1} \times \partial \Delta^n} Z_n \times \partial \Delta^n \longrightarrow Z_n \times \Delta^n$$

where $SZ_{n-1} \subseteq Z_n$ is the union of the degenerate sub $\mathcal{W}$-spaces $s_i(Z_{n-1})$ for $0 \leq i \leq n$. Thus, we see that the $\mathcal{W}$-space $Z$ is cofibrant provided that $Z_0$ is cofibrant and the maps $SZ_{n-1} \to Z_n$ are cofibrations. In the case at hand, $F^\mathcal{W}_0(X)$ is the geometric realization of a simplicial $\mathcal{W}$-space with $p$-simplices

$$\coprod_{k_0, \ldots, k_p} F^\mathcal{W}_{F(k_0)}(K(k_1, k_0) \times \cdots \times K(k_p, k_{p-1}) \times X(k_p)),$$

the coproduct being over all $(p+1)$-tuples $k_0, \ldots, k_p$ of objects in $\mathcal{K}$. Since the functors $F^\mathcal{W}_{F(k_0)}$ are left Quillen functors, the assumptions in the lemma ensure that this is a cofibrant $\mathcal{W}$-space for all $p$ and in particular for $p = 0$. Furthermore, the inclusions of the degenerate sub $\mathcal{W}$-spaces are obtained by applying the functors $F^\mathcal{W}_{F(k_0)}$ to the evident cofibrations in $\mathcal{Top}$, hence are cofibrations of $\mathcal{W}$-spaces. \qed

This lemma applies in particular to the identity functor on $\mathcal{W}$:

**Lemma 3.7.** Let $X$ be a $\mathcal{W}$-space and suppose that $X(n_1, n_2)$ is cofibrant as an object in $\mathcal{Top}$ for all $(n_1, n_2)$. Then the bar resolution $\overline{X}$ is a cofibrant $\mathcal{W}$-space. \qed

3.8. **The $\boxtimes$-product on $\mathcal{Top}^\mathcal{W}$.** Specializing the general theory in Section 2.2 to the permutative structure of $\mathcal{W}$, we get a symmetric monoidal product $\boxtimes$ on $\mathcal{Top}^\mathcal{W}$ with $U^\mathcal{W}$ as monoidal unit.

**Proposition 3.9.** If $X$ is a cofibrant $\mathcal{W}$-space, then the functor $X \boxtimes (-)$ preserves $\mathcal{W}$-equivalences.

**Proof.** Most parts of this argument are analogous to the corresponding statements for discrete indexing categories in [18, Proposition 8.5]. In particular, a cell reduction argument reduces the claim to the case where $X = F^\mathcal{W}_{(k_1, k_2)}(L)$. If $Y$ is a $\mathcal{W}$-spaces, an inspection of the coequalizer defining

$$\text{hocolim}(L \times \text{colim} Y) \cong Y((k_1, k_2) \otimes (l_1, l_2), (n_1, n_2))$$

if there exists an object $(l_1, l_2)$ with $(k_1, k_2) \otimes (l_1, l_2) = (n_1, n_2)$. If no such object exists, $(F^\mathcal{W}_{(k_1, k_2)}(L) \boxtimes Y) \cong Y((k_1, k_2) \otimes (l_1, l_2), (n_1, n_2))$ is a free $\mathcal{W}((l_1, l_2), (l_1, l_2))$-CW complex, it follows that $F^\mathcal{W}_{(k_1, k_2)}(L) \boxtimes (-)$ preserves level equivalences of $\mathcal{W}$-spaces. This implies that the bar resolution $\overline{Y} \to Y$ induces a level equivalence $F^\mathcal{W}_{(k_1, k_2)}(L) \boxtimes \overline{Y} \to F^\mathcal{W}_{(k_1, k_2)}(L) \boxtimes Y$. Writing $F = (k_1, k_2) \otimes (-) : \mathcal{W} \to \mathcal{W}$, we can identify $F^\mathcal{W}_{(k_1, k_2)}(-) \boxtimes (-)$ with the left Kan extension along $F$. Moreover, the isomorphism in [3] 3.1 Proposition[i] shows that the $\mathcal{W}$-space $F^\mathcal{W}_{(k_1, k_2)}(L) \boxtimes \overline{Y}$ is isomorphic to $L \times F^\mathcal{W}_h(Y)$, the product of $L$ with the homotopy left Kan extension of $Y$ along $F$. Since there also is a natural weak equivalence

$$\text{hocolim}(L \times F^\mathcal{W}_h(Y)) \cong L \times \text{hocolim} Y$$

it follows that $F^\mathcal{W}_{(k_1, k_2)}(-) \boxtimes (-)$ preserves $\mathcal{W}$-equivalences as claimed. \qed

**Corollary 3.10.** The $\mathcal{W}$-model structure satisfies the pushout-product axiom and the monoid axiom.
Proof. This is analogous to [18] Propositions 8.4 and 8.6, with the previous proposition replacing [18] Proposition 8.2.

We shall view the homotopy colimit functor \((-)_{hW}: \text{Top}^W \to \text{Top}\) as a lax monoidal functor with monoidal product defined by the natural map

\[
X_{hW} \times Y_{hW} \cong (X \times Y)_{h(W \times W)} \to (X \boxtimes Y \circ \otimes)_{h(W \times W)} \to (X \boxtimes Y)_{hW}
\]

Arguing as in the case of \(I\)-spaces [19] Lemma 2.25] we get the next result.

**Lemma 3.11.** The natural map \(X_{hW} \times Y_{hW} \to (X \boxtimes Y)_{hW}\) is a weak homotopy equivalence provided that either \(X\) or \(Y\) is cofibrant. □

We say that a map of \(W\)-spaces \(X \to Y\) is a positive \(W\)-fibration if the map \(X(n_1, n_2) \to Y(n_1, n_2)\) is a fibration for every object \((n_1, n_2)\) with \(n_1 \geq 1\), and if the square \((\ref{3.1})\) is homotopy cartesian for every morphism \((m_1, m_2) \to (n_1, n_2)\) with \(m_1 \geq 1\). Using Proposition \((\ref{3.2})\) one can show that the positive \(W\)-fibrations and the \(W\)-equivalences participate in a cofibrantly generated proper model structure which we shall refer to as the positive \(W\)-model structure. The cofibrations in this model structure are called positive cofibrations.

Since every positive cofibration is a cofibration in the (absolute) \(W\)-model structure of Proposition \((\ref{3.3})\), the statements of Proposition \((\ref{3.9})\) and Corollary \((\ref{3.10})\) also hold for the positive \(W\)-model structure. The relevance of the positive \(W\)-model structure comes from the following result.

**Theorem 3.12.** The category of commutative \(W\)-space monoids admits a positive \(W\)-model structure where a map is a fibration or a weak equivalence if and only if the underlying map of \(W\)-spaces is a positive fibration or \(W\)-equivalence.

**Proof.** Using the homotopy invariance of the \(\boxtimes\)-product from Proposition \((\ref{3.9})\), the construction of this model structure is completely analogous to its discrete counterpart in [18] Lemma 9.5. The only difference is that we here use the isomorphism \((\ref{3.2})\) considered in the proof of Proposition \((\ref{3.3})\) to see that the \(\Sigma_i\)-action on \(F^W_{(k_1, k_2)\mathcal{O} + \star}\) is free if \(k_1 \geq 1\). □

4. \(W\)-spaces and graded orthogonal spectra

In this section we set up the adjunction relating \(W\)-spaces to orthogonal spectra. This is the orthogonal analogue of the adjunction in [18] relating \(J\)-spaces to symmetric spectra. We write \(S^n\) for the one-point compactification of \(\mathbb{R}^n\) throughout.

4.1. Orthogonal spectra. Following [10], an orthogonal spectrum \(E\) is a sequence of based spaces \(E_n\) for \(n \geq 0\), together with a based left \(O(n)\)-action on \(E_n\) and a family of based structure maps \(E_n \wedge S^1 \to E_{n+1}\) for \(n \geq 0\) such that the iterated structure maps \(E_m \wedge S^n \to E_{m+n}\) are \((O(m) \times O(n))\)-equivariant. Here \(O(m) \times O(n)\) acts on \(E_m \wedge S^n\) via the monoidal structure map \(O(m) \times O(n) \to O(m+n)\). A map of orthogonal spectra \(E \to E'\) is a family of \(O(n)\)-equivariant based maps \(E_n \to E'_n\) that are compatible with the structure maps. We use the notation \(Sp^O\) for the (topological) category of orthogonal spectra and equip this with the stable model structure established in [10]. The sphere spectrum \(S\) is the orthogonal spectrum with \(n\)th space \(S^n\) and the obvious structure maps. It is proved in [10] that there is a smash product \(\wedge\) of orthogonal spectra making \(Sp^O\) a closed symmetric monoidal category with \(S\) as its monoidal unit.

The category \(Sp^O\) is related to the category of based spaces \(Top_*\) by a pair of adjoint functors \(F_d: Top_* \rightleftarrows Sp^O: Ev_d\) for each natural number \(d\). The evaluation
functor $E v_d$ takes an orthogonal spectrum to its $d$th space whereas the action of the left adjoint $F_d$ on a based space $K$ can be described explicitly by

$$F_d(K)_n = \mathcal{O}(d \oplus (n - d), n)_+ \wedge_{\mathcal{O}(n-d)} K \wedge S^{n-d}$$

if $n \geq d$ and $F_d(K)_n = *$ otherwise. (As usual the notation $(-)_+$ indicates the addition of a disjoint base point.) Here we think of $\mathbb{R}^{n-d}$ as the orthogonal complement of $\mathbb{R}^d$ included as the first $d$ coordinates in $\mathbb{R}^n$. The structure maps are defined by

$$F_d(K)_n \wedge S^1 \to F_d(K)_{n+1}, \quad [a, x, u, t] \mapsto [a \oplus \text{id}_{\mathbb{R}^t}, x, (u, t)]$$

for $a \in \mathcal{O}(d \oplus (n - d), n), x \in K, u \in S^{n-d}$, and $t \in S^1$. For $d = 0$ this construction gives the orthogonal suspension spectrum $F_0(K)$ with $n$th space $K \wedge S^n$. It follows from [10] Lemma 1.8 that the canonical family of maps

$$F_d(K)_m \wedge F_e(L)_n \to F_{d+e}(K \wedge L)_{m+n},$$

$$([a, x, u], [b, y, v]) \mapsto [a \oplus b \cdot \text{id}_{\mathbb{R}^e} \oplus \chi_e, m-d \oplus \text{id}_{\mathbb{R}^e}, (x, y), (u, v)]$$

induces an isomorphism of orthogonal spectra

$$(4.1) \quad F_d(K) \wedge F_e(L) \cong F_{d+e}(K \wedge L).$$

4.2. The adjunction to $W$-spaces. In order to set up the adjunction relating $W$-spaces to orthogonal spectra we first define a functor $W^{op} \to Sp^O$. On objects this is defined by mapping $(n_1, n_2)$ to $F_{n_1}(S^{n_2})$. Specifying the effect on morphism spaces

$$W((m_1, m_2), (n_1, n_2)) \to Sp^O(F_{n_1}(S^{n_2}), F_{m_1}(S^{m_2}))$$

amounts by adjointness to specifying a family of based maps

$$(4.2) \quad W((m_1, m_2), (n_1, n_2))_+ \wedge S^{n_2} \to F_{n_1}(S^{m_2})_{n_1}.$$  

Given a natural number $m$ such that $m_1 + m = n_1$ and $m_2 + m = n_2$, there is an $O(m)$-equivariant map

$$\left(\mathcal{O}(m_1 \oplus m, n_1) \times \mathcal{O}(m_2 \oplus m, n_2)\right)_+ \wedge S^{n_2} \to \mathcal{O}(m_1 \oplus m, n_1)_+ \wedge S^{m_2} \wedge S^{n_2}$$

defined by $(\sigma_1, \sigma_2, w) \mapsto ((\sigma_1, \sigma_2^{-1}(w))$. We define the map (4.2) to be the induced map of $O(m)$-orbit spaces. The following lemma is the orthogonal analogue of the corresponding result for symmetric spectra proved in [18] Section 4.21.

**Lemma 4.3.** The structure maps defined above make

$$F_-(S^-) : W^{op} \to Sp^O, \quad (n_1, n_2) \mapsto F_{n_1}(S^{n_2})$$

a strong symmetric monoidal functor.

**Proof.** Using the explicit description of the orthogonal spectra $F_{n_1}(S^{n_2})$, one checks that this construction indeed defines a continuous functor on $W^{op}$. The symmetric monoidal structure is given by the canonical isomorphism $S \to F_0(S^0)$ and the natural isomorphisms

$$F_{m_1}(S^{m_2}) \wedge F_{n_1}(S^{n_2}) \to F_{m_1+m_2}(S^{m_2+n_2})$$

defined as in (4.1). \qed

Now we apply a general principle for defining left adjoint functors out of diagram categories. For a $W$-space $X$, let $Sp^{W}[X]$ be the orthogonal spectrum defined as the coend

$$Sp^{W}[X] = \int_{(n_1, n_2) \in W} F_{n_1}(S^{n_2}) \wedge X(n_1, n_2)_+$$
of the indicated $W^\op \times W$-diagram of orthogonal spectra. Notice in particular that $S^W[F_{(d_1,d_2)}^W(*)]$ is naturally isomorphic to $F_d(S^{d_2})$. For an orthogonal spectrum $E$, let $\Omega^W(E)$ be the $W$-space defined by
\[ \Omega^W(E)(n_1,n_2) = S^P \circ (F_{n_1}(S^{n_2}), E). \]

By adjointness, the latter space can be identified with $\Omega^{m_2}(E_m)$. The functoriality of the construction assigns to a morphism $[m, \sigma_1, \sigma_2]: (m_1,m_2) \to (n_1,n_2)$ in $W$ the map $\Omega^{m_2}(E_{m_1}) \to \Omega^{n_2}(E_{n_1})$ that takes $f: S^{m_2} \to E_{m_1}$ to the composition
\[ S^{m_2} \xrightarrow{\sigma^{-1}_2} S^{m_2} \wedge S^m \xrightarrow{f \wedge \text{id}} E_{m_1} \wedge S^m \xrightarrow{\sigma_1} E_{n_1}. \]

In the next proposition we consider the absolute and positive stable model structures on $S^P$ introduced in [10].

**Proposition 4.4.** The functor $S^W$ is strong symmetric monoidal and the functor $\Omega^W$ is lax symmetric monoidal. These functors define a Quillen adjunction
\[ S^W: \text{Top}^W \rightleftarrows S^P : \Omega^W \]
with respect to the (positive) $W$-model structure on Top$^W$ and the (positive) stable model structure on $S^P$.

**Proof.** It is clear from the formal properties of the coend construction that $S^W$ is left adjoint to $\Omega^W$ (see e.g. [9] Section 2) for a general discussion of this phenomenon in a based topological context. For this to be a Quillen adjunction it suffices to show that $\Omega^W$ preserves fibrations and acyclic fibrations and this is clear from the characterization of stable fibrations in [10] Proposition 9.5. The statements about monoidality follow formally from the fact that the functor $F_\cdot(S \cdot)$ in Lemma [4,3] is strong symmetric monoidal, cf. [9] Proposition 2.14.

In more detail, the monoidal structure map $S^W[X] \wedge S^W[Y] \to S^W[X \wedge Y]$ is obtained by passage to coends from the natural maps
\[ (F_{m_1}(S^{m_2}) \wedge X(m_1,m_2)_+) \wedge (F_{n_1}(S^{n_2}) \wedge Y(n_1,n_2)_+) \]
\[ \to F_{m_1+n_1}(S^{m_2+n_2}) \wedge (X \boxtimes Y(m_1+n_1,m_2+n_2)_+) \]
whereas the monoidal structure of $\Omega^W$ is given by the natural maps
\[ \Omega^{m_2}(E_{m_1}) \wedge \Omega^{n_2}(E_{n_1}) \to \Omega^{m_2+n_2}(E_{m_1+n_1}) \]
where the first arrow takes a pair of maps to their smash product.

**4.5. Graded orthogonal spectra.** By a $(Z)$-graded orthogonal spectrum $E = \{E_d: d \in Z\}$ we understand a family of orthogonal spectra $E_{(d)}$ indexed by $d \in Z$. We write $\text{Grad}_Z S^P$ for the category of graded orthogonal spectra in which a morphism $f: D \to E$ is a family of maps of orthogonal spectra $f_{(d)}: D_{(d)} \to E_{(d)}$ indexed by $d \in Z$. The obvious “graded smash product” $D \wedge E$ with
\[ (D \wedge E)_{(d)} = \bigvee_{i+j=d} D_{(i)} \wedge E_{(j)} \]
makes $\text{Grad}_Z S^P$ a symmetric monoidal category with monoidal unit the graded orthogonal spectrum which is the sphere spectrum $S$ in degree 0 and the terminal spectrum $\ast$ in all other degrees. There are adjoint functors $t: \text{Grad}_Z S^P \rightleftarrows S^P : c$ defined by $t(E) = \bigvee_{d \in Z} E_{(d)}$ and $c(Z)_{(d)} = Z$. The functor $t$ is strong symmetric monoidal and $c$ is lax symmetric monoidal.

Now let us view $F_{n_1}(S^{n_2})$ as a graded orthogonal spectrum concentrated in degree $n_2 - n_1$. The functor $F_\cdot(S \cdot)$ from Lemma [4,3] then factors through a strong
symmetric monoidal functor \( F : \mathcal{S} \to \mathcal{W} \): \( \mathcal{W}^{op} \to \text{Grad}_\mathbb{Z} \mathcal{S}^0 \) such that the composition with \( t \) is the functor considered in the lemma. From this it follows formally that we have a pair of adjoint functors

\[
\mathcal{S}^0_{\mathbb{Z}} : \mathcal{W} \rightleftarrows \text{Grad}_\mathbb{Z} \mathcal{S}^0 : \Omega^W_{\mathbb{Z}}
\]

with \( \mathcal{S}^0_{\mathbb{Z}} \) strong symmetric monoidal and \( \Omega^W_{\mathbb{Z}} \) lax symmetric monoidal. By definition this fits in a commutative diagram of adjunctions

\[
\begin{array}{ccc}
\mathcal{W} & \xrightarrow{\mathcal{S}^0_{\mathbb{Z}}} & \text{Grad}_\mathbb{Z} \mathcal{S}^0 \\
\Omega^W_{\mathbb{Z}} & \xleftarrow{\mathcal{S}^0_{\mathbb{Z}}} & & \Omega^W_{\mathbb{Z}}
\end{array}
\]

For a \( \mathcal{W} \)-space \( X \), we have \( \mathcal{S}^0_{\mathbb{Z}}[X]_d = \mathcal{S}^W[X(d)] \) where \( X(d) \) denotes the \( \mathcal{W} \)-space which agrees with \( X \) on the \( d \)th component \( \mathcal{W}(d) \) and is the empty set otherwise.

**Proposition 4.6.** For a \( \mathcal{W} \)-space \( X \) and \( d \in \mathbb{Z} \), we have

\[
(4.3) \quad \mathcal{S}^W[X(d)]_n = \begin{cases} X(n,d+n)_+ \wedge_{\mathcal{W}(d+n)} S^{d+n} & \text{if } d + n \geq 0, \\ * & \text{if } d + n < 0. \end{cases}
\]

Here the notation indicates the coequalizer of the left \( O(d+n) \)-action on \( S^{d+n} \) and the right \( O(d+n) \)-action on \( X(n,d+n) \) obtained by letting \( a \in O(d+n) \) act as \( a^{-1} \) in the second variable. (This is the same as the orbit space of the diagonal left action). For \( 0 < -d < n \) we again think of \( \mathbb{R}^{d+n} \) as the orthogonal complement of \( \mathbb{R}^{-d} \) included as the first \(-d\) coordinates in \( \mathbb{R}^n \).

**Proof.** We first observe that the right hand side of (4.3) actually defines an orthogonal spectrum. The orthogonal group \( O(n) \) acts via the left action on \( X(n,d+n) \) in the first variable and the spectrum structure map

\[
X(n,d+n)_+ \wedge_{\mathcal{W}(d+n)} S^{d+n} \wedge S^1 \to X(n+1,d+n+1)_+ \wedge_{\mathcal{W}(d+n+1)} S^{d+n+1}
\]

is induced by the morphism

\[
[1, \text{id}_{\mathbb{R}^{d+n}}, \text{id}_{\mathbb{R}^{d+n+1}}] : (n,d+n) \to (n+1,d+n+1)
\]

and the canonical identification \( S^{d+n} \wedge S^1 = S^{d+n+1} \). For each \((n_1, n_2)\) in \( \mathcal{W}(d) \) this spectrum receives a spectrum map from \( F_{n_1}(S^{n_2}) \wedge X(n_1, n_2)_+ \) induced by the obvious map of spaces

\[
S^{n_2} \wedge X(n_1, n_2)_+ \to X(n_1, d+n_1)_+ \wedge_{\mathcal{W}(d+n_1)} S^{d+n_1}
\]

where \( n_2 = d+n_1 \). By the universal property of the coend, these spectrum maps assemble to a map from \( \mathcal{S}^W[X(d)] \) which gives the isomorphism in the proposition. The inverse is defined in spectrum degree \( n \) by factoring the composition

\[
X(n,d+n)_+ \wedge S^{d+n} \to F_n(S^{d+n})_n \wedge X(n,d+n)_+ \to \mathcal{S}^W[X(d)]_n
\]

over the coequalizer by the \( O(d+n) \)-action. \( \square \)

The right adjoint \( \Omega^W_{\mathbb{Z}} \) takes a graded orthogonal spectrum \( E \) to the \( \mathcal{W} \)-space \( \Omega^W_{\mathbb{Z}}(E) \) whose restriction to \( \mathcal{W}(d) \) equals the restriction of \( \Omega^W(E(d)) \) to \( \mathcal{W}(d) \).

Using that the adjoint functors \( \mathcal{S}^W_{\mathbb{Z}} \) and \( \Omega^W_{\mathbb{Z}} \) are (lax) symmetric monoidal, we get an induced adjunction between the corresponding categories of (commutative) monoids. In particular, a \( \mathcal{W} \)-space monoid \( M \) gives rise to the graded orthogonal ring spectrum \( \mathcal{S}^W_{\mathbb{Z}}[M] \) with graded multiplication

\[
\mathcal{S}^W_{\mathbb{Z}}[M(n)] \wedge \mathcal{S}^W_{\mathbb{Z}}[M(n)] \xrightarrow{\cong} \mathcal{S}^W_{\mathbb{Z}}[M(n) \boxtimes M(n)] \to \mathcal{S}^W_{\mathbb{Z}}[M(n+m+n)].
\]
With the explicit description in Proposition 4.13 this multiplication takes the form
\[
(M(m, d + m) + \wedge_{O(d + m)} S^{d+m}) \land (M(n, e + n) + \wedge_{O(e+n)} S^{e+n})
\]
\[
\to M(m + n, d + m + e + n) + \wedge_{O(d + m + e + n)} S^{d+m+e+n}
\]
\[
= M(m + n, d + e + m + n) + \wedge_{O(d + e + m + n)} S^{d+e+m+n}
\]
where the first map is given by the monoid structure of \( M \) and the canonical identification of \( S^{d+m} \land S^{e+n} \) with \( S^{d+m+e+n} \). The last equality indicates that the action of \( O(d + m + e + n) \) renders further permutations of the coordinates irrelevant.

4.7. Homotopy invariance of \( S^W \). The functor \( S^W \), being a left Quillen functor, takes \( \mathcal{W} \)-equivalences between cofibrant \( \mathcal{W} \)-spaces to stable equivalences. It is important for our applications that \( S^W \) is homotopically well-behaved on a larger class of \( \mathcal{W} \)-spaces.

**Definition 4.8.** A \( \mathcal{W} \)-space \( X \) is said to be \( \mathcal{S}^W \)-good if there exists a cofibrant \( \mathcal{W} \)-space \( X' \) and a \( \mathcal{W} \)-equivalence \( X' \to X \) such that \( S^W [X'] \to S^W [X] \) is a stable equivalence.

It is clear from the definition that cofibrant \( \mathcal{W} \)-spaces are \( \mathcal{S}^W \)-good. The terminal \( \mathcal{W} \)-space \( * \) is an example of a \( \mathcal{W} \)-space that is not \( \mathcal{S}^W \)-good. Using that \( S^W \) is a left Quillen functor we see that if \( X \) is \( \mathcal{S}^W \)-good and \( Y \to X \) is any \( \mathcal{W} \)-equivalence with \( Y \) cofibrant, then the induced map \( S^W [Y] \to S^W [X] \) is a stable equivalence. This in turn has the following consequence.

**Proposition 4.9.** The functor \( S^W \) takes \( \mathcal{W} \)-equivalences between \( \mathcal{W} \)-spaces that are \( \mathcal{S}^W \)-good to stable equivalences. \( \square \)

We proceed to establish some convenient criteria which ensure that a \( \mathcal{W} \)-space is \( \mathcal{S}^W \)-good. Let us write \( \text{Top}^{O(n)} \) for the category of left \( O(n) \)-spaces. It is well-known that this category admits a coarse model structure in which a map is a weak equivalence or fibration if and only if the underlying map of spaces is (this is also known as the “naive” model structure). We say that a map of \( O(n) \)-spaces is an \( O(n) \)-cofibration if it is a cofibration in the coarse model structure and an \( O(n) \)-space is \( O(n) \)-cofibrant if it is cofibrant in this model structure. It is clear that the orbit space/trivial action adjunction \( \text{Top}^{O(n)} \rightleftarrows \text{Top} \) is a Quillen adjunction when \( \text{Top}^{O(n)} \) is equipped with the coarse model structure. This implies in particular that a weak equivalence of \( O(n) \)-spaces \( X \to Y \) induces a weak equivalence of the orbit spaces \( X/O(n) \to Y/O(n) \) provided that \( X \) and \( Y \) are \( O(n) \)-cofibrant.

There also is a fine model structure on \( \text{Top}^{O(n)} \) in which a map is a weak equivalence (respectively a fibration) if and only if the induced map of \( G \)-fixed points is a weak equivalence (respectively a fibration) for all closed subgroups \( G \) in \( O(n) \) (this is also known as the “genuine” model structure). This model structure has more cofibrations than the coarse model structure and in particular any \( O(n) \)-equivariant CW complex is cofibrant. The next lemma is the \( O(n) \)-version of the analogous result for finite groups proved \[18\] Lemma 12.10.

**Lemma 4.10.** Let \( f: X_1 \to X_2 \) and \( g: Y_1 \to Y_2 \) be respectively a cofibration in the coarse and the fine model structure on \( \text{Top}^{O(n)} \). Then the pushout-product
\[
f \wedge g: X_1 \times Y_2 \cup X_1 \times Y_1, X_2 \times Y_1 \to X_2 \times Y_2
\]
is also an \( O(n) \)-cofibration. \( \square \)

As a consequence of the lemma we see that if \( X \) is \( O(n) \)-cofibrant and \( Y_1 \to Y_2 \) is a cofibration in the fine model structure, then \( X \times Y_1 \to X \times Y_2 \) is an \( O(n) \)-cofibration (since \( \emptyset \to X \) is an \( O(n) \)-cofibration).
Now let $X$ be a $W$-space and consider the action of $O(n_1) \times O(n_2)$ on $X(n_1, n_2)$. We say that $X$ is $O$-cofibrant in the second variable if forgetting the action of $O(n_1)$ makes $X(n_1, n_2)$ cofibrant in the coarse model structure on $\text{Top}^{O(n_2)}$ for all $(n_1, n_2)$.

**Lemma 4.11.** Let $X \to Y$ be a level equivalence of $W$-spaces that are $O$-cofibrant in the second variable. Then $S^W[X] \to S^W[Y]$ is a level equivalence.

**Proof.** By the explicit description of $S^W$ in Proposition 4.6 the statement in the lemma is equivalent to the maps

$$X(n_1, n_2)_+ \wedge_{O(n_2)} S^{n_2} \to Y(n_1, n_2)_+ \wedge_{O(n_2)} S^{n_2}$$

being weak homotopy equivalences for all $(n_1, n_2)$. We know from Lemma 4.10 that $X(n_1, n_2) \times S^{n_2}$ and $Y(n_1, n_2) \times S^{n_2}$ are $O(n_2)$-cofibrant and hence that the map of orbit spaces

$$X(n_1, n_2) \times_{O(n_2)} S^{n_2} \to Y(n_1, n_2) \times_{O(n_2)} S^{n_2}$$

is a weak homotopy equivalence. Since the inclusions of $X(n_1, n_2)/O(n_2)$ and $Y(n_1, n_2)/O(n_2)$ specified by the base point of $S^{n_2}$ are cofibrations by Lemma 4.10 we conclude that also the induced map of quotient spaces is a weak homotopy equivalence.

**□**

**Lemma 4.12.** Every cofibrant $W$-space is $O$-cofibrant in the second variable.

**Proof.** It is clear from the definition that $W((d_1, d_2), (n_1, n_2))$ is a smooth manifold with free $O(n_2)$-action and therefore a free $O(n_2)$-CW complex. This implies that in the appropriate sense, the generating cofibrations for the $W$-model structure on $\text{Top}^{W}$ are $O$-cofibrations in the second variable. The statement of the lemma follows from this since in general a cofibrant $W$-space is a retract of a cell complex constructed from the generating cofibrations.

**□**

**Proposition 4.13.** Let $X$ be a $W$-space that is $O$-cofibrant in the second variable. Then $X$ is $S^W$-good.

**Proof.** Let $Y$ be a cofibrant $W$-space and $Y \to X$ an acyclic fibration (that is, a cofibrant replacement of $X$). Then $Y \to X$ is a level-wise equivalence and so $S^W[Y] \to S^W[X]$ is a level equivalence by Lemmas 4.11 and 4.12.

Recall that the bar resolution $\overline{X}$ of a $W$-space $X$ is not necessarily cofibrant unless $X$ is level-wise cofibrant, cf. Lemma 3.7. The next result is therefore technically convenient.

**Proposition 4.14.** The bar resolution $\overline{X}$ of a $W$-space $X$ is $S^W$-good.

**Proof.** As in the proof of Proposition 4.13 we chose an acyclic fibration $Y \to X$ with $Y$ cofibrant. Then $\overline{Y}$ is cofibrant by Lemma 3.7, so it is enough to show that the map of bar resolutions $\overline{Y} \to \overline{X}$ induces a level equivalence $S^W(\overline{Y}) \to S^W(\overline{X})$. By the explicit description in Proposition 4.6 this is equivalent to showing that

$$\bigvee_{n^0, \ldots, n^p} \overline{W(n^0, (n_1, n_2))} \wedge_{O(n_2)} S^{n_2} \to \overline{X(n_1, n_2)_+} \wedge_{O(n_2)} S^{n_2}$$

is a weak homotopy equivalence for all $(n_1, n_2)$. Here the domain can be identified with the geometric realization of the simplicial space with $p$-simplices

$$\bigvee_{n^0, \ldots, n^p} \overline{W(n^0, (n_1, n_2))} \wedge_{O(n_2)} S^{n_2} \wedge (W(n^1, n^0) \times \cdots \times W(n^p, n^{p-1}) \times Y(n^p))_+$$

where $n^0, \ldots, n^p$ runs over all $(p + 1)$-tuples of object in $W$. There is a similar description of the domain with $X$ instead of $Y$ and the map in question is induced by the maps $Y(n^p) \to X(n^p)$. Clearly this is a weak homotopy equivalence in each simplicial degree. It is not difficult to check that these are good simplicial spaces.
in the sense of Segal [23, Appendix A] (e.g., using [6, Proposition 2.5]), hence the map of geometric realizations is also a weak equivalence. □

The above proposition gives us a useful criterion for a $\mathcal{W}$-space to be $\mathcal{S}^\mathcal{W}$-good.

**Corollary 4.15.** A $\mathcal{W}$-space $X$ is $\mathcal{S}^\mathcal{W}$-good if and only if the canonical map $\mathcal{X} \to X$ induces a stable equivalence $\mathcal{S}^\mathcal{W}[\mathcal{X}] \to \mathcal{S}^\mathcal{W}[X]$. □

### 4.16 Grouplike commutative $\mathcal{W}$-space monoids.

We say that a commutative $\mathcal{W}$-space monoid $M$ is grouplike if the underlying $E_\infty$ space $M_{h\mathcal{W}}$ is grouplike in the usual sense, that is, if $\pi_0(M_{h\mathcal{W}})$ is a group. This can also be expressed in terms of the unit $u \in M(0,0)$ and the map of $\mathcal{W} \times \mathcal{W}$-spaces

$$\mu: M(m_1, m_2) \times M(n_1, n_2) \to M((m_1, m_2) \oplus (n_1, n_2))$$

defining the multiplication.

**Lemma 4.17.** A commutative $\mathcal{W}$-space monoid $M$ is grouplike if and only if for each element $x \in M(m_1, m_2)$ there exists an element $y \in M(n_1, n_2)$ such that $m_2 - n_1 = m_1 - n_2$ and $\mu(x, y)$ belongs to the same path component as the image of the unit $u \in M(0,0)$ under the canonical morphism

$$[m_1 + n_1, id_{\mathbb{R}^{m_1+n_1}}, id_{\mathbb{R}^{m_2+n_2}}]: (0,0) \to (m_1 + n_1, m_2 + n_2).$$

**Proof.** Let $\pi_0\mathcal{W}$ be the category with the same objects as $\mathcal{W}$ and morphisms the path components of the morphism spaces in $\mathcal{W}$. Then it follows from the definition of $M_{h\mathcal{W}}$ as the realization of a simplicial space that $\pi_0(M_{h\mathcal{W}})$ can be identified with the colimit of the $\pi_0\mathcal{W}$-diagram defined by the path components $\pi_0(M(n_1, n_2))$. Restricting to the $d$th component $\mathcal{W}(d)$, we thus get that

$$\pi_0(M_{h\mathcal{W}(d)}) \cong \operatorname{colim}_n \pi_0(M(n, d + n))/\pi_0(\mathcal{O}(n, d + n))$$

where $\mathcal{O}(n, d + n) = \mathcal{O}(n) \times \mathcal{O}(d + n)$ and the colimit is over the ordered set of natural numbers $n$ such that $d + n \geq 0$. This easily gives the statement in the lemma. □

**Example 4.18** (Graded $\mathcal{W}$-space units). Let $R$ be a commutative orthogonal ring spectrum and assume for simplicity that $R$ is positive fibrant. We can then define the graded $\mathcal{W}$-space units $GL^\mathcal{W}_1(R)$ by mimicking the analogous construction for symmetric spectra in [18, Section 4]: Every path component in the space $\Omega^\mathcal{W}(R)(n_1, n_2)$ represents an element in the graded ring of homotopy groups $\pi_*(R)$, and we define $GL^\mathcal{W}_1(R)(n_1, n_2)$ to be the union of the path components that represent graded units. This defines a sub $\mathcal{W}$-space $GL^\mathcal{W}_1(R)$ of $\Omega^\mathcal{W}(R)$, and it is clear that $GL^\mathcal{W}_1(R)$ inherits the structure of a commutative $\mathcal{W}$-space monoid which is grouplike by Lemma 4.17.

In the next lemma we are given a commutative $\mathcal{W}$-space monoid $M$ and write $\check{x}: F^\mathcal{W}_{(d_1, d_2)}(\ast) \to M$ for the map of $\mathcal{W}$-spaces determined by a choice of element $x$ in $M(d_1, d_2)$.

**Lemma 4.19.** Let $M$ be a grouplike commutative $\mathcal{W}$-space monoid. Then every element $x$ in $M(d_1, d_2)$ with $d = d_2 - d_1$ induces a $\mathcal{W}$-equivalence

$$F^\mathcal{W}_{(d_1, d_2)}(\ast) \boxtimes M_{\{0\}} \xrightarrow{\check{x} \boxtimes \text{id}} M_{\{d\}} \boxtimes M_{\{0\}} \xrightarrow{\mu_0^\mathcal{W}} M_{\{d\}}.$$
Proof. Consider the commutative diagram of spaces

\[
\begin{array}{ccc}
(F^W_{(d_1, d_2)}(\ast) \boxtimes M)_{hW} & \xrightarrow{(\ast \boxtimes \text{id})_{hW}} & (M \boxtimes M)_{hW} \\
M_{hW} & \simeq & F^W_{(d_1, d_2)}(\ast)_{hW} \times M_{hW} \\
\end{array}
\]

in which the horizontal map on the bottom left is the inclusion specified by the point \([\text{id}_{d_1, d_2}]\) in \(F^W_{(d_1, d_2)}(\ast)_{hW}\). The latter is a weak homotopy equivalence since \(F^W_{(d_1, d_2)}(\ast)_{hW}\) is contractible (it can be identified with the classifying space of the comma category \(((d_1, d_2) \downarrow W)\)). We also know that the vertical map on the left is a weak homotopy equivalence, cf. Lemma 3.11. The long composition in the diagram multiplies an element in \(M_{hW}\) by the point \([x]\) in \(M_{hW}\) represented by \(x\). This is a weak homotopy equivalence since we assume \(M_{hW}\) to be grouplike. It follows that the composition in the upper row is a weak homotopy equivalence and restricting to the relevant components, we get the statement in the lemma. \(\square\)

Proposition 4.20. Let \(M\) be a grouplike commutative \(W\)-space monoid and assume in addition that \(M\) is \(S^W\)-good. Then every element \(x\) in \(M(d_1, d_2)\) with \(d = d_2 - d_1\) induces an equivalence

\[
\Sigma^d S^W[M(0)] \simeq F_{d_1}(S^{d_2}) \wedge S^W[M(0)] \cong S^W[F^W_{(d_1, d_2)}(\ast) \boxtimes M(0)] \cong S^W[M(d)]
\]

in the stable homotopy category.

Proof. Assuming \(M\) to be \(S^W\)-good ensures that \(M(n)\) is \(S^W\)-good for all \(n\). Since \(S^W\) is strong symmetric monoidal we conclude that \(F^W_{(d_1, d_2)}(\ast) \boxtimes M(0)\) is \(S^W\)-good, so \(S^W\) takes the \(W\)-equivalence in Lemma 4.19 to a stable equivalence. The first equivalence in the proposition arises from the fact that \(F_{d_1}(S^{d_2})\) represents the suspension \(\Sigma^d S\) as an object in the stable homotopy category:

\[
F_{d_1}(S^{d_2}) \cong \Omega^{d_1}(S^{d_1} \wedge F_{d_1}(S^{d_2})) \cong \Omega^{d_1}(S^{d_2} \wedge F_{d_1}(S^{d_1})) \cong \Omega^{d_1}(S^{d_2} \wedge S),
\]

where the last map is induced by the canonical stable equivalence \(F_{d_1}(S^{d_1}) \to S\), cf. [10] Lemma 8.6. \(\square\)

5. Graded Grassmannians and periodic cobordism spectra

In this section we show how the usual Stiefel manifolds fit in the setting of \(W\)-spaces. Using this, we introduce the oriented and unoriented graded Grassmannians in our context, and we show how the periodic real cobordism spectra \(MOP\) and \(MSOP_{ev}\) arise from commutative \(W\)-space monoids.

5.1. The Stiefel \(W\)-space \(V\). In the following \(\oplus^\infty \mathbb{R}^n\) denotes the infinite direct sum \(\bigoplus_{n=1}^\infty \mathbb{R}^n\), and we write \(V(\mathbb{R}^m, \oplus^\infty \mathbb{R}^n)\) for the space of linear isometries from \(\mathbb{R}^m\) to \(\oplus^\infty \mathbb{R}^n\). (This is a slight abuse of notation since \(\oplus^\infty \mathbb{R}^n\) is not an object of the category \(V\).) We shall define a “Stiefel” \(W\)-space \(V\) by

\[
V : W \to \text{Top}, \quad V(n_1, n_2) = V(\mathbb{R}^{n_2}, \oplus^\infty \mathbb{R}^{n_1}).
\]

Thus, \(V(n_1, n_2)\) can be identified with the space \(V_{n_2}(\oplus^\infty \mathbb{R}^{n_1})\) of orthogonal \(n_2\)-frames in \(\oplus^\infty \mathbb{R}^{n_1}\), topologized as the colimit of the finite dimensional Stiefel manifolds \(V_{n_2}(\oplus^k \mathbb{R}^{n_1})\). This is the empty set if and only if \(n_1 = 0\) and \(n_2 > 0\), and is otherwise contractible. In order to specify the action on morphisms, we fix a choice of isometries \(i_m : \mathbb{R}^m \to \oplus^\infty \mathbb{R}^m\) for \(m \geq 0\) by including \(\mathbb{R}^m\) as the first summand. Given a morphism \([m, \sigma_1, \sigma_2] : (m_1, m_2) \to (n_1, n_2)\) in \(W\), the induced map

\[
[m, \sigma_1, \sigma_2]_* : V(m_1, m_2) \to V(n_1, n_2)
\]
then takes an element \( \mathbf{v} \) in the domain to the isometry determined by the commutativity of the diagram

\[
\begin{array}{ccc}
\mathbb{R}^{n_2} & \xrightarrow{[m, \sigma_1, \sigma_2]_*} & \oplus^{\infty} \mathbb{R}^{n_1} \\
\sigma_2 \downarrow & & \downarrow \oplus^{\infty} \sigma_1 \\
\mathbb{R}^{m_2} \oplus \mathbb{R}^m & \xrightarrow{\oplus \iota_{m}} & (\oplus^{\infty} \mathbb{R}^{m_1}) \oplus (\oplus^{\infty} \mathbb{R}^m) \xrightarrow{2^m} \oplus^{\infty} (\mathbb{R}^{m_1} \oplus \mathbb{R}^m).
\end{array}
\]

Here the isomorphism on the bottom right is the obvious order preserving permutation of the coordinates. Clearly the projection of \( V \) onto the terminal \( \mathcal{W} \)-space defines a \( \mathcal{W} \)-equivalence \( V_{\mathcal{W}} \to BW \).

Next we define a natural map of \((\mathcal{W} \times \mathcal{W})\)-spaces

\[
\mu: V(m_1, m_2) \times V(n_1, n_2) \to V((m_1, m_2) \oplus (n_1, n_2))
\]
such that a pair \( v \in V(m_1, m_2) \) and \( w \in V(m_1, m_2) \) is mapped to the isometry

\[
\mathbb{R}^{m_2} \oplus \mathbb{R}^{n_2} \xrightarrow{\oplus \iota_{m_2}} (\oplus^{\infty} \mathbb{R}^{m_1}) \oplus (\oplus^{\infty} \mathbb{R}^{n_1}) \xrightarrow{2^m_1} \oplus^{\infty} (\mathbb{R}^{m_1} \oplus \mathbb{R}^{n_1})
\]
where we use the same order preserving permutation of the coordinates. By the universal property of the \( \mathbb{E} \)-product, this in turn gives rise to a map of \( \mathcal{W} \)-spaces \( \mu: V \boxtimes V \to V \). Checking from the definitions, we get the next result.

**Proposition 5.2.** The multiplication \( \mu: V \boxtimes V \to V \) makes \( V \) a grouplike commutative \( \mathcal{W} \)-space monoid with unit the unique map of \( \mathcal{W} \)-spaces \( U^{\mathcal{W}} \to V \). \( \square \)

### 5.3. The unoriented periodic cobordism spectrum \( MOP \)

We proceed to show that the corresponding commutative orthogonal ring spectrum \( \mathcal{S}^{\mathcal{W}}[V] \) is a model of the unoriented periodic cobordism spectrum. Using the description from Proposition 4.3, we get that

\[
\mathcal{S}^{\mathcal{W}}[V_{[d]}] = \begin{cases} V_{d+n}(\oplus^{\infty} \mathbb{R}^n)_+ \wedge_{O(d+n)} S^{d+n}, & \text{if } d + n \geq 0, \\
\ast, & \text{if } d + n < 0.
\end{cases}
\]

In this expression, the \( O(n) \)-action on the right hand side is induced from the diagonal left \( O(n) \)-action on \( \oplus^{\infty} \mathbb{R}^n \). The graded multiplication on \( \mathcal{S}^{\mathcal{W}}[V] \) is given by the maps

\[
\begin{align*}
(V_{d+m}(\oplus^{\infty} \mathbb{R}^m)_+ \wedge_{O(d+m)} S^{d+m}) \wedge (V_{e+n}(\oplus^{\infty} \mathbb{R}^n)_+ \wedge_{O(e+n)} S^{e+n}) & \to V_{d+m+e+n}(\oplus^{\infty} (\mathbb{R}^m \oplus \mathbb{R}^n)_+ \wedge_{O(d+m+e+n)} S^{d+m+e+n}) \\
= V_{d+e+m+n}(\oplus^{\infty} (\mathbb{R}^m \oplus \mathbb{R}^n)_+ \wedge_{O(d+e+m+n)} S^{d+m+e+n})
\end{align*}
\]

where we use the canonical identification of \( S^{d+m} \wedge S^{e+n} \) with \( S^{d+m+e+n} \), and map a pair of orthogonal frames

\[
(\{v^d_{i1}\}_{i1} \geq 1, \ldots, \{v^d_{i2}\}_{i2} \geq 1), \quad (\{w^e_{i1}\}_{i1} \geq 1, \ldots, \{w^e_{i2}\}_{i2} \geq 1)
\]
in \( \oplus^{\infty} \mathbb{R}^m \) and \( \oplus^{\infty} \mathbb{R}^n \) respectively, to the orthogonal frame

\[
(\{v^d_{i1}, 0\}_{i1} \geq 1, \ldots, \{v^d_{i2}, 0\}_{i2} \geq 1, \{0, w^e_{i1}\}_{i1} \geq 1, \ldots, \{0, w^e_{i2}\}_{i2} \geq 1)
\]
in \( \oplus^{\infty} (\mathbb{R}^m \oplus \mathbb{R}^n) \). For \( d = 0 \) we have \( \mathcal{S}^{\mathcal{W}}[V_{[0]}] = V_0(\oplus^{\infty} \mathbb{R}^n)_+ \wedge_{O(n)} S^n \), which can be identified with the Thom space of the canonical vector bundle over the Grassmannian \( G^n_r(\oplus^{\infty} \mathbb{R}^n) \) of \( n \)-planes in \( \oplus^{\infty} \mathbb{R}^n \). It follows that \( \mathcal{S}^{\mathcal{W}}[V_{[0]}] \) is a model of the unoriented real cobordism spectrum which justifies introducing the standard notation

\[
MO = \mathcal{S}^{\mathcal{W}}[V_{[0]}]
\]
in our context. Notice further that $V$ is $O$-cofibrant in the first variable and hence $S^W$-good by Proposition 5.13. Since $V$ is grouplike, it therefore follows from Proposition 5.20 that there is an equivalence $S^W[V] \simeq \bigvee_{n \in \mathbb{Z}} \Sigma^n MO$ in the stable homotopy category. This justifies introducing the standard notation

$$MOP = S^W[V]$$

for the unoriented periodic cobordism spectrum.

5.4. Graded Grassmannians. Before getting to the graded Grassmannians we are after, it is illuminating to discuss the passage from $W$-spaces to graded $V$-spaces in general. Let $\rho : W \to V$ be the functor that projects on the first variable, $\rho(n_1, n_2) = n_1$, and

$$\rho([m, \sigma_1, \sigma_2] : (m_1, m_2) \to (n_1, n_2)) = ([m, \sigma_1] : m_1 \to n_1),$$

where we identify the morphism space $V(m_1, n_1)$ with $\mathcal{O}(m_1 \otimes m, n_1)/\mathcal{O}(m)$. The functor $\rho$ gives rise to a Quillen adjunction $\rho_* : \text{Top}^W \rightleftarrows \text{Top}^V : \rho^*$, in which $\rho_*$ denotes the left Kan extension and $\rho^*$ pulls back a $V$-space to a $W$-space via $\rho$.

We shall be interested in a graded refinement of this and write $\text{Grad}_Z \text{Top}^V$ for the category of $\mathbb{Z}$-graded $V$-spaces $X = \{X(n) : d \in \mathbb{Z}\}$, equipped with the symmetric monoidal “graded” $\mathbb{E}$-product

$$(X \boxtimes Y)(d) = \coprod_{i+j=d} X_{\{i\}} \boxtimes Y_{\{j\}}.$$}

The graded $\mathbb{E}$-product has as its monoidal unit the graded $V$-space which is $U^V$ (the monoidal unit for $\text{Top}^V$) in degree 0 and the initial (that is, empty) $V$-space in all other degrees. As in the discussion of graded orthogonal spectra (cf. Section 5.5), we have adjoint functors $i : \text{Grad}_Z \text{Top}^V \rightleftarrows \text{Top}^V : c$, given by $i(X) = \coprod_{d \in \mathbb{Z}} X_{\{d\}}$ and $C(Y)(d) = Y$ for all $d \in \mathbb{Z}$. Now we lift $\rho_*$ to a functor $\rho^*_\mathbb{Z} : \text{Top}^W \to \text{Grad}_Z \text{Top}^V$ by letting

$$\rho^*_\mathbb{Z}(X)(d)(n) = \begin{cases} 
X(n, d+n)/\mathcal{O}(d+n), & \text{if } d+n \geq 0 \\
\emptyset, & \text{if } d+n < 0.
\end{cases}$$

We often write $\rho^*_\mathbb{Z}(X) = [\rho^*_\mathbb{Z}(X)](d)$. Given a morphism $[m', \sigma] : m \to n$ in $V$ with $m \geq 0$, the induced map $\rho^*_\mathbb{Z}(X)([m', \sigma])$ is defined by passage to orbit spaces

$$X(m, d+m)/\mathcal{O}(d+m) \to X(n, d+n)/\mathcal{O}(d+n), \quad x \mapsto [m', \sigma_1, \sigma_2](x),$$

where $[m', \sigma_1, \sigma_2] : (m, d+m) \to (n, d+n)$ denotes any morphism in $W$ that projects to $[m', \sigma]$ under $\rho$. It follows from the definitions that there is a commutative diagram of adjunctions

$$\begin{tikzcd}
\text{Top}^W \arrow[rr, \rho_*] \arrow[dr, \rho^*] & & \text{Grad}_Z \text{Top}^V \\
\text{Top}^V \arrow[ur, \rho^*_\mathbb{Z}] \arrow[rr, \rho_*] & & \text{Grad}_Z \text{Top}^V \\
\end{tikzcd}$$

Here all left adjoints are strong symmetric monoidal and all right adjoints are lax symmetric monoidal. The right adjoint $\rho^*_\mathbb{Z}$ takes a graded $V$-space $X$ to the $W$-space $\rho^*_\mathbb{Z}(X)$ whose restriction to $W_{\{d\}}$ equals $\rho^*(X_{\{d\}})$.

Recall the definition of the Stiefel $W$-space $V$ from Section 5.4.

**Definition 5.5.** The (unoriented) graded Grassmannian $Gr$ is the commutative graded $V$-space monoid defined by $Gr = \rho^*_\mathbb{Z}(V)$. 
Thus, writing $\text{Gr}_{d+n}(\oplus^\infty \mathbb{R}^n)$ for the Grassmannian of $(d+n)$-planes in $\oplus^\infty \mathbb{R}^n$, we have that

$$\text{Gr}_{(d]}(n) = \begin{cases} \text{Gr}_{d+n}(\oplus^\infty \mathbb{R}^n), & \text{if } d + n \geq 0 \\ \emptyset, & \text{if } d + n < 0 \end{cases}$$

with multiplication defined as in (5.1). It will be convenient to use the same notation $\text{Gr}$ for the corresponding (ungraded) $\mathcal{V}$-space $t(\text{Gr})$. The meaning will always be clear from the context.

5.6. **Oriented graded Grassmannian.** We again begin by discussing a general quotient construction on $\mathcal{W}$-spaces. Applying this construction to $\mathcal{V}$ will then give us the oriented graded Grassmannian we are after. Let $\mathcal{S}W$ be the topological subcategory of $\mathcal{W}$ with the same objects and morphism spaces

$$\mathcal{S}W((m_1, m_2), (n_1, n_2)) = \{ [m, \sigma_1, \sigma_2] : \det(\sigma_1) = \det(\sigma_2) \}$$

where $n_1 = m_1 + m$ and $n_2 = m_2 + m$. It is clear that $\mathcal{S}W$ is indeed a subcategory and one can check that its classifying space is a model of $B_{SO} \times \mathbb{Z}$. We write $\tilde{\rho} : \mathcal{S}W \to \mathcal{V}$ for the projections onto the first variable (that is, the composition of $\rho$ with the inclusion of $\mathcal{S}W$ in $\mathcal{W}$), and consider the corresponding adjunction $\tilde{\rho}_* : \text{Top}^{\mathcal{S}W} \rightleftarrows \text{Top}^\mathcal{V} : \tilde{\rho}^*$ given by left Kan extension and pullback along $\tilde{\rho}$. Since we are really interested in $\mathcal{W}$-spaces, we use the same notation for the composite functor

$$\tilde{\rho}_* : \text{Top}^\mathcal{W} \to \text{Top}^{\mathcal{S}W} \overset{\tilde{\rho}^*}{\to} \text{Top}^\mathcal{V}$$

obtained by first restricting a $\mathcal{W}$-space to $\mathcal{S}W$. As in the unoriented case, this lifts to a functor $\tilde{\rho}^*_Z : \text{Top}_Z^\mathcal{W} \to \text{Gr}_Z \text{Top}^\mathcal{V}$. Writing $\tilde{\rho}_Z^*(X) = \tilde{\rho}_Z^*(X)_{(d]}$ for $d \in \mathbb{Z}$, we set

$$\tilde{\rho}_Z^*(X)(n) = \begin{cases} X(n, d + n)/SO(d + n), & \text{if } d + n > 0 \\ O(n) \times_{SO(n)} X(n, 0), & \text{if } d + n = 0 \\ \emptyset, & \text{if } d + n < 0 \end{cases}$$

with the convention that $SO(0) = O(0)$ be the trivial group. Let $[m', \sigma] : m \to n$ be a morphism in $\mathcal{V}$ represented by $\sigma \in O(m \oplus m', n)$ and suppose that $d + m \geq 0$.

In case $d + n = 0$ (and thus $m = n = |d|$), the element $\sigma$ acts by left multiplication on $O(n)$. Now suppose that $d + n > 0$. If $d + m > 0$, then the map $\rho_Z^*(X)([m', \sigma])$ induced by $[m', \sigma]$ is defined by passage to orbit spaces

$$X(m, d + m)/SO(d + m) \to X(n, d + n)/SO(d + n), \quad x \to X([m', \sigma_1, \sigma_2])(x),$$

where $[m', \sigma_1, \sigma_2] : (m, d + m) \to (n, d + n)$ is any morphism in $\mathcal{S}W$ that projects to $[m', \sigma]$ under $\tilde{\rho}$. (The map of orbit spaces is independent of the choice.) If $d + m = 0$, then the map induced by $[m', \sigma]$ is given by

$$O(m) \times_{SO(m)} X(m, 0) \to X(n, d + n)/SO(d + n), \quad (a, x) \to X([m', \sigma(a \oplus \id_{\mathbb{R}^n})])(x),$$

where $a \in O(m)$, $x \in X(m, 0)$, and $[m', \tau_1, \tau_2] : (m, 0) \to (n, d + n)$ is any morphism in $\mathcal{S}W$ that projects to $[m', \sigma(a \oplus \id_{\mathbb{R}^n})]$ under $\tilde{\rho}$. The reason why the expression for $\tilde{\rho}_Z^*(X)$ depends on $d$ in this fashion is that the functor $\tilde{\rho}$ is surjective on morphism spaces except when applied to the endomorphisms of $([d], 0)$ in $\mathcal{S}W$ for $d < 0$. The reader may check that applying the expression for $\tilde{\rho}_Z^*(X)$ to the free $\mathcal{W}$-space $\mathcal{S}W(d_1, d_2, -)$ with $d = d_2 - d_1$, one gets the free $\mathcal{V}$-space $\mathcal{V}(d_1, -)$. It follows from the definitions that there is a commutative diagram of functors

$$\begin{array}{ccc}
\text{Top}^\mathcal{W} & \overset{\tilde{\rho}_*}{\longrightarrow} & \text{Grad}_Z \text{Top}^\mathcal{V} \\
\downarrow{\tilde{\rho}_*} & & \downarrow{t} \\
\text{Top}^\mathcal{V} & & \end{array}$$

where $\tilde{\rho}_* \circ \tilde{\rho}^* = t$.

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The fact that $\mathcal{SVW}$ is not a monoidal subcategory of $\mathcal{W}$ prevents the functors $\tilde{\rho}_*$ and $\tilde{\rho}_*^e$ from being lax symmetric monoidal. For this reason we often restrict to the full subcategory $\mathcal{W}_{ev}$ of $\mathcal{W}$ given by the objects $(n_1, n_2)$ such that $n_2 - n_1$ is even. Thus, $\mathcal{W}_{ev}$ is the coproduct of the subcategories $\mathcal{W}_{[d]}$ for even $d$. It is easy to check that the corresponding category $\mathcal{SW}_{ev}$ is a symmetric monoidal subcategory of $\mathcal{W}_{ev}$, which implies that the composite functor

$$\tilde{\rho}_*^e: \text{Top}^{\mathcal{W}_{ev}} \to \text{Top}^{\mathcal{SW}_{ev}} \xrightarrow{\mathcal{S}^e_{\mathcal{V}}} \text{Grad}_{\mathbb{Z}} \text{Top}^{\mathcal{V}}$$

is lax symmetric monoidal. We conclude that if $M$ is a (commutative) $\mathcal{W}$-space monoid which is concentrated on $\mathcal{W}_{ev}$, then $\tilde{\rho}_*^e(V)(X)$ inherits the structure of a (commutative) $\mathcal{V}$-space monoid. The individual terms $\tilde{\rho}_d^e(X)$ are as described above for $d$ even and the graded multiplication

$$\tilde{\rho}_d^e(X)(m) \times \tilde{\rho}_e^e(X)(n) \to \tilde{\rho}_{d+e}^e(X)(m+n)$$

is obtained by passage to orbit spaces with the addition that for $d + m = 0$ (respectively $e + n = 0$) it also uses the $O(m)$-action (respectively the $O(n)$-action) on $\tilde{\rho}_d^e(X)(m+n)$.

Now we apply the above to the Stiefel $\mathcal{W}$-space $V$ and write $V_{ev}$ for the restriction of $V$ to $\mathcal{W}_{ev}$.

**Definition 5.7.** The oriented graded Grassmmanian $\tilde{\mathcal{Gr}}$ is the graded $\mathcal{V}$-space defined by $\tilde{\mathcal{Gr}} = \tilde{\rho}_*^e(V)$, and the oriented evenly graded Grassmannian $\tilde{\mathcal{Gr}}_{ev}$ is the commutative graded $\mathcal{V}$-space monoid $\tilde{\mathcal{Gr}}_{ev} = \tilde{\rho}_*^e(V_{ev})$.

Let us write $\tilde{\mathcal{Gr}}_{m}(\oplus^\infty \mathbb{R}^n)$ for the Grassmannian of oriented $m$-planes in $\oplus^\infty \mathbb{R}^n$. This can be identified with the orbit space $V_m(\oplus^\infty \mathbb{R}^n)/SO(m)$, so we have that

$$\tilde{\mathcal{Gr}}_{(d)}(n) = \begin{cases} 
\tilde{\mathcal{Gr}}_{d+n}(\oplus^\infty \mathbb{R}^n), & \text{if } d + n > 0 \\
O(n)/SO(n), & \text{if } d + n = 0 \\
\emptyset, & \text{if } d + n < 0.
\end{cases}$$

As in the oriented case it will be convenient to simplify notation by writing $\tilde{\mathcal{Gr}}$ also for the corresponding $\mathcal{V}$-space $t(\tilde{\mathcal{Gr}})$. With this convention we define the oriented Stiefel $\mathcal{W}$-space $\tilde{V}$ by the pullback diagram

$$\begin{array}{ccc}
\tilde{V} & \longrightarrow & V \\
\tilde{\rho} \downarrow & & \downarrow \rho \\
\rho^* \tilde{\mathcal{Gr}} & \longrightarrow & \rho^* \mathcal{Gr}.
\end{array}$$

Spelling this out, we see that

$$(\ref{equation:stiefel_pullback}) \quad \tilde{V}(n_1, n_2) = \begin{cases} 
O(n_2) \times_{SO(n_2)} V_{n_2}(\oplus^\infty \mathbb{R}^{n_1}), & \text{if } n_2 > 0 \\
O(n_1)/SO(n_1), & \text{if } n_2 = 0.
\end{cases}$$

If $n_1 > 0$, then $V_{n_2}(\oplus^\infty \mathbb{R}^{n_1})$ is contractible and $\tilde{V}(n_1, n_2)$ decomposes in two contractible path components. This easily implies the next result.

**Proposition 5.8.** The $\mathcal{W}$-space $V$ is positive fibrant and $\tilde{V} \to V$ is a positive fibration. \hfill $\square$

We write $\tilde{V}_{ev}$ for the corresponding pullback over $\rho^* \mathcal{Gr}_{ev}$. This is a pullback of commutative $\mathcal{W}$-space monoids, so $\tilde{V}_{ev}$ is itself a commutative $\mathcal{W}$-space monoid. It follows from the previous proposition that $\tilde{V}_{ev} \to V_{ev}$ is a positive fibration and hence that $V_{ev}$ is positive fibrant.
Remark 5.9. We can give a more symmetric description of $\tilde{V}$ by writing

$$\tilde{V}(n_1, n_2) = O(n_1, n_2)^{(n_1, n_2)} V(n_1, n_2),$$

where $O(n_1, n_2) = O(n_1) \times O(n_2)$ and $\tilde{O}(n_1, n_2)$ denotes the subgroup given by the pairs $(a_1, a_2)$ such that $\det a_1 = \det a_2$. This shows that $\tilde{V}$ also can be obtained by first restricting $V$ to the subcategory $SW$ and then passing to the left Kan extension along the inclusion of $SW$ in $W$. From this point of view, a morphism $[m, σ_1, σ_2]$ from $(m_1, m_2)$ to $(n_1, n_2)$ in $W$ gives rise to the map

$$O(m_1, m_2) \times \tilde{O}(m_1, m_2) V(m_1, m_2) \to O(n_1, n_2) \times \tilde{O}(n_1, n_2) V(n_1, n_2)$$

taking $((a_1, a_2), ν)$ to $((σ_1(a_1 \oplus \text{id}), σ_2(a_2 \oplus \text{id})), [m, \text{id}, \text{id}]_\ast(ν))$.

Now consider the commutative graded orthogonal ring spectrum $S^W[\tilde{V}_{ev}]$. According to Proposition 4.6 we have $S^W[V_{[0]}] = \tilde{V}(n, n)^{\wedge} S^n$, which can be identified with the Thom space of the canonical vector bundle over $Gr_n(\oplus^\infty \mathbb{R}^n)$. Hence $S^W[V_{[0]}]$ is a model of the oriented real cobordism spectrum and we write $MSO = S^W[V_{[0]}]$. Since $V_{ev}$ is grouplike and $S^W$-good, it follows from Proposition 5.20 that there is a stable equivalence $S^W[\tilde{V}_{ev}] \simeq \bigvee_{n \in \mathbb{Z}} \Sigma^{2n} MSO$. Thus, $S^W[\tilde{V}_{ev}]$ is a model of the 2-periodic real oriented cobordism spectrum, and motivated by this way we introduce the notation

$$\text{MSOP}_{ev} = S^W[\tilde{V}_{ev}].$$

6. The Graded Thom Spectrum Functor

In the following we shall consider the comma category $\text{Top}^V / Gr$ of $V$-spaces over the Grassmannian $V$-space $Gr$. Since every $V$-space over $Gr$ inherits a grading from $Gr$, the latter category is isomorphic to the category $\text{Grad}_\mathbb{Z} \text{Top}^V / Gr$ of graded $V$-space over (the graded version of) $Gr$. Hence the distinction between graded and ungraded $V$-spaces becomes immaterial in this context. The functor $ρ$ gives rise to a pair of adjunctions

$$\text{Top}^W / V \overset{p}{\underset{ρ^{-1}}{\rightleftarrows}} \text{Top}^W / ρ^* Gr \overset{ρ}{\underset{ρ^{-1}}{\rightleftarrows}} \text{Top}^V / Gr$$

in which the first adjunction is given by post composition with and pullback along the adjunction unit map $p: V \to ρ^* Gr$. These are clearly Quillen adjunctions when we equip the participating comma categories with the induced model structures, cf. [3, Theorem 7.6.4]. Arguing as in the proof of [18, Proposition 13.4], we get the next result.

**Proposition 6.1.** The composite Quillen adjunction

$$ρ_\ast: \text{Top}^W / V \rightleftarrows \text{Top}^V / Gr: p^\ast$$

is a Quillen equivalence.  

We now consider graded Thom spectra arising from $V$-spaces over $Gr$, which we define using the composite functor

$$T: \text{Top}^V / Gr \overset{ρ}{\rightleftarrows} \text{Top}^W / V \overset{S^W}{\longrightarrow} Sp^P / MOP$$

where $p^\ast$ is the right adjoint in the Quillen equivalence from Proposition 6.1. In more detail, $p^\ast$ takes a map of $V$-spaces $f: X \to Gr$ to the pullback $p_\ast f(X)$ in

$$\begin{array}{ccc}
V & \xrightarrow{p} & \text{Top}^V / Gr \\
\downarrow & & \downarrow \\
p_\ast f(X) & \xrightarrow{ρ} & ρ^* Gr,
\end{array}$$
and we set \( T(f) = \mathbb{S}^W[p^*_X(X)] \). Let us momentarily restrict to maps of \( \mathcal{V} \)-spaces \( f: X \to Gr_{[0]} \) over the degree 0 part of \( Gr \). Then \( p^*_X(X)(n) \) is obtained by pulling back the principal \( O(n) \)-bundle \( V_n(\oplus^\infty \mathbb{R}^n) \to Gr_n(\oplus^\infty \mathbb{R}^n) \) to \( X(n) \), which implies that there is a pullback diagram of the associated vector bundles

\[
\begin{array}{ccc}
p^*_X(X)(n) \times_{O(n)} \mathbb{R}^n & \to & V_n(\oplus^\infty \mathbb{R}^n) \times_{O(n)} \mathbb{R}^n \\
f \downarrow & & \downarrow \\
X(n) & \xrightarrow{f_n} & Gr_n(\oplus^\infty \mathbb{R}^n).
\end{array}
\]

Hence we see from the explicit description in Proposition 6.3 that \( T(f)_n \) is homeomorphic to the Thom space of the vector bundle classified by \( f_n \). We conclude that applying the functor \( T \) to \( \mathcal{V} \)-spaces over \( Gr_{[0]} \) we get a model of the Thom spectrum for the corresponding sequence of vector bundles. The advantage of the current approach is that it extends to the graded setting and provides us with a lax symmetric monoidal graded Thom spectrum functor.

**Proposition 6.2.** The graded Thom spectrum functor \( T: \top^V/Gr \to Sp^V/MOP \) is lax symmetric monoidal.

**Proof.** The functor \( \rho \) is symmetric monoidal and the map \( \rho \) is a map of commutative \( \mathcal{V} \)-space monoids which together imply that the first functor \( p^* \) in the definition of \( T \) is lax symmetric monoidal. The second functor \( \mathbb{S}^V \) is strong symmetric monoidal by Proposition 4.3. \( \square \)

Notice in particular that by definition, applying the graded Thom spectrum functor to the canonical map \( Gr_{ev} \to Gr \), we get the evenly graded oriented real cobordism spectrum \( MSOP_{ev} \).

### 6.3. Lifting of space level data

We also want to construct graded Thom spectra from space level data and for this it will be convenient to use the homotopy colimit \( Gr_{hV} \) as a model of the classifying space \( BO \times \mathbb{Z} \). The point is that since the underlying \( \mathcal{V} \)-space of \( Gr \) is not cofibrant, we cannot apply the colimit functor to \( Gr \) directly. For this reason we instead consider the bar resolution \( \tilde{Gr} \) and the pair of Quillen adjunctions

\[
(6.1) \quad \text{Top}/Gr_{hV} \rightleftarrows \text{Top}^V/Gr \rightleftarrows \text{Top}^V/Gr
\]

in which the adjunction on the left is induced by the usual colim/constant functor adjunction, using the identification \( \text{colim}_V \tilde{Gr} = Gr_{hV} \), and the adjunction on the right is induced by the canonical level equivalence \( \tilde{Gr} \to Gr \).

**Proposition 6.4.** The Quillen adjunctions in (6.1) are Quillen equivalences. \( \square \)

Now we can define the graded Thom spectrum functor on \( \text{Top}/Gr_{hV} \) as the composition

\[
(6.2) \quad T: \text{Top}/Gr_{hV} \to \text{Top}^V/Gr \xrightarrow{ev} \text{Top}^W/V \xrightarrow{\mathbb{S}^V} Sp^D/MOP.
\]

It will always be clear from the context whether we view \( T \) as a functor on \( \text{Top}^V/Gr \) or on \( \text{Top}/Gr_{hV} \). In order for the graded Thom spectrum functor on \( \text{Top}/Gr_{hV} \) to be homotopically well-behaved, we follow Lewis [7 Section IX] by implicitly precomposing with a Hurewicz fibrant replacement functor. This is very similar to the strategy used in [21] and we refer to the latter paper for details. What we want to emphasize here is that this construction of the graded Thom spectrum functor has good multiplicative properties. Thus, let \( D \) be an operad augmented over the Barratt-Eccles operad, and write \( \text{Top}[D] \) for the category of \( D \)-algebras in \( \text{Top} \) and \( Sp^D[D] \) for the category of \( D \)-algebras in \( Sp^D \). The point of having \( D \)
Theorem 6.5. Let $\mathcal{D}$ be an operad augmented over the Barratt-Eccles operad. Then the graded Thom spectrum functor in $\text{Gr}_{hV}$ induces a functor of $\mathcal{D}$-algebras
$$T: \text{Top}[\mathcal{D}]/\text{Gr}_{hV} \to \text{Sp}^O[\mathcal{D}]/\text{MOP}$$

7. Orientations and the graded Thom isomorphism

In this section we shall discuss orientations and the resulting graded Thom isomorphisms. We first do this directly on the level of $W$-spaces and then derive the corresponding results for spaces over $\text{Gr}_{hV}$.

7.1. Orientations and the graded Thom isomorphism for $W$-spaces. In the discussion of orientations and the graded Thom isomorphism for $W$-spaces, it will be useful to introduce a graded version of the usual tensor structure on $\text{Sp}^O$ (see [10, Section 1]). By a graded space $W$ we understand a sequence of (unbased) spaces $K(\cdot)_{\cdot}$ indexed by $d \in \mathbb{Z}$. Given a graded orthogonal spectrum $E$ and a graded space $K$, we write $E \triangle K$ for the graded orthogonal spectrum with $d$th term
$$(E \triangle K)(\cdot)_d = E(\cdot)_d \wedge K(\cdot)_{\cdot}d.$$ For a $W$-space $X$ we shall view the corresponding colimit $X_W$ as a graded space with components $X_W(\cdot)_{\cdot}$ defined by evaluating the colimits over the subcategories $W(\cdot)_{\cdot}$. With this notation, every $W$-space $X$ gives rise to a “diagonal” map of graded orthogonal spectra
$$\delta: \mathcal{S}^W[X] \to \mathcal{S}^W[X] \triangle X_W \tag{7.1}$$ with $d$th component
$$\mathcal{S}^W[X(\cdot)_{\cdot}] \to \mathcal{S}^W[X(\cdot)_{\cdot} \times X_W(\cdot)_{\cdot}] \cong \mathcal{S}^W[X(\cdot)_{\cdot}] \wedge (X_W(\cdot)_{\cdot})$$ induced from the identity $X(\cdot)_{\cdot} \to X(\cdot)_{\cdot}$ and the projection $X(\cdot)_{\cdot} \to X_W(\cdot)_{\cdot}$. Now let us fix a commutative $W$-space monoid $W$ and consider the comma category $\text{Top}^W/W$ of $W$-spaces over $W$. Each object $X \to W$ gives rise to a map of graded orthogonal spectra
$$\delta: \mathcal{S}^W[X] \to \mathcal{S}^W[X] \triangle X_W \to \mathcal{S}^W[W] \triangle X_W \tag{7.2}$$ which in turn admits a canonical extension to a map of graded $\mathcal{S}^W[W(0)]$-modules
$$\delta_W: \mathcal{S}^W[W(0)] \wedge \mathcal{S}^W[X] \to \mathcal{S}^W[W] \wedge X_W.$$ Notice, that this is a natural transformation when we view the domain and codomain as functors on $\text{Top}^W/W$, and that the latter comma category inherits the structure of a symmetric monoidal category from $\text{Top}^W$ since we assume $W$ to be commutative. The fact that the functor $\mathcal{S}^W$ is strong symmetric monoidal implies that the domain in (7.2) is strong symmetric monoidal as a functor to the category of graded $\mathcal{S}^W[W(0)]$-modules. Furthermore, we may view the colimit functor on $\text{Top}^W$ as a strong symmetric monoidal functor, $X_W \times Y_W \cong (X \boxtimes Y)_W$, so that the codomain inherits the structure of a lax symmetric monoidal functor to graded $\mathcal{S}^W[W(0)]$-modules (but notice that $\mathcal{S}^W[W] \triangle X_W$ is not an $\mathcal{S}^W[W]$-module).

Lemma 7.2. The map $\delta_W$ of graded $\mathcal{S}^W[W(0)]$-modules defines a monoidal natural transformation between lax symmetric monoidal functors. It is a stable equivalence provided that $W$ is grouplike and $X$ is cofibrant.
Proof. For the statement about monoidality of \( \delta_W \), we remark that the map \( \delta \) in (7.4) may be viewed as a monoidal natural transformation between lax symmetric monoidal functors with values in \( \text{Gr} \text{ad} \text{z} \text{Sp}^P \). From this it follows that \( \delta_W \) is a composition of monoidal natural transformations, hence itself monoidal.

Now suppose that \( W \) is grouplike. In order to show that \( \delta_W \) is a stable equivalence for cofibrant \( X \) we first consider the case of a free \( W \)-space of the form \( F_{(d_1,d_2)}^W(*) \) and observe that in this case \( \delta_W \) can be identified with the map shown to be a stable equivalence in Proposition 4.20. It then follows by homotopy invariance that the lemma holds for \( W \)-spaces of the form \( F_{(d_1,d_2)}^W(D^n) \) and by induction on \( n \) that it holds for \( W \)-spaces of the form \( F_{(d_1,d_2)}^W(S^n) \). Here we have used that the stable model structure on \( \text{Sp}^P \) satisfies the gluing lemma [10, Theorem 7.4] and the monoid axiom [10, Proposition 12.5] in order to avoid making cofibrancy conditions on \( W \). It now follows by induction that the lemma holds for all cell complexes constructed from the generating cofibrations which gives the result since every cofibrant \( W \)-space is a retract of such a cell complex.

In the above lemma, the cofibrancy condition on \( X \) allowed us to avoid making cofibrancy conditions on \( W \). It will be important to have Thom isomorphism theorems for more general \( W \)-spaces and this requires us to make further assumptions on \( W \). For the rest of this section we fix a grouplike commutative \( W \)-space monoid \( W \) which we assume to be cofibrant and positive fibrant. This is not a serious restriction since the positive \( W \)-model structure on the commutative \( W \)-space monoids provided by Theorem 3.12 ensures that every grouplike commutative \( W \)-space monoid is \( W \)-equivalent to one that is cofibrant and positive fibrant.

**Definition 7.3.** A \( W \)-orientation of a \( W \)-space \( X \) is a diagram of the form

\[
X \leftarrow X' \rightarrow W
\]

where \( X' \) is a \( W \)-space and the first map is a \( W \)-equivalence as indicated.

A map of \( W \)-oriented \( W \)-spaces is defined in the obvious way as a pair of compatible maps relating the orientations. We say that \( X \) is \( W \)-orientable if it has a \( W \)-orientation. This notion is designed so as to be homotopy invariant: The positive fibrancy condition on \( W \) ensures that if \( X \) is \( W \)-orientable, then there exists a \( W \)-orientation in which \( X' \rightarrow X \) is a positive acyclic fibration. Since positive acyclic fibrations are preserved under pullback, this in turn implies that if \( Y \rightarrow X \) is a map of \( W \)-spaces and \( X \) is \( W \)-orientable, then also \( Y \) is \( W \)-orientable. Furthermore, if \( X \) is \( W \)-orientable and \( Y \rightarrow X \) is a \( W \)-equivalence, then clearly \( Y \) is also \( W \)-orientable.

Let \( X \) be a \( W \)-space equipped with a \( W \)-orientation as above and let as usual \( \overline{X'} \) denote the bar resolution on \( X' \). The given maps \( \overline{X'} \rightarrow X' \rightarrow W \) then defines an object in \( \text{Top}^W/W \) and we may form the composition

\[
S^W[W_0] \wedge S^W[\overline{X'}] \overset{\Delta_w}{\rightarrow} S^W[W] \triangle X_{hW} \rightarrow S^W[W] \triangle X_{hW}
\]

where the first map is the natural transformation (7.2) applied to \( \overline{X'} \) and the second map is induced by the map of homotopy colimits \( X_{hW} \rightarrow X_{hW} \). Here we identify the colimit \( \overline{X'}_{hW} \) with the homotopy colimit \( X_{hW}' \). Using the given maps \( \overline{X'} \rightarrow \overline{X} \rightarrow X \), we thus get a chain of maps of graded \( S^W[W_0] \)-modules

(7.3) \[ S^W[W_0] \wedge S^W[X] \leftarrow S^W[W_0] \wedge S^W[\overline{X'}] \rightarrow S^W[W] \triangle X_{hW}. \]

that are natural with respect to maps of \( W \)-oriented \( W \)-spaces.

**Lemma 7.4.** If the \( W \)-oriented \( W \)-space \( X \) is \( S^W \)-good, then the natural maps in (7.3) are stable equivalences.
Proof. We may assume without loss of generality that $X'$ is cofibrant. Then $X'$ is also cofibrant, so that the last map in (7.3) is a stable equivalence by Lemma 7.2. The first map is a stable equivalence since $X$ and the bar resolutions $X$ and $X'$ are $\mathbb{S}^W$-good. □

In the following we shall fix a map of commutative orthogonal ring spectra $\mathbb{S}^W[\mathbb{W} \langle 0 \rangle] \to R$ making $R$ a cofibrant $\mathbb{S}^W[\mathbb{W} \langle 0 \rangle]$-algebra in the sense of [11, Section 15]. The cofibrancy condition on $R$ is enough to ensure that extension of scalars $R \wedge \mathbb{S}^W[\mathbb{W} \langle 0 \rangle](-)$ from $\mathbb{S}^W[\mathbb{W} \langle 0 \rangle]$-modules to $R$-modules preserves stable equivalences. We define a periodic version of $R$ by writing $RP$ for the commutative graded orthogonal ring spectrum

$$RP = R \wedge \mathbb{S}^W[\mathbb{W} \langle 0 \rangle] \mathbb{S}^W[\mathbb{W}].$$

Thus, $RP$ depends on $W$ even though this is implicit in the notation. The terminology is justified by the fact that the stable equivalence in Proposition 4.20 gives rise to a stable equivalence $RP \simeq \bigvee_{n \in \mathbb{Z}} \Sigma^n R$, where $d$ denotes the periodicity of $W$ (that is, $d$ is a generator for the image of $\pi_0(W_\mathbb{W})$ in $\pi_0(BW) \cong \mathbb{Z}$). Applying the extension of scalars functor to the chain of stable equivalences in Lemma 7.2 we get the graded "Thom isomorphism" in the following theorem.

Theorem 7.5. Let $R$ be a cofibrant $\mathbb{S}^W[\mathbb{W} \langle 0 \rangle]$-algebra and let $X$ be an $\mathbb{S}^W$-good $\mathbb{W}$-space equipped with a $W$-orientation. Then there is a chain of natural stable equivalences $R \wedge \mathbb{S}^W[X] \simeq RP \triangledown X$ of graded $R$-modules. □

7.6. Multiplicative orientations. Let $\mathcal{D}$ be an operad of topological spaces in the sense of [11], and let $\mathcal{D}$ be the corresponding monad on $\mathbb{Top}^\mathbb{W}$ defined by

$$\mathbb{D}(X) = \coprod_{n \geq 0} \mathcal{D}(n) \times_{\Sigma_n} X^\otimes_n,$$

where $X^\otimes_0$ denotes the the monoidal unit $U^\mathbb{W}$. By definition, a $\mathcal{D}$-algebra in $\mathbb{Top}^\mathbb{W}$ is an algebra for the monad $\mathbb{D}$, and we write $\mathbb{Top}^\mathbb{W}[\mathcal{D}]$ for the category of $\mathcal{D}$-algebras. It follows from the universal property of the $\otimes$-product that a $\mathcal{D}$-algebra structure on a $\mathbb{W}$-space $X$ amounts to a sequence of natural transformations

$$\Theta_k : \mathcal{D}(k) \times X(n_1, n_2) \times \cdots \times X(n_k, n_k) \to X((n_1, n_2) \oplus \cdots \oplus (n_k, n_k))$$

of functors $\mathcal{W}^k \to \mathbb{Top}$, subjects to the usual associativity, unitality, and equivariance conditions, cf. [11] Lemma 1.4. The canonical projection of $\mathcal{D}$ onto the terminal "commutativity" operad allows us to view any commutative $\mathbb{W}$-space monoid as a $\mathcal{D}$-algebra.

Definition 7.7. Let $W$ be a grouplike commutative $\mathbb{W}$-space monoid that we assume to be cofibrant and positive fibrant. Then a $\mathcal{D}$-multiplicative $W$-orientation of a $\mathcal{D}$-algebra $X$ in $\mathbb{Top}^\mathbb{W}$ is a specified lift of a $W$-orientation $X \circlearrowleft X' \to W$ to a diagram in $\mathbb{Top}^\mathbb{W}[\mathcal{D}]$.

In connection with $\mathbb{W}$-spaces it is convenient to consider operads augmented over the Barratt-Eccles operad $\mathcal{E}$, that is, equipped with a map of operads $\mathcal{D} \to \mathcal{E}$. The reason for this comes from the following lemma which is proved by an argument similar to that used in the proof of the analogous result for $I$-spaces [21, Lemma 6.7].

Lemma 7.8. Let $\mathcal{D}$ be an operad augment over the Barratt-Eccles operad. Then the bar resolution $X \mapsto X$ induces a functor $\mathbb{Top}^\mathbb{W}[\mathcal{D}] \to \mathbb{Top}^\mathbb{W}[\mathcal{D}]$ such that the natural level equivalence $X \to X$ is a map of $\mathcal{D}$-algebras. □

Since the colimit functor on $\mathbb{Top}^\mathbb{W}$ is strong symmetric monoidal and we have the identification $X_\mathcal{W} = X_{\mathcal{E}\mathcal{W}}$, it follows from the lemma that the homotopy colimit functor induces a functor $\mathbb{Top}^\mathbb{W}[\mathcal{D}] \to \mathbb{Top}[\mathcal{D}]$, taking $X$ to $X_{\mathcal{E}\mathcal{W}}$. 
Theorem 7.5. An operad \( \mathcal{D} \) as above induces a monad \( \mathcal{D} \) on the category \( \text{Grad}_\mathbb{Z}\mathcal{S}^\mathcal{D} \) in the usual way,

\[
\mathcal{D}(E) = \bigvee_{n \geq 0} \mathcal{D}(n) \wedge \Sigma_n E^\wedge n,
\]

and we define a \emph{graded \( \mathcal{D} \)-algebra} to be an algebra for this monad. Given a commutative orthogonal ring spectrum \( R \), we shall use the term \emph{a graded \( \mathcal{D} \)-algebra under} \( R \) to mean a graded orthogonal spectrum \( A \) equipped with the structure of a graded \( \mathcal{D} \)-algebra and a map of graded \( \mathcal{D} \)-algebras \( R \to A \), where we view \( R \) as a graded \( \mathcal{D} \)-algebra concentrated in degree zero.

**Theorem 7.9.** Let \( R \) be a cofibrant commutative \( \mathcal{S}^W \)-algebra, let \( \mathcal{D} \) be an operad augmented over the Barratt-Eccles operad, and let \( X \) be an \( \mathcal{S}^W \)-good \( \mathcal{D} \)-algebra in \( \text{Top}^W \) equipped with a \( \mathcal{D} \)-multiplicative \( W \)-orientation. Then there is a chain of stable equivalences \( R \wedge \mathcal{S}^W[X] \simeq RP \triangle X_{h^W} \) of graded \( \mathcal{D} \)-algebras under \( R \).

**Proof.** We must show that with the given assumptions, the chain of stable equivalences in Theorem 7.3 is in fact a chain of maps of \( \mathcal{D} \)-algebras under \( R \). For this we use that the monoidal natural transformation \( \delta_W \) in Lemma 7.2 takes every \( \mathcal{D} \)-algebra over \( W \) to a map of graded \( \mathcal{D} \)-algebras under \( \mathcal{S}^W[W_{[0]}] \). By Lemma 7.8, this applies in particular to the \( \mathcal{D} \)-algebra \( X \to X' \to W \) given by the orientation. The result now follows since the extension of scalars along \( \mathcal{S}^W[W_{[0]}] \to R \) is a symmetric monoidal functor and hence preserves operad actions.

If the operad \( \mathcal{D} \) in Definition 7.7 is an \( E_\infty \)-operad, then we say that \( X \) has an \( E_\infty \) \( W \)-orientation. In case the \( E_\infty \) operad \( \mathcal{D} \) is not itself augmented over the Barratt-Eccles operad \( E \), we may replace it by the product operad \( \mathcal{D} \times E \), which is then an \( E_\infty \) operad augmented over \( E \).

**Corollary 7.10.** Suppose that the operad \( \mathcal{D} \) in Theorem 7.9 is an \( E_\infty \) operad (such that \( X \) has an \( E_\infty \) orientation). Then \( R \wedge \mathcal{S}^W[X] \simeq RP \triangle X_{h^W} \) as graded \( E_\infty \) \( R \)-algebras.

The above corollary applies in particular when \( M = X \) is a cofibrant commutative \( W \)-space monoid equipped with a map of commutative \( W \)-space monoids \( M \to W \), in which case we may set \( \mathcal{D} = E \). In general, the corollary shows that the graded Thom isomorphism is compatible with the Dyer-Lashof operations arising from the \( E_\infty \) structures.

For the algebraic version of the graded Thom isomorphism, it will be convenient to introduce some notation: Given a commutative orthogonal ring spectrum \( R \) and a \( W \)-space \( X \), we write

\[
R_\mathbb{S}(X_{h^W}) = \bigoplus_{n \in \mathbb{Z}} \Sigma^n R_\ast(X_{h^W(n)})
\]

where \( R_\ast(\cdot) \) is the unreduced homology theory defined by \( R \) and \( \Sigma^n R_\ast(X_{h^W(n)}) \) denotes the \( \mathbb{Z} \)-graded \( \pi_\ast(R) \)-module with \( k \)-th term \( R_{k-n}(X_{h^W(n)}) \).

**Corollary 7.11.** With \( R \) and \( X \) as in Corollary 7.10, there is an isomorphism of graded \( \pi_\ast(R) \)-algebras \( R_\mathbb{S}(\mathcal{S}^W[X]) \cong R_\mathbb{S}(X_{h^W}) \).

7.12. **Orientability with respect to the Stiefel \( W \)-spaces \( V \) and \( V_{ev} \).** Now we specialize to orientations with respect to the Stiefel \( W \)-spaces \( V \) and \( V_{ev} \) which are of course closely related to orientations with respect to the Eilenberg–Mac Lane spectra \( H\mathbb{Z}/2 \) and \( H\mathbb{Z} \). In the case of \( V \), we can realize the 0th Postnikov section \( \mathcal{S}^W[V_{[0]}] \to H\mathbb{Z}/2 \) as a map of commutative orthogonal ring spectra (see e.g. [22]). Using the positive stable model structure on commutative \( \mathcal{S}^W[V_{[0]}] \)-algebras, we
may further assume that $HZ/2$ is cofibrant and positive fibrant as an $S^W[V_{(0)}]$-algebra. Passing to the graded units, this in turn gives rise to a $W$-equivalence $V_{(0)} \to GL^W_1(HZ/2)$. Clearly, every $W$-space is orientable with respect to $V$ (or, according to our conventions, a cofibrant replacement of $V$) which gives the graded Thom isomorphism

$$H_*(S^W[X], Z/2) \cong HZ/2P_{\oplus}(X_{hW}) = \bigoplus_{n \in \mathbb{Z}} \Sigma^n H_*(X_{hW(2^n)}, Z/2)$$

where $HZ/2P$ denotes the commutative graded Eilenberg–Mac Lane spectrum defined as $HZ/2 \wedge_{S^W[V_{(0)}]} S^W[V]$.

In the oriented case of $\tilde{V}_{ev}$, the 0th Postnikov section $S^W[\tilde{V}_{(0)}] \to HZ$ may again be realised as a map of commutative orthogonal ring spectra making $HZ$ a cofibrant and positive fibrant $S^W[V_{(0)}]$-algebra. Passing to graded units, we now get a $W$-equivalence $\tilde{V}_{(0)} \to GL^W_1(HZ)$. This shows that the condition for a $W$-space to be orientable with respect to $\tilde{V}_{ev}$ (or according to our conventions, a cofibrant replacement of $\tilde{V}_{ev}$) is compatible with the usual notion of orientability. In this case we thus get an integral graded Thom isomorphism

$$H_*(S^W[X], Z) \cong HZP_{ev\oplus}(X_{hW}) = \bigoplus_{n \in \mathbb{Z}} \Sigma^{2n} H(X_{hW(2^n)}, Z)$$

where $HZP_{ev}$ denotes the commutative evenly graded Eilenberg–Mac Lane spectrum defined as $HZ \wedge_{S^W[V_{(0)}]} S^W[\tilde{V}_{ev}]$.

7.13. Orientations and graded Thom isomorphisms for spaces over $Gr_{hV}$.

The orientation theory for $W$-spaces is extended to spaces over $Gr_{hV}$ via the Quillen equivalences

$$Top/Gr_{hV} \simeq Top^V/Gr \simeq Top^W/V \simeq Top^W$$

resulting from Propositions 6.1 and 6.3. Thus, for $W$ a grouplike commutative $W$-space monoid, we say that $f: X \to Gr_{hV}$ is $W$-orientable if the corresponding $W$-space has a $W$-orientation in the sense of Definition 7.3. In the situation of Theorem 7.5 we then get a chain of stable equivalences

$$R \wedge T(f) \simeq RP \triangle X,$$

where we view $X$ as a $\mathbb{Z}$-graded space with nth term $X_{(n)} = f^{-1}(Gr_{hW(2^n)})$. It follows from Theorems 6.5 and 7.9 that this is multiplicative in the natural sense: If $f$ is a map of $\mathcal{D}$-algebras for an operad augmented over the Barratt-Eccles operad, then the stable equivalences in question are maps of $\mathcal{D}$-algebras. On the level of stable homotopy groups we get an isomorphism

$$R_*(T(f)) \cong R_{\oplus}(X) = \bigoplus_{n \in \mathbb{Z}} \Sigma^n R_*(X_{(n)}).$$

This applies in particular for $R = HZ$ provided that $f$ is homotopic to a map that factors through $Gr_{hV}$.

8. The graded Thom isomorphism for commutative $J$-space monoids

In this section we explain how the multiplicative Thom isomorphism for $W$-spaces studied in Section 7.9 gives rise to a Thom isomorphism for the graded suspension spectra of commutative $J$-space monoids. Moreover, we outline why this has interesting consequences for our joint work with John Rognes on logarithmic topological Hochschild homology [16].

Let $J$ be the category that is defined analogous to $W$, but with symmetric groups replacing orthogonal groups. The homotopy theory of $J$-spaces and commutative $J$-space monoid is extensively studied in [18 Section 4]. The category
of \(J\)-spaces is related to the category of symmetric spectra \(Sp^S\) by an adjunction \(S^J: Top^J \rightleftarrows Sp^S: \Omega^J\) that is analogous to the adjunction \((Sp^W, \Omega^W)\) from Proposition 4.4, see [18, Proposition 4.23].

To relate \(J\)- and \(W\)-spaces, we note that the strong symmetric monoidal functor \(\Sigma \to O\) sending a permutation to the associated isometry induces a functor \(\Psi: J \to W\). We will use the notation \((\Psi_\ast, \Psi^\ast)\) for both the resulting adjunction relating \(J\)- and \(W\)-spaces and the resulting adjunction relating symmetric and orthogonal spectra. These functors fit into a diagram of adjunctions

\[
\begin{array}{ccc}
Top^J & \overset{\Psi_\ast}{\longrightarrow} & Top^W \\
\Psi^\ast \downarrow & & \downarrow \Psi^\ast \\
Sp^E & \overset{\Psi_\ast}{\longrightarrow} & Sp^0
\end{array}
\]

where the square of right adjoints commutes and the square of left adjoints commutes up to isomorphism. Since all left adjoints are strong symmetric monoidal functors, there are corresponding squares of adjunctions for the associated categories of algebras over an operad \(\mathcal{D}\). Moreover, all adjunctions are Quillen adjunctions with respect to the absolute or positive model structures on the respective categories. We also note that as in the case of \(S^W\), there is a notion of \(S^J\)-good \(J\)-spaces on which \(S^J\) captures the homotopy type of the left derived functor. Useful criteria for \(S^J\)-goodness are developed in [18, Appendix A].

**Remark 8.1.** Although the lower horizontal adjunction in (8.1) is a Quillen equivalence [10, Theorem 10.4], this is not true for the upper horizontal adjunction relating \(J\)- and \(W\)-spaces. This follows by example by combining the facts that \(Top^J\) is Quillen equivalent to \(Top/BJ\) and that \(Top^W\) is Quillen equivalent to \(Top/BW\) with the observation that \(\Psi_\ast: BJ \to BW\) does not induce a Quillen equivalence between these comma categories.

Returning to the setup of Section 7.6, we let \(W\) be a grouplike commutative \(W\)-space monoid that is cofibrant and fibrant in the positive model structure on \(Top^W[C]\). Then \(\Psi^\ast(W)\) is a commutative \(J\)-space monoid, and implementing Definition 7.7 in the category \(Top^J[E]\) of \(E_\infty\) \(J\)-space monoids over the Barratt-Eccles operad provides a notion of a \(E_\infty\) \(\Psi^\ast(W)\)-orientation of a commutative \(J\)-space monoid.

In order to obtain a graded Thom isomorphism for such objects, we again fix a cofibrant \(S^W[W(0)]\)-algebra \(R\), let \(R^{cof} \to \Psi^\ast(R)\) be a cofibrant replacement of the underlying commutative symmetric ring spectrum, and denote the underlying commutative symmetric ring spectrum \(\Psi^\ast(RP)\) of \(RP\) also simply by \(RP\).

**Theorem 8.2.** If \(X \leftarrow X' \to \Psi^\ast(W)\) is an \(S^J\)-good \(E_\infty\) \(\Psi^\ast(W)\)-oriented commutative \(J\)-space monoid, then there is a natural chain of stable equivalences

\[R^{cof} \wedge S^J[X] \simeq RP \Delta X_{hJ}\]

of \(E_\infty\) symmetric spectra over the Barratt-Eccles operad.

**Proof.** We may without loss of generality assume that \(X' \to X\) is a cofibrant replacement in the absolute projective model structure on \(Top^J[E]\) provided by [18, Proposition 9.3(i)]. Then [18, Corollary 12.3] implies that the underlying \(J\)-space of \(X'\) is cofibrant in the absolute projective model structure on \(Top^J\). Hence it follows from the definition of \(S^J\)-goodness that \(S^J[X'] \to S^J[X]\) is a stable equivalence. Moreover, the fact that \(\Psi_\ast: Top^J \to Top^W\) is left Quillen with respect to the absolute model structures implies that \(\Psi_\ast(X')\) is cofibrant and hence in particular \(S^W\)-good.
Since $\Psi_*: \text{Sp}^\Sigma \rightleftarrows \text{Sp}^\Omega: \Psi^*$ is a Quillen equivalence whose right adjoint preserves stable equivalences between all objects, the adjoint $\Psi_*(R^{\text{cof}}) \to R$ of the above cofibrant replacement is a stable equivalence. This provides the following stable equivalence
\[
\Psi_*(R^{\text{cof}} \wedge S^J(X^*)) \cong \Psi_*(R^{\text{cof}}) \wedge S^W[\Psi_*(X^*)] \sim R \wedge S^W[\Psi_*(X^*)].
\]
By Theorem 7.9, $R^\Lambda \wedge S^W[\Psi_*(X^*)]$ and $RP \wedge (\Psi_*(X^*))_{hW}$ are naturally stably equivalent as orthogonal spectra with $E$-action. Therefore, another application of the above Quillen equivalence implies that $R^{\text{cof}} \wedge S^J[X^*]$ is naturally stably equivalent to $\Psi^*(RP \wedge (\Psi_*(X^*))_{hW})$. The fact that the latter object is stably equivalent to $RP \Delta X_{hJ}$ completes the proof. 

We now specialize to the case where $W$ is a cofibrant replacement $V_{ev}^{\text{cof}}$ of the even oriented Stiefel $W$-space $V_{ev}$. One can check from the definition of the homotopy colimit and the explicit description of $V_{ev}$ in (5.2) that the underlying commutative $J$-space monoid of $V_{ev}$ is grouplike.

In [17], the first author constructed a natural group completion $M \to M^\text{gp}$ for commutative $J$-space monoids. It has the universal property that any map of commutative $J$-space monoids extends over the group completion.

Let $(n_1, n_2)$ be an object of $J$ with $n_1 \geq 1$ and $n_2 - n_1$ even. We shall write $C(n_1, n_2)$ for the free commutative $J$-space monoid on a point in degree $(n_1, n_2)$. It has the universal property that maps of commutative $J$-space monoids $C(n_1, n_2) \to M$ correspond to points in $M(n_1, n_2)$. Since $\Phi^*(V_{ev})$ is non-empty in even degrees, there exists a map $C(n_1, n_2) \to \Phi^*(V_{ev})$ in $\text{Top}^J[C]$. Moreover, since $\Phi^*(V_{ev})$ is grouplike and positive fibrant, the universal property of the group completion implies that there exists a map of commutative $J$-space monoids
\[
(8.2) \quad C(n_1, n_2)^{\text{gp}} \to \Phi^*(V_{ev}).
\]
Since the group completions in [17] were set up in the context of $J$-spaces with values in simplicial sets and the paper [15] uses $J$-spaces with values in simplicial sets, we now implicitly apply the singular complex functor to pass to the simplicial set versions of all $J$-spaces and symmetric spectra considered for the rest of this section. (For $J$-spaces, this transition is explained in [18, Remark 9.7].)

Using the map $S^W[V_{ev}] \to HZ$ considered in Section 7.12, the above map (8.2) and Theorem 8.2 imply the following proposition.

**Proposition 8.3.** Let $(n_1, n_2) \in J$ with $n_2 - n_1$ even and let $M \to C(n_1, n_2)^{\text{gp}}$ be a commutative $J$-space monoid over $C(n_1, n_2)^{\text{gp}}$. Then $S^J[M] \wedge HZ$ and $\bigvee_{k \in \mathbb{Z}} (M_{hJ})_+ \wedge (HZP)_{even,k}$ are stably equivalent as $E_\infty$ symmetric spectra. The chain of stable equivalences is natural with respect to morphisms of commutative $J$-space monoids over $C(n_1, n_2)^{\text{gp}}$.

**Proof.** The only thing that does not immediately follow from Theorem 8.2 is the fact that we can skip the assumption on the $S^J$-goodness of $M$. Since we have switched to $J$-spaces based on simplicial sets, this condition follows from $C(n_1, n_2)^{\text{gp}}$ being cofibrant in $\text{sSet}^J[C]$ by [15, Lemma A.6]. 

The previous proposition is a useful tool for our calculations of logarithmic topological Hochschild homology ("log THH" for short) in joint work with John Rognes [16]. Log THH is an extension of the usual topological Hochschild homology to so-called pre-log ring spectra. By definition, a pre-log ring spectrum $(A, M, \alpha)$ is a commutative symmetric ring spectrum $A$ together with a map of commutative $J$-space monoids $M \to \Omega^J(A)$. We are interested in log THH for various reasons, for example because it allows for localization homotopy cofiber sequences that do
not exist for ordinary THH, and because it can be a useful tool for computations of ordinary THH.

Topological $K$-theory spectra like $ko$ or $ku$ can be extended to pre-log ring spectra whose underlying commutative $J$-space monoid is $J$-equivalent to $\mathbb{C}\langle n_1, n_2 \rangle_{\geq 0}^{sp}$, the non-negative part of $\mathbb{C}\langle n_1, n_2 \rangle^{sp}$, where $n_2 - n_1$ is the degree of the respective Bott element. When trying to compute the log THH of a pre-log ring spectrum $(A, M)$, it is useful to know the homology of the commutative symmetric ring spectra $S^T[M]$. When $M = \mathbb{C}\langle n_1, n_2 \rangle_{\geq 0}^{sp}$, the previous proposition identifies the homology of $S^T[M]$ with the the homology of the space $(\mathbb{C}\langle n_1, n_2 \rangle_{\geq 0}^{sp})_{hJ}$. This space turns out to be homotopy equivalent to $Q_{\geq 0} S^0$, the non-negative components of $Q S^0 = \Omega^\infty \Sigma^\infty S^0$, and its homology (with $F_p$-coefficients) is well understood.

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