Sums of prime element orders in finite groups
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ABSTRACT
Let $G$ be a finite group and $c^*(G)$ denote the sum of prime element orders of $G$. This paper presents some properties of $c^*$ and investigate the minimum value and the maximum value of $c^*$ on the set of groups of the same order.

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1. Introduction
Motivated by the works of Amiri, Jafarian Amiri and Isaacs [1–4] and Shen et al. [5] in the study of $\psi(G)$ – the sum of element orders of a finite group $G$, we will introduce, in this paper, another function denoted by $\psi_*(G)$, which is the sum of prime element orders. More precisely, the function $\psi_*$ is defined as follows:

\[ \psi_*(G) = \sum_{k \in V(G) | k \text{ is prime}} o(x), \]

where $o(x)$ is the order of the element $x$.

In Section 2, we give the preliminary definitions and results about the functions $\psi(G)$ and $\psi_*(G)$. Particularly, we will show that if $G_1$ and $G_2$ are two finite groups, then $\psi_*(G_1 \times G_2) = \psi_*(G_1) + \psi_*(G_2)$ if and only if the order of $G_1$ and that of $G_2$ are relatively prime, and as a consequence, we will prove that if $G$ is a nilpotent group of order $n$, then $\psi_*(H) \leq \psi_*(G)$ for every nilpotent group $H$ of order $n$ if and only if every Sylow subgroup of $G$ has prime exponent. Section 3 presents the main results of this work in the study of the minimum value and the maximum value of $\psi_*$ on the set of groups of the same order. More precisely, the main results are:

1. Let $G$ be a finite group. Then $\psi_*(G) \geq \psi_*(C)$ for every cyclic group $C$ of the same order as $G$.

2. Let $n$ be an integer which is not a nilpotent number and $\max\{|\psi_*(G) | |G| = n\} = \psi_*(K)$ for some group $K$ of order $n$. Then $K$ is not nilpotent.

2. Preliminaries and basic results
This section presents some results and notations that will be useful in the sequel. Given a finite group $G$, let:

\[ \Omega(G) \]
\[ \Omega_p(G) \]
\[ \Omega_1(G) = \{x \in G \mid o(x) = k\} \text{ for all } k \in \Omega(G). \]

Definition 2.1: We define the area of $G$ (the sum of element orders of $G$) as follows:

\[ \psi(G) = \sum_{k \in \Omega(G)} k|S_k|. \]

Definition 2.2: We define the prime area of $G$ (the sum of prime element orders of $G$) as follows:

\[ \psi_*(G) = \sum_{p \in \Omega_p(G)} k|S_p|. \]
Example 2.3: If \( o(G) = p \) is a prime number, then

1. \( \psi(G) = p(p - 1) + 1, \)
2. \( \psi_*(G) = p(p - 1). \)

Proposition 2.4: For every normal \( N \) subgroup of \( G \), we have

\[
\psi_*(G) \leq \psi_*(G/N) + \psi_*(N). \quad (4)
\]

Proof: Remark that if \( x \) is an element in \( G \) such that \( o(x) = p \) for some prime number \( p \), then \( (xN)^p = N \). Hence, \( o(xN) = p \) or \( o(xN) = 1 \) as an element in \( G/N \).

That means \( o(xN) \leq p \) or \( x \in N \). Therefore, \( |\mathcal{S}_p(G)\| \subseteq |\mathcal{S}_p(G/N)\| + |\mathcal{S}_p(N)\|. \)

Hence, \( \psi_*(G) \leq \psi_*(G/N) + \psi_*(N). \)

Proposition 2.5: Let \( G \) be a nonabelian group of order \( 2p \), where \( p \) is a prime number greater than or equal to 3.

1. \( \psi(G) = p(p + 1) + 1, \)
2. \( \psi_*(G) = p(p + 1). \)

Proof: It is well known that if \( G \) is a nonabelian group of order \( 2p \), where \( p \) is a prime number greater than or equal to 3, then \( G \cong D_{2p} \), and since the dihedral group \( D_{2p} \) has 1 element of order 1, \( p \) element of order 2, and \( p-1 \) element of order \( p \), we obtain \( \psi(G) = p(p + 1) + 1 \) and \( \psi_*(G) = p(p + 1). \)

Lemma 2.6: Let \( G_1 \) and \( G_2 \) be two finite groups, then

\[
\psi_*(G_1 \times G_2) = \psi_*(G_1) + \psi_*(G_2) + \sum p|\mathcal{S}_p(G_1)\| \cdot |\mathcal{S}_p(G_2)\|. \quad (5)
\]

Proof: Let \( (x, y) \in \mathcal{S}_p(G_1 \times G_2) \), then lcm \( (o(x), o(y)) = p \). Therefore, \( (o(x), o(y)) = (1, p) \) or \( (o(x), o(y)) = (p, 1) \) or \( (o(x), o(y)) = (p, p) \). Hence, for all prime \( p \), we have

\[
\mathcal{S}_p(G_1 \times G_2) = \mathcal{S}_p(G_1) \times \{e_2\} \cup \{e_1\} \times \mathcal{S}_p(G_2)
\]

where \( e_1 \) and \( e_2 \) are, respectively, the identity element of \( G_1 \) and \( G_2 \). Then, we have

\[
|\mathcal{S}_p(G_1 \times G_2)| = |\mathcal{S}_p(G_1)| + |\mathcal{S}_p(G_2)| + |\mathcal{S}_p(G_1)| \cdot |\mathcal{S}_p(G_2)|. \quad (6)
\]

Hence

\[
\psi_*(G_1 \times G_2) = \psi_*(G_1) + \psi_*(G_2) + \sum p|\mathcal{S}_p(G_1)\| \cdot |\mathcal{S}_p(G_2)\|. \quad (8)
\]

\[\square\]

Theorem 2.7: Let \( G_1 \) and \( G_2 \) be two finite groups. Then, the following statements are equivalent:

1. \( \psi_*(G_1 \times G_2) = \psi_*(G_1) + \psi_*(G_2). \)
2. \( \text{gcd}(|G_1|, |G_2|) = 1. \)

Proof: Since \( \text{gcd}(|G_1|, |G_2|) = 1 \), for all \( (x, y) \in (G_1 \times G_2) \) : \( o(x, y) = o(x) \cdot o(y) \). Then, \( o(x, y) \in \mathcal{S}_p(G_1 \times G_2) \) is equivalent to \( o(x) \cdot o(y) = p \).

That means \( (o(x), o(y)) = (1, p) \) or \( (o(x), o(y)) = (p, 1) \). Hence, for all prime \( p \), we have

\[
\mathcal{S}_p(G_1 \times G_2) = \mathcal{S}_p(G_1) \times \{e_2\} \cup \{e_1\} \times \mathcal{S}_p(G_2). \quad (9)
\]

It follows that

\[
|\mathcal{S}_p(G_1 \times G_2)| = |\mathcal{S}_p(G_1)| + |\mathcal{S}_p(G_2)|. \quad (10)
\]

Hence

\[
\psi_*(G_1 \times G_2) = \psi_*(G_1) + \psi_*(G_2). \quad (11)
\]

Reciprocally, assume that \( \psi_*(G_1 \times G_2) = \psi_*(G_1) + \psi_*(G_2) \). Using the previous lemma, we obtain

\[
\sum p|\mathcal{S}_p(G_1)| \cdot |\mathcal{S}_p(G_2)| = 0. \quad (12)
\]

Hence, for all prime number \( p \) we have \( |\mathcal{S}_p(G_1)| = 0 \) or \( |\mathcal{S}_p(G_2)| = 0 \). Consider the contrary that means \( |G_1|, |G_2| \neq 1 \). Then, there is a prime number \( p \) dividing both \( |G_1| \) and \( |G_2| \). Applying the Cauchy theorem, there are elements \( x \in G_1 \) and \( y \in G_2 \) such that \( |x| = |y| = p \). This is a contradiction to \( \mathcal{S}_p(G_1) = 0 \) or \( \mathcal{S}_p(G_2) = 0 \).

\[\square\]

Proposition 2.8: If \(|G| = pq\), where \( p \) and \( q \) are distinct prime numbers, then:

\[
\psi_*(G) = p(p - 1) + q(q - 1). \]

Proof: By the Cauchy theorem, there exists an element \( a \) (resp. \( b \)) in \( G \) of order \( p \) (resp. \( q \)). Let \( H = \langle a \rangle \) and \( K = \langle b \rangle \), then

\[
|HK| = \frac{|H| \cdot |K|}{|H \cap K|}. \quad (13)
\]

Since \( |H \cap K| = 1 \), we get \( HK = G \). Then, the map \( f : H \times K \to G \) defined by \( f(x, y) = xy \) is an
isomorphism. Applying Theorem 2.7, we obtain
\[ \psi_s(G) = \psi_s(H) + \psi_s(K) = p(p - 1) + q(q - 1). \] (14)

**Theorem 2.9:** Let \( G \) be a nilpotent group of order \( n \). Then, the following are equivalent:

1. \( \psi_s(H) \leq \psi_s(G) \) for every nilpotent group \( H \) of order \( n \).
2. Every Sylow subgroup of \( G \) has a prime exponent.

**Proof:** Put \( n = p_1^{n_1} \cdots p_r^{n_r} \) where \( p_1, \ldots, p_r \) are distinct primes and \( n_i \) are positive integers. Recall that a group is nilpotent if and only if it is the direct product of its Sylow subgroups [6,p.126]. Let \( H \) be a nilpotent group of order \( n \). Then
\[ H = P_1 \times P_2 \times \cdots \times P_r, \] (15)
where \( P_i \) is the Sylow \( p_i \)-subgroup of \( H \). From Theorem 2.7, we obtain
\[ \psi_s(H) = \sum_{i=1}^r \psi_s(P_i). \] (16)

Therefore
\[ \psi_s(H) \leq \sum_{i=1}^r p_i(p_i^{n_i} - 1). \] (17)

If every Sylow subgroup of \( G \) has a prime exponent, then
\[ \psi_s(G) = \sum_{i=1}^r p_i(p_i^{n_i} - 1). \] (18)

**Corollary 2.10:** Let \( G \) be a finite group of order \( n = p_1^{n_1} \cdots p_r^{n_r} \) where \( p_1, \ldots, p_r \) are distinct primes and \( n_i \) are positive integers. Then the following statements are equivalent:

1. \( G \) is not nilpotent,
2. \( \psi_s(G) > \sum_{i=1}^r p_i(p_i^{n_i} - 1) \),
3. \( \psi(G) < \prod_{i=1}^r p_i(p_i^{n_i} - 1) + 1 \).

**Proof:** The equivalence (1)\(\iff\)(2) is a direct consequence of the previous theorem, and the equivalence (1)\(\iff\)(3) is proved by Amiri and Jafari Amiri in [1, Corollary 2.2].

\[ \psi_s(G) \geq \psi_s(C) \] for every cyclic group of the same order as \( G \).

**Theorem 3.3:** Let \( G \) be a finite group. Then
\[ \psi_s(G) \geq \psi_s(C) \] for every cyclic group of the same order as \( G \).

**Proof:** Let \( \varphi \) be Euler’s phi-function. It is well known that if \( C \) is a cyclic group of order \( n \), and \( r \) is a positive divisor of \( n \), then the group \( C \) has \( \varphi(r) \) elements of order \( r \). Then
\[ \psi_s(C) = \sum_{p_i \in \Omega_s(C)} \psi(p_i) = \sum_{p_i \in \Omega_s(C)} \psi(p_i - 1). \] (21)

Using Corollary 3.2, we obtain that \( \psi_s(G) \geq \psi_s(C) \).

In the following, if \( d \) is a positive integer, we say that \( d \) satisfies the property \( N(d) \) if
\[ \max(\psi_s(G) \mid |G| = d) = \psi_s(K) \] for some group \( K \) of order \( d \), then \( K \) is not nilpotent.
Lemma 3.4: Let \( d \) be a positive integer that satisfies the property \( N(d) \). Then \( n \}=ds \) satisfy the property \( N(n) \) for all positive integers \( s \) such that \( \gcd(d, s) = 1 \).

Proof: Let \( G \) be a group of order \( n=ds \) such that 
\[
\psi_s(G) = \max(\psi_s(H) \mid |H| = n).
\]
If \( G \) is nilpotent, then \( G \) can be written as \( G = G_1 \times G_2 \), where \( |G_1| = d \) and \( |G_2| = s \). By hypothesis, there exists a not nilpotent group \( K \) of order \( d \) such that \( \psi_s(K) > \psi_s(G_1) \). Let \( H = K \times G_2 \). Then \( H \) is a not nilpotent group of order \( ds \) and 
\[
\psi_s(H) = \psi_s(K) + \psi_s(G_2) > \psi_s(G_1) + \psi_s(G_2) = \psi_s(G_1 \times G_2) = \psi_s(G).
\]

Definition 3.5 (\cite{7}): A positive integer \( n \) is called nilpotent number if every group of order \( n \) is nilpotent.

Lemma 3.6 (\cite{7}, Theorem 1): Let \( n = p_1^{n_1} \cdot \cdots \cdot p_r^{n_r} \) be an integer where \( p_1, \ldots, p_r \) are distinct primes and \( n_i \) are positive integers. Then \( n \) is nilpotent number if and only if \( p_i \neq 1 \mod p_j \) for all integers \( i,j \) and \( k \) with \( 1 < k < n_i \).

Theorem 3.7: Let \( n \) be an integer which is not a nilpotent number. Assume that \( \max|\psi_s(G) \mid |G| = n| = \psi_s(K) \) for some group \( K \) of order \( n \). Then \( K \) is not nilpotent.

Proof: The proof of this theorem is similar to that of Amiri and Jafarian Amiri mentioned in \cite{1}, Theorem. But we prefer to write it step by step to clarify certain changes for the reader. If \( n \) is not a nilpotent number, then there exists a group \( G \) not nilpotent of order \( n \), two prime numbers \( p \) and \( q \) in \( \Omega_n(G) \), and an integer \( i \) such that \( p \mid q^i - 1 \) but \( pp^i \) \( \cdot \) 1 for all \( j<i \). We can write the order of \( G \) as \( n = p^m q^k \), where \( \gcd(pq, k) = 1 \). As
\[
|\operatorname{Aut}((\mathbb{Z}_q)^i)| = \prod_{j=0}^{i-1}(q^j - q^i).
\]
we can find an element \( \varphi \) of order \( p \) in \( \operatorname{Aut}((\mathbb{Z}_q)^i) \). Let \( f : \mathbb{Z}_q \to \mathbb{Z}_q \) be the group homomorphism defined by \( f(a) = \varphi a \) where \( a \) is a generator of \( \mathbb{Z}_p \). The semidirect product \( \mathbb{Z}_p \rtimes (\mathbb{Z}_q)^i \), of \( \mathbb{Z}_p \) and \( (\mathbb{Z}_q)^i \) with respect to \( f \), is a not nilpotent group of order \( pq^i \). By assumption on \( p \) and \( q \), the group \( \mathbb{Z}_p \rtimes (\mathbb{Z}_q)^i \) has \( \varphi \) Sylow \( p \)-subgroup. Therefore, \( S_p(\mathbb{Z}_p \rtimes (\mathbb{Z}_q)^i) = q^i(p - 1) \) and \( S_q(\mathbb{Z}_p \rtimes (\mathbb{Z}_q)^i) = q^i - 1 \). Hence
\[
\psi_s(\mathbb{Z}_p \rtimes (\mathbb{Z}_q)^i) = pq^i(p - 1) + q(q^i - 1).
\]
Let \( T = (\mathbb{Z}_p \rtimes (\mathbb{Z}_q)^i) \times \mathbb{Z}_p^{m-1} \times \mathbb{Z}_q^{r-1} \). Then \( T \) is a not nilpotent group of order \( p^m q^r \). It is easy to see that \( \mathbb{S}_p(T) = q^i(p - 1)p^m-1 + p^m-1 - 1 \) and \( S_q(T) = q^i - 1 \). So
\[
\psi_s(T) = p(q^i(p - 1)p^m-1 + p^m-1 - 1) + q(q^i - 1).
\]
In addition
\[
\psi_s(\mathbb{Z}_p^{m-1} \times \mathbb{Z}_q^{r-1}) = \psi_s(\mathbb{Z}_p^m) + \psi_s(\mathbb{Z}_q^i) = p(p^m - 1) + q(q^i - 1).
\]
Therefore
\[
\psi_s(T) - \psi_s(\mathbb{Z}_p^{m-1} \times \mathbb{Z}_q^{r-1}) = p(q^i(p - 1)p^m-1 + p^m-1 - 1) - p(p^m - 1)
\]
\[
= p(q^i - 1) + p^m - p^m = q^i(q^i - 1) - 1.
\]
Since \( \mathbb{Z}_p^{m-1} \times \mathbb{Z}_q^{r-1} \) has the greatest \( \psi_s(H) \) among all nilpotent groups \( H \) of order \( d = p^m q^r \), the integer \( d = p^m q^r \) satisfy the property \( N(d) \). Lemma 3.4 completes the proof.

4. Conclusion

This paper determines the minimum value and the maximum value of \( \psi_s \) on the set of groups of the same order. More precisely, it is proved that a cyclic group \( G \) can be characterized by its order and the value of \( \psi_s \) at \( G \). That means, if \( C \) is a finite cyclic group, then \( \psi_s(G) < \psi_s(C) \) for all noncyclic groups \( G \) of the same order as \( C \). On the other hand, it is given in this paper a new characterization, for nilpotent groups, announced as follows: if \( n \) be an integer which is not a nilpotent number and \( \max|\psi_s(G) \mid |G| = n| = \psi_s(K) \) for some group \( K \) of order \( n \), then \( K \) is not nilpotent.

Disclosure statement

No potential conflict of interest was reported by the authors.

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