Stability Analysis of Linear Uncertain Systems via Checking Positivity of Forms on Simplices *

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Abstract: In this paper, we mainly study the robust stability of linear continuous systems with parameter uncertainties, a more general kind of uncertainties for system matrices is considered, i.e., entries of system matrices are rational functions of uncertain parameters which are varying in intervals. We present a method which can check the robust Hurwitz stability of such uncertain systems in finite steps. Examples show the efficiency of our approach.

Key words: linear uncertain system; stability

AMS subject classification(2000): 34D10, 34D20

1 Introduction

Given a continuous linear time-invariant system in the state space model, its Hurwitz stability is determined by the distribution of eigenvalues of the system matrix. When entries of the system matrix are uncertain, e.g., they are varying in intervals, the robust stability of such a system have been studied in a large amount of literatures. First, attempts were made to find a Kharitonov-like criterion [1] of the stability of an interval matrix which only checks some extreme matrices [2], but the criterion was found to be false [3]. Later, necessary and sufficient criterions were proposed for interval matrices with special properties (e.g., real symmetric interval matrices [7] or Hermitian interval matrices [8]). At the same time, various sufficient criterions were found to check the stability of interval matrices [4, 5, 6, 7].

In this paper, we study a more general kind of uncertainty of system matrices, i.e., entries of system matrices are rational functions of uncertain parameters which are bounded by intervals. We will present a complete method which can check the robust stability of such systems in finite steps.

*Partially supported by a National Key Basic Research Project of China (2004CB318000) and by National Natural Science Foundation of China (10571095)
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2 Main Results

Denote by \( \mathbb{R} \) the field of real numbers, the system matrix \( A \in \mathbb{R}^{n \times n} \) in \( \dot{x}(t) = Ax(t) \) is called Hurwitz stable if all its eigenvalues lie in the open left half complex plane. When \( A \) is continuously varying in \( \mathbb{R}^{n \times n} \), i.e., \( A \) is in a connected set \( A \subset \mathbb{R}^{n \times n} \), we say \( A \) is robustly Hurwitz stable if each \( A \in A \) is Hurwitz stable.

[14] showed that the system matrix with polytopic uncertainty is robustly Hurwitz stable if and only if a Hurwitz stable matrix exists and two forms (i.e., homogenous polynomials) are positive on the standard simplex. In fact, we could come to a similar conclusion for \( A \). Denote the characteristic polynomial of \( A \) by \( f_A(s) \triangleq \det(sI_n - A) = s^n + a_{n-1}s^{n-1} + \ldots + a_1s + a_0 \), (1)

and the Hurwitz matrix of \( f_A(s) \) by \( \Delta_A \), which is an \( n \times n \) matrix defined as

\[
\Delta_A = \begin{pmatrix}
a_{n-1} & a_{n-3} & a_{n-5} & \cdots & 0 \\
1 & a_{n-2} & a_{n-4} & \cdots & 0 \\
0 & a_{n-1} & a_{n-3} & \cdots & 0 \\
0 & 1 & a_{n-2} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]

The successive principal minors of \( \Delta_A \) are denoted by \( \Delta_k \), \( k = 1, 2, \ldots, n \). Then we have

**Theorem 1.** Suppose some \( A \in A \) is Hurwitz stable, then \( A \) is robustly Hurwitz stable if and only if

\[
a_0 > 0 \text{ and } \Delta_{n-1} > 0 \text{ for all } A \in A.
\]

(2)

**Proof.** The proof is exactly the same as that of Theorem 1 in [14]. \( \square \)

In this paper, we are interested in a type of matrix uncertainty in which case the entries of the matrix are rational functions of parameters varying in intervals, i.e.,

\[
A(q) = (a_{ij}(q))_{n \times n} \in A,
\]

where \( q = (q_1, \ldots, q_m)^T \), \( a_{ij}(q) \) are rational functions of \( q \), and \( q_k \in [q_k, \bar{q}_k] \), \( k = 1, \ldots, m \). We have

**Theorem 2.** The robust Hurwitz stability of the matrix set \( A \) can be checked in finite steps.

The proof of the above theorem will be given in Section 5.
3 Simplicial subdivision of the unit hypercubic

In our method of checking robust Hurwitz stability of $A$, we need transform this problem to a problem of checking positivity of forms on simplices. Since the uncertain parameters are varying in hypercubic, we first introduces the procedure \cite{10} of subdividing the unit hypercubic $[0, 1]^m$ into nonoverlapping simplices in this section.

Denote by $\Theta_m$ the set of all $m!$ permutations of $\{1, 2, \ldots, m\}$. Let $\theta = (k_1 k_2 \ldots k_m) \in \Theta_m$, a set of $m + 1$ vertexes $\{a_0, \ldots, a_m\}$ of $[0, 1]^m$ spanning a simplex can be formed using following equations.

$$a_0 = 0, \quad (3)$$
$$a_i = a_{i-1} + e_{k_i}, \quad i = 1, 2, \ldots, m. \quad (4)$$

Denote by $S_\theta$ the simplex spanned by $\{a_0, \ldots, a_m\}$, i.e.,

$$S_\theta = \{x \in \mathbb{R}^m : x = \sum_{i=0}^m \lambda_i a_i, \sum_{i=0}^m \lambda_i = 1, \lambda_i \geq 0, i = 0, \ldots, m\},$$

it could be readily shown that such constructed $S_\theta$ has the following equivalent definition

$$S_\theta = \{(x_1, \ldots, x_m)^T \in \mathbb{R}^m : 1 \geq x_{k_1} \geq x_{k_2} \geq \ldots \geq x_{k_m} \geq 0\}.$$

According to \cite{10}, these simplices have no common interior points with each other, and

$$[0, 1]^m = \bigcup_{\theta \in \Theta_m} S_\theta.$$

4 Positivity of forms on simplices

Denote by $\mathbb{N}$ the set of all nonnegative integers, let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{N}^m$, and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_m$. For a form of degree $d$

$$f(x_1, x_2, \ldots, x_m) = \sum_{|\alpha| = d} c_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m},$$

it is immediate that $f$ is strict positive on the standard $(m - 1)$-simplex $\tilde{S}_m$ if all $c_\alpha$ are positive, where

$$\tilde{S}_m = \{(t_1, \ldots, t_m) : \sum_{i=1}^m t_i = 1, t_i \geq 0, i = 1, \ldots, m\}.$$

In fact this condition is not only sufficient, but also necessary in the following sense.
Theorem 3 (Pólya’s Theorem, [15]). If a form \(f(x_1, \ldots, x_m)\) is strict positive on \(\tilde{S}_m\), then for sufficiently large integer \(N\), all coefficients of
\[
(x_1 + \ldots + x_m)^N f(x_1, \ldots, x_m)
\]
are positive.

[16] gave an explicit bound for \(N\), that is
\[
N > \frac{d(d - 1)}{2} \frac{L}{\lambda} - d, \tag{5}
\]
where
\[
L = \max \left\{ \frac{\alpha_1! \cdots \alpha_m!}{d!} |c_\alpha| : |\alpha| = d \right\},
\]
and \(\lambda\) is the minimum of \(f\) on \(\tilde{S}_m\).

A newly proposed method, i.e., the WDS (i.e., weighted difference substitution) method [12], can also be used to check positivity of forms efficiently, we will introduce this method below.

Suppose \(\theta = (k_1 k_2 \ldots k_m) \in \Theta_m\), let \(P_\theta = (p_{ij})_{m \times m}\) be the permutation matrix corresponding \(\theta\), that is
\[
p_{ij} = \begin{cases} 1, & j = k_i \\ 0, & j \neq k_i \end{cases}.
\]
Given \(T_m \in \mathbb{R}^{m \times m}\), where
\[
T_m = \begin{pmatrix}
1 & \frac{1}{2} & \ldots & \frac{1}{m} \\
0 & \frac{1}{2} & \ldots & \frac{1}{m} \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & \frac{1}{m}
\end{pmatrix}, \tag{6}
\]
let
\[
A_\theta = P_\theta T_m,
\]
and call it the WDS matrix determined by the permutation \(\theta\). The variable substitution \(x \leftarrow A_\theta x\) corresponding \(\theta\) is called a WDS.

In fact, each variable substitution corresponds an assumption of sizes of \(x_1, x_2, \ldots, x_m\) in \(\tilde{S}_m\). If for each \(\theta \in \Theta_m\), all coefficients of \(f(A_\theta x)\) are positive, then \(f(x)\) is positive on \(\tilde{S}_m\). More generally, if there exists \(k \in \mathbb{N}\), such that all forms in \(WDS^{(k)}(f)\) have no nonnegative coefficients, then \(f(x)\) is positive on \(\tilde{S}_m\), where
\[
WDS^{(k)}(f) = \bigcup_{\theta \in \Theta_m} \cdots \bigcup_{\theta_1 \in \Theta_m} \{f(A_{\theta_k} \cdots A_{\theta_1} x)\} \tag{7}
\]
is the \(k\)th WDS set of \(f(x)\). In fact, the reverse is also true.
Theorem 4 ([12]). If \( f(x_1, \ldots, x_m) \) is a form of degree \( d \), the magnitudes of its coefficients are bounded by \( M \), then \( f \) is positive on \( \tilde{S}_m \), if and only if there exists \( k \leq C_p(M, m, d) \), such that each form in \( \text{WDS}^{(k)}(f) \) has no nonnegative coefficients, where

\[
C_p(M, m, d) = \left[ \frac{\ln (d^{(n+1)d(n+1)(n-1)(n+2)L}) - \ln \lambda}{\ln m - \ln (m-1)} \right] + 2 \quad (8)
\]

Remark: The \( C_p(M, m, d) \) in (8) provides a theoretical upper bound of the number of steps of substitutions required to check positivity of an integral form. In practice, numbers of steps used are generally much smaller than this bound.

If coefficients of the form \( f(x_1, \ldots, x_m) \) are all integers with magnitude bounded by \( M \), and \( f \) is positive on \( \tilde{S}_m \), then an explicit positive lower bound of \( f \) on \( \tilde{S}_m \) exists \([17]\), i.e.,

\[
\lambda \geq (2M)^{-d^n n^{-d^{n+1} - d - nd^n}}. \quad (9)
\]

The bound in (9) was shown tight in [17], hence from (5) and (8), we know Pólya’s Theorem has a doubly exponential complexity, while the WDS method only has a power exponential complexity, as was shown in [14].

5 Proofs

Proof of Theorem 2. Let

\[
Q = \{ q : q_k \in [q_k, \overline{q}_k], k = 1, \ldots, m \}, \quad (10)
\]
then from Theorem 1, we know that the robust Hurwitz stability of \( A \) is equivalent to the positivity of rational functions \( a_0(q) \) and \( \Delta_{n-1}(q) \) on \( Q \), which is further equivalent to the positivity of polynomials \( f_1(q) \) and \( f_2(q) \) on \( Q \), where \( f_1(q) \) and \( f_2(q) \) are multiplications of the numerators and the denominators of \( a_0(q) \) and \( \Delta_{n-1}(q) \) respectively.

Without loss of generality, we can suppose \( Q = [0, 1]^m \). Otherwise, we can use translations and scale transforms of variables in \( f_1(q), f_2(q) \), and obtain new polynomials which are required to be positive on \( [0, 1]^m \).

The hypercubic \([0, 1]^m \) can be divided into \( m! \) nonoverlapping simplices according to the procedure in Section 3, each simplex corresponds a permutation \( \theta_j = (j_1 j_2 \ldots j_m), 1 \leq j \leq m! \) in \( \Theta_m \), and can be defined as

\[
S_{\theta_j} = \{ q : 1 \leq q_{j_1} \geq \ldots \geq q_{j_m} \geq 0 \}. \quad (11)
\]

\( f_1(q) \) or \( f_2(q) \) may be not homogenous on \( q \), if so, we need to homogenize them, i.e., we introduce a new variable \( q_0 \), and let

\[
h_1(q_0, q_1, \ldots, q_m) = q_0^{\deg(f_1)} f_1\left( \frac{q_1}{q_0}, \ldots, \frac{q_m}{q_0} \right),
\]

\[
h_2(q_0, q_1, \ldots, q_m) = q_0^{\deg(f_2)} f_2\left( \frac{q_1}{q_0}, \ldots, \frac{q_m}{q_0} \right).
\]
It is obvious that \( f_1(q) \) and \( f_2(q) \) are positive on \( S_{\theta_j} \) if and only if \( h_1(\hat{q}) \) and \( h_2(\hat{q}) \) are positive on \( \hat{S}_{\theta_j} \setminus \{0\} \), where

\[
\hat{q} = (q_0, q_1, \ldots, q_m)^T,
\]

and

\[
\hat{S}_{\theta_j} = \{(q_0, q_1, \ldots, q_m) : 1 \geq q_0 \geq q_1 \geq \ldots \geq q_m \geq 0\}. \tag{12}
\]

Denote by \( e_k \) the unit vector whose \( k \)th component is 1 and other components are all 0, and \( S_{m+1} \) the \((m + 1)\)-dimensional simplex in \( \mathbb{R}^m \) spanned by \( \{0, e_1, \ldots, e_m\} \), i.e.,

\[
S_{m+1} = \{(t_1, \ldots, t_{m+1}) : \sum_{i=1}^{m+1} t_i \leq 1, t_i \geq 0, i = 1, \ldots, m+1\}.
\]

Suppose vertices except 0 of the simplex \( \hat{S}_{\theta_j} \) are \( v_{j0}, \ldots, v_{jm} \), and the matrix \( V_j \) is defined as

\[
V_j = (v_{j0}, \ldots, v_{jm}),
\]

then through a nonsingular linear substitution of variables in \( h_i(\hat{q}) \), i.e., \( \hat{q} \leftarrow V_j \hat{q} \), we can transform \( \hat{S}_{\theta_j} \) to \( S_{m+1} \), and obtain a new form \( \hat{h}_{ij}(\hat{q}) = h_i(V_j \hat{q}) \). It is immediate that \( \hat{h}_{ij}(\hat{q}) \) is positive on \( \hat{S}_{\theta_j} \setminus \{0\} \) if and only if \( \hat{h}_{ij}(\hat{q}) \) is positive on \( S_{m+1} \setminus \{0\} \). Since \( \hat{h}_{ij}(\hat{q}) \) has the same positivity on \( S_{m+1} \setminus \{0\} \) and \( \tilde{S}_{m+1} \), we finally come to the following result.

**Lemma 1.** The matrix set \( A \) is robustly Hurwitz stable if and only if following \( 2m! \) conditions are satisfied:

\[
\hat{h}_{ij}(\hat{q}) > 0 \text{ for all } \hat{q} \in \hat{S}_{m+1}, \quad i = 1, 2, \quad j = 1, 2, \ldots, m!. \tag{13}
\]

From Equation (5) and (8) in Section 4, we know that the conditions in Lemma 1 can be checked in finite steps. Moreover, if the interval vertexes \( q_k, \tilde{q}_k, k = 1, \ldots, m \) are all rational numbers, then coefficients of \( \hat{h}_{ij}(\hat{q}) \) in Lemma 1 are all integers, and from Equation (9), we know that the bounds of steps required to check conditions in Lemma 1 can be explicitly expressed in \( m \) and the coefficient magnitudes and degrees of \( \hat{h}_{ij}(\hat{q}) \).

\[\square\]

### 6 Examples

**Example 1.** Consider the uncertain system matrix [18]

\[
\begin{pmatrix}
\frac{p_2}{1 + p_2} - 2.025 & 2 \\
\frac{p_2}{1 + p_1} & \frac{p_1}{1 + p_2} - 2.025
\end{pmatrix},
\]

where \( p_1 \in [1, 2], p_2 \in [0, 0.5] \). Since each of the four forms obtained in (13) has no nonnegative coefficients, this system is robustly stable according to Lemma 1.
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