EXTREMALS OF A LEFT-IN Variant SUB-FINSLER METRIC ON THE ENGEL GROUP

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Abstract: Using the Pontryagin maximum principle for the time-optimal problem in coordinates of the first kind, we find the extremals of an arbitrary left-invariant sub-Finsler metric on the Engel group defined by a distribution of rank 2.

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§ 1. Introduction

In [1], it is indicated that the shortest arcs of each left-invariant (sub-)Finsler metric $d$ on a Lie group $G$ are solutions of a left-invariant time-optimal problem with the closed unit ball $U$ of an arbitrary norm $F$ on a subspace $\mathfrak{p}$ of the Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ of the Lie group $G$ as a control domain. Moreover, the subspace $\mathfrak{p}$ generates $\mathfrak{g}$. The Pontryagin maximum principle gives the necessary conditions for optimal trajectories of the problem [2]; the curves satisfying these conditions, are called extremals. Apparently, for the first time the shortest arcs of any left-invariant sub-Finsler metric on Lie group $G$ have been found in paper [3] in the case of an arbitrary sub-Finsler metric $d$ on the Heisenberg group $H$. The quotient group of $H$ by its center $Z$ is isomorphic to the additive group $(\mathbb{R}^2, +)$. Moreover, the differential $dp$ of the canonical projection $p : H \to H/Z = \mathbb{R}^2$ is a linear isomorphism of the subspace $\mathfrak{p}$ onto $T_0 \mathbb{R}^2 = \mathbb{R}^2$. The identification of $\mathfrak{p}$ and $\mathbb{R}^2$ by means of $dp$ turns $\mathbb{R}^2$ into the normed vector space $(\mathbb{R}^2, F)$, the so-called Minkowski plane. In [3], with the help of the maximum principle, it is proved that the projection with respect to $p$ of any inclusion-maximal shortest curve in $(H, d)$ can be a part of (1) a metric straight line or (2) a (closed) isoperimetrix [4] of the Minkowski plane $(\mathbb{R}^2, F)$.

Earlier in [5], Busemann obtained the solution to the isoperimetric problem for the Minkowski plane. With a reference to [3], Noskov found in [6] the same shortest curves in $(H, d)$ on the base of [5] and some nontrivial argument. On the other hand, the statement in [7, 8] that Busemann found in [5] the shortest curves of the space $(H, d)$ is erroneous. This is not only because at that time there was no equivalent to sub-Finsler geometry, but also because the shortest paths of the above-mentioned first type are not connected with the isoperimetrix. The authors of [9] (see also [10]) supposed that they were the first who studied sub-Finsler manifolds. But with the other name (homogeneous) “nonholonomic Finsler manifolds,” they appeared yet in the three works of the first author published in 1988 and 1989, including [1], in connection with a characterization of general homogeneous manifolds with inner metric. Moreover, following the tradition of specialists in Finsler geometry, the authors of [9] and [10] impose additional strong conditions on the norm $F$ and apply the corresponding cumbersome apparatus.

In this paper we find the extremals of an arbitrary left-invariant sub-Finsler metric on the Engel group defined by a subspace $\mathfrak{p}$ of rank 2. In the papers [11–14] Ardentov and Sachkov investigated in detail the left-invariant sub-Riemannian metric on the Engel group in the coordinates different from these of the first kind which we apply.

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Only the classical methods and results of the monograph [2] are applied here. In [15] some new search methods are proposed for the normal extremals of the left-invariant (sub-)Finsler and (sub-)Riemannian metrics.

§ 2. The Campbell–Hausdorff Formula for the Engel Group

Let $X$, $Y$, $Z$, $V$ be a basis for the four-dimensional Engel algebra $\mathfrak{g}$ such that

\[
[X, Y] = Z, \quad [X, Z] = V, \quad [X, V] = Y, \quad [Y, V] = [Z, V] = [Y, Z] = 0. \tag{1}
\]

Thus $\mathfrak{g}$ is a three-step nilpotent Lie algebra with two generators $X$ and $Y$. Therefore, as it is known, there exists a connected simply connected nilpotent Lie group $G$ unique up to isomorphism with the Lie algebra $\mathfrak{g}$, the Engel group, and the exponential mapping $\exp : \mathfrak{g} \to G$ is a diffeomorphism. This diffeomorphism and the Cartesian coordinates $x$, $y$, $z$, $v$ in $\mathfrak{g}$ with the basis $X$, $Y$, $Z$, $V$ define the coordinates of the first kind on $G$ and thus a diffeomorphism $G \cong \mathbb{R}^4$.

**Proposition 1.** In the coordinates of the first kind, the multiplication on the Engel group $G \cong \mathbb{R}^4_{x,y,z,v}$ is given by the following rule:

\[
\begin{pmatrix}
  x_1 \\
  y_1 \\
  z_1 \\
  v_1
\end{pmatrix}
\times
\begin{pmatrix}
  x_2 \\
  y_2 \\
  z_2 \\
  v_2
\end{pmatrix} =
\begin{pmatrix}
  x_1 + x_2 \\
  y_1 + y_2 \\
  z_1 + z_2 + \frac{1}{2}(x_1 y_2 - x_2 y_1) \\
  v_1 + v_2 + \frac{1}{2}(x_1 z_2 - x_2 z_1) + \frac{1}{12}(x_1^2 y_2 - x_1 x_2 y_2 - x_1 x_2 y_1 + x_2^2 y_1)
\end{pmatrix}. \tag{2}
\]

**Proof.** Set $A_i = x_i X + y_i Y + z_i Z + v_i V$, $i = 1, 2$. Using (1), we consequently obtain

\[
[A_1, A_2] = (x_1 y_2 - x_2 y_1)[X, Y] + (x_1 z_2 - x_2 z_1)[X, Z] = (x_1 y_2 - x_2 y_1)Z + (x_1 z_2 - x_2 z_1)V;
\]

\[
[A_1, [A_1, A_2]] = x_1(x_1 y_2 - x_2 y_1)[X, Z] = x_1(x_1 y_2 - x_2 y_1)V;
\]

\[
[A_2, [A_2, A_1]] = [A_1, A_2], A_2 = x_2(x_1 y_2 - x_2 y_1)[Z, X] = x_2(x_2 y_1 - x_1 y_2)V.
\]

Since the Lie algebra $\mathfrak{g}$ is three-step, we have the following Campbell–Hausdorff formula (see [16]):

\[
\log(\exp(A_1) \exp(A_2)) = A_1 + A_2 + \frac{1}{2} [A_1, A_2] + \frac{1}{12} [A_1, [A_1, A_2]] + \frac{1}{12} [A_2, [A_2, A_1]].
\]

Therefore,

\[
\log(\exp(A_1) \exp(A_2)) = (x_1 + x_2)X + (y_1 + y_2)Y + \left( z_1 + z_2 + \frac{1}{2}(x_1 y_2 - x_2 y_1) \right)Z
\]

\[
+ \left( v_1 + v_2 + \frac{1}{2}(x_1 z_2 - x_2 z_1) + \frac{1}{12}(x_1^2 y_2 - x_1 x_2 y_2 - x_1 x_2 y_1 + x_2^2 y_1)(x_1 - x_2) \right)V.
\]

The last equality gives (2). \Box

It follows from the method we introduced the coordinates of the first kind and formulas (2) that the realization of the chosen basis for the Lie algebra $\mathfrak{g}$ as left-invariant vector fields on the Lie group $G$ in these coordinates has the form

\[
X = \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z} - \frac{z}{2} \frac{\partial}{\partial v} - \frac{x y}{12} \frac{\partial}{\partial v}, \quad Y = \frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial z} + \frac{x^2}{12} \frac{\partial}{\partial v},
\]

\[
Z = \frac{\partial}{\partial z} + \frac{x}{2} \frac{\partial}{\partial v}, \quad V = \frac{\partial}{\partial v}. \tag{3}
\]

It is easy to verify that these vector fields satisfy (1).
§ 3. The Left-Invariant Sub-Finsler Metric and the Optimal Control on the Engel Group

In [1], it is said that the shortest arcs of a left-invariant sub-Finsler metric $d$ on an arbitrary connected Lie group $G$ defined by a left-invariant bracket generating distribution $D$ and a norm $F$ on $D(e)$ coincide with the time-optimal solutions of the following control system:

$$\dot{g}(t) = d g(t)(u(t)), \quad u(t) \in U,$$

with measurable controls $u = u(t)$. Here $l_g(h) = gh$, the control domain is the unit ball

$$U = \{u \in D(e) \mid F(u) \leq 1\}.$$

Therein, by virtue of the Pontryagin maximum principle for the (local) time optimality of a control $u(t)$ and the corresponding trajectory $g(t)$, $t \in \mathbb{R}$, the existence is necessary of a nonvanishing absolutely continuous vector-function $\psi(t) \in T^* g(t) G$ such that for almost all $t \in \mathbb{R}$ the function $\mathcal{H}(g(t); \psi(t); u) = \psi(t)(d g(t)(u))$ of the variable $u \in U$ attains the maximum at the point $u(t)$:

$$M(t) = \psi(t)(d g(t)(u(t))) = \max_{u \in U} \psi(t)(d g(t)(u)).$$

Moreover, the function $M(t)$, $t \in \mathbb{R}$, is constant and nonnegative, $M(t) \equiv M \geq 0$.

In case when $M = 0$ (respectively, $M > 0$) the corresponding extremal; i.e., the curve, satisfying the Pontryagin maximum principle, is called abnormal (respectively, normal).

It follows from (1) that the left-invariant distribution $D$ on $G$ with the basis $X, Y$ for $D(e)$ is bracket generating. Let $F$ be an arbitrary norm on $D(e)$. Then the pair $(D(e), F)$ defines a left-invariant sub-Finsler metric $d$ on $G$, where $u_1 X(e) + u_2 Y(e)$ is identified with $u = (u_1, u_2)$ and $u_i \in \mathbb{R}$, $i = 1, 2$.

With regard to (2), the control system (4) is written as

$$\dot{x} = u_1, \quad \dot{y} = u_2, \quad \dot{z} = \frac{1}{2}(x u_2 - y u_1), \quad \dot{\psi} = -\frac{1}{2} \left( \frac{z + \frac{1}{6} x y}{12} \right) u_1 + \frac{1}{12} x^2 u_2,$$

where $(u_1, u_2) \in U$.

In consequence of the left-invariance of the metric $d$ we can assume that the trajectories initiate at the unit $e \in G$, i.e. $x(0) = y(0) = z(0) = \psi(0) = 0$.

The control $u = u(t) = (u_1(t), u_2(t)) \in U$, $t \in \mathbb{R}$, defined by the Pontryagin maximum principle, is bounded and measurable [2], and therefore integrable. Then the functions $x(t)$ and $y(t)$, $t \in \mathbb{R}$, defined by the first two equations in (6) are Lipschitz, the product of any finite number of these functions is Lipschitz, and its derivative is bounded and measurable on each compact segment of $\mathbb{R}$. Moreover, this derivative can be calculated by the usual differentiation rule of a product from the differential calculus of functions of one variable. Therefore, the last two equations of (6) can be integrated by parts, using the first two equations in (6) (see 2.9.21 and 2.9.24 in [17]). Considering $x(0) = y(0) = z(0) = \psi(0) = 0$, we sequentially get

$$z(t) = -\frac{1}{2} t x(t)g(t) + \int_0^t x(\tau) u_2(\tau) d\tau,$$

$$v(t) = -\frac{1}{2} t x(t) \left( z(t) + \frac{1}{3} t x(t) g(t) \right) + \frac{1}{2} \int_0^t x^2(\tau) u_2(\tau) d\tau$$

$$= \frac{1}{12} x^2(t) y(t) - \frac{1}{2} x(t) \int_0^t x(\tau) u_2(\tau) d\tau + \frac{1}{2} \int_0^t x^2(\tau) u_2(\tau) d\tau.$$

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By the Pontryagin maximum principle, system (6) corresponds to the function \( \mathcal{H}(x, y, z, v; \psi_1, \psi_2, \psi_3, \psi_4; u_1, u_2) \) defined by the formula

\[
\mathcal{H} = \psi_1 u_1 + \psi_2 u_2 + \frac{1}{2} \psi_3 (x u_2 - y u_1) - \frac{1}{2} \psi_4 \left( z + \frac{1}{3} x y \right) u_1 + \frac{1}{12} \psi_4 x^2 u_2 = h_1 u_1 + h_2 u_2,
\]

where

\[
h_1 = \psi_1 - \frac{1}{2} \psi_3 y - \frac{1}{12} \psi_4 x y - \frac{1}{2} \psi_4 z, \quad h_2 = \psi_2 + \frac{1}{2} \psi_3 x + \frac{1}{12} \psi_4 x^2. \tag{9}
\]

The absolutely continuous vector-function \( \psi = \psi(t) \) satisfies the conjugate of system (6) of ordinary differential equations

\[
\begin{cases}
\dot{\psi}_1 = \frac{1}{12} \psi_4 y u_1 - \left( \frac{1}{2} \psi_3 + \frac{1}{3} \psi_4 x \right) u_2,
\dot{\psi}_2 = \left( \frac{1}{2} \psi_3 + \frac{1}{12} \psi_4 x \right) u_1,
\dot{\psi}_3 = \frac{1}{2} \psi_4 u_1,
\dot{\psi}_4 = 0.
\end{cases} \tag{10}
\]

Assign an arbitrary set of initial data \( \psi_i(0) = \varphi_i, \ i = 1, 2, 3, 4, \) of (10). It follows from (10), the first equation of (6), and the initial condition \( x(0) = 0 \) that

\[
\psi_4 \equiv \varphi_4, \quad \psi_3 = \varphi_3 + \frac{1}{2} \varphi_4 x, \quad \psi_2 = \varphi_2 + \frac{1}{2} \varphi_3 x + \frac{1}{6} \varphi_4 x^2. \tag{11}
\]

Notice that \( \left( \frac{1}{2} x y + z \right) \cdot = x u_2, \left( \frac{1}{2} x y - z \right) \cdot = y u_1 \) by (6). With regard to (11) and (6) the first equation in (10) takes the form

\[
\dot{\psi}_1 = \frac{1}{12} \varphi_4 \left( \frac{1}{2} x y - z \right) \cdot - \frac{1}{2} \varphi_3 \dot{y} - \frac{5}{12} \varphi_4 \left( \frac{1}{2} x y + z \right) \cdot.
\]

Therefore, accounting for the initial data of (6) and (10), we get

\[
\psi_1 = \varphi_1 - \frac{1}{2} \varphi_3 y - \frac{1}{6} \varphi_4 (x y + 3 z). \tag{12}
\]

Inserting the last equality and (11) into (9), we find

\[
h_1 = \varphi_1 - \left( \varphi_3 + \frac{1}{2} \varphi_4 x \right) y - \varphi_4 z, \quad h_2 = \varphi_2 + \left( \varphi_3 + \frac{1}{2} \varphi_4 x \right) x. \tag{13}
\]

We notice that \( \psi_k, k = 1, 2, 3, 4, \) are the covector components of \( \psi = \psi(t) \) relative to the coordinate system \( (x, y, z, v) \); i.e.,

\[
\psi_1 = \psi \left( \frac{\partial}{\partial x} \right), \quad \psi_2 = \psi \left( \frac{\partial}{\partial y} \right), \quad \psi_3 = \psi \left( \frac{\partial}{\partial z} \right), \quad \psi_4 = \psi \left( \frac{\partial}{\partial v} \right). \tag{14}
\]

Put \( h_1 = \psi(X), h_2 = \psi(Y), h_3 = \psi(Z), \) and \( h_4 = \psi(V). \) Using (3), it is easy to verify that (11), (12), and (14) give the same \( h_1 \) and \( h_2, \) as in (13), and

\[
h_3 = \psi_3 + \psi_4 \frac{x}{2} = \varphi_3 + \frac{1}{2} \varphi_4 x + \varphi_4 \frac{x}{2} = \varphi_3 + \varphi_4 x, \quad h_4 = \psi_4 = \varphi_4. \tag{15}
\]

From (13) and (15) we obtain the two more integrals of (6) and (10):

\[
h_4 \equiv \varphi_4, \quad \mathcal{E} = \frac{h_3^2}{2} - h_2 h_4 \equiv \frac{\varphi_3^2}{2} - \varphi_2 \varphi_4, \tag{16}
\]

called in [7] the Casimir functions.
Now, using (6), (13), and (15), we compute
\[ \dot{h}_1 = -h_3u_2, \quad \dot{h}_2 = h_3u_1. \] (17)

By the Pontryagin maximum principle for the local time optimality of a control \( u(t) \) and the corresponding trajectory \((x(t), y(t), z(t), v(t))\), the existence is necessary of a nonvanishing absolutely continuous vector-function \( \psi(t) \) such that for almost all \( t \in \mathbb{R} \) the ODE system (10) is satisfied and the function \( \mathcal{H}(x(t), y(t), z(t), v(t); \psi_1(t), \psi_2(t), \psi_3(t), \psi_4(t); u_1, u_2) \) of \( u \in U \) attains the maximum at \( u(t) \):
\[ M(t) = h_1(t)u_1(t) + h_2(t)u_2(t) = \max_{u \in U}(h_1(t)u_1 + h_2(t)u_2) \equiv M \geq 0. \] (18)

Relations (6), (13), and (18) imply that under multiplication of the functions \( \psi_i(t), i = 1, 2, 3, 4 \), by a positive constant \( k \) the trajectory \((x(t), y(t), z(t), v(t))\) does not change, while \( M \) is multiplied by \( k \). Therefore, \textit{in the case when} \( M > 0 \) \textit{we will assume that} \( M = 1 \).

Further we will consider just this case in the sequel.

It follows from (18) that \( (h_1(t), h_2(t)) \) from (13) and \( (\varphi_1, \varphi_2) = (h_1(0), h_2(0)) \) lie on the boundary \( \partial U^* \) of the polar figure \( U^* = \{ h \mid F_U(h) \leq 1 \} \) to \( U \), where \( F_U \) is the norm on \( H = \{ h \} \) equal to the support Minkowski function of the body \( U \):
\[ F_U(h) = \max_{u \in U} h \cdot u. \]

Moreover, \((H, F_U)\) is the dual normed vector space to \((D(e), F)\) and \((U^*)^* = U\) in consequence of reflexivity of finite-dimensional normed vector spaces. Furthermore, using (17) and (18), we get
\[ h_1(t)\dot{h}_2(t) - \dot{h}_1(t)h_2(t) = h_3(t)(h_1(t)u_1(t) + h_2(t)u_2(t)) = h_3(t). \] (19)

Let \( r = r(\theta) \), \( \theta \in \mathbb{R} \), be the polar equation of the curve \( F_U(x, y) = 1 \). At every point \( \theta \in \mathbb{R} \) there exist one-sided derivatives of \( r = r(\theta) \) (and the usual derivative \( r'(\theta) \) exists with possible exclusion of at most countably many values \( \theta \)). For simplicity we will denote every value between these derivatives by \( r'(\theta) \). Then for \( \theta = \theta(t) \)
\begin{align*}
  h_1(t) = h_1(\theta) = r(\theta) \cos \theta, & \quad h_2(t) = h_2(\theta) = r(\theta) \sin \theta, \quad \theta = \theta(t), \quad (20) \\
  h_1'(\theta) = -(r(\theta) \sin \theta - r'(\theta) \cos \theta), & \quad h_2'(\theta) = r'(\theta) \sin \theta + r(\theta) \cos \theta. \quad (21)
\end{align*}

Independently of the existence of the usual derivatives (21), equality (19) implies the existence of the usual derivatives for the doubled oriented area
\[ \sigma(t) = 2S(\theta(t)) = \int_0^{\theta(t)} r^2(\theta)d\theta \]
of the sector counted from 0. Moreover, by (15) and (19)
\[ \dot{\sigma}(t) = \dot{h}_3(t) = \varphi_3 + \varphi_4 x(t) = r^2(\theta(t))\dot{\theta}(t), \quad \dot{\theta}(t) = \frac{\dot{\sigma}(t)}{r^2(\theta(t))}, \] (22)
If we square the second equality in (22), we get by (13)
\[ r^4(\theta)\dot{\theta}^2 = \varphi_3^2 + 2\varphi_4 \left( \varphi_3 + \frac{1}{2}\varphi_4 x \right) x = \varphi_3^2 + 2\varphi_4(h_2 - \varphi_2), \]
\[ \dot{\theta}^2 = \frac{\varphi_3^2 + 2\varphi_4(r(\theta) \sin \theta - \varphi_2)}{r^4(\theta)}. \] (23)
By (6), (17), and (22),

$$\dot{\sigma}(t) = \varphi_4 u_1(t),$$  \hspace{1cm} (24)

$$\dot{\epsilon} = \dot{\epsilon}(t) = \frac{1}{2}(\dot{\sigma}(t))^2 - h_2(t)h_4(t) = \frac{1}{2}(h_3(t))^2 - h_2(t)h_4(t) = \frac{\varphi_3^2}{2} - \varphi_2 \varphi_4.$$  \hspace{1cm} (25)

**Remark 1.** (25) is equivalent to (23).

**Remark 2.** In the notation of [8], equation (24) is written as

$$\ddot{\theta} = \varphi_4 \cos \Omega \theta.$$  \hspace{1cm} (26)

It is analogous to equation (5), when $\varphi_4 \neq 0$: $\ddot{\theta} = \sin \Omega \theta$ from [8]. In the paper $\Omega = U$, $\Omega^o = U^*$; $\theta^o$ is our $\sigma(\theta)$ for $\Omega^o$; in [8], $\theta$ plays a role of $\sigma$ for $\Omega$,

$$\cos \Omega \theta = u_1(\theta), \quad \sin \Omega \theta = u_2(\theta), \quad \cos \Omega^o \theta^o = h_1(\theta^o), \quad \sin \Omega^o \theta^o = h_2(\theta^o),$$

$$\cos \Omega^o \theta + \sin \Omega^o \theta^o \sin \theta = 1.$$

Fig. 8 in [8] shows a schematic representation of the phase portrait of a “generalized mathematical pendulum” (5). On the basis of the portrait, there is also given some general verbal but rather detailed information on the solutions to equation (5) and its application (including) to the Heisenberg, Cartan, and Engel groups with the given norm $F$ on $D(e)$ for the left-invariant two-dimensional totally nonholonomic distribution $D$ on these Lie groups. In other words, some analog of our function $\theta = \theta(t)$ for these Lie groups is described in sufficient detail in [8]. At the same time, the corresponding extremals on these groups are not searched in [8].

We claim that in the general case for $\theta = \theta(t)$

$$\dot{x}(t) = u_1(\theta) = \frac{r'(\theta) \sin \theta + r(\theta) \cos \theta}{r^2(\theta)}, \quad \dot{y}(t) = u_2(\theta) = \frac{r(\theta) \sin \theta - r'(\theta) \cos \theta}{r^2(\theta)},$$  \hspace{1cm} (27)

where $u_1(\theta) = h_2'(\theta)/r^2(\theta)$ and $u_2(\theta) = -h_1'(\theta)/r^2(\theta)$ due to (21). Indeed, the following two equalities must hold:

$$h_1(\theta)u_1(t) + h_2(\theta)u_2(\theta) = 1, \quad h_1'(\theta)u_1(t) + h_2'(\theta)u_2(\theta) = 0.$$

It is easy to see that the first of these equalities follows from (20) and (27), while the second is a corollary from (21) and (27).

It follows from (6) that

$$\left(3v + \frac{1}{2}xz\right)^\cdot = -\frac{3}{2}\dot{x}z + \frac{1}{2}x\dot{z} + \frac{1}{2}\dot{x}\dot{z} = x\dot{z} - \dot{x}z,$$  \hspace{1cm} (28)

and so by (6), (13), and (18) we get, omitting for brevity the variable $t$,

$$h_1u_1 + h_2u_2 = \varphi_1 \dot{x} + \varphi_2 \dot{y} + \left(\varphi_3 + \frac{1}{2}\varphi_4 x\right)(\dot{x}y - \dot{y}x) = \varphi_4 x + \varphi_2 y + 2\varphi_3 z,$$

$$+ \varphi_4 (x\dot{z} - z\dot{x}) = \left(\varphi_1 x + \varphi_2 y + 2\varphi_3 z + 3\varphi_4 v + \frac{1}{2}\varphi_4 xz\right)^\cdot = 1.$$

Taking into account the initial data of (6), we obtain

$$\varphi_1 x + \varphi_2 y + \left(2\varphi_3 + \frac{1}{2}\varphi_4 x\right)z + 3\varphi_4 v = t.$$  \hspace{1cm} (29)
§ 4. Search for Sub-Finsler Extremals

1. Let us consider an abnormal case. We have

**Proposition 2.** An abnormal extremal on the Engel group starting at the unit is a one-parameter subgroup

\[ x(t) \equiv 0, \quad y(t) = \pm \frac{t}{F(0,1)}, \quad z(t) \equiv 0, \quad v(t) \equiv 0 \]  

and is not strongly abnormal.

**Proof.** Assume that \( M = 0 \). Then we obtain from the maximum condition that \( h_1(t) = h_2(t) \equiv 0 \) and \( \varphi_1 = \varphi_2 = 0 \). Since \( u_1(t) \) and \( u_2(t) \) could not simultaneously vanish at any \( t \in \mathbb{R} \), \( \varphi_3 + \varphi_4 x(t) \equiv 0 \) by (17). This implies that \( \varphi_3 = 0 \) and \( x(t) \equiv 0 \) because \( x(0) = 0 \). Hence by (11) we get \( \psi_2(t) = \psi_3(t) \equiv 0 \), \( \psi_4(t) \equiv \varphi_4 \), and \( \psi_1(t) = \frac{1}{2} \varphi_4 z(t) \) by (12) and the first equality in (13). Thus, \( \varphi_4 \neq 0 \) because \( \psi(t) \) does not vanish.

Since \( x(t) \equiv 0 \), we have \( u_1(t) \equiv 0 \) according to the first equality of (6). Hence we obtain successively from the third and the fourth equations in (6) as well as of the initial data \( z(0) = v(0) = 0 \) that \( z(t) = v(t) \equiv 0 \).

Further, since \( u_1(t) \equiv 0 \) and \( F(u_1(t), u_2(t)) \equiv 1 \), \( u_2(t) \equiv \frac{1}{F(0,1)} \). This, the second equation in (6), and the initial condition \( y(0) = 0 \) imply that \( y(t) = \frac{1}{F(0,1)} \), and we get (30).

In consequence of (2), this extremal is one of the two one-parameter subgroups

\[ g_1(t) = \exp \left( \frac{tY}{F(0,1)} \right), \quad g_2(t) = g_1(-t) = g_1(t)^{-1}, \quad t \in \mathbb{R}, \]

satisfies (18) with \( M(t) \equiv 1 \) for the constant covector function

\[ \psi(t) = (0, \pm \varphi_2, 0, 0) = (0, \pm F(0,1), 0, 0) = (0, h_2(t), 0, 0), \]

subject to differential equations (10) and (17). Hence, the extremal is normal relative to this covector function, is not strongly abnormal, and is a geodesic; moreover, it is a metric straight line (see Proposition 3 below). □

2. Put \( M = 1 \).

**Theorem 1.** For every extremal on the Engel group starting at the unit,

\[ x(t) = \int_0^t \frac{|r'(\theta(\tau))\sin\theta(\tau) + r(\theta(\tau)) \cos\theta(\tau)|}{r^2(\theta(\tau))} d\tau, \]  

\[ y(t) = \int_0^t \frac{|r(\theta(\tau)) \sin\theta(\tau) - r'(\theta(\tau)) \cos\theta(\tau)|}{r^2(\theta(\tau))} d\tau \]  

with arbitrary measurable integrands of the above-indicated form and continuously differentiable function \( \theta = \theta(t) \) satisfying (22) and (23).

**Proof.** By Proposition 2, every extremal is normal for the corresponding control. By the above assertions, each control has the form of (27), which implies (31) and (32). □

Remark 3. On parts of the extremals with \( \dot{\theta}(t) \neq 0 \) for calculation of the functions \( x(t) \) and \( y(t) \) by (31) and (32) of any matter are only those values \( \theta = \theta(t) \) where the usual derivative \( r'(\theta) \) exists.

2.1. Let us assume that \( \varphi_3 = \varphi_4 = 0 \). The following proposition is true.
Proposition 3. For every extremal on the Engel group with the above conditions and the origin at the unit, \( \theta(t) \equiv \theta_0, t \in \mathbb{R} \), for some \( \theta_0 \). Moreover, each of these extremals is a one-parameter subgroup if and only if the usual derivative \( r'(\theta_0) \) exists. In general, each indicated extremal is a metric straight line.

Proof. The first statement follows from (22).

Moreover, by Theorem 1, every admissible control \( (u_1(t), u_2(t)) = (u_1(\theta_0), u_2(\theta_0)) \) from (27) is constant if and only if the usual derivative \( r'(\theta_0) \) exists and is equivalent to the condition that system (6) has unique solution, the one-parameter subgroup

\[
x(t) = u_1(\theta_0)t, \quad y(t) = u_2(\theta_0)t, \quad z(t) \equiv 0, \quad v(t) \equiv 0.
\]

Notice that there exist at most countably many values \( \theta_0 \) for which the second statement is false. For any such \( \theta_0 \) the functions \( x(t) \) and \( y(t) \), \( t \in \mathbb{R} \), are as in (31) and (32) with \( \theta(\tau) \equiv \theta_0 \) and arbitrary measurable integrands \( u_1(\tau) \) and \( u_2(\tau) \) are of the type indicated in Theorem 1, and the functions \( z(t) \) and \( v(t) \) are defined by formulas (7) and (8) respectively.

It follows from (6) that the length of each arc of the curve \( (x(t), y(t), z(t), v(t)) \) in \( (G, d) \) is equal to the length of the corresponding arc of its projection \( (x(t), y(t)) \) on the Minkowski plane. We can easily see that the projections of the curves are metric straight lines on the Minkowski plane. Therefore, the curves itself are metric straight lines. \( \square \)

Remark 4. The metric straight lines are obtained only in the case of Proposition 3; in particular, Proposition 2.

2.2. Let us consider the case \( \varphi_4 = 0, \varphi_3 \neq 0 \).

Proposition 4. Let \( (x, y, z, v)(t) \) with \( t \in \mathbb{R} \) be an extremal with conditions \( x(0) = y(0) = z(0) = v(0) = 0 \) on the Engel group such that \( \varphi_4 = 0 \) and \( \varphi_3 \neq 0 \). Then the functions \( \theta(t), h(t) = (h_1(t), h_2(t)), x(t), y(t) \) are periodic with common period \( L = 2S_0/|\varphi_3| \), where \( S_0 \) is the area of the figure \( U^* \). The projection \( (x, y)(t) \) of the extremal to the Minkowski plane \( z = v = 0 \) with the norm \( F \) has the form

\[
x(t) = \frac{h_2(t) - \varphi_2}{\varphi_3}, \quad y(t) = -\frac{h_1(t) - \varphi_1}{\varphi_3}, \quad \text{(33)}
\]

and is parametrized by the arclength periodic curve on an isoperimetric. Moreover, \( h_1 = h_1(\theta(t)) \) and \( h_2 = h_2(\theta(t)) \) are given by (20), \( \theta = \theta(t) \) is the inverse function of \( t(\theta) = \int_{\theta_0}^{\theta} (r^2(\xi)/\varphi_3)d\xi \),

\[
z(t) = \frac{t - \varphi_1x(t) - \varphi_2y(t)}{2\varphi_3}
\]

and \( z(t) \) is equal to the oriented area on the Euclidean plane with the Cartesian coordinates \( x, y \), traced by the rectilinear segment connecting the origin with the point \( (x(\tau), y(\tau)) \), \( \tau \in [0, t] \). Moreover, \( v = v(t) \) is defined by (8).

Proof. The statements about the function \( \theta(t) \) ensue from (22). It follows from (15) and (19) that by analogy to the second Kepler law the radius-vector-function \( h(\tau) = (h_1(\tau), h_2(\tau)) \in U^* \), \( t_1 \leq \tau \leq t_2 \), traces in the plane \( h_1, h_2 \) (or, if it is desired, \( u_1, u_2 \) or \( x,y \)) the standard Euclidean oriented area \( (\varphi_3/2)(t_2 - t_1) \). Consequently, \( h(t), t \in \mathbb{R} \), is a periodic function with period \( L = 2S_0/|\varphi_3| \), where \( S_0 \) is the area of \( U^* \). Furthermore, (15), (17), and (6) imply formulas (33); i.e., the projection \( (x,y)(t) \) of the curve \( (x,y,z,v)(t) \) lies on the boundary \( I(\varphi_1, \varphi_2, \varphi_3) \) of the figure obtained by rotation of \( U^*/|\varphi_3| \) by \( \pi/2 \) around the center (the origin of coordinates) with a consequent shift by the vector \( (-\varphi_1/2\varphi_3, \varphi_2/2\varphi_3) \). Thus, by analogy to the case of the Heisenberg group with left-invariant sub-Finsler metric, considered in [3], \( I(\varphi_1, \varphi_2, \varphi_3) \) is an isoperimetric of the Minkowski plane with the norm \( F \) [4].

By analogy to [3] from (33) it follows that \( (x(t), y(t)) \) is a periodic curve on \( I(\varphi_1, \varphi_2, \varphi_3) \) with period \( L \) indicated above. From (29) and (33) we see that

\[
z(t) = \frac{(t - \varphi_1x(t) - \varphi_2y(t))/(2\varphi_3)}{(2\varphi_3)} = \frac{(\varphi_3t - \varphi_1h_2(t) + \varphi_2h_1(t))/(2\varphi_3^2)}{2\varphi_3^2}, \quad \text{(34)}
\]

\[
z(L) = L/(2\varphi_3) = S_0/(|\varphi_3|\varphi_3). \quad \text{(35)}
\]
The statement of Proposition 4 on the function \( z(t) \) follows from (6). Since \( (x(t), y(t)) \) lies on the isoperimetric traversed clockwise (counterclockwise) if \( \varphi_3 < 0 \) \( (\varphi_3 > 0) \); therefore, \( z(t) \) is a monotone function. In particular, \( z(L) \) is the oriented area of the figure enveloped by the isoperimetric \( I(\varphi_1, \varphi_2, \varphi_3) \) or, which is the same, the area of the figure \( U^* / |\varphi_3| \) taken with the sign equal to the sign of \( z(L) \).

The last statement was proved earlier. □

2.3. Assume that \( \varphi_4 \neq 0 \).

**Lemma 1.** If the function \( \theta(t) \) is constant on some nondegenerate interval \( J \subset \mathbb{R} \), then on \( J \)

\[
x(t) \equiv -\frac{\varphi_3}{\varphi_4}, \quad y(t) = y_0 + \frac{2\varphi_4(t - t_0)}{2\varphi_2\varphi_4 - \varphi_3^2}, \quad z(t) = z_0 + \frac{\varphi_3(t - t_0)}{\varphi_3^2 - 2\varphi_2\varphi_4},
\]

\[
v(t) = v_0 + \frac{\varphi_3^2(t - t_0)}{6\varphi_4(2\varphi_2\varphi_4 - \varphi_3^2)},
\]

where \( y_0 = y(t_0), \; z_0 = z(t_0), \; v_0 = v(t_0), \; t_0 \) is a point of the interval \( J \) closest to zero; \( v_0 \) is calculated by \( x_0 = -\frac{\varphi_3}{\varphi_4}, \; y_0, \; z_0 \) and (29) for \( t = t_0 \).

In particular, for \( \varphi_3 = 0 \),

\[
x(t) \equiv 0, \quad y(t) = y_0 + \frac{t - t_0}{\varphi_2}, \quad z(t) \equiv z_0, \quad v(t) \equiv v_0 = (t_0 - \varphi_2y_0)/(3\varphi_4).
\]

**Proof.** If the function \( \theta(t) \) is equal to \( \theta_0 \) on some interval \( J \), then \( \dot{\theta}(t) \equiv 0 \) and \( x(t) \equiv -\varphi_3/\varphi_4 \),

\[
h_2(t) = \varphi_2 - \varphi_3^2/(2\varphi_4), \quad t \in J, \text{ due to (22) and (13). It follows from (6) and (18) that then } u_1(t) \equiv 0, \; u_2(t) = 1/h_2(t) \equiv (2\varphi_2\varphi_4 - \varphi_3^2)/2\varphi_4, \; t \in J, \text{ and the function } y(t) \text{ on the interval } J \text{ is determined by the second equality of (36). Owing to the calculated value } u_2 \text{ and (7), the function } z(t) \text{ on the interval } J \text{ is determined by the third equality of (36). Equality (37) follows from (8) and the first equality in (36). Equalities (38) ensue from (36), (37), and (29).} □

**Lemma 2.** Suppose that there exist \( \Theta_1, \Theta_2 \in \mathbb{R} \) such that \( \Theta_1 < \Theta_2 \) and the right-hand side of (23) is positive for \( \theta \in (\Theta_1, \Theta_2) \) and vanishes for \( \theta = \Theta_1 \) and \( \theta = \Theta_2 \), while \( \theta(t), t \in \mathbb{R}, \) is a function admissible by the maximum principle. Then \( \varphi_4u_1(\theta) > 0 \) \( (\varphi_4u_1(\theta) < 0) \) for \( \theta \in (\Theta_1, \Theta_2), \) sufficiently close to \( \Theta_1 \) (respectively, \( \Theta_2 \)), (27). If \( \varphi_4u_1(\theta(t_0)) = \Theta_1 \) \( (\theta(t_0) = \Theta_2) \) and \( \theta(t) \neq \Theta_1 \) \( (\theta(t) \neq \Theta_2) \) for all \( t < t_0 \) or \( t > t_0, \) sufficiently close to \( t_0 \), then \( \dot{\theta}(t)(t - t_0) > 0 \) \( (\dot{\theta}(t)(t - t_0) < 0) \) for these \( t \)’s.

**Proof.** We note that the second statement of the lemma is a consequence of the first and (22). □

**Theorem 2.** If \( \varphi_4 \neq 0 \) then each extremal on the Engel group starting at the unit is defined by equations (31) and (32) (with arbitrary measurable integrands of indicated view and continuously differentiable function \( \theta = \theta(t) \) satisfying (22) and (23)),

\[
z(t) = -\frac{1}{\varphi_4} \left( r(\theta(t)) \cos \theta(t) - \varphi_1 + \left( \varphi_3 + \frac{1}{2}\varphi_4 x(t) \right) y(t) \right), \quad (39)
\]

\[
v(t) = \frac{1}{3\varphi_4} \left( t - \varphi_1 x(t) - \varphi_2 y(t) - \left( 2\varphi_3 + \frac{1}{2}\varphi_4 x(t) \right) z(t) \right), \quad (40)
\]

Let us set

\[
\theta_0 := \theta(0), \quad \delta_0 = \max_{h \in U^*} (-\varphi_4 h_2), \quad \delta_{-1} = \min_{h \in U^*} (-\varphi_4 h_2).
\]

The following cases are possible:

1. Let \( \varphi_3 \neq 0 \) and \( \delta > \delta_0 \). Then \( \theta(t), t \in \mathbb{R}, \) is inverse to \( t(\theta) \) defined by the formula

\[
t(\theta) = \frac{\int_{\theta_0}^{\theta} \frac{r^2(\xi) d\xi}{\varphi_3 \sqrt{1 + (2\varphi_4/\varphi_3^2)(r(\xi) \sin \xi - \varphi_2)}}}{r(\theta_0) \sin \theta_0 = \varphi_2}.
\]

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2. Let \( \varphi_3 = 0 \) and \( \mathcal{E} = \mathcal{E}_{-1} \). Then \( \theta(t) \equiv \theta_0 \) and the desired extremal is the metric straight line (30).

3. Let \( \mathcal{E}_{-1} < \mathcal{E} < \mathcal{E}_0 \). Then we have for some numbers \( t_1, t_2, t_1 \neq t_2 \), for all \( t \in \mathbb{R} \) and \( k \in \mathbb{Z} \)

\[
\theta(t + 2k(t_2 - t_1)) = \theta(t), \quad \dot{\theta}(t_i + t) = -\dot{\theta}(t_i - t), \quad \theta(t_i + t) = \theta(t_i - t), \quad i = 1, 2.
\] (42)

3.1. If \( \varphi_3 \neq 0 \) then \( t_i = t(\theta_i), i = 1, 2 \), in equalities (42) are calculated by (41), where \( \theta_1 \neq \theta_2 \) are the values nearest to \( \theta_0 \) such that \( \varphi_3(\theta_2 - \theta_1) > 0 \) and the right-hand side in (23) vanishes.

3.2. If \( \varphi_3 = 0 \) then \( \theta_2 \neq \theta_1 = \theta_0 \) and \( t_1 = 0 \), \( t_2 = t(\theta_2) \) in (42), where

\[
t(\theta) = \pm \int_{\theta_0}^{\theta} \frac{r^2(\xi)\,d\xi}{\sqrt{2\varphi_4(r(\xi)\sin\xi + \varphi_2)}}, \quad r(\theta_0)\sin\theta_0 = \varphi_2,
\] (43)

and on the right-hand side stands + (respectively, −) if \( \varphi_4(\theta_2 - \theta_0) > 0 \) (respectively, \( \varphi_4(\theta_2 - \theta_0) < 0 \)).

Here \( \theta_2 \neq \theta_0 \) is a number such that \( h_2(\theta_0) = h_2(\theta_2) \) and \( \varphi_4(h_2(\theta) - h_2(\theta_0)) > 0 \) for any \( \theta \) from the interval \( I = (\min(\theta_0, \theta_2), \max(\theta_0, \theta_2)) \).

4. Let \( \varphi_3 \neq 0 \) and \( \mathcal{E} = \mathcal{E}_0 \). Then there exist some values \( \theta_1 \) and \( \theta_2 \) nearest to \( \theta_0 \) such that \( \theta_1 < \theta_0 < \theta_2 \) and the right-hand side of (23) vanishes for \( \theta = \theta_i, i = 1, 2 \). If improper integral (41) diverges for \( \theta = \theta_1 \) and \( \theta = \theta_2 \), then \( \theta(t) \in (\theta_1, \theta_2), t \in \mathbb{R} \), is the inverse function to the function \( t(\theta) \) defined by (41). If at least one of the improper integrals (43) is finite for \( \theta = \theta_1 \) and/or for \( \theta = \theta_2 \), then the function \( \theta(t) \) is not unique and can take constant values on some nondegenerate closed intervals of arbitrary length on which (36) and (37) are valid.

5. Let \( \varphi_3 = 0 \) and \( \mathcal{E} = \mathcal{E}_0 \). Then we have the largest segment \( [\theta_1, \theta_2], \theta_1 \leq \theta_2 \), such that \( \theta_0 \in [\theta_1, \theta_2] \) and \( h_2(\theta) = \varphi_2 \) for any \( \theta \in [\theta_1, \theta_2] \). If \( \theta_0 = \theta_2 \) (respectively, \( \theta_0 = \theta_1 \)) then further we denote by \( t(\theta) \) integral (41) for \( \theta \in [\theta_0, \theta_1 + 2\pi] \) (respectively, \( \theta \in [\theta_1 - 2\pi, \theta_0] \)) without + and −.

Then \( \theta(t) \equiv \theta_0 \) and the desired extremal is the metric straight line (30) in the following cases:

5.1. \( \theta_1 = \theta_0 = \theta_2 \) and \( t(\theta) = \infty \) as \( \theta \nrightarrow \theta_0 \) and \( \theta \searrow \theta_0 \);

5.2. \( \theta_1 < \theta_0 < \theta_2 \);

5.3. \( \theta_0 = \theta_1 < \theta_2 \) and \( t(\theta) = \infty \) as \( \theta \nrightarrow \theta_0 \);

5.4. \( \theta_0 = \theta_2 > \theta_1 \) and \( t(\theta) = \infty \) as \( \theta \searrow \theta_0 \).

In all other cases, the function \( \theta(t) \) is not unique and can take constant values on some closed intervals of arbitrary length on which (38) are valid.

**Proof.** The first statement follows from Theorem 1 and (13), (20), (29).

Let us prove the statements of the theorem about the function \( \theta(t) \). To this end, knowing the sign of the derivative \( \dot{\theta}(\theta) \), we find \( \dot{\theta}(\theta) \) from (23), where \( \dot{\theta} \) denotes the derivative with respect to \( t \). Integrating this function, we will find \( \theta(t) \).

1. The conditions mean that \( \varphi_3 \neq 0 \) and the right-hand side of (23) is positive for all \( \theta \in \mathbb{R} \). Then due to (22) and (23)

\[
\dot{\theta}(\theta) = \left( \frac{\varphi_3}{r^2(\theta)} \right) \sqrt{1 + \left( 2\varphi_4/\varphi_3^2 \right) (r(\theta)\sin\theta - \varphi_2)},
\] (44)

whence (41) follows. Moreover, by (22) and (44),

\[
x(t) = \left( \frac{\varphi_3}{\varphi_4} \right) \left( \sqrt{1 + \left( 2\varphi_4/\varphi_3^2 \right) (r(\theta(t))\sin\theta(t) - \varphi_2)} - 1 \right),
\] (45)

and \( y(t), t \in \mathbb{R} \), is defined by (32).

2. The conditions mean that \( \varphi_3 = 0 \) and the right-hand side of (23) is nonpositive for all \( \theta \in \mathbb{R} \). Then \( h_2(\theta_0) = \varphi_2 \) is a maximal (respectively, minimal) value of the second component for \( h \in U^* \) if \( \varphi_4 > 0 \) (respectively, \( \varphi_4 < 0 \)), \( \dot{\theta} \equiv 0, \theta(t) \equiv \theta_0 \), and due to (22) we get \( x(t) \equiv 0 \) and the metric straight lines (30).

We will use Lemma 2 implicitly to prove the remaining statements.
3. The conditions mean that the right-hand side of (23) takes both positive and negative values.

3.1. At first, let us consider the case $\varphi_3 \neq 0$. It is clear that there exist $\theta_1$ and $\theta_2$ as in the statement of item (3) of Theorem 2. In consequence of reflexiveness in passing to the dual normed space for finitedimensional case, for all values $\theta$ sufficiently close to $\theta_1$ (respectively, $\theta_2$), the first formula of (27) defines the values $u_1(\theta) \neq 0$ of the same sign, but of the opposite signs for $\theta_1$ and $\theta_2$. Then $\theta_1 < \theta_0 < \theta_2 < \theta_1 + 2\pi$ if $\varphi_3 > 0$, $\theta_2 < \theta_0 < \theta_1 < \theta_2 + 2\pi$ if $\varphi_3 < 0$, both values $t_i = t(\theta_i), i = 1, 2$, are finite and define the monotone continuously differentiable function $\theta(t), t_1 \leq t \leq t_2$, with zero one-sided derivatives at the ends. By the above, equalities (42) are valid which uniquely determine the function $\theta(t), t \in \mathbb{R}$.

3.2. It is clear that in the case $\varphi_3 = 0$ there exists $\theta_2$ for $\theta_1 = \theta_0$ as in the statement of 3.2 in Theorem 2. In view of the Taylor formula applied to the corresponding one-sided derivatives of the first order, the subradical function in the denominator of the integrand in (43) has order equal to 1/2 relative to $(\xi - \theta_0)^2$ and $(\xi - \theta_2)^2$ when $\xi \in I$ and $\xi \to \theta_0, \xi \to \theta_2$ respectively. Therefore, the function $t(\theta)$ calculated by (43) is finite for all $\theta \in \mathcal{T}, t_2 > 0$, and for $\varphi_4(\theta_2 - \theta_0) > 0$ (respectively, $\varphi_4(\theta_2 - \theta_0) < 0$) on the segment $[0, t_2]$ the increasing (respectively, decreasing) function $\theta(t)$ is defined. The function is inverse to the function $t(\theta)$, where $+$ (respectively, $-$) stands on the right-hand side of (43). By reflexiveness, for all values $\theta \in I$ sufficiently close to $\theta_0$ (respectively, $\theta_2$) the first formula in (27) defines $u_1(\theta) \neq 0$ of the same sign, but of the opposite signs for $\theta_0$ and $\theta_2$. Therefore, the function $\theta(t) \in \mathcal{T}, t \in \mathbb{R}$, is even, periodic with period $2t_2$, alternately increasing and decreasing on the respective segments of length $t_2$, and relations (42) are valid, uniquely determining the function $\theta(t), t \in \mathbb{R}$. According to the above, the function $x(t)$ for $t \in [0, t_2]$ is defined by the formula

$$x(t) = (\text{sgn}(\dot{\theta}(t)) / \varphi_3) \sqrt{2} \varphi_4 \left( r(\theta(t)) \sin \theta(t) - \varphi_2 \right),$$

(46)

$x(t)$ is odd, $x(t + 2kt_2) = x(t), k \in \mathbb{Z}$, and $x(t + t_2) = -x(t_2 - t), t \in \mathbb{R}$. The function $y(t), t \in \mathbb{R}$, is defined by (32).

4. The conditions mean that $\varphi_3 \neq 0$, the right-hand side in (23) is nonnegative, and there exists the only $h_2^0 \neq \varphi_2$ with some $h = (h_1, h_2^0) \in \partial U^*$ such that the right-hand side in (23) vanishes. It is clear that $h_2^0$ is minimal (respectively, maximal) value of the second component for points from $U^*$ if $\varphi_4 > 0$ (respectively, $\varphi_4 < 0$) and $h_2^0 = \varphi_2 - \varphi_2^2/2\varphi_4$ by (22) and (13)). Moreover, the vector $h \in \partial U^*$ with $h_2 = h_2^0$ is not unique if $\partial U^*$ is not strictly convex at the points $h$ with $h_2 = h_2^0$; in other words, if $\partial U$ is not differentiable at the point $u^0 = (0, 1/h_2^0)$. In any case there exist some values $\theta_1 < \theta_0$ and $\theta_2 > \theta_0$ nearest to $\theta_0$ such that $r(\theta_i) \sin \theta_i = h_2^0, i = 1, 2$.

4.1. If $t(\theta_i) = \pm \infty, i = 1, 2$, then $\theta(t) \in (\theta_1, \theta_2), t \in \mathbb{R}$, is the inverse function to the function $t(\theta)$ defined by (41). For example, this is true if the subradical function in the denominator of the integrand in (41) has order not less than 1 relative to $(\xi - \theta_1)^2$ and $(\xi - \theta_2)^2$ as $\xi \searrow \theta_1$ and $\xi \nearrow \theta_2$ respectively (which is satisfied if the usual second derivatives $r''(\theta_i), i = 1, 2, \exists$ exist). The conditions may fail even under the existence of the usual derivatives $r'(\theta_i), i = 1, 2$.

4.2. Let $t_i := t(\theta_i)$ be finite for $i = i_1$ and infinite for $i = i_2 \neq i_1$. Then $\dot{\theta}(t_i) = 0, t_{i_2} = \text{sgn}(\phi_3(t_2 - t_1)) \infty$ and the function $\theta(t)$ is defined in the interval $I$ between $t_{i_1}$ and $t_{i_2}$. For all $t \in \mathbb{R} - I$, the function could be defined as $\theta(t) = \theta(2t_{i_1} - t)$. It is possible also that $\theta(t) \equiv \theta_i$ if and only if $t$ belongs to the closure of some nonempty open interval $I_1 \subset \mathbb{R} - I$, where $\overline{I}_1 \cap \overline{I} = t_{i_1}$. If $I_1$ is finite then the graph of the function $\theta(\tau)$ on $\mathbb{R} - (\overline{I} \cup \overline{I}) := I_2$ of the last solution is obtained from the graph of the first solution by shifting the interval $\mathbb{R} - \overline{I}$ to $I_2$.

4.3. Let both $t_i, i = 1, 2$, be finite. Then $\dot{\theta}(t_1) = \dot{\theta}(t_2) = 0$ and on the segment $I$ with ends $t_1 \neq t_2$ the function $\theta = \theta(t)$ is uniquely defined. Under condition 4.3.1: $\theta_2 = \theta_1 + 2\pi$, the graph of a continuously differentiable function $\theta(t), t \in \mathbb{R}$, can admit some parts that are obtained from the graph of the function on $I$ or its reflection with respect to the straight line $t = t_2$ by a combination of the parallel vertical shifts by values equal to $2k\pi$ for some $k \in \mathbb{Z}$ and the parallel horizontal shifts with the adjacent closed intervals of arbitrary lengths of constancy of the function $\theta = \theta(t)$. Under condition 4.3.2: $\theta < \theta_1 + 2\pi$, admissible are all continuous functions $\theta(t), t \in \mathbb{R}$, whose graphs on some segments of the variable $t$
with length $|t_2 - t_1|$ are horizontal shifts of its graph on $I$ or its reflection in $t = t_2$ with the values $\theta_1$ and $\theta_2$ at ends of these segments, with the adjacent closed intervals of arbitrary lengths of constancy of the function $\theta = \theta(t)$.

5. The conditions mean that $\varphi_3 = 0$ and the right-hand side in (23) is nonnegative for all $\theta \in \mathbb{R}$. Then $h_2(\theta_0) = \varphi_2$ is a minimal (respectively, maximal) value of the second component for $h \in U^*$ if $\varphi_4 > 0$ (respectively, $\varphi_4 < 0$). Moreover, the vector $h \in OU^*$ with $h_2 = \varphi_2$ is not unique if $\partial U^*$ is not strictly convex at the points $h$ with $h_2 = \varphi_2$; in other words, if $\partial U$ is not differentiable at $v^0 = (0, 1/\varphi_2)$.

In general, there exists some largest segment $[\theta_1, \theta_2]$, $\theta_1 \leq \theta_2$, such that $\theta_0 \in [\theta_1, \theta_2]$ and $h_2(\theta) = \varphi_2$ for all $\theta \in [\theta_1, \theta_2]$.

Let us again consider improper integral (43). It is clear that $\theta(t) \equiv \theta_0$ and we obtain only one of the two metric straight lines (30) in each of the cases 5.1–5.4 of Theorem 2. In all other cases, there also may be such solutions.

We will indicate all other possible extremals in the remaining cases:

5.5. $\theta_1 = \theta_2 = \theta_0$, $\mu(\theta)$ is finite as $\theta \nearrow \theta_0$ and $\mu(\theta) = \infty$ as $\theta \searrow \theta_0$;

5.6. $\theta_1 = \theta_2 = \theta_0$, $\mu(\theta)$ is finite as $\theta \searrow \theta_0$ and $\mu(\theta) = \infty$ as $\theta \nearrow \theta_0$;

5.7. $\theta_1 < \theta_2 = \theta_0$ and $\mu(\theta)$ is finite as $\theta \nearrow \theta_0$;

5.8. $\theta_0 < \theta_1 < \theta_2$ and $\mu(\theta)$ is finite as $\theta \nearrow \theta_0$;

5.9. $\theta_1 = \theta_2 = \theta_0$, $\mu(\theta)$ is finite as $\theta \nearrow \theta_0$ and $\theta \nearrow \theta_0$.

In all these cases, it may be that $\theta(t) \equiv \theta_0$ on some finite or infinite closed interval $J$ including 0. Assume that $J \neq \mathbb{R}$ and $\theta(t) \neq \theta_0$ for $t \notin J$ close enough to $J$.

(a) Let us consider 5.7 under the condition $t(\theta_1 + 2\pi) = \infty$ and 5.5 (case 5.8 with condition $\mu(\theta_2 - 2\pi) = \infty$ and case 5.6). If $\sup(J) = t_2 < +\infty$ then on the interval $J_2$ between $t_2$ and $+\infty$ we have to define the function $\theta(t) = \Theta(t - t_2)$, where $\Theta(t)$, $t \geq 0$, is inverse to the function $t(\theta)$ from (43) with $+$ (respectively, $-$) on the right-hand side. If $\inf(J) = t_1 > -\infty$ then the function $\theta(t) = \Theta(t_1 - t)$ is defined on the interval $J_1$ between $t_1$ and $-\infty$.

With the same signs, owing to (22) and (44), the functions $x(t)$ and $y(t)$ are defined by (46) and (32) respectively.

(b) Let $t_3 = t(\theta_1 + 2\pi)$ be finite in case 5.7. Then $t_3 > 0$ and the function $\Theta(t)$, $t \in I = [0, t_3]$, is determined which is inverse to the function $t(\theta)$ from (43) with $+$ on the right-hand side. All continuous functions $\theta(t)$, $t \in \mathbb{R}$, with $\theta(0) = \theta_0$ are admissible whose graphs on some segments of the variable $t$ with length $t_3$ are horizontal shifts of the graph of the function $\Theta(t)$ on $I$ or its reflection in the vertical line $t = t_3$, with the adjacent closed intervals of arbitrary lengths on which $\theta$ takes constant values $\theta_0$ or $\theta_1 + 2\pi$. In all considered cases, the functions $x(t)$ and $y(t)$ are defined by (46) and (32) respectively. Case 5.8 with finite $t_2 := t(\theta_2 - 2\pi)$ is considered in similar way.

5.9. Assume now that $\theta_1 = \theta_2 = \theta_0$, $t(\theta)$ is finite as $\theta \nearrow \theta_0$, and $t_3 := t(\theta_1 + 2\pi)$. Then $t_3 > 0$ and all continuously differentiable functions $\theta(t)$, $t \in \mathbb{R}$, are admissible whose graphs contain the parts that are obtained from the graph of the function $\Theta(t)$ on $I$ from 5.7b) or its reflection in the straight line $t = t_3$ by combinations of the vertical parallel shifts by the values equal to $2k\pi$ for some $k \in \mathbb{Z}$ and the horizontal parallel shifts, with adjacent closed intervals with arbitrary lengths of constancy of $\theta$. \hfill \qed

§ 5. About the Cases with Square Control Domains

The norm $F_\alpha$ on $D(e)$ is defined by the formula

$$F_\alpha(u_1, u_2) = \max\{|u_1 \cos \alpha + u_2 \sin \alpha|, | - u_1 \sin \alpha + u_2 \cos \alpha|\}, \quad \alpha \in [0, \pi/2).$$

(47)

Remark 5. In the paper [7], these metrics were considered for $0 \leq \alpha < \pi/4$.

The unit ball $U_\alpha = \{(u_1, u_2) : F_\alpha(u_1, u_2) \leq 1\}$ is obtained from the unit ball $U := U_0$ of the norm $F(u_1, u_2) = \max\{|u_1|, |u_2|\}$ by rotation by the angle $\alpha$, and $\partial U_\alpha$ is a circumscribed square around the unit circle $S^1 = \{(u_1, u_2) : u_1^2 + u_2^2 = 1\}$ with four tangency points. The polar curve $U_{\alpha}^*$ is the convex hull of these points and $\partial U_{\alpha}^*$ is the isoperimetrix to the Minkowski plane $(D(e), F_\alpha)$.

The following proposition holds:
Proposition 5. The polar equation of the square $\partial U^*_\alpha$ has the form

$$r(\theta) = \frac{\sqrt{2}}{2\cos(\theta - \alpha - \frac{\pi}{4})}, \quad \alpha \leq \theta \leq \alpha + \frac{\pi}{2}, \quad r\left(\theta + \frac{\pi}{2}\right) = r(\theta), \quad \theta \in \mathbb{R}.$$  

Moreover,

$$r'(\theta) = \frac{\sqrt{2} \sin(\theta - \alpha - \frac{\pi}{4})}{2\cos^2(\theta - \alpha - \frac{\pi}{4})}, \quad \alpha < \theta < \alpha + \frac{\pi}{2},$$

$$-1 \leq r'(\alpha) \leq 1, \quad r'\left(\theta + \frac{\pi}{2}\right) = r'(\theta), \quad \theta \in \mathbb{R}.$$  

1. It follows from (47) that $F_\alpha(0,1) = \cos \alpha$ if $0 \leq \alpha \leq \pi/4$, and $F_\alpha(0,1) = \sin \alpha$ if $\pi/4 < \alpha < \pi/2$; in the abnormal case, equation (30) has the form

$$x(t) = z(t) = v(t) \equiv 0, \quad y(t) = \begin{cases} \pm \frac{t}{\cos \alpha}, & \text{if } 0 \leq \alpha \leq \pi/4, \\ \pm \frac{t}{\sin \alpha}, & \text{if } \pi/4 < \alpha < \pi/2. \end{cases}$$

2.1. The pair $(\varphi_1, \varphi_2) = (r(\theta_0)(\cos \theta_0, \sin \theta_0))$ is in $\partial U^*_\alpha$.

Assume first that the point $(\varphi_1, \varphi_2)$ is not a vertex of the square $\partial U^*_\alpha$. Then (27) gives the unique solution $(u_1(t), u_2(t)) = (u_1(\theta_0), u_2(\theta_0))$ that is a vertex of the square $\partial U^*_\alpha$ such that $\angle((u_1(\theta_0), u_2(\theta_0)), (\varphi_1, \varphi_2)) < \pi/4$, and system (6) has the only solution: namely, the one-parameter subgroup

$$x(t) = u_1(\theta_0)t, \quad y(t) = u_2(\theta_0)t, \quad z(t) \equiv 0, \quad v(t) \equiv 0.$$  

Now let the point $(\varphi_1, \varphi_2)$ be one of the vertices of the square $\partial U^*_\alpha$. In this case, there exists a segment $\Delta$ (a side of the square $\partial U_\alpha$) of solutions $(u_1, u_2) = (u_1(\theta_0), u_2(\theta_0))$ to equations (27) such that $\angle((u_1, u_2), (\varphi_1, \varphi_2)) \leq \pi/4$. Each measurable function $(u_1(t), u_2(t)) \in \Delta$ defines a curve (31), (32), (7), (8).

In any case, we get only metric straight lines.

Example 1. If $\alpha = \varphi_2 = \varphi_3 = \varphi_4 = 0, \varphi_1 = 1$, then $\theta(t) \equiv \theta_0 = 0, t \in \mathbb{R}$, an arbitrary vector–function of the kind

$$u(t) = (1, u_2(t)), \quad -1 \leq u_2(t) \leq 1, \quad t \in \mathbb{R},$$

with measurable real function $u_2 = u_2(t)$, and the covector function $\psi(t) = (1, 0, 0, 0)$ satisfy the Pontryagin maximum principle for $t \in \mathbb{R}$; moreover, the corresponding (extremal) trajectory $q = g(t), t \in \mathbb{R}$, with origin $g(0) = e$ is the metric straight line. Thus, in general case, when searching for extremals and even geodesics of a left-invariant sub-Finsler metric on a Lie group, it is impossible to exclude the control from the Hamiltonian system for the Pontryagin maximum principle. This statement is true for every nonstrictly convex control domain $U \subset D(e)$, in other words, for the polar figure $U^*$ with nondifferentiable boundary $\partial U^*$.

2.2. Arguing as in the proof of Proposition 4, we get that $h(t), u(t), x(t), y(t), t \in \mathbb{R}$, are periodic functions with period $L = 4/|\varphi_3|$ and equalities (33), (34), (35), and (38) hold; i.e., the projection $(x, y)(t)$ of the curve $(x, y, z, v)(t)$ lies on the square obtained from $\partial U_{\alpha}^*/|\varphi_3|$ by shifting its center to the point $(-\frac{\varphi_2}{\varphi_3}, \frac{\varphi_1}{\varphi_3})$. The control is piecewise constant on the complement to a countable set of isolated points.

2.3. Cases 1, 2, 3.1, and 3.2 of Theorem 2 are possible. Case 4 of Theorem 2 is possible only for the case of 4.3 considered in the proof of Theorem 2; moreover, the case of 4.3.2 is possible only for $\alpha = \pi/4$. Case 5 of Theorem 2 is possible for the cases of 5.2, 5.7(b), 5.8b) (for $\alpha = \pi/4$), 5.9 (for $\alpha \neq \pi/4$), considered in the proof of Theorem 2. The description of all possible cases for the function $\theta(t), t \in \mathbb{R}$, in these cases are given in the proof of Theorem 2, the functions $x(t)$ and $y(t)$ are found by (31), (32) with usage of Proposition 5, and the functions $z(t), v(t), t \in \mathbb{R}$, are found by (39) and (40).
§ 6. Extremals of a Left-Invariant Sub-Finsler Quasimetric on the Engel Group

The proofs and results of our paper are valid also for the case of a left-invariant sub-Finsler quasimetric on the Engel group. Quasimetrics have all properties of the metric, but possibly the symmetry property \( d(p, q) = d(q, p) \). To this end, we need to make only the following changes in the text:

1. As \( U \), we take an arbitrary convex (two-dimensional) figure containing 0, perhaps \( U \neq -U \).

2. Instead of references to the reflexiveness in passing to the dual normed vector space for the finite-dimensional case we must refer to the theorem on the bipolar figure \( U^{**} = U \) (see [18, Theorem 14.5]).

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