Geodesic stability for memoryless binary long-lived consensus

Cristina G. Fernandes* and Maya Stein†

January 12, 2013

Abstract

The determination of the stability of the long-lived consensus problem is a fundamental open problem in distributed systems. We concentrate on the memoryless binary case with geodesic paths. We offer a conjecture on the stability in this case, exhibit two classes of colourings which attain this conjectured bound, and improve the known lower bounds for all colourings. We also introduce a related parameter, which measures the stability only for certain geodesics, and for which we also prove lower bounds.

1 Introduction

The consensus problem in distributed systems consists of the following: given a set of values, each coming from a processor or sensor, decide on a representative value, meaning the consensus of the given values. The long-lived consensus problem consists of repeatedly solving related instances of the consensus problem. In [5], Dolev and Rajsbaum introduce the concept of stability of long-lived consensus, where one wishes the representative values, produced by an algorithm for a sequence of input instances, to change as few times as possible (there might be some cost associated to a change). So the question is how to choose the outputs in a way that they are stable in time. In the case with memory, the algorithm may use the value produced for the previous instances in the sequence to decide on the value of the current instance. That is not allowed in the so called memoryless case. See also [1].

*Instituto de Matemática e Estatística, Universidade de São Paulo, Brazil, cris@ime.usp.br. Partial support by CNPq 309657/2009-1 and 475064/2010-0.
†Centro de Modelamiento Matemático, Universidad de Chile, Santiago, Chile, mstein@dim.uchile.cl. Support by Fondecyt 11090141 and Fapesp 05/54051-9.
We will consider binary-valued consensus, with the input sequences being a geodesic path. The case with memory is completely solved in [5] and also, for the memoryless case, some bounds for the minimum number of changes are shown, which we will improve here. Davidovitch, Dolev, and Rajsbaum [3] consider multi-valued consensus. Becker et al. [2] study average instability for binary consensus using random walks instead of geodesic paths.

We need a few definitions in order to properly state the problem. The \( n \)-hypercube is \( H_n := \{0, 1\}^n \). Write \( 0_n \) for \((0, 0, \ldots, 0)\), and similar. The ball \( B_t(0_n) \) of radius \( t \) around \( 0_n \) consists of all elements of \( H_n \) with at most \( t \) entries identical to 1. In the same way, we define \( B_t(1_n) \).

A colouring of \( H_n \) is a function \( f : H_n \to \{0, 1\} \). We say that a colouring \( f \) respects \( B_t(0_n) \) and \( B_t(1_n) \) if \( f(x) = 0 \) for each \( x \) in \( B_t(0_n) \) and \( f(x) = 1 \) for each \( x \) in \( B_t(1_n) \). Observe that if \( n < 2t + 1 \), the two balls \( B_t(0_n) \) and \( B_t(1_n) \) intersect, and no colouring can respect \( B_t(0_n) \) and \( B_t(1_n) \). As we are not interested in this case, we say \( t \) is valid (for \( n \)) if \( n \geq 2t + 1 \).

A geodesic \( P \) (in \( H_n \)) is a sequence \((x_0, x_1, \ldots, x_n)\) with \( x_i \in H_n \) for \( i = 0, 1, \ldots, n \), so that there is a permutation \((p_0, p_1, \ldots, p_n)\) of \((0, 1, \ldots, n)\) such that the \( \ell \)th entry of \( x_j \) differs from the \( \ell \)th entry of \( x_{j-1} \) if and only if \( j = p_\ell \). We then say that \( P \) fixed the \( \ell \)th entry at time \( j \).

We denote by \( \text{inst}(f, P) \), for instability, the number of colour-jumps of \( P \) in the colouring \( f \), that is, the number of indices \( i \) where \( f(x_i) \neq f(x_{i-1}) \). Any such index \( i \) shall be called a jump of \( P \) (in \( f \)). Let \( \text{inst}(f) \) be the maximum value of \( \text{inst}(f, P) \) over all geodesics \( P \).

The connection of these concepts and the memoryless consensus problem in distributed systems is as follows. Each point of \( H_n \) represents a set of \( n \) input values (one from each sensor). A colouring of \( H_n \) corresponds to an assignment of a representative value for each possible set of input values. We prefer colourings that respect the balls of a certain radius as the output value should in some way be representative. A geodesic stands for a slowly changing system of inputs (one sensor at a time), and its instability is the number of changes of the representative value. We remark that, if one considers arbitrary paths instead of geodesics, there is no bound on the instability as the path might go back and forth between two points with a different output value (see [5]).

Now, a colouring that respects \( B_t(0_n) \) and \( B_t(1_n) \) and has low instability is a good candidate for a consensus algorithm. One is therefore interested in the lowest possible instability.
Problem 1.1 (Dolev & Rajsbaum [5]). Given $n \in \mathbb{N}$, and $t$ valid for $n$, find the minimum value $\text{inst}(n, t)$ for $\text{inst}(f)$ over all colourings $f$ of $\mathcal{H}_n$ that respect $B_t(0^n)$ and $B_t(1^n)$.

Dolev and Rajsbaum [5] prove several special cases: $\text{inst}(n, t) \geq 1$ for $n > 4t$, $\text{inst}(n, 0) = 1$, $\text{inst}(n, 1) = 3$, and $\text{inst}(2t + 1, t) = 2t + 1$.

In Section 4.1 we establish a lower bound of $\lceil \frac{t-1}{n-2t} \rceil + \lfloor \frac{t-1}{n-2t} \rfloor + 3$ on $\text{inst}(n, t)$ (Theorem 4.1) that holds for all values of $n$ and $t$. A similar lower bound also holds for the related parameter $\text{winst}(n, t)$, which measures the maximum instability of a colouring considering only a special class of geodesics. This parameter is introduced in Section 3.

In Section 4.2, we improve our bound to $\text{inst}(2t + 2, t) \geq t + 2 + (t + 1) \mod 2$ for $t \geq 1$ for the special case of $n = 2t + 2$ (Theorem 4.2). The basic tool for this result is Lemma 4.3 which serves to extend bounds for smaller values of $t$ to larger values of $t$. This tool is extended in Section 4.3 to arbitrary values of $n$.

In [5], it is also shown that $\text{inst}(n, t) \leq 2t + 1$. We conjecture that this bound is indeed the correct value.

Conjecture 1.2 (Main conjecture). Let $n \in \mathbb{N}$, and $t$ be valid for $n$. Then $\text{inst}(n, t) = 2t + 1$.

If one can solve Problem 1.1 it would be interesting to find all optimal colourings, i.e., colourings for which the bound $\text{inst}(n, t)$ is attained. In Section 2 we exhibit two new classes, $\text{maj}_t(k)$ and $b_t^k$, of colourings that have instability exactly $2t + 1$. Previously only one such colouring (namely $\text{maj}_t(2t + 1)$ in our language) was known [5].

2 Candidates for optimal colourings

We present two classes of colourings that respect the balls $B_t(0^n)$ and $B_t(1^n)$ and have instability $2t + 1$.

2.1 The majority colourings

For a positive odd value $k$, define $\text{maj}_t(k)$ to be the colouring that assigns to each point $x \in \mathcal{H}_n \setminus (B_t(0^n) \cup B_t(1^n))$ the colour that appears on the majority of the first $k$ entries of $x$. The balls $B_t(0^n)$ and $B_t(1^n)$ are coloured canonically with 0 and 1, respectively.

For a positive even value $k$, define the auxiliary class $\text{maj}_t'(k)$ as the class of colourings $f$ that assign to each point $x$ outside $B_t(0^n)$ and $B_t(1^n)$
(which are coloured canonically) the colour that appears on the majority of the first \( k \) entries of \( x \), and any colour if both colours appear equally often in the first \( k \) entries of \( x \). Let \( \text{maj}_i(k) \subseteq \text{maj}'_i(k) \) be the class of those colourings in \( \text{maj}'_i(k) \) for which \( f(x) \neq f(y) \) whenever \( x \) and \( y \) restricted to their first \( k \) entries are the complement of each other. In what follows, we often abuse notation and, for a positive even value \( k \), write \( \text{maj}_i(k) \) for an arbitrary element of \( \text{maj}_i(k) \).

**Proposition 2.1.** Let \( k, t, n \in \mathbb{N} \), with \( 0 < k \leq 2t + 1 \leq n \). Then \( \text{inst}(\text{maj}_i(k)) = 2t + 1 \).

The proof of Proposition 2.1 splits into two parts: in Lemma 2.2 we show that no geodesic jumps more than \( 2t + 1 \) in \( \text{maj}_i(k) \), and in Lemma 2.3 we present a geodesic that jumps that much.

Before we turn to these lemmas, let us remark that, when \( k > 2t + 1 \) and odd, it is easy to find a geodesic that jumps \( k \) times in \( \text{maj}_i(k) \). Indeed, we may start at the point \((01)^{\lfloor n/2 \rfloor}0\) and then at each step switch an entry, from the first to the last. This shows that \( \text{maj}_i(k) \), for \( k \) large and odd, has instability larger than \( 2t + 1 \).

The remainder of this section is devoted to the proof of Proposition 2.1, i.e., to Lemma 2.2 and Lemma 2.3. For a geodesic \( P = (x_0, x_1, \ldots, x_{n-1}, x_n) \), the path \( Q = (x_n, x_{n-1}, \ldots, x_1, x_0) \) is also a geodesic, and is called the reverse of \( P \). Clearly, \( \text{inst}(f, P) = \text{inst}(f, Q) \) for any colouring \( f \).

**Lemma 2.2.** If \( 0 < k \leq 2t + 1 \leq n \), then \( \text{inst}(\text{maj}_i(k)) \leq 2t + 1 \).

**Proof.** Suppose otherwise. Then there is a geodesic \( P = (x_0, x_1, x_2, \ldots, x_n) \) in \( H^n \) with \( \text{inst}(\text{maj}_i(k), P) \geq 2t + 2 \). Let \( m \) be so that the \( m \)th jump of \( P \) is the first jump that fixes one of the last \( n - k \) entries (as \( k < 2t + 2 \) there is such an \( m \), \( 1 \leq m \leq n \)). Suppose \( P \) is chosen such that \( m = m(P) \) is as large as possible. Our plan is to modify \( P \) to a geodesic \( P' \) with \( m(P') > m(P) \), thus obtaining a contradiction.

Let \( i + 1 \) be the first jump of \( P \), and let \( \ell \) be the \((2t + 2)\)nd jump of \( P \). We assume that \( \text{maj}_i(k)(x_\ell) = 1 \), and thus \( \text{maj}_i(k)(x_i) = 1 \). The other case is analogous.

As \( \ell \) is the \((2t + 2)\)nd jump of \( P \), there are \( t + 1 \) jumps \( j \) with \( j < \ell \) and \( \text{maj}_i(k)(x_j) = 0 \). Hence, \( P \) fixed \((t + 1) 0\)'s before time \( \ell \), and therefore \( x_\ell \notin B_t(1^n) \). Thus, since \( \text{maj}_i(k)(x_\ell) = 1 \), the majority of the first \( k \) entries of \( x_\ell \) is not 0: it is 1 or \( k \) is even and \( x_\ell \) has as many 0’s as 1’s in its first \( k \) entries. We can use the same argument on the reverse of \( P \) to obtain that
$x_i \notin B_t(1^n)$, and thus the majority of the first $k$ entries of $x_i$ is 1 or $k$ is even and $x_i$ has as many 0’s as 1’s in its first $k$ entries. Thus we showed that

\[ \text{the first k entries of } x_i \text{ contain at least as many 1’s as 0’s,} \quad (1) \]

and the same holds for $x_\ell$.

Because $maj_t(k)(x_i) = maj_t(k)(x_\ell) = 1$, the first $k$ entries of $x_i$ and of $x_\ell$ are not the complement of each other. So, by (1), at least one entry within the $k$ first, say the first entry, is 1 in both $x_i$ and $x_\ell$. This implies that all $x_j$ with $i \leq j \leq \ell$ start with a 1.

Let $S$ be the set of those of the first $k$ entries of $x_i$ that do not change in $P$ between $x_i$ and $x_\ell$. We have just seen that $s := |S| \geq 1$. Let $z_1$ be obtained from $x_i$ by changing the first entry to 0, and for $1 < j \leq s$ let $z_j$ be obtained from $z_{j-1}$ by changing another of the entries in $S$. Then

\[ \text{the first k entries of } z_s \text{ are the complement of the first k entries of } x_\ell. \quad (2) \]

Let $h$ be the $(2t + 1)$st jump of $P$. Then $maj_t(k)(x_{h-1}) = 1$ and $maj_t(k)(x_h) = 0$. There are $t$ jumps $j \leq h - 1$ with $maj_t(k)(x_j) = 1$, each fixing a 1 distinct from the first entry. Thus in total $x_h$ and $x_{\ell-1}$ have at least $(t+1)$ 1’s, and cannot be in $B_t(0^n)$. In the same way, we see that $x_{i+1} \notin B_t(0^n)$.

Consider $P' = (z_s, z_{s-1}, \ldots, z_1, x_i, x_{i+1}, \ldots, x_\ell, y_0, y_1, \ldots, y_{n-s-\ell+i-1})$, where the $y_j$’s are arbitrarily chosen to complete $P'$ to a geodesic. We claim that

\[ P' \text{ has a jump in its first } s + 1 \text{ steps.} \quad (3) \]

Then we are done because, by the choice of $P$, all the first $m+1$ jumps of $P'$ fix one of the first $k$ entries, contradicting our choice of $P$.

It remains to prove (3). As $x_{i+1} \notin B_t(0^n)$ and $maj_t(k)(x_{i+1}) = 0$, there are at least as many 0’s as 1’s among the first $k$ entries of $x_{i+1}$. So, since the first entry of $x_i$ is 1, but the first entry of $z_1$ is 0, there are also at least as many 0’s as 1’s among the first $k$ entries of $z_1$.

Now, as $x_i \notin B_t(1^n)$, also $z_1 \notin B_t(1^n)$. Hence, if the first $k$ entries of $z_1$ contain more 0’s than 1’s, it follows that $maj_t(k)(z_1) = 0$. As $maj_t(k)(x_i) = 1$, the geodesic $P'$ has the jump $x_i$, which is as desired for (3).

So we may assume that the first $k$ entries of $z_1$ contain exactly as many 0’s as 1’s. Thus by (1) and (2), and by the definition of $z_s$, it follows that $z_s$ has at least as many 0’s as 1’s in its first $k$ entries, and so at least as many
0’s as $z_1$ has. Therefore, $z_1 \notin B_t(1^n)$ implies that $z_s \notin B_t(1^n)$ and hence, $maj_t(k)(z_s) = 0$. This finishes the proof of (3), and thus the proof of the lemma.

Lemma 2.3. If $0 < k \leq 2t + 1 \leq n$, then $\text{inst}(maj_t(k)) \geq 2t + 1$.

Proof. We will prove the following stronger assertion.

There exists a $(t + 1)$-geodesic $P$ such that

$$\text{inst}(maj_t(k), P) \geq 2t + 1$$

and the last point of $P$ is coloured 0.

We shall proceed by induction on $k$. Observe that (4) holds for $k = 1$ and for $k = 2$. Indeed, for $k = 1$, consider the following $(t + 1)$-geodesic.

$$P = (t+1)0^n-t-1, \quad 1t00^n-t-1, \quad 1t010^n-t-2, \quad 1t-10210^n-t-2, \quad 1t-102120^n-t-3, \quad 1t-203120^n-t-3, \quad 1t-203130^n-t-4, \quad \ldots \quad 11t-11t-20^n-2t+1, \quad 11t-11t-10^n-2t, \quad 10t1t-10^n-2t, \quad 10t1t0^n-2t-1, \quad 0t+11t0^n-2t-1, \quad 0t+11t+10^n-2t-2, \quad 0t+11t+20^n-2t-3, \quad 0t+11t+30^n-2t-4, \quad \ldots \quad 0t+1^n-t-1) [0].$$

Note that $P$ jumps $2t + 1$ times. For $k = 2$, consider either $P$, or the $(t + 1)$-geodesic $P'$ obtained from $P$ by changing the two points
$10^t1^t0^{n-2t}$ and $10^t1^{0^{n-2t-1}}$ to $010^{t-1}1^t0^{n-2t}$ and $010^{t-1}1^t0^{n-2t-1}$. If $\text{maj}_t(k)(10^t1^t0^{n-2t}) = 0$, we choose $P$, otherwise we choose $P'$.

So suppose we are given a $k \geq 3$. Then $t \geq 1$ and $n \geq 3$. Consider $\text{maj}_t(k)$ on

$$\tilde{H}^n := \{ x \in H^n : x(1) = 0 \text{ and } x(2) = 1 \},$$

and observe that this is equivalent to considering $\text{maj}_{t-1}(k-2)$ on $H^{n-2}$. Hence, by induction, we know there exists a $t$-geodesic $\tilde{P}$ in $H^{n-2}$ that is as in (3) for $t-1$. In particular, $\tilde{P}$ jumps at least $2(t-1)+1 = 2t-1$ times. Abusing notation slightly, we shall consider $\tilde{P}$ as a path in $H^n$.

Now we extend $\tilde{P}$ to a geodesic in $H^n$ adding two more jumps. By (1), we know that $\tilde{P}$ ends in a point $y$ with $\text{maj}_t(k)(y) = 0$, and with exactly $t+1$ entries equal to 0 (among these the first entry).

Suppose the first point of $\tilde{P}$, let us call this point $a$, is coloured 1 in $\text{maj}_t(k)$. Then we add the points $y' := (1,1,y(3),y(4),\ldots)$ and $y'' := (1,0,y(3),y(4),\ldots)$ to the end of $\tilde{P}$ and obtain a geodesic $P$ as desired. Indeed, $y' \in B_t(1^n)$ as $y'$ has exactly $t$ 0's, hence $\text{maj}_t(k)(y') = 1$, and so we have our first extra jump. Note that $y''$ has exactly as many 1's as $y$ (in particular, $y'' \notin B_t(1^n)$), and moreover, $y''$ has exactly as many 1's in the first $k$ entries as $y$. Thus, $\text{maj}_t(k)(y'') \neq \text{maj}_t(k)(y)$ only if $k$ is even and $y$ and $y''$ have as many 0's as 1's in their first $k$ entries. But in this case, the definition of $\text{maj}_t(k)$ implies that $0 = \text{maj}_t(k)(y'') \neq \text{maj}_t(k)(a) = 1$. Therefore, $\text{maj}_t(k)(y'') = 0$, and we have the second extra jump, implying that $P$ is as in (4).

It remains to analyse the case where $\text{maj}_t(k)(a) = 0$. In this case, note that as $\tilde{P}$ starts and finishes with colour 0, it jumps an even number of times, that is, at least $2(t-1)+2 = 2t$ times. Thus we need to add only one more jump. If we build $P$ in the same way as above, $P$ jumps at least $2t+1$ times, but for the same reasons as above, it ends in a point coloured 1. So, instead, let $P$ be obtained from $\tilde{P}$ by adding at its beginning the two points $a'' := (1,0,a(3),a(4),\ldots)$ and $a' := (1,1,a(3),a(4),\ldots)$. Observe that since $a''$ is the complement of $y$, it has the opposite colour, i.e., $\text{maj}_t(k)(a'') = 1$. Hence between $a''$ and $a$ we have at least one jump. So $P$ is a well-ending $(t+1)$-geodesic with at least $2t+1$ jumps, as desired.

### 2.2 The partition colourings

We present a second class of colourings, the colourings $b^k_t$, which respect the balls $B_t(0^n)$ and $B_t(1^n)$ and have instability $2t+1$. Before that, we define the auxiliary colouring $a^Q_j$ that will be used in the definition of $b^k_t$. 


Let \( m, s \) and \( t \) be such that \( m \geq (s + 1)(t + 1) \). Let \( \mathcal{Q} \) be a partition of \([m]\) into \( s + 1 \) sets of size at least \( t + 1 \) each. For \( j = 0, 1 \), let \( a_j^\mathcal{Q} \) be the following colouring of \( \mathcal{H}_n \). Let \( a_j^\mathcal{Q}(x) = j \) if and only if, in at least one of the sets in \( \mathcal{Q} \), all entries are \( j \). It is not difficult to see that \( a_j^\mathcal{Q} \) respects both \( B_s(j^m) \) and \( B_t((1 - j)^m) \).

Consider a geodesic \( P = (x_0, x_1, \ldots, x_m) \) in \( \mathcal{H}_n \). Note that, if \( i \) is a jump of \( P \) in \( a_j^\mathcal{Q} \), then for some set \( Q \) in \( \mathcal{Q} \) we have that \( x_i(q) = j \) for all \( q \in Q \) either for \( \ell = j - 1 \) or for \( \ell = j \), but not for both. We say that the jump \( i \) is associated to this set \( Q \). Thus there are at most two jumps in \( P \) associated to the same set \( Q \) in \( \mathcal{Q} \). This implies that \( a_j^\mathcal{Q} \) jumps at most \( 2|\mathcal{Q}| = 2(s + 1) \) times.

Now, let \( k, s, t, \) and \( n \) be such that \( k \) is odd, \( s \geq -1, t = s + (k + 1)/2, \) and \( n \geq (s + 1)(t + 1) + k \). Note that \( k \leq 2t + 1 \) because \( s \geq -1 \). Let \( \mathcal{Q} \) be a partition of \([n - k]\) into \( s + 1 \) sets of size at least \( t + 1 \) each. (If \( s = -1 \), then \( n = k \) and \( \mathcal{Q} = \emptyset \).) We shall define the colouring \( b_k^\mathcal{Q} = b_k^\mathcal{Q}(Q) \) using \( a_0^\mathcal{Q} \) and \( a_1^\mathcal{Q} \). We abuse notation and assume that \( a_j^\mathcal{Q}(y) = 1 - j \) if \( \mathcal{Q} \) or \( y \) is empty.

For each point \( x \), if the majority of the first \( k \) entries of \( x \) is 1, then let \( b_k^\mathcal{Q}(x) = a_0^\mathcal{Q}(x') \), where \( x' \) is \( x \) without the first \( k \) entries. If the majority of the first \( k \) entries of \( x \) is 0, then let \( b_k^\mathcal{Q}(x) = a_1^\mathcal{Q}(x') \). In both cases, we sometimes abuse notation and write that \( b_k^\mathcal{Q}(x) \) respects the balls \( B_t(0^n) \) and \( B_t(1^n) \).

Indeed, let us suppose the majority of the first \( k \) entries of some point \( x \) is 1, and hence \( b_k^\mathcal{Q} = a_0^\mathcal{Q} \) (the other case is symmetric). If \( x \) has at most \( t \) entries equal to 0, clearly no set in \( \mathcal{Q} \) can only consist of 0’s, and so \( b_k^\mathcal{Q}(x) = 1 \). On the other hand, if \( x \) has at most \( t \) 1’s, then \( x' \) has at most \( t - (k + 1)/2 = s \) 1’s and therefore, as \( |\mathcal{Q}| = s + 1 \), there is a set in \( \mathcal{Q} \) that only consists of 0’s. Thus \( b_k^\mathcal{Q}(x) = 0 \) in this case. Hence, in either case, \( b_k^\mathcal{Q}(x) \) is as desired.

Observe that, for \( t = 0 \) and \( k = 1 \), we have \( s = -1 \), and hence \( n = 1 \). In this case, \( b_0^1 = \text{maj}_0(1) \).

**Proposition 2.4.** Let \( k, t, n \in \mathbb{N} \) be such that \( k \) is odd, \( k \leq 2t + 1 \) and \( n \geq (t + 1 - \frac{k+1}{2})(t + 1) + k = \frac{(t+1)(2t+1)-k(t-1)}{2} \). Then \( \text{inst}(b_k^\mathcal{Q}) = 2t + 1 \).

**Proof.** Let \( P \) be a geodesic in \( \mathcal{H}_n \). To prove that \( P \) jumps at most \( 2t + 1 \) times in \( b_k^\mathcal{Q} \), first note that at most \( k \) jumps of \( P \) are associated to its first \( k \) entries. Second, note that \( P \) has at most two jumps associated to each set \( Q \) in \( \mathcal{Q} \). Indeed, if \( P \) has one jump associated to \( Q \) while \( b_k^\mathcal{Q} = a_0^\mathcal{Q} \), then \( P \) has at most one more jump associated to \( Q \) while \( b_k^\mathcal{Q} = a_1^\mathcal{Q} \). Similarly, if \( P \) has
two jumps associated to \( Q \) while \( b^{k}_{t} = a^{Q}_{t-j} \), then \( P \) has no jumps associated to \( Q \) while \( b^{k}_{t} = a^{Q}_{t-j} \).

Also, it is not hard to find a geodesic in \( \mathcal{H}_{n} \) that jumps \( 2t+1 \) times in \( b^{k}_{t} \). Consider a point \( x_{0} \) with \( (k+1)/2 \) 1’s in the first \( k \) entries, and exactly one 1 in each of the sets in \( Q \). Then \( x_{0} \) has exactly \( t+1 \) entries equal to 1. Take a geodesic that starts in \( x_{0} \), and jumps \( k \) times by changing alternatively 1’s to 0’s and 0’s to 1’s within the first \( k \) entries. After that, we have that \( b^{k}_{t} = a^{Q}_{t} \). So we can jump twice per set \( Q \) in \( Q \) by changing all entries in \( Q \) to 1 first, and then changing the unique entry in \( Q \) that started with a 1 to a 0.

\[ \square \]

3  Well-ending geodesics and \( k \)-defined colourings

A geodesic in \( \mathcal{H}_{n} \) is called an \( m \)-geodesic if it starts in a point of \( \mathcal{H}_{n} \) which has exactly \( m \) entries equal to 1. (It then ends in a point which has exactly \( m \) entries that equal 0.)

Let \( f \) be a colouring of \( \mathcal{H}_{n} \) and \( t_{f} \) be the maximum \( t \) such that \( f \) respects \( B_{t}(0^{n}) \) and \( B_{t}(1^{n}) \). (If \( f(0^{n}) = 1 \) or \( f(1^{n}) = 0 \), then set \( t_{f} = -1 \).)

If \( P \) is a geodesic whose first point is coloured 1 in \( f \), or whose last point is coloured 0 in \( f \), we say \( P \) ends well (in \( f \)). Let \( \text{winst}(f) \) denote the maximum value of \( \text{inst}(f,P) \), taken over all well-ending \( (t_{f}+1) \)-geodesics \( P \).

In analogy to Problem 1.1 we ask the following.

**Problem 3.1.** Given \( t \) valid for \( n \), which is the smallest value \( \text{winst}(n,t) \) such that \( \text{winst}(n,t) = \text{winst}(f) \) for some colouring \( f \) with \( t_{f} = t \)?

Observe that \( \text{winst}(f) \leq \text{inst}(f) \) for every colouring \( f \). Moreover, \( \min_{t \leq s \leq (n-1)/2} \{ \text{winst}(n,s) \} \leq \text{inst}(n,t) \) for all \( t \) valid for \( n \).

Call a colouring \( f \) \( k \)-defined if there are \( k \) indices such that \( f(x) = f(y) \) for any two points \( x, y \in \mathcal{H}_{n} \setminus (B_{t}(0^{n}) \cup B_{t}(1^{n})) \) that coincide in all entries given by these \( k \) indices. A \( k \)-defined colouring that is not \((k-1)\)-defined is called strictly \( k \)-defined. For instance, \( \text{maj}_{t}(k) \) is strictly \( k \)-defined and \( a^{Q}_{0} \) is strictly \( n \)-defined.

Let \( t \) be valid for \( n \). For the next lemma, let \( F^{n}(t) \) denote the set of all strictly \( n \)-defined colourings \( f \) of \( \mathcal{H}_{n} \) with \( t_{f} = t \), and let \( F^{<n-2t}(t) \) denote the set of all strictly \( k \)-defined colourings \( f \) of \( \mathcal{H}_{n} \) with \( 0 \leq k < n - 2t \) and \( t_{f} = t \).

\[ ^{1} \text{We remark that in } [5, \text{pg. 39}], \text{ one-bit defined colourings are introduced. This definition differs from ours (for } k = 1) \text{ as we canonically colour the balls } B_{t}(0^{n}) \text{ and } B_{t}(1^{n}). \text{ For instance, } \text{maj}_{t}(1) \text{ is 1-defined, but not one-bit defined.} \]
Lemma 3.2. Let \( g : \mathbb{N} \rightarrow \mathbb{N} \) be such that \( g(s) + 2(t - s) \geq g(t) \) for all \( s \leq t \). If \( \text{winst}(f') \geq g(t') \) for all \( t' \) valid for \( n \) and all \( f' \in F^{n}(t') \), then \( \text{winst}(f) \geq \min\{g(t), 2t + 2\} \) for all \( t \) valid for \( n \) and all \( f \in F^{<n-2t}(t) \).

This lemma could be used as a step towards a solution of Problem 1.1. In fact, consider \( g(s) = 2s + 1 \) and note that such \( g \) satisfies the assumption of the lemma. If we could prove that \( \text{winst}(f) \geq 2t + 1 \) for all colourings \( f \) that are strictly \( k \)-defined with \( k \geq n - 2t \), then Lemma 3.2 would assure this bound holds for all colourings \( f \) of \( H_n \), and thus imply Conjecture 1.2.

The rest of this section is devoted to the proof of Lemma 3.2.

Proof of Lemma 3.2. Let \( t \) be valid for \( n \), and let \( f \in F^{<n-2t}(t) \). We assume that the defining entries of \( f \) are the first \( k < n - 2t \). Our aim is to find a well-ending \((t+1)\)-geodesic \( P \) for \( f \) that jumps at least \( \min\{g(t), 2t + 2\} \).

Consider the hypercube \( H_k \), and let \( f' \) be the colouring of \( H_k \) that assigns to each \( x \) in \( H_k \) the colour that \( f \) assigns to the point of \( H_n \) obtained from \( x \) by adding \((t+1)\) 1’s and \((n-k-t-1)\) 0’s at the end. Observe that

\[
\text{\( f' \) is strictly \( k \)-defined}, \quad (5)
\]
as \( f \) is. Now if \( f'(0^k) = f(0^k1^{t+1}0^{n-k-t-1}) = 1 \), then there exists a \((t+1)\)-geodesic in \( H_n \) that jumps at least \( 2t + 2 \) times. For example, consider

\[
Q = (0^k1^{t+1}0^{n-k-t-1}, [1],
\quad 0^k1^00^{n-k-t}, [0],
\quad 0^k1^t0^{n-k-t-1}, [1],
\quad 0^k1^{t-1}0^{n-k-t}, [0],
\quad \ldots
\quad 0^k1^{0^{n-k-t-1}t}, [1],
\quad 0^k0^{n-k-t}1^t, [0],
\quad 0^k0^{n-k-t-1}1^{t+1}, [1],
\quad 0^k0^{n-k-t-2}1^{t+2}, [1],
\quad 0^k0^{n-k-t-3}1^{t+3}, [1],
\quad \ldots
\quad 0^k0^{t+1}0^{n-k-t-1}, [1],
\quad 10^{k-1}0^{t+1}0^{n-k-t-1}, [?],
\quad \ldots
\quad 1^{k}0^{t+1}0^{n-k-t-1}) [?].
\]
Similarly, if $f'(1^k) = 0$, consider the geodesic obtained from $Q$ by swapping all 0’s and 1’s. Its reverse is a $(t + 1)$-geodesic that jumps at least $2t + 2$ times.

Therefore, we assume from now on that $f'(0^k) = 0$ and $f'(1^k) = 1$, in other words, that $f'$ respects $B_0(0^k)$ and $B_0(1^k)$ and thus $t_{f'} \geq 0$. By (5), we can use the assumption of the lemma for $s := t_{f'}$ and $f'$ to obtain an $(s+1)$-geodesic $P' = (p_0, p_1, \ldots, p_k)$ in $H_k$ such that $\text{inst}(f', P') \geq g(s)$. Furthermore, $P'$ starts with colour 1, or ends with colour 0, say the former (the other case is symmetric). Note that we can adjust $P'$ without decreasing its instability so that, when $P'$ is inside one of the balls, it has exactly $s$ 0’s or 1’s, respectively. That is, we may assume each point in $P'$ has at least $s$ 0’s and at least $s$ 1’s.

We shall now add a few 0’s and 1’s to each point in $P'$ in order to make it a path $P''$ in $H_n$. Then, we shall extend $P''$ to a $(t + 1)$-geodesic $P$ in $H_n$, and make it jump $2(t - s)$ additional times at the border of one of the balls $B_t(0^n)$ or $B_t(1^n)$. As we explain ahead, these two goals are achieved by the
geodesic

\[ P = (p_0^{1-t-s}0^{n-k-t+s}, \quad [1]) \]
\[ p_0^{1-t-s}0^{n-k-t+s+1}, \quad [0] \]
\[ p_0^{1-t-s}10^{n-k-t+s+1}, \quad [1] \]
\[ p_0^{1-t-s}20^{n-k-t+s+11}, \quad [0] \]
\[ p_0^{1-t-s}20^{n-k-t+s+12}, \quad [1] \]
\[ \ldots \]
\[ p_0^{10^{n-k-t+s+1}1^{t-s-2}}, \quad [0] \]
\[ p_0^{10^{n-k-t+s+1}1^{t-s-1}}, \quad [1] \]
\[ p_0^{0^{10^{n-k-t+s+1}1^{t-s}}}, \quad [0] \]
\[ p_0^{0^{n-k-t+s+1}1^{t-s}}, \quad [1] \]
\[ p_1^{0^{n-k-t+s+1}1^{t-s}}, \quad [?] \]
\[ p_2^{0^{n-k-t+s+1}1^{t-s}}, \quad [?] \]
\[ \ldots \]
\[ p_k^{0^{n-k-t+s+1}1^{t-s}}, \quad [?] \]
\[ p_k^{0^{n-k-t+s+1}1^{t-s+1}}, \quad [?] \]
\[ p_k^{0^{n-k-t+s+1}1^{t-s+2}}, \quad [?] \]
\[ \ldots \]
\[ p_k^{0^{1-t-s}1^{n-k-t+s}}) \quad [?] \].

The initial part of \( P \) jumps \( 2(t-s) \) times, since \( p_0 \) has exactly \( s+1 \) entries equal to 1. Moreover, the last part of \( P \) (where the colours are marked as “?”) jumps at least \( g(s) \) times. Indeed, as \( p_0, \ldots, p_k \) have at least \( s \) 0’s and at least \( s \) 1’s, the points \( p_i^{0^{n-k-t+s+1}1^{t-s}} \), for \( i = 0, \ldots, k \), have at least \( n - k - t + s + s = n - k - t + 2s > t + 2s \geq t \) 0’s and at least \( t \) 1’s, so, this part of \( P \) enters the balls exactly when \( P' \) does. Thus the part of \( P \) that goes through the points \( p_i^{0^{n-k-t+s+1}1^{t-s}} \) for \( i = 0, \ldots, k \) jumps exactly when \( P' \) does, that is, at least \( g(s) \) times. So, by our assumption on \( g \), it follows that \( P \) jumps at least \( 2(t-s) + g(s) \geq g(t) \) times, completing the proof of the lemma.
4 Lower bounds on $\text{inst}(n, t)$ and $\text{winst}(n, t)$

4.1 The zig-zag bound

In this section we prove lower bounds for $\text{inst}(n, t)$ and $\text{winst}(n, t)$. Recall that any lower bound on $\text{winst}(f)$ also serves as a lower bound for $\text{inst}(f)$. We start with a bound for all values of $n$ and valid $t$, which we obtain from a fairly basic zig-zag argument.

Later, in Theorem 4.2 and Proposition 4.5, the bounds from Theorem 4.1 will be improved for the special cases $n = 2t + 2$ and $n = 2t + 3$.

**Theorem 4.1** (The zig-zag bound). Let $n \in \mathbb{N}$ and let $t \geq 0$ be valid for $n$. Then

(a) $\text{winst}(n, t) \geq \left\lfloor \frac{t}{n-2t} \right\rfloor + \left\lceil \frac{t}{n-2t} \right\rceil + 1$,

(b) $\text{inst}(n, t) \geq \left\lfloor \frac{t-1}{n-2t} \right\rfloor + \left\lceil \frac{t-1}{n-2t} \right\rceil + 3$, if $t \geq 1$.

We remark that Theorem 4.1 (a) proves Conjecture 1.2 for $t = 0$ and Theorem 4.1 (b) proves Conjecture 1.2 for $t = 1$. This has been shown earlier in [5].

We dedicate the rest of this subsection to the proof of Theorem 4.1.

**Proof of Theorem 4.1.** Let $f$ be a colouring with $t_f \geq 0$ and let $t = t_f$. For (a), our aim is to find a well-ending $(t + 1)$-geodesic $P$ that jumps at least $\left\lfloor \frac{t}{n-2t} \right\rfloor + \left\lceil \frac{t}{n-2t} \right\rceil + 1$ times in $f$.

As $t = t_f$, there is a point $x \in \mathcal{H}^n$ that has exactly $(t + 1)$ 1’s or $(t + 1)$ 0’s, and that is coloured 1 or 0, respectively. Say the former holds for $x$ (the other case is symmetric).

We let $P$ start in $x$, then enter $B_t(0^n)$, then go to $B_t(1^n)$, come back to $B_t(0^n)$, go to $B_t(1^n)$ again, etc., until $P$ has used up all of its entries. For example, if $x = 1^{t+1}0^{n-t-1}$, we let $P$ pass next through $1^t0^{n-t}$ and then through $1^t1^{n-t-2}$, through $1^{t-(n-2t)}0^{n-t}1^{n-2t}$, through $1^{t-(n-2t)}0^{t}1^{2n-4t}$, and so on.

We can do this until one of the following two things happens. Firstly, coming from $B_t(0^n)$, we might end in the complement of $x$ with $(t + 1)$ 0’s (just before reaching $B_t(1^n)$). This will happen exactly when $n = \ell(n-2t)$ for some odd $\ell$, which is the case if and only if $n-2t$ divides $t$. Then we will have jumped at least $\ell$ times and

$$\ell = \frac{n}{n-2t} = \frac{2t}{n-2t} + 1 = \left\lfloor \frac{t}{n-2t} \right\rfloor + \left\lceil \frac{t}{n-2t} \right\rceil + 1.$$
Secondly, on our way from $B_t(1^n)$ to $B_t(0^n)$, we might reach a point of $\mathcal{H}_n \setminus (B_t(0^n) \cup B_t(1^n))$ which has no more unused 1’s. This happens if and only if $n - 2t$ does not divide $t$. Then we have to return in the direction of $B_t(1^n)$ to end in the complement of $x$ (if we are not already there). In this case, we have jumped at least

$$1 + 2 \cdot \left[ \frac{n - (n - 2t)}{2(n - 2t)} \right] + 1 = 2 \cdot \left[ \frac{t}{n - 2t} \right] + 2 = \left[ \frac{t}{n - 2t} \right] + \left[ \frac{t}{n - 2t} \right] + 1$$

times, because at least one jump is achieved during the first $n - 2t$ steps, then we get at least 2 jumps for every $2(n - 2t)$ steps, and finally we jump at least once more in the last part of $P$ when going through $B_t(1^n)$. Note that, by the construction of $P$, we have to end up in one of the two situations just described. This completes the proof of (a).

For (b), the proof is similar, the difference being that we let $P$ start inside $B_t(0^n)$, have $x$ as its second point, then re-enter $B_t(0^n)$, and then go on in a zig-zag fashion as before. We will obtain 2 jumps in the beginning, at least one jump during the next $n - 2t$ steps of $P$, and then 2 jumps every $2(n - 2t)$ steps. Finally we might ensure another jump depending on whether $n - 2 = \ell(n - 2t)$ for some odd $\ell$ or not. More precisely, if $n - 2 = \ell(n - 2t)$ for some odd $\ell$, that is, if $n - 2t$ divides $t - 1$, then we get

$$\ell + 2 = \frac{n - 2}{n - 2t} + 2 = \left[ \frac{t - 1}{n - 2t} \right] + \left[ \frac{t - 1}{n - 2t} \right] + 3$$

jumps, and otherwise, we also get

$$2 + 1 + 2 \cdot \left[ \frac{n - 2 - (n - 2t)}{2(n - 2t)} \right] + 1 = \left[ \frac{t - 1}{n - 2t} \right] + \left[ \frac{t - 1}{n - 2t} \right] + 3$$

jumps, which is as desired. Clearly, we need here that $t \geq 1$, because otherwise we could not enter $B_t(0^n)$ twice in the beginning. \( \square \)

### 4.2 Better bounds for one strip

In this subsection we will concentrate on the case when $\mathcal{H}_n$ contains, besides the balls, only one ‘strip’ of points which all have the same number of entries equal to 0 and equal to 1. That is, we treat the case $n = 2t + 2$.

From Theorem 4.1, we have that $\text{winst}(2t + 2, t) \geq t + 1$ and $\text{inst}(2t + 2, t) \geq t + 2$ for $t \geq 1$. The following result improves this bound.

**Theorem 4.2.** $\text{inst}(2t + 2, t) \geq \text{winst}(2t + 2, t) \geq t + 3$ for all $t \geq 2$. 

14
We will prove Theorem 4.2 by combining the next two lemmas. The first of these is a tool for extending bounds for small values of $t$ to larger values of $t$.

**Lemma 4.3.** Let $y_0$, $t_0$ and $t \in \mathbb{N}$ with $t \geq t_0$. If $\text{winst}(2t_0 + 2, t_0) \geq y_0$ for some $t_0 \geq 0$ then $\text{winst}(2t + 2, t) \geq y_0 + t - t_0$.

**Proof.** We proceed by induction on $t$. The base, for $t = t_0$, follows directly from the hypothesis of the lemma. For $t > t_0$, consider a colouring $f$ with $t_f = t$ of the hypercube $H_n$ of dimension $n = 2t + 2$.

Define a colouring $g$ of the hypercube $H_{n-2}$ by assigning to each $x'$ in $H_{n-2}$ the value $g(x') = f(01x')$. Then $g$ is such that $t_g = t - 1$. Indeed, any point of $H_{n-2} \setminus (B_{t-1}(0^{n-2}) \cup B_{t-1}(1^{n-2}))$ is a witness to this. We may thus apply the induction hypothesis to obtain a well-ending $t$-geodesic $\tilde{P}$ in $H_{n-2}$ that jumps at least $y_0 + t - 1 - t_0$ times in $g$. Extending each point $x'$ of $\tilde{P}$ to the point $01x'$ of $H_n$, we obtain a path $P'$ in $H_n$ that jumps at least $y_0 + t - 1 - t_0$ times in $f$.

Let $01a$ and $01z$ be the first and last point of $P'$ respectively. Note that since $\tilde{P}$ is well-ending, either $g(01a) = 1$ or $g(01z) = 0$ (or both). We extend $P'$ to $P$ by adding to its beginning the points $00a$ and $10a$, if $g(01a) = 1$, and the points $11z$ and $10z$ to its end otherwise. As we thus pass once more through either $B_t(0^n)$ or $B_t(1^n)$, our extension $P$ of $P'$ jumps at least once more than $P'$, that is, $y_0 + t - t_0$ times in $f$. Clearly, $P$ is a $(t + 1)$-geodesic, and so is its reverse, because $n = 2t + 2$. Now at least one of the two, $P$ or its reverse, has to be well-ending, which completes the proof of the lemma. 

\[\Box\]

The next lemma takes care of the base case $t = t_0$ for Lemma 4.3. It also confirms Conjecture 4.2 for $n = 2t + 2$ and small values of $t$.

**Lemma 4.4.** $\text{winst}(2t + 2, t) \geq 2t + 1$ for $t = 0, 1, 2$.

**Proof.** The case $t = 0$ is trivial. For $t = 1$, let $f$ be a colouring of $H_4$ such that $t_f = 1$. Note that there are two points $x$ and $y$ in $H_4$ with exactly $t + 1 = 2$ entries equal to 1, differing in exactly two entries (that is, such that $||x - y||^2 = 2$), and such that $f(x) = f(y)$. For example, two of the three points 1100, 1010, 1001 must have the same colour in $f$. Now it is easy to construct a well-ending 2-geodesic that starts in $x$ and jumps at least three times.

For $t = 2$, let $f$ be a colouring of $H_6$ such that $t_f = 2$. Observe that we only need to find three points $x, y, z$, all with exactly $t + 1 = 3$ entries equal to 1, such that $||x - y||^2 = ||y - z||^2 = 2$, $||x - z||^2 = 4$, and $f(x) = f(y) =$
Indeed, if we have such points, it is easy to construct a well-ending 3-geodesic that starts in $x$ and jumps at least five times.

The proof of the existence of $x$, $y$ and $z$ is a case analysis. By rearranging the order of the entries, we may assume the points $x = 111000$ and $y = 110100$ have the same colour $j$ in $f$. If one among $x' = 100110$, $y' = 100101$ and $z' = 010101$ has colour $j$, then we may take it as our third point $z$. If not, then $x'$, $y'$ and $z'$ all have colour $1 - j$ and form a triple of points as desired.

Proof of Theorem 4.2. The statement is an immediate consequence of Lemma 4.3 and Lemma 4.4 for $t = 2$. □

4.3 The extension method for more strips

We now extend the results from the previous subsection to the general case, when we have more ‘strips’. The main result of this subsection, Proposition 4.5, is an extension of Lemma 4.3.

Proposition 4.5. Let $n$, $y_0, t_0 \in \mathbb{N}$ and let $t \geq t_0$ be valid for $n$ and such that $n - 2t$ divides $t - t_0$. If $\text{winst}(n, t_0) \geq y_0$, then $\text{winst}(n, t) \geq y_0 + 2 \frac{t - t_0}{n - 2t}$.

Clearly, Proposition 4.5 can be used in the same way as Lemma 4.3 to improve Theorem 4.1. The next lemma takes care of the base case $t = t_0$ for Proposition 4.5 for the case $n = 2t + 3$. It also confirms Conjecture 1.2 for $n = 5$ and $t = 1$.

Lemma 4.6. $\text{winst}(5, 1) \geq 3$, $\text{winst}(7, 2) \geq 4$, and $\text{winst}(9, 3) \geq 4$.

Proof. We start proving that $\text{winst}(5, 1) \geq 3$. Let $f$ be a colouring of $\mathcal{H}_5$ with $t_f = 1$. We say a point $x$ in $\mathcal{H}_5$ is good (in $f$) if there is a $j \in \{0, 1\}$ so that $x$ has exactly two entries equal to $j$ and $f(x) = j$. Also, we say that two points $x$ and $y$ in $\mathcal{H}_5$ are neighbours if $||x - y||^2 = 2$ and they have the same number of entries equal to 1.

First of all, we observe that, if there are two good points $x$ and $y$ that are neighbours, then it is easy to construct a well-ending 2-geodesic that jumps the required number of times (in the same way as in Lemma 4.4). So we may assume that

if $x$ and $y$ are good in $f$, then they are not neighbours. (6)

Second, we may assume that

if $x$ is good in $f$ then its complement is not good in $f$. (7)
Indeed, if a point \( x \) and its complement are good in \( f \), then we may obtain a well-ending 2-geodesic as desired by starting out at \( x \), going to \( B_1(j^5) \), then going to \( B_1((1 - j)^5) \), and then ending at the complement of \( x \).

As \( t_f = 1 \), there is a point \( w \) that is good in \( f \). By symmetry, we can assume that \( w = 00011 \). Now, because of (5), at most one of the points 11000, 10100 and 01100 is good. So, at least one of them, say 11000, has colour 0 in \( f \). Consider the 2-geodesic

\[
(00011[1], 00001[0], 01001[0], 11001[?], 11000[0], 11100[1]).
\]

Its third point has colour 0 because of (6) and its last point has colour 1 because of (7). So this well-ending 2-geodesic only jumps less than three times if \( f(11001) = 0 \). But in this case, we use (6) to see that \( f(11010) = 1 \), and consider the well-ending 2-geodesic

\[
(00011[1], 00010[0], 10010[0], 11010[1], 11000[0], 11100[1]),
\]

that jumps 4 > 3 times. (Again, \( f(10010) = 0 \) because of (6).) This concludes the proof that \( \text{winst}(5, 1) \geq 3 \).

The idea for the other cases is similar to the one used in the proof of Lemma 4.3. We reduce the problem to 5 entries, obtaining as above a ‘partial’ geodesic that jumps at least three times, and extend it so that it jumps least four times, as needed.

For \( \text{winst}(7, 2) \), let \( f \) be a colouring of \( \mathcal{H}_7 \) with \( t_f = 2 \). Let \( w \) be a point in \( \mathcal{H}_7 \) with exactly 3 entries equal to \( j \) and such that \( f(w) = j \). By symmetry, we may assume that the first two entries of \( w \) are 01.

Define a colouring \( g \) of the hypercube \( \mathcal{H}_5 \) by assigning to each \( x' \) in \( \mathcal{H}_5 \) the value \( g(x') = f(01x') \). Then \( g \) is such that \( t_g = 1 \). Indeed, the point \( w' \) in \( \mathcal{H}_5 \) such that \( w = 01w' \) serves as a witness to this.

As \( \text{winst}(5, 1) \geq 3 \), there is a well-ending 2-geodesic \( \tilde{P} \) in \( \mathcal{H}_5 \) that jumps at least three times in \( g \). Extending each point \( x' \) of \( \tilde{P} \) to the point \( 01x' \) of \( \mathcal{H}_7 \), we obtain a path \( P' \) in \( \mathcal{H}_7 \) that jumps at least three times in \( f \). If \( \tilde{P} \) jumps exactly three times, then it ends well in both of its ends. Thus we can extend \( P' \) in one of its ends, passing by the neighbouring ball, so that it jumps once more, and the result will be a well-ending 3-geodesic as desired. If, on the other hand, \( \tilde{P} \) jumps at least four times, then we just extend it in any way so that the resulting geodesic is still well-ending. This completes the proof that \( \text{winst}(7, 2) \geq 4 \). The proof that \( \text{winst}(9, 3) \geq 4 \) is similar, so we omit it. \( \square \)
Corollary 4.7. Let \( t \geq 1 \). Then

\[
\text{winst}(2t + 3, t) \geq 2 + \frac{2t + (t \mod 3)}{3}.
\]

Proof. We obtain the bound by applying Proposition 4.5 to \( n = 2t + 3 \) and the base cases obtained from Lemma 4.6: \( t_0 = 1 \) with \( y_0 = 3, t_0 = 2 \) with \( y_0 = 4, \) and \( t_0 = 3 \) with \( y_0 = 4 \).

This bound improves by one the bound from Theorem 4.1 (a) for \( n = 2t + 3 \) and \( t \mod 3 = 0 \) or 1, and by two for \( t \mod 3 = 2 \).

The rest of this section is dedicated to the proof of Proposition 4.5.

Proof of Proposition 4.5. We proceed by induction on \( i = i(n, t) := \frac{t - t_0}{n - 2t} \).

The base, for \( i = 0 \) (i.e., \( t = t_0 \)), follows directly from the hypothesis of the lemma. For \( i > 0 \), consider a colouring \( f \) of the hypercube \( H_n \) with \( t_f = t \).

By the definition of \( t_f \), there is an \( x \in H_n \) with exactly \( t + 1 \) entries equal to \( f(x) \). As \( t \) is valid for \( n \), we know that \( x \) has at least \( t \) entries equal to \( 1 - f(x) \). So, as \( n - 2t \leq t - t_0 \leq t \), we may assume that \( x = 0^{n-2t} 1^{n-2t} x' \), where \( x' \in H_{n'} \) for \( n' := n - 2(n - 2t) \).

Define a colouring \( g \) of the hypercube \( H_{n'} \) by assigning to each \( x'' \) in \( H_{n'} \) the value \( g(x'') = f(0^{n-2t} 1^{n-2t} x'') \). Then \( g \) is such that \( t_0 = t' := t - (n - 2t) \). Indeed, \( g \) respects the balls \( B_l(0^n) \) and \( B_l(1^n) \) because \( f \) respects the balls \( B_{t_l}(0^n) \) and \( B_{t_l}(1^n) \), and the point \( x' \) has exactly \( t' = (n - 2t) + 1 = t' + 1 \) entries equal to \( g(x') = f(x) \).

Note that \( t' \) is valid for \( n' \) and that \( n' - 2t' = n - 2t \) divides \( t' - t_0 \). Moreover,

\[
i(n', t') = \frac{t - t_0 - (n - 2t)}{n - 2t} = i(n, t) - 1.
\]

So, we may apply the induction hypothesis to \( H_{n'} \) and \( g \) to obtain a well-ending \((t' + 1)\)-geodesic \( \tilde{P} \) in \( H_{n'} \) that jumps at least \( y_0 + 2 \frac{t - t_0}{n - 2t} - 2 \) times in \( g \). We suppose that the first point \( \tilde{a} \) of \( \tilde{P} \) is such that \( g(\tilde{a}) = 1 \). In other words, we suppose that \( \tilde{P} \) ends well in its first point. The other case is analogous.

Extending each point \( x'' \) of \( \tilde{P} \) to the point \( 0^{n-2t} 1^{n-2t} x'' \) of \( H_n \), we obtain a path \( P' \) in \( H_n \) that jumps at least \( y_0 + 2 \frac{t - t_0}{n - 2t} - 2 \) times in \( f \). Let \( z = 0^{n-2t} 1^{n-2t} z' \) be the last point of \( P' \). If \( f(z) = 0 \), then we extend \( P' \) to \( P \) by adding to its end the points

\[
0^{n-2t-1} 1^{n-2t+1} z',
0^{n-2t-1} 1101^{n-2t-1} z'.
\]
\[0^{n-2t-1}10^{n-2t-2}z',\]
\[\ldots\]
\[0^{n-2t-1}10^{n-2t}z',\]
\[10^{n-2t-2}10^{n-2t}z',\]
\[1210^{n-2t-3}10^{n-2t}z',\]
\[\ldots\]
\[1^{n-2t}0^{n-2t}z'.\]

As we thus pass once through \(B_t(1^n)\), and then through \(B_t(0^n)\), our geodesic \(P\) jumps at least two times more than \(P'\).

On the other hand, if \(f(z) = 1\), then we extend \(P'\) to \(P\) by adding to its end the points
\[0^{n-2t+1}1^{n-2t-1}z',\]
\[0^{n-2t+2}1^{n-2t-2}z',\]
\[0^{n-2t+3}1^{n-2t-3}z',\]
\[\ldots\]
\[0^{2n-4t-1}1z',\]
\[10^{2n-4t-2}1z',\]
\[120^{2n-4t-3}1z',\]
\[\ldots\]
\[1^{n-2t}0^{n-2t-1}1z',\]
\[1^{n-2t}0^{n-2t}z'.\]

We passed once through \(B_t(0^n)\), and then through \(B_t(1^n)\), thus again our geodesic \(P\) has at least two more jumps than \(P'\).

So, in either case, \(P\) jumps at least \(y_0 + 2 \frac{t-t_0}{n-2t}\) times in total. By construction, \(P\) is a well-ending \((t+1)\)-geodesic, as desired. \(\square\)

References

[1] Aspnes, J., Busch, C., Dolev, S., Fatourou, P., Georgiou, C., Shvartsman, A., Spirakis, P., and Wattenhofer, R., *Eight open problems in distributed computing*, Bulletin of the European Association for Theoretical Computer Science, Distributed Computing Column, 90:109–126, October 2006.

[2] Becker, F., Rajsbaum, S., Rapaport, I., and Rémiла, E., *Average long-lived binary consensus: Quantifying the stabilizing role played by mem-
ory, Theoretical Computer Science, Volume 411, Issues 14–15, pp.1558–1566, 2010.

[3] Davidovitch, L., Dolev, S., and Rajsbaum, S., *Stability of multivalued continuous consensus*, SIAM Journal on Computing, Vol. 37, Issue 4, pp.1057–1076, 2007.

[4] Dolev, S., and Hoch, E.N., *OCD: obsessive consensus disorder (or repetitive consensus)*, PODC’08 Proceedings of the Twenty-Seventh ACM Symposium on Principles of Distributed Computing, 356–404, 2008.

[5] Dolev, S., and Rajsbaum, S., *Stability of long-lived consensus*, Journal of Computer and System Sciences, Vol. 67, Num. 1, August 2003, pp.26–45.

[6] Rapaport, I., and Rémi, E., *Average long-lived memoryless consensus: the three-value case*, Lecture Notes in Computer Science, 2010, Volume 6058, 114–126.