Soliton propagation in relativistic hydrodynamics

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We study the conditions for the formation and propagation of Korteweg-de Vries (KdV) solitons in nuclear matter. In a previous work we have derived a KdV equation from Euler and continuity equations in non-relativistic hydrodynamics. In the present contribution we extend our formalism to relativistic fluids. We present results for a given equation of state, which is based on quantum hadrodynamics (QHD).

1. Introduction

Long ago [1] it was suggested that Korteweg-de Vries solitons might be formed in the nuclear medium. In a previous work [2] we have updated the early works on the subject introducing a realistic equation of state (EOS) for nuclear matter. We have found that these solitary waves can indeed exist in the nuclear medium, provided that derivative couplings between the nucleon and the vector field are included. These couplings lead to an energy density which depends on the Laplacian of the baryon density. For this class of equations of state, which is quite general as pointed out in [3], perturbations on the nuclear density can propagate as a pulse without dissipation.

During the analysis of several realistic nuclear equations of state, we realized that, very often the speed of sound $c_s^2$ is in the range $0.15 - 0.25$. Compared to the speed of light these values are not large but not very small either. This suggests that, even for slowly moving nuclear matter, relativistic effects might be sizeable. This concern justifies the extension of the formalism presented in [2].

2. Hydrodynamics

Euler equation and the continuity equation form the basis of hydrodynamics. In the non-relativistic regime and for a perfect fluid they are [2]:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\left(\frac{1}{M}\right)\vec{\nabla}h$$

$$\frac{\partial \rho_B}{\partial t} + \vec{\nabla} \cdot (\rho_B\vec{v}) = 0$$

where $\rho_B$, $M$, $h$ and $v$ are the baryon density, the nucleon mass, the enthalpy per nucleon and the fluid velocity respectively. In the relativistic case they are [4,5]:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{(\varepsilon + p)\gamma^2}\left(\vec{\nabla}p + \vec{\nabla}\frac{\partial p}{\partial t}\right)$$
\[ \frac{\partial}{\partial t}(\rho_B \gamma) + \nabla \cdot (\rho_B \gamma \vec{v}) = 0 \]  

(4)

where \( \gamma, \varepsilon \) and \( p \) are the usual Lorentz factor \( (\gamma = (1 - v^2)^{-1/2}) \), energy density and pressure respectively. We have deliberately written the above equations in a non-covariant way to make the comparison between the non-relativistic and relativistic cases easier. Using the definition of enthalpy per nucleon \([6]\) for a perfect fluid we find that \( dp = \rho_B dh \). Therefore, using the Gibbs relation at zero temperature \( \varepsilon + p = \mu_B \rho_B \) we can rewrite (3) as:

\[ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\left(1 - v^2\right) \frac{\mu_B}{\rho_B} \left( \nabla h + \vec{v} \frac{\partial h}{\partial t} \right) \]  

(5)

where \( \mu_B \) is the baryochemical potential. Since the enthalpy per nucleon may also be written as \([2,7]\) \( h = \frac{\partial \varepsilon}{\partial \rho_B} \) (6) it becomes clear that the “force” on the right hand side of the Euler equations (1) and (5) will be ultimately determined by the equation of state, i.e., by the function \( \varepsilon(\rho_B) \).

3. KdV equation and the nuclear equation of state

Equations (1) and (5) contain the gradient of the derivative of the energy density. If \( \varepsilon \) contains a Laplacian of \( \rho_B \), i.e., \( \varepsilon \propto \ldots + \ldots \nabla^2 \rho_B + \ldots \), then (1) and (5) will have a cubic derivative with respect to the space coordinate which will give rise to the Korteweg-de Vries equation for the baryon density. The most popular relativistic mean field models do not have higher derivative terms and, even if they have at the start, these terms are usually neglected during the calculations.

In [2] we have added a new derivative term to the usual non-linear QHD \([8]\), given by

\[ \mathcal{L}_M \equiv \frac{g_v}{m_v} \bar{\psi}(\partial_\nu \partial^\nu V_\mu) \gamma^\mu \psi \]  

(7)

where, as usual, the degrees of freedom are the baryon field \( \psi \), the neutral scalar meson field \( \phi \) and the neutral vector meson field \( V_\mu \), with the respective couplings and masses. The new term is designed to be small in comparison with the main baryon - vector meson interaction term \( g_v \bar{\psi} \gamma^\mu V_\mu \psi \). Following the standard steps of the mean field formalism we arrive at the following expression for the energy density:

\[ \varepsilon = \frac{g_v^2}{2m_v^2} \rho_B^2 + \frac{m_s^2}{2} \left[ \frac{(M^* - M)}{g_s} \right]^2 + \frac{\eta}{(2\pi)^3} \int_0^{k_F} d^3 k (k^2 + M^* k^2)^{1/2} \left( \frac{b}{3g_s^2} (M^* - M)^3 \right. \]  

\[ \left. + \frac{c}{4g_s^4} (M^* - M)^4 + \frac{g_v^2}{m_v^2} \rho_B \nabla^2 \rho_B \right) \]  

(8)

where \( \eta \) is the baryon spin-isospin degeneracy factor, \( M^* \) stands for the nucleon effective mass (given by \( M^* \equiv M - g_s \phi_0 \)) and the constants \( b, c, g_s \) and \( g_v \) are taken from [8].

Although Eq. (8) was obtained with the help of a specific Lagrangian taken from [8] and a prototype Laplacian interaction (7), the above form of the energy density follows quite naturally from an approach based on the density functional theory [9], regardless of the form of the underlying Lagrangian. Thus KdV solitons are a general consequence of many-body dynamics.
4. KdV solitons

We now repeat the steps developed in [1,2] and introduce dimensionless variables for the baryon density and velocity:
\[
\hat{\rho} = \frac{\rho_B}{\rho_0}, \quad \hat{v} = \frac{v}{c_s}
\]  

We next define the “stretched coordinates” \( \xi \) and \( \tau \) as in [1,7,10]:
\[
\xi = \sigma \frac{1}{2} \left( x - c_s t \right) R, \quad \tau = \sigma \frac{3}{2} c_s t R
\]  

where \( R \) is a size scale and \( \sigma \) is a small \((0 < \sigma < 1)\) expansion parameter chosen to be [10]:
\[
\sigma = \frac{|u - c_s|}{c_s}
\]

where \( u \) is the propagation speed of the perturbation in question. We then expand (9) around the equilibrium values:
\[
\hat{\rho} = 1 + \sigma \rho_1 + \sigma^2 \rho_2 + \ldots 
\]
\[
\hat{v} = \sigma v_1 + \sigma^2 v_2 + \ldots 
\]

After the expansion above (11), (2), (4) and (5) will contain power series in \( \sigma \) (in practice we go up to \( \sigma^2 \)). Since the coefficients in these series are independent of each other we get a set of equations, which, when combined, lead to KdV equations for \( \rho_1 \). In the non-relativistic case we have obtained [2]:
\[
\frac{\partial \rho_1}{\partial \tau} + 3 \rho_1 \frac{\partial \rho_1}{\partial \xi} + \left( \frac{g_v^2 \rho_0}{2M c_s^2 m_v^4 R^2} \right) \frac{\partial^3 \rho_1}{\partial \xi^3} = 0 
\]  

with the analytical solitonic solution:
\[
\hat{\rho}_1(x,t) = \frac{(u - c_s)}{c_s} \text{sech}^2 \left[ \frac{m_v^2}{g_v \sqrt{\frac{(u - c_s)c_s M}{2\rho_0}}} (x - ut) \right]
\]

where \( \hat{\rho}_1 \equiv \sigma \rho_1 \) and \( u \) is the velocity of propagation of the perturbation. The solution above is a bump with width \( \lambda \) given by:
\[
\lambda = \frac{g_v}{m_v^2} \sqrt{\frac{2\rho_0}{(u - c_s)c_s M}}
\]

Now, following the same sequence of steps, the combination of (5) and (4) leads to a similar KdV equation for the relativistic case:
\[
\frac{\partial \rho_1}{\partial \tau} + (3 - c_s^2) \rho_1 \frac{\partial \rho_1}{\partial \xi} + \left( \frac{g_v^2 \rho_0}{2M c^2 c_s^4 m_v^4 R^2} \right) \frac{\partial^3 \rho_1}{\partial \xi^3} = 0 
\]  

with the solution given by:
\[
\hat{\rho}_1(x,t) = \frac{3(u - c_s)}{c_s} (3 - c_s^2)^{-1} \text{sech}^2 \left[ \frac{m_v^2}{g_v \sqrt{\frac{(u - c_s)c_s M}{2\rho_0}}} (x - ut) \right]
\]

with the condition \( \mu_B = M \).

As a consistency check we take the non-relativistic limit, which, in this case, means taking a small speed of sound \( c_s^2 \to 0 \). In this limit \( (3 - c_s^2) \to 3 \), (17) reduces to (14) and (18) coincides with (15).
5. Conclusions

The existence of KdV solitons in nuclear matter has potential applications in nuclear physics at intermediate energies [1] and also possibly at high energies. The experimental measurements of jet quenching and related phenomena performed at RHIC [11] offer an unique opportunity of studying supersonic motion in hot and dense hadronic matter. With this scenario in mind we gave the first step in the adaptation of the KdV soliton formalism to the new environment. We have extended the results of our previous work [2], showing that it is possible to obtain the KdV solitons in relativistic hydrodynamics with an appropriate EOS. Taking the non-relativistic limit ($c_s^2 \to 0$) we were able to recover the previous results.

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