On the Weyl - Eddington - Einstein affine gravity in the context of modern cosmology

A.T. Filippov *

+ Joint Institute for Nuclear Research, Dubna, Moscow Region RU-141980

March 23, 2010

Abstract

We propose new models of an ‘affine’ theory of gravity in \( D \)-dimensional space-times with symmetric connections. They are based on ideas of Weyl, Eddington and Einstein and, in particular, on Einstein’s proposal to specify the space-time geometry by use of the Hamilton principle. More specifically, the connection coefficients are derived by varying a ‘geometric’ Lagrangian that is supposed to be an arbitrary function of the generalized (non-symmetric) Ricci curvature tensor (and, possibly, of other fundamental tensors) expressed in terms of the connection coefficients regarded as independent variables. In addition to the standard Einstein gravity, such a theory predicts dark energy (the cosmological constant, in the first approximation), a neutral massive (or, tachyonic) vector field, and massive (or, tachyonic) scalar fields. These fields couple only to gravity and may generate dark matter and/or inflation. The masses (real or imaginary) have geometric origin and one cannot avoid their appearance in any concrete model. Further details of the theory - such as the nature of the vector and scalar fields that can describe massive particles, tachyons, or even ‘phantoms’ - depend on the concrete choice of the geometric Lagrangian. In ‘natural’ geometric theories, which are discussed here, dark energy is also unavoidable. Main parameters - mass, cosmological constant, possible dimensionless constants - cannot be predicted, but, in the framework of modern ‘multiverse’ ideology, this is rather a virtue than a drawback of the theory. To better understand possible applications of the theory we discuss some further extensions of the affine models and analyze in more detail approximate (‘physical’) Lagrangians that can be applied to cosmology of the early Universe.

1 Introduction. Geometry and dynamics of the affine theory

In 1918-1923 H.Weyl [1], A.Eddington [2], [3], and A.Einstein [4], [5] proposed some generalizations of the Einstein gravity theory, which they considered as a ‘unified’ affine field theory of gravity and electromagnetism. The final formulation was given by Einstein in three beautiful and concise papers [4], later summarized by Eddington [2] and Einstein [5] and soon forgotten (see however brief discussions in [6], [7]). Recently, the simplest Einstein - Weyl model was reinterpreted in [8], where one can find a brief summary of the ideas and results of these papers that are of interest for today’s concerns. Here we start with formulating two main models predicting (in 1923!) the cosmological constant and a massive (or, tachyonic) vector particle (‘vecton’) accompanying graviton This particle may be interpreted as a sort of dark matter (or, alternatively, as an inflaton) interacting with gravity only\(^1\).

The great authors of these models did not study their beautiful new theory in detail and thus our first goal in [8] was to look for simplest spherically symmetric solutions and, especially, to cosmological ones. Here we propose a generalization of the Einstein models to any dimension and

\(^1\)If the vector field vanishes, both models reduce to the Einstein theory with nonvanishing cosmological constant.
argue that by its simple dimensional reduction we can also get massive (tachyonic) scalar mesons. We note that the second Einstein model (and its generalizations containing scalar inflatons) may be considered as the first approximation derived by expanding the square-root Lagrangian in powers of the vector field (see below). In this approximation we derive the general equations describing cosmologies and propose further approximation for which it is possible to find some exact analytic solutions (Section 3). Finally, we emphasize that Einstein’s approach is based on a clearly formulated volition giving a beautiful theory. In the end of the paper we briefly discuss possible extensions of his ideas having relation to modern cosmology but first we consider his original approach and discuss its immediate and most economic generalizations.

The main idea of Weyl was that a unification of gravity with electromagnetism requires using a non-Riemannian symmetric connection. He considered a special case of the connection depending on a symmetric tensor $g_{ij}$, which he identified with the Riemann metric, and on a vector $a_i$, which he tried to identify with the electromagnetic potential. It belongs to the simplest class suggested in [8]:

$$
\gamma^m_{kl} = \frac{1}{2} [g^{mn} (g_{nk,l} + g_{ln,k} - g_{kl,n}) + \alpha (\delta^m_k a_l + \delta^m_l a_k) - (\alpha - 2\beta) g_{kl} a^m],
$$

where the commas denote differentiations, $g_{ij} g^{jk} = \delta^i_j$ and $a_m = g_{mk} a^k$. The Weyl connection corresponds to $\alpha = 1, \beta = 0$, for the Einstein connection $\alpha = 1/3, \beta = -1/3$.

The curvature tensor can be defined without using any metric:

$$
r^i_{klm} = -\gamma^i_{kl,m} + \gamma^i_{nm} \gamma^m_{kl} + \gamma^i_{km,l} - \gamma^i_{nm} \gamma^m_{kl}.
$$

Then the Ricci-like (but non-symmetric) curvature tensor can be defined by contracting the indices $i, m$ (or, equivalently, $i, l$):

$$
r_{kl} = -\gamma^m_{kl,m} + \gamma^m_{nl} \gamma^l_{km} + \gamma^m_{km,l} - \gamma^m_{nm} \gamma^l_{kl}.
$$

(2)

(3)

(let us stress once more that $\gamma^m_{nl} = \gamma^m_{ln}$ but $r_{kl} \neq r_{lk}$). Using only these tensors and the anti-symmetric tensor density one can build up a rather rich geometric structure. In particular, Eddington discussed different sorts of tensor densities [3]. A notable scalar density is

$$
L \equiv \sqrt{-\det(r_{ij})} \equiv \sqrt{-r},
$$

which resembles the fundamental scalar density of the Riemannian geometry, $\sqrt{-\det(g_{ij})} \equiv \sqrt{-g}$.

For this and some other reasons Eddington suggested to identify the symmetric part of $r_{ij}$,

$$
s_{ij} \equiv \frac{1}{2} (r_{ij} + r_{ji}),
$$

with the metric tensor $g_{ij}$. The anti-symmetric part,

$$
a_{ij} \equiv \frac{1}{2} (r_{ij} - r_{ji}) = \frac{1}{2} (\gamma^m_{lm,j} - \gamma^m_{jm,i}),
$$

strongly resembles the electro-magnetic field tensor and it seemed natural to identify it with this tensor. Eddington tried to write consistent equations of the generalized theory but this problem was solved only by Einstein, with a different definition of the metric.

The starting point for Einstein (in his first paper of the series [4]) was to write the action principle and to suppose (4) to be the Lagrangian density depending on 40 connection functions

---

2 Following notation of Eddington we denote tensor densities by boldface Latin letters. Keeping clear distinction between tensor densities and tensors is important, especially, as far as there is no metric tensor at our disposal.
\[\gamma_{kl}^m.\] Varying the action w.r.t. these functions he derived 40 equations that allowed him to find the expression for \(\gamma_{kl}^m\) that is given by (1) with \(\alpha = -\beta = \frac{1}{3}\). In the second paper he proved that this result is valid for any Lagrangian density depending only on \(s_{ij}\) and \(a_{ij}\):

\[\mathcal{L} = \mathcal{L}(s_{ij}, a_{ij}).\]  

(7)

The most elementary example is \(\mathcal{L}(s_{ij} + \alpha a_{ij})\), where \(\alpha\) is a dimensionless constant.

Let us reproduce the main steps of the proof. Following Einstein we define the new tensor densities, \(g_{ij}\), \(f_{ij}\), by the relations

\[\frac{\partial \mathcal{L}}{\partial s_{ij}} \equiv g_{ij}, \quad \frac{\partial \mathcal{L}}{\partial a_{ij}} \equiv f_{ij},\]  

(8)

and also introduce a conjugate Lagrangian density, \(\mathcal{L}^* = \mathcal{L}^*(g_{ij}, f_{ij})\) by a Legendre transformation

\[s_{ij} = \frac{\partial \mathcal{L}^*}{\partial g_{ij}}, \quad a_{ij} = \frac{\partial \mathcal{L}^*}{\partial f_{ij}}.\]  

(9)

By varying \(\mathcal{L}\) in \(\gamma_{kl}^i\) and using the above definitions one can show that vanishing \(\delta \mathcal{L}/\delta \gamma_{kl}^i = 0\) gives the following 40 equations

\[2 \nabla^\gamma g_{kl} = \delta_i^l \nabla^\gamma (g_{km} + f_{km}) + \delta_i^k \nabla^\gamma (g_{lm} + f_{lm}),\]  

(10)

where \(\nabla^\gamma\) is the covariant derivative w.r.t. the affine connection \(\gamma\). Using the expression for the covariant derivative of \(f_{kl}\),

\[\nabla^\gamma f_{kl} = \partial_k f_{li} + \gamma_{lm}^i f_{ml} + \gamma_{im}^l f_{km} - \gamma_{im}^m f_{kl},\]  

(11)

we find an important relation allowing us to define the fundamental vector density \(a^k\):

\[\nabla^\gamma f_{ki} = \partial_i f_{ki} \equiv a^k.\]  

(12)

Then, from this relation and (10) we find that

\[\nabla^\gamma g_{ik} = -\frac{5}{3} a^k,\]  

(13)

and thus

\[\nabla^\gamma g_{kl} = -\frac{1}{3} (\delta_i^k a_i^l + \delta_i^l a_i^k).\]  

(14)

Now we can complete Einstein’s derivation of (1). To do this one has first to define the Riemann metric tensor \(g_{ij}\). It is natural to define it by the equations

\[g_{kl} \sqrt{-g} = g_{kl}, \quad g_{kl} g_{lm} = \delta_k^m.\]  

(15)

Using \(g_{ij}\) one can also define the covariant derivative \(\nabla_i\) w.r.t. the Riemann connection so that

\[\nabla_i g_{kl} = \nabla_i g_{kl} = 0.\]  

(16)

\(^3\)Einstein’s notation is somewhat casual because it tacitly assumes that \(g_{ij}\) is symmetric and \(f_{ij}\) - anti-symmetric while explicit derivations show that this is not automatically true. In fact, the derivatives in (8) - (9) should be properly symmetrized and in concrete derivations it is not difficult. More important is to note that these definitions tacitly assume that geometry has just one dimensional constant, e.g., the cosmological constant \(\Lambda\) having dimension \(L^{-2}\). Possibly, the characteristic dimensions for the symmetric and antisymmetric parts of geometry are different and then, to restore the correct dimension in (8), (9), we should multiply the densities \(g_{ij}, f_{ij}\) by \(\Lambda_x, \Lambda_y\) or, alternatively, introduce into geometry a new fundamental dimensionless constant, see next Section.
With these prerequisites, we can now use (14) to derive the expression for $\gamma_{ij}^k$ in terms of the metric tensor $g_{ij}$ and of the vector $a^k \equiv \bar{a}^k/\sqrt{-g}$ (and to find that in Eq.(1) $\alpha = -\beta = 1/3$):

$$\gamma_{ij}^k = \Gamma_{ij}^k + \frac{1}{6}(\delta_i^k a_j + \delta_j^k a_i) - \frac{1}{2}g_{ij} a^k,$$

where $\Gamma_{ij}^k$ is the Riemann connection corresponding to the metric $g_{ij}$ (the Christoffel symbol). We omit this simple but tedious derivation which is very similar to finding the standard Riemann connection in general relativity (for more details see [4]-[5]).

Returning to the definition of the nonsymmetric Ricci tensor $r_{ij}$ (3) we can now derive $s_{ij}$, $a_{ij}$ in terms of $\gamma_{ij}^k$ (and thus in terms of the metric $g_{ij}$ and of the vector field $a_k$):

$$s_{ij} = R_{ij} + \frac{1}{6}a_i a_j, \quad a_{ij} = \frac{1}{6}(a_{i,j} - a_{j,i}),$$

where $R_{ij} \equiv R_{ij}(g)$ is the standard Ricci tensor.

Up to now we did not use any information on $L$ except its dependence on $s_{ij}$ and $a_{ij}$ (correspondingly, $L^*$ depends only on $g^{ij}$ and $f^j$). To write the complete system of equations we should choose an analytic expression for the Lagrangian density. The best candidate is of course Eddington’s invariant (4). Using the fact that this invariant is the second-order homogeneous function of $r_{ij}$ we find that $L^*$ is equal to $L$. This means that it is sufficient to derive $L$ as a function of $g^{ij} + f^j$. It is clear (from (8)) that

$$g^{ij} + f^j = \frac{-1}{2\sqrt{-r} \bar{r}^{ij}}, \quad \sum_k r_{ik} \bar{r}^{jk} \equiv r \delta^j_i.$$ 

Then it follows that $\det(\bar{r}^{ij}) = r^3$ and thus

$$\det(g^{ij} + f^j) = (2\sqrt{-r})^{-4} r^3 = r/16.$$ 

Using this result one can find the final expression for $L$ (and thus for $L^*$)

$$L \equiv \sqrt{-\det(r_{ij})} \equiv \sqrt{-r} = 4\sqrt{-\det(g^{ij} + f^j)} \equiv 4\sqrt{-\det(g_{ij} + f_{ij})}.$$ 

This Lagrangian was first proposed by Eddington who tried (not quite successfully) to identify $s_{ij}$ with the metric tensor and $a_{ij}$ with the electromagnetic field tensor. Einstein gave a more correct interpretation of the intuitive Eddington idea and computed the same formula which has a different meaning.

Using this result it is possible to complete the equations (18) by expressing the $s_{ij}$ and $a_{ij}$ in terms of $g^{ij}$ and $f^j$ and finally to write equations for $g_{ij}$ and $f_{ij}$. In this report we will not go into this a bit cumbersome derivation and simply reproduce Einstein’s effective Lagrangian for the theory produced by the Lagrangian (4):

$$L_{eff} = -2\Lambda \sqrt{-\det(g_{ij} + f_{ij})} + \sqrt{-\det(g_{ij})} \left[R(g) + \frac{1}{6}g^{ij} a_i a_j\right],$$

which should be varied w.r.t. the metric and the vector field.\(^4\)

The first term in the Lagrangian (22) is essentially nonlinear and singular This probably was one of the reasons to forget it for long time. Its version with Minkowski metric reappeared in a simpler context of ‘nonlinear electrodynamics’ in 1934 [9] and nowadays is usually called the Born - Infeld

---

\(^4\)Here we modernize the original Einstein’s notation to make a different interpretation of his theory obvious. The cosmological constant $\Lambda$ is introduced for keeping the correct dimensions: $[\Lambda] = [R] = L^{-2}$ and $[a_i] = L^{-1}$. Dimensions which were completely neglected above will be restored in the next section.
Lagrangian. To the best of my knowledge the beautiful Einstein theory (22) was not seriously studied (I could not even find a reference to it in papers considering related topics). From today’s point of view it is of significant interest because it predicts dark energy (nonvanishing cosmological constant) and a new field - the massive or tachyonic vector field coupled to gravity only.

2 Interpretation and generalization of the affine theory

The meaning of \( g_{ij} \) and \( R(g) \) in (22) must be clear. To clarify the interpretation of \( a_i \) and \( f_{ij} \) let us expand the first term in powers of \( f \) up to the \( f^2 \)-order terms. Generalizing (22) by introducing a new dimensionless parameter \( \lambda \), we easily find

\[
\sqrt{\det(g_{ij} + \lambda f_{ij})} = \sqrt{-\frac{a}{g}} \sqrt{\det(\delta^i_j + \lambda f^i_j)} = \sqrt{-g} \left( 1 + \frac{1}{4}\lambda^2 f_{ij} f^{ij} + \ldots \right),
\]

and thus the approximate effective Lagrangian contains the cosmological constant term as well as the Proca Lagrangian [14] of the real massive (tachyonic) vector field \( a_i \) written several years after the Weyl and Einstein papers containing this theory as a part of the generalized gravity.

However, we should emphasize that this is only the first approximation. The exact theory is much more complex. Indeed, one can see that the exact determinant may have zeroes for large enough ‘electric’ vector field \( \vec{\epsilon}_i \equiv f_{0i} \). For example, if \( f_{ij} = 0 \) for \( i, j \neq 0 \) we have

\[
\det(\delta^i_j + \lambda f^i_j) = 1 - \lambda^2 |\vec{\epsilon}|^2,
\]

where \( |\vec{\epsilon}|^2 \equiv |g^{00}| \sum_i g^{ii} \epsilon_i^2 \). Therefore, in the exact theory there is a nontrivial interplay between dark energy and dark matter. The existence of the upper bound on \( |E| \) may give us a chance to establish the maximal ratio of dark matter to dark energy (supposing, of course, that the vector field gives a dominant contribution to the dark matter density). This will be possible only in a definite cosmological scenario presumably including inflation and some other sorts of matter. This motivates us at a certain generalization of the model.

Before we turn to this task, let us finish the interpretation of the basic Einstein models, which requires introducing physical fields with correct dimensions. To do this we just look into the third paper of Ref.[4], where Einstein writes the final equations for his second model

\[
\mathcal{L}^* = -2\Lambda \sqrt{-g} \left( 1 + \frac{1}{4}\lambda^2 f_{ij} f^{ij} \right).
\]

We will see that \( \lambda \) allows one to make the mass of the vecton absolutely arbitrary.

Now, the equations of motion can be written by calculating \( s_{ij} \) and \( a_{ij} \) using (18) and (9):

\[
R_{ij} - \Lambda g_{ij} = -\lambda^2 \Lambda \left[ f_{ik} f_j^k - \frac{1}{4} g_{ij} f_{kl} f^{kl} \right] - \frac{1}{6} \delta_i^j a_j,
\]

\[\text{In view of the above considerations (for a more detailed historical review see [8]), calling the Lagrangians like (22) the ‘Eddington - Born - Infeld’ or even ‘Born - Infeld - Einstein’ Lagrangians (see [10], [11]) is hardly justified. The models of these papers as well as some modern superstring ideas on inflation (see, e.g., [12]) are much closer in spirit to the Eddington and Einstein ideas than to the beautiful BI nonlinear electrodynamics, that is close in spirit to ideas of G. Mie [13]. Note also that the main emphasis of [9] is on quantum theory and it is not discussing GR.}\]

\[\text{Actually, we use this new parameter to disentangle the scale of the mass parameter of the vector field from the cosmological constant. We will see that for } \lambda = 1, \text{ i.e. for the original Eddington - Einstein Lagrangian (21), the mass parameter will be close to } \sqrt{\Lambda}.\]

\[\text{This configuration is realized in the spherically symmetric case. Then the nonvanishing potentials are } a_t \equiv a_0(t, r), \text{ } a_r \equiv a_1(t, r) \text{ and the only nonvanishing field component is } f_{01}.\]

\[\text{Einstein proposed this Lagrangian without relation to the nonlinear Lagrangian (21). Later he decided to take } \Lambda = 0 \text{ with finite } \lambda^2 \Lambda, \text{ but his final verdict was that this theory is also not describing ‘real physics’.}\]
\[ \Lambda f_{ij} = \frac{1}{6}(\partial_i a_j - \partial_j a_i), \]  

and the equation for the potential \( a_i \) immediately follows from (12) and (27). Here \( g_{ij}, f_{ij} \) are dimensionless while \([a_i] = L^{-1}\) and we use the system in which \( c = 1\). Now, let us multiply \( a_i^2 \) by some \( c_A^2 \) having the dimension \( M L^3 T^{-2} \) and define the standard fields \( F_{ij}, A_i \) having the correct CGS dimensions:

\[ 6c_A \Lambda f_{ij} \equiv F_{ij} = \partial_i A_j - \partial_j A_i, \quad A_i \equiv c_A a_i. \]

Rewriting Eq.(26) in terms of the physical fields we find:

\[ R_{ij} - \Lambda g_{ij} = -\kappa \left[ F_{ik} F^k_j - \frac{1}{4} g_{ij} F_{kl} F^{kl} - \mu^2 A_i A_j \right], \]

where \( \kappa^{-1} \equiv 36 \lambda^{-2} A c_A^2 \) and \( \mu^2 \equiv -6 \lambda^{-2} \Lambda \). Adjusting the parameter \( c_A \) we can identify \( \kappa \) with the gravitational constant \( G \) (in what follows we also take \( \kappa = 1 \)) and, identifying \( \kappa \equiv G/c^4 \), we have standard notation. The equations as well as the equations for \( A_i \) can be derived from the effective Lagrangian,\(^9\)

\[ \mathcal{L}_{eff} = \sqrt{-g} \left[ R - 2 \Lambda - \kappa \left( \frac{1}{2} F_{kl} F^{kl} + \mu^2 A_k A^k \right) \right], \]

by varying it in \( g_{ij} \) and \( A_i \). Quite similarly one can write the effective Lagrangian for the exact ‘square-root’ Lagrangian. In fact, one can simply restore the square root from the second and third terms in the approximate Lagrangian (30) and we leave this exercise to the reader.\(^10\)

Here it is appropriate to say a few words about the free parameters in the Lagrangian (we only discuss the case of the positive \( \mu^2 \)). The present value of \( \Lambda \) is estimated as \( \sim 10^{-56} \text{ cm}^{-2} \) (see, e.g., [15] - [19]). The de Broglie length of the vecton in smallest galaxies must be less than their dimension \( \sim 10^{21} \text{ cm} \). If the average velocity of the vecton is \( \sim 10^{-3} c \) we find that \( \lambda \Lambda^{1/2} < 10^{-3} 10^{21} \sim 10^{18} \text{ cm} \) and thus we must suppose that \( \lambda < 10^{-10} \). Such a naive estimate may be criticized and we do not insist on it.\(^11\) For these reasons, in what follows we consider the parameters as free. In cosmological applications, this is most natural in the context of the multiverse approach (see, e.g., discussions in [21]). The present status of dark energy, dark matter and inflation does not allow us to make more precise statements about the values of the parameters and we proceed with purely mathematical formulation not returning to quantitative estimates.

The most popular theories of inflation, require a few massive scalar mesons (see, e.g., [15] - [19]). In the frame of the ideas discussed above there may emerge several inflation mechanisms. For the approximate model (30) with \( \mu^2 > 0 \) there is no obvious inflation mechanism. In our study of a simplified cosmological model based on the vecton gravity (see [8]) it was shown that in spite of the fact that there exist some simple exponentially expanding solutions they hardly can be used in any inflationary scenario. On the other hand, a reasonable inflationary type solution is possible for \( \mu^2 < 0 \) (see [22]). Perhaps, it is easier to find a realistic inflation in gravity coupled to many massive vector mesons (see [23] - [28]) but most natural and transparent models of inflation use

---

\(^{9}\)Note that, according to Eq.(26), \( \mu^2 \) is negative and thus the vecton must be a tachyon. In two first papers of [4], Einstein apparently assumed that \( \mu^2 > 0 \) and in our paper [8] we called the Einstein model the corresponding effective Lagrangian with the positive \( \mu^2 \). Thus the geometric Einstein model does not completely correspond to his effective dynamical model.

\(^{10}\)The approximate Lagrangian (30) correctly describes properties of any affine theory with one vector field in the limit of small deviation of the affine connection from the Riemannian one. In the next order in \( A^2 \) (not speaking of the strong - field limit) the relevant physical model will essentially depend on the concrete structure of the chosen affine geometry.

\(^{11}\)If we do not wish to introduce additional very large/small dimensionless parameters and take \( \lambda \sim 1 \), the estimated mass of the vecton will be much less than the best present lower bound on the photon mass (\( \leq 10^{-51} \text{ g} \), see [20]) and, strictly speaking, the interpretation of the vecton as a massive photon cannot be completely excluded without further, more serious considerations.
massive scalar mesons (usually called inflatons). It is of interest to note that the most natural geometric theory with the square-root Lagrangian strongly resembles some recently studied ‘Born-Infeld’ models for inflation obtained in the frame of string theory (see, e.g. [12]). Although the geometric Einstein Lagrangian (even \( \lambda \)-modified) predicts negative \( \mu^2 \), it is possible that for a more general connection (1) this parameter is positive. We briefly discuss this possibility in Appendix.

We now demonstrate how such massive (or, tachyonic) scalar particles may be produced by the simplest dimensional reduction of a higher dimensional generalization of the Einstein affine model. In fact, the previous construction can easily be extended from the dimension \( D = 4 \) to any dimension \( D \neq 2 \). It easy to find that we must make the following replacements in the geometric equations: in Eq.(13) we replace \( 5/3 \) by \( (D + 1)/(D - 1) \), in Eq.(14) \(-1/3\) by \( (D - 1)^{-1} \), and Eqs.(17), (18) are replaced by:

\[
\gamma_{kl}^{m} = \Gamma_{ij}^{k} + \left( (D - 1)(D - 2) \right)^{-1} \left[ \delta_{k}^{m} a_{l} + \delta_{l}^{m} a_{k} - (D - 1)g_{kl} a_{m} \right], \quad (31)
\]

\[
s_{ij} = R_{ij}(g) + \left[ (D - 1)(D - 2) \right]^{-1} a_{i} a_{j}, \quad a_{ij} = \left[ (D - 1)(D - 2) \right]^{-1} (a_{i,j} - a_{j,i}). \quad (32)
\]

The equations related to dynamics, starting from (20), depend on \( D \) more significantly, as can be seen from the relation that can easily be derived:

\[
\mathcal{L}_{\text{eff}} \equiv \sqrt{-\det(r_{ij})} \sim \sqrt{-\tilde{g}} \left[ \det(\delta_{i}^{j} + f_{i}^{j}) \right]^{1/D-2}. \quad (33)
\]

(Note that this expression is meaningless when \( D = 2 \).) Now, by making the simplest dimensional reduction to \( D = 4 \), we can interpret the components of the vector field \( a_{k} \) with \( k \geq 4 \) as real massive scalar fields\(^{12}\) the geometric masses of which coincide with the vector mass. The components \( F_{ik} \) \( (k \geq 4, i \leq 3) \) give the kinetic terms of the scalar fields \( A_{k}(x^{0}, x^{1}, x^{2}, x^{3}) \). In the exact Lagrangian (22), \( \det(g_{ij} + f_{ij}) \) will depend both on derivatives of the vector and of the scalar fields. In the \( F^2 \)-approximation (30), we get the standard scalar terms. It is sufficient to write this expression with one scalar field:

\[
\mathcal{L}_{\text{eff}} = \sqrt{-\tilde{g}} \left[ R - 2\Lambda - \kappa \left( \frac{1}{2} F_{kl} F^{kl} + \mu^{2} A_{k} A^{k} + g^{ij} \partial_{i}\psi \partial_{j}\psi + m^{2} \psi^{2} \right) \right]. \quad (34)
\]

In next Section we discuss only this linearized (in \( A_{k} \) and \( \psi \)) model, which we further simplify by considering its spherically symmetric sector\(^{13}\).

3 Spherical symmetry and cosmology

In general, the spherically reduced theory is described by two-dimensional differential equations which are not integrable except very special cases (for many examples and references see, e.g., [29]-[37]). Following the approach to dimensional reduction and to resulting 1+1 dimensional dilaton gravity (DG) developed in papers [29]-[30], [32], [34]-[37] it is not difficult to derive these equations and the reader can find them in [8].

The dilaton gravity coupled to the massive vector field (I suggest to call it \textbf{vector-dilaton gravity, VDG}) is more complex than the well studied models of dilaton gravity coupled only to scalar fields and thus it requires a separate study. The first thing to do is to further reduce the theory to static or cosmological configurations.

The simplest way to derive the corresponding equations is to suppose that all the functions in the equations depend either on \( r \) or on \( t \). But this is not the most general dimensional reduction of

\(^{12}\)From now on we use the term ‘massive’ both for real and imaginary (tachyonic) masses. In Appendix we hint at the possibility of geometric models with real masses of the vecton and scalar particles.

\(^{13}\)The spherical reduction can equally be applied to the effective Lagrangian (22) and, with some caution, to the original Lagrangian (4).
the two-dimensional theory. There exist more general ones that allow one to simultaneously treat 
static states (black holes), cosmologies and some waves. These generalized reductions were proposed 
in papers [35], [36], which were devoted to dilaton gravity coupled to scalar fields and Abelian gauge 
fields. Here we only discuss the cosmological reductions (for more detailed presentation of static 
and cosmological solutions see [8]).

The simplest cosmology can be obtained by the same naive reductions as was used for static 
states in [8]). However, this is not the most general dimensional reduction giving all possible spher-
ically symmetric cosmological solutions. A more general procedure is described in [36]. Following 
this procedure we write the general spherically symmetric metric as:

$$ds^2 = e^{2\alpha}dr^2 + e^{2\beta}d\Omega^2(\theta, \phi) - e^{2\gamma}dt^2 + 2e^{2\delta}dr dt,$$  \hspace{1cm} (35)

where $\alpha, \beta, \gamma, \delta$ depend on $t$, $r$ and $d\Omega^2(\theta, \phi)$ is the metric on the 2-dimensional sphere $S^{(2)}$. Then the two-dimensional reduction of the four-dimensional theory (34) can easily be found (here the prime denotes differentiations in $r$ and the dot - in $t$:

$$\mathcal{L}^{(2)} = e^{2\beta}[e^{-\alpha-\gamma}(\dot{A}_1 - A_0')^2 - e^{-\alpha+\gamma}(\dot{\psi}^2 + \mu^2 A_1^2) + e^{\alpha-\gamma}(e^{-2\psi} + \mu^2 A_0^2) - e^{\alpha+\gamma}(V + 2\Lambda)] + \mathcal{L}_{gr},$$  \hspace{1cm} (36)

where $\dot{\psi} = \psi(t, r)$, $V = V(\psi)$, $A_i = A_i(t, r)$, $\dot{A}_1 - A_0' \equiv F_{10}$ and

$$\mathcal{L}_{gr} \equiv e^{-\alpha+2\beta+\gamma}(2\beta^2 + 4\dot{\beta}' \gamma') - e^{\alpha+2\beta-\gamma}(2\beta^2 + 4\dot{\beta}' \gamma) + 2ke^{\alpha+\gamma}$$  \hspace{1cm} (37)

is the gravitational Lagrangian, up to the omitted total derivatives that do not affect the equations of motion. Variations of this Lagrangian give all the equations of motion except one constraint,

$$-\beta' - \dot{\beta} \beta' + \dot{\alpha} \beta' + \dot{\beta} \gamma' = \frac{1}{2}[\dot{\psi} \psi' + A_0 A_1],$$  \hspace{1cm} (38)

which should be derived before we omit the $\delta$-term in the metric (taking the limit $\delta \to -\infty$). All other equations of motion can be obtained from the effective Lagrangian (36).

Now, the distinction between static and cosmological solutions is in the dependence of their ‘matter’ fields $A_i$ and $\psi$ on the space-time coordinates. We call static the solution for which $A_i = A_i(r)$ and $\dot{\psi} = \psi(t)$. If $A = A(t)$ and $\dot{\psi} = \psi(t)$ we call the solution cosmological. There may exist also the wave-like solutions for which $A$ and $\dot{\psi}$ depend on linear combinations of $t$ and $r$ but here we do not discuss this possibility (see, e.g., [37] and references therein). For both the static and cosmological solutions the gravitational variables in general depend on $t$ and $r$. To solve the equations of motion we may reduce them by separating $t$ and $r$. It is clear that to separate the variables $r$ and $t$ in the metric we should require that

$$\alpha = \alpha_0(t) + \alpha_1(r), \hspace{0.5cm} \beta = \beta_0(t) + \beta_1(r), \hspace{0.5cm} \gamma = \gamma_0(t) + \gamma_1(r).$$  \hspace{1cm} (39)

Inserting this into the equations of motion one can find the restrictions on the gravitational (and, possibly on the matter) variables that must be fulfilled. The details can be found in [36], where one can find the complete list of the static and cosmological spherically symmetric solutions when the vector field identically vanishes (we call this case the ‘scalar cosmology’). Here we only give a very brief summary and a simple generalization to nonvanishing vector field.

The naive cosmological reduction (that supposes all the fields to be independent of $r$) does not give the standard FRW scalar cosmology. As was shown in [36] (see also the earlier paper [32]), all homogeneous isotropic cosmologies should satisfy the following conditions:

$$\dot{\alpha} = \dot{\beta}, \hspace{0.5cm} \gamma' = 0, \hspace{0.5cm} \beta_1'' + ke^{-2\beta_1} = 0, \hspace{0.5cm} ke^{-2\beta_1} - 3\beta_1'^2 - 2\beta_1'' = C,$$  \hspace{1cm} (40)

where $C$ is a constant proportional to the 3-curvature (its time dependence is given by the factor $e^{-2\alpha_0}$) and the third equation is the isotropy condition. Neglecting inessential constant factors, we
also have chosen \( \alpha_1 = \gamma_1 = 0 \). We see that for the naive reduction the isotropy conditions in (40) can be satisfied only if \( k = 0 \) and that the first condition is not dictated by (38). Thus, the naive reduction gives, in general, a homogeneous non-isotropic cosmology.

For the FRW scalar cosmology \( \beta' \neq 0 \) and all the conditions (40) must be satisfied. Then the effective one-dimensional Lagrangian describing both the naive and FRW cosmology is

\[
\mathcal{L}^{(1)} = 6\bar{k}e^{\alpha+\gamma} - e^{2\beta}[e^{\alpha+\gamma}(V + 2\Lambda) - e^{\alpha-\gamma}(2\bar{\beta}^2 + 4\bar{\beta}\dot{\alpha} - \dot{\psi}^2)],
\]

where \( \bar{k} \) is a new real constant related to \( k \) and \( C \); \( \alpha, \beta, \gamma \) and \( \psi \) depend only on \( t \). Taking \( \alpha(t) = \beta(t) \) we get the Lagrangian of the FRW scalar cosmology, for which it is not difficult to derive the equations of motion.

We now write the general cosmological ‘Wedein’ Lagrangian supplemented by the minimally coupled scalar field (that may represent either matter or inflaton). At first sight, the dimensional reduction of the spherically symmetric Lagrangian (36)-(37) with the vector field must not differ from the usual one used for the scalar cosmology and can be written as:

\[
\mathcal{L}^{(1)} = 6\bar{k}e^{\alpha+\gamma} + e^{2\beta}[e^{-\alpha-\gamma}\dot{A}^2 - e^{-\alpha+\gamma}\mu^2 A^2 - e^{\alpha+\gamma}(V + 2\Lambda) - e^{\alpha-\gamma}(2\bar{\beta}^2 + 4\bar{\beta}\dot{\alpha} - \dot{\psi}^2)],
\]

where all the fields depend on \( t \). Then, taking \( \alpha = \beta \), we apparently obtain a FRW type cosmology with the vector field. However, unlike the scalar field, the two-dimensional vector field equations,

\[
\partial_0[e^{\alpha_0-\gamma_0+2\beta_1}\dot{A}_1] = -\mu^2 e^{\alpha_0+\gamma_0+2\beta_1} A_1,
\]

\[
\partial_1[e^{\alpha_0-\gamma_0+2\beta_1}\dot{A}_1] = -\mu^2 e^{3\alpha_0-\gamma_0+2\beta_1} A_0.
\]

do give additional constraints on \( \beta_1(r) \). The first equation does not depend on \( \beta_1 \), but the second one requires either \( \beta'_1(r) = 0 \) or \( \beta'_1(r) = \text{const} \). The second condition gives \( A_0 \sim \dot{A}_1 \) and so (38) is incompatible with the isotropy condition \( \dot{\alpha} = \dot{\beta} \). Therefore, there remains only the first case, \( \beta'_1(r) \equiv 0 \), from which it follows that \( k = \bar{k} = 0 \). Although the constraint (38) is identically satisfied (as we suppose that \( \gamma' = 0 \)) it does not give the necessary isotropy condition \( \dot{\alpha} = \dot{\beta} \) that automatically emerges in the scalar cosmology case. As we’ll see in a moment, this condition cannot be exactly satisfied in the vector cosmology and can only be approximate.

Summarizing this discussion, we consider the vector plus scalar cosmology described by the Lagrangian (42) with \( k = \bar{k} = 0 \) and \( A_1 \) being the \( A_z \) component of the four-dimensional vector field (it follows that the cosmology must be in general non-isotropic). To write the corresponding equations of motion in a most clear and compact form we introduce the temporal notation

\[
\rho \equiv \frac{1}{3}(\alpha + 2\beta), \quad \sigma \equiv \frac{1}{3}(\beta - \alpha), \quad A_\pm = e^{-2\rho+4\sigma}(\dot{A}^2 \pm \mu^2 e^{2\gamma} A^2), \quad \dot{V} \equiv V(\psi) + 2\Lambda.
\]

where \( A_1 \equiv A_z \equiv A \). Then the exact Lagrangian for vector plus scalar cosmology is:

\[
\mathcal{L}^{(1)} = e^{2\rho-\gamma}(\dot{\psi}^2 - 6\rho^2 + 6\dot{\sigma}^2) + e^{3\rho-\gamma}A_- - e^{3\rho+\gamma} \dot{V}(\psi).
\]

We see that \( A, \psi, \rho, \sigma \) are the dynamical variables and \( e^\gamma \) is the Lagrangian multiplier, variations of which give us the remaining energy constraint:

\[
\psi^2 - 6\rho^2 + 6\dot{\sigma}^2 + A_- + e^{2\gamma} \dot{V} = 0
\]

(the momentum constraint (38) is satisfied by construction). The other equations are:

\[
\ddot{A} + (\dot{\rho} + 4\dot{\sigma} - \dot{\gamma})\dot{A} + e^{2\gamma} \mu^2 A = 0,
\]

\[
4\ddot{\rho} + 6\dot{\rho}^2 - 4\dot{\rho}\dot{\gamma} - 6\dot{\sigma}^2 + \frac{1}{3}A_- + \psi^2 - e^{2\gamma} \dot{V} = o.
\]
\[
\ddot{\sigma} + 3\dot{\sigma}\dot{\rho} - \dot{\sigma}\dot{\gamma} - \frac{1}{3}\dot{A}_- = 0. \tag{50}
\]
\[
\ddot{\psi} + (3\dot{\rho} - \dot{\gamma})\dot{\psi} + \frac{1}{2}e^{2\gamma}\ddot{V}_\psi = 0, \tag{51}
\]

These equations are much more complex than the equations of the scalar cosmology. They are not integrable in any sense and rather difficult for a qualitative analysis.

They would be greatly simplified if it were possible to neglect the \(\sigma\)-field. Unfortunately, it is evident that this is in general impossible because then \(A_- = 0\) and the last condition is incompatible with the other equations. This means that the exact solutions of the ‘Wedein’ model (even with many scalar fields minimally coupled to gravity) should be non-isotropic\(^{14}\).

To get a simplified model (‘by brute force’) we may neglect Eq.(51) and take \(\sigma \equiv 0, \psi \equiv 0, V(\psi) \equiv 0\). Then \(\rho = \alpha = \beta\) and the approximate effective Lagrangian (46) becomes\(^{15}\)
\[
\mathcal{L}_a = -6\dot{\alpha}^2e^{3\alpha - \gamma} - 2\Lambda e^{3\alpha + \gamma} + \dot{A}^2e^{\alpha - \gamma} - \mu^2A^2e^{\alpha + \gamma}, \tag{52}
\]

The corresponding equations of motion are the three equations (47)-(49) with \(\sigma = \psi = V = 0\) and \(\rho = \alpha\). The first equation, (47), is equivalent to vanishing of the Hamiltonian. Denoting \(f \equiv e^\alpha\) and taking the gauge fixing condition \(\gamma = 0\), the ‘standard’ gauge, we have
\[
H_0^a \equiv f[-6\dot{f}^2 + 2\Lambda f^2 + \dot{A}^2 + \mu^2A^2] = 0. \tag{53}
\]

Another useful gauge, the \(\text{LC gauge}\), is \(\alpha = \gamma\). In this gauge, the effective Hamiltonian is:
\[
H_1^a \equiv -6\dot{f}^2 + 2\Lambda f^4 + \dot{A}^2 + f^2\mu^2A^2 = 0. \tag{54}
\]

It is also not difficult to write the equations independent of the gauge choice and we leave this as a simple exercise to the reader.

In [8] we constructed solutions of these equation using and iteration schemes. Recently, S. Vernov derived some analytically exact solutions for special values of the parameters.

4 Additional remarks and discussion

To confront the affine theory to ‘real physics’ we should first formulate and study in some detail the ‘really geometric’ theory with the square-root Lagrangian. I was not completely satisfied with the introduction of the \(\lambda\) parameter into this Lagrangian to solve the vecton mass problem, but recently I realized that there exists a much more beautiful solution. As was noted in the Appendix to the book [3], for the non-symmetric matrix \(r_{ik}\) there exists another scalar density similar to (4). If we replace \(\det r_{ik}\) by the following scalar density of the weight two (we may call it a ‘twisted’ determinant),
\[
\det'(r_{ij}) \equiv \frac{1}{4!}\epsilon^{ijkl}\epsilon^{mnrs}r_{im}r_{jn}r_{kr}r_{ls} \equiv \frac{1}{4!}\epsilon \cdot r \cdot r^T \cdot r^T \cdot r \cdot \epsilon; \quad r^T kr = r_{rk},
\]
we can write a more general Lagrangian:
\[
\mathcal{L} \equiv \alpha \sqrt{-\det(r_{ij})} + \alpha' \sqrt{-\det'(r_{ij})}. \tag{55}
\]

For this Lagrangian, the cosmological constant and the mass of the vecton can be made arbitrary.

\(^{14}\)If one would introduce other scalar fields non-minimally coupled to gravity, this statement may become not valid. At this stage of investigation, we are not ready to add other vector fields or fields with the the spin 1/2.

\(^{15}\)Above, we usually neglected the dimensions of all the variables and omitted the gravitational constant. Here we only need to restore one of the dimensions supposing that \([t^{-2}] = [k] = [\Lambda] = [\mu^2] = [L^{-2}].\)
In fact, in the four-dimensional theory there exists one more scalar density of the weight two,

\[ \det''(r_{ij}) \equiv \frac{1}{4!} \epsilon^{ijkl} \epsilon^{mnrs} r_{im} r_{jn} r_{kr} r_{ls} T^{T} \cdot \epsilon, \]

the square root of which could also be added to the ‘geometric’ Lagrangian (55). All the three densities are equally ‘fundamental’ from the geometric point of view but their dependence on the symmetric and antisymmetric parts of \( r_{ij} \) is significantly different. Indeed, the density

\[ \det''(r_{ij}) = \det(s_{ij}) - \det'(a_{ij}) \]

has no terms quadratic in \( a_{ij} \) while \( \det(r_{ij}) \) and \( \det'(r_{ij}) \) contain the \( a^{2} \)-order terms of opposite sign. These terms are easy to derive for the diagonal matrix \( s_{ij} \). Indeed, suppose that \( s_{ij} = \delta_{ij} s_{i} \) and introduce for the matrix \( a_{ij} \) the natural notation (\( i = 1, 2, 3 \)):

\[ e_{i} = a_{0i}, \quad \tilde{e}_{i} \equiv e_{i}/\sqrt{s_{0}s_{i}}, \]
\[ h_{i} \equiv \epsilon_{ijk} a_{jk}, \quad \tilde{h}_{i} \equiv h_{i}/\sqrt{s_{j}s_{k}}. \]

Then it is easy to derive the intuitively clear expressions for the three densities:

\[ \det(r_{ij}) = [1 - \tilde{e}^{2} + \tilde{h}^{2} + (\tilde{e} \cdot \tilde{h})^{2}] \prod_{i} s_{i}, \quad \det''(r_{ij}) = [1 - (\tilde{e} \cdot \tilde{h})^{2}] \prod_{i} s_{i}, \]
\[ \det'(r_{ij}) = [1 + \frac{1}{3} \tilde{e}^{2} - \frac{1}{3} \tilde{h}^{2} + (\tilde{e} \cdot \tilde{h})^{2}] \prod_{i} s_{i}. \]

A generalization of the Lagrangian (55) containing three square roots would define a rather complex and not very beautiful theory. An essentially equivalent but simpler Lagrangian can be made of the following three scalar densities of the weight two (\( a \) symbolizes the matrix \( a_{ij} \)):

\[ d_{0} \equiv \det(s_{ij}), \quad d_{2} \equiv \epsilon \cdot s \cdot s \cdot a \cdot a \cdot \epsilon, \quad d_{4} \equiv \epsilon \cdot a \cdot a \cdot a \cdot a \cdot \epsilon. \quad (56) \]

Then the geometric Lagrangian density,

\[ \mathcal{L} \equiv \alpha \sqrt{|d_{0}| + \alpha'd_{2} + \alpha''d_{4}}, \quad (57) \]

is conceptually as good as the Einstein’s one, although it is more difficult to work with. Depending on the signs of the numerical coefficients one could then obtain positive or negative cosmological constant, as well as the standard or exotic (phantom) sign of the vector field kinetic energy. Note that, in general, the Lagrangian has zeroes, like the simpler Lagrangian of Born and Infeld.

Let us summarize the main results, their possible generalizations, and applications. Einstein’s approach to constructing the generalized theory of gravity consists of two stages. At the first stage, it is supposed that the general symmetric connection should be restricted by applying the Hamilton principle to a general Lagrangian density depending either on \( r_{ij} \) (the first two papers) or separately on \( s_{ij} \) and \( a_{ij} \) (the third paper).\(^{16}\) He gave no motivation for this Ansatz but it is easy to see that, in the limit \( a_{ij} = 0 \), the resulting theory is consistent with the standard general relativity with a cosmological term. At this stage Einstein succeeded in deriving the remarkable expression (1) for

\(^{16}\) This idea was quite alien to Weyl and Eddington who started from formulating a particular geometry. Thus they postulated the connection (1) with \( \alpha = 1, \beta = 0 \) and then tried to write some equations generalizing the Einstein equations. Although they differ with Einstein in the general approach and in the connection coefficients, the equations have many features in common, e.g., the nonvanishing cosmological constant (exactly or in some approximation) and massive/tachyonic vector field.
the connection and the general equations (18) that introduce into the play a (massive/tachyonic) vector field $a_i$ (at the same time, he naturally introduced the metric $g^{ij}$).

At the next stage one should choose a concrete Lagrangian density $\mathcal{L}(s_{ij}, a_{ij})$. Einstein did not formulate any principle for selecting a Lagrangian, and from both geometrical and physical point of view his concrete choice looks sufficiently arbitrary, especially in the third paper.

Let us try to formulate properties we consider natural for the geometric Lagrangian density $\mathcal{L}$:

1. It must not depend on any dimensional constant. 2. Its integral over the $D$-dimensional space-time must be dimensionless. 3. Its analytic form is independent of $D$. 4. It should depend on tensors that have a direct geometrical meaning and a natural physical interpretation. 5. The most important requirement is that the resulting generalized theory must agree with the well established experimental consequences of the standard Einstein theory.

The last property is rather difficult to check without a detailed study of the theory. The fourth condition is somewhat vague and depends of our understanding of what is 'geometry' and what is 'physics'. Clearly, the variables $r_{ij}$, $s_{ij}$, and $a_{ij}$ satisfy the condition 4, and the Lagrangian densities (4), (55), (57) satisfy the conditions 1-4. However, the vector field $a_i$ is also fundamental (see Appendix) so that we can construct a density depending also on it and satisfying the first four conditions. For example, let us define one more density of the weight 2:

$$d_1 \equiv \epsilon \cdot s \cdot s \cdot \bar{a} \cdot \epsilon,$$

where $\bar{a}$ symbolizes the matrix $a_ia_j$. Then, replacing in (57) the density $d_4$ by $d_1$, we obtain a Lagrangian satisfying the conditions 1-4 and containing an additional mass term (it is obvious if we take a diagonal matrix $s_{ij}$).

One more question to Einstein’s approach is about the role of the metric tensor $g_{ij}$ in geometry and in physics. More generally, this is the question about the meaning of Einstein’s geometric Lagrangian. In Weyl’s geometric approach, a metric tensor is introduced from the very beginning but is defined up to a Weyl transformation. In Einstein’s approach, it is (seemingly uniquely) defined by the Hamilton principle. However, we know that it depends on our choice of the Weyl frame and that the vector field also depends on this choice. Einstein tacitly bypassed this question but we must try to get a deeper understanding of an interrelation between geometry and physics at this level. In particular, the role of the conformal transformation, of the Weyl frame choice and, especially, the significance of choosing different independent fields in the geometric Lagrangian must be carefully investigated and understood.

5 Appendix

Here we give a very brief summary of the main geometric facts used in his paper. As far as the author can judge, the best available introduction in the non-Riemannian geometry (for physicists) is the old book by Eisenhart [38]. Here we will speak about symmetric connections only, although many results are true or can be easily generalized to non-symmetric connections\textsuperscript{17}. The general symmetric connection can be written as:

$$\gamma^k_{ij} = \Gamma^k_{ij} + A^i_{jk},$$

where $\Gamma^k_{ij}$ is the Christoffel symbol for some $g_{ij}$ and $A^i_{jk}$ is an arbitrary tensor symmetric in the lower indices (more precisely, for any symmetric connection there exist a symmetric tensor $g_{ij}$ and

\textsuperscript{17}The earliest explorers of non-symmetric connections in relation to gravity theory were Eddington, E.Cartan, and Einstein. A concise and beautiful introduction into this subject was given by E.Schrödinger [6]. At present, one can see a revival of the interest to this field, see, e.g., [39] - [41] and references therein.
a tensor $A_{jk}^i = A_{kj}^i$ such that (58) is valid). Defining the vector $A_k \equiv A_{ik}^i$ one can find that the antisymmetric part of $r_{jk}$ is equal to the rotor of $A_k$, i.e
\[ a_{ij} = \frac{1}{2}(A_{i,j} - A_{j,i}). \] (59)
This equation, up to normalization, coincides with the second equation in (18) that was derived above in a rather complex way. It follows that the fundamental vector field and the metric can be defined in a general geometry, and Eq.(59) is generally true.

The really new result that requires using the Einstein variational principle is the reduction of the tensor $A_{jk}^i$ to one vector $a_k$ (in fact, $A_k = (2\alpha + \beta)a_k$). The expression for $s_{ij}$ generalizing r.h.s. of (18) can now be derived from the general equation:
\[ s_{jk} = \frac{1}{2}(\nabla_j A_k + \nabla_k A_j) - \nabla_i A_{jk}^i + A_{ij}^r A_{rk}^i - A_{rk}^r A_{jk}^i. \] (60)
The terms linear in $A$ are equal to
\[ \frac{1}{2}\left[(\alpha + \beta)(\nabla_j a_k + \nabla_k a_j) + (\alpha - 2\beta)\nabla_i a^i\right], \] (61)
and the quadratic terms are
\[ \frac{1}{4}\left[a_i a_k [(\alpha - 2\beta)^2 - 3\alpha^2] + 2g_{ik}a^2(\alpha - 2\beta)(\alpha + \beta)\right]. \] (62)
When $\alpha = -\beta = -\frac{1}{3}$, we reproduce Einstein’s expression for $s_{jk}$. One can see that the sign of the first term in (62) may be positive or negative but, in general, the second term in (62) and the linear terms in (60) do not vanish. Apparently, one could get models with the positive sign of $\mu^2$ but this requires a more careful consideration which will be published elsewhere.

Acknowledgment: It is a pleasure for the author to thank for useful remarks V. de Alfaro, A. Linde, V. Rubakov, A. Starobinsky, S. Vernov and E. Witten. The kind hospitality of M. Mueller-Preussker and J. Plefka at the Humboldt University (Berlin) is cordially appreciated.

This work was also supported in part by the Russian Foundation for Basic Research (Grant No. 09-02-12417 ofi-M).

References
[1] H. Weyl, Raum-Zeit-Materie, 5-th edition, 1923 (1-st ed. 1918; English translation 1950).
[2] A.S. Eddington, Proc. Roy. Soc. A99 (1919) 742.
[3] A.S. Eddington, The mathematical theory of relativity, Camb., 1923 (German translation of the 2-nd ed. 1925).
[4] A. Einstein, Sitzungsberichte der Preussischen Akademie der Wissenschaften, Physikalisch-mathematische Klasse, 1923, pp. 32-38, 76-77, 137-140. (‘Sobranie nauchnykh trudov’, vol.2, papers 72-74, Moscow, 1966 (in Russian)).
[5] A. Einstein, Nature, 112, pp. 448-449; Appendix to the book: A.Eddington, ‘Relativitäts theorie in mathematischer Behandlung’, Springer, Berlin, 1925. (‘Sobranie nauchnykh trudov’, vol.2, papers 75, 77, Moscow, 1966 (in Russian)).
[6] E. Schrödinger, Space - time structure, NY, 1950.

[7] W. Pauli, Relativitätstheorie (Enzykl. d. Math. Wiss., Bd.5), 1921 (Russian transl. 1947); Theory of Relativity, NY, 1958 (English transl. with author’s comments and addenda).

[8] A.T. Filippov, On Einstein - Weyl unified model of dark energy and dark matter, arXiv:0812.2616v2, gr-qc.

[9] M. Born, Proc. Roy. Soc. A143 (1933/34) 410; M. Born and L. Infeld, ibid. A144 (1934) 425; A147 (1934) 522; A150 (1935) 141.

[10] S. Deser and G. Gibbons, Born - Infeld -Einstein actions?, hep-th/9803049.

[11] M. Bañados, Eddington-Born-Infeld action for dark energy and dark matter, arXiv:0801.4103v4.

[12] D. Langlois, S. Renaux-Petel and D.A. Steer, Multi-field DBI inflation, arXiv:0902.2941v1.

[13] G. Mie, Ann. d. Phys. 37 (1913) 511, 39 (1913) 1, 40 (1913) 1.

[14] A. Proca, Journ. de Physique, 7 (1936) 347.

[15] V. Sahni and A. Starobinsky, Reconstructing dark energy, arXiv:astro-ph/0610026v3

[16] A. Linde, Particle physics and inflatary cosmology, hep-th/0503203v1.

[17] V. Mukhanov, Physical foundations of cosmology, Cambridge U. Press, 2005.

[18] S. Weinberg, Cosmology, Oxford U. Press, 2008.

[19] V. Rubakov and D. Gorbunov, Introduction into the theory of early Universe (Vvedenie v teoriyu rannej Vselevennoj, in Russian) vol.1,2, Moscow, 2008-2009.

[20] J. Luo, L.-C. Tu, Z.-K. Hu and E.-J. Luan, Phys. Rev. Lett. 90 (2003) 081801.

[21] Universe or Multiverse? (Editor B. Carr), Cambridge U. Press, 2007.

[22] L.H. Ford, Phys. Rev. D40 (1989) 967.

[23] M.C. Bento, O. Bertolami, P.V. Moniz, J.M. Mourão and P.M. Sá, On the cosmology of massive vector fields with SO(3) global symmetry, gr-qc/9302034.

[24] C. Armendariz-Picón, JCAP 0407 (2004) 007.

[25] A. Golovnev, V. Mukhanov and V. Vanchurin, Vector inflation, arXiv:0802.2068v3.

[26] T.S. Koivisto and D.F. Mota, JCAP 0808 (2008) 021.

[27] A. Golovnev and V. Vanchurin, Cosmological perturbations from vector inflation, arXiv:0903.2977v1.

[28] C. Germani and A. Kehagias, P-inflation: generating cosmic inflation with p-forms, arXiv:0902.3667.

[29] M. Cavaglià, V. de Alfaro and A.T. Filippov, IJMPD 4 (1995) 661; IJMPD 5 (1996) 227; IJMPD 6 (1996) 39.

[30] A.T. Filippov, MPLA 11 (1996) 1691; IJMPA 12 (1997) 13.
[31] D. Grumiller, W. Kummer and D. Vassilevich, Phys. Rep. 369 (2002) 327.

[32] V. de Alfaro and A.T. Filippov, Integrable low dimensional models for black holes and cosmologies from high dimensional theories, hep-th/0504101.

[33] G.A. Alekseev, Theor. Math. Phys. 143 (2005) 720.

[34] A.T. Filippov, Theor. Math. Phys. 146 (2006) 95; hep-th/0505060.

[35] V. de Alfaro and A.T. Filippov, Theor. Math. Phys. 153 (2006) 1709; hep-th/0612258v2.

[36] A.T. Filippov, Some unusual dimensional reductions of gravity: geometric potentials, separation of variables, and static - cosmological duality, hep-th/0605276.

[37] V. de Alfaro and A.T. Filippov, Multi - exponential models of (1+1)-dimensional dilaton gravity and Toda - Liouville integrable models, arXiv:0902.4445v.2.

[38] L.P. Eisenhart, Nonriemannian geometry, AMS Publ. N.Y., 1927.

[39] T. Damour, S. Deser and J. McCarthy, Nonsymmetric gravity has unacceptal globalasymp- totics, gr-qc/9312030.

[40] T. Janssen and T. Prokopec, Problems and hopes in nonsymmetric gravity, gr-qc/0611005.

[41] V.P. Nair, S. Randjbar-Daemi and V.A. Rubakov, Massive spin-2 fields of geometric origin in curved space-times, ArXiv:0811.3708.