Decomposition of tensor products
of modular irreducibles for $\text{SL}_2$

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Abstract

We use tilting modules to study the structure of the tensor product of two simple modules for the algebraic group $\text{SL}_2$, in positive characteristic, obtaining a twisted tensor product theorem for its indecomposable direct summands. Various other related results are obtained, and numerous examples are computed.

Introduction

We study the structure of $L \otimes L'$ where $L, L'$ are simple modules for the algebraic group $\text{SL}_2 = \text{SL}_2(k)$ over an algebraically closed field $k$ of positive characteristic $p$. The solution to this problem is well-known and easily obtained in characteristic zero, but in positive characteristic the problem is significant.

Given $L \otimes L'$, the initial question is to describe its indecomposable direct summands. This is answered in Theorem 2.1. It turns out that each such direct summand is expressible as a twisted tensor product of certain “small” indecomposable tilting modules where the structure of the latter is completely understood (see Lemma [1.1]). We note that for $p = 2$ the module $L \otimes L'$ is always indecomposable, in contrast to what happens for $p$ odd.

The indecomposable summands themselves are always contravariantly self-dual, with simple socle (and head), and they occur as subquotients of tilting modules (see Theorem 2.7). On the other hand, each tilting module occurs
as a direct summand of some $L \otimes L'$ (see Theorem 2.4). These results provide the starting point for calculating the examples in Section 5 and 6.

In fact, we suggest that the reader start by browsing through the examples in Sections 5 and 6. The close relationship between tensor products of simples and tilting modules will be apparent from these examples. Since examples of tilting module structure are rare in the literature, these computations should be of independent interest.

Our general results are given in Sections 2 through 4. In particular, we classify precisely which indecomposable summands of $L \otimes L'$ are tilting, and we obtain a result expressing certain indecomposable tilting modules as tensor products of two simple modules (usually in more than one way).

In Sections 3 and 4 we study in detail the tensor products $L \otimes L(1)$ where $L$ is arbitrary. Here we obtain a class of uniserial and biserial tilting modules and their “shifts”. The methods used here can be applied to obtain the structure of all tensor products $L \otimes L'$ where $L' = L(a)$ with $a \leq p - 1$, although we formulate the precise statement only for $L \otimes L(2)$.

The results of the paper are related to results of Alperin [A1], Brundan and Kleshchev [BK], and Erdmann and Henke [EH1, EH2]. Our main technical tools are Steinberg’s tensor product theorem and Donkin’s interpretation [Do], in the context of algebraic groups, of Ringel’s theory [R] of tilting modules.

1 Preliminaries

The set $X = X(T)$ of weights for a maximal torus $T$ in the algebraic group $\text{SL}_2$ will be identified with the set $\mathbb{Z}$ of integers, as usual. Then dominant weights correspond to nonnegative integers. If $r$ is such then we write $L(r)$ for the simple $\text{SL}_2$-module of highest weight $r$ and $\Delta(r)$ for the Weyl module of that same highest weight. We write $\nabla(r)$ for the transpose (contravariant) dual of $\Delta(r)$. By a theorem of Ringel (see [R]), there exists a unique indecomposable module $T(r)$ of highest weight $r$ such that $T(r)$ has both $\Delta$-filtration and $\nabla$-filtration. The modules $T(r)$ are the (partial) tilting modules.

If a module $M$ has a composition series $0 = M_0 \leq M_1 \leq \cdots \leq M_k = M$ with simple factors $S_i \cong M_i/M_{i-1}$ for $i = 1, \ldots, k$ then we denote that composition
series by writing

\[ [S_1, S_2, \ldots, S_k]. \]

We begin with some easy lemmas on tilting modules. Our first result describes the module structure of certain small tilting modules, which will turn out to be the basic material out of which all tilting modules and tensor products of simples are built up.

**Lemma 1.1**  
(a) For \(0 \leq u \leq p - 1\) we have \(T(u) = L(u) = \nabla(u) = \Delta(u)\).

(b) For \(p \leq u \leq 2p - 2\) the module \(T(u)\) is uniserial and its unique composition series has the form \([L(2p-2-u), L(u), L(2p-2-u)]\). Moreover, \(T(u)\) is a non-split extension of \(\Delta(2p-2-u)\) by \(\Delta(u)\) (or, dually, of \(\nabla(u)\) by \(\nabla(2p-2-u)\)).

**Proof.** Recall that \(\nabla(r) \cong S^r(E)\), the \(r\)th symmetric power of the natural module \(E\). Part (a) follows immediately from the fact that \(\nabla(u) = \Delta(u) = L(u)\) is simple for \(0 \leq u \leq p - 1\). This fact is well-known and follows for instance from the strong linkage principle, or from known results \([D1]\) on the structure of \(S^r(E)\).

Part (b) is a special case of \([EH2, Proposition 2.3]\). Or: Write \(G = \text{SL}_2\) and set \(G_1 = \text{Ker} F\) (the Frobenius kernel). As observed by Donkin \([Do, \S 2, Example 1]\), for \(0 \leq \lambda \leq p - 1\) we have an isomorphism \(T(2p-2-\lambda) \cong Q(\lambda)\), where \(Q(\lambda)\) is the unique (up to \(G\)-isomorphism) \(G\)-module such that \(Q(\lambda)|_{G_1}\) is isomorphic with the projective cover of \(L(\lambda)|_{G_1}\). The existence of this \(G\)-module lift follows from results of Jantzen, extending earlier results of Ballard. Now we can apply \([J, II, 11.4 Prop.]\) to compute the formal character of \(Q(\lambda)\). Setting \(u = 2p-2-\lambda\) and restricting to \(0 \leq \lambda \leq p - 2\), we can apply the aforementioned description of the \(\nabla^r\)'s as symmetric powers to obtain part (b) of the lemma. \(\Box\)

Let us call the tilting modules described in the preceding result fundamental. As we shall see (in Lemma 1.4 ahead) any tilting module for \(\text{SL}_2\) can be expressed as a twisted tensor product of fundamental tilting modules. Moreover, in Theorem 2.1 we shall see that the indecomposable summands of \(L \otimes L'\) can also be expressed as such a twisted tensor product. (The indecomposable summands are not necessarily tilting, however.) For \(0 \leq u \leq 2p - 2\), we
denote by \( \tilde{u} \) the highest weight of the socle (and head) of \( T(u) \), so that

\[
\tilde{u} = \begin{cases} 
  u & \text{if } u \leq p - 1, \\
  2p - 2 - u & \text{otherwise.}
\end{cases}
\tag{1.2}
\]

Most cases of the next lemma appear already in [EH1, Lemma 4].

**Lemma 1.3** Let \( L, L' \) be two simple modules in the bottom alcove, i.e. their highest weights are inclusively between 0 and \( p - 1 \). Then \( L \otimes L' \) is tilting, and isomorphic with the direct sum of \( T(u) \) as \( u \) varies over a set \( W(L, L') \) of weights which can be computed as follows. Let \( r \) (resp., \( s \)) be the larger (resp., smaller) of the highest weights of \( L, L' \). List the weights \( r + s, r + s - 2, \ldots, r - s \). For each \( u \geq p \) on this list, strike out the number \( 2p - 2 - u \) from the list. What remains is the set \( W(L, L') \). In other words, if \( S = \{r + s - 2i\}_{i=0}^{s} \) then

\[
W(L, L') = S - \{2p - 2 - u \mid u \in S, u \geq p\}.
\]

In particular, \( L \otimes L' \) is indecomposable if and only if \( s = 0 \) or \((r, s) = (p - 1, 1)\).

**Proof.** The module \( L(r) \otimes L(s) \) is tilting. It has a \( \nabla \)-filtration with sections \( \nabla(r + s), \nabla(r + s - 2), \ldots, \nabla(r - s) \). Each of these sections has either one or two composition factors. The statement about \( W(L, L') \) now follows from Lemma 1.1 and some simple bookkeeping.

It is clear that \( L \otimes L' \) is indecomposable in case \( s = 0 \) or \((r, s) = (p - 1, 1)\). For the converse, note that if \( s = 1 \) and \( r \neq p - 1 \) then \( W(L, L') = \{r + 1, r - 1\} \). If \( s > 1 \) then \( S \) contains at least three elements. If \( r + s - 2 \notin W(L, L') \) then \( r + s = p \) and \( r - s \in W(L, L') \), so \( W(L, L') \) either contains both \( r + s, r - s \) or both \( r + s, r + s - 2 \). This proves the indecomposability claim.

For later use we set \( W(a, b) = W(L, L') \) where \( L = L(a), L' = L(b) \) and \( 0 \leq a, b \leq p - 1 \). Note that \( W(a, b) = W(b, a) \).

We shall also need the following lemma from [EH1, Lemma 5], which describes how the tilting modules are built up as twisted tensor products of fundamental modules. We use a superscript \( F^i \) applied to a module to indicate twisting by the \( i \)th iterate of the Frobenius morphism.

**Lemma 1.4** Let \( u \in \mathbb{Z}_{\geq 0} \). Then \( u \) can be written uniquely in the form \( u = \sum_{i=0}^{m} u_i p^i \) where \( p - 1 \leq u_i \leq 2p - 2 \) for \( i < m \) and \( 0 \leq u_m \leq p - 1 \). Then \( T(u) \simeq \bigotimes_{i=0}^{m} T(u_i)F^{i} \).
The indecomposable summands of \( L \otimes L' \)

The next result follows easily from Lemmas 1.1 and 1.3, combined with Steinberg’s tensor product theorem. For \( r \in \mathbb{Z}_{\geq 0} \), write \( r = \sum_{i \geq 0} \delta_i(r)p^i \) where \( \delta_i(r) \in \{0, 1, \ldots, p-1\} \) (the \( p \)-adic expansion of \( r \)). In the following, when writing \( W(\delta_i(r), \delta_i(r')) \) we mean the set described in Lemma 1.3.

**Theorem 2.1** Let \( r, r' \) be arbitrary nonnegative integers. The tensor product \( L(r) \otimes L(r') \) can be expressed as a direct sum of twisted tensor products of fundamental tilting modules. In fact, we have

\[
L(r) \otimes L(r') \simeq \bigoplus_{u} \left( \bigotimes_{i} T(u_i)^{F^i} \right)
\]

where \( u = (u_0, \ldots, u_m) \) ranges over all elements of the finite Cartesian product

\[
W = W(\delta_0(r), \delta_0(r')) \times W(\delta_1(r), \delta_1(r')) \times \cdots \times W(\delta_m(r), \delta_m(r'))
\]

of the sets described in Lemma 1.3, and where \( m \) is the \( p \)-adic length of the largest of \( r, r' \). Given \( u \) as above, the corresponding indecomposable direct summand \( J(u) = \bigotimes_{i=0}^{m} T(u_i)^{F^i} \) is always contravariantly self-dual, with simple socle and head isomorphic with \( L(\sum_{i=0}^{m} \tilde{u}_i p^i) \) where \( \tilde{u}_i \) is defined as in equation (1.2).

**Proof.** For a given integer \( r \geq 0 \) we have its \( p \)-adic expansion \( r = \sum_{i \geq 0} \delta_i(r)p^i \) where each \( \delta_i(r) \) satisfies \( 0 \leq \delta_i(r) \leq p-1 \). By Steinberg’s tensor product theorem \( L(r) \simeq \bigotimes_{i} L(\delta_i(r))^{F^i} \). Thus (using Lemma 1.3) we have

\[
L(r) \otimes L(r') \simeq \bigotimes_{i} [L(\delta_i(r)) \otimes L(\delta_i(r'))]^{F^i} \simeq \bigotimes_{i} \bigoplus_{u \in W(\delta_i(r), \delta_i(r'))} T(u)^{F^i}
\]

and the stated isomorphism follows by interchanging products and sums. The statements about \( J(u) \) follow from Lemma 1.1 and Steinberg’s tensor product theorem. □

Note that the multiplicities of the composition factors of each \( J(u) \) are computable from Lemma 1.1 and Steinberg’s tensor product theorem. See the examples in the last two sections.

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We can prove the following criterion, classifying the summands of the tensor product that are tilting and/or irreducible:

**Theorem 2.2** For \( u, m \) as in the preceding theorem, the indecomposable summand \( J(u) \) is a tilting module if and only if \( u_i \geq p-1 \) for \( i = 0, 1, \ldots, m-1 \), in which case it is isomorphic with the tilting module \( T(\sum_{i=0}^m u_ip^i) \). Moreover, \( J(u) \) is irreducible if and only if \( u_i \leq p-1 \) for all \( i = 0, 1, \ldots, m \), in which case it is isomorphic with \( L(\sum_{i=0}^m u_ip^i) \).

**Proof.** The second claim follows immediately from Steinberg’s tensor product theorem and Lemma 1.1. The “if” part of the first claim follows immediately from Lemma 1.4. For the converse assertion, suppose that \( J(u) \) is tilting but there exists some index \( j < m \) for which \( u_j < p-1 \). The highest weight of \( J(u) \) is \( u' = \sum_{i=0}^m u_ip^i \), so \( J(u) \cong T(u') \). We express \( u' \) as in Lemma 1.4 and decompose \( T(u') \) into a twisted tensor product of tilting modules. Using Lemma 1.1 we can compute the highest weight of the socle of \( T(u') \). Now a computation shows that the highest weight of the socle of \( J(u) \) will differ from the highest weight of the socle of \( T(u') \), and thus we arrive at a contradiction. The details are as follows:

Let \( u' = \sum_{i=0}^t v_ip^i \) be an admissible decomposition as in Lemma 1.4. As \( \text{soc} J(u) = \text{soc} T(u') \) we obtain together with Theorem 2.1

\[
\sum_{i=0}^m \tilde{u}_i = \sum_{i=0}^t \tilde{v}_i p^i.
\]

As these are \( p \)-adic decompositions we obtain \( \tilde{v}_i = \tilde{u}_i \). Certainly \( t \leq m + 1 \) and if \( t = m + 1 \) then \( \tilde{v}_t = 0 = v_t \). We assume now that \( t = m \). (For \( t < m \) a similar calculation leads to a contradiction.) Define

\[
I = \{ i \mid u_i < p-1, i \neq t \} \quad \text{and} \quad R = \{ i \mid u_i \geq p-1, i \neq t \}.
\]

So by assumption \( I \) is not empty. We obtain

\[
\sum_{i=0}^m v_ip^i = u' = \sum_{i=0}^m u_ip^i = \sum_{i \in I} u_ip^i + \sum_{i \in R} u_ip^i + u_mp^m = \sum_{i \in I} (2p - 2 - v_i)p^i + \sum_{i \in R} v_ip^i + u_mp^m.
\]
which is equivalent to
\[ 2 \sum_{i \in I} v_i p^i + v_m p^m = 2 \sum_{i \in I} (p - 1)p^i + u_m p^m. \]

Either \( v_m = u_m < p - 1 \) or \( u_m = 2p - 2 - v_m \). In both cases we obtain \( v_i = p - 1 \) and hence \( u_i = v_i = p - 1 \). This contradicts the assumption that \( I \) is not empty.

We also have the following criterion for the tensor product \( L(r) \otimes L(r') \) to be indecomposable:

**Theorem 2.3** With \( m \) as above, \( L(r) \otimes L(r') \) is indecomposable if and only if, for each \( i = 0, \ldots, m \), either

(a) one of \( \delta_i(r) \) or \( \delta_i(r') \) is 0, or

(b) \( \delta_i(r) + \delta_i(r') = p \) and one of \( \delta_i(r) \) or \( \delta_i(r') \) is 1.

**Proof.** Combine Lemma 1.3 with Theorem 2.1. \( \square \)

Note that for \( p = 2 \) the conditions of the preceding theorem necessarily hold for any \( r, r' \), so \( L(r) \otimes L(r') \) is always indecomposable when \( p = 2 \). By combining the previous two results we obtain:

**Theorem 2.4** With \( m \) as above, necessary and sufficient conditions for the tensor product \( L(r) \otimes L(r') \) to be isomorphic to an indecomposable tilting module are that:

(a) \( p - 1 \leq \delta_i(r) + \delta_i(r') \leq p \), for all \( i = 0, \ldots, m - 1 \), and

(b) \( \delta_m(r) + \delta_m(r') \leq p \), and

(c) when \( \delta_i(r) + \delta_i(r') = p \), one of \( \delta_i(r) \) or \( \delta_i(r') \) is equal to 1, and

(d) when \( \delta_i(r) + \delta_i(r') < p \), one of \( \delta_i(r) \) or \( \delta_i(r') \) is equal to 0.

When these conditions hold we have the isomorphism \( L(r) \otimes L(r') \cong T(r + r') \).

**Proof.** This follows immediately from Theorems 2.2 and 2.3. \( \square \)

We also have the following criterion for when a given tilting module can be factored as a tensor product of two simple modules.
Theorem 2.5 Necessary and sufficient conditions for $T(u)$ to be factorizable as a tensor product of two simple modules are that when $u$ is expressed in the form $u = \sum_{i=0}^{m} u_i p^i$ with $u_i$ as in Lemma 1.4, we have $u_i = p$ or $p - 1$ for all $i = 0, \ldots, m - 1$. In that case, if $p > 2$, there are precisely $2^m$ or $2^{m-1}$ such factorizations, depending on whether $u_m > 0$ or $u_m = 0$, respectively. For $p = 2$ there are only $2^t$ such factorizations, where $t = |\{i \mid u_i = p - 1 = 1\}|$.

Proof. This follows from Theorem 2.4.

Note that for $p = 2$ every indecomposable tilting module has a factorization of the above form. This is not true for odd primes: for $p = 3$ there are no integers $r$, $r'$ for which the module $T(4)$ is isomorphic with $L(r) \otimes L(r')$. However, $T(4)$ is a direct summand of $L(2) \otimes L(2)$, and in fact we have the following statement in general.

Theorem 2.6 Every indecomposable tilting module occurs as a direct summand of some product $L \otimes L'$ for appropriate simple modules $L, L'$.

Proof. Given $T(u)$, express $u = \sum_{i=0}^{m} u_i p^i$ with $p - 1 \leq u_i \leq 2p - 2$ for $i < m$ and $0 \leq u_m \leq p - 1$. For each $i$ we can find integers $r_i, s_i$ in the range $0 \leq r_i, s_i \leq p - 1$ such that $u_i \in W(r_i, s_i)$. (One can, for instance, simply choose $r_i, s_i$ to satisfy $r_i + s_i = u_i$.) Then by Theorem 2.1 and Lemma 1.4 it follows that $T(u)$ is a direct summand of $L(\sum r_i p^i) \otimes L(\sum s_i p^i)$.

Theorem 2.7 The indecomposable summands of a tensor product of two simple modules $L(r)$ and $L(r')$ are always obtainable as subquotients of tilting modules, more precisely, of the direct summands of $T(r) \otimes T(r')$. In particular, if $L(r) \otimes L(r')$ is indecomposable then it is a subquotient of $T(r) \otimes T(r')$.

Proof. Given simple modules $L(r)$ and $L(r')$. As tilting modules have a $\Delta$-filtration we can choose submodules $M(r)$ and $M(r')$ of $T(r)$ and $T(r')$ respectively such that $L(r)$ and $L(r')$ occur in the head of $M(r)$ and $M(r')$ respectively. Hence $L(r) \otimes L(r')$ is a quotient of $M(r) \otimes M(r')$. Moreover, $M(r) \otimes M(r')$ is a submodule of the tilting module $T(r) \otimes T(r')$. As the head of each indecomposable summand of $L(r) \otimes L(r')$ is simple (by Theorem 2.4), we obtain each such summand as a subquotient of a tilting module $T$ where $T$ is a direct summand of the tilting module $T(r) \otimes T(r')$. The last claim follows by comparing highest weights.
3 Structure of tensor products

In this section we study the structure of the tensor product of two simple modules for \( SL_2 \). This is a more difficult problem than those considered so far, and our results are less than definitive.

Let \( r \geq 0 \). We use the term *biserial* to refer to a module that has precisely two distinct composition series. Our next two results describe the structure of \( M = L(r) \otimes L(1) \) for arbitrary \( r \). These results are related to results of Brundan and Kleshchev [BK]. The proof of the next two theorems is given in the next section.

**Theorem 3.1** The following cases occur for \( M = L(r) \otimes L(1) \):

(a) If \( r \equiv 0 \pmod{p} \) then \( M \) is simple and isomorphic with \( L(r+1) \).

(b) If \( r \equiv -1 \pmod{p} \) then \( M \) is an indecomposable module. Moreover, there exist unique positive integers \( t, a \) such that \( r \equiv ap^t - 1 \pmod{p^{t+1}} \) for \( 1 \leq a < p \), and \( M \) is uniserial when \( a = 1 \) and is biserial otherwise.

(c) In all other cases (that is \( r \not\equiv 0, -1 \pmod{p} \)) \( M \) is isomorphic to the direct sum \( L(r+1) \oplus L(r-1) \).

The proof for the cases (a) and (c) follows from Theorem 2.1, Lemma 1.1 and Steinberg’s tensor product theorem. Note that case (c) cannot occur when \( p = 2 \) and also the biserial part of case (b) never occurs when \( p = 2 \). Our next result describes the precise structure of \( M = L(r) \otimes L(1) \) in case (b).

**Theorem 3.2** Suppose that \( r \equiv -1 \pmod{p} \). Then \( r = (ap^t - 1) + p^{t+1}k \) for \( 1 \leq a \leq p - 1 \) and \( k \in \mathbb{Z} \). Set \( r_0 = r + 1 \) and \( r_i = r + 1 - 2p^{t-1} \) for \( i \in \mathbb{Z}_{>0} \). Then \( M = L(r) \otimes L(1) \) has the following precise structure:

(a) In case \( a = 1 \) the module \( M \) is the uniserial module of the form

\[ [L(r_1), \ldots, L(r_i), L(r_0), L(r_1), \ldots, L(r_1)] \].
(b) In all other cases $M$ is the biserial module with the structure diagram:

```
M = \begin{array}{c}
  r_1 \\
  \vdots \\
  r_{t-1} \\
  r_t
\end{array}
```

Finally, when $k = 0$ in both cases $M$ is isomorphic with the indecomposable tilting module of highest weight $r_0$ and has a $\nabla$-filtration the factors of which are the uniserial modules $\nabla(r_0)$ and $\nabla(r_1)$.

(The structure diagram of $M$ follows the conventions of Alperin [A2], in which vertices correspond to simple factors and edges to non-split extensions. In the diagram we identify simple composition factors $L(r_i)$ with their highest weight $r_i$.)

To describe the tensor products $M = L(r) \otimes L(a)$ for $2 \leq a \leq p - 1$ and $p > a$ is an application of the preceding results for $L(r) \otimes L(1)$. We illustrate this on the tensor products $M = L(r) \otimes L(2)$. Set $E = L(1)$, the natural module for $SL_2$.

Using the associativity of tensor products $(L \otimes E) \otimes E \simeq L \otimes (E \otimes E)$, the module $L \otimes E \otimes E$ can be calculated in two ways. As $E \otimes E \simeq L(0) \oplus L(2)$ for $p > 2$, this provides a way to calculate $M = L(r) \otimes L(2)$ for $p > 2$. The result (following immediately from the preceding results) is the following:

**Theorem 3.3** For $p = 3$ the following cases occur for $M = L(r) \otimes L(2)$:

(a) If $r \equiv 0 \pmod{p}$ then $M$ is isomorphic to $L(r + 2)$.

(b) If $r \equiv 1 \pmod{p}$ then $M$ is isomorphic to $L(r + 1) \otimes L(1) \simeq T(ap^t) \otimes L(k)^{p^{t+1}}$ where $r + 1 = (ap^t - 1) + kp^{t+1}$ and $a = 1, 2$.

(c) If $r \equiv -1 \pmod{p}$ then $M$ is isomorphic to $T(ap^t + 1) \otimes L(k)^{p^{t+1}} \oplus L(r)$ where $r = (ap^t - 1) + kp^{t+1}$ and $a = 1, 2$. 

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**Theorem 3.4** For $p \geq 5$ the following cases occur for $M = L(r) \otimes L(2)$:

(a) If $r \equiv 0 \pmod{p}$ then $M$ is isomorphic to $L(r + 2)$.

(b) If $r \equiv 1 \pmod{p}$ then $M$ is isomorphic to $L(r + 2) \oplus L(r)$.

(c) If $r \equiv -1 \pmod{p}$ then $M$ is isomorphic to $T(ap^t + 1) \otimes L(k)^{F^{t+1}} \oplus L(r)$ where $r = (ap^t - 1) + kp^{t+1}$ and $1 \leq a \leq p - 1$.

(d) If $r \equiv -2 \pmod{p}$ then $M$ is isomorphic to $T(ap^t) \otimes L(k)^{F^{t+1}} \oplus L(r - 2)$ where $r + 1 = (ap^t - 1) + kp^{t+1}$ and $1 \leq a \leq p - 1$.

(e) In all other cases $M$ is isomorphic to $L(r + 2) \oplus L(r) \oplus L(r - 2)$.

It should be noted that the submodule lattices of these tensor products can be deduced from Theorem 3.2.

**4 Proof of Theorems 3.1 and 3.2**

For a positive integer $t$ let $X_t$ be the set of $p^t$-restricted weights for $\text{SL}_2$; that is, the set of all $r \in \mathbb{Z}$ such that $0 \leq r < p^t$.

**Lemma 4.1** Let $M$ be an $\text{SL}_2$-module such that the highest weight of all composition factors of $M$ belongs to $X_t$. Then for any $\mu \in \mathbb{Z}_{\geq 0}$ the functor $(-) \otimes L(\mu)^{F^{t+1}}$ induces an isomorphism of submodule lattices between $M$ and $M \otimes L(\mu)^{F^{t+1}}$. Moreover, the composition factors of $M \otimes L(\mu)^{F^{t+1}}$ have the form $L(\omega + p^{t+1}\mu)$ as $\omega$ ranges over the set of highest weights of composition factors of $M$.

**Proof.** This follows immediately from Steinberg’s tensor product theorem, using induction on the length of a composition series for $M$. \[ \square \]

Recall the notation $\delta_i(m)$ for the $i$th digit in the $p$-adic expansion of $m$, so that $m = \sum_{i \geq 0} \delta_i(m)p^i$ with $0 \leq \delta_i(m) < p$ for all $i$. We have the following elementary result:

**Lemma 4.2** Let $r > 0$ be an integer with $r \equiv -1 \pmod{p}$. There exist unique positive integers $t$, $a$ such that $a < p$ and $r \equiv ap^t - 1 \pmod{p^{t+1}}$. 

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Proof. Let $t$ be the smallest index $i$ such that $\delta_i(r)$ is different from $p - 1$. By the hypothesis $t > 0$. Since $\delta_i(r) = p - 1$ for $i = 0, \ldots, t - 1$ we have

$$r \equiv \sum_{i=0}^{t-1} (p - 1)p^i + \delta_t(r)p^t \pmod{p^{t+1}}$$

$$\equiv (p^t - 1) + \delta_t(r)p^t \pmod{p^{t+1}}$$

$$\equiv ap^t - 1 \pmod{p^{t+1}}$$

where $a = 1 + \delta_t(r)$. This proves the existence of $t, a$ with the stated properties.

If $t_1, t_2, a_1, a_2$ are positive integers with $a_i < p$ such that $r \equiv a_ip^{t_i} - 1 \pmod{p^{t_i+1}}$ for $i = 1, 2$ then $r = (a_1p^{t_1} - 1) + k_1p^{t_1+1}$ for integers $k_1$. Hence

$$a_1p^{t_1} + k_1p^{t_1+1} = a_2p^{t_2} + k_2p^{t_2+1}.$$ 

If $t_1 < t_2$ then upon dividing by $p^{t_1}$ we obtain the contradiction that $p$ divides $a_1$. Similarly if $t_1 > t_2$. Thus $t_1 = t_2$ and so $a_1 + k_1p = a_2 + k_2p$ from which we obtain immediately that $a_1 = a_2, k_1 = k_2$. This proves the uniqueness statement. □

Lemma 4.3 For $r \in \mathbb{Z}_{\geq 0}$ of the form $r = ap^t - 1$ for $a \in \{1, \ldots, p - 1\}$ we have $\nabla(r) = L(r) = \Delta(r)$.

Proof. First, note that $\nabla(p^t - 1, 0) = L(p^t - 1, 0) = \Delta(p^t - 1, 0)$. This is well-known. It follows for $p$ odd from [1, II, 3.19(4)]. For general $p$ it follows for instance from [D2, Theorem 2].

Moreover, from [D2, Theorem 1] it follows that $\nabla_t(p^t - 1, 0) = L_t(p^t - 1, 0)$ where $\nabla_t(\lambda)$ (resp., $L_t(\lambda)$) is the induced $G_tB$-module $\text{ind}_B^{G_tB} \lambda$ (resp., the simple $G_tB$-module of highest weight $\lambda$). Here $G = \text{SL}_2$ and $G_t$ (resp., $G_tB$) is the kernel (resp., the inverse image of $B$) under the morphism (in the category of group schemes) $G \rightarrow G$ given by the $t$th iterate of the Frobenius.

Now by applying the same argument of [1, II, 3.19] one obtains the identity

$$\nabla((p^t - 1) + (a - 1)p^t) \cong \nabla(p^t - 1) \otimes \nabla(a - 1)^{F^t}.$$ 

Noting that $\nabla(a - 1) = L(a - 1)$ we obtain from Steinberg’s tensor product theorem that $\nabla(ap^t - 1) = L(ap^t - 1)$. The equality $\Delta(ap^t - 1) = L(ap^t - 1)$ follows by contravariant duality. □

As corollary to the above we have:
Lemma 4.4 For $r$ as above, $L(r) \otimes E$ has a good filtration, the sections of which are isomorphic with $\nabla(r + 1)$ and $\nabla(r - 1)$.

Proof. This follows from the Wang-Donkin-Mathieu theorem and Young’s rule since $L(r) \otimes E = \nabla(r) \otimes \nabla(1)$. One can also give a direct proof using properties of the induction functor. □

Lemma 4.5 For $r$ as above the modules $\nabla(r + 1)$ and $\nabla(r - 1)$ are uniserial modules whose unique composition series have the (respective) form

\[ [L(r_0), L(r_{t-1}), \ldots, L(r_1)] \]

and

\[ [L(r_1), \ldots, L(r_{t-1})] \quad \text{for } a = 1, \]

\[ [L(r_1), \ldots, L(r_{t-1}), L(r_i)] \quad \text{for } a \neq 1 \]

where $r_0 = r + 1$ and $r_i = r + 1 - 2p^{i-1}$ for $i \in \mathbb{Z}_{>0}$.

Proof. Uniserial (dual) Weyl modules for two-part partitions were classified in [EH2, Proposition 2.2]. The result also follows for instance from the results in [D1]. □

We can now prove Theorems 3.1 and 3.2. The first statement of Theorem 3.2 is contained already in Lemma 4.2. Suppose that $r$ has the form $(ap^t - 1) + p^{t+1}k$ for integers $t > 0$, $k \geq 0$ and $a \in \{1, \ldots, p - 1\}$. By Theorem 2.1, $M = L(r) \otimes E$ is indecomposable with socle isomorphic to $L(r - 1)$. Since simple modules are contravariantly self-dual, so is $M$. The description of the module structure given in the statement of Theorem 3.2 now follows from Lemmas 4.4 and 4.5. This completes the proof of the Theorems 3.1 and 3.2.

Corollary 4.6 For any $r \in \mathbb{Z}_{\geq0}$ of the form $r = ap^t - 1$ where $a \in \{1, \ldots, p - 1\}$ and $t > 0$ we have an isomorphism $T(r + 1) \cong L(r) \otimes E$.

Proof. As we have seen above, $L(r) \otimes E$ has in this case a good filtration. But simple modules are contravariantly self-dual, so $L(r) \otimes E$ is self-dual as well. Thus it has a $\Delta$-filtration. □

The preceding corollary considers the case $k = 0$. Now suppose that $k > 0$. Set $\tilde{r} = r_1 - kp^{t+1}$. Then $\tilde{r} \in \mathbb{Z}_{\geq0}$ and by the corollary $L(\tilde{r}) \otimes E$ is a
tilting module with module structure as described. In particular, all of its composition factors have highest weights that belong to the set $X_t$. Thus by Lemma 4.1 and Steinberg’s tensor product theorem tensoring by $L(k)^{F_{r+1}}$ induces an isomorphism of submodule lattices between $L(\tilde{r}) \otimes E$ and $L(r) \otimes E$.

Corollary 4.7 If $r \equiv -1 \pmod{p}$, the tensor product $L(r) \otimes E$ is either a tilting module or a ‘shift’ of a tilting module.

5 Examples: $p = 2$

In this section we give some examples of tensor product decompositions in the case $p = 2$. Unlike the situation for odd primes, at $p = 2$ a tensor product of simple modules is always indecomposable.

In this and the following section, when we write $M = [r_1, \ldots, r_k]$ we mean that $M$ is a uniserial module whose unique composition series has the indicated form. We shall always identify simple factors with their highest weight, and write $\cong$ to indicate isomorphism.

For modules that are not uniserial, we will give whenever possible the module structure diagram (in the sense of Alperin [2]), where the structure is depicted by a graph in which the vertices correspond with the simple factors, and an edge connects two vertices if and only if a non-split extension between those composition factors occurs as a subquotient of the module.

Module structure can be computed using known facts about good filtrations, along with the known structure of dual Weyl modules (see [1], [3]), along with other basic facts which can be found in [3]. We also make use of the fact that $\text{Ext}^1$ between two simple modules is at most 1-dimensional.

$[1] \otimes [1] = [0, 2, 0] = T(2)$.

$[2] \otimes [1] = [3] = T(3)$.

$[3] \otimes [1] = ([1] \otimes [2]) \otimes [1] = ([1] \otimes [1]) \otimes [2] = [0, 2, 0] \otimes [2] = [2, 0, 4, 0, 2] = T(4)$.

$[2] \otimes [2] = ([1] \otimes [1])^F = [0, 4, 0]$.
\[ 4 \otimes [1] = [5]. \]
\[ 3 \otimes [2] = [1] \otimes [2] \otimes [2] = [1] \otimes [0, 4, 0] = [1, 5, 1] = T(5). \]

\[ 5 \otimes [1] = [1] \otimes [1] \otimes [4] = [0, 2, 0] \otimes [4] = [4, 6, 4]. \]
\[ 4 \otimes [2] = [6]. \]
\[ 3 \otimes [3] = [1] \otimes [2] \otimes [1] \otimes [2] = [1] \otimes [1] \otimes [2] \otimes [2] = [0, 2, 0] \otimes [0, 4, 0] = T(6), \]
where \( T(6) \) has structure:

\[
\begin{array}{c}
\otimes \\
\otimes \\
\otimes \\
\otimes \\
\end{array}
\]

\[ 6 \otimes [1] = [1] \otimes [2] \otimes [4] = [7] = T(7). \]
\[ 5 \otimes [2] = [1] \otimes [2] \otimes [4] = [7] = T(7). \]
\[ 4 \otimes [3] = [1] \otimes [2] \otimes [4] = [7] = T(7). \]
\[ 7 \otimes [1] = [1] \otimes [1] \otimes [2] \otimes [4] = [0, 2, 0] \otimes [2] \otimes [4] = [2, 0, 4, 0, 2] \otimes [4] = [6, 4, 0, 8, 0, 4, 6] = T(8). \]
\[ 6 \otimes [2] = [2] \otimes [2] \otimes [4] = [0, 4, 0] \otimes [4] = [4, 0, 8, 0, 4]. \]
\[ 5 \otimes [3] = [1] \otimes [1] \otimes [2] \otimes [4] = T(8). \]
\[ 4 \otimes [4] = ([1] \otimes [1])^F = [0, 8, 0]. \]
\[ 8 \otimes [1] = [9]. \]
\[ 7 \otimes [2] = [1] \otimes [2] \otimes [2] \otimes [4] = [1] \otimes [0, 4, 0] \otimes [4] = [1] \otimes [4, 0, 8, 0, 4] = [5, 1, 9, 1, 5] = T(9). \]
\[ 6 \otimes [3] = [1] \otimes [2] \otimes [2] \otimes [4] = T(9). \]
\[ 5 \otimes [4] = [1] \otimes [4] \otimes [4] = [1] \otimes [0, 8, 0] = [1, 9, 1]. \]
\[ 9 \otimes [1] = [1] \otimes [1] \otimes [8] = [0, 2, 0] \otimes [8] = [8, 10, 8]. \]
\[ 8 \otimes [2] = [10]. \]
\[ [7] \otimes [3] = [1] \otimes [1] \otimes [2] \otimes [2] \otimes [4] = [4, 6, 4] \otimes [0, 4, 0] = T(10), \text{ where } T(10) \]

has structure:

\[
\begin{array}{c}
\otimes \\
\otimes \\
\otimes \\
\otimes \\
\end{array} 
\]

\[
T(10) = 
\begin{array}{cccc}
4 & 6 & 0 & 2 \\
0 & 8 & 10 & 0 \\
2 & 6 & 0 & 4 \\
4 & 2 & 10 & 2 \\
\end{array} 
\]

\[ [6] \otimes [4] = [2] \otimes [4] \otimes [4] = [2] \otimes [0, 8, 0] = [2, 10, 2]. \]

\[ [5] \otimes [5] = [1] \otimes [1] \otimes [4] \otimes [4] = [0, 2, 0] \otimes [0, 8, 0] = 
\begin{array}{c}
\otimes \\
\otimes \\
\otimes \\
\otimes \\
\end{array} 
\]

\[
\begin{array}{cccc}
0 & 12 & 0 & 10 \\
2 & 8 & 0 & 4 \\
10 & 0 & 4 & 0 \\
6 & 0 & 2 & 10 \\
\end{array} 
\]

\[ [10] \otimes [1] = [1] \otimes [2] \otimes [8] = [11]. \]

\[ [9] \otimes [2] = [1] \otimes [2] \otimes [8] = [11]. \]

\[ [8] \otimes [3] = [1] \otimes [2] \otimes [8] = [11]. \]

\[ [7] \otimes [4] = [1] \otimes [2] \otimes [4] \otimes [4] = [3, 11, 3] = T(11). \]

\[ [6] \otimes [5] = [1] \otimes [2] \otimes [4] \otimes [4] = [3, 11, 3] = T(11). \]

\[ [11] \otimes [1] = [1] \otimes [1] \otimes [2] \otimes [8] = [2, 0, 4, 0, 2] \otimes [8] = [10, 8, 12, 8, 10]. \]

\[ [10] \otimes [2] = [2] \otimes [2] \otimes [8] = [0, 4, 0] \otimes [8] = [8, 12, 8]. \]

\[ [9] \otimes [3] = [1] \otimes [1] \otimes [2] \otimes [8] = [10, 8, 12, 8, 10]. \]

\[ [8] \otimes [4] = [12]. \]

\[ [7] \otimes [5] = [1] \otimes [1] \otimes [2] \otimes [4] \otimes [4] = [2, 0, 4, 0, 2] \otimes [0, 8, 0] = T(12), \text{ where } \]

\[ T(12) \text{ has structure diagram:} \]

\[
\begin{array}{c}
\otimes \\
\otimes \\
\otimes \\
\otimes \\
\end{array} 
\]

\[
T(12) = 
\begin{array}{cccc}
0 & 4 & 2 & 10 \\
10 & 0 & 12 & 0 \\
2 & 8 & 0 & 4 \\
8 & 12 & 0 & 2 \\
\end{array} 
\]

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\[ [6] \otimes [6] = [2] \otimes [2] \otimes [4] \otimes [4] = [0, 4, 0] \otimes [0, 8, 0] = \]

\[ [12] \otimes [1] = [1] \otimes [4] \otimes [8] = [13]. \]
\[ [11] \otimes [2] = [1] \otimes [2] \otimes [2] \otimes [8] = [1, 5, 1] \otimes [8] = [9, 13, 9]. \]
\[ [10] \otimes [3] = [1] \otimes [2] \otimes [2] \otimes [8] = [1, 5, 1] \otimes [8] = [9, 13, 9]. \]
\[ [9] \otimes [4] = [1] \otimes [4] \otimes [8] = [13]. \]
\[ [8] \otimes [5] = [1] \otimes [4] \otimes [8] = [13]. \]
\[ [7] \otimes [6] = [1] \otimes [2] \otimes [2] \otimes [4] \otimes [4] = [1, 5, 1] \otimes [0, 8, 0] = T(13), \text{ where } T(13) \]
\[ \text{has structure:} \]
\[ T(13) = \]

\[ [13] \otimes [1] = [1] \otimes [1] \otimes [4] \otimes [8] = [0, 2, 0] \otimes [12] = [12, 14, 12] \]
\[ [12] \otimes [2] = [2] \otimes [4] \otimes [8] = [14]. \]
\[ [11] \otimes [3] = [1] \otimes [1] \otimes [2] \otimes [2] \otimes [8] = T(6) \otimes [8] = \]

\[ [10] \otimes [4] = [2] \otimes [4] \otimes [8] = [14]. \]
\[ [9] \otimes [5] = [1] \otimes [1] \otimes [4] \otimes [8] = [12, 14, 12]. \]
\[ [8] \otimes [6] = [2] \otimes [4] \otimes [8] = [14]. \]
\[ [7] \otimes [7] = [1] \otimes [1] \otimes [2] \otimes [2] \otimes [4] \otimes [4] = [0, 2, 0] \otimes [0, 4, 0] \otimes [0, 8, 0] = T(14). \]
\[ T(14) \text{ has 27 composition factors (counting multiplicities) and its diagram} \]
\[ \text{is a cube consisting of 27 vertices (standing on a vertex labelled 0) with the} \]

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following three ‘layers’:

When putting the layers together to a cube there are non-split extensions between 0, 8 and between 2, 10 (none between 4, 12 and 6, 14). The cube is rotated in space so as to have the two opposite vertices labelled 0 at the top and the bottom, and so that the socle layers are obtained by intersecting horizontal planes through the cube. We therefore obtain for $T(14)$ the diagram (only the three visible sides of the cube are shown)

It should be noted that $T(2^k - 2) = St_{k-1} \otimes St_{k-1}$ where $St_{k-1}$ stands for the $(k-1)$th Steinberg module. These tilting modules are projective indecomposable modules. For details see [EH], Section 3, where it was classified when a projective indecomposable module is a tilting module. Moreover, let $U_i$ be defined to be the uniserial module $U_i := L(2^i) \otimes L(2^i) = [0, 2^{i+1}, 0]$. Then $T := T(2^k - 2) \otimes U_{k-1}$ is a $(k-1)$-dimensional hypercube with $3^{k-1}$ vertices and with $(k-1)$ filtrations

$$T \supset T_1 \supset T_2 \supset 0$$

such that $T/T_1, T_1/T_2$ and $T_2/0$ are $(k-2)$-dimensional hypercubes, according to tensoring $T = V_i \otimes U_i$ where $V_i$ is given by

$$V_i = [1] \otimes [1] \otimes \ldots \otimes [2^{i-1}] \otimes [2^{i-1}] \otimes [2^{i+1}] \otimes [2^{i+1}] \ldots \otimes [2^{k-2}] \otimes [2^{k-2}]$$

In particular, in one of these filtrations, two copies of $T(2^{k-1} - 2)$ occur, one as a submodule and one as quotient module of $T(2^k - 2)$. 

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6 Examples: $p = 3$

In this section we give some examples of tensor product decompositions in the case $p = 3$. Unlike the situation for $p = 2$, in this case a tensor product of simple modules is not always indecomposable. We follow the notational conventions introduced at the beginning of the preceding section. For comparing these tilting modules to projective indecomposable modules we refer to $[EH1]$, section 4.

\[ [1] \otimes [1] = [2] \oplus [0] = T(2) \oplus T(0). \]
\[ [2] \otimes [1] = [1, 3, 1] = T(3). \]
\[ [3] \otimes [1] = [4]. \]
\[ [2] \otimes [2] = [0, 4, 0] \oplus [2] = T(4) \oplus T(2). \]
\[ [4] \otimes [1] = [5] \oplus [3] = T(5) \oplus [3]. \]
\[ [3] \otimes [2] = [5] = T(5). \]
\[ [5] \otimes [1] = [3] \otimes [2] \otimes [1] = [3] \otimes [1, 3, 1] = T(6), \text{ where } T(6) \text{ has structure:} \]
\[
\begin{array}{ccc}
4 & 0 \\
4 & 6
\end{array}
\]
\[ [4] \otimes [2] = [3] \otimes [1] \otimes [2] = T(6) \text{ (above).} \]
\[ [3] \otimes [3] = ([1] \otimes [1])^F = ([2] \oplus [0])^F = [6] \oplus [0]. \]
\[ [6] \otimes [1] = [7]. \]
\[ [5] \otimes [2] = [3] \otimes [2] \otimes [2] = [3] \otimes ([0, 4, 0] \oplus [2]) = T(7) \oplus [5], \text{ where } T(7) \text{ has structure:} \]
\[
\begin{array}{ccc}
3 & 1 \\
3 & 7
\end{array}
\]
\[ [4] \otimes [3] = [3] \otimes [3] \otimes [1] = ([6] \oplus [0]) \otimes [1] = [7] \oplus [1]. \]
\[ 7 \otimes [1] = [6] \otimes [1] \otimes [1] = [6] \otimes ([2] \oplus [0]) = [8] \oplus [6] = T(8) \oplus [6]. \]
\[ 6 \otimes [2] = [8] = T(8). \]
\[ 5 \otimes [3] = [3] \otimes [3] \otimes [2] = ([6] \oplus [0]) \otimes [2] = [8] \oplus [2] = T(8) \oplus T(2). \]
\[ 4 \otimes [4] = [3] \otimes [3] \otimes [1] \otimes [1] = ([6] \oplus [0]) \otimes ([2] \oplus [0]) = [8] \oplus [6] \oplus [2] \oplus [0]. \]
\[ 8 \otimes [1] = [6] \otimes [2] \otimes [1] = [6] \otimes [1, 3, 1] = [7, 3, 9, 3, 7] = T(9). \]
\[ 7 \otimes [2] = [6] \otimes [1] \otimes [2] = T(9) \text{ (above)}. \]
\[ 6 \otimes [3] = ([2] \otimes [1])^F = [1, 3, 1]^F = [3, 9, 3]. \]
\[ 5 \otimes [4] = [3] \otimes [3] \otimes [2] \otimes [1] = ([6] \oplus [0]) \otimes [1, 3, 1] = T(9) \oplus T(3). \]
\[ 9 \otimes [1] = [10]. \]
\[ 8 \otimes [2] = [6] \otimes [2] \otimes [2] = [6] \otimes ([0, 4, 0] \oplus [2]) = [6, 4, 10, 4, 6] \oplus [8] = T(10) \oplus T(8). \]
\[ 7 \otimes [3] = [6] \otimes [3] \otimes [1] = [3, 9, 3] \otimes [1] = [4, 10, 4]. \]
\[ 6 \otimes [4] = [6] \otimes [3] \otimes [1] = [4, 10, 4]. \]
\[ 5 \otimes [5] = [3] \otimes [3] \otimes [2] \otimes [2] = ([6] \oplus [0]) \otimes ([0, 4, 0] \oplus [2]) = [6, 4, 10, 4, 6] \oplus [8] \oplus [0, 4, 0] \oplus [2] = T(10) \oplus T(8) \oplus T(4) \oplus T(2). \]
\[ 10 \otimes [1] = [9] \otimes [1] \otimes [1] = [9] \otimes ([2] \oplus [0]) = [11] \oplus [9]. \]
\[ 9 \otimes [2] = [11]. \]
\[ 8 \otimes [3] = [6] \otimes [3] \otimes [2] = [3, 9, 3] \otimes [2] = [5, 11, 5] = T(11). \]
\[ 7 \otimes [4] = [6] \otimes [3] \otimes [1] \otimes [1] = [3, 9, 3] \otimes ([2] \oplus [0]) = T(11) \oplus [3, 9, 3]. \]
\[ 6 \otimes [5] = [6] \otimes [3] \otimes [2] = [5, 11, 5] = T(11). \]
\[ 11 \otimes [1] = [9] \otimes [2] \otimes [1] = [9] \otimes [1, 3, 1] = [10, 12, 10]. \]
\[ 10 \otimes [2] = [9] \otimes [2] \otimes [1] = [10, 12, 10]. \]
\[ 9 \otimes [3] = [12]. \]
\[ 8 \otimes [4] = [6] \otimes [3] \otimes [2] \otimes [1] = [3, 9, 3] \otimes [1, 3, 1] = T(12), \text{ where } T(12) \text{ has } \]
\[ T(12) = \]

\[
\begin{array}{ccc}
0 & 4 & 6 \\
12 & 10 & \\
0 & 4 & 6 \\
\end{array}
\]

\[ [7] \otimes [5] = [6] \otimes [3] \otimes [2] \otimes [1] = T(12) \text{ (see above).} \]

\[ [6] \otimes [6] = ([2] \otimes [2])^F = [0, 4, 0]^F \oplus [2]^F = [0, 12, 0] \oplus [6]. \]

\[ [12] \otimes [1] = [9] \otimes [3] \otimes [1] = [13]. \]

\[ [11] \otimes [2] = [9] \otimes [2] \otimes [2] = [9] \otimes ([0, 4, 0] \oplus [2]) = [9, 13, 9] \oplus [11]. \]

\[ [10] \otimes [3] = [9] \otimes [3] \otimes [1] = [13]. \]

\[ [9] \otimes [4] = [9] \otimes [3] \otimes [1] = [13]. \]

\[ [8] \otimes [5] = [6] \otimes [3] \otimes [2] \otimes [2] = [3, 9, 3] \otimes ([0, 4, 0] \oplus [2]) = [3, 9, 3] \otimes [0, 4, 0] \oplus [3, 9, 3] \otimes [2] = T(13) \oplus T(11), \text{ where } T(13) \text{ has structure:} \]

\[ T(13) = \]

\[
\begin{array}{ccc}
1 & 3 & 7 \\
13 & 9 & \\
1 & 3 & 7 \\
\end{array}
\]

\[ [7] \otimes [6] = [6] \otimes [6] \otimes [1] = ([0, 12, 0] \oplus [6]) \otimes [1] = [1, 13, 1] \oplus [7]. \]

\[ [13] \otimes [1] = [9] \otimes [3] \otimes [1] \otimes [1] = [9] \otimes [3] \otimes ([2] \oplus [0]) = [14] \oplus [12]. \]

\[ [12] \otimes [2] = [9] \otimes [3] \otimes [2] = [14]. \]

\[ [11] \otimes [3] = [9] \otimes [3] \otimes [2] = [14]. \]

\[ [10] \otimes [4] = [9] \otimes [3] \otimes [1] \otimes [1] = [14] \oplus [12]. \]

\[ [9] \otimes [5] = [9] \otimes [3] \otimes [2] = [14]. \]

\[ [8] \otimes [6] = [6] \otimes [6] \otimes [2] = ([0, 12, 0] \oplus [6]) \otimes [2] = [2, 14, 2] \oplus [8] = T(14) \oplus T(8). \]
\[ [7] \otimes [7] = [6] \otimes [6] \otimes [1] \otimes [1] = ([0, 12, 0] \oplus [6]) \otimes ([2] \oplus [0]) = [2, 14, 2] \oplus [8] \oplus [0, 12, 0] \oplus [6] = T(14) \oplus T(8) \oplus [0, 12, 0] \oplus [6]. \]

\[ [14] \otimes [1] = [9] \otimes [3] \otimes [2] \otimes [1] = [9] \otimes T(6) = 15 \]

\[ [13] \otimes [2] = [9] \otimes [3] \otimes [2] \otimes [1] \text{ (see above).} \]

\[ [12] \otimes [3] = [9] \otimes [3] \otimes [3] = [9] \otimes ([6] \oplus [0]) = [15] \oplus [9]. \]

\[ [11] \otimes [4] = [9] \otimes [3] \otimes [2] \otimes [1] \text{ (see above).} \]

\[ [10] \otimes [5] = [9] \otimes [3] \otimes [2] \otimes [1] \text{ (see above).} \]

\[ [9] \otimes [6] = ([3] \otimes [2])^F = [5]^F = [15]. \]

\[ [8] \otimes [7] = [6] \otimes [6] \otimes [2] \otimes [1] = ([0, 12, 0] \oplus [6]) \otimes [1, 3, 1] = [0, 12, 0] \otimes [1, 3, 1] + [6] \otimes [1, 3, 1] = T(15) \oplus T(9), \text{ where } T(15) \text{ has structure:} \]

\[ T(15) = \]

\[ [15] \otimes [1] = [9] \otimes [6] \otimes [1] = [16]. \]

\[ [14] \otimes [2] = [9] \otimes [3] \otimes [2] \otimes [2] = [9] \otimes (T(7) \oplus [5]) = 16 \]

\[ [13] \otimes [3] = [9] \otimes [3] \otimes [3] \otimes [1] = [9] \otimes ([6] \oplus [0]) \otimes [1] = [16] \oplus [10]. \]

\[ [12] \otimes [4] = [16] \oplus [10]. \]

\[ [11] \otimes [5] = [9] \otimes [3] \otimes [2] \otimes [2] = [9] \otimes (T(7) \oplus [5]) = [9] \otimes T(7) \oplus [14] \text{ (see } [14] \otimes [2] \text{ above).} \]

\[ [10] \otimes [6] = [9] \otimes [6] \otimes [1] = [16]. \]

\[ [9] \otimes [7] = [9] \otimes [6] \otimes [1] = [16]. \]

\[ [8] \otimes [8] = [6] \otimes [6] \otimes [2] \otimes [2] = ([0, 12, 0] \oplus [6]) \otimes ([0, 4, 0] \oplus [2]) = [0, 12, 0] \otimes [0, 4, 0] \oplus [0, 12, 0] \otimes [2] \oplus [6] \otimes [0, 4, 0] \oplus [6] \otimes [2] = T(16) \oplus T(14) \oplus T(10) \oplus T(8), \]
where $T(16)$ has structure:

$$T(16) = \begin{array}{c}
\begin{array}{c}
4 \\
0 \\
10 \\
12 \\
16
\end{array}
\end{array}$$

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