On the Basic Properties of $g$-Circulant Matrix via Generalized $k$-Horadam Numbers

Necati Taskara, Yasin Yazlik, Nazmiye Yilmaz
(1) Department of Mathematics, Faculty of Science, Selcuk University, Campus, 42075, Konya, Turkey
(2) Department of Mathematics, Faculty of Science and Art, Nevsehir University, 50300, Nevsehir, Turkey

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Abstract
In this paper, by considering the $g$-circulant matrix $C_{n,g}(H) = g$-circ($H_{k,1}, H_{k,2}, \ldots, H_{k,n}$) whose entries are the generalized $k$-Horadam numbers, we present a new generalization to compute spectral norm, determinant and inverse of $C_{n,g}(H)$. In fact the results in here are the most general statements to obtain the inverses and determinants in such matrices having the elements of all second order sequences.

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1 Introduction and Preliminaries

Many generalizations of the Fibonacci sequence have been introduced and studied [4-8,17]. Here we use the generalized $k$-Horadam numbers as follows [4] (see also [17]):

Let $k$ be any positive real number and $f(k), g(k)$ are scaler-value polynomials and $f^2(k) + 4g(k) > 0$. For $n \geq 0$, the generalized $k$-Horadam sequence $\{H_{k,n}\}_{n \in \mathbb{N}}$ is defined by

$$H_{k,n+2} = f(k)H_{k,n+1} + g(k)H_{k,n}, \quad H_{k,0} = a, \quad H_{k,1} = b. \quad (1)$$

where $a, b \in \mathbb{R}$. Obviously, if we choose suitable values on $f(k), g(k), a$ and $b$ in (1), then this sequence reduces to the special all second order sequences in the

*e mail: ntaskara@selcuk.edu.tr, yyazlik@nevsehir.edu.tr, nzyilmaz@selcuk.edu.tr
literature. For example, by taking \( f(k) = g(k) = 1, a = 0 \) and \( b = 1 \), then it is obtained the well known Fibonacci sequence.

Let \( r_1 \) and \( r_2 \) be the roots of the characteristic equation \( x^2 - f(k)x - g(k) = 0 \) of (1). Then the Binet formula of this sequence \( \{H_{k,n}\}_{n\in\mathbb{N}} \) have the form

\[
H_{k,n} = \frac{X r_1^n - Y r_2^n}{r_1 - r_2},
\]

where \( X = b - ar_2 \) and \( Y = b - ar_1 \). And, the summation of this sequence is given by

\[
\sum_{i=1}^{n} H_{k,i} = \frac{H_{k,n+1} + g(k) H_{k,n} - H_{k,1} - g(k) H_{k,0}}{f(k) + g(k) - 1},
\]

where \( f(k) + g(k) - 1 \neq 0 \).

The \( g \)-circulant matrices have been one of the most important and active research field of applied mathematic and computation mathematic increasingly. There are lots of examples from statistical and information theory illustrate applications of the \( g \)-circulant matrices, which emphasize how the asymptotic eigenvalue distribution theorem allows one to evaluate results for processes (see [1-3] and therein). In the last years, there have been several papers on circulant matrices [9-13,15]. For instance, Solak [10] defined the \( n \times n \) circulant matrices \( A = [a_{ij}] \) and \( B = [b_{ij}] \), where \( a_{ij} \equiv F_{(\text{mod}(j-i,n))} \) and \( b_{ij} \equiv L_{(\text{mod}(j-i,n))} \). Additionally, he investigated the upper and lower bounds of the matrices \( A \) and \( B \). In [11], it was studied the norms, eigenvalues and determinants of some matrices related to different numbers. Yazlık and Taskara [13,20] defined circulant matrix \( C_n(H) \) whose entries are the generalized \( k \)-Horadam numbers and computed the spectral norm, eigenvalues, determinant and the inverse of the this matrix. That is, authors gave the determinant and inverse of matrix \( C_n(H) \) as follows

\[
\det C_n (H) = H_{k,1} N^{n-1} + H_{k,1} M^{n-2} \sum_{i=1}^{n-1} \left( -\frac{H_{k,2} H_{k,i+1}}{H_{k,1}} + H_{k,i+2} \right) \left( \frac{N}{M} \right)^{i-1},
\]

and for \( n > 2 \),

\[
C_n^{-1} (H) = \text{circ} \left( \frac{1 + f(k) S_n (n-2) + g(k) S_n (n-3)}{h_n}, \frac{S_n (1)}{h_n}, \frac{S_n (2)}{h_n} - f(k) S_n (1)}{h_n}, \frac{S_n (3)}{h_n} - f(k) S_n (2)}{h_n}, \ldots, \frac{S_n (n-2) - f(k) S_n (n-3) - g(k) S_n (n-4)}{h_n} \right),
\]

where \( S_n (j) = \sum_{i=1}^{j} \left( H_{k,j+i-1} - \frac{H_{k,2} H_{k,j+1}}{h_n} \right) M^{i-1} (j = 1,2,\ldots,n-2), \) \( h_n = -\frac{H_{k,2} H_{k,n}}{H_{k,1}} + \ldots \)
\[H_{k,1} + \sum_{i=1}^{n-1} \left( \frac{-H_{k,i} H_{k,i+1} + H_{k,i+2}}{H_{k,i}} \right) \left( \frac{M}{N} \right)^{n-(i+1)} , \quad M = g(k)(H_{k,n} - H_{k,0}) \text{ and } N = H_{k,1} - H_{k,n+1}.\]

In addition, Alptekin, Mansour and Tuglu, [16], obtained the spectral norm and eigenvalues of circulant matrices with Horadam’s numbers. Also, they defined the semicirculant matrix with these numbers and give Euclidean norm of this matrix. In [9], authors defined the \(n \times n\) circulant matrices \(A_n = [a_{ij}]\) and \(B_n = [b_{ij}]\), where \(a_{ij} \equiv F_{(\text{mod}(j-i,n))}\) and \(b_{ij} \equiv L_{(\text{mod}(j-i,n))}\). Also, the inverses of matrices \(A_n\) and \(B_n\) were derived.

Now we give some preliminaries related our study. A, \(g\)-circulant matrix is an \(n \times n\) complex matrix with the following form

\[
A = \begin{pmatrix}
a_0 & a_1 & \cdots & a_{n-1} \\
a_{n-g} & a_{n-g+1} & \cdots & a_{n-g-1} \\
a_{n-2g} & a_{n-2g+1} & \cdots & a_{n-2g-1} \\
\vdots & \vdots & \ddots & \vdots \\
a_g & a_{g+1} & \cdots & a_{g-1}
\end{pmatrix},
\]

where \(g\) is nonnegative integer and each of the subscripts is understood to be reduced modulo \(n\).

The first row of \(A\) is \((a_0, a_1, \ldots, a_{n-1})\) and its \((j+1)\)-th row is obtained by giving \(j\)-th row a right circular shift by \(g\) positions. Note that, \(g = 1\) or \(g = n + 1\) yields the classical circulant matrix.

From [3], we further remind that, for a matrix \(A = [a_{i,j}] \in M_{m,n}(\mathbb{C})\), the spectral norm of \(A\) is given by

\[
\|A\|_2 = \sqrt{\max_{1 \leq i \leq n} \lambda_i (A^*A)}
\]

where \(A^*\) is the conjugate transpose of matrix \(A\).

**Lemma 1.1** [3]. Let \(A\) be an \(n \times n\) matrix with eigenvalues \(\lambda_1, \lambda_2, \ldots, \lambda_n\). If \(A\) is a normal matrix, then

\[
\|A\|_2 = \max_{1 \leq i \leq n} |\lambda_i|.
\]

**Lemma 1.2** [18]. An \(n \times n\) matrix \(Q_g\) is unitary if and only if

\[
(n, g) = 1,
\]

where \(Q_g\) is a \(g\)-circulant matrix with the first row \(e^* = (1, 0, \ldots, 0)\).

**Lemma 1.3** [18]. \(A\) is \(g\)-circulant matrix with the first row \((a_0, a_1, \ldots, a_{n-1})\) if and only if

\[
A = Q_g C,
\]

where \(C\) is circulant matrix, that is, \(C = \text{circ}(a_0, a_1, \ldots, a_{n-1})\).
In this paper, we consider \(g\)-circulant matrix \(C_{n,g}(H) = g\text{-circ}(H_{k,1}, H_{k,2}, \ldots, H_{k,n})\), where \(H_{k,n}\) is the generalized \(k\)-Horadam numbers. Firstly, we obtain the values of the spectral norm and determinant of this matrix can be expressed with only the generalized \(k\)-Horadam numbers. Also we formulate the inverse of \(g\)-circulant matrix \(C_{n,g}(H)\). In fact the results in here are the most general statements to obtain the spectral norms, determinants and inverses in such matrices having the elements of all second order sequences.

2 Main Results

Definition 2.1 An \((n \times n)\) \(g\)-circulant matrix with generalized \(k\)-Horadam numbers entries is defined by

\[
C_{n,g}(H) = \begin{pmatrix}
H_{k,1} & H_{k,2} & H_{k,3} & \cdots & H_{k,n} \\
H_{k,n-g+1} & H_{k,n-g+2} & H_{k,n-g+3} & \cdots & H_{k,n-g} \\
H_{k,n-2g+1} & H_{k,n-2g+2} & H_{k,n-2g+3} & \cdots & H_{k,n-2g} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
H_{k,g+1} & H_{k,g+2} & H_{k,g+3} & \cdots & H_{k,g}
\end{pmatrix},
\]

where \(g\) is nonnegative integer.

The following theorem gives us the values of the determinant of this matrix can be expressed by utilizing the generalized \(k\)-Horadam numbers.

Theorem 2.1 Let \(C_{n,g}(H) = g\text{-circ}(H_{k,1}, H_{k,2}, \ldots, H_{k,n})\) be circulant matrix as in (6). Then we have

\[
\|C_{n,g}(H)\|_2 = \frac{H_{k,n+1} + g(k)H_{k,n} - H_{k,1} - g(k)H_{k,0}}{f(k) + g(k) - 1},
\]

where \(f(k) + g(k) - 1 \neq 0\).

Proof. We express that \(g\)-circulant matrix is normal and irreducible (see [19]). So, the spectral norm of \(C_{n,g}(H)\) is given by the spectral radius of \(C_{n,g}(H)\). Also since \(C_{n,g}(H)\) is irreducible and entrywise nonnegative, its spectral radius is the same as its Perron value. Let \(u\) denote an all ones vector of order \(n\). Then \(C_{n,g}(H)u = (\sum_{i=1}^{n} H_{k,i})u\). As \(\sum_{i=1}^{n} H_{k,i}\) is an eigenvalue of \(C_{n,g}(H)\) associated with a positive eigenvector, it is necessarily the Perron value of \(C_{n,g}(H)\). Hence, from the Equation (3), we conclude that

\[
\|C_{n,g}(H)\|_2 = \frac{H_{k,n+1} + g(k)H_{k,n} - H_{k,1} - g(k)H_{k,0}}{f(k) + g(k) - 1}.
\]

Particular cases of the Theorem 2.1 are:

• If \(f(k) = 1, g(k) = 1, a = 0\) and \(b = 1\), for the Fibonacci sequence in ([19]), we obtain \(\|C_{n,g}(F)\|_2 = F_{n+2} - 1\),
• If \( f(k) = 1, g(k) = 1, a = 2 \) and \( b = 1 \), for the Lucas sequence in (19), we obtain \( \|C_{n,g}(L)\|_2 = L_{n+2} - 3 \).

• If \( f(k) = 2, g(k) = 1, a = 0 \) and \( b = 1 \), for the Pell sequence, we obtain
  \[
  \|C_{n,g}(P)\|_2 = \frac{P_{n+1} + P_n - 1}{2},
  \]

• If \( f(k) = 1, g(k) = 2, a = 0 \) and \( b = 1 \), for the Jacobsthal sequence, we obtain
  \[
  \|C_{n,g}(J)\|_2 = \frac{J_{n+2} - 1}{2}.
  \]

• Finally, we should note that choosing suitable values on \( f(k), g(k), a \) and \( b \) in Theorem 2.1, it is actually obtained the spectral norms of \( g \)-circulant matrix for the others second order sequences such as Pell-Lucas, Jacobsthal-Lucas, Horadam, etc.

**Theorem 2.2** Let \( C_{n,g}(H) = g\text{-circ} (H_{k,1}, H_{k,2}, \ldots, H_{k,n}) \) be circulant matrix as in (9). Then we have

\[
\det C_{n,g}(H) = \det Q_g \left[ H_{k,1} N^{n-1} + H_{k,1} M^{n-2} \sum_{i=1}^{n-1} \left( -\frac{H_{k,2} H_{k,i+1}}{H_{k,1}} + H_{k,i+2} \right) \left( \frac{N}{M} \right)^{i-1} \right],
\]

where \( M = g(k)(H_{k,n} - H_{k,0}), N = H_{k,1} - H_{k,n+1} \) and \( (n, g) = 1 \).

**Proof.** By using Lemma 1.2 ve 1.3, we can write

\[
C_{n,g}(H) = Q_g C_n(H),
\]

where \( (n, g) = 1 \), \( Q_g \) is a \( g \)-circulant matrix and \( C_n(H) \) is a circulant matrix with generalized \( k \)-Horadam number. From properties of determinant function and Equation (4), the proof is complete.

Particular cases of the Theorem 2.2 are:

• If \( f(k) = 1, g(k) = 1, a = 0 \) and \( b = 1 \), for the Fibonacci sequence, we obtain
  \[
  \det C_{n,g}(F) = \det Q_g \cdot \left[ (1 - F_{n+1})^{n-1} + F_n^{n-2} \sum_{i=1}^{n-1} F_i \left( \frac{1-F_{n+i+1}}{F_n} \right)^{i-1} \right]
  \]

• If \( f(k) = 1, g(k) = 1, a = 2 \) and \( b = 1 \), for the Lucas sequence, we obtain
  \[
  \det C_{n,g}(L) = \det Q_g \cdot \left[ (1 - L_{n+1})^{n-1} + (L_n - 2)^{n-2} \sum_{i=1}^{n-1} (L_{i+2} - 3L_{i+1}) \left( \frac{1-L_{n+i+1}}{L_n - 2} \right)^{i-1} \right]
  \]

• If \( f(k) = 2, g(k) = 1, a = 0 \) and \( b = 1 \), for the Pell sequence, we obtain
  \[
  \det C_{n,g}(P) = \det Q_g \cdot \left[ (1 - P_{n+1})^{n-1} + P_n^{n-2} \sum_{i=1}^{n-1} P_i \left( \frac{1-P_{n+i+1}}{P_n} \right)^{i-1} \right]
  \]

• If \( f(k) = 1, g(k) = 2, a = 0 \) and \( b = 1 \), for the Jacobsthal sequence, we obtain
  \[
  \det C_{n,g}(J) = \det Q_g \cdot \left[ (1 - J_{n+1})^{n-1} + 2^{n-1} J_n^{n-2} \sum_{i=1}^{n-1} J_i \left( \frac{1-J_{n+i+1}}{2J_n} \right)^{i-1} \right]
  \]
• Finally, we should note that choosing suitable values on \( f(k), g(k), a \) and \( b \) in Theorem 2.2, it is actually obtained the determinant of \( g \)-circulant matrix for the others second order sequences such as Pell-Lucas, Jacobsthal-Lucas, Horadam, etc.

**Proposition 2.1** Let \( C_{n,g}(H) = g\text{-circ}(H_{k,1}, H_{k,2}, \ldots H_{k,n}) \) be \( g \)-circulant matrix as in (5). For \( n > 2 \), \( C_{n,g}(H) \) is an invertible matrix.

**Proof.** By using Lemma 1.2 ve 1.3, we can write \( C_{n,g}(H) = Q_gC_n(H) \), where \((n, g) = 1, Q_g \) is a \( g \)-circulant matrix and \( C_n(H) \) is a circulant matrix with generalized \( k \)-Horadam number. From the Equation (5), \( C_n(H) \) is invertible for \( n > 2 \). Hence, \( C_{n,g}(H) \) is an invertible matrix, since \( C_n(H) \) and \( Q_g \) are invertible. ■

**Theorem 2.3** Let \( C_{n,g}(H) = g\text{-circ}(H_{k,1}, H_{k,2}, \ldots H_{k,n}) \) be \( g \)-circulant matrix as in (5), for \( n > 2 \), then we have

\[
C_{n,g}^{-1}(H) = \left[ \begin{array}{c}
\frac{1 + f(k)S_n^{(n-2)} + g(k)S_n^{(n-3)}}{h_n}, -\frac{H_{k,2} - H_{k,0}}{H_{k,1}}, -S_n^{(1)},
\frac{S_n^{(2)} - f(k)S_n^{(1)}}{h_n}, -\frac{g(k)S_n^{(n-2)} - H_{k,2} - H_{k,1}}{h_n}, -S_n^{(3)},
\frac{S_n^{(4)} - f(k)S_n^{(3)} - g(k)S_n^{(n-4)}}{h_n}, -\frac{H_{k,2} - H_{k,1}}{h_n}, -S_n^{(2)},
\end{array} \right] Q_g^T,
\]

where \( S_n^{(j)} = \sum_{i=1}^{j} \left( \frac{H_{k,1} - H_{k,i+1} + H_{k,i+2}}{N_i} \right) M^{i-1} \) \((j = 1, 2, \ldots, n-2)\), \( h_n = -\frac{H_{k,2}H_{k,n} + H_{k,1}}{H_{k,0}} \), \( H_{k,0} + \sum_{i=1}^{n-1} \left( \frac{H_{k,2}H_{k,i+1} + H_{k,i+2}}{H_{k,1}} \right) (M/N_i)^{n-(i+1)} \), \( M = g(k)(H_{k,n} - H_{k,0}) \) and \( N = H_{k,1} - H_{k,n+1} \).

**Proof.** The proofs of theorem can be done similarly by considering Proposition 2.1. ■

**Corollary 2.1** In Theorem 2.3, for special choices of \( a, b, f(k) \) and \( g(k) \), the following result can be obtained for well-known number sequences in literature:

• If \( f(k) = 1, g(k) = 1, a = 0 \) and \( b = 1 \), for the classic Fibonacci sequence, we obtain

\[
C_{n,g}^{-1}(F) = \left[ \begin{array}{c}
\frac{1}{f_n} \text{circ}(1 + \sum_{i=1}^{n-2} \frac{F_{n-i-1}^2}{(F_1-F_{n+1})^2}, -1 + \sum_{i=1}^{n-2} \frac{F_{n-i-1}^2}{(F_1-F_{n+1})^2}, -\frac{1}{(F_1-F_{n+1})^2},
\frac{F_n^2}{(F_1-F_{n+1})^2}, -\frac{F_n^2}{(F_1-F_{n+1})^2}, \ldots, -\frac{F_n^2}{(F_1-F_{n+1})^2},
\end{array} \right] Q_g^T,
\]

where \( f_n = F_1 - F_n + \sum_{i=1}^{n-2} F_i \left( \frac{F_n}{F_1-F_{n+1}} \right)^{n-(i+1)} \).
If \( f(k) = 1, g(k) = 1, a = 2 \) and \( b = 1 \), for the classic Lucas sequence, we obtain

\[
C_{n,g}^{-1}(L) = \begin{bmatrix}
\frac{1}{a} circ(1 + \sum_{i=1}^{n-2} \frac{(L_{i+2} - 3L_{i+1}) (L_{i+2} - 2)^{i-1}}{(L_1 - L_{n+1})^i)} \\
-3 + \sum_{i=1}^{n-2} \frac{(L_{i+2} - 3L_{i+1}) (L_{i+2} - 2)^{i-1}}{(L_1 - L_{n+1})^i) \cdot \frac{5}{L_1 - L_{n+1}}}
\end{bmatrix} \cdot Q_g^T,
\]

where \( l_n = L_1 - 3L_n + \sum_{i=1}^{n-2} (L_{i+2} - 3L_{i+1}) \left( \frac{L_{n+2} - 2}{L_1 - L_{n+1}} \right)^{(i+1)} \).

If \( f(k) = 2, g(k) = 1, a = 0 \) and \( b = 1 \), for the classic Pell sequence, we obtain

\[
C_{n,g}^{-1}(P) = \begin{bmatrix}
\frac{1}{a} circ(1 + \sum_{i=1}^{n-2} \frac{P_{n-i} p_{n-i-1}}{(P_{n-i} - P_{n-i+1})^i), -2 + \sum_{i=1}^{n-2} \frac{P_{n-i} p_{n-i-1}}{(P_{n-i} - P_{n-i+1})^i), \frac{p_{n-1}}{P_{n-1}}}
\end{bmatrix} \cdot Q_g^T,
\]

where \( p_n = P_1 - 2P_n + \sum_{i=1}^{n-2} P_i \left( \frac{P_n}{P_{n-1}} \right)^{n-(i+1)} \).

If \( f(k) = 1, g(k) = 2, a = 0 \) and \( b = 1 \), for the classic Jacobsthal sequence, we obtain

\[
C_{n,g}^{-1}(J) = \begin{bmatrix}
\frac{1}{s_n} circ(1 + \sum_{i=1}^{n-2} \frac{J_{n-i} (2J_n)^{i-1}}{(J_{n-i} - J_{n-i+1})^i), -1 + 4 \sum_{i=1}^{n-2} \frac{J_{n-i} (2J_n)^{i-1}}{(J_{n-i} - J_{n-i+1})^i), \frac{2}{J_{n-1}}}
\end{bmatrix} \cdot Q_g^T,
\]

where \( s_n = J_1 - J_n + 2 \sum_{i=1}^{n-2} J_i \left( \frac{2J_n}{J_{n-1}} \right)^{n-(i+1)} \).

If \( f(k) = p \) and \( g(k) = q \), for the classic Horadam sequence, we obtain

\[
C_{n,g}^{-1}(W) = \begin{bmatrix}
\frac{1}{z_n} circ(1 + \sum_{i=1}^{n-2} \frac{w_{n+2-i} - \frac{w_2}{w_1} w_{n+1-i}}{w_{n+1-i} - \frac{w_2}{w_1} w_{n+2-i}) A^{i-1}} \frac{4^{i-1}}{B^i},
-\frac{w_2}{w_1} + q \sum_{i=1}^{n-2} \frac{w_{n+1-i} - \frac{w_2}{w_1} w_{n+2-i}) A^{i-1}} \frac{4^{i-1}}{B^i}, - \frac{w_3 - \frac{w_2}{w_1}}{B^2}
\end{bmatrix} \cdot Q_g^T,
\]

where \( z_n = -\frac{w_2 w_{n-1}}{w_1} + w_1 + \sum_{i=1}^{n-2} \left( -\frac{w_2 w_{n+1-i}}{w_1} + w_{i+2} \right) \left( \frac{A}{B} \right)^{n-(i+1)}, A = q(w_n - w_0) \) and \( B = w_1 - w_{n+1} \).

Finally, we should note that choosing suitable values on \( f(k), g(k), a \) and \( b \) in Theorem 2.3, it is actually obtained inverse of \( g \)-circulant matrix for the others second order sequences such as Pell-Lucas, Jacobsthal-Lucas, \( k \)-Fibonacci sequences.
Conclusion 2.1 In this paper, we introduced the $g$-circulant matrix with the generalized $k$-Horadam numbers and presented some properties of this matrix. By the results in Sections 2 of this paper, we have a great opportunity to obtain norm, determinant and inverse of the circulant matrices with second order number sequences. Thus, we extend some recent result in the literature.

In the future studies on the circulant matrix for number sequences, we except that the following topics will bring a new insight. For example, it would be interesting to study the $g$-circulant matrix for third order number sequences.

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