BLOW-UP IN FINITE OR INFINITE TIME OF THE 2D CUBIC ZAKHAROV-KUZNETSOV EQUATION

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Abstract. We prove that near-threshold negative energy solutions to the 2D cubic ($L^2$-critical) focusing Zakharov-Kuznetsov (ZK) equation blow-up in finite or infinite time. The proof consists of several steps. First, we show that if the blow-up conclusion is false, there are negative energy solutions arbitrarily close to the threshold that are globally bounded in $H^1$ and are spatially localized, uniformly in time. In the second step, we show that such solutions must in fact be exact remodulations of the ground state, and hence, have zero energy, which is a contradiction. This second step, a nonlinear Liouville theorem, is proved by contradiction, with a limiting argument producing a nontrivial solution to a (linear) linearized ZK equation obeying uniform-in-time spatial localization. Such nontrivial linear solutions are excluded by a local-viral time estimate. The general framework of the argument is modeled on Merle [29] and Martel & Merle [24], who treated the 1D problem of the $L^2$-critical gKdV equation. Several new features are introduced here to handle the 2D ZK case.

1. Introduction

We consider the generalized Zakharov-Kuznetsov equation

\begin{equation}
\tag{gZK}
\partial_t u + \partial_x (\Delta u + u^p) = 0,
\end{equation}

where $u(t,x,y_1,\ldots,y_{N-1}) \in \mathbb{R}$, where $(x,y_1,\ldots,y_{N-1}) \in \mathbb{R}^N$, $t \in \mathbb{R}$, and $\Delta = \partial_x^2 + \partial_{y_1}^2 + \cdots + \partial_{y_{N-1}}^2$. Specifically, we consider two dimensions ($N = 2$) and with a specific power of nonlinearity $p = 3$. This equation is the higher-dimensional extension of the well-studied model describing, for example, the weakly nonlinear waves in shallow water, the Korteweg-de Vries equation:

\begin{equation}
\tag{KdV}
\partial_t u + \partial_x^2 u + \partial_x(u^p) = 0, \quad p = 2, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}.
\end{equation}

When other integer powers $p \neq 2$ are considered, it is referred to as the generalized KdV (gKdV). Despite its apparent universality, the gKdV equation is limited as a spatially one-dimensional model. While there are several higher dimensional generalizations of it, in this paper we are interested in the gZK equation (1.1). In the three dimensional setting and quadratic power ($N = 3$ and $p = 2$), the equation (1.1) was

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originally derived by Zakharov and Kuznetsov to describe weakly magnetized ion-acoustic waves in a strongly magnetized plasma [40], thus, the name of the equation. In two dimensions, it is also physically relevant; for example, with \( p = 2 \), it governs the behavior of weakly nonlinear ion-acoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field [31, 32]. Melkonian and Maslowe [28] showed that the equation (1.1) is the amplitude equation for two-dimensional long waves on the free surface of a thin film flowing down a vertical plane with moderate values of the surface fluid tension and large viscosity. Lannes, Linares & Saut in [18] derived the equation (1.1) from the Euler-Poisson system with magnetic field in the long wave limit, yet another derivation was carried by Han-Kwan in [11] from the Vlasov-Poisson system in a combined cold ions and long wave limit.

In this paper we consider \( H^1 \) solutions to the 2D cubic focusing Zakharov-Kuznetsov equation on \( \mathbb{R}^2 \)

\[
\partial_t u + \partial_x (\Delta u + u^3) = 0,
\]

where \( \Delta = \partial_x^2 + \partial_y^2 \) is the 2D Laplacian. Such solutions have a maximal forward lifespan \([0, T)\) where either \( T < +\infty \) or \( T = \infty \). It follows from the local well-posedness theory\(^1\)(see Faminski [6], Linares & Pastor [20], Ribaud & Vento [33]) that if \( T < +\infty \), then

\[
\lim_{t \uparrow T} \| \nabla u(t) \|_{L^2_{xy}} = +\infty.
\]

If \( T = +\infty \), then either

\[
\lim_{t \uparrow T} \| \nabla u(t) \|_{L^2_{xy}} = +\infty \quad \text{or} \quad \liminf_{t \uparrow T} \| \nabla u(t) \|_{L^2_{xy}} < +\infty
\]

is a priori possible. In either case \( T < +\infty \) or \( T = +\infty \), we say that \( u(t) \) blows-up at forward time \( T \) if

\[
\liminf_{t \uparrow T} \| \nabla u(t) \|_{L^2_{xy}} = +\infty
\]

During their lifespan \([0, T)\), solutions \( u(t, x, y) \) to ZK conserve mass and energy:

\[
M(u(t)) = \int_{\mathbb{R}^2} u^2(t) \, dx \, dy = M(u(0))
\]

and

\[
E(u(t)) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u(t)|^2 \, dx \, dy - \frac{1}{4} \int_{\mathbb{R}^2} u^4(t) \, dx \, dy = E(u(0)).
\]

\(^1\)In fact, Linares & Pastor [20] obtain local well-posedness in \( H^s(\mathbb{R}^2) \) for \( s > \frac{3}{4} \), and Ribaud & Vento [33] obtain local well-posedness for \( s > \frac{1}{4} \).
Similar to the gKdV equation, for solutions $u(t, x, y)$ decaying at infinity on $\mathbb{R}^2$ the following invariance holds

$$\int_{\mathbb{R}} u(t, x, y) \, dx = \int_{\mathbb{R}} u(0, x, y) \, dx,$$

which is obtained by integrating the original equation on $\mathbb{R}$ in the first coordinate $x$.

One of the useful symmetries in the evolution equations is the scaling invariance, which states that an appropriately rescaled version of the original solution is also a solution of the equation. For the equation (1.1) it is

$$u_\lambda(t, x, y) = \lambda^{\frac{2}{p-1}} u(\lambda^3 t, \lambda x, \lambda y).$$

This symmetry has the following effect on the Sobolev norm:

$$\| u_\lambda(0, \bullet, \bullet) \|_{H^s} = \lambda^{\frac{2}{p-1} + s - 1} \| u_0 \|_{H^s},$$

and the index $s$ gives rise to the critical-type classification of equations. For the gKdV equation (1.2) the critical index is $s = \frac{1}{2} - \frac{2}{p-1}$, and for the gZK equation (1.1) it is $s = 1 - \frac{2}{p-1}$. When $s > 0$ (in 2d gZK equation this corresponds to $p > 3$), the equation (1.1) is often referred as the $(L^2)$ supercritical equation. The gZK equation has other invariances such as space and time translation, time reversal symmetry, and sign invariance. We also note that gZK is not integrable.

The gZK equation has a family of traveling waves (or solitary waves, which sometimes are referred even as solitons), and observe that they travel only in $x$ direction

$$u(t, x, y) = Q_c(x - ct, y)$$

with $Q_c(x, y) \to 0$ as $|x| \to +\infty$. Here, $Q_c$ is the dilation of the ground state $Q:

$$Q_c(x, y) = c^{1/p-1}Q(c^{1/2}(x, y))$$

with $Q$ being a radial positive solution in $H^1(\mathbb{R}^2)$ of the well-known nonlinear elliptic equation $-\Delta Q + Q - Q^p = 0$. Note that $Q \in C^\infty(\mathbb{R}^2)$, $\partial_r Q(r) < 0$ for any $r = |(x, y)| > 0$ and for any multi-index $\alpha$

$$|\partial^\alpha Q(x, y)| \leq c(\alpha)e^{-r} \quad \text{for any } (x, y) \in \mathbb{R}^2.$$

It follows from energy and mass conservation and the Weinstein inequality that if $\|u\|_{L^2} < \|Q\|_{L^2}$, where $Q$ is the ground state, then

$$\|\nabla u(t)\|_{L^2}^2 \leq \frac{2E(u)}{1 - \frac{\|u\|_{L^2}^2}{\|Q\|_{L^2}^2}}.$$

Thus in this context solutions do not blow-up in finite or infinite time, so blow-up is only possible for solutions with $\|u\|_{L^2} \geq \|Q\|_{L^2}$.

In her study of dispersive solitary waves in higher dimensions, de Bouard [5] showed (in dimensions 2 and 3) that the travelling waves of the form (1.6) are stable for $p < p_c$ and unstable for $p > p_c$, where $p_c = 3$ in 2d. She followed the ideas developed for the
gKdV equation by Bona, Souganidis & Strauss [2] for the instability, and Grillakis, Shatah & Strauss [10] for the stability arguments. The first three authors of the present paper proved in [8] the instability of the traveling wave solutions of the form (1.6) for the \( p = 3 \) case, thus, completing the stability picture for the two-dimensional ZK equation (see also the alternative proof of instability for the \( L^2 \)-supercritical gZK, i.e., \( p > 3 \), in [7]). We also note that the more delicate questions about different types of stability have been previously studied; e.g., see Côte, Muñoz, Pilod & Simpson [4].

Once instability is demonstrated, one can consider whether solutions in fact blow-up. Unlike other dispersive equations such as the nonlinear Schrödinger equation (NLS), the KdV-type equations do not have a convenient virial identity which usually gives a straightforward proof of existence of blow-up solutions. The absence of a good virial identity makes it very difficult to prove the existence of finite time blow up solutions in the gKdV-type equations, although intuitively for large nonlinearities such solutions should exists, [10], also, see numerical investigations in [1]. (The exception is gKP-I, where a virial-type identity shows the existence of blow-up; for \( p \geq 4 \) see [37], also [34], a refined version down to the (critical) power \( p \geq 4/3 \) is in [21].) The breakthrough for the gKdV \( (p = 5) \) in terms of proving the existence of blow-up solutions was obtained by Merle and Martel & Merle in [24, 29], and further description in subsequent papers. In particular, they obtain the explicit construction of blow up solutions close to solitons. The proof of the finite time blow up was then developed in [25] provided initial data has certain spatial decay, an upper estimate on the blow up rate was also given; later the universality of blow up profile and lower bound estimates was obtained in [26].

It is worth noting that the blow up solutions were exhibited in the (usually difficult) critical borderline case \( p = 4 \), instead of supercritical case; one explanation is that the proof heavily relies on the spectral properties of the linearized operator which is well understood in the critical case \( p = 4 \) unlike the case when \( p > 4 \).

The only extension of this blow-up theory to other KdV-type equations is [14], where authors show the existence of blow-up solutions in the finite or infinite time for the dispersion generalized \( L^2 \)-critical Benjamin-Ono equation (dgBO) via a perturbation of the \( L^2 \)-critical gKdV theory.

The main result of this paper is

**Theorem 1.1 (main theorem).** There exists \( \alpha_0 > 0 \) such that the following holds. Suppose that \( u(t) \) is an \( H^1 \) solution to (ZK) with \( E(u) < 0 \) and

\[
\alpha(u) \overset{\text{def}}{=} \|u\|^2 - \|Q\|^2 \leq \alpha_0.
\]

Then \( u(t) \) blows-up in finite or infinite forward time.

Note that \( E(u) < 0 \) implies that \( \alpha(u) > 0 \) by the Weinstein inequality. Also note that by time reversal symmetry, the result applies to negative time as well.
The entire paper is devoted to the proof of Theorem 1.1, which consists of several steps. The first step, Prop. 1.2, under the assumption that the theorem is false, is the construction of a sequence of well-behaved solutions \( \tilde{u}_n \) of (ZK), which are ultimately shown not to exist in Prop. 1.3.

**Proposition 1.2** (reduction to uniformly-in-time \( H^1 \) bounded and spatially localized sequence). If the statement of Theorem 1.1 is false, then there exists a sequence of \( H^1 \) solutions \( \tilde{u}_n(t) \) to (ZK) such that \( E(\tilde{u}_n) < 0 \) and \( \alpha(\tilde{u}_n) \to 0 \) such that

1. (global existence and bounds) each \( \tilde{u}_n(t) \) exists globally in time such that for all \( t \),
   \[
   \frac{1}{2} \| \nabla Q \|_{L^2} \leq \| \tilde{u}_n(t) \|_{L^2} \leq 2 \| \nabla Q \|_{L^2},
   \]

2. (uniform-in-time spatial localization) there exists a path \((\tilde{x}_n(t), \tilde{y}_n(t))\) and a scale \( \tilde{\lambda}_n(t) \) such that the remainder function
   \[
   \tilde{\epsilon}_n(x, y, t) := \tilde{\lambda}_n(t) \tilde{u}_n(\tilde{\lambda}_n(t)x + \tilde{x}_n(t), \tilde{\lambda}_n(t)y + \tilde{y}_n(t)) - Q(x, y),
   \]
   satisfies the orthogonality conditions \( \langle \tilde{\epsilon}_n, \nabla Q \rangle = 0 \), \( \langle \tilde{\epsilon}_n, Q^3 \rangle = 0 \) and the uniform-in-time spatial localization: for each \( r > 0 \),
   \[
   \| \tilde{\epsilon}_n \|_{L_t^{\infty} L_{x,y}^2} \lesssim e^{-\omega r} \| \tilde{\epsilon}_n \|_{L_t^{\infty} L_{x,y}^2}
   \]
   where \( B(0, r) \) denotes the ball, centered at 0 and of radius \( r > 0 \) in \( \mathbb{R}^2 \), \( B(0, r)^c \) denotes the complement in \( \mathbb{R}^2 \), and \( \omega > 0 \) is some absolute constant.

First, a preliminary \( x \)-compactness property is obtained – there exists a path \( \tilde{x}_n(t) \) such that for all \( x \) and \( t \),
   \[
   \| \tilde{u}_n(x + \tilde{x}_n(t), y, t) \|_{L_y^2} \lesssim C_n^{1/2} e^{-|x|/32}
   \]
This is used to proof (1) in the statement of Prop. 1.2. With (1) on hand, the stronger localization in (2) is obtained. We next obtain the following rigidity statement (we keep the notation of a sequence \( \tilde{u}_n \) rather than \( u_n \) to remain consistent with the previous proposition).

**Proposition 1.3** (nonlinear Liouville property). Given a sequence of \( H^1 \) solutions \( \tilde{u}_n(t) \) to (ZK) such that \( 0 \leq \alpha(\tilde{u}_n) \to 0 \) satisfying properties (1) and (2) of Prop. 1.2, then the following holds. For \( n \) sufficiently large, there exists constants \( \tilde{\lambda}_n > 0 \), \((\tilde{x}_n, \tilde{y}_n) \in \mathbb{R}^2 \) such that
   \[
   \tilde{\lambda}_n \tilde{u}_n(\tilde{\lambda}_n x + \tilde{x}_n, \tilde{\lambda}_n y + \tilde{y}_n, t) = Q(x, y).
   \]
This proposition implies \( \tilde{\lambda}^2 E(\tilde{u}_n) = E(Q) = 0 \), which contradicts \( E(\tilde{u}_n) < 0 \) for Prop. 1.2, completing the proof of Theorem 1.1.

The proof of Prop. 1.3 involves two steps. First, assuming the conclusion of Prop. 1.3 is false, the proof of the convergence of remainders \( \tilde{\epsilon}_n \) (after passing to a
subsequence) to a nontrivial solution to a linear linearized ZK equation exhibiting uniform-in-time spatial localization. This is given in Prop. 12.1. On the other hand, any uniformly-in-time spatially localized solution to the linear linearized ZK equation must be trivial by Prop. 13.1, a linear Liouville property.

1.1. Outline. We start with an outline of the proof of Prop. 1.2. The arguments take place in the context of “near threshold and negative energy”, for which there is a modulational characterization of solutions given by Lemmas 4.1, 4.2, 4.3 below. It states that for every such solution \( u(t) \) of (ZK), there exists parameters of scale \( \lambda(t) > 0 \) and position \( (x(t), y(t)) \in \mathbb{R}^2 \) such that the remainder

\[
\epsilon(x, y, t) = \lambda(t)u(\lambda(t)x + x(t), \lambda(t)y + y(t), t) - Q(x, y)
\]

is small in \( H^1_{xy} \).

While we are following the general pattern of argument introduced by Merle \[29\] to address the corresponding problem for \( L^2 \)-critical gKdV equation, there are a few aspects in which we needed to make significant alternations to handle the 2D \( L^2 \)-critical ZK setting, which are indicated in the discussion below.

The negation of Theorem 1.1 yields a sequence of ZK solutions \( u_n \), which, after renormalization, have the property that \( \alpha(u_n) \to 0 \) as \( n \to \infty \) and for each \( n \), \( E(u_n) < 0 \) (they are near threshold and negative energy solutions) and for all \( t \geq 0 \),

\[
(1 - \frac{1}{n})\|\nabla Q\|_{L^2_{xy}} \leq \|\nabla u_n(t)\|_{L^2_{xy}}.
\]

There is no \textit{a priori} upper bound on \( \|\nabla u_n(t)\|_{L^2_{xy}} \), although we are granted the existence of sequence of times \( t_{n,m} \to +\infty \) such that

\[
\lim_{m \to +\infty} \|\nabla u_n(t_{n,m})\|_{L^2_{xy}} = \|\nabla Q\|_{L^2_{xy}}.
\]

For each \( u_n(t) \) and the time sequence \( t_{n,m} \to +\infty \), we can extract a weak limit

\[
u_n(\bullet + x(t_{n,m}), \bullet + y(t_{n,m}), t_{n,m}) \rightharpoonup \tilde{u}_n(\bullet, \bullet, 0).
\]

in \( H^1_{xy} \). The properties that \( \alpha(\tilde{u}_n) \leq \alpha(u_n) \) and \( E(\tilde{u}_n) < 0 \), proved in Lemma 5.1, are inherited from the weak limit and modulational characterization of the near threshold solutions.

Taking \( \tilde{u}_n(t) \) to be the ZK evolution of initial data \( \tilde{u}_n(0) \), we have a result on the stability of weak convergence (Lemma 5.3)

\[
u_n(x + x_n(t_{n,m} + t), y + y_n(t_{n,m} + t), t_{n,m} + t) \rightharpoonup \tilde{u}_n(x + \tilde{x}_n(t), y + \tilde{y}_n(t), t).
\]

We then show that the weak limiting process effectively strips away radiation leaving \( \tilde{u}_n(t) \) with uniform in time, exponential decay in the \( x \)-spatial direction. This is a consequence of monotonicity estimates (Lemma 6.2) analogous to that of Merle \[29\] for gKdV. It is key here that no upper bound on \( \|\nabla u_n(t)\|_{L^2_{xy}} \) is required (equivalently, no lower bound on \( \lambda_n(t) \)). To adapt the monotonicity estimates to the 2D context,
we needed a Gagliardo-Nirenberg estimate with a spatial weight in an external region (Lemma 6.1). The monotonicity estimates are applied on long time scales for the solutions $u_n(t)$, which has the implication of decay for the weak limit $\tilde{u}_n(t)$ owing to the convergence (1.9).

With the decay estimates on hand, we can now use an integral type conservation law. For smooth, rapidly decaying solutions to (ZK), one can verify by direct computation from the (ZK) equation that

$$\partial_t \int_x u(x, y, t) \, dx = 0$$

for each $y \in \mathbb{R}$, and thus,

$$\int_x u(x, y, t) \, dx$$

is constant in time.

Our solutions $\tilde{u}_n(t)$ do not have a high level of regularity, but at least do belong to $L^2_yL^1_x$ uniformly in time, from our decay estimates, and thus, the ability to approximate solutions by regular solutions (a consequence of the local theory machinery) yields that

$$\left\| \int_x u(x, y, t) \, dx \right\|_{L^2_y}$$

is constant in time.

This provides a means for controlling $\|\nabla \tilde{u}_n(t)\|_{L^2_y}$ from above (equivalently, the scale parameter $\tilde{\lambda}_n(t)$ from below). This gives all the properties of $\tilde{u}_n(t)$ claimed in Prop. 1.2.

Let us highlight, at the more technical level, two ways in which our proof adds new elements to the method of Merle [29]. The “stability of weak convergence” (Lemma 5.3) is the statement that if $v_m(0) \rightarrow v(0)$ weakly in $H^1$, then $v_m(t) \rightarrow v(t)$ weakly in $H^1$, where $v_m(t)$ and $v(t)$ are the ZK flows of $v_m(0)$ and $v(0)$, respectively. This was done in Appendix D of Martel & Merle [24] for gKdV, and they used the fact that an $L^2$ local theory is available, which removes the need for an $a$ priori bound on the $H^1$ norm of $v_m(t)$. In our case, an $L^2$ local theory for (ZK) is not available, and we need to assume an $a$ priori bound on the $H^1$ norms of $v_m(t)$, which means that we need to strengthen the bootstrap hypothesis on the admissible time scale $(-t_1(n), t_2(n))$, see (5.2), to include this assumption. In the end, the argument still works, since the convergence of $\lambda_m(t) \rightarrow \lambda(t)$ and local theory estimates as employed in Lemma 5.2 imply that a strengthening of the left part of the bootstrap hypothesis (5.2) implies a corresponding strengthening of the right part.

In the monotonicity estimates (Lemma 6.2), we need a Gagliardo-Nirenberg estimate (Lemma 6.1) with spatial weight in an external spatial region. For this we found a nice classical proof, interating the standard 1D estimate $\|f\|_{L^\infty}^2 \leq \|f\|_{L^2} \|f'\|_{L^2}$ with the weight appropriately distributed between the two terms. It was necessary to have an estimate that positioned the weight fully on the gradient term in the right-side
we could use the method employed by [4] of passing through $H^1$ estimates, since then the $\alpha$ threshold of the result would be $H^1$ upper bound dependent, which is not suitable for our purposes. It was suitable in the asymptotic stability context of [4] but not in our blow-up context, where there is no control on scale.

Let us now give some details on the proof of Prop. 1.3. We note that the conclusion of Prop. 1.3 is equivalent to $\bar{\epsilon}_n \equiv 0$. Indeed, if $\bar{\epsilon}_n \equiv 0$, we have for all $t$ that

$$\tilde{\lambda}_n(t) \tilde{u}_n(\tilde{\lambda}_n(t)x + \tilde{x}_n(t), \tilde{\lambda}_n(t)y + \tilde{y}_n(t), t) = Q(x, y).$$

Moreover, the estimates for parameters given in (4.4) yield, in this case, that $(\tilde{\lambda}_n)_t = 0, (\tilde{y}_n)_t = 0$ and $(\tilde{x}_n)_t = \tilde{\lambda}_n^{-2}$, from which it follows that $\tilde{\lambda}_n$ and $\tilde{y}_n$ are constant, and $\tilde{x}_n = \lambda_n^{-2}t + \bar{x}_n$. Substituting, we obtain the conclusion as stated in Prop. 1.3.

We begin the proof of Prop. 1.3 by negating the conclusion, obtaining that there exists a subsequence (still labeled with $n$-subscript) such that $\bar{\epsilon}_n \not\equiv 0$ for all $n$. We renormalize this sequence as follows. Let $b_n = \|\bar{\epsilon}_n\|_{L^\infty_t L^2_{x,y}}$, which by our construction makes $b_n > 0$ for all $n$. Let $t_n \in \mathbb{R}$ be a time so that $\|\bar{\epsilon}_n(t_n)\|_{L^2_{x,y}} \geq \frac{1}{2}b_n$. Then consider

$$(1.10) \quad w_n(t) = \frac{\bar{\epsilon}_n(t + t_n)}{b_n}.$$  

By Prop. 11.1, we in fact have that $\|\bar{\epsilon}_n\|_{L^\infty_t H^1_{x,y}} \sim \|\bar{\epsilon}_n\|_{L^\infty_t L^2_{x,y}}$, and hence, $\|w_n\|_{L^\infty_t H^1_{x,y}} \lesssim 1$ for all $n$. Moreover, by Prop. 1.2(2), it follows that $w_n$ has a uniform-in-time spatial localization: for each $r > 0$,  

$$\|w_n\|_{L^\infty_t L^2_{B(0,r)^c}} \lesssim e^{-\omega r},$$

where $B(0, r)$ denotes the ball, centered at 0 and of radius $r > 0$ in $\mathbb{R}^2$, $B(0, r)^c$ denotes the complement in $\mathbb{R}^2$, and $\omega > 0$ is some absolute constant.

By the Rellich-Kondrachov theorem, we can pass to a subsequence (still labeled with index $n$) so that $w_n(0)$ converges strongly in $L^2_{xy}$. Denoting the limit by $w_\infty(0)$, it follows that $\|w_\infty(0)\|_{L^2_{x,y}} \geq \frac{1}{2}$ from the fact that $\|\bar{\epsilon}_n(t_n)\|_{L^2_{x,y}} \geq \frac{1}{2}b_n$. Moreover, we prove, in Prop. 12.1 that for each $T > 0$, the strong convergence

$$w_n(t) \rightarrow w_\infty(t) \quad \text{in} \quad C([-T, T]; L^2_{x,y})$$

holds, where $w_\infty$ solves the linear linearized ZK equation (12.1). The orthogonality conditions and uniform-in-time spatial decay are inherited in the limit, i.e.,

$$\|w_\infty\|_{L^\infty_t L^2_{B(0,r)^c}} \lesssim e^{-\omega r}.$$  

A contradiction is obtained by appealing to the linear Liouville theorem, Prop. 13.1 which forces that $w_\infty \equiv 0$. The proof of Prop. 13.1 proceeds by deducing the additional orthogonality condition $\langle w_\infty, Q \rangle = 0$, from which it follows that $\langle Lw_\infty, w_\infty \rangle$ is constant in time (here, $L = 1 - \Delta - 3Q^2$ is the linearized operator), and thus, the result follows from a dispersive estimate, the local virial estimate Lemma 14.1

$$\|w_\infty\|_{L^2 H^3_{x,y}} \lesssim \|\langle x \rangle^{1/2} w_\infty\|_{L^\infty_t L^2_{x,y}}.$$
This type of estimate is ordinarily proved by a positive commutator argument leading to a spectral coercivity estimate for a Schrödinger operator \( \tilde{L} \) (different from \( L \)). If this is applied directly in this case, the corresponding \( \tilde{L} \) does not satisfy the needed coercivity estimate. To escape this problem, we pass to the adjoint problem \[ v = (1 - \delta \Delta)^{-1} L w_\infty, \]
for which the positive commutator argument yields an operator \( \tilde{\tilde{L}} \) that does indeed satisfy the coercivity estimate. This technique was introduced in the gKdV setting by Martel [23, p. 775]. The operator \( \tilde{\tilde{L}} \) is of standard Schrödinger type but with a rank 2 perturbation. The spectral coercivity estimate is checked numerically, as described in §16, giving the local viral estimate Lemma 14.2
\[ \|v\|_{L^2_t H^1_y} \lesssim \|\langle x \rangle^{1/2} v\|_{L^\infty_t L^2_y}. \]
The estimates allowing the conversion from \( w_\infty \) to \( v \) are given in §15.

Prop. 1.2 (2) combined with Prop. 1.3 were established as a “nonlinear Liouville theorem” in the case of \( L^2 \)-critical gKdV by Martel & Merle [24]. Our proof uses some of the same elements adapted to the 2D case – for example the comparability of \( L^2 \) and \( H^1 \) norms of the remainder functions \( \tilde{\epsilon}_n \) (see Prop. 11.1) proved by a virial-type estimate, and the convergence of renormalized remainders (1.10) to \( w_\infty \), solving the linear linearized equation (see Prop. 12.1), proved via adaptation of the local theory estimates in an appropriate reference frame. However, our proof differs in the follows aspects. The decay estimate with sharp coefficient (as in our Prop. 1.2 (2)) is proved in the gKdV case by Martel & Merle [24] by appealing to the \( L^2 \)-critical scattering theory available for that equation (Kenig, Ponce & Vega [15]). No such result is yet available for 2D ZK, so we instead prove a monotonicity estimate directly on the remainder \( \epsilon \) (rescaled to \( \eta \)) – see Lemma 10.1 and Corollary 10.2. This type of calculation was previously done by Martel & Merle [27, Claim 14] in the gKdV context, and we combine it here with our rotation method (see §9) for obtaining the decay in \( y \)-direction. Finally, we prove the linear virial estimate, Lemma 14.1, key to the proof of the linear Liouville property (Prop. 13.1), by transforming from \( w \) to \( v = (1 - \delta \Delta)^{-1} L w \), and addressing the analogous estimate for \( v \), which appears as Lemma 14.2. In [24], this method was not used and the estimate for \( w \) was proved directly, essentially by calculating \( \partial_y \int xw^2 \, dx \), which leads to a positivity estimate for a Schrödinger operator \( \tilde{L} \). If this were done in our case of 2D ZK, by computing \( \partial_y \int x w^2 \, dx \, dy \), we would obtain a Schrödinger operator \( \tilde{L} \), for which the positivity estimate seems to fail (as suggested by our numerics). The method of converting from \( w \) to \( v \) was introduced by Martel [23] in the gKdV context, where the transformation \( v = L w \) is used. The addition of the regularization operator, \( v = (1 - \delta \Delta)^{-1} L w \) was used by Kenig & Martel [13] in their treatment of asymptotic stability for the Benjamin-Ono equation. We use it here, together with a few additional commutator
arguments, that we establish in §15. For the 2D quadratic ZK, asymptotic stability was established by Côte, Muñoz, Pilod & Simpson [4] using a virial estimate established with the transformation $v = Lw$, without the use of regularization (and thus, requiring additional higher regularity estimates). A key difference between [4] and our paper is the choice of orthogonality conditions. The orthogonality conditions used in [4] do not seem to work in our context, as they are not sufficient to establish the spectral coercivity of $\tilde{L}$ for nonlinearities $\partial_x(|u|^{p-1}u)$ outside $1.8 < p < 2.1491$ as remarked in [4, Appendix A.2.2]. Our different choice of orthogonality conditions leads to a different linear operator $\tilde{L}$, which includes a rank two projection operator. We rely on numerical methods, as detailed in §16, and an angle lemma (Lemma 14.3) to confirm the positivity of $\tilde{L}$.

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2. Notation

We use

$$\alpha(u) = \|u\|^2_{L^2} - \|Q\|^2_{L^2}$$

consistently throughout the paper.

Many results apply generally for solutions $u(t)$ to ZK that are “near threshold negative energy”, that is, there exists $\alpha' > 0$ (small) such that for all $H^1$ solutions $u(t)$ such that $\alpha(u) < \alpha'$ and $E(u) < 0$, the result applies on the maximal lifespan of the solution $u(t)$ (perhaps together with some other hypotheses). For such solutions, Lemmas 4.1, 4.2 and 4.3 apply, giving a description of the solution in terms of modulation parameters of scale $\lambda(t) > 0$ and position $(x(t), y(t))$ and a remainder function

$$\epsilon(x, y, t) \overset{\text{def}}{=} \lambda(t)u(\lambda(t)x + x(t), \lambda(t)y + y(t), t) - Q(x, y)$$

with dynamical information about the parameters and remainder function given by Lemma 4.3. Results that apply in this general situation include the monotonicity estimate, Lemma 6.2 and the integral conservation law yielding scale control, Lemma
For each result of this type, one needs $\alpha(u)$ sufficiently small, so that we introduce a list of thresholds

$$\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 \geq \alpha_5 > 0.$$ 

Each new threshold $\alpha_j > 0$ is taken smaller than the previously introduced thresholds $\alpha_1, \ldots, \alpha_{j-1}$, so that the earlier results apply as well.

The linearized operator is defined as

$$\mathcal{L} \overset{\text{def}}{=} -\Delta + 1 - 3Q^2,$$

where $Q$ is the unique (up to translation) radial positive solution in $H^1(\mathbb{R}^2)$ of the well-known nonlinear elliptic equation

$$-\Delta Q + Q - Q^3 = 0.$$ 

We also introduce the generator $\Lambda$ of scaling symmetry

$$\Lambda f = f + x\partial_x f + y\partial_y f.$$ 

3. **Local theory estimates and ground state properties**

Let $U(t)\phi$ denote the solution to the linear homogenous problem

$$\begin{cases}
\partial_t \rho + \partial_x \Delta \rho = 0 \\
\rho(t_0, x, y) = \phi(x, y).
\end{cases}$$

Then

$$u(t) = U(t)\phi + \int_0^t U(t-t') \partial_x [u(t')^3] \, dt'.$$

**Lemma 3.1** (linear homogeneous estimates). We have

1. $\|U(t)\phi\|_{L^\infty_x H^1_y} \lesssim \|\phi\|_{H^1_y},$
2. $\|\partial_x U(t)\phi\|_{L^x_t L^2_y} \lesssim \|\phi\|_{L^2_y}.$

For $0 < T \leq 1$,

3. $\|U(t)\phi\|_{L^4_x L^\infty_y} \lesssim \|\phi\|_{H^1_y}.$

**Proof.** The first estimate is a standard consequence of Plancherel and Fourier representation of the solution. The second estimate (local smoothing) is Faminskii [6], Theorem 2.2 on p. 1004. The third estimate (maximal function estimate) is a special case ($s = 1$) of Faminskii [6], Theorem 2.4 on p. 1007. All of these estimates are used by Linares-Pastor [20], and quoted as Lemma 2.7 on p. 1326 of that paper. 

**Lemma 3.2** (linear inhomogeneous estimates). For $0 < T \leq 1$,

1. $\left\| \int_0^t U(t-t') \partial_x f(t') \, dt' \right\|_{L^\infty_x H^1_y \cap L^4_x L^\infty_y} \lesssim \|\partial_x f\|_{L^4_x L^2_y} + \|\partial_y f\|_{L^4_x L^2_y},$
2. $\left\| \int_0^t U(t-t') f(t') \, dt' \right\|_{L^\infty_x H^1_y \cap L^4_x L^\infty_y} \lesssim \|f\|_{L^4_x H^1_y}.$
Proof. These follow from Lemma \[3.1\] by duality, \(T^*T\), and the Christ-Kiselev lemma. The needed version of the Christ-Kiselev lemma is provided by Molinet & Ribaud \[30\]. It is stated as Lemma 3 on p. 287 and proved in Appendix A on p. 307–311 of that paper.

Let us now summarize the proof of the local well-posedness following from these estimates. We note that Linares-Pastor \[20\], in fact, achieved local well-posedness in \(H^s_x\) for \(s > \frac{3}{4}\), although we only need the \(s = 1\) case. Let \(X\) be the \(R\)-ball in the Banach space \(C([0, T]; H^1_x) \cap L^4_xL^\infty_y\), for \(T\) and \(R\) yet to be chosen. Consider the mapping \(\Lambda\) defined for \(u \in X\) by

\[
\Lambda u = U(t)\phi + \int_0^t U(t - t')\partial_x[u(t')^3] \, dt'.
\]

Then we claim that for suitably chosen \(R > 0\) and \(T > 0\), we have \(\Lambda : X \to X\) and \(\Lambda\) is a contraction. Indeed, by the estimates in Lemmas \[3.1\] and \[3.2\] we have

\[
\|\Lambda u\|_X \lesssim \|\phi\|_{H^1_y} + \|\partial_x(u^3)\|_{L^1_tL^2_x} + \|\partial_y(u^3)\|_{L^1_tL^2_y}.
\]

We estimate

\[
\|u_xu^2\|_{L^1_tL^2_y} \lesssim \|u_x\|_{L^2_tL^2_y} \|u\|^2_{L^\infty_y} \lesssim T^{1/2}\|u_x\|_{L^\infty_tL^2_y} \|u\|^2_{L^\infty_y},
\]

and similarly, for the \(x\) derivative replaced by the \(y\)-derivative. Consequently,

\[
\|\Lambda u\|_X \leq C\|\phi\|_{H^1_y} + CT^{1/2}\|u\|^3_X,
\]

for some constant \(C > 0\). By similar estimates,

\[
\|\Lambda u_2 - \Lambda u_1\|_X \leq CT^{1/2}\|u_2 - u_1\|_X \max(\|u_1\|_X, \|u_2\|_X)^2.
\]

We can thus take \(R = 2C\|\phi\|_{H^1_y}\) and \(T > 0\) such that \(CR^2T^{1/2} = \frac{1}{2}\) to obtain that \(\Lambda : X \to X\) and is a contraction. The fixed point is the desired solution.

For the uniqueness statement, we can take \(R \geq 2C\|\phi\|_{H^1_y}\) large enough so that the two given solutions \(u_1, u_2\) lie in \(X\), and then take \(T\) so that \(CR^2T^{1/2} = \frac{1}{2}\). Then \(u_1\) and \(u_2\) are both fixed points of \(\Lambda\) in \(X\), and since fixed points of a contraction are unique, \(u_1 = u_2\).

We now state the properties of the operator \(\mathcal{L} = -\Delta + 1 - 3Q^2\) (see Kwong \[17\] for all dimensions, Weinstein \[39\] for dimension 1 and 3, also Maris \[22\] and Chang et al. \[3\]).

**Theorem 3.3.** The following holds for an operator \(\mathcal{L}\) defined in \[2.1\]:

- \(\mathcal{L}\) is a self-adjoint operator and \(\sigma_{ess}(\mathcal{L}) = [1, +\infty)\),
- \(\ker \mathcal{L} = \text{span}\{Q_{y_1}, Q_{y_2}\}\),
- \(\mathcal{L}\) has a unique single negative eigenvalue \(-\lambda_0\) (with \(\lambda_0 > 0\)) associated to a positive radially symmetric eigenfunction \(\chi_0\). Without loss of generality, \(\chi_0\) can be chosen such that \(\|\chi_0\|_{L^2} = 1\). Moreover, there exists \(\delta > 0\) such that \(|\chi_0(x)| \lesssim e^{-\delta|x|}\) for all \(x \in \mathbb{R}^2\).
Lemma 3.4. The following identities hold

(i) $LQ = -2Q^3$,
(ii) $\mathcal{L}(\Lambda Q) = -2Q$, where $\Lambda$ is defined in (2.3). Moreover, $\int Q\Lambda Q = 0$.

In [8] we summarized several known positivity estimates for the operator $L$ following the works of Chang et al. [3] and Weinstein [39] (see Lemmas 3.3-3.6 in [8]). In this paper, we use the following set of orthogonality conditions to keep the quadratic form, generated by $L$, positive-definite.

Lemma 3.5. For any $f \in H^1(\mathbb{R}^2)$ such that (3.1) $\langle f, Q^3 \rangle = \langle f, Q_{x_j} \rangle = 0, \ j = 1, 2$, there exists an universal constant $C_1 > 0$ such that

$$\langle f, f \rangle \leq C_1 \langle Lf, f \rangle.$$

Proof. See Lemma 3.5 in [8]. □

4. Blow-up conclusion fails implies there exists renormalized $u_n$ sequence and time sequence $t_{n,m}$

We assume that the statement of Theorem 1.1 is false, and hence, there is a sequence of solutions $\bar{u}_n(t)$ of (ZK) such that for each given $n \in \mathbb{N}$, the solutions $\bar{u}_n(t)$ are defined for all times $t \geq 0$ with $E(\bar{u}_n) < 0$ and $\alpha_n = \alpha(\bar{u}_n) \to 0$ as $n \to \infty$, and moreover, for each $n$,

$$\ell_n = \liminf_{t \to +\infty} \|\nabla \bar{u}_n(t)\|_{L^2_{xy}} < +\infty.$$

We note that for each $n$, and for each $t$, by Gagliardo-Nirenberg

$$0 < -4E(\bar{u}_n) \leq \|\bar{u}_n(t)\|_{L^4_{xy}}^4 \lesssim \|\nabla \bar{u}_n(t)\|_{L^2_{xy}}^2 \|\bar{u}_n\|_{L^2_{xy}}^2,$$

and hence, $\ell_n > 0$. By definition of $\ell_n$, there exists $\bar{t}_n \geq 0$ such that

for all $t \geq \bar{t}_n$, \quad $\ell_n(1 - \frac{1}{n}) \leq \|\nabla \bar{u}_n(t)\|_{L^2_{xy}}$

and

$$\|\nabla \bar{u}_n(\bar{t}_n)\|_{L^2_{xy}} \leq \ell_n(1 + \frac{1}{n}).$$

Let us renormalize as

$$u_n(x, y, t) \overset{\text{def}}{=} \frac{\|\nabla Q\|_{L^2_{xy}}}{\ell_n} \bar{u}_n(x, y, t) \frac{\|\nabla Q\|_{L^2_{xy}}}{\ell_n} x, \frac{\|\nabla Q\|_{L^2_{xy}}}{\ell_n} y, \frac{\|\nabla Q\|_{L^2_{xy}}}{\ell_n} t + \bar{t}_n)$$

so that

for all $t \geq 0$, \quad $(1 - \frac{1}{n})\|\nabla Q\|_{L^2_{xy}} \leq \|\nabla u_n(t)\|_{L^2_{xy}}$

and

$$\|\nabla u_n(0)\|_{L^2_{xy}} \leq (1 + \frac{1}{n})\|\nabla Q\|_{L^2_{xy}}$$
as well as

\begin{equation}
\liminf_{t \nearrow +\infty} \|\nabla u_n(t)\|_{L^2_{xy}} = \|\nabla Q\|_{L^2_{xy}}.
\end{equation}

We in addition have $E(u_n) < 0$ for all $n$, and moreover,

\[ \alpha_n \overset{\text{def}}{=} \alpha(u_n) = \alpha(\bar{u}_n) \rightarrow 0 \text{ as } n \rightarrow \infty \]

as before. By (4.1), for each $n$, there exists a sequence $t_{n,m} \rightarrow +\infty$ as $m \rightarrow +\infty$ such that

\begin{equation}
\lim_{m \rightarrow +\infty} \|\nabla u_n(t_{n,m})\|_{L^2_{xy}} = \|\nabla Q\|_{L^2_{xy}}.
\end{equation}

By passing to a subsequence, we can assume without loss that, for each $n$, the gaps are expanding:

\[ \lim_{m \rightarrow +\infty} (t_{n,m+1} - t_{n,m}) = +\infty. \]

Lemma 4.1 (variational characterization and uniqueness of the ground state). For each $\eta > 0$ there exists $\alpha_1$ such that the following holds. For each $\phi \in H^1$ with $E(\phi) < 0$ and $\alpha(\phi) \leq \alpha_1$, there exists $x_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}$, $\lambda > 0$, $\mu \in \{-1, 1\}$ such that

\[ \|\mu \lambda \phi(\lambda x + x_0, \lambda y + y_0) - Q(x, y)\|_{H^1_{xy}} \leq \eta. \]

**Proof.** The proof is similar to that sketched in Lemma 1 of Merle [29], p. 563. It uses that any minimizer of the functional

\begin{equation}
\mu \rightarrow \frac{\|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2}{\|u\|_{H^1}^4}
\end{equation}

is a minimal mass solution to the ground state equation, and moreover, one has uniqueness of solutions to the ground state equation (of minimal mass) up to translation and phase. The lemma follows from these considerations plus concentration–compactness (or profile decomposition) lemmas applied to a minimizing sequence of the functional in (4.3).

Hence, we know that for each $n$ and $t \geq 0$, there exists $x_n(t) \in \mathbb{R}$, $y_n(t) \in \mathbb{R}$, $\mu_n(t) \in \{-1, 1\}$ and $\lambda_n(t) > 0$ such that for all $t \geq 0$,

\[ \|\mu_n(t) \lambda_n(t) u_n(\lambda_n(t)x + x_n(t), \lambda_n(t)y + y_n(t)) - Q(x, y)\|_{H^1_{xy}} \leq \delta_n \overset{\text{def}}{=} \delta(\alpha_n), \]

where $\delta(\alpha) > 0$ is some function such that $\delta(\alpha) \searrow 0$ as $\alpha \searrow 0$. By continuity of the ZK flow in $t$, we know that each $\mu_n(t) \in \{-1, 1\}$ is constant (independent of $t$). Thus, we might as well redefine $u_n(t)$ as $\mu_n u_n(t)$ so that we can drop the $\mu_n$ parameter entirely.
Lemma 4.2 (geometrical decomposition). There exists $\alpha_2 > 0$ such that if $E(u) < 0$ and $\alpha(u) \leq \alpha_2$, then there exist functions $\lambda(t) > 0$, $x(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$ such that the remainder function

$$\epsilon(x, y, t) \overset{\text{def}}{=} \lambda(t)u(\lambda(t)x + x(t), \lambda(t)y + y(t), t) - Q(x, y)$$

satisfies the orthogonality conditions

$$\langle \epsilon(t), \nabla Q \rangle = 0, \quad \langle \epsilon(t), Q^3 \rangle = 0.$$ 

Proof. Apply the implicit function theorem after invoking Lemma 4.1. (For a similar proof see Proposition 5.1 in [8].) □

For the solutions $u_n(t)$ discussed above, we apply Lemma 4.2 to modify the parameters $x_n(t)$, $y_n(t)$, $\lambda_n(t)$ so that the remainder function

$$\epsilon_n(x, y, t) \overset{\text{def}}{=} \lambda_n(t)u_n(\lambda_n(t)x + x_n(t), \lambda_n(t)y + y_n(t)) - Q(x, y)$$

satisfies the orthogonality conditions for all $t \geq 0$

$$\langle \epsilon_n(t), \nabla Q \rangle = 0, \quad \langle \epsilon_n(t), Q^3 \rangle = 0.$$ 

Lemma 4.3 (properties of remainder function and parameters). Let $u(t)$ solve ZK such that $E(u) < 0$ and $\alpha(u) \leq \alpha_2$ as in Lemma 4.2, and let $\lambda(t)$, $x(t)$, and $y(t)$ be the parameters given by Lemma 4.2 on the maximal time interval of existence of $u(t)$. Then $\lambda(t)$, $x(t)$, and $y(t)$ are $C^1$ functions and

$$|\lambda^2 \lambda_t| + |\lambda^2 x_t - 1| + |\lambda^2 y_t| \lesssim \|\epsilon(t)\|_{L^2}, \quad (4.4)$$

and moreover, there exist $\alpha_3 > 0$ such that if $\alpha(u) \leq \alpha_3$, then

$$\|\epsilon(t)\|_{H^3} \lesssim 1/\sqrt{\alpha(u)}. \quad (4.5)$$

Proof. The equation for $\epsilon$ is deduced by expressing $u$ in terms of $\epsilon$ and $Q$ and substituting into the equation for $u$ (the ZK equation), to obtain

$$\lambda^3 \partial_t \epsilon = \partial_x (\mathcal{L} \epsilon) + \lambda^2 \lambda_t \Delta Q + (\lambda^2 x_t - 1, \lambda^2 y_t) \cdot \nabla Q + \lambda^2 \lambda_t \Delta \epsilon + (\lambda^2 x_t - 1, \lambda^2 y_t) \cdot \nabla (\epsilon^3) - 3(\epsilon^2) x - (\epsilon^3)_x.$$ 

The estimates (4.4) follow from computing $\partial_t$ of the orthogonality conditions and substituting the above equation for $\epsilon$.

The equation (4.5) is a consequence of the following consideration. Let

$$Z(u) = \frac{1}{2} M(u) + E(u),$$

\footnote{We require that $0 < \alpha_2 \leq \alpha_1$ so that Lemma 4.1 applies.}

\footnote{We require that $0 < \alpha_3 \leq \alpha_2$ so that Lemmas 4.1, 4.2 apply as well}
and observe that $Z'(Q) = 0$ and $Z''(Q) = \mathcal{L} \overset{\text{def}}{=} 1 - \Delta - 3Q^2$, the linearization operator. Note that Taylor’s expansion

$$Z(u) = Z(Q + \epsilon) = Z(Q) + \langle Z'(Q), \epsilon \rangle + \frac{1}{2} \langle Z''(Q)\epsilon, \epsilon \rangle + O(\epsilon^3)$$

yields

$$\frac{1}{2} \langle Z''(Q)\epsilon, \epsilon \rangle = Z(u) - Z(Q) + O(\epsilon^3) = \alpha(u) + E(u) + O(\epsilon^3) \lesssim \alpha(u) + O(\epsilon^3),$$

where we have used that $E(u) < 0$. Spectral considerations imply that the orthogonality conditions yield positivity of $\mathcal{L}$, which together with elliptic regularity implies (4.5). □

Note that as a consequence of (4.5), we have the following. By scaling,

$$\lambda(t)^2 \| \nabla u(t) \|^2_{L^2} = \| \nabla (\epsilon + Q) \|^2_{L^2} = \| \nabla \epsilon(t) \|^2_{L^2} + 2 \langle \nabla \epsilon, \nabla Q \rangle + \| \nabla Q \|^2_{L^2},$$

from which we obtain

$$\left| \lambda(t)^2 \frac{\| \nabla u(t) \|^2_{L^2}}{\| \nabla Q \|^2_{L^2}} - 1 \right| \lesssim \| \epsilon(t) \|_{H^1} \lesssim \sqrt{\alpha(u)}.$$

This gives us conversion formulas

(4.6) \[ \frac{\| \nabla Q \|^2_{L^2}}{\| \nabla u(t) \|^2_{L^2}} (1 - C\sqrt{\alpha}) \leq \lambda(t) \leq \frac{\| \nabla Q \|^2_{L^2}}{\| \nabla u(t) \|^2_{L^2}} (1 + C\sqrt{\alpha}) \]

and

(4.7) \[ \lambda(t)^{-1} \| \nabla Q \|^2_{L^2} (1 - C\sqrt{\alpha}) \leq \| \nabla u(t) \|^2_{L^2} \leq \lambda(t)^{-1} \| \nabla Q \|^2_{L^2} (1 + C\sqrt{\alpha}) \]

for some absolute constant $C > 0$.

In later parts of the paper, we will need a more precise statement of the remainder equation and parameter dynamics.

**Lemma 4.4.** Suppose that $u(t)$ solves (ZK), $\alpha(u) \ll 1$ and $E(u) < 0$, so that the geometrical decomposition applies with orthogonality conditions $\langle Q^3, \epsilon \rangle = 0$ and $\langle \nabla Q, \epsilon \rangle = 0$. Rescaling time to $s(t)$, where $\frac{ds}{dt} = \lambda^{-3}$, we have that the remainder function satisfies the equation

(4.8) \[ \partial_s \epsilon = \partial_x \mathcal{L} \epsilon + \frac{\lambda_s}{\lambda} \Lambda Q + \left( \frac{x_s}{\lambda} - 1 \right) Q_x + \frac{y_s}{\lambda} Q_y + \frac{\lambda_s}{\lambda} \Lambda \epsilon + \left( \frac{x_s}{\lambda} - 1 \right) \epsilon_x + \frac{y_s}{\lambda} \epsilon_y \]

where, $\Lambda = 1 + x \partial_x + y \partial_y$ is the generator of scaling and $\mathcal{L} = I - \Delta - 3Q^2$ is the linearized operator.\(^4\)

\(^4\)We note, for the purposes of computation, that $\Lambda$ is skew-adjoint and $\mathcal{L}$ is self-adjoint.
Moreover, if we let \( b \overset{\text{def}}{=} \|\epsilon\|_{L^\infty_t L^2_y} \), then the parameters satisfy the equations

\[
\left| \frac{\lambda_s}{\lambda} - \langle f_1, \epsilon \rangle \right| \lesssim b^2, \quad \left| (\frac{x_s}{\lambda} - 1) - \langle f_2, \epsilon \rangle \right| \lesssim b^2, \quad \left| \frac{y_s}{\lambda} - \langle f_3, \epsilon \rangle \right| \lesssim b^2,
\]

where \( f_j \) are the smooth, rapidly decaying spatial functions

\[
f_1 = \frac{2}{\|Q\|_{L^4_x}^4} \mathcal{L}(Q^3)_x, \quad f_2 = \frac{1}{\|Q_x\|_{L^2_x}^2} \mathcal{L}Q_{xx}, \quad f_3 = \frac{1}{\|Q_y\|_{L^2_x}^2} \mathcal{L}Q_{xy}.
\]

**Proof.** Taking \( \partial_s \) of the orthogonality conditions, we obtain three equations, which are collectively expressed as

\[
(A + B(\epsilon)) \begin{bmatrix} \frac{\lambda_s}{\lambda} \\ \frac{x_s}{\lambda} - 1 \end{bmatrix} = \begin{bmatrix} \langle \mathcal{L}(Q^3)_x, \epsilon \rangle \\ \langle \mathcal{L}Q_{xx}, \epsilon \rangle \\ \langle \mathcal{L}Q_{xy}, \epsilon \rangle \end{bmatrix} + \begin{bmatrix} 3\langle (Q^3)_x Q, \epsilon^2 \rangle + \langle (Q^3)_x, \epsilon^3 \rangle \\ 3\langle Q_{xx} Q, \epsilon^2 \rangle + \langle Q_{xx}, \epsilon^3 \rangle \\ 3\langle Q_{xy} Q, \epsilon^2 \rangle + \langle Q_{xy}, \epsilon^3 \rangle \end{bmatrix},
\]

where

\[
A = \begin{bmatrix} \frac{1}{2} \|Q\|_{L^4}^4 \\ \|Q_x\|_{L^2_x}^2 \\ \|Q_y\|_{L^2_x}^2 \end{bmatrix} \quad \text{and} \quad B(\epsilon) = \begin{bmatrix} \langle \Lambda(Q^3), \epsilon \rangle & \langle (Q^3)_x, \epsilon \rangle & \langle (Q^3)_y, \epsilon \rangle \\ \langle \Lambda Q_x, \epsilon \rangle & \langle Q_{xx}, \epsilon \rangle & \langle Q_{xy}, \epsilon \rangle \\ \langle \Lambda Q_y, \epsilon \rangle & \langle Q_{xy}, \epsilon \rangle & \langle Q_{yy}, \epsilon \rangle \end{bmatrix}.
\]

Note that every entry \( b_{ij}(\epsilon) \) of the matrix \( B(\epsilon) \) satisfies \( |b_{ij}(\epsilon)| \lesssim \|\epsilon\|_{L^2} \). Using that

\[
(A + B(\epsilon))^{-1} = [A(I + A^{-1}B(\epsilon))]^{-1} = (I + A^{-1}B(\epsilon))^{-1}A^{-1}
\]

and Neumann expansion of \( (I + A^{-1}B(\epsilon))^{-1} \) that, if \( b = \|\epsilon\|_{L^\infty_t L^2_y} \ll 1 \), then \( (4.9) \) holds.

Since \( \frac{ds}{dt} = \frac{1}{\lambda t} \), we have the conversions

\[
\frac{\lambda_s}{\lambda} = \lambda^2 \lambda_t, \quad \frac{x_s}{\lambda} - 1 = \lambda^2 (x_t - \lambda^{-2}), \quad \frac{y_s}{\lambda} = \lambda^2 y_t.
\]

As a side remark, for use in subsequent sections, define

\[
\eta(x, y, t) = \lambda^{-1} \epsilon(\lambda^{-1}x, \lambda^{-1}y, t)
\]

and

\[
\tilde{f}_j(x, y) = \lambda^{-1} f_j(\lambda^{-1}x, \lambda^{-1}y).
\]

Then change of variables gives

\[
|\lambda^2 \lambda_t - (\tilde{f}_1, \eta)| \lesssim b^2, \quad |(\lambda^2 x_t - 1) - (\tilde{f}_2, \eta)| \lesssim b^2, \quad |\lambda^2 y_t - (\tilde{f}_3, \eta)| \lesssim b^2,
\]

which is used in sections \( 6 \) and \( 10 \).

Also, let

\[
\zeta(x, y, t) = b^{-1} \lambda^{-1} \epsilon(\lambda^{-1}(x - x(t), \lambda^{-1}(y - y(t), t)),
\]

where \( b = \|\epsilon\|_{L^\infty_t L^2_y} \), and

\[
\tilde{f}_j(x, y) = \lambda^{-1} f_j(\lambda^{-1}(x - x(t)), \lambda^{-1}(y - y(t))).
\]
Then change of variables gives
\[ |b^{-1}\lambda^2 \lambda_t - \langle f_1, \zeta \rangle| \lesssim b, \quad |b^{-1}2^2x_t - \langle f_2, \zeta \rangle| \lesssim b, \quad |b^{-1}\lambda^2 y_t - \langle f_3, \zeta \rangle| \lesssim b, \]
which is used in sections 11 and 12.

5. Extraction of “future state” weak limit \( \tilde{u}_n(0) \), introduction of time frame \((-t_1(n), t_2(n))\), stability of the weak limit and applications

By passing to a subsequence, we can assume that, for each \( n \),
\begin{align}
(5.1) \quad u_n(\cdot + x(t_{n,m}), \cdot + y(t_{n,m}), t_{n,m}) & \rightharpoonup \tilde{u}_n(\cdot, \cdot, 0) \text{ weakly in } H^1_{x,y} \text{ as } m \to \infty
\end{align}
for some \( \tilde{u}_n(0) \in H^1_{x,y} \).

**Lemma 5.1** (energy constraints on \( \tilde{u}_n \)).

1. For all \( n \), we have \( E(\tilde{u}_n(0)) \leq E(u_n(0)) \) and \( 0 < \alpha(\tilde{u}_n(0)) \leq \alpha(u_n(0)) \).
2. \( \tilde{u}_n(0) \to Q \) in \( H^1 \) strongly as \( n \to \infty \).

**Proof.** This follows the proof of Lemma 7 in Merle [29] given on p. 571-572 of that paper.

Let \( \tilde{u}_n(t) \) be the \( H^1_{x,y} \) evolution of initial data \( \tilde{u}_n(0) \) by ZK. Let \( \tilde{x}_n(t), \tilde{y}_n(t), \) and \( \tilde{\lambda}_n(t) \) be the geometrical parameters associated with \( \tilde{u}_n(t) \) on its maximal time interval of existence, as given by Lemma 5.2, noting that the corresponding remainder \( \tilde{\epsilon}_n \) and these parameters satisfy the properties delineated in Lemma 4.3.

Let \((-t_1(n), t_2(n))\) be the maximal time interval on which
\begin{align}
(5.2) \quad \frac{1}{2} \leq \tilde{\lambda}_n(t) \leq 2 \quad \text{and} \quad \frac{1}{2} \leq \liminf_{m \to \infty} \lambda_n(t_{n,m} + t) \leq \limsup_{m \to \infty} \lambda_n(t_{n,m} + t) \leq 2
\end{align}
hold. By (4.6) and (4.7), these can equivalently be viewed as upper and lower bounds on \( \|\nabla \tilde{u}_n(t)\|_{L^2} \) and \( \liminf_{m \to +\infty} \|\nabla u(t_{n,m} + t)\|_{L^2} \). We will ultimately show that for \( n \) sufficiently large, \( t_1(n) = t_2(n) = \infty \). But in the meantime, we can argue that the time interval \((-t_1(n), t_2(n))\) is nontrivial (Lemma 5.2), which is not \textit{a priori} obvious due to the limits that appear in the definition (5.2).

Let
\begin{align}
(5.3) \quad v_{n,m}(\cdot, \cdot, 0) = u_n(\cdot + x(t_{n,m}), \cdot + y(t_{n,m}), t_{n,m})
\end{align}
Let \( v_{n,m}(t) \) denote the nonlinear evolution by ZK of initial condition \( v_{n,m}(0) \). Then by spatial and time translation invariance of the ZK flow,
\begin{align}
(5.4) \quad v_{n,m}(x, y, t) = u_n(x + x(t_{n,m}), y + y(t_{n,m}), t_{n,m} + t).
\end{align}

We will drop the \( n \) subscript for a little while, since we consider fixed \( n \), and state and prove two general lemmas (Lemma 5.3 and Lemma 5.4) before applying Lemma
Lemma 5.2 (nontriviality of the time frame). For each $n$, we have
\[ t_1(n), t_2(n) \gtrsim 1. \]

Proof. Let $q(x,t) = Q(x-t)$ and $q_m(t) = v_m(t) - q(t)$. Note that (4.2) and (4.6) imply that $|\lambda_n(t_{n,m}) - 1| \lesssim \sqrt{\alpha(u_n)}$ and hence $\|q_m(0)\|_{H^1} \lesssim \alpha(u_n)^{1/2}$.

By substituting into the ZK equation for $v_m$, we obtain
\[ \partial_t q_m + \partial_x \Delta q_m + \partial_x ((q_m + q)^3 - q^3) = 0. \]

Expanding the nonlinear term, we get
\[ (q_m + q)^3 - q^3 = 3q_m q^2 + 3q^2_m q + q_m^3. \]

We set up (5.5) in Duhamel form and apply estimates in Lemmas 3.1, 3.2, using (5.5) to refer to each such subsequence. By the Rellich-Kondrachov compactness theorem, the following are equivalent for a sequence $\{v_m\} \subset H^1_{xy}$ bounded in $H^1_{xy}$, and $v \in H^1_{xy}$.

(1) $v_m \rightharpoonup v$ weakly in $H^1_{xy}$
(2) for each $k \in \mathbb{N}$, $v_m 1_{\leq k} \to v 1_{\leq k}$ strongly in $L^2_{xy}$. Equivalently, we state $v_m \to v$ strongly in $L^2_{loc}$.
(3) For each subsequence $v_{m'}$ of $v_m$, there exists a subsequence $v_{m''}$ of $v_{m'}$ such that the following holds: for each $k \in \mathbb{N}$, $v_{m''} 1_{\leq k} \to v 1_{\leq k}$ strongly in $L^2_{xy}$.
(4) For each subsequence $v_{m'}$ of $v_m$, there exists a subsequence $v_{m''}$ of $v_{m'}$ and radii $\rho_{m''} \to \infty$ such that the following holds: $v_{m''} 1_{\leq \rho_{m''}} \to v$ strongly in $L^2_{xy}$.

Proof. First, we note that (2) $\iff$ (3), which is a standard analytic fact proved by showing the negations are equivalent.

(1) $\implies$ (3). Our argument will involve successive passive to subsequences starting with $v_{m'}$, ultimately yielding $v_{m''}$, although for convenience we will use the notation $v_m$ to refer to each such subsequence. By the Rellich-Kondrachov compactness theorem,
for $k = 1$, there exists a subsequence in $m$ of $v_m 1_{\leq k}$ such that $v_m 1_{\leq k}$ converges strongly in $L^2_{x,y}$. Passing to a further subsequence, we can arrange that $v_m 1_{\leq k}$ converges strongly in $L^2_{x,y}$ for $k = 2$, and so forth. By taking the diagonal subsequence, we now have a subsequence such that for each $k \in \mathbb{N}$, the sequence $v_m 1_{\leq k}$ converges strongly in $L^2_{x,y}$ as $m \to \infty$. By the definition of weak convergence, it follows that, for fixed $k$, the value of this limit is $v 1_{\leq k}$.

(2) $\implies$ (4). Let $v_m'$ be a given subsequence of $v_m$. For convenience we relabel $v_m'$ as $v_m$. By (2) we know that for each $k \in \mathbb{N}$, $v_m 1_{\leq k}$ converges strongly in $L^2_{x,y}$. Now we will pass to a subsequence as follows. For each $\ell \in \mathbb{N}$, we will show that there exists $k_\ell$ and $m_\ell$ such that

$$
\|v_{m_\ell} 1_{\leq k_\ell} - v 1_{\leq k_\ell}\|_{L^2_{x,y}} \leq \frac{1}{\ell},
$$

and

$$
m_\ell \to +\infty \text{ and } k_\ell \to +\infty \text{ as } \ell \to +\infty.
$$

Indeed, for a given $\ell$, first take $k_\ell$ sufficiently large (also requiring $k_\ell \geq \ell$) so that

$$
\|v 1_{\leq k_\ell} - v\|_{L^2_{x,y}} \leq \frac{1}{2\ell}.
$$

Then for this $k_\ell$, find $m_\ell$ sufficiently large (also requiring $m_\ell \geq \ell$) so that

$$
\|v_{m_\ell} 1_{\leq k_\ell} - v 1_{\leq k_\ell}\|_{L^2_{x,y}} \leq \frac{1}{2\ell}.
$$

Combining the two gives (5.6). Now it is convenient to relabel (5.6) as the statement

$$
\|v_{m_\ell} 1_{\leq \rho_\ell} - v\|_{L^2_{x,y}} \leq \frac{1}{\ell m},
$$

which is achieved by replacing the original $v_m$ sequence with the subsequence constructed, and where now $\rho_\ell \to +\infty$ as $\ell \to +\infty$.

(4) $\implies$ (3). Straightforward, since for fixed $k$, for sufficiently large $m''$, we have

$$
(v_{m''} - v) 1_{\leq k} = (v_{m''} - v) 1_{\leq \rho_{m''}} 1_{\leq k}
$$

and the right side $\to 0$ in $L^2$ by (4).

(2) $\implies$ (1). Since we have assumed that $\{v_m\} \subset H^1$ is bounded, we can apply the density of $C^\infty_c(\mathbb{R}^2)$ in $H^1(\mathbb{R}^2)$ to reduce to a test function in $\phi \in C^\infty_c(\mathbb{R}^2)$. Let $k$ be any fixed integer larger than the support radius of $\phi$. Then

$$
|\langle v_m - v, \phi \rangle_{H^1}| = |\langle v_m - v, (1-\Delta)\phi \rangle_{L^2}| = |\langle (v_m - v) 1_{\leq k}, (1-\Delta)\phi \rangle_{L^2}|
\leq \|(v_m - v) 1_{\leq k}\|_{L^2} \|(1-\Delta)\phi\|_{L^2} \to 0
$$

by Cauchy-Schwarz and (2).
Lemma 5.4 (stability of weak limits under ZK flow). Suppose that \( v_m(0) \to v(0) \) weakly in \( H_{xy}^1 \) and that there exists \( k \in \mathbb{N} \) such that for all \( m \), \( \|v_m(0)\mathbf{1}_{\geq k}\|_{L^2_x} \leq \frac{1}{2}\|Q\|_{L^2_x} \). Letting \( v_m(t) \) and \( v(t) \) denote evolution of initial data \( v_m(0) \) and \( v(0) \), respectively, under the ZK flow, suppose that moreover

\[
\left\|v(t)\right\|_{L^\infty_{[−T_−,T_+]},H^1_{xy}} + \left\|v(t)\right\|_{L^\infty_{[−T_−,T_+]},H_y^1} < +\infty \quad \text{and} \quad \limsup_{m \to \infty} \left\|v_m(t)\right\|_{L^\infty_{[−T_−,T_+]},H^1_{xy}} < +\infty;
\]

where \( 0 \leq T_\pm < \infty \) (that is, \([-T_-, T_+]\) is a finite time interval). Then on \([-T_-, T_+]\), \( v_m(t) \to v(t) \) weakly in \( H_{xy}^1 \) and for each \( k \in \mathbb{N} \), \( v_m(t)\mathbf{1}_{\leq k} \to v(t)\mathbf{1}_{\leq k} \) strongly in \( C([-T_-, T_+]; L^2) \).

Proof. Let

\[
M = \max(\limsup_{m \to \infty} \left\|v_m(t)\right\|_{L^\infty_{[-T_-, T_+], H^1_{xy}}}, \left\|v(t)\right\|_{L^\infty_{[-T_-, T_+], H^1_{xy}}}).
\]

Let \( v_m'(0) \) be any subsequence of \( v_m(0) \), and invoke Lemma 5.3 (1) \( \implies \) (4) to obtain a subsequence \( v_m''(0) \) and \( \rho_m'' \to \infty \) such that \( v_m''(0)\mathbf{1}_{\leq \rho_m''} \to v(0) \) strongly in \( L^2 \). We will ultimately show that \( v_m''(t)\mathbf{1}_{\leq \rho_m''/2} \to v(t) \) strongly in \( C([-T_-, T_+]; L^2) \), and thus, can invoke Lemma 5.3 (4) \( \implies \) (1) to conclude that \( v_m(t) \to v(t) \) weakly in \( H^1 \) for each \( t \in [-T_-, T_+] \). Moreover, since \([-T_-, T_+]\) is compact, a statement similar to Lemma 5.3 holds in which \( L^2 \) is replaced by \( C([-T_-, T_+]; L^2) \) in (2), (3), (4), with the same proof of equivalence as given there.

Replace \( m'' \) by \( m \) for notational convenience, so that we have

\[
v_m(0) = v(0) + z_m(0) + q_m(0),
\]

where

\[
z_m(0) = v_m(0)\mathbf{1}_{\leq \rho_m} - v(0) \quad \text{and} \quad q_m(0) = v_m(0)\mathbf{1}_{\geq \rho_m}.
\]

Let \( z_m(t) \) and \( q_m(t) \) be the ZK (nonlinear) evolution of \( z_m(0) \) and \( q_m(0) \), respectively. Since \( \|z_m(0)\|_{L^2} \to 0 \) as \( m \to \infty \), we can restrict to \( m \) sufficiently large so that \( \|z_m(0)\|_{L^2} \leq \frac{1}{2}\|Q\|_{L^2} \), and thus, energy estimates imply that \( z_m(t) \) is globally bounded in \( H^1 \). Also, our assumptions imply that \( \|q_m(0)\|_{L^2} \leq \frac{1}{2}\|Q\|_{L^2} \), so that \( q_m(t) \) is globally bounded in \( H^1 \) by energy estimates (see Claim 1-2 below). Let \( r_m(t) \) be defined by

\[
v_m(t) = v(t) + z_m(t) + q_m(t) + r_m(t).
\]

We will prove that \( z_m \to 0 \) and \( r_m \to 0 \) strongly in \( C([-T_-, T_+]; L^2) \) and

\[
\|q_m(t)\mathbf{1}_{\leq \rho_m/2}\|_{L^2} \lesssim M^2 \max(T_-, T_+)/\rho_m,
\]

and hence,

\[
\lim_{m \to \infty} \|q_m(t)\mathbf{1}_{\leq \rho_m/2}\|_{C([-T_-, T_+]; L^2)} = 0.
\]

Thus,

\[
\lim_{m \to \infty} \|(v_m(t) - v(t))\mathbf{1}_{\leq \rho_m/2}\|_{C([-T_-, T_+]; L^2)} = 0,
\]

which will complete the proof. Now we present some of the details.
Claim 1. \( z_m(t), q_m(t) \) are global in \( H^1 \) with
\[
\| \nabla z_m(t) \|_{L^2} \leq 4M, \quad \| \nabla q_m(t) \|_{L^2} \leq 4M,
\]
where
\[
M = \max(\limsup_{m \to \infty} \| v_m(t) \|_{L^\infty_{[-T_-, T_+]}} H^1_{xy}, \| v(t) \|_{L^\infty_{[-T_-, T_+]} H^1_{xy}}).
\]

Proof of Claim 1. This is just use of the Weinstein inequality.

Claim 2. On time intervals \( I \) of length \( |I| \lesssim M^{-4} \), we have
\[
\| z_m \|_{L^4_t L^\infty_y}, \| q_m \|_{L^4_t L^\infty_y}, \| v_m \|_{L^4_t L^\infty_y}, \| v \|_{L^4_t L^\infty_y} \lesssim M.
\]

Proof of Claim 2. These estimates are inherited from the assumption of global \( H^1 \) bound of \( M \) and the local theory estimates in Lemmas 3.1, 3.2.

Claim 3. \( \| q_m(t) 1_{\rho_m/2} \|_{L^2_y}^2 \lesssim t \rho_m^{-1} M^2 \).

Proof of Claim 3. This is a localized mass estimate, which is obtained as follows: let \( \chi_m(x, y) = \chi(\frac{|x|}{\rho_m}) \). Then
\[
\partial_t \int \chi_m v_m(t)^2 \, dx \, dy = -2 \int \chi_m v_m (v_{m,xxx} + (v_m)_{xyy} + (v_m)^3) \, dx \, dy
\]
\[
= \int -3(\chi_m)_x (v_m)_x^2 - (\chi_m)_y (v_m)_x^2 - 2(\chi_m)_y (v_m)_x (v_m)_y + (\chi_m)_{xyy}v^2 + \frac{1}{2} (\chi_m)_x v_m^4 \, dx \, dy.
\]
Hence,
\[
\| \partial_t \int \chi_m v_m(t)^2 \, dx \, dy \| \lesssim \frac{1}{\rho_m}(\| \nabla v_m(t) \|_{L^2}^2 + \| v_m(t) \|_{L^2}^2 + \| v_m(t) \|_{L^4}^4).
\]
By Gagliardo-Nirenberg, we have
\[
\| \partial_t \int \chi_m v_m(t)^2 \, dx \, dy \| \lesssim \frac{M^2}{\rho_m}.
\]
Integration in time yields the claim, using that \( \chi_m q_m(0) = 0 \).

Claim 4. On time interval \( I = [t_\ell, t_r] \) of length \( |I| \lesssim M^{-4} \), we have
\[
\| r_m(t) \|_{L^2} \leq 2 \| r_m(t_\ell) \|_{L^2} + \omega(m) M^2,
\]
where \( \omega(m) \to 0 \) as \( m \to \infty \), uniformly in \( t \) over \( [T_-, T_+] \). Specifically,
\[
\omega(m) \sim M^{-2} \| q_m \|_{L^\infty_t L^2_y} + \| 1_{\rho_m/2} v \|_{L^4_y L^2_t} + \| z_m \|_{L^\infty_t L^2_y}.
\]
Before proceeding with the proof, let us note why \( \lim_{m \to \infty} \| 1_{\rho_m/2} v \|_{L^4_y L^2_t} = 0 \). Since
\[
\| v \|_{L^4_y L^2_t} = \| v \|_{L^4_t L^2_y} \leq |I|^{1/2} \| v \|_{L^\infty_t L^2_y} < \infty,
\]
we have
\[
\omega(m) \sim M^{-2} \| q_m \|_{L^\infty_t L^2_y} + \| 1_{\rho_m/2} v \|_{L^4_y L^2_t} + \| z_m \|_{L^\infty_t L^2_y}.
\]

it follow (by dominated convergence) that
\[ \lim_{k \to +\infty} \|1_{>k}v\|_{L^2_yL^2_t} = 0. \]

**Proof of Claim 4.** By plugging the decomposition into the equation for \( v_m \), we obtain an equation for \( r_m(t) \)
\[ \partial t r_m + \partial x \Delta r_m + \partial x F = 0, \]
where
\[ F = (r_m + v + z_m + q_m)^3 - v^3 - z_m^3 - q_m^3. \]
In the expansion of the nonlinearity
\[ F = \sum_j F_j \]
there are no pure cubic terms, except for \( r_m^3 \), otherwise, there are only cross terms. From Claim 1-2, we know that
\[ \|r_m(t)\|_{H^1_y}, \|r_m\|_{L^4_yL^\infty_t} \lesssim M. \]
We set up the \( r_m \) equation and apply the estimates in Lemmas 3.1, 3.2 at the \( L^2 \) level (instead of the \( H^1 \) level) and obtain
\[ \|r_m(t)\|_{L^2} \leq \|r_m(t_0)\|_{L^2} + C\|F\|_{L^1L^2_t} \]
for some absolute constant \( C > 0 \). For each term \( F_j = h_1h_2h_3 \) in the expansion of \( F \), we estimate as
\[ \|F_j\|_{L^1L^2_y} \leq \|h_1\|_{L^4L^\infty_y} \|h_2\|_{L^4L^\infty_y} \|h_3\|_{L^4L^\infty_y}. \]
For \( h_1 \), we could estimate as
\[ \|h_1\|_{L^4L^2_y} \leq |I|^{1/2}\|h_1\|_{L^\infty_yL^2_y}, \]
although for terms in Case C below, we do it differently.

We consider the following three cases:

- **Case A.** At least one \( r_m \) is present. Put \( h_1 = r_m \) and absorb on the left.
- **Case B.** No \( r_m \) is present, and at least one \( z_m \) is present. Put \( h_1 = z_m \).
- **Case C.** No \( r_m \) is present and no \( z_m \) is present. This consists of \( vq_m^2 \) and \( v^2q_m \) and we do an in/out spatial decomposition, as follows: for example, for \( vq_m^2 \), decompose as
\[ vq_m^2 = (1_{<\rho_m/2}q_m)vq_m + (1_{>\rho_m/2}q_m)^2, \]
so
\[ \|vq_m^2\|_{L^1_yL^2_t} \leq \|(1_{<\rho_m/2}q_m)vq_m\|_{L^1_yL^2_t} + \|(1_{>\rho_m/2}q_m)^2\|_{L^1_yL^2_t} \]
\[ \leq |I|^{1/2}\|1_{<\rho_m/2}q_m\|_{L^\infty_yL^2_t}\|v\|_{L^4L^\infty_y}\|q_m\|_{L^4L^\infty_y} + \|1_{>\rho_m/2}q_m\|_{L^4L^\infty_y}\|q_m\|_{L^4L^\infty_y} \]
\[ \leq (|I|^{1/2}\|1_{<\rho_m/2}q_m\|_{L^\infty_yL^2_t} + \|1_{>\rho_m/2}q_m\|_{L^4L^\infty_y})M^2, \]
which completes the proof of Claim 4.

Now we apply Claim 4 as follows. Decompose \([0, T_+]\) into intervals
\[
I_1 = [t_0, t_1], \quad I_2 = [t_1, t_2], \ldots, I_J = [t_{J-1}, t_J]
\]
of length \(\sim M^{-4}\) (thus, \(J \sim M^4 T_+\)) so that Claim 4 applies on each subinterval. We have
\[
\|r_m(t_1)\|_{L^2} \leq \omega_1(m)M^2,
\]
\[
\|r_m(t_2)\|_{L^2} \leq 2\|r_m(t_1)\|_{L^2} + \omega_2(m)M^2,
\]
\[
\|r_m(t_3)\|_{L^2} \leq 2\|r_m(t_2)\|_{L^2} + \omega_3(m)M^2,
\]
and thus, combining, we obtain
\[
\|r_m(t_3)\|_{L^2} \leq (4\omega_1(m) + 2\omega_2(m) + \omega_3(m))M^2.
\]
Finally, after reaching \(I_J\), we have
\[
\|r_m(T_+)\|_{L^2} \leq (2^{J-1}\omega_1(m) + \cdots + 2\omega_{J-1}(m) + \omega_J(m))M^2.
\]
Hence,
\[
\lim_{m \to \infty} \|r_m(t)\|_{L^2} = 0
\]
uniformly on \(0 \leq t \leq T_+\). A similar argument applies to \([-T_-, 0]\).

**Corollary 5.5** (application of stability of weak limits). For any fixed \(n\), take any \(t_1, t_2 < \infty\) such that \((-t_1, t_2) \subset (-t_1(n), t_2(n))\)\(^5\). Then for \(t \in [-t_1, t_2]\), we have
\[
u_n(t_n + t, x_n(t_n) + \bullet, y_n(t_n) + \bullet) \rightharpoonup \tilde{u}_n(t, \bullet, \bullet) \text{ weakly in } H^1 \text{ as } m \to \infty,
\]
and for each \(n\), for each \(k \in \mathbb{N}\),
\[
u_n(t_n + t, x_n(t_n) + \bullet, y_n(t_n) + \bullet, t_n + t) 1_{\leq k} \to \tilde{u}_n(\bullet, \bullet, t) 1_{\leq k}
\]
strongly in \(C([-t_1, t_2]; L^2)\) as \(m \to \infty\).

**Proof.** This is just Lemma 5.4 applied to \(v_{n,m}(t)\) as defined in (5.4), noting the definition of \(\tilde{u}_n(0)\) in (5.1), and \(\tilde{u}_n(t)\) defined as the ZK evolution of initial data \(\tilde{u}_n(0)\). Recall that the time frame \((-t_1(n), t_2(n))\) has been defined so that the hypotheses of Lemma 5.4 are satisfied for the time interval \([-t_1, t_2]\).\(^5\)

\(^5\)In other words, if \(t_2(n) < \infty\), then we can take \(t_2 = t_2(n)\), but if \(t_2(n) = +\infty\), then we can take \(t_2\) to be any finite positive number. Similarly, for \(t_1\) in relation to \(t_1(n)\).
Lemma 5.6 (convergence of geometric parameters). For any fixed $n$, take any $t_1, t_2 < \infty$ such that $(-t_1, t_2) \subset (-t_1(n), t_2(n))$. Then
\[
\lambda_n(t_{n,m} + t) \to \tilde{\lambda}_n(t),
\]
\[
x_n(t_{n,m} + t) - x_n(t_{n,m}) \to \tilde{x}_n(t),
\]
\[
y_n(t_{n,m} + t) - y_n(t_{n,m}) \to \tilde{y}_n(t)
\]
in $C([-t_1, t_2]; \mathbb{R})$ as $m \to \infty$.

Proof. This is a consequence of Lemma 4.3 and Corollary 5.5. \qed

Corollary 5.7. For any fixed $n$, take any $t_1, t_2 < \infty$ such that $(-t_1, t_2) \subset (-t_1(n), t_2(n))$. Then for $t \in [-t_1, t_2]$, we have
\[
u_n(t_{n,m} + t, x_n(t_{n,m} + t) + \bullet, y_n(t_{n,m} + t) + \bullet) \to \tilde{\nu}_n(t, \bullet + \tilde{x}_n(t), \bullet + \tilde{y}_n(t))
\]
weakly in $H^1$ as $m \to \infty$ and for each $n$, for each $k \in \mathbb{N}$,
\[
u_n(x_n(t_{n,m} + t) + \bullet, y_n(t_{n,m} + t) + \bullet, t_{n,m} + t)1_{\le k} \to \tilde{\nu}_n(\bullet + \tilde{x}_n(t), \bullet + \tilde{y}_n(t), t)1_{\le k}
\]
strongly in $C([-t_1, t_2]; L^2)$ as $m \to \infty$.

Proof. This follows from Corollary 5.5 and Lemma 5.6. \qed

6. $\tilde{\nu}_n$ has exponential decay in $x$, uniformly in time, via monotonicity

Lemma 6.1 (2D Gagliardo-Nirenberg with weight). Suppose that $\psi(x, y) \ge 0$ is differentiable with the pointwise bound $|\nabla \psi(x, y)| \lesssim \psi(x, y)$, and for any $R_0 \ge 0$, let
\[
B = \{(x, y) \mid |x| > R_0 \text{ or } |y| > R_0\}.
\]

Then
\[
\int_B \int_B \psi(x)u(x, y)^4 \, dx \, dy \lesssim \|u\|_{L^4_B}^2 \int_B \int_B \psi(x)(|\nabla u(x, y)|^2 + u(x, y)^2) \, dx \, dy
\]
with implicit constant independent of $R_0$ and $\psi$ (except for the implicit constant in the pointwise bound $|\nabla \psi(x, y)| \lesssim \psi(x, y)$).

Proof. We will apply the 1D inequality
\[
|v(x_0)|^2 \le 2\|v\|_{L^2_{x>x_0}}\|v_x\|_{L^2_x}
\]
or
\[
|v(x_0)|^2 \le 2\|v\|_{L^2_{x<x_0}}\|v_x\|_{L^2_x}
\]
in both $x$ (with $y$ fixed) and in $y$ (with $x$ fixed). Because the argument relies only on these estimates, it clearly localizes to the spatial region $|x| \ge R_0$ or $|y| \ge R_0$. For expository convenience we will ignore this spatial localization.
For fixed $y$, we have, where $\psi = \psi_1\psi_2$ and $\psi_1$ and $\psi_2$ are yet to be chosen, by “sup-ing out”
\begin{equation}
\int_x \psi u^4 \, dx \lesssim \|\psi_1 u^2\|_{L^\infty_x} \|\psi_2^{1/2} u\|_{L^2_x}^2.
\end{equation}

Apply the 1D estimate in $x$,
\[\|\psi_1 u^2\|_{L^\infty_x} \lesssim \int_x |(\psi_1)_x| u^2 \, dx + \int_x \psi_1 |u_x| |u| \, dx.\]

In the first term, use $|(\psi_1)_x| \lesssim \psi_1$. Split $\psi_1 = \psi_{11}\psi_{12}$, with $\psi_{11}$ and $\psi_{22}$ yet to be determined, and apply Cauchy-Schwarz to get
\[\|\psi_1 u^2\|_{L^\infty_x} \lesssim (\|\psi_{11} u_x\|_{L^2_x} + \|\psi_{11} u\|_{L^2_x})\|\psi_{12} u\|_{L^2_x}.\]

Plug this into (6.2) to obtain
\[\int_x \psi u^4 \, dx \lesssim (\|\psi_{11} u_x\|_{L^2_x} + \|\psi_{11} u\|_{L^2_x})\|\psi_{12} u\|_{L^2_x} \|\psi_{1/2} u\|_{L^2_x}^2.\]

Take $\psi_{12} = \psi_{2}^{1/2}$ to equalize the weights,
\[\int_x \psi u^4 \, dx \lesssim (\|\psi_{11} u_x\|_{L^2_x} + \|\psi_{11} u\|_{L^2_x})\|\psi_{1/2} u\|_{L^2_x}^3.\]

Apply the $y$ integral, and on the right-side Cauchy-Schwarz in $y$
\[\int_x \int_y \psi u^4 \, dx \, dy \lesssim (\|\psi_{11} u_x\|_{L^2_y} + \|\psi_{11} u\|_{L^2_y})\|\psi_{1/2} u\|_{L^2_y}^3 \|\psi_{1/2} u\|_{L^2_y}^3.\]

Apply Minkowski’s integral inequality to switch the norms in the last term
\begin{equation}
\int_x \int_y \psi u^4 \, dx \, dy \lesssim (\|\psi_{11} u_x\|_{L^2_y} + \|\psi_{11} u\|_{L^2_y})\|\psi_{1/2} u\|_{L^2_y}^3 \|\psi_{1/2} u\|_{L^2_y}^3.\end{equation}

Now we focus on the “inside” of the last term for fixed $x$.
\[\|\psi_{1/2} u\|_{L^6_y}^6 = \int \psi_{1/2} u^6 \, dy \lesssim \|\psi_{1/2} u^4\|_{L^\infty_y} \int u^2 \, dy \lesssim \|\psi_{1/2} u^2\|_{L^\infty_y}^2 \|\psi_{1/2} u\|_{L^2_y}^2.\]

Passing to the 1/6 power, we get
\[\|\psi_{1/2} u\|_{L^6_y}^6 \lesssim \|\psi_{1/2} u^2\|_{L^\infty_y} \|\psi_{1/2} u\|_{L^2_y}^2.\]

For this first term, we apply the 1D fundamental theorem of calculus estimate
\[\|\psi_{1/2} u\|_{L^6_y} \lesssim (\|\psi_{1/2} u_y\|_{L^1_y} + \|\psi_{1/2} u_y^2\|_{L^1_y})^{1/3} \|\psi_{1/2} u\|_{L^2_y}^{1/3}.\]

Distributing the $y$ derivative, we obtain
\[\|\psi_{1/2} u\|_{L^6_y} \lesssim (\|\psi_{1/2} u_y u\|_{L^1_y} + \|\psi_{1/2} (u_y u^2)\|_{L^1_y})^{1/3} \|\psi_{1/2} u\|_{L^2_y}^{1/3}.\]

Using that $|(\psi_2)_y| \lesssim \psi_2$ and Cauchy-Schwarz, we continue
\[\|\psi_{1/2} u\|_{L^6_y} \lesssim (\|\psi_{1/2} u_y u\|_{L^1_y} + \|\psi_{1/2} u\|_{L^2_y})^{1/3} \|\psi_{1/2} u\|_{L^2_y}^{1/3}.\]
Now apply the $L^2_x$ norm, and use Hölder with partition $\frac{1}{2} = \frac{1}{6} + \frac{1}{3}$ to obtain
$$
\|\psi_2^{1/2} u\|_{L^2_x L^6_y} \lesssim (\|\psi_2^{3/2} u_y\|_{L^2_y} + \|\psi_2^{3/2} u\|_{L^2_y})^{1/3} \|u\|_{L^2_y}^{2/3}.
$$

Insert this into (6.3) to obtain
$$
\int_x \int_y \psi u^4 dx dy \lesssim (\|\psi_1 u_x\|_{L^2_y} + \|\psi_1 u\|_{L^2_y})(\|\psi_2^{3/2} u_y\|_{L^2_y} + \|\psi_2^{3/2} u\|_{L^2_y}) \|u\|_{L^2_y}^2.
$$

Now let us review the weight partitions. We took $\psi = \psi_1 \psi_2$, $\psi_1 = \psi_1 \psi_1$, but we required that $\psi_1 = \psi_1^{1/2}$, so that in fact $\psi = \psi_1 \psi_2$, with $\psi_2$ yet to be determined. From the last inequality, we see that we would like to have $\psi_2 = \psi^{1/2}$ and $\psi_2^{3/2} = \psi^{1/2}$, and this in fact does meet the condition $\psi = \psi_1 \psi_2$. Thus, we have
$$
\int_x \int_y \psi u^4 dx dy \lesssim (\|\psi^{1/2} u_x\|_{L^2_y} + \|\psi^{1/2} u\|_{L^2_y})(\|\psi^{1/2} u_y\|_{L^2_y} + \|\psi^{1/2} u\|_{L^2_y}) \|u\|_{L^2_y}^2.
$$

\[\square\]

**Lemma 6.2 (I± estimates).** Let $t_{-1} < t_0 < t_1$ and suppose that $u(t)$ is an $H^1$ solution to ZK on $[t_{-1}, t_1]$ with $E(u) < 0$ and
$$
\forall t \in [t_{-1}, t_1] \quad \|\nabla u(t)\|_{L^2_y} \geq 0.9 \|\nabla Q\|_{L^2_y}.
$$

There exists an absolute constant $\alpha_4 > 0$ such that if
$$
\alpha(u) \overset{\text{def}}{=} \|u\|^2_{L^2} - \|Q\|^2_{L^2} \leq \alpha_4,
$$
we then have the following. Let
$$
I_{\pm, x_0, t_0}(t) = \iint u^2(x + x(t_0), y, t) \phi_\pm(x - x_0 - \frac{1}{2}(x(t) - x(t_0))) \, dx \, dy,
$$
where $\phi_-(x) = \phi_+(-x)$ and
$$
\phi_+(x) = \frac{2}{\pi} \arctan(e^{x/K}),
$$
so that $\phi_+(x)$ is increasing with $\lim_{x \to -\infty} \phi_+(x) = 0$ and $\lim_{x \to +\infty} \phi_+(x) = 1$, and $\phi_-(x)$ is decreasing with $\lim_{x \to -\infty} \phi_-(x) = 1$ and $\lim_{x \to +\infty} \phi_-(x) = 0$. Let $x_0 > 0$ and $K \geq 4$. For the increasing weight, we have two estimates that bound the future in terms of the past
\[6.4\]
for $t_{-1} < t_0$, \quad $I_{+, x_0, t_0}(t) \leq I_{+, x_0, t_0}(t_{-1}) + \theta_0 e^{-x_0/K}$,
\[6.5\]
for $t_0 < t_1$, \quad $I_{+, x_0, t_0}(t) \leq I_{+, x_0, t_0}(t_0) + \theta_0 e^{-x_0/K}$.

For the decreasing weight, we have two estimates that bound the past in terms of the future
\[6.6\]
for $t_{-1} < t_0$, \quad $I_{-, x_0, t_0}(t_{-1}) \leq I_{-, x_0, t_0}(t_0) + \theta_0 e^{-x_0/K}$,
Figure 6.1. Here $x_0 > 0$. The function $u(x + x(t_0), y, t)$ has the soliton component centered at position $(x(t) - x(t_0), y(t))$. In this figure we depict only the $x$ spatial direction horizontally, and time vertically. At time $t = t_0$ the soliton is centered in this frame of reference at position $x(t) - x(t_0) = 0$, and the soliton trajectory is the line $x(t) - x(t_0)$ with slope approximately 1. For $I_+$ the functional with increasing weight $\phi_+$, we can estimate forward in time from $t_0$ to $t_1$ or from $t_1$ to $t_0$. In the case of $t_0$ to $t_1$, the weight $\phi_+$ has transition centered along the right half-slope line $x_0 + \frac{1}{2}(x(t) - x(t_0))$ and in the case of $t_1$ to $t_0$, the weight $\phi_+$ has transition centered along the left half-slope line $-x_0 + \frac{1}{2}(x(t) - x(t_0))$. In either case, the essential aspect is that the soliton and weight center trajectories are separating in time as we move forward or backward. For $I_-$, the function with decreasing weight $\phi_-$, the trajectories are the same except the estimates are backwards in time from $t_1$ to $t_0$ or from $t_0$ to $t_-1$.

(6.7) \[ \text{for } t_0 < t_1, \quad I_{-,-x_0,t_0}(t_0) \leq I_{-,-x_0,t_0}(t_1) + \theta_0 e^{-x_0/K}. \]

Here, $\theta_0$ is some absolute constant. These are depicted in Figure 6.1, which shows the path of the “center” of transition of the weight $\phi_\pm$ through time.

Importantly, both $\alpha_4 > 0$ and $\theta_0 > 0$ are absolute constants, in particular, independent of the upper bound on $\|\nabla u(t)\|_{L^2_{x,y}}$ over the time interval $[t_-, t_1]$.

Proof. By time translation, it suffices to assume that $t_0 = 0$, and recall that $x_0 > 0$. For ease of reference, let us rewrite them in this case: For the increasing weight, we have two estimates that bound the future in terms of the past

(6.8) \[ \text{for } t_0 < 0, \quad I_{+,x_0,0}(0) \leq I_{+,x_0,0}(t_0) + \theta_0 e^{-x_0/K}, \]

(6.9) \[ \text{for } 0 < t_1, \quad I_{-,x_0,0}(t_1) \leq I_{-,x_0,0}(0) + \theta_0 e^{-x_0/K}. \]
For the decreasing weight, we have two estimates that bound the past in terms of the future
\begin{align}
(6.10) & \quad \text{for } t_{-1} < 0, \quad I_{-,x_0,0}(t_{-1}) \leq I_{-,x_0,0}(0) + \theta_0 e^{-x_0/K}, \\
(6.11) & \quad \text{for } 0 < t_1, \quad I_{-,x_0,0}(0) \leq I_{-,x_0,0}(t_1) + \theta_0 e^{-x_0/K}.
\end{align}

To compare the two estimates (6.8), (6.9) with the two estimates (6.10), (6.11), we temporarily augment the notation to include reference to the solution \( u \): \( I_{+,x_0,0,u}(t) \). Now if \( \tilde{u}(x,t) = u(-x,-y,-t) \), then the corresponding soliton parameters are \( \tilde{x}(t) = -x(-t) \) and \( \tilde{y}(t) = -y(-t) \). We obtain
\[
I_{+,x_0,0,u}(t) = \iint \tilde{u}^2(x + \tilde{x}(0), y, t) \phi_+(x - x_0 - \frac{1}{2}(\tilde{x}(t) - \tilde{x}(0))) \, dx \, dy.
\]

Change variable \( x \to -x \) and \( y \to -y \) to get
\[
= \iint u^2(x + x(0), y, -t) \phi_+(-x + x_0 + \frac{1}{2}(x(-t) - x(0))) \, dx \, dy
\]
\[
= \iint u^2(x + x(0), y, -t) \phi_-(x - x_0 - \frac{1}{2}(x(-t) - x(0))) \, dx \, dy
\]
\[
= I_{-,x_0,0,u}(-t).
\]

In summary, we have established the identity
\[
I_{+,x_0,0,u}(t) = I_{-,x_0,0,u}(-t).
\]

Suppose that we have proved (6.8), (6.9). Now applying (6.8) to \( \tilde{u} \), we get
\[
I_{+,x_0,0,u}(0) \leq I_{+,x_0,0,u}(t_{-1}) + \theta_0 e^{-x_0/K}.
\]

Applying the identity, this converts to
\[
I_{-,x_0,0,u}(0) \leq I_{-,x_0,0,u}(-t_{-1}) + \theta_0 e^{-x_0/K}.
\]

Taking \( t_1 = -t_{-1} \) and replacing \( -x_0 \) by \( x_0 \), we obtain (6.11).

Applying (6.9) to \( \tilde{u} \), we get
\[
I_{+,x_0,0,u}(t_1) \leq I_{+,x_0,0,u}(0) + \theta_0 e^{x_0/K}.
\]

Applying the identity,
\[
I_{-,x_0,0,u}(-t_1) \leq I_{-,x_0,0,u}(0) + \theta_0 e^{x_0/K}.
\]

Taking \( t_{-1} = t_1 \) and replacing \( x_0 \) by \(-x_0 \), we obtain (6.10).

Thus, we have shown that (6.8) and (6.9) imply (6.11) and (6.10), and it suffices for us to establish (6.8) and (6.9). For the remainder of the proof, we will drop the + subscript. Both (6.8) and (6.9) are proved by the same principle, which is to take the time derivative of \( I_{x_0,0}(t) \) and then note that it suffices to estimate the nonlinear term. The integration region is divided into two parts – near the soliton trajectory,
which is estimated using the smallness of $\phi'$, and away from the soliton trajectory, which is estimated using the smallness of $u$.

We start by noting that $0.9 \leq \lambda^2 x_t \leq 1.1$ and $\lambda \leq 1.1$ implies that

$$0.7 \leq \frac{0.9}{1.1^2} \leq \frac{0.9}{\lambda^2} \leq x_t.$$

We have

$$\phi'(x) = \frac{1}{\pi K} \text{sech}(x/K), \quad \phi''(x) = -\frac{1}{\pi K^2} \text{sech}(x/K) \tanh(x/K),$$

$$\phi'''(x) = \frac{1}{\pi K^3} (\text{sech}(x/K) \tanh^2(x/K) - \text{sech}^3(x/K)).$$

In particular, we have that

$$|\phi'''(x)| \leq \frac{1}{K^2} \phi'(x).$$

Let

$$I_{x_0,t_0}(t) = \int \int_{xy} u^2(x + x(t_0), y, t) \phi(x - x_0 - \frac{1}{2}(x(t) - x(t_0))) \, dx \, dy.$$

We compute

$$\frac{1}{2} \partial_t I_{x_0,t_0}(t) = \int \int_{xy} uu_t \phi \, dx \, dy - \frac{1}{2} x'(t) \int \int_{xy} u^2 \phi' \, dx \, dy$$

$$= \int \int_{xy} u(-u_{xxx} - u_{yy} - 3u^2u_x) \phi \, dx \, dy - \frac{1}{2} x'(t) \int \int_{xy} u^2 \phi' \, dx \, dy.$$

After several applications of integration by parts, we arrive at

$$\frac{1}{2} \partial_t I_{x_0,t_0}(t) = -\frac{3}{2} \int \int_{xy} u_x^2 \phi' - \frac{1}{2} \int \int_{xy} u_y^2 \phi' - \frac{1}{2} x'(t) \int \int_{xy} u^2 \phi'$$

$$+ \frac{1}{2} \int \int_{xy} u^2 \phi''' + \frac{3}{4} \int \int_{xy} u^4 \phi'.$$

Since $0.7 < x_t$, the term

$$-\frac{1}{2} x'(t) \int \int_{xy} u^2 \phi' \leq -\frac{1}{4} \int \int u^2 \phi'$$

as well as the first two terms in the first line of (6.13) can be used to absorb error terms.

By (6.12), we have, provided we fix $K \geq 4$,

$$\frac{1}{2} \int \int u^2 \phi''' \leq \frac{1}{32} \int \int u^2 \phi'.$$

We will go ahead and fix $K = 4$ at the end, since it seems there is no need to take it larger.

Now decompose into

$$B_1 = \{ (x, y) | |x - (x(t) - x(t_0))| > R_0 \text{ or } |y - y(t)| > R_0 \}$$ and
We have interval and \( \tilde{e} \) to ensure that \( \alpha \) and make \( \lambda \) small using that the coefficient of the first term is made small using that 

\[
\text{If } t < t, \text{ then } R(t) \text{ is an absolute constant.}
\]

Integrating in time, we have 

\[
\frac{3}{4} \int B u^4 \phi' = \int (x,y) \in B_1 u^4 \phi' dy dx + \int (x,y) \in B_2 u^4 \phi' dy dx.
\]

In \( B_1 \), we apply the weighted Gagliardo-Nirenberg \( (6.1) \), for any \( R_0 > 0 \), 

\[
(6.15) \quad \frac{3}{4} \int u^4 \phi' \leq \|u\|_{L^2_{x-x(t)>R_0 \text{ or } |y-y(t)|>R_0}}^2 \int (u^2 + u_x^2 + u_y^2) \phi' dx dy
\]

\[+ \frac{3}{4} \int (x,y) \in B_2 u^4 \phi' dy dx.\]

Note that the coefficient of the first term is made small using that \( \lambda u(\lambda x + x(t), \lambda y + y(t), t) - Q(x, y) \) is small in \( L^2 \), and \( Q \) is small in that region. This just requires that we use the upper bound on \( \lambda \) and not the lower bound on \( \lambda \) and that \( \|e\|_{H^1} \lesssim \sqrt{\alpha} \), and make \( \alpha \) smaller than some absolute constant. The \( R_0 \) just needs to be taken large to ensure that \( e^{-R_0/\lambda} \leq e^{-R_0/1.1} \) is sufficiently smaller than an absolute constant, so \( R_0 \) is an absolute constant.

Plug \( (6.15), (6.14) \) into \( (6.13) \) to obtain 

\[
\partial_t I_{x_0,t_0}(t) \lesssim - \int (|\nabla u|^2 + |u|^2) \phi' dx dy + \int_{(x,y) \in B_2} u^4 \phi' dx dy.
\]

Integrating in time, we have 

\[
I_{x_0,t_0}(t_0) \leq I_{x_0,t_0}(t_1) + \int_{t_1}^{t_0} \int (x,y) \in B_2 u^4 \phi' dx dy
\]

and 

\[
I_{x_0,t_0}(t_1) \leq I_{x_0,t_0}(t_0) + \int_{t_0}^{t_1} \int (x,y) \in B_2 u^4 \phi' dx dy.
\]

For \( B_2 \), we keep the spatial restriction on the weight \( \phi' \), the rest estimating using the (standard) Gagliardo-Nirenberg. Let \( \tilde{x} = x - x_0 - \frac{1}{2}(x(t) - x(t_0)) \) for the \([t_1, t_0] \) interval and \( \tilde{x} = x + x_0 - \frac{1}{2}(x(t) - x(t_0)) \) for the \([t_0, t_1] \) interval. Then 

\[
\int (x,y) \in B_2 u^4 \phi' dx dy \lesssim \left( \sup_{(x,y) \in B_2} \phi'(\tilde{x}) \right) \|u\|^2_{L^2_{\tilde{x}}} \|\nabla u\|^2_{L^2_{\tilde{x}}}
\]

We have \(|x - (x(t) - x(t_0))| < R_0 \) and thus writing \( \tilde{x} = (x - x(t_0) - x(t)) - x_0 - \frac{1}{2}x(t_0) + \frac{1}{2}x(t) \), we see that 

\[
|\tilde{x}| \geq |x_0 + \frac{1}{2}(x(t) - x(t_0)) - R_0|
\]

If \( t < t_0 \), then \( x(t) < x(t_0) \) and 

\[
|\tilde{x}| \geq x_0 + \frac{1}{2}(x(t_0) - x(t)) - R_0.
\]
If $t > t_0$, then $x(t) > x(t_0)$, and replacing $x_0$ by $-x_0$, we have

$$|\tilde{x}| \geq x_0 + \frac{1}{2}(x(t) - x(t_0)) - R_0.$$  

Since

$$\phi'(\tilde{x}) \leq \frac{1}{\pi K} \text{sech}(\tilde{x}/K) \leq \frac{2}{\pi K} e^{-|\tilde{x}|/K},$$

we obtain

$$\phi'(\tilde{x}) \leq \frac{1}{K} \exp(-x_0/K) \exp\left(\frac{R_0}{K}\right) \exp\left(-\frac{|x(t) - x(t_0)|}{2K}\right),$$

which is a bound independent of $x$. Plugging in, we obtain

$$\int \int_{(x,y) \in B_2} u^4 \phi' \, dx \lesssim \|u\|^2_{L^2_{E_y}} \|\nabla u\|^2_{L^2_{E_y}} \frac{1}{K} \exp(-x_0/K) \exp\left(\frac{R_0}{K}\right) \exp\left(-\frac{|x(t_0) - x(t)|}{2K}\right).$$

Using that $\|u\|^2_{L^2} \lesssim 1$ and $\|\nabla u(t)\|^2_{L^2_{E_y}} \sim \lambda^{-2} \lesssim x_t$,

$$\int \int_{(x,y) \in B_2} u^4 \phi' \, dx \lesssim \frac{1}{K} \exp(-x_0/K) \exp\left(\frac{R_0}{K}\right) x_t \exp\left(-\frac{|x(t_0) - x(t)|}{2K}\right).$$

Upon integrating in time, separately considering the cases $t < t_0$ and $t > t_0$, we obtain

$$\int_{t_0}^{t_0} \int \int_{(x,y) \in B_2} u^4 \phi' \, dx \lesssim \exp(-x_0/K) \exp\left(\frac{R_0}{K}\right)(1 - \exp\left(-\frac{t_0 - x(t_0)}{2K}\right))$$

and

$$\int_{t_0}^{t_1} \int \int_{(x,y) \in B_2} u^4 \phi' \, dx \lesssim \exp(-x_0/K) \exp\left(\frac{R_0}{K}\right)(1 - \exp\left(-\frac{t_1 - x(t_0)}{2K}\right)).$$

\[\square\]

**Lemma 6.3** (applying $I_\pm$ estimates to obtain exponential decay of $\tilde{u}_n$). For $x_0 > 0$, for all $-t_1(n) < t < t_2(n)$, we have

$$\|\tilde{u}_n(x + \tilde{x}(t), y, t)\|^2_{L^2_{|x| > x_0} L^2_{E_y}} \leq 24\theta_0 e^{-x_0/8},$$

where $\theta_0 > 0$ is an absolute constant.

**Proof.** Recall that $n$ is fixed in the proof. It suffices to prove the claim for any finite length interval $(-t_1, t_2) \subset (-t_1(n), t_2(n))$ (that is, for which $t_1, t_2 < \infty$).

First, we prove the decay on the right. We will in fact prove the following stronger statement: There exists $m(x_0)$ such that for all $m \geq m(x_0)$ and all $-t_1 \leq t \leq t_2$,

$$\|u_n(x + x_n(t, m + t), y, t)\|^2_{L^2_{|x| > x_0} L^2_{E_y}} \leq 6\theta_0 e^{-x_0/4}.$$  

Since by Corollary 5.7

$$u_n(\cdot + x_n(t, m + t), \cdot + y_n(t, m + t), t, m + t) \rightarrow \tilde{u}_n(\cdot + \tilde{x}(t), \cdot + \tilde{y}(t), t)$$  

weakly as $m \to \infty$, this will imply that

$$\|\tilde{u}_n(x + \tilde{x}(t), y, t)\|^2_{L^2_{|x| > x_0} L^2_{E_y}} \leq 6\theta_0 e^{-x_0/4}.$$  

Arguing by contradiction, if \[ (6.16) \] fails, there exists a subsequence \( m' \) and a corresponding sequence of times \( t_{m'} \) in \( -t_1 \leq t_{m'} \leq t_2 \) such that
\[
(6.18) \quad \| u_n(x + x_n(t_{n,m'} + t_{m'}), y, t_{n,m'} + t_{m'}) \|_{L^2_{x>x_0} L^2_y}^2 \geq 6\theta_0 e^{-x_0/4}.
\]

Passing to another subsequence so that \( t_{m'} \to t_* \), and using the uniform continuity\(^6\) of \( u_n(t) \) over \([t_{n,m} - t_1, t_{n,m} + t_2]\) for every \( m \), and the fact that \( x_n(t_{n,m'} + t_{m'}) - x_n(t_{n,m'} + t_*) \to 0 \),
\[
(6.19) \quad \| u_n(x + x_n(t_{n,m'} + t_*), y, t_{n,m'} + t_*) \|_{L^2_{x>x_0} L^2_y}^2 \geq 4\theta_0 e^{-x_0/4}.
\]

Restricting attention to this subsequence, relabeling it as \( m \), we can declare that there exists \( t_* \) with \( -t_1 \leq t_* \leq t_2 \) and \( m(x_0) \) such that for all \( m \geq m(x_0) \),
\[
\| u_n(x + x_n(t_{n,m} + t_*), y, t_{n,m} + t_*) \|_{L^2_{x>x_0} L^2_y}^2 \geq 4\theta_0 e^{-x_0/4}.
\]

We apply the \( I \) estimate with \( t_0 = 0 \) and \( t_0 = t_{n,m} + t_* \) (where the weight transition occurs on the right of the soliton trajectory). Now
\[
I_{-x_0,t_0}(t_0) = \iint u_n(x + x(t_0), y, t_0)^2 \phi_-(x - x_0) \, dx \, dy
\]
\[
= \iint u_n(x + x(t_0), y, t_0)^2 \, dx \, dy - \iint u_n(x + x(t_0), y, t_0)^2(1 - \phi_-(x - x_0)) \, dx \, dy.
\]

Using that \( \frac{1}{2} \leq (1 - \phi_-(x - x_0)) \) for \( x > x_0 \), we have
\[
I_{-x_0,t_0}(t_0) \leq M(u_n) - \frac{1}{2} \int \int_{x > x_0, y \in \mathbb{R}} u_n(x + x(t_0), y, t_0)^2 \, dx \, dy
\]
\[
= M(u_n) - \frac{1}{2} \int \int_{x > x_0, y \in \mathbb{R}} u_n(x + x(t_{n,m} + t_*), y, t_{n,m} + t_*)^2 \, dx \, dy,
\]
and hence,
\[
(6.20) \quad I_{-x_0,t_0}(t_0) \leq M(u_n) - 2\theta_0 e^{-x_0/4}.
\]

On the other hand,
\[
I_{-x_0,t_0}(t_0) = \iint u_n(x + x(t_0), y, t_0)^2 \phi_-(x - x_0 - \frac{1}{2}(x(t_0) - x(t_0))) \, dx \, dy
\]
\[
= \iint u_n(x + x(t_{n,m} + t_*), y, t_0)^2 \phi_-(x - x_0 - \frac{1}{2}(x(0) - x(t_{n,m} + t_*))) \, dx \, dy.
\]

Therefore,
\[
I_{-x_0,t_0}(t_0) = \iint u_n(x, y, t_0)^2 \phi_-(x - x_0 - \frac{1}{2}(x(0) + x(t_{n,m} + t_*))) \, dx \, dy.
\]

\(^6\)This can be proved using the local theory estimates in Lemmas 3.1, 3.2 together with the bootstrap assumption of (5.2).

\(^7\)This follows from Lemma 4.3, which gives the bound \( |\lambda_n(t)|^2 x_n(t) - 1| \lesssim \alpha(u_n)^{1/2} \), which implies a uniform upper bound on \( x_n(t) \) on \([t_{n,m} - t_1, t_{n,m} + t_2]\) for all \( m \).
Since \( u_n(\bullet, \bullet, 0) \) is a fixed function, and \( x(t_{n,m} + t) \rightarrow +\infty \) as \( m \rightarrow +\infty \), we have

\[
\lim_{m \rightarrow +\infty} I_{-x_0, t_0}(t) = \int \int u_n(x, y, 0)^2 \, dx \, dy = M(u_n).
\]

By Lemma 6.2 with \( K = 4 \),

\[
I_{-x_0, t_0}(t) \leq I_{-x_0, t_0}(t_0) + \theta_0 e^{-x_0/4}.
\]

By (6.20), (6.21) and sending \( m \rightarrow \infty \), we get

\[
M(u_n) \leq M(u_n) - \theta_0 e^{-x_0/4}
\]
a contradiction. This completes the proof of (6.16).

Before proving the decay on the left, let us note a consequence of (6.16). Since

\[
u_n(x + x_n(t_{n,m} + t), y + y_n(t_{n,m} + t), t_{n,m} + t) \rightarrow \tilde{u}_n(x + \tilde{x}_n(t), y + \tilde{y}_n(t), t)
\]

strongly in \( L^2_{\text{loc}}(\mathbb{R}^2) \), there exists \( m(x_0) \) such that for all \( m \geq m(x_0) \), we have

\[
\|u_n(x + x_n(t_{n,m} + t), y, t_{n,m} + t)\|_{L^2_{x \leq x_0 < \infty, \theta_y}}^2 \leq \|\tilde{u}_n(x + \tilde{x}_n(t), y + \tilde{y}_n(t), t)\|_{L^2_{x \leq x_0 < \infty, \theta_y}}^2 + \theta_0 e^{-x_0/4}.
\]

Note that

\[
\|u_n(x + x_n(t_{n,m} + t), y, t_{n,m} + t)\|_{L^2_{x > x_0 \theta_y}}^2 = \|u_n(x + x_n(t_{n,m} + t), y, t_{n,m} + t)\|_{L^2_{x \leq x_0 < \infty, \theta_y}}^2 + \|u_n(x + x_n(t_{n,m} + t), y, t_{n,m} + t)\|_{L^2_{x > x_0 \theta_y}}^2.
\]

By (6.22), (6.23) and (6.16), we have

\[
\|u_n(x + x_n(t_{n,m} + t), y, t_{n,m} + t)\|_{L^2_{x > x_0 \theta_y}}^2 \leq \|\tilde{u}_n(x + \tilde{x}_n(t), y + \tilde{y}_n(t), t)\|_{L^2_{x \leq x_0 < \infty, \theta_y}}^2 + 7\theta_0 e^{-x_0/4}
\]

\[
\leq M(\tilde{u}_n) - \|\tilde{u}_n(x + \tilde{x}_n(t), y + \tilde{y}_n(t), t)\|_{L^2_{x \leq x_0 < \infty, \theta_y}}^2 + 7\theta_0 e^{-x_0/4}.
\]

Next, we prove the decay on the left\(^8\) i.e., for all \( -t_1 \leq t \leq t_2 \),

\[
\|\tilde{u}_n(x + \tilde{x}_n(t), y, t)\|_{L^2_{x \leq x_0 < \infty, \theta_y}}^2 \leq 24\theta_0 e^{-x_0/8}.
\]

Arguing by contradiction, we assume that there exists \( t_* \) such that

\[
\|\tilde{u}_n(x + \tilde{x}_n(t_*), y, t_*)\|_{L^2_{x \leq x_0 < \infty, \theta_y}}^2 \geq 22\theta_0 e^{-x_0/8}.
\]

Combining with (6.24) gives

\[
\|u_n(x + x_n(t_{n,m} + t_*), y, t_{n,m} + t_*)\|_{L^2_{x > x_0 \theta_y}}^2 \leq M(\tilde{u}_n) - 15\theta_0 e^{-x_0/8}.
\]

\(^8\)Note that the bound is \( e^{-x_0/8} \) instead of \( e^{-x_0/4} \) as provided by Lemma 6.2 with \( K = 4 \). This is because we use a shift \( x_0/2 \) instead of \( x_0 \) – see below.
We apply the $I_+$ estimate for $K = 4$ with weight transition on the left, from $t_0 = t_{n,m(x_0)} + t_*$ to $t_1 = t_{n,m}$ for $m$ sufficiently large so that $t_{n,m} \geq t_{n,m(x_0)} + t_*$. 

We have 

$$I_{+, -x_0/2, t_0}(t_0) = \iint u_n(x + x_n(t_0), y, t_0)^2 \phi_+(x + \frac{1}{2}x_0) \, dx \, dy$$

$$= \iint u_n(x + x_n(t_{n,m(x_0)} + t_*) + x_n(t_{n,m(x_0)} + t_*)^2 \phi_+(x + \frac{1}{2}x_0) \, dx \, dy.$$ 

Since $\phi_+(x + \frac{1}{2}x_0) \leq \phi_+(\frac{-1}{2}x_0)$ for $x < -x_0$,

$$I_{+, -x_0/2, t_0}(t_0) \leq \phi_+(\frac{-1}{2}x_0) M(u_n) + \iint_{x > -x_0} u_n(x + x_n(t_{n,m(x_0)} + t_*), y, t_{n,m(x_0)} + t_*)^2 \, dx \, dy.$$ 

By (6.26),

$$I_{+, -x_0/2, t_0}(t_0) \leq M(\tilde{u}_n) - 15\theta_0 \phi_+(\frac{-1}{2}x_0) M(u_n).$$

Since $\phi_+(x) = \frac{2}{\pi} \arctan(e^{x/K})$, we have, for $x \to -\infty$, $\phi_+(x) \leq e^{x/K}$, and hence $\phi_+(\frac{-1}{2}x_0) \leq e^{-x_0/2K} = e^{-x_0/8}$.

Moreover, we can assume, without loss, that $\theta_0 > 0$ in Lemma 6.2 was taken large enough so that $\theta_0 \geq 2\|Q\|_{L_2}^2$. Then from (6.27), we obtain

$$I_{+, -x_0/2, t_0}(t_0) \leq M(\tilde{u}_n) - 12\theta_0 e^{-x_0/8}.$$ 

On the other hand,

$$I_{+, -x_0/2, t_0}(t_1) = \iint u_n(x + x_n(t_0), y, t_1)^2 \phi_+(x + \frac{1}{2}x_0 - \frac{1}{2}(x_n(t_1) - x(t_0))) \, dx \, dy$$

$$= \iint u_n(x + x_n(t_1), y + y_n(t_1), t_1)^2$$

$$\phi_+(x + \frac{1}{2}x_0 + \frac{1}{2}(x_n(t_1) - x_n(t_0))) \, dx \, dy$$

$$= \iint u_n(x + x_n(t_{n,m}), y + y_n(t_{n,m}), t_{n,m})^2$$

$$\phi_+(x + \frac{1}{2}x_0 + \frac{1}{2}(x_n(t_{n,m}) - x(t_{n,m(x_0)} + t_*))) \, dx \, dy.$$ 

Note that $\phi_+(x + \frac{1}{2}x_0 + \frac{1}{2}(x_n(t_{n,m}) - x_n(t_{n,m(x_0)} + t_*))) \to 1$ pointwise as $m \to \infty$. 

Since 

$$u_n(x + x(t_{n,m}), y + y(t_{n,m}), t_{n,m}) \to \tilde{u}_n(x + \tilde{x}_n(0), y + \tilde{y}_n(0), 0)$$ 

as $m \to \infty$, it follows that also

$$u_n(x + x(t_{n,m}), y + y(t_{n,m}), t_{n,m}) \phi_+(x + \frac{1}{2}x_0 + \frac{1}{2}(x_n(t_{n,m}) - x(t_{n,m(x_0)} + t_*)))^{1/2}$$

This is why it is $e^{-x_0/8}$ instead of $e^{-x_0/4}$. 


as \( m \to \infty \) (consider that the \( \phi_+ \) term times the test function converges to the test function). By the fact that the norm of the weak limit is less than or equal to the limit of the norms,

\[
M(\tilde{u}_n) = \|\tilde{u}_n(x + \tilde{x}_n(0), y + \tilde{y}_n(0), 0)\|_{L^2_y}^2 \leq \lim_{m \to \infty} I_{+, -x_0/2, t_0}(t_1).
\]

The \( I_+ \) estimate from Lemma 6.2 states

\[
I_{+, -x_0/2, t_0}(t_1) \leq I_{+, -x_0/2, t_0}(t_0) + \theta_0 e^{-x_0/4}.
\]

By (6.28), (6.29), taking \( m \to \infty \), we obtain

\[
M(\tilde{u}_n) \leq M(\tilde{u}_n) - 11\theta_0 e^{-x_0/8},
\]

which is a contradiction. This completes the proof of (6.25). Combining (6.17) and (6.25) completes the proof of lemma.

**Lemma 6.4** (pointwise-in-\( x \) estimates of \( \tilde{u}_n \) and \( \tilde{\epsilon}_n \)). For \( n \) sufficiently large, we have, uniformly in \( t \),

\[
\|\tilde{u}_n(x + \tilde{x}_n(t), y, t)\|_{L^2_y} \lesssim \tilde{\lambda}_n(t)^{-1/2} e^{-|x|/32}
\]

and

\[
\|\tilde{\epsilon}_n(x, y, t)\|_{L^2_y} \lesssim \alpha(\tilde{u}_n)^{1/4} e^{-\tilde{\lambda}_n(t)|x|/32}.
\]

**Proof.** For the proof let us instead write \( u \) for \( \tilde{u}_n \), \( x(t) \) for \( \tilde{x}_n(t) \), \( \lambda(t) \) for \( \tilde{\lambda}_n(t) \), and similarly, we just write \( \epsilon \) in place of \( \tilde{\epsilon}_n \). We have, for \( x_0 > 0 \),

\[
\|u(x_0 + x(t), y, t)\|_{L^2_y}^2 \lesssim \|u(x + x(t), y, t)\|_{L^2_{y > x_0}} \|u_x(x, y, t)\|_{L^2_y}.
\]

By Cauchy-Schwarz in \( y \) on the outside, we get

\[
\|u(x_0 + x(t), y, t)\|_{L^2_y} \lesssim \|u(x + x(t), y, t)\|_{L^2_{y > x_0}} \|u_x(x, y, t)\|_{L^2_y L^2_y}.
\]

By Lemma 6.3

\[
\|u(x_0 + x(t), y, t)\|_{L^2_y}^2 \lesssim \lambda(t)^{-1} e^{-x_0/16},
\]

where we have used that \( \lambda(t)^{-1} \sim \|\nabla u(t)\|_{L^2} \) from (4.6). A similar argument works for \( x_0 < 0 \).

Similar to the above, for \( x_0 \in \mathbb{R} \), we have

\[
\|\epsilon(x_0, y, t)\|_{L^2_y}^2 \lesssim \|\epsilon(x, y, t)\|_{L^2_{|x| > |x_0|} L^2_y} \|\epsilon_x(x, y, t)\|_{L^2_y}.
\]

The second term is bounded by \( \alpha(u)^{1/2} \). For the first term, we use the definition of \( \epsilon \) in terms of \( u \) and \( Q \) to estimate

\[
\|\epsilon(x, y, t)\|_{L^2_{|x| > |x_0|} L^2_y} \lesssim \|u(x + x(t), y, t)\|_{L^2_{|x| > \lambda|x_0|} L^2_y} + \|Q(x, y)\|_{L^2_{|x| > |x_0|} L^2_y}.
\]
By Lemma 6.3, we get
\[ \| \epsilon(x, y, t) \|_{L^2_{|x|>|x_0|}} \lesssim e^{-\lambda|x_0|/16} + e^{-|x_0|} \lesssim e^{-\lambda|x_0|/16}. \]
Plug this into (6.30), to obtain
\[ \| \epsilon(x, y, t) \|_{L^2_y}^2 \lesssim \alpha(u)^{1/2} e^{-\lambda|x|/16}. \]

7. Control of \( \tilde{\lambda}_n(t) \) via the \( L^1 \)-type invariance

Lemma 7.1 (integral conservation yields control on scale). For any solution \( u(t) \) with \( \alpha(u) \) defined as \( M(u) - M(Q) \) and \( E(u) < 0 \), let
\[ \epsilon(x, y, t) = \lambda(t) u(\lambda(t)x + x(t), \lambda(t)y + y(t), t) - Q(x, y) \]
with parameters \( \lambda(t), x(t), y(t) \) as given by Lemma 4.2 and \( \| \epsilon(t) \|_{H^1_{xy}} \lesssim \alpha(u)^{1/2} \).
Suppose that
\[ 0.9 \leq \lambda(0) \leq 1.1 \]
and for all \( -T_*^- < t < T_*^+ \), we have both
\[ 0 < \lambda(t) \leq 1.1 \]
and the \( x \)-pointwise estimate uniformly in \( t \)
\[ \| \epsilon(x, y, t) \|_{L^2_y} \lesssim \alpha(u)^{1/4} e^{-\lambda|x|/32}. \]
Let \( (-T^-, T^+) \) be the maximal time interval around 0 contained in \( (-T_*^-, T_*^+) \) such that for all \( -T^- < t < T^+ \), we have
\[ \lambda(t) \geq \frac{3}{4}. \]
Then there exists an absolute \( \alpha_5 > 0 \) such that for \( \alpha(u) \leq \alpha_5 \), we have \( (-T_*^-, T_*^+) = (-T^-, T^+) \).

Proof. First we note that by integrating (7.1) in \( x \), for each \( t \) such that \( -T^- < t < T^+ \), we have
\[ \| \epsilon(t, y) \|_{L^2_y L^1_x} \leq \| \epsilon(t, x, y) \|_{L^1_x L^2_y} \lesssim \alpha(u)^{1/4}. \]
Let
\[ F(t) \overset{\text{def}}{=} \left\| \int_x (Q(x, y) + \epsilon(x, y, t)) \, dx \right\|_{L^2_y}^2 - \left\| \int_x Q(x, y) \, dx \right\|_{L^2_y}^2. \]
By expanding the square and using Cauchy-Schwarz (in \( y \)), we get
\[ F(t) \leq 2\| Q \|_{L^2_y L^1_x} \| \epsilon(t) \|_{L^2_y L^1_x} + \| \epsilon(t) \|_{L^2_y L^1_x}^2 \lesssim \alpha(u)^{1/4}. \]
Substituting the definition of $\epsilon$, we let

$$F(t) \overset{\text{def}}{=} \left\| \int_x \lambda(t)u(\lambda(t)x + x(t), \lambda(t)y + y(t), t) \, dx \right\|_{L^2_y}^2 - \left\| \int_x Q(x, y) \, dx \right\|_{L^2_y}^2.$$

Scaling and translating in $x$ and $y$, we write

$$F(t) \overset{\text{def}}{=} \lambda(t)^{-1} \left\| \int_x u(x, y, t) \, dx \right\|_{L^2_y}^2 - \left\| \int_x Q(x, y) \, dx \right\|_{L^2_y}^2.$$

But

$$\left\| \int_x u(x, y, t) \, dx \right\|_{L^2_y} = \left\| \int_x u(x, y, 0) \, dx \right\|_{L^2_y},$$

and hence,

$$F(t) - F(0) = (\lambda(t)^{-1} - \lambda(0)^{-1}) \left\| \int_x u(x, y, t) \, dx \right\|_{L^2_y} \left\| \int_x u(x, y, 0) \, dx \right\|_{L^2_y}.$$

Solving (7.4) for $\left\| \int_x u(x, y, t) \, dx \right\|_{L^2_y}$, we obtain

$$\left\| \int_x u(x, y, t) \, dx \right\|_{L^2_y} = \lambda(t)^{1/2}(\|Q\|_{L^2_yL^1_x}^2 + F(t))^{1/2}.$$

Substituting this equation at time $t$ and at time 0 into (7.5), we obtain

$$F(t) - F(0) = (\lambda(t)^{-1} - \lambda(0)^{-1}) \lambda(t)^{1/2} \lambda(0)^{1/2} (\|Q\|_{L^2_yL^1_x}^2 + F(0))^{1/2} (\|Q\|_{L^2_yL^1_x}^2 + F(t))^{1/2}.$$ 

By (7.3)

$$\left| \left( \frac{\lambda(t)}{\lambda(0)} \right)^{1/2} - \left( \frac{\lambda(0)}{\lambda(t)} \right)^{1/2} \right| \leq \alpha(u)^{1/4}.$$ 

Thus, provided $\alpha(u) > 0$ is sufficiently small, then $\alpha(u)^{1/4}$ is also sufficiently small, and it follows that on $(-T_-, T_+)$, we have $\lambda(t) \geq \frac{7}{8} > \frac{3}{4}$. By continuity, since $(-T_-, T_+)$ is maximal within $(-T^*_-, T^*_+)$, it follows that $(-T_-, T_+) = (-T^*_-, T^*_+)$ as claimed.

\[ \square \]

8. Completion of part (1) of the proof of Proposition 1.2

For $n$ sufficiently large, $\alpha(u_n) \leq \alpha_\delta$, and thus, Lemmas 6.4 and 7.1 apply. Recall that the bootstrap time frame $(-t_1(n), t_2(n))$ is defined by (5.2), and also recall that Lemma 5.6 applies yielding the convergence of the parameters, in particular, that $\lambda_{n,m}(t) \to \lambda_n(t)$ as $m \to \infty$. By Lemma 7.1, (5.2) is reinforced so that in fact

$$\frac{3}{4} \leq \tilde{\lambda}_n(t) \leq \frac{5}{4}.$$
on \((-t_1(n), t_2(n))\). By Lemma 5.6
\[
\frac{3}{4} \leq \bar{\lambda}_n(t) = \liminf_{m \to \infty} \lambda_{n,m}(t) \leq \limsup_{m \to \infty} \lambda_{n,m}(t) = \bar{\lambda}_n(t) \leq \frac{5}{4}.
\]
By the method of proof of Lemma 5.2, if either \(t_1(n) < \infty\) or \(t_2(n) < \infty\), then a contradiction to the maximality in the definition of \((-t_1(n), t_2(n))\) is achieved, so we must have \(t_1(n) = t_2(n) = \infty\) as claimed.

9. Rotation and \(y\)-localization

We have established in Lemma 6.3 that for all \(x_0 > 0\),
\[(9.1) \quad \|\tilde{u}_n(x + \tilde{x}_n(t), y, t)\|_{L^2_{|x|>x_0}L^2_y} \lesssim e^{-x_0/8}.
\]
Here, we improve this to include \(y\)-decay.

**Lemma 9.1.** There exists \(\omega > 0\) such that
\[(9.2) \quad \|\tilde{u}_n(x + \tilde{x}_n(t), y + \tilde{y}_n(t), t)\|_{L^2_{B(0,r)} L^2_y} \lesssim e^{-\omega r},
\]
where \(B(0,r)\) is the ball with center 0 and radius \(r > 0\) in \(\mathbb{R}^d\), and \(B(0,r)^c\) denotes the complement.

**Proof.** For \(\theta\) constant and \(0 < |\theta| \ll 1\), define \(\tilde{v}_n\) in terms of \(\tilde{u}_n\) by
\[(9.3) \quad \tilde{u}_n(x, y, t) = \tilde{v}_n(\bar{x}, \bar{y}, t), \quad \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.
\]

**Figure 9.1.** The new decay estimate for \(\tilde{v}_n(\bar{x}, \bar{y}, t)\) gives a decay in terms of \(\bar{x}\). Above we depict the region \(|\bar{x}| \geq x_0\) for \(x_0 > 0\) for some small \(\theta > 0\).
Figure 9.2. Assume $0 < \theta < \frac{\pi}{2}$ (so that $\sin \theta > 0$ and $\cos \theta > 0$). If $(x, y)$ is inside the parallelogram $|x \cos \theta - y \sin \theta| < x_0$ and $|x| < x_0$, then $(x, y)$ is inside the ball of radius $r = x_0 / \sin \theta$. Hence, if $(x, y)$ is not in the ball $B(0, r)$, then either $|x| \geq x_0$ or $|x \cos \theta - y \sin \theta| \geq x_0$.

(the same $\theta$ works for all $n$, just chosen small in terms of an absolute constant). By revisiting the monotonicity calculations that led to the proof of (9.1) (see Lemmas 9.2, 9.3 below) we are able to prove in analogy that

$$\|\tilde{u}_n(x + x(t), y, t)\|_{L^2_{|x| > x_0} L^2_y} \lesssim e^{-x_0/8}.$$

When recast as a decay estimate for $\tilde{u}_n$ (see Figure 9.1), we obtain

$$\|\tilde{u}_n(x + x(t), y + y(t), t) 1_{|x \cos \theta - y \sin \theta| > x_0}\|_{L^2_x} \lesssim e^{-x_0/8}.$$  

Combining with (9.1) and taking $\bar{x}_0 = x_0$, we obtain (9.2) with

$$\omega = \frac{1}{8} \sin \frac{\theta}{2}.$$  

This can be seen by referencing Figure 9.2, where it is explained that $r = x_0 / \sin \frac{\theta}{2}$, and hence, $e^{-x_0/8} \leq e^{-\omega r}$.  

□
Now let us give some details on how to obtain the monotonicity estimate for the rotated function. Fix a $\theta > 0$ constant, for which it suffices to have $0 < \theta \ll 1$ (later when the formulae are derived, we see that any $0 < \theta < \frac{\pi}{3}$ works). Given $u$, let $u^\theta$ be defined in terms of $u$ by

$$u(x, y, t) = u^\theta(\bar{x}, \bar{y}, t), \quad \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. $$

Note that $u^\theta$ solves

$$(ZK_\theta) \quad 0 = \partial_t u^\theta + (\cos \theta \partial_x - \sin \theta \partial_y)[\Delta u^\theta + (u^\theta)^3].$$

Note that rotation does not affect mass or energy, thus,

$$E(u^\theta) = E(u) \quad \text{and} \quad M(u^\theta) = M(u).$$

Moreover, note that $u = u_0$ (the case $\theta = 0$).

Finally, we need to remark on the soliton center coordinates $(\bar{x}(t), \bar{y}(t))$ for $u^\theta$. We have

$$\begin{bmatrix} \bar{x}_t \\ \bar{y}_t \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix}. $$

The trajectory estimates

$$|x_t - \lambda^{-2}| \lesssim \|\epsilon\|_{L^2_{xy}}, \quad |y_t| \lesssim \|\epsilon\|_{L^2_{xy}},$$

which imply

$$|\bar{x}_t - (\cos \theta)\lambda^{-2}| \lesssim \|\epsilon\|_{L^2_{xy}}, \quad |\bar{y}_t - (\sin \theta)\lambda^{-2}| \lesssim \|\epsilon\|_{L^2_{xy}}.$$  

The following shows that the earlier monotonicity lemma generalizes to $u^\theta$ for all $|\theta| \leq \frac{\pi}{4}$.

For the purposes of the lemma, we write $(\bar{x}, \bar{y})$ instead of $(x, y)$.

**Lemma 9.2** (generalized $I_{\pm}^\theta$ estimates). Let $t_{-1} < t_0 < t_1$ and suppose that $|\theta| \leq \frac{\pi}{4}$ and $u^\theta(t)$ is an $H^1$ solution to $(ZK_\theta)$ on $[t_{-1}, t_1]$ with $E(u^\theta) < 0$ and $u^\theta(t)$ is an $H^1$ solution to $(ZK_\theta)$ on $[t_{-1}, t_1]$ with $E(u^\theta) < 0$ and

$$\forall t \in [t_{-1}, t_1], \quad \|\nabla u^\theta(t)\|_{L^2_{xy}} \geq 0.9 \|\nabla Q\|_{L^2_{xy}}.$$ 

There exists an absolute constant $\alpha_4 > 0$ such that if

$$\alpha(u^\theta) \overset{\text{def}}{=} \|u^\theta\|^2_{L^2} - \|Q\|^2_{L^2} \leq \alpha_4,$$

we then we have the following. Let

$$I_{\pm, x_0, t_0}^\theta(t) = \iint (u_\theta)(u^\theta)^2(x + x(t_0), y, t)\phi_\pm(x - x_0 - \frac{1}{2}(x(t) - x(t_0))) \, dx \, dy,$$

We only need $0 < |\theta| \ll 1$. The estimates can actually be pushed for $|\theta| < \frac{\pi}{3}$, but this would complicate the statement, since the other constants would depend on how close the quantity $1 - \frac{1}{\sqrt{3}} |\tan \theta| > 0$ is to zero. See the end of the proof of Lemma 9.2.
where $\phi_-(x) = \phi_+(-x)$ and
\[
\phi_+(x) = \frac{2}{\pi} \arctan(e^{x/K})
\]
so that $\phi_+(x)$ is increasing with $\lim_{x \to -\infty} \phi_+(x) = 0$ and $\lim_{x \to +\infty} \phi_+(x) = 1$, and $\phi_-(x)$ is decreasing with $\lim_{x \to -\infty} \phi_-(x) = 1$ and $\lim_{x \to +\infty} \phi_-(x) = 0$. Let $x_0 > 0$ and $K \geq 4$. For the increasing weight, we have two estimates that bound the future in terms of the past
\[
\begin{align*}
\text{for } t_{-1} < t_0, & \quad I_{+,x_0,t_0}^\theta(t_0) \leq I_{+,x_0,t_0}^\theta(t_{-1}) + \rho_0 e^{-x_0/K}, \\
\text{for } t_0 < t_1, & \quad I_{+,x_0,t_0}^\theta(t_1) \leq I_{+,x_0,t_0}^\theta(t_0) + \rho_0 e^{-x_0/K}.
\end{align*}
\]
For the decreasing weight, we have two estimates that bound the past in terms of the future
\[
\begin{align*}
\text{for } t_{-1} < t_0, & \quad I_{-,x_0,t_0}^\theta(t_{-1}) \leq I_{-,x_0,t_0}^\theta(t_0) + \rho_0 e^{-x_0/K}, \\
\text{for } t_0 < t_1, & \quad I_{-,x_0,t_0}^\theta(t_0) \leq I_{-,x_0,t_0}^\theta(t_1) + \rho_0 e^{-x_0/K}.
\end{align*}
\]
Here, $\theta, \alpha_4 > 0$ and $\rho_0 > 0$ are absolute constants, in particular, independent of the upper bound on $\|\nabla u(t)\|_{L^2}$ over the time interval $[t_{-1}, t_1]$.

Proof. The proof follows that of the earlier monotonicity lemma (Lemma 6.2) with minimal modification. Indeed, if $\phi$ depends only on $x$, then to calculate $\partial_1 I^\theta$, we need the following terms
\[
\begin{align*}
\mathcal{I}^\theta_1 &= \int \int \phi \theta \ u_{xx} \ dx \ dy = \frac{3}{2} \int \int \phi \theta (u_x^\theta)^2 \ dx \ dy - \frac{1}{2} \int \int \phi \theta \ (u^{\theta})^2 \ dx \ dy, \\
\mathcal{I}^\theta_2 &= \int \int \phi \theta \ u_{xy} \ dx \ dy = \frac{1}{2} \int \int \phi \theta \ (u_y^\theta)^2 \ dx \ dy,
\end{align*}
\]
which are the same as before, but now we also have
\[
\begin{align*}
\mathcal{I}^\theta_3 &= \int \int \phi \theta \ u_{xy} \ dx \ dy = \int \int \phi \theta \ u_{y}^\theta u^\theta_x \ dx \ dy, \\
\mathcal{I}^\theta_4 &= \int \int \phi \theta \ u_{yy} \ dx \ dy = 0.
\end{align*}
\]
We have (we drop the $\theta$ superscript)
\[
\frac{1}{2} \partial_1 I_{x_0,t_0}^\theta(t) = \int \int u u_x \phi \ dx \ dy - \frac{1}{2} \phi' \int \int u_x^2 \ dx \ dy
\]
\[
= \cos \theta \int \int u \left( -u_{xxx} - u_{yy} - 3u^2 u_x \right) \phi \ dx \ dy - \sin \theta \int \int u \left( -u_{xxy} - u_{yy} - 3u^2 u_y \right) \phi \ dx \ dy - \frac{1}{2} \phi' \int \int u_x^2 \ dx \ dy.
\]
\[^{11}\] Which actually means $\tilde{x}$ in the notation preceding the lemma.
After the indicated applications of integration by parts (recalling that $\phi$ is independent of $y$), we obtain

$$\frac{1}{2} \partial_t I^\theta_{x_0,t_0}(t) = -\frac{3}{2} \cos \theta \int \int u_x^2 \phi' - \frac{1}{2} \cos \theta \int \int u_y^2 \phi' - \frac{1}{2} x'(t) \cos \theta \int \int u^2 \phi'$$

$$+ \frac{1}{2} \cos \theta \int \int u^2 \phi'' + \frac{3}{4} \cos \theta \int \int u^4 \phi' - \sin \theta \int \int u_y u_x \phi' \, dx \, dy.$$ 

We can proceed exactly as in the proof of the earlier monotonicity lemma corresponding to $\theta = 0$ (Lemma 6.2) except that now we need to control the extra term

$$\left| - \sin \theta \int \int u_y u_x \phi' \, dx \, dy \right| \leq (1 - \delta) \left( \frac{3}{2} \cos \theta \int \int u_x^2 \phi' + \frac{1}{2} \cos \theta \int \int u_y^2 \phi' \right)$$

for some $\delta > 0$. Recalling the classical inequality $u_x u_y \leq \frac{1}{2} \omega u_x^2 + \frac{1}{2} \omega^{-1} u_y^2$ for any $\omega > 0$, we select $\omega > 0$ so that

$$\frac{1}{2} \omega \leq \frac{3}{2} \cot \theta (1 - \delta), \quad \frac{1}{2} \omega^{-1} \leq \frac{1}{2} \cot \theta (1 - \delta).$$

This has a solution only if (multiplying the two equations) $\frac{1}{3} \leq (1 - \delta)^2 \cot^2 \theta$, or equivalently, $|\tan \theta| \leq \sqrt{3} (1 - \delta)$. Thus, we require $|\theta| < \frac{\pi}{3}$ and then set $\delta = 1 - \frac{|\tan \theta|}{\sqrt{3}} > 0$ and $\omega = \sqrt{3}$. In the theorem statement, we have required $|\theta| \leq \frac{\pi}{4}$ so that $\delta$ can be chosen uniformly positive, and the statement of the theorem is not complicated by the dependence on $\delta > 0$ for $\delta$ close to zero. \(\square\)

**Lemma 9.3** (applying $I^\theta_\pm$ estimates to obtain exponential decay of $\tilde{v}_n$). Let $\tilde{v}_n$ be defined by (9.3), for $|\theta| \leq \frac{\pi}{4}$. For $x_0 > 0$, for all $-t_1(n) < t < t_2(n)$, we have

$$\|\tilde{v}_n(x + \tilde{x}(t), y, t)\|_{L^2_{|x|>|x_0|}, t_y^2} \leq 24 \rho_0 e^{-x_0/8},$$

where $\rho_0 > 0$ is an absolute constant.

**Proof.** Note that Lemma 6.3 is the $\theta = 0$ version of this lemma. This follows from Lemma 9.2 in analogy with the way in which Lemma 6.3 was deduced from Lemma 6.2. \(\square\)

10. **Spatial localization with sharp coefficient, completion of part (2) of proof of Proposition 1.2**

In this section, we substantially strengthen the decay estimate on $\tilde{u}_n$ by proving a monotonicity estimate directly on (a scaled version of) $\tilde{\epsilon}_n$. We state the result for general $u$ and $\epsilon$, although we will invoke it for $\tilde{u}_n$ and $\tilde{\epsilon}_n$ later in the compactness argument.
Lemma 10.1. Suppose that \( u(t) \) solves (ZK), \( \alpha(u) \ll 1 \) and \( E(u) < 0 \) (so that the geometrical decomposition applies), and \( \frac{1}{2} \leq \lambda(t) \leq 2 \) for all \( t \in \mathbb{R} \). Assume moreover that \( u(t) \) satisfies a weak decay estimate

\[
\lim_{t \to \pm \infty} \|u(x + x(t), y, t)\|_{L^2_y L^2_{|x| \geq \frac{1}{2} |t|}} = 0.
\]

Then for all \( t \in \mathbb{R} \) and each \( x_0 > 0 \),

\[
\|\epsilon(x, y, t)\|_{L^2_x L^2_{|x| \geq x_0}} \lesssim \|\epsilon\|_{L^\infty_t L^2_{xy} e^{-x_0/8}}.
\]

Note that the fact that the coefficient to the decay on the right side of (10.2) is \( \|\epsilon\|_{L^\infty_t L^2_{xy}} \), as opposed to \( \|\epsilon\|_{L^{\gamma}_t L^2_{xy}} \) for some \( \gamma < 1 \), is crucial for the compactness argument that follows.

Before starting the proof, we note that we have previously proved a monotonicity estimate on \( u \) using the functional

\[
I_{\pm, x_0, t_0}(t) = \int \int u^2(x + x(t_0), y + y(t_0), t) \phi_\pm(x - x_0 - \frac{1}{2}(x(t) - x(t_0))) \, dx \, dy.
\]

By a change of variable, this is equivalent to an estimate on

\[
I_{\pm, x_0, t_0}(t) = \int \int u^2(x + x(t), y + y(t), t) \phi_\pm(x - x_0 + \frac{1}{2}(x(t) - x(t_0))) \, dx \, dy,
\]

where now the soliton is centered at 0 for all times.

**Proof.** Define

\[
\eta(t, x, y) = \lambda^{-1} \epsilon(s(t), \lambda^{-1} x, \lambda^{-1} y),
\]

so that

\[
\eta(x, y, t) = u(x + x(t), y + y(t), t) - \lambda^{-1} Q(\lambda^{-1} x - \lambda^{-1} y).
\]

Let \( \tilde{Q}(x, y) = \lambda^{-1} Q(\lambda^{-1} x, \lambda^{-1} y) \). Then we find that \( \eta \) solves

\[
0 = \partial_t \eta - (x_t, y_t) \cdot \nabla \eta + \partial_x (\Delta \eta + (\eta + \tilde{Q})^3 - \tilde{Q}^3)
\]

\[
+ (\lambda^{-1})^2 \partial_{\lambda^{-1}} \tilde{Q} - (x_t - \lambda^{-2}, y_t) \cdot \nabla \tilde{Q}.
\]

Now we define the functional

\[
J_{\pm, x_0, t_0}(t) = \int \int \phi_\pm(x - x_0 + \frac{1}{2}(x(t) - x(t_0))) \eta^2(x, y, t) \, dx \, dy.
\]

As before, for the increasing weight \( \phi_+ \), we will prove an estimate of the future in terms of the past, and for the decreasing weight \( \phi_- \), we will prove an estimate of the past in terms of the future. However, the difference is that this time, we need \( \phi_\pm \) to be small near the origin, so we can only do \( \phi_+ \) estimates on the right and \( \phi_- \) estimates on the left. See Figure [10.1] for the new configuration, where \( x \)-space has been shifted so that the soliton is positioned at the origin.

Thus, we have, for some absolute constant \( C > 0 \),

\[
J_{-, x_0, t_0}(t_0) \leq J_{-, x_0, t_0}(t_1) + C e^{-x_0} \|\eta\|_{L^\infty_t L^2_{xy}}^2,
\]
Figure 10.1. The frame of reference in the proof of Lemma 10.1, where the soliton is at position $x = 0$, the $\phi_-$ transition occurs in the $x < 0$ region and is only used when $t > t_0$, the $\phi_+$ transition occurs in the $x > 0$ region and is only used for $t < t_0$.

(10.4) \[ J_{+,x_0,t_0}(t) \leq J_{+,x_0,t_0}(t-1) + Ce^{-x_0} \| \eta \|_{L_t^\infty L_x^2}^2. \]

By reflection symmetry, it suffices to prove the second one, so we take $\phi = \phi_+$ from here

\[ J'_{+,x_0,t_0}(t) = 2 \iint \eta \phi_+ \, dx \, dy + \frac{1}{2} x_t \iint \eta^2 \phi_+' \, dy. \]

By several applications of integration by parts, we obtain

\[
J'_{+,x_0,t_0}(t) = - \frac{1}{2} x_t \iint \phi_x \eta^2 - 3 \iint \phi_x \eta_x^2 - \iint \phi_x \eta_y^2 + \iint \phi_{xxx} \eta^2 \\
+ \iint (3\phi_x \tilde{Q}^2 - 6\phi \tilde{Q}\tilde{Q}_x) \eta^2 + \iint (4\phi_x \tilde{Q} - 2\phi \tilde{Q}_x) \eta^3 + \frac{3}{2} \iint \phi_x \eta^4 \\
+ 2\lambda^{-2} \lambda_t \iint \phi \partial_{\lambda^{-1}} \tilde{Q} \eta + 2(x_t - \lambda^{-2}) \iint \phi \tilde{Q}_x \eta + 2y_t \iint \phi \tilde{Q}_y \eta.
\]

In the first line, the first three terms all have the good sign, and the last term, $\iint \phi_{xxx} \eta^2$ is smaller than the first by taking $K \geq 4$, as before. In the second line, the last term $\iint \phi_x \eta^4$ is controlled by the weighted Gagliardo-Nirenberg estimate (without the need for spatial cutoff), as was done in the earlier monotonicity result.
By considering the effective support properties of $\phi$ and $\tilde{Q}$, and using that $\frac{1}{2} \leq \lambda \leq 2$, we have

$$
\|3\phi_x \tilde{Q}^2 - 6\phi \tilde{Q} \phi_x \|_{L^\infty} + \|4\phi_x \tilde{Q} - 2\phi \tilde{Q} \phi_x \|_{L^\infty} \lesssim e^{-x_0} e^{\frac{1}{2}(x(t) - x(t_0))},
$$

and thus, the first two terms on the second line can be handled by “suping out” the weight and, in the case of the middle term, following up with the Gagliardo-Nirenberg estimate $\int \eta^3 \lesssim \|\eta\|_{L^2}^2 \|\nabla \eta\|_{L^2}$. For the three terms in the last line, use

$$
\|\phi \partial_x \tilde{Q}\|_{L^2} + \|\phi \tilde{Q} x\|_{L^2} + \|\phi \tilde{Q} y\|_{L^2} \lesssim e^{-x_0} e^{\frac{1}{2}(x(t) - x(t_0))}
$$

and also the parameter bounds

$$
|\lambda t| + |x_t - \lambda^{-2}| + |y_t| \lesssim \|\eta\|_{L^2}.
$$

Thus, we have

$$
|J'_{+,t_0}(t)| \leq e^{-x_0} e^{\frac{1}{2}(x(t) - x(t_0))} \|\eta\|_{L^\infty L^2_{xy}}^2.
$$

Integrating from $t_{-1}$ to $t_0$ we obtain

$$
J_{+,x_0,t_0}(t) \leq J_{+,x_0,t_0}(t_{-1}) + e^{-x_0} \|\eta\|_{L^\infty L^2_{xy}}^2 \int_{t_{-1}}^{t_0} e^{\frac{1}{2}(x(t) - x(t_0))} dt.
$$

Using that $x(t) - x(t_0) \sim t - t_0$ we obtain $|10.4|$. We complete the proof by noting that, for fixed $t_0$, $|10.1|$ implies

$$
\lim_{t_1 \to +\infty} J_{-,x_0,t_0}(t_1) = 0
$$

in $|10.3|$ and

$$
\lim_{t_{-1} \to +\infty} J_{+,x_0,t_0}(t_{-1}) = 0
$$

in $|10.4|$. From the resulting limiting equations, we obtain

$$
\|\eta(x, y, t)\|_{L^2_{y_0 < x < x_0}}^2 \leq e^{-x_0} \|\eta\|_{L^\infty L^2_{xy}}^2 \quad \text{and} \quad \|\eta(x, y, t)\|_{L^2_{y_0 < x < x_0}}^2 \leq e^{-x_0} \|\eta\|_{L^\infty L^2_{xy}}^2,
$$

which yield $|10.2|$. □

**Corollary 10.2** (exponential decay of $\tilde{\epsilon}_n$ with sharp coefficient). For each $r > 0$, we have

$$
\|\tilde{\epsilon}_n\|_{L^\infty L^2_{B(0, r)^c}} \lesssim e^{-\omega r} \|\tilde{\epsilon}_n\|_{L^\infty L^2_{xy}},
$$

where $B(0, r)$ is the ball centered at 0 of radius $r$ in $\mathbb{R}^2$, and $B(0, r)^c$ denotes the complement.

**Proof.** This follows from the rotation method of Lemmas $|9.2|, |9.3|$ applied to the result of Lemma $|10.1|$. □

Note that Corollary $|10.2|$ completes part (2) of the proof of Prop. $|1.2|$. □
11. Comparibility of remainder norms

Proposition 11.1. Suppose that $E(u) < 0$ and $\alpha(u) \ll 1$ so that the geometrical decomposition applies, and $\frac{1}{2} \leq \lambda \leq 2$. Let $a = \|\epsilon\|_{L^\infty_x L^2_y}$ and $b = \|\epsilon\|_{L^\infty_x L^2_y}$. Moreover, suppose that the $x$-decay property holds:

$$\|\epsilon\|_{L^\infty_x L^2_y L^2_{|x|>x_0}} \lesssim \langle x_0 \rangle^{-1} a^\gamma b^{1-\gamma}$$

for some $0 \leq \gamma < 1$. Then $a \sim b$.

Proof. The proof will use Lemma 11.3, stated and proved below. Note that

$$\int_{y \in \mathbb{R}} \int_{x \in \mathbb{R}} |x||\epsilon(x,y,s)|^2\,dx\,dy = \int_{y \in \mathbb{R}} \int_{x \in \mathbb{R}} |\epsilon(x,y,s)|^2\,dz\,dx\,dy$$

$$= \int_{z=0}^{+\infty} \int_{y \in \mathbb{R}} \int_{|x|>z} |\epsilon(x,y,s)|^2\,dx\,dy\,dz \lesssim a^{2\gamma} b^{2-2\gamma} \int_{z=0}^{\infty} \langle z \rangle^{-2}\,dz \lesssim a^{2\gamma} b^{2-2\gamma}.$$ 

That is,

$$\langle x \rangle \int_{y \in \mathbb{R}} \int_{x \in \mathbb{R}} |x||\epsilon(x,y,s)|^2\,dx\,dy \lesssim a^{2\gamma} b^{2-2\gamma}. \tag{11.1}$$

Recalling the equation for $\epsilon$, (4.8), we now compute

$$-\frac{1}{2} \partial_s \int \int x \epsilon(x,y,s)^2\,dx\,dy$$

$$= \frac{1}{2} \|\epsilon\|^2_{L^2_x L^2_y} + \frac{3}{2} \|\epsilon_x\|^2_{L^2_x L^2_y} + \frac{1}{2} \|\epsilon_y\|^2_{L^2_x L^2_y} + \frac{3}{2} \int \int (2xQQ_x - Q^2)\epsilon^2\,dx\,dy$$

$$- \frac{\lambda_s}{\lambda} \langle \epsilon, x\Lambda Q \rangle - \langle \frac{x_s}{\lambda} - 1 \rangle \langle \epsilon, xQ_x \rangle - \frac{y_s}{\lambda} \langle \epsilon, xQ_y \rangle - \frac{\lambda_s}{\lambda} \int \int x\epsilon \Lambda \epsilon$$

$$- (\frac{x_s}{\lambda} - 1) \int \int x\epsilon \epsilon_x - \frac{\lambda_s}{\lambda} \int \int x\epsilon \epsilon_y - \int \int x(3Q\epsilon^2 + \epsilon^3)_x.$$ 

Using that

$$\int \int x\epsilon \Lambda \epsilon = -\frac{1}{2} \int \int x\epsilon^2, \quad \int \int x\epsilon \epsilon_x = -\frac{1}{2} \int \int \epsilon^2, \quad \int \int x\epsilon \epsilon_y = 0,$$

and the parameter bounds (4.9), we get

$$-\frac{1}{2} \partial_s \int \int x\epsilon(x,y,s)^2\,dx\,dy = \frac{3}{2} \|\epsilon_x\|^2_{L^2_x L^2_y} + \frac{1}{2} \|\epsilon_y\|^2_{L^2_x L^2_y} + E,$$

where

$$|E| \lesssim b^2 + b \int \int \epsilon^2 + \|\epsilon\|^3_{L^3_x L^2_y}.$$ 

Applying the Gagliardo-Nirenberg inequality

$$\|\epsilon\|^3_{L^3_x L^2_y} \lesssim \|\nabla \epsilon\|_{L^2_y} \|\epsilon\|^2_{L^2_x L^2_y} \lesssim b^2 \|\nabla \epsilon\|^2_{L^2_x L^2_y} + b^2$$
and integrating in \( s \) over \( s_0 - \sigma \leq s \leq s_0 + \sigma \), where \( s_0 \) and \( \sigma > 0 \) are as given in Lemma 11.3 below, we obtain

\[
\|\nabla \epsilon\|^2_{L^2_{\sigma - s_0, \sigma + s_0} L^2_{x,y}} \lesssim \sigma b^2 + \left\| \int_0^t x \epsilon^2 \right\|_{L^2_{\sigma - s_0, \sigma + s_0}}.
\]

By (11.1) and Lemma 11.3, we get

\[
\sigma a^2 \lesssim \|\epsilon\|^2_{L^2_{\sigma - s_0, \sigma + s_0} H^1_{x,y}} \lesssim \sigma b^2 + a^2 \gamma b^{2-2\gamma}.
\]

Dividing through by \( a^2 \), using that \( \sigma > 0 \) is an absolute constant, we now have

\[
1 \lesssim \left( \frac{b}{a} \right)^2 + \left( \frac{b}{a} \right)^{2-2\gamma} \lesssim \left( \frac{b}{a} \right)^{2-2\gamma},
\]

where, in the last inequality, we used that \( \frac{b}{a} \leq 1 \). Since \( \gamma < 1 \), this implies that \( a \lesssim b \).

\[\square\]

**Lemma 11.2.** Let \( Q \) be a dyadic decomposition of \( x \)-space – specifically take \( Q_{-1} = [-1, 1] \) and \( Q_{k} = [-2^{k+1}, -2^k] \cup [2^k, 2^{k+1}] \) for \( k \geq 0 \). If \( \frac{1}{2} \leq \lambda \leq 2 \), \( |x_1| \lesssim 1 \), \( |y_1| \lesssim 1 \), and \( t \) is restricted to a unit-size time interval, and

\[
(11.2) \quad \bar{g}(x,y) = \lambda^{-1} g(\lambda^{-1}(x-x(t), \lambda^{-1}(y-y(t)))).
\]

then for \( 1 \leq p \leq \infty \),

\[
(11.3) \quad \|\bar{g}\|_{L^p_{x,y} L^\infty_t} \lesssim \left\| 2^{j/p} \|g\|_{L^\infty_{x,y} L^p_t} \right\|_{\ell^j_j}.
\]

**Proof.** The proof is standard, starting with decomposing the outer \( x \)-integration into the \( Q_j \) regions. \[\square\]

**Lemma 11.3.** Suppose that \( E(u) < 0 \) and \( \alpha(u) \ll 1 \) so that the geometrical decomposition applies, and \( \frac{1}{2} \leq \lambda \leq 2 \). Let \( a = \|\epsilon\|_{L^\infty \mathcal{H}^1_{x,y}} \) and \( s_0 \in \mathbb{R} \) such that \( \|\epsilon(s_0)\|_{H^1_{x,y}} \geq \frac{1}{2} a \). There exists an absolute constant \( \sigma > 0 \) such that for all \( s_0 - \sigma \leq s \leq s_0 + \sigma \), we have \( \|\epsilon(s)\|_{H^1_{x,y}} \geq \frac{1}{16} a \).

**Proof.** Let

\[
\zeta(x,y,t) = \lambda^{-1} \epsilon(\lambda^{-1}(x-x(t)), \lambda^{-1}(y-y(t)), t)
\]

and let \( t_0 \) correspond to \( s_0 \) in the time transformation. Then \( \zeta \) solves

\[
\partial_t \zeta = -\partial_x \Delta \zeta - 3 \partial_x (Q^2 \zeta) + \langle \bar{f}_1, \zeta \rangle \lambda^{-3} \bar{Q} + \langle \bar{f}_2, \zeta \rangle \lambda^{-3} \bar{Q}_x + \langle \bar{f}_3, \zeta \rangle \lambda^{-3} \bar{Q}_y + \mu_1 \lambda^{-3} \bar{Q} + \mu_2 \lambda^{-3} \bar{Q}_x + \mu_3 \lambda^{-3} \bar{Q}_y - 3 \partial_x (\bar{Q} \zeta^2) - \partial_x (\zeta^3),
\]

where each \( |u_j| \lesssim \|\epsilon\|^2_{L^\infty \mathcal{H}^1_{x,y}} \), and the bar quantities \( \bar{Q}, \bar{Q}_x, \bar{Q}_y \), etc. are defined as in (11.2). Writing this equation in Duhamel form, and applying the \( H^1 \) theory local
estimates, where \( t \) is restricted to \([t_0 - \sigma, t_0 + \sigma]\) on both sides of the equation (\( \sigma \) to be determined), we have
\[
\|\zeta(t) - U(t)\zeta(t_0)\|_{L^\infty_t H^1_{x,y}} \lesssim \|\nabla(\bar{Q}^2\zeta)\|_{L^1_t L^2_{x,y}} + \|\langle \bar{f}_1, \zeta \rangle\|_{L^1_t L^2_{x,y}}
+ \|\langle \bar{f}_2, \zeta \rangle\|_{L^1_t L^2_{x,y}} + \|\langle \bar{f}_3, \zeta \rangle\|_{L^1_t L^2_{x,y}}
+ \|\mu_1\|_{L^1_t L^2_{x,y}} + \|\mu_2\|_{L^1_t L^2_{x,y}}
+ \|\mu_3\|_{L^1_t L^2_{x,y}} + \|\nabla(\bar{Q}^2\zeta)\|_{L^1_t L^2_{x,y}}
+ \|\nabla(\zeta^3)\|_{L^1_t L^2_{x,y}}.
\]
(On the right side there are also the (easier to estimate) copies of each term listed without the gradient.) For the first term,
\[
\|\nabla(\bar{Q}^2\zeta)\|_{L^1_t L^2_{x,y}} \lesssim \|\bar{Q}\nabla\bar{Q}\zeta\|_{L^1_t L^2_{x,y}} + \|\bar{Q}^2 \nabla\zeta\|_{L^1_t L^2_{x,y}}
\lesssim \|\bar{Q}\|_{L^2_t L^\infty_{x,y}} \|\nabla\bar{Q}\|_{L^1_t L^\infty_{x,y}} \|\zeta\|_{L^2_{x,y}} + \|\bar{Q}\|_{L^2_t L^\infty_{x,y}} \|\nabla\zeta\|_{L^2_{x,y}}
\lesssim \sigma^{1/2} \|\zeta\|_{L^\infty_t H^1_{x,y}},
\]
where the \( Q \) coefficient terms are controlled by (11.3). For the term \( \|\langle \bar{f}_1, \zeta \rangle\|_{L^1_t L^2_{x,y}} \), we use that
\[
\|\nabla \Lambda \bar{Q}\|_{L^1_t L^2_{x,y}} = \lambda^{-1} \|\nabla \Lambda \bar{Q}\|_{L^1_t L^2_{x,y}}
\lesssim \|\nabla \Lambda \bar{Q}\|_{L^1_t L^\infty_{x,y}}^{1/2} \|\nabla \Lambda \bar{Q}\|_{L^2_{x,y}}^{1/2}
\lesssim \sigma^{1/2} \|\nabla \Lambda \bar{Q}\|_{L^1_t L^\infty_{x,y}} \|\nabla \Lambda \bar{Q}\|_{L^\infty_t L^1_{x,y}},
\]
which is bounded by \( \sigma^{1/2} \) by (11.3). Thus, we have
\[
\|\langle \bar{f}_1, \zeta \rangle\|_{L^1_t L^2_{x,y}} \lesssim \|\bar{f}_1\|_{L^\infty_t L^2_{x,y}} \|\zeta\|_{L^\infty_t L^2_{x,y}} \sigma^{1/2} \lesssim \sigma^{1/2} \|\zeta\|_{L^\infty_t L^2_{x,y}}.
\]
The next five terms are treated similarly, which brings us to the next term \( \|\nabla(\bar{Q}^2\zeta)\|_{L^1_t L^2_{x,y}} \). Distributing the derivative and applying Hölder, we obtain
\[
\|\nabla(\bar{Q}^2\zeta)\|_{L^1_t L^2_{x,y}} \lesssim \|\bar{Q}\|_{L^2_t L^\infty_{x,y}} \|\zeta\|_{L^2_t L^2_{x,y}} \|\nabla\zeta\|_{L^2_{x,y}} \|\bar{Q}\|_{L^2_t L^\infty_{x,y}} \|\nabla\zeta\|_{L^2_{x,y}} \|\bar{Q}\|_{L^2_t L^\infty_{x,y}} \|\nabla\zeta\|_{L^2_{x,y}}
\lesssim \sigma^{1/2} \|\zeta\|_{L^\infty_t H^1_{x,y}},
\]
using that \( \|\zeta\|_{L^2_t L^\infty_{x,y}} \lesssim 1 \) and \( \|\zeta\|_{L^\infty_t H^1_{x,y}} \lesssim 1 \). Finally, for the last term, we have
\[
\|\nabla(\zeta^3)\|_{L^1_t L^2_{x,y}} \lesssim \|\zeta\|_{L^2_t L^\infty_{x,y}}^2 \|\nabla\zeta\|_{L^2_{x,y}} \lesssim \sigma^{1/2} \|\zeta\|_{L^2_t L^\infty_{x,y}} \|\nabla\zeta\|_{L^\infty_t L^2_{x,y}} \lesssim \sigma^{1/2} \|\nabla\zeta\|_{L^\infty_t L^2_{x,y}},
\]
where we have used that \( \|\zeta\|_{L^2_t L^\infty_{x,y}} \lesssim 1 \). In conclusion, we have obtained
\[
\|\zeta(t_0)\|_{H^1_{x,y}} - C \sigma^{1/2} \|\zeta\|_{L^\infty H^1_{x,y}} \lesssim \|\zeta(t) - U(t)\zeta(t_0)\|_{L^\infty H^1_{x,y}} \lesssim \sigma^{1/2} \|\zeta\|_{L^\infty H^1_{x,y}}.
\]
It follows that
\[
\|\zeta(t_0)\|_{H^1_{x,y}} - C \sigma^{1/2} \|\zeta\|_{L^\infty H^1_{x,y}} \lesssim \|\zeta\|_{L^\infty H^1_{x,y}} \lesssim \|\zeta(t_0)\|_{H^1_{x,y}} + C \sigma^{1/2} \|\zeta\|_{L^\infty H^1_{x,y}},
\]
and hence,
\[ \| \zeta(t_0) \|_{H^1_{t,y}} \leq \| \zeta(t) \|_{L^\infty_{t}H^1_{t,y}} \leq \| \zeta(t_0) \|_{H^1_{t,y}} \]

Taking \( \sigma \) small enough so that \( C\sigma^{1/2} \leq \frac{1}{2} \), we obtain the claim. \( \square \)

12. Convergence of renormalized remainders

**Proposition 12.1.** If the statement “for \( n \) sufficiently large, \( \bar{\epsilon}_n \equiv 0 \)” is false, then we can pass to a subsequence (still denoted \( \bar{\epsilon}_n \)) such that \( \bar{\epsilon}_n \neq 0 \) for all \( n \). Let \( b_n = \| \epsilon_n(t) \|_{L^\infty_{t}L^2_{x,y}} \), so that, by our assumption, \( b_n > 0 \) for all \( n \), and there exists \( s_n \in \mathbb{R} \) such that \( \| \epsilon_n(s_n) \|_{L^2_{x,y}} \geq \frac{1}{2} b_n \). Let
\[ w_n(x, y, s) = \frac{\bar{\epsilon}_n(x, y, s + s_n)}{b_n}. \]

Then, passing to another subsequence, \( w_n \to w_\infty \) in \( C_{\text{loc}}(\mathbb{R}; L^2(\mathbb{R}^2)) \), where \( w_\infty \) satisfies

1. \( w_\infty \in L^\infty_{s}H^1_{t,y} \) and solves
\[ \partial_s w_\infty = \partial_x L w_\infty + \alpha(s)\Lambda Q + \beta(s)Q_x + \gamma(s)Q_y, \]

2. \( w_\infty \) satisfies the orthogonality conditions \( \langle Q^3, w_\infty \rangle = 0 \) and \( \langle \nabla Q, w_\infty \rangle = 0 \),
3. \( w_\infty \) is nontrivial, in fact, \( \| w_\infty(0) \|_{L^2_{x,y}} \geq \frac{1}{2} \),
4. \( w_\infty \) satisfies a spatial localization, uniformly in time
\[ \| w_\infty(s) \|_{L^2_{B(0, r)^c}} \lesssim e^{-\omega r} \]

for some \( \omega > 0 \), and any radius \( r > 0 \); here, \( B(0, r) \) denotes the ball in \( \mathbb{R}^2 \) of center 0 and radius \( r \), and \( B(0, r)^c \) denotes the complement of this ball.

**Proof.** Let \( t_n \) correspond to \( s_n \) (in the time conversion). Since \( \frac{1}{2} \leq \bar{\lambda}_n(t) \leq 2 \) for all \( n \) and all \( t \in \mathbb{R} \), we can pass to a subsequence so that \( \bar{\lambda}_n(t_n) \to \lambda_\infty \). Let
\[ \zeta_n(x, y, t) = b_n^{-1}\bar{\lambda}_n^{-1}\bar{\epsilon}_n(\bar{\lambda}_n^{-1}(x - \bar{x}_n(t + t_n) + \bar{x}_n(t_n)), \bar{\lambda}_n^{-1}(y - \bar{y}_n(t + t_n) + \bar{y}_n(t_n)), t + t_n). \]

Every instance of \( \bar{\lambda}_n \) is evaluated at time \( t + t_n \), although this has been suppressed. Then \( \zeta_n \) satisfies
\[ b_n\zeta_n(x, y, t) = \bar{u}_n(x + \bar{x}_n(t_n), y + \bar{y}_n(t_n), t + t_n) - \lambda_n^{-1}Q(\bar{\lambda}_n^{-1}(x - \bar{x}_n(t + t_n) + \bar{x}_n(t_n)), \bar{\lambda}_n^{-1}(y - \bar{y}_n(t + t_n) + \bar{y}_n(t_n))). \]

By Corollary 10.2 (the exponential decay estimate on \( \bar{\epsilon}_n \)), and \( \frac{1}{2} \leq \bar{\lambda}_n \leq 2 \), we have, uniformly in \( n \),
\[ \| \zeta_n \|_{L^\infty_{t}L^2_{B(0, R)^c}} \lesssim e^{-\omega R} \]
for any ball of radius $R$. Moreover, by Prop. 11.1 (giving the comparability between $L^2_{xy}$ and $H^1_{xy}$ norms of $\epsilon_n$), we have, uniformly in $n$, 
\[ \|\zeta_n\|_{L^\infty H^1_{xy}} \lesssim 1. \]

Hence, at time $t = 0$, by the Rellich-Kondrachov compactness theorem, we can pass to a subsequence such that $\zeta_n(0) \to \zeta_\infty(0)$ strongly in $L^2_{xy}$, for some $H^1_{xy}$ function $\zeta_\infty(0)$. By relabeling, we assume that $\zeta_n$ follows this subsequence. We will now use local theory estimates to prove that this convergence holds at all times $t$. In fact, we show that for each $T > 0$,
\[ (12.2) \quad \|\zeta_n(t) - \zeta_\infty(t)\|_{L^\infty_{t < T} L^2_{xy}} \to 0 \]
as $n \to \infty$, where $\zeta_\infty$ solves the equation (12.11) with initial condition $\zeta_\infty(0)$. Since $\frac{1}{2} \leq \lambda_n \leq 2$, this implies the same convergence for $w_n$ to $w_\infty$ in the proposition statement.

Let $Q$ denote the second term, for convenience, in the definition of $\zeta_n$, which has implicit $t$ and $n$ dependence, i.e.,
\[ (12.3) \quad Q \overset{\text{def}}{=} \lambda_n^{-1} Q(\lambda^{-1}_n(x - \bar{x}_n(t + t_n) + \bar{x}_n(t_n)), \lambda_n^{-1}(y - \bar{y}_n(t + t_n) + \bar{y}_n(t_n))) . \]
Since the space and time shifts are by constants for each $n$, $\bar{u}_n(x + \bar{x}_n(t), y + \bar{y}_n(t), t + t_n)$ still solves (ZK), and we compute that $\zeta_n$ solves
\[ \begin{align*}
\partial_t \zeta_n + \partial_x \Delta \zeta_n + \partial_x[3\bar{Q}\zeta_n + 3\bar{Q} \zeta_n b_n + \zeta_n b_n^2] \\
&= -b_n^{-1}(\bar{\lambda}_n^{-1}), \partial_{\lambda^{-1}} \bar{Q} + b_n^{-1}(\bar{y}_n), \partial_y \bar{Q} + b_n^{-1}(\bar{x}_n) - \bar{\lambda}_n^{-2} \partial_x \bar{Q} .
\end{align*} \]

We rewrite the last line as

\[ (b_n^{-1} \bar{\lambda}_n^{-2}(\bar{\lambda}_n),) \bar{\lambda}_n^{-3} \bar{Q} + (b_n^{-1} \bar{\lambda}_n^{-2}(\bar{y}_n),) \bar{\lambda}_n^{-3} \partial_y \bar{Q} + (b_n^{-1} \bar{\lambda}_n^{-2}(\bar{x}_n), - 1)) \bar{\lambda}_n^{-3} \partial_x \bar{Q} , \]

where $\bar{Q}$, $\partial_x \bar{Q}$, and $\partial_y \bar{Q}$ are defined in terms of $\bar{Q}$, $\partial_x Q$, and $\partial_y Q$ in the same way as $Q$ is defined in terms of $Q$. We can now substitute the equations for the parameters at the expense of $O(b_n)$ terms. Let us write
\[ \begin{align*}
b_n(\mu_n)_1 &= b_n^{-1}(\bar{\lambda}_n t - \bar{f}_1, \zeta_n), \\
b_n(\mu_n)_2 &= b_n^{-1}(\bar{\lambda}_n(x), t - 1) - \bar{f}_2, \zeta_n), \quad \text{and} \\
b_n(\mu_n)_3 &= b_n^{-1}(\bar{\lambda}_n(y), t - \bar{f}_3, \zeta_n) ,
\end{align*} \]
so that each $\|\mu_n\| \lesssim 1$. Then
\[ (12.4) \quad \partial_t \zeta_n = -\partial_x \Delta \zeta_n - 3\partial_x(\bar{Q}^2 \zeta) + (\bar{f}_1, \zeta_n) \bar{\lambda}_n^{-3} \bar{Q} + (\bar{f}_2, \zeta_n) \bar{\lambda}_n^{-3} \partial_y \bar{Q} + (\bar{f}_3, \zeta_n) \bar{\lambda}_n^{-3} \partial_y \bar{Q} \\
+ b_n(\mu_n)_1 \bar{\lambda}_n^{-2} \bar{Q} + b_n(\mu_n)_2 \bar{\lambda}_n^{-2} \partial_y \bar{Q} + b_n(\mu_n)_3 \bar{\lambda}_n^{-2} \partial_y \bar{Q} - 3b_n \partial_x(\bar{Q}^2 \zeta_n) - b_n^2 \partial_x(\zeta_n^3) . \]

On the time interval $-T \leq t \leq T$, the parameter bounds imply the following. First, the fact that $|\bar{\lambda}_n(t)| \lesssim b_n$ implies
\[ (12.5) \quad |\bar{\lambda}_n(t + t_n) - \lambda_\infty| \lesssim b_n T + |\bar{\lambda}_n(t_n) - \lambda_\infty| . \]
Since \( \frac{1}{2} \leq \tilde{\lambda}_n \leq 2 \) and \( \frac{1}{2} \leq \lambda_{\infty} \leq 2 \), it follows that

\[
\lambda_n(t + t_n)^{-2} - \lambda_{\infty}^{-2} \lesssim b_n T + |\tilde{\lambda}_n(t_n) - \lambda_{\infty}|
\]
as well. The fact that \(|(\tilde{x}_n)_t - \tilde{\lambda}_n^{-2}| \lesssim b_n\) implies that

\[
|\tilde{x}_n(t + t_n) - \tilde{x}_n(t_n) - \int_0^t \tilde{\lambda}_n(\sigma + t_n)^{-2} d\sigma| \lesssim b_n t.
\]

Using (12.6), we obtain

\[
|\tilde{x}_n(t + t_n) - \tilde{x}_n(t_n) - \lambda_{\infty}^{-2} t| \lesssim b_n T + (b_n T + |\tilde{\lambda}_n(t_n) - \lambda_{\infty}|) T.
\]

Finally, since \(|(\bar{y}_n)_t| \lesssim b_n\), it follows that

\[
|\bar{y}_n(t + t_n) - \bar{y}_n(t_n)| \lesssim b_n T.
\]

The convergence statements (12.5), (12.7) and (12.8) will be used below. To deduce the equation for the expected limit \( \zeta_{\infty} \), we replace \( \bar{Q} \) with its limiting value

\[
Q(x, y, t) \overset{\text{def}}{=} \lambda_{\infty}^{-1} Q(\lambda_{\infty}^{-1} (x - \lambda_{\infty}^{-2} t), \lambda_{\infty}^{-1} y),
\]

and similarly for \( \tilde{\Lambda}Q, \tilde{\partial}_x \tilde{Q}, \tilde{\partial}_y \tilde{Q} \), and drop all \( O(b_n) \) terms. Note that (12.5), (12.7) and (12.8) imply that for any \( f \) (which could be \( Q, \Lambda Q, \partial_x Q, \) or \( \partial_y Q \) in the analysis), we have

\[
\| \tilde{f} - \hat{f} \|_{L_{x,T}^2 H_{xy}^1} \to 0
\]
as \( n \to \infty \). From this, we deduce the expected form for the equation for the limit \( \zeta_{\infty} \) from the equation (12.4) for \( \zeta_n \) and the above convergence assertions, thus, we have

\[
\partial_t \zeta_{\infty} = -\partial_x \Delta \zeta_{\infty} - 3\partial_x (\hat{Q}^2 \zeta_{\infty}) + \langle \hat{f}_1, \zeta_{\infty} \rangle \tilde{\lambda}_{\infty}^{-3} \hat{Q} + \langle \hat{f}_2, \zeta_{\infty} \rangle \tilde{\lambda}_{\infty}^{-3} \tilde{\partial}_x \hat{Q} + \langle \hat{f}_3, \zeta_{\infty} \rangle \tilde{\lambda}_{\infty}^{-3} \tilde{\partial}_y \hat{Q}.
\]

Thus, let us take \( \zeta_{\infty} \) to be the solution to this equation on \([-T, T]\) with the initial condition \( \zeta_{\infty}(0) \). We will now prove (12.2).

Into the first line of (12.4), we make the following replacements

\[
\begin{align*}
\zeta_n &= \rho_n + \zeta_{\infty} \\
\bar{Q} &= \hat{Q} + (\hat{Q} - \tilde{Q}) \\
\tilde{\Lambda}Q &= \tilde{\lambda}_{\infty}^3 \Lambda Q + (\Lambda Q - \tilde{\Lambda}Q) \\
\tilde{\partial}_x Q &= \tilde{\partial}_x \tilde{Q} + (\tilde{\partial}_x \tilde{Q} - \partial_x \tilde{Q}) \\
\tilde{\partial}_y Q &= \tilde{\partial}_y \tilde{Q} + (\tilde{\partial}_y \tilde{Q} - \partial_y \tilde{Q}) \\
\tilde{\lambda}_n &= \tilde{\lambda}_{\infty} + (\tilde{\lambda}_n - \tilde{\lambda}_{\infty})
\end{align*}
\]

and then use (12.11) to simplify the result. This gives us a \( \rho_n \) equation, that we estimate using the local theory estimates, (12.10), (12.5), and the fact that \( \rho_n(0) \to 0 \) strongly in \( L_{x,y}^2 \), and \( b_n \to 0 \). \( \square \)
13. Reduction of linear Liouville to linearized virial

**Proposition 13.1** (linear Liouville theorem). Suppose that \( w \in C^0(\mathbb{R}_t; H^1_{xy}) \cap C^1(\mathbb{R}_t; H^{-2}_{xy}) \) solves

\[
\partial_tw = \partial_xLw + \alpha \Lambda Q + \beta Q_x + \gamma Q_y
\]

for time-dependent coefficients \( \alpha, \beta, \) and \( \gamma. \) Suppose, moreover, that \( w \) satisfies the orthogonality conditions

\[
\langle w, Q^3 \rangle = 0, \quad \langle w, Q_x \rangle = 0, \quad \langle w, Q_y \rangle = 0,
\]

and is globally-in-time uniformly \( x \)-spatially localized

\[
\|\langle x \rangle^{1/2}w\|_{L^\infty_t L^2_{xy}} < \infty.
\]

Then \( w \equiv 0. \)

**Proof.** We can argue that the assumption that \( w \) is uniformly in time \( x \)-localized implies

\[
\langle w, Q \rangle = 0. \tag{13.1}
\]

Indeed, let

\[
F(x,y) = \int_0^x \Lambda Q(x',y) \, dx',
\]

which does not decay as \( x \to \pm \infty. \) Note that since \( \Lambda Q \) is even in both \( x \) and \( y, \) it follows that \( F(x,y) \) is odd in \( x \) and even in \( y. \) Let

\[
J(t) = \langle w, F \rangle.
\]

Despite the lack of decay in \( F \) as \( x \to \pm \infty, \) this quantity is finite due to the \( x \) decay assumption for \( w. \) Specifically,

\[
|J(t)|_{L^\infty_t} \leq \|\langle x \rangle^{1/2}w\|_{L^\infty_t L^2_{xy}} \|\langle x \rangle^{-1/2}F\|_{L^2_{xy}} < \infty, \tag{13.2}
\]

since

\[
\|\langle x \rangle^{-1/2}F\|_{L^2_{xy}} \leq \|\Lambda Q\|_{L^1_x L^2_y} < \infty.
\]

Since \( F \) is odd in \( x \) and \( \Lambda Q \) is even in \( x, \langle \Lambda Q, F \rangle = 0. \) By integration by parts in \( x, \langle Q_x, F \rangle = -\langle Q, \Lambda Q \rangle = 0. \) Since \( Q_y \) is even in \( x \) and \( F \) is odd in \( x, \langle Q_y, F \rangle = 0 \) (alternatively this holds since \( Q_y \) is odd in \( y \) and \( F \) is even in \( y \)). These orthogonality statements and the fact that \( L \Lambda Q = -2Q \) imply

\[
J'(t) = 2\langle w, Q \rangle.
\]

Since \( LQ_x = 0, \langle \Lambda Q, Q \rangle = 0, \langle Q_x, Q \rangle = 0, \) and \( \langle Q_y, Q \rangle = 0, \) we have

\[
J''(t) = 0.
\]

Thus \( J(t) = a_0 + a_1t, \) but \( J(t) \) is bounded by (13.2) so we must have \( a_1 = 0, \) i.e. \( (13.1) \) holds.
Now by (13.1) and $\mathcal{L}\Lambda Q = -2Q$, we deduce
\[ \partial_t \langle \mathcal{L}w, w \rangle = 0 \]
i.e., $\langle \mathcal{L}w, w \rangle$ is constant in time.

By the trivial estimate $\langle \mathcal{L}w, w \rangle \lesssim \|w\|_{H^1_{xy}}^2$ uniformly for all $t$ and the linearized virial estimate Lemma 14.1,
\[ \int_{-\infty}^{+\infty} \langle \mathcal{L}w, w \rangle \, dt \lesssim \|w\|_{L^2_{xy} H^1_{xy}}^2 \lesssim \|x\|^{1/2} w \|L^\infty_t L^2_{xy} \| < \infty. \]

Since $\langle \mathcal{L}w, w \rangle$ is constant in time, this implies that
\[ \langle \mathcal{L}w, w \rangle = 0 \]
for all $t \in \mathbb{R}$. By the orthogonality conditions, $\mathcal{L}$ is strictly positive definite (Lemma 3.5), and this implies $w \equiv 0$. \qed

### 14. The linearized virial estimate

**Lemma 14.1** (linearized virial estimate for $w$). Suppose that $w \in C^0(\mathbb{R}_t; H^1_{xy}) \cap C^1(\mathbb{R}_t; H^{-2}_{xy})$ solves

\[ \partial_t w = \partial_x \mathcal{L}w + \alpha \Lambda Q + \beta Q_x + \gamma Q_y \]

for time-dependent coefficients $\alpha$, $\beta$, and $\gamma$. Suppose, moreover, that $w$ satisfies the orthogonality conditions

\[ \langle w, Q^3 \rangle = 0, \quad \langle w, Q_x \rangle = 0, \quad \langle w, Q_y \rangle = 0. \]

Then
\[ \|w\|_{L^2_t H^1_{xy}} \lesssim \|x\|^{1/2} w \|L^\infty_t L^2_{xy}, \]

where $t$ is carried out globally $-\infty < t < \infty$.

**Proof.** We will reduce this to Lemma 14.2 below. For $\delta > 0$ to be chosen small later, let

\[ v = (1 - \delta \Delta)^{-1} \mathcal{L}w. \]

Since $\mathcal{L}Q_x = 0$ and $\mathcal{L}Q_y = 0$ and $\mathcal{L}\Lambda Q = -2Q$, we compute

\[ \partial_t v = (1 - \delta \Delta)^{-1} \mathcal{L} \partial_t w \]
\[ = (1 - \delta \Delta)^{-1} [\mathcal{L} \partial_x \mathcal{L}w + \alpha \mathcal{L}\Lambda Q] \]
\[ = (1 - \delta \Delta)^{-1} [\mathcal{L} \partial_x (1 - \delta \Delta)v - 2\alpha Q] \]
\[ = \mathcal{L} \partial_x v - 2\alpha Q + E_\delta v, \]

where $E_\delta$ is a zero-order operator with the property that

\[ |\langle x \rangle E_\delta v, v \rangle| \lesssim \delta \|v\|_{H^1_{xy}}^2. \]

\[ (14.2) \]
(sacrificing regularity one gains a power of $\delta$). Specifically, (14.2) is derived as follows. By definition, we have the formula

$$E_\delta v \overset{\text{def}}{=} [(1 - \delta \Delta)^{-1} L(1 - \delta \Delta) - L] \partial_x v - 2\alpha((1 - \delta \Delta)^{-1} - 1)Q.$$  

Note that via the orthogonality conditions on $w$, the equation for $w$, and the definition of $v$ in terms of $w$, we have $|\alpha| \lesssim \|v\|_{L^2_{x,y}}$. Substituting $L = 1 - \Delta - 3Q^2$, we get

$$E_\delta v = -3[(1 - \delta \Delta)^{-1} Q^2(1 - \delta \Delta) - Q^2] \partial_x v - 2\alpha((1 - \delta \Delta)^{-1} - 1)Q.$$  

Using the commutator identity

$$Q^2(1 - \delta \Delta) = (1 - \delta \Delta)Q^2 + \delta(\Delta Q^2) + 2\delta(\nabla Q)\nabla,$$

we reduce to

$$E_\delta v = -3\delta(1 - \delta \Delta)^{-1}[\Delta Q^2 + 2\nabla Q \cdot \nabla] \partial_x v - 2\alpha((1 - \delta \Delta)^{-1} - 1)Q.$$  

Finally, we use

$$(1 - \delta \Delta)^{-1} - 1 = \delta(1 - \delta \Delta)^{-1} \Delta$$

to simplify the second term:

$$E_\delta v = -3\delta(1 - \delta \Delta)^{-1}[\Delta Q^2 + 2\nabla Q \cdot \nabla] \partial_x v - 2\alpha(1 - \delta \Delta)^{-1} \Delta Q.$$  

From this formula, we see that (14.2) follows from Lemma 15.2. Each term has a $\delta$ factor, and each term has a $Q$-weight to absorb the $\langle x \rangle$ in (14.2).

Furthermore, we have

$$\langle v, (1 - \delta \Delta)Q_x \rangle = \langle Lw, Q_x \rangle = \langle w, LQ_x \rangle = 0,$$

$$\langle v, (1 - \delta \Delta)Q_y \rangle = \langle Lw, Q_y \rangle = \langle w, LQ_y \rangle = 0.$$  

By the orthogonality condition $\langle w, Q^3 \rangle = 0$,

$$\langle v, (1 - \delta \Delta)Q \rangle = \langle Lw, Q \rangle = \langle w, LQ \rangle = -2\langle v, Q^3 \rangle = 0.$$  

Thus, by spectral stability, the given estimate for the $E_\delta$ error, and the $v$-to-$w$ conversion estimates in Lemma 15.1 the estimate (14.1) is reduced to Lemma 14.2 below.

**Lemma 14.2** (linearized virial estimate for $v$). Suppose that $v \in C^0(\mathbb{R}_t; H^1_{x,y}) \cap C^1(\mathbb{R}_t; H^{-2}_{x,y})$ solves

$$\partial_t v = L \partial_x v - 2\alpha Q$$

for some time dependent coefficient $\alpha$, and moreover, $v$ satisfies the orthogonality conditions

$$\langle v, Q \rangle = 0, \quad \langle v, Q_x \rangle = 0, \quad \langle v, Q_y \rangle = 0.$$  

Then

$$(14.3) \quad \|v\|_{L^2_t H^1_{x,y}} \lesssim \|\langle x \rangle^{1/2} v\|_{L^\infty_t L^2_{x,y}},$$

where $t$ is carried out over all time $-\infty < t < \infty$.
Proof. Using the orthogonality condition \( \langle v, Q \rangle = 0 \), we compute
\[
0 = \partial_t \langle v, Q \rangle = \langle \mathcal{L} \partial_x v, Q \rangle - 2\alpha \langle Q, Q \rangle.
\]
This yields
\[
\alpha = \frac{\langle v, 3Q^2Q_x \rangle}{\langle Q, Q \rangle}
\]
so that
\[
\partial_t v = \mathcal{L} \partial_x v - \frac{\langle v, 6Q^2Q_x \rangle}{\langle Q, Q \rangle} Q.
\]
Now compute
\[
(14.4) - \frac{1}{2} \partial_t \int x v^2 = \langle B v, v \rangle + \langle P v, v \rangle,
\]
where
\[
B = \frac{1}{2} - \frac{3}{2} \partial_x^2 - \frac{1}{2} \partial_y^2 - \frac{3}{2} Q^2 - 3x QQ_x
\]
and \( P \) can be taken as the rank 2 self-adjoint operator
\[
P v = \frac{1}{2} \frac{6Q^2Q_x}{\langle Q, Q \rangle} \langle v, xQ \rangle + \frac{1}{2} \frac{xQ}{\langle Q, Q \rangle} \langle v, 6Q^2Q_x \rangle.
\]
The continuous spectrum of \( A = B + P \) is \([\frac{1}{2}, +\infty)\). We produced a numerical solver to find the eigenvalues and corresponding eigenfunctions below \( \frac{1}{2} \). The details of the numerical method are given in \([16]\) below.

We find two simple eigenvalues below \( \frac{1}{2} \), namely,
\[
\lambda_1 = -0.5368 \text{ and } \lambda_2 = -0.1075.
\]
Denoting the corresponding normalized eigenfunctions by \( f_1 \) and \( f_2 \), and \( g_1 = \frac{Q}{\|Q\|} \) and \( g_2 = \frac{Q_x}{\|Q_x\|} \), we find
\[
\langle f_1, g_1 \rangle = 0, \quad \langle f_1, g_2 \rangle = -0.8739, \\
\langle f_2, g_1 \rangle = -0.9902, \quad \langle f_2, g_2 \rangle = 0.
\]
Consider the closed subspace \( H_o \) of \( L^2(\mathbb{R}^2) \) given by functions that are odd in \( x \) (no constraint in \( y \)), and the closed subspace \( H_e \) of \( L^2(\mathbb{R}^2) \) given by functions that are even in \( x \) (no constraint in \( y \)). Note that \( L^2(\mathbb{R}^2) = H_o \oplus H_e \) is an orthogonal decomposition. Moreover, taking \( P_o \) and \( P_e \) to be the corresponding orthogonal projections, we have \( A P_o = P_o A \) and \( A P_e = P_e A \), and thus,
\[
\langle Av, v \rangle = \langle AP_o v, P_o v \rangle + \langle AP_e v, P_e v \rangle, \\
\langle v, v \rangle = \langle P_o v, P_o v \rangle + \langle P_e v, P_e v \rangle.
\]
Suppose that we can prove
\[
\langle AP_e v, P_e v \rangle \geq \mu_e \langle P_e v, P_e v \rangle, \\
\langle AP_o e, P_o v \rangle \geq \mu_o \langle P_o v, P_o v \rangle.
\]
Then the above combine to yield
\[ \langle Av, v \rangle \geq \min(\mu_e, \mu_o) \langle v, v \rangle. \]

We see that \( f_1 \) and \( g_2 \) belong to \( H_o \), while \( f_2 \) and \( g_1 \) belong to \( H_e \). Thus, \( A|_{H_o} \) has spectrum \( \{ \lambda_1 \} \cup [\frac{1}{2}, +\infty) \), and \( f_1 \) is the eigenfunction corresponding to \( \lambda_1 \). Applying Lemma 14.3 with \( H = H_o \) and \( \lambda_\perp = \frac{1}{2} \), noting that
\[ (\lambda_\perp - \lambda_1) \sin^2 \beta = (0.5 + 0.5368) * (1 - 0.8739^2) = 0.2450, \]
we find that
\[ \langle AP_o v, P_o v \rangle \geq (0.5000 - 0.2450) \langle P_o v, P_o v \rangle. \]

Also, \( A|_{H_e} \) has spectrum \( \{ \lambda_2 \} \cup [\frac{1}{2}, +\infty) \) with eigenfunction \( f_2 \) corresponding to eigenvalue \( \lambda_2 \). Applying Lemma 14.3 with \( H = H_e \), \( \lambda_\perp = \frac{1}{2} \), \( \lambda_2 \) replacing \( \lambda_1 \), noting that
\[ (\lambda_\perp - \lambda_2) \sin^2 \beta = (0.5000 + 0.10755) * (1 - 0.9902^2) = 0.0118, \]
we find
\[ \langle AP_e v, P_e v \rangle \geq (0.5000 - 0.0118) \langle P_e v, P_e v \rangle. \]

Thus \( A = B + P \) is positive assuming \( v \) satisfies the two orthogonality conditions, and we can integrate (14.4) in time and use elliptic regularity to obtain (14.3). □

A version of the following angle lemma was used in a similar way for spectral computations in [12, Lemma 4.9].

**Lemma 14.3** (angle lemma). Suppose that \( A \) is a self-adjoint operator on a Hilbert space \( H \) with eigenvalue \( \lambda_1 \) and corresponding eigenspace spanned by a function \( e_1 \) with \( \|e_1\|_{L^2} = 1 \). Let \( P_1 f = \langle f, e_1 \rangle e_1 \) be the corresponding orthogonal projection. Assume that \( (I - P_1)A \) has spectrum bounded below by \( \lambda_\perp \), with \( \lambda_\perp > \lambda_1 \). Suppose that \( f \) is some other function such that \( \|f\|_{L^2} = 1 \) and \( 0 \leq \beta \leq \pi \) is defined by \( \cos \beta = \langle f, e_1 \rangle \). Then if \( v \) satisfies \( \langle v, f \rangle = 0 \), we have
\[ \langle Av, v \rangle \geq (\lambda_\perp - (\lambda_\perp - \lambda_1) \sin^2 \beta) \|v\|_H^2. \]

**Proof.** It suffices to assume that \( \|v\|_H = 1 \). Decompose \( v \) and \( f \) into their orthogonal projection onto \( e_1 \) and its orthocomplement:
\[
\begin{align*}
v &= (\cos \alpha) e_1 + v_\perp, & \|v_\perp\|_H &= \sin \alpha, \\
f &= (\cos \beta) e_1 + f_\perp, & \|f_\perp\|_H &= \sin \beta
\end{align*}
\]
for \( 0 \leq \alpha, \beta \leq \pi \). Then
\[ 0 = \langle v, f \rangle = \cos \alpha \cos \beta + \langle v_\perp, f_\perp \rangle, \]
from which it implies
\[ |\cos \alpha \cos \beta| = |\langle v_\perp, f_\perp \rangle| \leq \|v_\perp\|_H \|f_\perp\|_H = \sin \alpha \sin \beta, \]
and thus, it follows that $|\cos \alpha| \leq \sin \beta$. Now

$$\langle Av, v \rangle = \lambda_1 \cos^2 \alpha + \langle Av_\perp, v_\perp \rangle$$

$$\geq \lambda_1 \cos^2 \alpha + \lambda_\perp \sin^2 \alpha$$

$$= \lambda_\perp - (\lambda_\perp - \lambda_1) \cos^2 \alpha$$

$$\geq \lambda_\perp - (\lambda_\perp - \lambda_1) \sin^2 \beta.$$  

\[\square\]

15. Conversion between $w$ and $v = (1 - \delta \Delta)^{-1} \mathcal{L}w$

In this section, we prove two lemmas (Lemma 15.2 and Lemma 15.3) that allow us to transfer the estimate (14.3) to (14.1) (from $v$ to $w$), as summarized in Lemma 15.1 below. The lemma provides spatial estimates (independent of, and hence, uniform in) time $t$.

**Lemma 15.1.** Suppose that $v = (1 - \delta \Delta)^{-1} \mathcal{L}w$ and $0 < \delta \leq 1$. Then

\[15.1\]

$$\|\langle x \rangle^{1/2} v\|_{L^2_{xy}} \lesssim \delta^{-1} \|\langle x \rangle^{1/2} w\|_{L^2_{xy}}.$$  

Suppose further that $\langle w, \nabla Q \rangle = 0$. Then there exists $\delta_0 > 0$ such that for all $0 < \delta \leq \delta_0$, we have

\[15.2\]

$$\|w\|_{L^2_{xy}}^2 + \delta^{1/4} \|
abla w\|_{L^2_{xy}}^2 \lesssim \|v\|_{L^2_{xy}}^2 + \delta \|
abla v\|_{L^2_{xy}}^2.$$  

**Proof.** These estimates are a consequence of Lemmas 15.2 and 15.3 below. Note that

$$v = (1 - \delta \Delta)^{-1} \mathcal{L}w = (1 - \delta \Delta)^{-1}(1 - \Delta - 3Q^2)w$$

$$= (1 - \delta \Delta)^{-1}w - \Delta(1 - \delta \Delta)^{-1}w - 3(1 - \delta \Delta)^{-1}Q^2w.$$  

In the middle term, use $-\Delta = -\delta^{-1} \delta \Delta = \delta^{-1}(1 - \delta \Delta) - \delta^{-1}$, which implies $-\Delta(1 - \delta \Delta)^{-1} = \delta^{-1} - \delta^{-1}(1 - \delta \Delta)^{-1}$ to get

$$v = (1 - \delta^{-1})(1 - \delta \Delta)^{-1}w + \delta^{-1}w - 3(1 - \delta \Delta)^{-1}Q^2w.$$  

By Lemma 15.2, we have

$$\langle x \rangle^{1/2}(1 - \delta \Delta)^{-1}w = \langle x \rangle^{1/2}(1 - \delta \Delta)^{-1} \langle x \rangle^{-1/2} \langle x \rangle^{1/2}w$$

so $\langle x \rangle^{1/2}(1 - \delta \Delta)^{-1}w \in L^2_{xy}$ and similarly for the last term in the expression for $v$ above. This completes the proof of (15.1).

Also, (15.2) is just a rephrasing of (15.3) in Lemma 15.3.

\[\square\]

**Lemma 15.2.** For any $\alpha \in \mathbb{R}$, $0 < \delta \leq 1$, the 2D operator $\langle x \rangle^{\alpha}(1 - \delta \Delta)^{-1} \langle x \rangle^{-\alpha}$ is $L^2_{xy} \to L^2_{xy}$ bounded with operator norm independent of $\delta$. Note the weight is only in $x$ (not both $x$ and $y$).
Thus,\footnote{The result then follows from Schur’s test. □} for \( \alpha > 0 \), we use \( \langle x \rangle^\alpha \lesssim \langle x - x' \rangle^\alpha + \langle x' \rangle^\alpha \) to obtain
\[
|K((x, y), (x', y'))| \leq \frac{1}{\delta} \left| k \left( \frac{x - x'}{\delta^{1/2}}, \frac{y - y'}{\delta^{1/2}} \right) \right| \left| \frac{x - x'}{\delta^{1/2}} \right|^\alpha + \frac{1}{\delta} \left| k \left( \frac{x - x'}{\delta^{1/2}}, \frac{y - y'}{\delta^{1/2}} \right) \right|.
\]
Thus,
\[
\|K((x, y), (x', y'))\|_{L^\infty_{x,y} L^1_{x',y'}} + \|K((x, y), (x', y'))\|_{L^\infty_{x',y'} L^1_{x,y}} \lesssim_\alpha 1.
\]
The result then follows from Schur’s test.

\[
\text{Lemma 15.3. There exists } \delta_0 > 0 \text{ such that if } \langle w, \nabla Q \rangle = 0 \text{ and } 0 < \delta \leq \delta_0, \text{ then}
\]
\[
\langle (1 - \delta^{1/4})w, w \rangle \lesssim \langle (1 - \delta)w, w \rangle,
\]
where \( v = (1 - \delta\Delta)^{-1}\mathcal{L}w \), and the implicit constant is independent of \( \delta \).

\[
\text{Proof. Replacing } v \text{ by its definition and taking } f = (1 - \delta^{1/4} \Delta)^{1/2}w, \text{ this is}
\]
\[
\langle f, f \rangle \lesssim \langle \mathcal{L}(1 - \delta^{1/4} \Delta)^{-1/2}f, (1 - \delta\Delta)^{-1}\mathcal{L}(1 - \delta^{1/4} \Delta)^{-1/2}f \rangle,
\]
which is operator inequality
\[
1 \lesssim (1 - \delta^{1/4} \Delta)^{-1/2}\mathcal{L}(1 - \delta\Delta)^{-1}\mathcal{L}(1 - \delta^{1/4} \Delta)^{-1/2}
\]
on the subspace given by orthogonality condition. The following identity can be checked by expanding out the commutators
\[
(1 - \delta^{1/4} \Delta)^{-1/2}\mathcal{L}(1 - \delta\Delta)^{-1}\mathcal{L}(1 - \delta^{1/4} \Delta)^{-1/2}
\]
\[
= (1 - \delta^{1/4} \Delta)^{-1/2}(1 - \delta\Delta)^{-1/2}\mathcal{L}^2(1 - \delta\Delta)^{-1/2}(1 - \delta^{1/4} \Delta)^{-1/2}
\]
\[
+ (1 - \delta^{1/4} \Delta)^{-1/2}[\mathcal{L}, (1 - \delta\Delta)^{-1/2}](1 - \delta\Delta)^{-1/2}\mathcal{L}(1 - \delta^{1/4} \Delta)^{-1/2}
\]
\[
- (1 - \delta^{1/4} \Delta)^{-1/2}(1 - \delta\Delta)^{-1/2}\mathcal{L}[\mathcal{L}, (1 - \delta\Delta)^{-1/2}](1 - \delta^{1/4} \Delta)^{-1/2}.
\]
Since \( (1 - \delta^{1/4} \Delta)^{-1/2}\mathcal{L}(1 - \delta^{1/4} \Delta)^{-1/2} \) is \( L^2 \to L^2 \) bounded with operator norm \( \leq \delta^{-1/4} \),
by Lemma \text{15.4} below, the second and third lines are \( L^2 \to L^2 \) bounded operators
with norm \( \leq \delta^{-1/4}\delta^{1/2} = \delta^{1/4} \). Thus, for sufficiently small \( \delta_0 \), the positivity reduces
to the question of positivity of the first line:
\[
1 \lesssim (1 - \delta^{1/4} \Delta)^{-1/2}(1 - \delta\Delta)^{-1/2}\mathcal{L}^2(1 - \delta\Delta)^{-1/2}(1 - \delta^{1/4} \Delta)^{-1/2}.
\]
Take
\[
g = (1 - \delta\Delta)^{-1/2}(1 - \delta^{1/4} \Delta)^{-1/2}w
\]
so that (15.4) can be written as
(15.5) \( \langle w, w \rangle \lesssim \langle \mathcal{L}^2 g, g \rangle \).

We claim that to prove (15.5), it suffices to establish
(15.6) \( \langle g, g \rangle \lesssim \langle \mathcal{L}^2 g, g \rangle \).

To see that (15.6) implies (15.5), note that
(15.7) \[ \| w \|_{L^2}^2 = \| (1 - \delta \Delta)^{1/2} (1 - \delta^{1/4} \Delta)^{1/2} g \|^2 \lesssim \| g \|_{L^2}^2 + \delta^{5/4} \| \Delta g \|_{L^2}^2. \]

Standard elliptic regularity estimates give
(15.8) \[ \| (1 - \Delta) g \|_{L^2}^2 \lesssim \langle \mathcal{L}^2 g, g \rangle + \| g \|_{L^2}^2. \]

Plugging (15.8) into (15.7) gives
(15.9) \[ \| w \|_{L^2}^2 \lesssim \| g \|_{L^2}^2 + \delta^{5/4} \langle \mathcal{L}^2 g, g \rangle. \]

Thus, if we assume (15.6), then (15.9) and (15.6) yields (15.5).

It therefore remains to prove (15.6). Note that the spectrum of \( \mathcal{L}^2 \) (being the square of the spectrum of \( \mathcal{L} \)) consists of 0 as an isolated eigenvalue with eigenspace spanned by \( \nabla Q \), and the rest of the spectrum of \( \mathcal{L}^2 \) lies in \([\alpha, +\infty)\) for some \( \alpha > 0 \). Since \( \langle w, \nabla Q \rangle = 0 \), (15.6) would follow immediately if \( g = w \). We can, however, use Lemma 15.5 to show that
(15.10) \[ \| \langle g, \nabla Q \rangle \| \lesssim \delta^{1/4} \| w \|_{L^2} \]

which suffices to establish (15.6) as follows. (We explain how to obtain (15.10) from Lemma 15.5 at the end of the proof.)

Let \( P_0 \) be the operator of orthogonal projection onto \( \nabla Q \), and let \( P_c = I - P_0 \). Explicitly,
\[ P_0 g = \frac{\langle g, \nabla Q \rangle \nabla Q}{\| \nabla Q \|_{L^2} \| \nabla Q \|_{L^2}}. \]

By (15.10) and (15.9), we get
(15.11) \[ \| P_0 g \|_{L^2}^2 \lesssim \delta^{1/2} (\| g \|_{L^2}^2 + \langle \mathcal{L}^2 g, g \rangle). \]

Since the spectrum of \( P_c \mathcal{L}^2 \) starts at \( \alpha > 0 \),
(15.12) \[ \alpha \| P_c g \|_{L^2}^2 \leq \langle \mathcal{L}^2 g, g \rangle. \]

We have
\[ \| g \|_{L^2}^2 = \| P_0 g \|_{L^2}^2 + \| P_c g \|_{L^2}^2. \]

Plugging in (15.11) and (15.12),
\[ \| g \|_{L^2}^2 \leq \frac{1}{\alpha} \langle \mathcal{L}^2 g, g \rangle + C \delta^{1/2} (\| g \|_{L^2}^2 + \langle \mathcal{L}^2 g, g \rangle) \]

for some constant \( C > 0 \). Taking \( \delta_0 \) sufficiently small gives (15.6).
Finally, we explain the proof of (15.10) from Lemma [15.5] Applying Lemma [15.5] to \( f = (1 - \delta^{1/4} \Delta)^{-1/2} \nabla Q \), we obtain
\[
\langle (1 - \delta^{1/4} \Delta)^{-1/2}(1 - \delta^{1/4} \Delta)^{-1/2} \nabla Q, w \rangle - \langle (1 - \delta^{1/4} \Delta)^{-1/2} \nabla Q, w \rangle \lesssim \delta \|w\|_{L^2}.
\]
Replacing \( \delta \) by \( \delta^{1/4} \) in Lemma [15.5] and taking \( f = \nabla Q \), yields
\[
\langle (1 - \delta^{1/4} \Delta)^{-1/2} \nabla Q, w \rangle - \langle \nabla Q, w \rangle \lesssim \delta^{1/4} \|w\|_{L^2}.
\]
Combining the above two estimates, we have
\[
\langle (1 - \delta \Delta)^{-1/2}(1 - \delta^{1/4} \Delta)^{-1/2} \nabla Q, w \rangle - \langle \nabla Q, w \rangle \lesssim \delta \|w\|_{L^2}.
\]
Since \( \langle \nabla Q, w \rangle = 0 \), this reduces to (15.10). \( \square \)

The following two commutator lemmas (Lemma [15.4] and [15.5]) were used in the proof of Lemma [15.3] above.

**Lemma 15.4** (1st commutator lemma). The compositions
\[(15.13) \quad (1 - \delta^{1/4} \Delta)^{1/2}[\mathcal{L}, (1 - \delta \Delta)^{-1/2}] (1 - \delta^{1/4} \Delta)^{-1/2}
\]
and
\[(15.14) \quad (1 - \delta^{1/4} \Delta)^{-1/2}[\mathcal{L}, (1 - \delta \Delta)^{-1/2}] (1 - \delta^{1/4} \Delta)^{1/2}
\]
are \( L^2 \to L^2 \) bounded with operator norm \( \lesssim \delta^{1/2} \).

**Proof.** In this proof, we use \( x \) to represent the 2D coordinate \( \mathbf{x} = (x, y) \). Since (15.14) is the adjoint of (15.13), it suffices to prove the claim for (15.13). For this, we start by showing that
\[(15.15) \quad [\mathcal{L}, (1 - \delta \Delta)^{-1/2}] \text{ is } L^2 \to L^2 \text{ bounded with norm } \lesssim \delta^{1/2}.
\]
Let \( \hat{k}(\xi) = (1 + |\xi|^2)^{-1/2} \) (in two dimensions). Then the Fourier transform of \( \delta^{-1} k(\delta^{-1/2} x) \) is \( (1 + \delta |\xi|^2)^{-1/2} \). Note that
\[
[\mathcal{L}, (1 - \delta \Delta)^{-1/2}] = -3[Q^2, (1 - \delta \Delta)^{-1/2}].
\]
Dropping the factor \(-3\), the kernel is \( K(x, x') \), where
\[
K(x, x') = \delta^{-1} k(\delta^{-1/2}(x - x'))(Q(x)^2 - Q(x')^2).
\]
Since \( |Q(x)^2 - Q(x')^2| \lesssim |x - x'| \), we have
\[
|K(x, x')| \lesssim \delta^{1/2} \cdot \delta^{-1} \hat{k}(\delta^{-1/2}(x - x')),
\]
where \( \hat{k}(x) = |x| k(x) \). Since \( \hat{k} \in L^1 \), we obtain
\[
\|K\|_{L^2_x L^1_y} \lesssim \delta^{1/2}, \quad \|K\|_{L^\infty_x L^1_y} \lesssim \delta^{1/2}.
\]
By Schur’s test, we obtain (15.15). To prove that (15.13) is \( L^2 \to L^2 \) bounded with operator norm \( \lesssim \delta^{1/2} \), it suffices to prove the following two statements:
\[(15.16) \quad [\mathcal{L}, (1 - \delta \Delta)^{-1/2}] (1 - \delta^{1/4} \Delta)^{-1/2} \text{ is } L^2 \to L^2 \text{ bounded with norm } \lesssim \delta^{1/2}
\]
and

\begin{equation}
\delta^{1/8} \nabla \left[ \mathcal{L}, (1 - \delta \Delta)^{-1/2} \right] (1 - \delta^{1/4} \Delta)^{-1/2} \text{ is } L^2 \to L^2 \text{ bounded with norm } \lesssim \delta^{1/2}.
\end{equation}

The claim \((15.16)\) follows immediately from \((15.15)\), since \((1 - \delta^{1/4} \Delta)^{-1/2}\) is \(L^2 \to L^2\) bounded with operator norm \(\lesssim 1\). For \((15.17)\), we use the notation \(K(x, x')\) and \(k\) introduced in the proof of \((15.15)\). The operator in \((15.17)\) applied to a function \(f\) takes the form

\begin{equation}
\delta^{1/8} \nabla_x \int_{x'} K(x, x')[(1 - \delta^{1/4} \Delta)^{-1/2} f](x') \, dx'.
\end{equation}

Substituting \(K(x, x') = \delta^{-1} k(\delta^{-1/2} (x - x'))(Q^2(x) - Q^2(x'))\) and distributing the \(x\) derivative, we write \((15.18)\) as

\[
= \delta^{1/8} \int_{x'} \left[ \nabla_x \left( \delta^{-1} k(\delta^{-1/2} (x - x')) \right) (Q^2(x) - Q^2(x')) \right] [(1 - \delta^{1/4} \Delta)^{-1/2} f](x') \, dx'
+ \delta^{1/8} \int_{x'} \delta^{-1} k(\delta^{-1/2} (x - x'))(\nabla_x Q^2(x))(1 - \delta^{1/4} \Delta)^{-1/2} f](x') \, dx'.
\]

In the first term, we can replace \(\nabla_x\) by \(-\nabla_{x'}\) and then integrate by parts to continue

\[
= \int_{x'} \delta^{-1} k(\delta^{-1/2} (x - x'))(Q^2(x) - Q^2(x')) \delta^{1/8} \nabla_{x'} [(1 - \delta^{1/4} \Delta)^{-1/2} f](x') \, dx'
+ \delta^{1/8} \int_{x'} \delta^{-1} k(\delta^{-1/2} (x - x'))(\nabla_x Q^2(x) - \nabla(Q^2)(x')) [(1 - \delta^{1/4} \Delta)^{-1/2} f](x') \, dx'.
\]

By the same argument that established \((15.15)\), the second line is an \(L^2 \to L^2\) bounded operator acting on \(f\) with operator norm \(\lesssim \delta^{1/8} \cdot \delta^{1/2}\). The first line is equal to the following operator acting on \(f\):

\begin{equation}
[\mathcal{L}, (1 - \delta \Delta)^{-1/2}] \delta^{1/8} \nabla (1 - \delta^{1/4} \Delta)^{-1/2}.
\end{equation}

Note that \(\delta^{1/8} \nabla (1 - \delta^{1/4} \Delta)^{-1/2}\) is \(L^2 \to L^2\) bounded with operator norm \(\lesssim 1\). This, combined with \((15.15)\), gives that \((15.19)\) is \(L^2 \to L^2\) bounded with operator norm \(\lesssim \delta^{1/2}\), completing the proof of \((15.13)\). \(\square\)

**Lemma 15.5 (2nd commutator lemma).** For any functions \(f\) and \(w\), we have the estimate

\[
|\langle (1 - \delta \Delta)^{-1/2} f, w \rangle - \langle f, w \rangle| \lesssim \delta \| f \|_{H^2} \|w\|_{L^2}.
\]

**Proof.** We have

\[
|\langle (1 - \delta \Delta)^{-1/2} f, w \rangle - \langle f, w \rangle| \lesssim \| [ (1 - \delta \Delta)^{-1/2} - 1 ] f \|_{L^2} \| w \|_{L^2}.
\]

As a Fourier multiplier, the operator \((1 - \delta \Delta)^{-1/2} - 1\) takes the form

\[
(1 + \delta |\xi|^2)^{-1/2} - 1 = \frac{-\delta |\xi|^2}{(1 + \delta |\xi|^2)^{1/2}(1 + (1 + \delta |\xi|^2)^{1/2})}.
\]
and hence,
\[ |(1 + \delta|\xi|^2)^{-1/2} - 1| \lesssim \delta|\xi|^2, \]
from which the conclusion follows.

16. Numerical method

We discuss the method for finding the eigenvalues and eigenfunctions of the following operator:

\[
2(B + P) \overset{\text{def}}{=} -3\partial_{xx} - \partial_{yy} + 1 - 3Q^2 - 6xQQ_x + 2P,
\]
where \(P\) is defined as the following operator with inner products:

\[
2Pv = \frac{6Q^2Q_x}{\|Q\|_2^2} \langle v, xQ \rangle + \frac{xQ}{\|Q\|_2^2} \langle v, 6Q^2Q_x \rangle.
\]

Similar to the process in [3], we numerically calculate the spectrum of the operator \(2(B + P)\) in the following steps:

1. We discretize the operator into the form of matrix.
2. We find the eigenvalues and corresponding eigenvectors of the matrix. Those eigenvalues are the spectrum of the operator \(2(B + P)\).
3. The eigenvalues which are less than 1 are the ones we are looking for, since we know \(2(B + P)\) has continuous spectrum from Lemma 14.2.

It has been shown in [3] that the matlab command `eig` or `eigs`, which incorporate ARPACK in [19], is an efficient way to compute the eigenvalues for the large matrices. Therefore, it is suffice to show how to discretize the operator \(2(B + P)\) into the matrix form.

The discretization of the operator \(B\) and imposing the homogeneous Dirichlet boundary conditions are standard. For that we follow the same procedure as in [36, Chapter 6, 9, 12].

We next describe how we discretize the projection operator \(P\). We also introduce a mapping to make the mapped Chebyshev collocation points to be more concentrated in the central region, where the functions (ground state \(Q\)) own the largest amplitude and gradient.

16.1. Discretization of the Projection term. In this part, we introduce our discretization of the projection term \(P\). We give a general formula for discretizing the operator \(P\) in the form

\[ Pu = \langle u, f \rangle g, \]
where \(u, f, g \in L^2(\mathbb{R}^d)\). For simplicity, we discuss the 1D case. One can easily extend the cases \(d \geq 2\) by standard numerical integration technique for multi-dimensions, e.g., see [36, Chapter 6, 12].
We use the notation \( f_i \) (and similar other variables) to be the discretized form of the function \( f(x) \) at the point \( x_i \), and the vector \( \vec{f} = (f_0, f_1, \cdots, f_N)^T \). We also denote the operation “\(*\)” to be the pointwise multiplication of the vectors or matrices with the same dimension, i.e., \( \vec{a} \ast \vec{b} = (a_0b_0, \cdots, a_Nb_N)^T \). And the notation “\( \ast \)” as the regular vector or matrix multiplication.

Let \( w(x) \) to be the weights for a given quadrature. For example, if we consider the composite trapezoid rule with step-size \( h \), we have
\[
\vec{w} = (w_0, w_1, \cdots, x_N)^T = \frac{h}{2}(1, 2, \cdots, 2, 1)^T,
\]

since the composite trapezoid rule can be written as
\[
\int_a^b f(x)dx \approx \sum_{i=0}^N f_iw_i = \vec{f}^T \ast \vec{w}.
\]

Similarly, if we want to evaluate Chebyshev Gauss-Lobatto quadrature, which is exactly what we use in this work, we have
\[
\int_{-1}^1 f(x)dx \approx \sum_{i=0}^N w_i f(x_i) = \vec{f}^T \ast \vec{w},
\]
where \( w_i = \frac{\pi}{N} \sqrt{1-x_i^2} \) for \( i = 1, 2, \cdots, N-1 \), \( w_0 = \frac{\pi}{2N} \sqrt{1-x_0^2} \), and \( w_N = \frac{\pi}{2N} \sqrt{1-x_N^2} \) are the weights together with the weighted functions.

Using this, we have
\[
Pu = (u, f)g = (\sum_{i=0}^N w_i f_i u_i) \vec{g} = \begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_N \end{bmatrix} \left( \sum_{i=0}^N w_i f_i u_i \right) = \begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_N \end{bmatrix} \left( \vec{w}^T \ast \vec{f}^T \ast \vec{u} \right) \quad := \mathbf{P} \vec{u},
\]
with the matrix
\[
(16.3) \quad \mathbf{P} = \begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_N \end{bmatrix} \left( \vec{w}^T \ast \vec{f}^T \right)
\]
to be the discretized approximation form of the projection operator \( P \).

One can easily see that the matrix \( \mathbf{P} \) is a dense matrix. This is the reason that we can only use limited number of Spectral collocation (e.g., Chebyshev collocation) points in our 2D computation. In the next subsection, we will introduce a mapping such that we change the domain from \([-1, 1]\) to \([-L, L]\), with more points concentrated around the center.
16.2. The Mapped Chebyshev collocation points. We know that $Q$ decays exponentially fast, therefore, we can take the computational square region $[-L, L] \times [-L, L]$ to approximate the real space $\mathbb{R}^2$.

We cannot assign too many grid points in each dimension, however, we do want to put more grid points in the center region of the computational domain $[-L, L] \times [-L, L]$, where the function $Q$ has the largest amplitude and gradient. We, therefore, redistribute the mesh grid based on the original Chebyshev collocation points.

Consider the mapping $T$ that maps the points more concentrated near the origin:

\begin{equation}
T : [-1, 1] \to [-L, L], \quad T(\xi) = x,
\end{equation}

here, $\xi$ represents the Chebyshev collocation points and $x$ is the grid points in the computational interval, which are more concentrated at the center that we use to discretize the ground state function $Q$ and the operator $B + P$. One possible mapping that we choose is

\begin{equation}
x(\xi) = L \frac{e^{a\xi} - e^{-a\xi}}{e^a - e^{-a}}
\end{equation}

for some constant parameter $a$.

During our computation, we take $L = 20$ and $a = 4$ or 5. Without this process, the stiffness and inaccuracy may occur, since we are bounded to using just a few points (say $N = 72$, which will lead to an eigenvalue problem for a $72^2 \times 72^2$ matrix) on each dimension to have a bearable computational cost. With this process, on the other hand, the number of grid points in each dimension could be taken to be equal to $N = 32, 48$ and 64. These three different choices lead to almost the same results.

Note that after the mapping is applied, we also need to apply the chain rule for both derivatives, $\partial_x$ and $\partial_y$. Denote by the vectors $\vec{x}_\xi$, $\vec{x}_{\xi\xi}$, $\vec{y}_\eta$ and $\vec{y}_{\eta\eta}$ – the vectors discretized from $x_\xi$, $x_{\xi\xi}$, $y_\eta$, $y_{\eta\eta}$. Denote by $\text{diag}(\vec{v})$ to be the diagonal matrix generated from the vector $\vec{v}$. The matrices $D_x^{(1)}$ and $D_x^{(2)}$ are the differential matrix of $\partial_\xi$ and $\partial_{\xi\xi}$ generated by the Chebyshev collocation differentiation. Similarly, the matrices $D_y^{(1)}$ and $D_y^{(2)}$ are for $\partial_\eta$ and $\partial_{\eta\eta}$.

By the chain rule, we have

\begin{equation}
\tilde{D}_x^{(1)} = \text{diag}(\frac{1}{\vec{x}_\xi})D_x^{(1)},
\end{equation}

and

\begin{equation}
\tilde{D}_x^{(2)} = \text{diag}(\frac{1}{\vec{x}_\xi^2})D_x^{(2)} + \text{diag}((D_x^{(1)} \ast \frac{1}{\vec{x}_\xi}) \ast \frac{1}{\vec{x}_\xi})D_x^{(1)}.
\end{equation}

Similarly, we can generate the differential matrices $\tilde{D}_y^{(1)}$ and $\tilde{D}_y^{(2)}$ in terms of $\eta$.

Finally, the operator $B + P$ is discretized in the form

\begin{equation}
M = -3\tilde{D}_x^{(2)} - \tilde{D}_y^{(2)} + \text{diag}(\vec{I} - 3 \ast \vec{Q}^2 - 6 \ast \vec{x} \ast \vec{Q} \ast \vec{Q}_x) + P,
\end{equation}
where $P$ is the matrix form for the inner products that can be discretized from the formula (16.3), and $\mathbf{1} = (1, 1, \cdots, 1)^T$ is the vector with the same size of the other variables, such as $\mathbf{Q}$.

16.3. **Numerical results.** After setting the operator $2(B + P)$ into the matrix form $M$ as in (16.6), we use the Matlab command `eigs` to find the first few smallest eigenvalues below $\frac{1}{2}$ and their corresponding eigenfunctions.

Denote by $\lambda_i$ the $i$th eigenvalue and its corresponding eigenfunction $\phi_i$. After computing the eigenfunctions $(\phi_1, \phi_2)$, we normalize them as well as the pair $(Q, Q_x)$ such that their $L^2$ norms are set to 1. The inner products of these four terms indicate the searched geometries.

The numerical result gives us two negative eigenvalues of $2(B + P)$:

\begin{equation}
\lambda_1 = -1.0735, \quad \lambda_2 = -0.2151.
\end{equation}

We obtain the following matrix of normalized inner products, i.e., angles

\begin{equation}
\begin{bmatrix}
\langle Q, \phi_1 \rangle & \langle Q, \phi_2 \rangle \\
\langle Q_x, \phi_1 \rangle & \langle Q_x, \phi_2 \rangle
\end{bmatrix} = \begin{bmatrix}
-0.0000 & 0.9902 \\
0.8739 & -0.0000
\end{bmatrix}.
\end{equation}

We note that these values are consistent when using $N = 32, 48, 64,$ or 72 collocation points up to the first 4 digits.

For accuracy and consistency, we tested other operators, for example, a well-studied operator $L = -\Delta + 1 - 3Q^2$ and obtained one negative eigenvalue $-5.4122$ and the double zero eigenvalue with the eigenfunctions $Q_x$ and $Q_y$.

We also tested the operator $2B$ (without the $2P$ term from the above). Since the operator $B$ can be discretized into sparse matrices, besides the mapped Chebyshev-collocation method, we could also use the finite difference discretization with $N = 512$ grid points assigned on each dimension. Both of these two different discretizations reach almost the same results: this operator has one negative eigenvalue $\lambda(B) = -0.2151$ with $[\langle Q, \phi \rangle, \langle Q_x, \phi \rangle] = [0.9902, -0.0000]$. Note that it has the same negative eigenvalue as the first one in $2(B + P)$.

We also check the operator $2P$ written in a non self-adjoint form (denoting as $2\bar{P}$), that is,

\begin{equation}
\langle 2\bar{P}v, v \rangle = \frac{4}{\|Q\|_2^2} \langle v, 3Q^2Q_x \rangle \langle v, xQ \rangle.
\end{equation}

Here, we discretize the $2\bar{P}$ by using the formula (16.3). We obtain the following numerical results for the eigenvalues

\begin{equation}
\lambda_1 = -0.2151, \quad \lambda_2 = 0.3580,
\end{equation}
as well as the angles

\[
\begin{bmatrix}
\langle Q, \phi_1 \rangle & \langle Q, \phi_2 \rangle \\
\langle Q_x, \phi_1 \rangle & \langle Q_x, \phi_2 \rangle
\end{bmatrix} = \begin{bmatrix}
0.9902 & 0.0000 \\
0.0000 & -0.9790
\end{bmatrix}.
\]

We observe that the first eigenvalue did not change, while the second did.

We then rewrite the $\bar{P}$ into the self-adjoint form (denoting by $\tilde{P}$) as

\[
\tilde{P}u = \frac{1}{2} \langle u, f \rangle g + \frac{1}{2} \langle u, f \rangle g,
\]

and then we get the same results as in (16.7) and (16.8). This shows that the self-adjoint representation is an important feature in the numerical calculations. We also note that the eigenvalue $\lambda = -0.2151$ is robust, and is directly dependent on the ground state $Q$.

16.4. Calculation of the ground state $Q$. While we can calculate the ground state directly in 2D space, in order to reduce the computational cost and numerical error, we do it from the 1D radial equation

\[
-R_{rr} - \frac{1}{r} R_r + R - R^3 = 0, \quad R_r(0) = 0, \quad R_r(\frac{3}{2}L) = 0,
\]

by using the renormalization method [9, Chapter 24], and then extend the solution to the 2D space by interpolation. We set $r \in [0, 1.5 \times L]$ as the coordinate in 1D, and $R$ to be the ground state calculated from the coordinate $r$.

We then generate the 2D function of $Q$ based on $R$. We take $\bar{x} \in [-L, L]$ to be the discretized collocation points that we use in 1D. Then, we generate the 2D mesh by matlab command `meshgrid` as follows

\[
[X, Y] = \text{meshgrid}(\bar{x}).
\]

Then, the matrix $Q$, describing the 2D ground state $Q$, can be obtained by shape-preserving cubic spline interpolation achieved by the matlab command ` interp1`, i.e.,

\[
Q = \text{ interp1}(r, R, \sqrt{X^2 + Y^2}, 'pchip').
\]

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