D-strings in unconventional type I vacuum configurations

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Abstract
We determine the spectrum of D-string bound states in various classes of generalized type I vacuum configurations with sixteen and eight supercharges. The precise matching of the BPS spectra confirms the duality between unconventional type IIB orientifolds with quantized NS-NS antisymmetric tensor and heterotic CHL models in $D = 8$. A similar analysis puts the duality between type II (4,0) models and type I strings without open strings on a firmer ground. The analysis can be extended to type II (2,0) asymmetric orbifolds and their type I duals that correspond to unconventional $K3$ compactifications. Finally we discuss BPS-saturated threshold corrections to the corresponding low-energy effective lagrangians. In particular we show how the exact moduli dependence of some $F^4$ terms in the eight-dimensional type II (4,0) orbifold is reproduced by the infinite sum of D-instanton contributions in the dual type I theory.

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1 Introduction

The standard $SO(32)$ heterotic - type I duality [1, 2] requires the existence of D-string bound states [3] in toroidally compactified type I theory that are mapped to fundamental heterotic winding states. In [4] the masses and multiplicities of these bound states carrying arbitrary quantum numbers was obtained by studying the effective $O(N)$ gauge theory on the type I D-string worldsheet in the infrared limit where it flows to an orbifold conformal field theory. In this paper we will extend these results to more general classes of dual pairs with sixteen supercharges [5] and to some dual pairs with eight supercharges. We will consider two types of models. In the first the type I side is modded by some orbifold group. In the second the orientifold $\Omega$ projection itself is modified by some orbifold group. In the first case the D-string effective action on the type I side is just the original $O(N)$ gauge theory modded by further orbifold group action. On the other hand in the second case one gets the D-string effective action by starting from type IIB D-strings and modding the open strings ending on them by the modified $\Omega$ projection. The resulting D-string effective action thus turns out to be a different $Z_2$ projection of the $(8, 8) U(N)$ gauge theory living on the type IIB D-strings.

More precisely, in section 3, we will discuss the simplest instance of type I duals to heterotic CHL models [6]. The former have been identified in [5, 7] with unconventional type IIB orientifolds with quantized NS-NS antisymmetric tensor, introduced long time ago [8] and termed $BPS$ unorientifolds in [5]. The observed rank reduction of the gauge group in these generalized toroidal compactifications admits a gauge theoretic interpretation in terms of non-commuting Wilson lines that represent the obstruction to defining a vector structure in the vacuum gauge bundle on the torus [9, 5, 7, 10].

In section 4 we study type II (4,0) models without D-branes that are dual to type I models without open strings [5, 7, 11, 12]. The former arise from projections that include $(-)^{F_L}$ and remove all massless R-R states to which D-brane usually couple [13]. The latter correspond to generalized $\Omega$-projections of the closed oriented type IIB string that give rise to vanishing massless tadpoles in the transverse channel Klein-bottle amplitudes thus preventing the introduction of D-banes and their open-string excitations [5, 7, 11, 12]. The duality relies on supersymmetry considerations (sixteen supercharges), matching of the massless spectra ($d = 10 - D$ matter vector multiplets) and S-duality invariance of the type IIB theory in $D = 10$ that maps the $(-)^{F_L}$ action to the $\Omega$ worldsheet parity operation through $(-)^{F_L} = S \Omega S^{-1}$ [14, 5]. We show that in the simplest $D = 9$ case the D-string bound-state spectrum in the type I models without open strings agrees with the BPS spectrum of fundamental strings in type II (4,0) models. The exact matching of the BPS spectra represents a precision test of the duality between these two classes of models. Eventually we extend our analysis to dual pairs of theories with eight supercharges corresponding to generalized orbifold/orientifold compactifications on $K3$. We will discuss in some detail the matching between a type
II (2,0) model \textit{without D-branes} and its dual type I model \textit{without open strings} in the \(T^4/Z_2\) orbifold limit of \(K3\) [14].

In preparation for the discussions in sections 3 and 4, in section 2 we recall the definition and explicitly compute the elliptic genera for some relevant orbifold CFT’s corresponding to symmetric product spaces [15].

As an application of the results in sections 3 and 4, in section 5, we compute D-string instanton contributions to BPS-saturated \(F^4\) eight dimensional couplings [16] and compare with the corresponding perturbative results on the dual side.

Section 6 contains our conclusions and some directions for further investigation.

2 Elliptic Genera and Symmetric Spaces

In this section we compute the elliptic genera for the class of conformal field theories relevant in the discussion of type I D-string bound states spectra and “BPS-saturated” threshold corrections. The dual pairs we will be interested in are contructed via orbifold/orientifold compactifications of the type IIB theory. The self-duality of the original type IIB string exchanges the NS-NS and R-R antisymmetric tensors, mapping fundamental string excitations with \(N\) units of winding to bound states of \(N\) D-strings. The information about charges, masses and degeneracies of these bound states is encoded in the elliptic genus of the low energy gauge theory describing \(N\) nearby D-strings in the generalized type I vacuum. Following [4], we compute this index in the infrared limit where the relevant gauge theories flow to \((8,0)\) orbifold conformal field theory in a symmetric target space \(M^N/S_N\). The details of the correspondence in each particular case will be discussed in the next sections where the complete agreement between the fundamental and bound-state spectra is established for theories with sixteen supercharges, such as the type I duals of the CHL models in \(D = 8\) or the type I duals of the type II (4,0) models in \(D = 9\) [5], as well as for a class of models with eight supercharges [14].

Given a two-dimensional CFT, the relevant elliptic genus for our present discussion, is the partition function with right moving fermions in the odd spin structure:

\[
\chi(q, \bar{q}) = Tr_R(-)^{F_R} q^{L_0-c/24} \bar{q}^{L_0-c/24} = Tr_R(-)^{F_R} e^{-2\pi \tau_2 H + 2\pi i \tau_1 \sigma} P_\sigma, \tag{2.1}
\]

where \(q = e^{2\pi i \tau}\) and \(\tau\) is the genus-one worldsheet modulus. The constant \(R\) is introduced for later convenience, \(H\) and \(P_\sigma\) are the Hamiltonian (t-evolution) and momentum along the \(\sigma\) direction respectively. The only dependence on \(q\) of \(\chi(q, \bar{q})\) arises from the integration over the bosonic zero modes since the trace in the odd spin structure receives contribution only from the right moving ground states. The elliptic genus (2.1) thus effectively counts the number of BPS states (ground states of the right moving supersymmetric sector), or more precisely the difference between the number of fermionic and bosonic supersymmetric ground-states.

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In the following, we will be interested in computing these elliptic genera for two-dimensional CFT's obtained by modding out $N$ copies of a given world-sheet theory (usually itself an orbifold theory with orbifold group $Z_K$) by the permutation group $S_N$. We will be interested in the situations where the orbifold group $Z_K$ preserves some number, let us say $\mathcal{N}$, of supersymmetries $^1$. The worldvolume theories for the D-strings under consideration are more like Green-Schwarz $\sigma$-models, where target-space supersymmetry is realized through the $Z_K$-invariant zero-modes of some periodic fermion which we will generically denote them by $S$ in the following. The full orbifold group acting on $N$ copies of the world sheet theory is therefore the semidirect product $G \equiv S_N \ltimes Z_K^N$.

Let us briefly review how the orbifold elliptic genus is computed [17, 15, 4]. The Hilbert space for a non-Abelian orbifold conformal field theory is built from the different twisted sectors associated to the conjugacy classes $[g]=[(g, \alpha)]$ of the orbifold group $G = S_N \ltimes Z_K^N$ (with $g \in S_N$ and $\alpha \in Z_K^N$). In each sector we project by the centralizer $C_g$ of the twist element $g$ in $G$. The elliptic genus for this CFT is zero, viz. $(2^N - 1 - 2^{N-1}) = 0$, due to the trace over the fermionic zero-modes associated to the center of mass combination $S_0^1 + S_0^2 + \cdots S_0^N$. That this combination is invariant under the orbifold group is clear, since $S_0^k$ are invariant under $Z_K$ and get permuted under $S_N$. Indeed, these zero modes generate the $2^{N-1}$ bosonic and $2^{N-1}$ fermionic components of short BPS supermultiplets. Being interested in computing the degeneracies of such supermultiplets, only sectors with no additional fermionic zero-modes will be relevant.

Let us start by identifying these sectors. In order to achieve this goal it is sufficient to consider the action of $S_N$, since as already stated $Z_K^N$ acts trivially on the $S$ fields. A general conjugacy class $[g]$ in $S_N$ is characterized by partitions $N_n$ of $N$ satisfying $\sum nN_n = N$, where $N_n$ denotes the multiplicity of the cyclic permutation $(n)$ in the decomposition

$$[g] = (1)^{N_1}(2)^{N_2}\cdots(s)^{N_s}. \quad (2.2)$$

The centralizer of an element in this conjugacy class takes the form

$$C_g = \prod_{n=1}^s S_{N_n} \times Z_n^{N_n}, \quad (2.3)$$

One immediately finds that if $[g]$ involves cycles of different lengths, say $(n)^a$ and $(m)^b$ with $n \neq m$, the corresponding twisted sector does not contribute to the elliptic genus. In order to see this, note that in this case there are at least two sets of zero modes of $S$, which can be expressed, by a suitable ordering of indices, as $(S^1 + S^2 + \cdots S^{na})$ and $(S^{ma+1} + \cdots S^{ma+mb})$, where the two factors $(n)^a$ and $(m)^b$ act on the two sets of indices in the obvious way. These zero modes survive the group projection because the centralizer of $g$ does not contain any element that mixes these two sets of indices with

$^1$Throughout the paper, we adhere to the standard, though somewhat confusing, notation of counting supersymmetries in four-dimensional units, so that four real supercharges correspond to each supersymmetry.
each other, thereby giving zero contribution to the elliptic genus. Thus we need only to consider those sectors with \([g] = (L)^M\) where \(N = LM\).

In the \([g] = (L)^M\) case the centralizer is \(C_g = \mathcal{S}_M \ltimes Z_L^M\). From the boundary condition along \(\sigma\) it is clear that there are \(M\) combinations of \(S\)’s that are periodic in \(\sigma\). By suitable ordering, they can be expressed as \(\hat{S}(\ell) = \sum_{i=\ell+1}^{L(\ell+1)} S^i\) for \(\ell = 0, \ldots, M - 1\). These zero modes have to be projected by the elements \(h\) in the centralizer \(C_g\). In particular, when \(h\) is the generator of \(Z_M \subset \mathcal{S}_M \subset C_g\), it acts on the zero modes \(\hat{S}(\ell)\) by a cyclic permutation. It is clear, therefore, that only the center of mass combination \(\sum_{\ell=0}^{M-1} \hat{S}(\ell)\) is periodic along the \(\tau\) direction. Hence, this sector contributes to the elliptic genus. More generally any \(h = (h, \beta) \in C_g = \mathcal{S}_M \ltimes Z_L^M\) will satisfy the above criteria provided \(h = (M) \in \mathcal{S}_M\) and \(\beta\) is some element in \(Z_L^M\). The number of such elements \(h\) is \((M - 1)! \times L^M\).

The full orbifold group \(G\) is specified by an element of \(\mathcal{S}_N\) (discussed above) together with an element of \(Z_K^N\). Let us consider a generic element \((g, \alpha)\). We denote by \(e^{2\pi i \phi(K)}\) the eigenvalue of a given field \(\phi\) under the \(Z_K\) action. The groups \(\mathcal{S}_N\) and \(Z_K^N\) form a semi-direct product, since \(\mathcal{S}_N\) acts as an automorphism in \(Z_K^N\) by permuting the various \(Z_K\) factors. Denoting this action by \(g(\alpha)\), the semi-direct product is defined in the usual way: \((g, \alpha) \cdot (g', \alpha') = (gg', \alpha(g(\alpha'))\). Twisted sectors will be labeled by conjugacy classes in \(G\). The relevant sectors, for the elliptic genus computation, as discussed above, are the conjugacy classes \([g]\) in \(\mathcal{S}_N\) of the form \([g] = (L)^M\) with \(N = LM\). One can easily verify that the various classes in \(G\) are labeled by \([\{g\}, \alpha]\) with \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_M)\) where each \(\alpha_i\) is a representative in \(Z_K^M\). It is easy to see that these representatives \(\alpha_i\)’s are labelled by the diagonal subgroup of \(Z_K^M\). Combining this with the condition we have found for \(h\) we may conclude that all the representatives \(\alpha_i\)’s must be equal (i.e. all \(\alpha_i\)’s must have the same eigenvalue under the diagonal subgroup of \(Z_K^M\)) in order for such an \(h\) to exist in the centralizer of \([\{g\}, \alpha]\) in \(G\). Different sectors are then characterized by a representative twist-element in the \(\sigma\)-direction

\[
g = (g; \alpha) = ((L)^M; \alpha_1 \cdot \alpha_2 \cdots \alpha_M) \tag{2.4}\]

with \(\alpha_1 = \alpha_2 = \ldots = \alpha_M = (e^{2\pi t_0 g}, 1, 1, \ldots, 1)\) and \(t_0 = 0, 1, \ldots, K - 1\) labelling the element of the diagonal subgroup of \(Z_K^M\). A sector twisted by a group element \((2.4)\) should be projected by the centralizer

\[
h = (h; \beta) = (Z_M \ltimes Z_L^M; \beta_1 \cdot \beta_2 \cdots \beta_M) \tag{2.5}\]

where \(\beta \in Z_K^N\) satisfies \(ah(\alpha) = \beta g(\beta)\). The number of independent \(\beta\)’s of this kind is \(K^M\) and therefore the order of the centralizer of \([\{g\}, \alpha]\) is \(M!L^M K^M\). The number of elements \(h\) in \((2.5)\) that give rise to a non-vanishing trace is \((M - 1)!L^M K^M\), and therefore these are the relevant elements for the computation of the elliptic genus.

However, not all the \(h\)’s of this form give different traces. Indeed, if \(h\) and \(h'\) are in the same conjugacy class in \(C_g\), they will give the same trace. We can choose again
a representative element \( h \). The orbifold group representatives (2.4) and (2.5) can be diagonalized with eigenvalues being given by:

\[
\begin{align*}
g &= e^{2\pi i (\frac{\ell_{\phi} t_g}{KL} + \frac{l}{L})} \\
h &= e^{2\pi i (\frac{\ell_{\phi} t_h}{KN} + \frac{s}{N} + \frac{r}{M})}
\end{align*}
\]  

(2.6)

with \((l, r) \ (l = 0, \cdots L - 1, \ r = 0, \cdots M - 1)\) denoting the \( N = M \cdot L \) copies of a generic field \( \phi \). Different orbifold sectors are denoted by \((s, t_g, t_h)\), where \( s = 0, \ldots, L - 1 \) labels the \( Z_L \) elements in \( h \) and \( t_g, t_h = 0, \ldots, K - 1 \) label the \( Z_K \) elements in the \( \sigma \) and \( \tau \) directions respectively. It is easy to verify that the number of elements in the centralizer \( \hat{C}_h \) in \( C_g \), for a relevant \( h \), is \( KML = KN \). As a result, the number of elements in the conjugacy class of such a \( h \) in \( C_g \) is \( |C_g|/|\hat{C}_h| = (M - 1)! L^{M-1} K^{M-1} \). Each of these classes appear with a prefactor, given by the number of elements divided by the order of \( C_g \), that is equal to \( 1/(KN) \).

We are now ready to compute the elliptic genus of the symmetric product of \( N \) copies of a worldsheet theory. We start with the elliptic genus of a single copy of the theory:

\[
\chi^{osc} \equiv \prod_\phi \chi_{\alpha_{\phi} \beta_{\phi}}[\alpha_{\phi} \beta_{\phi}]
\]

(2.7)

where

\[
\chi_{\alpha_{\phi} \beta_{\phi}} = \prod_{n=1}^{\infty} (1 + e^{2\pi i \beta_{\phi} q^{n-1/2} - \alpha_{\phi}}) \epsilon_{\phi}(2.8)
\]

is the oscillator-mode contributions of a generic, say right-moving, \( \epsilon_{\phi} = -1(1) \) bosonic (fermionic) field \( \phi \) with boundary conditions given by \( \alpha_{\phi}, \beta_{\phi} \) along \( \sigma \) and \( \tau \) directions respectively. We will omit in the following the zero mode contributions which are included at the end of the computation.

We can write the contribution to the elliptic genus of a given \( g, h \) sector with eigenvalues given by (2.6) as

\[
\chi^{s, t_g, t_h}_{L, M} \left[ \alpha_{\phi} \beta_{\phi} \right] (q) = \prod_{r=0}^{M-1} \prod_{l=0}^{L-1} \prod_{n=0}^{\infty} (1 + e^{2\pi i (l, r) q^{n-1/2} - \alpha_{\phi}(l)})^{\epsilon_{\phi}}
\]

(2.9)

with

\[
\begin{align*}
\alpha_{\phi}(l) &= \alpha_{\phi} + \frac{t_g t_h}{KL} + \frac{l}{L} \\
\beta_{\phi}(l, r) &= \beta_{\phi} + \frac{r}{M} + \frac{t_g t_h}{KM} + \frac{s}{N} \left( \frac{t_g}{K} + l \right).
\end{align*}
\]

Performing the products over \( r \) and \( l \) yields

\[
\chi^{s, t_g, t_h}_{L, M} \left[ \alpha_{\phi} \beta_{\phi} \right] (q) = \prod_{m=1}^{\infty} (1 + e^{2\pi i \beta_{\phi}(m, \tau) q^{m} e^{-2\pi i \frac{r}{M}}} q^{n-1/2} - \alpha_{\phi})^{\epsilon_{\phi}}
\]

(2.10)
in terms of the modified boundary conditions

\[
\alpha = L(\alpha + \frac{1}{2}) + t_q \frac{t_q}{K} + \frac{1}{2} \\
\beta = M(\beta + \frac{1}{2}) - s(\alpha + \frac{1}{2}) + t_h \frac{t_h}{K} + \frac{1}{2}.
\]  

(2.11)

Finally the zero-mode contributions to the elliptic genus depend on the bosonic or fermionic nature of the field under consideration. For fermions one finds:

\[
(2N-1 - 2^{N-1}) \left[ \prod_{j=1}^{M-1} (1 - e^{2\pi i j/M}) \right]^{2N} = (2N-1 - 2^{N-1})M^{N}.
\]  

(2.12)

For bosonic fields there is a further distinction depending on the compactness of the bosonic coordinate. For \( d \) compact bosons one has (in units of \( \alpha' = 1/2 \))

\[
\sum_{(p,\bar{p}) \in \Gamma_{d,d}} (q^M e^{-\frac{2\pi i}{2}})^{p^2/2} (\bar{q}^M e^{\frac{2\pi i}{2}})^{\bar{p}^2/2}.
\]  

(2.13)

where \( \Gamma_{d,d} \) is the even self-dual Lorentzian lattice of generalized momenta, combining K-K momenta and winding modes. For \( D \) non-compact bosons one gets

\[
L^D \tau_{-D/2} = L^D \int d^D p (q\bar{q})^{p^2/2}
\]  

(2.14)

We can now be more precise for some explicit choices of the initial worldsheet content.

The first example we will consider is defined by the \( Z_2 \)-orbifold of the worldvolume theory for \( N \) copies of the heterotic Green-Schwarz string \( (\phi = X^I, S^a, \chi^A) \) with \( \alpha_\phi = \beta_\phi = 1/2 \), where \( Z_2 \) is simply the GSO projection \( \chi^A \to -\chi^A \). Specializing (2.10-2.14) to the field content under consideration yields

\[
\chi_{-N}^I(q) = \frac{8 - 8}{\tau_2} \sum_{L,M} M^{-4} \frac{1}{2N} \sum_{s=0}^{L-1} \sum_{t_q, t_h=0,1} \frac{\vartheta \left[ \frac{t_q}{4} \right]}{\eta^4 (q^M e^{-2\pi i/2})^{16}}
\]  

(2.15)

where an overall factor \( \frac{1}{N^4} \) has been included so that the longest string sector appears with unit normalization in accordance with the fact that on \( (R^8)^N \) there is only one fixed plane under the \( Z_N \) action.

The second example start from a type IIB-like worldsheet theory \( (\phi = X^I, S^a, \dot{S}^\dot{a}) \) with \( X^I, S^a \) periodic \( (\alpha_\phi = \beta_\phi = 1/2) \) and \( \dot{S}^\dot{a} \) antiperiodic \( (\alpha_\phi = 0, \beta_\phi = 1/2) \) fields. In this case, (2.10-2.14) yield

\[
\tilde{\chi}_{-N}^I(q) = \frac{8 - 8}{\tau_2} \sum_{L,M} M^{-4} \frac{1}{N} \sum_{s=0}^{L-1} \frac{\vartheta \left[ \frac{t_q}{4} + \frac{1}{2} \right]}{\eta^{12} (q^M e^{2\pi i/2})^4}
\]  

(2.16)

where the same overall normalization \( \frac{1}{N^4} \) has been included.
3 Type I duals of heterotic CHL models

In [4] the spectrum of D-string bound states for toroidal compactifications of the type I theory was studied. Masses, multiplicities and charges of bound states were read from the elliptic genus of the effective $O(N)$ gauge theory describing $N$ nearby type I D-strings [18, 4]:

\[ S = \text{Tr} \int d^2\sigma \left\{ -\frac{1}{4g^2} F^2 + (DX_I)^2 + \Lambda_a \mathcal{D}_{RA^a}^a + S_a \mathcal{D}_{LA^a}^a + \chi_A \mathcal{D}_{RA^A}^a \right\} + g^2 ([X_I, X_J])^2 + \cdots \]

The fields transform in diverse representations of the gauge group $O(N)$. $X$ and $S$ transform as second rank symmetric, traceless tensors, while $\Lambda$ and $\chi$ transform in the adjoint and fundamental representations respectively. The singlet (trace) parts of $X$ and $S$ represent the collective super-coordinates of the center-of-mass motion and decouple from the rest. There is an $SO(8)$, R-symmetry group, under which $X$, $S$, $\Lambda$ and $\chi$ transform as an $8_V$ (labelled by $I$), an $8_S$ (labelled by $a$), an $8_C$ (labelled by $\hat{a}$) and a singlet, respectively. The $\chi$’s correspond to the Ramond ground states of open strings stretching between D1- and D9-branes [2] and transform under the vector representations of both $SO(32)$ and $O(N)$. As reminded by the subscript of the Dirac operators, $\Lambda$ and $\chi$ are negative chirality (right-moving) worldsheet fermions while $S$ are positive chirality (left-moving) fermions. Finally, $W^I_I$ are $SO(32)$ Wilson lines on the $I^{th}$ transverse direction, $T^\ell$ being the $SO(32)$ generators in the vector representation. The presence of this term in the worldvolume effective action can be deduced by explicitly computing the three-point function on the disk involving the vertex operators for two $\chi$ fields and one bosonic coordinate $X$ in the presence of $SO(32)$ Wilson lines $W^I_I$. In the canonical picture for the external fields ($-\frac{1}{2}$ for the spinors and $-1$ for the bosons):

\[ \langle e^{-\frac{\phi}{2}} S^+ \sigma(0) e^{-\frac{\phi}{2}} S^+ \bar{\sigma}(1) e^{-\varphi} \psi^I(\infty) \int dx (\partial X^J + ip \cdot \psi \psi^J) (x) \rangle \]

$\sigma, \bar{\sigma}$ represent the twist fields for the ND directions of the transverse $X$ coordinates, $S^+$ the two dimensional longitudinal spin field and $\varphi$ the scalar arising from the superghost bosonization. The $SO(32) \times O(N)$ group structure in (3.1) enters through the standard Chan-Paton (C-P) factor. From (3.2) one easily recognizes the last world-sheet coupling in (3.1). In [4], Wilson lines turned on the longitudinal $\sigma$-direction of the D-string worldsheet were considered. In that case the Wilson-line coupling reduces to a quadratic term in the $\chi$ fields.

As conjectured in [18] and supported by the results of [4] the $O(N)$ gauge theory described by (3.1) flows in the infrared to an $(8,0)$ orbifold conformal field theory given by the Green-Schwarz $\sigma$-model for $N$ copies of the heterotic string with target space $(R^8)^N/S_N$. It was shown how the longest-string sector of this orbifold theory reproduces the charges, masses and degeneracies required by the duality relation with the
fundamental heterotic BPS states. In this section we show how a similar analysis can be extended to more general vacuum configurations with sixteen supercharges. In particular we consider the type I dual [5, 7] of a CHL model [6] in \( D = 8 \), but it will be clear from the discussion that the arguments are rather general and apply to many, if not all, unconventional heterotic - type I dual pairs.

Let us consider, for example, a dual pair with gauge group \( SO(16) \) in \( D = 8 \). From the type I perspective, models of this kind were constructed long time ago [8] as open-string descendants of the type IIB string on tori with non-vanishing but quantized NS-NS antisymmetric tensor \( B^{NS} \). The presence of a quantized \( B^{NS} \) in these type I models, more concisely termed \textit{BPS unorientifolds} in [5], has been identified with the obstruction to defining a vector structure in the vacuum gauge bundle or equivalently with the presence of non-commuting Wilson lines on the torus [5, 7]. In \( D = 8 \), the dual heterotic CHL model [6] can be constructed by turning on non-commuting Wilson lines on the two-torus of a conventional \( SO(32) \) heterotic string compactification [9]. Non-commuting Wilson lines effectively modify the boundary conditions of the \( \chi \) fields representing the \( SO(32) \) algebra. We can therefore realize them as non-abelian orbifolds \( T^2/G \). Let us denote the generators of \( G \) as \( g_1, g_2 \). In order to keep an \( SO(16) \) component of the gauge group we decompose the 32 heterotic fermions \( \chi^A \) into two groups\(^2\) of 16 each \( \chi^A_{(1)} \) and \( \chi^A_{(2)} \) and choose

\[
\begin{align*}
g_1 & : X_8 \rightarrow X_8 + \pi R_8; \quad \chi^A_{(1)} \rightarrow +\chi^A_{(1)} \quad \chi^A_{(2)} \rightarrow -\chi^A_{(2)} \\
g_2 & : X_9 \rightarrow X_9 + \pi R_9 \quad \chi^A_{(1)} \rightarrow +\chi^A_{(2)} \quad \chi^A_{(2)} \rightarrow +\chi^A_{(1)} .
\end{align*}
\]

The first action in (3.3) breaks \( SO(32) \) to \( SO(16)^2 \) at level \( k = 1 \). The second projects on the diagonal \( SO(16) \) gauge group at level \( k = 2 \). The half shifts ensure that no new massless gauge bosons arise in the twisted sectors. Up to the original GSO projection on the world-sheet fermions, twisted sectors corresponds to the conjugacy classes \([1], [g_1], [g_2], [g_1 g_2] \) of \( G \). In each of these sectors we project by the centralizer \( C_g \), that only in the untwisted sector \(([1]) \) coincides with the full \( G \), because \( g_1 \) and \( g_2 \) do not commute.

Following [5] we can write the full CHL partition function as a sum of the contributions from the four sectors \([1], [g_1], [g_2], [g_1 g_2] \) that reads

\[
\begin{align*}
Z_{(1,1)} &= \frac{1}{8 \tau_2^2 \eta^8(q)} \frac{\vartheta_3^16 + \vartheta_4^16 + \vartheta_2^16}{\eta^{24}}(q) \sum_{P \in \Gamma_{2,2}} q^{p^2/2} \bar{q}^{p^2/2} \\
Z_{(1,g_1)} &= \frac{1}{4 \tau_2^2 \eta^8(q)} \frac{\vartheta_3^8 \vartheta_4^8}{\eta^{24}}(q) \sum_{P \in \Gamma_{2,2}} e^{i\pi P \cdot V_8} q^{p^2/2} \bar{q}^{p^2/2} \\
Z_{(1,g_2)} &= \frac{1}{4 \tau_2^2 \eta^8(q)} \frac{\vartheta_3^8 \vartheta_4^8}{\eta^{24}}(q) \sum_{P \in \Gamma_{2,2}} e^{i\pi P \cdot V_8} q^{p^2/2} \bar{q}^{p^2/2}
\end{align*}
\]

\(^2\)For lack of a better symbol, we continue to label the fermions in the two subsets with \( A = 1, \ldots, 16 \). We hope this would not cause confusion with the initial range of the same label \( A = 1, \ldots, 32 \).
We choose the order two “geometric” shifts generated by the level matching condition (3.14) simply reduces to the winding and momentum modes of the fundamental strings and the D-string windings around the eighth and nine directions. The former are encoded in the one-loop partition function that involves the Klein-bottle $K$, annulus $A$ and Möbius-strip $M$ amplitudes in addition to half the parent type IIB torus amplitude. For the $SO(16)$ type I model in $D = 8$ [8, 5] under consideration the various

$$Z_{(1,g_1g_2)} = \frac{1}{4\tau_2^4} \frac{Q}{\eta^8(q)} \frac{\vartheta_3^8(q^2) + \vartheta_4^8(q^2)}{\eta^8(q)\eta^8(q^2)} \sum_{P \in \Gamma_2,2} e^{i\pi P \cdot V_{g_9} q^2/2} q^2/2$$

(3.7)

$$Z_{(g_1,1)} = \frac{1}{4\tau_2^4} \frac{Q}{\eta^8(q)} \frac{\vartheta_3^8(q^2)}{\eta^8(q^2)} \sum_{P \in \Gamma_2,2} q^2/2 q^2/2$$

(3.8)

$$Z_{(g_1,g_1)} = \frac{1}{4\tau_2^4} \frac{Q}{\eta^8(q)} \frac{\vartheta_3^8(q^2)}{\eta^8(q^2)} \sum_{P \in \Gamma_2,2} e^{i\pi P \cdot V_{9} q^2/2} q^2/2$$

(3.9)

$$Z_{(g_2,1)} = \frac{1}{4\tau_2^4} \frac{Q}{\eta^8(q)} \frac{\vartheta_3^8(q^2)}{\eta^8(q^2)} \sum_{P \in \Gamma_2,2} q^2/2 q^2/2$$

(3.10)

$$Z_{(g_2,g_2)} = \frac{1}{4\tau_2^4} \frac{Q}{\eta^8(q)} \frac{\vartheta_3^8(q^2)}{\eta^8(q^2)} \sum_{P \in \Gamma_2,2} e^{i\pi P \cdot V_{9} q^2/2} q^2/2$$

(3.11)

$$Z_{(g_1g_2,g_1g_2)} = \frac{1}{4\tau_2^4} \frac{Q}{\eta^8(q)} \frac{\vartheta_3^8(q^2)}{\eta^8(q^2)} \sum_{P \in \Gamma_2,2} e^{i\pi P \cdot V_{9} q^2/2} q^2/2$$

(3.13)

where $Q = (\vartheta_3^4 - \vartheta_4^4 - \vartheta_2^4)/\eta^4$ is the ubiquitous “supersymmetric character” that vanishes in virtue of Jacobi’s $\textit{aequatio identica satis absurda}$ and $\Gamma_{2,2}$ is the even self-dual Lorentzian lattice of generalized momenta. As usual, the first subscript in the above amplitudes refers to the twist in the $\sigma$ direction and the second to the twist in the $\tau$ direction. We choose the order two “geometric” shifts generated by $V_8, V_9, V_{89} = V_8 + V_9$ with $V_i = (v_i, \bar{v}_i)$ such that $v_8 = -\bar{v}_8 = R_8/2$ and $v_9 = -\bar{v}_9 = R_9/2$. The $1/2$ BPS spectrum is defined by replacing $Q/\eta^8$ with its ground states $8_B - 8_F$ and imposing the level matching condition

$$\frac{1}{2} p^2 - \frac{1}{2} p^2 = m_{s8} n_8 + m_{g9} n_9 = N_L - c_L$$

(3.14)

where $m_i$ and $n_i$ (integers or half integers depending on the orbifold sector) represent the winding and momentum modes of the fundamental strings and $c_L$ the zero point energies in the corresponding sector. In particular for a string wrapping $N$ times around a single direction the level matching condition (3.14) simply reduces to $nN = (N_L - c_L)$ and the partition function for such states can be read off from the $N = 1$ partition function after replacing $q$ with $q^{1/N}$.

We can now compare the above BPS spectrum with the BPS spectrum of the conjectured dual type I model [5, 7]. The spectrum of $1/2$ BPS states in this unconventional type I toroidal compactification consists in the Kaluza-Klein (K-K) excitations of the massless type I string states and the D-string windings around the eighth and nine directions. The former are encoded in the one-loop partition function that involves the Klein-bottle $K$, annulus $A$ and Möbius-strip $M$ amplitudes in addition to half the parent type IIB torus amplitude. For the $SO(16)$ type I model in $D = 8$ [8, 5] under consideration the various
amplitudes read

\[ \mathcal{T} = \frac{1}{2} \frac{\mathcal{V}_8}{4\pi^2\alpha'} \int_F \frac{d^2 \tau}{\tau_2^2} \frac{1}{\tau_2^3} \left| Q \right|^2_{16} (\tau, \bar{\tau}) \sum_{p \in \Gamma_{2,2}} q^p q^\beta^{p/2}/q^{\delta^2/2} \]  

(3.15)

\[ \mathcal{K} = \frac{1}{2} \frac{\mathcal{V}_8}{4\pi^2\alpha'} \int_0^\infty \frac{dt}{t} \frac{1}{t^4} \bar{\eta}^8 (2it) \sum_{p \in \Gamma_{KK}} e^{-\pi tp^2} \]  

(3.16)

\[ \mathcal{A} = \frac{N^2 \mathcal{V}_8}{2} \int_0^\infty \frac{dt}{t} \frac{1}{t^4} \eta^8 \frac{i t}{2} \sum_{p \in \Gamma_{KK}+\epsilon} e^{-\pi tp^2} \]  

(3.17)

\[ \mathcal{M} = \frac{N^2 \mathcal{V}_8}{2} \int_0^\infty \frac{dt}{t} \frac{1}{t^4} \eta^8 \frac{i t}{2} + \frac{1}{2} \sum_{\epsilon} \gammale e^{-\pi tp^2} \]  

(3.18)

where \( \mathcal{V}_8 \) is a regularizing volume and \( \Gamma_{2,2} \) is a left-right symmetric even self-dual Lorentzian lattice of generalized momenta, combining K-K momenta and winding modes. The condition of left-right symmetry results in a quantization condition for the NS-NS antisymmetric tensor \( B_{16}^{NS} \) [8]. The corresponding restriction on the closed-string states flowing in the transverse channel halves the rank of the C-P group and induces the presence of order-two shifts \( \epsilon \) in the two dimensional lattice of K-K momenta \( \Gamma_{KK} \). The \( \mathbb{Z}_2 \) phases, \( \epsilon \), and the C-P multiplicities \( N \) are constrained by R-R charge neutrality. Indeed, tadpole cancellation implies \( \sum \gammale = -2 \) and \( N = 16 \). The four possible choices for \( \gammale \) correspond to three inequivalent \( SO(16) \) and one \( Sp(16) \) type I models [7]. The spectrum of perturbative BPS states can be read off after putting the supersymmetric part in one of its ground states. Up to the resulting overall multiplicity \( 8_B - 8_F \), one finds \( 8 \) \( SO(16) \) singlets from the K-K reduction of the supergravity sector (3.15,3.16), three \( 120 \) antisymmetric tensors from the sectors with K-K momenta shifted by \( \epsilon = (0,0), (0, \frac{1}{2}), (\frac{1}{2}, 0) \) with \( \gammale = -1 \) and finally one symmetric tensor \( (135+1) \) from the sector with K-K momenta shifted by \( \epsilon = (\frac{1}{2}, \frac{1}{2}) \) with \( \gammale = +1 \).

We can now compare the type I perturbative spectrum with its heterotic CHL counterpart defined by the fundamental strings with no-winding modes \( m_8 = m_9 = 0 \). Recalling that the \( V \)'s in (3.8-3.13) represent shifts in the winding heterotic modes, states with \( m_8 = m_9 = 0 \) only arise in the untwisted sector (3.4-3.7). Imposing the level matching condition (3.14) \( (q^0 \text{ order in the expansions of (3.4-3.7)}) \) gives rise to \( 8+120 \) supermultiplets for \( n_8, n_9 \) both even, \( 120 \) supermultiplets for one \( n \) even and the other odd, and \( 136 \) supermultiplets for \( n_8, n_9 \) both odd. The spectrum of heterotic BPS states with zero winding coincides with the previously found type I perturbative spectrum after the identifications \( R_8^H = 2R_8^I \) and \( R_9^H = 2R_9^I \).

The winding modes of the fundamental strings, on the other hand, are mapped to bound states of D-strings in the \( SO(16) \) type I model under consideration. For example the basic \( N = 1 \) unit (after the rescaling of the internal radius) of D-string winding in the \( 8^{th} \) direction will be represented in the dual theory by the fundamental heterotic strings in the sectors twisted by an element in \( [g_1] \) or in \( [g_1 g_2] \). In order to identify the relevant D-string gauge theory in the dual type I model we should combine the \( \Omega \) projection.
with the $g_1$, $g_2$ actions. For example for a single winding around the 8th direction the corresponding world-volume theory is defined in terms of the $\Omega g_1$-invariant fields

$$X^I, S^i, \chi^{A}_{(1)} \quad \text{for} \quad P_{\sigma} \text{ even}$$

$$\chi^{A}_{(2)} \quad \text{for} \quad P_{\sigma} \text{ odd}$$

(3.19)

After defining a new longitudinal variable $\bar{\sigma} = 2\sigma$, we are left with a CFT in terms of the $\bar{\sigma}$-periodic $X, S, \chi^{A}_{(1)}$ fields and the $\bar{\sigma}$-antiperiodic $\chi^{A}_{(2)}$ fields. Similarly for the $g_1 g_2$ $Z_4$-action we have a two dimensional free theory in terms of periodic $X, S$ fields and $\chi$ fields satisfying the boundary conditions $\chi^{A}_{\pm}(\bar{\sigma} + 1) = \pm i \chi^{A}_{\pm}(\bar{\sigma})$. For $N$ D-strings we should deal with the same $O(N)$ effective action (3.1) as in the standard type I theory now with the $\chi$ fields satisfying these new boundary conditions. The arguments of [4] directly apply to this D-string to show that the infrared limit of the gauge theory is governed by an orbifold conformal theory, defined by $N$ copies of the $N = 1$ D-strings above moving on the symmetric space $(\mathcal{M})^N/S_N$. The manifold $\mathcal{M}$ is defined by the $Z_2$ orbifold $R^7 \times (S^1)/g_2$ with $g_2$ defined as before. The symmetrization of this orbifold should be understood as follows: since the actions of $g_1$ and $g_2$ do not commute, a sensible $g_2$ action can be defined only in a sector with $N$ even. More precisely the symmetric space can be understood as the symmetrization of $N$ copies of the conformal theory defined by the $N = 1, 2$ sectors.

We can now apply formula (2.15) and check that the longest string sector $(M = 1, L = N)$ precisely yields the multiplicities of the D-string bound states wrapped $N$ times around the 8th direction. We start with the amplitudes that are $g_2$-twisted in neither two world-sheet directions. For the CFT defined by $N$ copies of the $[g_1]$-twisted heterotic strings ($\alpha_\phi = 0, \beta_\phi = \frac{1}{2}$ for $\phi = \chi^{A}_{(1)}$, $\alpha_\phi = \beta_\phi = \frac{1}{2}$ otherwise, $Z_K \equiv Z_2^{GSO}$) we find

$$\chi_{K}^L(q) = \frac{8 - 8}{r^2} \frac{1}{2N} \sum_{s=0}^{N-1} \sum_{t_p, t_h=0,1} \frac{1}{\eta^{24}(q)} \vartheta \left[ t_p + \frac{N}{2} \right] \vartheta \left[ t_h + \frac{N}{2} \right] (q)$$

(3.20)

with $q \equiv q^\frac{1}{2} e^{-2\pi t_{\frac{N}{2}}}$. Starting with $N$ copies of a heterotic string in the $[g_1 g_2]$ sector ($\alpha_\phi = \frac{1}{4}(\frac{3}{2}), \beta_\phi = \frac{1}{2}$ for $\phi = \chi^{A}_{(1)}$, $\alpha_\phi = \beta_\phi = \frac{1}{2}$ otherwise) yields

$$\chi_{N}^L(q) = \frac{8 - 8}{r^2} \frac{1}{2N} \sum_{s=0}^{N-1} \sum_{t_p, t_h=0,1} \frac{1}{\eta^{24}(q)} \left[ t_p + \frac{3N}{2} \right] \vartheta \left[ t_p + \frac{N}{2} \right] \vartheta \left[ t_h + \frac{N}{2} \right] (q)$$

(3.21)

For different values of $N$, (3.20) yields, in addition to the $(8s - 8_{F})$ factor associated to the center-of-mass fermionic zero-modes, degeneracies that are determined by the expansion of

$$s \text{ odd, } N \text{ odd} \quad \frac{1}{\eta^{24}(q)} \vartheta \left[ q^q \right] \vartheta \left[ q^{q_2} \right] \sum_{p \in \Gamma_{1,1}} q^{p^2/2} \vartheta \left[ q^{2p^2/2q} \right]$$

(3.22)

$$s \text{ even, } N \text{ odd} \quad \frac{1}{\eta^{24}(q)} \vartheta \left[ q^q \right] \vartheta \left[ q^{q_2} \right] \sum_{m_p \in Z} q^{p^2/2q} \vartheta \left[ q^{2p^2/2q} \right]$$

(3.23)
The sum over $s$ projects onto states which satisfy the heterotic level matching condition (3.14). Indeed for odd $N$ the above formula precisely matches with the degeneracies of the $g_1$-twisted heterotic sectors (3.8,3.9) while for $N$ even it matches with the multiplicities coming from the untwisted sectors (3.4, 3.5).

Similarly, starting with the partition function for $N$ copies of the $[g_1g_2]$-twisted heterotic string given in (3.21) one can easily show that they yield the degeneracies coming from (3.12) and its T-modular transform for $N$ odd and $s$ even and odd respectively; and (3.7,3.4) for $N$ even and $s$ odd and even respectively.

Finally for $N$ even we are allowed to perform the $g_2$-projection. This will reproduce the multiplicities in (3.6,3.10,3.11) in an obvious way since, as we have just discussed, $g_2$-untwisted sectors in both $\sigma$ and $\tau$ directions are given by the untwisted sector of the heterotic string.

4 Type IIB orbifold/orientifold dual pairs

The other classes of models we are going to discuss in this section correspond to $\mathcal{N} = (4,0)$ and $\mathcal{N} = (2,0)$ supersymmetric\footnote{The even more confusing notation $\mathcal{N} = (N_L, N_R)$ corresponds to the splitting $\mathcal{N} = N_L + N_R$, where $N_L$, respectively $N_R$, counts, in four-dimensional units, the “target-space” supersymmetries associated to left-moving, respectively right-moving, spin fields.} vacuum configurations of type IIB string [19]. These type II models are dual to type I models without open strings [5].

4.1 Type IIB on $T^d/(-)^{F_L}\sigma_V$ vs. Type IIB on $T^d/\Omega \sigma_V$

We will first consider $(4,0)$ models, which are obtained as asymmetric orbifolds [17] of $d$-dimensional tori $T^d$, where the orbifold group is generated by $(-)^{F_L} \times \sigma_V$. $F_L$ is the spacetime left-moving fermion number while $\sigma_V$ is a shift of order two in the $\Gamma_{d,d}$ lattice of generalized momenta of the torus [5, 19]. As a result of the orbifold projection, the left-moving supercharges are projected out in the untwisted sector. Due to the shift in the momentum lattice, no supercharge appears in the twisted sector. As remarked in the introduction, the orbifold projection removes all R-R states. Thus type II $(4,0)$ models are effectively type II models without D-branes in that, unlike the standard ones [13], D-branes in type II $(4,0)$ models do not give rise to BPS saturated states since there is no R-R counterpart of the NS-NS coupling to the graviton and dilaton. This line of reasoning leads one to conclude that, neglecting the states associated to wrapping
the NS 5-brane that only play a role in low enough dimensions \((D \leq 4)\), the spectrum of BPS states is completely perturbative much in the same way as in heterotic models. The BPS perturbative spectrum for the type II \((4,0)\) models can be read off from the elliptic genus
\[
TR(-)F_R e^{-2\pi \tau_2 H + 2\pi i R \tau_1 P_s},
\]
where the trace is computed in the right-moving Ramond sector: indeed in this case we are effectively setting the left-moving oscillator number \(N_L\) to its ground state value \(N_L = 0\). We are then left with the following non-trivial contributions from the different orbifold sectors \([5]\):

\[
Z_{++} = \frac{(8_v - 8_s)}{\tau_2^{4-d/2}} \frac{\vartheta_2(q)^4}{\eta(q)^8} \sum_{P \in \Gamma_{d,d}} e^{2\pi i V \cdot P} q^{p_2^2/2} \bar{q}^{\bar{p}_2^2/2}
\]

\[
Z_{+-} = \frac{8_v - 8_s}{\tau_2^{4-d/2}} \frac{\vartheta_4(q)^4}{\eta(q)^8} \sum_{P \in \Gamma_{d,d}+V} q^{p_2^2/2} \bar{q}^{\bar{p}_2^2/2}
\]

\[
Z_{--} = -e^{\pi i V \cdot V} \frac{(8_v - 8_s)}{\tau_2^{4-d/2}} \frac{\vartheta_3(q)^4}{\eta(q)^8} \sum_{P \in \Gamma_{d,d}+V} e^{2\pi i V \cdot P} q^{p_2^2/2} \bar{q}^{\bar{p}_2^2/2}.
\]

In the above expressions the first subscript \(\pm\) refers to the twists in the \(\sigma\) direction whereas the second to the twist in the \(\tau\) direction. The factor \(8_v - 8_s\) comes from the right-moving fermionic zero-modes and counts the 8 bosons and 8 fermions of the short \(N = 4\) supermultiplets, \(V\) is the shift vector generating the action of \(\sigma_V\). Finally \(P = (p, \bar{p})\) are the generalized momenta in the compact directions.

The only gauge fields in these models are those coming from the K-K reduction of the ten dimensional metric \(G_{MN}\) and NS-NS antisymmetric tensor \(B_{MN}^{NS}\), since, as discussed above, there are no gauge fields arising from the R-R sector. The corresponding charges belong to the (shifted) lattices appearing in eqs.(4.2) to (4.4).

For simplicity we will restrict the following discussion to the 9-dimensional case of compactification on a circle of radius \(R\). In this case the charges associated to the 9-dimensional gauge fields \(B_{\mu\theta}^{NS}\) and \(G_{\mu\theta}\) are the winding number \(N\) and K-K momentum \(k/R\), respectively. We will also take the “geometric” shift \(\sigma_V\) to be generated by the vector \(V = (v, \bar{v}) = (R/2, -R/2)\). The level matching condition then reduces to

\[
kN = N_L - c_L,
\]

where \(k\) is integer (half-integer) in the untwisted (twisted) sector and \(c_L\) denotes, as before, the zero point energy in the left moving sector. The multiplicity of these BPS states for given \(k\) and \(N\) is defined by the coefficient of \(q^{N_L}\) in the expansion of (4.2-4.4), with \(N_L\) given by (4.5).

The conjectured S-duality of the ten-dimensional type IIB string exchanges the NS-NS and R-R antisymmetric tensors and maps winding modes of the fundamental string into
D-string winding modes. We then conclude that in the system of \( N \) type I D-strings, each one wrapped once around the circle, there should exist bound states carrying a given K-K momentum \( k \), with mass and degeneracy given by the above relations through the duality map. As discussed in [4], in the context of type I - heterotic duality, these data about the bound states are encoded in the elliptic genus of the effective gauge theory describing the dynamics of \( N \) nearby type I D-strings.

In order to identify the effective gauge theory governing the dynamics of type I D-strings in the present situation, notice that, as shown in [14], S-duality maps \((-)^{F_L}\) into the world-sheet parity operator \( \Omega \). In \( D = 10 \) the two quotient theories are vastly different. On the one hand, projecting the type IIB theory by \((-)^{F_L}\) gives the type IIA theory since the twisted sector of the orbifold provides the extra (opposite-chirality) supercharges needed to restore (non-chiral) maximal supersymmetry. On the other hand, projecting the type IIB theory by \( \Omega \) gives the type I theory since the twisted sector of the parameter-space orbifold [20, 21, 22], now commonly termed orientifold, provides the \( SO(32) \) C-P multiplicities in terms of open-string excitations of the D9-branes needed to soak up the non-vanishing R-R charge of the O9-planes associated to the Klein-bottle \( \Omega \)-projection [13].

Nevertheless, accompanying the \( \mathbb{Z}_2 \)-projection by a shift in the compactification torus results in dual pairs in lower dimensions [14, 5, 19]. Closely following the analyses in [4] and in section 2, the relevant gauge theory will be obtained by projecting the \( U(N) \) gauge theory on the world-sheet of type IIB D-strings [3] onto \( \Omega\sigma_V \) invariant fields. The K-K momentum of the fundamental string corresponds to \( P_\sigma \) in the D-string system, and therefore, \( \sigma_V \), which is +1 or −1 depending on whether the K-K momentum is even or odd, corresponds to (anti-)periodic boundary conditions along the \( \sigma \)-direction on the D-string world-sheet. Recalling that in the action of \( \Omega \) on the \( U(N) \) C-P factors there is a relative sign between the transverse and longitudinal degrees of freedom, it is easy to obtain the field content resulting from the \( \Omega\sigma_V \) projection. Let us denote by \( A_\alpha, X^I, S^a \) and \( S^{\dot{a}} \) the gauge fields, transverse scalars, left- and right-moving fermions respectively, of the \( U(N) \) type IIB D-string system. The indices \( I, a, \dot{a} \) refer to the \( 8_v, 8_s \) and \( 8_c \) representations of the \( SO(8) \) R-symmetry group. The result of the \( \Omega\sigma_V \) projection is

\[
\begin{align*}
X^{(\pm)}_I, S^{(\pm)}_a, S^{(\mp)}_{\dot{a}}, A^{(-)}_\alpha & \quad \text{for } P_\sigma \text{ even} \\
X^{(-)}_I, S^{(-)}_a, S^{(+)}_{\dot{a}}, A^{(+)}_\alpha & \quad \text{for } P_\sigma \text{ odd}
\end{align*}
\]  

where \((\pm)\) and \((-)\) denote the symmetric and adjoint representations of \( O(N) \) respectively.

Given these preliminary observations, one can follow the approach of [4] where the elliptic genus of the gauge theory was computed by going to the infrared limit. In this limit the theory has been shown to flow to a superconformal orbifold field theory. Indeed, the charged fields get massive and can be integrated out leaving a free field theory in terms of the diagonal fields \( X^I_t, S^a_t \) and \( S^{\dot{a}}_t \), \((t = 1, \ldots, N)\) in (4.6). Finally, modding out by
the residual Weyl symmetry group, we are left with an orbifold theory with target space \((\mathbb{R}^8)^N/\mathcal{S}_N\).

Let us first analyze the free \(N = 1\) case. It is convenient to define a new variable, \(\tilde{\sigma} = 2\sigma\), in terms of which the field content (4.6) reduces to 8 \(\tilde{\sigma}\)-periodic bosons \(X_I\) and chiral fermions \(S_a\) and 8 \(\tilde{\sigma}\)-antiperiodic fermions \(\tilde{S}_a\) with all possible values of \(P_{\tilde{\sigma}}\) momenta.

The partition function (or elliptic genus, since we are in the odd spin structure for the \(S_a\) fields) is given by

\[
\frac{8 - 8 \vartheta_4^4(q)}{\tau_2^{4-d/2} \eta^{12}(q)} \sum_{\mathbf{P} \in \Gamma_{d,d}} \tilde{q}^{P^2/2} \tilde{\bar{q}}^{\bar{P}^2/2}
\]

(4.7)

where right-moving oscillators cancel out between the \(X_I\) and \(S_a\) supersymmetric fields. As before, the factor \((8 - 8)\) takes into account the ground-state multiplicities of the short BPS supermultiplets and \(\tilde{q} \equiv e^{2\pi i\tau}\). The partition function (4.7) reproduces the correct masses, charges and degeneracies coming from (4.3) once the level matching condition (4.5) for \(N = 1\) is implemented, by adding the \(\tau \rightarrow \tau + 1\) amplitude (4.4), and the radius of compactification of the dual theory \(\tilde{R}\) is identified with twice the radius \(R\) of the original one.

We now proceed to study the \(N > 1\) case which, as previously stated, corresponds to \(N\) copies of the \(N = 1\) field content modded out by the permutation group \(\mathcal{S}_N\). In this case we can use the results of section 2 for the elliptic genus of symmetric products. In [4] it was argued that only the longest string sector \([g] = (N)\) represents a truly one-particle bound state of \(N\) D-strings, while \((L)^M\) can be interpreted as an \(M\)-string state, each of which represents a threshold bound state of \(L\) strings. The contribution of the relevant sector \((M = 1, L = N)\) can be read off from (2.16), with \(\chi(q) \tilde{\chi}(\tilde{q})\) given by (4.7), to be

\[
\frac{8 - 8 \vartheta_4^4(q)}{\tau_2^{4-d/2} \eta^{12}(q)} \sum_{s=0}^{N-1} \vartheta_4^4(q^{1/2} \omega^s) \sum_{\mathbf{P} \in \Gamma_{d,d}} \omega^s \tilde{q}^{P^2/2} \tilde{\bar{q}}^{\bar{P}^2/2}
\]

(4.8)

with \(\omega = e^{-2\pi i/N}\) The sum over \(s\) projects on the modes that satisfy

\[
k = \frac{1}{N} \left(\frac{P^2}{2} + N_R\right) \in \mathbb{Z}
\]

(4.9)

which is just the level matching condition (4.5). The degeneracies can be found upon expanding (4.8) in powers of \(q^{1/2}\). In particular for \(N\) odd they come from the expansion of \(\vartheta_4(q^{1/2})\) reproducing the degeneracies arising from the sum of (4.3) and (4.4) once we apply the level matching condition (4.9). For \(N\) and \(s\) both even, additional fermionic zero-modes that we should omit in the sum (2.16) appear for the \(S_a\) field with \(\alpha_\phi = \frac{1}{2}\) and \(\beta_\phi = t_\phi = t_h = 0\) in (2.11). For the remaining values of \(s\) we find \(\vartheta_2(q^{1/2} \omega^s)\) \((\alpha = \frac{1}{2}, \beta = 0)\), which reproduces the degeneracies, masses and charges coming from (4.2).
4.2 Type IIB on \((K3 \times S^1)/(-)^{F_L}\sigma_V\)

Finally, we apply our previous analysis to a model with a lesser number of supersymmetries. We take as an example the type II \((2,0)\) model arising from the orbifolding of \((K3 \times S^1)\) by \((-)^{F_L}\sigma_V\), where \(\sigma_V\) is a shift of order two on the circle \(S^1\). As before left-moving supercharges are projected out by the orbifold group, while the \(\sigma_V\) shift ensures that no new supersymmetries appears from the twisted sector. In the effective five-dimensional theory only two gauge fields are left out by the projection: the K-K reduction of the metric \(G_{\mu \nu}\) and NS-NS antisymmetric tensor \(B_{\mu \nu}^{NS}\).

For simplicity, we will discuss in detail the orbifold limit \(T^4/I_4\) of \(K3\) with \(I_4\) the reflection of the four torus coordinates. The elliptic genus encoding masses, multiplicities and charges for the corresponding BPS perturbative spectrum is then given by

\[
\begin{align*}
Z_{JI} &= -\frac{(4 - 4) V}{\tau_2^{4-d/2}} \sum_{P \in \Gamma_{1,1} + V} q^{1-q_2^2+q_4^2/2} \\
Z_{KI} &= -\frac{(4 - 4) H}{\tau_2^{4-d/2}} \sum_{P \in \Gamma_{1,1} + V} q^{1-q_2^2+q_4^2/2} \\
Z_{KJ} &= -\frac{(4 - 4) H}{\tau_2^{4-d/2}} \sum_{P \in \Gamma_{1,1} + V} e^{2\pi i V \cdot P} q^{1-q_2^2+q_4^2/2} \\
Z_{IK} &= -\frac{(4 - 4) H}{\tau_2^{4-d/2}} \sum_{P \in \Gamma_{1,1} + V} e^{2\pi i V \cdot P} q^{1-q_2^2+q_4^2/2} \\
Z_{IJ} &= -\frac{(4 - 4) H}{\tau_2^{4-d/2}} \sum_{P \in \Gamma_{1,1} + V} e^{2\pi i V \cdot P} q^{1-q_2^2+q_4^2/2}
\end{align*}
\]

The subscripts refer as usual to the different twists in the \(\sigma\) and \(\tau\) direction respectively, \(J \equiv (-)^{F_L}\sigma_V\), \(I \equiv I_4\) and \(K = I \cdot J\). \(\Gamma_{1,1}\) denotes the even self-dual Lorentzian lattice of generalized momenta and \(V = (v, \bar{v}) = (R/2, -R/2)\). The sectors \(Z_{JI}\) and \(Z_{KJ}\) are the T-modular transforms \((\tau \rightarrow \tau + 1)\) of \(Z_{IK}\) and \(Z_{KI}\) respectively, ensuring the level matching condition. The \((4 - 4)\)'s factors with subscripts \(V\) and \(H\) are associated to the 4 bosons and fermions of the vector and hyper five-dimensional short \(\mathcal{N} = 4\) supermultiplets\(^4\) respectively. Notice that the above model allows the introduction of discrete torsion [23], \(i.e.\) an opposite sign in the \(J\) projection of the \(I\) sector. This would result in a reapparance of R-R massless states in the \(K\) sector. The same freedom is possible in the dual type I model, where it corresponds to retaining the (descendants of the 6D) tensor multiplets instead of the hypermultiplets in the twisted sector of the orbifold. Notice that the presence of R-R massless states in type II \((2,0)\) models opens new problems in the duality with type I models without open strings that we will not address here.

For this reason we will restrict our attention to the dual version of the type II \((2,0)\)

\(^4\)We are still adhering to the four-dimensional counting of supersymmetries.
model discussed above. The type I dual can be constructed as before mapping the $(-)^F \Omega_L$ operation to the $\Omega$-projection under type IIB S-duality [14]. Notice that the combination of $\Omega$ and $\sigma_V$ leads to a rather unconventional type I vacuum configuration in which no D9-branes and their open-string excitations can be introduced. More explicitly the Klein-bottle amplitude

$$K = Tr_c(\Omega \cdot \sigma_V q^H)$$

(4.15)

where the subscript $c$ denotes a trace in the unoriented closed-string spectrum does not give rise to unphysical massless tadpoles in the transverse channel. Indeed no massless state at all participate to the crosscap-to-crosscap amplitude. After target space T-duality along the circle $S^1$, this may be interpreted as saying that there are two oppositely charged O8-planes [7]. The overall vanishing of the R-R charge prevents the introduction of D-branes and their associated open-string amplitudes (annulus $A$ and Möbius-strip $M$) [12, 11, 5]. In some respect type I vacuum configurations without open strings may be considered as unconventional orientifolds without twisted sectors [20, 21, 22]. The presence of the $K3$ factor in the compactification manifold does not change the picture qualitatively with respect to the previous subsection. We are left with the same field content (4.6) as in the previous (toroidal) case now with target space $(R^4 \times K3)^N/S_N$. For $N = 1$ the partition function can be read off directly from the orbifolding of (4.7)

$$Z_{+-} = \frac{(4 - 4) \vartheta_2^3(q) \vartheta_3^2(q)}{\tau_2^{4-d/2} \eta^6(q) \vartheta_2^2(q)}$$

(4.16)

$$Z_{--} = \frac{(4 - 4) \vartheta_2^3(q) \vartheta_3^2(q)}{\tau_2^{4-d/2} \eta^6(q) \vartheta_2^2(q)}$$

(4.17)

where now $\pm$ refer to the $I_4$ twists. The spectrum of type I D-strings accounted for by (4.16) and (4.17) agrees with the spectrum of fundamental strings accounted for by (4.10) and (4.11) once the level matching condition is imposed. The remaining states (4.13) will appear for $N$ even. Indeed extracting the longest string contribution from expression (2.10) for $N$ odd, we recognize the same partition function (4.16-4.17) as before after the substitution $q \rightarrow q^{1/2 \omega^s}$ and the sum over $s = 0, \ldots, N - 1$ that ensures the level matching condition (4.9). For $N$ even, on the other hand, we find

$$Z_{--} = \frac{4 - 4}{\tau_2^{4-d/2}} \sum_{s=0}^{N-1} \vartheta_2^2(q^{1/2 \omega^s}) \vartheta_3^2(q^{1/2 \omega^s}) \eta^6(q^{1/2 \omega^s})$$

(4.18)

which agrees with the fundamental (4.13) spectrum of states.

5 Threshold corrections in type II string vacua

In the previous sections we provided some evidence for various equivalences between string theories by studying the spectrum of physical BPS states in lower dimensional
compactifications. We would now like to apply these results to the study of the low energy effective actions describing these string vacua. In particular, we are interested in studying the moduli dependence of the special (“BPS saturated”) $F^4$ terms in the $D = 8$ dimensional low energy effective action for the conjectured pair type II (4,0) string - type I without open strings. We closely follow a sequence of works [16], where a similar analysis for the threshold corrections in the context of type I - heterotic duality has been performed.

The interest in the study of these terms relies on the fact that they are believed to receive only one-loop corrections for toroidal heterotic compactifications to $D > 4$ dimensions. Supersymmetry protects these terms from higher-loop perturbative corrections while the only identifiable source of non-perturbative corrections (the 5-brane instanton) is infinitely heavy for compactifications to $D > 4$ dimensions. This seems to be the case for the type II (4,0) models too. For the model under consideration, the $(-)^F L \sigma V$ action removes the RR-fields leading to an effective type II theory without D-branes [12, 11, 5]. The only source of non-perturbative effects we can think of is again the 5-brane instanton which cannot enter the correction of a $D > 4$ dimensional effective action. On the type I side (type I without open strings or type IIB on $T^2/\Omega \sigma V$ [5]) however, D-string instantons are expected to correct the effective actions for $D \leq 8$ dimensions. Indeed, using the conformal description of the infrared limit for the $N$ D-instanton\(^5\) system we will be able to compute these non-perturbative corrections, showing the agreement with the one-loop exact formula found in the dual type II computation.

5.1 Threshold corrections in type IIB on $T^2/(−)^F L \sigma V$

For simplicity we will restrict our attention to the $F^4$ couplings in the low energy effective action. We will assume that, as it is the case for similar terms in toroidal heterotic compactifications to $D > 4$ dimensions, the moduli dependence for these terms in type IIB on $T^2/(−)^F L \sigma V$ receive only one-loop corrections. Let us recall how the arguments leading to this conclusion for the $SO(32)$ heterotic $F^4$ terms work. The $F^4$ terms we will study in the type II context involve the $O(2,2)$ gauge fields $G_{\mu i}, B_{\mu i}$ arising from the K-K reduction of the metric and NS-NS antisymmetric tensor. In toroidal compactifications of the heterotic string these gauge fields get mixed with the $SO(32)$ gauge fields by the action of the T-duality group $O(2,18,\mathbb{Z})$. It is therefore reasonable to believe that the same non-renormalization arguments apply to all the gauge fields that do not belong to the supergravity multiplet. As we will see, the latter, usually called “graviphotons”, stay on a different footing. For $SO(32)$ gauge fields, the $F^4$ terms (as well as some $R^4$ and $F^2R^2$ terms) can be obtained by dimensional

\(^5\)D-instantons in this context refer to instantons from the point of view of the eighth-dimensional effective action. The “D” recalls the origin of these contributions from D-strings wrapped on the $T^2$ torus.
reduction of ten dimensional superinvariants, whose bosonic parts read \[24\]

\[
I_1 = t_8 \text{tr} F^4 - \frac{1}{4} \varepsilon_10 B \text{tr} F^4
\]

\[
I_2 = t_8 (\text{tr} F^2)^2 - \frac{1}{4} \varepsilon_10 B (\text{tr} F^2)^2
\]  (5.1)

They are special because they contain CP-odd pieces related to the cancellation of gravitational and gauge anomalies in ten dimensions. Indeed in ten dimensions, the coefficients of these couplings are completely determined by supersymmetry and the anomaly cancelling mechanism. This is no longer true for compactifications to lower dimensions, where supersymmetry restricts, but does not completely fix, their dependence on the compactification moduli. For heterotic compactifications, the moduli dependence of the CP-odd pieces in (5.1) was studied in \[25\]. As shown there, they receive only one-loop perturbative corrections. Moreover, as argued before, non-perturbative corrections are ruled out in compactifications to \( D > 4 \) dimensions. It is then plausible to assume that no higher-order corrections are present for the supersymmetry-related CP-even \( F^4 \) terms, too.

A similar analysis for the type II models under study here, has not been done, but we expect similar result to be true. We will assume that this is the case, \textit{i.e.} that the one-loop formula we will obtain in this section for the moduli dependence of some \( F^4 \) terms in type IIB on \( T^2/(-)^{F_L} \sigma_V \) is exact. The non-perturbative results for similar terms in the low energy effective action of the dual type I string will support this assumption.

We will consider a compactification on a target space torus characterized by the complex moduli

\[
T = T_1 + iT_2 = \frac{1}{\alpha'} (B_{89}^{NS} + i\sqrt{G})
\]

\[
U = U_1 + iU_2 = (G_{89} + i\sqrt{G})/G_{88},
\]  (5.2)

where \( G_{ij} \) and \( B_{ij}^{NS} \) are the \( \sigma \)-model metric and NS-NS antisymmetric tensor.

We recall that by projecting the type IIB string with \( (-)^{F_L} \sigma_V \) we remove all R-R massless fields. The only eight-dimensional gauge bosons are then given by the \( G_{\mu8}, G_{\mu9} \) and \( B_{\mu8}, B_{\mu9}^{NS} \) components of the metric and NS-NS antisymmetric tensor. The corresponding vertex operators in the Green-Schwarz formalism can be written as

\[
V_{i}^R = \int d^2 z (G_{\mu i} - B_{\mu i}^{NS}) (\partial X^i - \frac{1}{4} p_{\nu} S_{\gamma}^{\nu i} S) (\bar{\partial} X^\mu - \frac{1}{4} \bar{p}_{\rho} \bar{S}_{\gamma}^{\rho \mu} \bar{S}) e^{ip \bar{X}} \]  (5.3)

\[
V_{i}^L = \int d^2 z (G_{\mu i} + B_{\mu i}^{NS}) (\partial X^\mu - \frac{1}{4} p_{\nu} S_{\gamma}^{\mu \nu} S) (\bar{\partial} X^i - \frac{1}{4} \bar{p}_{\rho} \bar{S}_{\gamma}^{i \rho} \bar{S}) e^{ip \bar{X}} \]  (5.4)

with \( i = 8, 9 \). The \( V_{i}^R \) are the vertices for the graviphotons, \textit{i.e.} the gauge fields sitting in the supergravity multiplet. It is easy to see that there are no one-loop corrections to \( F^4 \) terms involving the graviphotons. Indeed, after soaking the eight right moving zero-modes \( S_0 \) on the world-sheet torus with the fermionic piece \( p_{\nu} S_{\gamma}^{i \nu} S \) in (5.3), a
four-graviton amplitude already exposes the required fourth power of the external momenta. At this order in the momenta, only the bosonic piece in the left-moving parts of the vertices can enter, but their contractions unavoidably lead to total derivatives $\langle \partial X^\mu(z_1)\partial X^\mu(z_2) \rangle$ which vanish after the $z$-integrations. Corrections to the $F^4$ terms for the $G_{\mu i}$ and $B^{NS}_{\mu i}$ gauge fields then coincide and can be extracted from the four-point amplitude

$$A_\ell = \langle (V^L_8)^\ell (V^L_9)^{4-\ell} \rangle = t_8 F_8^\ell F_9^{4-\ell} \prod_{j=1}^{\ell} \int dz_j \partial X^8(z_j) \prod_{k=\ell+1}^{4} \int dz_k \partial X^9(z_k)$$

(5.5)

where $t_8$ is the tensor arising from the trace over the right-moving fermionic zero-modes. It is convenient to define a generating function for such terms that reads

$$A_\ell = t_8 F_8^\ell F_9^{4-\ell} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \left( \frac{\tau_2}{\pi} \right)^4 \frac{\partial^{4-\ell}}{\partial \nu^4} Z(\nu_i, \tau, \bar{\tau})$$

(5.6)

with $\mathcal{F}$ the fundamental domain for the world-sheet torus, and $Z(\nu_i, \tau, \bar{\tau})$ the partition function arising from a perturbed Polyakov action whose bosonic part reads

$$S(\nu_i) = \frac{2\pi}{\alpha'} \int d^2\sigma (\sqrt{g} G_{\mu\nu} \epsilon^{\alpha\beta} \partial_\alpha \Sigma^\mu \partial_\beta \Sigma^\nu + i B^{NS}_{\mu\nu} \epsilon^{\alpha\beta} \partial_\alpha \Sigma^\mu \partial_\beta \Sigma^\nu + \sqrt{g} \alpha' \nu_i \partial \Sigma^i)$$

(5.7)

and $\partial = \frac{1}{\tau_2} (\partial_{\sigma_2} - \bar{\tau} \partial_{\sigma_1})$. The partition function $Z(\nu_i, \tau, \bar{\tau})$ involves a sum over all possible world-sheet instantons

$$\begin{pmatrix} X^8 \\ X^9 \end{pmatrix} = M \begin{pmatrix} \sigma^1 \\ \sigma^2 \end{pmatrix} \equiv \begin{pmatrix} m_1 & n_1 \\ m_2 & n_2 \end{pmatrix} \begin{pmatrix} \sigma^1 \\ \sigma^2 \end{pmatrix}$$

(5.8)

with worldsheet and target space coordinates $\sigma_1, \sigma_2$ and $X^8, X^9$ respectively, both taking values in the interval $(0,1]$. The entries $m_1, n_1$ are integer or half-integer depending on the specific orbifold sector, while $m_2, n_2$ are always integers. We denote the three relevant sectors: $n_1$ half-integers, $m_1$ half-integers and both half integers by $\epsilon = +-, --, ---$ respectively. Clearly the untwisted sector, $\epsilon = ++$, will not contribute since it has too many zero modes to be soaked by the four vertex insertions at this order in the momenta.

In the following we use normalizations such that $\langle X^M X^N \rangle \sim G^{MN}$, with $G^{MN}$ the ten dimensional metric defined by

$$G_{ij} = \frac{\alpha' T_2}{U_2} \begin{pmatrix} 1 & U_1 \\ U_1 & |U|^2 \end{pmatrix}$$

(5.9)

in the compact space and the flat metric $G_{\mu\nu} = \eta_{\mu\nu}$ in $\mathbb{R}^8$. For the worldsheet metric we choose

$$g^{\alpha\beta} = \frac{1}{\tau_2^2} \begin{pmatrix} |\tau|^2 & -\tau_1 \\ -\tau_1 & 1 \end{pmatrix}.$$  

(5.10)

The generating function can then be written as

$$\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z(\nu_i, \tau, \bar{\tau}) = \frac{V_8}{2^{10} \pi^4} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_\epsilon \Gamma^{\epsilon}_{2,2}(\nu_i) A_\ell$$

(5.11)
\[
\Gamma_{\epsilon,2}(\nu_i) = \frac{T_2}{U} \sum_{M_\epsilon} e^{2\pi i T \det M} e^{-\frac{\pi}{4\sqrt{2}} |(1 U)M(\tau)|^2} e^{-\frac{\pi}{8}(\nu_8 \nu_9)M(\tau)}
\]  

(5.12)

and \(A_\epsilon\) the anti-holomorphic BPS partition functions (4.2-4.4)

\[
A_{\epsilon+} \equiv \frac{1}{2} \frac{\vartheta_2(q)}{\eta(q)}^4, \quad A_{\epsilon-} \equiv \frac{1}{2} \frac{\vartheta_4(q)}{\eta(q)}^4, \quad A_{-\epsilon} \equiv \frac{1}{2} \frac{\vartheta_3(q)}{\eta(q)}^4
\]  

(5.13)

Following Dixon, Kaplunovsky and Louis [26] we can express the sum over the \(M_\epsilon\) matrices in (5.12) as a sum over \(SL(2,\mathbb{Z})\) representative integrated in an unfolded domain. Notice that the complete generating function \(Z(\nu_i, \tau, \bar{\tau})\) has no definite modular transformation properties. This is not the case for the interesting term, the fourth \(\nu\)-derivative of \(Z(\nu_i, \tau, \bar{\tau})\) appearing in (5.5), which is indeed modular invariant. In the following we keep in mind that eventually we will only consider this term and, without further comments, perform modular manipulations which are only sensible on the final result. In this broad sense the generating function (5.11) is invariant under the combined \(SL(2,\mathbb{Z})\) actions

\[
\tau \to a\tau + b \quad \text{and} \quad M_\epsilon \to M_\epsilon \left( \begin{array}{cc} d & b \\ c & a \end{array} \right)
\]  

(5.14)

Different \(\epsilon\)-elements in the sum (5.12) get mixed in general by these transformations, but the sum is clearly invariant. An orbit is defined by the set of matrices \(M\), which can be related by some \(SL(2,\mathbb{Z})\) element \(V\) to a given representative \(M_0\) through \(M = M_0 V\).

By a change of variables in the \(\tau\) integration we can reduce the sum over matrices \(M\) in a given orbit to a single integration over an unfolded domain obtained as the union of \(V_i\) images of the fundamental domain through the modular transformations (5.14). These unfolded domain can be either the strip or the whole upper half plane depending on whether the matrix \(M\) is degenerate (\(\det M = 0\)) or non-degenerate (\(\det M \neq 0\)).

Let us consider first the degenerate case. Using the modular transformation properties of (5.12) and (5.13) we can write the contributions from the different orbifold sectors in (5.11) as

\[
\int_{\mathcal{F}_2} \frac{d^2\tau}{\tau_2^2} (\Gamma_2^{+,-} \mathcal{A}_{+} (\bar{\tau}) + \Gamma_2^{+,-} \mathcal{A}_{-} (\bar{\tau}) + \Gamma_2^{+,-} \mathcal{A}_{+} (\bar{\tau}')) = \int_{\mathcal{F}_2} \frac{d^2\tau}{\tau_2^2} \Gamma_2^{+,-} \mathcal{A}_{+} (\tau)
\]  

(5.15)

where \(\tau' = -\frac{1}{\tau}\) and \(\tau'' = \tau' + 1\). The new fundamental domain \(\mathcal{F}_2\) is the quotient of the upper half plane by a \(\Gamma_2\) subgroup of \(SL(2,\mathbb{Z})\) transformations defined as the set of elements \(V\) which keep invariant the form of an \(M_{+}\) matrix \((m_1 \in \mathbb{Z}, n_1 \in \mathbb{Z} + \frac{1}{2})\).

A generic element of \(\Gamma_2\) can be written as

\[
V = \left( \begin{array}{cc} a & b \\ 2c' & d \end{array} \right).
\]  

(5.16)

with \(ad - 2bc' = 1\). It is easy to see that acting with a \(V\) in this \(\Gamma_2\) subgroup we can always bring a matrix \(M_{+}\) with zero determinant to the form

\[
M = \left( \begin{array}{cc} 0 & j_1 - \frac{1}{2} \\ 0 & j_2 \end{array} \right).
\]  

(5.17)
The sum over $M_{+, -}$ matrices in (5.15) can then be written as a sum over the representatives (5.17) labeled by $(j_1, j_2)$. The $\tau$ integration runs over an unfolded domain defined by the union of all images of the $F_2$ fundamental domains under the $\Gamma_2$ actions (5.16). We should notice however that not all $\Gamma_2$ matrices define different $M$’s. Indeed, a representative (5.17) is invariant under the action

$$V = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad \text{i.e. } \tau \rightarrow \tau + b. \quad (5.18)$$

The unfolded domain is then the upper half plane modded out by these transformations, i.e. the strip $S = \{ |\tau_1| \leq \frac{1}{2}, \tau_2 > 0 \}$. Substituting (5.17) in (5.15) we are left with

$$\int_S d\tau_2^2 \sum_{(j_1, j_2) \neq (0, 0)} e^{-\frac{\pi}{2} |j_1 - \frac{1}{2} + j_2 U|^2} e^{-\frac{\pi}{2} (\nu_8 (j_1 - \frac{1}{2}) + \nu_9 n_2)} A_{+, -}(\bar{\tau}). \quad (5.19)$$

The $\tau_1$ integration picks the $q^0$ power in the expansion

$$A_{+, -}(\bar{\tau}) = \sum_{n = 0}^{\infty} A_{+, -}^n \bar{q}^n. \quad (5.20)$$

which is just $2^3$. After taking the $\nu$-derivatives and integrating in $\tau_2$ we are left with the final result

$$\langle (F_8)^\ell (F_9)^{4-\ell} \rangle_{\text{deg}} = \frac{V_8}{(4\pi^2 \alpha')^4} t_8 F_8^\ell F_9^{4-\ell} T_2 \int_0^\infty \frac{d\tau_2}{\tau_2^2} \sum_{(j_1, j_2) \neq (0, 0)} (j_1 - \frac{1}{2})^{\ell} j_2^{4-\ell} e^{-\frac{\pi}{2} |j_1 - \frac{1}{2} + j_2 U|^2} \quad (5.21)$$

Let us now consider the contributions from the non-degenerate orbits. In this case acting with an $SL(2, \mathbb{Z})$ transformation we can, at most, bring a matrix $M$ to the form

$$M = \begin{pmatrix} m_1 & n_1 \\ 0 & n_2 \end{pmatrix} \quad (5.22)$$

where $m_1, n_1$ are in $\mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$ as before depending on the orbifold sector. The $\tau \rightarrow \tau + b$ action (5.18) on this matrix shifts $n_1 \rightarrow n_1 + b m_1$. Therefore we can bring $n_1$ to the fundamental range $n_1 = 0(\frac{1}{2}), 1(\frac{1}{2}), \ldots, 0(\frac{1}{2}) + m_1 - 1$. The representatives of the non-degenerate matrices $M$ are then given by the matrices (5.22) with

$$\epsilon = + - : \quad m_1 \in \mathbb{Z}; \quad n_1 = \frac{1}{2}, \frac{3}{2}, \ldots, \frac{2m_1 - 1}{2}$$

$$\epsilon = - + : \quad m_1 \in \mathbb{Z} + \frac{1}{2}; \quad n_1 = 0, 1, \ldots, m_1 - 1$$

$$\epsilon = - - : \quad m_1 \in \mathbb{Z} + \frac{1}{2}; \quad n_1 = \frac{1}{2}, \frac{3}{2}, \ldots, \frac{2m_1 - 1}{2} \quad (5.23)$$

The unfolded domain is now the whole upper half complex plane since all $SL(2, \mathbb{Z})$ actions are allowed, i.e. distinct elements in a non-degenerate orbit are in one-to-one correspondence with the copies of the fundamental domain $\mathcal{F}$ of $\tau$ in the upper half.
plane. We should however notice that different replicas of the fundamental domain in the upper half plane come from different orbifold sectors since $SL(2, Z)$ transformations which allow us to bring a given matrix $M$ to a representative in (5.23) mixed the $M$, with different $\epsilon$’s. Therefore only the unfolding of the sum over $\epsilon$’s in (5.11) makes sense.

We can now perform the $\tau$ integrations. Expanding the antiholomorphic modular functions (5.13) as in (5.20) yields the integral

$$I_n = T_2 e^{2\pi T m_1 n_2} \int_{C^+} \frac{d^2 \tau}{\tau_2^2} e^{-\frac{\pi T}{\tau_2^2} \left| m_1 \tau - n_1 - n_2 U \right|^2} e^{-\frac{\pi}{\tau_2^2} \left( m_1 \tau - n_1 \nu_{8} - n_2 \nu_{9} \right)} e^{-2\pi \tau n}$$

(5.24)

which after the $\tau$ integrations can be written as

$$I_n = \frac{(U_2 T_2)^{\frac{1}{2}}}{m_1} e^{-2\pi i T m_1 n_2} e^{-2\pi i n (\frac{\nu_9 n_2}{m_1} - \nu_9 U n_2 - \frac{\nu_8^2 U_2}{4 T_2})} \int_0^\infty \frac{d\tau_2}{\tau_2^{3/2}} e^{-\frac{\beta}{\tau_2} - \gamma \tau_2}$$

(5.25)

$$= \frac{(U_2 T_2)^{\frac{1}{2}}}{m_1} e^{-2\pi i T m_1 n_2} e^{-2\pi i n (\frac{\nu_9 n_2}{m_1} - \nu_9 U n_2 - \frac{\nu_8^2 U_2}{4 T_2})} \int_0^\infty \frac{d\tau_2}{\tau_2^{3/2}} e^{-\frac{\beta}{\tau_2} - \gamma \tau_2}$$

(5.25)

with

$$\beta = \pi (n_2^2 U_2 T_2 + \nu_9 n_2 - \nu_8 U_1 n_2 - \frac{\nu_8^2 U_2}{4 T_2})$$

$$\gamma = \frac{\pi T_2}{U_2} (m_1 + \frac{n U_2}{T_2 m_1})^2.$$

(5.26)

In order to extract the $F^4$ term we should still act on (5.25) with four $\nu$-derivatives and finally set the sources $\nu$’s to zero. In the following we will restrict ourself to the leading behaviour in a $\frac{1}{T_2}$ expansion of this result. The interest in this particular expansion will become clear later. In this limit the four derivatives should hit one of the $\nu$-linear terms in the exponential of (5.25) leaving the final expression

$$\langle (F_8)^\ell (F_9)^{4-\ell} \rangle_{\text{nondeg}} =$$

$$= \frac{\nu_8}{(4 \pi^2 \alpha')^4} \int_8 F_8^{\ell} F_8^{4-\ell} \sum_{n, \epsilon} \sum_{m_1, n_1, n_2} A_{\epsilon} e^{-2\pi i n (\frac{\nu_9 n_2}{m_1})} e^{-2\pi i n_1 n_2 T} \frac{\partial^\ell}{\partial \nu_8} \frac{\partial^{4-\ell}}{\partial \nu_9} \frac{\partial_{\epsilon} \pi m_1}{U_2} (\nu_8 U - \nu_9)$$

(5.27)

where we have used the expansions (5.20) to reconstruct the modular forms (5.13), now evaluated at an induced modulus $U = \frac{n_1 + \bar{\nu}_{n_2}}{m_1}$.

6 Threshold corrections in type I without open strings

We now pass to the study of $F^4$ threshold corrections in the conjectured dual (type IIB on $T^2/\Omega_{L, \sigma}$) of the previously studied type II orbifold model. If the one-loop formulas for the moduli dependence of $F^4$ terms in type IIB on $T^2/(-)^F L, \sigma$ are exact, as argued
before, they should contain both the perturbative and non-perturbative corrections in the dual side. The aim of this section is to prove that this is the case. The results are in complete agreement with the predictions of the type IIB self-duality conjecture.

Let us begin by translating the one-loop exact results (5.21) and (5.27) in terms of the type I variables. The duality relations imply a rescaling of the lengths (in the $\sigma$ model variables) according to

$$L_F = \frac{L_I}{\sqrt{\lambda_I}}$$

(6.1)

where the subscripts “$F$” and “$I$” are used to distinguish orbifold ($T^2/(\sigma_F \sigma_V)$) and orientifold ($T^2/(\sigma_V \sigma_F)$) compactifications of the type IIB string. This implies in particular that the volume $T^2$ gets rescaled as $T^2_F = T^2_I / \lambda_I$ and therefore the expansion in $1/T^2$ of the exact result found in the previous section can be identified with the genus expansion as seen from the type I perspective. Taking into account also the scaling of the gauge field $A_F = G_F^\mu = G_I^\mu / \lambda_I = A_I^\mu / \lambda_I$, we find that the relevant $F^4$ terms in the eight dimensional effective action scales according to

$$f(T^2_F, \lambda_F) \int d^8 x \sqrt{G_F} F^4_F = \frac{1}{\lambda^4_I} f(T^2_I, \lambda_I) \int d^8 x \sqrt{G_I} F^4_I \, .$$

(6.2)

In the previous section we argued that the only non-trivial moduli dependence for these terms in the orbifold side are given by the one-loop ($\lambda^0_F$ order in the expansion of $f(T^2_F, \lambda_F)$) expressions (5.21) and (5.27). By plugging these results in (6.2) we can see that contributions from degenerate orbits ($f(T^2_F, \lambda_F) \sim (T^2_F)^{-4}$) correspond to one-loop effects (order $\lambda^0_F$) in the type I description, while those from non-degenerate matrices ($f(T^2_F) \sim e^{-2\pi T^2_F m_1 n_2 (1 + O(1/T^2_F))}$) should arise as D-instanton corrections with instanton number $N = m_1 n_2$. We will momentarily show how these corrections can be reproduced by a direct computation in the type I theory.

### 6.1 One loop threshold corrections

The one-loop effective action for a type I theory without open strings gets contribution from the torus and Klein bottle amplitudes. We are interested in the corrections to $F^4$ terms. As before there are four eight-dimensional gauge fields $G_\mu$ and $B^{RR}_\mu$, but only the former couple to elementary states (the K-K modes) in the type I string spectrum. As before the relevant vertex operators are defined by

$$V_i = \int d^2 z G_\mu \left( \partial X^\mu - \frac{1}{4} p_\rho S_{\gamma^{\mu\nu}} S \right) (\bar{\partial} X^i - \frac{1}{4} p_\rho \bar{S}_{\gamma^{ij \rho}} \bar{S}) e^{ip X},$$

(6.3)

since $\Omega \sigma_V$ is simply the worldsheet parity $\Omega$ when acting on a massless state. In a four-point amplitude, the sixteen fermionic zero-modes on the world-sheet torus, if soaked, would produce eight power of the external momenta. Therefore four-derivative terms
only receive contribution from the Klein-bottle amplitude, in which left- and right-moving zero-modes are identified. The Klein-bottle partition function is defined by

\[
K = -\frac{1}{2} \int_0^\infty \frac{dt}{t^6} \text{Tr} \Omega e^{\pi i k_8} e^{-\pi t(k_N k_M G^{MN} + M^2)}
\]

\[
= \frac{\mathcal{V}_8}{(4\pi^2\alpha')^4} (8 - 8) \int_0^\infty \frac{dt}{t^6} \sum_{n_i} e^{-\frac{\pi}{4}(n_i + \epsilon_i/2)(n_j + \epsilon_j/2)G^{ij}}
\]

(6.4)

where \(\epsilon_8 = 1/2\) and \(\epsilon_9 = 0\) and the metric \(G^{ij}\) is the inverse of (5.9). As always the factor \((8-8)\) comes from the trace over the fermionic zero-modes and \(\frac{1}{t^6}\) from the momentum integration in the non-compact directions. The second expression in (6.4) only involves a sum over the classical configurations since quantum bosonic and fermionic contributions cancel out by supersymmetry. The sign \(e^{\pi i k_8}\) defines the action of \(\sigma_V\) on a given state of K-K momentum \(k_8\) running in the loop. We have performed a Poisson resummation on the integers \(k_8, k_9\) expressing this projection as a half-shift in the Lagrangian mode.

Due to this shift in the Lagrangian mode (winding) no massless closed string state flow in the transverse channel. This implies in particular that the O9-planes do not carry R-R charge and therefore there is no room for the introduction of D9-branes and their open string string excitations [12, 11, 5].

The insertion of four vertex operators (6.3) in (6.4) will soak the eight left-right symmetric fermionic zero modes reproducing the correct momentum structure

\[
\langle (F_8)^{\ell} (F_9)^{4-\ell} \rangle_{\text{one-loop}} = \frac{\mathcal{V}_8}{(4\pi^2\alpha')^4} t_8 F_8^{\ell} F_9^{4-\ell} 2 \int_0^\infty \frac{dt}{t^6} \sum_{\langle j_1, j_2 \rangle \neq (0,0)} (j_1 - \frac{1}{2})^{\ell} j_2^{4-\ell} e^{-\frac{\pi t^2}{2\pi^2} |j_1 - \frac{1}{2} + j_2 U|^2}
\]

(6.5)

which precisely reproduces the contribution (5.21) of the degenerate orbits in the dual type IIB model on \(T^2/(\pm)^{\ell} \sigma_V\).

### 6.2 D-Instanton contributions

Let us now consider non-perturbative corrections in the type I description. In eight dimensions the only identifiable source of non-perturbative effects in the present model are the contributions from D-instantons, arising from wrapping the D-string worldsheet on the two-torus target-space. Since the insertion of four gauge vertices can soak up at most eight fermionic zero modes only \(\frac{1}{2}\)-BPS D-instantons can contribute. We can use the results of the previous section. There we studied the partition function for the \(N\)-wrapped D-string excitations, by going to the infrared limit where the theory flows to an orbifold conformal theory. Indeed we can read directly the BPS \(N\) D-instanton partition function from (2.16) once the one-loop \(\tau\)-parameter is identified with the complex structure \(U\) of the target torus. This summarizes the quantum contributions
to the partition function in the D-instanton background. In addition to this we should include the classical contribution arising from the $N$ D-instanton action for this model. The bosonic part of this action coincides with the one for $N$ type IIB D-strings and can be written as

$$S_{D-\text{inst}} = \frac{2\pi}{\alpha'} \sum_{t=1}^{N} \int d^2\sigma \left( \sqrt{g} \epsilon^{\alpha \beta} \frac{1}{\lambda_I} G_{\mu \nu} + iB_{\mu \nu}^{RR, \alpha \beta} D_\alpha X_t^\mu D_\beta X_t^\nu \right) \quad (6.6)$$

where $t$ labels the $N$ Cartan directions of the unbroken $U(1)^N$ gauge group and $D_\alpha$ represent the supersymmetric covariant derivatives, which can be written in a complex basis as

$$D X_t^\mu = \partial X_t^\mu - \frac{1}{4} p_\nu S_t^{\mu \nu} S_t$$

$$\bar{D} X_t^\mu = \bar{\partial} X_t^\mu - \frac{1}{4} p_\nu \bar{S}_t^{\mu \nu} \bar{S}_t.$$  

The $X$’s are always in the static gauge $X_t^8 = \sigma^1, X_t^9 = \sigma^2$ and the world-sheet modular parameter $\tau$ is identified with the complex structure $U$ of the target-space. Computing (6.6) for a background with only non-trivial components along $G_{ij}$ and $B_{ij}^{RR}$, we are simply left with $S_{D-\text{inst}} = 2\pi N T_I$ where $T_I$ is the “dual” complexified Kähler modulus

$$T_I = T_1 + i T_2 = \frac{1}{\alpha'} \left( B_{89}^{RR} + i \frac{\sqrt{G}}{\lambda_I} \right). \quad (6.7)$$

In order to study $F^4$ couplings for the $G_{\mu i}$ components of the metric, we can turn on a background $\nu_i$ for this field and extract the coupling from the fourth $\nu$-derivative. Notice that only the classical part of the D-instanton partition function will be modified by these insertions since quantum correlators are always given by total derivatives which drop out after the $z$-integrations. We can identify in (6.6) the relevant coupling as

$$S = 2\pi T_I N + \frac{N\pi}{\alpha' U_2} \left[ (G_{\mu 9} + U G_{\mu 8}) D_2 X^\mu + (G_{\mu 9} - \bar{U} G_{\mu 8}) \bar{D}_2 \bar{X}^\mu \right] + \cdots \quad (6.8)$$

where $z = \sigma_1 + U \sigma_2$ is the complex worldsheet coordinate. Each $G_{\mu i}$ insertion should soak up two right-moving fermionic zero modes. These fermionic modes only enter the term with $D_2$. Therefore the four $\nu$-derivatives always hit this term in (6.8) bringing a power of $NU/U_2$ for each $G_{\mu 8}$ insertion and a power of $N/U_2$ for each $G_{\mu 9}$ insertion. Collecting the different pieces:

- The classical contribution: $e^{2\pi i T_1 N}$
- The quantum contributions from (2.16) omitting the ubiquitous (8-8) factor and replacing $\tau$ by $U$
- The fermionic zero mode trace: $t_8 F_8^\ell F_9^{4-\ell}$
- A factor of $NU/U_2$ for each $G_{\mu 8}$ insertion and of $N/U_2$ for each $G_{\mu 9}$
yields the final result for the D-instanton contributions

\[
\langle (F_8)^\ell (F_9)^{4-\ell} \rangle_{D\text{-inst}} = \\
= \frac{V_8}{(4\pi^2\alpha')^4} t_8 F_8^{4-\ell} U^{\ell} \sum_{N,L,M} L^4 e^{2\pi iTN} 1 \frac{\eta^{L-1}}{\eta^N} \frac{q^{\frac{M}{2\pi i T}} e^{\frac{2\pi i T}{2}}}{(q^{\frac{M}{2\pi i T}} e^{\frac{2\pi i T}{2}})^4}
\]

which precisely reproduces the contributions of the non-degenerate orbits (5.27) after trivial identifications.

7 Conclusions and perspectives

In this paper we have analysed in detail dual pairs of unconventional models with 16 and 8 supercharges. The key ingredient in our discussion has been the correct identification of the D-string effective actions in the corresponding type I like models. In section 3, we have derived the one-loop partition function for both the SO(16) CHL model and for its candidate type I dual [5, 7]. Moreover we have also computed the elliptic genus for the effective \(O(N)\) theory governing the dynamics of \(N\) D-strings in this unconventional type I toroidal compactification. The perturbative type I BPS states were shown to be in one-to-one correspondence with heterotic BPS states with zero winding. The non-perturbative type I BPS states, identified with \(N\) D-string bound-states in the longest string sectors, were shown to be in one-to-one correspondence with the heterotic BPS states with \(N\) units of winding. In section 6, the perfect agreement between the BPS spectra of a type II (4,0) model in \(D = 8\) and its candidate dual type I model without open strings [5] has been the key ingredient in showing the matching between the thresholds corrections to some \(F^4\) terms in the two descriptions. The precise matching of the BPS spectra and the explicit computations performed in section 6 lead us to conclude that BPS-saturated thresholds must coincide for the \(SO(16)\) dual pair we have discussed as well as for any dual pair that passes the BPS precision test. Therefore, although we have not explicitly worked out the threshold corrections for CHL models and the corresponding duals, we do not expect any basic difficulties.

In this paper we have only analysed the leading term in the coupling constant expansion around D-string instanton sectors and have found that they agree with the results on the fundamental string side. However the latter calculation also predicts subleading corrections around the D-string instantons. Such corrections also exist in the standard heterotic-type I dual pairs [16] and their origin on D-string instanton side has not yet been analyzed. This problem is under investigation and it appears that the usual \(\sigma\)-model expansion around D-string instantons can account for such corrections [28].

The presence of stable non-BPS states may well play a role in the corrections to non BPS-saturated couplings that are present both in theories with a large amount of supersymmetry and in theories with lower or no supersymmetry at all. Computing non
BPS-saturated thresholds and finding precise agreement for some of them may help putting dualities for theories with lower (or none) supersymmetry on a firmer ground. Extending the present analysis to other D-brane bound-states, e.g. D5-branes, in order to compute threshold corrections in lower dimensions (e.g. $D = 4$) does not seem obvious at all and deserves a case-by-case analysis.

Another interesting direction of application of the results presented in this paper is the computation of the prepotential for theories with 16 supercharges in $D = 8$. As far as the dependence on the $O(2, 2)$-moduli is concerned the F-theory description [29] seems to be a very powerful competitor to our approach.

Let us conclude by sketching the steps needed in order to explicitly compute some of the BPS-saturated thresholds in the $SO(16)$ heterotic - type I dual pair. For the $F^4$ term involving the vector bosons of the $SO(16)$ gauge group, associated to the worldsheet current algebra at level $k = 2$, one has to extract the four-derivative term in the four-point amplitude on the torus with even spin structures of four massless vectors. The contribution of the odd spin structure is associated to the anomaly cancelling term in $D = 10$. The trick of the generating function used in the previous sections may be of great use in this respect. Supersymmetry considerations lead us to conclude that the relevant threshold gets only a one-loop contribution on the CHL heterotic side.

At the one-loop order, on the type I side, these terms only get contribution from the annulus $A$ and Möbius-strip $M$ amplitudes. It should be easy to realize that the contribution of the degenerate orbits to the heterotic threshold matches the contribution of the perturbative type I amplitudes $A$ and $M$. A somewhat involved computation may be required in order to match the contribution of the non-degenerate orbits to the heterotic thresholds with the D-instanton corrections to the type I thresholds. A similar discussion applies to the threshold corrections to the $F^4$ terms involving the K-K gauge fields. The analysis for the type I dual of the type II $(4,0)$ model may be carried over to the present situation after taking into account the presence of a non-vanishing contribution not only from the Klein-bottle amplitude $K$ but also from the annulus $A$ and the Möbius-strip $M$ amplitudes. All these surfaces allow for the closed-string insertions that are needed to extract the thresholds. A closely related discussion applies to the $R^4$ terms. The technical details needed to clarify the above issues are under investigation.

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