Numerical Simulation of Heat-like Models with Variable Coefficients by the Variational Iteration Method

A Sadighi, D D Ganji, M Gorji, N Tolou
Department of Mechanical Engineering, University of Mazandaran, Babol, Iran
E-mail: ddc_davood@yahoo.com

Abstract. In this study, the variational iteration method (VIM) is used to solve heat-like models with variable coefficients. These initial boundary value problems (IBVP) have significant importance in applied science and many fields of engineering. The exact solutions obtained by VIM are compared with those of Adomian’s decomposition method (ADM). The results show that although the obtained solutions are in excellent agreement, VIM is more convenient and easy-to-use than ADM. In this paper, this powerful method is introduced to overcome the difficulties arising in calculating Adomian’s polynomials.

1. Introduction
Initial boundary value problems (IBVP) with variable coefficients -such as heat-like models- can describe many physical problems in different field of science and engineering. These linear and nonlinear models play important roles in applied science, so finding their analytical solutions has fundamental significance in various field of science and engineering. Moreover, the method of separation of variables is not always usable to these problems. In this paper, a kind of analytical technique for linear and nonlinear problems called VIM is applied to solve heat-like models with variable coefficients. The effectiveness and accuracy of the method is revealed by obtaining the exact solutions of the models. VIM is to construct correction functionals using general Lagrange multipliers identified optimally via the variational theory, and the initial approximations can be freely chosen with unknown constants. This method was first proposed by He [1-6] and was successfully applied to a wide range of mathematical, physical and engineering problems by many authors [7-17].

Here, we consider a heat-like equation with variable coefficients described by a three-dimensional IBVP of the form [18]:

\[ u_t = f(x, y, z)u_{xx} + g(x, y, z)u_{yy} + h(x, y, z)u_{zz} \]  
\[ 0 < x < a, \quad 0 < y < b, \quad 0 < z < c, \quad t > 0 \]  

subject to the Neumann boundary conditions

\[ u_x(0, y, z, t) = f_1(y, z, t), \quad u_x(a, y, z, t) = f_2(y, z, t) \]  

(2a)

1 To whom any correspondence should be addressed
$u_x(x,0,z,t) = g_1(x,z,t), \quad u_y(x,0,z,t) = g_2(x,z,t)$ \hspace{1cm} (2b)

$u_z(x,y,0,t) = h_1(x,y,t), \quad u_{zz}(x,y,c,t) = h_2(x,y,t)$ \hspace{1cm} (2c)

and the initial condition of

$u(x,y,z,0) = \Phi(x,y,z)$ \hspace{1cm} (3)

2. Basic idea of He’s variational iteration method

To clarify the basic ideas of VIM, we consider the following differential equation:

$$Lu + Nu = g(t)$$ \hspace{1cm} (4)

where $L$ is a linear operator, $N$ a nonlinear operator and $g(t)$ an inhomogeneous term.

According to VIM, we can write down a correction functional as follows

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda' (\xi) \left( Lu_n(\xi) + Nu_n(\xi) - g(\xi) \right) d\xi$$ \hspace{1cm} (5)

where $\lambda$ is a general Lagrangian multiplier [1-6] which can be identified optimally via the variational theory. The subscript $n$ indicates the $n$th approximation and $\tilde{u}_n$ is considered as a restricted variation [1-6], i.e. $\delta \tilde{u}_n = 0$.

3. Application of VIM to heat-like equations

In order to assess the advantages and accuracy of VIM for solving heat-like equations, we will consider the following examples.

3.1. One-dimensional IBVP (first example)

First, we consider the one-dimensional heat-like model [18]

$$u_t = \frac{1}{2} x^2 u_{xx} \quad 0 < x < 1, \quad t > 0,$$ \hspace{1cm} (6)

subject to the boundary conditions of:

$$u(0,t) = 0 \quad u(1,t) = e^t$$ \hspace{1cm} (7)

and the initial condition of:

$$u(x,0) = x^2$$ \hspace{1cm} (8)

In order to solve Eq. (6) using VIM, we construct a correction functional, as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda' (\xi) \left[ u_{xx} - \frac{1}{2} x^2 \tilde{u}_{xx} \right] d\xi$$ \hspace{1cm} (9)

where $\delta \tilde{u}_{xx}$ is considered as restricted variation. Its stationary conditions can be obtained as follows:

$$\lambda'(\tau) = 0$$ \hspace{1cm} (10a)

$$1 + \lambda(\tau)|_{\tau=t} = 0$$ \hspace{1cm} (10b)
The Lagrange multiplier can therefore be simply identified as:

\[ \lambda = -1 \]  

and the following iteration formula can be obtained:

\[ u_{n+1}(x,t) = u_n(x,t) - \frac{1}{2} \int_0^t \left( u_{xx} - \frac{1}{2} x^2 u_{xx} \right) d\tau \]  

We start with an initial approximation \( u_0(x, t) = u(x, 0) \) given by Eq. (8). Using the above variational formula (12), we can obtain the following result:

\[ u_1(x, t) = u_0(x, t) - \frac{1}{2} \int_0^t \left( u_{0xx} - \frac{1}{2} x^2 u_{0xx} \right) d\tau \]  

Substituting \( u_0(x, t) \) into Eq. (20) and after some simplifications, we have:

\[ u_1(x, t) = x^2 (1 + t) \]  

Continuing in this manner, we can obtain the other components as follows:

\[ u_2(x, t) = x^2 \left( 1 + t + \frac{t^2}{2!} \right) \]  

\[ u_3(x, t) = x^2 \left( 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} \right) \]  

\[ u_4(x, t) = x^2 \left( 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} \right) \]  

\[ u_5(x, t) = x^2 \left( 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} \right) \]  

and so on. In the same manner the rest of components of the iteration formula (12) can be obtained. Therefore, the exact solution of \( u(x, t) \) in a closed form is

\[ u(x, t) = x^2 e^t \]  

which is exactly the same as that obtained by ADM [18].

3.2. Two-dimensional IBVP (second example)

We next consider the two-dimensional heat-like IBVP [18]

\[ u_t = \frac{1}{2} \left( y^2 u_{xx} + x^2 u_{yy} \right) \quad x > 0, \ y < 1, \ t > 0, \]  

subject to the boundary conditions of:

\[ u_x(0, y, t) = 0 \quad u_x(1, y, t) = 2 \sinh t \]  

(21a)
\( u_y(x,0,t) = 0 \quad u_y(x,1,t) = 2 \cosh t \) \hfill (21b)

and the initial condition of:

\[ u(x,y,0) = y^2 \] \hfill (22)

In order to solve Eq. (20) using VIM, we construct a correction functional, as follows:

\[ u_{n+1}(x,y,t) = u_n(x,y,t) + \int_0^t \lambda \left( u_{n0} - \frac{1}{2} \left( y^2 \tilde{u}_{xx} + x^2 \tilde{u}_{yy} \right) \right) d\tau \] \hfill (23)

where \( \delta \tilde{u}_{xx} \) and \( \delta \tilde{u}_{yy} \) are considered as restricted variation. Its stationary conditions can be obtained as follows:

\[ \dot{\lambda}(\tau) = 0 \] \hfill (24a)

\[ 1 + \dot{\lambda}(\tau)|_{\tau=t} = 0 \] \hfill (24b)

The Lagrange multiplier can therefore be simply identified as:

\[ \lambda = -1 \] \hfill (25)

We start with an initial approximation \( u_0(x,y,t) = u(x,y,0) \) given by Eq. (22). Using the above variational formula (26), we can obtain the following result:

\[ u_1(x,y,t) = u_0(x,y,t) + \int_0^t \left( u_{00} - \frac{1}{2} \left( y^2 \tilde{u}_{xx} + x^2 \tilde{u}_{yy} \right) \right) d\tau \] \hfill (26)

And therefore we can obtain the components of iteration formula as follows:

\[ u_1(x,y,t) = y^2 + x^2 t \] \hfill (27)

\[ u_2(x,y,t) = y^2 \left( 1 + \frac{t^2}{2!} \right) + x^2 t \] \hfill (28)

\[ u_3(x,y,t) = y^2 \left( 1 + \frac{t^2}{2!} + \frac{t^4}{4!} \right) + x^2 \left( t + \frac{t^3}{3!} \right) \] \hfill (29)

\[ u_4(x,y,t) = y^2 \left( 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} \right) + x^2 \left( t + \frac{t^3}{3!} + \frac{t^5}{5!} \right) \] \hfill (30)

\[ u_5(x,y,t) = y^2 \left( 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} \right) + x^2 \left( \frac{t^3}{3!} + \frac{t^5}{5!} \right) \] \hfill (31)

and so on. In the same manner the rest of components of the iteration formula (26) can be obtained.

Therefore, the exact solution of \( u(x,y,t) \) in a closed form is

\[ u(x,y,t) = x^2 \sinh t + y^2 \cosh t \] \hfill (32)
which is exactly the same as that obtained by ADM [18].

3.3. Three-dimensional IBVP (third example)
Consider the three-dimensional heat-like IBVP [18]

\[ u_t = x^4 y^4 z^4 + \frac{1}{36} \left( x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz} \right) \quad x > 0, \ y < 1, \ z < 1, \ t > 0, \]  

(33)

subject to the boundary conditions of:

\[ u(0, y, z, t) = 0 \quad u(1, y, z, t) = y^4 z^4 \left( e^t - 1 \right) \]  

(34a)

\[ u(x, 0, z, t) = 0 \quad u(x, 1, z, t) = x^4 z^4 \left( e^t - 1 \right) \]  

(34b)

\[ u(x, y, 0, t) = 0 \quad u(x, y, 1, t) = x^4 y^4 \left( e^t - 1 \right) \]  

(34c)

and the initial condition of:

\[ u(x, y, z, 0) = 0 \]  

(35)

To solve Eq. (33) using VIM, we construct a correction functional which reads

\[ u_{n+1}(x, y, z, t) = u_n(x, y, z, t) + \int_0^t \left\{ \lambda \left[ u_{nt} - x^4 y^4 z^4 - \frac{1}{36} \left( x^2 \delta_{u_{xx}} + y^2 \delta_{u_{yy}} + z^2 \delta_{u_{zz}} \right) \right] \right\} d\tau \]  

(36)

where \( \delta_{u_{xx}}, \delta_{u_{yy}} \) and \( \delta_{u_{zz}} \) are considered as restricted variation. Its stationary conditions can be obtained as follows:

\[ \lambda'(\tau) = 0 \]  

(37a)

\[ 1 + \lambda(\tau) \bigg|_{\tau=0} = 0 \]  

(37b)

which gives the Lagrange multiplier as \( \lambda = -1 \). Now we start with an initial approximation as follows:

\[ u_0(x, y, z, t) = x^4 y^4 z^4 t \]  

(38)

With the initial approximation given by Eq. (38) and using the above variational formula (36), we can obtain:

\[ u_1(x, y, z, t) = x^4 y^4 z^4 \left( t + \frac{t^2}{2!} \right) \]  

(37)

\[ u_2(x, y, z, t) = x^4 y^4 z^4 \left( t + \frac{t^2}{2!} + \frac{t^3}{3!} \right) \]  

(38)

\[ u_3(x, y, z, t) = x^4 y^4 z^4 \left( t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} \right) \]  

(39)
\[ u_4(x, y, z, t) = x^4 y^4 z^4 \left( t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} \right) \]

(40)

\[ u_5(x, y, z, t) = x^4 y^4 z^4 \left( t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \frac{t^6}{6!} \right) \]

(41)

and so on. In the same manner the rest of components of the iteration formula can be obtained. Therefore, the exact solution of \( u(x, y, z, t) \) in a closed form is

\[ u(x, y, z, t) = x^4 y^4 z^4 (e^t - 1) \]

(42)

which is exactly the same as that obtained by ADM [18].

4. Conclusion

In this paper, VIM has been successfully used to find the exact solution of Heat-like models which has fundamental importance in different field of engineering and applied science. Many of the results attained in this paper confirm the idea that VIM is powerful mathematical tool for solving different kinds of practical problems, having wide applications in engineering. The methods also can be introduced to overcome the difficulties arising in calculating Adomian polynomials.

References

[1] He J H, 1999 International Journal of Non-linear Mechanics, 34, 699

[2] He J H, 1998 Computer Methods in Applied Mechanics and Engineering, 167, 57

[3] He J H, 1998 Computer Methods in Applied Mechanics and Engineering, 167, 69

[4] He J H, 2000 Applied Mathematics and Computation, 114, 115

[5] He J H, Wu X H, 2006 Chaos Solitons & Fractals, 29, 108

[6] He J H, 2006 International Journal of Modern Physics B, 20 (10), 1141

[7] Khaleghi H, Ganji D D, Sadighi A., 2007 Numerical heat transfer, Part A, 52, 25

[8] Ganji D D, Sadighi A, 2007 Journal of computational and applied mathematics, 207, 24

[9] Sadighi A, Ganji D D, 2006 Exact solutions of nonlinear diffusion equations by variational iteration method, Computers and mathematics with applications, In press,

[10] Sweilam N H, Khader M M, 2007 Chaos, Solitons and Fractals, 32 (1), 145

[11] Wazwaz A M, 2006 The variational iteration method for solving two forms of Blasius equations on a half-infinite domain, Applied Mathematics and Computation, In press,

[12] Rafei M, Ganji D D, Daniali H, Pashayi H, 2007The variational iteration method for nonlinear oscillator with discontinuities, Journal of Sound and Vibration, In press,

[13] Yusufoglu E, 2007 International Journal of Nonlinear Science and Numerical Simulation, 8 (2), 153

[14] Tari H, Ganji D D, Rostamian M, 2007 International Journal of Nonlinear Science and Numerical Simulation, 8 (2) 230

[15] Bildik N, Konuralp A, 2006 International Journal of Nonlinear Science and Numerical Simulation, 7 (1), 65

[16] Odibat Z M, Momani S, 2006 International Journal of Nonlinear Science and Numerical Simulation, 7 (1), 27

[17] Yao H, Zou Y, Zhang G J, 2006 International Journal of Nonlinear Science and Numerical Simulation, 7 (4), 423

[18] Wazwaz A M, Gorguis A, 2004 Applied mathematics and computation, 19, 15