ON THE $V$-FILTRATION OF $\mathcal{D}$-MODULES

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Abstract. In this mostly expository note we give a down-to-earth introduction to the $V$-filtration of M. Kashiwara and B. Malgrange on $\mathcal{D}$-modules. We survey some applications to generalized Bernstein-Sato polynomials, multiplier ideals, and monodromy of vanishing cycles.

The $V$ filtration on $\mathcal{D}$-modules was introduced by M. Kashiwara and B. Malgrange to construct vanishing cycles in the category of (regular holonomic) $\mathcal{D}$-modules. Our aim is to give a down-to-earth introduction to this notion and describe some applications. The first application is to the generalized Bernstein-Sato polynomials introduced in [3]. Following G. Lyubeznik, we extend a finiteness result on the set of these polynomials. Then we describe applications to multiplier ideals ([4], [5]) and to monodromy of vanishing cycles and Hodge spectrum ([2], [4]).

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1. Basics

In this section we introduce the filtration $V$ and prove a few consequences assuming its existence. For a complete account on the $V$ filtration consult [12], [5], [8], [15].

Let $X$ be a smooth complex variety. The sheaf $\mathcal{D}_X$ of algebraic differential operators on $X$ is generated locally by multiplication by functions and by the tangent vector fields. If $X = \mathbb{A}^n$ is the affine $n$-space, then $\mathcal{D}_X$ is the Weyl algebra

$$A_n(\mathbb{C}) = \mathbb{C}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n],$$

where $\partial_i = \partial/\partial_{x_i}$ and $\partial_i x_j - x_j \partial_i = \delta_{i,j}$. In these notes we will consider only left $\mathcal{D}_X$-modules, the most important for our applications being $\mathcal{O}_X$. We will frequently work locally without specifically assuming that $X$ is affine.
The $V$-filtration of M. Kashiwara and B. Malgrange on $\mathcal{D}_X$-modules is defined with respect to some closed subvariety $Z \subset X$.

**Case $Z \subset X$ smooth.** First we consider smooth closed subvarieties $Z \subset X$. Let $\mathcal{I} \subset \mathcal{O}_X$ denote the ideal of $Z$. In local coordinates, write
\[ X = \{(x, t)\}, \quad Z = \{x\} = \{t = 0\}, \]
with $x = x_1, \ldots, x_n$, and $t = t_1, \ldots, t_r$. Then
\[ \mathcal{D}_X = \mathbb{C}[x, t, \partial_x, \partial_t]. \]

The $V$-filtration on $\mathcal{D}_X$ is defined by
\[ V^j \mathcal{D}_X = \{ P \in \mathcal{D}_X \mid P I_i \subset I_i + j \text{ for all } i \in \mathbb{Z} \}, \]
with $j \in \mathbb{Z}$ and $\mathcal{T}^i = \mathcal{O}_X$ for $i \leq 0$. Locally,
\[ V^j \mathcal{D}_X = \sum_{|\beta| - |\gamma| \geq j} h_{\alpha, \beta, \gamma}(x) \partial_x^\alpha t^\beta \partial_t^\gamma. \]

Here we use vectorial indices for monomials, and $|\beta| = \sum \beta_i$. A computation with local coordinates shows:

1. $V^{j_1} \mathcal{D}_X \cdot V^{j_2} \mathcal{D}_X \subset V^{j_1 + j_2} \mathcal{D}_X$, with equality if $j_1, j_2 \geq 0$;
2. $V^j \mathcal{D}_X = \mathcal{T}^j \cdot V^0 \mathcal{D}_X \cdot \mathcal{D}_{X, -j} = \mathcal{D}_{X, -j} \cdot V^0 \mathcal{D}_X \cdot \mathcal{T}^j$, where $\mathcal{D}_{X, j} \subset \mathcal{D}_X$ are the operators of order $\leq j$, and $\mathcal{T}^j = \mathcal{D}_{X, j} = \mathcal{O}_X$ for $j \leq 0$.

**Definition 1.1.** The filtration $V$ along $Z$ on a coherent left $\mathcal{D}_X$-module $M$ is an exhaustive decreasing filtration of coherent $V^0 \mathcal{D}_X$-submodules $V^\alpha := V^\alpha M$, such that:

1. $\{V^\alpha\}_\alpha$ is indexed left-continuously and discretely by rational numbers, i.e. $V^\alpha = \cap_{\beta < \alpha} V^\beta$, every interval contains only finitely many $\alpha$ with $\text{Gr}_V^\alpha \neq 0$, and these $\alpha$ must be rational. Here, $\text{Gr}_V^\alpha = V^\alpha / V^{\alpha+1}$, where $V^{\alpha+1} = \cup_{\beta > \alpha} V^\beta$.
2. $t_j V^\alpha \subset V^{\alpha+1}$, and $\partial_j V^\alpha \subset V^{\alpha-1}$ for all $\alpha \in \mathbb{Q}$, i.e. $(V^i \mathcal{D}_X)(V^\alpha M) \subset V^{\alpha+i} M$;
3. $\sum_j t_j V^\alpha = V^{\alpha+1}$ for $\alpha \gg 0$;
4. the action of $\sum_j \partial_j t_j - \alpha$ on $\text{Gr}_V^\alpha$ is nilpotent on $X$.

All conditions are independent of the choice of local coordinates.

**Theorem 1.2.** (M. Kashiwara, B. Malgrange) The filtration $V$ along $Z$ exists if $M$ is regular holonomic and quasi-unipotent.

It is beyond our scope to introduce the theory of holonomic systems of differential operators with regular singularities (see [1, 6]). It suffices to say that all the $\mathcal{D}$-modules considered in the applications are regular holonomic and quasi-unipotent.

**Proposition 1.3.** The $V$-filtration along $Z$ is unique.
Proof. Let \( \tilde{V} \) be another filtration on \( M \) satisfying Definition 1.1. By symmetry it suffices to show that \( V^\alpha \subset \tilde{V}^\alpha \) for every \( \alpha \). Suppose that \( \alpha \neq \beta \) and consider

\[
V^\alpha \cap \tilde{V}^\beta / (V^{>\alpha} \cap \tilde{V}^\beta) + (V^\alpha \cap \tilde{V}^{>\beta}).
\]

Since both filtrations satisfy 1.1-(iv), it follows that both \( (\sum_j \partial_1 t_j - \alpha) \) and \( (\sum_j \partial_j t_j - \beta) \) are nilpotent on this module. Hence the module is zero.

We show now that for every \( \alpha \) we have

\[
(1) \quad V^\alpha \subset V^{>\alpha} + \tilde{V}^\alpha.
\]

Fix \( w \in V^\alpha \). By exhaustion, there is \( \beta \ll 0 \) (in particular \( \beta < \alpha \)) such that \( w \in \tilde{V}^\beta \). By what we have already proved, we may write \( w = w_1 + w_2 \), with \( w_1 \in V^{>\alpha} \) and \( w_2 \in V^\alpha \cap \tilde{V}^{>\beta} \). If we replace \( w \) by \( w_2 \), then the class in \( V^\alpha / V^{>\alpha} \) remains unchanged, but we may choose a larger \( \beta \). We can repeat the process as long as \( \beta < \alpha \). Since the \( \tilde{V} \)-filtration is discrete, we can repeat the process until we have \( \beta \geq \alpha \).

Hence the class of \( w \) in \( V^\alpha / V^{>\alpha} \) can be represented by an element in \( \tilde{V}^\alpha \), and we get (1).

Since the \( V \)-filtration is discrete, a repeated application of (1) shows that for every \( \beta \geq \alpha \) we have \( V^\alpha \subset V^\beta + \tilde{V}^\alpha \). We deduce from 1.1-(iii) that if we fix \( \beta \gg 0 \), then

\[
(2) \quad V^\alpha \subset I^q \cdot V^\beta + \tilde{V}^\alpha
\]

for big enough \( q \), where \( I \subset \mathcal{O}_X \) is the ideal of \( Z \). By coherence, \( V^\beta = \sum V^\beta D_X \cdot u_i \) for finitely many \( u_i \). By exhaustion, there exists some \( \gamma \in \mathbb{Z} \) such that \( \tilde{V}^\gamma \) contains the \( u_i \), hence also \( V^\beta \). By 1.1-(ii), for \( q \) with \( q + \gamma \geq \alpha \) we have \( I^q \tilde{V}^\gamma \subset \tilde{V}^\alpha \). Thus \( I^q V^\beta \subset \tilde{V}^\alpha \). Hence by (2) we have \( V^\alpha \subset \tilde{V}^\alpha \).

\( \square \)

Case \( Z \subset X \) arbitrary. Now let \( X \) be a smooth complex variety and \( Z \neq X \) a closed subscheme. Suppose \( f_1, \ldots, f_r \in \mathcal{O}_X \) generate the ideal \( I \subset \mathcal{O}_X \) of \( Z \). Let \( i : X \to X \times \mathbb{A}^r = Y \) be the embedding \( x \mapsto (x, f_1(x), \ldots, f_r(x)) \). Let \( t_j : Y \to \mathbb{A}^1 \) be the projection with \( t_j \circ i = f_j \).

Let \( N \) be a \( \mathcal{D}_X \)-module. Let \( M = i_* N \), where \( i_* \) is the direct image for left \( \mathcal{D} \)-modules. Working out the definition of the direct image (e.g. [1]), one gets \( M = N \otimes \mathbb{C}[\partial_1, \ldots, \partial_r] \) with the left \( \mathcal{D}_Y \)-action given as follows. Let \( x_1, \ldots, x_n \) be local coordinates on \( X \). For \( g \in \mathcal{O}_X, m \in N, \) and \( \partial^m_i = \partial_1^m \ldots \partial_r^m \),
Suppose \( g(m \otimes \partial_i^r) = gm \otimes \partial_i^r \), \( \partial_{x_i}(m \otimes \partial_i^r) = \partial_{x_i}m \otimes \partial_i^r - \sum_j \frac{\partial f_j}{\partial x_i} m \otimes \partial_j \partial_i^r \),

\[ \partial_j(m \otimes \partial_i^r) = m \otimes \partial_j \partial_i^r, \quad t_j(m \otimes \partial_i^r) = f jm \otimes \partial_i^r - \nu_j m \otimes (\partial_i^r)_j, \]

where \((\partial_i^r)_j\) is obtained from \( \partial_i^r \) by replacing \( \nu_j \) with \( \nu_j - 1 \). If \( N \) satisfies the requirements of Theorem 1.2, then also \( M \) does.

**Definition 1.4.** The \( V \)-filtration along \( Z \) on \( N(= N \otimes 1) \) is defined by \( V^\alpha N = (N \otimes 1) \cap V^\alpha M \), for \( \alpha \in \mathbb{Q} \) and \( V \) on \( M \) taken along \( X \times \{0\} \).

**Proposition 1.5.** The definition above depends on the ideal \( I \) of \( Z \) in \( X \) and not on the particular generators chosen.

**Proof.** (cf. [3]-2.7) Suppose \( g_1, \ldots, g_r \in \mathcal{O}_X \) also generate \( I \), with \( g_j = \sum a_{ij} f_i \), \( a_{ij} \in \mathcal{O}_X \). Let \( i' : Y = X \times \mathbb{A}^r \to Y' = X \times \mathbb{A}^r \times \mathbb{A}^r \) be the embedding sending \((x,t)\) to \((x,t,t')\), where \( t'_j = \sum a_{ij}(x)t_i \), \( j = 1, \ldots, r' \). The crucial fact here is that the image of \( X \times \{0\} \) is \( X \times \{0\} \times \{0\} \). Working locally, we can assume that \( x, t, u \) is a local coordinate system on \( Y' \) such that \( Y = \{u = 0\} \), \( X = \{t = u = 0\} \). Hence \( M' = i'_* M \) can be written as \( M \otimes \mathbb{C}[\partial_{u_1}, \ldots, \partial_{u_r}] \) with left \( \mathcal{D}_{Y'^r} \)-action as above. Note that some simplifications occur: \( \partial_{x_i}(m \otimes \partial_i^r) = \partial_{x_i}m \otimes \partial_i^r \), \( \partial_{t_i}(m \otimes \partial_i^r) = \partial_{t_i}m \otimes \partial_i^r \), and \( u_j(m \otimes \partial_i^r) = -\nu_j m \otimes (\partial_i^r)_j \), where \( m \in M \), \( \partial_i^r = \partial_{u_1} \ldots \partial_{u_r} \), and \((\partial_i^r)_j\) is obtained from \( \partial_i^r \) by replacing \( \nu_j \) with \( \nu_j - 1 \).

The claim follows if we show that

\[ V^\alpha M' = \sum_{\nu \in N^r} V^{\alpha+|\nu|} M \otimes \mathbb{C}[\partial_i^r] \]

is the \( V \)-filtration on \( M' \) along \( X \times \{0\} \times \{0\} \).

Let us check the axioms for the \( V \)-filtration. In local coordinates, \( V^0 \mathcal{D}_{Y'^r} \) is generated over \( \mathcal{O}_{Y'} \) by the \( \partial_{x_i} \), and the \( v \partial_w \) with \( v, w \in \{t_1, \ldots, t_r, u_1, \ldots, u_r\} \). From definition, these actions are well-defined on \( V^\alpha M' \). To show that \( V^\alpha M' \) is coherent over \( V^0 \mathcal{D}_{Y'^r} \), it is enough to show that \( V^\alpha M' \) is locally finitely generated since \( V^0 \mathcal{D}_{Y'^r} \) is coherent. Since \( V^\alpha M \) is locally finitely generated over \( V^0 \mathcal{D}_Y \), we have that for \( c \gg 0 \), \( \sum_{|\nu| \leq c} V^{\alpha+|\nu|} M \otimes \mathbb{C}[\partial_i^r] \) is locally finitely generated. Also for \( c \gg 0 \), \( V^{\alpha+c+1} M = \sum_j t_j V^\alpha M \) by the axiom (iii) of \( 1.1 \). Therefore the rest of \( V^\alpha M' \) is recovered from \( \sum_{|\nu| \leq c} \) through the action of the \( t_i \partial_{u_j} \), hence \( V^\alpha M' \) is finitely generated.

The axioms (ii) and (iii) of \( 1.1 \) follow from the definition of \( V^\alpha M' \), the simplifications noted above in the \( \mathcal{D}_{Y'^r} \)-action on \( M' \), and the same axioms applied to \( V^\alpha M \).
The last property to show is the nilpotency of \( s - \alpha \) on \( \text{Gr}_V^\alpha M' \), where \( s = \sum_i \partial_i t_i + \sum_j \partial_u u_j \). Let \( m \otimes \partial'_u \in V^\alpha M' \) with \( m \in V^{\alpha+|\nu|} \). Then

\[
(s - \alpha)(m \otimes \partial'_u) = \left( \sum_i \partial_i t_i - |\nu| - \alpha \right) m \otimes \partial'_u.
\]

Hence \( (s - \alpha)^k(m \otimes \partial'_u) \in V^{\alpha+1}M' \) if \( k \) is the nilpotency order of \( (\sum_i \partial_i t_i - (\alpha + |\nu|)) \) on \( \text{Gr}_V^{\alpha+|\nu|} M' \). \( \Box \)

Examples will be provided in Section 3.

2. Bernstein-Sato polynomials

The \( V \)-filtration can be applied to show existence of quite general Bernstein-Sato polynomials, [3]. See [6] for an account of the classical version of these polynomials. Following G. Lyubeznik [11], we prove a finiteness result on the set of all polynomials that are Bernstein-Sato polynomials in a sense we make precise later.

We keep the notation from the previous section. Suppose first that \( Z \subset X \) is a smooth closed subvariety. To keep this article as concise as possible we take Theorem 1.2 for granted. Then the quickest way to proceed is by means of the following technical tool.

**Definition 2.1.** Let \( M \) be a coherent left \( D_X \)-module. For \( u \in M \), the Bernstein-Sato polynomial \( b_u(s) \) of \( u \) is the monic minimal polynomial of the action of \( s = -\sum_j \partial_j t_j \) on \( V^0D_Xu/V^1D_Xu \).

We suppressed from the notation the fact that \( b_u(s) \) also depends on \( Z \). Then we can make explicit the \( V \)-filtration as follows.

**Proposition 2.2.** (C. Sabbah [15]) If the \( V \)-filtration along \( Z \) exists on \( M \), then \( b_u(s) \) exists, it is non-zero for all \( u \in M \), and has rational coefficients. Moreover

\[
V^\alpha M = \{ u \in M \mid \alpha \leq c \text{ if } b_u(-c) = 0 \}.
\]

**Proof.** Suppose first that \( u \in V^\alpha M \). Recall that \( \sum_j \partial_j t_j - \beta \) is nilpotent on \( V^\beta/V^{>\beta} \) and \( V \) is indexed discretely. Then, for a given \( \beta \) there is a polynomial \( b(s) \) depending on \( \beta \), having all roots \( \leq -\alpha \) (and rational), and such that \( b(-\sum_j \partial_j t_j) \cdot u \in V^\beta \). Hence it is enough to show that there is \( \beta \) such that \( V^\beta \cap V^0D_Xu \subset V^1D_Xu \).

Let \( A = \bigoplus_{i \geq 0} V^iD_X\tau^{-i} \) and define \( F_k(A) = \bigoplus_{i \geq 0} (V^iD_X \cap D_{X,k})\tau^{-i} \). Then by [2.3] \( A \) is a noetherian ring. Now \( \bigoplus_{i \geq 0} V^iM \) is coherent over \( A \) because by axiom (iii) of [1.1] there exists \( i_0 \) such that \( V^iM \) is recovered from \( V^{i_0}M \) if \( i \geq i_0 \). Denote by \( N \) the \( V^0D_X \)-submodule \( V^0D_Xu \),
and let $U^i = V^i \cap N$ for $i \geq 0$. Then $\bigoplus_{i \geq 0} U^i N$ is also coherent over $A$ since $A$ is noetherian. It follows that $\bigoplus_{i \geq 0} \text{Gr}_i^U N$ is coherent over $\bigoplus_{i \geq 0} \text{Gr}_i^U \mathcal{D}_X$, in particular locally finitely generated. If $i$ is big compared with the degrees of local generators, we see that $U^i N \subset V^1 \mathcal{D}_X u$.

Conversely, fix an element $u \in M$ and suppose that $\alpha \leq c$ whenever $b_u(-c) = 0$. Let $\alpha_u = \max\{\beta \mid u \in V^\beta\}$. We need to show that $\alpha \leq \alpha_u$. It is enough to show that $b_u(-\alpha_u) = 0$. For $\beta \neq \alpha_u$, $(\sum_j \partial_j t_j - \beta)$ is invertible on $V^\alpha_u/V^{\alpha_u}$. But $b_u(-\sum_j \partial_j t_j)u \in V^{\alpha_u}$. Hence we must have $b_u(-\alpha_u) = 0$.

Lemma 2.3. (3-A.29.) Let $A$ be a be a filtered ring (sheaf on $X$). Assume that $F_0(A)$ and $\text{Gr}_A^F$ are noetherian rings, and that $\text{Gr}_k^F(A)$ are (locally) finitely generated $F_0(A)$-modules for all $k$. Then $A$ is noetherian.

Now let $Z \subset X$ be an arbitrary closed subset. Let $f_1, \ldots, f_r$ be generators of the ideal of $Z$, where $f_j \neq 0$ for any $j$. Then $\mathcal{D}_X$ acts naturally on $\mathcal{O}_X[\prod_i f_i^{-1}, s_1, \ldots, s_r] \prod_i f_i^{s_i}$, where the $s_i$ are independent variables. Define a $\mathcal{D}_X$-linear action of $t_i$ by $t_i(s_j) = s_j + 1$ if $i = j$, and $t_i(s_j) = s_j$ otherwise. Let $s_{ij} = s_{i}t_i^{-1}t_j$, and $s = \sum_i s_i$. We will see in Lemma 2.5 that under a well-defined isomorphism the $t_i$’s here correspond to the $t_i$’s introduced in the second part of section 1.

Definition 2.4. (3) The Bernstein-Sato polynomial $b_f(s)$ of $f := (f_1, \ldots, f_r)$ is defined to be the monic polynomial of the lowest degree in $s$ satisfying the relation

$$b_f(s) \prod_i f_i^{s_i} = \sum_j (P_j f_j \prod_i f_i^{s_i}),$$

where the $P_j$ belong to the ring generated by $\mathcal{D}_X$ and the $s_{ij}$. For $h \in \mathcal{O}_X$, define similarly $b_{f,h}(s)$ with $\prod_i f_i^{s_i}$ replaced by $\prod_i f_i^{s_i}h$.

Examples.

(i) $f = x^2 + y^3$. Then $b_f(s) = (s + 1)(s + 5/6)(s + 7/6)$ and $P = (\partial_y^3/27 + y \partial_x^2 \partial_y/6 + x \partial_x^3/8)$.

(ii) $f = (x_2x_3, x_1x_3, x_1x_2)$. Then $b_f(s) = (s + 3/2)(s + 2)^2$ and the $s_{ij}$ cannot be avoided by the operators $P_j$ in the above definition (see 3).

The polynomial $b_Z(s) := b_f(s - r)$ with $r = \text{codim}_X Z$ is shown in 3 to depend only on $Z$ and not on $f$. The existence of non-zero $b_{f,h}(s)$ follows from the following.
Lemma 2.5. With the notation as in 1.4, if \( M = i_* \mathcal{O}_X, \) \( u = h \otimes 1 \) with \( h \in \mathcal{O}_X, \) and the \( V \)-filtration is taken along \( X \times \{0\}, \) then \( b_u(s) = b_{f,h}(s). \)

Proof. It suffices to show that \( b_u(s) \) is the minimal polynomial of the action of \( s = \sum_j s_j \) on

\[
\mathcal{D}_X[s_{ij}] \prod_j f_j^{s_j}h / \sum_k \mathcal{D}_X[s_{ij}]f_k \prod_j f_j^{s_j}h,
\]

a quotient of submodules of \( \mathcal{O}_X[\prod_i f_i^{-1}, s_1, \ldots, s_r] \prod_i f_i^{s_i}h. \) We can check that \( \mathcal{D}_X[s_{ij}] \prod_j f_j^{s_j}h \) and \( \sum_k \mathcal{D}_X[s_{ij}]f_k \prod_j f_j^{s_j}h \) are isomorphic to \( V^0 \mathcal{D}_Y u \) and \( V^1 \mathcal{D}_Y u. \) The action of \( t_j \) is defined by \( s_j \mapsto s_j + 1, \prod_j f_j^{s_j}h \) corresponds to \( u, s_j \) corresponds to \( -\partial_{t_j} t_j, \) and \( s_{ij} = s_i t_i^{-1} t_j. \)

Hence it follows from 2.2 that \( b_{f,h}(s) \) are polynomials with rational coefficients. One is allowed to change the field of definition in the coefficients of the \( f_j \)'s.

Proposition 2.6. There exist non-zero Bernstein-Sato polynomials \( b_{f,h}(s) \) even if in 2.4 one replaces \( X, \mathcal{O}_X, \) and \( \mathcal{D}_X \) with \( \mathbb{A}_k^n, k[x_1, \ldots, x_n], \) and \( \mathbb{A}_n(k) \) (the Weyl algebra) respectively, for \( k \) a field of characteristic zero.

Proof. First, suppose that the coefficients of the \( f_j \)'s lie in a subfield \( K \) of \( \mathbb{C}. \) Then also the scalar coefficients of the \( P_j \)'s can be assumed to lie in \( K. \) Indeed, (3) implies, after equating coefficients of monomials in \( s_i \)'s and \( x_i \)'s, that certain \( K \)-linear relations (5) hold among the scalar coefficients of the \( P_j \)'s. Let \( L \) be the field generated by the coefficients of the \( P_j \)'s. Fix a basis \( S \) of \( L/K \) containing 1 and such that every scalar coefficient \( c \) which appears in a \( P_j \) can be written as a unique \( K \)-linear combination of a finite number of elements of \( S. \) Let \( c_1 \in K \) be the coefficient of 1 in \( c \) under this basis, and let \( P_{j,1} \) be the induced operator. Then the \( K \)-linear relations (5) hold with \( c_1 \) replacing \( c, \) and so (3) holds with \( P_{j,1} \) replacing \( P_j. \)

Now, going back to our proposition, the conclusion follows from the Lefschetz principle. Indeed, let \( K \) subfield of \( k \) generated over \( \mathbb{Q} \) by the coefficients of the \( f_j \)'s. Since \( \mathbb{C} \) has infinite transcendental dimension over \( \mathbb{Q}, \) \( K \) can be embedded into \( \mathbb{C}. \) Then the coefficients of the \( P_j \) are in \( K \subset k. \)

We extend a result of G. Lyubeznik [11] to the case of these more general Bernstein-Sato polynomials. The proof follows closely his proof.
Proposition 2.7. Fix $n$ and $d$ positive integers. The set of all polynomials which are of the form $b_f(s)$ for some $f = f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$ with $\deg f_i \leq d$ is finite even if $k$ is varying over all the fields of characteristic zero.

Proof. Let $N$ be the number of monomials in $x_1, \ldots, x_n$ of degree $\leq d$. Then the $f$'s are the closed $k$-rational points of $[A^N_k]^r$. Let $P = \sum_{|\alpha| \leq d} c_\alpha x^\alpha$ be the polynomial of $n$ variables of degree $d$ with undetermined coefficients. Then $B_k = k[\{c_\alpha\}]^r$ is the coordinate ring of $[A^N_k]^r$. Define

$$F_i = 1 \otimes \ldots \otimes P \otimes \ldots \otimes 1 \in B_Q[x_1, \ldots, x_n]$$

by placing $P$ in the $i$-th position. Here $\times$ and $\otimes$ mean over $Q$.

Let $Y$ be a reduced and irreducible closed subset of $[A^N_Q]^r$. Denote by $G = (G_i)_i$ the image of $F = (F_i)_i$ under the natural $Q$-algebra homomorphism $B_Q[x_1, \ldots, x_n] \to Q(Y)[x_1, \ldots, x_n]$, where $Q(Y)$ is the function field of $Y$. Let $Q[Y]$ be the coordinate ring of $Y$. Then, we have a functional equation

$$b_G(s) \prod_{i} G^s_i = \sum_{j} P_j G_j \prod_{i} G^s_i$$

with $s = \sum s_i$ and $P_j \in A_n(Q(Y))$ the common denominator. Denote by $U(c) \subset Y$ the subscheme whose coordinate ring is $Q[Y]_c$. By specializing (4), the $f$'s given by the closed $k$-rational points of $U(c) \times k$ have $b$-functions $b_f(s)$ dividing $b_G(s)$. Hence they are only finitely many such $b_f(s)$ even if $k$ varies.

We proceed now by induction on the dimension of $Y$ proving that the $k$-rational points of $Y \times k$ give only finitely many $b$-functions even if $k$ varies. For dimension zero, $c = 1$ and so $U(c) = Y$. In higher dimensions, $Y – U(c)$ is the union of reduced and irreducible closed subsets of smaller dimension. □

3. Multiplier ideals

The multiplier ideals introduced by A. Nadel encode the complexity of singularities via their resolutions. It turns out that they are essentially the same as the $V$ filtration on $O_X$.

Let $X$ be a smooth complex variety and $Z \neq X$ a closed subscheme. Let $\mu : X' \to X$ be a log resolution of $(X, Z)$. That is $\mu$ is proper birational, $X'$ is smooth, and $\text{Ex}(\mu) \cup \mu^{-1}Z$ is a divisor with simple normal crossings. Here $\text{Ex}(\mu)$ denotes the exceptional locus of $\mu$. Let $I \subset O_X$ denote the ideal of $Z$. Let $H$ be the effective divisor on $X'$ such that $\mu^{-1}(I) \cdot O_{X'} = O_{X'}(-H)$.
Definition 3.1. For $\alpha > 0$, the multiplier ideal of $(X, \alpha \cdot Z)$ is defined as

$$J(\alpha \cdot Z) = \mu_*(\omega_{X'/X} \otimes O_{X'}(-\mu \alpha \cdot H)).$$

Here $\omega_{X'/X} = \det \Omega^1_X \otimes \mu^*(\det \Omega^1_Y)^\vee$ is the sheaf of relative top-dimensional forms, and $\ll$ rounds down the coefficients of the irreducible divisors. One can extend this definition to a formal combination of closed subschemes $\sum_i \alpha_i \cdot Z_i$ by replacing $\alpha \cdot H$ with $\sum_i \alpha_i \cdot H_i$.

The original analytic definition of multiplier ideals is, locally, stratified locally conical divisors are determined in [13], and recursively [16]. Generally, stratified locally conical divisors are determined in [13], and recursively [16].

Theorem 3.2. ([3], [4]) For $\alpha > 0$, $V^\alpha O_X = J((\alpha - \epsilon) \cdot Z)$, where $0 < \epsilon \ll 1$ and the filtration $V$ of $O_X$ is taken along $Z$ as in [14].

The relation with Bernstein-Sato polynomials is then given by Proposition 2.4 and Lemma 2.5.
Corollary 3.3. For $\alpha > 0$,
\[ J(\alpha \cdot Z) = \{ h \in O_X \mid \alpha < c \text{ if } b_{f,h}(-c) = 0 \}, \]
where $f = f_1, \ldots, f_r$ is any set of generators of the ideal $I \subset O_X$ of $Z$.

In particular,
\[ \text{lc}(X, Z) = -(\text{biggest root of } b_f(s)), \]
since $b_f(s) = b_{f,1}(s)$ by definition [7], [10].

4. Monodromy of vanishing cycles

The initial scope of the $V$ filtration of M. Kashiwara and B. Malgrange was to construct vanishing cycles in the category of (regular holonomic) $\mathcal{D}$-modules.

Let $X$ be a smooth complex variety. Denote by $M_{rh}(\mathcal{D}_X)$ the abelian category of regular holonomic $\mathcal{D}_X$-modules, and by $D^b_{rh}(\mathcal{D}_X)$ the derived category of bounded complexes of $\mathcal{D}_X$-modules with regular holonomic cohomology. By A. Beilinson, this is equivalent with the bounded derived category of $M_{rh}(\mathcal{D}_X)$. Let $D^b_c(X)$ be the derived category of bounded complexes of sheaves (in the analytic topology of $X$) of $\mathbb{C}$-vector spaces with constructible cohomology. The Riemann-Hilbert correspondence generalizing the analogy between the $\mathcal{D}_X$-module $O_X$ and the constant sheaf $\mathbb{C}_X$ states (see [1]):

Theorem 4.1. (M. Kashiwara, Z. Mebkhout) Let $X$ be a smooth complex variety. There is a well-defined functor
\[ DR : D^b_{rh}(\mathcal{D}_X) \to D^b_c(X) \]
which is an equivalence of categories commuting with the usual six functors. $DR$ also defines an equivalence $M_{rh}(\mathcal{D}_X) \to \text{Perv}(X)$, where $\text{Perv}(X) \subset D^b_c(X)$ is the subcategory of perverse sheaves.

Let $f \in O_X$ be a regular function. The vanishing cycles functor $\phi_f$ on $D^b_c(X)$ and the monodromy action $T$ on it should then have a meaning only in terms of $\mathcal{D}$-modules since the shift $\phi_f[-1]$ restricts as a functor to $\text{Perv}(X)$. Let $M$ be a regular holonomic $\mathcal{D}_X$-module. Let $\tilde{M}$ be its direct image under the graph of $f$, as in section 2. If $M$ is also quasi-unipotent then there exists a $V$ filtration indexed by $Q$ on $\tilde{M}$ along $X \times \{0\}$. If $M$ is not quasi-unipotent, a close version of the following still holds:

Theorem 4.2. (M. Kashiwara, B. Malgrange) Let $\alpha \in [0,1)$ be a rational number. $\text{Gr}_V^\alpha \tilde{M}$ corresponds to the $\exp(-2\pi i\alpha)$-eigenspace of $\phi_f[-1](DR(M))$ with respect to the action of the semisimple part $T_s$ of the monodromy.
Combining this result with Theorem 3.2 and with additional structures such as mixed Hodge modules, one obtains a relation between multiplier ideals and the Hodge spectrum of hypersurface singularities. Let \( f : X \to \mathbb{A}^1 \) be a regular function. Recall that if \( i_x : x \mapsto f^{-1}(0) \) is a point, the Milnor fiber of \( f \) at \( x \) is is the Milnor fiber of the corresponding holomorphic germ \( f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0) \)

\[
M_{f,x} = \{ z \in \mathbb{C}^m \mid |z| < \epsilon \text{ and } f(z) = t \}
\]

for a fixed \( t \) with \( 0 < |t| < \epsilon \ll 1 \). Then

\[
H^i(i_x^* \phi_f \mathcal{O}_X) = \tilde{H}^i(M_{f,x}, \mathbb{C}),
\]

where \( \tilde{H} \) stands for reduced cohomology. These vector spaces are endowed with the monodromy action \( T \) and with mixed Hodge structures on which \( T \) acts as automorphism. Indeed, the mixed Hodge module theory of M. Saito on the left-hand side of (5) recovers the mixed Hodge structure of V. Navarro-Aznar from the right-hand side. As numerical invariants encoding the behaviour of the Hodge filtration \( F \) under \( T \) one has the generalized equivariant Euler characteristics

\[
n(i, \alpha) = \sum_j (-1)^j \dim \text{Gr}_F^j \tilde{H}^j(M_{f,x}, \mathbb{C})_{\alpha},
\]

where \( \alpha \in \mathbb{Q} \cap [0, 1) \), \( i \in \{0, \ldots, m-1\} \), and the subscript \( \alpha \) stands for the eigenspace of \( T \) with eigenvalue \( \exp(2\pi i \alpha) \). These invariants form the Hodge spectrum of \( f \) introduced by J. Steenbrink [17]. For \( \alpha \in (0, 1] \), let

\[
n_{\alpha,x}(f) = (-1)^{n-1} n(m-1, 1-\alpha),
\]

so that \( n_{\alpha,x}(f) \) describe the spectrum for the smallest piece of the Hodge filtration.

**Example.** If \( f = x^2 + y^3 \) and \( x = (0, 0) \in \mathbb{A}^2 \), then \( n_{\alpha,x}(f) \) is zero for \( \alpha \notin \{5/6, 7/6\} \), and is 1 otherwise.

On the other hand, for every jumping number \( \alpha \in (0, 1] \) of \((X, Z)\) where \( Z \) is the zero set of a regular function \( f \), define the inner jumping multiplicity at \( x \)

\[
n_{\alpha,x}(Z) = \dim \mathcal{J}((1-\epsilon)\alpha \cdot Z)/\mathcal{J}((1-\epsilon)\alpha \cdot Z + \delta \cdot x),
\]

where \( 0 < \epsilon \ll \delta \ll 1 \). It is proved in [2] that \( n_{\alpha,x}(Z) \) is finite and does not depend on \( \epsilon \) and \( \delta \). Let \( \tilde{\mathcal{O}}_X \) be the \( \mathcal{D} \)-module direct image of \( \mathcal{O}_X \) under the graph of \( f \). In connection with [5,2] it is crucial to observe that the smallest piece of the Hodge filtration on \( V^\alpha \tilde{\mathcal{O}}_X \) is exactly \( V^\alpha \mathcal{O}_X \). Then the above arguments lead to:
Theorem 4.3. \((2, 4)\) For \(\alpha \in (0, 1],\)
\[
    n_{\alpha,x}(f) = n_{\alpha,x}(Z).
\]

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