Symbolic Dynamics of Magnetic Bumps

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Abstract

For \( n \) convex magnetic bumps in the plane, whose boundary has a curvature somewhat smaller than the absolute value of the constant magnetic field inside the bump, we construct a complete symbolic dynamics of a classical particle moving with speed one.

1 Introduction and notation

The subject of chaotic scattering is mostly about scattering by obstacles and by potential bumps, see [Ga, Sm]. Here we consider the motion of a classical particle in the plane under the influence of a magnetic potential. In the case of motion in the plane, this has the form

\[
B := \sum_{\ell \in A} B_\ell \quad \text{with} \quad B_\ell := b_\ell \mathbb{1}_{\partial C_\ell} : \mathbb{R}^2 \to \mathbb{R},
\]

with the alphabet \( A := \{1, \ldots, n\} \), for mutually disjoint, convex and compact domains \( C_\ell \subseteq \mathbb{R}^2 \) with \( C^2 \) boundaries \( \partial C_\ell \), and field strengths \( b_\ell \in \mathbb{R} \setminus \{0\} \).

The Hamiltonian system \((P, \omega, H)\) with phase space \( P := T\mathbb{R}^2 \),

\[
H : P \to \mathbb{R} \quad , \quad H(q, v) = \frac{1}{2} \|v\|^2
\]

and (discontinuous) symplectic form

\[
\omega := dq_1 \wedge dv_1 + dq_2 \wedge dv_2 + B \ dq_1 \wedge dq_2
\]

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give rise to the Hamiltonian vector field $X : P \to TP$, $i_X \omega = dH$. The corresponding differential equation is

$$
\dot{q} = v, \quad \dot{v} = B(q) \mathbb{J} v, \quad \text{with } \mathbb{J} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
$$

The solution of the initial value problem for magnetic field $b \in \mathbb{R} \setminus \{0\}$ thus equals

$$
q(t) = q_0 + \frac{1}{b} \begin{pmatrix} \sin(bt) & \cos(bt) - 1 \\ 1 - \cos(bt) & \sin(bt) \end{pmatrix} v_0, \quad v(t) = \begin{pmatrix} \cos(bt) - \sin(bt) \\ \sin(bt) \cos(bt) \end{pmatrix} v_0.
$$

So the particle with energy $H = E$ has speed $\sqrt{2E}$. Outside the bumps it moves on a straight line, whereas inside the $\ell$th bump it moves on a segment of a circle of Larmor radius $\sqrt{2E}/|b_\ell|$. The sense of rotation is positive (counter-clockwise), if $b_\ell > 0$. The $t$–invariant center of the circle is

$$
c_\ell(v, q) := q + b_\ell^{-1} \mathbb{J} v. \quad (1.1)
$$

Without loss of generality we fix the energy to equal $E := 1/2$, so that we get unit speed on the energy surface

$$
\hat{\Sigma} := H^{-1}(E) \quad \text{with projection } \pi : \hat{\Sigma} \to \mathbb{R}^2, \ (q, v) \mapsto q.
$$

## 2 Single bumps

Consider for $\ell \in A$ the set $\hat{C}_\ell$ of points $c \in C_\ell$ whose minimal distance to the boundary $\partial C_\ell$ is larger than $|b_\ell|^{-1}$. This is a compact convex set, possibly void. We remove from $\hat{\Sigma}$ the sets $\Sigma_\ell := \{(v, q) \in \hat{\Sigma} \mid c_\ell(v, q) \in \hat{C}_\ell\}$. These compact sets are invariant under the $2\pi/|b_\ell|$–periodic Larmor flow for magnetic field $b_\ell$. At points $x \in \partial \Sigma_\ell$ corresponding to Larmor circles in the configuration plane that touch $\partial C_\ell$ in only one point, the boundary $\partial \Sigma_\ell$ is $C^2$. In particular this is the case if the curvature $\kappa_\ell \in C(\partial C_\ell, [0, \infty))$ of $\partial C_\ell$ is strictly smaller than $|b_\ell|$.

### 2.1 Definition

- We set $\kappa_\ell := \min_{x \in \partial C_\ell} \kappa_\ell(x)$ and $\kappa_\ell := \max_{x \in \partial C_\ell} \kappa_\ell(x)$ (so $0 \leq \kappa_\ell \leq \kappa_\ell > 0$).
- If $|b_\ell| < \kappa_\ell$ for all $\ell \in A$, the system has weak fields.
- If $|b_\ell| > \kappa_\ell$ for all $\ell \in A$, the system has strong fields.
- For $n \geq 2$ we set $d_\ell := \min_{m \in A \setminus \{\ell\}} \min_{c_\ell \in C_\ell, y \in C_m} \|x - y\|$ (so $d_\ell > 0$).
- If $|b_\ell| > 1/(d_\ell \alpha_{\min}) + 2\kappa_\ell$ for all $\ell \in A$, the system has very strong fields.\footnote{If $\hat{C}_\ell$ has positive area, its boundary is Lipschitz. However, e.g. for a stadion $C_\ell$ composed of two half-disks of radius $1/|b_\ell|$ and a rectangle, $\hat{C}_\ell$ is a line segment.}

\footnote{Here $\alpha_{\min}$ is a constant, related to the arrangement of the bumps in the plane, that will be discussed in more details in Section 4.}
2.2 Example (Single disks)
For \( n = 1 \) and a disk \( C \equiv C_1 \) of radius \( r > 0 \) the curvature of \( \partial C \) is constant and equals \( 1/r \). So the field \( B = b \mathbb{I}_C \) is weak iff \( |b| < 1/r \) and strong iff \( |b| > 1/r \).

Compare Figures 2.2 and 2.3 for dynamics near such a bump.

We will mainly consider the (very) strong fields cases.

The following lemma relates strong fields to some property of a single bump.

Our convention for the 'Jacobi equation' along the trajectory is to use the orthonormal basis \( e_1(t), e_2(t) \) of \( T_c(t)\mathbb{R}^2 \), given by \( e_1(t) := v(t) \) and \( e_2(t) := \mathbb{J}e_1(t). \) Then writing a vector field along \( c \) in the form \( t \mapsto I(t)e_1(t) + J(t)e_2(t) \), we obtain the linearized flow with

\[
J(t) = -\sin(bt)I(0) + \cos(bt)J(0) - (1 - \cos(bt))\dot{I}(0)/b + \sin(bt)\dot{J}(0)/b
\]

and

\[
\dot{J}(t) = -b\cos(bt)I(0) - b\sin(bt)J(0) - \sin(bt)\dot{I}(0) + \cos(bt)\dot{J}(0).
\]

2.3 Lemma

In the case of a strong field, along a trajectory \( c : I \rightarrow \mathbb{R}^2 \) entering the bump at \( c(0) \in \partial C_\ell \) and leaving it at time \( T > 0 \) \( (c(T) \in \partial C_\ell) \), the parallel incoming Jacobi field \( (J(0), \dot{J}(0)) = (1,0) \) has outgoing data \( J(T) < 0, \dot{J}(T) < 0 \).

Proof:
• Consider the unique disk \( D \) of radius \( r := 1/|b_\ell| \), whose boundary is tangent to \( \partial C_\ell \) at \( c(0) \) and which is entered by the trajectory at time \( t = 0 \). As the curvature \( |b_\ell| \) of its boundary is strictly larger than the curvature of \( \partial C_\ell \), we have \( D \subseteq C_\ell \), and \( c(0) \) is the only intersection between \( \partial D \) and \( \partial C_\ell \). We denote the center of \( D \) by \( z \in \mathbb{R}^2 \), see Figure 2.1 left.
• We claim that all trajectories \( d : I \rightarrow \mathbb{R}^2 \) entering \( D \) at time \( t = 0 \) with velocity \( \dot{d}(0) = \dot{c}(0) \) leave \( D \) at the same point \( f := z + \mathbb{J}\dot{c}(0)/b_\ell \): Clearly \( d(0) \) is a solution \( x \) of the equations

\[
\|x - z\|^2 = r^2 \quad \text{and} \quad \|x - d(0) - \mathbb{J}\dot{c}(0)/b_\ell\|^2 = r^2
\]

of \( \partial D \) and the Larmor circle \( \{1,1\} \). But the second intersection of these two circles equals \( f \).
• The union of the above orbit family is a Lagrangian manifold, and the vector in \( T_{c}\Sigma, x := (c(0), \dot{c}(0)) \) given Jacobi field data \( (J(0), \dot{J}(0)) \), is tangent to it. So the Jacobi field along \( c \) turns vertical at \( f = c(t_f) \), that is, \( J(t_f) = 0 \) and

\[\text{This is an abuse of language, since Hamilton's equation is not geodesic.}\]

\[\text{We assume w.l.o.g. that the component parallel to the direction } \dot{c}(0) \text{ vanishes.}\]
\[ \dot{J}(t_f) < 0. \] But \( f \), being a boundary point of \( D \) different from \( c(0) \), is an interior point of \( C_\ell \), so that \( t_f \in (0, T) \).

- Consider the line \( T \) in \( \mathbb{R}^2 \) that is tangent to \( \partial C_\ell \) at \( c(0) \). If we denote by \( \overline{D} \) the image of \( D \) under the reflection by the line through \( f \), perpendicular to \( T \) (see Figure 2.1 right), then the Larmor circle through \( c(0) \) and \( f \) intersects \( T \) for the second time at the unique point in \( T \cap \partial \overline{D} \). Again by reflection symmetry, at that point the Jacobi field along \( c \) is parallel.

- As \( C_\ell \) is strictly convex, it lies entirely on one side of \( T \). So by a comparison argument (see again Figure 2.1 right), \( J(T) < 0 \).

**2.4 Remark** If \( C_\ell \) is a disk, in the strong field case we conclude from Lemma 2.3, that for parallel incoming trajectories the envelope of the solution curves is a half-circle of radius \( r_\ell - 1/|b_\ell| \), see lower part of Figure 2.2.

**3 The degree for weak and strong fields**

Scattering by a bump is qualitatively different for weak and for strong force fields. To see this, consider oriented lines in configuration space \( \mathbb{R}^2 \). The set of these lines is naturally isomorphic to the cylinder \( TS^1 \cong S^1 \times \mathbb{R} \).

We assume that there is just one bump (so \( n = 1 \)), and consequently omit subindices. There is a problem in defining the flow on \( \Sigma \) in the case of glancing trajectories, that is, trajectories tangent to \( \partial C \) (consider the uppermost trajectory in Figure 2.2). Either we continue these incoming rays by just extending them
Figure 2.2: Dynamics for a disk $C$ and a strong field (with $b = -2r$)

Figure 2.3: Dynamics for a disk $C$ and a weak field (with $b = -r/2$)
to a straight line, or we extend them by a segment of the Larmor circle in the bump. This is a complete circle for a strong field $b$. So in that case we could attach the outgoing ray with the incoming direction.

None of these prescriptions leads to a continuous flow. However, it is important to notice that either prescription leads to a continuous map called

$$S : TS^1 \to TS^1,$$

sending incoming to outgoing oriented lines.$^5$

As shown in [Kn], see also [KK], such a map defines a topological index $\deg(C, b) \in \mathbb{Z}$. Using the bundle projection $\pi : TS^1 \to S^1$, it can be defined as follows. A family of incoming rays with the same direction is parameterized by the value of their angular momentum $L : \Sigma \to \mathbb{R}, \ (q, v) \mapsto \langle Jq, v \rangle$. This is obviously flow-invariant outside the bump. $S$ maps initial to final pairs, consisting of directions in $S^1$ and angular momenta in $\mathbb{R}$.

Given the initial direction $\varphi \in S^1$, the continuous map

$$T_{\varphi}S^1 \cong \mathbb{R} \to S^1 \ , \ \ell \to \pi \circ S(\ell, \varphi)$$

can be uniquely completed (using the Alexandrov compactification $\mathbb{R} \cup \{\infty\} \cong S^1$ of $\mathbb{R}$) to a continuous map $S^1 \to S^1$. The degree of that map is independent of $\varphi \in S^1$ and is called the scattering degree.

3.1 Lemma For a single bump $C$, the scattering degree $\deg(C, b)$ equals

- zero, for weak fields $b$,
- $\text{sign}(b)$, for strong fields $b$.

Proof: The total curvature of a trajectory equals the signed angular length of the segment of its Larmor circle. This is obviously zero if the trajectory does not intersect $C$. It is bounded away from $2\pi$ in absolute value for weak fields. For strong fields $b > 0$ it equals $+2\pi$ exactly for the glancing trajectories intersecting $\partial C$ with $C$ on their left hand side. It then decreases to zero when the angular momentum value of the incoming rays it decreased, until one arrives at a glancing trajectory with $C$ on its right hand side. For strong fields $b < 0$ the sides are interchanged. $\square$

$^5$For fields that are neither weak nor strong, it can happen that $S$ cannot be defined continuously.
4 A strictly invariant cone field

Assumption:

We assume that no three convex sets $C_\ell \subseteq \mathbb{R}^2$ lie on a line. For $n \geq 3$

$$\alpha_{\min} := \min_{k \neq \ell \neq m \neq k \in A} \min \left\{ \arccos \left( \frac{y-x}{\|y-x\|}, \frac{y-z}{\|y-z\|} \right) \mid x \in C_k, y \in C_\ell, z \in C_m \right\},$$

and for $n = 2$ we set $\alpha_{\min} := \pi/3$, say. As we assumed that no three bumps are intersected by a line, the angle $\alpha_{\min} \in (0, \pi/3]$ is positive. For a segment of a solution curve that intersects $C_k$, $C_\ell$ and $C_m$ in succession, the total curvature inside $C_\ell$ is bounded below by $\alpha_{\min}$. So the length of that sub-segment is at least $\alpha_{\min}/|b_m|$.

We now consider in the very strong force regime the set $\Lambda \subseteq \Sigma$ of initial conditions belonging to bounded orbits. We denote by

$$N(x) \in T_x \mathbb{R}^2 \quad (\ell \in A, x \in \partial C_\ell)$$

the unit outward normal vectors of $C_\ell$.

$\mathcal{H}^\pm := \bigcup_{\ell \in A} \mathcal{H}_\ell \subseteq \Sigma$ is the disjoint union of the Poincaré surfaces

$$\mathcal{H}^\pm_\ell := \{(v, q) \in \Sigma \mid q \in \partial C_\ell, \pm \langle v, N(q) \rangle > 0\}.$$ 

For arbitrary fields, the flow induces a Poincaré map $P^{(i)} : \mathcal{H}^- \to \mathcal{H}^+$ internal to the bumps, which is a diffeomorphism. On the other hand, there are maximal open subsets $V^\pm \subseteq \mathcal{H}^\pm$ so that the flow gives rise to a diffeomorphism $P^{(e)} : V^+ \to V^-$, external to the bumps. Setting $U^- := (P^{(i)})^{-1}(V^+)$, we obtain a diffeo

$$P := P^{(e)} \circ P^{(i)} : U^- \to V^-.$$

4.1 Lemma Assume that the fields $b_\ell$ obey the very strong fields inequalities

$$|b_\ell| > 1/(d_\ell \alpha_{\min}) + 2 \kappa_\ell \quad (\ell \in A). \quad (4.1)$$

Then the cone fields

$$C_\ell(x) := \{ \lambda^{(l)} e^{(l)} + \lambda^{(u)} e^{(u)} \mid \lambda^{(l)}, \lambda^{(u)} \in \mathbb{R}, \lambda^{(l)} \cdot \lambda^{(u)} > 0 \} \quad (x \in \mathcal{H}_\ell),$$

defined by the tangent vectors $e^{(l)}, e^{(u)} \in T \mathcal{H}_\ell$,

$$e^{(l)} := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e^{(u)} := \begin{pmatrix} d_\ell \\ 1 \end{pmatrix}$$

are strictly invariant under the linearized Poincaré map $TP$. 

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Proof:

- According to Lemma 2.3, the vector $e^{(l)} \in T_xU^-$ (with $x \in \mathcal{H}_x^-$) is mapped by $T_xP$ to a vector $T_yP(e^{(l)}) \in T_yV^-$ (with $y := P(x) \in \mathcal{H}_y^-$) of the form $(\frac{1}{1} \ 0 \ \frac{1}{J}) \ (\frac{0}{J})$ with $J, J \neq 0$, $d$ being the length of the trajectory segment between the points $\pi \circ P^{(i)}(x)$ of exit from $C_\ell$ resp. $\pi \circ P(x)$ of entrance in $C_m$. This shows that $e^{(l)}$ is contained in the cone field $C_\ell(x)$.

- We want to show a similar statement for the vector $e^{(u)} \in T_xU^-$, but we choose to work backwards in time. So we consider the vector $e^{(u)} = (\frac{d_m}{1}) \in T_yU^-$ and show that its preimage $T_y(P(e))^{-1}(e^{(l)})$ is not contained in $C_\ell(x)$. First of all, $T_y(P(e))^{-1}(e^{(l)}) = (\frac{d_n - d}{1})$ has non-positive first entry, since $T_y(P(e))^{-1} = (\frac{1}{1} \ -d)$ and $d \geq d_m$. For comparison we consider the vector $(\frac{0}{1}) \in T_x'$ with $x' := P^{(i)}(x)$ and show that its preimage $T_{x'}(P(e))^{-1}(\frac{0}{1})$ is not contained in $C_\ell(x)$.

We choose the disk $E \subseteq C_\ell$ of radius $1/\kappa_\ell$, whose boundary is tangent to $\partial C_\ell$ at $x'$. Likewise, $D \subseteq E$ is the disk of radius $1/|b|$, whose boundary is tangent to $\partial C_\ell$ (and $\partial E$) at $x'$; see Figure 4.1. Like in the proof of Lemma 2.3, at the second intersection $x''$ of $\partial D$ with the Larmor circle (gray in Figure 4.1), the family of Larmor solutions crossing at $x'$ has become parallel. We must show that by following the corresponding Jacobi field backwards from $x''$ to $x$, it is not contained in the cone field $C(x)$. Although we have no direct information about the length of the Larmor circle segment between $x''$ and $x$, we know (by definition of $\kappa_\ell$) that it is longer than the one between $x''$ and the intersection $x'''$ with $\partial E$.

So we must bound the length of that arc from below. The length of the arc between $x'$ and $x''$ equals $|b|\alpha$, $\alpha$ being the angle between the normal $N(x')$ and the center $L$ of the Larmor circle, seen from $x'$.

The Larmor angle between $x'$ and $x''$ equals $2 \arctan[r \sin(\alpha)/(1 + r \cos(\alpha))]$, with $r := \kappa_\ell/|b|$ being the ratio between the radii of $\partial D$ and $\partial E$. This identity follows when considering the line through $L$ and the center $e$ of $E$. That line bisects the Larmor angle. By elementary trigonometry the angle between that line and the one through $e$ and $x'$ equals $\arctan[r \sin(\alpha)/(1 + r \cos(\alpha))]$.

Thus the length of the Larmor arc between $x''$ and $x'''$ equals

$$f(\alpha) := \alpha - 2 \arctan[r \sin(\alpha)/(1 + r \cos(\alpha))],$$

multiplied by its radius $1/|b|$. As $f(0) = 0$, $f'(\alpha) = \frac{1-r^2}{r^2+2r\cos(\alpha)+1}$ with $f'(0) = \frac{1-r}{1+r} > 0$ and $f''(\alpha) = \frac{r(1-r^2)\sin(\alpha)}{(r^2+2r\cos(\alpha)+1)^2} \geq 0$, we can bound $f$ from below by

$$f(\alpha) \geq \frac{1-r}{1+r} \alpha = \frac{|b|\kappa_\ell}{|b|+\kappa_\ell} \alpha \quad (\alpha \in [0, \pi]). \quad (4.2)$$

As the Larmor arc belongs to a solution segment that intersects $C_k$, $C_\ell$ and $C_m$ in succession, the argument of $f$ is bounded below by $\alpha \geq \alpha_{\text{min}}$. 

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We have to check, whether then the inequality
\[ d^{-1}_\ell < \tan(|b_\ell| f(\alpha)) \]
holds true. By (4.2) we check the stronger inequality
\[ d^{-1}_\ell < \tan \left( |b_\ell| \frac{|\beta|-\pi_\ell}{|\beta|+\pi_\ell} \alpha \right). \]
In fact, we argue that even
\[ d^{-1}_\ell < |b_\ell| \frac{|\beta|-\pi_\ell}{|\beta|+\pi_\ell} \alpha. \]
The corresponding equation is quadratic in \( x := |b_\ell| \) and has the positive solution
\[ x = \frac{1}{2} \left( c + \pi_\ell + \sqrt{c^2 + 6c\pi_\ell + \pi_\ell^2} \right) \leq c + 2\pi_\ell \] with \( c := 1/(d_\ell \alpha_{\min}) \).

Figure 4.1: Proof of Lemma 4.1

5 Symbolic dynamics

We equip for the (discretely topologized) alphabet \( A = \{1, \ldots, n\} \) the sequence space
\[ \Xi_A := \{a : \mathbb{Z} \to A \mid \forall k \in \mathbb{Z} : a_{k+1} \neq a_k \} \]
with the product topology. Then the shift \( \sigma : \Xi_A \to \Xi_A, \sigma(a)_k = a_{k+1} \) is known to be a homeomorphism.
5.1 Theorem Let the very strong fields assumption (4.1) be valid for the \( n \geq 2 \) bumps. Then every bounded trajectory \( c : \mathbb{R} \to \Sigma \) intersects \( \mathcal{H} \) infinitely often. The set \( \Xi := \Lambda \cap \mathcal{H} \) is homeomorphic to \( \Xi_A \), under the homeomorphism \( \Phi : \Xi_A \to \Xi, a \mapsto \bigcap_{k \in \mathbb{Z}} \{ P^k(x) \mid x \in \mathcal{H}_{a_k} : P^k(x) \text{ is defined and in } \mathcal{H}_{a_k} \} \) to one-point sets. \( \Phi \) is a conjugacy for the Poincaré map \( P \) and the shift map \( \sigma \), i.e.,
\[
\Phi \circ \sigma = P \circ \Phi.
\]

Proof: • It is generally true that for a continuous, strictly invariant cone field there is at most one element in each set \( \Phi(a) \) of (5.1).
• Then we employ the technique of [Kn, Chapter 6] to show that the sets defining \( \Phi(a) \) are non-void. According to Lemma 3.1 we have degrees \( \deg(C_\ell, b_\ell) = \text{sign}(b_\ell) \in \{-1,1\} \) in the strong field case. Existence and non-vanishing of the degree is the condition of [Kn, Theorem 6.1].
• The conjugacy property follows, as \( P^k(P(x)) \in \mathcal{H}_{\sigma(a)_k} \) is equivalent to \( P^{k+1}(x) \in \mathcal{H}_{a_k+1} \) and to \( P^k(x) \in \mathcal{H}_{a_k} \) \( (k \in \mathbb{Z}) \). \( \square \)

5.2 Remarks

1. The result is quite different from the one in [KSS]. There, for each of the continuous radially symmetric bumps, one had assumed existence of an (unstable) periodic orbit. This then lead to arbitrarily large winding numbers. The symbolic dynamics of scattering then gave rise to a semi-conjugacy, see [KSS, Theorem 3.4].

2. Similarly, one may consider trapped orbits (with \( \lim_{t \to \pm \infty} \| q(t) \| = \infty \) but \( \limsup_{t \to \pm \infty} \| q(t) \| < \infty \) and scattering orbits (with \( \lim_{t \to + \infty} \| q(t) \| = \infty \) and \( \lim_{t \to - \infty} \| q(t) \| = \infty \)). There one uses half-infinite respectively finite symbol sequences. Prescribing initial and/or final directions of these orbits, one again obtains symbolic conjugacies. Then, however, one has to exclude directions that are given by oriented lines meeting two domains \( C_\ell, C_m \). \( \diamond \)

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\( ^6 \)identifying one-element sets with their elements
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