Short-Wave Approximation for Macroscopic Cosmology with Higgs Scalar Field

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Abstract—Based on the macroscopic equations of cosmological evolution obtained earlier by the author, a closed set of macroscopic Einstein equations in the short-wave approximation for perturbations of the scalar Higgs and gravitational fields has been obtained and examined. The resulting exact solutions of the macroscopic equations are determined by three microscopic parameters, depending on the spectrum of perturbations.

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I. INTRODUCTION

In [1], a closed set of equations was obtained that determines the macroscopic evolution of the Universe, consisting of locally fluctuating gravitational perturbations and perturbations of a classical scalar field with the Higgs interaction potential, taking into account both transversal and longitudinal field perturbations. The obtained macroscopic model consists of a set of ordinary linear homogeneous differential equations describing the evolution of microscopic perturbations of the gravitational and scalar fields against a macroscopic background, and a set of nonlinear ordinary inhomogeneous differential equations determined by macroscopic mean square values of the perturbations. In the same paper, a particular exact solution of this system of equations was obtained for the case of short-wave transversal perturbations, which describes a transition of the macroscopic Universe from the ultrarelativistic stage of expansion to the inflationary one.

The fundamental complexity of the obtained mathematical model lies in the fact that the macroscopic background relative to which perturbations evolve, according to the method of a self-consistent field, is determined by the same perturbations, therefore, the linear nature of the perturbations is misleading. In this regard, it is difficult to uncouple the resulting set of equations. This paper is devoted to derivation of a closed set of Einstein macroscopic equations based on the formulated model for the spatially flat Friedmann Universe in the short-wave approximation for local fluctuations of the scalar and gravitational fields, and to the solution of these equations.

Let us note that in [4–7] (2014–2016) and others, discussed was the back reaction of massless particle production on the background geometry of the early Universe. It should be mentioned that these papers were based on a semi-phenomenological approach in which particle production is described by a thermal spectrum with the temperature determined by curvature. These papers discussed the stability of the classical background metric with respect to back reaction, and different authors arrive at directly opposite conclusions. In particular, in [6] it is pointed out that the stability conclusions on the background metric that are based on a scheme that does not contain the averaging operation are therefore wrong. Let us also note that the cited papers did not take into account the back reaction of the perturbed metric on the particle production process, therefore, the models under study are not self-consistent and consider only a unilateral connection.

Thus one can state that, in quantum cosmology, the problem of quantum statistical description of the macroscopic early Universe¹ has not yet been studied on a sufficiently serious level. In this paper we will develop the classical theory of a macroscopic Universe. There is a certain semiclassical bridge that connects the classical statistical description of the early Universe with quantum theory. In the classical picture, random perturbations of the metric are determined by arbitrary initial conditions, while in quantum theory

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We would like to note note that such a description should be based on the density matrix, but the corresponding mathematical formalism for quantum gravity has not yet been developed.
these perturbations are determined by the instability of a vacuum state and have a deterministic nature, including quite a certain energy spectrum. It is this energy spectrum that can supplement the classical theory, eliminating its arbitrariness and thus playing the role of the above-mentioned connection between the quantum and classical theories.

Note that in the studies cited above on the back reaction of particle production processes on the background metric, the particle production factor is taken into account in the framework of the perturbation theory, therefore, it cannot lead to a significant change in the background. In this case, quantization is carried out using the classical scheme, against the background vacuum solution. In contrast to this approach, a self-consistent description of a macroscopic cosmological model can radically change the local fluctuations. The Universe, on one hand, is a macroscopic, classical object. On the other hand, at early stages of the evolution of the Universe, its local properties are determined by quantum processes. But these quantum processes, in turn, proceed against the classical (massive) gravitational background determined by macroscopic laws, which, due to Birkhoff’s theorem, provide the classical, inert nature of the gravitational background.

On avoiding misunderstanding, we emphasize once again the main goal of the present study, indicated above: on the basis of the macroscopic theory [1] formulated by the author, to derive a closed set of Einstein macroscopic equations for the spatially flat Friedmann Universe in short-wave approximation for local fluctuations of the scalar and gravitational fields, to solve these equations in a special case.

Note that the program of deriving macroscopic equations for the Friedmann Universe was previously implemented by the author for the Universe filled with radiation in the model with a cosmological constant [9] and for the Universe filled with microscopic black holes [14].

2. THE MACROSCOPIC MODEL OF AN ISOTROPIC SPATIALLY FLAT UNIVERSE

The macroscopic cosmological model formulated in [1] includes, first of all, linear equations for perturbations of the Friedmann metric

$$ds^2 = a^2(\eta)(d\eta^2 - dx^2 - dy^2 - dz^2)$$

(1)

and a scalar field. Further on, we call such equations the evolution equations for perturbations. Secondly, the macroscopic cosmological model includes a set of nonlinear equations for the macroscopic background scalar field $\Phi(\eta)$ and the macroscopic scale factor $a(\eta)$, which we will call the macroscopic equation of a scalar field and the macroscopic Einstein equations.

2.1. Mathematical Model of Averaging the Einstein Equations

The general ideology and the procedure of obtaining the macroscopic Einstein equations are presented in detail in [1] and [14]. Here we will present the necessary fundamentals of a model of averaging local plane-wave fluctuations in the case of the classical Higgs field $\Phi$ with the Lagrange function

$$L_s = \frac{1}{8\pi} \left( \frac{1}{2} g^{ik} \Phi_i \Phi_k - V(\Phi) \right),$$

(2)

where $V(\Phi)$ is the potential energy of the scalar field which, for the Higgs field, has the form

$$V(\Phi) = -\frac{\alpha}{4} \left( \Phi^2 - \frac{m^2}{\alpha} \right)^2,$$

(3)

$\alpha$ being the self-interaction constant, and $m$ the scalar boson mass.

The scalar field equation due to the Lagrangian function (2) has the form

$$\Box \Phi + V'_\Phi = 0.$$  (4)

Furthermore,

$$T^k_k = \frac{1}{8\pi} \left( \Phi^i \Phi_i - \frac{1}{2} \delta^i_k \Phi_j \Phi^j + \delta^i_k V(\Phi) \right)$$

(5)

is the energy-momentum tensor of the scalar field.

Let us note that the constant term in the potential energy $-m^4/4\pi\alpha$ may be omitted since it leads to a simple renormalization of the cosmological constant $\Lambda_0$,

$$\Lambda \rightarrow \Lambda_0 - \frac{m^4}{4\pi\alpha}.$$  (6)

In what follows, it is this quantity that will be meant to be the cosmological constant (see [1]).

We write the metric with gravitational perturbations with respect to the Friedmann metric in the synchronous reference frame (see, e.g., [8]):

$$ds^2 = ds^2_0 - a^2(\eta)h_{\alpha\beta} dx^\alpha dx^\beta.$$  (7)

2 Everywhere in this paper we adopt the Planck system of units $\hbar = c = G = 1$, the definition of the Riemann and Ricci tensors coincide with those from the book by Landau and Lifshitz [8], the metric signature is diag(−1, −1, −1, +1); the Latin indices take values 1, 4, the Greek ones 1, 3.
Herewith, the covariant perturbations of the metric are
\[ \delta g_{\alpha\beta} = -a^2(\eta)h_{\alpha\beta}. \]  
(8)

Furthermore,
\[ h^0_\alpha = h_{\gamma\beta} \delta_0^{\alpha\gamma} = \frac{1}{a^2} h_{\alpha\beta}, \]
\( h \equiv h^0_\alpha \equiv g^0_{\alpha\beta} h_{\alpha\beta} = -\frac{1}{a^2} (h_{11} + h_{22} + h_{33}). \)  
(10)

Each perturbation mode is written in the form
\[ h_{\alpha\beta}(r, \eta) = h_{\alpha\beta}(n, \eta) e^{imr + i\psi_0} + \text{CC}, \]
(11)
where \text{CC} means the complex conjugate quantity, and \( \psi_0 \) is an arbitrary constant phase for each harmonic \( n \). At every time instant \( \eta_0 \), a separate independent mode of metric perturbations is completely described by the three-dimensional tensor amplitudes \( h_{\alpha\beta}(n, \eta_0) \), whose classification by the directions of the spacelike vector \( n \) and the three-dimensional Kronecker tensor \( \delta_{\alpha\beta} \) has been described in the book by Lifshitz [8].

For transversal perturbations,
\[ h_{\alpha\beta} = e_{\alpha\beta} S e^{imr + i\psi_0} + \text{CC}, \]  
(12)
where \( S(n, \eta) \) is the perturbation amplitude, so that
\[ h^0_\alpha n_\alpha = 0; \]  
\( h = 0. \)  
(13)

For vector perturbations of the metric,
\[ h_{\alpha\beta} = V_\alpha n_\beta + V_\beta n_\alpha, \quad (nV) = 0, \]  
\[ V = ve^{imr + i\psi_0} + \text{CC}, \]  
(16)
where \( v(n, \eta), v(n, \eta) \) is the vector perturbation amplitude. In a similar way, for longitudinal perturbations,
\[ h_{\alpha\beta} = e^{imr + i\psi_0}(\lambda P_{\alpha\beta} + \mu Q_{\alpha\beta}) + \text{CC}, \]  
(17)
where \( \lambda(n, \eta), \lambda'(n, \eta), \mu(n, \eta), \mu'(n, \eta) \) are the amplitudes of scalar perturbations,
\[ P_{\alpha\beta} = \frac{1}{3} \delta_{\alpha\beta} - \frac{n_\alpha n_\beta}{n^2}; \quad Q_{\alpha\beta} = \frac{1}{3} \delta_{\alpha\beta}. \]  
(18)

Further, the scalar field potential \( \Phi(\eta, r) \) is presented in a similar form:
\[ \Phi(\eta, r) \rightarrow \Phi(\eta) + \phi(\eta) e^{imr + i\psi_0} + \text{CC}. \]  
(19)

### 2.2. Evolution Equations for Perturbations

The evolution equation for the amplitudes of perturbations of the scalar field \( \phi(\eta) \) has the form
\[ \phi'' + 2 \frac{a'}{a} \phi' + a^2 \phi \left( m^2 + \frac{n^2}{a^2} - 3\alpha \Phi^2 \right) \]
\[ + \frac{\mu'}{2} \Phi' = 0, \]
(20)
where \( \alpha \) is the self-interaction constant in the Higgs potential, \( m \) is the mass of the scalar field, \( n \) is the perturbation wave vector.

Let us write out the following nontrivial combinations of the Einstein equations as linearly independent evolution equations for the amplitudes of perturbations of the gravitational field \( \psi(\eta) \), \( S(\eta), \lambda(\eta), \mu(\eta) \) (the vector, transversal and scalar perturbations, respectively, in accordance with the Lifshitz classification; the notation of the perturbations coincides with the standard one [8]):
\[ v' = 0, \]
\[ S'' + 2 \frac{a'}{a} S' + n^2 S = 0, \]  
(21)
\[ \frac{1}{3}(\lambda + \mu)' + \phi \Phi' = 0, \]  
(22)
\[ \lambda'' + 2 \frac{a'}{a} \lambda' - \frac{n^2}{3}(\lambda + \mu) = 0, \]  
(23)
\[ \mu'' + 2 \frac{a'}{a} \mu' + \frac{n^2}{3}(\lambda + \mu) + 3\phi' \Phi' \]
\[ - 3\alpha^2 \Phi(\eta) (m^2 - \alpha \Phi^2) = 0. \]  
(24)
In this case Eqs. (21) and (22) for the amplitudes of vector and tensor perturbations are independent from other perturbations and coincide with the corresponding equations of the Lifshitz theory. Equation (24) coincides with the corresponding equation of the standard perturbation theory (see [8]), while the equation which is identical to (23) in the standard perturbation theory determines the longitudinal component of the fluid velocity. Equation (25) is also similar to the corresponding second-order differential equation of the standard perturbation theory (see [8]).

### 2.3. Macroscopic Scalar Field Equation and the Einstein Macroscopic Equations

In [1], the following macroscopic equation of the scalar Higgs field was obtained:
\[ \Phi'' + 2 \frac{a'}{a} \Phi' + a^2 \phi(m^2 - \alpha \Phi^2) \]

\[ 3 \text{ See the details in [1].} \]
\[
- \Phi \left[ (SS')' + \lambda \lambda' + \frac{1}{6} (\mu \mu')' \right] \\
- 6a^2 \alpha \phi \phi' + \frac{1}{2} \phi \mu'^2 + \phi^2 \mu' \\
- \frac{1}{2} n^2 (\phi \mu' + \phi' \mu) = 0, \tag{26}
\]

as well as the macroscopic Einstein equation with a \( \Lambda \) term for the \( \alpha \) component

\[
3 \frac{a^2}{a^4} \Phi^2 - \frac{2 a^2}{m^2} \Phi^2 + \frac{\alpha \Phi^4}{4} - \Lambda \\
= \frac{1}{2} \left[ n^2 \left( SS' \right)' + \frac{\lambda \lambda' + \lambda' \mu}{9} - \frac{\lambda \lambda'}{18} \right] \\
+ \frac{SS''}{2} + \frac{\lambda \lambda''}{6} - \frac{\mu'^2}{6} \\
+ 2 \frac{a'}{a} \left( SS' \right)' + \frac{\lambda \lambda'}{3} \left( \mu \mu' \right) \left( \frac{\phi \phi'}{a^2} \right) \\
+ \frac{\phi \phi'}{a^2} \left( n^2 + m^2 - 3 \alpha \Phi^2 \right), \tag{27}
\]

where \( \Lambda \) is the cosmological constant. Here the Einstein macroscopic equations for the spatial components \( \alpha \) are differential-algebraic consequences of the equations (20) and (22). Further, \( FF^* \) means the operation of averaging over the random phase of the oscillations \( \psi \) (see [9]):

\[
\overline{FF^*} = \frac{1}{2\pi} \int_0^{2\pi} FF^* d\psi. \tag{28}
\]

If there is not only a single isotropic mode of oscillations with the wave vector \( \mathbf{n} \), but a certain spectrum of oscillations \( f(n) \geq 0 \) is given, where

\[
\int_0^\infty f(n)dn = 1,
\]

one should also carry out averaging over frequencies in the macroscopic equations (26)–(27):

\[
\int_0^\infty F(n, \eta)f(n)dn.
\]

Let us note that the operation of averaging over the directions of the wave vector \( \mathbf{n} \) defined in [1] implies that the factor \( n^2 \), appearing in the spherical coordinate system, is transferred to the spectral function \( f(n) \). Therefore, when proceeding to averaging of specific perturbations, we should take into account this factor and determine the operation of averaging over the spectrum as follows:

\[
\overline{F} \equiv \langle FF* \rangle = \int_0^\infty F(n, \eta)f(n)n^2 dn. \tag{29}
\]

So, further on we determine the macroscopic cosmological model with a help of the evolution equations (20), (22)–(25) and the macroscopic equations (26), (27).

3. THE SOLUTION \( \Phi \approx \text{const} \)

As we noted above, the specifics of the self-consistent field method is that the microscopic and macroscopic equations should be solved together. Let us consider one of these possible solutions. We start with examining Eqs. (26), (20), (23) and (25). It can be noted that when discarding quadratic fluctuations in the inhomogeneous term of the macroscopic scalar field equation (26), a constant nonzero solution of this equation

\[
\Phi = \pm \frac{m}{\sqrt{\alpha}} \equiv \Phi_0 \quad (\alpha > 0) \tag{30}
\]

significantly simplifies the studied system of evolution equations (20), (23)–(25), for which in this case we get:

\[
\phi'' + 2 \frac{a'}{a} \phi' + \phi \left( n^2 - 2a^2 m^2 \right) = 0, \tag{31}
\]

\[
(\lambda + \mu)' = 0, \tag{32}
\]

\[
\lambda'' + 2 \frac{a'}{a} \lambda' - \frac{n^2}{3} (\lambda + \mu) = 0, \tag{33}
\]

\[
\mu'' + 2 \frac{a'}{a} \mu' + \frac{n^2}{3} (\lambda + \mu) = 0. \tag{34}
\]

Note that the constant solution (30) turns to zero the energy-momentum tensor of the scalar field (5) in the zero-order approximation, therefore, in this case the cosmological constant has no scalar component.

Thus, longitudinal perturbations of the metric and the scalar field in this case are independent, the equation for the transversal waves (22), as before, does not depend on the longitudinal perturbations of the metric and the scalar field. Equations (32), (33) are easily integrated:

\[
\lambda + \mu = C_1 = \text{const},
\]

\[
\lambda = C_1 \frac{n^2}{3} \int_0^\eta \frac{dy'}{a^2(\eta')} \int_0^\eta a^2(\eta'')d\eta'' + C_2 \int_0^\eta \frac{dy'}{a^2(\eta')} + C_3. \tag{35}
\]
As is known [8], solutions of the form (35) can be excluded by admissible transformation of the reference frame. Therefore, in the considered solution, we can put

$$\lambda = \mu = 0.$$  \hspace{1cm} (36)

Therefore, in the approximation under consideration, *longitudinal metric perturbations are not generated.*

Substituting further (30) and the obtained solution (36) into the macroscopic equation of the scalar field (26), we reduce it to the form

$$\Phi'' + \Phi' \left(2a'i - \frac{(SS')'}{a^2} \right) + a^2\Phi (m^2 - \alpha \Phi^2) = 6a^2\alpha \phi \phi^*.$$  \hspace{1cm} (37)

Substituting the solution (36) into the macroscopic Einstein equation (27), we reduce it to the form

$$\frac{3a'^2}{a^4} - \frac{Φ'^2}{2a^2} - \frac{m^2Φ^2}{2} + \frac{αΦ^4}{4} - \Lambda$$

$$= \frac{1}{a^2} \left[ \frac{n^2SS^*}{2} + \frac{SS'^2}{2} + \frac{2a'^2}{a} (SS')' \right]$$

$$+ \frac{Φ'^2}{a^2} + \frac{Φ^2}{a^2} \left( \frac{n^2}{a^2} - 2m^2 \right).$$  \hspace{1cm} (38)

Let us note that the solution (30), taken as the basis, corresponds to an *attracting focus* in the standard cosmological model based on the classical Higgs field [10].

\section*{4. WKB APPROXIMATION}

We now consider the WKB approximation in the evolution equations (22) and (31):

$$n > \frac{a'}{a}, \quad \Phi' > \frac{a'}{a} \Phi; \quad S' > \frac{a'}{a} S; \quad \phi' > \frac{a'}{a} \phi,$$  \hspace{1cm} (39)

presenting the solutions of these equations $f(η)$ in the form

$$f = \tilde{f}(η) \cdot e^{i\int u(η)dη + iψ} \quad (|u| > 1),$$  \hspace{1cm} (40)

where $ψ = $ const is a random phase of oscillations (see [9]), and $\tilde{f}(η)$ and $u(η)$ are functions weakly changing with the scale factor, such that:

$$\frac{a'}{a} \sim \frac{1}{ℓ}, \quad \tilde{f}' \sim \frac{1}{ℓ}, \quad u' \sim \frac{u}{ℓ}.$$  \hspace{1cm} (41)

Further on, we do not assume that the inequality $n > am$ is valid, assuming that the value $a(η)m$ can be sufficiently large.

\subsection*{4.1. Tensor Perturbations}

Let us first consider in detail the WKB approximation for tensor perturbations. Substituting the amplitude of tensor perturbations in the form (40) into Eq. (22), we obtain, dividing the quantities into orders in the WKB approximations:

$$(0) | (-u^2 + n^2)(S) = 0;$$

$$(1) | 2u\tilde{S}' + u'\tilde{S} + \frac{a'}{a} u\tilde{S} = 0.$$  \hspace{1cm} (43)

Thus we find in the zero-order approximation from (42)

$$u = \pm n.$$  \hspace{1cm} (44)

Substituting the solution (44) into (43), we obtain the equation:

$$\tilde{S}' + \frac{a'}{a} \tilde{S} = 0,$$

solving which, we finally obtain for the amplitude of tensor perturbations:

$$S = \frac{1}{a} S_0^+ e^{inη + iψ} + \frac{1}{a} S_0^- e^{-inη - iψ},$$  \hspace{1cm} (45)

where $S_0^±$ are constant amplitudes, so that $S_0^+ S_0^- = |S_0|^2$. Thus we find in the WKB approximation:

$$SS^* = \frac{1}{a^2} \left[ |S_0|^2 + 2S_0^0 S_0^± \cos(2nη + 2ψ) \right]$$

$$= \frac{S S^*}{a^2};$$  \hspace{1cm} (46)

$$S'S'^* = \frac{n^2}{a^2} \left[ |S_0|^2 - 2S_0^0 S_0^± \cos(2nη + 2ψ) \right]$$

$$+ \frac{a'^2}{a^4} \left[ |S_0|^2 + 2S_0^0 S_0^± \cos(2nη + 2ψ) \right]$$

$$= S S'^* \frac{1}{a^2} \left[ \frac{n^2}{a^2} |S_0|^2 + \frac{a'^2}{a^4} |S_0|^2 \right]$$

$$\simeq \frac{1}{a^2} n^2 |S_0|^2;$$  \hspace{1cm} (47)

$$(SS^*)' = -2 \frac{a'}{a^2} |S_0|^2 \simeq 0.$$  \hspace{1cm} (48)

\subsection*{4.2. Perturbations of the Scalar Field}

Let us proceed to scalar perturbations in our model. We now use the relations (39)–(41) in Eq. (31), dividing it by orders of the WKB approximation:

$$(0) | : \phi(u^2 - n^2 + 2a^2 m^2) = 0;$$

$$(1) | 2u(\alpha' + aφ') = 0.$$  \hspace{1cm} (50)

Thus we obtain a *dispersion relation* for scalar perturbations:

$$u = \pm u_0(n) = \pm \sqrt{n^2 - 2a^2 m^2} \quad (\simeq \pm n).$$  \hspace{1cm} (51)
Solving the simple differential equation (50) and substituting the solution into the formulas (39), we find the WKB solution of the equation for scalar field perturbations (31)

\[ \varphi = \frac{\phi_0^0}{a^2 u_0(n)} e^{iu_0(n)\eta} + \frac{\phi_0^0}{a^2 u_0(n)} e^{-iu_0(n)\eta}. \] (52)

Thus we find the averages:

\[ \overline{\phi\phi^*} \simeq \frac{1}{a^2} \left\langle \frac{|\phi_0(n)|^2}{n^2 - 2a^2m^2} \right\rangle, \] (53)
\[ \phi\phi^* \left( \frac{n^2}{a^2} - 2m^2 \right) \simeq \frac{1}{a^4} \left\langle |\phi_0(n)|^2 \right\rangle, \] (54)
\[ \overline{\phi'\phi'^*} \simeq \frac{1}{a^2} \left\langle \frac{n^2|\phi_0(n)|^2}{n^2 - 2m^2a^2} \right\rangle, \] (55)

where it holds

\[ |\phi_0(n)|^2 = (\phi_0^0)^2 + (\phi_0^0)^2, \]

and to simplify the cumbersome expressions, we replaced the sign of averaging over the spectrum as \( F \equiv \langle F \rangle. \)

4.3. Macroscopic Scalar Field Equation in the WKB Approximation

Substituting the relations (48) and (53) in the macroscopic field equation (37), let us reduce it to the explicit form

\[ \Phi'' + 2\frac{a'}{a}\Phi' + a^2\Phi(m^2 - \alpha\Phi^2) = \sigma, \] (56)

where

\[ \sigma = 6\alpha \left\langle \frac{|\phi_0(n)|^2}{n^2 - 2a^2m^2} \right\rangle. \] (57)

Thus, in the WKB approximation, the equation for a macroscopic scalar field becomes inhomogeneous, with a scalar source in the right part. Such a field can be interpreted as a charged scalar field, where the charge density \( \sigma \) is determined by the spectral function \( |\phi_0(n)|^2 \). Let us note that the cosmological models with charged scalar fields were investigated in [11]. In the range

\[ n^2 \gg 2a^2m^2, \] (58)

the effective density of the scalar charge in the WKB approximation tends to zero:

\[ \sigma \approx \sigma_0 \equiv 6\alpha \left\langle \frac{|\phi_0(n)|^2}{n^2} \right\rangle > 0, \quad (n \gg am), \] (59)

thus the approximate solution (30) becomes asymptotically exact. Not discarding the low charge density in the WKB approximation, we can find a new quasi-constant solution of the macroscopic field equation (56) which takes the form of a third-order algebraic equation:

\[ \alpha\Phi^3 - m^2\Phi + \frac{\sigma_0}{a^2} = 0. \] (60)

Equation (60), depending on the value of the dimensionless parameter

\[ \gamma^2 = \frac{\sigma_0\sqrt{\alpha}}{a^2m^4}, \] (61)

can have the following real solutions: one negative at \( \gamma^2 > 4/27 \), one negative and one doubly degenerate positive solution at \( \gamma^2 = 4/27 \), and finally, one negative and two positive solutions at \( \gamma^2 < 4/27 \). At \( n \gg 1 \) we get the approximate solutions

\[ \Phi \approx \Phi_0 - \frac{\sigma_0}{2m^2a^2}, \] (62)

which, with the growth of the scale factor, asymptotically tend to the unperturbed solutions (30).

At \( 1 \ll n^2 < 2a^2m^2 \) the effective density of the scalar charge becomes constant and negative:

\[ \Phi'' + \frac{a'}{a}\Phi' + a^2\Phi(m^2 - \alpha\Phi^2) = -6\frac{\alpha a^2}{m^4} \left\langle |\phi_0(n)|^2 \right\rangle. \] (63)

In this case, a strict renormalization of the unperturbed constant solutions becomes possible (30):

\[ \Phi \approx \Phi_0 + 3\frac{\alpha}{m^4} \left\langle |\phi_0(n)|^2 \right\rangle. \] (64)

4.4. Einstein Macroscopic Equation in the WKB Approximation

Substituting the expressions for the averages (48), (54) and (55) into Eq. (38), we reduce it to the form

\[ 3\frac{a'^2}{a^4} - \frac{\Phi'^2}{2a^2} - \frac{m^2\Phi^2}{2} + \frac{\alpha\Phi^4}{4} - \Lambda = \left\langle n^2|S_0(n)|^2 \right\rangle \] (65)

\[ + \frac{2}{a^4} \left\langle \frac{n^2 - m^2a^2}{n^2 - 2m^2a^2}|\phi_0(n)|^2 \right\rangle. \]

In particular, in the limit (58) we obtain the macroscopic Einstein equation in the WKB approximation

\[ 3\frac{a'^2}{a^4} - \frac{\Phi'^2}{2a^2} - \frac{m^2\Phi^2}{2} + \frac{\alpha\Phi^4}{4} - \Lambda = \frac{1}{a^4} \left[ \left\langle n^2|S_0(n)|^2 \right\rangle + 2 \left\langle |\phi_0(n)|^2 \right\rangle \right]. \] (66)
5. SOLUTIONS OF EINSTEIN’S MACROSCOPIC EQUATION

5.1. Solution in the WKB Limit

Using the unperturbed solution (30) in (66), we simplify this equation:

\[ 3a'^2 = \tilde{\Lambda}a + \langle n^2|S_0(n)|^2 \rangle + 2\langle |\phi_0(n)|^2 \rangle, \quad (67) \]

where

\[ \tilde{\Lambda} = \Lambda + \frac{m^4}{4\alpha}. \quad (68) \]

We proceed in the Einstein equation (67) to the differentiation with respect to the physical time \( t \) and the new nonnegative variable \( u(t) \):

\[ t = \int a(\eta)d\eta; \quad u(t) = a^2(t) \geq 0. \]

As a result, we obtain the equation

\[ \dot{u}^2 = \delta^2 u^2 + \xi^2, \quad (69) \]

where \( \dot{u} = du/dt \), and the following notations are introduced:

\[ \delta^2 = \frac{4}{3} \tilde{\Lambda}, \]
\[ \xi^2 = \frac{4}{3}\left[ \langle n^2|S_0(n)|^2 \rangle + 2\langle |\phi_0(n)|^2 \rangle \right]. \quad (70) \]

Equation (69) is elementarily integrated:

\[ \int_{a_0}^{a} \frac{du}{\sqrt{\delta^2 u^2 + \xi^2}} = t - t_0. \quad (71) \]

The integral (71) converges for any values of \( a_0, t_0 \) at \( \xi \neq 0 \), whence it is seen that one can always put \( a(t_0) \equiv a_0 = 0 \), i.e., the solution always contains a singularity at \( \xi \neq 0 \), to the left of which it does not continue. Using the freedom of \( t_0 \) choice, we choose this instant of time for the beginning of the time scale so that

\[ a(0) = 0, \quad (\xi \neq 0). \quad (72) \]

Then, reversing the integral (71), we find a solution of the Einstein equation:

\[ a(t) = \sqrt{\frac{\xi}{\delta}} \sinh \delta t \quad (\xi \neq 0). \quad (73) \]

In the absence of local perturbations, that is, at \( \xi = 0 \), we obtain the well-known inflationary solution from (71)

\[ a(t) = a_0 e^{\delta t/2}, \quad (74) \]

which has a singularity in the infinite past, where \( a(-\infty) = 0 \).

5.2. The Next WKB Approximation

Let us now keep the first nonvanishing term by smallness of \( a^2m^2/n^2 \) in the right-hand side of Eq. (65). As a result, instead of (69), we get the equation

\[ \dot{u}^2 = \delta^2 u^2 + \zeta^2 u + \xi^2, \quad (75) \]

where

\[ \zeta^2 = \frac{8}{3} m^2 \left\langle \frac{1}{n} \right\rangle \left\langle |\phi_0(n)|^2 \right\rangle. \quad (76) \]

Assuming \( \xi \neq 0, \zeta \neq 0 \), we find a solution of the equation (75), which is also singular:

\[ a(t) = \frac{1}{\delta} \sqrt{\sinh \frac{\delta t}{2} \left( 2\xi \cosh \frac{\delta t}{2} + \zeta^2 \sinh \frac{\delta t}{2} \right)}. \quad (77) \]

5.3. Analysis of the Solutions

The solution (77) at \( \zeta \to 0 \) continuously transfers into the solution (73), so we carry out further research with respect to this solution. Near the singularity \( t \to 0 \) we obtain the asymptotic of the solution (77)

\[ a(t) \bigg|_{t \to 0} \simeq \frac{1}{2} \sqrt{t(4\xi + \zeta^2 t)}, \quad (78) \]

the solution behaves like a cosmological one at the ultrarelativistic stage of expansion at \( \xi \neq 0 \), while the solution behaves like \( a \sim t \), that is, like quintessence at \( \xi = 0 \). Let us note that the asymptotic (78) is an exact solution of the Einstein equation at \( \delta = 0 \Rightarrow \tilde{\Lambda} = 0 \). As \( t \to \infty \), the solution (77) goes into the inflation type regime (74)

\[ a(t) \bigg|_{t \to \infty} \simeq \frac{\sqrt{2\delta^2 + \zeta^2}}{2\delta} e^{\delta t/2}. \quad (79) \]

For cosmology, two dynamic functions that are observable astronomical variables are important: the Hubble parameter \( H(t) \), and the invariant cosmological acceleration, \( w(t) \):

\[ H(t) = \frac{\dot{a}}{a}, \quad w(t) = \frac{a\ddot{a}}{a^2}. \quad (80) \]

Calculating the Hubble parameter for the solution (77), we get

\[ H(t) = \frac{\delta(2\xi \cosh(\delta t) + \zeta^2 \sinh(\delta t))}{4 \sinh \frac{\delta t}{2} \left( 2\xi \cosh \frac{\delta t}{2} + \zeta^2 \sinh \frac{\delta t}{2} \right)}. \quad (81) \]

In particular, as \( \delta \to 0 \), we obtain from (81) a formula that is also an asymptotic form near the singularity \( t \to 0 \):

\[ H(t) \bigg|_{\delta \to 0} \simeq \frac{2\xi + \zeta^2 t}{t(4\xi + \zeta^2 t)}. \quad (82) \]
As \( t \to \infty \), we get the asymptotic from (81)
\[
H(t) \big|_{t \to \infty} \simeq \frac{\delta}{2}.
\] (83)

The expression for the invariant cosmological acceleration is too cumbersome, therefore, we give only its expression for the case \( \delta = 0 \),
\[
w(t) \big|_{\delta \to 0} \simeq -\frac{4\xi^2}{(2\xi + \zeta^2 t)^2} \quad (w(0) = -1),
\] (84)
and for \( t \to \infty \):
\[
w(t) \big|_{t \to \infty} \simeq 1.
\] (85)

Let us notice that it is possible to get rid of one parameter in Eq. (75) at \( \delta \neq 0 \), introducing the dimensionless time variable
\[
\tau = \delta t.
\] (86)
Assuming further that \( \delta \neq 0 \), we reduce Eq. (75) to the dimensionless form
\[
u^2 = u^2 + \tilde{\zeta}^2 u + \tilde{\xi}^2,
\] (87)
where
\[
\tilde{\zeta} = \frac{\zeta}{\delta}, \quad \tilde{\xi} = \frac{\xi}{\delta},
\] (88)
and the time derivative \( \tau \) is denoted by a dot. The observable cosmological parameters (81) are transformed as follows:
\[
H(t) \rightarrow \delta h(\tau), \quad w(t) \rightarrow w(\tau),
\]
where \( h(\tau) = \dot{a}/a \). To interpret the results, we also introduce an invariant curvature \( K \) of the Friedmann space,
\[
K^2 = \frac{1}{6} R^{ijkl} R_{ijkl} = H^2 (1 + w^2).
\]
Thus we obtain for the invariant curvature in the time variable \( \tau \):
\[
K = \delta h \sqrt{1 + w^2},
\] (89)
where
\[
\lim_{\tau \to \infty} K(\tau) = \frac{\delta}{\sqrt{2}}.
\]

5.4. Graphic Illustration

Below are the plots illustrating the evolution of the quantities \( H(\tau) \) (Fig. 1), \( w(\tau) \) (Fig. 2) and \( K(\tau) \) (Fig. 3) depending on the parameters of the cosmological model \( P = [\tilde{\xi}, \tilde{\zeta}] \).

For a correct interpretation of these plots, it is necessary to take into account the relationship between the variable \( \tau \) and the physical time \( t \) (86) as well as the relations (88) connecting the dimensionless parameters \( \delta, \tilde{\xi}, \tilde{\zeta} \) with the physical parameters \( \Lambda \) (68), \( \xi \) (70), and \( \zeta \) (76).

DISCUSSION OF THE RESULTS

When interpreting the results, it is necessary to take into account the fact that the main parameters of the obtained cosmological model, \( \xi^2 \) and \( \zeta^2 \), should be determined with the help of spectrum averages.
that the time variable $\tau$ is actually measured in units $1/\sqrt{\Lambda}$, and the value of the effective cosmological constant $\tilde{\Lambda}$ is measured in Planck units according to the system of units accepted in [1]. In addition, this value of the cosmological constant refers to the early stages of the evolution of the Universe. If we accept the existing estimates of the radius of curvature of the early Universe at the primary inflation stage $t_k \sim 10^{10}\ell_{pl}$, we obtain an estimate of the value of the primary cosmological constant $\tilde{\Lambda} \sim 10^{-14} - 10^{-2}\ell_{pl}^{-2}$, where $\ell_{pl}$ is the Planck length. Thus the change in the acceleration mode can occur at the times $t \sim 10 - 10^7\ell_{pl}$. These times are not yet critical from the point of view of observations.

Secondly, let us note that the refinement of the parameters of the macroscopic model $\xi$ (90) and $\zeta$ (91) can be performed based on the results of quantum theory of generation of scalar and tensor perturbations [2] (see also [12]) which provides the necessary formulas for the spectra of the corresponding perturbations.

Thirdly, we note that we are not discussing here an application of the developed theory of macroscopic averaging of the classical Einstein—Higgs equations to the so-called standard scenario that assumes a vacuum initial state in which there are no initial perturbations of the geometry and physical fields. Here we are only implementing the program of consistent application of the Vlasov—Bogoliubov self-consistent field method to a locally fluctuating Universe filled with scalar fields and radiation. At the same time, it was important for us to find out the trend of the influence of collective interactions on the macroscopic picture of the Universe. We leave aside the issues of linking this theory to elements of the standard scenario.

Summing up the results of this paper, let us note that it completely implements the program of constructing a macroscopic cosmological model proposed by the first author in his previous paper [13]. In the present paper, which is a logical completion of [1], this program is implemented for the Universe generated by the fluctuating scalar Higgs field. For the Universe with a cosmological constant, filled with gravitational radiation, this program was implemented in [9], and for a Universe filled with a perfect fluid and black holes, in [14].

Let us also recall that the macroscopic quantities obtained using a self-consistent description can significantly differ from the “seed” quantities, which in our case would be represented by an inflationary model with a “seed” cosmological constant $\Lambda_0$. Illustrative examples demonstrating the collective effects can be the phenomena of screening of electric charges and confinement of quarks.
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