Bosonisation of the Complex-Boson Realisations of $W_\infty$

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ABSTRACT

We bosonise the complex-boson realisations of the $W_\infty$ and $W_{1+\infty}$ algebras. We obtain nonlinear realisations of $W_\infty$ and $W_{1+\infty}$ in terms of a pair of fermions and a real scalar. By further bosonising the fermions, we then obtain realisations of $W_\infty$ in terms of two scalars. Keeping the most non-linear terms in the scalars only, we arrive at two-scalar realisations of classical $w_\infty$.

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1. Introduction

Since the discovery of $W_\infty$ [1] and $W_{1+\infty}$ [2], it has become clear that there are many applications for these algebras. First of all, as an algebra containing generators $V^i$ with spin $i+2 \geq 2$, $W_\infty$ is a leading candidate for the universal $W$-algebra [3,10,11]. Secondly, the concept of 2D $W$-gravity and $W$-strings as a generalisation of ordinary 2D gravity and strings, which has been of much interest recently [4,16,17], may by itself be a very fruitful idea. Thirdly it has also been shown that $W_{1+\infty}$ algebra generates the $W_N$ constraints for the $N$-matrix model solutions for non-perturbative 2D quantum gravity coupled to a matter system with $c \leq 1$ [5,11], and has been further argued to be the underlying symmetry for these systems [6].

Various free-field realisations for $W_\infty$ [7], $W_{1+\infty}$ [8,9] and super-$W_\infty$ [8] have been found. These realisations provide a more transparent way to investigate issues concerning these algebras and their applications. For example in Ref. [10], the free-fermion realisation of $W_\infty$ at $c = -2$ has played a pivotal rôle in carrying out the reduction to $W_N$ from $W_\infty$. In Ref. [11], the $W_{1+\infty}$ constraints were obtained from a single-scalar realisation that is the bosonised form of the complex fermion realisation of $W_{1+\infty}$. In Ref. [15,16,17], realisations of $W_\infty$ provided the matter system in which framework the gauging of $W$ symmetries have been carried out.

There exists a common feature among the various complex free-field realisations of $W_\infty$, namely, the currents for these algebras are expressed in the bilinear form of the free-fields. In the language of field theory, they are termed linear realisations of the corresponding algebras. On the other hand, the one real-scalar realisations of $W_\infty$ and $W_{1+\infty}$ are different in that they are nonlinear realisations obtained from the bosonisation of the corresponding fermion realisation. Since one can bosonise complex bosons as well as complex fermions, it must be possible to achieve other nonlinear realisations by bosonising the complex-boson realisation of $W_\infty$. This is precisely what we will accomplish in this paper.

We shall first review various realisations of the $W_\infty$ algebras in the literature, and illustrate their interrelationships. We then proceed to carry out the bosonisation of the complex-boson realisations of the $W_\infty$ algebras, by the well-known procedure of Ref. [12]. The bosonised currents are expressed in terms of a pair of fermions and a scalar. Bosonising the fermions further, we obtain a realisation of $W_\infty$ in terms of two coupled scalars. We then extract a realisation of the classical $w_\infty$ by keeping the most nonlinear terms in the scalars. Finally we conclude with discussions.

2. Realisations of $W_\infty$

The first free-field realisation of the $W_\infty$ algebra was given in Ref. [7]. The $W_\infty$ currents $V^i$ with spin $s = i+2 \geq 2$ are realised as bilinears of a pair of scalars $\varphi(z)$ and $\varphi^*(z)$, with OPE

$$\varphi^*(z)\varphi(w) \sim -\log(z-w) \ ,$$

as follows:

$$V^i(z) = -\sum_{k=0}^{i+1} a_k(i, \frac{1}{2}) \partial^k \varphi^*(z) \partial^{i-k+2} \varphi(z) \ ,$$

(2)
where the constants \( a_k(i,\alpha) \) are given by

\[
a_k(i,\alpha) = \binom{i+1}{k} \frac{(i+2\alpha+2-k)k(2\alpha-i-1)i+1-k}{(i+2)i+1}.
\]

Here \((a)_n \equiv a(a+1)\cdots(a+n-1)\).

The \( W_{1+\infty} \) algebra can be realised in a similar fashion as bilinears of a complex fermion \( \psi \), with OPE

\[
\overline{\psi}(z)\psi(w) \sim \frac{1}{z-w}.
\]

There is a one-parameter family of basis in which the currents can be realised \([13,14]\). The currents \( \widetilde{V}^i(z) \) with spin \( i+2 \geq 1 \) in the basis corresponding to an arbitrary value of the parameter \( \alpha \) are given by \([14]\)

\[
\widetilde{V}^i(z) = \sum_{k=0}^{i+1} a_k(i,\alpha) \partial^k \overline{\psi}(z) \partial^{i+1-k} \psi(z).
\]

For a generic value of \( \alpha \), the central charges of the currents are not diagonalised. When \( \alpha = 0 \) they are diagonalised, and the corresponding realisation of \( W_{1+\infty} \) with \( c = 1 \) was described in Ref. \([8,9]\).

Since it has been shown that \( W_{\infty} \) can be embedded into \( W_{1+\infty} \) \([13]\), one can obtain a realisation of \( W_{\infty} \) from that of \( W_{1+\infty} \) by taking a suitable basis and truncating out the spin-1 current. Concretely, by taking \( \alpha = \frac{1}{2} \) in the above realisation \((5)\), one can show that \( \widetilde{V}^i \) with \( i \geq 0 \) form a closed algebra, namely the \( W_{\infty} \) algebra. The central charge of this realisation is \( c = -2 \), obeying the general relationship between the central charge of the embedded \( W_{\infty} \) algebra and that of \( W_{1+\infty} \) \([13]\).

It is worth noting that the currents \( V^i \) of \( W_{\infty} \) \((2)\), and \( \widetilde{V}^i \) of \( W_{1+\infty} \) \((5)\) with \( \alpha = 0 \), constitute a realisation for the bosonic sector of the \( N = 2 \) super-\( W_{\infty} \) algebra, while the fermionic currents can be constructed as mixed bilinears of \( \psi \) and \( \phi \) \([8]\).

There also exists a realisation of \( W_{1+\infty} \) in terms of a complex boson \( \phi \) with OPE \([14]\)

\[
\overline{\phi}(z)\phi(w) \sim -\frac{1}{z-w}.
\]

With respect to the stress tensor \( T \equiv \overline{V}^0 \) given by

\[
T = (\alpha + \frac{1}{2}) \partial \overline{\phi} \phi + (\alpha - \frac{1}{2}) \overline{\phi} \partial \phi,
\]

the \( \phi \) and \( \overline{\phi} \) have conformal spin \( \frac{1}{2} + \alpha \) and \( \frac{1}{2} - \alpha \). In general the currents of \( W_{1+\infty} \) in a basis with parameter \( \alpha \) are given by

\[
\widetilde{V}^i = \sum_{k=0}^{i+1} a_k(i,\alpha) \partial^k \overline{\phi} \partial^{i+1-k} \phi.
\]

In fact, this realisation has the same structure as the fermionic realisation of \( W_{1+\infty} \) with \( \phi \) and \( \overline{\phi} \) replacing \( \psi \) and \( \overline{\psi} \) respectively. However, the central charge in the complex-boson realisation differs from that of the complex-fermion realisation by a sign. In particular, in the diagonalised basis at \( \alpha = 0 \), the central charge of this realisation is \( c = -1 \).
For later convenience, we give the explicit expressions of the first few currents in this realisation of \( W_{1+\infty} \) at \( \alpha = 0 \):

\[
\begin{align*}
\tilde{V}^{-1} &= \bar{\phi} \phi, \\
\tilde{V}^{0} &= \frac{1}{2} \partial \bar{\phi} \phi - \frac{1}{2} \overline{\partial \phi}, \\
\tilde{V}^{1} &= \frac{1}{6} \partial^{2} \bar{\phi} \phi - \frac{1}{3} \partial \bar{\phi} \partial \phi + \frac{1}{6} \overline{\partial^{2} \phi}, \\
\tilde{V}^{2} &= \frac{1}{20} \partial^{3} \bar{\phi} \phi - \frac{9}{20} \partial^{2} \bar{\phi} \partial \phi + \frac{9}{20} \partial \bar{\phi} \partial^{2} \phi - \frac{1}{20} \overline{\partial^{3} \phi}.
\end{align*}
\] (9)

Once again, at \( \alpha = \frac{1}{2} \) one can truncate out the spin-1 current, thus obtaining a bosonic realisation of \( W_{\infty} \) at \( c = 2 \). In fact, this realisation is precisely the first realisation of \( W_{\infty} \) given in (2) [7], upon making the identifications of \( \phi \rightarrow -\partial \varphi \) and \( \overline{\varphi} \rightarrow \varphi^{*} \).

We have reviewed two independent realisations of \( W_{\infty} \) (and \( W_{1+\infty} \)): a complex-boson realisation and a complex-fermion realisation. They have identical structure in terms of their respective free fields, but opposite central charges, which can be summarised as follows.

| Bosonic | Fermionic |
|---------|-----------|
| \( W_{1+\infty} \) | \(-1\) | \(1\) |
| \( W_{\infty} \) | \(2\) | \(-2\) |

The common feature of these realisations is that the currents are bilinears of the free fields. Therefore the transformations generated by these currents on the free field are linear. There does exist a nonlinear realisation of \( W_{\infty} \) (and \( W_{1+\infty} \)) at \( c = -2 \) \((c = 1)\) in terms of a real scalar \( \chi \), with OPE \( \chi(z)\chi(w) \sim \log(z-w) \). It can be viewed as the bosonisation of the complex-fermion realisation given in (5) [11], with the following identification:

\[
\overline{\psi} \equiv e^{-\chi}, \quad \psi \equiv e^{\chi}.
\] (10)

Recently this realisation has been argued to furnish the quantum \( W_{\infty} \) symmetry, to which the (classical) \( w_{\infty} \) symmetry deforms in the process of quantisation, and has been used to extract the effective theory of \( W_{\infty} \) gravity [15].

It is known in the literature that complex bosons can also be “bosonised” [12]. Thus it must be possible to obtain another nonlinear realisation of \( W_{\infty} \) by bosonising the complex-boson realisation mentioned above. In next section we shall carry out this construction explicitly.

3. Bosonisation of the Complex-boson Realisations of \( W_{\infty} \)

Let us first recall the procedure of bosonising a pair of bosons \( \phi \) and \( \overline{\phi} \). The novelty involved in the procedure is the necessity of introducing a pair of fermions \( \eta \) and \( \xi \) in addition to a scalar field \( \sigma \) [12], which parallels the scalar \( \chi \) for bosonising fermions given in (10). These free fields satisfy the following OPEs:

\[
\sigma(z)\sigma(w) \sim -\log(z-w),
\] (11)

\[
\eta(z)\xi(w) \sim \frac{1}{z-w}.
\] (12)
Note that the OPE of \( \sigma \) differs from that of \( \chi \) by a sign. The bosonisation is furnished by
\[
\overline{\phi} \equiv \partial \xi e^{-\sigma} : , \quad \phi : \equiv \eta e^\sigma : ,
\]
where the symbol :: implies appropriate normal ordering.

To proceed with the bosonisation, we expand the appropriately normal ordered product of two bosonic operators: \( \phi(w) \overline{\phi}(w) \equiv \lim_{w \to z} \left( \phi(w) \phi(z) - \frac{1}{w-z} \right) \) in powers of \( (w-z) \). It reads
\[
\phi(w) \overline{\phi}(z) - \frac{1}{w-z} = \eta(w) e^{\sigma(w)} \partial \xi(z) e^{-\sigma(z)} - \frac{1}{w-z} - \left( \frac{1}{w-z} \right)^2 + \eta(w) \partial \xi(z) : (w-z) : e^{\sigma(w)-\sigma(z)} : - \frac{1}{w-z} \\
= \eta(w) \partial \xi(z) : (w-z) : e^{\sigma(w)-\sigma(z)} : + \frac{1}{w-z} : e^{\sigma(w)-\sigma(z)} : - \frac{1}{w-z} + \sum_{k,j=0}^{\infty} \frac{(w-z)^{k+j+1}}{k!j!} Q^{(k)}(z) P^{(j)}(z) \\
+ \sum_{j+1}^{\infty} \frac{(w-z)^{j+1}}{j!} P^{(j)}(z),
\]
where \( P^{(j)} \) and \( Q^{(k)} \) are given by
\[
\begin{align*}
P^{(j)}(z) & \equiv e^{-\sigma(z)} \partial^j e^{\sigma(z)} : , \\
Q^{(k)}(z) & \equiv \partial^k \eta(z) \partial \xi(z) : .
\end{align*}
\]
Here we have used the fact that: \( e^{\sigma(w)} : = (w-z) : e^{\sigma(w)-\sigma(z)} : \), which arises from eq. (11) and Baker-Campbell-Hausdorff formula. Thus this implies that, in the limit \( w \to z \),
\[
: \partial^r \phi(w) \partial^m \overline{\phi}(z) : = \partial^m \partial^r : \phi(w) \overline{\phi}(z) : = \\
\begin{align*}
&= \sum_{k,j=0}^{\infty} \sum_{r=0}^{m-r} \sum_{s=0}^{m-r} \frac{[k+j+1][l+r]}{k!j!} \frac{(-1)^r}{(l+r-1)!} \left( \frac{m}{r,s} \right) (w-z)^{k+j-l+r+1} \\
&\quad \times \partial^r Q^{(k)}(z) \partial^{m-r-s} P^{(j)}(z) \\
&\quad + \sum_{j=1}^{\infty} \sum_{r=0}^{m} \frac{[j-1][l+r]}{j!} \frac{(-1)^r}{(l+r-1)!} \left( \frac{m}{r,s} \right) (w-z)^{j-l+r} \partial^{m-r} P^{(j)}(z),
\end{align*}
\]
where \([a]_n \equiv a(a-1) \cdots (a-n+1)\) and \( \left( \frac{m}{r,s} \right) \) represents the multinomial expansion coefficient \( \frac{m!}{r!s!(m-r-s)!} \).

Considering the coefficient of \( (w-z)^0 \), we obtain
\[
: \partial^r \phi(z) \partial^m \overline{\phi}(z) : = \sum_{r=0}^{m} \sum_{s=0}^{m-r-l+r-1} \sum_{k=0}^{l+r-1} \frac{(-1)^r}{k!(l+r-k-1)!} \left( \frac{m}{r,s} \right) \\
\begin{align*}
&\times \partial^r Q^{(k)}(z) \partial^{m-r-s} P^{(l+r-k-1)}(z) \\
&\quad + \sum_{j=l+1}^{m+l+1} \frac{(-1)^{j-l-1}}{j} \left( \frac{m}{j-l-1} \right) \partial^{m-j+l+1} P^{(j)}(z).
\end{align*}
\]
Using this expression and (8), we have the bosonised currents of $W_{1+∞}$ given by

$$\tilde{V}^i = \tilde{V}^i_{FB} + \tilde{V}^i_B,$$

where

$$\tilde{V}^i_{FB} = \sum_{k=0}^{i+1} \sum_{r=0}^k \sum_{s=0}^{k-r} \sum_{j=0}^{i-k+r} (-)^r a_k(i, 0)(i-k+r+1)! \left( \begin{array}{c} k \\ r, s \end{array} \right) \partial^r Q(j) \partial^{k-r-s} \mathcal{P}(i+r-k-j),$$

$$\tilde{V}^i_B = \sum_{k=0}^{i+1} \sum_{j=i+2-k}^{i+2} (-)^{j+k-i} a_k(i, 0) \left( \begin{array}{c} k \\ j+k-i-2 \end{array} \right) \partial^{j+2-j} \mathcal{P}(j).$$

Although it is guaranteed that a correct bosonisation procedure will lead to a realisation of $W_{1+∞}$, it is instructive to see how it works. We shall mention some highlights in the closure of this realisation of $W_{1+∞}$ for the first few lower-spin currents.

The first few currents are the following:

$$\tilde{V}^{-1} = \partial \sigma,$$

$$\tilde{V}^0 = -\eta \partial \xi - \frac{1}{4} (\partial \sigma)^2,$$

$$\tilde{V}^1 = \partial \eta \partial \xi - \eta \partial^2 \xi + 2 \eta \partial \xi \partial \sigma + \frac{1}{3} (\partial \sigma)^3,$$

$$\tilde{V}^2 = -\frac{3}{5} \eta \partial^3 \xi + \frac{1}{2} \partial \eta \partial^2 \xi - \frac{3}{5} \eta \partial^2 \eta \partial \xi$$

$$+ 3 \eta \partial^2 \xi \partial \sigma - 3 \partial \eta \partial \xi \partial \sigma - 3 \eta \partial \xi (\partial \sigma)^2$$

$$- \frac{1}{4}(\partial \sigma)^4 - \frac{1}{10} \partial \sigma \partial^3 \sigma + \frac{3}{20} (\partial^2 \sigma)^2.$$

Note that with respect to the stress tensor $T \equiv V^0$ given above, $\eta$ and $\xi$ have conformal spin 1 and 0 respectively.

One characteristic of this realisation is that the currents are always bilinear in the fermions $\eta$ and $\xi$, as shown in the examples above and the general expression given in (19)-(21). Thus it is necessary that quartic terms of $\eta$ and $\xi$, which in principle might arise in the OPE of two currents, be absent. This condition is indeed satisfied in the OPEs of the first few currents, owing to the anti-commutativity of fermions and some remarkable cancellations. For example, a term quartic in fermions might exist in the OPE of $V^1(z)$ with $V^1(w)$, but it is zero because both $\eta(w) \partial \xi(w) \eta(w) \partial \xi(w) = 0$ and $\partial(\eta(w) \partial \xi(w)) \eta(w) \partial \xi(w) = 0$. Our second example is the OPE of $V^2(z)$ and $V^2(w)$, in which the absence of quartic fermion terms results from both the anti-commutativity of fermions and some very nice cancellations.

Although the currents in this realisation consist of two distinct parts, $\tilde{V}^i_{FB}$ and $\tilde{V}^i_B$ as given in (19), and in particular $\tilde{V}^i_B$ involves the bosonic field $\sigma$ only, neither one of them alone gives a realisation of $W_{1+∞}$. This seems to be contradictory to naive expectation, for there exists a single-scalar realisation of $W_{1+∞}$ [11] identical to $\tilde{V}^i_B$ in structure up to an overall factor $(-)^i$. But there is a crucial difference, namely the scalar $\sigma$ has opposite sign in the propagator (11) to that of $\chi$, which causes $\tilde{V}^i_B$ not to close. One should not be surprised, after all, since the OPEs among $\tilde{V}^i_{FB}$ give rise to terms of the $\tilde{V}^i_B$ form, which supply the additional terms needed for closure. Note, incidentally, that the central charges of $\tilde{V}^i_{FB}$ with odd $i$ vanish.
A different basis ($\alpha = \frac{1}{2}$) can be chosen for this realisation of $W_{1+\infty}$ in terms of $\eta$, $\xi$ and $\sigma$ for which the spin-1 current $\partial \sigma$ be truncated out. This procedure leads to a realisation of $W_\infty$. The following are the first few currents:

\begin{align*}
V^0 &= -\eta \partial \xi - \frac{1}{2} (\partial \sigma)^2 - \frac{1}{2} \partial^2 \sigma, \\
V^1 &= -\frac{3}{5} \eta \partial^2 \xi + \frac{2}{5} \partial \eta \partial \xi + 2 \eta \partial \xi \partial \sigma + \frac{1}{5} \partial^3 \sigma + \frac{1}{5} \partial \sigma \partial^2 \sigma + \frac{1}{12} \partial^3 \sigma, \\
V^2 &= -\frac{1}{5} \eta \partial^3 \xi + \frac{8}{5} \partial \eta \partial^2 \xi - \frac{6}{5} \partial^2 \eta \partial \xi \\
&\quad + 2 \eta \partial^2 \xi \partial \sigma - 4 \partial \eta \partial \xi \partial \sigma - 3 \eta \partial \xi (\partial \sigma)^2 - \eta \partial \xi \partial^2 \sigma \\
&\quad - \frac{1}{10} \partial^4 \sigma - \frac{1}{4} (\partial \sigma)^2 \partial^2 \sigma + \frac{1}{10} (\partial^2 \sigma)^2 - \frac{4}{10} \partial \eta \partial^3 \sigma - \frac{1}{10} \partial^4 \sigma, \\
V^3 &= -\frac{1}{14} \eta \partial^4 \xi + \frac{15}{14} \partial \eta \partial^3 \xi - \frac{15}{7} \partial^2 \eta \partial^2 \xi + \frac{7}{4} \partial^3 \eta \partial \xi \\
&\quad + \frac{7}{5} \eta \partial^3 \xi \partial \sigma - \frac{5}{7} \partial \eta \partial^2 \xi \partial \sigma + \frac{20}{7} \partial^2 \eta \partial \xi \partial \sigma \\
&\quad - \frac{5}{7} \eta \partial^2 \xi (\partial \sigma)^2 + \frac{12}{7} \partial \eta \partial \xi (\partial \sigma)^2 + 4 \eta \partial \xi (\partial \sigma)^3 \\
&\quad - \frac{22}{14} \eta \partial^2 \xi \partial^2 \sigma + \frac{12}{14} \partial \eta \partial \xi \partial^2 \sigma + 3 \eta \partial \xi \partial \sigma \partial^2 \sigma + \frac{11}{7} \eta \partial \xi \partial^3 \sigma \\
&\quad + \frac{3}{7} (\partial \sigma)^5 + \frac{1}{2} (\partial \sigma)^3 \partial^2 \sigma - \frac{3}{7} \partial \sigma (\partial^2 \sigma)^2 + \frac{11}{5} (\partial \sigma)^2 \partial^3 \sigma \\
&\quad - \frac{1}{28} \partial^2 \sigma \partial^3 \sigma + \frac{1}{14} \partial \sigma \partial^4 \sigma + \frac{1}{280} \partial^5 \sigma.
\end{align*}

(23)

Note that the conformal spins of $\eta$ and $\xi$ remain unchanged, while the conformal spins of $\phi$ and $\bar{\phi}$ change in correlation with those of $e^\sigma$ and $e^{-\sigma}$.

4. $w_\infty$ from $W_\infty$ in the Complex-boson Model

It is known that algebraically, $W_\infty$ admits a contraction to $w_\infty$ [1,2], which can be viewed as the classical limit of the quantum $W_\infty$ algebras. From the viewpoint of field theory, this contraction seemed to be less straightforward. In particular, it was not clear how $w_\infty$ gravity might arise in the context of $W_\infty$ gravity where the generating currents of $W_\infty$ were built from bilinears of complex boson [7,14] or fermion [8,9]. This puzzle was recently resolved for the model of complex fermion in Ref. [15].

The classical $w_\infty$ gravity was first introduced in the context of a single real scalar $\chi$ [16], whose quantisation deforms the $w_\infty$ symmetry to the quantum $W_\infty$ symmetry [15]. In the generating currents $V^i$ of $W_\infty$, the most nonlinear terms $(\partial \phi)^{i+2}$ furnish the original $w_\infty$ symmetry, when only single contractions are included in the OPEs, while additional terms are quantum corrections necessary for anomaly cancellations. To this end, a beautiful phenomenon occurs that the generating currents of $W_\infty$ can be interpreted as the bosonisation of the complex-fermion realisation of $W_\infty$. Therefore in order to contract the bilinear-fermion realisation of $W_\infty$ to a realisation of (classical) $w_\infty$, it is necessary to invoke an intermediate step, namely the bosonisation procedure, which makes it transparent how a classical limit of $W_\infty$ can emerge.

Analogously one can apply this treatment to the case of the complex-boson realisation of $W_\infty$, so as to obtain a classical limit, i.e. $w_\infty$. The bosonisation of the complex bosons $\phi$ and $\bar{\phi}$ has been carried out in the previous section, which sets the stage for extracting the $w_\infty$ algebra. However, the bosonised currents
given in (19)-(21) still involve a pair of fermions $\eta$ and $\xi$ in bilinear forms. For the same reason as mentioned above, the bilinear forms of fermion inhibit a direct extraction of $w_\infty$. Thus we need to bosonise $\eta$ and $\xi$ also.

Let $\rho$ be the scalar field that bosonises $\eta$ and $\xi$, according to

$$
\eta \equiv e^\rho, \quad \xi \equiv e^{-\rho},
$$

where $\rho$ satisfies OPE as follows,

$$
\rho(z)\rho(w) \sim \log(z-w).
$$

We find that the $Q^{(j)}$ appearing in (20) is given by

$$
Q^{(j)} = \frac{1}{j+1} \partial R^{(j+1)} - \frac{1}{j+2} R^{(j+2)},
$$

where

$$
R^{(j)} \equiv e^{-\rho} \partial^j e^\rho.
$$

Substituting this expression of $Q^{(j)}$ into (20), we have the currents $\tilde{V}^i$ in (19) expressed in terms of two scalar fields $\rho$ and $\sigma$. The first few currents are given by

$$
\tilde{V}^{-1} = \partial \sigma, \quad \tilde{V}^{0} = \frac{1}{2} (\partial \rho)^2 - \frac{1}{2} \partial^2 \rho - \frac{1}{2} (\partial \sigma)^2, \\
\tilde{V}^{1} = - \frac{2}{3} (\partial \rho)^3 - \partial \sigma (\partial \rho)^2 + \rho \rho' \rho + \partial \sigma \partial^2 \rho - \frac{1}{6} \partial^3 \rho + \frac{1}{6} (\partial \sigma)^3, \\
\tilde{V}^{2} = \frac{2}{3} (\partial \sigma)^2 (\partial \rho)^2 + 2 \partial \sigma (\partial \rho)^3 + \frac{2}{3} (\partial \rho)^4 \\
- \frac{3}{2} (\partial \sigma)^2 \partial^2 \rho - 3 \partial \sigma \partial \rho \partial^2 \rho - \frac{3}{2} (\partial \rho)^2 \partial^2 \rho \\
- \frac{3}{20} (\partial \rho)^2 \partial \sigma + \frac{3}{20} \partial \rho \partial^3 \rho - \frac{1}{20} \partial^4 \rho \\
- \frac{1}{4} (\partial \sigma)^4 - \frac{1}{12} \partial \sigma \partial^3 \sigma + \frac{1}{20} (\partial^2 \sigma)^2.
$$

Following the strategy of Ref. [15], we shall keep the most nonlinear terms in the scalars $\rho$ and $\sigma$ to obtain a realisation of the classical limit of $W_{1+\infty}$. For example, $R^{(j)}$ (26) can be rewritten as

$$
R^{(j)} = (\partial + \partial \rho)^j \cdot 1,
$$

of which the classical limit is $(\partial \rho)^j$. It follows that the classical limit of $Q^{(j)}$ (26) is given by

$$
Q^{(j)} = - \frac{1}{j+2} (\partial \rho)^{j+2}.
$$

Similarly one can work out the classical limit of $P^{(j)}$. The classical limit of the currents of $W_{1+\infty}$ then reads

$$
\tilde{v}^i = \sum_{k=0}^{i+1} \sum_{j=0}^{i} (-k) (k+1)(i+1)! \frac{1}{j! (i-j)! (j+2)!} (\partial \rho)^{j+2} (\partial \sigma)^{i-j} \\
+ \sum_{k=0}^{i+1} (-1)^k k! (i+1)! \frac{1}{i+2} (\partial \sigma)^{i+2}.
$$
This expression can be simplified by first performing the summation over $k$, which involves the summable $_2F_1$ hypergeometric function, and performing the summation over $j$ for the first term in (31). The final result reads

$$\tilde{v}^i = (-)^{i+1} \frac{1}{i+1} (\partial \rho + \partial \sigma)^{i+2} - (-)^i \partial \sigma (\partial \rho + \partial \sigma)^{i+1}. \tag{32}$$

For concreteness, we shall present the explicit expression of the first few currents.

$$\tilde{v}^{-1} = \partial \sigma,$$

$$\tilde{v}^0 = \frac{1}{2} (\partial \rho)^2 - \frac{1}{2} (\partial \sigma)^2,$$

$$\tilde{v}^1 = -\frac{2}{3} (\partial \rho)^3 - \partial \sigma (\partial \rho)^2 + \frac{1}{3} (\partial \sigma)^3,$$

$$\tilde{v}^2 = \frac{4}{3} (\partial \sigma)^2 (\partial \rho)^2 + 2 \partial \sigma (\partial \rho)^3 + \frac{4}{3} (\partial \rho)^4 - \frac{4}{3} (\partial \sigma)^4. \tag{33}$$

The generating currents given in (32) form a realisation of $w_{1+\infty}$ in the classical sense i.e. only single contraction in the OPE. One may explicitly check that this is indeed the case, bearing in mind a useful lemma that $\Pi(z)\Pi(w) \sim 0$, where $\Pi \equiv \partial \rho + \partial \sigma$. This closure in the classical sense can also be demonstrated on the transformations of $\rho$ and $\sigma$, which follow from the general rule for the transformations of a generic field $\Phi$ generated by the currents $\tilde{v}^i$ with parameter $k_i$

$$\delta \Phi = \oint \frac{dz}{2\pi i} k_i(z) \tilde{v}^i(z) \Phi. \tag{34}$$

For example, the first few transformations are given by

$$\delta_{-1} \rho = 0,$$

$$\delta_0 \rho = k_0 \partial \rho,$$

$$\delta_1 \rho = -2k_1 (\partial \rho)^2 - 2k_1 \partial \rho \partial \sigma,$$

$$\delta_2 \rho = 3k_2 (\partial \rho)^3 + 6k_2 (\partial \rho)^2 \partial \sigma + 3k_2 \partial \rho (\partial \sigma)^2. \tag{35}$$

$$\delta_{-1} \sigma = -k_{-1},$$

$$\delta_0 \sigma = k_0 \partial \sigma,$$

$$\delta_1 \sigma = k_1 (\partial \rho)^2 - k_1 (\partial \sigma)^2,$$

$$\delta_2 \sigma = -2k_2 (\partial \rho)^3 - 3k_2 (\partial \rho)^2 \partial \sigma + k_2 (\partial \sigma)^3.$$

For completeness, the general formulae for the transformations of $\rho$ and $\sigma$ are given by

$$\delta_{k,i} \rho = (-)^{l} (\partial + 1) k_{l} (\partial \rho + \partial \sigma)^{l} \partial \rho,$$

$$\delta_{k,i} \sigma = (-)^{l+1} k_{l} (\partial \rho + \partial \sigma)^{l} (\partial \rho - \partial \sigma). \tag{36}$$

It is easy to check the following closure relations of $w_{1+\infty}$

$$[\delta_{k,i}, \delta_{k,j}] \rho = \delta_{k_{i+j}} \rho, \quad [\delta_{k,i}, \delta_{k,j}] \sigma = \delta_{k_{i+j}} \sigma, \tag{37}$$

where

$$k_{i+j} = (j+1)k_i k_j - (i+1)k_i k_j. \tag{38}$$
One can also perform the same procedure to the case of $W_\infty$ to extract a realisation of $w_\infty$. In fact, since the rotation of basis needed for leaving out the spin-1 current in $W_{1+\infty}$ amounts to lower order terms only, which do not effect the most non-linear terms, we conclude that the realisation for $w_\infty$ is identical to the one given in (32).

5. Conclusions and Discussions

In this paper we have bosonised the complex-boson realisation of the $W_\infty$ algebras. We have found non-linear realisations of $W_\infty$ in terms of a scalar field and a pair of fermions. The couplings between the scalar and the fermions are rather subtle and crucial for the closure. We have also shown that, by bosonising the fermions further and keeping the most non-linear terms only, the classical limits of $W_\infty$ can be achieved. We explicitly demonstrate the closure of these limits, namely $w_\infty$ and $w_{1+\infty}$, in the classical sense.

Given the fact that the classical $w_\infty$ has been realised in terms of two scalars $\rho$ and $\sigma$, it would be interesting to find certain group theoretic structure for them, i.e. to view $\rho$ and $\sigma$ as components of certain multiplet, as the examples in Ref. [16].

It is also worth noting that the two-scalar realisation of $w_\infty$ can be used as a matter system coupled to $w_\infty$ gravity, whose quantisation presumably deforms the symmetry structure to quantum $W_\infty$, analogous to the quantisation of single scalar coupled to $w_\infty$ gravity [15]. The resulting theory of two scalar fields corresponds to the complex scalar $\varphi$ coupled to $W_\infty$ gravity given in [17].

It has been observed [10] that the single-scalar realisation of $W_\infty$ at $c = -2$ can be viewed as a realisation of $W_3$ or $W_N$ under appropriate normal ordering procedure with respect to the generating currents. This observation has lent support to the idea of $W_\infty$ being the universal $W$-algebra. It would be interesting to see some further evidence from this two-scalar realisation of $W_\infty$ at $c = 2$. 

10
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