Renormalization group for the probability distribution of magnetic impurities in a random-field $\phi^4$ model

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Extending the usual Ginzburg-Landau theory for the random-field Ising model, the possibility of dimensional reduction is reconsidered. A renormalization group for the probability distribution of magnetic impurities is applied. New parameters corresponding to the extra $\phi^4$ coupling constants in the replica Hamiltonian are introduced. Although they do not affect the critical phenomena near the upper critical dimension, they can when dimensions are lowered.

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I. INTRODUCTION

The random-field Ising model (RFIM) is the Ising model coupled to a random magnetic field. Although it has been extensively studied for about three decades, there remain many unresolved problems, one of which concerns dimensional reduction.

Dimensional reduction claims that the critical behavior of the $d$ dimensional RFIM is equivalent to the $d-2$ dimensional pure Ising model. Following this argument, phase transition does not occur in the three-dimensional RFIM. However, it is rigorously proved that the phase transition does occur in this model, so that dimensional reduction does not hold at least $d=3$. Various numerical computations are performed to obtain critical exponents in three dimensions. On the other hand, it is not understood whether dimensional reduction works in other dimensions lower than the upper critical dimension, which is believed to be six.

Since standard perturbation for the Ginzburg-Landau (GL) theory of the RFIM simply leads to the result consistent with dimensional reduction, other approaches were applied. Schwartz et al proposed modification of dimensional reduction, which indicates correspondence between the $d$ dimensional RFIM and the pure Ising model in $d'=d-2-\eta(d')$. Mezard and Young suggested the replica symmetry breaking of the RFIM by extrapolation from $1/m$ expansion, where $m$ is the components of the spin. Lancaster et al computed exponents, paying attention to many solutions of mean field equations. Although these works support the breakdown of dimensional reduction in other dimensions, it has not yet been settled.

The breakdown of standard perturbation may be caused by overlooked relevant operators. In $4+\epsilon$ dimensions, Fisher and Feldman pointed out that there are infinitely many relevant operators in $n$-component random spin models. Near the upper critical dimension, Brézin and de Dominicis investigated that the GL Hamiltonian corresponding to the RFIM has not only the usual $\phi^4$ interaction but also the following extra $\phi^4$ interactions:

$$\sum_{\alpha} \phi_\alpha^4,$$

where $\alpha$ denotes the replica index, but also the following extra $\phi^4$ interactions:

$$\sum_{\alpha,\beta} \phi_\alpha^3\phi_\beta, \sum_{\alpha,\beta} \phi_\alpha^2\phi_\beta^2, \sum_{\alpha,\beta} \phi_\alpha^2\phi_\beta\phi_\gamma, \sum_{\alpha,\beta,\gamma} \phi_\alpha\phi_\beta\phi_\gamma\phi_\delta.$$  

It means that the space of the coupling constants extends to the five dimensions and that a renormalization-group (RG) trajectory should be considered in five-dimensional space. They claimed that dimensional reduction does not work even near the upper critical dimension since the non-trivial fixed point of $O(\epsilon)$ becomes unstable in $d=6-\epsilon$.

However, their analysis has ambiguity that originates from the zero-replica limit. In fact, they proposed two different limiting procedures and derived two sets of beta functions quantitatively different each other.

In this paper, we circumvent the ambiguity and reconsider RG flow in the extended coupling-constant space. To this end, we use RG for the random probability distribution that controls random potentials including the magnetic field. This method was initiated by Harris and Lubensky in order to investigate a randomly diluted magnetic system with non-magnetic impurities. In this case, it is confirmed that the replica method is consistent with the Harris-Lubensky method. Here we extend the random probability distribution adopted in the previous literatures. It corre-
sponds to taking into account the replica interactions (2) as well as (1).

According to Refs. [17, 18], the coupling constants in the Hamiltonian are extended to inhomogeneous random potentials that are correlated to each other following the Hamiltonian are extended to inhomogeneous randomness as well as (1). Distribution $P$ is characterized by non-trivial cumulants with some parameters. Therefore the change of $P$ defines RG flow in that parameter space. We can investigate critical behavior from RG flow. An advantage of this method compared to the replica method is that any ambiguities originating from limiting procedures do not arise.

This paper is organized as follows: we show in the next section the RG for probability distribution of the Gaussian random magnetic impurities. We can naturally introduce five parameters that completely correspond to the five coupling constants for the interactions (1) and (2) in the replica Hamiltonian. In Sec. III we analyze the RG by perturbation and derive the recursion equations of the parameters that specify random-field distribution. We also compare the result with that of the replica method [17]. In Sec. IV we introduce expansion parameters of physical quantities that are convenient for analysis near the upper critical dimension. In the language of the replica method, it means to determine the scaling dimensions of the operators in (1) and (2). We show that the interaction (1) is the unique relevant operator near the upper critical dimension in all the $\phi^4$ interactions. Then arguments of dimensional reduction can survive near the upper critical dimension. However, as the dimensions are lowered, other interactions can become relevant, which cause the breakdown of the dimensional reduction. This picture is consistent with high-temperature expansion studied by Houghton et al [14]. The last section is devoted to summary and discussion.

II. RENORMALIZATION-GROUP TRANSFORMATION FOR THE RANDOM PROBABILITY DISTRIBUTION

In this section, we formulate the renormalization-group transformation (RGT) for the random probability distribution in the presence of magnetic impurities following Refs. [17, 18]. The GL Hamiltonian to the RFIM is usually described by (3):

$$\beta H = \frac{1}{2} \int_{k_1} (k_1^2 + t) \phi(k_1)\phi(-k_1) + \int_{k_1} v_1(k_1)\phi(k_1) + \frac{u_1}{4!} \int_{k_1, k_2, k_3} \phi(k_1)\phi(k_2)\phi(k_3)\phi(-k_1 - k_2 - k_3)$$

where the momenta $k_1, k_2, k_3$ belong to $\mathcal{K} = \{k : 0 \leq |k| \leq \Lambda\}$ (4) and the integral means

$$\int_{k_1 \ldots k_j} \equiv \int_{k_j \in \mathcal{K}} \frac{d^d k_j}{(2\pi)^d} \ldots \int_{k_j \in \mathcal{K}} \frac{d^d k_j}{(2\pi)^d}$$

(5)

The random magnetic field $v_1(k)$ follows the Gaussian distribution $P_0[v_1]$

$$P_0[v_1] = Ne^{-\frac{1}{2} \int_{k} v_1(k) v_1(-k)}$$

$$N \equiv \left( \int Dv_1 e^{-\frac{1}{2} \int_{k} v_1(k) v_1(-k)} \right)^{-1}$$

$$Dv_1 \equiv \prod_{k \in \mathcal{K}} dv_1(k).$$

(6)

We treat the random field $v_1$ as an external field for a while and examine the RGT. Namely, we first integrate higher-momentum components of $\phi$ and then perform the rescaling of the potentials and the fields appropriately. Let us observe correction terms that appear in higher-momentum integration. Let $G(q)$ be the free propagator: $G(q) = (q^2 + t)^{-1}$. By integrating the higher-momentum components by perturbation in $u_1$, we have the following correction to the $\phi^2$ term in the leading order:

$$u_1 \int_q v_1(q)G(q)v_1(p_1 + p_2 + q)G(p_1 + p_2 + q)\phi(p_1)\phi(p_2).$$

(7)

Here integration is carried out on higher-momentum space $\mathcal{K}_>$ and $\phi(p_1)$ is a lower-momentum component. Similarly, the $\phi^4$ term gets

$$u_1^2 \int_{q_1, q_2} \delta \left( \sum_{i=1}^4 p_i + \sum_{j=1}^2 q_j \right) \prod_{i=1}^4 v_1(q_i)G(q_i) \times G(p_1 + p_2 + q_1) \prod_{k=1}^4 \phi(p_i).$$

(8)

Further, a cubic interaction emerges through, for example,

$$u_1^2 \int_{q_1, q_2} \delta \left( \sum_{i=1}^3 p_i + \sum_{j=1}^3 q_j \right) \prod_{i=1}^3 v_1(q_i)G(q_i) \times G(p_1 + p_2 + q_1) \prod_{k=1}^3 \phi(p_i).$$

(9)

These corrections are depicted in FIG. 11 Eqs. (11, 12) and (13) respectively correspond to (a), (b) and (c) of the figure. The correction terms indicate that the coefficients of the $\phi^2$ and $\phi^4$ terms in the Hamiltonian no longer possess translational invariance. In order to absorb those terms, we need to extend $(k^2 + t)$ and $u_1$ to
Let us integrate over the higher-momentum components of the field. The remaining part is denoted \( S \) and \( \beta H \) are respectively defined as

\[
S[\phi; \rho] = \sum_{l=1}^{4} \frac{1}{l!} \int v_l(k_1, ..., k_l) \phi(k_1) \cdots \phi(k_l),
\]

where \( \rho \) represents all of the inhomogeneous potentials:

\[
\rho = (v_1, v_2, v_3, v_4).
\]

Starting with the Hamiltonian (10), we describe the RGT definitely. The higher and lower-momentum spaces are respectively defined as

\[
\mathcal{K}_{>} = \{ q : L^{-1} \Lambda < |q| \leq \Lambda \}
\]

and

\[
\mathcal{K}_{<} = \{ p : 0 < |p| \leq L^{-1} \Lambda \}
\]

with \( L > 1 \). The Hamiltonian \( S \) is decomposed into \( S_{<} \) and \( S_{>}, \) where \( S_{<} \) consists only of the lower momentum components of the field. The remaining part is denoted by \( S_{>}, \) i.e.,

\[
S = S_{<} + S_{>}.
\]

Let us integrate over the higher-momentum components \( \phi(q) \) in the partition function. Namely,

\[
Z = \int \prod_{k \in \mathcal{K}_{>}} d\phi(k) e^{-S_{>}}
\]

\[
= \int \prod_{p \in \mathcal{K}_{<}} d\phi(p) e^{-S_{<}} \int \prod_{q \in \mathcal{K}_{>}} d\phi(q) e^{-S_{>}}
\]

\[
= \int \prod_{p \in \mathcal{K}_{<}} d\phi(p) e^{-S_{<} - \delta S},
\]

where \( \delta S \) can be written as

\[
\delta S = -\ln \prod_{k \in \mathcal{K}_{>}} d\phi(k) e^{-S_{>}}
\]

\[
= \sum_{l=1}^{4} \frac{1}{l!} \int \delta v_l(p_1, ..., p_l) \phi(p_1) \cdots \phi(p_l)
\]

+ irrelevant terms.

Next the scaling transformation is performed as

\[
\phi(p) = L^\theta \phi'(k),
\]

where \( k \) is related to \( p \) by

\[
k = Lp.
\]

Defining the new inhomogeneous potentials \( v'_l \) by

\[
v'_l(k_1, ..., k_l) \equiv L^{(\theta - d)} (v_l(p_1, ..., p_l) + \delta v_l(p_1, ..., p_l)),
\]

the Hamiltonian turns back to the original form:

\[
S_{<} + \delta S = \sum_{l=1}^{4} \frac{1}{l!} \int v'_l(k_1, ..., k_l) \phi'(k_1) \cdots \phi'(k_l)
\]

\[
= S[\phi'; \rho'],
\]

where \( \rho' = (v'_1, v'_2, v'_3, v'_4) \). We then get the RGT for the potential \( \rho \rightarrow \rho' \). Furthermore, a correlation function transforms as

\[
\langle \phi(p_1), \cdots, \phi(p_n) \rangle_{S[\phi'; \rho']} = L^{n\theta} \langle \phi(k_1), \cdots, \phi(k_n) \rangle_{S[\phi; \rho]},
\]

where \( \langle \cdot \rangle_{S} \) means the thermal average using the Boltzmann weight \( e^{-S} \).

We proceed to the average over the random potentials. It should be noted that Eqs. (7), (8) and (9) show that \( v_2, v_3 \) and \( v_4 \) come to have non-trivial correlations with each other. Therefore we can regard them as random potentials that obey some probability distribution as well as \( v_1 \). We shall denote probability distribution by \( P[\rho] \). For example, if we take Eq. (9) as the initial Hamiltonian, the corresponding distribution \( P_1 \) is given by
Namely, potential average of the correlation function becomes pushed into change of variables with distribution $P$ where

$$F_X \rho \delta \left( \delta \left( v_4 (k_1, k_2, k_3, k_4) - u_1 (2\pi)^d \left( \sum_{j=1}^4 k_j \right) \right) \right).$$

(22)

The RGT for the random potential $\rho \rightarrow \rho'$ can be pushed into change of $P$ by the following rule:

$$\int F[\rho'] P[\rho] \mathcal{D}\rho = \int F[\rho] P'[\rho] \mathcal{D}\rho,$$

(23)

where $F$ is an arbitrary functional of $\rho$ and

$$\mathcal{D}\rho \equiv \prod_{i=1}^4 \prod_{k_1, \ldots, k_i} dv_i(k_1, \ldots, k_i).$$

(24)

Namely, $P'$ is formally written as

$$P'[\rho'] = P[\rho(\rho')] \left| \frac{\partial \rho(\rho')}{\partial \rho} \right|.\left(25\right)$$

Eq. (20) defines RGT $P \rightarrow P'$, keeping the Hamiltonian invariant. For instance, using Eq. (21), the random-potential average of the correlation function becomes

$$P[\rho] \langle \phi(p_1), \ldots, \phi(p_n) \rangle_{S[\phi; \rho]} \mathcal{D}\rho = \int P'[\rho] L^{n_0} \langle \phi(k_1), \ldots, \phi(k_n) \rangle_{S[\phi; \rho]} \mathcal{D}\rho.$$  

(26)

In the practical computation of the random-potential average, we give cumulants among random potentials instead of giving explicit form to the probability distribution. First, as in the original theory, we put

$$\begin{align*}
[v_1(k_1); v_2(k_2)]_{P[\rho]} &= \Delta (2\pi)^d \delta(k_1 + k_2) \\
[v_2(k_1, k_2)]_{P[\rho]} &= (k_1^2 + t) (2\pi)^d \delta(k_1 + k_2) \\
[v_4(k_1, k_2, k_3, k_4)]_{P[\rho]} &= u_1 (2\pi)^d \delta \left( \sum_{j=1}^4 k_j \right) \quad (27)
\end{align*}$$

where $[\cdot]_{P[\rho]}$ means to take the average over the random variables with distribution $P[\rho]$. The semicolon in the bracket means the cumulant product: e.g., $[X; Y]_{P[\rho]} \equiv [XY]_{P[\rho]} - [X]_{P[\rho]} [Y]_{P[\rho]}$. We assume that there is no long-range correlation between random variables. In this assumption, the following non-vanishing cumulants are added to Eq. (26):

$$\begin{align*}
[v_1(k_1); v_2(k_2); v_3(k_3, k_4)]_{P[\rho]} &= (k_1^2 + t) (2\pi)^d \delta(k_1 + k_2) \\
[v_2(k_1, k_2); v_3(k_3)]_{P[\rho]} &= (k_1^2 + t) (2\pi)^d \delta(k_1 + k_2) \\
[v_4(k_1, k_2, k_3, k_4)]_{P[\rho]} &= u_1 (2\pi)^d \delta \left( \sum_{j=1}^4 k_j \right).
\end{align*}$$

(28)

Note that we respect the symmetry

$$v_1 \rightarrow -v_1, \quad \phi \rightarrow -\phi,$$

(29)

so that

$$[v_1(k_1)]_{P[\rho]} = [v_3(k_1, k_2, k_3)]_{P[\rho]} = 0.$$  

(30)

The parameters $u_j (j = 1, \ldots, 5)$ have the scaling dimension $4 - d$ (measured by the inverse length). Cumulants other than Eqs. (26) and (28) are ignored since they are associated with higher-dimensional parameters. Probability distribution $P$ is characterized by parameters $(t, \Delta, u_1, \ldots, u_5) \equiv \mu$. Transformed distribution $P'$ is similarly characterized by $(t', \Delta', u_1', \ldots, u_5') \equiv \mu'$. Namely, $\mu'$ is obtained taking the random-potential average of Eqs. (27) and (28) in use of $P'[\rho]$ instead of $P[\rho]$.

Transformation $\mu \rightarrow \mu'$ is analyzed using the following formula:

$$\begin{align*}
[v_1(k_1), \ldots, k_1^1; \ldots; v_n(k_1^n, \ldots, k_1^n)]_{P[\rho]} &= \left[ v_1'(k_1, \ldots, k_1^n); \ldots; v_n'(k_1^n, \ldots, k_1^n) \right]_{P'[\rho]}.
\end{align*}$$

(31)

which is easily checked from Eq. (27). We apply it to the non-vanishing cumulants. By definition, the left-hand side can be written in $\mu'$. On the other hand, if the transformed potentials are described by the original ones employing Eq. (14), the right-hand side can be evaluated in terms of $\mu$. In this way, $\mu'$ is expressed in terms of $\mu$.

III. PERTURBATIVE RENORMALIZATION GROUP

A. Diagrammatic expansion

In this section we obtain transformation $\mu \rightarrow \mu'$ by perturbation. We consider all $u_j$s as small parameters
and express $\mu'$ up to $O(u_iu_j)$. On the other hand, we do not regard $t$ and $\Delta$ as small since they are relevant parameters of the dimension 2.

We can formulate the perturbative expansion by adopting the random-potential average of $v_2(q_1,q_2)$ and the linear term as the unperturbed Hamiltonian $S_0$:

$$
S_0 = \frac{1}{2} \int_{k_1,k_2} [v_2(k_1,k_2)] \rho_{(\rho)} \phi(k_1)\phi(k_2) + \int_k v_1(-k)\phi(k)
= \frac{1}{2} \int_k (k^2 + t)\varphi(-k) - \frac{1}{2} \int_k v_1(-k)G(k)v_1(k),
$$

where

$$
\varphi(k) \equiv \phi(k) + G(k)v_1(k)
$$

and $G(k)$ is the linear term as the unperturbed Hamiltonian 

$$
G(k) = \frac{1}{k^2 + t}.
$$

The remaining terms are denoted by $V$ and treated as the perturbative Hamiltonian. Let

$$
V = \sum_{j=2}^{4} V_j,
$$

where

$$
V_2 = \frac{1}{2} \int_{k_1,k_2} (v_2(k_1,k_2) - [v_2(k_1,k_2)] \rho_{(\rho)}) \phi(k_1)\phi(k_2)
V_j = \frac{1}{j!} \int_{k_1,\ldots,k_j} v_j(k_1,\ldots,k_j)\phi(k_1)\cdots\phi(k_j),
$$

where $j = 3, 4$.

As we have performed in Eq. (14), we divide $S_0$ and $V$ into

$$
S_0 = S_0^> + S_0^<
V = V^> + V^<
$$

respectively. The free part of the partition function is denoted by

$$
Z_0 = \int \prod_{q \in K_>} d\phi(q) e^{-S_0^>}.
$$

Writing

$$
\langle X \rangle > = \frac{1}{Z_0} \int \prod_{q \in K_>} d\phi(q) X e^{-S_0^>}.
$$

$\delta S$ in Eq. (16) is expanded as

$$
-\delta S = \log Z_0 + \log \langle e^{-V^>} \rangle >
= \log Z_0 - \langle V^> \rangle > + \frac{1}{2} \langle V^>; V^> \rangle >
- \frac{1}{3!} \langle V^>; V^>; V^> \rangle > + O(V^>^4),
$$

where the semicolon represents the cumulant with respect to the thermal average $\langle \rangle >$. Note that, in practical computation, $\varphi(q)$ in $V^>$ is understood as $\varphi(q) - G(q)v_1(q)$ and the Wick theorem is applied to $\varphi(q)$. Comparing Eqs. (16) and (39), we can obtain the perturbative expansion of $\delta v_1(l_1,\ldots,l_t)$, which leads to transformed potential $v'_l$ defined in Eq. (10). Inserting the resultant $v'_l$ into the right-hand side of Eq. (11), we can express the transformed parameters in terms of the original ones. In order to proceed with this program efficiently, we will use the diagrammatic technique.

Let us depict $v'_l$ by a shaded circle with $l$ dashed lines. FIG. 2 exemplifies the case of $l = 2$. The shaded circle contains $\delta v_1$, which is expanded by connected diagrams, according to Eq. (39). The random potential $v_1(k_1,\ldots,k_l)$ makes an $l$-point vertex. In particular, $v_1(q)(q \in K_>)$ is expressed by an open circle. It should be noted that total momentum is not conserved at the vertex $v_1(k_1,\ldots,k_l)$.

When we take the random-potential average, nonvanishing cumulants $[v_4]_{\rho(\rho)}$, $[v_3; v_1]_{\rho(\rho)}$, $[v_2; v_2]_{\rho(\rho)}$, $[v_2; v_1; v_1]_{\rho(\rho)}$ and $[v_1; v_1; v_1; v_1]_{\rho(\rho)}$ can be graphically represented as four-point vertices at which the total momenta, carried by the four lines, are conserved due to the delta function in Eq. (28). See FIG. 3.

Further, the cumulant $[v_1; v_1]_{\rho(\rho)}$ tells us that two internal lines ended with the open circles are merged by the random-potential average and produce $\Delta$. We here depict it by a cross, $\times$, as in FIG. 4. The cross reminds us that the two internal lines are not connected with respect to the thermal average.
FIG. 4: Merging two open circles

The right-hand side of Eq. (31) is graphically represented as the left-hand side of FIG. 5. When we expand the shaded circles in FIG. 5 we get a summation of the diagrams that contain several connected components, as shown in the right-hand side of the figure. If we want to know $u'_i$ in $O(u_j u_k)$, we need to consider the one-loop diagrams made of the vertices in FIG. 3 and $\times$, including the numerical factors of those diagrams. Note that a one-loop diagram that contains a connected component without external legs does not appear in the right-hand side of FIG. 5 because the shaded circles in the left-hand side consist of only connected diagrams. In particular, it is ruled out that there are two or more $\times$s on an internal line.

In order to write down all one-loop diagrams in the right-hand side of FIG. 5 we must first consider all the admissible diagrams without $\times$, and then put $\times$ on as many internal lines as possible. After we obtain the set of all one-loop diagrams, we have to compute their numerical factors. Although the procedure is fulfilled in appendix A, here we present the following examples.

B. Example

Let us consider FIG. 6(a). Putting $\times$ on the internal lines, (b), (c), and (d) in the figure are generated. However, (c) and (d) are excluded because they contain connected components without external lines. Counting the external lines of the connected components in FIG. 6(a), we find that it makes a contribution to $u'_4$, while FIG. 6(b) contributes to $u'_5$. First we compute the contribution from FIG. 6(a) to

$$u'_4 (2\pi)^d \delta \left( \sum_{k=1}^{4} k_i \right) = [v'_2(k_1, k_2); v'_1(k_3); v'_1(k_4)]_{P_{[p]}} \quad (40)$$

Using Eq. (19), FIG. 6(a) comes from

$$L^{4(\theta - d)} [\delta v_2(p_1, p_2); \delta v_1(p_3); v_1(p_4)]_{P_{[p]}} + (p_3 \leftrightarrow p_4) \quad (41)$$

The second term arises because the roles of $v'_2(k_1)$ and $v'_1(k_4)$ can interchange. The two terms in the right-hand side of Eq. (41) equally contribute to $u'_4$. In general, if a cumulant contains multiple potentials of the same kind, the number of ways of interchanging should be counted. We shall denote the number with $n_R$. As for the present case,

$$n_R = 2. \quad (42)$$

Now we compute the first term. Defining

$$V^{(m)}_i (p_1, \ldots, p_m) = \frac{1}{l!} \left( \begin{array}{c} l \\ m \end{array} \right) \times \int_{q_1, \ldots, q_{l-m}} v_i (p_1, \ldots, p_m, q_1, \ldots, q_{l-m}) \prod_{i=1}^{l-m} \phi(q_i), \quad (43)$$

FIG. 5: Perturbative expansion for cumulants contributing $u'_j$ ($j = 1, \ldots, 5$)

FIG. 6: Diagrams generated from (a) by putting $\times$. 

we show explicitly the terms in $\delta v_2(p_1, p_2)$ necessary for making FIG. 4(a) as

$$\delta v_2(p_1, p_2) = -2! \frac{1}{2!} \left\langle V_2^{(1)}(p_1); V_2^{(1)}(p_2) \right\rangle > + \cdots$$

$$= a_1 \int_{q_1} > v_2(p_1, q_1)v_2(p_2, -q_1)G(q_1) + \cdots. \tag{44}$$

Here the dots in the right-hand side indicate terms not employed for FIG. 4(a). In the first equality, the factor 2! originates from the normalization of $\delta v_2(p_1, p_2)$, while 1/2! comes from the expansion of $e^{-v_2}$. The minus sign is so put because it is a second-order perturbation term. Performing the thermal average, we can easily check that

$$a_1 = -1. \tag{45}$$

Similarly,

$$\delta v_1(p) = -\left\langle V_2^{(1)}(p) \right\rangle > + \cdots$$

$$= a_2 \int_{q_2} > v_2(p_3, q_2)v_1(-q_2)G(q_2) \tag{46}$$

with

$$a_2 = -1. \tag{47}$$

The factors $a_1$ and $a_2$ are associated with the connected components of the diagram we have computed. We shall denote the product of them with $n_A$, i.e.,

$$n_A = a_1a_2 = 1. \tag{48}$$

Next we go to the random-potential average. Inserting Eqs. 44 and 46 into the first term of Eq. 11, the random-potential average becomes

$$L^{4(\theta-d)} \int_{q_1, q_2} > G(q_1)G(q_2) [v_2(p_1, q_1)v_2(p_2, -q_1); v_2(p_3, q_2)v_1(-q_2); v_1(p_4)]_{P[p]}$$

$$= L^{4(\theta-d)} \int_{q_1, q_2} > G(q_1)G(q_2) [v_2(p_1, q_1); v_2(p_3, q_2)]_{P[p]} [v_2(p_2, -q_1); v_1(-q_2); v_1(p_4)]_{P[p]}$$

$$+ (v_2(p_1, q_1) \leftrightarrow v_2(p_2, -q_1))$$

$$= L^{4\theta-3d} u_3 u_4 (2\pi)^d \delta \left(\sum_{i=1}^4 k_i\right) \int_{q} > (G(q)G(p_2 + p_4 - q) + G(q)G(p_1 + p_4 + q)). \tag{49}$$

Note that there are two ways of contraction with respect to the random-potential average in the first equality. We denote the number of ways of the contraction by $n_C$. That is,

$$n_C = 2 \tag{50}$$

in the case of FIG. 4(a).

Thus the correction to $u_4$ by way of FIG. 4(a) is read from Eq. 19, combining with the second term of Eq. 41. Letting $p_i = 0 \ (i = 1, \ldots, 4)$ in $G$, we find that the numerical factor $n_F$ associated with the diagram is

$$n_F = n_A n_R n_C = 4. \tag{51}$$

Therefore the correction to $u_4$ is

$$n_F L^{4\theta-3d} u_3 u_4 \int_{q} > G^2(q) = 4L^{4\theta-3d} u_3 u_4 \int_{q} > G^2(q). \tag{52}$$

Let us turn to FIG. 4(b), which affects $u_5' \propto [v'_1(p_1); v'_1(p_2); v'_1(p_3); v'_1(p_4)]_{P[p]}$. For creating this diagram, we need three $\left\langle V_2^{(1)}(p) \right\rangle >$ and one $v_1(p)$. Obviously,

$$n_R = 4. \tag{53}$$

The numerical factor associated with the connected component was already computed in Eq. 16. Their product becomes

$$n_A = (-1)^3 \times 1 = -1. \tag{54}$$

Since the number of contractions is equal to the number of ways of contracting $v_2(p, q)$ and $v_1(p)$,

$$n_C = 3. \tag{55}$$
Thus the contribution to \( u'_5 \) of FIG. 3(b) is

\[
\begin{align*}
    n_F \Delta u_2 u_3 L^{49-3d} \int_q G(q)^3, \\
    \text{with} \\
    n_F = n_R n_C n_A = -12.
\end{align*}
\]

and equation in \( O(u_i u_j) \). The coefficients \( n_A, n_C, \) and \( n_R \) associated with each diagrams are presented in appendix A. Here we present the result.

C. Results

After writing all one-loop diagrams appearing in the right-hand side of FIG. 5 we can obtain the recursion with

\[
\begin{align*}
    \delta u_1 &= - \left( 3A_3 \Delta + \frac{3}{2} A_2 \right) u_1^2 + 6A_2 u_1 \tilde{u}_3 \\
    \delta u_2 &= -3A_3 \Delta u_1 \tilde{u}_3 - \frac{3}{2} A_2 u_1 u_2 + 6A_2 u_2 \tilde{u}_3 - 3A_2 u_2^2 + 3A_2 u_1 u_4 \\
    \delta u_3 &= \frac{1}{2} A_4 \Delta^2 u_1^2 - (2A_3 \Delta + A_2) u_1 \tilde{u}_3 + A_2 u_1 u_2 + A_2 u_1 u_4 + 4A_2 \tilde{u}_3^2 - 3A_2 u_2^2 - 6A_2 u_2 \tilde{u}_3 \\
    \delta u_4 &= 3A_3 \Delta u_1 \tilde{u}_3 - \left( 3A_3 \Delta + \frac{1}{2} A_2 \right) u_1 u_4 - 4A_3 \Delta \tilde{u}_3^2 + \frac{1}{2} A_2 u_1 u_5 - A_2 u_2 \tilde{u}_3 + 8A_2 \tilde{u}_3 u_4 \\
    \delta u_5 &= 6A_4 \Delta^2 u_3^2 - 36A_3 \Delta \tilde{u}_3 u_4 - 6A_2 u_2 u_4 + 6A_2 \tilde{u}_3 u_5 + 24A_2 u_2^2,
\end{align*}
\]

where we have defined

\[
A_n \equiv \int_q G(q)^n
\]

and

\[
\tilde{u}_3 \equiv u_2 + u_3.
\]

Similarly, \( \delta t \) and \( \delta \Delta \) are given as

\[
\begin{align*}
    \delta t &= \frac{1}{2} (A_2 \Delta + A_1) u_1 - A_1 \tilde{u}_3 \\
    &+ \left( \left( -\frac{1}{2} A_2 A_3 - \frac{1}{2} A_2 \right) \Delta^2 - \left( \frac{1}{2} A_1 A_3 + \frac{1}{2} B_1 + \frac{1}{4} A_2^2 \right) \Delta - \frac{1}{4} A_1 A_2 - \frac{1}{6} B_0 \right) u_1^2 \\
    &+ \frac{1}{2} A_1 A_2 u_1 u_2 + \left( (A_1 A_3 + A_2^2 + 2B_1) \Delta + A_1 A_2 + B_0 \right) u_1 \tilde{u}_3 \\
    &- (A_1 A_2 + B_0) u_1 u_4 - (A_1 A_2 + B_0) \tilde{u}_3^2 \\
    \delta \Delta &= A_2 \Delta \tilde{u}_3 + u_2 A_1 - 2u_4 A_1 + \frac{1}{6} B_3 \Delta^3 u_1^2 - \left( \frac{1}{2} A_2^2 \Delta + \frac{1}{2} A_1 A_2 + \frac{1}{3} B_0 ^2 \right) u_1 u_2 \\
    &- \left( (A_2 A_3 + 2B_2) \Delta^2 + (A_1 A_3 + B_1) \Delta \right) u_1 \tilde{u}_3 \\
    &+ \left( (A_2^2 + 3B_1) \Delta + A_1 A_2 + B_0 \right) u_1 u_4 - \frac{1}{3} B_0 u_1 u_5 - B_0 \tilde{u}_2^2 \\
    &+ (2A_1 A_2 + 2B_0) u_2 \tilde{u}_3 + (3B_1 + A_2^2 + 2A_1 A_3) \Delta \tilde{u}_3^2 - (4A_1 A_2 + 4B_0) \tilde{u}_3 u_4,
\end{align*}
\]

where

\[
\begin{align*}
    B_0 &= \int_{q_1, q_2} G(q_1)G(q_2)G(q_1 + q_2) \\
    B_1 &= \int_{q_1, q_2} G(q_1)^2G(q_2)G(q_1 + q_2)
\end{align*}
\]

D. Comparison to the replica method

We compare the recursion relations shown above to those obtained by the replica method, where the
perturbative Hamiltonian is
\[ S_{\text{int}}^{\text{rep}} = \frac{u_1^{\text{rep}}}{4!} \sum_{\alpha=1}^{n} \phi_{\alpha}^{4} + \frac{u_2^{\text{rep}}}{3!} \sum_{\alpha,\beta=1}^{n} \phi_{\alpha}^{3} \phi_{\beta}, \]
\[ + \frac{u_3^{\text{rep}}}{8} \sum_{\alpha,\beta=1}^{n} \phi_{\alpha}^{2} \phi_{\beta}^{2} + \frac{u_4^{\text{rep}}}{4!} \sum_{\alpha,\beta=1}^{n} \phi_{\alpha}^{2} \phi_{\beta} \phi_{\gamma}, \]
\[ + \frac{u_5^{\text{rep}}}{4!} \sum_{\alpha,\beta,\gamma,\delta=1}^{n} \phi_{\alpha} \phi_{\beta} \phi_{\gamma} \phi_{\delta}. \]  \hfill (64)

Brézin and de Dominicis proposed the two limiting procedures listed below:

(A) \( u_1^{\text{rep}}, \ldots, u_5^{\text{rep}} \) are fixed, then take \( n \to 0 \).

(B) \( u_1^{\text{rep}}, nu_2^{\text{rep}}, nu_3^{\text{rep}}, n^2u_4^{\text{rep}}, n^3u_5^{\text{rep}} \) are fixed and take \( n \to 0 \).

We can check that the recursion equations (63) - (66) are consistent with the procedure (A) by the following identification
\[ u_i = u_i^{\text{rep}}, \text{ for } i = 1, 4, \]
\[ u_j = -u_j^{\text{rep}}, \text{ for } j = 2, 3, 5. \]  \hfill (65)

More precisely, we pick the correction terms proportional to \( A_4 \) in Eq. (52), which is estimated as \( \log L \) in \( d = 6 - \epsilon \). Putting \( L = e^{\delta t} \) and taking the first order of \( \delta t \), we get the beta functions in Refs. [13, 22]. This is an expected result, because we can obtain the replicated Hamiltonian \( S'_{\text{rep}} \) from the cumulant expansion
\[ S'_{\text{rep}} = \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \left[ \sum_{\alpha} S[\phi_{\alpha}; \rho]; \cdots; \sum_{\beta} S[\phi_{\beta}; \rho] \right] p[\rho]. \]  \hfill (66)

with use of Eqs. (27) and (28). We find that \( S'_{\text{rep}} \) with Eq. (65) gives Eq. (64).

The procedure (B) contains some diagrams of \( O(n) \). These diagrams never appear by way of the present method, because those contain a connected component without external lines.

IV. EXPANSION PARAMETERS

A. Preliminaries

Since all \( u_i \)s have the dimension \( 4 - d \), they are apparently irrelevant when \( d > 4 \). However, coefficients of the perturbation series in \{\( u_i \)\} are expressed as polynomial of \( \Delta \), having the dimension 2, which can change the relevance of \( u_i \)s. That is, since \( \Delta \) is relevant for all dimensions, a term containing it can become larger as repeating the RGT. Hence, it is important to explore how \( \Delta \) appears in the perturbation series.

To make our argument clear, let us consider \( \delta t \) as an example. The coefficient of the first order in \( u_1 \), which appears in the first term of \( \delta t \) in Eq. (62), is depicted by the diagrams in FIG. 7. The internal line of (a) in the figure carries \( G(q) \), which behaves as \( 1/q^2 \) at the criticality, while the internal line of (b) brings \( \Delta G(q)^2 \sim \Delta/q^4 \). Namely, the dominant contribution is proportional to \( \Delta u_1 \), as pointed out in the previous literatures [3, 21].

We then introduce
\[ g_1 \equiv \Delta u_1. \]  \hfill (67)
as one of expansion parameters. It has the scaling dimension \( 6 - d \), which implies that the upper critical dimension is six. Therefore we cannot ignore \( u_1 \) in \( d \leq 6 \), even though \( u_1 \) itself is irrelevant when \( 4 < d \).

In addition, we define
\[ g_0 \equiv \Delta^{-1}. \]  \hfill (68)

FIG. 7(a) is then proportional to \( u_1 = g_0 g_1 \). Generally, if a diagram includes the irrelevant parameter \( g_0 \), its contribution is less relevant.

Next we discuss how \( \Delta \) couples with the other parameters \( u_2, \ldots, u_5 \). Let us look at the first-order terms in \( u_2 \) and \( u_3 \) respectively, which appear in the second term of \( \delta t \) in Eq. (62). (Note: we have defined \( \tilde{u}_3 \equiv u_2 + u_3 \).) It turns out that their coefficients do not carry \( \Delta \) be-

\[ u_2 \]
\[ (a) \]

\[ u_3 \]
\[ (b) \]

FIG. 8: Diagrams in (a) and (b) give leading correction to \( t \) by \( u_2 \) and \( u_3 \) respectively. However, diagrams in (c) and (d) do not contribute to \( \delta t \), because they both have two connected components before taking the random-field average.

\[ u_2 \]
\[ (c) \]

\[ u_3 \]
\[ (d) \]
that (c) and (d) do not appear in the perturbation series. Thus no $\Delta$ associates with $u_2$ and $u_3$, so that we choose
\begin{align}
g_2 & \equiv \Delta^0 u_2 = u_2 \\
g_3 & \equiv \Delta^0 u_3 = u_3
\end{align}
(69)
as expansion parameters. They have the scaling dimension $4 - d$.

Let us turn to the parameter $u_4$. It contributes to $\delta t$ by the combination $u_1 u_4$ in the lowest order. See FIG. [9] Here $\times$ is forbidden, for the same reason it is in FIG. [c].

FIG. 9: One of the leading contributions from $u_4$ to $\delta t$.

Therefore, this contribution is proportional to $u_1 u_4 = g_1 \left( \Delta^{-1} u_4 \right)$. Then we add
\begin{equation}
g_4 \equiv \Delta^{-1} u_4
\end{equation}
(70)
to the expansion parameters. It has the scaling dimension $2 - d$.

As for $u_5$, the leading correction to $\delta t$ comes from the order in $u_1^2 u_5 = g_1^2 \left( \Delta^{-2} u_5 \right)$, as we depicted in FIG. [10].

FIG. 10: One of the leading contributions from $u_5$ to $\delta t$.

Hence we define
\begin{equation}
g_5 = \Delta^{-2} u_5,
\end{equation}
(71)
which has the dimension $-d$.

In this way, we can express the perturbation series of $\delta t$ in terms of $g_0, ..., g_5$. Further, we can rewrite the recursion equations for $\Delta$ and $u_i$ ($i = 1, ..., 5$) into those for $g_\mu$ ($\mu = 0, ..., 5$). It is easily confirmed from the explicit form of Eqs. (51) and (2) that no $\Delta$ appears in these series at least in the second order in $\{g_\mu\}$. Therefore a naive dimensional analysis is expected to work. Since the scaling dimensions of the new parameters are
\begin{alignat}{3}
[g_0] & = -2, &\quad [g_1] & = 6 - d, &\quad [g_2] & = [g_3] = 4 - d, \\
[g_4] & = 2 - d, &\quad [g_5] & = -d
\end{alignat}
(72)
respectively, only $g_1$ is relevant near $d = 6$. This suggests that the extra parameters $g_2, ..., g_5$ do not play any important role for critical phenomena in the RFIM when $d$ is close to 6.

B. The Gaussian case

In order to make our argument more precise, we need to show that recursion relations for $(t, \{g_\mu\})$ do not contain positive powers of $\Delta$ for all orders.

To this end, it is instructive to consider the case where the probability distribution of impurities is Gaussian, i.e., $u_2, ..., u_5$ are ignored [3] [21]. Suppose that $u_1'$ is expressed as
\begin{equation}
u_1' = L^{4\theta - 3d} \sum_{a_1=1}^{\infty} f_{a_1} (\Delta, t) u_1^{a_1}.
\end{equation}
(73)
Here $f_{a_1} (\Delta, t)$ is obtained from the sum of all diagrams for $[v'_1 \rho]_p$ with $a_1 \phi^4$ vertices. Since the number of $\times$ in a diagram gives the power of $\Delta$, $f_{a_1} (\Delta, t)$ is a polynomial of $\Delta$ with a finite degree. Letting $\gamma_{a_1}$ be that degree, we can write
\begin{equation}
f_{a_1} (\Delta, t) = \sum_{n=0}^{\gamma_{a_1}} c_n \Delta^n,
\end{equation}
(74)
where $c_n$ is a coefficient that depends on $t$ and $a_1$. Hence $u_1'$ is written as
\begin{equation}
u_1' = L^{4\theta - 3d} \sum_{a_1=1}^{\infty} \sum_{n=0}^{\gamma_{a_1}} c_n \Delta^n u_1^{a_1} u_1^{n} = L^{4\theta - 3d} \sum_{a_1=1}^{\infty} \Delta^{\gamma_{a_1}} u_1^{a_1} \sum_{n=0}^{\gamma_{a_1}-n} c_n \Delta^{n}.
\end{equation}
(75)
Let us obtain $\gamma_{a_1}$. Since the diagrams for $u_1'$ have a single connected component, internal lines without $\times$ are needed at least $a_1 - 1$. Excluding external lines from the total lines $4a_1$, internal lines on which we can put $\times$ are at most
\begin{equation}
\frac{1}{2} (4a_1 - 4 - 2(a_1 - 1)) = a_1 - 1.
\end{equation}
(76)
This is nothing but $\gamma_1$. Thus
\begin{equation}
u_1' = L^{4\theta - 3d} \Delta^{-1} \left( \sum_{a_1=1}^{\infty} g_{a_1} \sum_{n=0}^{a_1-1} c_n g_{a_1} (a_1-1) - n \right).
\end{equation}
(77)
Note that the power of $g_0, a_1 - 1 - n$, is non-negative in the above summation. Similar observations of $\Delta'$ and $t'$ lead to the following form:
\begin{equation}
\Delta' = L^{2\theta - d} \Delta \left( 1 + \sum_{a_1=1}^{\infty} g_{a_1} \sum_{n=0}^{a_1+1} d_n g_{a_1} (a_1+1) - n \right)
\end{equation}
(78)
and
\begin{equation}
t' = L^{2\theta - d} \left( t + \sum_{a_1=1}^{\infty} \sum_{n=0}^{a_1} e_n g_{a_1} (a_1 - n) \right)
\end{equation}
(79)
where $\{d_n\}$ and $\{e_n\}$ are $t$-dependent coefficients. From Eqs. (77) and (78), one finds that $g_1' = \Delta' u_1', g_0' = (\Delta')^{-1}$ and $t'$ are expanded by $g_1$ and $g_0$ for all orders.
C. General case

Next, we extend the above argument to the case where the extra parameters $u_2, \ldots, u_5$ are included. Let $Z^+$ be the set of non-negative integers. Define

$$I \equiv \{ a : a = (a_1, \ldots, a_5), a_j \in Z^+, \forall j = 1, \ldots, 5, \sum_{j=1}^{5} a_j \geq 1 \}. \quad (80)$$

For $a \in I$, we use the notation

$$u_a^\alpha \equiv \prod_{j=1}^{5} u_{a_j}^{\alpha_j}. \quad (81)$$

The recursion equation for $u_j$ can be written as

$$u_j' = L^{4\theta - 3d} \sum_{a \in I} \Delta \gamma_a^j \sum_{n=0}^{\gamma_a^j} c_n^j \Delta^{n - \gamma_a^j}. \quad (82)$$

Here we compute $\gamma_a^j$, the degree of $\Delta$ for $u^a$ in $u_j'$. The number of connected components in the vertex corresponding to $u_j$ is denoted by $\alpha_j$, namely,

$$\alpha_1 = 1, \quad \alpha_2 = \alpha_3 = 2, \quad \alpha_4 = 3, \quad \alpha_5 = 4. \quad (83)$$

Here we consider a diagram proportional to $u_a^\alpha$ in Eq. (82). Before connecting the vertices in the diagram by internal lines, there are

$$\sum_{i=1}^{5} \alpha_i a_i \quad (84)$$

connected components. Since the diagram has $\alpha_j$ connected components after connecting the vertices, we need at least

$$\sum_{i=1}^{5} \alpha_i a_i - \alpha_j \quad (85)$$

no more internal lines. Similar computation for Eq. (76) leads to

$$\gamma_a^j = \frac{1}{2} \left( 4 \sum_{i=1}^{5} a_i - 4 - 2 \left( \sum_{i=1}^{5} \alpha_i a_i - \alpha_j \right) \right) = \beta_j - \sum_{i=1}^{5} \beta_i a_i. \quad (86)$$

where we have defined $\beta_j \equiv \alpha_j - 2$. Explicitly,

$$\beta_1 = -1, \quad \beta_2 = \beta_3 = 0, \quad \beta_4 = 1, \quad \beta_5 = 2. \quad (87)$$

According to the definition of the expansion parameters, we find

$$g_j = \Delta^{-\beta_j} u_j, \quad j = 1, \ldots, 5. \quad (88)$$

Applying the result of Eq. (80) to Eq. (82), we get

$$u_j' = L^{4\theta - 3d} \Delta \gamma_a^j \sum_{a \in I} g^a \sum_{n=0}^{\gamma_a^j} c_n^j \Delta^{n - \gamma_a^j}. \quad (89)$$

Similarly, we find that $\Delta'$ and $t'$ are expressed by the following expansion:

$$\Delta' = L^{2\theta - d} \left( 1 + \sum_{a \in I} g_a \Delta \sum_{n=0}^{\gamma_a} d_n g_a^{-n} \right) \quad (90)$$

with

$$\gamma_a = - \sum_{i=1}^{5} \beta_i a_i. \quad (91)$$

Using Eq. (80) and the above expansion for $\Delta'$, we conclude that $g_j' \equiv (\Delta')^{-\beta_j} u_j'$ is expanded by $\{g_a\}$. Further, it is obvious from Eq. (80) that $g_0$ and $t'$ also have perturbation series in terms of $\{g_a\}$.

It should be noted that some physical quantities, such as free energy, are proportional to the relevant parameter $\Delta$. Hence if we compute an exponent associated with those quantities, we have to know the singular behavior of $\Delta$ near the criticality. As we can see in the first line of Eq. (77), $\Delta$ is renormalized multiplicatively and its correction is expanded in terms of $\{g_a\}$. Therefore, we can compute the singular behavior within the framework of perturbation in terms of $\{g_a\}$.

D. Observation for dimensional reduction of the RFIM

We have shown that the transformed parameters $\{g'_a\}$ can be expressed as positive series in $\{g_a\}$. Since the expansion parameters other than $g_1$ are irrelevant near $d = 6$, we may ignore them.

As we have mentioned in Sec. III, our recursion relations are consistent with the replica method studied by Brézin and de Dominicis if we adopt the limiting procedure (A) explained in Sec. III. However, it is concluded that the non-trivial fixed point becomes unstable due to $u_2, \ldots, u_5$ in Ref. [15]. The discrepancy between this and our conclusion is resolved as follows: in Ref. [15], the recursion relations of parameters $\tilde{g}_j \equiv \Delta u_j$ are computed. Since $[\tilde{g}_j] = 6 - d$ for all $j$, it is possible that some of $\tilde{g}_j$ becomes larger as the RGT is repeated. Nevertheless, it does not mean that a diagram proportional to $\tilde{g}_j$ brings infrared divergence near the upper critical dimension.

In fact, if we write $\delta t$ in Eq. (82) by $\tilde{g}_j$ instead of $u_j$, one can easily check that $\tilde{g}_j$'s for $j = 2, \ldots, 5$ are always combined with $\Delta^{-1}$ or $\Delta^{-2}$. Thus, even though $\tilde{g}_j (j =$
2, ..., 5) behave as $g_j' \sim L^{c_j'} g_j$ with $c_j > 0$, the negative powers of $\Delta$ suppresses growth of terms proportional to $g_j$ in $\delta t$. The discussion in the previous subsection shows that the association with the negative powers of $\Delta$ occurs for all orders. Thus $\tilde{g}_j'$s $(j = 2, ..., 5)$ do not contribute to exponents.

Although the extra parameters remain irrelevant near the upper critical dimension, it is plausible that those parameters become relevant when $\epsilon$ exceeds some finite value $\epsilon_c$, which can cause the breakdown of the dimensional reduction in $d = 3$. The existence of such critical value is consistent with a high-temperature expansion by Houghton et al. They concluded that dimensional reduction occurs in $d = 5$ and 6 while the phase transition becomes first order in $d \leq 4$. It is strongly suggested that $1 < \epsilon_c < 2$. On the other hand, another high-temperature expansion performed by Gofman et al suggests that the breakdown of dimensional reduction in $d < 5$ is. It can be interpreted as $0 < \epsilon_c < 1$.

As we explained above, dimensional reduction can survive for sufficiently small $\epsilon$. In this case, the exponents $\nu$, $\eta$ and $\bar{\eta}$ are calculated as

$$\nu = \frac{1}{2} + \frac{1}{12} \epsilon + O(\epsilon^2), \quad \eta = \bar{\eta} = \frac{1}{34} \epsilon^2 + O(\epsilon^3)$$

(92)

However, other parameters not considered here may bring the breakdown of dimensional reduction for all $d < 6$. It is conjectured by Feldman that the apparently higher-dimensional operators $\sum_{ab}(\phi^a - \phi^b)^{2l}$ ($l > 1/\epsilon^2$) turn to relevant ones. In the Harris-Lubensky method, similar operators $\sum_{ab}(\phi^a \phi^b)^l$ are introduced, taking into account the following random average:

$$[v_l (k_1, ..., k_l); v_l (k'_1, ..., k'_l)]_{P^{|\rho|}} = \bar{u}_l \delta (k_1 + \cdots + k_l).$$

(93)

The parameter $\bar{u}_l$ is, itself, irrelevant. However, following the argument in Sec. XX, one can find an expansion parameter proportional to $\bar{u}_l$ is $(\Delta)^{-l-2} \bar{u}_l$, which has the canonical dimension $(l-1)(4-d) \sim (4-d)$ for large $l$. Feldman shows that they acquire anomalous dimension with $O(l^2 \epsilon^2)$ in the second-order perturbation. It means that $(\Delta)^{-l-2} \bar{u}_l$ can transmute a relevant parameter for sufficiently large $l$ satisfying $l > 1/\epsilon^2$. The conjecture should be checked by a non-perturbative means that is beyond the scope of this paper. Nevertheless it is consistent with our result in that the operator with $l = 2$ (i.e., $\sum (\phi^a \phi^b)^2$) is irrelevant for sufficiently small $\epsilon$.

V. SUMMARY AND DISCUSSION

In this paper, we have studied the critical phenomena of the extended Ginzburg-Landau theory for the random-field Ising model. We have employed the renormalization group for the probability distribution of the impurities. Probability distribution is characterized by non-trivial cumulants that bring extra parameters, which are essentially identical to the new coupling constants in the replica Hamiltonian introduced in Refs. 15, 16. In contrast to the replica method, our approach does not require any limiting procedures, and hence no artificial ambiguities arise. Thus we can definitely determine the scaling dimensions of expansion parameters. We have found that extra expansion parameters do not affect the critical phenomena in $d = 6 - \epsilon$ with sufficiently small $\epsilon$. On the other hand, those parameters could be an obstruction to the dimensional reduction at some finite $\epsilon$. This result indicates that we cannot rule out dimensional reduction near $d = 6$ by including the extra coupling constants in Eq. 28. It is consistent with the high-temperature expansion by Haughton et al. On the other hand, another high-temperature expansion by Gofman et al is not consistent with Ref. 16, which may indicate that the dimensional reduction does not occur in any $d < 6$.

It is important to resolve this discrepancy for clarifying mechanisms in the phase transition in the RFIM near the upper critical dimension.

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APPENDIX A: COMPUTATION OF THE RECURRENCE EQUATIONS

In this appendix, we show technical details of computing the correction terms Eq. 59 in $O(u_i u_j)$. Eq. 92 can be obtained in a similar manner.

We do not obtain an explicit form of $v_j'$. Instead, we derive all possible diagrams in the desired order that appear in the right-hand side of FIG. 5.

First, we consider diagrams without $\times$ on an internal line. As we have presented in FIG. 3, the cumulants can be expanded by the five kinds of vertices. The lowest-order correction to $u_j'$ comes from the vertices with two external legs. There are nine such vertices as depicted in FIG. 11. Merging two of them, we obtain diagrams that contribute to $u_j'(j = 1, ..., 5)$ up to the order $O(u_i u_j)$. We write the number $j$ in Table I if the resulting diagram contributes to $u_j'$. For instance, we can read from Table I that the diagram made of the vertices $A$ and $B$ contributes to $u_2'$. As we have mentioned in Sec. VA, any connected components of a diagram in the right-hand side of FIG. 5 must contain at least one external line. In Table I a blank means that the corresponding diagram does not satisfy this condition.

Now we calculate the numerical factor to each diagram, as we have demonstrated in 11B. In contrast to the pure $\phi^4$ theory, a one-loop diagram has multiple connected components in general. There is the combinatoric factor associated with each connected component, which can be computed in the same way as with the pure theory. Here we denote the product of the combinatoric factor of each
Finally, we have to take into account the number of ways of contraction in the random-potential average. Here it is denoted as

\[ n_C. \]  

(A4)

Then, the factor with a diagram \( n_F \) is computed as

\[ n_F = n_{ANRnC}. \]  

(A5)

In Tables II-VI we outline those factors. The first row denotes the power of \( \Delta \) of the diagrams. The second row shows the name of the vertices in FIG. 11 that are ingredients of the diagrams. The last row indicates the parameter dependence of the diagrams.

\[
\begin{array}{|c|c|c|c|}
\hline
& \Delta^0 & \Delta^1 \\
\hline
A & 6 & 6 & -3 \\
B & -\frac{3}{2} & 3 & 3 \\
C & 1 & 1 & 1 \\
D & -\frac{3}{2} & 3 & 3 \\
E & u_1 u_2 & u_3 u_4 & u_5 u_6 \\
F & u_7 u_8 & u_9 u_{10} & u_{11} u_{12} \\
\hline
\end{array}
\]

TABLE II: Diagrams for \( u'_1 \) and their numerical factors

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
& \Delta^0 & \Delta^1 & \Delta^2 \\
\hline
A & 1 & 1 & 1 \\
B & 2 & 2 & 2 \\
C & 1 & 1 & 1 \\
D & 1 & 1 & 1 \\
E & -1 & 1 & 2 \\
F & -1 & 1 & 2 \\
\hline
\end{array}
\]

TABLE III: Diagrams for \( u'_2 \) and their numerical factors

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
& \Delta^0 & \Delta^1 & \Delta^2 \\
\hline
n_A & -\frac{1}{2} & \frac{1}{2} & 1 \\
n_R & 2 & 2 & 2 \\
n_C & 1 & 1 & 1 \\
n_F & -1 & 1 & 2 \\
\hline
\end{array}
\]

TABLE IV: Diagrams for \( u'_3 \) and their numerical factors

It is worthwhile mentioning the relationship to the replica method[15]. From \( S_{\text{int}}^{\text{rep}} \) defined in Eq. (64),

\[
\frac{\partial^2 S_{\text{int}}^{\text{rep}}}{\partial \phi_\alpha \partial \phi_\beta} = \frac{u_1^{\text{rep}}}{2} \phi_\alpha^2 \delta_{\alpha \beta} + \frac{u_2^{\text{rep}}}{2} (\phi_\alpha^2 + \phi_\beta^2) + u_3^{\text{rep}} \phi_\alpha \phi_\beta + \frac{u_4^{\text{rep}}}{2} (\phi_\alpha + \phi_\beta) \sum_\mu \phi_\mu + \frac{u_5^{\text{rep}}}{2} \delta_{\alpha \beta} \sum_\mu \phi_\mu^2 + \frac{u_6^{\text{rep}}}{2} \sum_\mu \phi_\mu \phi_\nu + \frac{u_7^{\text{rep}}}{2} \sum_\mu \phi_\mu \phi_\nu \]  

(A6)
APPENDIX B: CRITICAL EXPONENTS NEAR THE UPPER CRITICAL DIMENSION

In this appendix, we show an outline of computing critical exponents in $d = 6 - \epsilon$.

Let us introduce the connected and the disconnected two-point function $g_c(k; \mu)$ and $g_d(k; \mu)$ by the following formulæ:

\[
\begin{align*}
\langle \phi(k_1) ; \phi(k_2) \rangle_{P[\rho]} &\equiv g_c(k_1; \mu) \delta(k_1 + k_2) \\
\langle \phi(k_1) ; \phi(k_2) \rangle_{P[\varrho]} &\equiv g_d(k_1; \mu) \delta(k_1 + k_2)
\end{align*}
\]

where $\mu$ represents a point in the parameter space $\mu = (t, g_0, ..., g_5)$. Since $g_c(k; \mu)$ is computed from the sum of connected diagrams that appear in $\delta t$, it can be expanded by $\{g_{\mu}\}$. On the other hand, $g_d(k; \mu)$ is made of diagrams having two connected components appearing in $\delta \Delta$. Namely, $g_d(k; \mu)$ can be written as

\[
g_d(k; \mu) = \Delta \tilde{g}_d(k; \mu)
\]

where $\tilde{g}_d(k; \mu)$ is expressed as a perturbative series of $\{g_{\mu}\}$. Suppose that $\Delta$ transforms as

\[
\Delta' \simeq L^\kappa \Delta.
\]

The correlation functions satisfy the following transformation law:

\[
\begin{align*}
g_c(p; \mu) &= L^{2\theta-d} g_c(Lp; \mu') \\
g_d(p; \mu) &= L^{2\theta-d+\kappa} \Delta \tilde{g}_d(Lp; \mu')
\end{align*}
\]

First, we focus on the critical exponents $\eta$ and $\tilde{\eta}$, which determine the small-momentum behavior of correlation functions at the criticality:

\[
\begin{align*}
g_c(p; \mu) &\simeq \frac{1}{p^{2+ \eta}} \delta(p_1 + p_2) \\
g_d(k; \mu) &\simeq \frac{1}{p^{4-\tilde{\eta}}} \delta(p_1 + p_2)
\end{align*}
\]

Suppose that we apply the RGT $n$ times at the criticality, where $n$ satisfies

\[
L^n p_1 = \Delta e_1,
\]

with $e_1$ being some $d$ dimensional unit vector. The probability distribution $P$ approaches the fixed-point distribution, $P_*$, characterized by $\mu_*$. We can evaluate $g_c(p; \mu)$ as

\[
\begin{align*}
g_c(p_1; \mu) &= L^{n(2\theta-d)} g_c(L^n p_1; \mu^{(n)}) \\
&\simeq \left( \frac{\Delta}{p_1} \right)^{2\theta-d} g_c(\Delta e_1; \mu^*)
\end{align*}
\]

where $\mu^{(n)}$ specifies the probability distribution of impurities after having applied the RGT $n$ times. Comparing the definition, we have

\[
\theta = \frac{1}{2} (2 + d - \eta).
\]
Applying the same argument to $g_d(p; \mu)$, we obtain
\[
\theta = \frac{1}{2} (4 + d - \kappa - \bar{\eta}) \quad (B9)
\]
and
\[
\kappa = 2 + \eta - \bar{\eta}. \quad (B10)
\]

The exponents $\gamma$ and $\bar{\gamma}$ are computed, respectively, from $g_d(0; \mu)$ and $\tilde{g}_d(0; \mu)$ in the disordered phase. In this case, the RGT is repeated $n$ times, where $L^n$ is equal to the correlation length $\xi$. Then
\[
\chi = g_c(0; \mu) \simeq g_c(0; \mu_i^{(n)}) \xi^{2d - d}. \quad (B11)
\]
Assuming that
\[
\xi \simeq (t - t_c)^{-\nu}, \quad (B12)
\]
we get, with the help of Eq. (B8),
\[
\chi \simeq (t - t_c)^{-\gamma}, \quad (B13)
\]
where
\[
\gamma = (2 - \eta)\nu. \quad (B14)
\]

Similar arguments for $g_d(0; \mu)$ lead to
\[
\tilde{\chi} = g_d(0; \mu) \simeq (t - t_c)^{-\bar{\gamma}} \quad (B15)
\]
with
\[
\bar{\gamma} = (4 - \bar{\eta})\nu. \quad (B16)
\]

Next we consider the exponent $\alpha$. The singular part of free energy $F(\mu)$ transforms as
\[
F(\mu) = F(\mu'), \quad (B17)
\]
hence its density $f$ transforms as
\[
f(\mu') \equiv \frac{F(\mu')}{V} = \frac{F(\mu)}{L^d V} = L^d f(\mu), \quad (B18)
\]
where $f(\mu)$ has the form of
\[
f(\mu) = \Delta \tilde{f}(\mu) \quad (B19)
\]
with $\tilde{f}(\mu)$ having the perturbative series of $\{g_\mu\}$. Thus
\[
f(\mu) \simeq L^{(d+\kappa)n} f(\mu^{(n)}) \simeq \xi^{-d+\kappa}, \quad (B20)
\]
which means that
\[
2 - \alpha = (d - 2 - \eta + \bar{\eta}). \quad (B21)
\]

Eq. (B20) shows that $\kappa$ is identical with the exponent of the singular part of the free energy in a correlation volume $\xi^d$, which is often denoted by $\theta^2$. We can perform the perturbation explicitly. We begin with the equality
\[
(k_1^2 + l') \delta (k_1 + k_2) = \left[ v_2 (k_1, k_2) \right]_{\mu_1^{(p)}} \rho_1^{(p)} \rho_2^{(p)} = \left[ v_2 (k_1, k_2) \right]_{\mu_1^{(p)}}. \quad (B22)
\]

Employing Eq. (B8) and denoting
\[
[\delta v_2 (p_1; p_2)]_{\mu_1^{(p)}} = \delta \Gamma_2 (p_1; \mu) \delta (p_1 + p_2), \quad (B23)
\]
we get
\[
[v_2' (k_1, k_2)]_{\mu_1^{(p)}} = L^{2d - d} (k_1^2 L^{-2} + t + \delta \Gamma_2 (p_1; \mu)) \delta (k_1 + k_2). \quad (B24)
\]
Comparing the coefficient of $k_1^2$ in Eqs. (B22) and (B24), we get
\[
L^\eta = 1 + \frac{\partial}{\partial p_1^{(p)}} \bigg|_{p_1 = 0} \delta \Gamma_2 (p_1^2; \mu_*) \quad (B25)
\]
which determines the exponent $\eta$. On the other hand, $\bar{\eta}$ is computed by a correction to $\Delta$. Define $\delta \Gamma_\Delta (p; \mu)$ by
\[
[v_2' (k_1); v_2' (k_2)]_{\mu_1^{(p)}} = L^{2(d - d)} \Delta (1 + \delta \Gamma_\Delta (p_1; \mu)) \delta (p_1 + p_2). \quad (B26)
\]

We can readily derive
\[
\Delta' = L^{2\eta - \bar{\eta}} \Delta (1 + \delta \Gamma_\Delta (0; \mu_*)). \quad (B27)
\]

Repeating similar calculations for deriving Eq. (B22), we obtain
\[
L^{2\eta - \bar{\eta}} = 1 + \delta \Gamma_\Delta (0; \mu_*). \quad (B28)
\]

When $d = 6 - \epsilon$, we can evaluate $\delta \Gamma_\Delta$ and $\delta \Gamma_\Delta$ by the $\epsilon$ expansion. Here we can ignore the irrelevant parameters $g_2, g_3, g_5$. In this case, we have
\[
\frac{\partial}{\partial p_1^{(p)}} \bigg|_{p_1 = 0} \delta \Gamma_2 (p_1^2; \bar{\mu}) = \delta \Gamma_\Delta (0; \bar{\mu}), \quad (B29)
\]
where $\bar{\mu}$ is a point in the parameter space satisfying $g_2 = \cdots = g_5 = 0$. The above equation indicates that
\[
\eta = \bar{\eta} \quad (B30)
\]
with Eqs. (B26) and (B28). This equality shows that
\[
\kappa = 2 \quad (B31)
\]
from Eq. (B10). It derives
\[
g_1' \simeq L^{\kappa + 4 - 2 \eta} (g_1 + \delta g_1) = L^{6 - 2 \eta} (g_1 + \delta g_1). \quad (B32)
\]

where $\delta g_1$ is shown to be identical with that of the $4 - \epsilon$ dimensional pure $\phi^4$ theory, with the coupling constant $g_1[3]$. Further, the correction term $\delta t$ is also equal to that of the pure theory in $4 - \epsilon$ dimensions. In this way, we can rederive the result of dimensional reduction near the upper critical dimension.
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[23] A similar Hamiltonian is also employed in Ref.16 for the high-temperature expansion of the RFIM.
[24] However, there are apparent higher-dimensional parameters that effectively have the same scaling dimensions as $u_0$ [14]. See Sec. IV D.
[25] A derivation of these results are outlined in appendix B.