ASYMPTOTIC BEHAVIOR OF STOCHASTIC COMPLEX GINZBURG-LANDAU EQUATIONS WITH DETERMINISTIC NON-AUTONOMOUS FORCING ON THIN DOMAINS

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(Communicated by Björn Schmalfuß)

Abstract. In this paper, we investigate the asymptotic behavior for non-autonomous stochastic complex Ginzburg-Landau equations with multiplicative noise on thin domains. For this aim, we first show that the existence and uniqueness of random attractors for the considered equations and the limit equations. Then, we establish the upper semicontinuity of these attractors when the thin domains collapse onto an interval.

1. Introduction. Let $I$ be a bounded interval in $\mathbb{R}$ and $0 < \varepsilon \leq 1$. Denote by

$$O_\varepsilon = \{ x = (x_1, x_2) | x_1 \in I, 0 < x_2 < \varepsilon g(x_1) \},$$

where $g \in C^2(I, (0, +\infty))$. Throughout this paper, we also write $O = I \times (0, 1)$.

In this paper, we study the limit of asymptotical behavior of non-autonomous stochastic complex Ginzburg-Landau equation with multiplicative noise on $O_\varepsilon$ : for any given $\tau \in \mathbb{R}$,

$$
\begin{aligned}
\frac{d\hat{u}_\varepsilon}{dt} &= \left[ (\lambda + i\alpha) \Delta \hat{u}_\varepsilon - (k + i\beta) |\hat{u}_\varepsilon|^2 \hat{u}_\varepsilon + \gamma \hat{u}_\varepsilon + G(t, x) \right] dt \\
&\quad + \rho \hat{u}_\varepsilon \circ dW(t), \quad x \in O_\varepsilon, \ t > \tau, \\
\frac{\partial \hat{u}_\varepsilon}{\partial \nu_\varepsilon} &= 0, \quad x \in \partial O_\varepsilon,
\end{aligned}
$$

with initial condition

$$\hat{u}_\varepsilon(\tau, x) = \hat{u}_\varepsilon^\tau(x), \quad x \in O_\varepsilon,$$

where $i$ is the unit of imaginary numbers such that $i^2 = -1$, $\hat{u}_\varepsilon$ is the unknown complex value function, $\nu_\varepsilon$ is the unit outward normal vector to $\partial O_\varepsilon$, $\lambda$, $\alpha$, $k$, $\beta$, $\gamma$, $\rho$ are real constants satisfying $k, \lambda, \rho, \gamma > 0$ and $k > |\beta|$, $G$ is a function defined later, and $W$ is two-sided real-valued Wiener processes on a probability

2010 Mathematics Subject Classification. Primary: 35B40; Secondary: 35B41, 37L30.

Key words and phrases. Thin domain, stochastic complex Ginzburg-Landau equation, random attractor, upper semicontinuity.

This work was supported by NSFC (11271270, 11601446 and 11331007) and Excellent Youth Scholars of Sichuan University (2016SCU04A15).

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The symbol $\circ$ means that the stochastic equation is understood in the sense of Stratonovich integration.

The domain $O_\varepsilon$ is the so-called thin domain when $\varepsilon$ is small. As $\varepsilon \to 0$, the thin domain $O_\varepsilon$ collapses to the interval $I$. We will study the limiting behavior of the equation (1) and (2). As in the case of deterministic Ginzburg-Landau equation [26], the limit equation that should correspond to (1) at $\varepsilon = 0$ is determined by

$$
\begin{align*}
\frac{du^0}{dt} &= \left[ (\lambda + i\alpha) \frac{1}{2} (g u^0_{y_1} y_1) - (k + i\beta) |u^0|^2 u^0 + \gamma u^0 + G(t, y_1, 0) \right] dt \\
+ \rho u^0 \circ dW(t), & y_1 \in I, \ t > \tau,
\end{align*}
$$

with initial condition

$$
u_0 = u^0_\tau (y_1), \ y_1 \in I,
$$

where $\nu_0$ is the unit outward normal vector to $\partial I$. Note that $u^0_{y_1}$ means $\frac{\partial u^0}{\partial y_1}$ in (3) and similar notation will be used throughout this paper.

Thin domains problems have been considered by many authors from different points of view, including modeling, control and homogenization of equations. Such problems have been investigated by many different approaches like asymptotic expansions and singular perturbations. The systematic study of the asymptotic behavior of deterministic dissipative systems on thin domains was initiated by Hale and Raugel [15, 16]. Later on, their results were extended to various problems, see for instance, [1, 3, 9, 7, 8, 17, 23, 24, 25, 26, 27].

Partial differential equations are often subject to white noise perturbations either from its surrounding environment or from intrinsic uncertainties. As an extension of the global attractor for deterministic equations the concept of pullback attractor for autonomous stochastic systems was defined in [11, 14, 29]. To handle the deterministic non-autonomous force and random force in a unified framework, Wang in [30] defined the concept of pullback random attractor for non-autonomous random dynamical system. There is a enormous number of publications on pullback random attractors, which, for instance, can be found in [4, 5, 10, 11, 12, 13, 14, 18, 19, 30, 31, 32, 35] for autonomous stochastic systems and in [31, 33, 34] for non-autonomous stochastic systems.

Recently, with the development of the theory of random dynamical systems, the limiting behavior of dynamics for stochastic partial differential equations on thin domains has been considered. Caraballo, Chueshov and Kloeden consider a semilinear parabolic stochastic partial differential equations with additive space-time noise on thin two layer domains. Limiting properties of the global random attractor are established as the thickness parameter of the domain tends to zero [6]. Under the frameworks of non-autonomous random dynamical systems, the limiting behavior of stochastic reaction-diffusion equations with deterministic non-autonomous terms defined on thin domains are studied in [21] and [22] for multiplicative noise and additive noise, respectively. Motivated by our previous works [21, 22], in this paper, we will study the limiting behavior of stochastic complex Ginzburg-Landau equations with deterministic non-autonomous forcing on thin domains.

The plan of this paper is as follows. In the next section, we establish the existence of a continuous cocycle in $L^2(O)$ for the stochastic equation converted from (1) and (2) and the existence of a continuous cocycle in $L^2(I)$ for the stochastic equation (3) and (4). Section 3 contains all necessary uniform estimates of the solutions. We then prove the existence and uniqueness of random attractors for the stochastic
equations in section 4, and analyze convergence properties of the solutions as well as the attractors in section 5.

2. Existence of continuous cocycle.

In the rest of this paper, we will study the existence and upper semicontinuity of tempered pullback attractors for stochastic complex Ginzburg-Landau equations defined on the thin domain $\mathcal{O}_\varepsilon$ with deterministic non-autonomous terms as well as multiplicative noise. Here we show that there is a continuous cocycle generated by such equations defined on $\mathcal{O}_\varepsilon$.

We now transfer problem (1) into boundary value problems on the fixed domain $\mathcal{O}$. To that end, we introduce a transformation $T_\varepsilon : \mathcal{O}_\varepsilon \to \mathcal{O}$ by $T_\varepsilon(x_1, x_2) = \left(x_1, \frac{x_2}{\varepsilon g(y_1)}\right)$ for $x = (x_1, x_2) \in \mathcal{O}_\varepsilon$. Let $y = (y_1, y_2) = T_\varepsilon(x_1, x_2)$. Then we have

$$x_1 = y_1, \quad x_2 = \varepsilon g(y_1)y_2.$$  

After some calculations, we find that the Jacobian matrix of $T_\varepsilon$ is given by

$$J = \frac{\partial (y_1, y_2)}{\partial (x_1, x_2)} = \begin{pmatrix} -\frac{1}{\varepsilon g(y_1)} & 0 \\ \frac{\varepsilon g(y_1)}{\gamma} & -\frac{1}{\varepsilon g(y_1)} \end{pmatrix}.$$  

The determinant of $J$ is $|J| = \frac{1}{\varepsilon g(y_1)}$. Let $J^*$ be the transport of $J$. Then we have

$$J J^* = \begin{pmatrix} \frac{1}{\varepsilon g(y_1)} & -\frac{\varepsilon g(y_1)}{\gamma} \\ -\frac{\varepsilon g(y_1)}{\gamma} & \frac{1}{\varepsilon g(y_1)} \end{pmatrix}.$$  

It follows from [20] (see also [15]) that the gradient operator and the Laplace operator in the original variable $x \in \mathcal{O}_\varepsilon$ and in the new variable $y \in \mathcal{O}$ are related by

$$\nabla_x \hat{u}(x) = J^* \nabla_y u(y) \quad \text{and} \quad \Delta_x \hat{u}(x) = |J| \text{div}_y \left(|J|^{-1} J J^* \nabla_y u(y)\right) = \frac{1}{\varepsilon g(y_1)} \text{div}_y (P_\varepsilon u(y)),$$

where we denote by $\hat{u}(x) = u(y)$, $\nabla_x$ and $\Delta_x$ are the gradient operator and the Laplace operator in $x \in \mathcal{O}_\varepsilon$ respectively, $\text{div}_y$ and $\nabla_y$ are the divergence operator and the gradient operator in $y \in \mathcal{O}$ respectively, and $P_\varepsilon$ is the operator given by

$$P_\varepsilon u(y) = \begin{pmatrix} \frac{1}{\varepsilon g(y_1)} & -\frac{\varepsilon g(y_1)}{\gamma} \\ -\frac{\varepsilon g(y_1)}{\gamma} & \frac{1}{\varepsilon g(y_1)} \end{pmatrix} \begin{pmatrix} g u_{y_1} - g y_1 y_2 u_{y_2} \\ -y_2 g_{y_1} u_{y_1} + \frac{1}{\varepsilon g(y_1)} \left(1 + (\varepsilon g_{y_1})^2\right) u_{y_2} \end{pmatrix}.$$  

For $y = (y_1, y_2) \in \mathcal{O}$, we denote $G_\varepsilon(t, y) = G(t, y_1, \varepsilon g(y_1)y_2)$ and $G_0(t, y_1) = G(t, y_1, 0)$. Then problem (1) is equivalent to the following system for $t > \tau$,

$$\begin{cases} du^\varepsilon = \left(\lambda + i\alpha\right)\frac{1}{\varepsilon g(y_1)} \text{div}_y (P_\varepsilon u^\varepsilon) - (k + i\beta)|u^\varepsilon|^2 u^\varepsilon + \gamma u^\varepsilon + G_\varepsilon(t, y) \right) dt \\
+ \rho u^\varepsilon \circ dW(t), \quad y \in \mathcal{O}, \\
P_\varepsilon u^\varepsilon \cdot \nu = 0, \quad y \in \partial \mathcal{O}, \end{cases}$$  

with initial condition

$$u^\varepsilon(\tau, y) = u^\varepsilon(\tau, y) = \hat{u}^\varepsilon(T_\varepsilon^{-1}(y)), \quad y \in \mathcal{O},$$  

where $\nu$ is the unit outward normal vector to $\partial \mathcal{O}$.

Now we want to write equation (5) as an abstract evolutionary equation. We first introduce the inner product $(\cdot, \cdot)_{H_\varepsilon(\mathcal{O})}$ on complex space $L^2(\mathcal{O})$ defined by

$$(u, v)_{H_\varepsilon(\mathcal{O})} = \int_{\mathcal{O}} g u \overline{v} dy, \quad \text{for all } u, v \in L^2(\mathcal{O})$$  

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and denote by $H_g(\mathcal{O})$ the space equipped with this inner product, where $v$ and $\overline{v}$ are conjugate. Since $g$ is a continuous function on $\mathcal{I}$, which implies that there exist positive constants $\gamma_1$ and $\gamma_2$ such that
\begin{equation}
\gamma_1 \leq g(x_1) \leq \gamma_2, \quad \forall x_1 \in \mathcal{I},
\end{equation}
one can easily show that $H_g(\mathcal{O})$ is a Hilbert space with equivalent norm of $L^2(\mathcal{O})$.

We will write $\gamma(x)$ for $\gamma(x;\mathcal{I})$, which denotes by
\begin{equation}
\gamma(x) = \langle \gamma(x;\mathcal{I}), \gamma(x) \rangle_{H_g(\mathcal{O})} = g(y_1) y_2 = \frac{1}{g} y_2.
\end{equation}

Using $\gamma(x)$, problem (5)-(6) can be written as
\begin{equation}
\begin{aligned}
&\Phi_2 \frac{d u_2}{dt} = -\left( \lambda + i\alpha \right) A_\varepsilon u_2 - \left( k + i\beta \right) |u_\varepsilon|^2 u_\varepsilon + \gamma u_\varepsilon \varepsilon + G_\varepsilon(t,y) + \rho u_\varepsilon \circ \frac{dW(t)}{dt}, \quad y \in \mathcal{O}, \ t > \tau, \\
&u_\varepsilon(\tau) = u^{\varepsilon}.
\end{aligned}
\end{equation}

To reformulate system (3)-(4), we introduce an inner product $(\cdot,\cdot)_{H_g(\mathcal{I})}$ on $L^2(\mathcal{I})$ as defined by
\begin{equation}
(u,v)_{H_g(\mathcal{I})} = \int_{\mathcal{I}} g u \overline{v} dy, \quad \text{for all } u, v \in L^2(\mathcal{I}),
\end{equation}
and denote by $H_g(\mathcal{I})$ the space equipped with this inner product. Let $a_0(\cdot,\cdot): H^1(\mathcal{I}) \times H^1(\mathcal{I}) \to \mathbb{C}$ be a bilinear form given by
\begin{equation}
a_0(u,v) = \int_{\mathcal{I}} g \nabla u \cdot \nabla \overline{v} dy_1.
\end{equation}
Denote by $A_0$ the unbounded operator on $H_g(\mathcal{I})$ as defined by
\begin{equation}
A_0 u = -\frac{1}{g} (gu_{y_1})_{y_1}, \quad u \in D(A_0),
\end{equation}
where $D(A_0) = \left\{ u \in H^2(\mathcal{I}), \frac{\partial u}{\partial n_0} = 0 \text{ on } \partial \mathcal{I} \right\}$. Thus we have
\begin{equation}
a_0(u,v) = (A_0 u,v)_{H_g(\mathcal{I})}, \quad \forall u \in D(A_0), \forall v \in H^1(\mathcal{I}).
\end{equation}
Using $A_0$, system (3)-(4) can be written as
\[
\begin{aligned}
    \frac{du^0}{dt} &= - (\lambda + i\alpha) A_0 u^0 - (k + i\beta) \left| u^0 \right|^2 u^0 + \gamma u^0 \\
    &+ G_0(t, y_1) + \rho u^0 \circ \frac{dW(t)}{dt}, \quad y_1 \in I, \quad t > \tau,
\end{aligned}
\]
(13)

We now specify the probability space. Denote by
\[
\Omega = \{ \omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0 \}.
\]

Let $\mathcal{F}$ is the Borel $\sigma$-algebra induced by the compact-open topology of $\Omega$, and $P$ the corresponding Wiener measure on $(\Omega, \mathcal{F})$. There is a classical group $\{\theta_t\}_{t \in \mathbb{R}}$ acting on $(\Omega, \mathcal{F}, P)$, which is defined by
\[
\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega, \quad t \in \mathbb{R}.
\]

Then $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$ is a metric dynamical system (see [2]). On the other hand, let us consider the one-dimensional stochastic differential equation
\[
dz + zd\tau = dW(t).
\]
(15)

This equation has a random fixed point in the sense of random dynamical systems generating a stationary solution known as the stationary Ornstein-Uhlenbeck process
\[
z(\theta_t \omega) = - \int_{-\infty}^{0} e^{s} (\theta_t \omega)(s) ds, \quad t \in \mathbb{R}.
\]

From [2] we know that there exists a $\theta_t$-invariant set $\tilde{\Omega} \subseteq \Omega$ of full measure such that $z(\theta_t \omega)$ is continuous in $t$ for every $\omega \in \tilde{\Omega}$, and the random variable $|z(\theta_t \omega)|$ is tempered. Let $\mathcal{F}_1$ and $P_1$ be the restrictions of $\mathcal{F}$ and $P$ on $\tilde{\Omega}$, respectively. For convenience, from now on, we will abuse the notation slightly and write the space $(\tilde{\Omega}, \mathcal{F}_1, P_1)$ as $(\Omega, \mathcal{F}, P)$.

We will define a continuous cocycle for problem (5)-(6) over $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$. To this end, we need to convert the stochastic equation into a deterministic non-autonomous one via the variable $z(\theta_t \omega)$. Let $v^\varepsilon(t, \tau, \omega, v_x^\varepsilon) = e^{-\rho z(\theta_t \omega)} v^\varepsilon(t, \tau, \omega, u_x^\varepsilon)$ where $v^\varepsilon$ is a solution of problem (12). Then $v^\varepsilon$ satisfies
\[
\begin{aligned}
    \frac{dv^\varepsilon}{dt} &= - (\lambda + i\alpha) A_x v^\varepsilon - (k + i\beta) e^{2\rho z(\theta_t \omega)} \left| v^\varepsilon \right|^2 v^\varepsilon \\
    &+ (\gamma + \rho z(\theta_t \omega)) v^\varepsilon + e^{-\rho z(\theta_t \omega)} G_x(t, \tau, y) \quad y \in \mathcal{O}, \quad t > \tau,
\end{aligned}
\]
(16)

Since (16) is a deterministic equation which is parametrized by $\omega \in \Omega$, by a Galerkin method, one can show that if $k > |\beta|$, then for every $\omega \in \Omega$, $t \in \mathbb{R}$ and $v_x^\varepsilon \in L^2(\mathcal{O})$, system (16) has a unique solution $v^\varepsilon(\cdot, \tau, \omega, v_x^\varepsilon) \in C([\tau, \infty), L^2(\mathcal{O})) \cap L^2([\tau, \tau+T], H^1(\mathcal{O}))$ for every $T > 0$. Moreover, one may show that $v^\varepsilon(t, \tau, \omega, v_x^\varepsilon)$ is $(\mathcal{F}, B(L^2(\mathcal{O})))$-measurable in $\omega \in \Omega$ and continuous in $v_x^\varepsilon$ with respect to the norm of $L^2(\mathcal{O})$. We now define a mapping $\Phi_\varepsilon : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times L^2(\mathcal{O}) \to L^2(\mathcal{O})$ for problem (12). Given $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $v_x^\varepsilon \in L^2(\mathcal{O})$, let
\[
\Phi_\varepsilon(t, \tau, \omega, v_x^\varepsilon) = u^\varepsilon(t + \tau, \theta_{-\tau} \omega, u_x^\varepsilon) = e^{\rho z(\theta_t \omega)} v^\varepsilon(t + \tau, \theta_{-\tau} \omega, v_x^\varepsilon),
\]
(17)

where $v_x^\varepsilon = e^{-\rho z(\omega)} u_x^\varepsilon$. As stated in [30], the mapping $\Phi_\varepsilon$ is a continuous cocycle on $L^2(\mathcal{O})$ over $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$.

Let $R_\varepsilon : L^2(\mathcal{O}_\varepsilon) \to L^2(\mathcal{O})$ be an affine mapping of the form
\[
(R_\varepsilon \hat{u})(y) = \hat{u}(T_\varepsilon^{-1} y), \quad \forall \hat{u} \in L^2(\mathcal{O}_\varepsilon).
\]
Given \( t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega \) and \( \hat{u}_t^\tau \in L^2(O) \), we can define a continuous cocycle \( \Phi_\varepsilon \) for problem (1) by the formula
\[
\Phi_\varepsilon(t, \tau, \omega, \hat{u}_t^\tau) = R_\varepsilon^{-1} \Phi_\varepsilon(t, \tau, \omega, R_\varepsilon \hat{u}_t^\tau),
\]
where \( \Phi_\varepsilon \) is the continuous cocycle for problem (12) on \( L^2(O) \) over \((\Omega, F, P, \{ \theta_t \}_{t \in \mathbb{R}})\).

A similar change of unknown variable \( \phi^0(t, \tau, \omega, u^0) = e^{-\rho_2(\theta_\omega)} u^0(t, \tau, \omega, u^0) \) can transform system (13) into the following equation on \( \mathcal{I} \)
\[
\begin{align*}
\frac{dv^0}{dt} = & \quad (\lambda + i\alpha) A_0 v^0 - (k + i\beta) e^{2\rho_2(\theta_\omega)} |v^0|^2 v^0 \\
& \quad + (\gamma + \rho z(\theta_\omega)) v^0 + e^{-\rho_2(\theta_\omega)} G_0(t, y), \quad y \in \mathcal{I}, \ t > \tau,
\end{align*}
\]
where \( \Phi_\varepsilon \) is called tempered (or subexponentially growing) if for every \( j > 0 \),
\[
\parallel \Phi_\varepsilon(t, \tau, \omega, \hat{u}_t^\tau) \parallel \leq e^{j|\tau|} \parallel \hat{u}_t^\tau \parallel,
\]
This definition is a straightforward extension of the concept of tempered random subsets for autonomous random dynamical systems. In the sequel, we denote by \( \mathcal{D}_j \) the collection of all families of tempered nonempty subsets of \( X_j \), for \( j = \varepsilon, 0 \) or 1, i.e.,
\[
\mathcal{D}_j = \{ B_j = \{ B_j(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} : B_j \text{ is tempered in } X_j \}.
\]

Our main purpose of the paper is to prove that the continuous cocycles \( \Phi_\varepsilon \) and \( \Phi_0 \) possess a unique \( \mathcal{D}_2 \)-pullback attractor \( A_\varepsilon \) in \( L^2(O) \) and \( \mathcal{D}_0 \)-pullback attractor \( A_0 \) in the space \( L^2(I) \), respectively. Furthermore \( A_\varepsilon \) is upper-semicontinuous at \( \varepsilon = 0 \), that is, for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),
\[
\lim_{\varepsilon \to 0} \sup_{u^0 \in A_\varepsilon} \inf_{u^0 \in A_0} \varepsilon^{-1} \int_{O_\varepsilon} |u^\varepsilon - u^0|^2 dx = 0.
\]

To prove (19), we only need to show that the cocycle \( \Phi_\varepsilon \) has a unique \( \mathcal{D}_1 \)-pullback attractor \( A_\varepsilon \) in \( L^2(O) \) and it is upper-semicontinuous at \( \varepsilon = 0 \) in the sense that for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),
\[
\lim_{\varepsilon \to 0} \text{dist}_{L^2(O)}(A_\varepsilon(\tau, \omega), A_0(\tau, \omega)) = 0,
\]
which will be established in the last section of the paper.

The following condition will be needed when deriving uniform estimates of solutions:
\[
\int_{-\infty}^{\tau} e^{\gamma s} \parallel G(s, \cdot) \parallel^2_{L^\infty(O)} ds < \infty, \quad \forall \tau \in \mathbb{R}.
\]

When constructing tempered pullback attractors, we will assume for any \( \sigma > 0 \)
\[
\lim_{r \to -\infty} e^{\sigma r} \int_{-\infty}^{0} e^{\gamma s} \parallel G(s, r, \cdot) \parallel^2_{L^\infty(O)} ds = 0.
\]

Both conditions (20) and (21) are needed when proving the existence of tempered pullback attractors for the cocycle \( \Phi_\varepsilon \).

In the sequel, we will use the following Agmon inequality (see [28] for instance):
\[
\parallel u \parallel_{L^\infty(O)} \leq c \parallel u \parallel_{L^2(O)}^\frac{1}{2} \parallel u \parallel_{H^2(O)}^\frac{1}{2}, \quad \forall u \in H^2(O), \quad O \subset \mathbb{R}^2,
\]
and
\[ \|u\|_{H^2(O)} \leq c \left( \|u\|_{L^2(O)} + \|\Delta u\|_{L^2(O)} \right), \quad \forall u \in H^2(O), \quad O \subset \mathbb{R}^2. \]

3. Uniform estimates of solutions.

In this section, we derive uniform estimates of solutions of problem (16) which are needed for proving the existence of $D_1$-pullback absorbing sets and the $D_1$-pullback asymptotic compactness of the continuous cocycle $\Phi_\varepsilon$. The estimates of solutions of problem (16) in $H_g(O)$ are provided below.

**Lemma 3.1.** Assume that (20) holds. Then there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D_1 = \{D_1(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D_1$, there exists $T = T(\tau, \omega, D_1) > 0$, independent of $\varepsilon$, such that for all $t \geq T$, the solution $v^\varepsilon$ of (16) with $\omega$ replaced by $\theta_\tau\omega$ satisfies
\[
\|v^\varepsilon(\tau, \tau-t, \theta_\tau\omega, v_{\tau-t}^\varepsilon)\|^2_{H^1_\varepsilon(O)} + \int_{-t}^0 e^{\frac{t}{2} \gamma s + 2p \int_0^s \tilde{z}(\theta_\omega) d\tau} \times a_\varepsilon(v^\varepsilon(s + \tau - t, \theta_\tau\omega, v_{\tau-t}^\varepsilon), v^\varepsilon(s + \tau - t, \theta_\tau\omega, v_{\tau-t}^\varepsilon)) ds
\leq M + M \int_{-\infty}^0 e^{\frac{t}{2} \gamma s + 2p \int_0^s \tilde{z}(\theta_\omega) d\tau} e^{-2p\varepsilon(\theta_\omega)} \left(1 + \|G(s + \tau, \cdot)\|^2_{L^\infty(\tilde{\Omega})}\right) ds, \quad (22)
\]
where $v_{\tau-t}^\varepsilon \in D_1(\tau - t, \theta_\tau\omega)$ and $M$ is a positive constant depending on $k$ and $\gamma$, but independent of $\tau$, $\omega$, $\varepsilon$ and $D_1$.

**Proof.** Taking the inner product of (16) with $v^\varepsilon$ in $H_g(O)$ and taking real parts, we find that
\[
\frac{1}{2} \frac{d}{dt} \|v^\varepsilon\|^2_{H^1_\varepsilon(O)} = -\lambda a_\varepsilon(v^\varepsilon, v^\varepsilon) - k e^{-2p\varepsilon(\theta_\omega)} \int_O g |u^\varepsilon|^4 \, dy
\]
\[+ (\gamma + \rho \varepsilon(\theta_\omega)) \|v^\varepsilon\|^2_{H^1_\varepsilon(O)} + e^{-\rho \varepsilon(\theta_\omega)} \text{Re}(G_\varepsilon(t, y), v^\varepsilon)_{H^1_\varepsilon(O)}. \quad (23)\]

By (10), we have
\[
k \frac{1}{2} e^{-2p\varepsilon(\theta_\omega)} \int_O g |u^\varepsilon|^4 \, dy \geq 2k e^{-2p\varepsilon(\theta_\omega)} \int_O g |u^\varepsilon|^2 \, dy - 2 \frac{1}{k} \gamma^2 e^{-2p\varepsilon(\theta_\omega)} \int_O g dy
\]
\[\geq 2 \gamma \|v^\varepsilon\|^2_{H^1_\varepsilon(O)} dy - 2 \frac{1}{k} \gamma^2 \|v^\varepsilon\|^2_{H^1_\varepsilon(O)} dy. \quad (24)\]

On the other hand, the last term on the right-hand side of (23) is bounded by
\[
e^{-\rho \varepsilon(\theta_\omega)} \text{Re}(G_\varepsilon(t, y), v^\varepsilon)_{H^1_\varepsilon(O)} \leq e^{-\rho \varepsilon(\theta_\omega)} \|G_\varepsilon(t, \cdot)\|^2_{H^1_\varepsilon(O)} \|v^\varepsilon\|^2_{H^1_\varepsilon(O)}
\[
\leq \frac{1}{4} \gamma \|v^\varepsilon\|^2_{H^1_\varepsilon(O)} + ce^{-2p\varepsilon(\theta_\omega)} \|G(t, \cdot)\|^2_{L^\infty(\tilde{\Omega})}. \quad (25)\]

Then it follows from (23)-(25) that
\[
\frac{d}{dt} \|v^\varepsilon\|^2_{H^1_\varepsilon(O)} + 2\lambda a_\varepsilon(v^\varepsilon, v^\varepsilon) + k \int_O g |u^\varepsilon|^4 \, dy
\leq -\left(\frac{3}{2} \gamma - 2\rho \varepsilon(\theta_\omega)\right) \|v^\varepsilon\|^2_{H^1_\varepsilon(O)} + ce^{-2p\varepsilon(\theta_\omega)}(1 + \|G(t, \cdot)\|^2_{L^\infty(\tilde{\Omega})}). \quad (26)\]
By Gronwall inequality and replacing $\omega$ by $\theta-\omega$, we get that for every $\omega \in \Omega$,

$$
\|v^\varepsilon(\tau, \tau - t, \theta-\omega, v^\varepsilon_{\tau-t})\|^2_{H^s(\omega)} + 2\lambda \int_{-t}^0 e^{2\gamma s + 2\rho \int_t^0 z(\theta, \omega) d\tau} \times a_v(v^\varepsilon(s + \tau, \tau - t, \theta-\omega, v^\varepsilon_{\tau-t}), v^\varepsilon(s + \tau, \tau - t, \theta-\omega, v^\varepsilon_{\tau-t})\) d\tau \\
\leq e^{-\frac{3}{2}\gamma t + 2\rho \int_t^0 z(\theta, \omega) d\tau} \|v^\varepsilon_{\tau-t}\|^2_{H^s(\omega)} + c \int_{-t}^0 e^{2\gamma s + 2\rho \int_t^0 z(\theta, \omega) d\tau} e^{-2\tau z(\theta, \omega)} (1 + \|G(s + \tau, \cdot)\|^2_{L^\infty(\Omega)}) d\tau.
$$

(27)

We now estimate the first term on the right-hand side of (27). By the properties of $z(\theta, \omega)$ we find that there exists $T_1 = T_1(\gamma, \omega) > 0$ such that for all $t \geq T_1$,

$$
|z(\theta, \omega)| \leq \frac{1}{8\rho} \gamma t \quad \text{and} \quad \int_{-t}^0 z(\theta, \omega) d\tau \leq \frac{1}{8\rho} \gamma t.
$$

(28)

By (28), $v^\varepsilon_{\tau-t} \in D_1(\tau - t, \theta-\omega)$ and $D_1$ is tempered, we find that there exists $T_2 = T(\tau, \omega, D_1) > T_1$ such that for all $t \geq T_2$,

$$
e^{-\frac{3}{2}\gamma t + 2\rho \int_t^0 z(\theta, \omega) d\tau} \|v^\varepsilon_{\tau-t}\|^2_{H^s(\omega)} \leq e^{-\frac{3}{2}\gamma t} \|v^\varepsilon_{\tau-t}\|^2_{H^s(\omega)} \leq e^{-\frac{3}{2}\gamma t} \|D_1(\tau - t, \theta-\omega)\|^2_{H^s(\omega)} \leq 1.
$$

(29)

By (28) and (20) we get, for all $t \geq T_1$

$$
\int_{-t}^{-T_1} e^{2\gamma s + 2\rho \int_t^0 z(\theta, \omega) d\tau} e^{-2\tau z(\theta, \omega)} (1 + \|G(s + \tau, \cdot)\|^2_{L^\infty(\Omega)}) d\tau \\
\leq \int_{-\infty}^{-T_1} e^{\gamma s} (1 + \|G(s + \tau, \cdot)\|^2_{L^\infty(\Omega)}) d\tau < \infty,
$$

which implies that

$$
\int_{-t}^0 e^{2\gamma s + 2\rho \int_t^0 z(\theta, \omega) d\tau} e^{-2\tau z(\theta, \omega)} (1 + \|G(s + \tau, \cdot)\|^2_{L^\infty(\Omega)}) d\tau < \infty.
$$

(30)

By (27), (29) and (30) we get, for all $t \geq T_2$,

$$
\|v^\varepsilon(\tau, \tau - t, \theta-\omega, v^\varepsilon_{\tau-t})\|^2_{H^s(\omega)} + 2\lambda \int_{-t}^0 e^{2\gamma s + 2\rho \int_t^0 z(\theta, \omega) d\tau} \times a_v(v^\varepsilon(s + \tau, \tau - t, \theta-\omega, v^\varepsilon_{\tau-t}), v^\varepsilon(s + \tau, \tau - t, \theta-\omega, v^\varepsilon_{\tau-t})\) d\tau \\
\leq 1 + c \int_{-\infty}^{-T_1} e^{2\gamma s + 2\rho \int_t^0 z(\theta, \omega) d\tau} e^{-2\tau z(\theta, \omega)} (1 + \|G(s + \tau, \cdot)\|^2_{L^\infty(\Omega)}) d\tau < \infty.
$$

(31)

From (31), the desired estimates follow immediately.

The following lemma is to derive the uniform estimates of solutions in $H^s_2(\omega)$.

**Lemma 3.2.** Assume that (20) holds. Then there exists $\varepsilon_1 > 0$ such that for every $0 < \varepsilon < \varepsilon_1$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D_1 = \{D_1(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in D_1$, there exists
\[ T = T(\tau, \omega, D_1) \geq 1, \text{ independent of } \varepsilon, \text{ such that for all } t \geq T, \text{ the solution } v^\varepsilon \text{ of (16)} \]

with \( \omega \) replaced by \( \theta - \tau, \omega \) satisfies

\[
\|v^\varepsilon(t, \tau - t, \theta - \tau, \omega, v_{\varepsilon-})\|_{H^1_0(\Omega)}^2 
\leq M + M \int_{-\infty}^0 e^{\frac{2}{\varepsilon}t + 2p\int_0^t z(\theta, \omega)dt} e^{-2\rho\varepsilon(\theta, \omega)}(1 + ||G(s + \tau, \cdot)||_{L^\infty(\Omega)}^2)ds,
\]

where \( v_{\varepsilon-} \in D_1(t - t, \theta - \omega) \) and \( M \) is a positive constant depending on \( k, \gamma \) and \( \lambda \), but independent of \( \tau, \omega, \varepsilon \) and \( D_1 \).

**Proof.** Taking the inner product of (16) with \( A_{\varepsilon} v^\varepsilon \) in \( H_\alpha(\Omega) \) and taking the real part, we find that

\[
\frac{1}{2} \frac{d}{dt} a_{\varepsilon}(v^\varepsilon, v^\varepsilon) = -\lambda \|A_{\varepsilon} v^\varepsilon\|_{H_\alpha(\Omega)}^2 - \frac{2}{\varepsilon} \frac{\partial}{\partial y_j} \left| \frac{\partial v}{\partial y_j} \right| v^\varepsilon + i \beta \frac{\partial}{\partial y_j} \Re \left( \partial \Re \right) v^\varepsilon \|A_{\varepsilon} v^\varepsilon\|_{H_\alpha(\Omega)}^2.
\]

We first estimate the nonlinear term in (33). By (8) we get

\[
(\|v\|^2, A_{\varepsilon} v^\varepsilon)_{H_\alpha(\Omega)} = (J^* \nabla \|v\|^2 v, J^* \nabla \|v\|^2 v)_{H_\alpha(\Omega)}
\]

\[
= \int_\Omega g(\|v\|^2 v)_{y_1} \nabla_{y_1} - \frac{g_{y_1}}{g} y_{2}(\|v\|^2 v, v_{y_2} - \frac{g_{y_1}}{g} y_{2}(\|v\|^2 v_{y_2} y_{y_1}
\]

\[
+ \frac{1}{\varepsilon g^2} (\|v\|^2 v_{y_2} y_{y_2} + \frac{1}{\varepsilon g^2} (\|v\|^2 v_{y_2} y_{y_2}
\]

By simple computations we have

\[
\text{Re}(k + i \beta)(\|v\|^2 v)_{y_1} \nabla_{y_1} = -k \|v\|^2 \frac{\partial v}{\partial y_j} \left| v \right|^2 - k \frac{\partial |v|^2}{\partial y_j}^2 + \beta \text{Im}(v) \left( \frac{\partial |v|^2}{\partial y_j}^2 \right)
\]

\[
\leq -(k - |\beta|) \|v\|^2 \left| \frac{\partial v}{\partial y_j} \right|^2 - k \frac{\partial |v|^2}{\partial y_j}^2, \forall j = 1, 2,
\]

and

\[
\text{Re}(k + i \beta) \left( (\|v\|^2 v)_{y_1} \nabla_{y_1} + (\|v\|^2 v)_{y_2} \nabla_{y_2} \right)
\]

\[
\leq k \frac{\partial |v|^2}{\partial y_1} \frac{\partial |v|^2}{\partial y_2} + |\beta| \frac{\partial |v|^2}{\partial y_1} \left| v \right|^2 \frac{\partial \Re}{\partial y_2} + 2k \frac{\partial |v|^2}{\partial y_1} \left| \frac{\partial v}{\partial y_1} \left| v \right|^2 \frac{\partial \Re}{\partial y_2} \right.
\]

Note that \( k > |\beta| \). By Young's inequality, there exists \( \varepsilon_1 > 0 \) such that for every \( 0 < \varepsilon < \varepsilon_1 \),

\[
-\text{Re}(k + i \beta) e^{2\rho \varepsilon(\theta, \omega)} \left( (\|v\|^2 v, A_{\varepsilon} v^\varepsilon)_{H_\alpha(\Omega)} \right) \leq 0.
\]

On the other hand, the last two terms on the right-hand side of (33) is bounded by

\[
(\gamma + \rho \varepsilon(\theta, \omega)) \text{Re}(v^\varepsilon, A_{\varepsilon} v^\varepsilon)_{H_\alpha(\Omega)} + e^{-\rho \varepsilon(\theta, \omega)} \text{Re}(G_{\varepsilon}(t, y), A_{\varepsilon} v^\varepsilon)_{H_\alpha(\Omega)}
\]

\[
\leq (\gamma + \rho \varepsilon(\theta, \omega)) a_{\varepsilon}(v^\varepsilon, v^\varepsilon) + \frac{\lambda}{2} \frac{2}{\varepsilon} \|A_{\varepsilon} v^\varepsilon\|_{H_\alpha(\Omega)}^2 + c e^{-2\rho \varepsilon(\theta, \omega)} ||G(t, \cdot)||_{L^\infty(\Omega)}^2.
\]

By (33)-(38) we get for \( 0 < \varepsilon < \varepsilon_1 \)

\[
\frac{d}{dt} a_{\varepsilon}(v^\varepsilon, v^\varepsilon) + \lambda \frac{\partial |v|^2}{\partial y_1} \left| v \right|^2 + (\gamma - 2\rho \varepsilon(\theta, \omega)) a_{\varepsilon}(v^\varepsilon, v^\varepsilon)
\]

\[
\leq \frac{7}{2} \gamma a_{\varepsilon}(v^\varepsilon, v^\varepsilon) + c e^{-2\rho \varepsilon(\theta, \omega)} ||G(t, \cdot)||_{L^\infty(\Omega)}^2.
\]
For $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, multiplying (39) by $e^{\int_0^\tau (\frac{2}{7} \gamma - 2 \rho(z(\theta, \omega))) \, ds}$ and integrating over $(\tau, \tau)$, we infer that

$$\begin{align*}
a_\varepsilon \left( v^\varepsilon \left( \tau, \tau - t, \omega, v^\varepsilon_{\tau-t} \right), v^\varepsilon \left( \tau, \tau - t, \omega, v^\varepsilon_{\tau-t} \right) \right) &\leq e^{\int_0^\tau (\frac{2}{7} \gamma - 2 \rho(z(\theta, \omega))) \, ds} a_\varepsilon \left( v^\varepsilon \left( r, \tau - t, \omega, v^\varepsilon_{\tau-t} \right), v^\varepsilon \left( r, \tau - t, \omega, v^\varepsilon_{\tau-t} \right) \right) \\
&\quad + \frac{7}{2} \gamma \int_\tau^\tau e^{\int_s^\tau (\frac{2}{7} \gamma - 2 \rho(z(\theta, \omega))) \, ds} a_\varepsilon \left( v^\varepsilon \left( s, \tau - t, \omega, v^\varepsilon_{\tau-t} \right), v^\varepsilon \left( s, \tau - t, \omega, v^\varepsilon_{\tau-t} \right) \right) \, ds \\
&\quad + c \int_\tau^\tau e^{\int_s^\tau (\frac{2}{7} \gamma - 2 \rho(z(\theta, \omega))) \, ds} e^{-2 \rho(z(\theta, \omega))} ||G(s, \cdot)||^2_{L^\infty(\hat{\Omega})} \, ds. \quad (40)
\end{align*}$$

Integrating (40) with respect to $r$ on $(\tau - 1, \tau)$, we obtain

$$\begin{align*}
a_\varepsilon \left( v^\varepsilon \left( \tau, \tau - t, \omega, v^\varepsilon_{\tau-t} \right), v^\varepsilon \left( \tau, \tau - t, \omega, v^\varepsilon_{\tau-t} \right) \right) &\leq (1 + \frac{7}{2} \gamma) \int_{\tau-1}^{\tau} e^{\int_s^{\tau} (\frac{2}{7} \gamma - 2 \rho(z(\theta, \omega))) \, ds} \\
&\quad \times a_\varepsilon \left( v^\varepsilon \left( r + \tau - t, \omega, v^\varepsilon_{\tau-t} \right), v^\varepsilon \left( r + \tau - t, \omega, v^\varepsilon_{\tau-t} \right) \right) \, dr \\
&\quad + c \int_{\tau-1}^{\tau} e^{\int_s^{\tau} (\frac{2}{7} \gamma - 2 \rho(z(\theta, \omega))) \, ds} e^{-2 \rho(z(\theta, \omega))} ||G(r + \tau, \cdot)||^2_{L^\infty(\hat{\Omega})} \, dr. \quad (41)
\end{align*}$$

Let $T$ be the constant in Lemma 3.1, and $T_0 = \max\{1, T\}$. Then for all $t \geq T_0$, we get from (41) and Lemma 3.1 that for $0 < \varepsilon < \varepsilon_1$

$$\begin{align*}
a_\varepsilon \left( v^\varepsilon \left( \tau, \tau - t, \omega, v^\varepsilon_{\tau-t} \right), v^\varepsilon \left( \tau, \tau - t, \omega, v^\varepsilon_{\tau-t} \right) \right) &\leq c + c \int_{\tau-\infty}^{\tau} e^{\int_s^{\tau} (\frac{2}{7} \gamma + 2 \rho(z(\theta, \omega)) \, ds} e^{-2 \rho(z(\theta, \omega))} (1 + ||G(s + \tau, \cdot)||^2_{L^\infty(\hat{\Omega})}) \, ds. \quad (42)
\end{align*}$$

This together with Lemma 3.1 completes the proof. 

The following estimates are needed when we derive the convergence of pullback attractors.

**Lemma 3.3.** Suppose (20) holds. Then there exists $\varepsilon_1 > 0$ such that for all $0 < \varepsilon < \varepsilon_1$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, $T > 0$ and $v^\varepsilon \in H^1(\Omega)$, the solution $v^\varepsilon$ of (16) satisfies, for all $t \in [\tau, \tau + T]$,

$$\begin{align*}
\int_\tau^t \|v^\varepsilon(s, \tau, \omega, v^\varepsilon_{\tau-t})\|^2_{L^2(\Omega)} + \|v^\varepsilon(s, \tau, \omega, v^\varepsilon_{\tau-t})\|^2_{H^2(\Omega)} \, ds \\
&\leq M ||v^\varepsilon||^2_{H^1(\Omega)} + M \int_\tau^{\tau + T} (1 + ||G(s, \cdot)||^2_{L^\infty(\hat{\Omega})}) \, ds, \quad (43)
\end{align*}$$

where $M$ is a positive constant depending on $\tau$, $\omega$, $\lambda$, $k$, $\gamma$ and $T$, but independent of $\varepsilon$.

**Proof.** It follows form (26) that

$$\frac{d}{dt} ||v^\varepsilon||^2_{H^1(\Omega)} + 2 \lambda a_\varepsilon(v^\varepsilon, v^\varepsilon) + \frac{\gamma}{2} ||v^\varepsilon||^2_{H^1(\Omega)} \leq c_1 ||v^\varepsilon||^2_{H^1(\Omega)} + c_2 (1 + ||G(t, \cdot)||^2_{L^\infty(\hat{\Omega})}),$$

where $c_1$ and $c_2$ are positive constants not depending on $\varepsilon$. 


Lemma 3.4. Suppose (20) holds. Then there exists \( \varepsilon < \varepsilon_1 \) and \( M \) and \( D \) where

\[
\int_{\tau}^{t} e^{\varepsilon_1(s-t)} a_\varepsilon (v^\varepsilon (s, \tau, \omega, v^\varepsilon_\tau), v^\varepsilon (s, \tau, \omega, v^\varepsilon_\tau)) ds + \frac{\gamma}{2} \int_{\tau}^{t} e^{\varepsilon_1(s-t)} \|v^\varepsilon (s, \tau, \omega, v^\varepsilon_\tau)\|_{H_{\sigma}(\partial)}^2 ds
\leq e^{\varepsilon_1(t-\tau)} \|v^\varepsilon_\tau\|_{H_{\sigma}(\partial)}^2 + c_2 \int_{\tau}^{t} e^{\varepsilon_1(s-t)} (1 + \|G(s, \cdot)\|_{L^\infty(\partial)}) ds.
\]

Integrating (39) on \((\tau, t)\), we get that for every \( \omega \in \Omega \) and \( t \in [\tau, \tau + T] \),

\[
a_\varepsilon (v^\varepsilon (t), v^\varepsilon (t)) + \lambda \int_{\tau}^{t} \|A_\varepsilon v^\varepsilon (s)\|_{H_{\sigma}(\partial)}^2 ds
\leq a_\varepsilon (v^\varepsilon (\tau), v^\varepsilon (\tau)) + c_3 \int_{\tau}^{t} a_\varepsilon (v^\varepsilon (s), v^\varepsilon (s)) ds + c_4 \int_{\tau}^{t} \|G(s, \cdot)\|_{L^\infty(\partial)}^2 ds,
\]

where \( c_3 \) and \( c_4 \) are positive constants not depending on \( \varepsilon \). Notice the definition of \( D(A_\varepsilon) \), we can complete the proof by (44) and (45).

Similarly, one can prove

**Lemma 3.4.** Suppose (20) holds. Then there exists \( \varepsilon_1 > 0 \) such that for all \( 0 < \varepsilon < \varepsilon_1, \tau \in \mathbb{R}, \omega \in \Omega, T > 0 \) and \( v^0_\tau \in H^1(\mathbb{I}) \), the solution \( v^0 \) of (18) satisfies, for all \( t \in [\tau, \tau + T] \),

\[
\int_{\tau}^{t} \|v^0(s, \tau, \omega, v^0_\tau)\|_{L^2(\mathbb{I})}^2 + \|v^0(s, \tau, \omega, v^0_\tau)\|_{H^1(\mathbb{I})}^2 ds
\leq M\|v^0_\tau\|_{H^1(\mathbb{I})}^2 + M \int_{\tau}^{\tau + T} (1 + \|G(s, \cdot)\|_{L^\infty(\partial)}^2) ds,
\]

where \( M \) is a positive constant depending on \( \tau, \omega, k, \gamma \) and \( T \), but independent of \( \varepsilon \).

4. Existence of pullback attractors.

In this section, we establish the existence of \( D_1 \)-pullback attractor for the cocycle \( \Phi_\varepsilon \) associated with the stochastic problem (12) and \( D_0 \)-pullback attractor for the cocycle \( \Phi_0 \) associated with the stochastic problem (13), respectively. We first show that problem (12) has a tempered pullback absorbing set as stated below.

**Lemma 4.1.** Suppose (20) and (21) holds. Then there exists \( \varepsilon_1 > 0 \) such that for all \( 0 < \varepsilon < \varepsilon_1 \), the continuous cocycle \( \Phi_\varepsilon \) associated with problem (12) has a closed measurable \( D_1 \)-pullback absorbing set \( K \in D_1 \) which is given by, for each \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \)

\[
K(\tau, \omega) = \left\{ u \in L^2(\mathcal{O}) : \|u\|_{L^2(\mathcal{O})}^2 \leq L(\tau, \omega) \right\},
\]

where

\[
L(\tau, \omega) = Me^{\rho z(\omega)} + Me^{\rho z(\omega)} \int_{-\infty}^{0} e^{2\gamma s + 2\rho} \int_{\theta}^{0} z(\theta, \omega) dl e^{-2\rho \theta} \|G(s + \tau, \cdot)\|_{L^\infty(\partial)}^2 ds
\]

and \( M \) is a positive constant depending on \( k, \gamma \) and \( \lambda \), but independent of \( \tau, \omega, \varepsilon \) and \( D_1 \).
Therefore, for every $\varepsilon \in \mathbb{R}$, assume that (20) and (21) hold. Then, the continuous cocycle

$$
\Phi(t, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \in \mathcal{D}_1,
$$

where $\mathcal{D}_1 = \{ D_1(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \}$, define a new family $\hat{D}_1$ for $D_1$ as

$$
\hat{D}_1 = \{ \hat{D}_1(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \},
$$

where $\hat{D}_1(\tau, \omega) = \{ v \in L^2(\Omega) : \| v \|_{L^2(\Omega)} \leq e^{-\rho_2(\omega)}\| D_1(\tau, \omega) \|_{L^2(\Omega)} \}$. For any $D_1 \in \mathcal{D}_1$, one can check that $\hat{D}_1$ also belongs to $\mathcal{D}_1$, i.e., $\hat{D}_1$ is tempered. For any $u_{\tau,\omega}^\varepsilon \in D_1(\tau - t, \theta_{-t}\omega)$, we find that $v_{\tau,\omega}^\varepsilon = e^{-\rho_2(\theta_{-t}\omega)}u_{\tau,\omega}^\varepsilon$ satisfies

$$
\| v_{\tau,\omega}^\varepsilon \|_{L^2(\Omega)} = e^{-\rho_2(\theta_{-t}\omega)}\| u_{\tau,\omega}^\varepsilon \|_{L^2(\Omega)} \leq e^{-\rho_2(\theta_{-t}\omega)}\| D_1(\tau - t, \theta_{-t}\omega) \|_{L^2(\Omega)}.
$$

By (47) we see that $v_{\tau,\omega}^\varepsilon \in \hat{D}_1(\tau - t, \theta_{-t}\omega)$. Since $\hat{D}_1 \subset \mathcal{D}_1$, by Lemmas 3.2, there exists $T = T(\tau, \omega, D_1) \geq 1$ such that for all $t \geq T$,

$$
\| v_{\tau,\omega}^\varepsilon \|_{L^2(\Omega)} \leq M + M \int_{-\infty}^{0} e^{\frac{1}{2} \gamma s + 2\rho \int_{-t}^{0} z(\theta_{-s}\omega)ds} e^{-\rho_2(\theta_{-t}\omega)} \| G(s + \tau, \omega) \|_{L^2(\Omega)} ds.
$$

Notice that $v_{\tau,\omega}^\varepsilon = e^{-\rho_2(\theta_{-t}\omega)}u_{\tau,\omega}^\varepsilon = e^{-\rho_2(\omega)}u_{\tau,\omega}^\varepsilon$. This implies

$$
v_{\tau,\omega}^\varepsilon = v_{\tau,\omega}^\varepsilon (\tau - t, \theta_{-t}\omega, v_{\tau,\omega}^\varepsilon),
$$

which along with (48) implies that for $u_{\tau,\omega}^\varepsilon \in D_1(\tau - t, \theta_{-t}\omega)$

$$
\| v_{\tau,\omega}^\varepsilon \|_{L^2(\Omega)} \leq L(\tau, \omega).
$$

Therefore, for every $\tau \in \mathbb{R}, \omega \in \Omega$, and $D_1 \in \mathcal{D}_1$ there exists $T(\tau, \omega, D_1) \geq 1$, independent of $\varepsilon$, such that for all $t \geq T$,

$$
\Phi_\varepsilon(t, \tau, \omega, D_1(\tau - t, \theta_{-t}\omega)) \in K(\tau, \omega).
$$

By the similar argument as in [31] we can obtain easily from (46) that $K = \{ K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \}$ is tempered. On the other hand, it is evident that, for each $\tau \in \mathbb{R}$, $L(\tau, \cdot) : \Omega \to \mathbb{R}$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable. Consequently, $K$ is a closed measurable $\mathcal{D}_1$-pullback absorbing set for $\Phi_\varepsilon$ in $\mathcal{D}_1$.

**Theorem 4.2.** Suppose (20) and (21) hold. Then there exists $\varepsilon_1 > 0$ such that for all $0 < \varepsilon < \varepsilon_1$, the cocycle $\Phi_\varepsilon$ has a unique $\mathcal{D}_1$-pullback attractor $A_\varepsilon = \{ A_\varepsilon(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D}_1$ in $L^2(\Omega)$.

**Proof.** First, we know from Lemma 4.1 that $\Phi_\varepsilon$ has a closed measurable $\mathcal{D}_1$-pullback absorbing set $K$. Thanks to (49) and the compact embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$, $\Phi_\varepsilon$ is $\mathcal{D}_1$-pullback asymptotically compact in $L^2(\Omega)$. Hence, the existence of a unique $\mathcal{D}_1$-pullback attractor for the cocycle $\Phi_\varepsilon$ follows from [30] immediately.

Analogous results also hold for the solutions of (13). In particular, we have:

**Theorem 4.3.** Assume that (20) and (21) hold. Then, the continuous cocycle $\Phi_0$ has a unique $\mathcal{D}_0$-pullback attractor $A_0 = \{ A_0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D}_0$ in $L^2(I)$.

5. **Upper-semicontinuity of random attractors.**

Given $u \in L^2(\Omega)$, let $Mu$ be the average function of $u$ in $y_2$ as defined by

$$
Mu = \int_{0}^{1} u(y_1, y_2) dy_2.
$$

Arguing as in [15], we can obtain following result on the average function.
Lemma 5.1. If \( u \in H^1(\mathcal{O}) \), then \( \mathcal{M}u \in H^1(\mathcal{I}) \) and
\[
\|u - \mathcal{M}u\|_{H^1(\mathcal{O})} \leq c\varepsilon \|u\|_{L^2(\mathcal{O})},
\]
where \( c \) is a constant, independent of \( \varepsilon \).

In the sequel, we further assume that
\[
\|G_\varepsilon(t, \cdot) - G_0(t, \cdot)\|_{L^2(\mathcal{O})} \leq \kappa_1(t)\varepsilon, \quad \text{for all } t \in \mathbb{R},
\]
where \( \kappa_1(t) \in L^2_{loc}(\mathbb{R}) \).

Since \( L^2(\mathcal{I}) \) can be embedded naturally into \( L^2(\mathcal{O}) \) as the subspace of functions independent of \( y_2 \), we can consider the cocycle \( \Phi_0 \) as a mapping from \( L^2(\mathcal{I}) \) into \( L^2(\mathcal{O}) \). In this sense, we can compare \( \Phi_0 \) and \( \Phi_\varepsilon \).

**Theorem 5.2.** Suppose (20), (21) and (50) hold. Given \( \tau \in \mathbb{R} \), \( \omega \in \Omega \) and a positive number \( \eta(\tau, \omega) \), if \( u_\varepsilon \in H^1(\mathcal{O}) \) such that \( \|u_\varepsilon\|_{H^1(\mathcal{O})} \leq \eta(\tau, \omega) \), then we have, for any \( t \geq \tau \),
\[
\lim_{\varepsilon \to 0} \|\Phi_\varepsilon(t, \tau, \omega, u_\varepsilon) - \Phi_0(t, \tau, \omega, \mathcal{M}u_\varepsilon)\|_{L^2(\mathcal{O})} = 0.
\]

**Proof.** Taking the inner product of (18) with \( gl \), where \( l \in H^1(\mathcal{I}) \), we find that
\[
\int_{\mathcal{I}} g \frac{dv_0}{dt} ldy_1 + (\lambda + i\alpha) \int_{\mathcal{I}} g v_0^0 l y_1 dy_1 = -(k + i\beta) e^{2\rho z(\theta, \omega)} \int_{\mathcal{I}} g |v_0|^2 v_0^0 dy_1
\]
\[
+ (\gamma + \rho z(\theta, \omega)) \int_{\mathcal{I}} g v_0^0 l y_1 dy_1 + e^{-\rho z(\theta, \omega)} (G_0(t, y_1, \zeta)) H^1(\mathcal{O}).
\]

As \( \int_0^1 \zeta(y_1, y_2) dy_2 \) belongs to \( H^1(\mathcal{I}) \) if \( \zeta \) is in \( H^1(\mathcal{O}) \), the above equality becomes, for any \( \zeta \in H^1(\mathcal{O}) \),
\[
(dv_0^0/dt, \zeta)_{H^1(\mathcal{O})} + (\lambda + i\alpha) (v_0^0, \zeta)_{H^1(\mathcal{O})} = -(k + i\beta) e^{2\rho z(\theta, \omega)} (|v_0|^2 v_0^0, \zeta)_{H^1(\mathcal{O})}
\]
\[
+ (\gamma + \rho z(\theta, \omega)) (v_0^0, \zeta)_{H^1(\mathcal{O})} + e^{-\rho z(\theta, \omega)} (G_0(t, y_1, \zeta))_{H^1(\mathcal{O})}.
\]

Since \( v_0^0 \) is independent of \( y_2 \), the above equality gives, for any \( \zeta \in H^1(\mathcal{O}) \) and \( 0 < \varepsilon \leq 1 \),
\[
(dv_\varepsilon^0/dt, \zeta)_{H^1(\mathcal{O})} + (\lambda + i\alpha) a_\varepsilon (v_\varepsilon^0, \zeta)
\]
\[
= -(k + i\beta) e^{2\rho z(\theta, \omega)} (|v_\varepsilon|^2 v_\varepsilon - |v_0|^2 v_0^0, \zeta)_{H^1(\mathcal{O})} + (\gamma + \rho z(\theta, \omega)) (v_\varepsilon^0, \zeta)_{H^1(\mathcal{O})}
\]
\[
+ e^{-\rho z(\theta, \omega)} (G_\varepsilon(t, y) - G_0(t, y_1), \zeta)_{H^1(\mathcal{O})} + (\lambda + i\alpha) \left( \frac{g_\varepsilon}{g} v_\varepsilon^0, y_2 \zeta_{y_2} \right)_{H^1(\mathcal{O})}.
\]

Due to (52) and (16), the function \( v_\varepsilon^0 - v_0^0 \) satisfies the equation, for any \( \zeta \in H^1(\mathcal{O}) \),
\[
(dv_\varepsilon^0/dt, \zeta)_{H^1(\mathcal{O})} + (\lambda + i\alpha) a_\varepsilon (v_\varepsilon^0 - v_0^0, \zeta)
\]
\[
= -(k + i\beta) e^{2\rho z(\theta, \omega)} (|v_\varepsilon|^2 v_\varepsilon - |v_0|^2 v_0^0, \zeta)_{H^1(\mathcal{O})} + (\gamma + \rho z(\theta, \omega)) (v_\varepsilon^0 - v_0^0, \zeta)_{H^1(\mathcal{O})}
\]
\[
+ e^{-\rho z(\theta, \omega)} (G_\varepsilon(t, y) - G_0(t, y_1), \zeta)_{H^1(\mathcal{O})} + (\lambda + i\alpha) \left( \frac{g_\varepsilon}{g} v_\varepsilon^0, y_2 \zeta_{y_2} \right)_{H^1(\mathcal{O})}.
\]
Setting $\zeta = v^\varepsilon - v^0$ in (53) and then taking the real part, we see that
\[
\frac{1}{2} \frac{d}{dt} \|v^\varepsilon - v^0\|^2_{L^2(\mathcal{O})} + \text{Re} (\lambda + i\alpha) a_x (v^\varepsilon - v^0, v^\varepsilon - v^0)
\]
\[
= -\text{Re} (k + i\beta) e^{2\rho z(\theta_\omega)} (\|v^\varepsilon\|^2 - |v^0|^2)_{L^2(\mathcal{O})} + (\gamma + \rho z(\theta_\omega)) \|v^\varepsilon - v^0\|^2_{L^2(\mathcal{O})}
\]
\[
+ e^{-\rho z(\theta_\omega)} \text{Re} (G_x (t, y) - G_0 (t, y_1), v^\varepsilon - v^0)_{H^s(\mathcal{O})}
\]
\[
+ \text{Re} (\lambda + i\alpha) \left( \frac{g_0}{g} v^0_{y_1}, y_2 (v^\varepsilon_{y_2} - v^0_{y_2}) \right)_{H^s(\mathcal{O})}.
\]

(54)

For the first term on the right side of the equality above, set $f(s) = s^2$ and we have
\[
-\text{Re} (k + i\beta) e^{2\rho z(\theta_\omega)} (\|v^\varepsilon\|^2 - |v^0|^2)_{L^2(\mathcal{O})} - \text{Re} (k + i\beta) e^{2\rho z(\theta_\omega)} \int_{\mathcal{O}} g |v^\varepsilon|^2 (v^\varepsilon - v^0) dy
\]
\[
\leq |k + i\beta| e^{2\rho z(\theta_\omega)} \left( \int_{\mathcal{O}} g |v^\varepsilon|^2 |v^\varepsilon - v^0|^2 dy + \int_{\mathcal{O}} g |f'(\xi)| |v^0| |v^\varepsilon - v^0|^2 dy \right)
\]
\[
\leq c e^{2\rho z(\theta_\omega)} (\|v^0\|_{L^\infty(\mathcal{O})}^{1/2} + \|f'(\xi)\|_{L^\infty(\mathcal{O})} \|v^0\|_{L^\infty(\mathcal{O})}) \int_{\mathcal{O}} g |v^\varepsilon - v^0|^2 dy,
\]
which together Agmon inequality implies that
\[
-\text{Re} (k + i\beta) e^{2\rho z(\theta_\omega)} (\|v^\varepsilon\|^2 - |v^0|^2)_{L^2(\mathcal{O})}
\]
\[
\leq c e^{2\rho z(\theta_\omega)} (\|v^0\|_{L^\infty(\mathcal{O})}^{1/2} + \|f'(\xi)\|_{L^\infty(\mathcal{O})} \|v^0\|_{L^\infty(\mathcal{O})}) \int_{\mathcal{O}} g |v^\varepsilon - v^0|^2 dy
\]
\[
\leq c e^{2\rho z(\theta_\omega)} (\|v^\varepsilon\|_{L^2(\mathcal{O})} + \|\varepsilon\|_{H^2(\mathcal{O})} + \|v^0\|_{L^2(\mathcal{O})} \|v^0\|_{H^2(\mathcal{O})}) \|v^\varepsilon - v^0\|^2_{H^s(\mathcal{O})}
\]
\[
\leq c e^{2\rho z(\theta_\omega)} (\|v^\varepsilon\|_{L^2(\mathcal{O})} + \|\varepsilon\|_{H^2(\mathcal{O})} + \|v^0\|_{L^2(\mathcal{O})} \|v^0\|_{H^2(\mathcal{O})}) \|v^\varepsilon - v^0\|^2_{H^s(\mathcal{O})}.
\]

(55)

By (50), we get
\[
e^{-\rho z(\theta_\omega)} \text{Re} (G_x (t, y) - G_0 (t, y_1), v^\varepsilon - v^0)_{H^s(\mathcal{O})}
\]
\[
\leq e^{-\rho z(\theta_\omega)} \|G_x (t, y) - G_0 (t, y_1)\|_{H^s(\mathcal{O})} \|v^\varepsilon - v^0\|_{H^s(\mathcal{O})}
\]
\[
\leq c k_1(s) e^{-\rho z(\theta_\omega)} \|v^\varepsilon - v^0\|_{H^s(\mathcal{O})}
\]
\[
\leq c \varepsilon e^{-\rho z(\theta_\omega)} k_1^2(t) + c \varepsilon \left( \|v^\varepsilon\|_{H^s(\mathcal{O})} + \|v^0\|_{H^s(\mathcal{O})} \right).
\]

(56)

Finally, by (9), we have
\[
\text{Re} (\lambda + i\alpha) \left( \frac{g_0}{g} v^0_{y_1}, y_2 (v^\varepsilon_{y_2} - v^0_{y_2}) \right)_{H^s(\mathcal{O})}
\]
\[
= \text{Re} (\lambda + i\alpha) \left( \frac{g_0}{g} v^0_{y_1}, y_2 (v^\varepsilon_{y_2} - v^0_{y_2}) \right)_{L^2(\mathcal{O})}
\]
\[
\leq c \varepsilon \|v^0\|_{H^1(\mathcal{I})} \|v^\varepsilon - v^0\|_{H^1(\mathcal{O})} \leq c \varepsilon \left( \|v^\varepsilon\|^2_{H^1(\mathcal{O})} + \|v^0\|^2_{H^1(\mathcal{I})} \right).
\]

(57)
By (55)-(57), we get from (54) that, for \( t \geq \tau \),
\[
\frac{d}{dt} \left\| v^\varepsilon - v^0 \right\|_{H_\varepsilon(\Omega)}^2 \leq \beta(t, \omega) \left\| v^\varepsilon - v^0 \right\|_{H_\varepsilon(\Omega)}^2 + c \varepsilon \left( \left\| v^\varepsilon \right\|_{H^1_\varepsilon(\Omega)}^2 + \left\| v^0 \right\|_{H^1(I)}^2 \right) + c \varepsilon e^{-2p_2(\theta, \omega)} \kappa_1^2(t),
\]
where
\[
\beta(t, \omega) = c\left(\left\| v^\varepsilon \right\|_{L^2(\Omega)}^2 + \left\| v^\varepsilon \right\|_{H^2(\Omega)}^2 + \left\| v^0 \right\|_{L^2(I)}^2 + \left\| v^0 \right\|_{H^2(I)}^2 \right) e^{2p_2(\theta, \omega)} + 2(\gamma + \rho |z(\theta, \omega)|).
\]
Integrating (58) on \((\tau, t)\) we obtain that for all \( t \in [\tau, \tau + T] \) with \( T > 0 \),
\[
\left\| v^\varepsilon(t) - v^0(t) \right\|_{H_\varepsilon(\Omega)}^2 \leq e^{\int_\tau^t \beta(s, \omega) ds} \left\| v^\varepsilon(\tau) - v^0(\tau) \right\|_{H_\varepsilon(\Omega)}^2 + c \varepsilon \int_\tau^t e^{\int_\tau^s \beta(x, \omega) d\xi} \left( \left\| v^\varepsilon(s) \right\|_{H^1_\varepsilon(\Omega)}^2 + \left\| v^0(s) \right\|_{H^1(I)}^2 \right) ds
\]
\[
+ c \varepsilon e^{\int_\tau^t \beta(s, \omega) ds} \left\| v^\varepsilon(s) \right\|_{H^1_\varepsilon(\Omega)}^2 + \left\| v^0(s) \right\|_{H^1(I)}^2 \right) ds
\]
\[
\leq e^{\int_\tau^{\tau + T} \beta(s, \omega) ds} \left\| v^\varepsilon(\tau) - v^0(\tau) \right\|_{H_\varepsilon(\Omega)}^2 + c \varepsilon e^{\int_\tau^{\tau + T} \beta(s, \omega) ds} \int_\tau^{\tau + T} \kappa_1^2(s) ds.
\]
By (59), Lemma 3.3, Lemma 3.4 and \( \left\| u^\varepsilon_\tau \right\|_{H^1_\varepsilon(\Omega)} \leq \eta(\tau, \omega) \) we find that there exists a positive constant \( \chi = \chi(\tau, \omega, \lambda, k, \gamma, T) \), independent of \( \varepsilon \), such that for all \( t \in [\tau, \tau + T] \),
\[
\left\| v^\varepsilon(t) - v^0(t) \right\|_{H_\varepsilon(\Omega)}^2 \leq \chi \left\| v^\varepsilon(\tau) - v^0(\tau) \right\|_{H_\varepsilon(\Omega)}^2 + e^{\int_\tau^{\tau + T} \beta(s, \omega) ds} \left\| v^\varepsilon(s) \right\|_{H^1_\varepsilon(\Omega)}^2 + \left\| v^0(s) \right\|_{H^1(I)}^2 \right) ds
\]
\[
+ \int_\tau^{\tau + T} \left( 1 + \kappa_1^2(s) + \left\| G(s, \cdot) \right\|_{L^\infty(\Omega)}^2 \right) ds.
\]
Then we have from Lemma 5.1, for all \( t \in [\tau, \tau + T] \),
\[
\left\| u^\varepsilon(t, \tau, \omega, u^\varepsilon_\tau) - u^0(t, \tau, \omega, \mathcal{M} u^\varepsilon_\tau) \right\|_{H_\varepsilon(\Omega)}^2
\]
\[
eq e^{2p_2(\theta, \omega)} \left\| v^\varepsilon(t, \tau, \omega, e^{-\rho_2(\theta, \omega)} u^\varepsilon_\tau) - v^0(t, \tau, \omega, e^{-\rho_2(\theta, \omega)} \mathcal{M} u^\varepsilon_\tau) \right\|_{H_\varepsilon(\Omega)}^2
\]
\[
\leq \chi e^{2\rho_2(\theta, \omega)} \left\| e^{-\rho_2(\theta, \omega)} u^\varepsilon_\tau - e^{-\rho_2(\theta, \omega)} \mathcal{M} u^\varepsilon_\tau \right\|_{H_\varepsilon(\Omega)}^2 \left\| v^\varepsilon \right\|_{H^1_\varepsilon(\Omega)}^2 + \left\| v^0 \right\|_{H^1(I)}^2 \right) ds
\]
\[
+ e^{\int_\tau^{\tau + T} \beta(s, \omega) ds} \left\| v^\varepsilon \right\|_{H^1_\varepsilon(\Omega)}^2 + \left\| v^0 \right\|_{H^1(I)}^2 \right) ds
\]
\[
\leq \chi e^{2\rho_2(\theta, \omega)} e^{-2p_2(\theta, \omega)} e^{\frac{2}{\varepsilon^2} \left\| u^\varepsilon_\tau \right\|_{H^1_\varepsilon(\Omega)}^2} \left\| v^\varepsilon \right\|_{H^1_\varepsilon(\Omega)}^2 + \left\| v^0 \right\|_{H^1(I)}^2 \right) ds
\]
\[
+ e^{\int_\tau^{\tau + T} \beta(s, \omega) ds} \left\| v^\varepsilon \right\|_{H^1_\varepsilon(\Omega)}^2 + \left\| v^0 \right\|_{H^1(I)}^2 \right) ds
\]
\[
+ \int_\tau^{\tau + T} \left( 1 + \kappa_1^2(s) + \left\| G(s, \cdot) \right\|_{L^\infty(\Omega)}^2 \right) ds.
\]
Then the desired result follows from the fact that \( \left\| u^\varepsilon_\tau \right\|_{H^1_\varepsilon(\Omega)} \leq \eta(\tau, \omega) \) and the above inequality. □
Now we are in a position to prove our main result.

**Theorem 5.3.** Assume that (20), (21) and (50) hold. Then the pullback attractors $A_\varepsilon$ are upper-semicontinuous at $\varepsilon = 0$, that is, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{\varepsilon \to 0} \text{dist}_{L^2(\mathcal{O})}(A_\varepsilon(\tau, \omega), A_0(\tau, \omega)) = 0.$$ 

**Proof.** For the proof, please see Theorem 5.2 in [21]. □

**Acknowledgments.** We would like to thank the referees for their useful suggestions and comments.

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Received November 2017; revised January 2018.

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