Strings and the Stable Cohomology of Mapping Class Groups

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2000 Mathematics Subject Classification: 57R20, 55P47, 32G15, 81T40.
Keywords and Phrases: Elliptic cohomology, Cohomology of moduli spaces, Infinite loop spaces, Cobordism theory.

1. Introduction

Twenty years ago, Mumford initiated the systematic study of the cohomology ring of moduli spaces of Riemann surfaces. Around the same time, Harer proved that the homology of the mapping class groups of oriented surfaces is independent of the genus in low degrees, increasing with the genus. The (co)homology of mapping class groups thus stabilizes. At least rationally, the mapping class groups have the same (co)homology as the corresponding moduli spaces. This prompted Mumford to conjecture that the stable rational cohomology of moduli spaces is generated by certain tautological classes that he defines. Much of the recent interest in this subject is motivated by mathematical physics and, in particular, by string theory. The study of the category of strings led to the discovery of an infinite loop space, the cohomology of which is the stable cohomology of the mapping class groups. We explain here a homotopy theoretic approach to Mumford’s conjecture based on this fact. As byproducts infinite families of torsion classes in the stable cohomology are detected, and the divisibility of the tautological classes is determined. An analysis of the category of strings in a background space leads to the formulation of a parametrized version of Mumford’s conjecture.

The paper is chiefly a summary of the author’s work and her collaboration with Ib Madsen. Earlier this year Madsen and Weiss announced a solution of Mumford’s conjecture. We touch on some of the consequences and the ideas behind this most exciting new development.

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2. Mumford’s conjecture

Let \( F^s_{g,n} \) be an oriented, connected surface of genus \( g \) with \( s \) marked points and \( n \) boundary components. Let \( \text{Diff}(F^s_{g,n}) \) be its group of orientation preserving diffeomorphisms that fix the \( n \) boundary components pointwise and permute the \( s \) marked points. By [2], for genus at least 2, \( \text{Diff}(F^s_{g,n}) \) is homotopic to its group of components, the mapping class group \( \Gamma^s_{g,n} \). Furthermore, if the surface has boundary, \( \Gamma^s_{g,n} \) acts freely on Teichmüller space. Hence,

\[
B\text{Diff}(F^s_{g,n}) \simeq B\Gamma^s_{g,n} \simeq \mathcal{M}^s_{g,n} \text{ for } n \geq 1, g \geq 2,
\]

where \( \mathcal{M}^s_{g,n} \) denotes the moduli space of Riemann surfaces appropriately marked. When \( n = 0 \), the action of the mapping class group on Teichmüller space has finite stabilizer groups and the latter is only a rational equivalence.

We recall Harer’s homology stability theorem [4] which plays an important role throughout the paper.

**Harer Stability Theorem 2.1.** \( H_*(B\Gamma^s_{g,n}) \) is independent of \( g \) and \( n \) in degrees \( 3* < g - 1 \).

Ivanov [5], [6] improved the stability range to \( 2* < g - 1 \) and proved a version with twisted coefficients. Glueing a torus with two boundary components to a surface \( F_{g,1} \) induces a homomorphism \( \Gamma_{g,1} \to \Gamma_{g+1,1} \). Let \( \Gamma_\infty := \lim_{g \to \infty} \Gamma_{g,1} \) be the stable mapping class group.

Mumford [12] introduced certain tautological classes in the cohomology of the moduli spaces \( \mathcal{M}_g \). Topological analogues were studied by Miller [10] and Morita [11]: Let \( E \) be the universal \( F \)-bundle over \( B\text{Diff}(F) \), and let \( T^e E \) be its vertical tangent bundle with Euler class \( e \in H^2 E \). Define

\[
\kappa_i := \int_F e_i^{i+1} \in H^{2i} B\text{Diff}(F).
\]

Here \( \int_F \) denotes “integration over the fiber” - the Gysin map. Miller and Morita showed independently that the rational cohomology of the stable mapping class group contains the polynomial ring on the \( \kappa_i \).

**Mumford Conjecture 2.2.** \( H^*(B\Gamma_\infty; \mathbb{Q}) \simeq \mathbb{Q}[\kappa_1, \kappa_2, \ldots] \).

2.1. Remark.

The stable cohomology of the decorated mapping class groups is known modulo \( H^* B\Gamma_\infty \) because of decoupling [1]. For example, let \( \Gamma^s_\infty := \lim \Gamma^s_{g,1} \). The following is a consequence of Theorem 2.1.

**Proposition 2.3.** \( (B\Gamma^s_\infty)^+ \simeq B\Gamma_\infty^+ \times B(\Sigma_s \wr S^1)^+ \).

Here \( Y^+ \) denotes Quillen’s plus-construction on \( Y \) with respect to the maximal perfect subgroup of the fundamental group. It is important to note that the plus construction does not change the (co)homology. In particular,

\[
H^* B\Gamma_\infty \simeq H^* B\Gamma_\infty^+.
\]
3. String category

The category underlying the quantum mechanics of a state space $X$ is the path category $\mathcal{P}X$. Its objects are particles represented by points in $X$. As time evolves a particle sweeps out a path. Thus a morphism between particles $a$ and $b$ is a continuous path in $X$ starting at $a$ and ending at $b$. Concatenation of paths defines the composition in the category.

$$\mathcal{P}X = \left\{ \begin{array}{c}
\text{objects : } a, b, \cdots \in X, \\
\text{morphisms : } \prod_{t > 0} \text{map}([0, t], X).
\end{array} \right. $$

In string theory, the point objects are replaced by closed loops in $X$. As time evolves these strings sweep out a surface. Thus the space of morphisms from one string to another is now described by a continuous map from an oriented surface $F$ to $X$. The parametrization of the path should be immaterial. To reflect this, take homotopy orbits under the action of $\text{Diff}(F)$.\footnote{Strings should also be independent of the parametrization. One should therefore take homotopy orbit spaces of the objects under the $S^1$ action. In that case the diffeomorphisms of the surfaces need not be the identity on the boundary. The resulting category has the same homotopy type as $\mathcal{S}X$ in the sense that its classifying space is homotopic to that of $\mathcal{S}X$.} Composition is given by concatenation of paths, i.e. by gluing of surfaces along outgoing and incoming boundary components.

To be more precise, let $LX = \text{map}(S^1, X)$ denote the free loop space on $X$. A cobordism $F$ is a finite union

$$F_{g_1, n_1, m_1} \cup \cdots \cup F_{g_k, n_k, m_k}$$

where $n = \Sigma_i n_i$ boundary components are considered incoming and $m = \Sigma_i m_i$ outgoing. For technical reasons we will assume $m_i > 0$. The category of strings in $X$ is then

$$\mathcal{S}X = \left\{ \begin{array}{c}
\text{objects : } \alpha, \beta, \cdots \in \prod_{n \geq 0} (LX)^n, \\
\text{morphisms : } \prod_{F} E \text{Diff}(F) \times_{\text{Diff}(F)} \text{map}(F, X).
\end{array} \right.$$  

The disjoint union is taken over all cobordisms $F$, one for each topological type.

3.1. Elliptic elements.

The category $\mathcal{S}X$ was first introduced by Segal [14]. A functor from the path category $\mathcal{P}X$ to the category of $n$-dimensional vector spaces and their isomorphisms defines a vector bundle on $X$ with connection. In particular, it defines an element in the $K$-theory of $X$. A functor from $\mathcal{S}X$ to an appropriate (infinite dimensional) vector space category is also referred to as a gerbe (or $B$-field) with connection. In [14], Segal proposes this as the underlying geometric object of elliptic cohomology. More recently, this notion has been refined by Teichner and Stolz.
3.2. Conformal field theory.

The category \( S := \mathcal{S}(\ast) \) is studied in conformal field theory \([15]\). Its objects are the natural numbers and its morphisms are Riemann surfaces. A conformal field theory (CFT) is a linear space \( H \) which is an algebra over \( S \). Thus each element in \( \mathcal{M}_{g,n+m} \) defines a linear map from \( H^\otimes n \) to \( H^\otimes m \). The physical states of a topological conformal field theory (TCFT) form a graded vector space \( A_\ast \). Each element of the homology \( H_{\ast}\mathcal{M}_{g,n+m} \) defines a linear map from \( A_\ast^\otimes n \) to \( A_\ast^\otimes m \).

3.3. Gromov-Witten theory.

Let \( X \) be a symplectic manifold. A model for the homotopy orbit spaces in the definition of \( S X \) is the fiber bundle \( \text{map}(F_{g,n}, X) \to \mathcal{M}_{g,n}(X) \to \mathcal{M}_{g,n} \) over the Riemann moduli space. In each fiber, \( F \) comes equipped with a complex structure and we may replace the continuous maps by the space of pseudo-holomorphic maps \( \text{hol}(F_{g,n}, X) \) yielding a category \( S^{\text{hol}} X \). This is the category relevant to Gromov-Witten theory. Note, for \( X \) a complex Grassmannian, a generalized flag manifold, or a loop group, the degree \( d \)-component of \( \text{hol}(F_{g,n}, X) \) approximates the components of \( \text{map}(F_{g,n}, X) \) in homology. The categories \( S X \) and \( S^{\text{hol}} X \) are therefore closely related.

4. From categories to infinite loop spaces

There is a functorial way to associate to a category \( \mathcal{C} \) a topological space \(|\mathcal{C}|\), the realization of its nerve. It takes equivalences of categories to homotopy equivalences. It is a generalization of the classifying space construction of a group: \(|G| = BG\) where \( G \) is identified with the category of a single object and endomorphism set \( G \). The path-category \( \mathcal{P}X \) of a connected space \( X \) is a many object group up to homotopy. The underlying “group” is the space \( \Omega X = \text{map}_*(S^1, X) \) of based loops in \( X \). Again one has \(|\mathcal{P}X| \simeq B(\Omega X) \simeq X \). A functor from \( \mathcal{P}X \) to \( n \)-dimensional vector spaces and their isomorphisms thus defines a map

\[ X \to BGL_n \mathbb{C}, \]

and hence an element in the \( K \)-theory of \( X \). Motivated by this, we would like to understand the classifying space of the string category \( S X \) and its relation to elliptic cohomology.

**Definition 4.1.** \( St(X) := \Omega|S X| \).

**Theorem 4.2.** \( St \) is a homotopy functor from the category of topological spaces to the category of infinite loop spaces.

We recall that \( Z \) is an infinite loop space if it is homotopic to some \( Z_0 \) such that successive based spaces \( Z_i \) can be found with homeomorphisms \( \gamma_i : Z_i \simeq \Omega Z_{i+1} \). Any infinite loop space \( Z \) gives rise to a generalized homology theory \( h_\ast \) which evaluated on a space \( Y \) is

\[ h_\ast Y := \pi_\ast \lim_{i \to \infty} \Omega^i(Z_i \wedge Y). \]
Infinite loop spaces are abelian groups up to homotopy in the strongest sense.

The proof of Theorem 4.2 can be sketched as follows, compare [16]. $S X$ is a symmetric monoidal category under disjoint union. Infinite loop space machinery (see for example [13]) implies that its classifying space $|S X|$ is a homotopy abelian monoid in the strongest sense. But $\pi_0|S X| = H_1 X$ is a group. Hence homotopy inverses exist and $|S X|$ is an infinite loop space.

Using another piece of infinite loop space machinery (a generalization of the group completion theorem) and Harer Stability Theorem 2.1, one can identify the string theory of a point as $\mathbb{Z} \times B \Gamma_*^\infty$. As an immediate consequence we have

**Corollary 4.3.** ([16]) $\text{St}(\ast) \simeq \mathbb{Z} \times B \Gamma_*^\infty$ is an infinite loop space.

### 5. CFT-operad

We offer now a different perspective on Theorem 4.1 and Corollary 4.3. Let $\mathcal{M} = \{M_n\}_{n \geq 0}$ with $M(n) = \bigsqcup_{g \geq 0} B \Gamma_{g,n+1}$ be the operad contained in the CFT category $S$. A space $X$ is an algebra over $\mathcal{M}$ if there are compatible maps $M(n) \times X^n \to X$. In particular $X$ has a monoid structure. Let $\mathcal{G}X$ be its group completion. $\mathcal{G}X$ is homotopic to $X$ if and only if $\pi_0 X$ is a group.

**Theorem 5.1.** ([18]) If $X$ is an algebra over $\mathcal{M}$ then its group completion $\mathcal{G}X$ is an infinite loop space.

CFT is therefore closely linked to the theory of infinite loop spaces. The crucial point of the proof is a decoupling result similar to Proposition 2.3. The corresponding statement for TCFT’s implies that Getzler’s Batalin-Vilkovisky algebra structure on the physical states $A_\ast$ is stably trivial, see [17]. The following examples illustrate the strength of Theorem 5.1.

**Example 5.1.**

Let $X_1$ be the disjoint union $\bigsqcup_{g \geq 0} B \Gamma_{g,1}$. It has a product induced by gluing $F_{g,1}$ and $F_{h,1}$ to the “legs” of a pairs of pants surface $F_{0,3}$. Miller observes in [10] that this induces a double loop space structure on the group completion $\mathcal{G}X_1 \simeq \mathbb{Z} \times B \Gamma_*^\infty$. This product extends to an $\mathcal{M}$-algebra structure. Hence Miller’s double loop space structure extends to an infinite loop space structure. Wahl [19] proved that it is equivalent to the infinite loop spaces structure implied by Corollary 4.3.

**Example 5.2.**

Let $X_2$ be the disjoint union of the Borel constructions

$$E_g := E \text{ Diff}(F_{g,1}) \times \text{ Diff}(F_{g,1}) \text{ map}(F_{g,1}, \partial; X, \ast).$$

As the functions to $X$ map the boundary to a point, they can be extended from $F_{g,1}$ to $F_{g+1,1}$ via the constant map. $X_2$ thus becomes an $\mathcal{M}$-algebra and $\mathcal{G}X_2 = \mathbb{Z} \times (\lim_{g \to \infty} E_g)^+$ is an infinite loop space. $\mathcal{G}X_2$ is homotopic to $\text{St}(X)$ when $X$ is simply connected.
Example 5.3.

Let $C(F_g; X)$ denote the space of unordered configurations in the interior of $F_g$ with labels in a connected space $X$. Let $C_g$ be its Borel construction. Their disjoint union defines an $\mathcal{M}$-algebra $X_3$. The following decoupling result determines its group completion.

Proposition 5.2. \(G X_3 \simeq \mathbb{Z} \times (\lim_{g \to \infty} C_g)^+ \simeq \mathbb{Z} \times B\Gamma_\infty^+ \times Q(BS^1 \land X_+).\)

Note though that the induced $\text{Diff}(F_g; 1)$-action from the left on the right is non-trivial on the sphere in the target space.

6. Refinement of Mumford’s conjecture

Infinite loop spaces are relatively rare and the question arises whether $\mathbb{Z} \times B\Gamma_\infty^+$ can be understood in terms of well-known infinite loop spaces. This question was addressed in joint work with Madsen.

Let $\mathbb{P}^l$ be the Grassmannian of oriented 2-planes in $\mathbb{R}^{l+2}$ and let $-L_l$ be the complement of the canonical 2-plane bundle $L$ over $\mathbb{P}^l$. The one-point compactification, the Thom space $\text{Th}(-L_l)$, restricts on the subspace $\mathbb{P}^{l-1}$ to the suspension of $\text{Th}(-L_{l-1})$. Taking adjoints yields maps $\text{Th}(-L_{l-1}) \to \Omega \text{Th}(-L_l)$, and we may define

$$\Omega^\infty \text{Th}(-L) := \lim_{l \to \infty} \Omega^l \text{Th}(-L_l).$$

More generally, for any space $X$, define

$$\Omega^\infty (\text{Th}(-L) \land X_+) := \lim_{l \to \infty} \Omega^l (\text{Th}(-L_l) \land X_+).$$

Conjecture 6.1. There is a homotopy equivalence of infinite loop space

$$\alpha : ST(X) \overset{\simeq}{\longrightarrow} \Omega^\infty (\text{Th}(-L) \land X_+).$$

Remarks 6.2.

For $X = \ast$, Conjecture 6.1 postulates a homotopy equivalence $\alpha : \mathbb{Z} \times B\Gamma_\infty^+ \to \Omega^\infty \text{Th}(-L)$. A proof of this has been announced by Madsen and Weiss, see Section 8. The Mumford conjecture follows from this as we will explain presently. Conjecture 6.1 claims in addition that $\text{St}(\omega)$ is a homology functor, i.e. $\pi_*$ $\text{St}(\omega)$ is the homology theory associated to the infinite loop space $\Omega^\infty \text{Th}(-L)$. 
The infinite loop space $\Omega^\infty \text{Th}(-L)$ is well-studied, more recently because of its relation to Waldhausen $K$-theory. The inclusion of $-L_l$ into the trivial bundle $L \oplus (-L_l) \simeq \mathbb{P}^l \times \mathbb{R}^{l+2}$ induces a map

$$\omega : \Omega^\infty \text{Th}(-L) \to Q(\mathbb{P}_+) .$$

$\omega$ has homotopy fibre $\Omega^2 Q(S^0)$. As the stable homotopy groups of the sphere are torsion in positive dimensions, $\omega$ is a rational equivalence. Let $\mathbb{P}^\infty \to BU$ be the map that classifies $L$. By Bott periodicity, the map can be extended to the free infinite loop space $L : Q_0(\mathbb{P}^\infty) \to BU$. The subscript here indicates the 0-component. $L$ has a splitting and is well known to be a rational equivalence:

$$H^*(\Omega^\infty \text{Th}(-L); \mathbb{Q}) \simeq \mathbb{Q}[c_1, c_2, \ldots].$$

The $\mathbb{Z}/p$-homology of $\Omega^\infty \text{Th}(-L)$ has recently been determined by Galatius [3].

\[ t = x + v \]

Figure 1: Surface $h(F) \subset \mathbb{R}^{l+2}$ with tubular neighborhood $U$.

To define a map $\alpha$ comes down to defining maps from the morphism spaces of $\mathcal{S}X$. (See also Example 5.2.) $\alpha$ is the homotopy theoretic interpretation of the formula defining $\kappa_i$ where the wrong way map $f_F$ is replaced by the (pre)transfer map of Becker and Gottlieb. We give now an explicit description of this map.

For simplicity, let $F$ be a closed surface. Consider the space of smooth embeddings $\text{Emb}(F, \mathbb{R}^{l+2})$. By Whitney’s embedding theorem, as $l \to \infty$, it may serve as a model for $E \text{Diff}(F)$. Let

$$(h, f) \in \text{Emb}(F, \mathbb{R}^{l+2}) \times \text{Diff}(F) \text{ map}(F, X).$$

Choose a tubular neighborhood $U$ of $h(F)$ such that every $t \in U$ can uniquely be written as $x + v$ with $x \in h(F)$ and $v$ normal to the tangent plane $T_x h(F)$. $\alpha$ sends $(h, f)$ to the continuous function $\alpha(h, f) : S^{l+2} \to \text{Th}(-L_l) \wedge X_+$ defined by

\[
\begin{align*}
t & \mapsto \\
& \begin{cases}
\infty & \text{if } t \not\in U, \\
((T_x h(F), v), f(h^{-1}(x)) & \text{if } t = x + v \in U.
\end{cases}
\end{align*}
\]

In [8], $\alpha$ is shown to be a 3-connected map of infinite loop spaces and the tautological classes are identified. Let $i! (ch_i) \in H^{2i} BU$ denote the $i$-th integral Chern character class. Then

$$\kappa_i = \alpha^* \omega^* \circ L^* (i! (ch_i)).$$
The main result of [8] is a partial splitting of the composition

$$\omega \circ \alpha : \mathbb{Z} \times B\Gamma_\infty^+ \to Q(\mathbb{P}_+^\infty) \simeq Q(S^0) \times Q(\mathbb{P}^\infty).$$

This is achieved by constructing a map $\mu$ from $\mathbb{P}_+^\infty$ to $\mathbb{Z} \times B\Gamma_\infty^+$. In order to construct $\mu$, approximate $\mathbb{P}_+^\infty \simeq BS^1$ by the classifying spaces of cyclic groups $C_{p^n}$ for $n \to \infty$, one prime $p$ at a time, as the cyclic groups can be mapped into suitable mapping class groups. However, this means that we have to work with $p$-completions.

Let $Y^\wedge_p$ denote the $p$-completion of $Y$ and $g \in \mathbb{Z}_p^\times$ be a topological generator of the $p$-adic units ($g = 3$ if $p = 2$). Denote by $\psi^k : \mathbb{P}_+^\infty \to (\mathbb{P}_+^\infty)^\wedge_p$ the map that represents $k$ times the first Chern class in $H^2(\mathbb{P}_+^\infty; \mathbb{Z}_p)$.

**Theorem 7.1.** [8]. There exists a map $\mu : (Q(S^0) \times Q(\mathbb{P}_+^\infty))^\wedge_p \to (\mathbb{Z} \times B\Gamma_\infty^+)^\wedge_p$ such that

$$\omega \circ \alpha \circ \mu \simeq \begin{pmatrix} -2 & * \\ 0 & g \psi^g \end{pmatrix}.$$

The map $1 - g\psi^g$ induces multiplication by $1 - g^{n+1}$ on $H_{2n}(\mathbb{P}_+^\infty; \mathbb{Z}_p)$ which is a $p$-adic unit precisely if $n \neq -1(\mod p - 1)$. The following applications of Theorem 7.1 are also found in [8]. There is a splitting $Q(\mathbb{P}_+^\infty)^\wedge_p \simeq E_0 \times \cdots \times E_{p-2}$ corresponding to the idempotent decomposition of $\mathbb{Z}[\mathbb{Z}/p^\infty] \subset \mathbb{Z}_p[\mathbb{Z}_p^\times]$.

**Corollary 7.2.** For some $W_p$, there is a splitting of infinite loop spaces

$$(\mathbb{Z} \times B\Gamma_\infty^+)^\wedge_p \simeq E_0 \times \cdots \times E_{p-3} \times W_p.$$
Corollary 7.3. For odd primes \( p \) and some space \( V_p \), there is a splitting of spaces

\[
(B\Gamma_\infty^+)^\wedge_p \simeq BU_p^\wedge \times V_p.
\]

This gives a \( \mathbb{Z}_p \)-integral version of Miller and Morita’s theorem: the polynomial algebra \( \mathbb{Z}_p[c_1,c_2,\ldots] \) is a split summand of \( H^*(B\Gamma_\infty;\mathbb{Z}_p) \). The divisibility of the tautological classes \( \kappa_i \) at odd primes \( p \) can also be deduced from the above diagram.

Corollary 7.4. If \( i = -1(\mod p - 1) \), then \( \kappa_i \) is divisible by \( p^{1+\nu_p(i+1)} \) where \( \nu_p \) is the \( p \)-adic valuation. Otherwise, \( p \) does not divide \( \kappa_i \).

In the light of [9] this result is sharp.

8. Geometric interpretation

\( \alpha : B\Gamma_\infty^+ \to \Omega_0^\infty Th(-L) \) is a homotopy equivalence if and only if it induces an isomorphism in oriented cobordism theory \( \Omega^*_X \). An element in \( \Omega^SO_X(B\Gamma_\infty^+) = \Omega^SO_X(B\Gamma_\infty) \) is a cobordism class of oriented surface bundles \( F \to E^{n+2} \xrightarrow{\pi} M^n \). An element in \( \Omega^SO_X(\Omega^\infty_X Th(-L)) \) is a cobordism class of pairs \([\pi : E^{n+2} \to M^n, \hat{\pi}]\) of smooth maps \( \pi \) and stable bundle surjections from \( TE \) to \( \pi^*TM \). (Upto cobordism one can assume that \( \hat{\pi} \) is a vector bundle surjection.) \( \alpha \) maps a bundle \([F \to E \xrightarrow{\pi} M]\) to the pair \([\pi : E \to M, D\pi]\) where \( D\pi \) denotes the differential of \( \pi \). Hence, \( \alpha \) is a homotopy equivalence if and only if each cobordism class of pairs \([\pi : E^{n+2} \to M^n, \hat{\pi}]\) contains a “unique” representative with \( \pi \) a submersion.

It is this geometric formulation that underpins the solution to the Mumford conjecture by Madsen and Weiss. A key ingredient of the proof is the Phillips-Gromov h-principle of submersion theory: A pair \((g : X \to M, \hat{g} : TX \to g^*TM)\) can be deformed to a submersion – provided \( X \) is open. \( E \) above, however, is closed. The approach taken in [9] is to replace \( \pi : E \to M \) by \( g = \pi \circ pr_1 : X = E \times \mathbb{R} \to M \). Now the submersion h-principle applies and \( g \) can be replaced by a submersion \( f \). The proof then consists of a careful analysis of the singularities of the projection \( pr_1 : X \to \mathbb{R} \) on the fibers of \( f \). At a critical point it uses Harer’s Stability Theorem 2.1.

Madsen-Weiss Theorem 8.1. The map \( \alpha : \mathbb{Z} \times B\Gamma_\infty^+ \to \Omega^\infty Th(-L) \) is a homotopy equivalence.

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