A Joint Criterion for Reachability and Observability of Nonuniformly Sampled Discrete Systems

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Abstract

A joint characterization of reachability (controllability) and observability (constructibility) for linear SISO nonuniformly sampled discrete systems is presented. The work generalizes to the nonuniform sampling the criterion known for the uniform sampling. Emphasis is on the nonuniform sampling sequence, which is believed to be an additional element for analysis and handling of discrete systems.

1 Introduction

The concepts of controllability and observability, first introduced by Kalman [7], are still two fundamental questions in modern control theory. On the other hand, the enormous increase in the use of digital computers has stimulated studies in the field of discrete systems. In particular, the problem of controllability (reachability) and observability of discrete systems has been already treated in the literature in a generalized form [1], [2], [6] - [10].

Most of the previous references are only concerned with uniformly sampled discrete systems. However, the general case of nonuniform sampling offers a wider range of situations in the analysis of these concepts for discrete systems. The present note tackles this problem and, in fact, a joint characterization of reachability and observability for linear single-input single-output nonuniformly sampled discrete systems has been developed. Under several conditions, a right choice of the sampling instants would guarantee the aforementioned internal properties.

This note emphasizes the importance of the nonuniform sampling sequence against other system parameters. Besides the above considerations, nonuniform

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sampling is believed to be an alternative solution to relevant problems such as propagation of measuring and/or rounding errors, communication delays in complex computer-controlled systems, identification, etc.

2 A Joint Criterion for Reachability and Observability of Nonuniformly Sampled Discrete Systems

2.1 Model Description

Consider a linear time-invariant SISO dynamic system

\[ \dot{X}(t) = AX(t) + bu(t) \quad X_0 = X(0) \]

\[ Y(t) = cX(t) \] (1)

where \( X \in \mathbb{R}^n \) denotes the state vector and \( u, y \in \mathbb{R} \) are the scalar input and output, respectively. \( A(n \times n), b(n \times 1), c(1 \times n) \) are real and constant matrices and \( n \) is the order of the system.

As additional assumption, the realization \((A, b, c)\) is required to be minimal.

2.2 The Reachability Problem

Let \((A, b, c)\) be an arbitrary minimal realization of order \( n \) for the kind of system under study. The solution of the state-space equation at the sampling instant \( t_n \) can be written as

\[ X(t_n) = \exp(At_n)X_0 + [G_{n-1}, \ldots, G_0] \begin{bmatrix} u_{n-1} \\ \vdots \\ u_0 \end{bmatrix} \] (2)

where

\[ G_i = \exp(A(t_n - t_i))b \quad (i = 0, \ldots, n - 1) \] (3)

and \( u_j \) is the value of the impulse input at time \( t_i \).

In mathematical terms, the condition of \( n \)-reachability will be accomplished if and only if the matrix \([G_{n-1}, \ldots, G_0]\) has full rank.

Vectors \( G_i \) can be rewritten as

\[ G_i = B \exp(J(t_n - t_i)) y_0 \] (4)

where \( J \) is the Jordan canonical form of the matrix \( A \), \( B \) is the invertible matrix of the change of basis, and

\[ y_0 = B^{-1}b. \] (5)

Therefore, in order to guarantee the \( n \)-reachability we must compute the value of
with

\[
\alpha_m = t_{n-1} - t_{n-m-1} \quad (\alpha_0 = 0)
\]

and ensure that such a determinant is nonnull.

First, we denote the components of \( y_0 \) by

\[
y_0 = (y_1^1, \ldots, y_{m_1}^1, \ldots, y_1^r, \ldots, y_{m_r}^r)^{\prime}
\]

where \( m_j \) \((j = 1, \ldots, r)\) is the multiplicity of the \( r \) different eigenvalues of the matrix \( A \) with \( r \leq n \). \( ^{\prime} \) denotes the transpose.

Then, we denote the reachability canonical form \[6\] of this arbitrary minimal realization by \((A_{re}, b_{re}, c_{re})\). Finally, the impulse response \[6\] for this kind of system can be written as

\[
h(t) = c exp(At)b = \sum_{i=1}^{n} c_i \varphi_i(t)
\]

where \( c_i \in \mathbb{C} \) are constant coefficients and \( \varphi_i: \mathbb{R} \rightarrow \mathbb{C} \) \((i = 1, \ldots, n)\) is the fundamental system of solutions of an \( n \)th-order homogeneous linear differential equation.

Now, making use of the Laplace’s expansion by minors, the determinant of \((6)\) can be factorized as follows:

\[
det[\exp(J\alpha_m)y_0] = N_1 N_2 \det[\varphi_i(\alpha_m)]. \tag{10}
\]

Let us consider separately each one of these factors

\[
N_1 = \frac{1}{0!} \cdots \frac{1}{(m_1 - 1)!} \cdots \frac{1}{0!} \cdots \frac{1}{(m_r - 1)!}
\]

The term \( N_1 \) is related to the multiplicity of the eigenvalues of the matrix \( A \) and will always be nonnull

\[
N_2 = det
\begin{bmatrix}
  \begin{bmatrix}
    y_1^1 & \cdots & y_{m_1}^1 \\
    \vdots & \ddots & \vdots \\
    y_{m_1}^1
  \end{bmatrix} \\
  \vdots \\
  \begin{bmatrix}
    y_1^r & \cdots & y_{m_r}^r \\
    \vdots & \ddots & \vdots \\
    y_{m_r}^r
  \end{bmatrix}
\end{bmatrix}
\]

By similarity transformations

\[
y_0 = B^{-1}b = B_{re}^{-1}b_{re} = (C_1, C_2, \ldots, C_n)^{\prime}
\]
where $B_{re}$ is the matrix of the change the basis of $A_{re}$ to the Jordan canonical form. The term $N_2$ is related to the weighting coefficients ($C_i$) of the characteristic modes in (9). Remark that $N_2$ will be nonnull if and only if

$$y_{mj}^j \neq 0 \quad (j = 1, \ldots, r).$$

(14)

This holds, according to the previous meaning of the components of $y_0$, because only minimal realizations are considered.

Finally, $[\varphi_i(\alpha_m)] \ (i = 1, \ldots, n; \ m = 0, \ldots, n - 1)$ is a $n \times n$ matrix involving characteristic modes and sampling instants. From (10), the following result is derived.

**Lemma 1:** An arbitrary minimal realization $(A, b, c)$ is completely $n$-reachable (reachable in $n$ steps) if and only if $n$ consecutive sampling instants are chosen in such a way that

$$\det[\varphi_i(\alpha_m)] \neq 0 \quad (i = 1, \ldots, n; \ m = 0, \ldots, n - 1).$$

(15)

**Proof:** The proof is evident from the factorization of (10).

Note that if the scalar input had been defined as a control of the form

$$u(t) = u(t_i) = u_i \quad t_i \leq t < t_{i+1}$$

(16)

which is more realistic, then the above result would still be valid. In fact, the presence of a data hold does not affect the system characteristic modes.

### 2.3 General Criterion

By duality, the $n$-observability characterization is straightforwardly derived. Indeed, a similar lemma (concerning the $n$-observability) can be stated by just changing the term reachability to observability in Lemma 1.

The results obtained previously can be unified as follows.

**Theorem:** An arbitrary minimal realization $(A, b, c)$ for the kind of system under study is jointly $n$-reachable and $n$-observable if and only if $n$ consecutive sampling instants, not necessarily equidistant, are chosen in such a way that

$$\det[\varphi_i(\alpha_m)] \neq 0 \quad (i = 1, \ldots, n; \ m = 0, \ldots, n - 1).$$

(17)

**Proof:** The proof is evident from the two previous lemmas. It must be noticed that the condition (17) depends exclusively on the system characteristic modes and the sampling instants.

Condition (17) imposes a rather weak restriction for the choice of the sampling instants. In fact, time intervals can be specified so that complete reachability and observability are preserved. Particular to the uniform case, (17) becomes the condition imposed on the sampling interval $T$ (see [9], [8]) when a uniform sampling is considered.

We can also remark that, for this kind of system, the $n$-reachability and $n$-observability are inseparable concepts. These systems are either $n$-reachable and $n$-observable or, if not, they are neither $n$-reachable nor $n$-observable.
On the other hand, the \( n \)-controllability (\( n \)-constructibility) of an arbitrary minimal realization can be considered as a weakening of the condition of \( n \)-reachability (\( n \)-observability) \cite{6}. Therefore, the pair \( n \)-controllability/\( n \)-constructibility can be characterized as a corollary of the preceding theorem. Indeed, both properties will be accomplished if and only if

\[
\exp(At_n)X_0 \in \mathbb{R}[G_{n-1}, \ldots, G_0]
\]  

(18)

where \( \mathbb{R}[\ldots] \) denotes the range space of the columns \( G_i \).

From the previous theorem and following a similar reasoning, the next corollary can be proved.

**Corollary:** An arbitrary minimal realization \((A, b, c)\) for the kind of system under study is jointly \( n \)-controllable and \( n \)-constructible if and only if \( n + 1 \) sampling instants, not necessarily equidistant, are chosen in such a way that

\[
(\varphi_1(\alpha_n), \ldots, \varphi_n(\alpha_n))' \in \mathbb{R}[(\varphi_1(\alpha_n), \ldots, \varphi_n(\alpha_n))']
\]  

(19)

(with \( \alpha_m, \varphi_i \) defined as before) \((m = 0, \ldots, n - 1)\)

\[
\alpha_n = t_n - t_0.
\]  

(20)

Remark that the condition of \( n \)-controllability / \( n \)-constructibility involves one sampling instant more than in the preceding characterization. Note also that the pair \( n \)-reachability / \( n \)-observability implies the pair \( n \)-controllability/\( n \)-constructibility, but the converse may not be true.

### 3 A Simple Strategy for Choosing Sampling Instants

In order to give more insight on the restriction imposed by (17), a simple example is presented. Consider a 2nd-order realization where the matrix \( A \) has a pair of complex eigenvalues

\[
a + jb \quad (b > 0).
\]  

(21)

Now, (6) can be written as

\[
det[Y_0, Y_1] = det[\exp(J\alpha_0)y_0, \exp(J\alpha_1)y_0]
\]  

(22)

where

\[
J = \begin{bmatrix}
a & -b \\
b & a
\end{bmatrix}
\]  

(23)

and \( \alpha_i, y_0 \) are defined as before.

A geometric interpretation is very easy. Indeed, the generic operator \( \exp(J\alpha) \) applied on the vector \( y_0 \) can be viewed as follows:

1. a counter-clockwise rotation through \( 5b \) rad;
2. a stretching (or shrinking) of the length of $y_0$ by a factor $\exp(a\alpha)$

Thus, a necessary and sufficient condition to preserve complete reachability and observability is that

$$b(t_1 - t_0) \neq k\Pi \quad k = 0, 1, \ldots .$$

(24)

Remark that given a first and arbitrary sampling instant $t_0$, just point values on the time axis will be forbidden. Thus, (17) is actually not very restrictive.

On the other hand, given three successive sampling instants $(t_0, t_1, t_2)$ and their corresponding vectors $(Y_0, Y_1, Y_2)$, the Fig. 1 can be interpreted as follows.

Case a: The sampling instants are chosen in such a way that the pair reachability /observability is preserved and also the pair controllability /constructibility.

Case b: The sampling instants are chosen in such a way that the pair reachability /observability is not preserved but the pair controllability /constructibility is preserved.

Case c: The pair reachability /observability is neither preserved nor the pair controllability /constructibility. Remark that this situation will never occur for uniform sampling. Indeed, if the two first vectors are linearly dependent, then the third vector will always be linearly dependent.
From Case b, it can be seen that the pair controllability/constructibility will always be guaranteed for any uniform sampling interval. Nevertheless in the nonuniform case, this statement may not be true.

More intervals for the choice of the sampling instants in systems of higher order can be found in [3] and [10]. According to the previous example and references, the underlying idea of this note is that, in spite of the restrictions of (17), there are intervals large enough on the time axis where the sampling instants can be chosen arbitrarily. Thus, this nearly free choice of the sampling instants can be conveniently used in conjunction with other additional criterion to optimize or improve different system aspects.

4 Conclusions

A joint characterization of reachability and observability for linear SISO nonuniformly sampled discrete systems has been developed. The classical characterization for uniformly sampled systems appears as a simple particularization of the general criterion. The nonuniform sampling offers a range of situations wider than the uniform one in the study of these properties.

The note stresses the nonuniform sampling sequence, which is believed to be an additional element for the analysis and handling of discrete systems.

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