Superradiant phase transition with flat bands in a circuit QED lattice

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(Dated: October 3, 2019)

We investigate the interplay of superradiant phase transition (SPT) and energy band physics in an extended Dicke-Hubbard lattice whose unit cell consists of a Dicke model coupled to an atomless cavity. We found in such a periodic lattice the critical point that occurs in a single Dicke model becomes a critical region that is periodically changing with the wavenumber k. In the weak-coupling normal phase of the system we observed a flat band and its corresponding localization that can be controlled by the ground-state SPT. Our work builds the connection between flat band physics and SPT, which may fundamentally broaden the regimes of many-body theory and quantum optics.

PACS numbers:

I. INTRODUCTION

A flat band is a non-dispersive energy band with a zero group velocity. Such a band supports a macroscopic number of degenerate localized states. This fascinating behavior has been manifested in various geometric structures, including Lieb, Kagome, and sawtooth lattices, and well as many other designed structures [1–16]. Flat bands arise due to destructive interference, and have recently been observed in experiments with exciton polariton condensates [17, 18], photonic lattices [3, 4, 19] and cold atom lattices [20]. The localization induced by flat bands opens the possibility for engineering strong nonlinear correlations [21], and for diffraction-free transmission of light [1, 22].

On the other hand, quantum phase transitions are driven by quantum fluctuations and occur at absolute zero temperature by altering coupling or external parameters [23]. Peculiarly, the Dicke model [24], which describes collective light-matter interactions, has been predicted to exhibit an intriguing superradiant phase transition (SPT) from normal phase to the superradiant phase at a critical atom-field coupling \( \lambda_c \) in the thermodynamic limit both theoretically [25–30] and experimentally [31–34]. Nevertheless, all these achievements are limited in view of the normal Dicke model (i.e., single lattice system), and thus the SPT in an extended Dicke-Hubbard lattice is lacking.

Motivated by recent progress in the experimental control of hybrid quantum systems, where atoms, spins, and superconducting qubits can readily interact with cavities or resonators [35–43], we here propose a circuit QED lattice in which two-level systems (e.g., NV center spins) are doped in every other cavity. This setup thus realizes a series of Dicke models coupled together through a set of atomless cavities. Increasing the spin-field coupling \( \lambda_c \) drives the system to undergo a SPT from a normal phase to a “superradiant” one. Due to the periodicity of the lattice, however, we find that the two phases are separated by a critical region rather than a critical point as in the single Dicke model. This critical region is periodically modulated by wavenumber \( k \) and is significantly affected by the bilinear coupling between two nearest-neighbor cavities.

Moreover, we show that the spectrum of this model exhibits a flat band, and we show how the SPT can control its appearance and further effectively control the localization of excitations. Remarkably, the localization induced by flat-band occurs in the normal phase but disappears in the superradiant phase. Our work paves a way for understanding the SPT theory in a multi-cavity system, and may inspire applications for manipulating slow light, designing optical switching, routing and storage.

II. SYSTEM HAMILTONIAN

The structure of quasi-one dimensional (Quasi-1D) hybrid circuit QED lattice consists of sublattices \( A \) and \( B \), see Fig. 1(a). As shown in Fig. 1(b), two-level systems (here as an example nitrogen-vacancy center spins) are doped in the sublattice \( A \), while sublattice \( B \) consists of an atomless cavity. We describe the system with the Hamiltonian

\[
\mathcal{H} = \sum_{n_A} \mathcal{H}_{n_A}^{\text{Dicke}} + \sum_{n_B} \mathcal{H}_{n_B}^{\text{Cavity}} + \mathcal{H}_{\text{int}}.
\]

Here, the sums over \( n_A \) and \( n_B \) run over all unit cells in each sublattice, distinguished with subscripts \( A \) and \( B \). Then \( \mathcal{H}_{n_A}^{\text{Dicke}} \) is the Hamiltonian on a Dicke model at site
FIG. 1: Schematic layout of hybrid quantum system implementation. (a) Quasi-one dimensional (Quasi-1D) extended Dicke-Hubbard lattice consists of two-level systems (e.g., an ensemble of NV center spins) in every other coplanar waveguide (CPW) resonator. (b) The unit cell is composed of two elements, i.e., sublattice $A$ and sublattice $B$. The spin ensemble is placed in a CPW resonator (blue) in sublattice $A$. The spins can be directly coupled to a CPW resonator with electric or magnetic field. The straight line arrows denote the magnetic field. Sublattice $B$ is an atomless cavity. Nearest-neighbor CPW resonators couple to each other with coupling $\zeta$. (c) The proposed system can be extended into higher dimensional structures, e.g., 2D honeycomb lattice.

with a collective coupling constant $\lambda$ that describes the strength of the $N$ spins coupled to the coplanar waveguide resonator mode. Here $a_{nA}$ is annihilation operator for the resonator mode in site $nA$ with frequency $\omega_A$, and the spins are described by the collective operators $J_{nA}^\pm = 1/2 \sum_N \sigma_{nA}^\pm$ and $J_{nA}^z = \sum_N \sigma_{nA}^z$, where $\sigma_{nA}^\pm$ are Pauli matrices and $N$ is the total number of spins. The $B$ sites are described by the Hamiltonian

$$\mathcal{H}^{\text{Cavity}}_{nB} = \omega_B a_{nB}^\dagger a_{nB}$$

with cavity-mode annihilation operators $a_{nB}$ and frequency $\omega_B$. Finally, we consider nearest-neighbor cavities to be coupled via an $x \times x$ interaction between the two, i.e.,

$$\mathcal{H}_{\text{int}} = - \zeta \sum_{\langle nA, mB \rangle} (a_{nA}^\dagger + a_{nA})(a_{mB}^\dagger + a_{mB}),$$

which can be readily realized in circuit QED systems [44, 45]. Note that $\langle nA, mB \rangle$ represents nearest-neighbor cavities.

FIG. 2: (a,b) The ground-state energy $E_G$ and its second-order derivative as a function of spin-field coupling $\lambda$ for a single unit cell. The solid lines denote the results for various finite values of $j = 1/2, 1, 3/2, 4$, whereas the dotted curve corresponds to the results in the thermodynamic limit $j \to \infty$. (c) The critical point $\lambda_c$ as a function of $\zeta$. Here we considered the resonant condition, $\Omega = \omega_A = \omega_B = 1$, and $\zeta = 0.24$ in (a,b).

III. SPT IN SINGLE UNIT CELL

We first consider the ground-state phase transition for a single unit cell, which is uncoupled from the rest of the lattice. Using the method of Ref. [25] in the thermodynamic limit $N \to \infty$, we can obtain the scaled ground-state energy,

$$\frac{E_G}{j} = \begin{cases} -\Omega, & \text{for } \lambda \leq \lambda_c, \\ \Omega \left( \frac{\lambda^2}{\lambda_c^2} + \frac{\lambda_c^2}{\lambda^2} \right), & \text{for } \lambda > \lambda_c, \end{cases}$$

with critical coupling

$$\lambda_c = \sqrt{\frac{\Omega(\omega_A - 4\omega_B^2)}{2}}.$$  

Note that $j = N/2$ corresponds to the maximum value of Dicke’s “cooperation number” [25]. We find that the critical point in a single unit cell is shifted due to the interaction between nearest-neighbor cavities. In Fig. 2 we plot the ground-state energy and its second-order derivative as a function as the spin-field coupling $\lambda$. Figs. 2(a,b) show that the unit cell undergoes a second-order phase transition from normal phase to the superradiant phase when the coupling goes through the critical point.
solutions related to the superradiant phase are given by
\[ \text{and this corresponds to the normal phase. The nontrivial work in the thermodynamic limit, } N \rightarrow \infty, \text{ respectively. Note that, here we work in the thermodynamic limit, } N \rightarrow \infty. \]

IV. SPT IN EXTENDED DICKE-HUBBARD LATTICE

To explore the phase transition in the extended Dicke-Hubbard lattice, we extend the method of Ref. [25] to the multi-cavity system. We first apply a Holstein-Primakoff representation on the spin modes, i.e., \( J^+_n = b_{nA}^\dagger \sqrt{2j-b_{nA}^\dagger b_{nA}} \), \( J^-_n = \sqrt{2j-b_{nA}^\dagger b_{nA}} b_{nA} \), and \( J^z_n = b_{nA}^\dagger b_{nA} - j \), where the introduced bosonic operators obey \([b_{nA}, b_{mA}^\dagger] = 1\). Then we make a displacement on bosonic modes, i.e., \( a_{nA}^\dagger \rightarrow c_{nA}^\dagger + \alpha, b_{nA} \rightarrow d_{nA}^\dagger - \beta, a_{nB} \rightarrow c_{nB}^\dagger + \gamma \). Expanding Hamiltonian \( \mathcal{H} \) and eliminating its linear terms, we can obtain the displacements \( \alpha, \beta \) and \( \gamma \) satisfy,
\[
2\lambda\beta \sqrt{\frac{(2j-\beta^2)}{2j}} - \omega_A \alpha + 2\zeta \gamma = 0, \\
4\lambda\alpha \sqrt{\frac{(2j-\beta^2)}{2j}} (j-\beta^2) - \Omega \beta = 0, \\
\omega_B \gamma - 2\zeta \alpha = 0. \tag{7}
\]
The trivial solution reads
\[
\alpha = \beta = \gamma = 0, \tag{8}
\]
and this corresponds to the normal phase. The nontrivial solutions related to the superradiant phase are given by
\[
\alpha = \pm \frac{\Omega}{2\mu\lambda} \sqrt{\frac{2}{2}} (1-\mu^2), \\
\beta = \pm \sqrt{j(1-\mu)}, \\
\gamma = \pm \frac{\zeta \Omega}{\mu \lambda \omega_B} \sqrt{\frac{j}{2}} (1-\mu^2) = \frac{2\zeta}{\omega_B} \alpha, \tag{9}
\]
here \pm represent two different directions of the displacement and we find that the solutions \( \alpha, \beta, \gamma \) have two sets of symmetry-broken branches. Here we have set
\[
\mu = \frac{\Omega (\omega_A - 4\zeta^2)}{4\omega_B^2}. \tag{10}
\]
In the normal phase, we transform the Hamiltonian into \( k \)-space by adopting a Fourier transformation \( O_k = \frac{1}{\sqrt{N}} \sum_n e^{-ik \cdot \vec{R}_n} O_n \), where \( O_n \) is an arbitrary operator, \( N \) is the number of unit cells and \( \vec{R}_n \) is the position of the lattice. The resultant effective Hamiltonian reads
\[
\mathcal{H}_{\text{norm}}(k) = \sum_{k_A} (\omega_A a^\dagger_{k_A} a_{k_A} + \Omega b^\dagger_{k_A} b_{k_A}) + \sum_{k_B} \omega_B a^\dagger_{k_B} a_{k_B} \\
+ \sum_{k_A} \lambda (a^\dagger_{k_A} b_{k_A} + a_{k_A} b_{-k_A} + \text{H.c.}) \\
- \sum_{k_A,k_B} 2\zeta \cos(k)(a^\dagger_{k_A} a_{k_B} + a_{k_A} a_{-k_B} + \text{H.c.}), \tag{11}
\]
where H.c. denotes Hermitian conjugation and the sums run over the first Brillouin zone. We have taken the lattice constant to be identical and neglected constant terms that yield an overall shift on the eigenvalues. We impose a periodic boundary condition, and in \( k \)-space the system Hamiltonian can be written as
\[
\mathcal{H}_{\text{norm}}(k) = \frac{1}{2} \sum_k [\psi^\dagger_k \mathcal{M}_{\text{norm}}(k) \cdot \psi_k], \tag{12}
\]
where \( \psi_k = [a_{kA}, a_{kB}, b_{kA}]^T \) denotes boson annihilation operators in \( k \) space and the superscript \( T \) is a transpose operation. The coefficients are collected into the \( 6 \times 6 \) matrix:
\[
\mathcal{M}_{\text{norm}}(k) = \begin{pmatrix}
\omega_A & f & \lambda & 0 & f & \lambda \\
f & \omega_B & 0 & f & \lambda & 0 \\
\lambda & 0 & \Omega & \lambda & 0 & 0 \\
0 & \lambda & \omega_A & f & \lambda & 0 \\
f & 0 & f & \omega_B & 0 & \lambda \\
\lambda & 0 & 0 & \lambda & 0 & \Omega
\end{pmatrix}, \tag{13}
\]
with \( f = -2\zeta \cos(k) \). Hamiltonian \( \mathcal{H}_{\text{norm}} \) is bilinear in bosonic operators. To diagonalize the coefficient matrix \( \mathcal{M}_{\text{norm}} \), we introduce the dynamic matrix \( D_{\text{norm}} = \tau_z \mathcal{M}_{\text{norm}} [46] \), where \( \tau_z = \text{diag}[1,1,1,-1,-1,-1] \). It is worth noting that the coefficient matrix \( \mathcal{M}_{\text{norm}} \) is Hermitian but the dynamic matrix \( D_{\text{norm}} \) is pseudo-Hermitian [47]. The dynamic matrix \( D_{\text{norm}}(k) \) can be diagonalized by applying a sympletic transformation,
\[
\mathcal{T}^{-1} D_{\text{norm}}(k) \mathcal{T} = \begin{pmatrix}
E(k) & 0 \\
0 & -E(-k)
\end{pmatrix}, \tag{14}
\]
Here \( \mathcal{T} \) is a pseudo-unitary matrix and it satisfies \( \mathcal{T}^\dagger \tau_z \mathcal{T} = \tau_z \). Under the condition \( \Omega = \omega_A = \omega_B = \omega \), the dynamic matrix \( D_{\text{norm}} \) can be analytically diagonalized [48], and the corresponding spectrum of \( \mathcal{M}_{\text{norm}} \) reads,
\[
E_{\text{norm}}(k) = \begin{pmatrix}
\sqrt{\omega^2 - 2\omega \sqrt{4\zeta^2 \cos^2(k) + \lambda^2}}; \\
\omega; \\
\sqrt{\omega^2 + 2\omega \sqrt{4\zeta^2 \cos^2(k) + \lambda^2}}.
\end{pmatrix} \tag{15}
\]
Evidently, the Hamiltonian $H_{\text{norm}}$ has 3 bands, each of which is doubly counted.

In the superradiant phase, both the resonator field and spin ensemble acquire macroscopic occupations [given by the non-trivial solutions of Eq. (7)]. After the displacement process, the effective Hamiltonian in the superradiant phase reads

$$H_{\text{sup}} = \sum_{n_a} \omega_A c_{n_a}^\dagger c_{n_a} + \chi d_{n_a}^\dagger d_{n_a} + \eta (d_{n_a}^\dagger + d_{n_a})^2 + \xi (c_{n_a}^\dagger + c_{n_a})(d_{n_a}^\dagger + d_{n_a}) + \sum_{n_B} \omega_B c_{n_B}^\dagger c_{n_B} - \zeta \sum_{\langle n_A, n_B \rangle} (c_{n_A}^\dagger + c_{n_A})(c_{n_B}^\dagger + c_{n_B}),$$

where we have set

$$\chi = \frac{\Omega}{2\mu}(1 + \mu),$$

$$\xi = \lambda \mu \sqrt{\frac{2}{1 + \mu}},$$

$$\eta = \frac{\Omega (1 - \mu)(3 + \mu)}{8 \mu (1 + \mu)}.$$  \hspace{1cm} (16)

As before, we Fourier transform Hamiltonian (16) to obtain a form similar to Eq. (12) but with coefficient matrix $M_{\text{sup}}(k)$ which reads

$$M_{\text{sup}}(k) = \begin{pmatrix}
\omega_A & f & \xi & 0 & f & \xi \\
f & \omega_B & 0 & f & 0 & 0 \\
\xi & 0 & \chi + 2\eta & \xi & 0 & 2\eta \\
f & f & \xi & \omega_A & f & \xi \\
f & 0 & f & \omega_B & 0 \\
\xi & 0 & 2\eta & \xi & 0 & \chi + 2\eta
\end{pmatrix}.$$  \hspace{1cm} (18)

In the superradiant phase, the diagonalization of $M_{\text{sup}}(k)$ is complicated, and the results below are obtained through numerical diagonalization.

V. INTERACTION-DEPENDENT CRITICAL REGION

In this section, we discuss the boundary between two phases of the extended Dicke-Hubbard lattice system. We have analytically diagonalized the Hamiltonian $H_{\text{norm}}(k)$ [see Eq. (15)]. Noticeably, when $\omega < 2 \sqrt{4\zeta^2 \cos^2(k) + \lambda^2}$, one of its eigenvalues is imaginary, which denotes that the diagonalization method is only valid when

$$\lambda^2 < \frac{\omega^2}{4} - 4\zeta^2 \cos^2(k).$$

This should be compared with the critical coupling of a single unit cell in Eq. (6) which shows that the periodicity of the lattice gives a $k$-dependent condition for validity, and this leads to a critical curve rather than critical point for the lattice system [see below]. Furthermore, since $\lambda^2 > 0$ and $\cos^2(k) \leq 1$, the proposed system is only well defined when

$$\left| \frac{\zeta}{\omega} \right| < \frac{1}{4},$$

in the resonant case ($\omega_A = \omega_B = \Omega = \omega$). Above this value of $\zeta/\omega$, the Hamiltonian does not possess normalizable eigenfunctions, and has no obvious physical meaning in the region considered [see more discussions in Refs. [49, 50]]. We note that this constraint condition on $\zeta$ is valid for both normal phase and superradiant phase. To verify our results, in Fig. 3 we plot the contour of the real part of the lowest energy spectra versus spin-field coupling $\lambda$ and wavenumber $k$ in normal phase and superradiant phase, respectively. The dashed (dotted) curve is the contour of $E(k) = 0$ which can be served as critical curve in normal phase (superradiant phase). Such critical curves distinguish the regions of $E(k)$ being real and imaginary. The blank zone between these two critical curves can be called as critical or unstable region. In this unstable region, one of eigenvalues of the Hamiltonian becomes imaginary. Unlike the standard Dicke model which has a single critical point, here there is a critical region separating the normal phase and superradiant phase. As coupling $\lambda$ increases, the system turns from normal phase to the critical region, and then finally reaches the superradiant phase. Figure 3 also shows that the critical region is modulated by wavenumber $k$ with a period of $\pi$. This periodical modulation causes the overlap (gap) of critical curves in the vicinity of $k = m\pi/2$ ($k = 1\pi$) and this overlap indicates that the system has an effective triple-well potential and allows first-order phase
from electromagnetically-induced transparency and cooperative interference between the two coupling processes to destructive interference between the two paths, one is \( \lambda(a_{nA}^\dagger + a_{nA})(J_{nA}^a + J_{nA}^b) \) and another is \( \zeta(a_{nA}^\dagger + a_{nA})(a_{nB}^\dagger + a_{nB}) \). Only if the eigenfrequencies of spins (described by \( J_{nA}^a \)) and cavity mode \( B \) (described by \( a_{nB} \)) are at resonance, i.e., \( \Omega = \omega_B \), then the destructive interference condition would be satisfied, and as a result the flat band would occur. This regime can be used to explain the case of normal phase. Another case is that these two effective frequencies are off-resonance, then the flat band would disappear. In the superradiant phase, both spin and cavity modes acquire macroscopic occupations. This fact causes the effective spin mode \( d_{nA} \) (after bosonic transformation, \( J_{nA}^\pm \rightarrow d_{nA}^{(1)} \)) is not resonant with the effective cavity mode \( c_{nB} \), i.e., \( \chi + 2\eta \neq \omega_B \) [see Eq. (18)]. Therefore the condition of destructive interference is broken up, which leads to the disappearance of flat band.

To explore the localization induced by the flat band, in Fig. 5 we numerically calculate the local density of states (LDOS),

\[
\rho_n(E) = \sum_{lk} |\langle \chi_n | \phi_{lk} \rangle|^2 \delta(E - E_{lk}),
\]

where the subscript \( n \) differentiates three physical modes, i.e., cavity mode \( A \), cavity mode \( B \) and spins. \( |\chi_n \rangle \) is the basis state corresponding to occupation of mode \( n \). The sum \( \sum_{lk} \) runs over various energy bands in the first Brillouin zone. Here \( E_{lk} \) is the eigenvalue related to eigenstate \( \phi_{lk} \), and \( \delta(E - E_{lk}) \) is a \( \delta \) function centered at \( E_{lk} \).

Figs. 5(a-f) show the LDOS of three different eigenmodes with respect to energy \( E \) in the normal phase (left column) and superradiant phase (right column), respectively. Three different shadow areas denote three bands. As shown in Fig. 5(a), in the normal phase, the LDOS of cavity mode \( A \) at \( E = 1 \) is zero but for cavity mode \( B \) and the spin mode, its LDOSs have regular Gaussian-like peaks [see Figs. 5(c) and 5(e)]. In such a regime, both cavity mode \( B \) and the spin mode are localized at \( E = 1 \), while cavity mode \( A \) remains completely dark in that the destructive interference between the two paths offsets the net flow of particles to cavity mode \( A \) [see Fig. 5(g)]. Nevertheless, in the superradiant phase, the effective frequencies of spins and cavity \( A \) are off-resonance, hence the destructive interference is destroyed, causing the disappearance of flat-band localization [see Figs. 5(b,d,f,h)]. This result manifests the flat-band localization is extremely sensitive to the off-resonance due to its infinite effective mass.

VI. SPT ASSOCIATED FLAT BAND

To build the connection between flat band physics and SPT, now we discuss the energy band property of system in the two phases above. In Fig. 4, we present the dispersion relation of \( E(k) \) in the first Brillouin zone for different values of spin-field coupling \( \lambda \). The dispersion relation of the lattice yields three bands, including a flat band in normal phase [see red-dotted curves in Fig. 4(a)] located at \( E(k) = \omega \) [see Eq. (15)]. With the increasing of coupling \( \lambda \), one of eigenvalues becomes completely imaginary for some certain \( k \), which indicates that the system becomes unstable. Despite this, the flat band persists. Once the spin-field coupling \( \lambda \) is large enough to surpass the unstable region, the system enters into superradiant phase. Here, the flat band turns to a dispersive one.

The two eigenvectors of matrix \( D_{\text{norm}} \) that correspond to the flat band are \((0, -\lambda, f, 0, 0, 0)^T\) and \((0, 0, 0, 0, -\lambda, f)^T\). The remarkable thing about these eigenstates is that they have zero occupancy of cavity mode \( A \) (despite its couplings to the other two modes). Thus, we see that these flat-bands occur due to destructive interference between the two coupling processes to mode \( A \). This is reminiscent of the dark states known from electromagnetically-induced transparency and coherent population trapping [53–56].

In our model, the bilinear interaction between two neighbor cavities [shown in Eq. (4)] is vital for the existence of the flat band. This flat band arises due to the destructive interference between two paths, one is \( \lambda(a_{nA}^\dagger + a_{nA})(J_{nA}^a + J_{nA}^b) \) and another is \( \zeta(a_{nA}^\dagger + a_{nA})(a_{nB}^\dagger + a_{nB}) \). Only if the eigenfrequencies of spins (described by \( J_{nA}^a \)) and cavity mode \( B \) (described by \( a_{nB} \)) are at resonance, i.e., \( \Omega = \omega_B \), then the destructive interference condition would be satisfied, and as a result the flat band would occur. This regime can be used to explain the case of normal phase. Another case is that these two effective frequencies are off-resonance, then the flat band would disappear. In the superradiant phase, both spin and cavity modes acquire macroscopic occupations. This fact causes the effective spin mode \( d_{nA} \) (after bosonic transformation, \( J_{nA}^\pm \rightarrow d_{nA}^{(1)} \)) is not resonant with the effective cavity mode \( c_{nB} \), i.e., \( \chi + 2\eta \neq \omega_B \) [see Eq. (18)]. Therefore the condition of destructive interference is broken up, which leads to the disappearance of flat band.

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VII. 2D HONEYCOMB LATTICE

Here we proposed a general model linked SPT and flat band physics. Remarkably, this construction can be extended to higher dimensional lattices. As an example, we
consider the 2D honeycomb structure [see Fig. 1(c)]. We assume the lattice constant $|a| = \sqrt{3}/3$ and thus basis vectors read $a_1 = (1, 0)$, and $a_2 = (1/2, \sqrt{3}/2)$. Different from 1D lattice, here every cavity mode has three nearest neighbors, thus the interaction between two nearest-neighbor cavities reads,

$$
\mathcal{H}'_{\text{int}} = -\zeta \sum_i \left( [a_{A,r_i}^+ + a_{A,r_i}](a_{B,r_i}^+ + a_{B,r_i} + e_1) \right) - \zeta \sum_i \left( [a_{A,r_i}^+ + a_{A,r_i}](a_{B,r_i}^+ + a_{B,r_i} + e_2) \right) - \zeta \sum_i \left( [a_{A,r_i}^+ + a_{A,r_i}](a_{B,r_i}^+ + a_{B,r_i} + e_3) \right),
$$

with

$$
e_1 = \left( 0, \frac{\sqrt{3}}{3} \right), \\
e_2 = \left( -\frac{1}{2} - \frac{\sqrt{3}}{6} \right), \\
e_3 = \left( \frac{1}{2} - \frac{\sqrt{3}}{6} \right).
$$

Here the sum $\sum_i$ runs over all unit cells and $a_{A,r_i}$, $(a_{B,r_i})$ is the annihilation operator for the cavity mode $A$ (cavity mode $B$) and $r_i$ is the position vector in the $i$th unit cell. Then the totally Hamiltonian is

$$
\mathcal{H}' = \sum_{n_A} \mathcal{H}_{n_A}^\text{Dicke} + \sum_{n_B} \mathcal{H}_{n_B}^\text{Cavity} + \mathcal{H}'_{\text{int}}.
$$

We apply the same Holstein-Primakoff representation and displacement process as before [see more details in Sec. IV], and perform a 2D Fourier transformation,

$$
a_{A,k} = \frac{1}{\sqrt{N}} \sum_i a_{A,r_i} e^{-i k \cdot r_i}, \\
a_{B,k} = \frac{1}{\sqrt{N}} \sum_i a_{B,r_i} e^{-i k \cdot r_i},
$$

then we could obtain the $6 \times 6$ coefficient matrix with the same form with Eq. (12) in the normal phase and Eq. (18) in the superradiant phase but with a new $f'$,

$$
f' = -\zeta [1 + \exp(i k \cdot a_1) + \exp(i k \cdot a_2)],
$$

where $k = (k_x, k_y)$. We numerically diagonalize the coefficient matrix and find its energy spectra own three doubly-degenerate bands as well. In Fig. 6, we plot energy band structure for the honeycomb lattice in normal phase (left panel) and superradiant phase (right panel), respectively. It is clearly shown that a flat band exhibits at $E = 1$ in the normal phase. But in the superradiant phase, all energy bands are dispersive. This result shows a well agreement with 1D circuit QED lattice.
VIII. CONCLUSIONS

We have presented a comprehensive picture of how the existence of a SPT in the the thermodynamic limit determines the energy band properties in a hybrid circuit-QED architecture. The extended Dicke-Hubbard lattice exhibits a phase transition from the normal phase, to an unstable phase and finally superradiant phase.

Our analysis of the extended-Dicke lattice in the thermodynamic limit initiated from a single unit cell in each of its normal and superradiant phase. It was shown that the critical point of the single unit cell was shifted because of the bilinear coupling between nearest cavities. Further we extended the method used in Ref. [25] to the extended Dicke-Hubbard lattice. Interestingly, we found that the critical point that occurs in a single Dicke model becomes a critical region in a lattice geometry that is periodically modulated by wavenumber $k$.

By performing a Fourier transformation, we explore the band structure of proposed Hamiltonian in the momentum space. The flat band, and the localization that engenders, are observed exclusively in the low-coupling normal phase of the system. In other words, the SPT can determine the existence of the flat band. In addition, it was reported that the flat band has been observed experimentally by collecting the emission of the polarization in a geometrically frustrated lattice of micropillar optical cavities, and the density profile in the real space can be reconstructed by spectrally filtering the image [18]. From this aspect, this work offers a flexible and intuitive method to distinguish the normal phase and superradiant phase via experimentally detectable energy spectrum.

Acknowledgements

G.-L. Zhu thanks L.-L. Wan, Z.-H. Li, M. Liang and G.-C. Lin for helpful discussions. This work is supported by the National Key Research and Development Program of China grant No. 2016YFA0301203 and the National Natural Science Foundation of China (Grant Nos. 11822502, 11374116, 11574104 and 11375067), and the Fundamental Research Funds for the Central Universities grant No. 2019kyfYXMBZ054.

[1] D. Guzmán-Silva et al., Experimental observation of bulk and edge transport in photonic Lieb lattices, New J. Phys. 16 063061 (2014).
[2] H. Ozawa, S. Taie, T. Ichinose, and Y. Takahashi, Interaction-Driven Shift and Distortion of a Flat Band in an Optical Lieb Lattice, Phys. Rev. Lett. 118, 175301 (2017).
[3] R. A. Vicencio et al., Observation of Localized States in Lieb Photonic Lattices, Phys. Rev. Lett. 114, 245503 (2015).
[4] S. Mukherjee, A. Spracklen, D. Choudhury, N. Goldman, P. Ølberg, E. Andersson, and R. R. Thomson, Observation of a Localized Flat-Band State in a Photonic Lieb Lattice, Phys. Rev. Lett. 114, 245504 (2015).
[5] S. Mukherjee and R. R. Thomson, Observation of localized flat-band modes in a quasi-one-dimensional photonic rhombic lattice, Opt. Lett. 40(23), 5443-5446 (2015).
[6] W.-X. Qiu, S. Li, J.-H. Gao, Y. Zhou, and F.-C. Zhang, Designing an artificial Lieb lattice on a metal surface, Phys. Rev. B 94, 241409(R) (2016).
[7] Y. Zong, S. Xia, L. Tang, D. Song, Y. Hu, Y. Pei, J. Su, Y. Li, and Z. Chen, Observation of localized flat-band states in Kagome photonic lattices, Opt. Lett. 24(8), 008878 (2016).
[8] Z. Lin et al., Flatbands and Emergent Ferromagnetic Ordering in Fe$_3$Sn$_2$ Kagome Lattices, Phys. Rev. Lett. 121, 096401 (2018).
[9] S. D. Huber, and E. Altman, Bose condensation in flat bands, Phys. Rev. B 82, 184502 (2010).
[10] S. Peotta, P. Törma, Superfluidity in topologically non-trivial flat bands, Nat. Commun. 6, 8944 (2015).
[11] L. Ge, Parity-time symmetry in a flat-band system, Phys. Rev. A 92, 052103 (2015).
[12] H. Ramezani, Non-Hermiticity-induced flat band, Phys. Rev. A 96, 011802(R) (2017).
[13] D. Leykam, S. Flach, and Y. D. Chong, Flat bands in lattices with non-Hermitian coupling, Phys. Rev. B 96, 064305 (2017).
[14] L. L. Wan, X.-Y. Lü, J. H. Gao, and Y. Wu, Controllable photon and phonon localization in optomechanical lieb lattices, Opt. Express, 25(15), 17364 (2017); Hybrid interference induced flat band localization in bipartite optomechanical lattices, Scientific Reports, 7(1), 15188 (2017).
[15] M. Tovmasyan, S. Peotta, L. Liang, P. Törma, and S. D. Huber, Preformed pairs in flat Bloch bands, Phys. Rev. B 98, 134513 (2018).
[16] R. Shen, L. B. Shao, B. Wang, and D. Y. Xing, Single Dirac cone with a flat band touching on line-centered-square optical lattices, Phys. Rev. B 81, 041410(R) (2010).
[17] T. Jacqmin, I. Carusotto, I. Sagnes, M. Abbarchi, D. D. Solnyshkov, G. Malpuech, E. Galopin, A. Lemaitre, J. Bloch, and A. Amo, Phys. Rev. Lett. 112, 114602 (2014).
[18] F. Baboux, L. Ge, T. Jacqmin, M. Biondi, E. Galopin, A. Lemaitre, L. Le Gratiet, I. Sagnes, S. Schmidt, H. E. Türeci, A. Amo, and J. Bloch, Bosonic Condensation and Disorder-Induced Localization in a Flat Band, Phys. Rev. Lett. 116, 066402 (2016).
[19] J. Li, T. P. White, L. O’Faolain, A. Gomez-Iglesias, and T. F. Krauss, Systematic design of flat band slow light in photonic crystal waveguides, Opt. Express, 16, (9) 6227 (2008).
[20] S. Taie, H. Ozawa, T. Ichinose, T. Nishio, S. Nakajima, and Y. Takahashi, Coherent driving and freezing of bosonic matter wave in an optical Lieb lattice, Sci. Adv. 1, 1500854 (2015).
[21] M. Biondi, E. P. L. van Nieuwenburg, G. Blatter, S. D. Huber, and S. Schmidt, Incompressible Polaritons in a Flat Band, Phys. Rev. Lett. 115, 143601 (2015).
