Summary. The linearized Boltzmann equation is considered to describe small spatial perturbations of the homogeneous cooling state. The corresponding macroscopic balance equations for the density, temperature, and flow velocity are derived from it as the basis for a hydrodynamic description. Hydrodynamics is defined in terms of the spectrum of the generator for the dynamics of the linearized Boltzmann equation. The hydrodynamic eigenfunctions and eigenvalues are calculated in the long wavelength limit. The results allow identification of the hydrodynamic part of the solution to the linearized Boltzmann equation. This contribution is used to calculate the fluxes in the macroscopic balance equations, leading to the Navier-Stokes equations and associated transport coefficients. The results agree with those obtained earlier by the Chapman-Enskog method. The implications of this analysis for application of methods of linear response to granular fluids and derivation of Green-Kubo expressions for transport coefficients are discussed.

1 Introduction

The derivation of hydrodynamic equations for granular gases and identification of expressions for the associated transport coefficients has been a problem of theoretical and practical interest for the last two decades [1–7]. Early direct extensions of the Chapman-Enskog method to inelastic systems have been refined recently so that the transport coefficients for the Boltzmann gas are now known accurately [5,8]. These refinements have clarified the role of the time dependent reference state, the local homogeneous cooling state (HCS), and its effects on the qualitative and quantitative characterization of the conditions for the validity of a hydrodynamic description. The extension of these results to higher densities also has been accomplished in the context of the phenomenological Boltzmann-Enskog equation [6]. However, more general conclusions regarding transport in dense granular fluids are quite limited. On the other hand, for elastic collisions the analysis of transport in fluids is quite sophisticated using, for instance, methods of linear response theory [9]. Such methods provide formally exact expressions for transport properties in terms of time dependent fluctuations in the equilibrium reference state. These representations have proven to be particularly useful due to the possibility of their direct evaluation by molecular dynamics simulation. In this context,
it is natural to ask to what extent the methods of linear response theory can be applied to granular fluids.

In applications of linear response for fluids with elastic collisions, the local conserved densities of mass, energy, and momentum play a special role. They represent the slow variables as the system approaches uniformity. More significantly, they are the hydrodynamic modes in the long wavelength limit, i.e., the formal eigenfunctions of the Liouville operator corresponding to the hydrodynamic eigenvalues. Standard linear response methods consider perturbations of the homogeneous state (the Gibbs ensemble) that are linear combinations of the conserved densities. In this way, hydrodynamic excitations are selected and, by formal manipulation of the response functions, Green-Kubo expressions for the transport coefficients are derived directly. A quite reasonable extension of this method to granular systems, using linear combinations of the local conserved densities as initial perturbations, has been described in Ref. [10]. The resulting Green-Kubo expressions are similar to those for systems with elastic collisions, but they do not agree at low density with those from the Chapman-Enskog method noted above. This difference has been made more precise by reformulating the Chapman-Enskog results in a Green-Kubo representation [11]. The resulting time correlation functions are not simply those for the fluxes of the conserved densities, as is the case for elastic collisions. The reason is that the hydrodynamic modes for granular systems are no longer simply related to the conserved densities, even in the long wavelength limit. Instead, the hydrodynamic modes and conserved densities are conjugate sets in the sense that the former generate the hydrodynamic excitation while the latter measure the response. An objective here is to clarify this subtle difference in the context of the linearized Boltzmann equation for inelastic hard particles, and to discuss the consequences for more general applications.

Clearly, an important first issue is the unambiguous definition of hydrodynamic modes. This is accomplished by considering the eigenvalue problem for the operator generating the dynamics for the Boltzmann equation, linearized around the reference homogeneous cooling state (see Section 3 below). The hydrodynamic part of its spectrum is identified first in the long wavelength limit. Hydrodynamics refers to systematic approximations to the macroscopic balance equations for the local relevant densities (here our choice is the deviations of number density, temperature, and flow velocity from their values in the homogeneous state). Consequently, we first derive these equations and find they are closed without approximation in the long wavelength limit. These asymptotic equations determine \( d+2 \) hydrodynamic eigenvalues, where \( d \) is the dimensionality of the system. For normal fluids these are all zero, while for granular systems there are three distinct eigenvalues, one of which is \( d \)-fold degenerate. The corresponding eigenvalues for the linearized Boltzmann equation then identify its asymptotic hydrodynamic spectrum. The hydrodynamic spectrum more generally is defined to be the eigenvalues that
are continuously connected to these asymptotic values in the long wavelength limit. The expression “hydrodynamic mode” means the hydrodynamic eigenfunction with an exponential time dependence, in the appropriate time units, characterized by the corresponding hydrodynamic eigenvalue. A primary new contribution here is the calculation of these hydrodynamic eigenfunctions in the long wavelength limit, showing their difference from the local conserved densities.

With the hydrodynamic eigenfunctions identified, the hydrodynamic part of the solution to the linearized Boltzmann equation can be extracted. This is the dominant contribution for large times and long wavelengths. More specifically, this hydrodynamic part is calculated to first order in the gradients of the hydrodynamic fields and the result is used to calculate the average fluxes in the macroscopic balance equations at finite wavelengths. This provides the Navier-Stokes hydrodynamics for granular gases, and allows identification of the transport coefficients. The results agree with those from the Chapman-Enskog method [5,8] and the Green-Kubo representation of Ref. [11]. However, their derivation here exposes more clearly the hydrodynamic eigenfunctions and their role as the origin of key differences from gases with elastic collisions.

In the next Section, the nonlinear Boltzmann equation for a granular gas is defined, and the exact balance equations for the density, temperature, and flow velocity are derived from it. The homogeneous state of these equations corresponds to global cooling, and the corresponding homogeneous cooling solution (HCS) to the nonlinear Boltzmann equation is described. This solution represents the homogeneous state about which small spatial perturbations are considered. The corresponding linear Boltzmann equation and linear balance equations for these small perturbations are then described. Next, the eigenvalue problem for the generator of the time dependence in the linearized Boltzmann equation is defined in Section 4, and the hydrodynamic eigenfunctions and eigenvalues are calculated in the long wavelength limit. These results are used in Section 5 to derive the Navier-Stokes hydrodynamic equations, with expressions for the transport coefficients in terms of the linearized Boltzmann collision operator. Their equivalence with previous results from the Chapman-Enskog method is demonstrated, and these expressions are also given a Green-Kubo representation. Finally, the analysis is recapitulated, and implications for a more general treatment of transport in rapid granular fluids beyond the low density context are discussed.
2 Nonlinear Boltzmann Equation and the Homogeneous Cooling State

The Boltzmann equation for a system of inelastic smooth hard disks \((d = 2)\) or spheres \((d = 3)\) of mass \(m\) and diameter \(\sigma\) has the form \([3,12]\)

\[
\left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) f(\mathbf{r}, \mathbf{v}, t) = J[f | f],
\]

where \(J[f | f]\) is the nonlinear inelastic Boltzmann collision operator,

\[
J[f | f] = \int d\mathbf{v} \bar{T}(\mathbf{v}, \mathbf{v}_1) f(\mathbf{r}, \mathbf{v}, t) f(\mathbf{r}, \mathbf{v}_1, t).
\]

Here \(\bar{T}(\mathbf{v}, \mathbf{v}_1)\) is the inelastic binary collision operator

\[
\bar{T}(\mathbf{v}, \mathbf{v}_1) = \sigma^{d-1} \int d\hat{\sigma} \Theta(g \cdot \hat{\sigma}) g \cdot \hat{\sigma} \left( \alpha^{-2} b^{-1}_\sigma - 1 \right),
\]

with \(g = \mathbf{v} - \mathbf{v}_1\), \(\Theta\) the Heaviside step function, \(\hat{\sigma}\) a unit vector joining the center of the two particles at contact, \(\alpha\) the coefficient of normal restitution, and \(b^{-1}_\sigma\) an operator replacing all the velocities \(\mathbf{v}\) and \(\mathbf{v}_1\) appearing to its right by their precollisional values \(\mathbf{v}^*\) and \(\mathbf{v}_1^*\), given by

\[
b^{-1}_\sigma \mathbf{v} = \mathbf{v}^* = \mathbf{v} - \frac{1 + \alpha}{2\alpha} (\hat{\sigma} \cdot g) \hat{\sigma}, \quad b^{-1}_\sigma \mathbf{v}_1 = \mathbf{v}_1^* = \mathbf{v}_1 + \frac{1 + \alpha}{2\alpha} (\hat{\sigma} \cdot g) \hat{\sigma}.
\]

For arbitrary functions \(g(\mathbf{v}, \mathbf{v}_1)\) and \(h(\mathbf{v}, \mathbf{v}_1)\), it is

\[
\int d\mathbf{v} \int d\mathbf{v}_1 g(\mathbf{v}, \mathbf{v}_1) \bar{T}(\mathbf{v}, \mathbf{v}_1) h(\mathbf{v}, \mathbf{v}_1) = \int d\mathbf{v} \int d\mathbf{v}_1 h(\mathbf{v}, \mathbf{v}_1) T(\mathbf{v}, \mathbf{v}_1) g(\mathbf{v}, \mathbf{v}_1),
\]

where

\[
T(\mathbf{v}, \mathbf{v}_1) = \sigma^{d-1} \int d\hat{\sigma} \Theta(g \cdot \hat{\sigma}) g \cdot \hat{\sigma} (b - 1).
\]

The operator \(b\) changes the velocities \(\mathbf{v}\) and \(\mathbf{v}_1\) to its right into their post-collisional values \(\mathbf{v}'\) and \(\mathbf{v}_1'\),

\[
b \mathbf{v} = \mathbf{v}' = \mathbf{v} - \frac{1 + \alpha}{2} (\hat{\sigma} \cdot g) \hat{\sigma}, \quad b \mathbf{v}_1 = \mathbf{v}_1' = \mathbf{v}_1 + \frac{1 + \alpha}{2} (\hat{\sigma} \cdot g) \hat{\sigma}.
\]

By using Eq. (5), the equalities

\[
\int d\mathbf{v} \left( \frac{m\mathbf{v}}{m} \right) J[f | f] = \begin{pmatrix} 0 \\ 0 \\ \frac{m}{2} \zeta[f] \end{pmatrix},
\]

are easily obtained. In the above expressions, \(p = nT\) is the pressure (Boltzmann’s constant has been set equal to unity as usual), \(n\) being the number
density and \( T \) the granular temperature, and \( \zeta \) is a a nonlinear functional of the distribution function,

\[
\zeta[f] = \frac{(1 - \alpha^2)\pi^{d/2}m^{d-1}}{4\Gamma(d+3/2)pd} \int dv \int dv_1 f(r, v, t)f(r, v_1, t)g^3.
\] (9)

The hydrodynamic fields are defined in the standard way in terms of velocity moments, i.e.

\[
n(r, t) = \int dv f(r, v, t),
\] (10)

\[
n(r, t)u(r, t) = \int dv v f(r, v, t),
\] (11)

\[
d \frac{1}{2}n(r, t)T(r, t) = \int dv m \frac{1}{2}V^2 f(r, v, t),
\] (12)

where \( V = v - u \) is the velocity of the particle relative to the local velocity flow \( u \). Their balance equations follow directly from Eq. (8),

\[
\frac{\partial n}{\partial t} + \nabla \cdot (nu) = 0,
\] (13)

\[
\frac{\partial u}{\partial t} + u \cdot \nabla u + \frac{1}{mn} \nabla \cdot P = 0,
\] (14)

\[
\frac{\partial T}{\partial t} + u \cdot \nabla T + \frac{2}{nd} (P : \nabla u + \nabla \cdot q) + T \zeta = 0.
\] (15)

The functionals giving the pressure tensor \( P \) and the heat flux \( q \) are:

\[
P(r, t) = m \int dv V V f(r, v, t) = p(r, t)I + \int dv D(V)f(r, v, t),
\] (16)

\[
q(r, t) = m \frac{1}{2} \int dv V^2 V f(r, v, t) = \int dv S(V)f(r, v, t).
\] (17)

Here \( I \) is the unit tensor, and

\[
D(V) = m \left(V V - \frac{1}{d} V^2 I\right), \quad S(V) = \left(\frac{m}{2} V^2 - \frac{d + 2}{2} T\right) V.
\] (18)

The Boltzmann equation (1) admits a special solution \( f_{HCS} \) describing the so-called homogeneous cooling state (HCS) and having the scaling form [3]

\[
f_{HCS}(v, t) = n_H v_0^{-d}(t) \chi_{HCS}(c), \quad c = \frac{v}{v_0(t)},
\] (19)

where

\[
v_0(t) = \left[\frac{2T_{HCS}(t)}{m}\right]^{1/2}
\] (20)
is the “thermal velocity” and $\chi_{HCS}(c)$ is an isotropic function of $c$. As indicated in Eq. (19), all the time dependence of $f_{HCS}$ occurs through the homogeneous temperature $T_{HCS}(t)$, which obeys the equation

$$\frac{\partial T_{HCS}(t)}{\partial t} + T_{HCS}(t)\zeta_{HCS}(t) = 0,$$

(21)

with

$$\zeta_{HCS}(t) = \frac{(1 - \alpha^2)\pi^{d-1}}{2\Gamma\left(\frac{d+3}{2}\right)} \int dc \int dc_1 \chi_{HCS}(c)\chi_{HCS}(c_1)|c-c_1|^3.$$

(22)

Substitution of the scaling form, Eq. (19), into the Boltzmann equation leads to

$$\frac{1}{2}\zeta_{HCS} \frac{\partial}{\partial c} \cdot (c\chi_{HCS}) = J'[\chi_{HCS}|\chi_{HCS}],$$

(23)

The explicit time dependence of the HCS can be eliminated by introducing a dimensionless time scale $s$ by

$$s = \int_0^{t'} \frac{v_0(t')}{\ell} dt',$$

(24)

where $\ell = (n_H\sigma^{d-1})^{-1}$ is proportional to the mean free path of the particles in the HCS. In terms of the $s$ variable, Eq. (21) becomes

$$\frac{\partial}{\partial s} T_{HCS}(s) = -\zeta_0 T_{HCS}(s),$$

(25)

with

$$\zeta_0 = \frac{\ell \zeta_{HCS}}{v_0(t)}.$$  

(26)

that does not depend on time. Equation (25) is the differential form of the Haff’s homogeneous cooling law [13]. Similarly, Eq. (23) takes the form

$$\frac{1}{2}\zeta_0 \frac{\partial}{\partial c} \cdot (c\chi_{HCS}) = J'[\chi_{HCS}|\chi_{HCS}],$$

(27)

$$J'[\chi_{HCS}|\chi_{HCS}] = \int dc_1 \tilde{T}(c,c_1)\chi_{HCS}(c)\chi_{HCS}(c_1),$$

(28)

$$\tilde{T}(c,c_1) = \int d\tilde{\sigma} \Theta \left( [(c - c_1) \cdot \tilde{\sigma}] (c - c_1) \cdot \tilde{\sigma} (\alpha^{-2}b_{\sigma}^{-1} - 1). \right.$$

(29)

The operator $b_{\sigma}^{-1}$ in the last equation is defined again by Eqs. (4), but substituting $(v,v_1)$ by $(c,c_1)$. The form of the distribution function $\chi_{HCS}$ given by the solution of Eq. (27) has been extensively studied. A summary of the results is given in Ref. [14].
3 Linearized Boltzmann Equation

Let us now consider a situation where the inelastic gas is very close to the HCS, so that we can write

\[ f(r, v, t) = f_{HCS}(v, t) + \delta f(r, v, t), \quad |\delta f(r, v, t)| \ll f_{HCS}(v, t). \quad (30) \]

Substitution of Eq. (30) into the Boltzmann equation (1), keeping only terms up to first order in \( \delta f \) yields

\[ \frac{\partial}{\partial t} \delta f(r, v, t) + v \cdot \nabla \delta f(r, v, t) = K \delta f(r, v, t). \quad (31) \]

The linearized Boltzmann collision operator is defined by

\[ K \delta f(r, v, t) = J [f_{HCS} | \delta f] + J [\delta f | f_{HCS}]. \quad (32) \]

Equation (31) can be written in dimensionless form, using the scaled velocity \( c \) defined in Eq. (19), the reduced time \( s \) introduced in Eq. (24), and the dimensionless length scale

\[ 1 = \frac{r}{T}. \quad (33) \]

Then, the linearized inelastic Boltzmann equation becomes

\[ \frac{\partial}{\partial s} \delta \chi(l, c, s) + c \cdot \frac{\partial}{\partial l} \delta \chi = \Lambda \delta \chi, \quad (34) \]

where

\[ \delta \chi(l, c, s) = n^{-1}_H v_0^d(t) \delta f(r, v, t) \quad (35) \]

and

\[ \Lambda \delta \chi = J'[\chi_{HCS} | \delta \chi] + J'[\delta \chi | \chi_{HCS}] - \frac{\zeta_0}{2} \frac{\partial}{\partial c} (c \delta \chi). \quad (36) \]

In this representation, there is not any explicit dependence on time due to the reference state, and all the stationary methods known from the case of elastic collisions [9] can now be applied to the analysis of Eq. (34).

We define relative deviations of the hydrodynamic fields from their values in the HCS by

\[ \rho(l, s) \equiv \frac{\delta n(r, t)}{n_H} = \int dc \delta \chi(l, c, s), \quad (37) \]

\[ \omega(l, s) \equiv \frac{\delta u(r, t)}{v_0(t)} = \int dc c \delta \chi(l, c, s), \quad (38) \]

\[ \theta(l, s) \equiv \frac{\delta T(r, t)}{T_{HCS}(t)} = \int dc \left( \frac{2c^2}{d} - 1 \right) \delta \chi(l, c, s), \quad (39) \]

where \( \delta y(r, t) \equiv y(r, t) - y_{HCS}(t) \) denotes the deviation of the local value of the macroscopic variable \( y \) from its value in the HCS. Taking velocity
moments in the Boltzmann equation (34), the linearized balance equations for the hydrodynamic fields are obtained,

\[
\frac{\partial}{\partial s} \rho(l, s) + \frac{\partial}{\partial l} \cdot \omega(l, s) = 0, \quad (40)
\]

\[
\left( \frac{\partial}{\partial s} - \frac{\zeta_0}{2} \right) \omega(l, s) + \frac{\partial}{\partial l} \cdot \Pi(l, s) = 0, \quad (41)
\]

\[
\frac{\partial}{\partial s} \theta(l, s) + \frac{2}{d} \frac{\partial}{\partial l} \cdot \omega(l, s) + \frac{2}{d} \frac{\partial}{\partial l} \cdot \Phi(l, s) + \delta \zeta_0(l, s) - \zeta_0 [\theta(l, s) + \rho(l, s)] = 0. \quad (42)
\]

The reduced linear pressure tensor \(\Pi\) and heat flux \(\Phi\) are defined as

\[
\Pi(l, s) = \frac{1}{2} [\theta(l, s) + \rho(l, s)] I + \int dc \Delta(c) \delta \chi(l, c, s), \quad (43)
\]

\[
\Phi(l, s) = \int dc \Sigma(c) \delta \chi(l, c, s), \quad (44)
\]

where \(\Delta\) and \(\Phi\) are the dimensionless forms of \(D\) and \(S\),

\[
\Delta(c) = c c - \frac{c^2}{d} I, \quad \Sigma(c) = \left( c^2 - \frac{d + 2}{2} \right) c, \quad (45)
\]

The quantities \(\Pi\) and \(\Phi\) are related with the linearization of the fluxes given in Eq. (16) and (17) by

\[
\Pi(l, s) = \frac{P(r, t)}{2n_H T_{HCS}(t)} - \frac{1}{2}, \quad \Phi(l, s) = \frac{m^{1/2} q(r, t)}{2^{1/2} n_H T_{HCS}^{3/2}(t)}. \quad (46)
\]

Finally, the term \(\delta \zeta_0(l, s)\) is the dimensionless linear deviation of the cooling term from its value in the HCS,

\[
\delta \zeta_0(l, s) = \frac{(1 - \alpha^2) \pi \frac{d+1}{d}}{\Gamma \left( \frac{d+3}{d} \right) d} \int dc \int dc_1 \chi_{HCS}(c) \delta \chi(l, c, s) |c - c_1|^3. \quad (47)
\]

Of course, Eqs. (40)–(42) can be derived also by linearizing the nonlinear balance equations (13)-(15). They are the starting point for deriving hydrodynamic equations, since they describe the evolution of the hydrodynamic fields in the linear approximation. This requires closing the equations, transforming the formal expressions of the pressure tensor, the heat flux, and the cooling rate into expressions in terms of the own hydrodynamic fields. These are the so-called constitutive relations.
4 Eigenvalue Problem

In this Section we explore solutions to the linearized Boltzmann equation and characterize the hydrodynamic modes. This is done through a formal analysis of the eigenvalue problem for the associated linear operator. Since the equation is linear and the collision operator $\Lambda$ does not change the space variable, it is sufficient to consider a single Fourier mode, i.e. we take

$$\delta \chi(l, c, s) = e^{ik \cdot l} \delta \chi(k, c, s).$$

(48)

The linearized Boltzmann equation, Eq. (34), then becomes

$$\frac{\partial}{\partial s} \delta \chi = (\Lambda - i k \cdot c) \delta \chi,$$

(49)

whose formal solution is

$$\delta \chi(k, c, s) = e^{s(\Lambda - i k \cdot c)} \delta \chi(k, c, 0).$$

(50)

The above expression shows most clearly that the possible excitations for the gas under these conditions of small initial perturbations are determined by the spectrum of the linear operator $\Lambda - i k \cdot c$. This suggests a study of the eigenvalue problem

$$(\Lambda - i k \cdot c) \xi_i(k, c) = \lambda_i(k) \xi_i(k, c).$$

(51)

The problem is posed in a Hilbert space of functions of $c$ with a scalar product given by

$$<g|h> = \int dc HCS^{-1}(c) g^*(c) h(c),$$

(52)

where $g^*$ denotes the complex conjugate of $g$. Of particular interest here are the eigenfunctions $\lambda_i^h \beta$ and eigenvalues $\chi_i^h \beta(c)$ associated with linear hydrodynamics. These can be identified as follows. First, at $k = 0$ they correspond to those eigenvalues of $\Lambda$ alone that coincide with the zeroth order in the gradients eigenvalues of the linearized balance equations (40)-(42), which are easily seen to be $0$, $\zeta_0/2$, and $-\zeta_0/2$, i.e.

$$A \xi_i^h \beta(c) = \lambda_i^h \beta \xi_i^h \beta(c), \quad \lambda_i^h \beta = 0, \pm \zeta_0/2,$$

(53)

where $\xi_i^h \beta(c)$ stands for $\xi_i^h \beta(k = 0, c)$. More generally, at finite $k$ the hydrodynamic modes are defined to be the solutions of Eq. (51) that are continuously connected to those of Eq. (53) as $k \to 0$. The eigenvalue problem (53) is solved below, so the existence of the hydrodynamic modes is assured. The formal solution (50) then can be decomposed in the form

$$\delta \chi(k, c, s) = \sum_\beta A_i^h \beta(k, s) \xi_i^h \beta(k, c) + \delta \chi_{\text{m}}(k, c, s).$$

(54)
The first term arises from the hydrodynamic modes while the second one represents all the other “microscopic” excitations. The hydrodynamic coefficients \( \tilde{a}_h^\beta(k,s) \) can be expressed as

\[
\tilde{a}_h^\beta(k,s) = e^{s\lambda_h^\beta(k)} < \tilde{\xi}_h^\beta(k,c) | \delta \chi(k,c,0) > \quad .
\]  

Since \( \Lambda - i k \cdot c \) is not self-adjoint, the functions \( \tilde{\xi}_h^\beta(k,c) \) differ from \( \xi_h^\beta(k,c) \) but together they must form a biorthogonal set. It will appear below that these coefficients are closely related to the hydrodynamic fields of the linearized balance equations.

The expected context for a hydrodynamic description is now clear. For given initial conditions both hydrodynamic and microscopic excitations contribute to (54). However, for long wavelengths (small \( k \)) it is expected that the microscopic excitations decay on time scales short compared to those for the hydrodynamic modes, leaving only the latter on long time scales. The relevant separation of time scales is expected to occur after a few collisions, so there is a large and interesting time scale for the hydrodynamic description. Preliminary analysis based on kinetic models for the Boltzmann equation [15–17] and also on low order matrix representations of the linearized Boltzmann collision operator [18], suggest that the separation of modes assumed here does occur, and hydrodynamics does dominate for wavelengths and times long compared with the mean free path and the mean free time, respectively. Furthermore, it appears from these models that the cooling rate \( \zeta_0 \) remains smaller than the collision frequency even for strong inelasticity, so that inclusion of the non-conserved energy among the hydrodynamic variables seems justified.

**Hydrodynamic modes for \( k = 0 \)**

We consider now the eigenproblem associated with the linearized homogeneous Boltzmann operator,

\[
\Lambda \xi_i(c) = \lambda_i \xi_i(c).
\]  

(56)

Of course, finding all the solutions of this equation is an impossible task, even in the elastic limit. Nevertheless, it is quite easy to obtain some particular solutions, which will turn out to be the relevant ones in the hydrodynamic regime. We begin by constructing the function

\[
F(c,\rho,\omega,\gamma) \equiv \rho \chi_{HCS}(\gamma C),
\]  

(57)

where \( C = c - \omega \). Although \( F \) has the functional form of a “local” HCS distribution, the parameters \( \rho \), \( \omega \), and \( \gamma \) do not need to have any particular meaning here, playing just the role of auxiliary parameters. From Eq. (27) it is obtained that \( F \) verifies

\[
\frac{\rho}{2 \zeta_0} \frac{\partial}{\partial c} \cdot (C F) = \gamma^{d+1} J' [F | F] .
\]  

(58)
Then, taking derivatives with respect to $\rho$, $\omega$, and $\gamma$ in the above equation and, afterwards, the limit $\rho = \gamma = 1$, $\omega = 0$, leads to the set of equations

$$A\psi_1(c) = -\frac{1}{2}\zeta_0\psi_3(c), \quad A\psi_2(c) = \frac{1}{2}\zeta_0\psi_2(c), \quad A\psi_3(c) = -\frac{1}{2}\zeta_0\psi_3(c),$$

(59)

where

$$\psi_1(c) \equiv \left(\frac{\partial F}{\partial \rho}\right)_0 = \chi_{HCS}(c),$$

(60)

$$\psi_2(c) \equiv \left(\frac{\partial F}{\partial \omega}\right)_0 = -\frac{\partial \chi_{HCS}(c)}{\partial c},$$

(61)

$$\psi_3(c) \equiv -\left(\frac{\partial F}{\partial \gamma}\right)_0 - \chi_{HCS}(c)d = -\frac{\partial}{\partial c}\left[c\chi_{HCS}(c)\right].$$

(62)

Here the zero subindex indicates that the derivatives are evaluated at $\rho = \gamma = 1$, $\omega = 0$. From Eqs. (59) it is easily seen that the set of functions

$$\{\xi_\beta(c)\} \equiv \{\psi_1(c) - \psi_3(c), \psi_2(c), \psi_3(c)\}$$

(63)

are solutions of Eq. (56) corresponding to the eigenvalues

$$\lambda_1 = 0, \quad \lambda_2 = \frac{\zeta_0}{2}, \quad \lambda_3 = -\frac{\zeta_0}{2},$$

(64)

respectively. The eigenvalue $\lambda_2$ is, therefore, $d$-fold degenerate. This confirms (53) above, indicating points in the spectrum at $k = 0$ corresponding to the linear balance equations. Hence, these are the hydrodynamic modes at longest wavelengths, $\xi_\beta^h(c)$.

The eigenfunctions $\xi_\beta^h(c)$ can be interpreted as follows. Consider a local form of the HCS distribution, $f_{HCS}^{(l)}$, whereby the hydrodynamics fields are replaced by their actual space and time dependent values in a general nonequilibrium state,

$$f_{HCS}^{(l)}(r, v, t) = n(r, t)v^{(l)-d}(t)\chi_{HCS}(C),$$

(65)

where

$$C = \frac{v - \delta u(r, t)}{v_0^{(l)}(t)}, \quad v_0^{(l)}(t) = \left[\frac{2T(r, t)}{m}\right]^{1/2}.$$ 

(66)

This is the analogous to the local equilibrium distribution for elastic collisions. It represents a state whose averages values of $1$, $v$, and $v^2$ are those of the nonequilibrium state. This seems a reasonable choice to describe an experimentally prepared state. The linearization of such an state around the HCS gives

$$f_{HCS}^{(l)}(r, v, t) \rightarrow f_{HCS}(v, t) + nhv^{(l)-d}(t)\delta\chi^{(l)}(c, l, s),$$

(67)
Thus, the eigenfunctions $\xi_3^h(c)$ describe the linear deviation of the local HCS, appearing then as the natural perturbations for a macroscopic preparation of the initial nonequilibrium state. This will be discussed further below.

It is easy to show that the functions $\{\xi_3^b(c)\}$ are linearly independent so that they define a $d + 2$ dimensional subspace of the Hilbert space. Although the functions $\psi_\beta(c)$ are orthogonal with the definition of scalar product in Eq. (52), the eigenfunctions $\xi_3^b(c)$ are not, as a consequence of the operator $\Lambda$ being non-Hermitian. This leads, in principle, to consider the left eigenvalue problem and, therefore, to study the adjoint operator $\Lambda^+$ defined by

$$< g|Ah >^* = < h|A^+g >,$$

(69)

for arbitrary $g(c)$ and $h(c)$. Using the property given in Eq. (5) it is found

$$A^+g(c) = \int dc_1 \chi_{HCS}(c)\chi_{HCS}(c_1)T(c, c_1) \left[\chi_{HCS}^{-1}(c)g(c) + \chi_{HCS}^{-1}(c_1)g(c_1)\right]$$

$$+ \frac{\zeta_0}{2} \chi_{HCS}(c)c \cdot \frac{\partial}{\partial c} \left[\chi_{HCS}^{-1}(c)g(c)\right].$$

(70)

The eigenfunctions of $\Lambda^+$ would provide the left eigenfunctions of $\Lambda$. We have not been able to find all the eigenfunctions corresponding to the eigenvalues in Eq. (64). Nevertheless, for our purposes here, and also for many other applications, it will suffice to identify a set of functions $\{\bar{\xi}_3^h(c), \beta = 1, 2, 3\}$ verifying the biorthonormality condition

$$< \xi_3^b|\xi_3^b > = 0,$$

(71)

$\beta, \beta' = 1, 2, 3$. A convenient choice is given by

$$\bar{\xi}_1^h(c) = \chi_{HCS}(c), \quad \bar{\xi}_2^h(c) = c\chi_{HCS}(c), \quad \bar{\xi}_3^h = \left(\frac{c^2}{d} + \frac{1}{2}\right)\chi_{HCS}(c).$$

(72)

The above functions span a dual subspace of that spanned by the right eigenfunctions $\{\xi_3^b(c)\}$. The fact that they are linear combinations of 1, c, and $c^2$ is closely related with the existence and approach to a hydrodynamic description, as it will be shown in the next Section. Let us stress that while $\bar{\xi}_1^h(c)$ and $\bar{\xi}_2^h(c)$ are eigenfunctions of $\Lambda^+$ corresponding to the eigenvalues 0 and $\zeta_0/2$, respectively, the function $\bar{\xi}_3^h(c)$ is not an eigenfunction of $\Lambda^+$. The important point is that for any linear combination

$$g(c) = \sum_{\beta=1}^3 a_\beta \bar{\xi}_\beta^h(c),$$

(73)

the coefficients $a_\beta$ are given by

$$a_\beta = < \bar{\xi}_\beta^h|g > = \int dc \chi_{HCS}^{-1}(c)\bar{\xi}_\beta^h(c)g(c).$$

(74)
It is instructive to consider what happens in the elastic limit $\alpha \to 1$. Then $\chi_{HCS}(c)$ becomes the Maxwellian $\chi_{MB}(c) = \pi^{-d/2} \exp(-c^2)$ and the right eigenfunctions are given by
\[
\left\{ \left[ 2c^2 - (d+1) \right] \chi_{MB}(c), 2c \chi_{MB}(c), (2c^2 - d) \chi_{MB}(c) \right\}.
\] (75)

Of course, these functions are not orthogonal and differ from those usually employed in the linear analysis of the elastic Boltzmann equation [19]. This is an important point, since when solving the eigenvalue problem associated with the inelastic homogeneous Boltzmann equation by means of a perturbation calculus around the elastic limit, the first step in the perturbation scheme is to remove the degeneracy of the elastic zeroth eigenvalue by exactly solving the perturbed eigenvalue problem within the subspace spanned by the degenerate eigenfunctions. The relevant conclusion is that the functions $\xi^h_\beta(c)$ appear in a natural way even when considering the small inelasticity limit.

5 Navier-Stokes and Green-Kubo expressions

In the last section the hydrodynamic modes were defined and calculated at asymptotically long wavelengths. It remains to identify the hydrodynamic coefficients in Eq. (54) as well as the hydrodynamic modes at finite $k$. Consider first the coefficients defined by Eq. (55) and evaluate the biorthogonal function $\bar{\xi}^h_\beta(k,c)$ to leading order in $k$, so that
\[
\tilde{a}^h_\beta(k,s) \to <\bar{\xi}^h_\beta(c) | \delta \chi(k,c,s)>.
\] (76)

Use of the expressions given in Eq. (72) shows that, in coordinate representation, these are just the hydrodynamic fields of the linear balance equations
\[
\left\{ a^h_\beta(1,s) \right\} \to \left\{ \rho(1,s), \omega(1,s), \frac{1}{2} \theta(1,s) + \rho(1,s) \right\}.
\] (77)

This completes the connection between the hydrodynamic modes and the dynamics determined from the linear balance equations. They are the same for long wavelengths and small perturbations, as expected.

5.1 Navier-Stokes Approximation

To obtain the hydrodynamic eigenvalues $\lambda^h_\beta(k)$ to Navier-Stokes ($k^2$) order, the eigenvalue problem (51) can be solved treating $ic \cdot k$ as a small perturbation of the operator $A$ [20]. This method implicitly restricts $k$ to be small relative to the deviation of the restitution coefficient $\alpha$ from unity, the elastic limit. A more uniform description, without restrictions on $k$ relative to $\alpha$, is obtained by calculating the fluxes in the balance equations as functions of
\( \tilde{a}_h^\beta(k, s) \), to first order in \( k \). The resulting Navier-Stokes hydrodynamic equations can then be used to determine \( \lambda^h_h(k) \). This is the approach used here. The first step is accomplished by using the identity
\[
e^{(A+B)s} = e^{As} + \int_0^s ds' e^{As'}Be^{(A+B)(s-s')},
\]
valid for arbitrary operators \( A \) and \( B \), in the formal solution (50) to get
\[
\delta \tilde{\chi}(k, c, s) = e^{s\Lambda} \delta \tilde{\chi}(k, c, 0) - \int_0^s ds' e^{s'\Lambda} i k \cdot c e^{-s'(A-ikc)} \delta \tilde{\chi}(k, c, s).
\]
Next, substitute the representation (54) to identify the dependence on \( \tilde{a}_h^\beta(k, s) \),
\[
\delta \tilde{\chi}(k, c, s) = e^{sA} \delta \tilde{\chi}(k, c, 0) - \int_0^s ds' e^{s'\Lambda} i k \cdot c e^{-s'(A-ikc)} \sum_{\beta} \tilde{a}_h^\beta(k, s) \xi_h^\beta(k, c) + \delta \tilde{\chi}^m(k, c, s)
\]
\[
\rightarrow e^{sA} \tilde{\chi}(k, c, 0) - \sum_{\beta} \left[ \int_0^s ds' e^{s'\Lambda} c e^{-s'(A-\lambda^h_h)} \xi_h^\beta(c) \right] \cdot i k \tilde{a}_h^\beta(k, s)
\]
\[
= e^{sA} \tilde{\chi}(k, c, 0) - \sum_{\beta} \left[ \int_0^s ds' e^{s'(A-\lambda^h_h)} c \xi_h^\beta(c) \right] \cdot i k \tilde{a}_h^\beta(k, s).
\]

In the second transformation, it has been assumed that \( k \) is sufficiently small so that only leading order terms need to be retained, and that \( s \) is sufficiently large to neglect the microscopic excitations relative to the hydrodynamic modes contributions. Moreover, in the last equality use has been made of the fact that \( \xi_h^\beta \) is an eigenfunction of \( A \). In coordinate representation this result becomes more explicitly
\[
\delta \chi(l, c, s) \rightarrow e^{sA} \delta \chi(l, c, 0) - \sum_{\beta=1}^3 F_\beta(c, s) \cdot \frac{\partial}{\partial l} \nu_\beta(l, s),
\]
where
\[
\{ \nu_\beta(l, s) \} \equiv \{ \rho(l, s), \omega(l, s), \theta(l, s) \}
\]
and
\[
F_1(c, s) = \int_0^s ds' e^{s'\Lambda} \xi_1(c) c + 2F_3(c, s),
\]
\[
F_{2,ij}(c, s) = \int_0^s ds' e^{s'(A-\frac{\partial}{\partial c})} \xi_{2,ij}(c) c_j.
\]
The pressure tensor and the heat flux up to first order in the gradients of the fields are now obtained by substituting Eq. (81) into Eqs. (43) and (44). The term involving the initial condition $\delta \xi(l, c, 0)$ does not contribute to these quantities since, in the long time limit, it is given by a linear combination of the functions $\chi_d(c)$, that are all orthogonal to both $\Delta(c)\chi_{HCS}(c)$ and $\Sigma(c)\chi_{HCS}(c)$. Taking into account the symmetry properties of the system, the resulting expressions can be written in the form

$$\Pi_{ij}(l, s) = \frac{1}{2} \left[ \theta(l, s) + \rho(l, s) \right] \delta_{ij}$$

$$\phi(l, s) = -\tilde{\kappa} \frac{\partial}{\partial l} \theta(l, s) - \tilde{\mu} \frac{\partial}{\partial l} \rho(l, s).$$

Equation (86) is the expected Navier-Stokes expression for the pressure tensor, involving the shear viscosity coefficient $\tilde{\eta}$, but Eq. (87) contains, besides the usual Fourier law characterized by the heat conductivity $\tilde{\kappa}$, an additional contribution proportional to the density gradient and with an associated transport coefficient $\tilde{\mu}$. This latter term has no analogue in elastic gases.

The expressions of the (time-dependent) transport coefficients are

$$\tilde{\eta}(s) = \frac{1}{d^2 + d - 2} \sum_{i,j} \int dc \Delta_{ij}(c) F_{2,ij}(c, s),$$

$$\tilde{\kappa}(s) = \frac{1}{d} \int dc \Sigma(c) \cdot F_3(c, s),$$

$$\tilde{\mu} = \frac{1}{d} \int dc \Sigma(c) \cdot F_1(c, s).$$

As already mentioned, the functions $\Delta_{ij}(c)\chi_{HCS}(c)$ and $\Sigma_i(c)\chi_{HCS}(c)$ are orthogonal to the subspace spanned by the right eigenfunctions $\xi_\delta(c)$. Therefore, the decay of the “correlation functions” in Eqs. (88)–(90) lies outside the spectrum of the slowest modes, and the $s$-time integral in the expression of the functions $F_\beta$ converges. Consequently, it is possible to set $s \to \infty$ for times scales long compared with the microscopic times. In this way, the reduced transport coefficients become time-independent. Returning to the original variables, the above expressions for the hydrodynamic fluxes are equivalent to

$$P_{ij}(r, t) = n_H T_{HCS}(t) \left[ 1 + \frac{\delta T(r, t)}{T_{HCS}(t)} + \frac{\delta n(r, t)}{n_H} \right] \delta_{ij}$$

$$-\eta \left[ \frac{\partial}{\partial r_i} \delta u_j + \frac{\partial}{\partial r_j} \delta u_i - \frac{2}{d} (\nabla \cdot \delta \mathbf{u}) \delta_{ij} \right],$$
\[ \mathbf{q}(\mathbf{r}, t) = -\kappa \nabla \delta T(\mathbf{r}, t) - \mu \nabla \delta n(\mathbf{r}, t), \]  
with the transport coefficients given by

\[ \eta = n_H m \ell v_0(t) \bar{\eta}, \]  
(93)

\[ \kappa = n_H \ell v_0(t) \bar{\kappa}, \]  
(94)

\[ \mu = \frac{m \ell v_0^2(t)}{2} \bar{\mu}. \]  
(95)

5.2 Green-Kubo Relations

It is possible to write Eqs. (88)–(90) in the form of Green-Kubo relations. Define an operator \( \tilde{\Lambda} \) by

\[ \left[ \int d\mathbf{c} g^*(\mathbf{c}) \Lambda h(\mathbf{c}) \right]^* = \int d\mathbf{c} h^*(\mathbf{c}) \tilde{\Lambda} g(\mathbf{c}), \]  
(96)

for arbitrary \( h \) and \( g \). Comparison with Eq. (69) yields

\[ \tilde{\Lambda} g(\mathbf{c}) = \chi_{HCS}^{-1}(\mathbf{c}) \Lambda^+ [g(\mathbf{c}) \chi_{HCS}(\mathbf{c})] \]
\[ = \int d\mathbf{c}_1 \chi_{HCS}(\mathbf{c}_1) T(\mathbf{c}, \mathbf{c}_1) [g(\mathbf{c}) + g(\mathbf{c}_1)] + \frac{\zeta_0}{2} \mathbf{c} \cdot \frac{\partial}{\partial \mathbf{c}} g(\mathbf{c}). \]  
(97)

Also define “correlation functions” in the HCS by

\[ < gh > \equiv \int d\mathbf{c} \chi_{HCS}(\mathbf{c}) g(\mathbf{c}) h(\mathbf{c}). \]  
(98)

Then, the expressions for the transport coefficients can be rewritten as

\[ \bar{\eta} = \frac{1}{d^2 + d - 2} \sum_{i,j} \int_0^\infty ds \ < \Delta_{ij}(s) \mathbf{\Phi}_{2,ij} > \ e^{-\zeta_0/2}, \]  
(99)

\[ \bar{\kappa} = \frac{1}{d} \int_0^\infty ds \ < \Sigma(s) \cdot \mathbf{\Phi}_3 > \ e^{\zeta_0/2}, \]  
(100)

\[ \bar{\mu} = 2\bar{\kappa} + \frac{1}{d} \int_0^\infty ds \ < \Sigma(s) \cdot \mathbf{\Phi}_1 >, \]  
(101)

with the definitions

\[ \mathbf{\Phi}_1 \equiv \mathbf{c} - 2\mathbf{\Phi}_3(\mathbf{c}), \]  
(102)

\[ \mathbf{\Phi}_{2,ij}(\mathbf{c}) = -c_j \frac{\partial \ln \chi_{HCS}(\mathbf{c})}{\partial c_i}, \]  
(103)

\[ \mathbf{\Phi}_3(\mathbf{c}) = -\frac{1}{2} \mathbf{c} \left[ d + \mathbf{c} \cdot \frac{\partial \ln \chi_{HCS}(\mathbf{c})}{\partial \mathbf{c}} \right]. \]  
(104)
The time dependence of the dynamical variables is given by
\[ g(c, s) = e^{s\Lambda}g(c). \] (105)

Equations (99)–(101) are the Green-Kubo formulae for a dilute granular gas described by the Boltzmann equation. They express the transport coefficients as integrals of time-correlations functions in the HCS. Although they are not identical to those derived in Ref. [11] by a different method, it is easy to show they are equivalent, since they differ in terms giving vanishing contributions.

Let us also compare them with the expressions derived from the nonlinear Boltzmann equation by the Chapmann-Enskog method [5,8]. Consider the reduced shear viscosity coefficient. It is given by Eq. (88) with \( F_{2,ij} \) defined in Eq. (84). In the limit of large \( s \), we have seen that the integral in the latter expression can be extended up to infinity. Performing then the integral, it is obtained
\[ \tilde{\eta} = -\frac{1}{d^2 + d - 2} \sum_{i,j} \int dc \Delta_{ij}(c) \left( \Lambda - \frac{\zeta_0}{2} \right)^{-1} \xi_{2,i}(c)c_j, \] (106)
or, equivalently,
\[ \tilde{\eta} = \frac{1}{d^2 + d - 2} \sum_{i,j} \int dc \Delta_{ij}(c)C_{ij}(c), \] (107)
where \( C_{ij} \) is the solution of the integral equation
\[ \left( \Lambda - \frac{\zeta_0}{2} \right) C_{ij}(c) = -\xi_{2,i}(c)c_j, \] (108)
being orthogonal to the subspace spanned by the lowest order right eigenfunctions of \( \Lambda \). Equations (107) and (108) are equivalent to those derived by the Chapman-Enskog procedure in Refs. [5,8], where the explicit expressions of the transport coefficients in the first Sonine approximation is obtained. A similar analysis leading to the same conclusion can be carried out for the other transport coefficients. The only conceptually relevant difference is in the reference state. In the present linearized analysis the transport coefficients are defined in the HCS, whereas in the Chapmann-Enskog method they are defined in a local HCS, characterized by the parameters \( n(r, t) \), \( u(r, t) \) and \( T(r, t) \). As a consequence, these are the fields appearing in the scaling in Eqs. (93)–(95).

It is also instructive to consider the elastic limit \( \alpha \to 1 \) of the Green-Kubo expressions derived here. In that case, we already know that the HCS distribution becomes the Maxwellian \( \chi_{MB} \), and \( \zeta_0 \to 0 \). Then, the functions defined in Eqs. (102)–(104) become
\[ \Phi_1(c) \to -2\Sigma(c), \quad \Phi_{2,ij}(c) \to 2c_i c_j, \quad \Phi_3(c) \to \left( c^2 - \frac{d}{2} \right) c. \] (109)
Therefore, the transport coefficient $\mu$, given by Eq. (101) vanishes in the elastic limit, as expected, and

$$\eta \to \eta_0 = \frac{2nmv_0(t)}{d^2 + d - 2} \sum_{i,j}^d \int_0^\infty ds <\Delta_{ij}(s)c_i c_j>$$

$$= \frac{2nmv_0(t)}{d^2 + d - 2} \sum_{i,j}^d \int_0^\infty ds <\Delta_{ij}(s)\Delta_{ij}> . \tag{110}$$

$$\kappa \to \kappa_0 = \frac{nLv_0(t)}{d} \int_0^\infty ds <\Sigma(s) \cdot \left(\mathbf{c}^2 - \frac{d}{2}\right)\mathbf{c}>$$

$$= \frac{nLv_0(t)}{d} \int_0^\infty ds <\Sigma(s) \cdot \Sigma> . \tag{111}$$

The last equalities in the above equations follow from the symmetry of the time correlation functions. For elastic systems, the temperature is constant to lowest order in the gradients of the hydrodynamic fields and, therefore, in the integrands appearing in the above expressions the definition of the scale $s$, given by Eq. (24), reduces to $s = v_0 t / \ell$, i.e. it is simply proportional to the actual time. Then, Eqs. (110) and (111) are the low density limit of the usual Green-Kubo expressions for the transport coefficients in terms of the autocorrelation functions of the microscopic fluxes of the relevant densities [9].

6 Discussion

The primary results of the above analysis have been the identification of hydrodynamic modes in the spectrum of the linearized Boltzmann operator, their leading order calculation in Section 4, and their use to derive the Navier-Stokes approximations to the heat and momentum fluxes. Of particular interest are the long wavelength forms of the hydrodynamic eigenfunctions $\xi^h_\beta$ given in Eq. (63), which are not simply linear combinations of the conserved densities $1, v, v^2$. It is clear from (54) that perturbations of the homogeneous state due to $\xi^h_\beta(c)$ generate the hydrodynamic excitations, while the response to these excitations is measured by the fields $\tilde{a}^h_\beta(s)$. These fields are averages of the biorthogonal set of functions $\bar{\xi}_\beta$, which are linear combinations of the conserved densities $1, v, v^2$, as required by the macroscopic balance equations. As anticipated in the Introduction, this “conjugate” relationship between the sets $\{\xi^h_\beta\}$ and $\{\bar{\xi}_\beta\}$ is a key difference between normal and granular gases. For elastic collisions, the two sets of fields are the same. This has important consequences for the derivation of transport properties as it has been illustrated here. For example, the reduction leading to the Navier-Stokes approximation in Eq. (80), depends critically on the fact that the functions $\xi^h_\beta$
are eigenfunctions of $\Lambda$. Any other choice for the initial perturbation would lead to a different time dependence, mixing microscopic and hydrodynamic excitations, and precluding direct identification of transport properties.

The identification of the hydrodynamic modes discussed here is important for the application of linear response methods [21] to granular systems. In order to illustrate it, let us consider again the formal solution to the linear Boltzmann equation given in (50) and choose for the initial condition the linearization of a local HCS as given by Eq. (67),

$$
\delta\tilde{\chi}(k, c, s) = e^{s(\Lambda - ik \cdot c)} \sum_{\beta} \langle \bar{\xi}_{\beta} | \delta\tilde{\chi}(0) > \xi_{\beta}(c)
$$

$$
= e^{s(\Lambda - ik \cdot c)} \sum_{\beta} a_{\beta}(k, s = 0) \xi_{\beta}(c).
$$

(112)

This initial state clearly represents a hydrodynamic perturbation. The response $a_{\beta}(k, s)$ is then given by

$$
a_{\beta}(k, s) = \sum_{\beta'} R_{\beta\beta'}(k, s) a_{\beta'}(k, s = 0), \quad R_{\beta\beta'}(k, s) = \langle \bar{\xi}_{\beta} | e^{s(\Lambda - ik \cdot c)} \xi_{\beta'} \rangle.
$$

(113)

The response functions $R_{\beta\beta'}(k, s)$ are time correlation functions comprised of the pair of the conjugate densities. It can be shown that application of standard linear response methods to these response functions leads to the correct Green-Kubo relations with the proper convergence properties for the time integrals. Conversely, the same procedure applied to correlation functions comprised of the $\{\bar{\xi}_{\beta}\}$ alone does not lead directly to well-posed Green-Kubo expressions. Let us also mention that linear perturbations of the kind considered in Eq. (112) have been used [22] in direct Monte Carlo simulations of the Boltzmann equation [23], to verify the validity of the linear hydrodynamic description and the explicit expressions of the transport coefficients.

Although the analysis here has been limited to the low density linearized Boltzmann equation, these considerations point the way for a proper application of linear response methods in the more general context of nonequilibrium statistical mechanics.

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