Time evolution of the free Dirac field in spatially flat FLRW spacetimes

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Abstract

The relativistic quantum mechanics of the free Dirac field in spatially flat FLRW spacetimes is considered as the framework for deriving the analytical solutions of the Dirac equation in different local charts of these manifolds. Different systems of commuting conserved operators are used for giving physical meaning to the integration constants as eigenvalues of these operators. Since these systems are incomplete there are integration constants that must be fixed by setting the vacuum either as the traditional adiabatic one or as the rest frame vacuum proposed recently. All the known solutions of the Dirac equation on these manifolds are discussed in all their details and a new type of spherical waves of given energy in the de Sitter expanding universe is reported.

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1 Introduction

In general relativity the Dirac field on Riemannian manifolds is less studied [1–9] since one prefers the scalar field that can be manipulated easier in the actual developments in astrophysics and cosmology. For this reason we devote this paper to the Dirac field on the spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) spacetimes paying a special attention to the framework we need for studying the analytical solutions of the Dirac equations that can be derived on such manifolds [10–16], including a new type of solution reported here for the first time.

In our opinion, the general relativistic quantum mechanics on curved spacetimes must respect ad litteram the principles of the traditional quantum mechanics which today is more than a coherent theory being in fact the source of new technologies. On the other hand, an independent relativistic quantum mechanics cannot be constructed since the wave functions are replaced here by free fields with different spins and specific equations. For this reason, the general relativistic quantum mechanics must be seen as the one particle restriction of the quantum field theory (QFT) at the level of the first quantization. This theory must be compatible with the geometry of any curved background keeping a balance between the local and global features.

We assume that the quantum states are prepared or measured by a global apparatus represented by algebra of the conserved operators (or observables) commuting with the operator of the field equation. The principal observables are the differential operators generated by the Killing vector fields but, in addition, there are observables that can be defined in different manners as we shall see in what follows. All these observables must be defined globally, independent on the local charts we use. The solutions of the field equations have to be determined completely or partially as common eigenstates of a system of commuting observables since then the integration constants get a physical meaning as eigenvalues of these operators. Thus we obtain complete systems of fundamental solutions, globally defined, representing the bases the different representations (reps.) of the quantum mechanics.

On the other hand, the time evolution can be described in many time evolution pictures which are strongly dependent on the local time coordinate. For this reason we assume that that two local charts may generate different time evolution pictures if their coordinates are related through a diffeomorphism transforming simultaneously the time and space coordinates. Thus we are in the apparently paradoxical situation to work in global reps. but with local time evolution pictures. In what follows we would like to inves-
igate exhaustively the solutions of the Dirac equation on the spatially flat FLRW manifolds, including the de Sitter (dS) one, in two different time evolution pictures, namely the natural picture (NP) in co-moving charts and the Schrödinger picture (SP) \([12]\) in the charts with de Sitter-Painlevé (dS-P) coordinates \([17, 18]\). We study the known plane \([1, 10]\) and spherical \([5, 11]\) waves depending on vector or scalar momentum in NP and the plane waves depending on energy in the SP of the dS expanding universe \([13]\). Moreover, on this manifold we derive for the first time the spherical waves depending on energy in SP where the spherical variables can be separated. For shortening the terminology we shall speak about plane and spherical P-waves in NP and plane and spherical E-waves in SP or NP of the dS background. These sets of fundamental solutions define different reps. among them those of plane P-waves and E-waves can be related in NP while for the spherical solutions we cannot obtain a similar result.

On the manifolds into consideration there are only incomplete sets of commuting observables such that there remain undetermined integration constants which must be fixed by using supplemental assumptions related mainly to the frequencies separation setting the vacuum. In what follows we consider the traditional adiabatic vacuum (a.v.) \([19, 20]\) and the new rest frame vacuum (r.f.v.) we proposed recently \([21]\). By using these vacua we can determine the integration constants of all the plane waves but there are some ambiguities in the case of the spherical waves we will discuss here.

We start in next section with the geometry of the spatially flat FLRW spacetimes defining the frames we need for writing the gauge covariant Dirac field \([22]\) whose conserved observables are briefly analysed showing how these may define the reps. of our relativistic quantum mechanics. In section III we introduce the time evolution pictures, NP and SP, we need in order to avoid coordinate transformations involving time. In this section we define, in addition, the energy operators and study the equivalence of these pictures. The next section is devoted to the solutions that can be obtained in NP presenting plane and spheric P-waves for which we define the a.v. and r.f.v. giving as examples a Milne-type manifold and the dS expanding universe. The plane and spherical E-waves, that can be derived exclusively in the SP of this last manifold, are presented in section V. We point out that these depend on the conserved energy which separates the frequencies as in special relativity but leaving undetermined integration constants. Finally we present our concluding remarks and present some technical details in four appendices.
2 Dirac field in FLR W spacetimes

We study here the free Dirac field (or perturbation) on the (1+3)-dimensional local Minkowskian spatially flat FLRW spacetimes whose geometries are given by a scale factor $a : D_t \rightarrow \mathbb{R}$ which is a smooth function defined on a given time domain $D_t$. We denote from now these FLRW spacetimes by $(M,a)$ for distinguish them from the general Riemannian manifolds of arbitrary metric $g$ denoted by $(M,g)$. The Minkowski spacetime will be denote by $(M,\eta)$ where $\eta = \text{diag}(1, -1, -1, -1)$.

2.1 Frames in FLRW spacetimes

The form of the Dirac equation depends on the choice of the thelocal coordinates and the unholonomic orthogonal local frames needed for describing the spin. On the FLRW manifolds $(M,a)$ there are many types of local charts (or natural frames) related to the standard co-moving FLRW one, $\{t, \vec{x}\}$, whose coordinates $x^\kappa$ (labelled by the natural indices $\kappa, \nu,... = 0,1,2,3$) are the proper (or cosmic) time, $t$, and the conformal Cartesian space coordinates $x^i$ ($i,j,k... = 1,2,3$) for which we use the vector notation $\vec{x} = (x^1, x^2, x^3)$. Another useful chart is that of the conformal time,

$$t_c = \int \frac{dt}{a(t)} \rightarrow a(t_c) = a[t(t_c)],$$

(1)

and the same Cartesian space coordinates, denoted by $\{t_c, \vec{x}\}$. The line elements of these charts are,

$$ds^2 = g_{\kappa\nu}(x)dx^\kappa dx^\nu = dt^2 - a(t)^2d\vec{x} \cdot d\vec{x}$$

(2)

$$= a(t_c)^2 (dt_c^2 - d\vec{x} \cdot d\vec{x}).$$

(3)

The advantage of the conformal chart is that here one can take over many results from the flat Minkowski spacetime $(M,\eta)$ through a simple conformal transformation [20].

A less used chart is the dS-P one, $\{t, \vec{x}\}$, with 'observed' space coordinates defined as [17],

$$\vec{x}^i = a(t)x^i.$$  

(4)

The line element

$$ds^2 = dt^2 \left(1 - \frac{\dot{a}(t)^2}{a(t)^2} \vec{x} \cdot \vec{x}\right) + 2 \frac{\dot{a}(t)}{a(t)} \vec{x} \cdot d\vec{x} dt - d\vec{x} \cdot d\vec{x},$$

(5)
\( \dot{a}(t) = \partial_t a(t) \), depends on Hubble function \( \frac{\dot{a}(t)}{a(t)} \) for which we do not use the symbol \( H \) since this is reserved for the energy operators. The corresponding chart \( \{ t_c, \vec{x} \} \), with the same space coordinates but with the conformal time, has the line element

\[
ds^2 = dt_c^2 \left( a(t_c)^2 - \frac{\dot{a}(t_c)^2}{a(t_c)^2} \vec{x} \cdot \vec{x} \right) + 2 \frac{\dot{a}(t_c)}{a(t_c)} \vec{x} \cdot d\vec{x} dt_c - d\vec{x} \cdot d\vec{x}, \tag{6}
\]

since after changing the time variable \( t \to t_c \) and denoting \( \dot{a}(t_c) = \partial_{t_c} a(t_c) \), we have to substitute \( dt \to a(t_c) dt_c \) and

\[
\partial_t \to \frac{1}{a(t_c)} \partial_{t_c}, \quad \frac{\dot{a}(t)}{a(t)} \to \frac{\dot{a}(t_c)}{a(t_c)^2}, \tag{7}
\]
in accordance with Eq. (1).

For studying problems with spherical symmetry we need local charts with spherical coordinates, \( \{ t, r, \theta, \phi \} \) and \( \{ t_c, r, \theta, \phi \} \), obtained from the charts \( \{ t, \vec{x} \} \) and \( \{ t_c, \vec{x} \} \) where we introduce the spherical coordinates \( \vec{x} \to (r, \theta, \phi) \) with \( r = |\vec{x}| \). In the charts with dS-P coordinates, \( \{ t, \vec{x} \} \) and \( \{ t_c, \vec{x} \} \), we consider the coordinates \( \vec{x} \to (\vec{r}, \theta, \phi) \) with \( \vec{r} = |\vec{x}| \) but the same angular coordinates.

For writing down the Dirac equation we need to fix the tetrad gauge giving the vector fields \( e_\alpha = e_\alpha^\kappa \partial_\kappa \) defining the local orthogonal frames, and the 1-forms \( \omega^\alpha = e_\alpha^\kappa dx^\kappa \) of the dual co-frames (labelled by the local indices, \( \hat{\kappa}, \hat{\nu}, \ldots = 0, 1, 2, 3 \)). The metric tensor of \( (M, g) \) can be expressed now as

\[
g_{\kappa\nu} = \eta_{\alpha\beta} e_\alpha^\kappa e_\beta^\nu. \]

Here we restrict ourselves to the diagonal tetrad gauge defined by the vector fields

\[
e_0 = \partial_t, \\
e_i = \frac{1}{a(t)} \partial_i = \partial_i + \frac{\dot{a}(t)}{a(t)} x^i \partial_t, \tag{8}
\]

and the corresponding dual 1-forms

\[
\omega^0 = dt, \\
\omega^i = a(t) dx^i = dx^i - \frac{\dot{a}(t)}{a(t)} x^i dt, \tag{9}
\]
in order to preserve the global \( SO(3) \) symmetry allowing us to use systematically the \( SO(3) \) vectors. When we use the charts with spherical coordinates we keep the same gauge rewritten in terms of these coordinates.
The frames \{x; e\} we need in the Dirac theory are formed by a local chart \{x\} and the local frames defined by the tetrads \(e\). These frames can be transformed, \{x; e\} \to \{x'; e'\}, with the help of diffeomorphisms, \(x \to x' = \phi(x)\), transforming the coordinates and by using local transformations \(\Lambda(x)\) of the Lorentz group, \(L^+_1\), for changing the tetrad gauge as

\[
e^\alpha(x) \rightarrow e'^\alpha(x) = \Lambda^\beta_\alpha(x) e^\beta(x),
\]

\[
\omega^\alpha(x) \rightarrow \omega'^\alpha(x) = \Lambda^\alpha_\beta(x) \omega^\beta(x).
\]

In general, any theory of fields with spin in general relativity must be gauge invariant in the sense that the above gauge transformations have to do not affect the physical meaning of the theory.

2.2 Covariant Dirac field

In any frame \{x; e\} of a curved spacetime \((M, g)\) the tetrad gauge invariant action of the Dirac field \(\psi\), of mass \(m\), minimally coupled to the background gravity, reads,

\[
S[e, \psi] = \int d^4x \sqrt{g} \left\{ \frac{i}{2} \bar{\psi} \gamma^\alpha \nabla_\alpha \psi - (\nabla_\alpha \psi) \gamma^\alpha \psi - m \bar{\psi} \psi \right\}
\]

where \(\bar{\psi} = \psi^* \gamma^0\) is the Dirac adjoint of \(\psi\) and \(g = |\det(g_{\mu\nu})|\). The Dirac matrices, \(\gamma^\alpha\) (with local indices) are self adjoint, \(\gamma^\mu = \gamma^0 \gamma^\alpha \gamma^0\), and satisfy the anti-commutation rules \(\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta}\) giving the \(SL(2, \mathbb{C})\) generators as,

\[
S^\alpha_\beta = \frac{i}{4} [\gamma^\alpha, \gamma^\beta].
\]

The notation \(^*\) stands for the Hermitian conjugation of matrices that has to do not be confused with the Hermitian conjugations with respect to the relativistic scalar products we will introduced later.

Mathematically speaking, the Dirac spinors transform according to the reducible rep. \(\rho_D = (\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\) of the \(SL(2, \mathbb{C})\) group but which is the simpler rep. allowing invariant bilinear forms. For this reason we consider here that \(\rho_D\) is just the rep. defining this group denoting its elements simply as \(\rho_D(A) \equiv A \in SL(2, \mathbb{C})\). We remind the reader that in the covariant parametrization with skew-symmetric parameters, \(\omega^\alpha_\beta = -\omega^\beta_\alpha\), the \(SL(2, \mathbb{C})\) transformations

\[
A(\omega) = \exp \left( -\frac{i}{2} \omega^\alpha_\beta S^\alpha_\beta \right),
\]
correspond through the canonical homomorphism to the transformation matrices \( \Lambda[A(\omega)] = \Lambda(\omega) \in L^+_\uparrow \) having the matrix elements \( \Lambda(\omega)_{\dot{\alpha}\hat{\beta}} = \delta_{\dot{\alpha}}^{\hat{\beta}} + \omega_{\dot{\alpha} \hat{\beta}} + \ldots \).

The covariance under the gauge transformation (10) and (11) produced by \( \Lambda[A(x)] \in L^+_\uparrow \), when the Dirac field transforms as

\[
\psi(x) \rightarrow \psi'(x) = A(x)\psi(x),
\]

is assured by the covariant derivatives

\[
\nabla_{\dot{\alpha}} = e_{\dot{\alpha}} + \frac{i}{2} \tilde{\Gamma}_{\dot{\alpha}\hat{\beta}} S_{\hat{\beta}},
\]

where \( \tilde{\Gamma}_{\dot{\mu}\hat{\nu}} = e^\alpha_\mu e^\beta_\nu (e^\gamma_\alpha \Gamma^\gamma_{\alpha\beta} - e^\beta_{\alpha\beta}) \) are the connection components in local frames (known as the spin connections) expressed in terms of tetrads and Christoffel symbols, \( \Gamma^\gamma_{\alpha\beta} \). From the action (12) it results the Dirac equation

\[
\left( i \gamma^{\dot{\alpha}} \nabla_{\dot{\alpha}} - m \right) \psi(x) = 0,
\]

that can be written in explicit form as

\[
\left( i \gamma^{\dot{\alpha}} e^\mu_\dot{\alpha} \partial_\mu \psi - m \psi + \frac{i}{2} \gamma^\alpha \partial_\alpha (\sqrt{g} e^\mu_\dot{\alpha}) \gamma^{\dot{\alpha}} \psi - \frac{1}{4} \{ \gamma^{\dot{\alpha}}, S_{\hat{\beta}} \} \tilde{\Gamma}^{\hat{\beta}}_{\dot{\alpha}\hat{\beta}} \psi = 0. \tag{18}
\]

The particular solutions of this equation form a vector space equipped with the conserved relativistic scalar product [10]

\[
\langle \psi, \psi' \rangle = \int_{\Sigma} d\sigma \sqrt{g} e^\mu_\dot{\alpha} \tilde{\psi}(x) \gamma^{\dot{\alpha}} \psi'(x), \tag{19}
\]

whose integral is performed on a space-like section \( \Sigma \subset M \).

Applying these general formulas to the Dirac field in the frames \{t, \vec{x}; e\} of a FLRW manifold \((M, a)\) we obtain the Dirac equation

\[
\left( i \gamma^0 \partial_t + \frac{1}{\alpha(t)} \gamma^i \partial_i + \frac{3i}{2} \frac{\dot{\alpha}(t)}{\alpha(t)} \gamma^0 - m \right) \psi(t, \vec{x}) = 0. \tag{20}
\]

In the frames \{t, \vec{x}; e\} the Dirac equation reads

\[
\left[ i \gamma^0 \partial_t + i \gamma^i \partial_i - m + i \gamma^0 \frac{\dot{\alpha}(t)}{\alpha(t)} \left( \frac{\vec{x}^i}{a(t)} \partial_i + \frac{3}{2} \right) \right] \psi(t, \vec{x}) = 0. \tag{21}
\]

Similar results can be written in the frames of conformal time performing the substitution (7) in the above equations. The versions of the Dirac equation in frames with spherical coordinates will be analysed later when the spherical modes will be studied.
2.3 Conserved observables

The general relativistic covariance under diffeomorphisms and gauge transformations is not able to generate conserved quantities via Noether theorem. Only the isometries can do that and for this reason these deserves a special attention.

In general, the spacetimes $\mathcal{M} = (M,g)$ of physical interest have isometries, $x \rightarrow x' = \phi_\xi(x)$, which are non-linear transformations preserving the metric. These form the isometry group $I(M)$ having the composition rule $\phi_\xi \circ \phi_\eta = \phi_{\xi \eta}$, $\forall \xi, \eta \in I(M)$ and the identity function $id = \phi_e$ as the unit element. In a given parametrization, $g = g(\xi)$ (with $e = g(0)$), the isometries

$$x \rightarrow x' = \phi_\xi(x) = x + \xi^a k_a(x) + ...$$

lay out the Killing vectors $k_a = \partial_{\xi^a} \phi_\xi|_{\xi=0}$ associated to the parameters $\xi^a$ ($a, b, ... = 1, 2, ... N$).

Since the isometries may change the relative position of the natural and local frames we proposed the theory of external symmetry [22, 23] introducing the combined transformations $(A_\xi, \phi_\xi)$ able to correct the positions of the local frames after each isometry. These transformations must preserve not only the metric but the tetrad gauge too, transforming the 1-forms as $\tilde{\omega}(x') = \Lambda[A_\xi(x)]\tilde{\omega}(x)$. Hereby, it results the form of the local transformations [22],

$$\Lambda_{\alpha\beta}[A_\xi(x)] = e^\alpha_\mu [\phi_\xi(x)] \frac{\partial \phi^\mu_\xi(x)}{\partial x^\nu} e^\beta_\nu (x),$$

which define the matrices $A_\xi(x)$ assuming, in addition, that $A_\xi=e(x) = 1 \in SL(2,\mathbb{C})$. The resulted combined transformations $(A_\xi, \phi_\xi)$ preserve the gauge, $e' = e$ and $\omega' = \omega$, transforming the Dirac field according to the covariant rep. $T : (A_\xi, \phi_\xi) \rightarrow T_\xi$ whose operators act as

$$(T_\xi \psi)[\phi_\xi(x)] = A_\xi(x)\psi(x).$$

We have shown that the pairs $(A_\xi, \phi_\xi)$ constitute a well-defined Lie group with respect to the new operation that can be seen as a rep. of the universal covering group of $I(M)$ denoted here by $S(M)$ [22]. In fact, these covariant reps. transfer the Lorentz covariance from special relativity to general relativity.

Given a parametrization, $g = g(\xi)$, for small values of $\xi^a$, we may expand the parameters of the transformation $A_\xi(\xi)(x) \equiv A[\omega_\xi(x)]$ as $\omega^\alpha_\beta (x) = ...
\[ \xi^a \Omega^{\hat{\alpha} \hat{\beta}}(x) + \cdots, \] in terms of the functions

\[ \Omega^{\hat{\alpha} \hat{\beta}}_a \equiv \frac{\partial \omega^{\hat{\alpha} \hat{\beta}}}{\partial \xi^a} \bigg|_{\xi=0} = \left( \bar{e}_\mu^\hat{\alpha} k^\mu_{a,\nu} + \bar{e}_\nu^\hat{\beta} k^\mu_{a,\mu} \right) e^\nu_\chi \eta^{\hat{\lambda} \hat{\beta}} \] (25)

that are skew-symmetric, \( \Omega^{\hat{\alpha} \hat{\beta}}_a = -\Omega^{\hat{\beta} \hat{\alpha}}_a \), only if \( k_a \) are Killing vectors. In this case we obtain the basis-generators of the covariant reps. \[22\],

\[ \psi(1) = -ik^\mu_a \partial_\mu + \frac{1}{2} \Omega^{\hat{\alpha} \hat{\beta}}_a S^{\hat{\alpha} \hat{\beta}}_a, \] (26)

which can be put in covariant form \[24, 25\]. These generators are the principal conserved observables of the quantum theory which commute with the operator of the Dirac equation.

In a given frame, any conserved operator \( X \) gives rise to the conserved quantity \( C[X] = \langle \psi, X\psi \rangle \) derived with the help of the scalar product (19). This quantity is interpreted as the expectation value at the level of the relativistic quantum mechanics and after the second quantization becomes the corresponding one particle operators of the QFT \[15, 16\]. We have shown that the relativistic scalar product (19) is invariant under isometries, \( \langle T_\theta \psi, T_\theta \psi' \rangle = \langle \psi, \psi' \rangle \), while all the conserved observables (26) are self-adjoint with respect to this scalar product, \( \langle X_a \psi, \psi' \rangle = \langle \psi, X_a \psi' \rangle \).

The spatially flat FLRW spacetimes, \((\mathcal{M}, a)\), have, in general, the Euclidean isometry group \( E(3) \) formed by space translations and space rotations. The basis-generators of these isometries are the components

\[ P^i = -i \partial_i = -i a(t) \partial_i, \] (27)

of the momentum operator and those of the total angular momentum,

\[ J_i = L_i + S_i, \quad S_i = \frac{1}{2} \varepsilon_{ijk} S_{jk}, \] (28)

where \( L_i = -i \epsilon_{ijk} x^j \partial_k = -i \epsilon_{ijk} x^j \partial_k \) are the components of the orbital angular momentum which is not conserved in the Dirac theory. The conserved observables, \( P^i \) and \( J_i \), can generate freely many other conserved observables as, for example, the Pauli-Lubanski operator

\[ W = \vec{P} \cdot \vec{J} = \vec{P} \cdot \vec{S}, \] (29)

which allows one to define the helicity (as in Appendix A).
A special case is the dS spacetime having the scale factor

\[ a(t) = e^{\omega t} \rightarrow \frac{\dot{a}(t)}{a(t)} = \omega, \quad (30) \]

where \( \omega \) is the Hubble dS constant in out notation. The dS isometry group \( SO(1,4) \) lays out ten basis-generators, \( P^i, J_i \), three more Abelian generators \( Q^i \) and the energy operator which has different forms \([26, 12]\),

\[ H = i \partial_t + \omega \vec{X} \cdot \vec{P}, \quad \mathcal{H} = i \partial_t, \quad (31) \]

in the local charts \( \{ t, \vec{x} \} \) and respectively \( \{ t, \vec{x} \} \). \( \vec{X} \) is the usual multiplicative operator acting as \( (X^i f)(\vec{x}) = x^i f(\vec{x}) \).

### 2.4 Representations

The Dirac theory lays out the natural \( U_{em}(1) \) internal symmetry, giving rise to the conserved electromagnetic current density \( \bar{\psi} \gamma^\mu \psi \) whose sign may be changed by the charge conjugation \( \psi \rightarrow \psi^c = C \psi^* \), with \( C = i \gamma^2 \), since the \( \gamma \)-matrices satisfy \( C \gamma^\mu C = -\gamma^\mu \). Consequently, for every particular solution \( U \) of the Dirac equation there exists the charge conjugated solution \( V = C U^* \) regardless the geometry of the background. Thus the space of the solutions is split into the set of solutions of positive frequencies associated to particles and the set of their charge conjugated solutions which are of negative frequencies describing antiparticles. The problem is how a such basis can be defined in the space of solutions, separating the frequencies without ambiguities.

The traditional method is to look for a complete system of commuting conserved observables \( \{ A \} = \{ A_1, A_2, ...A_n \} \) able to determine a (generalized) basis formed by the solutions of the Dirac equation which, in addition, solve the common eigenvalue problems

\[ A_i U_\alpha = a_i U_\alpha, \quad A_i V_\alpha = \pm a_i V_\alpha, \quad i = 1, 2...n, \quad (32) \]

corresponding to the eigenvalues \( \alpha = \{ a_1, a_2, ...a_n \} \) that form the spectrum \( S = S_d \cup S_c \) which may have a discrete part \( S_d \) and a continuous one \( S_c \). In this manner all the integration constants get physical meaning as eigenvalues of these operators that have a precise physical interpretation.

When the system of observables is complete, this determine a basis of the space of solutions said to be of the rep. \( \{ A \} \). Unfortunately, in the case of our geometries we do not find such complete systems of operators since the isometry groups of the FLRW spacetimes, including the dS one, do not have...
Cartan sub-algebras with more than three generators while for completing such systems we need at least four generators, as the components of the four-momentum operator of the Minkowski spacetime. Thus we must make do with incomplete systems of three operators resorting to supplemental hypotheses for setting all the integration constants we need for separating the frequencies, determining thus the vacuum.

In a given rep. \{A\} the Dirac field can be expanded as

$$
\psi(x) = \psi^+(x) + \psi^-(x)
$$

$$
= \int_{\alpha \in S} U_\alpha(x) a(\alpha) + V_\alpha(x) b^*(\alpha),
$$

where we sum over the discrete part \(S_d\) and integrate over the continuous part \(S_c\) of the spectrum \(S\). The functions \(a\) and \(b\) are the particle and respectively antiparticle wave functions of the rep. \{A\}. Under canonical quantization, these functions become field operators, \(a \rightarrow a\) and \(b^* \rightarrow b^\dagger\), satisfying the canonical anti-commutation relations

$$
\{a(\alpha), a^\dagger(\alpha')\} = \{b(\alpha), b^\dagger(\alpha')\}
$$

$$
= \delta(\alpha, \alpha') = \begin{cases}
\delta_\alpha, \alpha' \in S_d, \\
\delta(\alpha - \alpha'), \alpha, \alpha' \in S_c,
\end{cases}
$$

requested by the Fermi-Dirac statistics, while the fundamental spinors \(U_\alpha\) and \(V_\alpha\) have to form an orthonormal basis of the rep. \{A\} complying with corresponding orthonormalization relations. For example, in the chart \(\{t, \vec{x}\}\) these spinors are orthonormal,

$$
\langle U_\alpha, U_{\alpha'} \rangle = \langle V_\alpha, V_{\alpha'} \rangle = \delta(\alpha, \alpha')
$$

$$
\langle U_\alpha, V_{\alpha'} \rangle = \langle V_\alpha, U_{\alpha'} \rangle = 0,
$$

with respect to the scalar product \(\langle \cdot, \cdot \rangle\) and satisfy the completeness condition

$$
\int_{\alpha \in S} U_\alpha(t, \vec{x}) U^\dagger_\alpha(t, \vec{x}') = \frac{1}{a(t)^3} \delta^3(\vec{x} - \vec{x}'),
$$

associated to this scalar product \(\langle \cdot, \cdot \rangle\).

In what follows we focus only on various solutions of the Dirac equation remaining at the level of the relativistic quantum mechanics where the Dirac
field $\psi$ depends on that the wave functions $a$ and $b$ that can be derived by using the inversion formulas

$$a(\alpha) = \langle U_\alpha, \psi \rangle, \quad b(\alpha) = \langle \psi, V_\alpha \rangle.$$  

(38)

Note that these functions transform under isometries alike according to a unitary rep. of the isometry group such that Eq. (33) can be seen as defining the equivalence between the covariant rep. (24) and the unitary one transforming the functions $a$ and $b$ [26, 15, 16].

3 Time evolution pictures

Working with different local charts we may face with some difficulties since at the quantum level the time-dependent coordinate transformations (4) are not compatible with the usual quantum formalism where it is easier to change the time evolution pictures rather than transforming the coordinates as $\vec{x} \rightarrow \vec{x}$. For this reason we defined two different time evolution pictures which prevent us to work with such coordinate transformations, remaining with an unique set of space coordinates, $\vec{x}$.

3.1 Related pictures

The first picture is the mentioned NP which is just the usual theory in the frame $\{t, \vec{x}; e\}$ where the Dirac field $\psi$ satisfies Eq. (20) that can be rewritten as

$$(E_D(t) - m) \psi(t, \vec{x}) = 0$$  

(39)

pointing out its operator

$$E_D(t) = i\gamma^0 \partial_t + \frac{3i}{2} \frac{\dot{a}(t)}{a(t)} \gamma^0 - \frac{1}{a(t)} \vec{\gamma} \cdot \vec{P}.$$  

(40)

which depends explicitly on time apart from the dS case when the Hubble function (30) becomes constant. The scalar product derived from Eq. (19) reads

$$\langle \psi, \psi' \rangle = \int d^3 x \ a(t)^3 \bar{\psi}(x) \gamma^0 \psi'(x).$$  

(41)

The second picture is the SP which governs the time evolution of the new field [12],

$$\psi_S(t, \vec{x}) = \psi \left( t, \frac{1}{a(t)} \vec{x} \right),$$  

(42)
according to Eq. (21) in which we substitute
\[ \vec{x} \rightarrow \vec{x}, \quad \partial_i \rightarrow \partial_i, \quad (43) \]
obtaining the Dirac equation of SP,
\[ (E_S^S(t) - m) \psi_S(t, \vec{x}) = 0. \quad (44) \]
with the new operator
\[ E_S^S(t) = i\gamma^0 \partial_t - \gamma^0 \frac{\dot{\alpha}(t)}{\alpha(t)} \left( \vec{X} \cdot \vec{P} - \frac{3i}{2} \right) - \vec{\gamma} \cdot \vec{P}. \quad (45) \]
The relativistic scalar product of SP can be derived after performing the substitution (43) in Eq. (19) obtaining the expression (12)
\[ \langle \psi, \psi' \rangle_S = \int d^3x \bar{\psi}(t, \vec{x}) \gamma^0 \psi'(t, \vec{x}), \quad (46) \]
which has the same form as in \((M, \eta)\). Therefore, the scalar products of these pictures are different generating two different types of Hermitian conjugations denoted by \( ^\dagger \) for NP and by \( ^\ddagger \) for SP.

The advantage of this approach is of working with only one set of coordinate and momentum operators, \( \vec{X} \) and \( \vec{P} \), which satisfy the canonical commutation rule \([X^i, P^j] = i\delta_{ij}\). These operators are Hermitian with respect to both the relativistic scalar products of NP and SP, \( \vec{X}^\dagger = \vec{X}^\ddagger = \vec{X} \), and \( \vec{P}^\dagger = \vec{P}^\ddagger = \vec{P} \). With their help, and with the Dirac matrices \( \gamma^\alpha \), we can generate freely the operator algebra \( A(\vec{X}, \vec{P}, \gamma) \) of our pictures, constituted by all the analytic functions of the mentioned operators and matrices [12, 16].

Of a special interest is the dilation generator,
\[ D = \vec{X} \cdot \vec{P}, \quad (47) \]
which satisfies the commutation rules \([D, X^i] = -iX^i \) and \([D, P^i] = iP^i \).
This operator is non-Hermitian, \( D^\dagger = D^\ddagger = D - 3i \), but the operator
\[ \hat{D} = D - \frac{3i}{2}, \quad (48) \]
which satisfies the same commutation rules has this property, \( \hat{D} = \hat{D}^\dagger = \hat{D}^\ddagger \).
Consequently, the Dirac operator of SP is Hermitian, \( E_S^S(t)^\dagger = E_S^S(t) \) while that of NP does not have this property since
\[ E_D(t)^\dagger = E_D(t) + 3i \frac{\dot{\alpha}(t)}{\alpha(t)} \gamma^0, \quad (49) \]
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as it results from Eq. (40).

For relating the above defined pictures we need a transformation operator
\( T : NP \rightarrow SP \) acting as \( A_S = T A T^{-1} \) for any operators \( A \in NP \) and \( A_S \in SP \). We observe that the transformation \( \psi_S(t, \vec{x}) = T(t) \psi(t, \vec{x}) = \psi(t, \frac{1}{a(t)} \vec{x}) \),

generated by the dilation operator (47), can take over the role of the coordinates transformation (4) transforming any analytic function \( F(\vec{X}) \) or \( G(\vec{P}) \) as

\[
T(t)F(\vec{X})T(t)^{-1} = F\left(\frac{1}{a(t)} \vec{X}\right),
\]

\[
T(t)G(\vec{P})T(t)^{-1} = G\left(a(t) \vec{P}\right).
\]

Therefore, we may write the desired transformation

\[
\psi_S(t, \vec{x}) = T(t)\psi(t, \vec{x}) = \psi\left(t, \frac{1}{a(t)} \vec{x}\right),
\]

which is in accordance with Eq. (42) and defines the coordinate and momentum operators of SP as,

\[
\vec{X}_S = T(t)\vec{X}T(t)^{-1} = \frac{1}{a(t)} \vec{X} \in SP,
\]

\[
\vec{P}_S = T(t)\vec{P}T(t)^{-1} = a(t) \vec{P} \in SP.
\]

In addition, we obtain the important transformation rule

\[
T(t)i\partial_t T(t)^{-1} = i\partial_t - \frac{\dot{a}(t)}{a(t)} D,
\]

which enables us to relate the operators of the Dirac equations of our pictures as

\[
E_D^S(t) = T(t)E_D(t)T(t)^{-1},
\]

pointing out their equivalence.

Obviously, this equivalence is not a unitary one since the transformation operator (50) is non-unitary. Indeed, by using Eq. (49) it is not difficult to deduce the following rule \( T(t)^\dagger = T(t)^\dagger a(t)^3 T(t)^{-1} \),

which shows that \( T(t) \) is non-unitary. This is not an impediment since we can obtain an unitary equivalence if we replace NP by its Minkowskian projection we will discuss later.
3.2 Energy and Hamiltonian operators

The energy operator is conserved only in two particular spatially flat FLRW spacetimes, namely the Minkowski and the dS manifolds which have larger isometry groups, the Poincaré group and respectively the $SO(1,4)$ ones. The energy operator of the Minkowski geometry is just the time-like component of the four momentum, $P^0 = i\partial_t$.

In the case of the dS geometry the energy operator is a Killing vector field which is time-like only inside the null cone of an observer staying at rest in origin [26]. This takes different forms depending on the local coordinates and implicitly on the time evolution picture we adopt. In SP the dS energy operator has the simpler form

$$H \rightarrow H_S = i\partial_t,$$

as it results from Eq. (59).

In what follows we generalize this result assuming that the energy operator is defined by Eqs. (59) in SP of any FLRW spacetime $(M,a)$. Consequently, in NP the energy operator becomes

$$H = T(t)^{-1}H_ST(t) = i\partial_t + \frac{\dot{a}(t)}{a(t)}D,$$

having the algebraic properties

$$[H, X] = -i\frac{\dot{a}(t)}{a(t)}X, \quad [H, \vec{P}] = i\frac{\dot{a}(t)}{a(t)}\vec{P}.$$  

Then, bearing in mind that $D$ is related to the Hermitian operator $\hat{D}$ as in Eq. (18) and observing that

$$(i\partial_t)^\dagger = i\partial_t, \quad (i\partial_t)^\dagger = i\partial_t + 3\frac{\dot{a}(t)}{a(t)}$$

we understand that both these operators are Hermitian, $H = H^\dagger$ and $H_S = H_S^\dagger$, with respect to the specific scalar products of these pictures, (11) and (16).

The above defined energy operators give the related eigenvalues problems

$$HF = EF, \quad H_SF_S = EF_S,$$

which are solved by the eigenfunctions,

$$F_S(t, \vec{x}) = f(\vec{x})e^{-iEt},$$  

$$F(t, \vec{x}) = T(t)^{-1}F_S(t, \vec{x}) = f(a(t)\vec{x})e^{-iEt},$$

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where $E \in \mathbb{R}$ is the energy while $f$ is an arbitrary function of $\vec{x}$. In general, these eigenfunctions cannot be solutions of the Dirac equation since the energy operators do not commute with $E_D$ or $E_D^S$ apart from the dS case when the energy is conserved.

Now we may define the Hamiltonian operators of our pictures allowing us to bring the Dirac equations (39) and (44) in the Schrödinger form,

\begin{align}
NP : \quad H\psi(t, \vec{x}) &= H_D(t)\psi(t, \vec{x}), \quad (66) \\
SP : \quad H_S\psi_S(t, \vec{x}) &= H_D^S(t)\psi_S(t, \vec{x}). \quad (67)
\end{align}

According to Eqs. (40) and (45) we obtain the related Hamiltonian operators,

\begin{align}
H_D(t) &= \frac{1}{a(t)}\gamma^0 \gamma^i P^i + \gamma^0 m + \frac{\dot{a}(t)}{a(t)} \hat{D} \\
&= H_D'(t) + H_{\text{int}}(t) \quad (68) \\
H_D^S(t) &= \gamma^0 \gamma^i P^i + \gamma^0 m + \frac{\dot{a}(t)}{a(t)} \hat{D} \\
&= H_D^0 + H_{\text{int}}(t) \quad (69)
\end{align}

where $\hat{D}$ is the Hermitian dilation generator (48).

Similar time evolution pictures can be defined in the charts with conformal time by substituting $t \to t_c$ according to Eqs. (7) in the expressions of all the operators considered here. Thus we obtain the new pictures namely, the natural picture with conformal time, $NP_c$, and its associated Schrödinger picture, $SP_c$.

It is remarkable that in SP we obtain a typical structure of a problem of perturbations since the Hamiltonian is formed by the usual Hamiltonian of the free Dirac field on Minkowski spacetime, $H_D^0 = \gamma^0 \gamma^i P^i + \gamma^0 m$, and the time-dependent interaction Hamiltonians due to the background gravity, $H_{\text{int}} = \frac{\dot{a}(t)}{a(t)} \hat{D}$, $H_{\text{int}} = \frac{\dot{a}(t_c)}{a(t_c)^2} \hat{D}$, which are proportional to the Hubble function. Thus the unperturbed problem is the Dirac free field on the Minkowski spacetime while the gravitational affect is encapsulated in the interaction Hamiltonian.

In NP we do not have this opportunity since the energy operator has the unusual form (60), depending on the Hubble function, while $H'_D(t)$ cannot be interpreted as an unperturbed Hamiltonian as long as this depends explicitly on $a(t)$. This picture, in which the momentum operator is simpler, is suitable for studying the plane or spherical wave solutions depending on the vector or scalar momentum.
3.3 Minkowskian projection

Technically speaking there is a transformation of the fields and operators of NP

\[ \psi \rightarrow \hat{\psi} = a(t)^{\frac{3}{2}} \psi, \quad X \rightarrow \hat{X} = a(t)^{\frac{3}{2}} X a(t)^{-\frac{3}{2}}, \]  

which brings the Dirac operator in the simpler form

\[ \hat{E}_D(t) = i\gamma^{0} \partial_t - \frac{1}{a(t)} \vec{\gamma} \cdot \vec{P}, \]  

while the scalar product becomes identical to that of SP since,

\[ \langle \psi, \psi' \rangle = \langle \hat{\psi}, \hat{\psi}' \rangle_S, \]

taking the same form as in \((M, \eta)\). For this reason we say that this is the Minkowskian projection (MP) of NP observing that this offers us the advantages of a simpler Dirac equation and a common scalar product. Now the Hermitian conjugation of the MP is similar to that of SP and, consequently, will be denoted alike with \(^\dagger\). The Dirac operator (72) is now Hermitian with respect to this scalar product. The transformations (71) do not affect the coordinate and momentum operators but change the energy operator,

\[ H \rightarrow \hat{H} = H - \frac{3i}{2} \frac{\dot{a}(t)}{a(t)} = i\partial_t + \frac{\dot{a}(t)}{a(t)} \hat{D}, \]

which depends now on the Hermitian dilation operator (48) instead of \(D\) as in Eq. (60). Therefore, the operator \(\hat{H}\) is Hermitian with respect to the scalar product (19).

The principal feature of the MP is that this is equivalent to SP through the unitary transformation \(\psi_S(t, \vec{x}) = U(t) \hat{\psi}(t, \vec{x})\) whose operator

\[ U(t) = a(t)^{-\frac{3}{2}} T(t) = e^{-i \ln(a(t)) \hat{D}}, \]

generated now by the Hermitian dilation operator \(\hat{\Delta}\), is unitary with respect to the scalar product (19).

Another opportunity is of comparing the states of NP of \((M, a)\) with the states prepared in the Minkowski spacetime \((M, \eta)\). This can be done if we chose the same coordinates on both these manifolds since the scalar products giving the quantities with physical meaning are similar. Note that this choice is possible at any time since the manifolds \((M, a)\) are local Minkowskian. Therefore, given a Dirac states whose spinor in the MP is \(\hat{\psi}\) and a spinor \(\psi_M\) of a Minkowski state we can construct the time dependent quantity

\[ \mathcal{P}(t) = \left| \langle \hat{\psi}(t), \psi_M(t) \rangle_S \right|^2, \]
by using the scalar product \((19)\). This quantity can be interpreted as the probability of measuring at the time \(t\) the parameters of the state \(\psi_M\) in the state \(\hat{\psi}\) prepared in \((M, a)\). With their help we can imagine detectors measuring Minkowskian parameters on FLRW spacetimes. Thus the Minkowskian projection of NP can be a helpful auxiliary tool not only for solving the Dirac equation but for refining the physical interpretation too.

4 Representations in NP

As mentioned, the conserved generators \(\hat{P}\) and \(\hat{J}\) of the \(E(3)\) isometries of the FLRW manifolds \((M, a)\) are not able to generate freely an algebra rich enough for selecting complete systems of commuting operators. Therefore, we must make do with the incomplete system giving the plane waves of the momentum-helicity or those of the momentum-spin reps. and with the spherical waves of the total angular momentum rep..

4.1 Plane P-waves

In the frames \(\{t, \vec{x}; e\}\) of NP of the spacetimes \((M, a)\) the general solution of the Dirac equation \((39)\) may be written as a mode integral,

\[
\psi(t, \vec{x}) = \psi^{(+)}(t, \vec{x}) + \psi^{(-)}(t, \vec{x}) = \int d^3p \sum_{\sigma} \left[ U_{\vec{p}, \sigma}(x) a(\vec{p}, \sigma) + V_{\vec{p}, \sigma}(x) b^*(\vec{p}, \sigma) \right],
\]

in terms of the fundamental spinors \(U_{\vec{p}, \sigma}\) and \(V_{\vec{p}, \sigma}\) of positive and respectively negative frequencies which are plane waves solutions of the Dirac equation \((39)\), defined as common eigenspinors of the momentum components \(\{P^1, P^2, P^3\}\) corresponding to the eigenvalues \(\{p^1, p^2, p^3\}\) representing the components of the conserved momentum \(\vec{p}\). In addition, these solutions depend on a polarization \(\sigma\) that can be defined in different manners as we show in the Appendix A.

These spinors form an orthonormal basis with respect to the scalar product \((11)\) being related through the charge conjugation,

\[
V_{\vec{p}, \sigma}(t, \vec{x}) = U_{\vec{p}, \sigma}^c(t, \vec{x}) = i\gamma^2 \left[ U_{\vec{p}, \sigma}(t, \vec{x}) \right]^* ,
\]

and satisfying the orthogonality relations

\[
\langle U_{\vec{p}, \sigma}, U_{\vec{p}', \sigma'} \rangle = \langle V_{\vec{p}, \sigma}, V_{\vec{p}', \sigma'} \rangle = \delta_{\sigma \sigma'} \delta^3(\vec{p} - \vec{p}') \quad (79)
\]

\[
\langle U_{\vec{p}, \sigma}, V_{\vec{p}', \sigma'} \rangle = \langle V_{\vec{p}, \sigma}, U_{\vec{p}', \sigma'} \rangle = 0 \quad (80)
\]

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and a completeness condition of the form (37). This basis defines the momentum rep. that depends, in addition, on the way in which the polarization is defined.

In the standard rep. of the Dirac matrices (with diagonal $\gamma^0$) the general form of the fundamental spinors in momentum rep.,

\begin{align*}
U_{\vec{p},\sigma}(t, \vec{x}) &= \frac{e^{i\vec{p}\cdot\vec{x}}}{[2\pi a(t)]^{\frac{3}{2}}} \begin{pmatrix}
    u^+_{\vec{p}}(t) \xi_{\sigma} \\
    u^-_{\vec{p}}(t) \vec{\sigma} \cdot \vec{n}_p \xi_{\sigma}
\end{pmatrix}, \\
V_{\vec{p},\sigma}(t, \vec{x}) &= \frac{e^{-i\vec{p}\cdot\vec{x}}}{[2\pi a(t)]^{\frac{3}{2}}} \begin{pmatrix}
    v^+_{\vec{p}}(t) \vec{\sigma} \cdot \vec{n}_p \eta_{\sigma} \\
    v^-_{\vec{p}}(t) \eta_{\sigma}
\end{pmatrix},
\end{align*}

is determined by the time modulation functions (t.m.f.) $u^\pm_{\vec{p}}(t)$ and $v^\pm_{\vec{p}}(t)$ that depend only on $t$ and $p = |\vec{p}|$. The notation $\vec{n}_p$ stands for the unit vector of the momentum direction while $\xi_{\sigma}$ and $\eta_{\sigma} = i\sigma_3(\xi_{\sigma})^*$ are Pauli spinors supposed to be correctly normalized, $\xi^+_{\sigma} \xi^-_{\sigma'} = \eta^+_{\sigma} \eta^-_{\sigma'} = \delta_{\sigma\sigma'}$. In addition, they must satisfy the completeness condition

$$\sum_{\sigma} \xi_{\sigma} \xi^+_{\sigma} = \sum_{\sigma} \eta_{\sigma} \eta^+_{\sigma} = 1_{2\times2}. \quad (83)$$

The form of these spinors depends on the direction of the spin projection which can be chosen in many ways. In Ref. [10] we considered the Pauli spinors of the helicity basis but here we use the spin basis presented in Appendix A with polarizations along the third axis of the rest frame taking $\vec{n} = \vec{e}_3$ in Eqs. (A.1) and (A.2).

It is worth pointing out that the helicity is related to the eigenvalues of the Pauli-Lunabiski operator (29) while for the polarization $\sigma$ we do not have a differential operator since this is defined in the rest frame. Nevertheless, an operator of the spin projection, $S_3$, can be defined in QFT giving its spectral representation. For this reason we consider here that the momentum-spin rep. is given by the operators $\{\vec{P}, S_3\}$ whose common eigenspinors (S1) and (S2) depend on the eigenvalues $\{\vec{p}, \sigma\}$. However, this set of operators is not complete such that there remain some undetermined integration constants.

The t.m.f. $u^\pm_{\vec{p}}(t)$ and $v^\pm_{\vec{p}}(t)$ can be derived by substituting Eqs. (S1) and (S2) in the Dirac equation (20). Then, after a few manipulation, we find the systems of the first order differential equations

\begin{align*}
    a(t) (i\partial_t \mp m) u^\pm_{\vec{p}}(t) &= p u^\mp_{\vec{p}}(t), \quad (84) \\
    a(t) (i\partial_t \mp m) v^\pm_{\vec{p}}(t) &= -p v^\mp_{\vec{p}}(t), \quad (85)
\end{align*}
in the chart with the proper time or the equivalent system in the conformal chart,

\[
\begin{align*}
[i\partial_t \mp m a(t_c)] u_p^\pm (t_c) &= \pm p u_p^\mp (t_c), \quad (86) \\
[i\partial_t \mp m a(t_c)] v_p^\pm (t_c) &= -p v_p^\mp (t_c), \quad (87)
\end{align*}
\]

which govern the time modulation of the free Dirac field on any spatially flat FLRW manifold.

The solutions of these systems depend on integration constants that must be selected according to the charge conjugation (78) which gives the mandatory condition

\[
v_p^\pm = [u_p^\mp]^*. \quad (88)
\]

The remaining normalization constants can be fixed since the prime integrals of the systems (84) and (85), \( \partial_t(|u_p^\pm|^2 + |u_p^-|^2) = \partial_t(|v_p^\pm|^2 + |v_p^-|^2) = 0 \), allow us to impose the normalization conditions

\[
|u_p^\pm|^2 + |u_p^-|^2 = |v_p^\pm|^2 + |v_p^-|^2 = 1, \quad (89)
\]

which guarantee that Eqs. (79) and (80) are accomplished. In fact, we can focus only on the functions \( u_p^\pm \) since the functions \( v_p^\pm \) result from Eq. (88). These functions can be organized as a 2-dimensional space of complex valued vectors \( u_p = [u_p^+, u_p^-]^T \) and \( v_p = [v_p^+, v_p^-]^T \) equipped with the inner product

\[
(u_p, u'_p) = (v_p, v'_p) = (u_p^+)^* u'_p^+ + (u_p^-)^* u'_p^- . \quad (90)
\]

Two sets of t.m.f. \( u_p^a^\pm, a = 1, 2 \), solutions of the systems (84) or (86), which satisfy

\[
(u_p^a, u_p^b) = \delta_{ab}, a, b = 1, 2, \quad (91)
\]

are orthonormal generating a system of orthonormal fundamental spinors (81) and (82). Any linear combination

\[
u_p^\pm = c_1 u_p^1^\pm + c_2 u_p^2^\pm, \quad c_1, c_2 \in \mathbb{C}, \quad (92)
\]

give rise to normalized spinors only if

\[
|c_1|^2 + |c_2|^2 = 1. \quad (93)
\]

Thus we can control the orthogonality of the fundamental spinors exclusively at the level of the t.m.f., without calculating scalar products.
A special case is that of the rest frame where the Dirac equation in momentum-spin rep. for \( \vec{p} = 0 \) can be solved analytically carrying out the normalized fundamental spinors of the rest frame,

\[
U_{0,\sigma}(t, \vec{x}) = \frac{e^{-imt}}{[2\pi a(t)]^{\frac{1}{2}}} \begin{pmatrix} \xi_{\sigma} \\ 0 \end{pmatrix}, \quad (94)
\]

\[
V_{0,\sigma}(t, \vec{x}) = \frac{e^{*imt}}{[2\pi a(t)]^{\frac{1}{2}}} \begin{pmatrix} 0 \\ \eta_{\sigma} \end{pmatrix}. \quad (95)
\]

However, in the rest frames the energy operator of this picture (60) takes the simple form \( H = i\partial_t \) since \( D \to 0 \) when the momentum vanishes. Therefore, the rest frame spinors satisfy the eigenvalue problems \( HU_{0,\sigma}(t, \vec{x}) = E^+(t)U_{0,\sigma}(t, \vec{x}) \) and \( HV_{0,\sigma}(t, \vec{x}) = E^-(t)V_{0,\sigma}(t, \vec{x}) \) defining the time-dependent rest energies

\[
E^\pm_0(t) = \pm m - \frac{3i}{2} \frac{\dot{a}(t)}{a(t)}, \quad (96)
\]

whose real parts are just the rest energies \( \pm m \) of special relativity while the imaginary terms are due to the evolution of the background. Thus we generalize to any FLRW manifold the result we obtained for the dS spacetime [26].

Note that the study of the solutions in rest frames can be done only in the momentum-spin rep. since in the momentum-helicity one the helicity is not defined for vanishing momentum.

### 4.2 Spherical P-waves

The spherical waves are the solutions with spherical symmetry of the Dirac equation (39) that can be derived after introducing the spherical space coordinates of the frame \{t, r, \theta, \phi; e\} of \((M, a)\). This can be done by rewriting the Dirac operator (40) as [11]

\[
E_{D}(t) = i\gamma^0 \partial_t + \frac{3i}{2} \frac{\dot{a}(t)}{a(t)} \gamma^0 \\
+ \frac{1}{a(t)} \left( i \frac{1}{r^2} (\gamma^i x^i) (x^i \partial_i + 1) + i \frac{1}{r^2} \gamma^0 (\gamma^i x^i) K \right), \quad (97)
\]

where \( r = |\vec{x}| \), pointing out the angular Dirac operator,

\[
K = \gamma^0 \left( 2 \vec{L} \cdot \vec{S} + 1 \right), \quad (98)
\]
which encapsulates the action of all the angular operators allowing us to separate the spherical variables \((r, \theta, \phi)\) associated to \(\vec{x}\). The general solution of this equation,

\[
\psi(t, r, \theta, \phi) = \psi^{(+)}(t, r, \theta, \phi) + \psi^{(-)}(t, r, \theta, \phi)
\]

\[
= \int_0^\infty dp \sum_{\kappa_j, m_j} U_{p, \kappa_j, m_j}(t, r, \theta, \phi) a(p, \kappa_j, m_j)
\]

\[
+ \int_0^\infty dp \sum_{\kappa_j, m_j} V_{p, \kappa_j, m_j}(t, r, \theta, \phi) b^*(p, \kappa_j, m_j),
\]

where \(U_{p, \kappa_j, m_j}\) are the fundamental solutions of positive frequencies defined as common eigenspinors of the set \(\{\vec{P}^2, K, J_3\}\) corresponding to the eigenvalues \(\{p^2, - \kappa_j, m_j\}\) where \(\kappa_j = \pm (j + \frac{1}{2})\) \[34\]. The eigenspinors of negative frequencies,

\[
V_{p, \kappa_j, m_j}(t, r, \theta, \phi) = i \gamma^2 U_{p, \kappa_j, m_j}(t, r, \theta, \phi)^*,
\]

are defined with the help of the charge conjugation as in the case of the plane waves. All these spinors may be organized as an orthonormal basis satisfying,

\[
\langle U_{p, \kappa_j, m_j}, U_{p', \kappa_j', m_j'} \rangle = \langle V_{p, \kappa_j, m_j}, V_{p', \kappa_j', m_j'} \rangle = \delta_{\kappa_j, \kappa_j'} \delta_{m_j, m_j'} \delta(p - p'),
\]

\[
\langle U_{p, \kappa_j, m_j}, V_{p', \kappa_j', m_j'} \rangle = \langle V_{p, \kappa_j, m_j}, U_{p', \kappa_j', m_j'} \rangle = 0,
\]

with respect to the relativistic scalar product \[(41)\] that now reads

\[
\langle \psi, \psi' \rangle = \int r^2 dr \int_{S^2} d\Omega a(t)^3 \overline{\psi}(t, r, \theta, \phi) \gamma^0 \psi'(t, r, \theta, \phi),
\]

where \(d\Omega = d(\cos \theta)d\phi\) is the measure of integration on the sphere \(S^2\). These fundamental spinors form the basis of the rep. given by the set of operators \(\{\vec{P}^2, K, J_3\}\) which is not complete leaving undetermined some integration constants.

For solving the above eigenvalue problems it is convenient to separate the spherical variables looking for particular solutions of positive frequencies of the form

\[
U_{p, \kappa_j, m_j}(x) = \frac{1}{a(t)^\frac{3}{2} r} \left[ f^+_{p, \kappa_j}(t, r) \Phi^+_{\kappa_j, m_j}(\theta, \phi) \right.
\]

\[
+ \left. f^-_{p, \kappa_j}(t, r) \Phi^-_{\kappa_j, m_j}(\theta, \phi) \right],
\]

\(23\)
where $\Phi_{\kappa_j,m_j}^\pm$ are the orthonormal Dirac spherical spinors of special relativity that solve the eigenvalue problems of the operators $\vec{J}^2$, $K$ and $J_3$ as presented in Appendix B. Then, after a little calculation by using the identities (B.6) we derive the system

$$a(t) \left(i\partial_t \mp m\right)f_{p,\kappa_j}^\pm(t, r) = \left(\mp \partial_r + \frac{\kappa_j}{r}\right)f_{p,\kappa_j}^\mp(t, r),$$  \hspace{1cm} (105)$$

resulted from the Dirac equation (97). In addition, the eigenvalue problem of $\vec{P}^2$ leads to the supplemental radial equations

$$\left[-\partial^2_r + \frac{\kappa_j(\kappa_j \pm 1)}{r^2}\right]\rho_{p,\kappa_j}^\pm(t, r) = p^2 \rho_{p,\kappa_j}^\mp(t, r),$$  \hspace{1cm} (106)$$

since the spinors $\Phi_{\kappa_j,m_j}^\pm$ are eigenfunctions of $\vec{L}^2$ corresponding to the eigenvalues $\kappa_j(\kappa_j \pm 1)$.

Under such circumstances, Eqs. (105) and (106) can be solved separating the variables as,

$$f_{p,\kappa_j}^\pm(t, r) = u_{p}^\pm(t)\rho_{p,\kappa_j}^\pm(r),$$  \hspace{1cm} (107)$$

finding that the new functions satisfy

$$a(t) \left(i\partial_t \mp m\right)u_{p}^\pm(t) = p u_{p}^\mp(t),$$  \hspace{1cm} (108)$$

$$\left(\pm \partial_r + \frac{\kappa_j}{r}\right)\rho_{p,\kappa_j}^\pm(r) = p \rho_{p,\kappa_j}^\mp(r).$$  \hspace{1cm} (109)$$

Hereby it results that the t.m.f. $u_{p}^\pm$ are the same as in the case of the plane waves satisfying similar equations. Moreover, we assume that these are related as in Eq. (88) and satisfy the normalization conditions (89). Thus the spherical waves have the same t.m.f. as the plane ones as was expected since these have different space shapes but the same time evolution.

The radial equations (109) are independent on this time evolution such that we can solve them in terms of Bessel functions. There are two particular solutions

$$\rho_{p,\kappa_j}^{1\pm}(r) = \sqrt{pr}J_{\kappa_j \pm \frac{\pm}{2}}(pr),$$  \hspace{1cm} (110)$$

$$\rho_{p,\kappa_j}^{2\pm}(r) = \pm\sqrt{pr}J_{-\kappa_j \pm \frac{\pm}{2}}(pr)$$  \hspace{1cm} (111)$$

which satisfy

$$\int_{0}^{\infty} \rho_{p,\kappa_j}^{a\pm}(pr)\rho_{p',\kappa_j}^{a\pm}(p'r)dr = \delta(p - p'), \quad a = 1, 2,$$  \hspace{1cm} (112)$$

These solutions satisfy the boundary conditions for the spherical waves and the plane waves as expected.
thanks to the normalization factor \( \sqrt{p} \) introduced above. The general solution,

\[ \rho_{p,\kappa_j}^\pm = \hat{c}_1 \rho_{p,\kappa_j}^1 \pm \hat{c}_2 \rho_{p,\kappa_j}^2 \]  

keeps this property only if the new integration constants obey \( |\hat{c}_1|^2 + |\hat{c}_2|^2 = 1 \). We get thus a new integration constant that must be fixed by using supplemental criteria. For example, if we look for solution regular in origin then we must take

\[ \hat{c}_1 = \frac{1 + \text{sign} \kappa_j}{2}, \quad \hat{c}_2 = \frac{1 - \text{sign} \kappa_j}{2}, \]  

since the Bessel functions behave as in Eq. (C.8). Note that this radial solution is more general that in Ref. [11] where we adopted the particular version of Ref. [5].

Finally, by gathering all the above results and taking into account that the charge conjugation acts on the Dirac spherical spinors as in Eq. (B.8), we may write the definitive form of the fundamental spinors

\[ U_{p,\kappa_j,m_j}(x) = \frac{1}{a(t)^{\frac{3}{2}} r^1} \left[ u^+_p(t) \rho^+_{p,\kappa_j}(r) \Phi^+_{\kappa_j,m_j}(\theta, \phi) \right. \]

\[ + v^-_p(t) \rho^-_{p,\kappa_j}(r) \Phi^-_{\kappa_j,m_j}(\theta, \phi) \]  

\[ V_{p,\kappa_j,m_j}(x) = \frac{(-1)^m_j}{a(t)^{\frac{3}{2}} r^1} \left[ -u^+_p(t) \rho^-_{p,-\kappa_j}(r) \Phi^+_{-\kappa_j,-m_j}(\theta, \phi) \right. \]

\[ + v^-_p(t) \rho^+_{p,-\kappa_j}(r) \Phi^-_{-\kappa_j,-m_j}(\theta, \phi) \]  

which satisfy the charge conjugation symmetry (100) and the orthonormalization rules (101) and (102) if the t.m.f. comply with Eqs. (88) and (89). Moreover, if we have two sets of orthonormal t.m.f. in the sense of Eq. (91) then we may construct new solutions as linear combinations in similar conditions as in the case of the plane waves.

### 4.3 Adiabatic and rest frame vacua

The t.m.f. \( u_p^\pm(t) \) or \( u_p^\pm(t_c) \), are solutions of the systems (S1) or (S6) satisfying the conditions (S8) and (S9). Unfortunately, these are not enough for determining completely these functions such that a supplemental physical hypothesis is required. This is just the criterion of separating the positive and negative frequencies defining thus the vacuum in the momentum reps..

The vacuum usually considered in Dirac theories is the traditional adiabatic vacuum (a.v.) of the Bunch-Davies type similar to those intensively
studied in the case of the scalar fields \[19\]. This can be defined for any FLRW manifold for which the scale factor has the asymptotic behavior

\[\lim_{t_c \to -\infty} a(t_c) = 0.\]  

(117)

Then the asymptotic form of the system (86),

\[i\partial_{t_c} u^\pm_p(t_c) = pu^\pm_p(t_c),\]  

(118)

gives the behavior of the modulation functions,

\[u^\pm_p(t_c) \sim c_1 e^{-ipt_c} \pm c_2 e^{ipt_c},\]  

(119)

for \(t_c \to -\infty\). According to the common definition, the a.v. is set when \(c_2 = 0\) since then the modulation functions, \(u^\pm_p(t_c) = u^\pm_p(t_c)\), describe a massless particle assumed to be of genuine positive frequency. Thus the general condition of selecting the a.v. of the Dirac field on FLRW spacetimes takes the simple form \[21\]

\[u^p_p(t_c, m) = u^{-p}_{-p}(t_c, -m)\]  

(120)

and similarly for the functions \(v^\pm_p(t_c)\).

The major difficulty of the a.v. as defined above is that in the momentum-spin rep. we cannot reach the rest frame limit. Indeed, for \(p \to 0\) the condition (120) gives the normalized functions

\[\lim_{p \to 0} u^+_p(t) = \frac{1}{\sqrt{2}} e^{-imt}, \quad \lim_{p \to 0} u^-_p(t) = \frac{1}{\sqrt{2}} e^{imt},\]  

(121)

while the limit of \(\vec{\sigma} \cdot \vec{p}\) remains undetermined. Moreover, if we force this limit to zero we obtain different normalization factors \[28\] such that the limits of the fundamental spinors will differ from the correct rest spinors \[94\] and \[95\], mixing thus positive and negative frequencies. Another impediment is that the a.v. cannot be defined for manifolds whose scale factors do not have a suitable asymptotic behavior \[117\].

The solution was to define a new vacuum able to separate the frequencies in any the rest frame of the momentum-spin rep. imposing the conditions

\[\lim_{\vec{p} \to 0} U_{\vec{p},\sigma}(t, \vec{x}) = U_{0,\sigma}(t, \vec{x}) ,\]  

(122)

\[\lim_{\vec{p} \to 0} V_{\vec{p},\sigma}(t, \vec{x}) = V_{0,\sigma}(t, \vec{x}) ,\]  

(123)
according to Eqs. (94) and (95). These are accomplished if we require the normalized t.m.f. to satisfy

\[ \lim_{p \to 0} u_p^-(t) = \lim_{p \to 0} v_p^+(t) = 0, \]

(124)
since then the contribution of the matrix \( \vec{\sigma} \cdot \vec{p} \) is eliminated. We say that these conditions define the rest frame vacuum (r.f.v.) which, in general, is different from the a.v. apart from the Minkowski case in which the t.m.f.

\[ u_\pm(t) = v_\mp(t)^* = \sqrt{\frac{E(p) \pm m}{2E(p)}} e^{-iE(p)t}, \]

(125)
depending on the energy \( E(p) = \sqrt{p^2 + m^2} \), satisfy simultaneously both the conditions (120) and (124).

The above definitions of the a.v. and r.f.v. cannot be applied directly to the spherical waves even though these have the same t.m.f.. This is because we have, in addition, the integration constants of the radial functions which have to be fixed. In the case of the a.v. there are no restrictions upon the radial functions such that we can take the convenient t.m.f. according to Eq. (120) but any constants \( \hat{c}_1 \) and \( \hat{c}_2 \). This is an example in which the frequency separation is not enough for determining all the integration constants. In this case a good choice is as in Eq. (114) since then the radial functions are regular in \( r = 0 \).

In contrast, the restrictions imposed by the r.f.v. are more effective. This is because in the rest frame, for \( p \to 0 \), the angular momentum vanishes, \( \vec{L} = 0 \), and, consequently, \( K \to \gamma^0 0 \) which means that \( \kappa_j = -1 \) (corresponding to \( l = 0 \)). On the other hand, in this limit the radial function \( \rho^{-1}_{j,-1} \) is a constant which must remain finite while the other one, \( \rho^0_{-1} \), is eliminated if we impose the condition (124). Taking into account that the Bessel functions behave as in Eq. (C.8) we draw the conclusion that the only possible choice is

\[ \hat{c}_1 = 1, \quad \hat{c}_2 = 0. \]

(126)
Thus by setting the r.f.v. we determine all the integration constants.

### 4.4 Example I: Milne-type universe

Let us consider the simple example we proposed recently of a (1 + 3)-dimensional spatially flat FLRW manifold \((M, a)\) with the Milne-type scale factor \( a(t) = \omega t \) defined for \( t \in (0, \infty) \) such that the conformal time,

\[ t_c = \int_0^t \frac{dt}{a(t)} = \frac{1}{\omega} \ln(\omega t) \to a(t_c) \equiv a[t(t_c)] = e^{\omega t_c}, \]

(127)
can take any real value, \( t_c \in (-\infty, \infty) \), and \( a(t_c) \) satisfies the condition (117). Note that the free parameter \( \omega \), was introduced from dimensional considerations. We remind the reader that in the case of the genuine Milne’s universe (of negative space curvature but globally flat) one must set \( \omega = 1 \) for eliminating the gravitational sources [20].

In contrast, our spacetime \((M, a)\) is produced by isotropic gravitational sources, i.e. the density \( \rho \) and pressure \( p \), evolving in time as

\[
\rho = \frac{3}{8\pi G} \frac{1}{t^2}, \quad p = -\frac{1}{8\pi G} \frac{1}{t^2},
\]

and vanishing for \( t \to \infty \). These sources govern the expansion of \( M \) that can be better observed in the chart \( \{t, \vec{x}\} \), of ‘observed’ space coordinates \( \vec{x}^i = \omega t x^i \), where the line element

\[
ds^2 = \left(1 - \frac{1}{t^2} \vec{x} \cdot \vec{x}\right) dt^2 + 2 \vec{x} \cdot d\vec{x} \frac{dt}{t} - d\vec{x} \cdot d\vec{x},
\]

lays out an expanding horizon at \( |\vec{x}| = t \) and tends to the Minkowski spacetime when \( t \to \infty \) and the gravitational sources vanish.

On this manifold, we chose the frame \( \{t, \vec{x}; e\} \) where the system (84) can be analytically solved finding two particular solutions [30]

\[
u_1^\pm(t) = \sqrt{\frac{mt}{2\pi}} \left[K_{\nu_+}(p)(imt) \pm K_{\nu_-}(p)(imt)\right],
\]

\[
u_2^\pm(t) = \sqrt{\frac{mt}{2\pi}} \left[K_{\nu_+}(p)(-imt) \mp K_{\nu_-}(p)(-imt)\right],
\]

expressed in terms of the modified Bessel functions, \( K_{\nu_\pm}(p) \), of the orders \( \nu_\pm(p) = \frac{1}{2} \pm i \frac{p}{\omega} \), presented in Appendix C. These solutions satisfy Eqs. (91) as it results from the identity (C.6) with \( \mu = \frac{2}{\omega} \). Therefore, they are orthonormal such the general solution

\[
u_p^\pm = (\nu_p^\mp)^* = c_1 \nu_1^p \pm + c_2 \nu_2^p \pm,
\]

is normalized only if the integration constants satisfy the condition (93).

In the conformal chart where \( a(t_c) = e^{\omega t_c} \) satisfies the asymptotic condition (117), we can introduce the a.v. imposing the condition (120) which yields \( c_1 = c_2 = \frac{1}{\sqrt{2}} \). The r.f.v. is given by \( c_1 = 1 \) and \( c_2 = 0 \) since in the rest frame only the t.m.f. \( u_p^- \) satisfies the condition (121) vanishing in the rest limit. Thus the the t.m.f. of the a.v. and r.f.v. are defined for both the types of solutions, plane and spherical waves. In addition, for the r.f.v. of
the spherical waves we must set the radial integration constants as in Eq. (126) while in the a.v. these remain arbitrary.

It is worth pointing out that in the chiral rep. of the Dirac matrices (with diagonal $\gamma^5$) the fundamental spinors of the plane waves in r.f.v. take the simple form \[30\]

$$U_{\vec{p},\sigma}(x) = \sqrt{\frac{mt}{\pi}} e^{i\vec{p} \cdot \vec{x}} \begin{pmatrix} K_{\sigma-i\frac{p}{m}}(imt) \xi_\sigma(p) \\ K_{\sigma+i\frac{p}{m}}(imt) \xi_\sigma(p) \end{pmatrix}, \quad (133)$$

$$V_{\vec{p},\sigma}(x) = \sqrt{\frac{mt}{\pi}} e^{-i\vec{p} \cdot \vec{x}} \begin{pmatrix} K_{\sigma-i\frac{p}{m}}(-imt) \eta_\sigma(p) \\ -K_{\sigma+i\frac{p}{m}}(-imt) \eta_\sigma(p) \end{pmatrix}, \quad (134)$$

that can be used in applications \[30\].

4.5 Example II: de Sitter expanding universe

Another example is the well-studied dS expanding universe defined as the expanding portion of the dS manifold where the scale factor has the form \[30\]. In the frame $\{t_c, \vec{x}; e\}$ of the conformal time,

$$t_c = -\frac{1}{\omega} e^{-\omega t} \in (-\infty, 0] \quad \rightarrow \quad a(t_c) = -\frac{1}{\omega t_c}. \quad (135)$$

we may derive two particular solutions of the system \[86\],

$$u_{1,\pm}^1(t_c) = \sqrt{-\frac{pt_c}{\pi}} K_{\nu_{\pm}}(ipt_c), \quad (136)$$

$$u_{2,\pm}^2(t_c) = \pm \sqrt{-\frac{pt_c}{\pi}} K_{\nu_{\pm}}(-ipt_c), \quad (137)$$

expressed in terms of modified Bessel functions of the orders $\nu_{\pm} = \frac{1}{2} \pm i\mu$ with $\mu = \frac{m}{\omega}$. According to Eq. \[C.6\] we find that these t.m.f. are orthonormal, satisfying Eqs. \[91\], giving rise to the particular spinors

$$U_{\vec{p},\sigma}^{1/2}(t, \vec{x}) = (\omega t_c)^2 \frac{e^{i\vec{p} \cdot \vec{x}}}{(2\pi)^2 \sqrt{\frac{p}{\pi\omega}}} \sqrt{\frac{p}{\pi\omega}}$$

$$\times \begin{pmatrix} K_{\nu_{\pm}}(\pm ipt_c) \xi_\sigma \\ \pm K_{\nu_{\pm}}(\pm ipt_c) \vec{n}_{\mu} \cdot \vec{\xi}_\sigma \end{pmatrix}, \quad (138)$$

which are orthonormal. Then the linear combination \[92\] gives the general solution

$$U_{\vec{p},\sigma} = c_1 U_{\vec{p},\sigma}^1 + c_2 U_{\vec{p},\sigma}^2, \quad (139)$$
which are normalized only if the integration constants satisfy the condition (93). The corresponding spinors of negative frequencies can be obtained from Eq. (78).

The a.v. can be defined simply by choosing \( c_1 = 1 \) and \( c_2 = 0 \) as in Ref. [10] since then the condition (120) is accomplished as we can deduce from Eq. (C.7). This vacuum is different from the rest frame one which must comply with the condition (124). Taking into account that the Bessel functions behave as in Eq. (C.8) we find that now we obtain the constants

\[
 c_1 = \frac{e^{\pi \mu} p^{-i \mu}}{\sqrt{1 + e^{2 \pi \mu}}} , \quad c_2 = \frac{ip^{-i \mu}}{\sqrt{1 + e^{2 \pi \mu}}} .
\]  

(140)

that can be seen as the Bogolyubov coefficients of the transformation between the orthonormal bases corresponding to the a.v. and r.f.v. Furthermore, by using the connection formula (C.3), we obtain the definitive form of the t.m.f. (92) in the r.f.v. as [21]

\[
u_{\pm}^p(t_c) = \pm \frac{\sqrt{-\pi t_c}}{\sqrt{1 + e^{2 \pi \mu}}} I_{\frac{\pi}{2}}(i p t_c) \]  

(141)

which have the remarkable property

\[
\lim_{t_c \to 0} |u_{\pm}^p(t_c)| = 1 , \quad \lim_{t_c \to 0} u_{\pm}^p(t_c) = 0 ,
\]

(142)

that may be interpreted as an adiabatic condition for \( t \to \infty \) instead of \( t \to -\infty \). The t.m.f. of the negative frequencies have to be calculated according to Eq. (88). For the spherical waves we have to use the same t.m.f. with arbitrary radial integration constants in the a.v. or satisfying the condition (126) if we set the r.f.v..

We must specify that these t.m.f. are defined up to an arbitrary phase factor depending on \( p \) which may assure the correct flat limits of the plane or spherical waves, determining the forms of the one particle operators of the QFT [31].

5 Energy representations in de Sitter expanding universe

We studied so far the solutions of the Dirac equation that can be obtained separating the Cartesian or spherical variables in NP. All these solutions depend on the vector or scalar momentum which is conserved on any FLRW spacetime. However, on the dS expanding universe we have, in addition, a
conserved energy operator that in the SP takes the familiar form (59). This suggest us to look for states of given energy that could be derived on this time evolution picture of the dS manifold.

5.1 Plane E-waves in SP

We consider first the plane waves in the dS expanding universe where from Eqs. (61) and (30) we deduce the commutation relation \([H, P^i] = i\omega P^i\) showing that the energy and the momentum components cannot be measured simultaneously with a desired accuracy [26]. Nevertheless, this commutation relation does not affect the direction of the momentum operator which encourage us to define the new operators of SP, namely the scalar momentum, \(P_S\), and the operators \(N^i\) of the direction of propagation such that \(P^i = P_S N^i\) and

\[
[H_S, P_S] = iP_S, \quad [H_S, N^i] = 0.
\]

Then we can chose the system of commuting operators \(\{H_S, N^i\}\) for defining the energy rep. we look for. The difficulty is that the operators \(N^i\) are no longer differential operators such that their eigenvalues have to be pointed out indirectly when we construct the solutions.

Let us start with the Dirac equation (44) in the frame \(\{t, \vec{x}; e\}\) of SP of the dS manifold whose operator (45) becomes independent on time,

\[
E_D^S = i\gamma^0 \partial_t - \vec{\gamma} \cdot \vec{P} - \omega \gamma^0 \hat{D},
\]

as it results from Eq. (30). We assume that the solutions of this equation can be expanded in the most general form as

\[
\psi_S(x) = \psi^{(+)}_S(x) + \psi^{(-)}_S(x) = \int_0^\infty dE \int_{\mathbb{R}^3} d^3p \left[ \hat{\psi}^{(+)}_S(E, \vec{p}) e^{-i(Et - \vec{p} \cdot \vec{x})} + \hat{\psi}^{(-)}_S(E, \vec{p}) e^{i(Et - \vec{p} \cdot \vec{x})} \right],
\]

where \(\hat{\psi}^{(\pm)}_S\) are spinors which behave as tempered distributions on the domain \(\mathbb{R}^3\) such that the Green theorem may be used. This allows us to replace in the Dirac equation the momentum operators \(P^i\) by \(p^i\) and the coordinate operators \(X^i\) by \(i\partial_{p^i}\) obtaining the Dirac equation of SP in momentum rep.,

\[
\left[ \pm E\gamma^0 + \gamma^j p^j - m - i\gamma^0 \omega \left( p^j \partial_{p^j} + \frac{3}{2} \right) \right] \hat{\psi}^{(\pm)}_S(E, \vec{p}) = 0,
\]
where $E$ is the energy defined as the eigenvalue of the operator $H_S = i\partial_t$.

Denoting then $\vec{p} = p\vec{n}_p$ with $p = |\vec{p}|$, we observe that the differential operator of Eq. (146) is of radial type, $p^i\partial_{p^i} = p\partial_p$. Therefore, this operator acts on the functions which depend on $p$ while the functions which depend only on the momentum direction $\vec{n}_p$ behave as constants. This suggests us to look for solutions of the form [13]

\begin{align}
\hat{\psi}_S^+(E, \vec{p}) &= \sum_\sigma u^S(E, \vec{p}, \sigma) a(E, \vec{n}_p, \sigma), \\
\hat{\psi}_S^-(E, \vec{p}) &= \sum_\sigma v^S(E, \vec{p}, \sigma) b^*(E, \vec{n}_p, \sigma),
\end{align}

where the wave functions $a$ and $b$ play the role of constants as long as they do not depend on $p$.

Furthermore, we assume that, in the standard rep. of the Dirac matrices, the spinors of the momentum rep. complying with the charge conjugation symmetry have the form

\begin{align}
\xi_{\sigma}(E, \vec{p}, \sigma) &= f_+^{\sigma}(p) + E^{\sigma}(p) \sigma \cdot \vec{n}_p \xi_{\sigma}, \\
\eta_{\sigma}(E, \vec{p}, \sigma) &= f_-^{\sigma}(p) + E^{\sigma}(p) \sigma \cdot \vec{n}_p \eta_{\sigma},
\end{align}

where $\xi_{\sigma}$ and $\eta_{\sigma}$ are the Pauli spinors of an arbitrary spin basis. After a little calculation we find that, according to Eq. (146), the radial functions satisfy the system

\begin{align}
\left[ \frac{\mu}{\omega} \left( p \frac{d}{dp} + \frac{3}{2} \right) - (E \mp m) \right] f_E^{\pm}(p) &= -p f_E^{\mp}(p),
\end{align}

that is analytically solvable. Indeed, after denoting $\mu = \frac{m}{\omega}$ and $\epsilon = \frac{E}{\omega}$, we find that the general solutions are linear combinations,

\begin{align}
f_E^{\pm}(p) &= c_1 \phi_1^{\pm}(s) + c_2 \phi_2^{\pm}(s),
\end{align}

of the particular solutions [13]

\begin{align}
\phi_1^{\pm}(s) &= N_1 s^{-1-\epsilon} K_{\nu_{\mp}}(-is), \\
\phi_2^{\pm}(s) &= \pm N_2 s^{-1-i\epsilon} K_{\nu_{\mp}}(is),
\end{align}

depending on $s = \frac{p}{\omega}$ and $\nu_{\pm} = \frac{1}{2} \pm i\mu$. The normalization constants $N_{1,2}$ have to assure the normalization in the energy scale for each particular solution separately.
Collecting all the above results we can write down the final expression of the Dirac field (145) as

\[ \psi_S(x) = \int_0^\infty dE \int_{S^2} d\Omega_n \sum_\sigma \left[ U^S_{E,\vec{n}_p,\sigma}(t, \vec{x}) a(E, \vec{n}_p, \sigma) + V^S_{E,\vec{n}_p,\sigma}(t, \vec{x}) b^*(E, \vec{n}_p, \sigma) \right], \]  

(155)

where the integration covers the energy semi-axis and the sphere \( S^2 \). The notation \( U^S \) and \( V^S \) stands for the fundamental spinor solutions of positive and, respectively, negative frequencies of energy \( E \), momentum direction \( \vec{n}_p \) and polarization \( \sigma \). According to Eqs. (149), (152), (153) and (154) we can write the spinors of positive frequencies as the linear combination

\[ U^S_{E,\vec{n}_p,\sigma} = c_1 U_{1,\vec{n}_p,\sigma} + c_2 U_{2,\vec{n}_p,\sigma}, \]  

(156)

of the particular spinors defined by the integral reps. [13]

\[ U_{1,\vec{n}_p,\sigma}(t, \vec{x}) = e^{-iEt} \]

\[ \times \int_0^\infty s^2 ds \left( \frac{\phi_{1,2}^+(s) \xi_\sigma}{\phi_{1,2}^-(s) \bar{\sigma} \cdot \vec{n}_p \xi_\sigma} \right) e^{i\omega s \vec{n}_p \cdot \vec{x}}, \]  

(157)

while the negative frequencies ones are their charge conjugated spinors,

\[ V^S_{E,\vec{n}_p,\sigma} = (U^S_{E,\vec{n}_p,\sigma})^c = i\gamma^2 (U^S_{E,\vec{n}_p,\sigma})^*. \]  

(158)

After integrating over \( p \), the unit vector \( \vec{n}_p \) giving the direction of propagation becomes an independent parameter that will be denoted from now simply as \( \vec{n} \) considering that its components, \( n^i \), are just the eigenvalues of the operators \( N^i \). Note that in QFT these operators can be defined as one particle operators giving directly their spectral representations.

The functions \( f^{\pm}_{E} \) of Eq. (152) are defined up to the integration constants \( c_1 \) and \( c_2 \) and the normalization factors \( N_1 \) and \( N_2 \) which must be fixed in order to assure the orthonormalization relations

\[ \langle U^S_{E,\vec{n},\sigma}, U^S_{E,\vec{n}',\sigma}' \rangle_S = \langle V^S_{E,\vec{n},\sigma}, V^S_{E,\vec{n}',\sigma}' \rangle_S = \delta_{\sigma\sigma'} \delta (E - E'), \]  

\[ \delta^2 (\vec{n} - \vec{n}'), \]  

(159)

and the completeness condition of the form (37) corresponding to the scalar product (16). Moreover, we require the particular spinors \( U^S_{1,2,\vec{n},\sigma} \) to form
an orthonormal system applying the method of the Appendix D. Thus we obtain [13]

\[ N_1 = N_2 = N = \frac{1}{(2\pi)^{3/2}} \frac{\omega}{\sqrt{2\pi}}, \] (161)

and verify that \( U_{E,\mathbf{n},\sigma}^{S_1} \) and \( U_{E,\mathbf{n},\sigma}^{S_2} \) are orthogonal. This means that the integration constants must satisfy

\[ |c_1|^2 + |c_2|^2 = 1, \] (162)

in order to assure the correct normalization of the general spinors [156].

Hence we derived the most general system of fundamental spinors that form the generalized basis of the energy-spin rep. \( \{H_S, N^i, S_3\} \) in which the spinors depend on the eigenvalues \( \{E, \mathbf{n}, \sigma\} \). Obviously, these are determined up to an integration constant which has to be fixed according to supplemental criteria. In Ref. [13] we fixed \textit{a priori} for simplicity \( c_1 = 1 \) and \( c_2 = 0 \) without other arguments.

The integral rep. of the fundamental spinors is useful for calculating scalar products as in the Appendix D but the definitive form of these spinors may be obtained after performing these integrals. This is missing in Ref. [13] such that we perform this calculation for the first time here finding that the particular spinors [157] can be written as

\[ U_{E,\mathbf{n},\sigma}^{S_{1,2}}(t, \mathbf{x}) = e^{-iEt} \left( \frac{\mathcal{F}_{1,2}^{\pm}(z) \xi_{\sigma}}{\mathcal{F}_{1,2}^{\pm}(z) \mathbf{\sigma} \cdot \mathbf{n} \xi_{\sigma}} \right) \] (163)

in terms of the new functions,

\[ \mathcal{F}_1^{\pm}(z) = \frac{N}{2} e^{\pm \frac{z}{\pi}} \left[ A_{\pm}(\epsilon, \mu, z) - B_{\pm}(\epsilon, \mu, z) \right], \] (164)

\[ \mathcal{F}_2^{\pm}(z) = \pm \frac{N}{2} e^{-\frac{z}{\pi}} \left[ A_{\pm}(\epsilon, \mu, z) + B_{\pm}(\epsilon, \mu, z) \right], \] (165)

depending on the dimensionless variable \( z = \omega (\mathbf{n} \cdot \mathbf{x}) \), where the quantities

\[ A_{\pm}(\epsilon, \mu, z) = \frac{1}{2} \Gamma \left( -\frac{3}{4} - \frac{i\epsilon}{2} \pm \frac{i\mu}{2}, -\frac{1}{4} - \frac{i\epsilon}{2} \pm \frac{i\mu}{2}; \frac{1}{2} z^2 \right), \] (166)

\[ B_{\pm}(\epsilon, \mu, z) = z \Gamma \left( \frac{3}{4} - \frac{i\epsilon}{2} \mp \frac{i\mu}{2}, \frac{1}{4} - \frac{i\epsilon}{2} \mp \frac{i\mu}{2}; \frac{3}{2} z^2 \right), \] (167)
have the remarkable properties

\begin{align}
A_{\pm}(\epsilon, \mu, z) &= A_{\mp}(\epsilon, -\mu, z), \\
B_{\pm}(\epsilon, \mu, z) &= B_{\mp}(\epsilon, -\mu, z),
\end{align}

that could be of interest in interpreting the above solutions obtained in SP. Then Eq. (158) will give the corresponding particular spinors \(V_{E,\vec{n},\sigma}^{S,1,2}\). This closed form of the plane E-waves is obtained here for the first time.

The surprise in deriving these solutions is that, after separating the frequencies by using directly the eigenvalues of the energy operator, we remain with undetermined integration constants since in this case the integration constants are not related to the manner in which the vacuum is defined.

### 5.2 Plane E-waves in NP

The last step is to rewrite the final results in NP where the Dirac field,

\[ \psi(x) = T(t)^{-1} \psi_S(t, \vec{x}) = \psi_S(t, e^{i\epsilon t} \vec{x}) = \int_0^\infty dE \int_{S^2} d\Omega_n \sum_\sigma \left[ U_{E,\vec{n},\sigma}(t, \vec{x}) a(E, \vec{n}, \sigma) + V_{E,\vec{n},\sigma}(t, \vec{x}) b^*(E, \vec{n}, \sigma) \right], \]

depends on the fundamental spinors of the energy-spin basis of NP that may have the general form

\begin{align}
U_{E,\vec{n},\sigma} &= c_1 U_{E,\vec{n},\sigma}^1 + c_2 U_{E,\vec{n},\sigma}^2, \\
V_{E,\vec{n},\sigma} &= c_1 V_{E,\vec{n},\sigma}^1 + c_2 V_{E,\vec{n},\sigma}^2,
\end{align}

depending and the particular spinors

\begin{align}
U_{E,\vec{n},\sigma}^{1,2}(t, \vec{x}) &= U_{E,\vec{n},\sigma}^{S,1,2}(t, e^{i\epsilon t} \vec{x}) \\
&= e^{-iEt} \left( \frac{F_{1,2}^+(z e^{i\epsilon t}) \xi_\sigma}{F_{1,2}^-(z e^{i\epsilon t}) \vec{\sigma} \cdot \vec{n} \xi_\sigma} \right), \\
V_{E,\vec{n},\sigma}^{1,2}(t, \vec{x}) &= V_{E,\vec{n},\sigma}^{S,1,2}(t, e^{i\epsilon t} \vec{x}) \\
&= e^{iEt} \left( \frac{F_{1,2}^+(z e^{i\epsilon t})^* \vec{\sigma} \cdot \vec{n} \eta_\sigma}{F_{1,2}^+(z e^{i\epsilon t})^* \eta_\sigma} \right),
\end{align}

which have a more complicated time dependence since in NP the time and space variables are no longer separable.
The spinors (171) and (172) are eigenspinors of the energy operator (60) of NP corresponding to positive and negative frequencies which are separated as in special relativity according to the sign of the energy. For each pair of constants \( c_1 \) and \( c_2 \) satisfying the condition (93) they form a basis of the rep. \( \{H, K, J_3\} \) of NP but these constants remain unspecified playing the role of free parameters. Note that the criteria that fixed the a.v. or r.f.v. do not hold here since these conditions depend on momentum which is not defined in this rep.. We remain thus with an ambiguity that could be helpful in further developments.

The closed forms of the particular solutions (173) and (174) are not suitable for concrete calculations. For this reason we turn back to the integral representations that could offer us more flexibility. Let us focus on the particular solutions in the form (157) that in the frame \( \{t_c, \vec{x}; e\} \) of NP can be rewritten as

\[
U^{1/2}_{E, \vec{n}, \sigma}(t_c, \vec{x}) = U^{S1/2}(t_c, \vec{x}; e^{\omega t}) = N(\omega t_c)^2 \times \int_0^\infty ds s^{1-i\epsilon} \left( \frac{K_{\nu_-}(\pm s\omega t_c) \xi_{\sigma}}{\pm K_{\nu_+}(\pm s\omega t_c) \vec{\sigma} \cdot \vec{n} \xi_{\sigma}} \right) e^{i\omega s \vec{n} \cdot \vec{x}},
\]

after changing the integration variable as \( s \to se^{-\omega t} = -s \omega t_c \). With these integral representations we can calculate scalar products in NP.

We verify first that the spinors (173) and (174) are orthonormal with respect to the scalar product (41) of NP. For those of positive frequency we obtain

\[
\left\langle U^a_{E, \vec{n}, \sigma}, U^b_{E', \vec{n}', \sigma'} \right\rangle = \delta_{ab} \delta(E - E') \delta^2(\vec{n} - \vec{n}'),
\]

and similarly for the negative frequencies ones. Furthermore, we calculate the transition coefficients between the basis of the momentum and energy reps. of NP obtaining that the above particular spinors and those of the momentum-spin rep. (138) satisfy,

\[
\left\langle U^a_{P, \vec{\sigma}}, U^b_{E, \vec{n}, \sigma'} \right\rangle = \delta_{ab} \delta_{\sigma, \sigma'} \delta^2(\vec{n} - \vec{n}_p) \times \frac{1}{\sqrt{2\pi \omega^2}} \left( \frac{p}{\omega} \right)^{-\frac{3}{2}-i\frac{E}{2}}, \quad a, b = 1, 2,
\]

pointing our the isometry between the bases of P and E-plane waves having the same integration constants \( c_1 \) and \( c_2 \). Moreover, by using the inversion relations (38) we can relate the particle wave functions of these reps. in the
dS spacetime as
\[
a(\vec{p}, \sigma) = \int_0^\infty dE \int_{S^2} d\Omega \sum_{\sigma'} (U_{\vec{p}, \sigma}, U_{E, \vec{n}, \sigma'}) a(E, \vec{n}, \sigma') \\
= \frac{p^{-3/2}}{\sqrt{2\pi \omega}} \int_0^\infty dE \left( \frac{p}{\omega} \right)^{-iE} a(E, \vec{n}, \sigma),
\]
(178)
\[
a(E, \vec{n}, \sigma) = \int d^3p \sum_{\sigma'} (U_{E, \vec{n}, \sigma}, U_{\vec{p}, \sigma'}) a(\vec{p}, \sigma') \\
= \frac{1}{\sqrt{2\pi \omega}} \int_0^\infty dp \sqrt{p} \left( \frac{p}{\omega} \right)^{iE} a(p\vec{n}, \sigma),
\]
(179)
and similarly for the anti-particle wave functions \(b\). It is remarkable that these relations are very similar to those we found previously for the scalar \(^{32}\) and Maxwell \(^{33}\) fields.

Finally we note that this isometry does not solve the problem of the undetermined integration constants even though now we can take over the constants of P-waves for obtaining an isometric basis of E-waves. This is because the criteria used for defining the vacua of the P-waves become meaningless for the E-waves where the frequencies are already separated. In our opinion the problem of fixing these constants remains open.

5.3 Spherical E-waves

The solutions we present here as a premier are of the energy-angular momentum rep. that solve the Dirac equation \(^{41}\) in the frame \(\{t, r, \theta, \phi; e\}\) of SP of the dS manifold. In this frame the time-independent Dirac operator \(^{41}\) can be rewritten as
\[
E^S_D = i\gamma^0 \partial_t + i\gamma^0 \omega \left( x^i \partial_i + \frac{3}{2} \right) \\
+ i \frac{1}{r^2}(\gamma^i x^i) \left( x^i \partial_i + 1 \right) + i \frac{1}{r^2} \gamma^0(\gamma^i x^i) K,
\]
(180)
keeping the notation \(r = |\vec{x}|\) and \(K\) for the spherical Dirac operator \(^{98}\).

We look for for general solutions of the form
\[
\psi_S(t, r, \theta, \phi) = \psi_S^{(+)}(t, r, \theta, \phi) + \psi_S^{(-)}(t, r, \theta, \phi) \\
= \int_0^\infty dE \sum_{\kappa_j, m_j} U^S_{E, \kappa_j, m_j}(t, r, \theta, \phi) a(E, \kappa_j, m_j) \\
+ \int_0^\infty dE \sum_{\kappa_j, m_j} V^S_{E, \kappa_j, m_j}(t, r, \theta, \phi) b^*(E, \kappa_j, m_j),
\]
(181)
where \( U_{E,\kappa_j,m_j} \) are the fundamental solutions of positive frequencies defined as common eigenspinors of the set \( \{ H_S, K, J_3 \} \) corresponding to the eigenvalues \( \{ E, -\kappa_j, m_j \} \) where the energy \( E \) is the eigenvalue of the energy operator \( H_S = i\partial_t \) of this picture. The eigenspinors of negative frequencies,

\[
V_{E,\kappa_j,m_j}^S(t,r,\theta,\phi) = i\gamma^2 U_{E,\kappa_j,m_j}^S(t,r,\theta,\phi)^*, \tag{182}
\]

are defined with the help of the charge conjugation as in the case of the plane waves. All these spinors may be organized as the orthonormal angular momentum basis satisfying,

\[
\langle U_{E,\kappa_j,m_j}^S, U_{E,'\kappa'_j,m'_j}^S \rangle_S = \langle V_{E,\kappa_j,m_j}^S, V_{E,'\kappa'_j,m'_j}^S \rangle_S = \delta_{\kappa_j,\kappa'_j} \delta_{m_j,m'_j} \delta(E - E'), \tag{183}
\]

\[
\langle U_{E,\kappa_j,m_j}^S, V_{E,'\kappa'_j,m'_j}^S \rangle_S = \langle V_{E,\kappa_j,m_j}^S, U_{E,'\kappa'_j,m'_j}^S \rangle_S = 0, \tag{184}
\]

with respect to the relativistic scalar product (46) that now reads

\[
\langle \psi, \psi' \rangle_S = \int r^2 dr \int_{S^2} d\Omega \bar{\psi}_S(t,r,\theta,\phi)\gamma^0 \psi'_S(t,r,\theta,\phi), \tag{185}
\]

where we integrate on SPhere \( S^2 \) as in Eq. (103).

For solving the above eigenvalue problems it is convenient to separate the time and the spherical variables looking for particular solutions of positive frequencies of the form

\[
U_{E,\kappa_j,m_j}^S(x) = \frac{e^{-iEt}}{r} \left[ \rho_{E,\kappa_j}(r)\Phi_{\kappa_j}^+(\theta,\phi) + \rho_{E,\kappa_j}(r)\Phi_{\kappa_j}^-(\theta,\phi) \right], \tag{186}
\]

where \( \Phi_{\kappa_j,m_j}^\pm \) are the orthonormal Dirac spherical spinors of the Appendix B. Then, after a little calculation by using the identities (13) we derive the system

\[
\left( E \pm m + i\omega r \frac{d}{dr} + \frac{i\omega}{2} \right) \rho_{E,\kappa_j}^\pm = \left( \mp \frac{d}{dr} + \frac{\kappa_j}{r} \right) \rho_{E,\kappa_j}^\mp, \tag{187}
\]

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resulted from the Dirac equation (180) which can be rewritten as

\[
(1 - \omega^2 r^2) \frac{d\rho^+_{E,\kappa j}}{dr} = \left( 2\omega^2 \beta r + \omega^2 \kappa j r - \frac{\kappa j}{r} \right) \rho^+_{E,\kappa j} = 2i\omega \rho^-_{E,\kappa j}, \tag{188}
\]

\[
(1 - \omega^2 r^2) \frac{d\rho^-_{E,\kappa j}}{dr} = \left( 2\omega^2 \alpha r - \omega^2 \kappa j r + \frac{\kappa j}{r} \right) \rho^-_{E,\kappa j} = -2i\omega \beta \rho^+_{E,\kappa j}, \tag{189}
\]

keeping the previous notations, \( \mu = \frac{m}{\omega} \) and \( \epsilon = \frac{E}{\omega} \), and defining the parameters

\[
\alpha = \frac{1}{4} + \frac{\kappa j}{2} - \frac{i\epsilon}{2} - \frac{i\mu}{2}, \tag{190}
\]

\[
\beta = \frac{1}{4} - \frac{\kappa j}{2} - \frac{i\epsilon}{2} + \frac{i\mu}{2}. \tag{191}
\]

In this manner, after separating the time and angular variables, we remain with a radial problem in the spaces of the 2-dimensional vectors \( \mathcal{R}_{E,\kappa j} = \left[ \rho^+_{E,\kappa j}, \rho^-_{E,\kappa j} \right] \) which must satisfy the radial orthonormalization condition

\[
\langle \mathcal{R}_{E,\kappa j}, \mathcal{R}_{E',\kappa j} \rangle = \int_0^\infty dr \mathcal{R}_{E,\kappa j}(r) \mathcal{R}^+_{E',\kappa j}(r) = \delta(E - E'), \tag{192}
\]

resulted from Eqs. (183), (185) and the orthogonality of the spherical spinors. The system of Eqs. (188) and (189) can be solved analytically in terms of Gauss hypergeometric functions obtaining two particular solutions of the form

\[
\mathcal{R}^1_{E,\kappa j}(r)^T = N_1(\omega r)^{-\kappa j} \times \left[ \frac{2i\beta}{2\kappa j - 1} \omega r F \left( \alpha - \kappa j + \frac{1}{2}; \beta; \frac{1}{2} - \kappa j; \omega^2 r^2 \right) \right], \tag{193}
\]

\[
\mathcal{R}^2_{E,\kappa j}(r)^T = N_2(\omega r)^{\kappa j} \times \left[ \frac{2i\alpha}{2\kappa j + 1} \omega r F \left( \alpha + 1, \beta + \kappa j + \frac{1}{2}; \beta; \frac{3}{2} + \kappa j; \omega^2 r^2 \right) \right], \tag{194}
\]

39
defined up to the normalization factors $N_1$ and $N_2$.

The difficult task now is to derive these factors according to the condition (192) since we have not yet general rules for normalizing the mode functions expressed in terms of hypergeometric ones in the case of the continuous energy spectra. Nevertheless, here we can derive these quantities by using Eq. of Ref. [36] as an integral rep. as we show in Appendix D. Thus we obtain that the normalization condition (192) is accomplished if we take

$$N_1 = \left[ \sqrt{2 \omega \cosh \pi \mu \Gamma \left( \frac{1}{2} - \kappa_j \right)} \right]^{-1} \frac{\Gamma(\beta)}{\Gamma(\alpha)}, \quad (195)$$
$$N_2 = \left[ \sqrt{2 \omega \cosh \pi \mu \Gamma \left( \frac{1}{2} + \kappa_j \right)} \right]^{-1} \frac{\Gamma(\alpha)}{\Gamma(\beta)}. \quad (196)$$

Unfortunately, we cannot perform other integrals for investigating, for example, if the particular solutions are orthogonal or for calculating transition coefficients.

We remain thus with these results allowing us to write down the general fundamental spinors of this rep., $\{H_S, K, J_3\}$ selecting only the solutions regular in $r = 0$ as

$$\mathcal{R}_{E,\kappa_j} = \frac{1 - \text{sign} \kappa_j}{2} \mathcal{R}_{E,\kappa_j}^1 + \frac{1 + \text{sign} \kappa_j}{2} \mathcal{R}_{E,\kappa_j}^2. \quad (197)$$

Then, in order to use a compact notation, we introduce the matrix $\Phi_{\kappa_j, m_j} = \begin{bmatrix} \Phi_{\kappa_j, m_j}^+, \Phi_{\kappa_j, m_j}^- \end{bmatrix}^T$ helping us to write down the normalized particular solutions of positive frequencies simply as

$$U_{E,\kappa_j, m_j}^S(t, r, \theta, \phi) = \frac{e^{-iEt}}{r} \mathcal{R}_{E,\kappa_j}(r) \Phi_{\kappa_j, m_j}(\theta, \phi) \quad (198)$$

while the negative frequencies ones have to be derived by using the charge conjugation.

Finally, we transform this basis in the equivalent basis of the rep. $\{H, K, J_3\}$ of NP as

$$U_{E,\kappa_j, m_j}(t, r, \theta, \phi) = T(t)^{-1} U_{E,\kappa_j, m_j}^S(t, r, \theta, \phi)$$
$$= U_{E,\kappa_j, m_j}^S(t, re^{\omega t}, \theta, \phi)$$
$$= \frac{e^{-(\omega + iE)t}}{r} \mathcal{R}_{E,\kappa_j}(e^{\omega t}r) \Phi_{\kappa_j, m_j}(\theta, \phi) \quad (199)$$

and similarly for antiparticle spinors. We obtain again common eigenspinors having no separated variables expected to comply with a special time evolution.
6 Concluding remarks

We tried to present here exhaustively all the analytical solutions of the free Dirac field minimally coupled to the gravity of the spatially flat FLRW spacetimes. In addition, we report a new solution which completes our collections of pairs of plane and spherical waves. The next table resumes all the types of solutions discussed here.

| Basis          | Rep.          | Manifold | Picture | Refs. |
|----------------|---------------|----------|---------|-------|
| plane P-w.     | \{P^i, S_3\} | FLRW     | NP      | [1, 10] |
| spherical P-w. | \{P^2, K, J_3\} | FLRW     | NP      | [5, 11] |
| plane E-w.     | \{H_S, N^i, S_3\} | dS       | SP      | [13]   |
| spherical E-w. | \{H_S, K, J_3\} | dS       | SP      | Sec.V   |
| plane E-w.     | \{H, N^i, S_3\} | dS       | NP      | [13]   |
| spherical E-w. | \{H, K, J_3\} | dS       | NP      | Sec.V   |

We presented first the general theory of the plane and spherical P-waves in NP of the FLRW spacetime reducing the problem of finding solutions to the simple systems of equations which yield the t.m.f. governing the time evolution of the Dirac field. These may be solved in many concrete cases but here we restricted ourselves to present two examples, i.e. the Milne type universe and the dS expanding universe. This last manifold where the energy is conserved is the only FLRW spacetime laying out all the solutions listed above.

Technically speaking, we presented the framework in which the gauge covariant Dirac field can be studied on FLRW spacetimes. Moreover, we considered the time evolution pictures allowing us to derive the P-waves in NP and the dS E-waves in SP where the variables can be separated. Turning back in NP with the E-waves derived in SP we obtain special eigenspinors of the energy operator whose variables are no longer separated.

All the solutions presented here are determined by sets of commuting operators up to an integration constant that must be fixed according to supplemental criteria. For the P-waves this means to fix the vacuum by choosing between the traditional a.v. and the new r.f.v. However, for the E-waves whose frequencies are separated by construction these vacua are helpless such that we must look for alternative criteria. This problem remains open.

We must specify that apart the new solutions of section V we present here for the first time the definitions of the energy and Hamiltonian oper-
ators in FLRW spacetimes, the Minkowskian projection, the most general form of the P-waves, the closed form of the plane E-waves as well as the transition coefficients between the bases of plane P-waves and E-waves derived in section V.B.

Finally, we hope the results presented here will open new possibilities of integrating the Dirac field in a large QFT on curved spacetimes with applications in astrophysics and cosmology.

A Pauli spinors

Given an arbitrary direction of unit vector \( \vec{n} \), the Pauli spinors \( \xi_\sigma(\vec{n}) \) defined as

\[
\xi_+ \left( \vec{n} \right) = \sqrt{\frac{1 + n^3}{2}} \left( \begin{array}{c} 1 \\ \frac{n^2 + i n }{1 + n^3} \end{array} \right), \\
\xi_- \left( \vec{n} \right) = \sqrt{\frac{1 + n^3}{2}} \left( \begin{array}{c} -n^2 + i n \\ 1 + n^3 \end{array} \right),
\]

(A.1)

and the conjugated ones, \( \eta_\sigma(\vec{n}) = i \sigma_2 \xi_\sigma(\vec{n})^* \), form an arbitrary spin basis satisfying the eigenvalue equations

\[
(\vec{n} \cdot \vec{\sigma}) \, \xi_\sigma(\vec{n}) = 2 \sigma \xi_\sigma(\vec{n}), \\
(\vec{n} \cdot \vec{\sigma}) \, \eta_\sigma(\vec{n}) = -2 \sigma \eta_\sigma(\vec{n}),
\]

(A.3) (A.4)

where the polarization \( \sigma = \pm \frac{1}{2} \) gives the projection of SPin on the direction \( \vec{n} \). In the Dirac theory the direction \( \vec{n} \) is defined in the rest frame where \( \vec{p} = 0 \). In current applications one takes \( \vec{n} = \vec{e}_3 \) along the third axis of this frame. We must specify that in the momentum-spin rep. we do not have a corresponding differential operator since SPin projection is defined in the rest frames.

Another choice is the helicity basis where \( \vec{n} = \vec{n}_p \) is along the momentum direction. In this basis SPinors \( \xi_\lambda(\vec{n}_p) \) and \( \eta_\lambda(\vec{n}_p) = i \sigma_2 \xi_\lambda(\vec{n}_p)^* \) depend on the helicity \( \lambda = \pm \frac{1}{2} \) which is proportional to the eigenvalues of the Pauli-Lubanski operator \([29]\).

B Spherical Dirac spinors

The Dirac spherical spinors, \( \Phi_{\kappa, j, m_j} : S^2 \to \mathbb{C} \), solve the eigenvalue problems of the commuting operators \( \{ \vec{J}^2, J_3, K \} \) for the eigenvalues \( \{ j(j +
spherical spinors, Ψ

1), mj, −κj\} which can take the values \( j = \frac{1}{2}, \frac{3}{2}, \ldots, m_j = -j, -j + 1, \ldots, j \)
and \( \kappa_j = \pm \left( j + \frac{1}{2} \right) \). The operator eigenvalues \( l(l + 1) \) of the operator \( \hat{L}^2 \)
give the orbital angular quantum number \( l \) such that \( j = l \pm \frac{1}{2} \) \[34\]. The quantum numbers \( l \) and \( j \) do not appear explicitly since,

\[
\kappa_j = \begin{cases} 
  j + \frac{1}{2} = l & \text{for } j = l - \frac{1}{2}, \\
  -(j + \frac{1}{2}) = -l - 1 & \text{for } j = l + \frac{1}{2}
\end{cases}
\]

encapsulate all of them, \( j = |\kappa_j| - \frac{1}{2} \) and \( l = |\kappa_j| - \frac{1}{2} (1 - \text{sign } \kappa_j) \) \[34, 29\].

The above angular spinors are expressed in terms of the well-known Pauli spherical spinors, \( \Psi_{mj} \), as

\[
\Phi^+_{+(j+\frac{1}{2}),mj} = \begin{pmatrix} i\Psi_{mj}^j \frac{1}{2} & 0 \\
0 & 0 \end{pmatrix}, \quad \Phi^-_{+(j+\frac{1}{2}),mj} = \begin{pmatrix} 0 & \Psi_{mj}^j \frac{1}{2} \\
0 & 0 \end{pmatrix},
\]

forming an orthonormal set,

\[
\langle \Phi^\pm_{\kappa_j,j,m_j}, \Phi^\pm_{\kappa_j',j',m_j'} \rangle = \delta_{\kappa_j,\kappa_j'} \delta_{m_j,m_j'}, \quad \langle \Phi^\pm_{\kappa_j,j,m_j}, \Phi^\mp_{\kappa_j',j',m_j'} \rangle = 0,
\]

with respect to the angular scalar product

\[
\langle \Phi, \Phi' \rangle = \int_{S^2} d\Omega \Phi(\theta,\phi)^* \Phi'(\theta,\phi)
\]

defined on the sphere \( S^2 \).

The spherical spinors help us to separate the spherical variables \((r, \theta, \phi)\) associated to \( \vec{x} \) (with \( r = |\vec{x}| \)) by using the following identities \[34\],

\[
\frac{\vec{\sigma} \cdot \vec{x}}{r} \Psi_{mj}^j \frac{1}{2} = \Psi_{mj}^j \frac{1}{2} \rightarrow \frac{i}{r} \vec{\sigma} \cdot \vec{x} \Phi^\pm_{\kappa_j,j,m_j} = \Phi^\pm_{\kappa_j,j,m_j},
\]

and observing that \( \gamma^0 \Phi^\pm_{\kappa_j,j,m_j} = \pm \Phi^\pm_{\kappa_j,j,m_j} \).

For performing the charge conjugation we take into account that the spherical harmonics satisfy \((Y_l^m)^* = (-1)^m Y_{l-m}\) such that we can write

\[
i\sigma_2 \left( \Psi_{mj}^j \frac{1}{2} \right)^* = \mp (-1)^{m_j + \frac{1}{2}} \Psi_{mj}^{-m_j} \frac{1}{2}.
\]

from which we deduce:

\[
i\gamma^2 \left( \Phi^\pm_{mj,+(j+\frac{1}{2})} \right)^* = \pm (-1)^{m_j} \Phi^\mp_{mj,±(j+\frac{1}{2})}.
\]
C Some properties of Bessel functions

The Bessel functions $J_{\nu}(z)$ and $J_{-\nu}(z)$ form a satisfactory set of independent solutions whose Wronskian gives the identity:

$$J_{\nu+1}(z)J_{-\nu}(z) + J_{-\nu-1}(z)J_{\nu}(z) = -\frac{2}{\pi z} \sin \pi \nu. \quad (C.1)$$

With their help one defines the modified Bessel functions

$$I_{\nu}(z) = e^{\pm i\pi \nu/2} J_{\nu}\left(z e^{\pm i\pi/2}\right), \quad (C.2)$$

$$K_{\nu}(z) = K_{-\nu}(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_{\nu}(z)}{\sin \pi \nu}. \quad (C.3)$$

The functions $K_{\nu \pm}(z)$, with $\nu_\pm = \frac{1}{2} \pm i\mu$ are related among themselves through

$$[K_{\nu \pm}(z)]^* = K_{\nu \mp}(z^*), \quad \forall z \in \mathbb{C}, \quad (C.4)$$

satisfying the equations

$$\left(\frac{d}{dz} + \frac{\nu_\pm}{z}\right) K_{\nu \pm}(z) = -K_{\nu \mp}(z), \quad (C.5)$$

and the identities

$$K_{\nu \pm}(ix)K_{\nu \mp}(-ix) + K_{\nu \pm}(-ix)K_{\nu \mp}(ix) = \frac{\pi}{|x|}, \quad (C.6)$$

that guarantees the correct orthonormalization properties of the fundamental spinors.

For $|z| \to \infty$ we have

$$I_{\nu}(z) \to \sqrt{\frac{\pi}{2z}} e^z, \quad K_{\nu}(z) \to K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}, \quad (C.7)$$

for any $\nu$, while for $z \to 0$ these functions behave as

$$I_{\nu}(z) \sim \frac{1}{\Gamma(\nu + 1)} \left(\frac{z}{2}\right)^\nu, \quad K_{\nu}(z) \sim \frac{1}{2} \Gamma(\nu) \left(\frac{z}{2}\right)^{-\nu}. \quad (C.8)$$

D Normalization integrals

In spherical coordinates of the momentum space, $\vec{n}_p \sim (\theta_n, \phi_n)$, and the notation $\vec{p} = \omega s \vec{n}_p$, we have $d^3p = p^2 dp d\Omega_n = \omega^2 s^2 ds d\Omega_n$ with $d\Omega_n = \frac{4\pi}{2} \sin \theta_n d\theta_n d\phi_n.$
Moreover, we can write

\[
\delta^3(\mathbf{p} - \mathbf{p}') = \frac{1}{p^2} \delta(p - p') \delta^2(\mathbf{n}_p - \mathbf{n}'_p)
\]

\[
= \frac{1}{\omega^3 s^2} \delta(s - s') \delta^2(\mathbf{n}_p - \mathbf{n}'_p),
\]

(D.1)

where \(\delta^2(\mathbf{n}_p - \mathbf{n}'_p) = \delta(\cos \theta_n - \cos \theta'_n)\). Then the scalar products of the fundamental spinors of positive frequencies can be calculated according to Eqs. (81), (161), (C.6) and (D.1) as

\[
\langle U_{a,\mathbf{n},\sigma}^S E, \mathbf{n}, \sigma \rangle = \left. \int D d^3 x \right[ U_{a,\mathbf{n},\sigma}^S E, \mathbf{n}^\prime, \sigma^\prime(t, \mathbf{x}) \right] + U_{b,\mathbf{n}^\prime, \sigma^\prime}^S E, \mathbf{n}^\prime(t, \mathbf{x})
\]

\[
= e^{i(E - E')t} \left[ \frac{1}{2\pi \omega} \int_0^\infty ds \right] \delta_{ab} \delta_{\sigma\sigma'} \delta^2(\mathbf{n} - \mathbf{n}^\prime),
\]

\(a, b = 1, 2\). (D.2)

Eqs. (159) and (160) are deduced in the same manner.

For calculating the normalization condition (192) we use Eq. (6.574) of Ref. [36] as an integral representation,

\[
F(a, b; c; x^2) = x^{1-c} \frac{2^{c-b-a} \Gamma(c) \Gamma(1-b)}{\Gamma(a)} \times \int_0^\infty ds s^{a+b-c} J_{c-1}(xs) J_{a-b}(s),
\]

(D.3)

and perform first the radial integral according to Eqs. (112) and (C.1) obtaining an intermediate result proportional to \(\delta(s - s')\). Furthermore, we integrate over \(s'\) remaining with an integral over \(s\) giving just the Dirac \(\delta\)-function as in Eq. (D.2).

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