Some Notes on Quantum Information Theory and Emerging Computing Technologies

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Abstract

It is considered an interdependence of the theory of quantum computing and some perspective information technologies. A couple of illustrative and useful examples are discussed. The reversible computing from very beginning had the serious impact on the design of quantum computers and it is revisited first. Some applications of ternary circuits are also quite instructive and it may be useful in the quantum information theory.

1 Introduction

The theory of reversible computations produced important income to the quantum computing, because for an evolution of a quantum system described by Schrödinger equation the time inversion $t \rightarrow -t$ is valid operation. So the understanding of possibility to implement an universal computer using reversible devices was important for the development of first models of the quantum computing machines.

The reversible computation is also actively studied because according to the Landauer principle the heat generation may be reduced to an arbitrary small value only for the design with reversible, “conservative” logic gates. Yet, the Landauer limit $kT \ln 2$ for irreversible operation could be treated as an extremely small 50 years ago at the time of publication of the mentioned paper.

Even in 1984 in his report about the quantum-mechanical computers Feynman noted that because the actual dissipation is still much bigger (about $10^{10}kT$ for a transistor with about $10^{11}$ atoms at the moment of his talk) the discussion about such small quantities and logical elements with a few or single atom is rather “ridiculous” and “such nonsense is very entertaining to professors like me.” Nevertheless, nowadays logical gates with single atom, ion, electron or photon are already standard subject for real experiments in area of quantum computing and communications and due to some prognoses $kT$ limit may be actual for real processors to next decade or so [8, Fig. 17].

There were some discussions about the Landauer principle those above the scope of this presentation, but anyway the research of reversible computations is in the state of quite active development during an enough long time and it is
producing a mutually advantageous connection with the quantum information theory.

Some applications of a reversible design in the quantum information theory are discussed in this presentation. Examples of binary and ternary circuits together with a brief excursion to the many-valued logic are also provided.

2 Quantum Computations and Reversibility

There is very close relation between classical reversible computations and quantum information theory, because any reversible classical function directly corresponds to a quantum one [4,5]. If there is a discrete system with $N$ states, there are $N!$ reversible functions, corresponding to permutations of these states. A permutation $S$ has a standard representation by $N \times N$ matrix $\hat{S}$ with $N$ nonzero elements $\hat{S}_{ij} = 1$ for $S : i \mapsto j$, i.e., $\hat{S}_{ij} = \delta_{S_{ij}}$.

If $e_i$ denotes an element of the basis of the $N$-dimensional vector space, then due to such a definition $\hat{S} e_i = e_{S(i)}$ and it produces a standard linear representation of the permutation group. In the quantum information science the Dirac notation is often used for simplification: $|i\rangle$ — are $N$ basic vectors (instead of $e_i$), $|i\rangle \langle j|$ — is the matrix with only nonzero element $\hat{M}_{ij} = 1$ and $\langle i | j \rangle = (e_i, e_j) = \delta_{ij}$ — is the scalar product (for the basis it is the Kronecker delta). In such a notation an equation

\[ \hat{S} = \sum_i |S(i)\rangle \langle i|, \quad \hat{S} |i\rangle = |S(i)\rangle \]

is hold.

Roughly speaking, a model with reversible circuits may be “translated” into the language of the quantum information theory after a formal change of the notation $0, 1, \ldots$ to $|0\rangle, |1\rangle, \ldots$, but it is useful also to remember about specific properties of quantum systems. The qubit is a quantum analogue of bits, but besides it may be in any superposition of the basic states, i.e., $\alpha|0\rangle + \beta|1\rangle$, $|\alpha|^2 + |\beta|^2 = 1$. The qutrit is a ternary analogue of the qubit and most general state may be described as $\alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle$, $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$.

However, the problem of the realization of a classical algorithms on a quantum computer is also devoting an attention and it is actively discussed in the presented work. Let us consider a question about the representation of an arbitrary irreversible function or a “gate” using reversible one. Both in the quantum information theory and in the reversible computations a method with an auxiliary system is widely used [5]. Instead of the function (gate) $y = f(a)$ a gate with two “wires” is used

\[ g: (a, b) \mapsto (a, b + f(a)), \]

see Fig. 1. Such a gate has inverse: $(a, b) \mapsto (a, b - f(a))$ and for $b = 0$ reproduces initial function $(a, 0) \mapsto (a, f(a))$.

Really, due to the fixed size of a number in the computer representation instead of the addition in Eq. (2) the modular arithmetic should be used, e.g.,
Figure 1: Reversible gate for irreversible function

bitwise addition modulo 2 (so-called binary XOR, eXclusive OR operation) or addition modulo $N_{\text{max}}$ (some fixed value like $2^{32}$, $2^{64}$, etc.) may be applied.

Using Dirac notation Eq. (2) could be formally rewritten as

$$\hat{G}|x, z\rangle = |x, z + f(x)\rangle.$$  \hspace{1cm} (3)

Formally, Eq. (3) is a proper definition of $\hat{G}$, because for the definition of a linear operator it is enough to describe transformations of all basic vectors. On the other hand it is possible to produce “more constructive” description close to ideas of quantum control.

The CONTROLLED NOT gate may be considered as the first example of such approach. Already mentioned Feynman work about quantum-mechanical computers [4] discussed that gate. It may be written for two bits as

$$\text{CNOT}: (a, b) \mapsto (a, b \text{ XOR } a).$$

It corresponds to Eq. (2) for the trivial case of the identity function $f(a) = a$, but it is useful anyway as a simplest example of a controlled gate and due to numerous applications.

The algorithm of control for CNOT may be represented as

if $a$ then NOT $b$ else $b$

and there is an instructive method to represent such operations in the quantum computation. It is conditional quantum dynamics [10]. Using Dirac notation adopted in [10] the CNOT gate may be written as

$$\text{CNOT} = |0\rangle\langle 0| \otimes \hat{1} + |1\rangle\langle 1| \otimes \hat{\text{NOT}},$$  \hspace{1cm} (4)

where $\hat{1}$ is the identity operation, i.e., the unit matrix.

In the more general case there are more than two alternatives and both $a$ and $b$ may represent more than two binary values

```java
  case a of
    0 : F_0(b)
    1 : F_1(b)
    \ldots
    k : F_k(b)
end
```

3
and if instead of the reversible classical functions $F_k$ to use unitary quantum operators $\hat{U}_k$, then the conditional quantum dynamics may be written as [10]

$$\hat{U} = |0\rangle\langle 0| \otimes \hat{U}_0 + |1\rangle\langle 1| \otimes \hat{U}_1 + \cdots + |k\rangle\langle k| \otimes \hat{U}_k. \quad (5)$$

The quantum notation used in Eq. (5) almost directly corresponds to the “case control flow” example above. The tensor product signs $\otimes$ are used for construction of states and operators for composite systems in the quantum mechanics, i.e., notation like $|00\rangle$ or $|0\rangle|0\rangle$ could be rewritten in more pedantic way as $|0\rangle \otimes |0\rangle$. For operators sign $\otimes$ often should not be omitted to prevent confusion with usual multiplication (composition).

The terms $\hat{P}_k = |k\rangle\langle k|$ in Eq. (4) and Eq. (5) are projectors. We have $\hat{P}_k|k\rangle = |k\rangle$ and $\hat{P}_k|j\rangle = 0$, $j \neq k$ and so each term with $\hat{P}_a$ selects only necessary states $|a\rangle|b\rangle$ and applies $\hat{U}_a|b\rangle$ to the second variable.

Yet another example is the irreversible binary function AND. Because an argument here is the pair of bits, a reversible analog is

$$T: (a_1, a_2, b) \mapsto (a_1, a_2, b \text{ XOR } (a_1 \text{ AND } a_2)).$$

The T is called CONTROLLED CONTROLLED NOT or Toffoli gate [4, 9].

The Toffoli gate is important, because with CNOT gates it is still not possible to represent any logical circuit, but this problem may be resolved with Toffoli gates. So it is universal gate in the classical meaning [4, 9].

It corresponds to a formal algorithm

**if** ($a_1$ AND $a_2$) **then** NOT $b$ **else** $b$

and in quantum notation it is

$$\hat{T} = \hat{P}_1 \otimes \hat{P}_1 \otimes \hat{N}O\hat{T} + (\hat{P}_0 \otimes \hat{P}_0 + \hat{P}_0 \otimes \hat{P}_1 + \hat{P}_1 \otimes \hat{P}_0) \otimes \hat{I}.$$

So, two-bit reversible gates in the classical case are not enough to create any function and it is necessary to use three-bit gates. However, in the quantum case two-bit gates may me used for construction of any quantum circuit [11–14].

In the more general case Eq. (2) may be applied to arbitrary Boolean function with $N$ bits of input for $a$ and $M$ bits of output for $f(a)$. In such a case Eq. (2) requires $M$ auxiliary bits of input with zeros $b = 0$ and produces $N$ bits of “garbage” due to the copying of an initial state $a$.

Formally, in the classical case any presentation of numbers (“radix”) may be used, e.g., binary, ternary, decimal numbers and so on. In the quantum information theory some representation may be preferable, e.g., the qubit is most appropriate in many cases, sometimes prime numbers are more convenient than factored ones, but it is above the scope of this work.

Computers repeat some elementary set of operations many times and in each such step reversible circuits such as Eq. (2) need for clean zero bits and generate new bits with garbage. It may be visualized using idea of some tapes with initially zero values in each cell and with “garbage” or “history” data those are moving on each step of the computing device. Similar design was from very beginning used in models of reversible and quantum computing machines [11,2].
3 Qubits and Qutrits

For Toffoli gate a cost of the reversibility in comparison with AND is one extra zero bit and two bits of garbage. On the other hand it is possible to use more “economical” design if to work with nonbinary gates. For example, it is enough to let even one wire to represent three values instead of two to include the universal set of irreversible Boolean gates into a reversible system:

\[
\begin{array}{c|c|c}
\text{AND}_{23} & \text{OR}_{23} \\
\hline
a & b & a & b \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 2 & 1 & 2 \\
1 & 0 & 0 & 2 \\
1 & 1 & 1 & 0 \\
1 & 2 & 1 & 1 \\
\end{array}
\]

(6)

The boldface numbers in Eq. (6) mark the inclusion of AND, OR as subsets of suggested reversible operations. Eq. (6) may be represented as compositions of two reversible steps. The first one for both cases is the operation \( b \mapsto b - a \) (modulo 3). The second step is the application of (controlled) NOT gate to \( a \) either for \( b = 2 \) to implement AND\(_{23} \) or for \( b = 1 \) to do OR\(_{23} \).

Such implementation requires six states instead of eight (three bits) for Toffoli gate. Similar methods are known in reversible computation \[15,16\] and often for convenience and symmetry both values are ternary, yet for such a case formally there are nine states without a self-evident advantage in comparison with Toffoli gate.

If both values are ternary, then instead of Eq. (6) with six possible alternatives corresponding to an exchange of the values of \( b \), there are much more \((2160)\) variants of extensions of AND and OR gates. However, a reversible function with two values may be written in form \( f: (a, b) \mapsto (f_1(a, b), f_2(a, b)) \), there the second function \( f_2 \) is auxiliary.

It may be shown that there are only ten alternative for the first function \( f_1 \) appropriate for representation of AND, OR operations. Between them only two are symmetric and here is chosen one pair of such functions:

\[
\begin{array}{c|c|c|c|c}
\text{AND}\_3 & \text{OR}\_3 \\
\hline
0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 2 \\
2 & 2 & 1 & 2 & 2 \\
\end{array}
\]

(7)

The operations may be described using an idea of the selection “the previous or the same” \((\preceq_3)\) and “the next or the same” \((\succeq_3)\) between two values with respect to a nontransitive connected relation \( 0 \prec_3 1, 1 \prec_3 2, 2 \prec_3 0 \) depicted on Fig. 2. More precisely, \( a \text{AND}_3 b \) returns \( a \) if \( a \preceq_3 b \) and \( b \) otherwise. Conversely, \( a \text{ OR}_3 b \) selects \( a \) if \( a \succeq_3 b \) and \( b \) otherwise.

It is similar with definition of many-valued and real-valued Lukasiewicz logics with AND, OR expressed via MIN, MAX respectively \[17\] that is also relevant
to usual Boolean logic if to choose notation 0 for \textit{false}, 1 for \textit{true} and standard ordering $0 < 1$. For the case of a three-valued logic with third \textit{“unknown”} value $x$ in such MIN/MAX description the order (i.e., transitive relation) $0 \prec_3 1$, $0 \prec_3 x$, $x \prec_3 1$ should be used, see Fig. 3.

It corresponds to ordering in Lukasiewicz three-valued logic and so the notation $1/2$ sometimes is used for the third value (denoted here as $x$). But such a logic may not be used in construction of the reversible circuit with two ternary “wires.” It is enough to look on “truth tables” for ternary logic \cite{17}

\[
\begin{array}{c|ccc}
\text{AND}_3^L & 0 & 1 & x \\
\hline
0 & 0 & 0 & 0 \\
1 & 0 & 1 & x \\
x & 0 & x & x \\
\end{array}
\quad
\begin{array}{c|ccc}
\text{OR}_3^L & 0 & 1 & x \\
\hline
0 & 0 & 1 & x \\
1 & 1 & 1 & 1 \\
x & x & 1 & x \\
\end{array}
\quad (8)
\]

Any component of reversible function with two arguments should be “balanced,” i.e., each value should be presented an equal number of times like in Eq. (7). The property is consequence of possibility to represent a reversible function as some permutation.

So for a linear order such a method may not generate balanced table, because each element has different number of predecessors (and successors). But for the cyclic relation depicted on Fig. 2 a preceding or following element is always unique. It could be said, that \text{AND}_3^L, \text{OR}_3^L are functions derived from...
nontransitive (cyclic) arbitration relation in rock-paper-scissors kind games.

The operations $\text{AND}_3^c$, $\text{OR}_3^c$ are not associative and not distributive if expressions contain more than two different values. Yet,

$$\text{NOT}(a \ \text{AND}_3^c \ b) = (\text{NOT} \ a) \ \text{OR}_3^c (\text{NOT} \ b)$$

is valid for

$$\text{NOT} \ a = (1 - a) \mod 3.$$ 

It is possible to create the reversible ternary implementation of the binary AND, OR using expressions with $\text{AND}_3^c$, $\text{OR}_3^c$ defined by Eq. (7)

$$\text{AND}^c: (a, b) \mapsto \left(a \ \text{AND}_3^c b , \ (b - a) \mod 3\right)$$

$$\text{OR}^c: (a, b) \mapsto \left(a \ \text{OR}_3^c b , \ (b - a) \mod 3\right)$$

The OR$^c$ gate may be also used as a (binary) FANOUT gate, if to apply zero to the first input and to use only zero and unit for the second input. It may be checked also that the inverse of both gates in Eq. (9) may be used as a (ternary) FANOUT gate for arbitrary value on the first wire if to apply zero to the second one. Unlike Toffoli gate which is equivalent to its own inverse $T^{-1} = T$ presented operations have longer periods: $(\text{AND}^c)^{-1} = (\text{AND}^c)^6$, $(\text{OR}^c)^{-1} = (\text{OR}^c)^6$. The both operations represented in Eq. (9) may be performed with two steps. The first one is $b \mapsto b - a \mod 3$. The second step for AND$^c$ is controlled subtraction of unit (mod 3) from $a$ if $b = 2$. Contrary, for OR$^c$ it is controlled addition of unit (mod 3) to $a$ if $b = 1$. All such steps are reversible and resembles the method discussed after Eq. (6).

Let $X_3: a \mapsto (a + 1) \mod 3$ is a reversible operation with the property $X_3^{-1} = X_3^2$. In the quantum computation it may be represented as the matrix

$$\hat{X}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ \hat{X}_3^* = \hat{X}_3^{-1} = \hat{X}_3^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \ \ (10)$$

Then in the quantum case the first step discussed above may be expressed using $\hat{X}_3$ and projectors $\hat{P}_k = |k\rangle \langle k|$ as

$$\hat{C}_X^* = \hat{P}_0 \otimes \hat{1} + \hat{P}_1 \otimes \hat{X}_3^* + \hat{P}_2 \otimes \hat{X}_3$$
and operations used on second step are
\[
\hat{C}_2^* = \hat{I} \otimes (\hat{I} - \hat{P}_2) + \hat{X}_3^* \otimes \hat{P}_2,
\]
\[
\hat{C}_1 = \hat{I} \otimes (\hat{I} - \hat{P}_1) + \hat{X}_3 \otimes \hat{P}_1,
\]
respectively. Finally, in such notation it may be written
\[
\text{AND}^\circ = \hat{C}_2^* \hat{C}_X^*, \quad \text{OR}^\circ = \hat{C}_1 \hat{C}_X^*.
\]

4 Note on “Classical” Computations on Quantum Computer

After the early paper of Feynman [4] the question about doing usual computation on quantum computer is not widely discussed. Most attention is devoted to quantum phenomena like superposition, entanglement and to quantum algorithms providing a speedup in comparison with the classical case.

On the other hand, problems of an information processing by quantum systems may be very actual even for usual algorithms. A necessity for reversible operations was already mentioned, but even such gates should be considered in specific way for quantum systems. As an example may be mentioned the NOT gate. It simply swaps 0 and 1, but for quantum system it might be described by some process started at some time \( t_1 \) and finished at \( t_2 = t_1 + \Delta t \).

At some moment between \( t_1 \) and \( t_2 \) the system may not be described neither by state \( |0\rangle \) nor by state \( |1\rangle \) and for the simple example with an “ideal” qubit, i.e., a closed system with only two basic states, it may be expressed by some superposition \( \alpha t |0\rangle + \beta t |1\rangle \), \( |\alpha t|^2 + |\beta t|^2 = 1 \). So the notion about the specific quantum phenomena is reasonable even for the simplest classical algorithm, if it is implemented by a quantum system.

5 Conclusion

Some unconventional information technologies such as reversible and ternary circuits were discussed, which may be relevant to the development of the quantum computing. The main theme of the paper is realization of Boolean functions on a quantum computer using qubits and qutrits. Instead of a brute force method of implementation of Boolean AND, OR with ternary variables it is used some variation of a MIN/MAX approach known earlier due to Lukasiewicz many-valued logic. This construction is useful in theory of quantum and reversible computing, but also may have independent applications.

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