Neutrino masses and mixing with seesaw mechanism and universal breaking of extended democracy

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Abstract

In the framework of a minimal extension of the SM, where the only additional fields are three right-handed neutrinos, we suggest that the charged lepton, the Dirac neutrino and the right-handed Majorana neutrino mass matrices are all, to leading approximation, proportional to the democratic matrix. With the further assumption that the breaking of this extended democracy is universal for all leptonic mass matrices, a large mixing in the 2-3 sector can be obtained and is linked to the seesaw mechanism, together with the existence of a strong hierarchy in the masses of right-handed neutrinos. The structure of the resulting effective mass matrix of light neutrinos is stable against the RGE evolution, and a good fit to all solar and atmospheric neutrino data is obtained.
1 Introduction

The recent discovery of neutrino oscillations \[1\], pointing towards non-vanishing neutrino masses, provides special motivation to investigate the question of fermion masses and mixing, at present one of the major riddles of particle physics.

One of the striking features of the experimental evidence is the fact that large neutrino mixing is required at least in the 2-3 sector. This is to be contrasted to the situation in the quark sector, where the mixing is known to be small. In the search for a model which would naturally accommodate the experimental data, we find desirable to abide by the following principles:

(i) We consider the standard model (SM) with the addition of three right-handed neutrinos, but no extra Higgs doublets. The smallness of neutrino masses results then from the large right-handed neutrino masses through the seesaw mechanism \[2\].

(ii) We will treat all the fundamental mass matrices on equal footing. In particular, we will assume that there is a weak-basis (WB) where, in leading order, the mass matrix of charged leptons and both the Dirac and right-handed Majorana neutrino mass matrices are all of the “democratic type”, i.e., they are proportional to a matrix whose elements are, in leading approximation, all equal to unity. We will refer to this assumption as “extended democracy”.

(iii) We will assume that the breaking of extended democracy is small and it has the same pattern for all the fundamental mass matrices.

In this letter we present a class of models which abide by the above principles and where the large mixing in the 2-3 sector results from the seesaw mechanism and requires a strong hierarchy in the right handed neutrino masses.

It is worth emphasizing that assumption (ii) goes much beyond a simple choice of WB. Even if one assumes that both the neutrino Dirac and right-handed Majorana mass matrices have hierarchical eigenvalues, in general it does not follow that there is a WB where both the neutrino mass matrices as well as the charged lepton mass matrix are all, in leading approximation, proportional to the democratic matrix. If our framework is the correct one, it would mean that the appearance of large mixing in the leptonic sector originates in the seesaw mechanism. We recall that in the quark sector one can also obtain a good fit for the quark masses and mixings, assuming that both the up and down quark mass matrices are, to leading order, proportional to the democratic matrix, with a small perturbation generating the masses of the first two generations. The crucial new ingredient in the leptonic sector is the presence of the seesaw mechanism.

Motivated by the recent discovery of neutrino oscillations, there have been in the literature a large number of ansätze for the structure of leptonic mass matrices \[3\]. In most of them, a particular pattern is suggested directly for the effective left-handed neutrino mass matrix. A distinctive feature of the scheme we propose, is the fact that we suggest a universal pattern for all the fundamental leptonic mass matrices, which in our framework
are the charged lepton mass matrix, the Dirac neutrino mass matrix and the right-handed Majorana mass matrix. The structure of the effective left-handed neutrino mass matrix is then derived through the seesaw mechanism.

2 General framework

We consider the three generations SM, where three right-handed neutrino fields have been added, leading to the following charged lepton and neutrino mass terms:

\[
- \mathcal{L}_{\text{mass}} = \bar{l}_{iL} (M_L)_{ij} l_{jR} + \bar{\nu}_{iL} (M_D)_{ij} \nu_{jR} + \frac{1}{2} \nu_{iR}^T C (M_R)_{ij} \nu_{jR} + \text{h.c.},
\]

where the notation is obvious. Since the right-handed Majorana neutrino mass terms are \( SU(2)_L \times U(1) \) invariant, \( M_R \) is naturally large, not being protected by the low energy symmetry. Following our general principles stated in the introduction, we will assume that all the matrices \( M_L, M_D \) and \( M_R \) are, to leading order, proportional to the democratic matrix \( \Delta \) and, furthermore, that the breaking of the extended democracy has the same pattern for all the mass matrices. We thus write:

\[
M_k = c_k [\Delta + P_k],
\]

where

\[
\Delta = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad P_k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_k & 0 \\ 0 & 0 & b_k \end{pmatrix}, \quad k = l, D, R
\]

with \(|a_k|, |b_k| \ll 1\), so that all matrices are close to the democratic limit. As previously mentioned, the ansatz of Eqs. (2) and (3) is not just a choice of WB, together with the assumption of hierarchical masses for \( M_L, M_D \) and \( M_R \). Indeed if one assumes hierarchical masses, one can always choose, without loss of generality, a WB where, for example, both \( M_L \) and \( M_R \) are close to the democratic limit. However, once the \( \nu_R \) basis is fixed, the Dirac neutrino mass matrix \( M_D \) cannot in general be reduced to the quasi-democratic form by a choice of the \( \nu_L \) basis. Thus, it is not possible in general to choose a WB where all the three matrices are in leading order proportional to \( \Delta \). Therefore, the nontrivial content of our ansatz of Eq. (3) is the assumption that such a choice of WB is possible, implying an “alignment” of all three matrices in flavour space, and the suggestion that the breaking of the extended democracy has the same form for all three leptonic mass matrices.

Following the hints of some Grand Unified Theories (GUTs), we consider the mass spectrum of \( M_D \) similar to the one of the up-type quarks. This will allow us to establish the relations between \((a, b, c)_D\) and the quark masses \( m_u, m_c \) and \( m_t \). Taking into account that no Higgs triplets have been introduced, the effective mass matrix for the left-handed neutrinos is given by

\[
M_{\text{eff}} = -M_D M_R^{-1} M_D^T = -c_{\text{eff}} [\Delta + P_D] Z [\Delta + P_D]^T,
\]

2
where $c_{\text{eff}} = c_D^2/c_R$ and

$$Z \equiv [\Delta + P_R]^{-1} = \frac{1}{a_R b_R} \begin{pmatrix} a_R + b_R + a_R b_R & -b_R & -a_R \\ -b_R & b_R & 0 \\ -a_R & 0 & a_R \end{pmatrix}. \quad (5)$$

It is convenient to define the dimensionless matrix $M_0 \equiv -M_{\text{eff}}/c_{\text{eff}}$ which can be written as

$$M_0 = \Delta Z \Delta + \Delta Z P_D + P_D Z \Delta + P_D Z P_D. \quad (6)$$

Due to the form of $Z$, the first term in $M_0$ gives $\Delta Z \Delta = \left( \sum_{ij} Z_{ij} \right) \Delta = \Delta$, the second and third terms vanish, while the fourth term reduces to

$$P_D Z P_D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{pmatrix} \equiv P_{\text{eff}}, \quad (7)$$

where $x = a_D^2/a_R$ and $y = b_D^2/b_R$. The effective light neutrino mass matrix can then be written as

$$M_{\text{eff}} = -c_{\text{eff}} \left[ \Delta + P_{\text{eff}} \right]. \quad (8)$$

It is interesting to notice that this matrix has the same general form as the matrices $M_l$, $M_D$ and $M_R$, i.e. the seesaw mechanism preserves our ansatz. This is a remarkable feature of the scheme we propose in Eqs. (2) and (3).

## 3 Neutrino masses and mixing

We shall choose the values of the parameters in the mass matrix $M_{\text{eff}}$ so as to satisfy the experimental constraints on neutrino masses and mixings which can be summarized as follows. The Super-Kamiokande atmospheric neutrino data imply $\Delta m^2_{32} \simeq (2 - 6) \times 10^{-3}$ eV$^2$, $\sin^2 2\theta_{23} \geq 0.84$, and the combined data of the solar neutrino experiments lead to four domains of allowed values of $\Delta m^2_{21}$ and $\theta_{12}$ corresponding to the four neutrino oscillation solutions to the solar neutrino problem – large mixing angle MSW (LMA), small mixing angle MSW (SMA), vacuum oscillations (VO) and low-$\Delta m^2$ (LOW) solutions $\text{[4]} \text{[6]}$. For the remaining mixing angle $\theta_{13}$, which determines the element $U_{e3}$ of the lepton mixing matrix, only upper limits exist. The most stringent limit comes from the CHOOZ reactor neutrino experiment $\text{[4]}$, which together with the solar neutrino observations gives $|\sin \theta_{13}| \equiv |U_{e3}| \leq (0.22 - 0.14)$ for $\Delta m^2_{32} = (2 - 6) \times 10^{-3}$ eV$^2$.

\footnote{Most recent Super-Kamiokande data disfavour the SMA and VO solutions at 95% c.l. $\text{[6]}$. However these solutions cannot yet be considered as ruled out and so we discuss them here along with the LMA and LOW solutions.}
The hierarchical structure of the eigenvalues of the mass matrix of charged leptons $M_l$ implies that it is very close to the democratic form, i.e. $|a_l|, |b_l| \ll 1$. The democratic mass matrix $\Delta$ can be diagonalized as $F^T \Delta F = \text{diag}(0, 0, 3)$ with the real orthogonal matrix $F$:

$$F = \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\
-\frac{1}{\sqrt{2}} & \frac{v^2}{\sqrt{6}} & \frac{w}{\sqrt{3}} \\
0 & -\frac{2}{\sqrt{6}} & \frac{v^3}{\sqrt{3}}
\end{pmatrix}. \tag{9}$$

The matrix $U_l$ that diagonalizes $M_l$ can therefore be written as

$$U_l = FW, \tag{10}$$

where, due to the hierarchy $|a_l| \ll |b_l| \ll 1$, the matrix $W$ is close to the unit matrix. Next we will analyze two specific cases of the ansatz of Eq. (2).

### 3.1 The case of real mass matrices

Let us consider that all the parameters $a_k$ and $b_k$ in Eqs. (2) and (3) are real. It is instructive to consider first the limit when the matrix $W$ in Eq. (10) coincides with the unit matrix. The effective mass matrix of light neutrinos $\tilde{M}_{\text{eff}}$, in the basis where the mass matrix of charged leptons has been diagonalized, is then obtained from Eq. (8) through the rotation by the matrix $F$: $\tilde{M}_{\text{eff}} = F^T M_{\text{eff}} F$.

The first matrix on the r.h.s. of Eq. (8) becomes diagonal upon this rotation. Therefore, if it dominates over the second matrix (i.e. if $|x|, |y| \ll 1$), all lepton mixing angles, including $\theta_{23}$, are small. This is phenomenologically unacceptable. Therefore we shall require that the second matrix in Eq. (8), i.e. $P_{\text{eff}}$, either dominates or is of the same order as the first one. Since $P_{\text{eff}}$ is diagonal in the basis where $M_l$ has an (almost) democratic form, in the basis where $M_l$ has been diagonalized, $P_{\text{eff}}$ is non-diagonal and is diagonalized by the matrix $F^T$. This means that, when $P_{\text{eff}}$ dominates in Eq. (8), the lepton mixing matrix $U$ takes the form $U \simeq F^T$, with $F$ given by Eq. (9). Therefore, in this case the mixing angle $\theta_{12}$ responsible for the solar neutrino oscillations is $\theta_{12} = 45^\circ$. One obtains also $\sin^2 2\theta_{23} = 8/9$, which is within the range allowed by the Super-Kamiokande atmospheric neutrino data, and $\theta_{13} = 0$, in agreement with the CHOOZ limit. The value $\theta_{12} = 45^\circ$ is suitable for the VO and LOW solutions of the solar neutrino problem, but leads to slightly too high a value of $\sin^2 2\theta_{12}$ in the case the LMA solution which requires $\sin^2 2\theta_{12} < 0.97$ at 99% c.l. As we shall see, it is easy to satisfy this requirement if one takes into account a small contribution from the first matrix, $\Delta$, in Eq. (8). For the SMA solution, the contribution from $\Delta$ in Eq. (8) should be comparable to that of $P_{\text{eff}}$, otherwise $\theta_{12}$ will be too large.

We now proceed to analyze the effective mass matrix of the left-handed neutrinos. The parameters $a_k, b_k$ and $c_k$ in Eqs. (2) and (3) are related to the masses of charged leptons, up-type quarks and heavy Majorana neutrinos through

$$a_l \simeq 6 \frac{m_e}{m_\tau}, \quad b_l \simeq \frac{9}{2} \frac{m_\mu}{m_\tau},$$
where we have taken into account the mass hierarchies present in the charged leptons and up-type quark sectors. The heavy right-handed neutrino masses are denoted by $M_1$, $M_2$ and $M_3$.

The effective mass matrix of $\nu_L$ in the basis where $M_l$ has been diagonalized takes the form

$$\tilde{M}_{\text{eff}} = -c_{\text{eff}} y \begin{pmatrix} \frac{\varepsilon}{2} & -\frac{\varepsilon}{2\sqrt{3}} & -\frac{\varepsilon}{\sqrt{6}} \\ -\frac{\varepsilon}{2\sqrt{3}} & \frac{2}{3} + \frac{\varepsilon}{6} & -\frac{2}{3\sqrt{2}} + \frac{\varepsilon}{3\sqrt{2}} \\ -\frac{\varepsilon}{\sqrt{6}} & -\frac{2}{3\sqrt{2}} + \frac{\varepsilon}{3\sqrt{2}} & \frac{1}{3} + \frac{\varepsilon}{3} + \delta \end{pmatrix} \equiv -c_{\text{eff}} y \tilde{M}_0. \quad (12)$$

Here

$$|c_{\text{eff}}| y \simeq \frac{3}{2} \frac{m^2_2}{M_2}, \quad \varepsilon \equiv x/y \simeq \frac{4}{3} \left( \frac{m_u}{m_c} \right)^2 \frac{M_2}{M_1}, \quad \delta \equiv 3/y \simeq \frac{2}{3} \left( \frac{m_t}{m_c} \right)^2 \frac{M_2}{M_3}. \quad (13)$$

The above discussed requirement, that the first matrix in Eq. (8) does not dominate over the second one, reduces to the condition $|\delta| \lesssim \max\{|\varepsilon|, 1\}$. We also have to require $|\varepsilon|, |\delta| \ll 1$ in order to have the correct hierarchy $\Delta m^2_{21} \equiv \Delta m^2_{32} \ll \Delta m^2_{atm}$. The largest eigenvalue of the matrix $\tilde{M}_0$ in Eq. (12) is then always close to unity. Thus, the value of $c_{\text{eff}} y$ (and so of $M_2$) can be fixed by the requirement $m^2_3 \simeq \Delta m^2_{atm} = \Delta m^2_{32}$, which gives

$$M_2 \simeq \frac{3}{2} \frac{m^2_2}{\sqrt{\Delta m^2_{atm}}} \simeq 4 \times 10^{10} \text{ GeV}. \quad (14)$$

Using Eqs. (13), it is then easy to find

$$M_1 \simeq \frac{2}{\varepsilon} \frac{m^2_u}{\sqrt{\Delta m^2_{atm}}}, \quad M_3 \simeq \frac{1}{\delta} \frac{m^2_t}{\sqrt{\Delta m^2_{atm}}}. \quad (15)$$

The masses of heavy Majorana neutrinos $M_1$, $M_2$ and $M_3$ are the only free parameters in our model and so all the neutrino masses and leptonic mixing angles can be expressed through them (or, equivalently, through $a_R$, $b_R$ and $M_3 = 3|c_R|$).

Consider first the case $|\varepsilon| \ll |\delta| \ll 1$ relevant for the SMA solution of the solar neutrino problem. In this case the eigenvalues of the neutrino mass matrix $\tilde{M}_{\text{eff}}$ in Eq. (12) are

$$\{m_1, m_2, m_3\} \simeq -c_{\text{eff}} y \left( \frac{\varepsilon}{2}, \frac{2}{3} \delta + \frac{\varepsilon}{2}, 1 + \frac{\delta}{3} \right), \quad (16)$$
and the diagonalization of $M_{\text{eff}}$ results in the following lepton mixing matrix:

$$U \simeq \begin{pmatrix}
\frac{\sqrt{3}}{3} \varepsilon & -\frac{1}{\sqrt{3}} (1 + \frac{2}{3} \delta) & -\frac{\varepsilon \delta}{3\sqrt{2}} \\
\frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} (1 - \frac{4}{3} \delta) & \frac{1}{\sqrt{3}} (1 + \frac{2}{3} \delta) \\
\frac{\sqrt{\frac{2}{3}} \varepsilon}{3} & -\sqrt{\frac{2}{3}} (1 - \frac{4}{3} \delta) & \frac{1}{\sqrt{3}} (1 + \frac{2}{3} \delta)
\end{pmatrix}. \quad (17)$$

From Eqs. (16) and (17) we find

$$\varepsilon \simeq \sin 2\theta_{12} \sqrt{\frac{\Delta m_{21}^2}{\Delta m_{32}^2}}, \quad \delta \simeq \frac{3}{2} \sqrt{\frac{\Delta m_{21}^2}{\Delta m_{32}^2}}, \quad (18)$$

$$\sin^2 2\theta_{23} \simeq \frac{8}{9} \left(1 + \frac{2}{3} \delta\right), \quad \sin \theta_{13} = U_{e3} = -\frac{\varepsilon \delta}{3\sqrt{2}}. \quad (19)$$

Here we have used the fact that the smallness of $|U_{e3}|$ implies $\sin^2 2\theta_{12} \simeq 4U_{e1}^2 U_{e2}^2$, $\sin^2 2\theta_{23} \simeq 4U_{\mu 3}^2 U_{\tau 3}^2$. A sample choice of the values of $a_R$, $b_R$ and $M_3$ that yields the SMA solution and the resulting neutrino parameters are shown in table 1.

We shall now consider the case $|\delta| \simeq |\varepsilon| \ll 1$ which, as we shall see, is relevant for the VO, LOW and LMA solutions of the solar neutrino problem. The eigenvalues of the neutrino mass matrix $\tilde{M}_{\text{eff}}$ in Eq. (12) are

$$\{m_1, m_2, m_3\} \simeq -c_{\text{eff}} y \left\{\frac{\delta}{3} - \frac{\delta^2}{9\varepsilon}, \varepsilon + \frac{\delta}{3} + \frac{\delta^2}{9\varepsilon}, 1 + \frac{\delta}{3}\right\}. \quad (20)$$

The diagonalization of $\tilde{M}_{\text{eff}}$ results in the following lepton mixing matrix:

$$U \simeq \begin{pmatrix}
\cos \theta & \sin \theta & -\frac{\varepsilon \delta}{3\sqrt{2}} \\
\frac{1}{\sqrt{3}} \sin \theta (1 + \frac{2}{3} \delta) & -\frac{1}{\sqrt{3}} \cos \theta (1 + \frac{2}{3} \delta) & -\sqrt{\frac{2}{3}} (1 - \frac{4}{3} \delta) \\
\frac{\sqrt{\frac{2}{3}} \sin \theta (1 - \frac{4}{3} \delta)}{3} & -\sqrt{\frac{2}{3}} \cos \theta (1 - \frac{4}{3} \delta) & \frac{1}{\sqrt{3}} (1 + \frac{2}{3} \delta)
\end{pmatrix}, \quad (21)$$

where

$$\tan \theta = 1 - \frac{2}{3} \frac{\delta}{\varepsilon} + \frac{2}{9} \frac{\delta^2}{\varepsilon^2}. \quad (22)$$

In the limit $|\delta| \ll |\varepsilon|$, $\tan \theta \to 1$ and the mixing matrix $U$ of Eq. (21) becomes the matrix $F^T$ (up to the trivial sign changes due to the rephasing of the neutrino fields), as it should. Notice that for $\delta \neq 0$ the element $U_{e3}$ of the lepton mixing matrix is no longer zero.

From Eqs. (20) - (22) it is easy to find that, to leading order in $U_{e3}$,

$$\tan \theta_{12} = \tan \theta, \quad (23)$$

$$\varepsilon = \sqrt{\tan \theta_{12} \frac{\Delta m_{21}^2}{\Delta m_{32}^2}}, \quad \delta = \frac{3}{2} \left(1 - \sqrt{2} \tan \theta_{12} - 1\right) \sqrt{\tan \theta_{12} \frac{\Delta m_{21}^2}{\Delta m_{32}^2}}, \quad (24)$$

We use the parametrization of the leptonic mixing matrix $U$ which coincides with the standard parametrization of the quark mixing matrix 3.
which substitute for Eq. (18), whereas Eq. (19) remains valid in this case.

Eqs. (14) – (24) allow one to find the values of $a_R, b_R$ and $M_3$ necessary to obtain the relevant solutions of the solar neutrino problem. Sample choices of these parameters, along with the resulting values of neutrino masses and lepton mixing parameters, are shown in table 1. For example, if one chooses $a_R = 1.9 \times 10^{-10}, b_R = 8.0 \times 10^{-8}$ and $M_3 = 2 \times 10^{18}$ GeV, one obtains $\varepsilon = 5 \times 10^{-3}, \delta = 2.3 \times 10^{-4}$ which leads to the LOW solution of the solar neutrino problem. In particular, the solar neutrino mixing is nearly maximal, $\sin^2 2\theta_{12} = 0.999$.

The value $\theta_{12} \simeq 45^\circ$ is also appropriate for the VO solution, but for this solution the best fit corresponds to $\sin^2 2\theta_{12} \simeq 0.7$. Therefore, we require $\delta \simeq \varepsilon$ to have a smaller value of $\sin^2 2\theta_{12}$ than in the case of the LOW solution. Notice that, for the VO solution, we also have to require $\delta \simeq \varepsilon$ in order to avoid unacceptably large values of $M_3$ (see discussion in sec. 4). A choice for the parameters $a_R, b_R$ and $M_3$, very close to the best fit VO solution for $\Delta m_{21}^2$ and $\sin^2 2\theta_{12}$, is shown in the last column of table 1.

The LMA solution also requires $\delta \simeq \varepsilon$ in order for $\theta_{12}$ not to be too close to $45^\circ$. A possible choice of the values of $a_R, b_R$ and $M_3$ and the resulting neutrino parameters are shown in table 1. Notice that in this case $\sin^2 2\theta_{23} \simeq 0.91$ which is slightly larger than the value $8/9$ obtained for the LOW and VO solutions. These values are still slightly smaller than the ones obtained for the SMA solution. From Eq. (19), it is clear that this is due to the fact that, from all the solutions, the SMA requires the largest value of $\delta$.

We shall now take into account that the mass matrix of charged leptons is not exactly of the democratic form. Thus, the matrix $W$ in Eq. (10) deviates slightly from the unit matrix due to nonzero values of $m_e$ and $m_\mu$. Diagonalization of $\tilde M_l \equiv F^T M_l F$ gives

$$W \simeq \begin{pmatrix} 1 & -\frac{m_e}{\sqrt{3} m_\mu} & -\frac{\sqrt{2} m_e}{3 m_\tau} \\ \frac{m_e}{\sqrt{3} m_\mu} & 1 & -\frac{m_\mu}{\sqrt{2} m_\tau} \\ \frac{\sqrt{2}}{3} \frac{m_e}{m_\tau} & \frac{m_\mu}{\sqrt{2} m_\tau} & 1 \end{pmatrix}.$$  \hspace{1cm} (25)

The lepton mixing matrix is now obtained from Eq. (21) or Eq. (17) replacing $U \rightarrow W^T U$. This modifies the lepton mixing parameters. In the case $|\delta| \ll |\varepsilon| \ll 1$ relevant for the VO, LOW and LMA solutions of the solar neutrino problem one finds

$$\sin^2 2\theta_{12} = \sin^2 2\theta \left[ 1 - \frac{4 m_e}{3 m_\mu} \frac{\cos 2\theta}{1 - \cos 2\theta} \left( 1 + \frac{2}{3} \delta \right) \right],$$  \hspace{1cm} (26)

$$\sin^2 2\theta_{23} = \frac{8}{9} \left( 1 + \frac{2}{3} \delta \right) \left[ 1 + \frac{m_\mu}{m_\tau} (1 - 3 \delta) \right],$$  \hspace{1cm} (27)

$$\sin \theta_{13} = U_{e3} = -\frac{\varepsilon \delta}{3 \sqrt{2}} - \frac{\sqrt{2} m_e}{3 m_\mu} \left( 1 - \frac{\delta}{3} \right).$$  \hspace{1cm} (28)

In the limit $m_e, m_\mu \rightarrow 0$ the corresponding expressions of Eqs. (19) and (23) are recovered.
Analogously, in the case $|\varepsilon| \ll |\delta| \ll 1$ relevant for the SMA solution of the solar neutrino problem, one obtains from Eq. (17) and Eq. (25)

$$\sin^2 2\theta_{12} = 4 \left[ \frac{3 \varepsilon}{4 \delta} - \frac{1}{3} \frac{m_e}{m_\mu} \left( 1 + \frac{2}{3} \delta \right) \right]^2,$$

(29)

whereas $\sin^2 2\theta_{23}$ and $U_{e3}$ are again given by Eqs. (27) and (28).

The numerical values of the lepton mixing parameters with contributions from the charged lepton masses included are given in table 2. It is interesting to notice that these contributions tend to increase $\sin^2 2\theta_{23}$, bringing it closer to the Super-Kamiokande best fit value. They also increase significantly the value of the mixing parameter $|U_{e3}|$. Here, these contributions are dominant for all the solutions of the solar neutrino problem except, for LMA, in which case they constitute about 2/3 of $|U_{e3}|$.

### 3.2 Universal strength of Yukawa couplings

We shall now consider a special case of complex parameters $a_k$ and $b_k$, which are of the form

$$a_k = e^{i\alpha_k} - 1, \quad b_k = e^{i\beta_k} - 1,$$

(30)

with real $\alpha_k$ and $\beta_k$. In this case all matrix elements of the matrices $M_k$ in Eq. (2) have the same absolute values, i.e. this is the special case of the so called universal strength of Yukawa couplings (USY) \[8\].

The effective mass matrix of $\nu_L$ can again be written in the form of Eq. (8) with $P_{\text{eff}}$ given by Eq. (7), where $x$ and $y$ are now complex parameters:

$$x = \frac{a_D^2}{a_R} = \frac{2 \sin^2(\alpha_D/2)}{\sin(\alpha_R/2)} e^{i\rho}, \quad y = \frac{b_D^2}{b_R} = \frac{2 \sin^2(\beta_D/2)}{\sin(\beta_R/2)} e^{i\gamma}$$

(31)

with $\rho = \frac{\pi}{2} + \alpha_D - \frac{\alpha_R}{2}$ and $\gamma = \frac{\pi}{2} + \beta_D - \frac{\beta_R}{2}$.

The phases $\alpha_k$, $\beta_k$ and the parameters $c_k$ satisfy, to leading order, the relations of Eq. (11) with the substitution $a_k \rightarrow \alpha_k$, $b_k \rightarrow \beta_k$.

Next, we define the complex parameters $\varepsilon' = x/y$ and $\delta' = 3/y$, and denote $\varepsilon = |\varepsilon'|$, $\delta = |\delta'|$. Since $M_l$ is almost democratic, it can be approximately diagonalized by the matrix $F$. In this approximation, the effective left-handed neutrino mass matrix $\tilde{M}_{\text{eff}}$ in the basis where $\tilde{M}_l$ has been diagonalized, is given by Eq. (12) with $\varepsilon$ and $\delta$ substituted by $\varepsilon'$ and $\delta'$. The leptonic mixing matrix then coincides with the unitary matrix $U$ that diagonalizes $\tilde{M}_{\text{eff}}$. To find this matrix it is convenient to define the Hermitian matrix $\tilde{H}_{\text{eff}} = \tilde{M}_{\text{eff}} \tilde{M}_{\text{eff}}^\dagger = |c_{\text{eff}} y|^2 \tilde{M}_0 \tilde{M}_0^\dagger$, which can be diagonalized through $V^\dagger \tilde{H}_0 V$. The matrices $U$ and $V$ are then related by $U = V^* K$, where $K$ is a diagonal matrix of phases.

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8This approximation amounts to neglecting the small terms of the order of $m_e/m_\mu$, $m_e/m_\tau$ and $m_\mu/m_\tau$ in the unitary matrix $W$ defined in (10), setting it to the unit matrix.
Further simplification can be achieved by noticing that the phases $\rho$ and $\gamma$ are very close to $\pi/2$, which gives $\epsilon' \simeq \epsilon$ and $\delta' \simeq -i\delta$. The matrix $M_0$ is then obtained from Eq. (12) substituting $\delta \rightarrow -i\delta$. All its matrix elements except $(\tilde{M}_0)_{33}$ are real.

We shall first consider the case $|\epsilon| \ll |\delta| \ll 1$ which is relevant for the SMA solution of the solar neutrino problem. In this case, the eigenvalues of $\tilde{H}_0$ leads to the following lepton mixing matrix $U$:

$$
U \simeq \begin{pmatrix}
-i \left(1 + i \frac{3\delta}{\delta} \right) & -i \frac{3\epsilon}{\delta} \left(1 + i \frac{3\delta}{\delta} \right) & i \frac{\epsilon \delta}{\sqrt{3} \delta} \\
\frac{\sqrt{3}}{\delta} \left(1 - i\delta \right) & -\frac{1}{\sqrt{3}} \left(1 - i\delta \right) & -\frac{1}{\sqrt{3}} \left(1 - i\delta \right)
\end{pmatrix}.
$$

(33)

As in the case of real mass matrices, in the limit $|\epsilon| \ll |\delta| \ll 1$ the parameters $\epsilon$ and $\delta$ are related to physical observables through Eq. (13). For the mixing angles one obtains

$$
\sin^2 \theta_{12} \simeq \frac{9 \epsilon^2}{4 \delta^2}, \quad \sin^2 \theta_{23} \simeq \frac{8}{9}, \quad |\sin \theta_{13}| \simeq \frac{\epsilon \delta}{3\sqrt{2}}.
$$

Choosing for the SMA solution the same input parameters as in the case of real mass matrices, we get the results similar to those obtained in the previous subsection (see table 3).

Next, consider the case $|\delta| \lesssim |\epsilon| \ll 1$, relevant for the LMA, LOW and VO solutions. We get the following results for the eigenvalues of $\tilde{H}_0$:

$$
\{m_1^2, m_2^2, m_3^2\} \simeq |c_{\text{eff}} y|^2 \left\{ \frac{\delta^2}{9}, 1 - \frac{\delta^2}{3\epsilon^2} + \epsilon^2 \left(1 + \frac{\delta^2}{3\epsilon^2} + \frac{\delta^4}{27\epsilon^4}\right), 1 + \frac{5}{9}\delta^2 \right\}.
$$

(35)

Introducing $\kappa \equiv \delta/3\epsilon$ one finds for the mixing matrix $U$:

$$
U \simeq \begin{pmatrix}
\frac{1}{\sqrt{2}} \left(1 - 2i\kappa - 4\kappa^4\right) & \frac{1}{\sqrt{2}} \left(1 - 2i\kappa - 4\kappa^2 + 8i\kappa^3 + 12\kappa^4\right) & i \frac{\epsilon \delta}{3\sqrt{2}} \\
\frac{1}{\sqrt{6}} \left(1 - 2\kappa^2 + 2\kappa^4\right) & -\frac{1}{\sqrt{6}} \left(1 + 2\kappa^2 - 6\kappa^4\right) & -\frac{2}{\sqrt{6}} \left(1 - i\delta \right) \\
\frac{1}{\sqrt{3}} \left(1 - 2\kappa^2 + 2\kappa^4\right) & -\frac{1}{\sqrt{3}} \left(1 + 2\kappa^2 - 6\kappa^4\right) & \frac{1}{\sqrt{3}}
\end{pmatrix}.
$$

(36)

Notice that $\kappa \lesssim 1/3$. The mixing angles are

$$
\sin^2 2\theta_{12} \simeq 1 - 16\kappa^4 + 32\kappa^6, \quad \sin^2 2\theta_{23} \simeq \frac{8}{9}, \quad |\sin \theta_{13}| \simeq \frac{\epsilon \delta}{3\sqrt{2}}.
$$

(37)

In table 3 we give the results of the numerical diagonalization of the effective neutrino mass matrix (with no approximations made) and compare them with those obtained using the approximate analytic expressions.
The results for the USY case, presented so far, were obtained neglecting the corrections of the order \(m_e/m_\mu\), \(m_e/m_\tau\) and \(m_\mu/m_\tau\) coming from the diagonalization of \(M_l\), i.e. by replacing \(W\) by the unit matrix. Consider now these small corrections. The lepton mixing matrix will be \(U' = W^\dagger U\). The unitary matrix \(W\) is (up to a diagonal phase transformation) the matrix that diagonalizes \(H'_l = F^T M_l M_l^\dagger F\) as \(W^\dagger H'_l W = \text{diag}(m_e^2, m_\mu^2, m_\tau^2)\). We have obtained the following result:

\[
W \simeq \begin{pmatrix}
1 + \frac{3i}{2} \frac{m_e}{m_\tau} & \frac{m_e}{\sqrt{3} m_\mu} & -i \sqrt{\frac{2}{3}} \frac{m_e}{m_\tau} \\
-\frac{m_e}{\sqrt{3} m_\mu} - i \sqrt{\frac{3}{4}} \frac{m_e}{m_\tau} & 1 + \frac{3i}{4} \frac{m_e}{m_\tau} & i \sqrt{\frac{2}{3}} \frac{m_e}{m_\tau} \\
-i \sqrt{\frac{3}{2}} \frac{m_e}{m_\tau} & i \sqrt{\frac{2}{3} m_\tau} & 1
\end{pmatrix}.
\]

(38)

It is easy to show that, unlike in the case of real mass matrices, the corrections coming from \(W \neq 1\) give negligible contributions to the mixing angles \(\theta_{12}\) and \(\theta_{23}\). This can also be seen by comparing tables 3 and 4, which give the results of the numerical simulations performed with and without contributions of charged lepton masses. These corrections, however, constitute the main contributions to \(|\sin \theta_{13}|\). Indeed, taking them into account one finds

\[
|\sin \theta_{13}| \simeq \sqrt{\frac{2}{9} \left( \frac{m_e}{m_\mu} \right)^2 + \frac{\delta \varepsilon m_e}{6 m_\tau} + \frac{\delta^2 \varepsilon^2}{18}},
\]

(39)

which is valid for all the solutions (SMA, LMA, LOW and VO). In the limit \(m_e/m_\mu, m_e/m_\tau \to 0\) one recovers the result given in Eqs. (34) and (37). However, for the realistic values of parameters, the contributions of the order of \(m_e/m_\tau\) and \((m_e/m_\mu)^2\) are always important.

We shall now discuss briefly the leptonic CP violation. In general, in the case of three light Majorana neutrinos there is a Dirac-type CP violating phase \(\delta_{CP}\) and two additional Majorana-type phases. The latter cannot be observed in neutrino oscillation experiments and we shall not discuss them. The phase \(\delta_{CP}\) can be found from the invariant CP violating parameter \(|J| = |\text{Im} \left[ U_{ij} U_{kl} U_{kl}^* U_{ij}^* \right]|\) and the values of the mixing angles. The values of \(J\) and \(\delta_{CP}\), calculated numerically for all four solutions of the solar neutrino problem, are presented in tables 3 and 4.

### 3.3 Renormalization group effects

So far, we have analysed the phenomenological implications of an ansatz for the charged lepton and neutrino mass matrices, based on the hypothesis of extended democracy. In this section, we study the behaviour of the ansatz under the renormalization group. This analysis is especially important due to the fact that if the observed pattern of fermion masses...
and mixings reflects some flavour symmetry of the lagrangian, it is natural to assume that
this symmetry is manifest at a high energy scale and analyse its predictions at low energies
by studying its evolution under the renormalization group. In the SM context, the lowest
dimension operator that can generate a Majorana mass term for the left-handed neutrinos
is uniquely given by
\[ -\frac{1}{2}\kappa \nu^T \nu H H + h.c \] (40)
where \( \kappa \) is proportional to \( \tilde{M}_0 \) in Eq. (12), and \( H \) is the neutral component of the usual SM
Higgs doublet. Assuming that \( \kappa \) is defined at \( M_R \), we will study the stability of the implied
pattern of neutrino masses and mixings by analysing the renormalization group equation
(RGE) for the operator \( \kappa \) which can be written, at one loop level, as [10]:
\[
16\pi^2 \frac{d\kappa}{dt} \simeq \left[ 2\lambda + 6Y_t^2 - 3g_2^2 \right] \kappa - \frac{1}{2} \left[ \kappa Y_e^T Y_e + (Y_e^T Y_e)^T \kappa \right],
\]
where \( t = \log \Lambda \), and \( g_2, \lambda, Y_t, Y_e, \Lambda \) are the \( SU(2) \) gauge coupling, the quartic Higgs
coupling, the top Yukawa coupling, the charged leptons Yukawa couplings matrix and the
scale at which \( \kappa \) is evaluated, respectively. This equation allows us to relate the effective
mass matrix at \( M_R \) with the one at low energies, say at the scale of the \( Z \) boson mass \( M_Z \),
in the following way [11]:
\[
\tilde{M}_0(M_Z) \simeq A \text{diag}(1, 1, 1 + \eta) \tilde{M}_0(M_R) \text{diag}(1, 1, 1 + \eta),
\]
where \( \tilde{M}_0(M_Z) \), \( \tilde{M}_0(M_R) \) are the matrices \( \tilde{M}_0 \) at the scales of \( M_Z \) and \( M_R \), respectively, and
\( \eta \) is approximately given by:
\[
\eta \approx \frac{Y_t^2}{32\pi^2} \log \frac{\Lambda(M_R)}{M_Z} = \frac{1}{32\pi^2} \left( \frac{m_\tau}{v} \right)^2 \log \frac{\Lambda(M_R)}{M_Z},
\]
where \( m_\tau \) is the \( \tau \) mass and \( v \) denotes the vacuum expectation value of the neutral Higgs
doublet component \( \phi \). It is straightforward to see that the transformation defined in Eq. (12)
corresponds to multiplying the third row and the third column of \( \tilde{M}_0(M_R) \) by the parameter
\( 1 + \eta \). The overall factor \( A \) in Eq. (12) comes from the terms in the first square bracket
on the right-hand side of Eq. (11) and it will affect the values of the neutrino masses but
not the structure of the mass matrix. Taking into account the form of \( \tilde{M}_0(M_R) \) given in
Eq. (12), we get, neglecting second order terms:
\[
\tilde{M}_0(M_Z) \propto \tilde{M}_0(M_R) + \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -\frac{2\sqrt{2}}{3}\eta \\
0 & -\frac{2\sqrt{2}}{3}\eta & \frac{2}{3}\eta
\end{pmatrix}
\]
(44)
In the previous sections, we saw that our results implied a large hierarchy between the
right-handed neutrino masses which can lie between \( 10^6 \) GeV and \( 10^{18} \) GeV, corresponding

\(^5\) We consider \( m_\tau = 1.777 \text{ GeV}, M_Z = 91.187 \text{ GeV} \) and \( v = 174 \text{ GeV} \).
to the lightest and heaviest $\nu_R$, respectively (see tables 1 - 4). Using this range for the scale $\Lambda(M_R)$, we get $3.1 \times 10^{-6} \lesssim \eta \lesssim 1.2 \times 10^{-5}$. Since $10^{-4} \lesssim \epsilon, \delta \lesssim 10^{-1}$ (see tables 1 - 4), it can be readily verified that even for $\eta = 1.2 \times 10^{-5}$ the effect of the RGE on structure of the effective neutrino mass matrix is negligible. The point is that although the overall scale factor of $M_{\text{eff}}$ may run significantly, the structure of this matrix is quite stable. Therefore, the results we have obtained at low energies will still be valid if we impose the hypothesis of extended democracy, with universal breaking for all the leptonic mass matrices, at a high energy scale.

4 Discussion

We have proposed a model for lepton masses and mixing with a high predictive power. With just three free parameters (the masses of heavy right-handed neutrinos $M_1$, $M_2$ and $M_3$ or, equivalently, $a_R$, $b_R$ and $M_3$) we predict seven physical quantities – the masses of light neutrinos $m_1$, $m_2$, $m_3$, the mixing angles $\theta_{12}$, $\theta_{23}$ and $\theta_{13}$, and the CP-violating phase $\delta_{\text{CP}}$. Depending on the values of the input parameters, all four solutions of the solar neutrino problem (SMA, LMA, LOW and VO) can be obtained. Representative choices for the input parameters and the resulting values of light neutrino masses, lepton mixing angles and CP-violating parameters are given in tables 1 - 4. We have performed both exact numerical and approximate analytic diagonalizations of the light neutrino effective mass matrix. As follows from tables 1 - 4, our analytic expressions give a very accurate approximation to the exact results. We have also demonstrated that the structure of the effective mass matrix of light neutrinos is stable against the RGE evolution.

A characteristic feature of our scenario is a large hierarchy of all neutrino masses, including the masses of heavy Majorana neutrinos [12]. This is because we assume all the fundamental lepton mass matrices to have a nearly democratic form. It is interesting to notice that, for all the solutions of the solar neutrino problem, the values of $M_2$, i.e. the mass of the second heavy Majorana neutrino, are nearly the same. This stems from the fact that $M_2$, approximately given by Eq. (14), is related to $\Delta m_{\text{atm}}^2$, and is practically independent from $\Delta m_{\text{sol}}^2$ and $\theta_{12}$.

Physically, the mass $M_3$ of the heaviest right-handed Majorana neutrino must not exceed $M_{\text{Pl}} = 1.2 \times 10^{19}$ GeV. Eq. (15) then gives a lower bound on $|\delta|$: $|\delta| \geq 4.2 \times 10^{-5}$. However, for real lepton mass matrices, in the case of the LMA, LOW and VO solutions, it follows from the second equation in (24) that $|\delta|$ decreases with $\tan \theta_{12} \to 1$. Therefore values of $\theta_{12}$ too close to 45° are not allowed in our scenario. For the LMA and LOW solutions, this restriction is not severe (the values of $\sin^2 2\theta_{12}$ as large as $1 - 4 \cdot 10^{-8}$ for LMA and $1 - 3 \cdot 10^{-5}$ for LOW are allowed), but for the VO solution $\sin^2 2\theta_{12}$ must not exceed 0.967. Interestingly, in this case, the value of $\sin^2 2\theta_{12}$ giving the best fit of the solar neutrino data, is not close to 1. As follows from the first equality in (37), in the complex USY case $\sin^2 2\theta_{12}$ can be very close to unity.
It should be noticed that for LOW and VO solutions $M_3 \sim 10^{18}$ GeV (see tables 1 - 4), i.e. is close to the reduced Planck scale which is presumably the string scale. In these cases our results might be affected by new physics at this scale.

It can be seen, if one compares tables 1 and 2 (and Eqs. (19) and (27)), that in the case of real neutrino mass matrices the corrections of order $m_\mu/m_\tau$, coming from the deviation of the charged lepton mass matrix from the exact democratic form, increase the values of $\sin^2 2\theta_{23}$, bringing them closer to the Super-Kamiokande best-fit value $\sin^2 2\theta_{23} = 1$. At the same time, in the case of complex parameters the corrections to $\sin^2 2\theta_{23}$ are of the order $(m_\mu/m_\tau)^2$, i.e. they are negligible (compare tables 3 and 4). This comes about because the terms of the order $m_\mu/m_\tau$ in the matrix $W$ in Eq. (38) are purely imaginary, unlike those in Eq. (28).

The corrections to $\sin^2 2\theta_{12}$ due to nonzero $m_e$ and $m_\mu$ are small (of the order $m_e/m_\mu$) in the case of real lepton mass matrices and totally negligible in the case of complex USY matrices. In contrast to this, the corresponding contribution to $|U_{e3}| = |\sin \theta_{13}|$, though of the order $m_e/m_\mu$, is dominant (compare tables 1 and 2, 3 and 4 and also Eqs. (19) and (28)). For the SMA, LOW and VO solutions of the solar neutrino problem we find $|U_{e3}| \approx (\sqrt{2}/3)(m_e/m_\mu) \approx 2.3 \times 10^{-3}$, whereas for the LMA solution it is slightly larger. Unfortunately, these values are too small to be experimentally probed in currently planned long-baseline neutrino oscillation experiments.

The values for $|U_{e3}|$ that we have found are different from the predictions of Ref. [13] obtained under the assumption of no fine tuning between the elements $(\tilde{M}_{\text{eff}})_{12}$ and $(\tilde{M}_{\text{eff}})_{13}$ of the neutrino mass matrix in the basis where the mass matrix of charged leptons has been diagonalized. The reason for this is that, in our case, there is an approximate equality $(\tilde{M}_{\text{eff}})_{12} \sin \theta_{23} + (\tilde{M}_{\text{eff}})_{13} \cos \theta_{23} \approx 0$, which is exactly the kind of relation which was excluded from the consideration in [13]. This relation can be traced back to an approximate symmetry underlying $L_{\text{mass}}$ in Eqs. (11) - (3). Thus, our scheme provides an example of the case in which the predictions of [13] do not apply.

In the case of complex lepton mass matrices, we predict relatively large values for the CP-violating phase $\delta_{CP}$ in the case of SMA and VO solutions, and small values in the case of LMA and LOW solutions. The contributions due to nonzero $m_e$ and $m_\mu$ are very important in this case – they increase the CP-violating parameter $|J|$ by 2 - 6 orders of magnitude for the SMA, LOW and VO solutions and decrease it by 2 orders of magnitude for the LMA solution. Unfortunately, CP-violating effects in neutrino oscillations cannot be experimentally probed in our scheme because of the smallness of $|J|$, which is mainly due to the smallness of the mixing angle $\theta_{13}$.

In conclusion, we have suggested a simple structure for the leptonic mass matrices, with the remarkable feature that a simple explanation is provided for the large mixing in the leptonic sector, in contrast with the quark sector. It is well known that, in the quark sector, one may obtain the correct mass spectrum and mixing pattern [14] starting with a democratic matrix for the up and down quarks and adding a small perturbation which
generates the masses of the two light generations, as well as the small mixing present in the Cabibbo-Kobayashi-Maskawa matrix. In the scheme we have proposed, all leptonic mass matrices are treated in an entirely analogous way, i.e. they are, in leading order, all proportional to the democratic matrix, with a small universal breaking of democracy. The large mixing in the leptonic sector results from the seesaw mechanism, which is the crucial new ingredient, only present in the leptonic sector.

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### Table 1
- Results for exact numerical and approximate diagonalizations of $\tilde{M}_{ee}$ in the case of real lepton mass matrices, with sample choices of input parameters leading to the different solutions of the solar neutrino problem, not including the corrections due to nonzero $m_e$ and $m_\mu$.

| INPUT          | LMA          | SMA          | LOW          | VO           |
|----------------|--------------|--------------|--------------|--------------|
| $a_R$          | $1.5 \times 10^{-9}$ | $8.5 \times 10^{-3}$ | $1.9 \times 10^{-10}$ | $4.1 \times 10^{-9}$ |
| $b_R$          | $1.3 \times 10^{-5}$ | $2.2 \times 10^{-5}$ | $8.0 \times 10^{-8}$ | $3.9 \times 10^{-8}$ |
| $M_3$ (GeV)    | $1.3 \times 10^{-16}$ | $7.6 \times 10^{-15}$ | $2.0 \times 10^{-18}$ | $4.3 \times 10^{-18}$ |
| $\delta$      | $3.27 \times 10^{-2}$ | $6.30 \times 10^{-2}$ | $2.29 \times 10^{-4}$ | $1.12 \times 10^{-4}$ |
| $\sin^2 2\theta_{13}$ | Numerical: $0.95$ | $0.94$ | $5.1 \times 10^{-3}$ | $5.0 \times 10^{-3}$ |
| $\sin^2 2\theta_{32}$ | Numerical: $0.95$ | $0.96$ | $0.97$ | $0.94$ |
| $|U_{e3}|$     | $3.27 \times 10^{-3}$ | $3.17 \times 10^{-3}$ | $2.27 \times 10^{-3}$ | $2.28 \times 10^{-3}$ |

### Table 2
- Changes to table 1 if one includes the corrections due to nonzero $m_e$ and $m_\mu$.

| INPUT          | LMA          | SMA          | LOW          | VO           |
|----------------|--------------|--------------|--------------|--------------|
| $a_R$          | $1.5 \times 10^{-9}$ | $8.5 \times 10^{-3}$ | $1.9 \times 10^{-10}$ | $4.1 \times 10^{-9}$ |
| $b_R$          | $1.3 \times 10^{-5}$ | $2.2 \times 10^{-5}$ | $8.0 \times 10^{-8}$ | $3.9 \times 10^{-8}$ |
| $M_3$ (GeV)    | $1.3 \times 10^{-16}$ | $7.6 \times 10^{-15}$ | $2.0 \times 10^{-18}$ | $4.3 \times 10^{-18}$ |
| $\delta$      | $3.27 \times 10^{-2}$ | $6.30 \times 10^{-2}$ | $2.29 \times 10^{-4}$ | $1.12 \times 10^{-4}$ |
| $\sin^2 2\theta_{13}$ | Numerical: $0.95$ | $0.94$ | $5.1 \times 10^{-3}$ | $5.0 \times 10^{-3}$ |
| $\sin^2 2\theta_{32}$ | Numerical: $0.95$ | $0.96$ | $0.97$ | $0.94$ |
| $|U_{e3}|$     | $3.27 \times 10^{-3}$ | $3.17 \times 10^{-3}$ | $2.27 \times 10^{-3}$ | $2.28 \times 10^{-3}$ |
Table 3 – Results for exact numerical and approximate diagonalizations of \( \tilde{M}_{\text{eff}} \) in the case of complex USY lepton mass matrices, with sample choices of input parameters leading to the different solutions of the solar neutrino problem, not including the corrections due to nonzero \( m_\ell \) and \( m_u \). The CP-violating parameters \( J \) and \( |\delta_{CP}| \), were also computed.

Table 4 – Changes to table 3 if one includes the corrections due to nonzero \( m_\ell \) and \( m_u \).