Dissipative spin chains: Implementation with cold atoms and steady-state properties

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We propose a quantum optical implementation of a class of dissipative spin systems, including the XXZ and Ising model, with ultra-cold atoms in optical lattices. Employing the motional degree of freedom of the atoms and detuned Raman transitions we show how to obtain engineerable dissipation and a tunable transversal magnetic field, enabling the study of the dynamics and steady-states of dissipative spin models.

As an example of effects made accessible this way, we consider small spin chains and weak dissipation and show by numerical simulation that steady-state expectation values display pronounced peaks at certain critical system parameters. We show that this effect is related to degeneracies in the Hamiltonian and derive a sufficient condition for its occurrence.

I. INTRODUCTION

Quantum spin models play a fundamental role for the theoretical and experimental study of quantum many-body effects. They represent paradigmatic systems exhibiting, e.g., quantum phase transitions and peculiar forms of matter [1]. They also provide toy models for description of many solid state systems. Ultra-cold atoms in optical lattices [2] have emerged as a system that is especially suited to study the low-energy sector of quantum spin systems with the promise to eventually simulate theoretical models in large, controlled quantum systems.

To observe these effects, coupling to uncontrolled degrees of freedom has to be kept to a minimum, since it leads to dissipation and decoherence [3, 4] which can mask or destroy the quantum effects. But in recent years, it has been shown how the coupling to an environment can be harnessed to generate useful quantum states [5] or perform quantum information tasks [9, 10]. Moreover, the study of the phase diagram of open quantum systems has turned into a fruitful direction itself [11–16].

Our aim in the present work is twofold: In the first part of the paper, we propose a scheme to realize a quantum spin system using ultra-cold atoms in an optical lattice in which both coherent interaction and dissipation can be engineered and controlled, enabling the study the nonequilibrium and steady-state physics of open and driven spin systems. In the second part, we highlight a peculiar feature of the steady-state diagram for small spin chains: in the limit of weak dissipation, abrupt changes of steady-state expectation values for certain critical values of the system parameters are observed. We explain this feature and relate it to degeneracy properties of the system Hamiltonian and derive a sufficient condition for the occurrence of sharp peaks at critical system parameters.

II. PHYSICAL IMPLEMENTATION OF A ONE-DIMENSIONAL SPIN CHAIN UNDER DISSIPATION

Ultra-cold bosonic atoms in optical lattices are ideal candidates to simulate spin Hamiltonians. Different theoretical and experimental approaches [17] have been employed to simulate quantum spin chains in optical lattices, for example by optical driving of two hyperfine levels of cold bosons in the Hubbard regime [18]. Recently, a one-dimensional chain of interacting Ising spins has been implemented experimentally using a Mott-Insulator of spinless bosons in a tilted optical lattice [19].

In the following, we show theoretically how to add engineered dissipation to the toolbox of these systems [20, 21]. Specifically, we show how to implement a system with the following properties: (i) dissipative dynamics of Lindblad form, (ii) a tunable magnetic field in x-direction and (iii) an effective spin Hamiltonian such as, e.g., the XXZ, Heisenberg or Ising model. In the next Subsections, we first introduce the setup and explain qualitatively how such a one-dimensional spin chain in a tunable magnetic field under engineerable dissipation can be realized with cold atoms in optical lattices. In the subsequent Subsections we give specific requirements and parameters and details of the derivation for (i)-(iii).

A. Setup and qualitative description

The system we consider is an optical lattice populated with a single atomic bosonic species. We assume to be in the Mott-insulator regime with filling factor one, where the on-site interaction is much larger than the tunneling (hopping) between neighboring lattice sites. In this regime, the atoms are localized such that each lattice potential is occupied with one atom. We aim to use the motional ground and first excited state of the atom (denoted by $|0\rangle, |1\rangle$ [22], respectively) to realize an effective spin-1/2 system in each lattice site. To access the motional degree of freedom optically, we work in the Lamb-Dicke regime where the motion of the atom is restricted to a
region small compared with the laser wavelength. We make use of the anharmonicity of the lattice potential and, as explained in the following, of decay of the atoms. The dynamics to the two-dimensional subspace of $|0\rangle, |1\rangle$ (see Fig. 1). For the optical manipulation, we assume that the atoms have internal degrees of freedom that can be addressed with laser fields. We consider a Λ-scheme with two ground states $|g\rangle$ and $|r\rangle$ (both trapped by the same optical lattice potential) and an excited state $|e\rangle$. The level scheme of the internal states of the atoms is shown in Fig. 1. Off-resonant laser fields drive transitions between the two ground states $|g\rangle$ and $|r\rangle$ and the excited state $|e\rangle$. The system decays fast into the ground states, and, as we show below, effectively decays into the state $|g\rangle$. Therefore, the atoms are optically pumped to the state $|g\rangle \otimes |0\rangle$ and the states $|r\rangle$ and $|e\rangle$ can be adiabatically eliminated. Eliminating the excited state $|e\rangle$ leads to the effective two-level system in the lower part of Fig. 1 with designable decay rates. Further elimination of the state $|r\rangle$ leads to an effective description in the internal ground state $|g\rangle$. The optical couplings by laser fields give rise to effective Hamiltonians and effective dissipation (cooling) in the ground state $|g\rangle$ at each lattice site. Details are given in Section II B. In summary, we obtain an effective two-level system at each lattice site with Hilbert space spanned by $|g\rangle \otimes |0\rangle$ and $|g\rangle \otimes |1\rangle$ as depicted in Fig. 2.

In the following sections, we show that engineering the optical couplings as above leads to an effective master equation for the two-level system $|0\rangle, |1\rangle$ that describes (i) decay from $|1\rangle$ to $|0\rangle$ and (ii) an effective magnetic field in $x$-direction. In the Mott insulator regime, tunnel couplings between neighboring lattice wells can be treated as a perturbation, which [iii] leads to an effective spin Hamiltonian. The resulting master equation [24] is given by

$$\dot{\rho}_t = \sum_k A^- (2\sigma_k^- \rho_k \sigma_k^+ - \{\sigma_k^+ \sigma_k^- , \rho_k\}_+) - i [H, \rho_t].$$

Here, $\sigma_k^\dagger = |1\rangle \langle 0|_k$ is the operator that excites an atom at lattice site $k$ from the motional state $|0\rangle$ to state $|1\rangle$ and $\sigma_k^\dagger = (\sigma_k^\dagger)^\dagger$. The sum runs over all $N$ sites of the optical lattice potential. The first part in Eqn. (1) describes decay from state $|1\rangle$ into state $|0\rangle$ as depicted in Fig. 3. It is derived in Section II B. The decay parameter $A^-$ can be tuned by changing the Rabi frequencies of the lasers and the detunings and is given by Eqn. (18) in Section II E. The Hamiltonian is given by $H = H_B + H_{\text{spin}}$, where $H_B$ describes the magnetic field in $x$-direction defined by

$$H_B = \sum_k B_x (\sigma_k^+ + \sigma_k^-),$$

where $B_x$ is proportional to an effective magnetic field in $x$-direction. It is derived in Section II C. The Hamiltonian $H_{\text{spin}}$ describes the spin Hamiltonian

$$H_{\text{spin}} = \sum_k \alpha_1 (\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y) + \alpha_2 \sigma_k^z \sigma_{k+1}^z,$$

as derived in Section II D from a tunnel coupling between neighboring sites. The parameters $\alpha_1$ and $\alpha_2$ depend on the properties of the optical lattice potential and can be tuned. Therefore, the Hamiltonian $H_{\text{spin}}$ describes the XXZ model, the Ising model or the Heisenberg model. In the following three Sections, we employ a perturbative approach to derive a master equation comprising dissipation of Lindblad form (i) as in Eqn. (1), a magnetic field in $x$-direction (ii) as in Eqn. (2) and an effective spin Hamiltonian (iii) as in Eqn. (3). For the sake of clarity, we derive (i)-(iii) in three separate steps employing
the approximation of independent rates of variation as explained in [23].

B. Optical couplings of internal atomic states: dissipation of Lindblad form

In this Section, we show that optically addressing the atoms with suitably tuned lasers allows to engineer decay as in Eqn. (1).

We consider the internal levels $|g⟩$, $|r⟩$, $|e⟩$ of an atom at site $k$. The ground states $|g⟩$ and $|r⟩$ can be coupled via the excited state $|e⟩$ by a detuned Raman transition of two standing wave laser fields with Rabi frequencies $Ω_1$ and $Ω_2$. Eliminating the excited state $|e⟩$ leads to an effective coupling between $|g⟩$ and $|r⟩$ (see Fig. 1) with $Ω_{\text{eff}} = Ω_1Ω_2/δ_{\text{r}}$, where $δ_{\text{r}}$ is the detuning with respect to $|e⟩$ (for details see Appendix A). To induce controlled dissipation, we couple $|r⟩$ and $|e⟩$ by an additional off-resonant laser field (indicated by a red arrow) in Fig. 1. Then adiabatic elimination of the excited state $|e⟩$ leads to an effective two-level system (as shown in the lower part of Fig. 1) with states $|r⟩$ and $|g⟩$ which has designable decay rates $Γ$ and $γ$ as derived in [24] (see also Appendix A). Thereby, the excited state $|e⟩$ that is broadened by spontaneous emission is eliminated, and the effective two-level system $|g⟩, |r⟩$ allows to resolve the motional states $|0⟩$ and $|1⟩$ of the lattice potential (note that we are in the Lamb-Dicke regime), as can be seen in Fig. 2. Under appropriate resonance conditions that will be specified in the following, the atoms are excited from state $|1⟩ ⊗ |g⟩$ to state $|0⟩ ⊗ |r⟩$ and spontaneously decay into the state $|0⟩ ⊗ |g⟩$ as shown in Fig. 2. Adiabatically eliminating the state $|r⟩$, this corresponds to an effective decay from state $|1⟩ ⊗ |g⟩$ into $|0⟩ ⊗ |g⟩$. Thus the atoms effectively remain in the internal ground state $|g⟩$, such that the decay can be written as an effective decay from state $|1⟩$ to $|0⟩$ as depicted in Fig. 3.

In Appendix A we derive in a perturbative approach (that corresponds to an adiabatic elimination of the state $|r⟩$) a master equation that describes the dynamics of the two-level system $|0⟩$, $|1⟩$ of the atom. Assuming that the driving of level $|r⟩$ is sufficiently weak such that $|Ω_{\text{eff}}| ≪ Γ, γ, ν, |δ_{\text{r}}|$, and that the level broadening remains small

$$\Gamma + γ < ν,$$

the master equation is given by

$$\dot{ρ}_i = \sum_k A^− (2 σ^{−}_k ρ_i σ^{+}_k − \{ σ^{+}_k σ^{−}_k , ρ_i \} + ) + A^+ (2 σ^{+}_k ρ_i σ^{−}_k − \{ σ^{−}_k σ^{+}_k , ρ_i \} + ) − i[H^{(1)}_{\text{eff}}, ρ_i],$$

Here, $A^+ \approx A^−$ is the strength of the strength of the heating terms and $A^− \approx A^+$ the strength of the decay terms. For simplicity, $A^±$ are chosen to be independent of the lattice site $k$. $A^±$ can be made dependent on the lattice site $k$ by choosing different phases of the driving lasers as explained in Appendix A. Note that $A^+ ≪ A^−$ is required for the validity of the approximation that restricts to the $|0⟩$ and $|1⟩$ subspace. $A^−$ and $A^+$ are given by

$$A^± = Ω^2_{\text{eff}}η^2 \left( \frac{Γ + γ}{(Γ + γ)^2 + (δ_{\text{r}} ± ν)^2} \right).$$

Here, $δ_{\text{r}}$ is the effective detuning given by Eqn. (A5) in Appendix A. $η_1 = k_1/\sqrt{2 M ν}$ is the Lamb-Dicke parameter where $k_1$ is the wave number of the laser with Rabi frequency $Ω_1$, $M$ the atomic mass, and $ν$ denotes the energy difference between the motional state $|0⟩$ and $|1⟩$ of the lattice potential. The Hamiltonian $H^{(1)}_{\text{eff}}$ in the last term in Eqn. (6) is given by

$$H^{(1)}_{\text{eff}} = \sum_k ν |1⟩⟨1|_k + H_S,$$

where $H_S$ describes AC Stark shifts on the motional levels that are $≪ ν$ and are given in more detail in Appendix A. Now, we have everything at hand to implement dissipation. If

$$δ_{\text{r}} ≈ ν,$$

which can be achieved by choosing the laser frequency $ω_i$ in $δ_{\text{r}} = ω_i − ω_l$ accordingly, the strength of the dissipation is much larger than the strength of the heating:

$$A^+ ≪ A^−.$$

Then, the master equation has only decaying terms and is of the form

$$\dot{ρ}_i = \sum_k A^− (2 σ^{−}_k ρ_i σ^{+}_k − \{ σ^{+}_k σ^{−}_k , ρ_i \} + ) − i[H^{(1)}_{\text{eff}}, ρ_i],$$

It describes decay of the atoms from state $|1⟩$ into $|0⟩$, while the atoms effectively remain in the internal state $|g⟩$. By adiabatic elimination of the internal state $|r⟩$, we have thus shown that a master equation can be derived that can be tuned such that it describes almost pure decay.
while the atoms remain in the internal ground state \( |\text{g}\rangle \) and the excited state \( |e\rangle \) of the atom. Right: Adiabatic elimination of the excited state \( |e\rangle \) leads to an effective magnetic field in \( x \)-direction (see Section II C), which drives transitions between the motional states \( |0\rangle \) and \( |1\rangle \).

**C. Optical couplings of internal atomic states:**

**Effective magnetic field in \( x \)-direction**

To derive the effective magnetic field in \( x \)-direction, we consider a detuned Raman transition. Two standing wave laser fields with Rabi frequencies \( \Omega_a \) and \( \Omega_b \) couple the internal ground state \( |\text{g}\rangle \) and the excited state \( |e\rangle \) of the atoms as depicted in Fig. 4. The coupling is described by the Hamiltonian

\[
H_{ab} = \sum_k \Omega_a \cos (k_a x_k) |e\rangle \langle g|_k \\
+ \Omega_b \sin (k_b x_k) |e\rangle \langle g|_k + \text{h.c.},
\]

where \( k_a, k_b \) denote the wave numbers of the lasers and \( x_k \) the displacement from the equilibrium position of the atom at lattice site \( k \). As we are in the Lamb-Dicke regime, \( \sin (k_b x_k) \approx \eta_b (\sigma^-_k + \sigma^+_k) \) \cite{footnote1} and \( \cos (k_a x_k) \approx 1 \). Under the condition

\[
|\Omega_a|, |\Omega_b| \ll |\delta_e|,
\]

where \( \delta_e \) is the detuning of the driving lasers as depicted in Fig. 4, the excited state \( |e\rangle \) can be adiabatically eliminated and we get an effective Hamiltonian

\[
H^{(2)}_{\text{eff}} = H_B = \sum_k B_x \sigma^x_k,
\]

that describes a tunable magnetic field in \( x \)-direction, where \( B_x \) is proportional to the effective magnetic field strength in \( x \)-direction is given by

\[
B_x = \frac{2 \Omega_a \Omega_b \eta_b}{\delta_e}. \tag{12}
\]

FIG. 4. (color online) Level scheme and transitions used to implement the transverse magnetic field. Left: A detuned Raman transition couples the internal ground state \( |\text{g}\rangle \) and the excited state \( |e\rangle \) of the atom. Right: Adiabatic elimination of the excited state \( |e\rangle \) leads to an effective magnetic field in \( x \)-direction (see Section II C), which drives transitions between the motional states \( |0\rangle \) and \( |1\rangle \).

Thus, we have derived an effective magnetic field in \( x \)-direction that drives transitions between the motional states \( |0\rangle \) and \( |1\rangle \) (as depicted in Fig. 4 on the right), while the atoms remain in the internal ground state \( |\text{g}\rangle \).

**D. Effective spin Hamiltonian**

In the Mott-Insulator regime, bosonic atoms trapped by a lattice potential with two motional states are described by the two-band Bose-Hubbard model \cite{footnote2} (see Appendix B). We denote the on-site interaction by \( U_{01} \), \( U_{00} \) and \( U_{11} \) \cite{footnote2} and by \( t_0 \) (\( t_1 \)) the amplitudes for atoms in state \( |0\rangle \) (\( |1\rangle \)) to tunnel to neighboring lattice sites. We assume that the on-site interaction \( U_{01}, U_{00}, U_{11} \gg t_0, t_1 \) such that tunneling between neighboring wells that leads to states with two atoms in one lattice well can be treated as a perturbation (see Fig. 5). Using second order perturbation theory \cite{footnote2} (for a detailed derivation see Appendix B), we derive an effective spin Hamiltonian \( H_{\text{spin}} \) given by

\[
H^{(3)}_{\text{eff}} = H_{\text{spin}} + B_z \sum_k |1\rangle \langle 1|_k, \tag{14}
\]

with

\[
H_{\text{spin}} = \sum_k \alpha_1 (\sigma^+_k \sigma^-_{k+1} + \sigma^+_k \sigma^-_{k+1}) + \alpha_2 \sigma^+_k \sigma^-_{k+1}, \tag{15}
\]

where \( \alpha_1 = -4t_0 t_1 / U_{01}, \alpha_2 = 2 (t_0^2 + t_1^2) / (2U_{01}) - t_0^2 / U_{00} - t_1^2 / U_{11} \), the magnetic field in \( z \)-direction

\[
B_z = t_0^2 / U_{00} - t_1^2 / U_{11}. \tag{16}
\]

FIG. 5. (color online) Tunneling between neighboring lattice wells with tunnel amplitudes \( t_0 \) and \( t_1 \). States with two atoms per lattice well are treated in perturbation theory in Section II D as the on-site interaction is much larger than the tunneling amplitudes.

**E. Dissipative one-dimensional spin chain in a magnetic field**

In the previous Sections, we showed — for the sake of clarity in separate steps — that optical couplings of the internal levels can be engineered such that we obtain a
master equation of Lindblad form [Eqn. (6)] fulfilling the demands (i)-(iii) of tunable dissipation, spin-interaction, and tunable transverse field. Combining these results, one has to carefully consider the order of magnitude of each term. Doing so, we find that the magnetic field in $z$-direction $B_z$ in Eqn. (14) and the Stark shifts in Eqn. (8) can be of the same order of magnitude as $\nu$. Stark shifts and $B_z$ lead to an effective energy difference between the motional states $|0\rangle$ and $|1\rangle$ given by

$$\tilde{\nu} = \nu + B_z + s_- - s_+,$$

where $B_z$ is defined in Eqn. (15) and $s_-, s_+$ are the AC Stark shifts in Eqn. (8) (see Appendix A). Therefore, combining all results, the laser detuning $\delta_r$ that enters in the $A^{\pm}$ has to be adjusted to $\tilde{\delta}_r$ such that $\tilde{\delta}_r - \nu = \delta_r - \tilde{\nu}$ which means that $\delta_r = \tilde{\delta}_r \pm (B_z + s_- - s_+)$. Then, combining the results from Eqns. (6), (13) and (14), the master equation reads

$$\dot{\rho}_t = \sum_k A^+(2\sigma_k^+ \rho_k \sigma_k^- - \{\sigma_k^+ \sigma_k^-, \rho_k\})_+$$

$$+ A^- (2\sigma_k^- \rho_k \sigma_k^+ - \{\sigma_k^- \sigma_k^+, \rho_k\})_+ - i[H, \rho_t],$$

where the rates $A^\pm$ are modified by the renormalized $\tilde{\delta}_r$:

$$A^\pm = \Omega_{\text{eff}}^2 \eta \gamma (\Gamma + \gamma) \pm \delta_r \pm \nu)^2.$$  \hspace{1cm} (18)

The Hamiltonian part of the master equation is given by

$$H = H_{\text{spin}} + H_B + \tilde{\nu} \sum_k |1\rangle\langle 1|_k,$$  \hspace{1cm} (19)

where $H_{\text{spin}}$ is given by Eqn. (15) and $H_B$ by Eqn. (13). The magnetic field in $z$-direction and Stark shifts have been included in $\tilde{\nu}$. For $\delta_r \approx \nu$, as shown before, decay dominates over heating: $A^- \gg A^+$. Then, the master equation has only decaying terms and Eqn. (17) describes a dissipative XXZ spin chain in a magnetic field with both $x$ and $z$ components. However, only $B_x$ is fully tunable, while $B_z$ is large (compared to $B_x, A^{\pm}$) and required to be so by the conditions for adiabatic elimination, cf. Eqn. (1). However, an effective dissipative XXZ chain without any field in $z$-direction would be advantageous for observing critical behavior in the steady-state dynamics that we study in the next Sections. Therefore, we transform to a frame rotating with $\tilde{\nu}$. In the rotating frame, $H_B$ becomes time-dependent. To obtain a time-independent field in $x$-direction, the detuned Raman lasers that lead to the effective magnetic field $B_x$ have to be chosen time-dependent, adapted to the rotating frame (i.e., suitably detuned from the two-photon resonance). This then yields a time-independent transversal magnetic field, and the master equation in the rotating frame is then given by

$$\dot{\rho}_t = \sum_k A^+(2\sigma_k^+ \rho_k \sigma_k^- - \{\sigma_k^+ \sigma_k^-, \rho_k\})_+ - i[H_{\text{spin}} + H_B, \rho_t].$$  \hspace{1cm} (20)

It corresponds to the master equation given by Eqn. (1). In summary we have shown how to implement a one-dimensional spin chain with nearest-neighbor interaction described by the XXZ or the Ising model and a tunable effective magnetic field in $x$-direction under dissipation. This system is an ideal testbed for studying steady-state dynamics of dissipative spin models as discussed in the next Section. Note, that since we are in a rotating frame, observables other than the collective spin operator $\langle J_z \rangle$ become explicitly time-dependent.

### F. Steady-state behavior: Discontinuous steady-state behavior related to spectrum of Hamiltonian

A particular important characterization of dissipative dynamics is through their steady state: if it is unique (or distinguished by some conserved quantity) it allows for robust preparation of these states. Abrupt changes in the steady state as system parameters are varied may signal dissipative quantum phase transitions \cite{12,16,30,31}. We study the steady-state behavior of short spin chains under dissipation in a magnetic field in $x$-direction with numerical simulations. We find that the one-dimensional
FIG. 7. (Color online) Ising model with 4 spins with open boundary conditions in a transverse magnetic field $B_x$ and under local dissipation of form given by Eq. (23). Upper part: Steady-state expectation value $\langle J^x \rangle$ plotted versus $B_x/\alpha_3$. Peaks are observed that become more narrow for decreasing dissipation strength. Lower part: Spectrum of the Hamiltonian plotted versus $B_x/\alpha_3$. Peaks in the steady-state expectation value (upper part) appear at degeneracy points of the Hamiltonian that are marked with black circles.

XXZ model with 4 spins as given by Eqn. (15), where we chose as a typical example $\alpha_1 = \frac{1}{4} \alpha_2$, shows a surprising behavior: Changing the external magnetic field in $x$-direction, peaks occur in the steady-state expectation values of the collective spin operators $J^{x/z} = \sum_k \sigma_k^{x/z}$ for weak dissipation, see Fig. 6. Here, we have considered dissipation as in Eqn. (20) with equal dissipation strength on each spin. We find that decreasing the strength of the dissipation the peaks become more narrow and each peaks height approaches a finite value. For small $\gamma$ we observe very narrow peaks. This indicates a discontinuity in the steady-state expectation values of the spin operators. We find, that these narrow peaks appear exactly at points where the Hamiltonian becomes degenerate. In the following section we study this phenomenon in more generality.

FIG. 8. (Color online) Ising model with 6 spins with periodic boundary conditions in a magnetic field $B_x$ and under collective dissipation of the form given by Eq. (24) in the translation and reflection symmetric subspace $T = R = 1$. Upper part: Steady-state expectation value $\langle J^x \rangle$ plotted versus $B_x/\alpha_3$. Lower part: Spectrum of the Hamiltonian plotted versus $B_x/\alpha_3$.

III. DISCONTINUITIES IN THE STEADY-STATE DYNAMICS OF A GENERAL CLASS OF ONE-DIMENSIONAL SPIN MODELS UNDER DISSIPATION

In the previous section we saw that for the one-dimensional XXZ model, peaks in the steady-state expectation values of the collective spin operators appear, that are closely related to the spectrum of the Hamiltonian. In the following, we study in more generality, independent of a physical implementation, local one-dimensional spin Hamiltonians under dissipation of different kinds. We present a condition that elucidates the discontinuous behavior of the steady state at degeneracy points of the Hamiltonian. Then, we study and explain this condition in more detail for Ising Hamiltonians.

A. Numerical studies of discontinuous behavior in the steady state

We numerically simulate short spin chains. First, we study the one-dimensional Ising model with open boundary conditions, described by the Hamiltonian

$$H = H_{zz} + H_B$$

(21)
with
\[ H_{zz} = \alpha_3 \sum_k \sigma_k^+ \sigma_{k+1}^-, \quad (22) \]
and \( H_B \) as in Eqn. (23) subject to local or collective decay with Lindblad operators \( \sigma_k^+ \) or \( \sum_k \sigma_k^- \), respectively. The master equation describing the full system with local dissipation is given by
\[ \dot{\rho}_k = \sum_k \gamma_k \left( 2\sigma_k^- \rho_k \sigma_k^+ - \{ \sigma_k^+ \sigma_k^-, \rho_k \} + i[H, \rho_k] \right). \quad (23) \]

Changing the magnetic field \( B_x \), we find that for weak dissipation the steady-state expectation values of the spin operators \( \langle J^x \rangle \) and \( \langle J^z \rangle \) change abruptly at particular values of \( B_x \), see Fig. 7. Here, we have considered dissipation as in Eqn. (23) with equal dissipation strength on each spin, \( \gamma_k = \gamma \). Decreasing the strength of the dissipation, i.e., decreasing \( \gamma \), the peaks become more narrow and their height converges to some finite value, while the expectation value vanishes elsewhere. For \( \gamma \to 0 \), we observe very narrow peaks, which indicate discontinuities in the steady-state expectation values of the spin operators. We find, that these narrow peaks appear only at degeneracy points of the spectrum of the Hamiltonian. I.e., to every peak found at some value of \( B_x = x_0 \) for \( \gamma \to 0 \), at least one pair of degenerate eigenvalues \( \lambda_1, \lambda_2 \) of the local spin Hamiltonian \( H_{zz} \) can be found, i.e., \( \lambda_1(x) = \lambda_2(x) \) at \( x = x_0 \). Note that the discontinuities in the steady state at critical system parameters are only observed for \( \lambda \neq 0 \). I.e., the (weak) dissipation allows us to gain information about the Hamiltonian’s properties that is not readily accessible in the case of \( \gamma = 0 \).

This effect can be observed for different kinds of spin Hamiltonians such as, for example, the XXZ model (see Fig. 7), both for periodic and open boundary conditions. Moreover, changing the type of dissipation, the observed behavior does not change qualitatively. E.g., collective dissipation, which describes the dynamics of spins all coupled to the same bath and leads to the master equation
\[ \dot{\rho}_t = \gamma (2J^- \rho_t J^+ - \{ J^+ J^-, \rho_t \} + i[H, \rho_t]), \quad (24) \]
where \( J^\pm = \sum_k \sigma_k^\pm \), also leads to discontinuous behavior in the steady-state expectation values as shown in Fig. 8 for the Ising model. Choosing an “inhomogeneous” dissipation which is of the form of the dissipative part in Eqn. (23), where now the strengths of the dissipation \( \gamma_k \) are different for each spin, peaks can be observed for an even larger class of spin Hamiltonians: For \( \gamma_k = \gamma \), and \( H = H_H + H_B \), where \( H_H \) is the Heisenberg spin Hamiltonian, we do not observe any peaks. However, if we choose different dissipation strengths \( \gamma_k \) for each spin, we find peaks at the degeneracy points of the Hamiltonian, as can be seen in Fig. 9.

### B. General condition for discontinuities in the steady state

Since the Liouvillian depends smoothly on the system parameters, the observed discontinuities must be related to degeneracies in the spectrum of \( \mathcal{L} \). As we shall see, in the weak dissipation limit they are directly related to degeneracy points of the Hamiltonian.

We consider a system described by the master equation
\[ \dot{\rho}(t) = \mathcal{L} \rho \equiv [\mathcal{L}_0(x) + \gamma \mathcal{L}_1] \rho(t), \quad (25) \]
with a Hamiltonian \( H(x) \) depending (analytically) on a parameter \( x \). For simplicity, we consider the case that \( H_0(x) \) is non-degenerate for \( x \neq x_0 \). The term \( \mathcal{L}_1 \) contains dissipative terms and is independent of \( x \). We are interested in the limit of weak dissipation \( \gamma \to 0 \) and in the change of the steady state at the degeneracy point \( x = x_0 \).

The steady state \( \rho_{ss}(x) \) is determined by \( \mathcal{L}(x) \rho_{ss}(x) = 0 \) and can be determined perturbatively. The kernel of \( \mathcal{L}_0(x) \) is highly degenerate, being spanned by all eigen-projectors \( |\lambda_i(x)\rangle \langle \lambda_i(x)| \) of the (non-degenerate) \( H_0(x) \). This degeneracy is lifted by \( \mathcal{L}_1 \) and the steady state for \( \gamma \to 0 \) is for \( x \neq x_0 \) given by
\[ P^D(x) \mathcal{L}_1 P^D(x) \rho_{ss}(x) = 0, \quad (26) \]
where
\[ P^D(x) \rho = \sum_i |\lambda_i(x)\rangle \langle \lambda_i(x)| \rho |\lambda_i(x)\rangle \langle \lambda_i(x)|. \quad (27) \]

The possibility of discontinuous behavior of \( \rho_{ss}(x) \) at \( x = x_0 \) arises from the enlargement of the kernel of \( \mathcal{L}_0(x) \) at this point: if \( \lambda_i \) and \( \lambda_j \) become degenerate at \( x = x_0 \) then coherences between the corresponding eigenvectors (i.e., \( |\lambda_i(x)\rangle \langle \lambda_j(x)|, i \neq j \)) become stationary at \( x = x_0 \). We denote by \( P^\Delta \) the projector on these additional elements in the kernel of \( \mathcal{L}_0(x_0) \) [22]. As we show in Appendix C a discontinuity \( \rho_{ss}(x_0) \neq \lim_{x \to x_0} \rho_{ss}(x) \) arises if
\[ P^\Delta \mathcal{L}_1 \lim_{x \to x_0} \rho_{ss}(x) \neq 0, \quad (28) \]
i.e., if \( \mathcal{L}_1 \) couples the steady state to the newly available subspace \( P^\Delta \) in the kernel of \( \mathcal{L}_0 \). For simplicity, we made the assumption that the Hamiltonian is non-degenerate for \( x \neq x_0 \). If the Hamiltonian does have degeneracies outside \( x_0 \), but additional eigenvectors become degenerate at \( x = x_0 \) the argumentation follows identical lines, as also in this case, \( \mathcal{L}_1 \) can couple the steady state to a newly available subspace \( P^\Delta \).

Let us have another look at Figs. 6-9 in the light of the previous paragraph. Clearly, all the sharp isolated peaks occur for values of \( B_x \) (which plays the role of the
with small δB pressures how similar fidelity \[34\] between two density matrices, which measures all degeneracy points of the Hamiltonian when Eq.(28) holds. For weak dissipation this happens close to all degeneracy points of the Hamiltonian when Eq. (28) holds.

Note also, that in Figs. 6-9 a large feature appears in the steady-state expectation value \(\langle J^x \rangle\) around \(B_x = 0\). It narrows for decreasing \(\gamma\), but is not a sharp peak for any of the parameters used for \(\gamma\). This broad peak represents the effect of one (or several, cf. Figs. 6 and 9) unresolved degeneracies around \(B_x = 0\). Note that for all spin models considered, the degeneracy of their respective Hamiltonian is very high at \(B_x = 0\) and is lifted slowly (certain eigenvalues touch and do not cross as \(B_x \to 0\)). Therefore, the finite values of \(\gamma\) used in the plots are not much smaller than all energy differences and we are not in the weak dissipation limit. As \(\gamma\) is reduced, additional peaks are resolved (cf. Fig. 9).

C. Steady-state behavior for Ising Hamiltonians

To get a better insight into how the condition given by Eqn. \[28\] explains the peaks seen in Fig. 8, we now specialize to the Ising model under collective dissipation given by Eqn. \[24\]. Then we see that the steady state apart from the degeneracy points and the condition for discontinuity become very simple. For a detailed derivation of what follows, see Appendix.
The Hamiltonian in Eqn. [21] with periodic boundary conditions is in general degenerate due to translational and reflection symmetry. To obtain a non-degenerate Hamiltonian, we restrict our consideration to a specific subspace with eigenvalue 1 for the translation operator $T$ and the reflection operator $R$ [35]. Note that the Hamiltonian is also symmetric under the spin-flip operation $F = \sigma_z^N$, i.e., $FHF^\dagger = H$. Using the properties of $L_1$ and $F$-invariance of $H$, we find that if the system has a unique steady state of $\mathcal{L}(x)$, it is, in the limit of weak dissipation, given by the maximally mixed state $\propto \mathbb{1}$: plugging $\mathbb{1}$ into Eqn. [20] we obtain $P^D L_1(\mathbb{1}) = \sum_i |\lambda_i(x)\rangle \langle \lambda_i(x)| J^z |\lambda_i(x)\rangle \langle \lambda_i(x)|$ and flip invariance of $H$ implies $\langle \lambda_i(x)| J^z |\lambda_i(x)\rangle = 0$ for the eigenstates of a non-degenerate Hamiltonian $H$ (see Appendix [C]).

Thus if the steady state is unique, it is always maximally mixed outside degeneracy points and we see a discontinuity at $x = x_0$ if for the degenerate eigenstates $|\lambda_1(x_0)\rangle, |\lambda_2(x_0)\rangle$ we have

$$\langle \lambda_1| J^z |\lambda_2\rangle \neq 0.$$  \hfill (29)

This can be checked to hold for the points at which peaks are observed in Fig. [8].

For the Ising model in a transverse magnetic field, for larger $N$ the peaks decrease in height, and disappear in the limit $N \rightarrow \infty$. The spectrum for the Ising model in a transverse field is known analytically [35]. For large $N$, the spectrum is very dense and degeneracy points are so closely spaced that peaks are no longer resolvable (and vanish in the thermodynamic limit as bands develop). Nevertheless, for small spin systems, these features provide a method to dissipatively study degeneracies of the applicable Hamiltonian – anywhere in the spectrum, not just in the ground state. To the extent that $L_1$ is tunable, it even provides access to the nature of the degenerate states via Eqn. [28].

IV. CONCLUSIONS

We have shown that using cold atoms in an optical lattice in the Mott-insulator regime, dissipative spin chains with Hamiltonians such as the XXZ model, the Ising model or the Heisenberg model can be realized. Optical driving of internal atomic states allows for the realization of a tunable transversal magnetic field and engineered dissipation.

This system is an ideal testbed for studying steady-state dynamics of dissipative spin models. We have discovered a peculiar feature of the steady state diagram for small spin chains: in the limit of weak dissipation, the expectation values of the collective spin operators exhibit abrupt changes that hint at discontinuities in the steady state. These discontinuities occur at degeneracy points of the Hamiltonian. We have studied this phenomenon for different spin models with open and periodic boundary conditions subject to individual and collective dissipation. Finally, we have presented conditions that elucidate the discontinuous behavior of the steady state at degeneracy points of the Hamiltonian. Therefore, measurements of the steady state dynamics of cold atoms in optical lattices would allow to draw conclusions on the spectrum of the respective spin model.

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Appendix A: Derivation of effective dissipative master equation

The internal levels of the atom that we consider are $|g\rangle, |r\rangle, |e\rangle$. Adiabatically eliminating the excited state $|e\rangle$ as discussed in Section IIIB, we get an effective two-level system $|g\rangle$ and $|r\rangle$ that is coupled with the effective Rabi frequency $\Omega_{\text{eff}}$ as depicted in Fig. [1].

In the following, we derive in detail the master equation given by Eqn. [6] in Section IIIB. The internal levels of the atom that we consider are $|g\rangle$, $|r\rangle$, $|e\rangle$, as depicted in the upper part of Fig. [1]. The states $|g\rangle, |r\rangle$ are coupled by a detuned Raman transition via the excited state $|e\rangle$ by two standing wave laser fields. The coupling is described by the Hamiltonians

$$H_{11} = \sum_k \Omega_1 \cos (kx_0) (|e\rangle \langle g|)_k + \text{h.c.}, \quad \text{(A1)}$$

and

$$H_{12} = \sum_k \Omega_2 \sin (kx_0) (|r\rangle \langle g|)_k + \text{h.c.}, \quad \text{(A2)}$$

where $\Omega_1$ and $\Omega_2$ are the Rabi frequencies of the two lasers and $k_1$ and $k_2$ are the wave numbers of the lasers and $k$ denotes the lattice site. $x_0$ is the displacement from the equilibrium position $x_k$ of the atom at lattice site $k$. The phase of the lasers is for simplicity chosen such that $\cos [k_1 (x_k + x_0)] = \cos (k_1x_0)$ and $\cos [k_2 (x_k + x_0)] = \sin (k_2x_0)$. Choosing different phases of the lasers makes $A^\pm$ in Eqn. [7] dependent on the lattice site $k$. Adiabatic elimination of the excited state $|e\rangle$ leads to an effective coupling

$$H_1 = \sum_k \Omega_{\text{eff}} \eta_1 (\sigma^-_k + \sigma^+_k) (|r\rangle \langle g|)_k + \text{h.c.}, \quad \text{(A3)}$$

with $\Omega_{\text{eff}} = \Omega_1 \Omega_2 / \delta_\epsilon$ where $\delta_\epsilon$ is the detuning with respect to $|e\rangle$ and $\eta_1$ is the Lamb-Dicke parameter. Here, we have expressed the deviation from equilibrium position, $x_0$, in terms of harmonic oscillator operators truncated to the two lowest lying levels $\sin(k_1x_k) \approx \delta_\epsilon \sin(k_1x_0)$.
According to [26], the effective detuning and the effective frequency $\Omega$ between states $|r\rangle$ and $|g\rangle$ is given by

$$
\eta_{10}^H \approx \frac{(\Omega_{r_1}/2)^2}{\delta_1^2 + (\Gamma_{r_1} + \Gamma_{g})^2/4} \ll 1.
$$

(A4)

According to [26], the effective detuning and the effective decay rates are given by:

$$
\delta_r = \delta_{g_1} - \delta_{r_1} = \frac{(\Omega_{r_1}/2)^2}{[(\Gamma_{g_1} + \Gamma_{r_1})/2]^2 + \delta_{r_1}^2},
$$

(A5)

$$
\Gamma = \frac{(\Omega_{r_1}/2)^2}{[\Gamma_{r_1} + \Gamma_{g}]^2 + \delta_{r_1}^2 
$$

$$
\gamma = \frac{(\Omega_{r_1}/2)^2}{[\Gamma_{r_1} + \Gamma_{g}]^2 + \delta_{r_1}^2},
$$

(A6)

see also the lower part of Fig. 4. The effective two-level system $|g\rangle-|r\rangle$ with the effective decay rates $\Gamma$, $\gamma$ and the effective detuning $\delta_r$ is the starting point of the following discussion. The full Hamiltonian describing the system is given by

$$
H_{\text{full}} = H_1 + H_0,
$$

(A8)

where $H_1$ describes the atom-light interaction given by Eqn. (A3) and $H_0$ defines the energies of the system

$$
H_0 = \sum_k \delta_r |r\rangle \langle r|_k + \nu |1\rangle \langle 1|_k.
$$

(A9)

The effective dynamics of the system can be derived considering contributions to the Liouvillian up to second order in a perturbative approach. The full system is described by a Liouvillian given by:

$$
\rho(t) = (L_0 + L_1)\rho(t),
$$

(A10)

where $L_0$ is given by

$$
L_0 \rho(t) = \sum_k \Gamma \left( 2 |g\rangle \langle r|_k \rho(t) |r\rangle \langle g|_k - |r\rangle \langle r|_k , \rho(t) \right)_+ 
$$

$$
+ \gamma \left( 2 |r\rangle \langle r|_k \rho(t) |r\rangle \langle r|_k - |r\rangle \langle r|_k , \rho(t) \right)_+ 
$$

$$
- i [H_0, \rho(t)].
$$

(A11)

The first part of the Liouvillian is the decay part with the effective decay rate $\Gamma$ from state $|r\rangle$ to $|g\rangle$ and the dephasing rate $\gamma$. The projector

$$
P_g = |g\rangle \langle g| \otimes (|0\rangle \langle 0|_s + |1\rangle \langle 1|_s)
$$

(A12)

is stationary under $L_0$. The perturbative part of the Liouvillian is given by

$$
L_1 \rho(t) = -i \left[ H_1, \rho(t) \right],
$$

(A13)

where $H_1$ is given by Eqn. (A3) and describes the interaction of the two-level system with the effective laser field. Treating $L_1$ as a perturbation, we derive an effective Liouvillian in the stationary subspace of $L_0$. The projection onto this subspace reads

$$
\mathbb{P} \dot{\rho}(t) = \mathbb{P} L \mathbb{P} \rho(t) + \mathbb{P} L \mathbb{P} \rho(t),
$$

(A14)

where $\mathbb{P} \rho = |g\rangle \langle g| \otimes (|0\rangle \langle 0| + |1\rangle \langle 1|)$ and $Q = 1 - P$. Projecting onto the subspace we want to eliminate we get

$$
\mathbb{Q} \dot{\rho}(t) = \mathbb{Q} L \mathbb{Q} \rho(t).
$$

(A15)

In the following, we integrate Eqn. (A15) to get the time evolution of the density matrix in the fast space, $\dot{\rho}(t)$. We insert the result in Eqn. (A14) to get an equation of motion for the density matrix in the slow space. Therefore, we first go into the interaction picture, where the density matrix is given by $\dot{\rho}(t) = e^{-i\mathcal{L}t} \rho(t)$. The equation of motion in the fast space reads

$$
\mathbb{Q} \dot{\rho}(t) = \mathbb{Q} W_1(t) \dot{\rho}(t),
$$

(A16)

with $W_1(t) = e^{\mathcal{L}t} \mathcal{L} e^{\mathcal{L}(-t)}$. Solving this equation by iteration [37] we get

$$
\mathbb{Q} \rho(t) = e^{\mathcal{Q} \mathcal{L} t} \mathbb{Q} \rho(0) 
$$

$$
+ \int_0^t ds_1 \int_0^{s_1} ds W_1(s_1) W_1(s_2) \mathbb{Q} \rho(0).
$$

(A17)

At time $t = 0$, $\dot{\rho}(0) = \rho(0)$ and we assume that at $t = 0$, the population is in the ground state, i.e., $\rho(0) = \mathbb{P} \rho(0)$. Higher order integrals are neglected making the assumption that

$$
|\Omega_{\text{eff}}| \ll \Gamma, \gamma, |\delta_r|, \nu.
$$

(A18)

We denote the first integral in Eqn. (A17) by $R_1(t)$ and the second integral by $R_2(t)$ such that

$$
\mathbb{Q} \rho(t) = R_1(t) + R_2(t).
$$

(A19)

Inserting in Eqn. (A14) leads to

$$
\mathbb{P} \dot{\rho}(t) = \mathbb{P} L \mathbb{P} \rho(t) + \mathbb{P} L_0 R_1(t) + \mathbb{P} L_1 R_1(t)
$$

$$
+ \mathbb{P} L_0 R_2(t) + \mathbb{P} L_1 R_2(t).
$$

(A20)

The term $\mathbb{P} L_0 R_1(t) = 0$, and $\mathbb{P} L_1 R_2(t)$ is a third order term and can be neglected. Neglecting terms rotating with $\exp(\pm i\nu t)$ we get the master equation given by Eqn. (6) with AC Stark shifts given by

$$
H_S = s_- \sigma_+^k \sigma_-^k + s_+ \sigma_-^k \sigma_+^k,
$$

(A21)

where

$$
s_{\pm} = \Omega_{\text{eff}}^2 \eta_{10}^H \left( \delta_{10} \pm \nu \right)
$$

$$
\left( \Gamma + \gamma \right)^2 + \left( \delta_r \pm \nu \right)^2.
$$
Appendix B: Derivation of the spin Hamiltonian

In the Mott-Insulator regime, bosonic atoms trapped by a lattice potential with two motional states are described by the two-band Bose-Hubbard model

\[ H_{BH} = H_0 + H_t. \] (B1)

Here, the sum runs over the \( N \) sites \( k \) of the optical lattice. The unperturbed Hamiltonian \( H_0 \) is given by

\[ H_0 = \sum_k \left( \frac{\hbar w_0}{2} \hat{n}_{k0} \hat{n}_{k1} + \sum_{x=0,1} \frac{U_x}{2} \hat{n}_{kx} \left( \hat{n}_{kx} - 1 \right) + \nu \hat{n}_{k1} \right), \]

where \( U_{xx'} \) is the on-site repulsion of two atoms on lattice site \( k \), where one atom is in motional state \( |x\rangle \) and the other one in \( |x'\rangle \) with \( x, x' = 0, 1 \), respectively. The operator \( \hat{n}_{kx} = |x\rangle \langle x|_k \) counts the number of atoms at lattice site \( k \) in the motional states \( x = 0, 1 \) and \( \nu \) is the energy difference between ground and first excited motional states. We assume the system to be prepared in the ground state \( |0\rangle \). Due to the anharmonicity of the potential, we do not leave the subspace of \( n = 0 \) and \( n = 1 \) excitations.

The perturbative part of the Hamiltonian describes the tunneling between neighboring lattice sites and is given by

\[ H_t = \sum_k t_0 c_{k0}^\dagger c_{k+1,0} + t_1 c_{k,1}^\dagger c_{k+1,1} + \text{h.c.} \] (B2)

Here, the operators \( c_{kx} \) with \( x = 0, 1 \) are bosonic destruction operators for atoms in the two motional states \( |0\rangle \) and \( |1\rangle \) at lattice site \( k \). \( t_0, t_1 \) are the tunneling amplitudes from state \( |0\rangle \) \( (|1\rangle) \) at lattice site \( k \) to state \( |0\rangle \) \( (|1\rangle) \) at \( k + 1 \). As the on-site interaction \( U_{xx'} \gg t_0, t_1 \), tunneling between neighboring wells that leads to states with two atoms in one lattice well can be treated as a perturbation. For that, consider two neighboring lattice sites \( k \) and \( k + 1 \) and write the effective Hamiltonian in the basis of eigenvectors of \( H_0 \), \( |x_k, y_{k+1}\rangle \), where for example \( |0_k, 1_{k+1}\rangle \) is the notation for the state with one particle in well \( k \) in state \( |0\rangle \) and one particle in well \( k + 1 \) in state \( |1\rangle \). In perturbation theory [25], the second-order effective Hamiltonian can be evaluated in the following way:

\[ \langle x_k, y_{k+1} | H^{(3)}_{\text{eff}} | x'_{k}, y'_{k+1} \rangle = \frac{1}{2} \sum_{\chi} \langle x_k, y_{k+1} | H_t | \chi \rangle \frac{1}{E_{\chi}} \langle \chi | H_t | x'_{k}, y'_{k+1} \rangle, \] (B3)

where

\[ \frac{1}{E_{\chi}} = \frac{1}{E_{x\chi} - E_{\chi}} + \frac{1}{E_{x'\chi} - E_{\chi}}, \]

and \( |\chi\rangle \) are eigenstates of \( H_0 \) with two particles in one well (and no particle in the other one). \( E_{x\chi} = \langle x_k, y_{k+1} | H_0 | x_k, y_{k+1} \rangle \) and \( E_{\chi} = \langle \chi | H_0 | \chi \rangle \) are the unperturbed energies. Evaluating Eqn. [B3] leads to the effective spin Hamiltonian \( H^{(3)}_{\text{eff}} \) given by:

\[ H^{(3)}_{\text{eff}} = H_{\text{spin}} + B_z \sum_k |1\rangle \langle 1 |, \] (B4)

Here,

\[ H_{\text{spin}} = \sum_k \alpha_1 (\sigma^x_k \sigma^x_{k+1} + \sigma^y_k \sigma^y_{k+1}) + \alpha_2 \sigma^z_k \sigma^z_{k+1}. \] (B5)

Here, \( \alpha_1 = -\frac{4t_0 t_1}{U_{01}} \),

\[ \alpha_2 = 2 \left( \frac{t_0^2 + t_1^2}{U_{01}} - \frac{t_0^2}{U_{00}} - \frac{t_1^2}{U_{11}} \right), \]

and \( B_z \), the magnetic field in \( z \)-direction is

\[ B_z = \frac{t_0^2}{U_{00}} - \frac{t_1^2}{U_{11}}. \]

Thus, we have derived an effective XXZ-spin Hamiltonian with a magnetic field in \( z \)-direction.

Appendix C: Condition for discontinuous behavior

Here, we first derive a general condition for the discontinuous behavior in the steady state at a degeneracy point of a large class of spin Hamiltonians. Then, we focus on more specific Hamiltonians. We study the steady state of flip-invariant Hamiltonians outside the degeneracy point and, starting with the general condition for finding discontinuities in the steady state, we derive a more precise condition for flip-invariant Hamiltonians.

1. General condition for discontinuities in steady state

Here, we derive a general condition for discontinuous behavior in the steady state at the degeneracy point \( x = x_0 \) of a general Hamiltonian \( H \), where \( H = H(x) \) is an analytic function of \( x \). We consider a system described by the master equation

\[ \dot{\rho}(t) = (\mathcal{L}_0 + \mathcal{L}_1) \rho(t), \] (C1)

where the Hamiltonian part of the Liouvillian is given by \( \mathcal{L}_0(x) = \mathcal{L}_0 = -i[H(x), \cdot] \) and depends on a parameter \( x \), and the local decay Liouvillian is

\[ \mathcal{L}_1 \rho(t) = \sum_k \gamma_k \left( 2 \sigma_k^- \rho(t) \sigma_k^+ - \{ \sigma_k^+ \sigma_k^-, \rho(t) \} \right). \] (C2)
First, we want to describe the system outside the degeneracy point, i.e., for \( x \neq x_0 \). We assume that in the vicinity of \( x_0 \), the Hamiltonian is nondegenerate (for \( x \neq x_0 \)) and that the dissipation is weak. The steady state \( \rho_{ss}(x) \) defined by \((\mathcal{L}_0(x) + \mathcal{L}_1)\rho_{ss}(x) = 0\) is, in the limit \( \gamma \to 0 \) given by

\[
P^D(x)\mathcal{L}_1P^D(x)\rho_{ss}(x) = 0, \quad (C3)
\]

where \( P^D(x) \) is the projector onto the kernel(\( \mathcal{L}_0 \)). As the kernel of \( \mathcal{L}_0 \) is spanned by the eigeprojectors \(|\lambda_i(x)\rangle\langle\lambda_i(x)|\) of \( H \) we have for arbitrary \( A \):

\[
P^D A = \sum_i |\lambda_i(x)\rangle\langle\lambda_i(x)| A|\lambda_i(x)\rangle\langle\lambda_i(x)|, \quad (C4)
\]

where \(|\lambda_i(x)\rangle\) are eigenstates of the Hamiltonian \( H(x) \) which is assumed to be nondegenerate.

Now, let us consider the case that at \( x = x_0 \), the Hamiltonian has a degeneracy point at which two or more eigenvalues cross. At this degeneracy point, we expect a discontinuous behavior of the steady state that leads to the peaks we observe in our numerical simulation (see Figs. 6-9). At \( x = x_0 \) the projector onto the kernel of \( \mathcal{L}_0 \) has to be extended. It now also projects onto coherences between eigenstates of \( H: |\lambda_1\rangle,|\lambda_2\rangle \) which are eigenvectors to the degenerate eigenvalues \( \lambda_1 = \lambda_2 \). Therefore the projector on the coherences reads:

\[
P^\Delta A = |\lambda_1\rangle\langle\lambda_1| A|\lambda_2\rangle\langle\lambda_2| + h.c. \quad (C5)
\]

It is convenient to define a continuous extension of the projector \( P^D \) at \( x = x_0 \) which reads

\[
P^D(x_0) = \lim_{x \to x_0} P^D(x). \quad (C6)
\]

Thus, at \( x = x_0 \), the full projector onto the kernel of \( \mathcal{L}_0 \) reads \( P^D(x_0) + P^\Delta \). Now the condition for the steady state \( \rho_{ss}(x) = x_0 \) at the degeneracy point is given by

\[
[P^D(x_0) + P^\Delta] \mathcal{L}_1 [P^D(x_0) + P^\Delta] \rho_{ss}(x_0) = 0. \quad (C7)
\]

We want to find a sufficient condition for the steady state to change discontinuously. This means that

\[
\rho_{ss}(x_0) - \lim_{x \to x_0} \rho_{ss}(x) \neq 0, \quad (C8)
\]

where \( \lim_{x \to x_0} \rho_{ss}(x) \) is the continuous extension of \( \rho_{ss}(x) \) for any \( x \neq x_0 \) to \( x = x_0 \).

A discontinuity in the steady state as described by Eqn. (C8) can occur only if

\[
[P^D(x_0) + P^\Delta] \mathcal{L}_1 [P^D(x_0) + P^\Delta] \lim_{x \to x_0} \rho_{ss}(x) \neq 0, \quad (C9)
\]

holds, since otherwise the continuous extension \( \lim_{x \to x_0} \rho_{ss}(x) \) would be a steady state as well. The last part of Eqn. (C9) can be simplified using

\[
[P^D(x_0) + P^\Delta] \lim_{x \to x_0} \rho_{ss}(x) = \lim_{x \to x_0} \rho_{ss}(x),
\]

which holds since \( \lim_{x \to x_0} \rho_{ss}(x) \) is per definition in the space onto which \( P^D(x_0) \) projects and \( P^\Delta \) is orthogonal to that space. By Eqn. (C3) we then see that Eqn. (C9) reduces to the condition

\[
P^\Delta \mathcal{L}_1 \lim_{x \to x_0} \rho_{ss}(x) \neq 0. \quad (C10)
\]

If this condition is fulfilled, then \( \rho_{ss}(x_0) - \lim_{x \to x_0} \rho_{ss}(x) \neq 0 \) which means that the steady state shows discontinuous behavior at the degeneracy point \( x = x_0 \).

2. Condition for discontinuous behavior for Ising Hamiltonians

Here, we want to get a better insight how the condition given by Eqn. (C10) relates to the peaks observed in our numerical simulation. In the following, we will apply it to the Ising model in a transverse magnetic field. In the numerical simulation (see Fig. 8) for the Ising Hamiltonian with periodic boundary conditions under collective dissipation described by Eqn. (24), we restrict our consideration to a specific subspace with eigenvalue 1 for the translation operator \( T \) and the reflection operator \( R \): \( T = R = 1 \). First, we want to prove that if the steady state is unique, it is the fully mixed state outside the degeneracy points as indicated by our numerical simulation. Then, we show that starting from the condition given by Eqn. (C10), specialization to the Ising model allows to derive a more precise condition for finding a discontinuity in the steady state at the degeneracy points.

First, we show that \( 1 \) satisfies \( (\mathcal{L}_0 + \mathcal{L}_1)1 = 0 \) outside the degeneracy point \( x \neq x_0 \). Therefore, for systems with unique steady state, it is given by the fully mixed state for \( x \neq x_0 \) in the limit \( \gamma \to 0 \). The Hamiltonian given by Eqn. (24) is assumed to be non-degenerate for \( x \neq x_0 \) and invariant under the spin flip operator \( F = \sigma_i^x \), i.e., \( FHF^\dagger = H \). Thus, we want to show that for \( x \neq x_0 \):

\[
P^D \mathcal{L}_1 1 = 0, \quad (C11)
\]

where \( P^D \) is given by Eqn. (C4). Then,

\[
P^D \mathcal{L}_1 1 = 2\gamma (J^- J^+ - J^+ J^-) \propto \gamma J^2. \quad (C12)
\]

Therefore, Eqn. (C11) reads

\[
P^D \mathcal{L}_1 1 = P^D J^2 = \sum_i |\lambda_i\rangle \langle\lambda_i| J^2 |\lambda_i\rangle \langle\lambda_i| = 0 \quad (C13)
\]

If we can show that Eqn. (C13),

\[
|\lambda_i| J^2 |\lambda_i| = 0 \forall i, \quad (C14)
\]

then we have shown that the fully mixed state is a steady state of our system outside the degeneracy points of the Hamiltonian. As the Hamiltonian is nondegenerate and invariant under the spin flip operator \( F \), the eigenvectors of \( H \) are eigenvectors of \( F \): \( F |\lambda_i\rangle = \alpha_i |\lambda_i\rangle \). Let \( |\alpha\rangle \) denote an
arbitrary eigenvector of $H$ with $F$-eigenvalue $\alpha$. Since the spectrum of $F$ is $\{\pm 1\}$, we have $|\alpha\rangle = \alpha^2 |\alpha\rangle = \alpha F |\alpha\rangle$. Moreover, the flip $F$ changes the sign of $J^z$, i.e., $J^z$ and $F$ anticommute: $\{F, J^z\}^\dagger = 0$. Therefore, we can write Eqn. (C14) as

$$\langle \alpha | J^z | \alpha \rangle = \alpha \langle \alpha | J^z F | \alpha \rangle = -\alpha \langle \alpha | F J^z | \alpha \rangle = -\langle \alpha | J^z | \alpha \rangle,$$

where we have used that $\alpha^2 = 1$. It follows that

$$\langle \alpha | J^z | \alpha \rangle = 0.$$  

(C16)

Consequently, $P^D L_1 (\mathbb{I}) = 0$ and we have shown that in the limit of weak dissipation, the steady state, if it is unique, is the fully mixed state. For the Ising model with up to 8 atoms and collective dissipation, we know from our numerics that the steady state is unique.

To see that the steady state shows discontinuous behavior at the degeneracy point $x = x_0$, we need now only to show that the fully mixed state is not the steady state of the system. Thus, we need to show that

$$P^\Delta L_1 (\mathbb{I}) = P^\Delta J^z = \sum_{i,j,i \neq j} |\lambda_i \rangle \langle J^z | \lambda_j \rangle |\lambda_j \rangle \neq 0$$  

(C17)

where $P^\Delta$ is given by Eqn. (C5) and $\sum'$ sums over the labels of degenerate eigenvalues. Since the eigenvectors are orthogonal, Eqn. (C14) holds if $\exists i \neq j$ such that

$$\langle \lambda_i | J^z | \lambda_j \rangle \neq 0.$$  

(C18)

Therefore, Eqn. (C18) gives a condition for finding discontinuous behavior of the steady state of the Ising model in a transverse field under collective dissipation. Note that this derivation can be easily extended to all non-degenerate Hamiltonians that are flip-invariant.

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