On quark-lepton complementarity

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Abstract. Recent measurements of the neutrino solar mixing angle and the Cabibbo angle satisfy the empirical relation \( \theta_{\text{sol}} + \theta_{C} \approx \frac{\pi}{4} \). This relation suggests the existence of a correlation between the mixing matrices of leptons and quarks, the so called quark-lepton complementarity. Here, we examine the possibility that this correlation originates in the strong hierarchy in the mass spectra of quarks and charged leptons, and the seesaw mechanism that gives mass to the Majorana neutrinos. In a unified treatment of quarks and leptons in which the mass matrices of all fermions have a similar Fritzsch texture, we calculate the mixing matrices \( V_{\text{CKM}} \) and \( U_{\text{MNSP}} \) as functions of quark and lepton masses and only two free parameters, in very good agreement with the latest experimental values on masses and mixings. Three essential ingredients to explain the quark-lepton complementarity relation are identified: the strong hierarchy in the mass spectra of quarks and charged leptons, the normal seesaw mechanism and the assumption of maximal CP violation in the lepton sector.

Keywords: Quark and lepton masses and mixings, Neutrino masses and mixings, CKM matrix

PACS: 12.15.Ff, 14.60.Pq, 12.15.Hh, 14.60.St

INTRODUCTION

In the last few years, the neutrino oscillations between different flavour states were measured in a series of experiments with atmospheric neutrinos\(^1\), solar neutrinos\(^2\)-\(^3\), neutrinos produced in nuclear reactors \(^4\) and accelerators \(^5\). As a result, the difference of the squared neutrino masses and the mixing angles in the lepton mixing matrix, \( U_{\text{MNSP}} \), were determined:

\[
0.34 \leq \sin^2 \theta_{23} \leq 0.68, \quad 1.4 \times 10^{-3} \text{(eV)}^2 \leq \Delta m^2_{23} \leq 3.0 \times 10^{-3} \text{(eV)}^2, \quad (1)
\]

\[
0.29 \leq \sin^2 \theta_{12} \leq 0.40, \quad 7.1 \times 10^{-5} \text{(eV)}^2 \leq \Delta m^2_{12} \leq 8.9 \times 10^{-5} \text{(eV)}^2, \quad (2)
\]

\[
\sin^2 \theta_{13} \leq 0.046, \quad (3)
\]

at 90% confidence level \(^6\)-\(^7\). The CHOOZ experiment \(^8\) determined an upper bound for the \( \theta_{13} \) mixing angle. It was soon realized \(^9\) that the solar mixing angle \( \theta_{12}^{\text{MNSP}} \) and the Cabibbo angle \( \theta_{12}^{\text{CKM}} \), which is the corresponding angle in the quark sector, satisfy an interesting and intriguing numerical relation,

\[
\theta_{12}^{\text{MNSP}} + \theta_{12}^{\text{CKM}} = 45^\circ + 1^\circ \pm 2.4^\circ, \quad (4)
\]

with \( \theta_{12}^{\text{MNSP}} = 33.9^\circ \pm 2.4^\circ \) (1\(\sigma\)) and \( \theta_{12}^{\text{CKM}} = 12.8^\circ \pm 0.15^\circ \). Equation (4) relates the 1-2 mixing angles in the quark and lepton sectors, it is commonly called the quark-lepton complementarity relation (QLC) and, if not accidental, it could imply a quark-lepton symmetry (for a recent review see \(^10\)) or a quark-lepton unification \(^11\)-\(^14\).
A second QLC relation, $\theta_{23}^{MNSP} + \theta_{23}^{CKM} \approx \pi/4$, is also satisfied. However, this is not as interesting as (4) because $\theta_{23}^{CKM}$ is only about two degrees, and the corresponding QLC relation would be satisfied, within the errors, even if the angle $\theta_{23}^{CKM}$ had been zero, as long as $\theta_{23}^{MNSP}$ is close to the maximal value $\pi/4$. A third possible QLC relation is not realized at all, or at least not realized in the same way, since $\theta_{13}^{CKM} + \theta_{13}^{MNSP}$ is less than ten degrees. In this short note we will focus our attention on understanding the nature of the QLC relation shown in equation (4).

**UNIVERSAL FRITZSCH TEXTURE OF QUARKS AND LEPTONS**

The quark and lepton flavour mixing matrices, $U_{MNSP}$ and $V_{CKM}$, arise from the mismatch between diagonalization of the mass matrices of $u$ and $d$ type quarks and the diagonalization of the mass matrices of charged leptons and neutrinos,

$$U_{MNSP} = U_l^\dagger U_\nu, \quad V_{CKM} = U_u^\dagger U_d.$$  

Therefore, to get predictions for the flavour mixing angles and CP violating phases, we should specify the mass matrices.

In this work, we propose a unified treatment of quarks and leptons. Lepton and quark mass matrices could have the same mass texture from a universal flavour symmetry (exact at a certain energy scale). Imposing a flavour symmetry has been successful in reducing the number of parameters of the Standard Model. In particular, a permutational $S_3$ flavour symmetry and its sequential explicit breaking, allows us to represent the mass matrices as a modified Fritzsch texture:

$$M^{(F)} = \begin{pmatrix} 0 & A & 0 \\ A^* & B & C \\ 0 & C & D \end{pmatrix} \quad i = u, d, l, \nu.$$  

Some reasons to propose the validity of the modified Fritzsch texture as a universal mass texture for all fermions in the theory are the following:

1. The idea of $S_3$ flavour symmetry and its explicit breaking has been realized as a modified Fritzsch texture in the quark sector to interpret the strong mass hierarchy of up and down type quarks [13].

2. The quark mixing angles and the CP violating phase appearing in the $V_{CKM}$ mixing matrix were computed as explicit, exact functions of the four quark mass ratios $(m_u/m_t, m_c/m_t, m_d/m_b, m_s/m_b)$, one symmetry breaking parameter $Z^{1/2} = (81/32)^{1/2}$ and one CP violating phase $\phi_{u-d} = 90^\circ$, in very good agreement with experiment [16].

3. Since the mass spectrum of the charged leptons exhibits a similar hierarchy to the quark’s one, it would be natural to consider the same $S_3$ symmetry and its explicit breaking for the charged lepton mass matrix.

4. As for the Dirac neutrinos, we have no direct information about the absolute values or the relative values of the neutrino masses, but the Fritzsch texture can...
be incorporated in a $SO(10)$ neutrino model [17]. Therefore it would be sensible to assume that the Dirac neutrinos have a mass hierarchy similar to that of the $u$-quarks and it would be natural to take for the Dirac neutrino mass matrix also a modified Fritzsch texture.

5. The left handed Majorana neutrinos naturally acquire their mass through an effective seesaw mechanism of the form

$$M_{\nu_L} = M_{\nu_D}M_R^{-1}M_{\nu_D}^T,$$

where $M_{\nu_D}$ and $M_R$ denote the Dirac and right handed Majorana neutrino mass matrices. From our conjecture of a universal $S_3$ flavour symmetry it follows that $M_R$ could have the same texture as that of $M_{\nu_D}$ and $M_i$. Then, it is straightforward to show that $M_{\nu_L}$ has the same modified Fritzsch texture [18].

**MIXING MATRICES AS FUNCTIONS OF THE FERMION MASSES**

When the unitary matrices that diagonalize the mass matrices $M_i^{(F)}$ are written in polar form, $U_i = P_iO_i$ and $M_i^{(F)} = P_i^\dagger M_i P_i$, the expressions (5) for the mixing matrices take the form

$$U_{\text{MNSP}} = O_i^T P^{(l-v)} O_v K, \quad V_{\text{CKM}} = O_d^T P^{(u-d)} O_u,$$

where $O_i, i = u, d, v, l$, are the orthogonal matrices that diagonalize the real symmetric mass matrices $M_i^{(F)}$ and $P^{(u-d)} = \text{diag}[1, e^{i\phi}, e^{i\Phi}]$, $P^{(l-v)} = \text{diag}[1, e^{i\phi}, e^{i\Phi}]$, where $\phi = \phi_u - \phi_d$, and $\Phi = \Phi_l - \Phi_v$, are the matrices of the Dirac phases and $K$ is the diagonal matrix of the Majorana phases.

We reparametrized the matrices $\tilde{M_i}$ in terms of their eigenvalues. The orthogonal matrices are then expressed in terms of the mass eigenvalues of $M_i$:

$$O_i = \begin{pmatrix} \tilde{m}_{2f_{11}} D_1 & \tilde{m}_{2f_{12}} D_2 & \tilde{m}_{2f_{13}} D_3 \\ \tilde{m}_{1f_{11}} D_1 & \tilde{m}_{1f_{12}} D_2 & \tilde{m}_{1f_{13}} D_3 \\ \tilde{m}_{2f_{11}} D_1 & \tilde{m}_{2f_{12}} D_2 & \tilde{m}_{2f_{13}} D_3 \end{pmatrix}^{1/2},$$

$$f_{i1} = 1 - \tilde{m}_{i1} - \delta_i, \quad f_{i2} = 1 + \tilde{m}_{i2} - \delta_i, \quad f_{i3} = \delta_i,$$

$$D_{i1} = (1 - \delta_i)(\tilde{m}_{i1} + \tilde{m}_{i2})(1 - \tilde{m}_{i1}),$$

$$D_{i2} = (1 - \delta_i)(\tilde{m}_{i1} + \tilde{m}_{i2})(1 + \tilde{m}_{i2}),$$

$$D_{i3} = (1 - \delta_i)(1 - \tilde{m}_{i1})(1 + \tilde{m}_{i2}),$$

the small parameters $\delta_i$ are also functions of the mass ratios and the symmetry breaking parameter $Z^{1/2} = (81/32)^{1/2}$.

Substitution of the expressions (9) and (10)-(13) in (8) allows us to express the mixing matrices $U_{\text{MNSP}}$ and $V_{\text{CKM}}$ as explicit functions of the quark and lepton masses.
QUARK-LEPTON COMPLEMENTARITY

The resulting theoretical expression for the Cabibbo angle written to first order in \( m_u/m_c \) and \( m_d/m_s \), is

\[
\sin^2 \theta_C^\text{th} = |V_{ut}^\text{th}|^2 \approx \frac{\tilde{m}_d + \tilde{m}_s}{\tilde{m}_d \tilde{m}_s} - 2 \sqrt{\frac{\tilde{m}_d \tilde{m}_s}{\tilde{m}_d \tilde{m}_s}} \cos \phi \tag{14}
\]

Taking for the quark masses the values \( m_u = 2.75 \text{MeV}, m_c = 1310 \text{MeV}, m_d = 6.0 \text{MeV}, m_s = 120 \text{MeV} \) and maximal CP violation, \( \phi = 90^\circ \) \[16\], we reproduce the numerical value of the Cabibbo angle

\[
\sin \theta_C^\text{th} = 0.225 \quad \text{or} \quad \theta_C = 12.8^\circ,
\]

in very good agreement with the latest analysis of the experimental data \[19\].

The theoretical expression for the solar mixing angle is derived in a similar way. From \( |(U_{MNSP})_{12}|^2 / |(U_{MNSP})_{11}|^2 = \tan^2 \theta_1^\text{th} \), we obtain

\[
\tan^2 \theta_1^{th} = \frac{\tilde{m}_d \tilde{m}_s - 2 \sqrt{\tilde{m}_d \tilde{m}_s} \cos \Phi}{1 + \tilde{m}_d \tilde{m}_s + 2 \sqrt{\tilde{m}_d \tilde{m}_s} \cos \Phi} \tag{16}
\]

In the absence of experimental information, we assumed that CP violation is also maximal in the lepton sector \( i.e. \Phi = 90^\circ \). Taking for the masses of the left handed Majorana neutrinos a normal hierarchy with the numerical values \( m_{\nu_1} = 4.4 \times 10^{-3} \text{eV} \) and \( m_{\nu_2} = 1.0 \times 10^{-2} \text{eV} \), and for the charged lepton masses the values \( m_e = 0.5109 \text{MeV}, m_\mu = 105.685 \text{MeV} \) and \( m_\tau = 1776.99 \text{GeV} \), we obtain the following numerical value for the solar mixing angle

\[
\tan^2 \theta_{12}^{th} = 0.45 \quad \text{or} \quad \theta_{12}^{th} = 33.9^\circ.
\]

We may now address the question of the meaning of the quark-lepton complementarity relation as expressed in eq(4). The previous theoretical analysis allows us to calculate,

\[
\tan \left( \theta_C^\text{th} + \theta_{12}^\text{th} \right) = 1 + \Delta^\text{th}, \tag{18}
\]

\[
\Delta^\text{th} = \left( \frac{\tilde{m}_d + \tilde{m}_s}{\tilde{m}_d \tilde{m}_s} \right)^\frac{1}{2} \left[ \left( \frac{\tilde{m}_d + \tilde{m}_s}{\tilde{m}_d \tilde{m}_s} \right)^\frac{1}{2} + \left( \frac{\tilde{m}_d + \tilde{m}_s}{\tilde{m}_d \tilde{m}_s} \right)^\frac{1}{2} \right] + \left( \frac{1 + \tilde{m}_d \tilde{m}_s}{\tilde{m}_d \tilde{m}_s} \right)^\frac{1}{2} \left[ \left( \frac{1 + \tilde{m}_d \tilde{m}_s}{\tilde{m}_d \tilde{m}_s} \right)^\frac{1}{2} - 1 \right] \left( \frac{\tilde{m}_d + \tilde{m}_s}{\tilde{m}_d \tilde{m}_s} \right)^\frac{1}{2} \left( \frac{\tilde{m}_d + \tilde{m}_s}{\tilde{m}_d \tilde{m}_s} \right)^\frac{1}{2} \right]. \tag{19}
\]

After substitution of the numerical value of the mass ratios of quarks and leptons in \[19\], we obtain,

\[
\Delta^\text{th} = 0.061 \quad \text{and} \quad \theta_C^\text{th} + \theta_{12}^\text{th} = 45^\circ + 1.7^\circ.
\]

in very good agreement with the experimental value.
CONCLUSIONS

In this short communication, we outlined a unified treatment of masses and mixing of quarks and leptons in which the left handed Majorana neutrinos acquire their masses via the seesaw mechanism, and the mass matrices of all fermions have a similar Fritzsch texture and a normal hierarchy. In this scheme, we derived exact, explicit expressions for the Cabibbo and solar mixing angles as functions of the quark and lepton masses. The quark-lepton complementarity relation takes the form,

\[ \theta_{12}^{\text{CKM}} + \theta_{12}^{\text{MNSP}} = 45^\circ + \delta_{12}. \]  

(21)

The correction term, \( \delta_{12} \), is an explicit function of the ratios of quark and lepton masses, given in eq.(19), which reproduces the experimentally determined value, \( \delta_{12} \approx 1.7^\circ \), when the numerical values of the quark and lepton masses are substituted in (19) and maximal violation of CP in the lepton sector is assumed.

Three essential ingredients are needed to explain the correlations implicit in the small numerical value of \( \delta_{12} \):

1. The strong hierarchy in the mass spectra of the quarks and charged leptons, realized in our scheme through the explicit breaking of the \( S_3 \) flavour symmetry in the Fritzsch mass texture, explains the resulting small or very small quark mixing angles, the very small charged lepton mass ratios explain the very small \( \theta_{13}^{\text{MNSP}} \) which, in our scheme, is independent of the neutrino masses.

2. The normal seesaw mechanism that gives very small masses to the left handed Majorana neutrinos with relatively large values of the neutrino mass ratio \( m_{\nu_1}/m_{\nu_2} \) and allows for large \( \theta_{12}^{\text{MNSP}} \) and \( \theta_{23}^{\text{MNSP}} \) mixing angles.

3. The assumption of maximal CP violation in the lepton sector.

A more complete and detailed version of this work will be presented in a forthcoming publication [20].

ACKNOWLEDGEMENTS

We thank Dr. M. Mondragón for many inspiring discussions on this exciting problem and for a critical reading of the manuscript.

This work was partially supported by CONACyT Mexico under Contract No. 42026F, and DGAPA-UNAM Contract No. PAPIIT IN116202.

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