Channel Polarization through the Lens of Blackwell Measures

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Abstract

Each memoryless binary-input channel (BIC) can be uniquely described by its Blackwell measure, which is a probability distribution on the unit interval $[0,1]$ with mean $1/2$. Conversely, any such probability distribution defines a BIC. Viewing each BIC through the lens of its Blackwell measure, this paper provides a unified framework for analyzing the evolution of a variety of channel functionals under Arikan’s polar transform. These include the symmetric capacity, Bhattacharyya parameter, moments of information density, Hellinger affinity, Gallager’s reliability function, and the Bayesian information gain. An explicit general characterization is derived for the evolution of the Blackwell measure under Arikan’s polar transform. The evolution of the Blackwell measure is specified for symmetric BICs based on their decomposition into binary symmetric (sub)-channels (BSCs). As a byproduct, a simple algorithm is designed and simulated for computing the successive polarization of symmetric BICs. It is shown that all channel functionals that can be expressed as an expectation of a convex function with respect to the Blackwell measure of the channel polarize on the class of symmetric BICs. For this broad class, a necessary and sufficient condition is established which determines whether the bounded random process associated to a channel functional is a martingale, submartingale, or supermartingale. This condition is numerically verifiable for all known channel functionals.

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I. INTRODUCTION

Polar codes, introduced in a seminal paper of Arıkan [1], are a structured family of codes with low encoding and decoding complexity that provably achieve the capacity of symmetric binary-input channels (BICs). The polar code transforms $N := 2^n$ independent copies of a symmetric BIC $W$ into $N$ polarized channels whose individual capacities approach either 0 or 1 with increasing block length $N$. The fraction of perfect channels among the $N$ transformed channels approaches $I(W)$, the symmetric capacity of $W$. In Arıkan’s derivation, the polarization phenomenon is demonstrated for two channel functionals: the symmetric capacity $I(W)$ and the Bhattacharyya parameter $Z(W)$. More formally, the polarization phenomenon depends on the convergence of a martingale and a supermartingale, which are $[0, 1]$-valued random processes associated to the functionals $I(W)$ and $Z(W)$ respectively. Although extremely useful, existing proof techniques rely on a specialized set of channel characteristics. The objective of this paper is to develop a general framework to analyze random processes associated to a broad class of channel functionals.

A. Overview of contributions

We utilize a representation of channels that originates in the work of Blackwell on the comparison of statistical experiments [2], [3] (see [4] for a modern synthesis). The Blackwell measure is defined for BICs in Section II, and we discuss its relation to other representations, such as the information density and Neyman–Pearson regions arising from the theory of binary hypothesis testing. According to the Blackwell–Sherman–Stein theorem, the Blackwell measure uniquely specifies any BIC. We characterize the evolution of a broad class of channel functionals under Arıkan’s polar transform by tracking the evolution of the Blackwell measure. The following overview summarizes our key contributions:

- In Section III, it is shown that any measurable function $f$ on $[0, 1]$ induces a functional $I_f(W)$ of the channel $W$ through its Blackwell measure. Several non-trivial channel functionals may be derived in this manner, including the Hellinger affinity, moments of information density, Gallager’s reliability function, and the Bayesian information gain.
- In Section IV, BICs with output symmetry are analyzed as compound channels comprised by BSC sub-channels [5]. The Blackwell measure of any symmetric BIC may be written in terms of the Blackwell measures of its BSC sub-channels. We relate the Blackwell measure to the mutual information profile (MIP), which is a unique representation for symmetric BICs.
- In Section V, Arıkan’s polar transform is defined for two BICs $W_1$ and $W_2$. The transform yields two polarized channels, denoted by $W_1 \circledast W_2$ and $W_1 \boxast W_2$. The Blackwell measures of $W_1 \circledast W_2$ and $W_1 \boxast W_2$ are derived in terms of the Blackwell measures of $W_1$ and $W_2$. For channels with symmetry,
the Blackwell measures of $W_1 \otimes W_2$ and $W_1 \Box W_2$ may be derived in terms of the decomposition of $W_1$ and $W_2$ into BSC sub-channels.

- In Section VI, a simple algorithm is devised to compute the Blackwell measures of polarized symmetric channels after $n$ successive iterations of the polar transform. Channel representations are quantized to facilitate efficient numerical simulations. The effect of channel quantization on numerical accuracy is measured by theoretical bounds.
- In Section VII-A, it is shown that the polarization phenomenon is generic for the class of symmetric BICs. More precisely, for all channel functionals induced by a convex function, the Blackwell ordering of channels is preserved under Arıkan’s polar transform.
- Section VIII describes random processes associated to a broad class of channel functionals. For symmetric BICs, a necessary and sufficient condition is derived which indicates whether a polarization process is a martingale, submartingale, or supermartingale. This condition is numerically verifiable, and exploits the decomposition of symmetric BICs into BSC sub-channels.

B. Relation to prior work

The polarization phenomenon has been observed for several channel functionals beyond the symmetric capacity and Bhattacharyya parameter. Alsan and Telatar prove that the random process associated to Gallager’s reliability function $E_0$, which is related to various error exponents and cutoff rates, is a submartingale [6], [7]. Channel combining and splitting via Arıkan’s polar transformation increases and improves $E_0$. Similarly, Arıkan characterized the evolution of the variance of the information density, also named “varentropy” or “dispersion,” under the polar transform [8]. Arıkan’s proof establishes that the varentropy decreases after each iteration of the polar transform. Both Gallager’s reliability function and the second moment of information density related to dispersion are induced functionals in the framework of Blackwell measures. Consequently, we corroborate prior results within this framework.

Since Arıkan’s discovery of polarization, significant advances in theory have been made including: (i) multilevel and $q$-ary polarization [9], [10]; (ii) generalized $\ell \times \ell$ polarization matrices and algebraic constructions [11], [12]; (iii) refinements to the rate of polarization and finite-length scaling of polar codes [13]–[16]. Our work is complementary to these other generalizations of polarization which focus on Arıkan’s original martingale associated to the mutual information (or entropy) of random variables. We believe the framework of Blackwell measures could aid in defining other auxiliary random processes for analyzing both source and channel polarization.

Significant progress in theory and practice has led to the inclusion of polar codes in next-generation wireless systems. Efficient algorithms exist to construct polar codes for large blocklengths $N$ [17], [18].
Our explicit algorithm for computing the Blackwell measures of polarized channels given in Section VI is based on the parameterized decomposition of symmetric BICs into BSC sub-channels, and is extremely simple to implement for any symmetric BIC.

Lastly, although the focus of the present paper is channel polarization for point-to-point channels, the techniques developed here could potentially be extended to multi-user channels. Polar codes have been designed for multiple-access channels [19], broadcast channels [20], wiretap channels [21], as well as several other scenarios. If the notions of symmetry and information combining [5] could be extended to multi-user channels, a general measure-theoretic framework of polarization for multi-user channels could be developed.

C. Frequently used notation

The following mathematical notations are adopted in the sequel. For \( p, q \in [0,1] \), we let \( \bar{p} := 1 - p \) (for \( p \in \{0,1\} \), this is the Boolean NOT) and \( p \star q := pq + \bar{p}q \). For \( a, b \in \mathbb{R} \), we let \( a \land b := \min(a,b) \) and \( a \lor b := \max(a,b) \). Given a random object \( U \), we will denote by \( \mathcal{L}(U) \) its probability law. The closure of a set \( S \) is denoted by \( \text{cl}\{S\} \). The notation \( \delta_x \) denotes the Dirac measure centered on a fixed point \( x \) in a measurable space. The binary entropy function is denoted by \( h_2(x) := -x \log_2(x) - \bar{x} \log_2 \bar{x} \) for \( x \in [0,1] \).

II. Binary-input channels and their representations

In this work, we focus on binary-input channels (BICs) with finite output alphabets:

**Definition 1** (Binary-Input Channel (BIC)). A discrete binary-input channel (BIC) is a pair \((Y,W)\), where \( Y \) is the finite output alphabet and \( W = (W(\cdot|0), W(\cdot|1)) \) is a pair of probability distributions on \( Y \). For \( x \in \{0,1\} \), \( W(\cdot|x) \) is the probability distribution of the channel output when the channel input is equal to \( x \).

The channel transition matrix is the most familiar representation of a BIC:

**Definition 2** (Channel transition matrix). Given a BIC \((Y,W)\), let \( T_W \) denote the \( 2 \times |Y| \) matrix whose elements are \( W(y|x) \) for \((x,y) \in \{0,1\} \times Y\).

**Example 1** (Binary Erasure Channel \( \text{BEC}(\varepsilon) \)). The binary erasure channel with erasure probability \( \varepsilon \) is a BIC \((Y,W)\) with \( Y = \{0,1,\varepsilon\} \), \( W(\cdot|0) = \varepsilon \delta_0 + \varepsilon \delta_\varepsilon \), and \( W(\cdot|1) = \varepsilon \delta_1 + \varepsilon \delta_\varepsilon \). The transition matrix is

\[
T_{\text{BEC}(\varepsilon)} := \begin{bmatrix}
1 - \varepsilon & 0 & \varepsilon \\
0 & 1 - \varepsilon & \varepsilon \\
\end{bmatrix}.
\]
Example 2 (Binary Symmetric Channel BSC\((p)\)). The binary symmetric channel with bit-flip probability \(p\) is a BIC \((Y, W)\) with \(Y = \{0, 1\}\), \(W(\cdot | 0) = \text{Bern}(p)\), and \(W(\cdot | 1) = \text{Bern}(\bar{p})\). The transition matrix is
\[
T_{\text{BSC}(p)} := \begin{bmatrix}
1 - p & p \\
p & 1 - p
\end{bmatrix}.
\]
In the remainder of this section, we describe a number of alternative representations of BICs that will be used in the sequel.

A. The Blackwell measure

The Blackwell measure [2]–[4] particularized to BICs is defined as the distribution of the posterior probability of the binary input being 0, assuming a uniform input distribution to the channel:

Definition 3 (Blackwell measure of a BIC). Given a BIC \((Y, W)\), let \((X, Y)\) be a random couple on \(\{0, 1\} \times Y\) with \(P_X = \text{Bern}(1/2)\) and \(P_{Y|X} = W\). Define the function
\[
f_W(y) := \frac{W(y|0)}{W(y|0) + W(y|1)}.
\]
(1)
The random variable \(S = f_W(Y)\), which is equal to the posterior probability of \(X = 0\) given \(Y\), takes values in the unit interval \([0, 1]\) and has mean 1/2. The Blackwell measure of \(W\), which we will denote by \(m_W\), is the probability law of random variable \(S\).

Example 3 (Blackwell measures for BEC\((\varepsilon)\) and BSC\((p)\)). The Blackwell measures for the BEC\((\varepsilon)\) and BSC\((p)\) are
\[
m_{\text{BSC}(p)} = \frac{1}{2} \delta_p + \frac{1}{2} \delta_{\bar{p}},
\]
\[
m_{\text{BEC}(\varepsilon)} = \frac{\varepsilon}{2} \delta_0 + \frac{\varepsilon}{2} \delta_1 + \varepsilon \delta_{1/2}.
\]
Given two BICs \((Y, W)\) and \((Y', W')\), we say that \(W\) dominates \(W'\) (or is more informative than \(W'\)) in the sense of Blackwell [2]–[4] if there exists a random transformation \(K\) from \(Y\) to \(Y'\), such that \(W' = K \circ W\), i.e., for all \(x \in \{0, 1\}\) and all \(y' \in Y'\),
\[
W'(y'|x) = \sum_{y \in Y} K(y'|y)W(y|x),
\]
In other words, \(W\) dominates \(W'\) exactly when it is stochastically degraded with respect to \(W\). In that case, we write \(W \succeq W'\). We say that \(W\) and \(W'\) are equivalent if \(W \succeq W'\) and \(W' \succeq W\). In that case, we write \(W \equiv W'\). The fundamental nature of the Blackwell measure is evident from the following theorem:
Theorem 1 (Blackwell–Sherman–Stein). Consider two BICs \( W \) and \( W' \). Then:

1) \( W \equiv W' \) if and only if \( m_W = m_{W'} \) (that is, the Blackwell measure specifies the channel uniquely up to equivalence). Moreover, let \( \mathcal{M} \) denote the collection of all Borel probability measures on \([0,1]\) with mean \( 1/2 \). Then for any \( m \in \mathcal{M} \) there exists a BIC \( W \), unique up to equivalence, such that \( m = m_W \).\(^1\)

2) \( W \succeq W' \) if and only if

\[
\int_{[0,1]} f \, dm_W \geq \int_{[0,1]} f \, dm_{W'},
\]

for every convex \( f : [0,1] \to \mathbb{R} \).

Remark 1. There are one-to-one correspondences between the Blackwell measure \( m_W \) and other probabilistic objects associated to the BIC \( W \), such as its \( L \) - and \( D \) -distributions [22, Ch. 4]. The Blackwell measure is also a special case of the so-called \( \alpha \)-representation [8].

B. Information density

Information density furnishes another useful description of BICs. Let \((X,Y)\) be a random couple taking values in a finite product space \( X \times Y \). The information density is defined as

\[
i(x; y) := \log_2 \frac{P_{Y|X}(y|x)}{P_Y(y)},
\]

where \( P_Y(y) = \sum_{x \in X} P_X(x) P_{Y|X}(y|x) \). The expectation and the variance of the information density are the mutual information and the information variance:

\[
I(X; Y) = \mathbb{E}[i(X; Y)],
\]

\[
V(X; Y) = \mathbb{E}[i^2(X; Y)] - (I(X; Y))^2.
\]

We particularize this to BICs with equiprobable inputs:

Definition 4 (Information density for a BIC). Given a BIC \((Y, W)\), let \((X,Y)\) be a random couple on \( \{0,1\} \times Y \) with \( P_X = \text{Bern}(1/2) \) and \( P_{Y|X} = W \). The information density of \((X,Y)\) is given by

\[
i_W(x; y) = \log_2 \frac{W(y|x)}{\frac{1}{2}W(y|0) + \frac{1}{2}W(y|1)}.
\]

The expectation and variance of \( i_W(X; Y) \) with \( X \sim \text{Bern}(1/2) \) are known as the symmetric capacity \( I(W) \) and symmetric dispersion \( V(W) \), respectively. To express these parameters succinctly, we introduce the \( r \)th moment of the information density:

\[
M_r(W) := \mathbb{E}[i_W(X; Y)^r].
\]

\(^1\)The BIC \( W \) has a finite output alphabet if and only if \( m_W \) has finite support. This is precisely the setting of this paper.
Then \( I(W) = M_1(W) \) and \( V(W) = M_2(W) - I^2(W) \). For a BIC \((Y, W)\), it follows from Eqn. (3) and Eqn. (1) that
\[
i_W(x; y) = \begin{cases} 
1 + \log_2 f_W(y) & \text{if } x = 0; \\
1 + \log_2(1 - f_W(y)) & \text{if } x = 1;
\end{cases}
\]
for arbitrary \( y \in Y \). Therefore, the information density specifies a BIC uniquely up to equivalence.

C. The Neyman–Pearson region

Another useful representation of BICs arises from the theory of binary hypothesis testing (see, e.g., [23, Sec. 12.1 and 12.2]). Given a BIC \((Y, W)\), the Neyman–Pearson region \( \mathcal{R}_{NP}(W) \) is a subset of \([0, 1]^2\) consisting of all points \((\alpha, \beta)\), for which there exists some function \( f : Y \to [0, 1] \), such that
\[
\alpha = \sum_{y \in Y} f(y) W(y|0) \quad \text{and} \quad \beta = \sum_{y \in Y} f(y) W(y|1).
\]
(5)
The Neyman–Pearson region has the following properties:

1) It is a closed and convex subset of \([0, 1]^2\).
2) It contains the diagonal \( \mathcal{D} := \{(\alpha, \alpha) : \alpha \in [0, 1]\} \).
3) It is equal to the closed convex hull of all points \((\alpha, \beta)\) of the form given in (5) with \( f \) taking values in \{0, 1\}:
\[
\mathcal{R}_{NP}(W) = \text{cl} \left\{ \text{conv} \left\{ (W(A|0), W(A|1)) : A \subseteq Y \right\} \right\},
\]
where \( W(A|x) := \sum_{y \in A} W(y|x) \).

The following fundamental result is a consequence of the Blackwell–Sherman–Stein theorem:

**Theorem 2** (The Neyman–Pearson criterion for Blackwell dominance). Consider two BICs \( W \) and \( W' \). Then
\[
W \succeq W' \iff \mathcal{R}_{NP}(W) \supseteq \mathcal{R}_{NP}(W').
\]

For example, it is not hard to show that
\[
\mathcal{R}_{NP}(\text{BSC}(p)) = \text{cl} \left\{ \text{conv} \{(0, 0), (p, \bar{p}), (\bar{p}, p), (1, 1)\} \right\}.
\]
Then evidently \( \mathcal{R}_{NP}(\text{BSC}(0)) = [0, 1]^2 \), while \( \mathcal{R}_{NP}(\text{BSC}(1/2)) = \mathcal{D} \).
### TABLE I: Functionals of BICs

| Measurable function \( f : [0, 1] \to \mathbb{R} \) | Induced functional \( I_f(W) \) |
|--------------------------------------------------------|----------------------------------|
| \( f(s) = 1 - h_2(s) \)                              | \( I(W) \)                       |
| \( f(s) = \psi_r(s) \)                                | \( M_r(W) \)                     |
| \( f(s) = 2\sqrt{s(1-s)} \)                           | \( Z(W) \)                       |
| \( f(s) = 2s^\alpha(1-s)^{1-\alpha} \)               | \( H_\alpha(W) \)                |
| \( f(s) = 2^{-\rho} \left( s^{\frac{1}{1+\rho}} + (1-s)^{\frac{1}{1+\rho}} \right)^{1+\rho} \) | \( \exp(-E_\alpha(\rho, W)) \) |
| \( f(s) = \lambda \wedge \lambda - (2\lambda s) \wedge (2\lambda \bar{s}), \; \lambda \in [0, 1] \) | \( B_\lambda(W) \) |
| \( f(s) = |2s - 1| \)                                  | \( 1 - 2P_{\text{ML}}(W) \) |

### III. Functionals of BICs

Any measurable function \( f : [0, 1] \to \mathbb{R} \) induces a functional \( I_f \) on the collection of all BICs via

\[
I_f(W) := \int_{[0,1]} f d\mu_W = \mathbb{E}[f(S)],
\]

where \( S \sim \mu_W \). As summarized in Table I and explained in detail in this section, a variety of channel characteristics can be expressed in this way.

#### A. Symmetric capacity \( I(W) \)

With \( f(s) = 1 - h_2(s) \), where \( h_2(\cdot) \) is the binary entropy function, \( I_f(W) \) is equal to the symmetric capacity \( I(W) \) of \( W \) \cite{1}, i.e., the mutual information of \( W \) with uniform input distribution. Indeed, let \((X, Y)\) be a random couple with \( X \sim \text{Bern}(1/2) \) and \( P_{Y|X} = W \). Then

\[
I_f(W) = 1 - \mathbb{E}[h_2(S)]
\]

\[
= 1 + \mathbb{E} \left[ \frac{W(Y|0)}{2P_Y(Y)} \log_2 \frac{W(Y|0)}{2P_Y(Y)} + \frac{W(Y|1)}{2P_Y(Y)} \log_2 \frac{W(Y|1)}{2P_Y(Y)} \right]
\]

\[
= 1 + \mathbb{E} \left[ \log_2 \frac{W(Y|X)}{2P_Y(Y)} \bigg| X \right]
\]

\[
= D(P_{Y|X} \parallel P_Y|P_X)
\]

\[
\equiv I(W).
\]

#### B. The \( r \)th moment of information density \( M_r(W) \)

Let \( r \) be a positive integer. If we take

\[
f(s) = \psi_r(s) := s(1 + \log_2 s)^r + \bar{s}(1 + \log_2 \bar{s})^r,
\]

where \( \bar{s} = 1 - s \).
then $I_f(W)$ is equal to the $r$-th moment of the information density $M_r(W)$, assuming uniform input distribution, as defined in Eqn. (4). Note that $\psi_1(s) = 1 - h_2(s)$ and $M_1(W) = I(W)$. The following equalities establish our claim:

\[
M_r(W) := \mathbb{E}[(i_W(X; Y))^r] = \mathbb{E} \left[ \left( \log_2 \frac{2W(Y|X)}{W(Y|0) + W(Y|1)} \right)^r \right] = \mathbb{E} \left[ \left( 1 + \log_2 \frac{W(Y|X)}{W(Y|0) + W(Y|1)} \right)^r \right] = \mathbb{E} \left[ P_{X|Y}(0|Y) \left( 1 + \log_2 \frac{W(Y|0)}{W(Y|0) + W(Y|1)} \right)^r \right] + \mathbb{E} \left[ P_{X|Y}(1|Y) \left( 1 + \log_2 \frac{W(Y|1)}{W(Y|0) + W(Y|1)} \right)^r \right] = \mathbb{E} \left[ f_W(Y) \left( 1 + \log_2 f_W(Y) \right)^r \right] + \mathbb{E} \left[ (1 - f_W(Y)) \left( 1 + \log_2 (1 - f_W(Y)) \right)^r \right] = \mathbb{E} \left[ S(1 + \log_2 S)^r + \tilde{S}(1 + \log_2 \tilde{S})^r \right] = \mathbb{E} \left[ \psi_r(S) \right] \equiv I_f(W).
\]

The channel dispersion parameter $V(W)$ is defined as the variance of the information density, $V(W) := M_2(W) - (I(W))^2$, assuming a uniform input distribution. While the dispersion $V(W)$ cannot be expressed explicitly as an induced functional of the form $I_f(W)$ for any $f : [0, 1] \to \mathbb{R}$, we can use the variational representation of the variance as follows:

\[
V(W) = \text{Var}[i(X; Y)] = \min_{c \in \mathbb{R}} \mathbb{E}[(i(X; Y) - c)^2] = \min_{c \in \mathbb{R}} \mathbb{E}[(1 - h_2(S) - c)^2] = \min_{c \in \mathbb{R}} \mathbb{E}[(h_2(S) - c)^2].
\]

Thus, if we consider the family of functions $f_c(s) := h_2(s) - c, c \in \mathbb{R}$, we see that $V(W)$ can be expressed as

\[
V(W) = \min_{c \in \mathbb{R}} I_{f_c}(W).
\]
C. Hellinger affinity $H_\alpha(W)$

If we select $f(s) = 2s^\alpha(1-s)^{1-\alpha}$ for $\alpha \in [0, 1]$, the induced functional $I_f(W)$ is equal to the *Hellinger affinity* of order $\alpha$:

$$H_\alpha(W) := \sum_{y \in Y} W(y|0)^{\alpha}W(y|1)^{1-\alpha}.$$  

Indeed,

$$I_f(W) = 2\mathbb{E}[S^\alpha(1-S)^{\alpha}]$$

$$= 2\mathbb{E} \left[ \left( \frac{W(Y|0)}{2P_Y(Y)} \right)^{\alpha} \left( \frac{W(Y|1)}{2P_Y(Y)} \right)^{1-\alpha} \right]$$

$$= \mathbb{E} \left[ \frac{1}{P_Y(Y)} W(Y|0)^{\alpha}W(Y|1)^{1-\alpha} \right]$$

$$= \sum_{y \in Y} W(y|0)^{\alpha}W(y|1)^{1-\alpha}$$

$$\equiv H_\alpha(W).$$

In particular, if we set $\alpha = 1/2$, then we recover the *Bhattacharyya parameter* of $W$ [1],

$$H_{1/2}(W) = Z(W) := \sum_{y \in Y} \sqrt{W(y|0)W(y|1)}.$$  

D. Gallager’s function $E_0(\rho, W)$

Gallager’s $E_0$ function of a BIC $(Y, W)$ with input distribution $P_X$ is given by [24]

$$E_0(\rho, P, W) := -\log_2 \sum_{y \in Y} \left( P_X(0)W(y|0)^{\frac{1}{1+\rho}} + P_X(1)W(y|1)^{\frac{1}{1+\rho}} \right)^{1+\rho},$$

(6)

for any $\rho \geq 0$. In particular, we define

$$E_0(\rho, W) := E_0(\rho, \text{Bern}(1/2), W)$$

$$= -\log_2 \sum_{y \in Y} \left( \frac{1}{2} W(y|0)^{\frac{1}{1+\rho}} + \frac{1}{2} W(y|1)^{\frac{1}{1+\rho}} \right)^{1+\rho}.$$  

(7)

Choosing $f$ as

$$f(s) = 2^{-\rho} \left( s^{\frac{1}{1+\rho}} + (1-s)^{\frac{1}{1+\rho}} \right)^{1+\rho}$$

yields an induced functional

$$I_f(W) = \exp (-E_0(\rho, W)).$$
To see this, consider the following chain of equalities:

\[ I_f(W) := \mathbb{E}f(S) \]

\[ = \mathbb{E} \left[ 2^{-\rho} \left( S^{\frac{1}{1+\rho}} + (1 - S)^{\frac{1}{1+\rho}} \right)^{1+\rho} \right] \]

\[ = \mathbb{E} \left[ 2^{-\rho} \left( (f_W(Y))^{\frac{1}{1+\rho}} + (1 - f_W(Y))^{\frac{1}{1+\rho}} \right)^{1+\rho} \right] \]

\[ = \mathbb{E} \left[ 2^{-\rho} \left( \frac{(W(Y|0))^{\frac{1}{1+\rho}} + (W(Y|1))^{\frac{1}{1+\rho}}}{(W(Y|0) + W(Y|1))^{\frac{1}{1+\rho}}} \right)^{1+\rho} \right] \]

\[ = \mathbb{E} \left[ \left( \frac{1}{2}(W(Y|0))^{\frac{1}{1+\rho}} + \frac{1}{2}(W(Y|1))^{\frac{1}{1+\rho}} \right)^{1+\rho} \right] \]

\[ = \sum_{y \in Y} \left( \frac{1}{2}(W(y|0))^{\frac{1}{1+\rho}} + \frac{1}{2}(W(y|1))^{\frac{1}{1+\rho}} \right)^{1+\rho} \]

\[ \equiv \exp(-E_0(\rho, W)). \]

**E. Bayesian information gain \( B_\lambda(W) \)**

Given a BIC \((Y, W)\) and \(\lambda \in [0, 1]\), consider a random couple \((X, Y)\) with \(X \sim \text{Bern}(\lambda)\) and \(P_{Y|X} = W\). Define the minimum Bayes risk

\[ b_\lambda(W) := \min_{g: Y \to \{0, 1\}} \mathbb{P}[g(Y) \neq X] \]

\[ = \min_{g: Y \to \{0, 1\}} \left( \lambda \sum_{y \in Y} W(y|0)1_{g(y) = 1} + \lambda \sum_{y \in Y} W(y|1)1_{g(y) = 0} \right), \]

where the minimum is over all deterministic decoders \(g : Y \to \{0, 1\}\). The **Bayesian information gain** is defined as

\[ B_\lambda(W) := b_\lambda(\text{BSC}(1/2)) - b_\lambda(W). \quad (8) \]

We claim that \(B_\lambda(W) = I_{f_\lambda}(W)\) with \(f_\lambda(s) := \bar{\lambda} \wedge \lambda - (2\bar{\lambda}s) \wedge (2\lambda s)\). Moreover, since any convex \(f : [0, 1] \to \mathbb{R}\) can be approximated by a positive affine combination of such \(f_\lambda\)’s [4], it follows that \(W \succeq W’\) if and only if \(B_\lambda(W) \geq B_\lambda(W’)\) for all \(\lambda \in (0, 1)\).
To prove the claim, we first write down a closed-form expression for \( B_\lambda(W) \). For any decoder \( g \),

\[
P[g(X) \neq Y] = \mathbb{E}[\mathbb{E}[1_{g(X) \neq Y} | Y]]
\]

\[
= \mathbb{E} \left[ P_{X|Y}(0|Y) 1_{g(Y)=1} + P_{X|Y}(1|Y) 1_{g(Y)=0} \right]
\]

\[
= \mathbb{E} \left[ \frac{\lambda W(Y|0)}{\lambda W(Y|0) + \lambda W(Y|1)} 1_{g(Y)=1} + \frac{\lambda W(Y|1)}{\lambda W(Y|0) + \lambda W(Y|1)} 1_{g(Y)=0} \right]
\]

\[
= \sum_{y \in Y} \left( \frac{\lambda W(Y|0)}{\lambda W(Y|0) + \lambda W(Y|1)} 1_{g(Y)=1} + \lambda W(Y|1) 1_{g(Y)=0} \right),
\]

and the minimum over all \( g \) is evidently achieved by

\[
g^*(y) := \begin{cases} 
1, & \text{if } \lambda W(y|1) \geq \bar{\lambda} W(y|0) \\
0, & \text{if } \lambda W(y|1) < \bar{\lambda} W(y|0)
\end{cases}.
\]

This yields

\[
b_\lambda(W) = \sum_{y \in Y} (\bar{\lambda} W(y|0)) \wedge (\lambda W(y|1)).
\]

In particular, \( b_\lambda(\text{BSC}(1/2)) = \bar{\lambda} \wedge \lambda \). Moreover, using the identity \( a \wedge b = \frac{1}{2}(a + b - |a - b|) \), we can write

\[
B_\lambda(W) = \bar{\lambda} \wedge \lambda - \sum_{y \in Y} (\bar{\lambda} W(y|0)) \wedge (\lambda W(y|1))
\]

\[
= \frac{1}{2} \left( 1 - |1 - 2\lambda| - \sum_{y \in Y} (\bar{\lambda} W(y|0) + \lambda W(y|1) - |\bar{\lambda} W(y|0) - \lambda W(y|1)|) \right)
\]

\[
= \frac{1}{2} \sum_{y \in Y} |\bar{\lambda} W(y|0) - \lambda W(y|1)| - \frac{1}{2}|1 - 2\lambda|.
\]

We are now ready to prove the claim that \( B_\lambda(W) = I_{f_\lambda}(W) \). To that end, consider a random couple \((X, Y)\) with \( X \sim \text{Bern}(1/2) \) and \( P_{Y|X} = W \) and let \( S = f_\lambda(Y) \). Then, using the fact that \( \mathbb{E}[S] = 1/2 \), we have

\[
I_{f_\lambda}(W) = \mathbb{E}[f_\lambda(S)]
\]

\[
= \frac{1}{2} (1 - |1 - 2\lambda|) - \mathbb{E}[\bar{\lambda} S + \lambda \bar{S} - |\bar{\lambda} S - \lambda \bar{S}|]
\]

\[
= \mathbb{E}[|\bar{\lambda} S - \lambda \bar{S}|] - \frac{1}{2}|1 - 2\lambda|
\]

\[
= \frac{1}{2} \mathbb{E} \left[ \frac{1}{P_Y(Y)} |\bar{\lambda} W(Y|0) - \lambda W(Y|1)| \right] - \frac{1}{2}|1 - 2\lambda|
\]

\[
= \frac{1}{2} \sum_{y \in Y} |\bar{\lambda} W(y|0) - \lambda W(y|1)| - \frac{1}{2}|1 - 2\lambda|
\]

\[
\equiv B_\lambda(W).
\]
In particular, when \( \lambda = 1/2 \), the optimal decoder in (9) reduces to the maximum-likelihood (ML) rule, and

\[
B_{1/2}(W) = \frac{1}{4} \sum_{y \in Y} |W(y|0) - W(y|1)|.
\]

In that case, \( f_{1/2}(s) = \frac{1}{2} - s \wedge \bar{s} = \frac{1}{2} - \frac{1}{2}(1 - |2s - 1|) = \frac{1}{2}|2s - 1| \), and therefore

\[
B_{1/2}(W) = \frac{1}{2} - P_{e,ML}(W),
\]

where \( P_{e,ML}(W) \) denotes the probability of error of maximum-likelihood decoding of a single equiprobable bit sent through the channel \( W \) [6, Ch. 5]. This, in turn, shows that

\[
1 - 2P_{e,ML}(W) = I_f(W) \quad \text{with} \quad f(s) = |2s - 1|.
\]

IV. Output-Symmetric BICs

In his original paper introducing channel polarization [1], Arıkan analyzed BICs having the property of output symmetry:

**Definition 5** (Output-symmetric BIC). A BIC \((Y, W)\) is output-symmetric if there exists a bijection \( \pi : Y \rightarrow Y \), such that \( \pi^{-1} = \pi \) and \( W(\pi(y)|0) = W(y|1) \) for all \( y \in Y \).

From this point on, we will use the shorter phrase “symmetric BIC” instead of “output-symmetric BIC.”

A. Structural decomposition of symmetric BICs

Let us first recall the following definition:

**Definition 6** (Compound channel). Let \((Y_i, W_i), i \in [m] := \{1, \ldots, m\}, be a collection of BICs, and let \( \lambda = (\lambda_1, \ldots, \lambda_m) \) be a probability distribution on \([m]\). A compound channel with subchannels \( \{W_i\} \) and mixing distribution \( \lambda \) is a BIC \( W \) defined by transition probabilities

\[
W(i, y|x) = \lambda_i W_i(y|x),
\]

for all \( x \in \{0, 1\}, i \in [m], \) and \( y \in Y_i \). A compound channel will be denoted as \( W = \bigoplus_{i=1}^{m} \lambda_i W_i \).

The following structural result proved in [5, Thm. 2.1] establishes that any symmetric BIC is a compound channel with BSC subchannels:

**Theorem 3.** For any symmetric BIC \( W \), there exist a positive integer \( m \), a probability vector \( \lambda = (\lambda_1, \ldots, \lambda_m) \), and error parameters \( p_1, \ldots, p_m \in [0, 1] \), such that

\[
W \equiv \bigoplus_{i=1}^{m} \lambda_i \text{BSC}(p_i).
\]
B. Blackwell measures of symmetric BICs

It is not difficult to show that the Blackwell measure of a compound channel is given by the mixture of the Blackwell measures of the constituent subchannels:

\[ m_W = \sum_{i=1}^{m} \lambda_i m_{W_i}. \]

Thus, Thm. 3 shows that the Blackwell measure of any symmetric BIC is a mixture of Blackwell measures of BSCs. In particular, if \( W = \bigoplus_{i=1}^{m} \lambda_i \text{BSC}(p_i) \), then

\[ m_W = \sum_{i=1}^{m} \lambda_i m_{\text{BSC}(p_i)} = \sum_{i=1}^{m} \left( \frac{\lambda_i}{2} \delta_{p_i} + \frac{\lambda_i}{2} \delta_{\bar{p}_i} \right). \]

Thus, any symmetric BIC \( W \) that admits the decomposition (11) is specified, up to Blackwell equivalence, by the set

\[ C_W := \{ (\lambda_i, p_i) : i \in [m] \}. \]  

Moreover, if \( S \sim m_W \), then \( \bar{S} = 1 - S \sim m_W \) as well.

Another representation of BICs, proposed by Alsan [6, Ch. 2], is based on the quantity

\[ \Delta_W(y) := \frac{W(y|0) - W(y|1)}{W(y|0) + W(y|1)} \]

\[ \equiv 2f_W(y) - 1. \]

Thus, if \( (X, Y) \) is a random couple with \( P_X = \text{Bern}(1/2) \) and \( P_{Y|X} = W \), and \( S = f_W(Y) \), then the probability law of \( \Delta_W(Y) = 2S - 1 \) also specifies \( W \) uniquely up to Blackwell equivalence. Moreover, \( \Delta_W(Y) \) takes values in \([-1, 1]\), has mean zero, and is symmetric, i.e., \( \Delta_W(Y) \) and \(-\Delta_W(Y)\) have the same probability law. From this, it is not hard to show that, for symmetric BICs, the Blackwell ordering is equivalent to the symmetric convex ordering introduced by Alsan [6, Ch. 6], according to which \( W \) dominates \( W' \) if and only if

\[ E[f(\Delta_W(Y))] \geq E[f(\Delta_{W'}(Y'))] \]

for all convex and even functions \( f : [-1, 1] \to \mathbb{R} \), where \( P_X = \text{Bern}(1/2), P_{Y|X} = W, \) and \( P_{Y'|X} = W' \). (This equivalence was established in [6].)
C. Examples of properties of symmetric BICs

Theorem 3 also has implications for the computation of functionals of channels. Indeed, it follows directly from the definitions in Sec. III that, for any \( f : [0, 1] \to \mathbb{R} \),
\[
I_f(\text{BSC}(p)) = \frac{1}{2}f(p) + \frac{1}{2}f(\bar{p}).
\]
Thus, if \( W \) is a symmetric BIC that admits the decomposition (11), then
\[
I_f(W) = \sum_{i=1}^{m} \lambda_i I_f(\text{BSC}(p_i)) = \sum_{i=1}^{m} \lambda_i \left( f(p_i) + f(\bar{p}_i) \right).
\]
Therefore, induced functionals of the form \( I_f(W) \) (e.g., \( I(W), M_r(W), Z(W), \) etc.) summarized in Table I may be computed for any symmetric BIC \((Y, W)\) via the structural decomposition into BSC subchannels. The channel dispersion \( V(W) \) is not an induced functional, but may be derived from the induced functionals \( M_2(W) \) and \( I(W) \).

Example 4 (Channel dispersion of \( \text{BEC}(\varepsilon) \)). The channel dispersion of the binary erasure channel is (cf. [25, Thm. 53]),
\[
V(\text{BEC}(\varepsilon)) = \varepsilon \bar{\varepsilon}.
\]
We can also obtain this by observing that \( \text{BEC}(\varepsilon) \equiv \bar{\varepsilon} \text{BSC}(0) \oplus \varepsilon \text{BSC}(1/2) \). The second moment of the information density \( M_2(\text{BEC}(\varepsilon)) \) is computed by selecting \( f(s) = \psi_2(s) \), and applying Eqn. (14) as follows:
\[
M_2(\text{BEC}(\varepsilon)) = \frac{\varepsilon f(0)}{2} + \frac{\bar{\varepsilon} f(1)}{2} + \frac{\varepsilon f(1/2)}{2} + \frac{\bar{\varepsilon} f(1/2)}{2} = \bar{\varepsilon}.
\]
The dispersion parameter \( V(\text{BEC}(\varepsilon)) = M_2(\text{BEC}(\varepsilon)) - (I(\text{BEC}(\varepsilon)))^2 = \bar{\varepsilon} - \bar{\varepsilon}^2 = \varepsilon \bar{\varepsilon} \).

Example 5 (Channel dispersion of \( \text{BSC}(p) \)). The channel dispersion of the binary symmetric channel for \( p \notin \{0, \frac{1}{2}, 1\} \) is (cf. [25, Thm. 52]),
\[
V(\text{BSC}(p)) = p \bar{p} \left( \log_2 \frac{\bar{p}}{p} \right)^2.
\]
The channel dispersion for \( p \in \{0, \frac{1}{2}, 1\} \) approaches the limit of 0. We can also obtain this from Eqn. (14). The second moment of the information density \( M_2(\text{BSC}(p)) \) is computed by selecting \( f(s) = \psi_2(s) \):
\[
M_2(\text{BSC}(p)) = \frac{f(p)}{2} + \frac{f(\bar{p})}{2} = p(1 + \log_2 p)^2 + \bar{p}(1 + \log_2 \bar{p})^2.
\]
The dispersion parameter is computed as
\[ V(BSC(p)) := M_2(BSC(p)) - (I(BSC(p)))^2 \]
\[ = p\bar{p} \left( (\log_2 p)^2 + (\log_2 \bar{p})^2 - 2 \log_2 p \log_2 \bar{p} \right), \]
which is verified to be equivalent to Eqn. (16).

We can abstract these examples into the following general result:

**Lemma 1** (Channel dispersion of an arbitrary symmetric BIC). Consider a symmetric BIC \((Y, W)\) with decomposition \(W \equiv \bigoplus_{i=1}^{m} \lambda_i BSC(p_i)\). The channel capacity \(I(W)\) and channel dispersion \(V(W)\) may be written in terms of the capacities and dispersions of the subchannels:
\[
I(W) = \sum_{i=1}^{m} \lambda_i I(BSC(p_i)),
\]
\[
V(W) = \sum_{i=1}^{m} \lambda_i V(BSC(p_i)) + \sum_{i=1}^{m} \lambda_i (I(BSC(p_i)) - I(W))^2.
\]

**Proof.** Provided in Appendix A. \(\square\)

**Remark 2.** Lemma 1 shows precisely that \(I(W)\) is the average of the mutual information values of the subchannels of \(W\). Similarly, \(V(W)\) is a sum of the average channel dispersions of all subchannels and a variance term involving first-order mutual information values of subchannels.

**D. Mutual information profile**

The mutual information profile (MIP) (see [5, Chap. 2] for a detailed presentation) is based on the structural decomposition of symmetric BICs:

**Definition 7** (Mutual Information Profile). A symmetric BIC \((Y, W)\) with structural decomposition \(W \equiv \bigoplus_{i=1}^{m} \lambda_i BSC(p_i)\) as in Theorem 3 is uniquely characterized by a random variable \(\Phi\) that takes values in the unit interval \([0, 1]\) according to the probability law
\[
m_W^\Phi := \sum_{i=1}^{m} \lambda_i \delta_{I(BSC(p_i))}.
\]

The probability law \(m_W^\Phi\) is called the mutual information profile (MIP) of the channel \(W\).

Similar to the Blackwell measure which uniquely specifies an arbitrary BIC up to Blackwell equivalence, the MIP uniquely specifies BICs with the property of output symmetry. In fact, it is easy to see from Eqn (20) that the MIP \(m_W^\Phi\) is simply the probability law of \(1 - h_2(S)\) when \(S \sim m_W\). The lemma below follows immediately from this observation:
Lemma 2 (Mean and variance of $\Phi$). Consider a symmetric BIC $(Y, W)$ with structural decomposition $W \equiv \bigoplus_{i=1}^{m} \lambda_i \text{BSC}(p_i)$, and mutual information profile $\Phi$ with probability measure $m_\Phi$ as in Eqn. (20). The first and second moments of $\Phi$ are given by

$$E[\Phi] = \sum_{i=1}^{m} \lambda_i I(\text{BSC}(p_i)),$$

$$E[\Phi^2] = \sum_{i=1}^{m} \lambda_i (I(\text{BSC}(p_i)))^2.$$  

The variance $\text{Var}[\Phi] := E[\Phi^2] - (E[\Phi])^2$ may be written in the following form,

$$\text{Var}[\Phi] = \sum_{i=1}^{m} \lambda_i (I(\text{BSC}(p_i)) - I(W))^2.$$

Remark 3. The mean $E[\Phi]$ is equivalent to $I(W)$ given in Eqn. (18). The variance $\text{Var}[\Phi]$ is related to (but not equivalent to) the channel dispersion $V(W)$ given in Eqn. (19).

V. THE POLAR TRANSFORM

The polar transform maps a pair of BICs to another pair of polarized BICs via a Boolean XOR of the binary inputs of the original channels [1]. The Boolean XOR creates dependence between the random variables associated to the inputs and outputs of the original channels.

A. The polar transform

Definition 8 (The polar transform). The polar transform maps a pair of BICs $(Y_1, W_1)$ and $(Y_2, W_2)$ into another pair of BICs $(Y_1 \times Y_2, W_1 \boxoplus W_2)$ and $(Y_1 \times Y_2 \times \{0, 1\}, W_1 \varotimes W_2)$ as follows:

$$(W_1 \boxoplus W_2)(y_1, y_2|x) := \frac{1}{2} \sum_{u \in \{0, 1\}} W_1(y_1|u \oplus x)W_2(y_2|u)$$

$$(W_1 \varotimes W_2)(y_1, y_2, u|x) := \frac{1}{2} W_1(y_1|u \oplus x)W_2(y_2|x)$$

for all $x, u \in \{0, 1\}$ and all $(y_1, y_2) \in Y_1 \times Y_2$, where $\oplus$ is the Boolean XOR.

The polarized channel $W_1 \boxoplus W_2$ is “weaker” than both $W_1$ and $W_2$ as will be clarified in subsequent analysis. The polarized channel $W_1 \varotimes W_2$ is improved because it is equivalent to decoding based on two independent noisy versions of the binary input. A parallel broadcast of the binary input is formalized as follows:
**Definition 9** (Product BIC $W_1 \times W_2$). Given two BICs $(Y_1, W_1)$ and $(Y_2, W_2)$, we define the product BIC $(Y_1 \times Y_2, W_1 \times W_2)$ by

$$(W_1 \times W_2)(y_1, y_2|x) := W_1(y_1|x)W_2(y_2|x)$$

for all $x \in \{0, 1\}$ and all $(y_1, y_2) \in Y_1 \times Y_2$. In other words, $W_1 \times W_2$ is the parallel broadcast channel formed by $W_1$ and $W_2$ [26].

**B. Blackwell measures of polarized BICs**

Consider two Blackwell measures $m_1, m_2 \in \mathcal{M}$. The following operations $\boxplus$ and $\oplus$ on a pair of Blackwell measures yield two additional probability measures $m_1 \boxplus m_2$ and $m_1 \oplus m_2$ on $[0,1]$.

**Definition 10.** Let $m_1, m_2 \in \mathcal{M}$. Let $S_1 \sim m_1$ and $S_2 \sim m_2$ be two independent random variables. The probability measures $m_1 \boxplus m_2$ and $m_1 \oplus m_2$ are defined as follows: For any continuous bounded $f : [0,1] \to \mathbb{R}$, let

$$\int_{[0,1]} f d(m_1 \boxplus m_2) = \mathbb{E}[f(1 - S_1 \star S_2)]$$

and

$$\int_{[0,1]} f d(m_1 \oplus m_2) = \mathbb{E}\left[ (1 - S_1 \star S_2)f\left(\frac{S_1S_2}{1 - S_1 \star S_2}\right) \right] + (S_1 \star S_2)f\left(\frac{\bar{S}_1S_2}{S_1 \star S_2}\right).$$

**Lemma 3.** The probability measures $m_1 \boxplus m_2$ and $m_1 \oplus m_2$ are also Blackwell measures.

**Proof.** Setting $f(s) = s$ in Eqs. (24) and (25), and recalling that $S_1$ and $S_2$ are independent and both have mean $\frac{1}{2}$, we get $\int_{[0,1]} s(m_1 \boxplus m_2)(ds) = \mathbb{E}[1 - S_1 \star S_2] = \frac{1}{2}$. Similarly, $\int_{[0,1]} s(m_1 \oplus m_2)(ds) = \mathbb{E}[S_2] = \frac{1}{2}$. Thus, both $m_1 \boxplus m_2$ and $m_1 \oplus m_2$ are in $\mathcal{M}$. $\square$.

The Blackwell measures of polarized channels $W_1 \boxplus W_2$ and $W_1 \oplus W_2$ can be computed from those of $W_1$ and $W_2$:

**Theorem 4** (Evolution of Blackwell measures under polarization). The Blackwell measures of the polarized BICs $W_1 \boxplus W_2$ and $W_1 \oplus W_2$ introduced in Def. 8 are given by

$$m_{W_1 \boxplus W_2} = m_{W_1} \boxplus m_{W_2},$$

$$m_{W_1 \oplus W_2} = m_{W_1} \oplus m_{W_2},$$

where the operations $\boxplus$ and $\oplus$ on Blackwell measures were defined in Def. 10.
Proof. We first establish the formula for $W_1 \oplus W_2$. Let $(X_i, Y_i)$, for $i \in \{1, 2\}$, where $(X_1, Y_1)$ and $(X_2, Y_2)$ are independent, $P_{X_i} = P_{X_2} = \text{Bern}(1/2)$, and $P_{Y_i | X_i} = W_i$. Then, recalling the definition of $f_W$ in (1), we can write

$$(W_1 \boxast W_2)(y_1, y_2|0) = \frac{1}{2} (W_1(y_1|0)W_2(y_2|0) + W_1(y_1|1)W_2(y_2|1))$$

and

$$(W_1 \boxast W_2)(y_1, y_2|1) = \frac{1}{2} (W_1(y_1|1)W_2(y_2|0) + W_1(y_1|0)W_2(y_2|1))$$

Thus,

$$(W_1 \boxast W_2)(y_1, y_2|0) + (W_1 \boxast W_2)(y_1, y_2|1) = 2P_{Y_1}(y_1)P_{Y_2}(y_2),$$

which gives

$$f_{W_1 \boxast W_2}(y_1, y_2) = \frac{(W_1 \boxast W_2)(y_1, y_2|0)}{(W_1 \boxast W_2)(y_1, y_2|0) + (W_1 \boxast W_2)(y_1, y_2|1)}$$

$$= 1 - f_{W_1}(y_1) \ast f_{W_2}(y_2).$$

This shows that $S = f_{W_1 \boxast W_2}(Y_1, Y_2) = 1 - S_1 \ast S_2$, where $S_1 = f_{W_1}(Y_1)$ and $S_2 = f_{W_2}(Y_2)$ are independent. Thus, for any continuous $f : [0, 1] \to \mathbb{R}$,

$$\int_{[0,1]} f \, dm_{W_1 \boxast W_2} (ds) = \mathbb{E}[f(S)]$$

$$= \mathbb{E}[f(1 - S_1 \ast S_2)]$$

$$= \int_{[0,1]} f \, dm_{W_1} \oplus m_{W_2}.$$

We turn to $W_1 \oplus W_2$. From the definition of the polar transform given in Eqn. (23), it follows that

$$W_1 \oplus W_2 = \frac{1}{2} W^{(0)} + \frac{1}{2} W^{(1)}$$

with $W^{(0)} := W_1 \times W_2$ and $W^{(1)} := \bar{W}_1 \times W_2$, where $\bar{W}_1$ is the BIC related to $W_1$ via $\bar{W}_1(\cdot|x) = W_1(\cdot|x)$. Then the random variables $S^{(0)} \sim m_{W^{(0)}}$ and $S^{(1)} \sim m_{W^{(1)}}$ evidently satisfy

$$\mathbb{E}[f(S^{(0)})] = 2 \mathbb{E} \left[ (1 - S_1 \ast S_2) f \left( \frac{S_1 S_2}{1 - S_1 \ast S_2} \right) \right]$$
and
\[ E[f(S^{(1)})] = 2E \left[ (S_1 * S_2)f \left( \frac{\tilde{S}_1 S_2}{S_1 * S_2} \right) \right] \]
for every continuous \( f : [0, 1] \to \mathbb{R} \). Combining this result with (29) yields
\[
\int_{[0,1]} f dm_{W_1 \oplus W_2} = \frac{1}{2} E[f(S^{(0)})] + \frac{1}{2} E[f(S^{(1)})] \\
= E \left[ (1 - S_1 * S_2)f \left( \frac{S_1 S_2}{1 - S_1 * S_2} \right) \right. \\
+ (S_1 * S_2)f \left( \frac{\tilde{S}_1 S_2}{S_1 * S_2} \right) \bigg] \\
= \int_{[0,1]} f dm_{W_1 \otimes m_{W_2}}.
\]
Since \( f \) is arbitrary, we obtain the formula for \( m_{W_1 \otimes W_2} \).

\[ \square \]

C. Blackwell measures of polarized BSCs

Although Theorem 4 fully characterizes the Blackwell measure of arbitrary polarized BICs, further specialization is possible if the original BICs have the property of symmetry. Symmetric BICs are composed of BSC subchannels as established in Theorem 3. Thus, the essential aspect of polarization for symmetric BICs is the interaction of BSC subchannels which may have different probabilities of error.

The three lemmas below provide three complementary descriptions of the effect of the polar transform on a pair of BSCs:

**Lemma 4** (Polarization of BSCs — transition matrices). Let \((p, q) \in [0, 1] \times [0, 1]\). Consider the channels BSC\((p)\) and BSC\((q)\). As defined in Def. 8, let BSC\((p) \oplus\) BSC\((q)\) and BSC\((p) \otimes\) BSC\((q)\) represent the polarized channels. The transition matrices of the polarized channels have the following structure:

\[
\tilde{T}_{\text{BSC}(p) \oplus \text{BSC}(q)} = T_{\text{BSC}(p \ast q)} \\
= \begin{bmatrix}
1 - p \ast q & p \ast q \\
p \ast q & 1 - p \ast q
\end{bmatrix}, \\
(30)
\]
\[
\tilde{T}_{\text{BSC}(p) \otimes \text{BSC}(q)} = T_{\text{BSC}(p) \times \text{BSC}(q)} \\
= \begin{bmatrix}
\bar{p}q & pq & p\bar{q} & \bar{p}q \\
pq & \bar{p}q & \bar{p}q & pq
\end{bmatrix}, \\
(31)
\]
where \( \tilde{T}_W \) represents the transition matrix \( T_W \) with a reduced number of columns due to aggregating (i.e., grouping) output symbols of BIC \( W \).
Proof. Provided in Appendix B.

Lemma 4 establishes that, by grouping output symbols in a specific way, the polarized channels

\[ \text{BSC}(p) \boxplus \text{BSC}(q) \equiv \text{BSC}(p \ast q) \]

and

\[ \text{BSC}(p) \oslash \text{BSC}(q) \equiv \text{BSC}(p) \times \text{BSC}(q). \]

The parallel broadcast channel \( \text{BSC}(p) \times \text{BSC}(q) \) is an output-symmetric BIC. Due to Theorem 3, it has the structure of a compound channel with BSC subchannels.

**Lemma 5** (Polarization of BSCs — structural decomposition). Let \( (p, q) \in [0, \frac{1}{2}] \times [0, \frac{1}{2}] \). Assume \( (p, q) \neq (0, 0) \) so that \( p \ast q \neq 0 \). Consider the channels \( \text{BSC}(p) \) and \( \text{BSC}(q) \) as in Lemma 4, and define the following parameters:

\[
\alpha := \frac{pq}{1 - p \ast q}, \tag{32}
\]

\[
\beta := \frac{pq}{p \ast q}, \tag{33}
\]

where \( \alpha \in [0, \frac{1}{2}] \) and \( (\beta \land \bar{\beta}) \in [0, \frac{1}{2}] \). Then the polarized channels satisfy the following equivalences.

\[ \text{BSC}(p) \oplus \text{BSC}(q) \equiv \text{BSC}(p \ast q) \], \quad \tag{34} \]

\[ \text{BSC}(p) \circ \text{BSC}(q) \]

\[ \equiv \text{BSC}(p) \times \text{BSC}(q) \]

\[ \equiv (1 - p \ast q) \text{BSC}(\alpha) \oplus (p \ast q) \text{BSC}(\beta \land \bar{\beta}). \] \quad \tag{35} \]

Proof. Provided in Appendix C.

Lemmas 4 and 5 precisely characterize the interaction of a \( \text{BSC}(p) \) with a \( \text{BSC}(q) \) after one step of polarization. The equivalences are based on equivalences between corresponding transition matrices. In a more general approach based on Blackwell measures, the equivalences may be derived directly from Theorem 4 as a corollary:
Corollary 1 (Polarization of BSCs — Blackwell measures). Let \((p, q) \in [0, 1/2] \times [0, 1/2]\). Consider the channels \(\text{BSC}(p)\) and \(\text{BSC}(q)\). Then the Blackwell measures of the polarized channels \(\text{BSC}(p) \oplus \text{BSC}(q)\) and \(\text{BSC}(p) \otimes \text{BSC}(q)\) are given as follows:

\[
\begin{align*}
\text{m}_{\text{BSC}(p) \oplus \text{BSC}(q)} &= \text{m}_{\text{BSC}(p)} \oplus \text{m}_{\text{BSC}(q)} \\
&= \text{m}_{\text{BSC}(p \oplus q)}. \\
\text{m}_{\text{BSC}(p) \otimes \text{BSC}(q)} &= \text{m}_{\text{BSC}(p)} \otimes \text{m}_{\text{BSC}(q)} \\
&= \text{m}_{\text{BSC}(p \times q)}. \\
\end{align*}
\]

In addition, consider the parameters \(\alpha := \frac{pq}{1-p \times q}\) and \(\beta := \frac{pq}{p \times q}\) as defined in Lemma 5. Then for \((p, q) \neq (0, 0)\), the parallel broadcast channel \(\text{BSC}(p) \times \text{BSC}(q)\) has the Blackwell measure

\[
\begin{align*}
\text{m}_{\text{BSC}(p \times q)} &= (1 - p \times q)\text{m}_{\text{BSC}(\alpha)} + (p \times q)\text{m}_{\text{BSC}(\beta \wedge \bar{\beta})}. \\
\end{align*}
\]
Proof. Provided in Appendix D.

Remark 4. The polarized channel $\text{BSC}(p) \boxtimes \text{BSC}(q)$ is a BSC with a different probability of error, which is larger than both $p$ and $q$. The polarized channel $\text{BSC}(p) \otimes \text{BSC}(q)$ is a more complex channel. More precisely, it is a compound channel as defined in Def. 6 which is composed of two BSC components as subchannels. Fig. 1 depicts the polarization of BSCs.

D. Blackwell measures of polarized symmetric BICs

Building upon Corollary 1, it is possible to characterize the image of a pair of arbitrary symmetric BICs under the polar transformation. The polarization of symmetric BICs is characterized entirely by the polarization of BSC subchannels.

Corollary 2 (Polarization of symmetric BICs — Blackwell measures). Consider two symmetric BICs $(Y_1, W_1)$ and $(Y_2, W_2)$. As established by Theorem 3, there exist positive integers $m, k$, probability vectors $\lambda = (\lambda_1, \ldots, \lambda_m)$, $\mu = (\mu_1, \ldots, \mu_k)$, and tuples $p = (p_1, \ldots, p_m) \in [0,1]^m$, $q = (q_1, \ldots, q_k) \in [0,1]^k$ such that

$$W_1 \equiv \bigoplus_{i=1}^{m} \lambda_i \text{BSC}(p_i),$$

$$W_2 \equiv \bigoplus_{j=1}^{k} \mu_j \text{BSC}(q_j).$$

Then the polarized channels $W_1 \boxtimes W_2$ and $W_1 \otimes W_2$ have the following Blackwell measures which illustrate their structural decompositions:

$$m_{W_1 \boxtimes W_2} = \sum_{i=1}^{m} \sum_{j=1}^{k} \lambda_i \mu_j m_{\text{BSC}(p_i \otimes q_j)}.$$  \hspace{1cm} (39)

$$m_{W_1 \otimes W_2} = \sum_{i=1}^{m} \sum_{j=1}^{k} \lambda_i \mu_j m_{\text{BSC}(p_i) \times \text{BSC}(q_j)}.$$  \hspace{1cm} (40)

Proof. Provided in Appendix E.

As an illustration, we give an alternative derivation of the fact that the image of a pair of BECs under the polar transform is another pair of BECs:
Example 6 (Polarization of BECs — Blackwell measures). Consider two erasure channels, \( \text{BEC}(\varepsilon) \) and \( \text{BEC}(\tau) \), with erasure probabilities \( \varepsilon \in [0, 1] \) and \( \tau \in [0, 1] \). Applying Eqn. (39) of Corollary 2, the polarized channel \( \text{BEC}(\varepsilon) \boxast \text{BEC}(\tau) \) has the following Blackwell measure:

\[
\begin{align*}
m_{\text{BEC}(\varepsilon) \boxast \text{BEC}(\tau)} &= m_{\text{BEC}(\varepsilon)} \boxast m_{\text{BEC}(\tau)} \\
&= \left( \bar{\varepsilon}m_{\text{BSC}(0)} + \varepsilon m_{\text{BSC}(1/2)} \right) \boxast \left( \bar{\tau}m_{\text{BSC}(0)} + \tau m_{\text{BSC}(1/2)} \right) \\
&= \bar{\varepsilon}\bar{\tau}m_{\text{BSC}(0)} + \bar{\varepsilon}\tau m_{\text{BSC}(0)} + \varepsilon\bar{\tau}m_{\text{BSC}(1/2)} + \varepsilon\tau m_{\text{BSC}(1/2)} \\
&= \bar{\varepsilon}\bar{\tau}m_{\text{BSC}(0)} + \bar{\varepsilon}\tau m_{\text{BSC}(0)} + (1 - \bar{\varepsilon}\bar{\tau})m_{\text{BSC}(1/2)} \\
&= m_{\text{BEC}(1 - \bar{\varepsilon}\bar{\tau})}.
\end{align*}
\]

Similarly, applying Eqn. (40) of Corollary 2, the polarized channel \( \text{BEC}(\varepsilon) \boxast \text{BEC}(\tau) \) has the following Blackwell measure,

\[
\begin{align*}
m_{\text{BEC}(\varepsilon) \boxast \text{BEC}(\tau)} &= m_{\text{BEC}(\varepsilon)} \boxast m_{\text{BEC}(\tau)} \\
&= \left( \bar{\varepsilon}m_{\text{BSC}(0)} + \varepsilon m_{\text{BSC}(1/2)} \right) \boxast \left( \bar{\tau}m_{\text{BSC}(0)} + \tau m_{\text{BSC}(1/2)} \right) \\
&= \bar{\varepsilon}\bar{\tau}m_{\text{BSC}(0)} \times \text{BSC}(0) + \bar{\varepsilon}\tau m_{\text{BSC}(0)} \times \text{BSC}(1/2) \\
&\quad + \varepsilon\bar{\tau}m_{\text{BSC}(1/2)} \times \text{BSC}(0) + \varepsilon\tau m_{\text{BSC}(1/2)} \times \text{BSC}(1/2) \\
&= \bar{\varepsilon}\bar{\tau}m_{\text{BSC}(0)} + \bar{\varepsilon}\tau m_{\text{BSC}(0)} + \varepsilon\bar{\tau}m_{\text{BSC}(0)} + \varepsilon\tau m_{\text{BSC}(1/2)} \\
&= (1 - \varepsilon\tau)m_{\text{BSC}(0)} + \varepsilon\tau m_{\text{BSC}(1/2)} \\
&= m_{\text{BEC}(\varepsilon\tau)}.
\end{align*}
\]

Remark 5. As depicted in Fig. 1, the polarization of two binary erasure channels leads to two channels which are also binary erasure channels.

\[
\text{BEC}(\varepsilon) \boxast \text{BEC}(\tau) \equiv \text{BEC}(1 - \bar{\varepsilon}\bar{\tau}).
\]

\[
\text{BEC}(\varepsilon) \boxast \text{BEC}(\tau) \equiv \text{BEC}(\varepsilon\tau).
\]
VI. SUCCESSIVE CHANNEL POLARIZATION

For a given BIC \((Y, W)\), the polar transforms defined in Def. 8 may be applied successively, as originally analyzed by Arıkan [1]. Polarizing the original channel \(W\) successively over \(n\) iterations results in one of \(2^n\) possible channels.

**Definition 11** (Successive channel polarization). Consider a BIC \((Y, W)\). Let \(b = (b_1, b_2, \ldots, b_n)\) be an \(n\)-dimensional binary vector, \(b \in \{0, 1\}^n\). The channel \(W_b\) is obtained by successive polarization in the following manner:

\[
W_b = W_{(b_1, b_2, \ldots, b_n)} := \begin{cases} 
W_{(b_1, b_2, \ldots, b_{n-1})} \boxplus W_{(b_1, b_2, \ldots, b_{n-1})}, & \text{if } b_n = 0, \\
W_{(b_1, b_2, \ldots, b_{n-1})} \boxdot W_{(b_1, b_2, \ldots, b_{n-1})}, & \text{if } b_n = 1.
\end{cases}
\]

for all integers \(n > 1\). The base case is given by \(W_{b_1} = W \boxplus W\) if \(b_1 = 0\) and \(W_{b_1} = W \boxdot W\) if \(b_1 = 1\).

**A. Polar code construction: symmetric BICs**

The exact evolution of the Blackwell measure of arbitrary BICs over a single iteration of polarization was characterized in Theorem 4. Corollary 2 provides the exact evolution of the Blackwell measure for output-symmetric BICs. For the class of output-symmetric BICs, a simple explicit algorithm can be used to construct all \(2^n\) polarized channels over \(n\) iterations of successive polarization.

Recall the definition of the set \(C_W\) in Eqn. (12). Given the sets \(C_{W_1}\) and \(C_{W_2}\) for two output-symmetric BICs \((Y_1, W_1)\) and \((Y_2, W_2)\), Algorithm 1 and Algorithm 2 implement the operations \(W_1 \boxplus W_2\) and \(W_1 \boxdot W_2\) respectively. The pseudocode for these algorithms is based entirely on Corollary 1 and Corollary 2. The polar transforms may be applied successively as in Algorithm 3 to obtain the representation \(C_{W_b}\) for any polarized output-symmetric BIC \(W_b\) defined in Definition 11, where binary vector \(b \in \{0, 1\}^n\).

**B. Channel approximations: symmetric BICs**

Since the size of the output alphabet of polarized channels increases exponentially with the number of iterations \(n\) of Algorithm 3, the algorithm must be modified slightly to maintain computational tractability. In this section, a simple method is analyzed to approximate symmetric BICs based on the structural decomposition given in Theorem 3. More complex methods for channel approximation may be found in the literature; e.g., [17], [18].
Algorithm 1 Exact Polarization of Output-Symmetric BICs
1: function POLAR-$\boxast$($\mathcal{C}_W_1 = \{(\lambda_i, p_i)\}$, $\mathcal{C}_W_2 = \{(\mu_j, q_j)\}$)
\hspace{1em}\triangleright Require: $\sum_i \lambda_i = 1$, $\lambda_i > 0$, $p_i \in [0, \frac{1}{2}]$
\hspace{1em}\triangleright Require: $\sum_j \mu_j = 1$, $\mu_j > 0$, $q_j \in [0, \frac{1}{2}]$
2: $\mathcal{C}_W_1 \boxast W_2 = \{\}$
3: for $i = 1$ to $|\mathcal{C}_W_1|$ do
4: \hspace{1em}for $j = 1$ to $|\mathcal{C}_W_2|$ do
5: \hspace{2em}$\mathcal{C}_W_1 \boxast W_2 = \mathcal{C}_W_1 \boxast W_2 \cup \{(\lambda_i \mu_j, p_i \star q_j)\}$
6: \hspace{1em}end for
7: \hspace{1em}end for
8: return $\mathcal{C}_W_1 \boxast W_2$
9: end function

Definition 12 (Dyadic $\Delta$-interval). Define $\Delta := 2^{-L}$ where $L$ is a positive integer. For $k \in \{1, 2, \ldots, 2^L\}$, define the interval

$$D_{L,k} := \left[\frac{(k - 1)\Delta}{2}, \frac{k\Delta}{2}\right).$$

The collection of $2^L$ non-overlapping intervals $\{D_{L,k}\}$ each of uniform width $\frac{\Delta}{2}$ is the collection of dyadic subintervals of $[0, \frac{1}{2}]$.

Definition 13 (Quantization function). Consider the dyadic $\Delta$-intervals as defined in Def. 12. For any real number $x \in [0, \frac{1}{2}]$, define the quantization function $Q_{\Delta}(x) := \frac{\Delta}{2} \left\lceil \frac{2x}{\Delta} \right\rceil$.

Definition 14 (Channel approximation by $\Delta$-quantization). Consider the dyadic $\Delta$-intervals as defined in Def. 12, and the quantization function $Q_{\Delta}(x)$ defined in Def. 13. Consider a symmetric BIC $(Y, W)$. Due to Thm. 3, $W \equiv \bigoplus_{i=1}^m \lambda_i \text{BSC}(p_i)$ for some positive integer $m$, probability vector $\lambda = (\lambda_1, \ldots, \lambda_m)$, and tuple $p = (p_1, \ldots, p_m)$. Let $p_i \in [0, \frac{1}{2}]$ without loss of generality. As a quantized approximation to $(Y, W)$, define an output-symmetric BIC $(Y, Q_{\Delta}(W))$ as follows:

$$Q_{\Delta}(W) \equiv \bigoplus_{i=1}^m \lambda_i \text{BSC}(Q_{\Delta}(p_i)).$$

Lemma 6 (Accuracy of channel approximation). Consider a BIC $(Y, W)$ and quantized approximation $(Y, Q_{\Delta}(W))$ as in Def. 14. Then $Q_{\Delta}(W) \preceq W$, and the following bound on mutual information holds:

$$I(Q_{\Delta}(W)) \leq I(W) \leq I(Q_{\Delta}(W)) + h_2\left(\frac{\Delta}{2}\right).$$

(41)
Algorithm 2 Exact Polarization of Output-Symmetric BICs

1: function POLAR(@\(C_{W_1} = \{(\lambda_i, p_i)\}, \ C_{W_2} = \{(\mu_j, q_j)\}\))
   \[\triangleright\text{Require: } \sum_i \lambda_i = 1, \lambda_i > 0, \ p_i \in [0, \frac{1}{2}]\]
   \[\triangleright\text{Require: } \sum_j \mu_j = 1, \mu_j > 0, \ q_j \in [0, \frac{1}{2}]\]
2: \(C_{W_1} \oplus W_2 = \{\}\)
3: for \(i = 1\) to \(|C_{W_1}|\) do
4:   for \(j = 1\) to \(|C_{W_2}|\) do
5:     if \(p_i = 0\) or \(q_j = 0\) then
6:       \(C_{W_1} \oplus W_2 = C_{W_1} \oplus W_2 \cup \{(\lambda_i \mu_j, 0)\}\)
7:     else
8:       \(\alpha = \frac{p_i q_j}{1 - p_i \ast q_j}\)
9:       \(\beta = \frac{p_i q_j}{p_i \ast q_j}\)
10:      \(C_{W_1} \oplus W_2 = C_{W_1} \oplus W_2 \cup \{(\lambda_i \mu_j (1 - p_i \ast q_j), \alpha)\}\)
11:      \(C_{W_1} \oplus W_2 = C_{W_1} \oplus W_2 \cup \{(\lambda_i \mu_j (p_i \ast q_j), \beta \wedge \beta)\}\)
12:   end if
13: end for
14: end for
15: return \(C_{W_1} \oplus W_2\)
16: end function

Proof. Provided in Appendix F.

Remark 6. The quantized channel \(Q_\Delta(W)\) is stochastically degraded in relation to the original channel \(W\). Its symmetric capacity is nearly equal to that of \(W\) if the uniform width \(\frac{\Delta}{2}\) of the dyadic \(\Delta\)-intervals is chosen small enough. Critically, the approximation allows for computational tractability.

C. Channel approximations: polarized symmetric BICs

In this section, we show that the accuracy of the approximation derived in Lemma 6 is still maintained after successive iterations of the polar transform.

Lemma 7 (Blackwell ordering is preserved by the polar transform). Consider a BIC \((Y, W)\) and a BIC \((Y', W')\) such that \(W' \preceq W\). Then the stochastic degradation relation is preserved after the polar
Algorithm 3 Exact Polarization of Output-Symmetric BICs

1: function POLARIZE($C_W = \{(\lambda_i, p_i)\}, b = (b_1, \ldots, b_n)$)
   \hspace{1em} \triangleright \text{Require: } \sum_i \lambda_i = 1, \lambda_i > 0, p_i \in [0, \frac{1}{2}]
   \hspace{1em} \triangleright \text{Require: } b \in \{0, 1\}^n
2: $C_{W_b} = C_W$
3: for $i = 1$ to $n$ do
4: \hspace{1em} if $b_i = 0$ then
5: \hspace{2em} $C_{W_b} = \text{POLAR-⊗}(C_{W_b}, C_{W_b})$
6: \hspace{1em} else
7: \hspace{2em} $C_{W_b} = \text{POLAR-⊕}(C_{W_b}, C_{W_b})$
8: \hspace{1em} end if
9: end for
10: return $C_{W_b}$
11: end function

transform; i.e.,

\[
W' \boxtimes W' \preceq W \boxtimes W,
\]
\[
W' \odot W' \preceq W \odot W.
\]

Proof. Proved in [18, Lem. 5].

Lemma 8 (Accuracy of approximation after the polar transform). Consider a BIC $(Y, W)$ and quantized approximation $(Y, Q_\Delta(W))$ as in Def. 14. After the polar transform, the following bounds on mutual information values hold for the weak and strong polarized channels:

\[
0 \leq I(W \boxtimes W) - I(Q_\Delta(W) \boxtimes Q_\Delta(W)) \leq 2h_2 \left( \frac{\Delta}{2} \right),
\]
\[
0 \leq I(W \odot W) - I(Q_\Delta(W) \odot Q_\Delta(W)) \leq 2h_2 \left( \frac{\Delta}{2} \right).
\]

Proof. Provided in Appendix G.

Theorem 5 (Successive quantization and polarization). Consider a BIC $(Y, W)$. Let $b = (b_1, b_2, \ldots, b_n)$ be an $n$-dimensional binary vector, $b \in \{0, 1\}^n$. The channel $Q_\Delta^{(n)}(W_b)$ is obtained by $n$ levels of successive polarization as in Def. 11, with the inclusion of $n$ levels of quantization interleaved to ensure

Proof. Provided in Appendix G.
Polarization of a Hybrid Output-Symmetric BIC: $I(W) = 0.6280$.

Fig. 2: Experimental results for the polarization of a hybrid output-symmetric BIC with parameters $\epsilon_0 = 0.12$ and $\gamma_0 = 0.05$, with capacity $I(W) = (1 - \epsilon_0)(1 - h_2(\gamma_0)) \approx 0.6280$. Polar codes of block lengths $2^n$ were constructed for $n = 10, 11, 12$.

computational tractability. Then $Q^{(n)}_\Delta(W_b) \leq W_b$ and the following bounds hold:

$$0 \leq I(W_b) - I(Q^{(n)}_\Delta(W_b)) \leq 2n h_2 \left( \frac{\Delta}{2} \right).$$

Proof. The additive approximation for a single step of quantization and polarization is $2h_2 \left( \frac{\Delta}{2} \right)$. Therefore the theorem for $n$ levels of quantization and polarization follows directly from Lem. 8. \qed

D. Experimental results

To corroborate the theory, experimental evidence is provided regarding the successive quantization and polarization of a hybrid output-symmetric BIC. The hybrid BIC is a combination of BSC and BEC channels as formally stated in the following example.

Example 7 (Polarization of a hybrid output-symmetric BIC). Consider a BIC $(Y, W)$ with output alphabet $Y = \{0, 1, e\}$, and channel transition probabilities

$$W(e|0) = W(e|1) = \epsilon_0,$$
$$W(0|0) = W(1|1) = (1 - \epsilon_0)(1 - p_0),$$
$$W(1|0) = W(0|1) = (1 - \epsilon_0)p_0.$$
Consider explicit parameters $\epsilon_0 = 0.12$, $p_0 = 0.05$. In this case, the capacity of the hybrid channel is $I(W) = (1 - \epsilon_0) (1 - h_2(p_0)) \approx 0.6280$. Fig. 2 depicts the mutual information values of polarized channels sorted in descending order after $n = 10, 11, 12$ levels of successive polarization. Tbl. 7 lists both the theoretical and empirical approximation error for various choices of the block length $2^n$ and the quantization width $\Delta$. For $\Delta = 2^{-14}$, according to Thm. 5, the approximation error for each of the $2^n$ polarized channels is negligible. Thus, the average error is negligible as well.

Remark 7. The techniques developed in the present paper may be improved by using variable-length quantization intervals. The quantization intervals near the origin may be shortened to better approximate the mutual information function of channels. In its basic form, Alg. 3 is simple to implement, and universally applicable to all output-symmetric BICs. Without the inclusion of quantization, the algorithm is exact.

Remark 8. As elaborated in both [17], [18], there exist several methods for merging and shifting the point masses of the Blackwell measures of polarized channels. Such optimizations may be combined with Alg. 3. One of the ideas employed in [17] is to replace a polarized output-symmetric BIC with a BEC approximation for which subsequent polarization operations are exact.

VII. Polarization of Channel Functionals $I_f$

Informally speaking, the polar transform (23) replaces the original pair of BICs $W_1$ and $W_2$ with another pair, where one BIC $W_1 \boxast W_2$ is “worse” than both $W_1$ and $W_2$, and another BIC $W_1 \varoast W_2$ which is “better” than both $W_1$ and $W_2$. The polarization effect is responsible for the capacity-achieving

| Width, Block Length $(\Delta, 2^n)$ | Theoretical Accuracy $I(W) - 2n h_2(\frac{\Delta}{2})$ | Empirical Accuracy $\frac{1}{2^n} \sum_{b \in \{0,1\}^n} I(Q_b^n(W_b))$ |
|-----------------------------------|-------------------------------------------------|-------------------------------------------------|
| $(2^{-12}, 2^{10})$              | 0.5927                                          | 0.6235                                          |
| $(2^{-13}, 2^{10})$              | 0.6091                                          | 0.6256                                          |
| $(2^{-14}, 2^{10})$              | 0.6179                                          | 0.6268                                          |
| $(2^{-14}, 2^{12})$              | 0.6159                                          | 0.6265                                          |
| $(2^{-14}, 2^{14})$              | 0.6139                                          | 0.6262                                          |
performance of polar codes. The following definition makes precise the notion of polarization for a class of channels.

**Definition 15** (Polarization of channel functionals). Let $\mathcal{W}$ denote a class of BICs. A channel functional $\Psi$ associates a real number $\Psi(W)$ to every $W \in \mathcal{W}$. The functional $\Psi$ polarizes on $\mathcal{W}$ if, for any two BICs $W_1, W_2 \in \mathcal{W}$,

$$
\Psi(W_1 \boxast W_2) \leq \Psi(W_1) \land \Psi(W_2) \\
\leq \Psi(W_1) \lor \Psi(W_2) \leq \Psi(W_1 \boxdot W_2).
$$

The definition assumes that both $W_1 \boxast W_2, W_1 \boxdot W_2 \in \mathcal{W}$.

A. Polarization of a broad class of channel functionals

In this section, it is shown that the polarization phenomenon is generic; i.e., a broad class of channel functionals polarizes on the class of output-symmetric BICs. The class of channel functionals which polarizes as defined in Def. 15 is exactly the class $I_f$ described in Sec. III with restrictions on the function $f$. From the induced functionals listed in Tbl. I, the capacity $I_f(W) = I(W)$ with $f(s) = 1 - h_2(s)$ was shown by Arıkan to polarize [1]. Similarly, Arıkan showed that the Bhattacharyya parameter $I_f(W) = -Z(W)$ where $f(s) = -2 \sqrt{s(1-s)}$ polarizes. Since then, the polarization property has been demonstrated for other channel parameters, such as Gallager’s $E_0$ [6]. The following theorem establishes the polarization of $I_f$ for an arbitrary convex function $f$:

**Theorem 6** (Polarization of channel functionals). All channel functionals $I_f$ with a convex $f : [0, 1] \rightarrow \mathbb{R}$ polarize on the class of all symmetric BICs. That is, if $W_1, W_2$ are two symmetric BICs, then

$$
I_f(W_1 \boxast W_2) \leq I_f(W_1) \land I_f(W_2) \\
\leq I_f(W_1) \lor I_f(W_2) \leq I_f(W_1 \boxdot W_2).
$$
Proof. Let $S_1 \sim m_{W_1}$ and $S_2 \sim m_{W_2}$ be independent. Then, using Thm. 4, we can write

$$I_f(W_1 \boxast W_2)$$

$$= \int_{[0,1]} f dm_{W_1 \boxast W_2}$$

$$= \int_{[0,1]} f dm_{W_1 \boxast m_{W_2}}$$

$$= E[f(S_1 S_2 + (1 - S_1)(1 - S_2))]$$

$$= E \left[ E \left[ f(S_1 S_2 + (1 - S_1)(1 - S_2)) | S_2 \right] \right]$$

$$\leq E \left[ E \left[ S_2 f(S_1) + (1 - S_2)f(1 - S_1) | S_2 \right] \right] \tag{45}$$

$$= \frac{1}{2} E[f(S_1)] + \frac{1}{2} E[f(1 - S_1)] \tag{46}$$

$$= I_f(W_1), \tag{47}$$

where (45) is by Jensen’s inequality, (46) follows from the fact that $S_1$ and $S_2$ are independent with $E[S_1] = E[S_2] = \frac{1}{2}$, and (47) follows from the symmetry of $W_1$, which is equivalent to $\mathcal{L}(S_1) = \mathcal{L}(1 - S_1)$. This shows that $I_f(W_1 \boxast W_2) \leq I_f(W_1)$. Conditioning on $S_1$ instead of $S_2$, we prove that $I_f(W_1 \boxast W_2) \leq I_f(W_2)$.

Using Thm. 4 and Jensen’s inequality, we obtain

$$I_f(W_1 \ominus W_2) = \int_{[0,1]} f dm_{W_1 \ominus W_2}$$

$$= \int_{[0,1]} f dm_{W_1 \ominus m_{W_2}}$$

$$= E \left[ (1 - S_1 \ast S_2)f \left( \frac{S_1 S_2}{1 - S_1 \ast S_2} \right) 
+ (S_1 \ast S_2)f \left( \frac{S_1 S_2}{S_1 \ast S_2} \right) \right]$$

$$\geq E[f(S_2)]$$

$$= I_f(W_2).$$

By symmetry, the channels $W_1 \ominus W_2$ and $W_2 \ominus W_1$ are equivalent, so we also have $I_f(W_1 \ominus W_2) \geq I_f(W_1)$. \qed

Corollary 3 (Blackwell ordering of channels).

$$W_1 \boxast W_2 \leq_B W_1 \leq_B W_1 \boxast W_2$$

$$W_1 \ominus W_2 \leq_B W_2 \leq_B W_1 \ominus W_2.$$
Proof. The result is a direct consequence of Theorem 6 and the Blackwell–Sherman–Stein theorem (Theorem 1).

B. Polarization of $I_f$: convex vs non-convex $f$

All existing proofs of the polarization of channel functionals such as the capacity parameter, the Bhattacharyya parameter, and Gallager’s $E_0$, emerge as special cases of Theorem 6. Moreover, Theorem 6 implies the polarization phenomenon for the Bayes error functionals $B_\lambda(\cdot)$ described in Sec. III, which, to the best of our knowledge, has not been discussed previously. Moreover, as exhibited in Corollary 3, the polarization phenomenon is naturally linked to channel domination in the sense of Blackwell.

All channel functionals $I_f$ in Table I polarize on the class of output-symmetric BICs if $f$ is convex. The convexity of $f$ is a sufficient condition for polarization. To understand the case of functionals $I_f$ when $f$ is a non-convex function, consider the induced channel functional $I_f(W) = M_r(W)$ which represents the moments of the information density, corresponding to $f(s) = \psi_r(s)$ as defined in Sec. III.

Example 8 (Moments of information density). For a BIC $(Y, W)$, consider the induced functional $I_f(W) = M_r(W)$ with

$$f(s) = \psi_r(s) =: s(1 + \log_2 s)^\gamma + \bar{s}(1 + \log_2 \bar{s})^\gamma.$$  

Specifically $\psi_1(s)$ and $\psi_2(s)$ are given as follows:

$$\psi_1(s) = 1 - h_2(s),$$

$$\psi_2(s) = s(1 + \log_2 s)^2 + \bar{s}(1 + \log_2 \bar{s})^2.$$ 

Fig. 3 depicts functions $\psi_1(s)$ and $\psi_2(s)$ for $s \in [0, 1]$. It is clear that $\psi_1(s)$ is convex on the unit interval, while $\psi_2(s)$ is non-convex.

Example 9 (Polarization of BSCs — $M_1(W)$ and $M_2(W)$). Let $W = \text{BSC}(p)$ and consider the polarized channels $W \boxast W$ and $W \varoast W$. Applying Lemma 5,

$$\text{BSC}(p) \boxast \text{BSC}(p) \equiv \text{BSC}(p \ast p),$$

$$\text{BSC}(p) \varoast \text{BSC}(p) \equiv (1 - p \ast p) \cdot \text{BSC}\left(\frac{p^2}{1 - p \ast p}\right) \oplus (p \ast p) \cdot \text{BSC}\left(\frac{1}{2}\right).$$
The first moment $M_1(W) = I(W)$, i.e., the channel capacity. The induced functionals $M_1(W)$, $M_2(W)$, $M_1(W \boxast W)$, $M_2(W \boxast W)$, $M_1(W \varoast W)$ and $M_2(W \varoast W)$ may be computed using Eqn. (13). The second moment for a BSC is computed using Eqn. (17). For instance, let $p = 0.05$. Then
\[
M_1(\text{BSC}(0.05)) = 0.7136, \\
M_1(\text{BSC}(0.05) \boxast \text{BSC}(0.05)) = 0.5471, \\
M_1(\text{BSC}(0.05) \varoast \text{BSC}(0.05)) = 0.8801.
\]

It is clear that Theorem 6 and Corollary 3 hold for the first moment of information density. The second moments are given by
\[
M_2(\text{BSC}(0.05)) = 1.3664, \\
M_2(\text{BSC}(0.05) \boxast \text{BSC}(0.05)) = 1.2085, \\
M_2(\text{BSC}(0.05) \varoast \text{BSC}(0.05)) = 1.0359.
\]

In this particular example, the ordering of Corollary 3 does not hold for the second moments, and Thm. 6 is not guaranteed for $M_2(W)$ since $\psi_2(s)$ is non-convex.

VIII. PROPERTIES OF THE POLARIZATION PROCESS

An important method of analyzing successive polarization of channels in Def. 11 is through a certain random process referred to as the polarization process. Starting from a BIC $(Y, W)$, one level of
polarization yields either $W \oplus W$ or $W \otimes W$. As noted by Ankan [1], a random path over $n$ levels of polarization leads to randomly selecting a channel $W_b$ where $b \in \{0, 1\}^n$:

**Definition 16** (Channel polarization — random processes). Consider a BIC $(Y, W)$. Let $\{B_n\}_{n=1}^{\infty}$ be a sequence of i.i.d. Bern$(1/2)$ random variables. Let $W_0 = W$, and

$$W_n = \begin{cases} W_{n-1} \oplus W_{n-1}, & \text{if } B_n = 0 \\ W_{n-1} \otimes W_{n-1}, & \text{if } B_n = 1 \end{cases}$$

for $n \geq 1$. Define the random processes $\{I_n\}_{n=0}^{\infty}$ and $\{Z_n\}_{n=0}^{\infty}$ via $I_n = I(W_n)$ and $Z_n = Z(W_n)$. In general, a random process $\{I_f(W_n)\}_{n=0}^{\infty}$ is obtained for any induced functional listed in Table I.

**Example 10** (Properties of $\{I_n\}_{n=0}^{\infty}$ and $\{Z_n\}_{n=0}^{\infty}$). As shown in [1], for the class of output-symmetric BICs, $\{I_n\}$ is a nonnegative martingale, while $\{Z_n\}$ is a nonnegative supermartingale, both with respect to the natural filtration generated by $\{B_n\}$. More precisely,

$$\mathbb{E} [I_{n+1} | B_1, B_2, \ldots, B_n] = I_n,$$

$$\mathbb{E} [Z_{n+1} | B_1, B_2, \ldots, B_n] \leq Z_n.$$

In order to prove the above properties, consider any two BICs $(Y, W)$ and $(Y', W')$ from the class of output-symmetric BICs. As first noted by [1],

$$I(W \oplus W') + I(W \otimes W') = I(W) + I(W'),$$

$$Z(W \oplus W') + Z(W \otimes W') \leq Z(W) + Z(W').$$

The first relation is due to the conservation of mutual information. The second relation is due to the fact that $Z(W \oplus W') \leq Z(W) + Z(W') - Z(W)Z(W')$ and $Z(W \otimes W') = Z(W)Z(W')$, proven in [27, Chap. 2].

**A. The random processes $\{I_f(W_n)\}_{n=0}^{\infty}$**

As outlined in Def. 16, a random process $\{I_f(W_n)\}_{n=0}^{\infty}$ exists for any induced functional. In order to analyze the properties of the random process, the following relations are introduced:
**Definition 17** (f-relations). Consider two arbitrary BICs \((Y, W)\) and \((Y', W')\) from a given class \(W\) of BICs. Let \(f : [0, 1] \to \mathbb{R}\) be a continuous function. Then we say that the polarization process on the class \(W\) is:

- **f-preserving if**
  \[
  I_f(W \boxast W') + I_f(W \varoast W') = I_f(W) + I_f(W').
  \]  
- **f-improving if**
  \[
  I_f(W \boxast W') + I_f(W \varoast W') \geq I_f(W) + I_f(W').
  \]  
- **f-decreasing if**
  \[
  I_f(W \boxast W') + I_f(W \varoast W') \leq I_f(W) + I_f(W').
  \]

for all \(W_1, W_2 \in W\).

Consider the above conditions and bounded random processes. If (48) holds for all \(W, W' \in W\), then the random process \(\{I_f(W_n)\}_{n=0}^\infty\) is a martingale. If (49) holds, \(\{I_f(W_n)\}_{n=0}^\infty\) is a submartingale. Similarly, if (50) holds, \(\{I_f(W_n)\}_{n=0}^\infty\) is a supermartingale. If \(W\) is the class of all output-symmetric BICs, the following theorem shows that it suffices to verify the \(f\)-relations only on the subclass consisting of BSCs:

**Theorem 7** (f-relations for all symmetric BICs). The polarization process is \(f\)-preserving, \(f\)-improving, or \(f\)-decreasing as defined in Def. 17 on the class of output-symmetric BICs if and only if (48), (49), or (50) holds respectively for all pairs \((W, W') = (\text{BSC}(p), \text{BSC}(q)), \ (p, q) \in [0, \frac{1}{2}] \times [0, \frac{1}{2}]\).

**Proof.** Consider the \(f\)-improving relation for a class of BICs and induced functional \(I_f(\cdot)\). If (49) holds for all symmetric BICs, then it holds for all BSCs. To prove the converse, fix two symmetric BICs \(W, W'\). By Theorem 3, the following channel decompositions exist:

\[
W \equiv \bigoplus_{i=1}^m \lambda_i \text{BSC}(p_i),
\]
\[
W' \equiv \bigoplus_{j=1}^k \mu_j \text{BSC}(q_j).
\]

By Corollary 2, the Blackwell measures of the polarized channels \(W \boxast W'\) and \(W \varoast W'\) are given by

\[
m_{W \boxast W'} = \sum_{i=1}^m \sum_{j=1}^k \lambda_i \mu_j m_{\text{BSC}(p_i, q_j)},
\]
\[
m_{W \varoast W'} = \sum_{i=1}^m \sum_{j=1}^k \lambda_i \mu_j m_{\text{BSC}(p_i) \times \text{BSC}(q_j)}.
\]
Consequently, using the definitions for induced functionals in Sec. III, and the assumption that (49) holds for all BSCs, we have

\[
I_f(W \boxast W') + I_f(W \varoast W') \\
= \int f \, dm_{W \boxast W'} + \int f \, dm_{W \varoast W'} \\
= \sum_{i=1}^{m} \sum_{j=1}^{k} \lambda_i \mu_j \int f \left( dm_{BSC(p_i \ast q_j)} + dm_{BSC(p_i) \times BSC(q_j)} \right) \\
= \sum_{i=1}^{m} \sum_{j=1}^{k} \lambda_i \mu_j \left( I_f(BSC(p_i \ast q_j)) + I_f(BSC(p_i) \times BSC(q_j)) \right) \\
\geq \sum_{i=1}^{m} \sum_{j=1}^{k} \lambda_i \mu_j \left( I_f(BSC(p_i)) + I_f(BSC(q_j)) \right) \\
= \sum_{i=1}^{m} \lambda_i I_f(BSC(p_i)) + \sum_{j=1}^{k} \mu_j I_f(BSC(q_j)) \\
= I_f(W) + I_f(W').
\]

Thm. 7 is established in an identical manner for the \(f\)-preserving and \(f\)-decreasing relations.

**B. \(f\)-relations: The case of convex \(f\)**

It is tempting to conjecture that an \(f\)-relation such as the \(f\)-improving relation given in Eqn. (49) holds for all convex \(f\) on the class of output-symmetric BICs. However, the following counter-example proves that this conjecture is false.

**Example 11** (counter-example for convex \(f\)). If this were the case, then, by the Blackwell–Sherman–Stein theorem, the channel \(\frac{1}{2}(W_1 \boxast W_2) \oplus \frac{1}{2}(W_1 \varoast W_2)\) would dominate the channel \(\frac{1}{2}W_1 \oplus \frac{1}{2}W_2\). However, this conjecture turns out to be false. As a counterexample, consider the function \(f_\lambda(s) = \lambda \land \lambda - (2\lambda s) \land (2\lambda s)\) for \(\lambda \in (0, 1)\). Specifically, consider the case of \(\lambda = \frac{1}{4}\). Then a simple calculation shows that, for instance,

\[
I_{f_{1/3}}(BSC(1/4) \boxast BSC(1/4)) + \\
I_{f_{1/3}}(BSC(1/4) \varoast BSC(1/4)) \\
< 2I_{f_{1/3}}(BSC(1/4)).
\]

More generally, Figure 4 shows the Neyman–Pearson regions of \(\frac{1}{2}(W_1 \boxast W_2) \oplus \frac{1}{2}(W_1 \varoast W_2)\) and \(\frac{1}{2}W_1 \oplus \frac{1}{2}W_2\) when \(W_1 = W_2 = BSC(1/4)\). It is evident that the latter is not a subset of the former. A similar
calculation reveals

\[
I_{f_{1/3}}(\text{BSC}(3/8) \boxplus \text{BSC}(3/8)) + \\
I_{f_{1/3}}(\text{BSC}(3/8) \otimes \text{BSC}(3/8))
\]

\[
> 2I_{f_{1/3}}(\text{BSC}(3/8)).
\]

Thus, although \(f_{1/3}(s)\) is a convex function, the \(f\)-relations of Def. 17 do not hold consistently for all BSC pairs. In particular, the random process \(\{I_f(W_n)\}_{n=0}^{\infty}\) is not a submartingale, supermartingale, or ordinary martingale for the class of symmetric BICs.

C. \(f\)-relations: The case of non-convex \(f\)

The second-moment of the information density \(M_2(W)\) is equal to \(I_f(W)\) for \(f(s) = \psi_2(s)\), which is a non-convex function as plotted in Fig. 3. Prior to discussing \(M_2(W)\), we give the following second-order result:

**Lemma 9** (Squared mutual information). Consider any two arbitrary BICs \((Y, W)\) and \((Y', W')\). The polar transform of Def. 8 results in \(W \boxplus W'\) and \(W \otimes W'\). The following inequality holds:

\[
(I(W))^2 + (I(W'))^2 \leq (I(W \boxplus W'))^2 + (I(W \otimes W'))^2.
\]
Thus, the total sum of the squared mutual information values of channels increases due to the polar transform.

Proof. Provided in Appendix H.

Lemma 9 holds for arbitrary BICs. Using the machinery of induced functionals provided in Sec. III, the following example analyzes another second-order result involving the second moments of information density before and after polarization for the special case of all BSC pairs.

Example 12 (Polarization of all \((BSC(p), BSC(q))\) pairs — second moments of information density). Consider the channels \(W = BSC(p)\) and \(W' = BSC(q)\). Recall that \(\psi_2(s) := s(1 + \log_2(s))^2 + \bar{s}(1 + \log_2 \bar{s})^2\). The polar transform of Def. 8 results in \(W \ast W'\) and \(W \circ W'\). As shown previously in Eqn. (17), \(M_2(BSC(p)) = \psi_2(p)\). Using the structural decomposition of polarized BSCs in Lemma 5, the following second moments may be computed for all \((p, q) \in [0, \frac{1}{2}] \times [0, \frac{1}{2}]\) for which \((p, q) \neq (0, 0)\).

\[
\begin{align*}
M_2(W) &= \psi_2(p), \\
M_2(W') &= \psi_2(q), \\
M_2(W \oplus W') &= \psi_2(p \star q), \\
M_2(W \otimes W') &= (1 - p \star q)\psi_2(\alpha) + (p \star q)\psi_2(\beta).
\end{align*}
\]

The parameters \(\alpha := \frac{pq}{1 - p \star q}\) and \(\beta := \frac{\bar{pq}}{p \star q}\) were defined in Lemma 5. Consider the following difference of sums \((M_2(W) + M_2(W')) - (M_2(W \oplus W') + M_2(W \otimes W'))\) for all BSC pairs. Define the gap function

\[
\text{GAP}(p, q) := \psi_2(p) + \psi_2(q) - \psi_2(p \star q) - (1 - p \star q)\psi_2(\alpha) - (p \star q)\psi_2(\beta).
\]

Note that \(\text{GAP}(p, q) = \text{GAP}(q, p)\). In addition, along the boundaries, \(\text{GAP}(p, 0) = 0, \text{GAP}(0, q) = 0, \text{GAP}(p, \frac{1}{2}) = 0,\) and \(\text{GAP}(\frac{1}{2}, q) = 0\). Fig. 5 provides numerical evidence that \(\text{GAP}(p, q) \geq 0\) for all BSC pairs.

Ex. 12 shows numerically that the total sum of second moments of the information density decreases after polarization for all BSC pairs. Combined with Theorem 7, this would imply that such a result holds for all output-symmetric BICs. The exact analytic proof of a second-order result showing that \(M_2(W) + M_2(W') \geq M_2(W \oplus W') + M_2(W \otimes W')\) requires rigorous analysis. We refer the reader to the proofs in [8] which utilize variance-covariance decompositions and Chebyshev’s covariance inequality to establish the following related theorem regarding the second-order channel dispersion parameters:
**Theorem 8** (Varentropy decreases under the polar transform [8]). Consider any two arbitrary BICs \((Y, W)\) and \((Y', W')\). The polar transform of Def. 8 results in \(W \boxtimes W'\) and \(W \otimes W'\). The following inequality is true:

\[
V(W) + V(W') \geq V(W \boxtimes W') + V(W \otimes W'),
\]

where \(V(W) := M_2(W) - (I(W))^2\) for any BIC assuming a uniform input distribution. Thus, the sum of the channel dispersion parameters decreases due to the polar transform.

**Remark 9.** The result by Arıkan in [8] applies more generally to so-called binary data elements incorporating the input source distribution as well as each channel’s conditional distribution. Moreover, it is shown that the average varentropy decreases to zero asymptotically for independent and identically distributed binary data elements over successive iterations of the polar transform.

**Remark 10.** Theorem 7 establishes a simple method to check whether the polarization process for any induced functional is a martingale, submartingale, or supermartingale for the class of symmetric BICs. Such a property is not guaranteed to hold. Our method is consistent with Theorem 8 and provides verification for all other known channel functionals listed in Tbl. 1.

**IX. CONCLUSION**

A unified and general framework was presented for channel polarization based on the evolution of the Blackwell measure over successive iterations of Arıkan’s polar transform. The evolution of a broad class of channel functionals under Arıkan’s polar transform is characterized by tracking the evolution of the...
Blackwell measure. This is due to the fact that any measurable function $f$ induces a functional $I_f(W)$ on the channel $W$ through its Blackwell measure. The theoretical framework developed in this paper for BICs could lead to the discovery of new martingales and auxiliary random processes, generalizations to multi-level and multi-user channel polarization, and conceptually simplified algorithms for code construction.

**APPENDIX A**

**PROOF OF LEMMA 1**

Eqn. (18) is due to the fact that $I(W)$ is an induced functional, and Eqn. (13) applies for symmetric BICs. To derive Eqn. (19), note that the second moment $M_2(W)$ of the information density is also an induced functional. Therefore Eqn. (13) may be applied to decompose $M_2(W)$ for any symmetric BIC. Starting from the definition of the dispersion of a channel, after rearranging terms,

$$V(W) := M_2(W) - (I(W))^2$$

$$= \left( \sum_{i=1}^{m} \lambda_i M_2(BSC(p_i)) \right) - (I(W))^2$$

$$= \left( \sum_{i=1}^{m} \lambda_i \left( V(BSC(p_i)) + (I(BSC(p_i)))^2 \right) \right) - (I(W))^2$$

$$= \sum_{i=1}^{m} \lambda_i V(BSC(p_i)) + \sum_{i=1}^{m} \lambda_i \left( I(BSC(p_i)) - I(W) \right)^2.$$

**APPENDIX B**

**PROOF OF LEM. 4**

In this proof, it is noted that the two original channels have different probabilities of error. Denote the original channels by $W_1 \equiv BSC(p)$ and $W_2 \equiv BSC(q)$. The polar transform yields $W_1 \boxast W_2 \equiv BSC(p) \boxast BSC(q)$ and $W_1 \varoast W_2 \equiv BSC(p) \varoast BSC(q)$.

The output alphabet of $W_1 \boxast W_2$ is $\{0, 1\} \times \{0, 1\}$ with conditional distribution denoted as $(W_1 \boxast W_2)(y_1, y_2|u_1)$. The conditional probabilities given an input $u_1 = 0$ are as follows: $(W_1 \boxast W_2)(0, 0|0) = (W_1 \boxast W_2)(1, 1|0) = \frac{1}{2}(1 - p - q + 2pq)$; $(W_1 \boxast W_2)(0, 1|0) = (W_1 \boxast W_2)(1, 0|0) = \frac{1}{2}(p + q - 2pq)$. Similarly, the conditional probabilities for a binary input $u_1 = 1$ are as follows: $(W_1 \boxast W_2)(0, 0|1) = (W_1 \boxast W_2)(1, 1|1) = \frac{1}{2}(p + q - 2pq)$; $(W_1 \boxast W_2)(0, 1|1) = (W_1 \boxast W_2)(1, 0|1) = \frac{1}{2}(1 - p - q + 2pq)$. Consider the following disjoint sets of output pairs,

$$S^- = \{(0, 0), (1, 1)\},$$

$$T^- = \{(0, 1), (1, 0)\}.$$
The union $S^- \cup T^-$ contains all 4 output pairs. Viewing all output pairs grouped in each set $S^-$ and $T^-$ as super-symbols, the transition matrix $T_{W_1 \oplus W_2}$ is as claimed.

The proof regarding $W_1 \otimes W_2$ follows in an identical manner. The output alphabet of $W_1 \otimes W_2$ is \{0, 1\} $\times$ \{0, 1\} with conditional distribution denoted as $(W_1 \otimes W_2)(y_1, y_2, u_1 | u_2)$. The conditional probabilities for a binary input $u_2 = 0$ are as follows: $(W_1 \otimes W_2)(0, 0, 0 | 0) = (W_1 \otimes W_2)(1, 0, 1 | 0) = \frac{1}{2}(1 - p)(1 - q); (W_1 \otimes W_2)(0, 0, 1 | 0) = (W_1 \otimes W_2)(1, 1, 0 | 0) = \frac{1}{2}pq; (W_1 \otimes W_2)(0, 0, 1 | 0) = (W_1 \otimes W_2)(1, 0, 0 | 0) = \frac{1}{2}(p - q)$; $(W_1 \otimes W_2)(0, 1, 0 | 0) = (W_1 \otimes W_2)(1, 1, 0 | 0) = \frac{1}{2}(1 - p - q + pq); (W_1 \otimes W_2)(0, 1, 0 | 0) = (W_1 \otimes W_2)(1, 0, 1 | 0) = \frac{1}{2}(q - pq)$; $(W_1 \otimes W_2)(0, 1, 0 | 0) = (W_1 \otimes W_2)(1, 1, 1 | 1) = \frac{1}{2}(p - pq)$. Consider the following disjoint sets of output pairs,

$$S^+ = \{(0, 0, 0), (1, 0, 1)\},$$

$$T^+ = \{(0, 1, 1), (1, 1, 0)\},$$

$$B^+ = \{(0, 0, 1), (1, 0, 0)\},$$

$$G^+ = \{(0, 1, 0), (1, 1, 1)\}.$$  

The union $S^+ \cup T^+ \cup B^+ \cup G^+$ contains all 8 output pairs. Viewing all output pairs in the sets $S^+$, $T^+$, $B^+$ and $G^+$ as super-symbols, the transition matrix $T_{W_1 \otimes W_2}$ is as claimed.

The parallel broadcast channel $W_1 \times W_2 \equiv BSC(p) \times BSC(q)$ has an output alphabet \{0, 1\} $\times$ \{0, 1\} with conditional distribution denoted as $(W_1 \times W_2)(y_1, y_2 | x)$. The conditional probabilities for binary input $x = 0$ are as follows: $(W_1 \times W_2)(0, 0 | 0) = (1 - p)(1 - q); (W_1 \times W_2)(0, 1 | 0) = (1 - p)q; (W_1 \times W_2)(1, 0 | 0) = p(1 - q); (W_1 \times W_2)(1, 1 | 0) = pq$. Similarly the conditional probabilities for binary input $x = 1$ are as follows: $(W_1 \times W_2)(0, 0 | 1) = pq; (W_1 \times W_2)(0, 1 | 1) = p(1 - q); (W_1 \times W_2)(1, 0 | 1) = (1 - p)q; (W_1 \times W_2)(1, 1 | 1) = (1 - p)(1 - q)$. By comparing the transition probabilities, it is evident that $T_{BSC(p) \times BSC(q)} = T_{BSC(p) \oplus BSC(q)}$ as claimed.

**APPENDIX C**

**PROOF OF LEM. 5**

Eqn. (34) follows directly from the transition matrix Eqn. (30) of Lem. 4. To see why Eqn. (35) holds, consider the transition matrix of Eqn. (31). Assume $(p \times q) \neq 0$. Let error probabilities $\alpha$ and $\beta$ be defined
as in Eqn. 32 and Eqn. 33 respectively.

\[
\bar{T}_{BSC(p) \otimes BSC(q)} = T_{BSC(p) \times BSC(q)}
\]

\[
= \begin{bmatrix}
\bar{p}q & p\bar{q} & pq & \bar{p}q \\
pq & \bar{p}q & \bar{p}q & pq
\end{bmatrix}
\]

\[
= \begin{bmatrix}
(1-p \star q)\alpha & (1-p \star q)\alpha \\
(1-p \star q)\alpha & (1-p \star q)\alpha \\
(p \star q)\beta & (p \star q)\beta \\
(p \star q)\beta & (p \star q)\beta
\end{bmatrix}
\]

The above transition matrix for \(BSC(p) \otimes BSC(q)\) reveals the structural decomposition of the polarized channel as established by Thm. 3. More precisely, \(BSC(p) \otimes BSC(q)\) is a \(BSC(\alpha)\) with probability \((1-p \star q)\) and a \(BSC(\beta \wedge \bar{\beta})\) with probability \((p \star q)\). The transformed error probabilities are specified so that \(\alpha \in [0, \frac{1}{2}]\) and \((\beta \wedge \bar{\beta}) \in [0, \frac{1}{2}]\).

**APPENDIX D**

**PROOF OF COR. 1**

The corollary holds for \((p, q) = (0, 0)\) trivially as a degenerate case. Therefore, assume \((p, q) \neq (0, 0)\) so that \(p \star q \in (0, \frac{1}{2}]\). From Thm. 4,

\[
m_{BSC(p) \oplus BSC(q)} = m_{BSC(p)} \otimes m_{BSC(q)};
\]

\[
m_{BSC(p) \otimes BSC(q)} = m_{BSC(p)} \otimes m_{BSC(q)};
\]

where the operations \(\oplus\) and \(\otimes\) on Blackwell measures were defined in Def. 10. Thus, consider two independent random variables \(S_1 \sim m_{BSC(p)}\) and \(S_2 \sim m_{BSC(q)}\). The random variable \(S_1\) takes two equiprobable values \(p\) and \(\bar{p}\). The random variable \(S_2\) takes two equiprobable values \(q\) and \(\bar{q}\).

To prove Eqn. (36), consider Eqn. (24) of Def. 10. The random variable \(1 - S_1 \star S_2\) takes two equiprobable values, \(p \star q\) and \(1 - p \star q\). The integral of Eqn. (24) may be evaluated for any continuous \(f: [0, 1] \rightarrow \mathbb{R}\) as follows:

\[
\int_{[0,1]} f dm_{BSC(p) \oplus BSC(q)} = \frac{1}{2} f (p \star q) + \frac{1}{2} f (1-p \star q).
\]

The corresponding Blackwell measure of \(BSC(p) \oplus BSC(q)\) written as a weighted sum of Dirac measures is given by,

\[
m_{BSC(p) \oplus BSC(q)} = \frac{1}{2} \delta_{p \star q} + \frac{1}{2} \delta_{1-p \star q}.
\]
To prove Eqn. (37), consider Eqn. (25) of Def. 10. The integral of Eqn. (25) may be evaluated for any continuous $f : [0, 1] \to \mathbb{R}$ as follows:

$$\int_{[0,1]} f \, dm_{\text{BSC}(p) \otimes \text{BSC}(q)}$$

$$= \frac{1}{2}(1-p \ast q)f \left( \frac{pq}{1-p \ast q} \right)$$

$$+ \frac{1}{2}(1-p \ast q)f \left( \frac{pq}{1-p \ast q} \right)$$

$$+ \frac{1}{2}(p \ast q)f \left( \frac{pq}{p \ast q} \right)$$

$$+ \frac{1}{2}(p \ast q)f \left( \frac{pq}{p \ast q} \right).$$

The corresponding Blackwell measure of $\text{BSC}(p) \otimes \text{BSC}(q)$ written as a weighted sum of Dirac measures is given by

$$m_{\text{BSC}(p) \otimes \text{BSC}(q)} = (1-p \ast q) \left( \frac{1}{2} \delta_{\frac{pq}{1-p \ast q}} + \frac{1}{2} \delta_{\frac{pq}{1-p \ast q}} \right)$$

$$+ (p \ast q) \left( \frac{1}{2} \delta_{\frac{pq}{p \ast q}} + \frac{1}{2} \delta_{\frac{pq}{p \ast q}} \right).$$

As shown in the proof of Lem. 4 by direct computation, the above probability measure $m_{\text{BSC}(p) \otimes \text{BSC}(q)}$ is equivalent to the probability measure $m_{\text{BSC}(p) \times \text{BSC}(q)}$. In addition, Eqn. (38) follows from Eqn. (35) of Lem. 5.

**APPENDIX E**

**PROOF OF COROLLARY 2**

From Thm. 3, there exists a structural decomposition for output-symmetric BICs $W_1$ and $W_2$. The Blackwell measures of $W_1$ and $W_2$ may be written as follows:

$$m_{W_1} = \sum_{i=1}^{m} \lambda_i m_{\text{BSC}(p_i)} = \sum_{i=1}^{m} \lambda_i (\delta_{p_i} + \delta_{p_i})$$

$$m_{W_2} = \sum_{j=1}^{k} \mu_j m_{\text{BSC}(q_j)} = \sum_{j=1}^{k} \mu_j (\delta_{q_j} + \delta_{q_j})$$

for some choices of parameters $(m, \lambda, p)$ and $(k, \mu, q)$. Therefore, for two independent r.v.’s $S_1 \sim m_{W_1}$ and $S_2 \sim m_{W_2}$ and for any continuous $f : [0, 1] \to \mathbb{R}$ we have

$$\int_{[0,1]} f \, dm_{W_1 \oplus W_2} = \mathbb{E}[f(1 - S_1 + S_2)]$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{k} \lambda_i \mu_j \left( \frac{1}{2} f(p_i \ast q_j) + \frac{1}{2} f(1 - p_i \ast q_j) \right)$$
and

\[ \int_{[0,1]} f \, dm_{W_1 \otimes W_2} = E \left[ (1 - S_1 \ast S_2) f \left( \frac{S_1 S_2}{1 - S_1 \ast S_2} \right) \right. \]
\[ \left. + (S_1 \ast S_2) f \left( \frac{\tilde{S}_1 S_2}{S_1 \ast S_2} \right) \right] \]
\[ = \sum_{i=1}^{m} \sum_{j=1}^{k} \lambda_i \mu_j \left\{ \frac{1}{2} (1 - p_i \ast q_j) f \left( \frac{p_i q_j}{1 - p_i \ast q_j} \right) \right. \]
\[ + \frac{1}{2} (1 - p_i \ast q_j) f \left( \frac{\bar{p}_i \bar{q}_j}{1 - p_i \ast q_j} \right) \]
\[ + \frac{1}{2} (p_i \ast q_j) f \left( \frac{p_i q_j}{p_i \ast q_j} \right) \]
\[ + \frac{1}{2} (p_i \ast q_j) f \left( \frac{\bar{p}_i \bar{q}_j}{p_i \ast q_j} \right) \left\} \right. \]
\[ = \sum_{i=1}^{m} \sum_{j=1}^{k} \lambda_i \mu_j \left\{ \frac{1}{2} (1 - p_i \ast q_j) f \left( \frac{p_i q_j}{1 - p_i \ast q_j} \right) \right. \]
\[ + \frac{1}{2} (1 - p_i \ast q_j) f \left( \frac{\bar{p}_i \bar{q}_j}{1 - p_i \ast q_j} \right) \]
\[ + \frac{1}{2} (p_i \ast q_j) f \left( \frac{p_i q_j}{p_i \ast q_j} \right) \]
\[ + \frac{1}{2} (p_i \ast q_j) f \left( \frac{\bar{p}_i \bar{q}_j}{p_i \ast q_j} \right) \left\} \right. \]

Applying Cor. 1, we obtain Eqns. (39) and (40).

**APPENDIX F**

**PROOF OF LEMMA 6**

Assume without loss of generality that \( p_i \in [0, \frac{1}{2}] \). Consider the structural decompositions of \( W \) and \( Q_\Delta(W) \). For each pair of corresponding subchannels of \( W \) and \( Q_\Delta(W) \),

\[ \text{BSC}(Q_\Delta(p_i)) \preceq \text{BSC}(p_i), \text{ since } Q_\Delta(p_i) \geq p_i. \]

Since the subchannels are stochastically degraded, it follows that \( Q_\Delta(W) \preceq W \), and the mutual information bound \( I(Q_\Delta(W)) \leq I(W) \) holds. To prove Eqn. (42), we upper-bound the mutual information
difference for \(W\) and \(Q_\Delta(W)\) as follows:

\[
I(W) - I(Q_\Delta(W)) = \sum_{i=1}^{m} \lambda_i \left( I(BSC(p_i)) - I(BSC(Q_\Delta(p_i))) - I(BSC\left((p_i + \frac{\Delta}{2}) \wedge \frac{1}{2}\right)) \right)
\]

\[
\leq \sum_{i=1}^{m} \lambda_i \left( h_2\left((p_i + \frac{\Delta}{2}) \wedge \frac{1}{2}\right) - h_2(p_i)\right)
\]

\[
\leq \sum_{i=1}^{m} \lambda_i h_2\left(\frac{\Delta}{2} \wedge \frac{1}{2}\right)
\]

\[
= h_2\left(\frac{\Delta}{2}\right).
\]

In the above derivation, the first inequality is due to the fact that \(Q_\Delta(x)\) for \(x \in [0, \frac{1}{2}]\) rounds \(x\) up to the right endpoint of the \(\Delta\)-interval containing \(x\). The \(\Delta\)-interval has a maximum width of \(\frac{\Delta}{2}\). The second inequality is due to the fact that the binary entropy function \(h_2(x)\) is concave and monotonically increasing in the interval \([0, \frac{1}{2}]\).

**APPENDIX G**

**PROOF OF LEMMA 8**

Due to Lemma 6, \(Q_\Delta(W)\) \(\preceq W\) which implies \(I(Q_\Delta(W)) \leq I(W)\). Furthermore, as stated in Lem. 7, channel degradation is preserved after the polar transform. Thus,

\[
Q_\Delta(W) \boxdot Q_\Delta(W) \preceq W \boxdot W, \\
Q_\Delta(W) \otimes Q_\Delta(W) \preceq W \otimes W.
\]

The above channel degradation conditions directly imply the first parts of the inequalities Eqn. (43) and Eqn. (44). To fully establish the inequalities, note that, due to the conservation of mutual information after the polar transform,

\[
I(W \boxdot W) + I(W \otimes W) = 2I(W)
\]

and

\[
I(Q_\Delta(W) \boxdot Q_\Delta(W)) + I(Q_\Delta(W) \otimes Q_\Delta(W)) = 2I(Q_\Delta(W)).
\]
By Lemma 6,
\[ I(W \boxplus W) + I(W \boxast W) = 2I(W) \]
\[ \leq 2I(Q_\Delta(W)) + 2h_2\left(\frac{\Delta}{2}\right) \]
\[ = I(Q_\Delta(W) \boxplus Q_\Delta(W)) + I(Q_\Delta(W) \boxast Q_\Delta(W)) + 2h_2\left(\frac{\Delta}{2}\right). \]

Therefore, we obtain the following inequalities:
\[ I(W \boxplus W) - I(Q_\Delta(W) \boxplus Q_\Delta(W)) \leq 2h_2\left(\frac{\Delta}{2}\right) \]
and
\[ I(W \boxast W) - I(Q_\Delta(W) \boxast Q_\Delta(W)) \leq 2h_2\left(\frac{\Delta}{2}\right). \]

This completes the proof.

**APPENDIX H**

**PROOF OF LEMMA 9**

By the conservation of mutual information,
\[ I(W) + I(W') = I(W \boxplus W') + I(W \boxast W'). \]

Squaring both sides of the above equality yields
\[ (I(W'))^2 + (I(W'))^2 + 2I(W)I(W') = \]
\[ (I(W \boxplus W'))^2 + (I(W \boxast W'))^2 + 2I(W \boxplus W')I(W \boxast W'). \]
(53)

Define the following function:
\[ g(W, W') := (I(W \boxplus W'))^2 + (I(W \boxast W'))^2 \]
\[ - (I(W))^2 - (I(W'))^2. \]
(54)

Due to the equality in Eqn. (53),
\[ g(W, W') = 2I(W)I(W') - 2I(W \boxplus W')I(W \boxast W'). \]
(55)

Due to Thm. 6, the following ordering exists between mutual information values:
\[ I(W \boxplus W') \leq I(W) \land I(W') \]
\[ \leq I(W) \lor I(W') \leq I(W \boxast W'). \]
More precisely, there exists a $\delta \geq 0$ such that the following equations hold,

$$I(W \boxast W') = (I(W) \land I(W')) - \delta, \quad (56)$$

$$I(W \varoast W') = (I(W) \lor I(W')) + \delta. \quad (57)$$

The $\delta$-offset of Eqn. (56) is equal to the $\delta$-offset of Eqn. (57) due to the conservation of mutual information. Substituting the terms for $I(W \boxast W')$ and $I(W \varoast W')$ into Eqn. (55),

$$g(W, W') = 2\delta (I(W) \lor I(W')) - 2\delta (I(W) \land I(W')) + 2\delta^2.$$ 

Thus, $g(W, W') \geq 0$. Since $g(W, W')$ was defined in Eqn. (54), this proves the result that the total sum of the squared mutual information values increases after polarization.

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