THE GROUP OF UNITS ON AN AFFINE VARIETY

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Dedicated to Fook Loy

ABSTRACT. The object of study is the group of units $\mathcal{O}^*(X)$ in the coordinate ring of a normal affine variety $X$ over an algebraically closed field $k$. Methods of Galois cohomology are applied to those varieties that can be presented as a finite cyclic cover of a rational variety. On a cyclic cover $X \to \mathbb{A}^m$ of affine $m$-space over $k$ such that the ramification divisor is irreducible and the degree is prime, it is shown that $\mathcal{O}^*(X)$ is equal to $k^*$, the nonzero scalars. The same conclusion holds, if $X$ is a sufficiently general affine hyperelliptic curve. If $X$ has a projective completion such that the divisor at infinity has $r$ components, then sufficient conditions are given for $\mathcal{O}^*(X)/k^*$ to be isomorphic to $\mathbb{Z}^{(r-1)}$.

1. INTRODUCTION

1.1. Statement of problem and summary of results. Throughout, $X$ is a normal variety defined over an algebraically closed field $k$. The ring of regular functions on $X$ is denoted $\mathcal{O}(X)$ and the group of units, or invertible regular functions, is denoted $\mathcal{O}^*(X)$. If $X$ is a projective variety in $\mathbb{P}^m$, the ring $\mathcal{O}(X)$ is always equal to $k[[x_1, \ldots, x_m]]$, hence the group of units $\mathcal{O}^*(X)$ is equal to $k^*$. This confirms the intuitive notion that on a projective variety the only functions with no zero set and no pole set are the non-zero constants. For this reason, as indicated by the title, we focus on those varieties $X$ that are affine.

This article is concerned with the general question, “What is the group of units on an affine variety?” Our main results tend to be one of two types of statements, either sufficient conditions on $X$ such that the group of units is trivial, or sufficient conditions on $X$ such that the group of units is “large”. This terminology is explained in Section 1.2. We focus on those varieties $X$ that are either rational, or can be presented as a finite cyclic cover of a rational variety.

The group of units on a variety $X$ is intimately connected to the class group of Weil divisors, the Picard group, and the Brauer group. In fact, these connections are what led to the present study. For this reason, the strategy we employ is to exploit what is known about the latter groups in order to compute the former. Most of our computations are derived using Galois cohomology or étale cohomology.

In Section 2 the affine variety $X$ is a hypersurface in $\mathbb{A}^{m+1}$ defined by an equation of the form $z^n = f$, where $f$ is a polynomial in $k[x_1, \ldots, x_m]$. In Theorems 2.10 and 2.12 we show that $\mathcal{O}^*(X)$ is equal to $k^*$ when $f$ is irreducible and $n$ is either a prime or is equal to 4. The proof relies heavily on Galois cohomology.

In Section 3 the variety $X$ is an affine open subvariety of a finite cyclic covering $\pi : Y \to \mathbb{P}^m$. Section 3.1 treats the case where $\pi$ is unramified along the divisor at infinity of $X$. We apply these results to affine hyperelliptic curves in Section 3.2. If $\pi : Y \to \mathbb{P}^1$ is a ramified
cyclic cover of prime degree and \( k = \mathbb{C} \), then for a sufficiently general \( Q \in \mathbb{P}^1 \), the group of units on the affine curve \( X = \pi^{-1}(\mathbb{P}^1 - Q) \) is trivial (Theorem 3.5). The proofs are based on Galois cohomology. We see that for an affine plane curve over \( \mathbb{C} \) defined by an equation of the form \( y^n = (x - \lambda_1) \cdots (x - \lambda_n) \), where \( p \mid n \), the size of the group of units depends not only on the number of points at infinity, but on the choices of the scalars \( \lambda_i \). The proof relies on the fact that \( \mathbb{C} \) is uncountable and we mention here that this phenomenon does not occur over finite ground fields. For a nonsingular affine curve over a finite field, the group of units always has a free component of rank one less than the number of points at infinity (see for example, [16] Proposition 1.2). Section 3.3 considers the case where the ramification divisor of \( \pi \) is equal to the divisor at infinity of \( X \). In this case we show that the group of units is large (Proposition 3.12). The proof is based on \( \acute{e} \)tale cohomology. These results are applied in Section 4 to compute the group of units on an affine hypersurface \( X \) in \( \mathbb{A}^m \) defined by an equation of the form \( f_1 \cdots f_r = 1 \). In Theorem 3.14 we show that \( \mathcal{O}^*(X)/k^* \) is a free \( \mathbb{Z} \)-module with basis \( f_1, \ldots, f_{r-1} \) if the following conditions are satisfied: \( r \geq 2 \), \( m \geq 2 \), \( f_1, \ldots, f_r \) are distinct non-constant forms in \( k[x_1, \ldots, x_m] \), and the degrees \( \deg(f_1), \ldots, \deg(f_r) \) are relatively prime. Results of Section 3 have been applied in [8].

After reviewing some background results, Section 1 concludes with examples meant to provide motivation for the rest of the article.

1.2. Background results. For background definitions and theorems on homological algebra, we refer the reader to [18]. For \( \acute{e} \)tale cohomology, we suggest [14], and for all other unexplained terminology and notation, [13].

Lemma 1.1 reduces the computation of \( \mathcal{O}^*(X) \) as an abstract group to the determination of the rank of a free \( \mathbb{Z} \)-module. Given the affine variety \( X \) and a projective completion \( Y \), we see that the group of units on \( X \) is closely tied to the components of the divisor at infinity and the subgroup that they generate in the class group \( \text{Cl}(Y) \). If \( Y \) is a normal variety, the divisor class group \( \text{Cl}(Y) \) is the group of Weil divisors modulo the subgroup of principal Weil divisors [13] Section II.6]. The rank of the free \( \mathbb{Z} \)-module \( \mathcal{O}^*(X)/k^* \) is bounded above by \( r \), the number of irreducible components of the divisor at infinity. The group \( \mathcal{O}^*(X) \) is trivial, if it is equal to \( k^* \). The group \( \mathcal{O}^*(X) \) is said to be large, if the rank of \( \mathcal{O}^*(X)/k^* \) is greater than or equal to \( r - 1 \).

Lemma 1.1. [20] Lemme 1] If \( X \) is a normal affine variety, the group \( \mathcal{O}^*(X)/k^* \) is a finitely generated torsion free \( \mathbb{Z} \)-module.

Proof. We sketch Samuel’s proof. Embed \( X \) as an open affine subvariety of a normal projective variety \( Y \). Write the "divisor at infinity" as a union of prime divisors \( Y - X = Z_1 \cup \cdots \cup Z_r \). As previously remarked, \( \mathcal{O}^*(Y) = k^* \). The claim follows from the following version of the exact sequence

\[
1 \to \mathcal{O}^*(Y) \to \mathcal{O}^*(X) \xrightarrow{\text{div}} \bigoplus_{i=1}^r \mathbb{Z}Z_i \to \text{Cl}(Y) \to \text{Cl}(X) \to 0
\]

of Nagata (for example, [5] Theorem 1.1]). \( \square \)

Lemma 1.2 shows that the affine cone over a projective variety has trivial group of units. The proof can be found in [5].

Lemma 1.2. Let \( f \in k[x_0, \ldots, x_m] \) be a square-free homogeneous polynomial of degree \( d > 0 \), defining a projective hypersurface \( V \) in \( \mathbb{P}^m \). Let \( X = Z(f) \) be the affine cone over \( V \) in \( \mathbb{A}^{m+1} \). Then \( \mathcal{O}^*(X) = k^* \).
In Lemma 1.3 we compute the group of units on a basic Zariski open affine subset of \( \mathbb{A}^m \).

**Lemma 1.3.** Let \( A = k[x_1, \ldots, x_n] \) be the coordinate ring of \( \mathbb{A}^m \). Let \( f \in A \) be non-invertible and square-free with factorization \( f = f_1 \cdots f_v \) into irreducible elements of \( A \). If \( R = A[f^{-1}] \), then

(a) \( A^* = k[x_1, \ldots, x_n]^* = k^* \).

(b) \( R^* = k[x_1, \ldots, x_n][f^{-1}]^* = k^* \times \langle f_1 \rangle \times \cdots \times \langle f_v \rangle \).

**Proof.** The factorization of \( f \) is unique because \( A \) is a unique factorization domain. In addition, \( \text{Cl}(A) = \text{Cl}(R) = 0 \). The counterpart of (1) for the localization \( R = A[f^{-1}] \) is

\[
1 \to A^* \to R^* \xrightarrow{\text{div}} \bigoplus_{i=1}^v \mathbb{Z}F_i \to \text{Cl}(A) \to \text{Cl}(R) \to 0
\]

where \( F_i = Z(f_i) \), for \( i = 1, \ldots, v \). The sequence splits because \( \bigoplus \mathbb{Z}F_i \) is free. \( \square \)

We will apply Theorem 1.4 frequently. It is well known, but for convenience, as well as to establish notation, we state it here.

**Theorem 1.4.** Let \( G = \langle \sigma \rangle \) be a finite cyclic group of order \( n \) and \( M \) a \( G \)-module. In \( \mathbb{Z}G \) define elements \( D = \sigma - 1 \) and \( N = 1 + \sigma + \sigma^2 + \cdots + \sigma^{n-1} \). Let \( N_M = \{ x \in M \mid Nx = 0 \} \) denote the kernel of \( N : M \to M \). Then

\[
H^0(G, M) = M^G
\]

\[
H^{2i-1}(G, M) = N_M / N^i M
\]

\[
H^{2i}(G, M) = M^G / N^i M
\]

for all \( i \geq 1 \).

**Proof.** See [18, Theorem 10.35], or [4, § 17.2]. \( \square \)

**1.2.1. Background results from étale cohomology.** In this section we review some of the results from étale cohomology that will be utilized in Section 3.3. For any variety \( X \) over \( k \), we denote by \( G_m \) the sheaf of units for the étale topology. Then \( \mathcal{O}^+(X) = H^0(X, G_m) \) is the group of global units on \( X \). We identify \( \text{Pic}(X) \), the Picard group of \( X \), with \( H^1(X, G_m) \) [14, Proposition III.4.9]. The natural map \( \text{Pic}(X) \to \text{Cl}(X) \) is one-to-one [9, Corollary 18.5]. If \( X \) is regular, \( \text{Pic}(X) = \text{Cl}(X) \) [13, Corollary II.6.16].

The torsion subgroup of \( H^2(X, G_m) \) is denoted \( B'(X) \) and is called the cohomological Brauer group. If \( X \) is regular, \( H^2(X, G_m) \) is torsion [12, Proposition 1.4, p. 71]. The Brauer group of classes of \( \mathcal{O}(X) \)-Azumaya algebras is denoted \( B(X) \). There is a natural embedding \( B(X) \to B'(X) \) [11, (2.1), p. 51]. In this article, the Brauer group plays only a peripheral role. If \( X \) is regular and integral with field of rational functions \( K \), then the natural map

\[
H^2(X, G_m) \to B(K)
\]

is one-to-one [12, Corollary 1.8, p. 73].

Let \( v > 1 \) be an integer that is invertible in \( k \). The kernel of the \( v \)th power map on \( G_m \) is \( \mu_v \), the sheaf of \( v \)th roots of unity. By Kummer theory, the \( v \)th power map gives an exact sequence of sheaves

\[
1 \to \mu_v \to G_m \to \mu_v \to 1
\]
for the étale topology on $X$. The long exact sequence of cohomology associated to (4) is

$$
\cdots \to H^{i-1}(X, G_m) \xrightarrow{\partial^{i-1}} H^i(X, \mathbb{G}_m) \to H^i(X, G_m) \xrightarrow{\partial^i} H^i(X, G_m) \to \cdots.
$$

By $\nu\operatorname{Pic}(X)$ we denote the subgroup of $\operatorname{Pic}(X)$ annihilated by $\nu$. Then for $i = 1$, (5) gives rise to the short-exact sequence

$$
1 \to \mathcal{O}^*(X) / (\mathcal{O}^*(X))^\nu \to H^1(X, \mathbb{G}_m) \to \nu\operatorname{Pic}X \to 0.
$$

As shown in [14, Example II.2.18(b)], the cohomology group $H^1(X, \mathbb{G}_m)$ classifies the Galois group $\mathbb{G}_m$ over $\nu \cdot X$. We sketch here the results that will be needed. The reader is referred to [14, § VI.6] for the details. Let $X$ be a nonsingular integral variety over $k$ and $Z \subseteq X$ a nonsingular integral closed subvariety of codimension one. For any sheaf $F$ on $X$ the long exact sequence of cohomology with supports in $Z$ is

$$
\cdots \to H^{i-1}(X, F) \to H^{i-1}(X - Z, F) \xrightarrow{\partial^{i-1}} H^i_Z(X, F) \to H^i(X, F) \to H^i(X - Z, F) \to \cdots.
$$

For $F = G_m$, the lower degree terms of (7) are

$$
1 \to \mathcal{O}^*(X) \to \mathcal{O}^*(X - Z) \xrightarrow{\partial^0}
H^1_Z(X, G_m) \to \operatorname{Pic}(X) \to \operatorname{Pic}(X - Z) \xrightarrow{\partial^1}
H^2_Z(X, G_m) \to H^2(X, G_m) \to H^2(X - Z, G_m) \to \cdots.
$$

Since $X$ is regular, $\operatorname{Pic}(X) = \mathbb{C}l(X)$ and by (1) we know $\partial^1$ is the zero map. Using (3) for both $X$ and $X - Z$ we conclude that $H^2_Z(X, G_m) = \langle 0 \rangle$. Comparing (3) with (1) we identify $H^1_Z(X, G_m)$ with the infinite cyclic group $\mathbb{Z}$ and see that the generator of $H^1_Z(X, G_m)$ maps to the divisor class of $Z$ in $\operatorname{Pic}(X)$. The long exact sequence of cohomology associated to (4) simplifies to

$$
\xymatrix{
H^2_Z(X, G_m) \ar[r]^\nu & H^2_Z(X, G_m) \ar[r]^\partial^1 & H^2_Z(X, \mathbb{G}_m) & 1
}
$$

from which we conclude $H^2_Z(X, \mathbb{G}_m)$ is cyclic of order $\nu$. The generator of $H^2_Z(X, \mathbb{G}_m)$ corresponding to the image of $Z$ is called the fundamental class of $Z$ in $X$ and is denoted $s_Z / X$.

1.3. Motivational examples.

**Example 1.5.** Let $m > 1$ and $n > 1$. Let $A = k[x_1, \ldots, x_m]$ be the affine coordinate ring for $\mathbb{A}^m$. In this example we construct an affine variety $X$ in $\mathbb{A}^m$ of degree $n$ such that the group $\mathcal{O}^*(X) / k^*$ has rank $n - 1$. Let $f_1, \ldots, f_n$ be linear polynomials in $A$, chosen so that $f = f_1 f_2 \cdots f_n + 1$ is irreducible. Let $X = Z(f)$, $F_1 = Z(f_1), \ldots, F_n = Z(f_n)$ be the corresponding affine varieties in $\mathbb{A}^m$. Assume no two of the affine hyperplanes $F_1, \ldots, F_n$ are disjoint. Let $\mathbb{P}^n = \operatorname{Proj} k[x_0, x_1, \ldots, x_m]$ and write $(\cdot)^*$ for homogenization with respect to the variable $x_0$. The projective completions of $X$, $F_1$, $\ldots$, $F_n$ are $\bar{X} = Z(f^*)$, $\bar{F}_1 = Z(f_1^*)$, $\ldots$, $\bar{F}_n = Z(f_n^*)$. For the $\mathbb{G}_m$-étale topology on $X$ [14, Example II.2.18(b)]. The long exact sequence of cohomology associated to (4) is
\[ F_1 = Z(f_1^*), \ldots, F_n = Z(f_n^*). \] Let \( F_0 = Z(x_0) \) denote the hyperplane at infinity. The complement of \( X \) in \( \bar{X} \) is the zero set of \( x_0, f_1^* \cdots f_n^* \). Let \( L_i = F_i \cap \bar{X} = F_i \cap \bar{F}_0 \). Then each \( L_i \) is a hyperplane in \( F_0 \) and \( \bar{X} - X = L_1 + \cdots + L_n \). Using the Jacobian criterion \( [13, \text{Theorem I.5.1}] \), one can see that the singular locus of \( \bar{X} \) agrees with the singular locus of \( L_1 + \cdots + L_n \). But \( L_1, \ldots, L_n \) are distinct hyperplanes in \( F_0 \), by our assumption on \( f_1, \ldots, f_n \). Then \( \bar{X} \) is regular in codimension one and the Serre criteria \( [13, \text{Proposition II.8.23}] \) show \( \bar{X} \) is normal. In this notation, sequence \( (1) \) becomes

\[
1 \to k^* \to \mathcal{O}^*(X) \xrightarrow{\text{div}} \bigoplus_{i=1}^n ZL_i \xrightarrow{\chi} \text{Cl}(X) \to \text{Cl}(X) \to 0.
\]

Let \( H \) denote the image of \( \chi \), which is the subgroup of \( \text{Cl}(X) \) generated by the divisors \( L_1, \ldots, L_n \). We will prove the following.

(a) \( H \) is an infinite group.
(b) \( H \) is a homomorphic image of \( \mathbb{Z} \oplus (\mathbb{Z}/n)^{(n-2)} \).
(c) The image of \( \text{div} \) has rank \( n - 1 \).
(d) The elements \( f_1, \ldots, f_{n-1} \) generate a subgroup of finite index in \( \mathcal{O}^*(X)/k^* \).

The free group \( \mathbb{Z}L_i \) maps onto the subgroup of \( \text{Cl}(\bar{X}) \) generated by \( L_i \). Since \( L_i \) has degree one, the degree map \( D \mapsto \deg(D)L_i \) is a splitting map. Therefore (a) is true. This also shows the image of \( \text{div} \) has rank at most \( n - 1 \). At the generic point of \( \bar{X} \) we have

\[
f_1^* \cdots f_n^* = -1
\]

so the function \( f_1^*/x_0 \) represents a unit in the coordinate ring \( \mathcal{O}(X) \). The diagram

\[
\begin{array}{ccc}
1 & \longrightarrow & \prod_{i=1}^n \langle f_i^*/x_0 \rangle \\
\downarrow \quad \alpha & & \downarrow \beta \\
\mathcal{O}^*(X)/k^* & \xrightarrow{\text{div}} & \bigoplus_{i=1}^n ZL_i \\
1 & \longrightarrow & H \\
\end{array}
\]

commutes, where the second row comes from \( (10) \) and is exact. Since \( F_i \) intersects \( \bar{X} \) along \( L_i \) with intersection multiplicity \( n \), we see that the divisor of \( f_i^*/x_0 \) on \( \bar{X} \) is

\[
\text{div}(f_i^*/x_0) = nL_i - L_1 - \cdots - L_n.
\]

The matrix of the map \( \beta \) is:

\[
C = \begin{bmatrix}
n - 1 & -1 & \cdots & -1 & -1 \\
-1 & n - 1 & \cdots & -1 & -1 \\
-1 & -1 & \cdots & n - 1 & -1 \\
-1 & -1 & \cdots & -1 & n - 1
\end{bmatrix}.
\]

Using row and column operations, one can compute the invariant factors of \( C \). They are 1 and 0, each with multiplicity one, and \( n \) with multiplicity \( n - 2 \). The Snake Lemma \( [18, \text{Theorem 6.5}] \) applied to \( (12) \) gives the exact sequence

\[
0 \to \text{Coker } \alpha \to \text{Coker } \beta \to H \to 0.
\]

of finitely generated abelian groups. Since \( \text{Coker } \beta \cong \mathbb{Z} \oplus (\mathbb{Z}/n)^{(n-2)} \), (b) follows from \( (15) \). Since \( H \) is infinite, \( (15) \) implies the torsion free rank of \( H \) is one. The exact functor \( (\quad) \otimes \mathbb{Q} \) applied to the bottom row of \( (12) \) gives (c). The exact functor \( (\quad) \otimes \mathbb{Q} \) applied to \( (15) \) shows \( \text{Coker } \alpha \) is finite. Together with \( (11) \), this gives (d).
We have shown that $\mathcal{O}^*(X)/k^*$ is a torsion-free abelian group of rank $n - 1$ and the elements $f_1, \ldots, f_{n-1}$ generate a subgroup of finite index. We ask whether the elements $f_1, \ldots, f_{n-1}$ generate the group $\mathcal{O}^*(X)/k^*$. We do not know the general answer to this
question. Examples [1.6, 1.9] prove that the answer is yes for some particular cases. Another affirmative answer is given in Example 3.15, where we impose the additional hypotheses on $X$ that $X \to \mathbb{P}^{m-1}$ is cyclic, and the ramification divisor is $X - X$, the divisor at infinity.

Example 1.6. In the notation of Example 1.5, let $f_i = x_i$. If $n \leq m$, and $f = x_1 \cdots x_n + 1$, then in $\mathcal{O}(X)$ we have $x_1 = -x_2^{-1} \cdots x_n^{-1}$. The isomorphism

$$\mathcal{O}(X) = \frac{k[x_1, \ldots, x_m]}{(x_1 \cdots x_n + 1)} \cong k[x_2, \ldots, x_n][x_2^{-1}, \ldots, x_n^{-1}]$$

is defined by eliminating $x_1$. By Lemma 1.5, $\mathcal{O}^*(X)$ is equal to $k^* \times \langle f_2 \rangle \times \cdots \times \langle f_n \rangle$ which by symmetry is equal to $k^* \times \langle f_1 \rangle \times \cdots \times \langle f_{n-1} \rangle$.

Example 1.7. In the notation of Example 1.5, assume $n = m$. Moreover, assume $f_1, \ldots, f_n$ are linear polynomials in $k[x_1, \ldots, x_n]$ such that the hyperplanes $F_1 = Z(f_1), \ldots, F_n = Z(f_n)$ in $\mathbb{A}^n$ are in general position. After an affine change of coordinates $[10 \ \S \ 2.3, \ p. \ 40]$, we reduce to the case of Example 1.6. Therefore, $f_1, \ldots, f_{n-1}$ make up a free basis for the $\mathbb{Z}$-module $\mathcal{O}^*(X)/k^*$.

Example 1.8. In the notation of Example 1.5 suppose $n = 2$ and $m \geq 2$. Let $f_1, f_2$ be linear polynomials in $k[x_1, x_2, \ldots, x_m]$ such that the hyperplanes $F_1 = Z(f_1)$, and $F_2 = Z(f_2)$ in $\mathbb{A}^n$ have nontrivial intersection. Let $X$ be the hypersurface in $\mathbb{A}^n$ defined by $f_1 f_2 + 1 = 0$. After an affine change of coordinates, we can assume $f_1 = x_1, f_2 = x_2$. Therefore $\mathcal{O}(X)$ is isomorphic to $R[x_3, \ldots, x_m]$, where $R = k[x_1, x_2]/(x_1 x_2 + 1)$. By Example 1.7, the group $\mathcal{O}^*(X)$ is equal to $k^* \times \langle f_1 \rangle$.

Example 1.9. In the notation of Example 1.5, suppose $n = 3$ and $m = 2$. Let $f_1, f_2, f_3$ be linear polynomials in $k[x_1, x_2]$ defining three lines $F_1 = Z(f_1), F_2 = Z(f_2), F_3 = Z(f_3)$ in $\mathbb{A}^2$, no two of which are parallel. Let $X$ be the affine curve in $\mathbb{A}^2$ defined by $f_1 f_2 f_3 + 1 = 0$. In this case, $X$ is the nonsingular cubic curve in $\mathbb{P}^2$ defined by $f_1^2 f_2^2 f_3^2 + x_0^3 = 0$. The three points on $X$ where $x_0 = 0$ are denoted $L_1, L_2, L_3$. They generate the group $H$ in (12). Notice that (12) implies $H$ is generated by $L_3$ and $L_1 - L_3$. Since the genus of $X$ is one, $L_1 - L_3$ is not principal ([13] Example II.6.10.1). Therefore, $H$ is equal to the internal direct sum $\mathbb{Z}L_3 \oplus (\mathbb{Z}/3)(L_1 - L_3)$. It follows that $\text{Coker} \alpha = (0)$ in (15). This implies the group $\mathcal{O}^*(X)$ is equal to $k^* \times \langle f_1 \rangle \times \langle f_2 \rangle$.

Example 1.10. In the notation of Example 1.5, suppose $n = 3$ and $m \geq 2$. Then the group $\mathcal{O}^*(X)$ is equal to $k^* \times \langle f_1 \rangle \times \langle f_2 \rangle$. This is because by an affine change of coordinates we can reduce to the case where $\mathcal{O}(X)$ is a polynomial ring over either the ring in Example 1.7 or the ring in Example 1.9.

Example 1.11. A class of curves that have been widely studied are the affine Fermat curves $F = Z(x^n + y^n - 1)$ in $\mathbb{A}^2$, where $n \geq 2$. Assume $n$ is invertible in $k$. Since $k$ is algebraically closed, $x^n + y^n$ factors in $k[x, y]$ into a product of $n$ distinct linear forms. Then $F$ can be viewed as $Z(f_1 \cdots f_n + 1)$ and the argument in Example 1.5 shows that the group $\mathcal{O}^*(F)/k^*$ is free of rank $n - 1$. Let $\bar{F}$ denote the projective completion of $F$, and $X$ the affine open subset of $F$ where $xy \neq 0$. Let $H$ denote the subgroup of $\text{Cl}(\bar{F})$ generated by the $3n$ points in $\bar{F} - X$. If $H_0$ is the kernel of the degree homomorphism, then

$$0 \to H_0 \to H \xrightarrow{\deg} \mathbb{Z} \to 0$$
is a split exact sequence, because a point has degree one. As shown in [21], \( H_0 \) is a finite group which is annihilated by \( n \). Using Lemma 1.1 we find that \( O^*(X)/k^* \) has rank \( 3n - 1 \).

A basis for \( O^*(X)/k^* \) is computed in [17].

2. A CYCLIC COVER OF AFFINE SPACE

In this section we consider an affine variety \( X \) which is a finite cyclic cover of \( \mathbb{A}^m \). It will be helpful to set up some notation. Let \( A = k[x_1, \ldots, x_n] \) be the coordinate ring of \( \mathbb{A}^m \). Let \( f \) be a non-invertible square-free element of \( A \). Assume \( n \geq 2 \) is invertible in \( k \) and \( \zeta \) is a primitive \( n \)th root of unity in \( k \). Let \( T = A[z]/(z^n - f) \) denote \( O(X) \), the coordinate ring of the affine variety \( X \). Then \( T \) is a ramified cyclic extension of \( A \), and the associated morphism on varieties is \( \pi : X \to \mathbb{A}^m \). Let \( f = f_1 \ldots f_v \) be a factorization of \( f \) into irreducibles in \( A \). By Eisenstein’s criterion, for instance, \( T \) is an integral domain. The singular locus of \( X \) corresponds to the singular locus of the affine hypersurface \( F = Z(f) \subseteq \mathbb{A}^m \). Hence \( X \) is regular in codimension one. By the Serre criteria, \( X \) is normal, and \( T \) is an integrally closed integral domain. The assignment \( \sigma(z) = \zeta^m \) defines an \( A \)-algebra automorphism of \( T \). If \( G = \langle \sigma \rangle \), then \( T^G = A \). The morphism \( \pi \) ramifies only above the divisor \( F \). If we let \( R = A[f^{-1}] \) and \( S = T[z^{-1}] \), then \( S \) is a Galois extension of \( R \) with cyclic group \( G \). The rings defined so far make up this commutative diagram

\[
\begin{array}{ccc}
T = A[\sqrt[3]{]T}] & \longrightarrow & S = R[\sqrt[3]{]T}]
\end{array}
\]

(17)

where an arrow represents set inclusion. The norm

\[
N : T^* \to k^*
\]

is a homomorphism of abelian groups, defined by \( N(u) = u\sigma(u) \cdots \sigma^{n-1}(u) \). If \( a \in k^* \), then \( N(a) = a^n \) because \( k \) is a subfield of \( A \). Since \( k \) is algebraically closed, \( N \) is onto.

2.1. The group of units on a cyclic cover of affine space. The notation established in the preceding paragraph is used throughout this section.

**Proposition 2.1.** In the above context, the following are true.

(a) \( H^i(G, k^*) = \begin{cases} k^* & \text{if } i = 0 \\ \mu_n & \text{if } i = 1, 3, 5, \ldots \\ \langle 1 \rangle & \text{if } i = 2, 4, 6, \ldots \end{cases} \)

(b) \( H^i(G, R^*) = \begin{cases} R^* = k^* \times \langle f_1 \rangle \times \cdots \times \langle f_v \rangle & \text{if } i = 0 \\ \mu_n \langle f_1 \rangle \times \cdots \times \langle f_v \rangle & \text{if } i = 1, 3, 5, \ldots \\ \langle 1 \rangle \langle f_1 \rangle \times \cdots \times \langle f_v \rangle & \text{if } i = 2, 4, 6, \ldots \end{cases} \)

(c) \( H^i(G, R^*/k^*) = \begin{cases} R^*/k^* = \langle f_1 \rangle \times \cdots \times \langle f_v \rangle & \text{if } i = 0 \\ \langle 1 \rangle \langle f_1 \rangle \times \cdots \times \langle f_v \rangle & \text{if } i = 1, 3, 5, \ldots \\ \langle f_1 \rangle \langle f_2 \rangle \times \cdots \times \langle f_v \rangle & \text{if } i = 2, 4, 6, \ldots \end{cases} \)

**Proof.** The sequence of trivial \( G \)-modules

\[
1 \to k^* \to R^* \to R^*/k^* \to 1
\]

(19)

is exact. Use Lemma 1.3 and Theorem 1.4. \( \square \)
Lemma 2.2. In the context of Section 2.1, let \( H \) be a subgroup of \( G \) of order \( n \). If \( (T^*)^H = k^* \), then the following are true.

(a) \( (T^*/k^*)^H = (1) \).

(b) \( H^2(T^*/k^*) = (1) \), for all \( i > 0 \).

(c) There is an exact sequence

\[
1 \to \mu_n \to H^2(T^*/k^*) \to H^2(T^*/k^*) \to 1
\]

for all \( i > 0 \).

Proof. Begin with the exact sequence

\[
1 \to k^* \to T^* \to T^*/k^* \to 1.
\]

By Lemma 1.1, \( T^*/k^* \) is a finitely generated torsion free \( \mathbb{Z} \)-module. The long exact sequence associated to (20) is

\[
1 \to k^* \xrightarrow{\varphi^0} (T^*)^H \to (T^*/k^*)^H \xrightarrow{\partial^0} \\
H^1(H,k^*) \xrightarrow{\alpha^1} H^1(H,T^*) \to H^1(H,T^*/k^*) \xrightarrow{\partial^1} \\
H^2(H,k^*) \xrightarrow{\alpha^2} H^2(H,T^*) \to H^2(H,T^*/k^*) \xrightarrow{\partial^2} \ldots
\]

By hypothesis, \( \alpha^0 \) is an isomorphism, so \( \partial^0 \) is one-to-one. Since \( G \) acts trivially on \( k^* \), \( H^1(H,k^*) = \mu_n \), a finite group. Because \( (T^*/k^*)^H \) is a subgroup of the torsion free group \( T^*/k^* \), we conclude that (a) is true. Because \( H \) is a cyclic group, part (b) follows from Theorem 1.4 and part (a). By Proposition 2.1(a), for \( i > 0 \), \( H^2(H,k^*) = (1) \). Part (c) follows from (21), part (a) and periodicity. \( \square \)

Theorem 2.3. In the context of Section 2.1, assume \( n = p \) is a prime number. There is an isomorphism of \( \mathbb{Z}[G] \)-modules \( T^*/k^* \cong A_1 \oplus \cdots \oplus A_r \) where \( A_1, \ldots, A_r \) are \( \mathbb{Z}[\zeta] \)-ideals in \( \mathbb{Q}[\zeta] \). It follows that \( T^*/k^* \) is a free \( \mathbb{Z} \)-module of rank \( (p-1)t \) and there are isomorphisms

\[
H^1(G,T^*) \cong \mu_p \times (\mathbb{Z}/p)^t
\]

\[
H^1(G,T^*/k^*) \cong (\mathbb{Z}/p)^t
\]

Proof. Since \( (T^*)^G = k^* \), this follows from Lemma 2.2 and the structure theory for a module over a cyclic group of order \( p \), [3, Theorem 74.3]. A similar argument is given in [6, Proposition 2.17]. \( \square \)

Example 2.4. When \( v > 1 \), the group of units of \( T \) can be trivial. For instance, let \( f = (x-1)(x-\alpha_1) \cdots (x-\alpha_d) \), where \( \alpha_1, \ldots, \alpha_d \) are distinct elements in \( k^* \). As computed in [7, §3.3], the group of units in \( T = A[z]/(z^v - f) \) is \( k^* \).

The group \( G \) acts as a group of automorphisms of \( \text{Cl}(T) \). This action is induced on the group of divisors \( \text{Div}(T) \) by sending a height one prime ideal \( I \) to its conjugate \( \sigma(I) \).

By this same action, \( G \) acts as a group of automorphisms of \( \text{Pic}(S) \). Because \( \text{Spec} S \) is nonsingular, \( \text{Pic}(S) = \text{Cl}(S) \). Associated to the Galois extension \( S/R \) is

\[
1 \to H^1(G,\mathbb{Z}/p) \xrightarrow{\alpha} \text{Pic}(R) \xrightarrow{\alpha} (\text{Pic}(S))^G \xrightarrow{\alpha} H^1(G,\mathbb{Z}/p) \xrightarrow{\alpha} H^1(G,S^*) \xrightarrow{\alpha} B(S/R) \xrightarrow{\alpha} H^1(G,\text{Pic}(S)) \xrightarrow{\alpha} H^1(G,S^*)
\]

the so-called Chase-Harrison-Rosenberg exact sequence of cohomology groups [2, Corollary 5.5]. The group \( B(S/R) \) appearing in (22) is the kernel of the natural map on Brauer
groups $B(R) \to B(S)$. In this article it plays only a peripheral role. Since $\text{Pic}R = 0$, sequence (22) and periodicity implies

\[(23) \quad H^i(G, S^*) = \langle 1 \rangle\]

if $i > 0$ is odd.

**Proposition 2.5.** In the context of Section 2.7 the following are true.

(a) If $i \geq 0$ is even, then

\[H^i(G, S^*/R^*) = (S^*/R^*)^G \cong \mathbb{Z}/n\]

is generated by the coset containing $z$.

(b) If $i > 0$ is odd, there is an exact sequence

\[1 \to H^i(G, S^*/R^*) \to H^{i+1}(G, R^*) \to H^{i+1}(G, S^*) \to 1\]

of $\mathbb{Z}/n$-modules.

**Proof.** The exact sequence of $G$-modules

\[(24) \quad 1 \to R^* \to S^* \to S^*/R^* \to 1\]

gives rise to the exact sequence of cohomology

\[(25) \quad 1 \to R^* \to (S^*)^G \to (S^*/R^*)^G \xrightarrow{\partial^0} H^1(G, R^*) \to H^1(G, S^*) \to H^2(G, S^*/R^*) \xrightarrow{\partial^1} H^2(G, R^*) \to H^2(G, S^*) \to \ldots\]

Equation (23) implies $\partial^0$ and $\partial^2$ are onto. Since $R^* = (S^*)^G$, $\partial^0$ is an isomorphism. By Proposition 2.1, $H^i(G, R^*)$ is a cyclic group of order $n$, if $j$ is odd. The minimum polynomial of $z$ is $z^n - f$. In $S^*/R^*$, the coset containing $z$ has order $n$. This proves $(S^*/R^*)^G$ is cyclic of order $n$ and is generated by the coset containing $z$. The image of the norm map $N : S^*/R^* \to S^*/R^*$ is $\langle 1 \rangle$. By Theorem 1.4, if $i$ is even, $H^i(G, S^*/R^*) = (S^*/R^*)^G$. This proves (a). Since $\partial^2$ is also an isomorphism, we get (b). \qed

**Proposition 2.6.** In the context of Section 2.7 the following are true

(a) $(S^*/k^*)^G = \langle z \rangle \times (f_2) \times \cdots \times (f_N)$ is a free $\mathbb{Z}$-module of rank $\nu$.

(b) If $i > 0$ is odd, then

\[H^i(G, S^*/k^*) = \langle 1 \rangle.\]

(c) If $i > 0$ is even, there is an exact sequence

\[1 \to H^i(G, S^*) \to H^i(G, S^*/k^*) \to \mathbb{Z}/n \to 1\]

of $\mathbb{Z}/n$-modules.

**Proof.** In the exact sequence of $G$-modules

\[(26) \quad 1 \to R^*/k^* \to S^*/k^* \xrightarrow{\eta} S^*/R^* \to 1\]

$\eta(z) = z$. The sequence of cohomology associated to (26) is

\[(27) \quad 1 \to R^*/k^* \to (S^*/k^*)^G \xrightarrow{\eta} (S^*/R^*)^G \xrightarrow{\partial^0} H^1(G, R^*/k^*) \to \ldots\]

By Proposition 2.1 in (27), $\eta$ is onto. By Proposition 2.5, the image of $\eta$ is generated by $\eta(z)$. So $(S^*/k^*)^G$ is an extension of $R^*/k^*$ by the finite cyclic group $\langle z \rangle$. By Lemma 1.1, $(S^*/k^*)^G$ is a finitely generated torsion free $\mathbb{Z}$-module. By Lemma 1.3, $R^*/k^* = \langle f_1 \rangle \times$
The exact sequence of $G$-modules

\[ 1 \to k^* \to S^* \to S^*/k^* \to 1 \]

gives the exact sequence of cohomology

\[
\begin{align*}
\text{H}^{2i-1}(G, S^*) & \to \text{H}^{2i-1}(G, S^*/k^*) \xrightarrow{\partial^{2i-1}} \text{H}^2(G, k^*) \\Rightarrow \\
& \text{H}^2(G, S^*) \to \text{H}^2(G, S^*/k^*) \xrightarrow{\partial^2} \text{H}^{2i+1}(G, k^*) \to \text{H}^{2i+1}(G, S^*) \to \ldots
\end{align*}
\]

where $i > 0$ is arbitrary. Parts (b) and (c) follow from (29), (23), and Proposition 2.1(a).  

**Example 2.7.** When $v > 1$, the group of units of $T$ can be non-trivial. For instance, let $f = (xy - 1)(xy + 1) = f_1 f_2$. As computed in [7 § 3.2], the group of units in $T = A[z]/(z^2 - f)$ is

\[
T^* = k^* \times \langle z - xy \rangle
\]

and the group of units in $S = T[z^{-1}]$ is

\[
S^* = k^* \times \langle z - xy \rangle \times \langle z - xy + 1 \rangle \times \langle z - xy - 1 \rangle.
\]

Use the identity $\sigma(z - xy) = (z - xy)^{-1}$ to compute

\[
\text{H}^i(G, T^*) = \begin{cases} 
  k^* & \text{if } i = 0 \\
  \mu_2 \times \frac{\langle z - xy \rangle}{\langle (z - xy)^2 \rangle} & \text{if } i = 1, 3, 5, \ldots \\
  \langle 1 \rangle & \text{if } i = 2, 4, 6, \ldots
\end{cases}
\]

and

\[
\text{H}^i(G, T^*/k^*) = \begin{cases} 
  \langle 1 \rangle & \text{if } i \text{ is even} \\
  \frac{\langle z - xy \rangle}{\langle (z - xy)^2 \rangle} & \text{if } i \text{ is odd}
\end{cases}
\]

Use the identities

\[
\sigma(z - xy + 1) = (z - xy + 1)(z - xy)^{-1}
\]

\[
\sigma(z - xy - 1) = -(z - xy - 1)(z - xy)^{-1}
\]

\[
2z = (z - xy + 1)(z - xy - 1)(z - xy)^{-1}
\]

\[
-2(xy - 1) = (z - xy + 1)^2(z - xy)^{-1}
\]

\[
-2(xy + 1) = (z - xy - 1)^2(z - xy)^{-1}
\]

to compute

\[
\text{H}^i(G, S^*) = \begin{cases} 
  R^* = k^* \times \langle xy - 1 \rangle \times \langle xy + 1 \rangle & \text{if } i = 0 \\
  \langle 1 \rangle & \text{if } i > 0
\end{cases}
\]

and

\[
\text{H}^i(G, S^*/k^*) = \begin{cases} 
  \langle z \rangle \times \langle xy + 1 \rangle & \text{if } i = 0 \\
  \langle 1 \rangle & \text{if } i = 1, 3, 5, \ldots \\
  \frac{\langle z \rangle}{\langle z^2 \rangle} & \text{if } i = 2, 4, 6, \ldots
\end{cases}
\]

These results agree with Propositions 2.5 and 2.6.
2.2. The ramification divisor is irreducible. In addition to the notation established in the opening paragraph of Section 2 Assume that $T = A[z]/(z^n - f)$, where $f$ is irreducible in $A$.

**Theorem 2.8.** In the context above, the following are true.

(a) $\text{Cl}(T) = \text{Cl}(S)$.
(b) $\text{Cl}(T)^G = \text{Cl}(S)^G = \langle 0 \rangle$.
(c) $H^i(G, T^*) = \begin{cases} k^s & \text{if } i = 0 \\ \cong \mathbb{Z}/n & \text{if } i = 1, 3, 5, \ldots \\ \langle 1 \rangle & \text{if } i = 2, 4, 6, \ldots \end{cases}$
(d) $H^i(G, S^*) = \begin{cases} R^s = k^s \times \langle f \rangle & \text{if } i = 0 \\ \langle 1 \rangle & \text{if } i > 0 \end{cases}$
(e) $\text{B}(S/R) \cong H^1(G, \text{Pic}S)$.

**Proof.** Since $T/(z) = A/(f)$, the ideal $I = Tz$ is a height one prime. In $T$, $z^n = f$, so the ideal $Tf$ has only one minimal prime, namely the height one prime $I = Tz$. By Nagata’s Theorem [9, Theorem 7.1], the sequence

$$1 \to T^* \to S^* \xrightarrow{\text{div}} \mathbb{Z}I \to \text{Cl}(T) \to \text{Cl}(S) \to 0$$

is exact. The divisor of $z$ is $\text{div}(z) = I$, so (36) shows $\text{Cl}(T) = \text{Cl}(S)$. It also follows from (36) that $S^*/T^* = \langle z \rangle$. Since $\sigma(z) = \zeta z$ and $\zeta \in T^*$, we see that $S^*/T^*$ is a trivial $G$-module. Therefore,

$$H^i(G, S^*/T^*) = \begin{cases} \langle z \rangle & \text{if } i = 0, \\ \langle 1 \rangle & \text{if } i = 1, 3, 5, \ldots, \\ \langle z \rangle / \langle z^n \rangle & \text{if } i = 2, 4, 6, \ldots \end{cases}$$

by Theorem 1.4. The long exact sequence associated to $1 \to T^* \to S^* \to S^*/T^* \to 1$ is

$$1 \to (T^*)^G \to (S^*)^G \to S^*/T^* \to H^1(G, T^*) \to H^1(G, S^*) \to H^1(G, S^*/T^*) \to \ldots$$

As in Lemma 1.3, $\text{Pic}(R) = \langle 0 \rangle$. By sequence (22), $H^i(G, S^*) = \langle 1 \rangle$ for odd $i$. Again by Lemma 1.3, $(T^*)^G = A^* = k^s$. The norm map (18) is onto, so for $i = 2, 4, 6, \ldots$, we have $H^i(G, T^*) = 1$. By Lemma 1.3, $(S^*)^G = R^s = k^s \times \langle f \rangle$. The terms of lowest degree in (38) give rise to the short exact sequence

$$1 \to \langle f \rangle \to \langle z \rangle \to H^1(G, T^*) \to 1.$$ 

Therefore $H^i(G, T^*) = \langle z \rangle / \langle z^n \rangle \cong \mathbb{Z}/n$ for odd $i$, proving (c). On the element $z$, the norm map $N : S^* \to R^*$ is

$$N(z) = z\zeta z \cdots \zeta^{n-1}z = \zeta^n z^{n-1}/f.$$ 

Use this and the fact that (18) is onto to prove that $H^i(G, S^*) = 1$ for $i = 2, 4, 6, \ldots$, proving (d). Part (b) follows from (a), (d), and sequence (22). Part (e) follows from sequence (22).

**Conjecture 2.9.** If $f$ is irreducible, then $T^* = k^s$. A partial answer is given in Theorems 2.10 and 2.12.

**Theorem 2.10.** In the context of Section 2.2 assume $n = p$ is a prime number. Then $T^* = k^s$. 


Proof. This is a consequence of Theorem 2.3 and Theorem 2.8(c). \hfill \Box

Lemma 2.11. Let \( p \) be a prime and \( n = p^2 \). Let \( G = \langle \sigma \rangle \), \( H = \langle \sigma^p \rangle \). Let

\[
\begin{align*}
T_2 &= A[z]/(z^p - f), \\
S_2 &= R[z]/(z^p - f), \\
T_1 &= T_2^H = A[z^p], \\
S_1 &= S_2^H = R[z^p].
\end{align*}
\]

The following are true.

(a) \( H^i(H, S_2^*/T_2^*) = \begin{cases} 
\langle z \rangle & \text{if } i = 0 \\
\langle 1 \rangle & \text{if } i = 1, 3, 5, \ldots \\
\langle z \rangle / \langle z^p \rangle & \text{if } i = 2, 4, 6, \ldots 
\end{cases} \)

(b) \( H^0(H, T_2^*) = T_1^* = k^* \) and \( H^0(H, S_2^*) = S_1^* = k^* \times \langle z^p \rangle \).

(c) If \( i > 0 \) is even, then \( H^i(H, T_2^*) = 1 \) and \( H^i(H, S_2^*) = 1 \).

(d) If \( i > 0 \) is odd, there is a short exact sequence

\[
1 \to \mu_p \to H^i(H, T_2^*) \to H^i(H, S_2^*) \to 1.
\]

(e) The group \( H^1(H, S_2^*) \) is isomorphic to \( \text{Cl}(S_2/S_1) \), which is the kernel of the natural map \( i : \text{Cl}(S_1) \to \text{Cl}(S_2) \).

Proof. From Theorem 2.8, \( S_2^*/T_2^* = \langle z \rangle \) with trivial \( G \)-action. This gives (a). Since \( T_2^H = T_1 \), Theorem 2.10 implies \( (T_2^*)^H = T_1^* = k^* \) and \( (S_2^*)^H = S_1^* = k^* \times \langle z^p \rangle \), which is (b).

Since \( H \) is cyclic, if \( i > 0 \) is even, then \( H^i(H, T_2^*) \) is equal to \( k^*/NT_2^* \), which is trivial for the same reason that the norm \( (13) \) is onto. As in (10), one can check that \( z^p \) is in the image of the norm map \( S_2^* \to S_1^* \). Part (c) follows. The exact sequence

\[
1 \to (T_2^*)^H \to (S_2^*)^H \to S_2^*/T_2^* \to H^1(G, T_2^*) \to \\
H^1(H, S_2^*) \to H^1(H, S_2^*/T_2^*) \to H^2(H, T_2^*) \to H^2(H, S_2^*) \to \\
H^2(H, S_2^*/T_2^*) \to H^3(H, T_2^*) \to H^3(H, S_2^*) \to H^3(H, S_2^*/T_2^*) \to \ldots
\]

is the counterpart of (38) for the group \( H \). Use (41), parts (a), (b), (c), and periodicity to get (d). The groups in (d) are \( \mathbb{Z}/p \)-modules, so the sequence splits. Part (e) is proved in [7, Theorem 16.1]. \hfill \Box

Theorem 2.12. If \( n = 4 \), then \( T^* = k^* \).

Proof. In Lemma 2.11(e), the group \( H \) has order two. By [4, Corollary 17.27], the group \( \text{Cl}(S_2/S_1) \) is annihilated by two. Hence \( \text{Cl}(S_2/S_1) \) is a subgroup of \( 2 \text{Cl}(S_1) \). By [7, Theorem 2.3], \( 2 \text{Cl}(S_1) \) is equal to \( \text{Cl}(S_1)^G/H \), which by Theorem 2.8(b) is equal to \( 0 \). Lemma 2.11(e) implies \( H^1(H, S_2^*) = \langle 1 \rangle \) and Lemma 2.11(d) implies \( H^1(H, T_2^*) \cong \mu_2 \). By Lemma 2.2, \( H^1(H, T_2^*/k^*) = \langle 1 \rangle \). As in the proof of Theorem 2.10, \( T_2^*/k^* = \langle 1 \rangle \). \hfill \Box

Conjecture 2.13. If \( n = 2^s \), for any \( s > 0 \), then \( T^* = k^* \). To iterate the argument used in Theorem 2.12 it is necessary to know that elements of order two in \( \text{Cl}(S_1) \) are fixed by \( G \).
3. Localization of a Cyclic Cover of Projective Space

In this section we consider the group of units on a ramified cyclic cover \( \pi : X \rightarrow U \) of affine varieties which is the restriction of a cyclic cover of projective space \( \pi : Y \rightarrow \mathbb{P}^m \) to an open set. We treat two special cases. In Section 3.1 the localization of \( \pi \) is such that along the “divisor at infinity”, \( \pi \) is unramified. Section 3.3 considers the case where the “divisor at infinity” is equal to the ramification divisor. In Section 3.2 these results are applied to the group of units on an affine curve. In Section 3.4 the results of this section are applied to a special case of Example 1.5. Throughout Section 3, \( k \) is an algebraically closed field and if \( G \) is the cyclic group whose action on \( Y \) induces the quotient morphism \( \pi \), then we assume the order of \( G \) is invertible in \( k \).

### 3.1. Unramified at Infinity

Start with a normal projective variety \( Y \) of dimension \( m > 0 \), together with \( G = \langle \sigma \rangle \) a cyclic group of order \( n \) acting on \( Y \) such that the quotient morphism is \( \pi : Y \rightarrow \mathbb{P}^m \). Next, we restrict \( \pi \) to an affine open subset of \( \mathbb{P}^m \) which is the complement of a prime divisor that is split by \( \pi \). Specifically, let \( F \subseteq \mathbb{P}^m \) be an irreducible hypersurface and \( U = \mathbb{P}^m - F \) the affine open complement. Assume the irreducible hypersurface \( F \subseteq \mathbb{P}^m \) is split by \( \pi \). By this we mean \( \pi : Y \times \mathbb{P}^m \rightarrow F \) is unramified, and lying above \( F \) are \( n \) irreducible components. If we write \( \pi^{-1}(F) = F_1 \cup \cdots \cup F_n \), then for each \( i \), \( \pi : F_i \rightarrow F \) is an isomorphism. Taking \( X \) to be \( \pi^{-1}(U) \), the group \( G \) acts on \( X \) and \( \pi : X \rightarrow U \) is a ramified cyclic covering of affine varieties. The varieties defined so far make up the diagram:

\[
\begin{align*}
X & \xrightarrow{c} Y & F_1 + \cdots + F_n \\
\downarrow \pi & \downarrow \pi & \downarrow \pi \\
U & \xleftarrow{c} \mathbb{P}^m & F
\end{align*}
\]

(42)

In this section we study the groups of units on the affine varieties \( U \) and \( X \).

#### Lemma 3.1. [13, Example II.6.5.1] Let \( F \subseteq \mathbb{P}^m \) be an irreducible hypersurface of degree \( d \). For the open affine \( U = \mathbb{P}^m - F \),

(a) \( \mathcal{O}^+(U) \), the group of units, is equal to \( k^* \), and

(b) \( \text{Cl}(U) \), the class group, is a cyclic group of order \( d \).

**Proof.** The degree homomorphism \( \text{deg} : \text{Cl}(\mathbb{P}^m) \rightarrow \mathbb{Z} \) is an isomorphism. The exact sequence (1) becomes

\[
1 \rightarrow k^* \rightarrow \mathcal{O}^+(U) \xrightarrow{\text{div}} \mathbb{Z} \mathcal{F} \xrightarrow{\chi} \text{Cl}(\mathbb{P}^m) \rightarrow \text{Cl}(U) \rightarrow 0
\]

where \( \chi(F) = d \). Therefore, \( \chi \) is one-to-one. \( \square \)

#### Lemma 3.2. In the context of Section 3.1, the following are true.

(a) \( (\mathcal{O}^+(X)/k^*)^G = \langle 1 \rangle \).

(b) The \( \mathbb{Z} \)-module \( \mathcal{O}^+(X)/k^* \) is free and has rank less than or equal to \( n - 1 \).

**Proof.** Consider the commutative diagram

\[
\begin{array}{ccccccccc}
1 & \rightarrow & \mathcal{O}^+(U)/k^* & \xrightarrow{\text{div}} & \mathbb{Z} \mathcal{F} & \xrightarrow{\chi} & \text{Cl}(\mathbb{P}^m) & \rightarrow & \text{Cl}(U) & \rightarrow & 0 \\
\downarrow \pi^* & & \downarrow \pi^* & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & \mathcal{O}^+(X)/k^* & \xrightarrow{\text{div}} & \bigoplus_{i=1}^n \mathbb{Z} F_i & \xrightarrow{\chi} & \text{Cl}(Y) & \rightarrow & \text{Cl}(X) & \rightarrow & 0
\end{array}
\]

(44)
The top row of (44) is (43) and the second row is (1). The map \( \pi^* \) is defined by \( F \mapsto F_1 + \cdots + F_n \). The maps on class groups are the natural maps. The \( G \)-module \( \bigoplus_{i=1}^n \mathbb{Z}F_i \) is free of rank one. By [18, Lemma 10.34] the sequence

\[
\bigoplus_{i=1}^n \mathbb{Z}F_i \xrightarrow{N} \bigoplus_{i=1}^n \mathbb{Z}F_i \xrightarrow{D} \bigoplus_{i=1}^n \mathbb{Z}F_i \xrightarrow{\deg} \mathbb{Z} \to 0
\]

of \( G \)-modules is exact, where \( N \) and \( D \) are as in Theorem 1.4 and \( \deg(a_1F_1 + \cdots + a_nF_n) = a_1 + \cdots + a_n \). The image of \( N \) is the cyclic \( \mathbb{Z} \)-module generated by \( \pi^*(F) = F_1 + \cdots + F_n \).

It follows that

\[
H^i \left( G, \bigoplus_{i=1}^n \mathbb{Z}F_i \right) = \begin{cases} \langle \pi^*(F) \rangle = \langle F_1 + \cdots + F_n \rangle & \text{if } i = 0 \\ \langle 0 \rangle & \text{if } i > 0. \end{cases}
\]

Eq. (45) breaks up, yielding short exact sequences

\[
0 \to \text{Image} \left( D \right) \to \bigoplus_{i=1}^n \mathbb{Z}F_i \xrightarrow{\deg} \mathbb{Z} \to 0
\]

\[
0 \to \text{Image} \left( N \right) \to \bigoplus_{i=1}^n \mathbb{Z}F_i \to \text{Image} \left( D \right) \to 0
\]

of \( G \)-modules. As a sequence of \( \mathbb{Z} \)-modules, (47) is split-exact, so \( \text{Image}(D) \) is a free \( \mathbb{Z} \)-module of rank \( n - 1 \). A principal divisor on \( Y \) has degree 0 (see [13, Exercise II.6.2]), so in (44), the image of \( \text{div} \) is a subgroup of the kernel of the map \( \text{deg} \) of (47). Thus, \( \text{Image}(\text{div}) \) is a free \( \mathbb{Z} \)-module of rank at most \( n - 1 \), which proves (b). The image of \( N \) is \( \langle \pi^*(F) \rangle \), which is a trivial \( \mathbb{G} \)-module. Theorem 1.4 shows \( H^i \left( G, \langle \pi^*(F) \rangle \right) = \langle 0 \rangle \). Using (46), the long exact sequence of cohomology associated to (48) simplifies to

\[
0 \to \langle \pi^*(F) \rangle \xrightarrow{\partial} \langle \pi^*(F) \rangle \to \text{Image} \left( D \right) \xrightarrow{G} 0.
\]

Hence \( \langle \text{Image} \left( D \right) \rangle^G = \langle 0 \rangle \). Since \( \text{Image}(\text{div}) \subseteq \text{Image}(D) \), this proves (a). \( \square \)

**Remark 3.3.** The upper bound of Lemma 3.2(b) is sharp, as shown in Example 1.11

**Theorem 3.4.** In the above context, assume \( n = p \) is a prime number and \( H \) is the subgroup of \( \text{Cl}(Y) \) generated by the prime divisors \( F_1, \ldots, F_n \). For the group \( \mathcal{O}^*(X)/k^* \), there are two possibilities. The first is \( \mathcal{O}^*(X) = k^* \), in which case \( H \) is a free \( \mathbb{Z} \)-module of rank \( p \).

The second possibility is that \( \mathcal{O}^*(X)/k^* \) is a free \( \mathbb{Z} \)-module of rank \( p - 1 \) and in this case \( H \) is isomorphic to an extension of \( \mathbb{Z} \) by a finite group.

**Proof.** The second row of (44) gives rise to the exact sequence

\[
1 \to \mathcal{O}^*(X)/k^* \to \bigoplus_{i=1}^n \mathbb{Z}F_i \xrightarrow{\partial} H \to 0
\]

of \( G \)-modules. As in Theorem 2.3 the proof follows from Lemma 3.2, the structure theory for a module over a cyclic group of order \( p \), and (50). \( \square \)

3.2. **The group of units on an affine curve.** In this section we apply Theorem 3.4 to study the group of units on an affine curve over \( k = \mathbb{C} \), the field of complex numbers.

Let \( p \) be a prime number and \( n \geq 2 \) an integer such that \( p \mid n \). Let \( \lambda_1, \ldots, \lambda_n \) be distinct elements of \( k \) and set \( f(x) = (x - \lambda_1) \cdots (x - \lambda_n) \). Let \( X = \mathbb{Z}(y - f(x)) \), a nonsingular affine curve in \( \mathbb{A}^2 \). Let \( \pi : X \to \mathbb{A}^1 \) be the morphism induced by \( k[x] \to \mathcal{O}(X) \). Let \( Y \)
be the complete nonsingular model for $X$ and $\pi : Y \to \mathbb{P}^1$ the extension of $\pi$. Then $Y$ is a cyclic cover of $\mathbb{P}^1$ of degree $p$, and we are in the context of the introduction to Section 3.

Let $P_i$ denote the point on $X$ (and on $Y$) where $y = x - \lambda_i = 0$. Let $Q_i = \pi(P_i)$. The map $\pi$ ramifies at $P_i$ and the ramification index is $p$. These are the only points where $\pi$ is ramified. Solve the Riemann-Hurwitz Formula [13 Corollary IV.2.4] for the genus of $Y$ to get $g(Y) = (p - 1)(n - 2)/2$.

If $n = p = 2$, then $Y$ is a nonsingular plane curve of genus zero. It is an exercise [13 Exercise I.1.1] to show that $\mathcal{O}(X)$ is isomorphic to $k[x,y]/(xy + 1)$. This is the special case of Example 1.6 for which $n = 2$. It follows that $\mathcal{O}^*(X)/k^*$ is isomorphic to $\mathbb{Z}$. For the remainder of Section 3.2 we will assume $n \geq 3$, hence $g(Y) \geq 1$.

The degree map $\deg : \text{Cl}(Y) \to \mathbb{Z}$ is onto. The kernel of the degree map, denoted $\text{Cl}^0(Y)$, is the jacobian variety. By [15] p. 64, $\text{Cl}^0(Y)$ is an abelian variety of dimension $g(Y)$. By [15] (iv), p. 42, $\text{Cl}^0(Y)$ is a divisible group. Let $P_0$ be any closed point of $Y$. The group of divisors on $Y$ of degree zero, denoted $\text{Div}^0(Y)$, is generated by the set $\{P - P_0 \mid P \in \text{Div}(Y)\}$. In this context, the exact sequence (1) becomes

$$1 \to \mathcal{O}^*(Y) \to \mathcal{O}^*(Y - P_0) \to \mathbb{Z}P_0 \xrightarrow{\chi} \text{Cl}(Y) \to \text{Cl}(Y - P_0) \to 0.$$  

Since $P_0$ has degree one, the degree map is a splitting for $\chi$, so $\text{Cl}(Y - P_0)$ is isomorphic to $\text{Cl}^0(Y)$.

**Theorem 3.5.** Let $k = \mathbb{C}$ be the field of complex numbers and $p$ a prime number. Say $G = \langle \sigma \rangle$ is a cyclic group of order $p$ acting on a nonsingular projective curve $Y$ such that the quotient map is $\pi : Y \to \mathbb{P}^1$. Assume $Y$ has positive genus $g(Y) > 0$. For a sufficiently general $Q \in \mathbb{P}^1$, if $X = \pi^{-1}(\mathbb{P}^1 - Q)$, then $\mathcal{O}^*(X) = k^*$.

**Proof.** Let $P_0 \in Y$ be a point where $\pi$ is ramified. Let $Q_0 = \pi(P_0)$. For an arbitrary $P \in Y$, the divisor $\pi(P) - Q_0$ is a principal divisor on $\mathbb{P}^1$. Therefore,

$$\pi^*(\pi(P) - Q_0) = P + \sigma(P) + \cdots + \sigma^{p-1}(P) - pP_0$$

is a principal divisor on $Y$. Manipulation of (52) shows that

$$(P - \sigma(P)) + 2\sigma(P - \sigma(P)) + 3\sigma^2(P - \sigma(P)) + \cdots + p\sigma^{p-1}(P - P_0)$$

is a principal divisor on $Y$. The group $\text{Cl}^0(Y)$ is generated by the set of divisors $\{P - P_0 \mid P \in Y\}$ which is equal to the set $\{Q(P) - P_0 \mid P \in Y\}$. By [53] we see that $p\text{Cl}^0(Y)$ is generated by the set of divisors $\{P - \sigma(P) \mid P \in Y\}$. The group $\text{Cl}^0(Y)$ is an abelian variety of dimension $g(Y)$ over $\mathbb{C}$. The subgroup of torsion elements in $\text{Cl}^0(Y)$ make up a discrete group which is isomorphic to $(\mathbb{Q}/\mathbb{Z})^{2g}$. For a sufficiently general choice of $P$, $P - \sigma(P)$ generates an infinite subgroup of $\text{Cl}^0(Y)$. Fix one such $P$. Let $Q = \pi(P)$ and $X = Y - \pi^{-1}(Q) = \pi^{-1}(\mathbb{P}^1 - Q)$. Let $H$ denote the subgroup of $\text{Cl}(Y)$ generated by $P, \sigma(P), \ldots, \sigma^{p-1}(P)$. We are in the context of Theorem 3.4. Therefore $\mathcal{O}^*(X) = k^*$ if and only if $H$ is not an extension of $\mathbb{Z}$ by a finite group. Consider the degree map $\text{deg} : H \to \mathbb{Z}$, which splits since $\text{deg}(P) = 1$. By choice of $P$, the kernel of $\text{deg}$ contains the infinite subgroup $\{P - \sigma(P), \sigma(P) - \sigma^2(P), \ldots, \sigma^{p-1}(P) - P\}$. Since $H$ is not an extension of $\mathbb{Z}$ by a finite group, Theorem 3.4 says $H$ is a free $\mathbb{Z}$-module of rank $p$ and $\mathcal{O}^*(X) = k^*$. The proof follows. 

**Proposition 3.6.** Let $k = \mathbb{C}$ be the field of complex numbers. Let $p$ be a prime number, $n \geq 3$ an integer such that $p \mid n$, and $\lambda_1, \ldots, \lambda_n$ distinct elements in $k$. Let $f(x) = \prod_{i=1}^{n}(x - \lambda_i)$ and $X = \mathbb{Z}(y^p - f(x))$, an affine curve in $\mathbb{A}^2$. For a sufficiently general choice of $\lambda_1, \ldots, \lambda_n$, the group of units on $X$ is equal to $k^*$. 

□
Proof. Apply Theorem 3.5 to the nonsingular projective completion of $X$. \hfill $\square$

Proposition 3.7. Let $k = \mathbb{C}$, the field of complex numbers. Let $n \geq 3$ and $\lambda_1, \ldots, \lambda_n$ distinct elements in $k$. Let $f(x) = \prod_{i=1}^{n} (x - \lambda_i)$ and $X = \mathbb{A}^2$ the corresponding affine hyperelliptic curve in $\mathbb{A}^2$. For a sufficiently general choice of $\lambda_1, \ldots, \lambda_n$, the group of units on $X$ is equal to $k^*$.

Proof. Let $Y$ be the nonsingular projective completion for $X$. If $n$ is even, this follows from Proposition 3.6. If $n$ is odd, then $Y - X$ consists of only one point and we apply (51). \hfill $\square$

Example 3.8. In Proposition 3.7 if $X$ is arbitrary, the group of units can be non-trivial. For instance, let $f(x) = x^4 - 1$. In $\mathcal{O}(X)$, we have $1 = x^4 - y^2$, hence $x^2 - y$ is invertible and $\mathcal{O}^*(X) \neq k^*$. For another example, let $f = x^3 + x$ and $X = \mathbb{A}^2$. In $\mathcal{O}(X)$,

\[
\begin{align*}
(x^3 + 1/2 + xy)(x^3 + 1/2 - xy) &= (x^3 + 1/2)^2 - x^2y^2 \\
&= x^6 + x^3 + 1/4 - x^2(x^4 + x) \\
&= 1/4
\end{align*}
\]

Hence $\mathcal{O}^*(X) \neq k^*$.

3.3. The ramification divisor is at infinity. Start with a normal projective variety $Y$ of dimension $m > 0$, together with $G = \langle \sigma \rangle$ a cyclic group of order $n$ acting on $Y$ such that the quotient morphism is $\pi: Y \to \mathbb{P}^m$. Assume the ramification divisor of $\pi$ on $Y$ is the set of prime divisors $Q = \{Q_1, \ldots, Q_r\}$ and that the ramification index at each $Q_i$ is $n$. Let $\pi(Q) = P = \{P_1, \ldots, P_r\}$. Let $U = \mathbb{P}^m - P$ and $X = Y - Q$. Then $\pi: X \to U$ is a Galois cover with group $G$. The varieties defined so far make up the diagram:

\[
\begin{array}{c}
X \xrightarrow{\pi} Y \xleftarrow{\pi} Q = Q_1 + \cdots + Q_r \\
U \xrightarrow{\pi} \mathbb{P}^m \xleftarrow{\pi} P = P_1 + \cdots + P_r
\end{array}
\]

(54)

The Nagata sequences for $U$ and $X$ give the rows of the commutative diagram:

\[
\begin{array}{c}
1 \to \mathcal{O}^*(U)/k^* \longrightarrow \bigoplus_{i=1}^{r} \mathbb{Z}P_i \longrightarrow \text{Cl}(\mathbb{P}^m) \longrightarrow \text{Cl}(U) \to 0 \\
1 \to \mathcal{O}^*(X)/k^* \longrightarrow \bigoplus_{i=1}^{r} \mathbb{Z}Q_i \longrightarrow \text{Cl}(Y) \longrightarrow \text{Cl}(X) \to 0
\end{array}
\]

(55)

By the map $\delta$, $P_i$ is mapped to $nQ_i$. [9, p. 30]. We consider the group of units $\mathcal{O}^*(X)/k^*$ and the image of $\chi$, which is equal to the subgroup of $\text{Cl}(Y)$ generated by $Q_1, \ldots , Q_r$.

Proposition 3.9. In the above context, if $\text{Cl}(U) = \langle 0 \rangle$, then

(a) $\mathcal{O}^*(U)/k^*$ is a free $\mathbb{Z}$-module of rank $r - 1$.

(b) $\mathcal{O}^*(X)/k^*$ is a free $\mathbb{Z}$-module of rank $r - 1$, and

(c) $H^1(U, \mu_v)$ is a free $\mathbb{Z}$-$\mu$-module of rank $r - 1$, for all positive integers $v$ such that $v$ is invertible in $k$.

Proof. In the top row of (55), $\text{Cl}(U) = \langle 0 \rangle$ and $\text{Cl}(\mathbb{P}^m) \cong \mathbb{Z}$. The sequence

\[
1 \to \mathcal{O}^*(U)/k^* \to \bigoplus_{i=1}^{r} \mathbb{Z}P_i \to \text{Cl}(\mathbb{P}^m) \to 0
\]

is split-exact, which gives (a). Since $\pi^!$ in (55) is one-to-one, we know the rank of the free $\mathbb{Z}$-module $\mathcal{O}^*(X)/k^*$ is at least $r - 1$. On $Y$ a principal divisor has degree zero (see [13].
Exercise II.6.2], so the subgroup of Cl(Y) generated by the divisor Q₁ is infinite cyclic. In the second row of (55), the image of \( \chi \) is infinite. Therefore, the kernel of \( \chi \) is a free \( \mathbb{Z} \)-module of rank at most \( r-1 \), which proves (b). By [9 Corollary 18.5], Pic(U) = \( \langle 0 \rangle \) since Cl(U) = \( \langle 0 \rangle \). The exact sequence (5) simplifies to

\[
1 \to \Theta^+(U)/k^\times \to \Theta^+(U)/k^\times \to H^1(U, \mu_n) \to 0.
\]

Part (c) follows from (a) and (56). \( \square \)

**Proposition 3.10.** Let \( k \) be an algebraically closed field of characteristic \( p \), where \( p = 0 \) is allowed. Let \( n \) be a positive integer such that \( n \) is invertible in \( k \). Let \( \pi : X \to U \) be a cyclic Galois cover of degree \( n \) of integral varieties with group \( G \). Then modulo groups of \( p \)-torsion, the sequence

\[
0 \to \mathbb{Z}/n \to H^1(U, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\pi^*} H^1(X, \mathbb{Q}/\mathbb{Z})^G \to 0
\]

is exact. The kernel of \( \pi^* \) is generated by the class represented by the cyclic covering \( \pi : X \to U \).

**Proof.** We give the proof for \( p = 0 \). For positive characteristic, reduce all groups modulo \( p \)-torsion. The Hochschild-Serre spectral sequence [14 p. 105] for \( \pi : X \to U \) is

\[
H^i(G, H^j(X, \mathbb{Q}/\mathbb{Z})) \Rightarrow H^{i+j}(U, \mathbb{Q}/\mathbb{Z}).
\]

Since \( X \) is integral, the group of global sections of the constant sheaf is \( H^0(X, \mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z} \). Because \( G \) acts trivially on \( \mathbb{Q}/\mathbb{Z} \), \( H^0(X, \mathbb{Q}/\mathbb{Z})^G = \mathbb{Q}/\mathbb{Z} \). By Theorem 1.4

\[
H^i(G, \mathbb{Q}/\mathbb{Z}) = \begin{cases} \mathbb{Q}/\mathbb{Z} & \text{if } i = 0, \\ \langle 0 \rangle & \text{if } i > 0 \text{ is even}, \\ \mathbb{Z}/n & \text{if } i \text{ is odd.} \end{cases}
\]

Consider the filtration \( H^1(U, \mathbb{Q}/\mathbb{Z}) = E^1_0 \supset E^1_1 \supset 0 \). Then

\[
E^1_1 = E^{1,0}_1 = E^{1,0}_2 = H^1(G, H^0(X, \mathbb{Q}/\mathbb{Z})) \cong \mathbb{Z}/n
\]

and since \( d^{0,1}_{1} : H^1(X, \mathbb{Q}/\mathbb{Z})^G \to H^2(G, \mathbb{Q}/\mathbb{Z}) \) is the zero map, \( E^1_0/E^1_1 = E^{0,1}_1 = E^{0,1}_2 = H^1(X, \mathbb{Q}/\mathbb{Z})^G \). Therefore we have the short exact sequence

\[
0 \to \mathbb{Z}/n \to H^1(U, \mathbb{Q}/\mathbb{Z}) \to H^1(X, \mathbb{Q}/\mathbb{Z})^G \to 0.
\]

The inclusion \( E^1_1 \subseteq E^1_1 \) corresponds to the subgroup of \( H^1(U, \mathbb{Q}/\mathbb{Z}) \) generated by the cyclic Galois cover \( \pi : X \to U \). \( \square \)

**Proposition 3.11.** In the context of Section 3.3 the following are true.

(a) In the commutative square

\[
\begin{array}{ccc}
H^1(P^n, \mu_n) & \longrightarrow & H^1(U, \mu_n) \\
\downarrow & & \downarrow \pi^* \\
H^1(Y, \mu_n) & \longrightarrow & H^1(X, \mu_n)
\end{array}
\]

induced by the left side of (54), the image of \( \pi^* \) is a subgroup of the image of \( \rho \). We say that \( \pi^* \) splits the ramification of each \( \xi \in H^1(U, \mu_n) \).

(b) If \( r \geq 2 \) and Cl(U) = \( \langle 0 \rangle \), the image of \( \pi^* \) is isomorphic to \( (\mathbb{Z}/n)^{r-2} \).
Proof. By $\text{Sing} Y$ we denote the singular locus of $Y$. Since $Y$ is normal, the codimension of $\text{Sing} Y$ in $Y$ is at least two. The codimension of $\text{Sing} Q$ in $Y$ is at least two. Since $X \to U$ is unramified, we know $X$ is nonsingular. Let $\omega$ be a closed subset of $Q$, of codimension greater than or equal to two, such that $Y - \omega$ and $Q - \omega$ are nonsingular. Because $\pi$ is a finite morphism, $\pi(\omega)$ has codimension at least two in $\mathbb{P}^m$. By $[14, \text{Lemma VI.9.1}]$, $H^1(Y, \mu_n) = H^1(Y - \omega, \mu_n)$. In this proof, we are going to utilize the fundamental classes of the divisors $Q_i - \omega \subseteq Y - \omega$ and $P_i - \pi(\omega) \subseteq \mathbb{P}^m - \pi(\omega)$. For the background material, see Section 1.2.1. Proposition 3.12. In the notation of Section 3.3, assume $\text{Cl}(U) = (0)$ and $r \geq 2$.

(a) The Galois cover $X \to U$ corresponds to adjoining the $n$th root of an invertible function on $U$. That is, there exists a unit $f \in \mathcal{O}^*(U)$ such that $\mathcal{O}(X)$ is isomorphic to $\mathcal{O}(U)[z]/(z^n - f)$.

(b) Let $H$ denote the image of $\chi$ in $\mathbb{Z}$, which is the subgroup of $\text{Cl}(Y)$ generated by $Q_1, \ldots, Q_r$. As an abelian group, $H$ is isomorphic to $\mathbb{Z} \oplus (\mathbb{Z}/n)^{r-2}$. 

\begin{equation}
Z = H^1_{\partial_i - \pi(\omega)}(\mathbb{P}^m - \pi(\omega), G_m) \longrightarrow H^1(\mathbb{P}^m - \pi(\omega), G_m) = \text{Cl}(\mathbb{P}^m)
\end{equation}

(58)

$\xymatrix{Z = H^1_{\partial_i - \pi(\omega)}(\mathbb{P}^m - \pi(\omega), G_m) \ar[r]^-{\partial_i} \ar[d]_-{\alpha} & H^2_{\partial_i - \pi(\omega)}(\mathbb{P}^m - \pi(\omega), G_m) = Z/n \ar[d]_-{\beta_i} \\
Z = H^1_{\partial_i - \omega}(Y - \omega, G_m) \ar[r]^-{\partial_i} & H^2_{\partial_i - \omega}(Y - \omega, G_m) = Z/n \ar[u]_-{\alpha_i}}$

where $\alpha$ is the left side of (58) and the rows are from (9). Both of the connecting maps $\partial_i$ are onto. Thus $\beta$ is the zero map. The divisor $P - \pi(\omega)$ decomposes into the disjoint union $\cup_i(P_i - \pi(\omega))$. Therefore the cohomology with supports decomposes into a direct sum:

$\text{H}^1_{\partial_i - \pi(\omega)}(\mathbb{P}^m - \pi(\omega), G_m) = \bigoplus_{i=1}^{r} \text{H}^2_{\partial_i - \pi(\omega)}(\mathbb{P}^m - \pi(\omega), G_m).$

The similar decomposition occurs for $Q - \omega \subseteq Y - \omega$. Consider the commutative diagram

\begin{equation}
H^1(\mathbb{P}^m - \pi(\omega), G_m) \longrightarrow H^1(U, G_m) \longrightarrow \bigoplus_{i=1}^{r} H^2_{\partial_i - \pi(\omega)}(\mathbb{P}^m - \pi(\omega), G_m)
\end{equation}

(60)

$\xymatrix{H^1(Y - \omega, G_m) \ar[r]^-{\rho} \ar[d]_-{\pi^*} & H^1(U, G_m) \ar[d]_-{\pi^*} \ar[r]^-{\rho} & \bigoplus_{i=1}^{r} H^2_{\partial_i - \omega}(Y - \omega, G_m) \ar[d]_-{\beta = \beta_1 + \cdots + \beta_r} \\
H^1(Y - \omega, G_m) \ar[r]^-{\rho} & H^1(U, G_m) \ar[r]^-{\rho} & \bigoplus_{i=1}^{r} H^2_{\partial_i - \omega}(Y - \omega, G_m) \ar[d]_-{\beta = \beta_1 + \cdots + \beta_r} \\
H^1(Y - \omega, G_m) \ar[r]^-{\rho} & H^1(U, G_m) \ar[r]^-{\rho} & \bigoplus_{i=1}^{r} H^2_{\partial_i - \omega}(Y - \omega, G_m)}$

whose rows are from (7), hence are exact. The map $\beta$ is the sum of the maps $\beta_i$ in (59), hence is the zero map. This proves that the image of $\pi^*$ is a subgroup of the image of $\rho$, which gives (a). In part (b), Proposition 3.10 implies that the kernel of $\pi^*$ in (60) is cyclic of order $n$. By Proposition 3.9, the image of $\pi^*$ in (60) is isomorphic to $(\mathbb{Z}/n)^{r-2}$. 

\[ \square \]

Proposition 3.12. In the notation of Section 3.3, assume $\text{Cl}(U) = (0)$ and $r \geq 2$.

(a) The Galois cover $X \to U$ corresponds to adjoining the $n$th root of an invertible function on $U$. That is, there exists a unit $f \in \mathcal{O}^*(U)$ such that $\mathcal{O}(X)$ is isomorphic to $\mathcal{O}(U)[z]/(z^n - f)$.

(b) Let $H$ denote the image of $\chi$ in (55), which is the subgroup of $\text{Cl}(Y)$ generated by $Q_1, \ldots, Q_r$. As an abelian group, $H$ is isomorphic to $\mathbb{Z} \oplus (\mathbb{Z}/n)^{r-2}$. 

\[ \square \]
(c) The quotient group $\mathcal{O}^*(X) / \mathcal{O}^*(U)$ is cyclic of order $n$ and is generated by the element $z$ of Part (a).

Proof. We are given that $\text{Cl}(U) = \{0\}$. The exact sequence of Kummer Theory (6) implies any cyclic Galois extension of $\mathcal{O}(U)$ is of the form $\mathcal{O}(U)[\sqrt{f}]$ for some $f \in \mathcal{O}^*(U)$. This is (a). The diagram

\[
\begin{array}{ccccccc}
1 & \longrightarrow & \mathcal{O}^*(U) / k^* & \longrightarrow & \bigoplus_{i=1}^r \mathbb{Z} \mathcal{P}_i^* & \longrightarrow & \text{Cl}(\mathbb{P}^m) & \longrightarrow & 0 \\
\pi^i & \downarrow & \delta & \downarrow & \pi^r & \downarrow & \pi^r & \downarrow & \\
1 & \longrightarrow & \mathcal{O}^*(X) / k^* & \longrightarrow & \bigoplus_{i=1}^r \mathbb{Z} \mathcal{Q}_i^* & \longrightarrow & H & \longrightarrow & 0 \\
\end{array}
\]

(61) commutes, where the top two rows are from (55) and are exact. The bottom row of (61) is made up of the cokernels of the maps $\pi^2$, $\delta$, and $\pi^r$. Consider the map

\[
\mathcal{O}^*(X) / k^* \overset{\text{div}}{\longrightarrow} \bigoplus_{i=1}^r \mathbb{Z} \mathcal{Q}_i^*
\]

of the second row of (61). By Proposition 3.9, the groups $\mathcal{O}^*(U) / k^*$ and $\mathcal{O}^*(X) / k^*$ are free $\mathbb{Z}$-modules of rank $r - 1$. To prove (b), we show that the invariant factors of (62) are $1, n, \ldots, n$, where $n$ is the greatest common divisor of $\mathcal{Q}_i$. By assumption the ramification index of $\pi$ at each $Q_i$ is $n$. Therefore $z$ is a local parameter at each $Q_i$. This implies $\text{div}(z)$ generates a direct summand of $\bigoplus_{i=1}^r \mathbb{Z} \mathcal{Q}_i^*$. This proves 1 is an invariant factor of (62). Let $n_1, \ldots, n_{r-2}$ denote the other invariant factors of (62). The group $H$ contains the image of $\text{H}^1(U, \mu_n) \rightarrow \text{H}^1(X, \mu_n)$ which by Proposition 3.11 is a direct sum of $r - 2$ copies of $\mathbb{Z}/n$. This proves $n$ is a divisor of $n_i$, for $1 \leq i \leq r - 2$.

Since $\pi^2$ is one-to-one, its cokernel is finite. Since $\delta$ is one-to-one and the groups on the bottom row of (61) are finite, the Snake Lemma [18, Theorem 6.5] implies the kernel of $\pi^r$ is finite. Since $\text{Cl}(\mathbb{P}^m) \cong \mathbb{Z}$, we conclude that $\pi^r$ is one-to-one. The Snake Lemma implies the bottom row of (61) is exact. In particular, we know the groups $\mathcal{O}^*(X) / \mathcal{O}^*(U)$ and $H / \text{Image}(\pi^r)$ are both annihilated by $n$. The functor $() \otimes_Z Q / Z$ applied to the third column of (61) gives the exact sequence

\[
0 \rightarrow \text{Tor}_1(H, Q / Z) \rightarrow \text{Tor}_1(H / \text{Image}(\pi^r), Q / Z) \rightarrow Q / Z
\]

which shows the exponent of the torsion subgroup of $H$ is a divisor of $n$. This shows each invariant factor of (62) satisfies $n_i = n$, for $1 \leq i \leq r - 2$, which proves (b).

Apply the functor $() \otimes_Z Z / n$ to the third column of (61) to get the exact sequence

\[
0 \rightarrow \text{Tor}_1(H, Z / n) \rightarrow \text{Tor}_1(H / \text{Image}(\pi^r), Z / n) \rightarrow Z / n \overset{\pi^r \otimes 1}{\longrightarrow} H \otimes_Z Z / n \rightarrow (H / \text{Image}(\pi^r)) \otimes Z / n \rightarrow 0.
\]

In (61) the image of $\pi^r$ is a subgroup of $nH$, so in (64) the map $\pi^r \otimes 1$ is the zero map. By Part (b), $\text{Tor}_1(H, Z / n)$ is isomorphic to $(Z / n)^{r-2}$. Since $H / \text{Image}(\pi^r)$ is annihilated by $n$, the first part of (64) reduces to the exact sequence

\[
0 \rightarrow (Z / n)^{r-2} \rightarrow H / \text{Image}(\pi^r) \rightarrow Z / n \rightarrow 0
\]
which is split exact. Eq. (65) shows $H / \text{Image}(\pi^*)$ is isomorphic to $(\mathbb{Z}/n)^{(r-1)}$. It follows that the bottom row of (61) is split exact. Therefore, $\mathcal{O}^*(X) / \mathcal{O}^*(U)$ is a cyclic group of order $n$. By the computations above, we know the element $z$ is a generator for this group. This proves (c). □

In the notation of Section 3.3, Proposition 3.13 is the $r = 1$ case.

**Proposition 3.13.** Let $P$ be an irreducible divisor on $\mathbb{P}^m$ of degree $n > 1$ and $U = \mathbb{P}^m - P$. Let $d > 1$ be a divisor of $n$. Let $X \to U$ be a cyclic Galois cover of degree $d$ corresponding to a generator of $H^1(U, \mu_d)$. Then $\mathcal{O}^*(X) = \mathbb{C}^*$.

**Proof.** The Galois cover $X$ exists by Lemma 3.14 and Kummer Theory (6). Let $\pi : Y \to \mathbb{P}^m$ be the cyclic cover which ramifies along $P$ (for example, see [11, Section I.17]). Let $\pi^{-1}(P) = Q$. In Diagram (55), $\chi(Q)$ is non-zero because a principal divisor on $Y$ has degree zero (see [13, Exercise II.6.2]). □

3.4. The group of units on an affine variety. In this section we apply results from Section 3.3 to study the group of units on the affine variety defined by an equation of the form $f_1 \cdots f_r = 1$, where the polynomials $f_i$ are distinct irreducible forms in $k[x_1, \ldots, x_m]$.

**Theorem 3.14.** Let $m \geq 2$ and $r \geq 2$. Let $f_1, \ldots, f_r$ be irreducible forms in $k[x_1, \ldots, x_m]$ defining distinct varieties $Z(f_1), \ldots, Z(f_r)$ in $\mathbb{A}^m$. Assume the set $\{\deg(f_1), \ldots, \deg(f_r)\}$ generates the unit ideal in $\mathbb{Z}$. If $X$ denotes the affine variety in $\mathbb{A}^m$ defined by $f_1 \cdots f_r = 1$, then $\mathcal{O}^*(X)/\mathbb{C}^* = \langle f_1 \rangle \times \cdots \times \langle f_r-1 \rangle$.

**Proof.** For $i = 1, \ldots, r$, let $d_i = \deg(f_i)$. Let $n = d_1 + \cdots + d_r$. Let $Y$ be the projective variety in $\mathbb{P}^m = \text{Proj}(k[x_0, x_1, \ldots, x_m])$ defined by $f_1 \cdots f_r = x_0^n$. Let $Z_0 = Z(x_0)$ denote the hyperplane at infinity. For each $i = 1, \ldots, r$, let $F_i = Z(f_i)$ be the hypersurface in $\mathbb{P}^m$ defined by $f_i$ and set $L_i = F_i \cap Z_0$. Note that $L_i$ refers to a subvariety of $Z_0$ as well as of $Y$. The divisor at infinity on $Y$ is $L_1 + \cdots + L_r$. We are given that $L_1, \ldots, L_r$ are distinct prime divisors in $Z_0$. View $Y$ as a cyclic cover of $Z_0 = \text{Proj} k[x_1, \ldots, x_m]$ and let $\pi : Y \to Z_0$ be the projection. The ramification divisor of $\pi$ is the set $\{L_1, \ldots, L_r\}$, and the ramification index of $\pi$ at each $L_i$ is one. If we let $X$ be the open affine $Y - L_1 - \cdots - L_r$, then $\mathcal{O}(X) = k[x_1, \ldots, x_m]/(f_1 \cdots f_r - 1)$ is obtained by dehomogenizing with respect to $x_0$. If we let $U = Z_0 - L_1 - \cdots - L_r$, then $X \to U$ is a cyclic Galois covering. The varieties defined so far make up the commutative diagram:

$$
Y = \text{Proj} \left\{ \frac{k[x_0, x_1, \ldots, x_m]}{(f_1 \cdots f_r - x_0^n)} \right\} \xleftarrow{\pi} X = \text{Spec} \left\{ \frac{k[x_1, \ldots, x_m]}{(f_1 \cdots f_r - 1)} \right\} = Y - L_1 - \cdots - L_r
$$

(66)

$$
Z_0 = \text{Proj} k[x_1, \ldots, x_m] \xleftarrow{\pi} U = Z_0 - L_1 - \cdots - L_r
$$

At the generic point of $Y$ we have

$$
\frac{f_1}{x_0^d_1} \cdots \frac{f_r}{x_0^d_r} = 1
$$

(67)

so the function $f_i/x_0^d_i$ represents a unit in the coordinate ring $\mathcal{O}(X)$. The diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \prod_{i=1}^r \left\langle f_i/x_0^d_i \right\rangle & \overset{\text{div}}{\longrightarrow} & \bigoplus_{i=1}^r \mathbb{Z} \text{div}(f_i/x_0^d) & \longrightarrow & 0 \\
\downarrow \alpha & & \downarrow \beta & & \downarrow & & \downarrow \\
1 & \longrightarrow & \frac{\mathcal{O}^*(X)}{\mathbb{C}^*} & \overset{\text{div}}{\longrightarrow} & \bigoplus_{i=1}^r \mathbb{Z}L_i & \overset{\chi}{\longrightarrow} & H & \longrightarrow & 0
\end{array}
$$

(68)
commutes, where the second row comes from the middle row in \(61\) and is exact. Given \(67\) and Proposition 3.9, it is enough to show \(\alpha\) is onto. For this, we utilize the matrix of the map \(\beta\) in \(68\). Since \(Y\) is defined by \(f_1 \cdots f_r = x_0^n\), each \(F_i\) intersects \(Y\) along \(L_i\) with intersection multiplicity \(n\). It follows that the divisor of \(f_i/x_0^d\) on \(Y\) is

\[
\text{div}(f_i/x_0^d) = nL_i - d_1L_1 - \cdots - d_rL_r.
\]

The matrix of the map \(\beta\) is:

\[
C = \begin{bmatrix}
  n - d_1 & -d_2 & \cdots & -d_{r-1} & -d_r \\
  -d_1 & n - d_2 & \cdots & -d_{r-1} & -d_r \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  -d_1 & -d_2 & \cdots & n - d_{r-1} & -d_r \\
  -d_1 & -d_2 & \cdots & -d_{r-1} & n - d_r \\
\end{bmatrix}
\]

We compute the invariant factors of the matrix \(C\). First, replace column \(r\) with the sum of columns \(1, \ldots, r\). Second, for \(i = 1, \ldots, r - 1\), replace row \(i\) with row \(i\) minus row \(r\). After these column and row operations, the resulting matrix is

\[
\begin{bmatrix}
  n & 0 & \cdots & 0 & 0 \\
  0 & n & \cdots & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & \cdots & n & 0 \\
  0 & 0 & \cdots & 0 & n \\
\end{bmatrix}
\]

The ideal in \(\mathbb{Z}\) generated by \(d_1, d_2, \ldots, d_{r-1}, n\) is the unit ideal. Using the method of minors (see for example, \[19\] Theorem 9.64), one can compute the invariant factors of \(C\). They are 1 and 0, each with multiplicity one, and \(n\) with multiplicity \(r - 2\). Since \(d_1, \ldots, d_r\) are relatively prime, \(\text{Cl}(U) = \text{Cl}(Z_0 - L_1 - \cdots - L_r) = 0\). By Proposition 3.12 we know that the group \(H\) in \(68\) is isomorphic to \(\mathbb{Z} \oplus (\mathbb{Z}/n)^{(r-2)}\). By the Snake Lemma applied to \(68\), the sequence

\[
0 \to \text{Coker} \alpha \to \text{Coker} \beta \to H \to 0
\]

is exact. The second and third groups in \(71\) have the same invariant factors. The exact functor \((\cdot) \otimes_{\mathbb{Z}} \mathbb{Q}\) applied to \(71\) shows \(\text{Coker} \alpha\) is finite. Apply the functor \((\cdot) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}\) to \(71\). The exact Tor sequence which results shows that the torsion subgroup of \(\text{Coker} \beta\) maps onto the torsion subgroup of \(H\). This proves \(\text{Coker} \alpha = \{0\}\). Therefore \(\{f_1, \ldots, f_{r-1}\}\) is a basis for the group \(\partial^*(X)/k^*\).

**Example 3.15.** This is a special case of Example 1.5 as well as Theorem 3.14. Let \(m \geq 2\) and \(n \geq 2\). Let \(f_1, \ldots, f_n\) be distinct linear forms in \(k[x_1, \ldots, x_m]\). Let \(X\) be the affine variety in \(\mathbb{A}^m\) defined by \(f_1 \cdots f_n = 1\). Then \(\partial(X) = k[x_1, \ldots, x_m]/(f_1 \cdots f_n - 1)\). By Theorem 3.14 \(\{f_1, \ldots, f_{n-1}\}\) is a basis for the group \(\partial^*(X)/k^*\).

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