How does transverse (hydrodynamic) flow affect jet-broadening and jet-quenching?

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We give the modification of formulas for $p_\perp$-broadening and energy loss which are necessary to calculate parton interactions in a medium with flow. Arguments are presented leading to the conclusion that for large $p_\perp$-spectra observed in heavy ion collisions at RHIC, the influence of transverse flow on the determination of the "quenching power" of the produced medium is small. This leaves open the question of the interpretation of data in a consistent perturbative framework.

1. Introduction

Energy loss of a high transverse momentum parton travelling in the hot and dense medium created in ultrarelativistic collisions has been the subject in recent years of intense investigation, reviewed in [1–6]. With the aim of being able in particular to assert the existence of the QGP phase of matter [7].

It has become clear however, that an important issue is the influence of the medium evolution on the radiative energy loss, when following the BDMPS [8–10] - Zakharov [11, 12] - Wiedemann [13] approach. The medium cannot be described as being static [14] : longitudinal and transverse flow have to be considered [15–17]. They may alter substantially the distribution of matter before freeze out. The crucial parameter for energy loss is the local transport coefficient $\hat{q}(\tau)$, related to the squared average transverse momentum transfer – from the medium to the hard parton – per unit length.

For an ideal QGP, and in a comoving coordinate system, one may relate locally $\hat{q}$ with the energy density of the medium $\varepsilon : \hat{q} \simeq c \varepsilon^{3/4}$ with $c \simeq 2$ [18, 19].

Recently from various authors [20–23] came the determination of $c$ from data, taking into account Bjorken longitudinal expansion [24] but neglecting transverse flow. In [20] $c$ is found to be much larger than 2 ($c > 8 \cdots 19$, $c \simeq 10$), leading to speculations about a strongly coupled QGP [25–30] . A recent endeavour [31, 32] implementing a strong non-Bjorken expansion including radial flow [33], with a small value of primordial transverse velocity $v^t = 0.1$ has led to a "moderately optimistic" scenario reducing $c = 10$ to $c = 2$ compatible with perturbative estimates.

In the present note, we try to reformulate the problem. We show first how to determine $\hat{q}$ in the presence of transverse flow by applying a proper Lorentz boost. We assume a realistic (ideal) hydrodynamical description for heavy-ion collisions, based on longitudinal Bjorken expansion [24] and a radial flow with vanishing initial velocity.

We are led to the conclusion that the transverse flow is a small effect and does not provide the solution to the perturbative/non-perturbative dilemma.

2. Moving medium versus medium at rest : geometry

The problem is to find the relationship between $\hat{q}$ for a moving medium and $\hat{q}_0$ (medium at rest), locally in space and time. We focus on central collisions.

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We consider – in the transverse plane \((xy)\) of a collision – a medium moving with velocity \(\vec{v}\) along the \(y\) axis towards a parton which enters into it at the origin of our coordinate system at time \(\tau = 0\), at point \(A\), leaving it at point \(B\) with \(B^\mu = t(1, \sin \theta, \cos \theta, 0)\) where we denote by \(t\) the time spent by the parton in the medium. The dimension of the medium in the velocity direction being \(L\), one finds

\[
t = \frac{L}{v + \cos \theta}.
\]  

(2.1)

The medium at rest has length \(\frac{L}{\sqrt{1 - v^2}} \equiv L \cosh \zeta\) defining \(\zeta\) such that \(v = \tanh \zeta\). Entering at point \(A_0\), the parton leaves the medium at rest, at point \(B_0\) such that as a result of the Lorentz boost, we find

\[
B_0^\mu = t (\cosh \zeta (1 + \cos \theta \tanh \zeta), \sin \theta, \cosh \zeta (\cos \theta + \tanh \zeta), 0)
\]

(2.2)

with

\[
A_0B_0 = \frac{L \cosh \zeta}{\cos \theta_0} \equiv t \cosh \zeta (1 + \cos \theta \tanh \zeta)
\]

(2.3)

The angle \(\theta_0\) corresponds to the angle \(\theta\), but evaluated in the rest system of the medium, implying

\[
\cos \theta_0 = \frac{\tanh \zeta + \cos \theta}{1 + \cos \theta \tanh \zeta}
\]

(2.4)

and

\[
\sin \theta_0 = \frac{\sin \theta}{\cosh \zeta (1 + \cos \theta \tanh \zeta)}.
\]

(2.5)

3. Transverse momentum broadening

A. Local geometry

Let us define in the rest frame the initial momentum of the parton:

\[
p_0^\mu = p_0 (1, \sin \theta_0, \cos \theta_0, 0).
\]

(3.1)

After travelling on a small distance \(\delta \tau\) – we want to extract a local information – and experiencing \(p_\perp\)-broadening and energy loss, the parton has a momentum \(p_0^\mu\):

\[
p_0^\mu = (p_0 - \delta \varepsilon) (1, \sin (\theta_0 + \delta \theta_0) \cos \delta \varphi_0, \cos (\theta_0 + \delta \theta_0), \sin (\theta_0 + \delta \theta_0) \sin \delta \varphi_0)
\]

(3.2)

where \(\delta \varphi_0\) is the azimuthal angle in the \(zx\) plane. In the moving frame, the corresponding momenta are:

\[
p^\mu = p (1, \sin \theta, \cos \theta, 0)
\]

(3.3)

\[
\not{p} = (p - \delta \varepsilon) (1, \sin (\theta + \delta \theta) \cos \delta \varphi, \cos (\theta + \delta \theta), \sin (\theta + \delta \theta) \sin \delta \varphi).
\]

(3.4)

The momenta in both frames are related via a Lorentz boost, which leaves the \(x\) component unchanged, leading to

\[
p \sin \theta = p_0 \sin \theta_0.
\]

(3.5)

On the other hand \(\delta \varphi \equiv \delta \varphi_0\), since the Lorentz boost does not affect the projection on the \(zx\) plane. Using eqs. (2.4) and (2.5), we also find:

\[
\frac{\delta \theta}{\sin \theta} = \frac{\delta \theta_0}{\sin \theta_0}.
\]

(3.6)
B. Comparing frames

The corresponding transverse momentum broadening is easily obtained in both frames: we sum the squares of the momentum increase in the $z$ axis direction and in the direction orthogonal to the parton momentum:

$$\delta p_{0z}^2 = \delta p_0^2 + \delta p_0^2 \equiv p_0^2 [\sin^2 \theta_0 \delta \varphi^2 + \delta \theta_0^2].$$  \hspace{1cm} (3.7)

Similarly

$$\delta p_{\perp}^2 = p^2 [\sin^2 \theta \delta \varphi^2 + \delta \theta^2].$$  \hspace{1cm} (3.8)

Using eqs. (3.5) and (3.6), we find that $\delta p_{\perp}^2 = \delta p_{0\perp}^2$ and may immediately infer the relation between $\hat{q}_0$ and $\hat{q}$: indeed we get

$$\delta p_{\perp}^2 = \hat{q} \delta \tau$$  \hspace{1cm} (3.9)

$$\delta p_{0\perp}^2 = \hat{q}_0 \delta \tau_0 = \hat{q}_0 \delta \tau [\cosh \theta \sinh \theta \tau_0] ,$$  \hspace{1cm} (3.10)

leading to (locally):

$$\hat{q} \equiv \hat{q}_0 \cosh \theta \sinh \theta \tau_0 .$$  \hspace{1cm} (3.11)

Recently this transformation property has been independently derived in [34], and already used in [35]. $\hat{q}_0$ may be parametrized as discussed in the Introduction.

Writing the transverse flow velocity as $u^\mu = (\cosh \theta, 0, -\sinh \theta, 0)$ and with the parton momentum written as $p^\mu = (1, \sin \theta, \cos \theta, 0)$, we get

$$\hat{q} = \hat{q}_0 \frac{u^\mu \cdot p^\mu}{p_0} .$$  \hspace{1cm} (3.12)

A way to understand this relation is to write the transport coefficient as the rate of interaction – the number of interactions per unit time – multiplied by the typical invariant scale $\mu^2$ characterizing the momentum transfer to the parton in a parton-medium collision: for a medium at rest $\hat{q}_0 = R_0 \mu^2$ and for a moving medium $\hat{q} = R \mu^2$. The rate of interaction varies indeed like the inverse of the time spent in the medium:

$$\frac{R}{R_0} = \cosh \theta (1 + \cos \theta \sinh \theta) ,$$  \hspace{1cm} (3.13)

so that eq. (3.11) is recovered.

C. Parton transverse broadening

Using $\hat{q}_0(\tau)$ we may now calculate the transverse broadening [9] obtained when taking into account the transverse flow for a parton travelling in a thermalized medium at temperature $T$. This last condition allowing us to determine the $\tau$ dependence of $T$ and the transverse velocity $v$ by using hydrodynamical model codes.

In Fig. 1 we have schematized the situation in the transverse plane [20]: the vector $\vec{s}(s, \phi)$ defines the point in the plane where the hard parton enters the medium. The parton then follows a path along the unit vector $\hat{n}$ and along its way, encounters the radial flow along the direction $\vec{r}$ which depends on time $\tau$. 

3
We write $\vec{r} = \vec{s} + \tau \hat{n}$ and defining $\theta$ (according to the definition given in sect. 2) and $\phi$ as in Fig. 1, we find that

$$\cos \theta = \frac{s \cos \phi - \tau}{\sqrt{s^2 + \tau^2 - 2s \tau \cos \phi}}.$$

(3.14)

Writing $c_h \zeta$ as $1/\sqrt{1 - v^2} \equiv \gamma(v)$ and $\theta \zeta = v$, where $v$ depends on $r$ and $\tau$. We may now get the following expression for the transverse broadening of the parton for a cylindrical medium at central rapidity:

$$(\Delta p_{T}^2)_{Bj+\text{flow}} = \frac{1}{\pi R_A^2} \int_0^{R_A} \int_0^{2\pi} \int_{\tau_0}^{\tau_{\text{max}}} ds \, d\phi \, d\tau \, \hat{q}_0(T(r, \tau)) \gamma(v) (1 + v \cos \theta)$$

(3.15)

where $\tau_{\text{max}}$ is defined as the maximum in-medium path length compatible with geometry and taking into account the life-time of the thermalized medium $\tau_{\text{QGP}}$:

$$\tau_{\text{max}} = \min \{ \tau_{\text{geom}}^{\text{max}}, \tau_{\text{QGP}} \},$$

(3.16)

with

$$\tau_{\text{geom}}^{\text{max}} = s \cos \phi + \sqrt{s^2 \cos^2 \phi + R_A^2 - s^2},$$

(3.17)

and $\tau_0$ is the initial time. In the integral (3.15) the boundaries are simplified by neglecting the flow beyond the nucleus radius $R_A$ (c.f. Fig. 3). The life-time $\tau_{\text{QGP}}$ is estimated in the framework of Bjorken expansion [24], after having fixed the initial temperature and taking the final temperature as $T \simeq 200$ MeV. (In the following we take $\tau_{\text{QGP}} \simeq 4.5$ fm).

We actually calculate the ratio

$R_{\text{flow}} = (\Delta p_{T}^2)_{Bj+\text{flow}}/\Delta p_{T}^2_{Bj}$

(3.18)

in order to quantify the importance of the radial flow in addition to Bjorken (Bj) longitudinal expansion [24], for which

$$\hat{q}_0(T) = \hat{q}_0(T_0) \left(\frac{T}{T_0}\right)^3 = \hat{q}_0(\tau_0) \left(\frac{\tau_0}{\tau}\right)^{3c_s^2},$$

(3.19)

where $c_s$ denotes the sound velocity (taken to be $c_s = 1/\sqrt{3}$).

The calculation is done in the framework of ideal hydrodynamics, assuming initial conditions relevant for heavy-ion collisions, e.g. a vanishing flow velocity at initial time $\tau_0$ [36]. The result is shown as the solid curve in Fig. 2 for
a cylindrical medium of radius $R_A = 6.4$ fm (Au nucleus), fixing the initial time $\tau_0 = 0.5$ fm, as a function of the initial temperature $T_0$.

The curves are actually calculated using the approximate analytic expressions derived in [36] (see also [37]). In the ratio $R_{\text{flow}}$ the dependence on the initial temperature $T_0$ cancels. Ignoring the limit given by $\tau_{\text{QGP}}$ results in the dotted curve. We note that $R_{\text{flow}}$ is smaller than 1. This can be understood in the following way: the flow has a non negligible effect for large enough values of $r$, where $v$ differs significantly from 0. But this is only realized (see Fig. 1) when the jet is moving with the flow, since then $\cos \theta \approx -1$, and therefore $\gamma(v)(1 + v \cos \theta) < 1$. This effect is even larger for the dotted curve.

We note that the angular dependence of eq. (3.11), which is responsible for the reduction of broadening, and also quenching, is not present in the ansatz for $\hat{q}$ given in [33] and used in [31,32].

As an exercise we calculated $R_{\text{flow}}$ with the following set-up: we assume that $\hat{q}_0(T, \tau)$ follows Bjorken’s $\tau$ dependence (3.19), keeping the hydro velocity $v(r, \tau)$ in $\gamma(v)$, but without the $r$-dependence in the temperature, and dropping the $\cos \theta$ term. The ratio $R_{\text{flow}}$ becomes larger than one, which can be seen from the dashed curve in Fig. 2, because in this case the factor $\gamma(v)$ wins.

Two conclusions emerge:

a) the effect of the transverse flow is small and of the order $\leq 10\%$.

b) the ratio $R_{\text{flow}}$ is smaller than 1 which goes against the possibility that the transverse flow would allow us to solve the perturbative/non-perturbative dilemma discussed in the introduction.

The smallness of the effect is due to the fact that the transverse velocity takes a long time to reach an appreciable value, starting from $v = 0$ and cannot provide the possibility of compensating the cooling effect. This can be visualized in Fig. 3 where we have plotted the quantity $(T/T_0)^3 \cdot \gamma(v)$ – which is essentially the integrand in eq. (3.15) – for different values of $\tau$, as a function of $r$, using ideal hydrodynamics [36].

It is interesting to investigate the effects due to a non-vanishing shear viscosity compared to the ideal hydrodynamical description for the quantity $(T/T_0)^3 \cdot \gamma(v)$, which is plotted in Fig. 4. The details of this calculation may be found in [37].
Fig. 3. \((T/T_0)^3 \cdot \gamma(v)\) as a function of \(r\) for different values of \(\tau\) calculated from ideal hydrodynamics [36].

Fig. 4. Effects due to non-vanishing shear viscosity \(\eta\) [38].

No significant effects due to the transverse flow in the presence of shear viscosity are found, as maybe expected, when comparing the solid and dashed curves in Fig. 4, although cooling is slowed down at least for times \(\tau\) presented in this figure: \(R_{flow}\) stays of \(O(1)\).
4. Energy loss

The medium-induced radiative jet energy loss is determined by the transport coefficient $\hat{q}$. While the energy loss formalism has been previously developed only for a medium without flow it is apparent from the discussion at the end of section 3.B that if one uses $\hat{q}$, as given in eq. (3.11), to determine the interaction of QCD partons with the medium then there is no modification of the fundamental energy loss formulas. Thus eqs. (25) and (26) of [15] remain valid in the presence of flow. The basic quantity is the radiation spectrum of gluons emitted from the high energy parton. In the soft gluon energy limit, $\omega$ much smaller than the parton energy, the dominant mechanism consists in multiple scatterings of the radiated gluon off the "centers" describing the medium [8]. For static centers, as well as for a dense (thermal) medium which undergoes longitudinal Bjorken-type expansion [24] the gluon spectrum obeys the following scaling law in terms of a characteristic gluon energy $\omega_c$ [2, 16],

$$\frac{dI}{d\omega} = \tilde{I}(\omega/\omega_c) \ .$$

(4.1)

For static centers [8]

$$\omega_c = \frac{1}{2} \hat{q} L^2 \ ,$$

(4.2)

where $L$ denotes the path length of the energetic jet in the medium.

In the expanding case [15], eq. (4.2) is generalized to [16,17]

$$\omega_c = \int_0^L d\tau \hat{q}(\tau) \ ,$$

(4.3)

with $\hat{q}(\tau) \approx 1/\tau$, when following Bjorken [24].

However, when transverse flow is present in the medium this nice scaling property does not hold, since $\hat{q}$ appears at different times, corresponding to the interference product of the emission amplitude and the complex conjugate one (see eq. (25) in [15]). As for the case of $p_T$-broadening, we do not expect large effects due to transverse flow. Therefore, for simplicity, we calculate the ratio

$$R_{\omega_c} = \frac{(\omega_c)_{Bj + flow}}{(\omega_c)_{Bj}} \ .$$

(4.4)

In comparison with the calculation of $R_{\text{flow}}$, we only estimate the first moment with respect to the path length $\tau$ of the energetic jet, i.e.

$$(\omega_c)_{Bj + flow} = \frac{1}{\pi R_A} \int_0^{R_A} ds \int_0^{2\pi} d\phi \int_0^{r_{max}} d\tau \ \tau \ \hat{q}(T(r, \tau)) \gamma(v) (1 + v \cos \theta) \ ,$$

(4.5)

and accordingly for $(\omega_c)_{Bj}$.

As we expect from Fig. 2 the ratio $R_{\omega_c}$ does not differ significantly from 1, when calculated within (ideal) hydrodynamics and realistic initial conditions for the flow field $v(r, \tau_0)$: a typical value is $R_{\omega_c} \simeq 0.85$ for $\tau_0 = 0.5$ fm, $T_0 = 400$ MeV. From this estimate, we expect small effects of radial flow on the quenching of large transverse momentum hadrons produced in nucleus-nucleus collisions. This conclusion on $R_{\omega_c} < 1$ is confirmed in the recent analysis in [35] using eq. (3.11).

Our study seems to rule out the possibility of reducing the large value of $\hat{q}$ from model comparisons with data [20] by including radial flow with initial conditions, which we assume to be realistic and causal in heavy-ion collisions.

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