On the growth of real functions and their derivatives

Jürgen Grahl and Shahar Nevo

Abstract

We show that for any \( k \)-times continuously differentiable function \( f : [a, \infty) \to \mathbb{R} \), any integer \( q \geq 0 \) and any \( \alpha > 1 \) the inequality

\[
\liminf_{x \to \infty} \frac{x^k \cdot \log x \cdot \log 2 \cdot \ldots \cdot \log q x \cdot f^{(k)}(x)}{1 + |f(x)|^\alpha} \leq 0
\]

holds.

2000 Mathematics Subject Classification: 26D10

In [3], [6], [2], [5], [4] and [1] we had studied differential inequalities in the context of complex analysis, more precisely with respect to the question whether they constitute normality (or at least quasi-normality) in the sense of Montel.

**Theorem A** [2] Let \( \alpha > 1 \) and \( C > 0 \) be real numbers and \( k \geq 1 \) be an integer. Let \( \mathcal{F} \) be a family of meromorphic functions in some domain \( D \) in the complex plane such that

\[
\frac{|f^{(k)}(z)|}{1 + |f(z)|^\alpha} \geq C \quad \text{for all } z \in D \text{ and all } f \in \mathcal{F}.
\]

(1)

Then \( \mathcal{F} \) is normal.

This result doesn’t hold any longer if \( \alpha > 1 \) is replaced by \( \alpha = 1 \) as easy examples demonstrate. However, at least for \( k = 1 \) condition (1) implies quasi-normality if \( \alpha = 1 \) [6]. Furthermore, in [1] we had shown that the condition

\[
\frac{|f^{(k)}(z)|}{1 + |f(z)|^\alpha} \geq C \quad \text{for all } z \in D
\]

(2)

(where \( k > j \geq 0 \) are integers, \( \alpha > 1 \) and \( C > 0 \)) implies quasi-normality.

As to entire functions (i.e. functions analytic in the whole complex plane), it is almost obvious that they cannot satisfy a differential inequality like (1). Indeed, if \( f \) is entire and \( |f^{(k)}(z)| \geq C \cdot (1 + |f(z)|^\alpha) \) for all \( z \in \mathbb{C} \), then in particular \( |f^{(k)}(z)| \geq C \) for all \( z \in \mathbb{C} \), so \( f^{(k)} \) is constant by Picard’s (or Liouville’s) theorem. But then \( f \) is a non-constant polynomial, and one obtains a contradiction for \( z \to \infty \) provided that \( \alpha > 0 \).
These considerations motivated us to look at the differential inequality (1) in the context of real analysis, a problem that doesn’t seem to have been studied so far. For real-valued functions on unbounded intervals we have the following result which turns out to be sharp in a certain sense. Here, \( \log^p \) denotes the \( p \)-times iterated natural logarithm, defined recursively by \( \log^0 x := x \) and \( \log^p x := \log(\log^{p-1} x) \) for \( p \geq 1 \).

**Theorem 1** Let \( k \geq 1 \) and \( q \geq 0 \) be integers, \( \alpha > 1 \), \( a \in \mathbb{R} \) and \( f : [a, \infty) \rightarrow \mathbb{R} \) a \( k \)-times continuously differentiable function. Then

\[
\liminf_{x \to \infty} \frac{x^k \cdot \log x \cdot \log_2 x \cdot \ldots \cdot \log_q x \cdot f^{(k)}(x)}{1 + |f(x)|^\alpha} \leq 0 \tag{3}
\]

and

\[
\liminf_{x \to \infty} \frac{x^k \cdot \log x \cdot \log_2 x \cdot \ldots \cdot \log_q x \cdot |f^{(k)}(x)|}{1 + |f(x)|^\alpha} = 0. \tag{4}
\]

**Example 2** This result is best possible in the sense that it is not longer valid if \( \log^q x \) is replaced by \( (\log^q x)^\beta \) with any \( \beta > 1 \). This can be seen by considering the function \( f : [a, \infty) \rightarrow \mathbb{R} \) defined by

\[
f(x) := (-1)^{k-1} \cdot \int_a^x \int_{x_1}^\infty \ldots \int_{x_k}^\infty \frac{1}{x_1 \cdot \log x_1 \cdot \ldots \cdot \log_q x_1} \, dx_1 \ldots dx_k,
\]

where \( a > 0 \) is chosen sufficiently large\(^1\). Indeed, for \( x \geq a \) we have

\[
|f(x)| \leq \int_a^x \frac{1}{\log x \cdot \ldots \cdot \log_q x} \left( \int_{x_1}^\infty \ldots \int_{x_k}^\infty \frac{1}{x_1} \, dx_1 \ldots dx_k \right) \, dx_k = \frac{1}{(k-1)!} \int_a^x \frac{1}{\log x \cdot \ldots \cdot \log_q x} \cdot \frac{1}{x_k} \, dx_k = \frac{1}{(k-1)!} \cdot \log_{q+1} x
\]

and of course

\[
f^{(k)}(x) = \frac{1}{x^k \cdot \log x \cdot \ldots \cdot \log_q x},
\]

hence for any \( \alpha, \beta > 1 \)

\[
\frac{x^k \cdot \log x \cdot \log_2 x \cdot \ldots \cdot (\log_q x)^\beta \cdot f^{(k)}(x)}{1 + |f(x)|^\alpha} \geq \frac{(\log_q x)^{\beta-1}}{1 + \left( \frac{1}{(k-1)!} \cdot \log_{q+1} x \right)^\alpha} \to \infty \quad (x \to \infty).
\]

So (3) does not hold, and neither does (4).

---

\(^1\)Another, related example is \( f(x) := \log_{q+1} x \). However, it is more difficult to verify that it has the desired properties than for the example given above.
Of course, the appearance of the terms $\log x \cdot \log_2 x \cdot \ldots \cdot \log_q x$ in Theorem 1, where $\log_q x$ cannot be replaced by $(\log_q x)\beta$ with $\beta > 1$, is reminiscent of the well-known fact from basic calculus that for any natural number $q$ the infinite series $\sum_{k=k_0}^{\infty} (k \log k \cdot \ldots \cdot \log_{q-1} k \cdot (\log_q k)^\beta)^{-1}$ (where $k_0$ is chosen sufficiently large) is convergent for $\beta > 1$ and divergent for $0 < \beta \leq 1$ and that a corresponding result holds for the improper integral $\int_{x_0}^{\infty} (x \cdot \log x \cdot \ldots \cdot \log_{q-1} x \cdot (\log_q x)^\beta)^{-1} \, dx$. This resemblance seems to be more than coincidence as Case 3 of the proof of (3) reveals: It makes crucial use of the divergence of $\int_{x_0}^{\infty} (x \cdot \log x \cdot \ldots \cdot \log_q x)^{-1} \, dx$.

**Proof.** Our main efforts are required to prove (3). Then (4) will be an easy consequence from (3).

We want to prove (3) by induction w.r.t. $q$. However, the start of our induction is to consider $\frac{f^{(k)}(x)}{1 + |f(x)|^\alpha}$ rather than $\frac{x^{k-1} f^{(k)}(x)}{1 + |f(x)|^\alpha}$ (which would be the case $q = 0$). So we have to introduce a unifying notation first. For given $k \geq 1$, we set

$$P_{-1}(x) := 1 \quad \text{and} \quad P_q(x) := x^k \cdot \prod_{j=1}^{q} \log_j x \quad \text{for } q \geq 0.$$ 

In particular, $P_0(x) = x^k$. Then (3) has the form

$$\liminf_{x \to \infty} \frac{P_q(x) \cdot f^{(k)}(x)}{1 + |f(x)|^\alpha} \leq 0.$$ 

First we consider the case $q = -1$. Let’s assume the assertion is wrong. Then there is an $\varepsilon > 0$ and an $a_0 \geq 0$ such that

$$f^{(k)}(x) \geq \varepsilon \cdot (1 + |f(x)|^\alpha) \quad \text{for all } x \geq a_0.$$ 

From $f^{(k)}(x) \geq \varepsilon$ for all $x \geq a_0$ one easily sees that there is some $x_1 \geq a_0$ such that $f^{(k)}(x) > 0$, $f^{(k-1)}(x) > 0$, $f(x) > 0$, $f(x) > 0$ for all $x \geq x_1$. In particular, $f$ is strictly increasing (i.e. one-to-one) on $[x_1, \infty]$ and $\lim_{x \to \infty} f(x) = \infty$. We choose a natural number $n$ such that $(\alpha - 1) \cdot n > k - 1$. Then there is a natural number $j_0$ such that $f([x_1, \infty[)$ contains the interval $[j_0^n, \infty[$. For $j \geq j_0$ we set

$$r_j := f^{-1}(j^n).$$ 

Then $(r_j)_j$ is strictly increasing and unbounded, and by the mean value theorem, applied to $\varphi(t) := t^n$, we have

$$f(r_{j+1}) - f(r_j) = (j + 1)^n - j^n \leq n \cdot (j + 1)^{n-1} \quad \text{for all } j \geq j_0.$$ 

On the other hand, for $j \geq j_0$ we deduce from the fundamental theorem of calculus

$$f(r_{j+1}) - f(r_j) = \int_{r_j}^{r_{j+1}} f'(x_1) \, dx_1 = \int_{r_j}^{r_{j+1}} \cdots$$
\[
\begin{align*}
= & \int_{r_j}^{r_{j+1}} \left( f'(r_j) + \int_{r_j}^{x_1} f''(x_2) \, dx_2 \right) \, dx_1 \\
\ge & \int_{r_j}^{r_{j+1}} \int_{r_j}^{x_1} f''(x_2) \, dx_2 \, dx_1 \\
\ge & \ldots \\
\ge & \int_{r_j}^{r_{j+1}} \int_{r_j}^{x_1} \ldots \int_{r_j}^{x_{k-1}} f^{(k)}(x_k) \, dx_k \ldots dx_1 \\
\ge & \varepsilon \cdot \int_{r_j}^{r_{j+1}} \int_{r_j}^{x_1} \ldots \int_{r_j}^{x_{k-1}} (1 + f^\alpha(x_k)) \, dx_k \ldots dx_1 \\
\ge & \varepsilon \cdot \int_{r_j}^{r_{j+1}} \int_{r_j}^{x_1} \ldots \int_{r_j}^{x_{k-1}} f^\alpha(r_j) \, dx_k \ldots dx_1 \\
= & \varepsilon \cdot j^{\alpha n} \cdot \frac{1}{k!} \cdot (r_{j+1} - r_j)^k.
\end{align*}
\]

Combining these two estimates yields

\[
n \cdot (j + 1)^{n-1} \ge \frac{\varepsilon}{k!} \cdot j^{\alpha n} \cdot (r_{j+1} - r_j)^k,
\]

hence

\[
r_{j+1} - r_j \le \left( \frac{n \cdot k!}{\varepsilon} \cdot \frac{(j + 1)^{n-1}}{j^{\alpha n}} \right)^{1/k} \le \left( \frac{n \cdot k! \cdot 2^{n-1}}{\varepsilon} \right)^{1/k} \cdot j^{-(\alpha-1) \cdot n+1}/k.
\]

Here, by our choice of \( n \), \( (\alpha-1) \cdot n+1)/k \ge 1 \), so the series \( \sum_{j=j_0}^{\infty} j^{-(\alpha-1) \cdot n+1}/k \) converges. Hence also the telescope series \( \sum_{j=j_0}^{\infty} (r_{j+1} - r_j) = \lim_{j \to \infty} r_{j+1} - r_{j_0} \) converges, contradicting \( \lim_{j \to \infty} r_j = \infty \). This proves (3) for \( q = -1 \).

Now let some \( q \ge 0 \) be given and assume that (3) is true for \( q - 1 \) instead of \( q \) and for all \( k \)-times differentiable functions \( f : [0, \infty) \to \mathbb{R} \). We assume there is a \( k \)-times differentiable function \( f : [0, \infty) \to \mathbb{R} \) and an \( \varepsilon > 0 \) such that

\[
P_q(x) \cdot f^{(k)}(x) \ge \varepsilon \cdot (1 + |f(x)|^\alpha)
\]

holds for all \( x \) large enough. Then in particular \( f^{(k)}(x) > 0 \) for all large enough \( x \), so \( f^{(k-1)} \) is increasing, and we easily see by induction that \( f^{(k-1)}, f^{(k-2)}, \ldots, f', f \) are strictly monotonic on an appropriate interval \([x_0, \infty)\) (for large enough \( x_0 \)). So the limits

\[
L_j := \lim_{x \to \infty} f^{(j)}(x) \quad (j = 0, \ldots, k - 1)
\]

exist. (They might be \( +\infty \) or \( -\infty \).)
In the following we will apply the induction hypothesis to the function 
\[ g(t) := f(e^t) \]
and will use that 
\[ g(k)(t) = f(k)(e^t) = e^{kt} \sum_{j=1}^{k-1} c_j f^{(j)}(e^t) \cdot e^{jt} \]  
(6)
for certain constants \( c_j \geq 0 \). (This is easily seen by induction.)

By the mean value theorem, for all \( n \in \mathbb{N} \) there is a \( \zeta_n \in [n, 2n] \) such that 
\[ n \cdot |f^{(k)}(\zeta_n)| = |f^{(k-1)}(2n) - f^{(k-1)}(n)|. \]  
(7)
Here of course we have \( \lim_{n \to \infty} \zeta_n = \infty \).

Now we consider several cases.

**Case 1:** \( L_{k-1} \neq 0 \).
Since \( f^{(k-1)} \) is increasing, we either have \( L_{k-1} \in \mathbb{R} \) or \( L_{k-1} = +\infty \).

**Case 1.1:** \( L_{k-1} \in \mathbb{R} \), w.l.o.g. \( L_{k-1} > 0 \).
Then we have 
\[ \frac{1}{2} \cdot L_{k-1} \leq f^{(k-1)}(x) \leq 2L_{k-1} \quad \text{for large enough } x, \]
hence 
\[ \frac{1}{3(k-1)!} \cdot L_{k-1} \cdot x^{k-1} \leq f(x) \leq \frac{3}{(k-1)!} L_{k-1} \cdot x^{k-1} \quad \text{for large enough } x. \]

Using the lower estimate, we conclude that for large enough \( x \)
\[ 0 \leq P_q(x) \cdot \frac{1}{x} \cdot \frac{1}{1 + |f(x)|^\alpha} \leq \frac{x^{(k-1)(1+\alpha)/2}}{1 + |f(x)|^\alpha} \to 0 \quad (x \to \infty). \]  
(8)
(Here it is crucial that \( 1 < \frac{1}{2} \cdot (1 + \alpha) < \alpha \).) Furthermore,
\[ 0 \leq \zeta_n \cdot |f^{(k)}(\zeta_n)| \leq 2n \cdot |f^{(k)}(\zeta_n)| = 2 \cdot |f^{(k-1)}(2n) - f^{(k-1)}(n)| \to 0 \quad (n \to \infty) \]  
(9)
since \( L_{k-1} \) is finite. Multiplying (8) and (9) gives
\[ 0 \leq P_q(\zeta_n) \cdot \frac{|f^{(k)}(\zeta_n)|}{1 + |f^{(\zeta_n)}|^\alpha} \to 0 \quad (n \to \infty). \]
This is a contradiction to (5).
Case 1.2: \( L_{k-1} = +\infty \).

Then for large enough \( x \) we have \( f^{(k-1)}(x) \geq 1, f^{(k-2)}(x) \geq 1, \ldots, f'(x) \geq 1, f(x) \geq 1 \) (and \( L_{k-2} = \ldots = L_1 = L_0 = +\infty \)). By applying the induction hypothesis to \( g \), using (6) and substituting \( t = \log x \) we obtain

\[
0 \geq \liminf_{t \to +\infty} P_{q-1}(t) \cdot \frac{|g^{(k)}(t)|}{1 + |g(t)|^\alpha} = \liminf_{t \to +\infty} \prod_{j=1}^{q-1} \log_j t \cdot t^k \cdot \frac{f^{(k)}(e^t) \cdot e^{kt} + \sum_{j=1}^{k-1} c_j f^{(j)}(e^t) \cdot e^{jt}}{1 + |f(e^t)|^\alpha} = \liminf_{x \to +\infty} \prod_{j=1}^{q} \log_j x \cdot (\log x)^k \cdot \frac{f^{(k)}(x) \cdot x^k + \sum_{j=1}^{k-1} c_j f^{(j)}(x) \cdot x^j}{1 + |f(x)|^\alpha} \geq \liminf_{x \to +\infty} \prod_{j=2}^{q} \log_j x \cdot \frac{f^{(k)}(x) \cdot x^k}{1 + |f(x)|^\alpha} = \liminf_{x \to +\infty} P_q(x) \cdot \frac{f^{(k)}(x)}{1 + |f(x)|^\alpha},
\]

as desired.

Case 2: \( L_{k-1} = \ldots = L_{m+1} = 0 \), but \( L_m \neq 0 \) for some integer \( m \geq 0, m \leq k - 2 \).

Then for \( j = k-1, k-2, \ldots, m+1 \) and all large enough \( x \) there is a \( \zeta_x \in [x, 2x] \) such that

\[
x \cdot |f^{(j)}(2x)| \leq x \cdot |f^{(j)}(\zeta_x)| = |f^{(j-1)}(2x)| - |f^{(j-1)}(x)| \leq |f^{(j-1)}(x)|; \quad (10)
\]

here we have used that \( |f^{(j-1)}| \) is decreasing (since \( f^{(j-1)} \) is monotonic and \( L_{j-1} = 0 \)) and that \( f^{(j-1)}(2x) \) and \( f^{(j-1)}(x) \) have the same sign.

By induction we obtain for all \( x \) large enough

\[
x^{k-1} \cdot |f^{(k-1)}(2^{k-1-m}x)| \leq \frac{1}{2^{(k-1-m)(k-2-m)/2}} \cdot x^m \cdot f^{(m)}(x). \quad (11)
\]

Case 2.1: \( L_m \neq \pm \infty \), i.e. \( L_m \in \mathbb{R} \).

Then for all \( x \) large enough we have

\[
|f(x)| \geq \frac{x^m}{2m!} \cdot L_m,
\]

hence

\[
0 \leq \prod_{j=1}^{q} \log_j x \cdot \frac{x^m}{1 + |f(x)|^\alpha} \leq \prod_{j=1}^{q} \log_j x \cdot \frac{x^m}{1 + \left( \frac{x^m}{2m!} \cdot L_m \right)^\alpha} \to 0 \quad (x \to \infty). \quad (12)
\]
From (7) and (11) we conclude that for all $n$ large enough
\[
  n^k \cdot |f^{(k)}(\zeta_n)| = n^{k-1}|f^{(k-1)}(2n) - f^{(k-1)}(n)| \\
  \leq n^{k-1}|f^{(k-1)}(n)| \\
  = 2^{(k-1)(k-1)} \cdot \left( \frac{n}{2^{k-1-m}} \right)^{k-1} |f^{(k-1)}(n)| \\
  \leq 2^{(k-1)(k-1)} \cdot \left( \frac{n}{2^{k-1-m}} \right)^{k-1} |f^{(m)}\left( \frac{n}{2^{k-1-m}} \right)|.
\]
If we combine this estimate with (12) and observe that $f^{(m)}$ is bounded (since $L_m \in \mathbb{R}$), we obtain (with $C_m := 2^{(k-1-m)^2+k}$)
\[
  0 \leq \prod_{j=1}^{q} \log_j \zeta_n \cdot \frac{\zeta_n^k \cdot |f^{(k)}(\zeta_n)|}{1 + |f(\zeta_n)|^\alpha} \\
  \leq \prod_{j=1}^{q} \log_j \zeta_n \cdot 2^k \cdot \frac{n^k \cdot |f^{(k)}(\zeta_n)|}{1 + |f(\zeta_n)|^\alpha} \\
  \leq C_m \cdot \prod_{j=1}^{q} \log_j \zeta_n \cdot \frac{n^m}{1 + |f(\zeta_n)|^\alpha} \cdot |f^{(m)}\left( \frac{n}{2^{k-1-m}} \right)| \\
  \leq C_m \cdot \prod_{j=1}^{q} \log_j \zeta_n \cdot \frac{\zeta_n^m}{1 + |f(\zeta_n)|^\alpha} \cdot |f^{(m)}\left( \frac{n}{2^{k-1-m}} \right)| \longrightarrow 0 \quad (n \to \infty)
\]
for all $n$ large enough. This settles Case 2.1.

**Case 2.2:** $L_m = \pm \infty$, w.l.o.g. $L_m = +\infty$.

Then for all $x$ large enough we have $f^{(m)}(x) \geq m! + 1$, $f^{(m-1)}(x) \geq m! \cdot x + 1$, $\ldots$, $f'(x) \geq m \cdot x^{m-1} + 1$ and finally
\[
  f(x) \geq x^m,
\]
hence
\[
  \prod_{j=1}^{q} \log_j x \cdot \frac{x^m}{1 + |f(x)|^\alpha} \longrightarrow 0 \quad (x \to \infty).
\]
For $j = 1, \ldots, m$, by the Mean Value Theorem we find numbers $\zeta_x \in [x, 2x]$ such that for all $x$ large enough
\[
  f^{(j-1)}(2x) = f^{(j-1)}(x) + x \cdot f^{(j)}(\zeta_x) \geq 0 + x \cdot f^{(j)}(x),
\]
and by induction we conclude that
\[
  f(2^m x) \geq x^m \cdot f^{(m)}(x),
\]
(14)
provided that \( x \) is large enough. On the other hand, \( f^{(m+1)} \) is positive and decreases to 0, so for a suitably chosen \( x_0 \geq 0 \) and all \( x \geq 2x_0 \) we obtain

\[
\begin{align*}
f^{(m)}(2^m x) &\leq f^{(m)}(x_0 + 2^m x) = f^{(m)}(x_0) + \int_{x_0}^{x_0 + 2^m x} f^{(m+1)}(t) \, dt \\
&\leq f^{(m)}(x_0) + 2^{m+1} \cdot \int_{x_0}^{x_0 + \frac{x}{2}} f^{(m+1)}(t) \, dt \\
&= 2^{m+1} \cdot f^{(m)} \left( x_0 + \frac{x}{2} \right) - (2^{m+1} - 1) \cdot f^{(m)}(x_0) \\
&\leq 2^{m+1} \cdot f^{(m)}(x) + 0.
\end{align*}
\]

Combining this with (14), we obtain for all \( x \) large enough

\[
2^{m+1} \cdot f(2^m x) \geq x^m \cdot f^{(m)}(2^m x),
\]

hence (by replacing \( 2^m x \) with \( x \))

\[
2^{m+2+m} \cdot f(x) \geq x^m \cdot f^{(m)}(x). \tag{15}
\]

If we combine this estimate with (7), (11) and (13), as in Case 2.1 we obtain

\[
0 \leq P_q(\zeta_n) \cdot \frac{|f^{(k)}(\zeta_n)|}{1 + |f(\zeta_n)|^\alpha} \\
\leq C_m \cdot \prod_{j=1}^{q} \log_j \zeta_n \cdot \frac{n^m \cdot |f^{(m)} \left( \frac{n}{2^{-k-1-n+m}} \right) |}{1 + |f(\zeta_n)|^\alpha} \\
\leq C'_m \cdot \prod_{j=1}^{q} \log_j \zeta_n \cdot |f(\zeta_n)|^{-\alpha} \\
\leq C'_m \cdot \prod_{j=1}^{q} \log_j \zeta_n \cdot \zeta_n^{m(1-\alpha)} \to 0 \quad (n \to \infty),
\]

where \( C'_m \) is an appropriate constant. This settles this case as well.

**Case 3:** \( L_{k-1} = \ldots = L_0 = 0 \)

In this case, (11) holds as well (with \( m = 1 \)), i.e.

\[
|f'(x)| \geq x^{k-2} \cdot |f^{(k-1)} \left( 2^{k-2} x \right) |
\]

for all \( x \) large enough. Now we use

\[
|f^{(k)}(x)| \geq \frac{\varepsilon}{x^k \prod_{j=1}^{q} \log_j x}.
\]


(which is valid for all large enough \( x \)) and once more the Mean Value Theorem to obtain for all large enough \( x \)

\[
|f'(x)| \geq x^{k-2} \cdot |f^{(k-1)}(2^{k-2}x) - f^{(k-1)}(2^{k-1}x)|
= 2^{k-2} \cdot x^{k-1} \cdot |f^{(k)}(\xi_x)|
\geq \frac{2^{k-2} \cdot x^{k-1} \cdot \varepsilon}{\xi_x \cdot \prod_{j=1}^{q} \log_j \xi_x}
\geq \frac{2^{k-2} \cdot x^{k-1} \cdot \varepsilon}{(2^{k-1}x)^k \cdot \prod_{j=1}^{q} \log_j (2^{k-1}x)}
\geq c \cdot \frac{1}{x \cdot \prod_{j=1}^{q} \log_j x}
\]

with a suitable constant \( c > 0 \), hence by integration

\[
|f(x)| \geq c \cdot \log_{q+1} x + d \to \infty \quad (x \to \infty),
\]

(with some \( d > 0 \)), since \( \frac{d}{dx} \log_{q+1} x = \frac{1}{x \prod_{j=1}^{q} \log_j x} \). This contradicts \( L_0 = 0 \), i.e. this case cannot occur\(^2\).

This completes the proof of \( \text{(3)} \).

Now \( \text{(4)} \) is an easy consequence from \( \text{(3)} \) and from Darboux’ intermediate value theorem for derivatives. Indeed, if there is an \( x_0 \) such that \( f^{(k)}(x) \geq 0 \) for all \( x \geq x_0 \) or \( f^{(k)}(x) \leq 0 \) for all \( x \geq x_0 \), \( \text{(4)} \) follows immediately from \( \text{(3)} \), applied to either \( f \) or \( -f \). Otherwise, by Darboux’s theorem there is a sequence \( \{x_n\}_n \) tending to \( \infty \) such that \( f^{(k)}(x_n) = 0 \) for all \( n \), and \( \text{(4)} \) holds as well.

In view of Theorem \( \text{(1)} \) and the fact that the exponential function grows larger than every polynomial, the following fact certainly doesn’t come as a big surprise:

For every continuously differentiable function \( g : [a, \infty) \to \mathbb{R} \) we have

\[
\liminf_{x \to \infty} \frac{g'(x)}{e^{g(x)}} \leq 0.
\]  

Indeed, otherwise there would be an \( \varepsilon > 0 \) and an \( x_0 \geq a \) such that \( g'(x) \geq \varepsilon \cdot e^{g(x)} \) for all \( x \geq x_0 \). In particular, \( g' \) is positive on \([x_0, \infty)\), so \( g \) is increasing there, hence \( g'(x) \geq \varepsilon \cdot e^{g(x_0)} \) for all \( x \geq x_0 \), which implies \( \lim_{x \to \infty} g(x) = \infty \). This enables us to conclude that \( \frac{g(x)^2}{e^{g(x)}} \to \infty \) for \( x \to \infty \). Combining this with the fact that \( \liminf_{x \to \infty} \frac{g'(x)}{1+g(x)^2} \leq 0 \) by Theorem \( \text{(1)} \) gives the assertion.

\(^2\)In fact, Case 3 is the only part of the proof where it is crucial that in the assertion only the factors \( \log x \) and not \( (\log_j x)^{\beta} \) with \( \beta > 1 \) occur. It would not work with \( \beta > 1 \) since the improper integral

\[
\int_{x_0}^{\infty} \frac{1}{x \log x \cdots \log_{q+1} x} \, dx
\]

(with \( x_0 \) large enough) converges.
However, it might be a bit surprising that this no longer holds if $g'$ is replaced by higher derivatives of $g$, i.e. for $k \geq 2$ in general the estimate $\liminf_{x \to \infty} \frac{g^{(k)}(x)}{e^{g(x)}} \leq 0$ does not hold. This is demonstrated by the function $g(x) := -x^{k-3/2}$ which satisfies

$$
\frac{g^{(k)}(x)}{e^{g(x)}} = C \cdot \frac{x^{-3/2}}{\exp(-x^{k-3/2})} \to \infty \quad \text{for } x \to \infty
$$

with some $C > 0$.

On the other hand, for every $k$ times continuously differentiable function $g : [a, \infty) \to \mathbb{R}$ ($k \geq 1$) we have

$$
\liminf_{x \to \infty} \frac{g^{(k)}(x)}{1 + e^{g(x)}} \leq 0 \quad \text{and} \quad \liminf_{x \to \infty} \frac{g^{(k)}(x)}{e^{g(x)}} \leq 0.
$$

Both inequalities are proved by a similar reasoning as in the proof of (16), applying Theorem with (for example) $\alpha = 2$ and keeping in mind that $g^{(k)}(x) \geq \varepsilon$ for all $x \geq x_0$ would imply $g(x) \to \infty$ for $x \to \infty$ resp. that $x \mapsto \frac{e^{g(x)}}{1 + g(x)^2}$ is bounded away from zero.

References

[1] Bar, R.; Grahl, J.; Nevo, S.: Differential inequalities and quasi-normal families, Anal. Math. Phys. 4 (2014), 63-71

[2] Chen, Q.; Nevo, S.; Pang, X.-C.: A general differential inequality of the kth derivative that leads to normality, Ann. Acad. Sci. Fenn. 38 (2013), 691-695

[3] Grahl, J.; Nevo, S.: Spherical derivatives and normal families, J. Anal. Math. 117 (2012), 119-128

[4] Grahl, J.; Nevo, S.: An extension of one direction in Marty’s normality criterion, Monatsh. Math. 174 (2014), 205-217

[5] Grahl, J.; Nevo, S.; Pang, X.-C.: A non-explicit counterexample to a problem of quasi-normality, J. Math. Anal. Appl. 406 (2013), 386-391

[6] Liu, X.J., Nevo, S. and Pang, X.C.: Differential inequalities, normality and quasi-normality, Acta Math. Sin. (Engl. Ser.) 30 (2014), 277-282

Jürgen Grahl
University of Würzburg
Department of Mathematics
Würzburg
Germany
e-mail: grahl@mathematik.uni-wuerzburg.de

Shahar Nevo
Bar-Ilan University
Department of Mathematics
Ramat-Gan 52900
Israel
e-mail: nevosh@math.biu.ac.il