Topics on heavy baryon chiral perturbation theory in the large $N_c$ limit

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Abstract. We compute nonanalytical pion-loop corrections to baryon masses in a combined expansion in chiral symmetry breaking and $1/N_c$, where $N_c$ is the number of colors. Specifically, we compute flavor-$27$ baryon mass splittings at leading order in chiral perturbation theory. Our results, at the physical value $N_c = 3$, are compared with the expressions obtained in heavy baryon chiral perturbation theory with no $1/N_c$ expansion.

INTRODUCTION

Chiral perturbation theory has been a useful tool in the understanding of low-energy QCD hadron dynamics. Its application to baryons through a new formulation of the low-energy chiral effective Lagrangian in which the baryons appear as heavy static fields, was first introduced in Refs. [1, 2]. The chiral Lagrangian thus obtained was used to compute the leading nonanalytic in $m_s$ corrections to baryon axial currents [1, 2, 3], masses, and non-leptonic decays [1, 2], to name but a few.

Similarly, the $1/N_c$ expansion has proved to be useful in the analysis of the spin-flavor structure of baryons in QCD [4, 5, 6, 7, 8]. Evidence for the predictions of the $1/N_c$ expansion for baryons has been found in the analysis of masses [7, 8, 9], magnetic moments [7, 8, 10], and axial and vector currents [8, 11, 12].

The next natural step has been to combine both approaches so that baryon matrix elements are obtained in a simultaneous chiral and $1/N_c$ expansion [7, 5, 13, 14, 15]. The resulting approach, referred to as large-$N_c$ chiral perturbation theory, has proven to have a significant predictive power in the analysis of the baryon mass spectra [13, 15, 16], magnetic moments [10] and axial current [16].

Here we present an explicit calculation of flavor-$27$ mass splittings of the octet and decuplet, which are calculable and nonanalytic in the quark masses and baryon hyperfine mass splittings at leading order in chiral perturbation theory. This analysis illustrates how to implement the procedure.

BARYONS IN CHIRAL PERTURBATION THEORY

The heavy baryon chiral Lagrangian can be constructed in terms of the pion field $\Pi$, the baryon octet field $B_v$, and the baryon decuplet field $T^\mu_{abc}$, where the $\Pi$ and $B_v$ fields are...
I\ncontribution can be written as SU(3) multiplets of baryons.\nThe coupling of the pseudoscalar pion field with the baryon matter fields occurs through the vector and axial vector combinations $V^\mu = (1/2) (\xi \partial^\mu \xi^\dagger + \xi^\dagger \partial^\mu \xi)$ and $A^\mu = (i/2) (\xi \partial^\mu \xi^\dagger - \xi^\dagger \partial^\mu \xi)$, where $\xi = e^{\Pi f}/\Sigma$, $\Sigma = \xi^2 = e^{2\Pi f}/f$, and $f \approx 93$ MeV is the pion decay constant. Further considerations can be found in Refs. [1, 2].

The most general Lagrangian at lowest order is

$$\mathcal{L}_{\text{baryon}} = i \text{Tr} \bar{B}_v (v \cdot D) B_v - i \bar{T}^\mu_v (v \cdot D) T_{\nu \mu} + \Delta \bar{T}^\mu_v T_{\nu \mu} + 2 D \text{Tr} \bar{B}_v S^\mu_v \{A_\mu, B_v\} + 2 F \text{Tr} \bar{B}_v S^\mu_v \{A_\mu, B_v\} + C (\bar{T}^\mu_v A_\mu B_v + \bar{B}_v A_\mu T^{\nu}_{\dagger}) + 2 \mathcal{H} \bar{T}^\mu_v S^\nu_v A_\nu T_{\nu \mu},$$

where $D, F, C,$ and $\mathcal{H}$ are the baryon-pion couplings and $\Delta = m_T - m_B$ is the decuplet-octet mass difference. $\mathcal{L}_{\text{baryon}}$ describes massless pion fields interacting with degenerate SU(3) multiplets of baryons.

Three nonanalytic terms for the baryon masses are calculable, namely, the ones which vary as $m_3^3/2$, $m_8^2 \ln m_s$, and $(\Delta) m_i \ln m_s$. These contributions result from one-loop diagrams and have been computed in Ref. [17]. Here we analyze the leading nonanalytic contributions arising from the Feynman diagram displayed in Fig. 1. This diagram involves $\pi, K,$ and $\eta$ emission and reabsorption. The most general form of this contribution can be written as

$$-\delta M_i = I_1(\pi, K, \eta; \Delta) m_{i,1} + I_8(\pi, K, \eta; \Delta) m_{i,8} + I_{27}(\pi, K, \eta; \Delta) m_{i,27}. \quad (2)$$

$I_1(\pi, K, \eta; \Delta)$ are flavor lineal combinations of the integral over the loop, $F(m_{\Pi}, \Delta)$ [14]. Here $m_{\Pi}$ is the pion mass and $\Pi = \pi, K, \eta$. Specifically,

$$I_1(\pi, K, \eta; \Delta) = \frac{1}{8} [3F(\pi, \Delta) + 4F(K, \Delta) + F(\eta, \Delta)], \quad (3)$$

$$I_8(\pi, K, \eta; \Delta) = \frac{2\sqrt{3}}{5} \left(\frac{3}{2} F(\pi, \Delta) - F(K, \Delta) - \frac{1}{2} F(\eta, \Delta)\right), \quad (4)$$

$$I_{27}(\pi, K, \eta; \Delta) = \frac{1}{3} F(\pi, \Delta) - \frac{4}{3} F(K, \Delta) + F(\eta, \Delta) \quad (5)$$

with

$$m_{i,1} = \lambda_{i}^\pi + \lambda_{i}^K + \lambda_{i}^\eta, \quad m_{i,8} = \frac{1}{\sqrt{3}} (\lambda_{i}^\pi - \frac{1}{2} \lambda_{i}^K - \lambda_{i}^\eta)$$

$$m_{i,27} = \frac{3}{40} (3 \lambda_{i}^\pi - 2 \lambda_{i}^K + 9 \lambda_{i}^\eta). \quad (6)$$

FIGURE 1.
The $\lambda_i^{II}$ coefficients for the octet baryons read

\[
\begin{align*}
\tilde{\lambda}_N^\pi &= \frac{9}{4}(F + D)^2 + 2C^2, \\
\tilde{\lambda}_N^K &= \frac{1}{2}(9F^2 - 6FD + 5D^2 + C^2), \\
\tilde{\lambda}_N^\eta &= \frac{1}{4}(3F - D)^2, \\
\tilde{\lambda}_S^\pi &= \frac{9}{4}F^2 + \frac{1}{2}C^2, \\
\tilde{\lambda}_S^K &= \frac{3}{2}(F^2 + D^2) + \frac{5}{3}C^2, \\
\tilde{\lambda}_S^\eta &= D^2 + \frac{1}{2}C^2, \\
\tilde{\lambda}_A^\pi &= 3D^2 + \frac{3}{2}C^2, \\
\tilde{\lambda}_A^K &= 9F^2 + D^2 + C^2, \\
\tilde{\lambda}_A^\eta &= D^2.
\end{align*}
\]

(7)

whereas for the decuplet baryons one has

\[
\begin{align*}
\tilde{\lambda}_D^\Delta &= \frac{25}{36}\mathcal{H}^2 + \frac{1}{2}C^2, \\
\tilde{\lambda}_D^K &= \frac{5}{18}\mathcal{H}^2 + \frac{1}{2}C^2, \\
\tilde{\lambda}_D^\eta &= \frac{5}{36}\mathcal{H}^2, \\
\tilde{\lambda}_S^\Delta &= \frac{10}{27}\mathcal{H}^2 + \frac{5}{12}C^2, \\
\tilde{\lambda}_S^K &= \frac{5}{9}\mathcal{H}^2 + C^2, \\
\tilde{\lambda}_S^\eta &= \frac{1}{4}C^2, \\
\tilde{\lambda}_\Omega^\Delta &= \frac{5}{9}\mathcal{H}^2 + C^2.
\end{align*}
\]

(8)

Equation (2) can be used to analyze the two-flavor-27 combinations of baryon masses, namely, the Gell-Mann–Okubo combination for octet baryons and the equal spacing rule combination for decuplet baryons, which are respectively

\[
\begin{align*}
\frac{3}{4}\Lambda + \frac{1}{4}\Sigma - \frac{1}{2}(N + \Xi), \\
-\frac{4}{7}\Delta + \frac{5}{7}\Sigma^* + \frac{2}{7}\Xi^* - \frac{3}{7}\Omega,
\end{align*}
\]

(9)

where particle labels denote the corresponding masses. These relations are a direct consequence of the fact that SU(3) symmetry breaking is purely octet [1, 2].

The terms proportional to $m_{i,1}$ and $m_{i,8}$ in Eq. (2) have no effect on the above mass relations. The 27 piece produces the corrections

\[
\begin{align*}
\left[ -\frac{3}{4}(D^2 - 3F^2)\tilde{I}(0) + \frac{1}{8}C^2\tilde{I}(\Delta) \right], \\
\left[ \frac{5}{18}\mathcal{H}^2\tilde{I}(0) - \frac{1}{4}C^2\tilde{I}(-\Delta) \right]
\end{align*}
\]

(10)

to the Gell-Mann–Okubo mass relation and to the equal spacing rule, respectively, where $\tilde{I}(\Delta)$ is an abbreviation for $I_{27}(\pi, K, \eta; \Delta)/N_c$ [14]. The integral $I_{27}(\pi, K, \eta; \Delta)$, which is around 4 MeV, is highly suppressed relative to its singlet and octet counterparts. It explains, however, the small violation to the Gell-Mann–Okubo mass relation.
BARYONS IN LARGE $N_C$ CHIRAL PERTURBATION THEORY

The derivation of the $1/N_C$ baryon chiral Lagrangian is given in Ref. [14]. Recent developments on the large-$N_C$ formalism applied to baryons can be found in excellent reviews [18, 19]. One can also refer to Jenkins’ contribution to these proceedings.

The $1/N_C$ baryon chiral Lagrangian can be expressed as [14]

$$L_{\text{baryon}} = iD^0 - M_{\text{hyperfine}} + \text{Tr} (\lambda a) A^{ia} + \text{Tr} \left( \frac{2I}{\sqrt{6}} \right) A^i + \ldots$$  \hspace{1cm} (11)$$

Each term in Eq. (11) involves a baryon operator which can be given in terms of polynomials in the spin-flavor generators $J^i$, $T^a$, and $G^{ia}$ [8]. For instance, at the physical value $N_c = 3$, the baryon axial current $A^{ia}$ is expressed as

$$A^{ia} = a_1 G^{ia} + b_2 \frac{1}{N_c} J^i T^a + b_3 \frac{1}{N_c^2} D_3^{ia} + c_3 \frac{1}{N_c^2} O_3^{ia},$$  \hspace{1cm} (12)$$

where the operators $D_3^{ia}$ and $O_3^{ia}$ are defined in [8].

We now compute again the leading nonanalytic corrections to the baryon masses but now within the combined formalism in $1/N_C$ and chiral corrections. The computation is complicated by the presence of the hyperfine and quark mass splittings. In the chiral limit $m_i \to 0$ the baryon propagator is diagonal in spin and can be written as [14]

$$\frac{iP_j}{k^0 - \Delta_j},$$  \hspace{1cm} (13)$$

where $P_j$ is a spin projection operator for spin $J = j$. For $N_c = 3$ one has [14]

$$\begin{align*}
P_{\frac{j}{2}} &= -\frac{1}{3} \left( J^2 - \frac{15}{4} \right), & P_{\frac{3}{2}} &= \frac{1}{3} \left( J^2 - \frac{3}{4} \right). \hspace{1cm} (14)\end{align*}$$

On the other hand, $\Delta_j$ in Eq. (13) stands for the difference of the hyperfine mass splitting for spin $J = j$ and the external baryon.

The diagram in Fig. 1, given by the product of a baryon operator times the pion flavor tensor, can be expressed as

$$\frac{1}{N_c} \sum_j \left( A^{ia} P_j A^{ib} \right) \Pi^{ab}(\Delta_j),$$  \hspace{1cm} (15)$$

where $\Pi^{ab}$ is the symmetric tensor defined as

$$\begin{align*}
\Pi^{ab}(\Delta) &= I_1(\pi, K, \eta; \Delta) \delta^{ab} + I_8(\pi, K, \eta; \Delta) d^{ab8} \\
&+ I_{27}(\pi, K, \eta; \Delta) \left( \delta^{a8} \delta^{b8} - \frac{1}{8} \delta^{ab} - \frac{3}{5} d^{ab8} d^{888} \right). \hspace{1cm} (16)\end{align*}$$

For $N_c = 3$, Eq. (15) reduces to

$$\frac{1}{N_c} \left[ A^{ia} P_{\frac{j}{2}} A^{ib} \Pi^{ab}(\Delta_{\frac{j}{2}}) + A^{ia} P_{\frac{3}{2}} A^{ib} \Pi^{ab}(\Delta_{\frac{3}{2}}) \right].$$  \hspace{1cm} (17)$$
The evaluation of Eq. (17) involves the computation of the baryon operators $A^i a A^b \Pi^{ab}$ and $A^i J^8 A^b \Pi^{ab}$. One can follow the approach implemented by Jenkins [14] to perform the operator reduction of the spin operators involved in the latter expressions by using spin projection operators. However, we will use a simplified version of such analysis. After a long but otherwise standard calculation we find

$$A^i a A^i = \frac{3}{16} N_c (N_c + 6) a_1^2 + \frac{3}{4} \left(1 + \frac{6}{N_c} \right) a_1 c_3 + \left[ - \frac{5}{12} a_1^2 + \frac{2}{3} \left(1 + \frac{3}{N_c} \right) a_1 b_2 \\
+ \left( \frac{1}{2} + \frac{3}{N_c} + \frac{4}{N_c^2} \right) a_1 b_3 + \frac{1}{12} \left(1 + \frac{6}{N_c} \right) b_2^2 + \left( \frac{1}{2} + \frac{3}{N_c} - \frac{9}{N_c^2} \right) a_1 c_3 \right] J^2 \\
+ \frac{1}{N_c^2} \left[ \frac{2}{3} a_1 b_3 + b_2^2 - 2 a_1 c_3 + \frac{8}{3} \left(1 + \frac{3}{N_c} \right) b_2 b_3 \right] J^4 + O \left( \frac{1}{N_c^4} \right), \quad (18)$$

$$d^{iab} A^i A^b = \left[ \frac{3}{8} (N_c + 3) a_1^2 + \frac{3}{2N_c} \left(1 + \frac{3}{N_c} \right) a_1 c_3 \right] T^8 + \left[ - \frac{7}{12} a_1^2 + \frac{3}{N_c} a_1 b_3 \\
+ \frac{1}{6} \left(1 + \frac{3}{N_c} \right) a_1 b_2 - \frac{9}{2N_c^2} a_1 c_3 \right] \{ J^r , G^{i8} \} + \frac{1}{N_c} \left[ \frac{1}{6} a_1 b_2 \\
+ \frac{1}{2} \left(1 + \frac{3}{N_c} \right) \left( a_1 b_3 - \frac{1}{6} b_2^2 + a_1 c_3 \right) \right] \{ J^2 , T^8 \} + \frac{1}{N_c} \left[- \frac{1}{3} a_1 b_3 + \frac{1}{2} b_2^2 - a_1 c_3 \\
+ \left( \frac{1}{3} + \frac{1}{N_c} \right) b_2 b_3 \right] \{ J^2 , \{ J^r , G^{i8} \} \} + \frac{2}{3N_c^2} b_2 b_3 \{ J^4 , T^8 \} + O \left( \frac{1}{N_c^4} \right), \quad (19)$$

$$A^8 A^i = \frac{1}{2} a_1^2 \{ G^{i8} , G^{i8} \} + \frac{1}{2N_c} a_1 b_2 \{ T^8 , \{ J^r , G^{i8} \} \} + \frac{1}{N_c} a_1 b_3 \{ \{ J^r , G^{i8} \} , \{ J^r , G^{i8} \} \} + \frac{1}{N_c} a_1 c_3 \left[ \{ J^2 , \{ G^{i8} , G^{i8} \} \} + \left[ G^{i8} [ J^2 , G^{i8} ] \right] - \frac{1}{2} \{ \{ J^r , G^{i8} \} , \{ J^r , G^{i8} \} \} \right] \\
+ \frac{1}{4N_c^2} b_2^2 \{ J^2 , \{ T^8 , T^8 \} \} + \frac{1}{N_c} b_2 b_3 \{ J^2 , \{ T^8 , \{ J^r , G^{i8} \} \} \} + O \left( \frac{1}{N_c^4} \right). \quad (20)$$

Explicit expression for the $T^8$ and $G^{i8}$ operators can be found in Ref. [8]. Similarly,

$$A^i J^2 A^i = \frac{3}{8} N_c (N_c + 6) a_1^2 + \left( \frac{3}{2} + \frac{9}{N_c} \right) a_1 c_3 + \left[ \frac{3}{16} N_c (N_c + 6) - \frac{7}{2} \right] a_1^2 \\
+ \left( \frac{13}{4} \left(1 + \frac{6}{N_c} \right) - \frac{18}{N_c^2} \right) a_1 c_3 \right] J^2 + \left[ - \frac{5}{12} a_1^2 + \frac{2}{3} \left(1 + \frac{3}{N_c} \right) a_1 b_2 \\
+ \left( \frac{1}{2} + \frac{1}{N_c} \right) \left(1 + \frac{4}{N_c} \right) a_1 b_3 + \frac{1}{12} \left(1 + \frac{6}{N_c} \right) b_2^2 + \left( \frac{1}{2} + \frac{3}{N_c} - \frac{27}{N_c^2} \right) a_1 c_3 \right] J^4 \\
+ \frac{1}{N_c^2} \left[ \frac{2}{3} a_1 b_3 + b_2^2 - 2 a_1 c_3 + \frac{8}{3} \left(1 + \frac{3}{N_c} \right) b_2 b_3 \right] J^6 + O \left( \frac{1}{N_c^4} \right), \quad (21)$$

$$d^{iab} A^i J^2 A^b = \left[ \frac{3}{4} (N_c + 3) a_1^2 + \frac{3}{N_c} \left(1 + \frac{3}{N_c} \right) a_1 c_3 \right] T^8 - \left[ 2 a_1^2 + \frac{9}{N_c} a_1 c_3 \right] \{ J^r , G^{i8} \}.$$
\[ A^{i8} J^2 A^{i8} = \frac{1}{4} a_1^2 \left( \{ J^2, \{ G^{i8}, G^{i8} \} \} + 2[ G^{i8}, [ J^2, G^{i8} ] ] \right) + \frac{1}{4N_c} a_1 b_3 \left( \{ J^2, \{ J', G^{i8} \}, \{ J', G^{i8} \} \} + \frac{1}{8N_c^2} b_3^2 \{ J^2, \{ T^8, T^8 \} \} \right) + \frac{1}{2N_c^2} b_3 a_1 c_3 \left( \{ J^2, \{ J^2, \{ G^{i8}, G^{i8} \} \} \} + \{ J^2, [ G^{i8}, [ J^2, G^{i8} ] ] \} + 2[ G^{i8}, [ J^2, \{ J^2, G^{i8} \} ] \right) - \frac{1}{2} \{ J^2, \{ J', G^{i8} \}, \{ J', G^{i8} \} \} \right) + \frac{1}{2N_c^3} b_3^2 b_3 \left( \{ J^2, \{ J^2, \{ T^8, \{ J', G^{i8} \} \} \} \right) . \]

The full evaluation of the baryon operators leads to
\[ \frac{3}{4} \Delta + \frac{1}{4} \Sigma - \frac{1}{2} (N + \Xi) = \frac{1}{N_c} \left[ \left( \frac{1}{16} a_1^2 + \frac{3}{4} a_1 b_2 + \frac{9}{16} a_2 b_2 + \frac{3}{8} a_1 b_3 + \frac{9}{4} b_2 b_3 \right) I(0) + \left( \frac{1}{8} a_1^2 + \frac{9}{8} a_1 c_3 \right) I(\Delta) + O \left( \frac{1}{N_c^4} \right) \right] , \]

for the Gell-Mann Okubo mass formula and
\[ -\frac{4}{7} \Delta + \frac{5}{7} \Sigma^* + \frac{2}{7} \Xi^* - \frac{3}{7} \Omega = \frac{1}{N_c} \left[ \left( \frac{5}{8} a_1^2 + \frac{15}{4} a_1 b_2 + \frac{45}{8} a_2 b_2 + \frac{75}{4} a_1 b_3 + \frac{225}{4} b_2 b_3 \right) I(0) - \left( \frac{1}{4} a_1^2 + \frac{9}{4} a_1 c_3 \right) I(-\Delta) + O \left( \frac{1}{N_c^4} \right) \right] , \]

for the equal spacing rule. The above equations, at the physical value \( N_c = 3 \), can be straightforwardly compared with the analogous expressions obtained in heavy baryon chiral perturbation theory Eqs. (10) by using the identifications
\[
D = \frac{1}{2} a_1 + \frac{1}{6} b_3 , \quad C = -a_1 - \frac{1}{2} c_3 , \quad F = \frac{1}{3} a_1 + \frac{1}{6} b_2 + \frac{1}{9} b_3 , \quad \mathcal{H} = -\frac{3}{2} a_1 - \frac{3}{2} b_2 - \frac{5}{2} b_3 ,
\]

Both expressions agree.
CONCLUSIONS

We have exemplified how to compute corrections to baryon masses in a calculational scheme that simultaneously exhibits both the $m_q$ and $1/N_c$ expansions. Flavor-27 baryon mass splittings at leading order in chiral perturbation theory have been computed in detail to illustrate the procedure. Furthermore, our results for three colors have been compared with the ones obtained in heavy baryon chiral perturbation theory with no $1/N_c$ corrections and an agreement has been found term by term in the series.

Some other applications of the combined approach to baryon properties will be presented elsewhere.

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REFERENCES

1. Jenkins, E., and Manohar, A.V., Phys. Lett. B 255, 558-62 (1991).
2. Jenkins, E., and Manohar, A.V., Phys. Lett. B 259, 353-58 (1991).
3. Borasoy, B., Phys. Rev. D 59, 054021 (1999).
4. Carone, C.D., Georgi, H., and Osofsky, S.T., Phys. Lett. B 322, 227-32 (1994).
5. Luty, M.A., and March-Russell, J., Nucl. Phys. B 426, 71-93 (1994).
6. Dashen, R., and Manohar, A.V., Phys. Lett. B 315, 425-30 (1993); Phys. Lett. B 315, 438-40 (1993).
7. Dashen, R.F., Jenkins, E., and Manohar, A.V., Phys. Rev. D 49, 4713-38 (1994).
8. Dashen, R.F., Jenkins, E., and Manohar, A.V., Phys. Rev. D 51, 3697-3727 (1995).
9. Jenkins E., and Lebed, R., Phys. Rev. D 52, 282-94 (1995).
10. Luty, M.A., March-Russell, J., and White, M., Phys. Rev. D 51, 2332 (1995).
11. Dai, J., Dashen, R.F., Jenkins, E., and Manohar, A.V., Phys. Rev. D 53, 273 (1996).
12. Flores-Mendieta, R., Jenkins, E., and Manohar, A.V., Phys. Rev. D 58, 094028 (1998).
13. Luty, M.A., Phys. Rev. D 51, 2322-31 (1995).
14. Jenkins, E., Phys. Rev. D 53, 2625-44 (1996).
15. Bedaque, P.F., and Luty, M.A., Phys. Rev D 54, 2317-27 (1996).
16. Oh, Y., and Weise, W., Eur. Phys. J. A4 363-80 (1999).
17. Jenkins, E., Nucl. Phys. B 368, 190-203 (1992).
18. Manohar, A.V., “Large N QCD,” in Les Houches Session LXVIII, Probing the Standard Model of Particle Interactions, edited by F. David and R. Gupta, Amsterdam, Elsevier, 1998.
19. Jenkins, E., Annu. Rev. Nucl. Part. Sci. 48, 81-119 (1998).