**SO(N)_2 BRAID GROUP REPRESENTATIONS ARE GAUSSIAN**

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**Abstract.** We give a description of the centralizer algebras for tensor powers of spin objects in the pre-modular categories $SO(N)_2$ (for $N$ odd) and $O(N)_2$ (for $N$ even) in terms of quantum $(n-1)$-tori, via non-standard deformations of $U\mathfrak{so}_N$. As a consequence we show that the corresponding braid group representations are Gaussian representations, the images of which are finite groups. This verifies special cases of a conjecture that braid group representations coming from weakly integral braided fusion categories have finite image.

**1. Introduction**

The centralizer algebras for tensor powers $V^\otimes n$ of the $N$-dimensional representation $V$ of $U_q\mathfrak{sl}_N$, $U_q\mathfrak{so}_N$ and $U_q\mathfrak{sp}_N$ all have descriptions in terms of quotients of braid group algebras, namely Hecke and BMW-algebras ([16], [34]). By specializing $q$ to be a root of unity, these descriptions made it possible to analyze the unitary braid group representations associated with the corresponding objects in the modular categories $SU(N)_k$, $SO(N)_k$ and $Sp(N)_k$ ([5],[29]). These analyses provided evidence for the following conjecture (see [25, 27]):

**Conjecture 1.1.** Let $\mathcal{C}$ be a braided fusion category and let $X$ be a simple object in $\mathcal{C}$. The braid group representations $B_n$ on $\text{End}_\mathcal{C}(X^\otimes n)$ have finite image if and only if $\text{FPdim}(X)^2 \in \mathbb{Z}$.

Here $\text{FPdim}(X) \in \mathbb{R}$ is the Frobenius-Perron dimension, which coincides with the categorical dimension for unitary fusion categories. Categories with $\text{FPdim}(X)^2 \in \mathbb{Z}$ for all simple objects $X$ are called weakly integral. The "only if" part of the conjecture has been confirmed for a set of generating objects in the categories associated with quantum groups at roots of unity (see [28]), and the "if" part for all such quantum group categories except the infinite family $SO(N)_2$.

In $SO(N)_2$ the spinor objects $S$ for $N$ odd resp. $S_\pm$ for $N$ even have dimensions $\sqrt{N}$ resp. $\sqrt{N}/2$ so that the braid group representations on the centralizer algebras of $S^\otimes n$ and $S_\pm^\otimes n$ conjecturally have finite image. This can be verified for $N \leq 8$ using low rank coincidences and when $\sqrt{N}$ or $\sqrt{N}/2$ are integral (see [25]). However, for other values of $N$ no description in terms of braid group algebras is known. Indeed, the braid group image does not in general generate these centralizer algebras.

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We overcome these difficulties by the following approach: For ease of notation, we will denote by $S$ also the module $S_+ \oplus S_-$, which is irreducible over a certain semidirect product $U_q so_N \rtimes \mathbb{Z}_2$ for $N$ even. Then the centralizer algebras for $S_{\otimes n}$ are described in terms of a non-standard deformation $U'_q so_n$ of $U so_n$, for generic parameter $q$ in [35, Theorem 4.8], for both $N$ odd and even. On the other hand, we find in this paper that the algebra $U'_q so_n$ admits a homomorphism into the quantum $(n-1)$-torus $T_q(n)$, which contains an isomorphic copy of $\text{End}(S_{\otimes n})$ for $q$ a $2N$th root of unity. Using the representation theory of $U'_q so_n$, we can show that the braid representations in connection with the fusion category $SO(N)_2$ are equivalent to so-called Gaussian braid representations (see, e.g. [10]) which live in the quantum torus. The latter can easily be shown to have finite images, which implies the conjecture for $SO(N)_2$.

Here is a more detailed outline of the contents of this article. In Section 2 we review results about the centralizer algebras $\text{End}(S_{\otimes n})$ where $S$ is a spinor representation of $U_q so_N$ resp. $U_q so_N \rtimes \mathbb{Z}_2$, or the corresponding object in one of the associated fusion categories. Most of these results have already more or less appeared before in [11], [35]. In Section 3, we reprove and extend several results by Klimyk and his coauthors concerning the representation theory of $U'_q so_n$. In contrast with loc. cit., we use a Verma module approach which also has the advantage of proving (crucial, for our paper) uniqueness results at roots of unity, for certain types of modules. In Section 4 we construct representations of $U'_q so_n$ into algebras called quantum tori. The main result of this section is the identification of these representations with those in $\text{End}(S_{\otimes n})$ for fusion categories $SO(N)_2$ ($N$ odd) and $O(N)_2$ ($N$ even). This allows us to describe the corresponding tower of centralizer algebras in terms of the quantum tori using Jones’ basic construction. Finally, we identify the braid group representations corresponding to the object $S$ in $SO(N)_2$ (resp. $O(N)_2$) for $N$ odd (resp. $N$ even) with the so-called Gaussian braid representations found in the work of Jones and Goldschmidt [10],[18] for $N$ odd, and the easy generalization to $N$ even (worked out in [6]). From this one easily verifies Conj. 1.1 in our case.

2. Duality for spinor representations

2.1. Deformations of $U so_n$. The algebra $U'_q so_n$ is defined (see [8] via generators $B_1, \ldots, B_{n-1}$ satisfying the relations $B_i B_j = B_j B_i$ for $|i - j| \neq 1$ and the $q$-Serre relations:

$$B_i^2 B_{i \pm 1} - (q + q^{-1}) B_i B_{i \pm 1} B_i + B_{i \pm 1}^2 B_i = B_{i \pm 1}. \tag{2.1}$$

It is well-known that in the classical case $q = 1$ we obtain a presentation of the universal enveloping algebra $U so_n$ of the orthogonal Lie algebra $so_n$, and $U'_q so_n$ is sometimes called the non-standard deformation of $U so_n$. It follows from the definitions that the elements $B_1, B_3, \ldots, B_{n-1}$ resp. $B_{n-2}$, depending on whether $n$ is even or odd, generate an abelian subalgebra $A$ of $U'_q so_n$. We define a weight vector of a $U'_q so_n$-module $V$ to be a common eigenvector of the generators of $A$. We call a weight regular if all the eigenvalues of generators $B_{2i-1}$ of $A$ are of the form $[r]$ with $r$ an integer or a half integer, and $[r] = (q^r - q^{-r})/(q - q^{-1})$ the usual $q$-number.
In the following we denote by \( \mathbf{U} \) the semidirect product of the (standard) Drinfeld-Jimbo quantum group \( U_q\mathfrak{so}_N \) with \( \mathbb{Z}_2 \). We remark that our symmetric bilinear form on the root lattice is normalized so that \( \langle \beta, \beta \rangle = 2 \) for long roots for all \( N \), for uniformity's sake. For \( N \) odd, \( \mathbf{U} \) is just the direct sum of the corresponding \( \mathbb{C} \)-algebras, while in the \( N \) even case, the nontrivial element \( t \) of \( \mathbb{Z}_2 \) acts via the usual graph automorphism. This completely determines the defining relations for \( \mathbf{U} \). It is also easy to check that the map \( \Delta(t) = t \otimes t \) extends the bialgebra structure of \( U_q\mathfrak{so}_N \) to \( \mathbf{U} \). Indeed, by [24, Theorem 2.1] \( \mathbf{U} \) (called the smash product algebra in loc. cit.) is a ribbon Hopf algebra as the action of \( t \) preserves the braiding. For \( N \) odd, it is clear that \( \text{Rep}(\mathbf{U}) \cong \text{Rep}(U_q\mathfrak{so}_N) \boxtimes \text{Rep}(\mathbb{Z}_2) \) (Deligne tensor product) as ribbon categories. Note that by [7] \( \text{Rep}(\mathbf{U}) \) is the \( \mathbb{Z}_2 \)-equivariantization of \( \text{Rep}(U_q\mathfrak{so}_N) \). We shall also be interested in the case where \( q \) is a root of unity. In this case we consider the subcategory of tilting modules in \( \text{Rep}(\mathbf{U}) \) which is again a ribbon category. As such, we may consider the projective category by negligible morphisms (see [30, Section XI.4]) to obtain ribbon fusion categories which we describe below.

The algebra \( \mathbf{U} \) is obviously well-defined in the classical case \( q = 1 \), where its finite dimensional simple representations are in 1-1 correspondence with the simple representations of \( \text{Pin}(N) \). It is easy to check that we also obtain a well-defined representation of the quantum version on the spinor module \( S \) (where the matrices of the generators \( E_i, F_i \) and \( t \) do not depend on \( q \)). As any finite-dimensional simple \( \text{Pin}(N) \)-module does appear in some tensor power of \( S \), we can also make it into a \( U_q\mathfrak{so}_N \times \mathbb{Z}_2 \)-module. This extension also works for roots of unity.

Let \( S \) be a simple module of the Clifford algebra on \( V = \mathbb{C}^N \). It is well-known that in the classical case we get an irreducible representation \( S \) of \( \text{Pin}(N) \) which decomposes into a direct sum \( S \cong S_+ \oplus S_- \) of two irreducible representations of \( \text{Spin}(N) \) for \( N \) even. \( S_+ \) and \( S_- \) are both \( \mathbb{Z}_2 \)-equivariant. The representations \( S_+ \) and \( S_- \) correspond to the same \( \text{Spin}(N) \)-module, both of which correspond to isomorphic representations of \( \text{Spin}(N) \), in which case we will just denote them as \( S \), consistent with the notation above. We will also need the module \( \bar{S} = S_0 \oplus S_1 \) at some point.

We have analogous relationships between the spinor representations of \( \mathbf{U} \) and \( U_q\mathfrak{so}_N \). E.g. we have a \( \mathbf{U} \)-module \( S \) which is irreducible and decomposes as \( S \cong S_+ \oplus S_- \) as a \( U_q\mathfrak{so}_N \)-module for \( N \) even. We will give a description of \( \text{End}(S^{\otimes n}) \) in terms of \( U'_q\mathfrak{so}_n \) for both \( N \) even and odd, including the case \( q \) is a root of unity.

2.2. Classical case. We first check some well-known identities in the classical case, where \( \mathbf{U} \) is replaced by \( \text{Pin}(N) \) and \( U'_q\mathfrak{so}_n \) is replaced by \( SO(n) \). Most of these results have already more or less explicitly appeared, as special cases of a more general approach, see [11].

We consider the case where \( \text{Pin}(N) \) representations are also \( O(N) \) representations. Recall (see e.g. [37]) that simple \( O(N) \) representations are labeled by the Young diagrams \( \lambda \) for which \( \lambda'_1 + \lambda'_2 \leq N \) (here \( \lambda'_i \) denotes the number of boxes in the \( i \)-th column). The representations
of the Lie algebra $\mathfrak{so}_n$ for $n = 2j$ are labeled by the dominant integral weights $\mu = (\mu_i)_i$ such that $\mu_1 \geq \mu_2 \geq \ldots \mu_{j-1} \geq |\mu_j|$, where either all $\mu_i$ are integers or all $\mu_i \equiv 1/2 \mod \mathbb{Z}$. Then it is easy to check that the map

\[(2.2) \quad \lambda \mapsto \bar{\lambda}, \quad \text{where } \bar{\lambda}_i = N/2 - \lambda'_{j+1-i} \]
defines a bijection between the set of simple representations $V_\lambda$ of $O(N)$ for which $\lambda_1 \leq n/2 = j$ and the set of simple $\mathfrak{so}_n$ representations $\check{V}_\lambda$ for which $\lambda_1 \leq N/2$ and $N/2 - \bar{\lambda}_i$ is an integer for $1 \leq i \leq n/2$. Now consider the obvious action of $O(N) \times SO(n)$ on $\mathbb{C}^N \otimes \mathbb{C}^n$. This induces commuting actions of $O(N)$ and $SO(n)$ via automorphisms on $\text{Cliff}(\mathbb{C}^N \otimes \mathbb{C}^n)$, and hence to projective actions of these groups on a simple module $S_{Nn}$ of $\text{Cliff}(\mathbb{C}^N \otimes \mathbb{C}^n)$ i.e. proper actions of the corresponding covering groups, the spinor groups.

**Lemma 2.1.** (a) Let $n$ be even and let $S_{Nn}$ be a simple module of the simple algebra $\text{Cliff}(\mathbb{C}^N \otimes \mathbb{C}^n)$. Then $S_{Nn}$ decomposes as an $O(N) \times \text{Spin}(n)$ module as

\[S_{Nn} \cong \bigoplus_{\lambda} V_\lambda \otimes \check{V}_{\bar{\lambda}},\]

where $V_\lambda$ and $\check{V}_{\bar{\lambda}}$ are simple $O(N)$ and $\text{Spin}(n)$-modules and $\lambda$ runs through the set of Young diagrams as in Eq 2.2.

(b) If both $N$ and $n$ are even, $S^{\otimes n}$ is isomorphic as a $\text{Pin}(N) \times \text{Spin}(n)$ module to the module $S_{Nn}$ in (a). If $N$ is odd and $n$ is even, $\check{S}^{\otimes n}$ is isomorphic as a $\text{Pin}(N) \times \text{Spin}(n)$ module to the direct sum of $2^{n/2}$ copies of $S_{Nn}$ as in (a).

(c) Regardless of parity of $N$ and $n$, the irreducible representations of $\text{Spin}(n)$ in cases (a) and (b) are labeled by the dominant integrals weights $\mu$ satisfying $\mu_1 \leq N/2$ and such that $\mu_i - N/2$ is an integer for $1 \leq i \leq k$.

**Proof.** It suffices to calculate the $\text{Pin}(N) \times \text{Spin}(n)$ characters of the various modules. Let $n = 2k$ and $i = (i_1, \ldots, i_k) \in \mathbb{Z}_{\geq 0}^k$. We denote by $\omega(i)$ the $\text{Spin}(n)$ weight given by the vector $(i_j - N/2)_j$. Then we claim that the $\text{Spin}(n)$ character of a simple $\text{Cliff}(\mathbb{C}^N \otimes \mathbb{C}^n)$ module is given by

\[(2.3) \quad \chi^{S_{Nn}} = \sum_{i_1, \ldots, i_k = 0}^N (\prod_{j=1}^k \chi_i) e^{\omega(i)},\]

where $\chi_i$ is the $O(n)$ character for the $i$-th antisymmetrization $\wedge^i V$ of the vector representation of $O(N)$, for $0 \leq i \leq N$. This can be seen as follows: As $Nn$ is even by assumption, we can describe the character of the full spinor representation of $O(Nn)$ (which is a simple $\text{Cliff}(Nn)$-module) by

\[(z_1 z_2 \cdots z_{Nn/2})^{-1/2} \sum_{j=0}^{Nn/2} e_j(z),\]

where $e_j(z)$ is the $j$-th elementary symmetric function in the variables $z_1, \ldots, z_{Nn/2}$. To view this as a character of $\text{Spin}(n)$ we replace the $z$-variables by variables $x_i y_j$, $1 \leq i \leq n/2$,
$1 \leq j \leq N$. We regard the result as a polynomial in the $x_i$ variables over the ring of polynomials in the $y_i$ variables. As every $x_i$ variable comes with all possible $y_j$ variables, and our formula is obviously symmetric in the $z$-variables, and hence also in the $x$ and $y$ variables, a monomial in the $x$-variables containing the variable $x_i$ with the power $m_i$ must also have the factor $e_{m_i}(y)$, the elementary symmetric function in the variables $y_1, \ldots, y_N$. Now it is well-known that the elementary symmetric functions are the characters of the antisymmetrizations of the vector representation which remain irreducible as $O(N)$-modules. This proves Eq 2.3.

We can now prove statement (a) by induction with respect to inverse alphabetical order of the weights $\omega(i)$. It is clear that the highest possible weight occurring in Eq 2.3 is $\omega = N\varepsilon$. Then the coefficient of $e^\omega$ is equal to the trivial character, which proves (a) for $\lambda = 0$. The general claim follows by induction, using the formula

$$\prod \chi_{i_j} = \chi_\lambda + \text{lower characters},$$

where $i_j$ is a nonincreasing sequence of integers, $\lambda$ is the Young diagram whose $j$-th column has exactly $i_j$ boxes, and lower characters refers to a sum of simple $O(N)$ characters labeled by Young diagrams smaller than $\lambda$ in alphabetical order.

To prove the corresponding formulas for the tensor product representations, we check it first for $n = 2$. Here for $N$ even, the second tensor product of the spinor representation $S$ is a direct sum of all possible antisymmetrizations of the vector representation $C^N$. For $N$ odd, similarly we get that $S^\otimes 2$ decomposes into the direct sum of two copies of the exterior algebra of $C^N$. It was shown in [35] that the $i$-th antisymmetrization in $S^\otimes 2$ (resp. in $\tilde{S}^\otimes 2$, where it appears with multiplicity 2) is an eigenspace of the $\mathfrak{so}_2$ generator $B_1$ with eigenvalue $N/2 - i$. This proves that the $\mathfrak{so}_2$ character of $S^\otimes 2$ (resp. of $\tilde{S}^\otimes 2$) is given by Eq 2.3 for $N$ even (resp. by twice the value of Eq 2.3 for $N$ odd). For $n = 2k > 2$, we write $S^\otimes n = (S^\otimes 2)^\otimes k$ and observe that the $i$-th factor $S^\otimes 2$ gives us the eigenspaces of $B_{2i-1}$, to which we can apply the same arguments as before. Comparing with Eq 2.3 (with the $\chi_{i_j}$ evaluated at the identity element) we see that the $SO(n)$ character of $S^\otimes n$ is the same as the one for $S_{Nn}$ for $N$ even, and the $SO(n)$ character of $\tilde{S}^\otimes n$ is $2^n/2$ times the character of $S_{Nn}$ for $N$ odd. From this follow statements (b) and (c) (for $n$ even). For $n$ odd, the corresponding statements follow from the results for $n + 1$ from the restriction rules of representations of $\mathfrak{so}_{n+1}$. □

2.3. Quantum and fusion cases. By the main result of [35], we have commuting actions of $U = U_q\mathfrak{so}_N \rtimes \mathbb{Z}/2$ and $U'_q\mathfrak{so}_n$ on $S^\otimes n$ (for $N$ even) and $\tilde{S}^\otimes n$ for $N$ odd. Not surprisingly, the decomposition in the last lemma carries over to this setting if $q$ is not a root of unity. If $q$ is a primitive $2\ell-$th root of unity, we have a similar relationship in the corresponding ribbon fusion category $O(N)_r$. This is the quotient category of the (ribbon) category of tilting modules in $U = U_q\mathfrak{so}_N \rtimes \mathbb{Z}/2$ by negligible morphisms. Adopting the notation from the affine Lie algebra literature, we denote this category by $O(N)_r$ where $r = \ell + 2 - N$. In the case $N$ is odd, we have $O(N)_2 \cong SO(N)_r \boxtimes \text{Rep}(\mathbb{Z}/2)$, whereas in the case $N$ is even $O(N)_r$ is the $\mathbb{Z}/2$-equivariantization of $SO(N)_r$. The simple objects in $O(N)_r$ corresponding to $O(N)$-representations are labeled by Young diagrams $\lambda$ satisfying $\lambda'_1 + \lambda'_2 \leq N$ and $\lambda_1 + \lambda_2 \leq \ell+2-N$ and the additional Young diagram $\lambda = [\ell-N+1,1^{n-1}]$. The objects with half-integer spin
can be described by similar inequalities. A more explicit description is given below in the case $r = 2$. We will again denote the images of the corresponding tilting modules in $\mathbf{U}$ by $S$ resp. $\tilde{S}$ in the fusion category $O(N)_r$. We have the following results, most of which were already proved in [35]:

**Theorem 2.2.** (a) Let $n$ be even. Then we can define an action of $\mathbf{U} \times U'_q\mathfrak{so}_n$ on $S^\otimes n$ resp. $\tilde{S}^\otimes n$ whose decomposition into irreducibles is the same as in the classical case, if $q$ is not a root of unity.

(b) If $q$ is a primitive $2\ell$-th root of unity, then the objects $S^\otimes n$ resp. $\tilde{S}^\otimes n$ decompose in $O(N)_{\ell-N+2}$ as a direct sum $\bigoplus V_\lambda \otimes V_{\tilde{\lambda}}$. Here now $V_\lambda$ ranges over the objects as in the classical case, subject to the additional condition $\lambda_1 + \lambda_2 \leq \ell + 2 - N$, and the additional diagram $[\ell - N + 1, 1^{N-1}]$, and $V_{\tilde{\lambda}}$ is the (via 2.2) corresponding $U'_q\mathfrak{so}_n$ module with highest weight $\tilde{\lambda}$.

**Proof.** Part (a) follows from Lemma 2.1, using the explicit representations in [35] and the fact that for $q$ not a root of unity the representation theory of Drinfeld-Jimbo quantum groups is essentially the same as for the corresponding Lie algebra. For part (b) we just use the fact that tensor powers of $S$ resp $\tilde{S}$ can be written as a direct sum of indecomposable tilting modules; the objects in the fusion category are obtained by taking the quotient module by the tensor ideal generated by those tilting modules which have $q$-dimension equal to 0. The representations of $U'_q\mathfrak{so}_n$ into these tensor powers are still well-defined at a root of unity, and they factor over the fusion quotient. As these $U'_q\mathfrak{so}_n$ modules usually have smaller dimensions at a root of unity than in the generic case, we still need to check that they still have the same highest weight vector. But this follows from the restriction rule: restricting the action to $\mathfrak{so}_{n-1}$, the highest weight vector is again a highest weight vector in an $\mathfrak{so}_{n-1}$-module which also exists in the fusion category. The explicit combinatorics can be checked either directly by using Gelfand-Tsetlin bases for the orthogonal case (see e.g. [8]), or by using the tensor product rules for spinor representations (see e.g. [35]) via the correspondence 2.2. \qed

We use the notation $\varepsilon = (1/2, 1/2, \ldots, 1/2) \in \mathbb{R}^j$ and $e_i$ for the $i$-th standard basis vector of $\mathbb{R}^j$. We associate these vectors with weights of $\mathfrak{so}_n$ for $n = 2j$ or $n = 2j + 1$ in the usual way. Let $\rho$ be half the sum of the positive roots of $\mathfrak{so}_n$, and let $q^{2\rho}$ be the operator on a finite dimensional $\mathbf{U}$-module defined by $q^{2\rho}v_\mu = q^{(2,\rho,\mu)}v_\mu$ for a weight vector $v_\mu$ of weight $\mu$. We define, as usual, the $q$-dimension of a $\mathbf{U}$-module $V$ by $\dim_q V = Tr(q^{2\rho})$. As we have commuting actions of $\mathbf{U}$ and $U'_q\mathfrak{so}_n$ on $S^\otimes n$ resp. $\tilde{S}^\otimes n$, we can define the virtual $U'_q\mathfrak{so}_n$ character $\chi_n^\rho$ by

$$\chi_n^\rho(u) = Tr(ua^{2\rho}),$$

where $u$ is in the Cartan algebra of $U'_q\mathfrak{so}_n$, and $Tr$ is the usual trace of $S^\otimes n$ resp. $\tilde{S}^\otimes n$. The following lemma follows from the multiplicativity of the trace for tensor factors, using a similar argument as in the proof of Lemma 2.1.

**Lemma 2.3.** Let $N$ be even. Then the character $\chi_n^\rho$ is uniquely determined by

$$\chi_n^\rho(\prod B_{2i-1}^{2i-1}) = \prod \chi_2^\rho(B_{2i-1}^{2i-1}),$$
and by $\chi_2^\rho(B_{i}^\rho) = \sum_{j=1}^{N} \dim_q V[U]_{[N/2 - j]}^e$. If $N$ is odd, the same formulas hold, except that we have to add a factor 2 on the right hand side of the formula for each $\chi_2^\rho(B_{2i-1}^\rho)$, $1 \leq i < N/2$.

In the sequel we will mostly focus on the case $q = e^{\pi i/N}$ corresponding to $O(N)_2$. We record the following important:

**Corollary 2.4.** Let $q$ be a primitive $2N$-th root of unity. Then the representations $\Phi$ of $U_q^\prime \mathfrak{so}_n$ into the $O(N)_2$ centralizer algebras $\text{End}(S^\otimes n)$ resp. $\text{End}(\overline{S}^\otimes n)$ are labelled by the weights $N\varepsilon$ and $N\varepsilon - \varepsilon_j$ if $n = 2j + 1$ is odd, and by the weights $N\varepsilon - r\varepsilon_j$, $0 \leq r \leq N$, $N\varepsilon - \varepsilon_{j-1} - \varepsilon_j$ and $N\varepsilon - \varepsilon_{j-1} - (N-1)\varepsilon_j$ if $n = 2j$ is even. In particular, $\Phi$ is surjective.

**Proof.** This follows for $n$ even from the previous theorem, and for $n$ odd from the restriction rules for representations of $U_q^\prime \mathfrak{so}_n$ resp. tensor product rules of $U$. The surjectivity follows from a dimension count. \hfill \Box

### 2.4. Weakly Integral Cases

The special cases $O(N)_2$ correspond to the quotient by negligible morphisms of the categories of tilting $U$-modules for $q$ a $2N$th root of unity. These $O(N)_2$ are weakly integral unitary ribbon fusion categories, i.e. $(\dim_q V)^2 \in \mathbb{Z}$ for simple objects $V$.

The related categories $SO(N)_2$ (see e.g. [25]) obtained from $U_q^\prime \mathfrak{so}_N$ at $q = e^{\pi i/N}$ are also weakly integral modular categories and have simple objects labeled by highest weights for $\mathfrak{so}_N$. We will describe these categories in some detail.

Setting $N = 2k + 1$ for $N$ odd and $N = 2k$ for $N$ even, we denote the fundamental weights $\Lambda_1, \ldots, \Lambda_k$. No confusion should arise as we deal with $N$ even and $N$ odd separately. For later use we define for $0 \leq j \leq k$ the highest weight $\gamma_j = (1, \ldots, 1, 0, \ldots, 0)$ with the first $j$ entries equal to 1.

For $N$ odd $\Lambda_k = (1/2, \ldots, 1/2)$ labels the simple object $S$ associated with the fundamental spin representation for $\mathfrak{so}_N$ and $\Lambda_j = (1, \ldots, 1, 0, \ldots, 0)$ for $1 \leq j \leq k - 1$.

For $N$ even the two fundamental spin objects $S_\pm$ are labeled by $\Lambda_k = (1/2, \ldots, 1/2)$ and $\Lambda_{k-1} = (1/2, \ldots, 1/2, -1/2)$, while $\Lambda_j = (1, \ldots, 1, 0, \ldots, 0)$ for $1 \leq j \leq k - 2$.

#### 2.4.1. $N$ odd

The fusion category $SO(N)_2$ for $N$ odd has two simple (self-dual) objects $S = V_{\Lambda_k}$, $S' = V_{\Lambda_k + \Lambda_1}$ of dimension $\sqrt{N}$, 2 simple objects $1 = V_0$ and $V_{2\Lambda_1}$ of dimension 1 and $\frac{N}{2}$ simple objects $V_{\gamma_s}$ of dimension 2 where $1 \leq s \leq \frac{N-1}{2}$. Thus, for $N$ odd, the rank of $SO(N)_2$ is $\frac{N+1}{2}$ and the categorical dimension is $4N$.

As we have noted elsewhere, for $N$ odd $O(N)_2 \cong SO(N)_2 \boxtimes \text{Rep}(\mathbb{Z}_2)$ as ribbon fusion categories, so that the structure of $O(N)_2$ is easily determined from that of $SO(N)_2$.

#### 2.4.2. $N$ even

For $N$ even the fusion category $SO(N)_2$ has 4 simple objects $S_\pm$ (labeled by $\Lambda_k$ and $\Lambda_{k-1}$) and $S'_{\pm}$ (labeled by $\Lambda_k + \Lambda_1$ and $\Lambda_{k-1} + \Lambda_1$) of dimension $\sqrt{N/2}$, 4 simple objects $1, V_{2\Lambda_1}, V_{2\Lambda_k}$, and $V_{2\Lambda_{k-1}}$ of dimension 1 and $\frac{N}{2} - 1$ simple objects $V_{\gamma_s}$ of dimension 2 where $1 \leq s \leq \frac{N}{2} - 1$. Thus, for $N$ even, the rank of $SO(N)_2$ is $\frac{N}{2} + 7$ and the categorical dimension is $4N$. 


The simple objects in $O(N)_2$ are the images (under purification) of the simple $U_q\mathfrak{so}_N \times \mathbb{Z}_2$-tilting modules with non-zero $q$-dimension. Using [35, Section 3.4] we find that the simple objects in $O(N)_2$ are: $S$ and $S'$ of dimension $2\sqrt{N/2}$; $1, V_{[2]}, V_{[1]}$ and $V_{[1,1]}$ of dimension 1; and $V_{[1]}$ of dimension 2 with $1 \leq s \leq N - 1$. The restriction map $\text{Rep}(U_q\mathfrak{so}_N \times \mathbb{Z}_2) \to \text{Rep}(U_q\mathfrak{so}_N)$ induces a braided tensor functor $F : O(N)_2 \to SO(N)_2$ with images:

\[
F(S) = S_+ \oplus S_- \\
F(S') = S'_+ \oplus S'_- \\
F(V_{[2]}) = F(V_{[1,1]}) = V_{2\Lambda_1} \\
F(V_{[1]}) = F(1) = 1 \\
F(V_{[1^s]}) = F(V_{[1^{N-s}]}) = V_{\gamma_s}, \quad 1 \leq s \leq k - 1
\]

Observe that the objects $S$ and $St$ in $O(N)_2$ are self-dual, although $S_\pm$ are not.

2.5. $B_n$ representations on $\text{End}(S^{\otimes n})$. Denote by $\gamma_S : B_n \to \text{Aut}(S^{\otimes n})$ the representations of the braid group associated with the object $S$ in $SO(N)_2$ for $N$ odd or $O(N)_2$ for $N$ even. Explicitly, $\gamma_S$ is defined on generators by $\sigma_i \to \text{Id}_{S}^{\otimes (i-1)} \otimes c_{S,S} \otimes \text{Id}_{S}^{\otimes (n-i-1)}$.

For later use we compute the eigenvalues for the braiding operator $c_{S,S}$ for $SO(N)_2$ when $N$ is odd and $O(N)_2$ for $N$ even.

Remark 2.5. N.b. In the subsection only, we let $x = e^{\pi i/(2N)}$, and let $\langle , \rangle$ be the symmetric bilinear form on the weight lattice normalized so that $\langle \alpha, \alpha \rangle = 2$ for short roots. This is to conform with the standard results, and is only different for $N$ odd.

For $N = 2k + 1$ odd, we have

\[
S^{\otimes 2} \cong \bigoplus_{j=0}^{k} V_{\gamma_j}.
\]

The eigenvalues of $c_{S,S}$ are easily computed, and we record them in:

**Lemma 2.6.** Let $N = 2k + 1$ be odd. Up to an overall factor depending only on $N$, the eigenvalue of $c_{S,S}$ on the projection onto the simple object $V_{\gamma_s}$ is

\[
\Psi(N, s) := i^{(k-s)^2} e^{-\pi i s^2/(2N)}.
\]

**Proof.**

It follows from Reshetikhin’s formulas (see e.g. [22, Corollary 2.22]) that, up to an overall factor, $c_{S,S}$ acts on the projection onto $V_\lambda$ by the scalar $\zeta(\lambda)x^{\frac{2s}{N}}$ where $c_\lambda = \langle \lambda + 2\rho, \lambda \rangle$ for any weight $\lambda$ and the sign $\zeta(\lambda) = 1$ if the corresponding $\mathfrak{so}_N$ representation appears in the symmetric tensor square of the fundamental spin representation and $-1$ otherwise. Note that here $\langle , \rangle$ is twice the usual Euclidean inner product and $2\rho = (2k - 1, \ldots , 1)$. We compute $c_{\gamma_s} = 2(Ns - s^2)$ and note that

\[
\zeta(\gamma_s) = \begin{cases}
-1 & (k-s) \equiv 1, 2 \pmod{4} \\
1 & (k-s) \equiv 0, 3 \pmod{4}
\end{cases}
\]
The result follows.

In the case $N = 2k$ is even we have:

\[
S^{\otimes 2} = \bigoplus_{s=0}^{N} V_{[1^s]}
\]

and the eigenvalues of $c_{S,S}$ are given in:

**Lemma 2.7.** Let $N = 2k$ be even. Up to an overall factor depending only on $N$, the eigenvalue of the $O(N)_2$ braiding operator $c_{S,S}$ on the projection onto the simple object labeled by $[1^s]$ is:

\[
\eta(s) f(s), \quad \text{where } \eta(s) = e^{(N-2s)(N-2s+2)\pi i/8} \quad \text{and } f(s) = i^s e^{-\pi is^2/(2N)}.
\]

*Proof.* Since the functor $F : O(N)_2 \to SO(N)_2$ is a braided tensor functor we can compute the eigenvalues of $c_{S,S}$ from $F(c_{S,S})$. Up to signs these are just the eigenvalues of $c_{S_{+s},S_{-s}}$ and the square roots of the eigenvalues of $c_{S_{+s},S_{-s}}$. These can be computed up to an overall factor using Drinfeld’s quantum Casimir [4] (since $\langle \Lambda_k + 2\rho, \Lambda_k \rangle = \langle \Lambda_{k-1} + 2\rho, \Lambda_{k-1} \rangle$) as $q^{\frac{c_\Lambda}{2}}$ with $q = e^{\pi i/N}$ for any $V_\Lambda \in F(S^{\otimes 2})$. Up to signs, the eigenvalues corresponding to $V_{[1^s]}$ and $V_{[N-s]}$ are (both) $q^{\frac{N-s^2}{2}}$ for $0 \leq s \leq N/2$. We compute $c_{\gamma_s} = \langle \gamma_s + 2\rho, \gamma_s \rangle = Ns - s^2$ and set $f(s) = q^{\frac{N-s^2}{2}} = i^s e^{-\pi is^2/(2N)}$. Observe that $f(N-s) = f(s)$ so that $c_{S,S}$ has eigenvalue $\eta(s)f(s)$ on the projection onto $V_{[s]}$ for all $0 \leq s \leq N$, where $\eta(s)$ is a sign.

By continuity, it is enough to determine the signs for the classical case $q = 1$ for which the braiding is symmetric. One way to do this goes by induction on the dimension $N$, for even (a similar argument also works for the slightly easier case $N$ odd). One first observes that for $N = 4$ the signs are given by $\eta(0) = \eta(1) = \eta(4) = -1$ and $\eta(2) = \eta(3)$, using the fact that $Spin(4) \cong SU(2) \times SU(2)$.

The crucial observation now is that the sign for the representations $V_{[1^{N/2-s}]} \subset S_N^{\otimes 2}$ are the same as the ones for the representations $V_{[N/2-s-1]} \subset S_{N-2}^{\otimes 2}$, for $0 \leq |s| < N/2$; here $S_{2k}$ is the spinor representation in connection with $O(2k)$. This follows from the fact that $S_N$ decomposes as a $Pin(N-2)$ module into the direct sum of two modules isomorphic to $S_{N-2}$, see e.g. the discussion in [35], Lemma 2.1. Using the eigenspace decomposition of the permutation $R_s \in \text{End}(S^{\otimes 2})$, we obtain for the normalized trace $tr$ on $\text{End}(S^{\otimes 2})$

\[
\frac{1}{2^{N/2}} = tr(R_S) = \frac{1}{2^N} \sum_{s=0}^{N} \eta(N/2 - s) \dim V_{[1^{N/2-s}]}.
\]

We remark that a similar formula also holds for the odd-dimensional case $Spin(N+1)$, where now the summation only goes until $s = N/2$ and we have the antisymmetrizations of the $(N+1)$-dimensional vector representations on the right hand side. By induction assumption, $\eta(N/2 - s)$ is known for $s < N/2$, and $\dim V_{[1^{N/2-s}]}$ is equal to $\binom{N}{N/2-s}$. In the odd-dimensional case, we can now easily calculate the missing sign $\eta(0)$ from Eq 2.5, as adjusted for the odd-dimensional case. To calculate the two remaining signs in the even-dimensional case, we consider $Pin(N)$ as a subgroup of $Spin(N+1)$, which acts irreducibly.
via its spinor representation on the same vector space $S$; in particular, we can also identify the trivial subrepresentation in $S^\otimes 2$ for both groups, which hence has the same sign $\eta(0)$ for the permutation $R_S$ at $q = 1$. One now calculates $\eta(N)$ from Eq. 2.5. It is now easy to check that the signs can be given by the formula $\eta(s) = e^{(N-2s)(N-2s+2)s/n}$.

\[ \Box \]

3. Representation theory of $U'_q so_n$

We review and (re)prove certain results of the representation theory of $U'_q so_n$. Many of these results have already appeared in one form or another in work of Klimyk and his coauthors, see e.g. [8], [14]. However, in our case, we need these results also for roots of unity where the situation is more complicated. Hence we have decided to give our own, quite different proofs see e.g. [8], [14]. However, in our case, we need these results also for roots of unity where the results have already appeared in one form or another in work of Klimyk and his coauthors, of $SU$ proved in [14]; it also follows from the results in [35], as quoted in Theorem 2.2.

Lemma 3.2. Let $v$ be a vector in a $U'_q so_n$ module with weight $\mu$. Then

(a) $(B_{2i-1} - [\mu_i + 1])(B_{2i-1} - [\mu_i - 1])B_{2i}v = 0$,

(b) $(B_{2i+1} - [\mu_{i+1} \pm 1])(B_{2i-1} - [\mu_i + 1])B_{2i}v$ has weight $\mu - (\epsilon_i \pm \epsilon_{i+1})$, if it is nonzero. In particular, we can write $B_{2i}v$ as a sum of two eigenvectors of $B_{2i-1}$ (if $[\mu_i + 1] \neq [\mu_i - 1]$), and we can write $(B_{2i-1} - [\mu_i + 1])B_{2i}v$ as a sum of two weight vectors (if $[\mu_{i+1} + 1] \neq [\mu_{i+1} - 1]$).

Proof. These are straightforward calculations. E.g. for (a) we have

\[ B_{2i-1}^2 B_{2i}v = \begin{pmatrix} 2 \end{pmatrix} B_{2i-1} B_{2i} B_{2i-1} - B_{2i} B_{2i-1}^2 + B_{2i})v \]

\[ = \begin{pmatrix} 2 \end{pmatrix} \begin{pmatrix} \mu_i \end{pmatrix} B_{2i-1} B_{2i} v - \begin{pmatrix} \mu_i \end{pmatrix}^2 - 1) B_{2i}v. \]

(3.1)

We now get the claimed factorization in (a) using the identities $[2] [\mu_i] = [\mu_i + 1] + [\mu_i - 1]$ and $[\mu_i]^2 - 1 = [\mu_i + 1] [\mu_i - 1]$. For part (b) observe that a similar calculation also holds with $i$ replaced by $i+1$. The claim follows from this. \[ \Box \]
Lemma 3.3. Let \( U \) are not in \( I \) is spanned by the ordered products of the form 
\[
e_i \]
where the \( e_i \) are nonnegative integers which, for \( n = 3 \), are equal to 0 for those factors which are not in \( U_q\mathfrak{sl}_n \), and where \( v_0 \equiv 1 \mod I_\lambda \) is the highest weight vector.

Proof. The proof can be done via elementary, albeit somewhat tedious calculations. A more general result will be proved in \([36]\). We give a fairly detailed outline for a proof of this lemma for the skeptical reader as follows:

For \( n = 5 \), the idea is to move the generators \( B_4 \) as far to the right as possible. To make this mathematically precise, we define an order on words in the generators \( B_i \) first by the length of the word, and then by reversed alphabetical order e.g. \( B_1^2 < B_2B_1 < B_1B_2 \) etc. We first prove that the claim holds if we only apply generators \( B_i \), \( 2 \leq i \leq 4 \), to the highest weight vector. As a first step one shows that any vector generated this way is a linear combination of the form \( w(B_4B_3B_2)^{e_3}B_4^{e_4}v_0 \), with \( w \in \langle B_2, B_3 \rangle \). This follows by moving generators \( B_4 \) as far to the right as possible, using the relation
\[
B_4(B_4B_3B_2) = 2(B_4B_3B_2)B_4 - B_3B_3B_4^2 + B_3B_2.
\]
It is not hard to show that one can express \( B_3B_4^{e_3}v_0 \) as a linear combination of vectors \( B_4^{e_4}v_{\lambda} \), see Lemma 3.8 for details. Moreover, we also have the relation
\[
B_3(B_4B_3B_2) = (B_4B_3B_2)B_3 + [B_3, B_2B_3B_4].
\]
Using it, we can not only prove our claim, but can also show that we can assume \( w \) to end with a \( B_2 \), by induction on \( e_4 \) and \( e_3 \). It is now again an easy induction on the number of \( B_3 \)'s in \( w \) to prove that it can be expressed as a linear combination of words of the form \( B_2^{e_2}(B_3B_2)^{e_3} \) by moving the \( B_3 \)'s as far to the right as possible (taking into account that a \( B_3 \) on the right end of \( w \) will be absorbed, as just mentioned). To finish the proof for \( n = 5 \), it suffices to show that multiplying any of the words as in the statement by \( B_1 \) again results in a
linear combination of words without a $B_1$; this follows by a similar induction on the order of the words. The claims for $n = 4$ and $n = 3$ are proved similarly, with the proofs being much easier.

**Corollary 3.4.** A weight appears in the highest weight module $N_\lambda$ for $U'_q\mathfrak{so}_n$ with at most the multiplicity as in the Verma module $M_\lambda$ for the classical case $U\mathfrak{so}_n$ at $q = 1$, for $n \leq 5$.

**Proof.** We give an outline of the proof for the most difficult case $n = 5$. As $N_\lambda$ is a quotient of $M_\lambda$, it suffices to prove it for the latter module. It is standard to check that the elements $B_2, B_3B_2, B_4$ and $B_4B_3B_2$ form a basis of $(\mathfrak{so}_5 + I_\lambda)/I_\lambda$ for $q = 1$. Hence their ordered polynomials form a basis for $M_\lambda = U\mathfrak{so}_n/I_\lambda$.

Let us consider the subspaces $M_\lambda(f_1, f_2)$ spanned by all the monomials in the generators with at most $f_1$ and $f_2$ factors equal to $B_2$ and $B_4$ respectively. It follows from Lemma 3.3 and its proof that any such element can be written as a linear combination of words which also contain $\leq f_1$ factors equal to $B_1$ and $\leq f_2$ factors equal to $B_3$. Hence this space is a module of the Cartan algebra generated by $B_1$ and $B_3$. By Lemma 3.2, the zeroes of the characteristic polynomial of $B_{2i-1}$ acting on $M_\lambda(f_1, f_2)$ can only be of the form $[\lambda_i - j]$ for some integer $j$. Specializing at $q = 1$ gives us the estimates on the multiplicities of the zeroes (In fact, with a little more effort, one could show that our basis for $M_\lambda = U\mathfrak{so}_n/I_\lambda$ such that $(Bv, w) = (v, B_iw)$ for all $v, w \in M$ and $1 \leq i < n$. A $U'_q\mathfrak{so}_n$ module $M$ is called unitarizable if it allows a positive definite invariant form.

In the following, we will denote a highest weight module with highest weight $\lambda$ by $N_\lambda$. If $q$ is a root of unity, the action of the operators $B_i$ on $N_\lambda$ may no longer be diagonalizable. Moreover, we only have finitely many (generalized) weight spaces. For a weight $\mu$ we let $N_\lambda[\mu]$ be the generalized weight space of $N_\lambda$, i.e. the set of all vectors $v$ such that $(B_{2i-1} - [\mu_i]1)^k v = 0$ for sufficiently large $k$. Finally, if $q$ is a primitive $2\ell$-th root of unity, with $\ell \geq n$, we say that $\lambda$ is a restricted dominant weight for $U'_q\mathfrak{so}_n$ if $\lambda_1 \leq \ell/2$.

**Lemma 3.6.** Let $\lambda$ be a dominant integral weight with corresponding highest weight module $N_\lambda$ and highest weight vector $v_\lambda$.

(a) For $q$ not a root of unity, there is at most one invariant bilinear form $(\ , \ )$ on $N_\lambda$, up to scalar multiples.

(b) Let now $q$ be arbitrary, and suppose $N_\lambda$ admits an invariant bilinear form $(\ , \ )$. For $a = B_{i_1}B_{i_2}\cdots B_{i_k}$, set $a^t = B_{i_k}\cdots B_{i_2}B_{i_1}$. Then the value of $(av_\lambda, bv_\lambda)$ is uniquely determined.
by $(v_\lambda, v_\lambda)$ whenever $a^t b v_\lambda$ can be written as a linear combination of generalized weight vectors such that the $N_\lambda[\lambda]$ component is a multiple of $v_\lambda$.

Proof. Part (a) follows from the standard argument, which we recall for the reader’s convenience. Then it follows from invariance that

$$(a_1v_\lambda, a_2v_\lambda) = (v_\lambda, a'_1a_2v_\lambda).$$

If $q$ is not a root of unity, all the weight spaces are mutually orthogonal with respect to an invariant bilinear form. Hence the value of $(a_1v_\lambda, a_2v_\lambda)$ is given by the scalar of $v_\lambda$ in the expansion of $a'_1a_2v_\lambda$ as a linear combination of weight vectors, times $(v_\lambda, v_\lambda)$. Part (b) is proved the same way.

Remark 3.7. The strategy now will be to show that for certain dominant weights $\lambda$ there exists at most one unitarizable simple module with highest weight $\lambda$. The idea is to show that, loosely speaking, any additional vectors in the weight space of $M_\lambda$ already have to be in the annihilator ideal of a positive semidefinite form on $M_\lambda$.

3.4. $\mathfrak{so}_3$. We now give a classification of certain $U'_q\mathfrak{so}_3$ modules. The main reason for going over this in detail is that we need certain results when $q$ is a root of unity.

Lemma 3.8. Suppose $q$ is not a root of unity, and define $v_0 \in M_\lambda$ by $v_0 = 1 \mod I_\lambda$ for the $U'_q\mathfrak{so}_3$ weight $\lambda \in \mathbb{R}$. Then the set $(B^j_2v_0, j = 0, 1, 2, \ldots)$ forms a basis of $M_\lambda$. Moreover, $M_\lambda$ also has a basis of weight vectors $v_j$ with weight $[\lambda - j]$, $j = 0, 1, \ldots$ defined inductively by

$$v_{i+1} = B_2v_i - \alpha_{i-1,i}v_{i-1}, \quad \text{for } i > 1,$$

where

$$\alpha_{i-1,i} = \frac{i(2\lambda - i + 1)}{(q^{\lambda+1} + q^{\lambda-i})(q^{\lambda-i+1} + q^{\lambda-i-1})}.$$

In particular, if $\lambda$ is a half-integer, there exists a unique simple module with highest weight $\lambda$ whose dimension is $2\lambda + 1$, and on which both $B_1$ and $B_2$ act with the same set of eigenvalues $\{[\lambda - j], 0 \leq j \leq 2\lambda\}$. Finally, there is at most one invariant form $(\ , \ )$ on $M_\lambda$, up to scalar multiples. It is completely determined by $(v_j, v_i) = 0$ for $i \neq j$ and

$$(v_{j+1}, v_{j+1}) = \alpha_{j+1,j}(v_j, v_{j}).$$

Proof. Let us first consider a vector space $V$ with a basis denoted by $(\tilde{v}_j)$. We define an action of $B_1$ and $B_2$ on $V$ by substituting $v_j$ by $\tilde{v}_j$ in the claim, i.e. by $B_1\tilde{v}_j = [\lambda - j]\tilde{v}_j$ and by

$$B_2\tilde{v}_j = \tilde{v}_{j+1} + \alpha_{j-1,j}\tilde{v}_{j-1}.$$
easily that \( v_{2\lambda+1} \) generates an ideal spanned by the vectors \( v_j \) with \( j \geq 2\lambda + 1 \). As \( M_\lambda \) has a basis of weight vectors, the maximality of this ideal follows from a well-known standard argument.

To prove the statement about eigenvalues, we use the representations of \( U'_q\mathfrak{so}_3 \) in [35]. They are given by mapping \( B_1 \) to \( B \otimes 1 \) and \( B_2 \) to \( 1 \otimes B \), where \( B \in \text{End}(S^{\otimes 2}) \) and 1 stands for the identity of \( S \), with \( S \) the spinor representation as described in previous sections. It is well-known that \( B_1 \) and \( B_2 \) are conjugated via certain braiding morphisms, and these braiding morphisms are in the algebra generated by \( B_1 \) and \( B_2 \) (see Section ??).

Let \( ( , ) \) be an invariant form on \( M_\lambda \). If \( q \) is not a root of unity, then \( [\lambda - j] \neq [\lambda - i] \) for \( i \neq j \). Hence, by invariance, the \( v_j \) are pairwise orthogonal. But then we also have

\[
(v_{j+1}, v_{j+1}) = (B_2v_j - \alpha_{j-1,j}v_{j-1}, v_{j+1}) = (v_j, B_2v_{j+1}) = (v_j, v_{j+2} + \alpha_{j,j+1}v_j).
\]

The claim now follows from the fact that \( (v_{j-1}, v_{j+1}) = (v_j, v_{j+2}) = 0 \).

**Lemma 3.9.** Let \( q \) be a primitive \( 2\ell \)-th root of unity, and let \( 0 \leq \lambda \leq \ell/2 \), with \( \lambda \) being a half-integer. Then there exists a unique simple unitary \( U'_q\mathfrak{so}_3 \) module with highest weight \( \lambda \).

**Proof.** The proof goes along the lines of Lemma 3.8 by showing that any module as in the statement induces a unique form on \( M_\lambda \). The main problem now is that \( B_1 \) has large eigenspaces on \( M_\lambda \). First assume \( \lambda < \ell/2 \). Then we can construct vectors \( v_j \), \( 0 \leq j \leq 2\lambda + 1 \) with the same inner products as before. In particular, we have \( (v_{2\lambda+1}, v_{2\lambda+1}) = 0 \). As the pullback of the form \( ( , ) \) on \( M_\lambda \) is positive semidefinite, it follows that \( v_{2\lambda+1} \) is in its annihilator ideal. Hence also the vectors \( \tilde{v}_{2\lambda+1} = B_2v_{2\lambda+1} \) are in the annihilator ideal. As the vectors \( v_j \) resp. \( \tilde{v}_j \) are of the form \( B_2^jv_0 + \text{lower terms} \), the form is uniquely determined on \( M_\lambda \).

The same strategy also works for \( \lambda = \ell/2 \) until the construction of \( v_\ell \). We know from the generic case that, in \( M_\lambda \), we have \( v_{2\lambda+1} = \prod_{j=0}^{2\lambda}(B_2 - [\lambda - j])v_0 \), see Lemma 3.8. As \( B_2 \) acts via a diagonalizable matrix in a unitary representation \( W \), \( v_{2\lambda+1} \) must be in the annihilator ideal of the pull-back of the positive definite form on \( W \). So, in particular, also \( (v_{\ell+1}, v_{\ell-1}) = 0 \) if \( \lambda = \ell/2 \). Using this, we can prove the claim as before for \( \lambda < \ell/2 \).

3.5. \( \mathfrak{so}_4 \) and \( \mathfrak{so}_5 \). First recall the weight structures for Verma modules for \( \mathfrak{so}_4 \). We have seen in the last subsection that there exist polynomials \( P_j \) of degree \( j \) such that \( v_j = P_j(B_2)v_0 \) is a weight vector of weight \( \lambda - j \), where \( v_0 \) is the highest weight vector of the Verma module of \( U'_q\mathfrak{so}_3 \) with highest weight \( \lambda \). Then also \( B_3^kP_j(B_2)v_\lambda \) is an eigenvector of \( B_1 \) with eigenvalue \([\lambda_1 - j]\), where \( v_\lambda \) is the highest weight vector of a \( U'_q\mathfrak{so}_4 \) highest weight module. In view of Lemma 3.2, it follows by induction on \( j \) that the eigenvalues of \( B_3 \) are of the form \([\lambda_2 - j + 2i]\), \( 0 \leq i \leq j \). This can be written as

\[
\prod_{i=0}^{j}(B_3 - [\lambda_2 - j + 2i])P_j(B_2)v_\lambda = 0.
\]
Now leaving out the factor for a fixed $i = i_0$ gives us a weight vector of weight $(\lambda_1 - j, \lambda_2 - j + 2i_0)$, or, possibly the zero vector. As $(\lambda_2 - 1, \lambda_1 + 1)$ and $(-\lambda_2 - 1, -\lambda_1 - 1)$ are not weights of the simple $U_q^\lambda$ module with highest weight $\lambda = (\lambda_1, \lambda_2)$, the just mentioned expressions for these vectors have to be in an ideal of the Verma module. This means they are in the annihilator ideal of any invariant form in the generic case. Indeed, it follows from Harish-Chandra’s theorem (see e.g. [32], Theorem 4.7.3) that these vectors generate the maximal annihilator ideal of any invariant form in the generic case. In view of our explicit basis, this can also be checked directly for $U_q^\lambda$ in the classical case.

If $q$ is a primitive $2l$-th root of unity, and $0 \leq \lambda_2 \leq \lambda_1 \leq l/2$, it is straightforward to check that the weight vectors mentioned in the last paragraph are also in the annihilator ideal of any invariant form, using Lemma 3.6, except possibly if $\lambda_1 = l/2$ and $|\lambda_2|$ is equal to $l/2$ or $l/2 - 1$. In the first case, we basically have a $U_q^\lambda$ module, as, e.g. for $\lambda_2 = l/2$ we have $B_2B_2v_\lambda = [\lambda_2 - 1]B_2v_\lambda$ and the claim follows from the previous section. Similarly, if $\lambda_2 = l/2 - 1$, one considers the quotient of $M_\lambda$ modulo the vector of weight $(l/2 - 2, l/2 + 1)$. It is not hard to check that it is the sum of two $U_q^\lambda$ modules with highest weights $l/2$ and $l/2 - 1$, and the claim again follows from Lemma 3.9. We have shown most of the following lemma:

**Lemma 3.10.** Let $q = e^{\pm \pi i/\ell}$. There is at most one simple unitary $U_q^\lambda$ module with highest weight $\lambda$ for any restricted dominant weight $\lambda$. The same uniqueness statement holds for a unitary $U_q^\lambda$ module with highest weight $\lambda = (l/2, l/2)$ or $\lambda = (l/2, l/2 - 1)$, provided its restriction to $U_q^\lambda$ is isomorphic to the corresponding restriction for the $U_q^\lambda$ module in Corollary 2.4 with the same highest weight $\lambda$.

**Proof.** After the previous discussion, it only remains to check the claim for the two $U_q^\lambda$ modules. This can be done by a straightforward inspection as follows: One first checks that all the inner products for $U_q^\lambda$ highest weight vectors are uniquely determined by the value of $(v_\lambda, v_\lambda)$, by Lemmas 3.6 and 3.9. To do this, one deduces from the character formulas in Lemma 2.1 and Theorem 2.2 that for $\lambda = (l/2, l/2)$, the corresponding $U_q^\lambda$ module decomposes as a direct sum of simple $U_q^\lambda$ modules with highest weights $(l/2, j)$ and highest weight vectors $P_j(B_4)v_\lambda$, for which the inner products are known by Lemma 3.9. The same method works for $\lambda = (l/2, l/2 - 1)$, except for the submodules with highest weights $(l/2 - 1, \pm(l/2 - 1))$. In the latter exceptional cases, the uniqueness of the norm can be deduced using Lemma 3.6. The claim now follows from this and the already proven claim for unitary $U_q^\lambda$ modules. \qed

### 4. Quantum torus and braid representations

#### 4.1. Quantum torus.** Let $n > 1$ and let $A$ be an $(n-1) \times (n-1)$ integer matrix defined by $a_{ij} = (j - i)$ if $|i - j| = 1$ and by $a_{ij} = 0$ otherwise. The quantum $(n-1)$-torus associated with $A$ is:

$$
T_q(n) := \mathbb{C}\langle u_1^{\pm 1}, \ldots, u_{n-1}^{\pm 1} \rangle, \quad u_i u_j = q^{a_{ij} + 1} u_j u_i.
$$
For \( q \in \mathbb{C}^* \) we may specialize \( T_q(n) \) at \( q \). In this situation we can give \( T_q(n) \) the structure of a \( * \)-algebra by setting \( u_i^* = u_i^{-1} \).

We have the following elementary lemma:

**Lemma 4.1.** The algebra \( T_q(n) \) has a basis consisting of monomials \( u_1^{m_1} u_2^{m_2} \ldots u_{n-1}^{m_{n-1}} \) with \( m_j \in \mathbb{Z} \) for \( 1 \leq j < n \).

**Proof.** The spanning property is easy to check, using the fact that the generators \( u_i \) commute up to multiplication by a power of \( q \). To prove linear independence, we define an action of \( u_i \) on the space of Laurent polynomials \( \mathbb{C}[x_1^\pm 1, x_2^\pm 1, \ldots, x_{n-1}^\pm 1] \) by

\[
    u_i x^\vec{m} = q^{m_i} x_i x^\vec{m},
\]

where \( \vec{m} \in \mathbb{Z}^{n-1} \) and \( x^\vec{m} = x_1^{m_1} x_2^{m_2} \ldots x_{n-1}^{m_{n-1}} \). We leave it to the reader to check that this is indeed a representation of \( T_q(n) \). The linear independence follows from \( u_i^* 1 = x_i^\vec{m} \) and the linear independence of the vectors \( x_i^\vec{m} \). \( \square \)

**4.2. Finite dimensional representations.** If \( (\rho,V) \) is a \( d \)-dimensional representation of \( T_q(n) \) for \( n \geq 3 \) then \( u_i u_{i+1} u_i^{-1} = q u_{i+1} \) implies that Spec(\( \rho(u_i) \)) is invariant under multiplication by \( q \). This, in turn, implies that \( q^k = 1 \) for some \( k \) dividing \( d \). Moreover, it is easy to check that \( q^k = 1 \) if and only if \( u_i^k \) is in the center of \( T_q(n) \). We define for any \( z \in \mathbb{C}^* \), \( T_q(n,z) \) has a basis consisting of monomials \( u_1^{m_1} u_2^{m_2} \ldots u_{n-1}^{m_{n-1}} \) with \( m_j \in \mathbb{Z} \) for \( 1 \leq j < n \).

\[
    \text{Proposition 4.2. (a) The algebra } T_q(n) \text{ has nontrivial finite dimensional representations if and only if } q \text{ is a root of unity of finite order.}
\]

\[
    (b) \text{ The algebra } T_q^k(n,z) \text{ has dimension } k^{n-1}. \text{ It has one simple module of dimension } k^{(n-1)/2} \text{ for } n \text{ odd, and } k \text{ non-isomorphic simple modules of dimension } k^{(n-2)/2}.
\]

**Proof.** Part (a) has been proved already. It also follows easily that the dimension of \( T_q(n,z) \) is as stated above (a). To prove the remainder of (b), suppose first that \( n \) is odd so that \( T_q^k(n,z) \) has an even number of generators: \( u_1, \ldots, u_{n-1} \). Let \( V \) be a \( k^{(n-1)/2} \)-dimensional vector space with basis \( v(\vec{i}) \), where \( \vec{i} \in \{0,1,\ldots,k-1\}^{(n-1)/2} \). The action of \( u_{2s-1} \) on \( V \) is defined by \( u_{2s-1} v(\vec{i}) = z_{2s-1} q^{\vec{i} \cdot \vec{s}} v(\vec{i}) \). The action of \( u_{2s} \) is given by the rule (indices modulo \( k \)):

\[
    u_{2s}(v(i_1, \ldots, i_s, i_{s+1}, \ldots, i_{n-1})) = z_{2s} v(i_1, \ldots, i_s + 1, i_{s+1} - 1, i_{s+2}, \ldots, i_{n-1});
\]

in other words, the even indexed generators \( u_{2s} \) permute the vectors \( v(i_1, \ldots, i_{n-1}) \) by shifting the \( s \)th index up by 1 and the \((s+1)\)th index down by 1, except for \( s = (n-1)/2 \) where there is no index left for shifting down.

It is straightforward to check that \( V \) is a \( T_q^k(n,z) \)-module. Standard arguments show that if \( W \) is a submodule of \( V \), it must contain at least one common eigenvector of the elements \( u_{2s-1}, 1 \leq s < n/2 \), i.e. one of our basis vectors. It then follows for \( n \) odd that \( W \) contains...
all basis vectors, i.e. $W = V$ is simple. It follows that the image of $T_q(n, \vec{z})$ is the full matrix ring on $V$. This proves all the statements in (b) for $n$ odd.

For $n$ even, we look at the restriction of the just constructed representation of $T_q^k(n + 1, \vec{z})$ to $T_q^k(n, \vec{z})$. It obviously must be faithful. On the other hand, it decomposes into the direct sum of $V_r$, $0 \leq r < k$ of $T_q^k(n, \vec{z})$-modules, where each $V_r$ is the span of vectors $v(\vec{i})$ for which the sum of the indices $i_1 + i_2 + \cdots + i_{(n-1)/2}$ is congruent to $r$ mod $k$. From this follow the remaining statements of (b) for $n$ even.

In what follows we will only need to deal with the special case $\vec{z} = (1, \ldots, 1)$ for which we set $T_q^k(n) = T_q^k(n, (1, \ldots, 1))$.

4.3. $U'_q\mathfrak{so}_n$ Representations into the quantum torus. Let $B_i, 1 \leq i < n$ be the generators of $U'_q\mathfrak{so}_n$, as before.

In the following lemma, we consider the representations of the quantum torus, depending on the positive integer $N$ as follows:

**Lemma 4.3.** (a) The assignments $B_i \to \pm \frac{u_i - u_i^{-1}}{q - q^{-1}}$ and $B_i \to \pm \frac{u_i + u_i^{-1}}{q - q^{-1}}$ extend to algebra homomorphisms $U'_q\mathfrak{so}_n \to T_q^N(n)$ (for arbitrary $q$).

(b) For $q = e^{2\pi i/(2N)}$ the assignments in (a) extend to algebra homomorphisms $U'_q\mathfrak{so}_n \to T_q^{2N}(n)$.

(c) ($N = 2k$ Even case) Denote by $\Psi : U'_q\mathfrak{so}_n \to T_q^{2N}(n)$ the algebra homomorphism determined by $B_i \to b_i := i^{\frac{u_i + u_i^{-1}}{q - q^{-1}}}$ as in (b). Then $\prod_{j=-N}^N (B_i - [j]) \in \ker(\psi_N)$ where $[j] := \frac{q^j - q^{-j}}{q - q^{-1}}$. Moreover, the set $[j]$ are distinct for $-\frac{N}{2} \leq j \leq \frac{N}{2}$.

(d) ($N = 2k + 1$ odd case) Denote by $\Psi : U'_q\mathfrak{so}_n \to T_q^{2N}(n)$ the map determined by $B_i \to b_i := i^{\frac{u_i + u_i^{-1}}{q - q^{-1}}}$ as in (b). Then $\prod_{j=-k}^k (B_i - [j + \frac{1}{2}]) \in \ker(\Psi)$. In particular the image of the subalgebra of $U'_q\mathfrak{so}_n$ generated by $B_i^2$ factors through the algebra $U_q(\mathfrak{so}_n, k)$ (see [35, Definition 4.7(c)]), so that $\Psi$ induces $\tilde{\psi}_N : U_q(\mathfrak{so}_n, k) \to T_q^{2N}(n)$. Moreover, the $b_i$ eigenvalues $[j + \frac{1}{2}]$ for $-k - 1 \leq j \leq k$ and the $b_i^2$ eigenvalues $[j + \frac{1}{2}]^2$ for $0 \leq j \leq k$ are distinct.

**Proof.** Part (a) is a straight-forward calculation: the case $n = 3$ is sufficient since far-commutation is obvious, and writing out the $q$-Serre relations with $B_i = x(u_i \pm u_i^{-1})$ gives the specified values of $x$.

Part (b) is obvious since $q^2 \neq 1$.

For (c) first observe that for $-\frac{N}{2} \leq j \leq \frac{N}{2}$ the $N + 1$ numbers $[j]$ are distinct since $\sin(x)$ is increasing on $[-\pi, \pi]$. We have $u_i^{2N} = 1$ so $\text{Spec}(u_i) = \{q^j : 0 \leq j \leq 2N - 1\}$, where $q = e^{\pi i/N}$ and $i = q^{\frac{N}{2}}$. Thus $\text{Spec}(b_i) = \{q^{j/2} (q^{j} + q^{-j}) : 0 \leq j \leq 2N - 1\}$. Since $q^{N/2}q^{-j} = -q^{-j-N/2}$ and

$$\{j + N/2 \pmod{2N} : 0 \leq j \leq 2N - 1\} = \{j \pmod{2N} : 0 \leq j \leq 2N - 1\}$$
we have Spec($b_i$) = $\{[j] : -N/2 \leq j \leq N/2\}$. Since $b_i^* = -b_i$ the minimal polynomial of $b_i$ is a product of distinct (linear) factors.

For (d) we note as above that $\{[j+1/2] : -k-1 \leq j \leq k\}$ is a set of $2k+2$ distinct numbers and $\{[j+1/2]^2 : 0 \leq j \leq k\}$ is a set of $k+1$ distinct numbers (since $[j+1/2] = -[j-1/2]$).

We have $a_i^{2N} = 1$ so Spec($u_i$) = $\{q^j : 0 \leq j \leq 2N - 1\}$ (where $q = e^{2\pi i/(2N)}$) and $i = q^{N/2}$.

Thus $i(q^j + q^{-j}) = q^{j+N/2} - q^{-j-N/2}$, and $\{j+N/2 \ (\text{mod} \ 2N) : 0 \leq j \leq 2N - 1\} = \{j+1/2 \ (\text{mod} \ 2N) : 0 \leq j \leq 2N - 1\}$ so we have Spec($b_i$) = $\{[j+1/2] : -k-1 \leq j \leq k\}$. As in (c), the minimal polynomials of $b_i$ and $b_i^2$ are products of distinct linear factors.

\[ \square \]

4.4. Basics from subfactor theory. In order to compare the representations defined in this section with the ones defined before in connection with fusion categories we shall need a few basic results from Jones’ theory of subfactors (see [17, Section 3.1]). Let $\mathcal{A} \subset \mathcal{B}$ be finite or infinite dimensional unital von Neumann algebras with the same identity. Assume that $\mathcal{B}$ has a finite trace $tr$ satisfying $tr(1) = 1$ and $(b,b) = tr(b^*b) > 0$ for $b \neq 0$. Let $L^2(\mathcal{B}, tr)$ be the Hilbert space completion of $\mathcal{B}$ under the inner product $(\ , \ )$, and let $e_A$ be the orthogonal projection onto $L^2(\mathcal{A}, tr) \subset L^2(\mathcal{B}, tr)$. It can be shown that it maps any element $b \in \mathcal{B}$ to an element $e_A(b) \in \mathcal{A}$. The algebra $\langle \mathcal{B}, e_A \rangle$ is called Jones’ basic construction for $\mathcal{A} \subset \mathcal{B}$. If $\mathcal{A} \subset \mathcal{B} \subset \mathcal{C}$ are finite dimensional algebras and $e \in \mathcal{C}$ is such that $bee = e_A(b)e$ for all $b \in \mathcal{B}$ and the map $a \mapsto ae$ defines an algebra isomorphism between $\mathcal{A}$ and $\mathcal{A}e$, one can show that $\langle \mathcal{B}, e \rangle \cong BeB \oplus B'$, where $BeB$ is isomorphic to a Jones basic construction for $\mathcal{A} \subset \mathcal{B}$, and $B'$ is isomorphic to a subalgebra of $\mathcal{B}$.

4.5. Algebra isomorphisms. We consider the following set-up: Let $A_i, i \in \mathbb{N}$ be a sequence of self-adjoint operators acting on a Hilbert space, satisfying the following conditions:

1. We have $[A_i, A_j] = 0$ for $|i - j| > 1$, and $A_{i,j} = \langle A_i, A_{i+1}, \ldots, A_{j-1} \rangle$ is a finite-dimensional algebra for all $i < j$.
2. The map $A_i \mapsto A_{i+1}$, $1 \leq i < j - 2$ induces an isomorphism between $A_{1,j-1}$ and $A_{2,j}$.
3. There exists a unital trace on the algebra $\mathcal{A}$ generated by the elements $A_i, i \in \mathbb{N}$, and an $m > 0$ such that $A_{i,j+1} = \langle A_{i,j}, e_j \rangle$ is isomorphic to a Jones basic construction for $A_{i,j-1} \subset A_{i,j}$ whenever $j - i \geq m$, where $e_i$ is an eigenprojection of $A_i$.

Remark 4.4. The conditions above are satisfied for any self-dual object $X$ in a braided unitary fusion category for which $\text{End}(X^\otimes 2)$ is generated by an element $A$ for which $e$ is the projection onto the trivial object $1 \subset X^\otimes 2$, and where $\text{End}(X^\otimes n)$ is generated by the elements $A_i = 1_{i-1} \otimes A \otimes 1_{n-1-i}$ (see e.g. [35], Prop 2.2 and the references given there).

Lemma 4.5. Let $(A_i)$ and $(\tilde{A}_i)$ be operators satisfying conditions (1)–(3) above, with $m \in \mathbb{N}$ as in cond. (3). Assume the map $\Phi : A_i \mapsto \tilde{A}_i$, $1 \leq i \leq m$ defines an algebra isomorphism between $A_{1,m+1}$ and $\tilde{A}_{1,m+1}$. Then we can extend $\Phi$ to an algebra isomorphism between $A_{1,\infty}$ and $\tilde{A}_{1,\infty}$ such that $A_i$ is mapped to $\tilde{A}_i$ for $i \geq m$. We call this an inclusion-respecting isomorphism between these algebras.
Lemma 4.5) where \( \Psi : \) and 2.3. Hence the same irreducible characters of 1, \( \epsilon_i \) for the \( i \)-th standard basis vector of \( \mathbb{R}^j \). We associate these vectors with weights of \( \mathfrak{so}_n \) for \( n = 2k \) or \( n = 2k + 1 \) in the usual way.

**Theorem 4.6.** We have the following inclusion-respecting isomorphisms (in the sense of Lemma 4.5) where \( \Psi : U_q' \mathfrak{so}_n \to T_q^{2N}(n) \) is as in Lemma 4.3:

(a) For \( N \) even, \( \text{End}(S^{\otimes n}) \cong \Psi(U_q' \mathfrak{so}_n) = \langle 1, v_1 + v_1^{-1}, \ldots, v_{n-1} + v_{n-1}^{-1} \rangle \).

(b) For \( N \) odd, \( \text{End}(S^{\otimes n}) \cong \Psi(\langle 1, B_1^2, \ldots, B_{n-1}^2 \rangle) = \langle 1, v_1^2 + v_1^{-2}, \ldots, v_{n-1}^2 + v_{n-1}^{-2} \rangle \).

**Proof.** The theorem is proved by checking that conditions (1)-(3) of Subsection 4.2 and Lemma 4.5 are satisfied for \( A_i \) to \( A_i \) and \( \tilde{A}_i \) being the analogous images under \( \Psi \); here \( \Phi \) is the surjective map from \( U_q' \mathfrak{so}_n \) (for \( N \) even) resp. its subalgebra generated by the \( B_i^2 \) (for \( N \) odd) onto \( \text{End}(S^{\otimes n}) \) of Corollary 2.4. Conditions (1) and (2) are easy to check, using Remark 4.4 and the fact that \( v_i \mapsto v_{i+1} \) also induces a homomorphism in the quantum torus with \( q \) a root of unity. Indeed, \( S \) is a self-dual simple object in \( SO(N)_2 \) (resp. \( O(N)_2 \)) for \( N \) odd (resp. even) and the element \( A_1 \in \text{End}(S^{\otimes 2}) \) generates.

Observe that the representation \( \Psi \) of \( U_q' \mathfrak{so}_n \) into \( T_q^{2N}(n) \) for \( q = e^{\pi i/N} \) in the previous section has the same simple components (though not with the same multiplicities) as its representation \( \Phi \) into \( \text{End}(S^{\otimes n}) \) resp. \( \text{End}(\tilde{S}^{\otimes n}) \) in Cor. 2.4 for \( n \leq 5 \). Indeed, for \( n = 2 \) it suffices to calculate the eigenvalues of \( B_1 \), which was done in Lemma 4.3. They coincide with the ones in the fusion representation, see Lemmas 2.7 and 2.4. It is now easy to check that the usual trace for the standard representation of the quantum torus satisfies the same conditions as the functions \( \chi_n^\ell \) of Lemma 2.3. Hence the same irreducible characters of \( U_q' \mathfrak{so}_n \) for \( n \) even appear in its representation into the quantum torus as in its representation into \( \text{End}(S^{\otimes n}) \) resp. \( \text{End}(\tilde{S}^{\otimes n}) \). But as unitary representations are uniquely determined by their highest weights for \( n \leq 5 \), with the additional condition on the restriction for \( n = 5 \), see Lemma 3.10 (observe that all entries \( \mu_i \) of our weights have absolute value \( \leq \ell/2 \)), the irreducible representations of \( U_q' \mathfrak{so}_n \) in the quantum torus coincide with the ones in the fusion category, for \( n \leq 5 \).
This obviously also implies the isomorphism between the images of the representations of the subalgebras generated by the $B_i^2$, $i < 5$ for the case $N$ even.

Finally, condition (3) of Subsection 4.2 holds for the algebras $A_{i,j}$ with $m = 4$ by Remark 4.4 and, it was checked by Jones for the algebras $\hat{A}_{i,j}$, see [18].

But now the conditions of Lemma 4.5 are satisfied for $B_i$ (for $N$ even) resp. of $B_i^2$ (for $N$ odd) in End($S^\otimes n$), and $\hat{A}_i$ being the analogous images in the quantum torus, with $m = 4$. $\square$

4.7. Braid representations into quantum torus. The isomorphism in the last theorem transports the braid representations from the fusion categories to braid representations into the quantum torus. We determine precisely the images of the braid generators in these representations, up to an overall scalar factor.

Proposition 4.7. Let $q = e^{\pi i/N}$ and $\psi : B_n \rightarrow T_q^{2N}(n)$ the braid group representations obtained as compositions of $\gamma_S : B_n \rightarrow \text{Aut}(S^\otimes n)$ from Subsection 2.5 and the isomorphisms of Theorem 4.6. Then:

(a) For $N$ odd, we have $\psi(\sigma_i) = \frac{\gamma}{\sqrt{N}} \sum_{j=0}^{N-1} Q^j v_i^{2j}$ where $Q = q^2 = e^{2\pi i/N}$ and $\gamma$ is a scalar of norm 1.

(b) For $N$ even, we have $\psi(\sigma_i) = \frac{\gamma}{\sqrt{2N}} \sum_{j=0}^{2N-1} x^{\alpha j} v_i^{2j}$ where $x = e^{\pi i/(2N)}$, $\alpha = 1-N(-1)^{N/2}$ and $\gamma$ is a scalar of norm 1.

Proof. Clearly $\psi(\sigma_i)$ must be a polynomial in $b_i = \frac{v_i + v_i^{-1}}{q-q^{-1}}$ for $N$ even and $b_i^2$ for $N$ odd. Since the isomorphisms of Theorem 4.6 respect inclusions it is enough to prove that $\psi(\sigma_1) = R_o := \frac{\gamma}{\sqrt{N}} \sum_{j=0}^{N-1} Q^j v_1^{2j} \in T_q^{2N}(n)$ for $N$ odd and $\psi(\sigma_1) = R_e := \frac{\gamma}{\sqrt{2N}} \sum_{j=0}^{2N-1} \tau(x)^j v_1^{2j} \in T_q^{2N}(n)$ for $N$ even (for some scalars $\gamma$ and some $\tau \in \text{Aut}_Q(\mathbb{Q}(\tau(x)))$). Comparing the coefficients of $v_1^j$ and $v_1^{-j}$ one sees that $R_o$ resp. $R_e$ are indeed polynomials in $b_1$ resp. $b_1^2$. Since the number of distinct eigenvalues of $b_1$ and $b_1^2$ is equal to the dimension of End($S^\otimes 2$) (for $N$ even, resp. odd) it is enough to verify that the eigenvalues of $R_o$ and $R_e$ coincide with those of $c_{S,S}$ in Lemmas 2.7 and 2.4 on each $B_1$-eigenspace. The eigenvalues of $B_1$ are computed in [35, Lemma 4.2]: the eigenvalue of $B_1$ on the projection onto $V_{[1^{N/2-\sigma}, j]}$ is $[j]$ (note that in [35] the Young diagram in the subscript has a typo: $2k$ should be replaced by $k = N/2$ as we have here). For $N$ odd, we must verify that $R_o v = i^{(N/2-s)^2 + s} e^{-\pi i s} v$ for any eigenvector $v$ of $b_1^2$ with eigenvalue $[N/2-s]^2$, for $0 \leq s \leq (N-1)/2$ (up to a scalar independent of $s$) and for $N$ even $R_e v = \eta(N/2-s) f(N/2-s) v$ for any eigenvector $v$ of $b_1$ with eigenvalue $[s]$, for $-N/2 \leq s \leq N/2$ where $\eta$ and $f$ are functions defined in Lemma 2.7 (up to a scalar, independent of $s$, and some choice of $\tau$).

We will give the details in the $N$ even case and leave the $N$ odd case to the reader.
For $N$ even and $-N/2 \leq s \leq N/2$, $v_1$ acts on the $[s]$-eigenspace of $b_1$ by $x^{\pm (2s-N)}$. The corresponding eigenvalue of $\frac{1}{\sqrt{2N}} \sum_{j=0}^{2N-1} (x)^{2j} v_j$ is:

$$\frac{1}{\sqrt{2N}} \sum_{j=0}^{2N-1} x^{2\pm (2s-N)j}.$$  

Completing the square we have:

$$\frac{x^{-(s-N/2)^2}}{\sqrt{2N}} \sum_{j=0}^{4N-1} x^{j(s-N/2)^2}.$$  

Since $x$ is a $4N$th root of unity and the set of residues modulo $4N$ of $(j \pm (s-N/2))^2$ is the same for $0 \leq j \leq 2N-1$ and $2N \leq j \leq 4N-1$, we double the sum to obtain:

$$\frac{x^{-(s-N/2)^2}}{2\sqrt{2N}} \sum_{j=0}^{4N-1} x^{j} = \frac{x^{-(s-N/2)^2}(1+i)}{\sqrt{2}}.$$  

using Dirichlet’s improvement on Gauss’ result (see e.g. [2]).

Rescaling (independent of $s$) we obtain the eigenvalue $f(N/2-s)(-i)^{(N/2-s)}$ for $R_e$ on these spaces. The result now follows by verifying, for $\alpha = 1 - N(-1)^{N/2}$, that

$$\frac{[f(N/2-s)(-i)^{(N/2-s)}]^{\alpha}}{(\eta(N/2-s)f(N/2-s))}$$

is independent of $s$.

Remark 4.8. The “Gaussian” representations of $B_n$ in $T_{2N}^2(n)$ described in Prop. 4.7(a) are fairly well-known, going back to [10] and known to Jones in the case $N = 3$ in the early 1980s. In the case $N$ is even these representations seemed not to be known until recently [6], in which results of [18] are employed, and their properties are studied in some detail.

As a consequence we can prove (a generalized version of) [27, Conjecture 5.4]:

Theorem 4.9. The images of the braid group representations on $\text{End}_{SO(N)^2}(S^{\otimes n})$ for $N$ odd and $\text{End}_{SO(N)^2}(S^{\otimes n}_\pm)$ for $N$ even are isomorphic to images of braid groups in Gaussian representations; in particular, they are finite groups.

Proof. For $N$ odd it follows from the analysis of Gaussian representations in [6] that their image is finite. Hence the claim follows from Prop. 4.7. For $N$ even the same analysis implies that the braid group representation on $\text{End}_{O(N)^2}(S^{\otimes n})$ for $N$ even is a finite group. Since the forgetful functor $F : O(N)^2 = (SO(N)^2)^{\otimes 2} \to (SO(N)^2$ is a braided tensor functor and the braiding is functorial we conclude that the image of the braid group acting on $\text{End}_{SO(N)^2}(S^{\otimes n}_\pm)$ is a (finite) subquotient of the image of the braid group acting on $\text{End}_{O(N)^2}(S^{\otimes n})$.  \[\square\]
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