On the vanishing of local cohomology of the absolute integral closure in positive characteristic

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Abstract

The aim of this paper is to extend the main result of C. Huneke and G. Lyubeznik in [Absolute integral closure in positive characteristic, Adv. Math. 210 (2007), 498–504] to the class of rings that are images of Cohen-Macaulay local rings. Namely, let $R$ be a local Noetherian domain of positive characteristic that is an image of a Cohen-Macaulay local ring. We prove that all local cohomology of $R$ (below the dimension) maps to zero in a finite extension of the ring. As a direct consequence we obtain that the absolute integral closure of $R$ is a big Cohen-Macaulay algebra.

1 Introduction

Let $(R, \mathfrak{m})$ be a commutative Noetherian local domain with fraction field $K$. The absolute integral closure of $R$, denoted $R^+$, is the integral closure of $R$ in a fixed algebraic closure $\overline{K}$ of $K$.

A famous result of M. Hochster and C. Huneke says that if $(R, \mathfrak{m})$ is an excellent local Noetherian domain of positive characteristic $p > 0$, then $R^+$ is a (balanced) big Cohen-Macaulay algebra, i.e. every system of parameters in $R$ becomes a regular sequence in $R^+$ (cf. [8]). Furthermore, K.E. Smith in [12] proved that the tight closure of an ideal generated by parameters is the contraction of the extension of $I$ in $R^+$: $I^* = IR^+ \cap R$. This property is not true for every ideal $I$ in an excellent Noetherian domain since tight closure does not commute with localization (cf. [11]).

As mentioned above, $H^i_{\mathfrak{m}}(R^+) = 0$ for all $i < \dim R$ provided $R$ is an excellent local Noetherian domain of positive characteristic. Hence, the natural homomorphism $H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R^+)$, induced from the inclusion $R \to R^+$, is the zero map for all $i < \dim R$. In the case $R$ is an image of a Gorenstein (not necessarily excellent) 

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local ring, as the main result of [7], Huneke and G. Lyubeznik proved a stronger
conclusion that one can find a finite extension ring \( S \subseteq R \subseteq R^+ \), such that the
natural map \( H^i_m(R) \to H^i_m(S) \) is zero for all \( i < \dim R \). The techniques used in [7]
are the Frobenius action on the local cohomology, (modified) equation lemma (cf. [6], [12], [7]) and the local duality theorem (This is the reason of the assumption
that \( R \) is an image of a Gorenstein local ring). The motivation of the present paper
is our belief: If a result was shown by the local duality theorem, then it can be proven
under the assumption that the ring is an image of a Cohen-Macaulay local ring (for
example, see [10]). The main result of this paper extends Huneke-Lyubeznik’s re-
sult to the class of rings that are images of Cohen-Macaulay local rings. Namely,
we prove the following.

**Theorem 1.1.** Let \((R, \mathfrak{m})\) be a commutative Noetherian local domain containing a
field of positive characteristic \( p \). Let \( K \) be the fraction field of \( R \) and \( \overline{K} \) an algebraic
closure of \( K \). Assume that \( R \) is an image of a Cohen-Macaulay local ring. Let \( R' \)
be an \( R \)-subalgebra of \( \overline{K} \) (i.e. \( R \subseteq R' \subseteq \overline{K} \)) that is a finite \( R \)-module. Then there
is an \( R' \)-subalgebra \( R'' \) of \( \overline{K} \) (i.e. \( R' \subseteq R'' \subseteq \overline{K} \)) that is finite as an \( R \)-module such
that the natural map \( H^i_m(R') \to H^i_m(R'') \) is the zero map for all \( i < \dim R \).

As a direct application of Theorem 1.1 we obtain that the absolute integral
closure \( R^+ \) is a big Cohen-Macaulay algebra (cf. Corollary 3.2). The main result
will be proven in the last section. In the next section we recall the theory of attached
primes of Artinian modules.

## 2 Preliminaries

Throughout this section \((R, \mathfrak{m})\) be a commutative Noetherian local ring. We recall
the main result of [10] which is an illustration for our belief (mentioned in the in-
troduction).

I.G. Macdonald, in [9], introduced the theory of secondary representation for
Artinian modules, which is in some sense dual to the theory of primary decom-
position for Noetherian modules. Let \( A \neq 0 \) be an Artinian \( R \)-module. We say that
\( A \) is secondary if the multiplication by \( x \) on \( A \) is surjective or nilpotent for every
\( x \in R \). In this case, the set \( p := \sqrt{(\text{Ann}_RA)} \) is a prime ideal of \( R \) and we say that
\( A \) is \( p \)-secondary. Note that every Artinian \( R \)-module \( A \) has a minimal secondary
representation \( A = A_1 + \ldots + A_n \), where \( A_i \) is \( p_i \)-secondary, each \( A_i \) is not redundant
and \( p_i \neq p_j \) for all \( i \neq j \). The set \( \{p_1, \ldots, p_n\} \) is independent of the choice of the
minimal secondary representation of \( A \). This set is called the set of attached primes
of \( A \) and denoted by \( \text{Att}_RA \). Notice that if \( R \) is complete we have the Matlis dual
\( D(A) \) of \( A \) is Noetherian and \( \text{Att}_RA = \text{Ass}_RD(A) \).
For each ideal \( I \) of \( R \), we denote by \( \text{Var}(I) \) the set of all prime ideals of \( R \) containing \( I \). The following is easy to understand from the theory of associated primes.

**Lemma 2.1** ([9]). Let \( A \) be an Artinian \( R \)-module. The following statements are true.

(i) \( A \neq 0 \) if and only if \( \text{Att}_R A \neq \emptyset \).

(ii) \( A \neq 0 \) has finite length if and only if \( \text{Att}_R A \neq \{m\} \).

(iii) \( \min \text{Att}_R A = \min \text{Var}(\text{Ann}_R A) \). In particular,

\[
\dim(R/\text{Ann}_R A) = \max\{\dim(R/p) \mid p \in \text{Att}_R A\}.
\]

(iv) \( 0 \to A' \to A \to A'' \to 0 \) is an exact sequence of Artinian \( R \)-modules then

\[
\text{Att}_R A'' \subseteq \text{Att}_R A \subseteq \text{Att}_R A' \cup \text{Att}_R A''.
\]

Let \( \widehat{R} \) be the \( m \)-adic complete of \( R \). Note that every Artinian \( R \)-module \( A \) has a natural structure as an \( \widehat{R} \)-module and with this structure, each subset of \( A \) is an \( R \)-submodule if and only if it is an \( \widehat{R} \)-submodule. Therefore \( A \) is an Artinian \( \widehat{R} \)-module. So, the set of attached primes \( \text{Att}_{\widehat{R}} A \) of \( A \) over \( \widehat{R} \) is well defined.

**Lemma 2.2.** ([2, 8.2.4, 8.2.5]). \( \text{Att}_R A = \{P \cap R \mid P \in \text{Att}_{\widehat{R}} A\} \).

Let \( M \) be a finitely generated \( R \)-module. It is well known that the local cohomology module \( H^i_m(M) \) is Artinian for all \( i \geq 0 \) (cf. [2, Theorem 7.1.3]). Suppose that \( R \) is an image of a Gorenstein local ring. R.Y. Sharp, in [11], used the local duality theorem to prove the following relation

\[
\text{Att}_{R_p} \left( H^{i-\dim(R/p)}_{pR_p}(M_p) \right) = \left\{ q_{R_p} \mid q \in \text{Att}_R(H^i_m(M)), q \subseteq p \right\}
\]

for all \( p \in \text{Supp}(M) \) and all \( i \geq 0 \). Based on the study of splitting of local cohomology (cf. [4], [5]), L.T. Nhan and the author showed that the above relation holds true on the category of finitely generated \( R \)-modules if and only if \( R \) is an image of a Cohen-Macaulay local ring (cf. [10]). It worth be noted that \( R \) is an image of a Cohen-Macaulay local ring if and only if \( R \) is universally catenary and all its formal fibers are Cohen-Macaulay by T. Kawasaki (cf. [8, Corollary 1.2]). More precisely, we proved the following.

**Theorem 2.3.** The following statements are equivalent:

(i) \( R \) is an image of a Cohen-Macaulay local ring;
Throughout this section, let $(R, m, k)$ be a commutative Noetherian local ring that is an image of a Cohen-Macaulay local ring. The following plays the key role in our proof of the main result.

**Proposition 3.1.** Let $M$ and $N$ be finitely generated $R$-modules and $\varphi : M \to N$ a homomorphism. For each $i \geq 0$, $\varphi$ induces the homomorphism $\varphi^i : H^i_m(M) \to H^i_m(N)$. Suppose for all $p \in \text{Att}_R(H^i_m(M))$ and $p \neq m$, the map $\varphi_p : M_p \to N_p$ induces the zero map

$$\varphi^i_p : H^i_{pR_p}(M_p) \to H^i_{pR_p}(N_p),$$

where $t_p = \text{dim } R/p$. Then $\text{Im}(\varphi^i)$ has finite length.

**Proof.** Suppose $\text{Im}(\varphi^i)$ has not finite length. By Lemma 2.1 there exists $m \neq p \in \text{Att}_R(\text{Im}(\varphi^i))$. So $p \in \text{Att}_R(H^i_m(M))$ by Lemma 2.1 (iv). Consider $\text{Im}(\varphi^i)$ as an Artinian $R$-module. By Lemma 2.2, there exists $P \in \text{Att}_R(\text{Im}(\varphi^i))$ such that $P \cap R = p$. Hence we have $P \in \text{Att}_{\widehat{R}}(H^i_m(M))$ by Lemma 2.1 (iv) again. Since $\widehat{R}$ is an image of a Cohen-Macaulay local ring, Theorem 2.3 (iii) implies that $P \in \text{Ass}_{\widehat{R}}(\widehat{R}/p\widehat{R})$. Therefore $\dim \widehat{R}/P = \dim R/p$ by [3, Theorem 2.1.15]. We have $\widehat{R}$ is complete, so it is an image of a Gorenstein local ring $S$ (of dimension $n$). By local duality we have

$$D(\text{Ext}^{n-i}_{S}(\widehat{M}, S)) \cong H^i_{\widehat{m}}(\widehat{M}) \quad (\cong H^i_m(M) \otimes_R \widehat{R} \cong H^i_m(M)),$$

where $D = \text{Hom}_{\widehat{R}}(-, E_{\widehat{R}}(k))$ is the Matlis duality functor (cf. [2, Theorem 11.2.6]). Since $\widehat{R}$ is complete we have $\text{Ext}^{n-i}_{S}(\widehat{M}, S) \cong D(H^i_m(M))$.

We write the map $\varphi^i : H^i_m(M) \to H^i_m(N)$ as the composition of two maps

$$H^i_m(M) \to \mathcal{I} = \text{Im}(\varphi^i) \to H^i_m(N),$$

where the first of which is surjective and the second injective. Applying the Matlis duality functor $D$ we get the map $D(\varphi^i) : \text{Ext}^{n-i}_{S}(\widehat{N}, S) \to \text{Ext}^{n-i}_{S}(\widehat{M}, S)$ is the composition of two maps

$$\text{Ext}^{n-i}_{S}(\widehat{N}, S) \to D(\mathcal{I}) \to \text{Ext}^{n-i}_{S}(\widehat{M}, S)$$

3 Proof the main result

Throughout this section, let $(R, m, k)$ be a commutative Noetherian local ring that is an image of a Cohen-Macaulay local ring. The following plays the key role in our proof of the main result.

**Proposition 3.1.** Let $M$ and $N$ be finitely generated $R$-modules and $\varphi : M \to N$ a homomorphism. For each $i \geq 0$, $\varphi$ induces the homomorphism $\varphi^i : H^i_m(M) \to H^i_m(N)$. Suppose for all $p \in \text{Att}_R(H^i_m(M))$ and $p \neq m$, the map $\varphi_p : M_p \to N_p$ induces the zero map

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$$D(\text{Ext}^{n-i}_{S}(\widehat{M}, S)) \cong H^i_{\widehat{m}}(\widehat{M}) \quad (\cong H^i_m(M) \otimes_R \widehat{R} \cong H^i_m(M)),$$

where $D = \text{Hom}_{\widehat{R}}(-, E_{\widehat{R}}(k))$ is the Matlis duality functor (cf. [2, Theorem 11.2.6]). Since $\widehat{R}$ is complete we have $\text{Ext}^{n-i}_{S}(\widehat{M}, S) \cong D(H^i_m(M))$.

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$$H^i_m(M) \to \mathcal{I} = \text{Im}(\varphi^i) \to H^i_m(N),$$

where the first of which is surjective and the second injective. Applying the Matlis duality functor $D$ we get the map $D(\varphi^i) : \text{Ext}^{n-i}_{S}(\widehat{N}, S) \to \text{Ext}^{n-i}_{S}(\widehat{M}, S)$ is the composition of two maps

$$\text{Ext}^{n-i}_{S}(\widehat{N}, S) \to D(\mathcal{I}) \to \text{Ext}^{n-i}_{S}(\widehat{M}, S)$$
with the first of which is surjective and the second injective. We have \( D(I) \) is a finitely generated \( \hat{R} \)-module and \( P \in \text{Ass}_{\hat{R}} D(I) \) \((= \text{Att}_{\hat{R}} I)\). Let \( P' \) be the pre-image of \( P \) in \( S \). Localization at \( P' \) the above composition we get the composition

\[
\text{Ext}^{n-i}_{S'_{p'}}(\hat{N}_P, S_{p'}) \to (D(I))_P \to \text{Ext}^{n-i}_{S'_{p'}}(\hat{M}_P, S_{p'})
\]

with the first of which is surjective and the second injective. Since \((D(I))_P \neq 0\), we have the map

\[
D(\varphi^i)_P : \text{Ext}^{n-i}_{S'_{p'}}(\hat{N}_P, S_{p'}) \to \text{Ext}^{n-i}_{S'_{p'}}(\hat{M}_P, S_{p'})
\]

is a non-zero map. Notice that \( \dim S_{p'} = n - t_p \). Applying local duality (for \( S_{p'} \)) we have the map

\[
\varphi^{i-t_p}_P : H^{i-t_p}_{p\hat{R}_P}(\hat{M}_P) \to H^{i-t_p}_{p\hat{R}_P}(\hat{N}_P),
\]

induced from the map \( \varphi : \hat{M} \to \hat{N} \), is a non-zero map. Recalling our assumption that the map

\[
\varphi^{i-t_p}_P : H^{i-t_p}_{p\hat{R}_P}(M_P) \to H^{i-t_p}_{p\hat{R}_P}(N_P),
\]

induced from \( \varphi : M \to N \), is zero.

On the other hand, the faithfully flat homomorphism of local rings \((R, m) \to (\hat{R}, \hat{m})\) induces the faithfully flat homomorphism of local rings \((R_p, p\hat{R}) \to (\hat{R}_p, P\hat{R}_P)\) that satisfies \( \sqrt{p\hat{R}_P} = P\hat{R}_P \). Using the following commutative diagram of flat homomorphisms

\[
\begin{array}{ccc}
R & \longrightarrow & R_p \\
\downarrow & & \downarrow \\
\hat{R} & \longrightarrow & \hat{R}_p
\end{array}
\]

one can check that

\[
H^{i-t_p}_{p\hat{R}_P}(\hat{M}_P) \cong H^{i-t_p}_{p\hat{R}_P}(M_P) \otimes_{R_p} \hat{R}_P
\]

and \( \tilde{\varphi}^{i-t_p}_P \) is just \( \varphi^{i-t_p}_P \otimes_{R_p} \hat{R}_P \). Therefore we have the maps \( \varphi^{i-t_p}_P \) and \( \tilde{\varphi}^{i-t_p}_P \) are either zero or non-zero, simultaneously. This is a contradiction. The proof is complete. \( \square \)

We are ready to prove the main result of this paper. In the rest of this section, \((R, m)\) is a local domain of positive characteristic \( p \) that is an image of a Cohen-Macaulay local ring. Let \( I \) be an ideal of \( R \) and \( R' \) an \( R \)-algebra. Notice that the local cohomology, \( H^i_I(-) \), can be computed via the Čech co-complex of the generators of \( I \). The Frobenius ring homomorphism

\[
f : R' \longrightarrow R'; r \mapsto r^p
\]

induces a map \( f_* : H^i_I(R') \to H^i_I(R') \) on all \( i \geq 0 \) called the action of Frobenius on \( H^i_I(R') \). For an element \( \alpha \in H^i_I(R') \) we denote \( f_*(\alpha) \) by \( \alpha^p \).
Proof of Theorem 1.1. We proceed by induction on \( d = \dim R \). There is nothing to prove when \( d = 0 \). Assume that \( d > 0 \) and the theorem is proven for all smaller dimension. For each \( i < d \) and \( \mathfrak{p} \in \text{Att}_R H^i_m(R) \), \( \mathfrak{p} \neq \mathfrak{m} \), by the inductive hypothesis there is an \( R_\mathfrak{p}' \)-subalgebra \( \widetilde{R}_\mathfrak{p}^i \) that is finite as \( R_\mathfrak{p} \)-module such that the natural map

\[
H^i_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p}') \to H^i_{\mathfrak{p}R_\mathfrak{p}}(\widetilde{R}_\mathfrak{p}^i)
\]

is the zero map, where \( t_\mathfrak{p} = \dim R/\mathfrak{p} \). Let \( \widetilde{R}_\mathfrak{p}^i = R_\mathfrak{p}'[z_1, \ldots, z_k] \), where \( z_1, \ldots, z_k \in \overline{K} \) are integral over \( R_\mathfrak{p} \). Multiplying, if necessary, some suitable element of \( R \setminus \mathfrak{p} \), we can assume that each \( z_j \) is integral over \( R \). Set \( R_\mathfrak{p}^i = R'[z_1, \ldots, z_k] \). Clearly, \( R_\mathfrak{p}^i \) is an \( R' \)-subalgebra of \( \overline{K} \) that is finite as \( R \)-module.

Since the set \( \{ i \mid 0 \leq i < d \} \) and \( \text{Att}_R (H^i_m(R)) \) are finite, the following is a finite extension of \( R \)

\[
R^* = R[R^i_\mathfrak{p} \mid i < d, \mathfrak{p} \in \text{Att}_R (H^i_m(R)) \setminus \{ \mathfrak{m} \}].
\]

We have \( R^* \) is an \( R^i_\mathfrak{p} \)-subalgebra of \( \overline{K} \) for all \( i < d \) and all \( \mathfrak{p} \in \text{Att}_R (H^i_m(R)) \setminus \{ \mathfrak{m} \} \). The inclusions \( R' \to R^i_\mathfrak{p} \to R^* \) induce the natural maps

\[
H^i_{\mathfrak{p}R_\mathfrak{p}}(R'_\mathfrak{p}) \to H^i_{\mathfrak{p}R_\mathfrak{p}}(\widetilde{R}_\mathfrak{p}^i) \to H^i_{\mathfrak{p}R_\mathfrak{p}}(R^*_\mathfrak{p}).
\]

By the construction of \( \widetilde{R}_\mathfrak{p}^i \), we have the natural map

\[
H^i_{\mathfrak{p}R_\mathfrak{p}}(R'_\mathfrak{p}) \to H^i_{\mathfrak{p}R_\mathfrak{p}}(R^*_\mathfrak{p})
\]

is the zero map for all \( i < d \) and all \( \mathfrak{p} \in \text{Att}_R H^i_m(R) \setminus \{ \mathfrak{m} \} \). By Proposition 3.1 we have the natural map

\[
\varphi^i : H^i_m(R') \to H^i_m(R^*),
\]

induced from the inclusion \( \varphi : R' \to R^* \), has \( \ell(\text{Im}(\varphi^i)) < \infty \) for all \( i < d \).

Since the natural inclusion \( \varphi : R' \to R^* \) is compatible with the Frobenius homomorphism on \( R' \) and \( R^* \), we have \( \varphi^i \) is compatible with the Frobenius action \( f_\mathfrak{m} \) on \( H^i_\mathfrak{m}(R') \) and \( H^i_\mathfrak{m}(R^*) \). Therefore \( \text{Im}(\varphi^i) \) is an \( f_\mathfrak{m} \)-stable \( R \)-submodule of \( H^i_\mathfrak{m}(R^*) \), i.e. \( f_\mathfrak{m}(\alpha) \in \text{Im}(\varphi^i) \) for every \( \alpha \in \text{Im}(\varphi^i) \). Since \( \text{Im}(\varphi^i) \) has finite length, every \( \alpha \in \text{Im}(\varphi^i) \) satisfies the "equation lemma" of Huneke-Lyubeznik (cf. [7 Theorem 2.2]). Applying the "equation lemma" for all generators of \( \text{Im}(\varphi^i) \) (that can be chosen as a finite set) and for all \( i < d \), we get an \( R^* \)-subalgebra \( R'' \) of \( \overline{K} \) that is finite as \( R \)-module such that the natural map \( H^i_\mathfrak{m}(R') \to H^i_\mathfrak{m}(R'') \) is zero for all \( i < d \). The proof is complete. \( \square \)

Similar to [7 Corollary 2.3] we have the following.

Corollary 3.2. Let \((R, \mathfrak{m})\) be a commutative Noetherian local domain containing a field of positive characteristic \( p \) and \( R^+ \) the absolute integral closure of \( R \) in \( \overline{K} \). Assume that \( R \) is an image of a Cohen-Macaulay local ring. Then the following hold:
(i) $H^i_m(R^+) = 0$ for all $i < \dim R$.

(ii) Every system of parameters of $R$ is a regular sequence on $R^+$, i.e. $R^+$ is a big Cohen-Macaulay algebra.

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