ON THE EXISTENCE OF A GYSIN MORPHISM FOR THE BLOW-UP OF AN ORDINARY SINGULARITY

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Abstract. In this paper we characterize the Blowing-up maps of ordinary singularities for which there exists a natural Gysin morphism, i.e. a bivariant class $\theta \in \text{Hom}_{D(Y)}(R\pi_\ast \mathbb{Q}_X, \mathbb{Q}_Y)$, compatible with pullback and with restriction to the complement of the singularity.

Keywords: Bivariant theory, Gysin morphism, Blowing-up, Derived category, Borel-Moore Homology, Isolated singularities, Projective contractions.

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To the memory of Sacha

1. Introduction

One of the problems that have most attracted Sacha Lascu’s interest throughout his mathematical career, is the projective contractability of a smooth divisor inside a smooth projective variety (see e.g. [20], [21] and [16]). In this paper we are aimed at a topological problem closely related to this. More specifically, a consequence of the main result of [16] is that a smooth space curve $C \subseteq \mathbb{P}^3$ is contractable on a general surface of large degree $X$ containing $C$ iff $C$ is $\mathbb{Q}$-subcanonical. For such curves there exists a morphism $\pi : X \to Y$ contracting $C$ to the unique ordinary singular point of $Y$. The main result of this paper is that the morphism $\pi : X \to Y$ admits a Gysin map iff $C$ is rational, and this condition is equivalent to say that $Y$ is an homology manifold, hence Poincaré Duality holds true on $Y$ (compare with §4).

Consider a projective variety $Y \subseteq \mathbb{P}^N(C)$ with an ordinary singularity $\infty \in Y$, i.e. a singularity whose projective tangent cone $G \subseteq \mathbb{P}^{N-1}$ is smooth and connected. Set

$$X := \text{Bl}_\infty(Y) \xrightarrow{\pi} Y$$

the Blow-up at $\infty$. Of course, in this context we have a surjection of Chow groups

$$\mathbb{A}^\bullet(X) \to \mathbb{A}^\bullet(Y) \to 0,$$

but very seldom it happens that the push-forward for rational homology groups

$$\pi_* : H_* (X; \mathbb{Q}) \to H_* (Y; \mathbb{Q})$$

is surjective too. The surjectivity of push-forward $\pi_*$ is closely related to the existence of some kind of Gysin map, i.e. a “wrong way” morphism between rational
homology groups

\[ H_\bullet(Y; \mathbb{Q}) \longrightarrow H_\bullet(X; \mathbb{Q}) \]

or, dually, between rational cohomology groups

\[ H^\bullet(X; \mathbb{Q}) \longrightarrow H^\bullet(X; \mathbb{Q}). \]

The existence of natural morphisms like (2) or (3) had been extensively studied by Fulton and MacPherson in [15], where it is introduced the concept of \textit{bivariant theory}. These are “simultaneous generalizations of covariant group valued homology-like theories and contravariant ring valued cohomology-like theories” ([15, p. v]).

A bivariant theory from a category \( \mathcal{C} \) to abelian groups assigns to each morphism \( X \xrightarrow{f} Y \) in \( \mathcal{C} \) a (usually graded) group \( T(X \xrightarrow{f} Y) \). Such an assignment must satisfy appropriate axioms that ensure the existence of products, pullbacks and pushforwards. When \( \mathcal{C} \) is the category of complex algebraic varieties a \textit{Topological bivariant theory} can be defined in such a way that [15, §7], \[ T^i(X \xrightarrow{f} Y) := \text{Hom}_{D(Y)}(Rf^*\mathbb{Q}_X, \mathbb{Q}_Y[i]), \]

where \( X \xrightarrow{f} Y \) is a proper morphism of algebraic varieties, \( i \in \mathbb{Z} \), and \( D(Y) \) is the \textit{bounded derived category} of sheaves of \( \mathbb{Q} \)-vector spaces on \( Y \). An element

\[ \theta \in \text{Hom}_{D(Y)}(Rf^*\mathbb{Q}_X, \mathbb{Q}_Y[i]) \]

produces Gysin-like morphisms

\[ H_\bullet(Y; \mathbb{Q}) \longrightarrow H_{\bullet+i}(X; \mathbb{Q}), \quad H^\bullet(X; \mathbb{Q}) \longrightarrow H^{\bullet-i}(Y; \mathbb{Q}). \]

Such morphisms turn out to be defined only for particular maps of algebraic varieties. In particular, natural Gysin maps are defined when the target \( Y \) is smooth and when \( X \xrightarrow{f} Y \) is either local complete intersection or flat ([23, §4.5]). Unfortunately, our map (1) is neither local complete intersection nor flat, and in general it does not admit a Gysin morphism. Nevertheless, in particular cases (see §4) there is a Gysin morphism satisfying some natural conditions such as compatibility with pullback and restriction to the complement of the singularity (compare with Definition 2.3).

In this paper we prove a characterization of desingularizations like (1) admitting a natural Gysin morphism (see Theorem 3.5). What it turns out is that there exists a natural Gysin morphism if and only if we have a decomposition in \( D(Y) \):

\[ R\pi_*\mathbb{Q}_X \simeq \bigoplus_{h \geq 0} R^h\pi_*\mathbb{Q}_X[-h]. \]

Such a decomposition is very reminiscent of the Decomposition Theorem, as stated e.g. in [3, Remark 1.6.2, (3)], and of the Lefay-Hirsch Theorem (compare with the proof of Lemma 2.5 in [10]). Indeed, in view of Proposition 4.1, formula (4) could be proved as a consequence of the Decomposition Theorem. Nevertheless, in order to prove (4), we prefer to follow the (somewhat more direct) approach of §3.

One may ask whether our result holds true with \( \mathbb{Z} \)-coefficients. We have in mind to return on this question in a future paper.
2. Notations

Notations 2.1. (1) For any algebraic variety $X$ we will denote by $Q_X$ the constant sheaf on $X$, by $\mathcal{V}_X$ the category of sheaves of $Q_X$-modules, and by $D(X)$ the bounded derived category of $\mathcal{V}_X$.

(2) Consider a projective variety $Y \subseteq \mathbb{P}^N$ with an ordinary singularity $\infty \in Y$, i.e. a singularity whose projective tangent cone $G \subseteq \mathbb{P}^{N-1}$ is smooth. Set $X := Bl_\infty(Y) \xrightarrow{\pi} Y$ the Blow-up at $\infty$. Of course we have an inclusion $X \subseteq Bl_\infty(\mathbb{P}^N)$ and $G \xrightarrow{i} X$ coincides with the exceptional divisor. Furthermore, we set $U := X - G = Y - \{\infty\}$.

(3) Since the morphism $\pi : X \to Y$ is proper, for any sheaf $F \in \mathcal{V}_X$ the direct image with proper support $\pi_! F \in \mathcal{V}_Y$ [17, §2.6], [12, Definition 2.3.21] coincides with the ordinary direct image $\pi_* F \in \mathcal{V}_Y$.

Remark 2.2. By definition of direct image with proper support ([17 §2.6], [12 Definition 2.3.21]), the sheaf $k_! Q_U$ ($j_! Q_U$ resp.) can be identified with the sub-sheaf of $Q_Y$ (of $Q_X$ resp.) consisting of sections with support contained in $U$.

Definition 2.3. We will say that a graded morphism $\theta : H^\bullet(X; Q) \to H^\bullet(Y; Q)$ is natural if the following conditions are satisfied:

1. the composite of $\theta$ with the pullback $\theta \circ \pi^* : H^\bullet(Y; Q) \to H^\bullet(Y; Q)$ is the identity map;
2. $\theta$ is compatible with restrictions on $U$: $j^* = k^* \circ \theta : H^\bullet(X; Q) \to H^\bullet(U; Q)$.

Definition 2.4. Consider a (topological) bivariant class [15 §7], [2] $\theta \in Hom_D(Y)(R\pi_* Q_X, Q_Y)$.

By abuse of notations, we also denote by $\theta$ the map induced by such a class on the cohomology groups [15, 2]:

$\theta : H^\bullet(X; Q) \to H^\bullet(Y; Q)$.

According to Definition 2.3 we will say that $\theta$ defines a natural Gysin map if the last morphism is natural.
Lemma 2.5. Keep notations as above. Then pullbacks give isomorphisms

\[ H^h(Y; \mathbb{Q}) \simeq H^h(Y, \{\infty\}; \mathbb{Q}) \simeq H^h(X, G; \mathbb{Q}), \quad \forall h \geq 1. \]

Assume additionally that there exists a natural morphism:

\[ \theta : H^\bullet(X; \mathbb{Q}) \longrightarrow H^\bullet(Y; \mathbb{Q}). \]

Then the map

\[ (\theta, i^*) : H^\bullet(X; \mathbb{Q}) \longrightarrow H^\bullet(Y; \mathbb{Q}) \oplus H^\bullet(G; \mathbb{Q}) \]

is an isomorphism of graded groups in degree \( \geq 1 \).

Proof. The isomorphism \( H^h(Y, \{\infty\}; \mathbb{Q}) \simeq H^h(Y; \mathbb{Q}) \) follows from the long exact sequence:

\[ \cdots \longrightarrow H^h(Y, \{\infty\}; \mathbb{Q}) \longrightarrow H^h(Y; \mathbb{Q}) \longrightarrow H^h(\{\infty\}; \mathbb{Q}) \longrightarrow \cdots \]

As for the isomorphism

\[ \pi^* : H^h(Y, \{\infty\}; \mathbb{Q}) \simeq H^h(X, G; \mathbb{Q}) \]

we are going to give two different proofs.

**Topological proof:** consider a small open neighborhood \( \infty \in B \) and set \( T := \pi^{-1}(B) \). Of course \( T \) is a tubular neighborhood of \( G \) in \( X \) and we have \( \partial B \simeq \partial T \). Moreover, \( \{\infty\} \) and \( G \) are tautly imbedded in \( Y \) and \( X \) \[22\], p. 289]. Hence we have

\[ H^h(Y, \{\infty\}; \mathbb{Q}) \simeq H^h(Y, B; \mathbb{Q}) \simeq H^h(Y - B, \partial B; \mathbb{Q}) \]

\[ \simeq H^h(X - T, \partial T; \mathbb{Q}) \simeq H^h(X, T; \mathbb{Q}) \simeq H^h(X, G; \mathbb{Q}). \]

**Sheaf theoretic proof:** by \[1\] Theorem 12.1, \[12\] Remark 2.4.5, (ii)], and Remark \[2.2\] we have

\[ H^h(Y, \{\infty\}; \mathbb{Q}) \simeq H^h(Y, k_\mathbb{Q}U) = H^h(Y, \pi(j_\mathbb{Q}U)) \]

\[ \simeq H^h(Y, \pi_\ast(j_\mathbb{Q}U)) \simeq H^h(X, j_\mathbb{Q}U) \simeq H^h(X, G; \mathbb{Q}). \]

In order to prove \( \ref{6} \), look at the following long exact sequence:

\[ \cdots \longrightarrow H^h(Y; \mathbb{Q}) \longrightarrow H^h(X, G; \mathbb{Q}) \longrightarrow H^h(X; \mathbb{Q}) \longrightarrow H^h(G; \mathbb{Q}) \longrightarrow \cdots \]

By Definition \[2.3\] (1), the map \( \pi^* : H^h(Y; \mathbb{Q}) \longrightarrow H^h(X; \mathbb{Q}) \) is injective \( \forall h \geq 1 \), so we have:

\[ 0 \longrightarrow H^h(Y; \mathbb{Q}) \longrightarrow H^h(X; \mathbb{Q}) \longrightarrow H^h(G; \mathbb{Q}) \longrightarrow 0, \]

and we are done. \( \square \)
3. The main result

Notations 3.1. (1) Combining [17 I, Theorem 6.2] with [23 §7.3.2], we see that the natural morphism $Q_Y \rightarrow \pi_*Q_X$ in $\mathcal{V}_Y$ is induced by an element

$$i_0 \in \text{Hom}_{D(Y)}(Q_Y, R\pi_*Q_X).$$

(2) We denote by $K^\bullet$ an injective resolution of $Q_Y$.

(3) We denote by $I^\bullet$ an injective resolution of $Q_X$. By [17 II, Corollary 4.13], $J^\bullet := \pi_*I^\bullet$ can be identified as the derived direct image $R\pi_*Q_X$ in $D(Y)$. So, when $h \geq 1$, $R^h\pi_*Q_X = H^h(J^\bullet)$ is the skyscraper sheaf supported on $\infty$, with stalk at $\infty$ given by $H^h(G; Q)$. Furthermore, the morphism $i_0$ defined in (1) can be seen as an element in $[K^\bullet, J^\bullet]$.

We prove the following result which will be needed throughout the paper (compare with [5 Proposition 1.2]).

Proposition 3.2. Assume that, for any $h \geq 1$, the identity map $R^h\pi_*Q_X \rightarrow R^h\pi_*Q_X$ lifts to a morphism $i_h \in \text{Hom}_{D(Y)}(R^h\pi_*Q_X, R\pi_*Q_X[h])$. Then we have an isomorphism in $D(Y)$:

$$\bigoplus_{h \geq 0} i_h : Q_Y + \sum_{h \geq 1} R^h\pi_*Q_X[-h] \longleftarrow R\pi_*Q_X.$$

Proof. The above morphism is well defined by [31 (1)]. Since $G$ is connected we have $\pi_*Q_X = Q_Y$, i.e. $H^0(J^\bullet) = H^0(K^\bullet)$. Furthermore, the complex $R^h\pi_*Q_X[h]$ is injective for all $h \geq 1$, because $R^h\pi_*Q_X$ is a skyscraper sheaf. So we have

$$Q_Y + \sum_{h \geq 1} R^h\pi_*Q_X[-h] = K^\bullet + \sum_{h \geq 1} R^h\pi_*Q_X[-h]$$

in $D(Y)$. We are done because our hypothesis implies that

$$H^h \left( K^\bullet + \sum_{h \geq 1} R^h\pi_*Q_X[-h] \right) = H^h(J^\bullet) \forall h,$$

therefore the map $\bigoplus_{h \geq 0} i_h$ is a quasi-isomorphism.

Remark 3.3. By [5 Proposition 1.2], the hypothesis of Proposition 3.2 is equivalent to the fact that, for any cohomological functor $T$ from $\mathcal{V}_Y$ to an abelian category, the spectral sequence

$$E_2^{pq} = T(R^p\pi_*Q_X[p]) \Rightarrow RT(\pi_*Q_X[p + q])$$

degenerates at $E_2^{pq}$.

Proposition 3.4. Assume that there exists a natural morphism

$$H^\bullet(X; Q) \rightarrow H^\bullet(Y; Q).$$

Then the hypothesis of Proposition 3.2 is satisfied, i.e. the identity map $R^h\pi_*Q_X \rightarrow R^h\pi_*Q_X$ lifts to a morphism $i_h \in \text{Hom}_{D(Y)}(R^h\pi_*Q_X, R\pi_*Q_X[h])$ for any $h \geq 1$. 
Proof. Keep notations as in [3,1]. Set $\Gamma^* := \Gamma(J^*)$ and denote by $d^h : \Gamma^h \to \Gamma^{h+1}$ the differential. Then we have $H^h(X; \mathbb{Q}) = H^h(\Gamma^*)$, and by hypothesis any element of $H^h(G; \mathbb{Q})$ can be lifted to an element $\gamma \in Ker d^h$. We claim that any $\alpha \in H^h(G, \mathbb{Q})$ can be lifted to an element $\beta \in Ker d^h \subseteq \Gamma(J^h) = \Gamma^h$ which is supported on $\infty$. Of course, to prove our claim amounts to show that any $\alpha \in H^h(G; \mathbb{Q})$ can be lifted to an element $\beta \in Ker d^h \subseteq \Gamma(J^h) = \Gamma^h$ such that $\beta \mid_{\infty} = 0 \in \Gamma(J^h \mid_{\infty})$. But $\gamma \mid_{\infty}$ projects to a cohomology class living in $Im(H^h(X; \mathbb{Q}) \to H^h(U; \mathbb{Q}))$. By (2) of Definition 2.3, we have

$$Im(H^h(X; \mathbb{Q}) \to H^h(U; \mathbb{Q})) \subseteq Im(H^h(Y; \mathbb{Q}) \to H^h(U; \mathbb{Q})).$$

By Lemma 2.5, we find

$$Im(H^h(Y; \mathbb{Q}) \to H^h(U; \mathbb{Q})) = Im(H^h(X, G; \mathbb{Q}) \to H^h(U; \mathbb{Q})).$$

Since

$$H^h(Y, k_0\mathbb{Q}) \simeq H^h(Y, \infty; \mathbb{Q}) \simeq H^h(X, G; \mathbb{Q}) \simeq H^h(X, j_0\mathbb{Q})$$

([11, Theorem 12.1], [12, Remark 2.4.5, (ii)]), Remark 2.2 implies that there exists $\delta_U \in \Gamma(J^{h-1} \mid_{\infty})$ and $\sigma \in \Gamma(J^h)$ supported in $U$ such that

$$\gamma \mid_{\infty} - d^{h-1} (\delta_U) = \sigma \mid_{\infty}.$$  

Finally, there exists $\delta \in \Gamma(J^{h-1})$ with $\delta \mid_{\infty} = \delta_U$, because $J^{h-1}$ is injective (hence flabby). We conclude that the section

$$\gamma - \sigma - d^{h-1}(\delta) \in \Gamma(J^h)$$

is supported on $\infty$. Our claim is proved because $\sigma + d^{h-1}(\delta) \in \Gamma(J^h)$ vanishes in $H^h(G; \mathbb{Q})$.

To conclude the proof, fix a basis $\alpha_r \in H^h(G; \mathbb{Q})$ and lift any $\alpha_r$ to a $\beta_r \in Ker d^h \subseteq \Gamma(J^h) = \Gamma^h$ as in the claim. We get an isomorphism between $H^h(G; \mathbb{Q})$ and a subspace of $\Gamma(J^h)$ consisting of sections supported on $\infty$. We are done because such an isomorphism projects to a monomorphism of sheaves:

$$R^h \pi_* Q_X \hookrightarrow Ker(J^h \to J^{h+1}).$$

\[\square\]

Theorem 3.5. Keep notations as above. Then the following properties are equivalent:

1. there exists a natural Gysin map $\theta \in Hom_{D(Y)}(R \pi_* Q_X, Q_Y)$;
2. there exists a natural morphism of graded groups

$$H^\bullet(X; \mathbb{Q}) \longrightarrow H^\bullet(Y; \mathbb{Q});$$

3. we have an isomorphism in $D(Y)$:

$$\bigoplus_{h \geq 0} i_h : Q_Y \oplus \sum_{h \geq 1} R^h \pi_* Q_X [-h] \longleftrightarrow R \pi_* Q_X.$$
Proof. (1) $\Rightarrow$ (2) by Definitions 2.3 and 2.4. (2) $\Rightarrow$ (3) just combining Propositions 3.2 and 3.4.

(3) $\Rightarrow$ (1): If $\bigoplus_{h \geq 0} \iota_h$ is an isomorphism in $D(Y)$, the projection on the first summand of $Q_Y + \sum_{h \geq 1} R^h \pi_* Q_X[-h]$ represents a bivariant class $\theta \in \text{Hom}_{D(Y)}(R\pi_* Q_X, Q_Y)$ such that $\theta \circ \iota_0 = id \in \text{Hom}_{D(Y)}(Q_Y, Q_Y)$, and the first condition of Definition 2.3 is satisfied.

As for (2) of Definition 2.3, since the sheaves $R^h \pi_* Q_X$ are supported on $\infty$, we have $Q_Y|_U = ((\theta \circ \iota_0)Q_Y)|_U = \theta(Q_X)|_U = Q_X|_U$ in $D(U)$, and we are done. \hfill $\square$

4. Examples

Examples of Blow-up admitting a natural Gysin morphism are the following:

- any surface $Y$ with a node (cfr. [4], p. 127, Remark 3.3.3, and [19], p. 159, Example 2);
- the cone $Y \subseteq \mathbb{P}^N$ over a smooth projective variety $M \subseteq \mathbb{P}^{m-1}$ of dimension $m$ such that $H^\bullet(M) \cong H^\bullet(\mathbb{P}^m)$.

This follows from the Proposition 4.1 below, which gives further characterizations of the existence of a natural Gysin morphism. As for the equivalence of properties (2)-(5) in Proposition 4.1, we think they are certainly well-known. However we briefly give the proof for lack of a suitable reference (cfr. [4], p. 127, Remark 3.3.3). In the sequel we will denote by $IH(Y)$ the intersection cohomology of $Y$ (see e.g. [12], p. 154-159), by $IC^\bullet_Y$ the intersection cohomology complex of $Y$ (cfr. [12], p.159) and by $H^\text{BM}_k(U)$ the Borel-Moore homology of $U$. Furthermore, following [18] we say that a variety $Y$ of dimension $n$ is a $\mathbb{Q}$-intersection cohomology manifold when $IC^\bullet_Y \cong Q_Y[n]$, in $D(Y)$. We refer to [14], Appendix B, for some properties of Borel-Moore homology which we need in the proof. All cohomology and homology groups are with $\mathbb{Q}$-coefficients.

**Proposition 4.1.** Let $Y \subseteq \mathbb{P}^N$ be a projective irreducible variety of complex dimension $m + 1$, with a unique singular point $\infty \in Y$, which is an ordinary singularity. Let $\pi : X \rightarrow Y$ be the Blow-up at $\infty$, with smooth and connected exceptional divisor $G$. The following properties are equivalent.

1. There exists a natural morphism of graded groups $\theta : H^\bullet(X) \rightarrow H^\bullet(Y)$.
2. The duality morphism $H^\bullet(Y) \cong \text{Hom}_{D(Y)}(H^\bullet(Y), \mathbb{C})$ is an isomorphism (i.e. $Y$ satisfies Poincaré Duality).
3. $H^\bullet(G) \cong H^\bullet(\mathbb{P}^m)$. 
(4) The natural map map $H^\bullet(Y) \to IH^\bullet(Y)$ is an isomorphism.

(5) $Y$ is $\mathbb{Q}$-intersection cohomology manifold.

Proof. First we prove that (2) is equivalent to (3).

Since the singular locus of $Y$ is finite, by [19] we know that $Y$ satisfies Poincaré Duality if and only if $Y$ is a homology manifold, i.e. if and only if $H^h(Y, Y \setminus \{y\}) \cong H^h(\mathbb{R}^{2(m+1)}, \mathbb{R}^{2(m+1)} \setminus \{0\})$ for any $y \in Y$. This condition is certainly verified if $y$ is a regular point of $Y$. Choose a small closed ball $D \subseteq \mathbb{P}^N$ around $\infty$ and set $B := D \cap Y$. By excision we have $H^h(Y, Y \setminus \{\infty\}) \cong H^h(B, B \setminus \{\infty\})$. Recall that $B$ is homeomorphic to the cone over the link $K := \partial D \cap Y$ of the singularity $\infty \in Y$, with vertex at $\infty$ ([11], p. 23). In particular $B$ is contractible. Therefore, from the long exact sequence of cohomology of the couple $(B, B \setminus \{\infty\})$, it follows that $Y$ is a homology manifold if and only if $B \setminus \{\infty\}$ has the same $\mathbb{Q}$-homology type as a sphere $S^{2m+1}$. This in turn is equivalent to say that the link $K$ has the same $\mathbb{Q}$-homology type as a sphere $S^{2m+1}$, because $K$ is a deformation retract of $B \setminus \{\infty\}$.

On the other hand, via deformation to the normal cone, we may identify the link of the vertex of the projective cone over the exceptional divisor $G \subseteq \mathbb{P}^{N-1}$. Restricting the Hopf bundle $S^{2N-1} \to \mathbb{P}^{N-1}$ to $G$, we obtain an $S^1$-bundle $K \to G$ inducing the Thom-Gysin sequence ([22], p.260)

$$
\cdots \to H^h(G) \to H^h(K) \to H^{h-1}(G) \to H^{h+1}(G) \to H^{h+1}(K) \to \cdots
$$

And this sequence implies that $K$ has the same $\mathbb{Q}$-homology type as a sphere $S^{2m+1}$ if and only if $H^\bullet(G) \cong H^\bullet(\mathbb{P}^m)$.

Now we are going to prove that (2) is equivalent to (4).

First assume that property (2) holds true. Since the singular locus of $Y$ is finite, by ([12], p. 157) we already know that $H^h(Y) = IH^h(Y)$ if $h > m+1$. Moreover we know that $IH^{m+1} = \exists(H^{m+1}(Y) \to H^{m+1}(U))$, where $U = Y \setminus \{\infty\}$. On the other hand $H^h(U) = H_B^{BM}(U)$ ([13], p. 217, (26)), and from the natural exact sequence ([14], p. 219, Lemma 3)

$$
\cdots \to H_i(\{\infty\}) \to H_i(Y) \to H_i^{BM}(U) \to \cdots
$$

we see that $H_B^{BM}(2m+1-h)(U) = H_2(m+1-h)(Y)$ for $h < 2(m+1)$. In particular for $h = m+1$ we have $H^{m+1}(U) = H_{m+1}(Y)$, and therefore $IH^{m+1} = \exists(H^{m+1}(Y) \to H_{m+1}(Y)) = H_{m+1}(Y) = H^{m+1}(Y)$. Finally, when $h < m+1$ then we have $IH^h(Y) = H^h(U) = H_2(m+1-h)(Y) = H^h(Y)$. Conversely assume that property (4) holds true. Since intersection cohomology verifies Poincaré Duality ([12], p. 158), we have:

$$
H^h(Y) = IH^h(Y) = (IH^{2(m+1)-h}(Y))^\vee = (H^{2(m+1)-h}(Y))^\vee = H_2(m+1-h)(Y).
$$

Next we prove that property (2) implies (1).

To this purpose, consider the following commutative natural diagram:

$$
\begin{array}{cccc}
H^{h-1}(G) & \to & H^h(X, G) & \to & H^h(X) & \to & H^h(G) \\
\uparrow & & \uparrow \pi^* & & \uparrow & & \uparrow \\
H^{h-1}(\infty) & \to & H^h(Y, \infty) & \to & H^h(Y) & \to & H^h(\infty).
\end{array}
$$
Since $H^\bullet(G) \cong H^\bullet(P^m)$, it follows that the restriction map $H^{h-1}(X) \to H^{h-1}(G)$ is surjective for any $h$. Therefore from previous diagram we deduce that the pullback $\pi^*: H^h(Y) \to H^h(X)$ is injective, and that $H^h(Y) \cong \pi^*H^h(Y) = H^h(X)$ if $h$ is odd (either $h = 0$ or $h \geq 2(m+1)$), and that $\dim H^h(X) - \dim H^h(Y) = 1$ if $h$ is even with $2 \leq h \leq 2m$. Next put:

$$I^h := \ker(H^h(X) \to H^h(U)) = \Im(H^h(X, U) \to H^h(X)).$$

Since $X$ is smooth we have $H^h(X, U) \cong H_{2(m+1)-h}(G)$ ([22], p.351, Lemma 14), and the map $H^h(X, U) \to H^h(X)$ identifies with the push-forward:

$$H_{2(m+1)-h}(G) \to H_{2(m+1)-h}(X) \cong H^h(X).$$

Therefore $\dim I^h = 1$ if $h$ is even with $2 \leq h \leq 2m+1$, and $\dim I^h = 0$ otherwise. Now consider the following natural diagram, where all maps are restrictions:

$$
\begin{align*}
H^h(X) & \xrightarrow{\pi^*} H^h(Y) \\
\downarrow j^* & \quad \downarrow k^* \\
H^h(U)
\end{align*}
$$

By functoriality this diagram commutes:

$$k^* = j^* \circ \pi^*.$$

As before, when $h$ is even with $2 \leq h \leq 2m$, the restriction map $k^*: H^h(Y) \to H^h(U)$ identifies with the duality morphism $H^h(Y) \to H_{2(m+1)-h}(Y)$, which is bijective by our assumption. Since $I^h \cap \pi^*H^h(Y) = \pi^*(\ker k^*)$, it follows that

$$I^h \cap \pi^*H^h(Y) = 0.$$

Therefore, by dimensional reasons, we have

$$H^h(X) = \begin{cases} 
\pi^*H^h(Y) \oplus I^h & \text{if } h \text{ is even with } 2 \leq h \leq 2m \\
\pi^*H^h(Y) & \text{otherwise}.
\end{cases}$$

We are in position to define the natural morphism $\theta: H^h(X) \to H^h(Y)$. In fact, when $h$ is even with $2 \leq h \leq 2m$, for any $x = \pi^*(y) + i \in H^h(X) = \pi^*H^h(Y) \oplus I^h$ put:

$$\theta(x) := y,$$

and put

$$\theta := (\pi^*)^{-1}$$

otherwise. It is evident that $\theta \circ \pi^* = \id_{H^h(Y)}$, and that $k^* \circ \theta = j^*$ when $\theta = (\pi^*)^{-1}$, because $j^* \circ \pi^* = k^*$. When $h$ is even with $2 \leq h \leq 2m$, taking into account that $I^h = \ker j^*$, we have: $(k^* \circ \theta)(x) = k^*(y) = (j^* \circ \pi^*)(y) = j^*(\pi^*(y) + i) = j^*(x)$.

Finally we are going to prove that property (1) implies (2).

Since $\theta \circ \pi^* = \id_{H^h(Y)}$ we get the decomposition

$$H^h(P) = \pi^*H^h(Y) \oplus \ker \theta.$$

Hence $\theta$ is compatible with restrictions on $U$ if and only if

$$\ker \theta \subseteq I^h.$$
(in fact for any $x \in H^h(X)$, $x = \pi^*y + c$, with $c \in \ker \theta$, we have $(k^* \circ \theta)(x) = k^*(y) = (j^* \circ \pi^*)(y)$; therefore $(k^* \circ \theta)(x) = j^*(x)$ if and only if $j^*(x) = j^*(\pi^*y + c) = (j^* \circ \pi^*)(y) + j^*(c) = (j^* \circ \pi^*)(y)$, hence if and only if $j^*(c) = 0$). On the other hand, combining Lemma 2.5 with (8), we get for $h > 0$

$$\dim \ker \theta = \dim H^h(G),$$

and therefore (cfr. [7]), for any $h > 0$

$$\dim H^h(G) \leq \dim I^h \leq \dim H_{2(m+1)-h}(G) = \dim H^{h-2}(G).$$

Finally, for the equivalence between (2) and (5) we refer the reader to [19] and [18].

Remark 4.2. From previous Proposition we see that, if existing, the natural morphism $\theta : H^*(X) \rightarrow H^*(Y)$ is unique and identifies with the push-forward via Poincaré Duality:

$$H^*(X) \cong H_{2(m+1)-*}(X) \rightarrow H_{2(m+1)-*}(Y) \cong H^*(Y).$$

In fact $\theta = (k^*)^{-1} \circ j^*$, and $k^* : H^h(Y) \rightarrow H^h(U)$, for $h < 2(m+1)$, is nothing but the duality morphism because $H^h(U) \cong H_{2(m+1)-h}(Y)$.

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