Finiteness and anomalies in (4, 0) supersymmetric sigma models

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ABSTRACT

Power-counting arguments based on extended superfields have been used to argue that two-dimensional supersymmetric sigma models with (4,0) supersymmetry are finite. This result is confirmed up to three loop order in perturbation theory by an explicit calculation using (1,0) superfields. In particular, it is shown that the finite counterterms which must be introduced into the theory in order to maintain (4,0) supersymmetry are precisely the terms that are required to establish ultra-violet finiteness.
1. Introduction

It has been known for some time that there several examples of supersymmetric quantum field theories which are finite to all orders in perturbation theory (for a review see, e.g. [1]). Included in the list of finite theories are some two-dimensional non-linear sigma models; in particular, it was shown by Alvarez-Gaumé and Freedman [2] that supersymmetric models with hyperkähler target spaces have this property (see also [3,4]). These sigma models have $N = 4$ supersymmetry, or in chiral notation, $(4,4)$ supersymmetry. Using (conventional) extended superfield power-counting arguments, the authors of refs. [5,6] argued that sigma models with $(4,p)$ supersymmetry, $p = 0, \ldots, 4$, are also be finite. In the case of $(4,0)$ supersymmetry this result was confirmed by harmonic superspace methods in ref. [7] and by an analysis of the $(4,0)$ Ward identities in ref. [8].

Two-dimensional supersymmetric sigma models with four-dimensional target spaces are of interest in the context of solutions to heterotic string theory for which spacetime has the form of a product of six-dimensional Minkowski space and a non-trivial four-dimensional Riemannian space. This type of solution is identified as a “five-brane” soliton in string theory and has been studied in a series of papers by Callan, Harvey and Strominger [9,10,11,12,13]. (For further results on this subject see refs. [14,15].) If the solution preserves six-dimensional spacetime supersymmetry, then the sigma model whose target space is the four-dimensional space transverse to the five-brane should be $(4,0)$ supersymmetric [16,17,11]. In references [9,10,11,12,13] several explicit solutions have been studied, with particular emphasis being placed on those which have $(4,4)$ worldsheet supersymmetry.

In order to get a solution of string theory it is necessary that the sigma model be superconformally invariant, a requirement automatically met by finite sigma models. As we have remarked $(4,4)$ sigma models have this property [2], including those with torsion [5]. According to the results quoted above, sigma models with $(4,0)$ supersymmetry are also finite, but the authors of refs. [11,13] have claimed that this is not the case, so that there appears to be a contradiction. It is the
purpose of this article to clarify the issues that have arisen and to resolve this contradiction. We shall confirm the finiteness of (4,0) sigma models explicitly up to three loop order in perturbation theory and identify the corrections to the metric as arising from finite local counterterms which must be added to the theory in order to maintain (4,0) supersymmetry.

As we understand it, the argument that (4,0) sigma models are not finite given in refs [11,13] is based on the fact that the spacetime supersymmetry transformations, computed for example in ref. [18] in the effective ten-dimensional supergravity theory for the massless modes of the string, do not automatically vanish for background spacetimes corresponding to (4,0) sigma models. This, it is claimed, shows that there must be higher order corrections to the sigma model beta-functions. However, the world-sheet sigma model for the heterotic string is naturally only (1,0) supersymmetric; if a solution is (4,0) supersymmetric at lowest order in \( \alpha' \) it does not follow that such a sigma model remains (4,0) supersymmetric at higher orders in perturbation theory. In fact, in general, such a sigma model does not remain (4,0) supersymmetric unless finite local counterterms are added. The introduction of the correct finite corrections at \( L-1 \) loop order should, and indeed does, ensure that the beta-functions vanish at the \( L^{th} \) loop order. Thus, for (4,0) theories there are higher-order corrections to the metric, as the authors of refs. [11,13] have stated, and when taken into account they correct the spacetime supersymmetry transformations, whereas this is not necessary in the (4,4) case. However, these corrections come from finite local counterterms and are moreover computable in terms of the tree level background fields. From refs. [5,6,7,8] we know that (4,0) superconformal invariance can be implemented at all orders, so that (4,0) sigma models do provide solutions to heterotic string theory of the type we have been discussing, and the explicit forms of the background fields can be found in principle by implementing (4,0) supersymmetry. This is somewhat different to the case of \( N = 2 \) sigma models on Calabi-Yau spaces, where the corrections to the metric that are required to restore superconformal invariance involve solving differential equations of progressively higher order on the target space [19].
The organisation of the paper is as follows: in the next section we briefly review (4,0) sigma models, in particular those for which the target space is four-dimensional; in section 3 we analyse the one-loop anomaly and show how it is cancelled; in section 4 we explicitly compute the metric beta-function up to three-loop order using the results of ref. [20], and show that it vanishes to this order modulo sigma model field redefinitions (diffeomorphisms of the target space). We make some concluding remarks in section 5.

2. (4,0) sigma models

The study of the interplay between the geometry of the target space and extended supersymmetry in two-dimensional supersymmetric sigma models began with the realisation that $N = 2$ supersymmetry requires the target space to be a Kähler manifold [21], and was followed by the result that $N = 4$ requires hyperkähler target manifolds [2]. Models with torsion and heterotic supersymmetry involve different geometries, and these have been discussed by many authors; see, for example, refs. [22,23,5,6,24]. We shall follow [5,6] here.

The classical action for the $(1,0)$ sigma model is

$$S[g, b, \phi] = -i \int d^2 x d\theta^+ (g_{ij} + b_{ij}) D_+ \phi^i \partial_+ \phi^j \quad (2.1)$$

where $\phi$ is a map from $(1,0)$ superspace, $\Sigma^{(1,0)}$, with real (light-cone) co-ordinates $(x^+, x^-, \theta^+)$, to the target space $M$ given in local co-ordinates by $\phi^i(x^+, x^-, \theta^+)$, $i = 1, \cdots, \dim M$. $M$ is equipped with a metric $g$ and a locally defined two-form $b$. The fermionic derivative $D_+$ satisfies

$$D_+^2 = i \partial_+ \quad (2.2)$$

The action (2.1) is $(4,0)$ supersymmetric if $M$ admits three complex structures.
\( I_r \) \((r = 1, 2, 3)\) obeying the algebra of the imaginary unit quaternions,

\[
I_r I_s = -\delta_{rs} + \epsilon_{rst} I_t, \tag{2.3}
\]

if \( g \) is Hermitian with respect to all three complex structures, and if each of these is covariantly constant with respect to \( \Gamma^{(+)} \):

\[
g_{ij} I^i_{(r)k} I^j_{(r)l} = g_{kl} \tag{2.4}
\]

\[
\nabla_k^{(+)} I^i_{ri} = 0, \tag{2.5}
\]

where repeated indices in parentheses are not summed over and

\[
\Gamma^{(\pm) i}_{jk} = \Gamma^i_{jk} \pm \frac{1}{2} H^i_{jk}; \quad H_{ijk} = 3 \partial_i [b_{jk}]. \tag{2.6}
\]

If \( M \) is \( n \)-dimensional, then \( n = 4m \) for some integer \( m \). The additional supersymmetry transformations are given by

\[
\delta_r \phi^i = i \zeta_{(r)I} I^i_{(r)j} D_+ \phi^j \tag{2.7}
\]

where \( \zeta_r, r = 1, 2, 3 \) are the anticommuting parameters of the extended supersymmetry transformations which we take to be constant. If \( E \) is a real rank \( q \) vector bundle over \( M \) with connection \( A \) and \( \psi \) a section of \( \phi^* E \otimes S_- \) where \( S_- \) is a bundle of right-handed spinors over \( \Sigma^{(1,0)} \), then we can generalise the \((1,0)\) action \((2.3)\) by adding to it a fermionic sector with the action

\[
S_F = - \int d^2 x d\theta^+ \psi_-^A \nabla_+ \psi_-^A; \tag{2.8}
\]

where

\[
\nabla_+ \psi_-^A = D_+ \psi_-^A + D_+ \phi^i A^A_{iB} \psi_-^B; \quad A = 1, \ldots q. \tag{2.9}
\]

This is not the most general action of this type, but will suffice for the present purpose.
The action \((2.8)\) is \((4,0)\) supersymmetric under

\[
\delta_r \psi^A_r = -i \zeta_r I^j_{(r)j} D + \phi^j A^A_{iB} \psi^B_r + i \zeta_r \hat{I}^A_{(r)B} \nabla^B \psi^B_r
\]  \hspace{1cm} (2.10)

if the set of three complex structures \(\hat{I}_r\) obeys the algebra of the imaginary unit quaternions and if all three are covariantly constant with respect to \(A\). In addition, the curvature of \(A\) must be \((1,1)\) with respect to all three complex structures,

\[
F_{ij} = F_{kl} I^k_{(r)i} I^l_{(r)j}
\]  \hspace{1cm} (2.11)

The covariant constancy of the \(\hat{I}\)'s implies that the holonomy of the connection \(A\) must be a subgroup of \(Sp(q/4)\),

\[
\hat{I}^A_{rC} F^C_{ijB} = F^A_{ijC} \hat{I}^C_{rB}.
\]  \hspace{1cm} (2.12)

If \(M\) is a four-dimensional manifold then the equation (2.11) is familiar from instanton physics; it is the condition for \(F\) to be self-dual. In addition it was shown in reference [11] that the metric \(g\) is conformally equivalent to a Ricci flat metric \(g^0\), i.e.

\[
g = \exp \Omega g^0
\]  \hspace{1cm} (2.13)

where \(\Omega\) is harmonic with respect to the Laplacian of the metric \(g\) and \(H = *d\Omega\), where \(*\) denotes the Hodge dual. If \(M\) is a compact manifold without boundary, \(\Omega\) is constant and the torsion \(H\) vanishes. In this paper, we shall be concerned with two classes of models: for type one we will take the target space \(M\) to be compact without boundary, and hence hyperkähler (i.e. a \(K3\) surface), with a self-dual \(Sp(q/4)\) gauge field, chosen so that the instanton number is equal to the Euler number of \(M\) (\(K3\) has been studied in the context of string compactifications to six dimensions previously [25]); for type two we shall consider instantons on \(R^4\). We shall be able to discuss the two cases together, since for both of them the Ricci tensor and the torsion vanish at the zeroth order in \(\alpha'\).
Since the algebra of supersymmetry transformations (2.7) and (2.10) closes without the use of the field equations, we have an off-shell representation of (4,0) supersymmetry [5, 6]. It is therefore possible to write (4,0) sigma models in terms of (4,0) superfields, and thus to apply extended superfield non-renormalisation theorems to them. In [5, 6] it was shown that this leads one to conclude that these models should be finite. At first sight this might seem to be surprising as (2,2) superspace has the same number of $\theta$'s as (4,0) superspace but (2,2) sigma models are not finite. The crucial difference between the two cases is that the (4,0) superspace measure is not two-dimensionally Lorentz invariant, so that any counterterm Lagrangian will also not be a Lorentz scalar. For a counterterm to be constructed from scalar fields, for example, it follows that there must be derivatives in the counterterm Lagrangian, and this immediately gives improved power-counting behaviour compared to the (2,2) case. This property of the measure was also used in the harmonic superspace analysis of the ultra-violet behaviour of (4,0) sigma models [7].

3. Anomalies

It is convenient when discussing the anomalies to introduce a vielbein $e^a_i$ on $M$; the spin connection for the local orthonormal frame rotations which corresponds to the Levi-Civita connection is $\omega_{ih}^a$ and we define, as usual,

$$\omega_i^{(\pm)ab} = \omega_i^{ab} \pm \frac{1}{2} H_i^{ab}. \quad (3.1)$$

The action (2.1) can be regarded as a function of $\phi$, $e$ and $b$. It is invariant under the (generalised) transformations

$$\delta_l e_i^a = l_b^a (\phi) e_i^b, \quad \delta_l b_{ij} = 0, \quad \delta_l \phi^i = 0 \quad (3.2)$$

$$\delta_v e_i^a = v^k \partial_k e_i^a + \partial_i v^k e_k^a, \quad \delta_v \phi^i = -v^i$$

$$\delta_v b_{ij} = v^k \partial_k b_{ij} + \partial_i v^k b_{kj} - \partial_j v^k b_{ki} \quad (3.3)$$
\[ \delta_{m} b_{ij} = \partial_{i} m_{j} - \partial_{j} m_{i} \] (3.4)

where the parameters \( l_{ab}(\phi) (-l_{ba}(\phi)) \), \( v^{i}(\phi) \) and \( m_{i}(\phi) \) correspond to local frame rotations, diffeomorphisms and antisymmetric gauge transformations respectively.

For the fermion sector we have the gauge transformations

\[
\begin{align*}
\delta_{u} \psi^{A} &= u_{B}^{A} \psi_{B}, \quad \delta_{u} \phi^{i} = 0 \\
\delta_{u} A_{iB}^{A} &= -\partial_{i} u_{B}^{A} + u_{C}^{A} A_{iB}^{C} - A_{iC}^{A} u_{B}^{C}
\end{align*}
\] (3.5)

The above transformations are not symmetries in the usual field-theoretic sense since they involve transforming the parameters (couplings) (as well as the sigma model field in the case of diffeomorphisms). Nevertheless, they lead to identities which are satisfied by the classical action and which one wishes to preserve at the quantum level in order to maintain the geometric interpretation of the parameters as fields on the target space. For the (4,0) models we also have the additional supersymmetry transformations (2.7) and (2.10) which commute with the gauge transformations.

Sigma models are most easily quantised using the background field method. The quantum field \( \xi^{a}(x, \theta) \) is a section of \( \phi^{*}TM \) where \( \phi \) is the background field. The background-quantum split introduces a new shift symmetry, but a regularization scheme which preserves this symmetry and the anti-symmetric tensor gauge invariance can always be found. A Ward identity proof of the non-anomalous nature of the shift symmetry has also been given [26]. Furthermore, by a suitable choice of counterterms the theory can be made invariant under diffeomorphisms. Thus we are led to consider the one-loop background effective action \( \Gamma[e, b, \phi] \) as a function of the parameters \( e, b \) and the background field \( \phi \) which satisfies

\[ \delta_{u} \Gamma = \delta_{m} \Gamma = 0 \] (3.6)

and

\[ \delta_{l} \Gamma = \Delta(l) \] (3.7)

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\[ \delta_{\zeta_r} \Gamma = \Delta(\zeta_r) \quad (3.8) \]

where \( \Delta(l) \) and \( \Delta(\zeta_r) \) are the potential obstructions; they are integrated local functionals of the background field \( \phi \). The Wess-Zumino consistency conditions [27] include

\[ \delta l_1 \Delta(l_2) - \delta l_2 \Delta(l_1) = \Delta([l_1, l_2]) \quad (3.9) \]

\[ \delta l \Delta(\zeta_r) - \delta \zeta_r \Delta(l) = 0 \quad (3.10) \]

\[ \delta \zeta_r \Delta(\zeta_s) - \delta \zeta_s \Delta(\zeta_r) = 0. \quad (3.11) \]

Equation (3.9) is the usual Wess-Zumino equation for the gauge anomaly; its solution is

\[ \Delta(l) = -ik \int d^2x d\theta^+ Q^1_{ij}(l, \omega) D_+ \phi^i \partial_- \phi^j \quad (3.12) \]

where \( \omega \) is an \( SO(n) \) connection which can be arranged to be equal to \( \omega^{(-)} \) [28]. The constant \( k = \frac{\alpha'}{2} \) by an explicit calculation [28]. (In terms of two-dimensional quantum field theory, \( \alpha' = \frac{\hbar}{2\pi} \).) \( Q^1_2 \) is the usual two-dimensional anomaly occurring in the descent sequence [27]

\[ P_4 \rightarrow Q^0_3 \rightarrow Q^1_2 \rightarrow Q^2_1 \rightarrow Q^3_0; \quad P_4 = \text{Tr}R^2(\omega). \quad (3.13) \]

(3.10) then implies that

\[ \Delta(\zeta_r) = 3ik \int d^2x d\theta^+ \delta \zeta_r \phi^i Q^0_{ijk}(\omega^{(-)}) D_+ \phi^i \partial_- \phi^k + (l - \text{invariant}) \quad (3.14) \]

The identities (3.11) for \( r = s \) will be satisfied by (3.14) with no \( l \)-invariant term because \( P_4(\omega^{(-)}) \) is a \( (2, 2) \) form with respect to all three complex structures. The cross-identities, \( r \neq s \), of (3.11) are also satisfied. The absence of \( l \)-invariant terms in (3.14) can be confirmed by an explicit one loop calculation.
It is straightforward to include the fermion sector. For the gauge anomaly, the only changes are that \( l \) is replaced by \( u \) and \( \omega \) by \( A \); there is also a sign change for \( \Delta(u) \) as compared to \( \Delta(l) \). Furthermore, since \( P_4(A) \) is \((2,2)\) with respect to all three complex structures the consistent (potential) supersymmetry anomaly is given by (3.14) with \( Q_3^0(\omega(-)) \) replaced by \( Q_3^0(\omega(-)) - Q_3^0(A) \).

Having identified the potential obstructions to implementing the symmetries \((3.2) \rightarrow (3.5)\) quantum mechanically, we need to examine whether or not they can be removed by finite local counterterms, i.e. we need to find a finite local counterterm whose variation under supersymmetry gives precisely (3.14).

Since \( P_4 \) is a \((2,2)\) form it can be written in three different ways as

\[
P_4 = dd_{(r)}Y_{(r)}
\]  

(3.15)

where \( Y_r \) is a \((1,1)\) form with respect to \( I_r \). The derivative \( d_r \) is defined as follows: for each complex structure we can define holomorphic and anti-holomorphic derivatives \( \partial_r \) and \( \bar{\partial}_r \); the exterior derivative \( d \) is then the sum of the two whereas \( d_r \) is \( i(\partial_r - \bar{\partial}_r) \). The Chern-Simons three-form \( Q_3^0 \) can also be written in three different ways as

\[
Q_3^0 = dX_r + d_{(r)}Y_{(r)}
\]  

(3.16)

where the \( X_r \) are two-forms. The necessary and sufficient conditions for the cancellation of supersymmetry anomalies is the existence of a symmetric second rank tensor \( T \) which is hermitian with respect to all the complex structures and a two-form (possibly locally defined on \( M \)) \( X \) such that

\[
Y_{r ij} = T_{ik}I_{r j}^k \quad \text{and} \quad X = X_r
\]  

(3.17)

Indeed, in this case there is a finite local counterterm for which the action, \( S_c \), is

\[
S_c = -\frac{i\alpha'}{2} \int d^2xd\theta^+ (T + X)_{ij}D_+\phi^i\partial_\pm\phi^j
\]  

(3.18)

This finite local counterterm, if it exists, cancels the extended supersymmetry anomalies and the \( SO(n) \) anomaly as well. This can be shown by a straightforward
computation, and is similar to the anomaly cancellation process in (2, 0) models [29,30]. The \( SO(n) \) transformation property of \( X \) is

\[
\delta_l X = -Q_2^1(l, \omega^{(-)})
\]  

(3.19)

so that if \( b \) is redefined by \( b \rightarrow \bar{b} = b + \alpha' X \), the field \( \bar{b} \) has the familiar “anomalous” transformation rule \( \delta \bar{b} = -\alpha' Q_2^1(l, \omega^{(-)}) \) [31] (and similarly for gauge transformations). There may be some remaining symmetries in the theory like the holomorphic symmetries for the (2, 0) case [30]. However, these do not cause any problems; we shall comment on them a little later. Finally, there is a global reparametrisation anomaly in the theory which is cancelled provided that \( P_4 \) is an exact form on \( M \).

The analysis given so far is applicable to any (4,0) target space. We shall now focus on the case of four-dimensional target spaces where there are some simplifications. For both types of model we shall consider we can write \( P \equiv P_4(\omega) - P_4(A) \) in the form

\[
P = d \ast df
\]  

(3.20)

where \( f \) is a differentiable function on the target space. This is not difficult to see in the compact case, using the Hodge decomposition, and may be explicitly verified in the instanton case. For this latter case \( f \) is given by

\[
f = \triangle \text{Tr} \log h
\]  

(3.21)

where \( \triangle \) is the ordinary Laplacian on \( R^4 \). The matrix \( h \), which is a \( k \times k \) matrix for an instanton with instanton number \( k \), is given by

\[
h^{-1} = a_{ij} y^i y^j - b_i y^i + c
\]  

(3.22)

where \( a \), \( b \) and \( c \) are \( k \times k \) matrices with entries which are the instanton parameters, and \( y^i \) are Cartesian co-ordinates on \( R^4 \). In both cases the symmetric tensor \( T \) is
given by

\[ T_{ij} = -3fg_{ij} \quad (3.23) \]

and is clearly hermitian with respect to all three complex structures. We can also show that there is an antisymmetric tensor \( X \) that satisfies (3.17). To prove this observe that for four-dimensional manifolds the second term in the right-hand-side of (3.16) can be written in such a way that the complex structures do not appear explicitly by using the \( \epsilon \)-tensor. Hence the supersymmetry and gauge anomalies are cancelled in this case. The remaining symmetry of the theory is the ambiguity in specifying the instanton parameters of a given self-dual connection. This is a rigid symmetry and the finite local counterterm is invariant under it.

4. The beta function

We now turn to the evaluation of the metric \( \beta \)-function. Our strategy is as follows: the (1,0) sigma model \( \beta \)-function for the metric has been computed up to two [32,23] and three loops [20]. We shall express this \( \beta \)-function in terms of parameters \( \bar{g}, \bar{b} \) and then correct it order by order in \( \alpha' \) by demanding (4,0) supersymmetry. Thus at lowest order, \( \bar{g} = g \) and \( \bar{b} = b = 0 \), where \( g \) is the (4,0) metric, while at first order, we must add the finite local counterterm (3.18) in order to cancel the anomalies. This amounts to a redefinition of the parameters by

\[ \bar{g}_{ij} = g_{ij} - \frac{3}{2}\alpha'fg_{ij} \quad (4.1) \]

and

\[ \bar{b}_{ij} = b_{ij} + \frac{\alpha'}{2}X_{ij} \quad (4.2) \]

We then substitute these expressions into the two-loop \( \beta \)-function and evaluate it to order \( (\alpha')^2 \) in terms of the parameters \( g, b \). Having done this, we go to the next order which will entail adding \( (\alpha')^2 \) terms to the right-hand sides of (4.1) and (4.2).
In principle, we could calculate these second-order terms by looking at the two-loop anomaly, but we have not done this. However, we know they must exist, and it is easy to guess the correct form for the metric from the three-loop $\beta$-function. Because $b = 0$, the second order correction to $b$ is not needed for the order to which we are working. In this way we are able to calculate the $\beta$-function (for $\bar{g}$) order by order in $\alpha'$ in terms of the “known” metric $g$ (as well as the gauge field $A$). Not surprisingly, it vanishes up to a Lie derivative of the metric which can be removed by a sigma model field renormalisation. When this last step has been carried out, the $\beta$-function for $\bar{g}$ is zero (to this order) and so the $\beta$-function for $g$ is also zero. By the general results of refs [5,6,7,8] we know this can be done to all orders, which means that there is a renormalisation scheme, using the parameters $g, b, A$, in which the metric $\beta$-function vanishes order by order in perturbation theory, i.e. the theory is finite.

We now turn to the details of the three-loop calculation outlined above. Up to two loops the $(1,0)$ sigma model metric $\beta$-function is [23]

$$\beta_{ij}(\bar{g}) = -\alpha' R_{(ij)}^{(+)} - \frac{(\alpha')^2}{4} \text{Tr} \left( R_{ik}^{(-)} R_{jk}^{(-)} - F_{ik} F_{jk} \right)$$

(4.3)

expressed in terms of parameters $\bar{g}, \bar{b}$ (and $\bar{A}$, which we can set equal to $A$). The curvature tensors $R^{(+)}$ and $R^{(-)}$ are computed from the connections $\Gamma^{(+)}$ and $\Gamma^{(-)}$ respectively. They obey the identity

$$R_{ij,kl}^{(+)} = R_{kl,ij}^{(-)}$$

(4.4)

where the first pair of indices are the Lie algebra indices, and the second pair the differential form indices. The trace in (4.3) is over the Lie algebra indices. Because of (4.4), the Ricci tensors are not symmetric and this necessitates the symmetrisation in the one-loop contribution. Alternatively, we can write

$$R_{(ij)}^{(+)} = R_{ij} - \frac{1}{4} \bar{H}_{i}^{kl} \bar{H}_{jkl}$$

(4.5)
where $\tilde{H}$ is the Chern-Simons corrected torsion, i.e

$$\tilde{H} = d\bar{b} - \frac{\alpha'}{2}(Q_3^0(\omega^{(-)}) - Q_3^0(A))$$  (4.6)

The (+) and (−) connections appearing in equations (4.3)(4.6) above now involve the corrected torsion $\tilde{H}$, and so must be found iteratively. It is usually assumed that the Chern-Simons terms appear in this way, and this has not been explicitly verified except at two loops [33]; for our purposes, though, it will only be necessary to consider the Chern-Simons term for the standard spin connection of the metric $g$. In fact, because $b$ is zero, the $H^2$ term does not play any role until $O(\alpha')^3$. Following the strategy outlined above, it is clear that the $O(\alpha')$ contribution to the beta-function vanishes as $g$ is Ricci-flat. To the next order, we have to expand $R_{ij}(\bar{g})$ to order $\alpha'$ and include the lowest order contribution from the two-loop term. Using the easily proved relation,

$$\text{Tr}(R_{ik}R^{k}j - F_{ik}F^{k}j) = \frac{1}{4}g_{ij}\text{Tr}(R^2 - F^2)$$  (4.7)

valid for self-dual curvature tensors, one can show that the $\beta$-function reduces to

$$\beta_{ij}(\bar{g}) = -\frac{3}{2}(\alpha')^2\nabla_i\nabla_j f$$  (4.8)

to this order. This remaining $O(\alpha')^2$ contribution is thus of the form $\nabla_i(v_j)$ and so can be removed by making a suitable redefinition of the sigma model field. Hence the $\beta$-function vanishes to two-loop order.

We now turn to the three-loop calculation. This was carried out by Foakes et al [20], and their result agrees with the calculation carried out from a string theory point of view by Metsaev and Tseytlin [34]. The complete expression for a general (1,0) model is quite complicated, but for our purposes we need only evaluate it for the metric $g$, with $b$ set equal to zero. In this case nearly all the terms drop out.
due to Ricci-flatness, and we are left with the contribution

$$\beta_{ij}^{(3)} = -\frac{(\alpha')^3}{32} \Delta \text{Tr} \left( R_{ik} R^k_j - F_{ik} F^k_j \right)$$  \hspace{1cm} (4.9)$$

Due to the identity (4.7), this can be written, to the desired degree of accuracy, as

$$\beta_{ij}^{(3)} = 3(\alpha')^3 \Delta^2 f g_{ij}.$$  \hspace{1cm} (4.10)$$

Clearly, this term can be removed by a two-loop adjustment to the metric which should be taken to be $-\frac{3}{16} (\alpha')^2 \Delta f g_{ij}$. Notice again that it is a finite adjustment at the two-loop level which produces the three loop cancellation in the $\beta$-function. However, we are not finished yet as we should also take into account the effect of the one-loop counterterms at three loops.

There are three contributions: the first is from the $O(\alpha')^2$ terms in $R_{ij}(\bar{g})$, the second from the $H^2$ term which is contained in $R_{+}^{ij}$, and which contributes first at three-loop order, and the third is from making the first-order shift in the two-loop $\beta$-function, not forgetting the term (4.8). When all these contributions are summed the complete result, to three-loop order, is

$$\beta_{ij}(\bar{g}) = \bar{\nabla}_{(i} v_{j)}$$  \hspace{1cm} (4.11)$$

where

$$v_i = \bar{\nabla}_i \left( -\frac{3(\alpha')^2}{2} f - \frac{9(\alpha')^3}{8} f^2 - \frac{3}{16} (\alpha')^3 \Delta f \right)$$  \hspace{1cm} (4.12)$$

Thus the three loop result reduces to a Lie derivative which can be removed by a wave-function renormalisation. Hence we have explicitly verified the superconformal invariance of the model up to three loops and have shown that the finite counterterms required to restore $(4,0)$ supersymmetry are responsible for the finiteness of the theory, at least in the metric sector of the theory. The expression for
the metric $\bar{g}$ is, to this order

$$\bar{g}_{ij} = g_{ij} - \frac{3}{2}\alpha' f g_{ij} - \frac{3}{16}(\alpha')^2 \Delta f g_{ij}. \quad (4.13)$$

To our knowledge, there does not exist a complete three-loop calculation of all of the $\beta$-functions in a general $(1,0)$ sigma model, but a three-loop calculation has been made of $\beta$-function for $A$ in the case of a background gauge field only (i.e. a trivial metric) [35]. This result covers the case of the gauge solution, and it is a short computation to verify that the $\beta$-function for $A$ vanishes up to three loops when $F$ is self-dual. Explicitly, the $\beta$-function for $A$ is

$$\beta_{i}^{AB} = -\frac{1}{4}(\alpha')\nabla^{j}F_{ij}^{AB} - \frac{3}{64}(\alpha')^{3}\left[\nabla_{k}F_{ij}^{AB}F_{CD}^{k} + 2F_{kAB}^{ij}\nabla_{(k}F_{l)CD}^{j}\right]F_{jlCD}. \quad (4.14)$$

The first term clearly vanishes when $F$ is self-dual, and it is not difficult to see that the three-loop contribution does also. Further, the conformal rescaling of the metric induced by the finite local counterterms does not affect this result.

5. Conclusions

In this article we have argued that there exists a renormalisation scheme for the $(4,0)$ sigma model in which the metric $\beta$-function vanishes order by order in perturbation theory, and we have explicitly confirmed this up to three loop order. This scheme is the natural one from the point of view of $(4,0)$ world-sheet supersymmetry, whereas the scheme using $\bar{g}$ and $\bar{b}$ is more natural from the point of view of $(1,0)$ supersymmetry. The difference between two methods of calculation in a field theory should only result in a change of scheme as far as the $\beta$-functions are concerned; thus, although it is more difficult in the $(4,0)$ formalism to see explicitly what is going on in terms of the metric, etc., since this method of calculation manifestly preserves $(4,0)$ supersymmetry, it must be the case that the (finite) counterterms which arise in the $(1,0)$ computation are automatically
included in the (4,0) calculation. On the other hand, from the point of view of string theory, it is the $\bar{g}$ metric which is more natural, since the heterotic string is in general only (1,0) supersymmetric. The $\beta$-function for this metric is not zero order by order, as we have seen, although it does vanish when summed. To this extent, the authors of refs. [11,13] have a point, although we would maintain that it is the metric $g$ which is natural in (4,0) supersymmetry.

The parameters $\bar{g}$ and $\bar{b}$ do not satisfy the (4,0) conditions given in section 2, but instead satisfy more complicated constraints. It would be of interest to determine these equations and to compare them with spacetime results, and this is under investigation. Finally, we have concentrated in this paper on the case of four-dimensional target spaces as these are both topical and simpler to deal with. However, we believe that these results can be extended to the general case, where the dimension of the target space is a multiple of four, and our belief is fortified by the general results of refs. [5,6,7,8].

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