Fluid turbulence and eddy viscosity in relativistic heavy-ion collisions

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The eddy viscosity for a turbulent compressible fluid with a relativistic equation of state is derived. Compressibility allows for sound modes, but the eddy viscosity in the shear mode is found to be the same as for incompressible fluids. For two space dimensions (which is the relevant case for the dynamics of relativistic heavy-ion collisions) the eddy viscosity in the shear mode is negative, reducing the effective viscosity below its microscopic value. This could explain the tiny viscosity found at RHIC. Implications for the experimentally accessible elliptic flow coefficient at the LHC are speculated on.

Efforts to understand the bulk physics of the ongoing experimental program at the Relativistic Heavy-Ion Collider (RHIC) have led to an interest in the theory of relativistic fluid dynamics, both with and without shear viscosity. The curious fact that ideal fluid dynamics is able to describe the experimentally measured particle spectra amazingly well [1, 2, 3, 4, 5] has lead to the hypothesis that the shear viscosity $\eta$, or rather the dimensionless quantity $\eta/s$ involving the entropy density $s$, has to be extremely small [3, 7]. Indeed, values of $\eta/s \sim 1$ (in natural units $k_B = h = c = 1$) obtained by solving Quantum-Chromodynamics (QCD) in the weak coupling expansion [8] were seemingly ruled out [9]. Interest in QCD but for the smaller values [13]). While addressing the caveats outlined in [12] will ultimately allow to decide whether RHIC to be extremely small [6, 7]. Indeed, values of $\eta/s$ to description the experimentally measured particle spectra violate the bound until recently, when it was shown [12] that the experimental data from top RHIC energies actually favored $\eta/s \sim \frac{1}{T}$ (another group argues for even smaller values [13]). While addressing the caveats outlined in [12] will ultimately allow to decide whether RHIC violates this bound [12] (and for even smaller values [13]).

The Reynolds number for a relativistic fluid at temperature $T$ can be estimated as $\text{Re} \sim \frac{\eta}{T} L$, where $L$ is a typical length scale. For gold collisions at RHIC, taking $L$ to be the radius of a gold nucleus $L \sim 6 \text{ fm}$ and the QCD scale as temperature ($T \sim 200 \text{ MeV}$) imply

$$\text{Re}_{RHIC} \sim \frac{6}{\eta} \sim 48\pi \gg 1,$$

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The natural starting point are the relativistic fluid dynamic equations in the presence of shear viscosity

$$(\epsilon + p)Du^\mu = \nabla^\mu p - \Delta^\mu \beta \Pi^{\alpha\beta},$$

$$D\epsilon = - (\epsilon + p) \nabla_\mu u^\mu + \frac{1}{2} \Pi^{\mu\nu} (\nabla_\nu u_\mu), \quad (2)$$

where $\epsilon, p$ are the energy density and pressure, $u^\mu$ is the fluid four velocity obeying $u^\mu u_\mu = 1$ and $\Pi^{\mu\nu}$ is the shear tensor which to lowest order in gradients [37] is $\Pi^{\mu\nu} = \eta (\nabla_\nu u^\mu)$. The remaining symbols in Eq. (2) are: $D_\mu$, the covariant derivative, $D \equiv u^\mu D_\mu$, $D_\mu \equiv \Delta_\mu D_\nu$, $\Delta_\mu \equiv g^{\mu\nu} - u^\mu u^\nu$ and $(\nabla_\mu u^\nu) \equiv \nabla_\mu u^\nu + \nabla^\nu u^\mu - \frac{1}{2} \Delta_\mu \nabla_\nu u^\nu$. The metric signature is $g^{\mu\nu} = (\pm, \pm, \cdots)$ and $(\pm, \pm, \cdots)$ for $d = 2$ and $d = 3$ space dimensions, respectively. The equation of state $p = p(\epsilon)$ is used to close the system (2).

In what follows, an equation of state with a constant viscosity $\eta/s$ will be considered.

$$\ln \epsilon(t, x) \equiv \int \frac{d\omega d\mathbf{k}}{(2\pi)^{d+1}} \exp[-i\omega t + i\mathbf{k} \cdot \mathbf{x}] \ln \epsilon(\omega, \mathbf{k}), \quad (3)$$

and similarly for $u^i$, one finds (in flat space) from Eq. (2):

$$(\epsilon_0^{-1}(K))^{ij} u^j_K = \frac{i}{2} \int_Q u^m_K \varphi^{ilm}(K, Q) + O(\alpha^3)$$

(4)

$$G_0^{-1}(K) \ln \epsilon_K = (1+c_2^2) \int_Q u^i_K \varphi^{ij}(K, Q) + O(\alpha^3)$$

(5)

$$D_0^{-1}(K) t^{ij}_K = \left[ (-i\omega + \nu k^2) \delta^{ij} + k^i k^j \left( \nu \frac{d-2}{d} + i\frac{c_2^2}{\omega} \right) \right]$$

(6)

$$G_0^{-1}(K) \left[ -\omega^2 + c_2^2 k^2 - \frac{2(d-1)}{d} i\nu k^2 \right]$$

(7)

$$\varphi^{ilm}(K, Q) = c_2^2 k^i k^j \varphi^{ilm} + \delta^{ij} (q^m - k^m) - \delta^{im} q^j$$

(8)

$$Q^{ij}(K, Q) = \left( \frac{\omega^2}{2} + \frac{d-1}{d} i\nu k^2 \right) \delta^{ij} - k^i (k^j - q^j),$$

(9)
where $K = (\omega, \mathbf{q})$, and $\eta/(\epsilon + p)$ was approximated by its space-time average, $\nu \equiv \int dt dx \eta/(\epsilon + p)$. This approximation – though not strictly justified – will allow to connect to literature on non-relativistic turbulence, where $\nu$ is referred to as kinematic viscosity.

Following the Yakhut-Orszag approach to turbulence [18], Eq. (3) is replaced by a more general equation

\[
(D_0^{-1}(K))^{ij} u^j_K = f^i(K) + \frac{i}{2} \int Q P^{ilm}(K,Q),
\]

where $f^i$ is a random force used to model turbulent stirring at high wavenumbers $k > \Lambda$. Its correlator in $d$ space dimensions is taken to be Gaussian

\[
< f^i(K) f^j(K') >= 2a \omega^d d^{d+1} \delta^{ij} \delta(K + K')
\]

with strength $a$ and zero mean $< f^i(K) >= 0$. Splitting $u^j_K = u^j_K^{<} + u^j_K^{>} \to$ a low and high wavenumber part ($u^j_K^{<} = u^j_K \theta(\Lambda - k)$), the aim is then to derive an averaged equation for the geometric flow $u^i^{<}$ in the presence of the fluctuating $u^i^{>}$,

\[
u^j_K^{<} = \frac{i}{2} D^{lj}_0(K) \theta(\Lambda - k) \int Q P^{ilm}(K,Q) \times \left( u^{m <}_Q u^{l <}_K + u^{m >}_Q u^{l >}_K \right) .
\]

To calculate the mean $< u^{m <}_Q u^{l >}_K >$, one solves Eq. (5) for $u^r$ recursively in $u^r <$. Using Eq. (10) and focusing on the term in $u^r <$ it follows that

\[
u^j_K^{<} = 4 i a \omega^d \int Q P^{ilm}(K,Q) \times D^{lj}_0(K,Q) \beta^{ij}_K \delta^{<}(K - Q) \theta(\Lambda - k) \theta(|k - q| - \Lambda).
\]

Defining the projectors $A^{ij}_K = \delta^{ij} - B^{ij}_K$ and $B^{ij}_K = k^ik^j/q^2$, one can decompose the propagator $D_0$ into a shear and a sound mode (similar to the Kovasznay modes [19]),

\[
u^j_K^{<} = \alpha_K A^{ij}_K + \beta_K B^{ij}_K
\]

The inversion of $D_0^{-1}$ is then straightforward and one finds

\[
u^j_K^{<} = \int Q P^{ilm}(K,Q) u^{m <}_Q u^{l <}_K
\]

\[
u^j_K^{<} = \int Q P^{ilm}(K,Q) \times A^{ij}_K
\]

\[
u^j_K^{<} = \int Q P^{ilm}(K,Q) \times B^{ij}_K
\]

where $\int Q \equiv \int Q \theta(q - \Lambda) \theta(|k - q| - \Lambda)$. The frequency integrations are readily performed and in the small wavenumber limit $\omega, k \to 0$ one finds

\[
u^j_K^{<} = \int Q \theta(q - \Lambda) \theta(|k - q| - \Lambda).
\]

where only terms larger than $O(\nu) \sim O(o)$ were kept. Again, $R^{ia}$ can be decomposed as $R^{ia} = r_A A^{ia}_K + r_B B^{ia}_K$. From Eq. (3) this implies that

\[
u^j_K^{<} = \alpha_K A^{ij}_K + \beta_K B^{ij}_K.
\]

To evaluate $r_A$ in the small $k$ limit, one contracts $R^{ia} A^{ia}_K$ and upon shifting $\mathbf{q} \to \mathbf{q} + \frac{\Lambda}{k}$ and using the symmetries of the integral obtains (see also [18])

\[
u^j_K^{<} = \alpha \nu^2 k^2 \frac{2 \pi d/2}{(2\pi)^d \Gamma(d/2)} \frac{d^2 - d - 4}{8d(d + 2)} \Lambda^4 + O(k^3)
\]

Interestingly, this result is identical to that for incompressible fluids [18] and modifications for compressible fluids only arise in the sound channel of the propagator (which is absent for incompressible fluids). The equation for the full propagator Eq. (3) then implies that

\[
u^j_K^{<} = \alpha \nu^2 k^2 \frac{2 \pi d/2}{(2\pi)^d \Gamma(d/2)} \frac{d^2 - d - 4}{8d(d + 2)} \Lambda^4 + O(k^3)
\]

where the effective turbulent viscosity $\nu_{eff} = \nu + \nu_{edd}$ being the sum of microscale and eddy viscosity

\[
u_{edd} = \alpha \nu^2 k^2 \frac{2 \pi d/2}{(2\pi)^d \Gamma(d/2)} \frac{d^2 - d - 4}{8d(d + 2)} \Lambda^4
\]

has been introduced. $\nu_{edd}$ reflects the effect of turbulence acting on length scales smaller than $\Lambda^{-1}$ onto the geometric flow at scales larger than $\Lambda^{-1}$. Curiously, note that $\nu_{edd}$ is positive for $d = 3$ space dimensions, while it is negative for $d = 2$. In other words, the effective viscosity in a system exhibiting two-dimensional turbulence is smaller than the microscopic viscosity, due to the presence of a negative eddy viscosity. This interesting phenomenon has been known for decades, starting with the work of Starr and Kraichnan [20, 21] (see also [22, 23] for other approaches).
In heavy-ion collisions, one typically follows Bjorken in assuming boost-invariance in the longitudinal direction \([24]\). Therefore, the dynamics of relevance for fluid dynamics is effectively two-dimensional. Hence, if fluid turbulence develops, one expects the eddy viscosity to be negative and thus the fluid at RHIC would appear more ideal than it is based on its microscopic viscosity.

The physics of the appearance of a negative eddy viscosity seem to be tied to the phenomenon of inverse energy cascade. While in three dimensional turbulence, energy generally cascades down to smaller and smaller length scales until it is finally dissipated into heat (regular cascade), in two dimension this process can seemingly reverse: energy is transferred from smaller to larger length scales (see also the discussion in Ref. \([21]\)). A key ingredient seems to be that vorticity is conserved in two-dimensional incompressible fluids, allowing small eddies to convey energy to larger eddies. For compressible relativistic fluids, vorticity is no longer conserved \([12]\), and hence one can expect the inverse cascade to “leak” energy. Indeed, working out the turbulent correction to the dimensionless coupling parameter \(\nu\) one has to replace

\[
\frac{d^2 - d - 4}{d(d + 2)} - \frac{2(d^2 - 1) - 2c_s^2(d - 1)(d + 2)}{d(d + 2)} + \frac{d(1 - c_s^2)^2}{2(d - 1)^2}.
\]

This implies that \(\nu_B > 0\) for both \(d = 2, 3\) and \(c_s^2 \sim \frac{4}{3}\).

It seems that in two-dimensional compressible fluid turbulence the shear mode gets less dissipative while for sound the converse is true. This means that it could be difficult to decide whether to expect an increase or decrease of effective viscosity in different experimental observables. The very small apparent viscosity at RHIC has been extracted mainly from the experimental observable called “elliptic flow” \([12, 25]\). Invoking Morkovin’s Hypothesis that under certain conditions in compressible turbulence “the essential dynamics of these shear flows will follow the incompressible pattern” \([10]\), in the following it will be speculated that elliptic flow is mostly influenced by the negative eddy viscosity in the shear mode. Then, is it possible to verify or falsify the idea of fluid turbulence in relativistic heavy-ion collisions?

Clearly, some other experimentally accessible prediction from Eq. \((16)\) is needed. To do so, first introduce the dimensionless coupling parameter \(\lambda = \frac{\alpha}{\nu^2 \Lambda^2}\) so that \(\nu_{\text{eff}} = \nu(1 - \frac{\lambda}{64\pi})\), where \(d = 2\) has been used in Eq. \((16)\). Now a renormalization group improvement is performed by solving the set of equations \(\text{(18)}\)

\[
\frac{d\nu_{\text{eff}}}{d\Lambda} = \nu_{\text{eff}} \frac{\lambda}{16\pi\Lambda}, \quad \lambda = \frac{\alpha}{\nu_{\text{eff}}^2 \Lambda^2}, \quad (17)
\]

where to leading order in \(\lambda\), \(\nu\) has been replaced by \(\nu_{\text{eff}}\) and \(\alpha\) was assumed to be independent of \(\Lambda\) \([18]\). Using \(\nu_{\text{eff}}|_{\lambda=0} = \nu\), the result is \(\nu_{\text{eff}} = \nu \left(1 - \frac{3\lambda}{64\pi}\right)^{1/3}\). In other words, one expects the effective ratio \(\eta/s\) to behave as

\[
\left(\frac{\eta}{s}\right)_{\text{eff}} = \left(\frac{\eta}{s}\right) \left[1 - \lambda \left(\frac{\eta}{s}\right)^{-3}\right]^{1/3}, \quad (18)
\]

where for convenience \(\lambda = \frac{3\alpha T^3}{64\pi^2 A}\) has been introduced. It may be that \(\lambda\) still depends on \(\eta/s\): for the following qualitative discussion, this should not matter unless \(\lambda\) is proportional to \(\eta/s\) with a power greater or equal three. \(\lambda\) will typically be rather small: since \(\alpha\) has to be proportional to the temperature (the only other dimensionful scale except \(\Lambda\)), one expects \(\lambda \sim (T/\Lambda)^3\). Assuming \(\Lambda\) corresponds to the smallest resolved length scale of current heavy-ion collision simulations \([12]\) one has \(\lambda \sim 1\) GeV, so using again \(T \sim 200\) MeV gives \(\lambda \sim 10^{-3}\).

The qualitative behavior of the effective viscosity is shown in Fig. \(1\) for \(\lambda = 10^{-3}\). It can be seen that close to the critical viscosity the effective viscosity becomes very small and there is some indication that even slightly negative values of effective viscosities can be achieved. It is expected that higher non-linearities subsequently render the effective viscosity again positive \([23]\). Since these terms have been ignored in the above calculation, the result in Eq. \((18)\) cannot be trusted below the critical viscosity. Nevertheless, the strong suppression of the effective viscosity close to the critical viscosity suggested by Eq. \((18)\) should be experimentally accessible, e.g. by a measurement of the elliptic flow at the upcoming Large Hadron Collider (LHC). The magnitude of elliptic flow is known to decrease for increasing \(\eta/s\). Turning the argument around, if fluid turbulence is in operation at RHIC, and elliptic flow is indeed mostly affected by the negative eddy viscosity in the shear mode, the effect at the LHC should be even more pronounced: from Fig. \(1\) one would expect the elliptic flow to increase beyond the
RHIC values, maybe even beyond the “ideal hydrodynamic limit”, if a negative effective viscosity is realized for an extended period of time. Note that more conventional approaches \cite{26,27} call for a decrease of elliptic flow at the LHC, while extrapolations from existing data do indicate an increase \cite{28}.

It should be stressed that the above calculation of the eddy viscosity for fluid turbulence in relativistic heavy-ion collisions can be regarded as qualitative at best. However, given interest in the community, there are many ways to improve or strengthen Eq. \cite{13}, e.g. by relaxing various assumptions made in the derivation or building upon the considerable knowledge from studying two-dimensional incompressible fluid turbulence (see e.g. \cite{29} and references therein).

Finally, it should be pointed out that the concept of anomalous turbulent viscosity in the context of heavy-ion collisions has already suggested in Ref. \cite{30}. However, the discussion in \cite{30} is based on assuming the presence of plasma turbulence, which is somewhat different than the fluid turbulence picture outline here. (Non-Abelian) Plasma turbulence seems to occur as a consequence of plasma instabilities \cite{31,32,33,34}. However, given current estimates of initial plasma parameters it seems somewhat unlikely that plasma instabilities set in early enough \cite{34,35} for plasma turbulence to be relevant at RHIC. Nevertheless, the spectrum of initial fluctuations studied in this context \cite{30} could also be of relevance for fluid turbulence.

To summarize, following the Yakhot-Orszag approach to turbulence the eddy viscosity in a compressible fluid with a relativistic equation of state was calculated. The result differs from incompressible fluids by the presence of a sound channel in the fluid dynamic propagator. In the sound channel, the eddy viscosity is positive for both two and three dimensions. In the shear channel, the eddy viscosity is found to be identical to that of incompressible fluids, and happens to be negative for two space dimensions. Given the Reynolds number for RHIC experiments is expected to be much larger than unity, and that the relevant fluid dynamics is essentially two-dimensional, it may be that this explains the apparently tiny viscosity over entropy density ratio extracted from experimental data. If this is the case, the model would predict a substantially increased elliptic flow at the LHC, which is experimentally falsifiable in the near future.

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