Abelian $BF$-Theory and Spherically Symmetric Electromagnetism

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Abstract

Three different methods to quantize the spherically symmetric sector of electromagnetism are presented: First, it is shown that this sector is equivalent to Abelian $BF$-theory in four spacetime dimensions with suitable boundary conditions. This theory, in turn, is quantized by both a reduced phase space quantization and a spin network quantization. Finally, the outcome is compared with the results obtained in the recently proposed general quantum symmetry reduction scheme. In the magnetically uncharged sector, where all three approaches apply, they all lead to the same quantum theory.

1 Introduction

Recently, H. Kastrup and the author proposed a general framework for a quantum symmetry reduction procedure of diffeomorphism invariant theories of connections [1]. In the case of a reduction of electromagnetism to its spherically symmetric sector an explicit expression for the quantum symmetry reduction and for the observables of the reduced theory was obtained. In the present paper we study this sector in more detail by providing another approach to symmetry reduction and quantization which, however, has the drawbacks of being applicable for this special theory only and of requiring a vanishing magnetic charge (or an explicit coupling to an external one). Nevertheless, the methods involved are more standard and this alternative quantization can serve as a simple test of the general quantum symmetry reduction of Ref. [1].

The new approach makes use of a novel identification of an Abelian $BF$-theory [2] with the spherically symmetric sector of electromagnetism. More precisely, it is proved that a partial gauge fixing of an Abelian $BF$-theory with suitable boundary conditions is equivalent to this symmetric sector upon a straightforward identification of their variables. The two constraints of the $BF$-theory provide the Gauß constraint of electromagnetism,

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and a second constraint which on the one hand serves to perform after gauge fixing a symmetry reduction of the theory and on the other hand constrains the magnetic charge to vanish.

At first sight it may be surprising that we identify a sector of electromagnetism with a topological field theory, but this causes no problems because the kinematics of the spherically symmetric sector is indeed diffeomorphism invariant, which leads to boundary observables only (the electric charge and its conjugate momentum) after solving the Gauß constraint. But the dynamics is not diffeomorphism invariant because we need a background metric to construct the Hamiltonian. Alternatively, we can couple the spherically symmetric sector to gravitation, thereby rendering the metric dynamical and restoring diffeomorphism invariance. This leads to an interpretation of the spherically symmetric sector of electromagnetism as the electromagnetic part of a Reissner-Nordstrøm gravitational system (which was our original motivation to study this sector). However, as the gravitational degrees of freedom complicate the theory considerably, we will not study their dynamics in this article.

In order to be able to ignore the gravitational degrees of freedom, and at the same time maintain diffeomorphism invariance (which is necessary to employ a spin network quantization) we regard the electromagnetic sector as a spherically symmetric sector of electromagnetism coupled to gravity, but on a degenerate gravitational sector. Thereby the electromagnetic degrees of freedom are decoupled and we can study them isolated from the complicated gravitational dynamics. Note that a degeneracy of the metric does not prevent electric and magnetic flows, and the densitized electric and magnetic fields from being well-defined. The usual electromagnetic Hamiltonian, however, vanishes, so we can study a static electromagnetism only.

The plan of the paper is as follows: In Section 2 we will prove the central assertion of this paper, namely the equivalence of the Abelian $BF$-theory with the spherically symmetric sector of electromagnetism, discuss the boundary conditions and present the reduced phase space quantization. The corresponding spin network quantization, which is also useful for a general $BF$-theory and not just for the special case related to spherically symmetric electromagnetism, is developed in Section 3 leading to the same results as in the reduced phase space quantization of Section 2. Finally, we will recall results obtained in the general symmetry reduction scheme of Ref. [1] and compare them with the approaches of the present paper. In the appendices we will describe the dimensions used here, and recall the classical reduction from Ref. [3] as well as basics about $U(1)$-spin networks.

2 $BF$-Theory

As already noted above, the quantization of spherically symmetric electromagnetism presented here is related to a $BF$-theory which requires a vanishing magnetic field $\mu^a$. We thus are lead to study an Abelian $BF$-theory [4, 11] which has, besides the Gauß constraint analogous to that of electromagnetism, a second constraint $F_{ab} = \epsilon_{abc}\mu^c \approx 0$ which constrains the curvature of a $U(1)$-connection to vanish.
2.1 Action and Constraints

The theory we start with has as variables a two-form $B$ and a $U(1)$-connection $\omega$ on a four-dimensional spacetime manifold of the form $M = \Sigma \times \mathbb{R}$. The curvature of $\omega$ is given by $F = d\omega$ and appears in the $BF$-action (for the definitions of our dimensions and the constants $q$ and $\alpha$ see Appendix A)

$$S_{BF} = \frac{q}{\alpha} \int_M B \wedge d\omega.$$  

We now insert the $3 + 1$-decomposition $B = \frac{1}{2} B_{ab} dx^a \wedge dx^b + B_{0a} dt \wedge dx^a$ and $\omega = \omega_0 dt + \omega_a dx^a$ to obtain the Hamiltonian formulation (a dot denotes a time derivative)

$$S_{BF} = \frac{1}{2} \frac{q}{\alpha} \int_{\Sigma \times \mathbb{R}} d^3 x \ dt \epsilon^{abc} (B_{ab} \dot{\omega}_c - B_{ab} \partial_c \omega_0 + B_{0a} F_{bc})$$

$$= \frac{q}{\alpha} \int_{\Sigma \times \mathbb{R}} d^3 x \ dt \left( \epsilon^c \dot{\omega}_c - \epsilon^c \partial_c \omega_0 + \frac{1}{2} \epsilon^{abc} \xi_a F_{bc} \right).$$

In the last step we introduced the field $\epsilon^a := \frac{1}{2} \epsilon^{abc} B_{bc}$, which will later be identified with the electric field. These field components are canonically conjugate to $\omega_a$ with the Poisson structure given by Eq. (A.1). The remaining components $\xi^a := B_{0a}$ of $B$ and $\omega_0$ are Lagrange multipliers which upon variation lead to the constraints

$$G[\omega_0] := \int_{\Sigma} d^3 x \omega_0 \partial_a \epsilon^a \approx 0,$$

$$F_0[\xi_a] := \frac{1}{2} \int_{\Sigma} d^3 x \xi_a \epsilon^{abc} F_{bc} \approx 0.$$

Boundary conditions, which are important because we had to integrate by parts, are discussed in the next subsection.

The Gauß constraint generates gauge transformations of $\omega$ as in electromagnetism, whereas the new constraints $F_0$ constrains the magnetic field to vanish. More important here is the fact that $F_0$ generates gauge transformations of $\epsilon^a$ which effect the symmetry reduction. This is stated as

**Lemma 1** Let $\Sigma$ be a three-dimensional manifold carrying an action of the group $SO(3)$, and $(x, \vartheta, \varphi)$ be a (local) system of polar coordinates adapted to the spherical symmetry.

The set of all spherically symmetric (time-dependent) fields $\epsilon^a = (\epsilon(t, x), 0, 0)$ with the boundary condition $\epsilon(t, \infty) = 0$ is a set of representatives of the gauge equivalence classes of ‘electric’ fields in the $BF$-theory Eq. (A) vanishing at infinity.

**Proof:** Because of $\{F_0[\xi_a], \omega_b\} = 0, \{F_0[\xi_a], \epsilon^a\} = \alpha q^{-1} \epsilon^{abc} \partial_b \xi_c$ the gauge transformations generated by $F_0$ lead to the addition of an exact two-form to the dual two-form $\epsilon_{abc} \epsilon^c$ of $\epsilon^a$.

First we show that two different symmetric electric fields $\epsilon_1^a$ and $\epsilon_2^a$ cannot lie in the same $F_0$-gauge class. By assumption the difference $\delta \epsilon^a = \epsilon_1^a - \epsilon_2^a$ of these fields fulfills
\[ \delta e^\theta = \delta e^\varphi = 0 \quad (x, \vartheta \text{ and } \varphi \text{ are spherical coordinates}). \] If that difference was a gauge transformation generated by some \( \xi_a \), this function had to obey the equations

\[ \delta e^\theta = \partial_\varphi \xi_x - \partial_x \xi_\varphi = 0 \quad , \quad \delta e^\varphi = \partial_x \xi_\theta - \partial_\theta \xi_x = 0. \]

This, in turn, would imply

\[ \partial_x \delta e^x = \partial_x (\partial_\theta \xi_\varphi - \partial_\varphi \xi_\theta) = \partial_\theta \partial_\varphi \xi_x - \partial_\varphi \partial_\theta \xi_x = 0, \]

i.e., the difference of the electric fields would be a constant which had to vanish due to the boundary condition. Therefore, each gauge class contains at most one spherically symmetric electric field.

We now prove that each class contains at least one spherically symmetric electric field. In order to show this we need a vector field \( N^a \) on \( \Sigma \) which is spherically symmetric, i.e.,

\[ N^a = (N(t, x), 0, 0), \]

and which is subject to the conditions \( \partial_a N^a = 0 \) and \( \int_{S_x} d^2 S_a N^a = 1 \) for all \( SO(3) \)-orbits \( S_x \) in \( \Sigma \). Such a field exists because, for the symmetry reduced theory to be non-trivial, we have to assume that there is at least one spherically symmetric electric field \( \epsilon^a_0 \), by means of which we can construct

\[ N^a(t, x) := \left( \int_{S_x} d^2 S_a \epsilon^a_0 \right)^{-1} \epsilon^a_0. \]

The existence of such a non-trivial field \( \epsilon^a_0 \) depends on the topology of \( \Sigma \). It always exists in the manifolds of Appendix [3]. The properties of \( N^a \) postulated above follow from the ones of \( \epsilon^a_0 \) and the fact that \( \int_{S_x} d^2 S_a \epsilon^a_0 \) does not depend on \( x \) (due to \( \partial_a \epsilon^a_0 = 0 \)).

Let \( \epsilon^a(t, x, \vartheta, \varphi) \) now be a field vanishing at infinity. Due to the properties of \( N^a \) the averaged field \( \overline{\epsilon}^a(t, x) := N^a \int_{S_x} d^2 S_b \epsilon^b \) is spherically symmetric and fulfills the Gauß constraint \( \partial_a \overline{\epsilon}^a = 0 \). Furthermore, we have \( \int_{S_x} d^2 S_a (\epsilon - \overline{\epsilon})^a = 0 \), which remains valid after replacing \( S_x \) by an arbitrary closed surface. According to de Rham duality of homology and cohomology groups the difference of the two fields is cohomologically trivial and, therefore, exact: \( \epsilon^a - \overline{\epsilon}^a = \epsilon^{abc} \partial_b \xi_c \) with an appropriate \( \xi_c \). An electric field and its spherically symmetric average, therefore, lie in the same gauge class.

Summarizing, we have proved that each gauge class contains exactly one spherically symmetric electric field. \( \Box \)

The meaning of this lemma is that the spherical symmetry reduction of electromagnetism in its magnetically uncharged sector can be viewed as gauge fixing of the transformations generated by the constraint \( F_0 \) of the associated Abelian \( BF \)-theory. The remaining Gauß constraint has the same meaning in both theories generating the gauge transformations \( \omega_a \mapsto \omega_a + \partial_a \omega_0 \). Using \( F_{ab} = 0 \) and a fixed basis \( \{[\omega^{(k)}]\} \) of \( H^1(\Sigma, \mathbb{R}) \) any connection can be written as \( \omega_a = \omega^{(k)}_a + \partial_a l \) with some function \( l: \Sigma \to \mathbb{R} \). Each function \( l \) can be gauged to the spherically symmetric value \( l = 0 \) by the gauge transformation \( l \mapsto l + \omega_w \) with \( \omega_0 := -l \). Analogously, \( B_{ab} dx^a \wedge dx^b \) is closed due to the Gauß constraint \( G \), and \( F_0 \) generates an additional exact two-form added to \( B_{ab} dx^a \wedge dx^b \). This shows
that on a manifold without boundary the reduced phase space of BF-theory is given by the product $H^1(\Sigma, \mathbb{R}) \times H^2(\Sigma, \mathbb{R})$ of first and second de Rham cohomology groups \[2\].

However, in the manifolds used in the spherically symmetric context (Appendix B) we have $H^1(\Sigma, \mathbb{R}) = 0$, whereas $H^2(\Sigma, \mathbb{R})$ does not need to be even dimensional and is, therefore, inappropriate as phase space. This is possible because we use manifolds with boundary where $H^1(\Sigma, \mathbb{R}) = 0$ and $H^2(\Sigma, \mathbb{R})$ do not have necessarily the same dimension. Furthermore, the constraints are affected by the presence of a boundary and the consideration of the preceding paragraph cannot be applied unaltered: The function $l$ can now be gauged to be zero only in the interior of $\Sigma$, whereas it remains arbitrary at the boundary. Taking the boundary properly into account will thus lead to new boundary degrees of freedom, which will render the reduced phase space even dimensional.

### 2.2 Surface Terms and Boundary Degrees of Freedom

Before discussing boundary conditions we will slightly generalize in the BF-theory context the manifolds defined in Appendix B by increasing the number of boundary components. Because we are interested mainly in the boundary degrees of freedom, we will confine ourselves to manifolds with a trivial first homology group only.

Besides the wormhole manifold $W^3 := \mathbb{R} \times S^2$ with $H_2(W^3, \mathbb{Z}) = \mathbb{Z}$ and boundary

$$\partial W^3 =: \partial_\infty W^3 =: \partial_+ W^3 \cup \partial_- W^3 \cong S^2 \cup S^2$$

with two boundary components at positive and negative infinity (the boundary is a disjoint union of two $S^2$), we will use the punctured manifolds $P^3_n := \mathbb{R}^3 \setminus \{p_1, \ldots, p_n\}$ with $H_2(P^3_n, \mathbb{Z}) = \mathbb{Z}^n$. Equivalently, we can cut out of $\mathbb{R}^3$ a small ball centered in each of the points $p_i$ resulting in new boundary components $\partial_i P^3_n \cong S^2$, the full boundary

$$\partial P^3_n = \partial_\infty P^3_n \cup \bigcup_{i=1}^n \partial_i P^3_n$$

having $n + 1$ components. Similar to the wormhole manifold above we denote with $\partial_\infty P^3_n$ the part of the boundary lying at infinity, which topologically is just a specification of a distinguished boundary component. Due to the non-trivial second homology groups these manifolds allow topological electric charge, and therefore we do not have to couple matter fields as sources of charge. Of course, $W^3$ is homeomorphic to $P^3_1$, but the interpretation is different as a spacelike section in the Reissner-Nordstrøm manifold as opposed to a point charge sitting in the origin.

For the constraints to be functionally differentiable we have to impose boundary conditions, and to correct the action by boundary terms. Boundary conditions for BF-theories have already been discussed in Refs. \[4, 5, 6, 7, 8\], but here we choose different ones adapted to the interpretation as spherically symmetric electromagnetism.

The variation of the constraints is

$$\delta G[\omega_0] = \int_\Sigma d^3x \omega_0 \partial_a \delta \epsilon^a = - \int_\Sigma d^3x \delta \epsilon^a \partial_a \omega_0 + \int_{\partial \Sigma} d^2S_a \omega_0 \delta \epsilon^a, \quad (5)$$

$$\delta F_0[\xi_a] = \int_\Sigma d^3x \xi_a \epsilon^{abc} \partial_b \delta \omega_c = - \int_\Sigma d^3x \epsilon^{abc} (\partial_b \xi_a) \delta \omega_c + \int_{\partial \Sigma} d^2S_b \epsilon^{abc} \xi_a \delta \omega_c. \quad (6)$$
In order to achieve functional differentiability the surface integrals have to vanish or to be compensated by appropriate boundary terms in the action. This can be enforced, first for the variation of \(G\), by the condition \(\omega_0|_{\partial\Sigma} = 0\) for gauge transformations. If we have instead \(\omega_0 = O(1)\) on \(\partial\Sigma\), the generated transformation is viewed as symmetry transformation. For the surface integral to vanish in this case we must require \(\delta \epsilon = O(\mathbf{r}^{-(2+\delta)})\), \(\delta > 0\) on \(\partial\infty \Sigma\) and \(\delta \epsilon|_{\partial\Sigma} = 0\) which is also necessary for symmetry transformations not to change the charge. Surface variables are given by

\[
O[\omega_0] := \int_{\Sigma} d^3x \, \varepsilon^a \partial_a \omega_0 = \int_{\partial\Sigma} d^2S_a \omega_0 \varepsilon^a, \quad \omega_0 = O(1),
\]

further constrained by \(F_0\), however.

According to Lemma 1 the transformations generated by \(F_0\) are necessary for a symmetry reduction. Therefore, we want to regard them as gauge transformations in any case without specifying further boundary conditions on \(\xi_a\). We need a surface term in the action (2) with a variation eliminating the surface integral in Eq. (6). The corrected action is

\[
S := S_{BF} - \frac{q}{\alpha} \int_{\partial \Sigma \times \mathbb{R}} d^2S_b d\tau \varepsilon^{abc} \xi_a \omega_c,
\]

leading to the Hamiltonian

\[
H = \frac{q}{\alpha} \int_{\Sigma} d^3x \left( \varepsilon^a \partial_a \omega_0 - \frac{1}{2} \varepsilon^{abc} \xi_a F_{bc} \right) + \frac{q}{\alpha} \int_{\partial\Sigma} d^2S_b \varepsilon^{abc} \xi_a \omega_c
\]

\[
= -\frac{q}{\alpha} \int_{\Sigma} d^3x \left( \omega_0 \partial_a \varepsilon^a + \frac{1}{2} \varepsilon^{abc} \xi_a F_{bc} \right) + \frac{q}{\alpha} \int_{\partial\Sigma} d^2S_b (\omega_0 \varepsilon^b + \varepsilon^{abc} \xi_a \omega_c).
\]

The boundary values of \(\omega_0\) on \(\partial\Sigma\) are prescribed functions, which are determined by an external observer, depending on the time variable \(t\). In contrast, \(\xi_a\) is regarded as Lagrange multiplier also at the boundary leading to the corrected curvature constraint

\[
F[\xi_a] = F_0 - \int_{\partial\Sigma} d^2S_b \varepsilon^{abc} \xi_a \omega_c.
\]

Note that we did not specify boundary conditions for \(\xi_a\) in the context of \(F\). Therefore, variation of the boundary values leads to the so-called natural boundary conditions. Thereby we obtain the surface constraints \(n_a \varepsilon^{abc} \omega_c|_{\partial\Sigma} \approx 0\) (\(n_a\) being the normal on \(\partial\Sigma\)), which yield that \(l\) is locally constant on the boundary, i.e., constant on each boundary component, after inserting \(\omega_c = \partial_c l\). Together with Lemma 1 we can now see full equivalence to spherically symmetric electromagnetism of Appendix B (for the manifolds \(W^3\) or \(P_1^3\)):

**Theorem 1** The partially reduced phase space of the Abelian BF-theory obtained after solving only the constraint \(F\) is equivalent to the phase space of spherically symmetric electromagnetism.
Before reducing the theory completely we check the algebra of constraints. Because $G$ and $F$ contain either $\epsilon^a$ or $\omega_a$, we have

$$\{G[\omega_0], G[\omega'_0]\} = \{F[\xi_a], F[\xi'_b]\} = 0. \quad (10)$$

The mixed Poisson bracket is

$$\{G[\omega_0], F[\xi_a]\} = -\frac{\alpha}{q} \int_\Sigma d^3x (\partial_c \omega_0) \partial_b \xi_a \epsilon^{abc} = \frac{\alpha}{q} \int_\Sigma d\omega_0 \wedge d\xi$$

$$= -\frac{\alpha}{q} \int_\Sigma d(d\omega_0 \wedge \xi) = -\frac{\alpha}{q} \int_{\partial\Sigma} d\omega_0 \wedge \xi, \quad (11)$$

which vanishes for gauge transformations because then $\omega_0$ has to vanish on the boundary. Therefore, the constraints are first class. For the Poisson bracket to vanish we must have $d\omega_0|_{\partial\Sigma} = 0$ which is also fulfilled for some special symmetry transformations (for which $\omega_0|_{\partial\Sigma} \neq 0$, but locally constant). Because $\xi$ is arbitrary at the boundary the surface variables $O[\omega_0]$ are observables exactly if $\omega_0|_{\partial\Sigma}$ is locally constant:

$$\{O[\omega_0], F[\xi_a]\} = 0. \quad (12)$$

The observables $O[\omega_0]$ (with unrestricted $\omega_0$) have already appeared in Ref. [6], together with additional surface observables which are integrals of $\omega_a$ associated with boundary values of $\xi_a$. These latter observables are excluded here by our special boundary conditions (free boundary values of the Lagrange multiplier $\xi_a$). It also leads to the restriction of $\omega_0$ in $O[\omega_0]$ to be locally constant on $\partial\Sigma$. The special treatment of $\xi_a$, leading to these two effects, is crucial for the identification with the spherically symmetric sector of electromagnetism (see also Eq. (13) below); and we will see that the remaining surface observables are just the correct ones for this application.

### 2.3 Reduction and Quantization

The constraints are easy to solve: $F$ forces the connection $\omega_a$ to be flat, i.e., $\omega_a = \partial_a l$ for some $l: \Sigma \to \mathbb{R}$. Since $G$ generates the gauge transformation $l \mapsto l + \omega_0$ with an arbitrary $\omega_0$ vanishing on the boundary, only the boundary values of $l$ have physical meaning. Furthermore, due to the boundary constraints

$$C[\xi_b|_{\partial\Sigma}] := \int_{\partial\Sigma} dS_a \epsilon^{abc} \xi_b \omega_c = \int_{\partial\Sigma} dS_a \epsilon^{abc} \xi_b \partial_c l \approx 0 \quad (13)$$

$l$ has to be constant on each boundary component because the $\xi_a$ are arbitrary at $\partial\Sigma$. This is the most important consequence of our special boundary conditions introduced above.

The physical degrees of freedom associated with $\epsilon^a$ can also be localized at the boundary and given by integrals

$$p^A := \int_{\partial\Lambda^\Sigma} d^2S_a \epsilon^a \quad (14)$$
over each of the \( n + 1 \) boundary components, i.e. \( p^A = \mathcal{O}[\omega^A_0] \) with \( \omega^A_0|_{\partial_B \Sigma} = \delta^A_B \). They are not all independent, however, because of \( \sum_A p^A = \int_{\partial \Sigma} d^2 S_a e^a = \int_{\Sigma} d^3 x \partial_a e^a = 0 \) as a consequence of the Gauß constraint. As above, the constraints imply that the class of \( \epsilon_{abc} c^c \) in the second de Rham cohomology group represents the physical degree of freedom specified by its evaluation on all classes of the second homology group. Choosing as representatives for a basis of the second homology groups of the two manifolds \((\partial_+ \Sigma)\) and \((\partial_i \Sigma)_{1 \leq i \leq n}\), respectively, we arrive at the independent observables \( p^+ \) and \( p^1, \ldots, p^n \).

From Eq. (9) we derive the reduced Hamiltonian: At first we insert the bulk constraints to obtain

\[
H' = \frac{q}{\alpha} \int_{\partial \Sigma} d^2 S_a (\omega_0 e^a - \epsilon_{abc} \xi^b \omega_c) = \frac{q}{\alpha} \int_{\partial \Sigma} d^2 S_a (\dot{l} e^a - \epsilon_{abc} \xi^b \partial_c l).
\]

In the second part of this equation we provided a time dependence for \( l \) by defining \( \dot{l} := \omega_0 \), which formally extends the relation \( \omega_a = \partial_a l \) to the four-dimensional connection on \( \Sigma \times \mathbb{R} \).

The boundary values of \( \xi_a \) are the remaining Lagrange multipliers, and their variation leads to the boundary constraints \( \mathcal{C} \) of Eq. (13). The solution of these requires a locally constant \( l \) on the boundary eliminating the second term in \( H' \):

\[
H'' = \frac{q}{\alpha} \int_{\partial \Sigma} d^2 S_a \dot{l} e^a = \frac{q}{\alpha} \sum_A l_A \int_{\partial_A \Sigma} d^2 S_a e^a = \frac{q}{\alpha} \sum_A l_A p^A,
\]

where \( A \) runs over all \( n + 1 \) boundary components and \( l_A \) is the constant value of \( l \) on the component \( \partial_A \Sigma \).

Up to now the constraints are not solved completely: There remains the condition \( \sum_A p^A = \int_{\partial \Sigma} d^2 S_a e^a \approx 0 \) implied by \( G \approx 0 \). Therefore, only \( n \) of the \( n + 1 \) boundary variables \( p^A \) are independent. At the same time, the remaining constraint generates the gauge transformation \( l_A \mapsto l_A + c \) with some \( c \) being constant on the full boundary (not just locally constant). One of the \( l_A \) can thereby be gauged to zero, and we end up with only \( n \) independent values of the \( l_A \). In our manifolds we will choose the gauge fixing \( l_- = 0 \) in \( W^3 \) and \( l_\infty = 0 \) in \( P^3_n \), respectively. Finally we obtain the reduced Hamiltonian

\[
H_{\text{red}} = \frac{q}{\alpha} \sum_A \dot{l}_A p^A, \quad A = + \text{ or } A \in \{1, \ldots, n\}.
\]

A comparison with Appendix B shows that this is, on the manifolds \( W^3 \) or \( P^3_1 \), the reduced Hamiltonian of spherically symmetric electromagnetism with the prescribed function of \( t \) being \( \dot{l} = U \), which reveals that its boundary dynamics is equivalent to that of \( BF \)-theory, too. The canonical variables

\[
(p^A, qa^{-1} \Phi_A)_{A=+ \text{ or } A \in \{1, \ldots, n\}}
\]

are action-angle coordinates of the reduced Hamiltonian. The equations of motion are solved by

\[
p^A = c_A, \\
\Phi_A = c'_A - l_A(t)
\]
with constants $c_A, c'_A$ to be specified by initial values.

This system with phase space $T^*\mathbb{R}^n$ can be quantized without problems. As Hilbert space we choose $L^2(\mathbb{R}^n, d^nx)$, $n + 1$ being the number of boundary components. In the $\Phi$-representation states are given by $\psi(\Phi_1, \ldots, \Phi_n)$, acted on by $\hat{\Phi}_A$ and $\hat{p}_A$ as usually:

\[
\hat{\Phi}_A \psi = \Phi_A \psi, \quad \hat{p}_A \psi = \frac{\hbar}{i q} \frac{\partial}{\partial \Phi_A} \psi = \frac{q}{i} \frac{\partial}{\partial \Phi_A} \psi.
\]

This quantization leads to a continuous spectrum of the charges $p_A$, but a quantization condition can be imposed by $\Phi_A \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$, justified by the fact that $\Phi$ represents a $\mathbb{L}U(1)$ element: $l$ and $l+2\pi$ yield the same element $\exp il = \exp i(l+2\pi)$ of the gauge group. This periodical identification of the phase space leads to charge quantization: Simultaneous eigenstates of the $\hat{p}_A = -iq \frac{\partial}{\partial \Phi_A}$ are given by $\psi_{(K_A)}(\Phi_1, \ldots, \Phi_n) = \prod_A \exp iK_A \Phi_A$ with eigenvalue $qK_A$ of $\hat{p}_A$. The periodic identification demands $K_A \in \mathbb{Z}$ leading to

\[
p_A \in q\mathbb{Z} \quad \text{for all } A
\]

which is the observed charge quantization with a ‘fundamental charge’ $q$. Its value, however, cannot be determined because the theory contains one free parameter $\alpha$. In the present context the periodic identification of $\Phi_A$ looks somewhat ad-hoc, but the quantization condition (19) will arise in the following spin network quantization more naturally.

### 3 Spin Network Quantization of BF-Theory

In the preceding section we arrived at a quantization of Abelian BF-theory, which can be interpreted as a quantum theory of spherically symmetric electromagnetism. In order to compare with the results of the quantum symmetry reduction procedure it is, however, more instructive to present a spin network quantization, too. The solution of the constraints will be given in the same order as in the symplectic reduction of the previous section: We first solve the Gauß constraint, then the curvature constraint to arrive at a boundary theory, and finally the boundary constraints. Our notation for $U(1)$-spin networks, which are extensively used in this section, is described in Appendix C.

#### 3.1 Gauß Constraint

Of course, the Gauß constraint can be solved by using only gauge invariant spin networks, i.e., those with $k_v = 0$ for each vertex $v$, but for the sake of completeness we will give also a quantization of the classical constraint $G$.

Let $\gamma$ be a graph and $f_\gamma$ be a cylindrical function which depends on a connection $\omega_a$ only via the edge holonomies $\eta_e$ for all $e \in E(\gamma)$. On that function the Gauß constraint acts as

\[
\hat{G}[\omega_0] f_\gamma = \int_\Sigma d^3 x \omega_0 \delta_a e^a f_\gamma = \frac{\hbar}{i q} \int_\Sigma d^3 x \omega_0(x) \partial_a \frac{\delta}{\delta \omega_a(x)} f_\gamma.
\]
\[
\dot{Q}[S] = \int_S d^2 y n_a \dot{e}^a = \int_S d^2 y \sum_{e \in E(\gamma)} \int_\epsilon dt n_a \dot{e}^a \delta(y, e(t)) \eta_e \frac{\partial}{\partial \eta_e}
\]

Recall the definition of \(\text{sgn}(v, e)\) given in Appendix which implies that in the last sum only edges incident in \(v\) contribute.

The \(\dot{G}[\omega_0]\) commute with one another which means that the classical algebra of constraints is represented anomaly-free. Applied to a spin network state \(T_{\gamma k}\) the constraint yields \(\dot{G}[\omega_0]T_{\gamma k} = -q \sum_{v \in V(\gamma)} \omega_0(v) k_v T_{\gamma k}\), implying that the solution space of the constraint is given by gauge invariant spin networks with \(k_v = 0\) for all vertices \(v \not\in \partial \Sigma\), as anticipated. This constrains, however, only vertices in the interior of \(\Sigma\) because at the boundary we had to demand \(\omega_0|_{\partial \Sigma} = 0\) in the constraint. Therefore, \(k_v\) for \(v \in \partial \Sigma \cap V(\gamma)\) is arbitrary meaning that at the boundary edges of a spin network can end. This is analogous to the boundary observables \(O\) appearing in the second section, and it makes possible electric charge. The electric charge \(Q[S]\) enclosed by a closed surface \(S \subset \Sigma\) is given by the integral \(\int_S d^2 S \, e^a\) in the classical case, which depends only on the homology class of \(S\). This lead us to use topologies of \(\Sigma\) with non-trivial second homology groups to allow electric charge, and it follows classically from the Stokes theorem in a well-known fashion: If \(S_1\) and \(S_2\) are the equally oriented boundary components of a domain \(B \subset \Sigma\), we have \(0 = \int_B d^3 x \partial_a e^a = \int_{S_1} d^2 S_a e^a - \int_{S_2} d^2 S_a e^a\) as a consequence of the Gauß constraint \(\partial_a e^a \approx 0\).

The reason for our dwelling on that point is that in the quantum theory there is an analogous, but quite differently, namely topologically realized version. Here, the Gauß constraint manifests itself in the condition \(k_v = 0\). At first we quantize the charge functional applied to a function cylindrical with respect to a graph \(\gamma\) which is chosen such that all the intersection points \(v \in V(\gamma \cap S)\) are vertices of \(\gamma\):

\[
\dot{Q}[S] = \int_S d^2 y n_a \dot{e}^a = \int_S d^2 y \sum_{e \in E(\gamma)} \int_\epsilon dt n_a \dot{e}^a \delta(y, e(t)) \eta_e \frac{\partial}{\partial \eta_e}
\]

Here, \(\text{sgn}(e, S)\) is the intersection number of \(e\) with \(S\) which is defined to be \(\frac{1}{2}\) if \(e \cap S \subset \partial e\). The charge of a spin network state \(T_{\gamma k}\) is proportional to \(\sum_{e \in S \neq \emptyset} \text{sgn}(e, S) k_v\) which can be interpreted as the intersection number of \(S\) with a curve associated to \(T_{\gamma k}\). This curve,
which is disconnected in general, can be constructed by stacking $|k_e|$ copies of each edge $e \in E(\gamma)$ on top of each other. All such copies incident in a vertex $v$ can be linked there to form curves with no endpoints in $v$ if and only if $k_v = 0$. In the interior of a domain $B$ as above we therefore obtain pieces of curves ending only at the boundary $\partial B = S_1 \cup S_2$ ($S$ is $S$ in opposite orientation) if and only if the spin network state is gauge invariant. If there are only divalent vertices at $\partial B$, the charges $\hat{Q}[S_1]T_{\gamma,k}$ and $\hat{Q}[S_2]T_{\gamma,k}$ are equal being given by intersection numbers of homologically equivalent closed surfaces with a closed curve: Each curve entering $B$ through $S_1$ has to leave $B$ either again through $S_1$, which does not contribute to both charges measured by $S_1$ and $S_2$, or it runs through $S_2$ contributing to the two charges the same amount.

3.2 Curvature Constraint

As opposed to the Gauß constraint the curvature constraint cannot be solved in a subspace of the space of cylindrical functions, but it has to be solved by means of a rigging map [9]. This map can be written formally as multiplication with a delta function supported on the space of flat connections which will be constructed in this subsection.

3.2.1 The Space $\mathcal{A}_0$ of Pure Gauge Connections

In the simply connected topologies used here $F = 0$ means that $\omega_a = \partial_a l$ is pure gauge. Holonomies associated with an edge $e$ are

$$\eta_e(l) = \exp \left( i \int e \, dl \right) = \exp \left( i(l(e(1)) - l(e(0))) \right),$$

which depend on $l$ only in the starting point $e(0)$ and the end point $e(1)$ of $e$. A spin network state, therefore, depends only on the values of the gauge potential $l$ in its vertices:

$$\prod_e \exp (i \sgn(v,e) k_e l(v)) = \exp (i k_v l(v)).$$

By using the pure gauge connection each point $v \in \Sigma$ is mapped to a $U(1)$ group element $\lambda(v) := \exp (i l(v))$, and a spin network state evaluated in this connection takes the form $\prod_{v \in V(\gamma)} \lambda(v)^{k_v}$. The function $\lambda: \Sigma \to U(1)$ is smooth for a classical connection, but it will be generalized to an arbitrary function in the course of quantization:

Definition 1 $\mathcal{A}_0 := \{\lambda: \Sigma \to U(1) \text{ smooth}\}$ is the space of pure gauge connections.

The space of generalized pure gauge connections is $\overline{\mathcal{A}}_0 := \{\lambda: \Sigma \to U(1)\}$.

Analogously to $\overline{G}$ in Ref. [9], the space $\overline{\mathcal{A}}_0$ can be constructed as a projective limit with index set being the set of all finite subsets $\sigma \subset \Sigma, |\sigma| \in \mathbb{N}_0$ of $\Sigma$, and with cylindrical spaces $\mathcal{A}_{0,\sigma} := U(1)^{\sigma}$ being the spaces of all maps from $\sigma$ to $U(1)$ and projections $p_{\sigma' \sigma} (\lambda_\sigma) = \lambda_\sigma|_{\sigma'}$ for $\lambda_\sigma \in \mathcal{A}_{0,\sigma}$. Then we have

$$\overline{\mathcal{A}}_0 = \proj_{\sigma \subset \Sigma} \mathcal{A}_{0,\sigma} = \proj_{\sigma \subset \Sigma} U(1)^{\sigma} \equiv U(1)^{\Sigma}.$$
A cylindrical basis of functions on $\mathcal{A}_0$ associated to the index set of all finite subsets $\sigma$ together with labelings $k: \sigma \to \mathbb{Z}\setminus\{0\}$ is given by the functions $t_{\sigma,k}(\lambda) := \prod_{p \in \sigma} \lambda(p)^{k_p}$. For $\sigma$ fixed these are the monomials in the finitely many variables $\lambda(p), p \in \sigma$, which certainly span the space of functions on $\mathcal{A}_{0,\sigma}$ modulo functions which are constant in some $\lambda(p)$, i.e., functions on a space $\mathcal{A}_{0,\sigma'}$ with $\sigma' \subset \sigma$. Analogously to the Ashtekar-Lewandowski measure we can define a measure on $\mathcal{A}_0$ cylindrically:

**Lemma 2** A diffeomorphism invariant probability measure on $\mathcal{A}_0$ is given by

$$\mu(f) := \int_{\mathcal{A}_0} \, d\mu := \mu_\sigma(f_\sigma) := \int_{U(1)^{|\sigma|}} \mu^{|\sigma|}_H(\lambda_1, \ldots, \lambda_{|\sigma|}) f_\sigma(\lambda_1, \ldots, \lambda_{|\sigma|})$$

for some representative $f_\sigma$ of $f$. $\text{Diff}(\Sigma)$ acts on $\mathcal{A}_{0,\sigma}$ by $U(\phi)f_\sigma = f_{\phi(\sigma)}$.

The monomials $t_{\sigma,k}$ form an orthonormal basis with respect to this measure.

**Proof:** The cylindrical consistency condition for the measure and its normalization as well as diffeomorphism invariance follow from properties of the Haar measure.

If $t_{\sigma,k}$ and $t_{\sigma',k'}$ are two monomials, then

$$\langle t_{\sigma,k}, t_{\sigma',k'} \rangle = \mu(t_{\sigma,k} t_{\sigma',k'}) = \int_{U(1)^{|\sigma|\cup\sigma'|}} \mu^{|\sigma|\cup\sigma'|}_H(\lambda) \prod_{p \in \sigma\cup\sigma'} \lambda(p)^{k_p-k_p} = \delta_{\sigma\sigma'} \delta_{kk'} \quad (21)$$

proving orthonormality. □

For $\sigma = V(\gamma)$ and $k$ being the vertex labeling of a spin network state $T_{\gamma,k'}$ we have $t_{\sigma,k}(\lambda) = T_{\gamma,k'}(\lambda^{-1} \, d\lambda)$ showing that the $t_{\sigma,k}$ emerge by restriction of spin network states to pure gauge connections. To each spin network $T_{\gamma,k'}$ we can associate a monomial $t_{\sigma,k} = \partial T_{\gamma,k'}$ and continue the operation $\partial$ to the space $\Phi$ of all cylindrical functions on $\mathcal{A}$. Formally, we can write $t_{\sigma,k} = \delta(F) T_{\gamma,k'}$ with

$$\delta(F) := \prod_{e \in \Sigma} \prod_{p \in \Sigma_e} \int_{U(1)} \mu_H(\lambda(p)) \delta\left(A_e, \lambda(e(0))^{-1} \lambda(e(1))\right).$$

Given a cylindrical function $f_\gamma$ we have to interpret $\delta(F) f_\gamma$ as a distribution on the space of cylindrical functions, and equivalently $\delta(F): \Phi \to \Phi'$ as a rigging map according to

$$\langle \delta(F) f_\gamma, g_{\gamma'} \rangle := \int_{\mathcal{A}_0} \mu(\lambda) \prod_{e \in E(\gamma) \cup E(\gamma')} \int_{U(1)} \mu_H(A_e) \delta\left(A_e, \lambda(e(0))^{-1} \lambda(e(1))\right) T_{\gamma} g_{\gamma'} = \int_{\mathcal{A}_0} \mu(\lambda) \mathcal{F}_\gamma g_{\gamma'} \quad (22)$$

This map solves the constraint $F$ on a subspace of $\Phi'$. 

\[12\]
3.2.2 Boundary Spin Networks

Up to now we solved the constraints $G$ and $F$ separately. To solve them together we have to investigate the space $A_0/G$. In contrast to $A_0$, the space $G$ consists of functions $g: \Sigma \to U(1)$ which have to become unity on the boundary $\partial \Sigma$, and it acts on $A_0$ as $(g \cdot \lambda)(p) = g(p)\lambda(p)$ (note that $\lambda$ is the exponentiated gauge potential of a connection; therefore, $g$ does not act by conjugation). For $\Sigma$ without boundary we have $A_0/G = \{1\}$, and similar to the boundary observables in the classical theory non-trivial states emerge only in presence of a boundary. We have the projective spaces

$$A_{0,\sigma}/G_\sigma = \{\lambda: \sigma \to U(1)|\lambda(p) = 1 \text{ if } p \notin \partial \Sigma\}$$

with limit $A_0/G = A_0/G_\sigma = \{\lambda: \partial \Sigma \to U(1)\}$.

All degrees of freedom are localized at the boundary of $\Sigma$ motivating

**Definition 2** The space of functions on the space $A_0/G$ of gauge invariant pure gauge connections is spanned by boundary spin networks $t_{\sigma,k}$ with finite sets $\sigma \subset \partial \Sigma$ and labelings $k: \sigma \to \mathbb{Z}\setminus\{0\}$. The associated boundary state is given by $t_{\sigma,k}(\lambda) := \prod_{p \in \sigma} \lambda(p)^{k_p}$.

These functions $t_{\sigma,k}$ span the boundary Hilbert space when completed with respect to the measure $\mu$ of Lemma 3.

The fundamental operations can also be projected down from the spin network basis by inserting pure gauge connections. The holonomy to an edge $e$ in $\Sigma$ with $e \cap \partial \Sigma = e(1) =: p$ is

$$\eta_e = \lambda(p)/\lambda(e(0))$$

Gauge invariant is only $\lambda(p)$ leading to the multiplication operator $\lambda_p := \lambda(p)$ instead of $\eta_e$. Its action on a boundary spin network $t_{\sigma,k}$ is to increase $k_p$ by 1.

The other fundamental operator associated to $e$ is the derivative operator $\kappa_e$:

$$\kappa_et_{\sigma,k} = \eta_e \partial_{\eta_e} t_{\sigma,k} = \frac{\lambda(p)}{\lambda(e(0))} \frac{\partial}{\partial \lambda(e(0))^{-1}\lambda(p)} t_{\sigma,k} = \lambda(p) \frac{\partial}{\partial \lambda(p)} t_{\sigma,k} = k_p t_{\sigma,k},$$

where we used independence of $t_{\sigma,k}$ on $\lambda(e(0))$. This operator also acts only in the point $p$ and can be written as

$$\kappa_p := \lambda_p \frac{\partial}{\partial \lambda_p}.$$
3.3 Boundary Constraints

The bulk constraints \( G \) and \( F \) are now solved on \( \mathcal{A}_0/\mathcal{G} \). However, the boundary constraints \( C \), which force \( l \) to be constant on each of the \( n+1 \) boundary components, still remain to be solved. We have to impose them on boundary spin network states by restricting these functions to locally constant \( \lambda = \lambda_A \) on each \( \partial_A \Sigma \). A restricted state is completely determined by an integer

\[
K_A := \sum_{p \in \sigma \cap \partial_A \Sigma} k_p
\]

for each boundary component \( \partial_A \Sigma \), which can be seen from the calculation

\[
t_{\sigma,k}(\lambda)|_{c=0} = \prod_A \lambda_A^{\sum_{p \in \sigma \cap \partial_A \Sigma} k_p} = \prod_A \lambda_A^{K_A} = : \langle K_1, \ldots, K_{n+1} \rangle.
\]  

As in the classical reduction there remains one last condition following from gauge invariance. The boundary spin networks descend from gauge invariant spin networks in the course of constraint reduction, which implies \( \sum_A K_A = 0 \). Again, only \( n \) of the \( n+1 \) numbers \( K_A \) are to be chosen freely. Accordingly, the \( \lambda_A \) can be multiplied by some \( \lambda_0 \) which is constant on the whole boundary, because states change then by multiplication with a factor \( \lambda_0^{\sum_A K_A} = 1 \). This freedom can be fixed by imposing the condition \( \lambda_A = 1 \) for some fixed boundary component \( A \), analogous to the classical case, and discarding its charge \( K_A \).

The states \( |K_1, \ldots, K_n\rangle \) labeled by the remaining \( n \) charges build an orthonormal basis of the physical Hilbert space \( \mathcal{H}_{\text{phys}} = L^2(U(1)^n, d\mu_H) \) with the inner product descending from the space of boundary spin networks:

\[
\langle K_1, \ldots, K_n | K'_1, \ldots, K'_n \rangle := \langle t_{\sigma,k}, \delta(C) t_{\sigma',k'} \rangle := \prod_A \int_{U(1)} d\mu_H(\lambda_A) \delta(\lambda_A)^{K'_A - K_A} = \prod_A \delta_{K_A,K'_A}
\]

where, formally,

\[
\delta(C) := \prod_A \prod_{p \in \partial_A \Sigma} \int_{U(1)} d\mu_H(\lambda_A) \delta(\lambda_p, \lambda_A).
\]  

Finally, we need a representation of the Poisson algebra of the canonical variables \( (p^A, \Phi_A) \) on the physical Hilbert space. The operators are to be built from the boundary operators \( \lambda_p \) and \( \kappa_p \), and they can be deduced from their action on three-dimensional spin network states.

According to Appendix [3], \( p^A \) and \( \Phi_A \) are in generalization from the spherically symmetric case, i.e., from the manifolds \( W^3 \) or \( P^3 \), given by the charge on the boundary component \( \partial_A \Sigma \) and by the holonomy associated with a radial curve \( B_A \) ending on \( \partial_A \Sigma \):

\[
p^A = \int_{\partial_A \Sigma} d^2S \epsilon^a, \quad \Phi_A = -\int_{B_A} \omega = i \log \eta_{B_A}.
\]

These expressions are the same as the reduced phase space variables of the preceding section. To obtain independent variables we have to choose \( n \) out of the \( n+1 \) boundary
components (as in Eq. (17), for instance), the index $A$ running over them in the following. The excluded component can be used to provide a starting point for the curves $B_A$ (for $\Phi_A$ to be gauge invariant $B_A$ cannot start in the interior of $\Sigma$). In this way, each curve intersects only one of the distinguished components $\partial A \Sigma$.

$p^A$ is quantized by using Eq. (20). The surface $S$ in this equation is chosen to be homologically equivalent to the boundary component $\partial A \Sigma$ and lying in the interior of $\Sigma$. $S$ must not be the boundary component itself because this would introduce a factor of $\frac{1}{2}$ since all edges would end on $S$. Note that charges are defined in the classical calculation also by choosing a surface in the interior and computing the limit where this surface approaches the boundary at infinity (which is, of course, only necessary if there is no Gauß law, as e.g. for the ADM mass in a theory of gravity). However, for a boundary around a point charge we could equally well integrate over the boundary in the classical theory. In quantum theory this is no longer the case due to the distributional nature of generalized connections. On boundary spin networks we obtain

$$\hat{p}^A t_{\sigma,k} = q \sum_{p \in \sigma \cap \partial A \Sigma} \lambda_p \frac{\partial}{\partial \lambda_p} t_{\sigma,k} = q \sum_{p \in \sigma \cap \partial A \Sigma} k_p t_{\sigma,k} = q K_A t_{\sigma,k},$$

which is to be projected into the physical Hilbert space:

$$\hat{p}^A | K_1, \ldots, K_n \rangle = q K_A | K_1, \ldots, K_n \rangle.$$  (27)

On spin network states the basic multiplication operator is not $\Phi_A$, but the holonomy

$$\eta_A := \eta_B = \exp \left( i \int_{B_A} \omega \right) = \exp(-i \Phi_A)$$

with operation

$$\exp(-i \hat{\Phi}_A) | K_1, \ldots, K_n \rangle = \hat{\eta}_A | K_1, \ldots, K_n \rangle = | K_1, \ldots, K_A + 1, \ldots, K_n \rangle$$  (28)

because $\eta_A$ reduces to multiplication with $\lambda(B_A(1))$ (see Subsection 3.2.2, $B_A(1)$ denotes the endpoint of $B_A$). The operator $\hat{\eta}_A$ can be interpreted as shifting charge from the excluded boundary component to the component $\partial A \Sigma$ along the curve $B_A$ (or rather its diffeomorphism class). In this way, the total charge situated on all the boundary components remains constant, namely zero.

We can now quote from Ref. [1] the following

**Theorem 2** The Equations (27) and (28) define a representation of the classical Poisson $\ast$-algebra on $\mathcal{H}_{\text{phys}}$.

**Proof.** The proof is the same as in Ref. [1] except for an obvious generalization to $n$ variables.  \qed
We note that the adjointness relations $\hat{p}^A$ being self-adjoint and $\hat{\eta}_A$ being unitary – uniquely (up to a constant factor) determine the inner product (24) which was derived by descending from the Ashtekar-Lewandowski measure. Moreover, holonomy variables of spin network quantization turn out to be well suited to represent the classical algebra of observables. As opposed to Ref. [7] we did not have to use a normal ordering to define charge creation (or rather shifting) operators: Due to the basic assumption of every spin network quantization, namely that holonomies are well defined in quantum theory, the operators $\hat{\eta}_A$ are perfectly well defined in our Hilbert space.

In complete analogy to the reduced phase space method we arrived at the same quantum theory with one degree of freedom per boundary component given by the electric charge. Moreover, we obtain automatically a discrete charge spectrum (this has already been observed in Ref. [10] in case of unreduced $U(1)$-spin networks) with eigenvalues $qK_A$ of the charge operator belonging to the boundary component $\partial A \Sigma$ being integer multiples of $q$, which is however undetermined. This leads again to the charge spectrum (19).

### 3.4 Rigging Map

As noted already, the curvature and boundary constraints cannot be solved in a subspace of the space $\Phi$ spanned by spin network states, but they have according to refined algebraic quantization [9] to be solved in its topological dual $\Phi'$. What we have to do now is to present a rigging map implementing the constraints. This can be constructed by using partial rigging maps corresponding to the curvature and boundary constraints, respectively.

The basic ingredient for the rigging map $\eta_1 : \Phi \to \Phi'$ has already been given in Eq. (22). There we named it more pictorially $\eta_1 f_\gamma := \delta(F) f_\gamma$.

In a second step we have to implement the boundary constraint $C$. With the help of $\eta_1$ we went to the space $\Psi$ of boundary spin networks $t_{\sigma,k}$, which is interpreted as a subspace of $\Phi'$ analogously to Eq. (22). Now we have to start from $\Psi$ to go over to its dual $\Psi'$. Again, this can formally be done by means of a delta distribution, $\delta(C)$ in Eq. (25), which enforces $\lambda$ to take the constant value $\lambda_A$ at each boundary component $A$. The result of a multiplication with $\delta(C)$ is a function depending only on the $n$ variables $\lambda_A$ which we denoted above as $|K_1, \ldots, K_n\rangle = \prod_A \lambda_A^{K_A}$ (see Eq. (23)). This function is in the dual of $\Psi$ by means of

$$|K_1, \ldots, K_n\rangle \langle t_{\sigma,k}| = \prod_A \delta_{K_A; \sum p \in \sigma \cap \partial A \Sigma} k_p$$

$$= \int_{U(1)^n} d^n \mu_H(\lambda_1, \ldots, \lambda_n) \prod_A \lambda_A^{-K_A} \cdot \prod_{p \in \sigma \cap \partial A \Sigma} \int_{U(1)} d\mu_H(\lambda_p) \delta(\lambda_p; \lambda_A) t_{\sigma,k}(\lambda).$$

In the last step it is written as an integral of the two functions multiplied with the delta
distribution $\delta(C)$. Eq. (29) leads to the second rigging map
\[ \eta_2: \Psi \to \Psi', \quad t_{\sigma,k} \mapsto \left| K_A := \sum_{p \in \sigma \cap \partial A} k_p \right> \],
which has to be extended anti-linearly, implementing the boundary constraints.

The composition of the two maps, $\eta_2 \circ \eta_1: \Phi \to \Psi'$, cannot be used as a rigging map $\eta: \Phi \to \Phi'$ because it has the wrong domain as its image. It can, however, easily be extended to such a map by interpreting $\sigma$ (the labels of functions in $\Psi$) as $V(\gamma) \cap \partial \Sigma$ for a graph $\gamma$ labeling a function in $\Phi$. This leads us to the rigging map
\[ \eta: T_{\gamma,k} \mapsto \left| K_A = \sum_{v \in V(\gamma) \cap \partial A} k_v \right> \]
by extending anti-linearly. Now $|K_1, \ldots, K_n\rangle$ is interpreted as a distribution in $\Phi'$ analogously to Eq. (23):
\[ |K_1, \ldots, K_n\rangle (T_{\gamma,k}) := \prod_A \delta_{K_A \sum_{v \in V(\gamma) \cap \partial A} k_v} \].
This rigging map solves both constraints $\mathcal{F}$ and $\mathcal{C}$ at once by incorporating both delta expressions. Moreover, it is real and positive: $(\eta \phi_1)(\phi_2) = \overline{(\eta \phi_2)(\phi_1)}$ and $(\eta \phi_1)(\phi_1) \geq 0$ for $\phi_1, \phi_2 \in \Phi$. Finally, $\eta$ commutes with physical observables $O$ by construction of the observables $\hat{p}^A$ and $\hat{n}_A$ via $\kappa_p$ and $\lambda_p$:
\[ (\eta \phi_1)(O \phi_2) = (\eta O^* \phi_1)(\phi_2) \].

The inner product in the solution space $\eta(\Phi)$ is given by
\[ \langle \eta(T_{\gamma,k}), \eta(T_{\gamma',k'}) \rangle_{\text{phys}} = \langle \eta(T_{\gamma,k}))(T_{\gamma',k'}) = \prod_A \delta_{\sum_{v \in V(\gamma) \cap \partial A} k_v} \sum_{v' \in V(\gamma') \cap \partial A} k_{v'} \].

The same properties are fulfilled for the partial rigging maps $\eta_1$ and $\eta_2$ yielding the inner products (21) and (24).

4 Quantum Symmetry Reduction

The application of the general quantum symmetry reduction scheme to the case treated here has already been carried out in Section 4.2 of Ref. [1], and it suffices to recall the main results and to compare with the methods of the present paper.

This framework implements a symmetry reduction procedure at the quantum level, i.e., the theory is spin network quantized first followed by singling out symmetric states which are distributional and represented by one-dimensional spin networks in case of spherical symmetry. The results for spherically symmetric electromagnetism are the following:
There is one degree of freedom given by the electric charge. The physical Hilbert space is spanned by states $|K\rangle$ exactly as in the preceding section. Recall that in the interpretation of $BF$-theory as spherically symmetric electromagnetism there is only one independent boundary component leading to $n = 1$ in the formulae above. Moreover, the observables are represented in the same way as derived here yielding the same quantum theory. Furthermore, the application of a symmetric spin network state, which is a generalized state in $\Phi'$, on a non-symmetric one

$$\sigma_g((\eta_B)^K)(T_{\gamma k}) = \beta_{\gamma,k}^g \delta_{K,\pi(k)}$$

is reminiscent of the rigging map $\eta$. Here $(\eta_B)^K$ is a one-dimensional spin network with charge $K$ in the radial manifold $B$, $g$ is the magnetic charge, $\beta_{\gamma,k}$ a phase factor, and $\pi(k)$ is a labeling of a one-dimensional spin network projected from the labeling $k$ which yields for gauge invariant spin networks the charge (see Ref. [1] for details). Note however that there appears no rigging map in that paper: In general there will be no constraint implementing the symmetry reduction as the constraint $F$ here. Therefore, no rigging map, which solves gauge as opposed to symmetry conditions, is needed. Indeed the methods for tackling symmetry developed in Ref. [1] are quite different from those to deal with gauge. The present paper shows that in the electromagnetic example both these methods apply and lead to the same quantum theory.

In particular, both approaches manage to reduce the infinitely many degrees of freedom of the non-symmetric field theory to only one in quite different ways.

There are two main advantages of the general method: First, magnetic charge is included there from the outset leading to superselection sectors labeled by the magnetic charge. In the $BF$-theory approach we could implement magnetic charge by coupling one in the $BF$-action. However, this had to be done by hand, whereas all sectors arise directly in the quantum symmetry reduction. The constraint reduction in spin network quantization then had to be performed by evaluating spin network states in connections of the form $\omega_a = \omega_a^{(g)} + \partial_a l$, where $\omega_a^{(g)}$ is a fixed connection with magnetic charge $g$, leading again to the same boundary spin networks. The magnetic charge then appears only in a phase factor which is given by the value of a spin network in the fixed connection $\omega^{(g)}$ and which depends on its geometry (compare Eq. (30)).

The second, and more important advantage of the quantum symmetry reduction is that it is a general procedure which applies to any theory with compact and semisimple (up to $U(1)$-factors) gauge group and any compact symmetry group (the condition of compactness can be relaxed). In particular, it applies to symmetry reduction of general relativity in the real Ashtekar formulation (the gauge group being $SU(2)$) which was our main motivation to develop that procedure.

The treatment in the present article also allows a comparison of spin network techniques with the standard Fock space methods used in Ref. [3, 7] in the context of $BF$-theory. If we do not impose the surface constraints to provide a more direct comparison with these two articles, we can use the boundary theory obtained in Subsection 3.2.2. The Hilbert space of boundary spin network functions obtained there has some advantages over the Fock
space quantization: The full group of boundary diffeomorphisms can be represented, and operators creating charge do not have to be normal ordered, but they are well defined from the outset in the spin network context. Thus we see that the spin network representation is well suited for the kinematical sector of $BF$-theory.

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Appendices

A Dimensions

In the present article we use electromagnetic dimensions which are unconventional, but analogous to the geometrical ones used in the gravitational part of the theory. This means that coordinates as well as the $U(1)$-connection $\omega_a$ are dimensionless. The electric field $e^a$ integrated over a surface yields the enclosed charge, and therefore it should carry the dimension of electric charge. (We reserve the letters $A$ and $E$ for the respective gravitational fields, although they will not appear in this paper. For the electromagnetic fields we use $\omega$, $\epsilon$, and $\mu$ as in Ref. [3].)

A Liouville form with the dimension of an action is given by

$$\frac{q}{\alpha} \int_{\Sigma} d^3 x e^a \delta \omega_a$$

where $\alpha$ is a dimensionless constant which fixes the norm of the electromagnetic part of the action, and $q$ is a unit of electric charge providing the correct dimension of an action. This leads to the symplectic structure

$$\{\omega_a(x), e^b(y)\} = \frac{\alpha}{q} \delta^b_a \delta(x,y). \quad \text{(A.1)}$$

Up to now we have two constants, $\alpha$ and $q$, which provide the norm and the dimension, respectively, of the action. We can fix one of them to obtain a theory with only one undetermined parameter. This will be done by choosing $q$ in such a way that $\alpha = q^2 \hbar^{-1}$ becomes a fine structure constant, and thereby the only parameter. This leads to the commutator $[\hat{\omega}_a(x), \hat{e}^b(y)] = iq \delta^b_a \delta(x,y)$ in a quantum theory.

B Classical Reduction

Here we recall the main formulae from Ref. [3] which are used in the present paper. The basic fields are the electric field $e^a$ with density weight 1 and the $U(1)$-connection $\omega_a$ which
are conjugate to one another. The Gauß constraint reads $\partial_a e^a \approx 0$. Symmetry reduction is done by imposing the restrictions

$$(e^x, e^\theta, e^\varphi) = (\epsilon(x,t), 0, 0) \quad (B.2)$$

and

$$(\omega_x, \omega_\theta, \omega_\varphi) = (\omega(x,t), 0, 0) \quad (B.3)$$

in spherical coordinates $(x, \theta, \varphi)$ provided by the given $SO(3)$-action on the spacelike section $\Sigma$. Here we demand that there is no magnetic charge. If we use the electric flow $p := 4\pi \epsilon$, we obtain the two conjugate fields $(\omega, q\alpha^{-1}p)$ on a radial manifold subject to the Gauß constraint $p' \approx 0$.

For simplicity we restrict $\Sigma$ to be simply connected and to be either the wormhole manifold $\mathbb{R} \times S^2$, which is the case for a Reissner-Nordstrøm black hole, or $\mathbb{R}^3 \setminus \{0\} \cong \mathbb{R}^+ \times S^2$, which simulates the presence of a non-dynamical point charge in the origin. These manifolds are most interesting in the context of spherical symmetry, but are generalized slightly in the $BF$-theory approach. Due to simple connectedness and vanishing of the magnetic field we have $\omega_a = \partial_a l$ with a function $l: \Sigma \to \mathbb{R}$. Symmetry reduction implies that $l$ is spherically symmetric, i.e., locally constant on the boundary.

After solving the Gauß constraint, conjugate variables on the reduced phase space are found to be $p$, which is constrained to be constant, and $\Phi := -\int dx \omega$. The reduced Hamiltonian accounting for boundary dynamics is $H_{\text{red}} = q\alpha^{-1}pU$ with a prescribed function $U(t)$ which is the value of the Lagrange multiplier of the Gauß constraint at infinity.

## C $U(1)$-Spin Networks

To fix our notation we present in this appendix the basic definitions of $U(1)$-spin networks. Due to Abelianess and the simple representation theory of $U(1)$ they are more easy to deal with than $SU(2)$-spin networks. They also appeared in Ref. [10].

The irreducible representations of the Abelian $U(1)$ are all one-dimensional and given by $\rho^k: U(1) \to \mathbb{C}^*, g \mapsto g^k$ for all $k \in \mathbb{Z}$. The dual representation of $\rho^k$ is given by $\rho^{-k}$, and the tensor product of two representations is $\rho^{k_1} \otimes \rho^{k_2} = \rho^{k_1 + k_2}$. A $U(1)$-spin network is a graph $\gamma$ with a labeling $k \in (\mathbb{Z}\setminus\{0\})^{E(\gamma)}$ of its edge set $E(\gamma)$ with irreducible, non-trivial $U(1)$-representations. Since the representations are not self-dual, an inverted edge has to be labeled with the dual representation: $k_{e^-1} = -k_e$. Contrary to the case of $SU(2)$-spin networks, we do not need contractors in the vertex set $V(\gamma)$, because intertwiners of $U(1)$-representations are unique up to a constant. If we do not restrict to gauge invariant spin networks, a coloring of the vertices which determines the transformation of the spin network under gauge transformations in that vertex can be computed from the edge labeling by $k_v = \sum_{e \in E(\gamma)} \text{sgn}(v,e)k_e$ where $v$ is a vertex of $\gamma$ and $\text{sgn}(v,e)$ is defined to be 1 if $e$ is an edge starting at $v$, $-1$ if $e$ ends in $v$, and 0 otherwise, i.e., if $v$ is not contained in $e$. Given a graph $\gamma$ with edge labeling $k$ we can form the spin network state as a function on the
space of generalized $U(1)$-connections given by
\[ T_{\gamma k}(\omega) := \prod_{e \in E(\gamma)} \rho^{k_e}(\omega(e)), \]
which transforms under a gauge transformation $g: \Sigma \to U(1)$ by multiplication with the $U(1)$-element $\prod_{v \in V(\gamma)} \rho^{k_v}(g(v))$. Of course, gauge invariant spin networks are obtained if in all vertices we have $k_v = 0$.

The basic operators are multiplication by a holonomy $\eta_e(\omega) := \exp i \int_e \hat{\omega}^a \omega_a$ along an edge $e$ which changes the edge labeling by $k_e \mapsto k_e + 1$, and a derivative operator which is the invariant vector field $i\kappa_e(\eta_e) := i\eta_e \partial_{\eta_e}$ on $U(1)$. Again for $U(1)$ being Abelian these operators are much simpler than their analogs in $SU(2)$. Spin network states are eigenvectors of $\kappa_e(\eta_e)$ with eigenvalue $k_e$: $\kappa_e(\eta_e)T_{\gamma k} = k_e T_{\gamma k}$.

References

[1] M. Bojowald and H. A. Kastrup, Quantum Symmetry Reduction for Diffeomorphism Invariant Theories of Connections, hep-th/9907042, to appear in Class. Quantum Grav.

[2] G. T. Horowitz, Exactly Soluble Diffeomorphism Invariant Theories, Commun. Math. Phys. 125 (1989) 417–437

[3] T. Thiemann, The reduced phase space of spherically symmetric Einstein-Maxwell theory including a cosmological constant, Nucl. Phys. B 436 (1995) 681–720

[4] M. Blau and G. Thompson, Topological Gauge Theories of Antisymmetric Tensor Fields, Ann. Phys. 205 (1991) 130–172

[5] S. Wu, Topological Quantum Field Theories on Manifolds with a Boundary, Commun. Math. Phys. 136 (1991) 157–168

[6] A. P. Balachandran and P. Teotonio-Sobrinho, The Edge States of the $BF$ System and the London Equations, Int. J. Mod. Phys. A 8 (1993) 723–752

[7] A. P. Balachandran and P. Teotonio-Sobrinho, Vertex Operators for the $BF$ System and its Spin-Statistics Theorems, Int. J. Mod. Phys. A 9 (1994) 1569–1629

[8] V. Husain and S. Major, Gravity and $BF$ theory defined in bounded regions, Nucl. Phys. B 500 (1997) 381–401, gr-qc/9703043

[9] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourão, and T. Thiemann, Quantization of diffeomorphism invariant theories of connections with local degrees of freedom, J. Math. Phys. 36 (1995) 6456–6493, gr-qc/9504018

[10] A. Corichi and K. Krasnov, Loop Quantization of Maxwell Theory and Electric Charge Quantization, Mod. Phys. Lett. A13 (1998) 1339–1346, hep-th/9703177