New Flavor-Kinematics Dualities and Extensions of Nonlinear Sigma Models

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Nonlinear sigma model (nlσm) based on the coset SU(N)×SU(N)/SU(N) exhibits several intriguing features at the leading \(\mathcal{O}(p^2)\) in the derivative expansion, such as the flavor-kinematics duality and an extended theory controlling the single and triple soft limits. In both cases the cubic biadjoint scalar theory plays a prominent role. We extend these features in two directions. First we uncover a new extended theory for SO(N+1)/SO(N) nlσm at \(\mathcal{O}(p^2)\), which is a cubic bifundamental/biadjoint scalar theory. Next we provide evidence for flavor-kinematics dualities up to \(\mathcal{O}(p^3)\) for both SU(N) and SO(N) nlσm’s. In particular, we introduce a new duality building block based on the symmetric tensor \(\delta^{ab}\) and demonstrate several flavor-kinematics dualities for 4-point amplitudes, which precisely match the soft blocks employed to soft-bootstrap the nlσm’s up to \(\mathcal{O}(p^3)\).

I. INTRODUCTION

The nlσm [1] is the prototype of effective field theories, with the general construction based on an arbitrary coset \(G/H\) given in Refs. [2, 3] exactly half-a-century ago. However, it was relatively recently when several new and surprising aspects were uncovered. They include:

(−) An infrared (IR) construction without reference to the broken group \(G\), implemented either at the Lagrangian level through shift symmetries [4, 5] or at the on-shell level through the soft recursion relation [6–8]. The IR universality may have important implications for models attempting to understand the origin of electroweak symmetry breaking [9, 10].

(→) For SU(N) nlσm there exists a larger, extended theory of biadjoint cubic scalars interacting with the Nambu-Goldstone bosons (NGB), which controls both the single soft [11, 12] and triple soft [13] limits at leading order in the derivative expansion.

(Ξ) Again for SU(N) nlσm at \(\mathcal{O}(p^2)\), tree amplitudes exhibit flavor-kinematics duality [14–20] that is most transparent in the Cachazo-He-Yuan (CHY) [21–24] or the positive geometry [25] representations of nlσm amplitudes. The cubic biadjoint scalar theory also plays a prominent role in these constructions.

More broadly, these new developments belong to recent advances to understand quantum field theories from the IR [26–38] and/or on-shell perspectives [39–46]. Origins of many of the recently discovered features are still poorly understood, even for gauge theories and gravity, although certain insights can be obtained by supersymmetrization or taking the low-energy limit of string theories. In this regard, scalar theories such as the nlσm may serve as a simpler playing field than gauge theories and gravity.

In this work we will focus on the flavor-kinematics duality and the extended theory mentioned above. Our goal is two-fold: to understand whether these features persist outside the SU(N)×SU(N)/SU(N) coset and/or beyond the leading \(\mathcal{O}(p^2)\) in the derivative expansion. We study both the SU(N) and the SO(N) nlσm’s. The latter, based on the SO(N+1)/SO(N) coset, is a new example for “soft-bootstrap” introduced in Ref. [47], where the NGB transforms as the fundamental of SO(N). In Yang-Mills theories the gauge field always transforms as the adjoint representation. The SO(N) nlσm, which contains particles only in the fundamental representation, is a new venue for exploration.

Two previous works on flavor-kinematics duality applied the \(\mathcal{O}(p^2)\) Born-Carrasco-Johansson (BCJ) relations [14] to SU(N) nlσm amplitudes with higher derivatives and concluded that the duality fails already at \(\mathcal{O}(p^4)\) [8, 48]. It is further claimed that at \(\mathcal{O}(p^6)\) there is a single operator satisfying the \(\mathcal{O}(p^2)\) BCJ relation, which corresponds to a low-energy limit of open strings called the abelian Z theory [49, 50]. Recently Ref. [51] explored a more general construction of color-kinematics duality and applied to a special class of higher derivative operators in gravity and gauge theories. Does color/flavor-kinematics duality hold for generic higher dimensional operators from an effective field theory point of view? We will present evidence for a positive proposition, at least for nlσm’s at \(\mathcal{O}(p^4)\).

In this note we will present several results and sketch their derivations, which include the existence of a new extended theory for SO(N) nlσm involving cubic interactions of bifundamental/biadjoint scalars, as well as evidence for new flavor-kinematics dualities up to \(\mathcal{O}(p^3)\) for both SU(N) and SO(N) nlσm’s. Many details are saved for a future publication.

II. ALL ABOUT THAT BASIS

In this section we discuss the three flavor bases that will be used in this work: the single trace, the Del Duca-Dixon-Maltoni (DDM) [52] and the pair basis [47]. While the first two bases are commonly employed for amplitudes in Yang-Mills theories, we adapt them to the general \(G/H\) nlσm. The pair basis, on the other hand, is specific to theories containing the fundamental of SO(N).

For a general coset \(G/H\) the unbroken generators \(T^i\) in \(H\) and the broken generators \(X^a\) in \(G/H\) satisfy

\[
[T^i, T^j] = i f^{ijk} T^k, \quad [T^i, X^a] = i f^{iab} X^b, \\
[X^a, X^b] = i f^{abc} T^c.
\]  \hspace{1cm} (1)

The unbroken generators \(T^i\) transform under the adjoint representation of \(G\) and the broken generators \(X^a\) transform under the symmetric tensor \(\delta^{ab}\).
In a symmetric coset $f^{abc} = 0$, which we will focus on. For internal symmetry there is an NGB $\pi^a$ for each broken internal generator $X^a$. Therefore, the NGB carries the adjoint index of the broken group $G$. From this perspective, tree amplitudes of NGBs at the leading order can then be written naturally in the “single trace” basis [53–55]:

$$M_n^{\alpha_1\cdots\alpha_n}(p_1, \cdots, p_n) = \sum_{\alpha \in S_{n-1}} \text{Tr}(X^a_{\alpha_1} \cdots X^a_{\alpha_{n-1}} X^a_\alpha) M_n(\alpha, n), \quad (2)$$

where $\alpha$ is a permutation of $\{1, \cdots, n-1\}$ and $M(\alpha, n)$ is the flavor-ordered partial amplitudes. Because of the cyclic invariance of the trace, there are $(n-1)!$ independent trace structures. Nevertheless, the partial amplitudes satisfy the Kleiss-Kuijf (KK) [56] and BCJ relations [14, 57], which reduce the independent partial amplitudes to $(n-2)!$ and $(n-3)!$, respectively. Using the KK relations, the full amplitude can be written in the more minimal DDM basis, which for n-loop amplitudes is

$$M_n^{\alpha_1\cdots\alpha_n} = \sum_{\alpha \in S_{n-2}} (-1)^{n/2 - 1} f^{\alpha_1\alpha_{n-1}}_1 \times \left( \prod_{j=1}^{n/2-2} f^{\alpha_{2j-1}\alpha_{2j}}_j f^{\alpha_{2j+1}\alpha_{2j+2}}_{j+1} \right)^{f^{\alpha_{n-2}\alpha_n}}_n M_n(1, \alpha, n), \quad (3)$$

where $\alpha$ is a permutation of $\{2, 3, \cdots, n-1\}$. The DDM basis contains only $(n-2)!$ independent flavor structures.

For the SO$(N+1)/$SO$(N)$ nlsm, the SO$(N+1)$ structure constants $f^{iab}$ satisfy the relation

$$f^{iab} f^{ecd} = -\frac{1}{2} \left( \delta^{ad}\delta^{bc} - \delta^{ac}\delta^{bd} \right), \quad (4)$$

which is secretly the completeness relation for generators in the fundamental representation of the SO$(N)$ group. The relation can be used to further simplify the DDM basis:

$$M_n^{\alpha_1\cdots\alpha_n}(p_1, \cdots, p_n) = \sum_{\alpha \in P_n} \left( \prod_{j=1}^{n/2} \delta^{\alpha_{2j-1}\alpha_{2j}} \right) M_n(\hat{\alpha}_1 \hat{\alpha}_2 | \hat{\alpha}_3 \hat{\alpha}_4 | \cdots | \hat{\alpha}_{n-1} \hat{\alpha}_n), \quad (5)$$

where $P_n$ is the partition of $\{12 \cdots n\}$ into $n/2$ subsets, $\{\hat{\alpha}_1 \hat{\alpha}_2\}, \{\hat{\alpha}_3 \hat{\alpha}_4\}, \cdots, \{\hat{\alpha}_{n-1} \hat{\alpha}_n\}$, where each subset contains two elements contracted by Kronecker delta $\delta^{\alpha_{2j-1}\alpha_{2j}}$ and therefore symmetric in $\hat{\alpha}_{2j-1} \leftrightarrow \hat{\alpha}_{2j}$. The argument of the partial amplitude $M_n(\hat{\alpha})$ then contains $n/2$ non-ordered pairs of external particle indices. Since we are considering a symmetric coset, the amplitude contains an even number of external legs. The right-hand side (RHS) of the above is a sum of $(n-1)!$ independent flavor factors and forms what we call the “pair basis.” Since $\delta^{ab} = \text{tr} (T^a T^b)$, the pair basis is a multi-trace basis. The amplitude in this basis is invariant under exchanging the positions of different traces, as well as exchanging the two particle labels in each trace. For $n \geq 8$ the pair basis becomes smaller than the BCJ basis. In addition, it is possible to express partial amplitudes in the pair basis as linear combinations of those in the single trace basis. This relation leads to a CHY representation for all tree amplitudes in the pair basis of SO$(N)$ nlsm.

### III. NEW EXTENSION OF NLSM

For a set of scalars $\pi^a$ transforming under the fundamental representation of the SO$(N)$ group, the leading $\mathcal{O}(p^2)$ Lagrangian satisfying the Adler’s zero condition and describing the SO$(N+1)/$SO$(N)$ nlsm is [47]

$$\mathcal{L}^{(2)} = \frac{1}{2} F_\mu^{ab} \partial_\mu \pi^a \partial_\mu \pi^b - \frac{1}{4 f^2 R} [F_\mu(R) - 1] (\pi^a \partial_\mu \pi^a)^2, \quad (6)$$

where $F_\mu(x) = (1/x) \sin^2 \sqrt{x}$, $R = \pi^a \pi^a/(2 f^2)$ and $f$ is the Goldstone decay constant. The flavor structure of NGB’s in the Lagrangian is such that all flavor indices are contracted pair-wise by the Kronecker deltas, leading to the pair basis introduced in Sect. II. Eq. (6) is invariant under the nonlinear shift symmetry,

$$\pi^a \rightarrow \pi^a + \varepsilon^a + \frac{1}{2 f^2 R} [F_\mu(R) - 1] (\pi^a \varepsilon^a - \pi \cdot \varepsilon \pi^a), \quad (7)$$

where $F_\mu(x) \equiv \sqrt{x}$ cot $\sqrt{x}$. The conserved current corresponding to the shift symmetry is

$$J_\mu^a = \partial_\mu \pi^a + \frac{1}{2 f^2 R} \sum_{k=1}^{\infty} \frac{(-4)^k}{(2n+1)!} \left[ R^k \partial_\mu \pi^a - \frac{R^{k-1}}{2 f^2} (\pi \cdot \partial_\mu \pi) \pi^a \right], \quad (8)$$

and the corresponding Ward identity is

$$i \partial^\mu (\Omega) [J_\mu^a(x) \prod_{i=1}^{n} \pi^a_i(x_i)] / \Omega \sim 0, \quad (9)$$

where we have neglected Schwinger terms on the RHS since they do not contribute when all particles become on-shell.

Following the same procedure as in Ref. [12], we perform the LSZ reduction and take the on-shell limit on Eq. (9). Then the $\partial_\mu \pi^a$ term in $J_\mu^a$ contributes a single-particle pole and gives rise to the $(n+1)$-point (pt) on-shell amplitude, which is related to the matrix elements of the higher-order operators in Eq. (8). More explicitly,

$$M_{n+1}^{a_1 \cdots a_n+1}(p_1, \cdots, p_{n+1}) = \sum_{k=1}^{\infty} \langle 0 | \hat{O}_k^a(p_{n+1}) \pi^a_1(p_1) \cdots \pi^a_n(p_n) | \rangle, \quad (10)$$

where

$$\hat{O}_k^a(p_{n+1}) = \int d^4 x e^{-ip_{n+1} x} \left( \frac{(-4)^k}{(2k+1)!} \right) \times \partial_\mu \left[ R^k \partial_\mu \pi^a - \frac{R^{k-1}}{2 f^2} (\pi \cdot \partial_\mu \pi) \pi^a \right]. \quad (11)$$
Notice that $\tilde{\phi}_k^n(p_{n+1})$ gives rise to a $(2k+1)$-pt vertex, which in the pair basis can be written as

$$V(q_1|q_2q_3|\cdots|q_{2k}q_{2k+1}) = \frac{(-4)^k(k-1)!}{2f^{2k}(2k)!}p_{n+1}\cdot q_1,$$

and when entering Eq. (10) at tree level, the $q_i$ in the above are sum of subsets of the external momenta $p_1,\cdots,p_n$. In taking the single soft limit, $p_{n+1} \to \tau p_{n+1}$, $\tau \to 0$, Eq. (12) starts at the linear order in $\tau$, which makes the Adler's zero manifest and leads to the single soft theorem,

$$M_{n+1}(n+1,1|23|45|\cdots|n-1,n) = \tau \frac{n}{2} \sum_{k=2}^n s_{k,n+1} \sum_{j=2}^n M_n(1jjp|\phi^{(j)}||\bar{1}j\bar{k}),$$

where $j_p = j + (-1)^j$, $\phi^{(j)}$ is the partition $23|45|\cdots|n-1,n$ with the pair $(j,j_p)$ removed. In the above $M_n(1jjp|\phi^{(j)}||\bar{1}j\bar{k})$ denotes the $n$-pt amplitude of an extended theory containing two bifundamental scalars, charged under SO(N)×SO(N) in particle-$k$ as well as a biadjoint scalar in particle-$j$. The ordering to the right of $||$ denotes the flavor structure under SO(N), where two external bifundamental scalars $\phi^{a\bar{a}r,r=1,k}$, and one biadjoint scalar $\Phi^{ij}$ are contracted by $(T^j)_{a\bar{a}r}$. The ordering to the left of $||$ denotes the flavor structure under SU(N), where the pair-wise indices in $\phi^{(j)}$ are contracted by Kronecker deltas. Eq. (13) is to be contrasted with the result in SU(N) nπσ [11],

$$M_{n+1}(1\cdots n+1) = \tau \frac{n-1}{2} \sum_{i=2}^n s_{n+1,i} M_n(1\cdots n||\bar{1}i\bar{n}),$$

which is controlled by an extended theory with cubic biadjoint scalars charged under SU(N)×SU(N).

In the SO(N) extended theory, odd-pt vertices containing two $\phi$‘s, one $\Phi$ and an even number of NGB’s are given by

$$(T_i)^{ab}(T_i^\dagger)_{ab} \phi^{a\bar{a}} \phi^{\bar{b}b} \Phi^{i\bar{i}} \sum_{\alpha} (-4)^n 2(2n+1)! \left(\frac{n^\alpha n^{\bar{i}\bar{\alpha}}}{2f^{2n}}\right)^n,$$

where $(T_i)^{ab}$ and $(T_i^\dagger)_{ab}$ are generators of SO(N) and SO(N), respectively. Eq. (15) leads to the amplitudes in the extended theory appearing in Eq. (13), if one further assumes that all even-pt vertices involved in these amplitudes are identical to those in the SO(N) nπσ, much like in the extension of the SU(N) theory.

IV. NEW FLAVOR-KINEMATICS DUALITIES

The flavor-kinematics duality hinges on, for the 4-pt cubic graph shown in Fig. 1, a function of three flavor indices $j(123)$ which satisfies anti-symmetry and the Jacobi identity [20],

$$j(123) = j(213)+j(312) = 0.$$

Using Mandelstam variables we have $(s,t,u) = (s_{12},s_{23},s_{13})$, where $s_{ij} = 2p_i \cdot p_j$. Then $j_i = j(123), j_k = j(231)$ and $j_s = j(312)$. In general $j(123)$ could be a function of both flavor factors and kinematic invariants [51].

Below we consider such functions that are local in momenta. At the lowest mass dimension, $j$ only contains flavor factors and the most well-known example is

$$f_{d}(123) = f_{1a1}f_{a3a4},$$

which involves the structure constant of a Lie group $H$ and is commonly employed for particles transforming under the adjoint representation such as Yang-Mills theories [20]. Eq. (17) is the special case of a more general situation. Consider an arbitrary representation $r$ of $H$ and choose a basis for the generator $T_r$ such that it is purely imaginary and anti-symmetric. If $(T_r)^{ab}$ satisfies the following “Closure Condition” [4]

$$(T_r)^{a1a2}(T_r)^{a3a4}+(T_r)^{a2a3}(T_r)^{a1a4}+(T_r)^{a3a1}(T_r)^{a2a4} = 0,$$

then the flavor factor

$$f_{r}(123) = (T_r)^{a1a2}(T_r)^{a3a4},$$

also satisfies anti-symmetry and Jacobi identity. When $r$ is the adjoint representation, Eq. (19) reduces to $f_{d}$. The Closure Condition is the condition that $r$ of $H$ can be embedded into a symmetric coset $G/H$ [4]. For example, the fundamental representation of SO(N) is isomorphic to SO(N+1)/SO(N) and the fundamental representation of SU(N) parameterizes SU(N+1)/SU(N)×U(1).

At this mass dimension, however, there is a second possibility for $j(123)$ that has yet to be considered in the literature. It involves the rank-2 symmetric tensor $\delta^{ab}$,

$$f_{d}(123) = \delta^{a1a2}\delta^{a3a4} - \delta^{a1a4}\delta^{a2a3},$$

which exists for any representations. For the fundamental representation of SO(N), $f_{d} \propto f_{r}$ upon the completeness relation in Eq. (4). For other groups/representations, it is a new building block.

At the next order in mass dimension, there is one building block which contains only kinematic invariants,

$$n^{m}(123) = t - u.$$
This turns out to be the kinematic numerator for the single-flavor Yang-Mills scalar theory.

These simple building blocks allow us to construct more complicated $j$'s. One way is to just multiply existing blocks with objects invariant under permutations of $(1234)$. For local building blocks, there are two such objects at the lowest mass dimension, without any kinematic invariants,

\[
d_{4}^{abcd} = \sum_{\sigma \in S_{3}} \text{Tr} \left(T^{a_{1}\sigma(1)} T^{a_{2}\sigma(2)} T^{a_{3}\sigma(3)} T^{a_{4}} \right) ,
\]

\[
d_{2}^{abcd} = \frac{1}{2} \sum_{\sigma \in S_{3}} \delta^{a_{1}(1)\sigma(2)} \delta^{a_{3}(3)\sigma(4)} .
\]

Notice that $d_{4}$ is only for adjoint representations, while $d_{2}$ applies to any representation. For an arbitrary representation $d_{4}$ can be generalized to the rank-4 totally symmetric tensor that is independent of $d_{2}$, if it exists. For example, the fundamental of $SO(N)$ does not have such a rank-4 symmetric tensor. Going to higher orders in mass dimension one can write down two permutation invariant building blocks,

\[
X = stu , \quad Y = s^{2} + t^{2} + u^{2} ,
\]

which are first pointed out in Ref. [51].

A second way to generate new numerators is to take two existing blocks $j$ and $j'$, and define $J_{i}(j, j') = i - j - j'$. Then $j^{i}(1, 2, 3) = J_{i}(j, j')$ is a valid BCJ numerator [51]. Using this technique one can build more numerators only containing momenta. For example,

\[
J_{i}(n^{ss}, n^{ss}) \propto \frac{1}{3} s(t - u) \equiv n^{(1,2,3)} ,
\]

which is the BCJ kinematic numerator for $n^{ss}m$ at $O(p^{2})$ [49, 58, 59]. However, other numerators constructed this way that only contain kinematic invariants can all be written as a linear combination of $n^{ss}$ and $n^{(1,2,3)}$ with powers of $X$ and $Y$ [51].

Focusing on 4-pt amplitudes in the remainder of this work, flavor-kinematics duality at the leading $O(p^{2})$ for both $SU(N)$ and $SO(N)$ nlm's can be written as

\[
M_{k}^{(4)}(2) = \frac{1}{f^{2}} \left( \frac{f_{s}n_{s}^{u}}{s} + \frac{f_{t}n_{t}^{u}}{t} + \frac{f_{u}n_{u}^{u}}{u} \right) ,
\]

where $r = \text{Adj}$ for $SU(N)$ and Fundamental for $SO(N)$. For the $SO(N)$ nlm, $f_{s}$ is also interchangeable with $f_{u}$. In both cases replacing $f_{s}$ by another copy of $n^{u}$ gives rise to the 4-pt amplitude, $M_{k}^{(4)} \propto s t u$, in special Galileon theory [24, 27]. Alternatively, one can realize the double copy structure using the Kawai-Lewellen-Tye (KLT) relations [60].

At higher orders in derivative expansion, previous works aimed to generate local 4-pt amplitudes by modifying the kinematic factors $n^{u}$ while keeping the flavor factors $f_{s}$ intact, which turned out to be impossible until $O(p^{4})$ [8, 48]. Even at that order, the corresponding operator is a special one, belonging to the low-energy $\alpha'$ expansion of open strings [49]. In the following we will demonstrate a more general procedure to implement flavor-kinematics duality, which is valid for generic operators up to $O(p^{6})$.

It is convenient to recall that 4-pt amplitudes at $O(p^{4})$ in $n^{\sigma}m$ are characterized by four “soft blocks” [47], which are 4-pt local partial amplitudes in the single or double trace basis:

\[
S_{1}^{(4)}(1234) = \frac{u^{2}}{X^{2}f^{2}} , \quad S_{2}^{(4)}(1234) = \frac{st}{X^{2}f^{2}} ,
\]

\[
S_{1}^{(4)}(12|34) = \frac{s^{2}}{X^{2}f^{2}} , \quad S_{2}^{(4)}(12|34) = \frac{tu}{X^{2}f^{2}}
\]

where $\Lambda$ is a dimensionful parameter controlling the convergence of the derivative expansion. It turns out that there are a total of four possibilities to replace $f_{s}$ that result in local 4-pt amplitudes at $O(p^{4})$, which correspond to the four soft blocks above. They are

\[
\hat{f}_{1} = \frac{1}{\Lambda^{2}} J(f_{s}, n^{ss}) , \quad \hat{f}_{2} = \frac{1}{\Lambda^{2}} d_{4}^{a_{1}a_{2}a_{3}a_{4}}n^{ss} ,
\]

\[
\hat{f}_{3} = \frac{1}{\Lambda^{2}} J(f_{s}, n^{ss}) , \quad \hat{f}_{4} = \frac{1}{\Lambda^{2}} d_{4}^{a_{1}a_{2}a_{3}a_{4}}n^{ss} .
\]

Replacing $f_{s}$ in Eq. (26) by the above gives rise to four different $O(p^{4})$ amplitudes, whose partial amplitudes are related to the soft blocks via

\[
M_{k}^{(4)}(1234) = S_{1}^{(4)}(1234) + 2 S_{2}^{(4)}(1234) ,
\]

\[
M_{k}^{(4)}(1234) = 2 \left[ S_{1}^{(4)}(1234) - S_{2}^{(4)(1234)} \right] ,
\]

\[
M_{k}^{(4)}(12|34) = S_{1}^{(4)}(12|34) + 2 S_{2}^{(4)(12|34)} ,
\]

\[
M_{k}^{(4)}(12|34) = 2 \left[ S_{1}^{(4)}(12|34) - S_{2}^{(4)(12|34)} \right] .
\]

Since the soft blocks generate tree amplitudes for the complete set of parity-even operators at $O(p^{4})$, so does our construction of flavor-kinematics duality.

V. CONCLUSION AND OUTLOOKS

In this work we have demonstrated that several intriguing features persist beyond the leading $O(p^{2})$ effective action based on the $SU(N)$ nlm. We supplied a new example of extended theory in the $SO(N)$ nlm, also at the $O(p^{4})$, and presented evidence for flavor-kinematics duality at $O(p^{4})$ for both $SU(N)$ and $SO(N)$ nlm's, by providing a new duality building block based on $\delta^{ab}$, which could also potentially be applied to Yang-Mills theories.

Several future directions are currently being undertaken. They include amplitude relations and double-copy procedures in the pair basis, corrections to the single and double soft theorems from $O(p^{4})$ operators, the possible existence of an extended theory at $O(p^{4})$, and flavor-kinematics duality for $O(p^{4})$ amplitudes at higher multiplicities, just to name a few. The outcome, as well as detailed derivations of results outlined in this note, will be reported in a future publication.
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