Effects of flow fluctuations and partial thermalization on $v_4$

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The second and fourth Fourier harmonic of the azimuthal distribution of particles, $v_2$ and $v_4$, have been measured in Au+Au collisions at the Relativistic Heavy Ion Collider (RHIC). The harmonic $v_4$ is mainly induced from $v_2$ as a higher-order effect. However, the ratio $v_4/(v_2)^2$ is significantly larger than predicted by hydrodynamics. Effects of partial thermalization are estimated on the basis of a transport calculation, and are shown to increase $v_4/(v_2)^2$ by a small amount. We argue that the large value of $v_4/(v_2)^2$ seen experimentally is mostly due to elliptic flow fluctuations. However, the standard model of eccentricity fluctuations is unable to explain the large magnitude of $v_4/(v_2)^2$ in central collisions.

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I. INTRODUCTION

The azimuthal distribution of particles emitted in ultrarelativistic nucleus-nucleus collisions at RHIC is a sensitive tool in understanding the bulk properties of the matter produced in these collisions (see [11] for a recent review). It is generally written as a Fourier series

$$\frac{dN}{d\phi} \propto 1 + 2v_2 \cos 2\phi + 2v_4 \cos 4\phi + \cdots$$  \hspace{1cm} (1)

where $\phi$ is the azimuthal angle with respect to the direction flow. In this paper, we consider analyses done near the center-of-mass rapidity, so that odd harmonics vanish by symmetry. The large magnitude of elliptic flow, $v_2$, suggests that the lump of matter formed in an Au-Au collision at RHIC is close to local thermal equilibrium and expands as a relativistic fluid. Elliptic flow is large at high $p_t$ (up to 0.25 for baryons), which motivated the idea to study the higher-order harmonic $v_4$ [2, 3]. Several analyses of $v_4$ have been reported [4, 5, 6, 7]. Experimental results give $v_4 \sim (v_2)^2$, while the ideal-fluid picture generally predicts $v_4 = \frac{1}{4} (v_2)^2$ [8]. This discrepancy has not yet been explained. In this paper, we investigate the sensitivity of $v_4$ to two effects: viscous deviations from the ideal-fluid picture (Sec. III), and elliptic flow fluctuations (Sec. IV).

II. IDEAL HYDRODYNAMICS

We first briefly recall the prediction of relativistic hydrodynamics. In this theory, the $\phi$ dependence of particle distribution results from a similar $\phi$ dependence of the fluid 4-velocity [8, 9]:

$$u(\phi) = U \left( 1 + 2V_2 \cos 2\phi + 2V_4 \cos 4\phi \cdots \right),$$ \hspace{1cm} (2)

where $\phi$ is the azimuthal angle of the fluid velocity with respect to the minor axis of the participant ellipse [10] (see Fig. 1). This is due to the fact that the overlap area between the two colliding nuclei is elliptic, which results in anisotropic pressure gradients. For a semi-central Au-Au collision at RHIC, $V_2 \sim 4\%$, and one expects $V_4$ to be of much smaller magnitude, typically $V_4 \sim (V_2)^2$.

The fluid expands, becomes dilute and eventually transforms into particles. As argued in Ref. [8], fast particles are produced where the fluid velocity is maximum, and parallel to the particle momentum. The resulting momentum distribution is a boosted thermal distribution. Neglecting quantum statistics (this is justified in the transverse momentum range where $v_4$ is measured), the momentum distribution for a given particle of mass $m$ is

$$\frac{dN}{p_t dp_t d\phi} \propto e^{-p_t u/T} \exp \left( -\frac{m_t u_0(\phi) - p_t u(\phi)}{T} \right),$$ \hspace{1cm} (3)

where $m_t = \sqrt{p_t^2 + m^2}$, $u_0(\phi) = \sqrt{1 + u(\phi)^2}$, and $\phi$ is the azimuthal angle of the particle. Inserting Eq. (2) into Eq. (3), expanding to leading order in $V_2$, $V_4$ and identifying with Eq. (1), one obtains [8]

$$v_2(p_t) = \frac{V_2 U}{T} (p_t - m_t v)$$
$$v_4(p_t) = \frac{1}{2} v_2(p_t)^2 + \frac{V_4 U}{T} (p_t - m_t v),$$ \hspace{1cm} (4)

where $v \equiv U/\sqrt{1 + U^2}$. The higher harmonic $v_4$ is the
sum of two contributions: an “intrinsic” $v_4$ proportional to the $\cos 4\theta$ term in the fluid velocity distribution, $V_4$, and a contribution induced by elliptic flow itself, which turns out to be exactly $\frac{1}{2}(v_2)^2$. The latter contribution becomes dominant as $p_t$ increases.

In order to confirm these qualitative results, we solve numerically the equations of ideal relativistic hydrodynamics. The fluid is initially at rest. We choose a gaussian initial entropy density profile, with rms widths $\sigma_x = 2$ fm and $\sigma_y = 3$ fm. The equation of state is that of an two-dimensional ideal gas of massless particles, $s \propto T^4$, for reasons to be explained below. The normalization has been fixed in such a way that the average transverse momentum per particle is $\langle p_t \rangle = 0.42$ GeV/c, which is roughly the value of $p_t$ for pions in a central Au-Au collision at RHIC [12]. Fig. 2 displays the variation of $v_4/(v_2)^2$ with the particle transverse momentum $p_t$. For massless particles, $m_t = p_t$ and Eq. (4) gives $v_4/(v_2)^2 = 0.5 + k/p_t$, where $k$ is independent of $p_t$. To check the validity of this formula, our numerical results are fitted over the interval $0.5 < p_t < 2.5$ GeV/c by the simple formula

$$\frac{v_4(p_t)}{v_2(p_t)^2} = A + B \frac{p_t}{p_t},$$

where we have introduced the average transverse momentum $\langle p_t \rangle$ in such a way that the coefficient $B$ is dimensionless. We refer to $A$ (resp. $B$) as to the induced (resp. intrinsic) $v_4$. We find $A = 0.557$ and $B = 0.479$. The value of $A$ is close to the expected value 0.5. The small discrepancy is due to the fact that Eqs. (4) are only valid for small values of $v_2$ and $v_4$. This approximation breaks down at the upper end of our fitting interval, where $v_2(2.5$ GeV/c) = 0.51. This large value is due to the fact that the equation of state is that of an ideal gas. For large $p_t$, however, the intrinsic $V_4$ term in Eq. (4) can be neglected, because it is linear in $p_t$ while the other term is quadratic in $p_t$. Neglecting this term, the Fourier expansion in Eq. (1) can be done exactly. This yields

$$v_{2n}(p_t) = \frac{I_n(x)}{I_0(x)},$$

where $x = 2v_2U(p_t - m_t)/T$, and $I_n(x)$ is the modified Bessel function. Inverting Eq. (6) with $n = 1$ and $v_2 = 0.51$, one obtains $x = 1.19$. Eq. (6) with $n = 2$ then gives $v_4/(v_2)^2 = 0.552$, in better agreement with our numerical result.

We have systematically investigated the sensitivity of our hydrodynamical results to initial conditions. With a smaller initial eccentricity ($\sigma_x = 2$ fm and $\sigma_y = 2.5$ fm), the value of $A$ is closer to 0.5, as expected from the discussion above. We have also repeated the calculation with a more realistic density profile corresponding to a Au-Au collision at RHIC, obtained using an optical Glauber model calculation. We expected that $B$, which we understand as the “intrinsic” $v_4$, would be sensitive to the change in initial conditions, but the changes in both $A$ and $B$ were insignificant.

Experimental results are also shown in Fig. 2. The value of $v_4/v_2^2$ is constant, even at relatively low $p_t$: a fit to these results using Eq. (5) gives $B = 0.01 \pm 0.04$, compatible with zero. The other fit parameter is $A = 0.89 \pm 0.02$, significantly larger than the value 0.5 predicted by hydrodynamics. Some of the discrepancies between our model calculation and data can be attributed to the equation of state, which is much softer in QCD near the transition region than in our hydrodynamical calculation. More specifically, the coefficient $B$ representing the intrinsic $v_4$ may depend on the equation of state. It would be interesting to investigate whether the small value of $B$ seen experimentally can be attributed to the softness of the equation of state. On the other hand, our argument leading to $A = \frac{1}{2}$ is quite general, so that the discrepancy with data cannot be attributed to the equation of state. In this paper, we investigate the possible origins of this discrepancy.

III. PARTIAL THERMALIZATION

It has been argued [14] that if interactions among the produced particles are not strong enough to produce local thermal equilibrium, so that the hydrodynamic description breaks down, the resulting value of $v_4/v_2^2$ is higher. This is confirmed by transport calculations within the AMPT model [13]. This naturally raises the

1 Note, however, that STAR results for charged particles [12] clearly display an intrinsic $v_4$ component, although smaller than in our calculation.
question of how \( v_4 \) reaches the hydrodynamic limit \[10\]. We investigate this issue systematically by solving numerically a relativistic Boltzmann equation, where the mean free path \( \lambda \) of the particles can be tuned by varying the elastic scattering cross section \( \sigma \). The degree of thermalization is characterized by the Knudsen number

\[
K = \frac{\lambda}{R},
\]

where \( R \) is a measure of the system size. We consider massless particles moving in the transverse plane (no longitudinal motion) \[10\]. In the limit \( K \to 0 \), this Boltzmann equation is expected to be equivalent to ideal hydrodynamics, with the equation of state of a two-dimensional ideal gas. For sake of consistency with our hydrodynamical calculation, the initial phase space distribution of particles is locally thermal: \( \frac{dN}{d^2x dp_t} \propto \exp(-p_t/T(x,y)) \), where the temperature profile \( T(x,y) \) is the same as in the hydrodynamical calculation. The Knudsen number is normalized as in Ref. \[17\]:

\[
K = \frac{\lambda}{R} = \frac{4\pi \sqrt{\sigma_x^2 + \sigma_y^2}}{N\sigma},
\]

where \( N \) is the total number of particles in the Monte-Carlo simulation, and \( \sigma \) the scattering cross section, which has the dimension of a length in two dimensions.

Fig. \[3\] displays our results for two values of \( K \). The results for \( K = 0.05 \) are almost identical to the results from ideal hydrodynamics, as expected. For \( K = 0.5 \), \( v_4/(v_2)^2 \) is larger, as anticipated in Ref. \[14\]. Although the fit formula \[10\] is inspired by hydrodynamics, the quality of the fit is equally good for the Boltzmann calculation. In particular, the ratio \( v_4/(v_2)^2 \) quickly saturates with increasing \( p_t \), which means that the scaling \( v_4 \propto (v_2)^2 \) still holds if the system does not reach local thermal equilibrium, as already observed in previous transport calculations \[18\].

The sensitivity of \( v_4 \) to the Knudsen number \( K \) is seen more clearly in Fig. \[3\] which displays the variation of the fit parameters \( A \) and \( B \) with \( K \). A linear extrapolation of our Boltzmann results to the limit \( K = 0 \) gives \( A = 0.524 \pm 0.008 \) and \( B = 0.508 \pm 0.012 \), to be compared with our results from ideal hydrodynamics \( A = 0.557 \) and \( B = 0.479 \), in good agreement \[2\].

These transport results may be sensitive to the choice of initial conditions. We have assumed a locally thermal distribution. Now, the prediction \( v_4/(v_2)^2 \) from hydrodynamics originates precisely from the assumption that momentum distributions are thermal in the rest frame of the fluid, see Eq. \[3\]. Replacing the exponential in this equation with a more general function \( f(p \cdot u) \) leads to \( v_4/(v_2)^2 = f f''/(2 f') \). With a Levy distribution \( f(x) = (1 + x/n/T)^{-n} \), the value of \( v_4/(v_2)^2 \) is enhanced by a factor \( (1 + n)/n \). Values of \( n \) inferred from \( p_t \) spectra of particles produced in p-p collisions are close to 10 \[19\], which yields a slight increase from the prediction of hydrodynamics.

Realistic values of the Knudsen number \( K \), inferred from the centrality dependence of \( v_2 \) \[20\], are in the range \( 0.3 \sim 0.5 \) for semi-central collisions. For these values, Fig. \[3\] shows that \( v_4/(v_2)^2 \) is at most 0.6, still significantly below the experimental value 0.9. We conclude that partial thermalization alone cannot explain experimental data.

\[2\] There is a small residual discrepancy of a few percent between Boltzmann and ideal hydrodynamics, which we do not understand.
RHIC experiments have analyzed in detail the centrality dependence of $v_4/(v_2)^2$. Preliminary results from STAR [21] and PHENIX [22] are presented in Fig. 4. The values of $v_4/(v_2)^2$ are larger than 0.8 for all centralities, and increase up to 1.6 for central collisions. Both experiments observe a similar centrality dependence of $v_4/(v_2)^2$. STAR obtains values slightly higher than PHENIX. This difference may be due to nonflow effects, which are smaller for PHENIX than for STAR because the reaction plane detector is in a different rapidity window than the central arm detector [7]. Nonflow effects contribute both to $v_2$ and $v_4$. We now estimate the order of magnitude of the error on $v_4$. We consider for simplicity the case when $v_4$ is analyzed from three-particle correlations. The corresponding estimate of $v_4$, denoted by $v_4\{3\}$ [23], is defined by

$$v_4\{3\} \equiv \frac{\langle \cos(4\phi - 2\phi_2 - 2\phi_3) \rangle}{(v_2)^2}$$

(9)

where $\phi_j$ are azimuthal angles of outgoing particles and angular brackets denote an average over triplets of particles belonging to the same event. In Eq. (9), $v_2$ must be obtained from another analysis. Nonflow effects arise when particles 1 and 2 come from the same source [4]. Assuming that the source flows with the same $v_2$ as the daughter particles, we obtain

$$\langle \cos(4\phi - 2\phi_2 - 2\phi_3) \rangle = v_4(v_2)^2 + \delta_{nf}(v_2)^2,$$

(10)

where $\delta_{nf}$ is the nonflow correlation. The latter can be estimated [24] using the azimuthal correlation $\delta_{pp}$ measured in proton-proton collisions [25] and scaling it down by the number of participants: $\delta_{nf} = 2\delta_{pp}/N_{\text{part}}$. Dividing by $(v_2)^4$, we obtain the corresponding error on $v_4/(v_2)^2$:

$$\delta \left( \frac{v_4}{(v_2)^2} \right)_{nf} = \frac{2\delta_{pp}}{N_{\text{part}}(v_2)^2}.$$

(11)

In practice, the analysis is done using the event-plane method rather than three-particle correlations, but this changes little the magnitude of nonflow effects [24]. The error (11) varies with centrality like $1/\chi^2$, where $\chi = v_2\sqrt{N}$ is the resolution parameter entering the flow analysis. The numerical value $\delta_{pp} = 0.0145$ has been used in Ref. [23] to subtract nonflow effects from $v_2$. It was obtained by integrating the azimuthal correlation in proton-proton collisions over $p_t$. The error bar on STAR results in Fig. 4 is obtained using Eq. (11) with $\delta_{pp} = 0.0145$. The agreement with PHENIX is much improved. However, this may be a coincidence: in the case of $v_4$, which is measured at relatively large $p_t$, nonflow effects are likely to be larger; on the other hand, nonflow contributions to $v_2$ tend to increase $v_2$ and decrease the ratio $v_4/(v_2)^2$, which goes in the opposite direction. Finally, we must keep in mind that even with a rapidity gap as in the PHENIX analysis, there may be a residual nonflow error of a similar magnitude.

V. FLOW FLUCTUATIONS

The scaling $v_4 = 0.5(v_2)^2$ predicted by ideal hydrodynamics only holds for identified particles at a given transverse momentum $p_t$ and rapidity $y$, for a given initial geometry. In order to increase the statistics, however, experimental results for $v_2$ and $v_4$ are averaged over some of these quantities before computing the ratio $v_4/(v_2)^2$. The averaging process increases the ratio. For instance, the results shown in Fig. 2 are averaged over a large centrality interval 20-60%. Even within a narrow centrality class, the initial geometry varies significantly due to fluctuations in the initial state [26, 27].

We now discuss the influence of these fluctuations on $v_2$ and $v_4$. We assume for simplicity that $v_2$ and $v_4$ are analyzed using two-particle correlations and three-particle correlations, respectively. The case where the analysis is done using the event-plane method is more complex and will be discussed in Sec. VI. The estimate of $v_2$ from two-particle correlations is denoted by $v_2\{2\}$ and defined by

$$v_2\{2\} \equiv \langle \cos(2\phi - 2\phi_2) \rangle.$$  

(12)

Similarly, if $v_4$ and $v_2$ fluctuate, $\langle \cos(4\phi - 2\phi_2 - 2\phi_3) \rangle = \langle v_4(v_2)^2 \rangle$. We thus obtain

$$\frac{v_4\{3\}}{v_2\{2\}^2} = \frac{v_4(v_2)^2}{(v_2)^4} = \frac{1}{2} \left( \frac{\langle v_2 \rangle^4}{\langle v_2 \rangle^2} \right)^2,$$

(13)

where, in the last equality, we have assumed that the prediction of hydrodynamics $v_4 = (v_2)^2/2$ holds for a given value of $v_2$. If $v_2$ fluctuates, $\langle (v_2)^4 \rangle > \langle (v_2)^2 \rangle^2$, which shows that elliptic flow fluctuations increase the observed $v_4/(v_2)^2$. We now estimate quantitatively the magnitude of these fluctuations.

A. Flow fluctuations from $v_2$ analyses

The magnitude of $v_2$ fluctuations can be inferred from the difference between estimates of $v_2$, which is dominated by flow fluctuations except for very peripheral collisions [23]. The estimate from 2-particle correlations, $v_2\{2\}$, gives directly $\langle (v_2)^2 \rangle$, while the estimate of $v_2$ from 4-particle cumulants, denoted by $v_2\{4\}$, involves $\langle (v_2)^4 \rangle$ [28]:

$$v_2\{4\}^4 = 2 \left( \frac{\langle v_2 \rangle^2}{\langle v_2 \rangle^2} \right)^2 - \langle (v_2)^4 \rangle.$$  

(13)

Inverting this relation and inserting into Eq. (12), one obtains an estimate of the effect of $v_2$ fluctuations on $v_4$:

$$\frac{v_4\{3\}}{v_2\{2\}^2} = \frac{1}{2} \left( 2 - \frac{\langle v_2\{4\} \rangle^4}{\langle v_2\{2\} \rangle^4} \right).$$  

(14)

We use $v\{2\}$ from [29], instead of $v_2\{4\}$, we use the more recent measurement $v_2\{LYZ\}$ using Lee-Yang zeros [30, 31], which is expected to have a similar sensitivity to flow fluctuations. Data on $v_2\{LYZ\}$ are only
available for semi-central collisions. The resulting prediction for $v_4/(v_2)^2$ is shown in Fig. 3. The agreement with data is much improved when fluctuations are taken into account. We have checked numerically that our results do not change significantly if nonflow effects are subtracted from $v_2[2]$ using the parametrization introduced in Ref. [24].

**B. Flow fluctuations from eccentricity fluctuations**

Since there are no data on $v_2[LYZ]$ for the most central and peripheral bins, we need a model of $v_2$ fluctuations to cover the whole centrality range. We use the standard model of eccentricity fluctuations [11,28]. The idea is that the overlap area between the colliding nuclei (see Fig. 1) is not smooth: positions of nucleons within the nucleus fluctuate from one event to another, even for a fixed impact parameter. Therefore, the participant eccentricity, $\epsilon_{PP}$, which is the eccentricity of the ellipse defined by the positions of participant nucleons, also fluctuates. Assuming that $v_2$ in a given event scales like $\epsilon_{PP}$, Eq. (12) gives

$$v_4(3) = \frac{1}{2} \frac{v_2(2)^2}{v_2(2)} \frac{\langle \epsilon_{PP}^4 \rangle}{\langle \epsilon_{PP}^2 \rangle^2}.$$  \hspace{1cm} (15)

We estimate this quantity using the Monte-Carlo Glauber model [32] provided by the PHOBOS collaboration [33]. In each event, the participant eccentricity is defined by

$$\epsilon_{PP} = \sqrt{(\sigma_y^2 - \sigma_x^2)^2 + 4\sigma_y^2 \sigma_x^2}.$$  \hspace{1cm} (16)

where $\sigma_x^2 = \{ \{x^2\} - \{x\}^2 \}$ and $\sigma_{xy} = \{xy\} - \{x\} \{y\}$, and $\{\cdots\}$ denotes event-by-event averages over participant nucleons. Each participant is given a weight proportional to the number of particles it creates:

$$w = (1 - x) + x N_{coll/part},$$  \hspace{1cm} (17)

where $N_{coll/part}$ is the number of binary collisions of the nucleon. The sum of weights scales like the multiplicity:

$$\frac{dN_{ch}}{d\eta} = n_{pp} \left[ \frac{(1 - x)}{2} N_{part} + x N_{coll} \right].$$  \hspace{1cm} (18)

where $N_{part}$ and $N_{coll}$ are respectively the number of participants and of binary collisions of the considered event. We choose the value $x = 0.13$ which best describes the charged hadron multiplicity observed experimentally [33]. We define the centrality according to the multiplicity $N_{part}$ [15]. We evaluate eccentricity fluctuations in centrality classes containing 5% of the total number of events.

Our results are presented in Fig. 4. For peripheral and semi-central collisions, the estimates from eccentricity fluctuations are in good agreement with the earlier estimate from the difference between $v_2$ analyses, in line with the observation that this difference is mostly due to eccentricity fluctuations [24]. For the most central bin, however, eccentricity fluctuations only increase $v_4/(v_2)^2$ by a factor 2, while a factor 3 would be needed to match STAR and PHENIX data. This factor 2 can be simply understood. For central collisions, eccentricity fluctuations are well described by a two-dimensional gaussian distribution [34]:

$$\frac{dN}{d\epsilon_{PP}} = \frac{\epsilon_{PP}}{\sigma^2} \exp \left( -\frac{\epsilon_{PP}^2}{2\sigma^2} \right).$$  \hspace{1cm} (19)

This implies $\langle \epsilon_{PP}^4 \rangle/\langle \epsilon_{PP}^2 \rangle^2 = 2$.

We now combine the effects of flow fluctuations and partial thermalization, discussed in Sec. III. We take partial thermalization into account using the linear fit to the coefficient $K$ from Eq. 3:

$$\frac{v_4}{v_2^2} = \frac{1}{2} + 0.18 K.$$  \hspace{1cm} (20)

This modifies Eq. (15) into the following equation:

$$\frac{v_4(3)}{v_2(2)^2} = \left( \frac{1}{2} + 0.18 K \right) \frac{\langle \epsilon_{PP}^4 \rangle}{\langle \epsilon_{PP}^2 \rangle^2}.$$  \hspace{1cm} (21)

The value of $K$ can be evaluated using the centrality dependence of elliptic flow. We borrow our estimates from Ref. 24. This study has recently been corrected and refined [35], but the resulting estimates of $K$ differ little from the original ones. Results are shown in Fig. 4. Partial thermalization is a small effect. Agreement with data is significantly improved for semicentral collisions, not for central collisions. For peripheral collisions, our calculation overshoots PHENIX data. Note that Eq. (20) was derived using the results of a Boltzmann transport calculation, which only applies to a dilute gas. With a realistic, soft equation of state, the coefficient in front of $K$ could be different.

**C. A toy model of Gaussian flow fluctuations**

In order to illustrate the sensitivity of $v_4$ to the statistics of $v_2$ fluctuations, we finally consider a toy model where the distribution of $v_2$ at fixed impact parameter $b$ is Gaussian:

$$\frac{dN}{dv_2} = \frac{1}{\sigma_v \sqrt{2\pi}} \exp \left( -\frac{(v_2 - \kappa v_{\text{RP}}(b))^2}{2\sigma_v^2} \right),$$  \hspace{1cm} (22)

where $v_{\text{RP}}$ is the reaction-plane eccentricity obtained using an optical Glauber model (smooth initial density profile), and $\kappa$ a proportionality constant. We assume that $\sigma_v$ scales like $N_{\text{part}}^{-1/2}$, as generally expected for initial state fluctuations, and we adjust the proportionality constant so as to match the difference between $v_2(2)$ and $v_2(4)$ for midcentral collisions. The result is displayed in Fig. 5. We have checked that a similar result is
Similarly, one can write

\[ \frac{v_4(3)}{v_2(2)^2} = \frac{1}{2} \left( 1 + \frac{\sigma_n}{(v_2)^2} \right). \]

(23)

Similarly, one can write

\[ \frac{v_4(EP)}{v_2(EP)^2} = \frac{1}{2} \left( 1 + \alpha \frac{\sigma_n^2}{(v_2)^2} \right), \]

(24)

where \( \alpha \) depends on the reaction plane resolution. A similar parametrization has been introduced for the fluctuations of \( v_2 \) [11]. The expression of \( \alpha \) is derived in Appendix A using the same methods as in Ref. [24].

Fig. 6 displays the variation of \( \alpha \) with the event-plane resolution for elliptic flow. One sees that \( \alpha < 1 \), which means that the effect of fluctuations is always smaller for \( v_4(EP) \) than for \( v_4(3) \); this is confirmed by the experimental observation \( v_4(3) > v_4(EP) \) [1]. The resolution is 1 when the reaction plane is reconstructed exactly. In this limit, \( v_2(EP) = \langle v_2 \rangle, v_4(EP) = \frac{1}{2} \langle (v_2)^2 \rangle \), which implies \( \alpha = 1 \). In practice, however, the maximum resolution for mid-central collisions is 0.84 for STAR [29] and 0.74 for PHENIX [7]. In the case of PHENIX, \( \alpha \) is larger than 3.2 for all centralities, which means that the effect of fluctuations is decreased at most by 20% compared to our estimates in the previous section.

VI. FLUCTUATIONS AND FLOW METHODS

In practice, \( v_2 \) and \( v_4 \) are analyzed using the event-plane method [38, 39]. The corresponding estimates are denoted by \( v_2(EP) \) and \( v_4(EP) \). In this Section, we argue that flow fluctuations have almost the same effect on \( v_4(EP) \) as on \( v_4(3) \). We limit our study to small fluctuations for simplicity, in the same spirit as in Ref. [24]. We write \( v_2 = \langle v_2 \rangle + \delta v \), with \( \langle \delta v \rangle = 0 \) and \( \langle \delta v^2 \rangle = \sigma_n^2 \), where \( \sigma_n \) characterizes the magnitude of flow fluctuations. Expanding Eq. (12) to leading order in \( \sigma_n \), we obtain

\[ \frac{v_4(3)}{v_2(2)^2} = \frac{1}{2} \left( 1 + 4 \frac{\sigma_n^2}{(v_2)^2} \right). \]

(23)

We have shown that experimental data on \( v_4 \) are rather well explained by combining the prediction \( v_4 = \frac{3}{2} (v_2)^2 \) from ideal hydrodynamics with elliptic flow fluctuations. If this scenario is correct, then \( v_4 \) should be independent of particle species and rapidity for fixed \( p_t \) and centrality. This is confirmed by preliminary results from PHENIX, which give the same value for pions, kaons and protons [7]. Ideal hydrodynamics, which fails to describe \( v_2(p_t) \) for \( p_t > 1.5 \) GeV/c, seems to describe well \( v_4/v_2^2 \) at least up to \( p_t \sim 3 \) GeV/c.

Note that our scenario does not support the picture of hadron formation through quark coalescence at large \( p_t \) [40]. We find values of \( v_4/v_2^2 \) below 1 as a result of the hydrodynamic expansion, which is believed to take place in the quark phase. But coalescence requires that \( v_4/(v_2)^2 \) for the underlying quark distribution is much higher, around 2 [41].

The centrality dependence of \( v_4 \) offers a sensitive probe of the mechanism underlying flow fluctuations. Eccentricity fluctuations have been shown to explain quantita-

FIG. 5: (Color online) Results using a toy model of gaussian fluctuations. STAR and PHENIX data as in Fig. 4. Dashed line: ideal hydrodynamics + gaussian flow fluctuations. Full line: gaussian flow fluctuations and partial thermalization.
tively $v_2$ data in Au-Au and Cu-Cu collisions. We find that they also explain most of the results on $v_2$ for peripheral and semi-central collisions. However, they are unable to explain the steep rise of $v_4/(v_2)^2$ for the most central bins, which is clearly seen both by STAR and PHENIX. Data suggest that $\langle (v_2)^4 \rangle / \langle (v_2)^2 \rangle^2 \simeq 3$ for the most central bin, while eccentricity fluctuations give 2. Impact parameter fluctuations only increase $v_4/v_2^2$ by a few percent. We cannot exclude a priori that the large experimental value is due to large errors in the extraction of $v_4$: if we multiply the nonflow error estimated in Sec. IV by a factor 4, data agree with our calculation for central collisions; however, the agreement is spoilt for peripheral collisions. It therefore seems unlikely that the discrepancy is solely due to nonflow effects. These results suggest that initial state fluctuations do not reduce to eccentricity fluctuations, as recently shown by a study of transverse momentum fluctuations \cite{42}. Interestingly, the direct measurement of $v_2$ fluctuations attempted by PHOBOS \cite{43}, which agrees with the prediction from eccentricity fluctuations, does not extend to the most central bin.

An independent confirmation that $\langle (v_2)^4 \rangle / \langle (v_2)^2 \rangle^2 \simeq 3$ for central collisions could be obtained from the 4-particle cumulant analysis. Interestingly, there is no published value of $v_2$ \cite{44} for the most central bin: the reason is probably that $v_2$ \cite{44} cannot be defined using Eq. (13), because the right-hand side is negative. This indicates that $\langle (v_2)^4 \rangle / \langle (v_2)^2 \rangle^2 > 2$. It would be interesting to repeat the cumulant analysis for central collisions, and to scale the right-hand side of Eq. (14) by $v_2(2)^4$. The ratio should be around $-1$ if $\langle (v_2)^4 \rangle / \langle (v_2)^2 \rangle^2 \simeq 3$. This would give invaluable information on the mechanism driving elliptic flow fluctuations.

**APPENDIX A: EFFECT OF FLUCTUATIONS ON THE EVENT-PLANE $v_4$**

In this Appendix, we derive the expression of $\alpha$ in Eq. (24). This parameter measures the effect of fluctuations on $v_4/(v_2)^2$ when flow is analyzed using the event-plane method. The event plane $v_4$ is defined by

$$v_4\{\text{EP}\} = \frac{\langle \cos 4(\phi - \Psi_R) \rangle}{R_4}, \quad (A1)$$

where $\phi$ is the azimuthal angle of the particle, $\Psi_R$ is the angle of the event plane, and $R_4$ is the event-plane resolution in the fourth harmonic. Using Eq. (A1), the relative variation of $v_4/(v_2)^2$ due to eccentricity fluctuations can be decomposed as the sum of three contributions

$$\frac{\delta(v_4/(v_2)^2)}{(v_4/(v_2)^2)} = \frac{\delta(\cos 4(\phi - \Psi_R))}{\langle \cos 4(\phi - \Psi_R) \rangle} \frac{\delta R_4}{R_4} - 2\frac{\delta v_2}{v_2}, \quad (A2)$$

The first term on the right-hand side is the contribution of fluctuations to the correlation with the event plane, the second term is the contribution of fluctuations to the resolution, the last term is the contribution of fluctuations to $v_2\{\text{EP}\}$. The definition of $\alpha$, Eq. (24), can be rewritten as

$$\frac{\delta(v_4/(v_2)^2)}{(v_4/(v_2)^2)} = \frac{\sigma_\alpha^2}{\langle v_2 \rangle^2}, \quad (A3)$$

The three terms in Eq. (A2) give additive contributions to $\alpha$, which we evaluate in turn.

We start with the correlation with the event-plane. The event plane $\Psi_R$ is determined from elliptic flow \cite{38}. Even flow harmonics $v_{2n}$ are analyzed by correlating particles with this event plane: $\langle \cos 2n(\phi - \Psi_R) \rangle = v_{2n}\mathcal{R}_{2n}(\chi)$, where the resolution $\mathcal{R}_{2n}$ is given by \cite{38}

$$\mathcal{R}_{2n}(\chi) = \frac{\sqrt{\pi} e^{-\chi^2/2} \left( I_{n-1} \left( \frac{\chi^2}{2} \right) + I_{n+1} \left( \frac{\chi^2}{2} \right) \right)}{\sqrt{\chi^2 n}}, \quad (A4)$$

where $\chi$ is the resolution parameter, which is estimated using the correlation between two subevents. For $n = 2$, this equation reduces to

$$\mathcal{R}_4(\chi) = \frac{e^{-\chi^2} - 1 + \chi^2}{\chi^2} \frac{1}{\chi^2}. \quad (A5)$$

These relations are derived neglecting flow fluctuations. If $v_2$ fluctuates, the resolution parameter $\chi$ scales like $v_2$, $\chi = v_2$. Assuming in addition that $v_4$ scales like $(v_2)^2$, the relative change due to fluctuations is, to leading order in $\sigma_v$,

$$\frac{\delta(\cos 4(\phi - \Psi_R))}{\langle \cos 4(\phi - \Psi_R) \rangle} = \frac{\sigma_v^2}{2} \frac{d^2}{(d\chi)^2} \left( \frac{\langle v_2 \rangle^2 \mathcal{R}_4(v_2) \mathcal{R}_4(r_2) \right)}{\mathcal{R}_4(\chi)}, \quad (A6)$$

where the right-hand side is evaluated for $\chi \equiv r(v_2)$, the average resolution parameter. Using Eq. (A5), one obtains

$$\frac{1}{\mathcal{R}_4(\chi)} \frac{d^2}{d\chi^2} \left( \chi^2 \mathcal{R}_4(\chi) \right) = \frac{2e^2(e^2 + 2\chi^2 - 1)}{1 + e^2(\chi^2 - 1)}. \quad (A7)$$

Inserting into Eqs. (A6) and (A2), and identifying with Eq. (A3), we obtain the contribution to $\alpha$ from the correlation with the event plane:

$$\alpha_{\text{EP}} = \frac{\chi^2(e^2 + 2\chi^2 - 1)}{1 + e^2(\chi^2 - 1)}. \quad (A8)$$

We now evaluate the second term in Eq. (24), namely, the shift in the resolution from fluctuations. The resolution is defined as $R_4 = \mathcal{R}_4(\chi_{\text{exp}})$, where $\chi_{\text{exp}}$ is determined from the correlation between subevents. Flow fluctuations shift the estimated resolution. Writing $\chi_{\text{exp}} = \chi + \delta \chi$, one obtains, to leading order in $\delta \chi$,

$$\delta R_4 \frac{\mathcal{R}_4(\chi) \delta \chi}{\mathcal{R}_4(\chi) \chi} = \frac{\chi R_4(\chi) \delta \chi}{\mathcal{R}_4(\chi) \chi}. \quad (A9)$$
Eq. (A9) gives
\[
\frac{\chi R'(\chi)}{R_\chi(\chi)} = \frac{2(e^{r^2} - \chi^2 - 1)}{1 + e^{r^2}(\chi^2 - 1)}.
\] (A10)

The shift in the resolution to fluctuations is given by Eq. (A7) of Ref. [24]
\[
\frac{\delta \chi}{\chi} = \frac{\sigma^2}{2 \langle v^2 \rangle} \left( 1 - 2\chi_s^2 + \frac{4r^2}{i_0^2 - i_1^2} \right).
\] (A11)

where \(i_{0,1}\) is a shorthand notation for \(I_{0,1}(\chi_s^2/2)\), and \(\chi_s\) denotes the resolution parameter of a subevent: \(\chi_s = \chi/\sqrt{2}\) in the usual case when the event plane consists of two subevents [38], and \(\chi_s = \chi\) if the event plane has only one subevent [11]. Inserting Eqs. (A10) and (A11) into (A9) and (A2), and identifying with Eq. (A3), we obtain the contribution to \(\alpha\) from the resolution:
\[
\alpha_{\text{res}} = \frac{e^{r^2} - \chi^2 - 1}{1 + e^{r^2}(\chi^2 - 1)} \frac{1 - 2\chi_s^2 + \frac{4r^2}{i_0^2 - i_1^2}}{1 - 2\chi_{s}^2 + \frac{4r^2}{i_0^2 - i_1^2}}
\] (A12)

Finally, the third term in Eq. (A2) is
\[
2\frac{\delta v_2}{v_2} = \frac{\sigma^2}{\langle v^2 \rangle^2} (\alpha_{v_2} - 1)
\] (A13)

where \(\alpha_{v_2}\) is given by Eq. (23) of Ref. [24]:
\[
\alpha_{v_2} = 2 - \frac{I_0 - I_1}{I_0 + I_1} \left( 2\chi^2 - 2\chi_s^2 + \frac{4r^2}{i_0^2 - i_1^2} \right),
\] (A14)

where \(I_{0,1}\) is a shorthand notation for \(I_{0,1}(\chi^2/2)\).

The final result is obtained by summing the three contributions from Eqs. (A8), (A12) and (A14):
\[
\alpha = \alpha_{\text{ep}} - \alpha_{\text{res}} - (\alpha_{v_2} - 1).
\] (A15)

The limit of low resolution \(\chi \to 0\) (resp. high resolution \(\chi \to \infty\)) is \(\alpha_{\text{ep}} = 6\) (resp. 1), \(\alpha_{\text{res}} = 1\) (resp. 0), \(\alpha_{v_2} = 2\) (resp. 1), \(\alpha = 4\) (resp. 1).

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