Mathieu functions computational toolbox implemented in Matlab

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The Mathieu functions are used to solve analytically some problems in elliptical cylinder coordinates. A computational toolbox was implemented in Matlab. Since the notation and normalization for Mathieu functions vary in the literature, we have included sufficient material to make this presentation self contained. Thus, all formulas required to get the Mathieu functions are given explicitly. Following the outlines in this presentation, the Mathieu functions could be readily implemented in other computer programs and used in different domains. Tables of numerical values are provided.

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I. INTRODUCTION

Some problems regarding the elliptical cylinders can be solved by using an analytical approach like that applied to circular cylinders: one separates the variables and the exact solution is given by expansions involving angular and radial Mathieu functions. These functions have been introduced by Emile Mathieu in 1868 by investigating the vibrating modes in an elliptic membrane [1]. Details (tables or relations) concerning the Mathieu functions can be found for example in [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. For circular cylinders the solutions involve readily available trigonometric and Bessel functions, while for elliptical cylinders there are still controversial and incomplete algorithms for computing the Mathieu functions. Largely applied computer programs provide only few or none routines referring to the Mathieu functions.

A computational toolbox for Mathieu functions was implemented in Matlab [13]. Since not all people are familiarized with the Matlab program, in this presentation the mathematics is outlined. Tables of numerical values are provided. Following the outlines in this presentation, it would be a readily task to implement the Mathieu functions in other computer programs and use them in different domains.

One reason for the lack of algorithms for Mathieu functions was probably the complicated and various notation existent in the literature. A main purpose for us was to simplify as much as possible the notation. With a simplified and self-contained notation, the use of Mathieu functions should be as simple as the use of Bessel functions. We largely followed the notations used by Stratton [6] and Stamnes [11, 12], but we introduced further simplifications. Since the notation and normalization for Mathieu functions vary in the literature, we have included sufficient material to make this presentation self contained. Thus, all formulas required to get the Mathieu functions are given explicitly. Tables of numerical values are provided. Examples of Mathieu functions applied to plane wave scattering by elliptical cylinders are given in [14, 15].
II. FUNDAMENTALS

A. Elliptical cylinder coordinates

Let consider an ellipse in the plane \((x, y)\) defined by equation \((x/x_0)^2 + (y/y_0)^2 = 1\) with \(x_0 > y_0\). The semifocal distance \(f\) is given by \(f^2 = x_0^2 - y_0^2\) and the eccentricity is \(e = f/x_0 < 1\). The elliptic cylindrical coordinates \((u, v, z)\) are defined by relations

\[
    x = f \cosh u \cos v, \quad y = f \sinh u \sin v, \quad z = z
\]

with \(0 \leq u < \infty\) and \(0 \leq v \leq 2\pi\). In terms of \((\xi, \eta, z)\), with \(\xi = \cosh u\) and \(\eta = \cos v\), the elliptic cylindrical coordinates are defined by relations

\[
    x = f \xi \eta, \quad y = f \sqrt{(\xi^2 - 1)(1 - \eta^2)}, \quad z = z.
\]

The contours of constant \(u\) are confocal ellipses (of semiaxes \(x_0 = f \xi\) and \(y_0 = f \sqrt{\xi^2 - 1}\)) and those of constant \(v\) are confocal hyperbolas. The \(z\) axis coincides with the cylinder axis. The scale factors \(h_j\), with \(j = \xi, \eta, z\), are defined like as for any coordinate transformation \([6]\),

\[
    h_\xi = f \frac{\sqrt{\xi^2 - \eta^2}}{\sqrt{\xi^2 - 1}}, \quad h_\eta = f \frac{\sqrt{\xi^2 - \eta^2}}{\sqrt{1 - \eta^2}}, \quad h_z = 1.
\]

B. Wave equation in elliptic cylindrical coordinates

The scalar wave equation \((\nabla^2 + k^2)U(\mathbf{r}) = 0\), where \(\mathbf{r}\) is the position vector, \(k\) is the wave number, \(k = 2\pi\sqrt{\epsilon/\lambda}\), \(\epsilon\) is the permittivity, and \(\lambda\) is the wavelength in vacuum, when expressed in elliptic cylindrical coordinates becomes

\[
    \left[ \frac{2}{f^2(\cosh 2u - \cos 2v)} \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + \frac{\partial^2}{\partial z^2} + k^2 \right] U(u, v, z) = 0.
\]

Using a solution of the form \(U = Z(z)S(v)R(u)\) gives

\[
    \left( \frac{d^2}{dz^2} + k_z^2 \right) Z(z) = 0,
\]

\[
    \left[ \frac{d^2}{dv^2} + (a - 2q \cos 2v) \right] S(v) = 0,
\]

\[
    \left[ \frac{d^2}{du^2} - (a - 2q \cosh 2u) \right] R(u) = 0,
\]
where \( k_z \) is the wave vector component on \( z \) direction, \( q = k_r^2 f^2 / 4 \), with \( k_r^2 = k^2 - k_z^2 \), and \( a \) is separation constant. Equation (5) has solution \( Z(z) = \exp(ik_z z) \). Equations (6) and (7) are known as the angular and radial Mathieu equations, respectively.

III. ANGULAR MATHIEU FUNCTIONS

In this presentation, only the periodic solutions of period \( \pi \) or \( 2\pi \) are considered. For a given order \( n \), there are four categories of periodic solutions satisfying (6),

1. even-even: \( S_{ee}(v, q, n) = \sum_{j=0}^{\infty} A_{ee}^{(2j)}(q, n) \cos(2jv) \),
2. even-odd: \( S_{eo}(v, q, n) = \sum_{j=0}^{\infty} A_{eo}^{(2j+1)}(q, n) \cos((2j+1)v) \),
3. odd-even: \( S_{oe}(v, q, n) = \sum_{j=1}^{\infty} A_{oe}^{(2j)}(q, n) \sin(2jv) \),
4. odd-odd: \( S_{oo}(v, q, n) = \sum_{j=0}^{\infty} A_{oo}^{(2j+1)}(q, n) \sin((2j+1)v) \).

\( A_{pm} \) with \( p, m = e, o \) are expansion coefficients. In the following, the angular Mathieu functions are denoted \( S_{pm}(v, q, n) \), with \( p, m = e, o \). Instead of two angular Mathieu functions, even \( S_{ep} \) and odd \( S_{op} \), with \( p = e, o \) [12], a single angular Mathieu function \( S_{pm} \), with \( p, m = e, o \), is considered referring to all the four categories. For a given value of \( q \) there exist four infinite sequences of characteristic values (eigenvalues) \( a \), for either value of \( a \) corresponding an infinite sequence (eigenvector) of expansion coefficients.

A. Characteristic values and coefficients

By substituting \( (8) \) in \( (6) \), the following recurrence relations among the expansion coefficients result

1. even-even:
   \[
   a A_{ee}^{(0)} - q A_{ee}^{(2)} = 0,
   \]
   \[
   (a - 4) A_{ee}^{(2)} - q[2 A_{ee}^{(0)} + A_{ee}^{(4)}] = 0,
   \]
   \[
   [a - (2j)^2] A_{ee}^{(2j)} - q[A_{ee}^{(2j-2)} + A_{ee}^{(2j+2)}] = 0, \quad j = 2, 3, 4 \ldots
   \]
2 even-odd:

\[(a - 1)A_{eo}^{(1)} - q[A_{eo}^{(1)} + A_{eo}^{(3)}] = 0,\]

\[a - (2j + 1)^2]A_{eo}^{(2j+1)} - q[A_{eo}^{(2j-1)} + A_{eo}^{(2j+3)}] = 0, \quad j = 1, 2, 3 \cdots \quad (10)\]

3 odd-even:

\[(a - 4)A_{oe}^{(2)} - qA_{oe}^{(4)} = 0,\]

\[a - (2j)^2]A_{oe}^{(2j)} - q[A_{oe}^{(2j-2)} + A_{oe}^{(2j+2)}] = 0, \quad j = 2, 3, 4 \cdots \quad (11)\]

4 odd-odd:

\[(a - 1)A_{oo}^{(1)} + q[A_{oo}^{(1)} - A_{oo}^{(3)}] = 0,\]

\[a - (2j + 1)^2]A_{oo}^{(2j+1)} - q[A_{oo}^{(2j-1)} + A_{oo}^{(2j+3)}] = 0, \quad j = 1, 2, 3 \cdots \quad (12)\]

The recurrence relations can be written in matrix form \[\text{[11]},\]

\[
\begin{pmatrix}
-a & q & 0 & 0 & 0 & 0 & \cdots \\
2q & 2^2 - a & q & 0 & 0 & 0 & \cdots \\
0 & q & 4^2 - a & q & 0 & 0 & \cdots \\
0 & 0 & q & 6^2 - a & q & 0 & \cdots \\
& & & & & & \\
& & & & & & \\
\end{pmatrix}
\begin{pmatrix}
A_{ee}^{(0)} \\
A_{ee}^{(2)} \\
A_{ee}^{(4)} \\
A_{ee}^{(6)} \\
& \\
& \\
\end{pmatrix} = 0,
\]

\[
(13)
\]

\[
\begin{pmatrix}
1 + q - a & q & 0 & 0 & 0 & 0 & \cdots \\
q & 3^2 - a & q & 0 & 0 & 0 & \cdots \\
0 & q & 5^2 - a & q & 0 & 0 & \cdots \\
0 & 0 & q & 7^2 - a & q & 0 & \cdots \\
& & & & & & \\
& & & & & & \\
\end{pmatrix}
\begin{pmatrix}
A_{eo}^{(1)} \\
A_{eo}^{(3)} \\
A_{eo}^{(5)} \\
A_{eo}^{(7)} \\
& \\
& \\
\end{pmatrix} = 0,
\]

\[
(14)
\]

\[
\begin{pmatrix}
2^2 - a & q & 0 & 0 & 0 & 0 & \cdots \\
q & 4^2 - a & q & 0 & 0 & 0 & \cdots \\
0 & q & 6^2 - a & q & 0 & 0 & \cdots \\
0 & 0 & q & 8^2 - a & q & 0 & \cdots \\
& & & & & & \\
& & & & & & \\
\end{pmatrix}
\begin{pmatrix}
A_{oo}^{(2)} \\
A_{oo}^{(4)} \\
A_{oo}^{(6)} \\
A_{oo}^{(8)} \\
& \\
& \\
\end{pmatrix} = 0,
\]

\[
(15)
\]
The matrices are real, tridiagonal, and symmetric for all categories, with the exception of the “1 even-even” category where the matrix is slightly non-symmetric. The eigenvalue problem is accurately solved in Matlab. In other computer programs it could be necessary to transform the slightly non-symmetric matrix in a symmetric one [11]. Both the eigenvalues \( a \) and the corresponding eigenvectors \( (A_{pm}, \text{with } p,m = e,o) \) are determined for either category at any order \( n \). The order \( n \) takes different values for each category of Mathieu functions. For the purpose of avoiding any confusion, a distinction must be done between the \( n^{th} \) order (in the succession of all orders) and the true value of that order. Thus, let denote \( n \) the order in the succession of all orders, and \( t \) the true value of order \( n \). The values of \( n \) and \( t \) for the four categories of Mathieu functions are

1. even-even: \( n = 0, 1, 2 \cdots \), \( t = 0, 2, 4 \cdots \),
2. even-odd: \( n = 0, 1, 2 \cdots \), \( t = 1, 3, 5 \cdots \),
3. odd-even: \( n = 1, 2, 3 \cdots \), \( t = 2, 4, 6 \cdots \),
4. odd-odd: \( n = 0, 1, 2 \cdots \), \( t = 1, 3, 5 \cdots \).

Note that, if the notation is self-contained by all routines of Mathieu functions, there is no need to determine the specific values of \( n \) and \( t \) for either category of Mathieu functions since it is done automatically.

### B. Normalization and orthogonality

Following [6, 11], the angular Mathieu functions are normalized by requiring that

\[
S_{ep}(0, q, n) = 1, \quad \left[ \frac{dS_{op}(v, q, n)}{dv} \right]_{v=0} = 1, \quad p = e, o.
\]
These requirements imply that,

1. even-even: \( \sum_{j=0}^{\infty} A_{ee}^{(2j)}(q, n) = 1 \),

2. even-odd: \( \sum_{j=0}^{\infty} A_{eo}^{(2j+1)}(q, n) = 1 \),

3. odd-even: \( \sum_{j=1}^{\infty} 2j A_{oe}^{(2j)}(q, n) = 1 \),

4. odd-odd: \( \sum_{j=0}^{\infty} (2j+1) A_{oo}^{(2j+1)}(q, n) = 1 \).

(18)

The orthogonality relation for the angular Mathieu functions is

\[
\int_0^{2\pi} S_{pm}(v, q, n) S'_{pm'}(v, q, n) \, dv = N_{pm} \delta_{mm'}, \quad p, m, m' = e, o,
\]

(19)

where \( N_{pm} \) is normalization factor, \( \delta_{mm'} \) equals 1 if \( m = m' \) and equals 0 otherwise. Then, the following relations for the normalization factor result,

1. even-even: \( N_{ee}(q, n) = 2\pi [A_{ee}^{(0)}(q, n)]^2 + \pi \sum_{j=1}^{\infty} [A_{ee}^{(2j)}(q, n)]^2 \),

2. even-odd: \( N_{eo}(q, n) = \pi \sum_{j=0}^{\infty} [A_{eo}^{(2j+1)}(q, n)]^2 \),

3. odd-even: \( N_{oe}(q, n) = \pi \sum_{j=1}^{\infty} [A_{oe}^{(2j)}(q, n)]^2 \),

4. odd-odd: \( N_{oo}(q, n) = \pi \sum_{j=0}^{\infty} [A_{oo}^{(2j+1)}(q, n)]^2 \).

(20)

Since different normalization schemes have been adopted in the literature, much attention should be paid when numerical results provided by different authors are compared against the others.

C. Correlation factors

Let consider two regions of different permittivities, \( \epsilon \) and \( \epsilon' \). The parameter \( q \) being different in the two regions, \( q \neq q' \), the characteristic values and expansion coefficients are also different. Let \( S_{pm} \) and \( S'_{pm} \) be the respective angular Mathieu functions. The correlation
factors $C_{pm}(q, q', n)$, with $p, m = e, o$, between the angular Mathieu functions $S_{pm}$ and $S'_{pm}$ are defined by relation

$$C_{pm}(q, q', n) = \delta_{mm'} \int_0^{2\pi} S'_{pm}(v, q, n) S_{pm}(v, q', n) dv, \quad p, m, m' = e, o. \quad (21)$$

Using (8) gives

1 even-even: 
$$C_{ee}(q, q', n) = 2\pi A_{ee}^{(0)}(q, n) A'_{ee}^{(0)}(q', n) + \pi \sum_{j=1}^{\infty} A_{ee}^{(2j)}(q, n) A'(2j)_{ee}(q', n),$$

2 even-odd: 
$$C_{eo}(q, q', n) = \pi \sum_{j=0}^{\infty} A_{eo}^{(2j+1)}(q, n) A'(2j+1)_{eo}(q', n), \quad (22)$$

3 odd-even: 
$$C_{oe}(q, q', n) = \pi \sum_{j=1}^{\infty} A_{oe}^{(2j)}(q, n) A'(2j)_{oe}(q', n),$$

4 odd-odd: 
$$C_{oo}(q, q', n) = \pi \sum_{j=0}^{\infty} A_{oo}^{(2j+1)}(q, n) A'(2j+1)_{oo}(q', n).$$

D. Derivatives of angular Mathieu functions

The derivatives of the angular Mathieu functions follow readily from (8).

1 even-even: 
$$\frac{dS_{ee}(v, q, n)}{dv} = -\sum_{j=1}^{\infty} 2j A_{ee}^{(2j)}(q, n) \sin(2jv),$$

2 even-odd: 
$$\frac{dS_{eo}(v, q, n)}{dv} = -\sum_{j=0}^{\infty} (2j + 1) A_{eo}^{(2j+1)}(q, n) \sin[(2j + 1)v], \quad (23)$$

3 odd-even: 
$$\frac{dS_{oe}(v, q, n)}{dv} = \sum_{j=1}^{\infty} 2j A_{oe}^{(2j)}(q, n) \cos(2jv),$$

4 odd-odd: 
$$\frac{dS_{oo}(v, q, n)}{dv} = \sum_{j=0}^{\infty} (2j + 1) A_{oo}^{(2j+1)}(q, n) \cos[(2j + 1)v].$$

IV. RADIAL MATHIEU FUNCTIONS

Solutions of (7) can be obtained from (8) by replacing $v$ by $iu$. Instead of $\sin v$ and $\cos v$, the terms of the series now involve $\sinh u$ and $\cosh u$. The convergence is low unless $|u|$ is small. Better convergence of series results by expressing the solutions of (7) in terms of Bessel functions associated with the same expansion coefficients that are determined once for
both the angular and radial Mathieu functions. Either pair of angular and radial Mathieu functions are proportional to one another [6],

\[ S_{ep}(iu, q, n) = \sqrt{2\pi g_{ep}(q, n)} J_{ep}(u, q, n), \quad p = e, o, \]  

(24)

where \( J_{ep} \) are even radial Mathieu functions of the first kind and \( g_{ep} \) are joining factors. When \( u = 0 \),

\[ S_{ep}(0, q, n) = 1, \quad J_{ep}(0, q, n) = \frac{1}{\sqrt{2\pi g_{ep}(q, n)}}, \quad p = e, o. \]  

(25)

Thus, one obtains,

1. even-even: \( g_{ee}(q, n) = \frac{(-1)^r}{\pi A_{ee}^{(0)}(q, n)} S_{ee}(\pi/2, q, n), \quad r = t/2, \) 
2. even-odd: \( g_{eo}(q, n) = \frac{(-1)^r}{\pi \sqrt{q} A_{eo}^{(1)}(q, n)} \left[ \frac{dS_{eo}(v, q, n)}{dv} \right]_{v=\pi/2}, \quad r = (t - 1)/2. \)  

(26)

Similarly [6],

\[ -i S_{op}(iu, q, n) = \sqrt{2\pi g_{op}(q, n)} J_{op}(u, q, n), \quad p = e, o. \]  

(27)

When \( u = 0 \),

\[ J_{op}(0, q, n) = 0, \quad \left[ \frac{dJ_{op}(u, q, n)}{du} \right]_{u=0} = \frac{1}{\sqrt{2\pi g_{op}(q, n)}}, \quad p = e, o. \]  

(28)

Thus, one obtains,

3. odd-even: \( g_{oe}(q, n) = \frac{(-1)^r}{\pi q A_{oe}^{(2)}(q, n)} \left[ \frac{dS_{oe}(v, q, n)}{dv} \right]_{v=\pi/2}, \quad r = t/2, \) 
4. odd-odd: \( g_{oo}(q, n) = \frac{(-1)^r}{\pi \sqrt{q} A_{oo}^{(1)}(q, n)} S_{oo}(\pi/2, q, n), \quad r = (t - 1)/2. \)  

(29)

Remember that \( t \) is the true value of order \( n \).

A. Radial Mathieu functions of the first kind

Since rapidly converging series are those expressed in terms of products of Bessel functions [10, 11], in the following relations refer only to them. Similarly to the angular Mathieu functions, one may distinguish four categories of radial Mathieu functions of the first kind.
which are denoted \( J_{pm}(u, q, n) \), with \( p, m = e, o, \)

1 even-even: \[ J_{ee}(u, q, n) = \sqrt{\frac{\pi}{2}} \frac{(-1)^r}{A_{ee}^{(0)}(q, n)} \sum_{j=0}^{\infty} (-1)^j A_{ee}^{(2j)}(q, n) J_j(v_1) J_j(v_2), \]

\[ r = t/2, \]

2 even-odd: \[ J_{eo}(u, q, n) = \sqrt{\frac{\pi}{2}} \frac{(-1)^r}{A_{eo}^{(1)}(q, n)} \sum_{j=0}^{\infty} (-1)^j A_{eo}^{(2j+1)}(q, n) [J_j(v_1) J_{j+1}(v_2) \]

\[ + J_j(v_2) J_{j+1}(v_1)], \quad r = (t - 1)/2, \]

3 odd-even: \[ J_{oe}(u, q, n) = \sqrt{\frac{\pi}{2}} \frac{(-1)^r}{A_{oe}^{(2)}(q, n)} \sum_{j=1}^{\infty} (-1)^j A_{oe}^{(2j)}(q, n) [J_{j-1}(v_1) J_{j+1}(v_2) \]

\[ - J_{j-1}(v_2) J_{j+1}(v_1)], \quad r = t/2, \]

4 odd-odd: \[ J_{oo}(u, q, n) = \sqrt{\frac{\pi}{2}} \frac{(-1)^r}{A_{oo}^{(1)}(q, n)} \sum_{j=0}^{\infty} (-1)^j A_{oo}^{(2j+1)}(q, n) [J_j(v_1) J_{j+1}(v_2) \]

\[ - J_j(v_2) J_{j+1}(v_1)], \quad r = (t - 1)/2, \]

where \( v_1 = \sqrt{q} \exp(-u) \) and \( v_2 = \sqrt{q} \exp(u) \). The derivatives of the radial Mathieu functions of the first kind are

1 even-even: \[ r = t/2, \]

\[ \frac{dJ_{ee}(u, q, n)}{du} = \sqrt{\frac{\pi}{2}} \frac{(-1)^r}{A_{ee}^{(0)}(q, n)} \sum_{j=0}^{\infty} (-1)^j A_{ee}^{(2j)}(q, n) v_1 J_{j+1}(v_1) J_j(v_2) \]

\[ - v_2 J_j(v_1) J_{j+1}(v_2), \]

2 even-odd: \[ r = (t - 1)/2, \]

\[ \frac{dJ_{eo}(u, q, n)}{du} = \sqrt{\frac{\pi}{2}} \frac{(-1)^r}{A_{eo}^{(1)}(q, n)} \sum_{j=0}^{\infty} (-1)^j A_{eo}^{(2j+1)}(q, n) \left\{ (v_2 - v_1) [J_j(v_1) J_j(v_2) \]

\[ - J_{j+1}(v_1) J_{j+1}(v_2)] + (2j + 1) [J_{j+1}(v_1) J_j(v_2) - J_j(v_1) J_{j+1}(v_2)] \right\}, \]

3 odd-even: \[ r = t/2, \]

\[ \frac{dJ_{oe}(u, q, n)}{du} = \sqrt{\frac{\pi}{2}} \frac{(-1)^r}{A_{oe}^{(2)}(q, n)} \sum_{j=0}^{\infty} (-1)^{j+1} A_{oe}^{(2j+2)}(q, n) (4j + 4) \left\{ J_j(v_1) J_j(v_2) \right. \]

\[ + \cosh 2u J_{j+1}(v_1) J_{j+1}(v_2) - j + 1 \left\{ \frac{J_{j+1}(v_1) J_j(v_2)}{v_1} + \frac{J_j(v_1) J_{j+1}(v_2)}{v_2} \right\} \] \hspace{2cm} (31)
odd-odd: \[ r = (t - 1)/2, \]
\[
\frac{dJ_{oo}(u, q, n)}{du} = \sqrt{\frac{\pi}{2}} \frac{(-1)^r}{A_{oo}^{(1)}(q, n)} \sum_{j=0}^{\infty} (-1)^j A_{oo}^{(2j+1)}(q, n) \left\{ (v_1 + v_2) [J_j(v_1)J_j(v_2) + J_{j+1}(v_1)J_{j+1}(v_2)] - (2j + 1) [J_{j+1}(v_1)J_j(v_2) + J_j(v_1)J_{j+1}(v_2)] \right\}.
\]

B. Radial Mathieu functions of the second kind

A second independent solution of (7) is obtained by replacing the Bessel functions of the first kind \( J_n(v_2) \) in (30) by the Bessel functions of the second kind \( Y_n(v_2) \). This solution is denoted \( Y_{pm}(u, q, n) \), with \( p, m = e, o \).

1 even-even: \( Y_{ee}(u, q, n) = \sqrt{\frac{\pi}{2}} \frac{(-1)^r}{A_{ee}^{(0)}(q, n)} \sum_{j=0}^{\infty} (-1)^j A_{ee}^{(2j)}(q, n) [v_1 J_{j+1}(v_2) - v_2 J_j(v_2)], \]
\[ r = t/2, \]

2 even-odd: \( Y_{eo}(u, q, n) = \sqrt{\frac{\pi}{2}} \frac{(-1)^r}{A_{eo}^{(1)}(q, n)} \sum_{j=0}^{\infty} (-1)^j A_{eo}^{(2j+1)}(q, n) [J_j(v_1)Y_{j+1}(v_2) + Y_j(v_2)J_{j+1}(v_1)], \]
\[ r = (t - 1)/2, \] (32)

3 odd-even: \( Y_{oe}(u, q, n) = \sqrt{\frac{\pi}{2}} \frac{(-1)^r}{A_{oe}^{(2)}(q, n)} \sum_{j=1}^{\infty} (-1)^j A_{oe}^{(2j)}(q, n) [J_{j-1}(v_1)Y_{j+1}(v_2) - Y_{j-1}(v_2)J_{j+1}(v_1)], \]
\[ r = t/2, \]

4 odd-odd: \( Y_{oo}(u, q, n) = \sqrt{\frac{\pi}{2}} \frac{(-1)^r}{A_{oo}^{(1)}(q, n)} \sum_{j=0}^{\infty} (-1)^j A_{oo}^{(2j+1)}(q, n) [J_j(v_1)Y_{j+1}(v_2) - Y_j(v_2)J_{j+1}(v_1)], \]
\[ r = (t - 1)/2, \]

The derivatives of the radial Mathieu functions of the second kind are

1 even-even: \[ r = t/2, \]
\[
\frac{dY_{ee}(u, q, n)}{du} = \sqrt{\frac{\pi}{2}} \frac{(-1)^r}{A_{ee}^{(0)}(q, n)} \sum_{j=0}^{\infty} (-1)^j A_{ee}^{(2j)}(q, n) [v_1 J_{j+1}(v_2)Y_j(v_2) - v_2 J_j(v_1)Y_{j+1}(v_2)],
\]
even-odd: \[ r = (t - 1)/2, \]
\[
\frac{dY_{eo}(u, q, n)}{du} = \sqrt{\frac{\pi}{2}} A_{eo}^{(1)}(q, n) \sum_{j=0}^{\infty} (-1)^j A_{eo}^{(2j+1)}(q, n) \left\{ (v_2 - v_1)[J_j(v_1)Y_j(v_2) - J_{j+1}(v_1)Y_{j+1}(v_2)] + (2j + 1)[J_{j+1}(v_1)Y_j(v_2) - J_j(v_1)Y_{j+1}(v_2)] \right\},
\]

odd-even: \[ r = t/2, \]
\[
\frac{dY_{oe}(u, q, n)}{du} = \sqrt{\frac{\pi}{2}} A_{oe}^{(2)}(q, n) \sum_{j=0}^{\infty} (-1)^{j+1} A_{oe}^{(2j+2)}(q, n)(4j + 4) \left\{ J_j(v_1)Y_j(v_2) + \cosh 2uJ_{j+1}(v_1)Y_{j+1}(v_2) - (j + 1)[\frac{1}{v_1}J_{j+1}(v_1)Y_j(v_2) + \frac{1}{v_2}J_j(v_1)Y_{j+1}(v_2)] \right\},
\]

odd-odd: \[ r = (t - 1)/2, \]
\[
\frac{dY_{oo}(u, q, n)}{du} = \sqrt{\frac{\pi}{2}} A_{oo}^{(1)}(q, n) \sum_{j=0}^{\infty} (-1)^j A_{oo}^{(2j+1)}(q, n) \left\{ (v_1 + v_2)[J_j(v_1)Y_j(v_2) + J_{j+1}(v_1)Y_{j+1}(v_2)] - (2j + 1)[J_{j+1}(v_1)Y_j(v_2) + J_j(v_1)Y_{j+1}(v_2)] \right\}.
\]

C. Radial Mathieu functions of the third and the fourth kinds

Radial Mathieu functions of the third kind, analogous to the Hankel functions of the first kind are defined as follows [6, 11]
\[
H_{pm1}(u, q, n) = J_{pm}(u, q, n) + iY_{pm}(u, q, n), \quad p, m = e, o.
\]

Similarly, radial Mathieu functions of the fourth kind, analogous to the Hankel functions of the second kind are defined as follows [6, 11]
\[
H_{pm2}(u, q, n) = J_{pm}(u, q, n) - iY_{pm}(u, q, n), \quad p, m = e, o.
\]

V. IMPLEMENTATION OF MATHIEU FUNCTIONS IN MATLAB

Following the notation of the four categories of angular Mathieu functions, the implementation in Matlab or in any other computer program is readily done by introducing a function code $KF$. The first step in any algorithm of Mathieu function computation is to find the characteristic values (eigenvalues) and the expansion coefficients (eigenvectors). In [13], this is done by routine “eig_Spm” which has $q$ as input parameter.
(see Table I). Besides \( q \), the function code \( KF \) should be specified. Thus, if \( KF = 1 \), the routine \texttt{eig.Spm} solves the eigenvalue problem for category “1 even-even” of Mathieu functions, if \( KF = 2 \) for category “2 even-odd”, and so on. The number of expansion coefficients is the same, it is set equal to 25, for all categories of Mathieu functions. Concerning the outputs of routine \texttt{eig.Spm}, \( va \) is a line vector representing the characteristic values \( a \) for all the 25 orders; \( mc \) is \( 25 \times 25 \) matrix, where the columns represent the eigenvectors (that is, the expansion coefficients) for all orders; \( vt \) is a column vector specifying the true value \( t \) for all orders. Note that the eigenvectors in \( mc \) were processed to obey equation \eqref{18}. For the purpose to save the time of computation, all the other routines have \( mc \) as input (see Table I), the routine \texttt{eig.Spm} being called once, at the beginning of the computation, for any values of coordinates \( u \) and \( v \) that intervene in that computation. Since in many cases the convergence is assured by the first several orders, all the other routines have \( nmax \leq 25 \) as input. It means that those routines take into account only the first \( nmax \) orders, but for either order the length of the corresponding eigenvector is the same, equal to 25. The routine \texttt{extract.one.value} can be used to get a single value, and the routine \texttt{extract.one.column} to get a single eigenvector, corresponding to the order \( t \). The derivatives of \( S_{pm} \), with \( p, m = e, o \), are computed by routine \texttt{dSpm}. For both \( “Spm” \) and \( “dSpm” \), \( v \) is expressed in radians, with values in interval \((0, 2\pi)\). The normalization, correlation, and joining factors are computed by routines \texttt{Npm}, \texttt{Cpm}, and \texttt{gpm}, respectively. The four kinds of radial Mathieu functions, \( J_{pm}, Y_{pm}, H_{pm1}, \) and \( H_{pm2} \), with \( p, m = e, o \), are computed by routines \texttt{Jpm}, \texttt{Ypm}, \texttt{Hpm1}, and \texttt{Hpm2}, respectively, and their derivatives with respect to \( u \) by routines \texttt{dJpm}, \texttt{dYpm}, \texttt{dHpm1}, and \texttt{dHpm2}, respectively.

Numerical values of the separation constant \( a \), of the angular Mathieu functions \( S_{pm} \) and their derivatives \( S'_{pm} \), with \( p, m = e, o \), where the prime denotes differentiation with respect to \( v \), are given in Tables II–IV. They can be compared with data in \cite{2}. With the purpose to facilitate the comparison, since in \cite{2} the normalization \( N_{pm} = \pi \) is applied, the data of \( S_{pm} \) and \( S'_{pm} \) in Tables II–IV are multiplied by \( \sqrt{\pi/N_{pm}} \).

Concerning the radial Mathieu functions, numerical values of \( S_{ep}(iu, q, n) \) and \(-iS_{op}(iu, q, n)\) are given for \( u = 0.5 \) in Tables \texttt{V} and \texttt{VI}. They are multiplied by \( \sqrt{\pi/N_{pm}} \) and compared with data in \cite{9}. Note that \( S_{ep} \) is correlated to the radial Mathieu function of the first kind \( J_{ep} \) by Eq. \eqref{24}, whereas \( S_{op} \) is correlated to \( J_{op} \) by Eq. \eqref{27}. We found
that, for parameters in [9], the values of $S_{ep}(iu, q, n)$ and $-iS_{op}(iu, q, n)$ calculated with Eqs. (24) and (27) differ from those obtained with Eq. (8) by less than $7.5 \times 10^{-12}$.

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TABLE I: Routines comprised in the toolbox [13].

| Name of routine | Routine call | What the routine computes |
|-----------------|--------------|---------------------------|
| eig_Spm         | $[va, mc, vt]=\text{eig}_Spm(KF, q)$ | Vector of characteristic values $va$, matrix of coefficients $mc$, and vector of orders $vt$, at given function code $KF$ and elliptical parameter $q \geq 0$. |
| Spm             | $y=\text{Spm}(KF, v, mc, nmax)$ | Angular Mathieu functions $S_{pm}$, [Eq. (8)]. |
| dSpm            | $y=\text{dSpm}(KF, v, mc, nmax)$ | Derivatives with respect to $v$ of $S_{pm}$, [Eq. (23)]. |
| Npm             | $y=\text{Npm}(KF, mc, nmax)$ | Normalizing factors of angular Mathieu functions $S_{pm}$, [Eqs. (19) and (20)]. |
| Cpm             | $y=\text{Cpm}(KF, mc, mc', nmax)$ | Correlation factors of $S_{pm}$ and $S'_{pm}$, having matrices of coefficients $mc$ and $mc'$, [Eqs. (21) and (22)]. |
| Jpm             | $y=\text{Jpm}(KF, u, q, mc, nmax)$ | Radial Mathieu functions of the first kind $J_{pm}$, [Eq. (30)]. |
| dJpm            | $y=\text{dJpm}(KF, u, q, mc, nmax)$ | Derivatives with respect to $u$ of $J_{pm}$, [Eq. (31)]. |
| gpm             | $y=\text{gpm}(KF, q, mc, nmax)$ | Joining factors for pairs of angular, $S_{pm}$ and radial, $J_{pm}$ Mathieu functions, [Eqs. (24)–(29)]. |
| Ypm             | $y=\text{Ypm}(KF, u, q, mc, nmax)$ | Radial Mathieu functions of the second kind $Y_{pm}$, [Eq. (32)]. |
| dYpm            | $y=\text{dYpm}(KF, u, q, mc, nmax)$ | Derivatives with respect to $u$ of $Y_{pm}$, [Eq. (33)]. |
| Hpm1            | $y=\text{Hpm1}(KF, u, q, mc, nmax)$ | Radial Mathieu functions of the third kind $H_{pm1}$, [Eq. (34)]. |
| dHpm1           | $y=\text{dHpm1}(KF, u, q, mc, nmax)$ | Derivatives with respect to $u$ of $H_{pm1}$. |
| Hpm2            | $y=\text{Hpm2}(KF, u, q, mc, nmax)$ | Radial Mathieu functions of the fourth kind $H_{pm2}$, [Eq. (35)]. |
| dHpm2           | $y=\text{dHpm2}(KF, u, q, mc, nmax)$ | Derivatives with respect to $u$ of $H_{pm2}$. |
| extract_one_column | $y=\text{extract_one_column}(KF, t, mc)$ | Extracts one column from $mc$ at given $t$. |
| extract_one_value | $y=\text{extract_one_value}(KF, t, vec)$ | Extracts one value from $vec$ at given $t$. |
TABLE II: Values of $S_{ee}$ multiplied by $\gamma_{ee} = \sqrt{\pi/N_{ee}}$ to be compared with data in [2]

|   | q   | a   | $\gamma_{ee}S_{ee}(0,q,n)$ | $\gamma_{ee}S_{ee}(\pi/2,q,n)$ |
|---|-----|-----|----------------------------|----------------------------------|
| 0 | 0   | 0   | 0.7071067811865           | 0.7071067811865                  |
| 5 | -5.8000460208515  | 0.0448001816519 | 1.3348486746980               |
| 10| -13.9369799566589 | 0.0076265175709 | 1.4686604707129               |
| 15| -22.5130377608640 | 0.0019325083152 | 1.5501081466866               |
| 20| -31.313900703364  | 0.0006037438292 | 1.6098908573959               |
| 25| -40.2567795465667 | 0.0002158630184 | 1.6575102983235               |
|   | 4.00000000000000 | 1.00000000000000 | -1.00000000000000            |
| 5 | 7.4491097395292  | 0.7352943084007 | -0.7244881519677              |
| 10| 7.7173698497796  | 0.245883492913  | -0.9267592641263              |
| 15| 5.0779831975435  | 0.0787928278464 | -1.0199662260303              |
| 20| 1.1542828852468   | 0.0286489431471 | -1.0752932287797              |
| 25| -3.5221647271583 | 0.0115128663309 | -1.1162789532953              |
| 10| 100.0000000000000 | 1.00000000000000 | -1.00000000000000           |
| 5 | 100.1263692161636 | 1.0259950270894 | -0.9753474872360             |
| 10| 100.5067700246816 | 1.0538159921009 | -0.9516453181790             |
| 15| 101.1452034473016 | 1.0841063118392 | -0.9285480638845             |
| 20| 102.0489160244372 | 1.1177886312594 | -0.9057107845941             |
| 25| 103.2302048044949 | 1.1562399186322 | -0.8826919105637             |
TABLE III: Values of $S_{eo}$ and $S'_{eo}$ multiplied by $\gamma_{eo} = \sqrt{\pi/N_{eo}}$ to be compared with data in [2]

| t  | q | $a$       | $\gamma_{eo}S_{eo}(0,q,n)$ | $\gamma_{eo}S'_{eo}(\pi/2,q,n)$ |
|----|---|----------|----------------------------|---------------------------------|
| 1  | 0 | 1.00000000000000 | 1.00000000000000 | -1.00000000000000 |
| 5  | 1 | 1.858187541547816 | 0.25654287932244 | -3.46904200340571 |
| 10 | -2.399142400036319 | 0.055398747747232 | -4.85043830449644 |
| 15 | -8.101105131641816 | 0.015040066453805 | -5.76420643905101 |
| 20 | -14.491301425174816 | 0.00505181376477 | -6.49056578258007 |
| 25 | -21.314899690665719 | 0.00191105150671 | -7.10674123529011 |
| 5  | 0 | 25.00000000000000 | 1.00000000000000 | -5.00000000000000 |
| 5  | 25.549971749981619 | 1.12480725063854 | -5.39248615498821 |
| 10 | 27.703768733939319 | 1.25801994130834 | -5.32127654116094 |
| 15 | 31.957821252172919 | 1.19343223041313 | -5.1191498840644 |
| 20 | 36.644989734132819 | 0.93657553142267 | -5.7786752500644 |
| 25 | 40.050190985807719 | 0.61069431005071 | -7.05988429165531 |
| 15 | 0 | 225.00000000000000 | 1.00000000000000 | 15.00000000000000 |
| 5  | 225.055812476709619 | 1.01129373252961 | 15.16365747206002 |
| 10 | 225.223356974964419 | 1.02287828243823 | 15.31988030566234 |
| 15 | 225.502956244654119 | 1.03479365223693 | 15.46874350328304 |
| 20 | 225.895153416207919 | 1.04708434416294 | 15.61027852323804 |
| 25 | 226.400720044748119 | 1.05980044181391 | 15.74447250506791 |
### TABLE IV: Values of $S_{op}$ and $S_{op}'$ multiplied by $\sqrt{\pi/N_{op}}$, $p = e, o$ (see [2])

| t  | q   | $a$          | $\sqrt{\pi/N_{oe}S_{oe}'(0,q,n)}$ | $\sqrt{\pi/N_{oe}S_{oe}'(\pi/2,q,n)}$ |
|----|-----|--------------|-----------------------------------|----------------------------------------|
| 2  | 0   | 4.0000000000000 | 2.0000000000000 | -2.0000000000000 |
|    | 5   | 2.0994604454587 | 0.7331661960372 | -3.6405178524082 |
| 10 | -2.3821582359570 | 0.2488228403985 | -4.8634220691653 |
| 15 | -8.0993467988959 | 0.0918197143696 | -5.7655737717278 |
| 20 | -14.4910632559807 | 0.0370277776852 | -6.490752240373 |
| 25 | -21.3148606222498 | 0.0160562170491 | -7.106719073739 |
|    | 10  | 100.0000000000000 | 10.0000000000000 | -10.0000000000000 |
| 5  | 100.1263692156019 | 9.7341731518695 | -10.2396462566908 |
| 10 | 100.5067694628784 | 9.4404054347686 | -10.453975316485 |
| 15 | 101.145172929092 | 9.1157513395126 | -10.6428998776563 |
| 20 | 102.0483928609361 | 8.7555450801360 | -10.805721781325 |
| 25 | 103.2256800423735 | 8.3526783655914 | -10.9413538308191 |
| 1  | 0   | 1.0000000000000 | 1.0000000000000 | 1.0000000000000 |
|    | 5   | -5.7900805986378 | 0.1746754006198 | 1.3374338870223 |
| 10 | -13.9365524792501 | 0.0440225659111 | 1.4687556641029 |
| 15 | -22.5130034974235 | 0.0139251347875 | 1.5501150743576 |
| 20 | -31.3133861669129 | 0.0050778849001 | 1.6098915926038 |
| 25 | -40.2567789846842 | 0.002043593656 | 1.6575103983745 |
| 5  | 0   | 25.0000000000000 | 5.0000000000000 | 1.0000000000000 |
|    | 5   | 25.5108160463032 | 4.3395700104946 | 0.9060779302024 |
| 10 | 26.7664263604801 | 3.4072267604013 | 0.8460384335355 |
| 15 | 27.9678805967175 | 2.4116664728002 | 0.8379493400125 |
| 20 | 28.4682213251027 | 1.5688968684857 | 0.8635431218534 |
| 25 | 28.0627658994543 | 0.9640716219024 | 0.8992683245108 |
| 15 | 0   | 225.0000000000000 | 15.0000000000000 | -1.0000000000000 |
|    | 5   | 225.0558124767096 | 14.8287889732852 | -0.9889607027406 |
| 10 | 225.2233569749643 | 14.6498600449581 | -0.9781423471832 |
| 15 | 225.5029562446537 | 14.4630006940372 | -0.9675137031855 |
| 20 | 225.8951534161767 | 14.2679460909928 | -0.9570452540613 |
| 25 | 226.4007200438825 | 14.0643732956172 | -0.9467086958781 |
TABLE V: Values of $S_{ep}(iu,q,n)$ for $u = 0.5$ multiplied by $\sqrt{\pi/N_{ep}}$, where $p = e, o$, compared with data in [9]

| $t$ | $q$ | Values at $p = e$ | Data in [9] | $t$ | $q$ | Values at $p = o$ | Data in [9] |
|-----|-----|-------------------|-------------|-----|-----|-------------------|-------------|
| 0   | 5   | -0.019325304910071| -0.01932    | 1   | 5   | 0.021440743185527 | 0.02144    |
| 10  | -0.007055239716193| -0.00705    | 10          | -0.038634237458525 | -0.03863 |
| 20  | -0.000169411415735| -0.00016    | 20          | -0.003373888309642 | -0.00337 |
| 2   | 5   | 0.446937465741068 | 0.44693    | 3   | 5   | 1.205528267066838 | 1.2055     |
| 10  | -0.063855921612085| -0.06385    | 10          | 0.235940782144547 | 0.23594 |
| 20  | -0.024916657795101| -0.02491    | 20          | -0.09735461080731 | -0.09738 |
| 4   | 5   | 2.234088244534832 | 2.2341     | 5   | 5   | 3.864089377116713 | 3.8641     |
| 10  | 1.039103163573830 | 1.0391     | 10          | 2.285610444204526 | 2.2856    |
| 20  | -0.143991090269732| -0.14399    | 20          | 0.274270780278172 | 0.27427   |

TABLE VI: Values of $-iS_{op}(iu,q,n)$ for $u = 0.5$ multiplied by $\sqrt{\pi/N_{op}}$, with $p = e, o$, compared with data in [9]

| $t$ | $q$ | Values at $p = e$ | Data in [9] | $t$ | $q$ | Values at $p = o$ | Data in [9] |
|-----|-----|-------------------|-------------|-----|-----|-------------------|-------------|
| 2   | 5   | 0.238342768735937 | 0.23834    | 1   | 5   | 0.036613617783886 | 0.03661    |
| 10  | 0.028675814044625| 0.02867    | 10          | 0.000750806874015 | 0.00075    |
| 20  | -0.003176296415956| -0.00317   | 20          | -0.00053825853937 | -0.00053   |
| 4   | 5   | 1.883560277440876 | 1.8836     | 3   | 5   | 0.806555153528872 | 0.80655    |
| 10  | 0.769679129538722 | 0.76968    | 10          | 0.204495885546638 | 0.20449    |
| 20  | 0.040515136278697 | 0.04051    | 20          | -0.005279473480675| -0.00527   |
| 6   | 5   | 6.6066602369876   | 6.6067     | 5   | 5   | 3.667530204538722 | 3.6675     |
| 10  | 4.1161420952367   | 4.1161     | 10          | 1.972361938552091 | 1.9724     |
| 20  | 1.1805904286267   | 1.1806     | 20          | 0.32039885594192 | 0.32040    |