The powers of smooth words over arbitrary 2-letter alphabets

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Abstract. A. Carpi (1993) and A. Lepistö (1993) proved independently that smooth words are cube-free for the alphabet \{1, 2\}, but nothing is known on whether for the other 2-letter alphabets, smooth words are $k$-power-free for some suitable positive integer $k$. This paper establishes the derivative formula (Theorem 10) of the concatenation of two smooth words and power derivative formula of smooth words over arbitrary 2-letter alphabets. And by making use of power derivative formula (Theorem 12), for arbitrary 2-letter alphabet \{a, b\} with $a$, $b$ being positive integers and $a < b$, we prove that smooth words of length larger than or equal to 2 are $h(a, b)$-power-free, which means that the power-free index of smooth words is $\delta(a, b)$ (Theorem 14), where

$$h(a, b) = \begin{cases} 
  b + 2, & a = 1, \ b = 3 \\
  \frac{b+4}{2}, & 2 \mid b \\
  \frac{b+5}{2}, & 2 \nmid b, \text{ and } a = 1, b \neq 3 \\
  \frac{b+3}{2}, & 2 \nmid b, \text{ and } a \geq 2
\end{cases}, \quad \delta(a, b) = \begin{cases} 
  b + 2, & a = 1, b = 3 \\
  b + 1, & \text{or else}
\end{cases}. $$

Moreover, we give the number $\gamma_{a,b}(n)$ of smooth words of form $w^n$ with $a$ and $b$ having the same parity (Theorem 16). That is,

$$\gamma_{1,3}(n) = \begin{cases} 
  0, & n \geq 5 \\
  \infty, & n < 5
\end{cases}, \quad \text{in other cases, } \gamma_{a,b}(n) = \begin{cases} 
  0, & n > b \\
  2, & h(a, b) \leq n \leq b \\
  \infty, & n < h(a, b)
\end{cases}. $$

Thus, we obtain unexpectedly that smooth words are quintic-free, and there are infinitely many smooth biquadrates for the alphabet \{1, 3\}.

Keywords: Kolakoski sequence; derivative; derivative formula; power derivative formula; derivative closure; closure of a word; smooth words; power-free index.
1. Introduction

The Kolakoski sequence $K$ [19] is the infinite sequence over the alphabet $\Sigma = \{1, 2\}$, which starts with 1 and equals the sequence defined by its run lengths:

$$K = 22 11 21 12 22 11 22 11 21 12 22 11 21 12 22 \cdots$$

Here, a run is a maximal subsequence of consecutive like symbols. The curious Kolakoski sequence $K$ has received a remarkable attention by showing some intriguing combinatorial properties, constituting mainly a series of conjectures (see C. Kimberling [18], F. M. Dekking [12]). F. M. Dekking [10, 11] and W. D. Weakely [26] have studied the finite words which occur in $K$. Dekking [12] gave a survey in which he collected the known properties of the Kolakoski sequence $K$ and introduced the Kolakoski measure which presumably describes the frequency behavior of all subwords of the Kolakoski sequence.

B. Steinsky [25] gave a recursive formula for the $n$th term of the Kolakoski sequence. Using this formula, it is easy to find recursions for the number of ones in the first $n$ terms and for the sum of the first $n$ terms of the Kolakoski sequence.

M. S. Keane [17] asked whether the density of 1’s in $K$ is 0.5. V. Chvátal [9] proved that the upper density of 1’s as well as the upper density of 2’s in $K$ is less than 0.501. R. Steacy [24] studied the structure in the kolakoski sequence $K$ and obtained some conditions which are equivalent to Keane’s problem.

M. Baake and B. Sing [1] and B. Sing [22, 23] established a connection between generalized Kolakoski sequences and model sets.

S. Brlek, and A. Ladouceur [5] set up a link between the existence of arbitrary long palindromes and some well-known open problems on Kolakoski sequence. V. Berthé, S. Brlek, and P. Choquette [2], S. Brlek, S. Dulucq, A. Ladouceur and L. Vuillon [3], and S. Brlek, G. Melançon and G. Paquin [6] obtained many significative results of smooth words.

Y. B. Huang [14, 15] explored the complexity of $C^\infty$-words of form $\tilde{u}vw$ and two sided infinitely $C^\infty$-words of form $\tilde{u}vw$.

G. Păun [21] conjectured that the kolakoski sequence $K$ contains only squares of bounded length and that it is cube-free. A. Carpi [7] and A. Lepistö [20] independently solved the conjecture respectively. A. Carpi [8] further showed that for any positive integer $n$, only finitely many words can occur twice, at distance $n$, in a $C^\infty$-word, which generalized the results in [7].
Lately S. Brlek, D. Jamet and G. Paquin [4] investigated smooth infinite words on 2-letter alphabets having same parity and showed that all smooth infinite words are recurrent; that the closure of the set of factors under reversal holds for odd alphabets only; that the frequency of letters in extremal words is 1/2 for even alphabets, and for $a = 1$ with $b$ odd, the frequency of $b$’s is $1/(\sqrt{2b-1}+1)$; that the minimal word is an infinite Lyndon word if and only if either $a = 1$ and $b$ are odd, or $a$, $b$ are even; and provided a linear time algorithm computing the extremal words, w.r.t. lexicographic order.

Recently Huang [13] established the derivative formula of the concatenation of two smooth words and provides a general framework to convert the problems of $D$-closure smooth words of form $uwv$ with gap $|w| \leq n$ into the corresponding ones with gap $|w| \leq m$ ($5 \leq m < n$), which enables us easily to calculate the number of smooth words of form $uvu$ with gap $|v| = n$. Moreover, making use of this method, we proved that for arbitrary nonnegative integers $n$, $k$, one has $|\Gamma_{n^{-k}}| < 10380 \cdot n^m$, which generalizes a result due to A. Carpi, where $m = \log 5 / \log \frac{10}{7}$, $\Gamma_{n^{-k}} = \{uwv \in C^\infty : D^k(u) = D^k(v) \text{ and } |w| \leq n\}$. In addition, we introduce the notion of minimal non-smooth words and prove that if $u$ is not a smooth word, then $\sigma_w(u) \notin C^\infty$ for any nonempty smooth word $w$, which generalizes the result that smooth words are cube-free, where $\sigma_w$ is a homomorphism from $\Sigma^*$ to $\{w, \bar{w}\}^*$ such that $\sigma_w(1) = w, \sigma_w(2) = \bar{w}$.

A naturally arising question is whether or not the similar results to Huang [13] still hold for smooth words over arbitrary 2-letter alphabet $\{a, b\}$ with $a < b$. This paper is a study of derivative formula of the concatenation of two smooth words, power derivative formula of smooth words and the property of smooth words being $k$-power-free for some suitable positive integers. However, an extension of the important result, which smooth words are cube-free for the alphabet $\{1, 2\}$, to arbitrary 2-letter alphabets leads to difficulties if we attempt to follow A. Carpi’s [8] or our former method. But fortunately, the power derivative formula provides us the required method to establish the power-free index of smooth words over arbitrary 2-letter alphabets.

The paper is structured as follows. In Section 2, we shall first fix some notations and introduce some notions. Secondly in Section 3, we establish the derivative formula of the concatenation of two smooth words (Theorem 10) and power derivative formula of smooth words (Theorem 12) on any 2-letter alphabets. And then in Section 4,
without machine computation, we give a new proof of smooth words being cube-
free for the alphabet \( \{1, 2\} \). In Section 5, for arbitrary alphabet \( \{a, b\} \), we prove
that smooth words of length \( \geq 2 \) are \( h(a, b) \)-power-free and that all smooth words are
\( \delta(a, b) \)-power-free, where if \( a = 1, b = 3 \), then \( \delta(a, b) = b + 2 \), otherwise, \( \delta(a, b) = b + 1 \).
We naturally expect to know about what reasons cause the differences of values of
\( \delta(a, b) \) between \( a = 1, b = 3 \) and the other cases. To do so, in Section 7, we give
an explanation to this phenomenon. In Section 6, we obtain the number of smooth
words of form \( u^n \) over 2-letter alphabet \( \{a, b\} \) having the same parity, where \( n \) is a
positive integer. Finally, in Section 8, we give some open problems on smooth words
which are deserved to attention.

2. Definitions and notation

Definitions and notation are introduced in this section, which are mainly borrowed
from Refs [5, 11, 26]. Let \( \Sigma = \{a, b\} \) with \( a < b \) and \( a, b \) being positive integers, \( \Sigma^* \)
denotes the free monoid over \( \Sigma \), with \( \varepsilon \) as the empty word (the identity element of
the monoid), and \( \Sigma^+ \) denotes \( \Sigma^* - \{\varepsilon\} \). A finite word over \( \Sigma \) is an element of \( \Sigma^* \). If
\( w = w_1w_2 \cdots w_n, w_i \in \Sigma \) for \( i = 1, 2, \cdots, n \), then \( n \) is called the length of the word \( w \)
and is denoted by \( |w| \). Let \( |w|_\alpha \) be the number of \( \alpha \) which occur in \( w \), where \( \alpha = a, b \),
then \( |w| = |w|_a + |w|_b \). In addition, for a set \( A \), the cardinal number of \( A \) is denoted
by \( |A| \).

The set of all right infinite words is denoted by \( \Sigma^\omega \), the set of all left infinite
words is denoted by \( \Sigma^l\omega \), and the set of all two sided infinite words is denoted by \( \Sigma^{b\omega} \). Given a word \( w \in \Sigma^* \), a factor \( u \) of \( w \) is a word \( u \in \Sigma^* \) satisfying \( \exists x, y \in \Sigma^* \) such
that \( w = xuy \), and \( \Gamma(w) \) denotes the set of all factors of \( w \). If \( x = \varepsilon \) (resp. \( y = \varepsilon \))
then \( u \) is called prefix (resp. suffix). A run (or block) is a maximal factor of form
\( u = \alpha^k, \alpha \in \Sigma \). \( \text{Pref}(w) \) denotes the set of all prefixes of \( w \). Finally, \( N^*, N^\omega, N^l\omega \) and
\( N^{b\omega} \) denote respectively the free monoid, the set of all right infinite words, the set of
all left infinite words and the set of all two sided infinite words over \( N \), where \( N \) is the
set of all positive integers.

The reversal (or mirror image) of \( u = u_1u_2 \cdots u_n \in \Sigma^* \) is the word \( \tilde{u} = u_nu_{n-1} \cdots u_2 
\ u_1 \). Similarly, we can define the reversal of an infinite word, and it is obvious that
\( u \in \Sigma^{l\omega} \iff \tilde{u} \in \Sigma^\omega \). A palindrome is a word \( P \) such that \( P = \tilde{P} \). The complement
(or permutation) of \( u = u_1u_2 \cdots u_n \in \Sigma^* \) is the word \( \bar{u} = \bar{u}_1 \bar{u}_2 \cdots \bar{u}_n \), where
\( \bar{a} = b, \bar{b} = a. \)

An infinite word \( w \) is closed under reversal (or reversal invariant) if \( \forall u \in F(w) \Rightarrow \bar{u} \in F(w) \). An infinite word \( w \) is closed under complementation (or complementation invariant) if \( \forall u \in F(w) \Rightarrow \bar{u} \in F(w) \). An infinite word \( w \) is recurrent if every factor has infinitely many occurrences. And an infinite word \( w \) is uniformly recurrent if it satisfies

\[
\forall n \in N, \exists R(n) \in N \text{ such that } [u \in F_n(w) \text{ and } v \in F_{R(n)}(w)] \Rightarrow u \in F(v).
\]

We see that every word \( w \in \Sigma^* \) can be uniquely written as a product of factors as follows:

\[
w = \alpha_{i_1} \alpha_{i_2} \alpha_{i_3} \alpha_{i_4} \cdots, \text{ where } i_j > 0.
\]

The operator giving the size of the blocks appearing in the coding, which is called run-length encoding and is a simple and effective data-compression method, is a function,

\[
\Delta : \Sigma^* \rightarrow N^*, \text{ defined by}
\]

\[
\Delta(w) = i_1 i_2 i_3 \cdots = \prod_{k \geq 1} i_k
\]

which is easily extended to infinite words and two sided infinite words respectively.

For any \( w \in \Sigma^* \) (or \( \Sigma^\omega \)), \( \text{first}(w) \) denotes the first letter of the word \( w \). For each \( w \in \Sigma^* \) (or \( \Sigma^\omega \)), \( \text{last}(w) \) denotes the last letter of the word \( w \). It is clear that the operator \( \Delta \) satisfies the property: \( \Delta(uv) = \Delta(u)\Delta(v) \) if and only if \( \text{last}(u) \neq \text{first}(v) \).

The function \( \Delta \) is not bijective because \( \Delta(w) = \Delta(\bar{w}) \) for every word \( w \). However, pseudo-inverse functions

\[
\Delta^{-1}_a, \Delta^{-1}_b : \Sigma^b\omega \rightarrow \Sigma^b\omega, \quad u = \cdots u_{-3}u_{-2}u_{-1}u_0u_1u_2u_3\cdots
\]

can be defined by

\[
\Delta^{-1}_a(u) = a^{u_1}b^{u_2}a^{u_3}b^{u_4}\cdots
\]
\[
\Delta^{-1}_b(u) = b^{u_1}a^{u_2}b^{u_3}a^{u_4}\cdots
\]

which is easily extended to \( \Sigma^\omega \) and \( \Sigma^{b\omega} \).

But if \( \Delta^{-1}_a \) is extended from \( \Sigma^* \) to \( \Sigma^{b\omega} \) in a similar way as follows:

\[
\Delta^{-1}_a, \Delta^{-1}_b : \Sigma^{b\omega} \rightarrow \Sigma^{b\omega}, \quad u = \cdots u_{-3}u_{-2}u_{-1}u_0u_1u_2u_3\cdots
\]
\[
\Delta^{-1}_a(u) = \cdots b^{u_{-3}}a^{u_{-2}}b^{u_{-1}}a^{u_0}b^{u_1}a^{u_2}b^{u_3}\cdots
\]
\[
\Delta^{-1}_b(u) = \cdots a^{u_{-3}}b^{u_{-2}}a^{u_{-1}}b^{u_0}a^{u_1}b^{u_2}a^{u_3}\cdots
\]

If \( w = \cdots w_{-3}w_{-2}w_{-1}w_0w_1w_2w_3\cdots \) and \( w = u \), then there exists \( k \in \mathbb{Z} \) such that \( w_i = u_{i+k} \) for all \( i \in \mathbb{Z} \). Clearly, if \( k \) is odd, then \( \Delta^{-1}_a(w) = \Delta^{-1}_a(u) \); if \( k \) is even, then \( \Delta^{-1}_a(w) = \Delta^{-1}_a(u) \) for \( \alpha = a, b \). Hence \( \Delta^{-1}_a(\alpha = a, b) \) is not a function from \( \Sigma^{b\omega} \) to \( \Sigma^{b\omega} \). But \( \Delta^{-1}_a(w) \) is unambiguous for a fixed \( w \in \Sigma^{b\omega} \). The following property is
immediate:
\[ \forall u \in \Sigma^* (\Sigma^\omega, \Sigma^l, \Sigma^b) : \quad \Delta^{-1}_a(u) = \Delta^{-1}_\alpha(u). \]
\[ \forall u \in \Sigma^* (\Sigma^b) : \quad \Delta^{-1}_\alpha(\tilde{u}) = \Delta^{-1}_\beta(u), \text{ where if } |u| \text{ is odd then } \beta = \alpha, \]
\[ \text{if } |u| \text{ is even then } \beta = \breve{\alpha}. \]

The operator \( \Delta \) over \( \Sigma^\omega \) has exactly two fixpoints, that is, \( \Delta(K_{a,b}) = K_{a,b}, \Delta(K_{b,a}) = K_{b,a} \), where \( K_{a,b} \) (or \( K_{b,a} \)) is an infinite sequence over the alphabet \( \Sigma = \{a, b\} \), which starts with \( a \) (or \( b \)) and equals the sequence defined by its run lengths. If \( a > 1 \) then
\[ K_{a,b} = a^a b^a \cdots a^a \cdot \hat{\alpha} \hat{\alpha} \hat{\alpha} \hat{\alpha} \cdots b^b, \]
where \( \alpha = \beta \) if \( a \) is even, otherwise \( \alpha = a, b \),
\[ K_{b,a} = b^b a^b \cdots b^b \cdot \hat{\beta} \hat{\beta} \hat{\beta} \hat{\beta} \cdots a^a, \]
where \( \beta = \alpha \) if \( b \) is even, or else \( \beta = b \).

Since \( \Delta(\tilde{K}_{b,a}K_{a,b}) = K_{a,b}K_{b,a} \), if \( a > 1 \) then \( \tilde{K}_{b,a}K_{a,b} \) and \( K_{a,b}K_{b,a} = \tilde{K}_{b,a}K_{a,b} \) are two fixpoints of \( \Delta \) over \( \Sigma^b \) (see Proposition 8). But, if \( a > 1 \) we don’t know whether \( \Delta \) has exactly the two fixpoints over \( \Sigma^b \).

Now we generalize the definition of differentiable words given by Dekking [11] from over the alphabet \( \{1, 2\} \) to over arbitrary 2-letter alphabet \( \{a, b\} \).

For \( w \in \Sigma^* \), \( r(w) \) denotes the number of runs of \( w \), \( fr(w) \) and \( lr(w) \) denote the first run and last run of \( w \) respectively, and \( lfr(w) \) and \( llr(w) \) denote the length of the first run and last run of \( w \) respectively. For example, if \( w = a^2 b^2 a^3 b^3 \), then \( fr(w) = a^2 \), \( lr(w) = b^3 \), \( lfr(w) = 2 \) and \( llr(w) = 3 \). Now we introduce the notion of the closure of a word \( w \in \Sigma^* \).

**Definition 1.** Let \( w \in \Sigma^* \) and
\[ w = \alpha^{t_1} \hat{\alpha}^{t_2} \cdots \beta^{t_k}, \text{ where } \alpha, \beta \in \Sigma, 1 \leq t_i \leq b \text{ for } 1 \leq i \leq k; \quad (2.1) \]
\[ \tilde{w} = \begin{cases} 
  w, & lfr(w) \leq a \text{ and } llr(w) \leq a \\
 a^{b-t_1} w, & lfr(w) > a \text{ and } llr(w) \leq a \\
 w \beta^{b-t_k}, & lfr(w) \leq a \text{ and } llr(w) > a \\
 a^{b-t_1} w \beta^{b-t_k}, & lfr(w) > a \text{ and } llr(w) > a 
\end{cases}. \]

Then \( \tilde{w} \) is said to be the closure of \( w \).

For example, let \( w = 33113331313331133333311333, u = 3313133311, \) then \( u \) is a factor of \( w \), and \( \tilde{w} = 3311333131333111333333311333, \tilde{u} = 3331313331111. \) Thus \( \tilde{u} \) is a factor of \( \tilde{w} \), which also holds in general (see Lemma 5 (1)).

**Definition 2.** Let \( w \in \Sigma^* \) be of form Eq. (2.1). If the length of every run of \( w \) takes only \( a \) or \( b \) except for the length of first and last runs, then we call that \( w \) is
differentiable, and its derivative, denoted by \( D(w) \), is the word whose \( j \)th symbol equals the length of the \( j \)th run of \( w \), discarding the first and/or the last run if its length is less than \( b \).

If \( \hat{w} \) is differentiable, then we call that \( w \) is closurely differentiable. If a finite word \( w \) is arbitrarily often closurely differentiable, then we call \( w \) a \( C_{a,b}^\infty \)-word or a smooth word over the alphabet \( \{a, b\} \), and the set of all smooth words over the alphabet \( \{a, b\} \) is denoted by \( C_{a,b}^\infty \) or \( C^\infty \).

Let \( \rho(w) = D(\hat{w}) \), then it is clear that \( w \) is a smooth word if and only if there is a positive integer \( k \) such that \( \rho^k(w) = \varepsilon \).

Note that if \( b = a + 1 \) then \( \hat{w} = w \). Thus, \( w \) is differentiable if and only if \( w \) is closurely differentiable, which suggests that \( w \) is a smooth word if and only if there is a positive integer \( k \) such that \( D^k(w) = \varepsilon \).

By the definition 2, it is apparent that if \( b - a \geq 2 \) and \( a \neq 2 \), then \( a^{b-1}b^a a^b a^{b-1} \) is differentiable but not closurely differentiable. Moreover, it is clear that \( D \) is an operator from \( \Sigma^* \) to \( \Sigma^* \), \( r(w) \leq |D(w)| + 2 \) and

\[
D(w) = \begin{cases}
\varepsilon, & \Delta(w) = yz, \text{ where } y + z \geq 1, y, z < b \text{ or } w = \varepsilon \\
\Delta(w), & \Delta(w) = bxb \text{ or } \Delta(w) = b \\
xb, & \Delta(w) = yxb \text{ and } 1 \leq y < b \\
bx, & \Delta(w) = bxz \text{ and } 1 \leq z < b \\
x, & \Delta(w) = yxz \text{ and } 1 \leq y, z < b
\end{cases}, \quad (2.2)
\]

\[
D(\hat{w}) = \begin{cases}
bD(w), & b > lfr(w) > a \text{ and } llr(w) \leq a \\
D(w)b, & b > llr(w) > a \text{ and } lfr(w) \leq a \\
bD(w)b, & b > lfr(w) > a \text{ and } b > llr(w) > a \\
D(w), & \text{otherwise}
\end{cases}. \quad (2.3)
\]

By Eq. (2.3), it is obvious that if \( w \) is closurely differentiable, then it must be differentiable.

A word \( v \) such that \( D(v) = w \) is said to be a primitive of \( w \). The two primitives of \( w \) having minimal length are the shortest primitives of \( w \). For example, \( b \) have \( 2b^2 \) primitives of form \( \alpha^i \hat{\alpha}^b \alpha^j \), where \( \alpha = a, b, i, j = 0, 1, \ldots b - 1 \), and \( a^b, b^b \) are the shortest primitives. It is easy to see that for any word \( w \in C^\infty \), there are at most \( 2b^2 \) primitives, and the difference of lengths of two primitives of \( w \) is at most \( 2(b - 1) \).
The height of a $C^\infty$-word $w$ is the smallest integer $k$ such that $D^{k+1}(w) = \varepsilon$. We write $ht(w)$ for the height of $w$. For example, if $w = 32^332^23^22^32^332^23^22$, then $ht(w) = 3$.

Obviously, if $w$ is a $C^\infty$-word and $|w| > 0$, then $|D(w)| < |w|$. Moreover, $D$ and $\Delta$ can be both iterated.

**Definition 3.** (1) $w \in \Sigma^\omega$ is a $C^\omega_{a,b}$-word or a smooth infinite word if for all $k \in \mathbb{N}$, $\Delta^k(w) \in \Sigma^\omega$. The class of smooth infinite words is denoted by $C^\omega_{a,b}$.
(2) $w \in \Sigma^{l\omega}$ is a $C^{l\omega}_{a,b}$-word or a smooth left-infinite word if for all $k \in \mathbb{N}$, $\Delta^k(w) \in \Sigma^{l\omega}$. The class of smooth left-infinite words is denoted by $C^{l\omega}_{a,b}$.
(3) $w \in \Sigma^{b\omega}$ is a $C^{b\omega}_{a,b}$-word or a smooth bi-infinite word if for all $k \in \mathbb{N}$, $\Delta^k(w) \in \Sigma^{b\omega}$. The class of smooth bi-infinite words is denoted by $C^{b\omega}_{a,b}$.

It is easy to see that finite factors of $C^\omega_{a,b}$-words, $C^{l\omega}_{a,b}$-words and $C^{b\omega}_{a,b}$-words are all $C^\infty_{a,b}$-words. Thus finite smooth words [2] are $C^\infty_{a,b}$-words, and the converse is also true by Lemma 7 (2), which means that finite smooth words [2] are equivalent to our (finite) smooth words. Clearly, $K_{a,b}, \overline{K}_{a,b}, K_{b,a}, \overline{K}_{b,a} \in C^\omega_{a,b}$, $\overline{K}_{a,b}, \overline{K}_{b,a} \in C^{l\omega}_{a,b}$ and $\overline{K}_{b,a}K_{a,b}, \overline{K}_{a,b}K_{b,a} \in C^{b\omega}_{a,b}$.

For simplifying the notations, if without confusion, we shall use $C^\infty$, $C^\omega$, $C^{l\omega}$ and $C^{b\omega}$ to stand for $C^\infty_{a,b}$, $C^\omega_{a,b}$, $C^{l\omega}_{a,b}$ and $C^{b\omega}_{a,b}$ respectively.

It is easy to check that $\Delta$, $D$ and $\overline{\phantom{w}}$ commute with the mirror image (\overline{\phantom{w}}) and are stable for the permutation($\overline{\phantom{w}}$) over arbitrary 2-letter alphabet $\{a, b\}$. Thus Proposition 4 in [5] still holds over arbitrary 2-letter alphabets $\{a, b\}$.

**Lemma 4** (Proposition 4 in [5]). (1) For all $u \in \Sigma^*$, $D(\overline{u}) = \overline{D(u)}$, $D(\overline{u}) = D(u)$;
(2) For all $u \in \Sigma^*$ $(\Sigma^\omega, \Sigma^{l\omega}, \Sigma^{b\omega})$, $\Delta(\overline{u}) = \overline{\Delta(u)}$, $\Delta(\overline{u}) = \Delta(u)$.

These properties indicate that $C^\infty$, $C^\omega$, $C^{l\omega}$ and $C^{b\omega}$ are all closed under these operators:

$w \in C^\infty \iff \overline{w}, \overline{\overline{w}} \in C^\infty$;
$w \in C^\omega \iff \overline{w} \in C^\omega$;
$w \in C^{l\omega} \iff \overline{w} \in C^{l\omega}$;
$w \in C^{l\omega} \iff \overline{w} \in C^\omega$;
$w \in C^{b\omega} \iff \overline{w}, \overline{\overline{w}} \in C^{b\omega}$. 

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3. Derivative Formula

The following simple results are important in what follows. First of all, we discuss the relations among the operators mirror image, complement, closure, derivative and run length-encoding.

Lemma 5. Let \( w \) be a differentiable word and \( u \) is a factor of \( w \). Then

1. \( \hat{u} \) and \( w \) are both factors of \( \hat{w} \);
2. \( \hat{w} = \tilde{w}, \tilde{w} = \tilde{w}, \rho(\Delta(w)) = \Delta(\rho(w)) \);
3. \( D(u) \) is a factor of \( D(w) \), \( D(w) \) is a factor of \( \Delta(w) \);
4. If \( w \) is closurely differentiable, then \( \rho(u) \) and \( D(w) \) are both factors of \( \rho(w) \), \( \rho(\tilde{w}) = \rho(w) \) and \( \rho(\tilde{w}) = \tilde{\rho(w)} \).

Proof. (1) From the definitions 1-2 of the closure of a word and a differentiable word, it easily follows the assertion (1).

(2) It immediately follows from the definition 1 of the closure of a word \( w \), the definitions of complement and mirror image of a word \( w \) (see page 5) and the definitions of the operators \( \rho \) and \( \Delta \) (see page 6 and 8).

(3) Since \( u \) is a factor of \( w \), by the definition 2 of derivative of a word \( w \), we easily see that \( D(u) \) is a factor of \( D(w) \); From Eq. (2.2), it follows that \( D(w) \) is a factor of \( \Delta(w) \).

(4) Since \( w \) is closurely differentiable and \( \rho(w) = D(\hat{w}) \), by the assertion (1), \( \hat{u} \) and \( w \) are both factors of \( \hat{w} \). Moreover by the assertion (3), we see that \( D(\hat{u}) \) and \( D(w) \) are factors of \( D(\hat{w}) \), that is, both \( \rho(u) \) and \( D(w) \) are factors of \( \rho(w) \). Finally, by Lemma 4 (1) and the assertion (2), we have \( \rho(\tilde{w}) = D(\hat{w}) = D(\hat{w}) = D(\tilde{w}) = \rho(w) \).

Similarly, \( \rho(\tilde{w}) = D(\tilde{w}) = D(\tilde{w}) = D(\tilde{w}) = \tilde{\rho(w)} \). □

Secondly, we need to establish the corresponding results to Lemma 1 and Proposition 2 in Weakly [26]. From the definitions 1-2, it easily follows that

Lemma 6. Let \( w = w_1 w_2 \cdots w_n \) be a differentiable word with \( n \geq a + 1 \).

1. If \( lfr(w) = b \) then \( w_1 w \) is not differentiable word and \( D(\tilde{w}_1^i w) = D(w) \) for \( i \leq b - 1 \);
2. If \( lfr(w) < b \) then \( D(w_1^{b-lfr(w)} w) = bD(w) \);
3. If \( lfr(w) \leq a \) and \( r(w) > 1 \) then \( D(w_1 a^{-lfr(w)} w) = aD(w) \). □

Lemma 7. (1) Let \( w = w_1 w_2 \cdots w_n \) be a \( C^\infty \)-word. Then any factor of \( w \) is also a \( C^\infty \)-word;
(2) Any $C^\infty$-word $w = w_1w_2 \cdots w_n$ has both a left $C^\infty$-extension and a right $C^\infty$-extension;

(3) If $w \in \Sigma^*$ and $\Delta(w) \in C^\infty$, then $w \in C^\infty$.

**Proof.** (1) If $w$ is a $C^\infty$-word and $u$ is a factor of $w$, then note that $w \in C^\infty \iff \rho^k(w) = \varepsilon$ for some positive integer $k$, by Lemma 5 (4), we obtain that $\rho^i(u)$ is a factor of $\rho^j(w)$ for any positive integer $i \leq k$. And hence $\rho^k(w) = \varepsilon$ suggests $\rho^k(u) = \varepsilon$, so that $u$ is a $C^\infty$-word.

(2) We verify the assertion (2) by induction on $|w|$. Since $D(\bar{w}) = \overline{D(w)}$, we only need to verify that $w$ has a left $C^\infty$-extension. It is clear that if $r(w) \leq 1$, where $r(w)$ is the number of runs of $w$, then the assertion (2) holds. We proceed to the induction step. Assume now that $r(w) \geq 2$ and the assertion (2) holds for $C^\infty$-words shorter than $w$.

If $lfr(w) \leq a$ then by Lemma 6 (2-3), we have $D(\bar{w}_1w_1^{a-lfr(w)}) = aD(w)$ and $D(\bar{w}_1w_1^{b-lfr(w)}) = bD(w)$. Thus by $|D(w)| < |w|$, we see that at least one of $aD(w)$ and $bD(w)$ is a $C^\infty$-word, which means that $w$ has a left $C^\infty$-extension.

If $b > lfr(w) > a$, then by $w \in C^\infty$, we obtain that $\bar{w}$ is a left $C^\infty$-extension of $w$.

If $b = lfr(w)$, then by Lemma 6 (1), we see that $\bar{w}_1w$ is a left $C^\infty$-extension of $w$.

(3) By Lemma 5 (2), $\rho^k(\Delta(w)) = \Delta(\rho^k(w))$. Note that $\Delta(w) = \varepsilon \iff w = \varepsilon$. Thus $\Delta(w) \in C^\infty \implies$ there is a positive integer $k$ such that $\rho^k(\Delta(w)) = \varepsilon \iff \Delta(\rho^k(w)) = \varepsilon \implies \rho^k(w) = \varepsilon$. Therefore, the assertion (3) holds. □

**Proposition 8.** If $a > 1$ then $\bar{K}_{b,a}K_{a,b}$ and $\bar{K}_{a,b}K_{b,a}$ are two different fixpoints of $\Delta$ over $\Sigma = \{a, b\}$.

**Proof.** Assume to the contrary that $\bar{K}_{b,a}K_{a,b} = \bar{K}_{a,b}K_{b,a}$, then $K_{a,b} = xK_{b,a}$ or $K_{b,a} = yK_{a,b}$.

If $K_{a,b} = xK_{b,a}$ and $|x|$ is minimal then $\text{last}(x) \neq \text{first}(K_{b,a})$. Thus $K_{a,b} = \Delta(K_{a,b}) = \Delta(x)\Delta(K_{b,a}) = \Delta(x)K_{b,a}$. Since $|x|$ is minimal and $|\Delta(x)| \leq |x|$ we have $|\Delta(x)| = |x|$. Therefore the length of all runs of $x$ must be equal to 1, which means that $a = 1$, a contradiction to the hypothesis.

If $K_{b,a} = yK_{a,b}$ and $|y|$ is minimal, then $\text{last}(y) = a$ or $\text{last}(y) \neq a$.

**case 1.** $\text{last}(y) \neq a$. Then $\bar{K}_{b,a} = \Delta(K_{b,a}) = \Delta(y)\Delta(K_{a,b}) = \Delta(y)K_{a,b}$. Since $|y|$ is minimal and $|\Delta(y)| \leq |y|$, we have $|\Delta(y)| = |y|$. Thus the length of all runs of $y$ must be 1, which means that $a = 1$, a contradiction to the hypothesis.

**case 2.** $\text{last}(y) = a$. Then by $yK_{a,b} \in C_{a,b}^\omega$, we obtain $y = za^{b-a}$ and $K_{b,a} = \Delta^2(K_{b,a}) = \Delta(z(\overline{a^{b-a}})) = \Delta(z(\overline{a^{b-a}}))$, which is a contradiction.
\[
\Delta(\Delta(z)\Delta(a^{b-a}K_{a,b})) = \Delta(\Delta(z)\Delta(a^b\theta^a\cdot\cdot\cdot\alpha^a\beta^b\cdot\cdot\cdot)) = \Delta(\Delta(z)ba^{a-1}b\cdot\cdot\cdot) = (a-1)\cdot\cdot\cdot,\text{ a contradiction.} \]

Let \( \Gamma \) be a nonempty subset of \( C^\infty \), \( \bar{\Gamma} = \{ \bar{w} : w \in \Gamma \} \) and \( \tilde{\Gamma} = \{ \tilde{w} : w \in \Gamma \} \). In order to prove Derivative Formula, we need the following corresponding results of Lemmas 4-5 [16].

**Lemma 9.** Let \( \Gamma \) be a nonempty subset of \( C^\infty \) and \( x \) be a fixed \( C^\infty \)-word. Assume that if \( uxv \in C^\infty \) then there exists a \( w \in \Gamma \) such that \( D(uxv) = D(u)wD(v) \). Then one has

(a) If \( uxv \in C^\infty \) then there is a \( z \in \Gamma \) such that \( D(uxv) = D(u)zD(v) \);

(b) If \( uxv \in C^\infty \) then there exists a \( z \in \Gamma \) such that \( D(uxv) = D(u)zD(v) \).

**Proof.** If \( uxv \in C^\infty \) then by Lemma 4, we obtain \( \tilde{uxu} = \tilde{uxv} \in C^\infty \) and \( D(\tilde{uxv}) = D(uxv) \). Thus by the hypothesis, there is a \( z \in \Gamma \) such that \( D(\tilde{uxu}) = D(\tilde{uxv}) = D(\tilde{uxu}) = D(\tilde{uxu}) = D(\tilde{uxv}) \). Similarly, one can check the case (a). \( \Box \)

Now we can check the case (a).

**Theorem 10** (Derivative Formula). For any \( x \in D_\Sigma \), if \( uxv \in C^\infty \) then there exists an element \( w \in D_\Sigma \) such that \( D(uxv) = D(u)wD(v) \), where \( D_\Sigma = D_{a,b} \) and

\[
D_{1,2} = \{ \varepsilon, 1, 2, 12, 21, 11, 22, 112, 211, 121, 122, 221, 121, 112, 121, 121, \\
2121, 2112, 1221, 1122, 2211, 11211 \}; \quad (3.1)
\]

\[
D_{1,3} = \{ \varepsilon, 1, 3, 13, 11, 33, 113, 311, 131, 313, 111, 1111, 1113, 1131 \}; \quad (3.2)
\]

\[
D_{1,4} = \{ \varepsilon, 1, 4, 14, 41, 11, 44, 111, 411, 114, 414, 1111, 4111, 1114 \}; \quad (3.3)
\]

\[
D_{1,b} = \{ \varepsilon, 1, b, 1b, b1, 11, bb, 11b, b11, 1111, 11111 \}, \text{ where } b \geq 5; \quad (3.4)
\]

\[
D_{2,b} = \{ \varepsilon, 2, b, 2b, 2b, 22, bb, 222 \}; \quad (3.5)
\]

\[
D_{a,b} = \{ \varepsilon, a, b, aa, bb, ab, ba \}, \text{ where } a \geq 3. \quad (3.6)
\]

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Proof. We call the $D_\Sigma$ in Theorem 10 the derivative closure of $D$ on $\Sigma$ and divide the proof of Theorem 10 into the following six cases. In what follows, $\alpha \in \Sigma$.

**Case 1.** $\Sigma = \{1, 2\}$. Then by Theorem 1 [13], we see that the assertion holds. For the convenience of the referees, note that $\tilde{D}_{1,2} = D_{1,2}$, by Lemma 9, the following Eqs. (3.7-3.17) provide the simplified calculation process of $D_{1,2}$.

\[
D(uv) = \begin{cases} 
D(u)D(v) \\
D(u)1D(v) \\
D(u)2D(v) \\
D(u)11D(v)
\end{cases} \quad (3.7)
\]

\[
D(u\alpha v) = \begin{cases} 
D(u)D(v) \\
D(u)1D(v) \\
D(u)2D(v) \\
D(u)11D(v) \\
D(u)12D(v) \\
D(u)21D(v)
\end{cases} \quad (3.8)
\]

\[
D(u\alpha\alpha v) = \begin{cases} 
D(u)2D(v) \\
D(u)12D(v) \\
D(u)21D(v) \\
D(u)121D(v)
\end{cases} \quad (3.9)
\]

\[
D(u\alpha\bar{\alpha}v) = \begin{cases} 
D(u)D(v) \\
D(u)1D(v) \\
D(u)2D(v) \\
D(u)11D(v) \\
D(u)12D(v) \\
D(u)21D(v) \\
D(u)22D(v) \\
D(u)112D(v) \\
D(u)211D(v)
\end{cases} \quad (3.10)
\]
\[ D(\bar{u}\bar{v}) = \left\{ \begin{array}{l} D(u)2D(v) \\ D(u)22D(v) \\ D(u)12D(v) \\ D(u)21D(v) \\ D(u)211D(v) \\ D(u)121D(v) \\ D(u)122D(v) \\ D(u)1211D(v) \end{array} \right. \] 

(3.11)

\[ D(\bar{u}v) = \left\{ \begin{array}{l} D(u)1D(v) \\ D(u)11D(v) \\ D(u)12D(v) \\ D(u)21D(v) \\ D(u)212D(v) \\ D(u)211D(v) \\ D(u)112D(v) \end{array} \right. \] 

(3.12)

\[ D(\bar{u}\bar{v}\bar{v}) = \left\{ \begin{array}{l} D(u)12D(v) \\ D(u)112D(v) \\ D(u)212D(v) \\ D(u)121D(v) \\ D(u)1121D(v) \\ D(u)2121D(v) \end{array} \right. \] 

(3.13)

\[ D(\bar{u}\bar{v}\bar{v}) = \left\{ \begin{array}{l} D(u)11D(v) \\ D(u)211D(v) \\ D(u)112D(v) \\ D(u)2112D(v) \end{array} \right. \] 

(3.14)
\[ D(\alpha\alpha\alpha\alpha\alpha) = \begin{cases} 
D(u)2D(v) \\
D(u)12D(v) \\
D(u)21D(v) \\
D(u)22D(v) \\
D(u)211D(v) \\
D(u)121D(v) \\
D(u)122D(v) \\
D(u)221D(v) \\
D(u)112D(v) \\
D(u)1211D(v) \\
D(u)1121D(v) \\
D(u)1122D(v) \\
D(u)2211D(v) \\
D(u)11211D(v) 
\end{cases} \] (3.15)

\[ D(\alpha\alpha\alpha\alpha\alpha) = \begin{cases} 
D(u)22D(v) \\
D(u)122D(v) \\
D(u)221D(v) \\
D(u)1221D(v) 
\end{cases} \] (3.16)

\[ D(u11211) = \begin{cases} 
D(u)212D(v) \\
D(u)2121D(v) \\
D(u)1212D(v) 
\end{cases} \] (3.17)

**Case 2.** \( \Sigma = \{1, 3\} \). Then similarly, we only need to compute the derivative closure of \( D \) over the alphabet \( \{1, 3\} \). Note that \( \hat{D}_{1,3} = D_{1,3} \), by Lemma 9, the following Eqs. (3.18-3.26) provide the computing process of \( D_{1,3} \).

\[ D(uv) = \begin{cases} 
D(u)D(v) \\
D(u)1D(v) \\
D(u)3D(v) \\
D(u)11D(v) 
\end{cases} \] (3.18)
\[ D(u\alpha v) = \begin{cases} 
D(u)D(v) \\
D(u)3D(v) \\
D(u)1D(v) \\
D(u)11D(v) \\
D(u)13D(v) \\
D(u)31D(v) \\
D(u)111D(v) \\
\end{cases} \] (3.19)

\[ D(u\alpha\alpha v) = \begin{cases} 
D(u)D(v) \\
D(u)1D(v) \\
D(u)3D(v) \\
D(u)31D(v) \\
\end{cases} \] (3.20)

\[ D(u\alpha\bar{\alpha}v) = \begin{cases} 
D(u)D(v) \\
D(u)1D(v) \\
D(u)3D(v) \\
D(u)11D(v) \\
D(u)13D(v) \\
D(u)31D(v) \\
D(u)111D(v) \\
D(u)113D(v) \\
D(u)311D(v) \\
\end{cases} \] (3.21)

\[ D(u111v) = \begin{cases} 
D(u)3D(v) \\
D(u)13D(v) \\
D(u)31D(v) \\
D(u)131D(v) \\
\end{cases} \] (3.22)

\[ D(u\alpha\alpha v) = \begin{cases} 
D(u)D(v) \\
D(u)1D(v) \\
D(u)3D(v) \\
D(u)11D(v) \\
D(u)33D(v) \\
D(u)111D(v) \\
D(u)311D(v) \\
\end{cases} \] (3.23)
Case 3. $\Sigma = \{1, 4\}$. Analogously, we only need to compute the derivative closure of $D$ over the alphabet $\{1, 4\}$. Note that $\tilde{D}_{1,4} = D_{1,4}$, by Lemma 9, the following Eqs. (3.27-3.35) provide the computing process of $D_{1,4}$.

$$D(u131v) = \begin{cases} 
D(u)1D(v) \\
D(u)11D(v) \\
D(u)13D(v) \\
D(u)31D(v) \\
D(u)313D(v) \\
D(u)111D(v) \\
D(u)311D(v) \\
D(u)113D(v) \\
D(u)3111D(v) \\
D(u)1113D(v) 
\end{cases} \quad (3.24)$$

$$D(u3111v) = \begin{cases} 
D(u)3D(v) \\
D(u)13D(v) \\
D(u)31D(v) \\
D(u)33D(v) \\
D(u)131D(v) \\
D(u)113D(v) \\
D(u)331D(v) \\
D(u)1131D(v) 
\end{cases} \quad (3.25)$$

$$D(u1131v) = \begin{cases} 
D(u)1D(v) \\
D(u)11D(v) \\
D(u)31D(v) \\
D(u)13D(v) \\
D(u)111D(v) \\
D(u)311D(v) \\
D(u)3111D(v) 
\end{cases} \quad (3.26)$$

$$D(uv) = \begin{cases} 
D(u)D(v) \\
D(u)1D(v) \\
D(u)4D(v) \\
D(u)11D(v) 
\end{cases} \quad (3.27)$$
\[ D(uv) = \begin{cases} 
D(u)D(v) \\
D(u)1D(v) \\
D(u)4D(v) \\
D(u)11D(v) \\
D(u)14D(v) \\
D(u)41D(v) \\
D(u)111D(v) 
\end{cases} \]  \hspace{1cm} (3.28)

\[ D(u\alpha v) = \begin{cases} 
D(u)D(v) \\
D(u)1D(v) \\
D(u)4D(v) \\
D(u)14D(v) \\
D(u)41D(v) 
\end{cases} \]  \hspace{1cm} (3.29)

\[ D(u\alpha \bar{\alpha}v) = \begin{cases} 
D(u)D(v) \\
D(u)1D(v) \\
D(u)4D(v) \\
D(u)14D(v) \\
D(u)41D(v) \\
D(u)44D(v) \\
D(u)411D(v) \\
D(u)114D(v) \\
D(u)111D(v) \\
D(u)1111D(v) 
\end{cases} \]  \hspace{1cm} (3.30)

\[ D(u11v) = \begin{cases} 
D(u)D(v) \\
D(u)1D(v) \\
D(u)4D(v) \\
D(u)14D(v) \\
D(u)41D(v) 
\end{cases} \]  \hspace{1cm} (3.31)

\[ D(u111v) = \begin{cases} 
D(u)4D(v) \\
D(u)41D(v) \\
D(u)14D(v) \\
D(u)141D(v) 
\end{cases} \]  \hspace{1cm} (3.32)
Case 4. \( \Sigma = \{1, b\} \). Similarly, we only need to compute the derivative closure of \( D \) over the alphabet \( \{1, b\} \). Note that \( \tilde{D}_{1,b} = D_{1,b} \), by Lemma 9, the following Eqs. (3.36-3.42) give the computing process of \( D_{1,b} \).

\[
D(uv) = \begin{cases} 
D(u)D(v) \\
D(u)1D(v) \\
D(u)4D(v) \\
D(u)11D(v) \\
D(u)41D(v) \\
D(u)411D(v) \\
D(u)4111D(v) \\
\end{cases}
\] (3.36)
$$D(u\alpha \nu) = \begin{cases} 
D(u)D(v) \\
D(u)1D(v) \\
D(u)bD(v) \\
D(u)11D(v) \\
D(u)1bD(v) \\
D(u)b1D(v) \\
D(u)111D(v) 
\end{cases} \quad (3.37)$$

$$D(u\alpha \alpha \nu) = \begin{cases} 
D(u)D(v) \\
D(u)1D(v) \\
D(u)bD(v) \\
D(u)1bD(v) \\
D(u)b1D(v) 
\end{cases} \quad (3.38)$$

$$D(u\alpha \alpha \bar{\nu}) = \begin{cases} 
D(u)D(v) \\
D(u)1D(v) \\
D(u)bD(v) \\
D(u)11D(v) \\
D(u)1bD(v) \\
D(u)b1D(v) \\
D(u)111D(v) 
\end{cases} \quad (3.39)$$

$$D(u11\nu) = \begin{cases} 
D(u)D(v) \\
D(u)1D(v) \\
D(u)bD(v) \\
D(u)1bD(v) \\
D(u)b1D(v) 
\end{cases} \quad (3.40)$$

$$D(u111\nu) = \begin{cases} 
D(u)D(v) \\
D(u)1D(v) \\
D(u)bD(v) \\
D(u)1bD(v) \\
D(u)b1D(v) \\
D(u)1bD(v) 
\end{cases} \quad (3.41)$$
\[ D(ub1v) = \begin{cases} 
D(u)D(v) \\
D(u)bD(v) \\
D(u)1bD(v) \\
D(u)11D(v) \\
D(u)bbD(v) \\
D(u)11bD(v) 
\end{cases} \quad (3.42) \]

Case 5. \( \Sigma = \{2, b\} \). Then analogously, we only need to compute the derivative closure of \( D \) over the alphabet \( \{2, b\} \). Note that \( \tilde{D}_{2,b} = D_{2,b} \), by Lemma 9, the following Eqs. (3.43-3.47) provide the computing process of \( D_{2,b} \).

\[ D(uv) = \begin{cases} 
D(u)D(v) \\
D(u)2D(v) \\
D(u)bD(v) \\
D(u)22D(v) 
\end{cases} \quad (3.43) \]

\[ D(u\bar{\alpha}v) = \begin{cases} 
D(u)D(v) \\
D(u)2D(v) \\
D(u)bD(v) \\
D(u)22D(v) \\
D(u)2bD(v) \\
D(u)b2D(v) 
\end{cases} \quad (3.44) \]

\[ D(u\alpha\bar{\alpha}v) = \begin{cases} 
D(u)D(v) \\
D(u)2D(v) \\
D(u)bD(v) \\
D(u)2bD(v) \\
D(u)bD(v) \\
D(u)2bD(v) \\
D(u)22D(v) \\
D(u)222D(v) 
\end{cases} \quad (3.45) \]

\[ D(u\alpha\bar{\alpha}v) = \begin{cases} 
D(u)D(v) \\
D(u)2D(v) \\
D(u)bD(v) \\
D(u)2bD(v) \\
D(u)b2D(v) \\
D(u)2bD(v) 
\end{cases} \quad (3.46) \]
\[ D(u222v) = \begin{cases} 
D(u)bD(v) \\
D(u)2bD(v) \\
D(u)b2D(v) 
\end{cases} \]  \hspace{1cm} (3.47)

**Case 6.** \( \Sigma = \{a, b\} \). Then similarly, we only require to compute the derivative closure of \( D \) over the alphabet \( \{a, b\} \). Note that \( \tilde{D}_{a,b} = D_{a,b} \), by Lemma 9, the following Eqs. (3.48-3.51) present the computing process of \( D_{a,b} \).

\[ D(uv) = \begin{cases} 
D(u)D(v) \\
D(u)aD(v) \\
D(u)bD(v) \\
D(u)aaD(v)
\end{cases} \]  \hspace{1cm} (3.48)

\[ D(u\alpha v) = \begin{cases} 
D(u)D(v) \\
D(u)aD(v) \\
D(u)bD(v) \\
D(u)aaD(v) \\
D(u)abD(v) \\
D(u)baD(v)
\end{cases} \]  \hspace{1cm} (3.49)

\[ D(u\alpha\alpha v) = \begin{cases} 
D(u)D(v) \\
D(u)aD(v) \\
D(u)bD(v) \\
D(u)abD(v) \\
D(u)baD(v) \\
D(u)aaD(v) \\
D(u)baD(v) \\
D(u)abD(v)
\end{cases} \]  \hspace{1cm} (3.50)

\[ D(u\alpha\check{\alpha}v) = \begin{cases} 
D(u)D(v) \\
D(u)aD(v) \\
D(u)bD(v) \\
D(u)aaD(v) \\
D(u)bbD(v) \\
D(u)baD(v) \\
D(u)abD(v)
\end{cases} \]  \hspace{1cm} (3.51)

Now we can generalize Theorem 10 to the following more general form.
Theorem 11. Let $u_i \in C^\infty$ for $i = 1, 2, 3, \ldots, n \ (\geq 2)$. Then for any positive integer $k$, there exist $w_j \in D_\Sigma$ for $j = 1, 2, 3, \ldots, n - 1$ such that $D^k(u_1 u_2 \cdots u_n) = D^k(u_1) w_1 D^k(u_2) w_2 \cdots D^k(u_{n-1}) w_{n-1} D^k(u_n)$.

Proof. First of all, we prove that the assertion holds for $n=2$. For this, we only need to proceed by induction on $k$. If $k = 1$, in view of $u_1 u_2 \in C^\infty$ and taking $x = \varepsilon$ in Theorem 10, one sees that Theorem 11 holds for $k = 1$.

Now we suppose that Theorem 11 holds for all $k \leq m \ (\geq 1)$, i.e. there exists a $w_1 \in D_\Sigma$ such that $D^m(u_1 u_2) = D^m(u_1) x_1 D^m(u_2)$. Thus $D^{m+1}(u_1 u_2) = D(D^m(u_1) x_1 D^m(u_2))$, in view of Theorem 10, one sees that there is a $w_1 \in D_\Sigma$ such that $D(D^m(u_1) x_1 D^m(u_2)) = D^{m+1}(u) w_1 D^{m+1}(v)$, which implies that Theorem 11 holds for $n = 2$.

From the above discussion it immediately follows that Theorem 11 holds for any positive integer $n \geq 2$. □

Note that if $u$ has at least two runs, then $D(u^2) = D(u) x D(u)$, where $x$ is uniquely determined by the last($u$) and first($u$). Thus from Theorem 11, we obtain the following useful result.

Theorem 12 (Power Derivative Formula). Let $u \in C^\infty_{a,b}$, $n \ (\geq 2)$ and $k$ be positive integers. If $D^{k-1}(u)$ has at least two runs, then for any positive integer $1 \leq j \leq k$, there exists a $w \in D_{a,b}$ such that $D^j(u^n) = (D^j(u) w)^{n-1} D^j(u)$.

4. A new proof of $C^\infty_{1,2}$-words being cube-free

A. Carpi [7, 8] and A. Lepistö [20] independently proved the following interesting result.

Proposition 13. Smooth words are cube-free over the alphabet \{1, 2\}.

Now we are in a position to give a new proof of Proposition 13 without machine computation.

Proof. Assume on the contrary that there is a $u \in C^\infty_{1,2}$ with $|u| \geq 1$ such that $u^3 \in C^\infty_{1,2}$. Then we easily see that $u$ contains at least two runs. Let $k$ be the maximal positive integer such that $D^{k-1}(u)$ has at least two runs. Then by Power Derivative Formula, we have

$$(D^j(u) x)^2 D^j(u) = D^j(u^3) \in C^\infty_{1,2} \text{ for } 1 \leq j \leq k, \ x \in D_{1,2}$$

(4.1)
and $D^k(u)$ has at most one run, which means that

$$D^k(u) = \alpha^i, \ i = 0, 1, 2.$$  

**Case 1.** $D^k(u) = \varepsilon$. Then since $D^{k-1}(u)$ has at least two runs, we obtain

$$D^{k-1}(u) = \alpha \bar{\alpha}, \quad (4.2)$$

and by Eq. (4.1), we have

$$D^k(u^3) = x^2, \text{ where } x \in D_{1,2}. \quad (4.3)$$

By Eq. (4.3), we easily obtain

$$D^k(u^3) = \varepsilon, 1^2, 2^2, (12)^2, (21)^2, (112)^2, (211)^2, (121)^2, (212)^2, (122)^2, (221)^2. \quad (4.4)$$

From Eq. (4.1) it follows that

$$(D^{k-1}(u)x)^2 D^{k-1}(u) = D^{k-1}(u^3), \text{ where } x \in D_{1,2}.$$  

which by Eqs. (4.2, 4.4), implies that there are $y \in D_{1,2}$ and $\beta \in \Sigma$ such that

$$(\alpha \bar{\alpha} y)^2 \alpha \bar{\alpha} = \beta^i \bar{\beta}^j, \ 0 \leq i, j \leq 1, \ i + j \geq 1;$$  

$$= \beta \bar{\beta} \beta \bar{\beta};$$  

$$= \beta^i \bar{\beta}^j \beta^2 \bar{\beta}^i, \ 0 \leq i, j \leq 1;$$  

$$= \beta \bar{\beta} \beta \bar{\beta}^2 \beta \bar{\beta}, \ 0 \leq j \leq 1;$$  

$$= \beta \bar{\beta} \beta \bar{\beta}^2 \beta \bar{\beta} \beta \bar{\beta}, \ 0 \leq i \leq 1;$$  

$$= \beta \bar{\beta} \beta \bar{\beta}^2 \beta \bar{\beta} \beta \bar{\beta} \beta \bar{\beta};$$  

$$= \beta^i \bar{\beta}^j \beta^2 \bar{\beta}^i \beta \bar{\beta}^2 \beta \bar{\beta}, \ 0 \leq j \leq 1;$$  

$$= \beta \bar{\beta} \beta \bar{\beta}^2 \beta \bar{\beta} \beta \bar{\beta} \beta \bar{\beta}, \ 0 \leq j \leq 1;$$  

$$= \beta \bar{\beta} \beta \bar{\beta}^2 \beta \bar{\beta} \beta \bar{\beta} \beta \bar{\beta}, \ 0 \leq i \leq 1. \quad (4.5)$$

$$= \beta \bar{\beta} \beta \bar{\beta}^2 \beta \bar{\beta} \beta \bar{\beta} \beta \bar{\beta};$$  

$$= \beta^i \bar{\beta}^j \beta^2 \bar{\beta}^i \beta \bar{\beta}^2 \beta \bar{\beta} \beta \bar{\beta}, \ 0 \leq i \leq 1;$$  

$$= \beta \bar{\beta} \beta \bar{\beta}^2 \beta \bar{\beta} \beta \bar{\beta} \beta \bar{\beta}, \ 0 \leq j \leq 1;$$  

$$= \beta \bar{\beta} \beta \bar{\beta}^2 \beta \bar{\beta} \beta \bar{\beta} \beta \bar{\beta}, \ 0 \leq i \leq 1. \quad (4.6)$$

We can verify that the other equations except for Eqs. (4.5-4.6) all do not hold and

$$D^{k-1}(u^3) = \alpha \bar{\alpha} \alpha^2 \bar{\alpha} \alpha; \quad (4.7)$$  

$$= \alpha \bar{\alpha} \alpha^2 \alpha \bar{\alpha}. \quad (4.8)$$

By Eq. (4.2), we have a $\beta \in \Sigma$ such that

$$D^{k-2}(u) = \beta \bar{\beta} \beta^2 \bar{\beta}^j, \ j = 0, 1; \quad (4.9)$$  

$$= \beta^i \bar{\beta}^j \beta \bar{\beta}, \ i = 0, 1. \quad (4.10)$$
Thus from Eqs. (4.1, 4.7 and 4.8), we obtain a $z \in \Sigma$ such that

$$
(D^{k-2}(u)z)^2 D^{k-2}(u)(= D^{k-2}(u^3)) = \gamma \bar{\gamma} \gamma^2 \gamma \bar{\gamma} \gamma^2 \gamma \bar{\gamma}, j = 0, 1; \quad (4.11)
$$

$$
= \gamma^i \gamma^2 \gamma^2 \gamma \bar{\gamma} \gamma^2 \gamma \bar{\gamma}, i = 0, 1; \quad (4.12)
$$

$$
= \gamma^i \gamma^2 \gamma^2 \gamma \bar{\gamma} \gamma^2 \gamma \bar{\gamma}, j = 0, 1; \quad (4.13)
$$

$$
= \gamma^i \gamma^2 \gamma \bar{\gamma} \gamma^2 \gamma \bar{\gamma}, i = 0, 1. \quad (4.14)
$$

By Eqs. (4.9-4.10), we see that the left sides of Eqs. (4.11-4.14) contains at least three disjoint occurrences of the same factor $\beta \bar{\beta} \beta^2$ (or $\bar{\beta} \beta \beta$), which contradict the right sides of the corresponding ones.

**Case 2.** $D^k(u) = \alpha^i$, where $i = 1, 2$. Then by Eq. (4.1) we have

$$
\alpha^i x \alpha^i x \alpha^i = D^k(u^3) \in C_{1,2}^\infty, \text{where } x \in D_{1,2}. \quad (4.15)
$$

If $i = 2$ then by Eq. (4.15), we can check that there is no $x \in D_{1,2}$ such that $\alpha^2 x \alpha^2 x \alpha^2 \in C_{1,2}^\infty$, a contradiction.

If $i = 1$ then on the one hand, from $D^k(u) = \alpha$ we obtain

$$
D^{k-1}(u) = \beta \bar{\beta} \beta; \quad (4.16)
$$

$$
= \beta \bar{\beta}^2; \quad (4.17)
$$

$$
= \bar{\beta}^2 \beta; \quad (4.18)
$$

$$
= \beta \bar{\beta}^2 \beta. \quad (4.19)
$$

On the other hand, by Eq. (4.15), we have

$$
D^k(u^3) = (D^k(u)x)^2 D^k(u) = \alpha \bar{\alpha} \alpha \bar{\alpha} \alpha, \alpha \bar{\alpha} \alpha \bar{\alpha} \alpha, \alpha \bar{\alpha} \alpha \bar{\alpha} \alpha,
$$

which means that

$$
(D^{k-1}(u)x)^2 D^{k-1}(u)(= D^{k-1}(u^3)) = \gamma \bar{\gamma} \gamma^2 \gamma \bar{\gamma} \gamma^2 \gamma \bar{\gamma}; \quad (4.20)
$$

$$
= \gamma^i \gamma^2 \gamma \bar{\gamma} \gamma^2 \gamma \bar{\gamma}, 0 \leq i, j \leq 1; \quad (4.21)
$$

$$
= \gamma \bar{\gamma} \gamma^2 \gamma \bar{\gamma} \gamma^2 \gamma \bar{\gamma}; \quad (4.22)
$$

$$
= \gamma \bar{\gamma} \gamma^2 \gamma \bar{\gamma} \gamma^2 \gamma \bar{\gamma}; \quad (4.23)
$$

$$
= \gamma \bar{\gamma} \gamma^2 \gamma \bar{\gamma} \gamma^2 \gamma \bar{\gamma}; \quad (4.24)
$$

$$
= \gamma \bar{\gamma} \gamma^2 \gamma \bar{\gamma} \gamma^2 \gamma \bar{\gamma}; \quad (4.25)
$$

**Case 2.1.** $D^{k-1}(u) = \beta \bar{\beta} \beta$. Then comparing the first and second runs of both sides of Eqs. (4.21, 4.23 and 4.25) approaches a contradiction. And a more detailed comparison of factor of form $\beta \bar{\beta} \beta$ in both sides of Eqs.(4.20, 4.22 and 4.24) also leads
a contradiction.

**Case 2.2.** \( D^{k-1}(u) = \beta \bar{\beta}^2 \). Then comparing the last runs of two sides of Eqs. (4.20, 4.22 and 4.24) get a contradiction. And a more detailed comparison of factor of form \( \beta \bar{\beta}^2 \) in both sides of Eqs. (4.21, 4.23 and 4.25) also leads to a contradiction.

**Case 2.3.** \( D^{k-1}(u) = \beta^2 \bar{\beta} \). Then \( D^{k-1}(\tilde{u}) = \bar{\beta} \beta^2 \), by Case 2.2, \( \tilde{u}^3 \notin C_{1,2}^{\infty} \), which implies \( u^3 \notin C_{1,2}^{\infty} \), a contradiction.

**Case 2.4.** \( D^{k-1}(u) = \beta \bar{\beta}^2 \bar{\beta} \). Then note that the left sides of Eqs. (4.20-4.25) contain at least three disjoint occurrences of the same factor \( \beta \bar{\beta}^2 \bar{\beta} \), comparing both sides of Eqs. (4.20-4.25) reachs a contradiction. □

## 5. The power-free index of smooth words

Let \( n \) be a positive integer, if \( u^n \notin C_{a,b}^{\infty} \) for any nonempty \( C_{a,b}^{\infty} \)-word \( u \), then we call that \( C_{a,b}^{\infty} \)-words are \( n \)-power-free. And the minimal positive integer \( n \) such that \( C_{a,b}^{\infty} \)-words are \( n \)-power-free is said to be the power-free index of \( C_{a,b}^{\infty} \)-words.

In this section, we establish the power-free index of smooth words over arbitrary 2-letter alphabet \( \{a, b\} \), of which the Proposition 13 is a special case.

**Theorem 14.** Let

\[
\begin{align*}
  h(a, b) &= \begin{cases} 
    b + 2, & a = 1 \text{ and } b = 3 \\
    \frac{b+4}{2}, & 2 \mid b \\
    \frac{b+5}{2}, & 2 \nmid b, \text{ and } b \neq 3, \ a = 1 \\
    \frac{b+3}{2}, & 2 \nmid b, \ a \geq 2 
  \end{cases} \\
  \delta(a, b) &= \begin{cases} 
    b + 2, & a = 1 \text{ and } b = 3 \\
    b + 1, & \text{otherwise} 
  \end{cases}
\end{align*}
\]

then

1. Smooth words of length \( \geq 2 \) are \( h(a, b) \)-power-free over arbitrary 2-letter alphabet \( \{a, b\} \);
2. The power-free index of smooth words is \( \delta(a, b) \) for the alphabet \( \{a, b\} \).

**Proof.** It is clear that \( h(a, b) \geq 3 \). Note that \( \tilde{D}_\Sigma = D_\Sigma \). We easily see that the following assertions hold.

\[
(\alpha^i x)^{h(a, b)-1} \alpha^i \in C_{a,b}^{\infty} \iff (\alpha^i x)^{h(a, b)-1} \alpha^i \in C_{a,b}^{\infty}, \quad (5.3)
\]

If \( u \) has at least two runs then \( D(u^{h(a, b)}) \neq \varepsilon \). \( (5.4) \)
(1) By Proposition 13, it suffices to check the assertion (1) for the cases except for \(a = 1, b = 2\).

**Case 1.** \(\Sigma = \{1, 3\}\). Assume on the contrary that there exists a \(u \in C_{1,3}^\infty\) with \(|u| \geq 1\) such that \(u^5 \in C_{1,3}^\infty\). Then \(u\) has at least two runs. Let \(k\) be the maximal integer such that \(D^{k-1}(u)\) has at least two runs. Thus by Power Derivative Formula, we have

\[
(D^j(u)x)^4D^j(u) = D^j(u^5) \in C_{1,3}^\infty \text{ for } 1 \leq j \leq k, \text{ where } x \in D_{1,3}, \tag{5.5}
\]

and \(D^k(u)\) has at most one run, which implies that

\[
D^k(u) = \alpha^i, \text{ where } i = 0, 1, 2, 3.
\]

**Case 1.1.** \(D^k(u) = \varepsilon\). Then since \(D^{k-1}(u)\) has at least two runs, we have

\[
D^{k-1}(u) = \alpha^s\bar{\alpha}^t, \text{ where } s, t = 1, 2, 3, \tag{5.6}
\]

and by Eq. (5.5) \((j = k)\), we can get

\[
D^k(u^5) = x^4, \text{ where } x \in D_{1,3}. \tag{5.7}
\]

From Eq. (5.7), a direct verification leads to

\[
D^k(u^5) = \varepsilon.
\]

Note that \(s, t = 1, 2\), by Eqs. (5.6) and (5.5) \((j = k - 1)\), we easily see that \(D^{k-1}(u^5)\) has at least ten runs, which means that \(D^k(u^5)\) has at least eight runs, a contradiction to \(D^k(u^5) = \varepsilon\).

**Case 2.** \(\Sigma = \{1, 4\}\). Suppose on the contrary that there exists a \(u \in C_{1,4}^\infty\) with \(|u| \geq 2\) such that \(u^4 \in C_{1,4}^\infty\). Then \(u\) has at least two runs. Let \(k\) be the maximal integer such that \(D^{k-1}(u)\) has at least two runs. Then by Power Derivative Formula, we obtain

\[
(D^j(u)x)^3D^j(u) = D^j(u^4) \in C_{1,4}^\infty \text{ for } 1 \leq j \leq k, \text{ where } x \in D_{1,4}, \tag{5.8}
\]

and \(D^k(u)\) has at most one run, which implies that

\[
D^k(u) = \alpha^i, \text{ where } i = 0, 1, 2, 3, 4.
\]

**Case 2.1.** \(D^k(u) = \varepsilon\). Then since \(D^{k-1}(u)\) has at least two runs, we have

\[
D^{k-1}(u) = \alpha^s\bar{\alpha}^t, \text{ where } s, t = 1, 2, 3, \tag{5.9}
\]

and by Eq. (5.8) we have
\[ D^k(u^4) = x^3, \] where \( x \in D_{1,4}. \)

A direct verification leads to
\[ D^k(u^4) = \varepsilon, 1^3, 4^3, (14)^3, (41)^3. \] (5.10)

From Eq. (5.8) it follows that
\[ (D^{k-1}(u)x)^3 D^{k-1}(u) = D^{k-1}(u^4). \] (5.11)

Hence by Eqs. (5.9-5.11), we obtain
\[
(\alpha^s \bar{\alpha}^t x)^3 \alpha^s \bar{\alpha}^t = \begin{cases} \\
= \alpha^i \bar{\alpha}^j, & i, j = 1, 2, 3 \\
= \alpha^i \bar{\alpha} \alpha \bar{\alpha}^j, & i, j = 1, 2, 3; \\
= \alpha^i \bar{\alpha}^4 \alpha \bar{\alpha}^4 \alpha^j, & i, j = 0, 1, 2, 3; \\
= \alpha^i \bar{\alpha}^4 \alpha \bar{\alpha}^4 \alpha \bar{\alpha}^4 \alpha^j, & 1 \leq i \leq 3, 0 \leq j \leq 3; \\
= \alpha^i \bar{\alpha}^4 \alpha \bar{\alpha}^4 \alpha \bar{\alpha}^4 \alpha \bar{\alpha}^4 \alpha^j, & 0 \leq i \leq 3, 1 \leq j \leq 3.
\end{cases} \] (5.12) (5.13)

Since \( 1 \leq s, t \leq 3 \), we see that the left side of the above equations has at least eight runs, which implies that the other equations except for Eqs. (5.12-5.13) all do not hold.

**Case 2.1.1.** If Eq. (5.12) holds, then from Eq. (5.9) we obtain
\[ D^{k-1}(u) = \alpha^i \bar{\alpha} \text{ and } x = \alpha^{4-i} \in D_{1,4}, i = 1, 2, 3 \] (5.14)

**Case 2.1.1.1.** \( i = 1 \). Then by \( x = \alpha^3 \in D_{1,4} \), we have \( \alpha = 1 \). Thus from Eqs. (5.12, 5.14) we obtain
\[
D^{k-1}(u) = 14; \\
D^{k-1}(u^4) = 141^4 41^4 4. \\
\] (5.15) (5.16)

From Eqs. (5.15-5.16) it follows that
\[
D^{k-2}(u) = \beta^k \bar{\beta}^4 \beta^l, & 1 \leq k \leq 3, 0 \leq l \leq 3; \\
D^{k-2}(u^4) = \gamma^s \bar{\gamma}^4 \gamma \bar{\gamma}^4 \gamma \bar{\gamma}^4 \gamma \bar{\gamma}^4 \gamma, & 1 \leq s \leq 3, 0 \leq t \leq 3. \] (5.17) (5.18)

By Eq. (5.8), we have
\[ D^{k-2}(u^4) = (D^{k-2}(u)x)^3 D^{k-2}(u), \text{ where } x \in D_{1,4}. \] (5.19)
Thus, Eq. (5.17) means that the right side of Eq. (5.19) has at least four same runs of form $\beta^4$, a contradiction to Eq. (5.18).

**Case 2.1.1.2.** i=2. Then by Eqs. (5.12, 5.14), we have

\[
D^{k-1}(u) = \alpha^2 \bar{\alpha}; \\
D^{k-1}(u^4) = \alpha^2 \bar{\alpha} \alpha^4 \bar{\alpha} \alpha^4 \bar{\alpha}, \; \alpha = 1, 4. \tag{5.20}
\]

If $\alpha = 1$ then from Eqs. (5.20-5.21), we obtain

\[
D^{k-2}(u) = \beta^k \bar{\beta} \beta^4 \bar{\beta}^l, \; 1 \leq k \leq 3, \ 0 \leq l \leq 3; \tag{5.22}
\]

\[
D^{k-2}(u^4) = \gamma^s \bar{\gamma} \gamma^4 \bar{\gamma} \gamma^4 \bar{\gamma} \gamma^4 \bar{\gamma} \gamma^4 \bar{\gamma}, \; 1 \leq s \leq 3, \ 0 \leq t \leq 3. \tag{5.23}
\]

An argument similar to Case 2.1.1.1 leads to a contradiction to Eq. (5.23).

If $\alpha = 4$ then from Eqs. (5.20-5.21), we obtain

\[
D^{k-2}(u) = \beta^k \bar{\beta} \beta^4 \bar{\beta}^l, \; 0 \leq k \leq 3, \ 1 \leq l \leq 3; \tag{5.24}
\]

\[
D^{k-2}(u^4) = \gamma^s \bar{\gamma} \gamma^4 \bar{\gamma} \gamma^4 \bar{\gamma} \gamma^4 \bar{\gamma} \gamma^4 \bar{\gamma}, \; 0 \leq s \leq 3, \ 1 \leq t \leq 3. \tag{5.25}
\]

where $0 \leq s \leq 3, \ 1 \leq t \leq 3$. Thus Eq. (5.24) implies that the right side of Eq. (5.19) has at least four same factors of form $\beta^4 \beta^4 \bar{\beta}^4$, a contradiction to Eq. (5.25).

**Case 2.1.1.3.** i=3. Then from Eqs. (5.12, 5.14), we obtain

\[
D^{k-1}(u) = \alpha^3 \bar{\alpha}; \\
D^{k-1}(u^4) = \alpha^3 \bar{\alpha} \alpha^4 \bar{\alpha} \alpha^4 \bar{\alpha}, \; \alpha = 1, 4. \tag{5.26}
\]

If $\alpha = 1$ then from Eqs. (5.26-5.27), we have

\[
D^{k-2}(u) = \beta^l \bar{\beta} \beta^4 \bar{\beta}^k, \; 1 \leq l \leq 3, \ 0 \leq k \leq 3; \tag{5.28}
\]

\[
D^{k-2}(u^4) = \gamma^s \bar{\gamma} \gamma^4 \bar{\gamma} \gamma^4 \bar{\gamma} \gamma^4 \bar{\gamma} \gamma^4 \bar{\gamma}, \; 1 \leq s \leq 3, \ 0 \leq t \leq 3. \tag{5.29}
\]

An argument similar to Case 2.1.1.1 leads to a contradiction to Eq. (5.29).

If $\alpha = 4$ then from Eqs. (5.26-5.27), we get

\[
D^{k-2}(u) = \beta^l \bar{\beta} \beta^4 \bar{\beta}^k, \; 0 \leq k \leq 3, \ 1 \leq l \leq 3; \tag{5.30}
\]

\[
D^{k-2}(u^4) = \gamma^s \bar{\gamma} \gamma^4 \bar{\gamma} \gamma^4 \bar{\gamma} \gamma^4 \bar{\gamma} \gamma^4 \bar{\gamma}, \; 0 \leq s \leq 3, \ 1 \leq t \leq 3. \tag{5.31}
\]

where $0 \leq s \leq 3, \ 1 \leq t \leq 3$. Therefore, Eq. (5.30) suggests that the right side of Eq. (5.19) has at least four same factors of form $\beta^4 \beta^4 \bar{\beta}^4$, a contradiction to Eq. (5.31).

**Case 2.1.2.** If Eq. (5.13) holds, then from Eq. (5.9), we obtain

\[
D^{k-1}(u) = \alpha \bar{\alpha}^j, \; 1 \leq j \leq 3; \\
D^{k-1}(u^4) = \alpha \bar{\alpha}^4 \alpha \bar{\alpha}^4 \alpha \bar{\alpha}^4 \alpha \bar{\alpha}^j, \; 1 \leq j \leq 3, \; \alpha = 1, 4.
\]
Thus, by $\widetilde{D}(w) = D(\tilde{w})$, we have

\begin{align*}
D^{k-1}(\tilde{u}) &= \tilde{\alpha}^j \alpha, \ 1 \leq j \leq 3; \\
D^{k-1}(\tilde{u}^4) &= \tilde{\alpha}^j \alpha \tilde{\alpha}^4 \alpha \tilde{\alpha}^4 \alpha, \ 1 \leq j \leq 3,
\end{align*}

which means that $D^{k-1}(\tilde{u})$ and $D^{k-1}(\tilde{u}^4)$ satisfy Eqs. (5.12, 5.14). So by Case 2.1, we can arrive at a contradiction.

**Case 2.2.** $D^k(u) = \alpha$. Then by Eq. (5.8), we have

$$(\alpha x)^3 \alpha = D^k(u^4) \in C_{1,4}^\infty,$$

where $x \in D_{1,4}$, which implies that

\begin{align*}
D^k(u) &= \alpha, \quad (5.32) \\
D^k(u^4) &= \alpha^4, \ \alpha = 1, 4; \quad (5.33)
\end{align*}

or

\begin{align*}
D^k(u) &= 4; \quad (5.34) \\
D^k(u^4) &= (41111)^3; \quad (5.35)
\end{align*}

or

\begin{align*}
D^k(u) &= 1; \quad (5.36) \\
D^k(u^4) &= (11114)^3; \quad (5.37) \\
&= 1(41111)^3. \quad (5.38)
\end{align*}

**Case 2.2.1.** If Eqs. (5.32-5.33) hold then

**Case 2.2.1.1.** $\alpha = 1$. Then since Eqs. (5.32-5.33) satisfy Eq. (5.8), we have

\begin{align*}
D^{k-1}(u) &= \beta^i \tilde{\beta} \beta^j, \ 1 \leq i, j \leq 3; \\
D^{k-1}(u^4) &= \beta^i \tilde{\beta} \beta \tilde{\beta} \beta^j, \ 1 \leq i, j \leq 3, \ \beta = 1, 4. \quad (5.39, 5.40)
\end{align*}

Thus Eqs. (5.11, 5.39) mean that $D^{k-1}(u^4)$ has at least eight runs, contradicts Eq. (5.40).

**Case 2.2.1.2.** $\alpha = 4$. Then similarly, we have

\begin{align*}
D^{k-1}(u) &= \beta^i \tilde{\beta}^4 \beta^j, \ 0 \leq i, j \leq 3; \\
D^{k-1}(u^4) &= \beta^i \tilde{\beta}^4 \beta^4 \beta^j \beta^4 \beta^j, \ 0 \leq i, j \leq 3. \quad (5.41, 5.42)
\end{align*}
Thus Eqs. (5.11, 5.41) mean that $\beta^4$ occurs at least four times in $D^{k-1}(u^4)$, contradicts Eq. (5.42).

**Case 2.2.2.** If Eqs. (5.34-5.35) hold, then we obtain

$$D^{k-1}(u^4) = a^i\bar{a}a^j\alpha^i, 0 \leq i, j \leq 3$$

$$D^{k-1}(u^4) = a^i\bar{a}a^4\alpha^4\bar{a}a^i, 0 \leq i, j \leq 3$$

Hence, an argument similar to Case 2.2.1.2 leads to a contradiction.

**Case 2.2.3.** If Eqs. (5.36-5.37) hold then Eq. (5.39) holds, and

$$D^{k-1}(u^4) = a^i\alpha^4\bar{a}a^4\alpha^4\bar{a}a^4\alpha^4\bar{a}a^4a^i, 1 \leq i, j \leq 3.$$ (5.45)

Thus, Eqs. (5.11, 5.39, 5.45) mean that $\alpha = \beta$, and the last runs of Eqs. (5.39, 5.45) are same, which leads to a contradiction.

**Case 2.2.4.** If Eqs. (5.36, 5.38) hold then Eq. (5.39) holds, and

$$D^{k-1}(u^4) = a^i\alpha^4\bar{a}a^4\alpha^4\bar{a}a^4\alpha^4\bar{a}a^4a^i, 1 \leq i, j \leq 3.$$ (5.46)

Eqs. (5.11, 5.39, 5.46) suggest that $\alpha = \beta$, and the last runs of Eqs.(5.39, 5.46) are same, which leads to a contradiction.

**Case 2.3.** $D^k(u) = a^2$. Then by Eq. (5.8), we obtain

$$(\alpha^2x)^3\alpha^2 = D^k(u^4) \in C_{1,4}^{\infty}, \text{ where } x \in D_{1,4}.$$ 

A direct verification leads to

$$D^k(u) = 11;$$ (5.47)

$$D^k(u^4) = (1^4)^31^2;$$ (5.48)

$$= 1^2(4^4)^3;$$ (5.49)

$$= 1^3(4^4)^24^3;$$ (5.50)

or

$$D^k(u) = 44;$$ (5.51)

$$D^k(u^4) = 4^3(14^4)^21^3.$$ (5.52)

**Case 2.3.1.** If Eqs. (5.47 and 5.48, or 5.49, or 5.50) hold, then we obtain

$$D^{k-1}(u) = a^i\bar{a}a^i, 1 \leq i, j \leq 3;$$ (5.53)

$$D^{k-1}(u^4) = a^i\bar{a}a^4\alpha^4\bar{a}a^4\alpha^4\bar{a}a^4\alpha^4\bar{a}a^4a^i, 1 \leq i, j \leq 3;$$ (5.54)

$$= a^i\bar{a}a^4\alpha^4\bar{a}a^4\alpha^4\bar{a}a^4\alpha^4\bar{a}a^4\alpha^4\bar{a}a^4\alpha^4\bar{a}a^4a^i, 1 \leq i, j \leq 3;$$ (5.55)

$$= a^i\bar{a}a^4\alpha^4\bar{a}a^4\alpha^4\bar{a}a^4\alpha^4\bar{a}a^4\alpha^4\bar{a}a^4\alpha^4\bar{a}a^4\alpha^4\bar{a}a^4a^i, 1 \leq i, j \leq 3.$$ (5.56)
So, Eq. (5.11) implies that $D^{k-1}(u)$ and $D^{k-1}(u^4)$ have the same last run. But by Eqs. (5.53) and (5.54-5.56), this cannot occur.

Case 2.3.2. If Eqs. (5.51-5.52) hold, then we obtain

$$D^{k-1}(u) = \alpha^i\alpha^j\alpha^i, \ 0 \leq i, j \leq 3; \quad (5.57)$$
$$D^{k-1}(u^4) = \alpha^i\alpha^j\alpha^i\alpha^i\alpha^j, \ 0 \leq i, j \leq 3. \quad (5.58)$$

An argument similar to Case 2.3.1 leads to a contradiction.

Case 2.4. $D^k(u) = \alpha^3$. Then by Eq. (5.8) we have

$$(\alpha^3x)^3x^3 = D^k(u^4) \in C_{1,4}^\infty,$$

which means that

$$D^k(u^4) = (\alpha^3)^3x^3; \quad (5.59)$$
$$= \alpha^3(\alpha\alpha)^3. \quad (5.60)$$

Case 2.4.1. $\alpha = 1$. Then from Eqs. (5.59-5.60) and $D^k(u) = \alpha^3$, we obtain

$$D^{k-1}(u) = \beta^i\beta^j\beta^i\beta^j, \ 1 \leq i, j \leq 3; \quad (5.61)$$
$$D^{k-1}(u^4) = \beta^i\beta^j\beta^i\beta^j\beta^i\beta^j\beta^i\beta^j, \ 1 \leq i, j \leq 3; \quad (5.62)$$
$$= \beta^i\beta^j\beta^i\beta^j\beta^i\beta^j\beta^i\beta^j, \ 1 \leq i, j \leq 3. \quad (5.63)$$

An argument similar to Case 2.3.1 leads to a contradiction.

Case 2.4.2. $\alpha = 4$. Then by Eqs. (5.59-5.60) and $D^k(u) = \alpha^3$, we have

$$D^{k-1}(u) = \beta^i\beta^j\beta^i\beta^j, \ 0 \leq i, j \leq 3; \quad (5.64)$$
$$D^{k-1}(u^4) = \beta^i\beta^j\beta^i\beta^j\beta^i\beta^j\beta^i\beta^j\beta^i\beta^j\beta^i\beta^j\beta^i\beta^j, \ 1 \leq i, j \leq 3; \quad (5.65)$$
$$= \beta^i\beta^j\beta^i\beta^j\beta^i\beta^j\beta^i\beta^j\beta^i\beta^j\beta^i\beta^j\beta^i\beta^j, \ 1 \leq i, j \leq 3. \quad (5.66)$$

where $0 \leq s, t \leq 3$. An argument similar to Case 2.3.1 leads to a contradiction.

Case 2.5. $D^k(u) = \alpha^4$. Then by Eq. (5.8), we have

$$(\alpha^4x)^3x^4 = D^k(u^4) \in C_{1,4}^\infty,$$

A direct examination shows that there is no $x \in D_{1,4}$ such that Eq. (5.67) holds.

Case 3. $\Sigma = \{1, b\}, b \geq 5$. Assume on the contrary that there exists a $u \in C_{1,b}^\infty$ with $|u| \geq 2$ such that $u^{h(1,b)} \in C_{1,b}^\infty$. Then $u$ has at least two runs. Let $k$ be the maximal integer such that $D^{k-1}(u)$ has at least two runs. Then by Power Derivative Formula, we obtain

$$(D^j(u)x)^{h(1,b)-1}D^j(u) = D^j(u^{h(1,b)}) \in C_{1,b}^\infty \text{ for } 1 \leq j \leq k, \text{ where } x \in D_{1,b}, \quad (5.68)$$
and \( D^k(u) \) has at most one run. Therefore
\[
D^k(u) = \alpha^i, \text{ where } i = 0, 1, \ldots, b.
\]

**Case 3.1.** \( D^k(u) = \varepsilon \). Then from Eq. \((5.68)\) we obtain
\[
D^k(u^{h(1,b)}) = x^{h(1,b)} - 1 \in C^\infty_{1,b}, \text{ where } x \in D_{1,b}. \tag{5.69}
\]

**Case 3.1.1.** \( 2 \nmid b \). Then since \( b \geq 5 \), we have \( h(1,b) = \frac{b^4 + 5}{2} \leq b \) and \( 2(h(1,b) - 1) - 2 = b + 1 \). By virtue of Eq. \((5.69)\), a direct verification leads to
\[
D^k(u^{h(1,b)}) = \varepsilon, 1^{h(1,b)} - 1, b^{h(1,b)} - 1. \tag{5.70}
\]

Hence, on the one hand, in view of \( r(w) \leq |D(w)| + 2 \) (see page 8), from Eq. \((5.70)\) we obtain
\[
r(D^{-1}(u^{h(1,b)})) \leq |D^k(u^{h(1,b)})| + 2 \leq b + 1. \tag{5.71}
\]

On the other hand, from Eq. \((5.68)\) \((j = k - 1)\), it follows that
\[
D^{-1}(u^{h(1,b)}) = (D^{-1}(u)x)^{h(1,b) - 1}D^{-1}(u) \in C^\infty_{1,b}, \text{ where } x \in D_{1,b}. \tag{5.72}
\]

Since \( D^{-1}(u) \) has at least two runs, by Eq. \((5.72)\), we have
\[
r(D^{-1}(u^{h(1,b)})) \geq 2h(1,b) = b + 5,
\]

contradicts Eq. \((5.71)\).

**Case 3.1.2.** \( 2 \mid b \). Then since \( b \geq 5 \), we have \( h(1,b) = \frac{b^4 + 4}{2} < b \) and \( 2(h(1,b) - 1) - 2 = b \). Thus from Eq. \((5.69)\), a direct check leads to
\[
D^k(u^{h(1,b)}) = \varepsilon, 1^{h(1,b)} - 1, b^{h(1,b)} - 1, (1b)^{h(1,b)} - 1, (b1)^{h(1,b)} - 1. \tag{5.73}
\]

By Case 3.1.1, we only need to verify the last two cases. Since \( D^{-1}(u) \) has at least two runs, we have
\[
D^{-1}(u) = \alpha^i\bar{\alpha}^j, 1 \leq i, j \leq b - 1, \tag{5.74}
\]

and from the last two cases of Eq. \((5.73)\), we obtain
\[
D^{-1}(u^{h(1,b)}) = \beta^s(\beta^b)^{h(1,b) - 1} \beta^t, 1 \leq s \leq b - 1, 0 \leq t \leq b - 1; \tag{5.75}
\]
\[
= \beta^s(\beta^b)^{h(1,b) - 1} \beta^t, 0 \leq s \leq b - 1, 1 \leq t \leq b - 1. \tag{5.76}
\]

**Case 3.1.2.1.** If Eqs. \((5.74-5.75)\) hold, then since Eqs. \((5.74-5.75)\) satisfy Eq. \((5.72)\), we have \( \alpha = \beta, i = s, j = t = 1 \) and \( x = \alpha^{b-s} \in D_{1,b} \). Thus, from Eqs. \((5.74, 5.75)\) it follows that
\[
D^{-1}(u) = \alpha^s \bar{\alpha}, 1 \leq s \leq b - 1; \tag{5.77}
\]
\[
D^{-1}(u^{h(1,b)}) = \alpha^s(\alpha \bar{\alpha})^{h(1,b) - 1} \alpha, 1 \leq s \leq b - 1 \tag{5.78}
\]
Therefore, by \( x = \alpha^{b-s} \in D_{1,b} \) and \( 1 \leq s \leq b - 1 \), we can get \( b - s = 1, 2, 3, 4 \).

**Case 3.1.2.1.1.** \( b - s = 1 \). Then \( s = b - 1 \), and from Eqs. (5.77-5.78), we obtain

\[
D^{k-1}(u) = \alpha^{b-1}\bar{\alpha}; \\
D^{k-1}(u^{h(1,b)}) = \alpha^{b-1}(\bar{\alpha}\alpha^{b})^{h(1,b)-1}\bar{\alpha}. \tag{5.79}
\]

From Eq. (5.68) \( (j = k - 2) \) we obtain

\[
(D^{k-2}(u)x)^{h(1,b)-1}D^{k-2}(u) = D^{k-2}(u^{h(1,b)}) \in C^{\infty}_{1,b}, \text{ where } x \in D_{1,b}. \tag{5.81}
\]

If \( \alpha = 1 \) then since \( D^{k-2}(u) \) and \( D^{k-2}(u^{h(1,b)}) \) satisfy Eq. (5.81), by Eqs. (5.79-5.80), we get

\[
D^{k-2}(u) = \beta^{i}\beta^{j}\ldots\beta^{b}\beta\beta^{j}, \ 1 \leq i \leq b - 1, \ 0 \leq j \leq b - 1. \tag{5.82}
\]

If \( 2 \mid (h(1,b) - 1) \) then

\[
D^{k-2}(u^{h(1,b)}) = \beta^{i}\beta^{j}\ldots\beta^{b}\beta\beta^{j}, \ 1 \leq i \leq b - 1, \ 0 \leq j \leq b - 1.
\]

If \( 2 \nmid (h(1,b) - 1) \) then

\[
D^{k-2}(u^{h(1,b)}) = \beta^{i}\beta^{j}\ldots\beta^{b}\beta\beta^{j}, \ 1 \leq i \leq b - 1, \ 0 \leq j \leq b - 1. \tag{5.83}
\]

where \( 1 \leq i \leq b - 1, \ 0 \leq j \leq b - 1 \).

If \( 2 \mid (h(1,b) - 1) \) then

\[
D^{k-2}(u^{h(1,b)}) = \beta^{i}\beta^{j}\ldots\beta^{b}\beta\beta^{j}, \ 1 \leq i \leq b - 1, \ 0 \leq j \leq b - 1. \tag{5.84}
\]

where \( 1 \leq i \leq b - 1, \ 0 \leq j \leq b - 1 \). Thus, in virtue of Eqs. (5.82) and (5.83 or 5.84), and comparing the number of the factor \( \beta^{b} \) of two sides of Eq. (5.81), we approach a contradiction.

If \( \alpha = b \) then similarly, by Eqs. (5.79-5.80), we get

\[
D^{k-2}(u) = \beta^{i}\beta^{b}\beta^{i}\ldots\beta^{b}\beta\beta^{j}, \ 0 \leq i \leq b - 1, \ 1 \leq j \leq b - 1. \tag{5.85}
\]

If \( 2 \mid (h(1,b) - 1) \) then

\[
D^{k-2}(u^{h(1,b)}) = \beta^{i}\beta^{b}\beta^{i}\ldots\beta^{b}\beta\beta^{j}, \ 1 \leq i \leq b - 1, \ 0 \leq j \leq b - 1.
\]

If \( 2 \nmid (h(1,b) - 1) \) then

\[
D^{k-2}(u^{h(1,b)}) = \beta^{i}\beta^{b}\beta^{i}\ldots\beta^{b}\beta\beta^{j}, \ 1 \leq i \leq b - 1, \ 0 \leq j \leq b - 1. \tag{5.86}
\]

where \( 0 \leq i \leq b - 1, \ 1 \leq j \leq b - 1 \).

If \( 2 \mid (h(1,b) - 1) \) then

\[
D^{k-2}(u^{h(1,b)}) = \beta^{i}\beta^{b}\beta^{i}\ldots\beta^{b}\beta\beta^{j}, \ 1 \leq i \leq b - 1, \ 0 \leq j \leq b - 1. \tag{5.87}
\]
where \(0 \leq i \leq b - 1, 1 \leq j \leq b - 1\). Thus, by virtue of Eqs. (5.85) and (5.86 or 5.87), and comparing the number of the disjoint factor \(\overline{b^\beta b^\beta \cdots b^\beta \beta \bar{\beta}}\) of two sides of Eq. (5.81), we can get a contradiction.

**Case 3.1.2.1.2.** \(b - s = 2\). Then \(s = b - 2\), and from Eqs. (5.77-5.78), we obtain

\[
D^{k-1}(u) = \alpha^{b-2} \bar{\alpha}; \quad (5.88)
\]
\[
D^{k-1}(u^{h(1,b)}) = \alpha^{b-2}(\bar{\alpha} \alpha^b)^{h(1,b)-1} \bar{\alpha}. \quad (5.89)
\]

If \(\alpha = 1\) then since \(D^{k-2}(u)\) and \(D^{k-2}(u^{h(1,b)})\) satisfy Eq. (5.81), by Eqs. (5.88-5.89) we get

\[
D^{k-2}(u) = \, \beta^i \beta^\beta \cdots \beta^b \beta^\beta, 1 \leq i \leq b - 1, 0 \leq j \leq b - 1. \quad (5.90)
\]

If \(2 \mid (h(1,b) - 1)\) then

\[
D^{k-2}(u^{h(1,b)}) = \beta^i \beta^\beta \cdots \beta^b \beta^\beta (\beta \beta \cdots \beta \beta \beta \beta \cdots \beta) \frac{b^{h(1,b)-1}}{2} \beta^\beta \beta^\beta, \quad (5.91)
\]

where \(1 \leq i \leq b - 1, 0 \leq j \leq b - 1\).

If \(2 \nmid (h(1,b) - 1)\) then

\[
D^{k-2}(u^{h(1,b)}) = \beta^i \beta^\beta \cdots \beta^b \beta^\beta (\beta \beta \cdots \beta \beta \beta \beta \cdots \beta) \frac{b^{h(1,b)-2}}{2} \beta^\beta \beta^\beta \beta^\beta, \quad (5.92)
\]

where \(1 \leq i \leq b - 1, 0 \leq j \leq b - 1\). Thus by Eqs. (5.90) and (5.91 or 5.92), and comparing the number of the factor \(\beta^b\) of two sides of Eq. (5.81), we arrive at a contradiction.

If \(\alpha = b\) then analogously, by Eqs. (5.88-5.89), we have

\[
D^{k-2}(u) = \, \beta^i \beta^b \beta^b \cdots \beta^b \beta^b, 0 \leq i \leq b - 1, 1 \leq j \leq b - 1. \quad (5.93)
\]

If \(2 \mid (h(1,b) - 1)\) then

\[
D^{k-2}(u^{h(1,b)}) = \beta^i \beta^b \beta^b \cdots \beta^b \beta^b (\beta \beta \cdots \beta \beta \beta \beta \cdots \beta) \frac{b^{h(1,b)-1}}{2} \beta^b \beta^b \beta^b, \quad (5.94)
\]

where \(0 \leq i \leq b - 1, 1 \leq j \leq b - 1\).

If \(2 \nmid (h(1,b) - 1)\) then

\[
D^{k-2}(u^{h(1,b)}) = \beta^i \beta^b \beta^b \cdots \beta^b \beta^b (\beta \beta \cdots \beta \beta \beta \beta \cdots \beta) \frac{b^{h(1,b)-2}}{2} \beta^b \beta^b \beta^b \beta^b \beta^b, \quad (5.95)
\]
where \(0 \leq i \leq b - 1, 1 \leq j \leq b - 1\). Thus, in view of Eqs. (5.93) and (5.94 or 5.95), and comparing the number of the disjoint factor \(\beta^b \beta^b \cdots \beta^b \beta^b \beta^b\) of two sides of Eq. (5.81), we can reach a contradiction.

**Case 3.1.2.1.3.** \(b - s = 3\). Then \(s = b - 3, \alpha = 1\), and from Eqs. (5.77-5.78), we obtain

\[
D^{k-1}(u) = 1^{b-3}b; \\
D^{k-1}(u^{(1,b)}) = 1^{b-3}b^{1^{h(1,b)}-1}b. 
\]

(5.96)

(5.97)

Since \(D^{k-2}(u)\) and \(D^{k-2}(u^{h(1,b)})\) satisfy Eq. (5.81), from Eqs. (5.96-5.97), we get

\[
D^{k-2}(u) = \beta^i \beta^i \beta^b \cdots \beta^b \beta^i, \quad 1 \leq i \leq b - 1, \quad 0 \leq j \leq b - 1. 
\]

(5.98)

If \(2 \mid (h(1,b) - 1)\) then

\[
D^{k-2}(u^{h(1,b)}) = \beta^i \beta^i \beta^b \cdots \beta^b \beta^{h(1,b)-1} \beta^b \beta^i, 
\]

where \(1 \leq i \leq b - 1, \quad 0 \leq j \leq b - 1\).

If \(2 \nmid (h(1,b) - 1)\) then

\[
D^{k-2}(u^{h(1,b)}) = \beta^i \beta^i \beta^b \cdots \beta^b \beta^{h(1,b)-2} \beta^b \beta \cdots \beta^b \beta^i, 
\]

where \(1 \leq i \leq b - 1, \quad 0 \leq j \leq b - 1\). Thus, in view of Eqs. (5.98) and (5.99 or 5.100), and comparing the number of the factor \(\beta^b\) of two sides of Eq. (5.81), we attain a contradiction.

**Case 3.1.2.1.4.** \(b - s = 4\). Then \(s = b - 4, \alpha = 1\), and from Eqs. (5.77-5.78), we obtain

\[
D^{k-1}(u) = 1^{b-4}b; \\
D^{k-1}(u^{(1,b)}) = 1^{b-4}b^{1^{h(1,b)}-1}b. 
\]

(5.101)

(5.102)

Since \(D^{k-2}(u)\) and \(D^{k-2}(u^{h(1,b)})\) satisfy Eq. (5.81), from Eqs. (5.101-5.102), we get

\[
D^{k-2}(u) = \beta^i \beta^i \beta^b \cdots \beta^b \beta^i, \quad 1 \leq i \leq b - 1, \quad 0 \leq j \leq b - 1. 
\]

(5.103)

If \(2 \mid (h(1,b) - 1)\) then

\[
D^{k-2}(u^{h(1,b)}) = \beta^i \beta^i \beta^b \cdots \beta^b \beta^{h(1,b)-1} \beta^b \beta^i, 
\]

(5.104)
where $1 \leq i \leq b - 1$, $0 \leq j \leq b - 1$.

If $2 \nmid (h(1, b) - 1)$ then

$$D^{k-2}(u^{h(1,b)}) = \beta^4 \beta \cdots \beta(\beta^b \beta \cdots \beta) \frac{h(1, b) - 2}{2} \beta^b \beta \cdots \beta \beta^b \beta^j, \quad (5.105)$$

where $1 \leq i \leq b - 1$, $0 \leq j \leq b - 1$. Thus, by virtue of Eqs. (5.103) and (5.104 or 5.105), and comparing the number of the factor $\beta^b$ of two sides of Eq. (5.81), we can arrive at a contradiction.

**Case 3.1.2.2.** If Eq. (5.76) holds, then by Eqs. (5.74, 5.76) and $\overline{D}_{1,b} = D_{1,b}$, we easily see that $D^{k-1}(\overline{u})$ and $D^{k-1}((\overline{u})^{h(1,b)})$ are of form Eqs. (5.74, 5.75) and satisfy the Eq. (5.72). Thus, by Case 3.1.2.1, we also reach a contradiction.

**Case 3.2.** $D^k(u) = \alpha^i$, where $1 \leq i \leq b$. Then from Eq. (5.68) ($j = k$) it follows that

$$(\alpha^i x)^{h(1,b) - 1} \alpha^i \in C_{1,b}^\infty, \text{ where } x \in D_{1,b}, \ i = 1, 2, \ldots, b. \quad (5.106)$$

**Case 3.2.1.** $|x| = 1$. Then by Eq. (5.106), we have $x = \bar{\alpha}$ and $i = 1$ or $b$, which means that

$$(\alpha \bar{\alpha})^{h(1,b) - 1} \alpha \text{ or } (\alpha^b \bar{\alpha})^{h(1,b) - 1} \alpha^b \in C_{1,b}^\infty.$$ 

Since $1^{2(h(1,b) - 1) - 1} = D((\alpha \bar{\alpha})^{h(1,b) - 1} \alpha) = D((b1)^{h(1,b) - 1} b) = D^2((\alpha b \bar{\alpha})^{h(1,b) - 1} \alpha^b) \in C_{1,b}^\infty$ and $h(1,b) \geq \frac{b+1}{2}$, so in any case, we obtain $b + 1 \leq 2(h(1,b) - 1) - 1 \leq b$, a contradiction.

**Case 3.2.2.** $|x| = 2$. Then by Eq. (5.106) and $b \geq 5$, we have

$$x = \bar{\alpha} \alpha, \ \alpha \bar{\alpha} \text{ and } i = b - 1,$$

which by Eq. (5.68) ($j = k$), means that

$$D^k(u) = \alpha^{b-1}; \quad (5.107)$$
$$D^k(u)^{h(1,b)} = \alpha^{b-1}(\bar{\alpha} \alpha)^{h(1,b) - 1}; \quad (5.108)$$
$$= (\alpha^b \bar{\alpha})^{h(1,b) - 1} \alpha^{b-1}. \quad (5.109)$$

Since $1^{2(h(1,b) - 2)} = D^2((\alpha^b \bar{\alpha})^{h(1,b) - 1} \alpha^{b-1}) = D^2(\alpha^{b-1}(\bar{\alpha} \alpha)^{h(1,b) - 1} \alpha^b) \in C_{1,b}^\infty$, we obtain $2(h(1,b) - 2) \leq b$. Thus if $b$ is an odd integer, then $b + 1 = 2(h(1,b) - 2) \leq b$, a contradiction.

If $b$ is an even number then $b \geq 6$.

**Case 3.2.2.1.** $\alpha = 1$ and $2 \mid (h(1,b) - 1)$. Then since $D^{k-1}(u)$ and $D^{k-1}(u^{h(1,b)})$
satisfy Eq. (5.72), from Eqs. (5.107-5.109), we obtain

\[ D^{k-1}(u) = \beta^i \beta \beta \cdots \beta \beta^j; \]  

\[ D^{k-1}(u^{h(1,b)}) = \beta^i \beta \beta \cdots \beta (\beta^b \beta \beta \cdots \beta \beta^b \beta \beta \cdots \beta)^{h(1,b)-1} \beta^j; \]  

\[ = \beta^i (\beta \beta \cdots \beta \beta^b \beta \beta \cdots \beta \beta^b \beta^b \beta^b \beta^b \beta^b \beta \beta \cdots \beta^j), \]  

where \(1 \leq i, j \leq b - 1\).

If Eqs. (5.110-5.111) hold, then from Eqs. (5.110-5.111) satisfying Eq. (5.72), and comparing the \((b+1)\)-th run of the right sides of Eqs. (5.110 and 5.111), it follows that \(j = b\), a contradiction to the fact \(1 \leq j \leq b - 1\).

If Eqs. (5.110-5.112) hold, then one easily see that \(D^{k-1}(\bar{u})\) and \(D^{k-1}(\bar{u}^{h(1,b)})\) are respectively of form Eqs. (5.110-5.111). Thus, we again arrive at a contradiction.

**Case 3.2.2.2.** \(\alpha = 1\) and \(2 \nmid (h(1,b) - 1)\). Then similarly, from Eqs. (5.108-5.109), we obtain

\[ D^{k-1}(u^{h(1,b)}) = \beta^i \beta \beta \cdots \beta (\beta^b \beta \beta \cdots \beta \beta^b \beta \beta \cdots \beta)^{h(1,b)-2} \beta \beta \cdots \beta \beta^j; \]  

\[ = \beta^i (\beta \beta \cdots \beta \beta^b \beta \beta \cdots \beta \beta^b \beta \beta \cdots \beta \beta^j), \]  

where \(1 \leq i, j \leq b - 1\). Therefore, an argument similar to Case 3.2.2.1 leads to a contradiction.

**Case 3.2.2.3.** \(\alpha = b\) and \(2 \mid (h(1,b) - 1)\). Then since \(D^{k-1}(u)\) and \(D^{k-1}(u^{h(1,b)})\) satisfy Eq. (5.72), by Eqs. (5.107-5.109), we get

\[ D^{k-1}(u) = \beta^i \beta^b \beta \beta^b \cdots \beta \beta^b; \]  

\[ D^{k-1}(u^{h(1,b)}) = \beta^i (\beta^b \beta \beta^b \cdots \beta \beta^b \beta \beta \cdots \beta \beta^b \beta^b \beta^b \beta^b \beta^b \beta \beta \cdots \beta^{h(1,b)-1} \beta \beta^b \cdots \beta \beta^j; \]  

\[ = \beta^i \beta^b \beta^b \cdots \beta (\beta \beta^b \beta \beta^b \cdots \beta \beta^b \beta \beta \cdots \beta \beta^b \beta^b \beta^b \beta^b \beta^b \beta \beta \cdots \beta^{h(1,b)-1} \beta \beta^j), \]  

where \(0 \leq i, j \leq b - 1\).

If Eqs. (5.115-5.116) hold, then (5.115, 5.116) satisfy Eq. (5.72). So by comparing the \((n+1)\)-th runs of Eqs. (5.115-5.116), we have \(x = \beta^{b-j} \beta \beta^b \beta \beta^b \cdots \beta \beta^b \beta \beta \cdots \beta \beta^b \beta \beta \cdots \beta \beta^b \beta \beta \cdots \beta \beta^b \beta \beta \cdots \beta \beta^b \beta \beta \cdots \beta \beta^j), \) contradicts the fact that the elements of \(D_{1,b}\) has at most two runs. Similarly, Eqs. (5.115, 5.117) also lead to a contradiction.
**Case 3.2.2.4.** $\alpha = b$ and $2 \nmid (h(1, b) - 1)$. Then analogously, from Eqs. (5.108-5.109), we obtain

\[
D^{k-1}(u^{h(1,b)}) = \beta^i(\beta \beta^b \cdots \beta^b \beta \beta \beta^b \cdots \beta^b \beta)^{\frac{h(1,b)-2}{2}} \beta^b \beta^b \cdots \beta^b \beta^j;
\]

where $0 \leq i, j \leq b - 1$. An argument similar to Case 3.2.2.3 leads to a contradiction.

**Case 3.2.3.** $|x| = 3$. Then by Eq. (5.106) and $b \geq 5$, we have

\[
x = 1^2b, \ b^1 \text{ and } i = b - 2,
\]

which by Eq. (5.68) ($j = k$), means that

\[
D^k(u) = 1^{b-2}; \quad (5.118)
\]

\[
D^k(u^{h(1,b)}) = (1^b b)^{h(1,b)-1} 1^{b-2}; \quad (5.119)
\]

\[
= 1^{b-2}(b^1 b)^{h(1,b)-1}. \quad (5.120)
\]

Thus, by Eqs. (5.119 and 5.120), we have $1^{2(h(1,b)-2)} = D^2((1^b b)^{h(1,b)-1} 1^{b-2}) = D^2((1^{b-2}(b^1 b)^{h(1,b)-1}) = D^{k+2}(u^{h(1,b)}) \in C_{1,b}^\infty$, which means that $2(h(1,b) - 2) \leq b$.

Therefore, if $b$ is an odd integer, then by Eq. (5.1), $b + 1 = 2(h(1,b) - 2) \leq b$, a contradiction.

If $b$ is an even number then

**Case 3.2.3.1.** $2 \mid (h(1, b) - 1)$. Then since $D^{k-1}(u)$ and $D^{k-1}(u^{h(1,b)})$ satisfy Eq. (5.72), from Eqs. (5.118-5.120), we obtain

\[
D^{k-1}(u) = \beta^i \beta \beta^b \cdots \beta^b \beta^j; \quad (5.121)
\]

\[
D^{k-1}(u^{h(1,b)}) = \beta^i(\beta^b \beta \beta^b \cdots \beta^b \beta^b \beta^b \cdots \beta^b \beta)^{\frac{h(1,b)-1}{2}} \beta \beta \cdots \beta^b \beta^j; \quad (5.122)
\]

\[
= \beta^i \beta \beta^b \cdots \beta(\beta^b \beta^b \cdots \beta^b \beta^b \beta^b \cdots \beta)^{\frac{h(1,b)-1}{2}} \beta^j, \quad (5.123)
\]

where $1 \leq i, j \leq b - 1$. Since Eqs. (5.121, 5.122) and Eqs. (5.121, 5.123) satisfy Eq. (5.72), by comparing the $b$-th runs of the right sides of Eqs. (5.121, 5.122), we have $j = 1$ and $x = \beta \beta^b y \in D_{1,b}$, contradicts the fact which the length of the elements of $D_{1,b}$ is less than 5. Similarly, from Eqs. (5.121, 5.123), we can arrive at a contradiction.
Case 3.2.3.2. \(2 \mid (h(1, b) - 1)\). Then similarly, from Eqs. (5.119-5.120), we obtain

\[
D^{k-1}(u^{h(1,b)}) = \beta^i(\bar{\beta} \cdots \bar{\beta} \bar{\beta}^b \bar{\beta} \cdots \bar{\beta}^b)^{\frac{h(1,b)-2}{2}} \bar{\beta} \cdots \bar{\beta} \bar{\beta} \cdots \bar{\beta}^i;
\]

where \(1 \leq i, j \leq b - 1\). Thus, an argument similar to Case 3.2.3.1 leads to a contradiction.

Case 3.2.4. \(|x| = 4\). Then by \(x \in D_{1,b}\), we see that \(x = 1^4\). Since \(b \geq 5\), it is obvious that \((\alpha i^4)^{b-1} \notin C_{1,b}^\infty\) for \(i = 1, 2, \ldots, b\), a contradiction.

Case 4. \(\Sigma = \{2, b\}\). Suppose to the contrary that there exists a \(u \in C_{2,b}^\infty\) with \(|u| \geq 2\) such that \(u^{h(2,b)} \in C_{2,b}^\infty\). Then \(u\) has at least two runs. Let \(k\) be the maximal integer such that \(D^{k-1}(u)\) has at least two runs. Then by Power Derivative Formula, we obtain

\[
(D^i(u)x)^{h(2,b)-1}D^j(u) = D^i(u^{h(2,b)}) \in C_{2,b}^\infty\text{ for }1 \leq j \leq k, \text{ where } x \in D_{2,b}, \quad (5.124)
\]

and \(D^k(u)\) has at most one run. Therefore

\[
D^k(u) = \alpha^i, \text{ where } i = 0, 1, \ldots, b.
\]

Case 4.1. \(D^k(u) = \varepsilon\). Then by Eq. (5.124), we have

\[
x^{h(2,b)-1} = D^k(u^{h(2,b)}) \in C_{2,b}^\infty, \text{ where } x \in D_{2,b}. \quad (5.125)
\]

Note that \(b \geq 3\) and \(h(2,b) \geq \frac{b+3}{2} \geq 3\). From Eq. (5.125), it follows that

\[
D^k(u^{h(2,b)}) = \varepsilon, \ 2^{h(2,b)-1}, \ b^{h(2,b)-1}. \quad (5.126)
\]

From Eqs. (5.124) \((j = k - 1)\) and (5.126), we obtain

\[
(D^{k-1}(u)x)^{h(2,b)-1}D^{k-1}(u) = \alpha^i \alpha^{2}, \ 0 \leq i, j \leq b - 1; \quad (5.127)
\]

\[
= \alpha^i \alpha^{2} \alpha^2 \cdots \alpha^2 \bar{\beta}^j, \ 1 \leq i, j \leq b - 1; \quad (5.128)
\]

\[
= \alpha^i \alpha^b \alpha^b \cdots \bar{\beta}^j, \ 0 \leq i, j \leq b - 1. \quad (5.129)
\]

Since \(D^{k-1}(u)\) has at least two runs, the left sides of Eqs. (5.127-5.129) have at least \(2h(2,b)\) runs, but the right sides of Eqs. (5.127-5.129) have at most \(h(2,b) + 1\) runs, which means that \(h(2,b) \leq 1\), a contradiction to \(h(2,b) \geq 3\).

Case 4.2. \(D^k(u) = \alpha^i, 1 \leq i \leq b\). Then by Eq. (5.124) \((j = k)\), we have

\[
(\alpha^i x)^{h(2,b)-1} \alpha^i = D^k(u^{h(2,b)}) \in C_{2,b}^\infty, \text{ where } x \in D_{2,b}. \quad (5.130)
\]

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Note that \( i \geq 1 \) and \( h(2, b) \geq 3 \). From Eq. (5.130), it follows that \( x = \varepsilon \) and \( i = 1 \), that is,
\[
D^k(u) = \alpha; \quad D^k(u^{h(2,b)}) = \alpha^{h(2,b)} .
\] (5.131) (5.132)
Thus, on the one hand, by Eq. (5.132), we see that \( D^{k-1}(u^{h(2,b)}) \) has at most \( h(2, b) + 2 \) runs. On the other hand, since \( D^{k-1}(u) \) has at least two runs, by Eq. (5.124) \((j = k - 1)\), we see that \( D^{k-1}(u^{h(2,b)}) = (D^{k-1}(u)x)^{h(2,b)-1}D^{k-1}(u) \) has at least \( 2h(2, b) \) runs. Therefore, \( h(2, b) \leq 2 \), contradicts \( h(a, b) \geq 3 \).

**Case 5.** \( \Sigma = \{a, b\} \), where \( b > a \geq 3 \). Suppose on the contrary that there exists a \( u \in C_{a,b}^{\infty} \) with \( |u| \geq 2 \) such that \( u^{h(a,b)} \in C_{a,b}^{\infty} \). Then \( u \) has at least two runs. Let \( k \) be the maximal integer such that \( D^{k-1}(u) \) has at least two runs. Then by Power Derivative Formula, we obtain
\[
(D^j(u)x)^{h(a,b)-1}D^j(u) = D^j(u^{h(a,b)}) \in C_{a,b}^{\infty} \text{ for } 1 \leq j \leq k, \text{ where } x \in D_{a,b}, \quad (5.133)
\]
and \( D^k(u) \) has at most one run. Therefore
\[
D^k(u) = \alpha^i, \text{ where } i = 0, 1, \ldots, b.
\]

**Case 5.1.** \( D^k(u) = \varepsilon \). Then by Eq. (5.133) \((j = k)\), we have
\[
D^k(u^{h(a,b)}) = x^{h(a,b)-1} \in C_{a,b}^{\infty}, \text{ where } x \in D_{a,b}. \quad (5.134)
\]
Note that \( b \geq 4 \) and \( h(a,b) \geq 4 \). From Eq. (5.134), it follows that
\[
D^k(u^{h(a,b)}) = \varepsilon, \quad a^{h(a,b)-1}, \quad b^{h(a,b)-1}. \quad (5.135)
\]
From Eqs. (5.133) \((j = k - 1)\) and (5.135), we obtain
\[
(D^{k-1}(u)x)^{h(a,b)-1}D^{k-1}(u) = \alpha^i \bar{\alpha}^j, \quad 0 \leq i, j \leq b - 1;
\]
\[
= \alpha^i \bar{\alpha}^j \underbrace{\alpha^b \cdots \beta^b}_{h(a,b)-1} \bar{\beta}^j, \quad 1 \leq i, j \leq b - 1;
\]
\[
= \alpha^i \bar{\alpha}^j \underbrace{\alpha^b \cdots \beta^b}_{h(a,b)-1} \bar{\beta}^j, \quad 0 \leq i, j \leq b - 1. \quad (5.136)
\]
Since \( D^{k-1}(u) \) has at least two runs, the left sides of Eqs. (5.136-5.138) have at least \( 2h(2, b) \) runs, but the right sides of Eqs. (5.136-5.138) have at most \( h(a,b) + 1 \) runs, which means that \( h(a,b) \leq 1 \), a contradiction to \( h(a,b) \geq 4 \).

**Case 5.2.** \( D^k(u) = \alpha^i, \quad 1 \leq i \leq b \). Then by Eq. (5.133) \((j = k)\), we have
\[
(\alpha^i x)^{h(a,b)-1} \alpha^i = D^k(u^{h(2,b)}) \in C_{a,b}^{\infty}, \text{ where } x \in D_{a,b}. \quad (5.139)
\]
Note that \( i \geq 1 \) and \( h(a, b) \geq 4 \). From Eq. (5.139), it follows that \( x = \varepsilon \) and \( i = 1 \), that is,
\[
D^k(u) = \alpha; \\
D^k(u^{h(a,b)}) = \alpha^{h(a,b)}.
\] (5.140) (5.141)

Thus, on the one hand, by Eq. (5.141), we see that \( D^{k-1}(u^{h(a,b)}) \) has at most \( h(a, b)+2 \) runs. On the other hand, since \( D^{k-1}(u) \) has at least two runs, by Eq. (5.133) \((j = k-1)\), we see that \( D^{k-1}(u^{h(a,b)}) = (D^{k-1}(u)x^{h(a,b)})^{-1}D^{k-1}(u) \) has at least \( 2h(a, b) \) runs. Therefore, \( h(a, b) \leq 2 \), contradicts \( h(a, b) \geq 4 \).

(2) From Eq. (5.1), it immediately follows that if \( b \geq 5 \) then \( h(a, b) \leq b \), and \( h(1, 2) = 3 \), \( h(1, 3) = 5 \), \( h(1, 4) = 4 \), \( h(2, 3) = 3 \), \( h(2, 4) = 4 \), \( h(3, 4) = 4 \). Hence, except for the case \( a = 1 \), \( b = 3 \), we have \( h(a, b) \leq b+1 \). Moreover, since \( (1^33131^33)^4 \in C_{1,3}^\infty \) and \( 1^b \in C_{a,b}^\infty \). Thus, by Eq. (5.2), the assertion (2) holds. □

6. The number of smooth power words

Note that if \(|u|\) is an even number, then first\((\Delta^{-1}_a(u))\) ≠ last\((\Delta^{-1}_a(u))\). Thus from the definition of smooth words, we can obtain the useful result of the operator \( \Delta^{-1}_a (\alpha = a \) or \( b) \) as below.

**Lemma 15.** Let \( u \in \Sigma_{a,b}^+ \), where \( a, b \) have the same parity, and \( u \) has even length. Then

1. \( \Delta^{-k}_a(u^n) = (\Delta^{-k}_a(u))^n \) and \( \Delta^{-k}_a(u) \) has even length for \( \forall k, n \in N \);
2. If \( u^n \in C_{a,b}^\infty \) then \( (\Delta^{-k}_a(u))^n \in C_{a,b}^\infty \) for \( \forall k, n \in N \).

**Proof.** (1) Since \( u \) has even length and \( a, b \) have the same parity, we readily see that \( \Delta^{-1}_a(u) \) also has even length and \( \Delta^{-1}_a(u^n) = (\Delta^{-1}_a(u))^n \) for \( \alpha \in \Sigma \), which suggest that the assertion (1) holds for \( k = 1 \).

Now suppose that the assertion (1) holds for \( k = m (\geq 1) \). Then we easily see that \( \alpha^{-m}(u) \) has even length and \( \Delta^{-m}_a(u^n) = (\Delta^{-m}_a(u))^n \), which imply that \( \Delta^{-1}_a(u) = \Delta^{-1}_a(u^{-m}(u)) \) still has even length and
\[
\Delta^{-1}(u^n) = \Delta^{-1}_a(\Delta^{-m}_a(u^n)) \\
= \Delta^{-1}_a((\Delta^{-m}_a(u))^n) \\
= (\Delta^{-1}_a(\Delta^{-m}_a(u)))^n \\
= (\Delta^{-1}_a(\Delta^{-m}_a(u)))^n
\]
that is, the assertion (1) also holds for $k = m + 1$.

(2) Since $u^n = \Delta^k(\Delta^{-k}((u^n)))$ and $u^n \in C_{a,b}^\infty$, by Lemma 7 (3), we can get $\Delta^{-k}(u^n) \in C_{a,b}^\infty$. □

Now we are in a position to prove the following significative result.

**Theorem 16.** Let $\gamma_{a,b}(n)$ denote the number of smooth words of form $u^n$ over the 2-letter alphabet $\{a, b\}$, then

1. if $a, b$ have the same parity, then except for $a = 1, b = 3$,

$$\gamma_{a,b}(n) = \begin{cases} 0, & n > b \\ 2, & h(a,b) \leq n \leq b \\ \infty, & n < h(a,b) \end{cases} \quad (6.1)$$

2. if $a = 1, b \neq 3$ and $2 \nmid b$, then

$$\gamma_{1,b}(n) = \begin{cases} 0, & n \geq 4 \\ \infty, & n \leq 4 \end{cases} \quad (6.2)$$

3. if $a, b$ have the different parity, then

$$\gamma_{a,b}(n) = \begin{cases} 0, & n > b \\ 2, & h(a,b) \leq n \leq b \\ \infty, & n = 1 \end{cases} \quad (6.3)$$

4. if $a = 1, b \neq 3$ and $2 \nmid b$, then

$$\gamma_{1,b}(n) = \begin{cases} 0, & n \geq 3 \\ 46, & n = 2 \\ \infty, & n = 1 \end{cases} \quad (6.4)$$

**Proof.** By Theorem 14, we easily comprehend that

$$\gamma_{a,b}(n) = \begin{cases} 0, & n \geq \delta(a,b) \\ 2, & h(a,b) \leq n < \delta(a,b) \end{cases} \quad (6.5)$$

(1) Note that by the definitions of $h(a,b)$ (Eq. 5.1), if $2 \mid b$ then $2(h(a,b) - 1) - 2 = b$, if $2 \nmid b$ and $a \geq 2$ then $2(h(a,b) - 1) - 2 = b - 1$, and if $2 \nmid b$ and $a = 1, b \neq 3$ then $2(h(a,b) - 2) - 1 = b$. It immediately follows that if $a \geq 2$ then $(\alpha^a \bar{\alpha}^a)^{h(a,b)-1} \in C_{a,b}^\infty$ and if $a = 1, b \neq 3$ and $2 \nmid b$, then $(1b^b)^{h(a,b)-1} \in C_{a,b}^\infty$. Note that if $a, b$ have same parity, then $\alpha^a \bar{\alpha}^a$ and $1b^b$ have both even length. Thus, by Lemma 15 (2), we get that $(\Delta^{-k}(\alpha^a \bar{\alpha}^a))^{h(a,b)-1}$ and $(\Delta^{-k}(1b^b))^{h(a,b)-1}$ are both smooth words for $\forall k \in N$,
which mean that \( \gamma_{a,b}(n) = \infty \) for \( n < h(a,b) \). Therefore, since \( \delta(a,b) = b+1 \), by Eq. (6.5), we see that the assertion (1) holds.

(2) Note that \((31^3131^3)^4 \in C_{1,3}^\infty \), by Lemma 15 (2), we see that \([\Delta_a^{-k}(31^3131^3)]^4 \in C_{1,3}^\infty \) for every positive integer \( k \), which means that \( \gamma_{1,3}(4) = \infty \). By Theorem 14, we have \( \delta(1,3) = h(1,3) = 5 \), which suggests \( \gamma_{1,3}(n) = 0 \) for \( \forall n \geq 5 \). Thus, the assertion (2) holds.

(3) If \( a, b \) have different parity, then since \( \delta(a,b) = b+1 \), from Eq. 6.5 immediately follows the assertion (3).

(4) By Proposition 13 and [13, Lemma 6], we easily arrive at the desired result.

From the above we easily see that \( \gamma_{a,b}(n) = \gamma_{a,b}(n) \) for same parity alphabets. \( \square \)

7. Reasons for invalidating smooth words over \( \{1, 3\} \)

being biquadrate-free

By Theorem 14, we see that \( C_{a,b}^\infty \)-words are \((b+1)\)-power-free except for \( a = 1 \) and \( b = 3 \), and the proof depends on the fact that the number of \( C_{a,b}^\infty \)-words of form \( u^b \) is finite and no \( C_{a,b}^\infty \)-words \( u \) satisfy \( D^k(u^{b+1}) = (D^k(u)x)^bD^k(u) \), where \( x \in D_{a,b} \). Next Proposition 17 gives us an explanation of the reasons for invalidating \( C^\infty \)-words being 4-power-free on the alphabet \( \{1, 3\} \).

**Proposition 17.** Let \( u_1 = 31 \), \( v_1 = (u_11131)^3u_1 \), \( u_{n+1} = 3\Delta_1^{-1}(u_n)1 \) and \( v_{n+1} = 3\Delta_1^{-1}(v_n)1 \) for \( n = 1, 2, \ldots \). Then

(1) \( v_n = (u_n x)^3 u_n \), where \( 2 \mid |u_n| \), \( 2 \mid |v_n| \), and if \( 2 \nmid n \) then \( x = 1131 \), or else \( x = 3111 \), where \( x \in D_{1,3} \);

(2) \( D(u_{n+1}) = u_n \), \( D(v_{n+1}) = v_n \). Thus \( u_n, v_n \in C_{1,3}^\infty \);

(3) \( (u_n x)^4 \in C_{1,3}^\infty \) for \( n \geq 2 \), where \( x \) is stated as (1).

**Proof.** (1) Induction on \( n \). It is obvious that the assertion (1) holds for \( n = 1 \). Assume now that the assertion (1) holds for \( n = k \geq 1 \).
If $2 \nmid k$ then $v_k = (u_k1131)^3u_k$. Thus, since $2 \mid |u_k|$, we have

$$v_{k+1} = 3\Delta_1^{-1}(v_k)1$$
$$= 3\Delta_1^{-1}((u_k1131)^3u_k)1$$
$$= 3(\Delta_1^{-1}(u_k)\Delta_1^{-1}(1131))^3\Delta_1^{-1}(u_k)1$$
$$= 3(\Delta_1^{-1}(u_k)131^33)\Delta_1^{-1}(u_k)1$$
$$= (3\Delta_1^{-1}(u_k)131^3)^3\Delta_1^{-1}(u_k)1$$
$$= ((3\Delta_1^{-1}(u_k)1)1131)^3(3\Delta_1^{-1}(u_k))1$$
$$= (u_{k+1}1131)^3u_{k+1},$$

which prove the assertion (1).

(2) By the assertion (1), we see that $|u_n|$ and $|v_n|$ are even numbers. Thus the last letters of $\Delta_1^{-1}(u_n)$ and $\Delta_1^{-1}(v_n)$ are both 3, which means that $D(u_{n+1}) = u_n$ and $D(v_{n+1}) = v_n$. In addition, it is apparent that $\hat{u}_n = u_n$, $\hat{v}_n = v_n$ for all $n \in N$. Therefore, by $u_1$, $v_1 \in C_{1,3}^\infty$, we see that the assertion (2) holds.

(3) We proceed by induction on $n$. If $n = 2$ then by the proof of Theorem 16 (2), we see that $(u_2x)^4 = (31^33131)^4 \in C_{1,3}^\infty$ which suggests that the assertion (3) holds for $n = 2$. Assume now that the assertion (3) holds for $n = m \geq 2$.

If $2 \mid m$ then by the definition of $u_m$, we have $(u_{m+1}x)^4 = (3\Delta_1^{-1}(u_m)1x)^4 = (3\Delta_1^{-1}(u_m)1^331)^4$, where $x = 1131$ (because of $2 \nmid m + 1$). Thus $D((u_{m+1}x)^4) = D((3\Delta_1^{-1}(u_m)1^331)^4) = (D(u_{m+1})31^3)D(u_{m+1})31 = (u_m31^3)^3u_m31$, which means that $D((u_{m+1}x)^4)$ is a factor of $(u_m31^3)^4$.

If $2 \nmid m$ then by the definition of $u_m$, we have $(u_{m+1}x)^4 = (3\Delta_1^{-1}(u_m)1x)^4 = (3\Delta_1^{-1}(u_m)131^3)^4$, where $x = 31^3$. Thus $D((u_{m+1}x)^4) = D((3\Delta_1^{-1}(u_m)131^3)^4) = (D(u_{m+1})1^231)^3D(u_{m+1})1^23 = (u_m1^231)^3u_m1^23$, which means that $D((u_{m+1}x)^4)$ is a
factor of \((u_m1^231)^4\). Hence the inductive hypothesis and Lemma 7 (1) mean that the assertion (3) also holds for \(n = m + 1\). □

Let \(u_n\) and \(v_n\) be stated as in the Proposition 17, \(a_1 = 13, b_1 = (a_11311)^3a_1, a_{n+1} = 1\Delta_1^{-1}(a_n)3\) and \(b_{n+1} = 1\Delta_1^{-1}(b_n)3\) for \(n = 1, 2, \ldots\), then since \(\overline{D(w)} = D(\overline{w})\), we easily see that \(a_n = \overline{u}_n\) and \(b_n = \overline{v}_n\) for \(n \in N\).

Moreover, in the proof of Theorem 16, we have shown that \((31^33131^3)^4 \in C_{1,3}^\infty\). In fact, by machine computation, we easily see that the smallest length of \(C_{1,3}^\infty\)-words \(w\) such that \(w^4 \in C_{1,3}^\infty\) is 10 and there are exactly four \(C_{1,3}^\infty\)-words \(w\) with length 10 satisfying the required conditions. Actually, they are \(1^33131^3, 13^3131^3, 31^3313^3\) and \(3^3131^3\) respectively. Furthermore, by machine computation, we can obtain the number \(\varphi(|w|)\) of \(C_{1,3}^\infty\)-words \(w\) such that \(w^4 \in C_{1,3}^\infty\) with length \(|w|\) \((\leq 100)\).

The distribution of \(\varphi(|w|)\) on its length \(|w|\)

| \(|w|\) | 10 | 14 | 16 | 24 | 26 | 32 | 38 | 42 | 54 | 58 |
|-------|----|----|----|----|----|----|----|----|----|----|
| \(\varphi(|w|)\) | 4  | 4  | 16 | 24 | 36 | 16 | 44 | 116| 76 | 32 |
| \(|w|\) | 62 | 66 | 68 | 70 | 74 | 86 | 90 | 94 | 98 | 100|
| \(\varphi(|w|)\) | 92 | 76 | 120| 76 | 64 | 116| 12 | 100| 152| 168|

Table 1

Finally, we list all smooth biquadrates of length 14, that is, \(w = 1^231^33^31^331, 131^33^31^331^2, 313^33^31^33^2, 3^213^31^33^313\).

8. Concluding remarks

By Theorem 16, if \(a\) and \(b\) have the different parity, then we do not know the value of \(\gamma_{a,b}(n)\) for \(1 < n < h(a, b)\) except for the case \(a = 1, b = 2\).

Open problem 1. Compute the values of \(\gamma_{a,b}(n)\) for \(1 < n < h(a, b)\), where \(a\) and \(b\) have the different parity.

Proposition 17 implies that there are infinitely many smooth biquadrates for the alphabet \(\{1, 3\}\). Table 1 provides the distribution of smooth biquadrates on its length \((\leq 100)\) over the alphabet \(\{1, 3\}\). Obviously, if we could completely know about the distribution of smooth biquadrates on its length, then it would help us to reveal more secrets of structure of \(C_{1,3}^\infty\)-words, which differs from \(C^\infty\)-words over 2-letter other alphabets.

Open problem 2. Determine the values of \(\varphi(n)\), where \(\varphi(n)\) is stated as Section 7 (Page 46).
By Theorem 14, we see that $C_{a,b}^\infty$-words are $(b + 1)$-power-free except for $a = 1, b = 3$, which show that $C_{a,b}^\infty$-words have the property $a^{b+1} \notin C_{a,b}^\infty \implies u^{b+1} \notin C_{a,b}^\infty$ for any $C_{a,b}^\infty$-word $u$. Recently Huang [13] proved that if $u$ is not a $C_{1,2}^\infty$-word then $\sigma_w(u) \notin C_{1,2}^\infty$ for any nonempty $C_{1,2}^\infty$-word $w$, where $\sigma_w$ is a homomorphism from $\Sigma^*$ to $\{w, \bar{w}\}^*$ such that $\sigma_w(1) = w$, $\sigma_w(2) = \bar{w}$. To establish the corresponding results over arbitrary 2-letter alphabets is a significative research direction. The case of larger $k$-letter alphabets is also challenging. Their description is beyond the scope of this paper and we shall give them in a forthcoming paper.

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