Numerical analysis of distributed optimal control problems governed by elliptic variational inequalities

Mariela Olguín * and Domingo A. Tarzia †

Abstract

A continuous optimal control problem governed by an elliptic variational inequality was considered in Boukrouche-Tarzia, Comput. Optim. Appl., 53 (2012), 375-392 where the control variable is the internal energy $g$. It was proved the existence and uniqueness of the optimal control and its associated state system. The objective of this work is to make the numerical analysis of the above optimal control problem, through the finite element method with Lagrange’s triangles of type 1. We discretize the elliptic variational inequality which define the system and the corresponding cost functional, and we prove that there exists a unique discrete optimal control and its associated discrete state system for each positive $h$ (the parameter of the finite element method approximation). Finally, we show that the discrete optimal control and its associated state system converge to the continuous optimal control and its associated state system when the parameter $h$ goes to zero. From our point of view, a result of this type is the first time which is obtained by the numerical approximation of an optimal control problem governed by elliptic variational inequalities being the cornerstone of our proof an inequality between the discrete solution of a convex combination of two data and the convex combination of the discrete solutions of the corresponding two data.

Key words: Elliptic variational inequalities, distributed optimal control problems, numerical analysis, convergence of the optimal controls, free boundary problems.

2010 AMS Subject Classification 35J86, 35R35, 49J20, 49J40, 49M25, 65K15, 65N30.

1 Introduction

We consider a bounded domain $\Omega \in \mathbb{R}^n$ whose regular boundary $\partial \Omega = \Gamma_1 \cup \Gamma_2$ consists of the union of two disjoint portions $\Gamma_1$ and $\Gamma_2$ with $\text{meas} (\Gamma_1) > 0$. We consider the following free boundary problem $(S)$:

\begin{align*}
    u &\geq 0; \quad u(-\Delta u - g) = 0; \quad -\Delta u - g \geq 0 \quad \text{in} \quad \Omega; \\
    u &\equiv b \quad \text{on} \quad \Gamma_1; \quad -\frac{\partial u}{\partial n} = q \quad \text{on} \quad \Gamma_2;
\end{align*}

\[u \geq 0; \quad u(-\Delta u - g) = 0; \quad -\Delta u - g \geq 0 \quad \text{in} \quad \Omega;\]

\[u \equiv b \quad \text{on} \quad \Gamma_1; \quad -\frac{\partial u}{\partial n} = q \quad \text{on} \quad \Gamma_2;\]
where the function $g$ in the inequality (1.1) can be considered as the internal energy in $\Omega$, $b$ is the constant temperature on $\Gamma_1$ and $q$ is the heat flux on $\Gamma_2$. The variational formulation of the above problem is given as: Find $u = u_g \in K$ such that

$$a(u, v - u) \geq (g, v - u)_H - \int_{\Gamma_2} q(v - u) \, ds, \quad \forall v \in K,$$

where

$$V = H^1(\Omega), \quad K = \{ v \in V : v \geq 0 \text{ in } \Omega, v/\Gamma_1 = b \}, \quad V_0 = \{ v \in V : v/\Gamma_1 = 0 \},$$

$$H = L^2(\Omega), \quad Q = L^2(\Gamma_2), \quad (u, v)_Q = \int_{\Gamma_2} u v \, ds \quad \forall u, v \in Q,$$

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx \quad \forall u, v \in V, \quad (u, v)_H = \int_{\Omega} u v \, dx \quad \forall u, v \in H.$$
In Section 2, we establish the discrete elliptic variational inequality (2.3) which is the discrete formulation of the continuous elliptic variational inequality (1.3), and we obtain that these discrete problems have unique solutions for all positive $h$. Moreover, on the adequate functional spaces these solutions are convergent when $h \to 0^+$ to the solutions of the continuous elliptic variational inequality (1.3). Moreover, we obtain the inequality (2.11) which is the discrete inequality of the corresponding continuous inequality given in [12, 44]. The inequality (2.11) says that the discrete solution of a convex combination of two data is less than or equal to the convex combination of the discrete solutions of the corresponding two data; and it is the cornerstone to prove our main result.

In Section 3, we define the discrete optimal control problem (3.2) corresponding to continuous optimal control problem (1.5). We obtain that the discrete cost functional is a strictly convex application by using the discrete inequality (2.11) and therefore we prove the existence and the uniqueness of its solution for each parameter $h$ and we obtain the convergence of this family to the continuous optimal control problem $(P)$.

2 Discretization of the problem $(P)$

Let $\Omega \subset \mathbb{R}^n$ a bounded polygonal domain; $b$ a positive constant and $\tau_h$ a regular triangulation with Lagrange triangles of type 1, constituted by affine-equivalent finite elements of class $C^0$ over $\Omega$ being $h$ the parameter of the finite element approximation which goes to zero [13, 19]. We take $h$ equal to the longest side of the triangles $T \in \tau_h$ and we can approximate the sets $V$ and $K$ by:

$$V_h = \{ v_h \in C^0(\Omega) : v_h/T \in \mathbb{P}_1(T), \ \forall \ T \in \tau_h \}$$

$$V_{h0} = \{ v_h \in C^0(\Omega) : v_h/\Gamma_1 = 0; \ v_h/T \in \mathbb{P}_1(T), \ \forall \ T \in \tau_h \}$$

and

$$K_h = \{ v_h \in C^0(\Omega) : v_h \geq 0, \ v_h/\Gamma_1 = b, \ v_h/T \in \mathbb{P}_1(T) \ \forall \ T \in \tau_h \}$$

where $\mathbb{P}_1(T)$ is the set of the polynomials of degree less than or equal to 1 in the triangle $T$. Let $\Pi_h : V \to V_h$ be the corresponding linear interpolation operator and $c_0 > 0$ a constant (independent of the parameter $h$) such that [13]:

$$\| v - \Pi_h(v) \|_H \leq c_0 h^r \| v \|_r \ \forall \ v \in H^r(\Omega), \ 1 < r \leq 2 \quad (2.1)$$

$$\| v - \Pi_h(v) \|_V \leq c_0 h^{r-1} \| v \|_r \ \forall \ v \in H^r(\Omega), \ 1 < r \leq 2. \quad (2.2)$$

The discrete variational inequality formulation $(S_h)$ of the system $(S)$ is defined as: Find $u_{hg} \in K_h$ such that

$$a(u_{hg}, v_h - u_{hg}) \geq (g, v_h - u_{hg})_H - \int_{\Gamma_2} q(v_h - u_{hg})d\gamma, \ \forall v_h \in K_h. \quad (2.3)$$

Theorem 2.1. Let $g \in H$, $b > 0$ and $q \in Q$ be, then there exist unique solution of the problem $(S_h)$ given by the elliptic variational inequality (2.3).

Proof. It follows from the application of Lax-Milgram Theorem [38, 40].
Lemma 2.1. Let \( g_1, g_2 \in H \), and \( u_{h,g_1}, u_{h,g_2} \in K_h \) be the solutions of \((S_h)\) for \( g_1 \) and \( g_2 \) respectively, then we have that:

a) there exist a constant \( C \) independent of \( h \) such that:

\[
\|u_{h,g}\| \leq C, \quad \forall \ h > 0; \tag{2.4}
\]

b) \[
\|u_{h,g_2} - u_{h,g_1}\| \leq \frac{1}{\lambda}\|g_2 - g_1\|_H \quad \forall \ h > 0; \tag{2.5}
\]

c) if \( g_1 \geq g_2 \in \Omega \) then \( u_{h,g_1} \geq u_{h,g_2} \) in \( \Omega \);

d) if \( g_n \to g \) in \( H \) weak, then \( u_{h,g_n} \to u_h \) in \( V \) strong for each fixed \( h > 0 \).

Proof. a) If we consider \( v_h = b \in K_h \) in the discrete elliptic variational inequality \[(2.3)\] we have:

\[
\lambda \|u_{h,g} - b\|^2 \leq a(u_{h,g}, u_{h,g} - b) \leq (g, u_{h,g} - b)_H + (q, b - u_{h,g})_Q
\]

\[
\leq (\|g\|_H + \|q\|_Q \|\gamma_0\|)\|u_{h,g} - b\|_V
\]

where \( \gamma_0 \) is the trace operator and therefore \[(2.4)\] holds.

b) As \( u_{h,g_1} \) and \( u_{h,g_2} \) are respectively the solutions of discrete elliptic variational inequalities \[(2.3)\] for \( g_1, g_2 \), we have:

\[
a(u_{h,g_i}, v_h - u_{h,g_i}) \geq (g_i, v_h - u_{h,g_i})_H - (q, v_h - u_{h,g_i})_Q, \quad \forall v_h \in K_h \tag{2.6}
\]

for \( i = 1, 2 \). By coerciveness of \( a \) we deduce:

\[
\lambda\|u_{h,g_2} - u_{h,g_1}\|_V^2 \leq a(u_{h,g_2} - u_{h,g_1}, u_{h,g_2} - u_{h,g_1}) \leq (g_2 - g_1, u_{h,g_2} - u_{h,g_1})_H
\]

\[
\leq \|g_2 - g_1\|_H \|u_{h,g_2} - u_{h,g_1}\|_V \quad \forall \ h > 0,
\]

thus \[(2.5)\] holds.

c) By considering \( z_h = u_{h,g_1} - u_{h,g_2} \) and \( v_{h,i} = u_{h,g_i} + (-1)^i z_h^- \), \( (i = 1, 2) \) in \[(2.6)\] respectively, we obtain:

\[
a(u_{h,g_1}, z_h^-) \geq (g_1, z_h^-)_H - (q, z_h^-)_Q,
\]

and

\[
a(u_{h,g_2}, -z_h^-) \geq (g_2, -z_h^-)_H - (q, -z_h^-)_Q.
\]

If we add these both inequalities, it result that \( \lambda\|z_h^-\|_V^2 \leq a(z_h^-, z_h^-) \leq (g_2 - g_1, z_h^-)_H \leq 0 \), that is to say \( \|z_h^-\|_V = 0 \) and in consequence \( u_{h,g_1} \geq u_{h,g_2} \) in \( \Omega \).

d) Let \( h > 0 \) be. From item a) we have that \( \|u_{h,g_i}\| \leq C \ \forall \ n \), then there exist \( \eta \in V \) such that \( u_{h,g_n} \to \eta \) in \( V \) weak (in \( H \) strong). If we consider the discrete elliptic inequality \[(2.3)\] we have:

\[
a(u_{h,g_n}, v_h - u_{h,g_n}) \geq (g_n, v_h - u_{h,g_n})_H - (q, v_h - u_{h,g_n})_Q
\]
and using that $a$ is a lower weak semicontinuous application then, when $n$ goes to infinity, we obtain that:

$$a(\eta, v_h - \eta) \geq (g, v_h - \eta)_H - (q, v_h - \eta)_Q$$

and from uniqueness of the solution of problem $(S_h)$, we deduce that $\eta = u_{hg} \in K_h$.

Now, it is easily to see that:

$$a(u_{hg_n} - u_{hg}, u_{hg_n} - u_{hg}) \leq -(g - g_n, u_{hg_n} - u_{hg})_H$$

and from the coerciveness of $a$ we obtain

$$\lambda \| u_{hg_n} - u_{hg} \|_V \leq (g - g_n, u_{hg_n} - u_{hg})_H.$$ 

As $u_{hg_n} \to u_{hg}$ in $H$ and $g_n \rightharpoonup g$ in $H$, by pass to the limit when $n \to \infty$ in the previous inequality, we obtain

$$\lim_{n \to \infty} \| u_{hg_n} - u_g \|_V = 0.$$ 

Henceforth we will consider the following definitions \[12\]: Given $\mu \in [0, 1]$ and $g_1, g_2 \in H$, we have the convex combinations of two data

$$g_3(\mu) = \mu g_1 + (1 - \mu) g_2 \in H,$$ 

the convex combination of two discrete solutions

$$u_{h3}(\mu) = \mu u_{hg_1} + (1 - \mu) u_{hg_2} \in K_h$$

and we define $u_{h4}(\mu)$ as the associated state system which is the solution of the discrete elliptic variational inequality \[2.3\] for the control $g_3(\mu)$.

Then, we have the following properties:

**Lemma 2.2.** Given the controls $g_1, g_2 \in H$, we have that:

a) $$\| u_{h3} \|_H^2 = \mu \| u_{hg_1} \|_H^2 + (1 - \mu) \| u_{hg_2} \|_H^2 - \mu (1 - \mu) \| u_{hg_2} - u_{hg_1} \|_H^2$$ (2.9)

b) $$\| g_3(\mu) \|_H^2 = \mu \| g_1 \|_H^2 + (1 - \mu) \| g_2 \|_H^2 - \mu (1 - \mu) \| g_2 - g_1 \|_H^2$$ (2.10)

**Proof.** a) From the definition \[2.8\] we get

$$\| u_{h3} \|_H^2 = \mu^2 \| u_{hg_1} \|_H^2 + (1 - \mu)^2 \| u_{hg_2} \|_H^2 + 2 \mu (1 - \mu) (u_{hg_1}, u_{hg_2})_H$$

and

$$\| u_{hg_2} - u_{hg_1} \|_H^2 = \| u_{hg_2} \|_H^2 + \| u_{hg_1} \|_H^2 - 2(u_{hg_1}, u_{hg_2})_H,$$

then we conclude \[2.9\].

b) It follows from a similar method to the part a). 

Now, we will obtain the cornerstone of our main result.
Theorem 2.2. Given the controls $g_1, g_2 \in H$, we obtain that:

$$0 \leq u_{h4}(\mu) \leq u_{h3}(\mu) \text{ in } \Omega, \quad \forall \mu \in [0, 1], \forall h > 0.$$  \hspace{1cm} (2.11)

Proof. Let $\mu \in [0, 1]$. From the definition, the state system $u_{h4}(\mu)$ verifies the elliptic variational inequality

$$a(u_{h4}(\mu), v_h - u_{h4}(\mu)) \geq (g_3(\mu), v_h - u_{h4}(\mu)) - (q, v_h - u_{h4}(\mu))Q, \quad \forall v_h \in K_h.$$  \hspace{1cm} (2.12)

If we define $z_h(\mu) = u_{h3}(\mu) - u_{h4}(\mu)$ and consider $v_h = u_{h\eta} + z_h^-(\mu)$ for $i=1, 2$ in (2.6), we obtain:

$$a(\mu u_{h\eta}, z_h^-(\mu)) \geq (\mu g_1, z_h^-(\mu)) - (q, z_h^-(\mu))Q$$  \hspace{1cm} (2.13)

and

$$a((1 - \mu) u_{h\eta}, z_h^-(\mu)) \geq ((1 - \mu) g_2, z_h^-(\mu)) - (1 - \mu)(q, z_h^-(\mu))Q.$$  \hspace{1cm} (2.14)

By adding (2.13) and (2.14), we have:

$$a(u_{h3}(\mu), z_h^-(\mu)) \geq (g_3(\mu), z_h^-(\mu)) - (q, z_h^-(\mu))Q.$$  \hspace{1cm} (2.15)

Now, by using $v_h = u_{h4}(\mu) - z_h^-\mu) in (2.10)$ it results that:

$$a(u_{h4}(\mu), -z_h^-\mu)) \geq (g_3(\mu), -z_h^-\mu)) + (q, z_h^-\mu)Q.$$  \hspace{1cm} (2.16)

Again, by adding (2.15) and (2.16) we have $a(u_{h3}(\mu) - u_{h4}(\mu), z_h^-\mu)) \geq 0$ then, by the coerciveness of the application $a$, we have

$$\lambda \|z_h^-\mu)\|_V^2 \leq a(z_h^-\mu), z_h^-\mu)) \leq 0,$$

then $\|z_h^-\mu\|_V = 0$ and $z_h(\mu) \geq 0$ and, in consequence, the inequality (2.11) holds. \hfill \Box

Theorem 2.3. Let $u_g$ and $u_{hg}$ be the solutions of the elliptic variational inequalities (1.3) and (2.3) respectively for the control $g \in H$. Then, if $u_g \in H^r(\Omega)$ and $u_{hg} \in H^r(\Omega) \forall h > 0$, $u_{hg}$ converge to $u_g$ in $V$ strong when $h \to 0^+.$

Proof. From Lemma 2.1 we have that there exist a constant $C > 0$ independent of $h$ such that $\|u_{hg}\|_V \leq C \forall h > 0$, then we conclude that there exists $\eta \in V$ so that $u_{hg} \to \eta$ in $V$ weak as $h \to 0^+$ and $\eta \in K$. On the other hand, given $v \in K$ there exist $v_h^* \mu)$ such that $v_h^* \in K_h$ for each $h$ and $v_h^* \to v$ in $V$ strong when $h$ goes to zero. Now, by considering $v_h^* \in K_h$ in the discrete elliptic variational inequality (2.3) we get:

$$a(u_{hg}, u_{hg}) \leq a(u_{hg}, v_h^*) - (g, v_h^* - u_{hg}) + (q, v_h^* - u_{hg})Q$$  \hspace{1cm} (2.17)

and when we pass to the limit as $h \to 0^+$ in (2.17) by using that the bilinear form $a$ is lower weak semicontinuous in $V$ we obtain:

$$a(\eta, \eta) \leq a(\eta, v) - (g, v - \eta) + (q, v - \eta)Q$$

that it is to say:
and, from the uniqueness of the solution of the discrete elliptic variational inequality (1.3), we obtain that \( \eta = u_g \).

Now, we will prove the strong convergence. If we consider \( v = u_{hg} \in K_h \subset K \) in the elliptic variational inequality (1.3) and \( v_h = \Pi_h(u_g) \in K_h \) in (2.3), from the coerciveness of \( a \) and by some mathematical computation, we obtain that:

\[
\lambda \| u_{hg} - u_g \|_V^2 \leq a(u_{hg} - u_g, u_{hg} - u_g) \\
\leq a(u_{hg}, \Pi_h(u_g) - u_g) - (g, \Pi_h(u_g) - u_g) + (q, \Pi_h(u_g) - u_g)Q
\]

(2.18)

then by pass to the limit when \( h \to 0^+ \) it results that \( \lim_{h \to 0^+} \| u_{hg} - u_g \|_V = 0 \).

\[ \square \]

3 Discretization of the optimal control problem

Now, we consider the continuous optimal control problem which was established in (1.5). The associated discrete cost functional \( J_h : H \to \mathbb{R}_0^+ \) is defined by the following expression:

\[
J_h(g) = \frac{1}{2} \| u_{hg} \|_H^2 + \frac{M}{2} \| g \|_H^2
\]

(3.1)

and we establish the discrete optimal control problem as: Find \( g_{oph} \in H \) such that

\[
J_h(g_{oph}) = \min_{g \in H} J_h(g)
\]

(3.2)

where \( u_{hg} \) is the associated state system solution of the problem \( (S_h) \) which was described for the discrete elliptic variational inequality (2.3) for a given control \( g \in H \).

**Theorem 3.1.** Given the control \( g \in H \), we have:

a) \( \lim_{\| g \|_H \to \infty} J_h(g) = \infty \).

b) \( J_h(g) \geq M \| g \|_H^2 - C \| g \|_H \) for some constant \( C \) independent of \( h \).

c) The functional \( J_h \) es a lower weakly semicontinuous application in \( H \).

d) The quadratic cost functional \( J_h \) defined by (2.16) is a strictly convex application.

e) There exists unique solution of the discrete optimal control problem (2.17) for all \( h > 0 \).

**Proof.** a) From the definition of \( J_h(g) \) we obtain a) and b).

c) Let \( g_n \rightharpoonup g \) in \( H \) weak, then by using the equality \( \| g_n \|_H^2 = \| g_n - g \|_H^2 - \| g \|_H^2 + 2(g_n, g)_H \) we obtain that \( \| g \|_H \leq \liminf_{n \to \infty} \| g_n \|_H^2 \). Therefore, we have

\[
\liminf_{n \to \infty} J_h(g_n) \geq \frac{1}{2} \| u_{hg} \|_H^2 + \frac{M}{2} \| g \|_H^2 = J_h(g).
\]

d) Let \( \mu \in (0, 1) \); \( g_1, g_2 \in H \) and \( u_{hgi} \) be the solution of the variational inequality (2.3) for the control \( g_i \) \( (i = 1, 2) \) respectively. If we consider \( g_3(\mu) = \mu g_1 + (1 - \mu)g_2 \) and
$u_{h4}$ is the solution of $(P_h)$ associated to $g_3(\mu)$, and $u_{h3} = \mu u_{hg_1} + (1 - \mu)u_{hg_2}$ then it results:

$$
\mu J_h(g_1) + (1 - \mu)J_h(g_2) - J_h(g_3(\mu)) = \frac{1}{2} \mu \|u_{hg_1}\|_H^2 + (1 - \mu)\|u_{hg_2}\|_H^2 - \|u_{h4}\|_H^2 + \frac{M}{2} \big(\mu \|g_1\|_H^2 + (1 - \mu)\|g_2\|_H^2 - \|g_3(\mu)\|_H^2\big)
$$

$$
= \frac{\mu(1 - \mu)}{2}\|u_{hg_2} - u_{hg_1}\|_H^2 + \frac{M}{2} \mu(1 - \mu)\|g_2 - g_1\|_H^2 + \frac{1}{2}\|u_{h3}\|_H^2 - \|u_{h4}\|_H^2\
\geq \frac{\mu(1 - \mu)}{2}\|u_{hg_2} - u_{hg_1}\|_H^2 + \frac{M}{2} \mu(1 - \mu)\|g_2 - g_1\|_H^2 > 0
$$

by using the inequality (2.11).

e) It follows from [41].

**Lemma 3.1.** If the continuous state system has the regularity $u_g \in H^r(\Omega) \ (1 < r \leq 2)$ then we have the following estimations $\forall g \in H$:

a) $\|u_{hg} - u_g\|_V \leq Ch^{r-1}$, \hspace{1cm} (3.3)

b) $|J_h(g) - J(g)| \leq Ch^{r-1}$, \hspace{1cm} (3.4)

where C’s are constants independents of h.

**Proof.** a) As $u_g \in K$, we have that $\Pi_h(u_g) \in K_h \subset K$. If we consider $v_h = \Pi_h(u_g)$ in (2.3), by using the inequalities (2.18), we obtain:

$$
\lambda \|u_{hg} - u_g\|_V^2 \leq a(u_{hg} - u_g, u_{hg} - u_g)
$$

$$
\leq a(u_{hg}, \Pi_h(u_g) - u_g) - (g, \Pi_h(u_g) - u_g) + \int_{\Gamma_2} q(\Pi_h(u_g) - u_g) d\gamma
$$

$$
\leq C\|\Pi_h(u_g) - u_g\|_V \leq C\|u_g\|_r h^{r-1} \leq Ch^{r-1},
$$

and then (3.3) holds.

b) From the definitions of $J$ and $J_h$, it results:

$$
J_h(g) - J(g) = \frac{1}{2}\|u_{hg}\|_H^2 - \|u_g\|_H^2 = \frac{1}{2}\|u_{hg} - u_g\|_H^2 + (u_g, u_{hg} - u_g)
$$

and therefore

$$
|J_h(g) - J(g)| \leq \left(\frac{1}{2}\|u_{hg} - u_g\|_H + \|u_g\|_H\right)\|u_{hg} - u_g\|_H \leq Ch^{\frac{r-1}{2}}.
$$

\□
Theorem 3.2. Let $u_{\text{op}} \in K$ be the continuous state system associated to the optimal control $g_{\text{op}} \in H$ which is the solution of the continuous distributed optimal control problem (1.3), and for each $h > 0$, $u_{h g_{\text{op}}} \in K_h$ is the discrete state system corresponding to the control $g_{\text{op}} \in H$ which is the solution of the discrete distributed optimal control problem (3.2). Then we obtain that:

$$u_{h g_{\text{op}}} \rightarrow u_{\text{op}} \quad \text{on } V \quad \text{strong and} \quad g_{\text{op}} \rightarrow g_{\text{op}} \quad \text{on } H \quad \text{strong when } h \rightarrow 0^+.$$  

Proof. Let $h > 0$ and $g_{\text{op}}$ the solution of (3.2) and $u_{h g_{\text{op}}}$ the associated discrete optimal states system which are solution of the problem defined in (2.3) for each $h > 0$. From (3.1) we have that for all $g \in H$:

$$J_h(g_{\text{op}}) = \frac{1}{2} \|u_{h g_{\text{op}}}\|_H^2 + \frac{M}{2} \|g_{\text{op}}\|_H^2 \leq \frac{1}{2} \|u_{g_{\text{op}}}\|_H^2 + \frac{M}{2} \|g\|_H^2.$$  

Then, if we consider $g = 0$ and $u_{h 0}$ his corresponding associated state system, it results that:

$$J_h(g_{\text{op}}) = \frac{1}{2} \|u_{h g_{\text{op}}}\|_H^2 + \frac{M}{2} \|g_{\text{op}}\|_H^2 \leq \frac{1}{2} \|u_{h 0}\|_H^2.$$  

From the Lemma 2.1 we have that $\|u_{h 0}\|_H \leq C$ $\forall$ $h$, then we can obtain:

$$\|u_{h g_{\text{op}}}\|_H \leq C \quad \forall \quad h$$  

and

$$\|g_{\text{op}}\|_H \leq \frac{1}{M} \|u_{h 0}\|_H \leq \frac{1}{M} C \quad \forall \quad h.$$  

If we consider $v_h = b \in K_h$ in the inequality (2.3) for $g_{\text{op}}$, we obtain:

$$a(u_{h g_{\text{op}}}, b - u_{h g_{\text{op}}}) \geq (g_{\text{op}}, b - u_{h g_{\text{op}}}) - (q, b - u_{h g_{\text{op}}})Q,$$  

therefore:

$$a(u_{h g_{\text{op}}}, b - u_{h g_{\text{op}}}) \leq (g_{\text{op}}, u_{h g_{\text{op}}} - b) - (q, u_{h g_{\text{op}}} - b)Q,$$  

and from the coerciveness of the application $a$ we have that $\|u_{h g_{\text{op}}} - b\|_V \leq C$ and in consequence $\|u_{h g_{\text{op}}}\|_V \leq C$.

Now we can say that there exist $\eta \in V$ and $f \in H$ such that $u_{h g_{\text{op}}} \rightarrow \eta$ in $V$ weak (in $H$ strong), and $g_{\text{op}} \rightarrow f$ in $H$ weak when $h \rightarrow 0^+$. Then, $\eta/\Gamma_1 = b$ and $\eta \geq 0$ in $\Omega$ i.e., $\eta \in K$.

Let given $v \in K$, there exist $v_h \in K_h$ such that $v_h \rightarrow v$ in $V$ strong when $h \rightarrow 0^+$. Then, if we consider the variational elliptic inequality (2.3) for $g = g_{\text{op}}$ we have:

$$a(u_{h g_{\text{op}}}, v_h) \geq a(u_{h g_{\text{op}}}, u_{h g_{\text{op}}}) + (g_{\text{op}}, v - u_{h g_{\text{op}}}) - (q, v - u_{h g_{\text{op}}})Q.$$  

Taking into account that the application $a$ is a lower weak semicontinuous application in $V$ and by pass to the limit when $h$ goes to zero in (3.9) we obtain that:

$$a(\eta, v - \eta) \geq (f, v - \eta) - (q, v - \eta)Q, \quad \forall \quad v \in K$$
and by the uniqueness of the solution of the problem given by the elliptic variational inequality (1.3), we deduce that \( \eta = u_f \).

Finally, the norm on \( H \) is a lower semicontinuous application in the weak topology, then we can prove that:

\[
J(f) \leq \liminf_{h \to 0^+} J_h(g_{op_h}) \leq \liminf_{h \to 0^+} J_h(g) = \frac{1}{2} \lim_{h \to 0^+} \|u_{h_0}\|_H^2 + \frac{M}{2} \|g\|_H^2
\]

\[
= \frac{1}{2} \|g\|_H^2 + \frac{M}{2} \|g\|_H^2 = J(g), \quad \forall g \in H
\]

and because the uniqueness of the optimal problem (1.5), it results that \( f = g_{op} \) and \( \eta = u_{g_{op}} \).

Now, if we consider \( v = u_{h,g_{op_h}} \in K_h \subset K \) in the elliptic variational inequality (1.3) for the control \( g_{op} \) and we define \( z_h = u_{h,g_{op_h}} - u_{g_{op}} \), we have that:

\[
a(z_h, z_h) \leq a(u_{h,g_{op_h}}, u_{g_{op}}) - a(u_{h,g_{op_h}}, u_{g_{op}}) - (g_{op}, u_{h,g_{op_h}} - u_{g_{op}}) + (q, u_{h,g_{op_h}} - u_{g_{op}})Q,
\]

and by consider \( v = \Pi_h(u_{g_{op}}) \in K_h \) for \( g = g_{op} \) in the inequality (2.3) we obtain:

\[
a(u_{h,g_{op_h}}, u_{h,g_{op_h}}) \leq -(g_{op}, \Pi_h(u_{g_{op}}) - u_{g_{op}}) + (q, \Pi_h(u_{g_{op}}) - u_{g_{op}})Q + a(u_{h,g_{op_h}}, \Pi_h(u_{g_{op}})),
\]

and then by the coerciveness of \( a \) we get

\[
\lambda \|z_h\|_V^2 \leq (q, \Pi_h(u_{g_{op}}) - u_{g_{op}})Q + a(u_{h,g_{op_h}}, \Pi_h(u_{g_{op}}) - u_{g_{op}}) + (g_{op} - g_{op}, u_{h,g_{op_h}} - u_{g_{op}}) - (g_{op}, \Pi_h(u_{g_{op}}) - u_{g_{op}})
\]

\[
\leq \liminf_{h \to 0^+} \|u_{g_{op}} - u_{h,g_{op_h}}\|_V = 0.
\]

The strong convergence of the optimal controls \( g_{op_h} \) to \( g_{op} \) is obtained by using Theorem 3.1 and \( g_{op_h} \rightharpoonup g_{op} \) weakly on \( H \), i.e.

\[
J(g_{op}) = \frac{1}{2} \|u_{g_{op}}\|_H^2 + \frac{M}{2} \|g_{op}\|_H^2 \leq \liminf_{h \to 0^+} J_h(g_{op_h})
\]

\[
\leq \liminf_{h \to 0^+} J_h(g_{op}) = \liminf_{h \to 0^+} \frac{1}{2} \|u_{g_{op_h}}\|_H^2 + \frac{M}{2} \|g_{op}\|_H^2 = J(g_{op}),
\]

then \( \lim_{h \to 0} \|g_{op_h}\|_H = \|g_{op}\|_H \) and therefore \( \lim_{h \to 0^+} \|g_{op_h} - g_{op}\|_H = 0 \).

\[
\square
\]

4 Conclusions

We have proved, for the first time from our point of view, the convergence of the discrete to the continuous optimal control problems governed by elliptic variational inequalities by using the finite element method with Lagrange’s triangles of type 1. This result can be mainly obtained by using an inequality between the discrete solution of a convex combination of two data and the convex combination of the discrete solutions of the corresponding two data. Moreover, it is an open problem to obtain the error estimates as a function of the parameter \( h \) of the finite element method.
5 Acknowledgements

This paper has been partially sponsored by Project PIP # 0534 from CONICET-UA, Rosario, Argentina, and AFOSR-SOARD Grant FA9550-14-1-0122.

References

[1] Abergel F. (1988) *A non-well posed problem in convex optimal control*, Appl. Math. Optim., 17:133-175.

[2] Adams D.R., Lenhart S.M. and Yong J. (1998) *Optimal control of the obstacle for an elliptic variational inequality*, Appl. Math. Optim., 38:121-140.

[3] Ait Hadi K. (2008) *Optimal control of an obstacle problem: optimality conditions*, IMA J. Math. Control Inform, 23:325-334.

[4] Barbu V. (1984) *Optimal Control of Variational Inequalities*, Research Notes in Mathematics No100, Pitman, London.

[5] Ben Belgacem F., El Fekih H. and Metoui H. (2003), *Singular perturbations for the Dirichlet boundary control of elliptic problems*, ESAIM: M2AN, 37:833-850.

[6] Bergounioux M. (1997) *Use of augmented Lagrangian methods for the optimal control of obstacle problems*, J. Optim. Theory Appl., 95:101-126.

[7] Bergounioux M. and Kunisch K. (1997) *Augmented Lagrangian techniques for elliptic state constrained optimal control problems*, SIAM J. Control Optim., 35:1524-1543.

[8] Bergounioux M. (1997), *Optimal control of an obstacle problem*, Appl. Math. Optim., 36:147-172.

[9] Bergounioux M. and Mignot F. (2000) *Optimal control of obstacle problems: existence of lagrange multipliers*, ESAIM: Control, Optimization and Calculus of Variations, 5:45-70.

[10] Beuchler S., Pechstein C. and Wachsmuth D. (2012), *Boundary concentrated finite elements for optimal boundary control problems of elliptic PDEs*, Comput. Optim. Appl., 51:883-908.

[11] Boukrouche M. and Tarzia D.A. (2011), *Existence, uniqueness, and convergence of optimal control problems associated with parabolic variational inequalities of the second kind*, Nonlinear Analysis: Real World Appl., 12:2211-2224.

[12] Boukrouche M. and Tarzia D. (2012) *Convergence of distributed optimal control problems governed by elliptic variational inequalities*, Comput. Optim. Appl. 53:375-393.

[13] Brenner S. and Scott L. (1994) *The mathematical theory of finite elements*, Springer, Berlin.

[14] Casas E. and Mateos M. (2002), *Uniform convergence of the FEM. Applications to state constrained control problems*, Comput. Appl. Math., 21:67-100.
[15] Casas E., Mateos M. and Tröltzsch F.(2005), Error estimates for the numerical approximation of boundary semilinear elliptic control problems, Comput. Optim. Appl., 31:193-219.

[16] Casas E. and Raymond J.P.(2006), Error estimates for the numerical approximation of Dirichlet boundary control for semilinear elliptic equations, SIAM J. Control Optim., 45:1586-1611.

[17] Casas E. and Mateos M.(2008), Error estimates for the numerical approximation of Neumann control problems, Comput. Optim. Appl., 39:265-295.

[18] Casas E. and Mateos M.(2011), Dirichlet control problems in smooth and nonsmooth convex plain domains, Control Cybernetics, 40:931-955.

[19] Ciarlet P. (2002) The finite element method for elliptic problems, SIAM, Philadelphia.

[20] De Los Reyes J. (2011) Optimal control of a class of variational inequalities of the second kind, SIAM Journal of Control and Optimization, 49:1629-1658.

[21] De Los Reyes J.C. and Meyer C.(2014), Strong stationarity conditions for a class of optimization problems governed by variational inequalities of the second kind, arXiv : 1404.4787v2 [math.OC] 7 July 2014.

[22] Deckelnick K., Gunther A. and Hinze M.(2009), Finite element approximation of elliptic control problems with constraints on the gradient, Numer. Math., 111:335-350.

[23] Deckelnick K. and Hinze M.(2007), Convergence of a finite element approximation to a state-constrained elliptic control problem, SIAM J. Numer. Anal., 45:1937-1953.

[24] Gamallo P., Hernández E. and Peters A.(2011), On the error estimates for the finite element approximation of a class of boundary optimal control systems, Numer. Funct. Anal. Optim., 32:383-396.

[25] Gariboldi C.M. and Tarzia D.A.(2003), Convergence of distributed optimal controls on the internal energy in mixed elliptic problems when the heat transfer coefficient goes to infinity, Appl. Math. Optim., 47:213-230.

[26] Gariboldi C.M. and Tarzia D.A.(2008), Convergence of boundary of optimal control problems with restrictions in mixed elliptic Stefan like problems, Adv. Diff. Eq. Control Processes, 1:113-132.

[27] Haller-Dintelmann R., Meyer C. Rehberg J. and Schiela A. (2009) Hölder continuity and optimal control for nonsmooth elliptic problems, Applied Mathematics and Optimization, 60:397-428.

[28] Haslinger J. and Roubicek T. (1986) Optimal control of variational inequalities. Approximation theory and numerical realization, Applied Mathematics and Optimization, 14:187-201.

[29] Hintermüller M. (2001) Inverse coefficient problems for variational inequalities: optimality conditions and numerical realization, Math. Modeling Numer. Anal., 35:129-152.
[30] Hintermüller M. (2008) An active-set equality constrained Newton solver with feasibility restoration for inverse coefficient problems in elliptic variational inequalities, Inverse Problems, 24:034017 (23pp).

[31] Hintermüller M. and Kopacka I. (2009) Mathematical programs with complementarity constraints in function space: C- and strong stationarity and a path-following algorithm, SIAM J. Optim., 20:868-902.

[32] Hintermüller M. and Hinze M.(2009), Moreau-Yosida regularization in state constrained elliptic control problems: Error estimates and parameter adjustment, SIAM J. Numer. Anal., 47:1666-1683.

[33] Hintermüller M. and Loebhard C.(2013), Solvability and stationarity for the optimal of variational inequalities with point evaluations in the objective functional, PAMM, 13:459-460.

[34] Hinze M.(2005), A variational discretization concept in control constrained optimization: The linear-quadratic case, Comput. Optim. Appl., 30:45-61.

[35] Hinze M.(2009), Discrete concepts in PDE constrained optimization, in M. Hinze, R. Pinnau, R. Ulbrich, S. Ulbrich (Eds.), Optimization with PDE constrained, Chapter 3, Springer, New York.

[36] Hinze M. and Matthes U.(2009), A note on variational discretization of elliptic Neumann boundary control, Control Cybernetics, 38:577-591.

[37] Ito K. and Kunisch K., (2000) Optimal control of elliptic variational inequalities, Appl. Math. Optim., 41:343-364.

[38] Kinderhlerer D. and Stampacchia G. (1980) An introduction to variational inequalities and their applications, Academic Press, New York.

[39] Kunisch K. and Wachsmuth D. (2012) Path-following for optimal control of stationary variational inequalities, Comput. Optim. Appl., 41:1345-1373.

[40] Lions J.L. and Stampacchia G. (1967) Variational inequalities, Comm. Pure Appl. Math., 20:493-519.

[41] Lions J.L. (1968) Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles, Dunod, Paris.

[42] Menaldi J.L. and Tarzia D.A.(2007), A distributed parabolic control with mixed boundary conditions, Asymptotic Analysis, 52:227-241.

[43] Mermri E.B. and Han W.(2012), Numerical approximation of a unilateral obstacle problem, J. Optim. Th. Appl., 153:177-194.

[44] Mignot F. (1976) Control dans les inéquations variationnelles elliptiques, J. Funct. Anal., 22:130-185.

[45] Mignot F. and Puel P. (1984) Optimal control in some variational inequalities, SIAM J. Control Optim., 22:466-476.
[46] Tarzia D.A. (1996) *Numerical analysis for the heat flux in a mixed elliptic problem to obtain a discrete steady-state two-phase Stefan problems*, SIAM J. Numer. Anal., 33-4:1257-1265.

[47] Tarzia D.A. (1999) *Numerical analysis of a mixed elliptic problem with flux and convective boundary conditions to obtain a discrete solution of non-constant sign*, Numer. Meth. Partial Diff. Eq., 15:355-369.

[48] Tarzia D.A. (2014) *A commutative diagram among discrete and continuous Neumann boundary optimal control problems*, Adv. Diff. Eq. Control Processes, In Press.

[49] Tröltzsch F.(2010), *Optimal control of partial differential equations*, Amer. Math. Soc., Providence.

[50] Yan M., Chang L. and Yan N.(2012), *Finite element method for constrained optimal control problems governed by nonlinear elliptic PDEs*, Math. Control Related Fields, 2:183-194.

[51] Ye Y., Chan C.K., Leung B.P.K. and Chen Q.(2004), *Bilateral obstacle optimal control for a quasilinear elliptic variational inequality with a source term*, Nonlinear Analysis, 66:1170-1184.

[52] Ye Y. and Chen Q. (2004) *Optimal control of the obstacle in a quasilinear elliptic variational inequality*, J. Math. Anal. Appl., 294:258-272.

[53] Ye Y., Chan C.K. and Lee H.W.J.(2009), *The existence results for obstacle optimal control problems*, Appl. Math. Comput., 214:451-456.