Singularity confinement and projective resolution of triangulated category

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Published February 21, 2014

We proposed, in our previous paper, to characterize the Hirota–Miwa equation (HM equation) by means of the theory of triangulated category. This correspondence was expected because the HM equation describes a large amount of integrable phenomena in physics while the category theory provides a unified description of various fields in mathematics. For this purpose we introduced the notion of “difference” forms on a lattice space where the HM equation is embedded, so that an iteration of the dynamical mapping was identified with a complex object. It was also suggested that the phenomenon called singularity confinement, which was supposed to characterize integrable maps, could be associated with the resolution in the theory of triangulated category. We extend our argument in this paper to support these ideas. We refine the notion of difference geometry on the lattice space. In particular, we show in detail how singularity confinement takes place within the framework of the projective resolution of the triangulated category.

Subject Index
A10, A13, A30, A34

1. Introduction

Let us consider a dynamical system in which a rule is fixed to change the state of an object, say X, at every step. We call such a system deterministic. If we can predict the state of X at any time, when the initial state of X is given as arbitrary, we call the system integrable. The Kepler motion is an integrable system such that an orbit is determined entirely if the initial position and initial velocity are given. Such an integrable system is, however, quite unusual, because the probability of encountering an integrable system is almost zero when a deterministic system is given as arbitrary.

We are interested in finding a way to discriminate integrable systems from nonintegrable ones. From the viewpoint of category theory in mathematics [1] an integrable system is a category in which objects are the states of X and the morphism is the change of the states. In our previous paper [2] we discussed this problem and proposed that the triangulated category might characterize the Hirota–Miwa equation (HM equation), a completely integrable difference equation from which infinitely many soliton equations of the Kadomtsev-Petviashvili (KP) hierarchy can be derived [3,4].

We would like to extend our previous argument in this paper. In particular, we discuss in detail how the singularity confinement, a phenomenon that was proposed to characterize integrable maps [5,6], can be associated with the projective resolution of the triangulated category.

†Part of this work was reported by one of the authors (S.S.) at the conference on “Nonlinear Mathematical Physics: Twenty Years of JNMP”, held at Nordfjordeid, Norway, from June 4 to June 14, 2013.
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The plan of this paper is as follows. In Sect. 2 we discuss the geometrical features of the HM equation. It is convenient to describe the HM equation by means of the discrete geometry on a lattice space, such that the notion of distinguished triangles becomes manifest. This idea has already been explained briefly in our previous paper [2] but we extend the idea in much more detail in this paper. Based on this geometrical structure, we fix in Sect. 3 the deterministic rule of information transfer from one place to another on the lattice space. It will be shown in Sect. 4 that all these rules can be described in terms of the triangulated category.

We proposed in Ref. [2] a possible interpretation of the singularity confinement as a projective resolution of the triangulated category. Our main subject in this paper is an investigation of this conjecture in detail, which will be presented in Sect. 5.

2. Geometrical features of the HM equation

Before starting our argument it will be worthwhile to briefly review some features of the HM equation [3,4]. We consider the following formulas throughout this paper:

\[
\begin{align*}
    a_{14}a_{23}\tau_{14}(p)\tau_{23}(p) &- a_{24}a_{13}\tau_{24}(p)\tau_{13}(p) + a_{34}a_{12}\tau_{34}(p)\tau_{12}(p) = 0, \\
    \tau(p) &\in \mathbb{C}, \quad p = (p_1, p_2, p_3, p_4) \in \mathbb{C}^4, \quad a_{ij} = -a_{ji} \in \mathbb{C}.
\end{align*}
\]

Above and hereafter we use abbreviations, such as

\[
\begin{align*}
    \tau_j(p) := D_j \tau(p) &= \tau(p + \delta_j), \\
    \tau_{ij}(p) := D_i D_j \tau(p) &= \tau(p + \delta_i + \delta_j), \\
    \delta_j := (\delta_{1j}, \delta_{2j}, \delta_{3j}, \delta_{4j})
\end{align*}
\]

with the Kronecker symbol $\delta_{ij}$.

(1) This is the simplest but nontrivial Plücker relation identically satisfied by the determinants.

(2) If we substitute

\[
\tau(p) = \prod_{i,j} \left( \frac{E(z_i, z_j)}{z_i - z_j} \right)^{p_i p_j} \theta \left( \zeta + \sum_j p_j w(z_j) \right), \quad a_{ij} = z_i - z_j
\]

into (1), we obtain an identity, called Fay’s trisecant formula, for the hyperelliptic function $\theta$ and the prime form $E(z_i, z_j)$ defined on a Riemann surface of arbitrary genus [7,8].

(3) This formula characterizes the Jacobi varieties among the Abel varieties [9,10].

(4) From this single equation all soliton equations in the KP hierarchy can be derived, corresponding to various continuous limits of independent variables [3,4].

(5) The solutions of (1) were identified with the points of the universal Grassmannian by Sato and are known as $\tau$ functions [11,12].

2.1. Nature of the $\tau$ functions

It was shown in Ref. [13] that the $\tau$ functions can be represented by means of the tachyon correlation functions of string theory. Since it provides the most convenient formulation in our argument, we
will use the notion of string theory in what follows. The 4-point string (tachyon) correlation function is given by

\[ \Phi(p, z; G) = \langle 0 | V(p_1, z_1) V(p_2, z_2) V(p_3, z_3) V(p_4, z_4) | G \rangle, \]

(4)

where \( z = (z_1, z_2, z_3, z_4) \in \mathbb{Z}^4 \) is a set of parameters determined by \( a_{ij} \) in Eq. (1). Here

\[ V(p_j, z_j) = : \exp(i p_j X(z_j)) : \]

is the vertex operator of momentum \( p_j \) of a string attached at \( z_j \) of the string world sheet specified by the state vector \( |G\rangle \). The string coordinate \( X(z) \) is an operator that acts on the state \( |\cdot\rangle \), while the symbol \( : \) means the normal order product. It was proved in Ref. [13] that the substitution of the ratio

\[ \tau(p) = \frac{\Phi(p, z; G)}{\Phi(p, z; 0)} \]

(5)

into (1) exactly yields Fay’s formula associated with the Riemann surface shown in Fig. 1, corresponding to the world sheet \( |G\rangle \). Hence the point \( z_j \) on the world sheet is a puncture of the Riemann surface. Notice that, since we do not integrate over \( z_j \), and thus avoid the problem of divergence, we define the vertex operator \( V(p, z) \) with no ghost field \( c \).

Although every solution of (1) is obtained by specifying the state \( |G\rangle \) of the general solution (5), we do not discuss explicit forms of \( |G\rangle \) in this paper. Therefore we simply write \( \Phi(p, z; G) = \Phi(p, z) \) unless it is necessary. On the other hand, the main property of the \( \tau \) functions is determined by the nature of the vertex operators, as we will see now. Since they satisfy

\[ V(p, z)V(p', z') = (-1)^{pp'} V(p', z')V(p, z), \]

(6)

we see immediately that the fields \( \psi_{\pm}(z) := V(\pm1, z) \) have the properties

\[ \psi_{\pm}(z)\psi_{\pm}(z') = -\psi_{\pm}(z')\psi_{\pm}(z), \quad \psi_{\pm}(z)\psi_{\mp}(z') = -\psi_{\mp}(z')\psi_{\pm}(z), \]

(7)

and, in particular, the following holds:

\[ \psi_{+}(z)\psi_{+}(z) = 0, \quad \psi_{-}(z)\psi_{-}(z) = 0. \]

(8)

Hence \( \psi_{\pm}(z) \) are Grassmann fields.

By taking this property into account we define operators \( \hat{D}^{\pm1}_s \) by

\[ \hat{D}^{\pm1}_s \Phi(p, z) := \langle 0 | V(p_1, z_1) V(p_2, z_2) V(p_3, z_3) V(p_4, z_4) \psi_{\pm}(z_s) | G \rangle \]

(9)

to describe the insertion of \( \psi_{\pm}(z_s) \) into \( \Phi(p, z) \) of (4). When \( z_s \) of \( \psi_{\pm}(z_s) \) coincides with one instance of \( z = (z_1, z_2, z_3, z_4) \) in (9), say \( s = j \), the insertion of \( \psi_{\pm}(z_s) \) is equivalent to changing \( p_j \) to \( p_j \pm 1 \),
up to a phase factor that comes from an exchange of the order of $\psi_\pm(z_j)$ with vertex operators. If we use notation such as
\[ \Phi_j(p, z) = \hat{D}_j \Phi(p, z), \quad \Phi_{ij}(p, z) = \hat{D}_i \hat{D}_j \Phi(p, z), \quad i, j = 1, 2, 3, 4 \]
in these particular cases, we obtain
\[ \Phi_{ij}(p, z) = -\Phi_{ji}(p, z), \quad i, j = 1, 2, 3, 4, \]
and hence
\[ \Phi_{ii}(p, z) = 0, \quad i = 1, 2, 3, 4. \]
This is an expression of (8). Thus we have found that the zero of $\Phi(p, z)$ is associated with a coincidence of two punctures on the Riemann surface.

In the theory of KP hierarchy [14–17] the operators $\exp \psi_\pm(z_j) = 1 + \psi_\pm(z_j)$ are known as elements of the symmetry group $GL(\infty)$ that act on the state $|G\rangle$. Correspondingly we call
\[ \hat{D}_j = e^{\hat{D}_j} - 1 \]
a “difference” operator, in analogy with the differential operator $\partial_j$.

In spite of this odd behavior of the correlation function $\Phi$ under the operation of $\hat{D}_j^\pm$, the solutions (5) of the HM equation behave regularly. This is owing to the fact that both $\Phi(p, z; G)$ and $\Phi(p, z; 0)$ are shifted by $\hat{D}_j$ simultaneously in $\tau(p)$. The extra phase factors arising from the exchange of the order of vertex operators and $\psi_\pm(z_j)$ in (9) cancel exactly. As a result, we find
\[ \hat{D}_j \tau(p) = D_j \tau(p) := \tau(p + \delta_j) =: \tau_j(p), \]
in agreement with our previous notation (2). Nevertheless, it is important to notice that the zero point of the $\Phi$ function (10) is an indeterminate point of $\tau(p)$. This fact will play a central role in our analysis of the singularity confinement in Sect. 5.

2.2. Difference geometry on lattice spaces

Although the variable $p$ of the $\tau$ function is on $\mathbb{C}^4$, the solutions of the HM equation are on a lattice space $\mathbb{Z}^4$ embedded in $\mathbb{C}^4$, which is fixed once an “initial point” $p_0 \in \mathbb{C}^4$ of $\tau$ is fixed. Let us call this lattice space
\[ \Xi_4(p_0) := \left\{ p \in \mathbb{C}^4 \middle| p - p_0 \in \mathbb{Z}^4 \right\}. \]
Since, however, the “initial point” $p_0$ does not appear explicitly in our discussion, we simply write $\Xi_4(p_0)$ as $\Xi_4$. Moreover, we often write $p - p_0$ as $p \in \Xi_4$ unless there could be confusion.

In order to study the HM equation within the framework of the theory of category, it will be useful to study its geometrical features on the lattice space $\Xi_4$. For this purpose we introduce the notion of “difference form” on the lattice space in this subsection.

2.2.1. 4D “difference” forms. Let us define [2] an exterior “difference” operator $d_B$, which acts on a form $\omega(p)$ of arbitrary degree, by
\[ d_B \omega(p) := \sum_{j=1}^{4} D_j \omega(p) \wedge dp_j, \quad (11) \]
in analogy with the differential forms. Notice that the action of $d_B$ on the correlation functions $\Phi$ is in analogy with the de Rham complex, while such a correspondence is unclear when it acts on the $\tau$ functions because the denominator of (5) obscures the Grassmann nature of $\psi(z)$. 

Fig. 2. $\Xi_4^{(n)}$.

We must emphasize that the form $D_j \omega(p)$ is on $\Xi_4$ if $p \in \Xi_4$, but, in contrast to the differential form, it is not at the same point $p$ but at $p + \delta_j$. In particular, the operation of $d_B$ to $\omega(p)$ increases the value of the sum $p_1 + p_2 + p_3 + p_4$ of the components of $p$ by 1. To describe the situation more precisely, we define a subspace of $\Xi_4$ by

$$\Xi_4^{(n)} := \{ p \in \Xi_4 \mid p_1 + p_2 + p_3 + p_4 = n \in \mathbb{Z} \},$$

so that

$$\bigcup_{n \in \mathbb{Z}} \Xi_4^{(n)} = \Xi_4.$$

We notice that $\Xi_4^{(n)}$ is a lattice hyperplane in $\Xi_4$. In particular, $\Xi_4^{(1)}$ is the hyperplane that includes the four points $\{ \delta_1, \delta_2, \delta_3, \delta_4 \}$. All other hyperplanes are parallel to $\Xi_4^{(1)}$.

Each hyperplane is embedded in a 3D lattice space $\mathbb{Z}^3$. In fact, the points of $\Xi_4^{(n)}$ occupy all corners of the octahedra that fill $\mathbb{Z}^3$ together with the tetrahedra, as is illustrated in Fig. 2.

If $p \in \Xi_4^{(n)}$, the forms $D_j \omega(p) = \omega_j(p)$ are on $\Xi_4^{(n+1)}$ for all $j$, hence

$$D : \Xi_4^{(n)} \to \Xi_4^{(n+1)},$$

with $D = (D_1, D_2, D_3, D_4)$. Since all functions in the HM equation (1) are of the form $\tau_{ij}(p) = D_i D_j \tau(p)$, they are on $\Xi_4^{(n+2)}$ if $p \in \Xi_4^{(n)}$. Moreover, the six functions $\tau_{ij}(p)$ in (1) are at the six corners of the octahedron whose center is at $p$. Hence the HM equation determines the relations between functions on $\Xi_4^{(n+2)}$. In other words, the solutions of the HM equation are different if they are on different hyperplanes. We note that this is a result of the fact that the HM equation is a Plücker relation.

Let us denote by $\omega^{(n)}$ a difference form on $p \in \Xi_4^{(n)}$. We then naturally obtain a graded algebra

$$\bigoplus_{n \in \mathbb{Z}} \mathbb{C} \omega^{(n)},$$

on which $d_B$ acts by

$$d_B^{(n)} : \omega^{(n)} \to \omega^{(n+1)}.$$ (13)

This can be interpreted as follows. On the right-hand side $\omega^{(n+1)}$ is an $m + 1$ form obtained from an $m$ form $\omega^{(n)}$ on $\Xi_4^{(n)}$. At the same time it is an $m$ form on $\Xi_4^{(n+1)}$ by definition.

2.2.2. 3D “difference” forms. From Fig. 2 we can see that the lattice space $\Xi_4^{(n+2)}$ consists of parallel planes, each filled by triangles of octahedra, as illustrated in Fig. 3. If we fix the direction of
the planes parallel to the direction of $-p_4$, we can specify them by the values of $t := p_1 + p_2 + p_3$. Such a plane is defined by

$$\mathbb{S}_3^{(t,n+2)} := \{(p_1, p_2, p_3) \in \mathbb{Z}^3 | p_1 + p_2 + p_3 = t \in \mathbb{Z}, (p_1, p_2, p_3, p_4) \in \mathbb{S}_4^{(n+2)}\},$$

$$\bigcup_{t \in \mathbb{Z}} \mathbb{S}_3^{(t,n+2)} = \mathbb{S}_4^{(n+2)}.$$

Since $n + 2 (= p_1 + p_2 + p_3 + p_4) = t + p_4$ is fixed, the planes are perpendicular to the direction of $-p_4$.

We denote a point on $\mathbb{S}_3^{(t,n+2)}$ by $p = (p_1, p_2, p_3) \in \mathbb{Z}^3$. Corresponding to $d_B$ of (13), we define the 3D exterior “difference” operator $d_T$ by

$$d_T \omega(p) = \sum_{a=1}^{3} D_a \omega(p) \wedge dp_a,$$

which we call the shift operator. Since, similar to the 4D case, $\omega(p)$ and $D_a \omega(p)$ are on different planes, we define graded forms of degree $t$ by $\omega[t]$ when $p \in \mathbb{S}_3^{(t,n+2)}$. The graded algebra

$$\bigoplus_{t \in \mathbb{Z}} \mathbb{C} \omega[t]$$

is generated by

$$d^t_T : \omega[t] \rightarrow \omega[t + 1]. \quad (14)$$

3. Dynamical features of the HM equation

Based on the geometrical structure of the HM equation, which we studied in the previous section, we discuss the dynamical features of the HM equation in this section.
3.1. Linear Bäcklund transformation

In order to study the dynamical features of the HM equation, we consider a pair of 4D “difference” 2-forms [2]:

\[
F := \sum_{i,j=1}^{4} F_{ij} dp_i \wedge dp_j, \quad F_{ij}(p) = a_{ij} \tau_{ij}(p),
\]

\[
\tilde{F} := \sum_{i,j=1}^{4} \tilde{F}_{ij} dp_i \wedge dp_j, \quad \tilde{F}_{ij}(p) = *a_{ij} \tau_{ij}(p)
\]

with \(*a_{ij} = \sum_{k,l} \epsilon_{ijkl} a_{kl}\), where \(\epsilon_{ijkl}\) is the Levi-Civita symbol. We can easily check that each of the following [18]

\[
det F_{ij} = 0, \quad det \tilde{F}_{ij} = 0
\]

is equivalent to the HM equation (1). For this reason, we call \(F\) of (15) the HM 2-form in this paper.

Let us consider, for any \(\tau(p)\) and \(\sigma(p)\), the following forms:

\[
\tilde{F} \wedge d_B \sigma = \sum_{j,k,l} *a_{kl} \tau_{kl} \sigma_j dp_k \wedge dp_l \wedge dp_j = \sum_{i,j} \tilde{F}_{ij} \sigma_j *dp_i,
\]

\[
d_B(\tilde{F} \wedge d_B \sigma) = \sum_{i,j} a_{ij} *\tau_j \sigma_{ij} dp_i \wedge *dp_i = \sum_{i,j} G_{ij} *\tau_j \delta_i, \quad (G_{ij} := a_{ij} \sigma_{ij})
\]

where we use the notation \(*dp_i := \epsilon_{ijkl} dp_j \wedge dp_k \wedge dp_l\) and \(\sum_{kl} \epsilon_{ijkl} a_{kl} \tau_{kl} = a_{ij} *\tau_j\). If we require \(\tilde{F} \wedge d_B \sigma = 0\), it is compatible iff \(d_B(\tilde{F} \wedge d_B \sigma) = 0\), or equivalently

\[
\sum_j \tilde{F}_{ij} \sigma_j = 0, \quad i = 1, 2, 3, 4,
\]

\[
\sum_j G_{ij} *\tau_j = 0, \quad i = 1, 2, 3, 4.
\]

Now suppose that \(\tau^{(0)}(p)\) is a solution of the HM equation (1). If it is substituted for \(\tilde{F}\) in (18), we can solve the linear equation for \(\sigma\), because \(\tilde{F}\) satisfies (17). Let us denote by \(\tau^{(1)}(p)\) the solution of (18) that is just obtained dependent on \(\tau^{(0)}\). Then we can substitute \(\tau^{(1)}\) into \(G\) of (19). Because (18) and (19) are the same equation, Eq. (19) for \(\tau\) certainly has a solution. This means that \(det G\) must vanish, so that, by (17) again, \(\tau^{(1)}(p)\) is a solution of the HM equation.

As \(\tau^{(1)}(p)\) is substituted, the linear equation (19) has solutions other than \(\tau^{(0)}\) in general. Let \(\tau^{(2)}\) be one of them that is independent from \(\tau^{(0)}\). \(\tau^{(2)}\) is again a solution of the HM equation, so that we can substitute it into (18) to find \(\tau^{(3)}\), and so on. Here we must mention that if the initial function \(\tau^{(0)}(p)\) is on \(\Xi_4^{(n)}\), the solution \(\tau^{(1)}(p)\) is on \(\Xi_4^{(n+1)}\). We can repeat this procedure and obtain a sequence of the Bäcklund transformation [19]:

\[
\begin{align*}
\tau^{(0)} & \quad d_B^{(0)} \quad \tau^{(1)} \\
& \quad d_B^{(1)} \quad \tau^{(2)} \\
& \quad d_B^{(2)} \quad \tau^{(3)} \\
& \quad \cdots 
\end{align*}
\]

If we choose, in (20), another solution of the HM equation for the initial function \(\tau^{(0)}\), we should have different series of solutions. In fact, in the string theory, we can add loops of closed strings...
so that the holes in the world sheet increase, as depicted in Fig. 1. There is also a vertex operator that substitutes a D-brane, so that the world sheet is attached to a boundary [20]. All such variations change the topology of the state |G⟩, and correspond to different initial solutions τ(0) of the HM equation. If we denote the complex (20) by BTG, corresponding to the state |G⟩, we will find a family of complexes of the Bäcklund transformation. The precise meaning of (20), however, will become clear in Sect. 4.2.

3.2. Dynamical evolution

In this subsection we fix the hyperplane Ξ(n+2)/3; hence, we simply denote Ξ(tn+2)/3 by Ξ(t), and study the behavior of a particular solution of the HM equation. For this purpose we rewrite the HM 2-form by using the operator dT of (14) as

\[ F = \sum_{i,j=1}^{4} F_{ij} dp_i \wedge dp_j = dT \left( F_4(p) dp_4 + \sum_{b=1}^{3} F_b(p) dp_b \right) \]

and see that the six components Fij split into two parts

\[ dT F_4(p) = \sum_{a=1}^{3} F_{a4}(p) dp_a, \]

\[ dT \sum_{b=1}^{3} F_b(p) dp_b = \sum_{b,c=1}^{3} F_{bc}(p) dp_b \wedge dp_c, \]

corresponding to the 3D 1-form and 2-form, respectively.

Let us denote by \( \tilde{S}_j \) and \( S_j \) the triangles in an octahedron that are perpendicular to \( p_j \) and parallel with each other. Then we can see that \{F_{a4}\} and \{F_{bc}\} form triangles

\[ \tilde{S}_4 := (\tau_{14}, \tau_{24}, \tau_{34}), \quad S_4 := (\tau_{23}, \tau_{31}, \tau_{12}), \]

which are perpendicular to \( p_4 \). We denote the octahedron, which consists of these triangles, by \( O = (\tilde{S}_4, S_4) \), as illustrated in Fig. 4.

From Fig. 3 we see that every octahedron is put between two nearest planes, such that two parallel triangles are on each plane. In fact, the triangle \( \tilde{S}_4 \) is on \( \Xi(t+1) \) and \( S_4 \) is on \( \Xi(t+2) \) if \( \tau(p) \) is on \( \Xi(t) \). Define \( O[t] = (\tilde{S}_4[t+1], S_4[t+2]) \) with

\[ \tilde{S}_4[t+1] = (\tau_{14}[t+1], \tau_{24}[t+1], \tau_{34}[t+1]), \]

\[ S_4[t+2] = (\tau_{23}[t+2], \tau_{31}[t+2], \tau_{12}[t+2]) \]
and let \( \{O\}[t] \) be all sets of octahedra put between \( \Xi_3^{[t+1]} \) and \( \Xi_3^{[t+2]} \). Then the procedure to solve the initial value problem is to determine the following sequence:

\[
\{O\} \xrightarrow{d_r} \{O\}[1] \xrightarrow{d_r} \{O\}[2] \xrightarrow{d_r} \{O\}[3] \xrightarrow{d_r} \cdots \tag{23}
\]

when information \( \{O\} \) at initial time \( t = 0 \) is given.

### 3.3. Deterministic rule for the flow of information

We want to know how information on \( \{O\} \) transfers to other octahedra as \( t \) increases. The HM equation determines the relation between \( \tilde{S}_4 \) and \( S_4 \). But it does not tell us how the information of \( \tilde{S}_4 \) transfers to \( S_4[1] \). In order to solve the initial value problem, we must know some deterministic rules that decide uniquely the local flow of information. In this subsection we set up a rule for the flow of information in an octahedron.

Let us consider two lattice points \( p \) and \( p' \) on \( \Xi_4 \), which are separated by

\[
p' - p = m \in \mathbb{Z}^4.
\]

If they are on the same hyperplane \( \Xi_4^{(n)} \), the separation \( m \) must satisfy

\[
m_1 + m_2 + m_3 + m_4 = 0. \tag{24}
\]

If \( p \) and \( p' \) are neighbors of an octahedron,

\[
m = \delta_i - \delta_j, \quad i \neq j. \tag{25}
\]

corresponding to the edge parallel to the vector \( p_i - p_j \). There will be many possible routes along which information can transfer between two points \( p \) and \( p' \) fixed arbitrary. Since an addition of a path of the type (25) does not change the condition (24), any routes can connect \( p \) and \( p' \) as long as they are connected by the edges of octahedra.

When we decide the rule of transfer of information we must keep in mind the following items:

- Information at \( p + \delta_i + \delta_j \) is transferred properly to \( p' + \delta_k + \delta_l \) if all operators corresponding to all possible routes \( \{r\} \) change \( \tau_{ij}(p) \) to \( \tau_{kl}(p') \) uniquely:

\[
D_r(ij, kl)_{p, p'} : \quad \tau_{ij}(p) \xrightarrow{r} \tau_{kl}(p'), \quad \forall r.
\]

- Our system is deterministic if a rule of transfer of information is fixed along the edges of an octahedron, and is the same for all octahedra.
- The system is integrable if this rule is sufficient to predict the values of \( \tau \) on \( \Xi_3^{[t]} \) for all \( t \) when the values on \( \Xi_3^{(0)} \) are given arbitrary.

There are 12 edges in an octahedron \( O \). Every object \( \tau_{ij} \) at a corner has connections to its four neighbors but no direct connection to its diagonal one. Since we are interested in the flow of information from one corner to another of \( O \), we must decide the direction of (and hence a rule for)
the flow. In other words, we fix the order of points in \( O \). A natural way is the cyclic ordering of suffixes, i.e.,

\[
3 > 2 > 1 > 4 > 3 > 2, \quad \text{or} \quad 1 > 4 \land 2 < 3.
\]

We notice in Fig. 4 that every pair of corners connected by an edge has a common suffix, such as \((\tau_{14}, \tau_{12})\). Therefore we define the direction of transfer by

\[
D(ij, jk) : \tau_{ij} \rightarrow \tau_{jk}, \quad \text{iff} \quad i > j > k. \tag{27}
\]

This rule fixes arrows along three edges of the triangle \( \tilde{S}_j \) for all \( j \). For example, when \( j = 1 \), three sets of order \( 2 > 1 > 4 \), \( 3 > 1 > 4 \), and \( 2 > 1 > 3 \) are compatible with (26) corresponding to \( D(21, 14) \), \( D(31, 14) \), and \( D(21, 13) \), respectively. On the other hand \( 3 > 1 > 2 \) is excluded by the rule, so that \( D(31, 12) \) is not allowed. In this way we can define uniquely the directions that are necessary to connect all objects in \( O \). In Fig. 5(a) the directions allowed by this rule (27) are shown by arrows. If we project the diagram along \( p_3 \), we have Fig. 5(b).

The action of \( D(ij, kl) \) is to remove punctures at \( z_i \) and \( z_j \) from \( |G\rangle \) and insert other punctures at \( z_k \) and \( z_l \). It will be convenient to represent this action explicitly by means of the operators

\[
D(ij, kl) = D_k D_l D_i^{-1} D_j^{-1}. \tag{28}
\]

We can easily check the rule of product,

\[
D(ij, kl) \circ D(kl, mn) = D(ij, mn).
\]

From

\[
D(ij, kl) \circ D(kl, ij) = D(ij, ij) = id
\]

we see that \( D(ij, kl) \) is an isomorphism.

Since corners connected by an edge have a common suffix, (28) simplifies to

\[
D(ij, jk) = D_k D_i^{-1},
\]

whereas the morphism connecting the diagonals of an octahedron can be obtained by a product of morphisms, for instance,

\[
D(32, 21) \circ D(21, 14) = D(32, 14) = D_2^{-1} D_3^{-1} D_4 D_1.
\]
3.4. Transfer of information along a chain of octahedra

In this subsection, we want to see how information flows along a chain of octahedra. Let us denote by \( \mathcal{O}(p) = (\tilde{S}_4(p), S_4(p)) \) the octahedron whose center is at \( p \in \Sigma_3^{(n)} \). There are three octahedra that share three edges of the triangle \( S_4(p) \). Since these three neighbors are on the same hyperplane \( \Sigma_3^{(n+2)} \), their centers must be at \( p + \delta_1 - \delta_4, \quad p + \delta_2 - \delta_4, \quad p + \delta_3 - \delta_4 \), respectively. Let \( \mathcal{O}(p) \) be one of them, say \( \mathcal{O}(p + \delta_3 - \delta_4) \). Then these octahedra are connected as illustrated in Fig. 6, where we use the abbreviations

\[
\tilde{S}_4 = (X, Y, Z) = (\tau_{14}, \tau_{24}, \tau_{34}), \quad S_4 = (X', Y', Z') = (\tau_{23}, \tau_{31}, \tau_{12}).
\]

For the information on \( \mathcal{O} \) to transfer to its neighbor \( \mathcal{O}[1] \) properly, we must impose the following conditions:

\[
\tau_{14}[1] = \tau_{13}, \quad \tau_{24}[1] = \tau_{23}. \tag{29}
\]

Repeating this procedure, we can define a chain of octahedra by

\[
\mathcal{O}(p)[t] = \mathcal{O}(p + t\delta_3 - t\delta_4), \quad t \in \mathbb{Z}. \tag{30}
\]

We recall that \( \tilde{S}_4(p) \) is on \( \Sigma_3^{(t+1)} \) and \( S_4(p) \) is on \( \Sigma_3^{(t+2)} \), if \( p_1 + p_2 + p_3 = t \). Because \( d_T \) transfers information on \( \Sigma_3^{(t)} \) to \( \Sigma_3^{(t+1)} \), we see that the information on \( \mathcal{O} \) can be transferred along the chain of octahedra,

\[
\mathcal{O} \xrightarrow{d_T} \mathcal{O}[1] \xrightarrow{d_T} \mathcal{O}[2] \xrightarrow{d_T} \mathcal{O}[3] \xrightarrow{d_T} \cdots, \tag{31}
\]

if the connection conditions (29) are satisfied at every site of connection. This is certainly compatible with (23), and explains the local flow of information.

4. View from the category theory

We are now ready to summarize our result in the previous section in terms of the category theory.

4.1. Triangulated category

We have pointed out in our previous paper [2] a possible explanation for the flow of information by means of the triangulated category. In order to see this correspondence in more detail let us first recall the axioms of the triangulated category [21,22].

**Definition 1.** Let \( \mathcal{D} \) be an additive category, \( X, Y, Z, X', Y', Z' \) be objects, and \( u, v, w \) be morphisms of \( \mathcal{D} \). The structure of a triangulated category on \( \mathcal{D} \) is defined by the shift functor \( T \) and the class of distinguished triangles satisfying the following axioms:
Tr1 (1) Any triangle of the form
\[ X \xrightarrow{\text{id}} X \rightarrow 0 \rightarrow T(X) \]
is in the class of distinguished triangles.

(2) Any triangle isomorphic to a distinguished triangle is distinguished.

(3) Any morphism \( u : X \rightarrow Y \) can be completed to a distinguished triangle
\[ X \xrightarrow{u} Y \rightarrow C(u) \rightarrow T(X) \]
by the object \( C(u) \) obtained by morphism \( u \).

Tr2 The triangle
\[ X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X) \]
is a distinguished triangle if and only if
\[ Y \xrightarrow{v} Z \xrightarrow{u} T(X) \xrightarrow{T(u)} T(Y) \]
is a distinguished triangle.

Tr3 Suppose there exists a commutative diagram of distinguished triangles,
\[ X \rightarrow Y \rightarrow Z \rightarrow T(X) \]
\[ u \downarrow \quad v \downarrow \quad \downarrow T(u) \]
\[ X' \rightarrow Y' \rightarrow Z' \rightarrow T(X') \]
This diagram can then be completed to a commutative diagram by a (not necessarily unique) morphism \( w : Z \rightarrow Z' \).

Tr4 (the octahedron axiom) Let \( X \xrightarrow{u} Y \xrightarrow{v} Z \) be a triangle. Then the following commutative diagram holds:

Comparing this set of axioms Tr1–Tr4 with our arguments in Sects. 3.3 and 3.4, we naturally find the following correspondence:

\[
\begin{array}{ccc}
\text{objects} & \leftrightarrow & \tau \text{ functions}, \\
\text{morphism} & \leftrightarrow & D(ij, kl), \\
T : A \rightarrow T(A) & \leftrightarrow & dT : A \rightarrow A[1], \\
\text{octahedron axiom} & \leftrightarrow & \text{flow diagram (Fig. 6)}.
\end{array}
\]
For example, there are flows of information that start from $A \in (X, Y, Z)$ and then end at $A[1] \in (X[1], Y[1], Z[1])$ as

\begin{align*}
X &\rightarrow Y \rightarrow Z \rightarrow X[1], \\
X &\rightarrow Y \rightarrow Z' \rightarrow X[1], \\
X &\rightarrow Z \rightarrow Y' \rightarrow X[1], \\
Y &\rightarrow Z \rightarrow X' \rightarrow Y[1], \\
Z &\rightarrow Y' \rightarrow X' \rightarrow Z[1].
\end{align*}

(34)

They are all compatible with the axioms, as we can easily check. Here the $id$ in (34) results from the condition (29). Via these routes, information in $\mathcal{O}$ can transfer to $\mathcal{O}[1]$. The correspondence (33) is, however, not yet complete, because we have not defined null object 0, which appears in the first axiom, Tr1. This is the main subject of this paper, which we are going to discuss in Sect. 5. We should also mention that, in contrast to the usual setting of a category in mathematics, our category has only one morphism $D_{ijkl}$, and thus has no Abelian structure. Nevertheless, as we have shown, this is sufficient to guarantee all axioms of the triangulated category.

4.2. The category theory of global behavior

Let us denote the triangulated category (33) as $\mathcal{HM}_G^{(n)}$ in this paper. The suffix $(n)$ is to remind us that a solution of the HM equation is given separately for each lattice space $\Xi_4^{(n)}$, while the suffix $G$ is to remind us that the solution is dependent on the initial state $|G\rangle$.

From the above argument, we see that the objects of the category $\mathcal{HM}_G^{(n)}$

$$\text{Ob}(\mathcal{HM}_G^{(n)}) = \{\tau_{ij}^{(n)}(p) \mid p \in \Xi_4^{(n)}\}$$

are nothing but a solution of the HM equation (1), which we denote as

$$\mathcal{HM}_G^{(n)} \simeq \tau_G^{(n)}$$

when $|G\rangle$ is given.

On the other hand, we learned in Sect. 3.1 that a series of Bäcklund transformations forms a complex $\mathcal{BT}_G$ of the solutions of the HM equation associated with the state $|G\rangle$, i.e.,

$$d_B : \tau_G^{(n)} \rightarrow \tau_G^{(n+1)}.$$

If we combine this with the result (35), we obtain

$$\mathcal{BT}_G = \left\{\mathcal{HM}_G^{(n)}_{n \in \mathbb{Z}}, d_B\right\}.$$

Therefore $\mathcal{BT}_G$ is a category of categories and $d_B$ is a functor. This is the precise meaning of (20). Moreover, as we mentioned in Sect. 3.1, the change of the state $|G\rangle$,

$$|G\rangle \rightarrow |G'\rangle = \mathcal{F}|G\rangle,$$

will generate a different complex $\mathcal{BT}_{G'}$ of the Bäcklund transformations. Therefore we obtain another functor:

$$\mathcal{F} : \mathcal{BT}_G \rightarrow \mathcal{BT}_{G'}.$$

We have thus found various types of categories that are related to each other. Among other points, our interpretation of the information flow in the HM equation by means of the triangulated category is the most fundamental.
5. Localization and singularity confinement

It is well known that a localization of a triangulated category is also a triangulated category. We show in this section that the singularity confinement of a rational map obtained from the $\tau$ function of the HM equation can be described in terms of the localization of the triangulated category.

5.1. Reduction of the lattice space

We have studied the difference geometry of 4 and 3 dimensions in Sect. 2. Our concern in this section is a 2D lattice space.

If we fix $n = p_1 + p_2 + p_3 + p_4 - 2$ and $t = p_1 + p_2 + p_3$, we are left with a 2D lattice space, which we parameterize by

$$q = p_1 + p_3, \quad j := p_2 - p_1.$$  

Like the higher-dimensional lattice cases, we define the 2D lattice space by

$$\Xi_2^{(q,t,n+2)} := \{ (p_1, p_2) \in \mathbb{Z}^2 \mid p_1 + p_2 = q \in \mathbb{Z}, \ (p_1, p_2, p_3) \in \Xi_3^{(t,n+2)} \}.$$  

and the exterior difference operator $d_Q$ by

$$d_Q \tau(p) = D_1 \tau(p) dp_1 + D_2 \tau(p) dp_2,$$

which displaces the lattice space

$$D : \Xi_2^{(q,t,n+2)} \to \Xi_2^{(q+1,t,n+2)}.$$  

Because $t = q + p_3$, we can fix $p_3$ instead of $t$. In this coordinate frame, the change of $t$ is exactly the same as the change of $q$. We recall that Fig. 6(b) was the projection of Fig. 6(a) along $p_3$. Since $j = p_2 - p_1 \in \mathbb{Z}$ is still free, we denote $\tau_j^{[t]} := \tau(p)$ and consider the lattice space

$$\left\{ (j, t) \in \mathbb{Z}^2 \mid \Delta t = \Delta q, \ p_2 - p_1 = j \in \mathbb{Z}, \ (p_1, p_2) \in \Xi_2^{(q,t,n+2)} \right\}.$$  

In our previous section we discussed the transfer of information along the chain of octahedra. We now extend the study to consider a transfer of information of many octahedra linked along a line in the $j$ direction.

We have to mention that the connection condition (29) is already taken into account in this expression. Moreover, the projection along $p_3$ enforces the degeneration of $Z'[t]$ in $O[t]$ and $Z[t + 2]$ in $O[t + 2]$. Therefore, the shift operation $d_T$ brings $\tilde{S}_4$ directly to $\tilde{S}_4[1]$, so that $\tau_j^{[t]}$ is determined uniquely for all $t$ and $j$, as we can see in Fig. 7.

In the theory of KP hierarchy, it is known that we can either truncate the function $\tau_j^{[t]}$, or impose periodicity in the direction of $j$ at any value, with no violation of integrability. For example, we can impose

$$\tau_j^{[t]} = \tau_j^{[t]}$$  

(36)

to obtain a reduced map of $d$ dimensions. In this case it is more convenient to consider

$$\tau^{[t]} = (\tau_0^{[t]}, \tau_1^{[t]}, \tau_2^{[t]}, \tau_3^{[t]}, \ldots, \tau_{d-1}^{[t]}),$$

instead of a triangle $\tilde{S}_4$ of each octahedron separately. When $d = 3$, the chain of Fig. 7 becomes a chain of triangles.
5.2. Localization of a triangulated category

The theory of triangulated category tells us that, if there is a null system, the theory can be localized such that the localized theory again satisfies the axioms of triangulated category [21,23]. A null system $N \subset T$ of the triangulated category $T$ is a set of objects defined by

1. $0 \in N$.
2. $Z \in N$ if $X, Y \in N$ and $X \rightarrow Y \rightarrow Z \rightarrow X[1]$ is a distinguished triangle.
3. $X[1] \in N$ iff $X \in N$.

For any triangulated category $T$ and a null system $N \subset T$, we define a multiplicative system by

$$S(N) := \{g | X \xrightarrow{g} Y \rightarrow Z \rightarrow X[1], \ X, Y \in T, \ Z \in N\}.$$ 

Then the localization is defined by the functor $T \rightarrow T/S(N)$. The following theorem is known in the theory of triangulated category:

**Theorem 1.** $T/S(N)$ is again a triangulated category whose null object is 0 itself.

Therefore, in order to discuss the localization of our system we must know the null system of our triangulated category. Our objects are solutions $\{\tau_{ij}\}$ of the HM equation that are assigned at corners of each octahedron. They are generically finite, because the $\tau$ functions are defined by the ratios (5) of correlation functions such that the zeros of correlation functions cancel each other out.

As we explained in Sect. 2.1, the correlation functions $\{\Phi_{ij}\}$ vanish by themselves when two punctures encounter each other on the Riemann surface. It is, however, important to notice that, when the correlation functions $\Phi(p, z; G)$ and $\Phi(p, z; 0)$ vanish simultaneously, the cancellation of their zeros does not mean that the value of their ratio is definite. Let $\lambda(p)$ be the ratio of the correlation functions at the point where they vanish, i.e.,

$$\lambda(p) := \left\{ \frac{\Phi(p, z; G)}{\Phi(p, z; 0)} \right\} \Phi(p, z; G) = \Phi(p, z; 0) = 0,$$

then $\lambda(p)$ is indeterminate in general, and hence can take any value. Since zero is not excluded in (37) we call the zero of $\lambda(p)$ the null object and denote it by 0, i.e.,

$$0 \in \lambda(p).$$

We now focus our attention on this subtle object in the following discussion and show how the localization of triangulated category resolves the subtlety.
The localization of our system will be introduced by considering rational maps of the $\tau$ functions. To be specific, we consider some reduced flow diagrams of Fig. 7 that satisfy the condition (36). In particular, we study in detail rational maps defined by the following variables:

\[
x^\tau_j = \begin{cases} 
\frac{[t_j]}{\tau^j_{j+\epsilon}} & \text{if } j, \epsilon = 1, 2, 3, \ldots, d, \text{ LV}, \\
\frac{[r_{j+1}]}{\tau^j_{j+\epsilon}} & \text{if } j, \epsilon = 1, 2, 3, \ldots, d, \text{ KdV}.
\end{cases}
\] (38)

The maps that are obtained by the new variables LV and KdV are called the Lotka–Volterra map and the Korteweg–de Vries map, respectively [24,25].

An important feature of the variables (38) is that they are invariant under the local gauge transformation of $\tau_j^\tau$,

\[
g : \tau_j^\tau \rightarrow \exp \left( \int^\prime v(t', j) dt' + \int^\prime \mu(t, j') dj' \right) \tau_j^\tau,
\] (39)

where $v(t, j)$ and $\mu(t, j)$ are arbitrary functions. For example, we can write the right-hand side of (38) as

\[
x^\tau_j = \begin{cases} 
\Phi_j^{[t]}(p, z) \Phi_j^{[r]}(p, z) & \text{if } j, \epsilon = 1, 2, 3, \ldots, d, \text{ LV}, \\
\Phi_j^{[r]}(p, z) \Phi_j^{[r]}(p, z) & \text{if } j, \epsilon = 1, 2, 3, \ldots, d, \text{ KdV}.
\end{cases}
\] (40)

This follows from the fact that the denominator of $\tau_{j+k}^{[t+u]}(p)$ is given by

\[
\Phi_j^{[r]}(p, z, 0) = (z_1 - z_2)^{(p_1 - k)(p_2 + k)}(z_3 - z_4)^{(p_3 + u)(p_4 - u)}
\]

so that all denominators of $\tau$ functions in (38) are eliminated exactly from the expression.

We notice that we cannot distinguish a change of $\lambda(p)$ in (37) with the gauge transformation

\[
g : \lambda(p) \rightarrow \lambda'(p).
\] (41)

This means that, if $\Lambda(\infty)$ is the set of all possible $\lambda(p)$, i.e.,

\[
\Lambda(\infty) := \left\{ \lambda(p) \middle| p \in \mathbb{Z}_4^{(u+2)} \right\},
\] (42)

$\Lambda(\infty)$ is invariant under the gauge transformation.

Now suppose that $\tau_j^{[t+1]}$ (or $\tau_j^{[r+1]}$ in the KdV case) in the denominator of $x_j^{[t]}$ in (38) is the null object, and hence takes the value zero. Then $x_j^{[t]}$ and also $x_j^{[r+1]}$ diverge while all components of $x_j^{[t+u]}$ with $u \geq 2$ are finite, as long as the other $\tau$ functions are finite. This is owing to the fact that the same $\tau$ function does not propagate beyond two steps. There is no way to determine the values of $\tau^{[t+1]}$ because the null object is invariant under the gauge (41), i.e.,

\[
g : 0 \rightarrow g0 = 0.
\]

In other words, the null object is transferred to an indeterminate object,

\[
d_T : 0 \rightarrow \lambda(p),
\] (43)

which is an element of $\Lambda(\infty)$. It should be emphasized that $\Lambda(\infty)$ does not appear in the localized theory, because the localized variables $x_j^{[t]}$ are gauge invariant. This fact strongly suggests us the
identification of $\Lambda(\infty)$ with the null system $\mathcal{N}$ of our map:

$$\mathcal{N} = \Lambda(\infty).$$

(44)

Our argument in the rest of this paper will be devoted to supporting this conjecture.

5.3. Singularity confinement

To advance our argument further, we consider the case $d = 3$ and $\epsilon = 1$ in (38) for simplicity. Then the HM equation becomes the following rational maps:

$$\begin{align*}
\text{LV map} & \quad x_1^{[t+1]} = x_1^{[t]} \frac{1 - x_2^{[t]} + x_2^{[t]} x_3^{[t]}}{1 - x_3^{[t]} + x_2^{[t]} x_1^{[t]}}, \\
& \quad x_2^{[t+1]} = x_2^{[t]} \frac{1 - x_3^{[t]} + x_2^{[t]} x_1^{[t]}}{1 - x_1^{[t]} + x_2^{[t]} x_2^{[t]}}, \\
& \quad x_3^{[t+1]} = x_3^{[t]} \frac{1 - x_1^{[t]} + x_2^{[t]} x_2^{[t]}}{1 - x_2^{[t]} + x_2^{[t]} x_3^{[t]}}.
\end{align*}$$

(45)

$$\begin{align*}
\text{KdV map} & \quad x_1^{[t+1]} = x_1^{[t]} \frac{1 + x_1^{[t]} x_3^{[t]} + x_1^{[t]} x_2^{[t]} (x_3^{[t]})^2}{1 + x_1^{[t]} x_2^{[t]} + (x_1^{[t]})^2 x_2^{[t]} x_3^{[t]}}, \\
& \quad x_2^{[t+1]} = x_2^{[t]} \frac{1 + x_1^{[t]} x_2^{[t]} + (x_1^{[t]})^2 x_2^{[t]} x_3^{[t]}}{1 + x_2^{[t]} x_3^{[t]} + x_1^{[t]} (x_2^{[t]})^2 x_3^{[t]}}, \\
& \quad x_3^{[t+1]} = x_3^{[t]} \frac{1 + x_2^{[t]} x_3^{[t]} + x_1^{[t]} (x_2^{[t]})^2 x_3^{[t]}}{1 + x_1^{[t]} x_3^{[t]} + x_1^{[t]} x_2^{[t]} (x_3^{[t]})^2}.
\end{align*}$$

(46)

These maps have two invariants. If $x = (x_1, x_2, x_3)$ denotes the initial value $x^{[0]}$, the invariants are given by

$$\text{LV map} \quad r = x_1 x_2 x_3, \quad s = (1 - x_1)(1 - x_2)(1 - x_3),$$

(47)

$$\text{KdV map} \quad r = x_1 x_2 x_3, \quad s = (1 + x_1 x_2)(1 + x_2 x_3)(1 + x_3 x_1),$$

(48)

respectively.

We can solve the initial value problem of the HM equation according to the algorithm:

A1 Fix initial values $\tau^{[0]} = (\tau_0^{[0]}, \tau_1^{[0]}, \tau_2^{[0]})$ and $\tau^{[1]} = (\tau_0^{[1]}, \tau_1^{[1]}, \tau_2^{[1]})$ by hand to determine $x = (x_1, x_2, x_3)$.

A2 $x^{[t]} := (x_1^{[t]}, x_2^{[t]}, x_3^{[t]}), \quad t \geq 1$ are obtained as functions of $x$ iteratively by the maps (45) or (46).

A3 $\tau^{[t+1]} = (\tau_0^{[t+1]}, \tau_1^{[t+1]}, \tau_2^{[t+1]}), \quad t \geq 1$ are determined by (38) from $\tau^{[t]}$ and $x^{[t]}$.

Needless to say, this procedure to solve the HM equation (1) is compatible with the flow of information through the chain of octahedra, since the rational maps (45) and (46) are derived from the HM equation by the transformation of dependent variables (38). The algorithm is certainly deterministic, since values of $\tau^{[t]}$ for all $t \geq 2$ are determined if the initial values $\tau^{[0]}$ and $\tau^{[1]}$ are fixed. As we will show in the following, however, it becomes unclear if the null object appears during the procedure.

The singularity confinements of the LV map (45) and KdV map (46) have been studied in detail [26–28]. To see what happens, we review this problem from the viewpoint of the theory of category.
Since we are interested in studying the singularity confinement we fix the initial conditions such that $x^{[1]}$ is divergent. Without loss of generality, this condition is satisfied by requiring the denominator of $x^{[1]}_1$ to vanish. Let us solve the 3D LV map case, by following our algorithm.

A1 We fix the initial values $\tau^{[1]}$ at

$$\tau^{[1]} = (\lambda_0, \lambda_1, \lambda_2)$$

and, instead of fixing $\tau^{[0]}$ by hand, we require that

(a) the denominator of $x^{[1]}_1$ vanishes:

$$1 - x_3 + x_3 x_1 = 0,$$

(b) the invariants $r, s$ are fixed by

$$r = x_1 x_2 x_3, \quad s = (1 - x_1)(1 - x_2)(1 - x_3),$$

from which we obtain

$$x := \left( \frac{r - s}{r + 1}, \frac{s + 1}{r + 1}, \frac{r + 1}{s + 1} \right),$$

and

$$\tau^{[0]} = \left( (r - s)\lambda_1, (s + 1)\lambda_2, (r + 1)\lambda_0 \right).$$

A2 Iteration of the map (45) yields the following sequence of singularity confinement:

$$x \rightarrow (\infty, 0, 1) \rightarrow (1, 0, \infty) \rightarrow x^{[3]} \rightarrow x^{[4]} \rightarrow \cdots$$

where

$$x^{[3]} = \left( \frac{\alpha^{(2)}}{\gamma^{(2)}}, \frac{\beta^{(2)}}{\alpha^{(2)}}, \frac{\beta^{(2)}}{\alpha^{(2)}} \right), \quad x^{[4]} = \left( \frac{\alpha^{(2)} \alpha^{(3)}}{\beta^{(2)} \gamma^{(3)}}, \frac{\beta^{(2)} \beta^{(3)}}{\gamma^{(2)} \alpha^{(3)}} \right),$$

$$\alpha^{(2)} := r + 1, \quad \beta^{(2)} := r - s, \quad \gamma^{(2)} := s + 1, \quad \alpha^{(3)} := r^2 - 3rs - s - rs^2, \quad \beta^{(3)} := sr^2 + 3rs - s^2 + r,$$

$$\gamma^{(3)} := r^2 + s^2 + r + s - rs + 1.$$  

A3 (a) From $x^{[1]} = (\infty, 0, 1)$, we find $\tau^{[2]}_1 = 0$ and $\tau^{[2]}_2 \lambda_2 = \tau^{[2]}_0 \lambda_1$. Since an overall factor is irrelevant, we obtain

$$\tau^{[2]} = (\lambda_2, 0, \lambda_1).$$

(b) From $x^{[2]} = (1, 0, \infty)$ we find $\tau^{[3]}_0 \lambda_2 = \tau^{[3]}_1 \lambda_1$, but $\tau^{[3]}_2$ is undetermined.

(c) Since the $x^{[r]}$ are finite for all $r \geq 3$, the rest of the $\tau^{[r+1]}$ are determined for all $t \geq 3$; thus, we obtain, up to the overall factors,

$$\tau^{[0]} \rightarrow (\lambda_0, \lambda_1, \lambda_2) \rightarrow (\lambda_2, 0, \lambda_1) \rightarrow (\lambda_0', \lambda_1', \lambda_2') \rightarrow \left( \lambda_2' \alpha^{(2)}, \lambda_0' \gamma^{(2)}, \lambda_1' \beta^{(2)} \right)$$

$$\rightarrow \left( \lambda_1' \alpha^{(3)}, \lambda_2' \gamma^{(3)}, \lambda_0' \beta^{(3)} \right) \rightarrow \cdots$$

Here we have defined new functions:

$$\left( \lambda_0', \lambda_1', \lambda_2' \right) := \left( \tau^{[3]}_0, \tau^{[3]}_1, \tau^{[3]}_2 \right),$$

which are free as long as

$$\lambda_0' \lambda_2 = \lambda_1' \lambda_1$$

is satisfied.
From this result it is clear how the singularity confinement undergoes the test. The singularities of $x^{[1]}$ and $x^{[2]}$ in (50) come from $\tau^{[2]}_{t_1} = 0$. This null object is the source of the singularities. This information, however, can transfer only to its neighbor since $\tau^{[2]}_{t_1}$ does not appear beyond $x^{[2]}$. Hence it does not transfer directly to remote objects.

We can extend the sequence of (53) to the left, if we apply the inverse map of (45) to $x$. We find, with $\lambda_{t+3} = \lambda_t$, $\lambda'_{t+3} = \lambda'_t$,

$$
\tau^{[t+2]} = \begin{cases} 
(\lambda'_{t-1}^{(t)}, \lambda_2^{(t)}, \lambda_1^{(t)}, \lambda_0^{(t)}), & t \geq 2, \\
(\lambda_0^{(t)}, \lambda_1^{(t)}, \lambda_2^{(t)}), & t = 1, \\
(\lambda_2^{(t)}, 0, \lambda_1^{(t)}), & t = 0, \\
(\lambda_0^{(t)}, \lambda_1^{(t)}, \lambda_2^{(t)}), & t = -1, \\
(\lambda_2^{(t)}, \lambda_1^{(t)}), & t \leq -2.
\end{cases}
$$

(55)

Here we denote by $\gamma^{[t]}$ the product

$$\gamma^{[t]} := \prod_{\{v\}} \gamma^{(v)}$$

and $\{v\}$ means the set of all prime numbers that divide $t$. Explicit forms of $\alpha^{(t)}$, $\beta^{(t)}$, $\gamma^{(t)}$ are given in the Appendix and in Eq. (59).

We can summarize the result of this subsection by the diagram in Fig. 8. In this figure, we have used the notation $\alpha^{(t)}_\lambda$, $\beta^{(t)}_\lambda$, and $\gamma^{[t]}_\lambda$ to represent the elements of $\tau$ in (55).

From this analysis we have learned that the existence of a null object introduces free functions $(\lambda_0^{(t)}, \lambda_1^{(t)}, \lambda_2^{(t)})$ in addition to the initial values $(\lambda_0, \lambda_1, \lambda_2)$. Since it is constrained by (54) and the overall factor is irrelevant, there is only one degree of freedom. As we have seen, this is owing to the gauge invariance of the null object $g0 \sim 0$.

5.4. Projective resolution

We are now ready to apply the localization theorem of the category theory discussed in Sect. 5.2. The diagram of Fig. 8 shows how the effect of this gauge freedom propagates as $t$ increases. It is important to notice that all the objects in the three triangles $\tilde{S}_4[1], \tilde{S}_4, \tilde{S}_4[1]$ are indeterminate. In other words,

$$\tilde{S}_4[1], \tilde{S}_4, \tilde{S}_4[1] \subset \Lambda(\infty).$$
If we accept the conjecture (44) to identify \( \mathcal{N} = \Lambda(\infty) \), we naturally define our multiplicative system by the set of gauge transformations

\[
S(\mathcal{N}) := \{ g \mid g : \lambda(p) \to \lambda'(p), \ 0 \to g0 = 0 \},
\]

so that our local theory is obtained simply by gauge fixing all \( \lambda_j \) and \( \lambda'_j \) at 1; see Fig. 9.

Figure 10 represents a subchain of Fig. 8, which can be obtained by iteration of the map \( x^t \to x^{t+k} \). In particular, in Fig. 11, the morphism \( u : \gamma^{[k]}_\lambda \to \lambda'_0 \) passes the epimorphism \( \pi : \lambda'_1 \to \lambda'_0 \).

Hence the object \( \gamma^{[k]}_\lambda \) is a projective object for all \( k \), and the exact sequence

\[
P := \cdots \to \gamma^{[4]}_\lambda \to \gamma^{[3]}_\lambda \to \gamma^{[2]}_\lambda \to \lambda_1 \to \lambda'_0 \to 0
\]

is a projective resolution of \( \lambda'_0 \).

This is a result of the theory of triangulated category in mathematics [23]. It tells us that infinitely many projections by \( \gamma^{[t]}_\lambda \) constitute the object \( \lambda'_0 \). But it is certainly unclear at this moment if it has something to do with the integrability of the map. To explain their relationship we must recall some results from our previous work.

Fig. 9.

Fig. 10.

Fig. 11.
5.5. Invariant varieties of periodic points (IVPP)

5.5.1. Generation of IVPPs. Let us consider a $d$-dimensional rational map whose initial point is at $x = (x_1, x_2, \ldots, x_d)$. The period $t$ condition of the map is a set of $d$ equations

$$x^{[t]}(x) - x = 0.$$  \hfill (57)

The following theorem—the IVPP theorem—was then proved in Refs. [26–28]:

**Theorem 2.** All periodic points of the map form a variety for each period if the map has more than $d/2$ invariants and there is no Julia set.

The varieties are called invariant varieties of periodic points (IVPPs) because they are algebraic varieties that are determined by the invariants alone. Namely, if we denote the variety by

$$v^{[t]} = \{ x | \gamma^{(t)} = 0 \},$$  \hfill (58)

the functions $\gamma^{(t)}$ are polynomial functions of the invariants. They can be derived from (57) directly by means of the Gröbner base method, although it is not easy.

In our recent papers [28,29] we have shown, however, that, if we use the singularity confinement method, we can derive them quite easily. In fact, we have already encountered $\gamma^{(2)}$ in (51) and $\gamma^{(3)}$ in (52) for the 3D LV map in the previous subsection:

$$\gamma^{(2)} = s + 1, \quad \gamma^{(3)} = r^2 + s^2 + r + s - rs + 1.$$  

If we continue the map we will find

$$\gamma^{(4)} = (r - s)^3 - s(r + 1)^3$$  
$$\gamma^{(5)} = -(r - s)^3 + (r - 2)(r + 1)(s + 1)(r - s)^4$$  
$$+ (2s + r)(r + 1)^2(s + 1)^2(r - s)^2 - s^2(r + 1)^3(s + 1)^3$$  
$$\gamma^{(6)} = -3(r - s)^4 - (r + 1)(s + 1)((r + 1)(s + 1) - 3)(r - s)^2$$  
$$- 3(r + 1)(s + 1)(r^3 + s^3 - r^2s^2 - 2r^2s - 2rs^2 - r^2 - s^2)$$  

etc.

This can be understood as follows. Since $v^{[2]} = 0$, a point satisfying $v^{[t]} = 0$ must include periodic points of period $t$. On the other hand, $v^{[t+2]} \propto \gamma^{[t]}$ holds, as we can see from (55). Hence the points on $v^{[t]}$ of (58) are periodic points of period $t$. If we repeat the procedure explained in the previous subsections, we derive a sequence of polynomial functions of the invariants. Thus we have found an important fact:

*P of Eq. (56) is a chain of polynomial functions whose zero sets are IVPPs.*

5.5.2. Degeneration of IVPPs. Now we must call attention to another remarkable feature of IVPPs. In Refs. [27,28] we found that IVPPs of all periods intersect on a variety that we called a variety of singular points (VSP). It is a variety of singular points for two reasons.

(1) Every point on the VSP is singular because it is a point that is occupied by periodic points of all periods simultaneously.

(2) It is an indeterminate point of the map. Namely, the VSP is a set of points on which the denominators and the numerators of the rational map vanish simultaneously.

In order to explore this odd behavior on the VSP, we studied the deformation of integrable maps by introducing a parameter, say $a$, by hand. When $a \neq 0$, periodic points form sets of discrete points for
each period, including the Julia set. We have shown that, as $a$ becomes small, some of these isolated points approach IVPPs, while a large number of them approach the VSP and crash there altogether in the $a = 0$ limit [27,28].

Moreover, in our recent paper [30] we have shown, by studying a simple 2D map, how the Julia set approaches the VSP. We have found that the periodic points move along an algebraic curve for each period as $a$ changes its value. As $a$ becomes zero the points approach a singular locus of the curve for each period. This locus is common to all curves of different periods. Hence periodic points of all periods degenerate at this point.

In all cases, the VSP itself is a variety of fixed points when $a \neq 0$, while it turns to a set of indeterminate points as $a$ becomes zero. Therefore, from these observations, we learn that a large number of periodic points of a generic map approach a set of fixed points as the map becomes integrable and the fixed points turn to the VSP.

Now we can translate the information on the VSP into the language of $\tau$ functions using the formula (38). In the case of the LV map (45) the points $x^{[1]} = (\infty, 0, 1)$ and $x^{[2]} = (1, 0, \infty)$ are on the VSP. From the algorithm in Sect. 5.3 they correspond exactly to the three triangles $(\tilde{S}_4[-1], \tilde{S}_4, \tilde{S}_4[1])$ of the chain in Fig. 8. The latter belongs to $\Lambda(\infty)$, which is another set of indeterminate points, but of the $\tau$ functions defined by (5) instead of the map functions (45).

The correspondence of the VSP with $\Lambda(\infty)$ is clear because the indeterminacy of both functions comes from the same source, i.e., the zero set of correlation functions $\Phi(p, z; G)$. Moreover, from this correspondence we see that $\lambda'_0$ and $\lambda_1$ in $P$ of Eq. (56) must be objects in which all IVPPs are degenerate. All these facts support the following conjecture:

$$\Lambda(\infty) = VSP$$

which provides us, together with (44), an interpretation of our null set as a source of IVPPs, and hence of the projective resolution.

6. Conclusion

Before we close this paper we would like to briefly discuss the characterization of integrability.

The Julia set is a closure of the set of unstable periodic points of iterations of a map [31]. If there is a Julia set the map is nonintegrable, because the map has some chaotic orbits whose behavior cannot be predicted. The existence of the Julia set is not necessary but sufficient for a map being nonintegrable.

Theorem 2, the IVPP theorem, which we explained in Sect. 5.5.1, tells us that a Julia set and an IVPP of any period cannot exist in one map simultaneously. This, however, does not mean that the existence of an IVPP is sufficient to guarantee the integrability of a rational map in general, because the theorem was proved only for maps with a sufficient number of invariants. Nevertheless, it will be worthwhile to study conditions for IVPPs being generated from the null object, in order to clarify the notion of integrability.

The rational maps, which we studied in this paper, belong to the KP hierarchy. The dynamical variables of these maps are related to the $\tau$ functions of the HM equation by the formulas (38) of the form

$$x^{[t]}_j = \frac{\tau_a^{[t]} \tau_b^{[t+1]}}{\tau_c^{[t]} \tau_d^{[t+1]}}.$$  

This particular form of transformation is the key to all our arguments.
In order to explain what this means, let us consider an arbitrary rational map \( x \to f(x) \). As we repeat the map \( t \) times, we will obtain a rational function \( f^{[t]}(x) \) whose degree increases exponentially as \( t \) increases, unless cancellations of factors in the numerator and the denominator happen to take place. If an initial condition was such that one of the factors of the denominators vanishes, the map will diverge. This particular factor appears in the map repeatedly since it has no opportunity of cancellation. In other words, the singularity confinement does not take place in general.

In the case of the map of the KP hierarchy, on the other hand, the function \( f^{[t]}(x) \) is always factorized in the form (61). This fact guarantees that the same factor is not transferred beyond two steps, and hence the singularity confinement becomes possible. We can understand this phenomenon from the very construction of the \( \tau \) functions. Namely, the zero of a \( \tau \) function does not force its neighbor to vanish, because of the Grassmann nature of the correlation function \( \Phi(p, z; G) \).

Now we want to rephrase this phenomenon in terms of the category theory. The IVPPs are generated from the null set of the triangulated category as the resolution of an object. This is a natural consequence of the localization of the \( \tau \) functions. The localization itself is supported by the axioms of the triangulated category as long as there exists a null object. Therefore the integrability of the KP hierarchy is guaranteed by the axioms of distinguished triangles.

\[ \alpha^{(t)} \text{ and } \beta^{(t)} \]

\[
\alpha^{(2)} = r + 1
\]
\[
\alpha^{(3)} = -rs^2 + r^2 - 3rs - s
\]
\[
\alpha^{(4)} = r^4s + 3r^3s + r^2s^2 + 6r^2s + r^2 - r^3s^2 - s^4r - 6rs^2 - 3r^3s - 2s^2
\]
\[
\alpha^{(5)} = -s^6r - 3rs^5 + 3r^2s^5 + 6s^4r - 6r^3s^4 + 3s^4r^2 + 2^2s^3 - 3r^3s^3 + 10r^3s^3
\]
\[
\quad - 10rs^3 - s^3 + 3^3r^3 + 21r^2s^3 + 2r^4s^2 + 27r^3s^2 + 6r^2s + 3r^3s - 3r^4s
\]
\[
\quad - 6r^5s + r^5 + r^4 + r^3 + r^6
\]
\[
\beta^{(2)} = r - s
\]
\[
\beta^{(3)} = 3rs + r - s^2 + r^2s
\]
\[
\beta^{(4)} = r^4 - 3r^3s + r^2s^3 + 6r^2s^2 + r^2 + r^3 + 3rs + 6rs^2 + r - s^3
\]
\[
\beta^{(5)} = -s^6r^2 + s^5 - 6r^2s^5 + s^5r^3 - 21s^4r^2 - s^4r^4 + 3s^3r^4 - 10s^4r + 3r^4s^3
\]
\[
\quad - r^2s^3 - 6rs^3 + 27r^3s^2 + 3s^3r^3 + 3r^3s^2 - 3rs^2 - 2r^4s^2 - r^3s^2 - 6r^4s^2
\]
\[
\quad - r^6s^2 - rs - 3r^2s - 6r^3s - 10r^4s + r^5,
\]

etc.

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