On the equivalence of lossy evolution and POVM generalized quantum measurements

PACS numbers: 03.65.Aa, 03.65.-w, 03.67.-a

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Abstract. Loss induced generalized measurements have been introduced years ago as a mean to implement generalized quantum measurements (POVM). Here the original idea is extended to a complete equivalence of lossy evolution and a certain widely used class of POVM. This class includes POVM used for unambiguous state discrimination and entanglement concentration. One implication of this equivalence is that unambiguous state discrimination schemes based on $\mathcal{PT}$ symmetric and non-Hermitian Hamiltonians have the same performance as those of standard POVM. After discussing several key points of this equivalence we illustrate our findings in two elementary physical realizations. Finally, we discuss several implications of this equivalence.
1. Introduction and motivation

Positive Operator Valued Measure (POVM) \cite{1,2} refers to a set of measurements within the standard theory of quantum mechanics that generalizes the standard projective measurement. POVM’s can be viewed as standard projective quantum measurements in an augmented Hilbert space (Neumark dilation \cite{1}). Using POVM’s it is possible to extract information that cannot be accessed by standard measurements. Perhaps one of the most impressive examples of the power of POVM is the unambiguous state discrimination (USD) problem (\cite{3,4,5,6,7} and references therein). USD has fundamental importance in quantum information and quantum cryptography (\cite{3,4}). Consider a system that is prepared in one of the states $|\alpha_i\rangle$, but not in a superposition of them. If the states are not orthogonal ($\langle \alpha_i | \alpha_j \rangle \neq 0$), there is no standard projective measurement that can detect the state of the system without an error that depends on the overlap of the states. In contrast, POVM, can insure a zero error probability. As expected, this does not come without a price. It turns out that there is an intrinsic nonzero chance that the POVM will give an “inconclusive result”. Namely, a result that cannot be uniquely assigned to one of the input states (yet, this is not considered an error).

In \cite{8} Hutnner et al. suggested and demonstrated experimentally the use of lossy evolution followed by projective measurements to perform a state discrimination. In contrast to the dilated projectors used in the Neumark scheme, their measurement is carried out in the original Hilbert space. The implementation in \cite{8} is based on embedding the lossy evolution in a larger Hilbert space where a unitary evolution takes place. Bender et al. \cite{9} suggested a different scheme. Instead of embedding, a PT symmetric Hamiltonian is used to generate a special non-unitary evolution that makes the states orthogonal at the end of the evolution. Once the states are orthogonal, a regular projective measurement can be used for discrimination. This procedure also leads to zero error probability but so far it has not been clarified how its performance (the inconclusive result probability) compares to that of the best POVM. In a recent paper \cite{10}, the Hamiltonian resources needed for USD based on $\mathcal{PT}$ symmetric/non-Hermitian Hamiltonian, have been quantified. In this work, we focus on the success probability, and show that the lossy (or $\mathcal{PT}$ symmetric/non-Hermitian generated) evolution and the POVM state discrimination schemes are equivalent, and therefore yield the same results.

It should be mentioned that lossy evolution associated with non-Hermitian Hamiltonians, is not a just a mathematical curiosity. It appears naturally when the particle number in a subsystem of interest is not conserved. In the past few years, non-Hermitian inspired non-unitary evolution has been intensively studied experimentally and theoretically in the context of $\mathcal{PT}$ symmetric Hamiltonians \cite{11-19}. For other studies of non-unitary evolution generated by more general Hamiltonians see \cite{20-28}.

We start by showing that given a lossy evolution there exists a USD POVM that yields the same measurement results. This holds for any lossy evolution regardless of state discrimination capabilities. Later for completeness we repeat in greater detail and from a slightly different point of view the converse direction that was studied in \cite{8}. That is, given a USD-like POVM set, we show an explicit construction of the equivalent lossy evolution operator. Several key implications of this equivalence are shortly described at the end.
2. Preliminaries

2.1. Projective measurements and optimal POVM

In this section we describe POVM from the perspective of the pure states USD problem. For a more general point of view we refer the reader to [1, 2]. If a system is prepared in a normalized state \(|\alpha_i\rangle\) with probability \(P_i\) the density matrix is given by
\[
\rho_0 = \sum_{i=1}^{N} P_i |\alpha_i\rangle \langle \alpha_i |
\]
In the USD problem the set \(\{|\alpha_i\rangle\}_{i=1}^{N}\) is not orthogonal. For generality we assume the complete USD problem where in an \(N\)-level system, \(N\) states should be discriminated. If there are \(L < N\) vectors of interest, other linearly-independent vectors can be added artificially. Alternatively, the vectors of interest can be unitarily rotated to a subspace of dimension \(L\) where once again the number of vectors is equal to the dimension of the Hilbert space.

Standard Von Neumann measurements are given by orthogonal projection operators of the form
\[
\Pi_i = |\psi_i\rangle \langle \psi_i |
\]
where \(\langle \psi_i | \psi_j \rangle = \delta_{ij}\). From the orthogonality of the states it follows that:
\[
\Pi_i \Pi_j = \Pi_i \delta_{ij}.
\]
In addition \(\sum_{i=1}^{N} \Pi_i = I\). The probability to find the system in state the \(|\psi_i\rangle\) is given by:
\[
p_i = \text{tr}[\rho \Pi_i].
\]
Since the system is prepared in one of the non-orthogonal states \(|\alpha_i\rangle\), there will be at least one state that will have a nonzero overlap with more than one operator \(\Pi_i\). Thus, when using \(\Pi_i\), an error in detecting the state of the system is inevitable. Yet, it is possible to find a different set of rank-one operators that will not overlap with more than one state. For this purpose we introduce the bi-orthogonal set of states \(\{|\alpha_{i\perp}\rangle\}_{i=1}^{N}\) that satisfies:
\[
\langle \alpha_{i\perp} | \alpha_{j\perp} \rangle = \delta_{ij},
\]
The set \(\{|\alpha_{i\perp}\rangle\}\) can be obtained from \(\{|\alpha_i\rangle\}\) in the following way. Let \(A\) be a matrix whose columns are \(\{|\alpha_i\rangle\}\). Since \(A^{-1}A = I\), the \(\{|\alpha_{i\perp}\rangle\}\) vectors are given by the rows of \(A^{-1}\). The transverse vectors are not orthogonal to each other \(\langle \alpha_{i\perp} | \alpha_{j\perp} \rangle \neq 0\). Furthermore, while the vectors \(|\alpha_i\rangle\) are normalized, the vectors \(|\alpha_{i\perp}\rangle\) are not, and their amplitude is determined by \(2\). In general, transverse vectors are not uniquely defined. Yet, if \(N\) transverse vectors are needed in an \(N\)-dimensional Hilbert space, then the orthogonal vectors are well defined (up to normalization) provided that the original vector are linearly independent.

From these transverse states we construct the rank one positive operators:
\[
F_{i\leq N} = \lambda_i |\alpha_{i\perp}\rangle \langle \alpha_{i\perp} |,
\]
where \(0 < \lambda_i\). We define another operator \(F_{N+1} = I - \sum \lambda_i |\alpha_{i\perp}\rangle \langle \alpha_{i\perp} |\) so that together with \(F_{i\leq N}\) we get:
\[
\sum_{i}^{N+1} F_i = I.
\]
† This rotation can be accomplished by the following procedure. First a Graham-Schmidt orthogonalization on the input states is performed, starting with the \(L\) vectors of interest. Then a unitary rotation is used to rotate the new first \(L\) orthogonal unit vectors (that span the original subspace \(L\)) to the computational basis \(\{e_i\}_{i=1}^{L}\). This transformation will also rotate the \(L\) vectors of interest to the subspace \(\{e_i\}_{i=1}^{L}\).
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The $\lambda_i$ cannot be chosen completely arbitrarily since $F_{N+1}$ must be a positive operator as well. A set of positive operators that satisfies (4) is called a POVM. In analogy to $\Pi_i$, the posterior probability to find the system in the state $|\alpha_i\rangle$ (not $|\alpha_i^\perp\rangle$) by performing a POVM is:

$$p_i = \text{tr}[\rho F_i]. \quad (5)$$

The completeness relation (4) insures that $\sum p_i = 1$ for any density matrix. Notice that relation (1) does not hold for POVM’s, since the vectors $|\alpha_i\rangle$ are not orthogonal to each other. The $F_{i\leq N}$ constructed here are the POVM operators needed for the USD of states $\{|\alpha_i\rangle\}$. The extra operator $F_{N+1}$ yields the probability of an inconclusive result, $p_{N+1}$. A set of $\lambda_i$ that minimizes the inconclusive results is called optimal [7]. In this work, the explicit values of $\lambda_i$ are not needed. Notice that while $F_{i\leq N}$ are rank one operators, while the rank of $F_{N+1}$ is typically larger than one.

In this work “USD POVM” refers to a POVM set $\{F_i\}_{i=1}^{N+1}$ over Hilbert space of dimension $N$, that has at least $N$ linearly independent rank one operators. The purpose for which the POVM is actually used may be different, but it still has the potential to perform an $N$-state unambiguous discrimination.

2.2. A lossy, single Kraus operator non-unitary evolution

“Non-unitary” evolution may refer to any evolution that is not unitary. In this paper, however, we always refer to a specific type of non-unitary evolution associated with losses (or in principle with gain as well). Let $K(T) = K$ be an evolution operator so that a state evolves from $t = 0$ to $t = T$ according to: $|\psi(T)\rangle = K |\psi(0)\rangle$, or in density matrix formalism:

$$\rho(T) = K \rho K^\dagger. \quad (6)$$

$$K^\dagger \neq K^{-1} \quad (7)$$

A more general class of non-unitary evolution includes a sum over different $K$’s. The most common scenario of this more general non-unitary evolution, arises when decoherence terms are included in the Lindblad equation [2]. Yet, as shown here, a single Kraus operator (lossy evolution) is enough to establish a complete mapping between USD POVM and non-unitary evolution.

Unlike a general Kraus map, a “Single Kraus operator” evolution (6), can always be generated by the Schrödinger equation with some non-Hermitian Hamiltonian. For example $\mathcal{PT}$ symmetric Hamiltonians generates evolution operators of the form (6). More generally, non-Hermitian Hamiltonians often appear in the study of resonances and metastable systems [20]. Typical scenarios include particle leakage from the system of interest (e.g. by tunneling) or the presence of absorption in the medium. In optics, for example, some photons are absorbed and converted into phonons. If only the photons are of interest their effective description leads to a non-Hermitian Hamiltonian (complex refractive index).

Although in optics the “wavefunction” can actually be amplified, in quantum mechanics it is not so easily done. Therefore, we focus here on passive systems, which cannot amplify the magnitude of the state. The $K$ associated with such systems is characterized [10] by $||K||_{sp} \leq 1$, where the spectral norm [29], $||\cdot||_{sp}$, is equal to the largest singular value of $K$ (see matrix norm and singular value decomposition in [29]):

$$||K||_{sp} = \sqrt{\max[\text{eigenvalues}(K^\dagger K)]} \quad (8)$$
The passiveness condition becomes more apparent if an alternative (yet equivalent) definition of the spectral norm is used. Let $|\psi_i\rangle$ be some initial state and $|\psi_f\rangle = K|\psi_i\rangle$ be a final state. The spectral norm is given by:

$$\|K\|_{sp} = \max_{|\psi_i\rangle} \sqrt{\frac{\langle \psi_f | \psi_f \rangle}{\langle \psi_i | \psi_i \rangle}}.$$  

(9)

Namely, the spectral normal is the maximal amplitude amplification $K$ can generate from all possible input states.

Another reason for looking at passive systems is that it makes the performance comparison sensible. Otherwise, by controlling the signal amplification, the detection probability can be effectively increased. A lossy evolution can also be realized by embedding in a larger unitary system [8, 30, 31, 32, 33]. When only some part of Hilbert space is measured it appears as if the evolution is non-unitary. In embedding schemes the condition $\|K\|_{sp} \leq 1$ is automatically satisfied.

3. Equivalence of lossy evolution and USD POVM

3.1. The results of lossy coherent evolution can be reproduced by a POVM.

Given some lossy evolution operator $K$, initial density matrix $\rho_0$, and projective measurement operators $\Pi_i$, we want to show that the probabilities $p_i = \text{tr}[\rho_f \Pi_i] = \text{tr}[K \rho_0 K^\dagger \Pi_i]$ can be obtained by:

$$p_i = \text{tr}[\rho_0 F_i],$$  

(10)

where $\{F_i\}_{i=1}^N$ is a part of a POVM set $\{F_i \leq N, F_{N+1}\}$ that satisfies the positivity and completeness requirements. As explained before we assume $K$ corresponds to a passive system (without gain) so $\|K\|_{sp} \leq 1$. This can always be arranged by setting $K \rightarrow K/\Gamma$ where $\Gamma \geq \|K\|_{sp}$. The probabilities at the end of the evolution are:

$$p_i = \text{tr}[\rho_f \Pi_i] = \text{tr}[K \rho_0 K^\dagger \Pi_i] = \text{tr}[\rho_0 K^\dagger \Pi_i K] \equiv \text{tr}[\rho_0 F_i],$$

Clearly the operators $F_i$ reproduce the same probabilities as obtained by $K$. It remains to show that when complemented with another operator $F_{N+1}$, the set $\{F_n\}_{n=1}^N$ constitutes a legitimate POVM. We start by verifying that the $F_i$ operators are positive. Using $\Pi_i^2 = \Pi_i$ and $\Pi_i^\dagger = \Pi_i$:

$$F_i = K^\dagger \Pi_i K = (\Pi_i K)^\dagger \Pi_i K,$$  

(11)

it becomes clear that $F_i$ is positive since the RHS has the generic form of a positive operator. Next, in order to satisfy (4) we define:

$$F_{N+1} = I - \sum_{i=1}^N F_i = I - K^\dagger K.$$  

(12)

Now (4) is trivially satisfied but it must be verified that $F_{N+1}$ is a positive operator as well. In the diagonal basis, $F_{N+1}$ is equal to $I - S^2$, where $S$ is a diagonal positive matrix whose elements are the singular values of $K$. Since in passive systems the largest singular value ($\|K\|_{sp}$) must be one or less, it follows that $F_{N+1}$ is positive (all of its eigenvalues are non negative). This is a general feature; the passiveness requirement
of $K$ is equivalent to the POVM completeness requirement. In summary, we conclude that:

$$\{ F_{i \leq N}, F_{N+1} \} = \text{POVM}. \quad (13)$$

The lossy evolution outcome is the same as that obtained from the POVM set $\{ 11 \}, (12)$. The immediate consequence of the results above is that the performance of the $\mathcal{PT}$ symmetric discrimination scheme presented in [9] is the same as the performance of the POVM scheme. Before concluding we point out a feature of a POVM constructed from $K$. If the system is marginally passive, $\| K \|_{sp} = 1$, then rank $(F_{N+1}) < N$. As a result the Kraus operator $M_{N+1}$ defined by $F_{N+1} = M_{N+1}^\dagger M_{N+1}$ operator has a rank smaller than $N$. The new density matrix after an inconclusive result is obtained, is $\rho? = M_{N+1} \rho M_{N+1}^\dagger / \text{tr}(M_{N+1} \rho M_{N+1}^\dagger)$. Since the rank of $\rho?$ is also smaller than $N$, we get that the states in $\rho?$ are no longer linearly independent $\S$. This means that the inconclusive density matrix cannot be used to perform another Unambiguous state discrimination $[34]$. 

3.2. USD POVM results can be reproduced by a lossy coherent evolution.

In this subsection we show that any USD POVM in Hilbert space of size $N$, is equivalent to a single lossy evolution operator (LEO) $K_{N \times N}$ followed by a projective measurement. This should be contrasted from the Neumark dilation scheme $[1]$ where POVM is interpreted/implemented as projective measurements in a Hilbert space larger than $N$. To avoid overlap with $[8]$ and to take a slightly more general point of view we take a slightly different approach. Let $\{ F_i \}_{i=1}^{N+1}$ be a given USD POVM set where the first $N$ operators have rank one. We forget for now about the state discrimination problem and consider a bit more general problem. We want to replace $\{ F_i \}_{i=1}^{N+1}$ by an equivalent lossy evolution operator $K$ for any density matrix. That is, the density matrix to be measured may not be a statistical mixture of the states $\{ F_i \}_{i=1}^{N+1}$ can discriminate. Although the $\lambda_i$ and the $| \alpha_i \rangle$ normalization can be calculated it is not explicitly needed. Therefore we can simply write:

$$F_{i \leq N} = | \beta_i \rangle \langle \beta_i |, \quad (14)$$

where the normalization of $| \beta_i \rangle$ is determined by the given $F_i$. The extra operator satisfies $F_{N+1} = I - \sum_{i=1}^{N} F_i$.

Given a density matrix, $\rho_0$, the probability to detect the $i$-th result associated with the POVM operator $F_i$ is given by $p_i = \text{tr}[\rho_0 F_i]$. Our goal is to show that the same probabilities can be obtained by a lossy evolution $K$:

$$p_i = \text{tr}[\rho_f \Pi_i] = \text{tr}[(K \rho_0 K^\dagger) \Pi_i], \quad (15)$$

where $\{ \Pi_i \}_{i=1}^{N}$ is a set of projective measurement (i.e. satisfies $[1]$) and $\rho_f$ is the final density operator generated by $K$. Next we find an explicit expression for $K$ as a function of the chosen $\Pi_i$. From $[6]$ and $[15]$:

$$F_i = K^\dagger \Pi_i K.$$ 

Direct substitution verifies that the $K$ that satisfies this relation is:

$$K = \sum_{i=1}^{N} \pi_i F_i e^{\Phi_i} / \sqrt{\text{tr}(F_i \pi_i)}, \quad (16)$$

$\S$ One may suspect that the rank reduction of $\rho_f$ with respect to $\rho$ in the $\| K \| = 1$ case is an indication the one of the input states simply does not appear $\rho_f$ and as a result further USD is possible. However in $[35]$ there is an argument that explains why this is impossible.
where the \( \phi_i \) are arbitrary phase degrees of freedom that do not affect the measurement results or the measurement basis. Nonetheless \( \phi_i \) effects the eigenvalues of \( K \) and other properties of \( K \).

### 3.3. Properties of \( K \)

In [8], it is mentioned that \( K \) is diagonalizable operator and that eigenvalues of \( K \) have moduli smaller than one. It what follows we clarify that the first statement is not necessary, and that the second statement is not sufficient. The only limitations on \( K \) are that \( \|K\|_{sp} \leq 1 \) and that \( K \) is invertible. The invertibility follows from the following argument. \( K \) takes \( N \) non-orthogonal linear independent vectors and transforms them into \( N \) orthogonal linearly independent vectors. Writing the vector in column matrix \( G \) we have \( G_{\text{out}} = KG_{\text{in}} \). Since \( \det(G_{\text{in}}) \neq 0, \det(G_{\text{out}}) \neq 0 \) if follows that \( K \) must be invertible.

As an example of a legitimate non-diagonalizable lossy evolution operator consider the following Jordan form evolution operator:

\[
K_J = \begin{pmatrix}
\alpha & 1/2 \\
0 & \alpha
\end{pmatrix}
\]  

The passiveness condition is: \( |\alpha| \leq 1/\sqrt{2} \). The states that can be discriminated are given by the columns of \( (K_J^{-1})^\dagger \). Furthermore, notice that the eigenvalues are just \( \alpha \) and for \( 1/\sqrt{2} < |\alpha| \leq 1 \) their modulus is smaller than one. Yet, in this regime the spectral norm is larger than one and the evolution operator is not passive anymore. In particular non-passive systems cannot be embedded in a unitary evolution as suggested in [8, 31, 32, 33]. The necessary and sufficient equivalence condition \( \|K\|_{sp} \leq 1 \) does implies that the moduli of the eigenvalues of \( K \) are smaller than one, but the converse is not true.

The lossy evolution operator (16) can be written in a more intuitive form. Using \( \pi_i = |\psi_i\rangle \langle \psi_i| \) one can see that:

\[
K = \sum_{i=1}^{N} a_i |\psi_i\rangle \langle \beta_i|,
\]  

where \( a_i \) are some complex coefficients. Essentially, \( K \) converts the non-orthogonal vectors which are bi-orthonormal to \( |\beta\rangle \) to the orthogonal states \( |\psi_i\rangle \). Alternatively \( K^\dagger \) takes the orthogonal states \( |\psi_i\rangle \) to the non-orthogonal states \( |\beta_i\rangle \). This reflects the two complimentary points of view on USD: one can think of \( K \) as orthogonalization operator that acts on the density matrix, or alternatively as an operator that transform a standard projective measurement into a POVM. Further aspects and properties of \( K \) which are beyond the scope of this paper are studied in [35].

### 4. Illustrative physical examples

In this section, we study two optical systems that demonstrate the close kinship lossy evolution and POVM. For other USD implementations in optics see [8, 30, 31, 32, 37, 38] and references therein. In the first example, we show how a lossy evolution can implement a POVM without extending the Hilbert space, while in the second example we examine an implementation of USD that does resort to Hilbert space dilation (embedding scheme).

Figure 1 shows a very simple optical realization of a POVM using a non-unitary element. The system consists of a 50-50 beam splitter and an attenuator \( \gamma \) that
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Figure 1. a non-unitary optical implementation of a POVM measurement that performs an unambiguous state discrimination (USD). A click at detectors C or D will correctly indicate if the input is in state $|\alpha_1\rangle$ or $|\alpha_2\rangle$, even though $\langle \alpha_2 | \alpha_1 \rangle \neq 0$. This is possible only due to the presence of the attenuation plate $\gamma$ that breaks unitarity. Since part of the light is absorbed, in some cases there will be no click at the detectors. This is a manifestation of the “inconclusive result” that appears in POVM based USD.

attenuates light by a factor $0 < \gamma < 1$. The evolution operator is:

$$K = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \gamma \\ -1 & \gamma \end{pmatrix}. \quad (18)$$

We wish to find two non-orthogonal input vectors, $|\alpha_{1,2}\rangle$, that at the output will populate exclusively either waveguide no. 1 (for $|\alpha_1\rangle$) or waveguide no. 2 (for $|\alpha_2\rangle$). These vectors are given by the columns of $K^{-1}$ since they need to satisfy:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = K \{ |\alpha_1\rangle, |\alpha_2\rangle \}. \quad (19)$$

The LHS constitutes a choice of the measurement basis $\pi_i$. For a different choice of $\pi_i = |\psi_i\rangle \langle \psi_i|$, the identity matrix should be replaced by a matrix whose columns are the $|\psi_i\rangle$ vectors. After normalization we get: $|\alpha_{1,2}\rangle = \frac{1}{\sqrt{1 + |\gamma|^2}} (\pm \gamma, 1)^T$ where $'T'$ stands for transposition. Upon applying $K$ to these vectors, the output is $\frac{\sqrt{2\gamma}}{\sqrt{1 + |\gamma|^2}} (1, 0)^T$ for $|\alpha_1\rangle$, and $\frac{\sqrt{2\gamma}}{\sqrt{1 + |\gamma|^2}} (0, 1)^T$ for $|\alpha_2\rangle$. Since, at the output the vectors are orthogonal, they can easily be discriminated by detectors C and D. Notice that as the input vectors become almost identical ($\gamma \rightarrow 0$) the detection probability goes to zero, since the output is proportional to $\gamma$ for these specific input vectors. Though understandable, we find it beautiful that loosing part of the input signal gives access to information that lies outside the reach of unitary evolution.

Figure 2 shows another optical system that consists of three parallel waveguides equally spaced from each other. If the waveguides are not too close to each other, this system is well described by the tight binding Hamiltonian:

$$H = a \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \quad (20)$$
Figure 2. In this system, the non-unitarity is achieved by coupling the light to an auxiliary waveguide (no. 3) that is not initially populated. The effective evolution of light in the subsystem of waveguides 1 and 2 is non-unitary, and USD becomes possible. A click at detectors 1 or 2 means a successful discrimination while a click at the third detector means an inconclusive result.

where the components of the state vector are the peak amplitudes in waveguides 1, 2 and 3. Without loss of generality, we set $a = 1$ (it just rescales the propagation coordinate). Quantum mechanically, this Hamiltonian can describe three potential wells arranged in an equilateral triangular where only the ground state interaction is dominant (i.e. there is a large energy gap to the next level). We denote by '1' and '2' the input ports and the output ports of interest. Waveguide no. 3 will be used as an auxiliary waveguide that is initially not populated. The unitary evolution of the three waveguides is given by $U = \exp(-iHz)$, where $z$, the propagation coordinate, plays the role of time. By applying $U$ to $(1, 0, 0)$ and to $(0, 1, 0)$, we can construct a reduced two-waveguide evolution operator $K = K_{2 \times 2}(z)$. This $K$ will give the correct output of waveguides 1 and 2 for any input state that does not initially populate the third waveguide. In general, $K$ is not unitary, since part of the optical power goes to waveguide no. 3. According to (19), the input states the system is able to discriminate, can be obtained from $K^{-1}$. To see the evolution in the whole three-waveguide system, we apply $U$ (instead of $K$) on these two input states. One can show that the output is of the form:

$$U(|u_1^T, 0)^T = (\beta, 0, \sqrt{1 - |\beta|^2}),$$

$$U(|u_2^T, 0)^T = (0, \beta, \sqrt{1 - |\beta|^2}).$$

The factor $|\beta| \leq 1$ becomes smaller as $|\langle \alpha_1 | \alpha_2 \rangle|$ becomes larger (i.e. when the input vectors are more similar to each other). To complete the USD scheme, a photon detector is placed at the output of each port. If there is a hit at no. 1 (no. 2) we infer the system was in state $|\alpha_1\rangle$ ($|\alpha_2\rangle$). If there is a hit at detector no. 3, we cannot tell what was the state of the system (follows from the form of (21) and (22)). This is exactly the POVM inconclusive result. We conclude that this simple apparatus successfully implements USD for ports 1 and 2. Inspecting the output vectors [21].
and (22), we see that, as expected, they remained non-orthogonal when all three components are considered, since $U$ is unitary. Yet, when only the subspace 1 and 2 is considered, the two vectors look orthogonal at the output.

Notice that there is a relation to the first example. If the attenuator is replaced by a beam splitter with transmittance $\gamma$, then the inconclusive result can be detected by monitoring the reflected photon. By adding this extra port, the system is now described by a larger Hilbert space just like in the second example.

5. Concluding remarks

The implications of the results presented here extend beyond the formal equivalence of two different approaches to unambiguous state discrimination. In [33] we discuss the resources needed for embedding a lossy evolution in a larger Hilbert space where a unitary (zero-loss) evolution takes place. Together with the findings presented here we obtain a non-trivial relation between energy and generalized measurements. We find what are the minimal Hamiltonian resources needed to embed a USD POVM in a unitary evolution.

A second implication concerns the general theory of multiple quantum state discrimination. The representation of a USD POVM set of operators by a single lossy evolution operator reveals new features of multiple state discrimination that are very difficult to deduce directly from the original POVM set of operators [35].

Acknowledgments

The author thanks Omri Gat for useful comments.

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