3-D Stochastic Model for Turbulent Silt-laden Flows

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Abstract: A 3-D stochastic model is developed for turbulent silt-laden flows in the Xiaolangdi river. Based on the stochastic theory of turbulent flows, Reynolds stresses for anisotropic turbulent flows, a pressure Poisson equation for free surface flow and a refined wall function for solid wall boundaries are obtained. In body-fitted coordinates, the finite volume method (FVM) is used to discretize the turbulent equations together with a staggered grid system, and the SIMPLEC algorithm is applied to simulate the flow over the complex geometry field. Finally, this mathematical model is applied to simulate water levels, silt concentrations and velocity distributions at the various cross-sections in this river, and all the computed results agreed well with the observed data, indicating this model and this computational schedule are reliable.

1. Introduction

2D sediment mathematical models have been shown to provide useful results to meet the requirements of hydraulic engineering. However, more often 3D problems of water flow and sediment motion are observed characteristic of a natural river and the area near a hydraulic building. To understand thoroughly these concerns, 2D numerical models are sometimes not sufficient, and therefore 3D mathematical models have become one of the most intensively studied subjects in the field of river dynamics [1].

Now several 3D mathematical models for sediment transport are available. Van Rijn [2] proposed a combined model, in which the water flow is calculated by a 2D depth-averaged water model with the assumption of a vertical logarithmic velocity profile, and further developed a combined 3D mathematical model. Shimizu et al. [3] predicted water flows and bed deformation of meandering channels using a 3D sediment model. Olsen et al. [4, 5] suggested a 3D mathematical model for sediment flow in a sand trap tank and estimations of maximum local scour depth. Lin et al. [6] proposed a 3D model to simulate suspended sediment in estuarine and coastal waters under the assumption of static water pressure. Wu et al. [7] put forth a 3D mathematical model for total sediment (suspended sediment and bed load) based on Reynolds-averaged Navier-Stokes equations together with $k-\varepsilon$ turbulence model. Based no a non-equilibrium sediment transport mode, Fang et al. [8] developed a 3D numerical model for turbulent secondary flows and suspended load motions using a non-orthogonal curvilinear system. Bijan [9] used the RNG $k-\varepsilon$ turbulence model and a non-

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equilibrium wall function to investigate sediment transport patterns of the natural river in the southwest part of Sweden. Based on $k - \varepsilon$ turbulence model, Olsen [10] proposed a three-dimensional CFD model to compute the formation of the meandering pattern in an initially straight alluvial channel, and successfully replicated many of the meander characteristics, including secondary currents, cross-sectional profiles, meander planform, meander wavelength, downstream meander migration, and chute formation.

Most theories and modeling techniques for sediment motions are based on the coherent structures of turbulence flows [11]. The above-mentioned models are isotropic based on the Boussinesq assumption [12]. Though the $k - \varepsilon$ turbulence model and other models are widely applied to hydraulic engineering projects, such models are not robust enough to predict complex problems, such as buoyant flows, rotational flows, open flows in meandering channels and streamline curved flows. On the contrary, the stochastic theory of turbulent flows overcomes the shortcomings of Boussinesq assumption and has been validated by various applications [13]. In this paper, a 3D stochastic model was proposed for turbulent silt-laden flows in the river reach of Xiaolangdi Project. In order to verify the effectiveness of the numerical model, which combines the stochastic model and a Poisson equation model for free surface flow, the simulated results are investigated and compared with the experimental data from a physical model [14].

2. Governing equations

The sediment mathematical model can be written in the tensor form as follows.

2.1. Continuity equations for water flows

$$\frac{\partial u_i}{\partial x_j} = 0$$  \hspace{1cm} (1)

2.2. Momentum equations for water flows

$$\frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_j)}{\partial x_j} = f_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left( \nu \frac{\partial u_i}{\partial x_j} - \frac{\partial (u_i u_j)}{\partial x_j} \right)$$  \hspace{1cm} (2)

where, $f_i$ is the unit mass force, $u_i$ is the flow velocity, $\rho$ is the water density, $\nu$ is the kinematic viscosity, $u_i'$ the fluctuating velocity, and $\frac{\partial (u_i u_j)}{\partial x_j}$ is the Reynolds stress term.

In order to the Reynolds stress term, the correlation of fluctuating velocity can be written as follow based on stochastic theory of turbulent flows [13].

$$u_i' u_j' = -\beta_j \alpha_j \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$  \hspace{1cm} (3)

where, $\beta_j = \frac{2}{3} k + 2 C_n \frac{k^3}{\varepsilon^2} \left( \frac{\partial u_i}{\partial x_m} \frac{\partial u_j}{\partial x_j} - \frac{Q}{48} - \frac{R}{3} \right)$, $R = (3 \delta_{lm} - 1) \left( \frac{\partial u_n}{\partial x_m} + \frac{\partial u_m}{\partial x_n} \right) \left( \frac{\partial u_n}{\partial x_l} + \frac{\partial u_l}{\partial x_n} \right)$, 

$$Q = \frac{\partial u_n}{\partial x_m} \frac{\partial u_i}{\partial x_l} , \hspace{0.5cm} \alpha_j = C_i \frac{k^2}{\varepsilon} - C_n \frac{k^3}{8 \varepsilon^3} (3 \delta_{lm} - 1) \left( \frac{\partial u_n}{\partial x_m} + \frac{\partial u_m}{\partial x_n} \right) \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) , \hspace{0.5cm} k$$ is the turbulent energy, $\varepsilon$ is the turbulent dissipation rate, $C_i$ and $C_n$ are empirical coefficients, subscripts $m$, $n$ and $l$ ($=1, 2, 3$) meet summations.

Transport equations for $k$ and $\varepsilon$ are
\[
\frac{\partial k}{\partial t} + u_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j} \left( \frac{\nu_t}{\sigma} \frac{\partial k}{\partial x_j} \right) + G - \varepsilon \tag{4}
\]
\[
\frac{\partial \varepsilon}{\partial t} + u_j \frac{\partial \varepsilon}{\partial x_j} = \frac{\partial}{\partial x_j} \left( \frac{\nu_t}{\sigma} \frac{\partial \varepsilon}{\partial x_j} \right) + \frac{\varepsilon}{k} (C_1 G - C_2 \rho \varepsilon) \tag{5}
\]

where, \( G = \nu_t \left[ 2 \left( \frac{\partial u_i}{\partial x_j} \right)^2 + \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 \right] \) is the product term of turbulent energy, and \( \nu_t = C_s \rho k^2 / \varepsilon \) is turbulent viscosity based on Boussinesq assumption.

\( \alpha_j \) is equivalent to the turbulent viscosity \( \nu_t \), but \( \alpha_j \) stands for a 2nd-order tensor, so the above equations can reflect general flow with an anisotropic turbulent viscosity.

The turbulence parameters are available in the literature [13] as shown in Table 1.

| Table 1. Parameters of stochastic model |
|-----------------------------------------|
| \( C_1 \)  | 0.09 | 0.08583 | 0.08386 | 0.0817 |
| \( C_n \)  | 0.00 | 0.01 | 0.0147 | 0.02 |

It can be seen from Table 1 that the present model can be simplified as standard \( k - \varepsilon \) model when \( C_n = 0 \). In this paper, \( C_1 = 0.08386 \) and \( C_n = 0.0147 \) are used here.

2.3. Non-equilibrium transport equations for suspended load

Non-uniform suspended load can be classified into \( n_0 \) groups according to their grain sizes. \( S_L \) and \( P_{SL} \) stand for the sediment concentration and its percentage of size group \( L \), respectively. Then we have

\[
S_L = P_{SL} S, \quad S = \sum_{L=1}^{n_0} S_L.
\]

Transport equation for 3D non-equilibrium suspended load of grain size group \( L \) can be expressed as

\[
\frac{\partial S_L}{\partial t} + u_j \frac{\partial S_L}{\partial x_j} - \omega_L \frac{\partial S_L}{\partial z} = \frac{\partial}{\partial x_j} \left( \frac{\nu_t}{\sigma_S} \frac{\partial S_L}{\partial x_j} \right) \tag{6}
\]

where, \( \omega_L \) is the particle settling velocity of this group, and \( \sigma_S \) is the turbulence Schmiten number and \( \sigma_S = 1 \).

2.4. Riverbed deformation equations

\[
\gamma_s \frac{\partial Z_{bl}}{\partial t} = \omega_L (S_{bl} - S'_{bl}) \tag{7}
\]

where, \( \gamma_s \) is the sediment dry bulk density, \( S_{bl} \) and \( S'_{bl} \) denote sediment concentration and carrying capacity of grain size \( L \) near the bed surface, respectively.

Then the total riverbed deformation can be written as \( Z_b = \sum_{L=1}^{n_0} Z_{bl} \)
2.5. Equations for bed-load gradation composition

Mass conservation equation for mixed-layer of grain size class is as follows.

\[ \gamma_s \frac{\partial E_m}{\partial t} + \omega_h (S_{bl} - S_{bl}^*) + [\epsilon_1 P_{ml} + (1 - \epsilon_1) P_{ml0}] \gamma_s (\frac{\partial Z_{bl}}{\partial t} - \frac{\partial E_m}{\partial t}) = 0 \]  \hspace{1cm} (8)

where, \( E_m \) is the thick value of mixing layer, \( P_{ml0} \) stands for the size-\( L \) fraction of the initial bed material, and \( P_{ml} \) for the size-\( L \) fraction of the mixing layer.

Equation (8) is established by extending CARICHAR mixing layer model [15] to the 3D sediment model. The third term on the left-hand side has the physical process of sediment erosion and deposition, that is, the lower boundary of the mixing layers continuously erodes riverbed to supplement the mixing layers when scour happens, which can avoid sediment deficiency in the mixing layers, simultaneously \( \epsilon_1 = 0 \) when the mixing layers affect the original riverbed, otherwise \( \epsilon_1 = 1 \).

3. Boundary conditions

Free Surface Boundary Conditions: the gradients of the velocity components are assumed to be zero and the Poisson equation of the free surface level is employed here to treat the free surface, which is based on 2D depth-averaged momentum equations [7].

Wall Boundary Conditions: the stochastic turbulent model requires the velocity derivative so a refined wall function is used here [16].

Inlet Boundary Condition: the velocity components are obtained with logarithmic distribution and wall function, \( S_L \) is obtained with exponential distribution; \( k = 1.5(0.05u_0)^2 \), \( \varepsilon = C_\mu^3/4k^{3/2}/\ell \), where \( u_0 \) is the mean velocity at the inlet and \( \ell \) is the nearest distance between computational node and wall surface.

Outlet Boundary Condition: the gradients of the other variables are assumed to be zero.

Sediment carrying capacity \( S_{bl}^* \) near the bed surface: based on vertical distribution of concentration of suspended load [17], \( S_{bl}^* \) is determined.

\[ S_{bl}^* = S_L^* \int_{\eta_b}^{\eta_s} \exp\{\omega_l[f(\eta) - f(\eta_b)]/\kappa u_*\} d\eta \]

where \( S_L^* = k_0(U^3 H)^m / \sum_{L=1}^{m} P_{ml}^m \), \( k_0 = 0.016 \), \( U \) is the depth-averaged velocity, \( H \) is the total water depth, \( m = 0.92 \); \( \eta = 1 - \frac{z}{H} \), \( f(\eta) \) is the vertical distribution function of sediment concentration, \( \eta_s \) and \( \eta_b \) are the relative position away from the bed surface and water surface, respectively; \( \kappa \) is the von Karman constant; \( u_* \) is the friction velocity.

Sediment concentration \( S_{bd} \) near the bed surface: during simulation, based on a known sediment concentration \( S_{bd} \) at a certain node above the bed surface, \( S_{bl} \) can be derived.

\[ S_{bl} = S_{bd} + S_{bl}^*[1 - e^{-\sigma_k(\eta_b - \delta_b)/\nu}] \]

where, \( \delta_b \) is the thickness of bed sediment exchanging layer and \( \delta_b = 2D_{50} \), where \( D_{50} \) is volume median-diameter of bed material.

4. Numerical methods
Body-fitted coordinates (BFC) [18] are used to simulate flows over complex geometries of river channel. The above governing equations of turbulent silt-laden flows, expressed in the BFC system $(\xi, \eta, \zeta)$, can be represented by the following transport model equation in the conservative form.

$$\frac{\partial \phi}{\partial t} + \frac{\partial \left(U_i \phi\right)}{\partial x_i} = \frac{\partial}{\partial x_i} \left( \Gamma_{\phi} \frac{\partial \phi}{\partial x_i} \right) + S_{\phi}(\xi, \eta, \zeta)$$

where $\phi$ is all the dependent variables representing $u_i, k, \varepsilon, S_L, Z_{bl}, E_m$ and so on, respectively, $U_i$ are the contravariant velocity components in BFC, $\Gamma_{\phi}$ is the diffusion coefficient concerning $\phi$, $g_{ij}$ are the factors from the coordinate transformation and $S_{\phi}$ is the source term.

The FVM [19] is used to discretize the governing equations together with a staggered grid system and the SIMPLEC [20] algorithm is applied for the solution of the discretized turbulent equations.

In general, it is desirable to increase the grid resolution in regions where the flow variables exhibited large gradients. Therefore the grid topology is non-uniform where the grid is packed near the solid wall, near the free surface and riverbed so that reliable calculated results could be obtained. Several computational trials are run with various grids in order to ensure that the final grid is sufficiently refined.

5. Computational domain

The physical model [14] investigated the river reach of Xiaolangdi Project with a complex morphology before constructed and riverbed mean elevation is about 130 m. The model can well repeat the process of sediment-laden flows and the prototype channel deformation. The field data from Jun. to Sep. of 1990 include the sediment-laden flow process and erosion/deposition in the river reach between these two hydrological cross-sections (HCS) of Tuyandi CS2 and Xiaolangdi CS14. In the process of sediment-laden flow with a discharge $1840 \text{ m}^3/\text{s}$, water level, flow velocity and sediment concentration are measured. Computational domain is about 4 km long from Dayuhe embouchure to Xiaolangdi debouchure shown in Figure 1. The grid consists of $I \times J \times K = 150 \times 61 \times 39$ cells ($I$ is in the streamwise direction from the inlet section to the outlet section, $J$ in the wide direction from the right bank to the left bank, and $K$ in the deep direction from the riverbed to the free surface).

![Figure 1. River bed elevation (Unit m)](image_url)

6. Results and discussions

6.1. Comparisons between water levels
Water surface curve along the river downstream shown in Figure 2 is obtained using gages arranged at eight positions along the surveyed river reach. The calculated results are in fairly good agreement with experimental data and both have the same changing tendency along the river downstream, and the error is less than 0.25 m.

6.2. Momentum equations for water flows

It can be seen from Figure 3 that the velocity decreases from the water surface to the riverbed and the riverbed has both dip slope and adverse slope, that is, the slope value of the river bed is probably positive or negative. Whether there are velocity distributions or not indicates that the riverbed topography is very complex and is in a process of dynamical readjustment. The computed results are in accordance with the rule of water flows and riverbed evolution in a river.

6.3. Comparisons between vertical velocities

Figure 4 illustrates that the velocity distributions along left and right vertical lines at the hydrological cross-section CS2 of Tuyandi, where the relative coefficient of water depth is defined as and is the water depth at a given position, LD stands for the distance from initial point. The velocity in the middle line of watercourse is higher than that near the riverside while the velocities decrease gradually from the water surface to the river bottom. The calculated velocity distributions agree well with the experiment.
Figure 5. Velocity distribution at CS14

Figure 5 illustrates that the velocity distributions along left and right vertical lines at the hydrological cross-sections CS14 of Xiaolangdi, the velocity distributions are similar to that at CS2. However, there is a small error between the simulation and the experiment.

6.4. Comparisons between water levels

Figure 6. Sediment concentration distribution at CS2

Figure 6 shows that the sediment concentration distributions along left and right vertical lines at Tuyandi CS2, and both are satisfactory with the experimental data. However, there is a small difference between the simulation and the experiment near the riverbed, which is likely due to the sediment concentration formula, which is derived using uniform sediment concentration expression [17].

Figure 7. Sediment concentration distribution at CS14

Figure 7 shows that the sediment concentration distributions along left and right vertical lines at Xiaolangdi CS14, and both agree well with the experimental data.

6.5. Comparisons between water levels

Table 2. Sediment erosion & deposition along the river downstream (Unit $10^5$ m$^3$)

| HCS Num. | Simulation | Experiment |
|----------|------------|------------|
| CS2      | +0.5416    | +0.59      |
| CS4      | +0.7273    | +0.18      |
| CS6      | -1.1547    | -2.00      |
| CS9      | +0.4130    | +0.80      |
| CS11     | -1.2968    | +1.24      |
| CS14     | -2.2918    | -4.7       |
| Total    | -3.0613    | -3.26      |

Note: + for deposition, - for erosion
It can be seen from Table 2 that the calculation has the similar distribution of sediment erosion and deposition to the experiment, that is, both have either scouring phenomena or depositing phenomena at the same river reach and the calculated sum of total erosion and deposition is close to the experiment one. The relative computational error is 6.1 or so. Therefore, numerical simulation repeats well the process of sediment erosion and deposition along the river downstream.

7. Conclusion
Firstly, a 3-D stochastic model was proposed for the silt-laden turbulence flows. This modeling technique includes anisotropic Reynolds stress model, water level model, and sorting equation of bed material, a freezing method and a refined wall function and so on.

Then this mathematical model was successfully applied to the river reach of Xiaolangdi Project. The body-fitted coordinate system was used here to overcome the complexity of river course, the FVM was employed to discretize the turbulent equations together with a staggered grid system, and the SIMPLEC algorithm was applied to simulate the water flows.

Finally, the water surface curve along the river downstream, vertical sediment concentrations and velocity distributions at the various cross-sections were predicted. The numerical simulation repeats well the process of erosion and deposition along the river downstream, and all the simulated results are in fairly good agreement with the experimental data, which indicate this model, the numerical algorithm are practicable and credible.

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