Non-Malleable Secret Sharing against Affine Tampering

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Abstract

Non-malleable secret sharing was recently studied by Goyal and Kumar in independent tampering and joint tampering models for threshold scheme (STOC18) and secret sharing with general access structure (CRYPTO18). We study non-malleable secret sharing in a natural adaptive tampering model, where the share vector is tampered using a function, in a given tampering family, chosen adaptively according to any unauthorised set of shares. Intuitively, the passive privacy adversary of secret sharing and the active adversary characterized by the given tampering family collude. We then focus on the tampering family of affine functions and construct non-malleable secret sharing in the adaptive tampering model. The constructions are modular with an erasure code and an extractor that provides both privacy and non-malleability. We make use of randomness extractors of various flavours, including the seeded/seedless non-malleable extractors. We discuss our results and open problems.

1 Introduction

Secret sharing, introduced independently by Blakley [8] and Shamir [39], is a fundamental cryptographic primitive with far-reaching applications; e.g., a major tool in secure multiparty computation (cf. [18]). The goal in secret sharing is to encode a secret $s$ into a number of shares $c_1, \ldots, c_P$ that are distributed among a set $\mathcal{P} = \{1, \ldots, P\}$ of players such that the access to the secret through collaboration of players can be accurately controlled. An authorized subset of players is a set $A \subseteq [P]$ such that the shares with indices in $A$ can be pooled together to reconstruct the secret $s$. On the other hand, $A$ is an unauthorized subset if the knowledge of the shares with indices in $A$ reveals no information about the secret. The set of authorized and unauthorized sets define an access structure, where the most widely used is the so-called threshold structure. A threshold secret sharing scheme is defined with respect to an integer parameter $r$ and satisfies the following property: Any set $A \subseteq [P]$ with $|A| < r$ is an unauthorized set and any set $A \subseteq [P]$ with $|A| \geq r$ is an authorized set. When the share lengths are below the secret length, the threshold guarantee that requires all subsets of participants be either authorized, or unauthorized can no longer be attained. Instead, the notion can be relaxed to ramp secret sharing which allows some subset of participants to learn some information about the secret. A $(t, r, P)$-ramp scheme is defined with respect to two

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thresholds, $t$ and $r$. The knowledge of any $t$ shares or fewer does not reveal any information about the secret. On the other hand, any $r$ shares can be used to reconstruct the secret. The subsets of size $\geq t + 1$ or $\leq r - 1$ shares, may reveal some information about the secret. Most constructions of secret sharing are algebraic. The homomorphic property of the algebraic secret sharing is crucial for their application in secure multiparty computation. Also nice algebraic structures allow for efficient sharing and reconstruction. But for protection against active adversary, such structures could be exploited by the adversary. For example, by exploiting the additive homomorphic property of such schemes, an arbitrary offset can be added to the secret through corrupting a few shares (without violating the privacy) and [19] showed an elegant counter measure called Algebraic Manipulation Detection (AMD) code for such simple tampering attacks. But once the active adversaries become more sophisticated, for example, adversaries whose tampering is defined by a class of functions that contains the constant functions, it was not known what guarantee is still available and how to achieve it.

Protection against sophisticated tampering is studied in the literature of non-malleable codes, proposed by Dziembowski, Pietrzak and Wichs [25]. A code with a randomized encoder and a deterministic decoder is said to provide non-malleability with respect to a family $\mathcal{F}$ of tampering functions if decoding the tampered codeword yields the original message or a value that follows a background distribution, where the probability of the first case and the probability distribution in the second case are dictated by the particular tampering function $f \in \mathcal{F}$ alone (all probabilities are taken over the randomness of the encoder). Intuitively, non-malleable coding prevents the adversary from tampering with the protected message in a message-specific way, which is the essence of non-malleable cryptography [22]. In fact, non-malleability is a coding guarantee that includes other coding guarantees as special cases and is achievable for the most general tampering families [25].

Perhaps the most widely studied function family for non-malleability is the $C$-split state model, where for a constant integer $C$, a tampering function is described by $f = (f_1, \ldots, f_C)$, for arbitrary functions $f_i : \{0, 1\}^{n/C} \to \{0, 1\}^{n/C}$. The most difficult problem in $C$-split state model is the extreme case when $C = 2$, which has attracted a lot of attention (just to name a few and restricted to information-theoretic security)[23, 3, 4, 2, 12, 33, 34]. Despite the tremendous efforts, an explicit constant rate (the message length is a constant fraction of the codeword length) construction of 2-split state non-malleable codes remains elusive even today. An important theoretical discovery in constructions of non-malleable codes is the connection between non-malleable codes and invertible seedless non-malleable extractors by Cheraghchi and Guruswami [17]. A seedless non-malleable extractor is defined with respect to a family of tampering functions, which are applied to the input of the extractor. Non-malleability here means that the output corresponding to the original input is independent of the output of a tampered input. Intuitively, if one uses the extractor as the decoder then non-malleability of the obtained code follows naturally from the independence of the two outcomes. This connection plays an important role in the construction of $C$-split state non-malleable codes [15, 12, 33, 34]. Recently, seedless non-malleable extractors with respect to affine tampering functions are constructed, yielding non-malleable codes with respect to non-compartmentalized (do not belong to $C$-split state) tampering families [14]. (We will discuss various types of extractors and the non-compartmentalized tampering in more details later in Our constructions.)

Goyal and Kumar initiated a systematic study of non-malleable secret sharing [26, 27] with inspirations from the non-malleable codes. Their study starts with the observation in [4] that a 2-split state non-malleable code is a non-malleable secret sharing in the sense that independently tampering with the two states/shares is non-malleable and each state/share is statistically indistinguishable (the proof is non-trivial, see [4]). Unfortunately, this connection does not generalise to $C$-split state for $C > 2$. A $C$-split state non-malleable code for $C > 2$ is not a threshold secret
sharing with \( C \) players, due to the absence of privacy guarantee. A threshold secret sharing in general can have a reconstruction threshold \( r < C \), while the decoder of a \( C \)-split state non-malleable code always need all the \( C \) shares. This second issue is easier to resolve. Non-malleability of secret sharing with respect to a tampering family \( \mathcal{F} \) now means reconstructing from any reconstruction set \( R \) of tampered shares yields the original secret or a value that follows a background distribution, where the probability of the first case and the probability distribution in the second case are dictated by the particular tampering function \( f \in \mathcal{F} \) and the particular reconstruction set \( R \) (all probabilities are taken over the randomness of the sharing algorithm). Goyal and Kumar study two tampering models for threshold secret sharing and construct them for \( C > 2 \) players and any threshold \( r \leq C \) [26]. The so-called independent tampering model of non-malleable secret sharing is essentially a secret sharing with \( C \) players, which is non-malleable with respect to the \( C \)-split state tampering family. The so-called joint tampering model allows the adversary to group any \( r \) shares into two subsets of different size and tamper jointly with the shares within each group but independently across the two groups. In their follow up work [27], non-malleability for secret sharing with general access structure are studied. In the independent tampering model, they construct a transformation that takes any secret sharing and outputs a non-malleable secret sharing with the same access structure. In the joint tampering model, more powerful adversaries that can group shares into two overlapping subsets, as long as no authorized set is jointly tampered, are considered for threshold secret sharing and for the special case of \( C \)-out-of-\( C \) threshold scheme, explicit constructions are given. The 2-split state non-malleable code also plays a crucial role in the constructions of non-malleable secret sharing in [26, 27], which follow the same idea of constructing a 2-split state non-malleable code adversary. Given that the construction of 2-split state non-malleable codes is a difficult problem in its own right, one would wonder if there is a natural model for non-malleable secret sharing against strong tampering adversary (at least as strong as the currently strongest jointly tampering model), for which there is a simple construction without relying on 2-split state non-malleable codes.

Our model.

We consider the following natural way of combining secret sharing (a primitive for passive adversary) with non-malleable codes (a primitive for active adversary). We view the \( P \) shares \( c_1, \ldots, c_P \), where \( c_i \in \mathbb{F}_q \), of the secret sharing scheme as an \( N \)-bit string \((c_1||\ldots||c_P)\), where \( c_i \in \{0,1\}^{\log q} \) and \( N = P \log q \). An adversary chooses any unauthorized set of shares to read. After that this adversary chooses, adaptively according to the value of the unauthorized set of shares, a tampering function \( f : \{0,1\}^N \rightarrow \{0,1\}^N \) and apply it to the whole share vector. Obviously, the tampering of the share vector needs to be restricted to some tampering family \( \mathcal{F} \) to prevent trivial attacks such as exploiting the functionalities of the secret sharing to reconstruct the original secret then re-share a related secret. We term this non-malleable secret sharing in adaptive tampering model as opposed to the independent tampering and joint tampering models in [26, 27].

To fix ideas, we restrict our study to \((t,r,P)\)-ramp schemes. We allow \( t = r - 1 \), which is the threshold secret sharing. For a family \( \mathcal{F} \) of tampering functions from \( \{0,1\}^N \) to \( \{0,1\}^N \). Let \( \mathcal{F}_{O_t/P}^{t/P} = \{ (A, \sigma) | \text{all } A, \text{ all } \sigma \} \) be the family of tampering functions, where each function is described by specifying a leakage adversary \( A \) and a tampering strategy \( \sigma \). A \( t/P \)-leakage adversary \( A \) is a process of adaptively reading \( t \) blocks out of \( P \) blocks. A \( \mathcal{F} \)-tampering strategy (associate with \( A \)) is a metafunction \( \sigma : \{0,1\}^{t \log q} \rightarrow \mathcal{F} \) describing that on seeing the value \( \alpha \in \{0,1\}^{t \log q} \) during the leaking phase, a corresponding function \( f^\alpha \in \mathcal{F} \) is to be applied to the share vector. We define non-malleability of a \((t,r,P)\)-ramp scheme with respect to a tampering family \( \mathcal{F} \) as follows. Reconstructing from any reconstruction set \( R \) of the tampered shares yields the original secret or a value that follows a background distribution, where the probability of the first case and
the probability distribution in the second case are dictated by the particular leakage adversary $A$, the particular tampering strategy $\sigma$ and the particular reconstruction set $R$ (all probabilities are taken over the randomness of the sharing algorithm). This non-malleability guarantee should hold for any $t/P$-leakage adversary $A$, any $F$-tampering strategy $\sigma$ and any reconstruction set $R$ of size $r$ (see Definition 14).

Our constructions.

We focus on the family $F_{\text{affine}}$ of affine tampering functions and give constructions of non-malleable secret sharing in the adaptive tampering model. For binary strings of length $N$, an $F_2$-affine tampering function is a function from $\{0,1\}^N$ to $\{0,1\}^N$, where each output bit is an affine function of the input bits. Affine tampering family is non-compartmentalized tampering, since the tampering at one bit can depend on all the $N$ bits, which is different from, for example, the 2-split state model, where the tampering at the bits in the left half can not dependent on the bits in the right half. Affine tampering functions include the important tampering family considered in [5, 6], where non-malleable codes against the composition of permutations and Bit-wise Independent Tampering $F_{\text{BIT}}$ [25, 17] were shown to give a reduction from non-malleable string commitment to non-malleable bit commitments.

Our constructions start with an extractor based construction of secret sharing scheme and strengthen the extractor towards obtaining non-malleability. An extractor is a function that turns non-uniform distributions (called source) over the domain into an almost uniform distribution over the range. An affine source is a flat distribution on an affine subspace and an extractor for affine sources is called an affine extractor. An extractor is invertible if there is an efficient algorithm that, given an extractor output, samples a pre-image for that output uniformly at random. Very recently, Lin et.al. [35] proposed a construction of secret sharing through combining an invertible affine extractor and a linear erasure correcting code. In the construction, the secret is the output of the affine extractor. The sharing algorithm first uses the inverter of the extractor to sample a random pre-image for the secret. The key observation is if we start with a uniformly distributed secret, the inverter will output a distribution that is uniform over the domain of the extractor. The privacy analysis is focused on this uniform pre-image. Now this pre-image is further encoded using the erasure correcting code to yield the share vector. But since the erasure correcting code is linear, knowing several components of its codeword amounts to putting several linear equations on the uniform pre-image, which is now flatly distributed on an affine sub-space of the domain of the extractor, hence an affine source. If this affine source has enough entropy, then the distribution of the uniform secret conditioned on the adversary’s view remains uniform. This means that the adversary’s view and the secret are independent and hence privacy is provided. Using this construction, ramp secret sharing families with statistical privacy and probabilistic reconstruction over binary shares can be constructed, given any relative privacy threshold $\tau$ and relative reconstruction threshold $\rho$, for arbitrary constants $0 \leq \tau < \rho \leq 1$. Now given a privacy threshold $t$ and a reconstruction threshold $r$ for a ramp scheme with $P$ players, we set $\tau = t/P$ and $\rho = r/P$, and obtain a family of binary ramp schemes with $N$-bit share vector, where $N$ is a multiple of $P$. We then divide the $N$-bit share vector into $P$ blocks and call each block a share of a $(t, r, P)$-ramp scheme.

Now we strengthen the affine extractor in the above construction to an affine non-malleable extractor, which is an affine extractor that has non-malleability guarantee for all affine tampering functions. We should have a $(t, r, P)$-ramp scheme, since an affine non-malleable extractor is in particular an affine extractor. We can further show that the scheme is non-malleable for any $t/N$-leakage adversary $A$, any affine tampering strategy $\sigma$ and any reconstruction set $R$. The analysis is again focused on the uniform pre-image (of a uniform secret) generated by the inverter of the extractor. As argued before, conditioned on a view $v$ of the $t/N$-leakage adversary $A$, the uniform
pre-image becomes an affine source. Under the same conditioning, the affine tampering strategy $\sigma$ outputs the corresponding affine tampering function $f^\sigma$ that is applied to the share vector. Due to the linearity of the erasure correcting code, this $f^\sigma$ induces an affine tampering function $g$ that is applied to the pre-image we are investigating. If the extractor can non-malleably (with respect to affine functions) extract from the affine source, the tampered outcome is independent of the original secret. We then obtain a clean reduction from affine non-malleable secret sharing to affine non-malleable extractors (see Theorem 16).

We can simply plug in the off-the-shelf affine non-malleable extractors in [14] together with appropriate explicit erasure correcting codes to obtain explicit non-malleable secret sharing schemes with respect to $F_{\text{affine}}$. A major drawback of the construction of affine non-malleable extractors in [14] is that it crucially relies on high entropy of the affine source. This means, in particular, the amount of leakage that can be tolerated by the obtained non-malleable secret sharing scheme is very limited.

Our second construction considers the subset $F_{\text{BIT}} \subset F_{\text{affine}}$ and can tolerate a constant fraction of leakage. It uses the same high level ideas as the first construction but with a seeded non-malleable extractor to provide non-malleability instead of a seedless one. A seeded extractor is a function that takes a second input (called the seed) which is uniform and independent of the source input. Seeded non-malleable extractors were proposed for application in privacy amplification over public unauthenticated discussion [21]. A seeded non-malleable extractor is very different from its seedless counterpart and the only thing that these two objects have in common is to achieve independence of the original extractor output from the tampered extractor output. The first difference lies in what is tampered. The source of the seeded extractor is not tampered, it is its seed that is tampered. The second difference lies in what tampering is allowed. The seed tampering of the seeded extractor is not restricted by a family of functions, but not allowed to have any fixed points. We overcome the first problem through suitably conditioning on some event such that the tampered source is equal to the original source adding a constant offset, thanks to restriction to $F_{\text{BIT}} \subset F_{\text{affine}}$. Since the seeded non-malleable extractor is linear, we can separate the constant offset from the tampered source completely and reduce to the same source situation. We overcome the second problem through detecting the tampering, whenever the tampered seed coincides the original seed, using an AMD pre-coding of the secret. We can not guarantee that the tampered share vector always leads to a seed different from the original seed. But when the two do coincide, as mentioned a few lines ago, the linearity of the non-malleable extractor allows for separating out an additive offset. This results in reconstructing an (obliviously) additively tampered secret, which is easily detected using, for example, the AMD code [19].

Given a linear seeded non-malleable extractor that can extract from a constant fraction of entropy, we do obtain non-malleable secret sharing in adaptive tampering model with constant relative thresholds $0 < \tau < \rho < 1$ (see Theorem 22). Seeded non-malleable extractors is well studied topic and there are many good constructions. But when restricted to linear case, as far as we know, there is only the inner product based construction [32], which requires entropy rate around half. This entropy rate around half barrier existed in the literature of (non-linear) non-malleable extractor constructions [20], but was quickly overcome [13]. We prove the existence of the required linear seeded non-malleable extractors using a probabilistic argument (see Theorem 24) and leave the explicit construction as an interesting open question.

In the technique aspect, we use a classical paradigm of combining a seedless extractor with a seeded extractor [38]. The seedless extractor extracts a short seed for the seeded extractor and the combination is in effect a seedless extractor with the good properties of the seeded extractor. The error bounding techniques in the second construction, if we drop the non-malleability requirement
and use a plain linear seeded extractor (Trevisan’s extractor [41]), have independent interest in secret sharing over small constant share size [11, 9, 35]. We in fact obtain an explicit secret sharing against an adaptive leakage adversary with significantly better parameters than [35] (see Related works). The improvement comes from making good use of the linearity of the seeded extractor and a more efficient way of inverting the extractor that exploits this hybrid structure.

Related works.

Soon after the publication of [26, 27], there are new results on preprint continuing the study of independent tampering model and independent tampering with add-on properties. The works in [7, 40] achieves constant rate in the independent tampering model making use of the recent constant rate 3-split-state non-malleable codes [29]. There are works on independent tampering with leakage model [30, 1]. The leakage resilience in these works refer to powerful leakage that is beyond the standard secret sharing model (and fall into the leakage resilient secret sharing paradigm started from [24]), such as, leaking from all shares independently (even allow groups of jointly leaked shares) but the number of bits leaked is bounded. In [7, 1], there is also a concurrent/multiple independent tampering model, where the tampering is continuous instead of one-shot. All the works above consider the $C$-split state tampering family only. We define our adaptive tampering with respect to general ramp secret sharing a general tampering family $F$ (in theory, could be any family studied in non-malleable code literature). Interestingly, there seems to be little intersection between our model and all other works. A thorough study of the connections between each pair of models is obviously beyond the scope of this work.

Another line of works related to the current work is the study of ramp secret sharing over a constant share size $q$ (note that in the current work, the number $P$ of players is fixed while the share size $q = 2^{N/P}$ is growing as $N$ grows.) The main characteristics of this line of works are fixed share size $q$, unconstrained number of players ($P = N$) and minimizing the relative threshold gap $\gamma = \frac{q}{N} = \frac{r-t}{N}$. It is shown in [11] and [9] that for $0 < t < N - 1$,

$$g \geq \frac{(N + 2)}{(2q - 1)}.$$

This means that once $q$ is fixed, the relative gap $\gamma = \frac{q}{N} > \frac{1}{2q-1}$. In particular, when $q = 2$, we must have $\gamma > 1/3$. This constraint is recently showed avoidable once the perfect privacy and perfect reconstruction of the ramp secret sharing are relaxed to statistical privacy (any $t$ shares from a pair of secrets have a statistical distance negligible in $N$) and probabilistic reconstruction (reconstruction with $r$ shares has a failure probability that is negligible in $N$), respectively [35]. It is shown that for any $0 \leq \tau < \rho \leq 1$, ramp secret sharing families (with relaxed privacy and reconstruction) can be explicitly constructed such that the privacy threshold $t = \tau N$ and the reconstruction threshold $r = \rho N$. The non-perfect privacy brings out the distinction between an adaptive reading adversary and a non-adaptive reading adversary. The authors then give two constructions for these two types of reading adversaries, respectively. In particular, the construction for non-adaptive adversary shares a secret of $N(\rho - \tau - o(1)) \log q$ bits, which they show is optimal. The construction for adaptive adversary does not achieve this secret length and the authors leave improving the secret length as an open problem. As mentioned previously, the tools developed in our second construction can be used to significantly improve the secret length of the construction in [35].

The rest of the paper is organised as follows. Section 2 contains the definitions of various randomness extractors that appear in this work. Section 3 defines our adaptive tampering model for non-malleable secret sharing. Section 4 proves a reduction from affine non-malleable secret sharing to affine non-malleable extractors. Section 5 gives a variation using seeded non-malleable extractors.
2 Preliminaries

Coding schemes define the basic properties for codes (schemes) that are used in cryptography. Let ⊥ denote a special symbol that means detection.

**Definition 1** ([25]). A \((k,n)\)-coding scheme consists of two polynomial-time functions: a randomised encoding function \(\text{Enc} : \{0,1\}^k \rightarrow \{0,1\}^n\), where the randomness is implicit, and a deterministic decoding function \(\text{Dec} : \{0,1\}^n \rightarrow \{0,1\}^k \cup \{\bot\}\) such that, for each \(m \in \{0,1\}^k\), \(Pr[\text{Dec}(\text{Enc}(m)) = m] = 1\) (correctness), and the probability is over the randomness of the encoding algorithm.

The **statistical distance** of two random variables (their corresponding distributions) is defined as follows. For \(X,Y \leftarrow \Omega\),

\[
\text{SD}(X;Y) = \frac{1}{2} \sum_{\omega \in \Omega} |\Pr(X = \omega) - \Pr(Y = \omega)|.
\]

We say \(X\) and \(Y\) are \(\epsilon\)-close (denoted \(X \sim_\epsilon Y\)) if \(\text{SD}(X,Y) \leq \epsilon\).

A **tampering function** for a \((k,n)\)-coding scheme is a function \(f : \{0,1\}^n \rightarrow \{0,1\}^n\).

**Definition 2** ([25]). Let \(F\) be a family of tampering functions. For each \(f \in F\) and \(m \in \{0,1\}^k\), define the tampering-experiment

\[
\text{Tamper}_m^f = \begin{cases} 
\{ x \leftarrow \text{Enc}(m), \tilde{x} = f(x), \tilde{m} = \text{Dec}(\tilde{x}) \} & \text{Output } \tilde{m}, \\
\text{Output } m \text{ if } \tilde{m} = \text{same}^*, \text{ and } \tilde{m} \text{ otherwise; } & \end{cases}
\]

which is a random variable over the randomness of the encoding function \(\text{Enc}\). A coding scheme \((\text{Enc}, \text{Dec})\) is **non-malleable with respect to** \(F\) if for each \(f \in F\), there exists a distribution \(D_f\) over the set \(\{0,1\}^k \cup \{\bot, \text{same}^*\}\), such that, for all \(m \in \{0,1\}^k\), we have:

\[
\text{Tamper}_m^f \sim \begin{cases} 
\tilde{m} \leftarrow D_f & \text{Output } m \text{ if } \tilde{m} = \text{same}^*, \text{ and } \tilde{m} \text{ otherwise; } \\
\end{cases}
\]

and \(D_f\) is efficiently samplable given oracle access to \(f(\cdot)\).

The right hand side of (1) is sometimes denoted by \(\text{Copy}(D_f, m)\). Using this notation, (1) can be written as,

\[
\text{Tamper}_m^f \sim \text{Copy}(D_f, m). \quad (1')
\]

The following coding scheme, originally proposed for constructing robust secret sharing, is frequently used as a building block for constructing non-malleable codes.

**Definition 3** ([19]). Let \((\text{AMDenc}, \text{AMDdec})\) be a coding scheme with \(\text{AMDenc} : \{0,1\}^k \rightarrow \{0,1\}^n\). We say that \((\text{AMDenc}, \text{AMDdec})\) is a \(\delta\)-secure Algebraic Manipulation Detection (AMD) code if for all \(m \in \{0,1\}^k\) and all non-zero \(\Delta \in \{0,1\}^n\), we have \(\Pr[\text{AMDdec}(\text{AMDenc}(m) + \Delta) \in \{m, \bot\}] \leq \delta\), where the probability is over the randomness of the encoding.

An explicit optimal construction of AMD code is given in [19] that in fact gives a tamper detection code [28]. We say an AMD code achieves \(\delta\)-tamper detection security if for all \(\Delta \neq 0^n\), \(\Pr[\text{AMDdec}(\text{AMDenc}(m) + \Delta) \neq \bot] \leq \delta\).

We use various types of randomness extractors to construct the AONT that is used in our NM-codes construction. Randomness extractors extract close to uniform bits from input sequences.
that are not uniform but have some guaranteed entropy. See [36] and references there in for more information about randomness extractors.

A **randomness source** is a random variable with lower bound on its min-entropy, which is defined by \( H_\infty(X) = -\log \max_x \{\Pr[X = x]\} \). We say a random variable \( X \leftarrow \{0, 1\}^n \) is a \((n, k)\)-source, if \( H_\infty(X) \geq k \). For well structured sources, there exist deterministic functions that can extract close to uniform bits. An affine \((n, k)\)-source is a random variable that is uniformly distributed on an affine translation of some \(k\)-dimensional sub-space of \(\{0, 1\}^n\). Let \( U_m \) denote the random variable uniformly distributed over \(\{0, 1\}^m\).

**Definition 4.** A function \( a\mathrm{Ext} : \{0, 1\}^n \rightarrow \{0, 1\}^m \) is an affine \((k, \varepsilon)\)-extractor if for any affine \((n, k)\)-source \( X \), we have

\[
\text{SD}(a\mathrm{Ext}(X) ; U_m) \leq \varepsilon.
\]

We will use Bourgain’s affine extractor (or the alternative [31] due to Li) in our second construction.

**Lemma 5** ([10]). For every constant \( 0 < \mu \leq 1 \), there is an explicit affine extractor \( a\mathrm{Ext} : \{0, 1\}^n \rightarrow \{0, 1\}^m \) for affine \((n, n\mu)\)-sources with output length \( m = \Omega(n) \) and error at most \( 2^{-\Omega(n)} \).

For general \((n, k)\)-sources, there does not exist a deterministic function that can extract close to uniform bits from all of them simultaneously. A family of deterministic functions are needed.

**Definition 6.** A function \( \mathrm{Ext} : \{0, 1\}^d \times \{0, 1\}^n \rightarrow \{0, 1\}^m \) is a strong seeded \((k, \varepsilon)\)-extractor if for any \((n, k)\)-source \( X \), we have

\[
\text{SD}(S, \mathrm{Ext}(S, X) ; S, U_m) \leq \varepsilon,
\]

where \( S \) is chosen uniformly from \(\{0, 1\}^d\). A seeded extractor \( \mathrm{Ext}(\cdot, \cdot) \) is called linear if for any fixed seed \( S = s \), the function \( \mathrm{Ext}(s, \cdot) \) is a linear function.

There are linear seeded extractors that extract all the randomness, for example, the Trevisan’s extractor [41]. In particular, we use the following improvement of this extractor due to Raz, Reingold and Vadhan [37].

**Lemma 7** ([37]). There is an explicit linear strong \((k, \varepsilon)\)-extractor \( \mathrm{Ext} : \{0, 1\}^d \times \{0, 1\}^n \rightarrow \{0, 1\}^m \) with \( d = O(\log^3 (n/\varepsilon)) \) and \( m = k - O(d) \).

Non-malleability of randomness extractors captures their tolerance against tampering. It was first defined for seeded extractors by Dodis and Wichs [21] with application in privacy amplification over public and unauthenticated discussion. The tampering considered is an arbitrary seed tampering that does not have any fixed point.

**Definition 8** ([21]). A seeded \((k, \varepsilon)\)-non-malleable extractor is a function \( \text{nmExt} : \{0, 1\}^d \times \{0, 1\}^n \rightarrow \{0, 1\}^m \) such that given any \((n, k)\)-source \( X \), an independent uniform seed \( Z \in \{0, 1\}^d \), for any (deterministic) function \( A : \{0, 1\}^d \rightarrow \{0, 1\}^d \) such that \( A(z) \neq z \) for any \( z \), we have

\[
\text{SD}(Z, \text{nmExt}(A(Z), X), \text{nmExt}(Z, X) ; Z, \text{nmExt}(A(Z), X), U_m) \leq \varepsilon.
\]

Non-malleable seedless extractors were proposed by Cheraghchi and Guruswami for constructing non-malleable codes. The tampering now is a source tampering and is restricted to a particular tampering family.
Lemma 10. We first give the restricted form of the construction, where the tampering function does not act with respect to a class $U$ where the two copies of $x$ are converted into a seedless non-malleable extractor with respect to an affine function of the input bits. The affine non-malleable extractors in Lemma 10 can be easily converted into a seedless non-malleable extractor with respect to $F$. We have

$$\text{SD}(\text{nmExt}(f(X), \text{nmExt}(X)); \text{Copy}(Y, U_m), U_m) \leq \epsilon,$$

where the two copies of $U_m$ denote the same random variable and $\text{Copy}(y, u) = y$ always except when $y = \text{same}^*$, in which case it outputs $u$.

We will use Chattopadhyay and Li’s affine non-malleable extractor in our first construction. We first give the restricted form of the construction, where the source tampering function does not have any fixed points.

Lemma 11. For all $n, k > 0$, any $\delta > 0$ and $k \geq n - n^{\delta}$, there exists an efficient function $\text{nmExt}: \{0, 1\}^n \rightarrow \{0, 1\}^m$, $m = n^{\Omega(1)}$, such that if $X$ is an affine $(n, k)$-source and $\mathcal{A}: \{0, 1\}^n \rightarrow \{0, 1\}^n$ is an affine function with no fixed point, then

$$\text{SD}(\text{nmExt}(\mathcal{A}(X)), \text{nmExt}(X); \text{nmExt}(\mathcal{A}(X)), U_m) \leq 2^{-n^{\Omega(1)}}.$$

Let $F_{\text{affine}}$ be the set of tampering functions from $\{0, 1\}^n$ to $\{0, 1\}^n$ where each output bit is an affine function of the input bits. The affine non-malleable extractors in Lemma 10 can be easily converted into a seedless non-malleable extractor with respect to $F_{\text{affine}}$.

Lemma 12. Let $\text{nmExt}: \{0, 1\}^n \rightarrow \{0, 1\}^m$ be a $(k - \eta, \epsilon)$-non-malleable extractor for affine sources, with respect to affine tampering functions with on fixed points. Then $\text{nmExt}$ is a $(k, \epsilon + (n + 1)2^{-n})$-non-malleable extractor for affine sources, with respect to $F_{\text{affine}}$.

Explicit constructions of randomness extractors have efficient forward direction of extraction. In some applications, we usually need to efficiently invert the process: Given an extractor output, sample a random pre-image. This is not necessarily efficient if the extractor is not a linear function, in which case we need to explicitly construct an invertible extractor. If the extractor is linear, sampling a random pre-image can be done in polynomial time. In general,

Definition 12. Let $f$ be a mapping from $\{0, 1\}^n$ to $\{0, 1\}^m$. For $v \geq 0$, a function $\text{Inv}: \{0, 1\}^m \times \{0, 1\}^r \rightarrow \{0, 1\}^m$ is called a $v$-inverter for $f$ if the following conditions hold:

- (Inversion) Given $y \in \{0, 1\}^m$ such that its pre-image $f^{-1}(y)$ is nonempty, for every $r \in \{0, 1\}^r$ we have $f(\text{Inv}(y, r)) = y$.
- (Uniformity) $\text{Inv}(U_m, U_r)$ is $\mu$-close to $U_n$.

A $\mu$-inverter is called efficient if there is a randomized algorithm that runs in worst-case polynomial time and, given $y \in \{0, 1\}^m$ and $r$ as a random seed, computes $\text{Inv}(y, r)$. We call a mapping $\mu$-invertible if it has an efficient $\mu$-inverter, and drop the prefix $\mu$ from the notation when it is zero. We abuse the notation and denote the inverter of $f$ by $f^{-1}$.

Finally, we need the following simple lemma whose proof can be found in Appendix A.

Lemma 13. Let $V, V'$ be two random variables distributed over the set $\mathcal{V}$ and $W, W'$ over $\mathcal{W}$ satisfying $\text{SD}(V, W; V', W') \leq \epsilon$. Let $\mathcal{E} \subseteq \mathcal{W}$ be an event. Then we have the following,

$$\text{SD}(V|W \in \mathcal{E}; V'|W' \in \mathcal{E}) \leq \frac{2\epsilon}{\Pr[W' \in \mathcal{E}]}.$$
3 Non-malleable secret sharing in adaptive tampering model

In this section, we define the non-malleable secret sharing model studied in this work.

A leakage oracle is a machine $O(\cdot)$ that takes as input an $N$-bit string $c \in \{0,1\}^N$ and then answers the leakage queries of the type $I_j$, for $I_j \subset [N]$, $j = 1, 2, \ldots, m$. Each query $I_j$ is answered with $c_{I_j}$. An interactive machine $A$ that issues the leakage queries is called a leakage adversary. Let $A_c = \bigcup_{j=1}^m I_j$ denote the union of all the index sets chosen by $A$ when the oracle input is $c$. Assume for simplicity that $N = P \log q$ for a constant integer $P$ and $c$ is viewed as $P$ blocks. The oracle is called $t/P$-bounded, denoted by $O_{t/P}(\cdot)$, if it only accepts leakage queries from $A$ that is block-wise and there does not exist any $c \in \{0,1\}^N$ such that $|A_c| > t \log q$. An adaptive leakage adversary decides the index set $I_{j+1}$ according to the oracle’s answers to all previous queries $I_1, \ldots, I_j$. Let $\text{View}_A^{O_{t/P}(\cdot)}$ denote the view of the leakage adversary $A$ interacting with a $t/P$-bounded leakage oracle.

A $\mathcal{F}$-tampering strategy associated with $A$ is a metafunction $\sigma : \{0,1\}^{tN/P} \to \mathcal{F}$ that takes as input a view $\text{View}_A^{O_{t/P}(\text{Share}(s))} = v$ and outputs a tampering function $f^v \in \mathcal{F}$.

Definition 14. For integers $0 \leq t < r \leq P$, a $(\varepsilon(N), \delta(N))$-nmSSS for ramp parameters $(t, r, P)$, with respect to a family $\mathcal{F}$ of tampering functions from $\{0,1\}^N$ to $\{0,1\}^N$ with $P|N$, is a pair of polynomial-time algorithms $(\text{Share}, \text{Recst})$,

$$\text{Share} : \{0,1\}^{\ell(N)} \times \mathcal{R} \to \{0,1\}^N,$$

where $\mathcal{R}$ denote the randomness set, and for any $R \subset [P]$ of size $|R| = r$, there is a

$$\text{Recst}_R : \{0,1\}^{rN/P} \to \{0,1\}^{\ell(N)} \cup \{\bot\},$$

that satisfy the following properties.

- Reconstruction: Given any $r$ out of the $P$ blocks of the share vector $\text{Share}(s)$, the reconstruct algorithm $\text{Recst}$ reconstructs the secret $s$ with probability at least $1 - \delta(N)$.

  When $\delta(N) = 0$, we say the scheme has perfect reconstruction.

- Privacy:

  For any $s_0, s_1 \in \{0,1\}^{\ell(N)}$ and any adaptive leakage adversary $A$ interacting with a $t/P$-bounded leakage oracle $O_{t/P}(\cdot)$,

  $$\text{SD} \left( \text{View}_A^{O_{t/P}(\text{Share}(s_0))}, \text{View}_A^{O_{t/P}(\text{Share}(s_1))} \right) \leq \varepsilon(N). \tag{4}$$

  When $\varepsilon(N) = 0$, we say the scheme has perfect privacy.

- Non-malleability: For any adaptive leakage adversary $A$, any $\mathcal{F}$-tampering strategy $\sigma$, any $R \subset [P]$ of size $|R| = r$ and any secret $s$, define the tampering-experiment

  $$\text{Tamper}_s^{A,\sigma,R} = \left\{ \begin{array}{c} c \leftarrow \text{Share}(s), v = \text{View}_A^{O_{t/P}(\cdot)}, \bar{c} = f^v(\cdot), \bar{s} = \text{Recst}_R(\text{block}^P(\bar{c}), R) \end{array} \right\},$$

  which is a random variable over the randomness of the share algorithm $\text{Share}$, where $\text{block}^P(\cdot)$ packs an $N$-bit vector into $P$ blocks. We say the scheme is $\varepsilon(N)$-non-malleable if for any $A$,
erasures. Given any $0 \leq \text{ECCenc}$ be a linear erasure correcting code with encoder
its inverter that maps an $s \in \{0,1\}^\ell$ to one of its pre-images chosen uniformly at random. Let ECC
be a linear erasure correcting code with encoder $\text{ECCenc}: \{0,1\}^n \to \{0,1\}^N$ that tolerates $(1-\rho)N$
erasures. Given any $0 \leq \tau < \rho \leq 1$, the following pair of sharing and reconstructing algorithms give
a $(\tau N, \rho N, N)$-ramp scheme with binary shares, if $\text{aExt}$ can extract from affine $(n, n-\tau N)$-source.

$$\begin{align*}
\text{Share}(s) &= \text{ECCenc}(\text{aExt}^{-1}(s)); \\
\text{Recst}_R(y) &= \text{aExt}(\text{ECCdec}_R(y)),
\end{align*}$$

where $y = c_R$ is the projection of a share vector $c$ on $R \subset [N]$ with $|R| = \rho N$, and $\text{ECCdec}_R$ is the
the erasure code decoding algorithm with respect to $R$.

Intuitively, ECC enables the reconstruction from any $\rho N$ shares. The privacy for any $\tau N$ shares
is not as straightforward. Firstly, since $\text{ECCenc}$ is a linear function, the view $V$ of the adversary is the
image of a linear function, whose domain is the message space $\{0,1\}^n$ of ECC. This linear function
induces a partition of the message space $\{0,1\}^n$ into cosets each corresponding to a particular
value of the adversary’s view $V = \nu$. Given that the adversary observes $V = \nu$, the message of ECC
can only be one element in the coset corresponding to $\nu$. Secondly, the secret and the message of
ECC (as two random variables) are connected by an invertible affine extractor. According to the
definition of an inverter, if the secret has uniform distribution $U_\ell$, then the message of ECC has
uniform distribution $U_n$.

$$U_n \overset{\mu}{\sim} \text{aExt}^{-1}(U_\ell).$$

So given that $V = \nu$, the adversary can deduce that the ECC message belongs to the coset
responding to $\nu$, which is the support of an affine source. But the adversary can not deduce any
information about the secret, since the vectors in the coset are mapped to each secret evenly by
$\text{aExt}$ according to the functionality of an affine extractor. Since this is true for any $V = \nu$, one has the following

$$(V, \text{aExt}(U_n)) \overset{\varepsilon_A}{\sim} (V, U_\ell),$$

where $\varepsilon_A$ is the error (measured in statistical distance) of the extractor $\text{aExt}$. Finally, the privacy
error of SSS is the statistical distance between two views $V_0$ and $V_1$ that are corresponding to a

## 4 Construction using affine non-malleable extractors

The general approach we take in this work is to start with binary SSS constructions and consider
how to strengthen them for providing non-malleability.

We first recall the construction of binary SSS against adaptive adversary in [35], which uses an
invertible affine extractor and a linear (more generally stochastic affine) erasure correcting code. Let $\text{aExt}: \{0,1\}^n \to \{0,1\}^\ell$ be a $\mu$-invertible affine extractor and $\text{aExt}^{-1}: \{0,1\}^\ell \times R \to \{0,1\}^n$ be
its inverter that maps an $s \in \{0,1\}^\ell$ to one of its pre-images chosen uniformly at random. Let ECC
be a linear erasure correcting code with encoder $\text{ECCenc}: \{0,1\}^n \to \{0,1\}^N$ that tolerates $(1-\rho)N$
erasures. Given any $0 \leq \tau < \rho \leq 1$, the following pair of sharing and reconstructing algorithms give
a $(\tau N, \rho N, N)$-ramp scheme with binary shares, if $\text{aExt}$ can extract from affine $(n, n-\tau N)$-source.

$$\begin{align*}
\text{Share}(s) &= \text{ECCenc}(\text{aExt}^{-1}(s)); \\
\text{Recst}_R(y) &= \text{aExt}(\text{ECCdec}_R(y)),
\end{align*}$$

where $y = c_R$ is the projection of a share vector $c$ on $R \subset [N]$ with $|R| = \rho N$, and $\text{ECCdec}_R$ is the
the erasure code decoding algorithm with respect to $R$.

Intuitively, ECC enables the reconstruction from any $\rho N$ shares. The privacy for any $\tau N$ shares
is not as straightforward. Firstly, since $\text{ECCenc}$ is a linear function, the view $V$ of the adversary is the
image of a linear function, whose domain is the message space $\{0,1\}^n$ of ECC. This linear function
induces a partition of the message space $\{0,1\}^n$ into cosets each corresponding to a particular
value of the adversary’s view $V = \nu$. Given that the adversary observes $V = \nu$, the message of ECC
can only be one element in the coset corresponding to $\nu$. Secondly, the secret and the message of
ECC (as two random variables) are connected by an invertible affine extractor. According to the
definition of an inverter, if the secret has uniform distribution $U_\ell$, then the message of ECC has
uniform distribution $U_n$.

$$U_n \overset{\mu}{\sim} \text{aExt}^{-1}(U_\ell).$$

So given that $V = \nu$, the adversary can deduce that the ECC message belongs to the coset
responding to $\nu$, which is the support of an affine source. But the adversary can not deduce any
information about the secret, since the vectors in the coset are mapped to each secret evenly by
$\text{aExt}$ according to the functionality of an affine extractor. Since this is true for any $V = \nu$, one has the following

$$(V, \text{aExt}(U_n)) \overset{\varepsilon_A}{\sim} (V, U_\ell),$$

where $\varepsilon_A$ is the error (measured in statistical distance) of the extractor $\text{aExt}$. Finally, the privacy
error of SSS is the statistical distance between two views $V_0$ and $V_1$ that are corresponding to a
pair of secrets $s_0$ and $s_1$, respectively. One can use the above bound for uniform secret to obtain the following bound for any secret $s$ (using Lemma 13 for example).

$$(V|_{\mathbf{aExt}}(U_n) = s) \overset{2^{\ell\cdot\varepsilon_A}}{\sim} V.$$  

Observe that $V_0 = (V|_{\mathbf{aExt}}(U_n) = s_0)$ and $V_1 = (V|_{\mathbf{aExt}}(U_n) = s_1)$. They are both $(2^\ell \cdot \varepsilon_A + \mu)$-close to the distribution of $V$, which is some $t$ components of $ECCenc(U_n)$. It then follows that the privacy error is $\varepsilon = 2^{\ell+1} \cdot \varepsilon_A + 2\mu$.

We now describe how we pack the bits in the $N$-bit share vector into shares to obtain a family of $(t, r, P)$-ramp schemes for given integer tuple $(t, r, P)$. We set $\tau = t/P$ and $\rho = r/P$, and obtain a family of binary ramp schemes with $N$-bit share vector, from which we a sub-family where $N$ is a multiple of $P$. We then divide the $N$-bit share vector into $P$ blocks and call each block of length $N/P$ a share of the $(t, r, P)$-ramp scheme thus obtained.

We next consider whether non-malleability as defined in Definition 14 can be realized within this framework. The above analysis is focused on the message of $nmSSS$ that involves an erasure correcting code. We should also take that into account. We first formerly define the concept of an erasure correcting code

**Definition 15.** Let $ECC$ be a linear erasure correcting code with an encoder $ECCenc: \{0,1\}^n \rightarrow \{0,1\}^N$ and a decoding algorithm $ECCdec$. Let $\sigma$ be an $F$-tampering strategy. Let $R \subset [P]$ be of size $|R| = r$ and $\Pi_R$ denotes the block-wise projection function on the block index set $R$. The induced tampering $g^\sigma_{\sigma,R} : \{0,1\}^n \rightarrow \{0,1\}^n$ at a particular view value $View_A^{\rho \cdot \sigma(R)} = v$ for given $ECC$, $\sigma$ and $R$ is defined as follows.

$$g^\sigma_{\sigma,R} : = ECCdec_R \circ \Pi_R \circ f^\sigma \circ ECCenc,$$  

where $\sigma(v) = f^\sigma \in F$.

We are now at a good position to show a reduction of affine nmSSS to affine non-malleable extractors.

**Theorem 16.** Let $\mathbf{amExt} : \{0,1\}^n \rightarrow \{0,1\}^{\ell}$ be a $\mu$-invertible affine non-malleable $(n - tN/P, \varepsilon_A)$-extractor and $\mathbf{amExt}^{-1} : \{0,1\}^{\ell} \times R \rightarrow \{0,1\}^n$ be its inverter that maps an $s \in \{0,1\}^{\ell}$ to one of its pre-images chosen uniformly at random. Let $ECCenc : \{0,1\}^n \rightarrow \{0,1\}^N$ be the encoder of a linear erasure correcting code $ECC$ that tolerates $N - rN/P$ erasures with decoding error $\delta$. Let

$$\begin{align*}
\text{Share}(s) &= ECCenc(\mathbf{amExt}^{-1}(s)) \\
\text{Recst}_R(y) &= \mathbf{amExt}(ECCdec_R(y)),
\end{align*}$$  

where $y = \text{block}^\rho(c)_R$ is the block-wise projection of a share vector $c \in \{0,1\}^N$ on block index set $R \subset [P]$ with $|R| = r$. Then $(\text{Share}, \text{Recst})$ is an affine $(2^\ell+1\cdot\varepsilon_A + \mu, \delta)$-nmSSS for ramp parameters $(t, r, P)$.
Proof. Reconstruction from any \( r \) shares follows trivially from the functionality of ECC. We next show privacy and non-malleability. Our analysis starts with sharing a uniform secret. According to the definition of a \( \mu \)-invertible extractor, we have
\[
U_n \overset{\mu}{\leftarrow} \text{amnExt}^{-1}(U_{\ell}).
\] (7)
Without loss of generality, we will assume the message of the erasure correcting code ECC is \( U_n \) at the cost of an increase of \( \mu \) in the final error parameter. For any \( t/P \)-leakage adversary \( A \), let \( V : = \text{View}_{\ell}(\text{ECCenc}(U_n)) \) be the view of the adversary on the encoding of a uniform source. Since \( \text{ECCenc} \) is a linear function, \( V \) is the image of an affine function. This shows that \((U_n|V = v)\) is an affine source with at least \( n - tN/P \) entropy. The affine non-malleable \((n - tN/P, \varepsilon_A)\)-extractor \( \text{amnExt} \) is in particular an affine \((n - tN/P, \varepsilon_A)\)-extractor, which yields
\[
((V, \text{amnExt}(U_n))|V = v) \overset{s_d}{\approx} ((V, U_{\ell})|V = v) \text{ or simply } (V, \text{amnExt}(U_n)) \overset{s_d}{\approx} (V, U_{\ell}).
\]
This together with Lemma 13 with respect to the event \( \text{amnExt}(U_n) = s \) for any secret \( s \) gives a privacy error of \( 2^{\ell+1} \varepsilon_A \).

For any affine tampering strategy \( \sigma \) and \( R \subset [P] \) with \( |R| = r \), let \( W : = g_{\sigma,R}^V(U_n) \) denote the tampered source of \( \text{amnExt} \). According to Definition 15, the induced tampering \( g_{\sigma,R}^V \) is an affine function for any \( V = v \). The functionality of the affine non-malleable \((n - tN/P, \varepsilon_A)\)-extractor asserts that there is a distribution \( D_{g_{\sigma,R}^v} \) such that
\[
((\text{amnExt}(W), \text{amnExt}(U_n))|V = v) \overset{s_d}{\approx} (\text{Copy}(D_{g_{\sigma,R}^v}, U_{\ell}), U_{\ell}),
\]
where the two copies of \( U_{\ell} \) are the same random variable and are independent of \( D_{g_{\sigma,R}^v} \).

Let \( D_{A,\sigma,R} \) be the convex combination of \( \{D_{g_{\sigma,R}^v}|v \in V\} \) with coefficients \( \{\Pr[V = v]|v \in V]\) , where \( V \) is the range of the affine leakage function. We then have
\[
(\text{amnExt}(W), \text{amnExt}(U_n)) \overset{s_d}{\approx} (\text{Copy}(D_{A,\sigma,R}, U_{\ell}), U_{\ell}),
\] (8)
where the two copies of \( U_{\ell} \) are the same random variable and are independent of \( D_{A,\sigma,R} \).

Applying Lemma 13 to (8) with respect to the event \( \text{amnExt}(U_n) = s \) for any secret \( s \) yields
\[
(\text{amnExt}(W)|\text{amnExt}(U_n) = s) \overset{2^{\ell+\varepsilon_A}}{\sim} \text{Copy}(D_{A,\sigma,R}, s),
\]
where \( D_{A,\sigma,R} \) is independent of \( s \).

Finally, since the tampering experiment with respect to \( A, \sigma, R \) and \( s \) is \( \mu \)-close to \((\text{amnExt}(W)|\text{amnExt}(U_n) = s)\) according to (7), we have
\[
\text{Tamper}_{A,\sigma,R}^s \overset{\mu + 2^{\ell+\varepsilon_A}}{\approx} \text{Copy}(D_{A,\sigma,R}, s).
\]
\[\square\]

Theorem 16 gives a clean reduction from an affine \((\varepsilon, 0)\)-nmSSS with thresholds \((t, k)\) to an invertible affine non-malleable \((n - t, \varepsilon_A)\)-extractor and a linear code that correct \( k \) erasures. Note that we can use any explicit constructions of invertible affine non-malleable extractors and erasure correcting codes. Any improvement in the construction of the building blocks will lead to affine nmSSS with better parameters.
Remark 17. The constructions of affine non-malleable extractors (Lemma 10 and Lemma 11) require source entropy $n - n^\xi/2$ and have output length $\ell = n^{\Omega(1)}$ with extractor error $\varepsilon_A = 2^{-n^{\Omega(1)}} + n^{-n^\xi/2}$, for some $0 < \xi < 1$. According to [14], they can be made invertible with $\mu = \varepsilon_A$. This means that the privacy threshold $t$ must satisfy

$$n - tN/P \geq n - n^\xi/2 \Rightarrow \frac{\tau N}{n} \leq \frac{n^\xi/2}{n} \Rightarrow \tau \leq \frac{n^\xi}{2n},$$

and the non-malleability error is $(2^\ell + 1)\cdot \varepsilon_A$. The construction in [14] crucially relies on high entropy of the source (entropy rate $\frac{n - n^\xi/2}{n} \rightarrow 1$). This means that the affine non-malleable extractors in [14] does not yield families of nmSSS with relative threshold $\tau > 0$. On the other hand, by replacing the linear erasure correcting code ECC with a stochastic affine code (those used in [35]), we can reconstruct the secret with any $\rho$ fraction of share vector with negligible error probability at rate $R_{ECC} = \frac{k}{n} \sim \frac{k}{n}$. And this replacement does not affect the analysis of non-malleability in Theorem 16. In particular, the induced tampering $g_{\sigma, R}$ in (6) becomes

$$g_{\sigma, R}^v = ECC_{dec}^\tilde{r} \circ \Pi_R \circ f^v \circ ECC_{enc}^r,$$

(6')

where $r$ and $\tilde{r}$ denote the randomness of the stochastic code and its tampered version, respectively. But since the stochastic code is affine, which means for any fixing of its randomness $r$ both $ECC_{enc}^r$ and $ECC_{dec}^\tilde{r}$ are affine functions, the induced tampering $g_{\sigma, R}^v$ is still an affine function. This means that we can obtain families of nmSSS with arbitrary relative reconstruction threshold $\rho$. Finally, the output length of the affine non-malleable extractor is $\ell = n^{\Omega(1)}$ and the non-malleability error bound from Theorem 16 is $(2^\ell + 1)\cdot \varepsilon_A$. In this case, we can not use all $\ell$ bits for secrets. A way to control the non-malleability error is to use $\ell - a$ bits for the real secret and append $a$ random bits. This, however, significantly reduces the secret length.

5 Construction using seeded non-malleable extractors

In this section, we restrict to bit-wise tampering strategies, a special case of affine tampering strategies, and give a construction that has relative thresholds $0 < \tau < \rho < 1$. Our approach is again to start with binary secret sharing and then strengthen it for providing non-malleability.

5.1 New construction of binary secret sharing

As a result of independent interest, we obtain a new construction of binary secret sharing against adaptive adversary that has better coding rate than the construction in [35].

We first recall a classical framework of constructing seedless extractors from seeded extractors. Seeded extractors are known to explicitly extract all the entropy and are not restricted by source structures. Moreover, there are known constructions of linear seeded extractors perform almost as well as the best seeded extractors. The elegant idea of this framework is to use a seedless extractor to extract a short output from the structured source, which then serves as the seed for a seeded extractor to extract all the entropy from the same source. For this idea to work, the dependence of the extracted seed on the source has to be carefully analyzed (and removed).

Lemma 18 ([38]). Let $C$ be a class of distributions over $\{0, 1\}^n$. Let $E : \{0, 1\}^n \rightarrow \{0, 1\}^d$ be a seedless extractor for $C$ with error $\epsilon$. Let $F : \{0, 1\}^d \times \{0, 1\}^n \rightarrow \{0, 1\}^m$. Let $X$ be a distribution in
\( C \) and assume that for every \( z \in \{0,1\}^d \) and \( y \in \{0,1\}^m \), the distribution \((X|F(z,X) = y)\) belongs to \( C \). Then

\[
SD(E(X), F(E(X), X); U_d, F(U_d, X)) \leq 2^{d+3} \epsilon.
\]

An example of such a class of distributions is the affine source, in which case we can use an affine extractor \( F = aExt \) and a linear seeded extractor \( E = Ext \). An affine source \( X \) conditioned on \( Ext(z,X) = y \), which amounts to a set of linear equations, is still an affine source for \( aExt \). With appropriate choice of parameters, we obtain a better affine extractor \( aExt'(X) = Ext(aExt(X), X) \). With an increase of \( d \) bits in the input, we have the following invertible affine extractor.

\[
aExt''(Sd||X) = Ext(aExt(X) + Sd, X),
\]
whose inverter is \((aExt'')^{-1}(s) = (aExt(Ext^{-1}(Z, s)) + Z||Ext^{-1}(Z, s))\), where \( Z \) is \{0,1\}^d.

**Theorem 19.** Let \( aExt: \{0,1\}^n \to \{0,1\}^d \) be a \((n−τN−\ell, ε_A)\)-affine extractor. Let \( Ext: \{0,1\}^d \times \{0,1\}^N \to \{0,1\}^d \) be a linear \((n−τN−d, ε_E)\)-strong extractor with \( ε_E < \frac{3}{N} \). Let \( ECC: \{0,1\}^d + n \to \{0,1\}^N \) be the encoder of a linear erasure correcting code ECC that corrects \((1−ρ)N\) erasures with probability \( δ \). Let

\[
\begin{align*}
\text{Share}(s) &= ECCEnc(Sd||X), \text{ where } X \sim Ext^{-1}(Z, s) \text{ and } \\
Sd &= Z + aExt(X) \text{ with } Z \sim \{0,1\}^d \\
\text{Recst}_R(y) &= Ext(aExt(x) + Sd, x), \text{ where } (sd||x) = ECCdec_R(y),
\end{align*}
\]

where \( y = \mathcal{c}_R \) is the projection of a share vector \( c \) on \( R \subset [N] \) with \( |R| = ρN \). Let \( ε = 2^{(ℓ+1)+(d+4)+2ε_A} + 8ε_E \). Then we have a \((ε, δ)\)-SSS over binary shares with relative threshold \((τ, ρ)\) against an adaptive adversary.

**Proof.** Reconstruction from any \( k \) shares follows from the functionality of ECC and the invertibility guarantee of the invertible extractor, which insures that any correctly recovered pre-image is mapped back to the original secret.

We next prove privacy. Consider a uniform secret \( U_\ell \). By the uniformity guarantee of the inverter, we have \( \text{Share}(U_\ell) = ECCEnc(Sd||U_n) \). Our analysis is done for any fixed \( Sd = sd \). This captures a stronger adversary who on top of adaptively reading \( t \) shares, also has access to \( Sd \) through an oracle. It is easy to see that the fixing of \( Sd = sd \) does not alter the distribution of the source \( U_n \), which remains uniform over \( \{0,1\}^n \). Let \( V = \text{View}_A^{O_{ε}}(ECCEnc(sd||U_n)) \) denote the view of the adversary \( A \) on the encoding of a uniform source. Let \( Z = aExt(U_n) + sd \) denote the seed of the strong linear extractor \( Ext \). Let \( S = Ext(Z, U_n) \). We study the random variable tuple \((V, Z, S)\) to complete the proof.

The tuple \((Z, S)|V = v\) for any fixed \( V = v \) is by definition \((aExt(U_n) + sd, Ext(aExt(U_n) + sd, U_n))|V = v\). Since \((U_n|V = v)\) is an affine source with at least \( n−t \) entropy, according to Lemma 18, we have

\[
(Z, S)|V = v \sim 2^{d+3ε_A} (U_d, Ext(U_d, U_n))|V = v.
\]

Our concern is the relation between \( S \) and \( V \), and therefore would like to further condition on values of \( Z \). In this step, we crucially use the linearity of \( Ext \) and the underlying linear space structure of the affine source \( U_n|V = v \) to claim that there is a subset \( G \subset \{0,1\}^d \) of good seeds such that \( Pr[U_d \in G] \geq 1−4ε_E \) and for any \( z \in G \), the distribution of \( Ext(z, U_n)|V = v \) is exactly uniform. This is true because \( Ext(z, U_n)|V = v \) is an affine source. If its entropy is \( ℓ \), then it is exactly uniform. If its entropy is less than \( ℓ \), its statistical distance \( ε_E^2 \) from uniform is at least \( \frac{1}{2} \). Using
an averaging argument we have that at least $1 - 4\varepsilon_E$ fraction of the seeds should satisfy $\varepsilon_E^2 < \frac{1}{4}$, and hence $\varepsilon_E = 0$. We then use Lemma 13 with respect to the event $Z \in \mathcal{G}$ to claim that

$$(S|(V = v, Z \in \mathcal{G})) \frac{2^{d+4\varepsilon_A}}{1-4\varepsilon_E} \approx (\text{Ext}(U_d, X)|(V = v, U_d \in \mathcal{G})),$$

where the right hand side is exactly $U_\ell$. Note that the subset $\mathcal{G}$ is determined by the indices of the $t$ shares chosen by the leakage adversary $\mathcal{A}$ and hence remains the same for any value of $V = v$. We then have

$$(\{V, S\}|Z \in \mathcal{G}) \frac{2^{d+4\varepsilon_A}}{1-4\varepsilon_E} \approx (V, U_\ell).$$

Another application of Lemma 13 with respect to the event $S = s$ gives

$$(V|(Z \in \mathcal{G}, S = s)) \frac{2^{(\ell+1)+(d+4)\varepsilon_A}}{1-4\varepsilon_E} \approx V.$$

We finally bound the privacy error as follows.

$$\begin{align*}
\text{SD}((V|S = s_0); (V|S = s_1)) &\leq 2\text{SD}((V|S = s); V) \\
&= 2\Pr[Z \in \mathcal{G}] \cdot \text{SD}((V|(Z \in \mathcal{G}, S = s); V) + 2\Pr[Z \notin \mathcal{G}] \cdot \text{SD}((V|(Z \notin \mathcal{G}, S = s); V) \\
&\leq 2 \left( 1 \cdot \frac{2^{(\ell+1)+(d+4)\varepsilon_A}}{1-4\varepsilon_E} + (4\varepsilon_E + \varepsilon_A) \cdot 1 \right) \\
&< 2^{(\ell+1)+(d+4)2\varepsilon_A} + 8\varepsilon_E.
\end{align*}$$

\[\square\]

**Remark 20.** Note that in the error bound $2^{(\ell+1)+(d+4)+2\varepsilon_A} + 8\varepsilon_E$ above, the exponential term $2^{(\ell+1)+(d+4)+2}$ only affects $\varepsilon_A$, the error of aExt. There are known constructions of affine extractor that can extract from any constant fraction of entropy with error exponentially small in the entropy (see Lemma 5). Instantiate aExt with such an affine extractor and Ext with Trevisan’s seeded extractor (see Lemma 7), we have an explicit construction that provide negligible error with seed length $d$ negligible in $\ell$.

This explicit binary SSS has better coding rate than the adaptive binary SSS construction given in [35]. The authors in [35] use the aExt in Lemma 5 with output length $\ell$ and error $\varepsilon_A = 2^{-\Omega(\ell)}$. Instead of using aExt to extract a seed for Ext, they directly make aExt invertible using a One-Time-Pad trick that costs $\ell$ bits increase in the input. So their coding rate is $\frac{\ell}{(\ell + n)/R_{ECC}}$, where $R_{ECC}$ is the rate of the erasure correcting code. Recall that making aExt$^t(\cdot) = \text{Ext}(\text{aExt}^t(\cdot), \cdot)$ invertible only costs $d$ bits, which is negligible in $\ell$ if we use the linear seeded extractor from Lemma 7. We then have coding rate $\frac{\ell}{d + n}/R_{ECC} \approx \frac{\ell}{n}/R_{ECC}$, for the same level of privacy and reconstruction errors.

### 5.2 Strengthened to obtain non-malleability

We again consider strengthening the new construction of binary SSS to obtain binary nmSSS.

The candidate is a linear seeded non-malleable extractor nmExt: $\{0, 1\}^d \times \{0, 1\}^n \rightarrow \{0, 1\}^\ell$, which by definition gives independence guarantee for two applications of the extractor, with respect to two different seeds, to the same source.

Intuitively, we want to replace the linear seeded extractor Ext in Theorem 19 with a linear seeded non-malleable extractor nmExt. Using a seeded non-malleable extractor in the construction
of non-malleable codes has many challenges (as far as we know this has not been considered in the literature). First of all, the tampered source and the original source are not the same. We should first reduce the different sources situation to a same source situation in order to be able to use the functionality of \texttt{nmExt}. Secondly, seeded non-malleable extractors allow the seed to be arbitrarily tampered, but impose a condition that the tampered seed should never be the same as the original seed (the seed tampering function has no fixed point). Lemma 18 only shows that the original seed and the tampered seed are both uniform and independent of the original source and tampered source, respectively. But the two seeds could be related in an arbitrary way, for example, collide with any probability. When the tampered seed coincides the original seed, we don’t have independence guarantee for the two copies of outputs. In fact, they are nicely related. We then exploit this relation and use an AMD pre-coding of the secret to detect the tampering. Besides the challenges from using a seeded non-malleable extractor, to be able to invoke Lemma 18, the tampered source should have enough entropy. But we know the adversary of non-malleable secret sharing can overwrite many shares and leave negligible amount of entropy in the tampered source. Luckily, in this case, we can simply consider the tampered source as a leakage and make the source itself independent of the secret. To address these challenges in a systematic fashion, we define the entropy of an affine function with respect to an affine source and use it to separate our discussion into two cases.

The entropy of a function is the entropy of its output when the input is uniform. Recall that our analysis is focused on induced tampering (see Definition 15) that is applied to the source of the invertible affine extractor. Since the induced tampering \( g_{\sigma, R} \) is applied only under the condition that the view value is \( v \), we then have to consider the entropy of a function when its input is not uniform. We then define the entropy of a function \( g \) with respect to a source \( X \).

**Definition 21.** The entropy of a function \( g \) with respect to a source \( X \) is the quantity \( H_\infty(g(X)) \).

From now on, we consider a linear erasure correcting code \texttt{ECC} with encoder \texttt{ECCenc}: \( \{0, 1\}^{d+n} \rightarrow \{0, 1\}^N \). Let the input to \texttt{aExt''} be \( (\text{sd}|\text{U}_n) \). We refer to the first \( d \) bits as the \textit{seed indicator} and only consider \( \text{U}_n \) as the source of \texttt{aExt''}. In fact, in the security analysis, we always consider a fixed \( \text{Sd} = \text{sd} \). For any non-adaptive/adaptive leakage adversary \( A \), let \( V = (\text{View}_A^{O_T, P^{(\text{ECCenc}(\text{sd}|\text{U}_n))}}) \) denote the view of the adversary on the encoding of a uniform source. We have that \( (\text{U}_n|V=v) \) is an affine source with at least \( n-t \) entropy. For any tampering strategy \( f \) and reconstruction set \( K \subset [N] \) with \( |K| = k \), let

\[
(\tilde{\text{sd}}||W) = g_{\sigma, R}^{V}(\text{sd}||\text{U}_n)
\]

denote the tampered source of \texttt{aExt''}. According to Definition 15, the induced tampering \( g_{\sigma, R}^{V} \) is an affine function for any \( V = v \). We call the entropy of \( g_{\sigma, R}^{V} \) with respect to the source \( (\text{U}_n|V=v) \) the \textit{entropy of} \( g_{\sigma, R}^{V} \) for short. The entropy of an affine function \( g \) with respect to an affine source \( X \) is equal to the dimension of the support of the affine source \( g(X) \). The entropy of \( g_{\sigma, R}^{V} \) is then an integer. It is easier to consider \( g_{\sigma, R}^{V} \) as a function defined over the support of the distribution \( \text{U}_n|V=v \) (instead of \( \{0, 1\}^n \)). Then we have that the entropy of \( g_{\sigma, R}^{V} \) is \( H_\infty(W|V=v) = \dim(\text{Im}(g_{\sigma, R}^{V})) \). Now the fundamental theorem of linear algebra yields

\[
n - H_\infty(V) = \dim(\text{Ker}(g_{\sigma, R}^{V})) + H_\infty(W|V=v).
\]

The quantity \( \dim(\text{Ker}(g_{\sigma, R}^{V})) \) characterizes the remaining entropy of \( (\text{U}_n|V=v) \) after revealing \( W = w \) for some particular \( w \).

We now strengthen the linear seeded extractor \texttt{Ext} in Theorem 19 to a linear non-malleable extractor \texttt{nmExt} and show that this together with an AMD pre-coding of the secret provides non-malleability. Recall that our goal is to replace the \texttt{nmExt} in (8) with an invertible affine extractor.
**Theorem 22.** Let $aExt: \{0,1\}^n \to \{0,1\}^d$ be a $(n-tN/P, \varepsilon_A)$-affine extractor. Let $nmExt: \{0,1\}^d \times \{0,1\}^n \to \{0,1\}^\ell$ be a linear $(\frac{n-tN/P}{2} - d, \varepsilon_E)$-strong extractor with error $\varepsilon_E < 2^{-(d+3)}$. Let $ Ecc: \{0,1\}^{d+n} \to \{0,1\}^N$ be the encoder of a linear erasure correcting code $ Ecc $ that corrects $ N - tN/P $ erasures with probability $ \delta $. Let $ (AMDenc, AMDdec) $ be an AMD code with detection error $ \varepsilon_{AMD} $. Let

\[
\begin{align*}
\text{Share}(s) & = Ecc(Sd | X), \text{ where } X \overset{\$}{\leftarrow} nmExt^{-1}(Z, AMDenc(s)) \text{ and } \\
Sd & = Z + aExt(X) \text{ with } Z \overset{\$}{\leftarrow} \{0,1\}^d \\
Rec_{R}(y) & = AMDdec(nmExt(aExt(\tilde{x}) + sd, \tilde{x})), \text{ where } (sd || \tilde{x}) = Eccdec_{R}(y),
\end{align*}
\]
where \( y = \text{block}^P(c)_R \) is the block-wise projection of a share vector \( c \in \{0,1\}^N \) on block index set \( R \subset [P] \) with \(|R| = r \). Let \( \varepsilon = 2^{d+7}\varepsilon_A + 4\varepsilon_E + \varepsilon_{AMD} \). Then we have a \((\varepsilon, \delta)\)-nmSSS with ramp parameters \((t, r, P)\).

**Proof.** Reconstruction from any \( r \) shares follows from the functionality of ECC and the invertibility guarantee of the invertible extractor, which assures that any correctly recovered pre-image is mapped back to the original secret.

We next prove non-malleability. Consider a uniform secret \( U_\ell \). By the uniformity guarantee of the inverter, we have \( \text{Share}(U_\ell) = \text{ECCenc}(\text{SD}(U_n)) \). Our analysis is done for any fixed \( \text{SD} = \text{sd} \). This captures a stronger adversary who on top of adaptively reading \( t \) shares, also has access to \( \text{SD} \) through an oracle. It is easy to see that the fixing of \( \text{SD} = \text{sd} \) does not alter the distribution of the source \( U_n \), which remains uniform over \( \{0,1\}^n \). Let \( V = \text{View}_{\mathcal{A}}^O(\text{ECCenc}(\text{SD}(U_n))) \) denote the view of the adversary \( \mathcal{A} \) on the encoding of a uniform source. Let \( (\text{sd}|\text{W}): = g^v_{\sigma,R}(\text{sd}|U_n) \) denote the tampered source of the affine extractor \( a_{\text{Ext}}(\text{sd}|\cdot): = \text{Ext}(a_{\text{Ext}}(\cdot) + \text{sd}, \cdot) \). Let \( Z = a_{\text{Ext}}(U_n) + \text{sd} \) denote the original seed of \( \text{nmExt} \), which is in particular a strong linear extractor. Let \( S = \text{nmExt}(Z, U_n) \). We study the random variable tuple \((V, W, Z, S)\) to complete the proof.

**Handling the low entropy case.** We assume the induced tampering \( g^v_{\sigma,R} \) has entropy at most \( \frac{n-t(N/P)}{2} \). This means that \((U_n|(V = v, W = w)) \) has entropy at least \( \frac{n-t(N/P)}{2} \), according to (9).

The tuple \((Z, S)| (V = v, W = w)\) for any fixed \( V = v \) and \( W = w \) is by definition \((a_{\text{Ext}}(U_n) + \text{sd}, \text{Ext}(a_{\text{Ext}}(U_n) + \text{sd}, U_n))(V = v, W = w) \). Since \((U_n|(V = v, W = w)) \) is an affine source with at least \( \frac{n-t(N/P)}{2} \) entropy, according to Lemma 18, we have

\[
(Z, S)|(V = v, W = w) \sim \left( U_d, \text{Ext}(U_d, U_n) \right)(V = v, W = w).
\]

Our concern is the relation between \( S \) and \( W \), and therefore would like to further condition on values of \( Z \). In this step, we crucially use the linearity of \( \text{nmExt} \) and the underlying linear space structure of the affine source \((U_n|(V = v, W = w))\) to claim that there is a subset \( \mathcal{G} \subset \{0,1\}^d \) of good seeds such that \( \Pr[U_d \in \mathcal{G}] \geq 1 - 4\varepsilon_E \) and for any \( z \in \mathcal{G} \), the distribution of \( \text{nmExt}(z, U_n)|(V = v, W = w) \) is exactly uniform. This is true because \( \text{nmExt}(z, U_n)|(V = v, W = w) \) is an affine source. If its entropy is \( \ell \), then it is exactly uniform. If its entropy is less than \( \ell \), its statistical distance \( \varepsilon_E^z \) from uniform is at least \( \frac{1}{4} \). Using an averaging argument we have that at least \( 1 - 4\varepsilon_E \) fraction of the seeds should satisfy \( \varepsilon_E^z < \frac{1}{4} \), and hence \( \varepsilon_E^z = 0 \). We then use Lemma 13 with respect to the event \( Z \in \mathcal{G} \) to claim that

\[
S|(V = v, W = w, Z \in \mathcal{G}) \sim \text{nmExt}(U_d, X)|(V = v, W = w, U_d \in \mathcal{G}),
\]

where the right hand side is exactly \( U_\ell \). Note that the subset \( \mathcal{G} \) is determined by the indices of the \( t \) shares chosen by the leakage adversary \( \mathcal{A} \) and the induced tampering function \( g^v_{\sigma,R} \), hence remains the same for any value of \( W = w \). We then have

\[
((W, S)|(V = v, Z \in \mathcal{G})) \sim ((W, U_\ell)|V = v).
\]

Another application of Lemma 13 with respect to the event \( S = s \) gives

\[
(W|(V = v, Z \in \mathcal{G}, S = s)) \sim (W|V = v).
\]
We finally bound the non-malleability error as follows.

\[
\begin{align*}
SD(W|(V = v, S = s); (W|V = v)) \\
= \Pr[Z \in G] \cdot SD((W|(V = v, S = s, Z \in G)); (W|V = v)) \\
+ \Pr[Z \notin G] \cdot SD((W|(V = v, S = s, Z \notin G)); (W|V = v)) \\
\leq 1 \cdot \frac{2^{(\ell+1)+(d+4)}}{E_{\Xi}} + (4\varepsilon_E + \varepsilon_A) \cdot 1 \\
< 2^{(\ell+1)+(d+4)} + 1 \varepsilon_A + 4\varepsilon_E.
\end{align*}
\]

Handling the high entropy case. We assume the induced tampering \(g_{\sigma,R}^\vee\) has entropy at least \(\frac{n - tN/P}{2}\).

Note that for any bit-wise independent function \(f^\vee\), we can define a difference function \(\Delta f^\vee\) such that for any \(c \in \{0,1\}^N\),

\[
f^\vee(c) = c + \Delta f^\vee(c).
\]

The difference function \(\Delta f^\vee\) also induces a source tampering \(\Delta g_{\sigma,R}^\vee\). Now since the erasure correcting code ECC is linear, we must have for any \(m \in \{0,1\}^{d+n}\),

\[
g_{\sigma,R}^\vee(m) = m + \Delta g_{\sigma,R}^\vee(m).
\]

Let \(\Delta W : = \Delta g_{\sigma,R}^\vee(sd||U_n)\) be the tapered source induced by the difference function \(\Delta f^\vee\). We immediately have

\[
W = U_n + \Delta W. \quad (12)
\]

Moreover, since the overwrite bit functions of \(f^\vee\) become non-overwrite bit functions of \(\Delta f^\vee\), we then have

\[
H_\infty(\Delta W | V = v) = n - H_\infty(V) - H_\infty(W | V = v).
\]

This means that the dimension of the kernel space of \(\Delta f^\vee\) restricted to the support of \((U_n | V = v)\) satisfies the following.

\[
\dim(\text{Ker}(\Delta g_{\sigma,R}^\vee)) = n - H_\infty(V) - H_\infty(\Delta W | V = v) = H_\infty(W | V = v) \geq \frac{n - tN/P}{2}. \quad (13)
\]

The quantity \(\dim(\text{Ker}(\Delta g_{\sigma,R}^\vee))\) characterises the remaining entropy in \(U_n\) after conditioning on \(V = v\) and \(\Delta W = \Delta w\), for any particular \(\Delta w\).

Now since by assumption \(H_\infty(W | V = v) \geq \frac{n - tN/P}{2}\), Lemma 18 says that

\[
((a\text{Ext}(W) + \tilde{s}, a\text{Ext}''(sd||W)) | V = v) \overset{2d+3\varepsilon_A}{\sim} ((Z', \text{nmExt}(Z', W)) | V = v), \quad (14)
\]

where \(Z'\) is a uniform seed independent of \(W\). We next use (12) and the linearity of \(\text{nmExt}\) to claim that

\[
((Z', \text{nmExt}(Z', W)) | V = v) = ((Z', \text{nmExt}(Z', U_n) + \text{nmExt}(Z', \Delta W)) | V = v).
\]

We next show that the additive term \(\text{nmExt}(Z', \Delta W)\) can be ignored in the subsequent analysis of comparing \(\text{nmExt}(Z', W)\) against \(\text{nmExt}(Z, U_n)\). Since the remaining entropy in \(U_n\) after conditioning on \(V = v\) and \(\Delta W = \Delta w\) is at least \(\frac{n - tN/P}{2}\) (see (13)), we have according to the functionality of \(\text{nmExt}\) that

\[
((Z, \text{nmExt}(T(Z), U_n)), \text{nmExt}(Z, U_n)) | (V = v, \Delta W = \Delta w) \\
\overset{\varepsilon_E}{\sim} ((Z, \text{nmExt}(T(Z), U_n)), U_t) | (V = v, \Delta W = \Delta w),
\]

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where $\mathcal{T}(\cdot)$ is a seed tampering function without fixed point. Let $\mathcal{E}_{g_{s}, R}$ denote the event that $Z \neq Z'$ and w.l.o.g. assume $0 < \Pr[\mathcal{E}_{g_{s}, R}] < 1$. Applying Lemma 13 with respect to the event $\mathcal{E}_{g_{s}, R}$ yields

$$
\Pr[\mathcal{E}_{g_{s}, R}] \leq \Pr[\mathcal{E}_{g_{s}, R}] + \Pr[\varepsilon_{\mathcal{E}_{g_{s}, R}}] = \varepsilon_{\mathcal{E}_{g_{s}, R}}.
$$

Now for any original seed $z$ and its tampered version $z'$, we always have that $(\text{nmExt}(z, U_n))((V = v, \Delta W = \Delta w, \mathcal{E}_{g_{s}, R}^v)) = \varepsilon_{\mathcal{E}_{g_{s}, R}}$, for any $\tilde{s}$, is an affine source. Its statistical distance to uniform is then either 0 or at least $\frac{1}{2}$. Using an averaging argument, we have for at most

$$\frac{4\varepsilon_{E}}{\Pr[\mathcal{E}_{g_{s}, R}]},$$

fraction of such seeds, the above statistical distance exceeds $\frac{1}{4}$. Let $B$ denote these bad seeds. We then have

$$
((\text{nmExt}(Z', W), \text{nmExt}(Z, U_n)))((V = v, \Delta W = \Delta w, \mathcal{E}_{g_{s}, R}^v, Z \notin B)) = ((\text{nmExt}(Z', W), U_{\ell})))((V = v, \Delta W = \Delta w, \mathcal{E}_{g_{s}, R}^v, Z \notin B)).
$$

Taking the error that incurs transforming from seedless extractor to seeded extractor $(14)$ into account, we have that when the event $\mathcal{E}_{g_{s}, R}^v$ occurs, the non-malleability error is upper bounded as follows.

$$
\varepsilon_{\mathcal{E}_{g_{s}, R}} \leq 1 \cdot \frac{2^{(\ell+1)+(d+4)}\varepsilon_{A}}{1 - \frac{4\varepsilon_{E}}{\Pr[\mathcal{E}_{g_{s}, R}}] + \varepsilon_{A}} \cdot 1
$$

where the second inequality follows from the fact that $\Pr[\mathcal{E}_{g_{s}, R}] \geq 2^{-d}$ once $\Pr[\mathcal{E}_{g_{s}, R}] > 0$ and the last inequality follows from the assumption that $\varepsilon_{E} < 2^{-(d+3)}$.

On the other hand, if the complimentary event $\mathcal{E}_{g_{s}, R}^c$ occurs, then

$$
((Z, \text{nmExt}(Z, W), S)))((V = v, \Delta W = \Delta w)) = ((Z, S + \text{nmExt}(Z, \Delta w), S)))((V = v, \Delta W = \Delta w)).
$$

This means that the tampering results in turning $S$ into $S + \text{nmExt}(Z, \Delta w)$, where the offset $\text{nmExt}(Z, \Delta w)$ is independent of $S$. In this case, let $S$ be the AMD codeword of the real secret with fresh independent encoding randomness. The decoder of the AMD code outputs $\bot$ with $\varepsilon_{AMD}$. Taking the error that incurs transforming from seedless extractor to seeded extractor $(14)$ into account, we have that when the complimentary event $\mathcal{E}_{g_{s}, R}^c$ occurs, the non-malleability error is upper bounded as follows.

$$
\varepsilon_{\mathcal{E}_{g_{s}, R}} \leq 1 \cdot \frac{2^{(\ell+1)+(d+4)}\varepsilon_{A}}{1 - \Pr[\mathcal{E}_{g_{s}, R}^c] + \varepsilon_{AMD}} \cdot 1.
$$

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Finally, the total non-malleability error is
\[
\varepsilon \leq \Pr[\mathcal{E}^g_{\sigma,R}] \cdot \varepsilon_{\mathcal{E}^g_{\sigma,R}} + \left(1 - \Pr[\mathcal{E}^g_{\sigma,R}]\right) \cdot \varepsilon_{\bar{\mathcal{E}}^g_{\sigma,R}} \\
< \left(2^{(\ell+2)+(d+4)}\varepsilon_A + 4\varepsilon_E\right) + \left(2^{(\ell+1)+(d+4)}\varepsilon_A + \varepsilon_{AMD}\right) \\
< 2^{d+7}\varepsilon_A + 4\varepsilon_E + \varepsilon_{AMD}.
\]

\[\square \quad \square\]

**Remark 23 (On Explicit Constructions of Linear Non-malleable Extractors).** The only linear non-malleable extractors we found in the literature is an inner product based construction $IP(X, \text{enc}(Z))$, where $IP(\cdot, \cdot)$ denotes the inner product of vectors over finite field $\mathbb{F}_q$ and $\text{enc}(Z)$ is a specific encoding of the seed $Z$ [32]. Let $q = 2^k$. We can have a non-malleable extractor that outputs $\ell$ bits with exponentially small error, if the source $X \leftarrow \mathbb{F}_q^n$ has more than half entropy rate. This extractor is $\mathbb{F}_2$-linear because for any seed $Z = z$, we have $IP(X + X', \text{enc}(z)) = IP(X, \text{enc}(z)) + IP(X', \text{enc}(z))$. This linear non-malleable extractor’s output is a constant fraction of $n$ and error is exponentially small in $n$. But this extractor requires a source entropy rate bigger than half, which makes it not applicable in our construction since the entropy requirement of $\text{nmExt}$ is $\frac{n - tN/P}{2} - d < \frac{n}{q}$.

This entropy rate around half barrier existed in the literature of (non-linear) non-malleable extractor constructions [20], but was quickly overcome [13], being only a technical barrier (not inherent). We next show that to output a $\Omega(\log n)$ number of uniform bits with negligible error, at most $\phi n$ bits of entropy suffices, for any constant $\phi > 0$. This is shown using a probabilistic argument (see Appendix B for its proof) and we leave the explicit construction as an interesting open problem.

We conclude this section by stating an existence result for the linear seeded non-malleable extractors with our required parameters.

**Theorem 24.** For all integers $n, d, m$ and positive parameters $k, \varepsilon$, there is a linear seeded non-malleable $(k, \varepsilon)$-extractor $E : \{0,1\}^d \times \{0,1\}^n \to \{0,1\}^m$ provided that
\[
\begin{align*}
\{ d & \geq \log(n/\varepsilon^2) + O(1), \\
2m & \leq \log(k + \log\varepsilon) - \log(1/\varepsilon^2) - \log d - O(1).
\end{align*}
\]

## 6 Conclusion

We proposed a natural non-malleable secret sharing model called adaptive tampering, as opposed to the known independent tampering and joint tampering. We considered the affine tampering family and constructed non-malleable secret sharing in adaptive tampering model.

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Appendices

A Proof for Lemma 13

Proof. Assume by contradiction that \( \text{SD}(V \mid W \in \mathcal{E}; V' \mid W' \in \mathcal{E}) > \frac{2\varepsilon}{\Pr[W \in \mathcal{E}]} = \varepsilon_0 \). W.l.o.g. there is an event \( \Omega \subset \mathcal{V} \) (complementing \( \Omega \) if necessary) , such that

\[
\Pr[V \in \Omega \mid W \in \mathcal{E}] - \Pr[V' \in \Omega \mid W' \in \mathcal{E}] > \varepsilon_0.
\]

Now consider the event \( \Omega \times \mathcal{E} \subset \mathcal{V} \times \mathcal{W} \). We have

\[
\begin{align*}
\left\{ \begin{array}{c}
\Pr[(V, W) \in \Omega \times \mathcal{E}] = \Pr[V \in \Omega \mid W \in \mathcal{E}] \cdot \Pr[W \in \mathcal{E}]; \\
\Pr[(V', W') \in \Omega \times \mathcal{E}] = \Pr[V' \in \Omega \mid W' \in \mathcal{E}] \cdot \Pr[W' \in \mathcal{E}].
\end{array} \right.
\end{align*}
\]

On the other hand, we have \( \text{SD}(W; W') \leq \text{SD}(V, W; V', W') \leq \varepsilon \) and hence

\[
\Pr[W \in \mathcal{E}] \geq \Pr[W' \in \mathcal{E}] - \varepsilon.
\]

We then can derive the following contradiction.

\[
\begin{align*}
\Pr[(V, W) \in \Omega \times \mathcal{E}] - \Pr[(V', W') \in \Omega \times \mathcal{E}] &\geq \Pr[W' \in \mathcal{E}] \cdot (\Pr[V \in \Omega \mid W \in \mathcal{E}] - \Pr[V' \in \Omega \mid W' \in \mathcal{E}]) - \varepsilon \\
&> \Pr[W' \in \mathcal{E}] \cdot \varepsilon_0 - \varepsilon \\
&= \varepsilon.
\end{align*}
\]

This concludes the proof. \( \square \)
B Proof for Theorem 24

*Proof.* We adapt the proof of [21] to show the existence of non-malleable extractors that are linear; i.e., the extractor is a linear function for every fixed seed. This will however result in much weaker parameters than non-linear counterparts.

For a function $E : \{0,1\}^d \times \{0,1\}^n \rightarrow \{0,1\}^m$, distinguisher $D : \{0,1\}^d \times \{0,1\}^m \rightarrow \{0,1\}^m$, seed tampering adversary $A : \{0,1\}^d \rightarrow \{0,1\}^d$, and error parameter $\varepsilon$, call an input $x \in \{0,1\}^n$ bad for the tuple $(E,A,D)$ if it violates the following condition for a uniform random seed $S \xleftarrow{\$} \{0,1\}^d$:

$$|Pr[D(S,E(A(S),x),E(S,x)) = 1] − Pr[D(S,E(A(S),x),U_m) = 1]| ≤ \varepsilon.$$

Let $B(E,A,D,\varepsilon)$ denote the set of all bad inputs for $(E,A,D)$ for the parameter $\varepsilon$. We have the following.

**Lemma 25.** Suppose $|B(E,A,D,\varepsilon)| ≤ \varepsilon 2^k$ for all distinguishers $D$ and adversaries $A$. Then $E$ is a non-malleable $(k,2\varepsilon)$-extractor.

*Proof.* Consider any source $X$ of min-entropy at least $k$, any distinguisher $D$ and adversary $A$. Then,

$$Pr[X \in B(E,A,D,\varepsilon)] ≤ |B(E,A,D,\varepsilon)|2^{-k} ≤ \varepsilon.$$

Let $\Delta = |Pr[D(S,E(A(S),X),E(S,X)) = 1] − Pr[D(S,E(A(S),X),U_m) = 1]|$. We have

$$\Delta ≤ Pr[X \in B(E,A,D,\varepsilon)] + \varepsilon ≤ 2\varepsilon,$$

where the first inequality follows from the definition of the bad inputs. The result follows. \(\square\)

Adapting the notation of [21], the Martingale-based argument of [21] proves the following:

**Lemma 26 ([21], Implicit in Theorem 37).** Let $x \in \{0,1\}^n$ be fixed and $E : \{0,1\}^d \times \{0,1\}^n \rightarrow \{0,1\}^m$ be any random function such that $E(s,x)$ is uniformly random and independent for all choices of $s \in \{0,1\}^d$. Then, for any distinguisher $D$, adversary $A$, and error $\varepsilon > 0$,

$$Pr[x \text{ is bad for } (E,A,D)] ≤ 4 \exp(-2^{d-4} \varepsilon^2),$$

where the probability is over the randomness of $E$.

We now consider a random function $E : \{0,1\}^d \times \{0,1\}^n \rightarrow \{0,1\}^m$. This time, however, the random function is linear. That is for every seed $s$, we independently sample a random $m \times n$ matrix $M_s$ over $\mathbb{F}_2$ and define $E(s,x) = M_s x$. Consider an adversary that perturbs a seed $s$ to $A(s)$, and a distinguisher $D$.

Let $X \subset \{0,1\}^n$ be any set of size $\varepsilon 2^k$. Then $X$ must have a subset $I(X) \subset X$ of size at least $\log |X| = k + \log \varepsilon$ such that the elements of $I(X)$ are linearly independent. This means that the random variables $E(s,x)$ for all $x \in I(X)$ and $s \in \{0,1\}^d$ are jointly independent. In particular, the events “$x$ is bad for $(E,A,D)$” are also jointly independent for $x \in I(X)$. Therefore, using Lemma 26,

$$Pr[\text{all } x \in X \text{ are bad for } (E,A,D)] ≤ Pr[\text{all } x \in I(X) \text{ are bad for } (E,A,D)]$$

$$≤ 4^{|I(X)|} \exp(-2^{d-4} \varepsilon^2 |I(X)|)$$

$$< \exp(2 |I(X)| - 2^{d-4} \varepsilon^2 |I(X)|).$$
Now, using the above bound and the fact that $|I(\mathcal{X})| = k + \log \varepsilon$, we have

\[
\Pr[|B(E, A, D, \varepsilon)| > \varepsilon^{2k}] \leq \Pr[\exists x \in \mathcal{X}: \text{all } x \in \mathcal{X} \text{ are bad for } (E, A, D)] \\
< \exp \left( (2 - 2^{d-4} \varepsilon^2)|I(\mathcal{X})| \right) \cdot \left( \frac{2^n}{|I(\mathcal{X})|} \right) \\
= \exp \left( (2 - 2^{d-4} \varepsilon^2)(k + \log \varepsilon) \right) \cdot \left( \frac{2^n}{k+\log \varepsilon} \right),
\]

where in the last inequality we have used a union bound over all possibilities of $I(\mathcal{X})$. Now, by using a union bound over all choices of $D$ and $A$ and using Lemma 25, we conclude that

\[
\Pr[E \text{ is not a non-malleable } (k, \varepsilon)\text{-extractor}] \\
\leq \exp \left( (2 - 2^{d-4} \varepsilon^2)(k + \log \varepsilon) \right) \cdot 2^n(k+\log \varepsilon + 2^{d+2m} + d2^d).
\]

The right hand side can be made less than 1, hence ensuring the existence of a linear non-malleable $(k, \varepsilon)$-extractor provided that (15) holds. \qed