ALMOST MATHIEU OPERATORS WITH COMPLETELY RESONANT PHASES

WENCAI LIU

Abstract. Let $\alpha \in \mathbb{R} \backslash \mathbb{Q}$ and $\beta(\alpha) = \limsup_{n \to \infty} (\ln q_{n+1})/q_n < \infty$, where $p_n/q_n$ is the continued fraction approximations to $\alpha$. Let $(H_{\lambda,\alpha,\theta}u)(n) = u(n+1) + u(n-1) + 2\lambda \cos 2\pi (\theta + n\alpha)u(n)$ be the almost Mathieu operator on $l^2(\mathbb{Z})$, where $\lambda, \theta \in \mathbb{R}$. Avila and Jitomirskaya [2] conjectured that for $2\theta \in \alpha \mathbb{Z} + \mathbb{Z}$, $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization if $|\lambda| > e^{2\beta(\alpha)}$. In this paper, we developed a method to treat simultaneous frequency and phase resonances and obtain that for $2\theta \in \alpha \mathbb{Z} + \mathbb{Z}$, $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization if $|\lambda| > e^{3\beta(\alpha)}$.

1. Introduction

The almost Mathieu operator (AMO) is the (discrete) quasi-periodic Schrödinger operator on $l^2(\mathbb{Z})$: 

$$(H_{\lambda,\alpha,\theta}u)(n) = u(n+1) + u(n-1) + 2\lambda \cos 2\pi (\theta + n\alpha)u(n),$$

where $\lambda$ is the coupling, $\alpha$ is the frequency, and $\theta$ is the phase.

The AMO is the most studied quasi-periodic Schrödinger operator, arising naturally as a physical model. We refer the readers to [34, 40] and the references therein for physical background. Most recently, there are a lot of interesting topics related to AMO, e.g. [4, 27, 28, 31, 33, 35, 45].

We say phase $\theta \in \mathbb{R}$ is completely resonant with respect to frequency $\alpha$ if $2\theta \in \alpha \mathbb{Z} + \mathbb{Z}$. In this paper, we always assume $\alpha \in \mathbb{R} \backslash \mathbb{Q}$.

Conjecture 1: Avila and Jitomirskaya [1, 2] assert that for $2\theta \in \alpha \mathbb{Z} + \mathbb{Z}$, $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization if $|\lambda| > e^{2\beta}$, where

$$\beta = \beta(\alpha) = \limsup_{n \to \infty} \frac{\ln q_{n+1}}{q_n},$$

and $p_n/q_n$ is the continued fraction approximations to $\alpha$.

Completely resonant phases of quasi-periodic operators correspond to the rational rotation numbers with respect to frequency in the Aubry dual model. We refer the readers to [13, 18, 29] for the Aubry duality. The (quantitative) reducibility of Schrödinger cocycles with rational rotation numbers is related to many topics in quasi-periodic operators. For example, it is a good approach to show that all the spectral gaps $G_m$ labeled by gap labeling theorem[7, 32] are open (named after dry Ten Martini Problem for the almost Mathieu operator). The dry Ten Martini Problem[2] is stronger than Ten Martini Problem (the latter one was finally solved by Avila and Jitomirskaya [2]). It is also related to the Hölder continuity of Lyapunov exponents, rotation numbers and the integrated density of states.

---

1The rotation number $\rho$ on gap $G_m$ satisfies $2\rho = m\alpha \mod \mathbb{Z}$.

2The dry Ten Martini Problem is still open for all parameters. The non-critical coupling case has been solved by Avila-You-Zhou [2].

1
The reducibility of the Schrödinger cocycles with rational rotation numbers was first established by Moser and Pöschel [41], who modified the proof of reducibility of cocycles with Diophantine rotation numbers [14]. See [13, 19] for more precise results. It was first realized by Puig [32, 43] that localization at completely resonant phases leads to reducibility for Schrödinger cocycles with rational rotation numbers for the dual model. The argument was significantly developed in [3, 20, 36, 39].

For completely resonant phases, Jitomirskaya-Koslover-Schulte [25] proved localization for $H_{\lambda,\alpha,\theta}$ with an arithmetic condition (DC) [6, 21, 24, 26, 27]. Later, universal (reflective) hierarchical structure of parameters. In particular, phase transitions happen in positive Lyapunov exponent regime for $\phi$ is an eigenfunction, that is

\[ \lambda, \alpha, \theta \in \mathbb{R}\setminus\mathbb{Q} \text{ satisfies } \beta(\alpha) < \infty. \] Then the almost Mathieu operator $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization if $2\theta \in \alpha\mathbb{Z} + \mathbb{Z}$ and $|\lambda| > e^{3\beta(\alpha)}$. Moreover, if $\phi$ is an eigenfunction, that is $H_{\lambda,\alpha,\theta} \phi = E \phi$, we have

\[ \limsup_{k \to \infty} \frac{\ln(\phi^2(k) + \phi^2(k-1))}{2|k|} \leq -(\ln \lambda - 3\beta). \]

**Remark 1.2.** For $\alpha$ with $\beta(\alpha) = +\infty$, $H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum [17, 14] if $|\lambda| > 1$.

Now we will discuss the histories of Conjecture 1 and also our approach to the proof of Theorem 1.1. We state another related conjecture first. Define

\[ \delta(\alpha, \theta) = \limsup_{n \to \infty} -\frac{\ln ||2\theta + na||}{|n|}. \]

**Conjecture 2:** Jitomirskaya [22] conjectured that

1. **(Diophantine phase)** $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization if $|\lambda| > e^{\beta(\alpha)}$ and $\delta(\alpha, \theta) = 0$, and $H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum for all $\theta$ if $1 < |\lambda| < e^{\beta(\alpha)}$.

2. **(Diophantine frequency)** Suppose $\beta(\alpha) = 0$. $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization if $|\lambda| > e^{\beta(\alpha)}$ and has purely singular continuous spectrum if $1 < |\lambda| < e^{\delta(\alpha, \theta)}$.

Notice that $\beta(\alpha) = 0$ for almost every $\alpha$, and $\delta(\alpha, \theta) = 0$ for almost every $\theta$ and fixed $\alpha$.

The case $\beta(\alpha) = 0$ and $\delta(\alpha, \theta) = 0$ of Conjecture 2 was solved by Jitomirskaya in her pioneering paper [22]. Avila and Jitomirskaya [2] proved the localization part for Diophantine phases in the regime $|\lambda| > e^{\frac{16}{7}\beta}$, which was a key step to solve the Ten Martin Problem. Liu and Yuan followed their proof and extended the result to $|\lambda| > e^{\frac{12}{7}\beta}$ [37]. Liu and Yuan [38] further developed Avila-Jitomirskaya’s technics in [2] and verified the Conjecture 1 in regime $|\lambda| > e^{\frac{7}{7}\beta}$. Here, $\frac{3}{7}$ and 7 are the limit of the method of [2].

Recently, Avila-You-Zhou [6] proved the singular continuous spectrum part of 2a, as well as the measure-theoretic version of 2a: $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization for $|\lambda| > e^{\delta}$ and

\[ ||k\alpha|| \geq \kappa |k|^{-\tau} \text{ for any } k \in \mathbb{Z} \setminus \{0\}, \]

where $||x|| = \text{dist}(x, \mathbb{Z})$.  

\[ \frac{3}{7} \]
almost every $\theta$. See also [24]. Diophantine frequency (2b) and localization part of Diophantine phase (2a) were proved by Jitomirskaya and Liu [27, 28], who developed Avila-Jitomirskaya’s scheme and found a better way to deal with the phase and frequency resonances.

One of the ideas of [27, 28] is that they treat the values of the generalized eigenfunction at resonant points as variables and obtain the localization via solving the equations of resonant points, not just using block expansion and the exponential decay of the Green functions. We should mention that the Green’s functions are not necessarily exponential decay in [27, 28] and also in the present paper.

We want to explain the motivations for Conjectures 1 and 2, and also explain the new challenge for completely resonant phases. For Diophantine frequency $\beta(\alpha) = 0$, the resonant points come from the phase resonance $\theta$. For Diophantine phase $\delta(\alpha, \theta) = 0$, the resonant points come from the frequency resonance $\delta$. Phase resonances lead to reflective repetitions of potential [30] and frequency resonances lead to repetitions of potential [17, 44]. Indeed, all known proofs of localization, for example [10–12, 16], are based, in one way or another, on avoiding resonances and removing resonance-producing parameters. For AMO and $|\lambda| > 1$, the Lyapunov exponent is $\ln |\lambda|$. Conjecture 2 says that the competition between the Anderson localization and the singular continuous spectrum is actually the competition between the Lyapunov exponent and the strength of the resonance. 2a says that without phase resonances, if the Lyapunov exponent beats the frequency resonance, then Anderson localization follows. Otherwise, $H_{\lambda, \alpha, \theta}$ has purely singular continuous spectrum. 2b says that without frequency resonances, if the Lyapunov exponent beats the phase resonance, then Anderson localization follows. Otherwise, $H_{\lambda, \alpha, \theta}$ has purely singular continuous spectrum.

For completely resonant phases $2\theta \in \alpha \mathbb{Z} + \mathbb{Z}$, $\delta(\alpha, \theta) = \beta(\alpha)$. Thus phase resonances and frequency resonances happen at the same time. Conjecture 1 says that if the Lyapunov exponent beats the frequency resonance plus the phase resonance, then the Anderson localization follows. This is the first challenge in our paper since we need to deal with frequency and phase resonances simultaneously. The second challenge is to avoid the complete resonance. In dealing with Conjecture 1, the original arguments of Jitomirskaya [23] do not work directly since there is the complete resonance. In [25], Jitomirskaya-Koslover-Schulteis found a trick to avoid the complete resonance by shrinking the size of the interval around 0 (we refer it as “shrinking scale” technic). Later, the shrinking scale technic was fully explored in [3, 20, 38, 39]. It is a natural idea to develop the shrinking scale technic and the localization arguments in [27, 28] to treat our situation. Since we shrink the scale, there is one phase resonance and one frequency phase resonance in one half scale. It is different from the situation in Conjecture 2, where there is one phase resonance or one frequency resonance in one scale. Using full strength of the localization proof of [27, 28] to treat both phase resonances and frequency resonances, one can only obtain the Anderson localization for $|\lambda| > e^{4\beta}$ in Conjecture 1, where 4 is the non-trivial technical limit in such approach. We bring several new ingredients that go beyond the technique of [3, 20, 25, 27, 28, 38, 39] and allow us to improve the constant to 3, thus going well beyond the previous technical limit. In particular, instead of using Lagrange interpolation uniformly, we treat Lagrange interpolation individually during the process of finding the points without “small divisors”. This gives us significantly more varieties to construct Green functions. We believe our method has a wider applicability to Anderson localization.

---

4Roughly speaking, if $||2\theta + k\alpha||$ is small, $k$ is called a phase resonance.

5Roughly speaking, if $||k\alpha||$ is small, $k$ is called a frequency resonance.
2. SOME NOTATIONS AND KNOWN FACTS

It is well known that in order to prove Anderson localization of $H_{\lambda,\alpha,\theta}$, we only need to show the following statements [8]: assume $\phi$ is a generalized function, i.e.,

$$H\phi = E\phi,$$

and $|\phi(k)| \leq 1 + |k|$, for some $E$.

then there exists some constant $c > 0$ such that

$$|\phi(k)| \leq Ce^{-|k|} \text{ for all } k.$$

It suffices to consider $\alpha$ with $0 < \beta(\alpha) < \infty$. Without loss of generality, we assume $\lambda > e^{3\beta}$, $\theta \in \{\frac{\pi}{2}, \frac{3\pi}{2} + \frac{1}{2}, 0, \frac{1}{2}\}$ (shift is a unitary operator). In order to avoid too many notations, we still use $2\theta \in \alpha\mathbb{Z} + \mathbb{Z}$ to represent $\theta \in \{\frac{\pi}{2}, \frac{3\pi}{2} + \frac{1}{2}, 0, \frac{1}{2}\}$. We also assume $E \in \Sigma_{\lambda,\alpha}$ (denote by $\Sigma_{\lambda,\alpha}$ the spectrum of operator $H_{\lambda,\alpha,\theta}$ since the spectrum does not depend on $\theta$). For simplicity, we usually omit the dependence on parameters $E, \lambda, \alpha, \theta$.

Given a generalized eigenfunction $\phi$ of $H_{\lambda,\alpha,\theta}$, without loss of generality assume $\phi(0) = 1$. Our objective is to show that there exists some specific $c > 0$ such that

$$|\phi(k)| \leq e^{-c|k|} \text{ for } k \to \infty.$$

Let us denote

$$P_k(\theta) = \det(R_{[0,k-1]}(H_{\lambda,\alpha,\theta} - E)R_{[0,k-1]}).$$

It is easy to see that $P_k(\theta)$ is an even function of $\theta + \frac{1}{2}(k - 1)\alpha$ and can be written as a polynomial of degree $k$ in $\cos 2\pi(\theta + \frac{1}{2}(k - 1)\alpha)$:

$$P_k(\theta) = \sum_{j=0}^{k} c_j \cos^{2j} 2\pi(\theta + \frac{1}{2}(k - 1)\alpha) \triangleq Q_k(\cos 2\pi(\theta + \frac{1}{2}(k - 1)\alpha)).$$

Lemma 2.1. (p. 16, [2]) The following inequality holds

$$\lim_{k \to \infty} \sup_{\theta \in \mathbb{R}} \frac{1}{k} \ln |P_k(\theta)| \leq \ln \lambda.$$

By Cramér’s rule (see p.15, [3] for example) for given $x_1$ and $x_2 = x_1 + k - 1$, with $y \in I = [x_1, x_2] \subset \mathbb{Z}$, one has

$$(2) \quad |G_I(x_1, y)| = \left|\frac{P_{x_2-y}(\theta + (y+1)\alpha)}{P_k(\theta + x_1\alpha)}\right|,$$

$$(3) \quad |G_I(y, x_2)| = \left|\frac{P_{y-x_1}(\theta + x_1\alpha)}{P_k(\theta + x_1\alpha)}\right|.$$  

By Lemma 2.1 the numerators in (2) and (3) can be bounded uniformly with respect to $\theta$. Namely, for any $\varepsilon > 0$,

$$(4) \quad |P_n(\theta)| \leq e^{(\ln \lambda + \varepsilon)n}$$

for large enough $n$.

Definition 2.2. Fix $t > 0$. A point $y \in \mathbb{Z}$ will be called $(t, k)$ regular if there exists an interval $[x_1, x_2]$ containing $y$, where $x_2 = x_1 + k - 1$, such that

$$|G_{[x_1, x_2]}(y, x_1)| \leq e^{-t|y-x_i|} \text{ and } |y - x_i| \geq \frac{1}{t}k \text{ for } i = 1, 2.$$
It is easy to check that (p. 61 [9])

\[ \phi(x) = -G_{[x_1,x_2]}(x_1,x)\phi(x_1 - 1) - G_{[x_1,x_2]}(x,x_2)\phi(x_2 + 1), \]

where \( x \in I = [x_1, x_2] \subset \mathbb{Z} \).

Given a set \( \{\theta_1, \cdots, \theta_{k+1}\} \), the lagrange Interpolation terms \( La_i, i = 1, 2, \cdots, k + 1 \), are defined by

\[ La_i = \ln \max_{x \in [-1,1]} \prod_{j=1,j \neq i}^{k+1} \frac{|x - \cos 2\pi \theta_j|}{\cos 2\pi \theta_i - \cos 2\pi \theta_j}. \]

The following lemma is another form of Lemma 9.3 in [2].

**Lemma 2.3.** Given a set \( \{\theta_1, \cdots, \theta_{k+1}\} \), there exists some \( \theta_i \) in set \( \{\theta_1, \cdots, \theta_{k+1}\} \) such that

\[ P_k(\theta_i - \frac{k-1}{2} \alpha) \geq \frac{e^{k \ln \lambda - La_i}}{k+1}. \]

**Proof.** Otherwise, for all \( i = 1, 2, \cdots, k + 1 \),

\[ Q_k(\cos 2\pi \theta_i) = P_k(\theta_i - \frac{k-1}{2} \alpha) < \frac{e^{k \ln \lambda - La_i}}{k+1}. \]

By (11), we can write the polynomial \( Q_k(x) \) in the Lagrange interpolation form at points \( \cos 2\pi \theta_i, i = 1, 2, \cdots, k + 1 \). Thus

\[ |Q_k(x)| = \left| \sum_{i=1}^{k+1} Q_k(\cos 2\pi \theta_i) \prod_{j \neq i} (x - \cos 2\pi \theta_j) \right| \]

\[ < (k+1) \frac{e^{k \ln \lambda - La_i}}{k+1} \]

for all \( x \in [-1,1] \). By (11) again, \( |P_k(x)| < e^{k \ln \lambda} \) for all \( x \in \mathbb{R} \). However, by Herman’s subharmonic function methods (see p.16 [9]), \( \int_{\mathbb{R}/\mathbb{Z}} |P_k(x)| dx \geq k \ln \lambda \). This is impossible. \( \square \)

Fix a sufficiently small constant \( \eta \), which will be determined later. Let \( b_n = n\eta \). For any \( y \neq 0 \), we will distinguish between two cases:

(i) dist\((y, q_n\mathbb{Z} + \frac{3\eta}{4}\mathbb{Z}) \leq b_n \), called \( n \)-resonance.

(ii) dist\((y, q_n\mathbb{Z} + \frac{3\eta}{4}\mathbb{Z}) \geq b_n \), called \( n \)-nonresonance.

**Theorem 2.4.** ([38]) Assume \( \theta_0 \in \alpha \mathbb{Z} + \mathbb{Z} \) and \( \lambda > 1 \).

Suppose either

i) \( b_n \leq |y| < Cb_{n+1} \) for some \( C > 1 \) and \( y \) is \( n \)-nonresonant

or

ii) \( |y| \leq Cq_n \) and \( \text{dist}(y, q_n\mathbb{Z} + \frac{3\eta}{4}\mathbb{Z}) > b_n \).

Let \( n_0 \) be the least positive integer such that \( 4q_{n_0} \leq \text{dist}(y, q_n\mathbb{Z} + \frac{3\eta}{4}\mathbb{Z}) - 2 \). Let \( s \in \mathbb{N} \) be the largest number such that \( 4sq_{n_0} \leq \text{dist}(y, q_n\mathbb{Z} + \frac{3\eta}{4}\mathbb{Z}) - 2 \). Then for any \( \varepsilon > 0 \) and sufficiently large \( n \), \( y \) is \( (\ln \lambda - \varepsilon, 6sq_{n_0} - 1) \) regular.

The proof of Theorem 2.4 builds on the ideas used in the proof of Lemma B.4 in [27], which is original from [2]. However, it requires some modifications to avoid the completely resonant phases. Thus we give the proof in the Appendix.

The following lemma can be proved directly by block expansion and Theorem 2.3, which is similar to the proof of Lemma 4.1 in [27]. We also give the proof in the Appendix.
Lemma 2.5. Suppose \( k \in [jq_n, (j + \frac{1}{2})q_n] \) or \( k \in [(j + \frac{1}{2})q_n, (j + 1)q_n] \) with \( 0 \leq |j| \leq C \frac{ln \frac{n}{\eta}}{q_n} + C \), and \( \text{dist}(k, q_nZ + \frac{2q_n}{n}Z) \geq 10\eta q_n \). Let \( d_t = |k - tq_n| \) for \( t \in \{ j, j + \frac{1}{2}, j + 1 \} \). Then for sufficiently large \( n \),

\[
|\phi(k)| \leq \max\{r_j \exp\{-(\ln \lambda - \eta)(d_j - 3\eta q_n)\}, r_{j+\frac{1}{2}} \exp\{-(\ln \lambda - \eta)(d_{j+\frac{1}{2}} - 3\eta q_n)\}\},
\]
or

\[
|\phi(k)| \leq \max\{r_{j+\frac{1}{2}} \exp\{-(\ln \lambda - \eta)(d_{j+\frac{1}{2}} - 3\eta q_n)\}, r_{j+1} \exp\{-(\ln \lambda - \eta)(d_{j+1} - 3\eta q_n)\}\}.
\]

3. Proof of Theorem 1.1

We always assume \( n \) is large enough and \( C \) is a large constant below. Denote by \( \lfloor x \rfloor \) the largest integer less or equal than \( x \).

Let

\[
r_j = \sup_{|r| \leq 10\eta} |\phi(jq_n + rq_n)|,
\]

and

\[
r_{j+\frac{1}{2}} = \sup_{|r| \leq 10\eta} |\phi(jq_n + \lfloor \frac{q_n}{2} \rfloor + rq_n)|.
\]

We prove a crucial theorem first.

Theorem 3.1. Let \( |\ell| \leq \frac{2n+1}{q_n} + 3 \). Then except \( r_0 \), we have

\[
r_{\ell} \leq \exp\{-(\ln \lambda - 3\beta - C\eta)|\ell|q_n\},
\]

and

\[
r_{\ell-\frac{1}{2}} \leq \exp\{-(\ln \lambda - 3\beta - C\eta)|\ell - \frac{1}{2}|q_n\}.
\]

Lemma 3.2. For any \( |j| \leq \frac{2n+1}{q_n} + 16 \), the following holds,

\[
r_{j+\frac{1}{2}} \leq \exp\{-\frac{1}{2}(\ln \lambda - 2\beta - C\eta)q_n\} \max\{r_j, r_{j+1}\}.
\]

Proof. Take \( \phi(jq_n + \lfloor \frac{q_n}{2} \rfloor + rq_n) \) with \( |r| \leq 10\eta \) into consideration. Without loss of generality assume \( j \geq 0 \). Let \( n_0 \) be the least positive integer such that

\[
\frac{1}{\eta}q_{n-n_0} \leq (\frac{1}{6} - 2\eta)q_n.
\]

Let \( s \) be the largest positive integer such that \( sq_{n-n_0} \leq (\frac{1}{6} - 2\eta)q_n \). Then

\[
s \geq \frac{1}{\eta}.
\]

By the fact \( (s+1)q_{n-n_0} \geq (\frac{1}{6} - 2\eta)q_n \), one has

\[
(\frac{1}{6} - 3\eta)q_n \leq sq_{n-n_0} \leq (\frac{1}{6} - 2\eta)q_n.
\]

Set \( I_1, I_2 \subset Z \) as follows

\[
I_1 = [-2sq_{n-n_0} - 1],
\]

\[
I_2 = [jq_n + \lfloor \frac{q_n}{2} \rfloor - s + \lfloor \eta s \rfloor]q_{n-n_0}, jq_n + \lfloor \frac{q_n}{2} \rfloor + (s + \lfloor \eta s \rfloor)q_{n-n_0} - 1],
\]

and let \( \theta_m = \theta + m\alpha \) for \( m \in I_1 \cup I_2 \). The set \( \{\theta_m\}_{m \in I_1 \cup I_2} \) consists of \( (4s + 2\lfloor \eta s \rfloor)q_{n-n_0} \) elements. Let \( k = (4s + 2\lfloor \eta s \rfloor)q_{n-n_0} - 1 \).
By modifying the proof of [2, Lemma 9.9] or [38, Lemma 4.1], we can prove the claim (Claim 1): for any \( \varepsilon > 0 \), \( m \in I_1 \), one has \( La_m \leq \varepsilon a_n \); and for any \( m \in I_2 \), one has \( La_m \leq q_n(\beta + \varepsilon) \). We also give the proof in the Appendix.

By Lemma [2, 3] there exists some \( j_0 \in I_1 \) such that \( P_k(\theta_{j_0} - \frac{k-1}{2} \alpha) \geq e^{k \ln \lambda - \varepsilon q_n} \), or some \( j_0 \in I_2 \) such that \( P_k(\theta_{j_0} - \frac{k-1}{2} \alpha) \geq e^{k \ln \lambda - (\beta + \varepsilon) q_n} \).

Suppose \( j_0 \in I_1 \), i.e., \( P_k(\theta_{j_0} - \frac{k-1}{2} \alpha) \geq e^{k \ln \lambda - \varepsilon q_n} \). Let \( I = [j_0 - 2s q_{n-n_0} - |sn| q_{n-n_0} + 1, j_0 + 2s q_{n-n_0} + |sn| q_{n-n_0} - 1] = [x_1, x_2] \). Denote by \( x_1' = x_1 - 1 \) and \( x_2' = x_2 + 1 \).

By (2), (3) and (4), it is easy to verify
\[
|G_I(0, x_i)| \leq e^{(\ln \lambda + \varepsilon)(k-2-|x_i|)-k \ln \lambda + \varepsilon q_n} \leq e^{-|x_i| \lambda + C \varepsilon q_n}.
\]

Using (5) and noticing that \( |x_i| \geq \frac{na}{\beta} q_{n-n_0} \), we obtain
\[
|\phi(0)| \leq \sum_{i=1,2} e^{-\frac{a}{\beta} q_{n-n_0} \ln \lambda + C \varepsilon q_n} |\phi(x_i')| < 1,
\]
where the second inequality holds by (11). This is contradictory to the fact \( \phi(0) = 1 \).

Thus there exists \( j_0 \in I_2 \) such that \( P_k(\theta_{j_0} - \frac{k-1}{2} \alpha) \geq e^{k \ln \lambda - (\beta + \varepsilon) q_n} \). Let \( I = [j_0 - 2s q_{n-n_0} - |sn| q_{n-n_0} + 1, j_0 + 2s q_{n-n_0} + |sn| q_{n-n_0} - 1] = [x_1, x_2] \). By (2), (3) and (4) again, we have
\[
|G_I(p, x_i)| \leq e^{(\ln \lambda + \varepsilon)(k-2-|p-x_i|)-k \ln \lambda + \beta q_n + \varepsilon q_n},
\]
where \( p = jq_n + \frac{na}{\beta} \) + \( r q_n \). Using (5), we obtain
\[
|\phi(p)| \leq \sum_{i=1,2} e^{(\beta + C \eta) q_n} |\phi(x_i')| e^{-|p-x_i| \ln \lambda}.
\]

Let \( d_{i,1,i_2} = |x_i - i q_n - i_2 \frac{na}{\beta}| \), where \( i = 1, 2, i_1 \in \mathbb{Z} \) and \( i_2 = 0, 1 \). If \( d_{i,1,i_2} \geq 10 \eta q_n \), then we replace \( \phi(x_i) \) in (14) with (7) (or (5)). If \( d_{i,1,i_2} \leq 10 \eta q_n \), then we replace \( \phi(x_i') \) in (14) with \( r_{i,1} + \frac{\eta}{\beta} \). Then we have
\[
\begin{align*}
\frac{1}{2} < \max\{ \exp\left(-\frac{1}{2}(\ln \lambda - 2\beta - C \eta) q_n \right) & r_j, \ \exp\left(-\frac{1}{2}(\ln \lambda - 2\beta - C \eta) q_n \right) r_{j+1}, \\
& \exp\left(-2s q_{n-n_0} \ln \lambda + \beta q_n + C \eta q_n \right) r_j, \ \exp\left(-2s q_{n-n_0} \ln \lambda + \beta q_n + C \eta q_n \right) r_{j+1}\}. \\
\end{align*}
\]

By (11), one has
\[
-2s q_{n-n_0} \ln \lambda + \beta q_n + C \eta q_n < \left(-\frac{\ln \lambda}{3} + \beta + C \eta \right) q_n < 0,
\]
for small \( \eta \). This implies
\[
\frac{1}{2} < \exp\left(-2s q_{n-n_0} \ln \lambda + \beta q_n + C \eta q_n \right) r_{j+\frac{1}{2}}
\]
cannot happen.

Thus (15) becomes
\[
\begin{align*}
\frac{1}{2} < \max\{ \exp\left(-\frac{1}{2}(\ln \lambda - 2\beta - C \eta) q_n \right) & r_j, \ \exp\left(-\frac{1}{2}(\ln \lambda - 2\beta - C \eta) q_n \right) r_{j+1}\}. \\
\end{align*}
\]

\textbf{Lemma 3.3.} For \( 1 \leq |j| \leq \frac{4 \ln \alpha + 12}{q_n} \), the following holds
\[
r_j \leq \max\{ \max\{ \exp\left(-|t| \ln \lambda - \beta - C \eta \right) q_n \} r_{j+t}, \ \exp\left(-\ln \lambda - 3\beta - C \eta \right) q_n \} r_{j+\frac{1}{2}}\},
\]
where \( O = \{ \pm \frac{1}{2}, \pm \frac{1}{2} \} \).
Proof. It suffices to estimate $\phi(jq_n + rq_n)$ with $|j| \geq 1$ and $|r| \leq 10\eta$. Without loss of generality assume $j \geq 1$. Let $n_0$ be the least positive integer such that

$$\frac{1}{\eta}q_{n-n_0} \leq \frac{q_n}{6} - 2.$$

Let $s$ be the largest positive integer such that $sq_{n-n_0} \leq \frac{q_n}{6} - 2$. Then $s \geq \frac{1}{\eta}$.

Set $J_1, J_2, J_3 \subset \mathbb{Z}$ as follows

$$J_1 = [-2sq_{n-n_0}, -1],$$
$$J_2 = [jq_n - 3sq_{n-n_0}, jq_n - 2sq_{n-n_0} - 1] \cup [jq_n + 2sq_{n-n_0}, jq_n + 3sq_{n-n_0} - 1],$$
$$J_3 = [jq_n - 2sq_{n-n_0}, jq_n + 2sq_{n-n_0} - 1],$$
and let $\theta_m = \theta + m\alpha$ for $m \in J_1 \cup J_2 \cup J_3$. The set $\{\theta_m\}_{m \in J_1 \cup J_2 \cup J_3}$ consists of $8sq_{n-n_0}$ elements. By modifying the proof of [2, Lemma 9.9] or [38, Lemma 4.1] again, we can prove the claim (Claim 2) that for any $m \in J_1 \cup J_3$ and any $\varepsilon > 0$, $La_m \leq 2(\beta + \varepsilon)q_n$, and for any $m \in J_2$, $La_m \leq (\beta + \varepsilon)q_n$. We also give the details of proof in the Appendix.

Applying Lemma 2.3 there exists some $j_0$ with $j_0 \in J_1 \cup J_3$ such that

$$P_{8sq_{n-n_0}-1}(\theta_{j_0} - (4sq_{n-n_0} - 1)\alpha) \geq e^{8sq_{n-n_0}ln \lambda - 2\beta q_n - \xi q_n},$$

or there exists some $j_0$ with $j_0 \in J_2$ such that

$$P_{8sq_{n-n_0}-1}(\theta_{j_0} - (4sq_{n-n_0} - 1)\alpha) \geq e^{8sq_{n-n_0}ln \lambda - \beta q_n - \xi q_n}.$$

If $j_0 \in J_2$, let $I = [j_0 - 4sq_{n-n_0} + 1, j_0 + 4sq_{n-n_0} - 1] = [x_1, x_2]$, then

$$|G_I(jq_n + rq_n, x_i)| \leq e^{(\ln \lambda + \eta)(8sq_{n-n_0} - 2(jq_n + rq_n - x_i)) - 8sq_{n-n_0}ln \lambda + \beta q_n + C\eta q_n}.$$

Using (50), we obtain

$$\phi(jq_n + rq_n) \leq \sum_{i=1,2} e^{(\beta + C\eta)q_n} |\phi(x'_i)| e^{-|jq_n + rq_n - x_i| ln \lambda}.$$

Recall that

$$d_{i_1, i_2} = |x_i - i_1 q_n - i_2 \frac{q_n}{6}|,$$

where $i = 1, 2, i_1 \in \mathbb{Z}$ and $i_2 = 0, 1$. If $d_{i_1, i_2} \geq 10\eta q_n$, then we replace $\phi(x'_i)$ in (19) with (12) or (51). If $d_{i_1, i_2} \leq 10\eta q_n$, then we replace $\phi(x'_i)$ in (19) with $r_{i_1, i_2}.\frac{q_n}{6}.$

Then by (19), we have

$$r_j \leq \exp(\beta q_n + C\eta q_n) \max_{t \in O} \{\exp(-|t| q_n ln \lambda) r_{j+t}, \exp(-2sq_{n-n_0} ln \lambda) r_j\},$$

where $O = \pm \frac{q_n}{6}, \pm 1, \pm \frac{q_n}{6}$.

Noting $sq_{n-n_0} \geq (1 - \eta)\frac{q_n}{6}$ (using $(s + 1)q_{n-n_0} > \frac{q_n}{6} - 2$ and $s \geq \frac{1}{\eta}$), then

$$r_j \leq \exp(\beta q_n + C\eta q_n) \exp(-2sq_{n-n_0} ln \lambda) r_j.$$

can not happen since $ln \lambda > 3\beta$.

Thus

$$r_j \leq \max_{t \in O} \{\exp(\beta q_n + C\eta q_n - |t| q_n ln \lambda) r_{j+t}\},$$

where $O = \pm \frac{q_n}{6}, \pm 1, \pm \frac{q_n}{6}$. This implies (17).

If $j_0 \in J_3$, by the same arguments, we have

$$r_j \leq \max_{t \in \{\pm 1, \pm \frac{q_n}{6}\}} \{\exp(2\beta q_n + C\eta q_n - |t| q_n ln \lambda) r_{j+t}\}.$$

Using the estimate of $r_{j \pm \frac{q_n}{6}}$ in Lemma 3.2, we have

$$r_j \leq \exp(-ln \lambda - 3\beta - C\eta) q_n \max\{r_{j \pm 1}, r_j\}.$$
By the same reason, 
\[ r_j \leq \exp\{-\ln \lambda - 3\beta - C\eta\} r_j \]
can not happen. Thus 
\[ r_j \leq \exp\{-\ln \lambda - 3\beta - C\eta\} r_{j+1}. \tag{20} \]
This also implies \((17)\).
If \(j_0 \in J_1\), then \((20)\) holds for \(j = 0\), which will lead to \(|\phi(0)| < 1\). This is impossible. \(\square\)

**Proof of Theorem 3.1**

**Proof.** By Lemmas 3.2 and 3.3 for \(1 \leq j \leq \frac{b_{n+1}}{q_n} + 4\), we have
\[ r_{j-\frac{1}{2}} \leq \exp\{-\frac{1}{2}\ln \lambda - 3\beta - C\eta\} \max\{r_{j-1}, r_j\}, \tag{21} \]
and
\[ r_j \leq \max_{t \in O}\{\exp\{-|t|\ln \lambda - 3\beta - C\eta\} r_{j+t}\}, \tag{22} \]
where \(O = \{\pm \frac{3}{2}, \pm 1, \pm \frac{1}{2}\}\). For \(-\frac{b_{n+1}}{q_n} - 3 \leq j \leq -1\), we have
\[ r_{j+\frac{1}{2}} \leq \exp\{-\frac{1}{2}\ln \lambda - 3\beta - C\eta\} \max\{r_{j+1}, r_j\}, \tag{23} \]
and
\[ r_j \leq \max_{t \in O}\{\exp\{-|t|\ln \lambda - 3\beta - C\eta\} r_{j+t}\}. \tag{24} \]
Suppose \(\ell > 0\). Let \(j = \ell\) in \((22)\) and \((21)\), and iterate \(2\ell\) times or until \(j \leq 1\), we obtain
\[ r_\ell \leq (2\ell + 2)q_n \exp\{-\ln \lambda - 3\beta - C\eta\ell q_n\}, \tag{25} \]
and
\[ r_{\ell-\frac{1}{2}} \leq (2\ell + 2)q_n \exp\{-\ln \lambda - 3\beta - C\eta\ell q_n\}. \tag{26} \]
Notice that we have used the fact that \(|r_j| \leq (|j| + 2)q_n\) and \(|r_{j-\frac{1}{2}}| \leq (|j - \frac{1}{2}| + 2)q_n\).
Suppose \(\ell < 0\). Let \(j = \ell\) in \((24)\) and \((23)\), and iterate \(2|\ell|\) times or until \(j \geq -1\), we obtain
\[ r_\ell \leq (2\ell + 2)q_n \exp\{-\ln \lambda - 3\beta - C\eta\ell q_n\}, \tag{27} \]
and
\[ r_{\ell+\frac{1}{2}} \leq (2\ell + 2)q_n \exp\{-\ln \lambda - 3\beta - C\eta\ell q_n\}. \tag{28} \]
Now Theorem 3.1 follows from \((25)\), \((26)\), \((27)\) and \((28)\). \(\square\)

**Proof of Theorem 1.1**

**Proof.** Without loss of generality, we assume \(k > 0\). Let \(\eta > 0\) be much smaller than \(\ln \lambda - 3\beta\).
For any \(k\), let \(n\) be such that \(b_n < k < b_{n+1}\).
Case 1: \(\text{dist}(k,q_nZ + \frac{4\eta}{n}Z) \leq 10\eta q_n\).
In this case, applying Theorem 3.1 one has
\[ |\phi(k)|, |\phi(k-1)| \leq \exp\{-\ln \lambda - 3\beta - C\eta\}|k|\}. \tag{29} \]
Case 2: \(\text{dist}(k,q_nZ + \frac{4\eta}{n}Z) \geq 10\eta q_n\).
Let \(0 \leq j \leq \frac{b_{n+1}}{q_n}\) such that \(k \in [j, q_n, (j + \frac{1}{2})q_n]\) or \(k \in [(j + \frac{1}{2})q_n, (j + 1)q_n]\).
By Lemma 2.3 and Theorem 3.1 one also has
\[ |\phi(k)|, |\phi(k-1)| \leq \exp\{-\ln \lambda - 3\beta - C\eta\}|k|\}. \tag{30} \]
By (29), (30) and letting \( \eta \to 0 \), we have
\[
\limsup_{k \to \infty} \frac{\ln(\phi^2(k) + \phi^2(k - 1))}{2|k|} \leq -(\ln \lambda - 3\beta).
\]
We finish the proof.

\[\square\]

\textbf{Appendix A. Proof of Theorem 2.4 Claims 1 and 2}

Let \( \frac{b_n}{q_n} \) be the continued fraction approximations to \( \alpha \), then
\[
\forall 1 \leq k < q_{n+1}, \text{dist}(k\alpha, \mathbb{Z}) \geq |q_n\alpha - p_n|,
\]
and
\[
\frac{1}{2q_{n+1}} \leq |q_n\alpha - p_n| \leq \frac{1}{q_{n+1}}.
\]

\textbf{Lemma A.1. (Lemma 9.7, [2])} Let \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \), \( x \in \mathbb{R} \) and \( 0 \leq \ell_0 \leq q_n - 1 \) be such that \( |\sin \pi(x + \ell_0\alpha)| = \inf_{0 \leq \ell \leq q_n - 1} |\sin \pi(x + \ell\alpha)| \), then for some absolute constant \( C > 0 \),
\[
-C \ln q_n \leq \sum_{\ell=0}^{q_n-1} \ln |\sin \pi(x + \ell\alpha)| + (q_n - 1) \ln 2 \leq C \ln q_n.
\]

\textbf{Proof of Theorem 2.4}

\[\text{Proof.}\] We only give the proof of case 1: \( b_n \leq |y| \leq Cb_{n+1} \) is non-resonant.

By the definition of \( s \) and \( n_0 \), we have \( 4sq_{n_0} \leq \text{dist}(y, q_n\mathbb{Z}) - 2 \) and \( 4q_{n_0+1} > \text{dist}(y, q_n\mathbb{Z}) - 2 \). This leads to \( sq_{n_0} \leq q_{n_0+1} \). Set \( I_1, I_2 \subset \mathbb{Z} \) as follows
\[
I_1 = [-2sq_{n_0} - 1, 1], \\
I_2 = [y - 2sq_{n_0}, y + 2sq_{n_0} - 1],
\]
and let \( \theta_j = \theta + j\alpha \) for \( j \in I_1 \cup I_2 \). The set \( \{\theta_j\}_{j \in I_1 \cup I_2} \) consists of \( 6sq_{n_0} \) elements.

Let \( k = 6sq_{n_0} - 1 \). We estimate \( L_{a_j} \) first. For this reason, let \( x = \cos 2\pi a \), and take the logarithm in (6), one has
\[
\ln \prod_{j \in I_1 \cup I_2, j \neq i} \frac{|\cos 2\pi a - \cos 2\pi \theta_j|}{|\cos 2\pi \theta_i - \cos 2\pi \theta_j|} = \sum_{j \in I_1 \cup I_2, j \neq i} \ln |\cos 2\pi a - \cos 2\pi \theta_j| - \sum_{j \in I_1 \cup I_2, j \neq i} \ln |\cos 2\pi \theta_i - \cos 2\pi \theta_j|.
\]
We start to estimate \( \sum_{j \in I_1 \cup I_2, j \neq i} \ln |\cos 2\pi a - \cos 2\pi \theta_j| \). Obviously,
\[
\sum_{j \in I_1 \cup I_2, j \neq i} \ln |\cos 2\pi a - \cos 2\pi \theta_j| = \sum_{j \in I_1 \cup I_2, j \neq i} \ln |\sin \pi(a + \theta_j)| + \sum_{j \in I_1 \cup I_2, j \neq i} \ln |\sin \pi(a - \theta_j)| + (6sq_{n_0} - 1) \ln 2
\]
\[= \Sigma_+ + \Sigma_- + (6sq_{n_0} - 1) \ln 2.
\]
Both \( \Sigma_+ \) and \( \Sigma_- \) consist of \( 6s \) terms of the form of (58), plus \( 6s \) terms of the form
\[
\ln \min_{j=0, \cdots, sq_{n_0} - 1} |\sin \pi(x + j\alpha)|,
\]
and $\ln |\sin \pi (\alpha \pm \theta_i)|$. Thus, using $|s|$ 6 times of $\Sigma_+ \text{ and } \Sigma_-$ respectively, one has

$$
\sum_{j \in i_1 \cup i_2 \atop j \neq i} \ln |\cos 2\pi a - \cos 2\pi \theta_j| \leq -6s q_{n-n_0} \ln 2 + C s \ln q_{n-n_0}.
$$

Let $a = \theta_i$, we obtain

$$
\sum_{j \in i_1 \cup i_2 \atop j \neq i} \ln |\cos 2\pi \theta_i - \cos 2\pi \theta_j| = \sum_{j \in i_1 \cup i_2 \atop j \neq i} \ln |\sin \pi (\theta_i + \theta_j)| + \sum_{j \in i_1 \cup i_2 \atop j \neq i} \ln |\sin \pi (\theta_i - \theta_j)| + (6s q_{n-n_0} - 1) \ln 2
$$

$$
= \Sigma_+ + \Sigma_- + (6s q_{n-n_0} - 1) \ln 2,
$$

where

$$
\Sigma_+ = \sum_{j \in i_1 \cup i_2 \atop j \neq i} \ln |\sin \pi (2\theta + (i+j)\alpha)|,
$$

and

$$
\Sigma_- = \sum_{j \in i_1 \cup i_2 \atop j \neq i} \ln |\sin \pi (i - j)\alpha|.
$$

We will estimate $\Sigma_+$. Set $J_1 = [-2s, -1]$ and $J_2 = [0, 4s - 1]$, which are two adjacent disjoint intervals of length $2s$ and $4s$ respectively. Then $I_1 \cup I_2$ can be represented as a disjoint union of segments $B_j$, $j \in J_1 \cup J_2$, each of length $q_{n-n_0}$.

Applying (33) to each $B_j$, we obtain

$$
\Sigma_+ \geq -6s q_{n-n_0} \ln 2 + \sum_{j \in J_1 \cup J_2} \ln |\sin \pi \theta_j| - C s \ln q_{n-n_0} - \ln |\sin 2\pi (\theta + i\alpha)|,
$$

where

$$
|\sin \pi \theta_j| = \min_{\ell \in B_j} |\sin \pi (2\theta + (\ell + i)\alpha)|.
$$

By the construction of $I_1$ and $I_2$, one has

$$
2\theta + (\ell + i)\alpha = \pm (m q_{n} \alpha + r_1 \alpha) \mod \mathbb{Z}
$$
or

$$
2\theta + (\ell + i)\alpha = \pm r_2 \alpha \mod \mathbb{Z},
$$

where $0 \leq m \leq C \frac{\ln q_{n}}{q_{n}}$ and $1 \leq r_i < q_{n}$, $i = 1, 2$.

By (31) and (32), it follows

$$
\min_{\ell \in J_1 \cup J_2} \ln |\sin \pi (2\theta + (\ell + i)\alpha)| \geq C \ln(||r_i \alpha||_{\mathbb{R}/\mathbb{Z}} - \Delta_{n-1}/2)
$$

$$
\geq C \ln (\Delta_{n-1} - \Delta_{n-1}/2)
$$

$$
\geq \ln C \Delta_{n-1}/2 \geq -C \ln q_{n},
$$

since $||m q_{n} \alpha||_{\mathbb{R}/\mathbb{Z}} \leq C \frac{\ln q_{n}}{q_{n}} \Delta_{n} \leq \Delta_{n}/2$.

By the construction of $I_1$ and $I_2$, we also have

$$
\min_{j \neq i, j \in i_1 \cup i_2} \ln |\sin \pi (j - i)\alpha| \geq -C \ln q_{n}.
$$
Next we estimate \( \sum_{j \in J_1} \ln |\sin \pi \hat{\theta}_j| \). Assume that \( \hat{\theta}_{j+1} = \hat{\theta}_j + q_{n-n_0} \alpha \) for every \( j, j+1 \in J_1 \). In this case, for any \( i, j \in J_1 \) and \( i \neq j \), we have
\[
|\hat{\theta}_i - \hat{\theta}_j|_{\mathbb{R}/\mathbb{Z}} \geq |\|q_{n-n_0} \alpha\|_{\mathbb{R}/\mathbb{Z}}|.
\]

By the Stirling formula, (40) and (42), one has
\[
\sum_{j \in J_1} \ln |\sin 2\pi \hat{\theta}_j| > \frac{2}{q_{n-n_0+1}} \left( \sum_{j=1}^{s} \ln (j \Delta_n) - C \ln q_n \right) > 2s \ln \frac{s}{q_{n-n_0+1}} - C \ln q_n - C s.
\]

In the other cases, decompose \( J_1 \) into maximal intervals \( T_\kappa \) such that for \( j, j+1 \in T_\kappa \), we have \( \hat{\theta}_{j+1} = \hat{\theta}_j + q_{n-n_0} \alpha \). Notice that the boundary points of an interval \( T_\kappa \) are either boundary points of \( J_1 \) or satisfy \( \| \hat{\theta}_j \|_{\mathbb{R}/\mathbb{Z}} + \Delta_n \geq \frac{\Delta_n \cdot q_{n-n_0}}{2} \). This follows from the fact that if \( 0 < |z| < q_{n-n_0} \), then \( \| \hat{\theta}_j + q_{n-n_0} \alpha \|_{\mathbb{R}/\mathbb{Z}} \leq \| \hat{\theta}_j \|_{\mathbb{R}/\mathbb{Z}} + \Delta_n \), and \( \| \hat{\theta}_j + (z + q_{n-n_0}) \alpha \|_{\mathbb{R}/\mathbb{Z}} \geq \| z \alpha \|_{\mathbb{R}/\mathbb{Z}} - \| \hat{\theta}_j \|_{\mathbb{R}/\mathbb{Z}} \geq \Delta_n \). Assuming \( T_\kappa \neq J_1 \), then there exists \( j \in T_\kappa \) such that \( \| \hat{\theta}_j \|_{\mathbb{R}/\mathbb{Z}} \geq \frac{\Delta_n - q_{n-n_0}}{2} \).

If \( T_\kappa \) contains some \( j \) with \( \| \hat{\theta}_j \|_{\mathbb{R}/\mathbb{Z}} < \frac{\Delta_n - q_{n-n_0}}{10} \), then
\[
|T_\kappa| \geq \frac{\Delta_n - q_{n-n_0}}{2} - \Delta_n \geq \frac{\Delta_n - q_{n-n_0}}{10}.
\]

Similarly,
\[
\sum_{j \in T_\kappa} \ln |\sin \pi \hat{\theta}_j| \geq |T_\kappa| \ln \frac{|T_\kappa|}{q_{n-n_0+1}} - C |T_\kappa|.
\]

If \( T_\kappa \) does not contain any \( j \) with \( \| \hat{\theta}_j \|_{\mathbb{R}/\mathbb{Z}} \leq \frac{\Delta_n - q_{n-n_0}}{10} \), then by (42)
\[
\sum_{j \in T_\kappa} \ln |\sin \pi \hat{\theta}_j| \geq -|T_\kappa| \ln q_{n-n_0} - C |T_\kappa|.
\]

By (45) and (46), one has
\[
\sum_{j \in J_1} \ln |\sin \pi \hat{\theta}_j| \geq 2s \ln \frac{s}{q_{n-n_0+1}} - C s - C \ln q_n.
\]

Similarly,
\[
\sum_{j \in J_2} \ln |\sin \pi \hat{\theta}_j| \geq 4s \ln \frac{s}{q_{n-n_0+1}} - C s - C \ln q_n.
\]

Putting (36), (47) and (48) together, we have
\[
\Sigma_+ > -6s q_{n-n_0} \ln 2 + 6s \ln \frac{s}{q_{n-n_0+1}} - C s \ln q_{n-n_0} - C \ln q_n.
\]

Now we start to estimate \( \Sigma_- \).
Replacing (10) with (11), and following the proof of (49), we obtain,
\begin{equation}
\Sigma_\eta > -6q_{n-n_0} \ln 2 + 6s \ln \frac{s}{q_{n-n_0}+1} - C \ln q_n.
\end{equation}

By (55), (19) and (50), we obtain
\[\sum_{j \in I_1 \cup I_2, j \neq 1} \ln |\cos 2\pi \theta_i - \cos 2\pi \theta_j|\]
\begin{equation}
\geq -6q_{n-n_0} \ln 2 + 6s \ln \frac{s}{q_{n-n_0}+1} - C \ln q_n.
\end{equation}

By (52) and (51), we have for any \(i \in I_1 \cup I_2,\)
\begin{equation}
\prod_{j \in I_1 \cup I_2, j \neq 1} \frac{|x - \cos 2\pi \theta_j|}{|\cos 2\pi \theta_i - \cos 2\pi \theta_j|} \leq e^{q_{n-n_0} \epsilon}.
\end{equation}

Using the fact \(4(s+1)q_{n-n_0} > \eta q_n - 2,\) one has for any \(i \in I_1 \cup I_2,\)
\begin{equation}
|\sin \pi (2\theta + (\ell + i) \alpha)| \geq -C \ln q_n.
\end{equation}

Applying Lemma 2.3 there exists some \(j_0\) with \(j_0 \in I_1 \cup I_2\) such that
\[P_{\ell-1}(\theta_{j_0} - \frac{k-1}{2} \alpha) \geq e^{(\ln \lambda - \epsilon)k}.
\]

Firstly, we assume \(j_0 \in I_2.\)

Set \(I = [j_0 - 3q_{n-n_0} + 1, j_0 + 3q_{n-n_0} - 1] = [x_1, x_2].\) By (31) and (32) again, one has\n\[|G_I(y, x_i)| \leq \exp\\{\ln \lambda + \epsilon\(6q_{n-n_0} - 1 - |y - x_i|\) - 6q_{n-n_0}(\ln \lambda - \epsilon)\}.
\]

Notice that \(|y - x_i| \geq q_{n-n_0},\) we obtain
\begin{equation}|G_I(y, x_i)| \leq \exp\\{-(\ln \lambda - \epsilon)|y - x_i|\}\).
\end{equation}

If \(j_0 \in I_1,\) we may let \(y = 0\) in (53). By (50), we get\n\[|\phi(0)| \leq 6q_{n-n_0} \exp\\{-(\ln \lambda - \epsilon)sq_{n-n_0}\}\).
\]

This contradicts \(\phi(0) = 1.\) Thus \(j_0 \in I_2,\) and the theorem follows from (53). \(\square\)

**Proof of Claim 1**

**Proof.** By the construction of \(I_1\) and \(I_2\) in Claim 1, (31) and (32), we have for \(i \in I_1,\)
\begin{equation}
\min_{\ell \in I_1 \cup I_2} \ln |\sin \pi (2\theta + (\ell + i) \alpha)| \geq -C \ln q_n,
\end{equation}
and
\begin{equation}
\min_{j \in I_1 \cup I_2, j \neq 1} \ln |\sin \pi (j - i) \alpha)| \geq -C \ln q_n.
\end{equation}

Replacing (10) with (11) and (11) with (53), and following the proof of (52), we can show that for any \(i \in I_1,\)
\[\prod_{j \in I_1 \cup I_2, j \neq 1} \frac{|x - \cos 2\pi \theta_j|}{|\cos 2\pi \theta_i - \cos 2\pi \theta_j|} \leq e^{q_{n-n_0}}.
\]

This implies for \(i \in I_1,\) \(La_i \leq \epsilon q_n.\)
By the construction of $I_1$ and $I_2$ in Claim 1, (31) and (32) again, we have for $i \in I_2$,

\begin{equation}
\min_{\ell \in I_1 \cup I_2} \ln |\sin \pi (2\theta + (\ell + i)\alpha)|_{\mathbb{R}/\mathbb{Z}} \geq -\beta q_n - C \ln q_n,
\end{equation}

and

\begin{equation}
\min_{j \neq i, j \in I_1 \cup I_2} \ln |\sin (j - i)\alpha| \geq -C \ln q_n.
\end{equation}

We should mention that, for each $i \in I_2$, there is exactly one $j \in I_1 \cup I_2$ such that the lower bound of (56) can be achieved.

Replacing (40) with (56) and (41) with (57), and following the proof of (52), we can show that for any $i \in I_1$,

\begin{equation}
\prod_{j \neq i, j \in I_1 \cup I_2} \frac{|x - \cos 2\pi \theta_j|}{|\cos 2\pi \theta_i - \cos 2\pi \theta_j|} \leq e^{\varepsilon q_n - n_0 + \beta q_n}.
\end{equation}

This implies for any $i \in I_2$, $L_{a_i} \leq q_n (\beta + \varepsilon)$.

\textbf{Proof of Claim 2}

\textbf{Proof.} Let $J_3^1 = [jq_n - 2s_{q_n - n_0}, jq_n - 1]$ and $J_3^2 = [jq_n + 2s_{q_n - n_0} - 1]$ so that $J_3 = J_3^1 \cup J_3^2$.

Let $I = J_1 \cup J_2 \cup J_3$.

\textbf{Case 1:} $i \in J_1 \cup J_3^1$

By the construction of $J_1$, $J_2$ and $J_3$ in Claim 2, and (31), (32), we have

\begin{equation}
\min_{\ell \in I} \ln |\sin \pi (2\theta + (\ell + i)\alpha)| \geq -\beta q_n - C \ln q_n,
\end{equation}

and

\begin{equation}
\min_{j \neq i, j \in I} \ln |\sin (j - i)\alpha| \geq -C \ln q_n.
\end{equation}

Moreover, there are exactly two $\ell, j \in I$ such that the lower bound of (58) can be achieved for $\ell$ and the lower bound of (59) can be achieved for $j$.

\textbf{Case 2:} $i \in J_1 \cup J_3^2$

By the same reason, we have

\begin{equation}
\min_{\ell \in I} \ln |\sin \pi (2\theta + (\ell + i)\alpha)| \geq -\beta q_n - C \ln q_n,
\end{equation}

and

\begin{equation}
\min_{j \neq i, j \in I} \ln |\sin (j - i)\alpha| \geq -C \ln q_n.
\end{equation}

Moreover, there are exactly two $\ell_1, \ell_2 \in I$ such that the lower bound of (60) can be achieved for both $\ell_1$ and $\ell_2$.

\textbf{Case 3:} $i \in J_2$

By the same reason, we have

\begin{equation}
\min_{\ell \in I} \ln |\sin \pi (2\theta + (\ell + i)\alpha)| \geq -\beta q_n - C \ln q_n,
\end{equation}

and

\begin{equation}
\min_{j \neq i, j \in I} \ln |\sin (j - i)\alpha| \geq -C \ln q_n.
\end{equation}

Moreover, there is exactly one $\ell \in I$ such that the lower bound of (62) can be achieved for $\ell$.

Now following the proof of the Claim 1, we can prove Claim 2.
Appendix B. Proof of Lemma 2.5

Without loss of generality, we assume $k \in [jq_n, (j + \frac{1}{2})q_n]$ and $j \geq 0$. Let $d_j = k - jq_n$ and $d_{j+\frac{1}{2}} = (j + \frac{1}{2})q_n - k$.

For any $y \in [jq_n + \eta q_n, (j + \frac{1}{2})q_n - \eta q_n]$, by Theorem 2.4 $y$ is regular with $\tau = \ln \lambda - \eta$. Therefore there exists an interval $I(y) = [x_1, x_2] \subset [jq_n, (j + \frac{1}{2})q_n]$ such that $y \in I(y)$ and

\begin{equation}
\text{dist}(y, \partial I(y)) \geq \frac{1}{l}|I(y)| \geq \frac{q_{n-n_0}}{2}
\end{equation}

and

\begin{equation}
|G_{I(y)}(y, x_i)| \leq e^{-\left(\ln \lambda - \eta\right)|y - x_i|}, \quad i = 1, 2,
\end{equation}

where $\partial I(y)$ is the boundary of the interval $I(y)$, i.e., $\{x_1, x_2\}$, and $|I(y)|$ is the size of $I(y) \cap \mathbb{Z}$, i.e., $|I(y)| = x_2 - x_1 + 1$. For $z \in \partial I(y)$, let $z'$ be the neighbor of $z$, (i.e., $|z - z'| = 1$) not belonging to $I(y)$.

If $x_2 + 1 \leq (j + \frac{1}{2})q_n - \eta q_n$ or $x_1 - 1 \geq jq_n + \eta q_n$, we can expand $\phi(x_2 + 1)$ or $\phi(x_1 - 1)$ using (5). We can continue this process until we arrive to $z$ such that $z + 1 > (j + \frac{1}{2})q_n - \eta q_n$ or $z - 1 < jq_n + \eta q_n$, or the iterating number reaches $\lfloor \frac{4q_n}{q_{n-n_0}} \rfloor$. Thus, by (5)

\begin{equation}
\phi(k) = \sum_{s ; z_i + 1 \in \partial I(z_i')} G_{I(k)}(k, z_1)G_{I(z_1)}(z_1', z_2) \cdots G_{I(z_s)}(z_s', z_{s+1})\phi(z_{s+1}'),
\end{equation}

where in each term of the summation one has $jq_n + \eta q_n + 1 \leq z_i \leq (j + \frac{1}{2})q_n - \eta q_n - 1$, $i = 1, \cdots, s$, and either $z_{s+1} \notin [jq_n + \eta q_n + 1, (j + \frac{1}{2})q_n - \eta q_n - 1]$, $s + 1 < \lfloor \frac{4q_n}{q_{n-n_0}} \rfloor$; or $s + 1 = \lfloor \frac{4q_n}{q_{n-n_0}} \rfloor$. We should mention that $z_{s+1} \in [jq_n, (j + \frac{1}{2})q_n]$.

If $z_{s+1} \in [jq_n, jq_n + \eta q_n]$, $s + 1 < \lfloor \frac{4q_n}{q_{n-n_0}} \rfloor$, this implies

\[|\phi(z_{s+1})| \leq r_j.\]

By (65), we have

\[|G_{I(k)}(k, z_1)G_{I(z_1)}(z_1', z_2) \cdots G_{I(z_s)}(z_s', z_{s+1})\phi(z_{s+1}')| \leq \]

\[r_j \leq e^{-\left(\ln \lambda - \eta\right)(k - z_1 + \sum_{i=1}^s |z_i' - z_{i+1}|)} \leq \]

\[r_j e^{-\left(\ln \lambda - \eta\right)(k - z_{s+1} - (s + 1))} \leq \]

\[e^{-\left(\ln \lambda - \eta\right)\left\{d_j - 2q_n - 4 - \frac{4q_n}{q_{n-n_0}} \right\}}.\]

If $z_{s+1} \in [(j + \frac{1}{2})q_n - \eta q_n, (j + \frac{1}{2})q_n]$, $s + 1 < \lfloor \frac{4q_n}{q_{n-n_0}} \rfloor$, by the same arguments, we have

\[|G_{I(k)}(k, z_1)G_{I(z_1)}(z_1', z_2) \cdots G_{I(z_s)}(z_s', z_{s+1})\phi(z_{s+1}')| \leq r_{j+\frac{1}{2}} e^{-\left(\ln \lambda - \eta\right)d_{j+\frac{1}{2}} - 2q_n - 4 - \frac{4q_n}{q_{n-n_0}}}.\]

If $s + 1 = \lfloor \frac{4q_n}{q_{n-n_0}} \rfloor$, using (64) and (65), we obtain

\[|G_{I(k)}(k, z_1)G_{I(z_1)}(z_1', z_2) \cdots G_{I(z_s)}(z_s', z_{s+1})\phi(z_{s+1}')| \leq e^{-\left(\ln \lambda - \eta\right)\frac{1}{2}q_{n-n_0} - \frac{4q_n}{q_{n-n_0}}} |\phi(z_{s+1}').\]
Notice that the total number of terms in (65) is at most $2^\left\lfloor \frac{n}{\eta q_n} \right\rfloor$ and $d_j, d_j + \frac{1}{2} \geq 10\eta q_n$. By (67), (68) and (69), we have
\[
|\phi(k)| \leq \max\{r_j e^{-(\ln \lambda-\eta)(d_j-3\eta q_n)}, r_j + \frac{1}{2} e^{-(\ln \lambda-\eta)(d_j+\frac{1}{2}-3\eta q_n)}, e^{-(\ln \lambda-\eta)q_n} \max_{p \in [jq_n, (j+\frac{1}{2})q_n]} |\phi(p)| \}.
\]
Now we will show that for any $p \in [jq_n, (j + \frac{1}{2})q_n]$, one has $|\phi(p)| \leq \max\{r_j, r_j + \frac{1}{2}\}$. Then (70) implies Lemma 2.5. Otherwise, by the definition of $r_j$, if $|\phi(p')|$ is the largest one of $|\varphi(z)|, z \in [jq_n + 10\eta q_n + 1, (j + \frac{1}{2})q_n - 10\eta q_n - 1]$, then $|\phi(p')| > \max\{r_j, r_j + \frac{1}{2}\}$. Applying (70) to $\phi(p')$ and noticing that $\text{dist}(p', q_n\mathbb{Z}) \geq 10\eta q_n$, we get
\[
|\phi(p')| \leq e^{-\eta q_n} \max\{r_j, r_j + \frac{1}{2}, |\phi(p')| \}.
\]
This is impossible because $|\phi(p')| > \max\{r_j, r_j + \frac{1}{2}\}$.

Acknowledgments

I would like to thank Svetlana Jitomirskaya for comments on earlier versions of the manuscript. This research was supported by the AMS-Simons Travel Grant 2016-2018, NSF DMS-1401204 and NSF DMS-1700314. The author is grateful to the Isaac Newton Institute for Mathematical Sciences, Cambridge, for its hospitality, supported by EPSRC Grant Number EP/K032208/1, during the programme Periodic and Ergodic Spectral Problems where this work was started.

References

[1] A. Avila and S. Jitomirskaya. Solving the ten Martini problem. In Mathematical physics of quantum mechanics, volume 690 of Lecture Notes in Phys., pages 5–16. Springer, Berlin, 2006.
[2] A. Avila and S. Jitomirskaya. The Ten Martini Problem. Ann. of Math. (2), 170(1):303–342, 2009.
[3] A. Avila and S. Jitomirskaya. Almost localization and almost reducibility. J. Eur. Math. Soc. (JEMS), 12(1):93–131, 2010.
[4] A. Avila, S. Jitomirskaya, and Q. Zhou. Second phase transition line. Math. Ann., 370(1-2):271–285, 2018.
[5] A. Avila, J. You, and Q. Zhou. Dry ten Martini problem in non-critical case. Preprint.
[6] A. Avila, J. You, and Q. Zhou. Sharp phase transitions for the almost Mathieu operator. Duke Math. J., 166(14):2697–2718, 2017.
[7] J. Bellissard, R. Lima, and D. Testard. Almost periodic Schrödinger operators. In Mathematics+ Physics: Lectures on Recent Results (Volume 1), pages 1–64. World Scientific, 1985.
[8] J. M. Berezanski. Expansions in eigenfunctions of self-adjoint operators. Translations of Mathematical Monographs, vol. 17. American Mathematical Society, Providence, RI, 1968.
[9] J. Bourgain. Green’s function estimates for lattice Schrödinger operators and applications, volume 158 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2005.
[10] J. Bourgain and M. Goldstein. On nonperturbative localization with quasi-periodic potential. Annals of Mathematics, 152(3):835–879, 2000.
[11] J. Bourgain, M. Goldstein, and W. Schlag. Anderson localization for Schrödinger operators on $\mathbb{Z}^2$ with quasi-periodic potential. Acta Math., 188(1):41–86, 2002.
[12] J. Bourgain and S. Jitomirskaya. Absolutely continuous spectrum for 1D quasiperiodic operators. Invent. Math., 148(3):453–463, 2002.
[13] V. Chulaevsky and F. Delyon. Purely absolutely continuous spectrum for almost Mathieu operators. J. Statist. Phys., 55(5-6):1279–1284, 1989.
[14] E. Dinaburg and Y. G. Sinai. The one-dimensional Schrödinger equation with a quasiperiodic potential. Functional Analysis and Its Applications, 9(4):279–289, 1975.
[15] L. H. Eliasson. Floquet solutions for the 1-dimensional quasi-periodic Schrödinger equation. Comm. Math. Phys., 146(3):447–482, 1992.
[16] J. Fröhlich, T. Spencer, and P. Wittwer. Localization for a class of one dimensional quasi-periodic Schrödinger operators. Communications in mathematical physics, 132(1):5–25, 1990.
[17] A. Y. Gordon. The point spectrum of the one-dimensional Schrödinger operator. Uspekhi Matematicheskikh Nauk, 31(4):257–258, 1976.
[18] A. Y. Gordon, S. Jitomirskaya, Y. Last, and B. Simon. Duality and singular continuous spectrum in the almost Mathieu equation. Acta Math., 178(2):169–183, 1997.
[19] S. Hadj Amor. Hölder continuity of the rotation number for quasi-periodic co-cycles in $SL(2, \mathbb{R})$. Comm. Math. Phys., 287(2):565–588, 2009.
[20] R. Han. Dry Ten Martini problem for the non-self-dual extended Harper's model. Trans. Amer. Math. Soc., 370(1):197–217, 2018.
[21] R. Han and S. Jitomirskaya. Full measure reducibility and localization for quasiperiodic Jacobi operators: A topological criterion. Adv. Math., 319:224–250, 2017.
[22] S. Jitomirskaya. Almost everything about the almost Mathieu operator. II. XIth International Congress of Mathematical Physics (Paris, 1994), 373–382, Int. Press, Cambridge, MA (1995).
[23] S. Jitomirskaya. Metal-insulator transition for the almost Mathieu operator. Annals of Mathematics, 150(3):1159–1175, 1999.
[24] S. Jitomirskaya and I. Kachkovskiy. $L^2$-reducibility and localization for quasiperiodic operators. Math. Res. Lett., 23(2):431–444, 2016.
[25] S. Jitomirskaya, D. A. Koslover, and M. S. Schulteis. Localization for a family of one-dimensional quasiperiodic operators of magnetic origin. Ann. Henri Poincaré, 6(1):103–124, 2005.
[26] S. Jitomirskaya and W. Liu. Universal hierarchical structure of quasiperiodic eigenfunctions. Ann. of Math. (2), 187(3):721–776, 2018.
[27] S. Jitomirskaya and W. Liu. Universal hierarchical structure of quasiperiodic eigenfunctions and sharp spectral transition in phase. arXiv preprint [arXiv:1802.00781], 2018.
[28] S. Jitomirskaya and B. Simon. Operators with singular continuous spectrum. III. Almost periodic Schrödinger operators. Comm. Math. Phys., 163(1):201–205, 1994.
[29] S. Jitomirskaya and S. Zhang. Quantitative continuity of singular continuous spectral measures and arithmetic criteria for quasiperiodic Schrödinger operators. arXiv preprint arXiv:1510.07086, 2015.
[30] R. Johnson and J. Moser. The rotation number for almost periodic potentials. Communications in Mathematical Physics, 84(3):403–438, 1982.
[31] I. Krasovsky. Central spectral gaps of the almost Mathieu operator. Comm. Math. Phys., 351(1):419–439, 2017.
[32] Y. Last, Spectral theory of Sturm-Liouville operators on infinite intervals: a review of recent developments. In Sturm-Liouville theory, pages 99–120. Birkhäuser, Basel, 2005.
[33] Y. Last and M. Shamis. Zero Hausdorff dimension spectrum for the almost Mathieu operator. Comm. Math. Phys., 348(3):729–750, 2016.
[34] W. Liu and X. Yuan. Anderson localization for the completely resonant phases. J. Funct. Anal., 268(3):732–747, 2015.
[35] W. Liu and X. Yuan. Spectral gaps of almost Mathieu operators in the exponential regime. J. Fractal Geom., 2(1):1–51, 2015.
(Wencai Liu) Department of Mathematics, University of California, Irvine, California 92697-3875, USA

E-mail address: liuwencai1226@gmail.com