Direct Systems of Spherical Functions
and Representations

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Abstract. Spherical representations and functions are the building blocks for harmonic analysis on riemannian symmetric spaces. Here we consider spherical functions and spherical representations related to certain infinite dimensional symmetric spaces $G_\infty/K_\infty = \lim_{n\to \infty} G_n/K_n$. We use the representation theoretic construction $\varphi(x) = \langle e, \pi(x)e \rangle$ where $e$ is a $K_\infty$-fixed unit vector for $\pi$. Specifically, we look at representations $\pi_\infty = \lim_{n\to \infty} \pi_n$ of $G_\infty$ where $\pi_n$ is $K_n$-spherical, so the spherical representations $\pi_n$ and the corresponding spherical functions $\varphi_n$ are related by $\varphi_n(x) = \langle e_n, \pi_n(x)e_n \rangle$ where $e_n$ is a $K_n$-fixed unit vector for $\pi_n$, and we consider the possibility of constructing a $K_\infty$-spherical function $\varphi_\infty = \lim \varphi_n$. We settle that matter by proving the equivalence of (i) $\{e_n\}$ converges to a nonzero $K_\infty$-fixed vector $e$, and (ii) $G_\infty/K_\infty$ has finite symmetric space rank (equivalently, it is the Grassmann manifold of $p$-planes in $F_\infty$ where $p < \infty$ and $F$ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$). In that finite rank case we also prove the functional equation

$$\varphi(x)\varphi(y) = \lim_{n\to \infty} \int_{K_n} \varphi(xky)dk$$

of Faraut and Olshanski, which is their definition of spherical functions. We use this, and recent results of M. Rössler, T. Koornwinder and M. Voit, to show that in the case of finite rank all $K_\infty$-spherical representations of $G_\infty$ are given by the above limit formula. This in particular shows that the characterization of the spherical representations in terms of highest weights is still valid as in the finite dimensional case.

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1. Introduction

Representation theory and harmonic analysis on symmetric spaces is by now well understood. The building blocks are the spherical representations and the

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corresponding spherical functions. For the case of a compact symmetric space \( G/K \) the spherical representations are parameterized by a certain semi-lattice \( \Lambda \). When \( G/K \) is simply connected, \( \Lambda \) is described by the Cartan-Helgason theorem.

For each \( \mu \in \Lambda \) let \( (\pi_\mu, V_\mu) \) denote the corresponding irreducible representation. Then the space \( V^K_\mu \) of \( K \)-fixed vectors has dimension 1. Let \( e_\mu \) be a unit vector in \( V^K_\mu \). The function

\[
\psi_\mu(x) = \langle e_\mu, \pi_\mu(x)e_\mu \rangle \tag{1.1}
\]
does not depend on the choice of \( e_\mu \), and \( \{\sqrt{\dim V_\mu}\psi_\mu\}_{\mu \in \Lambda} \) is an orthonormal basis for \( L^2(G/K)^K \). In particular

\[
f = \sum_{\mu \in \Lambda} \dim V_\mu (f, \psi_\mu) \psi_\mu
\]

for every \( f \in L^2(G/K)^K \). Here the sum is taken in the \( L^2 \)-sense. Similarly, every \( f \in L^2(G/K) \) can be written as

\[
f = \sum_{\mu \in \Lambda} \dim V_\mu (\pi_\mu(f)e_\mu, \pi_\mu(\cdot)e_\mu)
\]

where \( \pi_\mu(f) = \int_G f(gK)\pi_\mu(g) d(gK) \).

The function \( \psi_\mu \) is spherical in the sense that

\[
\int_K \psi_\mu(xky) dk = \psi_\mu(x)\psi_\mu(y) \quad \text{for all } x, y \in G. \tag{1.2}
\]

Here \( dk \) is normalized Haar measure on the compact group \( K \). Every positive definite spherical function on \( G \) is obtained in this way from an irreducible unitary representation of \( G \).

It is natural to extend the study to infinite dimensional Lie groups and symmetric spaces. The simplest case is \( G_\infty = \lim G_n, \ K_\infty = \lim K_n \) and \( M_\infty = \lim G_n/K_n \) where \( G_n \subseteq G_{n+1} \) is a sequence of compact Lie groups such that \( K_n = G_n \cap K_{n+1} \). The basic theory was developed by G. Olshanskii (see [9, 10]) for the classical direct limits and for a very important class of representations; by L. Natarajan, E. Rodriguez–Carrington and one of us for more general direct limits (see [6], [7] and [8]); and by S. Strătilă & D. Voiculescu (see their survey [17]) and D. Pickrell’s paper [15] for the factor representation viewpoint. See J. Faraut [2] for further information and references.

The equation (1.2) does not make sense here because there is no invariant measure on \( K_\infty \). The replacement is the functional equation

\[
\lim_{n \to \infty} \int_{K_n} \psi(xk_ny) dk_n = \psi(x)\psi(y) \quad \text{for all } x, y \in G_\infty. \tag{1.3}
\]

Again, see [2], the function \( \psi \) is spherical if and only if there is an irreducible unitary representation \( (\pi, V) \) of \( G_\infty \) and \( e \in V^K_\infty \) with \( \|e\| = 1 \), such that \( \psi \) is given by (1.1).

On the other hand, limits of irreducible spherical representations for a strict direct system \( \{M_n = G_n/K_n\} \) of compact symmetric spaces were studied by the
last named author in a series of articles [18], [19], and [20], and then later by
the last two authors in [12] and [13]. In particular, in [12], [13] and [14] they
introduced the notion of propagation of symmetric spaces. In short, if the $G_n$ are
compact and connected, and $\pi_n$ is a spherical representation of $G_n$, then there
exists in a canonical way a spherical representation $\pi_{n+1}$ of $G_{n+1}$ such that $\pi_n$ is a
subrepresentation of $\pi_{n+1}|_{G_n}$ with multiplicity 1. Furthermore, if $u_{n+1}$ is a highest
weight vector for $\pi_{n+1}$ then $\pi_n$ is realized as $\pi_{n+1}|_{G_n}$ acting on the space generated
by $\pi_{n+1}(G_n)u_{n+1}$. The system $\{(\pi_n, V_n)\}$ is injective and $(\pi_\infty, V_\infty) := \lim_{n\to\infty}(\pi_n, V_n)$
is an irreducible unitary representation of $G_\infty$.

In this article we address the question of whether the representation $(\pi_\infty, V_\infty)$
is spherical. Our main result is Theorem 4.5 below. It states that $V_\infty^{K_\infty} \neq \{0\}$
if and only if the symmetric space ranks of the compact riemannian symmetric
spaces $M_n$ are bounded. Thus $V_\infty^{K_\infty} \neq \{0\}$ only for the symmetric spaces
$\text{SO}(p + \infty)/\text{SO}(p) \times \text{SO}(\infty)$, $\text{SU}(p + \infty)/\text{SU}(p) \times \text{U}(\infty)$, and
$\text{Sp}(p + \infty)/\text{Sp}(p) \times \text{Sp}(\infty)$ where $0 < p < \infty$.

We then show that if $\{e_n\}$ is a sequence of $K_n$-invariant vectors in $V_n$ of norm 1
and $e = \lim_{n \to \infty} e_n \in V_\infty^{K_\infty}$, then the function $\psi_\infty(x) := \langle e, \pi_\infty(x)e \rangle$
is spherical in the sense of (1.3), and
\[
\psi_\infty(x) = \lim_{n \to \infty} \psi_n(x)
\]
where $\psi_n(x) = \langle e_n, \pi_n(x)e_n \rangle$. See Theorem 7.1.

Further discussion of the finite rank case is given in Section 8. Using result a
from [16] we show that in the case of finite rank all $K_\infty$-spherical representations of
$G_\infty$ are given by the limit construction in Section 4. This in particular shows that
the characterization of the spherical representations in terms of highest weights is
still valid in the finite rank case.

2. Propagation of Symmetric Spaces

In this section we give a short overview of injective limits and propagation of
compact symmetric spaces, as needed for our considerations on limits of spherical
representations. We refer to [13] and [20] for details.

Let $M = G/K$ be a riemannian symmetric space of compact type. Thus $G$
is a connected semisimple compact Lie group with an involution $\theta$ such that
\[
(G^\theta)_0 \subseteq K \subseteq G^\theta
\]
where $G^\theta = \{x \in G \mid \theta(x) = x\}$ and the subscript $_0$ denotes the connected
component containing the identity element. For simplicity we assume that $M$
is simply connected.

Denote the Lie algebra of $G$ by $\mathfrak{g}$. By abuse of notation we write $\theta$ for
the involution $d\theta : \mathfrak{g} \to \mathfrak{g}$. As usual $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ where $\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta(X) = X\}$
is the Lie algebra of $K$ and $\mathfrak{s} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}$. Fix a maximal abelian
subspace $\mathfrak{a} \subset \mathfrak{s}$. For $\alpha \in \mathfrak{a}_C^*$ let
\[
\mathfrak{g}_{C,\alpha} = \{X \in \mathfrak{g}_C \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}_C\}.
\]
If $\mathfrak{g}_{C,\alpha} \neq \{0\}$ then $\alpha$ is called a (restricted) root. Denote by $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a}) \subset \mathfrak{i}\mathfrak{a}^*$ the set of roots. Let $\Sigma_0 = \Sigma_0(\mathfrak{g}, \mathfrak{a}) = \{\alpha \in \Sigma \mid 2\alpha \not\in \Sigma\}$, the set of
nonmultipliable roots. Then \( \Sigma_0 \) is a root system in the usual sense and the Weyl group corresponding to \( \Sigma(g, a) \) is the same as the Weyl group generated by the reflections \( s_\alpha, \alpha \in \Sigma_0 \). Furthermore, \( M \) is irreducible as a riemannian symmetric space if and only if \( \Sigma_0 \) is irreducible as a root system.

Let \( \Sigma^+ \subset \Sigma \) be a positive system and \( \Sigma^+_0 = \Sigma^+ \cap \Sigma_0 \). Then \( \Sigma^+_0 \) is a positive system in \( \Sigma_0 \). Denote by \( \Psi = \{\alpha_1, \ldots, \alpha_r\} \), \( r = \dim a \), the set of simple roots in \( \Sigma^+_0 \). Since we will be dealing with direct limits we may assume that \( \Sigma \), and hence \( \Sigma_0 \), is one of the classical root systems. In order to facilitate considerations of direct limits, we number the simple roots in the following way:

\[
\begin{array}{c|c|c|c}
\Psi &=& & r \\
A_r &=& \alpha_1 & r \geq 1 \\
B_r &=& \alpha_2 & r \geq 2 \\
C_r &=& \alpha_3 & r \geq 3 \\
D_r &=& \alpha_4 & r \geq 4 \\
\end{array}
\]

The classical irreducible symmetric spaces are given by the following table. For the grassmannians we always assume that \( p \leq q \) and we let \( n = p + q \). For \( \alpha \in \Sigma \) we write \( m_\alpha = \dim g_{\xi, \alpha} \), and for the simple roots we write \( m_j = m_{\alpha_j} \). For the realization of each root system see [1], [3, Chapter 10] or [12]. In all these classical cases \( m_{\alpha_j/2} = 0 \) for \( j > 1 \). We will go into more detail in Section 3.

| \( G \) | \( K \) | \( \Psi \) | \( m_j \) \( j > 1 \) | \( m_1 \) | \( m_{\alpha_j/2} \) |
|----------|----------|----------|----------------|---------|----------------|
| 1 SU\( (n) \times SU(n) \) | \( \text{diag} \, SU(n) \) | \( A_{n-1} \) | 2 | 2 | 0 |
| 2 Sp\( (2n+1) \times Sp(2n+1) \) | \( \text{diag} \, \text{Spin}(2n+1) \) | \( B_n \) | 2 | 2 | 0 |
| 3 \( \text{Spin}(2n) \times \text{Spin}(2n) \) | \( \text{diag} \, \text{Spin}(2n) \) | \( D_n \) | 2 | 2 | 0 |
| 4 Sp\( (n) \times \text{Sp}(n) \) | \( \text{diag} \, \text{Sp}(n) \) | \( C_n \) | 2 | 2 | 0 |
| 5 SU\( (n) \) | \( \text{S}(U(p) \times U(q)) \) | \( C_p \) | 2 | 1 | \( 2(q-p) \) |
| 6 SU\( (n) \) | \( \text{SO}(n) \) | \( A_{n-1} \) | 1 | 1 | 0 |
| 7 SU\( (2n) \) | \( \text{Sp}(n) \) | \( A_{n-1} \) | 4 | 4 | 0 |
| 8 SO\( (n) \) | \( \text{SO}(p) \times \text{SO}(q) \) | \( B_{p,q} \) | 1 | \( q-p \) | 0 |
| 9 Sp\( (n) \) | \( \text{SO}(4n) \) | \( U(2n) \) | \( C_n \) | 4 | 1 | 0 |
| 10 Sp\( (n) \) | \( \text{Sp}(p) \times \text{Sp}(q) \) | \( C_p \) | 4 | 3 | \( 4(q-p) \) |

Cases (5), (8), and (10) are the grassmannians of \( p \)-planes in \( \mathbb{F}^n \), \( n = p + q \), where \( \mathbb{F} = \mathbb{C}, \mathbb{R} \) or \( \mathbb{H} \), respectively. In cases (5) and (10), \( m_{\alpha_j/2} = (q-p)d \) and \( m_{\alpha_1} = d-1 \) where \( d = \dim \mathbb{R} \mathbb{F} \). It is therefore more natural to view (8) as of type \( C_p \) with \( m_{\alpha_1/2} = q-p \) and \( m_{\alpha_1} = d-1 = 0 \).
We now assume that \( \{ M_k = G_k / K_k \}_{k \geq 1} \) is a sequence of compact symmetric spaces such that \( G_n \subseteq G_k \) and \( K_n = G_n \cap K_k \) for \( n \leq k \). Then \( M_n \subseteq M_k \).

We write \( \Sigma_n, \Sigma^+_n, \Sigma^0_n, \Psi_n \), etc. when we need to indicate dependence on the symmetric space \( M_n \). We say that \( M_k \) propagates \( M_n \) if (i) \( a_n = a_k \), or (ii) by choosing \( a_n \subseteq a_k \) we obtain the Dynkin diagram in Table 2.1 for \( \Psi_k \). If \( \Sigma_k \) is of nonmultipliable roots. Both definitions are equivalent.

When \( g_k \) propagates \( g_n \), and \( \theta_k \) and \( \theta_n \) are the corresponding involutions with \( \theta_k | G_n = \theta_n \), the corresponding eigenspace decompositions \( g_k = \mathfrak{t}_k \oplus s_k \) and \( g_n = \mathfrak{t}_n \oplus s_n \) give us

\[
\mathfrak{t}_n = \mathfrak{t}_k \cap g_n, \quad \text{and} \quad s_n = g_n \cap s_k \quad \text{for} \quad k \geq n.
\]

We recursively choose maximal commutative subspaces \( a_k \subseteq s_k \) such that \( a_n \subseteq a_k \) for \( k \geq n \). We then have \( \Sigma_n \subseteq \Sigma_k | a_n \setminus \{ 0 \} \). We choose the positive ordering such that \( \Sigma^+_n \subseteq \Sigma^+_k | a_n \setminus \{ 0 \} \).

Note that fixing a row in Table 2.2 but let \( n \) increase toward infinity we have a propagation of symmetric spaces. In all cases except (5), (8) and (10) the multiplicities remain constant, in fact less or equal to 4.

We set

\[
G_\infty = \lim_{n \to \infty} G_n, \quad K_\infty = \lim_{n \to \infty} K_n, \quad \text{and} \quad M_\infty = \lim_{n \to \infty} M_n = G_\infty / K_\infty.
\]

In this paper we consider the question of whether the inductive limit of \( K_n \)–spherical representations of \( G_n \) is \( K_\infty \)–spherical. For that we need to recall the construction of inductive limits of spherical representations, the theory of highest weights of spherical representations and the Harish–Chandra \( c \)–function of the noncompact dual of \( G_n \).

### 3. Spherical Representations of Compact Groups

In this section we give a short overview of spherical representations, their highest weights, and connections to propagation of symmetric spaces. As our groups \( G \) and \( G_k \) are always assumed compact, there is no restriction in always assuming that the representations under consideration are unitary as we will always do in this article. Most of the material can be found in [12], [13], [20], [19] and [18]. The notation will be as in Section 2, and \( G \) or \( G_n \) will always stand for a connected compact group. If \( k \geq n \) then we assume that \( G_n \subseteq G_k \) and that \( M_k \) propagates \( M_n \). We also assume that each of the symmetric spaces \( M_n \) is simply connected.

We denote by \( r_k \) and \( r_n \) the respective real ranks of \( M_k \) and \( M_n \). As always we fix compatible \( K_k \)– and \( K_n \)–invariant inner products on \( s_k \), respectively \( s_n \).

For a representation \(( \pi, V ) \) of \( G \) let \( V^K = \{ u \in V | ( \forall k \in K ) \pi(k) u = u \} \).

If \( ( \pi, V ) \) is irreducible, then we say that \(( \pi, V ) \), or simply \( \pi \), is \( K \)–spherical, or just spherical, if \( V^K \neq \{ 0 \} \). It is well known that \( \pi \) is spherical if and only if \( \dim V^K = 1 \). Furthermore, in that case the highest weight of \( \pi \) is contained in
Let $\pi = \pi_\mu$ denote the irreducible spherical representation with highest weight $\mu$. Define linear functionals $\xi_j \in i\mathfrak{a}^*$ by

$$
\frac{\langle \xi_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{i,j} \quad \text{for} \quad 1 \leq j \leq r .
$$

Then $\xi_1, \ldots, \xi_r \in \Lambda^+$ and

$$
\Lambda^+ = \mathbb{Z}^+ \xi_1 + \ldots + \mathbb{Z}^+ \xi_r = \left\{ \sum_{j=1}^r n_j \xi_j \bigg| n_j \in \mathbb{Z}^+ \right\} .
$$

The weights $\xi_j$ are called the class 1 fundamental weights for $(\mathfrak{g}, \mathfrak{k})$. Set $\Xi = \{\xi_1, \ldots, \xi_r\}$. Let $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$. We always have

$$
\rho = \sum_{j=1}^r \rho_j \xi_j \quad \text{with} \quad \rho_j = \frac{1}{2} \left( m_{\alpha_j} + \frac{m_{\alpha_j/2}}{2} \right) .
$$

We write $a = \frac{1}{2} \left( m_{\alpha_1} + \frac{m_{\alpha_1/2}}{2} \right)$. Using $m_{\alpha_j} = m_{\alpha_i}$ for $i, j \geq 2$ we set $b = \frac{1}{2} m_{\alpha_j}$, $j \geq 2$. Then

$$
\rho = a \xi_1 + b \sum_{j=2}^r \xi_j .
$$

We will need a particular formulation for each classical root system. We identify $\mathfrak{a}$ with $\mathbb{R}^r$ so that, as usual, $\mathfrak{a} = \{(x_{r+1}, \ldots, x_1) \mid x_1 + \ldots + x_{r+1} = 0\}$ if $\Psi = A_r$ and otherwise $\mathfrak{a} = \mathbb{R}^r$. Set $f_1 = (0, \ldots, 0, 1)$, $f_2 = (0, \ldots, 0, 1, 0)$, \ldots, $f_n = (1, 0, \ldots, 0)$ where $n = r + 1$ for $A_r$ and otherwise $n = r$. We view the vectors $f_j$ also as elements in $\mathfrak{a}^*$ via the standard inner product in $\mathbb{R}^{r+1}$ in the case $\Psi = A_r$ and otherwise $\mathbb{R}^r$. Note that in the case $\Psi = A_r$ the so defined map $\mathbb{R}^{r+1} \to \mathfrak{a}^*$ is not injective.
For $\Psi = A_r$ we have $\Sigma_0^+ = \{f_j - f_i \mid 1 \leq i < j \leq n\}$, and $\alpha_j = f_{j+1} - f_j$, $j = 1, \ldots, r$. We have
\[
\xi_j = 2 \sum_{i=j+1}^{r+1} f_i .
\]
Thus
\[
\Lambda^+ \simeq \{(m_r, m_{r-1}, \ldots, m_1, 0) \in (2\mathbb{Z}^*)^{r+1} \mid m_i \leq m_j \text{ if } i < j\}.
\]
The multiplicities are constant, equal to $m = 1, 2$ or $4$. Hence $a = b = 1/2, 1$, or $2$ and we have
\[
\rho = a \sum_{j=1}^{r} \xi_j = m(r, r-1, \ldots, 1, 0) = 2a \sum_{j=1}^{r+1} (j-1)f_j .
\]
(3.4)
If $\Psi$ is of type $B_r$ then we have $\Sigma_0^+ = \{f_j \mid j=1, \ldots, r\} \cup \{f_j \pm f_i \mid 1 \leq i < j \leq r\}$ and $\Psi = \{\alpha_1 = f_1\} \cup \{\alpha_i = f_i - f_{i-1} \mid i = 2, \ldots, r\}$. Thus
\[
\xi_1 = \sum_{j=1}^{r} f_j \text{ and } \xi_j = 2 \sum_{i=j}^{r} f_i , \ j > 1 .
\]
In particular
\[
\Lambda^+ \simeq \{(m_r, \ldots, m_1) \in (\mathbb{Z}^*)^r \mid m_i \leq m_j \text{ and } m_j - m_i \text{ even for } i < j\} .
\]
Finally we have
\[
\rho = \sum_{j=1}^{n} \xi_j = (2r-1, 2r-3, \ldots, 3, 1) = \sum_{j=1}^{r} (2j-1)f_j .
\]
(3.5)
in case (2). Case (8), which is the other possibility for $\Psi$ of type $B$, will be considered in the discussion of $C_r$, as explained just after Table 2.2.
If $\Psi$ is of type $C_r$ then we have $\Sigma_0^+ = \{2f_j \mid j=1, \ldots, r\} \cup \{f_j \pm f_i \mid 1 \leq i < j \leq r\}$ and $\Psi = \{\alpha_1 = 2f_1\} \cup \{\alpha_i = f_j - f_{i-1} \mid j = 2, \ldots, r\}$. Thus
\[
\xi_j = 2 \sum_{i=j}^{r} f_i
\]
and
\[
\rho = 2a \sum_{j=1}^{r} f_j + 2b \sum_{\nu=2}^{r} \sum_{j=\nu}^{r} f_j = 2 \sum_{j=1}^{r} (a + b(j-1))f_j .
\]
(3.6)
There is just one case where $\Psi$ is of type $D_r$. There $a = b = 1$. In that case we have $\alpha_1 = f_1 + f_2$ and $\alpha_j = f_j - f_{j-1}$ for $j \geq 2$. Thus
\[
\xi_1 = \sum_{i=1}^{r} f_i , \ \xi_2 = -f_1 + \sum_{j=2}^{r} f_j , \text{ and } \xi_j = 2 \sum_{i=j}^{r} f_i \text{ for } j \geq 3 .
\]
That gives us
\[ \rho = 2 \sum_{j=2}^{r} (j-1) f_j . \] (3.7)

Fix a \( \mu \in \Lambda^+ \) and let \( (\pi_\mu, V_\mu) \) be the corresponding spherical representation of \( G \). Fix a highest weight vector \( u_\mu \in V_\mu \) and a \( K \)-fixed vector \( e_\mu \). We assume that \( ||u_\mu|| = ||e_\mu|| = 1 \). For the following it is important to evaluate the inner product \( \langle u_\mu, e_\mu \rangle \) in a systematic way so that we can control it as we consider inductive limits of spherical representations in the next section. The following is well known, but we include the proof for completeness.

First of all we always have \( \langle u_\mu, e_\mu \rangle \neq 0 \). We choose \( u_\mu \) and \( e_\mu \) so that \( \langle u_\mu, e_\mu \rangle > 0 \).

Let \( g' = k \oplus i s \). As \( G \) is compact it is a linear group, thus contained in a complex linear group \( G_c \) with Lie algebra \( g_c \). Let \( G' \) be the analytic subgroup of \( G_c \) with Lie algebra \( g' \). Note that the holomorphic extension of \( \theta \) to \( g_c \) restricted to \( g' \) defines a Cartan involution on \( g' \). We also denote this involution and the corresponding Cartan involution on \( G' \) by \( \theta \). Let \( \overline{N} = \theta(N) \). Then \( G' \) has a Iwasawa decomposition (recall that we are assuming \( G \) simply connected, in particular \( K \) is connected)
\[ G' = KA'N \] (3.8)

where \( A' = \exp(i a) \) and the Lie algebra of \( N \) is \( n = \bigoplus_{\alpha \in \Sigma^+} g'_\alpha \).

For \( x \in G' \) write \( x = k(x)a(x)n(x) \) according to the Iwasawa decomposition (3.8). We normalize the Haar measure on \( \overline{N} \) such that
\[ \int_{\overline{N}} a(\overline{n})^{-2\rho} \, d\overline{n} = 1 , \]
Then the integral
\[ c(\lambda) = \int_{\overline{N}} a(\overline{n})^{-\lambda - \rho} \, d\overline{n} \]
converges for all \( \lambda \in a_c^* \) such that \( \text{Re} \langle \lambda, \alpha \rangle > 0 \) for all \( \alpha \in \Sigma^+ \). The function \( c(\lambda) \) is the Harish–Chandra \( c \)-function. It has a meromorphic continuation to all of \( a_c^* \) and is given by
\[ c(\lambda) = 'c(\lambda) \]
where \( 'c(\lambda) \) is explicitly given by the Gindikin–Karpelevich product formula. In terms of \( \Sigma^+_0 \), we have
\[ 'c(\lambda) = \prod_{\alpha \in \Sigma^+_0} 'c_\alpha(\lambda_\alpha) \] (3.9)

where
\[ 'c_\alpha(\lambda_\alpha) = \frac{2^{-2\lambda_\alpha}}{\Gamma \left( \lambda_\alpha + \frac{m_\alpha/2 + 1}{2} \right) \Gamma \left( \lambda_\alpha + \frac{m_\alpha/2 + m_\alpha}{2} \right) }, \quad \lambda_\alpha = \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \] (3.10)

where \( \Gamma \) is the Euler \( \Gamma \)-function \( \Gamma(x) := \int_0^\infty e^{-t}t^{x-1} \, dt \). Formula (3.10) looks slightly different from the usual formula for the \( c \)-function as found for instance
in [4], Ch. IV, Theorem 6.4, where it is written in terms of positive indivisible roots \((α ∈ \Sigma^+ \text{ with } α/2 ∉ \Sigma^+)\) rather than in terms of positive nonmultipliable roots. The formula (3.9) was used in [11]. The equivalence of the two formulas follows from the doubling formula \(\sqrt{π Γ(2x)} = 2^{2x−1}Γ(x)Γ(x + \frac{1}{2})\) for the Gamma function.

Let \(M = Z_K(a)\). The following can be found in any standard reference on symmetric spaces.

**Lemma 3.2.** Let \(f ∈ L^1(K/M)\). Then

\[
\int_{K/M} f(kM)dk = \int_N f(k(\bar{n})M) a(\bar{n})^{-2\rho} d\bar{n}.
\]

Clearly the vector \(\int_K \pi_\mu(k)u_\mu dk\) is \(K\)-invariant, and by taking the inner product with \(e_\mu\) one gets \(\int_K \pi_\mu(k)u_\mu dk = ⟨u_\mu, e_\mu⟩e_\mu\).

**Lemma 3.3.** \(⟨\int_K \pi_\mu(k)u_\mu dk, u_\mu⟩ = \int_K ⟨\pi_\mu(k)u_\mu, u_\mu⟩ dk = c(μ + ρ)\).

**Proof.** The proof is a simple calculation using Lemma 3.2. To simplify the notation we write \(u = u_\mu\) and \(π\) for \(π_\mu\). The representation \(π\) extends to a holomorphic representation of \(G_{C'}\). Hence \(π(x)\) is well defined for \(x ∈ G_{C'}\). We note that \(π(mn)u = u\) for all \(n ∈ N\) and \(m ∈ M\) (see [4], Theorem 4.1, p. 535). In particular \(⟨π(\bar{n})u, u⟩ = ⟨u, π(θ(n)^{-1})u⟩ = ⟨u, u⟩ = 1\).

\[
\int_{K/M} ⟨π(k)u, u⟩ dk = \int_N ⟨π(k(\bar{n}))u, u⟩ a(\bar{n})^{-2\rho} d\bar{n}
\]

\[
= \int_N ⟨π(\bar{n}a(\bar{n})^{-1}n')u, u⟩ a(\bar{n})^{-2\rho} d\bar{n} \quad \text{where } n' ∈ N
\]

\[
= \int_N ⟨π(\bar{n})u, u⟩ a(\bar{n})^{-\mu-2\rho} d\bar{n}
\]

\[
= \int_N a(\bar{n})^{-\mu-2\rho} d\bar{n} = c(μ + ρ).
\]

That proves the Lemma. ■

**Theorem 3.4.** Let \(u_\mu\) and \(e_\mu\) be as above. Then \(⟨u_\mu, e_\mu⟩ = \sqrt{c(μ + ρ)}\). In particular \(\int_K \pi_\mu(k)u_\mu dk = \sqrt{c(μ + ρ)} e_\mu\).

**Proof.** We use the same notation as in Lemma 3.3. By that lemma, we see that

\[
c(μ + ρ) = \langle \int_K \pi_\mu(k)u_\mu dk, u_\mu⟩
\]

\[
= ⟨⟨u_\mu, e_\mu⟩e_\mu, u_\mu⟩
\]

\[
= ⟨u_\mu, e_\mu⟩^2
\]

Thus we see that \(⟨u_\mu, e_\mu⟩ = \sqrt{c(μ + ρ)}\). The rest of the result follows when we recall that \(\int_K \pi_\mu(k)u_\mu dk = ⟨u_\mu, e_\mu⟩e_\mu\). ■
Theorem 3.5. For \( \alpha \in \Sigma_0^+ \) let \( x_\alpha := \frac{1}{4}(m_{\alpha/2} + 2) \) and \( y_\alpha = \frac{1}{4}(m_{\alpha/2} + 2m_{\alpha}) \). Let \( \mu \in \Lambda^+ \). Then \( \mu_\alpha \in \mathbb{Z}^+ \) for all \( \alpha \in \Sigma_0^+ \) and

\[
c(\mu + \rho) = \prod_{\alpha \in \Sigma_0^+} \left( 1 + \frac{x_\alpha}{\rho_\alpha} \right)^{-\mu_\alpha} \prod_{j=0}^{\mu_\alpha-1} \frac{1 + j}{1 + \frac{j}{x_\alpha + \rho_\alpha}} \frac{1 + \frac{\mu_\alpha + j}{2\rho_\alpha}}{1 + \frac{\mu_\alpha + j}{y_\alpha + \rho_\alpha}} (3.11)
\]

where the product \( \prod_{j=0}^{\mu_\alpha-1} \) is understood to be 1 if \( \mu_\alpha = 0 \).

Proof. By (3.9) and (3.10) we can write \( c(\mu + \rho) = \prod_{\alpha \in \Sigma_0^+} c_\alpha(\mu_\alpha + \rho_\alpha) \) with

\[
c_\alpha(\mu_\alpha + \rho_\alpha) = \frac{2^{-2\mu_\alpha} \Gamma(2(\mu_\alpha + \rho_\alpha))}{\Gamma(2\rho_\alpha)} \frac{\Gamma(\rho_\alpha + x_\alpha)}{\Gamma(\mu_\alpha + \rho_\alpha + x_\alpha)} \frac{\Gamma(\rho_\alpha + y_\alpha)}{\Gamma(\mu_\alpha + \rho_\alpha + y_\alpha)}
\]

Now, using that \( \mu_\alpha \in \mathbb{Z}^+ \) and \( \Gamma(x+1) = x\Gamma(x) \), we get for \( \mu_\alpha \neq 0 \):

\[
\Gamma(2(\mu_\alpha + \rho_\alpha)) = \left( \prod_{j=1}^{2\mu_\alpha} (2(\mu_\alpha + \rho_\alpha) - j) \right) \Gamma(2\rho_\alpha)
\]

\[
= \left( \prod_{j=0}^{2\mu_\alpha-1} (2\rho_\alpha + j) \right) \Gamma(2\rho_\alpha)
\]

\[
= 2^{2\mu_\alpha} \rho_\alpha^{2\mu_\alpha} \left( \prod_{j=0}^{\mu_\alpha-1} \left( 1 + \frac{j}{2\rho_\alpha} \right) \left( 1 + \frac{\mu_\alpha + j}{2\rho_\alpha} \right) \right) \Gamma(2\rho_\alpha).
\]

Similarly

\[
\frac{\Gamma(\rho_\alpha + x_\alpha)}{\Gamma(\rho_\alpha + y_\alpha)} = \prod_{j=0}^{\mu_\alpha-1} \frac{1}{(\rho_\alpha + x_\alpha + j)(\rho_\alpha + y_\alpha + j)}
\]

and the claim follows.

\[\blacksquare\]

4. Inductive Limits of Spherical Representations

In this section we recall the construction of inductive limits of spherical representations [20, Section 3] and [13]. We always assume that \( k \geq n \) and that we have a sequence \( \{M_k = G_k/K_k\} \) such that \( M_k \) propagates \( M_n \). The index \( k \) (respectively \( n \)) will indicate objects related to \( G_k \) (respectively \( G_n \)). As in [18] or [20], our description of the root system and the fundamental weights gives

Lemma 4.1. Assume that \( M_k \) propagates \( M_n \). Let \( \Psi_n = \{\alpha_{n,1}, \ldots, \alpha_{n,r_n}\} \) and \( \Xi_n = \{\xi_{n,1}, \ldots, \xi_{n,r_n}\} \) and similarly for \( M_k \). Assume that \( j \leq r_n \). Then

1. \( \alpha_{k,j} \) is the unique element of \( \Psi_k \) whose restriction to \( a_n \) is \( \alpha_{n,j} \).
2. If \( \mu_n = \sum_{j=1}^{r_n} k_j \xi_{n,j} \in \Lambda_n^+ \), then \( \mu_k := \sum_{j=1}^{r_n} k_j \xi_{k,j} \in \Lambda_k^+ \) and \( \mu_k|_{\Lambda_n} = \mu_n \).

For \( I = (k_1, \ldots, k_{r_n}) \in (\mathbb{Z}^+)^{r_n} \) define \( \mu_I := k_1 \xi_{n,1} + \ldots + k_{r_n} \xi_{n,r_n} \). Lemma 4.1 allows us to form a direct system of representations, as follows. For \( \ell \in \mathbb{N} \) denote by \( 0_\ell = (0, \ldots, 0) \) the zero vector in \( \mathbb{R}^\ell \). For \( I_n = (k_1, \ldots, k_{r_n}) \in (\mathbb{Z}^+)^{r_n} \) let

\[
\begin{align*}
\cdot \mu_{I,n} &= \mu(I_n) = \sum_{j=1}^{r_n} k_j \xi_{n,j} \in \Lambda_n^+; \\
\cdot \pi_{I,n} &= \pi_{\mu(I_n)} \text{ the corresponding spherical representation}; \\
\cdot V_{I,n} &= V_{\mu(I_n)} \text{ a fixed Hilbert space for the representation } \pi_{I,n}; \\
\cdot u_{I,n} &= u_{\mu(I_n)} \text{ a highest weight unit vector in } V_{I,n}, \|u_{I,n}\| = 1; \\
\cdot e_{I,n} &= e_{\mu(I_n)} \text{ a } K_n \text{-fixed unit vector in } V_{I,n} \text{ such that } \langle u_{I,n}, e_{I,n} \rangle > 0
\end{align*}
\]

Later, \( I_n \) will be fixed and we will write \( \pi_n, V_n, \mu_n, u_n, \) and \( e_n \) without further comments.

**Theorem 4.2.** Assume that \( M_k \) propagates \( M_n \). Let \( (\pi_{I,n}, V_{I,n}) \) be an irreducible spherical representation of \( G_n \) with highest weight \( \mu_{I,n} \in \Lambda_n^+ \). Let \( I_k = (I_n, 0_{r_n-r_n}) \). Then the following hold.

1. The \( G_n \)-submodule of \( V_{I,k} \) generated by \( u_{I,k} \) is irreducible and isomorphic to \( V_{I,n} \).

2. The multiplicity of \( \pi_{I,n} \) in \( \pi_{I,k}|_{G_n} \) is 1.

**Remark 4.3.** From this point on, when \( n \leq k \) we will always assume that the Hilbert space \( V_{I,n} \) is realized inside \( V_{I,k} \) as the span of \( \pi_{I,k}(G_n)u_{I,k} \), in other words by identification of highest weight unit vectors. Thus we can then assume that \( u_{I,n} = u_{I,k} \). On the other hand we almost never have \( e_{I,n} = e_{I,k} \) under this inclusion. But we can always assume that \( e_{I,k} = q(k,n)e_{I,n} + e_{k,n}^\perp \) where \( q(k,n) = \langle e_{I,k}, e_{I,n} \rangle > 0 \) and \( \langle e_{I,n}, e_{k,n}^\perp \rangle = 0 \). One of our aims is to evaluate \( \langle e_{I,k}, e_{I,n} \rangle \) in terms of \( c \)-functions.

Theorem 4.2 allows us to define an inductive limit of spherical representations starting with a given spherical representation \( (\pi_{I,n}, V_{I,n}) \) of \( G_n \). We have isometric embeddings \( V_n = V_{I,n} \hookrightarrow V_{n+1} = V_{I,n+1} \) defined by the map \( u_n = u_{I,n} \mapsto u_{I,n+1} = u_{I,n+1} \). As \( u_{I,n} \) is independent of \( n \) we simply write \( u \) for the fixed highest weight vector. Then the algebraic inductive limit \( \lim V_n \) is a pre–Hilbert space with inner product \( \langle v, w \rangle = \langle v, w \rangle_{V_k} \) if \( v, w \in V_k \). This inner product is well defined as the embeddings \( V_n \hookrightarrow V_k \) are isometric. We denote by \( \mu_\infty = \lim \mu_{I,n} \in \mathfrak{i}a^*_\infty \) and \( V_\infty, \mu_\infty = V_\infty \) the Hilbert space completion of \( \lim V_n \). Notice that \( u \in V_\infty \).

The following lemma is a special case of results proved in [5, Theorem 1] and [10, §1.17].
Lemma 4.4. The action of $G_\infty$ on $\lim V_n$ is an algebraically irreducible representation, and the action of $G_\infty$ on $V_\infty$ is an irreducible unitary representation.

The main result in this article is the following theorem:

Theorem 4.5. Let the notation be as above and assume that $\mu \neq 0$. Then $V^K_{\infty} \neq \{0\}$ if and only if the ranks of the compact riemannian symmetric spaces $M_n$ are bounded. Thus for the limit of the classical compact symmetric spaces in table II, $V^K_{\infty} \neq \{0\}$ only for the finite rank infinite Grassmann manifolds $SO(p + \infty)/SO(p) \times SO(\infty)$ (real), $SU(p + \infty)/SU(p) \times U(\infty)$ (complex) and $Sp(p + \infty)/Sp(p) \times Sp(\infty)$ (quaternionic) where $0 < p < \infty$.

Here is our strategy. First, if $V^K_{\infty} \neq \{0\}$ let $e_\infty \in V^K_{\infty}$ be a unit vector. Then consider the projection $\text{proj}_{\infty,n}(e_\infty) \in V^K_n \setminus \{0\}$. Let $e_n = \text{proj}_{\infty,n}(e_\infty)/\|\text{proj}_{\infty,n}(e_\infty)\|$. Then $\{e_n\}$ is a Cauchy sequence such that $e_n$ is a unit vector in $V^K_n$ and $\{e_n\} \to e_\infty$. On the other hand if $\{e_n\}$ is a Cauchy sequence in $V_\infty$ such that $e_n \in V^K_n$ and $\|e_n\| = 1$, then $e_\infty = \lim e_n$ is a nonzero element of $V^K_{\infty}$. Thus $V^K_{\infty} \neq \{0\}$ if and only if we can find a Cauchy sequence $\{e_n\}$ such that $e_n \in V^K_n$ is a unit vector.

Recursively choose $K_{n+1}$-fixed unit vectors $e_{n+1}$ so that the orthogonal projection of $V_{n+1}$ onto $V_n$ sends $e_{n+1}$ to a positive real multiple of $e_n$ as mentioned before. Then $\text{proj}_{m,n}(e_m) = q(m,n)e_n$ for $m \geq n$ where the $q(m,n)$ are real with $0 < q(m,n) \leq 1$. Since $\text{proj}_{m,\ell} = \text{proj}_{n,\ell} \circ \text{proj}_{m,n}$ we have $q(n,\ell)q(m,n) = q(m,\ell)$. Furthermore, for a fixed $n$ the function $m \mapsto q(m,n)$ is decreasing. Also, choosing $e_1$ such that $\langle u, e_1 \rangle$ is positive real we have $\langle u, e_n \rangle$ positive real for all $n$.

Theorem 4.6. Let $m \geq n$. Then $\langle e_m, e_n \rangle = \sqrt{c_m(\mu_m + \rho_m)/c_n(\mu_n + \rho_n)}$.

Proof. To simplify the notation we write $\mu$ for both $\mu_m$ and $\mu_n$ and $c_n(\mu_n + \rho_n)$ for $c_n(\mu_n + \rho_n)$. From Theorem 3.4 we have $e_m = c_m(\mu + \rho)^{-1/2} \int_{K_m} \pi_m(k) u \, dk$ and similarly for $e_n$. So

$$\langle e_m, e_n \rangle = (c_m(\mu + \rho)c_n(\mu + \rho))^{-1/2} \int_{K_m} \int_{K_n} \langle \pi_m(h)u, \pi_n(k)u \rangle \, dk \, dh$$

$$= (c_m(\mu + \rho)c_n(\mu + \rho))^{-1/2} \int_{K_m} \int_{K_n} \langle \pi_m(k^{-1}h)u, u \rangle \, dh \, dk$$

$$= (c_m(\mu + \rho)c_n(\mu + \rho))^{-1/2} \int_{K_m} \langle \pi_m(h)u, u \rangle \, dh \quad \text{as } K_n \subseteq K_m$$

$$= (c_m(\mu + \rho)c_n(\mu + \rho))^{-1/2} c_m(\mu + \rho)$$

$$= \sqrt{c_m(\mu + \rho)/c_n(\mu + \rho)}$$

as asserted.

Theorem 4.7. The limit $\lim_{m \to \infty} c(\mu_m + \rho_m)$ exists and is non-negative. Let the sequence $\{e_n\}$ be as before. Then $\{e_n\}$ converges to a nonzero element $e \in V^K_{\infty}$ if and only if $\lim c_m(\mu_m + \rho_m) > 0$.
Proof. We start by observing
\[ \| e_m - e_n \|^2 = \| e_n \|^2 - 2 \langle e_m, e_n \rangle + \| e_m \|^2 = 2(1 - \langle e_m, e_n \rangle). \]
Hence \( \{ e_n \} \) is a Cauchy sequence if and only if
\[ \lim_{m,n \to \infty} \langle e_m, e_n \rangle = \lim_{m,n \to \infty} \sqrt{c_m(m + n)/c_n(m + n)} = 1. \]
For fixed \( n \) the sequence \( \langle e_m, e_n \rangle \geq 0 \) is decreasing and bounded below by zero. Hence \( \ell_n := \lim_m \langle e_m, e_n \rangle \) exists. This implies that the limit \( \lim_m c(m + n) \) exists and is non-negative.

The sequence \( 0 \leq \ell_n \leq 1 \) is either zero or increasing and hence \( \lim \ell_n \) exists. It follows that \( \lim_{m,n \to \infty} \sqrt{c_m(m + n)/c_n(m + n)} \) exists (and thus has to be equal to 1) if and only if \( \lim_{m \to \infty} c_m(m + n) > 0 \).

5. The Finite Rank Cases

In this section we prove Theorem 4.5 for the finite rank cases, i.e. the cases where \( M_n = \text{SO}(n)/\text{SO}(p) \times \text{SO}(q) \), \( \text{SU}(n)/\text{S(U}(p) \times \text{U}(q)) \) or \( \text{Sp}(n)/\text{Sp}(p) \times \text{Sp}(q) \) with \( p \) fixed and \( n = p + q \). We may assume \( q \geq p \), so all the \( M_n \) have the same finite rank \( p \). The cardinality of \( \Sigma_0^+ \) is constant.

We use the notation from the previous section. Recall \( d = \dim_{\mathbb{R}} \mathbb{F} \). View the real grassmannian \( \text{SO}(n)/\text{SO}(p) \times \text{SO}(q) \) as of type \( C_p \) with \( m_{\alpha_1} = 0 \) as explained after Table 2.2.

Note that \( q \to \infty \) with \( n \). Furthermore, the highest weight \( \mu = \sum_{\nu=1}^{p} k_{\nu,\xi_{\nu}} \) is independent of \( n \). Now, in view of Theorem 3.5, it suffices to prove that
\[ \frac{x_{\alpha}}{\rho_{\alpha}} = \frac{1}{4} \left( \frac{2}{\rho_{\alpha}} + \frac{m_{\alpha/2}}{\rho_{\alpha}} \right) \quad \text{and} \quad \frac{y_{\alpha}}{\rho_{\alpha}} = \frac{1}{4} \left( \frac{2m_{\alpha}}{\rho_{\alpha}} + \frac{m_{\alpha/2}}{\rho_{\alpha}} \right) \]
are bounded for all \( \alpha \).

For that we only need to consider where \( \alpha \) is in the Weyl group orbit of \( \alpha_1 \), because in all other cases \( m_{\alpha} \) and \( m_{\alpha/2} \) are bounded (in fact \( \leq 4 \)). We have
\[ m_{\alpha_1} = d - 1, \quad m_{\alpha_1/2} = d(q - p) \quad \text{and} \quad \rho_{\alpha_1} = \frac{1}{2} \left( d - 1 + \frac{(q-p)d}{2} \right). \]
Thus \( \frac{m_{\alpha_1/2}}{\rho_{\alpha_1}} \) is bounded. That completes the proof of Theorem 4.5 for the finite rank cases.

6. The Infinite Rank Cases.

In this section we prove Theorem 4.5 for the infinite rank cases, i.e., the cases where rank \( M_n \) is unbounded. We may pass to a subsequence of \( \{ M_n \} \), and then of \( \{ n \} \), and assume that rank \( M_n = n \). Now we start the proof by reducing it to the case where \( \mu_n = 1 \) in (3.11).

Lemma 6.1. Assume that \( \mu_m = \sum_{j=1}^{n} k_{j,\xi_{m,j}} \) with \( k_n > 0 \). Then
\[ c_m(m + \mu_m) \leq c_m(\xi_{m,n} + \mu_m). \]
Proof. By [4, Corollary 6.6, Ch IV] we have \( \mu_{m,n}(\log(a(n))) \leq 0 \), so \( c_m(\mu_m + \rho_m) \) is a decreasing function of \( \mu_m \).

We will also need the following well known and simple fact:

**Lemma 6.2.** Assume that \( \epsilon, \delta > 0 \). Let \( a_j \geq \epsilon \) and \( 0 \leq x_j \leq \delta \). Then

\[
\lim_{N \to \infty} \prod_{j=L}^{N} \left( 1 + \frac{a_j}{x_j+j} \right)^{-1} = 0.
\]

**Proof.** If \( x > 0 \) is small enough then \( 1 + x \leq e^{x/2} \). Hence

\[
\prod_{j=L}^{N} \left( 1 + \frac{a_j}{x_j+j} \right) \geq \exp \left( \epsilon \sum_{j=L}^{N} \frac{1}{\delta+j} \right) \to \infty \quad \text{as} \quad N \to \infty
\]

and the claim follows.

The idea of the proof of Theorem 4.5, for the unbounded rank cases, is to find a sequence of roots \( \alpha_n \) such that \( \xi_{n,k,\alpha_n} = 1 \) and \( \rho_{n,\alpha_n} \) is affine linear in \( n \). Then, if \( \alpha_n/2 \) is not a root, \( x_{\alpha_n} = 1/2 \) and the expression for \( c_{\alpha_n}(\xi_{n,k,\alpha_n} + \rho_{n,\alpha_n}) \) in (3.11) reduces to

\[
c_{\alpha_n}(\xi_{n,k,\alpha_n} + \rho_{n,\alpha_n}) = \left( 1 + \frac{y_{\alpha_n}}{\rho_{n,\alpha_n}} \right)^{-1}.
\]

It will then follow from Lemma 6.2 that

\[
\lim_{N \to \infty} \prod_{n=k}^{N} c_{\alpha_n}(\xi_{n,k,\alpha_n} + \rho_{n,\alpha_n}) = 0.
\]

That will finish the proof because \( c_{n,\alpha}(\xi_{n,k,\alpha} + \rho_{n,\alpha}) \leq 1 \) for all \( n \) and all positive roots.

In the case \( \Psi = A_n \) we let \( \alpha_n = f_{n+1} - f_1 \). Then \( \rho_{n,\alpha_n} = tn, \ t = 1/2, 1 \) or \( 2, \ \xi_{n,k,\alpha_n} = 1, \ x_{\alpha_n} = 1/2, \) and \( y_{\alpha_n} = t, \) and the claim follows by the argument indicated above.

If \( \Psi = B_n \) we take \( \alpha_n = f_n - f_1 \) when \( k \neq 1 \) and \( \alpha_n = f_n \) when \( k = 1 \). Then \( x_{\alpha_n} = 1/2, \ \xi_{n,k,\alpha_n} = 1, \ \rho_{n,\alpha_n} \) is affine linear in \( n \), and the claim follows as in the \( A_n \) case using the argument indicated above.

When \( \Psi = C_n \) we take \( \alpha_n = f_n + f_1 \). Then both the multiplicities and the \( \rho_{n,\alpha_n} \) increase affinely in \( n \), to the claim follows as above.

In the one \( D_n \) case we take \( \alpha_n = f_n + f_2 \) for \( n \) large and the same argument goes through. This completes the proof of Theorem 4.5.

---

7. The Connection to Spherical Functions on \( G_\infty \)

Finally, we discuss the connection with the theory of \( K_\infty \)-spherical functions on \( G_\infty \) as developed by Olshanskii, Faraut and coworkers. See [2] for references.
Those authors define a nonzero continuous function \( \varphi : G_\infty \to \mathbb{C} \) to be \textit{spherical} if for all \( x, y \in G_\infty \)

\[
\lim_{n \to \infty} \int_{K_n} \varphi(xky) \, dk = \varphi(x) \varphi(y).
\]  

(7.1)

By taking \( x \), respectively \( y \) to be the identity it is clear that a spherical function is \( K_\infty \)-biinvariant and takes the value 1 at the identity.

**Theorem 7.1.** Assume that rank \( G_\infty/K_\infty \) is finite and \( V_\infty = \lim V_{n,\mu} \). Let \( \{e_n\}_n \) be a Cauchy sequence in \( V_\infty \) such that for all \( n \) \( \|e_n\| = 1 \), \( e_n \in V_{n,\mu}^K \) and \( e_n \to e_\infty \in V_{\infty}^K \). Then

\[
\varphi_{\mu}(x) := \langle e_\infty, \pi_{\infty,\mu}(x)e_\infty \rangle = \lim_{n \to \infty} \langle e_n, \pi_{n,\mu}(x)e_n \rangle
\]

is a positive definite \( K_\infty \)-spherical function on \( G_\infty \) in the sense of (7.1).

**Proof.** Write \( e_\infty = e_n + e_\perp^1 \). Let \( x, y \in G_{j_o} \). Then, for \( j \geq j_o \),

\[
\varphi_{\mu}(x) = \langle e_j, \pi_j(x)e_j \rangle = \varphi_{\mu}(x) + \langle e_j^\perp, \pi_j(x)e_j^\perp \rangle
\]

because \( V_j \) and \( V_j^\perp \) are \( K_j \)-invariant. Thus

\[
|\langle e_j^\perp, \pi_j(x)e_j^\perp \rangle| \leq \|e_j^\perp\|^2 \to 0.
\]

Hence \( \varphi_{\mu}(x) \to \varphi_{\mu}(x) \), i.e., \( \varphi_{\mu} \to \varphi_{\mu} \) pointwise. Similarly, for \( x, y \in G_j \),

\[
\lim_{j \to \infty} \int_{K_j} \varphi_{\mu}(xky) \, dk = \lim_{j \to \infty} \left( \int_{K_j} \varphi_{\mu}(xky) \, dk + \int_{K_j} \langle e_j^\perp, \pi_j(xky)e_j^\perp \rangle \, dk \right)
\]

\[
= \lim_{j \to \infty} \varphi_{\mu}(x) \varphi_{\mu}(y) + \lim_{j \to \infty} \int_{K_j} \langle e_j^\perp, \pi_j(xky)e_j^\perp \rangle \, dk
\]

\[
= \varphi_{\mu}(x) \varphi_{\mu}(y)
\]

because

\[
\left| \int_{K_j} \langle e_j^\perp, \pi_j(xky)e_j^\perp \rangle \, dk \right| \leq \|e_j^\perp\|^2 \int_{K_j} \, dk = \|e_j^\perp\|^2 \to 0. \quad \blacksquare
\]

The definition and construction of spherical functions in the infinite rank case remains to be clarified.

**8. The Finite Rank Case**

In this section we discuss the case of the finite rank Grassmannian in more detail. In this case we can assume that \( a_n = a \) is fixed for all \( n \). Then \( \Sigma_n = \Sigma \) is fixed, and only the root multiplicities change as \( n \) grows.

We start with the case of the sphere. Let \( X_n = S^n = SO(n+1)/SO(n) \) where the inclusions are given by \( S^n \hookrightarrow S^{n+1}, \ u \mapsto (u, 0) \) and

\[
SO(n) \hookrightarrow SO(n+1), \ A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.
\]
Let $\mathcal{H}_n(k)$ be the space of homogeneous harmonic polynomials on $\mathbb{R}^{n+1}$ of degree $k$ with inner product
\[ \langle p, q \rangle = \frac{1}{k!} \partial_q(p)(0), \]
and let $\pi_{k,n}$ denote the natural representation of $G_n = SO(n+1)$ on $\mathcal{H}_n(k)$. Then $\pi_{k,n}$ is a spherical representation of $G_n$ and every spherical representation is constructed this way. Clearly $\mathcal{H}_n(k) \subset \mathcal{H}_m(k)$ if $n \leq m$ and the natural inclusion is an isometry. We can take $u(x) = (x_1 - ix_2)^k$ as a highest weight vector. It is independent of $n$. Use polar coordinates $\cos(\theta)e_1 + \sin(\theta)u$, $u \in S^{n-1}$. Then the $K_n$-invariant functions corresponds to functions of one variable $P(\cos \theta)$. Using the radial component of the Laplacian on $S^n$, the spherical function associated to $\mathcal{H}_n(k)$ is a solution to the initial value problem
\[ \left( \frac{d^2}{d\theta^2} + (n-1) \cot(\theta) \frac{d}{d\theta} \right) p_{n,k}(\cos \theta) = -k(k + n - 1)p_{n,k}(\cos \theta) \]
$\ p_{n,k}(1) = 1$. Putting $t = \cos(\theta)$ and dividing by $n$,
\[ \left( \frac{1-t^2}{n} \frac{d^2}{dt^2} - t \frac{d}{dt} \right) p_{n,k}(t) = -\left( \frac{k(k-1)}{n} + k \right) p_{n,k}(t), \quad p_{n,k}(1) = 1 \]
Letting $n \to \infty$ we see that the corresponding spherical function $\varphi_\infty$ is a solution to the first order differential equation
\[ t \frac{d}{dt} p_{\infty,k}(t) = kp_{\infty,k}(t), \quad p_{\infty,k}(1) = 1 \]
Thus $\varphi_{\infty,k}(t) = t^k$ or in coordinates on the sphere $\varphi_{\infty,k}(x) = x_1^k$. According to [2], p. 11, every $K_\infty$-spherical function on $G_\infty$ is given by
\[ g \mapsto \langle e_1, g(e_1) \rangle^k \text{ for some integral } k \geq 0. \]
Thus all spherical functions or $G_\infty$ are constructed as in our limit theorem. In particular, all irreducible spherical representations are obtained by the limit construction. The question is whether that is also the case for the other finite rank Grassmannians. We have been informed that this question has been answered affirmative in [16]: Every spherical function in the sense of Olshanskii is a limit of spherical functions on $X_n$ obtained by letting the root multiplicities go to infinity. Combining our construction with this result gives the following theorem, which in principle states that the theory of highest weights remains valid for the finite rank case.

**Theorem 8.1.** Assume that the rank of $X_\infty$ is finite. Let $\pi$ be an unitary irreducible $K_\infty$-spherical representation of $G_\infty$. Then there exists $\mu \in \Lambda^+$ such that $\pi \simeq \pi_{\mu,\infty}$. In particular the set of equivalence classes of unitary $K_\infty$-spherical representation of $G_\infty$ is isomorphic to $\Lambda^+$. 
This gives us in particular a natural embedding of $V_{\mu,\infty}$ into $C_b(X_\infty)$, the space of bounded continuous functions on $X_\infty$ by $u \mapsto \langle u, \pi_{\mu,\infty}(x)e_{\mu,\infty} \rangle$. We can then ask the question: If $(\pi, V)$ is a unitary irreducible representation of $G_\infty$ with $V \subset C_b(X_\infty)$ does there exists a $\mu \in \Lambda^+$ such that $(\pi, V) \simeq (\pi_{\mu,\infty}, V_{\mu,\infty})$?

It is also natural to ask what happens in the infinite rank case. For that we would like to point the following out. Fix $\mu \in \Lambda^+$ and $e_n^* \in (V_n^*)^{K_n}$, $\|e_n\| = 1$. Then construct the inductive sequence $(\pi_m, V_m)$ as before. Let $e_m^* \in (V_m^*)^{K_m}$ be so that the projection of $e_m^*$ onto $V_n^*$ is our fixed $K_n$-invariant functional $e_n^*$. Then the sequence $\{e_m^*\}$ defines an element $e_\infty^* \in \varprojlim V_m^* \simeq (\varprojlim V_m)^*$, where the limit is now with respect to the inductive/projective limit topology and not in the category of Hilbert spaces. In particular, $e_\infty^*$ defines a linear form on $\bigcup V_n$ given by $u \mapsto \langle u, e_n^* \rangle$ if $u \in V_n$. It therefore defines a linear $G_\infty$-equivariant embedding of $\bigcup V_n$ into $C_b(X_\infty)$.

Restricting $\pi_{\mu,\infty}$ to $G_n$ gives a unitary representation of $G_n$. Let $V_{\mu,\infty}^{n,\infty}$ denote the space of smooth vectors for this representation and let

$$V_{\mu,\infty}^\infty = \bigcap_n V_{\mu,\infty}^{n,\infty}.$$ 

Then $V_{\mu,\infty}^\infty$ is a locally convex topological vector space in the usual way. A continuous linear form $\nu : V_{\mu,\infty}^\infty \to \mathbb{C}$ is a distribution vector. We denote the space of distribution vectors by $V_{\mu,\infty}^{-\infty}$. The question now is whether $e_\infty^* : v \mapsto \langle v, e_\infty^* \rangle$ is a distribution vector.
References

[1] Araki, S., *On root systems and an infinitesimal classification of irreducible symmetric spaces*, J. Math. Osaka City Univ. **13** (1962), 1–34.

[2] Faraut, J., “Infinite Dimensional Spherical Analysis,” COE Lecture Note **10** (2008), Kyushu University.

[3] Helgason, S., “Differential Geometry, Lie Groups, and Symmetric Spaces,” Academic Press, New York, 1978.

[4] —, “Groups and Geometric Analysis”, Academic Press, New York, 1984.

[5] Kolomycev, V. I., and Ju. S. Samoilenko, *On irreducible representations of inductive limits of groups*, Ukrainian Math. J. **29** (1977), 402–405.

[6] Natarajan, L., E. Rodríguez-Carrington, and J. A. Wolf, *Differentiable structure for direct limit groups*, Lett. Mat., Physics **23** (1991), 99–109.

[7] —, *Locally convex Lie groups*, Nova J. Algebra and Geometry, **2** (1993), 59–87.

[8] —, *The Bott-Borel-Weil theorem for direct limit groups*, Trans. Amer. Math. Soc. **353** (2001), 4583–4622.

[9] Olshanskii, G., *Unitary representations of the infinite-dimensional classical groups U(p,∞), SO_o(p,∞), Sp(p,∞), and of the corresponding motion groups*, Functional Anal. Appl. **12** (1978), 185–195.

[10] —, *Unitary representations of infinite dimensional pairs (G,K) and the formalism of R. Howe*, in: A. M. Vershik, and D. P. Zhelobenko, Eds., “Representations of Lie Groups and Related Topics,” Adv. Stud. Contemp. Math. **7**, Gordon and Breach, 1990.

[11] Ólafsson, G., and A. Pasquale, *Ramanujan’s Master Theorem for Riemannian Symmetric Spaces*, J. Funct. Anal. **262** (2012), 4851–4890.

[12] Ólafsson, G., and J. A. Wolf, *Weyl group invariants and application to spherical harmonic analysis on symmetric spaces*, Preprint, {arXiv:0901.4765}.

[13] —, *The Paley-Wiener Theorem and limits of symmetric spaces*, J. Geom. Anal., to appear.

[14] —, *Extension of symmetric spaces and restriction of Weyl groups and invariant polynomials*, in: “New Developments in Lie Theory and Its Applications,” Contemporary Mathematics **544** (2011), 85–100.

[15] Pickrell, D., *Separable representations for automorphism groups of infinite symmetric spaces*, J. Funct. Anal. **90** (1990), 1–26.

[16] Rösler, M., T. Koornwinder, and M. Voit, *Limit transition between hypergeometric functions of type BC and type A* {arXiv:1207.0487}.
[17] Strătilă, S., and D. Voiculescu, *A survey of the representations of the unitary group $U(\infty)$*, in: “Spectral Theory,” Banach Center Publ. 8, Warsaw, 1982.

[18] Wolf, J. A., *Infinite dimensional multiplicity free spaces I: Limits of compact commutative spaces*, in: K.-H. Neeb and A. Pianzola, Eds., “Developments and Trends in Infinite Dimensional Lie Theory,” Progress in Math. 288, Birkhäuser 2011, 459–481.

[19] —, *Infinite dimensional multiplicity free spaces II: Limits of commutative nilmanifolds*, Contemporary Mathematics 491 (2009), 179–208.

[20] —, *Infinite dimensional multiplicity free spaces III: Matrix coefficients and regular function*, Math. Ann., 349 (2011), 263–299.