Almost $\eta$-Ricci and almost $\eta$-Yamabe solitons with torse-forming potential vector field

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Abstract

We provide properties of almost $\eta$-Ricci and almost $\eta$-Yamabe solitons on submanifolds isometrically immersed into a Riemannian manifold $\left(\tilde{M}, \tilde{g}\right)$ whose potential vector field is the tangential component of a torse-forming vector field on $\tilde{M}$, treating also the case of a minimal or pseudo quasi-umbilical hypersurface. Moreover, we give necessary and sufficient conditions for an orientable hypersurface of the unit sphere to be an almost $\eta$-Ricci or an almost $\eta$-Yamabe soliton in terms of the second fundamental tensor field.

1 Introduction

In 1982, R. S. Hamilton introduced the intrinsic geometric flows, Ricci flow \[24\]
\[
\frac{\partial}{\partial t} g(t) = -2 \text{Ric}(g(t))
\]
and Yamabe flow \[23\]
\[
\frac{\partial}{\partial t} g(t) = -\text{scal}(t) \cdot g(t)
\]
which are evolution equations for Riemannian metrics. In a 2-dimensional manifold, Ricci flow and Yamabe flow are equivalent, but for higher dimensions, we do not have such a relation.

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Almost $\eta$-Ricci and $\eta$-Yamabe solitons

Ricci solitons and Yamabe solitons correspond to self-similar solutions of Ricci flow and Yamabe flow, respectively. Therefore, on an $n$-dimensional smooth manifold $M$, a Riemannian metric $g$ and a non-vanishing vector field $V$ is said to define a Ricci soliton \cite{23} if there exists a real constant $\lambda$ such that

\begin{equation}
\frac{1}{2} \mathcal{L}_V g + \text{Ric} = \lambda g,
\end{equation}

respectively, a Yamabe soliton \cite{23} if there exists a real constant $\lambda$ such that

\begin{equation}
\frac{1}{2} \mathcal{L}_V g = (\text{scal} - \lambda) g,
\end{equation}

where $\mathcal{L}_V$ denotes the Lie derivative operator in the direction of the vector field $V$, Ric and scal denote the Ricci curvature tensor field and respectively the scalar curvature of $g$. A Ricci soliton (or a Yamabe soliton) $(V, \lambda)$ on a Riemannian manifold $(M, g)$ is said to be shrinking, steady or expanding according as $\lambda$ is positive, zero or negative, respectively.

Remark that Ricci solitons are natural generalizations of Einstein metrics, any Einstein metric giving a trivial Ricci soliton.

Different generalizations of Ricci and Yamabe solitons have been lately considered. If $\lambda$ is a smooth function on $M$, then (3) defines an almost Ricci soliton \cite{27} and (4) defines an almost Yamabe soliton \cite{1}. Moreover, for a given 1-form $\eta$ on $M$, if there exist two real constants $\lambda$ and $\mu$ such that

(i)

\begin{equation}
\frac{1}{2} \mathcal{L}_V g + \text{Ric} = \lambda g + \mu \eta \otimes \eta
\end{equation}

we call $(V, \lambda, \mu)$ an $\eta$-Ricci soliton \cite{18}, and if

(ii)

\begin{equation}
\frac{1}{2} \mathcal{L}_V g = (\text{scal} - \lambda) g + \mu \eta \otimes \eta
\end{equation}

we call $(V, \lambda, \mu)$ an $\eta$-Yamabe soliton \cite{16}. If $\lambda$ and $\mu$ are smooth functions on $M$, then (5) defines an almost $\eta$-Ricci soliton \cite{5} and (6) defines an almost $\eta$-Yamabe soliton \cite{26}.

If the potential vector field $V$ is of gradient type, $V = \text{grad}(\sigma)$, for $\sigma$ a smooth function on $M$, then $(V, \lambda, \mu)$ is called a gradient soliton (see \cite{23}, \cite{27}, \cite{18} and \cite{5}). If $\sigma$ is a constant, then the gradient soliton $(V, \lambda, \mu)$ is trivial.
Motivated by the above studies, in the present paper, we establish some properties of almost $\eta$-Ricci and almost $\eta$-Yamabe solitons on Riemannian manifolds and on submanifolds isometrically immersed into a Riemannian manifold $\left(\tilde{M}, \tilde{g}\right)$ whose potential vector field is the tangential component of a torse-forming (in particular, concircular or recurrent) vector field on $\tilde{M}$, treating also the case of a minimal or pseudo quasi-umbilical hypersurface. Moreover, we provide necessary and sufficient conditions for an orientable hypersurface of the unit sphere to be an almost $\eta$-Ricci or an almost $\eta$-Yamabe soliton in terms of the second fundamental tensor field. A partial study on this topic has been begun in [4]. Remark also that almost Yamabe solitons on submanifolds were studied in [30].
2 Preliminaries

A non-flat Riemannian manifold \((M, g)\) \((n \geq 3)\) is called

a) \textit{mixed generalized quasi-Einstein manifold} \([3]\) if its Ricci tensor field is not identically zero and verifies

\begin{equation}
\text{Ric} = \alpha g + \beta A \otimes A + \gamma B \otimes B + \delta (A \otimes B + B \otimes A),
\end{equation}

where \(\alpha, \beta, \gamma\) and \(\delta\) are smooth functions and \(A, B\) are 1-forms on \(M\) such that the corresponding vector fields to the 1-forms \(A\) and \(B\) are \(g\)-orthogonal. In particular, the manifold \(M\) is called:

i) \textit{generalized quasi-Einstein} \([6]\) if \(\delta = 0\);

ii) \textit{almost quasi-Einstein} \([9]\) if \(\beta = \gamma = 0\);

iii) \textit{quasi-Einstein} \([7]\) if \(\beta = \delta = 0\) or \(\gamma = \delta = 0\);

iv) \textit{Einstein} \([2]\) if \(\beta = \gamma = \delta = 0\);

b) \textit{pseudo quasi-Einstein} \([25]\) if its Ricci tensor field is not identically zero and verifies

\[
\text{Ric} = \alpha g + \beta A \otimes A + \gamma E,
\]

where \(\alpha, \beta\) and \(\gamma\) are smooth functions, \(A\) is a 1-form and \(E\) is a symmetric \((0,2)\)-tensor field with vanishing trace on \(M\). In particular, if \(\gamma = 0\), then the manifold \(M\) is \textit{quasi-Einstein}.

A vector field \(V\) on a (pseudo)-Riemannian manifold \((M, g)\) is called \textit{torse-forming} \([32]\) if

\[
\nabla_X V = aX + \psi(X)V,
\]

where \(a\) is a smooth function, \(\psi\) is a 1-form and \(\nabla\) is the Levi-Civita connection of \(g\). Moreover, if \(\psi(V) = 0\), then \(V\) is called a \textit{torqued} vector field.

In particular, if \(\psi = 0\), \(V\) is called a \textit{concircular vector field} \([22]\) and if \(a = 0\), \(V\) is called a \textit{recurrent vector field} \([28]\).
3 Solitons with torse-forming potential vector field

3.1 Almost $\eta$-Ricci solitons

Remark that any concircular vector field with $a(x) \neq 0$, for any $x \in M$, is of gradient type, namely

$$V = \frac{1}{2a} \text{grad}(|V|^2)$$

whose divergence is $\text{div}(V) = an$, with $n = \text{dim}(M)$. Moreover,

$$R(X,V)V = X(a)V - V(a)X$$

for any $X \in \chi(M)$ and

$$\text{Ric}(V,V) = (1 - n)V(a).$$

If $(M,g)$ is an almost $\eta$-Ricci soliton with the potential vector field $V$ and $\eta$ is the $g$-dual of $V$, then

$$\text{Ric} = (\lambda - a)g + \mu \eta \otimes \eta,$$

$$\text{scal} = (n - 1) \left[ (\lambda - a) - \frac{V(a)}{|V|^2} \right]$$

and we can state:

**Proposition 3.1.** If a Riemannian manifold $(M,g)$ is an almost $\eta$-Ricci soliton $(V,\lambda,\mu)$ with concircular potential vector field $V$ and $a$ is a non zero constant, $\eta$ is the $g$-dual of $V$, then

i) $M$ is a quasi-Einstein manifold with associated functions $(\lambda - a)$ and $\mu$;

ii) $\text{grad}(\lambda)$, $\text{grad}(\mu)$ and $\text{grad}(\text{scal})$ are collinear with $V$.

**Proof.** Since $R(X,V)V = 0$, for any $X \in \chi(M)$, we get $\text{Ric}(V,V) = 0$ and

$$(\lambda - a) + |V|^2\mu = 0.$$  

Differentiating the previous relation, using $d\text{scal} = 2\text{div}(\text{Ric})$ and taking into account that

$$(\text{div}(\mu(\eta \otimes \eta)))(X) = \mu g(\nabla_VV,X) + \mu \eta(X)\text{div}(V) + \eta(X)d\mu(V),$$

we get:

$$(n - 3)\text{grad}(\lambda) = 2[(n + 1)a\mu + g(\text{grad}(\mu),V)]V$$
and applying the gradient to the same relation, we have:

$$\text{grad}(\lambda) = -2\mu V - \text{grad}(\mu)|V|^2,$$

which combined give:

$$\text{grad}(\mu) = -\frac{2}{(n-3)|V|^2}[2(n-2)\mu + g(\text{grad}(\mu),V)]V.$$

Taking now the inner product with $V$, we get:

$$g(\text{grad}(\mu),V) = -\frac{4(n-2)}{n-1}\mu,$$

therefore

$$\text{grad}(\mu) = -\frac{4(n-2)}{n-1}\mu\frac{V}{|V|^2}.$$

Also, we get

$$\text{grad}(\lambda) = \frac{2(n-3)}{n-1}\mu V$$

and

$$\text{grad}(\text{scal}) = 2(n-3)\mu V.$$

If $M$ has constant scalar curvature and $n > 3$, then $\mu = 0$, $\lambda = a$, Ric = 0 and scal = 0, therefore:

**Corollary 3.2.** Under the hypotheses of Proposition [3.4], if $M$ is of constant scalar curvature and $n > 3$, then $M$ is a Ricci-flat manifold.

For the almost Ricci solitons, we can state:

**Proposition 3.3.** If an $n$-dimensional Riemannian manifold $(M, g)$, with $n > 3$, is an almost Ricci soliton $(V, \lambda, \mu)$ with concircular potential vector field $V$ and $a$ is a non zero constant, then $M$ is a Ricci-flat manifold.

**Proposition 3.4.** If a Riemannian manifold $(M, g)$ is an almost $\eta$-Ricci soliton $(V, \lambda, \mu)$ with torqued potential vector field $V$ and $\eta$ is the $g$-dual of $V$, then $M$ is a mixed generalized quasi-Einstein manifold with associated functions $(\lambda - a), \mu, 0$ and $-\frac{1}{2}$. 
Proof. From the condition for $V$ to be torqued, we get
\[ \frac{1}{2}(\mathcal{L}_V g)(X,Y) = \frac{1}{2}[g(\nabla_X V, Y) + g(\nabla_Y V, X)] = ag(X,Y) + \frac{1}{2}[\psi(X)\eta(Y) + \eta(X)\psi(Y)] \]
and from the soliton equation, we obtain
\[ \text{Ric}(X,Y) = (\lambda - a)g(X,Y) + \mu\eta(X)\eta(Y) - \frac{1}{2}[\psi(X)\eta(Y) + \eta(X)\psi(Y)]. \]

Let $U$ be the $g$-dual vector field of $\psi$. Then
\[ \eta(U) = \psi(V) = 0, \]
hence the conclusion.

Corollary 3.5. Under the hypotheses of Proposition 3.4, if $\psi = \mu\eta$, then $M$ is an Einstein manifold. In this case:

i) $V$ is a geodesic vector field if and only if $\mu = -\frac{a}{|V|^2}$;

ii) $V$ is concircular and the soliton is given by $(V, a, 0)$.

Proof. From $\psi(V) = \mu\eta(V)$, we have
\[ \nabla_V V = (a + \mu|V|^2)V, \]
hence the two statements.

If the potential vector field is torse-forming, we get the following two results for almost $\eta$-Ricci solitons similar to those proved in [15] for Ricci solitons with concurrent potential vector field.

Theorem 3.6. Let $(M, g)$ be an $n$-dimensional Riemannian manifold and let $V$ be a torse-forming vector field satisfying $\text{Ric}_M(V,V) = 0$. Then $(V, \lambda, \mu)$ is an almost $\eta$-Ricci soliton with $\eta$ the $g$-dual of $V$ if and only if the following conditions hold:

i) the soliton is given by $\left(\lambda, \psi(V) - \frac{\lambda - a}{|V|^2}\right)$;

ii) $M$ is an open part of a warped product manifold $(I \times_s F, ds^2 + s^2g_F)$, where $I$ is an open real interval with arclength $s$ and $F$ is an $(n-1)$-dimensional Einstein manifold with $\text{Ric}_F = (n-2)g_F$. 
Almost $\eta$-Ricci and $\eta$-Yamabe solitons

**Proof.** Following the same steps like in [15], we get ii). In our case

$$\frac{1}{2}L_V g = ag + \frac{1}{2}(\psi \otimes \eta + \eta \otimes \psi)$$

and from [3], we have:

$$\text{Ric}_M = (\lambda - a)g + \mu \eta \otimes \eta - \frac{1}{2}(\psi \otimes \eta + \eta \otimes \psi).$$

By using $\text{Ric}_M(V, V) = 0$, we obtain i). 

For almost Ricci solitons with concurrent potential vector field, we can state:

**Proposition 3.7.** An $n$-dimensional Riemannian manifold $(M, g)$ is an almost Ricci soliton $(V, \lambda)$ with concurrent potential vector field $V$ if and only if the following conditions hold:

i) the soliton is a shrinking Ricci soliton with $\lambda = 1$;

ii) $M$ is an open part of a warped product manifold $(I \times_s F; ds^2 + s^2 g_F)$, where $I$ is an open real interval with arclength $s$ and $F$ is an $(n-1)$-dimensional Einstein manifold with $\text{Ric}_F = (n-2)g_F$.

Therefore, there do not exist proper almost Ricci solitons (i.e. with non-constant $\lambda$) with concurrent potential vector field.

It was proved in [10] that the gradient of a non-constant smooth function $\sigma$ on a Riemannian manifold $(M, g)$ is a torse-forming vector field with

$$\nabla_X (\text{grad}(\sigma)) = aX + \psi(X) \text{grad}(\sigma)$$

if and only if

$$\text{Hess}(\sigma) = ag + \delta d\sigma \otimes d\sigma,$$

where $\psi = \delta d\sigma$ and we can state:

**Proposition 3.8.** If a Riemannian manifold $(M, g)$ is an almost $\eta$-Ricci soliton $(V, \lambda, \mu)$ with torse-forming potential vector field $V = \text{grad}(\sigma)$ and $\eta$ is the $g$-dual of $V$, then $M$ is a quasi-Einstein manifold with associated functions $(\lambda - a)$ and $(\mu - \delta)$.

**Proof.** From [3], we get

$$\text{Ric} = (\lambda - a)g + (\mu - \delta)d\sigma \otimes d\sigma.$$  

So $M$ is a quasi-Einstein manifold with associated functions $(\lambda - a)$ and $(\mu - \delta)$. □
Corollary 3.9. With the above notations, if a Riemannian manifold \((M, g)\) is an almost \(\eta\)-Ricci soliton \((V, a, \delta)\) with torse-forming potential vector field \(V\) of gradient type and \(\eta\) is the \(g\)-dual of \(V\), then \(M\) is a Ricci-flat manifold.

Example 3.10. Let \(M = \{(x, y, z) \in \mathbb{R}^3, z > 0\}\), where \((x, y, z)\) are the standard coordinates in \(\mathbb{R}^3\). Set
\[
V := -z \frac{\partial}{\partial z}, \quad \eta := -\frac{1}{z} dz,
\]
\[
g := \frac{1}{z^2} (dx \otimes dx + dy \otimes dy + dz \otimes dz).
\]

Consider the linearly independent system of vector fields:
\[
E_1 := z \frac{\partial}{\partial x}, \quad E_2 := z \frac{\partial}{\partial y}, \quad E_3 := -z \frac{\partial}{\partial z}.
\]

Then:
\[
\eta(E_1) = 0, \quad \eta(E_2) = 0, \quad \eta(E_3) = 1,
\]
\[
[E_1, E_2] = 0, \quad [E_2, E_3] = E_2, \quad [E_3, E_1] = -E_1
\]
and the Levi-Civita connection \(\nabla\) is deduced from Koszul’s formula
\[
2 g(\nabla_X Y, Z) = X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) -
\]
\[
-g([X, Y], Z) + g([Y, Z], X) + g([Z, X], Y),
\]
precisely
\[
\nabla_{E_1} E_1 = -E_3, \quad \nabla_{E_1} E_2 = 0, \quad \nabla_{E_1} E_3 = E_1, \quad \nabla_{E_2} E_1 = 0,
\]
\[
\nabla_{E_2} E_2 = -E_3, \quad \nabla_{E_2} E_3 = E_2, \quad \nabla_{E_3} E_1 = 0, \quad \nabla_{E_3} E_2 = 0, \quad \nabla_{E_3} E_3 = 0.
\]

Then the Riemann and the Ricci curvature tensor fields are given by:
\[
R(E_1, E_2) E_2 = -E_1, \quad R(E_1, E_3) E_3 = -E_1, \quad R(E_2, E_1) E_1 = -E_2,
\]
\[
R(E_2, E_3) E_3 = -E_2, \quad R(E_3, E_1) E_1 = -E_3, \quad R(E_3, E_2) E_2 = -E_3,
\]
\[
\text{Ric}(E_1, E_1) = \text{Ric}(E_2, E_2) = \text{Ric}(E_3, E_3) = -2.
\]

Writing the \(\eta\)-Ricci soliton equation in \((E_i, E_i)\) we obtain:
\[
g(\nabla_{E_i} E_3, E_i) + \text{Ric}(E_i, E_i) = \lambda g(E_i, E_i) + \mu \eta(E_i) \eta(E_i),
\]
for all \(i \in \{1, 2, 3\}\) and we get that for \(\lambda = \mu = -1\), the data \((V, \lambda, \mu)\) define an \(\eta\)-Ricci soliton on \((M, g)\). Moreover, it is a gradient \(\eta\)-Ricci soliton, since the potential vector field \(V\) is a torse-forming vector field of gradient type, \(V = \text{grad}(f)\), where \(f(x, y, z) := -\ln z\).
Example 3.11. Let $M = \mathbb{R}^3$, $(x, y, z)$ be the standard coordinates in $\mathbb{R}^3$ and $g$ be the Lorentzian metric:

$$g := e^{-2z}dx \otimes dx + e^{2x-2z}dy \otimes dy - dz \otimes dz.$$ 

Consider the vector field $V$ and the 1-form $\eta$:

$$V := \frac{\partial}{\partial z}, \quad \eta := dz.$$ 

For the orthonormal vector fields:

$$E_1 = e^z \frac{\partial}{\partial x}, \quad E_2 = e^{z-2} \frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z},$$

we get:

$$\nabla_{E_1} E_1 = -E_3, \quad \nabla_{E_1} E_2 = 0, \quad \nabla_{E_1} E_3 = -E_1, \quad \nabla_{E_2} E_1 = e^z E_2,$$

$$\nabla_{E_2} E_2 = -e^z E_1 - E_3, \quad \nabla_{E_2} E_3 = -E_2, \quad \nabla_{E_3} E_1 = 0, \quad \nabla_{E_3} E_2 = 0, \quad \nabla_{E_3} E_3 = 0$$

and the Riemann tensor field and the Ricci tensor field are given by:

$$R(E_1, E_2)E_2 = (1 - e^{2z}) E_1, \quad R(E_1, E_3)E_3 = -E_1, \quad R(E_2, E_1)E_1 = (1 - e^{2z}) E_2,$$

$$R(E_2, E_3)E_3 = -E_2, \quad R(E_3, E_1)E_1 = E_3, \quad R(E_3, E_2)E_2 = E_3,$$

$$\text{Ric}(E_1, E_1) = \text{Ric}(E_2, E_2) = 2 - e^{2z}, \quad \text{Ric}(E_3, E_3) = -2.$$ 

Then the data $(V, \lambda, \mu)$, for $\lambda = 1 - e^{2z}$ and $\mu = -1 - e^{2z}$, define an almost $\eta$-Ricci soliton on $(M, g)$. Moreover, it is a gradient almost $\eta$-Ricci soliton, since the potential vector field $V$ is of gradient type, $V = \text{grad}(f)$, where $f(x, y, z) := -z$.

3.2 Almost $\eta$-Yamabe solitons

Assume that $V$ is a torse-forming vector field on an $n$-dimensional Riemannian manifold $(M, g)$, $\nabla_X V = aX + \psi(X)V$, for any $X \in \chi(M)$, where $\nabla$ is the Levi-Civita connection of $g$. If $\eta$ is the $g$-dual 1-form of $V$ and $U$ is the $g$-dual vector field of $\psi$, then

$$\eta(U) = \psi(V)$$

and

$$\text{div}(V) = an + \eta(U) = an + \psi(V).$$
Also
\[ \text{div}(\psi \otimes \eta + \eta \otimes \psi) = \text{div}(V)\psi + \text{div}(U)\eta + i_{\nabla V}g + i_{\nabla U}y. \]

Since
\[ \nabla U = aU + |U|^2 V, \]
we obtain:
\[ \text{div}(\psi \otimes \eta + \eta \otimes \psi) = \left[ (n+1)a + \psi(V) \right] \psi + \left[ |U|^2 + \text{div}(U) \right] \eta + i_{\nabla V}g. \]

Let \((V, \lambda, \mu)\) be an almost \(\eta\)-Yamabe soliton with \(\eta\) the \(g\)-dual 1-form of the torse-forming vector field \(V\). Then
\[ \frac{1}{2}(\psi \otimes \eta + \eta \otimes \psi) = (\text{scal} - \lambda - a)g + \mu \eta \otimes \eta \]
and taking the divergence, we get
\[ (8) \quad d(\text{scal} - \lambda - a) = \frac{1}{2} i_{\nabla V}g \]
\[ + \frac{1}{2} \left[ (n+1)a + \psi(V) \right] \psi + \left[ |U|^2 \right] + \frac{\text{div}(U)}{2} - (\mu + n)a - (\mu + 1)\psi(V) - V(\mu) \right] \eta. \]

PROPOSITION 3.12. Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold and let \((V, \lambda, \mu)\) be an almost \(\eta\)-Yamabe soliton with \(\eta\) the \(g\)-dual 1-form of the torse-forming vector field \(V\). Then \((V, \lambda, \mu)\) is an almost Yamabe soliton with \(\lambda = \text{scal} - \frac{1}{2|V|^2} V(|V|^2) \) or \(\left[ \psi(V) \right]^2 = |V|^2 \cdot |U|^2 \).

PROOF. From the soliton equation (8), taking \(X = Y = V\), we get
\[ (9) \quad \text{scal} - \lambda - a - \psi(V) + \mu |V|^2 = 0 \]
and taking \(X = Y = U\), we obtain
\[ (10) \quad [\text{scal} - \lambda - a - \psi(V)] |U|^2 + \mu [\psi(V)]^2 = 0. \]

Replacing (9) in (10), we get:
\[ -\mu |V|^2 |U|^2 + \mu [\psi(V)]^2 = 0, \]
which implies \(\mu = 0\) (which yields an almost Yamabe soliton) or \([\psi(V)]^2 = |V|^2 \cdot |U|^2 \).
Almost $\eta$-Ricci and $\eta$-Yamabe solitons

If $\mu = 0$, from (9), we get

$$\text{scal} - \lambda - a - \psi(V) = 0$$

and since

$$\frac{1}{2}V(|V|^2) = g(\nabla_V V, V) = [a + \psi(V)]|V|^2,$$

we obtain $\lambda = \text{scal} - \frac{1}{2|V|^2}V(|V|^2).$ 

**Proposition 3.13.** Let $(M, g)$ be an $n$-dimensional Riemannian manifold and let $(V, \lambda, \mu)$ be an almost $\eta$-Yamabe soliton with $\eta$ the $g$-dual 1-form of the concircular vector field $V$. Then $V$ is $\nabla$-parallel or the soliton is given by

$$(\lambda, \mu) = (\text{scal} - a + n|V|^2, n).$$

**Proof.** Taking $\psi = 0$ and $U = 0$ in (8), we get

$$d(\text{scal} - \lambda - a) = [-(\mu + n)a - V(\mu)]\eta$$

and by differentiating (9), we obtain:

$$d(\text{scal} - \lambda - a) = -\mu d(|V|^2) - |V|^2 d\mu.$$

Replacing the second relation in the previous one and computing it in $V$, we get

$$\mu V(|V|^2) = (\mu + n)a|V|^2.$$

Also

$$V(|V|^2) = 2g(\nabla_V V, V) = 2a|V|^2,$$

which combined with the previous relation implies either $a = 0$ (i.e. $V$ is $\nabla$-parallel) or $\mu = n$, which from (9) gives $\lambda = \text{scal} - a + n|V|^2.$ 

**Proposition 3.14.** Let $(M, g)$ be an $n$-dimensional mixed generalized quasi-Einstein manifold and let $(V, \lambda, \mu)$ be an almost $\eta$-Yamabe soliton with $\eta$ the $g$-dual 1-form of the torqued vector field $V$. If the Ricci tensor field of $M$ is of the form $\text{Ric} = \alpha g + \beta \eta \otimes \eta + \gamma (\eta \otimes \psi + \psi \otimes \eta)$, then

$$(11) \quad \lambda = \left(\beta + \frac{\mu}{n}\right)|V|^2 + \alpha n - a.$$
PROOF. Since $V$ is a torqued vector field and $\eta$ the $g$-dual 1-form of $V$, we have

$$(\mathcal{L}_V g)(X, Y) = 2ag(X, Y) + \eta(X)\psi(Y) + \psi(X)\eta(Y).$$

On the other hand,

$$\text{Ric}(X, Y) = \alpha g(X, Y) + \beta\eta(X)\eta(Y) + \gamma[\eta(X)\psi(Y) + \psi(X)\eta(Y)],$$

gives us

$$\text{scal} = \alpha n + \beta |V|^2 + 2\gamma \psi(V) = \alpha n + \beta |V|^2.$$

Then (6) turns into

$$ag(X, Y) + \frac{1}{2} [\eta(X)\psi(Y) + \psi(X)\eta(Y)] = [\alpha n + \beta |V|^2 - \lambda]g(X, Y) + \mu\eta(X)\eta(Y).$$

So by a contraction in the last equation, we obtain (11).

**Example 3.15.** Let $M = \{(x, y, z) \in \mathbb{R}^3, z > 0\}$, where $(x, y, z)$ are the standard coordinates in $\mathbb{R}^3$. Set

$V := -z \frac{\partial}{\partial z}, \quad \eta := \frac{1}{z} dz,$

$$g := \frac{1}{z^2} (dx \otimes dx + dy \otimes dy + dz \otimes dz)$$

and consider the linearly independent system of vector fields:

$$E_1 := z \frac{\partial}{\partial x}, \quad E_2 := z \frac{\partial}{\partial y}, \quad E_3 := -z \frac{\partial}{\partial z}.$$

According to Example 3.10, the Riemann and the Ricci curvature tensor fields are given by:

$$R(E_1, E_2)E_2 = -E_1, \quad R(E_1, E_3)E_3 = -E_1, \quad R(E_2, E_1)E_1 = -E_2,$$

$$R(E_2, E_3)E_3 = -E_2, \quad R(E_3, E_1)E_1 = -E_3, \quad R(E_3, E_2)E_2 = -E_3,$$

$$\text{Ric}(E_1, E_1) = \text{Ric}(E_2, E_2) = \text{Ric}(E_3, E_3) = -2$$

and the scalar curvature is $\text{scal} = -6$.

Writing the $\eta$-Yamabe soliton equation in $(E_i, E_i)$ we obtain:

$$g(\nabla_{E_i} E_3, E_i) = (-6 - \lambda)g(E_i, E_i) + \mu\eta(E_i)\eta(E_i),$$

for all $i \in \{1, 2, 3\}$ and we get that for $\lambda = -7$ and $\mu = -1$, the data $(V, \lambda, \mu)$ define an $\eta$-Yamabe soliton on $(M, g)$. Moreover, it is a gradient $\eta$-Yamabe soliton, since the potential vector field $V$ is a torse-forming vector field of gradient type, $V = \text{grad}(f)$, where $f(x, y, z) := -\ln z$. 
**Example 3.16.** Let \( M = \mathbb{R}^3 \), \((x, y, z)\) be the standard coordinates in \( \mathbb{R}^3 \), let \( g \) be the Lorentzian metric:

\[
g := e^{-2z}dx \otimes dx + e^{2x-2z}dy \otimes dy - dz \otimes dz.
\]

Consider the vector field \( V \) and the 1-form \( \eta \):

\[
V := \frac{\partial}{\partial z}, \quad \eta := dz.
\]

For the orthonormal vector fields:

\[
E_1 = e^z \frac{\partial}{\partial x}, \quad E_2 = e^{z-x} \frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z},
\]

according to Example [3.11] the Riemann tensor field, the Ricci tensor field and the scalar curvature are given by:

\[
\begin{align*}
R(E_1, E_2)E_2 &= (1 - e^{2z})E_1, \\
R(E_1, E_3)E_3 &= -E_1, \\
R(E_2, E_1)E_1 &= (1 - e^{2z})E_2, \\
R(E_2, E_3)E_3 &= -E_2, \\
R(E_3, E_1)E_1 &= E_3, \\
R(E_3, E_2)E_2 &= E_3, \\
\text{Ric}(E_1, E_1) &= \text{Ric}(E_2, E_2) = 2 - e^{2z}, \\
\text{Ric}(E_3, E_3) &= -2, \\
\text{scal} &= 2(3 - e^{2z}).
\end{align*}
\]

Then the data \((V, \lambda, \mu)\), for \( \lambda = 7 - 2e^{2z} \) and \( \mu = -1 \), define an almost \( \eta \)-Yamabe soliton on \((M, g)\). Moreover, it is a gradient almost \( \eta \)-Yamabe soliton, since the potential vector field \( V \) is of gradient type, \( V = \text{grad}(f) \), where \( f(x, y, z) := -z \).

### 4 Almost \( \eta \)-Ricci and almost \( \eta \)-Yamabe solitons on submanifolds

Let \( M \) be a submanifold isometrically immersed into a Riemannian manifold \((\tilde{M}, \tilde{g})\). Denote by \( g \) the Riemannian metric induced on \( M \), by \( \nabla \) and \( \tilde{\nabla} \) the Levi-Civita connections on \((M, g)\) and \((\tilde{M}, \tilde{g})\). The Gauss and Weingarten formulas corresponding to \( M \) are given by:

\[
\begin{align*}
\tilde{\nabla}_X Y &= \nabla_X Y + h(X, Y), \\
\tilde{\nabla}_X N &= -B_N(X) + \nabla^\perp_X N,
\end{align*}
\]

where \( h \) is the second fundamental form and \( B_N \) is the shape operator.
where $h$ is the second fundamental form and $B_N$ is the shape operator in the direction of the normal vector field $N$ defined by $\tilde{g}(B_N(X), Y) = \tilde{g}(h(X, Y), N)$ for $X, Y \in \chi(M)$ [11].

A Riemannian submanifold $M$ is called \textit{minimal} if its mean curvature vanishes.

A submanifold $M$ in a Riemannian manifold $(\tilde{M}, \tilde{g})$ is called

i) \textit{ξ-umbilical} [8] (with respect to a normal vector field $ξ$) if its shape operator satisfies $B_ξ = \varphi I$, where $\varphi$ is a function on $M$ and $I$ is the identity map;

ii) \textit{totally umbilical} [11] if it is umbilical with respect to every unit normal vector field.

An $n$-dimensional hypersurface $M$, $n \geq 4$, in a Riemannian manifold $(\tilde{M}, \tilde{g})$ is called

i) \textit{2-quasi-umbilical} [20] if its second fundamental tensor field $H$ satisfies

$$H = \alpha g + \beta \varpi \otimes \varpi + \gamma \eta \otimes \eta,$$

where $\varpi$ and $\eta$ are 1-forms and $\alpha$, $\beta$ and $\gamma$ are smooth functions on $M$ such that the corresponding vector fields to the 1-forms $\varpi$ and $\eta$ are $g$-orthogonal. In particular, if $\gamma = 0$, then $M$ is called \textit{quasi-umbilical} [11];

ii) \textit{pseudo quasi-umbilical} [25], if its second fundamental tensor field $H$ satisfies

$$H = \alpha g + \beta \varpi \otimes \varpi + E,$$

where $\varpi$ is a 1-form, $\alpha$ and $\beta$ are smooth functions on $M$ and $E$ is a symmetric $(0, 2)$-tensor field with vanishing trace. If $E$ vanishes, then $M$ is quasi-umbilical.

If $V$ is a concurrent vector field on $\tilde{M}$, then for any $X \in \chi(M)$, we have

$$\nabla_X V^T = X + B_{VT}(X), \quad \nabla_X V^\perp = -h(X, V^T),$$

therefore

$$\frac{1}{2} \text{grad}(|V^\perp|^2) = -B_{VT}(V^T), \quad \frac{1}{2} \text{grad}(|V|^2) = V^T$$

and we can state:

**Proposition 4.1.** Every almost $\eta$-Ricci and every almost $\eta$-Yamabe soliton $(V^T, \lambda, \mu)$ on a submanifold $M$, which is isometrically immersed into a Riemannian manifold $(\tilde{M}, \tilde{g})$, is of gradient type with the potential function $\frac{1}{2} \text{grad}(|V|^2)$, where $V^T \in \chi(M)$ is the tangential component of the concurrent vector field $V \in \chi(\tilde{M})$ and $\eta$ is the $g$-dual of $V^T$. 
In particular, if $V$ is of constant length, then the almost $\eta$-Ricci soliton $(M, g)$ is a quasi-Einstein manifold with associated functions $\lambda$ and $\mu$.

Next, we shall characterize almost $\eta$-Ricci and almost $\eta$-Yamabe solitons on $M$ whose potential vector field is the tangential component of a torse-forming vector field on $\tilde{M}$.

**Theorem 4.2.** Let $M$ be a submanifold isometrically immersed into a Riemannian manifold $(\tilde{M}, \tilde{g})$, let $V$ be a torse-forming vector field on $\tilde{M}$ and let $\eta$ be the $g$-dual of $V^T$. Then

i) $(M, g)$ is an almost $\eta$-Ricci soliton $(V^T, \lambda, \mu)$ if and only if the Ricci tensor field of $M$ satisfies:

$$\text{Ric}_M(X, Y) = (\lambda - a)g(X, Y) - \tilde{\nabla}(h(X, Y), V^\perp) + \mu\eta(X)\eta(Y) - \frac{1}{2}[\psi(X)\eta(Y) + \eta(X)\psi(Y)],$$

for any $X, Y \in \chi(M)$;

ii) $(M, g)$ is an almost $\eta$-Yamabe soliton $(V^T, \lambda, \mu)$ if and only if:

$$(\text{scal} - \lambda - a)g(X, Y) - \tilde{\nabla}(h(X, Y), V^\perp) + \mu\eta(X)\eta(Y) - \frac{1}{2}[\psi(X)\eta(Y) + \eta(X)\psi(Y)] = 0,$$

for any $X, Y \in \chi(M)$.

**Proof.** For any $X \in \chi(M)$, we have:

$$aX + \psi(X)V^T + \psi(X)V^\perp = \tilde{\nabla}_X V = \nabla_X V^T + h(X, V^T) - B_{V^\perp}(X) + \nabla^\perp_X V^\perp$$

and by the equality of the tangent components, we get:

$$\nabla_X V^T = aX + \psi(X)V^T + B_{V^\perp}(X).$$

Then

$$(L_{V^T}g)(X, Y) = g(\nabla_X V^T, Y) + g(\nabla_Y V^T, X) = 2[a\psi(X, Y) + \tilde{g}(h(X, Y), V^\perp)] + \psi(X)\eta(Y) + \eta(X)\psi(Y).$$

i) Suppose that there exist smooth functions $\lambda$ and $\mu$ on $M$ such that the condition in the hypotheses holds. Then we obtain

$$\frac{1}{2}(L_{V^T}g)(X, Y) + \text{Ric}_M(X, Y) = \lambda g(X, Y) + \mu\eta(X)\eta(Y).$$

Hence the submanifold $(M, g)$ is an almost $\eta$-Ricci soliton. The converse is trivial.
ii) Suppose that there exist smooth functions $\lambda$ and $\mu$ on $M$ such that the condition in the hypotheses holds. Then we obtain
\[
\frac{1}{2}(\mathcal{L}_{V^T}g)(X,Y) = (\text{scal} - \lambda)g(X,Y) + \mu \eta(X)\eta(Y).
\]
Hence the submanifold $(M,g)$ is an almost $\eta$-Yamabe soliton. The converse is trivial. □

If $M$ is a minimal submanifold, then we can state the following corollary:

**Corollary 4.3.** Let $M$ be an $n$-dimensional isometrically immersed minimal submanifold of a Riemannian manifold $(\tilde{M},\tilde{g})$, let $V$ be a concircular vector field on $\tilde{M}$ and let $\eta$ be the $g$-dual of $V^T$.

i) If $(V^T,\lambda,\mu)$ is an almost $\eta$-Ricci soliton on $M$, then $\text{scal}_M = n(\lambda - a) + \mu |V^T|^2$.

ii) If $(V^T,\lambda,\mu)$ is an almost $\eta$-Yamabe soliton on $M$, then $\text{scal}_M = \lambda + a - \frac{\mu}{n} |V^T|^2$.

When $M$ is a $V^\perp$-umbilical submanifold, we have:

**Corollary 4.4.** Let $M$ be an $n$-dimensional $V^\perp$-umbilical submanifold isometrically immersed into an $(n+d)$-dimensional Riemannian manifold $(\tilde{M},\tilde{g})$. If $V$ is a concircular vector field on $\tilde{M}$, then $M$ is an almost $\eta$-Ricci soliton with potential vector field $V^T$, for $\eta$ the $g$-dual of $V^T$ if and only if $M$ is a quasi-Einstein submanifold with associated functions $(\lambda - a - \phi)$ and $\mu$.

For a hypersurface, since
\[
\tilde{g} \left(h(X,Y),V^\perp\right) = g(B(X),Y)g(N,V^\perp) = H(X,Y)g(N,V^\perp),
\]
where $N$ is the unit normal vector field of $M$ and $H$ is the second fundamental tensor field, if we denote by $\rho = g(N,V^\perp)$, then we can state:

**Corollary 4.5.** Let $M$ be an $n$-dimensional hypersurface isometrically immersed into an $(n+1)$-dimensional Riemannian manifold $(\tilde{M},\tilde{g})$. If $V$ is a torse-forming vector field on $\tilde{M}$ and $\eta$ is the $g$-dual of $V^T$, then

i) $(M,g)$ is an almost $\eta$-Ricci soliton with potential vector field $V^T$ if and only if there exist two smooth functions $\lambda$ and $\mu$ on $M$ such that the Ricci tensor field of $M$ satisfies
\[
\text{Ric}_M = (\lambda - a)g - \rho H + \mu \eta \otimes \eta - \frac{1}{2}(\psi \otimes \eta + \eta \otimes \psi);
\]

ii) $(M,g)$ is an almost $\eta$-Yamabe soliton with potential vector field $V^T$ if and only if there exist two smooth functions $\lambda$ and $\mu$ on $M$ such that
\[
(\text{scal} - \lambda - a)g - \rho H + \mu \eta \otimes \eta - \frac{1}{2}(\psi \otimes \eta + \eta \otimes \psi) = 0.
\]
Almost $\eta$-Ricci and $\eta$-Yamabe solitons

**Theorem 4.6.** Let $M$ be an $n$-dimensional hypersurface isometrically immersed into an $(n+1)$-dimensional Riemannian manifold $(\widetilde{M}(c), \widetilde{g})$ of constant curvature $c$. If $V$ is a torse-forming vector field on $\widetilde{M}$ and $\eta$ is the $g$-dual of $V^T$, then

i) $(M, g)$ is an almost $\eta$-Ricci soliton with potential vector field $V^T$ if and only if there exist two smooth functions $\lambda$ and $\mu$ on $M$ such that the second fundamental tensor field $H$ of $M$ satisfies

$$H^2 = [\rho + tr(H)]H + [(n-1)c - \lambda + a]g - \mu\eta \otimes \eta + \frac{1}{2}(\psi \otimes \eta + \eta \otimes \psi);$$

ii) $(M, g)$ is an almost $\eta$-Yamabe soliton with potential vector field $V^T$ if and only if there exist two smooth functions $\lambda$ and $\mu$ on $M$ such that the second fundamental tensor field $H$ of $M$ satisfies

$$\rho H = [n(n-1)c + (tr(H))^2 - tr(H^2) - \lambda - a]g + \mu\eta \otimes \eta - \frac{1}{2}(\psi \otimes \eta + \eta \otimes \psi).$$

**Proof.** From the Gauss equation, we have

$$\text{Ric}_M(X, Y) = tr(H)H(X, Y) - H^2(X, Y) + (n-1)cg(X, Y).$$

i) Then comparing (18) and (14), we get

$$(\lambda - a)g(X, Y) - \rho H(X, Y) + \mu\eta(X)\eta(Y) - \frac{1}{2}[\psi(X)\eta(Y) + \eta(X)\psi(Y)]$$

$$= tr(H)H(X, Y) - H^2(X, Y) + (n-1)cg(X, Y),$$

which gives us

$$H^2(X, Y) = [\rho + tr(H)]H(X, Y) + [(n-1)c - \lambda + a]g(X, Y) - \mu\eta(X)\eta(Y)$$

$$+ \frac{1}{2}[\psi(X)\eta(Y) + \eta(X)\psi(Y)].$$

Conversely, assume that (16) is satisfied. Then by the Gauss equation, we have

$$\text{Ric}_M(X, Y) = (\lambda - a)g(X, Y) - \rho H(X, Y) + \mu\eta(X)\eta(Y) - \frac{1}{2}[\psi(X)\eta(Y) + \eta(X)\psi(Y)],$$

so by Corollary 4.5, $(M, g)$ is an almost $\eta$-Ricci soliton with potential vector field $V^T$.

ii) By a contraction in (18), we find

$$\text{scal} = (tr(H))^2 - tr(H^2) + n(n-1)c$$
and replacing scal in (15), we get

\[ n(n-1)c + (\text{tr}(H))^2 - \text{tr}(H^2) - \lambda - a]g - \rho H + \mu \eta \otimes \eta - \frac{1}{2}(\psi \otimes \eta + \eta \otimes \psi) = 0. \]

Conversely, assume that (17) is satisfied. Then by a contraction, we find

(20) \[ \text{scal} = (\text{tr}(H))^2 - \text{tr}(H^2) + n(n-1)c, \]

so by Corollary 4.5, \((M,g)\) is an almost \(\eta\)-Yamabe soliton with potential vector field \(V_T\).

**Proposition 4.7.** Let \(M\) be an \(n\)-dimensional quasi-Einstein hypersurface isometrically immersed into an \((n+1)\)-dimensional Riemannian manifold \((\widetilde{M},\tilde{g})\). Assume that the Ricci tensor field of \(M\) is of the form \(\text{Ric} = \alpha g + \beta \eta \otimes \eta\). If \(V\) is a concircular vector field on \(\widetilde{M}\), then \((M,g)\) is an almost \(\eta\)-Ricci soliton with potential vector field \(V^T\), for \(\eta\) the \(g\)-dual of \(V^T\), if and only if it is a quasi-umbilical hypersurface with associated functions \(\frac{\lambda - a - \alpha}{\rho}\) and \(\frac{\mu - \beta}{\rho}\).

**Proof.** Assume that \(M\) is a quasi-Einstein hypersurface whose Ricci tensor field \(\text{Ric}\) is of the form \(\text{Ric} = \alpha g + \beta \eta \otimes \eta\). If \(V\) is a concircular vector field on \(\widetilde{M}\), then from Theorem 4.2, we can write

\[ \alpha g(X,Y) + \beta \eta(X)\eta(Y) = (\lambda - a) g(X,Y) - \rho H(X,Y) + \mu \eta(X)\eta(Y), \]

which gives us

\[ H(X,Y) = \frac{\lambda - a - \alpha}{\rho} g(X,Y) + \frac{\mu - \beta}{\rho} \eta(X)\eta(Y). \]

Hence \(M\) is a quasi-umbilical hypersurface with associated functions \(\frac{\lambda - a - \alpha}{\rho}\) and \(\frac{\mu - \beta}{\rho}\). The converse is trivial.

It is known that an \(n\)-dimensional hypersurface \(M, n \geq 4\), in a Riemannian manifold \((\widetilde{M}(c),\tilde{g})\) of constant curvature \(c\) is conformally flat if and only if it is quasi-umbilical [29]. So we have:

**Corollary 4.8.** Let \(M\) be an \(n\)-dimensional quasi-Einstein hypersurface isometrically immersed into an \((n+1)\)-dimensional Riemannian manifold \((\widetilde{M},\tilde{g})\). Assume that the Ricci tensor field of \(M\) is of the form \(\text{Ric} = \alpha g + \beta \eta \otimes \eta\). If \(V\) is a concircular vector field on \(\widetilde{M}\), then \((M,g)\) is an almost \(\eta\)-Ricci soliton with potential vector field \(V^T\), for \(\eta\) the \(g\)-dual of \(V^T\) and \(M\) is a conformally flat hypersurface.
Let \( \varphi : M \to \mathbb{S}^{n+1}(1) \) be an immersion. We denote by \( g \) the induced metric on the hypersurface \( M \) as well as that on the unit sphere \( \mathbb{S}^{n+1}(1) \). Let \( N \) and \( B \) be the unit normal vector field and the shape operator of the hypersurface \( M \) in the unit sphere \( \mathbb{S}^{n+1}(1) \) and we denote by \( \langle \cdot, \cdot \rangle \) the Euclidean metric on the Euclidean space \( \mathbb{E}^{n+2} \). Assume that \( V \) is a torse-forming vector field on \( \mathbb{E}^{n+2} \). If we denote by \( N_S \) the unit normal vector field of the unit sphere \( \mathbb{S}^{n+1}(1) \) in the Euclidean space \( \mathbb{E}^{n+2} \), we can define the smooth functions \( \delta, \rho \) on the hypersurface \( M \) by
\[
\delta = \langle V, N \rangle \big|_M \quad \text{and} \quad \rho = \langle V, N_S \rangle \big|_M .
\]
Hence the restriction of the torse-forming vector field \( V \) to the hypersurface \( M \) can be written as
\[
V \big|_M = U + \delta N + \rho N_S ,
\]
where \( U \in \chi(M) \).

Then as an extension of Theorem 3.3 given in [31], we can state:

**Theorem 4.9.** Let \( M \) be an orientable hypersurface of the unit sphere \( \mathbb{S}^{n+1}(1) \), with immersion \( \varphi : M \to \mathbb{S}^{n+1}(1) \) and let \( V \) be a torse-forming vector field on the Euclidean space \( \mathbb{E}^{n+2} \). Denote by \( \xi \) the tangential component of \( V \) on the unit sphere \( \mathbb{S}^{n+1}(1) \) and by \( U \) the tangential component of \( \xi \) on \( M \). Then

i) the hypersurface \( (M, g) \) is an almost \( \eta \)-Ricci soliton \((U, \lambda, \mu)\) if and only if
\[
H^2(X, Y) = [\text{tr}(H) + \delta] H(X, Y) + (n - 1 - \rho - \lambda + \alpha) g(X, Y)
\]
\[
- \mu \eta(X) \eta(Y) + \frac{1}{2} [\psi(X) g(U, Y) + \psi(Y) g(U, X)] ,
\]
for any \( X, Y \in \chi(M) \);

ii) the hypersurface \( (M, g) \) is an almost \( \eta \)-Yamabe soliton \((U, \lambda, \mu)\) if and only if
\[
\delta H(X, Y) = [n(n - 1) + (\text{tr}(H))^2 - \text{tr}(H^2) + \rho - \lambda - \alpha] g(X, Y)
\]
\[
+ \mu \eta(X) \eta(Y) - \frac{1}{2} [\psi(X) g(U, Y) + \psi(Y) g(U, X)] ,
\]
for any \( X, Y \in \chi(M) \). In this case, \( M \) is a pseudo quasi-umbilical hypersurface.

**Proof.** Let \( \nabla, \nabla \) and \( D \) denote the Levi-Civita connections on \( M, \mathbb{S}^{n+1}(1) \) and \( \mathbb{E}^{n+2} \), respectively. Then we can write
\[
V \big|_{\mathbb{S}^{n+1}(1)} = \xi + \rho N_S ,
\]
and for any \( X \in \chi(M) \), by taking the covariant differential w.r.t. \( X \), we have
\[
a X + \psi(X) V = a X + \psi(X) \xi + \psi(X) \rho N_S = D_X V = D_X \xi + X(\rho) N_S + \rho D_X N_S .
\]
By using the Gauss and Weingarten formulas, we find
\[ aX + \psi(X)\xi + \psi(X)gN_S = \nabla_X\xi - g(X,\xi)N_S + X(g)N_S + gX. \]
By the equality of the tangent and the normal components, we get
\[(21)\]
\[ \nabla_X\xi + (\varrho - a)X = \psi(X)\xi \]
and
\[ X(\varrho) - g(X,\xi) = \psi(X)\varrho. \]
The vector field \( \xi \) on \( S^{n+1}(1) \) can be written as
\[ \xi = U + \delta N. \]
So from \((21)\), we have
\[ \nabla_X(U + \delta N) + (\varrho - a)X = \psi(X)(U + \delta N). \]
By using Gauss and Weingarten formulas again, we find
\[ \psi(X)U + \psi(X)\delta N - (\varrho - a)X = \nabla_XU + g(B(X), U)N + X(\delta)N - \delta B(X). \]
Then by the equality of the tangent and the normal components, we have
\[(22)\]
\[ \nabla_XU = \psi(X)U - (\varrho - a)X + \delta B(X) \]
and
\[ \psi(X)\delta = g(B(X), U) + X(\delta). \]
So
\[ (\mathcal{L}_U g)(X, Y) = g(\nabla_XU, Y) + g(\nabla_YU, X) \]
\[(23)\]
\[ = \psi(X)g(U, Y) + \psi(Y)g(U, X) - 2(\varrho - a)g(X, Y) + 2\delta H(X, Y). \]
On the other hand, the Gauss equation for a hypersurface \( M \) in \( S^{n+1}(1) \) gives us
\[(24)\]
\[ \text{Ric}(X, Y) = (n - 1)g(X, Y) + tr(H)H(X, Y) - H^2(X, Y). \]
i) Then combining \((21)\) and \((23)\), we find
\[ \frac{1}{2} (\mathcal{L}_U g)(X, Y) + \text{Ric}(X, Y) = (n - 1 - \varrho + a)g(X, Y) + [tr(H) + \delta]H(X, Y) \]
\[-H^2(X, Y) + \frac{1}{2} [\psi(X)g(U, Y) + \psi(Y)g(U, X)]. \]
Suppose that there exist smooth functions $\lambda$ and $\mu$ on $M$ such that the condition in the hypothesis holds. Then we obtain \( \frac{1}{2} (\mathcal{L}_U g)(X,Y) + \text{Ric}(X,Y) = \lambda g(X,Y) + \mu \eta(X) \eta(Y) \). Hence the hypersurface $M$ is an almost $\eta$-Ricci soliton. The converse is trivial.

ii) By a contraction in (24), we find

\[
\text{scal} = n(n-1) + (tr(H))^2 - tr(H^2).
\]

Then combining (25) and (23), we get

\[
\frac{1}{2} (\mathcal{L}_U g)(X,Y) - (\text{scal} - \lambda) g(X,Y) = \frac{1}{2} [\psi(X) g(U,Y) + \psi(Y) g(U,X)]
\]

\[
- [n(n-1) + (tr(H))^2 - tr(H^2) + g - \lambda - a] g(X,Y) + \delta H(X,Y).
\]

Suppose that there exist smooth functions $\lambda$ and $\mu$ on $M$ such that the condition in the hypothesis holds. Then we obtain \( \frac{1}{2} (\mathcal{L}_U g)(X,Y) = (\text{scal} - \lambda) g(X,Y) + \mu \eta(X) \eta(Y) \). Hence the hypersurface $M$ is an almost $\eta$-Yamabe soliton and moreover, $M$ is a pseudo quasi-umbilical hypersurface. The converse is trivial.

**Corollary 4.10.** Under the conditions of Theorem 4.9, if the 1-form $\psi$ is the $g$-dual of $U$, then

i) the hypersurface $(M, g)$ is an almost $\eta$-Ricci soliton $(U, \lambda, \mu)$ if and only if

\[
H^2 = [tr(H) + \delta] H + (n - 1 - g - \lambda + a) g - \mu \eta \otimes \eta + \psi \otimes \psi;
\]

ii) the hypersurface $(M, g)$ is an almost $\eta$-Yamabe soliton $(U, \lambda, \mu)$ if and only if

\[
\delta H = [n(n-1) + (tr(H))^2 - tr(H^2) + g - \lambda - a] g + \mu \eta \otimes \eta - \psi \otimes \psi.
\]

**Corollary 4.11.** Let $M$ be an orientable hypersurface of the unit sphere $\mathbb{S}^{n+1}(1)$ and let $V$ be a $\nabla$-parallel or constant vector field on the Euclidean space $\mathbb{E}^{n+2}$. Then

i) the hypersurface $(M, g)$ is an almost $\eta$-Ricci soliton $(U, \lambda, \mu)$ if and only if

\[
H^2 = [tr(H) + \delta] H + (n - 1 - g - \lambda) g - \mu \eta \otimes \eta;
\]

ii) the hypersurface $(M, g)$ is an almost $\eta$-Yamabe soliton $(U, \lambda, \mu)$ if and only if

\[
\delta H = [n(n-1) + (tr(H))^2 - tr(H^2) + g - \lambda] g + \mu \eta \otimes \eta.
\]

It is known that a pseudo quasi-umbilical hypersurface of a Riemannian manifold of constant curvature $(\tilde{M}(c), \tilde{g})$ is a pseudo quasi-Einstein hypersurface [25]. So we have:

**Corollary 4.12.** Let $M$ be an orientable hypersurface of the unit sphere $\mathbb{S}^{n+1}(1)$ and let $V$ be a torse-forming vector field on the Euclidean space $\mathbb{E}^{n+2}$. If the hypersurface $(M, g)$ is an almost $\eta$-Yamabe soliton $(U, \lambda, \mu)$, then it is a pseudo quasi-Einstein hypersurface.
Almost $\eta$-Ricci and $\eta$-Yamabe solitons

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