In this paper, we study the classical limit of quantum cosmology by applying the Weyl-Wigner-Groenewold-Moyal formalism of deformation quantization to quantum cosmology of Friedmann-Lemaître-Robertson-Walker (FLRW) universe with positive spatial curvature and conformally coupled scalar field. The corresponding quantum cosmology is described by the Moyal-Wheeler-DeWitt equation which has an exact solution in Moyal phase space, resulting in Wigner quasiprobability distribution function, peaking over the classical solutions. We showed that for a large value of quantum number $n$, the emerged classical universe is filled with radiation with quantum mechanical origin.

PACS numbers: 98.80.Qc, 04.60.Ds, 98.80.Jk

I. INTRODUCTION

Since the dawn of quantum physics, a question has been posed that how quantum mechanical wave formalism has the classical description. The generally accepted explanation is that quantum mechanics is the fundamental theory and the classical behavior is only a limit for large systems, i.e. a quantum system with enough large quantum number. Generalizing this scope, one may ask how the classical limit of our world can emerge from a quantum cosmology theory? Actually it is not easy at all to address this question. A viewpoint regards the peaking of the wave function which provides a way to study the classical limit, by preparing a comparison between the classical and quantum dynamics in the phase space. This implies that for a classical limit to have existed, the quantum world should be peaked to occur around a classical trajectory in the phase space [1].

But, while dealing with the wave function, which is obtained by Wheeler-DeWitt (WDW) equation or path integral, a problem is arisen where it is necessary to know how to construct a proper wave packet, peaked around the original classical cosmological model [2]. In ordinary quantum mechanics, where one describes the dynamics of an ensemble of identical systems, the wave packet reduction in the Copenhagen interpretation leads to no practical problem. But in cosmology, the observer is an element of the Universe itself, i.e. there is only one Universe as a system. Therefore, the corresponding wave function of the state of the Universe is not clear.

Another problem arises here which is called the measurement problem. During an observation in the standard quantum mechanics, the quantum system interacts with a classical domain where it is a necessity that a classical domain comes from the way in which it solves the measurement problem [3]. In a conventional measurement, the wave function plus measuring device splits into non-overlapping branches, containing the measured system in an eigenstate of the measured observable, while the measuring device indicating the corresponding eigenvalue. Therefore, the wave function collapses into an eigenstate of the observable, and the other branches disappear. This is due to the short duration and strong coupling interaction between the measured system and the classical measuring device. In the Copenhagen interpretation, a real collapse of the wave function cannot be described by the unitary quantum evolution, and the fundamental measuring process should occur outside the quantum system, i.e. in a classical realm. But this is problematic, since in quantum cosmology, as a quantum theory of the whole Universe, there is no place for classical domain outside of that.

Hence, an improved scheme needed to be applied to quantum cosmology. Some models such as de Broglie-Bohm interpretation of quantum cosmology [4], quantum Hamilton-Jacobi cosmology [5], and deformation quantization of cosmology [6, 7], are proposed to overcome the quantum cosmological difficulties about the measurement problem while maintaining the universality of quantum theory and emergence of the classical universe.

Here, we talk over the deformation quantization, which is known as the phase space formalism of quantum mechanics. Deformation quantization is based on the Wigner quasi-distribution function, and Weyl correspondence between
quantum mechanical operators and ordinary phase space functions \[8-10\].

In this formalism, observables are not represented by operators and are defined as the functions of phase space variables. To get a quantum mechanical description, the algebra of phase space functions is changed via replacing common point-wise product between observables with an associative, but noncommutative, pseudo-differential star-product \[2,11,12\]. Thus, instead of changing the nature of classical phase space functions, deformation quantization only deforms the structure of the corresponding algebra.

Applying this approach to pass from classical to quantum cosmology has the advantage of making quantum cosmology calculations similar to the Hamiltonian formalism of classical cosmology, to stay away from doing arduous operator calculations \[6,13\]. As we know, quantum cosmology formalism is based on Wheeler-DeWitt (WDW) equation, which represents the wave function of the whole universe \[14,15\]. Hence, constructing wave functions, obtained from the solutions of WDW equation, is a suitable approach to investigate quantum cosmology.

In this article, by applying Arnowitt-Deser-Misner (ADM) formalism of quantum cosmology \[16\], we construct the Wigner function of a conformally coupled scalar field model. In this way, the extremum of quasi-distribution probability function and classical trajectories are coincided. In classical cosmology scenarios, a conformal (nonminimal) coupling is usually referred as the coupling of a scalar field and the Ricci scalar \[17\], whereas it could have different types of interactions in different cosmological themes \[18-20\]. Indeed, the conformal coupling of the scalar field seems to be interesting for several reasons \[21-24\]. For instance, it allows us to explore the exact solutions of simple models, and at the same time, it is rich enough to be considered as a significant modification of quantum cosmology \[25-27\]. Moreover, having a model with a coupled scalar field and gravity in hand, one can provide a more precise explanation for the effects of the curvature on the very early universe \[28\].

This article is organized as follows. In section II, we explain a classical model as a non-minimally coupling of free scalar field and gravity with a positive curvature FLRW background. In section III we analyze the quantum cosmology of the model and find that the Wigner function of the model is made of two independent Laguerre functions. Furthermore, we checked the compatibility of classical and quantum solutions, and finally, we showed that the classical universe emerged is radiation dominated, and its entropy with quantum cosmology origin is estimated.

**II. THE CLASSICAL MODEL**

In the ADM formalism, the 4D spacetime \(\mathcal{M}\) is splitted (or decomposed) into a family of spacelike three-hypersurfaces \(\Sigma_t\), and the spacetime curvature scalar is expressed through the curvature \((3) R\) of \(\Sigma_t\), its induced metric \(h_{ab}\) \((a, b = 1, 2, 3)\), its extrinsic curvature tensor \(K_{ab}\), the lapse function \(N\) and the shift vector \(N^a\). The ADM action functional of a non-minimally coupled scalar field \(\Phi\) in natural units, \((\hbar = c = 1)\), is given by \[29\]

\[
S = \int_{t_i}^{t_f} dt \int_{\Sigma_t} \left[ \frac{N\sqrt{h}}{2} \left( M_p^2 - \zeta \Phi^2 \right) \left( (3) R + K_{ab}K^{ab} - K^2 \right) - 2\sqrt{h}\zeta \Phi K^2 - 2\sqrt{h}\zeta \Phi K_{ab} \left( KN^a - \sqrt{h}h^{ab}N_b \right) - \frac{\sqrt{h}K}{2N} \left( \frac{\Phi^2}{N^2} + h^{ab}\Phi_a\Phi_b + V(\Phi) \right) \right] d^3x,
\]

where \(M_p = 1/\sqrt{4\pi G}\) is the reduced Planck mass and \(\zeta\) is a dimensionless coupling constant which is valued as \(\zeta = 0\) for minimal coupling, and \(\zeta = \frac{1}{6}\) for conformal \((V(\Phi) = 0)\) coupling \[30\]. Here, we apply the non-minimal value for \(\zeta\) with \(V(\Phi) = 0\) to have a conformally invariant \(\Phi\).

Let us consider a classical model which is consisted of a cosmological system, presented by action \(\Pi\), and a FLRW minisuperspace model with a constant positive curvature with the line element

\[
dS^2 = -N^2(t) \ dt^2 + a^2(t) \left[ \frac{dr^2}{1-r^2} + r^2 \left( d\theta^2 + sin^2\theta d\phi^2 \right) \right],
\]

where, \(a(t)\) is the scale factor and the lapse function is identified by \(N(t)\). Assuming the scalar field, \(\Phi = \Phi(t)\), to be homogeneous, for a conformally coupled case we substitute the Eq.\(2\) into the action functional \(\Pi\) and rescaling lapse function as \(N(t) = 12\pi^2 M_p a(t)\dot{N}(t)\) and introducing new variables \(x_1(t) = a(t), x_2(t) = \frac{a(t)\Phi(t)}{\sqrt{6}M_p}\) we get

\[
S = -\int dt \left( \frac{M_p}{2N} (x_1^2 - x_2^2) + \frac{1}{2}M_p\omega^2 \dot{N}(x_1^2 - x_2^2) \right),
\]

with \(\omega = 12\pi^2 M_p\). To construct the Hamiltonian of the model, we consider the conjugate momenta of \(\{x_1, x_2\}\) defined
by
\[ \Pi_1 = -\frac{M_p}{\tilde{N}} \dot{x}_1, \Pi_2 = -\frac{M_p}{\tilde{N}} \dot{x}_2. \] (4)

The corresponding Hamiltonian in terms of 2D minisuperspace \( \{x_1, x_2\} \) is
\[ H = \tilde{N} \mathcal{H} := \tilde{N} (\mathcal{H}_1 - \mathcal{H}_2), \] (5)

where \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are the superhamiltonians of the gravitational and scalar field parts respectively which are introduced by
\[ \mathcal{H}_1 := \frac{1}{2M_p} \Pi_1^2 + \frac{1}{2} M_p \omega^2 x_1^2, \quad x_1 \geq 0, \]
\[ \mathcal{H}_2 := \frac{1}{2M_p} \Pi_2^2 + \frac{1}{2} M_p \omega^2 x_2^2 \quad -\infty < x_2 < \infty, \] (6)

The lapse function \( \tilde{N} \) acts as a Lagrange multiplier. The variation of the above Hamiltonian with respect to \( \tilde{N} \) yields to the minisuperspace superhamiltonian constraint
\[ \mathcal{H} \approx 0. \] (7)

In conformal time gauge fixing, \( \tilde{N} = 1 \), the classical solutions of field equations are given by
\[ x_1 = a_{\text{max}} \sin(\omega t), \quad x_2 = \eta a_{\text{max}} \sin(\omega t + \delta), \] (8)

where \( \eta = \pm 1 \) is imposed via using the superhamiltonian constraint (7) and \( a_{\text{max}} \) is the maximum value of the scale factor \( a(t) \). The above solution implies that the classical solution to be displayed as following trajectories
\[ x_1^2 + x_2^2 - 2\eta \cos \delta x_1 x_2 = a_{\text{max}}^2 \sin^2 \delta. \] (9)

### III. QUANTUM COSMOLOGY AND EMERGED CLASSICAL UNIVERSE

In deformation quantization observables are represented by phase space functions. Consequently the principal element of deformation quantization, that lays in the algebraic structure of the theory, is an associative and noncommutative pseudo-differential star-product [7, 31]. The quasi-probability distribution Wigner function, which corresponds to the state of the system, is a prominent component of phase space quantization and allows us to calculate expectation values and probabilities [32]. The Wigner function in a 2D-dimensional phase space \( (x^i, \Pi_j) \), \( i, j = 1, 2, \ldots, D \) is introduced by
\[ W_n(x, \Pi) = C \int dy^D e^{-i\Pi y} \psi_n^*(x - \frac{y}{2}) \psi_n(x + \frac{y}{2}) \int d^D x d^D \Pi W_n(x, \Pi) = 1, \] (10)

where \( C \) is a constant and \( \psi_n(x) \) denotes a general state. In addition, the concept of star-product was introduced by Gerstenhaber [10]. To apply it into quantum mechanics we should consider the Moyal-Groenewold star product, \( *_{M} \), of two observables, say \( f(x, \Pi) \) and \( g(x, \Pi) \), on a Poisson manifold as
\[ f(x, \Pi) *_{M} g(x, \Pi) := f(x, \Pi) \exp \left\{ \frac{i}{2} \left( \frac{\partial f}{\partial \Pi} \overrightarrow{\partial x} - \frac{\partial g}{\partial x} \overrightarrow{\partial \Pi} \right) \right\} g(x, \Pi). \] (11)

To obtain a quantum cosmological theme, we introduce the formal form of the Moyal star-product between super-hamiltonian function \( H(x, \Pi) \) in (5) and the Wigner function \( W(x, \Pi) \) with the Moyal-Wheeler-DeWitt as
\[ H(x, \Pi) *_{M} W(x, \Pi) = 0, \] (12)

in which the ordinary product of the observables in phase space is replaced by the Moyal product. To be in the form of Bopp’s shift formula [33], the Moyal-Wheeler-DeWitt (12) becomes
\[ H \left( x_i + \frac{i}{2} \overrightarrow{\partial_{\Pi_i}}, \Pi_i - \frac{i}{2} \overrightarrow{\partial_{x_i}} \right) W(x_i, \Pi_i) = 0, \] (13)
with \(i = 1, 2\), stating two modes of the superhamiltonian (6). This is equivalent to

\[
\left[ \frac{(\Pi_1 - \frac{i}{2} \partial_{\pi_1})^2}{2M_p} + \frac{M_p \omega^2}{2} (x_1 + \frac{i}{2} \partial_{\theta_1})^2 - \frac{(\Pi_2 - \frac{i}{2} \partial_{\pi_2})^2}{2M_p} - \frac{M_p \omega^2}{2} (x_2 + \frac{i}{2} \partial_{\theta_2})^2 \right] W(x_1, x_2, \Pi_1, \Pi_2) = 0. \tag{14}
\]

Since the Wigner function is a real valued function, one can separate the real and imaginary parts of (14) and obtain two coupled partial differential equations with the real part identified as

\[
\left[ \frac{1}{2M_p} (\Pi_1^2 - \Pi_2^2) + \frac{M_p \omega^2}{2} (x_1^2 - x_2^2) - \frac{1}{8M_p} (\partial_{x_1}^2 + \partial_{x_2}^2) - \frac{M_p \omega^2}{8} (\partial_{\Pi_1}^2 + \partial_{\Pi_2}^2) \right] W(x_1, x_2, \Pi_1, \Pi_2) = 0, \tag{15}
\]

and the imaginary part as

\[
\left[ \frac{1}{2M_p} (\Pi_1 \partial_{x_1} - \Pi_2 \partial_{x_2}) + \frac{M_p \omega^2}{2} (x_1 \partial_{\Pi_1} - x_2 \partial_{\Pi_2}) \right] W(x_1, x_2, \Pi_1, \Pi_2) = 0. \tag{16}
\]

The imaginary part (16) enforces a special symmetry that the Wigner function depending only on the superhamiltonians (6), \(W = W(H_1, H_2)\). This leads

\[
\begin{align*}
\frac{\partial^2 W}{\partial H_i} f(H_i) &= M_p \omega^2 \partial_{H_i} f(H_i) + M_p^2 \omega^4 x_i^2 \partial_{H_i}^2 f(H_i), \\
\frac{\partial^4 W}{\partial H_i^2} f(H_i) &= \frac{M_p^2 \omega^4}{M_p^2} \partial_{H_i}^2 f(H_i) + \frac{M_p^2 \omega^6}{M_p^2} \partial_{H_i}^4 f(H_i).
\end{align*}
\]

Substituting them into (15) we obtain the following differential equation

\[
\left[ \left( H_1 - \frac{\omega^2}{4} \partial_{H_1} - \frac{\omega^2}{4} H_1 \partial^2_{H_1} \right) - \left( H_2 - \frac{\omega^2}{4} \partial_{H_2} - \frac{\omega^2}{4} H_2 \partial^2_{H_2} \right) \right] W(H_1, H_2) = 0, \tag{18}
\]

which leads to the separable Wigner function

\[
W(H_1, H_2) = W_1(H_1) W_2(H_2), \tag{19}
\]

each of them satisfying

\[
H_i \frac{\partial^2 W_i(H_i)}{\partial H_i^2} + \frac{\partial W_i(H_i)}{\partial H_i} - \left( \frac{\omega^2}{4} H_i - \frac{4E}{\omega^2} \right) W_i(H_i) = 0, \tag{20}
\]

where \(E\) is a separating constant. The equations (20) give the following solutions in terms of Laguerre polynomials, \(L_n\)

\[
W_i(H_i) = C_i e^{-\frac{2H_i}{\omega}} L_n \left( \frac{4H_i}{\omega^2} \right), \tag{21}
\]

with natural numbers \(n\) and \(E = \omega(n + \frac{1}{2})\). The integration constant \(C_i = \frac{2}{\omega}\) is also obtained via the unit quasi-probability distribution,

\[
\int_0^\infty W dH_1 dH_2 = 1. \tag{22}
\]

Hence, the Wigner function (19) is

\[
W_n(H_1, H_2) = \frac{4}{\omega^2} e^{-\frac{1}{2}(H_1 + H_2)} L_n \left( \frac{4}{\omega} H_1 \right) L_n \left( \frac{4}{\omega} H_2 \right). \tag{23}
\]

Figure (1) shows a plot of Wigner function \(W_{20}(H_1, H_2)\) with classical Hamiltonian constraint (7) superimposed on it. It will be observed that among the quantum fluctuations there is a pattern of extremum in the vicinity of the classical loci. To see the classical-quantum correspondence we use the large values of quantum number \(n\). Then by
FIG. 1: The Wigner function of conformally coupled scalar field cosmology obtained in (23) for $\omega = 1$ and $n = 20$.

The corresponding classical universe (straight red line) is $H_1 - H_2 = 0$.

using the asymptotic expansion of the Laguerre function, the Wigner function (23) reduces to

$$W(H_1, H_2) \simeq \frac{2}{\pi \omega^{3/2} \sqrt{n (H_1 H_2)^{1/4}}} \cos \left( 4 \sqrt{\frac{n H_1}{\omega}} - \frac{\pi}{4} \right) \cos \left( 4 \sqrt{\frac{n H_2}{\omega}} - \frac{\pi}{4} \right). \quad (24)$$

The locus of extremums of the above Wigner function are given by the following simultaneous conditions

$$H_1 = \frac{\pi \omega}{16 n} \left( m_1 + 1 \right)^2, \quad H_2 = \frac{\pi \omega}{16 n} \left( m_2 + 1 \right)^2, \quad (25)$$

with $m_1, m_2 = 0, 1, 2, \ldots$. The quantum deformed superhamiltonian for the most probable universes is then given by

$$\frac{\Pi_1^2}{2 M_p} + \frac{1}{2} M_p \omega^2 x_1^2 - \frac{\Pi_2^2}{2 M_p} - \frac{1}{2} M_p \omega^2 x_2^2 - \mathcal{E} = 0, \quad (26)$$

where

$$\mathcal{E} := \frac{\omega \pi^2 (m_1 - m_2)}{32 n} (1 + 2m_1 + 2m_2). \quad (27)$$

Considering $m_1 \simeq n \gg m_2$, we get

$$\mathcal{E} = \frac{\pi^2 \omega n}{16}. \quad (28)$$

On the other hand, one can show that the quantum corrections are manifested in classical emerged universe as a perfect fluid of radiation type, as it is shown in [34, 35]

$$\mathcal{E} = 2 \pi^2 \omega \rho x_1 = \frac{\omega}{8} \left( \frac{1215}{\pi^4} \right)^{1/3} S_\gamma^{1/3}, \quad (29)$$

where $\rho$ and $S_\gamma$ are energy density and entropy of radiation, respectively. The equality of relations (28) and (29) gives the estimation for the radiation entropy as $S_\gamma \sim n^{2/3}$.

Using the current value of the entropy of radiation, i.e. $S_\gamma \sim 10^{88}$, we can estimate the approximate value of the quantum number $n$ as $n \sim 10^{117}$ which is in agreement with [26, 34].
IV. CONCLUSION

In the present work, we studied quantum cosmology of a conformally coupled scalar field in a positive curvature background of a FLRW type universe. We have solved the MWDW equation exactly for this model and obtained the quasi-probability distribution Wigner function that relate to classical solutions without recourse to WKB approximation techniques. This equation, which is plotted in figure (1), shows that there existed a peak over the classical trajectory, which indicates a good coincidence between classical and the most probable quantum states. As we know, CMBR is landmark evidence of the Big Bang origin of the universe. We showed for large values of quantum number \( n \), the classical universe arose from the quantized model includes radiation perfect fluid.
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