The Energy-Weighted and Non Energy-Weighted Gamow-Teller Sum Rules in Relativistic Random Phase Approximation

Haruki Kurasawa\(^1\) and Toshio Suzuki\(^2\)

\(^1\) Department of Physics, Faculty of Science, Chiba University, Chiba 263-8522, Japan
\(^2\) Department of Applied Physics, Fukui University, Fukui 910-8507, Japan

RIKEN, 2-1 Hirosawa, Wako-shi, Saitama 351-0198, Japan

The non-energy-weighted Gamow-Teller (GT) sum rule is satisfied in relativistic models, when all nuclear density-dependent terms, including Pauli blocking terms from nucleon-antinucleon excitations, are taken into account in the RPA correlation function. The no-sea approximation is equivalent to this approximation for the giant GT resonance state and satisfies the sum rule, but each of the total \(\beta_-\) and \(\beta_+\) strengths is different in the two approximations. It is also shown that the energy-weighted sum of the GT strengths for the \(\beta_-\) and \(\beta_+\) transitions in RPA is equal to the expectation value of the double commutator of the nuclear Hamiltonian with the GT operator, when the expectation value is calculated with the ground state in the mean field approximation. Since the present RPA neglects renormalization of the divergence, however, the energy-weighted strengths outside of the giant GT resonance region become negative. These facts are shown by calculating in an analytic way the GT strengths of nuclear matter.

PACS numbers: 21.60.-n, 21.60.Jz, 21.65.+f

I. INTRODUCTION

Recently, the present authors have investigated Ikeda-Fujii-Fujita(IFF) sum rule\(^3\) of Gamow-Teller(GT) transitions in relativistic models.\(^2\)\(^4\). It has been shown that the sum of the GT strengths in the nucleon sector is quenched by about 6% in finite nuclei\(^2\) and 12% in nuclear matter.\(^2\) This fact is owing to the small component of nucleon wave functions, and the quenched amount is taken by nucleon-antinucleon states. Although the nucleon-antinucleon states are far from the giant GT resonance region, if there is a coupling with particle-hole states, they may contribute to nuclear excitations in the giant resonance region as virtual states. In the previous paper\(^2\)\(^3\), the effects of the coupling were estimated with use of the random phase approximation(RPA) in relativistic nuclear models. RPA correlation functions were calculated by neglecting the nuclear density-independent terms in order to avoid the divergence, but by keeping all the density-dependent terms including the Pauli blocking ones from nucleon-antinucleon excitations. This method was frequently used in the study of no charge-exchange excitations.\(^6\)\(^7\)\(^8\). From now on let us call this approximation no free term approximation(NFA). In this calculation, we found that effects of the coupling are negligible on the excitation energy and strength of the GT state in the nucleon sector.

The purpose of the present paper is threefold. First, we will study whether or not NFA satisfies IFF sum rule which is the non energy-weighted sum rule for the difference between the \(\beta_-\) and \(\beta_+\) transition strengths. When we discuss excitation strengths, the used approximation should satisfy at least the model-independent sum rule. It seems not to be trivial for NFA to satisfy IFF sum rule, since for the \(\beta_-\) transitions, the backward amplitudes in the particle-hole sector do not contribute to the strength as in Tamm-Dankoff approximation, while in the nucleon-antinucleon excitations, they are comparable to the forward ones in the present model. We will show, however, that the sum rule is satisfied in NFA. The proof will be done in an analytic way for nuclear matter.

Second, we will discuss the sum rule in the no-sea approximation(NSA), which is also used frequently in no charge-exchange excitations.\(^6\)\(^7\)\(^8\). It will be shown that NSA also satisfies IFF sum rule, but that each total strength of the \(\beta_-\) and \(\beta_+\) transitions is different from those of NFA. As far as the giant GT resonance state, however, its excitation energy and strength are predicted in the same way as in NFA.

Third, we will investigate the energy-weighted sum of the GT strengths in RPA, which have not been discussed so far, as far as the authors know. For no charge-exchange excitations, there is a famous theorem in the non-relativistic framework that the energy-weighted sum of the excitation strengths in RPA is equal to the expectation value of the double commutator of the nuclear Hamiltonian with the relevant operator.\(^9\) Here the expectation value is calculated with the Hartree-Fock ground state. It will be shown in relativistic models that the same theorem holds for charge-exchange excitations with respect to the sum of the GT strengths for \(\beta_-\) and \(\beta_+\) transitions. In both NFA and NSA, however, the energy-weighted strength outside of the giant GT resonance region is negative. This is due to the fact that divergent terms are simply neglected in the two approximations. We definitely need the renormalization to solve this problem in future work.

We note here that such detailed discussions as mentioned in the above are required not only from a theoretical point of view, but also from the recent experiment. It has been shown experimentally that the sum rule value is quenched by about 10% from the non-relativistic analysis.
of \((p, n)\) reaction\textsuperscript{10}. So far, all the 10% quenching was assumed to be due to the coupling of the particle-hole states with \(\Delta\)-hole states\textsuperscript{11,12}. If the 6\% quenching is due to the relativistic effects, it may conclude that only 4\% stems from the contribution of the \(\Delta\) degrees of freedom. Such a weak coupling of the particle-hole states with the \(\Delta\)-hole states may yield the extremely small value of Landau-Migdal parameter \(\gamma_{\Delta}\) which dominates the critical density of the pion condensation\textsuperscript{13} and other spin-dependent response functions\textsuperscript{14}. Thus we need to discuss GT strengths carefully, before studying spin-dependent structure of nuclei in detail.

In the next section, we will briefly review NFA and NSA in no charge-exchange excitations. In section \[\text{III}\] the mean field correlation function of the GT excitations will be derived. In section \[\text{IV}\] we will discuss the GT sum rules in RPA within the framework of NFA and NSA. The final section will be devoted to discussions and conclusions of the present paper.

II. NFA AND NSA

The fundamental problem in relativistic nuclear models is how to renormalize the divergence due to the antinucleon degrees of freedom. Only few attempts of the renormalization have been reported so far\textsuperscript{14,16}. Nevertheless, it has been shown for the past 30 years that relativistic models explain phenomenologically very well nuclear structure and reactions without renormalization\textsuperscript{16,17}. In those studies, there are some cases where it is necessary to take into account a part of the antinucleon degrees of freedom. In RPA based on the mean field approximation, the continuity equation of the baryon current is violated, if the configuration space is limited to the nucleon sector only. At present, there are two ways to avoid the violation. The one(NFA) is to neglect the density-independent terms in the RPA correlation function which are divergent, but to keep all the density-dependent terms including the Pauli blocking terms from nucleon-antinucleon excitations\textsuperscript{4,5,6}. The other is called the no-sea approximation(NSA), where the Dirac sea is assumed to be empty, and the antinucleon states are treated as particle states with negative energies in the configuration space of RPA\textsuperscript{4}. NSA is described by changing the sign of the imaginary part of the Green function of the antinucleon. It is shown in this way that the divergence of the RPA correlation function disappears and that the continuity equation is not violated\textsuperscript{4}.

In this section, we briefly review the above two approximations in nuclear matter, since their structure in charge-exchange excitations is a little different from that in no charge-exchange ones discussed so far.

The Green function of the Dirac particle in nuclear matter is written as\textsuperscript{4,16},

\[
G(p) = (1 - \theta_{p})G_{p}(p) + \theta_{p}G_{h}(p) + G_{N}(p),
\]

where the particle, hole and antinucleon parts are given, respectively, as

\[
G_{p}(p) = \frac{A_{+}(p)}{p_{0} - E_{p} + i\varepsilon}, \quad G_{h}(p) = \frac{A_{+}(p)}{p_{0} - E_{p} - i\varepsilon}, \quad G_{N}(p) = -\frac{A_{-}(p)}{p_{0} + E_{p} - i\varepsilon}.
\]

The projection operators in the above equation are defined as usual,

\[
A_{+}(p) = \sum_{\alpha} u_{\alpha}(p)\bar{\sigma}_{\alpha}(p) = \frac{\hat{p} + M^{*}}{2E_{p}}, \quad A_{-}(p) = -\sum_{\alpha} v_{\alpha}(-p)\bar{\sigma}_{\alpha}(-p) = \frac{\hat{p} + M^{*}}{2E_{p}}
\]

with

\[
p^{\mu} = (E_{p}, p), \quad \hat{p}^{\mu} = (-E_{p}, p), \quad E_{p} = \sqrt{M^{*2} + p^{2}}.
\]

The step function is expressed by using the abbreviation: \(\theta_{p} = \theta(k_{F} - |p|), k_{F}\) being the Fermi momentum. The Green function Eq.\textsuperscript{11} can be also expressed in terms of density-dependent and -independent parts as,

\[
G(p) = G_{D}(p) + G_{F}(p),
\]

\[
G_{D}(p) = \theta_{p}(G_{h}(p) - G_{p}(p)),
\]

\[
G_{F}(p) = G_{p}(p) + G_{N}(p).
\]

The mean field correlation function is described as

\[
\Pi = -\frac{1}{2\pi i} \int d^{4}p \text{Tr}(\gamma_{a}G(p + q)\gamma_{b}G(p)),
\]

where \(\gamma_{a}\) and \(\gamma_{b}\) stand for the \(4 \times 4\) matrix of the external field. If Eq.\textsuperscript{13} is inserted into Eq.\textsuperscript{8}, the mean field correlation function is divergent, because of the density-independent term which contains \(G_{F}G_{F}\). In NSA, it is simply neglected without renormalization, to have

\[
\Pi_{D} = -\frac{1}{2\pi i} \int d^{4}p \text{Tr}(\gamma_{a}G_{D}(p + q)\gamma_{b}G_{D}(p) + G_{F}(p + q)\gamma_{b}G_{D}(p) + G_{D}(p + q)\gamma_{b}G_{F}(p)).
\]

Keeping the terms which remain after integration over \(p_{0}\) in the complex plane, the above equation is separated into
two parts,
\[ \Pi_D = \Pi_{ph} + \Pi_{Pauli}. \] (10)

The first term represents the particle-hole correlation function, while the second term the nucleon-antineucleon one,
\[ \Pi_{ph} = -\frac{1}{2\pi i} \int d^4 p \text{Tr} \gamma_a ((1 - \theta_{p+q}) \theta_p G_p(p + q) \gamma_b G_{h}(p) + (1 - \theta_p) \theta_{p+q} G_{h}(p + q) \gamma_b G_p(p)), \] (11)
\[ \Pi_{Pauli} = \frac{1}{2\pi i} \int d^4 p \text{Tr} \gamma_a (\theta_{p+q} G_p(p + q) \gamma_b G_{N}(p) + \theta_p G_{N}(p + q) \gamma_b G_p(p)). \] (12)

Thus, \( \Pi_{Pauli} \) is composed of the Pauli blocking terms. In NFA, however, these terms are necessary for keeping the continuity equation of the baryon current, and for satisfying the GT sum rule, as shown later.

In NSA, Green function of the antinucleon in Eq.(14) is artificially modified by changing a sign of the imaginary part,
\[ G_{no}(p) = -\frac{\Lambda_-(p)}{p_0 + E_p + i\varepsilon}. \] (13)

Then density-independent terms have no contribution to the integration over \( p_0 \) in Eq.(15), and we can avoid the divergence problem without violating the continuity equation. The terms which should be kept in NSA are written as
\[ \Pi = \Pi_{ph} + \Pi_{no-sea}, \] (14)

where the particle-hole correlation function is the same as in NFA, but the antinucleon-dependent part is given by
\[ \Pi_{no-sea} = -\frac{1}{2\pi i} \int d^4 p \text{Tr} \gamma_a (\theta_{p+q} G_h(p + q) \gamma_b G_{no}(p) + \theta_p G_{no}(p + q) \gamma_b G_h(p)). \] (15)

After integration over \( p_0 \), the mean field correlation functions in NFA and NSA are finally written, respectively, as
\[ \Pi_{ph} = \int d^3 p \text{Tr} (\gamma_a A_+(p + q) \gamma_b A_+(p)) \left( \frac{(1 - \theta_{p+q}) \theta_p}{E_{p+q} - E_p - q_0 - i\varepsilon} - \frac{(1 - \theta_p) \theta_{p+q}}{E_{p+q} - E_p - q_0 + i\varepsilon} \right), \] (16)
\[ \Pi_{Pauli} = \int d^3 p \text{Tr} \left( \theta_{p+q} \frac{\gamma_a A_+(p + q) \gamma_b A_+(p)}{E_{p+q} + E_p - q_0 - i\varepsilon} + \theta_p \frac{\gamma_a A_(p + q) \gamma_b A_+(p)}{E_{p+q} + E_p + q_0 + i\varepsilon} \right), \] (17)
\[ \Pi_{no-sea} = \int d^3 p \text{Tr} \left( \theta_{p+q} \frac{\gamma_a A_+(p + q) \gamma_b A_-(p)}{E_{p+q} + E_p - q_0 + i\varepsilon} + \theta_p \frac{\gamma_a A_(p + q) \gamma_b A_-(p)}{E_{p+q} + E_p + q_0 + i\varepsilon} \right). \] (18)

Thus, the real parts are the same in NFA and NSA, while the imaginary parts of \( \Pi_{Pauli} \) and \( \Pi_{no-sea} \) have an opposite sign to each other. This fact implies that both approximations yield the same excitation energies for the discrete states in RPA, but there is a possibility that their response functions and sum values of the excitation strengths are different from each other. An example will be shown later in the case of the GT strengths.

We note that the relationship between the Landau-Migdal parameters and the correlation functions is the same in NFA and NSA, since it depends on the only real part of the correlation functions, as shown in ref.16. All physical quantities expressed in terms of the Landau-Migdal parameters, therefore, must be the same in the two approximations. The reason why NSA as well as NFA does not violate the continuity equation is because it is also independent of the imaginary part. This fact is verified by showing explicitly
\[ \Pi_{ph} + \Pi_{Pauli} = \Pi_{ph} + \Pi_{no-sea} = 0 \] (19)
for the operator \( \gamma_b = \frac{\gamma}{\gamma} \). Each of them for \( \gamma_b = \frac{\gamma}{\gamma} \) is written, respectively, as
\[ \Pi_{ph} = \int d^3 p \frac{\theta_p}{4 E_p E_{p+q}} \text{Tr} (\gamma_a \gamma_0 (\hat{p}' \hat{p} - \hat{p} \hat{p}')), \]
\[ \Pi_{Pauli} = \Pi_{no-sea} = \int d^3 p \frac{\theta_p}{4 E_p E_{p+q}} \text{Tr} (\gamma_a \gamma_0 (\hat{p}' \hat{p} - \hat{p} \hat{p}')), \]
using the notations:
\[ p'^\mu = (E_{p+q}, p + q), \quad \tilde{p}'^\mu = (-E_{p+q}, p + q). \]
III. THE MEAN FIELD CORRELATION FUNCTIONS FOR GT EXCITATIONS

When we discuss GT excitations in $N \neq Z$ nuclei, we need the following replacement in Eqs. (16), (17) and (18),

$$\gamma_a \rightarrow \gamma_a = \gamma_5 \gamma_y \tau^+, \quad \gamma_b \rightarrow \gamma_b = \gamma_5 \gamma_y \tau^-,$$

$$\theta_p \rightarrow \frac{1 - \tau_z}{2} \theta_p^{(p)} + \frac{1 + \tau_z}{2} \theta_p^{(n)},$$

where we have defined the isospin operators for the convenience as

$$\tau_\pm = (\tau_x \pm i \tau_y) / \sqrt{2}, \quad \tau_0 = \tau_z,$$

and the step function with respect to the proton($k_p$) and neutron Fermi momentum($k_n$) :

$$\theta_p^{(i)} = \theta(k_i - |p|), \quad i = p, n.$$

Then, for the $\beta_-$ excitation at $q = 0$ in $N > Z$ nuclei, Eq. (16) becomes to be

$$\Pi_{ph} = -4 \int d^3 p \frac{P^2 - P_y^2}{E_p^2} \left[ \theta_p^{(p)} - \theta_p^{(n)} \right],$$

and Eqs. (17) and (18) are described, respectively, as

$$\Pi_{Pauli} = 4 \int d^3 p \frac{P^2 - P_y^2}{E_p^2} \left[ \theta_p^{(p)} - \theta_p^{(n)} \right],$$

$$\Pi_{no-sea} = -4 \int d^3 p \frac{P^2 - P_y^2}{E_p^2} \left[ \theta_p^{(n)} - \theta_p^{(p)} \right].$$

From the above equations, we see, at this stage, first that the present model has only the forward amplitudes for particle-hole pairs. Second, in NFA, the forward amplitudes of the nucleon-antinucleon pairs have an opposite sign to the one of the particle-hole pairs. This means that the GT strength from the Pauli blocking terms is negative. Third, in NSA, the excitation energy of the antinucleon-hole pairs is negative, as mentioned before. Finally we can see that the density dependence of NSA, expressed by the step functions, is different from the one of NFA. As a result, their GT strengths are different from each other, although the difference between the strengths of $\beta_-$ and $\beta_+$ transitions is the same in the two approximations.

The correlation functions for the $\beta_+$ transitions are obtained by changing the sign of $q_0$ in Eqs. (20) to (22). In this case, there are no forward amplitudes for particle-hole pairs, but backward ones.

By performing the integration, Eq. (20) is written as

$$\Pi_{ph} = -\frac{\alpha_{ph}}{q_0 + i \epsilon}, \quad \alpha_{ph} = \frac{16\pi}{3} \left( Q(k_n) - Q(k_p) \right),$$

where $Q(k_i)$ is defined by

$$Q(k_i) = \frac{3}{4\pi} \int_0^{k_i} d^3 p \frac{M^2 + P_y^2}{E_p^2}$$

$$= \frac{k_i^3}{3} + 2k_iM^2 - 2M^3 \tan^{-1} \frac{k_i}{M}. \quad (23)$$

Eqs. (21) and (22) are expressed by separating into the real and imaginary parts:

$$\Pi_{Pauli}^{(R)} = \Pi_{no-sea}^{(R)}$$

$$\Pi_{Pauli}^{(I)} = -\Pi_{no-sea}^{(I)},$$

which are related with each other as

$$\Pi_{Pauli}^{(R)} = \Pi_{no-sea}^{(R)} \quad \Pi_{Pauli}^{(I)} = -\Pi_{no-sea}^{(I)}.$$

The explicit forms of $\Pi_{Pauli}^{(R)}$ and $\Pi_{Pauli}^{(I)}$ are described as

$$\Pi_{Pauli}^{(R)} = -\frac{16\pi}{3} \left[ P_c(k_p, -q_0) + P_c(k_n, q_0) \right],$$

$$\Pi_{Pauli}^{(I)} = I_{Pauli}(k_p, q_0) + I_{Pauli}(k_n, -q_0),$$

where we have defined the two functions:

$$P_c(k_p, q_0) = \frac{3}{4\pi} \int_0^{k_p} d^3 p \frac{P^2 - P_y^2}{E_p^2 + q_0},$$

$$I_{Pauli}(k_p, q_0) = -\frac{16\pi}{3} \int_0^{k_p} d^3 p \frac{P^2 - P_y^2}{E_p^2} \delta(q_0 - 2E_p).$$

The second function is simply expressed as

$$I_{Pauli}(k_p, q_0)$$

$$= \begin{cases} -\frac{4\pi^2 (q_0^2 - 4M^2)^{3/2}}{3q_0}, & 2M^* < q_0 < 2E_F, \\ 0, & \text{otherwise}, \end{cases} \quad (24)$$

with $E_F = \sqrt{M^2 + k_F^2}$, while the first one is written as

$$P_c(k_F, q_0) = \frac{k_F E_F}{2} - \frac{q_0 k_F}{2} - \frac{6M^2 - q_0^2}{4} \log \frac{k_F + E_F}{M^*}$$

$$+ 2M^3 \tan^{-1} \frac{k_F}{M^*} + \frac{4M^2 - q_0^2}{4q_0} I_B,$$

where $I_B$ is given by

$$I_B = \int_{E_F - k_F}^{M^*} dt \frac{q_0^2 - 4M^2}{t^2 + q_0^2 + M^2}$$

$$= \begin{cases} -4M^* \sqrt{1 - x^2} \tan^{-1} y, & x^2 < 1 \\ 2M^* \sqrt{x^2 - 1} \log \frac{1 + y}{1 - y}, & x^2 > 1 \end{cases}$$
Then, the total GT strength in the mean field approximation from the particle-hole states to the strength of \( V \) contains \( \eta \) with
\[
\begin{align*}
F_0 & = \frac{3}{2} v_F^2 - \frac{3}{4} v_F^4 + \cdots \\
& = \frac{k_F^2 v_F^2}{3} \left( 1 + \frac{2}{3} v_F^2 + \cdots \right),
\end{align*}
\]
where \( v_F \) denotes the Fermi velocity \( k_F/E_F \). Thus, in contrast to the imaginary part, the real part of \( \Pi^{\text{no-sea}} \) and \( \Pi^{\text{Pauli}} \) may yield a small contribution to the excitation energy and strength of the giant GT resonance state in the region \( q_0 \ll 2M^* \).

The response function \( R(q_0) \) is defined with the above correlation functions as
\[
R(q_0) = \frac{V}{\pi} \frac{E}{(2\pi)^3} \text{Im} \Pi(q_0),
\]
where \( V \) standing for the volume of the system \( (3\pi^2/2k_F^3)A \). Then, the total GT strength in the mean field approximation is obtained by integrating it over \( q_0 \). The contribution from the particle-hole states to the strength of the \( \beta^- \) transitions is
\[
S_{\text{ph}}^{(-)} = \frac{1}{\pi} \frac{V}{(2\pi)^3} \lim_{q_0 \to 0} \int_0^\infty dq_0 \text{Im} \Pi_{\text{ph}}(q_0) = \frac{4V}{(2\pi)^3} \int d^3 p \left( \theta_{\text{p}}^{(n)} - \theta_{\text{p}}^{(p)} \right) \frac{M^2 + p_p^2}{E_p^2},
\]
where the variable \( \eta \) is introduced, since \( \text{Im} \Pi_{\text{ph}}(q_0) \) contains \( \delta(q_0) \). This can be written with use of Eq. (23) as
\[
S_{\text{ph}}^{(-)} = \frac{A}{k_F^2} (Q(k_n) - Q(k_p)).
\]
The function \( Q(k_n) - Q(k_p) \) can be expanded in terms of \( (k_n - k_p) \),
\[
Q(k_n) - Q(k_p) \approx \frac{dQ(k_F)}{dk_F} (k_n - k_p) + \cdots,
\]
where we have
\[
dQ(k_F) = 3k_F^2 \left[ 1 - \frac{2}{3} v_F^2 \right].
\]
When utilizing as usual the relationship
\[
k_n - k_p \approx \frac{2}{3} k_F N - \frac{Z}{A},
\]
the GT strength in the nucleon sector is approximately written as
\[
S_{\text{ph}}^{(-)} \approx \left( 1 - \frac{2}{3} v_F^2 \right) 2(N - Z).
\]
In the present definition of the spin-isospin operators, the well-known IFF sum rule is described as
\[
\langle | \frac{Q^+}{Q^-} - | \frac{Q^-}{Q^+} \rangle = 2(N - Z)
\]
with
\[
Q_{\pm} = \sum_i A \tau_{\pm i} \sigma_{yi}.
\]
This sum rule is nothing but a result of the commutation relation: \( [\tau_{+} \sigma_{y}, \tau_{-} \sigma_{y}] = 2r_2 \). If we assume that \( Q_{+} = 0 \), then we simply obtain
\[
\langle | \frac{Q^+}{Q^-} | \rangle = 2(N - Z).
\]
Comparing Eq. (23) with the above equation, it is seen that the GT strength in the nucleon sector of the relativistic model is quenched by a factor \( (1 - \frac{2}{3} v_F^2) \). In fact, as shown below, the quenched amount is taken by the nucleon-antinucleon pair excitations. In the present model, there is no GT strength of the \( \beta^- \) transitions in the particle-hole sector,
\[
S_{\text{ph}}^{(+)} = \frac{1}{\pi} \frac{V}{(2\pi)^3} \lim_{q_0 \to 0} \int_0^\infty dq_0 \text{Im} \Pi_{\text{ph}}(q_0) = 0.
\]
The total GT strength of the proton-antineutron(\( \beta^- \)) excitations in NFA is given by
\[
S_{\text{Pauli}}^{(-)} = \frac{1}{\pi} \frac{V}{(2\pi)^3} \int_0^\infty dq_0 \text{Im} \Pi_{\text{Pauli}}(q_0),
\]
which is negative, as mentioned before. The one of the neutron-antiproton(\( \beta^+ \)) excitations is obtained in the same way by changing the sign of \( q_0 \) in Eq. (24).
\[
S_{\text{Pauli}}^{(+)} = \frac{1}{\pi} \frac{V}{(2\pi)^3} \int_0^\infty dq_0 \text{Im} \Pi_{\text{Pauli}}(-q_0) = \frac{4V}{(2\pi)^3} \int d^3 p \left( \theta_p^{(n)} \frac{D^2 - p_p^2}{E_p^2} \right).
\]
From Eqs. (24) and (26), we obtain the GT sum value in the mean field approximation of NFA,
\[
S_{\text{ph}}^{(-)} + S_{\text{Pauli}}^{(-)} - S_{\text{Pauli}}^{(+)} = 2(N - Z),
\]
which is just IFF sum rule. Thus the quenched amount of the GT strength in the nucleon sector is taken by the nucleon-antinucleon pairs, and the Pauli blocking terms are necessary for NFA to satisfy the sum rule.

In NSA, the GT strength of the particle-hole pairs is the same as in NFA, but the strengths from the antinucleon degrees of freedom are different. For the \( \beta^- \) transitions, the integration of Eq. (24) over negative excitation energy gives
\[
S_{\text{no-sea}}^{(-)} = \frac{1}{\pi} \frac{V}{(2\pi)^3} \int_{-\infty}^0 dq_0 \text{Im} \Pi_{\text{no-sea}}(q_0) = \frac{4V}{(2\pi)^3} \int d^3 p \left( \theta_p^{(n)} \frac{D^2 - p_p^2}{E_p^2} \right).
This is equal to $-S_{\text{Pauli}}^{(+)}$ as seen in Eq.(30),

$$S_{\text{no-sea}}^{(-)} = -S_{\text{Pauli}}^{(+)}.$$

In the same way, we obtain the relationship:

$$S_{\text{no-sea}}^{(+)} = -S_{\text{Pauli}}^{(-)}.$$

where the l.h.s denotes the total strength of the $\beta_+$ antinucleon-hole excitations in NSA. In NSA, each strength is positive, but the energy-weighted sum becomes negative. Although the strengths of the $\beta_-$ and $\beta_+$ transitions in NSA are different from those in NFA, their difference satisfies the sum rule, as in Eq. (31).

$$S_{\text{ph}}^{(-)} + S_{\text{no-sea}}^{(-)} - S_{\text{no-sea}}^{(+)} = 2(N-Z).$$ (32)

### IV. THE GT SUM RULE IN RELATIVISTIC RPA

In non-relativistic models, RPA correlations in the GT states are assumed to be induced by Landau-Migdal(LM) parameter $g'$. In the relativistic model, we also introduce $g'$ which is provided in the nuclear Lagrangian as a contact term with the pseudo vector coupling:

$$L = \frac{g'}{2} \overline{\psi} \gamma^\mu \psi \gamma_\mu \psi,$$ (33)

where

$$\Gamma^\mu = \gamma_5 \gamma^\mu \tau_1, \quad g_5 = \left( \frac{f_\pi}{m_\pi} \right)^2 g'.$$

Although it is not unique how to introduce $g'$ in the relativistic model, we have shown that the above Lagrangian yields the known expression for the excitation energy of the GT state in the non-relativistic limit. We note that, for example, the GT state can not be described in relativistic models by putting the LM parameter into the meson propagators.

For the Lagrangian Eq. (33), the RPA correlation function $\Pi_{\text{RPA}}$ is written in terms of the mean field one $\Pi$ as,

$$\Pi_{\text{RPA}}(q_0) = \frac{\Pi(q_0)}{1 + \chi_5 \Pi(q_0)}, \quad \chi_5 = \frac{g_5}{(2\pi)^3}.$$

Then the RPA response functions for the $\beta_-$ and $\beta_+$ transitions are given, respectively, by

$$R_{\text{RPA}}^{(\pm)}(q_0) = \frac{V}{\pi (2\pi)^3} \text{Im} \Pi_{\text{RPA}}(\pm q_0).$$

In NFA, $\Pi(q_0)$ is provided by Eqs. (20) and (21) as,

$$\Pi(q_0) = -\frac{\alpha_{\text{ph}}}{q_0 + i\varepsilon} - 4 \int d^3p \frac{p^2 - p_y^2}{E_p^2} \left( \frac{\theta_p^{(p)}}{2E_p - q_0 - i\varepsilon} + \frac{\theta_p^{(n)}}{2E_p + q_0 - i\varepsilon} \right),$$ (34)

while the one in NSA is given by Eqs. (20) and (21) as,

$$\Pi(q_0) = -\frac{\alpha_{\text{ph}}}{q_0 + i\varepsilon} - 4 \int d^3p \frac{p^2 - p_y^2}{E_p^2} \left( \frac{\theta_p^{(p)}}{2E_p - q_0 + i\varepsilon} + \frac{\theta_p^{(n)}}{2E_p + q_0 + i\varepsilon} \right).$$ (35)

### A. The Non Energy-Weighted GT Sum Rule

In NFA, the non energy-weighted GT sum value in RPA is given by

$$S_{\text{RPA}}^{(0)} = \lim_{\eta \to 0} \left\{ \int_\eta^{\infty} dq_0 R_{\text{RPA}}^{(\pm)}(q_0) - \int_{-\infty}^\eta dq_0 R_{\text{RPA}}^{(\pm)}(q_0) \right\}.$$

In order to calculate the r.h.s., we expand $\Pi_{\text{RPA}}$ in terms of $\chi_5$,

$$S_{\text{RPA}}^{(0)} = \frac{1}{\pi (2\pi)^3} \frac{V}{\chi_5} \sum_{n=0}^{\infty} (-\chi_5)_n \lim_{\eta \to 0} \int_{\eta}^{\infty} I_n^{(0)},$$ (36)

where $I_n^{(0)}$ is defined as

$$I_n^{(0)} = \int_{-\infty}^{\eta} dq_0 \text{Im} \left( \Pi(q_0) \right)^{n+1} - \int_{\eta}^{\infty} dq_0 \text{Im} \left( \Pi(-q_0) \right)^{n+1}.$$ (37)
Since \( \Pi(\pm q_0) \) has no pole in the first quadrant of the complex plane, the integration on the closed contour provides us with:

\[
\int_{-\eta}^{\eta} dq_0 \left( \Pi(q_0) \right)^{n+1} - i \int_{0}^{R} dq_0 \left( \Pi(iq_0 - \eta) \right)^{n+1} + \int_{C} dq_0 \left( \Pi(q_0) \right)^{n+1} = 0, \tag{38}
\]

\[
\int_{\eta}^{\pi} dq_0 \left( \Pi(-q_0) \right)^{n+1} - i \int_{0}^{R} dq_0 \left( \Pi(-iq_0 - \eta) \right)^{n+1} + \int_{C} dq_0 \left( \Pi(-q_0) \right)^{n+1} = 0, \tag{39}
\]

where \( C \) indicates the contour on \( q_0 = Re^{i\theta} \), \( (0 \leq \theta \leq \pi/2) \). Using the above two equations together with the fact that \( \Pi^*(-iq_0 - \eta) = \Pi(iq_0 - \eta) \), \( I_n \) can be expressed as

\[
I_n^{(0)} = -\text{Im} \int_{C} dq_0 \left[ (\Pi(q_0))^{n+1} - (\Pi(-q_0))^{n+1} \right]. \tag{40}
\]

When \( |q_0| \to \infty, \Pi(q_0) \) in Eq. (33) behaves as

\[
\Pi(q_0) = -\frac{2(2\pi)^3 N - Z}{q_0}. \tag{41}
\]

Hence, the integration on the contour gives

\[
\int_{C} dq_0 \left( \Pi(\pm q_0) \right)^{n+1} = \frac{i}{\pi R^n} \left( \pm 2(2\pi)^3 \frac{N - Z}{V} \right)^{n+1} \int_{0}^{\pi/2} d\theta e^{-in\theta} \to 0, \ (R \to \infty, \ n \geq 1). \tag{42}
\]

From Eqs. (40) and (42), finally we obtain

\[
I_n^{(0)} = 0, \quad (n \geq 1). \]

Hence, the GT sum value of RPA in NFA is equal to the one in the mean field approximation Eq. (31),

\[
S_{\text{RPA}} = \frac{1}{\pi} \frac{V}{(2\pi)^3} \lim_{\eta \to 0} I_0^{(0)} = 2(N - Z).
\]

Thus, RPA in NFA satisfies IFF sum rule.

We can calculate in the same way the GT sum value of RPA in NSA, but by defining \( J_n^{(0)} \) instead of \( I_n^{(0)} \) in Eq. (33),

\[
J_n^{(0)} = \int_{-\infty}^{\eta} dq_0 \text{Im} \left( \Pi(q_0) \right)^{n+1} - \int_{-\infty}^{\eta} dq_0 \text{Im} \left( \Pi(-q_0) \right)^{n+1},
\]

where \( \Pi(q_0) \) is given by Eqs. (33). By taking a closed contour with no pole in the second quadrant in this case, we can prove that \( J_n^{(0)} \) is \( 0 \) \((n \geq 1)\), and therefore, NSA also satisfies IFF sum rule.

In the above, we have shown that both NFA and NSA satisfy IFF sum rule. This means that the difference between total GT strengths for the \( \beta_- \) and \( \beta_+ \) transitions is independent of the value of LM parameter \( \ell' \). Of course, however, the strength of the giant GT resonance state and each total strength of the \( \beta_- \) and \( \beta_+ \) transitions depend on the value of the parameter. Let us estimate each strength separately.

We divide the excitation-energy region into two parts: \( |q_0| < 2M^* \) and \( |q_0| \geq 2M^* \). In the first region \( |q_0| < 2M^* \), we have a discrete state which corresponds to the giant GT resonance state in both NFA and NSA.

In this region, we have \( \Pi_{\text{Pauli}}^{(I)} = \Pi_{\text{no-sea}}^{(I)} = 0 \), so that the RPA response function is described as

\[
R_{\text{RPA}}(q_0) = -\frac{1}{\chi_5} \frac{V}{8\pi^4} \text{Im} \left( \frac{q_0}{F_T(q_0) + i\varepsilon} \right),
\]

with

\[
F_T(q_0) = q_0 + \chi_5 \left( -\alpha_{ph} + q_0 \Pi_{\text{Pauli}}^{(R)}(q_0) \right).
\]

In the case of NSA, \( \Pi_{\text{Pauli}}^{(R)}(q_0) \) is replaced with \( \Pi_{\text{no-sea}}^{(R)}(q_0) \), but they are the same, as mentioned before.

The excitation energy \( \omega_0 \) of the giant GT resonance state is given by the solution of the equation:

\[
F_T(\omega_0) = 0.
\]

When we expand \( F_T(q_0) \) at \( \omega_0 \) as,

\[
F_T(q_0) = F_T'(\omega_0)(q_0 - \omega_0) + O((q_0 - \omega_0)^2),
\]

we can describe the response function in the form:

\[
R_{\text{RPA}}(q_0) = \frac{1}{\chi_5} \frac{V}{8\pi^4} \frac{\omega_0}{F_T'(\omega_0)} \delta(q_0 - \omega_0).
\]

From this equation, we obtain the excitation strength of the GT state:

\[
S_{\text{GT}} = \frac{1}{\chi_5} \frac{V}{8\pi^4} \frac{1}{F_T'(\omega_0)}, \tag{43}
\]
\[ D_T(\omega_0) = \chi_5 \left( \frac{\alpha_{ph}}{\omega_0^2} + \frac{d}{d\omega_0} I_{\text{Pau}}^{(R)}(\omega_0) \right), \]

using the dinemic function \( D_T(q_0) \) defined by

\[ F_T(q_0) = q_0 D_T(q_0). \]

If we neglect \( I_{\text{Pau}}^{(R)} \), which is small at \( |q_0| \ll 2M^*, \omega_0 \)

\[ \text{TABLE I: Dependence of the excitation energy } \omega_0 \text{ and strength } S_{GT} \text{ of the GT state on Landau-Migdal parameter } g'. \text{ For details, see the text.} \]

| \( g' \) | \( \omega_0 \) | \( S_{GT/V} \) | \( S_{\text{Pau}/V} \) | \( S_{\text{total}/V} \) |
|-------|-------|-------|-----------|-----------|
| MeV   | fm\(^{-3}\) | fm\(^{-3}\) | fm\(^{-3}\) | fm\(^{-3}\) |
| 0.0   | 0.00672 | 0.00733 | -0.01524 | 0.06863   |
| 0.5   | 0.06116 | 0.00750 | -0.01497 | 0.06863   |
| 1.0   | 0.06161 | 0.00767 | -0.01470 | 0.06863   |
| 2.0   | 0.06250 | 0.00804 | -0.01417 | 0.06863   |
| 3.0   | 0.06341 | 0.00844 | -0.01366 | 0.06863   |
| 4.0   | 0.06432 | 0.00887 | -0.01318 | 0.06863   |

Thus, the value of \( S_{GT} \) is almost independent of the value of the LM parameter.

\[ \omega_0 \approx \chi_5 \alpha_{ph}, \quad S_{GT} \approx \frac{V}{(2\pi)^3} \alpha_{ph}. \quad (44) \]

The GT strengths of the region \( \beta_- \) transitions:

\[ S_{\text{NFA}}^{(-)} = \int_{2M^*} \omega_0 \omega_0 R_{\text{RPA}}(q_0) = \frac{V}{8\pi^4} \int_{2M^*} dq_0 \frac{I_{\text{Pau}}(k_\rho, q_0)}{(1 + \chi_5 \Pi_r(q_0))^2}, \]

and that of the \( \beta_+ \) transitions:

\[ S_{\text{NFA}}^{(+)} = \int_{2M^*} dq_0 R_{\text{RPA}}(-q_0) = \frac{V}{8\pi^4} \int_{2M^*} dq_0 \frac{I_{\text{Pau}}(k_\rho, q_0)}{(1 + \chi_5 \Pi_r(-q_0))^2}. \]

where we have used the notations:

\[ E_{Fi} = \sqrt{M^* + k_i^2}, \quad (i = p, n). \]

The bounds of integral are determined by Eq. (24).

The GT strengths of the region \( |q_0| \geq 2M^* \) in NSA are also calculated in the same way. They are obtained in terms of those in NFA as,

\[ S_{\text{NSA}}^{(-)} = -S_{\text{NFA}}^{(+)} \quad \text{and} \quad S_{\text{NSA}}^{(+)} = -S_{\text{NFA}}^{(-)}. \quad (47) \]

Thus, each total strength of the \( \beta_- \) and \( \beta_+ \) transitions in NSA is different from the one in NFA. We note that the above relationships are obtained by performing the integration of the NSA response functions over negative energy, as we did before.

In Table I we list the excitation energy and strength \( S_{GT} \) of the giant GT resonance state as a function of \( g' \), and compare the strength of each energy region with the total RPA strength in NFA:

\[ S_{\text{total}} = S_{GT} + S_{\text{NFA}}^{(-)} - S_{\text{NFA}}^{(+)}. \]

In addition to \( (\mu_\pi/m_\pi)^2 = 392 \text{MeV} \cdot \text{fm}^3 \), we have employed the values of parameters, as an example, \( M^*/M = 0.6, \quad k_n = 1.4 \text{fm}^{-1} \) and \( k_p = 1.2 \text{fm}^{-1} \). These give the sum rule value in the unit of the nuclear volume as

\[ 2\frac{N - Z}{V} = \frac{2}{3\pi^2} (k_n^3 - k_p^3) = 0.06863 \text{fm}^{-3}. \quad (48) \]

It is seen in the last column in Table I that this value is reproduced in NFA. Table II also shows that the strength of the giant GT resonance state does not depend strongly on the value of \( g' \), as expected from the previous discussions. Comparing with sum rule value, \( S_{GT} \) is quenched by 12% \( \sim 10\% \) for \( g' = 0 \sim 1 \). From the experimental value for the excitation energy of the giant GT resonance state, \( g' \) is estimated to be about 0.6 [11]. In the present case, the approximate values given by Eq. (44) are

\[ \omega_0 \approx 23.801 \text{MeV}, \quad \frac{S_{GT}}{V} \approx 0.06072 \text{fm}^{-3}. \]
B. The Energy-Weighted GT Sum Rule

In no charge-exchange excitations, there is a famous theorem on the energy-weighted sum value of the excitation strengths in RPA by Thouless\[12\]. According to the RPA theorem, the sum value is equal to the expectation value of the double commutator of the nuclear Hamiltonian with the excitation operator. Here the expectation value is calculated by the ground state in the Hartree-Fock approximation. If the double commutator is a c-number, the sum value is given by the model-independent sum rule. Even if it is not a c-number, the theorem has been frequently used for a justification of the numerical results. In this subsection, we will show that the previous subsection, we expand $I_{\beta}$ second term that of the RPA theorem, the sum value is equal to the expectation value of the double commutator of the nuclear Hamiltonian with the excitation operator. Here the expectation value is calculated using Eq. (51),

$$I_{\beta}^{(1)} = -8\pi \int d\theta p \frac{p^2 - p_\beta^2}{E_p} (\theta_p^{(\beta)} + \theta_p^{(\alpha)}) ,$$

(52)

while $I_{\beta}^{(1)}$ is obtained directly from Eq. (51),

$$I_{\beta}^{(1)} = -\pi \left( \frac{2 \pi^3 N - Z}{V} \right)^2 .$$

(53)

We note that Eq. (51) should not be used for $I_{\beta}^{(1)}$ which requires the $1/q$ term in Eq. (51). Finally, we obtain the energy-weighted sum of the $\beta$- and $\beta'$ transition strengths in RPA of NFA,

$$S_{\beta,\beta'}^{(1)}_{\text{RPA}} = \frac{V}{(2\pi)^3} \chi_5 \alpha_{\text{ph}}^2 .$$

(55)

This is equal to $\omega_0 S_{\text{GT}}$ from Eqs. (54). In fact, even if we take into account the coupling between the particle-hole states and the nucleon-antinucleon states, the energy-weighted strength of the giant GT resonance state is approximately given by the above equation,

$$\omega_0 S_{\text{GT}} \approx \frac{V}{(2\pi)^3} \chi_5 \alpha_{\text{ph}}^2 ,$$

(56)

since the coupling is weak. This implies that the energy-weighted sum of the strengths in the giant GT resonance region is quenched by a factor $(1 - 2v_F^2/3)^2$, compared with the non-relativistic one. Its value is about 0.77 for $v_F = 0.43$ obtained from $M^* = 0.6M$ and $k_F = 1.36$ fm$^{-1}$.

It is worthwhile noting that we can do formally the above calculations including the divergent terms with $G_F G_F$. In this case, $\theta_p^{(\beta)}$ is replaced with $\theta_p^{(\gamma)} - 1$ in Eq. (52), so that we have

$$S_{\beta,\beta'}^{(1)}_{\text{RPA}} = \frac{V}{\pi^2} \int d\theta p \frac{p^2 - p_\beta^2}{E_p} (2 - \theta_p^{(\beta)} - \theta_p^{(\alpha)})$$

$$+ 4g_5 (N - Z)^2 \frac{(N - Z)^2}{V} .$$

(57)
We can see that the negative contribution in Eq. 54 stems from neglect of the divergent term. A part of contribution from the Pauli blocking terms is included in the second term of Eq. 54 also, as seen from Eqs. 55 and 56. In NSA, we can perform the same calculation, but by defining \( J_n^{(1)} \) instead of \( I_n^{(1)} \) in Eq. 50 for NFA,

\[
J_n^{(1)} = \int_{-\infty}^{0} dq_0 \, q_0 \text{Im} \left( \Pi(q_0) \right)^{n+1} + \int_{-\infty}^{-\eta} dq_0 \, q_0 \text{Im} \left( \Pi(-q_0) \right)^{n+1},
\]

with \( \Pi(q_0) \) given in Eq. 58. We can show that the energy-weighted sum in NSA is the same as Eq. 54 for NFA.

Now we will show that Eq. 54 is equal to the expectation value of the double commutator of the Hamiltonian with the GT operator, when the expectation value is calculated by the ground state in the mean field. The nuclear field in the present model is written as

\[
\psi(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \sum_\alpha \left( u_\alpha(p) \exp(i p \cdot x) a_\alpha(p) + v_\alpha(p) \exp(-i p \cdot x) \right),
\]

where the first term in the parentheses describes the particles and the second the antiparticles. The suffix \( \alpha \) denotes the spin and isospin quantum numbers as \( u_\alpha(p) = u_\sigma(p) | \tau \rangle, ( \alpha = \sigma, \tau ) \), and the positive and negative spinors are given by

\[
u_\sigma(p) = \left[ \frac{E_p + M^*}{2E_p} \right]^{1/2} \left( \frac{\sigma \cdot p}{E_p + M^*} \right) \xi,
\]

with the Pauli spinor \( \xi \). The creation and annihilation operators, \( a_\alpha(p) \) and \( b_\alpha^\dagger(p) \), are defined as usual. Then, the mean field Hamiltonian, the interaction and the GT field operator are described, respectively, as

\[
H_0 = \int d^3x \, \overline{\psi}(x) \left( -i \gamma \cdot \nabla + M^* \right) \psi(x),
\]

\[
V = -\frac{g_5}{2} \int d^3x \, \overline{\psi}(x) \Gamma_\mu^\dagger \psi(x) \overline{\psi}(x) \Gamma_\mu \psi(x),
\]

\[
F_\pm = \int d^3x \, \overline{\psi}(x) \gamma_5 \gamma_\mu \tau \pm \psi(x).
\]

We are assuming throughout the present paper that Dirac particles are bound in the Lorentz scalar and vector potentials, but in the above mean field Hamiltonian, we have deleted the Lorentz vector potential which does not appear explicitly in the present discussions, while the Lorentz scalar potential is included in the nucleon effective mass \( M^* \).

The double commutator of the mean field Hamiltonian with the GT operator is easily calculated. By using the relationship for arbitrary operators \( A \) and \( B \),

\[
[ \psi^\dagger(x) A \psi(x), \psi^\dagger(y) B \psi(y) ] = \psi^\dagger(x) [ A, B ] \psi(x) \delta(x - y),
\]

we obtain

\[
[ F_+, [ H_0, F_- ] ] = 4i \int d^3x \, \psi^\dagger(x) \gamma_0 \left( \gamma \cdot \nabla - \gamma_y \frac{\partial}{\partial y} \right) \psi(x).
\]

Its expectation value of the ground state in the mean field becomes

\[
\langle \psi^\dagger [ F_+, [ H_0, F_- ] ] \psi \rangle = \frac{-4V}{(2\pi)^3} \int d^3p \left[ \left( \theta^{(p)} - \theta^{(p)}_{\uparrow} \right) \right. \left. \frac{1}{p_0^2} \frac{1}{E_p} \left( 2 - \theta^{(p)}_{\downarrow} - \theta^{(p)}_{\uparrow} \right) \right].
\]

The double commutator of \( V \) with the GT operator is calculated in the same way. Using the abbreviations:

\[
f^\mu_q = \gamma_0 \gamma_5 \gamma^\mu r_q, \quad X^\mu_q = [ f^\mu_q, f^2_\mu ], \quad ( q = \pm, 0 ),
\]

it is described as

\[
[ F_+, [ V, F_- ] ] = \frac{-g_5}{2} \int d^3x \left( \psi^\dagger (X^\mu_q) \psi \psi^\dagger X^\mu_q \psi + \psi^\dagger f^\mu_q \psi \psi^\dagger \right. \left. + f^\mu_q \psi \psi^\dagger \left[ f^2_+, X^\mu_q \right] \psi \right.
\]

\[
+ \psi^\dagger \left[ f^2_+, X^\mu_q \right] \psi \psi^\dagger + \psi^\dagger X^\mu_q \psi \psi^\dagger (X^\mu_q) \psi \right).\]
In the mean field approximation, the expectation value is calculated neglecting the exchange terms of the matrix elements. Keeping the only direct terms, we have

\[
\left\langle \left[ \left[ F_+ , [V,F_-] \right] \right] \right\rangle = -\frac{g_5}{2} \int d^3x \left( \left| \psi^\dagger (X_{\mu+})^\dagger \psi \right| \psi^\dagger X_+^\mu \psi \right) + \left( \left| \psi^\dagger X_+^\mu \psi \right| \psi^\dagger (X_{\mu+})^\dagger \psi \right) + \left( \left| \psi^\dagger f_{\mu\nu} \psi \right| \right) \left( f_+^2, X_0^\mu \right) \psi \right) + \left( \left| \psi^\dagger \right| \right) \left( f_+^2, X_0^\mu \right) \psi \right) f_{\mu\nu} \psi \right). 
\]

The straightforward calculation of the above matrix elements yields

\[
\left\langle \left[ \left[ F_+ , [V,F_-] \right] \right] \right\rangle = g_5 \int d^3x \left( \left| \psi^\dagger (X_+^2)^\dagger \psi \right| \psi \right) = g_5 \int d^3x \left( \left| \psi^\dagger \right| \psi \right) \frac{V}{(2\pi)^3} \left( 4 \int d^3p \left( \theta_p^{(p)} + \frac{\theta_p^{(n)}}{\theta_p^{(p)}} \right) \right)^2 = 4g_5 \frac{(N - Z)^2}{V}. \tag{60}
\]

Thus, the sum of Eqs. (59) and (60) is just equal to the energy-weighted sum of the strengths for the \( \beta_- \) and \( \beta_+ \) transitions in RPA given in Eq. (57),

\[
S^{(1)}_{RPA} = \left\langle \left[ \left[ F_+ , [H_0 + V,F_-] \right] \right] \right\rangle,
\]

including the divergent term. In Eq. (59) for NFA, the divergent term has been simply neglected.

In NSA, the nuclear field is given by replacing the creation operator of the antiparticles with the annihilation one in Eq. (57) for NFA. Because of this change, on the one hand, we obtain, instead of Eq. (59),

\[
\left\langle \left[ \left[ F_+ , [H_0,F_-] \right] \right] \right\rangle = \frac{\pi^3}{V} \int d^3x \left( \theta_p^{(p)} + \frac{\theta_p^{(n)}}{\theta_p^{(p)}} \right),
\]

which does not contain the divergent term. On the other hand, the expectation value of the double commutator as for \( V \) is the same as Eq. (60). Thus, in NSA also, the energy-weighted sum of the GT strengths in RPA is equal to the expectation value of the double commutator, when the expectation value is calculated with the ground state in the mean field.

Formally we have proved that the RPA theorem holds in charge-exchange excitations also, but with respect to the sum of the strengths for the \( \beta_- \) and \( \beta_+ \) transitions. In the present relativistic model, however, we have also shown that the energy-weighted sum value itself is divergent. If we neglect simply the divergent terms as in NFA and NSA, the sum value becomes negative, owing to the strengths of the Pauli blocking terms or the antinucleon-hole excitations. Generally speaking, all previous calculations in NFA and NSA have the same problem. In order to solve this problem, we need definitely the renormalization of the divergence.

V. DISCUSSIONS AND CONCLUSIONS

The relativistic model has been extensively used as a phenomenological model of nuclei for the past 30 years\textsuperscript{16, 17}. In particular, it explains very well the ground-state properties of nuclei with the mean field approximation, where antinucleon degrees of freedom are neglected\textsuperscript{16, 17}. In RPA based on the mean field approximation, however, it is known that we can not describe the excited states within the nucleon space only, and should include at least Pauli blocking terms from nucleon-antinucleon excitations in the correlation functions. We have called this approximation NFA. The Pauli blocking terms are required for RPA to keep the continuity equation\textsuperscript{18, 19}, and to reproduce the correct Landau-Migdal(LM) parameters, etc.\textsuperscript{20}. The reason why we need the Pauli blocking terms may be partially because of the fact that in relativistic models, the complete set needs antinucleon degrees of freedom.

In this paper, we have shown that the Pauli blocking terms are also necessary for RPA to satisfy the Gamow-Teller(GT) sum rule with respect to the difference between the strengths of the \( \beta_- \) and \( \beta_+ \) transitions which is called Ikeda-Fujii-Fujita(IFF) sum rule\textsuperscript{21}. This fact has been shown in an analytic way for nuclear matter. If the configuration space is limited to the nucleon one, the sum rule value is exhausted only by about 88%. When adding the Pauli blocking terms to the RPA correlation function, the sum rule value is reproduced. The coupling of the particle-hole states with the nucleon-antinucleon states is weak for reasonable values of the LM parameter \( g' \). As a result, the GT strength which is distributed over the giant GT resonance region remains to be quenched by about 10 to 12%.

In the previous paper\textsuperscript{22}, the GT strength in the nucleon sector was estimated for finite nuclei in the mean field approximation. The value of the total strength was about 94% of the sum rule value. The reduction of the quenching is due to a larger value of the effective mass in finite nuclei than in nuclear matter. Since the coupling between the particle-hole and particle-antiparticle states is weak, the about 6% quenching of the GT strength is also expected in RPA of NFA for finite nuclei.

The IFF sum rule in the no-sea approximation(NSA)\textsuperscript{23} has been also investigated. In NSA, the Dirac sea is
assumed to be empty, and the antinucleon states are treated as particle states with negative energy. In this way, one can avoid the divergence problem without violating the continuity equation as in NFA. We have shown that NSA also satisfies IFF sum rule, and predicts the same quenching of the strength for the giant GT resonance state. It has been shown, however, that each strength of the total $\beta_-$ and $\beta_+$ transitions is different from those in NFA.

The energy-weighted sum of the GT strengths in RPA is also studied in NFA and NSA. It has been shown that the sum of the energy-weighted strengths for the $\beta_-$ and $\beta_+$ transitions is equal to the expectation value of the double commutator of the Hamiltonian with the GT operator, when the expectation value is calculated with the ground state in the mean field and the divergent terms are deleted. Thus the well-known RPA theorem by Thouless holds for charge-exchange excitations also.

We should note finally that renormalization of the divergence should be investigated in the future study of relativistic models. So far most of the nuclear observables are well reproduced phenomenologically without the renormalization. NFA and NSA, which satisfy various conservation laws, are such examples. As shown in the present paper, however, NFA and NSA provide us with unphysical results also, as a price of neglecting the divergence. In the case of the GT excitations, the non-energy weighted strengths themselves in NFA and the energy-weighted strengths in NSA are negative outside of the giant resonance region. All previous calculations using NFA and NSA may have the same problems.

It may depends on nuclear observables whether or not effects of the renormalization are important. In the previous studies, on the one hand, it was shown that the LM parameter $F_1$ depends on antinucleon degrees of freedom, only through the Pauli blocking terms. These facts were shown by renormalized calculations in the $\sigma - \omega$ model. As a result, some physical quantities which are dominated by $F_1$ can be described using approximation with the Pauli blocking terms. Indeed, for example, as well as the center of mass motion, the space part of the nuclear current, which is responsible for the orbital part of the nuclear magnetic moments was shown to be described well in NFA and NSA. On the other hand, effects of the antinucleon degrees of freedom on the LM parameter $F_0$ are not represented by the Pauli blocking terms only, but the contribution from other nucleon-antinucleon excitations is more important, as shown in ref. Another example is the Coulomb sum rule, where the renormalization provides us with a strong quenching of the sum rule value at high momentum transfer.

Thus, there may be some cases where the renormalization is not essential for description of the observables, but generally speaking, we should investigate in the future how effects of the renormalization change previous results in relativistic models. This may be also true for the GT strengths discussed here, although the coupling between the particle-hole terms and the Pauli blocking terms is weak in RPA.

[1] K. Ikeda, S. Fujii and J.I. Fujita, Phys. Lett. 3, 271 (1963).
[2] H. Kurasawa, T. Suzuki and N. Van Giai, Phys. Rev. Lett. 91, 062501 (2003).
[3] H. Kurasawa, T. Suzuki and N. Van Giai, nucl-th/0306080.
[4] S.A. Chin, Ann. of Phys. 108, 301 (1977).
[5] H. Kurasawa and T. Suzuki, Nucl. Phys. A445, 685 (1985).
[6] H. Kurasawa and T. Suzuki, Phys. Lett. B474, 262 (2000).
[7] J.F. Dawson and R.J. Furnstahl, Phys. Rev. C42, 2009 (1990).
[8] Z. Ma, N. Van Giai, A. Wandelt, D. Vretenar and P. Ring, Nucl. Phys. B686, 173 (2001).
[9] D.J. Thouless, Nucl. Phys. B22, 78 (1961).
[10] T. Wakasa et al., Phys. Rev. C55, 2909 (1997); Phys. Lett. B426, 257 (1998); H. Sakai and K. Yako, to appear in Nucl. Phys. A.
[11] T. Suzuki and H. Sakai, Phys. Lett. B455, 25 (1999).
[12] A. Arima, W.Bentz, T. Suzuki and T. Suzuki, Phys. Lett. 499, 104 (2001).
[13] H. Kurasawa, T. Suzuki and N. Van Giai, to appear in Nucl. Phys. A.
[14] K. Kawahigashi, K. Nishida, A. Itabashi and M. Ichimura, Phys. Rev. C63, 044609 (2001).
[15] H. Kurasawa and T. Suzuki, Nucl. Phys. A490, 571 (1988).
[16] B. D. Serot and J.D. Walecka, Adv. Nucl. Phys. 16, 1 (1986).
[17] P. Ring, Prog. Part. Nucl. Phys. 37, 193 (1996).
[18] H. Kurasawa and T. Suzuki, Nucl. Phys. A54, 527 (1986).
[19] A. Bohr and B.R. Mottelson, Phys. Lett. B100, 10 (1981).
[20] T. Suzuki, Nucl. Phys. A379, 110 (1982); T. Suzuki and M. Kohno, Prog. Theor. Phys. 68, 690 (1982).
[21] G.F. Bertsch and S.F. Tsai, Phys. Rep. 18, 125 (1975).
[22] H. Kurasawa and T. Suzuki, Phys. Lett. 165B, 234 (1985); S. Nishizaki, H. Kurasawa and T. Suzuki, Nucl. Phys. A462, 687 (1987).