COMPACT PLANETARY SYSTEMS PERTURBED BY AN INCLINED COMPANION. II.
STELLAR SPIN–ORBIT EVOLUTION

GWENAËL BOUÉ1,2 AND DANIEL C. FABRYCKY1
1 Department of Astronomy and Astrophysics, University of Chicago, 5640 South Ellis Avenue, Chicago, IL 60637, USA; boue@imcce.fr
2 Astronomie et Systèmes Dynamiques, IMCCE-CNRS UMR 8028, Observatoire de Paris, UPMC, 77 Av. Denfert-Rochereau, F-75014 Paris, France

ABSTRACT

The stellar spin orientation relative to the orbital planes of multiplanet systems is becoming accessible to observations. Here, we analyze and classify different types of spin–orbit evolution in compact multiplanet systems perturbed by an inclined outer companion. Our study is based on classical secular theory, using a vectorial approach developed in a separate paper. When planet–planet perturbations are truncated at the second order in eccentricity and mutual inclination, and the planet–companion perturbations are developed at the quadrupole order, the problem becomes integrable. The motion is composed of a uniform precession of the whole system around the total angular momentum, and in the rotating frame, the evolution is periodic. Here, we focus on the relative motion associated with the oscillations of the inclination between the planet system and the outer orbit and of the obliquities of the star with respect to the two orbital planes. The solution is obtained using a powerful geometric method. With this technique, we identify four different regimes characterized by the nutation amplitude of the stellar spin axis relative to the orbital plane of the planets. In particular, the obliquity of the star reaches its maximum when the system is in the Cassini regime where planets have more angular momentum than the star and where the precession rate of the star is similar to that of the planets induced by the companion. In that case, spin–orbit oscillations exceed twice the inclination between the planets and the companion. Even if the mutual inclination is only $\sim 20^\circ$, this resonant case can cause the spin–orbit angle to oscillate between perfectly aligned and retrograde values.

Key words: celestial mechanics – methods: analytical – methods: numerical – planets and satellites: dynamical evolution and stability – planets and satellites: general – planet–star interactions

Online-only material: color figures

1. INTRODUCTION

Hot or eccentric Jupiters only constitute a small fraction of the exoplanets discovered to date. Among those with short orbital periods, most are smaller, less massive, and part of compact multiplanet systems (Howard et al. 2010, 2012; Howard 2013; Petigura et al. 2013). Due to the smaller planetary radii and larger orbital periods, the efficiency of the standard method to measure spin–orbit angle in systems with hot Jupiters, based on the Rossiter–McLaughlin effect (Holt 1893; Rossiter 1924; McLaughlin 1924) decreases significantly in multiplanet systems. Only two multiplanet systems, called KOI-94 and Kepler-25, have been studied with this technique (Hirano et al. 2012; Albrecht et al. 2013). Two other methods have been implemented to measure the spin–orbit angle in multiplanet systems: the stellar spot crossing technique on Kepler-30 (Sanchis-Ojeda et al. 2012) and asteroseismology on Kepler-50, Kepler-65 (Chaplin et al. 2013), and Kepler-410A (Van Eylen et al. 2014). The six systems prove to be compatible with perfect spin–orbit alignments, a priori suggesting that multiplanet systems are preferentially in the equatorial plane of their star. A seventh system, Kepler-56 (Huber et al. 2013), shows coplanarity between two transiting planets but misalignment with the star; a distant companion, detected by radial velocity, may be responsible. Therefore, we are led to ask what happens to the stellar obliquity if a coplanar multiplanet system is accompanied by a distant planetary companion or is embedded in a binary stellar system?

Dynamically, large stellar obliquity in an isolated close-in planet system can be the outcome of either planet–planet scattering (Nagasawa et al. 2008; Beaugé & Nesvorný 2012) or Lidov–Kozai excitation by an outer inclined perturber (Fabrycky & Tremaine 2007; Correia et al. 2011; Naoz et al. 2011, 2012; Petrovich 2004). Moreover, if the inner eccentricity is large, even a coplanar outer object can flip the planet’s orbit by $180^\circ$ (Li et al. 2014). More generally, in single as well as in multiplanetary systems, spin–orbit misalignment may also result from the magnetic interaction between the protostar and its circumstellar disk (Lai et al. 2011) or from the solid precession of the protoplanetary disk induced by an inclined companion (Batygin 2012; Batygin & Adams 2013; Lai 2014).

In multiplanet systems surrounded by an outer stellar companion, apsidal precession frequencies are dictated by the companion and by the planet–planet interactions. As a consequence, even at high inclination, if the planet system is sufficiently packed, planet–planet interactions dominate the apsidal motion, the evolution is stabilized with respect to the Lidov–Kozai mechanism, eccentricities remain small, and all planets move in concert (Innanen et al. 1997; Takeda et al. 2008; Saleh & Rasio 2009). Such systems are classified as dynamically rigid.3 Although the Lidov–Kozai evolution is quenched, the planetary mean plane still precesses if it is inclined relative to the orbit of the companion and can eventually lead to spin–orbit misalignment with the central star. Kaib et al. (2011) applied this idea to the 55 Cancri multiplanet system, which has a stellar companion, and concluded that the planets are likely misaligned with respect to the stellar equator. However, the results only hold as long as the stellar spin axis is weakly coupled to the companion.

3 Note that our definition of dynamically rigid is more stringent than that of Takeda et al. (2008) who also include case where planet eccentricities increase in concert.
planets’ orbit. We show here that this condition is not satisfied for the 55 Cancri system unless the semi-minor axis of the perturber is very small, of the order of 180 au (periastron distance \( \lesssim 30 \text{ au} \)), whereas the projected separation is 1065 au (Mugrauer et al. 2006). Nevertheless, the required conditions for this mechanism may have been met earlier in the history of this system in particular or other systems in general. Indeed, in our own solar system, for instance, the Sun is weakly coupled to the ecliptic and its obliquity of \( 7^\circ \) might be the signature of an earlier tilt of the planet system (Tremaine 1991). Moreover, analyzing a similar problem where a protoplanetary disk takes the place of the compact planet system, Batygin (2012) showed that this mechanism is able to tilt, forming planetary systems around slow rotator T Tauri pre-main sequence stars.

Here, we revisit the problem composed of a dynamically rigid system perturbed by a stellar or a planetary companion on a wide and inclined orbit. The inner planets are assumed to have low eccentricities and mutual inclinations comparable to or lower than those of our own solar system. According to these assumptions, orbital evolution induced by tides is expected to be weak and is neglected. These hypotheses are motivated by statistical studies of compact exoplanet systems detected by Kepler or by radial velocity (e.g., Tremaine & Dong 2012; Figueira et al. 2012; Fabrycky et al. 2012; Wu & Lithwick 2013). However, we allow the overall plane of planets to tilt by an arbitrary angle. A hierarchical companion is included, which is allowed to have any eccentricity and inclination. The main goal of this study is to follow the evolution of the inclination of the planet system with respect to the spin axis of the parent star. Thus, the interaction between the stellar spin axis and the orbital motion of the inner planets is taken into account.

For this study, we exploit the results of the so-called “three-vector problem” which has been solved geometrically in Boué & Laskar (2006, hereafter BL06) and in Boué & Laskar (2009, hereafter BL09). The three-vector problem aims to model the secular evolution of three coupled angular motions such as the lunar problem with the planet spin and the orbital angular momenta of the satellite and the star (BL06) or the binary asteroid problem with two spin axes and their mutual orbital motion (BL09). Here, the three vectors are the spin of the star, the total orbital angular momentum of the planet system, and that of the companion. In Section 2, we recall the main results of the three-vector problem, and we also provide a new integral expression for the precession frequency. Then, in Section 3, we employ the vectorial formalism of the classical secular theory that we described in a previous paper (Boué & Fabrycky 2014 hereafter BF14) and we show how the three-vector problem emerges from this general secular model. The validity of the simplification is also discussed. In this work, we thus consider two different models that correspond to two levels of approximation. On the one hand, the perturbing function is expanded at the fourth order in planet eccentricity and mutual inclination and at the octupole order in the interaction between each planet and the outer companion. This model provides accurate results but is non-integrable and has to be solved numerically. On the other hand, the system is described by the integrable three-vector problem which gives deeper geometrical insight. In the following, we refer to the former as the numerical model and to the latter as the analytical model. In Section 4, the two models are compared in their application to real exoplanet systems. We exploit more deeply the possibilities of the analytical model to span the parameter space and identify four different regimes of evolution in Section 5. The conclusions are given in the last section.

2. THREE-VECTOR PROBLEM

This section summarizes a few key results associated with the so-called “three-vector problem” described in (BL06; BL09). In the context of this paper, the three vectors are the angular momentum of the star \( \mathbf{L} = Ls \), the orbital angular momenta of the planet system \( \mathbf{G} = G\mathbf{w} \), and that of the companion \( \mathbf{G}' = G'\mathbf{w}' \), where \( s \), \( \mathbf{w} \), and \( \mathbf{w}' \) are unit vectors. The three-vector problem assumes that the evolution is governed by a Hamiltonian of the form

\[
H = -\frac{\alpha}{2}(s \cdot \mathbf{w})^2 - \frac{\beta}{2}(s \cdot \mathbf{w})^2 - \frac{\gamma}{2}(\mathbf{w} \cdot \mathbf{w})^2,
\]

where \( \alpha \), \( \beta \), and \( \gamma \) are constant parameters representing the coupling between the planetary system and both the stellar rotation and the binary orbit, respectively. Their expression will be derived in the subsequent section. Note that, in contrast to the more general three-vector problem, here we neglect the direct interaction between the stellar spin and the orbit of the companion, i.e., we set \( \beta = 0 \). The equations of motion are

\[
\frac{ds}{dt} = -\frac{1}{Ls} s \times \nabla_s H;
\]
\[
\frac{d\mathbf{w}}{dt} = -\frac{1}{G} \mathbf{w} \times \nabla_w H;
\]
\[
\frac{d\mathbf{w}'}{dt} = -\frac{1}{G'} \mathbf{w}' \times \nabla_{w'} H,
\]

which lead to

\[
\frac{ds}{dt} = -\frac{\alpha}{L}(s \cdot \mathbf{w})\mathbf{w} \times s;
\]
\[
\frac{d\mathbf{w}}{dt} = -\frac{\alpha}{G}(s \cdot \mathbf{w})s \times \mathbf{w} - \frac{\gamma}{G}(\mathbf{w}' \cdot \mathbf{w})\mathbf{w} \times \mathbf{w};
\]
\[
\frac{d\mathbf{w}'}{dt} = -\frac{\gamma}{G'}(\mathbf{w}' \cdot \mathbf{w})\mathbf{w} \times \mathbf{w}'.
\]

For the subsequent analysis, we set

\[
\nu_1 = \alpha / L, \quad \nu_2 = \alpha / G, \quad \nu_3 = \nu_4 = \gamma / G'.
\]

These quantities are important as they are the characteristic precession frequencies of \( s \) around \( \mathbf{w} \), of \( \mathbf{w} \) around \( s \) and \( \mathbf{w}' \), and of \( \mathbf{w}' \) around \( \mathbf{w} \), respectively.

2.1. Integrability

The three-vector problem is integrable (BL06; BL09). Let

\[
W = Ls + G\mathbf{w} + G'\mathbf{w}'
\]

be the total angular momentum of the system. The general solution is a uniform rotation of the three vectors around the total angular momentum combined with a periodic motion in the rotating frame (BL06; BL09). The evolution is thus characterized by two frequencies or periods. Hereafter, the uniform rotation is referred to as the precession motion with
period $P_{\text{pre}}$, and the periodic loops described in the rotating frame are equally qualified as nutation in reference to the Earth–Moon problem, or simply as the relative motion with period $P_{\text{nut}}$.

The relative motion can be solved elegantly with geometric arguments (BL06; BL09). It is also very important for our study for two reasons: it enables us (1) to check if any system can be misaligned and (2) to evaluate the timescale of the secular spin–orbit evolution which can then be compared to the lifetime of the system. Next, we recall its solution and main properties as derived in (BL06; BL09). Then, we present a new integral of the system. Next, we recall its solution and main properties for two reasons: it enables us (1) to check if any system can be misaligned and (2) to evaluate the timescale of the secular spin–orbit evolution which can then be compared to the lifetime of the system.

2.2. Relative Motion

In order to get the relative evolution of the system described by the Hamiltonian (1), we follow the same derivation as in BL06 and BL09. We denote

\[
x = s \cdot w,
\]

\[
y = s \cdot w',
\]

\[
z = w \cdot w'.
\]

(6)

Sometimes, we will also use the corresponding angles defined by $x = \cos \theta_x$, $y = \cos \theta_y$, and $z = \cos \theta_z$. In this coordinate system, the Hamiltonian reads as

\[
H = -\frac{\alpha}{2} x^2 - \frac{\gamma}{2} z^2.
\]

(7)

The conservation of the norm of each angular momentum $L$, $G$, and $G'$, as well as the total angular momentum of the system $W$, Equation (5), lead to the second constant of the motion

\[
K = \frac{\|W\|^2 - L^2 - G^2 - G'^2}{2} = LGx + LG'y + GG'z.
\]

(8)

Each trajectory of the relative motion in the $(x, y, z)$ frame is at the intersection of a cylinder defined by $H(x, y, z) = h$ and a plane defined by $K(x, y, z) = k$, where $h$ and $k$ are two constants given by the initial conditions. Trajectories are thus subsets of ellipses defined by the values $h$ and $k$ of the two first integrals of the motion. Hereafter, we denote them as $E = \{ (x, y, z) \in \mathbb{R}^3 \mid H(x, y, z) = h, K(x, y, z) = k \}$. These ellipses can be parameterized as follows

\[
x(\tau) = A_x \cos \tau,
\]

\[
z(\tau) = A_z \sin \tau,
\]

\[
y(\tau) = \frac{1}{LG'}(k - LGx(\tau) - GG'z(\tau)),
\]

(9)

where

\[
A_x = \sqrt{-\frac{2h}{\alpha}}, \quad A_z = \sqrt{-\frac{2h}{\gamma}}.
\]

(10)

The change of time $t \mapsto \tau$ leading to the parameterization (9) will be made explicit in Section 2.4.

In general, systems do not cover the full ellipses. Indeed, $x$, $y$, and $z$ are dot products of unit vectors and the evolution is restricted inside the cube $C = \{(x, y, z) \in [-1, 1]^3\}$. There is also a more stringent additional constraint (BL06). Let

\[
V = s \cdot (w \times w').
\]

(11)

$V$ represents the oriented volume of the parallelepiped generated by the vectors $s$, $w$, and $w'$. In terms of $x$, $y$, $z$, the square of the volume $V$ is given by the Gram determinant

\[
V^2 = \begin{vmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{vmatrix} = 1 - x^2 - y^2 - z^2 + 2xyz.
\]

(12)

When the three vectors $s$, $w$, and $w'$ are coplanar, $V^2(x, y, z) = 0$. This is the equation of a cubic surface known as Cayley’s nodal cubic. The restriction of this surface to the cube $C$ is displayed in Figure 1. In BL09, this restriction is called Cassini Berlingot. The word Berlingot comes from a French hard candy with a similar shape. The name Cassini has been added in reference to the “Cassini states” characterized by the coplanarity of the same three vectors as in this problem. Thus, “Cassini states” are located at the surface $V^2(x, y, z) = 0$. Because the cubic (12) represents the square of the volume $V$, it must be positive. As a consequence, the evolution of the system is restricted inside the Berlingot $B = \{(x, y, z) \in C \mid V^2(x, y, z) \geq 0\}$. Here, $B$ denotes the inside of the Berlingot, and $\partial B$ its surface.

In Figure 1, important diagonals $D_x$, $D_y$, and $D_z$ are represented. They all belong to the intersection $\partial B \cap \partial C$, in which the three vectors $(s, w, w')$ lie in the same plane. The point $(1, 1, 1)$ corresponds to the configuration where all three vectors are aligned and in the same direction. In that case, the system is fully coplanar with only prograde orbits. At the point $(1, -1, -1)$, the three vectors are still collinear, but $w'$ is pointing in the opposite direction as $s$ and $w$. In that case, the system is also coplanar, but the outer companion is on a retrograde orbit. Along the diagonal $D_z$ joining these two points, the planet...
system remains in the equatorial plane of the star, and the companion’s inclination \( \theta_y = \theta_x \) increases from 0° at \((1, 1, 1)\) to 180° at \((-1, -1, -1)\). In the same way, along the diagonal \( D_y \), the angle \( \theta_z = \cos^{-1}(s \cdot w') \) is constant and equal to 0°. Thus, the equator of the star remains in the plane of the outer body, while the planet system tilts from 0° to 180° at \((-1, -1, -1)\). Finally, along the diagonal \( D_z \), \( s \cdot w' = \cos \theta_z = 1 \), the planet system remains in the plane of the outer perturber and the stellar obliquity \( \theta_z = \theta_x \) increases from 0° at \((1, 1, 1)\) to 180° at \((-1, -1, 1)\). Two classes of solutions, drawn schematically in Figure 2, exist (BL06). The first class comprises special solutions, so called because the three vectors \( s, w, \) and \( w' \) never get coplanar and always remain almost orthogonal to each other, which is not usual. In this case, the ellipse \( E \) is fully contained inside the Berlingot \( B \). The solutions of the second class are called general solutions. They are such that \( s, w, \) and \( w' \) periodically lie in the same plane, which happens when \( E \) crosses the surface of the Berlingot \( \partial B \). In the following, we limit our analysis to systems belonging to the second class only. For a detailed description of the special solutions, see BL06 and BL09.

As stated before, the relative motion of the three vectors is integrable. In particular, the trajectory of any general solution in the \((x, y, z)\) frame is a continuous piece of ellipse. More precisely, trajectories are connected sections of equations \( \partial E \) with extrema at the surface \( \partial B \) of the Berlingot. These elliptic sections are swept back and forth indefinitely, each round trip defining a nutation oscillation, unless the ellipse \( E \) is tangent to the Berlingot \( \partial B \), in which case the point of tangency is a fixed point of the relative motion (BL06). The stability of each relative equilibrium is easily obtained as follows. If the ellipse is tangent to the Berlingot “from the outside” (Figure 3(a)), the trajectory is a singleton and thus the relative equilibrium is stable. On the other hand, if the ellipse is tangent to the Berlingot “from the inside” (Figure 3(b)), the trajectory goes within the Berlingot, and the relative equilibrium is unstable. We stress again that because the fixed points are located at the surface \((V^2(x, y, z) = 0)\) of the Berlingot, in the equilibrium states the three vectors \( s, w, \) and \( w' \) lie in the same plane as in the Cassini states.

### 2.3. Amplitudes of Nutation

Precession motion is a solid rotation of the three vectors \( s, w, w' \), leaving their mutual angles unchanged. Conversely, oscillations of inclinations and spin–orbit angles constitute the nutation motion. The amplitude of these angles is thus dictated by the relative motion taking place in the space \((x, y, z)\). More precisely, the extrema of the stellar obliquity \( \theta_x \) with respect to the mean plane of the stellar obliquity \( \theta_z \) with respect to the outer orbit and of the mutual inclination \( \theta_x \) between the planet mean plane and the outer orbit correspond to the extrema of \( x, y, z \) reached during one nutation period. In the following, most of the derivation is done for \( \theta_x \), but the method is equivalent for \( \theta_y \) and \( \theta_z \). With the parameterization (9), the extrema of \( x \) are attained when \( dx/d\tau = 0 \) or when the three vectors lie in the same plane, i.e., \( V^2(\tau) = V^2(x(\tau), y(\tau), z(\tau)) = 0 \). In the last case, although \( dx/d\tau \) is not necessarily zero, the trajectory on the ellipse \( E \) makes a bounce because the point \( M = (x, y, z) \) reaches the surface \( \partial B \) of the Berlingot.

Using the parameterization (9), the Gram determinant \( V^2(\tau) \) becomes a third-degree polynomial in \( \cos \tau \) and \( \sin \tau \). The zeros of \( V^2(\tau) \) can thus be deduced from the roots of a sixth-degree polynomial in \( \cos \tau \). Given that for non-stationary general solutions, the volume \( V(\tau) \) (and the Gram determinant \( V^2(\tau) \)) only cancels at two different values, \( \tau_1 \) and \( \tau_2 \), once the roots of \( V^2(\tau) \) are known, one has to select \( \tau_1 \) and \( \tau_2 \) as the closest ones surrounding the initial condition \( \tau_0 = \text{atan2}(z_0/A_x, x_0/A_x) \), where \( x_0 = x(t = 0) \) and \( z_0 = z(t = 0) \). Once the roots \( (\tau_0)_{k=1,2} \) are known, the amplitudes of variation of \( \theta_x, \theta_y, \) and \( \theta_z \) are straightforward. The angle \( \theta_x \) varies between \( \theta_{x,-1} = \cos^{-1}(x(\tau_1)) \) and \( \theta_{x,2} = \cos^{-1}(x(\tau_2)) \), idem for \( \theta_y \) and \( \theta_z \).

Nevertheless, the amplitude of \( \theta_x, \theta_y, \) or \( \theta_z \) can be larger than what has just been calculated if a value of \( \tau \) exists between \( \tau_1 \) and \( \tau_2 \) such that \( dx/d\tau = 0 \), \( dy/d\tau = 0 \), or \( dz/d\tau = 0 \), respectively. The derivative \( dx/d\tau \) cancels at

\[
\tau_x^\pm = \frac{\pi}{2} \pm \frac{\pi}{2},
\]

the quantity \( dz/d\tau \) vanishes at

\[
\tau_z^\pm = 0 \pm \frac{\pi}{2},
\]

and finally, \( dy/d\tau = 0 \) for

\[
\tau_y^\pm = \text{atan2}(LA_x, -G'A_x) \pm \frac{\pi}{2}.
\]

All values \( \theta_x, \theta_y, \) or \( \theta_z \) are extrema. Thus, if more than two such values exist (\( \tau_x^+ \) or \( \tau_x^- \)) falls between \( \tau_1 \)
and $\tau_2$, one has to sort them to get the maximal amplitude of $\theta_x$. The same goes for $\theta_y$ and $\theta_z$. For example, in the systems studied in the following sections, the condition $dy/d\tau = 0$ is met several times with $\theta_x(\tau_1) < \theta_x(\tau_2) < \theta_x(\tau^-_1)$. In these cases, the variations of $\theta_y$ are thus bounded by $\theta_y(\tau_1)$ and $\theta_y(\tau^-_1)$.

2.4. Nutation Period

The equations of motion (3) expressed in terms of $x$, $y$, $z$ (6) are

$$
\begin{align*}
\dot{x} & = v_3 V z; \\
\dot{y} & = v_1 V x - v_4 V z; \\
\dot{z} & = -v_2 V x,
\end{align*}
$$

(16)

where $(v_k)_{k=1,\ldots,4}$ are frequencies defined in (4). It is then straightforward to check that the change of time leading to the parameterization (9) satisfies

$$
d\tau = \sqrt{v_2 v_3} V dt. 
$$

(17)

The nutation period $P_{nut}$ corresponds to the oscillation period of $x(t)$, $y(t)$, and $z(t)$, which is twice the time needed to reach $\tau_2$ starting from $\tau_1$, the two roots of $V^2(\tau) = 0$ (BL06). Using the definition of $\tau$ (17), the result is

$$
P_{nut} = \frac{2}{\sqrt{v_2 v_3}} \int_{\tau_1}^{\tau_2} \frac{d\tau}{V(\tau)},
$$

(18)

where $V(x, y, z)$ is given by (12) and the parameterization $x(t)$, $y(t)$, and $z(t)$ written in (9). In the vicinity of the integral boundaries $(\tau_k)_{k=1,2}$, $V(\tau)$ evolves like $\sqrt{\tau - \tau_k}$, unless the ellipse $\mathcal{E}$ is tangent to the Berlingot, a case we discard for the moment. The integral (18) is thus convergent (BL06). Nevertheless, to avoid the singularities at the boundaries, one should perform the following change of variable

$$
\tau = \frac{\tau_2 - \tau_1}{2} \sin u + \frac{\tau_2 + \tau_1}{2},
$$

(19)

leading to

$$
P_{nut} = \frac{\tau_2 - \tau_1}{\sqrt{v_2 v_3}} \int_{-\pi/2}^{\pi/2} \frac{\cos u du}{V(u)}. 
$$

(20)

If the trajectory of $(x, y, z)$ is tangent to the Berlingot “from the inside,” the point of tangency is an unstable fixed point. In that case, the nutation period $P_{nut}$ becomes infinite, as expected for an evolution to or from an equilibrium. On the other hand, if the trajectory is tangent “from the outside,” the system is in the middle of a libration zone. In that case, the nutation period is derived from a linearization of the equations of motion (3).

2.5. Precession Period

As stated before, the precession period is more difficult to derive. We have reached the limit of the geometrical approach. It is now necessary to return to a more conventional description based on action/angle variables which we denote $(L_z, G_z, G_z', \psi, \Omega, \Omega')$. The three actions are the third components of the angular momenta in a given reference frame while $\psi$, $\Omega$, and $\Omega'$ are the precession angle of the star and the longitudes of the ascending nodes of the planet system and of the companion, respectively. $(L_z, \psi)$ are Andoyer conjugate variables; $(G_z, \Omega)$ and $(G_z', \Omega')$ are subsets of Delaunay variables.

The system thus has a priori three degrees of freedom and the Hamiltonian written in these variables reads

$$
H = -\frac{\alpha}{2} \left( \sin \epsilon \sin l \cos(\psi - \Omega) + \cos \epsilon \cos l \right)^2 \\
- \frac{\gamma}{2} \left( \sin l \sin l' \cos(\Omega - \Omega') + \cos l \cos l' \right)^2,
$$

(21)

with $\cos \epsilon = L_z/L$, $\cos l = G_z/G$, and $\cos l' = G_z'/G'$. We recall that $(L, G, G')$ are constants of motion. After a usual reduction of the node (Malige et al. 2002), the problem reduces to two degrees of freedom. The associated symplectic transformation is

$$
\begin{align*}
\tilde{\psi} & = \psi, \\
\tilde{\Omega} & = \Omega - \psi, \\
\tilde{\Omega}' & = \Omega' - \psi,
\end{align*}
$$

(22)

and the Hamiltonian expressed in the new variables reads

$$
\tilde{H} = -\frac{\alpha}{2} \left( \sin \epsilon \sin l \cos \tilde{\Omega} + \cos \epsilon \cos l \right)^2 \\
- \frac{\gamma}{2} \left( \sin l \sin l' \cos(\tilde{\Omega} - \tilde{\Omega}') + \cos l \cos l' \right)^2,
$$

(23)

with $\cos \epsilon = (\tilde{L}_z - \tilde{G}_z - \tilde{G}_z')/L$, $\cos l = \tilde{G}_z/G$, and $\cos l' = \tilde{G}_z'/G'$. The variable $\tilde{\psi}$ is cyclic; its conjugated momentum $\tilde{L}_z$ is constant. $\tilde{L}_z$ represents the third component of the total angular momentum of the system. The two remaining degrees of freedom are carried by $(\tilde{G}_z, \tilde{\Omega})$ and $(\tilde{G}_z', \tilde{\Omega}')$. The problem is further simplified when the reference plane is chosen to be the invariant plane orthogonal to the total angular momentum $\mathbf{W}$. Indeed, in that case, the first two components of $\mathbf{W}$ are zero and thus the two angle variables $\tilde{\Omega}$ and $\tilde{\Omega}'$ are related by

$$
L \sin \epsilon + G \sin l \exp(i\tilde{\Omega}) + G' \sin l' \exp(i\tilde{\Omega}') = 0.
$$

(24)

Hence, the relative motion described by the Hamiltonian (23) has only one degree of freedom when expressed in the invariant plane. This agrees with the fact that the relative motion is characterized by a single frequency called the nutation frequency, but although the Hamiltonian (23) is integrable, the general solution does not seem easy to derive in action/angle variables. Next, we thus rely on the solution obtained in the variables $(x, y, z)$.

The expression of the precession period $P_{prec}$ is then obtained by solving the evolution of $\tilde{\psi} = \psi$. Indeed, consider a three-vector system in a given configuration at time $t = 0$. After a nutation period $P_{nut}$, the system returns to its initial configuration (same mutual distances) but its overall orientation is changed by an angle $\psi(P_{nut}) - \psi(0)$ around the total angular momentum. Hence,

$$
P_{prec} = \frac{2\pi}{\Delta \psi} P_{nut},
$$

(25)

with $\Delta \psi = |\psi(P_{nut}) - \psi(0)|$ is deduced from the equation of motion

$$
\frac{d\psi}{dt} = \frac{d\psi}{dt} = \frac{\partial \tilde{H}}{\partial L_z} = \frac{\partial \tilde{H}}{L \partial \cos \epsilon}.
$$

(26)

Substituting the expression of the Hamiltonian (23), and remembering that the first term in parentheses is precisely $x = s \cdot \mathbf{w}$, we get

$$
\frac{d\psi}{dt} = -\frac{\alpha}{L} \left( \frac{\cos \epsilon}{\sin \epsilon} \sin l \cos \tilde{\Omega} + \cos l \right).
$$

(27)
This expression is further simplified after expressing \( \cos \tilde{\Omega} \) in terms of \( x, e, \) and \( I \) as
\[
\frac{d\psi}{dt} = -\frac{\alpha}{L} x \cos I - x \cos e.
\] (28)

Moreover, \( \cos e \) and \( \cos I \) are functions of \( (x, y, z) \) given by
\[
\cos e = \frac{W \cdot s}{W} = \frac{L + Gx + G'y}{W},
\]
\[
\cos I = \frac{W \cdot w}{W} = \frac{Lx + G + G'z}{W},
\] (29)

where \( W = \|W\| \) is the norm of the total angular momentum of the system and is a constant of the motion. Thus,
\[
\frac{d\psi}{dt} = -\frac{\alpha}{L} Wx \left( \frac{G(1 - x^2) + G'(z - xy)}{W^2 - (L + Gx + G'y)^2} \right).
\] (30)

Finally, using the change of time \( t \mapsto \tau \) (17) and the fact that the oriented volume \( V(\tau) \) changes its sign at a rebound, we get
\[
\Delta \psi = \int_{\tau_1}^{\tau_2} \frac{d\psi}{dt} \frac{d\tau}{V(\tau)} = \frac{2}{\sqrt{2}v_3} \int_{\tau_1}^{\tau_2} \frac{d\psi}{dt} \frac{d\tau}{V(\tau)}.
\] (31)

The expression of the precession period is thus
\[
P_{\text{prec}} = \pi \frac{1}{\sqrt{2}v_3} \frac{1}{P_{\text{tot}}} \int_{\tau_1}^{\tau_2} \frac{d\psi}{dt} \frac{d\tau}{V(\tau)}^{-1},
\] (32)

with \( d\psi/dt \) given by (30).

3. N-BODY PROBLEM
3.1. Numerical Model

The generic system we wish to study is composed of \( p \) packed planets orbiting an oblate star with a companion on a wide eccentric inclined orbit. The planet system is supposed to be relatively coplanar with low eccentricities. Furthermore, we assume that mean-motion resonances are not dominating the secular motion. In that case, using the formalism developed in (BF14), the secular Hamiltonian of the system can be approximated by
\[
H_{\text{tot}} = \sum_{1 \leq j < k \leq p} \tilde{H}_{\text{close}}(j, k) + \sum_{j=1}^{p} (\tilde{H}_{\text{bierac}}(j, p + 1) + \tilde{H}_{\text{spin}}(0, j) + \tilde{H}_{\text{relac}}(j)).
\] (33)

In this equation, \( \tilde{H}_{\text{close}}(j, k) \) represents the secular interaction between planets \( j \) and \( k \) expanded up to the fourth order in eccentricity and mutual inclination. \( \tilde{H}_{\text{bierac}}(j, p + 1) \) denotes the secular interaction between a planet \( j \) and the outer companion. This term is developed in semi-major axis ratio at the octupole. \( \tilde{H}_{\text{spin}}(0, j) \) is the quadrupolar secular spin–orbit interaction between the stellar spin-axis and the planet \( j \). Finally, \( \tilde{H}_{\text{relac}}(j) \) describes the relativistic precession of the pericenter of the planet \( j \) induced by the central star. The expression of each of these terms as well as the corresponding equations of motion are detailed in (BF14). Hereafter, we refer to numerical integrations of \( H_{\text{tot}} \) as the numerical model. Note that because the Hamiltonian \( H_{\text{tot}} \) is averaged over all mean anomalies, the \( N \)-body problem (with \( N = p + 2 \)) has already been changed into an \( N \)-ring problem (including the equator of the central star).

In the following, we show how the numerical \( N \)-ring problem can be further simplified into a three-vector problem which will be referred to as the analytical model. Then, we discuss the validity of the latter and perform a few comparisons of the two models.

3.2. Analytical Model

The three-vector problem presented at the beginning provides the solution of the secular evolution of obliquity and inclinations only. Conversely, the numerical model also accounts for the evolution of eccentricities. The analytical model is thus more limited than the numerical model. It uses the fact that eccentricities and inclinations are decoupled in Laplace–Lagrange linear theory, and that the quadrupole expansion of hierarchical interaction is independent of the direction of the outer body’s pericenter, which is known as a happy coincidence (Lidov & Ziglin 1976). At this order, and neglecting planet eccentricities, the Hamiltonian \( H_{\text{tot}} \) given in (BF14) reduces to
\[
H_{\text{inc}}^{(2)} = -\sum_{j=1}^{p} \frac{\alpha_j}{2} (s \cdot w_j)^2 - \frac{\gamma_j}{2} (w_j \cdot w_j)^2 - \sum_{1 \leq j < k \leq p} \beta_{jk} (w_j \cdot w_k),
\] (34)

where \( w_j \) is the unit vector normal to the orbit \( j \) and
\[
\alpha_j = \frac{3}{2} \frac{G m_0 m_j J_2 R_0^2}{a_j}.
\]
\[
\gamma_j = \frac{3}{4} \frac{G m_j m'_j a_j^2}{a_j^3 (1 - e_j^2)^{3/2}},
\]
\[
\beta_{jk} = \frac{1}{4} \frac{G m_j m_k a_j a_k}{a_k^3 (1 - e_j^2)^{1/2}} (a_j a_k). \] (35)

Hereafter, \( m \) denotes mass, \( a \) is the semi-major axis, and \( e \) is the eccentricity. Subscripts 0 and \( j = 1, \ldots, p \) stand for the central star and for the \( p \) planets, respectively. The companion’s parameters are written with a prime. \( G \) is the gravitational constant. The star has a radius \( R_0 \), a second fluid Love number \( k_2 \), a rotation speed \( \omega_0 \), a moment of inertia \( C \), and a quadrupole coefficient \( J_2 \) given by (e.g., Lambeck 1988)
\[
J_2 = k_2 \frac{\omega_0^2 R_0^3}{3 G m_0}.
\] (36)

The \( b_k^{(j)} \) are Laplace coefficients. Note that all notations are summarized in Table 1.

Let \( L = L_S = C \omega_0 s, G_j = G_j w_j, \) and \( G' = G'w' \) be the angular momenta of the star, of the orbit of planet \( j \), and of the orbit of the companion, respectively. Within the same level of approximation as in Equation (34), \( e' \) and thus \( G' \) are constant and the equations of motion are
Table 1

| Variable | Ref. | Description |
|----------|------|-------------|
| Hamiltonian | $H_{\text{tot}}$ | Equation (33) secular Hamiltonian of the numerical system |
| $H$ | Equations (1, 7) | secular Hamiltonian of the analytical model |
| $K$ | Equation (8) | first integral of the analytical problem |
| $\alpha, \gamma$ | Equation (35) | coefficients of the analytical Hamiltonian $H$: $\sum \alpha_k$ and $\sum \gamma_k$, respectively |
| $b_{ij}^{(k)}$ | Laplace coefficient |
| Timescales | $v_1$ | Equation (41) precession frequency $a/L$ of $x$ relative to $w$ |
| $v_2$ | Equation (41) | precession frequency $a/G$ of $w$ relative to $s$ |
| $v_3$ | Equation (41) | precession frequency $y/G$ of $w$ relative to $w'$ |
| $v_4$ | Equation (41) | precession frequency $y'/G'$ of $w'$ relative to $w$ |
| $v_{1d}, v_{2d}, v_3, v_4$ | permutation of $v_1, v_2, v_3, v_4$ |
| $P_{\text{nut}}$ | Equation (20) | nutation period |
| $P_{\text{prec}}$ | Equation (32) | precession period |
| Stellar parameters | $m_0$ | mass |
| $R_0$ | radius |
| $J_2$ | Equation (36) | quadrupole gravitational harmonic |
| $k_2$ | second fluid Love number |
| $C$ | moment of inertia along the short axis |
| $\omega_0$ | rotation speed |
| $P_0$ | rotation period $2\pi/\omega_0$ |
| $e$ | obliquity relative to the companion's orbit |
| $\psi$ | precession angle |
| $s$ | stellar spin axis |
| $L$ | stellar angular momentum $C_{00}$s |
| $j$th planet and companion | $m_j$ | mass |
| $a_j$ | $a'$ | semi-major axis |
| $b_j$ | $b'$ | semi-minor axis $a' (1 - e^2)^{1/2}$ |
| $P_j$ | $P'$ | revolution period |
| $e_j$ | $e'$ | eccentricity |
| $I_j$ | $I'$ | absolute inclination (with respect to the reference plane) |
| $\Omega_j$ | $\Omega'$ | longitude of the ascending node |
| $e_j$ | eccentricity vector |
| $J_j$ | dimensionless orbital angular momentum $(1 - e^2)^{1/2}w_j$ |
| $w_j$ | $w'$ | unit vector of orbital angular momentum |
| $G_j$ | $G'$ | orbital angular momentum |
| other variables | $w$ | unit vector of $G$ |
| $G$ | Equation (38) | total angular momentum of the planet system $G = \sum G_j w_j$ |
| $W$ | Equation (5) | total angular momentum $L + G + G'$ |
| $x = \cos \theta_x$ | Equation (6) | cosine of the stellar obliquity relative to the planets plane $s \cdot w$ |
| $y = \cos \theta_y$ | Equation (6) | cosine of the stellar obliquity relative to the companion’s orbit $s \cdot w'$ |
| $z = \cos \theta_z$ | Equation (6) | cosine of the mutual inclination between the planets and the companion $w \cdot w'$ |
| $\mathcal{E}$ | eccentricity orbit in $(x, y, z)$ satisfying $H(x, y, z) = \mathcal{E}$ and $K(x, y, z) = k$ for two reals $(h, k)$ |
| $C, \partial C$ | Cassini Berlingot defined by $V^2(x, y, z) \geq 0$ and its boundary, respectively |
| $B, \partial B$ | Figure 1 | diagonals which are the intersections of $\partial C$ and $\partial B$ |
| $D_x, D_y, D_z$ | Figure 1 | hyperbolic surface equal to the union of all elliptic trajectories intersecting $D_i$ |
| $S$ | Figure 10 | oriented volume generated by $(s, w, w')$ |
| $V$ | Equations (11, 12) | quadric function defining the surface $S$ |
| $A_x, A_z$ | Equation (10) | length scales of the elliptic orbit in $(x, y, z)$ |
| $\mathcal{G}$ | gravitational constant |
| $c$ | speed of light |

In a last step, we consider the total angular momentum $G$ of the planet system

$$
G = \sum_{j=1}^{p} G_j w_j. \tag{38}
$$

We also denote $G = \|G\|$ and $w = G/G$. The latter represents the normal of the planet system. Using the equations of

$$
\frac{ds}{dt} = -\frac{1}{L} s \times \nabla_s H_{\text{inc}}^{(2)},
$$

$$
\frac{d w_j}{dt} = -\frac{1}{G_j} w_j \times \nabla_{w_j} H_{\text{inc}}^{(2)},
$$

$$
\frac{d w'}{dt} = -\frac{1}{G'} w' \times \nabla_w H_{\text{inc}}^{(2)}. \tag{37}
$$
motion (37), we get

\[
\frac{dG}{dt} = -\sum_{j=1}^{p} (\alpha_j (s \cdot w_j) s + \gamma_j (w' \cdot w_j) w') \times w_j. \tag{39}
\]

As expected, all the interactions internal to the planet system (involving $\beta_{jk}$ and $\beta_{ij}$) cancel out in the evolution of $G$. Furthermore, assuming that each planet $j$ remains close to the planet system (low mutual inclination), $w_j$ is replaced by $w$ in (37) and (39). As a result, $G$ becomes constant and the equations of motion read

\[
\begin{align*}
\frac{ds}{dt} &= -\frac{\alpha}{L} (s \cdot w) w \times s, \\
\frac{dw}{dt} &= -\frac{\alpha}{G} (s \cdot w) s \times w + \frac{\gamma}{G} (w' \cdot w) w' \times w, \\
\frac{dw'}{dt} &= -\frac{\gamma}{G} (w' \cdot w) w' \times w', \tag{40}
\end{align*}
\]

with $\alpha = \sum_{j} \alpha_j$ and $\gamma = \sum_{j} \gamma_j$. These equations are exactly those of the three-vector problem (3). We can thus use the results of the three-vector problem to study the spin–orbit evolution in compact planetary systems perturbed by an inclined companion. The expressions of the characteristic frequencies $\nu_k$, Equation (4), are

\[
\begin{align*}
\nu_1 &= \frac{\pi k_2 m_0 R_0^2}{P_0} A_1, \\
\nu_2 &= \frac{\pi k_2 A_1}{P_0} A_2, \\
\nu_3 &= \frac{3\pi}{2P'} \frac{m'}{m_0 + m'} \frac{P_0 A_3}{P'A_2} \frac{1}{(1 - e'^2)^{3/2}}, \\
\nu_4 &= \frac{3\pi}{2P'} \left( \frac{R_0}{a'} \right)^2 \frac{A_3}{(1 - e'^2)^2}, \tag{41}
\end{align*}
\]

with

\[
\begin{align*}
A_1 &= \sum_{j=1}^{p} \frac{m_j}{m_0} \left( \frac{R_0}{a_j} \right)^3, \\
A_2 &= \sum_{j=1}^{p} \frac{m_j}{m_0 + m_j} \frac{P_0}{P_j} \left( \frac{a_j}{R_0} \right)^2, \\
A_3 &= \sum_{j=1}^{p} \frac{m_j}{m_0} \left( \frac{a_j}{R_0} \right)^2. \tag{42}
\end{align*}
\]

In (41) and (42), $P_0$, $P_j$, and $P'$ are the rotation period of the central star, the orbital period of the planet $j$, and the orbital period of the companion, respectively.

It must be stressed that although the model (40) has been obtained for a non-resonant planet system, the result is more general. Indeed, Equation (39) holds whatever are the interactions between the planets: the evolution of $G$ depends solely on external perturbations (those raised by the stellar oblateness and by the companion). As a consequence, the three-vector approximation also applies to problems with a resonant planetary system as long as their eccentricities and mutual inclinations remain low, but the same is not true for the numerical model because these planetary orbital elements are explicitly integrated using non-resonant secular equations.

### 3.3. Validity of the Analytical Model

Here, we assume that the planet system is non-resonant. Then, both the numerical and the analytical models are valid as long as the eccentricities and the mutual inclinations of the planets remain small. This condition is easy to check with the numerical model because it includes the evolution of the planet eccentricities and their coupling with inclinations. Conversely, the analytical model completely discards planetary eccentricities and individual inclinations. As a result, the latter does not provide any information on validity. It would thus be convenient to have a simple criterion saying whether or not eccentricities might evolve significantly. In other words, the criterion should distinguish among all systems that are dynamically rigid.

In the case with only two planets, the Lidov–Kozai cycles are suppressed as long as the periapsis precession period of the outer planet $\tau_{pp}$ is shorter than the Lidov–Kozai timescale $\tau_{Koz}$ (Takeda et al. 2008). The study of the general case is beyond the scope of this paper. We thus assume that this is also true in the $p$-planet case.

Under the approximation of the Laplace–Lagrange theory, the secular frequencies of multiplanet systems are those of the eigenmodes. However, in order to get a simple criterion and to avoid the problem of assigning an eigenfrequency to a specific planet, we only consider the secular interaction between the two outermost planets. More precisely, within the Laplace–Lagrange linear model, the secular evolution of $e_p$, induced by the planet $p - 1$, with the notation of (BF14), reads

\[
\frac{de_p}{dt} = \frac{Gm_p-m_p}{a_p G_p} \left( c_2 w \times e_p + c_3 w \times e_{p-1} \right) \tag{43}
\]

where $G_p = m_0 m_p / (m_0 + m_p) \sqrt{G(m_0 + m_p) a_p}$ and $c_2$ and $c_3$ are two functions of $(a_{p-1}/a_p)$. We then set $\tau_{pp}$ as the inverse of the coefficient of $w \times e_p$ in (43). At the lowest order in $(a_{p-1}/a_p)$, the result is

\[
\tau_{pp} \approx \frac{2}{3\pi} \frac{m_0}{m_{p-1}} \left( \frac{a_p}{a_{p-1}} \right)^2, \tag{44}
\]

with $P_p$ the orbital period of the (outermost) planet $p$. On the other hand, the Lidov–Kozai timescale (Lidov & Ziglin 1976), written in canonical astrocentric elements is

\[
\tau_{Koz} = \frac{2}{3\pi} \frac{m_0}{m'} \left( \frac{b'}{a_p} \right)^3, \tag{45}
\]

where $b' = a' \sqrt{1 - e'^2}$ is the semi-minor axis of the companion. The criterion of validity of the analytical model is thus

\[
\tau_{pp} \ll \tau_{Koz}, \tag{46}
\]

where $\tau_{pp}$ and $\tau_{Koz}$ are given by (44) and (45), respectively.

We have checked this criterion on four-body problems through 4800 integrations of the numerical model. The simulated system is composed of two planets with mass $10^{-5} \, M_\odot$ orbiting a 1 $M_\odot$ star. We chose three different masses for the companion: $m' = 10^{-4}, 10^{-2}, 10^{-1} \, M_\odot$; two eccentricities $e' = 0, 0.5$; and two inclinations relative to the initial planet plane $i'_0 = 50', 89'$. For each combination $(m', e', i'_0)$, the semi-major axis of the inner planet and the semi-minor axis of the companion take 20 values uniformly distributed in the ranges $0.1 \leq a_{p-1}/a_p \leq 0.9$ and $10 \leq b'/a_p \leq 100$, respectively. Neither spin–orbit interaction nor general relativity are included in
this analysis. For \( I_0' = 89^\circ \), we found a sharp transition between low \( \max(e_p) < 0.01 \) and high \( \max(e_p) > 0.8 \) eccentricity regimes at \( \varepsilon_{Koz} \sim 1.5 \varepsilon_{pp} \) leading to the criterion

\[
\frac{b'}{a_p} \gtrsim 1.5 \left( \frac{m'}{m_{p-1}} \right)^{1/3} \left( \frac{a_{p-1}}{a_p} \right)^{-2/3}
\]  

represented in Figure 4. For \( I_0' = 50^\circ \), the amplitude of \( \max(e_p) \) (not shown) is smaller than in the \( I_0' = 89^\circ \) case. Nevertheless, a sharp transition is still observed although at lower values of \( b' \) (the limits obtained with the two inclinations differ by 25%). The constraint given by Equation (47) is thus more stringent.

Finally, we would like to stress that the criterion (47) displayed in Figure 4 has only been verified in limited configurations. For instance, we have only considered inner systems composed of two planets with equal mass. We expect the criterion to remain informative in other cases, but close to its threshold, we recommend integrations of the numerical model to certify the validity of the analytical one. Moreover, the criterion has to be adapted if the system contains several well separated compact groups of planets. It is indeed necessary to check whether they all behave coherently or not.

4. NUMERICAL TESTS AND APPLICATIONS

In a previous paper (BF14), we tested the numerical model against N-body integrations, finding them to hold well in for a system like the solar system, with low eccentricities and inclinations, and a hierarchical system that is highly inclined and undergoing Kozai–Lidov oscillations. In this section, we assume the numerical model can well represent the motion of planetary systems, and we use it to test the analytic model in a few cases involving two observed exoplanet systems.

The first one, 55 Cancri, is composed of five planets orbiting one member of a wide binary system. This system is actually the prototype for which the formalism of the previous section has been developed. The second system, HD 20794, contains three super-Earths, but no wide perturber detected to date. We have thus added an arbitrary planetary companion to it. This latter system has been chosen because of the low angular momentum in the planets’ orbit in comparison to the 55 Cancri system. The two systems allow us to explore relatively different regions of the parameter space.

4.1. 55 Cancri

The system 55 Cnc is our first example because it is actually a binary system and five planets orbit the most massive component 55 Cnc A. Moreover, the evolution of the inclination of the planet system has already been studied in Kaib et al. (2011) (hereafter noted as K11) using another approach. The masses and the orbital parameters of the five planets are summarized in Table 2. Planets b and c are close to the 3:1 mean-motion resonance (Fischer et al. 2008), but we assume that this proximity does not significantly affect the secular motion derived from the numerical model. As in K11, we introduce small initial inclinations to avoid purely coplanar evolutions of the planet system.

The central star 55 Cnc A has a mass \( m_0 = 0.905 \, M_\odot \) and a radius \( R_0 = 0.943 \, R_\odot \) (von Braun et al. 2011). Its rotation period determined photometrically is \( P_0 = 42.7 \) days (Fischer et al. 2008). The internal structure parameters, \( k_2 = 0.028 \) and \( C = 0.09 \, m_0 R_0^2 \), have been taken from stellar models (Landin et al. 2009, Table 1a). The stellar companion is at a projected separation of 1065 au and has a mass \( m'' = 0.26 \, M_\odot \) (Mugrauer et al. 2006). In all the following studies, the initial obliquity of the central star relative to the planet plane \( \theta_i(t = 0) \) is set to zero.

As a first test, we reproduce the evolution presented in K11’s Figure 1. The orbital parameters of the companion are \( a'' = 1250 \, au, e'' = 0.93, \) and \( I'' = 115 \) deg. For this simulation, we use the numerical Hamiltonian expressed in vectorial form.
Thus planet–planet interactions are modeled at the fourth order in eccentricity inclination and the interaction with the companion is modeled at the third order in semi-major axis ratio. Relativistic precession is included, but the central star is considered to be a point mass; we thus neglect the interactions with the stellar figure. The result of the simulation is displayed in Figure 5. The numerical evolution agrees very well with the \( n \)-body integration performed in K11. All the planet orbital inclinations follow the same track. The planetary orbital plane tilts periodically like a rigid body. The main difference between our work and K11’s result is a slight shift in the evolution of the planet’s inclination.

Now, we focus our attention on the evolution of the spin axis of the central star. Figure 5, showing the solid rotation of the planetary system, justifies the analytical model of the previous section: the planetary system remains almost coplanar. The numerical values of the precession frequencies, Equation (41), are \( v_1 \approx 323 \text{ deg Myr}^{-1} \), \( v_2 \approx 0.49 \text{ deg Myr}^{-1} \), \( v_3 \approx 9.8 \text{ deg Myr}^{-1} \), and \( v_4 \approx 0.030 \text{ deg Myr}^{-1} \).

The equations of motion are thus

\[
\frac{ds}{dt} = -323 \cos \theta_z \; \mathbf{w} \times s;
\]

\[
\frac{d\mathbf{w}}{dt} = -0.49 \cos \theta_z \; s \times \mathbf{w} - 9.8 \cos \theta_z \; \mathbf{w}' \times \mathbf{w};
\]

\[
\frac{d\mathbf{w}'}{dt} = -0.030 \cos \theta_z \; \mathbf{w} \times \mathbf{w}'
\]

in deg Myr\(^{-1}\). The precession motion of the spin axis \( s \) of the central star relative to the pole \( \mathbf{w} \) of the planet mid-plane is about 80 times faster than any of the other secular motions (when taking into account \( \cos \theta_z \approx -0.42 \)). In good approximation, the evolution of \( s \) can thus be computed assuming all the other vectors to be constant. As a result, the obliquity \( \theta_z \) of the star with respect to the orbital plane of the planets should remain constant and equal to its initial value. If it begins aligned, it will remain aligned. A numerical integration similar to that of Figure 5, but including the evolution of the orientation of the central star (represented by the vector \( s \)) confirms this analytical conclusion. The trajectory of the unit vectors \( s, \mathbf{w}, \) and \( \mathbf{w}' \) are plotted in Figures 6(a) and (c) as well as \( \mathbf{w}_1 \) to check whether the

in the innermost planet of the system follows the evolution of the outer planets. These trajectories are a combination of a solid rotation (precession motion) around the total angular momentum and quasi-periodic nutation motions (BL06; BL09). In Figures 6(b) and (d)–(f), the trajectories, displayed in the frame rotating with the main precession frequency, show that the stellar spin axis \( s \) never moves away from \( \mathbf{w} \) and \( \mathbf{w}_1 \) by more than about 2\(^\circ\). For comparison, the analytical model (Section 2.3) predicts a spin–orbit amplitude of 1\(^\circ\). The star thus never moves along with the planetary orbits as their inclinations oscillate. As a consequence, the sole presence of an inclined stellar companion in the 55 Cnc system is not enough to generate a spin–orbit misalignment.

The above conclusions have been derived for a specific set of physical and orbital parameters. Are there any other choices producing effective misalignment? In this system, the star follows the tilt of the orbits of the planets because \( v_1 \gg v_3 \).

To cancel this effect, one should decrease \( v_1 \) or increase \( v_3 \).

Except for the planet \( e \) observed in transit, only the minimum masses \( m_k \sin i_k \), where the inclinations \( i_k \) are measured with respect to the plane of the sky, are known. Nevertheless, the frequency \( v_k \) depends linearly on \( m_k \) while \( v_3 \) is not affected by a rescaling of the planet masses. Thus, increasing the planet masses would only cause the star to be more strongly coupled to the planets.

In the expression of \( v_1 \) (41), the next less known parameter is the Love number \( k_2 \) of the star. Based on the results of the internal structure model of Landin et al. (2009), we estimate the error on \( k_2 \) to be of the order of 20\%, but even if \( k_2 \) is wrong by a factor of two, it is not enough to compensate for the large ratio \( v_1/v_3 \sim 33 \).

On the contrary, the precession frequency \( v_3 \) is much more undetermined. Indeed, \( v_3 \) scales as \( m^4/[a^3(1-e^2)^{5/2}] \). The mass \( m^4 = 0.26 M_\odot \) has been derived by Mugrauer et al. (2006) from the absolute infrared magnitude using the evolutionary models of Baraffe et al. (1998), whereas the semi-major axis \( a = 1025 \text{ au} \) and the eccentricity \( e' = 0.93 \) are only constrained by the projected separation \( d = 1065 \text{ au} \). There is thus a full range of semi-major axes and eccentricities compatible with the observations. The closer and the more eccentric the perturber’s orbit is, the larger is the frequency \( v_3 \). Fixing the distance of the pericenter at \( q = 35 \text{ au} \), and assuming that the projected separation corresponds to the distance of the apecenter \( Q = 1065 \text{ au} \), we get \( a' = 550 \text{ au} \) and \( e'' = 0.936 \). Moreover, since \( v_3 \) is proportional to \( \cos \theta_z \), where \( \theta_z \) is the orbital inclination of the stellar companion relative to the planetary mid-plane, we chose a lower initial inclination \( I'(t = 0) = \theta_z(t = 0) = 30^\circ \) only (we recall that \( I' \) is measured with respect to the reference plane, which is also the planet plane at \( t = 0 \), while \( \theta_z \) is the mutual inclination between the orbits of the companion and of the planets). With this set of parameters, \( v_3 \cos \theta_z \) increases from 4.2 deg Myr\(^{-1}\) to 114 deg Myr\(^{-1}\). This quantity is still less that \( v_1 \) but it is of the same order of magnitude. The trajectories of the unit vectors computed with this new set of initial conditions are plotted in Figure 7. We indeed observe that the stellar axis and the planets’ orbital plane become periodically misaligned from 0\(^\circ\) to about 50\(^\circ\). In this case, the amplitude of 32\(^\circ\) given by the analytical model is underestimated by about 36\%. This discrepancy is mainly due to the relatively weak coupling between 55 Cnc d and the four innermost planets.

The semi-major axis ratio between the two outermost planets is about 7.4. There is a better agreement on the nutation period derived from the numerical integration 2.3 Myr, and analytically, 1.8 Myr.
We have shown on this example (55 Cancri) that the central star can be strongly linked to the motion of the orbital plane of its planets. This is mainly due to the fact that most of the angular momentum is in the orbit of the planets rather than in the rotation of the star. Nevertheless, if the interaction with the outer companion is strong enough, the evolution of the stellar spin axis can be decoupled from the motion of its planet. A detailed analysis of the statistical distribution of the spin–orbit misalignment in the 55 Cnc system is beyond the scope of the present paper. Nevertheless, a global analysis of the maximal spin–orbit misalignment of compact systems perturbed by a wide companion is performed in Section 5.

4.2. HD 20794

The system HD 20794 is composed of three super-Earths orbiting a G8V type star with mass $m_0 = 0.85 M_\odot$ and radius $R_0 = 0.9 R_\odot$ (Pepe et al. 2011; Bernkopf et al. 2012). In comparison to 55 Cancri, this system is not known to harbor any outer companion. We nevertheless choose this system because the angular momentum is more balanced between the star and the planets than in the 55 Cancri system, and this difference allows new kinds of spin–orbit angle evolutions. The rotation period of the star is $P_0 = 33.2$ days (Pepe et al. 2011), and its internal structure constants inferred from Landin et al. (2009) are $k_2 = 0.017$ and $C = 0.08 m_0 R_0^2$. The parameters of the three super-Earths are summarized in Table 3.

Because no stellar companion has been detected in this system until now, we arbitrarily add a perturber: a giant planet with mass $m' = 1 M_J$ at $a' = 20$ au with an eccentricity $e' = 0.1$ on an orbit initially inclined by $30^\circ$ with respect to the plane of the

\begin{table}
\centering
\begin{tabular}{ccccccc}
\hline
\hline
Planet & $m$ & $a$ & $e$ & $i_0$ & $f$ & $\Omega$ \\
& ($M_\text{Jup}$) & (au) & & (°) & (°) & (°) \\
\hline
b & 3.28 & 0.1288 & 0.00 & 0 & 0.10 & 0 \\
c & 2.91 & 0.2172 & 0.01 & 90 & 0.30 & 0 \\
d & 5.83 & 0.3733 & 0.03 & 270 & 180 & 0 \\
\hline
\end{tabular}
\caption{Masses and Orbital Elements of the Planets of the HD 20794 System}
\end{table}

Notes. Data are derived from Pepe et al. (2011), masses and semi-major axis have computed for a central mass $m_0 = 0.85 M_\odot$ instead of $0.70 M_\odot$. The eccentricity and the inclination are introduced solely to avoid circular and coplanar evolutions.
Figure 7. Same as Figure 6 but with the stellar companion 55 Cnc B on a closer and less inclined orbit: $a' = 550$ au, $e' = 0.936$, and $I' = 30^\circ$, initially. In panel (d), $w_2$, $w_3$, and $w_4$ follow $w_1$, while $w_5 \sim w$.

(A color version of this figure is available in the online journal.)

planets. This planet would have a 97 yr orbital period and a radial velocity semi-amplitude $K = 7.25 \, \text{m s}^{-1}$. For an observation time span of 7 yr (Pepe et al. 2011), the amplitude of the drift induced by the new planet would range between 0.19 m s$^{-1}$ and 3.3 m s$^{-1}$, depending on the orbital phase. Although the upper limit is well above the 0.82 m s$^{-1}$ rms of the residuals in Pepe et al. (2011), the lower limit is significantly below and such a planet could have been missed. In this system, the central star and the planet system have very similar angular momenta. Thus, the frequencies $\nu_1$ and $\nu_2$ are almost equal. The equations of motion are

$$\frac{ds}{dt} = -0.211 \; w \times s ;$$
$$\frac{d\,w}{dt} = -0.164 \; s \times w - 6.12 \; w' \times w ;$$
$$\frac{d\,w'}{dt} = -0.025 \; w \times w'$$

in deg Myr$^{-1}$. From (49), we deduce that the outer giant planet possesses most of the angular momentum of the system, its orbit remains close to the invariant plane of the system. Equations (49) also tell us that the angular momentum of the planet system $w$ precesses around $w'$ about 26 times faster than the star $s$ around $w$ (with $\cos \theta \approx 0.87$). As a consequence, the star precesses around the averaged orbital pole $\langle w \rangle$ which is collinear to $w'$, but at a much slower frequency than $w$. Figure 8 shows the trajectories of $s$, $w_1$, $w$, and $w'$ in the frame rotating at the precession frequency of the stellar axis $s$ around the total angular momentum. Since the precession motion of the planets’ pole is faster than that of the star, the planets mid-plane is still precessing around the total angular momentum of the system in the rotating frame. As a consequence, the spin–orbit angle oscillates periodically between $0^\circ$ and $60^\circ$, the latter being equal to twice the initial inclination $I'(t = 0)$. This amplitude is in perfect agreement with the analytical model as well as the nutation period, which is equal to 67.4 Myr.

In a last experiment, we reduce the mass $m'$ of the perturbing companion by a factor of 25, such that the frequencies $\nu_3$ becomes of the same order of magnitude as $\nu_1$ and $\nu_2$. As a result, $w$ is expected to describe a nutation motion around a center located halfway between $s$ and $w'$. Moreover, the amplitude of
the nutation of $s$ should be similar to that of $w$. This is confirmed by Figure 9 showing the trajectories of the unit angular momenta in the frame rotating at the precession frequency. Because of the obliquity $\theta_x$, moreover, considering the hypothesis where the initial stellar effect of initial inclinations on the behavior of those systems.

In particular, we abandon studies of individual motions in favor of more global analysis of group of trajectories. In all cases, we assume that the initial spin–orbit angle $\theta_s(t = 0)$ is nil.

5.1. Hyperboloid of Trajectories

In the analytical approximation, Section 2, systems composed of packed planets surrounded by a perturber on a wide orbit have fixed physical and orbital parameters except inclinations and obliquities. The constant orbital parameters are the semi-major axes of the planets and the semi-minor axis of the perturber $b' = a'\sqrt{1-e'^2}$. It is thus natural to keep masses, semi-major axes, and eccentricities at given values and study the effect of initial inclinations on the behavior of those systems. Furthermore, considering the hypothesis where the initial stellar obliquity $\theta_s(t = 0)$ is nil, i.e., $x = 1$ and $y = z$ ($D_3$ in Figure 1), evolutions are only characterized by the initial inclination of the perturber $I'_3 = I'(0) = \theta_s(0) = \theta_s(0)$ with respect to the inner system. Thus, in the following, we consider the surface resulting from the union of the trajectories in the $(x, y, z)$ frame with $0 \leq I'_3 \leq 180^\circ$, or equivalently, $-1 \leq z_0 \leq 1$. More generally, we define $S$ as the set of all $(x, y, z) \in \mathbb{R}^3$ such that a $z_0 \in \mathbb{R}$ exists verifying

$$H(x, y, z) = H(1, z_0, z_0)$$

$$K(x, y, z) = K(1, z_0, z_0)$$

(50)

where $H(7)$ and $K(8)$ are the two first integrals of the motion. The second equation of (50) provides the expression of $z_0$ as a function of $(x, y, z)$

$$z_0 = p(x - 1) + qy + rz,$$

(51)

with

$$p = \frac{LG}{(L + G)G},$$

$$q = \frac{L}{L + G},$$

$$r = \frac{G}{L + G},$$

(52)

The substitution into the first equation of (50) gives

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid S(x, y, z) = 0\},$$

(53)

with

$$S = \frac{\alpha}{\gamma}(x^2 - 1) + z^2 - (p(x - 1) + qy + rz)^2.$$

(54)

$S$ being a quadratic function of $(x, y, z)$, $S$ is a quadric surface. Moreover, the diagonal $D_3$ belongs to $S$. Therefore, $S$ is an hyperboloid of one sheet (neither an ellipsoid nor a hyperboloid of two sheets contains a straight line). For some peculiar values of the parameters, $S$ can also be a cylinder or a cone, but this set of parameters is negligible in the sense that its measure is

**Figure 8.** Trajectory of the unit angular momenta of the system HD 20794 to which a giant planet with mass $m' = 1 M_J$, semi-major axis $a' = 20$ au, eccentricity $e' = 0.1$, and inclination $I' = 30^\circ$ has been added. The colors are the same as in Figure 6. The frame is rotating at the stellar precession frequency. (A color version of this figure is available in the online journal.)

**Figure 9.** Same as Figure 8 but with a lighter perturbing planet $m' = 0.04 M_J$ comparable to Uranus. (A color version of this figure is available in the online journal.)
5.2. Intersection with the Berlingot and Maximal Obliquity

Now that the quadric $S$ is determined, we examine its intersections with the surface of the Berlingot $\partial B$. The maximal amplitudes of the spin–orbit angle $\theta_0$ are then deduced from these intersections. To make the approach more concrete, we again use the two examples of Section 4.

The two surfaces $S$ and $\partial B$ are plotted in Figure 11 with parameters of 55 Cancri for $b' = 190, 182, 180$, and 170 au. As in Figure 10, the initial conditions are the thick black diagonal $\Delta_x$. In the case $b' = 190$ au (panel (a)), and for prograde orbits like the red curve, elliptic trajectories evolve toward the Berlingot and the planet system the companion. Trajectories exit the Berlingot after a relatively short distance and re-enter the Berlingot at negative values of $x$ and $y$ (retrograde rotation of the star) before exiting again. The evolution of the system is restricted to the interior of the Berlingot, the maximal spin–orbit angles correspond to the first exit from the Berlingot. This topology is equivalent to the case studied by Kaib et al. (2011). As the distance of the companion decreases (panel (b), (c), and (d)), the hyperboloid shrinks so that the first exit from the Berlingot occurs later and later. As a consequence, the maximal obliquity increases more and more. Furthermore, from panel (b) to (c), we see a modification of the topology of the intersection $S \cap \partial B$. If we imagine $B$ as an ocean and $S$ as a continent, panel (c) is a strait which gets larger as the companion’s semi-minor axis decreases (panel (d)). Once the strait is open, trajectories passing through it reach the retrograde side of the Berlingot and produce high spin–orbit misalignment.

The intersections between the two surfaces for each distance of the perturber have been computed with Maple. Then, for each point $(x, y, z)$ of the intersections, the initial condition $I_0 = \cos^{-1} z_0$ are computed from (51), and the spin–orbit angle from $\theta_0 = \cos^{-1} x$. The results are displayed in Figure 12 for $0 \leq \theta_0 \leq 90^\circ$ and $0 \leq I_0 \leq 90^\circ$. In Figure 12(a), the dashed curves represent the intersections $S \cap \partial B$ associated to the different values of the semi-minor axis $b'$, and the shaded areas show the regions $S \cap B$ where the hyperboloid is inside the Berlingot. The darkest shaded areas, associated with the largest distances of the perturber, are disconnected. In these conditions, systems starting with zero spin–orbit angle must stay in the lowest regions and cannot reach misalignment larger than about $50^\circ$. On the contrary, when the strait is open, shaded areas become connected and at the vertical of the strait, systems reach high obliquities of the order of $100^\circ$. Nevertheless, in the $(I_0', \theta_0)$ plane, evolutions are vertical, thus systems conserve relatively low obliquities at both sides of the strait. Figure 12(b) shows the maximal reachable spin–orbit misalignments deduced from Figure 12(a). Interestingly, we see that when the companion’s semi-minor axis is of the order of 170 au (lightest gray area), even an inclination as small as $10^\circ$ is enough to generate a spin–orbit misalignment of about $100^\circ$.

It is worth noting that, in the limit where most of the angular momentum is in the orbit of the companion, the hyperboloid $S$ and the Berlingot $B$ are invariant by the transformation $(x, y, z) \to (x, -y, -z)$ according to (55) and (12). The spin–orbit angle $\theta_0$ is thus the same whether the orbit of the outer companion is prograde or retrograde. Hence, Figure 12 stays unchanged if the abscissa $I_0'$ is replaced by $180^\circ - I_0'$.

Once the intersections between the Berlingot and the hyperboloid are known, the rotation periods $P_{\text{rot}}$ associated with the
oscillation of \( \theta_s \) are obtained numerically from Equation (20). These periods are plotted in Figures 12(c)–(f). When the perturber is close \( (b' = 170, 180 \text{ au}, \text{Figures 12(c) and (d)}) \), \( P_{\text{nut}} \) shows spikes at the exact position of the borders of the strait. As the perturber semi-minor axis increases (Figures 12(e) and (f)), the strait disappears as well as the spikes. The spikes are actually associated with trajectories along separatrices where the period is infinite. Indeed, these trajectories are tangent to the intersection \( S \cap \partial B \) and thus, they are tangent to the surface of the Berlingot \( \partial B \). Moreover, from Figure 11(d), it is manifest that the tangency is “from the inside,” as in Figure 3(b), so these trajectories actually pass through a hyperbolic fixed point. Besides those unstable fixed points, two stable fixed points exist in all panels of Figure 12. They are at \( I_0' = 0 \) and 90°. In Figure 11, these initial conditions are located at the extremity (1, 1, 1) and at the middle (1, 0, 0) of the diagonal \( D_x \), respectively. In contrast to the previous fixed points, these ones belong to orbits that are tangent to the surface of the Berlingot “from the outside,” as in Figure 3(a). They are thus stable, and indeed, the amplitude of nutation at these two points (Figure 12(b)) is nil.

To check the analytical results, we performed numerical integrations with \( a' = 170 \text{ au} \) and \( e' = 0 \) starting at \( 1° \leq I_0' \leq 89° \) with 1° increments. The results are shown in Figures 12(b) and (c). The strait is actually present in the numerical simulations, although its width derived analytically was underestimated by about 10 deg. The nutation period obtained numerically has the expected shape with two spikes at the border of the strait except that a small discrepancy exists inside the strait where the analytical method slightly overestimates the period. Nevertheless, the qualitative behavior is well recovered, and the quantitative differences must be due to the weak coupling between the inner planet and the outer ones as observed in Figure 7.

The same exercise has been carried out on the system HD 20794, with \( m'=1\ M_J \) and \( b' \in \{70, 60, 58.6, 55, 50\} \text{ au} \). Results are plotted in Figure 13. The two systems present some qualitative similarities: (1) for a large semi-minor axis of the perturber, \( S \cap \partial B \) is disconnected, the inner system is quite decoupled from the outer companion, and the stellar obliquity \( \theta_s \) remains low whatever the inclination of the perturber; (2) at a small semi-minor axis, a strait forms, spikes appear in the nutation period, and some initial conditions lead to very high spin–orbit angle amplitudes. Nevertheless, behaviors are quantitatively different. In the case of the system 55 Cnc, the strait opens at low inclination \( I_0' \sim 30° \) and high values of \( I_0' \) do not produce any strong spin–orbit misalignments, while for HD 20794, the strait is created at larger inclination \( I_0' \sim 60° \) and a high initial inclination is required to produce...
high stellar obliquity. Moreover, in contrary to 55 Cancri, the maximal obliquity $\theta_x$ of HD 20794 never exceeds twice the initial inclination $I_0'$ (dash-dotted line in Figure 13(b)).

The origin of these differences will be discussed in the following. For this system, numerical simulations done at $b' = 58.6 \text{ au}$ (Figures 13(b) and (e)) are in perfect agreement with the analytical results because the planets are well coupled.

5.3. Spanning the Parameter Space

The relative evolution described by the analytical model (Section 2) depends on four parameters: the frequencies $v_1$, $v_2$, $v_3$, $v_4$ (see Equation (16)). If we discard the temporal evolution, we can analyze all these frequencies by $v_4$, for example, and reduce the dimension of the parameter space to three. This dimension is still too large to be spanned entirely. Moreover, for each set of frequencies, we wish to analyze the curve representing the maximal obliquity of the star as a function of the initial inclination which adds one more dimension.

To solve this issue, we limit the study to asymptotic cases described as follows. Let $(v_2, v_b, v_c, v_d)$ be a permutation of $(v_1, v_2, v_3, v_4)$. We distinguish 8 classes of parameters—A, B, C, D, E, F, G, and H—defined as

class A : $v_a \ll v_b \ll v_c \ll v_d$,
class B : $(v_a \sim v_b) \ll v_c \ll v_d$,
class C : $v_a \ll (v_b \sim v_c) \ll v_d$,
class D : $v_a \ll v_b \ll (v_c \sim v_d)$,
class E : $(v_a \sim v_b \sim v_c) \ll v_d$,
class F : $(v_a \sim v_b) \ll (v_c \sim v_d)$,

high stellar obliquity. Moreover, in contrary to 55 Cancri, the maximal obliquity $\theta_x$ of HD 20794 never exceeds twice the initial inclination $I_0'$ (dash-dotted line in Figure 13(b)). The origin of these differences will be discussed in the following. For this system, numerical simulations done at $b' = 58.6 \text{ au}$ (Figures 13(b) and (e)) are in perfect agreement with the analytical results because the planets are well coupled.

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class D : $v_a \ll v_b \ll (v_c \sim v_d)$,
class E : $(v_a \sim v_b \sim v_c) \ll v_d$,

Figure 12. (a) Intersections $S \cap B$ seen in the plane $(I_0', \theta_1)$. (b) Range of accessible spin–orbit angles $\theta_x$ with initial condition $\theta_1(0) = 0$ (x-axis). (c–f) Nutation periods computed from Equation (20). Parameters are those of the 55 Cancri system (same as in Figure 11), colors from dark to light correspond to $b' = 190$, 182, 180, and 170 au, respectively. Points in the panels (b) and (c) are the results of numerical integrations with $b' = 170 \text{ au}$.

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class B : $(v_a \sim v_b) \ll v_c \ll v_d$,
class C : $v_a \ll (v_b \sim v_c) \ll v_d$,
class D : $v_a \ll v_b \ll (v_c \sim v_d)$,
class E : $(v_a \sim v_b \sim v_c) \ll v_d$,

5.3.1. Cassini Regime

We call systems in the Cassini regime those satisfying $(v_2, v_4) \ll (v_1 \sim v_3)$. This hierarchy of frequencies implies that (1) the planet system is not affected by the orientation of the central star ($v_2$ is small), (2) the plane of the outer orbit is fixed ($v_4$ is small), and (3) the precession frequency of the star matches that of the planet system ($v_1 \sim v_3$). These hypotheses are those leading to the well-known Cassini problem (e.g., Colombo 1966; Peale 1969; Henrard & Murigande 1987; Ward & Hamilton 2004). Among the 75 different configurations, 3 of them fulfill the condition $(v_2, v_4) \ll (v_1 \sim v_3)$, with $v_2 \ll v_4$, $v_2 \sim v_4$, or $v_4 \ll v_2$. They all present similar nutation amplitudes $\theta_x^{\text{max}}$ as a function of the inclination $I_0'$ (Figure 14(a)). Interestingly, the maximal stellar obliquity
Figure 13. Same as Figure 12 with the parameters of HD 20794, $m' = 1 M_J$, and $b' = 70, 60, 58.6, 55$, and 50 au from dark to light gray, respectively. A series of numerical simulations with $b' = 58.6$ au are plotted in panels (b) and (e), showing that the analytic model performs excellently for this system. The dash-dotted line in panel (b) corresponds to $\theta_x = 2 I_0'$.

relative to the planet plane exceeds twice the initial inclination $I_0'$ between the planet plane and the outer orbit.

As an example, the system 55 Cnc where the perturber is put on a rather close-in orbit $b' \lesssim 170$ au belongs to the Cassini regime with $\nu_3 \ll \nu_2 \ll (\nu_1, \nu_3)$. The dynamics of this type of system are usually analyzed in a frame rotating at the precession frequency (as in Figures 6(d), 7(d), 8, and 9), by plotting the projected trajectories of the spin axis on the invariant plane for different initial obliquities. Numerical integrations of the 55 Cnc system with $b' = 170$ au and $I_0' = 15^\circ$ are displayed in Figure 15. To make the comparison easier, the axes are the same as in Ward & Hamilton (2004). Trajectories in Figure 15 present small amplitude oscillations that are due to the individual motion of the orbit of each planet. Nevertheless, Cassini states 1, 2, and 4 are easily identifiable, and the large increase of the obliquity observed in Figures 11 and 14(a) is due to the wide oscillation of $s$ around the Cassini state 2 along the trajectory passing through the point labeled $w$ in Figure 15 (the initial condition $\theta_x(0) = 0$ implies $s(0) = w(0)$).

5.3.2. Pure Orbital Regime

When the frequencies associated with the spin–orbit coupling $v_1$ and $v_2$ are much smaller than those of the planet–companion interaction $v_3$ and $v_4$, the evolution of the system becomes purely orbital. The stellar orientation remains fixed and does not perturb the motion of the orbital planes. According to the distribution of angular momentum between the planets and the companion, this regime leads to different spin-axis evolutions displayed in Figures 14(b), (c), and (k).

If the companion contains most of the angular momentum, its orbital plane is almost invariant and the planet system precesses around it at constant inclination. In this case, $(v_1, v_2, v_4) \ll v_3$ and the spin–orbit angle oscillates between 0 and $\min(I_0', 180^\circ - I_0')$ (Figure 14(b)). The period associated with this motion is simply $P_{\text{nut}} = 2\pi/v_3$. In this regime, we can cite in particular Kepler-56, the only multiplanet system with an observed spin–orbit misalignment (Huber et al. 2013). Also, the five planets of the 55 Cancri system, with the outer one playing the role of the perturber and the others coupled together, lie in this regime. In that case, the inclination of the innermost planet with respect to the plane of the sky is known by transit to be $82.5 \pm 1.4^\circ$ (Gillon et al. 2012) while the outermost is measured at $53 \pm 7^\circ$ by astrometry (McArthur et al. 2004). According to these observations, the orbits should be mutually inclined by at least $20^\circ$ which would give rise to a periodic misalignment of $\gtrsim 40^\circ$.

When the angular momentum is equally distributed between the planets and the companion, $\nu_3 \sim \nu_4$ and both orbital planes precess around the total orbital angular momentum $G + G'$. As a consequence, if the initial obliquity $\theta_x(0)$ is nil, the two angles $\theta_x$ and $\theta_y$ oscillate in antiphase between 0 and $I_0'$ (Figure 14(c)). The corresponding nutation period is $P_{\text{nut}} = \pi/(v_3 \cos I_0' \cos(I_0'/2))$.

In the last case, when planets have significantly more angular momentum than the companion, the orbital frequencies satisfy $v_3 \ll v_4$. As a result, the planet’s plane does not move while the outer body precesses around it at a frequency $v_4$. Once the latter motion is averaged out, the remaining interaction in the
system is between the stellar equatorial bulge and the planets, but the angular momenta are aligned, so this configuration does not produce any spin–orbit misalignment (Figure 14(k)). Such would be the case if a hot Jupiter were perturbed by an Earth analogue.

More generally, within the pure orbital regime and considering all orbital angular momentum distributions, the amplitude of the spin–orbit angle $\theta_s$ varies from 0 to $2I'_0$ as $G'/G$ increases from zero to infinity, the mutual inclination $\theta_z$ remains constant equal to $I'_0$, and the associated nutation period is

$$P_{nut} = \frac{2\pi G'G}{\gamma \cos \theta_z \sqrt{G'^2 + G^2 + 2G'G \cos \theta_z}}.$$  

(57)

5.3.3. Laplace Regime

We now consider the case where the frequency $\nu_2$ is one of the largest frequencies in the system, but $\nu_1$ remains much smaller. This occurs only if the star has more angular momentum than the planets ($L \gg G$). With these parameters, the stellar orientation does not evolve and the motion of the planet plane is dictated by the inclination of the Laplace surface with respect to the equator of the star. The Laplace surface, commonly used in satellite problems, is such that inclinations measured relative to it remain roughly constant (BL06; Tremaine et al. 2009). For close-in planets with $\nu_2 \gg \nu_3$, the Laplace surface is aligned with the stellar equator and the spin–orbit angle remains constant (Figure 14(k)). In systems with longer period planets satisfying $\nu_3 \gg (\nu_2, \nu_4)$, the Laplace surface is that of the companion’s orbit. The latter configuration falls into the abovementioned pure orbital regime characterized by $\theta_s,\max = I'_0$. In between the two extremes ($\nu_2 \sim \nu_3$, Figures 14(d) and (e)), the Laplace surface is situated halfway between the stellar equator and the outer orbit. In particular, when $(\nu_1, \nu_2) \ll (\nu_2 \sim \nu_3)$ (Figure 14(d)), $\theta_s$ and $\theta_z$ oscillate in antiphase between 0 and $\min(I'_0, 180° - I'_0)$. The evolution is more complicated when all the three frequencies $\nu_2, \nu_3, \nu_4$ are similar (Figure 14(e)).

5.3.4. Hybrid Regime

The hybrid regime gather systems in which the stellar precession frequency $\nu_1$ is larger than or of the same order as the dominant orbital one(s). Thus, both the equator of the star and the orbits have a nutation motion. The Cassini regime ($\nu_2, \nu_4 \ll (\nu_1 \sim \nu_3)$ fulfills this criterion, but given its importance for spin–orbit misalignment, it has been studied separately. Other configurations are $\nu_2 \ll \nu_3 \ll (\nu_1 \sim \nu_4)$ (Figure 14(f)), $\nu_2 \ll \nu_3 \ll (\nu_1 \sim \nu_4)$ (Figure 14(g)), $\nu_2 \ll \nu_3 \ll (\nu_1 \sim \nu_4)$ (Figure 14(h)), $\nu_4 \ll (\nu_1 \sim \nu_2 \sim \nu_3)$ (Figure 14(i)), and $\nu_1 \sim \nu_2 \sim \nu_3 \sim \nu_4$ (Figure 14(j)). These cases lead to intermediate spin–orbit angle amplitudes. All the others do not produce any significant misalignment (Figure 14(k)).

Figure 14. Nutation amplitude $\theta_{s,\max}$ as a function of the initial inclination $I'_0$ of the perturber in different regimes. Similar behaviors are grouped together in the same panels. (a) Cassini regime. (b)–(c) Pure orbital regime. (d)–(e) Laplace regime. (f)–(j) Hybrid regime. (k) All remaining configurations leading to low nutation amplitudes. The dashed line and the dotted line represent $\theta_{s,\max} = 2I'_0$ and $\theta_{s,\max} = I'_0$, respectively.
example, using the parameters of Kaib et al. (2011), we obtained $v_1 \gg (v_2, v_3, v_4)$ which is a particular case of the hybrid regime that does not produce any increase in the stellar obliquity.

6. CONCLUSION

The dynamics of compact multiplanet systems perturbed by an outer companion are very rich. These systems can be represented by three unit vectors, $s$, $w$, and $w'$, along the angular momentum of the star, of the planet system, and of the companion’s orbit, respectively. Their dynamics are mainly the combination of a uniform rotation of the three vectors around the total angular momentum and of a periodic motion in the rotating frame. These two evolutions are called precession and nutation in reference to the Earth–Moon problem perturbed by the Sun. The relative motion described in the frame $(x, y, z)$, with $x = s \cdot w$, $y = s \cdot w'$, and $z = w \cdot w'$, is integrable whatever the four precession constants $v_1, v_2, v_3, v_4$ (see Equations (41)) involved in the problem. The solutions are derived geometrically as in BL06 and BL09.

In this work, we went even further in the geometrical approach. In particular, gathering all evolutions starting with a spin–orbit alignment and an arbitrary inclination of the companion, nutation amplitudes (and in particular the amplitude of the spin–orbit angle) are obtained altogether without numerical simulations from the intersection of a Berlingot-shaped surface defined in BL06 with a hyperboloid of one sheet equal to the union of all trajectories in the $(x, y, z)$ frame. The parameter space of the problem controlled by the precession constants is three-dimensional. It is thus not possible to explore the full dynamics. Nevertheless, using simple enumerative combinations, we identified 75 different asymptotic configurations. We computed the nutation amplitudes for all these cases and distinguished four different regimes:

1. The Cassini regime, where the stellar spin evolution is well described by the Cassini approximation and the obliquity can exceed twice the initial inclination of the perturber. This occurs when the star has much less angular momentum than the planets and the companion.
2. The pure orbital regime, where the stellar orientation remains fixed and does not perturb the orbital evolution. In this case, due to the motion of the planet system, the stellar nutation amplitude varies between zero and twice the initial inclination of the perturber as the orbital angular momentum ratio of the latter over that of the planet system increases from zero to infinity.
3. The Laplace regime where the stellar spin-axis is fixed and planets precess around a Laplace plane between the equator of the star and the orbit of the companion. In this regime, the amplitude of the spin–orbit angle varies between zero and the initial inclination of the perturber. The maximum is reached when the equatorial bulge and the outer body produce similar torques on the planets.
4. The hybrid regime where the precession rate of the stellar equator is larger than or similar to those of the orbital planes. Systems in the Cassini regime are excluded since they have already been studied. The hybrid regime is characterized by intermediate nutation amplitudes ranging from zero to almost twice the initial inclination of the outer body.

This formalism applied to the 55 Cancri system showed that the central star is linked to the planets unless the outer companion is on a very high eccentric orbit ($e' \gtrsim 0.95$). Indeed, most of the angular momentum is in the planets and the evolution of the stellar spin-axis is dictated by the precession rate of the planet system around the binary orbital plane. Below the eccentricity limit, the perturbation by the outer body is weak enough that the stellar spin-axis follows the planets’ motion, but close to the threshold, the system is in the Cassini regime characterized by the resonance between the stellar and the planetary precession frequencies. In that case, the stellar obliquity exceeds twice the inclination of the companion.

The geometrical analysis has been made possible thanks to the vectorial formalism detailed in BF14. Thus we have shown spin–orbit mechanics, in which the orbital mechanics are far from trivial, as a particular application of that approach.

Taking both papers together, we first provided the formalism to analyze dynamically rigid planet systems interacting with the spin axis of their parent star and with an outer companion and then we gave a general view of all possible evolutions. The next step is to develop and analyze different scenarios that may produce spin–orbit misalignment according to the results of this study. Particularly, we expect that evolution in the spin rate of the star or orbital migration may excite spin–orbit resonances which leave the star tilted. We also hope the present formalism may reveal a deeper understanding of trends in spin–orbit measurements or at least interesting application to individual systems with such measurements.

This work benefited greatly from G.B.’s PhD thesis done with Jacques Laskar during which the three-vector problem was solved geometrically for the first time. G.B. is thus particularly grateful to Jacques Laskar. G.B. also thanks Philippe Robutel and Alain Chenciner for many helpful discussions about the vectorial approach and Cayley’s nodal cubic surface.

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Figure 15. Evolution of $s$ obtained by numerical integration of the 55 Cnc system with $m^* = 0.26 M_{\odot}$, $b' = 170$ au, and $I_{\odot} = 15^\circ$. Trajectories are plotted in the invariant plane in a frame rotating with the precession period of the system. The Cassini states are labeled 1, 2, and 4. The point labeled $w$ represents the position of $w$ which evolves very slightly in this frame.
