ON SOME GEOMETRIC PROPERTIES OF NORMALIZED WRIGHT FUNCTIONS

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ABSTRACT. The main purpose of the present paper is to determine the radii of starlikeness and convexity associated with lemniscate of Bernoulli and the Janowski function, $(1 + A z)/(1 + B z)$ for $-1 \leq B < A \leq 1$, of normalized Wright functions. The key tools in the proof of our main results are the infinite product representation of Wright function and properties of real zeros of the Wright function and its derivative.

1. Introduction and the main results

Let $\mathbb{D}_r$ be the open disk $\{z \in \mathbb{C} : |z| < r\}$ with the radius $r > 0$ and let $\mathbb{D} = \mathbb{D}_1$. Let $f : \mathbb{D}_r \to \mathbb{C}$ be the function defined by

$$f(z) = z + \sum_{n \geq 2} a_n z^n,$$

where $r$ is less or equal than the radius of convergence of the above power series. Let $\mathcal{A}$ be the class of analytic functions of the form (1.1), that is, normalized by the conditions $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ denote the class of functions belonging to $\mathcal{A}$ which are of univalent in $\mathbb{D}_r$. A function $f \in \mathcal{A}$ is said to be starlike function if $f(\mathbb{D})$ is starlike domain with the respect to the origin. It is well known fact that various subclasses of starlike function can be unified by making use of the concept of subordination. A function $f \in \mathcal{A}$ subordinate to a function $g \in \mathcal{A}$, written as $f(z) \prec g(z)$, if there exist a Schwarz function $w$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. In addition, we know that if $g$ is a univalent function, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. For an analytic function $\varphi$, let $\mathcal{S}^*(\varphi)$ denote the class of all analytic functions satisfying $1 + zf'(z)/f(z) \prec \varphi(z)$. By $\mathcal{K}(\varphi)$ we mean the class of all analytic functions satisfying $1 + zf''(z)/f'(z) \prec \varphi(z)$. It is worth mentioning that these classes include respectively several famous subclasses of starlike and convex functions. For instance, the class $\mathcal{S}^*_L := \mathcal{S}^*(\sqrt{1 + z})$ denotes the class of lemniscate starlike functions introduced and investigated by Sokół and Stankiewich [14] and the class $\mathcal{K}_L := \mathcal{K}(\sqrt{1 + z})$ represents the class of lemniscate convex functions. Moreover, for $-1 \leq B < A \leq 1$, the class $\mathcal{S}^*[A, B] := \mathcal{S}^*((1 + A z)/(1 + B z))$ is the class of Janowski starlike functions and $\mathcal{K}[A, B] := \mathcal{K}((1 + A z)/(1 + B z))$ is the class of Janowski convex functions [10].

Given a class of functions $\mathcal{M} \subset \mathcal{A}$ and a function $f \in \mathcal{A}$, the $\mathcal{M}$--radius of the function $f$ is the largest number $r$ with $0 \leq r \leq 1$ such that $f_r \in \mathcal{M}$, where $f_r(z) := f(r z)/r$. If we choose $\mathcal{M} = \mathcal{S}^*_L$, the $\mathcal{M}$--radius of the function $f$, which is represented by $r^*_L(f)$, is called the radius of lemniscate starlikeness. It is indeed the largest $r$ with $0 \leq r \leq 1$ such that

$$\left| \left( \frac{zf'(z)}{f(z)} \right)^2 - 1 \right| < 1 \quad (|z| < r).$$
If we choose $M = K_C$, the $M$–radius of the function $f$, which is represented by $r^*_M(f)$, is called as the radius of lemniscate convexity. It is indeed the largest $r$ with $0 \leq r \leq 1$ such that

$$\left| \left( 1 + \frac{zf''(z)}{f'(z)} \right)^2 - 1 \right| < 1 \quad (|z| < r).$$

If we take $M = S^*[A, B]$ or $M = K[A, B]$, the respective $M$–radii, which are represented by $r^*_{A,B}(f)$ and $r^c_{A,B}(f)$, are called as the radii of Janowski starlikeness and Janowski convexity. These are respectively the largest $r$ with $0 \leq r \leq 1$ such that

$$\left| \frac{(zf'(z)/f(z)) - 1}{A - Bzf'(z)/f(z)} \right| < 1 \quad \text{and} \quad \left| \frac{zf''(z)/f'(z)}{A - B(1 + zf''(z)/f'(z))} \right| < 1 \quad (|z| < r).$$

Recently, there has been a vivid interest on some geometric properties such as univalency, starlikeness, convexity and uniform convexity of various special functions such as hyper-Bessel, Wright, $q$–Bessel and Mittag-Leffler functions (see [11, 2, 3, 5, 7, 16]). More details on the radius problems, one may consult on [4], [9], [11], [15], [17]. Moreover, in [12] the authors determined the radii of starlikeness and convexity associated with lemniscate of Bernoulli and the Janowski function $(1 + Az)/(1 + Bz)$. Motivated by the above series of papers on geometric properties of special functions, in this paper our aim is to determine the radii of lemniscate starlikeness, lemniscate convexity, Janowski starlikeness and Janowski convexity of normalized Wright functions.

Let us consider the Wright function defined as

$$\phi(\rho, \beta, z) = \sum_{n \geq 0} \frac{z^n}{n! \Gamma(n\rho + \beta)},$$

where $\rho > -1$ and $z, \beta \in \mathbb{C}$. This function was introduced by Wright for $\rho > 0$ in connection with his investigations on the asymptotic theory of partitions [13], see also [14] for further details. Furthermore, it is important to mention that the Wright function is an entire function of $z$ for $\rho > -1$.

From [7] we know that if $\rho > 0$ and $\beta > 0$, then the function $z \mapsto \lambda_{\rho, \beta}(z) = \phi(\rho, \beta, -z^2)$ has infinitely many zeros which are all real. Denoting by $\lambda_{\rho, \beta, n}$ the $n$th positive zero of $\phi(\rho, \beta, -z^2)$, under the same conditions the Weierstrassian decomposition

$$\Gamma(\beta)\phi(\rho, \beta, -z^2) = \prod_{n \geq 1} \left( 1 - \frac{z^2}{\lambda_{\rho, \beta, n}^2} \right)^{\phi(\rho, \beta, -z^2)}$$

is valid, and this product is uniformly convergent on compact subsets of the complex plane. Moreover, if we denote by $\zeta'_{\rho, \beta, n}$ the $n$th positive zero of $\Psi_{\rho, \beta}$, where $\Psi_{\rho, \beta}(z) = z^2 \lambda_{\rho, \beta}(z)$, then the positive zeros of $\lambda_{\rho, \beta}$ (or the positive real zeros of the function $\Psi_{\rho, \beta}$) are interlaced with those of $\Psi'_{\rho, \beta}$. In other words, the zeros satisfy the chain of inequalities

$$\zeta'_{\rho, \beta, 1} < \lambda_{\rho, \beta, 1} < \zeta'_{\rho, \beta, 2} < \lambda_{\rho, \beta, 2} < \ldots.$$ 

Observe that the function $z \mapsto \phi(\rho, \beta, -z^2)$ does not belong to $A$, and thus first we perform some natural normalization. We define three functions originating from $\phi(\rho, \beta, \cdot)$:

$$f_{\rho, \beta}(z) = \left( z^2 \Gamma(\beta)\phi(\rho, \beta, -z^2) \right)^{\frac{\beta}{2}}, \quad (1.3)$$
$$g_{\rho, \beta}(z) = z \Gamma(\beta)\phi(\rho, \beta, -z^2), \quad (1.4)$$
$$h_{\rho, \beta}(z) = z \Gamma(\beta)\phi(\rho, \beta, -z). \quad (1.5)$$

Obviously these functions belong to the class $A$. Of course, there exist infinitely many other normalization, the main motivation to consider the above ones is the fact that their particular cases in terms of Bessel functions appear in literature or are similar to the studied normalization in the literature.
1.1. Lemniscate starlikeness and lemniscate convexity of normalized Wright functions. This section is devoted to determine the radii of lemniscate starlikeness and lemniscate convexity of the normalized Wright functions.

**Theorem 1.1.** Let \( \rho > 0 \) and \( \beta > 0 \).

a. The radius of lemniscate starlikeness \( r^*_\rho(f_{\rho,\beta}) \) is the smallest positive root of the transcendental equation

\[
r^2(\lambda'_{\rho,\beta}(r))^2 - 2 \beta \lambda'_{\rho,\beta}(r) \lambda_{\rho,\beta}(r) - \beta^2 (\lambda_{\rho,\beta}(r))^2 = 0.
\]

b. The radius of lemniscate starlikeness \( r^*_\rho(g_{\rho,\beta}) \) is the smallest positive root of the transcendental equation

\[
r^2(\lambda'_{\rho,\beta}(r))^2 - 2r \lambda'_{\rho,\beta}(r) \lambda_{\rho,\beta}(r) - (\lambda_{\rho,\beta}(r))^2 = 0.
\]

c. The radius of lemniscate starlikeness \( r^*_\rho(h_{\rho,\beta}) \) is the smallest positive root of the transcendental equation

\[
r(\lambda_{\rho,\beta}(\sqrt{r}))^2 - 4 \sqrt{r} \lambda'_{\rho,\beta}(\sqrt{r}) \lambda_{\rho,\beta}(\sqrt{r}) - (\lambda_{\rho,\beta}(\sqrt{r}))^2 = 0.
\]

**Proof.** The infinite product representation of the function \( z \mapsto \lambda_{\rho,\beta}(z) \) given in \((1.2)\) implies

\[
\frac{z \lambda'_{\rho,\beta}(z)}{\lambda_{\rho,\beta}(z)} = -\sum_{n \geq 1} \frac{2z^2}{\lambda_{\rho,\beta,n}^2 - z^2}.
\]

Keeping in view the normalizations \((1.3), (1.4), (1.5)\) and by making use of \((1.6)\) we obtain the following equations

\[
\frac{zf'_{\rho,\beta}(z)}{f_{\rho,\beta}(z)} = 1 + \frac{1}{\beta} \left( \frac{z \lambda'_{\rho,\beta}(z)}{\lambda_{\rho,\beta}(z)} \right) = 1 - \frac{1}{\beta} \sum_{n \geq 1} \frac{2z^2}{\lambda_{\rho,\beta,n}^2 - z^2},
\]

\[
\frac{zg'_{\rho,\beta}(z)}{g_{\rho,\beta}(z)} = 1 + \frac{2z^2}{\lambda_{\rho,\beta}(z)} = 1 - \sum_{n \geq 1} \frac{2z^2}{\lambda_{\rho,\beta,n}^2 - z^2},
\]

\[
\frac{zh'_{\rho,\beta}(z)}{h_{\rho,\beta}(z)} = 1 + \frac{1}{2} \left( \frac{\sqrt{z} \lambda'_{\rho,\beta}(\sqrt{z})}{\lambda_{\rho,\beta}(\sqrt{z})} \right) = 1 - \sum_{n \geq 1} \frac{z}{\lambda_{\rho,\beta,n}^2 - z}.
\]

By means of \((1.7), (1.8), (1.9)\), we deduce that the following inequalities

\[
\left| \frac{zf'_{\rho,\beta}(z)}{f_{\rho,\beta}(z)} \right|^2 - 1 \leq \frac{1}{\beta^2} \left( \sum_{n \geq 1} \frac{2|z|^2}{\lambda_{\rho,\beta,n}^2 - |z|^2} \right) \left( \sum_{n \geq 1} \frac{2|z|^2}{\lambda_{\rho,\beta,n}^2 - |z|^2} + 2\beta \right)
\]

\[
= \left| \frac{zf'_{\rho,\beta}(|z|)}{f_{\rho,\beta}(|z|)} \right|^2 - 4 \left( \frac{|z| f'_{\rho,\beta}(|z|)}{f_{\rho,\beta}(|z|)} \right) + 3,
\]

\[
\left| \frac{zg'_{\rho,\beta}(z)}{g_{\rho,\beta}(z)} \right|^2 - 1 \leq \sum_{n \geq 1} \frac{2|z|^2}{\lambda_{\rho,\beta,n}^2 - |z|^2} \left( \sum_{n \geq 1} \frac{2|z|^2}{\lambda_{\rho,\beta,n}^2 - |z|^2} + 2 \right)
\]

\[
= \left| \frac{zg'_{\rho,\beta}(|z|)}{g_{\rho,\beta}(|z|)} \right|^2 - 4 \left( \frac{|z| g'_{\rho,\beta}(|z|)}{g_{\rho,\beta}(|z|)} \right) + 3,
\]

\[
\left| \frac{zh'_{\rho,\beta}(z)}{h_{\rho,\beta}(z)} \right|^2 - 1 \leq \sum_{n \geq 1} \frac{|z|}{\lambda_{\rho,\beta,n}^2 - |z|} \left( \sum_{n \geq 1} \frac{|z|}{\lambda_{\rho,\beta,n}^2 - |z|} + 2 \right)
\]

\[
= \left| \frac{zh'_{\rho,\beta}(|z|)}{h_{\rho,\beta}(|z|)} \right|^2 - 4 \left( \frac{|z| h'_{\rho,\beta}(|z|)}{h_{\rho,\beta}(|z|)} \right) + 3.
\]
are valid for $|z| < \lambda_{p,\beta,1}$, $\rho > 0$ and $\beta > 0$.

For the sake of simplicity, we consider the function $z \mapsto \phi_{p,\beta}$ which collectively represents the functions $f_{p,\beta}, g_{p,\beta}$ and $h_{p,\beta}$. Suppose that $r^*$ is the smallest positive root of the equation

$$
\left( \frac{r\phi'_{p,\beta}(r)}{\phi_{p,\beta}(r)} \right)^2 - 4 \frac{r\phi'_{p,\beta}(r)}{\phi_{p,\beta}(r)} + 2 = 0,
$$

then the inequality

$$
\left| \frac{z\phi'_{p,\beta}(z)}{\phi_{p,\beta}(z)} \right|^2 - 1 \leq 1
$$
is valid for $|z| < r^*$. As a result of the equations (1.7), (1.8) and (1.9), we conclude that the zeros of the above mentioned equation for the functions $f_{p,\beta}$, $g_{p,\beta}$ and $h_{p,\beta}$ coincide with those of equations, respectively

$$
r^2 (\lambda'_{p,\beta}(r))^2 - 2\beta \lambda'_{p,\beta}(r) \lambda_{p,\beta}(r) - \beta^2 (\lambda_{p,\beta}(r))^2 = 0,
$$

and

$$
r (\lambda'_{p,\beta}(\sqrt{r}))^2 - 4\sqrt{r} \lambda'_{p,\beta}(\sqrt{r}) \lambda_{p,\beta}(\sqrt{r}) - \lambda_{p,\beta}(\sqrt{r}) \lambda_{p,\beta}(\sqrt{r}) = 0.
$$

That means that the radii of lemniscate starlikeness $r^*_F(f_{p,\beta})$, $r^*_G(g_{p,\beta})$ and $r^*_L(h_{p,\beta})$ are the smallest positive roots of the above mentioned equations.

In order to end the proof we need to show that each of the above roots is in $(0, \lambda_{p,\beta,1})$. To reach our aim, let us consider the functions $F_{p,\beta}, G_{p,\beta}, H_{p,\beta} : (0, \lambda_{p,\beta,1}) \to \mathbb{R}$ defined by

$$
F_{p,\beta}(r) = \left( \frac{rf'_{p,\beta}(r)}{f_{p,\beta}(r)} \right)^2 - 4 \left( \frac{rf'_{p,\beta}(r)}{f_{p,\beta}(r)} \right) + 2,
$$

$$
G_{p,\beta}(r) = \left( \frac{rg'_{p,\beta}(r)}{g_{p,\beta}(r)} \right)^2 - 4 \left( \frac{rg'_{p,\beta}(r)}{g_{p,\beta}(r)} \right) + 2,
$$

$$
H_{p,\beta}(r) = \left( \frac{rh'_{p,\beta}(r)}{h_{p,\beta}(r)} \right)^2 - 4 \left( \frac{rh'_{p,\beta}(r)}{h_{p,\beta}(r)} \right) + 2.
$$

Each of the above functions are strictly increasing functions of $r$ since

$$
F'_{p,\beta}(r) = \frac{4}{r} \sum_{n \geq 1} \left( \frac{2r^2}{\lambda_{p,\beta,n}^2} \right) \left( 1 + \frac{1}{\beta} \sum_{n \geq 1} \frac{2r^2}{\lambda_{p,\beta,n}^2} \right) > 0,
$$

$$
G'_{p,\beta}(r) = \frac{4}{r} \sum_{n \geq 1} \left( \frac{2r^2}{\lambda_{p,\beta,n}^2} \right) \left( 1 + \sum_{n \geq 1} \frac{2r^2}{\lambda_{p,\beta,n}^2} \right) > 0,
$$

$$
H'_{p,\beta}(r) = 2 \sum_{n \geq 1} \left( \frac{\lambda_{p,\beta,n}}{\lambda_{p,\beta,n}^2} \right) \left( 1 + \sum_{n \geq 1} \frac{r}{\lambda_{p,\beta,n}^2} \right) > 0.
$$

Observe also that

$$
\lim_{r \searrow 0} F_{p,\beta}(r) = \lim_{r \searrow 0} G_{p,\beta}(r) = \lim_{r \searrow 0} H_{p,\beta}(r) = -1 < 0,
$$

and

$$
\lim_{r \searrow \lambda_{p,\beta,1}} F_{p,\beta}(r) = \lim_{r \searrow \lambda_{p,\beta,1}} G_{p,\beta}(r) = \lim_{r \searrow \lambda_{p,\beta,1}} H_{p,\beta}(r) = \infty.
$$
Theorem 1.2. Let \( r > 0 \) and \( \beta > 0 \).

a. The radius of lemniscate convexity \( r_L^*(f_{\rho,\beta}) \) is the smallest positive root of the transcendental equation

\[
\left( \frac{z\Psi''_{\rho,\beta}(z)}{\Psi'_{\rho,\beta}(z)} + \frac{1}{\beta} - 1 \right) \frac{z\Psi'_{\rho,\beta}(z)}{\Psi_{\rho,\beta}(z)} - 2 \left( \frac{z\Psi''_{\rho,\beta}(z)}{\Psi'_{\rho,\beta}(z)} + \frac{1}{\beta} - 1 \right) \frac{z\Psi'_{\rho,\beta}(z)}{\Psi_{\rho,\beta}(z)} - 1 = 0.
\]

b. The radius of lemniscate convexity \( r_L^*(g_{\rho,\beta}) \) is the smallest positive root of the transcendental equation

\[
\left( \frac{r^2\lambda''_{\rho,\beta}(r) + 2r\lambda'_{\rho,\beta}(r)}{\lambda'_{\rho,\beta}(r) + r\lambda'_{\rho,\beta}(r)} \right)^2 - 2 \left( \frac{r^2\lambda''_{\rho,\beta}(r) + 2r\lambda'_{\rho,\beta}(r)}{\lambda'_{\rho,\beta}(r) + r\lambda'_{\rho,\beta}(r)} \right) - 1 = 0.
\]

c. The radius of lemniscate convexity \( r_L^*(h_{\rho,\beta}) \) is the smallest positive root of the transcendental equation

\[
\left( \frac{3\sqrt{r}\lambda'_{\rho,\beta}(\sqrt{r}) - r\lambda''_{\rho,\beta}(\sqrt{r})}{4\lambda'_{\rho,\beta}(\sqrt{r}) + 2\sqrt{r}\lambda''_{\rho,\beta}(\sqrt{r})} \right)^2 - 2 \left( \frac{3\sqrt{r}\lambda'_{\rho,\beta}(\sqrt{r}) - r\lambda''_{\rho,\beta}(\sqrt{r})}{4\lambda'_{\rho,\beta}(\sqrt{r}) + 2\sqrt{r}\lambda''_{\rho,\beta}(\sqrt{r})} \right) - 1 = 0.
\]

Proof. a. It is easy to check that

\[
1 + \frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)} = 1 + \frac{z\Psi''_{\rho,\beta}(z)}{\Psi'_{\rho,\beta}(z)} + \frac{1}{\beta} - 1 \frac{z\Psi'_{\rho,\beta}(z)}{\Psi_{\rho,\beta}(z)}.
\]

From (7) we have the following infinite product representations

\[
\Psi_{\rho,\beta}(z) = \frac{\zeta_{\rho,\beta}^{\beta}}{\Gamma(\beta)} \prod_{n \geq 1} \left( 1 - \frac{z^2}{\zeta_{\rho,\beta,n}^2} \right), \quad \Psi'_{\rho,\beta}(z) = \frac{\zeta_{\rho,\beta}^{\beta-1}}{\Gamma(\beta)} \prod_{n \geq 1} \left( 1 - \frac{z^2}{\zeta_{\rho,\beta,n}^2} \right)
\]

where \( \zeta_{\rho,\beta,n} \) and \( \zeta'_{\rho,\beta,n} \) are the \( n \)th positive roots of \( \Psi_{\rho,\beta} \) and \( \Psi'_{\rho,\beta} \), respectively. The logarithmic differentiation on both sides of the above relations yields

\[
(1.10) \quad 1 + \frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)} = 1 - \sum_{n \geq 1} \frac{2z^2}{\zeta_{\rho,\beta,n}^2 - z^2} - \frac{1}{\beta} - 1 \sum_{n \geq 1} \frac{2z^2}{\zeta_{\rho,\beta,n}^2 - z^2}.
\]

Suppose that \( \beta \in (0,1) \), then \( 1/\nu - 1 \geq 0 \). In light of equation (1.10) and triangle’s inequality for \( |z| < \zeta'_{\rho,\beta,1} < \zeta_{\rho,\beta,1} \), we obtain

\[
\left| 1 + \frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)} \right|^2 - 1 \leq \left( \sum_{n \geq 1} \frac{2|z|^2}{\zeta_{\rho,\beta,n}^2 - |z|^2} + \frac{1}{\beta} - 1 \sum_{n \geq 1} \frac{2|z|^2}{\zeta_{\rho,\beta,n}^2 - |z|^2} \right)^2 + 2 \left( \sum_{n \geq 1} \frac{2|z|^2}{\zeta_{\rho,\beta,n}^2 - |z|^2} + \frac{1}{\beta} - 1 \sum_{n \geq 1} \frac{2|z|^2}{\zeta_{\rho,\beta,n}^2 - |z|^2} \right).
\]

By virtue of equation (1.10), the above inequality yields

\[
(1.11) \quad \left| 1 + \frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)} \right|^2 - 1 \leq \left( \frac{|z| f''_{\rho,\beta}(|z|)}{f'_{\rho,\beta}(|z|)} \right)^2 - 2 \left( \frac{|z| f''_{\rho,\beta}(|z|)}{f'_{\rho,\beta}(|z|)} \right).
\]

Moreover, due to the relation [3] Lemma 2.1

\[
(1.12) \quad \left| \frac{z}{a} - \frac{z}{b} \right| \leq \frac{|z|}{a - |z|} - \lambda \frac{|z|}{b - |z|}
\]
for $|z| \leq r < a < b$ and $0 \leq \lambda < 1$ we deduce that the inequality (1.11) holds for the case when $\beta > 1$ as well. That means that the relation (1.11) is valid for $\beta > 0$ and $|z| < \zeta_{p,\beta,1}$. Therefore, the function $f_{p,\beta}(z)$ is strictly increasing for $|z| < r_1$, where $r_1$ is the smallest positive root of

$$
\left( \frac{r f''_{p,\beta}(r)}{f'_{p,\beta}(r)} \right)^2 - 2 \left( \frac{r f''_{p,\beta}(r)}{f'_{p,\beta}(r)} \right) - 1 = 0.
$$

We need to show that the above equation has a unique root in $(0, \zeta'_{p,\beta,1})$. To reach our aim, let us consider the function $u_{p,\beta} : (0, \zeta'_{p,\beta,1}) \to \mathbb{R}$ defined by

$$
u_{p,\beta}(r) = \left( \frac{r f''_{p,\beta}(r)}{f'_{p,\beta}(r)} \right)^2 - 2 \left( \frac{r f''_{p,\beta}(r)}{f'_{p,\beta}(r)} \right) - 1.
$$

It is obvious that this function is strictly increasing for $\beta > 0$ since

$$u'_{p,\beta}(r) > 8r \sum_{n \geq 1} \left( \frac{\zeta^2_{p,\beta,n}}{\left( \zeta^2_{p,\beta,n} - r^2 \right)^2} - \frac{\zeta^2_{p,\beta,n}}{\left( \zeta^2_{p,\beta,n} - r^2 \right)^2} \right)
\left( 2r^2 \sum_{n \geq 1} \frac{1}{\zeta^2_{p,\beta,n} - r^2} - \frac{1}{\zeta^2_{p,\beta,n} - r^2} \right) > 0.
$$

Here we tacitly used the relation $\zeta^2_{p,\beta,n} \left( \zeta^2_{p,\beta,n} - r^2 \right)^2 < \zeta^2_{p,\beta,n} \left( \zeta^2_{p,\beta,n} - r^2 \right)^2$ for $r < \sqrt{\zeta_{p,\beta,n} \zeta'_{p,\beta,n}}$ and $\beta > 0$. Observe also that

$$\lim_{r \searrow 0} u_{p,\beta}(r) = -1 < 0 \text{ and } \lim_{r \nearrow \zeta'_{p,\beta,1}} u_{p,\beta}(r) = \infty.$$

Therefore, the root is unique in $(0, \zeta'_{p,\beta,1})$ and the radius of lemniscate convexity of the function $f_{p,\beta}(z)$, denoted by $r_{f_{p,\beta}}^2$, is the unique root of the equation (1.13) in $(0, \zeta'_{p,\beta,1})$.

b. We now ascertain the radii of lemniscate convexity of normalized Wright function $g_{p,\beta}(z)$. We know from [7, Theorem 5] that the Weierstrassian canonical representation of the function $g'_{p,\beta}(z)$ can be stated as

$$g'_{p,\beta}(z) = \prod_{n \geq 1} \left( 1 - \frac{z^2}{\vartheta^2_{p,\beta,n}} \right),$$

where $\vartheta_{p,\beta,n}$ is the $n$th positive zero of the function $g'_{p,\beta}(z)$. The logarithmic derivation of both sides yields

$$1 + \frac{2z g''_{p,\beta}(z)}{g'_{p,\beta}(z)} = 1 - \sum_{n \geq 1} \frac{2z^2}{\vartheta^2_{p,\beta,n} - z^2}.$$ 

By making use of the similar approach of the proof of Theorem 1.1 for $|z| < \vartheta_{p,\beta,1}$ we get

$$\left| \left( 1 + \frac{2z g''_{p,\beta}(z)}{g'_{p,\beta}(z)} \right)^2 - 1 \right| \leq \left( \frac{2z g''_{p,\beta}(z)}{g'_{p,\beta}(z)} \right)^2 - 2 \left( \frac{2z g''_{p,\beta}(z)}{g'_{p,\beta}(z)} \right).$$
Therefore, it follows that the radius of lemniscate convexity \( r^c_L(g_{\rho,\beta}) \) is the unique positive root of the equation

\[
(1.14) \quad \left( \frac{r g''_{\rho,\beta}(r)}{g'_{\rho,\beta}(r)} \right)^2 - 2 \left( \frac{r g''_{\rho,\beta}(r)}{g'_{\rho,\beta}(r)} \right) - 1 = 0
\]

in \((0, \vartheta_{\rho,\beta,1})\). In order to end the proof we need to show that the above equation has a unique root in \((0, \vartheta_{\rho,\beta,1})\). To do this, let us deal with the function \( v_{\rho,\beta}(r) : (0, \vartheta_{\rho,\beta,1}) \rightarrow \mathbb{R} \) defined by

\[
v_{\rho,\beta}(r) = \left( \frac{r g''_{\rho,\beta}(r)}{g'_{\rho,\beta}(r)} \right)^2 - 2 \left( \frac{r g''_{\rho,\beta}(r)}{g'_{\rho,\beta}(r)} \right) - 1
\]

which is strictly increasing in the same interval since

\[
v'_{\rho,\beta}(r) = 8r \left( \sum_{n \geq 1} \frac{\vartheta^2_{\rho,\beta,n} \vartheta^{2}_{\rho,\beta,n} - r^2}{\vartheta^2_{\rho,\beta,n} - r^2} \right) \left( \sum_{n \geq 1} \frac{2r^2}{\vartheta^2_{\rho,\beta,n} - r^2 + 1} \right) > 0.
\]

Observe also

\[
\lim_{r \searrow 0} v_{\rho,\beta}(r) = -1 < 0 \quad \text{and} \quad \lim_{r \nearrow \vartheta_{\rho,\beta,1}} v_{\rho,\beta}(r) = \infty.
\]

That means that the radius of lemniscate convexity \( r^c_L(g_{\rho,\beta}) \) is the unique root of equation \((1.14)\) in \((0, \vartheta_{\rho,\beta,1})\).

c. From [7] we know that the infinite product representation of the function \( h'_{\rho,\beta}(z) \) can be written as

\[
h'_{\rho,\beta}(z) = \prod_{n \geq 1} \left( 1 - \frac{z}{\tau_{\rho,\beta,n}} \right),
\]

where \( \tau_{\rho,\beta,n} \) is the \( n \)th positive zero of the function \( h'_{\rho,\beta} \). Consequently, by repeating the same calculations in the part \((b)\), we say that the radius of lemniscate convexity \( r^c_L(h_{\rho,\beta}) \) is the unique root of equation stated in the part \((c)\) of the theorem.

\[\square\]

1.2. **Janowski starlikeness and Janowski convexity of normalized Wright functions.**

In this section, we focus on determining the radii of Janowski starlikeness and Janowski convexity of the normalized Wright functions \( f_{\rho,\beta}(z) \), \( g_{\rho,\beta}(z) \) and \( h_{\rho,\beta}(z) \).

**Theorem 1.3.** Let \( \rho > 0 \) and \( \beta > 0 \).

a. The radius of Janowski starlikeness \( r_{_{A,B}}^\lambda(f_{\rho,\beta}) \) is the smallest positive root of the equation

\[
\frac{r \lambda'_{\rho,\beta}(r)}{\lambda_{\rho,\beta}(r)} + \beta \left( \frac{A - B}{1 + |B|} \right) = 0.
\]

b. The radius of Janowski starlikeness \( r_{_{A,B}}^\lambda(g_{\rho,\beta}) \) is the smallest positive root of the equation

\[
\frac{r \lambda'_{\rho,\beta}(r)}{\lambda_{\rho,\beta}(r)} + \frac{A - B}{1 + |B|} = 0.
\]

c. The radius of Janowski starlikeness \( r_{_{A,B}}^\lambda(h_{\rho,\beta}) \) is the smallest positive root of the equation

\[
\sqrt{r} \lambda'_{\rho,\beta}(\sqrt{r}) + 2 \frac{A - B}{1 + |B|} = 0.
\]
Proof. In order to determine the radius of Janowski starlikeness of the normalization $f_{p, \beta}(z)$ of $\phi(p, \beta, \cdot)$ given by (1.3), we need to find a real positive real number $r^*$ such that

$$\left| \frac{zf'_{p, \beta}(z)}{f_{p, \beta}(z)} - 1 \right| < 1, \quad \text{for } |z| < r^*.$$  

By virtue of Eq. (1.7) and using triangle inequality, we deduce that the inequality

$$\frac{zf'_{p, \beta}(z)}{f_{p, \beta}(z)} - 1 \leq \frac{\frac{1}{\beta} \sum_{n \geq 1} \frac{2|z|^2}{\lambda^2_{p, \beta, n} - |z|^2}}{A - B - |B| \frac{1}{\beta} \sum_{n \geq 1} \frac{2|z|^2}{\lambda^2_{p, \beta, n} - |z|^2}}, \quad \text{for } |z| < \lambda_{p, \beta, 1}$$

holds for $\beta > 0$ with equality at $z = r$. It is obvious that the above inequality yields

$$\frac{zf'_{p, \beta}(z)}{f_{p, \beta}(z)} - 1 \leq \frac{1 - \frac{|zf''_{p, \beta}(z)|}{f''_{p, \beta}(z)}}{A - B - |B| \frac{|zf''_{p, \beta}(z)|}{f''_{p, \beta}(z)} - 1}.$$  

(1.15)

Let $r^*$ be the smallest positive root of the equation

$$\frac{rf'_{p, \beta}(r)}{f_{p, \beta}(r)} + \frac{A - B}{1 + |B|} - 1 = 0,$$

then the inequality (1.15) implies that the function $f_{p, \beta}(z)$ is Janowski starlike for $|z| < r^*$. We need to show that the above equation has a unique root in $(0, \lambda_{p, \beta, 1})$. To do this, let us deal with the function $u_{p, \beta} : (0, \lambda_{p, \beta, 1}) \to \mathbb{R}$ defined by

$$u_{p, \beta}(r) = \frac{rf'_{p, \beta}(r)}{f_{p, \beta}(r)} + \frac{A - B}{1 + |B|} - 1.$$  

Since

$$u'_{p, \beta}(r) = -\frac{1}{\beta} \sum_{n \geq 1} \frac{4r\lambda^2_{p, \beta, n}}{(\lambda^2_{p, \beta, n} - r^2)^2} < 0$$

the function $u_{p, \beta}$ is strictly decreasing on $(0, \lambda_{p, \beta, 1})$. Observe also

$$\lim_{r \to 0} u_{p, \beta}(r) = -\infty \quad \text{and} \quad \lim_{r \to \lambda_{p, \beta, 1}} u_{p, \beta}(r) = \frac{A - B}{1 + |B|} > 0.$$  

Therefore, we conclude that the Janowski starlikeness radius of $f_{p, \beta}$, denoted by $r^*_{A,B}(f_{p, \beta})$, is the unique zero of equation (1.10) in $(0, \lambda_{p, \beta, 1})$ which is desired result.

It is clear that similar equations hold for the normalizations $g_{p, \beta}$ and $h_{p, \beta}$ given in (1.4) and (1.5), respectively for $\beta > 0$. Hence, the radii of Janowski starlikeness $r^*_{A,B}(g_{p, \beta})$ and $r^*_{A,B}(h_{p, \beta})$ are the unique roots of the equations stated in the parts (b) and (c) of the theorem, respectively.

\[ \Box \]

**Theorem 1.4.** Let $\rho > 0$ and $\beta > 0$.

a. The radius of Janowski starlikeness $r^*_{A,B}(f_{p, \beta})$ is the smallest positive root of the equation

$$\frac{rf''_{p, \beta}(r)}{f_{p, \beta}(r)} + \left( \frac{1}{\beta} - 1 \right) \frac{r \Psi'_{p, \beta}(r)}{\Psi_{p, \beta}(r)} + \frac{A - B}{1 + |B|} = 0.$$  

b. The radius of Janowski starlikeness $r^*_{A,B}(g_{p, \beta})$ is the smallest positive root of the equation

$$\frac{r^2 \lambda''_{p, \beta}(r) + 2r \lambda'_{p, \beta}(r)}{\lambda_{p, \beta}(r) + r \lambda'_{p, \beta}(r)} + \frac{A - B}{1 + |B|} = 0.$$  

The radius of Janowski starlikeness $r_{A,B}(h_{\rho,\beta})$ is the smallest positive root of the equation

$$3\sqrt{\bar{T}^n_{\rho,\beta}(\sqrt{r})} - r\lambda^n_{\rho,\beta}(\sqrt{r}) + \frac{A - B}{1 + |B|} = 0.$$ 

Proof. By definition, we know that in order to be Janowski convex in the open disk $D_r$ of the function $f_{\rho,\beta}$, the inequality

$$\left| \frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)} \right| < 1$$

must hold for $|z| < r$. By using Eq. (1.10), it is easy to see that for $0 \leq \beta < 1$ the function $f_{\rho,\beta}(z)$ satisfies the inequality

$$\left| \frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)} \right| \leq \frac{\sum_{n \geq 1} \frac{2z^2}{\rho^2_{\rho,\beta,n}z^2} + \left( \frac{1}{\beta} - 1 \right) \sum_{n \geq 1} \frac{2z^2}{\rho^2_{\rho,\beta,n}z^2}}{A - B + |B| \left\{ \sum_{n \geq 1} \frac{2z^2}{\rho^2_{\rho,\beta,n}z^2} + \left( \frac{1}{\beta} - 1 \right) \sum_{n \geq 1} \frac{2z^2}{\rho^2_{\rho,\beta,n}z^2} \right\}},$$

for $|z| < \zeta_{\rho,\beta,1}$ with $z = |z| = r$. In light of the inequality (1.12), the above inequality remains valid for $\beta \geq 1$ as well. With the help of Eq. (1.10), replacing $z$ by $|z|$, we get

$$\left| \frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)} \right| \leq \frac{\sum_{\rho,\beta,n} \frac{2z^2}{(\rho^2_{\rho,\beta,n}z^2)^2}}{A - B + |B| \left\{ \sum_{\rho,\beta,n} \frac{2z^2}{(\rho^2_{\rho,\beta,n}z^2)^2} \right\}},$$

for $\beta > 0$. Therefore, we deduce that the radius of Janowski convexity $r_{A,B}(f_{\rho,\beta})$ is the smallest positive root of the equation

$$\frac{rf''_{\rho,\beta}(r)}{f'_{\rho,\beta}(r)} + \frac{A - B}{1 + |B|} = 0.$$ 

In order to finish the proof, we need to show that equation (1.17) has a unique root in $(0, \zeta_{\rho,\beta,1}')$. To reach our aim, let us consider the function $u_{\rho,\beta} : (0, \zeta_{\rho,\beta,1}') \to \mathbb{R}$, defined by

$$u_{\rho,\beta}(r) = \frac{rf''_{\rho,\beta}(r)}{f'_{\rho,\beta}(r)} + \frac{A - B}{1 + |B|},$$

which is continuous on $(0, \zeta_{\rho,\beta,1}')$ and is strictly decreasing in $(0, \zeta_{\rho,\beta,1}')$ since

$$u'_{\rho,\beta}(r) < 4r \sum_{n \geq 1} \left( \frac{\zeta_{\rho,\beta,n}^2}{(\zeta_{\rho,\beta,n}^2 - r^2)^2} - \frac{\zeta_{\rho,\beta,n}^2}{(\zeta_{\rho,\beta,n}^2 - r^2)^2} \right) < 0.$$

Here we used the interlacing property $\zeta_{\rho,\beta,n}^2 (\zeta_{\rho,\beta,n}^2 - r^2)^2 < \zeta_{\rho,\beta,n}^2 (\zeta_{\rho,\beta,n}^2 - r^2)^2$ for $r < \sqrt{\zeta_{\rho,\beta,n}^2 \zeta_{\rho,\beta,n}^2}$ and $\beta > 0$. Observe also

$$\lim_{r \to 0} u_{\rho,\beta}(r) = \frac{A - B}{1 + |B|} > 0 \quad \text{and} \quad \lim_{r \to \zeta_{\rho,\beta,1}'} u_{\rho,\beta}(r) = -\infty.$$

Therefore, we say that the radius of Janowski starlikeness $r_{A,B}(f_{\rho,\beta})$ is the unique positive root of equation (1.17) in $(0, \zeta_{\rho,\beta,1}')$. This is desired result.

Since it can be obtained results presented in part (b) and part (c) by repeating the same calculations in the previous theorem and by keeping in view the infinite product representations of the functions $g'_{\rho,\beta}(z)$ and $h'_{\rho,\beta}(z)$ (see [7]) we omit the proof of part b and part c. □
References

[1] Aktaş İ. and Baricz Á., Bounds for radii of starlikeness of some $q$–Bessel functions, *Results Math.*, **72**(1): 947–963, 2017.

[2] Aktaş İ., Baricz Á. and Singh S., Geometric and monotonic properties of hyper-Bessel functions, *Ramanujan J.*, 1–21, doi: 10.1007/s11139-018-0105-9, 2019.

[3] Aktaş İ., Toklu E. and Orhan H., Radii of uniform convexity of some special functions. *Turk J Math.*, **42**(6): 3010–3024, 2018.

[4] Ali R. M., Jain N. K. and Ravichandran V., Radii of starlikeness associated with the lemniscate of Bernoulli and the left-half plane, *Appl. Math. Comput.*, **218** (2012), no. 11, 65576565.

[5] Baricz Á. and Prajapati A. Radii of starlikeness and convexity of generalized Mittag-Leffler functions. arXiv preprint arXiv:1901.04333, 2019.

[6] Baricz Á. and Szász R. The radius of convexity of normalized Bessel functions of the first kind, *Anal. Appl.*, **12**(5), 485–509, 2014.

[7] Baricz Á., Toklu E. and Kadıoğlu E., Radii of starlikeness and convexity of Wright functions. *Math. Commun.*, **23**: 97–117, 2018.

[8] Chaggara H. and Romdhane N.B., On the zeros of the hyper-Bessel function, *Integr. Transf. Spec. Funct.*, **26**(2), 96–101, 2015.

[9] Goodman A. W., *Univalent functions*. Vol. I, Mariner Publishing Co., Inc., Tampa, FL, 1983.

[10] Janowski W., Extremal problems for a family of functions with positive real part and for some related families, *Ann. Polon. Math.*, **23** (1970/1971), 159177.

[11] Madaan V., Kumar A., and Ravichandran V., Lemniscate Convexity and Other Properties of Generalized Bessel Functions. arXiv preprint arXiv:1902.04277, 2019.

[12] Madaan V., Kumar A., and Ravichandran V., Radii of starlikeness and convexity of Bessel functions. arXiv preprint arXiv:1906.05647, 2019.

[13] R. Gorenflo, Y. Luchko and F. Mainardi, Analytical properties and applications of the Wright function, *Fract. Calc. Appl. Anal.*, **2**(4), 383–414, 1999.

[14] Sokól J. and Stankiewicz J., Radius of convexity of some subclasses of strongly starlike functions, *Zeszyty Nauk. Politech. Rzeszowskiej Mat.*, No. **19**, 101105, 1996.

[15] Toklu E., Radii of starlikeness and convexity of $q$–Mittag-Leffler functions. *Turk J Math.*, **43**(5): 2610–2630, 2019.

[16] Toklu E., Aktaş İ. and Orhan H., Radii problems for normalized $q$–Bessel and Wright functions. *Acta Univ Sapientiae Mathematica*, **11**(1): 193–213, 2019.

[17] Verma S. and Ravichandran V., Radius problems for ratios of Janowski starlike functions with their derivatives, *Bull. Malays. Math. Sci. Soc.*, **40** (2017), no. 2, 819840, 2017.

[18] E.M. Wright, On the coefficients of power series having exponential singularities, *J. Lond. Math. Soc.*, 71–79, 1933.

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