BEHAVIORS OF THE ENERGY OF SOLUTIONS OF TWO COUPLED WAVE EQUATIONS WITH NONLINEAR DAMPING ON A COMPACT MANIFOLD WITH BOUNDARY.

M. DAOULATLI

Abstract. In this paper we study the behaviors of the energy of solutions of coupled wave equations on a compact manifold with boundary in the case of indirect nonlinear damping. Only one of the two equations is directly damped by a localized nonlinear damping term. Under geometric conditions on both the coupling and the damping regions we prove that the rate of decay of the energy of smooth solutions of the system is determined from a first order differential equation.

1. Introduction and Statement of the results

Let \((\Omega, g_0)\) be a \(C^\infty\) compact connected \(n\)-dimensional Riemannian manifold with boundary \(\Gamma\). We denote by \(\Delta\) the Laplace-Beltrami operator on \(\Omega\) for the metric \(g_0\). We consider a system of coupled wave equations with nonlinear damping

\[
\begin{aligned}
\partial_t^2 u - \Delta u + b(x)v + a(x)g(\partial_t u) &= 0 \quad \text{in } \mathbb{R}_+^* \times \Omega \\
\partial_t^2 v - \Delta v + b(x)u &= 0 \quad \text{in } \mathbb{R}_+^* \times \Omega \\
u = v = 0 \quad \text{on } \mathbb{R}_+^* \times \Gamma \\
(u(0,x), \partial_t u(0,x)) = (u_0, u_1) \text{ and } (v(0,x), \partial_t v(0,x)) = (v_0, v_1) \quad \text{in } \Omega,
\end{aligned}
\]

(1.1)

where \(g: \mathbb{R} \to \mathbb{R}\) is a continuous, monotone increasing function, \(g(0) = 0\). In addition we assume that

\[
\begin{aligned}
g(y)y &\leq M_0 y^2, \quad |y| < 1 \\
m g^2 &\leq g(y)y \leq M g^2, \quad |y| \geq 1 \\
\|g^\prime\|_{L^\infty} &\leq M_1,
\end{aligned}
\]

for some positive real numbers \(M_0, m, M\) and \(M_1\). In this paper, we deal with real solutions, the general case can be treated in the same way. With the system above we associate the energy functional given by

\[
E_{u,v}(t) = \frac{1}{2} \int_\Omega |\nabla u(t,x)|^2 + |\nabla v(t,x)|^2 + |\partial_t u(t,x)|^2 + |\partial_t v(t,x)|^2 \, dx \\
+ \int_\Omega b(x) u(t,x) v(t,x) \, dx.
\]

(1.2)

We assume that \(a\) and \(b\) are two nonnegative smooth functions such that

\[
\|b\|_{\infty} \leq \frac{1 - \delta}{\lambda^2},
\]

(1.3)

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for some $\delta > 0$, where $\lambda$ is the Poincaré’s constant on $\Omega$. Under these assumptions we have
\[
E_{u,v}(0) = \frac{1}{2} \int_\Omega |\nabla u_0(x)|^2 + |\nabla v_0(x)|^2 + |u_1(x)|^2 + |v_1(x)|^2 \, dx + \int_\Omega b(x) u_0(x) v_0(x) \, dx \\
\geq \frac{\delta}{2} \int_\Omega |\nabla u_0(x)|^2 + |\nabla v_0(x)|^2 + |u_1(x)|^2 + |v_1(x)|^2 \, dx,
\]
for all $(u_0, v_0, u_1, v_1) \in \mathcal{H} = H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$.

The nonlinear evolution equation (1.1) can be rewritten under the form
\[
\begin{aligned}
\frac{d}{dt} U + A U + B U &= 0 \\
U(0) &= U_0 \in \mathcal{H}
\end{aligned}
\] (1.5)
where
\[
U = \begin{pmatrix}
u \\
v \\
\partial_t u \\
\partial_t v
\end{pmatrix}, \\
U_0 = \begin{pmatrix} u_0 \\
v_0 \\
u_1 \\
v_1
\end{pmatrix},
\]
and the unbounded operator $A$ on $\mathcal{H}$ is defined by
\[
A = \begin{pmatrix}
0 & 0 & -Id & 0 \\
0 & 0 & 0 & -Id \\
-\Delta & b & 0 & 0 \\
b & -\Delta & 0 & 0
\end{pmatrix}
\]
with domain
\[
D(A) = \{ U \in \mathcal{H}; AU \in \mathcal{H} \} = (H_0^1(\Omega) \cap H^2(\Omega)) \times (H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega) \times H_0^1(\Omega),
\]
and
\[
B U = \begin{pmatrix}
0 \\
0 \\
a(x) g(\partial_t u) \\
0
\end{pmatrix}
\]
Under our assumptions and from the nonlinear semi-group theory (see for example [5]), we can infer that for $U_0 \in \mathcal{H}$, the problem (1.5) admits a unique solution $U \in C^0(\mathbb{R}_+, \mathcal{H})$. Moreover we have the following energy estimate
\[
E_{u,v}(t) - E_{u,v}(0) = - \int_0^t \int_\Omega a(x) g(\partial_t u(s,x)) \partial_t u(s,x) \, dxds 
\]
(1.6)
for all $t \geq 0$. In addition, since $g' \in L^\infty(\mathbb{R})$, then if $U_0 \in D(A)$, we have $U \in C(\mathbb{R}_+, D(A))$ and
\[
E_{\partial_t u, \partial_t v}(t) - E_{\partial_t u, \partial_t v}(0) = - \int_0^t \int_\Omega a(x) g' \left( \partial_t u(s,x) \right) \left| \partial_t^2 u(s,x) \right|^2 \, dxds.
\]
(1.7)

The systems like (1.1) appear in many physical situations. Indirect damping of reversible systems occurs in several applications in engineering and mechanics. In general it is impossible or too expansive to damp all the components of the state, so it is important to study stabilization properties of coupled systems with a reduced number of feedbacks.

The case of a linear damping and constant coupling $b$ in (1.1) has already been treated in [3]. They showed that the System (1.1) cannot be exponentially stable and that the
energy decays polynomially. In [2] Alabau et al generalized these results to cases for which the coupling $b = b(x)$ and the damping term $a = a(x)$ satisfy the Piecewise Multipliers Geometric Condition (PMGC) [2]. This geometric assumption is a generalization of the usual multiplier geometric condition (or $\Gamma$-condition) of [2,16] and is much more restrictive than the sharp Geometric Control Condition (GCC). In [1] Alabau et al generalized this result and they proved that the system (1.1) is polynomially stable when the regions $\{a > 0\}$ and $\{b > 0\}$ both satisfy the Geometric Control Condition and the coupling term satisfies a smallness assumption. This result has generalized by Aloui et al [7], by assuming a more natural smallness condition on the infinity norm of the coupling term $b$.

Finally we quote the result of Fu [13] in which he shows the logarithmic decay property without any geometric conditions on the effective damping domain.

The problem of the indirect nonlinear damping has been studied by Alabau et al [5] when the system is coupled by the velocity. In this case they show that the energy of these kinds of system decays as fast as that of the corresponding scalar nonlinearly damped equation. Hence, the coupling through velocities allows a full transmission of the damping effects. To our knowledge no results seems to be known in the case of indirect nonlinear damping for a coupled system coupled in displacements.

The goal of this paper is to determine the rate of decay of the energy of coupled wave system with indirect nonlinear damping and coupled in displacements. More precisely, we prove, under some geometric conditions on the localized damping domain and the localized coupling domain, that the rate of decay of the energy is determined from a first order differential equation. In addition, we obtain that if the behavior of the damping is close to the linear case, then the linear and the nonlinear case has the same rate of decay. In the other case we find that the rate of decay of the coupled system is close to the one obtained for a single damped wave equation.

The optimality of our results is an open question. Lower energy estimates have been established in [4, 5] and [6] for scalar one-dimensional wave equations, scalar Petrowsky equations in two-dimensions and one-dimensional wave systems coupled by velocities. These results can be extended to the case of one-dimensional wave systems coupled by displacement. In our case we obtain a quasi-optimal energy decay formula when the behavior of the damping is not close to the linear one.

A natural necessary and sufficient condition to obtain controllability for wave equations is to assume that the control set satisfies the Geometric Control Condition (GCC) defined in [8,18]. For a subset $\omega$ of $\Omega$ and $T > 0$, we shall say that $(\omega, T)$ satisfies GCC if every geodesic traveling at speed one in $\Omega$ meets $\omega$ in a time $t < T$. We say that $\omega$ satisfies GCC if there exists $T > 0$ such that $(\omega, T)$ satisfies GCC. We also set $T_0 = \inf \{T > 0; (\omega, T) \text{satisfies GCC}\}$.

We denote by $\omega = \{a(x) > 0\}$ the control set and by $\mathcal{O} = \{b(x) > 0\}$ the coupling set.

**Assumption (A1):** Unique continuation property:

There exists $T_0 > 0$, such that the only solution of the system

\[
\begin{align*}
\partial_t^2 u_1 - \Delta u_1 + b(x)u_2 &= 0 \quad \text{in } (0, T_0) \times \Omega \\
\partial_t^2 u_2 - \Delta u_2 + b(x)u_1 &= 0 \quad \text{in } (0, T_0) \times \Omega \\
u_1 &= u_2 \quad \text{on } (0, T_0) \times \Gamma \\
\alpha(x)u_1 &= 0 \quad \text{on } (0, T_0) \times \Omega \\
u_1 \in H^1((0, T_0) \times \Omega) \quad \text{and } u_2 \in L^2((0, T_0) \times \Omega),
\end{align*}
\]

is the null one $u_1 = u_2 = 0$. 

\[\text{(1.8)}\]
Note that the unique continuation assumption above is valid if we assume that \( \omega \cap \mathcal{O} \) satisfies the GCC (see [7]). Also according to Alabau et al. [11 Proposition 4.7] we have the following result: We assume that \( \omega \) and \( \mathcal{O} \) satisfy the GCC, then if \( \|b\|_\infty \leq \min \left( \frac{1}{5\lambda}, \frac{1}{50\lambda \sqrt{c_{\mathcal{T}}}} \right) \), there exists \( T^* \geq \max \left( T_\omega, T_\mathcal{O} \right) \) such that if \( T_0 > T^* \) then the only solution of the system (1.8) is the null one.

In order to characterize decay rates for the energy, we need to introduce several special functions, which in turn will depend on the growth of \( g \) near the origin. According to [14] there exists a concave continuous, strictly increasing function \( h \), and there exist \( \beta > 1 \) and \( \alpha_0 > 0 \), such that

\[
\lim_{s \to 0} h^{-1}(s) = \lim_{s \to 0} (h^{-1})'(s) = \lim_{s \to 0} (h^{-1})''(s) = \lim_{s \to 0} (h^{-1})'''(s) = 0, \tag{1.11}
\]

and there exist \( \beta > 1 \) and \( \alpha_0 > 0 \), such that

\[
\lim_{s \to \infty} (h^{-1})'(1/s^\beta) = \alpha_0, \\
(h^{-1})'(s) \leq \beta s (h^{-1})''(s), \text{ for all } s \in [0, r_0], \\
(\beta^2 - \beta) s (h^{-1})''(s) + \beta^2 s^2 (h^{-1})'''(s) \geq 0, \text{ for all } s \in [0, r_0]. \tag{1.12}
\]

Moreover, we assume that if \( \beta s (h^{-1})''(s) - (h^{-1})'(s) > 0 \), for all \( s \in (0, r_0] \), then there exists \( \alpha_1 > 0 \), such that

\[
\frac{(h^{-1})'(s) (s^2 - \beta s h^{-1})''(s) + \beta^2 s^2 (h^{-1})'''(s))}{\beta s (h^{-1})'(s - (h^{-1})'(s))} \leq \alpha_1, \text{ for all } s \in [0, r_0]. \tag{1.13}
\]

We know that in the case of linear damping we have

\[
E_{u,v}(t) \leq \frac{1}{\lambda} \sum_{i=0}^{1} E_{\partial_t^i u, \partial_t^i v}(0), \quad t > 0, \\
U_0 = (u_0, v_0, u_1, v_1) \in (H^1_0(\Omega) \cap H^2(\Omega))^2 \times (H^1_0(\Omega))^2,
\]

so we cannot expect to obtain a better rate of decay in the case of nonlinear damping. More precisely we have the following result.

**Theorem 1.** We suppose that \( a \) and \( b \) are two smooth non-negative functions and the conditions (1.3) and the assumption A2 hold. In addition, we assume that \( \omega \) and \( \mathcal{O} \) satisfy the GCC and the assumption A1 holds. The solution \( U(t) = (u(t), v(t), \partial_t u(t), \partial_t v(t)) \) of the system (1.1) then satisfies

\[
E_{u,v}(t) \leq C \left( 1 + \sum_{i=0}^{1} E_{\partial_t^i u, \partial_t^i v}(0) \right) (\varphi(t))^{-1}, \quad t > 0, \\
U_0 = (u_0, v_0, u_1, v_1) \in (H^1_0(\Omega) \cap H^2(\Omega))^2 \times (H^1_0(\Omega))^2, \tag{1.14}
\]
where $C$ is positive constant and $\varphi$ is a solution of the following ODE
\[
\frac{d\varphi}{dt} - \frac{c_0}{2C_1} \varphi \left(h^{-1}\right)' \left(\varphi^\beta\right) = 0, \ 0 < \varphi^{-\beta}(0) \leq r_0 \text{ such that } (h^{-1})' \left(\varphi^{-\beta}(0)\right) < \inf \left(\frac{2C_1\delta}{c_0} \left(8C + 2\lambda^2 + 1\right)^{-1}, \frac{1}{c_0} \left(\frac{1}{m} + M^2\right)^{-1}\right),
\]
where $C_1 = C(T, ||b||_\infty, ||a||_\infty)$. Moreover,
\[
\lim_{t \to \infty} E_{u,v}(t) = 0.
\]

Remark 1. The smallness condition on the infinity norm of $b$ is required to ensure that the only solution of the system
\[
\begin{align*}
-\Delta u + b(x)v &= 0 \\
-\Delta v + b(x)u &= 0 \\
u &= v = 0 \text{ on } \Gamma
\end{align*}
\]
is the null one.

Remark 2. We note that $\theta(t) = \frac{1}{\varphi(t)}$ is a solution of the following ODE
\[
\frac{d\theta}{dt} + \frac{c_0}{2C_1} \theta \left(h^{-1}\right)' \left(\theta^\beta\right) = 0.
\]

To prove our result it is sufficient to show the integrability of $\varphi' E_{u,v}$ on $(0, \infty)$. For this purpose we show an estimate on a functional $X(t)$ which is equivalent to the weighted energy functional (See [11] for similar idea). Also we prove a weighted observability estimate for the wave equation with a potential. In addition, we use the unique continuation hypotheses (A1) to prove a weak observability estimate of the weighted $L^2$-norm of the solution.

1.1. Some examples of decay rates and lower energy estimates. We give some examples of feedback growths together with the resulting energy decay rate when applying our results. For clarity of exposition we will deal with the damping which satisfies strict bounds, i.e. by saying $g(s)s \simeq f(s)$ we will mean there are constants $m, M$ so that $mf(s) \leq g(s)s \leq Mf(s)$. In the sequel $C$ denotes a generic positive constant which is independent of the energy of the initial data and setting $E_0 = 1 + \sum_{i=0}^{1} E_{\partial_t u, \partial_t v}(0)$. Below we assume that $\varphi(0)$ verifies the condition (1.15).

First we give an explicit upper bound of solutions of the ordinary differential equation (1.18).

Lemma 1. We assume that there exists $0 < r_0 \leq 1$ such that the function $h^{-1} \in C^2((0, r_0])$, monotone increasing and strictly convex. In addition we suppose that there exists $\alpha > 0$, such that
\[
(h^{-1})'(s) \leq \alpha (h^{-1})''(s), \text{ for all } s \in (0, r_0].
\]
Let $\theta$ be a solution of the following ODE
\[
\frac{d\theta}{dt} + C \theta (h^{-1})' \left(\theta^\beta\right) = 0, \text{ such that } 0 < \theta(0) \leq r_0,
\]
where $C$ is a positive constant and $\beta \geq 1$. We have
\[
\theta(t) \leq \left(\left(h^{-1}\right)' - \frac{\alpha}{\beta C t + k_0}\right)^{1/\beta}, \text{ for all } t \geq 0,
\]
where
\[ \frac{\alpha/\beta C}{(h-1)'(r_0)} \leq k_0 \leq \frac{\alpha/\beta C}{(h-1)'(\theta^\beta(0))}. \]  

(1.20)

Proof. Let
\[ \psi(t) = \left( \left( (h-1)' \right)^{-1} \left( \frac{\alpha/\beta C}{t + k_0} \right) \right)^{\frac{1}{\beta}}, \] for all \( t \geq 0. \)

Direct computations and (1.12), give
\[ \psi'(t) \geq -\frac{\alpha}{\beta t + k_0}, \] for all \( t \geq 0. \)

On the other hand, it is easy to see that
\[ \psi(t) \left( (h-1)'(\psi(t)) \right) = \frac{\alpha}{\beta C t + k_0}, \] for all \( t \geq 0. \)

Therefore, using (1.20), we conclude that
\[ \frac{d\psi}{dt} + C\psi(h-1)'(\psi^\beta) \geq 0, \] \( \psi(0) \geq \theta(0). \)

The desired result follows from [10, Lemma 1]. □

Now we give some examples.

**Example 1** (Linearly bounded case). Suppose \( g(s) \approx s^2 \). According to (1.9), auxiliary function \( h_0 \) which may be defined as \( h_0(y) = (cy)^{\gamma} \) with \( 1/2 < \gamma < 1 \) and for suitable constant \( c > 0 \). We use the ODE
\[ \frac{d\varphi}{dt} - \frac{c_0}{2C\gamma}\varphi(h_0^{-1})' \left( \frac{1}{m_u(\Omega)} \varphi^{-\frac{\gamma}{1-\gamma}} \right) = 0. \]

Consequently,
\[ E_{u,v}(t) \leq \frac{\hat{C}E_0}{t^{1+1}}, \quad t \geq 0. \]

**Example 2.** Suppose \( g(s) \approx s^2 (\ln (1/s))^{-p} \), \( 0 < \frac{1}{2} < 1 \) for some \( p > 0 \). According to (1.9), auxiliary function \( h_0 \) which may be defined as \( h_0(y) = (cy)^{\gamma} \) with \( 1/2 < \gamma < 1 \) and for suitable constant \( c > 0 \). We use the ODE
\[ \frac{d\varphi}{dt} - \frac{c_0}{2C\gamma}\varphi(h_0^{-1})' \left( \frac{1}{m_u(\Omega)} \varphi^{-\frac{\gamma}{1-\gamma}} \right) = 0. \]

Consequently,
\[ E_{u,v}(t) \leq \frac{\hat{C}E_0}{t^{1+1}}, \quad t \geq 0. \]

**Example 3** (The Polynomial Case). Suppose \( g(s) \approx |s|^{p+1} \), \( 0 < |s| < 1 \) for some \( p > 1 \). According to (1.9), auxiliary function \( h_0 \) which may be defined as \( h_0(y) = (cy)^{2/(p+1)} \) for suitable constant \( c > 0 \) (determined by the coefficients in the polynomial bound on the damping \( g(s) \)). We use the ODE
\[ \frac{d\varphi}{dt} - \frac{c_0}{2C\gamma}\varphi(h_0^{-1})' \left( \frac{1}{m_u(\Omega)} \varphi^{-\frac{\gamma}{1-\gamma}} \right) = 0. \]

Consequently,
\[ E_{u,v}(t) \leq \frac{\hat{C}E_0}{(t+1)^{2/(p-1)}}, \quad t \geq 0, \]
where

\[ \beta > 1 \quad \text{if } p \geq 3 \]
\[ \beta = \frac{2}{p-1} \quad \text{if } p < 3. \]

**Example 4** (Exponential damping at the origin). Assume: \( g(s) = s^3 e^{-1/s^2} \), \( 0 < |s| < 1 \). First we need to determine \( h_0 \) according to (1.9). Setting \( h_0(g(y)y) = cy^2 \), we see that

\[ h_0^{-1}(y) = \sqrt{y/c} \cdot g \left( \sqrt{y/c} \right) = c^{-2}y^2 \exp(-cy) \]

We use the ODE

\[ \frac{d\varphi}{dt} - \frac{c_0}{2c_1} \varphi \left( h_0^{-1} \right)' \left( \frac{1}{m_0(\Omega)} \varphi^{-\beta} \right) = 0, \]

to obtain

\[ E_{u,v}(t) \leq \frac{\hat{C}E_0}{(\ln(t+2))^{1/\beta}}, \quad t \geq 0, \]

for all \( \beta > 1 \).

**Example 5** (Exponential damping at the origin). Assume: \( g(s) = s^3 e^{-1/s^2} \), \( 0 < |s| < 1 \). First we need to determine \( h_0 \) according to (1.9). Setting \( h_0(g(y)y) = cy^2 \), we see that

\[ h_0^{-1}(y) = \sqrt{y/c} \cdot g \left( \sqrt{y/c} \right) = c^{-2}y^2 \exp(-\exp(c/y)) \]

We use the ODE

\[ \frac{d\varphi}{dt} - \frac{c_0}{2c_1} \varphi \left( h_0^{-1} \right)' \left( \frac{1}{m_0(\Omega)} \varphi^{-\beta} \right) = 0, \]

to obtain

\[ E_{u,v}(t) \leq \frac{\hat{C}E_0}{(\ln\ln(t+e^2))^{1/\beta}}, \quad t \geq 0, \]

for all \( \beta > 1 \).

We finish this part by giving a result on the lower estimate of the energy of the one-dimensional coupled wave system.

**Proposition 1.** We suppose that \( \Omega = (0,1) \) and \( g \) is a odd function. We set

\[ h^{-1}(s) = g(\sqrt{s}) \sqrt{s}, \quad \text{for } s \geq 0. \]

We assume that \( a \) and \( b \) are two smooth non-negative functions and the conditions (1.3) and the assumption A2 hold. In addition, we suppose that \( \omega \) and \( c \) satisfy the GCC and the assumption A1 holds. Let \( U(t) = (u(t), v(t), \partial_t u(t), \partial_t v(t)) \) be the solution of the system (1.1), then there exists \( T_0 > 0 \) such that

\[ E_{u,v}(t) \geq \left( \frac{\psi(t)}{4\sqrt{E_{\partial_t u, \partial_t v}(0)}} \right)^2, \quad t \geq T_0, \]

\[ U_0 = (u_0, v_0, u_1, v_1) \in \left( H^1_0(\Omega) \cap H^2(\Omega) \right)^2 \times \left( H^1(\Omega) \right)^2, \]

where \( \psi \) is a solution of the following ODE

\[ \frac{d\psi}{dt} + \|a\|_{L^\infty} \psi \left( h^{-1} \right)'(\psi) = 0, \quad 0 < \psi(0) \leq 4\sqrt{\frac{E_{\partial_t u, \partial_t v}(0)}{E_{u,v}(T_0)}}. \quad (1.21) \]
Proof. We proceed as in [5] and using the fact that
\[ h^{-1}(s) \leq s (h^{-1})'(s), \]
we see that there exists \( T_0 > 0 \) such that
\[ d\sqrt{E_{u,v}} \frac{dt}{dt} + \|a\|_{L^\infty} \sqrt{E_{u,v}} (h^{-1})' \left( 4\sqrt{E_{\partial_t u, \partial_t v}}(0) \sqrt{E_{u,v}} \right) \geq 0, \quad t \geq T_0. \]
Since \( \psi \) is a solution of (1.21), then using [10, Lemma 1] we conclude that
\[ \sqrt{E_{u,v}}(t) \geq \frac{\psi(t)}{4\sqrt{E_{\partial_t u, \partial_t v}}(0)}, \quad t \geq T_0. \]

2. Proof of Theorem 1

First we give the following weighted observability estimate for the wave equation with potential.

**Proposition 2.** Let \( \gamma, \delta > 0 \) and \( \chi \in L^\infty(\Omega) \) satisfying
\[ \bullet \ \chi \geq 0 \text{ or else}, \]
\[ \bullet \ |\chi|_{L^\infty} \leq \frac{\gamma^2 - \delta}{\chi^2}. \]

Let \( \phi \) be a positive function in \( C^2(\mathbb{R}_+) \) such that
\[ \begin{align*}
\text{for every } I \subset \subset \mathbb{R}_+, \text{ there exist } m, M > 0 & \quad m \leq \phi(s) \leq M, \text{ for all } s \in I. \tag{2.1} \\
\text{In addition, if } \phi' \text{ is not the null function, we assume that there exists a positive constant } K & \quad \sup_{\mathbb{R}_+} \left| \frac{\phi''(t)}{\phi'(t)} \right| \leq K. \tag{2.2} \\
\text{Moreover we suppose that the function } t & \mapsto \left| \frac{\phi'(t)}{\phi(t)} \right| \text{ is decreasing and } \lim_{t \to +\infty} \left| \frac{\phi'(t)}{\phi(t)} \right| = 0. \tag{2.3} \\
\end{align*} \]

We consider also \( \psi \) nonnegative smooth function on \( \Omega \) such that the set \( (V := \{ \psi(x) > 0 \}, T) \) satisfies the GCC. Then there exists \( C_T > 0 \), such that for all \( (u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega), \ f \in L^2_{loc}(\mathbb{R}_+, L^2(\Omega)) \) and all \( t > 0 \), the solution of
\[ \begin{align*}
\begin{cases}
\partial_t^2 u - \gamma^2 \Delta u + \chi(x) u = f & \text{in } \mathbb{R}_+^* \times \Omega \\
u = 0 & \text{on } \mathbb{R}_+^* \times \Gamma \\
u(0, x), \partial_t u(0, x) = (u_0, u_1) & \text{in } \Omega \\
\end{cases} \tag{2.4} \\
\end{align*} \]
satisfies with
\[ E_u(t) = \frac{1}{2} \int_\Omega \gamma^2 |\nabla u(t, x)|^2 + |\partial_t u(t, x)|^2 + \chi(x) |u(t, x)|^2 \, dx, \]
the inequality
\[ \int_t^{t+T} \phi(s) E_u(s) \, ds \leq C_T \left( \int_t^{t+T} \phi(s) \int_\Omega \left( \psi(x) |\partial_t u(s, x)|^2 + |f(s, x)|^2 \right) \, dx \, ds \right). \tag{2.5} \]
Proof. First we remark that
\[ E_u(t) \approx \frac{1}{2} \int_{\Omega} |\nabla u(t, x)|^2 + |\partial_t u(t, x)|^2 \, dx. \]

To prove the estimate (2.5), we argue by contradiction. We assume that there exist a positive sequence \((t_n)\), a sequence of functions \(f_n \in L^2_{loc} (\mathbb{R}_+, L^2(\Omega))\) and a sequence \((u_n(t))\) of solutions of the system (2.4) with initial data \((u_{0,n}, u_{1,n}) \in H^1_0(\Omega) \times L^2(\Omega)\), such that
\[
\int_{t_n}^{t_n+T} \phi(s) E_{u_n}(s) \, ds \\
\geq n \left( \int_{t_n}^{t_n+T} \phi(s) \int_{\Omega} \psi(x) |\partial_t u_n(s, x)|^2 + |f_n(s, x)|^2 \, dx \, ds \right).
\]

1st case: \(t_n \to +\infty\). Setting
\[
\lambda_n^2 = \int_{t_n}^{t_n+T} \phi(s) E_{u_n}(s) \, ds \quad \text{and} \quad v_n(t) = \frac{1}{\lambda_n} (\phi(s))^{1/2} u_n(t+t_n),
\]
Therefore \(v_n\) is a solution of the following system
\[
\begin{cases}
\partial^2 v_n - \gamma^2 \Delta v_n + \chi(x) v_n = \frac{1}{\lambda_n} (\phi(t_n+t))^1/2 f_n(t_n+t,x) + f_n^1(t,x) & \text{in } \mathbb{R}_+^+ \times \Omega \\
v_n = 0 & \text{on } \mathbb{R}_+^+ \times \Gamma \\
(v_n(0,x), \partial_t v_n(0,x)) = (v_{n,0}, v_{n,1}) & \text{in } \Omega
\end{cases}
\]
where
\[
f_n^1(t,x) = \frac{1}{2} \phi''(t+t_n)(\phi(t+t_n))^{-1} v_n - \frac{1}{4} (\phi'(t+t_n))^2 \phi(t+t_n)^{-2} v_n \\
+ \frac{1}{\lambda_n} \phi'(t+t_n)(\phi(t+t_n))^{-1/2} \partial_t u_n.
\]

Thanks to (2.6), we get
\[
\int_0^T \int_{\Omega} |\nabla v_n(s,t,x)|^2 + \frac{\phi(s+t_n)}{\lambda_n^2} |\partial_t u_n(s+t_n,x)|^2 \, dx \, ds = 1, \\
\frac{1}{\lambda_n} \int_0^T \int_{\Omega} \phi(s+t_n) \left( \psi(x) |\partial_t u_n(t_n+s,x)|^2 + |f_n(t_n+s,x)|^2 \right) dx \, ds \leq \frac{1}{n}.
\]

Using Poincare’s inequality we deduce that
\[
\int_0^T \int_{\Omega} |v_n(s, x)|^2 \, dx \, ds \leq \lambda^2.
\]

Utilizing the first part of (2.7) and the estimate above, we deduce that there exists \(\alpha_1 > 0\), such that
\[
\int_0^T E_{v_n}(s) \, ds \leq \alpha_1.
\]

A combination of the first part of (2.7), (2.2) and (2.3), gives
\[
\lim_{n \to \infty} \int_0^T \int_{\Omega} |f_n^1(t+n+s, x)|^2 \, dx \, ds = 0.
\]
It is easy to see that
\[
\int_0^T \int_\Omega \psi(x) |\partial_t v_n(s,x)|^2 dxds
\]
\[
\leq \int_0^T \int_\Omega \frac{1}{\lambda_n^2} \phi(t_n+s) \psi(x) |\partial_t u_n(t_n+s,x)|^2 + \left| \frac{\phi'(t_n)}{\phi(t_n)} \right|^2 \psi(x) |v_n(s,x)|^2 dxds
\]
\[
\leq \int_0^T \int_\Omega \frac{1}{\lambda_n^2} \phi(t_n+s) \psi(x) |\partial_t u_n(t_n+s,x)|^2 dxds + \alpha_2 \left| \frac{\phi'(t_n)}{\phi(t_n)} \right|^2 \rightarrow 0,
\]
noting that in the last result we have used the second part of (2.7) and (2.8).

On the other hand, According to [7], we know that
\[
\int_0^T \int_\Omega \psi(x) |\partial_t v_n(s,x)|^2 + \frac{1}{\lambda_n} \phi(t_n+s) |f_n(t_n+s,x)|^2 + |f_n^1(s,x)|^2 dxds
\]
and the contradiction follows from the fact that the RHS of the estimate above goes to zero as \( n \) goes to infinity and
\[
1 = \int_0^T \int_\Omega \left| \nabla v_n(s,x) \right|^2 + \frac{\phi(s+t_n)}{\lambda_n^2} |\partial_t u_n(s+t_n,x)|^2 dxds
\]
\[
\leq 2 \int_0^T \int_\Omega \left| \nabla v_n(s,x) \right|^2 + |\partial_t v_n(s,x)|^2 dxds + \alpha_2 \left| \frac{\phi'(t_n)}{\phi(t_n)} \right|^2.
\]

2\textsuperscript{nd} case: The sequence \((t_n)\) is bounded. Setting
\[
\lambda_n^2 = \int_{t_n}^{t_n+T} E_u_n(s) ds \quad \text{and} \quad u_n(t) = \frac{1}{\lambda_n} u_n(t+t_n).
\]
Using the fact that the sequence \((t_n)\) is bounded and (2.11), we infer that there exist \( \alpha_0, \alpha_1 > 0 \), such that
\[
0 < \alpha_0 \leq \int_0^T E_u_n(s) ds \leq \alpha_1,
\]
and
\[
\int_0^T \int_\Omega \psi(x) |\partial_t v_n(s,x)|^2 + \frac{1}{\lambda_n^2} |f_n(s+t_n,x)|^2 dxds \rightarrow 0.
\]

To finish the proof we need to proceed as in the first case.

The result below is a week weighted observability inequality and we need it to control the \( L^2 \) norm of the solution.

**Proposition 3.** We assume that \( \omega \) and \( \mathcal{O} \) satisfy the GCC and the assumption A1 holds. Let \( T > \max \{ T_\omega, T_\mathcal{O}, T_0 \} \) and \( \alpha > 0 \). Let \( \phi \) be a positive function in \( C^2(\mathbb{R}_+) \) such that
\[
\text{for every } I \subset \subset \mathbb{R}_+, \text{ there exist } m, M > 0 \quad m \leq \phi(s) \leq M, \text{ for all } s \in I.
\]
In addition, if \( \phi' \) is not the null function, we assume that there exists a positive constant \( K \) such that
\[
\sup_{\mathbb{R}_+} \left| \frac{\phi''(t)}{\phi'(t)} \right| \leq K.
\]
Moreover we suppose that
\[
\text{the function } \ t \mapsto \left| \frac{\phi'(t)}{\phi(t)} \right| \text{ is decreasing and } \lim_{t \to +\infty} \left| \frac{\phi'(t)}{\phi(t)} \right| = 0. \tag{2.10}
\]

Then there exists \( C_{T,\alpha} > 0 \), such that for all \((u_0, v_0, u_1, v_1) \in \mathcal{H} \), and all \( t > 0 \), the solution of the system
\[
\begin{align*}
\begin{cases}
\partial^2_t u - \Delta u + b(x)v + a(x)g(\partial_t u) &= 0 &\text{in } \mathbb{R}^+ \times \Omega \\
\partial^2_t v - \Delta v + b(x)u &= 0 &\text{in } \mathbb{R}^+ \times \Omega \\
u = v &= 0 &\text{on } \mathbb{R}^+ \times \Gamma \\
(u(0, x), \partial_t u(0, x)) = (u_0, u_1) \quad \text{and} \quad (v(0, x), \partial_t v(0, x)) = (v_0, v_1) &\text{in } \Omega
\end{cases}
\end{align*}
\tag{2.11}
\]
satisfies the inequality
\[
\int_t^{t+T} \phi(s) \left( \left| a(x) \left( |\partial_t u(s, x)|^2 + |\partial_t v(s, x)|^2 \right) \right| \right) dx ds + \alpha \int_t^{t+T} \phi(s) E_{u,v}(s) ds.
\tag{2.12}
\]

Proof. To prove the estimate (2.12), we argue by contradiction. We assume that there exist a positive sequence \((t_n)\) and a sequence \((U_n(t) = (u_n(t), v_n(t), \partial_t u_n(t), \partial_t v_n(t)))\) of solutions of the system (2.11) with initial data \((u_{0,n}, v_{0,n}, u_{1,n}, v_{1,n}) \in \mathcal{H}\), such that
\[
\int_{t_n}^{t_n+T} \phi(s) \left( |v_n(s, x)|^2 + |u_n(s, x)|^2 \right) dx ds
\geq n \int_{t_n}^{t_n+T} \phi(s) \left( |\partial_t u_n(s, x)|^2 + |\partial_t v_n(s, x)|^2 \right) dx ds + \alpha \int_{t_n}^{t_n+T} \phi(s) E_{u_n,v_n}(s) ds.
\tag{2.13}
\]

1st case: The sequence \((t_n)\) is bounded. Setting
\[
\lambda_n^2 = \int_{t_n}^{t_n+T} \left( |v_n(s, x)|^2 + |u_n(s, x)|^2 \right) dx ds \quad \text{and} \quad V_n(t) = (w_n(t), y_n(t), \partial_t w_n(t), \partial_t y_n(t)) = \frac{1}{\lambda_n} U_n(t + t_n),
\]
using the fact that the sequence \((t_n)\) is bounded, (2.8) and (2.13), we infer that there exist \( \alpha_0, \alpha_1, \alpha_2 > 0 \), such that
\[
\int_0^T \int_{\Omega} \left( |w_n(s, x)|^2 + |y_n(s, x)|^2 \right) dx ds \geq \alpha_0 > 0,
\int_0^T \int_{\Omega} a(x) \left( |\partial_t w_n(s, x)|^2 + \frac{1}{\lambda_n} |g(\partial_t u_n(t_n + s, x))|^2 \right) dx ds \leq \frac{\alpha_1}{\lambda_n}
\]
and
\[
\int_0^T E_{w_n,y_n}(s) ds \leq \alpha_1.
\]
To finish the proof we use the unique continuation hypotheses (A1) and we proceed as in [7] Proof of lemma 7.

2nd case: \( t_n \to +\infty \). Setting
\[
\lambda_n^2 = \int_{t_n}^{t_n+T} \phi(s) \left( |v_n(s, x)|^2 + |u_n(s, x)|^2 \right) dx ds \quad \text{and} \quad V_n(t) = (w_n(t), y_n(t), \partial_t w_n(t), \partial_t y_n(t)) = \frac{1}{\lambda_n} (\phi(t + t_n))^{1/2} U_n(t + t_n),
\]
Therefore
\[
\begin{align*}
\frac{\partial^2 w_n}{\partial t^2} + \Delta w_n + b(x) y_n + \frac{1}{n} a(x) (\phi (t_n + t))^{1/2} g (\partial_t u_n) &= f_n (t, x) \quad \text{in } \mathbb{R}^*_+ \times \Omega, \\
\frac{\partial^2 y_n}{\partial t^2} + \Delta y_n + b(x) w_n &= f_n^t (t, x) \quad \text{in } \mathbb{R}^*_+ \times \Omega, \\
w_n = y_n = 0 \quad \text{on } \mathbb{R}^*_+ \times \Gamma, \\
\begin{cases}
(w_n (0, x), \partial_t w_n (0, x)) = (w_{n,0}, w_{n,1}) \quad \text{and} \\
(y_n (0, x), \partial_t y_n (0, x)) = (y_{n,0}, y_{n,1}) \quad \text{in } \Omega
\end{cases}
\end{align*}
\]
where
\[
\begin{align*}
f_n &= \frac{1}{2} \phi'' (t + t_n) (\phi (t + t_n))^{-1} w_n - \frac{1}{4} (\phi' (t + t_n))^2 \phi (t + t_n)^{-2} w_n \\
&\quad + \frac{1}{n} \phi' (t + t_n) (\phi (t + t_n))^{-1/2} \partial_t u_n, \\
f_n^t &= \frac{1}{2} \phi'' (t + t_n) (\phi (t + t_n))^{-1} y_n - \frac{1}{4} (\phi' (t + t_n))^2 \phi (t + t_n)^{-2} y_n \\
&\quad + \frac{1}{n} \phi' (t + t_n) (\phi (t + t_n))^{-1/2} \partial_t v_n.
\end{align*}
\]
Thanks to (2.13) we get
\[
\begin{align*}
\int_0^T \int_\Omega |w_n (s, x)|^2 + |y_n (s, x)|^2 \, dx \, ds = 1,
\end{align*}
\[
\begin{align*}
\frac{1}{\lambda^2} \int_0^T \int_\Omega a(x) \phi (t_n + s) \left( |\partial_t u_n (t_n + s, x)|^2 + |g (\partial_t u_n (s + t_n, x))|^2 \right) \, dx \, ds \leq \frac{1}{n} \tag{2.14}
\end{align*}
and
\[
\begin{align*}
\int_0^T \frac{\phi (t_n + s)}{\lambda^2} E_{u_n, v_n} (s) \, ds \leq 1/\alpha.
\end{align*}
\]
Now using the estimates above, we deduce that there exist \( \alpha_1 > 0 \), such that
\[
\int_0^T E_{w_n, y_n} (s) \, ds \leq \alpha_1.
\]
Utilizing (2.14), (2.8) and (2.10), we can show that
\[
\lim_{n \to \infty} \int_0^T \int_\Omega |f_n (s, x)|^2 \, dx \, ds = \lim_{n \to \infty} \int_0^T \int_\Omega |f_n^t (s, x)|^2 \, dx \, ds = 0.
\]
On the other hand, it is easy to see that
\[
\begin{align*}
\int_0^T \int_\Omega \phi (t_n + s) \psi (x) |\partial_t v_n (s, x)|^2 \, dx \\
\leq \int_0^T \int_\Omega \frac{1}{\lambda^2} \phi (t_n + s) \psi (x) |\partial_t u_n (t_n + s, x)|^2 + \left| \frac{\phi' (t_n)}{\phi (t_n)} \right|^2 \psi (x) |w_n (s, x)|^2 \, dx \, ds \\
\leq \int_0^T \int_\Omega \frac{1}{\lambda^2} \phi (t + s) \psi (x) |\partial_t u_n (s, x)|^2 \, dx \, ds + \alpha_2 \left| \frac{\phi' (t_n)}{\phi (t_n)} \right|^2 \, dx \to 0, \quad n \to \infty
\end{align*}
\]
noting that in the last result we have used the second part of (2.14) and (2.10). To finish the proof we use the unique continuation hypotheses (A1) and we proceed as in [7] Proof of lemma 7.

\[\square\]

**Lemma 2.** Let \( \phi \in C^2 (\mathbb{R}_+) \) and non-decreasing. Let \( u \) be a solution of the system (1.1) with initial data \( U_0 = (u_0, v_0, u_1, v_1) \in D (A) \). We set
\[
\begin{align*}
X (t) &= \phi' (t) \int_\Omega u (t, x) \partial_t u (t, x) + v (t, x) \partial_t v (t, x) \, dx \\
&\quad + k \phi' (t) \int_\Omega \partial_t^2 u (t, x) \partial_t v (t, x) - \partial_t^2 v (t, x) \partial_t u (t, x) \, dx \\
&\quad + \phi (t) E_{u, v} (t) + k \phi' (t) E_{\partial_t u, \partial_t v} (t), \quad \text{(2.15)}
\end{align*}
\]
where $k$, $T$ and $k_1$ are positive constants. Then

$$X(t+T) - X(t) + \int_t^{t+T} \varphi'(s) E_{u,v}(s) \, ds + \int_t^{t+T} a(x) \varphi(s) g(\partial_t u(x), \partial_x u(x)) \partial_t u(x) \, dx \, ds$$

$$+ k \int_t^{t+T} \int_\Omega a(x) \varphi'(s) g'(\partial_t u(x)) |\partial_t^2 u(x)|^2 \, dx \, ds$$

$$\leq 2 \int_t^{t+T} \varphi'(s) \int_\Omega |\partial_t u(x)|^2 + |\partial_t v(x)|^2 \, dx \, ds$$

$$+ \int_t^{t+T} \varphi'(s) \int_\Omega a(x) \left( |g(\partial_t u(x))|^2 + |u(x)|^2 \right) \, dx \, ds$$

$$+ (k E_{\partial_t u, \partial_t v}(0) + k_0 E_{u,v}(0)) \int_t^{t+T} |\varphi''(s)| \, ds$$

$$+ k_1 \left[ \varphi'(t) \int_\Omega \partial_t^2 u(s,x) \partial_t v(s,x) - \partial_t^2 v(s,x) \partial_t u(s,x) \, dx \right]_{s=t}^{s=t+T}.$$

(2.16)

**Proof.** We differentiate $X(t)$ with respect to $t$, we obtain

$$X'(t) = \varphi'(t) \int_\Omega |\partial_t u(t,x)|^2 + |\partial_t v(t,x)|^2 \, dx$$

$$+ \varphi'(t) \int \Omega u(t,x) \partial_t^2 u(t,x) + v(t,x) \partial_t^2 v(t,x) \, dx$$

$$+ \varphi''(t) \int \Omega u(t,x) \partial_t u(t,x) + v(t,x) \partial_t v(t,x) \, dx$$

$$+ k_1 \frac{d}{dt} \left[ \varphi'(t) \int_\Omega \partial_t^2 u(t,x) \partial_t v(t,x) - \partial_t^2 v(t,x) \partial_t u(t,x) \, dx \right] + \varphi'(t) E_{u,v}(t)$$

$$- \varphi(t) \int_\Omega a(x) g(\partial_t u(t,x)) \partial_t u(t,x) \, dx - k \varphi'(t) \int_\Omega a(x) g'(\partial_t u(t,x)) \partial_t^2 u(t,x) \, dx$$

$$+ k \varphi''(t) E_{\partial_t u, \partial_t v}(t).$$

(2.17)

Using the first and the second equations of (1.1), we infer that

$$\varphi'(t) \int_\Omega (u(t,x) \partial_t^2 u(t,x) + v(t,x) \partial_t^2 v(t,x)) \, dx$$

$$= - \varphi'(t) \int_\Omega \nabla u(t,x)^2 + \nabla v(t,x)^2 + a(x) g(\partial_t u(t,x)) u(t,x) + 2 b(x) u(t,x) v(t,x) \, dx$$

$$= -2 \varphi'(t) E_{u,v}(t) + \varphi'(t) \int_\Omega |\partial_t u(t,x)|^2 + |\partial_t v(t,x)|^2 \, dx - \varphi'(t) \int_\Omega a(x) g(\partial_t u(t,x)) u(t,x) \, dx.$$

Thanks to Young’s inequality we get

$$\varphi'(t) \int_\Omega a(x) g(\partial_t u(t,x)) u(t,x) \, dx \leq \frac{\varphi'(t)}{2} \int_\Omega a(x) \left( |g(\partial_t u(t,x))|^2 + |u(t,x)|^2 \right) \, dx.$$

To estimate the third term of the RHS of (2.17), we use Poincare’s inequality and the fact that the energy is decreasing

$$\varphi''(t) \int_\Omega u(t,x) \partial_t u(t,x) + v(t,x) \partial_t v(t,x) \, dx \leq k_0 |\varphi''(s)| E_{u,v}(0).$$

For the last term of the RHS of (2.17), we use the fact that

$$E_{\partial_t u, \partial_t v}(t) \leq E_{\partial_t u, \partial_t v}(0).$$
we obtain (2.16).

Let

\[ H(x) = \begin{cases} h^{-1}(x) & \text{on } \mathbb{R}_+ \\ \infty & \text{on } \mathbb{R}_-. \end{cases} \quad (2.18) \]

Noting that according to [4], if \( h^{-1} \) is a strictly convex \( C^1 \) function from \([0, r_0]\) to \( \mathbb{R} \) such that \( h^{-1}(0) = (h^{-1})'(0) = 0 \). Then the convex conjugate function of \( H \) is defined by

\[ H^*(x) = x \left( (h^{-1})' \right)^{-1}(x) - h^{-1} \left( (h^{-1})' \right)^{-1}(x) \text{, on } [0, (h^{-1})'(r_0)]. \quad (2.19) \]

**Lemma 3.** We assume that the assumption A2 holds. Let \( \varphi \) be a solution of the following ODE

\[ \frac{d\varphi}{dt} - \frac{\epsilon_0}{2C_1} \varphi (h^{-1})' (1/\varphi^\beta) = 0, \quad 0 < \varphi^{-\beta} (0) \leq r_0. \quad (2.20) \]

where \( \epsilon_0 \) and \( C_1 \) are positive constant. Then we have \( \varphi \) is a concave strictly increasing function in \( C^3(\mathbb{R}_+) \). In addition we have

1. \( \lim_{t \to \infty} \varphi(t) = \infty. \)
2. \( \lim_{t \to \infty} \varphi'(t) = \frac{\epsilon_0}{2C_1} \) and \( \lim_{t \to \infty} \frac{\varphi''(t)}{\varphi'(t)} = 0. \)
3. The function \( t \mapsto \left| \frac{\varphi''(t)}{\varphi'(t)} \right| \) is decreasing.
4. If \( \varphi'' \) is not the null function, then there exists \( K > 0 \) such that \( \sup_{\mathbb{R}_+} \left| \frac{\varphi''(t)}{\varphi'(t)} \right| \leq K. \)
5. \( \int_0^\infty |\varphi''(s)| \, ds = \varphi'(0) - \frac{\epsilon_0}{2C_1}. \)
6. \( \frac{\epsilon_0}{2C_1} \int_0^\infty \varphi(s) H^* \left( \frac{2C_1 \varphi'(s)}{\epsilon_0 \varphi(s)} \right) \, ds \leq \frac{1}{\beta-1} \varphi^{1-\beta}(0). \)

**Proof.** Using the second part of (1.12) and the fact that

\[ 0 < \varphi^{-\beta}(t) \leq \varphi^{-\beta}(0) \leq r_0, \text{ for all } t > 0, \quad (2.21) \]

we obtain

\[ \varphi''(t) = \frac{\epsilon_0}{2C_1} \varphi'(t) \left( (h^{-1})' (\varphi^{-\beta}(t)) - \beta \varphi^{-\beta}(t) (h^{-1})'' (\varphi^{-\beta}(t)) \right) \leq 0, \quad (2.22) \]

for all \( t \in \mathbb{R}_+ \), which means that the function \( \varphi \) is a concave on \( \mathbb{R}_+ \).

It is easy to see that the function \( \varphi \in C^3(\mathbb{R}_+). \)

1. First we note that

\[ \varphi^{-1}(t) = \frac{2C_1}{\epsilon_0} \int_{\varphi(0)}^t \frac{ds}{s \left( (h^{-1})' (1/s^\beta) \right)}. \]

Therefore, using the fact that

\[ s \left( (h^{-1})' (1/s^\beta) \right) \leq s \left( (h^{-1})' (1/\varphi^\beta (0)) \right), \text{ for all } s \in [0, \infty), \]

we deduce that

\[ \lim_{t \to \infty} \varphi^{-1}(t) = \infty, \]

thus

\[ \lim_{t \to \infty} \varphi(t) = \infty. \quad (2.23) \]
Using (2.28) and (1.12) we get
\[
\lim_{t \to \infty} \varphi'(t) = \frac{\epsilon_0 \alpha_0}{2C_1}. \tag{2.24}
\]

Direct computations and using (1.11), yield
\[
\lim_{t \to \infty} \left| \frac{\varphi''(t)}{\varphi'(t)} \right| = 0.
\]

From (2.22), we deduce that
\[
\left| \frac{\varphi''(t)}{\varphi'(t)} \right| = \frac{\epsilon_0 \alpha_0}{2C_1} \left( - (h^{-1})' \left( \varphi^\beta (t) \right) + \beta \varphi^- \beta (t) \left( h^{-1} \right)'' \left( \varphi^\beta (t) \right) \right).
\]

We differentiate the estimate above and making some arrangements, we obtain
\[
\frac{d}{dt} \left| \frac{\varphi''(t)}{\varphi'(t)} \right| = - \frac{\epsilon_0 \alpha_0}{2C_1} \frac{\varphi'(t)}{\varphi'(t)} \left( \beta^2 - \beta \right) \left( h^{-1} \right)'' \left( \varphi^\beta (t) \right) + \beta^2 \varphi^- \beta (t) \left( h^{-1} \right)''' \left( \varphi^\beta (t) \right).
\]

From the estimate above, (2.21) and (1.12), we see that the function \( t \mapsto \left| \frac{\varphi''(t)}{\varphi'(t)} \right| \) is decreasing.

We differentiate the identity (2.22) and making some arrangements, we obtain
\[
\frac{\varphi'''(t)}{\varphi'(t)} = \frac{\epsilon_0 \alpha_0}{2C_1} \left( (h^{-1})' \left( \varphi^\beta (t) \right) - \beta \varphi^- \beta (t) \left( h^{-1} \right)'' \left( \varphi^\beta (t) \right) \right)
+ \frac{\epsilon_0 (\varphi'(t))^2}{2C_1 \varphi'(t)} \left( \beta^2 - \beta \right) \varphi^- \beta (t) \left( h^{-1} \right)'' \left( \varphi^\beta (t) \right) + \beta^2 \varphi^- \beta (t) \left( h^{-1} \right)''' \left( \varphi^\beta (t) \right).
\]

On the other hand, from (2.21) and (2.22), we infer that
\[
\left| \frac{(\varphi'(t))^2}{\varphi'(t)} \right| = \frac{(h^{-1})' \left( \varphi^\beta (t) \right)}{\beta \varphi^- \beta (t) \left( h^{-1} \right)'' \left( \varphi^\beta (t) \right) - (h^{-1})' \left( \varphi^\beta (t) \right)}.
\]

Combining the two estimates above, we see that
\[
\left| \frac{\varphi'''(t)}{\varphi'(t)} \right| \leq \frac{\epsilon_0 \alpha_0}{2C_1} \left( \beta \varphi^- \beta (t) \left( h^{-1} \right)'' \left( \varphi^\beta (t) \right) - (h^{-1})' \left( \varphi^\beta (t) \right) \right)
+ \frac{\epsilon_0 (h^{-1})' \left( \varphi^\beta (t) \right) \left( \beta^2 - \beta \right) \varphi^- \beta (t) \left( h^{-1} \right)'' \left( \varphi^\beta (t) \right) + \beta^2 \varphi^- \beta (t) \left( h^{-1} \right)''' \left( \varphi^\beta (t) \right)}{\beta \varphi^- \beta (t) \left( h^{-1} \right)'' \left( \varphi^\beta (t) \right) - (h^{-1})' \left( \varphi^\beta (t) \right)}.
\]

So from Assumption A2, we conclude that there exists \( K > 0 \) such that
\[
\sup_{\mathbb{R}_+} \left| \frac{\varphi'''(t)}{\varphi'(t)} \right| \leq K.
\]

Using (2.22) and (2.24), we obtain
\[
\int_0^\infty |\varphi''(s)| \, ds = - \int_0^\infty \varphi''(s) \, ds = \varphi'(0) - \frac{\epsilon_0 \alpha_0}{2C_1}.
\]

Thanks to (2.19) and (2.20), we see that
\[
\varphi(s) H^* \left( \frac{2C_1 \varphi'(s) \epsilon_0}{\epsilon_0 \varphi(s)} \right) \leq \frac{2C_1}{\epsilon_0} \varphi'(s) \left( (h^{-1})' \right)^{-1} \left( \frac{2C_1 \varphi'(s) \epsilon_0}{\epsilon_0 \varphi(s)} \right) = \frac{2C_1}{\epsilon_0} \varphi'(s) \varphi^\beta(s),
\]

therefore integrating the estimate above between zero and infinity and using (2.23) and the fact that \( \beta > 1 \) we obtain
\[
\frac{\epsilon_0}{2C_1} \int_0^\infty \varphi(s) H^* \left( \frac{2C_1 \varphi'(s) \epsilon_0}{\epsilon_0 \varphi(s)} \right) \, ds \leq \frac{1}{\beta-1} \varphi^{1-\beta}(0).
\]
2.1. **Proof of Theorem 1.** We assume that \( \omega \) and \( \mathcal{O} \) satisfy the GCC and the assumption A1 holds. Let \((u, v)\) be a solution of the system \((1.1)\) with initial data \(U_0 = (u_0, v_0, u_1, v_1) \in D(A)\). Let \( T > \max(\{T_\omega, T_\mathcal{O}, T_0\})\).

We have \(u\) is a solution of the nonhomogeneous wave equation with a localized nonlinear damping and \((\omega, T)\) satisfies the GCC. In addition, taking into account of lemma \((3.3)\), we see that \(\varphi'\) satisfies the required assumptions of proposition \((2.4)\). Therefore, using the observability estimate \((2.5)\) and \((2.16)\), we deduce that

\[
X(t+T) - X(t) + \int_t^{t+T} \varphi'(s) E_{u,v}(s) \, ds + \int_t^{t+T} \int_\Omega a(x) \varphi(s) g(\partial_t u(s,x)) \partial_t u(s,x) \, dx \, ds \\
+ k \int_t^{t+T} \int_\Omega a(x) \varphi'(s) g'(\partial_t u(s,x)) |\partial_t^2 u(s,x)|^2 \, dx \, ds \\
\leq 2 \int_t^{t+T} \varphi'(s) \int_{\Omega} |\partial_t v(s,x)|^2 \, dx \, ds \\
+ (4C_T + 1) \left( \int_t^{t+T} \varphi'(s) \int_\Omega \left(a(x) \left| g(\partial_t u(s,x)) \right|^2 + |\partial_t u(s,x)|^2 \right) + |b(x) v(s,x)|^2 \right) \, dx \, ds \\
+ 3 \int_t^{t+T} \varphi'(s) \int_\Omega a(x) |u(s,x)|^2 \, dx \, ds + \left(k E_{\partial_t u, \partial_t v}(0) + k_0 E_{u,v}(0)\right) \int_t^{t+T} |\varphi''(s)| \, ds \\
+ k_1 \left[ \varphi'(t) \int_{\Omega} \partial_t^2 u(s,x) \partial_t v(s,x) - \partial_t^2 v(s,x) \partial_t u(s,x) \, dx \right]_{s=t}^{s=t+T}. \\
\tag{2.25}
\]

To estimate \(\int_t^{t+T} \varphi'(s) \int_{\Omega} |\partial_t v(s,x)|^2 \, dx \, ds\), we first use the fact that \(v\) is a solution of the nonhomogeneous wave equation and \((\mathcal{O}, T)\) satisfies the GCC, then from the observability estimate \((2.5)\), we infer that

\[
2 \int_t^{t+T} \varphi'(s) \int_{\Omega} |\partial_t v(s,x)|^2 \, dx \, ds \leq 4C_T \left( \int_t^{t+T} \varphi'(s) \int_{\Omega} b(x) |\partial_t v(s,x)|^2 + |b(x) u(s,x)|^2 \, dx \, ds \right). \\
\tag{2.26}
\]

Now we estimate \(\int_t^{t+T} \varphi'(s) \int_{\Omega} b(x) |\partial_t v(s,x)|^2 \, dx \, ds\). We have

\[
\begin{cases}
\partial_t^2 (\partial_t u) - \Delta (\partial_t u) + b(x) (\partial_t v) + a(x) g'(\partial_t u) \partial_t^2 u = 0 & \text{in } \mathbb{R}^*_+ \times \Omega \\
\partial_t^2 (\partial_t v) - \Delta (\partial_t v) + b(x) (\partial_t u) = 0 & \text{in } \mathbb{R}^*_+ \times \Omega \\
\partial_t u = \partial_t v = 0 & \text{on } \mathbb{R}^*_+ \times \Gamma. \\
\end{cases} \tag{2.27}
\]

We multiply the first equation of \((2.27)\) by \((\varphi'(t) \partial_t v)\) and the second equation by \((\varphi'(t) \partial_t u)\) and integrating the difference of these results over \(\Omega\), we obtain

\[
\begin{align*}
\varphi'(t) \int_{\Omega} b(x) |\partial_t v(t,x)|^2 \, dx &= -\frac{d}{dt} \left( \varphi'(t) \int_{\Omega} (\partial_t v(t,x) \partial_t^2 u(t,x) - \partial_t u(t,x) \partial_t^2 v(t,x)) \, dx \right) \\
- \varphi''(t) \int_{\Omega} (\partial_t v(t,x) \partial_t^2 u(t,x) - \partial_t u(t,x) \partial_t^2 v(t,x)) \, dx \\
+ \varphi'(t) \int_{\Omega} b(x) |\partial_t u(t,x)|^2 \, dx - \varphi'(t) \int_{\Omega} a(x) g'(\partial_t u(t,x)) \partial_t^2 u(t,x) \partial_t v(t,x) \, dx.
\end{align*}
\]
Using Young’s inequality, we infer that
\[ \varphi'(t) \int_{\Omega} b(x) |\partial_t v(s, x)|^2 \, dx \leq \epsilon \varphi'(t) \int_{\Omega} a(x) |\partial_t v(s, x)|^2 \, dx \]
\[- \frac{d}{dt} \left( \varphi'(t) \int_{\Omega} (\partial_t u(t, x) \partial_t^2 v(t, x) - \partial_t v(t, x) \partial_t^2 u(t, x)) \, dx \right) \]
\[ + \varphi''(t) \int_{\Omega} (\partial_t u(t, x) \partial_t^2 v(t, x) - \partial_t v(t, x) \partial_t^2 u(t, x)) \, dx \]
\[ + \varphi'(t) \int_{\Omega} b(x) |\partial_t u(s, x)|^2 \, dx + \frac{\epsilon}{2} \varphi'(t) \int_{\Omega} a(x) |g'(\partial_t u(t, x))| \partial_t^2 u(s, x) |^2 \, dx. \]

Integrating the estimate above between \( t \) and \( t + T \), we obtain
\[ \int_{t}^{t+T} \varphi'(s) \int_{\Omega} b(x) |\partial_t v(s, x)|^2 \, dx ds \]
\[ \leq \epsilon \int_{t}^{t+T} \varphi'(s) \int_{\Omega} a(x) |\partial_t v(s, x)|^2 \, dx ds + \int_{t}^{t+T} \varphi'(s) \int_{\Omega} b(x) |\partial_t u(s, x)|^2 \, dx ds \]
\[ + \frac{1}{\epsilon} \int_{t}^{t+T} \varphi'(s) \int_{\Omega} a(x) |g'(\partial_t u(s, x))| \partial_t^2 u(s, x) |^2 \, dx ds \]
\[ + \int_{t}^{t+T} \varphi''(s) \int_{\Omega} (\partial_t u(t, x) \partial_t^2 v(t, x) - \partial_t v(t, x) \partial_t^2 u(t, x)) \, dx ds \]
\[ - \left[ \varphi'(s) \int_{\Omega} (\partial_t^2 u(s, x) \partial_t v(s, x) - \partial_t^2 v(s, x) \partial_t u(s, x)) \, dx \right]_{s=t}^{s=t+T}. \]

Now using the observability estimate (2.26) and taking \( \epsilon = \frac{1}{4\|a\|_{\infty} C_T} \), we get
\[ \int_{t}^{t+T} \varphi'(s) \int_{\Omega} b(x) |\partial_t v(s, x)|^2 \, dx ds \]
\[ \leq 2 \int_{t}^{t+T} \varphi'(s) \int_{\Omega} b(x) |\partial_t u(s, x)|^2 + |b(x) u(s, x)|^2 \, dx ds \]
\[ + 16 \|a\|_{\infty} C_T \int_{t}^{t+T} \varphi'(s) \int_{\Omega} a(x) |g'(\partial_t u(s, x))| \partial_t^2 u(s, x) |^2 \, dx ds \]
\[ + 2 \int_{t}^{t+T} \varphi''(s) \int_{\Omega} (\partial_t u(t, x) \partial_t^2 v(t, x) - \partial_t v(t, x) \partial_t^2 u(t, x)) \, dx ds \]
\[ - 2 \left[ \varphi'(s) \int_{\Omega} (\partial_t^2 u(s, x) \partial_t v(s, x) - \partial_t^2 v(s, x) \partial_t u(s, x)) \, dx \right]_{s=t}^{s=t+T}. \]

Combining the estimate above and (2.26), we find that
\[ 2 \int_{t}^{t+T} \varphi'(s) \int_{\Omega} |\partial_t v(s, x)|^2 \, dx ds \]
\[ \leq 8 C_T \int_{t}^{t+T} \varphi'(s) \int_{\Omega} b(x) |\partial_t u(s, x)|^2 + |b(x) u(s, x)|^2 \, dx ds \]
\[ + 64 \|a\|_{\infty} (C_T)^2 \|g'\|_{L^\infty} \int_{t}^{t+T} \varphi'(s) \int_{\Omega} a(x) |g'(\partial_t u(s, x))| \partial_t^2 u(s, x) |^2 \, dx ds \]
\[ + 8 C_T \left( \sum_{i=0}^{1} E_{\partial_t^2 u, \partial_t v}(0) \right) \int_{t}^{t+T} |\varphi''(s)| \, ds \]
\[ - 8 C_T \left[ \varphi'(s) \int_{\Omega} (\partial_t^2 u(s, x) \partial_t v(s, x) - \partial_t^2 v(s, x) \partial_t u(s, x)) \, dx \right]_{s=t}^{s=t+T}. \]
Now using the observability estimate \((2.3)\), we infer that

\[
2 \int_t^{t+T} \varphi'(s) \int_\Omega |\partial_t v(s, x)|^2 \, dx \, ds \\
\leq 32C_T^2 \|b\|_\infty \left( \int_t^{t+T} \varphi'(s) \int_\Omega \left( a(x) \left( |g(\partial_t u(s, x))|^2 + |\partial_t u(s, x)|^2 \right) \right) \, dx \, ds \right) \\
+ \left( 8C_T \|b\|_\infty + 32C_T^2 \|b\|_\infty^2 \right) \int_t^{t+T} \varphi'(s) \int_\Omega |u(s, x)|^2 + |v(s, x)|^2 \, dx \, ds \\
+ 64 \|a\|_\infty (C_T)^2 \|g'\|_{L^\infty} \int_t^{t+T} \varphi'(s) \int_\Omega a(x) g'(\partial_t u(s, x)) |\partial_t^2 u(s, x)|^2 \, dx \, ds \\
+ 8C_T \left( \sum_{i=0}^1 E_{\partial_t^i u, \partial_t^i v}(0) \right) \int_t^{t+T} |\varphi''(s)| \, ds \\
- 8C_T \left[ \varphi'(s) \int_\Omega (\partial_t^2 u(s, x) \partial_t v(s, x) - \partial_t^2 v(s, x) \partial_t u(s, x)) \, dx \right]_{s=t}^{s=t+T}.
\]

Utilizing \((2.25)\), \((2.29)\) and making some arrangements, we obtain

\[
X(t + T) - X(t) + \int_t^{t+T} \varphi'(s) E_{u,v}(s) \, ds + \int_t^{t+T} \varphi(s) \int_\Omega a(x) g(\partial_t u(s, x)) \partial_t u(s, x) \, dx \, ds \\
+ k \int_t^{t+T} \varphi'(s) \int_\Omega a(x) g'(\partial_t u(s, x)) |\partial_t^2 u(s, x)|^2 \, dx \, ds \\
\leq (4C_T + 32C_T^2 \|b\|_\infty + 1) \left( \int_t^{t+T} \varphi'(s) \int_\Omega \left( a(x) \left( |g(\partial_t u(s, x))|^2 + |\partial_t u(s, x)|^2 \right) \right) \, dx \, ds \right) \\
+ (\|a\|_\infty + 12C_T \|b\|_\infty + 32C_T^2 \|b\|_\infty^2) \int_t^{t+T} \varphi'(s) \int_\Omega |v(s, x)|^2 + |u(s, x)|^2 \, dx \, ds \\
+ 64 \|a\|_\infty (C_T)^2 \|g'\|_{L^\infty} \int_t^{t+T} \varphi'(s) \int_\Omega a(x) g'(\partial_t u(s, x)) |\partial_t^2 u(s, x)|^2 \, dx \, ds \\
+ (k_1 - 8C_T) \left[ \varphi'(s) \int_\Omega (\partial_t^2 u(s, x) \partial_t v(s, x) - \partial_t^2 v(s, x) \partial_t u(s, x)) \, dx \right]_{s=t}^{s=t+T} \\
+ (8C_T + k) \left( \sum_{i=0}^1 E_{\partial_t^i u, \partial_t^i v}(0) \right) \int_t^{t+T} |\varphi''(s)| \, ds.
\]

We take \(k_1 = 8C_T\), we conclude that

\[
X(t + T) - X(t) + \int_t^{t+T} \varphi'(s) E_{u,v}(s) \, ds + \int_t^{t+T} \varphi(s) \int_\Omega a(x) g(\partial_t u(s, x)) \partial_t u(s, x) \, dx \, ds \\
+ \left( k - 64 \|a\|_\infty (C_T)^2 \|g'\|_{L^\infty} \right) \int_t^{t+T} \varphi'(s) \int_\Omega a(x) g'(\partial_t u(s, x)) |\partial_t^2 u(s, x)|^2 \, dx \, ds \\
\leq (\|a\|_\infty + 12C_T \|b\|_\infty + 32C_T^2 \|b\|_\infty^2) \int_t^{t+T} \varphi'(s) \int_\Omega |v(s, x)|^2 + |u(s, x)|^2 \, dx \, ds \\
+ \left( 32 (C_T)^2 \|b\|_\infty + 4C_T + 1 \right) \left( \int_t^{t+T} \varphi'(s) \int_\Omega a(x) \left( |\partial_t u(s, x)|^2 + |g(\partial_t u(s, x))|^2 \right) \, dx \, ds \right) \\
+ (8C_T + k + k_0) \left( \sum_{i=0}^1 E_{\partial_t^i u, \partial_t^i v}(0) \right) \int_t^{t+T} |\varphi''(s)| \, ds.
\]
Now using (2.12) with \( \alpha = \frac{1}{2(\|a\|_{\infty} + 12C_T \|b\|_{\infty} + 32C_T \|b\|_{\infty}^2)} \), we get

\[
X(t + T) + \frac{1}{2} \int_t^{t+T} \varphi'(s) E_{u,v}(s) \, ds + \int_t^{t+T} \varphi(s) \left( \int_{\Omega} a(x) g(\partial_t u(s, x)) \, dx \right) \, ds \\
+ \left( k - 64 \|a\|_{\infty} (C_T)^2 \|g\|_{L^\infty} \right) \int_t^{t+T} \varphi'(s) \left( \int_{\Omega} a(x) g'(\partial_t u(s, x)) \right) \left| \partial_t^2 u(s, x) \right| \, ds \\
\leq C_1 \left( \int_t^{t+T} \varphi'(s) \left( \int_{\Omega} a(x) \left( \left| \partial_t u(s, x) \right|^2 + \left| g(\partial_t u(s, x)) \right|^2 \right) \, dx \right) \, ds \\
+ X(t) + (8C_T + k + k_0) \left( \sum_{i=0}^{1} E_{\partial_t^i u, \partial_t^i v}(0) \right) \int_t^{t+T} \left| \varphi''(s) \right| \, ds.
\]

(2.31)

where

\[
C_1 = C(T, \|b\|_{\infty}, \|a\|_{\infty}).
\]

(2.32)

Now we have to estimate the first term of RHS of the estimate above by the third term of the LHS. We set for all fixed \( s \geq 0 \), \( \Omega^s = \{ x, |\partial_t u(s, x)| < 1 \} \). Thanks to (1.9), we have

\[
\int_t^{t+T} \varphi'(s) \int_{\Omega^s} a(x) \left( \left| \partial_t u(s, x) \right|^2 + \left| g(\partial_t u(s, x)) \right|^2 \right) \, dx \, ds \\
\leq \frac{1}{\alpha} \int_t^{t+T} \varphi'(s) \int_{\Omega^s} h_0 \left( g(\partial_t u(s, x)) \partial_t u(s, x) \right) a(x) \, dx \, ds \\
\leq \frac{1}{\alpha} \int_t^{t+T} \varphi'(s) \int_{\Omega^s} h_0 \left( g(\partial_t u(s, x)) \partial_t u(s, x) \right) a(x) \, dx \, ds
\]

Since \( h_0 \) is concave, we can use (the reverse) Jensen’s inequality and we obtain

\[
\int_t^{t+T} \varphi'(s) \int_{\Omega^s} a(x) \left( \left| \partial_t u(s, x) \right|^2 + \left| g(\partial_t u(s, x)) \right|^2 \right) \, dx \, ds \\
\leq \frac{1}{\alpha} \int_t^{t+T} \varphi'(s) h \left( \int_{\Omega} g(\partial_t u(s, x)) \partial_t u(s, x) a(x) \, dx \right) \, ds.
\]

Under our assumptions the function \( H^* \) defined by (2.18) is convex and proper. Hence, we can apply Young’s inequality [19]

\[
\int_t^{t+T} \varphi'(s) \int_{\Omega^s} a(x) \left( \left| \partial_t u(s, x) \right|^2 + \left| g(\partial_t u(s, x)) \right|^2 \right) \, dx \, ds \\
\leq \left( \frac{\alpha}{2C_1} \int_t^{t+T} \varphi(s) H^* \left( \frac{\alpha C_1 \varphi(s)}{\alpha C_1} \right) \, ds + \frac{\alpha}{2C_1} \int_t^{t+T} \varphi(s) \left( \int_{\Omega} a(x) g(\partial_t u(s, x)) \partial_t u(s, x) \, dx \right) \, ds \right)
\]

where \( H^* \) is the convex conjugate of the function \( H \).

On the other hand, using the fact that the function \( g \) is linearly bounded near infinity, we infer that

\[
\int_t^{t+T} \varphi'(s) \int_{\Omega} a(x) \left( \left| \partial_t u(s, x) \right|^2 + \left| g(\partial_t u(s, x)) \right|^2 \right) \, dx \, ds \\
\leq \frac{1}{2C_1} \int_t^{t+T} \varphi(s) H^* \left( \frac{\alpha C_1 \varphi(s)}{\alpha C_1} \right) \, ds + \frac{1}{2C_1} \int_t^{t+T} \varphi(s) \left( \int_{\Omega} a(x) g(\partial_t u(s, x)) \partial_t u(s, x) \, dx \right) \, ds \\
+ \left( \frac{1}{m} + M^2 \right) \int_t^{t+T} \varphi'(s) \left( \int_{\Omega} a(x) g(\partial_t u(s, x)) \partial_t u(s, x) \, dx \right) \, ds.
\]
The estimate above combined with (2.31), give

\[
X(t + T) + \frac{1}{T} \int_t^{t+T} \varphi'(s) E_{u,v}(s) \, ds \\
+ \frac{1}{T} \int_t^{t+T} \left( \frac{1}{2} - \left( \frac{1}{m} + M^2 \right) C_1 \frac{\varphi'(s)}{\varphi(s)} \right) \varphi(s) \int_\Omega a(x) g(\partial_t u(s,x)) \partial_t u(s,x) \, dx \, ds \\
+ \left( k - 64 \|a\|_\infty (C_T)^2 \|g'\|_{L^\infty} \right) \int_t^{t+T} \varphi'(s) \int_\Omega a(x) g'(\partial_t u(s,x)) \left| \partial^2_t u(s,x) \right|^2 \, dx \, ds \\
\leq X(t) + (8C_T + k + k_0) \left( \sum_{i=0}^{n-1} E_{\partial_t^i u, \partial_t^i v}(0) \right) \int_t^{t+T} |\varphi''(s)| \, ds + \frac{1}{T} \int_t^{t+T} \varphi(s) H^* \left( \frac{2C_1 \varphi'(s)}{\epsilon_0 \varphi(s)} \right) \, ds.
\]

We remind that

\[
X(t) = \varphi'(t) \int_\Omega u(t,x) \partial_t u(t,x) + v(t,x) \partial_t v(t,x) \, dx \\
+ k_1 \varphi'(t) \int_\Omega \partial^2_t u(t,x) \partial_t v(t,x) - \partial^2_t v(t,x) \partial_t u(t,x) \, dx + \varphi(t) E_{u,v}(t) + k \varphi'(t) E_{\partial_t u, \partial_t v}(t),
\]

with \( k_1 = 8C_T \). Using Young’s inequality and Poincaré inequality, it is easy to see that

\[
X(t) \leq \left( \varphi(t) + \frac{\varphi'(t)}{\varphi(t)} (k_1 + 2\lambda^2 + 1) \right) E_{u,v}(t) + \varphi'(t) \left( k + \frac{k_1}{\delta} \right) E_{\partial_t u, \partial_t v}(t)
\]

\[
X(t) \geq \left( 1 - \frac{\varphi'(t)}{\varphi(t)} (k_1 + 2\lambda^2 + 1) \right) \varphi(t) E_{u,v}(t) + \varphi'(t) \left( k - \frac{k_1}{\delta} \right) E_{\partial_t u, \partial_t v}(t)
\]

(2.34)

So, taking \( k \) such that

\[
k \geq \max \left( \frac{8C_T}{\delta}, 64 \|a\|_\infty (C_T)^2 \|g'\|_{L^\infty} \right),
\]

using (2.33) and the fact that

\[
\left( \frac{1}{2} - C_1 \left( \frac{1}{m} + M^2 \right) \frac{\varphi'(t)}{\varphi(t)} \right) \geq 0 \text{ and } 1 - \frac{\varphi'(t)}{\varphi(t)} (8C_T + 2\lambda^2 + 1) \geq 0, \text{ for all } t \geq 0,
\]

we deduce that \( X(t) \geq 0 \) and

\[
X(t + T) + \frac{1}{T} \int_t^{t+T} \varphi'(s) E_{u,v}(s) \, ds \\
\leq X(t) + 8C_T \left( \sum_{i=0}^{n-1} E_{\partial_t^i u, \partial_t^i v}(0) \right) \int_t^{t+T} |\varphi''(s)| \, ds + \frac{1}{T} \int_t^{t+T} \varphi(s) H^* \left( \frac{2C_1 \varphi'(s)}{\epsilon_0 \varphi(s)} \right) \, ds.
\]

(2.35)

for all \( t \geq 0 \). Thus

\[
\sum_{i=0}^{n-1} \left( X((i + 1)T) - X(iT) + \frac{1}{T} \int_{iT}^{(i+1)T} \varphi'(s) E_{u,v}(s) \, ds \right) \\
\leq (8C_T + k + k_0) \left( \sum_{i=0}^{n-1} E_{\partial_t^i u, \partial_t^i v}(0) \right) \int_0^{nT} |\varphi''(s)| \, ds + \frac{1}{T} \int_0^{nT} \varphi(s) H^* \left( \frac{2C_1 \varphi'(s)}{\epsilon_0 \varphi(s)} \right) \, ds,
\]
and this gives

\[
X(nT) + \frac{1}{2} \int_0^{nT} \varphi'(s) E_{u,v}(s) \, ds \\
\leq X(0) + (8C_T + k + k_0) \left( \sum_{i=0}^1 E_{\partial_t u, \partial_t v}(0) \right) \int_0^{nT} |\varphi''(s)| \, ds \\
+ \frac{1}{2} \int_0^{nT} \varphi(s) H^* \left( \frac{2C_1 \varphi'(s)}{\epsilon_0 \varphi(s)} \right) \, ds, \quad \text{for all } n \in \mathbb{N}.
\]

Utilizing lemma 3, we conclude that there exists a positive constant \( C \) such that

\[
\int_{0}^{+\infty} \varphi'(s) E_{u,v}(s) \, ds \leq C \left( 1 + \sum_{i=0}^1 E_{\partial_t u, \partial_t v}(0) \right).
\]  

(2.36)

Since

\[
\varphi(t) E_{u,v}(t) \leq \varphi(0) E_{u,v}(0) + \int_{0}^{+\infty} \varphi'(s) E_{u,v}(s) \, ds, \quad t \geq 0,
\]

then (2.36), gives (1.14).

Finally, using the density of \( D(A) \) in \( \mathcal{H} \), we obtain (1.16).

**References**

[1] Alabau-Boussouira, Fatiha; Léautaud, Matthieu, Indirect controllability of locally coupled wave-type systems and applications. J. Math. Pures Appl. (9) 99 (2013), no. 5, 544–576.

[2] F. Alabau-Boussouira, M. Léautaud, Indirect stabilization of locally coupled wave-type systems, ESAIM Control Optim. Calc. Var. 18 (2012) 548–582.

[3] Alabau, Fatiha; Cannarsa, Piermarco; Komornik, Vilmos, Indirect internal stabilization of weakly coupled evolution equations. J. Evol. Equ. 2 (2002), no. 2, 127–150.

[4] Alabau-Boussouira, Fatiha, A unified approach via convexity for optimal energy decay rates of finite and infinite dimensional vibrating damped systems with applications to semi-discretized vibrating damped systems. J. Differ. Equations 248, No. 6, 1473-1517 (2010).

[5] Alabau-Boussouira, Fatiha, Wang Zhiqiang and Yu Lixin, A one-step optimal energy decay formula for indirectly nonlinearly damped hyperbolic systems coupled by velocities. ESAIM: COCV 23 (2017) 721-749.

[6] F. Alabau-Boussouira, New trends towards lower energy estimates and optimality for nonlinearly damped vibrating systems. J. Differential Equations, 249: 1145-1178, 2010.

[7] Aloui, L.; Daoulatli, M., Stabilization of two coupled wave equations on a compact manifold with boundary. J. Math. Anal. Appl. 436, No. 2, 944-969 (2016).

[8] Bardos, C., Lebeau, G., Rauch, J.: Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. SIAM J. Control Optim. 30, 1024–1065 (1992).

[9] L. Bociu and I. Lasiecka, Uniqueness of weak solutions for the semilinear wave equations with supercritical boundary/interior sources and damping, Discrete Contin. Dyn. Syst., 22 (2008), 835-860.

[10] Daoulatli, M. Behaviors of the energy of solutions of the wave equation with damping and external force. J. Math. Anal. Appl. 389 (2012), no. 1, 205–225.

[11] Daoulatli, M. Energy decay rates for solutions of the wave equation with linear damping in exterior domain. Evol. Equ. Control Theory 5 (2016), no. 1, 37–59.

[12] Dehman, Belhassen; Le Rousseau, Jérôme; Léautaud, Matthieu, Controllability of two coupled wave equations on a compact manifold. Arch. Ration. Mech. Anal. 211 (2014), no. 1, 113–187.

[13] X. Fu, Sharp decay rates for the weakly coupled hyperbolic system with one internal damping, SIAM J. CONTROL OPTIM., 50 (2012), 1643-1660

[14] I. Lasiecka and D. Tataru, Uniform boundary stabilization of semi-linear wave equation with nonlinear boundary dissipation, Differential Integral Equations 6 (1993), 507–533.

[15] Léautaud, M. Spectral inequalities for non-selfadjoint elliptic operators and application to the null-controllability of parabolic systems. J. Funct. Anal. 258 (2010), no. 8, 2739–2778.
[16] J.-L. Lions. Controlabilite exacte, perturbations et stabilisation de systemes distribues. Tome 1, volume 8 of Recherches en Mathematiques Appliquees. Masson, Paris, 1988.

[17] M. Nakao, Energy decay for the linear and semilinear wave equations in exterior domains with some localized dissipations, Math. Z. 238 (2001) 781–797.

[18] Rauch, J., Taylor, M.: Exponential decay of solutions to hyperbolic equations in bounded domains. Indiana Univ. Math. J. 24, 79–86 (1974).

[19] R. T. Rockafellar, Convex Analysis Princeton University Press, Princeton, NJ, 1970.

University of Dammam, King Saudi Arabia & University of Carthage, Tunisia

E-mail address, M. Daoulatli: moez.daoulatli@infcom.rnu.tn