STRICHTHARTZ AND UNIFORM SOBOLEV INEQUALITIES
FOR THE ELASTIC WAVE EQUATION

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Abstract. We prove dispersive estimate for the elastic wave equation by which we extend the known Strichartz estimates for the classical wave equation to those for the elastic wave equation. In particular, the endpoint Strichartz estimates are deduced. For the purpose we diagonalize the symbols of the Lamé operator and its semigroup, which also gives an alternative and simpler proofs of the previous results on perturbed elastic wave equations. Furthermore, we obtain uniform Sobolev inequalities for the elastic wave operator.

1. Introduction

Let \( n \geq 2 \) and let \( f, g : \mathbb{R}^n \rightarrow \mathbb{C}^n \) and \( F : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{C}^n \) be vector fields. We consider the inhomogeneous elastic wave equation

\[
\begin{aligned}
\left\{
\begin{array}{l}
(\partial_t^2 - \Delta^*)u(t, x) = F(t, x), \\
u(0, x) = f(x), \quad \partial_t u(0, x) = g(x),
\end{array}
\right.
\end{aligned}
\tag{1.1}
\]

where \( \Delta^* \) denotes the Lamé operator (acting in the spatial variable \( x \)) defined by

\[
\Delta^* u = \mu \Delta u + (\lambda + \mu)\nabla \text{div} u,
\]

and the Laplacian \( \Delta \) acts on a vector field component-wise. The Lamé constants \( \lambda, \mu \in \mathbb{R} \) satisfy the standard condition

\[
\mu > 0, \quad \lambda + 2\mu > 0,
\tag{1.2}
\]

which guarantees the ellipticity of \( \Delta^* \). The equation has been used to model wave propagation in an elastic medium, where \( u \) denotes the displacement field of the medium (see e.g., [23, 24]).
**Dispersive estimate and Strichartz estimate.** We begin by introducing the notations. For a vector-valued function \( f = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{C}^n \) we use the norms

\[
\|f\|_{L^r(\mathbb{R}^n)} = \left\{ \frac{\sum_{j=1}^n \|f_j\|_{L^r(\mathbb{R}^n)}^r}{\max_{1 \leq j \leq n} \|f_j\|_{L^\infty(\mathbb{R}^n)}} \right\}^{\frac{1}{r}}, \quad 1 \leq r < \infty,
\]

\[
\|f\|_{L^\infty(\mathbb{R}^n)} = \max_{1 \leq j \leq n} \|f_j\|_{L^\infty(\mathbb{R}^n)}, \quad r = \infty.
\]

(1.3)

For a time-dependent vector field \( u = (u_1, \ldots, u_n) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}^n \) we denote

\[
\|u\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^n)} = \left\| \|u(t, \cdot)\|_{L^r(\mathbb{R}^n)} \right\|_{L^q_t(\mathbb{R})}.
\]

Besides, we use the standard notations \( \|f\|_{H^s_x} = \|\nabla^s f\|_{L^r_x} \) and \( \|f\|_{\dot{H}^s_x} = \|f\|_{\dot{H}^s_x} \).

For \( q, r \geq 2, r \neq \infty \) and \( (q, r, n) \neq (2, \infty, 3) \), we say \((q, r)\) is wave-admissible if \( \frac{1}{q} = \frac{n-1}{2} \left( 1 - \frac{1}{r} \right) \). We also call \((q, r)\) sharp wave-admissible if \( \frac{1}{q} = \frac{n-1}{2} \left( 1 - \frac{1}{r} \right) \).

Our first result is the following dispersive estimate.

**Theorem 1.1.** Suppose \( \hat{f} \) is supported in the annulus \( \{ \xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2 \} \). Then

\[
\|e^{it\sqrt{-\Delta}} f\|_{L^\infty(\mathbb{R}^n)} \lesssim |t|^{-\frac{n-1}{4}} \|f\|_{L^1(\mathbb{R}^n)}.
\]

(1.4)

Once we have the dispersive estimate (1.4) we can prove the Strichartz estimate for the elastic wave equation whenever \((q, r)\) is wave-admissible using the abstract framework due to Keel–Tao [15]. Indeed, we combine the frequency-localized dispersive estimate (1.4) and the \( L^2 \) estimate (see (3.3) in Section 3) to get the estimate for frequency-localized initial data, then by scaling and the Littlewood–Paley inequality we obtain the homogeneous Strichartz estimates with an arbitrary initial data \((f, g) \in \dot{H}^s(\mathbb{R}^n) \times \dot{H}^{s-1}(\mathbb{R}^n)\). The estimates and the standard \( TT^* \)-argument give the following.

**Theorem 1.2.** Let \((q, r)\) and \((\tilde{q}, \tilde{r})\) be wave-admissible pairs with \( r, \tilde{r} < \infty \). If \( u \) is a solution to the Cauchy problem (1.1), then we have

\[
\|u\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^n)} \lesssim \|f\|_{\dot{H}^s(\mathbb{R}^n)} + \|g\|_{\dot{H}^{s-1}(\mathbb{R}^n)} + \|F\|_{L_t^{\tilde{q}} L_x^{\tilde{r}}(\mathbb{R} \times \mathbb{R}^n)}
\]

(1.5)

provided that

\[
\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - s = \frac{1}{\tilde{q}} + \frac{n}{\tilde{r}} - 2.
\]

Remark 1.3. When \( f = g = 0 \) in (1.1), the estimate (1.5) is called the inhomogeneous Strichartz estimate, which holds for a wider range of pairs \((q, r)\) and \((\tilde{q}, \tilde{r})\) than that of the wave-admissible pairs. As the precise description of the pairs is rather complicated, we provide the detailed statement in Section 3.1 (see Theorem 3.1).

The estimates (1.4) and (1.5) are in complete analogue with the dispersive estimate and Strichartz estimate, respectively, for the classical wave equation

\[
\begin{align*}
\left\{ \begin{array}{l}
\partial_t^2 u - \Delta u(t, x) = F(t, x), \\
u(0) = f, \quad \partial_t u(0) = g.
\end{array} \right.
\end{align*}
\]

(1.6)

In context of the wave equation (1.6) all the above results are standard, and there has been a large body of literature concerning the Strichartz estimates for (1.6). Among others, the diagonal case \( q = r \) was obtained in [30] in connection with the restriction
theorems for the cone. This was later extended to mixed norms $L^q_t L^r_x$ independently by Ginibre and Velo \[11\] and Lindblad and Sogge \[24\]. The remaining endpoints where $q = 2$ were later settled by Keel and Tao \[15\].

**Perturbed elastic wave equation.** Now we turn to the perturbed elastic wave equation

$$\begin{aligned}
\begin{cases}
(\partial^2_t - \Delta + V(x))u(t, x) = 0, \\
u(0, x) = f(x), \quad \partial_t u(0, x) = g(x).
\end{cases}
\end{aligned} \tag{1.7}$$

In \[11\] Barceló et al. studied \[1.7\] in three spatial dimension $n = 3$. Under the assumptions that $V : \mathbb{R}^3 \to \mathcal{M}_{3 \times 3}(\mathbb{R})$ is symmetric, $|x|^2 |V(x)| \leq c$ for a small constant $c$, $\mu > 0$ and $3\lambda + 2\mu > 0$, they obtained the Strichartz estimates for \(1.7\)

$$\|u\|_{L^q_t H^{\frac{1}{2}}_r} \leq \|f\|_{H^{\frac{1}{2}}_r} + \|g\|_{H^{-\frac{1}{2}}_r}$$

whenever $(q, r)$ is sharp wave-admissible.

In \[17\] the authors improved the result to a more general class of potentials $V$ under the weaker assumption \[1.2\] on the Lamé coefficients. To facilitate the statement, let us recall the Fefferman–Phong class defined by

$$\mathcal{F}^p := \left\{ V : \mathbb{R}^n \to \mathcal{M}_{n \times n}(\mathbb{C}) : \|V\|_{\mathcal{F}^p} = \sup_{x \in \mathbb{R}^n, r > 0} r^{2 - \frac{n}{p}} \left( \int_{B(x, r)} |V(y)|^p dy \right)^{\frac{1}{p}} < \infty \right\},$$

for $1 \leq p \leq n/2$. Here $B(x, r)$ denotes the open ball in $\mathbb{R}^n$ centered at $x$ with radius $r$ and $|V| = \left( \sum_{i,j=1}^n |V_{ij}|^2 \right)^{1/2}$. If $1 \leq p < n/2$ the class $\mathcal{F}^p$ includes the weak space $L^{\frac{n}{2}}$, and in particular, the critical inverse-square potential $|x|^{-2}$. For related results on the classical wave equation, see \[4, 5, 18\] and references therein.

**Theorem 1.4 \[17\].** Let $n \geq 3$ and $V \in \mathcal{F}^p$ for $p > \frac{n-1}{2}$. Let $u$ be a solution to \(1.7\) with $(f, g) \in \dot{H}^{1/2}(\mathbb{R}^n) \times \dot{H}^{-1/2}(\mathbb{R}^n)$. If $\|V\|_{\mathcal{F}^p} \leq c$ for $c > 0$ small enough, then

$$\|u\|_{L^q_t H^\sigma} \lesssim \|f\|_{H^{1/2}} + \|g\|_{H^{-1/2}}$$

whenever $(q, r)$ is wave-admissible, $q > 2$ and $\sigma = \frac{1}{q} + \frac{n}{r} - \frac{n-1}{2}$.

In \[17\], instead of using the spectral theoretic approach of \[1\], the authors took an approach based on harmonic analysis technique. They focused on the Fourier multiplier of $\sqrt{-\Delta + V}$ after viewing the solution as a sum of the Duhamel term given by $V(x)u(t, x)$ and the solution to the free case.

The approach in \[17\] is significantly different from that in \[1\] and leads to improvements on assumptions on $\lambda$, $\mu$ and $V$. However, it still has a similar flavor in that it followed a strategy making use of the Helmholtz decomposition (i.e., Leray projection) of vector fields $f = f_S + f_P$, where $f_S$ is a divergence-free field and $f_P$ is a gradient field. To carry out the strategy, it was necessary to control $\|f_S\|_{H^s} + \|f_P\|_{H^s}$ and $\|F_S\|_{L^2(u)} + \|F_P\|_{L^2(u)}$ with $\|f\|_{H^s}$ and $\|F\|_{L^2(u)}$, respectively. Thus, the authors had to use $L^2$-orthogonality between the Leray projections (\[17\] Lemma 2.1) and elliptic regularity estimates (\[17\] Lemma 4.1)].
In this paper, we develop a new approach to analyze (1.1) and (1.7). Rather than relying on the Helmholtz decomposition of vector fields, we make use of diagonalization of the Lamé operator. As the rotations resulting from the diagonalization process are smooth and homogeneous of degree zero, they satisfy the Mikhlin condition, hence, the classical multiplier theorems become available. Therefore we can utilize the estimates for the classical wave equations. The diagonalization argument completely replaces the role of the Helmholtz decomposition, for instance, we can prove Theorem 1.4 in a simpler way without using the Helmholtz decomposition (see Section 5). Similarly, the proofs of all the other results in [1, 17] can be simplified. This direct and Fourier-analytic approach is likely to be useful in different problems because neither any orthogonality property of the Leray projections nor elliptic regularity estimate is necessary.

Uniform Sobolev inequality. We can adapt the diagonalization method to obtain the uniform Sobolev inequality for the elastic wave operator $\partial_t^2 - \Delta^*$. For the wave operator $\partial_t^2 - \Delta$, Kenig, Ruiz and Sogge [16] proved the uniform Sobolev inequality
\[
\|u\|_{L^q(\mathbb{R}^{1+n})} \leq C\|\partial_t^2 - \Delta + a \cdot \nabla + z\|_{L^p(\mathbb{R}^{1+n})} \tag{1.8}
\]
with $C$ independent of $a \in C^{1+n}$ and $z \in C$, when $(p, q) = \left(\frac{2(n+1)}{n+3}, \frac{2(n+1)}{n-1}\right)$. As a consequence, they obtained a unique continuation result for $|\nabla (\partial_t^2 - \Delta) u| \leq |Vu|$ via the Carleman inequality
\[
\|e^{v(t,x)} u\|_{L^q(\mathbb{R}^{1+n})} \leq C\|e^{v(t,x)} (\partial_t^2 - \Delta) u\|_{L^p(\mathbb{R}^{1+n})}, \tag{1.9}
\]
which follows from (1.8). Later, the range of $p, q$ on which the uniform Sobolev inequality (1.8) and Carleman inequality (1.9) hold was completely characterized in [13] and [14], respectively, where the authors proved that both (1.8) and (1.9) hold if and only if
\[
\frac{1}{p} - \frac{1}{q} = \frac{2}{n+1}, \quad \frac{2n(n+1)}{n^2 + 4n - 1} < p < \frac{2n}{n+1}.
\]
In proving the uniform Sobolev inequality (1.8), uniform resolvent estimate (1.8) with $a = 0$ that is seemingly weaker than (1.8) is a main ingredient. So, the two are more or less equivalent.

However, three of the authors [22] recently proved that if the wave operator $\partial_t^2 - \Delta$ is replaced by the $((n+1)$-dimensional) Lamé operator $\Delta^*_{\mathbb{R}^{n+1}}$ in the above (1.8) and (1.9)), then the uniform Sobolev inequality (1.8) and Carleman inequality (1.9) fail, while the uniform (and even non-uniform sharp) resolvent estimates are available in the general context of [21] (also, see [2, 8]). This shows a fundamental difference between $\Delta^*$ and $\Delta$.

We aim to investigate in this direction for the elastic wave operator $\partial_t^2 - \Delta^*$.

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1If we consider the Laplacian $\Delta_{\mathbb{R}^{1+n}}$ instead of $\partial_t^2 - \Delta$, the ranges of $p, q$ for the uniform Sobolev and Carleman inequalities do not coincide. We refer the interested readers to [14] for details.
Theorem 1.5. There exists $C$ independent of $(a, z) \in \mathbb{C} \times \mathbb{C}$ such that

$$
\|u\|_{L^p(\mathbb{R}^{1+n})} \leq C\|\partial_t^2 - \Delta^* + a\partial_t - z\|_{L^p(\mathbb{R}^{1+n})}
$$

if and only if

$$
\frac{1}{p} - \frac{1}{q} = \frac{2}{n+1}, \quad \frac{2n(n+1)}{n^2 + 4n - 1} < p < \frac{2n}{n+1}.
$$

We close the introduction with a few remarks. By the famous argument in [16] certain unique continuation property follows from the uniform Sobolev inequality (1.10). See Corollary 6.1 in Section 6. In view of [22], if the first order spatial derivatives $\nabla_x$ is involved in the right side of (1.10), it seems that any uniform estimate cannot hold true. However, we do not pursue this issue here.

Organization. In the next section, we diagonalize the Lamé operator. In Section 3, we prove the dispersive estimate (Theorem 1.1). In Section 4, we prove the Strichartz inequalities (Theorem 1.2). In Section 5, we prove Theorem 1.4. In the last section, we prove Theorem 1.5 and deduce a unique continuation property of the elastic wave operator $\partial_t^2 - \Delta^* + V$.

2. Diagonalization of the Lamé operator

In this section, we diagonalize $\sqrt{-\Delta^*}$. To do so, we need to rotate the associated multiplier and decompose the frequency space into two parts to make the involved rotations well-defined.

2.1. Rotations in the frequency space. Let $e_1 = (1, 0, \ldots, 0)^t \in \mathbb{R}^n$ and $S_\pm = \{\omega \in S^{n-1} : -\frac{1}{\sqrt{2}} \leq \omega \cdot (\pm e_1) \leq 1\}$. For every $\omega \in S_\pm \setminus \{\pm e_1\}$ we set

$$
C_\pm(\omega) = S_\pm \cap \text{span}\{e_1, \omega\}.
$$

In other words, $C_\pm(\omega)$ is the intersection of $S_\pm$ and the great circle passing through $e_1$ and $\omega$. We take $\rho_\pm(\omega) \in SO(n)$ so that its transpose $(\rho_\pm(\omega))^t$ is the unique rotation mapping $\omega$ to $\pm e_1$ along the arc $C_\pm(\omega)$ and satisfying $(\rho_\pm(\omega))^t y = y$ whenever $y \in \text{span}\{e_1, \omega\}^\perp$. When $\omega = \pm e_1$ we set $\rho_\pm(\omega) = I_n$.

It is clear that the mapping $\rho_\pm : S_\pm \to SO(n)$ is smooth and $\omega = \pm \rho_\pm(\omega) e_1$ if $\omega \in S_\pm$. Now, let us set $\mathbb{R}_\pm^n = \{\xi \in \mathbb{R}^n \setminus \{0\} : \xi/|\xi| \in S_\pm\}$ and define $R_\pm : \mathbb{R}_\pm^n \to SO(n)$ by

$$
R_\pm(\xi) = \rho_\pm(\xi/|\xi|), \quad \xi \in \mathbb{R}_\pm^n.
$$

In each of $\mathbb{R}_+^n$ and $\mathbb{R}_-^n$, the matrices $R_+$ and $R_-$ satisfy the Mikhlin type condition:

**Lemma 2.1.** The mapping $R_\pm : \mathbb{R}_\pm^n \to SO(n)$ is smooth and homogeneous of degree zero. Thus, every $(j, k)$-component $r_{jk}^\pm(\xi)$ of $R_\pm(\xi)$ satisfies

$$
|\partial_\xi^\alpha r_{jk}^\pm(\xi)| \lesssim |\xi|^{-|\alpha|}, \quad \xi \in \mathbb{R}_\pm^n
$$

for all multi-indices $\alpha \in \mathbb{N}_0^n$. 

Proof. By the definition of $R_{\pm}$ and our choice of $\rho_{\pm}$, the smoothness, homogeneity and boundedness of $r_{j,k}^{\pm}$ are clear. Hence, we need only to show the inequality \eqref{2.1} when $|\alpha| > 0$. By the standard spherical coordinate $(r, \theta) \rightarrow \xi$ and the chain rule, we see that

$$\frac{\partial}{\partial \xi_j} = a_j(\theta) \frac{\partial}{\partial r} + A_j(\theta, \partial_{\theta}) \frac{1}{r},$$

where $a_j$ and $A_j$ are smooth functions independent of $r$. The inequality \eqref{2.1} follows by homogeneity.

Consequently, Mikhlin’s multiplier theorem imply the following. Let $\{\varphi_+, \varphi_-\}$ be a smooth partition of unity on the unit sphere $S^{n-1}$ subordinate to the covering $\{\text{int} S_+, \text{int} S_-\}$ and let $\mathcal{P}_+$ and $\mathcal{P}_-$ be projections to $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$, respectively, defined by

$$\hat{\mathcal{P}}_{\pm} f(\xi) = \varphi_{\pm}(\xi/|\xi|) \hat{f}(\xi), \quad \xi \in \mathbb{R}^n \setminus \{0\}.$$ 

We denote $D = -i\nabla$ and $m(D) f(x) = (m\hat{f})^\vee(x)$.

\textbf{Lemma 2.2.} Let $1 < r < \infty$. Then, for any vector-valued function $f$, we have

$$\|R_{\pm}(D) \mathcal{P}_{\pm} f\|_{L^r(\mathbb{R}^n)} \approx \|\mathcal{P}_{\pm} f\|_{L^r(\mathbb{R}^n)} \lesssim \|f\|_{L^r(\mathbb{R}^n)}. \quad (2.2)$$

\textit{Proof.} The last inequality in the above is a direct consequence of Mikhlin’s multiplier theorem. It follows from the definition \eqref{1.3}, Lemma 2.1 and Mikhlin’s multiplier theorem that

$$\|R_{\pm}(D) \mathcal{P}_{\pm} f\|_{L^r(\mathbb{R}^n)} \lesssim \|\mathcal{P}_{\pm} f\|_{L^r(\mathbb{R}^n)}.$$ 

Considering $(R_{\pm}(\xi))^t$ instead of $R_{\pm}(\xi)$ we also get the reverse inequality

$$\|\mathcal{P}_{\pm} f\|_{L^r(\mathbb{R}^n)} \lesssim \|R_{\pm}(D) \mathcal{P}_{\pm} f\|_{L^r(\mathbb{R}^n)},$$

which shows the first estimate in \eqref{2.2}. \hfill \Box

2.2. Diagonalization. By taking the Fourier transform it is easy to see that the multiplier associated with the differential operator $-\Delta^* - z$, $z \in \mathbb{C}$, is the matrix-valued function $L_z: \mathbb{R}^n \rightarrow \mathcal{M}_{n \times n}(\mathbb{C})$ defined by

$$L_z(\xi) := (\mu|\xi|^2 - z)I_n + (\lambda + \mu)\xi \xi^t.$$ 

See \cite{22} Proof of Lemma 2.1. In particular, the matrix $L(\xi) := L_0(\xi)$ is the multiplier of $-\Delta^*$.

If we write $\xi \in \mathbb{R}^n_+$ as $\xi = |\xi|\omega$ with $\omega \in S_\pm$ we have $(R_{\pm}(\xi))^t \xi = |\xi|(\rho_{\pm}(\omega))^t \omega = \pm |\xi|e_1$ by the definition of $\rho_{\pm}$. Hence we see that, if $\xi \in \mathbb{R}^n_+$ then

$$(R_{\pm}(\xi))^t L_z(\xi) R_{\pm}(\xi) = (\mu|\xi|^2 - z)(R_{\pm}(\xi))^t R_{\pm}(\xi) + (\lambda + \mu)(R_{\pm}(\xi))^t \xi \xi^t R_{\pm}(\xi)$$

$$= (\mu|\xi|^2 - z)I_n + (\lambda + \mu)|\xi|^2 e_1 e_1^t$$

$$= \text{diag}(\lambda + 2\mu|\xi|^2 - z, \mu|\xi|^2 - z, \ldots, \mu|\xi|^2 - z),$$

where $\text{diag}(a_1, \ldots, a_n)$ denotes the $n \times n$ diagonal matrix whose $j$-th diagonal entry is $a_j$. Thus, for any $\xi \in \mathbb{R}^n$

$$\det(L(\xi) - zI_n) = ((\lambda + 2\mu)|\xi|^2 - z)(\mu|\xi|^2 - z)^{n-1},$$
so the eigenvalues of $L(\xi)$ are $(\lambda + 2\mu)|\xi|^2$ and $\mu|\xi|^2$. The latter is of multiplicity $n - 1$. Setting $\Lambda(\xi) := (R_{\pm}(\xi))^{t} L(\xi) R_{\pm}(\xi) = \text{diag}((\lambda + 2\mu)|\xi|^2, \mu|\xi|^2, \ldots, \mu|\xi|^2)$ we have

$$L(\xi) = R_{\pm}(\xi)(R_{\pm}(\xi))^{t} L(\xi) R_{\pm}(\xi)(R_{\pm}(\xi))^{t} = R_{\pm}(\xi)\Lambda(\xi)(R_{\pm}(\xi))^{t}$$

on $\mathbb{R}^{n}_{\pm}$, and therefore

$$-\Delta = L(D) = \sum_{\pm} L(D)P_{\pm} = \sum_{\pm} R_{\pm}(D)\Lambda(D)(R_{\pm}(D))^{t}P_{\pm}.$$  \hspace{1cm} (2.3)

Clearly, we can take the square root in the above equations under the assumption $|\xi| > 1$. So, $\sqrt{L}(\xi) = R_{\pm}(\xi)\sqrt{\Lambda}(\xi)(R_{\pm}(\xi))^{t}$ on $\mathbb{R}^{n}_{\pm}$ and

$$\sqrt{-\Delta} = \sqrt{L}(D) = \sum_{\pm} \sqrt{L}(D)P_{\pm} = \sum_{\pm} R_{\pm}(D)\sqrt{\Lambda}(D)(R_{\pm}(D))^{t}P_{\pm},$$  \hspace{1cm} (2.4)

where $\sqrt{\Lambda}(\xi) = \text{diag}(\sqrt{\lambda + 2\mu}|\xi|, \sqrt{\mu}|\xi|, \ldots, \sqrt{\mu}|\xi|)$.

3. Proofs of the dispersive estimate and Strichartz estimate

Considering the matrix exponential, the above diagonalization process enables us to express the semigroup as

$$e^{it\sqrt{-\Delta}} = \sum_{\pm} e^{it\sqrt{\Lambda}(D)}P_{\pm} = \sum_{\pm} R_{\pm}(D)e^{it\sqrt{\Lambda}(D)}(R_{\pm}(D))^{t}P_{\pm},$$  \hspace{1cm} (3.1)

where

$$e^{it\sqrt{\Lambda}(D)} = \text{diag}(e^{it\sqrt{-(\lambda + 2\mu)|\xi|}}, e^{it\sqrt{-\mu|\xi|}}, \ldots, e^{it\sqrt{-\mu|\xi|}}).$$

Let $P_0$ be a projection defined by $P_{0} f(\xi) = \beta(|\xi|)\hat{f}(\xi)$ where $\beta$ is a smooth function supported in the interval $(1/4, 4)$ such that $0 \leq \beta \leq 1$ and $\beta(t) = 1$ if $t \in [1/2, 2]$. As the functions

$$\xi \mapsto \beta(|\xi|)\rho_{\pm}(\xi/|\xi|)\varphi_{\pm}(\xi/|\xi|), \quad \xi \mapsto (\rho_{\pm}(\xi/|\xi|))^{t}\varphi_{\pm}(\xi/|\xi|)\beta(|\xi|)$$

are smooth and compactly supported we have the uniform bounds

$$\|P_{0}R_{\pm}(D)P_{\pm}h\|_{L^{p}} \lesssim \|h\|_{L^{p}}, \quad \|(R_{\pm}(D))^{t}P_{\pm}P_{0}h\|_{L^{q}} \lesssim \|h\|_{L^{q}}$$  \hspace{1cm} (3.2)

for any $1 \leq p \leq q \leq \infty$. Therefore, if $\hat{f}$ is supported in the annulus $\{\xi : 2/3 \leq |\xi| \leq 2\}$, it follows from (3.1) and (3.2) that

$$\|e^{it\sqrt{-\Delta}}P_{0}f\|_{L^{\infty}(\mathbb{R}^{n})} \lesssim \sum_{\pm} \|P_{0}R_{\pm}(D)e^{it\sqrt{\Lambda}(D)}(R_{\pm}(D))^{t}P_{\pm}P_{0}f\|_{L^{\infty}(\mathbb{R}^{n})} \lesssim \sum_{\pm} \|e^{it\sqrt{\Lambda}(D)}(R_{\pm}(D))^{t}P_{\pm}P_{0}f\|_{L^{\infty}(\mathbb{R}^{n})}.$$  

By the well-known dispersive estimate for $e^{it\sqrt{-\Delta}}$ and (3.2), this is estimated by

$$C \sum_{\pm} |t|^{-\frac{n-4}{2}} \|(R_{\pm}(D))^{t}P_{\pm}P_{0}f\|_{L^{1}(\mathbb{R}^{n})} \lesssim |t|^{-\frac{n-4}{2}} \|f\|_{L^{1}(\mathbb{R}^{n})},$$

and the proof of Theorem 1.1 is complete.
On the other hand, by Lemma 2.2 and the Plancherel theorem it is easy to see that
\[ \|e^{it\sqrt{-\Delta}} f\|_{L^2(\mathbb{R}^n)} \leq \sum_{\pm} \|R_{\pm}(D)e^{it\sqrt{\mathcal{A}}}(R_{\pm}(D))'P_{\pm}f\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)}. \] (3.3)
Therefore, by the well-known theorem of Keel–Tao in [13], combining the $L^2$ estimate and the dispersive estimate (1.4) implies Theorem 1.2.

3.1. Further results on the inhomogeneous Strichartz estimates. As was mentioned earlier in Remark 1.3, if $f = g = 0$ then the estimates (1.5) are available for a wider range of the Lebesgue exponent pairs $(q, r)$ and $(\tilde{q}, \tilde{r})$.

For the classical wave equation (1.6), this phenomenon has been observed by Harmse [12] and Oberlin [27] for the diagonal case $q = r$ and $\tilde{q} = \tilde{r}$. Later, Foschi [10] followed the scheme of Keel–Tao [13] and obtained the inhomogeneous estimates for the currently known widest range with $q \neq r$ and $\tilde{q} \neq \tilde{r}$.

Furthermore, Taggart [31] obtained more estimates involving the Besov spaces. More recently, Bez, Cunanan and the third author [3] obtained certain weak type (in temporal variable) estimates in borderline cases. All of these results are essentially based on the dispersive estimate (1.4).

In this direction, analogous results are also available for the elastic wave equation (1.1) as we now have the dispersive estimate (1.4). To facilitate the description let us say that $(q, r)$ is wave-acceptable if

\[ 1 \leq q < \infty, \quad 2 \leq r \leq \infty, \quad \frac{1}{q} < (n - 1)\left(\frac{1}{2} - \frac{1}{r}\right), \quad \text{or} \quad (q, r) = (\infty, 2). \]

Applying the result of Foschi [10, Theorem 1.4] combined with the $L^2$ estimate (3.3) and the dispersive estimate (1.4) we have the following.

**Theorem 3.1.** Let $(q, r)$ and $(\tilde{q}, \tilde{r})$ be wave-acceptable and suppose that $r, \tilde{r} < \infty$\(^2\) and

\[ \frac{1}{q} + \frac{1}{r} = \frac{n - 1}{2}(1 - \frac{1}{r} - \frac{1}{\tilde{r}}). \] (3.4)

If $n > 3$ we further assume that

\[
\begin{align*}
\left\{ \begin{array}{ll}
\frac{n - 3}{r} \leq \frac{n - 1}{r}, & \frac{n - 3}{r} \leq \frac{n - 1}{r} \\
\frac{n - 3}{r} < \frac{n - 1}{r}, & \frac{n - 3}{r} < \frac{n - 1}{r}
\end{array} \right. & \quad \text{when} \quad \frac{1}{q} + \frac{1}{r} < 1,
\end{align*}

\[
\left\{ \begin{array}{ll}
\frac{n - 3}{r} < \frac{n - 1}{r}, & \frac{n - 3}{r} < \frac{n - 1}{r} \\
\frac{n - 3}{r} \leq \frac{n - 1}{r}, & \frac{n - 3}{r} \leq \frac{n - 1}{r}
\end{array} \right. & \quad \text{when} \quad \frac{1}{q} + \frac{1}{r} = 1.
\]

Then, we have

\[ \left\| \int_0^t \sin \left( (t - s)\sqrt{-\Delta}\right) \sqrt{-\Delta}^{-1} F(s, \cdot) ds \right\|_{L^q_t L^r_x(\mathbb{R}^n)} \lesssim \|F\|_{L_{t,x}^{q', r'}(\mathbb{R}^n \times \mathbb{R}^n)}. \] (3.5)

\(^2\)For related results on the Schrödinger equation we refer to [32, 19, 20].

\(^3\)In distinction to the statement of [10, Theorem 1.4], the condition $r, \tilde{r} < \infty$ is necessary in all dimensions since we have the frequency-localized dispersive estimate (1.4) and need to use the Littlewood–Paley inequalities to obtain global estimates.
Remark 3.2. If \((1/q^*, 1/r^*)\) is the midpoint between the points \((1/q, 1/r)\) and \((1/\tilde{q}, 1/\tilde{r})\) in the theorem, then it is a sharp wave-admissible pair. This fact follows from the gap condition \((3.4)\) since \(1/q + 1/\tilde{q} \leq 1\).

4. AN ALTERNATIVE PROOFS OF THEOREMS 1.2 AND 3.1

It would be interesting to notice that the diagonalization argument provides another proofs of Theorems 1.2 and 3.1 without passing through the dispersive estimate \((4.4)\).

Let us first consider the homogeneous part of \((1.1)\)

\[
\begin{cases}
(\partial_t^2 - \Delta^*)u(t, x) = 0, \\
u(0, x) = f(x), \quad \partial_t u(0, x) = g(x),
\end{cases}
\]

and prove

\[
\|u\|_{L_t^1 L_x^\infty(\mathbb{R} \times \mathbb{R}^n)} \lesssim \|f\|_{\dot{H}^{s}(\mathbb{R}^n)} + \|g\|_{\dot{H}^{-s}(\mathbb{R}^n)}
\]

for \(q, r\) and \(s\) given as in Theorem 1.2.

We break \((4.1)\) into

\[
\begin{cases}
(\partial_t^2 - \Delta^*)u_\pm(t, x) = 0, \\
u_\pm(0, x) = \mathcal{P}_\pm f(x), \quad \partial_t u_\pm(0, x) = \mathcal{P}_\pm g(x).
\end{cases}
\]

By the diagonalization \((2.4)\) the solutions \(u_\pm\) are written as\(^4\)

\[
\begin{align*}
\hat{u}_\pm &= \cos(t R_\pm \sqrt{\Lambda} R^t_\pm) \mathcal{P}_\pm f + \sin(t R_\pm \sqrt{\Lambda} R^t_\pm)(R_\pm \sqrt{\Lambda} R^t_\pm)^{-1} \hat{\mathcal{P}}_\pm g, \\
&= R_\pm \left( \cos(t \sqrt{\Lambda}) R^t_\pm \mathcal{P}_\pm f + \sin(t \sqrt{\Lambda}) \sqrt{\Lambda}^{-1} R^t_\pm \hat{\mathcal{P}}_\pm g \right),
\end{align*}
\]

from which it is easy to see that \(\text{supp} \hat{u}_\pm \subset \mathbb{R}^n_+\). It is also clear that \(u = u_+ + u_-\).

Now it is straightforward to prove the estimate \((4.2)\). Indeed, by Lemma 2.2 and the classical homogeneous Strichartz estimate \(30\) \(11\) \(15\) we deduce

\[
\|u\|_{L_t^1 L_x^\infty(\mathbb{R} \times \mathbb{R}^n)} \leq \|u_+\|_{L_t^1 L_x^\infty(\mathbb{R} \times \mathbb{R}^n)} + \|u_-\|_{L_t^1 L_x^\infty(\mathbb{R} \times \mathbb{R}^n)}
\]

\[
\lesssim \sum_{\pm} \|\cos(t \sqrt{\Lambda}(D))(R_\pm (D))^t \mathcal{P}_\pm f + \sin(t \sqrt{\Lambda}(D)) \sqrt{\Lambda}^{-1} (D)(R_\pm (D))^t \mathcal{P}_\pm g\|_{L_t^1 L_x^\infty(\mathbb{R} \times \mathbb{R}^n)}
\]

\[
\lesssim \sum_{\pm} \|(R_\pm (D))^t \mathcal{P}_\pm f\|_{\dot{H}^{s}(\mathbb{R}^n)} + \|(R_\pm (D))^t \mathcal{P}_\pm g\|_{\dot{H}^{-s}(\mathbb{R}^n)}
\]

\[
\lesssim \|f\|_{\dot{H}^{s}(\mathbb{R}^n)} + \|g\|_{\dot{H}^{-s}(\mathbb{R}^n)}.
\]

The last inequality follows from Lemma 2.2 since \(|\xi|^s I_n\) (i.e., the multiplier of \(|\nabla|^s\) acting on \(n\)-dimensional vector-valued functions) commutes with all matrices.

It remains to consider the inhomogeneous part of \((1.1)\)

\[
\begin{cases}
(\partial_t^2 - \Delta^*)u(t, x) = F(t, x), \\
u(0, x) = 0, \quad \partial_t u(0, x) = 0,
\end{cases}
\]

\(^4\)From now on, for notational convenience, we sometimes suppress the frequency variable \(\xi\) when doing so does not cause confusion.
and prove the inhomogeneous estimates \(3.5\). The strategy is similar to the homogeneous part. As before we break \(4.4\) into

\[
\left\{ \begin{array}{l}
(\partial_t^2 - \Delta^\ast)u_\pm(t, x) = \mathcal{P}_\pm F(t, x), \\
u_\pm(0, x) = 0, \quad \partial_t u_\pm(0, x) = 0.
\end{array} \right.
\]

By the diagonalization \(2.4\) and Duhamel’s formula we have

\[
\tilde{u}_\pm(t) = \int_0^t \sin \left( (t - s) \sqrt{\Lambda} R_\pm \right) R_\pm^{-1} \mathcal{P}_\pm F(s) ds
\]

so it is clear that \(\text{supp} \tilde{u}_\pm \subset \mathbb{R}^+\) and \(u = \sum_\pm u_\pm\).

By \(4.5\), Lemma \(2.2\) and the inhomogeneous Strichartz estimates for the wave equation \([12, 27, 15, 10]\) we have, for \((q, r)\) and \((\tilde{q}, \tilde{r})\) given in Theorem \(3.1\) that

\[
\langle u \rangle_{L_q^q L_r^r(\mathbb{R} \times \mathbb{R}^n)} \lesssim \sum_\pm \left\| \int_0^t \sin \left( (t - s) \sqrt{\Lambda} (D) \right) R_\pm^{-1} (D) \mathcal{P}_\pm F(s) ds \right\|_{L_q^q L_r^r(\mathbb{R} \times \mathbb{R}^n)}
\]

This completes the proof of the inhomogeneous Strichartz estimate \(3.5\).

5. Perturbed equations

In this section, we provide a new proof of Theorem \(1.4\) making use of the diagonalization argument rather than using the Helmholtz decomposition.

By Duhamel’s principle, we first write the solution to \(1.7\) as

\[
u(t, x) = \cos(t\sqrt{-\Delta^\ast})f + \sin(t\sqrt{-\Delta^\ast})\sqrt{-\Delta^\ast}^{-1} g
\]

\[
+ \int_0^t \sin((t - s)\sqrt{-\Delta^\ast})\sqrt{-\Delta^\ast}^{-1} [Vu(s, \cdot)] ds.
\]

The estimate \(4.2\) gives the following a priori estimate

\[
\| \cos(t\sqrt{-\Delta^\ast})f + \sin(t\sqrt{-\Delta^\ast})\sqrt{-\Delta^\ast}^{-1} g \|_{L_q^q L_r^r(\mathbb{R} \times \mathbb{R}^n)} \lesssim \| f \|_{\dot{H}^s(\mathbb{R}^n)} + \| g \|_{\dot{H}^{-s}(\mathbb{R}^n)}
\]

for \(q, r\) and \(s\) as in Theorem \(1.2\). In this estimate if we replace \(f\) and \(g\) with \(|\nabla|^s f\) and \(|\nabla|^s g\), respectively, then (since the multiplier of \(|\nabla|^s\) commutes with all matrices) we obtain

\[
\| \cos(t\sqrt{-\Delta^\ast})f + \sin(t\sqrt{-\Delta^\ast})\sqrt{-\Delta^\ast}^{-1} g \|_{\dot{H}_s^s} \lesssim \| f \|_{\dot{H}_s^{1/2}} + \| g \|_{\dot{H}_s^{-1/2}}
\]

with \(\sigma = 1/2 - s = 1/q + \frac{n}{2} - \frac{1}{2}\). Hence, for the Duhamel part, we will show

\[
\left\| \int_0^t \sin((t - s)\sqrt{-\Delta^\ast})\sqrt{-\Delta^\ast}^{-1} [Vu(s, \cdot)] ds \right\|_{\dot{H}_s^s} \lesssim \| V \|_{L_{2s}^2} \| u \|_{L_{2s}^2(|V|)}
\]
and
\[ ||u||_{L^2_x,\gamma(V)} \lesssim ||V||_{L^2}^{1/2} (||f||_{H^{1/2}} + ||g||_{\dot{H}^{-1/2}}), \tag{5.4} \]
which are sufficient to prove Theorem 1.4.

In order to prove the estimates (5.3) and (5.4) we need the following weighted \( L^2 \) inequalities.

**Proposition 5.1.** Let \( n \geq 3 \) and \( V \) be as in Theorem 1.4. Then we have
\[ \| \cos(t \sqrt{-\Delta^s}) f \|_{L^2_x,\gamma(V)} \lesssim ||V||_{L^2}^{1/2} ||f||_{H^{1/2}}, \tag{5.5} \]
\[ \| \sin(t \sqrt{-\Delta^s}) \sqrt{-\Delta^s}^{-1} g \|_{L^2_x,\gamma(V)} \lesssim ||V||_{L^2}^{1/2} ||g||_{\dot{H}^{-1/2}} \tag{5.6} \]
and
\[ \left\| \int_0^t \sin((t-s) \sqrt{-\Delta^s}) \sqrt{-\Delta^s}^{-1} [Vu(s,\cdot)] ds \right\|_{L^2_x,\gamma(V)} \lesssim ||V||_{L^2}^{1/2} ||u||_{L^2_x,\gamma(V)}. \tag{5.7} \]

Let us hold off the proof of the proposition for the moment and first prove the estimates (5.3) and (5.4).

**Proofs of (5.3) and (5.4).** Applying Proposition 5.1 (with \( F = Vu \)) to (5.1), we see that
\[ ||u||_{L^2_x,\gamma(V)} \lesssim ||V||_{L^2}^{1/2} (||f||_{H^{1/2}} + ||g||_{\dot{H}^{-1/2}}) + ||V||_{H^\infty} ||u||_{L^2_x,\gamma(V)}. \]
Since \( ||V||_{L^2} \) is small we obtain the estimate (5.4).

The other estimate (5.3) follows from
\[ \left\| \int_{-\infty}^{\infty} \sin((t-s) \sqrt{-\Delta^s}) \sqrt{-\Delta^s}^{-1} [Vu(s,\cdot)] ds \right\|_{L^1_t \dot{H}^{-\gamma}} \lesssim ||V||_{L^2}^{1/2} ||u||_{L^2_x,\gamma(V)}. \tag{5.8} \]
by the Christ–Kiselev lemma (see [7]). Furthermore, by the formula \( \sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta \), it is enough to show (5.8) with \( \sin(t \sqrt{-\Delta^s}) \cos(s \sqrt{-\Delta^s}) \) and \( \cos(t \sqrt{-\Delta^s}) \sin(s \sqrt{-\Delta^s}) \) in place of \( \sin((t-s) \sqrt{-\Delta^s}) \).

Making use of the estimates (5.2) and the dual form of (5.5), we see that
\[ \left\| \sin(t \sqrt{-\Delta^s}) \sqrt{-\Delta^s}^{-1} \int_{-\infty}^{\infty} \cos(s \sqrt{-\Delta^s}) [Vu(s,\cdot)] ds \right\|_{L^1_t \dot{H}^{-\gamma}} \lesssim \left\| \int_{-\infty}^{\infty} \cos(s \sqrt{-\Delta^s}) [Vu(s,\cdot)] ds \right\|_{\dot{H}^{-1/2}} \lesssim ||V||_{L^2} ||u||_{L^2_x,\gamma(V)}. \]
Similarly, from (5.2) and the dual to estimate (5.6) we deduce
\[ \left\| \cos(t \sqrt{-\Delta^s}) \int_{-\infty}^{\infty} \sin(s \sqrt{-\Delta^s}) \sqrt{-\Delta^s}^{-1} [Vu(s,\cdot)] ds \right\|_{L^1_t \dot{H}^{-\gamma}} \lesssim \left\| \int_{-\infty}^{\infty} \sin(s \sqrt{-\Delta^s}) \sqrt{-\Delta^s}^{-1} [Vu(s,\cdot)] ds \right\|_{\dot{H}^{-1/2}} \lesssim ||V||_{L^2}^{1/2} ||u||_{L^2_x,\gamma(V)}. \]

Thus the proof of (5.3) is complete. \( \square \)
In the rest of this section we prove Proposition 5.1. For the purpose we make use
of weighted $L^2$ boundedness for Mikhlin multipliers and maximal functions involving
the Muckenhoupt $A_q$ weights. We first recall the following (see e.g., [26, Theorem
7.21]).

**Lemma 5.2.** Let $1 < q < \infty$ and $w \in A_q$. If $m$ is a smooth function defined on
$\mathbb{R}^n \setminus \{0\}$ satisfying
$$|\partial^\alpha m(\xi)| \lesssim |\xi|^{-|\alpha|}$$
for all multi-indices $\alpha \in \mathbb{N}^n_0,$ then we have
$$\|m(D)f\|_{L^q(w)} \lesssim \|f\|_{L^q(w)}.$$ 

Let us then recall some useful facts on the Hardy–Littlewood maximal operator $M$
on the class $F_p$ (see [6, Lemma 1]): If $V \in F_p$ then for any $1 < a < p$
$$w = (M(|V|^a))^{1/a} \in A_1 \cap F_p$$
and
$$\|w\|_{F_p} \lesssim \|V\|_{F_p}. \quad (5.9)$$
It is also clear that $|V| \leq w$ almost everywhere. In particular, $w \in A_2.$

**Proof of Proposition 5.1.** Let us first prove (5.5). Setting $w = (M(|V|^a))^{1/a} \in A_2$
and using Lemma 5.2, we have
$$\|R_{\pm} f\|_{L^2(w)} \approx \|\mathcal{P}_{\pm} f\|_{L^2(w)} \lesssim \|f\|_{L^2(w)}. \quad (5.10)$$
Since $|V| \leq w,$ by using (4.3) with $g = 0$ and (5.10), we get
$$\|\cos(t\sqrt{-\Delta^*})f\|_{L^2_\epsilon((V))} \leq \sum_{\pm} \|R_{\pm} (D) \cos(t\sqrt{\Lambda})(R_{\pm} (D))^4 \mathcal{P}_{\pm} f\|_{L^2_\epsilon,(w)} \lesssim \sum_{\pm} \|\cos(t\sqrt{\Lambda})(R_{\pm} (D))^4 \mathcal{P}_{\pm} f\|_{L^2_\epsilon,(w)}.$$ 
Now we use the analog [29, (2.11)] of (5.5) (with $\Delta^*$ replaced by $\Delta$), (2.2) and (5.9)
to estimate this by
$$C\|w\|_{F_p}^{1/2} \sum_{\pm} \|(R_{\pm} (D))^4 \mathcal{P}_{\pm} f\|_{H^{1/2}} \lesssim \|V\|_{F_p} \|f\|_{H^{1/2}}.$$ 
Hence, we obtain (5.5). The proof of (5.6) is similar, so we shall omit it.

To show (5.7) we make use of (4.3) and (5.10) to see that
$$\left\| \int_0^t \sin((t-s)\sqrt{-\Delta^*})\sqrt{-\Delta^*}^{-1} F(s,\cdot)ds \right\|_{L^2_\epsilon,(w)} \lesssim \sum_{\pm} \left\| \int_0^t \sin((t-s)\sqrt{\Lambda})(\sqrt{\Lambda}^{-1}(R_{\pm} (D))^4 \mathcal{P}_{\pm} F(s,\cdot)ds \right\|_{L^2_\epsilon,(w)}.$$ 
We recall the wave equation analog [29, Proposition 4.2] of (5.7) and utilize (5.9) and
(5.10) to dominate this by
$$C\|w\|_{F_p} \sum_{\pm} \|(R_{\pm} (D))^4 \mathcal{P}_{\pm} F(t,\cdot)\|_{L^2_{\epsilon,w-1}} \lesssim \|V\|_{F_p} \|F\|_{L^2_{\epsilon,w-1}}.$$
Since $|V| \leq w$ the estimate (5.7) follows. The proof of Proposition 5.1 is complete. □

Finally, we present some difficult aspects and open problems related to the endpoint issue ($q = 2$) in Theorem 1.4.

**Further discussion.** We discuss the endpoint issue $q = 2$. We need the assumption $q > 2$ in the proof when we apply the Christ–Kiselev lemma to handle the Duhamel term. One might be motivated to try a simple approach to use (a Lorentz space variant of) the endpoint Strichartz estimate (1.5) (applied to (5.1) with $V(x) = c|x|^{-2}$, $|c| \ll 1$)

$$\|u\|_{L^2_tL^r_x} \lesssim \|f\|_{H^s} + \|g\|_{H^{s-1}} + \|Vu\|_{L^1_tL^{\tilde{r}}_x},$$

(5.11)

for wave-admissible pairs $(2, r)$ and $(2, \tilde{r})$ satisfying

$$\frac{1}{2} + \frac{n}{r} = \frac{n}{2} - s = \frac{1}{2} + \frac{n}{\tilde{r}} - 2,$$

(5.12)

and argue as the following. If (5.11) were true, then from (5.12) combined with O’Neil’s inequality ([28]) it follows that

$$\|u\|_{L^2_tL^r_x} \lesssim \|f\|_{H^s} + \|g\|_{H^{s-1}} + \|Vu\|_{L^{n/2,\infty}} \|u\|_{L^2_tL^r_x},$$

and we can ignore the last term since $\|V\|_{L^{n/2,\infty}} \ll 1$. However, unfortunately, such pairs $(2, r)$ and $(2, \tilde{r})$ do not exist.

For the wave equation perturbed by the inverse square potential, Burq et al. [4] obtained the endpoint case (see Theorem 9 in [4]). The framework in [4] does not seem to be accessible in the elastic case because the differential operator $\nabla \text{div} = (\partial^2/\partial x_i \partial x_j)_{1 \leq i, j \leq n}$ has variable coefficients in the spherical coordinate. In fact,

$$\frac{\partial^2}{\partial x_i \partial x_j} = a_i a_j \frac{\partial^2}{\partial r^2} + \left( \frac{a_j}{r} A_i + \frac{a_i}{r} A_j - \frac{a_i a_j}{r^2} \right) \frac{\partial}{\partial r} + \frac{1}{r^2} A_i A_j,$$

where $a_j$ and $A_j$ are functions of $\theta$ and $\partial \theta$ as in the proof of Lemma 2.1. In this regard it would be an interesting open question to ask whether the endpoint estimates hold for the elastic case.

6. Uniform Sobolev inequality

In this final section, we prove the uniform Sobolev inequality (1.10), which follows from corresponding result on the wave operator in [13] once we diagonalize the Lamé operator $\Delta^s$ as in the previous sections. As a corollary, the uniform inequality yields temporal unique continuation for $$(\partial_t^2 - \Delta^s)u = Vu$$ whenever $V \in L^{\frac{n}{n-2}}(\mathbb{R}^{1+n})$.

**Proof of Theorem 1.5.** If we denote by $\mathcal{F}$ the space-time Fourier transform and $\mathcal{F}^{-1}$ the inverse of $\mathcal{F}$, then in terms of Fourier multiplier the inequality (1.10) is equivalent to

$$\left\| \mathcal{F}^{-1} \left\{ \left( (-\tau^2 + i \alpha \tau - z) I_n + L(\xi) \right) ^{-1} \mathcal{F} f(\tau, \xi) \right\} \right\|_{L^p(\mathbb{R}^{1+n})} \leq C \|f\|_{L^p(\mathbb{R}^{1+n})},$$

5See, for example, [8], p. 282, for a similar argument concerning perturbed Schrödinger equations.
By (2.3) we diagonalize the multiplier and see that
\[
((\tau^2 + i\alpha - z)I_n + L(\xi))^{-1}Ff(\tau, \xi) = \sum_{\pm} R_{\pm}(\xi)((\tau^2 + i\alpha - z)I_n + \Lambda(\xi))^{-1}(R_{\pm}(\xi))^t\varphi_{\pm}(\xi/|\xi|)Ff(\tau, \xi)
\]

Therefore, by making use of Lemma 2.2 and the uniform Sobolev inequalities for the wave operator \([13, \text{Theorem 1.1}]\) with \(d = 1 + n\) we have the uniform estimate
\[
\left\|((\partial_t^2 - \Delta^*) + a\partial_t - z)^{-1}f\right\|_{L^q(\mathbb{R}^{1+n})} \\
= \left\|\mathcal{F}^{-1}\left\{((\tau^2 + i\alpha - z)I_n + L(\xi))^{-1}Ff(\tau, \xi)\right\}\right\|_{L^q(\mathbb{R}^{1+n})} \\
\lesssim \sum_{\pm} \left\|\mathcal{F}^{-1}\left\{((\tau^2 + i\alpha - z)I_n + \Lambda(\xi))^{-1}(R_{\pm}(\xi))^t\varphi_{\pm}(\xi/|\xi|)Ff(\tau, \xi)\right\}\right\|_{L^q(\mathbb{R}^{1+n})} \\
\lesssim \sum_{\pm} \left\|\mathcal{F}^{-1}\left\{(R_{\pm}(\xi))^t\varphi_{\pm}(\xi/|\xi|)Ff(\tau, \xi)\right\}\right\|_{L^q(\mathbb{R}^{1+n})} \\
\lesssim \|f\|_{L^p(\mathbb{R}^{1+n})}
\]

whenever \((p, q)\) lies in the uniform boundedness range \((1.11)\). This completes the proof of the uniform Sobolev inequalities \((1.10)\). \(\square\)

We now turn to show the following form of Carleman inequalities with weights in the temporal variable: If \(p, q\) satisfy \((1.11)\) we have
\[
\|e^{\nu t}u\|_{L^q(\mathbb{R}^{1+n})} \leq C\|e^{\nu t}(\partial_t^2 - \Delta^*)u\|_{L^p(\mathbb{R}^{1+n})},
\]
where \(C\) is independent of \(\nu \in \mathbb{R}\). These inequalities are direct consequences of the uniform Sobolev inequalities \((1.10)\) since
\[
e^{\nu t}(\partial_t^2 - \Delta^*)e^{-\nu t} = \partial_t^2 - 2\nu\partial_t + \nu^2 - \Delta^*.
\]
As a consequence, by the well-known argument in \([16]\) p. 343] we obtain temporal unique continuation for the differential inequality
\[
|\partial_t^2 - \Delta^*)u(t, x)| \leq |V(t, x)u(t, x)|.
\]

**Corollary 6.1.** Let \(p\) satisfy the second condition in \((1.11)\) and suppose that \(V \in L^{\frac{1+n}{2}}(\mathbb{R}^{1+n})\) and \(u \in W^{2,p}(\mathbb{R}^{1+n})\). For some \(t_0 \in \mathbb{R}\), if the support of \(u\) is contained in one side of the hyperplane \(\{t, x) \in \mathbb{R}^{1+n}; t = t_0\}\) and \(u\) satisfies \((6.1)\) almost everywhere, then \(u \equiv 0\) on \(\mathbb{R}^{1+n}\).

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