NONCONFORMING ELEMENTS OF CLASS $L^2$ FOR
HELMHOLTZ TRANSMISSION EIGENVALUE PROBLEMS

JIAYU HAN*
School of Mathematical Sciences, Guizhou Normal University
Guiyang, 550001, China

(Communicated by Zhimin Zhang)

Abstract. For solving the Helmholtz transmission eigenvalue problem, we use the mixed formulation of Cakoni et al. to construct a new nonconforming element discretization. Based on the discretization, this paper first discuss the nonconforming element methods of class $L^2$, and prove the error estimates of the discrete eigenvalues obtained by the cubic tetrahedron element, incomplete cubic tetrahedral element and Morley element et al. We report some numerical examples using the nonconforming elements mixed with linear Lagrange element to show that our discretization can obtain the transmission eigenvalues of higher accuracy in 3D domains than the nonconforming element discretization in the existing literature.

1. Introduction. The transmission eigenvalue problem arising in the inverse scattering theory is an important problem in acoustic scattering [6, 11, 21]. Due to its important physical background, exploring its efficient numerical methods is a hot topic in computational mathematics. Since Colton et al. [10] proposed three numerical methods, the related scholars sequentially developed many numerical methods, including conforming element methods [22, 7, 25, 16, 14], nonconforming element methods [27], spectral methods [1], $C^0$IPG methods [12] and boundary element methods [17, 28]. Regarding the study of conforming elements, early in 2011 Sun [22] proposed the iterative methods using conforming elements to solve the problem. Afterwards, Cakoni et al. [7] used linearization technique to derive a linear weak formulation of the transmission eigenvalue problem, and both $H^1$ conforming element space and $H^2$ conforming element space are adopted to discretize the linear weak formulation. More recently Yang et al. [25] used a different way to linearize the problem and then propose a different conforming element method.

However, the conforming element methods is usually difficult to realize in 3D domains. To overcome this defect, the mixed methods in [15, 26] decrease the order of the problem by introducing an auxiliary variable. The nonconforming element methods in [27] make use of the weak continuity and $C^0$ continuity of the finite element spaces, and the $C^0$IPG methods [12] introduce the inner penalty terms to keep the ellipticity of the bilinear form associated with the problem. Note that...
the nonconforming element methods in [27] do not cover the highly nonconforming elements of class \(L^2\) such as the Morley element, cubic tetrahedron element and incomplete cubic tetrahedron element [23]. To fill the gap of the nonconforming element methods in [27], this paper continues to develop the new nonconforming element methods so that the corresponding finite element spaces could be discontinuous. The new nonconforming element methods of class \(L^2\) in this paper adopt the mixed linear formulation in [7] to discretize the eigenvalue problem. Due to the discontinuity of nonconforming element spaces, we have to apply the spectral approximation theory of [2] in the union of the piecewise smooth Sobolev spaces, which needs to be completed. This increases the difficulty of our theory analysis, however, we overcome it and thus explore a new analysis approach of applying the spectral approximation theory in piecewise smooth Sobolev spaces. Another obvious feature of our error analysis is the application of the projector-mean operator in [20], which plays a crucial role in the proof of the error of consistence terms and eigenvalues.

In the numerical experiment, we present several numerical examples using cubic tetrahedron element in 3D to solve the problem. The numerical results show that the discretization of this paper using the cubic tetrahedron element mixed with linear element can capture the eigenvalues of higher accuracy than the discretization in [27] using the Morley-Zienkiewicz element. We think the reason lies in that the finite element space associated with the cubic tetrahedron element contains total piecewise polynomials of degree less or equal to three, which is higher than that of Morley-Zienkiewicz element, and meanwhile the former is of the same degrees of freedom as the latter.

In this paper we refer to [4, 9, 18, 20] for the basic theory of finite element methods.

Throughout this paper, \(C\) denotes a positive constant independent of \(h\), which may not be the same constant in different places. For simplicity, we use the symbol \(a \lesssim b\) to mean that \(a \leq Cb\).

2. The weak formulation and nonconforming element approximation.

Consider the Helmholtz transmission eigenvalue problem: Find \(k \in \mathbb{C}, \ w, \sigma \in L^2(D), \ w - \sigma \in H^2(D)\) such that

\[
\begin{align*}
\Delta w + k^2nw &= 0, \text{ in } D, \\
\Delta \sigma + k^2\sigma &= 0, \text{ in } D, \\
w - \sigma &= 0, \text{ on } \partial D, \\
\frac{\partial w}{\partial \nu} - \frac{\partial \sigma}{\partial \nu} &= 0, \text{ on } \partial D,
\end{align*}
\]

where \(D \subset \mathbb{R}^d\) (\(d=2,3\)) is a bounded inhomogeneous medium, \(\nu\) is the unit outward normal to \(\partial D\) and the index of refraction \(n(x)\) is positive.

Let \(W^{s,p}(D)\) denote the usual Sobolev space with norm \(\|\cdot\|_{s,p}\), \(H^s(D) = W^{s,2}(D)\) with norm \(\|\cdot\|_{s,2} = \|\cdot\|_s\), \(H^0(D) = L^2(D)\) with the inner product \((u, v)_0 = \int_D u v dx\).

Let \(\pi_h\) be a shaped-regular mesh with size \(h\) and define the piecewise Sobolev space \(H^1_h(D) = \{v : v \in H^s(\kappa), \forall \kappa \in \pi_h\}\) with norm \(\|v\|_{H^1_h} = (\sum_{\kappa} |v|_{s,\kappa}^2)^{1/2}\). We write \(H^1_h := H^1_h(D)\) and \(L^2 := L^2(D)\) for short. Define Hilbert space \(H^1_h \times L^2\) with norm \(\|(v, z)\|_h = \|v\|_{H^1_h} + \|z\|_0\) and \(H = H^1_0(D) \times H^1_0(D)\) with norm \(\|(v, z)\|_H = \|v\|_2 + \|z\|_1\).
In this paper, we suppose that \( n = n(x) \in W^{1,\infty}(D) \) is strictly greater than 1, although the analysis is similar for \( n \) strictly less than 1.

It is known that the problem (2.1)-(2.4) can be written as an equivalent fourth order equation for \( u = w - \sigma \in H^1_0(D) \) (see [5, 19]):

\[
(\Delta + k^2)u \frac{1}{n} \frac{1}{n-1} (\Delta + k^2)u = 0.
\]

Then the weak formulation for the transmission eigenvalue problem (2.1)-(2.4) can be stated as follows: Find \( k \in \mathbb{C}, u \in H^1_0(D) \) such that

\[
\left( \frac{1}{n-1} \Delta u, \Delta v \right)_0 = -k^2(\Delta u, \frac{n}{n-1} v)_0 - k^2(\frac{1}{n-1} u, \Delta v)_0
\]

\[-k^4(\frac{n}{n-1} u, v)_0, \quad \forall v \in H^1_0(D).
\]

(2.5)

Introduce an auxiliary variable \( \omega \in H^1_0(D) \) satisfying

\[
(\Delta \omega, \omega)_0 = k^2 \frac{n}{n-1} u.
\]

(2.6)

Thus, we arrive at a linear weak formulation [7]: Find \( (k^2, u, \omega) \in \mathbb{C} \times H^1_0(D) \times H^1_0(D) \) such that

\[
\left( \frac{1}{n-1} \Delta u, \Delta v \right)_0 = k^2(\Delta u, \frac{n}{n-1} v)_0 + k^2(\Delta \omega, \nabla(\frac{n}{n-1} v))_0
\]

\[+k^2(\nabla \omega, \nabla v)_0, \quad \forall v \in H^1_0(D),
\]

(2.7)

\[\nabla \omega, \nabla z)_0 = -k^2(\frac{n}{n-1} u, z)_0, \quad \forall z \in H^1_0(D).
\]

(2.8)

Let

\[
A((u, \omega), (v, z)) = ((\frac{1}{n-1} - \mu_1) \Delta u, \Delta v)_0 + (\mu_1 \Delta u, \Delta v)_0 + (\nabla \omega, \nabla z)_0
\]

\[= ((\frac{1}{n-1} - \mu_1) \Delta u, \Delta v)_0
\]

\[+ \mu_1 \int_{\Omega} \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 \nabla}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_1 \partial x_2} \frac{\partial^2 \nabla}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial x_2^2} \frac{\partial^2 \nabla}{\partial x_2^2})dx + (\nabla \omega, \nabla z)_0,
\]

\[
B((u, \omega), (v, z)) = (\nabla(\frac{1}{n-1} u), \nabla v)_0 + (\nabla u, \nabla(\frac{n}{n-1} v))_0
\]

\[+(\nabla \omega, \nabla v)_0 - (\frac{n}{n-1} u, z)_0.
\]

where \( \mu_1 \leq \frac{1}{n-1} \) is a good approximation of \( \inf \frac{1}{n-1} \). It is obvious that \( A(\cdot, \cdot) \) is a self-adjoint, coercive form on \( \mathbf{H} \times \mathbf{H} \) satisfying

\[
A((v, z), (v, z)) \geq \mu_1 \|v\|_2^2 + \|z\|_1 \geq \|(v, z)\|^2_{\mathbf{H}}.
\]

(2.9)

Let \( \lambda = k^2 \), then (2.7)-(2.8) can be rewritten as: Find \( \lambda \in \mathbb{C}, (u, \omega) \in \mathbf{H} \) such that

\[
A((u, \omega), (v, z)) = \lambda B((u, \omega), (v, z)), \quad \forall (v, z) \in \mathbf{H},
\]

(2.10)

With a weak formulation different from (2.7)-(2.8), we have studied some nonconforming finite element approximations (see [25]). However, the theoretical analysis in [25] requires the \( H^1 \)-conformity of the finite element spaces and cannot be suited to the discontinuous nonconforming elements. For this purpose, we continue to explore the error estimates of the nonconforming elements of class \( L^2 \) with the formulation (2.7)-(2.8). In view of the discontinuity of nonconforming element considered, we have to give the linear boundedness of the piecewise form of \( B(\cdot, \cdot) \).
Let $\mathbb{H}$ be a sequence of mesh sizes which are decreasing and converging to 0 such that if $h, l \in \mathbb{H}$ and $h < l$ then $\pi_l$ is obtained by refining $\pi_l$. We define the linear form $B_h(\cdot, \cdot)$ by replacing the gradient $\nabla$ with the piecewise gradient $\nabla_h$ in the definition of $B(\cdot, \cdot)$, namely
\[
B_h((f, g), (v, z)) = (\nabla_h(\frac{1}{n-1} f), \nabla_h v)_0 + (\nabla_h f, \nabla_h(\frac{n}{n-1} v))_0 + (\nabla_h g, \nabla_h v)_0
- (\frac{n}{n-1} f, z)_0
\]
When $n \in W^{1, \infty}(D)$, let $h, l \in \mathbb{H}$ then by taking $s = \min(h, l)$
\[
|B_s((f, g), (v, z))| \lesssim \|f\|_{H^1}^2 \|v\|_{H^1} + \|g\|_{H^1}^2 \|v\|_{H^1} + \|f\|_0 \|z\|_0
\lesssim (\|f\|_{H^1}^2 + \|g\|_{H^1}^2)(\|v\|_{H^1} + \|z\|_0)
\lesssim \|f, g\|_{H^1 \times H^1}^2 \|(v, z)\|_{H^1 \times L^2}, \forall (f, g) \in H^1 \times H^1, (v, z) \in H^1 \times L^2.
\]
(2.11)

We equip $\bigcup_{l \in \mathbb{H}} H^1_l$ ($s = 1, 2$) with the norm $\lim_{l \to 0}\|\cdot\|_{H^1_l}$. For simplicity of notations, we still use $B((f, g), (v, z))$ to denote $\lim_{l \to 0} B_h((f, g), (v, z))$. Then if $(f, g) \in \bigcup_{l \in \mathbb{H}} H^1_l \times H^1_l = \bigcup_{l \in \mathbb{H}} H^1_l \times \bigcup_{l \in \mathbb{H}} H^1_l, (v, z) \in \bigcup_{l \in \mathbb{H}} H^1_l \times L^2$
\[
|B((f, g), (v, z))| \lesssim \lim_{l \to 0} \|f\|_{H^1_l}^2 \|v\|_{H^1_l} \lim_{l \to 0} \|(v, z)\|_{H^1_l \times L^2}.
\]
(2.12)

That is, $B(\cdot, \cdot)$ is a continuous linear form on $\bigcup_{l \in \mathbb{H}} H^1_l \times \bigcup_{l \in \mathbb{H}} H^1_l \times \bigcup_{l \in \mathbb{H}} H^1_l \times L^2$. We denote $\bigcup_{l \in \mathbb{H}} H^1_l$ the completion of $\bigcup_{l \in \mathbb{H}} H^1_l$ with norm $\lim_{l \to 0}\|\cdot\|_{H^1_l}$. Then for $(f, g) \in H^1_h \times L^2, B((f, g), (v, z))$ has a continuous extension to $(f, g) \in \bigcup_{l \in \mathbb{H}} H^1_l \times \bigcup_{l \in \mathbb{H}} H^1_l$ such that (2.12) holds; in fact, for $(f, g) \in \bigcup_{l \in \mathbb{H}} H^1_l \times \bigcup_{l \in \mathbb{H}} H^1_l$ there is a Cauchy sequence
\[
\{(\tilde{f}_i, \tilde{g}_i)\}_{i=1}^\infty \subset \bigcup_{l \in \mathbb{H}} H^1_l \times \bigcup_{l \in \mathbb{H}} H^1_l
\] such that $\|((\tilde{f}_i, \tilde{g}_i) - g)\|_{\bigcup_{l \in \mathbb{H}} H^1_l \times \bigcup_{l \in \mathbb{H}} H^1_l} \to 0$ as $i \to \infty$
and
\[
|B((\tilde{f}_i, \tilde{g}_i) - (\tilde{f}_j, \tilde{g}_j), (v, z))| \lesssim \|((\tilde{f}_i - \tilde{f}_j, \tilde{g}_i - \tilde{g}_j))\|_{\bigcup_{l \in \mathbb{H}} H^1_l \times \bigcup_{l \in \mathbb{H}} H^1_l} \|(v, z)\|_{H^1_l \times L^2},
\]
which implies $\{(B((\tilde{f}_i, \tilde{g}_i), (v, z))\}_{i=1}^\infty$ is Cauchy in $C$ and so has a limit in $C$, still denoted by $B((f, g), (v, z))$. Then for $(v, z) \in H^1_h \times L^2$ since $f \in \bigcup_{l \in \mathbb{H}} H^1_l \subset L^2$ we denote
\[
B((f, g), (v, z)) = < F, v > - (\frac{n}{n-1} f, z)_0,
\]
(2.13)
where $< F, v > = \lim_{l \to 0}((\nabla_l(\frac{1}{n-1} f), \nabla_l v)_0 + (\nabla_l f, \nabla_l(\frac{n}{n-1} v))_0 + (\nabla_l g, \nabla_l v)_0, \forall (f, g) \in \bigcup_{l \in \mathbb{H}} H^1_l \times \bigcup_{l \in \mathbb{H}} H^1_l$; then for $v \in H^1_h < F, v >$ has a continuous extension to $(f, g) \in \bigcup_{l \in \mathbb{H}} H^1_l \times \bigcup_{l \in \mathbb{H}} H^1_l$. From Lax-Milgram theorem we can define the corresponding solution operator $T : \bigcup_{l \in \mathbb{H}} H^1_l \times \bigcup_{l \in \mathbb{H}} H^1_l \to H$ by
\[
A(T(f, g), (v, z)) = B((f, g), (v, z)), \forall (v, z) \in H.
\]
(2.14)
Consider the dual problem of (2.10): Find $\lambda^* \in \mathbb{C}$, $(u^*, \omega^*) \in \mathbf{H}$ such that
\[
A((v, z), (u^*, \omega^*)) = \overline{\lambda^*} B((v, z), (u^*, \omega^*)), \quad \forall (v, z) \in \mathbf{H}. \tag{2.15}
\]
We define
\[
B_h^*((f, g), (v, z)) = (\nabla_h (\frac{1}{n-1} v), \nabla_h f)_0 + (\nabla_h v, \nabla_h (\frac{n}{n-1} f))_0 - (\frac{n}{n-1} v, g)_0 + (\nabla_h z, \nabla_h f)_0
\]
and use $B^*((f, g), (v, z))$ to denote $\lim_{h \to 0} B_h^*((f, g), (v, z))$. Define the corresponding solution operator $T^* : \bigcup_{t \in \mathbb{H}} H^2_t \times \bigcup_{t \in \mathbb{H}} H^1_t \to \mathbf{H}$ by
\[
A((v, z), T^*(f, g)) = B^*((f, g), (v, z)), \quad \forall (v, z) \in \mathbf{H}. \tag{2.16}
\]
Then (2.10) and (2.15) have the equivalent operator forms
\[
T(u, \omega) = \lambda^{-1}(u, \omega), \quad T^*(u^*, \omega^*) = \lambda^{*-1}(u^*, \omega^*), \tag{2.17}
\]
and $T^*$ is the adjoint operator of $T$ in the sense of inner product $A(\cdot, \cdot)$. In fact, from (2.14) and (2.16) we have
\[
A(T(u, \omega), (v, z)) = B((u, \omega), (v, z)) = A((u, \omega), T^*(v, z)), \quad \forall (u, \omega), (v, z) \in \mathbf{H}. \tag{2.18}
\]

**Definition 2.1** [Section 5.1 in [20]]. The finite element space $Y_h$ is said having \( m \)th order weak continuity if for any $v_h \in Y_h$, any $\alpha$ with $|\alpha| < m$ and arbitrary interelement face $F$ of $\pi_h$, the jump $[\partial^\alpha v_h]$ through $F$ vanishes at a point on $F$ at least. $Y_h$ is said having $m$th order weak nullity on $\partial \Omega$ if for any $v_h \in Y_h$, any $\alpha$ with $|\alpha| < m$ and arbitrary free face $F$ of $\pi_h$, the jump $[\partial^\alpha v_h]$ through $F$ vanishes at a point on $F$ at least.

Let $S_h = U_h \times V_h \subset L^2 \times L^2$ be nonconforming element space defined on $\pi_h$. As is well known, many nonconforming element space satisfies the following assumption (see Section 2.6 in [20]) and they will be very useful for us to give the estimates of the consistence terms in the next section.

(A) $U_h$ has the 2nd order weak continuity and 2nd order weak nullity on $\partial \Omega$ and $V_h$ has the 1st order weak continuity and 1st order weak nullity on $\partial \Omega$. Furthermore, if $F$ is the common face of elements $\kappa$ and $\kappa'$ in $\pi_h$, then
\[
\int_F \nabla(v|_\kappa) ds = \int_F \nabla(v|_{\kappa'}) ds, \quad \forall v \in U_h, \quad and \quad \int_F u|_\kappa ds = \int_F u|_{\kappa'} ds, \quad \forall u \in V_h; \tag{2.19}
\]
if $F$ is a face of element $\kappa$ and $F \in \partial \Omega$, then
\[
\int_F \nabla(v|_\kappa) ds = 0, \quad \forall v \in U_h, \quad and \quad \int_F u|_\kappa ds = 0, \quad \forall u \in V_h. \tag{2.20}
\]

The paper [27] has studied the nonconforming element of class $C^0$ including the Morley-Zienkiewicz element [20], modified Zienkiewicz element [23], 12-parameter triangle plate element and 15-parameter triangle plate element [20]. Here in this paper using the formulation (2.7)-(2.8) we continue to develop the theoretical results in [27] so that the discontinuous nonconforming element space $U_h \notin C^0(D)$ satisfying (2.19)-(2.20) can be covered. For example, $U_h \subset L^2(D)$ is the finite element space associated with one of the cubic tetrahedron element, incomplete cubic tetrahedral element (see [24]) and Morley element. Meanwhile $V_h$ can be taken as the finite element space associated with one of the linear Lagrange element, cubic
tetrahedron element and incomplete cubic tetrahedral element. Now we are in a
d position to give some discrete forms of such elements.

Define the piecewise \( A_h((u_h, \omega_h), (v, z)) = \sum_{\kappa \in \text{tetrahedron element}} \int \left( \frac{1}{n-1} - \mu_1 \right) \Delta u_h \Delta v \)
\[ + \mu_1 \left( \frac{\partial^2 u_h}{\partial x_1^2} \frac{\partial^2 v}{\partial x_1^2} + 2 \frac{\partial^2 u_h}{\partial x_1 \partial x_2} \frac{\partial^2 v}{\partial x_1 \partial x_2} + \frac{\partial^2 u_h}{\partial x_2^2} \frac{\partial^2 v}{\partial x_2^2} \right) dx + (\nabla \omega_h, \nabla z)_0. \]

From [9] and Lemma 5.4.3 of [20], we know \( A_h((., .)) \) satisfies the uniform \( S_h \)-ellipticity:

\[ \| (v, z) \|^2 := \| v \|^2_{H^2} + \| z \|^2_{H^2} \leq A_h((v, z), (v, z)), \forall (v, z) \in S_h. \]

The nonconforming element approximation of (2.10) is given by the following:

Find \( \lambda_h \in \mathbb{C} \), \((u_h, \omega_h) \in S_h \) such that

\[ A_h((u_h, \omega_h), (v, z)) = \lambda_h B_h((u_h, \omega_h), (v, z)), \forall (v, z) \in S_h. \] (2.21)

We introduce the corresponding solution operator: \( T_h : \bigcup_{t \in \mathbb{H}} H^2_t \times \bigcup_{t \in \mathbb{H}} H^1_t \rightarrow S_h:\n
\[ A_h(T_h(f, g), (v, z)) = B((f, g), (v, z)), \forall (v, z) \in S_h. \] (2.22)

Before ending this section, we shall introduce some interpolation operators that
will be used in the next section.

The projection-mean operators in [20] are powerful tools for making the error
estimates for nonconforming element space. Here in this paper, we shall introduce
two projection-mean operators. Let \( I^2_k : L^2(D) \rightarrow H^2_0(D) \) be the projection-mean
operator corresponding to the simplex element of degree 2(see page 132 in [20]),
and let \( I^2_k : L^2(D) \rightarrow U_h \) the projection-mean operator corresponding to the nonconforming element(see page 140 in [20]).

The argument of Lemma 5.2.3 in [20] and inverse estimate indicate that if \( q = \frac{p}{p-1} \)
with \( p \in [1, 2] \) then for any \( v \) piecewise polynomial defined on \( \pi_h \) satisfying \( mn \) weak continuity and \( mth \) weak nullity on \( \partial \Omega \) there holds

\[ \| I^2_k v - v \|_{0,q,\kappa} \leq h_\kappa^{\frac{d}{2} - \frac{d}{q}} \| I^2_k v - v \|_{0,\kappa} \lesssim h_\kappa^{m + \frac{d}{2} - \frac{d}{q}} \| v \|_{H^m(\omega_\kappa)}, \] (2.23)

where \( \omega_\kappa \) is the union of elements sharing at least one node with \( \kappa \). When \( \psi \in H^s_0(D) \cap W_{3,p}(D) \) \((p \in (\frac{d}{2}, 2]) \) and \( \varphi \in H^r_0(D) \cap H^{1+r'}(D) \) \((r' = 1 + \frac{d}{p} - \frac{d}{p'}, p' \in (\frac{d}{2}, 2)] \), noting that \( W_{3,p}(D) \hookrightarrow H^{2+r}(D) \) \((r = 1 + \frac{d}{p} - \frac{d}{p'}, \text{ Lemma 3.5.8 or Lemma 5.4.1 in [20]) shows that}

\[ \| I^2_k \psi - \psi \|_{H^s} \lesssim h^{2+s-r} \| \psi \|_{2+r} \lesssim h^{3-s+\frac{d}{2} - \frac{d}{q}} \| \psi \|_{3,p}, s = 0, 1, 2, \] (2.24)

\[ \| I^2_k \varphi - \varphi \|_{H^r} \lesssim h^{1+r'} \| \varphi \|_{1+r'} \lesssim h^{1+\frac{d}{2} - \frac{d}{p'}} \| \varphi \|_{2,p'}. \] (2.25)

3. Error estimates for consistency terms and source problems. In this
section we aim to derive the convergence relations \( T_h \rightarrow T \) in \( \bigcup_{t \in \mathbb{H}} H^2_t \times \bigcup_{t \in \mathbb{H}} H^1_t \)
so that we can use the spectral approximation theory in [2] to obtain the error
estimates of nonconforming element eigenpairs.

Throughout this section, we denote \((\psi, \varphi) := T(f, g), (\psi_h, \varphi_h) := T_h(f, g) \) and
\((\psi^*, \varphi^*) := T^*(f, g)\). Define the consistency terms: For any \((v, z) \in S_h,\)
\[ D_h((\psi, \varphi), (v, z)) = B((f, g), (v, z)) - A_h((\psi, \varphi), (v, z)), \]
\[ D_h^*(((v, z), (\psi^*, \varphi^*)) = B^*((f, g), (v, z)) - A_h^*((v, z), (\psi^*, \varphi^*)). \] (3.1) (3.2)

As the beginning of our analysis, the following lemma is a generalization of Strang Lemma (1972), which can be found in [27].

**Lemma 3.1.** There holds the following error estimates
\[ \| (\psi, \varphi) - (\psi_h, \varphi_h) \|_h \]
\[ \lesssim \inf_{(v, z) \in S_h} \| (\psi, \varphi) - (v, z) \|_h + \sup_{(v, z) \in S_h \setminus \{0\}} \frac{D_h((\psi, \varphi), (v, z))}{\| (v, z) \|_h}. \] (3.3)

Based on the above lemma, the first step of this section is to analyze the consistency terms, which plays a crucial role in our analysis. Define the face and element average interpolation operators
\[ P^0_F f = \frac{1}{\text{meas}(F)} \int_F f ds, \quad R^0_F f = f - P^0_F f, \]
\[ P^0_\kappa f = \frac{1}{\text{meas}(\kappa)} \int_{\kappa} f dx, \quad R^0_\kappa f = f - P^0_\kappa f, \]
where element \( \kappa \in \pi_h \) and \( F \) is an arbitrary element face of \( \pi_h \).

Now we shall give the following estimates for the consistence terms.

**Theorem 3.1.** Suppose that \( \psi \in W^{3,p}(D) \) and \( \varphi \in W^{2,p'}(D)(p, p' \in (\frac{4}{3}, 2)) \). Let \( v \) and \( z \) be any piecewise polynomial defined on \( \pi_h \) satisfying (2.19)-(2.20), the 2nd and 1st order weak continuity, and the 2nd and 1st order weak nullity on \( \partial\Omega \), respectively. Then
\[ |D_h((\psi, \varphi), (v, z))| \]
\[ \lesssim h^{1+\frac{d}{2}-\frac{d}{\min(p, p')}} (\| (f, g) \|_{\bigcup_{l \in \mathcal{H}} H^2_l} + \| \psi \|_{3,p} + \| \varphi \|_{2,p'}) \| (v, z) \|_h. \] (3.4)

Suppose \( \psi^* \in W^{3,p}(D), \varphi^* \in W^{2,p'}(D) \) and \( (f, g) \in W^{2,p}(D) \times L^2(D)(p, p' \in (\frac{4}{3}, 2)) \) then
\[ |D_h^*((v, z), (\psi^*, \varphi^*))| \]
\[ \lesssim h^{1+\frac{d}{2}-\frac{d}{\min(p, p')}} ((\| (f, g) \|_{W_2,p \times L^2} + \| \psi^* \|_{3,p} + \| \varphi^* \|_{2,p'}) \| (v, z) \|_h. \] (3.5)

**Proof.** The proof follows the way in Theorem 3.1 in [25] except for using the projection-mean operator \( I^*_h \). Since \( (I^*_h v, I^*_h z) \in H^1_h(D) \times H^1(D) \), by (2.14) and the integration by part we deduce
\[ B((f, g), (I^*_h v, I^*_h z)) = \sum_{\kappa \in \pi_h} \int_{\kappa} -\nabla \left( \frac{1}{n-1} \Delta \psi \right) \cdot \nabla I^*_h v + \nabla \varphi \cdot \nabla I^*_h z dx. \] (3.6)

For any \( (f, g) \in \bigcup_{l \in \mathcal{H}} H^2_l \times \bigcup_{l \in \mathcal{H}} H^1_l \), by the definition of \( B(\cdot, \cdot) \) and (2.11) we have
\[ |B((f, g), (v, z) - (I^*_h v, I^*_h z))| \lesssim \| (f, g) \|_{\bigcup_{l \in \mathcal{H}} H^2_l} \| (v - I^*_h v, z - I^*_h z) \|_{H^2_h \times L^2}
\[
\lesssim h \| (f, g) \|_{\bigcup_{l \in \mathcal{H}} H^2_l} \| (v, z) \|_{H^2_h \times L^1}. \] (3.7)
Since the integration by part leads to
\[
\sum_{\kappa \in \pi_h} \int \Delta \psi \Delta \tau - \left( \frac{\partial^2 \psi}{\partial x_1^2} \frac{\partial^2 \tau}{\partial x_1^2} + 2 \frac{\partial^2 \psi}{\partial x_1 \partial x_2} \frac{\partial \tau}{\partial x_2} + \frac{\partial^2 \psi}{\partial x_2^2} \frac{\partial^2 \tau}{\partial x_2^2} \right) dx
\]
\[
= \sum_{\kappa \in \pi_h} \int \left( \frac{\partial^2 \psi}{\partial x_1^2} \frac{\partial \tau}{\partial x_2} + \frac{\partial^2 \psi}{\partial x_2^2} \frac{\partial \tau}{\partial x_1} - 2 \frac{\partial^2 \psi}{\partial x_1 \partial x_2} \frac{\partial \tau}{\partial x_1 \partial x_2} \right) dx
\]
\[
= \sum_{\kappa \in \pi_h} \int \left\{ \frac{\partial^2 \psi}{\partial x_1 \partial x_2} \nu_2 + \frac{\partial^2 \psi}{\partial x_2 \partial x_1} \nu_1 \right\} ds
\]
\[
- \sum_{\kappa \in \pi_h} \int \left\{ \frac{\partial^2 \psi}{\partial x_1 \partial x_2} - \frac{\partial^2 \psi}{\partial x_1 \partial x_2} \right\} \nu_1 + \frac{\partial^2 \psi}{\partial x_1 \partial x_2} \nu_2 \} ds
\]
from (3.6), interpolation estimate (2.23) and inverse estimate we have
\[
B((f, g), (I_h^1 v, I_h^1 z)) - A_h((\psi, \varphi), (v, z))
\]
\[
= \sum_{\kappa \in \pi_h} \int -\nabla \left( \frac{1}{n - 1} \Delta \psi \right) \cdot \nabla I_h^1 v + \nabla \varphi \cdot \nabla I_h^1 dx - \sum_{\kappa \in \pi_h} \int \left\{ (\frac{1}{n - 1} - \mu_1) \Delta \psi \Delta \tau \right\}
\]
\[
+ \mu_1 \left( \frac{\partial^2 \psi}{\partial x_1^2} \frac{\partial \tau}{\partial x_1} + 2 \frac{\partial^2 \psi}{\partial x_1 \partial x_2} \frac{\partial \tau}{\partial x_2} + \frac{\partial^2 \psi}{\partial x_2^2} \frac{\partial \tau}{\partial x_1} \right) dx - (\nabla \varphi, \nabla z)_0
\]
\[
= \sum_{\kappa \in \pi_h} \int -\nabla \left( \frac{1}{n - 1} \Delta \psi \right) \cdot \nabla I_h^1 v - v dx - \int_{\partial \kappa} \frac{1}{n - 1} \Delta \psi \frac{\partial \nu}{\partial \nu} ds
\]
\[
+ (\nabla \psi, \nabla (I_h^1 z - z))_0 + \mu_1 \sum_{\kappa \in \pi_h} \int \Delta \psi \Delta \tau
\]
\[
- \left( \frac{\partial^2 \psi}{\partial x_1^2} \frac{\partial \tau}{\partial x_1} + 2 \frac{\partial^2 \psi}{\partial x_1 \partial x_2} \frac{\partial \tau}{\partial x_2} + \frac{\partial^2 \psi}{\partial x_2^2} \frac{\partial \tau}{\partial x_1} \right) dx
\]
\[
\lesssim h^{1 + (\frac{1}{d} - \frac{1}{2})d} \|\psi\|_{1,p} \|v\|_{H^1_0}
\]
\[
+ \mu_1 \sum_{\kappa \in \pi_h, \partial \kappa} \int \left( \frac{\partial^2 \psi}{\partial x_1 \partial x_2} \frac{\partial \nu}{\partial x_1} \nu_1 \right) dx - \int_{\partial \kappa} \frac{1}{n - 1} \Delta \psi \frac{\partial \nu}{\partial \nu} ds
\]
\[
- \mu_1 \sum_{\kappa \in \pi_h, \partial \kappa} \int \left\{ \frac{\partial^2 \psi}{\partial x_1 \partial x_2} \frac{\partial \nu}{\partial x_1 \partial x_2} \nu_1 + \frac{\partial^2 \psi}{\partial x_1 \partial x_2} \frac{\partial \nu}{\partial x_1 \partial x_2} \nu_2 \right\} ds + (\nabla \varphi, \nabla (I_h^1 z - z))_0
\]
(3.8)

Due to the weak continuity (2.19) and (2.20), we deduce that
\[
\sum_{\kappa \in \pi_h, \partial \kappa} \int \frac{\partial^2 \psi}{\partial x_1 \partial x_2} \frac{\partial \nu_1}{\partial x_1} \frac{\partial \nu_2}{\partial x_2} = \sum_{\kappa \in \pi_h, \partial \kappa} \int R_{\kappa}^1 \left( \frac{\partial^2 \psi}{\partial x_1} \right) R_{\kappa}^2 \left( \frac{\partial \nu}{\partial x_2} \right) ds.
\]
(3.9)

Let \( \hat{\kappa} \) be a reference element, \( \kappa \) and \( \hat{\kappa} \) be affine-equivalent. When \( \hat{w} \in W^{1,1}(\hat{\kappa}) \) and \( \hat{g} \in [1, \frac{d-1}{d-1}], \) by the trace theorem we get \( W^{1,1}(\hat{\kappa}) \leftrightarrow L^p(\partial \hat{\kappa}) \), thus we deduce the following trace inequality:
\[
\int_{\partial \kappa} |\hat{w}|^g ds = \int_{\partial \hat{\kappa}} |\hat{w}|^g \left( \frac{\partial \hat{\kappa}}{\partial \kappa} \right) ds \lesssim h_{\kappa}^{d-1} ||\hat{w}||_{0, g, \partial \kappa} \lesssim h_{\kappa}^{d-1} ||\hat{w}||_{1, \hat{\kappa}}^{g} \lesssim h_{\kappa}^{d-1} \left( h_{\kappa}^{g - \frac{ad}{d-1}} ||w||_{0, \partial \kappa}^{g} + h_{\kappa}^{g - \frac{ad}{d-1}} ||w||_{1, \kappa}^{g} \right)
\]
(3.10)
Since $g \in \mathcal{P}_{d-1}^\text{tr}$, the combination of (2.19) and (2.20) yields
\begin{align*}
\sum_{\kappa \in \pi_h} \sum_{\partial \kappa} \left| \int \frac{\partial^2 \psi}{\partial x_1^2} \frac{\partial \kappa}{\partial x_2} \right| & \lesssim \sum_{\kappa \in \pi_h} \sum_{\partial \kappa} \left| R_{\psi,0}^\kappa \left( \frac{\partial^2 \psi}{\partial x_1^2} \right) \right|_{\partial \kappa} + \sum_{\kappa \in \pi_h} \left| R_{\psi,0}^\kappa \left( \frac{\partial^2 \psi}{\partial x_1^2} \right) \right|_{\partial \kappa} \\
& \lesssim \sum_{\kappa \in \pi_h} \left( h^{d-\frac{d}{p'}} \right) \left( \frac{\partial^2 \psi}{\partial x_1^2} \right)_{\partial \kappa} + \sum_{\kappa \in \pi_h} \left( \frac{\partial^2 \psi}{\partial x_1^2} \right)_{\partial \kappa} \\
& \lesssim \sum_{\kappa \in \pi_h} \left( h^{d-\frac{d}{p'}} \right) \left( \frac{\partial^2 \psi}{\partial x_1^2} \right)_{\partial \kappa} + \sum_{\kappa \in \pi_h} \left( \frac{\partial^2 \psi}{\partial x_1^2} \right)_{\partial \kappa} \\
& \lesssim h^{1+\left(\frac{d}{2} - 1\right)d} \left\| \psi \right\|_{3,p} \left\| v \right\|_{H^2_k}. \quad (3.11)
\end{align*}

By the same argument it is clear that the third, fourth and fifth terms in (3.8) have similar estimates as above; also, similarly by (2.19) and (2.20)
\begin{align*}
\sum_{\kappa \in \pi_h} \int_{\partial \kappa} \nabla \varphi \cdot \nu z \lesssim h^{1+\frac{d}{2} - 1} \left\| \varphi \right\|_{2,p'} \left\| z \right\|_{H^1_k}, \quad \text{for } p' \in \left(\frac{d}{2}, 2\right]. \quad (3.12)
\end{align*}

On the other hand, if $\varphi \in W^{2,p'}(D)$ with $p' \in \left(\frac{d}{2}, 2\right]$ then we deduce from (3.12) and (2.23)
\begin{align*}
\left( \nabla \varphi, \nabla I^1_h z - \nabla hz \right)_0 = \sum_{\kappa \in \pi_h} \int_{\kappa} -\Delta \varphi (I^1_h z - z) + \int_{\kappa} \nabla \varphi \cdot \nu z \lesssim h^{1+\frac{d}{2} - 1} \left\| \varphi \right\|_{2,p'} \left\| z \right\|_{H^1_k}.
\end{align*}

Using the above two estimates, therefore the combination of (3.7) and (3.8) gives (3.4).

At last we shall prove (3.5). If $(f, g) \in W^{2,p}(D) \times L^2(D)$ then
\begin{align*}
\left( \nabla h (z - I^1_h z), \nabla f \right)_0 = - (z - I^1_h z, \Delta f)_0 + \sum_{\kappa \in \pi_h} \int_{\kappa} z \nabla f \cdot \nu d\kappa;
\end{align*}
both terms at the right-hand side above are bounded by $h^{1+\frac{d}{2} - 1} \left\| f \right\|_{2,p} \left\| z \right\|_{H^1_k}$; meanwhile
\begin{align*}
& \left| \left( \nabla h \left( \frac{1}{1-\tau} (v - I^1_h v) \right), \nabla f \right)_0 + \left( \nabla h (v - I^1_h v), \nabla \left( \frac{1}{n-\tau} f \right) \right)_0 - \left( \frac{n}{n-\tau} (v - I^1_h v), g \right)_0 \right| \\
& \lesssim h^{1+\frac{d}{2} - 1} \left\| (f, g) \right\|_{W_{2,p} \times L^2} \left\| (v, z) \right\|_{H^1_k \times H^1_k}.
\end{align*}
Then the combination of the two estimates above yields
\begin{align*}
\left| B^* ((f, g), (v, z) - (I^1_h v, I^1_h z)) \right| \lesssim h^{1+\frac{d}{2} - 1} \left\| (f, g) \right\|_{W_{2,p} \times L^2}.
\end{align*}
The left argument of (3.5) is similar as that of (3.4). \hfill \Box
Theorem 3.3. Suppose that $\psi \in W^{3,p}(D)$, $\varphi \in W^{2,p'}(D)(p,p' \in (\frac{d}{2},2])$. Then for $h$ small enough

$$
\| (\psi, \varphi) - (\psi_h, \varphi_h) \|_h \lesssim h^{1 + \frac{d}{2} - \min(p,p')} (\| (f,g) \|_{\bigcup_{l \in \mathbb{H}} H^2_l \times \bigcup_{l \in \mathbb{H}} H^1_l} + \| \psi \|_{3,p} + \| \varphi \|_{2,p'}) .
$$

(3.13)

Proof. By the interpolation error estimates (2.24)-(2.25) and

$$
\inf_{(\psi,\varphi) \in S_h} \| (\psi, \varphi) - (v, z) \|_h \leq \| \psi - I_h^2 \psi \|_{H^2_h} + \| \varphi - I_h^2 \varphi \|_{H^1_h}
$$

substituting (3.4) into (3.3) yields (3.13). The proof is completed. \qed

4. Error analysis of numerical eigenpair. Let $(\lambda, u, \omega)$ and $(\lambda^*, u^*, \omega^*)$ be the eigenpair of (2.10) and (2.15), respectively. Then from (3.1) and (3.2) we get that for any $(v,z) \in S_h$,

$$
D_h((u,\omega),(v,z)) = B_h(\lambda(u,\omega),(v,z)) - A_h((u,\omega),(v,z)),
$$

(4.1)

$$
D_h^*(((v,z),(u^*,\omega^*))) = B_h((v,z),\lambda^*(u^*,\omega^*)) - A_h((v,z),(u^*,\omega^*)). \tag{4.2}
$$

We need the following two regularity assumptions.

**R1(D).** For any $\xi \in H^{-1}(D)$, there exists $\psi \in W^{3,p_0}(D) \cap H^2_0(D)$ satisfying

$$
\Delta(\frac{1}{n-1}\Delta\psi) = \xi, \text{ in } D,
$$

and

$$
\| \psi \|_{3,p_0} \leq C_{R1}\| \xi \|_{-1}, \tag{4.3}
$$

where $p_0 \in (\frac{d}{2},2]$, $C_{R1}$ denotes the prior constant dependent on $D$ but independent of the right-hand side $\xi$ of the equation.

It is well known that (4.4) is valid when $n$ and $\partial D$ are appropriately smooth. For example, when $D \subset \mathbb{R}^2$ is a convex polygon, from Theorem 2 in [3], we can get that $p_0 = 2$.

**R2(D).** For any $\xi \in L^2(D)$, there exists $\psi \in W^{2,p_0'}(D) \cap H^2_0(D)$ satisfying

$$
\Delta\varphi = \xi, \text{ in } D,
$$

and

$$
\| \varphi \|_{2,p_0'} \leq C_{R2}\| \xi \|_0, \tag{4.4}
$$

where $p_0' \in (\frac{d}{2},2]$ depends on the largest inner angle of the domain $D$, $C_{R2}$ denote the prior constants dependent on $D$ but independent of the right-hand side $\xi$ of the equation (see [13]).

For any $(f,g) \in \bigcup_{l \in \mathbb{H}} H^2_l \times \bigcup_{l \in \mathbb{H}} H^1_l$ due to (2.13) the source problem (2.14) gives

$$
(\Delta(\frac{1}{n-1}\Delta\psi),v)_0 = \langle F,v \rangle, \forall v \in C_0^\infty(D),
$$

$$
(\Delta\varphi,z)_0 = \langle \frac{n}{n-1}f,z \rangle_0, \forall z \in C_0^\infty(D).
$$

Note that the right-hand side of the equations above belongs to $H^{-1}(D)$ and $L^2(D)$, respectively. According to **R1(D)** and **R2(D),

$$
\| \psi \|_{3,p_0} \lesssim \| (f,g) \|_{\bigcup_{l \in \mathbb{H}} H^2_l \times \bigcup_{l \in \mathbb{H}} H^1_l}, \| \varphi \|_{2,p_0'} \lesssim \| f \|_{\bigcup_{l \in \mathbb{H}} H^2_l} . \tag{4.5}
$$
Theorem 4.1. Assume $n \in W^{1,\infty}(D)$, $R1(D)$ and $R2(D)$ hold. Then
\[
\|T - T_h\|_{\bigcup_{i \in I} \mathcal{H}_i^2 \times \bigcup_{i \in I} \mathcal{H}_i^1} \lesssim h^{1 + \frac{d}{2} - \frac{d}{\min(p, p')}} \to 0 \text{ as } h \to 0.
\] (4.6)

Proof. We deduce from (3.13) and (4.5)
\[
\|T - T_h\|_{\bigcup_{i \in I} \mathcal{H}_i^2 \times \bigcup_{i \in I} \mathcal{H}_i^1} = \sup_{(f,g) \in \bigcup_{i \in I} \mathcal{H}_i^2 \times \bigcup_{i \in I} \mathcal{H}_i^1} \|(T - T_h)(f,g)\|_{\mathcal{H}_i^2 \times \mathcal{H}_i^1} \leq h^{1 + \frac{d}{2} - \frac{d}{\min(p, p')}}.
\]
The uniform convergence (4.6) follows. \hfill \square

In this paper, we suppose that $\lambda$ is an eigenvalue of (2.10) with the algebraic multiplicity $q$ and the ascent $\alpha$. Then $\lambda^* = \bar{\lambda}$ is an eigenvalue of (2.15). Since $\|T_h - T\|_{\bigcup_{i \in I} \mathcal{H}_i^2 \times \bigcup_{i \in I} \mathcal{H}_i^1} \to 0$, $q$ eigenvalues $\lambda_1, \ldots, \lambda_q$ of (2.21) will converge to $\lambda$. Let $E$ be the spectral projection associated with $T$ and $\lambda$, then $\text{Ran}(E) = N((\lambda - T))$ is the space of generalized eigens functions associated with $\lambda$ and $T$, where $\text{Ran}$ denotes the range and $N$ denotes the null space. In view of the adjoint problem (2.15), the definitions of $E^*$, $\text{Ran}(E^*)$ are analogous to $E$, $\text{Ran}(E)$(see [2]).

The spectral approximation theory in [2] leads to the following results.

Theorem 4.2. Assume that the conditions of Theorem 4.1 are valid. Let $(u_h, \omega_h)$ be eigenfunction corresponding to $\lambda_{i,h}$ ($i = 1, 2, \ldots, q$) and $\|(u_h, \omega_h)\|_h = 1$. Then there exists eigenfunction $(u, \omega)$ corresponding to $\lambda$, such that
\[
\|(u_h, \omega_h) - (u, \omega)\|_h \lesssim \|(T - T_h)\|_{\text{Ran}(E)} \bigg\|_{\bigcup_{i \in I} \mathcal{H}_i^2 \times \bigcup_{i \in I} \mathcal{H}_i^1},
\] (4.7)
\[
|\lambda_{i,h} - \lambda| \lesssim \|(T - T_h)\|_{\text{Ran}(E)} \bigg\|_{\bigcup_{i \in I} \mathcal{H}_i^2 \times \bigcup_{i \in I} \mathcal{H}_i^1}. \tag{4.8}
\]
\[
\left|\frac{1}{q} \sum_{i=1}^{q} \lambda_{i,h}^{-1} - \lambda\right| \lesssim \|(T - T_h)\|_{\text{Ran}(E)} \bigg\|_{\bigcup_{i \in I} \mathcal{H}_i^2 \times \bigcup_{i \in I} \mathcal{H}_i^1}. \tag{4.9}
\]
Furthermore assume $R(E) \subset W^{3,p}(D) \times W^{2,p'}(D)$ ($p, p' \in (\frac{d}{2}, 2)$), then
\[
\|(T - T_h)\|_{\text{Ran}(E)} \bigg\|_{\bigcup_{i \in I} \mathcal{H}_i^2 \times \bigcup_{i \in I} \mathcal{H}_i^1} \lesssim h^{1 + \frac{d}{2} - \frac{d}{\min(p, p')}} \tag{4.10}
\]
and
\[
\|(u_h, \omega_h) - (u, \omega)\|_h \lesssim h^{\frac{1}{2} + \left(\frac{1}{\min(p, p')} - \frac{1}{d}\right)}, \tag{4.11}
\]
with $\|(u, \omega)\|_h = 1$.

Proof. From Theorem 4.1 we know $\|T - T_h\|_{\bigcup_{i \in I} \mathcal{H}_i^2 \times \bigcup_{i \in I} \mathcal{H}_i^1} \to 0$ ($h \to 0$), thus from Theorem 7.4, Theorem 7.3 and Theorem 7.2 of [2] we get (4.7), (4.8) and (4.9),
respectively. From Theorem 3.3 we get
\[
\| (T - T_h) |_{\text{Ran}(E)} \|_{\bigcup_{l \in \mathcal{K}} H_l^2 \times \bigcup_{l \in \mathcal{K}} H_l^1} \\
= \sup_{(u, \omega) \in \text{Ran}(E), \| (u, \omega) \|_H = 1} \| (T - T_h)(u, \omega) \|_{H^2_h \times H^1_h} \\
\lesssim h^{1 + \frac{d}{2} - \min\{p, q\}}.
\]
(4.12)
Substituting (4.12) into (4.7), we get
\[
\| (u_h, \omega_h) - (u, \omega) \|_H \lesssim h^{1 + \frac{d}{2} - \min\{p, q\}} + \frac{d}{2}.
\]
(4.13)
It is easy to know that when replacing \((u, \omega)\) by \(\frac{(u, \omega)}{\| (u, \omega) \|_H}\), (4.13) also holds.

Obviously the estimate (4.9) for the eigenvalues is not optimal. In what follows we shall improve it by using the following identity which can be found in [27].

**Lemma 4.1.** Let \((\lambda, u, \omega)\) and \((\lambda^*, u^*, \omega^*)\) be the eigenpairs of (2.10) and (2.15) respectively. Then for any \((v, z), (v^*, z^*) \in S_h\), when \(B_h((v, z), (v^*, z^*)) \neq 0\) it is valid that
\[
\frac{A_h((v, z), (v^*, z^*))}{B_h((v, z), (v^*, z^*))} - \frac{A_h((u, \omega) - (v, z), (u^*, \omega^*) - (v^*, z^*))}{B_h((v, z), (v^*, z^*))} \\
- \lambda \frac{B_h((u, \omega) - (v, z), (u^*, \omega^*) - (v^*, z^*))}{B_h((v, z), (v^*, z^*))} + \frac{D_h((u, \omega), (v^*, z^*))}{B_h((v, z), (v^*, z^*))} + \frac{D_h((v, z), (u^*, \omega^*))}{B_h((v, z), (v^*, z^*))} = 0.
\]
(4.14)

**Theorem 4.3.** Assume that the conditions of Theorem 4.2 are valid, \(\text{Ran}(E), \text{Ran}(E^*) \subset W^{3,p}(D) \times W^{2,p'}(D)\) \((p, p' \in \left\{\frac{d}{2}, 2\right\})\), and the ascent \(\alpha\) of \(\lambda\) is equal to 1, then
\[
|\lambda_h - \lambda| \lesssim h^{2 + 2d\left(\frac{1}{2} - \min\{p, q\}\right)}.
\]
(4.15)

**Proof.** From \(\alpha = 1\), we know \(\text{Ran}(E^*)\) is the space of eigenfunctions associated with \(\lambda^*\). Let \((u, \omega)\) and \((u_h, \omega_h)\) satisfy (4.7) and (4.11). Since \((u, \omega) \in \text{Ran}(E)\), from Hahn-Banach Theorem we know that there exists \((u', \omega') \in \text{Ran}(E)^*\) such that
\[
A((u, \omega), (u', \omega')) = \| (u, \omega) \|_A = 1, \| (u', \omega') \|_A = 1.
\]
(4.16)
Since \(\mathbf{H} = \text{Ran}(E) \oplus N(E)\), any \((v, z) \in \mathbf{H}\) can be written as \((v, z) = (v, z)_1 + (v, z)_2\) with \((v, z)_1 \in \text{Ran}(E)\) and \((v, z)_2 \in N(E)\). Define
\[
A((v, z), (u^*, \omega^*)) = A((v, z)_1, (u', \omega')).
\]
Clearly \((u^*, \omega^*)\) so extended, is bounded, i.e., \((u^*, \omega^*) \in \mathbf{H}^*\). Now, for any \((v, z) \in \mathbf{H}\),
\[
A((v, z), (\lambda^{-1} - T)^{\alpha}(u^*, \omega^*)) = A((\lambda^{-1} - T)^{\alpha}(v, z), (u^*, \omega^*)) \\
= A((\lambda^{-1} - T)^{\alpha}(v, z)_1, (u^*, \omega^*)) + A((\lambda^{-1} - T)^{\alpha}(v, z)_2, (u^*, \omega^*)) = 0,
\]
i.e., \((\lambda^{-1} - T)^{\alpha}(u^*, \omega^*) = 0\). Hence \((u^*, \omega^*) \in \text{Ran}(E^*)\). By (4.16) we have
\[
A((u, \omega), (u^*, \omega^*)) = A((u, \omega), (u', \omega')) = \| (u, \omega) \|_A = 1.
\]
(4.17)
Then, from (4.11), (2.24), (2.25) and (4.17), when \(h\) is small enough, \(|A_h((u_h, \omega_h), (I_h^2 u^*, I_h^2 \omega^*))|\) has a positive lower bound uniformly with respect to \(h\), thus there is a positive constant \(C_0\) independent of \(h\) such that
\[
|B_h((u_h, \omega_h), (I_h^2 u^*, I_h^2 \omega^*))| = |\lambda^{-1}_h A_h((u_h, \omega_h), (I_h^2 u^*, I_h^2 \omega^*))| \geq C_0.
\]
(4.18)
then according to Lemma 4.1

\[
|\lambda_h - \lambda| \lesssim \|(u, \omega) - (u_h, \omega_h)\|_h \|(u^*, \omega^*) - (I_h^0 u^*, I_h^0 \omega^*)\|_h \\
+ |D_h((u, \omega), (I_h^0 u^*, I_h^0 \omega^*))| + |D_h((u_h, \omega_h), (u^*, \omega^*))|. \tag{4.19}
\]

Let \( P_h : H^2_0(D) \to H^2_0(D) \) be the orthogonal projection from \( H^2_0(D) \) to \( C^1 \)-conforming finite element space defined on \( \pi_h \) in the sense of \( A(\cdot, \cdot) \). By (2.24) and (2.25), the estimate (3.4) gives

\[
|D_h((u, \omega), (I_h^0 u^*, I_h^0 \omega^*))| = |D_h((u, \omega), (I_h^0 u^*, I_h^0 \omega^*) - (P_h u^*, P_h \omega^*))| \\
\lesssim h^{1.5 - \frac{1}{\min(p, r)} d} \|(u, \omega)\|_{W^{3,p} \times W^{2,r'}} \\
-\|(I_h^0 u^*, I_h^0 \omega^*) - (u^*, \omega^*) - (P_h u^*, P_h \omega^*)\|_h \tag{4.20}
\]

and by (4.11) the estimate (3.5) gives

\[
|D_h((u_h, \omega_h), (u^*, \omega^*))| = |D_h((u_h, \omega_h) - (P_h u, P_h \omega), (u^*, \omega^*))| \\
\lesssim h^{2.5 - \frac{1}{\min(p, r)} d} \|(u, \omega)\|_{W^{3,p} \times W^{2,r'}} \|(u^*, \omega^*)\|_{W^{3,p} \times W^{2,r'}}. \tag{4.21}
\]

Substituting (4.11), (2.24), (2.25), (4.20) and (4.21) into (4.19), we get (4.15). □

5. Numerical experiment. In this section, we will report some numerical experiments using nonconforming finite element discretizations to validate our theoretical results. We use Matlab 2012a to solve (2.1)-(2.4) on a Lenovo G480 PC with 4G memory. Our program is implemented using the package of iFEM [8].

First of all, in order to derive our discretization in matrix form some notations shall be introduced. We set the finite element eigenfunctions \( u_h = \sum_{j=1}^{N_h} u_j \xi_j \) and \( \omega_h = \sum_{j=1}^{N_h} \omega_j \phi_j \), where \( \{\xi_j\}_{j=1}^{N_h} \) and \( \{\phi_j\}_{j=1}^{N_h} \) are bases of \( U^h \) and \( V^h \), respectively. Denote \( \vec{u} = (u_1, \cdots, u_{N_h})^T \) and \( \vec{\omega} = (\omega_1, \cdots, \omega_{M_h})^T \). To describe our algorithm, we specify the following matrices in the discrete case.

| Matrix | Dim | Definition |
|--------|-----|------------|
| \( A_h \) | \( N_h \times N_h \) | \( a_{l,j} = \sum_n (\frac{\pi^2 \xi_l \xi_j}{\pi - \xi_l} - \mu_l \Delta \xi_l \Delta \xi_j + \mu_1 \frac{\pi^2 \xi_l \xi_j}{\pi - \xi_l} + \mu_2 \frac{\pi^2 \xi_l \xi_j}{\pi - \xi_l} + \mu_3 \frac{\pi^2 \xi_l \xi_j}{\pi - \xi_l} + \mu_4 \frac{\pi^2 \xi_l \xi_j}{\pi - \xi_l}) \) |
| \( B_h \) | \( N_h \times N_h \) | \( b_{l,j} = \sum_n (\nabla (\frac{\pi \xi_l - \xi_j}{\pi - \xi_j}) \cdot \nabla \xi_l + \nabla \xi_j \cdot \nabla (\frac{\pi \xi_l - \xi_j}{\pi - \xi_j})) dx \) |
| \( C_h \) | \( N_h \times M_h \) | \( c_{l,j} = \sum_n \nabla \xi_l \cdot \phi_j dx \) |
| \( D_h \) | \( M_h \times N_h \) | \( d_{l,j} = -\int_D \nabla \xi_l \cdot \phi_j dx \) |
| \( E_h \) | \( M_h \times M_h \) | \( e_{l,j} = -\int_D \nabla \xi_l \cdot \phi_j dx \) |

where \( N_h = \dim(U_h) \) and \( M_h = \dim(V_h) \). Then (2.21) can be written in the generalized eigenvalue problem

\[
\begin{pmatrix}
A_h & 0 \\
0 & D_h
\end{pmatrix}
\begin{pmatrix}
\vec{u} \\
\vec{\omega}
\end{pmatrix}
= \lambda_h
\begin{pmatrix}
B_h & C_h \\
D_h & 0
\end{pmatrix}
\begin{pmatrix}
\vec{u} \\
\vec{\omega}
\end{pmatrix}. \tag{5.1}
\]

Next we consider the model problems (2.1)-(2.4) with the refraction index \( n = 8 + x_1 - x_2 \) or \( n = 16 \) on the unit square \((0, 1)^2\), L-shaped \((-1, 1)^2 \setminus [0, 1] \times (-1, 0] \), thick L-shaped \((-1, 1)^2 \setminus (-1, 0)^2 \) \times \((0, 1)\), cube \((0, 1)^3\), sphere with radius \( \frac{1}{2} \) and center \((0, 0, 0)\). To solve the problem on 2D/3D domain we take \( V_h \) to be the linear Lagrange element space, and take \( U_h \) to be the Morley-Zienkiewicz(MZ) element space [20] on 2D domains or the cubic tetrahedron element on 3D domains. In computation, quasi-uniform meshes are adopted both on 2D/3D domains. Next let us recall the definition of the Cubic tetrahedron(CT) element which was put up forward in [24]. Its finite element space is defined as follows:
\[ U_h = \{ v \in U_h[v \text{ and } \nabla v \text{ vanish at all vertices on } \partial \Omega, \text{ and at the barycenter of face } F \subset \partial \Omega \frac{\partial v}{\partial \nu} \text{ vanishes} \}, \]

where

\[ U_h = \{ v \in L^2(\Omega)|v|_{\kappa} \in P_3(\kappa), \forall \kappa \in \pi_h; v \text{ and } \nabla v \text{ are continuous at all vertices of } \pi_h, \text{ and at the barycenter of each face } F \text{ of } \pi_h \frac{\partial v}{\partial \nu} \text{ is continuous} \}. \]

Next we consider the numerical implementation of our discretion. In our computation, for all the domains mentioned above we set \( \mu_1 = \frac{1}{9} \) when the refraction index \( n = 8 + x_1 - x_2 \) and set \( \mu_1 = \frac{1}{15} \) when the refraction index \( n = 16 \). The numerical eigenvalues on 2D domains and 3D domains computed by MZ element mixed with linear element and CT element mixed with linear element are listed in Table 1 and Tables 2, 4, 6 and 8 respectively, while the error curves corresponding to numerical eigenvalues on the square and L-shaped are depicted in Figure 1. For reading conveniently, in our tables and figures we use the notation \( k_{\text{Dom}}^{h,j} = \sqrt{\lambda_{\text{Dom}}^{h,j}} \) to denote the \( j \)th eigenvalue on the domain \( \text{Dom} = S, L, TL, C, Sp \) obtained by (2.21) on \( \pi_h \), where the symbols \( S, L, TL, C, Sp \) denote the domains square, L-shaped, thick L-shaped, cube, sphere, respectively. Note that these numerical eigenvalues are listed in sequence from small magnitude to large one.

It is seen from Figure 1 that the convergence orders of the numerical eigenvalues on the unit square computed by MZ element mixed with linear element are around 2, which coincides with the theoretical result. Nevertheless, the convergence orders on the L-shaped domain of the numerical eigenvalues \( k_{1,h}, k_{2,h}, k_{5,h}, k_{6,h} \) with \( n = 8 + x_1 - x_2 \) and \( k_{1,h}, k_{3,h}, k_{4,h} \) with \( n = 16 \) are less than 2. This phenomenon is caused by the singularities of the eigenfunctions corresponding to these eigenvalues on the L-shaped domain.

Comparing the results in Tables 1-2 of [27] with Table 1, we see that in 2D domains the solving accuracy of the discretization (5.1) using MZ element mixed with linear element is less than that of the discretization using MZ element in [27]. However, this is not true for the case of 3D domains when \( U_h \) is taken as CT element space. We continue to show the advantage of CT element mixed with linear element for solving the problem in 3D. Tables 2, 4, 6 and 8 list some numerical eigenvalues computed by CT element on the Cube, the tetrahedron, the thick L-shaped and the sphere, respectively. Using the same meshes as CT element does, we also adopt MZ element using the discretization in [27] to compute the numerical eigenvalues which are listed in Tables 3, 5, 7 and 9. For comparative purpose, some reference values can be found in the existing literatures(see [26, 14]) and we list them as follows.

| Domain       | \( n \) | Eigenvalue | Ref.                  |
|--------------|---------|------------|-----------------------|
| cube         | 16      | \( k_1, k_2, k_5 \) | 2.0672, 2.5849, 2.9870 |
| cube         | 8+\( x_1 - x_2 \) | \( k_1, k_2, k_3, k_4 \) | 3.0257, 3.7221, 3.7247, 3.7248 |
| thick L-shaped | 16     | \( k_1, k_2, k_3 \) | 1.8076, 1.8706, 1.9292 |
| thick L-shaped | 8+\( x_1 - x_2 \) | \( k_1, k_2, k_3 \) | 2.6422, 2.6464, 2.8419 |
| tetrahedron  | 16      | \( k_1, k_2, k_5 \) | 2.7808, 3.3469, 3.9443 |
| sphere       | 16      | \( k_1, k_2, k_5 \) | 2.3048, 2.9271, 3.5345 |

By comparison, we see CT element mixed with linear element can capture the numerical eigenvalues on 3D domains of higher accuracy than MZ element of using the discretization in [27]. It worths noticing that the number of degrees of freedom of CT element is same as MZ element. We think the reason of the higher accuracy
Table 1. The eigenvalues obtained by MZ element mixed with linear element in 2D domains.

| $h$ | $j$ | $k_{i,h}^S$ | $k_{i,h}^L$ | $n = 16$ | $n = 8 + x_1 - x_2$ |
|-----|-----|-------------|-------------|----------|-------------------|
| 1   | 1   | 1.870983492 | 1.4773077   | 1        | 2.819407434       |
| 1   | 1   | 1.879447918 | 1.4767038   | 1        | 2.821437358       |
| 1   | 1   | 1.879554183 | 1.4769944   | 1        | 2.821996985       |
| 2   | 2   | 2.443168293 | 1.5694361   | 2        | 3.534452084       |
| 2   | 2   | 2.443929469 | 1.5696511   | 2        | 3.537524631       |
| 2   | 2   | 2.444156467 | 1.5697071   | 2        | 3.538394853       |
| 3   | 2   | 2.443168293 | 1.7049725   | 5        | 4.498450          |
| 3   | 2   | 2.443929469 | 1.7051045   | 5        | 4.496942          |
| 3   | 2   | 2.444156467 | 1.7050978   | 5        | 4.496644          |
| 4   | 2   | 2.865221584 | 1.7827065   | 6        | 4.498450          |
| 4   | 2   | 2.866032777 | 1.7830026   | 6        | 4.496942          |
| 4   | 2   | 2.866819734 | 1.7836069   | 6        | 4.496644          |

Table 2. The eigenvalues obtained by CT element mixed with linear element on the cube.

| Dof  | $j$  | $k_{i,h}^C(n = 16)$ | Dof  | $j$  | $k_{i,h}^C(n = 8 + x_1 - x_2)$ |
|------|-----|--------------------|------|-----|-------------------|
| 1895 | 1   | 2.0449             | 1895 | 1   | 2.9579            |
| 1895 | 2,3,4| 2.5461             | 1895 | 3   | 3.0162            |
| 1895 | 3   | 2.5714             | 1895 | 4   | 3.6098            |
| 1895 | 4   | 2.5785             | 1895 | 5   | 3.6853            |
| 1895 | 5,6,7| 2.9402             | 1895 | 6   | 3.9049            |
| 55903| 1   | 2.0641             | 55903| 2   | 3.6098            |
| 55903| 2   | 2.0641             | 55903| 3   | 3.6853            |
| 55903| 3   | 2.0641             | 55903| 4   | 3.9049            |
| 55903| 4   | 2.0641             | 55903| 5   | 3.6098            |

Table 3. The eigenvalues obtained by MZ element on the cube.

| Dof  | $j$  | $k_{i,h}^C(n = 16)$ | Dof  | $j$  | $k_{i,h}^C(n = 8 + x_1 - x_2)$ |
|------|-----|--------------------|------|-----|-------------------|
| 3608 | 1   | 2.1651             | 3608 | 1   | 3.2181            |
| 3608 | 2   | 2.0935             | 3608 | 2   | 4.0750            |
| 3608 | 3,4 | 2.7794,2.7803,2.7807| 3608 | 3   | 4.0750            |
| 3608 | 3   | 2.6350,2.6350,2.6351| 3608 | 4   | 3.8098            |
| 3608 | 4   | 2.6007             | 3608 | 5   | 3.7596            |
| 3608 | 5,6,7| 3.2927,3.2959,3.2973| 3608 | 6   | 3.4079,4.0800     |
| 3608 | 6   | 3.0680,3.0682,3.0684| 3608 | 7   | 3.8126,3.8126     |
| 3608 | 7   | 3.0234             | 3608 | 8   | 3.7622,3.7623     |

of CT element is due to its higher degree of polynomial space than MZ element.

Acknowledgments. We cordially thank the referees and the editor for their valuable comments and suggestions that led to the insignificant improvement of this paper.
Table 4. The eigenvalues obtained by CT element mixed with linear element on the tetrahedron, $n = 16$.

| Dof     | $k_{1,h}^T$ | $k_{2,h}^T, k_{3,h}^T, k_{4,h}^T$ | $k_{5,h}, k_{6,h}, k_{7,h}^T$ |
|---------|-------------|----------------------------------|-------------------------------|
| 1071    | 2.7560      | 3.3119, 3.3119, 3.3617            | 3.9151, 3.9623, 3.9738        |
| 9955    | 2.7650      | 3.3220, 3.3239, 3.3239            | 3.9144, 3.9151, 3.9151        |
| 85963   | 2.7758      | 3.3371, 3.3396, 3.3396            | 3.9320, 3.9320, 3.9342        |

Table 5. The eigenvalues obtained by MZ element on the tetrahedron, $n = 16$.

| Dof     | $k_{1,h}^T$ | $k_{2,h}^T, k_{3,h}^T, k_{4,h}^T$ | $k_{5,h}, k_{6,h}, k_{7,h}^T$ |
|---------|-------------|----------------------------------|-------------------------------|
| 2072    | 3.2333      | 4.0873, 4.0873, 4.1929            | 5.1997, 5.1997, 5.2041        |
| 19000   | 2.8870      | 3.5128, 3.5128, 3.5666            | 4.2089, 4.2528, 4.2557        |
| 162936  | 2.8051      | 3.3852, 3.3852, 3.3980            | 4.0036, 4.0185, 4.0185        |

Table 6. The eigenvalues obtained by CT element mixed with linear element on the thick L-shaped.

| Dof     | $j$ | $k_{j,h}^T (n = 16)$ | $j$ | $k_{j,h}^T (n = 8 + x_1 - x_2)$ |
|---------|-----|---------------------|-----|--------------------------------|
| 257     | 1   | 2.4603              | 1   | 3.3865                          |
| 2575    | 1   | 1.9756              | 1   | 2.8223                          |
| 23171   | 1   | 1.8209              | 1   | 2.6248                          |
| 257     | 2   | 2.5244              | 2   | 3.4304                          |
| 2575    | 2   | 2.0359              | 2   | 2.8372                          |
| 23171   | 2   | 1.8801              | 2   | 2.6415                          |
| 257     | 3   | 2.5311              | 3   | 3.5598                          |
| 2575    | 3   | 2.0974              | 3   | 3.0510                          |
| 23171   | 3   | 1.9431              | 3   | 2.8327                          |

Table 7. The eigenvalues obtained by MZ element on the thick L-shaped.

| Dof     | $j$ | $k_{j,h}^T (n = 16)$ | $j$ | $k_{j,h}^T (n = 8 + x_1 - x_2)$ |
|---------|-----|---------------------|-----|--------------------------------|
| 504     | 1   | 3.6272              | 1   | 4.5809 - 2.0747i                |
| 4952    | 1   | 2.5041              | 1   | 3.9589                          |
| 44088   | 1   | 1.9305              | 1   | 2.8511                          |
| 504     | 2   | 3.9842              | 2   | 4.5809 + 2.0747i                |
| 4952    | 2   | 2.7611              | 2   | 4.5904                          |
| 44088   | 2   | 1.9943              | 2   | 2.8939                          |
| 504     | 3   | 4.6844              | 3   | 5.2114 - 2.2381i                |
| 4952    | 3   | 2.8373              | 3   | 4.9593                          |
| 44088   | 3   | 2.0851              | 3   | 3.1157                          |

REFERENCES

[1] J. An and J. Shen, A spectral-element method for transmission eigenvalue problems, J. Sci. Comput., 57 (2013), 670–688.
[2] I. Babuska, J. E. Osborn, Eigenvalue Problems, in: P.G. Ciarlet, J.L. Lions,(Ed.), Finite Element Methods (Part 1), Handbook of Numerical Analysis, Elsevier Science Publishers, North-Holland, 2 (1991), 641–787.
[3] H. Blum and R. Rannacher, On the boundary value problem of the biharmonic operator on domains with angular corners, Math Method Appl Sci, 2 (1980), 556–581.
[4] S. C. Brenner and L. R. Scott, The Mathematical Theory of Finite Element Methods, 2nd ed., Springer-Verlag, New York, 2002.
TABLE 8. The eigenvalues obtained by CT element mixed with linear element on the sphere, \( n = 16 \).

| Dof | \( k_{1,h}^{SP} \) | \( k_{2,h}^{SP}, k_{3,h}^{SP}, k_{4,h}^{SP} \) | \( k_{5,h}^{SP}, k_{6,h}^{SP}, k_{7,h}^{SP} \) |
|-----|-----------------|-------------------------------|-------------------------------|
| 1608 | 2.3431          | 2.9533, 2.9550, 2.9564       | 3.5575, 3.5597, 3.5600 |
| 16643| 2.3075          | 2.9250, 2.9252, 2.9254       | 3.5257, 3.5260, 3.5261 |
| 51277| 2.3035          | 2.9229, 2.9229, 2.9230       | 3.5261, 3.5262, 3.5263 |

TABLE 9. The eigenvalues obtained by MZ element on the sphere, \( n = 16 \).

| Dof | \( k_{1,h}^{SP} \) | \( k_{2,h}^{SP}, k_{3,h}^{SP}, k_{4,h}^{SP} \) | \( k_{5,h}^{SP}, k_{6,h}^{SP}, k_{7,h}^{SP} \) |
|-----|-----------------|-------------------------------|-------------------------------|
| 3076 | 2.4860          | 3.3110, 3.3259, 3.3441       | 4.2252, 4.2297, 4.2574 |
| 31568| 2.3306          | 2.9875, 2.9886, 2.9905       | 3.6451, 3.6471, 3.6520 |
| 97048| 2.3147          | 2.9520, 2.9521, 2.9529       | 3.5817, 3.5825, 3.5848 |

Figure 1. Error curves computed by MZ element mixed with linear element for \( n = 16 \) on the unit square (left top) and on the L-shaped (right top), for \( n = 8 + x_1 - x_2 \) on the unit square (left bottom) and on the L-shaped (right bottom).

[5] F. Cakoni and H. Haddar, On the existence of transmission eigenvalues in an inhomogeneous medium, Appl. Anal., 88 (2009), 475–493.
[6] F. Cakoni, D. Colton, P. Monk and J. Sun, The inverse electromagnetic scattering problem for anisotropic media, Inverse Problems, 26 (2010), 074004, 14pp.
[7] F. Cakoni, P. Monk and J. Sun, Error analysis for the finite element approximation of transmission eigenvalues, Comput. Meth. Appl. Math., 14 (2014), 419–427.
[8] L. Chen, iFEM: An integrated finite element method package in MATLAB., Technical Report, University of California at Irvine, 2009.
[9] P. G. Ciarlet, Basic error estimates for elliptic problems, in: P.G. Ciarlet, J.L. Lions, (Ed.), Finite Element Methods (Part1), Handbook of Numerical Analysis, vol.2, Elsevier Science Publishers, North-Holland, 1991, 21–343.
[10] D. Colton, P. Monk and J. Sun, Analytical and computational methods for transmission eigenvalues, Inverse Problems, 26 (2010), 045011, 16pp.
[11] D. Colton, L. Päivärinta and J. Sylvester, The interior transmission problem, Inverse Problem Imaging, 1 (2007), 13–28.
[12] H. Geng, X. Ji, J. Sun and L. Xu, C0IP methods for the transmission eigenvalue problem, J. Sci. Comput., 68 (2016), 326–338.
[13] P. Grisvard, Elliptic Problems in Nonsmooth Domains, Pitman, London, 1985.
[14] J. Han and Y. Yang, An adaptive finite element method for the transmission eigenvalue problem, J. Sci. Comput., 69 (2016), 1–22.
[15] X. Ji, J. Sun and T. Turner, Algorithm 922: A mixed finite element method for Helmholtz transmission eigenvalues, ACM Transaction on Math. Soft., 38 (2012), Art. 29, 8 pp.
[16] X. Ji, J. Sun and H. Xie, A multigrid method for Helmholtz transmission eigenvalue problems, J. Sci. Comput., 60 (2014), 276–294.
[17] A. Kleefeld, A numerical method to compute interior transmission eigenvalues, Inverse Problems, 29 (2013), 104012, 20pp.
[18] J. T. Oden and J. N. Reddy, An Introduction to the Mathematical Theory of Finite Elements, New York-London-Sydney, 1976.
[19] B. P. Rynne and B. D. Sleeman, The interior transmission problem and inverse scattering from inhomogeneous media, SIAM J. Math. Anal., 22 (1991), 1755–1762.
[20] Z. Shi and M. Wang, Finite Element Methods, Beijing, Scientific Publishers, 2013.
[21] J. Sun, Estimation of transmission eigenvalues and the index of refraction from Cauchy data, Inverse Probl, 27 (2011), 015009, 24pp.
[22] J. Sun, Iterative methods for transmission eigenvalues, SIAM J. Numer. Anal., 49 (2011), 1860–1874.
[23] M. Wang, Z. Shi and J. Xu, A new class of Zienkiewicz-type nonconforming element in any dimensions, Numer. Math., 106 (2007), 335–347.
[24] M. Wang and J. Xu, Nonconforming tetrahedral finite elements for fourth order elliptic equations, Math. Comp., 76 (2007), 1–18.
[25] Y. Yang, J. Han, H. Bi, Error estimates and a two grid scheme for approximating transmission eigenvalues, arXiv: 1506.06486 V2 [math. NA] 2 Mar 2016.
[26] Y. Yang, H. Bi, H. Li and J. Han, Mixed method for the helmholtz transmission eigenvalues, SIAM J. Sci. Comput., 38 (2016), A1383–A1403.
[27] Y. Yang, J. Han and H. Bi, Non-conforming finite element methods for transmission eigenvalue problem, Comput. Methods Appl. Mech. Engrg., 307 (2016), 144–163.
[28] F. Zeng, J. Sun and L. Xu, A spectral projection method for transmission eigenvalues, Sci China Math, 59 (2016), 1613–1622.

Received June 2017; revised November 2017.
E-mail address: hanjiayu126@126.com