RELATIVISTIC BGK MODEL FOR MASSLESS PARTICLES IN THE FLRW SPACETIME

BYUNG-HOON HWANG
Department of Mathematics
Sungkyunkwan University, Suwon 440-746, Republic of Korea

HO LEE
Department of Mathematics and Research Institute for Basic Science
Kyung Hee University, Seoul, 02447, Republic of Korea

SEOK-BAE YUN
Department of Mathematics
Sungkyunkwan University, Suwon 440-746, Republic of Korea

(Communicated by Tao Luo)

Abstract. In this paper, we address the Cauchy problem for the relativistic BGK model proposed by Anderson and Witting for massless particles in the Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime. We first derive the explicit form of the Jüttner distribution in the FLRW spacetime, together with a set of nonlinear relations that leads to the conservation laws of particle number, momentum, and energy for both Maxwell-Boltzmann particles and Bose-Einstein particles. Then, we find sufficient conditions that guarantee the existence of equilibrium coefficients satisfying the nonlinear relations and we show that the condition is satisfied through all the induction steps once it is verified for the initial step. Using this observation, we construct explicit solutions of the relativistic BGK model of Anderson-Witting type for massless particles in the FLRW spacetime.

1. Introduction.

1.1. Relativistic BGK model. We consider the Anderson–Witting model \[1\]:

\[ p^\alpha \frac{\partial F}{\partial x^\alpha} - \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \frac{\partial F}{\partial p^\alpha} = - \frac{U^\mu p_\mu}{c^2} (J(F) - F). \] (1.1)

Here \( F = F(x^\alpha, p^\alpha) \) is the distribution function representing a number density of particles in the phase space spanned by the spacetime coordinates \( x^\alpha \) and four-momentum \( p^\beta \). The Greek indices run from 0 to 3 and the repeated Greek indices are assumed to be summed over its whole range. In (1.1), \( \Gamma^\alpha_{\beta\gamma} \) denote the Christoffel symbols, \( U^\mu \) is the Landau-Lifshitz four-velocity defined by the moments of \( F \) (See

2020 Mathematics Subject Classification. 76P05, 83C10, 82C40, 35A01.
Key words and phrases. Kinetic theory of gases, relativistic Boltzmann equation, relativistic BGK model, Anderson-Witting model, FLRW spacetime.
(1.9) below), $c$ is the speed of light, $\tau$ is the characteristic time of order of the mean free time, and $J(F)$ is the Jüttner distribution [16, 17] which takes the form of

$$J(F) = \begin{cases} \frac{g_s}{\hbar^3} \exp \left\{ \frac{\mu_E}{kT_E} + \frac{U_{E\alpha}}{kT_E} \right\} & \text{for the relativistic Maxwell-Boltzmann statistics,} \\ \exp \left\{ -\frac{\mu_E}{kT_E} - \frac{U_{E\alpha}}{kT_E} \right\} & \text{for the relativistic Bose-Einstein statistics,} \end{cases}$$

where $\hbar$ is the Planck constant, $k$ is the Boltzmann constant, $g_s$ is the degeneracy factor, $\mu_E$ is the chemical potential, $U_{E\alpha}$ is the four-velocity, and $T_E$ is the equilibrium temperature. Note that $\mu_E$, $U_{E\alpha}$ and $T_E$ are functions of $t$ and $x$ determined by the relations

$$U^\mu \int_{\mathbb{R}^3} p_\alpha (J(F) - F) \sqrt{-|\eta|} \frac{dp}{p^0} = 0, \quad U^\mu \int_{\mathbb{R}^3} p_\alpha p^\nu (J(F) - F) \sqrt{-|\eta|} \frac{dp}{p^0} = 0,$$

where $|\eta|$ denotes the determinant of the metric tensor $\eta_{\alpha\beta}$ and $dp = dp^1 dp^2 dp^3$, so that the conservation laws of particle number, momentum and energy for (1.1) hold true. To be precise, the particle four-flow $N^\alpha$ and the energy-momentum tensor $T^{\alpha\beta}$ are defined by

$$N^\alpha = \int_{\mathbb{R}^3} p^\alpha F \sqrt{-|\eta|} \frac{dp}{p^0}, \quad T^{\alpha\beta} = \int_{\mathbb{R}^3} p^\alpha p^\beta F \sqrt{-|\eta|} \frac{dp}{p^0}, \quad (1.2)$$

and the following conservation laws hold:

$$N^{\alpha;\alpha} = \partial_\alpha N^\alpha + \Gamma^{\alpha}_{\alpha\mu} N^\mu = 0, \quad T^{\alpha\beta;\alpha} = \partial_\alpha T^{\alpha\beta} + \Gamma^{\alpha}_{\alpha\mu} T^{\mu\beta} + \Gamma^{\beta}_{\alpha\mu} T^{\alpha\mu} = 0. \quad (1.3)$$

The BGK model [7, 30] is the most well-known model equation of the classical Boltzmann equation. Three relativistic generalizations have been proposed respectively by Marle [24, 23], Anderson and Witting [1], and recently by Pennisi and Ruggeri [27], which have been widely applied to various physical problems [9, 10, 15, 25, 26, 22]. The first mathematical analysis for relativistic BGK models was carried out in [5] where the unique determination of equilibrium variables, the scaling limits, and the linearization problem were studied for the Marle model. For the existence theory of the nonlinear Marle model, we refer to [6] for near-equilibrium solutions, [14] for stationary solutions, and [8] for weak solutions. In the case of the Anderson-Witting model, the unique determination of equilibrium variables, and the global existence and asymptotic behavior of near-equilibrium solutions were studied in [12], and the stationary problem was covered in [13]. Recently, the Pennisi-Ruggeri model for polyatomic gases was studied in [11] where the unique determination of equilibrium variables, and the global existence and asymptotic behavior of near-equilibrium solutions were addressed. To the best knowledge of authors, the BGK model has not been much studied in the context of cosmology, which is the main motivation of the current work.

1.2. **The FLRW spacetime with massless particles.** In this paper, we study the BGK model in the Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime. This is a simple cosmological model which describes the evolution of the universe. It assumes that the universe is spatially homogeneous and isotropic, and we assume further in this paper that the universe is spatially flat. The metric tensor $\eta_{\mu\nu}$ is given by

$$\eta_{00} = \eta^{00} = -1, \quad \eta_{ij} = R^2 \delta_{ij}, \quad \eta^{ij} = R^{-2} \delta^{ij}$$
for $1 \leq i, j \leq 3$, where $R = R(t) > 0$ is the cosmic scale factor, $\delta_{ij}$ and $\delta^{ij}$ are the Kronecker delta, and $\eta^{\mu\nu}$ is the inverse of $\eta_{\mu\nu}$. The Einstein equations reduce to equations for $R$:

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi\mu + \Lambda}{3}, \quad \frac{\ddot{R}}{R} = -\frac{4\pi(\mu + 3P) + \Lambda}{3}.$$  

where the over dots denote the derivatives with respect to $t$, $\mu$ and $P$ are matter terms which should be induced by a suitable matter model, and $\Lambda \geq 0$ is the cosmological constant. In this paper, we are interested in the BGK model (1.1) so that $\mu$ and $P$ are given by

$$\mu = T^{00}, \quad P = \frac{1}{3}T^{ij}\eta_{ij},$$  

where $T^{\alpha\beta}$ is the energy-momentum tensor (1.2). One may consider the coupled Einstein-BGK equations by collecting (1.1), (1.4) and (1.5), but the equations decouple in the case of massless particles. By the mass shell condition

$$\eta_{\alpha\beta}p^\alpha p^\beta = -m^2$$  

with $m = 0$, we have $T^{\alpha\beta}\eta_{\alpha\beta} = 0$ so that $P = \mu/3$. Then, the Einstein equations can be rewritten as

$$\frac{\dot{R}^2}{R^2} + \frac{\ddot{R}}{R} = \frac{2\Lambda}{3},$$

and this can be solved explicitly as follows:

$$R = \begin{cases} 
C_1 t^2 & \text{for } \Lambda = 0, \\
C_2 \sqrt{\sinh(2\Lambda t)} & \text{for } \Lambda > 0, 
\end{cases} \quad (1.6)$$

where $C_1$ and $C_2$ are some positive constants depending on initial data, and $\Lambda = 3H^2$ (see [20] for more details). Hence, we may assume that an FLRW background is given and only need to study the BGK model. Our main result, Theorem 1.1, concerns only the Cauchy problem of the BGK model in a given spatially flat FLRW background, but it extends to the coupled Einstein-BGK equations due to the decoupling of the equations for massless particles. Throughout the paper, we assume that the scale factor $R$ is given by (1.6), and we set all physical constants to be unity except for particle mass ($m = 0$) for brevity.

1.3. Main result. In this paper, we are concerned with the Cauchy problem of the Anderson-Witting model (1.1) for massless particles in a given FLRW spacetime. As such, the distribution function is also assumed to be spatially homogeneous and isotropic. The four momentum $p^\alpha$ and its covariant components $v_\alpha := \eta_{\alpha\beta}p^\beta(= p_\alpha)$ are defined by

$$p^\alpha = (R|p|, p), \quad v_\alpha = (-p^0, R^2p) = (-R^{-1}|v|, v),$$

due to the mass shell condition with $m = 0$. To define the macroscopic quantities, we consider the particle four-flow and the energy-momentum tensor (1.2). In the Landau-Lifshitz frame [18] with the isotropy assumption, both $N^\alpha$ and $T^{\alpha\beta}$ are decomposed as

$$N^\alpha = nU^\alpha, \quad T^{\alpha\beta} = (en + P)U^\alpha U^\beta + P\eta^{\alpha\beta}.$$
Here the particle number density $n$, the Landau-Lifshitz four-velocity $U^\alpha$, the internal energy per particle $e$, and the pressure $P$ are defined as follows

$$
n = R^3 \int_{\mathbb{R}^3} F \, dp,
$$

$$
U^\alpha = \frac{R^3}{n} \int_{\mathbb{R}^3} \rho^\alpha \frac{F \, dp}{p^0} = (1, 0, 0, 0),
$$

$$
e = \frac{R^4}{n} \int_{\mathbb{R}^3} |p| F \, dp = R \int_{\mathbb{R}^3} |p| F \, dp,
$$

$$
P = \frac{R^5}{3} \int_{\mathbb{R}^3} |p|^2 F \, dp,
$$

where we used $\sqrt{-|\eta|} = R^3$ and (1.7). In the FLRW case, the nonzero Christoffel symbols are

$$
\Gamma^0_{ij} = R \dot{R} \delta_{ij}, \quad \Gamma^i_{j0} = \Gamma^i_{0j} = \dot{R} \frac{R^2}{R} \delta^i_j
$$

for $1 \leq i, j \leq 3$. In conclusion, the Cauchy problem of the Anderson-Witting model (1.1) for massless particles in the FLRW spacetime is reduced into

$$
\partial_t F - 2 \frac{\dot{R}}{R} p \cdot \nabla_p F = J(F) - F, \quad F(0, p) = F_0(p).
$$

(1.10)

We then consider the characteristic curve of (1.10):

$$
\frac{dp}{dt} = -2 \frac{\dot{R}}{R} p, \quad p(0) = y
$$

(1.11)

which can be solved explicitly:

$$
p(t) = R^{-2}(t) y.
$$

Therefore, in terms of the covariant variable $v = R^2(t)p$, (1.10) can be simplified further into

$$
\partial_t F(t, v) = J(F)(t, v) - F(t, v), \quad F(0, v) = F_0(v),
$$

(1.12)

where the Jüttner distribution $J(F)$ is written as

$$
J(F) = \exp \left\{ \frac{\mu}{T_E} + \frac{U_E^\alpha v_\alpha}{T_E} \right\}
$$

(1.13)

for the relativistic Maxwell-Boltzmann statistics, and

$$
J(F) = \frac{1}{\exp \left\{ -\frac{\mu}{T_E} - \frac{U_E^\alpha v_\alpha}{T_E} \right\} - 1}
$$

(1.14)

for the relativistic Bose-Einstein statistics. The Cauchy problem (1.12) is not as simple as it looks: Due to the specific structure of $J(F)$, the Cauchy problem (1.12) must be understood to be coupled to the nonlinear relation (2.5) in the case of the relativistic Bose-Einstein statistics:

$$
\partial_t F = \frac{1}{\exp\{c + \gamma|v|\} - 1} - F,
$$
where $c$ is implicitly determined by the following relation (for the precise definition of $\beta$, see Lemma 2.2):

$$\beta(c) = \frac{\rho(F)}{(3T(F))^\gamma},$$

and $\gamma$ is defined by

$$\gamma = \left\{ \begin{array}{cc}
\rho(F)^{\frac{1}{3}} \left( \int_{\mathbb{R}^3} \frac{1}{\exp\{c + |v|\} - 1} \, dv \right)^{\frac{1}{3}}, & \\
\exp\{-|v|\} & \text{for the relativistic Maxwell-Boltzmann statistics,} \\
\exp\{1 + |v|\}^{-1} & \text{for the relativistic Bose-Einstein statistics.}
\end{array} \right.$$

This makes the problem (1.12) a highly nontrivial one (See Lemma 2.2 and (3.3)). To state our main result, we define two types of global equilibrium:

$$J^0 = \left\{ \begin{array}{cc}
\exp\{-|v|\} & \text{for the relativistic Maxwell-Boltzmann statistics,} \\
\exp\{1 + |v|\}^{-1} & \text{for the relativistic Bose-Einstein statistics.}
\end{array} \right. \quad (1.15)$$

Our main result is as follows.

**Theorem 1.1.** Assume $F_0 = F_0(v)$ is nonnegative, and $F_0$ and $J^0$ share the total particle number and energy in the following sense:

$$\int_{\mathbb{R}^3} F_0 \, dv = \int_{\mathbb{R}^3} J^0 \, dv, \quad \int_{\mathbb{R}^3} |v|F_0 \, dv = \int_{\mathbb{R}^3} |v|J^0 \, dv. \quad (1.16)$$

Then the Cauchy problem (1.12) is explicitly solved as follows

$$F(t, v) = \exp(-t)F_0(v) + \{1 - \exp(-t)\} J^0.$$

**Remark 1.2.** (1) Theorem 1.1 shows that the solution to the Anderson-Witting model (1.12) converges exponentially to the global Jüttner distribution function with the same particle number density and internal energy as the initial data. (2) The moments of $J^0$ in (1.16) can be explicitly computed in the case of Maxwell-Boltzmann case with $8\pi$, $24\pi$ for the mass and the energy respectively. This explicit value is, of course, not crucial since it depends on normalization. Such explicit values are not available for the Bose-Einstein case.

A typical choice of matter model for the FLRW cosmology is a perfect fluid. This is a simple matter model defined by the energy-momentum tensor in the form of (1.8) with a suitable equation of state, which is usually linear, i.e., $P = \mu \mu$ for some constant $w$. Two important cases are radiation ($w = 1/3$) and dust ($w = 0$), where $\mu$ decreases as $R^{-4}$ for radiation and $R^{-3}$ for dust (see Chapter 5 of [29] for more details about the FLRW cosmology with a perfect fluid). In our case, the energy-momentum tensor is defined by (1.2), but it is reduced to the perfect fluid form (1.8) by the FLRW symmetry, and we obtain $P = \mu/3$ by the assumption of massless particles. This corresponds to a radiation-filled universe, and the work of this paper is to use the BGK model to describe the evolution of it at a mesoscopic level. One can see from Theorem 1.1 that the energy density, $\mu = en$ defined in (1.9), decreases as $R^{-4}$.

As is mentioned above, mathematical analysis for BGK models in a cosmological framework has never been made in the literature. We refer to [22] for the study of exact solutions and to [3, 4] in the Boltzmann case. For an introduction to the Boltzmann or Vlasov equation for massless particles, we refer to [2, 19, 20, 21, 28].

This paper is organized as follows. In Section 2, we find an explicit form of Jüttner distribution of (1.13) and (1.14) for massless particles in the FLRW spacetime. In Section 3, we investigate the iteration scheme for (1.12) to prove Theorem 1.1.
2. **Determination of $J(F)$**. Recall from (1.13) and (1.14) that the Jüttner distribution $J(F)$ has unknown variables $\mu_E, U_E$ and $T_E$, and using the covariant variable $v$ of (1.7), unknown variables are determined through the relation

$$U^\alpha \int_{\mathbb{R}^3} v_\alpha (J(F) - F) \frac{1}{\sqrt{-\eta}} \frac{dv}{v_0} = 0,$$

$$U^\alpha \int_{\mathbb{R}^3} v_\alpha \eta^\beta v_\mu (J(F) - F) \frac{1}{\sqrt{-\eta}} \frac{dv}{v_0} = 0$$

so that the conservation laws (1.3) hold true. In the following lemma, we investigate the explicit form of $J(F)$ using the relation (2.1) for the case of massless particles in the FLRW spacetime.

**Lemma 2.1.** *The explicit form of (1.13) satisfying (2.1) is given by*

$$J(F) = \frac{\rho}{8\pi T^3} \exp \left\{ - |v| \frac{T}{T} \right\}$$

*where $\rho$ and $T$ denote*

$$\rho = \int_{\mathbb{R}^3} F dv, \quad 3T = \frac{\int_{\mathbb{R}^3} |v| F dv}{\int_{\mathbb{R}^3} F dv}.$$  

*Proof.* Taking $U^\alpha_E = U^\alpha$ (see (1.9)), then (1.13) becomes

$$J(F) = \exp \left\{ \frac{\mu_E}{T_E} - \frac{|v|}{RT_E} \right\}$$

and (2.1) reduces to

$$\int_{\mathbb{R}^3} J(F) dv = \int_{\mathbb{R}^3} F dv,$$

$$\left( \int_{\mathbb{R}^3} |v| J(F) dv, 0, 0, 0 \right) = \left( \int_{\mathbb{R}^3} |v| F dv, 0, 0, 0 \right)$$

due to the isotropic property of $F$. Inserting (2.2) into (2.3), we have

$$8\pi R^3 T_E^3 \exp \left\{ \frac{\mu_E}{T_E} \right\} = \rho,$$

$$24\pi R^4 T_E^4 \exp \left\{ \frac{\mu_E}{T_E} \right\} = 3\rho T,$$

which leads to

$$T_E = R^{-1} T, \quad \exp \left\{ \frac{\mu_E}{T_E} \right\} = \frac{\rho}{8\pi T^3}.$$  

*Putting (2.4) into (2.2) gives the desired result.*

**Lemma 2.2.** *The explicit form of (1.14) satisfying (2.1) is given by*

$$J(F) = \frac{1}{\exp \{ c + \gamma |v| \} - 1}$$
where \( c \) and \( \gamma \) are determined by the relations
\[
\beta(c) \equiv \left\{ \frac{1}{\exp\{c+|v|\}-1} \right\} dv = \frac{\rho}{(3T)^3},
\]
\[
\gamma = \rho^{-\frac{1}{3}} \left( \int_{\mathbb{R}^3} \frac{1}{\exp\{c+|v|\}-1} dv \right)^{-\frac{1}{3}}.
\]
(2.5)

If \( c \) is positive, and \( F \) satisfies
\[
0 < \frac{\rho}{(3T)^3} < \frac{8\pi}{27} \left\{ \sum_{k=1}^{\infty} \frac{1}{k^3} \right\}^4 \left\{ \sum_{k=1}^{\infty} \frac{1}{k^4} \right\}^3,
\]
(2.6)
then (2.5) determines unique \( c \) and \( \gamma \).

**Proof.** • The explicit form of (1.14): As in (2.2) of Lemma 2.1, we can see that
\[
J(F) = \frac{1}{\exp\{c+\gamma|v|\}-1},
\]
(2.7)
where \( c \) and \( \gamma \) denote
\[
c = -\frac{\mu E}{T_E}, \quad \gamma = \frac{1}{RT_E}.
\]
Inserting (2.7) into (2.3) gives
\[
\gamma^3 \int_{\mathbb{R}^3} \frac{1}{\exp\{c+|v|\}-1} dv = \rho,
\]
\[
\gamma^4 \int_{\mathbb{R}^3} \frac{|v|}{\exp\{c+|v|\}-1} dv = 3\rho T
\]
(2.8)
from which we can derive (2.5).

• Unique determination of \( c \) and \( \gamma \): The integrals of the Bose-Einstein distribution can be represented by
\[
\int_0^\infty \frac{r^n}{\exp\{r\}-1} dr = n! \sum_{k=1}^{\infty} \frac{z^k}{k^{n+1}}
\]
(2.9)
which converges for \( n \in \mathbb{N} \) and \( z < 1 \). For \( c > 0 \), we put \( z = e^{-c} \) and apply (2.9) to \( \beta(c) \) of (2.5) to see
\[
\beta(c) = 4\pi \left\{ \frac{1}{\exp\{c+r\}-1} \right\}^4 dr = 4\pi \left\{ 2! \sum_{k=1}^{\infty} \frac{\exp\{-ck\}}{k^2} \right\}^4 \left\{ 3! \sum_{k=1}^{\infty} \frac{\exp\{-ck\}}{k^3} \right\}^3
\]
(2.10)
\[
= \frac{8\pi}{27} \left\{ \sum_{k=1}^{\infty} \frac{\exp\{-ck\}}{k^3} \right\}^4 \left\{ \sum_{k=1}^{\infty} \frac{\exp\{-ck\}}{k^4} \right\}^3.
\]
Now we differentiate \( \beta(c) \) with respect to \( c \) and use the following relation
\[
\frac{\partial}{\partial c} \left\{ \sum_{k=1}^{\infty} \frac{\exp\{-ck\}}{k^n} \right\} = -\sum_{k=1}^{\infty} \frac{\exp\{-ck\}}{k^{n-1}}
\]
and it satisfies (2.3):
\[
\int_{\mathbb{R}^3} J(F) - F \, dv = 0, \quad \int_{\mathbb{R}^3} |v|(J(F) - F) \, dv = 0.
\]
In terms of (1.12), this leads to the following conservation laws
\[
\frac{d}{dt} \int_{\mathbb{R}^3} F \, dv = 0, \quad \frac{d}{dt} \int_{\mathbb{R}^3} |v| F \, dv = 0. \tag{3.2}
\]
Since we assumed
\[
\int_{\mathbb{R}^3} F_0 \, dv = \int_{\mathbb{R}^3} J^0 \, dv (= 8\pi), \\
\int_{\mathbb{R}^3} |v| F_0 \, dv = \int_{\mathbb{R}^3} |v| J^0 \, dv (= 24\pi),
\]
combining (3.1) and (3.2) gives
\[
\rho = 8\pi, \quad T = 1,
\]
and hence
\[
J(F) = J^0.
\]
Therefore (1.12) is explicitly solved as
\[
F(t, v) = \exp(-t) F_0(v) + (1 - \exp(-t)) J^0.
\]
In the case of relativistic Bose-Einstein statistics, it is not obvious because, by Lemma 2.2, \(J(F)\) is well-defined in a way to satisfy (3.1) only when the solution \(F\) satisfies (2.6). For this, we consider the following iteration scheme that for \(n \geq 0\),
\[
\partial_t F^{n+1} = \frac{1}{\exp\{c^n + \gamma^n |v|\} - 1} - F^{n+1},
\]
\[
\beta(c^n) = \frac{\rho(F^n)}{(3T(F^n))^3},
\]
\[
\gamma^n = \{\rho(F^n)\}^{-\frac{1}{3}} \left( \int_{\mathbb{R}^3} \frac{1}{\exp\{c^n + |v|\} - 1} \, dv \right)^\frac{1}{3},
\]
where we set \(F^n(0, v) = F_0(v)\) and \(c^n > 0\). Then it follows from (1.16) that
\[
\int_{\mathbb{R}^3} F^0 \, dv = \int_{\mathbb{R}^3} J^0 \, dv, \quad \int_{\mathbb{R}^3} |v| F^0 \, dv = \int_{\mathbb{R}^3} |v| J^0 \, dv.
\]
Inserting \(F^0\) into the definition of \(\rho/(3T)^3\), this gives
\[
0 < \frac{\rho(F^0)}{(3T(F^0))^3} = \frac{\left( \int_{\mathbb{R}^3} F^0 \, dv \right)^4}{\left( \int_{\mathbb{R}^3} |v| F^0 \, dv \right)^3} = \frac{\left( \int_{\mathbb{R}^3} J^0 \, dv \right)^4}{\left( \int_{\mathbb{R}^3} |v| J^0 \, dv \right)^3} = \beta(1).
\]
Since \(\beta(c)\) is strictly decreasing on \(c \in (0, \infty)\) (see the proof of Lemma 2.2), we have
\[
0 < \frac{\rho(F^0)}{(3T(F^0))^3} = \beta(1) < \lim_{c \to 0} \beta(c) = \frac{8\pi}{27} \left( \sum_{k=1}^{\infty} \frac{1}{k^3} \right)^4,
\]
which says that \(F^0\) satisfies (2.6). So we conclude by Lemma 2.2 that \(c^0\) is equal to 1 and
\[
\gamma^0 = \left( \int_{\mathbb{R}^3} J^0 \, dv \right)^{-\frac{1}{3}} \left( \int_{\mathbb{R}^3} \frac{1}{\exp\{1 + |v|\} - 1} \, dv \right)^\frac{1}{3} = 1,
\]
and hence
\[
\frac{1}{\exp\{c^0 + \gamma^0 |v|\} - 1} = J^0.
\]
Thus $F^1$ of (3.3) satisfies
\[
\int_{\mathbb{R}^3} \left( \frac{1}{|v|} \right) F^1(t,v) \, dv = \exp(-t) \int_{\mathbb{R}^3} \left( \frac{1}{|v|} \right) F_0(v) \, dv + \left( 1 - \exp(-t) \right) \int_{\mathbb{R}^3} \left( \frac{1}{|v|} \right) J^0 \, dv
\]
\[
= \int_{\mathbb{R}^3} \left( \frac{1}{|v|} \right) J^0 \, dv.
\]
In the last line, we used (1.16). Applying the same argument, we get
\[
\frac{1}{\exp\{c_1 + \gamma_1|v|\} - 1} = J^0.
\]
Proceeding in the same manner leads to
\[
\int_{\mathbb{R}^3} \left( \frac{1}{|v|} \right) F^n(t,v) \, dv = \int_{\mathbb{R}^3} \left( \frac{1}{|v|} \right) F_0 \, dv,
\]
and hence
\[
\frac{1}{\exp\{c^n + \gamma^n|v|\} - 1} = J^0
\]
for all $n \in \mathbb{N}$. Therefore, the iteration (3.3) can be explicitly solved as
\[
F^{n+1}(t,v) = \exp(-t)F_0(v) + \left( 1 - \exp(-t) \right) J^0
\]
which gives the desired result.

**Acknowledgments.** Byung-Hoon Hwang was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2019R1A6A1A10073079). Ho Lee was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT & Future Planning (NRF-2018R1A1A1A05078275). Part of this work was done while Ho Lee was visiting the Laboratoire Jacques-Louis Lions. Seok-Bae Yun was supported by Samsung Science and Technology Foundation under Project Number SSTF-BA1801-02.

**REFERENCES**

[1] J. L. Anderson and H. R. Witting, A relativistic relaxation-time model for the Boltzmann equation, *Physica*, 74 (1974), 466–488.

[2] H. Barzegar, D. Fajman and G. Heißel, Isotropization of slowly expanding spacetimes, *Phys. Rev. D.*, 101 (2020), 044046.

[3] D. Bazow, G. S. Denicol, U. Heinz, M. Martinez and J. Noronha, Analytic solution of the Boltzmann equation in an expanding system, *Phys. Rev. Lett.*, 116 (2016), 022301.

[4] D. Bazow, G. S. Denicol, U. Heinz, M. Martinez and J. Noronha, Nonlinear dynamics from the relativistic Boltzmann equation in the Friedmann-Lemaître-Robertson-Walker spacetime, *Phys. Rev. D.*, 94 (2016), 125006.

[5] A. Bellouquid, J. Calvo, J. Nieto and J. Soler, On the relativistic BGK-Boltzmann model: Asymptotics and hydrodynamics, *J. Stat. Phys.*, 149 (2012), 284–316.

[6] A. Bellouquid, J. Nieto and L. Urrutia, Global existence and asymptotic stability near equilibrium for the relativistic BGK model, *Nonlinear Anal.*, 114 (2015), 87–104.

[7] P. L. Bhatnagar, E. P. Gross and M. L. Krook, A model for collision processes in gases. I. Small amplitude processes in charged and neutral one-component systems, *Phys. Rev.*, 94 (1954), 511–525.

[8] J. Calvo, P.-E. Jabin and J. Soler, Global weak solutions to the relativistic BGK equation, *Comm. Partial Differential Equations*, 45 (2020), 191–229.

[9] W. Florkowski, R. Ryblewski and M. Strickland, Anisotropic hydrodynamics for rapidly expanding systems, *Nucl. Phys. A.*, 916 (2013), 249–259.

[10] W. Florkowski, R. Ryblewski and M. Strickland, Testing viscous and anisotropic hydrodynamics in an exactly solvable case, *Phys. Rev. C.*, 88 (2013), 024903.
[11] B.-H. Hwang, T. Ruggeri and S.-B. Yun, On a relativistic BGK model for polyatomic gases near equilibrium, Preprint; \texttt{arXiv:2102.00462}.

[12] B.-H. Hwang and S.-B. Yun, Anderson-Witting model of the relativistic Boltzmann equation near equilibrium, \textit{J. Stat. Phys.}, \textbf{176} (2019), 1009–1045.

[13] B.-H. Hwang and S.-B. Yun, Stationary solutions to the Anderson–Witting model of the relativistic Boltzmann equation in a bounded interval, \textit{SIAM J. Math. Anal.}, \textbf{53} (2021), 730–753.

[14] B.-H. Hwang and S.-B. Yun, Stationary solutions to the boundary value problem for the relativistic BGK model in a slab, \textit{Kinet. Relat. Models}, \textbf{12} (2019), 749–764.

[15] A. Jaiswal, R. Ryblewski and M. Strickland, Transport coefficients for bulk viscous evolution in the relaxation time approximation, \textit{Phys. Rev. C.}, \textbf{90} (2014), 044908.

[16] F. Jüttner, Das Maxwellsche gesetz der geschwindigkeitsverteilung in der relativtheorie, \textit{Ann. Physik}, \textbf{339} (1911), 856–882.

[17] F. Jüttner, Die relativistische Quantentheorie des idealen Gases, \textit{Zeitschr. Physik}, \textbf{47} (1928), 542–566.

[18] L. D. Landau and E. M. Lifshitz, \textit{Fluid Mechanics}, Pergamon Press., 1959.

[19] H. Lee, The spatially homogeneous Boltzmann equation for massless particles in an FLRW background, \textit{J. Math. Phys.}, \textbf{62} (2021), 031502, 15 pp.

[20] H. Lee, E. Nungesser and P. Tod, The massless Einstein-Boltzmann system with a conformal-gauge singularity in an FLRW background, \textit{Classical Quantum Gravity}, \textbf{37} (2020), 035005.

[21] H. Lee, E. Nungesser and P. Tod, On the future of solutions to the massless Einstein-Vlasov system in a Bianchi I cosmology, \textit{Gen. Relativity Gravitation}, \textbf{52} (2020), no. 48.

[22] R. Maartens and F. P. Wolvaardt, Exact non-equilibrium solutions of the Einstein-Boltzmann equations, \textit{Classical Quantum Gravity}, \textbf{11} (1994), 203–225.

[23] C. Marle, Modele cinétique pour l'établissement des lois de la conduction de la chaleur et de la viscosité en théorie de la relativité, \textit{C. R. Acad. Sci. Paris}, \textbf{260} (1965), 6539–6541.

[24] C. Marle, Sur l’établissement des équations de l’hydrodynamique des fluides relativistes dissipatifs, I. L’équation de Boltzmann relativiste, \textit{Ann. Inst. Henri Poincaré}, \textbf{10} (1969), 67–127.

[25] M. Mendoza, I. Karlin, S. Succi and H. J. Herrmann, Relativistic lattice Boltzmann model with improved dissipation, \textit{Phys. Rev. D.}, \textbf{87} (2013), 065027.

[26] E. Molnár, H. Niemi and D. H. Rischke, Derivation of anisotropic dissipative fluid dynamics from the Boltzmann equation, \textit{Phys. Rev. D.}, \textbf{93} (2016), 114025.

[27] S. Pennisi and T. Ruggeri, A new BGK model for relativistic kinetic theory of monatomic and polyatomic gases, \textit{J. Phys. Conf. Ser.}, \textbf{1035} (2018), 012005.

[28] K. P. Tod, Isotropic cosmological singularities: Other matter models, \textit{Class. Quantum Grav.}, \textbf{20} (2003), 521–534.

[29] R. M. Wald, \textit{General Relativity}, University of Chicago Press, Chicago, IL, 1984.

[30] P. Walender, On the temperature jump in a rarefied gas, \textit{Ark. Fys.}, \textbf{7} (1954), 507–553.

Received April 2021; 1st revision July 2021; 2nd revision August 2021; early access September 2021.

\textit{E-mail address:} bhh0116@skku.edu
\textit{E-mail address:} holee@khu.ac.kr
\textit{E-mail address:} sbyun01@skku.edu