A method for proving lower bounds for the 2-limited broadcast domination number on grid graphs

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October 19, 2021

Abstract

We establish lower bounds for the 2-limited broadcast domination number of various grid graphs, in particular the Cartesian product of two paths, a path and a cycle, and two cycles. Our lower bounds are derived by computational techniques. Some of the lower bounds are periodically best possible, and yield exact values.

1 Introduction

Suppose there is a transmitter located at each vertex of a graph $G$. A $k$-limited broadcast $f$ on $G$ is an assignment $f(v) \in \{0, 1, \ldots, k\}$, for each $v \in V(G)$, where $k \leq \text{rad}(G)$ and $\text{rad}(G)$ is the radius of $G$. The assignment $f(v)$ to vertex $v$ represents the strength of the broadcast from $v$, where $f(v) = 0$ means the transmitter at $v$ is not broadcasting. A broadcast of positive strength $f(v)$ from $v$ is heard by all vertices at distance at most $f(v)$ from $v$. A broadcast $f$ is dominating if each vertex of $G$ hears the broadcast from some vertex. The cost of a broadcast $f$ is $\sum_{v \in V(G)} f(v)$. The $k$-limited broadcast domination number $\gamma_{b,k}(G)$ of $G$ is the minimum cost of a $k$-limited dominating broadcast.

The $k$-limited broadcast domination number $\gamma_{b,k}(G)$ can also be formulated as an integer linear program. Let $G$ be a graph and fix $1 \leq k \leq \text{rad}(G)$. For each vertex $i \in V(G)$ and $\ell \in \{1, 2, \ldots, k\}$ let

$$x_{i,\ell} = \begin{cases} 1 & \text{if vertex } i \text{ is broadcasting at strength } \ell \text{ and} \\ 0 & \text{otherwise}. \end{cases}$$

The $k$-limited broadcast domination number $\gamma_{b,k}(G)$ of a graph $G$ is the cost of an optimal solution to

$$\text{Minimize: } \sum_{\ell=1}^{k} \sum_{i \in V(G)} \ell \cdot x_{i,\ell} \tag{ILP 1.1}$$

$$\text{Subject to: } (1) \sum_{\ell=1}^{k} \sum_{i \in V(G)} x_{i,\ell} \geq 1, \text{ for each vertex } j \in V(G),$$

$$\quad \quad \quad \quad \text{for each vertex } j \in V(G) \text{ and } \ell \in \{1, 2, \ldots, k\}. $$

Here $d(x, v)$ is the distance between $x$ and $v$ in $G$.

The $k$-limited broadcast domination number of a graph was first mentioned, although not explored, in Erwin’s Ph.D. thesis [4]. There are known bounds for the $k$-limited broadcast domination number on trees [3]. Specific to 2-limited broadcast domination, there are known bounds for cubic graphs (with some restrictions)

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in [9] and the Cartesian product of two paths, a path and a cycle, and two cycles in Slobodin’s Masters Thesis [7] (see [8] also).

For each fixed positive integer $k$, the problem of deciding whether there exists a $k$-limited dominating broadcast of cost at most a given integer $B$ is NP-complete [3]. There are polynomial time algorithms to compute $\gamma_{b,k}(G)$ in several graph classes in [9]. A detailed summary of bounds and complexity results for the $k$-limited broadcast domination number can be found in [8].

The $k$-limited broadcast domination problem is a restriction of the broadcast domination problem (which was also introduced in [4]). The broadcast domination number of a graph can be obtained from [ILP 1] by setting $k = \text{rad}(G)$. A survey of results on broadcast domination can be found in [6].

1.1 Outline
Upper and lower bounds for the 2-limited broadcast domination numbers of the Cartesian product of two paths, a path and a cycle, and two cycles have been recently established in [8]. Some of the bounds produced yield optimal or periodically optimal values for the 2-limited broadcast domination numbers of these graphs. This paper is devoted to improving the lower bounds found in [8] by computational methods. Section 2 provides an intuition behind our method. Section 3 describes our algorithm in full and Section 4 presents the results found. Section 5 concludes with some future research questions.

2 Eliminating Induced Sub-broadcasts of Fixed Cost

Our approach completes an exhaustive search of all possible small induced sub-broadcasts of given costs on a graph. A computational approach is used to eliminate cases which provably cannot be part of an optimal broadcast. For example, cases where the same region can be dominated by a broadcast with less cost. Section 2.1 provides an informal example of one of the tactics used in our search. Section 2.2 describes two main subroutines used.

2.1 Intuition
This section presents a method to prove lower bounds for $\gamma_{b,2}(P_m \Box C_n)$ and $\gamma_{b,2}(C_m \Box C_n)$. Throughout the description of our method, we use examples specific to $P_5 \Box C_n$ to assist understanding. We recall the following result.

Proposition 2.1. [8] Theorem 4.1]For $n \geq 3$,

$$
\gamma_{b,2}(P_5 \Box C_n) \leq n + \begin{cases} 
0 & \text{for } n \equiv 0 \text{ or } 2 \pmod{4} \text{ and } \\
1 & \text{for } n \equiv 1 \text{ or } 3 \pmod{4}.
\end{cases}
$$

Suppose we wish to establish $n \leq \gamma_{b,2}(P_5 \Box C_n)$ to yield periodically optimal values for $\gamma_{b,2}(P_5 \Box C_n)$. By computation, we have that $n \leq \gamma_{b,2}(P_5 \Box C_n)$ for $3 \leq n \leq 16$. Suppose the bound does not hold and let $n$ be the smallest integer greater than 16 such that there exists an optimal 2-limited dominating broadcast $f$ on $P_5 \Box C_n$ of cost at most $n - 1$. If every sequence of eight consecutive columns of $f$ has cost at least eight, then the total cost of $f$ is at least $n$. By assumption then, there must exist some sequence of eight consecutive columns with cost at most seven. Let $C$ be the subgraph of $P_5 \Box C_n$ induced by the vertices appearing in a minimum cost set of eight consecutive columns with respect to $f$. For each integer $x \leq 7$, we consider all possible broadcasts of cost $x$ within $C$ and try to reach a contradiction. If a contradiction is discovered for all such broadcasts, then this establishes $n \leq \gamma_{b,2}(P_5 \Box C_n)$ as desired.

We require the following natural definitions.
Definition 2.2. Let \( f \) be a 2-limited broadcast on the graph \( G \) and let \( X \subseteq V(G) \). Define the sub-broadcast \( g \) induced by \( X \) by
\[
g(x) = \begin{cases} 
  f(x) & \text{if } x \in X \text{ and } \\
  0 & \text{otherwise}. 
\end{cases}
\]

Definition 2.3. Given two 2-limited broadcasts \( f \) and \( g \) on a graph \( G \), define \( h = f \oplus g \) by \( h(x) = \max\{f(x), g(x)\} \) for all \( x \in V(G) \). Similarly, define \( h = f \ominus g \) for all \( x \in V(G) \) by
\[
h(x) = \begin{cases} 
  0 & \text{if } g(x) > 0 \text{ and } \\
  f(x) & \text{otherwise}. 
\end{cases}
\]

Definition 2.4. If \( f \) is a broadcast on \( G \), then we say \( f \) dominates \( y \in V(G) \) if there exists a vertex \( x \) such that \( y \) hears the broadcast from \( x \) under \( f \). Further, we say that \( f \) dominates \( X \subseteq V(G) \) if it dominates every vertex \( x \in X \).

Definition 2.5. Let \( f \) be a broadcast on \( G \). The broadcast range of \( f \) is the set of vertices which hear a broadcast under \( f \).

Continuing with the previously mentioned assumption, suppose we are considering sub-broadcasts of cost at most seven induced by \( V(C) \) on a 2-limited dominating broadcast \( f \) on \( P_5 \Box C_n \) of cost at most \( n - 1 \). One such possible broadcast is given by \( g \) in Figure 1. Here, a region containing the vertices of \( C \) is depicted by the thick black rectangle. Black circles with a black inner fill indicate vertices of \( C \) broadcasting at a non-zero strength. In this example, \( C \) has one vertex broadcasting at strength two in one of the centre columns and bottom row and five vertices broadcasting at strength one. The thick red dotted lines indicate the broadcast ranges of the broadcasting vertices at their centres.

Let \( f' = f \oplus g \). Let \( R \) be the range of \( g \). The broadcast \( f' \) dominates all vertices of \( P_5 \Box C_n \) with the possible exception of the vertices of \( R \). Suppose we delete four columns from the grid, as indicated in Figure 2 and “patch” the grid back together in the natural way such that the resulting graph is \( P_5 \Box C_{n-4} \). Note that, in this way, \( V(P_5 \Box C_n) \) has a natural correspondence with \( V(P_5 \Box C_{n-4}) \). Let \( f'' \) be the broadcast formed by restricting \( f' \) to this reduced graph in the natural way. Let \( R' = \{ v \in P_5 \Box C_{n-4} : v \in R \} \). The vertices of \( R' \) are indicated by the green circles in Figures 2 and 3. By the same logic as before, \( f'' \) dominates all of \( P_5 \Box C_{n-4} \) with the possible exception of the vertices of \( R' \). However, \( R' \) can be dominated by the broadcast \( h \) of cost three on \( P_5 \Box C_{n-4} \) defined by the broadcast in Figure 3. Let \( f''' = f'' \oplus h \). The broadcast \( f''' \) is a 2-limited dominating broadcast on \( P_5 \Box C_{n-4} \).
Figure 3: Broadcast of cost three which dominates $R'$ from Figure 2.

broadcast on $P_5 \square C_{n-4}$. The cost of $f'''$ is at most

$$\text{cost}(f''') + 3 = \text{cost}(f \ominus g) + 3 \leq (n-1) - 7 + 3 = n - 5.$$  

However, this is a contradiction as, by the assumption of the minimality of $n$, $n-4 \leq \gamma_{b,2}(P_5 \square C_{n-4})$. Therefore, if there exists a 2-limited dominating broadcast $f$ on $P_5 \square C_n$ of cost at most $n-1$, it cannot contain, as a sub-broadcast induced by $V(C)$, the broadcast $g$ in Figure 1.

This type of contradiction is formalized in Algorithm 2 in Section 2.2. Additional methods for yielding contradictions are also described in Section 3.

2.2 Applying Induction to Reach Contradictions

Our method repeatedly utilizes two main ways of obtaining contradictions. The first of these tactics, as described in Algorithm 1, HasBroadcastScheme, provides a routine which, given a graph $G$ and set of vertices $R \subseteq V(G)$, returns the truth value of the statement “$R$ can be dominated with cost at most $x$.” Example 2.6 provides an application of HasBroadcastScheme on $P_5 \square C_n$.

**Algorithm 1:** Routine to determine whether a given set of vertices on a graph can dominated with cost at most $x$.

1 function HasBroadcastScheme $(G, R, x)$:
   2 Input : A graph $G$, a set of vertices $R \subseteq V(G)$, and desired cost $x$ to dominate $R$.
   3 Output: Value of the truth statement: “$R$ can be dominated with cost at most $x$.”
   4 Let $f$ be a minimum cost 2-limited dominating broadcast of $R$ on $G$;
   5 if $\text{cost}(f) \leq x$ then
   6 return True;
   7 end
   8 return False;

**Example 2.6.** Consider the broadcast $g$ of cost six on $P_5 \square C_n$, for some fixed $n \geq 8$, defined by the left-hand figure in Figure 4. Let $R$ be the range of $g$. Observe that $R$ can be dominated with cost five, as depicted in the right-hand figure of Figure 4. Therefore, HasBroadcastScheme$(P_5 \square C_n, R, 5)$ is true. Hence, no optimal broadcast on $P_5 \square C_{n \geq 8}$ contains $g$ as an induced sub-broadcast.

Our second method, as described in Algorithm 2, ResultsInMinimalityContradiction, is a generalization of the example described in Section 2.1. Prior to a formal description of ResultsInMinimalityContradiction, we include a rigorous example to aid in its explanation.
Example 2.7. By computation, $B(n) = n \leq \gamma_{n,2}(P_5 \square C_n)$ for $3 \leq n \leq 16$. Suppose that $B(n)$ does not hold for all $n$ and let $n$ be the smallest integer greater than 16 such that there exists an optimal 2-limited dominating broadcast $f$ on $P_5 \square C_n$ of cost at most $n - 1$. Suppose $f$ contains the induced sub-broadcast $g$ of cost 12 on $P_5 \square C_n$ as defined by Figure 5. Let $R$ be the range of $g$.

Figure 5: Assumed induced sub-broadcast $g$ of $f$ of cost 12.

Let $f' = f \ominus g$. The broadcast $f'$ dominates all of $P_5 \square C_n$ with the possible exception of the vertices of $R$. Suppose we delete six columns of the grid, as indicated in Figure 6 and “patch” the grid back together in the natural way such that the resulting graph is $P_5 \square C_{n-6}$. Note that $V(P_5 \square C_n)$ has a natural correspondence with $V(P_5 \square C_{n-6})$. The columns deleted could contain vertices broadcasting with non-zero strength under $f'$. Our goal is to create a new broadcast $f''$ such that every vertex of $P_5 \square C_{n-6}$, which heard a broadcast under $f'$, hears a broadcast under $f''$. Furthermore, we want the cost of $f''$ to be at most the cost of $f'$.

Figure 6: Procedure which reduces $P_5 \square C_n$ to $P_5 \square C_{n-6}$.

For each vertex $v \in V(P_5 \square C_{n-6})$ that is not in one of the deleted columns, define the broadcast $f''$ on $P_5 \square C_{n-6}$ by $f''(v) = f'(v)$. For each vertex $v$ in a deleted column, pick a vertex $u$ in the same row as $v$ and in a nearest column undeleted to $v$ and let $f''(u) = \max\{f'(u), f'(v)\}$. For example, suppose $f'$ contains the vertex $v$ broadcasting at strength one in the leftmost deleted column in the left-hand figure of Figure 7. After deleting these columns, the broadcast of $v$ is re-allocated to one column to the left, as depicted in the right-hand figure in Figure 7. We make the following observation about this process of redistribution.

Observation 1. Suppose $f$ is a 2-limited dominating broadcast on $G = (P_m \text{ or } C_m) \square C_n$. Let $g$ be an induced sub-broadcast of $f$ and let $R$ be the range of $g$. Define $f' = f \ominus g$. Let $G'$ be the graph formed by deleting columns $c_{\alpha_1}, c_{\alpha_2}, \ldots, c_{\alpha_z}$ which contain vertices of $R$ from $G$ and “patching” the graph together in the natural way. Let $f''$ be the broadcast formed by restricting $f'$ to this reduced graph in the natural way. For each deleted vertex $v$ broadcasting with non-zero strength under $f'$, pick a vertex $u$ in the same row as $v$ and in a nearest column undeleted to $v$ and let $f''(u) = \max\{f'(u), f'(v)\}$. The cost($f''$) $\leq$ cost($f'$) and the vertices of $G'$ that
belong to the set \((V(G) \setminus [V(c_{a_1}) \cup V(c_{a_2}) \cup \cdots \cup V(c_{a_z})]) \setminus R\) each hear a broadcast under \(f''\) (possibly other vertices do too).

For this example, by Observation 1, \(f''\) dominates all the vertices of \(P_5 \square C_{n-6}\) with the possible exception of the vertices indicated by the green circles in Figure 6. Let \(R' = \{v \in V(P_5 \square C_{n-6}) : v \in R\}\). The vertices of \(R'\) might not be dominated by \(f''\). The vertices in \(R'\) can be dominated by the broadcast \(h\) of cost six on \(P_5 \square C_{n-6}\), as defined in Figure 8. Let \(f'' = f'' \oplus h\). The broadcast \(f''\) is a 2-limited dominating broadcast of

\[
\begin{array}{|c|c|c|}
\hline
\text{Column} & \text{Column} & \text{Column} \\
\hline
\times & \times & \times \\
\hline
\end{array}
\]

Figure 8: Broadcast \(h\) of cost six which dominates the vertices circled in green in Figure 6

\(P_5 \square C_{n-6}\) of cost at most

\[
\text{cost}(f'') + 6 \leq \text{cost}(f \ominus g) + 6 \leq (n - 1) - 12 + 6 = n - 7.
\]

However, this is a contradiction since, by the assumption of the minimality of \(n\), \(n - 6 \leq \gamma_{b,2}(P_5 \square C_{n-6})\).

There are several important things to take away from Example 2.7. Given some assumed induced sub-broadcast \(g\) of a broadcast \(f\) whose broadcast range is contained within \(k\) consecutive columns on a larger graph, we contradict the minimality of \(n\) by “patching” the graph back together after deleting some number of columns and inferring the existence of a dominating broadcast of a bounded cost on a smaller graph. To derive a contradiction, we need not know what the broadcast \(f\) looks like, we simply need to know its cost. The example in Section 2.1 and Example 2.7 delete four and six columns, respectively, to yield contradictions. In general, given some induced sub-broadcast \(g\) whose broadcast range \(R\) is contained within \(k\) columns, we attempt to find a contradiction of minimality after deleting one selection of \(i\) columns, for each \(i\) from 1 to \(k\). Our method could be generalized to delete all selections of \(i\) columns, however, for our results, deleting one such selection was sufficient. If we are trying to prove that \(B(n) \leq \gamma_{b,2}(P_m \square C_n)\) or \(\gamma_{b,2}(C_m \square C_n)\), if the vertices of \(R\) not in the \(i\) columns that are deleted can be dominated with cost at most

\[
\text{cost}(g) - (B(n) - B(n - i))
\]

then this implies a contradiction of the minimality of \(n\). Given some fixed \(k\), our approach first verifies that the lower bound holds for \(n \leq k + 2\) by computation. We then assume that \(n \geq k + 3\). This assumption ensures that, after deleting at most \(k\) columns, the reduced graph has at least three columns and therefore this process is well defined.

ResultsInMinimalityContradiction takes as an input, a graph \(G_{m,k} = (P_m \text{ or } C_m \square P_k)\), some set of vertices \(R \subseteq V(G_{m,k})\), and the previous cost \(x\) used to dominate \(R\) by some induced sub-broadcast \(g\) (of a broadcast \(f\)) whose broadcast range is contained within \(G_{m,k}\). The graph \(G_{m,k}\) is some set of \(k\) consecutive columns of a larger graph \((P_m \text{ or } C_m) \square C_n\) where \(n > k + 2\). Moreover, \(n\) is assumed to be the smallest integer greater than \(k + 2\) such that there exists an optimal 2-limited dominating broadcast on this larger graph \((P_m \text{ or } C_m) \square C_n\) of cost at most \(B(n) - 1\) where \(B(n)\) is the bound we are attempting to prove (dependent upon \(P_m\) or \(C_m\)). To determine whether or not the cost \(x\) of this induced sub-broadcast \(g\), which dominates \(R\) on \(G_{m,k}\), results in a contradiction of minimality, the values \(m_1, m_2, \ldots, m_k\), defined subsequently, are also passed to ResultsInMinimalityContradiction. For each \(i\) from 1 to \(k\), define \(m_i\) as the maximum value, over all \(n > k + 2\), of \(B(n) - B(n - i)\).

For instance, given \(B(n) = 4 \left\lfloor \frac{n}{6} \right\rfloor\) and \(k = 14\), the values of \(m_i\) are given in the bottom row of Table 1. Table 1 also includes the calculation of the \(m_i\)’s, broken down according to the least residue of \(n\) modulo 6.
The values $m_1, m_2, \ldots, m_k$ are used in ResultsInMinimalityContradiction as follows. Suppose we delete $i \leq k$ columns of $G_{m,k}$ and “patch” the grid together in the natural way to obtain $G' = G_{m,k-i}$. Let $R' = \{v \in R : v \in G'\}$. If $R'$ can be dominated with cost $x - m_i$, then there exists a broadcast of $G_{m,n-i}$ that has cost less than $B(n-i)$. This follows as, by our choices of $m_i$, and the arguments discussed above, this implies there exists a 2-limited dominating broadcast on $G'$ of cost strictly less than $B(n-i)$. This contradicts the minimality of $n$. To determine whether $R'$ can be dominated with cost at most $x - m_i$, it suffices to determine whether $\text{HasBroadcastScheme}(G_{m,k-i}, R', x - m_i)$ is true.

Algorithm 2 provides the pseudo-code for ResultsInMinimalityContradiction and Theorem 2.8 proves the correctness of this approach.

**Theorem 2.8.** Fix $k \geq 1$ and let $B(n)$ be a lower bound for the 2-limited broadcast domination number of $G_{m,n} = (P_m \text{ or } C_m) \square C_n$ which holds for all $n \leq k+2$. Suppose the bound does not hold and let $n$ be the smallest integer greater than $k+2$ such that there exists an optimal 2-limited dominating broadcast $f$ on $G_{m,n}$ of cost at most $B(n)-1$. Let $g$ be an induced sub-broadcast of $f$ whose broadcast range is contained in some $k$-column induced subgraph $G_{m,k}$ of $G_{m,n}$ which is isomorphic to $(P_m \text{ or } C_m) \square P_k$. Let $R$ be the range of $g$. Define $m_i$ (for each $i = 1$ to $k$) as the maximum value over all $n > k+2$ of $B(n) - B(n-i)$. If

$$\text{ResultsInMinimalityContradiction}(G_{m,k}, R, \text{cost}(g), m_1, m_2, \ldots, m_k)$$

is true, there exists a $1 \leq i \leq k$ such that the graph $G_{m,n-i} = (P_m \text{ or } C_m) \square C_{n-i}$ has a 2-limited dominating broadcast of cost less than $B(n-i)$.

**Proof.** Assume the conditions of Theorem 2.8 and suppose

$$\text{ResultsInMinimalityContradiction}(G_{m,k}, R, \text{cost}(g), m_1, m_2, \ldots, m_k)$$

is true. Let $f' = f \ominus g$. Let $S$, $G$, and $R'$ be defined as in Algorithm 2. Let $G'$ be the resulting graph $G_{m,n-i}$ constructed by removing the set of columns $S$ that results in ResultsInMinimalityContradiction returning true on line 11 and “patching” the grid back together in the natural way such that the resulting graph is $(P_m \text{ or } C_m) \square C_{n-i}$. The resulting set of vertices $R'$ of $G$ defined on line 9 can be dominated by an 2-limited broadcast $h$ on $G'$ with cost $x - m_i$ since the function call on line 10 returns true. Let $f''$ be the broadcast formed by restricting $f'$ on this reduced graph in the natural way and redistributing the broadcasts of vertices deleted as described in Observation 1. By Observation 1 $f''$ dominates all of $G'$ with the possible exception of the vertices of $R'$. Let $f''' = f'' \ominus h$. The broadcast $f'''$ is a 2-limited dominating broadcast on $G'$. The cost of such a broadcast is at most

$$\text{cost}(f \ominus g) + x - m_i < B(n) - B(n) + B(n-i) = B(n-i).$$

This establishes $\gamma_{b,2}(G') < B(n-i)$. This contradicts the minimality of $n$.

\[\square\]
Algorithm 2: Routine to determine a minimality contradiction.

1 function ResultsInMinimalityContradiction \( (G_{m,k}, R, x, m_1, m_2, \ldots, m_k) \):

\[ \textbf{Input} : \text{A graph } G_{m,k} = (P_m \text{ or } C_m) \square P_k \text{ with columns labelled from left to right by } c_1, c_2, \ldots, c_k, \]
and rows from 1 to \( m \), a set of vertices \( R \subseteq V(G_{m,k}) \), the cost \( x \) used to dominate \( R \) by some broadcast \( g \) whose broadcast range lies entirely within \( G_{m,k} \), and \( m_i \) (for each \( i = 1 \) to \( k \))
defined as the maximum values over all \( n > k + 2 \) of \( B(n) - B(n - i) \) where \( B(n) \) is the bound we are attempting to prove.

\[ \textbf{Output} : \text{Value of the truth statement: “given the cost } x \text{ used to dominate a region } R \text{ of a larger graph with } n > k + 2 \text{ columns with a broadcast cost less than } B(n), \text{ there is a } 1 \leq i \leq k \text{ such that there exists a broadcast on a graph with } n - i \text{ columns with a broadcast cost less than } B(n - i).” \]

2 Create a sorted list \( L \) of the columns that contain at least one vertex of \( R \) so that they are first ordered from maximum to minimum according to the number of vertices that are in \( R \). Resolve ties by sorting so that \( c_i \) comes before \( c_j \) if \( i < j \);
3 Let the vertices of \( G_{m,k} \) be labelled according to their row number and column label;
4 for \( i \) from 1 to \( \text{length}(L) \) do
5 \hspace{1em} Let \( S \) be the set of columns contained in the first \( i \) entries of list \( L \);
6 \hspace{1em} Let \( G \) be the graph \( G_{m,k-i} \);
7 \hspace{1em} Number the columns of \( G \) using the set of column labels \( \{c_1, c_2, \ldots, c_k\} \setminus S \) going from left to right in order of the column indices;
8 \hspace{1em} Let the vertices of \( G \) be labelled according to their row number and column label;
9 \hspace{1em} Set \( R' = \{v \in R : v \in G\} \);
10 if \( \text{OptimalityContradiction}(G, R', x - m_i) \) then
11 \hspace{1em} return True;
12 end
13 end
14 return False;
3 Computation Proof Method

Fix $m$ and suppose we wish to establish $B(n)$ as a lower bound for the 2-limited broadcast domination number of $G_{m,n}$ where $G_{m,n}$ is either $P_m \triangledown C_n$ or $C_m \triangledown C_n$. Our computational approach starts by assuming there is a minimum value of $n$ for which there is a broadcast $f$ of cost less than $B(n)$. A positive fixed number $r \geq 5$ of columns is chosen. This constraint on $r$ is not necessary in general but is chosen here as it is sufficient for our methods and makes the explanation of our subroutines simpler. Let $C$ be the subgraph of $G_{m,n}$ induced by the vertices appearing in a minimum cost set (with respect to $f$) of $r$ consecutive columns of $G_{m,n}$. Our approach completes an exhaustive search of all possible sub-broadcasts $g$ induced by $V(C)$ of $f$ and subjects each broadcast $g$ to a series of tests to try and reach a contradiction.

Given $C$, four columns are added to both the left and right of $C$ in order to ensure that the subgraph considered is large enough to include the region dominated by any vertex that could potentially dominate some vertex in $C$. That is, our algorithm takes as an input, $(P_m \cup C_m) \triangle P_k$ centred at $C$ where $k = r + 8$.

All cases with $n \leq k + 2$ are first checked by computation to ensure that the bound holds. As we assume $n > k + 2$, deleting up to $k$ columns is well defined.

By our choice of $C$, we can bound the possible costs of the sub-broadcasts $g$ induced by $C$ between some fixed values $s$ and $t$. Let $C' \subseteq V(C)$ be the set of the vertices of $C$ which are distance greater than one from the leftmost and rightmost columns in $C$. Informally, $C'$ can be thought of as the “middle” of $C$. Vertices in $C'$ can only hear the broadcasts from vertices in $C$. For example, given $G_{m,k} = P_5 \triangledown P_{16}$, $C$ is formed by the eight consecutive columns centred in $G_{m,k}$ and $C'$ contains the vertices in the four consecutive columns centred in $C$. Figure 9 depicts $G_{m,k}$. $C$ (indicated by the thick black rectangle), $C'$ (indicated by the shaded grey region), and possible broadcast vertices in and outside of $C$. The value of $s$ can be set as the minimum cost of a broadcast required to dominate $C'$. The value of $t$ is set to equal the maximum possible cost of a minimum $r$ consecutive column induced sub-broadcast for a graph $G_{m,n}$ that hypothetically has a broadcast of cost less than $B(n)$. For example, given $B(n) = 4 \left\lceil \frac{n}{6} \right\rceil$, if every six consecutive column induced subgraph has cost at least four, then the total cost is at least $\left\lceil \frac{n}{6} \right\rceil$ and hence the bound $B(n)$ holds. But if a broadcast has one or more six consecutive column induced subgraphs with cost at most three, then it is possible that the total cost is less than $B(n)$. So for this example, $t = 3$.

Let $\mathcal{C}$ be the set of all possible sub-broadcasts induced by $V(C)$ of $f$ of costs $s$ to $t$. The following five functions are designed to try and yield a contradiction for each $g \in \mathcal{C}$.

3.1 Dominating the Middle Contradiction

Since the vertices of $C'$ can only hear a broadcast from a vertex in $C$, they must be dominated by $g$. The function $\text{DoesNotDominateMiddle}$ returns true if any vertices of $C'$ are not dominated by $g$, and false otherwise.

3.2 Forbidden Broadcasts Structure Contradiction

If $g$ contains a vertex broadcasting at strength one within distance one of a vertex broadcasting at strength two, then $f$ is trivially not optimal. If there is an optimal broadcast $h$ of cost $< B(n)$ that has two vertices broadcasting at strength one within distance two of each other, then there is another optimal broadcast $h'$ such that $\text{cost}(h) = \text{cost}(h')$ where these two vertices are replaced by a vertex broadcasting at strength two. Proving

![Figure 9: Depiction of $C$ embedded in $P_5 \triangle P_{16}$ to illustrate that the middle four columns of $C$ must be dominated by vertices within $C$.](image-url)
Algorithm 3: Routine to determine if a broadcast dominates the “middle” of a graph.

```plaintext
1. function DoesNotDominateMiddle(G_{m,k}, g);
   Input: A graph G_{m,k} = (P_m or C'_m) □ P_k with columns labelled from left to right by c_1, c_2, ..., c_k, where k ≥ 13, and a broadcast g.
   Output: Value of the truth statement: “g does not dominate C’.”
2. Let C’ be formed by the vertices of columns c_7 to c_{k-6};
3. if C’ is not a subset of the range of g then
4.    return True;
5. end
6. return False;
```

that there is no optimal broadcast like h’ of cost < B(n) proves that there is no optimal broadcast like h as h’ can be modified to construct h. Therefore, if g contains two strength one vertices within distance two of each other, g need not be considered. The function ForbiddenBroadcast returns true if g contains either such broadcast structure, and false otherwise.

Algorithm 4: Routine to determine if a broadcast contains a forbidden structure.

```plaintext
1. function ForbiddenBroadcast(G, g);
   Input: A graph G and a broadcast g.
   Output: Value of the truth statement: “g contains a forbidden structure.”
2. if g contains a vertex broadcasting at strength one within distance one of a vertex broadcasting at strength two then
3.    return True;
4. end
5. if g contains two vertices broadcasting at strength one within distance two of each other then
6.    return True;
7. end
8. return False;
```

3.3 Necessary Broadcasts Contradiction

Let D be the set of vertices at distance one from either the leftmost or rightmost column in C which are not in R. For example, given G_{m,k} = P_5 □ P_{16}. Figure 10 depicts such a sub-broadcast g induced by V(C), where the vertices of D are indicated by thick orange circles. As the vertices broadcasting at non-zero strength under g are the only vertices broadcasting at non-zero strength under f when restricted to C, the only vertices whose broadcast can be heard by the vertices of D are vertices broadcasting at strength two in the same row, respectively, at distance one from either the leftmost or rightmost column of C. Figure 11 depicts the necessary vertices broadcasting at strength two forced by the vertices of D of Figure 10. Therefore, for each v ∈ D, there exists a vertex u, in the same row as v and at distance one from the leftmost or rightmost column of

Figure 10: Sub-broadcast g induced by V(C) and undominated vertices D.
Let $g'$ be the sub-broadcast induced by $V(C) \cup U$, where $U$ is the set of vertices at distance one from the leftmost or rightmost column of $C$ that are necessarily broadcasting at strength two under $f$ given $g$ (as defined formally in Section 3.3).

Suppose there exists vertices of $V(C)$ which do not hear a broadcast under $g'$. For example, given $G_{m,k} = P_5 \Box P_6$, Figure 12 depicts a possible sub-broadcast $g$ induced by $V(C)$ which, as shown by the circled vertex in Figure 13 does not result in a sub-broadcast $g'$ induced by $V(C) \cup U$ which dominates $V(C)$.
Let \( h \) be a minimum cost broadcast on \( G_{m,k} \) which extends the broadcast of \( g' \) to dominate \( V(C) \). For example, Figure 14 depicts a possible minimum cost broadcast \( h \) given \( g' \) in Figure 13. Let \( h' \) be the minimum cost induced sub-broadcast of \( f \) which dominates \( V(C) \). Note that, though we do not know what \( h' \) looks like, we know that it contains \( g' \) as a sub-broadcast. Therefore, by our choice of \( h \), \( \text{cost}(h) \leq \text{cost}(h') \). For each vertex \( u \notin V(C) \) which is within distance two of a vertex \( v \in V(C) \) not dominated by \( g' \), let \( h''(u) = 2 \). Let \( R'' \) be the range of \( h'' \). Observe that the range \( R' \) of \( h' \) is a subset of \( R'' \). Therefore, if \( R'' \) can be dominated with cost less than \( \text{cost}(h) \), then \( R' \) can be dominated with cost less than \( \text{cost}(h') \). This means that \( h' \) cannot be an induced sub-broadcast of an optimal broadcast \( f \). This contradicts the choice of \( h' \). For example, Figure 15 depicts \( h'' \) as determined from \( g' \) in Figure 13. The range \( R'' \) of \( h'' \) can be dominated with cost 14, as seen in Figure 16. This broadcast costs less than \( h \) as found in Figure 14.

The routine \( \text{MaximumRangeContradiction} \) returns true if \( h \) and \( h'' \) lead to a contradiction, and false otherwise.

### 3.5 Contradiction when Considering All Possible Sub-Cases

Let \( g' \) be the sub-broadcast induced by \( V(C) \cup \mathcal{U} \), where \( \mathcal{U} \) is the set of vertices at distance one from the leftmost or rightmost column of \( C \) that are necessarily broadcasting at strength two under \( f \) given \( g \) (as defined formally in Section 3.3).

Let \( D' \) denote the vertices of \( C \) which do not hear a broadcast under \( g' \). As the broadcast of \( f \) beyond \( V(C) \cup \mathcal{U} \) is not defined explicitly, there are many possible ways \( f \) could dominate \( D' \). We consider all possible ways \( f \) could dominate \( D' \). That is, we construct every possible induced sub-broadcast \( g'' \) of \( f \) which extends the broadcast of \( g' \) to dominate \( D' \). Let \( C' \) be the collection of all possible induced sub-broadcasts of \( f \) formed by extending the broadcast of \( g' \) until \( D' \) is dominated. For example, given \( G_{m,k} = P_3 \Box C_{16} \), Figure 19 depicts a possible induced sub-broadcast \( h \in C' \) constructed from \( g' \) in Figure 18 which was forced by \( g \) in Figure 17. The routine \( \text{NecessaryBroadcastsContradiction} \) returns true if, for all \( h \in C' \), \( h \) leads to a contradiction, and false otherwise.
Algorithm 6: Routine to determine if a contradiction arises from the maximum range test.

1 function MaximumRangeContradiction \((G_{m,k}, g, m_1, m_2, \ldots, m_k)\):

Input: A graph \(G_{m,k} = (P_m \text{ or } C_m) \square P_k\) with columns labelled from left to right by \(c_1, c_2, \ldots, c_k\), where \(k \geq 13\), a broadcast \(g\), and \(m_i\) (for each \(i = 1\) to \(k\)) defined as the maximum values over all \(n > k + 2\) of \(B(n) - B(n - i)\) where \(B(n)\) is the bound we are attempting to prove.

Output: Value of the truth statement: “maximum range test force a contradiction.”

2 Construct \(g'\) as described in Algorithm 5;
3 Let \(D'\) be the vertices in columns \(c_5\) and \(c_{k-4}\) which do not hear a broadcast under \(g'\);
4 Let \(h\) be a minimum cost broadcast which extends the broadcast of \(g'\) to dominate \(D'\) constructed by altering the broadcasts of vertices in \(V(G_{m,k}) \setminus [V(c_5) \cup V(c_6) \cup \cdots \cup V(c_{k-4})]\);
5 Set \(h'' = g'\);
6 foreach \(v \in D'\) do
7    foreach \(x \in V(G_{m,k}) \setminus [V(c_5) \cup V(c_6) \cup \cdots \cup V(c_{k-4})]\) do
8        if \(d(x, v) \leq 2\) then
9            \(h''(x) = 2;\)
10        end
11    end
12 end
13 Let \(R''\) be the range of \(h''\);
14 if HasBroadcastScheme\((G_{m,k}, R'', \text{cost}(h) - 1)\) then
15    return True;
16 end
17 if ResultsInMinimalityContradiction\((G_{m,k}, R'', \text{cost}(h), m_1, m_2, \ldots, m_k)\) then
18    return True;
19 end
20 return False;
Figure 18: Sub-broadcast $g'$ induced by $V(C) \cup \mathcal{U}$ forced by $g$ in Figure 17 which does not dominate $V(C)$.

Figure 19: A possible induced sub-broadcast $h \in C'$ given $g'$ in Figure 18.

**Algorithm 7:** Routine to determine if every possible sub-case yields a contradiction.

```plaintext
1 function ContradictionForEverySubCase ($G_{m,k}, g, m_1, m_2, \ldots, m_k$);
   Input : A graph $G_{m,k} = (P_m$ or $C_m) \square P_k$ with columns labelled from left to right by $c_1, c_2, \ldots, c_k$, where $k \geq 13$, a broadcast $g$, and $m_i$ (for each $i = 1$ to $k$) defined as the maximum values over all $n > k + 2$ of $B(n) - B(n - i)$ where $B(n)$ is the bound we are attempting to prove.
   Output: Value of the truth statement: “every possible sub-case yields a contradiction.”
   2 Construct $g'$ as described in Algorithm 5;
   3 Let $D$ be the set of vertices which do not hear a broadcast under $g$ in columns $c_6$ and $c_{k-5}$;
   4 Let $C'$ be the collection of broadcasts formed by extending the broadcast of $g'$ by altering the broadcasts of vertices in $V(G_{m,k}) \setminus [V(c_5) \cup V(c_6) \cup \cdots \cup V(c_{k-4})]$ until every vertex of $D'$ is dominated;
   5 foreach $h \in C'$ do
      Let $R$ be the range of $h$;
      7 If $\text{HasBroadcastScheme}(G_{m,k}, R, \text{cost}(h) - 1)$ then break;
      8 If $\text{ResultsInMinimalityContradiction}(G_{m,k}, R, \text{cost}(h), m_1, m_2, \ldots, m_k)$ then break;
      9 return False;
   10 end
   11 return True;
```

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3.6 Algorithm to Prove Lower Bounds

Algorithm 8: ProvedLowerBound implements the battery of tests described in Section 3 and Sections 3.1 through 3.5 to try and find contradictions for all possible sub-broadcasts induced on $r$ consecutive columns on $P_m \Box P_k$ or $C_m \Box P_k$. The routine ProvedLowerBound returns true if every such possible induced sub-broadcast results in a contradiction and the bound is proven, and false otherwise.

Algorithm 8: Routine to prove lower bound.

1 function ProvedLowerBound $(G_{m,k}, s, t, m_1, m_2, \ldots, m_k)$;

   Input : A graph $G_{m,k} = (P_m \text{ or } C_m) \Box P_k$, where $k \geq 13$, with columns labelled from left to right by $c_1, c_2, \ldots, c_k$, the minimum $s$ and maximum $t$ possible costs of a sub-broadcast $g$ of $G_{m,k}$ in the $r = k - 8$ consecutive columns centred in $G_{m,k}$, and $m_i$ (for each $i = 1 \text{ to } k$) defined as the maximum values over all $n > k + 2$ of $B(n) - B(n - i)$ where $B(n)$ is the bound we are attempting to prove.

   Output: Value of the truth statement: “lower bound $B(n)$ proven.”

2 Let $C$ be all possible sub-broadcasts $g$ of costs $s$ to $t$ induced by the vertices of the $r = k - 8$ consecutive columns centred in $G_{m,k}$;

3 foreach $g \in C$ do

4   if DoesNotDominateMiddle($G_{m,k}, g$) then break;
5   if ForbiddenBroadcast($G, g$) then break;
6   Let $R$ be the range of $g$;
7   if HasBroadcastScheme($G_{m,k}, R, \text{cost}(g) - 1$) then break;
8   if ResultsInMinimalityContradiction($G_{m,k}, R, \text{cost}(g), m_1, m_2, \ldots, m_k$) then break;
9   if NecessaryBroadcastsContradiction($G_{m,k}, g, m_1, m_2, \ldots, m_k$) then break;
10  if MaximumRangeContradiction($G_{m,k}, g, m_1, m_2, \ldots, m_k$) then break;
11  if ContradictionForEverySubCase($G_{m,k}, g, m_1, m_2, \ldots, m_k$) then break;
12 return False;

13 return True;

4 Implementation

We have implemented ProvedLowerBound in Python 3.9 for $P_m \Box C_n$ and $C_m \Box C_n$. The code is compiled with Cython and all ILP calls are run with a Gurobi solver. All computations in this section were run on one of Slobodin’s 2012 4GB MacBook Pro with a 2.5 GHz Intel Dual-Core i5 processor.

Our implementation of ProvedLowerBound was first designed for proving lower bounds on $P_m \Box C_n$. Given this, we included a canonicity test for $P_m \Box P_n$ so as to only consider the set $C$ of all possible sub-broadcast induced by the vertices of some set of $r$ consecutive columns with costs $s$ to $t$ up to isomorphism. This test was done by checking that each broadcast (when expressed as a sequence) was the least lexicographically when compared to all broadcasts isomorphic to it. When adapting the code to work on $C_m \Box C_n$, we did not update the canonicity test to reduce the number of cases up to isomorphism on $C_m \Box C_n$ from $P_m \Box C_n$. Fortunately, for the bounds we tested, this redundancy was acceptable in terms of run times.

Our implementation of ProvedLowerBound has allowed us to prove Theorems 4.1 and 4.5 and their respective corollaries. The values of $|C|$ in Tables 2 and 3 have been verified by Pólya’s Theorem (see [2, Theorem 14.3.3]) as 2-limited broadcasts can simply be thought of as three-colourings of a graph (note that the colouring need not be proper).
Theorem 4.1. For $n \geq 3$,
\[ \gamma_{b,2}(C_3 \Box C_n) = \left\lceil \frac{2n}{3} \right\rceil. \]

Proof. Theorem 5.1 of [8] states that $\gamma_{b,2}(C_3 \Box C_n) \leq \left\lceil \frac{2n}{3} \right\rceil$. Fix $r = 6$. Let $k = 14 = r + 8$. By computation, we know that $\gamma_{b,2}(C_3 \Box C_n) = \left\lceil \frac{2n}{3} \right\rceil$ for all $3 \leq n \leq 16$. Suppose the bound does not hold and let $n$ be the smallest integer greater than 16 such that there exists an optimal 2-limited dominating broadcast $f$ on $C_3 \Box C_n$ of cost $\leq \left\lceil \frac{2n}{3} \right\rceil - 1$. As $B(n) = \left\lceil \frac{2n}{3} \right\rceil$ is the bound we hope to prove, for $1 \leq i \leq 14 = k$, the $m_i$'s take on the following values:
\[ (m_1, m_2, \ldots, m_{14}) = (1, 2, 2, 3, 4, 4, 5, 6, 6, 7, 8, 8, 9, 10). \]

As $r - 4 = 2$ and $\gamma_{b,2}(C_3 \Box P_2) = 2$, set $s = 2$. Set $t = 3$. Observe if $t > 3$ then $\left\lceil \frac{4n}{6} \right\rceil = \left\lceil \frac{2n}{3} \right\rceil \leq \gamma_{b,2}(C_3 \Box C_n)$.

Table 2: Cases considered in the proof of Theorem 4.1.

| Case | Cost 2 | Cost 3 |
|------|--------|--------|
| DoesNotDominateMiddle: | 54 | 302 |
| ForbiddenBroadcast: | 48 | 231 |
| ResultsInMinimalityContradiction: | 4 | 45 |
| NecessaryBroadcastsContradiction + HasBroadcastScheme: | 0 | 8 |
| NecessaryBroadcastsContradiction + ResultsInMinimalityContradiction: | 2 | 3 |
| ContradictionForEverySubCase + HasBroadcastScheme: | 0 | 336 |
| ContradictionForEverySubCase + ResultsInMinimalityContradiction: | 0 | 96 |

Observe the following corollaries of Theorem 4.1.

Corollary 4.2. For $n \geq 3$,
\[ \left\lceil \frac{4n}{6} \right\rceil \leq \gamma_{b,2}(C_4 \Box C_n) \leq 4 \left\lceil \frac{n}{6} \right\rceil + \begin{cases} 
0 & \text{for } n \equiv 0 \pmod{6}, \\
2 & \text{for } n \equiv 1 \text{ or } 2 \pmod{6}, \\
3 & \text{for } n \equiv 3 \text{ or } 4 \pmod{6}, \text{ and} \\
4 & \text{for } n \equiv 5 \pmod{6}.
\end{cases} \]

Proof. The upper bound of Corollary 4.2 is stated in Theorem 5.1 of [8]. Any 2-limited dominating broadcast on $C_4 \Box C_n$ can be modified to give a 2-limited dominating broadcast on $C_3 \Box C_n$. Therefore $\gamma_{b,2}(C_3 \Box C_n) \leq \gamma_{b,2}(C_4 \Box C_n)$ and the result follows from Theorem 4.1.

Corollary 4.3. For $m \geq 23$,
\[ \gamma_{b,2}(P_m \Box C_3) = \left\lceil \frac{2m}{3} \right\rceil. \]
Proof. The upper bound of Corollary 4.3 is stated in Theorem 4.3 of [8]. Any 2-limited dominating broadcast on $P_m \square C_3$ is a 2-limited dominating broadcast on $C_m \square C_3$. Therefore $\gamma_{b,2}(C_3 \square C_m) \leq \gamma_{b,2}(P_m \square C_3)$ and the result follows from Theorem 4.1.

Corollary 4.4. For $m \geq 23$,

$$\left\lfloor \frac{4m}{6} \right\rfloor \leq \gamma_{b,2}(P_m \square C_4) \leq 4 \left\lfloor \frac{m}{6} \right\rfloor + \begin{cases} 1 & \text{for } m \equiv 0 \pmod{6}, \\ 2 & \text{for } m \equiv 1 \text{ or } 2 \pmod{6}, \\ 3 & \text{for } m \equiv 3 \pmod{6}, \text{ and} \\ 4 & \text{for } m \equiv 4 \text{ or } 5 \pmod{6}. \end{cases}$$

Proof. The upper bound of Corollary 4.4 is stated in Theorem 4.3 of [8]. Any 2-limited dominating broadcast on $P_m \square C_4$ is a 2-limited dominating broadcast on $C_m \square C_4$. Therefore $\gamma_{b,2}(C_3 \square C_m) \leq \gamma_{b,2}(P_m \square C_4)$ and the result follows from Theorem 4.1.

Theorem 4.5. For $n \geq 5$,

$$\gamma_{b,2}(C_5 \square C_n) = n.$$ 

Proof. Theorem 5.1 of [8] states that $\gamma_{b,2}(C_5 \square C_n) \leq n$. Fix $r = 8$. Let $k = 16 = r + 8$. By computation, we know that $\gamma_{b,2}(C_5 \square C_n) = n$ for all $3 \leq n \leq 18$. Suppose the bound does not hold and let $n$ be the smallest integer greater than 18 such that there exists an optimal 2-limited dominating broadcast $f$ on $C_5 \square C_n$ of cost $\leq n - 1$. As $B(n) = n$ is the bound we hope to prove, for $1 \leq i \leq 16 = k$, the $m_i$’s take on the following values:

$$(m_1, m_2, \ldots, m_{16}) = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16).$$

As $r - 4 = 4$ and $\gamma_{b,2}(C_5 \square P_4) = 5$, set $s = 4$. Set $t = 7$. Observe if $t > 7$ then $n \leq \gamma_{b,2}(C_5 \square C_n)$. As

$$ProvedLowerBound(G_{5,16}, 5, 7, m_1, m_2, \ldots, m_{16})$$

is true, this proves the theorem. Running ProvedLowerBound for the above values took under eight minutes. See Table 3 for a summary of the number of broadcasts rejected at each step of the algorithm per considered cost. Steps with zero cases are omitted from Table 3. Note that, as ContradictionForEverySubCase considers all possible induced sub-broadcast of a given case, the total number of cases considered will be at least $|\mathcal{C}|$.

| $|\mathcal{C}|$: | Cost 5 | Cost 6 | Cost 7 |
|------------------|--------|--------|--------|
| DoesNotDominateMiddle: | 264,148 | 1,925,104 | 12,162,548 |
| ForbiddenBroadcast: | 264,115 | 1,922,880 | 12,103,722 |
| HasBroadcastScheme: | 8 | 1423 | 48,899 |
| ResultsInMinimalityContradiction: | 0 | 161 | 5,198 |
| NecessaryBroadcastsContradictionResultsInMinimalityContradiction: | 25 | 632 | 4,696 |
| ContradictionForEverySubCase: | + | 0 | 5 | 27 |
| HasBroadcastScheme: | + | 0 | 27 | 0 |
| ResultsInMinimalityContradiction: | + | 0 | 48 | 30 |

Table 3: Cases considered in the proof of Theorem 4.5.
Observe the following corollaries of Theorem 4.5:

**Corollary 4.6.** For $n \geq 6$,

$$n \leq \gamma_{b,2}(C_6\square C_n) \leq n + \begin{cases} 0 & \text{for } n \equiv 0 \pmod{4} \text{ and} \\ 1 & \text{for } n \equiv 1, 2, \text{ or } 3 \pmod{4}. \end{cases}$$

*Proof.* The upper bound of Corollary 4.6 is stated in Theorem 5.1 of [8]. Any 2-limited dominating broadcast on $C_6\square C_n$ can be modified to give a 2-limited dominating broadcast on $C_5\square C_n$. Therefore $\gamma_{b,2}(C_6\square C_n) \leq \gamma_{b,2}(C_5\square C_n)$ and the result follows from Theorem 4.5. 

**Corollary 4.7.** For $n \geq 3$,

$$n \leq \gamma_{b,2}(P_5\square C_n) \leq n + \begin{cases} 0 & \text{for } n \equiv 0 \pmod{2} \text{ and} \\ 1 & \text{for } n \equiv 1 \pmod{2}. \end{cases}$$

*Proof.* For $n = 3$ and 4, the bound is easily verified by computation. For $n \geq 5$, the upper bound of Corollary 4.7 is stated in Theorem 4.1 of [8]. Any 2-limited dominating broadcast on $P_5\square C_n$ is a 2-limited dominating broadcast on $C_5\square C_n$. Therefore $\gamma_{b,2}(P_5\square C_n) \leq \gamma_{b,2}(C_5\square C_n)$ and the result follows from Theorem 4.5. 

**Corollary 4.8.** For $m \geq 23$,

$$m \leq \gamma_{b,2}(P_m\square C_5) \leq m + 1.$$

*Proof.* For $n \geq 23$, the upper bound of Corollary 4.8 is stated in Theorem 4.3 of [8]. Any 2-limited dominating broadcast on $P_m\square C_5$ is a 2-limited dominating broadcast on $C_5\square C_m$. Therefore $\gamma_{b,2}(P_m\square C_5) \leq \gamma_{b,2}(C_5\square C_m)$ and the result follows from Theorem 4.5. 

**Corollary 4.9.** For $m \geq 23$,

$$m \leq \gamma_{b,2}(P_m\square C_6) \leq m + 1.$$

*Proof.* For $n \geq 23$, the upper bound of Corollary 4.9 is stated in Theorem 4.3 of [8]. Any 2-limited dominating broadcast on $P_m\square C_6$ can be modified to give a 2-limited dominating broadcast on $C_5\square C_m$. Therefore $\gamma_{b,2}(C_5\square C_m) \leq \gamma_{b,2}(P_m\square C_6)$ and the result follows from Theorem 4.5. 

**Corollary 4.10.** For $n \geq 2$,

$$n \leq \gamma_{b,2}(P_5\square P_n) \leq n + 1.$$

*Proof.* For $n = 2, 3, \text{ and } 4$, the bound is easily verified by computation. For $n \geq 5$, the upper bound of Corollary 4.10 is stated in Theorem 3.1 of [8]. Any 2-limited dominating broadcast on $P_5\square P_n$ is a 2-limited dominating broadcast on $C_5\square C_n$. Therefore $\gamma_{b,2}(C_5\square C_n) \leq \gamma_{b,2}(P_5\square P_n)$ and the result follows from Theorem 4.5. 

5 Future Work

This paper presents a method for computationally proving lower bounds for the 2-limited broadcast domination of special sub-families of the Cartesian product of two paths, a path and a cycle, and two cycles. Given this work, we note the follow rather natural questions.

**Problem 5.1.** Can this method be optimized further to prove bounds on larger graphs?
Problem 5.2. Can this method be generalized to prove bounds on graphs other the Cartesian product of two paths, a path and a cycle, and two cycles?

Problem 5.3. Although the \(k\)-limited broadcast domination problem is NP-complete, does there exist polynomial time algorithms for this problem for the Cartesian products of two paths, a path and a cycle, and two cycles?

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