Stability of Utility Maximization in Nonequivalent Markets

Kim Weston

October 6, 2014

Abstract

Stability of the utility maximization problem with random endowment and indifference prices is studied for a sequence of financial markets in an incomplete Brownian setting. Our novelty lies in the nonequivalence of markets, in which the volatility of asset prices (as well as the drift) varies. Degeneracies arise from the presence of nonequivalence. In the positive real line utility framework, a counterexample is presented showing that the expected utility maximization problem can be unstable. A positive stability result is proven for utility functions on the entire real line.

Keywords: Expected utility theory, Incompleteness, Random endowment, Market stability, Nonequivalent markets
Mathematics Subject Classification (2010): 91G80, 93E15, 60G44
JEL Classification: G13, D81

1 Introduction

As part of Hadamard’s well-posedness criteria, stability of the utility maximization problem with random endowment is studied with respect to perturbations in both volatility and drift. Specifically, we seek to answer the question:

What conditions on the utility function and modes of convergence on the sequence of volatilities and drifts guarantee convergence of the corresponding value functions and indifference prices?

Perhaps surprisingly, convergence can fail even in the tamest of settings when the utility function is finite only on \( \mathbb{R}_+ \) and volatility can vary. We present a simple counterexample to convergence in the basis risk setting with power utility. When the utility function is finite only on \( \mathbb{R}_+ \), the admissibility criterion is harsh: negative values in terminal wealth plus random endowment equate to minus infinity in utility. When volatility can vary, a

\[1\]Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213
e-mail: kimberly@andrew.cmu.edu
contingent claim that is replicable only in the limiting market requires strictly more initial capital in every pre-limiting market in order to avoid a minus infinity contribution towards expected utility. As part of the counterexample, we prove a positive convergence result in which the limiting market adopts an additional admissibility condition that is implicitly present in each pre-limiting market.

When the investor’s utility function is finite on the entire real line, the admissibility criterion is different. Our main result provides conditions on the utility function and on the sequence of markets so that we have convergence of the value functions and indifference prices. We consider a similar setup to [18], and our main assumptions are analogous to theirs. The only non-standard assumption we require is an assumption on the limiting market. The significant difficulty stems from the growth of the dual utility function at infinity because in contrast to utility on $\mathbb{R}_+$, the conjugate of real line utility grows strictly faster than linearly at infinity. We provide two sufficient conditions. These conditions include:

1. The first condition applies to a contingent claim that is replicable in the limiting market yet not replicable in any pre-limiting market. The corresponding stability problem is relevant when a claim’s underlying asset is not liquidly traded but is closely linked to a liquidly traded asset. This situation arises, e.g., when hedging weather derivatives by trading in related energy futures or when an executive wants to hedge his position in company stock options but is legally restricted from liquidly trading his own company’s stock. Practical and computational aspects of this problem are considered by [5], [19], and in more generality by [9].

2. The second sufficient condition requires exponential preferences and additional regularity of the limiting market but places no restrictions on the claim’s replicability. This case covers a general incomplete Brownian market structure under a mild BMO condition on the limiting market. The connection between BMO and exponential utility is long established. See, for example, [6] and [10].

The questions of existence and uniqueness for the optimal investment problem from terminal wealth are thoroughly studied. The surrounding literature is vast, and only a small subset of work is mentioned here. For general utility functions on $\mathbb{R}_+$ in a general semimartingale framework, [16] finish a long line of research on incomplete markets without random endowment. In [4], this work is extended to include bounded random endowment, while [12] study the unbounded random endowment case. For utility functions on $\mathbb{R}$ in a locally bounded semimartingale framework, [21] studies the case with no random endowment, while [20] handle the unbounded random endowment case. In [2], the authors study the non-locally bounded semimartingale setting without random endowment and unify the framework...
for utilities on $\mathbb{R}$ and $\mathbb{R}_+$. 

Stability with respect to perturbations in the market price of risk for fixed volatility is first studied in [18] for utility on $\mathbb{R}_+$ and later in [1] for exponential utility. Both works consider risky assets with continuous price processes and no random endowment. For a locally bounded asset and an investor with random endowment, [14] study a market stability problem in which the financial market and random endowment stay fixed while the subjective probability measure and utility function vary. A BSDE stability result is used in [8] to study a specific stability problem for an exponential investor related to the indifference price formulas derived in [9]. Using this BSDE stability result, [8]'s market stability result extends to a case with a fixed market price of risk and a varying underlying correlation factor between the traded and nontraded securities. In contrast to these previous works, we seek to prove a stability result for a general utility function on $\mathbb{R}$ allowing for varying both volatility and market price of risk with the presence of random endowment.

The structure of the paper is as follows. Section 2 presents a counterexample for a power investor in the basis risk setting. Section 3 lays out the model assumptions and states the main result. The main result is proven in Section 4. Finally, Section 5 provides a counterexample showing the necessity of a nondegeneracy assumption and provides sufficient conditions on the structure of the dual problem for the assumption to hold.

## 2 Stability Counterexample for Power Utility

When an investor’s preferences are described by utility on the positive real line and random endowment is present, the admissibility condition provides an additional implicit constraint. As we will prove, this constraint can create a discontinuity in the value function and indifference prices for markets with varying martingale drivers. The following are simple incomplete Brownian models with a contingent claim that can only be replicated in the limiting market.

We let $B$ and $W$ be independent Brownian motions on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ is the natural filtration of $(B, W)$ completed with $\mathbb{P}$-null sets and $\mathcal{F} = \mathcal{F}_T$. We consider market models with correlation parameter $\rho \in (-1, 1)$ driven by stocks $S^\rho$ where

$$dS^\rho_t = S^\rho_t \left( dt + \sqrt{1 - \rho^2} dB_t + \rho dW_t \right), \quad S^\rho_0 := 1. \quad (2.1)$$

Let $Z^\rho_t := \mathcal{E} \left( -\sqrt{1 - \rho^2} B - \rho W \right)$ for $t \in [0, T]$, where $\mathcal{E} (\cdot)$ refers to the stochastic exponential. The random variable $Z^\rho_T$ is the minimal martingale density corresponding to the $S^\rho$-market. Each $\rho$ market also has a bank account with zero interest rate.
A contingent claim $f$ is defined by $f := \phi(B_T)$, where $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is a bounded, continuous, non-constant function. The claim $f$ is replicable in the $\rho = 0$ market; however, it is not replicable for any other market. We define $\phi_{\text{min}} := \inf \phi$, which corresponds to the subreplication price of $f$ in the $\rho \neq 0$ markets.

### 2.1 Optimal Investment Problem

An investor is modeled by power utility $U(x) = x^p/p$ for $x \geq 0$ with $p \in (0, 1)$. As a convention, $U(x) = -\infty$ for $x < 0$. The investor begins with initial capital $x > -\phi_{\text{min}}$.

A progressively measurable process $H$ is integrable if $\int_0^T H_t^2 dt < \infty$, a.s. For any $\rho$, an integrable $H$ is called $S^\rho$-admissible if there exists a finite constant $K = K(H)$ such that $(H \cdot S^\rho)_t \geq -K$ for all $t \in [0, T]$. We define the primal optimization set by

$$C(\rho) := \{X \in L^\infty(\mathbb{P}) : X \leq (H \cdot S^\rho)_T \text{ for some } \rho\text{-admissible } H\}.$$ 

For $\rho \in (-1, 1)$, the primal value function is defined by

$$u(x, \rho) := \sup_{X \in C(\rho)} \mathbb{E}[U(x + X + f)], \quad x > -\phi_{\text{min}}. \quad (2.2)$$

**Remark 2.1.** For $\rho = 0$, $u(\cdot, 0)$ is well-defined for a larger $x$-domain than $(-\phi_{\text{min}}, \infty)$. Yet the $x$-domain is tight for every $\rho \neq 0$. This discontinuity in the domains at $\rho = 0$ hints at the issue of (dis)continuity with respect to $\rho$ in the primal problem. See [4] for more details on the primal domain definition.

An inherent admissibility constraint is present for each of the $\rho \neq 0$ markets. For each $\rho \in (-1, 1)$, we define the dual domain by

$$\mathcal{D}(\rho) := \left\{ \text{measures } \mathbb{Q} \ll \mathbb{P} : \mathbb{E}\left[ \frac{d\mathbb{Q}}{d\mathbb{P}} \right] = 1 \text{ and } \mathbb{E}^\mathbb{Q}[X] \leq 0 \; \forall X \in C(\rho) \right\}.$$ 

Fix $\rho \neq 0$ and $x > -\phi_{\text{min}}$. For any $\rho$-admissible strategy $H$ such that $x + (H \cdot S^\rho)_T + f \geq 0$, and any $\mathbb{Q} \in \mathcal{D}(\rho)$, the stochastic integral $(H \cdot S^\rho)$ is a $\mathbb{Q}$-supermartingale. Hence for any $t \in [0, T)$, we have $0 \leq \mathbb{E}^\mathbb{Q}[x + (H \cdot S^\rho)_T + f | \mathcal{F}_t] \leq x + (H \cdot S^\rho)_t + \mathbb{E}^\mathbb{Q}[f | \mathcal{F}_t]$. Taking the infimum over all $\mathbb{Q} \in \mathcal{D}(\rho)$ yields, $0 \leq \inf_{\mathbb{Q}} \left(x + (H \cdot S^\rho)_t + \mathbb{E}^\mathbb{Q}[f | \mathcal{F}_t]\right) = x + (H \cdot S^\rho)_t + \phi_{\text{min}}$. Continuity with respect to time produces

$$x + (H \cdot S^\rho)_T \geq -\phi_{\text{min}}. \quad (2.3)$$

We consider a different optimization problem for $\rho = 0$ with an additional admissibility
constraint motivated by \((2.3)\). For any \(x > -\phi_{\min}\), we define the admissibly-constrained primal optimization sets in the \(\rho = 0\) market by

\[
C_c(x) := \{ X = x + (H \cdot S^0)_T : H \text{ is } S^0\text{-admissible and } x + (H \cdot S^0)_T \geq -\phi_{\min}\}.
\]

The corresponding admissibly-constrained primal value function is defined by

\[
u_c(x) := \sup_{X \in C_c(x)} \mathbb{E}[U(X + f)], \quad x > -\phi_{\min}.
\] (2.4)

The following is the main result of the section.

**Theorem 2.1.** Assume the market dynamics \((2.1)\) and utility function \(U(x) = x^p/p\), for \(x \geq 0\), with \(p \in (0, 1)\). Assume the random endowment function \(\phi\) is continuous, bounded, and non-constant, and the initial endowment is \(x > -\phi_{\min}\). Let the \(u\) and \(u_c\) be as in \((2.2)\) and \((2.4)\), respectively. Then,

\[
\lim_{\rho \to 0} u(x, \rho) = u_c(x).
\]

The proofs of Theorem \(2.1\) and Corollary \(2.3\) below will follow in Subsection \(2.2\). The corollary says that indifference prices for \(f\) do not converge to the unique arbitrage-free price in the \(\rho = 0\) market as \(\rho \to 0\). For any \(\rho \in (-1, 1)\), we define the value function without random endowment by

\[
w(x, \rho) := \sup_{X \in C(\rho)} \mathbb{E}[U(X + f)], \quad x > 0.
\] (2.5)

**Definition 2.2.** Given \(x > -\phi_{\min}\) and \(\rho \in (-1, 1)\), \(p = p(x, \rho) \in \mathbb{R}\) is called the indifference price for \(f\) at \(x\) in the \(\rho\)-market if \(w(x + p, \rho) = u(x, \rho)\).

Of course, for \(\rho = 0\), the indifference price corresponds to the unique arbitrage-free price for the bounded replicable claim, \(f\). Also notice that since indifference prices are arbitrage-free prices, then \(p(x, \rho) > \phi_{\min}\) for every \(x > -\phi_{\min}\).

**Corollary 2.3.** Under the assumptions of Theorem \(2.1\): For \(x > -\phi_{\min}\), the indifference prices for \(f\) do not converge to the arbitrage-free price in the \(\rho = 0\) market. Indeed, \(\lim_{\rho \to 0} \sup p(x, \rho) < p(x, 0)\).
2.2 Dual Problem

For \( y > 0 \), define \( V(y) := \sup_{x > 0} \{ U(x) - xy \} \). For \( U(x) = x^p / p \) with \( x \geq 0 \) and \( p \in (0, 1) \), we have \( V(y) = \frac{1-p}{p} y^{p/(p-1)} \). For \( y > 0 \) and \( z \geq \phi_{\min} \), we define

\[
V_c(y, z) := \sup_{x > -\phi_{\min}} \{ U(x + z) - xy \} = \begin{cases} V(y) + yz, & \text{for } y < U'(z - \phi_{\min}), \\ U(z - \phi_{\min}) + y\phi_{\min}, & \text{otherwise.} \end{cases}
\]

We can then define a constrained form of the dual value function for \( \rho \in (-1, 1) \) by,

\[
v_c(y, \rho) := \inf_{Q \in D(\rho)} \mathbb{E} \left[ V_c \left( y dQ \right), f(\rho) \right], \quad y > 0. \tag{2.6}
\]

**Remark 2.2.** For \( \rho \neq 0 \), [17] prove that the constrained form of the dual value function, (2.6), is in fact equal to the dual value function as it is defined in [4], Equation (3.1). (See [17] Theorem 4.2.)

**Lemma 2.4.** Let the assumptions of the model be as in Theorem 2.1. For \( y > 0 \),

\[
\limsup_{\rho \to 0} v_c(y, \rho) \leq \mathbb{E} \left[ V_c(y Z_0^T, f) \right],
\]

where \( Z_0^T \) is the minimal martingale density for the \( S^0 \) market.

**Proof.** In order to show the result, it is convenient to view this problem as one of converging contingent claims, \( f(\rho) \), and a fixed market (\( \rho = 0 \)). For \( \rho \in (-1, 1) \), define the Brownian motion \( B^{(\rho)} := \sqrt{1 - \rho^2} B + \rho W \) and claim \( f^{(\rho)} := \phi(B^{(\rho)}_T) \). When \( \rho = 0 \), we have \( f^{(0)} = f = \phi(B_T) \). Notice that for any \( \rho \in (-1, 1) \) and \( y > 0 \),

\[
v_c(y, \rho) = \inf_{Q \in D(0)} \mathbb{E} \left[ V_c \left( y dQ \right), f(\rho) \right].
\]

The collection \( \{ f(\rho) \}_\rho \) is uniformly bounded from above and below, and \( f^{(\rho)} \to f^{(0)} \) a.s. as \( \rho \to 0 \). We define the sets \( A^{(\rho)} := \{ y Z_T^0 < U' (f^{(\rho)} - \phi_{\min}) \} \), where \( Z_T^0 \) is the \( S^0 \) market’s minimal martingale density. Let \( 1_A \) denote the indicator function of a set \( A \in \mathcal{F} \). Notice that \( \{ U (f^{(\rho)} - \phi_{\min}) \}_\rho \) and \( \{ 1_A^{(\rho)} \}_\rho \) are also uniformly bounded from above and below and
converge a.s. as $\rho \to 0$. Finally, for any $y > 0$,

\[
\mathbb{E} \left[ V_c \left( yZ_0^T, f^{(0)} \right) \right] = \mathbb{E} \left[ (V(yZ_0^T) + yZ_0^T f^{(0)}) I_{A(0)} + (U(f^{(0)} - \phi_{\min}) + yZ_0^T \phi_{\min}) I_{(A(0))^c} \right] \\
= \lim_{\rho \to 0} \mathbb{E} \left[ (V(yZ_0^T) + yZ_0^T f^{(\rho)}) I_{A(\rho)} + (U(f^{(\rho)} - \phi_{\min}) + yZ_0^T \phi_{\min}) I_{(A(\rho))^c} \right] \\
= \lim_{\rho \to 0} \mathbb{E} \left[ V_c \left( yZ_0^T, f^{(\rho)} \right) \right] \\
\geq \limsup_{\rho \to 0} v_c(y, \rho).
\]

\[\square\]

**Lemma 2.5.** Let the assumptions of the model be as in Theorem 2.1. Let $u$ and $u_c$ be as defined in (2.2) and (2.4), respectively. For any $x > -\phi_{\min}$, $u_c(x) \leq \liminf_{\rho \to 0} u(x, \rho)$.

**Proof.** As in the proof of Lemma 2.4, it is convenient to work with varying contingent claims, $f^{(\rho)}$, and a fixed market ($\rho = 0$). As before, define Brownian motions $B^{(\rho)} := \sqrt{1 - \rho^2} B + \rho W$ and claims $f^{(\rho)} := \phi(B^{(\rho)}_T)$. Notice that for any $\rho \in (-1, 1)$ and $x > -\phi_{\min}$, we have

\[u(x, \rho) = \sup_{X \in C(0)} \mathbb{E} \left[ U(x + X + f^{(\rho)}) \right].\]

For all $N \in \mathbb{N}$ and $x + (H \cdot S_0^0)_T \in C_c(x)$, we have $(H \cdot S_0^0)_T \wedge N \in C(0)$ and $x + (H \cdot S_0^0)_T \wedge N \geq -\phi_{\min}$, which implies that for all $\rho \in (-1, 1)$,

\[x + (H \cdot S_0^0)_T \wedge N + f^{(\rho)} \geq x + (H \cdot S_0^0)_T \wedge N + \phi_{\min} \geq 0.\]

By applying Fatou’s Lemma twice, we obtain

\[
\mathbb{E} \left[ U \left( x + (H \cdot S_0^0)_T + f^{(0)} \right) \right] \leq \liminf_{N \to \infty} \liminf_{\rho \to 0} \mathbb{E} \left[ U \left( x + (H \cdot S_0^0)_T \wedge N + f^{(\rho)} \right) \right] \\
\leq \liminf_{N \to \infty} \liminf_{\rho \to 0} u(x, \rho) \\
= \liminf_{\rho \to 0} u(x, \rho).
\]

Taking the supremum over all such $x + (H \cdot S_0^0)_T$ in $C_c(x)$ now yields the result. \[\square\]

**Proof of Theorem 2.1** Fix $\rho \neq 0$. For $x > -\phi_{\min}$, $X \in C(\rho)$ such that $x + X \geq -\phi_{\min}$,
$y > 0$, and $Q \in D(\rho)$, we have

$$
\mathbb{E} [U(x + X + f)] \leq \mathbb{E} \left[ V_c \left( y \frac{dQ}{dP}, f \right) + y \frac{dQ}{dP} (x + X) \right] 
\leq \mathbb{E} \left[ V_c \left( y \frac{dQ}{dP}, f \right) \right] + xy.
$$

This strengthening of Fenchel’s inequality relies on the bound $x + X \geq -\phi_{\min}$ in order to replace $V$ with $V_c(\cdot, f)$. Next, we take the supremum over all $X \in C(\rho)$ with $x + X \geq -\phi_{\min}$ and the infimum over all $Q \in D(\rho)$, which yields that for any $x > -\phi_{\min}$ and $y > 0$,

$$
u(x, \rho) \leq v_c(y, \rho) + xy.
$$

This inequality along with Lemmas 2.4 and 2.5 shows that for any $x > -\phi_{\min}$ and $y > 0$,

$$
\sup_{\rho \to 0} u(x, \rho) \leq \lim \inf_{\rho \to 0} u(x, \rho) \leq \lim \sup_{\rho \to 0} v_c(y, \rho) + xy \leq \mathbb{E} [V_c(yZ_0^0, f)] + xy.
$$

Next, we show that $u_c(\cdot)$ and $v_c(\cdot, 0)$ are conjugates. We let $y > 0$ be given and define the candidate optimizer $\hat{X}$ by

$$
\hat{X} := \begin{cases} 
-V'(yZ_0^0) - f, & \text{if } yZ_0^0 \leq U'(f - \phi_{\min}), \\
-\phi_{\min}, & \text{otherwise}.
\end{cases}
$$

For $\frac{dQ^0}{dP} := Z_T^0 = \mathcal{E}(-B)_T$, we have that $\hat{X} \in L^2(Q^0)$. By martingale representation and $S^0$ being a geometric Brownian motion under $Q^0$, we may write $\hat{X} = \mathbb{E}^{Q^0}[\hat{X}] + (H \cdot S^0)_T$ for some integrable $H$. Since $\hat{X} \geq -\phi_{\min}$ and $(H \cdot S^0)$ is a $Q^0$-martingale, we know that $(H \cdot S^0)_t \geq -\phi_{\min} - \mathbb{E}^{Q^0}[\hat{X}] > -\infty$ for all $t \in [0, T]$. Thus, $H$ is $S^0$-admissible.

We define $\hat{x} := \mathbb{E}^{Q^0}[\hat{X}] > -\phi_{\min}$ so that $\hat{X} \in C_c(\hat{x})$. Recall that $U$ is of power type for $p \in (0, 1)$, which yields $U \geq 0$ and allows for the use of Fatou’s Lemma. For any $y > 0$,

$$
\mathbb{E} \left[ V_c(yZ_0^0, f) \right] = \mathbb{E} \left[ U \left( \hat{X} + f \right) - yZ_0^0 \hat{X} \right] 
\leq \sup_{x > -\phi_{\min}} \left\{ \sup_{X \in C_c(x)} \mathbb{E} [U(X + f)] - xy \right\} 
= \sup_{x > -\phi_{\min}} \left\{ u_c(x) - xy \right\}.
$$

Since the other direction of the inequality holds by (2.4), we obtain that for any $y > 0$,
\( \mathbb{E}[V_c(yZ^0_T, f)] = \sup_{x > -\phi_{\min}} \{ u_c(x) - xy \} \). Since \( u_c(\cdot) \) is convex and lower semicontinuous on \((-\phi_{\min}, \infty)\), we have \( u_c(x) = \inf_{y > 0} \{ \mathbb{E}[V_c(yZ^0_T, f)] + xy \} \) for \( x > -\phi_{\min} \). Strict convexity of \( y \mapsto \mathbb{E}[V_c(yZ^0_T, f)] \) implies the differentiability of \( u_c(\cdot) \). (See, e.g., Proposition 6.2.1 on page 40 of [11].) Now for any \( x > -\phi_{\min} \), choosing \( y = \frac{\partial}{\partial x} u_c(x) \) yields equality in (2.7).

Finally, we show that indifference prices do not converge as \( \rho \to 0 \).

**Proof of Corollary 2.3.** Let \( x > -\phi_{\min} \) be given. For any \( \rho \in (-1, 1) \), \( w(x, \rho) = w(x, 0) \). Suppose that for \( \rho_n \to 0 \), we have \( p(x, \rho_n) \to \bar{p} \) as \( n \to \infty \). Being the limit of arbitrage-free prices in the \( \{ \rho_n \}_n \) models, \( \bar{p} \in [\inf \phi, \sup \phi] \).

For \( x > -\phi_{\min} \), we first note that \( u_c(x) < u(x, 0) \). This result can be obtained, for example, by Theorem 2.2 of [16] and \( f \)'s replicability in the \( S^0 \) market, which imply that \( u(x, 0) = \mathbb{E}[U(I(\frac{\partial}{\partial x} u(x, 0)Z^0_T))] \) where \( \mathbb{P} \left( \frac{\partial}{\partial x} u(x, 0)Z^0_T < f - \phi_{\min} \right) > 0 \). By Theorem 2.1

\[
\lim_{n} u(x, \rho_n) = u_c(x) < u(x, 0) = w(x + p(x, 0), 0).
\]

Since \( w \) is continuous in its first argument and constant in its second,

\[
\lim_{n} w(x + p(x, \rho_n), \rho_n) = \lim_{n} w(x + p(x, \rho_n), 0) = w(x + \bar{p}, 0),
\]

which implies that \( w(x + \bar{p}, 0) < w(x + p(x, 0), 0) \). Since \( w(\cdot, 0) \) is strictly increasing, we conclude that \( \bar{p} < p(x, 0) \). \( \square \)

### 3 Utility Functions on \( \mathbb{R} \)

Modeling investor preferences on the entire real line removes the fixed admissibility lower bound, which prevents the degeneracy of Theorem 2.1 from occurring. The remainder of this work is devoted to studying conditions that guarantee stability for real line utility functions.

Let \((\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})\) be a filtered probability space with the filtration generated by \( d \)-dimensional Brownian motion \( B = (B^1, \ldots, B^d) \). We assume that \( \mathbb{F} \) is completed with all \( \mathbb{P} \) null sets and \( \mathcal{F} = \mathcal{F}_T \), for a fixed time horizon \( T \in (0, \infty) \).

We consider a sequence of financial market models with stocks \( S^n \) valued in \( \mathbb{R} \), for \( 1 \leq n \leq \infty \),

\[
dS^n_t = \lambda^n_t d\langle M^n \rangle_t + dM^n_t, \quad S^n_0 = 0, \tag{3.1}
\]

where the \( M^n \) are \( \mathbb{R} \)-valued \( \mathbb{P} \)-martingales. For a martingale \( N \) and \( p \geq 1 \), let \( \mathcal{L}^p(N) := \{ \text{progressively measurable } \theta : \int_0^T |\theta_t|^p d\langle N \rangle_t < \infty \text{ a.s.} \} \). We assume that \( \lambda^n \in \mathcal{L}^2(M^n) \) for
$1 \leq n \leq \infty$. Since the filtration is generated by $B$, each $M^n$ is continuous. Each market is assumed to have a bank account with a zero interest rate.

For $1 \leq n \leq \infty$, we let $Z^n_t := \mathcal{E}(-\lambda^n \cdot M^n)_t$, $t \in [0, T]$, denote each market’s minimal martingale density process. A $\mathbb{P}$-local martingale, $N$, is said to be in $H^2_0(\mathbb{P})$ provided $N_0 = 0$ and $\mathbb{E}[\langle N \rangle_T] < \infty$, in which case $N$ is a martingale. A sequence of martingales $\{N^n\}_{1 \leq n < \infty} \subseteq H^2_0(\mathbb{P})$ converges to $N$ in $H^2_0(\mathbb{P})$ if $\mathbb{E}[\langle N^n - N \rangle_T] \to 0$ as $n \to \infty$. The following assumption captures the necessary market regularity and the convergence of a sequence of markets.

**Assumption 3.1.** The collections $\{M^n\}_{1 \leq n \leq \infty}$ and $\{((\lambda^n \cdot M^n))\}_{1 \leq n \leq \infty}$ are in $H^2_0(\mathbb{P})$ and satisfy the convergence relations:

$$M^n \to M^\infty \quad \text{and} \quad (\lambda^n \cdot M^n) \to (\lambda^\infty \cdot M^\infty) \text{ in } H^2_0(\mathbb{P}) \text{ as } n \to \infty.$$  

Furthermore, each minimal martingale density process, $Z^n$, for $1 \leq n \leq \infty$, is a $\mathbb{P}$-martingale.

Under the minimal martingale measure $Q^n$, where $\frac{dQ^n}{d\mathbb{P}} = Z^n_T$, $S^n$ is a local martingale and any $\mathbb{P}$-local martingale $N$ such that $\langle N, M^n \rangle_t = 0$ for $t \in [0, T]$ remains a local martingale under $Q^n$. We refer to [7] for a survey on minimal martingale measures and their use in mathematical finance.

**Remark 3.1.** Under Assumption 3.1, $((\lambda^n \cdot M^n))_T \to (\lambda^\infty \cdot M^\infty)_T$ in $L^2(\mathbb{P})$ as $n \to \infty$, which implies that $((\lambda^n)^2 \cdot \langle M^n \rangle)_T \to ((\lambda^\infty)^2 \cdot \langle M^\infty \rangle)_T$ in $L^1(\mathbb{P})$ as $n \to \infty$. Hence, $Z^n_T \to Z^\infty_T$ in probability as $n \to \infty$. Since each minimal martingale density process is a true martingale, Scheffe’s Lemma implies the seemingly stronger fact that $Z^n_T \to Z^\infty_T$ in $L^1(\mathbb{P})$ as $n \to \infty$.

A further non-degeneracy assumption is needed on the limiting market. A counterexample showing that this condition is in some sense necessary is provided in Section 5.

**Assumption 3.2.** The dynamics of $\langle M^\infty \rangle$ can be expressed as

$$d\langle M^\infty \rangle_t = \sigma_t^2 dt,$$

for $\sigma \in L^2(B^1)$ such that $\sigma_t \neq 0$ ($\mathbb{P} \times \text{Leb}$)-a.e., where Leb denotes the Lebesgue measure on $[0, T]$.

**Remark 3.2.** Assumptions 3.1 and 3.2 are satisfied by the markets $\{S^{\rho_n}\}_{1 \leq n \leq \infty}$ of Section 2 for any $\rho_n \to \rho \in [-1, 1]$ as $n \to \infty$. Finally, a contingent claim $f \in L^\infty(\mathbb{P})$ is given and is independent of $n \in \mathbb{N}$. We make no assumption on the replicability of $f$ at this time.
3.1 Optimal Investment Problem

An investor is modeled by preferences \(U : \mathbb{R} \to \mathbb{R}\), which is finite on the entire real line. \(U\) is assumed to be continuously differentiable, strictly increasing, strictly concave and satisfies the Inada conditions at \(-\infty\) and \(+\infty\):

\[
U'(\infty) := \lim_{x \to \infty} U'(x) = \infty \quad \text{and} \quad U'(0) := \lim_{x \to 0} U'(x) = 0. \tag{3.2}
\]

Additionally, we assume that \(U\) satisfies the reasonable asymptotic elasticity conditions of [16] and [21]:

\[
AE_{-\infty}(U) := \liminf_{x \to -\infty} \frac{xU'(x)}{U(x)} > 1 \quad \text{and} \quad AE_{+\infty}(U) := \limsup_{x \to \infty} \frac{xU'(x)}{U(x)} < 1. \tag{3.3}
\]

The utility function’s Fenchel conjugate is defined by \(V(y) := \sup_{x \in \mathbb{R}} \{U(x) - xy\}\) for \(y > 0\). \(V\) is strictly convex and continuously differentiable. Without loss of generality, we assume that \(U(0) > 0\). When \(U(0) > 0\), we have \(V(y) > 0\) for all \(y > 0\).

We introduce the following notions of primal admissibility, similar to [20]. For \(1 \leq n \leq \infty\), a process \(H\) is \(S^n\)-integrable if \(H \in L^2(M^n)\). Cauchy-Schwartz’s inequality produces \(H\lambda^n \in L^1(M^n)\). The \(S^n\) market’s admissible strategies are defined by

\[
\mathcal{H}^n_{adm} := \{H : H \text{ is } S^n\text{-integrable, } \exists K = K(H), (H \cdot S^n)_t \geq -K, \forall t\}.
\]

Let \(\mathcal{M}^n\) denote the set of probability measures \(Q\) such that \(Q \ll P\) and \(S^n\) is a local martingale under \(Q\). We are primarily interested in such measures that have finite \(V\)-entropy: \(\mathbb{E}[V(\frac{dQ}{dP})] < \infty\). Let \(\mathcal{M}^n_V\) denote those measures \(Q \in \mathcal{M}^n\) having finite \(V\)-entropy.

The admissible class of strategies is too small to attain a solution to the optimal investment problem. To this end, we introduce the permissible strategies, as in [20].

**Definition 3.3.** A strategy \(H\) is \(S^n\)-permissible if it is \(S^n\)-integrable and \((H \cdot S^n)\) is a \(Q\)-supermartingale for every \(Q \in \mathcal{M}^n\). We write \(\mathcal{H}^n_{perm}\) for the set of \(S^n\)-permissible strategies.

For \(1 \leq n \leq \infty\), the primal value function is defined by

\[
u_n(x) := \sup_{H \in \mathcal{H}^n_{perm}} \mathbb{E}[U(x + (H \cdot S^n)_T + f)], \quad x \in \mathbb{R}, \tag{3.4}
\]

while the dual value function is defined for the \(S^n\) market by

\[
u_n(y) := \inf_{Q \in \mathcal{M}^n_V} \mathbb{E}\left[V\left(y \frac{dQ}{dP}\right) + y \frac{dQ}{dP} f\right], \quad y > 0. \tag{3.5}
\]
Similar to [18], [14], and [1], we make the following assumption:

**Assumption 3.4.** The collection of random variables \( \{V(Z^n_T)\}_{1 \leq n \leq \infty} \), where \( Z^n_T \) is the minimal martingale density for the \( S^n \) market, is uniformly integrable.

By using Proposition 3.2 of [18], we can rewrite any \( Q \in M^\infty \) as

\[
\frac{dQ}{dP} = Z^\infty_T E[L_T],
\]

where \( L \) is a local martingale null at 0 such that \( \langle L, M^\infty \rangle_t = 0 \) for all \( t \in [0, T] \). We need to make a further assumption in order to ensure a “nice” structure of the limiting market’s dual domain. Let \( \mathcal{B} \) be defined by

\[
\mathcal{B} := \{ \text{local martingales } L : L_0 = 0, \langle L, M^\infty \rangle_t = 0, \forall t \in [0, T], \exists \text{ constant } C = C(L), \mathcal{E}(L)_t \leq C, \forall t \in [0, T] \}.
\]

**Assumption 3.5.** For \( n = \infty \), the dual problem, (3.5), can be expressed as

\[
v_\infty(y) = \inf_{L \in \mathcal{B}} E[V(yZ_\infty^T \mathcal{E}(L)_T) + yZ_\infty^T \mathcal{E}(L)_T f], \quad y > 0,
\]

where \( Z_\infty^T \) is the minimal martingale density in the \( S^\infty \) market.

This assumption is non-trivial to verify in general due to the fact that \( V \) is increasing strictly faster than linearly as \( y \longrightarrow +\infty \). Section 5 provides two sufficient conditions. The first condition covers the original motivation for our stability problem, where the contingent claim is replicable in the (incomplete) limiting market but not replicable in any pre-limiting market. In this case, the limiting market consists of a driving Brownian motion, a replicable claim, and additional independent Brownian noise. The second condition makes no assumptions on the claim’s replicability; however, it requires exponential preferences and imposes a mild BMO condition on the limiting market.

The following is the main result.

**Theorem 3.6.** Suppose that the sequence of markets satisfies Assumptions 3.1 and 3.4. Suppose that the limiting market satisfies Assumptions 3.2 and 3.5. Then, for \( x_n \longrightarrow x \) as \( n \longrightarrow \infty \),

\[
\lim_{n \rightarrow \infty} u_n(x_n) = u_\infty(x).
\]

For \( 1 \leq n \leq \infty \), the value function without random endowment is defined by

\[
w_n(x) := \sup_{H \in \mathcal{H}^n_{\text{perm}}} E[U(x + (H \cdot S^n)_T)], \quad x \in \mathbb{R}.
\]

**Definition 3.7.** Given \( 1 \leq n \leq \infty \) and \( x \in \mathbb{R} \), \( p_n = p_n(x) \) is called the *indifference price* for \( f \) at \( x \) in the \( S^n \) market if \( w_n(x + p_n) = u_n(x) \).
Corollary 3.8. Let the assumptions be as in Theorem 3.6. Then for \( x \in \mathbb{R} \), the indifference prices for \( f \) converge; that is, \( \lim_{n \to \infty} p_n(x) = p_\infty(x) \).

Remark 3.3. The results in Theorem 3.6 and Corollary 3.8 remain true (with only minor notational changes to the proofs) in the case with varying random endowment. Specifically, the random endowments \( \{f_n\}_{1 \leq n \leq \infty} \) corresponding to the \( \{S^n\}_{1 \leq n \leq \infty} \) markets need to satisfy

\[
\sup_n \|f_n\|_{L^\infty} < \infty \quad \text{and} \quad f_n \longrightarrow f_\infty \quad \text{in probability as } n \to \infty
\]

(3.8)

in order for the results to hold. This additional flexibility allows us to consider the case of a varying quantity of contingent claims and also contingent claims that depend on the individual markets. For example, if \( g : \mathbb{R} \to \mathbb{R} \) is bounded and continuous, then \( f_n := g(S^n_T) \) will satisfy (3.8).

4 Proofs

The proof of the main result follows Lemmas 4.1 and 4.3, which establish lower and upper semicontinuity-type results for the sequence of primal and dual value functions, respectively.

Lemma 4.1. Suppose that the sequence of markets satisfies Assumption 3.1 and \( \mathcal{M}_\infty \neq \emptyset \). Then for \( x \in \mathbb{R} \) and \( x_n \longrightarrow x \) as \( n \to \infty \),

\[
 u_\infty(x) \leq \liminf_{n \to \infty} u_n(x_n).
\]

Significant difficulty in proving Lemma 4.1 stems from the nonequivalence of markets (the martingale drivers, \( M^n \), differ). The idea behind the proof of Lemma 4.1 is that since the pre-limiting markets are “close” to the \( S^\infty \)-market, strategies in the \( S^\infty \)-market are “close” to being strategies in the pre-limiting markets. This idea will be made precise by appropriate approximation and stopping. First, we need a helper lemma.

Lemma 4.2. Let Assumption 3.1 hold, and let \( H \) be progressively measurable and uniformly bounded in \( (\omega, t) \). Then for \( 1 \leq n \leq \infty \), \( H \) is \( S^n \)-integrable and

\[
 \sup_{t \leq T} |(H \cdot S^n)_t - (H \cdot S^\infty)_t| \longrightarrow 0 \quad \text{in } L^1(\mathbb{P}) \quad \text{as } n \to \infty.
\]

Proof. We let \( K \in (0, \infty) \) be the uniform bound, \( |H| \leq K \). Boundedness and progressive measurability of \( H \) implies \( S^n \)-integrability for each \( n \). We have \( (H \cdot M^n) \longrightarrow (H \cdot M^\infty) \) in
$H_0^2$ as $n \to \infty$ since
\[
\mathbb{E} \left[ \int_0^T H^2 \langle M^n - M^\infty \rangle \right] \leq K^2 \mathbb{E} \left[ \langle M^n - M^\infty \rangle_T \right] \to 0 \quad \text{as } n \to \infty.
\]

The Burkholder-Davis-Gundy inequality implies that
\[
\sup_{t \leq T} |(H \cdot M^n)_t - (H \cdot M^\infty)_t| \to 0 \quad \text{in } L^2(\mathbb{P}) \quad \text{as } n \to \infty. \tag{4.1}
\]

Our setting is generated by $d$-dimensional Brownian motion $(B^1, \ldots, B^d)$, and so we may write $M^n = \sum_{i=1}^d (\sigma^{n,i} \cdot B^i)$ for progressively measurable $\sigma^{n,i}$ such that $\mathbb{E} \left[ \int_0^T (\sigma^{n,i}_t)^2 dt \right] < \infty$ for all $1 \leq n \leq \infty$, $1 \leq i \leq d$. The conditions $M^n \to M^\infty$ and $(\lambda^n \cdot M^n) \to (\lambda^\infty \cdot M^\infty)$ in $H_0^2(\mathbb{P})$ translate to:

1. $\sigma^{n,i} \to \sigma^{\infty,i}$ in $L^2(\mathbb{P} \times \text{Leb})$ as $n \to \infty$ for each $i = 1, \ldots, d$;
2. $\lambda^n \sigma^{n,i} \to \lambda^\infty \sigma^{\infty,i}$ in $L^2(\mathbb{P} \times \text{Leb})$ as $n \to \infty$ for each $i = 1, \ldots, d$.

Cauchy-Schwartz’s inequality implies that
\[
\lambda^n (\sigma^{n,i})^2 \to \lambda^\infty (\sigma^{\infty,i})^2 \quad \text{in } L^1(\mathbb{P} \times \text{Leb}) \quad \text{as } n \to \infty \quad \text{for each } i = 1, \ldots, d.
\]

Then,
\[
\sup_{t \leq T} \left| (H \lambda^n \cdot \langle M^n \rangle)_t - (H \lambda^\infty \cdot \langle M^\infty \rangle)_t \right| \leq K \sum_{i=1}^d \int_0^T \left| \lambda^n (\sigma^{n,i}_t)^2 - \lambda^\infty (\sigma^{\infty,i}_t)^2 \right| dt \to 0
\]

in $L^1(\mathbb{P})$ as $n \to \infty$. This calculation along with the convergence of the martingale terms in (4.1) imply the desired result.

\[\square\]

Proof of Lemma 4.1. First, we show that the supremum in the limiting primal optimization problem, (3.4), can be taken over all admissible wealth processes whose integrands are bounded. Using that $\mathcal{M}^\infty_{\nu} \neq \emptyset$ and $f \in L^\infty(\mathbb{P})$, by Theorem 1.2 in [20], we have
\[
u_{\infty}(x) = \sup_{H \in \mathcal{H}^\infty_{\text{adm}}} \mathbb{E}[U(x + (H \cdot S^\infty)_T + f)].
\]

Let $H \in \mathcal{H}^\infty_{\text{adm}}$ be given, and let $K \in (0, \infty)$ be such that $(H \cdot S^\infty)_t \geq -K$ for all $t \in [0, T]$. For $N \geq 1$, we define integrands $\hat{H}^N := H\mathbb{I}_{(|H| \leq N)}$, where $\mathbb{I}_A$ denotes the indicator function of a set $A \subseteq \Omega \times [0, T]$. For $N \geq 1$, we define stopping times
\[
\sigma_N := \inf\{t \leq T : (\hat{H}^N \cdot S^\infty)_t \leq -2K\}.\]
Then \( \tilde{H}^N_{[0,\sigma_N]} \in \mathcal{H}^\infty_{adm} \) with \{\( (\tilde{H}^N_{[0,\sigma_N]} \cdot S^n) \)\} \( N \) sharing the same lower admissibility bound, \(-2K\). Moreover, \( \sup_t |((\tilde{H}^N - H) \cdot S^n)_t| \to 0 \) in probability as \( N \to \infty \) by Lemma 4.11 and Remark (ii) following Definition 4.8 in [3]. This convergence implies that \( \mathbb{P}(\sigma_N = T) \to 1 \) and hence \( (\tilde{H}^N \cdot S^n)_{\sigma_N} \to (H \cdot S^n)_T \) in probability as \( N \to \infty \). By Fatou’s Lemma,

\[
\mathbb{E}[U(\bar{x} + (H \cdot S^n)_{T} + f)] \leq \liminf_{N \to \infty} \mathbb{E}\left[U\left(\bar{x} + (\tilde{H}^N \cdot S^n)_{\sigma_N} + f\right)\right].
\]

Therefore, it suffices to take the supremum in (3.4) over all \( H \in \mathcal{H}^\infty_{adm} \) such that \( H \) is uniformly bounded in \( t \) and \( \omega \). That is,

\[
u_\infty(x) = \sup_{H \in \mathcal{H}^\infty_{adm}, H \text{ bdd}} \mathbb{E}\left[U\left(\bar{x} + (H \cdot S^n)_{T} + f\right)\right].
\] (4.2)

Now let \( H \in \mathcal{H}^\infty_{adm} \) be given such that \( H \) is uniformly bounded in \( t \) and \( \omega \) by a constant \( K \in (0, \infty) \). Even though \( H \) is \( S^n \)-admissible and \( S^n \)-integrable for every \( n \), it is not necessarily admissible (or permissible) for each \( S^n \) market. The following choice of stopping times mitigates this issue while providing a lower admissibility bound uniform in \( n \). For each \( 1 \leq n < \infty \), we define stopping times \( \tau_n \) by

\[
\tau_n := \inf\{t \leq T : (H \cdot S^n)_t \leq -3K\}.
\]

By the definition of \( \tau_n \), we have \( H_{[0,\tau_n]} \in \mathcal{H}^n_{adm} \subseteq \mathcal{H}^n_{perm} \). Moreover, Lemma 4.2 implies that

\[
\mathbb{P}(\tau_n < T) = \mathbb{P}(\exists t' \leq T : (H \cdot S^n)_{t'} \leq -3K) \\
\leq \mathbb{P}\left(\sup_{t \leq T} |(H \cdot (S^n - S^n)_t)| \geq K\right) + \mathbb{P}(\exists t' \leq T : (H \cdot S^n)_{t'} < -K) \\
\leq \mathbb{P}\left(\sup_{t \leq T} |(H \cdot (S^n - S^n)_t)| \geq K\right) + 0 \\
\to 0 \quad \text{as } n \to \infty.
\]

Lemma 4.2 along with \( \mathbb{P}(\tau_n = T) \to 1 \) as \( n \to \infty \) implies that \( (H \cdot S^n)_{\tau_n} \to (H \cdot S^n)_{T} \) in probability as \( n \to \infty \). For each \( 1 \leq n < \infty \), the integrals \( (H_{[0,\tau_n]} \cdot S^n) \) share a uniform admissibility bound: \( (H \cdot S^n)_{\tau_n \wedge t} \geq -3K \) for \( t \in [0, T] \). Applying Fatou’s Lemma gives us that

\[
\mathbb{E}[U(\bar{x} + (H \cdot S^n)_{T} + f)] \leq \liminf_{n \to \infty} \mathbb{E}[U(\bar{x}_n + (H \cdot S^n)_{\tau_n} + f)] \\
\leq \liminf_{n \to \infty} u_n(x_n).
\]
Taking the supremum over all uniformly bounded $H \in \mathcal{H}_{adm}^\infty$, as in (4.2), yields the result.

We next proceed to the second main lemma, which establishes an upper-semicontinuity result for the dual problem.

**Lemma 4.3.** Let the assumptions of the model be as in Theorem 3.6. Then for $\{y_n\}_{1 \leq n < \infty} \subseteq (0, \infty)$ such that $y_n \rightarrow y > 0$ as $n \rightarrow \infty$,

$$v_\infty(y) \geq \limsup_{n \rightarrow \infty} v_n(y_n).$$

Using Assumption 3.5, the following lemma will further refine the collection $\mathcal{B}$ over which the infimum is taken in the limiting market’s dual problem. We define $\mathcal{B}'$ by

$$\mathcal{B}' := \{L \in \mathcal{B} : \exists \text{ constants } c = c(L), d = d(L),
\quad 0 < c \leq \mathcal{E}(L)_t \leq d < \infty, \forall t \in [0, T], \text{ and } \langle L \rangle_T \leq d\}$$

The following lemma builds on Corollary 3.4 in [18].

**Lemma 4.4.** Suppose that the limiting market’s dual problem satisfies Assumption 3.5 and that $\mathbb{E}[V(Z_\infty^\infty)] < \infty$, where $Z_\infty^\infty$ is the minimal martingale density for $S^\infty$. Let $\mathcal{B}'$ be defined as in (4.3). Then for $y > 0$,

$$v_\infty(y) = \inf_{L \in \mathcal{B}'} \mathbb{E}[V(yZ_\infty^\infty \mathcal{E}(L)_T) + yZ_\infty^\infty \mathcal{E}(L)_T f].$$

**Proof.** The first part of the proof is based on the proof of Corollary 3.4 of [18]. Let $L \in \mathcal{B}$ be given. By the convexity of $V$, we have

$$\mathbb{E} \left[ V \left( yZ_\infty^\infty \left( \frac{1}{n} + \frac{n-1}{n} \mathcal{E}(L)_T \right) \right) + yZ_\infty^\infty \left( \frac{1}{n} + \frac{n-1}{n} \mathcal{E}(L)_T \right) f \right]$$

$$\leq \frac{1}{n} \mathbb{E} [V(yZ_\infty^\infty) + yZ_\infty^\infty f] + \frac{n-1}{n} \mathbb{E} [V(yZ_\infty^\infty \mathcal{E}(L)_T) + yZ_\infty^\infty \mathcal{E}(L)_T f]$$

$$\rightarrow \mathbb{E} [V(yZ_\infty^\infty \mathcal{E}(L)_T) + yZ_\infty^\infty \mathcal{E}(L)_T f] \quad \text{as } n \rightarrow \infty,$n

because $V(yZ_\infty^\infty) \in L^1(\mathbb{P})$ by the assumption that $\mathbb{E}[V(Z_\infty^\infty)] < \infty$ and reasonable asymptotic elasticity, (3.3). For each $n \geq 1$, we let $L^n$ denote the element $L^n \in \mathcal{B}$ such that $\frac{1}{n} + \frac{n-1}{n} \mathcal{E}(L) = \mathcal{E}(L^n)$.

Let $\varepsilon > 0$ be given, and choose $N$ sufficiently large such that

$$\mathbb{E} [V(yZ_\infty^\infty \mathcal{E}(L^n)_T) + yZ_\infty^\infty \mathcal{E}(L^n)_T f] \leq \mathbb{E} [V(yZ_\infty^\infty \mathcal{E}(L)_T) + yZ_\infty^\infty \mathcal{E}(L)_T f] + \varepsilon.$$
We define the sequence of stopping times \( \{ \tau_k \}_{1 \leq k < \infty} \) by \( \tau_k := \inf \{ t \leq T : \langle L^N \rangle_t \geq k \} \). Then \((L^N)^{\tau_k} \in \mathcal{B}'\) for each \( k \). By continuity of \( L^N \) and finiteness of \( \langle L^N \rangle_T \), we have that \( \mathcal{E}(L^N)_{\tau_k} \to \mathcal{E}(L^N)_T \) in probability as \( k \to \infty \). Scheffe’s Lemma implies that the \( L^1(\mathbb{P}) \)-limit is \( \lim_k Z_T^\infty \mathcal{E}(L^N)_{\tau_k} = Z_T^\infty \mathcal{E}(L^N)_T \), which implies that \( \lim_k \mathbb{E} \left[ y Z_T^\infty \mathcal{E}(L^N)_{\tau_k} f \right] = \mathbb{E} \left[ y Z_T^\infty \mathcal{E}(L^N)_T f \right] \).

Convergence in probability of \( \{ \mathcal{E}(L^N)_{\tau_k} \}_{1 \leq k < \infty} \) also implies that \( V(y Z_T^\infty \mathcal{E}(L^N)_{\tau_k}) \to V(y Z_T^\infty \mathcal{E}(L^N)_T) \) in probability as \( k \to \infty \). Let \( C \) be the bound on \( \mathcal{E}(L^N) \) from above given to us in definition of \( \mathcal{B} \). Since \( \frac{1}{\mathcal{E}(L^N)_t} \leq \mathcal{E}(L^N)_t \leq C \) for all \( t \), we have for all \( k \) that \( V(y Z_T^\infty \mathcal{E}(L^N)_{\tau_k}) \leq \max (V(\frac{1}{N} Z_T^\infty), V(C Z_T^\infty)) \), where \( \max (V(\frac{1}{N} Z_T^\infty), V(C Z_T^\infty)) \) is in \( L^1(\mathbb{P}) \) by reasonable asymptotic elasticity, \( (3.3) \). Thus, \( V(y Z_T^\infty \mathcal{E}(L^N)_{\tau_k}) \to V(y Z_T^\infty \mathcal{E}(L^N)_T) \) in \( L^1(\mathbb{P}) \) as \( k \to \infty \).

We may choose \( K \) sufficiently large so that \( \mathbb{E} \left[ V(y Z_T^\infty \mathcal{E}(L^N)_{\tau_K}) + y Z_T^\infty \mathcal{E}(L^N)_{\tau_K} f \right] \leq \mathbb{E} \left[ V(y Z_T^\infty \mathcal{E}(L^N)_T) + y Z_T^\infty \mathcal{E}(L^N)_T f \right] + \varepsilon \), which then implies that

\[
\mathbb{E} \left[ V(y Z_T^\infty \mathcal{E}(L^N)_{\tau_K}) + y Z_T^\infty \mathcal{E}(L^N)_{\tau_K} f \right] \leq \mathbb{E} \left[ V(y Z_T^\infty \mathcal{E}(L^N)_T) + y Z_T^\infty \mathcal{E}(L^N)_T f \right] + 2\varepsilon.
\]

Since \( \varepsilon > 0 \) and \( L \in \mathcal{B} \) are arbitrary, Assumption \ref{3.5} allows us to conclude the desired result.

Establishing an upper-semicontinuity property for the dual problem is difficult because with small changes in the limiting market, we must produce a dual element of a pre-limiting market with appropriately small changes. Lemma \ref{4.4} helps us to overcome this issue because it allows us to take the infimum in the dual value function over martingales \( L \) that lie in \( H_0^2(\mathbb{P}) \). Using this additional regularity on \( L \), we establish an \( H_0^2(\mathbb{P}) \)-convergence result for a decomposition of \( L \) in terms of strongly orthogonal components based on the varying martingale drives, \( M^n \). For \( M, N \in H_0^2(\mathbb{P}) \), we say that \( M \) and \( N \) are strongly orthogonal if \( \langle M, N \rangle_t = 0 \) for all \( t \in [0, T] \).

**Lemma 4.5.** Let \( \{ M^n \}_{1 \leq n \leq \infty} \) be \( H_0^2(\mathbb{P}) \)-martingales such that \( M^n \to M^\infty \) in \( H_0^2(\mathbb{P}) \) as \( n \to \infty \), and suppose that \( M^\infty \) satisfies Assumption \ref{3.2}. Let \( L \in H_0^2(\mathbb{P}) \) be strongly orthogonal to \( M^\infty \). Then for \( 1 \leq n < \infty \), \( L \) can be decomposed into

\[
L = L^n + (H^n \cdot M^n),
\]

where \( L^n \) and \( (H^n \cdot M^n) \) are in \( H_0^2(\mathbb{P}) \), \( L^n \) is strongly orthogonal to \( M^n \), and \( L^n \to L \) in \( H_0^2(\mathbb{P}) \) as \( n \to \infty \).

**Proof.** The filtration \( \mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T} \) is the (\( \mathbb{P} \)-completed) filtration generated by the \( d \)-dimensional Brownian motion \( (B^1, \ldots, B^d) \) on \( (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}) \) with \( \mathcal{F} = \mathcal{F}_T \). For notational
concreteness, we denote

$$M^n = (\sigma^{n,1} \cdot B^1) + \ldots + (\sigma^{n,d} \cdot B^d)$$

and

$$L = (\nu^1 \cdot B^1) + \ldots + (\nu^d \cdot B^d),$$

for $\sigma^{n,k}, \nu^k \in L^2(B^k), 1 \leq n \leq \infty, 1 \leq k \leq d$. For $x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in \mathbb{R}^d$, we let $|x|$ denote the Euclidean norm, $|x| = \sqrt{x_1^2 + \ldots + x_d^2}$, and let the inner product be given by $x \cdot y := x_1y_1 + \ldots + x_dy_d$. We define the vectors $\nu := (\nu^1, \ldots, \nu^d)$ and $\sigma^n := (\sigma^{n,1}, \ldots, \sigma^{n,d})$.

For $1 \leq n < \infty$, we define

$$H^n := \frac{\nu \cdot \sigma^n}{|\sigma^n|^2} \mathbb{I}_{\{|\sigma^n| \neq 0\}}.$$

Then $H^n$ is progressively measurable and $M^n$-integrable with $(H^n \cdot M^n) \in H^2_0(\mathbb{P})$:

$$\mathbb{E} \left[ \int_0^T (H^n)^2 d\langle M^n \rangle_t \right] = \mathbb{E} \left[ \int_0^T (H^n)^2 |\sigma^n|^2 dt \right] = \mathbb{E} \left[ \int_0^T (\nu \cdot \sigma^n)^2 |\sigma^n|^2 \mathbb{I}_{\{|\sigma^n| \neq 0\}} dt \right] \leq \mathbb{E} \left[ \int_0^T |\nu|^2 dt \right] = \mathbb{E} \left[ \langle L \rangle_T \right] < \infty.$$

We define $L^n := L - (H^n \cdot M^n) \in H^2_0(\mathbb{P})$. Strong orthogonality of $L^n$ and $M^n$ follows from:

$$\langle L^n, M^n \rangle_t = \langle L - (H^n \cdot M^n), M^n \rangle_t = \int_0^t (\nu - H^n \sigma^n) \cdot \sigma^n ds = \int_0^t \left( \nu - \frac{\nu \cdot \sigma^n}{|\sigma^n|^2} \mathbb{I}_{\{|\sigma^n| \neq 0\}} \sigma^n \right) \cdot \sigma^n ds = 0,$$

for $t \in [0,T]$. Since $L^n$ and $M^n$ are strongly orthogonal, $L^n \rightarrow L$ in $H^2_0(\mathbb{P})$ if and only if $(H^n \cdot M^n) \rightarrow 0$ in $H^2_0(\mathbb{P})$ as $n \rightarrow \infty$. For $1 \leq n < \infty$, we have that

$$\mathbb{E} \left[ \langle H^n \cdot M^n \rangle_T \right] = \mathbb{E} \left[ \int_0^T \frac{(\nu \cdot \sigma^n)^2}{|\sigma^n|^2} \mathbb{I}_{\{|\sigma^n| \neq 0\}} dt \right].$$

Since $L \in H^2_0(\mathbb{P})$, we have for $1 \leq n < \infty$,

$$\frac{(\nu \cdot \sigma^n)^2}{|\sigma^n|^2} \mathbb{I}_{\{|\sigma^n| \neq 0\}} \leq |\nu|^2 \in L^1(\mathbb{P} \times \text{Leb}).$$

The assumption that $M^n \rightarrow M^\infty$ in $H^2_0(\mathbb{P})$ as $n \rightarrow \infty$ implies that for $1 \leq k \leq d$, 

18
\[ \sigma^{n,k} \longrightarrow \sigma^\infty,k \text{ in } (\mathbb{P} \times \text{Leb})\text{-measure as } n \to \infty. \] Assumption 3.2 ensures that \(|\sigma^\infty| \neq 0\) (\(\mathbb{P} \times \text{Leb}\))-a.e., and hence,

\[
\frac{(\nu \cdot \sigma^n)^2}{|\sigma^n|^2} \mathbb{I}_{|\sigma^n| \neq 0} \longrightarrow 0 \text{ in } (\mathbb{P} \times \text{Leb})\text{-measure as } n \to \infty.
\]

Thus dominated convergence implies that \(\mathbb{E}[\{H^n \cdot M^n\}_T] \longrightarrow 0\) as \(n \to \infty\), which completes the proof of the claim. \(\square\)

**Proof of Lemma 4.3.** We let \(\mathcal{B}'\) be defined as in (4.3) and let \(L \in \mathcal{B}'\) be given. Let \(K \in (0, \infty)\) be the constant given in the definition of \(\mathcal{B}'\) such that \(|L_t| \leq K\) for all \(t\) and \(\langle L \rangle_T \leq K\).

We let \(L^n\) be given as in Lemma 4.3. Then \(L^n \longrightarrow L\) in \(H_0^2\) as \(n \to \infty\). For \(1 \leq n < \infty\), define stopping times \(\tau_n := \inf\{t \leq T : |L^n_t - L_t| \geq 1 \text{ or } \langle L^n \rangle_t \geq K + 1\}\). The \(H_0^2(\mathbb{P})\) convergence of \(\{L^n\}_{1 \leq n < \infty}\) implies that \(\langle L^n \rangle_T \longrightarrow \langle L \rangle_T\) in \(L^1(\mathbb{P})\) as \(n \to \infty\), while the Burkholder-Davis-Gundy inequalities additionally give us that \(\mathbb{P}(\sup_t |L^n_t - L_t| \geq 1) \longrightarrow 0\) as \(n \to \infty\). Hence, \(\mathbb{P}(\tau_n = T) \longrightarrow 1\) as \(n \to \infty\). We conclude that \(L^n_{\tau_n} \longrightarrow L_T\) and \(\langle L^n \rangle_{\tau_n} \longrightarrow \langle L \rangle_T\) in probability as \(n \to \infty\), which yields

\[
\mathcal{E}(L^n)_{\tau_n} \longrightarrow \mathcal{E}(L)_T \text{ in probability as } n \to \infty.
\]

Furthermore, the definition of \(\tau_n\) provides upper and lower bounds on \(\mathcal{E}(L^n)_{\tau_n}\), which are independent of \(n\):

\[
e^{-2K-2} \leq \mathcal{E}(L^n)_{\tau_n} \leq e^{K+1}.
\]

As mentioned in Assumption 1.2(i) of [20], the reasonable asymptotic elasticity condition (3.3) along with the \(U(0) > 0\) is equivalent to the following: for all \(\alpha > 0\) there exists \(C > 0\) such that \(V(\lambda y) \leq CV(y)\) for all \(y \geq 0\). Then for \(1 \leq n < \infty\),

\[
0 \leq V \left( y_n Z^n_T \mathcal{E}(L^n)_{\tau_n} \right)
\]

\[
\leq V \left( y_n Z^n_T e^{K+1} \mathbb{I}_{\{y_n Z^n_T \mathcal{E}(L^n)_{\tau_n} \geq V'(0)\}} \right) + V \left( y_n Z^n_T e^{-2K-2} \mathbb{I}_{\{y_n Z^n_T \mathcal{E}(L^n)_{\tau_n} \leq V'(0)\}} \right)
\]

\[
\leq V \left( \left( \sup_m y_m \right) e^{K+1} Z^n_T \right) + V \left( \left( \inf_m y_m \right) e^{-2K-2} Z^n_T \right)
\]

\[
\leq (C_1 + C_2) V(Z^n_T),
\]

where \(C_1, C_2\) are the constants produced by the reasonable asymptotic elasticity of \(U\). The constants \(C_1, C_2\) depend on the choice of \(L, K, \inf_m y_m, \text{ and } \sup_m y_m\) but not on \(n\). Assumption 3.2 now guarantees the uniform integrability of \(\{V \left( y_n Z^n_T \mathcal{E}(L^n)_{\tau_n} \right)\}_{1 \leq n < \infty}\).

Convergence in probability plus uniform integrability implies that \(V \left( y_n Z^n_T \mathcal{E}(L^n)_{\tau_n} \right) \longrightarrow V(y Z^T_T \mathcal{E}(L)_T)\) in \(L^1(\mathbb{P})\) as \(n \to \infty\). Moreover, the convergence in probability of \(\{y_n Z^n_T \mathcal{E}(L^n)_{\tau_n}\}_{1 \leq n < \infty}\)
along with Scheffe’s Lemma imply $y_n Z^n_T \mathcal{E}(L^n)_{t_n} \to y Z^n_T \mathcal{E}(L_T)$ in $L^1(\mathbb{P})$ as $n \to \infty$. By using that $f \in L^\infty(\mathbb{P})$,

$$
\mathbb{E}[V(y Z^n_T \mathcal{E}(L^n)_{t_n} + y N^n_T \mathcal{E}(L^n)_{t_n}) f] \geq \limsup_n v_n(y_n).
$$

Taking the infimum over all $L \in B'$ and applying Lemma 4.4 yields $v_\infty(y) \geq \limsup_n v_n(y_n)$. \hfill \Box

Proof of Theorem 3.6. We first note that the assumption that $\mathcal{M}^\infty_x \neq \emptyset$ of Lemma 4.1 is satisfied by Assumption 3.4. For $x_n \to x \in \mathbb{R}$ and $y = y(x)$, Lemmas 4.1 and 4.3 imply

$$
u_\infty(x) \leq \liminf_{n \to \infty} u_n(x_n) \leq \limsup_{n \to \infty} v_n(y) + x_n y \leq v_\infty(y) + xy = u_\infty(x).
$$

The last equality can be shown by Theorem 1.1 of [20] by taking $\mathcal{E} = x + f$ and $y = \mathbb{E}\left[\frac{d\tilde{\mu}(x)}{df}\right]$. Here, $\mathcal{E}$ and $\tilde{\mu}(x)$ refer to the notation used in [20]. \hfill \Box

Proof of Corollary 3.8. Let $\{p_n(x)\}_{1 \leq n < \infty}$ be a convergent subsequence of $\{p_n(x)\}_{1 \leq n < \infty}$ with $\lim_k p_{n_k}(x) = p \in \mathbb{R}$. By Theorem 3.6

$$
u_\infty(x) = \lim k u_{n_k}(x),
$$

while $w_{n_k}(x + p_{n_k}(x)) = u_{n_k}(x)$ for each $k \geq 1$ by the definition of the indifference price. Next, we take the contingent claim to be 0 and note that $\lim_k x + p_{n_k}(x) = x + p$, which allows us to conclude from Theorem 3.6 that

$$
u_\infty(x + p) = \lim k w_{n_k}(x + p_{n_k}(x)),
$$

which implies that $p = p_\infty(x)$. Since $f \in L^\infty(\mathbb{P})$, $\{p_n(x)\}_n$ is bounded, hence any subsequence has a further subsequence that converges to $p_\infty(x)$. Therefore, $\lim_n p_n(x)$ exists and equals $p_\infty(x)$. \hfill \Box

5 Examples

The first example shows that Assumption 3.2 is necessary in the sense that its absence can allow Theorem 3.6’s conclusion to fail.

Example 5.1. Let $d = 1$, so that the probability space is generated by a 1-dimensional Brownian motion, $B$. We define the martingales $M^n := \frac{1}{n} B$ for $1 \leq n < \infty$ and $M^\infty := 0$. 

20
Let $\lambda^n := 0$ for all $1 \leq n \leq \infty$ so that $S^n_t = 0$ for all $t \in [0, T]$ and for $1 \leq n < \infty$, $S^n$ has the dynamics
\[ dS^n = \frac{1}{n} dB, \quad S^n_0 = 0. \]
The stock markets satisfy Assumption 3.1, but the limiting market does not satisfy Assumption 3.2.

Let the contingent claim be given by $f := I_{\{B_T \geq 0\}}$. By the martingale representation theorem and the boundedness of $f$, there exists $H \in \mathcal{L}^2(B)$ such that $f = \frac{1}{2} + (H \cdot B)_T$ and the stochastic integral $(H \cdot B)$ is bounded. Hence, we can conclude by Theorem 2.1 of [21] that for all $1 \leq n < \infty$ and $x \in \mathbb{R},$
\[ u_n(x) = U \left( x + \frac{1}{2} \right). \]
Yet for all $x \in \mathbb{R}$, Jensen’s inequality implies that $u_\infty(x) = \mathbb{E}[U(x + f)] < U \left( x + \frac{1}{2} \right)$.

The following two examples provide sufficient conditions on the limiting market for Assumption 3.5 to hold.

**Example 5.2.** This example covers the original motivation for this work, where the contingent claim is replicable in the (possibly incomplete) limiting market. In this case, the limiting market consists of a driving Brownian motion, a replicable claim, and additional independent Brownian noise.

Recall that $(B^1, \ldots, B^d)$ is the $d$-dimensional Brownian motion generating the completed filtration, $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$. Let $(\mathcal{F}^1_t)_{0 \leq t \leq T}$ denote the filtration generated by $B^1$, completed with all $\mathbb{P}$-null sets. The risky asset, $S_\infty$, has dynamics as in (3.1) and is $(\mathcal{F}^1_t)_{0 \leq t \leq T}$-adapted. The contingent claim, $f \in L^\infty(\Omega, \mathcal{F}^1_T, \mathbb{P})$, is replicable: there exists an integrand, $H$, and constant, $c$, such that $H \in \mathcal{L}^2(M^\infty)$ and $f = c + (H \cdot S_\infty)_T$.

**Proposition 5.3.** Suppose that $S_\infty$ is $(\mathcal{F}^1_t)_{0 \leq t \leq T}$-adapted with dynamics (3.1) and satisfies Assumption 3.2. Suppose that $f \in L^\infty(\Omega, \mathcal{F}^1_T, \mathbb{P})$ is replicable. Then Assumption 3.5 is satisfied.

**Proof.** Let $y > 0$ and $\mathbb{Q} \in \mathcal{M}_V^\infty$ be given. Write $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_\infty \mathcal{E}(L)_T$ for its Radon-Nikodym density. We have that $Z_\infty \in \mathcal{F}^1_T$, while Assumption 3.2 implies that $(L, B^1)_t = 0$ for $t \in [0, T]$. Note that $\mathbb{E}[\mathcal{E}(L)_T|\mathcal{F}^1_T] = 1$, $\mathbb{P}$-a.s., since
\[ 1 = \mathbb{E}[Z_\infty \mathcal{E}(L)_T] = \mathbb{E}[Z_\infty \mathbb{E}[\mathcal{E}(L)_T|\mathcal{F}^1_T]] \leq \mathbb{E}[Z_\infty] = 1, \quad \mathbb{E}[\mathcal{E}(L)_0|\mathcal{F}^1_T] = 1. \]
with equality holding if and only if \( \mathbb{E}[\mathcal{E}(L)_T|\mathcal{F}_T^1] = 1 \), \( \mathbb{P} \)-a.s. By Jensen’s inequality,

\[
\mathbb{E}[V(yZ_T^\infty \mathcal{E}(L)_T)] = \mathbb{E}\left[\mathbb{E}\left[V(yz\mathcal{E}(L)_T)|\mathcal{F}_T^1\right]|z = Z_T^\infty\right] \\
\geq \mathbb{E}\left[V(yz\mathbb{E}[\mathcal{E}(L)_T]|\mathcal{F}_T^1)|z = Z_T^\infty\right] \\
= \mathbb{E}[V(yZ_T^\infty)].
\]

Since \( f \) is bounded and replicable, \( \mathbb{Q} \mapsto \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} f\right] \) is constant on \( \mathcal{M}_\infty^V \). Hence, for all \( \mathbb{Q} \in \mathcal{M}_\infty^V \),

\[
\mathbb{E}[V(yZ_T^\infty) + yZ_T^\infty f] \leq \mathbb{E}\left[V\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right) + y\frac{d\mathbb{Q}}{d\mathbb{P}} f\right],
\]

which implies that \( Z_T^\infty \) is the density of the dual minimizer, and so Assumption 3.5 is satisfied. \( \square \)

**Example 5.4 (Exponential Investors).** For the exponential investor, Assumption 3.5 is satisfied, under an easier-to-verify BMO assumption. We refer to [15] for additional details on BMO martingales.

**Definition 5.5.** A \( \mathbb{P} \)-local martingale \( N \) is said to be in \( \text{BMO}(\mathbb{P}) \) if

\[
\sup_{\tau} \left\| \mathbb{E}^\mathbb{P}[|N_T - N_\tau| |\mathcal{F}_\tau]\right\|_\infty < \infty,
\]

where the supremum is taken over stopping times \( \tau \leq T \).

**Assumption 5.6.** \( (\lambda^\infty \cdot M^\infty) \in \text{BMO}(\mathbb{P}) \).

For the remainder of this section, we let \( U(x) = -\exp(-\alpha x) \) for a positive constant \( \alpha \). The conjugate to \( U \) is \( V(y) = \frac{y}{\alpha} \left(\log \frac{y}{\alpha} - 1\right), y > 0 \). We have the following relationships for \( c \in \mathbb{R} \) and \( y > 0 \):

\[
V'(cy) = V'(y) + \frac{1}{\alpha} \log c, \tag{5.1}
\]

\[
V(y) + yc = y \left(V'(y e^{oc}) - \frac{1}{\alpha}\right). \tag{5.2}
\]

For a set \( A \in \mathcal{F} \) and random variable \( X \in L^1(\mathbb{P}) \), we adopt the notation \( \mathbb{E}[X; A] := \mathbb{E}[X1_A] = \int_A Xd\mathbb{P} \).

**Theorem 5.7.** Let \( U(x) = -\exp(-\alpha x) \) for a positive constant \( \alpha \) and assume that Assumption 5.6 holds. Let \( \mathbb{Q}^\infty \) denote the minimal martingale measure, \( \frac{d\mathbb{Q}^\infty}{d\mathbb{P}} := Z_T^\infty = \mathcal{E}(-\lambda^\infty \cdot M^\infty)_T \), and suppose that \( \mathbb{Q}^\infty \in \mathcal{M}_\infty^V \). Then Assumption 3.5 is satisfied.

**Proof.** Let \( x \in \mathbb{R} \) and \( Z_T^\infty \mathcal{E}(L)_T = \mathcal{E}(-(\lambda^\infty \cdot M^\infty) + L)_T \in \mathcal{M}_\infty^V \) be the dual optimizer for the dual problem [3.5] with \( n = \infty \) and \( y := u'_\infty(x) \). For \( 1 \leq n < \infty \), we define the stopping times \( \tau_n := \inf\{t \leq T : \mathcal{E}(L)_t \geq n\} \). Using that \( V(0) = 0 \) and the definition of \( \tau_n \), it is
not difficult to verify that each probability density \( Z_t^\infty \mathcal{E}(L)_{\tau_n} \) corresponds to a martingale measure in \( \mathcal{M}^\infty_V \).

Using that \( f \in L^\infty(\mathbb{P}) \), Theorem 1.2(i) of [20] plus Theorems 2.1 and 2.2 of [13] imply that there exists \( \hat{H} \in \mathcal{H}^\infty_{\text{perm}} \) such that \( \hat{H} \) is optimal for (3.4) with \( n = \infty \) and \( \hat{H} \cdot S^\infty \) is a martingale with respect to every measure \( Q \in \mathcal{M}^\infty_V \). Proposition 4.1 from [20] implies that 
\[ x + (\hat{H} \cdot S^\infty)_T + f = -V'(yZ_T^\infty \mathcal{E}(L)_T). \]
Hence, for any \( Q \in \mathcal{M}^\infty_V \), (5.1) with \( c = x \) implies that
\[
\mathbb{E} \left[ \frac{dQ}{d\mathbb{P}} V'(yZ_T^\infty \mathcal{E}(L)_T e^{\alpha f}) \right] = \mathbb{E} \left[ Z_T^\infty \mathcal{E}(L)_T V'(yZ_T^\infty \mathcal{E}(L)_T e^{\alpha f}) \right].
\] (5.3)

Then,

\[
0 \leq \mathbb{E} \left[ V(yZ_T^\infty \mathcal{E}(L)_{\tau_n}) + yZ_T^\infty \mathcal{E}(L)_{\tau_n} f \right] - v_\infty(y) \\
= \mathbb{E} \left[ V(yZ_T^\infty \mathcal{E}(L)_{\tau_n}) + yZ_T^\infty \mathcal{E}(L)_{\tau_n} f \right] - \mathbb{E} \left[ V(yZ_T^\infty \mathcal{E}(L)_T) + yZ_T^\infty \mathcal{E}(L)_T f \right] \\
= \mathbb{E} \left[ yZ_T^\infty \mathcal{E}(L)_{\tau_n} V'(yZ_T^\infty \mathcal{E}(L)_{\tau_n} e^{\alpha f}) - yZ_T^\infty \mathcal{E}(L)_T V'(yZ_T^\infty \mathcal{E}(L)_T e^{\alpha f}) \right] \quad \text{by (5.2)} \\
= \mathbb{E} \left[ yZ_T^\infty \mathcal{E}(L)_{\tau_n} (V'(yZ_T^\infty \mathcal{E}(L)_{\tau_n} e^{\alpha f}) - V'(yZ_T^\infty \mathcal{E}(L)_T e^{\alpha f})) \right] \quad \text{by (5.3)} \\
= \frac{y}{\alpha} \mathbb{E} \left[ Z_T^\infty \mathcal{E}(L)_{\tau_n} \left( \log \mathcal{E}(L)_{\tau_n} - \log \mathcal{E}(L)_T \right) \right] \quad \text{by (5.1)} \\
= \frac{y}{\alpha} \mathbb{E}^{\mathbb{Q}^\infty} \left[ n \log \left( \frac{n}{\mathcal{E}(L)_T} \right) \mid \{ \tau_n < T \} \right] \\
= \frac{y}{\alpha} \left( n \log n \mathbb{Q}^\infty(\tau_n < T) - n \mathbb{E}^{\mathbb{Q}^\infty} [\log \mathcal{E}(L)_T \mid \{ \tau_n < T \}] \right).
\]

In order to show Assumption 3.5, it now suffices to show
\[
n \log n \mathbb{Q}^\infty(\tau_n < T) - n \mathbb{E}^{\mathbb{Q}^\infty} [\log \mathcal{E}(L)_T \mid \{ \tau_n < T \}] \longrightarrow 0 \quad \text{as } n \to \infty. \tag{5.4}
\]

Showing \( n \log n \mathbb{Q}^\infty(\tau_n < T) \longrightarrow 0 \) as \( n \to \infty \) will employ Doob’s submartingale inequality, whereas \( n \mathbb{E}^{\mathbb{Q}^\infty} [\log \mathcal{E}(L)_T \mid \{ \tau_n < T \}] \longrightarrow 0 \) relies on the assumption that \( (\lambda^\infty \cdot M^\infty) \in \text{BMO}(\mathbb{P}) \).

Let \( \phi(y) := y \log y \). We have that \( \phi \) is convex, \( \phi \geq -1/e \), and \( \phi \) is increasing on \([1/e, \infty)\). Using that \( Z_T^\infty \mathcal{E}(L)_T \) is the dual optimizer, it is not difficult to check that \( \phi(\mathcal{E}(L)_t) \in L^1(\mathbb{Q}^\infty) \) for each \( t \in [0, T] \). Convexity of \( \phi \) implies that \( \phi(\mathcal{E}(L)) \) is a \( \mathbb{Q}^\infty \)-submartingale. (Note that \( \mathcal{E}(L) \) is a \( \mathbb{Q}^\infty \)-martingale since \( \mathbb{E}^{\mathbb{Q}^\infty} [\mathcal{E}(L)_T] = \mathbb{E}[Z_T^\infty \mathcal{E}(L)_T] = 1 \).)

For a process \( Y \), we let \( Y^* := \sup_{0 \leq t \leq T} Y_t \). For any \( n > 1 \),
\[
\mathcal{E}(L)^* \geq n \quad \text{if and only if} \quad \phi(\mathcal{E}(L))^* = (\mathcal{E}(L) \log \mathcal{E}(L))^* \geq n \log n.
\]
Doob’s submartingale inequality implies that for \( n > 1 \),
\[
 n \log n \mathbb{Q}^\infty(\mathcal{E}(L)^* \geq n) = n \log n \mathbb{Q}^\infty(\phi(\mathcal{E}(L))^* \geq n \log n) \\
\leq E^\mathbb{Q}^\infty[\phi(\mathcal{E}(L)_T)^+; \{\phi(\mathcal{E}(L))^* \geq n \log n\}] \\
= E^\mathbb{Q}^\infty[\phi(\mathcal{E}(L)_T)^+; \{\mathcal{E}(L)^* \geq n\}].
\]

Since \( \phi(\mathcal{E}(L)_T) \in L^1(\mathbb{Q}^\infty) \), we have that
\[
\limsup_{n \to \infty} n \log n \mathbb{Q}^\infty(\tau_n < T) \leq \limsup_{n \to \infty} n \log n \mathbb{Q}^\infty(\mathcal{E}(L)^* \geq n) \\
\leq \limsup_{n \to \infty} E^\mathbb{Q}^\infty[\phi(\mathcal{E}(L)_T)^+; \{\mathcal{E}(L)^* \geq n\}] \\
= 0.
\]

Now suppose that Assumption 5.6 holds. Then by Lemma 3.1 of [6] the density of the dual optimizer, \( Z^\infty \mathcal{E}(L) \), satisfies \( \mathcal{R}_{L \log L}(\mathbb{P}) \); that is, \( Z^\infty \mathcal{E}(L) \) is a \( \mathbb{P} \)-martingale and
\[
\sup_{\tau} \left\| E^\mathbb{P} \left[ \frac{Z^\infty \mathcal{E}(L)_T}{Z^\infty \mathcal{E}(L)_\tau} \log \left( \frac{Z^\infty \mathcal{E}(L)_\tau}{Z^\infty \mathcal{E}(L)_T} \right) | \mathcal{F}_\tau \right] \right\|_\infty < \infty,
\]
where the supremum is taken over all stopping times \( \tau \leq T \). Lemma 2.2 of [10] shows that \( -(\lambda^\infty \cdot M^\infty) + L \in \text{BMO}(\mathbb{P}) \), which then implies that \( L \in \text{BMO}(\mathbb{P}) \).

Since \( \langle -\lambda^\infty \cdot M^\infty, L \rangle_t = 0 \) for all \( t \in [0, T] \), then Theorem 3.6 of [15] implies that \( L = L - \langle -\lambda^\infty \cdot M^\infty, L \rangle \in \text{BMO}(\mathbb{Q}^\infty) \). Then by Theorem 2.4 of [15], \( L \) satisfies
\[
\sup_{\tau} \left\| E^\mathbb{Q}^\infty \left[ \log^+ \left( \frac{\mathcal{E}(L)_\tau}{\mathcal{E}(L)_T} \right) | \mathcal{F}_\tau \right] \right\|_\infty < \infty,
\]
where the supremum is taken over all stopping times \( \tau \leq T \). Re-writing (5.5), and considering only the stopping times \( \tau_n \) for \( n \geq 1 \), we have
\[
K := \sup_n \left\| E^\mathbb{Q}^\infty \left[ (\log \mathcal{E}(L)_{\tau_n} - \log \mathcal{E}(L)_T) 1_{\{\mathcal{E}(L)_{\tau_n} \geq \mathcal{E}(L)_T\}} | \mathcal{F}_{\tau_n} \right] \right\|_\infty < \infty.
\]
For each \( n \geq 1 \), \( \{\tau_n < T\} \in \mathcal{F}_{\tau_n} \) and \( \mathcal{E}(L)_{\tau_n} = n \) on \( \{\tau_n < T\} \). Then,
\[
-E^\mathbb{Q}^\infty[\log \mathcal{E}(L)_T; \{\mathcal{E}(L)_{\tau_n} \geq \mathcal{E}(L)_T\} \cap \{\tau_n < T\}] \\
\leq E^\mathbb{Q}^\infty[\log \mathcal{E}(L)_{\tau_n} - \log \mathcal{E}(L)_T; \{\mathcal{E}(L)_{\tau_n} \geq \mathcal{E}(L)_T\} \cap \{\tau_n < T\}] \\
= E^\mathbb{Q}^\infty \left[ E^\mathbb{Q}^\infty \left[ (\log \mathcal{E}(L)_{\tau_n} - \log \mathcal{E}(L)_T) 1_{\{\mathcal{E}(L)_{\tau_n} \geq \mathcal{E}(L)_T\}} | \mathcal{F}_{\tau_n} \right] \right] ; \{\tau_n < T\} \\
\leq K \mathbb{Q}^\infty(\tau_n < T).
\]
Thus,

\[-n \mathbb{E}_{Q}^\infty [\log \mathcal{E}(L)_T; \{\tau_n < T\}]\]

\[= -n \mathbb{E}_{Q}^\infty [\log \mathcal{E}(L)_T; \{\mathcal{E}(L)_T > n\} \cap \{\tau_n < T\}]\]

\[= -n \mathbb{E}_{Q}^\infty [\log \mathcal{E}(L)_T; \{\mathcal{E}(L)_T \leq n\} \cap \{\tau_n < T\}]\]

\[\leq 0 + nK \mathbb{Q}^\infty(\tau_n < T).\]

Equation (5.4) now follows from

\[0 \leq n \log n \mathbb{Q}^\infty(\tau_n < T) - n \mathbb{E}_{Q}^\infty [\log \mathcal{E}(L)_T; \{\tau_n < T\}]\]

\[\leq n \log n \mathbb{Q}^\infty(\tau_n < T) + nK \mathbb{Q}^\infty(\tau_n < T)\]

\[\rightarrow 0, \text{ as } n \rightarrow \infty.\]

\[\square\]

**References**

[1] Erhan Bayraktar and Ross Kravitz. Stability of exponential utility maximization with respect to market perturbations. *Stochastic Processes and their Applications*, 123:1671–1690, 2013.

[2] Sara Biagini and Marco Frittelli. A unified framework for utility maximization problems: An orlicz space approach. *The Annals of Applied Probability*, 18(3):929–966, 2008.

[3] Alexander Cherny and Albert Shiryaev. Vector stochastic integrals and the fundamental theorems of asset pricing. *Proceedings of the Steklov Mathematical Institute*, 237:12–56, 2002.

[4] Jakša Cvitanić, Walter Schachermayer, and Hui Wang. Utility maximization in incomplete markets with random endowment. *Finance and Stochastics*, 5(2):259–272, 2001.

[5] Mark H. A. Davis. Optimal hedging with basis risk. 2000.

[6] Freddy Delbaen, Peter Grandits, Thorsten Rheinländer, Dominick Samper, Martin Schwizer, and Christophe Stricker. Exponential hedging and entropic penalties. *Mathematical Finance*, 12(2):99–123, April 2002.

[7] Hans Föllmer and Martin Schweizer. Minimal martingale measure. *Encyclopedia of Quantitative Finance*, pages 1200–1204, 2010.
[8] Christoph Frei. Convergence results of the indifference value based on the stability of bsdes. *Stochastics*, 85:464–488, 2013.

[9] Christoph Frei and Martin Schweizer. Exponential utility indifference valuation in two brownian settings with stochastic correlation. *Advances in Applied Probability*, 40:401–423, 2008.

[10] Peter Grandits and Thorsten Rheinländer. On the minimal entropy martingale measure. *Annals of Probability*, 30(3):1003–1038, 2002.

[11] J.B. Hiriart-Urruty and C. Lemarechal. *Convex Analysis and Minimization Algorithms I: Part 1: Fundamentals*. Grundlehren der mathematischen Wissenschaften. Springer, 1996.

[12] Julien Hugonnier and Dmitry Kramkov. Optimal investment with random endowments in incomplete markets. *Annals of Applied Probability*, 14(2):845–864, 2004.

[13] Yuri M. Kabanov and Christophe Stricker. On the optimal portfolio for the exponential utility maximization: Remarks to the six-author paper. *Mathematical Finance*, 12(2):125–134, April 2002.

[14] Constantinos Kardaras and Gordan Žitković. Stability of the utility maximization problem with random endowment in incomplete markets. *Mathematical Finance*, 21(2):313–333, 2011.

[15] Norihiko Kazamaki. *Continuous Exponential Martingales and BMO*. Springer-Verlag, 1994.

[16] Dmitry Kramkov and Walter Schachermayer. The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Annals of Applied Probability*, 9(3):904–950, 1999.

[17] Kasper Larsen, H. Mete Soner, and Gordan Žitković. Facelifting in Utility Maximization. *ArXiv e-prints*, April 2014.

[18] Kasper Larsen and Gordan Žitković. Stability of utility-maximization in incomplete markets. *Stochastic Processes and their Applications*, 117(11):1642–1662, 2007.

[19] Michael Monoyios. Performance of utility-based strategies for hedging basis risk. *Quantitative Finance*, 4:245–255, 2004.
[20] Mark Owen and Gordan Žitković. Optimal investment with an unbounded random endowment and utility-based pricing. *Mathematical Finance, 19*(1):129–159, January 2009.

[21] Walter Schachermayer. Optimal investment in incomplete markets when wealth may become negative. *Annals of Applied Probability, 11*(3):694–734, 2001.