Mathematical model of the polymer concrete by fractional calculus with respect to a spatial variable

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Abstract. The article presents a solution of the vibration string equation containing the fractional derivative concerning the spatial variable, in which the fractional derivative is defined by Caputo. This model is utilized for characterizing the oscillation mechanical process in a viscoelastic medium. The theoretical solution is presented, and the fractional derivative parameter is determined. We compared the theoretical solution and the experimental data for polymer concrete samples. A sample of the problem of structural mechanics has been considered. This sample allows the demonstration of some advantages of the application of the suggested approach to solve the fractional vibration equation.

1. Introduction

The partial differential equations (PDEs) for fractional order arise in many different areas of engineering and science. The use of fractional calculus has much attention to developing the improved mathematical modeling of several real-world problems. The theoretical development and application of the fractional calculus have been discussed by many authors in the previous works. Fractional differential equations (FDEs) are more realistic to present natural phenomena, and they have been used in several areas of applied mathematics. In engineering applications, fractional derivatives are advantageous for characterizing the happening of vibrations. In numerous researches, studies concerning fractional calculus and its applications to mechanical problems show widely [1].

Firstly, we notice that the fractional derivatives concerning time could be used to model some processes with a "memory". Particular attention must be paid to the equation of the form:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + c_1 \cdot D_{\alpha}^\alpha u + c_0 \cdot D_{\alpha}^\beta u + F$$

That is used to characterize the vibration of string in a medium with fractal geometry considering friction, where this equation is used in this work to model changes in polymer concrete's deformation-strength characteristics under loading. This model is intrinsically a form of multi term space-time fractional wave equations. So far, there exist many works on analytical and numerical methods for multi term space-time fractional vibration string equation and fractional wave equations, see [2, 3]. The remnants of the article are arranged as follows. In section 2, we apply the separation of variables
scheme based on the Fourier method for solving equation (1). In section 3, Parameter identification and experimental Studies are devoted.

2. Vibration string equation and Fourier method

In the region $D = \{0 < x < L, t > 0\}$ assume the vibration string equation with respect to fractional derivative of order $\alpha$ and partial variable

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + c_1 \cdot D_{0x}^\alpha u,$$

with boundary conditions

$$u(0, t) = u(1, t) = 0,$$

and initial conditions

$$u(x, 0) = \varphi(x);$$
$$u'(x, 0) = \psi(x),$$

where, $0 < \alpha < 1$, is the viscoelasticity parameters of the medium, $c_1$ is the viscosity modulus of resin, $D_{0x}^\alpha$ is the Caputo fractional derivative of order $\alpha$ with respect to $x$. $u(x, t)$ is the displacement.

The Caputo fractional derivative of the function $f(t) \in C^m_{-1}$ with order $\alpha > 0$, $m \in N \cup \{0\}$, is defined by the formula

$$D_{0x}^\alpha f(x) = \frac{1}{\Gamma(m - \alpha)} \int_0^x f^{(m)}(t) dt (x - t)^{\alpha - m}, m - 1 < \alpha < m.$$  

(5)

To solve the vibration string equations (1)-(4) by using the method of separating variables, assume

$$u(x, t) = \chi(x) T(t),$$

(6)

and substituting from equation (6) in equation (1), we get on an ordinary linear differential equation for $T(t)$:

$$T''(t) = \lambda T(t),$$

(7)

where $\lambda$ is a positive constant parameter (eigenvalues), then the solution of the equation (7) is written as

$$T(t) = a \cos(t\sqrt{-\lambda}) + b \sin(t\sqrt{-\lambda}).$$

(8)

Also, for $\chi(x)$ we obtain on a fractional ordinary linear differential equation with the Caputo derivative:

$$\chi''(x) + c_1 \cdot D_{0x}^\alpha \chi(x) = \lambda \chi(x);$$
$$\chi(0) = 0; \chi(1) = 0.$$ 

(9)

Then the solution of the problem (1)-(4) can be written in the standard way:

$$u(x, t) = \sum_{m=1}^{\infty} (a_m \cos(t\sqrt{-\lambda_m}) + b_m \sin(t\sqrt{-\lambda_m})) \chi_m(x).$$

(10)

The number $\lambda$ is an eigenvalue of problem (9) - (10), if and only if $\lambda$ is the zero of the function

$$\omega(\lambda) = \sum_{n=0}^{\infty} \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{(-c_1)^k \lambda^{n-k}}{\Gamma(2n + 4 - k\alpha)}.$$ 

(11)
The next figure shows a graph of the function $\omega(\lambda)$ for the case $0 < \alpha < 1$ and $c_1 = 1.8$ on the interval $\lambda \in (1; 610)$.

Figure 1. The graph of the function ($\lambda$) for $0 < \alpha < 1$ and $c_1 = 1.8$ on the interval $\lambda \in (1; 610)$.

Now to seek a solution of equation (8), let us take

$$\chi''(x) = Y(x)$$  \hspace{1cm} (12)

integrate both sides from 0 to $x$

$$\chi'(x) = \int_0^x Y(t) \, dt + \chi'(0),$$  \hspace{1cm} (13)

here we used the Cauchy formula for repeated integration:

$$\int_{x_0}^{x_n} \int_{x_0}^{x_{n-1}} \ldots \int_{x_0}^{x_1} f(x) \, dx = \frac{1}{(n-1)!} \int_{x_0}^{x} (x-z)^{n-1} f(z) \, dz,$$  \hspace{1cm} (14)

then we integrate both sides of equation (13) from 0 to $x$ we obtain

$$\chi(x) = \int_0^x (x-t)Y(t) \, dt + \chi'(0) + x. \chi''(0),$$  \hspace{1cm} (15)

substituting from equation (5), equation (12) and equation (15) in equation (8)

$$Y(x) + \frac{c_1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} \left[ \int_0^t Y(\xi) \, d\xi + \chi'(0) \right] \, dt$$
$$= \lambda \int_0^x (x-t)Y(t) \, dt + \chi'(0) + x. \chi''(0),$$  \hspace{1cm} (16)

let’s set $\chi'(0) = A$, $\chi(0) = B$. So

$$Y(x) + \int_0^x \left[ \frac{c_1}{\Gamma(2-\alpha)} (x-t)^{1-\alpha} - \lambda(x-t) \right] Y(t) \, dt = \lambda.B + \lambda.A.x - A. \frac{c_1}{\Gamma(2-\alpha)} x^{1-\alpha}.$$  \hspace{1cm} (17)

The solution of equation (17) is

$$Y(x) = y_0(x) + \cdots + y_n(x),$$  \hspace{1cm} (18)

where:
and
\[
y_n(x) = \int_0^x \left\{ \frac{(-c_1)}{\Gamma(2 - \alpha)} (x - t)^{1-\alpha} + \lambda (x - t) \right\} y_{n-1}(t) \, dt. 
\] (20)

By integrating equation (20), and consider equation (10), so we get on
\[
y_0(x) = A \frac{c_1}{\Gamma(2 - \alpha)} x^{1-\alpha}, 
\] (21)
\[
y_n(x) = A \left[ \sum_{k=0}^{n+1} \frac{(n+1)(-c_1)^k \lambda^{n-k}}{\Gamma(2n + 2 - k\alpha)} x^{2n+1-k\alpha} \right], 
\] (22)
then,
\[
Y(x) = A \left[ \sum_{n=0}^{\infty} \sum_{k=0}^{n+1} \frac{(n+1)(-c_1)^k \lambda^{n+1-k}}{\Gamma(2n + 2 - k\alpha)} x^{2n+1-k\alpha} \right], 
\] (23)
\[
\chi'(x) = A \left[ 1 + \sum_{n=0}^{\infty} \sum_{k=0}^{n+1} \frac{(n+1)(-c_1)^k \lambda^{n+1-k}}{\Gamma(2n + 3 - k\alpha)} x^{2n+3-k\alpha} \right], 
\] (24)
and
\[
\chi(x) = A \left[ x + \sum_{n=0}^{\infty} \sum_{k=0}^{n+1} \frac{(n+1)(-c_1)^k \lambda^{n+1-k}}{\Gamma(2n + 4 - k\alpha)} x^{2n+4-k\alpha} \right], 
\] (25)

which is a solution of equation (8), then substituting from equation (26) into equation (11), deduce the solution of equations (1)-4. We find the first several eigenfunctions of the boundary value problem (8) - (9) numerically using the high-level technical calculation language Wolfram Mathematica, and plot their graphs, taking \( 0 < \alpha < 1 \) and \( c_1 = 1.8 \), where the first five eigenvalues [4] are written below in table1:

| \( \lambda \) | \( \lambda_1 \) | \( \lambda_2 \) | \( \lambda_3 \) | \( \lambda_4 \) | \( \lambda_5 \) |
|-------------|-------------|-------------|-------------|-------------|-------------|
|             | -16.51      | -59.49      | -125.13     | -213.33     | -323.27     |

The eigenfunctions of problem (8) - (9) can be written as
\[
\chi_m(x) = x + \sum_{n=0}^{\infty} \sum_{k=0}^{n+1} \frac{(n+1)(-c_1)^k \lambda_m^{n+1-k}}{\Gamma(2n + 4 - k\alpha)} x^{2n+4-k\alpha}; \ m = 1, 2, ... 
\] (26)

The system of eigenfunctions (26) is complete [5], but not orthogonal. Thus, we construct a system that will be biorthogonal to system (26); therefore, we consider the following problem associated with problem (9) - (10).
\[
\chi''(x) + c_1 \cdot D_{\chi} \chi(x) = \lambda \chi(x), 
\] (27)
where: $D_{x_1}^{\alpha}$ is the fractional differentiation operator which is a start from $x$ to 1, i.e. $D_{x_1}^{\alpha}$ is conjugated with operator of fractional differentiation $D_{0x}^{\alpha}$ [6]

$$D_{x_1}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_x^1 f'(t) dt \frac{1}{(t-x)^\alpha}.$$  \hspace{1cm} (28)

The solution of problem (28) leads to the following relation

$$\tilde{\chi}_m(x) = \chi_m(1-x), \hspace{1cm} m = 1; 2; ...$$ \hspace{1cm} (29)

The graphs of the first five functions $\chi_m(x)$ and the biorthogonal $\tilde{\chi}_m(x)$ functions are presented in the following figures.

Figure 2 show the graph of the functions $\chi_m(x), \tilde{\chi}_m(x); m = 1, 2, ..., 0 < \alpha < 1, c_1=1.8.$

$\alpha = 0.1$

$\alpha = 0.2$

$\alpha = 0.3$

$\alpha = 0.4$
Figure 2. The graph of the functions $\chi_m(\alpha), \tilde{\chi}_m(\alpha); m = 1; 2; \ldots, 0 < \alpha < 1; c_1 = 1.8.$

Now we determine the constants of the equation (10), as following,
So, the solution of the problem (1)-(4) takes the form:

$$\begin{align*}
A_m &= \frac{1}{\langle \chi_m(x), \chi_m(x) \rangle} \cdot \langle \varphi(x), \chi_m(x) \rangle, \\
B_m &= \frac{1}{\langle \chi_m(x), \chi_m(x) \rangle} \cdot \langle \psi(x), \chi_m(x) \rangle \sqrt{-\lambda_m}. 
\end{align*}$$

(30)

So, the solution of the problem (1)-(4) takes the form:

$$u(x, t) \approx u_{(5)}(x, t) = \sum_{m=1}^{5} \chi_m(x) \left[ a_m \cos(t \sqrt{-\lambda_m}) + b_m \cdot \sin(t \sqrt{-\lambda_m}) \right].$$

(31)

3. Parameter identification and experimental Studies

We use the experimental data obtained in [6], to validate the technique. The stress-strain state of these samples was well defined by models (9) and (10), and their parameter was also found to be $c_1 = 1.8$ [7], The values for polymer concrete samples based on polyester resin (Dian- and dihloangidride-1,1-dichloro-2,2-diethylene) are shown below in Table 2. The advantage of this technique of parametric identification compared with different descriptive techniques [8] consists of the quantitative accurate estimation of selecting the search parameter. The relation between the theoretical solution (31) and the experimental data is shown below in figure 3, when

$$\varphi(x) = 0; \psi(x) = 1; t \in [0,5].$$

**Table 2.** Experimental data for polymer concrete samples.

| $x_i$ | 0.25 | 0.5 | 0.75 | 1 |
|-------|------|-----|------|---|
| $U_i$ | 0.05 | -0.04 | -0.01 | 0.02 |

4. Conclusion

This comparison between the experimental data and the theoretical solution at the various value of the parameter $0 < \alpha < 1$, allows us to deduce that the order of the fractional derivative in the model (1) is $\alpha = 0.9$, where we see from the comparison that the theoretical solution is in very good agreement with the experimental data at this value, that allows us to conclude that it is a suitable value. Knowing the parameter of the model lets us, predict the stress-strain characteristics of the material (asphalitic concrete, polymer concrete, etc.) when to undergo loading.

![Graphs for different values of \( \alpha \)](image-url)
Figure 3. The comparison of the experimental data and the theoretical solution at various values of the parameter $0 < \alpha < 1$.

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