Exploration-Exploitation in Constrained MDPs

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Abstract

In many sequential decision-making problems, the goal is to optimize a utility function while satisfying a set of constraints on different utilities. This learning problem is formalized through Constrained Markov Decision Processes (CMDPs). In this paper, we investigate the exploration-exploitation dilemma in CMDPs. While learning in an unknown CMDP, an agent should trade-off exploration to discover new information about the MDP, and exploitation of the current knowledge to maximize the reward while satisfying the constraints. While the agent will eventually learn a good or optimal policy, we do not want the agent to violate the constraints too often during the learning process. In this work, we analyze two approaches for learning in CMDPs. The first approach leverages the linear formulation of CMDP to perform optimistic planning at each episode. The second approach leverages the dual formulation (or saddle-point formulation) of CMDP to perform incremental, optimistic updates of the primal and dual variables. We show that both achieve sublinear regret w.r.t. the main utility while having a sublinear regret on the constraint violations. That being said, we highlight a crucial difference between the two approaches; the linear programming approach results in stronger guarantees than in the dual formulation based approach.

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1 Introduction

Markov Decision Processes (MDPs) have been successfully used to model several applications, including video games, robotics, recommender systems and many more. However, MDPs do not take into account additional constraints that can affect the optimal policy and the learning process. For example, while driving, we want to reach our destination but we want to avoid to go off-road, overcome the speed limits, collide with other cars [García and Fernández, 2015]. Constrained MDPs [Altman, 1999] extend MDPs to handle constraints on the long term performance of the policy. A learning agent in a CMDP has to maximize the cumulative reward while satisfying all the constraints. Clearly, the optimal solution of a CMDP is different than the one of an MDP when at least one constraint is active. Then, the optimal policy, among the set of policies which satisfies the constraint, is stochastic.

In this paper, we focus on the online learning problem of CMDPs. While interacting with an unknown MDP, the agent has to trade-off exploration to gather information about the system and exploration to maximize the cumulative reward. Performing such exploration in a CMDP may be unsafe since may lead to numerous violations of the constraints. Since the constraints depend on the long term performance of the agent and the CMDP is unknown, the agent cannot exactly evaluate the constraints. It can only exploit the current information to build an estimate of the constraints. The objective is thus to design an algorithm with a small number of violations of the constraints.

Objective and Contributions. The objective of this technical report is to provide an extensive analysis of exploration strategies for tabular constrained MDPs with finite-horizon cost. Similar to [Agrawal and Devanur, 2019], we allow the agent to violate the constraints over the learning process but we require the cumulative cost of constraint violations to be small (i.e., sublinear). Opposite to [Zheng and Ratliff, 2020], we consider the CMDP to be unknown, i.e., the agent does not know the transition kernel, the reward function and the constraints.

The performance of the learning agent is measured through the regret, that accounts for the difference in executing the optimal policy and the learning agent. We define two regrets: i) the regret w.r.t. to the main objective (as in standard MDP), ii) the regret w.r.t. the constraint violations. These terms account for both convergence to the optimal policy and cumulative cost for violations of the constraints.

We introduce and analyze the following exploration strategies:

- **OptCMDP** leverages the ideas of UCRL2 [Jaksch et al., 2010]. At each episodes, it builds a set of plausible CMDPs compatible with the observed samples, and plays the optimal policy of the CMDP with the lowest cost (i.e., optimistic CMDP). To solve this planning problem, we introduce an extended linear programming (LP) problem in the space of occupancy measures. The important property is that there always exists a feasible solution of this extended LP.

- **OptCMDP-bonus** merges the uncertainties about costs and transitions used by OptCMDP into an exploration bonus. As a consequence, OptCMDP-bonus solves a single (optimistic) CMDP rather than planning in the space of plausible CMDPs. This leads to a more computationally efficient algorithm. In fact, this planning problem can be solved through an LP with $O(SAH)$ constraints and decision variables, a factor $O(S)$ smaller than the LP solved by OptCMDP.

- **OptDual-CMDP** leverages the saddle-point formulation of constrained MDP [e.g., Altman, 1999]. It solves this problem using an optimistic version of the dual projected sub-gradient algorithm (e.g., Beck 2017). At each episode, OptDual-CMDP solves an optimistic MDP defined using the estimated Lagrangian multiplier. Then, it uses the computed solution to update the Lagrange multipliers via
### 1.1 Related Work

The problem of online learning under constraints (with guarantees) have been analyzed both in bandits and in RL. Conservative exploration focuses on the problem of learning an optimal policy while satisfying a constraint w.r.t. a predefined baseline policy. This problem can be seen as a specific instance of CMDPs where the constraint is that the policy should perform (in the long run) better than a predefined baseline policy. Conservative exploration has been analyzed both in bandits \cite{wu2016conservative,kazerouni2017soft,garcelon2020optimistic} and in RL \cite{garcelon2020optimistic}. All these algorithms are able to guarantee that the performance of the learning agent is at least as good as the one of the baseline policy with high probability at any time. While they enjoy strong theoretical guarantees, they perform poorly in practice since are too conservative. In fact, the idea of these algorithms is to build a policy that (by playing the baseline policy) in order to be able to take standard exploratory actions. Consequently to this paper, \cite{zheng2020optimistically} has extended conservative exploration to CMDP with average reward objective. They assume that the transition functions are known, but the rewards and costs (i.e., the constraints) are unknown. The goal is thus to guarantee that, at any time, the policy executed by the agent satisfies the constraints with high probability. These requirement poses several limitations. Similarly to \cite{garcelon2020optimistic}, they need to assume that the MD is ergodic and that the initial policy is safe (i.e., satisfies the constraints). Furthermore, despite the theoretical guarantees, this approach is not practical due to these strong requirements/assumptions. \cite{agrawal2019conservative} studied the exploration problem for bandits under constraints as well as bandits with knapsack constraints \cite{badanidiyuru2013bandits}. Algorithms OptCMDP and OptCMDP-bonus can be understood as

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### Table 1: Summary of the regret bounds obtained in this work.

| Algorithm            | Optimality Regret         | Constraint Regret          |
|----------------------|---------------------------|----------------------------|
| OptCMDP              | $\tilde{O}(\sqrt{SNH^2K})$ | $\tilde{O}(\sqrt{SNH^2K})$ |
| OptCMDP-bonus        | $\tilde{O}(\sqrt{SNH^2K})$ | $\tilde{O}(\sqrt{SNH^2K})$ |
| OptDual-CMDP         | $\tilde{O}(\sqrt{(SNH^2 + \rho^2\chi^2)H^2K})$ | $\tilde{O}((1 + \frac{1}{\rho})\sqrt{ISNH^2K + I\sqrt{H^2K}})$ |
| OptPrimalDual-CMDP   | $\tilde{O}(\sqrt{(SNH^2 + \rho^2\chi^2)H^2K})$ | $\tilde{O}((1 + \frac{1}{\rho})\sqrt{ISNH^2K + I\sqrt{H^2K}})$ |

Notice that different types of regrets are bounded (see Section 2 for definitions).
generalizing their bandit setting to a CMDP setting. That being said, in the following we derive regret bounds on a stronger type of regret relatively to Agrawal and Devanur [2019] (see Remark 1).

There are several approaches in the literature that have focused on (approximately) solving CMDPs. These methods are mainly based on Lagrangian formulation [Bhatnagar and Lakshmanan, 2012, Chow et al., 2017, Tessler et al., 2019, Patermain et al., 2019] or constrained optimization [Achiam et al., 2017]. Lagrangian-based methods formulate the CMDP optimization problem as a saddle-point problem and optimize it using primal-dual algorithms. While these algorithms may eventually converge to the true policy, they have no guarantees on the policies recovered during the learning process. Constrained Policy Optimization (CPO) [Achiam et al., 2017] leverages the intuition behind conservative approaches [e.g., Kakade and Langford, 2002] to force the policy to improve overtime. This is a practical implementation of conservative exploration where the baseline policy is updated at each iteration.

Another way to solve CMDPs and guarantee safety during learning is through Lyapunov functions [Chow et al., 2018, 2019]. Despite the fact that some of these algorithms are approximately safe over the learning process, analysing the convergence is challenging and the regret analysis is lacking. Other approaches use Gaussian processes to model the dynamics and/or the value function [Berkenkamp et al., 2017, Wachi et al., 2018, Koller et al., 2018, Cheng et al., 2019] in order to be able to estimate the constraints and (approximately) guarantee safety over learning.

A related approach is the literature about budget learning in bandits [e.g., Ding et al., 2013, Combes et al., 2013]. In this setting, the agent is provided with a budget (known and fix in advance) and the learning process is stopped as soon as the budget is consumed. The goal is to learn how to efficiently handle the budget in order to maximize the cumulative reward. A widely studied case of budget bandit is bandit with knapsack [e.g., Agrawal and Devanur, 2014, Badanidiyuru et al., 2018]. In our setting, we do not have a “real” concept of budget and the length of the learning process does not depend on the total cost of constraint violations. This paper is also related to learning with fairness constraints [e.g., Joseph et al., 2016]. Similarly to conservative exploration, fairness constraints can be sometimes formulated as a specific instance of CMDPs.

2 Preliminaries

We start introducing finite-horizon Markov Decision Processes (MDPs) and their constrained version. We define $[N] := \{1, \ldots, N\}$, for all $N \in \mathbb{N}$.

2.1 Finite-Horizon Constrained MDPs

Finite Horizon MDPs. We consider finite-horizon MDPs with time-dependent dynamics [Puterman, 1994]. A finite-horizon constraint MDP is defined by the tuple $M = (\mathcal{S}, \mathcal{A}, c, p, s_1, H)$, where $\mathcal{S}$ and $\mathcal{A}$ are the state and action spaces with cardinalities $|\mathcal{S}|$ and $|\mathcal{A}|$, respectively. The non-stationary immediate cost for taking an action $a$ at state $s$ is a random variable $C_h(s, a) \in [0, 1]$ with expectation $E[C_h(s, a)] = c_h(s, a)$. The transition probability is $p_h(s' | s, a)$, the probability of transitioning to state $s'$ upon taking action $a$ at state $s$ at time-step $h$. The initial state in each episode is chosen to be the same state $s_1$ and $H \in \mathbb{N}$ is the horizon. Furthermore, $N := \max_{s,a,h} |\{s' : p_h(s' | s, a) > 0\}|$ is the maximum number of non-zero transition probabilities across the entire state-action pairs.

A Markov non-stationary randomized policy $\pi = (\pi_1, \pi_2, \ldots, \pi_H) \in \Pi^{|\mathcal{A}|}$ where $\pi_i : \mathcal{S} \rightarrow \Delta_\mathcal{A}$ maps states to probabilities $\Delta_\mathcal{A}$ on the action set $\mathcal{A}$. We denote by $a_h \sim \pi(s_h, h) := \pi_h(s_h)$, the action taken at time $h$ at state $s_h$ according to a policy $\pi$. For any $h \in [H]$ and $(s, a) \in \mathcal{S} \times \mathcal{A}$, the state-action value function of a non-stationary policy $\pi = (\pi_1, \ldots, \pi_H)$ is defined as

$$Q^\pi_h(s, a) = c_h(s, a) + \mathbb{E} \left[ \sum_{j=h+1}^H c(s_j, a_j) \mid s_h = s, a_h = a, \pi, p \right]$$

where the expectation is over the environment and policy randomness. The value function is $V^\pi_h(s) = \sum_a \pi_h(a | s) Q^\pi_h(s, a)$. Since the horizon is finite, under some regularity conditions, [Shreve and Bertsekas, 1978], there always exists an optimal Markov non-stationary deterministic policy $\pi^\star$ whose value and action-value functions are defined as $V^*_{\pi^\star}(s) := \max_{\pi} V^\pi_h(s)$ and $Q^\pi_{\pi^\star}(s, a) := Q^\pi_h(s, a)$. The Bellman principle of optimality (or Bellman optimality equation) allows to efficiently compute the optimal solution of an MDP using backward induction:

$$V^*_{\pi^\star}(s) = \min_{a \in \mathcal{A}} \left\{ c_h(s, a) + \mathbb{E}_{a' \sim p_h(\cdot | s, a)} [V^*_{\pi^\star}(s')] \right\}, \quad Q^*_{\pi^\star}(s, a) = c_h(s, a) + \mathbb{E}_{a' \sim p_h(\cdot | s, a)} [V^*_{\pi^\star}(s')]$$ (1)
where \( V_{H+1}^*(s) := 0 \) for any \( s \in S \) and \( V^*_h(s) = \min_a Q^*_h(s,a) \), for all \( s \in S \). The optimal policy \( \pi^*_h \) is thus greedy w.r.t. \( V^*_h \) [e.g., Puterman, 1994]. Notice that by boundedness of the cost, for any \( h \) and \( (s,a) \), all functions \( Q^*_h, V^*_h, Q^*_h, V^*_h \) are bounded in \([0,H-h+1]\).

We can reformulate the optimization problem by using the occupancy measure [e.g., Puterman, 1994, Altman, 1999]. The occupancy measure \( q^\pi \) of a policy \( \pi \) is defined as the set of distributions generated by the policy \( \pi \) in the finite-horizon MDP \( M \) [e.g., Zimin and Neal, 2013]:

\[
q^\pi_h(s,a;p) := \mathbb{P}\{s_h = s,a_h = a \mid s_1 = s_1, p, \pi\} = \mathbb{P}\{s_h = s,a_h = a \mid s_1 = s_1, p, \pi\}.
\]

For ease of notation, we define the matrix notation \( \mathbf{q}^\pi_h(s,a;p) \) where \( \mathbf{q}^\pi_h(s,a;p) \) element is given by \( q^\pi_h(s,a;p) \). This implies the following relation between the occupancy measure and the value of a policy:

\[
V^\pi_t(s_1;p,c) = \sum_{h,s,a} q^\pi_h(s,a;p)c_h(s,a) = c^Tq^\pi(p).
\]

where \( c \in \mathbb{R}^{HSA} \) such that element \( (s,a,h) \) element is given by \( c_h(s,a) \).

Proof. The value function \( V^\pi_t(s_1;p,c) \) is given by the following equivalent relations.

\[
\mathbb{E} \left[ \sum_{h=1}^H c_h(s_h,a_h) \mid s_1 = s_1, \pi, p \right] = \mathbb{E} \left[ c_h(s_h,a_h) \mid s_1 = s_1, \pi, p \right] = \mathbb{E} \left[ \sum_{h=1}^H \sum_{s,a} c_h(s,a) \mathbb{P}\{s_h = s,a_h = a \mid s_1 = s_1, p, \pi\} \right] = \mathbb{E} \left[ \sum_{h=1}^H \sum_{s,a} c_h(s,a)q^\pi_h(s,a;p) = c^Tq^\pi(p) \right],
\]

where the first relation holds by linearity of expectation.

Finite Horizon Constraint MDPs. A constraint MDP [Altman, 1999] is an MDP supplied with a set of \( I \) constraints \( \{d_i, \alpha_i\}_{i=1}^I \), where \( d_i \in \mathbb{R}^{SAH} \) and \( \alpha_i \in [0,H] \). The immediate \( i^{th} \) constraint when taking an action \( a \) from state \( s \) at time-step \( h \) is random variable \( D_i(s,a) \in [0,1] \) with expectation \( \mathbb{E}[D_i(s,a)] = d_i(s,a) \). The expected cost of the \( i^{th} \) constraint violation from state \( s \) at time-step \( h \) is defined as

\[
V^\pi_h(s;p,d_i) := \mathbb{E} \left[ \sum_{h'=h}^H d_i(h',s_h',a_h') \mid s_h = s, p, \pi \right].
\]

Similarly to \( \mathbf{2} \), we can rewrite the constraint in terms of occupancy measure:

\[
V^\pi_h(s;p,d_i) = d_i^Tq^\pi(p).
\]

Notice that by boundedness of the constraint cost, for any \( h,i \) and \( (s,a) \), all functions \( Q^*_h(s,a;d_i,p) \), \( V^*_h(s;d_i,p) \) are bounded in \([0,H-h+1]\). The objective of a CMDP is to find a policy minimizing the cost while satisfying all the constraints. Formally,

\[
\pi^* \in \arg \min_{\pi \in \Pi^{MR}} \mathbb{E} \left[ c^Tq^\pi(p) \right] \quad \text{s.t.} \quad Dq^\pi(p) \leq \alpha,
\]

where \( D \in \mathbb{R}^{I \times SAH} \) and \( \alpha \in \mathbb{R}^I \) such that

\[
D = \begin{bmatrix} d_1^T \\ \vdots \\ d_I^T \end{bmatrix}, \quad \alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_I \end{bmatrix},
\]

The optimal value is the value of \( \pi^* \) from the initial state, i.e., \( V^\pi_1(s_1) := V^\pi_1(s_1;p,c) \).

Assumption 1 (Feasibility). The unknown CMDP is feasible, i.e., there exists an unknown policy \( \pi \in \Pi^{MR} \) which satisfies the constraints. Thus, an optimal policy exists as well.

It is important to stress that the optimal policy of a CMDP may be stochastic [e.g., Altman, 1999], i.e., may not exist an optimal deterministic policy. In fact, due to the constraints, the Bellman optimality principle, see Eq. \( \mathbf{1} \) may not hold anymore. This means that we cannot leverage backward induction and the greedy operator. Altman [1999] showed that it is possible to compute the optimal policy of a constrained problem by using linear programming. We will review this approach in Sec. 2.3.
2.2 The Learning Problem.

We consider an agent which repeatedly interacts with a CMDP in a sequence of \( K \) episodes of fixed length \( H \) by playing a non-stationary policy \( \pi_k = (\pi_{1k}, \ldots, \pi_{hk}) \) where \( \pi_{hk} : S \rightarrow \Delta_A \). Each episode \( k \) starts from the fixed initial state \( s_k^1 = s_1 \). The learning agent does not know the transition or reward functions, and it relies on the samples (i.e., trajectories) observed over episodes to improve its performance over time.

The performance of the agent is measured using multiple objectives: \( i) \) the regret relatively to the value of the best policy, and \( ii) \) the amount of constraint violations. In sections 2.3 and 2.4, we analyze algorithms with guarantees on the following type of regrets

\[
\text{Reg}_+(K; c) = \sum_{k=1}^{K} \left[ V^\pi_k(s_1; p, c) - V^*_1(s_1) \right]_+ \tag{4}
\]

\[
\text{Reg}_+(K; d) = \max_{i \in [I]} \left[ \sum_{k=1}^{K} \left[ V^\pi_k(s_1; p, d_i) - \alpha_i \right]_+ \right], \tag{5}
\]

where \( [x]_+ := \max\{0, x\} \). The term \( \text{Reg}_+(K; d) \) represents the maximum cumulative cost for violations of the constraints.

We later continue and analyze algorithms with reduced computational complexity in sections 5.1 and 5.2. For these algorithms, we supply regret guarantees for all \( K' \in [K] \) with respect to a weaker measure of regrets defined as follows.

\[
\text{Reg}(K; c) = \sum_{k=1}^{K} V^\pi_k(s_1; p, c) - V^*_1(s_1) \tag{6}
\]

\[
\text{Reg}(K; d) = \max_{i \in [I]} \left[ \sum_{k=1}^{K} V^\pi_k(s_1; p, d_i) - \alpha_i \right]. \tag{7}
\]

**Remark 1.** Note that in our setting, the immediate regret \( V^\pi_k(s_1; p, c) - V^*_1(s_1) \) might be negative since policy \( \pi_k \) might violate the constraints. For this reason, bounding the regret as \( \text{Reg}_+(K; c) \) is stronger than bounding \( \text{Reg}_+(K; c) \) in the sense that the a bound on the first implies a bound on the latter; but not vice-versa.

Similar relation holds between the two definitions of the constraint violations types of regret; a bound on \( \text{Reg}_+(K; d) \) implies a bound on \( \text{Reg}(K; d) \), but the opposite does not holds. In words, a bound on the first implies a bound on the absolute sum of constraint violations where the latter bounds the cumulative constraint violations, and, thus, allows for “error cancellations”.

2.3 Linear Programming for CMDPs

In Sec. 2.4, we have seen that the cost criteria can be expressed as the expectation of the immediate cost w.r.t. to the occupancy measure. The convexity and compactness of this space is essential for the analysis of constrained MDPs. We refer the reader to [Altman, 1999, Chap. 3 and 4] for an analysis in infinite horizon problems.

We start stating two basic properties of an occupancy measure \( q \). In this section, we remove the dependence on the model \( p \) to ease the notation. It is easy to see that the occupancy measure of any policy \( \pi \) satisfies [e.g., Zimin and Neu, 2013; Bhattacharya and Kharoufeh, 2017]:

\[
\sum_a q^\pi_h(s, a) = \sum_{s', a'} p_{h-1}(s|s', a') q^\pi_{h-1}(s', a') \quad \forall s \in S \tag{8}
\]

\[
q^\pi_h(s, a) \geq 0 \quad \forall s, a
\]

for all \( h \in [H] \setminus \{1\} \). For \( h = 1 \) and an initial state distribution \( \mu \), we have that

\[
q^\pi_1(s, a) = \pi_1(a|s) \cdot \mu(s) \quad \forall s, a
\]

Notice that \( \sum_{a} q^\pi_1(s, a) = 1 \). As a consequence, by summing the first constraint in (8) over \( s \) we have that \( \sum_{s,a} q^\pi_h(s, a) = 1 \), for all \( h \in [H] \). Thus the \( q^\pi \) satisfying the constraints are probability measures.

We denote by \( \Delta^\mu(M) \) the space of occupancy measures.

Since the set \( \Delta^\mu(M) \) can be described by a set of affine constraints, we can state the following property. Please refer to [e.g., Puterman, 1994; Altman, 1999; Mannor and Tsitsiklis, 2005] for more details.
Algorithm 1 \textsc{OptCMDP}

\textbf{Require:} $\delta \in (0,1)$
\textbf{Initialize:} $n^0_b(s,a) = 0$, $\mathcal{P}^b_0(s' | s,a) = 1/\mathcal{S}$ and $\mathcal{P}^a_0(s,a) = 0$

\textbf{for} $k = 1,\ldots,K$ \textbf{do}
- Define $\hat{c}^k$ and $d^k$ as in \textbf{[L]}
- Compute the solution of (14) through the extended LP
- Execute $\pi_k$ and collect a trajectory $(s^k_h, a^k_h, c^k_i, d^k_{i,h})$ for $h \in [H]$
- Update counters and empirical model (i.e., $n^k, \hat{c}^k, \hat{d}^k, \hat{p}^k$) as in \textbf{[9]}
\textbf{end for}

\textbf{Proposition 1.} The set $\Delta^\mu(M)$ of occupancy measure is convex.

An important consequence of the linearity of the cost criteria and of the structure of $\Delta(M)$ is that the original control problem can be reduced to a Linear Program (LP) where the optimization variables are measures. Furthermore, optimal solutions of the LP define the optimal Markov policy through the occupancy measure. In fact, a policy $\pi^q$ generates an occupancy measure $q \in \Delta(M)$ if

$$q^y_h(a|s) = \frac{q_h(s,a)}{\sum_b q_h(s,b)}, \quad \forall (s,a,h) \in S \times A \times [H].$$

The constrained problem \textbf{[3]} is equivalent to the LP:

$$\min_q \sum_{s,a,h} q_h(s,a)c_h(s,a)$$

s.t.

$$\sum_{s,a,h} q_h(s,a)d_{i,h}(s,a) \leq \alpha_i, \quad \forall i \in [I]$$

$$\sum_{a} q_h(s,a) = \sum_{s',a'} p_{h-1}(s|s',a')q_{h-1}(s',a'), \quad \forall h \in [H] \setminus \{1\}$$

$$\sum_{a} q_1(s,a) = \mu(s), \quad \forall s \in S$$

$$q_h(s,a) \geq 0, \quad \forall (s,a,h) \in S \times A \times [H]$$

The constraint $\sum_{s,a} q_h(s,a) = 1$ is redundant.

2.4 Notations and Definitions.

Throughout the paper, we use $t \in [H]$ and $k \in [K]$ to denote time-step inside an episode and the index of an episode, respectively. The filtration $\mathcal{F}_k$ includes all events (states, actions, and costs) until the end of the $k$-th episode, including the initial state of the $k+1$ episode. We denote by $n^k_b(s,a)$ the number of times that the agent has visited state-action pair $(s,a)$ at the $h$-th step, and by $\sum_{s,a} h$, the empirical average of a random variable $X$. Both quantities are based on experience gathered until the end of the $k$-th episode and are $\mathcal{F}_k$ measurable. Since $\pi_k$ is $\mathcal{F}_{k-1}$ measurable, so is $q^\pi_h(s,a;p)$. Furthermore, from this definition we have that for any $X$ which is $\mathcal{F}_{k-1}$ measurable

$$\mathbb{E}[X(s^k_h, a^k_h) | \mathcal{F}_{k-1}] = \sum_{s,a} q^\pi_h(s,a;p)X(s,a).$$

We use $\tilde{O}(X)$ to refer to a quantity that depends on $X$ up to a poly-log expression of a quantity at most polynomial in $S, A, K, H$ and $\delta^{-1}$. Similarly, $\preceq$ represents $\leq$ up to numerical constants or poly-log factors. We define $X \lor Y \triangleq \max\{X,Y\}$.

3 Upper Confidence Bounds for CMDPs

We start by considering a natural adaptation of UCRL2 \cite{jaksch2010near} to the setting of CMDPs which we call \textsc{OptCMDP} (see Algorithm\textbf{[I]}).
Let $n_h^{k-1}(s,a) = \sum_{k'=1}^{k-1} \mathbb{1}(s_h^{k'} = s, a_h^{k'} = a)$ denote the number of times a pair $(s,a)$ was observed before episode $k$. At each episode, OptCMDP estimates the transition model, cost function and constraint cost function by their empirical average:

$$ \tilde{P}_h^{k-1}(s' | s,a) = \frac{\sum_{k'=1}^{k-1} \mathbb{1}(s_h^{k'} = s, a_h^{k'} = a, s_{h+1}^{k'} = s')} {n_h^{k-1}(s,a) \lor 1} $$

$$ \tilde{\pi}_h^{k-1}(s,a) = \frac{\sum_{k'=1}^{k-1} c_h^{k'} \cdot \mathbb{1}(s_h^{k'} = s, a_h^{k'} = a)} {n_h^{k-1}(s,a) \lor 1} $$

$$ \forall i \in [I], \quad \tilde{d}_{i,h}^{k-1}(s,a) = \frac{\sum_{k'=1}^{k-1} d_{i,h}^{k'} \cdot \mathbb{1}(s_h^{k'} = s, a_h^{k'} = a)} {n_h^{k-1}(s,a) \lor 1} $$

Following the approach of *optimism-in-the-face-of-uncertainty* we would like to act with an optimistic policy. To this end, we generalize the notion of optimism from the bandit setup presented in [Agrawal and Devanur, 2019] to the RL setting. Specifically, we would like for our algorithm to satisfy the following demands:

(a) *Feasibility of $\pi^*$ for all episodes*. The optimal policy $\pi^*$ should be contained in the feasible set in every episode.

(b) *Value optimism*. The value of every policy should be optimistic relatively to its true value, $V_{\pi}^\tau(s_1; \tilde{c}, \tilde{p}) \leq V_{\pi}^\tau(s_1; c,p)$ where $\tilde{c}, \tilde{p}$ are the optimistic cost and model by which the algorithm calculates the value of a policy.

Indeed, optimizing over a set which satisfy (a) while satisfying (b) results in an optimistic estimate of $V_{\pi}^*(s_1)$.

Similar to UCRL2, at the beginning of each episode $k$, OptCMDP constructs confidence intervals for the costs and the dynamics of the CMDP. Formally, for any $(s,a) \in S \times A$ we define

$$ B_{h,k}^p(s,a) = \left\{ \tilde{p}(\cdot | s,a) \in \Delta_S : \forall s' \in S, \ |\tilde{p}(\cdot | s,a) - \tilde{p}_{h,k}^{k-1}(\cdot | s,a)\rangle \leq \beta_{h,k}^p(s,a,s') \right\} $$

$$ B_{h,k}^c(s,a) = \left\{ \tilde{c}_h^{k-1}(s,a) - \beta_{h,k}^c(s,a), \tilde{c}_h^{k-1}(s,a) + \beta_{h,k}^c(s,a) \right\} $$

$$ B_{i,h,k}^d(s,a) = \left\{ \tilde{d}_{i,h}^{k-1}(s,a) - \beta_{i,h,k}^d(s,a), \tilde{d}_{i,h}^{k-1}(s,a) + \beta_{i,h,k}^d(s,a) \right\} $$

where the size of the confidence intervals is built using empirical Bernstein inequality [e.g., Audibert et al., 2007, Maurer and Pontil, 2003] for the transitions and Hoeffding inequality for the costs:

$$ \beta_{h,k}^p(s,a,s') \leq \sqrt{\frac{\text{Var}(\tilde{p}_{h,k}^{k-1}(s'|s,a))} {n_h^{k-1}(s,a) \lor 1}} + \frac{1} {n_h^{k-1}(s,a) \lor 1} $$

$$ \beta_{h,k}^c = \beta_{i,h,k}^d \leq \sqrt{\frac{1} {n_h^{k-1}(s,a) \lor 1}} $$

where $\text{Var}(\tilde{p}_{h,k}^{k-1}(s'|s,a)) = \tilde{p}_{h}^{k-1}(s'|s,a) \cdot (1 - \tilde{p}_{h}^{k-1}(s'|s,a))$ [e.g., Dann and Brunsick, 2015]. The set of plausible CMDPs associated with the confidence intervals is then $M_k = \{ M = (S, A, \tilde{c}, \tilde{d}, \tilde{p}) : \tilde{c}_h(s,a) \in B_{h,k}^c(s,a), \tilde{d}_{i,h} \in B_{i,h,k}^d(s,a), \tilde{p}(\cdot | s,a) \in B_{h,k}^p(s,a) \}$. Once $M_k$ been computed, OptCMDP finds a solution to the optimization problem

$$ (M_k, \pi_k) = \arg \min_{(\tilde{c}, \tilde{d}, \tilde{p}) \in M_k} \sum_{h,s,a} c_h^*_h(s,a)q_h^*(s,a;\tilde{p}) $$

$$ \text{s.t.} \sum_{h,s,a} \tilde{d}_{i,h}^*(s,a)q_h^*(s,a;\tilde{p}) \leq \alpha_i, \quad \forall i \in [H] $$

While this problem is well-defined and feasible, we can simplify it and avoid to optimize over the sets $B_{h,k}^c$ and $B_{i,h,k}^d$. We define

$$ \tilde{c}_h^*(s,a) = \tilde{c}_h^{k-1}(s,a) - \beta_{h,k}^c(s,a) $$

$$ \tilde{d}_{i,h}^*(s,a) = \tilde{d}_{i,h}^{k-1}(s,a) - \beta_{i,h,k}^d(s,a) $$
to be the lower confidence bounds on the costs. Then, we can solve the following optimization problem

$$\min_{\tilde{\pi} \in B_k^c, \pi \in \Pi_{\text{MR}}} \sum_{h,s,a} \tilde{c}_h^k(s,a)q_h^c(s,a;\tilde{p})$$

$$\text{s.t.} \quad \sum_{h,s,a} d_h^k(s,a)q_h^c(s,a;\tilde{p}) \leq \alpha_i, \quad \forall i \in [H]$$ (14)

Consider a feasible solution $M' = (S, A, c', d', p')$ and $\pi'$ of problem (14). We can replace $c'$ with $c_k$ and $d'$ with $d_k$ as in (13) and still have a feasible solution. This holds since $c' \geq c_k$ and $d' \geq d_k$ componentwise.

We can now state some property of (14).

**Proposition 2.** The optimization problem (14) is feasible. Denote by $\pi_k$ the policy recovered solving (14) and by $M_k = (S, A, c_k, \tilde{d}_k, \tilde{p}_k)$ the associated CMDP. Then, policy $\pi_k$ is optimistic, i.e.,

$$V^\pi_k(s_1; \tilde{c}_k, \tilde{p}_k) := \tilde{c}_h^k q^\pi_k(\tilde{p}_k) \leq c^\pi q^\pi(p) := V^\pi_c(s_1; c, p)$$

**Proof.** The proof of optimism is reported in Lem. (9) and the feasibility is proven in Lem. (10). \(\square\)

**The extended LP problem.** Problem (14) is similar to (3), the crucial difference is that the true costs and dynamics are unknown. Since we cannot directly optimize this problem, we propose to rewrite (14) as an extended LP problem by considering the state-action-state occupancy measure $z^\pi(s, a, s'; p)$ defined as $z_h^c(s, a, s', p) = \tilde{p}_h(s'|s, a)q_h^c(s, a; p)$. We leverage the Bernstein structure of $B_{h,k}^p$ (see Eq. (10)) to formulate the extended LP over variable $z$:

$$\min_{\pi, z} \sum_{h,s,a,s'} z_h(s,a,s')c_h(s,a)$$

$$\text{s.t.} \quad \sum_{h,s,a,s'} z_h(s,a,s')d_h(s,a) \leq \alpha_i \quad \forall i \in [H]$$

$$\sum_{a,s'} z_h(s,a,s') = \sum_{s',a'} z_{h-1}(s',a',s) \quad \forall h \in [H] \setminus \{1\}$$

$$\sum_{a,s'} z_1(s,a,s') = \mu(s) \quad \forall s \in S$$

$$z_h(s,a,s') \geq 0 \quad \forall (s, a, s', h) \in S \times A \times S \times [H]$$

$$z_h(s,a,s') - (\bar{\pi}_h^{-1}(s'|s,a) + \beta_{h,k}^p(s,a,s')) \sum_y z_h(s,a,y) \leq 0 \quad \forall (s, a, s', h) \in S \times A \times S \times [H]$$

$$- z_h(s,a,s') + (\bar{\pi}_h^{-1}(s'|s,a) - \beta_{h,k}^p(s,a,s')) \sum_y z_h(s,a,y) \leq 0 \quad \forall (s, a, s', h) \in S \times A \times S \times [H]$$

This LP has $O(S^2HA)$ constraints and $O(S^2HA)$ decision variables. Such an approach was also used in [Jin et al. 2019] in a different context. Notice that $B_{h,k}^p$ can be chosen by using different concentration inequalities, e.g., $L_1$ concentration inequality for probability distributions. [Rosenberg and Mansour 2019] showed that even in that case we can formulate an extended LP.

Once we have computed $z$, we can recover the policy and the transitions as

$$\tilde{p}_h^k(s'|s,a) = \frac{z(s,a,s')}{\sum_y z(s,a,y)} \quad \text{and} \quad \tilde{c}_h^k(s,a) = \frac{\sum_{s'} z(s,a,s')}{\sum_y z(s,a,y)}$$

**Proposition 2** shows that (a) and (b) are satisfied and the solution is optimistic. This allows us to provide the following guarantees.

**Theorem 3** (Regret Bounds for OptCMDP). Fix $\delta \in (0,1)$. With probability at least $1 - \delta$ for any $K' \in [K]$ the following regret bounds hold

$$\text{Reg}_c(K';c) \leq \tilde{O}\left(\sqrt{SNH^4K} + (\sqrt{N} + H)H^2SA\right),$$

$$\text{Reg}_c(K';d) \leq \tilde{O}\left(\sqrt{SNH^4K} + (\sqrt{N} + H)H^2SA\right).$$
requires to solve a single CMDP. To this extent, it has to solve an LP problem with constraints and decision variables. In this section, we present a bonus-based algorithm for exploration in CMDPs that we call OptCMDP-bonus. This algorithm can be seen as a generalization of UCBVI [Azar et al. 2017] to constrained MDPs. The main advantage of OptCMDP-bonus is that it requires to solve a single CMDP. To this extent, it has to solve an LP problem with $O(SH)$ constraints and decision variables.

At each episode $k$, OptCMDP-bonus builds an optimistic CMDP $M_k := (S, A, \hat{c}^k, \hat{d}^k, \hat{p}^k)$ where

$$\hat{c}^k(s, a) = \tilde{c}^k(s, a) - b^k(s, a), \quad \hat{d}^k(s, a) = \tilde{d}^k(s, a) - b^k(s, a),$$

while $\tilde{c}^k$, $\tilde{d}^k$ and $\tilde{p}^k$ are the empirical estimates defined in (9). The term $b^k$ integrates the uncertainties about costs and transitions into a single exploration bonus. Formally,

$$b^k(s, a) = \beta_h^k(s, a) + H \sum_{s'} \beta_{h,k}^p(s, a, s')$$

where $\beta_h$ and $\beta_p$ are defined as in (11). Then, OptCMDP-bonus solves the following optimization problem

$$\min_{\pi \in \Pi} \sum_{h, s, a} \hat{c}^k(s, a) q^\pi(s, a; \hat{p}^k)$$

subject to

$$\sum_{h, s, a} \hat{d}^k(s, a) q^\pi(s, a; \hat{p}^k) \leq \alpha_i, \quad \forall i \in [H]$$

This problem can be solved using the LP described in Sec. 2.3. In App. B.2, we show that $\pi_k$ is an optimistic policy, i.e., $V_1^{\pi_k}(s_1; \tilde{c}^k, \tilde{p}^k) \leq V_1^*(s_1).

**Theorem 4 (Regret Bounds for OptCMDP-bonus).** Fix $\delta \in (0, 1)$. With probability at least $1 - \delta$ for any $K' \in [K]$ the following regret bounds hold

$$\text{Reg}_c(K'; c) \leq \tilde{O}\left(\sqrt{SNH^4K} + S^2H^4A(NH + S)\right),$$

$$\text{Reg}_d(K'; d) \leq \tilde{O}\left(\sqrt{SNH^4K} + S^2H^4A(NH + S)\right).$$

The regret bounds of OptCMDP-bonus include the same $\tilde{O}\left(\sqrt{SNH^4K}\right)$ term as of OptCMDP. However, the constant term in the regret bounds of OptCMDP-bonus has worst dependence w.r.t. $S, H, N$. This suggests that in the limit of large state space the bonus-based approach for CMDPs have worse performance relatively to the optimistic model approach.

**Remark 2.** The origin of the worst regret bound comes from the larger bonus term ([10]) we need to add to compensate on the lack of knowledge of the transition model. This bonus term, allows us to replace the optimistic planning w.r.t. a set of transition models (as in OptCMDP) by using the empirical transition model. However, it leads to a value function which is not bounded within $[0, H]$ but within $[-\sqrt{S}H^2, H]$. To circumvent this problem, a truncated Bellman operator has been used [e.g., Azar et al.].
OptDual-CMDP learning. In this sense, we can interpret projected sub-gradient algorithm (e.g., Beck [2017]). It can also be interpreted through the lens of online λ manner where the first player (the agent, We start by describing the optimistic dual approach for CMDPs.

5.1 Optimistic Dual Algorithm for CMDPs

For details).

Stage it solves the following optimistic problem:

\[ Q_h^k(s, a; \tilde{c}_k, \tilde{p}_{k-1}) = \max \{0, \tilde{c}_k(s, a) + \tilde{p}_{k-1}^{k-1}(s, h)V_{h+1}^\pi(\cdot | s, a)\} \]

\[ V_h^k(s; \tilde{c}_k, \tilde{p}_{k-1}) = (Q_h^k(s, \cdot; \tilde{c}_k, \tilde{p}_{k-1}), \pi_\delta(\cdot | s)) \]  

However, plugging this idea into the CMDP problem (Sec. 2.3) is not simple. In particular, it is not clear how to enforce truncation in the space of occupancy measures. Thus, reduction to LP seems problematic to obtain. At the same time, using dynamic programming to solve CMDP is problematic due to the presence of constraints (and the lack of Bellman optimality principle). We leave it for future work to devise a polynomial algorithm to solve this problem, or establishing it is a “hard-problem” to solve. If solved, it would result in an algorithm with similar performance to that of OptCMDP (up to polylog and constant factors).

5 Optimistic Dual and Primal-Dual Approaches for CMDPs

In previous sections, we analyzed algorithms which require access to a solver of an LP with at least \( \Omega(SHA) \) decision variables and constraints. In the limit of large state space, solving such linear program is expected to be prohibitively expensive in terms of computational cost. Furthermore, most of the practically used RL algorithms [e.g., Achiam et al. 2017, Tessler et al. 2019] are motivated by the Lagrangian formulation of CMDPs.

Motivated by the need to reduce the computational cost, we follow the Lagrangian approach to CMDPs in which the dual problem to CMDP (3) is being solved. Introducing Lagrange multipliers \( \lambda \in \mathbb{R}_+^* \), the dual problem to (3) is given by

\[ L^* = \max_{\lambda \in \mathbb{R}_+^*} \min_{\pi \in \Delta_S} \{c^T q^\pi(p) + \lambda^T (D q^\pi(p) - \alpha)\} \]  

(18)

With this in mind, a natural way to solve a CMDP is to use a dual sub-gradient algorithm [see e.g., Beck 2017] or a primal-dual gradient algorithm. Viewing the problem in this manner, a CMDP can be solved by playing a game between two-player; the agent \( \pi \) and the Lagrange multiplier \( \lambda \). This process is expected to converge to the Nash equilibrium with value \( L^* \). Furthermore, strong duality is known to hold for CMDP [e.g., Altman, 1999] and thus the expected value of this game is expected to converge to \( L^* = V^*_\pi(s_1) \). This general approach is also followed in the line of works on online learning with long-term constraints [e.g., Mahdavi et al. 2012, Yu et al. 2017]. There, the problem does not have a decision horizon \( H \) nor state space as in our case.

As the environment is unknown, and the agents gathers its experience based on samples, the algorithm should use an exploration mechanism with care. To handle the exploration, we use the optimism approach. In the following sections, we formulate and establish regret bounds for optimistic dual and primal-dual approaches to solve a CMDP. These algorithms are computationally easier than the algorithms of previous sections. Unfortunately, the regret bounds obtained in this section are weaker. We establish bounds on \( \text{Reg}(K; c) \) (resp. \( \text{Reg}(K; d) \)) instead of \( \text{Reg}_+(K; c) \) (resp. \( \text{Reg}_+(K; d) \)) as in previous section (see Sec. 2.2 for details).

5.1 Optimistic Dual Algorithm for CMDPs

We start by describing the optimistic dual approach for CMDPs. OptDual-CMDP is based upon the dual projected sub-gradient algorithm (e.g., Beck 2017). It can also be interpreted through the lens of online learning. In this sense, we can interpret OptDual-CMDP as solving a two-player game in a decentralized manner where the first player (the agent, \( \pi \)) applies “be-the-leader” algorithm, and the second player (the Lagrange multiplier, \( \lambda \)) uses projected gradient-descent.

Algorithm OptDual-CMDP (see Alg. 3) acts by performing two stages in each iteration. At the first stage it solves the following optimistic problem:

\[ \pi_k, \tilde{p}_k \in \arg \min_{\pi \in \Pi^{ph}, \mu \in B^p_k} (\tilde{c}_k + \tilde{D}_k^T \tilde{\lambda}_k)^T q^\pi(p') - \lambda_k^T \alpha \]

where \( \tilde{c}_k, \tilde{d}_k, \) and \( B^p_k \) are the same as in Sec. 3 (refer to (10) and (13)). This problem corresponds to finding the optimal policy (denoted \( \pi_k^* \)) of the following extended MDP \( \mathcal{M}_k = \{M = (\mathcal{S}, \mathcal{A}, r^*, p^+) : r_h^k(s, a) = \tilde{c}_h^k(s, a) + \sum_i (d_i^k(s, a) - \alpha_i) \lambda^{i}_h, p_h^k(\cdot | s, a) \in B^p_h(s, a)\} \). Since this is an extended MDP and not
a CMDP, we can use standard dynamic programming techniques. One possibility is to use the extended LP similar to the one introduced in Sec. 3. Otherwise, we can use backward induction to compute \( Q_k \) with \( Q_{k+1}(s, a) = 0 \) for all \( s, a \). Then, \( \pi_k^\ast(s) \in \arg \min_a Q_k^k(s, a) \). To compute \( q_k^\ast(s, a) \) we can use Alg. 3 in [Jin et al., 2019].

At the second stage, OptDual-CMDP updates the Lagrange multipliers proportionally to the violation of the “optimistic” constraints: \( \lambda_{k+1} = \left[ \lambda_k + 1/(\bar{D}_k q^\pi_k - \alpha) \right]_+ \).

The following assumption is standard for the analysis of dual projected sub-gradient method which we make as well. This assumption is quite mild and demands a policy which satisfy the constraint with equality exists. For example, a policy with zero constraint-cost (from state \( s_1 \)) exists this assumption hold.

**Assumption 2 (Slater Point).** We assume there exists an unknown policy \( \pi \) for which \( d_k^Tq^\pi(p) < \alpha_i \) for all the constraints \( i \in [I] \). Set

\[
\rho = \frac{c^Tq^\pi(p) - c^Tq^\pi(p)}{\min_{i=1,...,I}(\alpha_i - d_i^Tq^\pi(p))}.
\]

The following theorem establishes guarantees for both the performance and the total constraint violation (see App. C for the proof).

**Theorem 5 (Regret Bounds for OptDual-CMDP).** For any \( K' \in [K] \) the regrets the following bounds hold

\[
\text{Reg}(K'; c) \leq \tilde{O}\left(\sqrt{SNH^4K} + \rho \sqrt{H^2IK} + (\sqrt{N} + H)H^2SA\right)
\]

\[
\text{Reg}(K'; d) \leq \tilde{O}\left(\left(1 + \frac{1}{\rho}\right)\sqrt{NSN^4K} + (\sqrt{N} + H)\sqrt{TH^2SA}\right).
\]
the exploration bonus $b_h^k(s, a)$ defined in Eq. (13) (see also Eq. (14)). Then, it applies a Mirror Descent (MD) [Beck and Teboulle, 2003] update on the weighted Q-function

$$Q_h^k(s, a) = Q_h^k(s, a; \tilde{c}_k, \tilde{p}_{k-1}) + \sum_{i=1}^I \lambda_k, Q_h^\pi(s, a; \tilde{d}_k, \tilde{p}_{k-1})$$

and updates the dual variables, i.e., the Lagrange multipliers $\lambda$, by a projected gradient step. Since we optimize over the simplex and choose the Bregman distance to be the KL-divergence, the update rule of $\lambda$ is

$$\lambda_{k+1} = \max \left\{ \lambda_k + \frac{1}{\lambda_k} (D_k - q_k - \alpha), 0 \right\}$$

$$\lambda_{k+1} = \min \{ \lambda_{k+1}, \rho I \}$$

Execute $\pi_k$ and collect a trajectory $(s_h, a_h, c_h, d_h, h_{h+1})$ for $h \in [H]$

Update counters and empirical model (i.e., $n_k^i, \pi_k^i, c_k^i, d_k^i$) as in (13)

end for

end for

Theorem 6 (Regret Bounds for OptPrimalDual-CMDP). For any $K' \in [K]$ the regrets the following bounds hold

$$\text{Reg}(K', c) \leq \tilde{O} \left( \sqrt{SNH^4K} + \sqrt{H^4(1 + I \rho)^2K} + (\sqrt{N} + H)H^2SA \right)$$

$$\text{Reg}(K', d) \leq \tilde{O} \left( 1 + \frac{1}{\rho} \left( \sqrt{1SNH^4K} + (\sqrt{N} + H)\sqrt{TH^2SA} \right) + I\sqrt{H^4K} \right).$$
Algorithm 5 Truncated Policy Evaluation

Require: \( \forall s, a, s', h, l^h(s, a), \tilde{p}^h(s' | s, a), \pi^h(a | s) \)
\( \forall s \in S, V^\pi_{h+1}(s) = 0 \)

for \( \forall h = H, \ldots, 1 \) do
  for \( \forall s, a \in S \times A \) do
    \( \hat{Q}^h(s, a; \hat{l}, \hat{p}) = \max \left\{ \tilde{l}^h(s, a) + \tilde{p}^h(s, a) \hat{V}^\pi_{h+1}(s; \hat{l}, \hat{p}), 0 \right\} \)
  end for
  for \( \forall s \in S \) do
    \( \hat{V}^h(s; \hat{l}, \hat{p}) = \langle \hat{Q}^h(s; \cdot; \hat{l}, \hat{p}), \pi^h(\cdot | s) \rangle \)
  end for
end for

return \( \{ \hat{Q}^h(s, a) \}_{h,s,a} \)

Observe that Theorem 6 has worst performance relatively to Theorem 5 w.r.t. the terms multiplying the \( \sqrt{K} \) term. However, its constant term has similar performance to the constant term in Theorem 5.

6 Conclusions and Summary

In this work, we formulated and analyzed different algorithms by which safety constraints can be combined in the framework of RL by combining learning in CMDPs. We investigated both UCRL-like approaches (Sec. 3 and 4) motivated by UCRL2 [Jaksch et al., 2010], as well as, optimistic dual and primal-dual approaches, motivated by practical successes of closely related algorithms [e.g., Achiam et al., 2017, Tessler et al., 2019]. For all these algorithms, we established regret guarantees for both the performance and constraint violations.

Interestingly, although the dual and primal-dual approaches are nowadays more practically acceptable, we uncovered an important deficiency of these methods; these have ‘weaker’ performance guarantees (Reg) relatively to UCRL-like algorithms (Reg+). This fact highlights an important practical message if an algorithm designer is interested in good performance w.r.t. Reg+. Furthermore, the primal-dual algorithm (section 5.2), which is computationally easier, has worse performance relatively to the optimistic dual algorithm (section 5.1). In light of these observations, we believe an important future venue is to further study the computational-performance tradeoff in safe RL. This would allow algorithm designers better understanding into the types of guarantees that can be obtained when using different types of safe RL algorithms.

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## A Optimistic Algorithm based on Bounded Parameter CMDPs

In this section, we establish regret guarantees for OptCMDP (Alg. 1). As a first step, we recall the algorithm and we formally states the confidence intervals. The empirical transition model, cost function and constraint cost functions are defined as in (10). We recall that OptCMDP constructs confidence intervals for the costs and the dynamics of the CMDP. Formally, for any \((s,a) \in \mathcal{S} \times \mathcal{A}\) we define

\[
B_{p,h,k}(s,a) = \left\{ \tilde{p}(s',a) \in \Delta_S : \forall s \in \mathcal{S}, |\tilde{p}(s,a) - \pi_h^{k-1}(s,a)| \leq \beta^{p}_{h,k}(s,a,s') \right\},
\]

\[
B_{c,h,k}(s,a) = \left\{ c_{h,k}^{-1}(s,a) - \beta_{h,k}^{c}(s,a) \leq c_{h,k}^{-1}(s,a) + \beta_{h,k}^{c}(s,a) \right\},
\]

\[
B_{d,h,k}(s,a) = \left\{ d_{h,k}^{-1}(s,a) - \beta_{h,k}^{d}(s,a) \leq d_{h,k}^{-1}(s,a) + \beta_{h,k}^{d}(s,a) \right\},
\]

where

\[
\beta^{p}_{h,k}(s,a,s') := 2 \sqrt{\frac{\text{Var}(\tilde{p}^{-1}(s'|s,a)) L^{p}_{\delta}}{n_{h,k}^{-1}(s,a) \lor 1}} + \frac{14/3 L^{p}_{\delta}}{n_{h,k}^{-1}(s,a) \lor 1},
\]

\[
\beta^{c}_{h,k} = \beta^{d}_{h,k} := \sqrt{\frac{L_{\delta}}{n_{h,k}^{-1}(s,a) \lor 1}},
\]

with \(L^{p}_{\delta} = \ln\left(\frac{8SAH_{\delta}}{\delta}\right)\), \(L_{\delta} = 2 \ln\left(\frac{8SAH_{\delta}(1+\gamma K)}{\delta}\right)\) and \(\text{Var}(\tilde{p}^{-1}(s'|s,a)) = \frac{\pi_h^{k-1}(s',a)}{\pi_h^{k-1}(s,a)} \cdot (1 - \pi_h^{k-1}(s'|s,a))\).

The set of plausible CMDPs associated with the confidence intervals is then \(\mathcal{M}_k = \{ M = (\mathcal{S}, \mathcal{A}, \tilde{c}, \tilde{d}, \tilde{p}) : \tilde{c}_{h,k}(s,a) \in B_{c,h,k}(s,a), \tilde{d}_{h,k}(s,a) \in B_{d,h,k}(s,a), \tilde{p}(s,a) \in B_{p,h,k}(s,a) \}\). In the next section, we define the good event under which \(M^* \in \mathcal{M}_k \) w.h.p.
A.1 Failure Events

Define the following failure events.

\[ F_k^p = \left\{ \exists s, a, s', h : |p_h(s' | s, a) - p_h^{k-1}(s' | s, a)| \geq \beta_{h,k}^p(s, a, s') \right\} \]

\[ F_k^N = \left\{ \exists s, a, h : n_h^{k-1}(s, a) \leq \frac{1}{2} \sum_{k < h} q_h^{n}(s, a | p) - H \ln \frac{S_A}{\delta} \right\} \]

\[ F_k^c = \left\{ \exists s, a, h : |F_h(s, a) - c_h(s, a)| \geq \beta_{h,k}^c(s, a) \right\} \]

\[ F_k^d = \left\{ \exists s, a, h, i \in [I] : |d_{i,h}(s, a) - d_{i,k}(s, a)| \geq \beta_{i,h,k}^d(s, a) \right\} \]

Furthermore, the following relations hold by standard arguments.

- Let \( F_{cd} = \bigcup_{k=1}^{K} F_k^c \cup F_k^d \). Then \( \Pr\{F_{cd}\} \leq \delta' \), by Hoeffding’s inequality, and using a union bound argument on all \( s, a, \) all possible values of \( n_h(s, a) \) for all \( i \in [I] \) and \( k \in [K] \). Furthermore, for \( n(s, a) = 0 \) the bound holds trivially since \( C, D_i \in [0, 1] \).

- Let \( F_P = \bigcup_{k=1}^{K} F_k^p \). Using Thm. 4 in [Maurer and Pontil, 2009], for every fixed \( s, a, h, k \) and value of \( n_h^k(s, a) \), we have that

\[
\Pr\{|p_h(s' | s, a) - p_h^{k-1}(s' | s, a)| \geq \epsilon_1 \} \leq \delta'',
\]

where

\[
\epsilon_1 = \sqrt{\frac{2 \text{Var}[p_h^{k-1}(s | s, a)] \ln \left( \frac{2}{\delta''} \right)}{n_h^{k-1}(s, a) - 1} + \frac{7 \ln \left( \frac{1}{\delta''} \right)}{3(n_h^{k-1}(s, a) - 1) - 1}}.
\]

See that for any \( n_h^k(s, a) \geq 2 \), we use Theorem 4 in [Maurer and Pontil, 2009], and for \( n_h^k(s, a) \in \{0, 1\} \) the bound holds trivially. This also implies that

\[
\Pr\{|p_h(s' | s, a) - p_h^{k-1}(s' | s, a)| \geq \epsilon_2 \} \leq \delta'',
\]

where

\[
\epsilon_2 = \sqrt{\frac{2 \text{Var}[p_h^{k-1}(s | s, a)] \ln \left( \frac{2}{\delta''} \right)}{n_h^{k-1}(s, a) - 1} + \frac{7 \ln \left( \frac{1}{\delta''} \right)}{3(n_h^{k-1}(s, a) - 1) - 1}}.
\]

since \( \epsilon_1 \leq \epsilon_2 \). Applying union bound on all \( s, a, h, \) and all possible values of \( n_h(s, a) \) and \( k \in [K] \) and set \( \delta'' = \frac{1}{(SAH_{KD})} \) we get that \( \Pr\{F_P\} \leq \delta' \). This analysis was also used in [Jin et al., 2019].

- Let \( F_N = \bigcup_{k=1}^{K} F_k^N \). Then \( \Pr\{F_N\} \leq \delta' \). The proof is given in [Dann et al., 2017, Cor. E.4].

Remark 3. Boundness of of immediate cost and constraints cost. Notice that we assumed that the random variables \( C_h(s, a) \in [0, 1] \) and \( D_{i,h}(s, a) \in [0, 1] \) for any \( s, a, h \).

Lemma 7 (Good event of OptCMDP). Setting \( \delta' = \frac{\delta}{4} \) then \( \Pr\{\mathcal{C}\} \leq \delta \) where

\[
\mathcal{C} = F_c \cup F_d \cup F_p \cup F_N = F_{cd} \cup F_P \cup F_N.
\]

When the failure events does not hold we say the algorithm is outside the failure event, or inside the good event \( G \) which is the complement of \( \mathcal{C} \).

The fact \( F_P \) holds conditioning on the good event implies the following result [e.g., Jin et al., 2019, Lem. 8].

Lemma 8. Conditioned on the basic good event, for all \( k, h, s, a, s' \) there exists constants \( C_1, C_2 > 0 \) for which we have that

\[
|F_h^{k-1}(s' | s, a) - p_h(s' | s, a)| = C_1 \sqrt{\frac{p_h(s' | s, a) L_{\delta,p}}{n_h^k(s, a) - 1}} + C_2 \frac{L_{\delta,p}}{n_h^k(s, a) - 1},
\]

where \( L_{\delta,p} = \ln \left( \frac{SAH_{KD}}{\delta} \right) \).
A.2 Optimism

Recall that $\bar{D} \in \mathbb{R}^{K \times SAH}$ and $\alpha \in \mathbb{R}^I$ such that $\bar{D} = [\bar{d}_1^T, \ldots, \bar{d}_I^T]^T$ and $\alpha = [\alpha_1, \ldots, \alpha_I]^T$, with $\bar{d}_k$ and $\alpha_k$ defined in [13].

Lemma 9 (Optimism). Conditioning on the good event, for any $\pi$ there exists a transition model $p' \in B_k^p$ for which (i) $\bar{D}_k q^\pi(p') \leq D q^\pi(p)$, and, (ii) $\tilde{c}_k^T q^\pi(p') \leq c^T q^\pi(p)$.

Proof. Conditioning on the good event, the true model $p$ is contained in $B_k^p$. Furthermore, conditioned on the good event $\bar{D}_k \leq D$ and $\tilde{c}_k \leq c$ component-wise. Thus, setting $p' = p \in B_k^p$ we get

$$\bar{D}_k q^\pi(p') = \bar{D}_k q^\pi(p) \leq D q^\pi(p)$$

$$\tilde{c}_k^T q^\pi(p') = c_k^T q^\pi(p) \leq c^T q^\pi(p),$$

where we used the fact that $q^\pi(p) \geq 0$ component-wise.

Lemma 10 ($\pi^*$ is Feasible Policy). Conditioning on the good event, $\pi^*$ is a feasible policy for any $k \in [K]$, i.e.,

$$\pi^* \in \{ \pi \in \Delta_A^S : \bar{D}_k q^\pi(p') \leq \alpha, p' \in B_k^p \}.$$ 

Proof. Denote $\Pi_D = \{ \pi : D q^\pi(p) \leq \alpha \}$ as the set of policies which does not violate the constraint on the true model. Furthermore, let

$$\Pi_D^k = \{ \pi : \bar{D}_k q^\pi(p') \leq \alpha, p' \in B_k^p \}$$

be the set of policies which do not violate the constraint w.r.t. all possible models at episode $k$. Observe that $\Pi_D^k$ is the set of feasible policies at episode $k$ for OptCMDP.

Conditioning on the good event, by Lemma 9 $D q^\pi(p) \leq \alpha$ implies that exists $p' \in B_k^p$ such that $\bar{D}_k q^\pi(p') \leq \alpha$. Thus,

$$\Pi_D \subseteq \Pi_D^k. \quad (21)$$

Since $\pi^* \in \Pi_D$ it implies that $\pi^* \in \Pi_D^k$.

From the two lemmas we arrive to the following important corollary

Corollary 11. Conditioning on the good event (i) $V_1^{\pi_k}(s_1; \tilde{c}_k, \bar{p}_k) \leq V_1^*(s_1)$, and, (ii) $V_1^{\pi_k}(s_1; \tilde{c}_k, \bar{p}_k) \leq V_1^{\pi_k}(s_1; c, p)$.

Proof. The following relations hold.

$$V^*(s_1) = \min_{\pi \in \Delta_A^S} \{ c^T q^\pi(p) \mid \pi \in \Pi_D \}$$

$$\geq \min_{\pi \in \Delta_A^S, p' \in B_k^p} \{ c^T q^\pi(p) \mid \pi \in \Pi_D^k \}$$

$$= \min_{\pi \in \Delta_A^S, p' \in B_k^p} \{ c^T q \mid \bar{D}_k q^\pi(p') \leq \alpha \}$$

$$\geq \min_{\pi \in \Delta_A^S, p' \in B_k^p} \{ \tilde{c}_k^T q^\pi(p') \mid \bar{D}_k q^\pi(p') \leq \alpha \} = V_1^{\pi_k}(s_1; \tilde{c}_k, \bar{p}_k).$$

The second relation holds by Lemma 9 and the forth relation holds by Lemma 9.

A.3 Proof of Theorem 3

In this section, we establish the following regret bounds for OptCMDP (see Alg. 1).

Theorem 3 (Regret Bounds for OptCMDP). Fix $\delta \in (0, 1)$. With probability at least $1 - \delta$ for any $K' \in [K]$ the following regret bounds hold

$$\text{Reg}_c(K'; c) \leq \bar{O} \left( \sqrt{SNH^4K} + (\sqrt{N} + H)H^2SA \right),$$

$$\text{Reg}_d(K'; d) \leq \bar{O} \left( \sqrt{SNH^4K} + (\sqrt{N} + H)H^2SA \right).$$
Proof. We start by conditioning on the good event. By Lem. it holds with probability at least 1 − δ.

We now analyze the regret relatively to the cost c. The following relations hold for any \( K^I \in [K] \).

\[
\text{Regret}^+(K^I; c) = \sum_k [V_1^{\pi_k}(s_1; c, p) - V_1^*(s_1; c, p)]_+ \leq \sum_k [V_1^{\pi_k}(s_1; c, p) - V_1^{\pi_k}(s_1; \tilde{c}_k, \tilde{p}_k)]_+ \\
= \sum_k V_1^{\pi_k}(s_1; c, p) - V_1^{\pi_k}(s_1; \tilde{c}_k, \tilde{p}_k) \\
\leq \tilde{O}(\sqrt{SNH^4K + (\sqrt{N} + H)H^2SA}).
\]

The second and third relations hold by optimism, i.e., Cor. The forth relation holds by Lem. See that assumptions 1, 2, 3 of Lem. are satisfied conditioning on the good event.

We now turn to prove the regret bound on the constraint violation. For any \( i \in [I] \) and \( K^I \in [K] \) the following relations hold.

\[
\sum_{k=1}^{K'} [V_1^{\pi_k}(s_1; d_i, p) - \alpha_i]_+ = \sum_{k=1}^{K'} \left[ V_1^{\pi_k}(s_1; d_i, p) - V_1^{\pi_k}(s_1; \tilde{d}_k^i, \tilde{p}_k) + V_1^{\pi_k}(s_1; \tilde{d}_k^i, \tilde{p}_k) - \alpha_i \right]_+ \\
\leq \sum_{k=1}^{K'} V_1^{\pi_k}(s_1; d_i) - V_1^{\pi_k}(s_1; \tilde{d}_k^i, \tilde{p}_k) \\
\leq \tilde{O}(\sqrt{SNH^4K + (\sqrt{N} + H)H^2SA}).
\]

The first relation holds since \( V_1^{\pi_k}(s_1; \tilde{d}_k^i, \tilde{p}_k) \leq \alpha \) as the optimization problem solved in every episode is feasible (see Lem. ). Furthermore, by optimism \( V_1^{\pi_k}(s_1; \tilde{d}_k^i, \tilde{p}_k) \leq V_1^{\pi_k}(s_1; d_i, p) \) (see the first relation of Lem. ). The third relation holds by applying Lem. See that assumptions (a), (b) and (c) of Lem. are satisfied conditioning on the good event (see also Lem. ).

B  Optimistic Algorithm based on Exploration Bonus

In this section, we establish regret guarantees for \( \text{OptCMDP-bonus} \) (see Alg. ). The main advantage of this algorithm w.r.t. \( \text{OptCMDP} \) is the computational complexity. While \( \text{OptCMDP} \) requires to solve an extended CMDP through an LP with \( O(S^2AH) \) constraints and decision variable, \( \text{OptCMDP-bonus} \) requires to find the solution of a single CMDP by solving an LP with \( O(SAH) \) constraints and variables.

At each episode \( k \), \( \text{OptCMDP-bonus} \) builds an optimistic CMDP \( M_k := (S, A, \tilde{c}_k^i, \tilde{d}_k^i, \tilde{p}_k^i) \) where

\[
\tilde{c}_k^i(s, a) = c_k^i(s, a) - b_k^i(s, a) \quad \text{and} \quad \tilde{d}_k^i(s, a) = d_k^i(s, a) - b_k^i(s, a),
\]

while \( \tilde{c}_k^i, \tilde{d}_k^i \) and \( \tilde{p}_k^i \) are the empirical estimates defined in (11). The exploration bonus \( b_k^i \) is defined as

\[
b_k^i(s, a) := \beta_k^c(s, a) + H \sum_{s'} \beta_k^p(s', s) + \sum_{s'} \beta_{k, a}^p(s', a)
\]

where \( \beta^c \) and \( \beta^p \) are defined as in (20).

The policy by which we act at episode \( k \) is given by solving the following optimization problem

\[
\pi_k, \tilde{p}_k = \arg \min_{\pi \in \Delta_A} \tilde{c}_k^T \pi q_{\pi}(\tilde{p}_{k-1}) \\
\text{s.t.} \quad \tilde{D} q_{\pi}(\tilde{p}_{k-1}) \leq \alpha
\]

where \( \tilde{D} = [\tilde{d}_1^T, \ldots, \tilde{d}_{H}^T]^T \) and \( \tilde{c}^T \) is defined as in (15). Solving this problem can be done by solving an LP, much similar to the LP by which a CMDP is solved (Section 2.3).

Before supplying the proof of Theorem we formally defining the set of good events which we show holds with high probability. Conditioning on the good, we establish the optimism of \( \text{OptCMDP-bonus} \) and then regret bounds for \( \text{OptCMDP-bonus} \).
B.1 Failure Events

We define the same set of good events as for \( \text{OptCMDP} \) (App. A.1). We restate this set here for convenience.

\[
F^p_k = \left\{ \exists s, a, s', h : |p_h(s' | s, a) - \bar{p}_h^{k-1}(s' | s, a)| > \beta_{h,k}^0(s, a, s') \right\}
\]
\[
F^N_k = \left\{ \exists s, a, h : n_h^k(s, a) \leq \frac{1}{2} \sum_{j<k} q_{h}^s(s, a | p) - H \ln \frac{SAH}{\delta^p} \right\}
\]
\[
F^c_k = \{ \exists s, a, h : |c_h(s, a) - c_h(s, a)| \geq \beta_{h,k}^0(s, a) \}
\]
\[
F^d_k = \{ \exists s, a, h, i \in [I] : |d_i(s, a, i) - c_h(s, a)| \geq \beta_{i,h,k}^d(s, a) \}
\]

As in App. A.1 the union of these events hold with probability greater than \( 1 - \delta \).

Lemma 12 (Good event of \( \text{OptCMDP} \)-bonus). Setting \( \delta' = \frac{\delta}{2} \) then \( \Pr(\overline{G}) \leq \delta \) where

\[
\overline{G} = F^c \cup F^d \cup F^p \cup F^N.
\]

When the failure events does not hold we say the algorithm is outside the failure event, or inside the good event \( G \) which is the complement of \( \overline{G} \).

Lemma 13. Conditioned on the basic good event, for all \( k, h, s, a, s' \) there exists constants \( C_1, C_2 > 0 \) for which we have that

\[
|\bar{p}_h^{k-1}(s' | s, a) - p_h(s' | s, a)| = C_1 \sqrt{p_h(s' | s, a)L_{\delta,p}} + \frac{C_2 L_{\delta,p}}{n_h^k(s, a) \vee 1},
\]

where \( L_{\delta,p} = \ln(\frac{SAH}{\delta^p}) \).

B.2 Optimism

Lemma 14 (Per-State Optimism.). Conditioning on the good event, for any \( \pi, s, a, h, k, i \in [I] \) it holds that

\[
\tilde{c}_h(s, a) - c_h(s, a) - \sum_{s'} (p_h - \bar{p}_h^{k-1})(s' | s, a)V_{h+1}(s'; c, p) \leq 0,
\]

and

\[
\tilde{d}_h(s, a) - d_h(s, a) - \sum_{s'} (p_h - \bar{p}_h^{k-1})(s' | s, a)V_{h+1}(s'; d, p) \leq 0.
\]

Proof. For any \( s, a, h, k \), conditioning on the good event,

\[
\sqrt{\pi_h(s, a) - c_h(s, a)} - b_{h,k}^{\pi}(s, a) \leq \frac{\sqrt{\pi_h(s, a) - c_h(s, a)} - b_{h,k}^{\pi}(s, a)}{\beta_{h,k}^0(s, a)} \leq 0
\]

by the choice of the bonus \( b_{h,k}^{\pi} \).

Furthermore, for any \( s, a, h, k \)

\[
(p_h - \bar{p}_h^{k-1})(s' | s, a)V_{h+1}(s') - b_{h,k}^{\pi}(s, a)
\]

\[
\leq \sum_{s'} |(p_h - \bar{p}_h^{k-1})(s' | s, a)||V_{h+1}(s'; d_i)| - b_{h,k}^{\pi}(s, a)
\]

\[
\leq H \sum_{s'} |(p_h - \bar{p}_h^{k-1})(s' | s, a)| - b_{h,k}^{\pi}(s, a)
\]

\[
\leq 2H \sum_{s'} \sqrt{p_h^{k-1}(s' | s, a)L_{p,\delta}} + \frac{14L_{p,\delta}}{3\left((n_h^k(s, a) - 1) \vee 1\right)} - b_{h,k}^{\pi}(s, a)
\]

\[
= b_{h,k}^{\pi}(s, a) - b_{h,k}^{\pi}(s, a) = 0,
\]
where the forth relation holds conditioning on the good event, and the fifth relation by the choice of the bonus $b_k^p(s, a)$.

Combining (23) and (24) we get that

$$\tilde{c}_h(s, a) - c_h(s, a) - (p_h - \tilde{\pi}_h^{k-1}):(s, a)V_{h+1}^\pi(\cdot; c, p) \leq 0.$$  

Repeating this analysis while replacing $c, \tilde{c}_k$ with $d_i, \tilde{d}_i, k$ we conclude the proof of the lemma. □

Lemma 15 (Optimism). Conditioning on the good event, for any $\pi, s, h, k, i$ it holds that (i) $V_h^\pi(s; \tilde{c}_k, \tilde{\pi}_k) \leq V_h^\pi(s; c, p)$, and, (ii) $V_h^\pi(s; \tilde{d}_i, \tilde{\pi}_k) \leq V_h^\pi(s; d_i, p)$.

Proof. For any $k \in [K]$ we have that

$$\begin{align*}
V^\pi(s_1; \tilde{c}_k, \tilde{\pi}_k) - V^\pi(s_1; c, p) &= \mathbb{E} \left[ \sum_{h=1}^H \tilde{c}_h(s_h, a_h) - c_h(s_h, a_h) - (p_h - \tilde{\pi}_h^{k-1})(s_h, a_h)V_{h+1}^\pi(\cdot; c, p) | s_1, \pi, \tilde{\pi}_k^{-1} \right] \\
&\leq \mathbb{E} \left[ \sum_{h=1}^H \tilde{d}_h(s_h, a_h) - d_h(s_h, a_h) - (p_h - \tilde{\pi}_h^{k-1})(s_h, a_h)V_{h+1}^\pi(\cdot; c, p) | s_1, \pi, \tilde{\pi}_k^{-1} \right] \\
&= V^\pi(s_1; \tilde{d}_i, \tilde{\pi}_k) - V^\pi(s_1; d_i, p)
\end{align*}$$

where we used the value difference lemma (see Lem. 33). Applying the first statement of Lem. 14 which hold for any $s, a, h, k$ (conditioning on the good event) we conclude the proof of the first claim.

The second claim follows by the same analysis on the difference $V_h^\pi(s; \tilde{d}_i, \tilde{\pi}_k) - V_h^\pi(s; d_i, p)$, i.e., using the value difference lemma and the second claim in Lem. 14. □

The following lemma shows that the problem solved by OptCMDP-bonus is always feasible. This lemma follows the same idea used to prove the feasibility for OptCMDP (see Lem. 16).

Lemma 16 ($\pi^*$ is Feasible Policy.). Conditioning on the good event, $\pi^*$ is a feasible policy for any $k \in [K]$, i.e.,

$$\pi^* \in \left\{ \pi \in \Delta^S : \tilde{D}_kq^\pi(\tilde{\pi}_k^{-1}) \leq \alpha \right\}.$$  

Proof. Denote $\Pi_D = \{ \pi : Dq^\pi(p) \leq \alpha \}$ as the set of policies which does not violate the constraint on the true model. Furthermore, let

$$\Pi_D^k = \{ \pi : \tilde{D}_kq^\pi(\tilde{\pi}_k^{-1}) \leq \alpha \}$$

be the set of policies which do not violate the constraint w.r.t. all possible models at the $k^{th}$ episode.

Conditioning on the good event, by Lem. 15 $Dq^\pi(p) \leq \alpha$ implies that $\tilde{D}_kq^\pi(\tilde{\pi}_k^{-1}) \leq \alpha$. Thus,

$$\Pi_D \subseteq \Pi_D^k.$$  

(25)

Since $\pi^* \in \Pi_D$ it implies that $\pi^* \in \Pi_D^k$. □

From the two lemmas we arrive to the following corollary as

Corollary 17. Conditioning on the good event (i) $V_1^{\pi_k}(s_1; \tilde{c}_k, \tilde{\pi}_k^{-1}) \leq V_1^*(s_1)$, and, (ii) $V_1^{\pi_k}(s_1; \tilde{c}_k, \tilde{\pi}_k^{-1}) \leq V_1^{\pi_k}(s_1; c, p)$.

Proof. The following relations hold.

$$\begin{align*}
V^*(s_1) &= \min_{\pi \in \Pi_D} \{ c^Tq^\pi(p) \mid \pi \in \Pi_D \} \\
&\geq \min_{\pi \in \Delta^S} \{ c^Tq^\pi(p) \mid \pi \in \Pi_D^k \} \\
&\geq \min_{\pi \in \Delta^S} \{ c^Tq^\pi(\tilde{\pi}_k^{-1}) \mid \tilde{D}_kq^\pi(\tilde{\pi}_k^{-1}) \leq \alpha \} \\
&\geq \min_{\pi \in \Delta^S} \{ c^Tq^\pi(\tilde{\pi}_k^{-1}) \mid \tilde{D}_kq^\pi(\tilde{\pi}_k^{-1}) \leq \alpha \} = V_1^{\pi_k}(s_1; \tilde{c}_k, \tilde{\pi}_k^{-1}).
\end{align*}$$

The second relation holds by Lem. 16 and the forth relation holds by Lem. 15. □
B.3 Proof of Theorem 4

In this section, we establish the following regret bounds for OptCMDP-bonus algorithm.

**Theorem 4 (Regret Bounds for OptCMDP-bonus).** Fix $\delta \in (0, 1)$. With probability at least $1 - \delta$ for any $K' \in [K]$ the following regret bounds hold

\[
\begin{align*}
\text{Reg}_+(K'; c) &\leq \hat{O}\left(\sqrt{SNH^4K} + S^2H^4A(NH + S)\right), \\
\text{Reg}_+(K'; d) &\leq \hat{O}\left(\sqrt{SNH^4K} + S^2H^4A(NH + S)\right).
\end{align*}
\]

Unlike the proof of the OptCMDP-bonus algorithm (Thm. 3), the value function is not constraint to be within $[0, H]$. However, since the bonus is bounded, the estimated value function is bounded in the range of $[-\sqrt{SH^2}, H]$. Although this discrepancy, in the following we are able to reach similar dependence in $\sqrt{K}$. The fact the estimated value is bounded in OptCMDP-bonus differently then in OptCMDP results in worse constant term as Thm. 4 exhibits (see Remark 2).

**Proof.** We start by conditioning on the good event. By Lem. 7, it holds with probability at least $1 - \delta$. We now analyze the regret relatively to the cost $c$. The following relations hold for any $K' \in [K]$:

\[
\begin{align*}
\text{Reg}_+(K'; c) &= \sum_k \left[ V^\pi_k(s_1; c, p) - V^*_i(s_1; c, p) \right]_+ \leq \sum_k \left[ V^\pi_k(s_1; c, p) - V^\pi_k(s_1; \tilde{c}_k, \tilde{p}_{k-1}) \right]_+ \\
&\leq \hat{O}\left(\sqrt{SNH^4K} + S^2H^4A(NH + S)\right).
\end{align*}
\]

The second and third relations hold by optimism, see Cor. 17. The forth relation holds by Lem. 31. See that assumptions 1,2,3 of Lem. 31 are satisfied conditioning on the good event. Assumption 4 of Lem. 31 holds by the optimism of the value estimate (see Lem. 15). Assumption 5 of Lem. 31 holds by Lem. 17.

We now turn to prove the regret bound on the constraint violation. For any $i \in [I]$ and $K' \in [K]$ the following relations hold.

\[
\begin{align*}
\sum_{k=1}^{K'} \left[ V^\pi_k(s_1; d_i) - \alpha \right]_+ &= \sum_{k=1}^{K} \left[ V^\pi_k(s_1; d_i, p) - V^\pi_k(s_1; \tilde{d}_i, \tilde{p}_{k-1}) \right]_+ \\
&\leq \sum_{k=1}^{K} V^\pi_k(s_1; d_i, p) - V^\pi_k(s_1; \tilde{d}_i, \tilde{p}_{k-1}) \\
&\leq \hat{O}\left(\sqrt{SNH^4K} + S^2H^4A(NH + S)\right).
\end{align*}
\]

The first relation holds since $V^\pi_k(s_1; d_i, \tilde{p}_{k-1}) \leq \alpha$ as the optimization problem solved in every episode is feasible, see Lem. 16. Furthermore, by optimism $V^\pi_k(s_1; \tilde{d}_i, \tilde{p}_{k}) \leq V^\pi_k(s_1; d_i, p)$ (see the first relation of Lem. 15). The third relation holds by applying Lem. 31. See that assumptions 1,2,3 of Lem. 31 are satisfied conditioning on the good event (see also Lem. 13).

\[ \square \]

C Constraint MDPs Dual Approach

In this section, we establish regret guarantees for OptDual-CMDP by proving Theorem 5. Unlike both previous sections, OptDual-CMDP does not require an LP solver, but repeatedly solves MDPs with uncertainty in their transition model.

Before supplying the proof of Theorem 5 we formally define the set of good events which we show holds with high probability. Conditioning on the good, we establish the optimism of OptDual-CMDP and then regret bounds for OptDual-CMDP. The regret bound of OptDual-CMDP relies on results from constraint convex optimization with some minor adaptations which we establish in Appendix 4.
C.1 Definitions

We introduce a notation that will be used across the proofs of this section. Following this notation allows us to apply generic results from convex optimization to the problem.

- The optimistic and true constraints valuation are denoted by
  \[ \tilde{g}_k = (\tilde{D}_k q^{\alpha}(\tilde{g}_k) - \alpha) \]
  \[ g_k = (Dq^{\alpha}(p) - \alpha). \]

- The optimistic value, true value, and optimal value are denoted by
  \[ \tilde{f}_k = c^T q^{\alpha}(\tilde{g}_k) \]
  \[ f_k = c^T q^{\alpha} \]
  \[ f_{opt} = V^*_1(s_1) = c^T q^*. \]

C.2 Failure Events

We define the same set of good events as for OptDual-CMDP (Appendix A.1). We restate this set here for convenience.

For any \( \delta, p \):

\[ F^p_k = \left\{ \exists s, a, s', h : |p_h(s' | s, a) - \tilde{p}_h^{k-1}(s' | s, a)| \geq \beta^p_{h,k}(s, a, s') \right\} \]

\[ F^N_k = \left\{ \exists s, a, h : n_h^{k-1}(s, a) \leq \frac{1}{2} \sum_{j<k} q_h^{c_j}(s, a | p) - H \ln \frac{SAH}{\delta} \right\} \]

\[ F^c_k = \left\{ \exists s, a, h : |c_h(s, a)| \geq \beta^c_{h,k}(s, a) \right\} \]

\[ F^d_k = \left\{ \exists s, a, h, i \in [I] : |d_{i,h}(s, a) - d_{i,h}(s, a)| \geq \beta^d_{i,h,k}(s, a) \right\} \]

As in Appendix A.1 the union of these events hold with probability greater than \( 1 - \delta \).

**Lemma 18** (Good event of OptDual-CMDP). Setting \( \delta' = \frac{\delta}{2} \) then \( \Pr(\overline{G}) \leq \delta \) where

\[ G = F^c \cup F^d \cup F^p \cup F^N. \]

When the failure events do not hold we say the algorithm is outside the failure event, or inside the good event \( G \) which is the complement of \( \overline{G} \).

**Lemma 19.** Conditioned on the basic good event, for all \( k, h, s, a, s' \) there exists constants \( C_1, C_2 > 0 \) for which we have that

\[ |\tilde{p}_h^{k-1}(s' | s, a) - p_h(s' | s, a)| = C_1 \sqrt{\frac{p_h(s' | s, a)L_{\delta,p}}{n_h^{k}(s, a) \vee 1} + \frac{C_2 L_{\delta,p}}{n_h^{k}(s, a) \vee 1}}, \]

where \( L_{\delta,p} = \ln \left( \frac{SAH}{\delta} \right) \).

C.3 Proof of Theorem 5

In this section, we establish the following regret bound for OptDual-CMDP.

**Theorem 5** (Regret Bounds for OptDual-CMDP). For any \( K' \in [K] \) the regrets the following bounds hold

\[ \text{Reg}(K'; e) \leq \hat{O} \left( \sqrt{SNH^4K} + \rho \sqrt{H^2TK} + (\sqrt{N} + H)H^2SA \right) \]

\[ \text{Reg}(K'; d) \leq \hat{O} \left( (1 + \frac{1}{\rho}) \left( \sqrt{SNH^4K} + (\sqrt{N} + H)\sqrt{TH^2SA} \right) \right). \]

We start by proving several useful lemmas on which the proof is based upon.

**Lemma 20** (Dual Optimism). Conditioning on the good event, for any \( k \in [K] \)

\[ \tilde{f}_k - f_{opt} \leq -\lambda_k^T g_k \]
Proof. We have that
\[ f_{opt} = c^T q^*(p) \geq c^T q^*(p) + \lambda^T_k (Dq^*(p) - \alpha) \]
\[ \geq \min_{\pi \in \Delta_k, p' \in P_k} c^T \pi' q^*(p') + \lambda^T_k (\tilde{D}_k q^*(\pi') - \alpha) \]
\[ = c^T \pi \tilde{q}^k (\tilde{\pi}_k) + \lambda^T_k (\tilde{D}_k q^*(\tilde{\pi}_k) - \alpha) \]
\[ = \tilde{f}_k + \lambda^T_k \tilde{\pi}_k. \]

The first relation holds since \(\pi^*\) satisfies the constraint (Assumption 1) which implies that \((Dq^*(p) - \alpha) \leq 0\), and that \(\lambda_k \geq 0\) by the update rule. The second relation holds since conditioning on the good event the true model is contained in \(B_k^c\) as well as \(\tilde{c}_k \leq c\).

**Lemma 21** (Update Rule Recursion Bound). For any \(\lambda \in \mathbb{R}^I_+\) and \(K' \in [K]\)
\[ \sum_{k=1}^{K'} (-\tilde{g}_k^T \lambda_k) + \sum_{k=1}^{N} \tilde{g}_k^T \lambda \leq \frac{t_\lambda}{2} \|\lambda_1 - \lambda\|_2^2 + \frac{1}{2t_\lambda} \sum_{k=1}^{K'} \|\tilde{\pi}_k\|^2 \]

Proof. For any \(\lambda \in \mathbb{R}^I_+\) by the update rule we have that
\[ \|\lambda_{k+1} - \lambda\|_2^2 = \|[\lambda_k + \frac{1}{t_\lambda} \tilde{g}_k]_+ - |\lambda|_+\|_2^2 \]
\[ = \|\lambda_k + \frac{1}{t_\lambda} \tilde{g}_k - \lambda\|_2^2 \]
\[ = \|\lambda_k - \lambda\|_2^2 + \frac{2}{t_\lambda} \tilde{g}_k^T (\lambda_k - \lambda) + \frac{1}{t_\lambda} \|\tilde{g}_k\|^2. \]

Summing this relation for \(k \in [K']\) and multiplying both sides by \(t_\lambda/2\) we get
\[ -\frac{t_\lambda}{2} \lambda_1 - \lambda \|_2^2 \leq \frac{t_\lambda}{2} \lambda_{k+1} - \lambda \|_2^2 - \frac{t_\lambda}{2} \lambda_1 - \lambda \|_2^2 \]
\[ \leq \sum_{k=1}^{K'} \tilde{g}_k^T (\lambda_k - \lambda) + \frac{1}{2t_\lambda} \sum_{k=1}^{K'} \|\tilde{g}_k\|^2. \]

Rearranging we get,
\[ \sum_{k=1}^{N} (-\tilde{g}_k^T \lambda_k) + \sum_{k=1}^{N} \tilde{g}_k^T \lambda \leq \frac{t_\lambda}{2} \|\lambda_1 - \lambda\|_2^2 + \frac{1}{2t_\lambda} \sum_{k=1}^{K'} \|\tilde{g}_k\|^2 \]
for any \(\lambda \in \mathbb{R}^I_+\).

We are now ready to establish Theorem 5.

**Proof.** Plugging Lemma 20 into Lemma 21 we get
\[ \sum_{k=1}^{K'} (\tilde{f}_k - f_{opt}) + \sum_{k=1}^{K'} \tilde{g}_k^T \lambda \leq \sum_{k=1}^{K'} (-\tilde{g}_k^T \lambda_k) + \sum_{k=1}^{K'} \tilde{g}_k^T \lambda \leq \frac{t_\lambda}{2} \|\lambda_1 - \lambda\|_2^2 + \frac{1}{2t_\lambda} \sum_{k=1}^{K'} \|\tilde{g}_k\|^2. \]

Adding, subtracting \(\sum_{k=1}^{K'} \tilde{g}_k^T \lambda\), \(\sum_{k=1}^{K'} f_k\) and rearranging we get
\[ \sum_{k=1}^{K'} (f_k - f_{opt}) + \sum_{k=1}^{K'} \tilde{g}_k^T \lambda \]
\[ \leq \frac{t_\lambda}{2} \|\lambda_1 - \lambda\|_2^2 + \frac{1}{2t_\lambda} \sum_{k=1}^{K'} \|\tilde{g}_k\|^2 + \sum_{k=1}^{K'} \|f_k - \tilde{f}_k\|^2 \]
\[ \leq \frac{t_\lambda}{2} \|\lambda_1 - \lambda\|_2^2 + \frac{1}{2t_\lambda} \sum_{k=1}^{K'} \|\tilde{g}_k\|^2 + \frac{1}{2} \left( \sum_{k=1}^{K'} \left( \|g_k - \tilde{g}_k\|^2 \right) \right) \|\lambda\|_2^2 + \sum_{k=1}^{K'} (f_k - \tilde{f}_k) \]
(26)
for any $\lambda \in \mathbb{R}_+^I$, where the last relation holds by Cauchy Schwarz inequality.

We now bound each term in \ref{eq:26}. Notice that $\tilde{g}_{k,i} = V^{x_k}(s_1; \tilde{d}_{k,i}, \tilde{p}_k) - \alpha_i \in [-L_H H, H]$ (where $L_H = 2\ln \left(\frac{6SAH}{\delta} \max I K \| \lambda \|_2 \right)$); it is a value function defined on an MDP with immediate cost in $[-L_H H, H]$ and $\alpha \in [0, H]$. Thus, we have that

$$\frac{1}{2t_\lambda} \sum_{k=1}^{K'} \| \tilde{g}_k \|^2 \lesssim \frac{H^2 I K}{2t_\lambda}.$$  

Applying Lemma \ref{lem:main} (see that assumptions (a), (b) and (c) hold conditioning on the good event), we get that

$$\left| \sum_{k=1}^{K'} (f_k - \tilde{f}_K) \right| = \sum_{k=1}^{K'} (V^{x_k}(s_1; c, p) - \tilde{V}^{x_k}(s_1; \tilde{c}_k, \tilde{p}_k)) \leq \tilde{O}(\sqrt{SNH^4 K} + (\sqrt{N} + H)H^2 S A),$$

$$\left| \sum_{k=1}^{K'} (g_{k,i} - \tilde{g}_{k,i}) \right| = \sum_{k=1}^{K'} (V^{x_k}(s_1; d_i, p; \tilde{c}_k) - \tilde{V}^{x_k}(s_1; \tilde{d}_{k,i}, \tilde{p}_k)) \leq \tilde{O}(\sqrt{SNH^4 K} + (\sqrt{N} + H)H^2 S A),$$

which implies that

$$\sqrt{\frac{1}{I} \sum_{i=1}^{I} \left( \sum_{k=1}^{K'} (g_{k,i} - \tilde{g}_{k,i}) \right)^2} \leq \tilde{O}(\sqrt{ISN H^4 K} + (\sqrt{N} + H)\sqrt{IH^2 S A}).$$

Plugging these bounds back into \ref{eq:26} and setting $t_\lambda = \sqrt{\frac{H^2 I K}{\rho}}$, we get

$$\sum_{k=1}^{K'} (f_k - f_{opt}) + \sum_{k=1}^{K'} g_k \lambda \lesssim \rho + \frac{\| \lambda \|_2^2}{\rho} \sqrt{H^2 I K} + \left( \sqrt{ISN H^4 K} + (\sqrt{N} + H)\sqrt{IH^2 S A} \right) \| \lambda \|_2$$

$$+ \left( \sqrt{SN H^4 K} + (\sqrt{N} + H)H^2 S A \right),$$

for any $\lambda \in \mathbb{R}_+^I$.

**First claim of Theorem 5.** Setting $\lambda = 0$ (see that $\lambda \in \mathbb{R}_+^I$) in \ref{eq:27} we get

$$\sum_{k=1}^{K'} V^{x_k}(s_1; c, p) - V^*(s_1) = \sum_{k=1}^{K'} f_k - f_{opt} \lesssim \tilde{O}(\sqrt{ISN H^4 K} + \sqrt{H^2 I K} + (\sqrt{N} + H)H^2 S A).$$

**Second claim of Theorem 5.** Fix $i \in [I]$ and let

$$\overline{\lambda}_i = \begin{cases} \rho c_i & \frac{\sum_{k=1}^{K'} g_k}{K'} \downarrow < 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $c_i(j) = 1$ and $c_i(j) = 0$ for $j \neq i$, and $\rho$ is given in Assumption \ref{lem:main}. See that $\overline{\lambda}_i \in \mathbb{R}_+^I$ and that, by the definition,

$$\| \overline{\lambda}_i \|_2^2 \leq \rho^2.$$  

Setting $\lambda = \overline{\lambda}_i$ in \ref{eq:27} we get

$$\sum_{k=1}^{K'} (f_k - f_{opt}) + \rho \sum_{k=1}^{K'} g_{k,i} \lesssim \tilde{O}\left( (1 + \rho) \left( \sqrt{ISN H^4 K} + \sqrt{H^2 I K} + (\sqrt{N} + H)\sqrt{IH^2 S A} \right) \right) := \epsilon(K).$$
Now, by the convexity of the state-action frequency (see Proposition \ref{prop:conv_objective}) function there exists a policy \( \pi_{K'} \) which satisfies
\[
q_{K'} \quad \text{for any} \quad s, a, h, i
\]
we have that
\[
\sum_{k=1}^{K'} \sum_{i,k} g_k \leq \max_{i \in [I]} \left[ \sum_{k=1}^{K'} g_k \right]
\]
for any \( K' \in [K] \).

**Remark 4 (Convexity of the RL Objective Function).** Although it is common to refer to the objective function in RL as non-convex, in the state action visitation polytope the objective is linear and, hence, convex (however, the problem is constraint to the state action visitation polytope). Thus, we can use Theorems \[42\] and Cor. \[44\] which are valid for constraint convex problems.

## D Constraint MDPs Primal Dual Approach

In this section we establish regret guarantees for OptPrimalDual-CMDP by proving Theorem \[6\]. Unlike for OptDual-CMDP, OptPrimalDual-CMDP requires an access to a (truncated) policy estimation algorithm which returns \( \tilde{Q}_k^g(s, a; \pi_h) \), \( Q_k^g(s, a; d_{i,k}, \pi_h) \), i.e., the \( Q \)-function w.r.t. to the empirical transition model and optimistic cost and constraint cost. This reduces the computational complexity of OptPrimalDual-CMDP. However, it results in worse performance guarantees relatively to OptDual-CMDP.

Before supplying the proof of Theorem \[6\] we formally define the set of good events which we show holds with high probability. Conditioning on the good, we establish the optimism of OptPrimalDual-CMDP and then regret bounds for OptPrimalDual-CMDP. The regret bounds of OptPrimalDual-CMDP relies on results from constraint convex optimization with some minor adaptations which we establish in Appendix \[C\].

### D.1 Failure Events

We define the same set of good events as for UCRL-OptCMDP (Appendix \[A.1\]). We restate this set here for convenience.

\[
F_k^p = \left\{ \exists s, a, s', h : |p_h(s' | s, a) - \tilde{p}_h^{-1}(s' | s, a)| \geq \beta_{p,h,k}(s, a) \right\}
\]
\[
F_k^N = \left\{ \exists s, a, h : n_h^{-1}(s, a) \leq \frac{1}{2} \sum_{j < k} q_j^g(s, a | p) - H \ln \frac{SAH}{\delta} \right\}
\]
\[
F_k^c = \left\{ \exists s, a, h : |\tilde{c}_h(s, a) - c_h(s, a)| \geq \beta_{c,h,k}(s, a) \right\}
\]
\[
F_k^{d} = \left\{ \exists s, a, h, i \in [I] : |d_i^j(s, a) - d_{i,h}(s, a)| \geq \beta_{i,h,k}^d(s, a) \right\}
\]

As in Appendix \[A.1\] the union of these events hold with probability greater than 1 - \( \delta \).
Lemma 22 (Good event of OptimalDualCMDP). Setting \( \delta' = \frac{\delta}{4} \) then \( \Pr\{\mathcal{G}\} \leq \delta \)

\[
\mathcal{G} = F^e \cup F^d \cup F^p \cup F^N.
\]

When the failure events does not hold we say the algorithm is outside the failure event, or inside the good event \( \mathcal{G} \) which is the complement of \( \mathcal{G} \).

Lemma 23. Conditioned on the basic good event, for all \( k, h, s, a, s' \) there exists constants \( C_1, C_2 > 0 \) for which we have that

\[
\left| \tilde{p}_{h}^{k-1}(s' | s, a) - p_h(s' | s, a) \right| = C_1 \sqrt{\frac{p_h(s' | s, a) L_{\delta, p}}{n_h^k(s, a) \vee 1}} + \frac{C_2 L_{\delta, p}}{n_h^k(s, a) \vee 1},
\]

where \( L_{\delta, p} = \ln \left( \frac{4 H K}{\delta A H K} \right) \).

D.2 Optimality and Optimism

Lemma 24 (On Policy Optimality.). Conditioning on the good event, for any \( k \in [K'] \)

\[
\sum_{k=1}^{K'} \tilde{f}_k + \lambda_k^T \tilde{g}_k - f_{\pi^*} - \lambda_k^T g_{\pi^*} \leq \tilde{O}(\sqrt{H^4(1 + I_p)^2 K})
\]

Proof. By definition,

\[
f_{\pi^*} + \lambda_k^T g_{\pi^*} = V_1^\pi(s_1; c, p) + \sum_{i=1}^{I} \lambda_k(V_1^\pi(s_1; d_i, p) - \sum_{i=1}^{I} \lambda_k \alpha_i
\]

\[
\tilde{f}_k + \lambda_k^T \tilde{g}_k = \tilde{V}_1^\pi(s_1; \tilde{c}_k, \tilde{p}_k) + \sum_{i=1}^{I} \lambda_k(V_1^\pi(s_1; \tilde{d}_{k,i}, \tilde{p}_k) - \sum_{i=1}^{I} \lambda_k \alpha_i
\]

Let

\[
Q_h^k(s, a) := Q_h^\pi(s, a; \tilde{c}_k, \tilde{p}_{k-1}) + \sum_{i=1}^{I} \lambda_{k,i} Q_h^\pi(s, a; \tilde{d}_{k,i}, \tilde{p}_{k-1})
\]

\[
V_h^k(s_1) := (Q_h^k(s, \cdot), \pi_h^k)
\]

Applying the extended value difference lemma \[\text{in the text}\] we get that

\[
\sum_{k=1}^{K'} \tilde{f}_k + \lambda_k^T \tilde{g}_k - f_{\pi^*} - \lambda_k^T g_{\pi^*}
\]

\[
= \sum_{k=1}^{K'} V_k^k(s_1) - V_1^\pi(s_1; c + \lambda_k \tilde{d}, p)
\]

\[
= \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}\left[ \langle Q_h^k(s_h, \cdot), \pi_h^k(\cdot | s_h) - \pi_h^\pi(\cdot | s_h) \rangle | s_1 = s_1, \pi^*, p \right] \tag{1}
\]

\[
+ \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}\left[ Q_h^k(s_h, a_h) - c_h(s_h, a_h) - \sum_{i=1}^{I} \lambda_{h,i} d_{h,i}(s_h, a_h) - p_h(\cdot | s_h, a_h) V_{h+1}^k | s_1 = s_1, \pi^*, p \right]. \tag{2}
\]

To bound (1), we apply Lemma \[\text{in the text}\] while setting \( \pi = \pi^* \).

\[
(i) = \sum_{k=1}^{K'} \sum_{h=1}^{H} \mathbb{E}\left[ \langle Q_h^k(s_h, \cdot), \pi_h^k(\cdot | s_h) - \pi_h^\pi(\cdot | s_h) \rangle | s_1 = s_1, \pi^*, p \right] \lesssim \sqrt{H^4(1 + I_p)^2 K}, \tag{29}
\]
To bound (ii), observe that by Lemma 25 for all $s, a, h, k$ it holds that

$$Q_h^k (s,a) - c_h (s,a) - \sum_{i=1}^{I} \lambda_k d_{h,i} (s,a) - p_h (\cdot | s,a) V_{h+1}^k \leq 0.$$ 

This implies that

$$(ii) \leq 0$$

(30)

since (ii) is an expectation over negative terms. Combining (29) and (30) we conclude that

$$\sum_{k=1}^{K'} \left( \tilde{f}_k + \lambda_k T \tilde{g}_k \right) \leq \tilde{f}_1 + \lambda_1 T \tilde{g}_1 - \lambda_1 g_{\pi^*} \leq \sum_{k=1}^{K'} V_{h+1}^k (s_1) - V_{h}^k (s_1; c + \lambda_k \tilde{d}, p) \leq \sqrt{H^4(1+I\rho)^2K}.$$ 

\[ \square \]

**Lemma 25 (Policy Estimation Optimism).** Conditioning on the good event, for any $s, a, h, k$ the following bound holds

$$Q_h^k (s,a) - c_h (s,a) - \sum_{i=1}^{I} \lambda_k d_{h,i} (s,a) - p_h (\cdot | s,a) V_{h+1}^k \leq 0,$$

where

$$Q_h^k (s,a) = Q_h^{\pi_k} (s,a; \tilde{c}_k, \bar{\pi}_{k-1}) + \sum_{i=1}^{I} \lambda_k, i, Q_h^{\pi_k} (s,a; \tilde{d}_{k,i}, \bar{\pi}_{k-1}),$$

$$V_h^k (s) = (Q_h^k (s,a), \pi_h^k (\cdot | s)).$$

See that $Q_h^{\pi_k} (s,a; \tilde{c}_k, \bar{\pi}_{k-1}), Q_h^{\pi_k} (s,a; \tilde{d}_{k,i}, \bar{\pi}_{k-1})$ are defined in the update rule of OptPrimalDual-CMDP (Algorithm 4).

**Proof.** For all $s, a, h, k$ the following relations hold.

$$Q_h^k (s,a) - c_h (s,a) - \sum_{i=1}^{I} \lambda_k d_{h,i} (s,a) - p_h (\cdot | s,a) V_{h+1}^k$$

$$= Q_h^{\pi_k} (s,a; \tilde{c}_k, \bar{\pi}_{k-1}) + \sum_{i=1}^{I} \lambda_k, i, Q_h^{\pi_k} (s,a; \tilde{d}_{k,i}, \bar{\pi}_{k-1})$$

$$- c_h (s,a) - \sum_{i=1}^{I} \lambda_k d_{h,i} (s,a) - p_h (\cdot | s,a) \left( V_{h+1}^k (\cdot ; \tilde{c}_k, \bar{\pi}_{k-1}) + \sum_{i=1}^{I} \lambda_k, i, V_h^{\pi_k} (\cdot ; \tilde{d}_{k,i}, \bar{\pi}_{k-1}) \right),$$

(33)

where $V_h^{\pi_k} (\cdot ; \tilde{c}_k, \bar{\pi}_{k-1}) := (Q_h^{\pi_k} (s,\cdot; \tilde{c}_k, \bar{\pi}_{k-1}), \pi_h^{\pi_k} (\cdot | s)), V_h^{\pi_k} (\cdot; \tilde{d}_{k,i}, \bar{\pi}_{k-1}) := (Q_h^{\pi_k} (s,\cdot; \tilde{d}_{k,i}, \bar{\pi}_{k-1}), \pi_h^{\pi_k} (\cdot | s)).$

Furthermore, see that

$$Q_h^{\pi_k} (s,a; \tilde{c}_k, \bar{\pi}_{k-1}) = \max \{ 0, \tilde{c}_{h,-1} (s,a) + \bar{\pi}_{h,-1} (\cdot | s,a) V_h^{\pi_{h+1}} (\cdot ; \tilde{c}_k, \bar{\pi}_h) \}$$

$$= \max \{ 0, \tilde{c}_{h,-1} (s,a) - b_{h,k-1} (s,a) - b_{h,k-1} (s,a) + \bar{\pi}_{h,-1} (\cdot | s,a) V_h^{\pi_{h+1}} (\cdot ; \tilde{c}_k, \bar{\pi}_h) \}$$

$$\leq \max \{ 0, \tilde{c}_{h,-1} (s,a) - b_{h,k-1} (s,a) \}$$

$$+ \max \{ 0, -b_{h,k-1} (s,a) + \bar{\pi}_{h,-1} (\cdot | s,a) V_h^{\pi_{h+1}} (\cdot ; \tilde{c}_k, \bar{\pi}_h) \},$$

(34)

since $\max \{ 0, a + b \} \leq \max \{ 0, a \} + \max \{ 0, b \}$. Similarly, for any $i \in [I],$

$$Q_h^{\pi_k} (s,a; \tilde{d}_{i,k}, \bar{\pi}_{k-1}) \leq \max \{ 0, \tilde{d}_{i,-1,k} (s,a) - b_{h,k-1} (s,a) \}$$

$$+ \max \{ 0, -b_{h,k-1} (s,a) + \bar{\pi}_{h,-1} (\cdot | s,a) V_h^{\pi_{h+1}} (\cdot ; \tilde{d}_{i,k}, \bar{\pi}_h) \}.$$

(35)
Proof. Lemma 26

Conditioning on the good event. The forth relation holds by the choice of $\pi_k(s, a) = \frac{\nabla_{\pi_k}^{\theta_1} \rho(s_i, a) + \sum_{h=1}^{H} \nabla_{\pi_k}^{\theta_2} \rho(s_i, a)}{\sum_{h=1}^{H} \nabla_{\pi_k}^{\theta_2} \rho(s_i, a)}$.

Plugging (34) and (35) into (33) we get

$$Q_h^k(s, a) - c_h(s, a) - p_h(\cdot | s, a)V_{h+1}^k$$

$$\leq \max\left\{0, \pi_k^{-1}(s, a) - b_{h, k-1}(s, a)\right\} - c_h(s, a)$$

$$+ \max\left\{0, -b_{h, k-1}(s, a) + \pi_k^{-1}(s, a) - d_h, p_h(\cdot | s, a)\right\} - p_h(\cdot | s, a)V_{h+1}^k$$

$$+ \sum_{i=1}^{I} \lambda_k, i \left(\max\left\{0, \pi_k^{-1}(s, a) - b_{h, k-1}(s, a)\right\} - d_h, p_h(\cdot | s, a)\right)$$

$$+ \sum_{i=1}^{I} \lambda_k, i \left(\max\left\{0, -b_{h, k-1}(s, a) + \pi_k^{-1}(s, a) - d_h, p_h(\cdot | s, a)\right\} - p_h(\cdot | s, a)V_{h+1}^k\right)$$

We now show each of these terms is negative conditioning on the good event.

$$\max\left\{0, \pi_k^{-1}(s, a) - b_{h, k-1}(s, a)\right\} - c_h(s, a)$$

$$- b_{h, k-1}(s, a) + \pi_k^{-1}(s, a) - d_h, p_h(\cdot | s, a)\right\} - p_h(\cdot | s, a)V_{h+1}^k$$

$$\leq b_{h, k-1}(s, a) + \sum_{s'} (\pi_k^{-1}(s) - p_h(s' | s, a)) |V_{h+1}^k(s') - (\pi_k^{-1}(s))|$$

$$= b_{h, k-1}(s, a) + 2H \sum_{s'} (\pi_k^{-1}(s) - p_h(s' | s, a)) |V_{h+1}^k(s') - (\pi_k^{-1}(s))|$$

The second relation holds since $V_{h+1}^k(s') = (\pi_k^{-1}(s'))(|\pi_k^{-1}(s)), 0) \leq b_{h, k-1}(s, a)$ by the update rule (OptPrimalDual-$\pi$MDP uses truncated policy evaluation, see Algorithm 2). The third relation holds conditioning on the good event. The forth relation holds by the choice of $\pi_k^{-1}(s, a)$.

Similarly, we get that each term in the sums at (33), (39) is non-positive. Since $\lambda_k \geq 0$ we conclude that both (38) and (39) are non-positive. Thus, we establish that

$$Q_h^k(s, a) - c_h(s, a) - p_h(\cdot | s, a)V_{h+1}^k \leq 0.$$

Lemma 26 (OMD Term Bound). Conditioned on the good event, we have that for any $\pi$

$$\sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[\left(Q_h^k(s_h, \cdot), \pi_h^k(\cdot | s_h) - \pi_h(\cdot | s_h)\right) | s_1 = s, \pi, \rho\right] \leq 2H^2(1 + I\rho)^2K \log A.$$

Proof. This term accounts for the optimization error, bounded by the OMD analysis.

By standard analysis of OMD [Orabona, 2019] with the KL divergence used as the Bregman distance (see Lemma 20) we have that for any $s, h$ and for policy $\pi$,

$$\sum_{k=1}^{K} \mathbb{E} \left[Q_h^k(\cdot | s), \pi_h^k(\cdot | s) - \pi_h(\cdot | s)\right] \leq \frac{\log A}{t_K} + \frac{t_K}{2} \sum_{k=1}^{K} \sum_{a} \pi_h^k(a | s) (Q_h^k(a | s))^2$$

(41)
where $t_K$ is a fixed step size.

By the form of $Q^h$, we get that $Q^h \geq 0$ since it is a sum of positive terms (policy evaluation is done with truncated policy evaluation, see Algorithm 4). Furthermore, we upper bound $Q^h$ for any $s, a, h, k$ as follows,

\[
Q^h_k(s, a) := Q^h(s, a; \tilde{c}_k, \tilde{p}_{k-1}) + \sum_{i=1}^I \lambda_{k,i}Q^h(s, a; \tilde{d}_{k,i}, \tilde{p}_{k-1}) \\
\leq H + H \sum_{i=1}^I \lambda_{k,i} \leq H + H \rho.
\]

The second relation holds by the fact that $Q^h(s, a; \tilde{c}_k, \tilde{p}_{k-1}), Q^h(s, a; \tilde{d}_{k,i}, \tilde{p}_{k-1}) \leq H$ by the update rule (both $\tilde{c}_k, \tilde{d}_{k,i} \leq 1$, thus, an expectation over an $H$ such terms is smaller than $H$) and the fact $\lambda_k \geq 0$ (by the update rule).

Plugging this bound into we get that for any $s, a, h$

\[
\sum_{k=1}^{K'} \langle Q^h_k(s, \cdot), \pi^h_k(\cdot | s) - \pi_h(\cdot | s) \rangle \leq \frac{\log A}{t_K} + \frac{t_K H^2 (1 + \rho)^2 K}{2}.
\]  

Thus, the following relations hold.

\[
\sum_{k=1}^K \sum_{h=1}^H \mathbb{E}[(Q^h_k(s_h, \cdot), \pi^h_k(\cdot | s_h) - \pi_h(\cdot | s_h)) | s_1 = s, \pi, p]
\]

\[
= \sum_{h=1}^H \mathbb{E} \sum_{k=1}^K \langle Q^h_k(s_h, \cdot), \pi^h_k(\cdot | s_h) - \pi_h(\cdot | s_h) \rangle | s_1 = s, \pi, p
\]

\[
\leq \sum_{h=1}^H \mathbb{E} \left[ \frac{\log A}{t_K} + t_K H^2 K | s_1 = s, \pi \right] = \frac{H \log A}{t_K} + \frac{t_K H^2 (1 + \rho)^2 K}{2}.
\]

See that the first relation holds as the expectation does not depend on $k$. Thus, by linearity of expectation, we can switch the order of summation and expectation. The second relation holds since holds for any $s$.

Finally, by choosing $t_K = \sqrt{2 \log A / (H^2 (1 + \rho)^2 K)}$, we obtain

\[
\sum_{k=1}^K \sum_{h=1}^H \mathbb{E}[(Q^h_k(s_h, \cdot), \pi^h_k(\cdot | s_h) - \pi_h(\cdot | s_h)) | s_1 = s, \pi, p] \leq \sqrt{2 H^4 (1 + \rho)^2 K \log A}.
\]

\[
\square
\]

### D.3 Proof of Theorem 6

In this section, we establish the following regret bound for OptPrimalDual-CMDP.

**Theorem 6** (Regret Bounds for OptPrimalDual-CMDP). For any $K' \in [K]$ the regrets the following bounds hold

\[
\text{Reg}(K'; c) \leq \bar{O}\left(\sqrt{SNH^4 K} + \sqrt{H^4 (1 + \rho)^2 K} + (\sqrt{N} + H)H^2 SA\right)
\]

\[
\text{Reg}(K'; d) \leq \bar{O}\left((1 + \frac{1}{\rho})\left(\sqrt{1SNH^4 K} + (\sqrt{N} + H)\sqrt{TH^2 SA}\right) + 1 \sqrt{H^4 K}\right).
\]

We start by proving several useful lemmas on which the proof is based upon.

**Lemma 27** (Dual Optimism). Conditioning on the good event, for any $k \in [K']$

\[
\tilde{f}_k - f_{opt} \leq -\lambda_k^T \hat{g}_k + \left(\tilde{f}_k + \lambda_k^T \hat{g}_k - f_{opt} - \lambda_k^T \hat{g}_{opt} \right)
\]

\[
= \left(\tilde{f}_k - f_{opt} \right) - \lambda_k^T \hat{g}_k + \left(\lambda_k^T \hat{g}_k - f_{opt} - \lambda_k^T \hat{g}_{opt} \right).
\]
Proof. We have that
\[ f_{opt} = c^T q^* (p) \geq c^T q^* (p) + \lambda^T_k (Dq^* (p) - \alpha) \]
\[ = f_k + \lambda^T_k g_{\pi^*} \]
\[ = \tilde{f}_k + \lambda^T_k \tilde{g}_k + f_{\pi^*} + \lambda^T_k g_{\pi^*} - \tilde{f}_k - \lambda^T_k \tilde{g}_k. \]

The first relation holds since \( \pi^* \) satisfies the constraint (Assumption 1) which implies that \( (Dq^* (p) - \alpha) \leq 0 \), and that \( \lambda_k \geq 0 \) by the update rule.

We now state a lemma which corresponds to Lemma 11 from previous section.

**Lemma 28** (Update Rule Recursion Bound Primal-Dual). For any \( \lambda \in \{ \lambda \in \mathbb{R}^I : 0 \leq \lambda \leq \rho 1 \} \) and \( K' \in [K] \)
\[ \sum_{k=1}^{K'} (-g^T_k \lambda_k) + \sum_{k=1}^{K'} \tilde{g}^T_k \lambda \leq \frac{t_n}{2} \| \lambda_1 - \lambda \|^2 + \frac{1}{2t_n} \sum_{k=1}^{K'} \| \tilde{g}_k \|^2. \]

**Proof.** Similar proof to Lemma 11 while using the fact that projection to the set \( \{ \lambda \in \mathbb{R}^I : 0 \leq \lambda \leq \rho 1 \} \) is non-expansive operator as the operator \([x]_+\).

We are now ready to establish Theorem 6.

**Proof.** Applying Lemma 11 into Lemma 28 we get
\[ \sum_{k=1}^{K'} (\tilde{f}_k - f_{opt}) + \sum_{k=1}^{K'} \tilde{g}^T_k \lambda \]
\[ \leq \sum_{k=1}^{K'} (-\tilde{g}^T_k \lambda_k) + \sum_{k=1}^{K'} \tilde{g}^T_k \lambda + \sum_{k=1}^{K'} \tilde{f}_k + \lambda^T_k \tilde{g}_k - f_{\pi^*} - \lambda^T_k g_{\pi^*}. \]
\[ \leq \frac{t_n}{2} \| \lambda_1 - \lambda \|^2 + \frac{1}{2t_n} \sum_{k=1}^{K'} \| \tilde{g}_k \|^2 + \sum_{k=1}^{K'} \tilde{f}_k + \lambda^T_k \tilde{g}_k - f_{\pi^*} - \lambda^T_k g_{\pi^*}. \]

Adding, subtracting \( \sum_{k=1}^{K'} g^T_k \lambda \), \( \sum_{k=1}^{K'} f_k \) and rearranging we get
\[ \sum_{k=1}^{K'} (f_k - f_{opt}) + \sum_{k=1}^{K'} \tilde{g}^T_k \lambda \]
\[ \leq \frac{t_n}{2} \| \lambda \|^2 + \frac{1}{2t_n} \sum_{k=1}^{K'} \| \tilde{g}_k \|^2 + \sum_{k=1}^{K'} (g_k - \tilde{g}_k)^T \lambda + \sum_{k=1}^{K'} (f_k - \tilde{f}_k) \]
\[ + \sum_{k=1}^{K'} \tilde{f}_k + \lambda^T_k \tilde{g}_k - f_{\pi^*} - \lambda^T_k g_{\pi^*}. \]
\[ \leq \frac{t_n}{2} \| \lambda \|^2 + \frac{1}{2t_n} \sum_{k=1}^{K'} \| \tilde{g}_k \|^2 + \left( \sum_{i=1}^I \left( \sum_{k=1}^{K'} (g_{k,i} - \tilde{g}_{k,i}) \right) \right)^2 \| \lambda \|_2 + \sum_{k=1}^{K'} (f_k - \tilde{f}_k) \]
\[ + \sum_{k=1}^{K'} \tilde{f}_k + \lambda^T_k \tilde{g}_k - f_{\pi^*} - \lambda^T_k g_{\pi^*}. \]

for any \( \lambda \in \mathbb{R}^I_+ \), where the last relation holds by Cauchy Schwartz inequality.

We now bound each term in (44). Since \( \tilde{g}_k \in [-H, H] \)
\[ \frac{1}{2t_n} \sum_{k=1}^{K'} \| \tilde{g}_k \|^2 \leq \frac{H^2 IK}{2t_n}. \]
Applying Lemma 33 (see that assumptions (1),(2),(3) hold conditioning on the good event), we get that
\[
\left| \sum_{k=1}^{K'} (f_k - \hat{f}_k) \right| = \sum_{k=1}^{K'} \left| (V^{\pi_k}(s_1; c, p) - \hat{V}^{\pi_k}(s_1; \tilde{c}_k, \tilde{\pi}_k)) \right| \leq \tilde{O} \left( \sqrt{SNH^4K} + (\sqrt{N} + H)H^2SA \right)
\]
which implies that
\[
\left| \sum_{k=1}^{K'} (g_{k,i} - \bar{g}_{k,i}) \right| = \sum_{k=1}^{K'} \left| (V^{\pi_k}(s_1; d_i, p) - \hat{V}^{\pi_k}(s_1; \tilde{d}_{k,i}, \tilde{\pi}_k)) \right| \leq \tilde{O} \left( \sqrt{SNH^4K} + (\sqrt{N} + H)H^2SA \right),
\]
Lastly, by Lemma 24
\[
\sum_{k=1}^{K'} \tilde{f}_k + \lambda_k^T \bar{g}_k - f_{\pi^*} - \lambda_k^T g_{\pi^*} \lesssim \sqrt{H^4(1 + \rho)^2K}.
\]
Plugging these bounds back into \((44)\) and setting \(t_\lambda = \sqrt{\frac{H^4K}{\rho^2}}\) we get
\[
\sum_{k=1}^{K'} (f_k - f_{opt}) + \sum_{k=1}^{K'} g_k^T \lambda \\
\lesssim (\rho + \frac{\|\lambda\|_2^2}{\rho}) \sqrt{H^2IK} \left( \sqrt{ISN^4K} + (\sqrt{N} + H)\sqrt{H^2SA} \right) \|\lambda\|_2 \\
+ \left( \sqrt{SNH^4K} + (\sqrt{N} + H)H^2SA \right) + \sqrt{H^4(1 + \rho)^2K}, \tag{45}
\]
for any \(0 \leq \lambda \leq \rho 1.\)

**First claim of Theorem 8**. Fix \(\lambda = 0\) which satisfies \(0 \leq \lambda \leq \rho 1\) in \((45)\) we get
\[
\sum_{k=1}^{K'} V^{\pi_k}(s_1; c, p) - V^*(s_1) = \sum_{k=1}^{K'} f_k - f_{opt} \\
\leq \tilde{O} \left( \sqrt{SNH^4K} + (\sqrt{N} + H)\sqrt{H^2SA} \right).
\]

**Second claim of Theorem 8**. Fix \(i \in [I]\) and let
\[
\lambda_i = \begin{cases} 
\rho e_i & \text{if } \sum_{k=1}^{K'} g_{i,k} \neq 0 \\
0 & \text{otherwise}
\end{cases}
\]
where \(e_i(j) = 1\) and \(e_i(j) = 0\) for \(j \neq i\), and \(\rho\) is given in Assumption 5. See that \(0 \leq \lambda_i \leq \rho 1.\) Furthermore, it holds that
\[
\|\lambda_i\|_2^2 \leq \rho^2 \tag{46}
\]
Set \(\lambda = \lambda_i \) in \((45)\) we get
\[
\sum_{k=1}^{K'} (f_k - f_{opt}) + \rho \left[ \sum_{k=1}^{K'} g_{i,k} \right] \\
\lesssim (1 + \rho) \left( \sqrt{ISN^4K} + (\sqrt{N} + H)\sqrt{H^2SA} \right) + \sqrt{H^4(1 + \rho)^2K} := \epsilon(K) \tag{47}
\]
32
where we applied (46) in the second relation. Since the bound \((47)\) holds for any \(i\) we get that

\[
\max_{i \in [l]} \sum_{k=1}^{K'} (f_k - f_{opt}) + \rho \left[ \sum_{k=1}^{K'} g_{s,k} \right] = \sum_{k=1}^{K'} (f_k - f_{opt}) + \rho \max_{i \in [l]} \left[ \sum_{k=1}^{K'} g_{s,k} \right] + \\
= \sum_{k=1}^{K'} (f_k - f_{opt}) + \rho \max_{i \in [l]} \left[ \sum_{k=1}^{K'} g_{s,k} \right] + \\
= \sum_{k=1}^{K'} (f_k - f_{opt}) + \rho \left[ \sum_{k=1}^{K'} g_k \right] \leq \epsilon(K).
\]

Now, by the convexity of the state-action frequency function (Proposition\([1]\)) there exists a policy \(\pi_{K'}\) which satisfies \(q^{\pi_{K'}}(p) = \frac{1}{K'} \sum_{s,a} q^{\pi_k}(p)\) for any \(K'\). Since both \(f\) and \(g\) are linear in \(\frac{1}{K'} \sum_{s,a} q^{\pi_k}(p)\) we have that

\[
\frac{1}{K'} \left( \sum_{k=1}^{K'} (f_k - f_{opt}) + \rho \left[ \sum_{k=1}^{K'} g_k \right] \right) = f_{\pi_{K'}} - f_{opt} + \rho \left[ g_{\pi_{K'}} \right] \leq \frac{1}{K'} \epsilon(K).
\]

Applying Corollary\([41]\) and Theorem\([42]\) we conclude that

\[
\max_{i \in [l]} \left[ \sum_{k=1}^{K'} g_k \right] \leq \max_{i \in [l]} \left[ \sum_{k=1}^{K'} g_k \right] + \left[ \sum_{k=1}^{K'} g_k \right] \leq \frac{\epsilon(K)}{\rho}
\]

for any \(K' \in [K]\).

E  Bounds of On-Policy Errors

**Lemma 29** (On Policy Errors for Optimistic Model). Let \(l_h(s,a), \tilde{l}_h(s,a)\) be a a cost function, and its optimistic cost. Let \(p\) be the true transition dynamics of the MDP and \(\tilde{p}_h\) be an estimated transition model \(l,p\) and \(\tilde{l}_h, \tilde{p}_h\) respectively. Assume the following holds for all \(s,a,h,k \in [K]\):

(a) \(\left| \tilde{l}_h(s,a) - l_h(s,a) \right| \leq \frac{1}{\sqrt{n_h^{-1}(s,a)}}\).

(b) \(\left| \tilde{p}_h(s' | s,a) - p_h(s' | s,a) \right| \leq \frac{p_h(s' | s,a)}{n_h^{-1}(s,a) \vee 1} + \frac{1}{n_h^{-1}(s,a) \vee 1}\).

(c) \(n_h^{-1}(s,a) \leq \frac{1}{2} \sum_{j<k} q_{h}^{\pi_k}(s,a | p) - H \ln \frac{SAH}{\delta}\).

Furthermore, let \(\pi_k\) be the policy by which the agent acts at the \(k^{th}\) episode. Then, for any \(K' \in [K]\)

\[
\sum_{k=1}^{K'} \left| V_{1}^{\pi_k}(s_1;l,p) - V_{1}^{\pi_k}(s_1;\tilde{l}_h,\tilde{p}_h) \right| \leq \tilde{O} \left( \sqrt{SNH^4K} + (\sqrt{N} + H)H^2SA \right).
\]
Proof. The following relations hold.

\[
\sum_{k=1}^{K'} |V_{1}^{\pi_{k}}(s_{1}; l, p) - V_{1}^{\pi_{k}}(s_{1}; \bar{I}_{k}, \bar{p}_{k})| \\
= \sum_{k=1}^{K'} \left| \sum_{h=1}^{H} l_{h}(s_{h}, a_{h}) - \bar{I}_{h}(s_{h}, a_{h}) \right| + (p_{h} - \bar{p}_{h})(s' \mid s_{h}, a_{h}) V_{h+1}^{\pi_{k}}(s_{1}, p, \pi_{k}) \\
\leq \sum_{k=1}^{K'} \left| \sum_{h=1}^{H} l_{h}(s_{h}, a_{h}) - \bar{I}_{h}(s_{h}, a_{h}) \right| + s_{1}, p, \pi_{k} \\
+ \sum_{k=1}^{K'} \sum_{s' \mid p_{h}} |(p_{h} - \bar{p}_{h})(s' \mid s_{h}, a_{h}) V_{h+1}^{\pi_{k}}(s'; \bar{I}_{k}, \bar{p}_{k})| \right| s_{1}, p, \pi_{k},
\]

where the first relation holds by the value difference Lem. 35. We now bound the terms (i) and (ii).

Bound on (i). To bound (i) we use the assumption (1) and get,

\[
(i) \leq \sum_{k=1}^{K'} \sum_{h=1}^{H} E \left[ \frac{1}{n_{h}^{k}(s_{h}, a_{h})} \mid s_{1}, p, \pi_{k} \right] \\
= \sum_{k=1}^{K'} \sum_{h=1}^{H} E \left[ \frac{1}{n_{h}^{k}(s_{h}, a_{h})} \mid F_{k-1} \right] \leq \bar{O}\left( \sqrt{SAH^{2}K + SAH} \right).
\]

The first relation holds by assumption (a). The second relation holds since \(\pi_{k}\) is the policy by which the agent acts at episode \(k\) in the true MDP. The third relation holds by Lem. 36.

Bound on (ii). To bound (ii) use the fact that

\[
|V_{\pi_{k}}^{k}(s; \bar{I}_{k}, \bar{p}_{k})| \leq H
\]

for every \(s\) since the immediate cost is bounded in \(|\bar{I}_{h}(s, a)| \leq l_{h}(s, a) + \frac{1}{\sqrt{n_{h}^{k-1}(s, a)}} \leq l_{h}(s, a)\) component-wise up to constants, since the second term is bounded by \(\bar{O}(1)\). Thus,

\[
(ii) \leq H \sum_{k=1}^{K'} \sum_{h=1}^{H} E \left[ \frac{1}{n_{h}^{k}(s_{h}, a_{h})} \sqrt{1} \sum_{s'} \sqrt{p_{h}(s' \mid s_{h}, a_{h}) + \frac{S}{n_{h}^{k}(s_{h}, a_{h})} \sqrt{1}} \mid s_{1}, p, \pi_{k} \right] \\
\leq \sum_{k=1}^{K'} \sum_{h=1}^{H} E \left[ \frac{1}{n_{h}^{k}(s_{h}, a_{h})} \sqrt{1} \sum_{s'} \sqrt{p_{h}(s' \mid s_{h}, a_{h}) + \frac{S}{n_{h}^{k}(s_{h}, a_{h})} \sqrt{1}} \mid s_{1}, p, \pi_{k} \right] \\
= H \sum_{k=1}^{K'} \sum_{h=1}^{H} E \left[ \frac{1}{n_{h}^{k}(s_{h}, a_{h})} \sqrt{1} + \frac{S}{n_{h}^{k}(s_{h}, a_{h})} \sqrt{1} \mid s_{1}, p, \pi_{k} \right] \\
= \sum_{k=1}^{K'} \sum_{h=1}^{H} E \left[ \frac{1}{n_{h}^{k}(s_{h}, a_{h})} \sqrt{1} + \frac{S}{n_{h}^{k}(s_{h}, a_{h})} \sqrt{1} \mid F_{k-1} \right] \\
\leq \sqrt{SNH^{2}K + \sqrt{NH^{2}}SA + S H^{2}A} \leq \bar{O}\left( \sqrt{SNH^{2}K + (\sqrt{N} + H)H^{2}SA} \right).
\]

The first relation holds by plugging the bound (48) and assumption (b) into (ii). The second relation holds by Jensen’s inequality. The third relation holds since \(p_{h}\) is a probability distribution. The forth relation holds since \(\pi_{h}\) is the policy with which the agent interacts with the true CMDP. The fifth relation holds by Lem. 36 (its assumption holds by assumption (c)).

Combining the bounds on (i) and (ii) we conclude the proof.
Lemma 30 (On Policy Errors for Truncated Policy Estimation). Let \( l_h(s, a), \tilde{l}_h(s, a) \) be a cost function, and its optimistic cost. Let \( p \) be the true transition dynamics of the MDP and \( \tilde{p}_h \) be an estimated transition dynamics. Let \( V_h^\pi(s; \tilde{l}, p) \) be the value of a policy \( \pi \) according to the cost and transition model \( l, p \). Furthermore, let \( V_h^\pi(s; \tilde{l}_h, \tilde{p}_h) \) be a value function calculated by a truncated value estimation (see Algorithm 5) by the cost and transition model \( \tilde{l}_h, \tilde{p}_h \). Assume the following holds for all \( s, a, h, k \in [K] \):

1. \( |\tilde{l}_h(s, a) - l_h(s, a)| \lesssim \frac{1}{\sqrt{n_h^{k-1}(s, a)}} \).
2. \( |\tilde{p}_h(s' | s, a) - p_h(s' | s, a)| \lesssim \frac{1}{\sqrt{n_h^{k-1}(s, a)}} + \frac{1}{n_h^{k-1}(s, a)} \).
3. \( n_h^{k-1}(s, a) \leq \frac{1}{2} \sum_{j<k} q_h^{\pi_k}(s, a) | p ) - H \ln \frac{SAH}{\delta} \).

Furthermore, let \( \pi_k \) be the policy by which the agent acts at the \( k \)th episode. Then, for any \( K' \in [K] \)

\[
\sum_{k=1}^{K'} |V_h^\pi(s_1; l, p) - \hat{V}_h^\pi(s_1; \tilde{l}_h, \tilde{p}_h)| \leq \tilde{O}\left( \sqrt{SNH^4K} + (\sqrt{N} + H)H^2SA \right).
\]

Proof. The following relations hold.

\[
\sum_{k=1}^{K'} \left| V_h^\pi(s_1; l, p) - \hat{V}_h^\pi(s_1; \tilde{l}_h, \tilde{p}_h) \right| = \sum_{k=1}^{K'} \left| \mathbb{E} \sum_{h=1}^{H} \left( l_h(s_h, a_h) - p_h(\cdot | s_h, a_h) \hat{V}_h^{\pi_k} - Q^{\pi_k}(s_h, a_h; \tilde{l}_h, \tilde{p}_h) \right) | s_1, p, \pi_k \right| \leq \tilde{O}\left( \sqrt{SNH^4K} + (\sqrt{N} + H)H^2SA \right).
\]

Observe that

\[
-Q^{\pi_k}(s_h, a_h; \tilde{l}_h, \tilde{p}_h) = \min \left\{ 0, -l_h(s_h, a_h) - \tilde{p}_h(\cdot | s_h, a_h) \hat{V}_h^{\pi_k} \right\},
\]

where the first relation holds by the extended value difference lemma [34]. Plugging back to (50) we get

\[
\left( i \right) \leq \sum_{k=1}^{K'} \left| \sum_{h=1}^{H} \mathbb{E} \left[ l_h(s_h, a_h) - \tilde{l}_h(s_h, a_h) \right] | s_1, p, \pi_k \right| + \sum_{k=1}^{K'} \left| \sum_{h=1}^{H} \mathbb{E} \left[ p_h - \tilde{p}_h \right] (s' | s_h, a_h) \hat{V}_h^{\pi_k} | s_1, p, \pi_k \right|,
\]

We now bound the terms (i) and (ii).

Bound on (i). To bound (i) we use the assumption (1) and get,

\[
\left( i \right) \lesssim \sum_{k=1}^{K'} \sum_{h=1}^{H} \mathbb{E} \left[ \frac{1}{n_h^{k-1}(s_h, a_h)} \right] | s_1, p, \pi_k |
\]

\[
= \sum_{k=1}^{K'} \sum_{h=1}^{H} \mathbb{E} \left[ \frac{1}{n_h^{k-1}(s_h, a_h)} \right] | s_{k-1}, p_k | \leq \tilde{O}\left( \sqrt{SAH^2K} + SAH \right).
\]

The first relation holds by assumption (1). The second relation holds since \( \pi_k \) is the policy by which the agent acts at the \( k \)th episode at the true MDP. The third relation holds by Lemma 36.
Bound on (ii). To bound (ii) use the fact that

$$|\hat{V}_{k+1}^{\pi_k}(s; \tilde{I}_k, \tilde{p}_k)| \lesssim H$$

(51)

for every $s$ since the immediate cost is bounded in $|\hat{l}_h(s, a)| \lesssim l_h(s, a) \leq 1 + \frac{1}{\sqrt{n_{h^{-1}}(s, a)}} \lesssim l_h(s, a)$ component-wise up to constants, since the second term is bounded by $\mathcal{O}(1)$. Thus,

$$(ii) \lesssim H \sum_{k=1}^{K'} \sum_{l=1}^{H} \mathbb{E}\left[ \frac{1}{n_{h}(s_{h}, a_{h}) \vee 1} \sum_{s'} \sqrt{n_{h}(s_{h}, a_{h}) \vee 1} \sqrt{N} \sum_{s'} p_{h}(s' | s_{h}, a_{h}) + \frac{S}{n_{h}(s_{h}, a_{h}) \vee 1} \mid s_{1}, p, \pi_{k} \right]$$

$$\leq H \sum_{k=1}^{K'} \sum_{l=1}^{H} \mathbb{E}\left[ \frac{1}{n_{h}(s_{h}, a_{h}) \vee 1} \sqrt{N} \sum_{s'} p_{h}(s' | s_{h}, a_{h}) + \frac{S}{n_{h}(s_{h}, a_{h}) \vee 1} \mid s_{1}, p, \pi_{k} \right]$$

$$= H \sum_{k=1}^{K'} \sum_{l=1}^{H} \mathbb{E}\left[ \frac{1}{n_{h}(s_{h}, a_{h}) \vee 1} \sqrt{N} + \frac{S}{n_{h}(s_{h}, a_{h}) \vee 1} \mid s_{1}, p, \pi_{k} \right]$$

$$= H \sum_{k=1}^{K'} \sum_{l=1}^{H} \mathbb{E}\left[ \frac{1}{n_{h}(s_{h}, a_{h}) \vee 1} \sqrt{N} + \frac{S}{n_{h}(s_{h}, a_{h}) \vee 1} \mid \mathcal{F}_{k-1} \right]$$

$$\lesssim \sqrt{SNH^4K} + \sqrt{N}H^2SA + SH^3A \leq \mathcal{O} \left( \sqrt{SNH^4K} + (\sqrt{N} + H)H^2SA \right).$$

The first relation holds by plugging the bound (51) and assumption (2) into (ii). The second relation holds since $p$ is a probability distribution. The third relation holds since $\pi_k$ is the policy with which the agent interacts with the true MDP $p$. The fifth relation holds by Lemma 31 (its assumption holds by assumption (3)).

Combining the bounds on (i) and (ii) we conclude the proof. □

Lemma 31 (On Policy Errors for Bonus Based Optimism). Let $l_h(s, a), \hat{l}_h(s, a)$ be a cost function, and its optimistic cost. Let $p$ be the true transition dynamics of the MDP and $\overline{p}_{k-1}$ be an estimated transition dynamics. Let $V_h^{\pi}(s; l, p), \hat{V}_h^{\pi}(s; \tilde{l}_k, \tilde{p}_k)$ be the value of a policy $\pi$ according to the cost and transition model $l, p$ and $\tilde{l}_k, \tilde{p}_k_{k-1}$, respectively. Assume the following holds for all $s, a, s', h, k \in [K]$: 

1. $|\hat{l}_h^{k-1}(s, a) - l_h(s, a)| \lesssim \sqrt{\frac{1}{n_{h^{-1}}(s, a) \vee 1}} + \sum_{s'} H \sqrt{\frac{\overline{p}_{h}^{k-1}(s' | s, a)}{n_{h}^{-1}(s, a) \vee 1} + \frac{HS}{(n_{h}^{-1}(s, a) - 1) \vee 1}}.$

2. $|\overline{p}_{h}^{k-1}(s' | s, a) - p_h(s' | s, a)| \lesssim \sqrt{\frac{\overline{p}_{h}^{k-1}(s' | s, a)}{(n_{h}^{-1}(s, a) - 1) \vee 1}} + \sqrt{\frac{1}{(n_{h}^{-1}(s, a) - 1) \vee 1}}.$

3. $n_{h}^{-1}(s, a) \leq \frac{1}{2} \sum_{j < k} q_{h}^{\pi_k}(s, a; p) - H \ln \frac{SH}{p}.$

4. $V_{h}^{\pi_k}(s; l, k, \overline{p}_{k-1}) \leq V_{h}^{\pi_k}(s; l, p).$

5. $l_h(s, a) - \hat{l}_h^{k-1}(s, a) + (p_h(s' | s, a) - \overline{p}_{h}^{k-1}(s' | s, a))V_{h}^{\pi_k}(s' | l, p) \geq 0.$

Let $\pi_k$ be the policy by which the agent acts at episode $k$. Then, for any $K' \in [K]$

$$\sum_{k=1}^{K'} V_{h}^{\pi_k}(s_1; l, p) - V_{h}^{\pi_k}(s_1; \tilde{l}_k, \tilde{p}_{k-1}) \leq \tilde{O} \left( \sqrt{SNH^4K} + S^2H^2A(NH + S) \right).$$

Proof. Denote for any $s, h \ V_{h}^\pi(s) = V_{h}^\pi(s; l_k, \overline{p}_{k-1})$ and $\hat{V}_{h}^\pi(s) = V_{h}^\pi(s; l, p)$. The following relations
where the first relation holds by the value difference lemma (see Lem. 35).

**Bound on (i) and (ii).** Since 0 ≤ \( V_{h+1}^\pi (s; l, p) \) ≤ \( H \) (the value of the true MDP is bounded in [0, \( H \)]), we can bound both (i) and (ii) by the same analysis as in Lem. 29. Thus, (i) + (ii) ≤ \( \sqrt{SNH^4K} + (\sqrt{N} + H)H^2SA \).

**Bound on (iii).** Applying Lem. 32 we obtain the following bound

\[
(iii) \lesssim \sqrt{SNH^4K} + S^2H^4A(NH + S) + \sqrt{NSH^{5/2}A} \sqrt{\sum_k (V_{1}^\pi_k(s_1) - \tilde{V}_{1}^\pi_k(s_1))}.
\]

Plugging the bounds on terms (i), (ii), and (iii) into (52) we get

\[
\sum_{k=1}^{K'} V_{1}^\pi_k(s_1) - \tilde{V}_{1}^\pi_k(s_1)
\lesssim \sqrt{SNH^4K} + S^2H^4A(NH + S) + \sqrt{NSH^{5/2}A} \sqrt{\sum_k (V_{1}^\pi_k(s_1) - \tilde{V}_{1}^\pi_k(s_1))}.
\]

Denoting \( X = \sum_{k=1}^{K'} V_{1}^\pi_k(s_1) - \tilde{V}_{1}^\pi_k(s_1) \) this bound has the form 0 ≤ \( X \) ≤ \( a + b\sqrt{X} \), where

\[
a = \sqrt{SNH^4K} + S^2H^4A(NH + S)
\]

\[
b = \sqrt{NSH^{5/2}A}.
\]

Applying Lem. 38 by which \( X \) ≤ \( a + b^2 \), we get

\[
\sum_{k=1}^{K'} V_{1}^\pi_k(s_1) - \tilde{V}_{1}^\pi_k(s_1) \lesssim \sqrt{SNH^4K} + S^2H^4A(NH + S).
\]

**Lemma 32.** Let the assumptions of Lem. 37 hold. Then, for any \( K' \in [K] \)

\[
\sum_{k=1}^{K'} \sum_{l=1}^{H} \mathbb{E} \left[ (p_h - \tilde{p}_h^{k-1})(s; s_h^k, a_h^k) (V_{h+1}^\pi_k(s; l, p)) - V_{h+1}^\pi_k(s; l, p) \right] | \mathcal{F}_{k-1}
\lesssim S^2H^4A(NH + S) + \sqrt{NSH^{5/2}A} \sqrt{\sum_k (V_{1}^\pi_k(s_1) - \tilde{V}_{1}^\pi_k(s_1))}.
\]

\[\Box\]
Proof. Denote for any \( s, h \) \( \tilde{V}_h^\pi_k(s) = V_h^\pi_k(s; \tilde{T}_k, \hat{\pi}_{k-1}) \) and \( V_h^\pi_k(s) = V_h^\pi_k(s; t, p) \). The following relations hold:

\[
\sum_k E \left[ \sum_{l=1}^{H} (p_h - \tilde{p}_h^{k-1})(\cdot | s_h, a_h)(\tilde{V}_h^\pi_k - V_h^\pi_k) \mid s_1, \pi_k, p \right] = \sum_{k, h, s, a} q_h^\pi(s, a; p) \left( p_h - \tilde{p}_h^{k-1}(\cdot | s, a)(\tilde{V}_h^\pi_k - V_h^\pi_k) \right) \leq \sum_{k, h, s, a} q_h^\pi(s, a; p) \sum_{s'} \sqrt{p_h(s' | s, a)} \left| \tilde{V}_h^\pi_k(s') - V_h^\pi_k(s') \right| + \sum_{k, h, s, a} q_h^\pi(s, a; p) \frac{H^2 \sigma^2}{n_h(s, a)}. \tag{53}
\]

In the third relation we used assumption (2) of Lem. 31 as well as bounding

\[
\left| \tilde{V}_h^\pi_k(s) - V_h^\pi_k(s) \right| \leq SH^2 \tag{54}
\]

since \( \tilde{V}_h^\pi_k(s) \in [-SH^2, H] \) by the assumption on its instantaneous cost (assumption (1) of Lem. 31).

Note that \( V_{h+1}^\pi_k(s) \in [0, H] \) as usual.

Term \((ii)\) is bounded as follows

\[
(ii) = H^2 \sigma^2 \sum_{k, h} E \left[ \frac{1}{n_h^k(s_h, a_h)} | s_1, \pi_k, p \right] = H^2 \sigma^2 \sum_{k, h} E \left[ \frac{1}{n_h^k(s_h, a_h)} | F_{k-1} \right] \leq H^4 \sigma^3 A, \tag{55}
\]

by Lem. 37.

We now bound term \((i)\) as follows.

\[
(i) \leq \sum_{k} \sum_{s, a, h} q_h^\pi(s, a; p) \sqrt{N} \sum_{s'} p_h(s' | s, a)(\tilde{V}_h^\pi_k(s') - V_h^\pi_k(s'))^2 \sqrt{n_h^k(s, a)} \leq \sqrt{N} \sum_{k} \sum_{s, a, h} q_h^\pi(s, a; p) \frac{1}{n_h^k(s, a)} \sum_{k} \sum_{s', h} q_h^\pi(s', a; p) p_h(s' | s, a)(\tilde{V}_h^\pi_k(s') - V_h^\pi_k(s'))^2 \leq \sqrt{N} \sum_{k} \sum_{s, a, h} q_h^\pi(s, a; p) \frac{1}{n_h^k(s, a)} \sum_{k} \sum_{s', h} q_h^\pi(s', a; p)(\tilde{V}_h^\pi_k(s') - V_h^\pi_k(s'))^2 \leq \sqrt{N} \sum_{k} \sum_{s, a, h} q_h^\pi(s, a; p)(V_h^\pi_k(s) - \tilde{V}_h^\pi_k(s)) \leq \sqrt{N} \sum_{k} (V_h^\pi_k(s_1) - \tilde{V}_h^\pi_k(s_1)) + \sum_{k, h, s, a} q_h^\pi(s, a; p) (p_h - \tilde{p}_h)(\cdot | s, a)(V_h^\pi_k - V_{h+1}^\pi_k) \leq \sqrt{N} \sum_{k} (V_h^\pi_k(s_1) - \tilde{V}_h^\pi_k(s_1)) + \sum_{k, h, s, a} q_h^\pi(s, a; p) (p_h - \tilde{p}_h)(\cdot | s, a)(\tilde{V}_h^\pi_k - V_h^\pi_k), \tag{56}
\]

The first relation holds by Jensen’s inequality while using the fact that \( p_h(\cdot | s, a) \) has at most \( N \) non-zero terms. The second relation holds by Cauchy-Schwartz inequality. The third relation follows from properties of the occupancy measure (see Eq. 59). In particular, \( \sum_{s, a} p_h(s' | s, a) q_h(s, a; p) = \sum_a q_{h+1}(s', a; p) \). The forth relation holds by applying Lem. 37 and bounding \( (\tilde{V}_h^\pi_k(s) - V_h^\pi_k(s))^2 \leq SH^2(V_h^\pi_k(s) - \tilde{V}_h^\pi_k(s))^2 \) due to (54) and \( V_{h+1}^\pi_k(s) - \tilde{V}_{h+1}^\pi_k(s) \geq 0 \) due to optimism (assumption (4) of Lem. 31). The fifth relation holds by Lemma 33 (see that its assumption holds by assumption (5)). The sixth relation holds by \( \sqrt{a + b} \leq \sqrt{a} + \sqrt{b} \).
Plugging the bounds on term (i), \((55)\), and term (ii), \((56)\), into \((53)\) we get
\[
\sum_{k,h,s,a} q^π_k(s,a;p)\left| (p_h - \overline{p}_h)(\cdot \mid s,a)(\tilde{V}^π_{k+1} - V^π_{k+1}) \right|
\leq H^4 S^3 A + \sqrt{N} S H^{5/2} \sqrt{A} \sqrt{\sum_k (V^π_{k+1}(s_1) - \tilde{V}^π_{k+1}(s_1))}
\]
\[
+ \sqrt{N} S H^{5/2} \sqrt{A} \sum_{k,h,s,a} q^π_k(s,a;p)\left| (p_h - \overline{p}_h)(\cdot \mid s,a)(\tilde{V}^π_{k+1} - V^π_{k+1}) \right|
\]

Denoting \(X = \sum_{k,h,s,a} q^π_k(s,a;p)\left| (p_h - \overline{p}_h)(\cdot \mid s,a)(\tilde{V}^π_{k+1} - V^π_{k+1}) \right|\) this bound has the form \(0 \leq X \leq a + b \sqrt{X}\), where
\[
a = H^4 S^3 A + \sqrt{N} S H^{5/2} \sqrt{A} \sqrt{\sum_k (V^π_{k+1}(s_1) - \tilde{V}^π_{k+1}(s_1))}
\]
\[
b = \sqrt{N} S H^{5/2} \sqrt{A}.
\]

Applying Lem. \((38)\) by which \(X \leq a + b^2\), we get
\[
\sum_{k,h,s,a} q^π_k(s,a;p)\left| (p_h - \overline{p}_h)(\cdot \mid s,a)(\tilde{V}^π_{k+1} - V^π_{k+1}) \right|
\leq H^4 S^3 A + \sqrt{N} S H^{5/2} \sqrt{A} \sqrt{\sum_k (V^π_{k+1}(s) - \tilde{V}^π_{k+1}(s))} + NS^2 H^5 A
\]
\[
\leq S^2 H^4 A(NH + S) + \sqrt{N} S H^{5/2} \sqrt{A} \sqrt{\sum_k (V^π_{k+1}(s) - \tilde{V}^π_{k+1}(s))}
\]

\(\square\)

**Lemma 33.** Let \(l_h(s,a), \tilde{l}_h(s,a)\) be a cost function and its optimistic cost. Let \(p, \overline{p}\) be two transition probabilities. Let \(V^π_h(s) := V^π_h(s;l,p)\) and \(\tilde{V}^π_h(s) := V^π_h(s;l_k, \overline{p})\) be the value of a policy \(\pi\) according to the cost and transition model \(l, p\) and \(l, \overline{p}\), respectively. Assume that
\[
l_h(s,a) - \tilde{l}_h(s,a) + (p_h(\cdot \mid s,a) - \overline{p}_h(\cdot \mid s,a))V^π_h(s_i) \geq 0, \tag{57}
\]
for any \(s,a,h\). Then, for any \(\pi\) and \(s\)
\[
\sum_{h=1}^{H} \mathbb{E} \left[ V^π_h(s_h) - \tilde{V}^π_h(s_h) \mid s_1 = s, \pi, p \right]
\]
\[
\leq H \left( V^π_1(s) - \tilde{V}^π_1(s) \right) + H \sum_{h=1}^{H} \mathbb{E} \left[ (p_h(\cdot \mid s_h,a_h) - \overline{p}_h(\cdot \mid s_h,a_h))(\tilde{V}^π_{h+1} - V^π_{h+1}) \mid s_1 = s, \pi, p \right]
\]

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Proof. By definition
\[ V^n_\pi(s) - \tilde{V}^\pi_1(s) \]
\[ = E\left[ V^n_\pi(s_1) - l_1(s_1, a_1) - p_1(\cdot \mid s_1, a_1)\tilde{V}^\pi_2 \mid s_1 = s, \pi, P \right] \]
\[ + E\left[ l_1(s_1, a_1) + p_1(\cdot \mid s_1, a_1)\tilde{V}^\pi_2 - \tilde{V}^\pi_1(s) \mid s_1 = s, \pi, P \right] \]
\[ = E\left[ V^n_2(s_2) - \tilde{V}^\pi_2(s_2) \mid s_1 = s, \pi, P \right] \]
\[ + E\left[ l_1(s_1, a_1) - \tilde{l}_1(s_1, a_1) + (p_1(\cdot \mid s_1, a_1) - \tilde{p}_1(\cdot \mid s_1, a_1))\tilde{V}^\pi_2 \mid s_1 = s, \pi, P \right] \]
\[ = E\left[ V^n_2(s_2) - \tilde{V}^\pi_2(s_2) \mid s_1 = s, \pi, P \right] \]
\[ + E\left[ (p_1(\cdot \mid s_1, a_1) - \tilde{p}_1(\cdot \mid s_1, a_1))(\tilde{V}^\pi_2 - V^n_2) \mid s_1 = s, \pi, P \right] \]
\[ + E\left[ l_1(s_1, a_1) - \tilde{l}_1(s_1, a_1) + (p_1(\cdot \mid s_1, a_1) - \tilde{p}_1(\cdot \mid s_1, a_1))\tilde{V}^\pi_2 \mid s_1 = s, \pi, P \right] \]
\[ \geq E\left[ V^n_2(s_2) - \tilde{V}^\pi_2(s_2) \mid s_1 = s, \pi, P \right] \]
\[ + E\left[ (p_1(\cdot \mid s_1, a_1) - \tilde{p}_1(\cdot \mid s_1, a_1))(\tilde{V}^\pi_2 - V^n_2) \mid s_1 = s, \pi, P \right], \quad (58) \]

where the first relation holds by the value difference lemma and the last relation holds due to the assumption

Iterating on this relation we get that for any \( h \in \{2, \ldots, H\} \)
\[ V^n_\pi(s) - \tilde{V}^\pi_1(s) \]
\[ \geq E\left[ V^n_h(s_h) - \tilde{V}^\pi_h(s_h) \mid s_1 = s, \pi, P \right] \]
\[ + \sum_{h=1}^{h-1} E\left[ (p_{h'}(\cdot \mid s_{h'}, a_{h'}) - \tilde{p}_{h'}(\cdot \mid s_{h'}, a_{h'}))(\tilde{V}^\pi_{h+1} - V^n_{h+1}) \mid s_1 = s, \pi, P \right]. \]

By summing this relation for \( h \in \{2, \ldots, H\} \) and rearranging we get
\[ H\left( V^n_\pi(s) - \tilde{V}^\pi_1(s) \right) - \sum_{h=2}^{H} \sum_{h'=1}^{h-1} E\left[ (p_{h'}(\cdot \mid s_{h'}, a_{h'}) - \tilde{p}_{h'}(\cdot \mid s_{h'}, a_{h'}))(\tilde{V}^\pi_{h+1} - V^n_{h+1}) \mid s_1 = s, \pi, P \right] \]
\[ \geq \sum_{h=2}^{H} E\left[ V^n_h(s_h) - \tilde{V}^\pi_h(s_h) \mid s_1 = s, \pi, P \right]. \]

Thus,
\[ \sum_{h=2}^{H} E\left[ V^n_h(s_h) - \tilde{V}^\pi_h(s_h) \mid s_1 = s, \pi, P \right] \]
\[ \leq H\left( V^n_\pi(s) - \tilde{V}^\pi_1(s) \right) + \sum_{h=2}^{H} \sum_{h'=1}^{h-1} E\left[ (-p_{h'}(\cdot \mid s_{h'}, a_{h'}) - \tilde{p}_{h'}(\cdot \mid s_{h'}, a_{h'}))(\tilde{V}^\pi_{h+1} - V^n_{h+1}) \mid s_1 = s, \pi, P \right] \]
\[ \leq H\left( V^n_\pi(s) - \tilde{V}^\pi_1(s) \right) + \sum_{h=2}^{H} \sum_{h'=1}^{h-1} E\left[ (p_{h'}(\cdot \mid s_{h'}, a_{h'}) - \tilde{p}_{h'}(\cdot \mid s_{h'}, a_{h'}))(\tilde{V}^\pi_{h+1} - V^n_{h+1}) \mid s_1 = s, \pi, P \right] \]
\[ \leq H\left( V^n_\pi(s) - \tilde{V}^\pi_1(s) \right) + H \sum_{h=1}^{H} E\left[ (p_h(\cdot \mid s_h, a_h) - \tilde{p}_h(\cdot \mid s_h, a_h))(\tilde{V}^\pi_{h+1} - V^n_{h+1}) \mid s_1 = s, \pi, P \right]. \]

\[ \square \]

F Useful Lemmas

We start stating the value difference lemma (a.k.a. simulation lemma). This lemma has been used in several papers [e.g., Cai et al. 2019, Efroni et al. 2020]. The following lemma is central for the analysis of OptPrimalDual-CMDP.
Lemma 34 (Extended Value Difference). Let $\pi, \pi'$ be two policies, and $\mathcal{M} = (S, A, \{p_h\}_{h=1}^H, \{c_h\}_{h=1}^H)$ and $\mathcal{M}' = (S, A, \{p'_h\}_{h=1}^H, \{c'_h\}_{h=1}^H)$ be two MDPs. Let $\tilde{Q}_h^\pi(s; a, p)$ be an approximation of the $Q$-function of policy $\pi$ on the MDP $\mathcal{M}$ for all $h, s, a$, and let $\tilde{V}_h^\pi(s; c, p) = \left\langle \tilde{Q}_h^\pi(s; c, p), \pi_h(\cdot | s) \right\rangle$. Then,

$$\tilde{V}_h^\pi(s; c, p) - \tilde{V}_h'^\pi(s; c', p') = \sum_{h=1}^H \mathbb{E}\left[\left(\tilde{Q}_h^\pi(s_h, \cdot; c, p), \pi'_h(\cdot | s_h) - \pi_h(\cdot | s_h)\right) \mid s_1, \pi, p\right] + \sum_{h=1}^H \mathbb{E}\left[\tilde{Q}_h^\pi(s_h, a_h; c, p) - \tilde{c}'_h(s_h, a_h) - \tilde{p}'_h(\cdot | s_h, a_h)\tilde{V}_h'^\pi(\cdot; c, p) \mid s_1, \pi, p\right]$$

where $\tilde{V}_h'^\pi(s; c', p')$ is the value function of $\pi'$ in the MDP $\mathcal{M}'$.

The following lemma is standard [see e.g., Dann et al. 2017, Lem. E.15], and can be seen as a corollary of the extended value difference lemma.

Lemma 35 (Value difference lemma). Consider two MDPs $\mathcal{M} = (S, A, \{p_h\}_{h=1}^H, \{c_h\}_{h=1}^H)$ and $\mathcal{M}' = (S, A, \{p'_h\}_{h=1}^H, \{c'_h\}_{h=1}^H)$. For any policy $\pi$ and any $s, h$ the following relation holds.

$$V_h^\pi(s; c, p) - V_h'^\pi(s; c', p') = \mathbb{E}\left[\sum_{h'=h}^H (c_h(s_h, a_h) - \tilde{c}'_h(s_h, a_h)) + (p_h - \tilde{p}'_h)(\cdot | s_h, a_h)V_h'^\pi(\cdot; c, p) \mid s_h = s, \pi, p'\right]$$

The following lemmas are standard. There proof can be found in Dann et al. 2017; Zanette and Brunskill 2019; Efroni et al. 2019 (e.g., Efroni et al. 2019, Lem. 38).

Lemma 36. Assume that for all $s, a, h, k \in [K]$

$$n_h^{-1}(s, a) > \frac{1}{2} \sum_{j<k} n_h^{\pi_k}(s, a; p) - H \ln \frac{SAH}{\delta^r},$$

then

$$\sum_{k=1}^K \sum_{h=1}^H \mathbb{E}\left[\frac{1}{n_h^{-1}(s_h^k, a_h^k) \vee 1} \mid F_{k-1}\right] \leq \tilde{O}(\sqrt{SAH^2} + SAH).$$

Lemma 37 (e.g., Zanette and Brunskill 2019, Lem. 13). Assume that for all $s, a, h, k \in [K]$

$$n_h^{-1}(s, a) > \frac{1}{2} \sum_{j<k} n_h^{\pi_k}(s, a; p) - H \ln \frac{SAH}{\delta^r},$$

then

$$\sum_{k=1}^K \sum_{h=1}^H \mathbb{E}\left[\frac{1}{n_h^{-1}(s_h^k, a_h^k) \vee 1} \mid F_{k-1}\right] \leq \tilde{O}(SAH^2).$$

Lemma 38 (Consequences of Self Bounding Property). Let $0 \leq X \leq a + b\sqrt{X}$ where $X, a, b \in \mathbb{R}$. Then, $X \leq a + b^2$.

Proof. We have that

$$X - b\sqrt{X} - a \leq 0.$$

Since $X \geq 0$ this implies that

$$\sqrt{X} \leq \frac{b}{2} + \sqrt{\frac{1}{4}b^2 + 4a} \leq \frac{b}{2} + \sqrt{\frac{b^2/4}{4} + 4a} \leq b + 2\sqrt{a},$$
where we used the relation $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$.

Since $\sqrt{X} \geq 0$ by squaring the two sides of the later inequality we get

$$X \leq (b + 2\sqrt{a})^2 \leq 2b^2 + 4a \leq b^2 + a,$$

where in the second relation we used the relation $(a+b)^2 \leq 2a^2 + 2b^2$.

\[\square\]

F.1 Online Mirror Descent

In each iteration of Online Mirror Descent (OMD) the following problem is solved:

$$x_{k+1} \in \arg \min_{x \in C} t_K \langle g_k, x - x_k \rangle + B_{\omega}(x, x_k),$$

(59)

where $t_K$ is a stepsize, and $B_{\omega}(x, x_k)$ is the bregman distance.

When choosing $B_{\omega}(x, x_k)$ as the KL-divergence, and the set $C$ is the unit simplex OMD has the following closed form,

$$x_{k+1} \in \arg \min_{x \in C} \{t_K \langle \nabla f_k(x), x - x_k \rangle + d_{KL}(x||x_k)\},$$

The following lemma [Orabona, 2019, Theorem 10.4] provides a fundamental inequality which will be used in our analysis.

**Lemma 39** (Fundamental inequality of Online Mirror Descent). Assume $g_{k,i} \geq 0$ for $k = 1, ..., K$ and $i = 1, ..., d$. Let $C = \Delta_d$. Using OMD with the KL-divergence, learning rate $t_K$, and with uniform initialization, $x_1 = [1/d, ..., 1/d]$, the following holds for any $u \in \Delta_d$,

$$\sum_{k=1}^K \langle g_k, x_k - u \rangle \leq \frac{\log d}{t_K} + \frac{t_K}{2} \sum_{k=1}^K \sum_{i=1}^d x_{k,i} g_{k,i}.$$

In our analysis, we will be solving the OMD problem for each time-step $h$ and state $s$ separately,

$$\pi_{h+1}^k(\cdot | s) \in \arg \min_{\pi \in \Delta_A} t_K \langle Q_h^k(s, \cdot), \pi - x_h^k(\cdot | s) \rangle + d_{KL}(\pi||\pi_h^k(\cdot | s)).$$

(60)

Therefore, by adapting the above lemma to our notation, we get the following lemma,

**Lemma 40** (Fundamental inequality of Online Mirror Descent for RL). Let $t_K > 0$. Let $\pi_h^k(\cdot | s)$ be the uniform distribution for any $h \in [H]$ and $s \in S$. Assume that $Q_h^k(s, a)$ is [0, M] for all $s, a, h, k$. Then, by solving (60) separately for each $k \in [K], h \in [H]$ and $s \in S$, the following holds for any stationary policy $\pi$,

$$\sum_{k=1}^K \langle Q_h^k(\cdot | s), \pi_h^k(\cdot | s) - \pi_h(\cdot | s) \rangle \leq \frac{\log A}{t_K} + \frac{t_K M^2 K}{2}.$$

**Proof.** First, observe that for any $h, k, s$, we solve the optimization problem defined in (60) which is the same as (60). By the fact that the estimators used in our analysis are non-negative, we can apply Lemma 39 separately for each $h, s$ with $g_k = Q_h^k(s, \cdot)$ and $x_k = \pi_h^k(s, \cdot)$. Lastly, bounding $(Q_h^k(s, a))^2 \leq M^2$ and $\sum_a \pi_h^k(a | s) = 1$ for all $s$ concludes the result.

\[\square\]

G Useful Results from Constraint Convex Optimization

In this section we enumerate several results from constraint convex optimization which are central to establish the bounds for the dual algorithms. To keep the generality of discussion, we follow results from Beck [2017], Chapter 3, and consider a general constraint convex optimization problem

$$f_{\text{opt}} = \min_{x \in \mathbb{R}^n} \{f(x) : g(x) \leq 0, Ax + b = 0\},$$

(61)

where $g(x) := (g_1(x), ..., g_l(x))^T$, and $f, g_1, ..., g_l : \mathbb{R} \to (-\infty, \infty)$ are convex real valued functions, $A \in \mathbb{R}^{p \times n}, b \in \mathbb{R}^p$. By defining the vector of constraints.
We define a value function associated with (61)

\[ v(u, t) = \min_{x \in X} \{ f(x) : g(x) \leq u, Ax + b = t \}, \]

Furthermore, we define the dual problem to (61). The dual function is

\[ q(\lambda, \mu) = \min_{x \in X} \{ L(x, \lambda, \mu) = f(x) + \lambda^T g(x) + \mu^T (Ax + b) \}, \]

where \( \lambda \in \mathbb{R}^m_+ \), \( \mu \in \mathbb{R}^p \) and the dual problem is

\[ q_{\text{opt}} = \max_{\lambda \in \mathbb{R}^m_+, \mu \in \mathbb{R}^p} \{ q(\lambda, \mu) : (\lambda, \mu) \in \text{dom}(-q) \}. \] (62)

Where \( \text{dom}(-q) = \{ (\lambda, \mu) \in \mathbb{R}^m_+, \mu \in \mathbb{R}^p : q(\lambda, \mu) > -\infty \} \). Furthermore, denote an optimal solution of (62) by \( \lambda^*, \mu^* \).

We make the following assumption which will be verified to hold. The assumption implies strong duality, i.e., \( q_{\text{opt}} = f_{\text{opt}} \).

**Assumption 3.** The optimal value of (61) is finite and exists a slater point \( \bar{x} \) such that \( g(\bar{x}) < 0 \) and exists a point \( \bar{x} \in \text{ri}(X) \) satisfying \( A\bar{x} + b = 0 \), where \( \text{ri}(X) \) is the relative interior of \( X \).

The following theorem is proved in [Beck 2017].

**Theorem 41** ([Beck 2017], Theorem 3.59.). \( (\lambda^*, \mu^*) \) is an optimal solution of (62) iff

\[ - (\lambda^*, \mu^*) \in \partial v(0, 0). \]

Where \( \partial f(x) \) denotes the set of all sub-gradients of \( f \) at \( x \).

Using this result we arrive to the following theorem, which is a variant of [Beck 2017], Theorem 3.60.

**Theorem 42.** Let \( \lambda^* \) be an optimal solution of the dual problem (62) and assume that \( 2\|\lambda^*\|_1 \leq \rho \). Let \( \bar{x} \) satisfy \( A\bar{x} + b = 0 \) and

\[ f(\bar{x}) - f_{\text{opt}} + \rho \| g(\bar{x}) \|_\infty \leq \delta, \]

then

\[ \| g(\bar{x}) \|_\infty \leq \frac{\delta}{\rho}. \]

**Proof.** Let

\[ v(u, t) = \min_{x \in X} \{ f(x) : g(x) \leq u, Ax + b = t \}. \]

Since \( - (\lambda^*, \mu^*) \) is an optimal solution of the dual problem it follows by Theorem 41 that \( - (\lambda^*, \mu^*) \in \partial v(0, 0) \). Therefore, for any \( (u, 0) \in \text{dom}(v) \)

\[ v(u, 0) - v(0, 0) \geq - \langle \lambda^*, u \rangle. \] (63)

Set \( u = \bar{u} := [g(\bar{x})]_+ \). See that \( \bar{u} = 0 \) which implies that

\[ v(\bar{u}, 0) \leq v(0, 0) = f_{\text{opt}} \leq f(\bar{x}). \]

Thus, (63) implies that

\[ f(\bar{x}) - f_{\text{opt}} \geq - \langle \lambda^*, \bar{u} \rangle. \] (64)

We obtain the following relations.

\[ (\rho - \| \lambda^* \|_1) \| \bar{u} \|_\infty = - \| \lambda^* \|_1 \| \bar{u} \|_\infty + \rho \| \bar{u} \|_\infty \leq - \langle \lambda^*, \bar{u} \rangle + \rho \| \bar{u} \|_\infty \leq f(\bar{x}) - f_{\text{opt}} + \rho \| \bar{u} \|_\infty \leq \delta, \]

where the last relation holds by (64). Rearranging, we get

\[ \| g(\bar{x}) \|_\infty = \| \bar{u} \|_\infty \leq \frac{\delta}{\rho - \| \lambda^* \|_1} \leq \frac{2\delta}{\rho}, \]

by using the assumption \( 2\|\lambda^*\|_1 \leq \rho \).
Lastly, we have the following useful result by which we can bound the optimal dual parameter by the properties of a slater point. This result is an adjustment of [Beck 2017, Theorem 8.42].

**Theorem 43.** Let $\mathbf{x} \in X$ be a point satisfying $g(\mathbf{x}) < 0$ and $A\mathbf{x} + b = 0$. Then, for any $\lambda, \mu \in \{\lambda \in \mathbb{R}^m_+, \mu \in \mathbb{R}^p_+: q(\lambda, \mu) \geq M\}$

$$\|\lambda\|_1 \leq \frac{f(\mathbf{x}) - M}{\min_{j=1,\ldots,m} -g_j(\mathbf{x})}.$$  

**Proof.** Let

$$S_M = \{\lambda \in \mathbb{R}^m_+, \mu \in \mathbb{R}^p_+: q(\lambda, \mu) \geq M\}.$$  

By definition, for any $\lambda, \mu \in S_M$ we have that

$$M \leq q(\lambda, \mu)$$

$$= \min_{x \in X} \left\{f(x) + \lambda^T g(x) + \mu^T (Ax + b)\right\}$$

$$\leq f(\mathbf{x}) + \lambda^T g(\mathbf{x}) + \mu^T (A\mathbf{x} + b)$$

$$= f(\mathbf{x}) + \sum_{j=1}^{m} \lambda_j g_j(\mathbf{x}).$$

Therefore,

$$- \sum_{j=1}^{m} \lambda_j g_j(\mathbf{x}) \leq f(\mathbf{x}) - M,$$

which implies that for any $(\lambda, \mu) \in S_M$

$$\sum_{j=1}^{m} \lambda_j = \|\lambda\|_1 \leq \frac{f(\mathbf{x}) - M}{\min_{j=1,\ldots,m} (-g_j(\mathbf{x}))}.$$  

From this theorem we get the following corollary.

**Corollary 44.** Let $\mathbf{x} \in X$ be a point satisfying $g(\mathbf{x}) < 0$ and $A\mathbf{x} + b = 0$, and $\lambda^*$ be an optimal dual solution. Then,

$$\|\lambda^*\|_1 \leq \frac{f(\mathbf{x}) - M}{\min_{j=1,\ldots,m} -g_j(\mathbf{x})}.$$  

**Proof.** Since $(\lambda^*, \mu^*) \in S_{f_{opt}}$ be an optimal solution of the dual problem [62].