Integrable Models And The Toda Lattice Hierarchy

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Abstract

A pedagogical presentation of integrable models with special reference to the Toda lattice hierarchy has been attempted. The example of the KdV equation has been studied in detail, beginning with the infinite conserved quantities and going on to the Lax formalism for the same. We then go on to symplectic manifolds for which we construct the Lax operator. This formalism is applied to Toda Lattice systems. The Zakharov Shabat formalism aimed at encompassing all integrable models is also covered after which the zero curvature condition and its fallout are discussed. We then take up Toda Field Theories and their connection to W algebras via the Hamiltonian reduction of the WZNW model. Finally, we dwell on the connection between four dimensional Yang Mills theories and the KdV equation along with a generalization to supersymmetry.

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1 Introduction: Non-Linear Equations

Linear partial differential equations, in particular the Schroedinger, Klein-Gordon and Dirac equations, have been known in field theory over a long time, and have been used in many different problems with great success. Non-linear equations, i.e., equations where the potential term is non-linear in the field \( S \), have been known for some time as well. These equations and their solutions are the topic of the present Article.

The earliest non-linear wave equations known in physics were the Liouville and Sine-Gordon equations. The Liouville equation arose in the context of a search for a manifold with constant curvature. Pictorially, such parametrizations may be likened to covering a surface with a fishing net. Since the knots on the fishing net do not move, the arc length is constant. The threads in the net correspond to a local coordinate system on the surface.

The Liouville manifolds may be reparametrized locally so as to have a metric of the form:

\[ A = \begin{pmatrix} \exp \rho & 0 \\ 0 & \exp \rho \end{pmatrix} \]

so as to be conformally equivalent to a flat space metric. The study of such manifolds with constant curvature led J.Liouville \([1]\) to the equation known by his name:

\[ \frac{\partial^2 \rho}{\partial x \partial y} = \exp \rho \]  

\( x \) and \( y \) being local orthogonal coordinates. Interest in this equation was renewed in the 70’s and 80’s due to its appearance in string theories \([2,3,4]\).

The Sine-Gordon equation, named after a pun on the Klein-Gordon equation, is an equation for the angle \( \omega \) between two coordinate lines when the total curvature is constant and negative. This equation first appeared in the work of Enneper in 1870, and has the form:

\[ \frac{\partial^2 \omega}{\partial x \partial y} = \sin(\omega) \]

where \( x \) and \( y \) are coordinates in a system with constant arc length.

The Sine-Gordon equation has some interesting solutions known as solitons and breathers. A soliton satisfies three conditions. First, a single soliton must have constant shape and velocity. Secondly, it must be localized, and its derivative must vanish at infinity. Thirdly, if two solutions collide, they should survive the collision with their shapes unchanged.

Principally, there are two types of solitons, one which increases by a fixed amount (say \( 2\pi \)), and is called a ‘kink’; the other which decreases by the same amount, and is called an ‘anti-kink’.

A breather is a localized solution that varies periodically, and could be considered as a permanently bound system of a kink and anti-kink.

An interesting property of the Sine-Gordon equation is that its solutions can be mapped into others through the Baecklund transformation \([5]\), and can thus be used to create new solutions from known solutions. It is however impossible to generate a complete set of solutions from one original solution, via the Baecklund transformation \([5]\).

A third non-linear equation which we shall study in some detail, was discovered in 1895 by D.J.Korteweg and G. de Vries \([6]\), while trying to describe the motion of water-
waves in a canal. It has the form:

\[ u_t - 6uu_x + u_{xxx} = 0 \]  \hspace{1cm} (1.4)

and is also known as the KdV equation. It has been extensively studied, and many of the properties of non-linear wave equations that are known today, were discovered in connection with its solution. This equation was solved by Gardner, Greene, Kruskal, and Miura in 1967 [7-13]. Along with N.J. Zabusky and C. H. Su, they also found many interesting properties of the same. One of these is that the KdV equation has an infinite number of conservation laws, and that the conserved quantities of each of these laws can be used as a Hamiltonian for an integrable system. This collection of Hamiltonians is called the KdV hierarchy.

There exists a theorem of classical mechanics, which states that if a Hamiltonian system with \(2n\) degrees of freedom has \(n\) functionally independent conserved quantities such that the Poisson bracket of any two of them vanishes, i.e., the integrals of motion are in ‘involution’, the system is completely integrable. It is clear that solutions of systems with an infinite number of conserved quantities must be infinitely restricted. A soliton is precisely such a solution: it is a localized wave which retains its shape even after collisions. Intuitively, it is clear that for this to happen, there must be an infinite number of conservation laws, and therefore an infinite number of conserved quantities. The terms ‘integrable models’ and ‘solitons’ are often used synonymously.

A system of coupled equations of motion describing a 1-dimensional crystal with non-linear coupling between nearest neighbour atoms, was introduced by M. Toda [14] in 1967. The equations of motion are

\[
m \frac{d^2 r_n}{dt^2} = a[2e^{-r_n} - e^{-r_{n-1}} - e^{-r_{n+1}}]
\]  \hspace{1cm} (1.5)

where \(r_n = u_{n+1} - u_n\), and \(u_n(t)\) is the longitudinal displacement of the \(n\)-th atom with mass \(m\) from its equilibrium position, \(a\) being a constant. These models admit soliton solutions which have been studied experimentally on an electrical network by K. Hitota and K. Suzuki [15]. In the continuum limit, these equations reduce to the KdV equation [5].

We see that models with exponential interactions are a source of non-linear equations, the Liouville and Sine-Gordon equations being examples. The Liouville equation could be generalized to include a mass term:

\[
\frac{\partial^2 \phi}{\partial x \partial y} + m^2 \phi = e^\phi
\]  \hspace{1cm} (1.6)

while the Sine-Gordon equation could be generalized to the ”Sinh-Gordon” equation with the replacement \(\omega \rightarrow i\omega\). Thus

\[
\frac{\partial^2 \omega}{\partial x \partial y} + m^2 \omega = \sinh \omega
\]  \hspace{1cm} (1.7)

We also have the Toda Field Theory equations

\[
\frac{\partial^2 \phi_i}{\partial x \partial y} = -e^{k_{ij} \phi_j}
\]  \hspace{1cm} (1.8)
Here $k_{ij}$ is the Cartan matrix for some complex Lie Algebra. The simplest of these field theories is the $A_r$ Toda field theory, and it includes the Liouville field theory for the special case $r = 1$. There exist generalizations of the Toda equations called ”Affine Toda Equations”, and have an extra term on the RHS, taking the form:

$$\frac{\partial^2 \phi_i}{\partial x \partial y} = -e^{k_{ij} \phi_j} + \gamma R_i e^{k_0 \phi_j} \quad (1.9)$$

Here $K$ is an affine Cartan matrix, and $R_i$ the right null vector for this matrix when $R_0$ is normalized to unity.

These models include the Sinh-Gordon equation as a special case. Both the Toda and Affine Toda field theories have an infinite number of conserved quantities [16]. They admit soliton solutions with an imaginary $\phi_i$ [17]. Both models have been formally solved by Leznov and Saveliev [18].

The Toda field theories can be obtained from the Toda Lattice by setting

$$\psi_i = (\phi_i - \phi_{i-1}) - (\phi_{i+1} - \phi_i) \quad (1.10)$$

whence

$$\frac{\partial^2 \psi_i}{\partial t^2} - \frac{\partial^2 \psi_i}{\partial x^2} = -[2e^{\psi_i} - e^{\psi_{i-1}} - e^{\psi_{i+1}}] \quad (1.11)$$

for $SU(n + 1)$, showing that the space-independent solutions of (1.11) satisfy (1.5).

Since the Toda field theories are the $\gamma = 0$ limits of the Affine Toda field theories, they could be used to classify 2-dimensional models with a second order phase transition, with the Toda field theory describing the model at the critical point where it has to be conformally invariant [19]. Hence the great interest in (Affine) Toda field theories. However the precise connection is still unclear. Central charges and critical exponents have been calculated and compared. One hopes that the Affine Toda field theories are perturbations that correspond to the physical model away from the critical point. However, more explicit connections are yet to be found.

The method originally used for solving non-linear equations, and especially the $KdV$ equation, was the inverse scattering method originated by Gelfand and Levitan [20]. This involved looking for a linear equation related to the original non-linear equation, and studying the evolution of the latter. In 1968, P.Lax provided this method within a solid theoretical framework [21]. The Lax equation is

$$L_t + [L, M] = 0 \quad (1.12)$$

where $L$ and $M$ are operators satisfying

$$L \psi = \lambda \psi; \quad (1.13)$$

and

$$\psi_t = M \psi \quad (1.14)$$

where $\lambda$ is a scalar, and $\psi$ a solution of a linear equation which is just the Schroedinger equation for the $KdV$ case! The Lax equation was generalized to the form of a zero curvature condition which facilitates greatly the form of the transition matrix from the initial to the final state.
In what follows, we attempt to give a pedagogical presentation of Integrable Systems with special emphasis on the \( KdV \) and Toda systems. After an introduction to the \( KdV \) equation and its properties, we show how an infinite number of conserved quantities arise via the Muira [8] transformation, while detailed calculations are referred to ref.[22]. We then dwell on solutions of the \( KdV \) equation via the inverse scattering method and the Lax formalism [21], after which we obtain the Lax operator for symplectic manifolds, using the Toda Lattice as an example. The group structure of the Toda equations for \( SU(N) \) is also studied. The Lax transformation was later generalized by Zakharov and Shabat [23] to a first order formalism which was used by Ablowitz, Kamp, Newell and Segur (AKNS) [24], for a unified description of other integrable models. The essential features of this approach are also discussed. A fall-out of the above is the ‘zero curvature condition’ that facilitates the transition to the quantum case. However, the treatment we follow is strictly classical.

Next we take up the Toda field theories, and after reporting briefly the connection with conformal invariance, dwell on the Hamiltonian reduction of the WZNW model to the Toda field theory, which in effect transforms an affine Lie Algebra to a W-Algebra. (Most calculational details are skipped, but may be found in the literature [25]). Finally we refer to the interesting connection between the 4D self-dual Yang-Mills theory and 2D Integrable models, and the generalization to SuperSymmetry.

The material is presented as follows. In Sect.2, we introduce the \( KdV \) equation and its conserved quantities. In Sect.3, solutions of non-linear equations are taken up, in particular the inverse scattering method and the Lax formalism. In Sect.4, we digress to Symplectic Manifolds and construct conserved quantities for these manifolds. Sect.5 applies the above framework to the Toda Lattice where the group structure of the Toda equations is also discussed. In Sect.6, we take up the unifying first order formalism of Zakharov and Shabat [23], continuing in Sect.7 to the zero curvature formalism and its ramifications. In Sect.8, we take up Conformal Invariance, and introduce Toda Field Theories which are constructed independently of the Toda Lattice. In Sect.9, we carry out the Hamiltonian reduction of the WZNW model to Toda Field Theories. Finally in Sect.10, we take up the connection of Toda Field Theories with Self-dual Yang-Mills models. Sect.11 contains some concluding remarks.

## 2 The \( KdV \) Equation

The \( KdV \) equation was formulated to explain the solitary water waves observed by J.Scott Russell in the Edinburgh Glasgow canal. It is a non-linear equation in one space and one time dimension and possesses soliton solutions. Of this, however, nothing was known at the time of its formation.

The \( KdV \) equation after an initial scaling takes the form

$$\frac{\partial u}{\partial t} = u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}$$  \hspace{1cm} (2.1)

This equation is Galilean invariant, but not Lorentz invariant. It can be derived from the Hamiltonian

$$H(u) = \int_{-\infty}^{+\infty} \left[ \frac{u^3}{6} - \frac{1}{2} \left( \frac{\partial u}{\partial x} \right)^2 \right] dx$$  \hspace{1cm} (2.2)
where the $u(x)$ satisfy the Poisson bracket relations

$$[u(x), u(y)] = \partial_x \delta(x - y)$$  \hspace{1cm} (2.3)

However, the Lagrangian from which it can be derived, is non local:

$$L_{KdV} = \frac{1}{2} \int_{-\infty}^{+\infty} dx dy u(x) \frac{\partial u(y)}{\partial t} - \int dx \left[ \frac{u^3}{6} - \frac{1}{2} \left( \frac{\partial u}{\partial x} \right)^2 \right]$$  \hspace{1cm} (2.4)

where

$$\epsilon(x - y) = \theta(x - y) - \frac{1}{2},$$  \hspace{1cm} (2.5)

$\theta$ being the step function. Ergo, one cannot write down a local Lagrangian whose Euler-Lagrange equations yield the KdV equation.

Solutions of the KdV equation can be shown to be soliton solutions which travel without any change of shape. It is the non-linear term which is responsible for the above property.

What is most interesting about the KdV equation is that it admits of an infinite number of conserved quantities as was shown by Miura [8]. This procedure is explained below.

The KdV equation is related to another equation called the modified KdV (MKdV) equation, viz.,

$$\frac{\partial v}{\partial t} = v^2 \frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3}$$  \hspace{1cm} (2.6)

where $v$ is related to $u$ in the KdV equation through the Riccati transformation

$$u = v^2 + i\sqrt{6} \frac{\partial v}{\partial x}$$  \hspace{1cm} (2.7)

The MKdV equation is however not Galilean invariant. Under the transformation

$$t \to t; \quad x \to x + \frac{3t}{2\epsilon^2}; \quad u \to u + \frac{3}{2\epsilon^2}; \quad v \to \frac{v\epsilon}{\sqrt{6}} + \frac{\sqrt{6}}{2\epsilon}$$  \hspace{1cm} (2.8)

it reduces to

$$\partial_t v = (\frac{\epsilon^2 v^2}{6} + v) \partial_x v + \partial_x^3 v = \partial_x \left[ \frac{\epsilon^2 v^3}{18} + \frac{v^2}{2} + \partial_x^2 v \right]$$  \hspace{1cm} (2.9)

This yields a solution of the KdV equation through the transformation

$$u = \epsilon^2 v^2/6 + v + i\epsilon \partial_x v$$  \hspace{1cm} (2.10)

The second form of (2.9) is in the nature of a continuity equation, so that we can identify

$$K = \int_{-\infty}^{+\infty} dx v(x(t))$$  \hspace{1cm} (2.11)

as the conserved quantities. $v$ can be inverted in terms of $u$ as

$$v = \sum_{0}^{\infty} \epsilon^n v_n(u(x, t))$$  \hspace{1cm} (2.12)

and this yields $v_n(u(x, t))$ as the conserved densities, since each power of $\epsilon$ must independently satisfy a continuity equation. That these are also in involution can also be checked, being explicitly shown by Das [22]. Some of the conserved quantities are

$$v_1 = -i \partial_x u_1; \quad v_2 = -\frac{u^2}{6} - \partial_x^2 u; \quad v_3 = i\partial_x \left[ \frac{u^2}{3} + \partial_x^2 u \right]$$  \hspace{1cm} (2.13)
3 The Lax Framework

Linear Hamiltonian systems with fixed initial value problems can be solved using the Laplace or Fourier transformations. Such methods are inapplicable for the nonlinear equations and new methods must be found. Gardner, Green, Krushal and Miura [9] managed to solve the initial value problem for the KdV equation in a very ingenious way. In subsequent years, this method has become the standard method for solving non-linear systems and goes by the name of inverse scattering theory [20,21]. This method is outlined in Fig 1.

The initial value for the partial differential equation is used as the potential in a 1-dimensional scattering problem for a linear equation, e.g. the Schroedinger equation. One then finds the so called scattering data, i.e. discrete spectrum, normalization constants, reflection constants (as a function of the wave number) for this scattering problem. Using the partial differential equation \((pde)\) evaluated for \(|x|\) asymptotically large, (and hence the \(pde\) becomes a linear equation because the potential is assumed to vanish at spatial infinity), the values of the scattering data can be found for all later times. Finally, the scattering data allow one to reconstruct the potential, and hence the solution of the \(pde\) for any later time.

One would intuitively like a better understanding of the origin and relevance of the linear Schroedinger equation. One way to see this is through a generalized Riccati relation of the form:

\[
u + 6\lambda = v^2 + i\sqrt{6}\frac{\partial v}{\partial x}
\]  

(3.1)

so that the \(KdV\) relation (2.1) reduces to

\[
\frac{\partial v}{\partial t} - (v^2 - 6\lambda) \frac{\partial v}{\partial x} - \frac{\partial^3 v}{\partial x^3} = 0
\]

(3.2)

As mentioned earlier, a solution of the \(MKdV\) equation yields a solution of the \(KdV\) equation through the Riccati relation. The simplest way to attempt an inversion of the
Riccati relation is to linearize it. To that end we define
\[ v = i\sqrt{6}\psi_x/\psi \] (3.3)
so that (3.1) takes the form
\[ u + 6\lambda = -6\psi_{xx}/\psi, \] (3.4)
or equivalently,
\[ \psi_{xx} + \left(\frac{u}{6} + \lambda\right)\psi = 0 \] (3.5)
which is, in fact, the time-independent Schroedinger equation. There exists however a more formal theory due to Lax [21], which we now elaborate.

Given a linear equation described by a time-independent Hamiltonian \( H \), and an operator \( A \) whose expectation values are time independent, \( A(t) \) is unitarily equivalent to \( A(0) \):
\[ U\dagger(t)A(t)U(t) = A(0) \] (3.6)
where \( U(t) \) is the time-evolution operator with the form
\[ U(t) = \exp[-iHt] \] (3.7)
Differentiating (3.6) gives
\[ U\dagger(t)\left(\frac{\partial A}{\partial t} - i[A,H]\right)U(t) = 0 \] (3.8)
which implies
\[ \frac{\partial A}{\partial t} = i[A,H] \] (3.9)
Thus for the expectation value of \( A(t) \) to be time independent, the standard time evolution relation (3.8) must be satisfied. Further, from eq.(3.7) follows the relation
\[ \frac{\partial U(t)}{\partial t} = -iHU(t) = BU(t) \] (3.10)
where
\[ B = -iH \] (3.11)
is an anti-Hermitian operator.

This argument is mimicked in the case of a non-linear evolution equation. Let
\[ L(u(x,t)) = L(t) \] (3.12)
denote the linear operator we seek. We assume it to be Hermitian, and to have eigen-values independent of \( t \). For this to be true, one must have \( u\dagger(t)L(t)u(t) = L(0) \). Differentiating both sides w.r.t. \( t \), we obtain
\[ \frac{\partial U\dagger(t)}{\partial t}L(t) + U\dagger(t)\frac{\partial L(t)}{\partial t}U(t) + U\dagger(t)L(t)\frac{\partial U(t)}{\partial t} \] (3.13)
Unlike the linear case, we do not know the form of \( U(t) \). However, \( U \) is unitary, so
\[ U\dagger U = 1 \Rightarrow \frac{\partial U\dagger(t)}{\partial t}U(t) + U\dagger\frac{\partial U(t)}{\partial t} = 0 \] (3.14)
Thus we can write
\[ \frac{\partial U(t)}{\partial t} = B(t)U(t) \] (3.15)
where anti-hermiticity must be imposed on \( B \). Substitution in (3.12), and a little simplification, yields
\[ \frac{\partial L(t)}{\partial t} = [B(t), L(t)] \] (3.16)
which is similar to (3.8), except for the fact that we do not yet know the form of \( B \). However, let us assume that \( L(t) \) is linear in \( u(x,t) \). Consequently, the LHS of (3.14) is a multiplicative operator, proportional to the time evolution operator of \( u(x,t) \). This would ensure that the eigen-values \( \lambda \) of \( L(t) \) would be time-independent, i.e.,
\[ L(t)\psi(t) = -\lambda \psi(t) \] (3.17)
Further, \( \psi(t) \) must be unitarily related to its value at \( t = 0 \), i.e.,
\[ \psi(t) = U(t)\psi(0) \] (3.18)
and its evolution w.r.t. time would take the form
\[ \frac{\partial \psi(t)}{\partial t} = \frac{\partial U(t)}{\partial t} \psi(0) = B(t)\psi(t) \] (3.19)
The operators \( L(t) \) and \( B(t) \), when they exist, are known as the Lax pair, corresponding to a given non-linear evolution equation, and play a fundamental role in determining the solution. For the \( KdV \) equation, \( L(t) \) is obtained from the linear form of the Schroedinger equation
\[ L(t) = D^2 + \frac{1}{6}u(x,t); \quad D \equiv \frac{\partial}{\partial x}. \] (3.20)
By trial and error, \( B(t) \) can be chosen so that (3.15) is satisfied, and a possible solution is
\[ B(t) = 4D^3 + \frac{1}{2}(Du + uD) \] (3.21)
The solution for \( \psi \) w.r.t. \( t \) follows from (3.18) and (3.20) to be
\[ \psi_t = 4\psi_{xxx} + \frac{1}{2}u_x\psi + u\psi_x + \text{const.}\psi \] (3.22)
which yields, using the Schroedinger equation (3.5):
\[ \psi_t + \frac{1}{6}u_x\psi - \frac{1}{3}u\psi_x + 4\lambda \psi_x = \text{const.}\psi \] (3.23)
A. Lenard [26], in an unpublished report, further displayed the relation between the Schroedinger equation and the \( KdV \) relation by elegantly deriving the latter from the former, using only the assumption that the spectral parameter \( \lambda \) in (3.4) is time-independent.

The \( KdV \) equation exhibits also a fascinating symmetry, i.e., that of the group \( SL(2, R) \). Consider a group element
\[ g = \exp[i\theta^a T_a] \] (3.24)
where $T_a$ is a generator of $SL(2, R)$, and define
\[ A_\mu \equiv g^{-1}\partial_\mu g \] (3.25)
Then the KdV equation follows from the fact that the Maurer-Cartan equation
\[ \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu] = 0 \] (3.26)
is satisfied for a special choice of gauge, e.g.,
\[ A_1^1 = -\sqrt{\lambda}; \quad A_3^3 = 6; \quad A_1^2 = -\frac{1}{36}u(x, t); \quad A_0^3 = A(u(x, t)) \] (3.27)

4 Lax Formalism On Symplectic Manifolds

In this Section, we conclude the above study of the KdV equation with a short discussion on symplectic geometry, which is directly relevant for application to the Toda Lattice.

A symplectic manifold is one with a preferred 2-form $f_{\mu\nu}$ which is non-degenerate and closed. The phase space of an integrable model corresponds to a very special symplectic manifold, since it possesses a dual Poisson bracket structure. We assume that there exist two distinct 2-forms which are both non-degenerate and closed. One way of expressing the existence of two distinct symplectic structures is to require that the same dynamical equation be described by two distinct first order Lagrangians $L_0$ and $L$, where
\[ L_0 = \theta_\mu^{(0)}(y)\dot{y}^\mu - H_0(y); \] (4.1)
\[ L = \theta_\mu(y)\dot{y}^\mu - H(y) \] (4.2)
where
\[ \dot{y}^\mu = \frac{dy^\mu}{dt}; \quad [\mu = 1, 2, ...2N] \] (4.3)
The Euler-Lagrangian equations following from (4.1-2) are
\[ f_{\mu\nu}(y)y^\nu = \partial_\mu H_0(y) \] (4.4)
\[ F_{\mu\nu}(y)y^\nu = \partial_\mu H(y) \] (4.5)
where
\[ f_{\mu\nu} = \partial_\mu \theta_\nu^{(0)}(y) - \partial_\nu \theta_\mu^{(0)}(y) \] (4.6)
\[ F_{\mu\nu} = \partial_\mu \theta_\nu(y) - \partial_\nu \theta_\mu(y) \] (4.7)
It is easy to see that the two forms $f$ and $F$ are closed, where
\[ f = \frac{1}{2}f_{\mu\nu}dy^\mu \wedge dy^\nu \] (4.8)
\[ F = \frac{1}{2}F_{\mu\nu}dy^\mu \wedge dy^\nu \] (4.9)
since $f_{\mu\nu}$ and $F_{\mu\nu}$ satisfy the Bianchi identities
\[ \partial_\lambda f_{\mu\nu} + \partial_\mu f_{\nu\lambda} + \partial_\nu f_{\lambda\mu} = 0; \] (4.10)
and
\[ \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0. \] (4.11)
Besides, they must also be non-degenerate since (4.4) and (4.5) describe the same dynamical system. Let their universes be \( f_{\mu\nu} \) and \( F_{\mu\nu} \), i.e.,
\[ f_{\mu\nu} f^{\nu\lambda} = F_{\mu\nu} F^{\nu\lambda} = \delta_\lambda^\mu \] (4.12)
so that (4.4-5) take the forms
\[ \dot{y}^\nu = f^{\nu\mu} \partial_\mu H_0(y) \] (4.13)
\[ \dot{y}^\nu = F^{\nu\mu} \partial_\mu H(y). \] (4.14)
We can also construct a nontrivial \((1,1)\) tensor \( S^\nu_\mu \) as
\[ S^\nu_\mu = F_{\mu\lambda}(y) f^{\lambda\nu}(y). \] (4.15)
Consistency of (4.4) and (4.5) further requires that
\[ \partial_\mu \partial_\nu H_0(y) - \partial_\nu \partial_\mu H_0(y) = 0 \] (4.16)
so that after a little algebra, one can show that
\[ \frac{df_{\mu\nu}(y)}{dt} = -U^{\nu}_\mu f_{\lambda\nu} + U^{\lambda}_\nu f_{\lambda\mu} \] (4.17)
where
\[ U^\nu_\mu = \partial_\mu y^\nu = \partial_\mu [f^{\nu\lambda} \partial_\lambda H_0(y)] = \partial_\mu [F^{\nu\lambda} \partial_\lambda H(y)] \] (4.18)
with a corresponding relation for \( F^{\mu\nu} \), i.e.,
\[ \frac{dF_{\mu\nu}(y)}{dt} = -U^{\nu}_\mu F_{\lambda\nu} + U^{\lambda}_\nu F_{\lambda\mu} \] (4.19)
involving the same \( U \)-tensor. The corresponding equations for the inverses \( f^{\mu\nu} \) and \( F^{\mu\nu} \) follow from (4.17) and (4.19), and have the forms
\[ \frac{df^{\mu\nu}(y)}{dt} = f^{\mu\lambda} U^\nu_\lambda - f^{\nu\lambda} U^\mu_\lambda \] (4.20)
\[ \frac{dF^{\mu\nu}(y)}{dt} = F^{\mu\lambda} U^\nu_\lambda - F^{\nu\lambda} U^\mu_\lambda \] (4.21)
We can finally show that
\[ \frac{dS^\nu_\mu}{dt} = S^\lambda_\mu U^\nu_\lambda - U^\lambda_\mu S^\nu_\lambda \] (4.22)
which in matrix notation
\[ \frac{dS}{dt} = [S, U] \] (4.23)
can be recognized as a Lax equation (3.15), thus providing a Lax representation of the dynamical equations (4.13) and (4.14). One important consequence of (4.23) is that the set of quantities
\[ K_n = \frac{1}{n} Tr S^n \] (4.24)
and
\[ K_0 = \ln |\det S| \]  
(4.25)
can be shown to be invariants since
\[ \frac{dK_n}{dt} = Tr[P(S) \frac{dS}{dt}] = Tr[P(S)[S,U]] = 0 \]  
(4.26)

\( P(S) \) is a polynomial in \( S \). That these are in involution can easily be checked, as done explicitly in ref.[22]. Applied to the KdV equation, the two Poisson structures of that equation are given by the correspondence:

\[ F_{\mu\nu} \rightarrow D; \]  
(4.27)
\[ f_{\mu\nu} \rightarrow D^3 + \frac{1}{3}(Du + uD) \]  
(4.28)

Going to the coordinate bases we have

\[ F(x,y) = <y | D | x> = \partial_x \delta(x - y) \]  
(4.29)
\[ f(x,y) = \frac{\partial^3}{\partial x^3} + \frac{1}{3}(\partial_x u + u\partial_x)\delta(x - y) \]  
(4.30)

so that

\[ F^{-1}(x,y) = \epsilon(x - y) = \theta(x - y) - \frac{1}{2} \]  
(4.31)

However \( f^{-1}(x - y) \) cannot be expressed in a closed form. The Lax operator \( S \) takes the form

\[ S = D^2 + \frac{2}{3}u + \frac{1}{3}(Du)D^{-1}, \]  
(4.32)

and with a little algebra, (4.23) can be shown to be reduced to the KdV equation, with consequently an infinite \# of conserved quantities. This is described in detail in ref.[22].

5 The Toda Lattice

The model of the KdV equation that has been studied so far is a continuum model. A finite dimensional system with a finite \# of degrees of freedom is simpler to study. The Toda Lattice is such a system to which the symplectic approach of the above Section is especially applicable. We now study the Toda Lattice and its integrability from a symplectic point of view, following it up with a group theoretical treatment.

The Toda Lattice describes the motion of \( N \) point masses on the line, under the influence of an exponential interaction. The Hamiltonian equations in terms of the canonical coordinates \( Q_i \) and momenta \( P_i \) are given by

\[ \dot{Q}_i = P_i; \quad (i = 1, 2, \ldots, N); \]  
\[ \dot{P}_j = e^{-(Q_j - Q_{j-1})} - e^{-(Q_{j+1} - Q_j)}, \quad (j = 2, 3, \ldots, N - 1); \]  
\[ \dot{P}_1 = -e^{-(Q_2 - Q_1)}; \quad \dot{P}_N = e^{-(Q_N - Q_{N-1})}. \]  
(5.1)

The equations can be cast into a more symmetrical form by enlarging the system to \((N + 2)\) point masses, with end points at spatial infinity. In that case, the Hamiltonian equations take the form:

\[ \dot{Q}_i = P_i; \quad (i = 1, 2, \ldots, N); \dot{P}_i = e^{-(Q_i - Q_{i+1})} - e^{-(Q_{i+1} - Q_i)}. \]  
(5.2)
We can choose
\[ y^i = Q_i; \quad y^{N+i} = P_i; \quad (i = 1, 2, ..N). \] (5.3)

Applying the geometrical method of the previous Section, two choices of the Lagrangian
are as follows:
\[ L_0 = \sum_{i=1}^{N} \left[ \frac{1}{2} (P_i \dot{Q}_i - Q_i \dot{P}_i) - \frac{1}{2} P_i^2 + e^{-(Q_{i+1} - Q_i)} \right]; \] (5.4)
\[ L = \sum_{i=1}^{N} \left[ \frac{1}{2} (P_i^2 + e^{-(Q_{i+1} - Q_i)}) \dot{Q}_i + \pi_i(P) \dot{P}_i \right] - H(Q, P) \] (5.5)

where
\[ \pi_i(P) = \frac{1}{2} \sum_{j=1}^{N} \epsilon(i - j) \dot{P}_j; \] (5.6)
\[ H(Q, P) = \sum_{i=1}^{N} \left[ \frac{P_i^3}{3} + (P_i + P_{i+1}) e^{-(Q_{i+1} - Q_i)} \right] \] (5.7)

\( f_{\mu\nu} \) turns out to have the canonical Poisson bracket structure
\[ f_{\mu\nu} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \] (5.8)
so that
\[ f^{\mu\nu} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \] (5.9)

\( F_{\mu\nu} \) can be shown to have the form [22]
\[ F_{\mu\nu} = \begin{pmatrix} A & B \\ B & e \end{pmatrix} \] (5.10)

where
\[ A_{ij} = \delta_{i+1,j} e^{-(Q_{i+1} - Q_i)} - \delta_{i,j+1} e^{-(Q_{j+1} - Q_j)} \] (5.11)
\[ B_{ij} = P_i \delta_{ij}; \quad e_{ij} = \epsilon(j - i) \]

The \((1, 1)\) tensor \( S_\mu^{\nu} \) thus takes the form
\[ S_\mu^{\nu} = \begin{pmatrix} B & A \\ -e & B \end{pmatrix} \] (5.12)
and the conserved quantities are
\[ TrS = 2TrB = 2 \sum_{i=1}^{N} P_i; \] (5.13)
\[ \frac{1}{2} TrS^2 = Tr[2B^2 - (Ae + eA)] = \sum_{i=1}^{N} \left[ \frac{P_i^2}{2} + e^{-(Q_{i+1} - Q_i)} \right] \equiv H_0(Q, P); \] (5.14)
\[ \frac{1}{6} TrS^3 = \sum_{i=1}^{N} \frac{P_i^3}{3} + (P_i + P_{i+1}) e^{-(Q_{i+1} - Q_i)} \equiv H(Q, P) \] (5.15)
The Lax representation (4.23) for the Toda equation takes the form of the following matrix equations

\[
\frac{dA}{dt} = -[B, D];
\]

\[
\frac{dB}{dt} = A - De = \frac{1}{2}[e, D]
\]

which reduce to the Toda equations \( \dot{Q}_i = P_i \) and \( \dot{P}_i = e^{-(Q_i - Q_{i-1})} - e^{-(Q_{i+1} - Q_i)} \) respectively.

### 5.1 Group Structure of Toda Equations

Eq.(5.1) can be differentiated and put in the form

\[
\ddot{Q}_1 = -e^{-(Q_2 - Q_1)}
\]

\[
\ddot{Q}_i = \dot{P}_i = e^{-(Q_i - Q_{i-1})} - e^{-(Q_{i+1} - Q_i)}
\]

\[
\ddot{Q}_N = \dot{P}_N = e^{-(Q_N - Q_{N-1})}
\]

It is easily checked that

\[
\sum_{i=1}^{N} \dot{Q}_i = \sum_{i=1}^{N} \dot{P}_i = 0
\]

i.e., the total momentum is conserved, and therefore the centre of mass motion can be separated and the dynamics of the system expressed in terms of \((N - 1)\) coordinates and momenta. Defining

\[
q_a = Q_{a+1} - Q_a; \quad a = 1, 2...N - 1,
\]

the second order equations satisfied by the \(q_a\)'s can be written as

\[
\ddot{q}_1 = 2e^{-q_1} - e^{-q_2}
\]

\[
\ddot{q}_a = -e^{-q_{a-1}} + 2e^{-q_a} - e^{-q_{a+1}}; \quad a = 1,..N - 1
\]

\[
\ddot{q}_N = -e^{-q_{N-1}} + 2e^{-q_N}
\]

which can be compactly written as

\[
\ddot{q}_a = \sum_{b=1}^{N-1} K_{ab}e^{-q_b}
\]

\(K_{ab}\) being the Cartan matrix for \(SU(N)\). Eq.(5.22) generalizes for the other Lie Algebras as well.

The Lagrangian giving rise to the above Euler-Lagrangian equations can be written as

\[
L = \sum_{a=1}^{N} \sum_{b=1}^{N} \frac{1}{2} \dot{q}_a K_{ab}^{-1} \dot{q}_b - \sum_{a=1}^{N} e^{-q_a}
\]

\(K_{ab}^{-1}\) being the inverse of the Cartan matrix. The momenta conjugate to \(q_a\) are defined as

\[
p_a = \frac{\partial L}{\partial \dot{q}_a} = \sum_{b=1}^{N-1} K_{ab}^{-1} \dot{q}_b
\]

and it is easily checked that

\[
\{q_a, p_b\} = \delta_{ab}
\]
so that \( \{q_a, p_a\} \) constitute a canonical coordinate system.

That the group structure entering above is not just accidental, can be seen by defining the following Lax operators:

\[
S = \frac{1}{2} \sum_{a=1}^{N} [p_a H_a + (E_a + E_{-a}) e^{-q_a/2}] ;
\]

\[
U = -\frac{1}{2} \sum_{a=1}^{N-1} e^{-q_a/2} [E_a - E_{-a}]
\]

where \( H_a \) and \( E_a \) are the generators of \( SU(N) \) in the Chevalley basis.

The Lax equation (4.23) can be seen to be satisfied, since \( \frac{dS}{dt} - [S, U] \) reduces to

\[
\frac{1}{2} \sum_{a,b=1}^{N-1} H_a K_{ab}^{-1} [\tilde{q}_b - \sum_{c=1}^{N-1} K_{bc} e^{-q_c}]
\]

which is zero by virtue of the Toda equations (5.22). Hence the quantities

\[
K_n = \frac{1}{n} \text{Tr} S^n
\]

must be conserved under the flow of the Toda equations. Since \( S \) belongs to the \( SU(N) \) algebra, the number of independent conserved quantities can equal \( (N - 1) \), which is the rank of \( SU(N) \). The total number of conserved quantities is thus \( N \), if we add the total momentum. It can be shown that these are also in involution [22]. This treatment is due to Leznov and Saveliev [18].

6 Zakharov-Shabat Formalism

So far we have only studied two integrable models, viz., the continuum \( KdV \) and the finite dimensional Toda Lattice. In trying to understand the non-linear Schrödinger equation which is also integrable, Zakharov and Shabat [23] obtained a description which was later generalized by AKNS [24] to describe various other integrable models. This approach uses a Lax operator which is first order in the derivative \( \partial_x \), in contrast to the second order formalism in eq.(3.19). Besides describing various integrable models in a unified manner, this approach has the additional advantage that the inverse scattering method generalizes readily to the quantum case. In what follows, we describe the first order formulation of the Lax operator, and elucidate the essential features of this approach. It is easily checked that if

\[
L(t) \psi(t) = -\lambda \psi(t);
\]

\[
\partial_t L(t) = [B(t), L(t)],
\]

where

\[
\frac{\partial \psi(t)}{\partial t} = B(t) \psi(t),
\]

then

\[
\frac{\partial \lambda(t)}{\partial t} = 0.
\]
We can invert the argument to identify the Lax pair in the following way. Namely, if
\[ L(t)\psi(t) = -\lambda\psi(t); \frac{\partial\psi(t)}{\partial t} = B(t)\psi(t), \text{ with } \frac{\partial\lambda(t)}{\partial t} = 0 \]

i.e., if the compatibility condition of (6.1) and (6.3) yield the system under study, then
\( L(t) \) and \( B(t) \) can be identified as the Lax pair of the system. We would like \( L(t) \) to be linear in \( \partial_x \). Using the analogy between the Klein-Gordon and Dirac equations, we define a two-component column matrix
\[
\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}
\]
(6.5)
and generalize the two equations (6.1) and (6.3) to first order matrix equations as

\[
\frac{\partial \phi}{\partial x} = (q\sigma_+ + r\sigma_- i\rho\sigma_3)\phi; \quad (6.6) \\
\frac{\partial \phi}{\partial t} = (P\sigma_+ Q\sigma_+ R\sigma_3)\phi
\]

where \( \sigma_\pm \) and \( \sigma_3 \) are the Pauli spin matrices. The dynamical variables \( q(x, t) \) and \( r(x, t) \) do not depend on the spectral parameter \( \rho \) which is assumed to be independent of \( x \) and \( t \). The coefficient functions \( P \), \( Q \) and \( R \) on the other hand, do depend on \( \rho \), and are functionals of \( q \) and \( r \). Demanding that the partial derivatives of \( \phi \) w.r.t. \( x \) and \( t \) commute, we obtain the compatibility conditions to be

\[
\frac{\partial R}{\partial x} = qQ - rP; \quad (6.7) \\
\frac{\partial r}{\partial t} = \frac{\partial Q}{\partial x} - 2rR - 2i\rho Q; \quad (6.8) \\
\frac{\partial q}{\partial t} = \frac{\partial P}{\partial x} + 2qR + 2i\rho P \quad (6.9)
\]
i.e., if (6.7-9) describe the non-linear evolution of a system, then (6.6) describes the Lax pair appropriate for such a system. Explicitly

\[
L = \partial_\lambda - q\sigma_+ - r\sigma_- \quad (6.10) \\
B = P\sigma_+ + Q\sigma_- + R\sigma_3 \quad (6.11)
\]
so that (6.2) is satisfied.

The choice of \( r = 6 \) yields the \( KdV \) equation, and the choice \( r - q = -i\nu/\sqrt{6} \), the \( MKdV \) equation. The choice \( q = \sqrt{k}\psi^* \) and \( r = \sqrt{k}\psi \), \( k \) being an arbitrary constant parameter, yields the non-linear Schroedinger equation:

\[
i\partial_\psi \psi = -\psi_{xx} + 2k|\psi|^2\psi \quad (6.12)
\]
and the choice \( r = -q = \frac{1}{2}\omega_x \) with

\[
P = Q = \frac{i}{4\rho}\sin\omega
\]
yields the sine-Gordon equation.
The operator \((L + \lambda)\) in (6.1) can be rewritten as \(v(x, t, \lambda) + \partial_x\), where

\[
v = -q\sigma_+ - r\sigma_3 + i\rho\sigma_3 \tag{6.13}
\]

If one knows the solution of the associated Schrödinger equation at some other point \((x, t)\) by multiplying the solution by a hermitian matrix \(T(x, y, t, \lambda)\), i.e.,

\[
\psi(x, t, \lambda) = T(x, y, t, \lambda)\psi(y, t, \lambda) \tag{6.14}
\]

where \(T(x, y, t, \lambda)\) is a solution of

\[
\partial_x T(x, y, t, \lambda) = -(q\sigma_+ - r\sigma_- + i\rho\sigma_3)T(x, y, t, \lambda) \tag{6.15}
\]

with the initial condition \(T(x, x, t, \lambda) = I\).

### 7 The Zero Curvature Condition

The Lax condition (6.2) can be written as

\[
[\partial_t - B, L] = 0 \tag{7.1}
\]

Using

\[
L = \partial_x - A(x) \tag{7.2}
\]

we obtain the form

\[
[(\partial_t - B), (\partial_x - A)] = 0 \tag{7.3}
\]

which is like a zero-curvature condition for

\[
F_{01} = [(\partial_0 - A_0), (\partial_1 - A_1)] \tag{7.4}
\]

with the identification

\[
A_0 = -B(x, \rho); A_1 = -A(x, \rho) \tag{7.5}
\]

The importance of the zero curvature condition stems from the fact that (6.6) may be solved, using

\[
\psi(x) = T(x, y, \rho)\psi(y) \tag{7.6}
\]

where the transformation

\[
T(x, y, \rho) = P_r \exp[-\int_y^x A_1(z)dz] \tag{7.7}
\]

where \(P_r\) denotes path ordering.

It is easy to see that \(T(x, y, \rho)\) translates solutions of the problem along the \(x\)-axis for a fixed time, i.e.,

\[
T(x, y, \rho)T(y, z, \rho) = T(x, z, \rho); \quad T^{-1}(x, y, \rho) = T(y, x, \rho); \quad T(x, x, \rho) = 1 \tag{7.8-7.10}
\]

Setting

\[
U_r(x_2, t_2; x_1, t_1) = P_r \exp[-\int_{x_1, t_1}^{x_2, t_2} A_\mu dx^\mu] \tag{7.11}
\]
and taking the product of two such exponents, it is easy to see that
\[ U_{r_1}(x_2, t_2; x_1, t_1)U_{r_2}(x_1, t_1; x_2, t_2) = \exp[-\frac{1}{2} \oint_C d\sigma^{\mu\nu} F_{\mu\nu}], \quad (7.12) \]
using the Baker-Campbell-Hausdorff formula and the Stokes theorem, the integration being done over the area enclosed by the closed path \( r_1 + r_2 \). As the curvature \( F_{\mu\nu} \) vanishes,
\[ U_{r_1}(x_2, t_2; x_1, t_1)U_{r_2}(x_1, t_1; x_2, t_2) = 1 \quad (7.13) \]
and so
\[ U_{r}^{-1}(x_2, t_2; x_1, t_1) = U_{r}(x_1, t_1; x_2, t_2) \quad (7.14) \]
so that
\[ U_{r_1}(x_2, t_2; x_1, t_1) = U_{r_2}(x_2, t_2; x_1, t_1) \quad (7.15) \]

ergo, \( U \) is independent of the path taken. For a closed path, \( U(x, t; x, t) = 1 \). Hence path ordering drops out of the transition matrix \( T(x, y, \rho) \).

Returning to the time evolution of the transition matrix, it can be shown that
\[ \partial_t T(x, y, \rho) = [B(x, \rho), T(x, y, \rho)] \quad (7.16) \]
which is the form of a Lax equation, so that all quantities of the form
\[ K_n = \frac{1}{n} Tr[T(\rho)]^n; \quad K_0 = \ln[detT(\rho)] \quad (7.17) \]
are conserved. We thus have an infinite number of conserved quantities when the zero curvature conditions are fulfilled. That this holds also for Toda Field Theories was shown by Olive and Turok [16].

8 From Conformal Invariance To Toda Field Theory

That the \( KdV \) equation has a hidden conformal symmetry can be seen by making a Fourier expansion with Fourier coefficients
\[ u_n = -\frac{1}{4} \int_0^{2\pi} u(x)e^{-inx} \frac{dx}{2\pi} \delta_{n0} \quad (8.1) \]
It can be shown that the Poisson brackets of the \( u_n \) satisfy the Virasoro Algebra (up to trivial factors), i.e.,
\[ -2i\pi\{u_n, u_m\} = -(n - m)u_{m+n} + \frac{1}{2}n(n^2 - 1)\delta_{n+m} \quad (8.2) \]
Higher order terms in the \( KdV \) hierarchy have a hidden \( \omega \) symmetry.

We now digress to take a look at Toda Field Theories. These are essentially the only class of integrable, interacting, conformally invariant field theories in two space-time dimensions. To see this, we start with the generic action
\[ S = \int \left[ \frac{1}{2} \partial_{\mu} \phi_i \partial^{\mu} \phi_i - V(\phi_i) \right] d^2z \quad (8.3) \]
The trace of the naive conserved energy-momentum tensor becomes

$$T^\mu_\mu = 2V.$$  \hspace{1cm} (8.4)

As the trace of the energy-momentum tensor is required to vanish in a conformally invariant theory, it seems that if $V \neq 0$, the theory is not conformal. However there is an ambiguity in the definition of the energy-momentum tensor. If we attempt to improve the naive energy-momentum tensor without violating the conservation property, we could choose

$$\theta_{\mu\nu} = T_{\mu\nu} + \left[ \partial_\mu \partial_\nu - \eta_{\mu\nu} \partial_2^2 \right] f(\phi_i)$$ \hspace{1cm} (8.5)

whence the trace of the modified energy-momentum tensor is

$$\theta_{\mu\mu} = 2V + \partial_+ \partial_- f$$ \hspace{1cm} (8.6)

\(\pm\) being the light cone directions. If the second term is to cancel the first, we somehow need to get rid of the derivatives. This can be done, using the equations of motion. Without knowing the explicit equations of motion, the most general expression for $f(\phi_i)$ is $\sum c_i \phi_i$. Using the equations of motion resulting from varying the action, the tracelessness condition becomes

$$2V + \sum_i c_i \frac{\partial V}{\partial \phi_i} = 0$$ \hspace{1cm} (8.7)

Eq.(8.7) is easily solved, with the result that the trace of the energy-momentum tensor vanishes if the potential is of the form

$$V(\phi_i) = \sum_j d_j \exp[\sum b_{ij} \phi_j],$$ \hspace{1cm} (8.8)

satisfying the requirement

$$\sum_i c_i b_{ij} = -2$$ \hspace{1cm} (8.9)

We choose $b_{ij}$ to be related to the Cartan matrix of a simple Lie Algebra. The resulting field theories are called Toda Field Theories, and are described by the action

$$S_{\text{Toda}} = \int \sum \frac{1}{2} (\partial_\mu \phi, \partial^\mu \phi) - \frac{m^2}{\beta^2} \exp (\beta \langle \alpha^{(i)}, \phi \rangle) d^2 x$$ \hspace{1cm} (8.10)

where $\langle,\rangle$ is the scalar product in the root space, and $\phi$ takes its values in the root space of the simple Lie Algebra on hand.

The equations of motion obtained from (8.10) are

$$\beta \partial_\mu \partial^\mu \phi_i + m^2 \exp (\sum K_{ij} \phi_j) = 0$$ \hspace{1cm} (8.11)

Specializing for the $SU(n)$ group, and setting $m = \beta = 1$, this becomes

$$\partial_+ \partial_- \phi_i = -\exp (K_{ij} \phi_j)$$ \hspace{1cm} (8.12)

With $\phi_0 = 0$ and $\phi_{i+1} = 0$, this reduces to

$$\partial_+ \partial_- \phi_i = -\exp (2\phi_i - \phi_{i-1} - \phi_{i+1})$$ \hspace{1cm} (8.13)
Setting
\[ \psi_i = (\phi_i - \phi_{i-1}) - (\phi_{i+1} - \phi_i), \]

after Mikhailov [27], we get the equation
\[ \partial_t^2 \psi_i - \partial_x^2 \psi_i = -[2e^{\psi_i} - e^{\psi_{i-1}} - e^{\psi_{i+1}}] \] (8.14)

which is easily seen to be related to the Toda equations (5.20). One expects that the Toda Field Theories are integrable, and it turns out that they are indeed so (see ref.[8]). The calculation rests upon the existence of a zero curvature condition for certain group theoretical combinations of \( \phi \), which can be chosen as gauge fields.

As mentioned earlier, the Toda Field Theories have been completely solved for simple \( g \) by Leonov and Saveliev [18]. They have also been solved for affine \( g \) by Olive and Turok [16].

Quantization of the Toda Field Theories is more problematic since the potential has no local minimum, the latter being attained at infinity, using the gauge group \( A_1 \). A lucid discussion of the problems encountered in the theory is given in ref.[29].

The central charge of the Toda theories can be constructed using free field technology, and is found to be [30]
\[ C = \frac{h}{2\pi} + 12 \left[ \frac{\hbar \beta \rho}{4\pi} + \frac{\rho^\vee}{\beta} \right]^2, \] (8.15)
r being the rank of the algebra, \( \rho \) being half the sum of the positive roots, and \( \rho^\vee \) its dual. Eq.(8.15) gives an indication that a quantum Toda theory with a strong coupling constant is equivalent to another Toda theory with a weak coupling constant, obtained by replacing \( \beta \) by \( 4\pi/h\beta \), and interchanging roots and ”coroots”.

Incidentally, strong/weak coupling duality has recently become a subject of immense study in relation to string theories.

It is possible to obtain the minimal models from the Toda Field Theories. For a particular value of \( \beta \), the central charges can be made to agree. However this is not enough. A complication arises from the fact that not all primary fields in the minimal models are actually present in the Toda theory. However, because of the duality in the theory, we can add another part of the potential with the coupling constant replaced by its dual; see Mansfield [31]. This modification is sufficient to give complete agreement.

9 W-Algebras: Hamiltonian Reduction of WZNW

Another fact which makes the conformally invariant Toda theories interesting is that to each such Toda theory, there corresponds a \( W \)-algebra. The \( W \)-algebras are an extension of the Virasoro algebra by adding primary fields primary fields of spin higher than \( Z \), and were introduced by Zamolodchikov [32] as a pointer to conformal field theories with a larger overall symmetry. Zamolodchikov [32] investigated the case in which a primary field \( w(r) \) of weight 3 is added to the Virasoro algebra. In order for the algebra to be close, it had to be made ‘non-linear’, and hence lost its linear Lie Algebra character.

Balog et al [33-35] showed that the Liouville and Toda Field Theories can be obtained as conformally reduced \( WZNW \) theories. This reduction can be viewed as a gauge procedure, and the Toda field theory can be obtained as the gauge invariant content of a gauged \( WZNW \) theory. The Liouville theory is obtained for the special case of the \( SL(2, R) \) gauge group.
The most powerful method of constructing $\mathcal{W}$-algebras is through the so-called quantum Drinfeld-Sokolov reduction. In this, one starts with an affine Lie Algebra, and reduces it by imposing some constraint on its generators. At the classical level, this procedure which leads to the so-called Gelfand-Dickey algebras [36], was pioneered by Drinfeld and Sokolov [37].

It is thus clear that under the reduction that takes a WZNW field theory to a Toda field theory, the affine Lie Algebra that characterizes the WZNW theory reduces to a $\mathcal{W}$-algebra that is associated to a Toda field theory. This approach is also readily generalizable to the supersymmetric case where various new $\mathcal{W}$-superlagebras have been found as symmetry algebras of supersymmetric Toda field theories. We refer the interested reader to ref.[38] for further progress in this area.

In what follows, we review the essential steps of the Lagrangian reduction of the WZNW model. The WZNW action for a non-compact group $G$ in 2D Minkowski space-time is

$$S(g) = -\frac{k}{8\pi} \int_{S^2} d^2\rho \eta^{\mu\nu} Tr(g^{-1}\partial_\mu g)(g^{-1}\partial_\nu g) + \frac{k}{12\pi} \int_B Tr(g^{-1}dg)^3$$

(9.1)

where $B$ is the volume occupied by $S^2$. The left and right Affine Kac-Moody [AKM] symmetries of this theory are generated by the Noether currents

$$J(\lambda) = \kappa Tr[\lambda(\partial_+ g)g^{-1}]; \quad \tilde{J}(\lambda) = -\kappa Tr[\lambda g^{-1}(\partial_- g)]$$

(9.2)

where $\kappa = -\frac{k}{4\pi}$, and $\lambda$ is an element of the Lie Algebra $g$. The WZNW equations of motion are known to be equivalent to the current conservation

$$\partial_- J = \partial_+ \tilde{J} = 0.$$ 

(9.3)

We now choose the following Gauss decomposition of an arbitrary element $g=ABC$, e.g.,

$$A = \exp\left[ \sum_{\alpha \in \Delta^+} x^\alpha E_\alpha \right];$$

$$B = \exp\left[ \frac{1}{2} \sum_{\alpha \in \Delta} \phi^\alpha H_\alpha \right];$$

$$C = \exp\left[ \sum_{\alpha \in \Delta^-} y^\alpha E_\alpha \right];$$

where Cartan-Weyl root vectors $E_\alpha$, Cartan subalgebra generators $H_\alpha = [E_\alpha, E_{-\alpha}]$, and a set of positive (negative) roots $\Delta^\pm$ have been introduced with the following properties

$$K_{\alpha\beta} = \alpha(H_\beta) = \frac{2\alpha.\beta}{|\alpha|^2}; \quad \alpha, \beta \in \Delta; \quad |\alpha_{long}|^2 = 2;$$

(9.5)

$$Tr(H_\alpha \dot{H}_\beta) = \frac{2}{|\alpha|^2} K_{\alpha\beta} \equiv C_{\alpha\beta};$$

(9.6)

$$Tr(E_\alpha \dot{E}_\beta) = \frac{2}{|\alpha|^2} \delta_{\alpha,-\beta}; \quad Tr[E_\alpha, H_\beta] = 0.$$ 

(9.7)

We also introduce the Polyakov-Wiegmann identity

$$S(ABC) = S(A) + S(B) + S(C) + \kappa \int d^2\rho Tr[(A^{-1}\partial_- A)(\partial_+ B)B^{-1}$$

$$+(B^{-1}\partial_- B)(\partial_+ C)C^{-1} + (A^{-1}\partial_- A)(B(\partial_+ C)C^{-1}B^{-1})]$$

(9.8)
We now see, using eqs. (9.4-9.8), that the generalized constraints

\[ J(E_\alpha) = \kappa c_1^\alpha; \quad \bar{J}(E_{-\alpha}) = -\kappa c_2^\alpha; \quad \alpha \in \Delta^+ \] (9.9)

with some real numbers \( c_1^\alpha, c_2^\alpha \) whose values do not vanish only for primitive roots \( \alpha \in \Delta \), are enough to reduce the G- based WZNW theory to the Toda Field Theory defined by the Lagrangian

\[ L_{\text{Toda}} = -\frac{k}{8\pi} \left[ \frac{1}{4} C_{\alpha\beta} \partial_+ \phi^\alpha \partial_- \phi^\beta - \sum_{\alpha \in \Delta} \alpha (u^2)^\alpha e^{\frac{1}{2} K_{\alpha\beta} \phi^\beta} \right] \] (9.10)

where \((u^2)^\alpha = |\alpha|^2 c_1^\alpha c_2^\alpha\).

Due to \( c_1^\alpha \neq 0 \) for the primitive roots, the constraint (9.7) can be re-written in terms of the Gauss decomposition (9.5-7) as follows:

\[ A^{-1} \partial_- A = B \left[ \sum_{\alpha \in \Delta} \frac{1}{2} |\alpha|^2 c_2^\alpha E_\alpha \right] B^{-1} \]

\[ = \sum_{\alpha \in \Delta} \frac{1}{2} |\alpha|^2 c_2^\alpha E_\alpha \exp\left[ \frac{1}{2} K_{\alpha\beta} \phi^\beta \right]; \]

\[ (\partial_+ C) C^{-1} = B^{-1} \left[ \sum_{\alpha \in \Delta} \frac{1}{2} |\alpha|^2 c_1^\alpha E_{-\alpha} \right] B \]

\[ = \sum_{\alpha \in \Delta} \frac{1}{2} |\alpha|^2 c_1^\alpha E_{-\alpha} \exp\left[ \frac{1}{2} K_{\alpha\beta} \phi^\beta \right]; \]

In the WZNW equations of motion, \( A \) and \( C \) occur only in the combinations given in (9.11-12), so that they can be eliminated in favour of \( B \) or \( \phi^\alpha \). The remaining equation is just the Toda equation [25,34,35]:

\[ \partial_+ \partial_- \phi^\alpha + \frac{1}{2} |\alpha|^2 (u^2)^\alpha \exp\left[ \frac{1}{2} K_{\alpha\beta} \phi^\beta \right] = 0; \] (9.13)

(see also ref.[25] for details).

As mentioned earlier, the Toda Field Theory possesses an extended symmetry represented by a classical W-algebra. These W-algebras can be obtained as the quantum versions of the so-called Gelfand-Dickey algebras [36] known in the theory of KdV equations. For instance, the Poisson bracket associated with the KdV equation in (8.2), results in the classical version of the Virasoro algebra which is the simplest W-algebra. Moreover, the Lax representation of the KdV equation (3.15), defines the third order differential operator \( B = w^{(3)} \). The Fourier components of \( B \), along with those of the KdV field, form the Gelfand-Dickey [36] algebra that generalizes to \( w^{(3)} \) in the quantum case.

Now regarding the Toda theory as a constrained WZNW theory, the Hamiltonian structure can be obtained by a classical Drinfeld-Sokolov reduction from the constrained phase space of the AKM algebra. In the Hamiltonian formalism, the AKM symmetry of the WZNW theory is represented by first class constraints. The W-algebra of the Toda theory arises as the Poisson bracket algebra of gauge-invariant polynomials of the
constrained AKM currents and their derivatives. In what follows, we summarize the arguments supporting these statements.

Let \( g(z, \bar{z}) \) be the \( G \)-valued WZNW fields and \( J(z) \) the corresponding AKM currents having the form
\[
g(z, \bar{z}) = g(z)g(\bar{z}); \quad \partial g(z) = J(z)g(z)
\] (9.14)

Let \( \text{dim}g \) be the dimension of \( G \); \( l \) its rank; \( k \) the level of the associated AKM algebra \( \hat{g} \); \( g \) the dual Coxeter number of \( G \); \( \rho \) the half sum of the positive roots; and \( \beta \) the dual of \( \rho \).

The constrained WZNW theory is specified by (9.9). After a suitable choice of constants \( c_i \), the currents \( J(z) \) can be decomposed as
\[
J(z) = I_- + j(z); \quad I_- = \sum_{i=1}^{l} E_{-\alpha_i}; \quad j(z) = \sum_{i=1}^{l} j^i(z)H_i + \sum_{\phi \in \Delta^+} E_\phi
\] (9.15)

where \( \{E_{\alpha_i}\} \) are \( l \) simple roots of \( g \). The maximal subgroup of \( \hat{G} \) leaving this form of currents invariant, is the maximal nil-potent subgroup generated by \( E_\phi \), \( (\phi \in \Delta^+) \), and implemented by the \((\text{dim}g - l)/2\) constrained AKM currents \( J^\phi(z) \). This allows us to interpret the constrained WZNW theory as the gauge theory in which all but \( l \) of the \((\text{dim}g + l)/2\) components of \( J \) are gauge components [33-35].

The current \( j(z) \) and the gauge transformations corresponding to \( E_\phi \) act on each column of the WZNW field \( g(z) \) separately, while each column contains only one gauge-invariant component \( e \) (of the highest weight), satisfying \( E_\phi e = 0 \). The gauge degrees of freedom corresponding to the other elements of each column can be eliminated by a gauge fixing in favour of \( e \). Because of (9.15), this leads to a linear pseudo-differential equation \( De = 0 \), where \( D \) is a polynomial pseudo-differential operator whose coefficients are gauge invariant polynomials in the currents \( J \). This operator \( D \) can now be used to define a classical \( \mathcal{W} \)-algebra by choosing a Drinfeld-Sobolov gauge in which one has
\[
j_{DS} = \sum w^P(z)F_P
\] (9.16)

where \( P \)'s are the orders of \( l \) independent Casimir operators of \( g \), and \( F_P \) generators with \( H \) weights \((P - 1)\), so that the gauge-fixed current (9.16) has only one non-vanishing component in each of the \( l \) irreducible representations in a decomposition of the adjoint of \( g \) w.r.t. one of its sub-groups \( SL(2, \mathbb{R}) \). The Poisson brackets between the different polynomials \( w^P \) define a classical \( \mathcal{W} \)-algebra.

We close this Section by noting that Toda field theories also play an important role in the discussion of \( \mathcal{W} \)-gravity, where they arise as effective quantum theories [39,40] for the \( \mathcal{W} \)-gravity degrees of freedom in the conformal gauge. For a quantum version of the WZNW \( \rightarrow \) Toda conformal reduction, see [34, 41].

# 10 Self-Dual Y-M Theories: 2D Integrable Models

The self-dual Yang-Mills (SDYM) theory appears to be a master theory for a whole variety of 2D integrable systems, as we are now going to explain. Though there is no general
proof, the statement can be checked on a case by case basis. The main point is that the 4D self-duality condition admits of a zero curvature representation underlying a Hamiltonian description of SDYM descendents in lower dimensions. This makes it possible to apply the inverse scattering method for integration of the SDYM equations. Simultaneously, it explains the origin of gauge symmetries in integrable systems of the \textit{KdV} type, since the SDYM theory in both gauge and conformally invariant in 4D. And last but not least, this connection provides us with a systematic way to associate the \textit{KdV} type hierarchy with any simple Lie Algebra.

SDYM solutions invariant by the action of a subgroup with two conformal generators satisfy a 2D differential equation, since each 1D subgroup reduced the number of independent variables by one. This allows us to describe the invariant SDYM solutions in terms of a 2D integrable system. All known 2D integrable systems seem to be derivable this way, by appropriate truncations of a 4D self-dual gauge theory. This is true, in particular, for the \textit{KdV} and non-linear Schrödinger equations, the Liouville and Toda equations, as well as other integrable in 2 and 3 dimensions. Our presentation in this section is only illustrative; we give one explicit example of embedding of the \textit{KdV} equation into the 4D SDYM theory \cite{42}, and a supersymmetric generalization.

Let $x^a = (x, y, z, t)$ be the coordinates of a flat 4D space-time of signature $(+, +, -, -)$. The invariant metric reads

$$ds^2 = 2dxdz - 2dydt$$  \hspace{1cm} (10.1)

The SDYM equations in 2+2 dimensions ($\epsilon_{xyzt} = 1$) read as

$$F_{ab} = \frac{1}{2} \epsilon_{abcd} F^{cd}$$  \hspace{1cm} (10.2)

and are equivalently represented by 3 equations of the form

$$F_{tx} = F_{yz} = F_{ty} + F_{xz} = 0$$  \hspace{1cm} (10.3)

After a dimensional reduction which is equivalent to setting

$$\partial_y = \partial_z - \partial_x = 0,$$  \hspace{1cm} (10.4)

(10.3) takes the form

$$[\partial_t - H, \partial_x - Q] = [P, B] = 0; \quad [H, B] = [\partial_x - Q, \partial_x - P]$$  \hspace{1cm} (10.5)

where

$$A_t = H; \quad A_x = Q; \quad A_y = -B; \quad A_z = P$$

It is clear that the first equation in (10.5) is a zero curvature equation. We now choose the non-compact group and an embedding pattern in the form

$$B = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix};$$ \hspace{2cm} (10.6)

$$Q = \begin{pmatrix} \lambda & 1 \\ -u & -\lambda \end{pmatrix}$$ \hspace{2cm} (10.7)

where $\lambda$ is a constant and $u = u(t, x + z)$. We can expand the Lie Algebra-valued fields $H$ and $P$ as

$$H = H_+ \tau_+ + H_- \tau_- + H_3 \tau_3; \quad P = P_- \tau_+ + P_+ \tau_- + P_3 \tau_3$$ \hspace{2cm} (10.8)
where $\tau_{\pm} = (\tau_1 \pm i\tau_2)/2$, and $\tau_{1,2,3}$ are the Pauli spin matrices. It is clear that the second equation of (10.5) gives

$$P_- = P_3 = 0,$$  \hspace{1cm} (10.9)

while the third equation of (10.5) gives

$$H_- = -P_+; \quad H_3 = -\frac{1}{2}\partial_x(u + P_3') - \lambda P_+$$  \hspace{1cm} (10.10)

where primes denote derivatives w.r.t. $x$.

Finally, the first equation of (10.5) yields 3 equations

$$H_+ = uP_+ - \lambda \partial_x P_+ - \frac{1}{2}\partial_x \partial_x (u + P_+); \quad \partial_x (u + 2P_+) = 0;$$  \hspace{1cm} (10.11)

$$\dot{u} = \frac{1}{2}\partial_x \partial_x \partial_x (u + P_+) + (u - P_+)(\partial_x u + 2\lambda^2 P_+).$$

It follows that

$$P_+ = -\frac{1}{2}u; \quad H_+ = -\frac{1}{2}u^2 + \frac{\lambda}{2}u_x - \frac{1}{4}u_{xx};$$  \hspace{1cm} (10.12)

$$\dot{u} = \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x - \lambda^2 u_x.$$

Changing the notation as

$$u \rightarrow u + \frac{2}{3}\lambda^2; \quad t \rightarrow 4t; \quad x + y \rightarrow x,$$

one obtains the $KdV$ equation

$$u_t = u_{xxx} + 6uu_x.$$  

This example may be relevant towards an ultimate unification of 2D integrable models and 2D conformal field theories, as well as within the 4D SDYM theories which are also closely related to $N + 2$ strings.

### 10.1 Self-Duality and Supersymmetry

Extended Supersymmetry is compatible with self-duality in 2+2 dimensions. Therefore the Supersymmetric self-dual Yang-Mills theory (SSDYM) is capable of generating Supersymmetric 2D integrable models. However a Supersymmetric generalization of the SDYM theory is not unique. One could either replace a gauge group by its graded version, or a 2+2 dimensional space-time by superspace.

Supersymmetric generalizations of the $KdV$ equation in 1+1 dimensions were obtained independently by Manin and Radul [43], Mathieu [44], Bilal and Gervais [45]. These equations have two dynamical variables, one bosonic $u(x,t)$, and one fermionic $\psi(x,t)$, and read

$$\partial_t u = \frac{1}{2}u_{xxx} + 3u\partial_x u + \frac{3}{2}(\psi_{xx})\psi$$  \hspace{1cm} (10.13)

$$\partial_t \psi = \frac{1}{2}\psi_{xxx} + \frac{3}{2}\partial_x (u\psi).$$
They are invariant under the $N = 1$ Supersymmetry transformations
\[ \delta u = \epsilon \partial_x \psi; \quad \delta \psi = \epsilon u \] (10.14)
$\epsilon$ being a constant Grassmann parameter. Eqs.(10.13) are integrable, and can be obtained from the zero curvature condition associated with the graded Lie Algebra $osp(2, 1)$
\[ \partial_t A_x - \partial_x A_t + [A_t, A_x] = 0 \] (10.15)
when the following ansatz is used for 2D Yang-Mills potentials [45]:
\[ 2A_t(x, t) = \begin{pmatrix} u_x & u_{xx} + 2u^2 + \psi_x \psi & -i\psi_{xx} - 2iu\psi \\ -2u & -u_x & i\psi_x \\ i\psi_x & i\psi_{xx} + 2iu\psi & 0 \end{pmatrix} \] (10.16)
\[ A_x = \begin{pmatrix} 0 & u & -i\psi \\ -1 & 0 & 0 \\ 0 & i\psi & 0 \end{pmatrix} \] (10.17)

The 2D Super $KdV$ can be embedded into the self-duality equations by choosing the $osp(2/1)$-valued matrices $H, Q$ as $H = A_t(x, t)$, $Q = A_x(x, t)$, and $B, P$ as $3 \times 3$ matrices as
\[ B = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \] (10.18)
\[ P = \begin{pmatrix} 0 & \frac{i}{2} & -\frac{3i}{4}\psi \\ 0 & 0 & 0 \\ 0 & \frac{3i}{4}\psi & 0 \end{pmatrix} \] (10.19)

It can also be shown that the $N = 1$ and $N = 2$ Super $KdV$ equations, as well as the $N = 1$ Super-Liouville and Super-Toda equations, can all be obtained from the $N = 2$ SSDYM theory by dimensional reductions and truncations [46]. A detailed analysis is however outside the scope of this Article.

## 11 Conclusions

It has been our aim to present a bird’s eye view of the important developments in Integrable Systems over the past few decades. What has been achieved is possibly a more subjective viewpoint, related to building connections between sundry topics of immediate interest. It has certainly not been possible to delve more deeply into the fascinating developments in affine Toda Field Theory which seems to be a thrust area of research today. We refer to the excellent lecture series by Corrigan [48] on this subject. Neither is it possible to present an account of the interesting link between the $KdV$ theory and Matrix models, encompassing thus 2D gravity; (see ref.[25] for a readable account).Supersymmetric Toda Field Theories have also been given the go by. They were first studied by Evans and Hollowood [49], as well as by Leites et al [50]. It seems to be possible to construct Toda Field Theories based on Lie Superalgebras with one proviso, namely, that it is necessary that the Lie Superalgebra admits a purely fermionic root system. This is only possible for the following algebras:

$A(n, n - 1); B(n, n); B(n - 1, n); D(n,n - 1); D(n, n); and D(2, 1, \alpha)$. 
In the generic case, $N = 1$ Supersymmetric theories are obtained, which can be formulated in $N = 1$ superspace. There is one special case, namely, the $sl(n,n - 1)$ theories have in fact $N = 2$ Supersymmetry; see [49]. Recently, Brink and Vasiliev [51] have proposed a model generalizing $A_N$ Toda Field Theories based on a continuous parameter, such that when this parameter takes on certain discrete values, the model reduces to the ordinary $A_N$ Toda Theories. More recently, Wyllard [52] has worked out a WZNW reduction of these generalized theories, and has also attempted a Supersymmetric generalization [53] of the same.

One could also picture the affine Toda theories as integrable deformations of the conformal Toda theories. As an example, by adding an extra simple root to the $A_1$ Toda theory, one obtains the affine Toda theory, which is also the sinh-Gordon theory. General integrable deformations have been investigated by Zamolodchikov, among others, as an interesting field. Toda theories also appear in many other diverse areas of theoretical physics, e.g., 1D discrete versions appear in the physics of monopoles [54]. Further, certain 3D continuous Toda systems are relevant to the classification of hyper-Kahler metrics in 4D [55]. Finally, it also appears that Toda Field Theories are relevant to $M$-Theory, the conjectured all-in-all Theory encompassing all String Theories [56].

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