DIRECTED RIEMANNIAN MANIFOLDS OF POINTWISE CONSTANT RELATIVE SECTIONAL CURVATURE

GEORGI GANCHEV AND VESSELKA MIHOVA

Abstract. We study a class of Riemannian manifolds with respect to the covariant derivative of their curvature tensors. We introduce geometrically the class of directed Riemannian manifolds of pointwise constant relative sectional curvature and give a tensor characterization for such manifolds. We prove that all rotational hypersurfaces are directed and find the rotational hypersurfaces of pointwise constant relative sectional curvature. For the class of directed Riemannian manifolds of pointwise constant relative sectional curvature having a totally umbilical scalar distribution we prove a structural theorem and a theorem of Schur’s type.

1. Introduction

Let \((M, g)\) be a Riemannian manifold with Levi-Civita connection \(\nabla\) and curvature tensor \(R\). If \(V\) is an \(n\)-dimensional vector space identified with the tangent space at an arbitrary point in \(M\), denote by \(\mathcal{R}(V)\) the linear space of all tensors of type \((0,4)\) over \(V\) having the symmetries of \(R\).

According to the general theory of group representations \([6]\) there exists a splitting of \(\mathcal{R}(V)\) into irreducible components under the action of \(O(n)\). Singer and Thorpe \([4]\) and Nomizu \([3]\) give explicitly a decomposition of \(\mathcal{R}(V)\) and describe it geometrically in terms of the well-known classes of Riemannian manifolds of constant sectional curvature, Einstein manifolds and conformally flat Riemannian manifolds.

In the case when \((M, g, J)\) is an almost Hermitian manifold with almost complex structure \(J\) and \(V\) is a \(2n\)-dimensional Hermitian vector space, Tricerri and Vanhecke give in \([5]\) a complete explicit decomposition of \(\mathcal{R}(V)\) under the action of \(U(n)\). In this case the splitting of \(\mathcal{R}(V)\) gives many new classes of almost Hermitian manifolds with respect to \(R\) and leads to the problem of their geometrical description.

Following this scheme of studying Riemannian manifolds it seems natural to investigate the linear space \(\nabla \mathcal{R}(V)\) of all tensors of type \((0,5)\) over \(V\) having the symmetries of the covariant derivative \(\nabla R\) of the curvature tensor \(R\) of a Riemannian manifold \((M, g)\). A complete explicit decomposition of \(\nabla \mathcal{R}(V)\) under the action of \(O(n)\) has been given by Gray and Vanhecke in \([2]\). The zero space of this splitting (\(\nabla R = 0\)) leads to the class of locally symmetric Riemannian manifolds and this class corresponds to the class of locally flat Riemannian manifolds which is the zero class (\(R = 0\)) in the splitting of \(\mathcal{R}(V)\). However we have to mention that for the classes of Riemannian manifolds with respect to \(\nabla R\) \((\nabla R \neq 0)\) it is not known very much.

Conformally flat Riemannian manifolds \((M, g, d\tau)\) with metric \(g\) and scalar 1-form \(d\tau\) \((\tau\) being the scalar curvature of \((M, g))\) have been studied in \([1]\).

In this paper we consider the class of Riemannian manifolds whose covariant derivative \(\nabla R\) of the curvature tensor is constructed only by the metric \(g\) and the scalar 1-form \(d\tau\). This

\begin{itemize}
  \item 1991 Mathematics Subject Classification. Primary 53B20, Secondary 53A05.
  \item Key words and phrases. Riemannian manifolds with scalar distribution, sectional 1-form, directed Riemannian manifolds, pointwise constant relative sectional curvature.
\end{itemize}
class corresponds to the class of Riemannian manifolds of constant sectional curvature (in the splitting of $\mathcal{R}(V)$). We introduce geometrically the class of directed Riemannian manifolds of pointwise constant relative sectional curvature and prove that these manifolds form the class of Riemannian manifolds with special covariant derivative $\nabla R$ of the curvature tensor mentioned above. We prove that any rotational hypersurface is a directed Riemannian manifold and find all rotational hypersurfaces of pointwise constant relative sectional curvature. For the special subclass of the directed Riemannian manifolds of pointwise constant relative sectional curvature whose distribution is totally umbilical we prove a structural theorem and a theorem of Schur’s type.

2. Directed Riemannian manifolds of pointwise constant relative sectional curvature

Let $(M, g)$ be a Riemannian manifold with Levi-Civita connection $\nabla$. The Riemannian curvature operator $R$ is given by $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ and the corresponding curvature tensor of type (0,4) is defined by $R(X, Y, Z, U) = g(R(X, Y)Z, U)$ for arbitrary differentiable vector fields $X, Y, Z, U$. Further the algebra of all differentiable vector fields on $M$ will be denoted by $\mathcal{X}M$.

The covariant derivative $\nabla R$ of the curvature tensor $R$ has the following symmetries:

$$\nabla W R(X, Y, Z, U) = -\nabla W R(Y, X, Z, U) = -\nabla W R(X, Y, U, Z);$$

$$\sigma_{XYZ}(\nabla W R)(X, Y, Z, U) = 0;$$

$$\sigma_{WXY}(\nabla W R)(X, Y, Z, U) = 0,$$

where $W, X, Y, Z, U \in \mathcal{X}M$ and $\sigma$ denotes the corresponding cyclic summation.

We denote by $\tau$ the scalar curvature of the manifold $(M, g)$ and by $\pi$ the tensor $\pi(X, Y, Z, U) = g(Y, Z)g(X, U) - g(X, Z)g(Y, U); \quad X, Y, Z, U \in \mathcal{X}M$.

We recall that a Riemannian manifold of constant sectional curvatures is characterized by the equality

$$R = \frac{\tau}{n(n-1)}\pi,$$

i.e. the curvature tensor of a Riemannian manifold of constant sectional curvatures is constructed only by the metric $g$.

Let $\omega$ be a 1-form on the Riemannian manifold $(M, g)$. We consider the tensor

$$\Pi(\omega)(W, X, Y, Z, U) = 2\omega(W)\pi(X, Y, Z, U) + \omega(X)\pi(W, Y, Z, U) + \omega(Y)\pi(X, W, Z, U) + \omega(Z)\pi(X, Y, W, U) + \omega(U)\pi(X, Y, Z, W).$$

It is easy to check that the tensor $\Pi(\omega)$ has the symmetries (1) of the tensor $\nabla R$.

Our aim in this paper is to study the class of Riemannian manifolds characterized by the condition

$$\nabla R = \frac{1}{2(n-1)(n+2)}\Pi(\omega).$$

With respect to $\nabla R$ this class formally corresponds to the class of Riemannian manifolds of constant sectional curvatures. In terms of the decomposition of $\nabla R$ the condition (3) means that $\nabla R$ coincides with its component in the space $\mathcal{R}_I$.

In this section we characterize the equality (3) geometrically.
Let \( E = \text{span}\{X, Y\} \) be a 2-plane in the tangent space \( T_pM \) at a point \( p \) in \( M \) and \( \{X, Y\} \) be an orthonormal basis of \( E \). The tensor \( \nabla R \) generates the 1-form \( \varphi_E \) defined on \( E \) as follows:

\[
\varphi_E(Z) = (\nabla_Z R)(X, Y, Y, X), \quad Z \in E.
\]

Because of the properties (1) of \( \nabla R \) the 1-form \( \varphi_E \) does not depend on the orthonormal basis of \( E \).

The 1-form \( \varphi_E \) defined on \( E \) by (4) is said to be a sectional 1-form.

Let now \( \eta \) be a unit 1-form on \((M, g)\) and \( \Delta \) be the distribution of \( \eta \), i.e.

\[
\Delta(p) = \{X \in T_pM : \eta(X) = 0\}, \quad p \in M.
\]

**Definition 2.1.** A Riemannian manifold \((M, g)\) is said to be directed if there exists a unit 1-form \( \eta \) on \( M \) such that

i) \( \varphi_E = k(E, p) \eta|_E \) for all \( E \not\subset \Delta \);

ii) \( \varphi_E = 0 \) for all \( E \subset \Delta \).

For any 2-plane \( E \not\subset \Delta \) the function \( k(E, p) \) is said to be a relative sectional curvature.

The condition i) means that all sectional 1-forms \( \varphi_E, E \not\subset \Delta \) are collinear with the restriction of the 1-form \( \eta \) to the 2-plane \( E \). We say that \((M, g)\) is directed by the 1-form \( \eta \).

**Definition 2.2.** A directed Riemannian manifold \((M, g)\) is said to be of pointwise constant relative sectional curvatures if the relative sectional curvature \( k(E, p) \) of any 2-plane \( E \not\subset \Delta \) does not depend on \( E \).

In order to find a tensor characterization for the Riemannian manifolds described in Definition 2.2 we need the following

**Lemma 2.3.** Let \( L \) be a tensor of type \((0, 5)\) satisfying the following equalities

i) \( L(W, X, Y, Z, U) = -L(W, Y, X, Z, U) = -L(W, X, Y, U, Z) \);

ii) \( \sigma_{XYZ} L(W, X, Y, Z, U) = 0 \);

iii) \( \sigma_{WXY} L(W, X, Y, Z, U) = 0 \)

for all \( W, X, Y, Z, U \in \mathcal{X}M \).

If \( L(X, X, Z, Z, X) = 0 \) for arbitrary \( X, Z \in \mathcal{X}M \), then \( L \equiv 0 \).

**Proof.** Substituting successively \( X \) by \( X + Y \) and by \( X - Y \) into the equality

\[
L(X, X, Z, Z, X) = 0
\]

and taking into account the properties of \( L \), we obtain

\[
L(X, Y, Z, Z, Y) + 2L(Y, X, Z, Z, Y) = 0.
\]

This implies that

\[
L(Y, X, Z, Z, Y) = L(Z, X, Y, Y, Z).
\]

Applying the condition iii) to \( L(X, Y, Z, Z, Y) \) and taking into account (6) we find

\[
L(X, Y, Z, Z, Y) - 2L(Y, X, Z, Z, Y) = 0.
\]

The last equality combined with (5) implies \( L(X, Y, Z, Z, Y) = 0 \) for all \( X, Y, Z \in \mathcal{X}M \). Now it follows in a standard way that \( L \equiv 0 \).

We give a tensor characterization for directed manifolds of pointwise constant relative sectional curvatures.
Theorem 2.4. Let \((M, g)\) be a Riemannian manifold directed by the unit 1-form \(\eta\). Then \((M, g)\) is of pointwise constant relative sectional curvatures \(k(p)\) if and only if
\[
\nabla R = \frac{1}{4} k(p) \Pi(\eta).
\]
The function \(k(p)\) satisfies the equality
\[
d\tau = \frac{(n-1)(n+2)}{2} k \eta,
\]
where \(\tau\) is the scalar curvature of the manifold.

Proof. To prove the first implication we put
\[
L = \nabla R - \frac{1}{4} k \Pi(\eta).
\]

Under the conditions of the theorem it is easy to check that \(L(X, X, Y, Y, X) = 0\) for all \(X, Y \in \mathcal{X}M\). Applying Lemma 2.3 we obtain (7).

The inverse is an easy verification.

The equality (8) follows from (7) by two contractions. \qed

Theorem 2.4 implies immediately Corollary 2.5. Let \((M, g)\) be a directed Riemannian manifold of pointwise constant relative sectional curvatures. Then, \((M, g)\) is locally symmetric if and only if \(d\tau = 0\).

Considering directed Riemannian manifolds of pointwise constant relative sectional curvatures \(k\) and of nonconstant scalar curvature \(\tau\), i.e. \(d\tau \neq 0\) on \(M\), we compute (up to an orientation of \(\eta\)) from (8)
\[
k = \frac{2 \|d\tau\|}{(n-1)(n+2)}, \quad \eta = \frac{1}{\|d\tau\|} d\tau.
\]

Hence, the 1-form \(\eta\) is uniquely determined by the metric \(g\).

3. Examples

In this section we give examples of the manifolds introduced in the previous section among the rotational hypersurfaces.

First we need some formulas.

Let \((M, g)\) be a rotational hypersurface in the Euclidean space \(\mathbb{R}^{n+1}\) with a rotational axis oriented by a unit vector \(e\). We consider \(M\) as a 1-parameter family of spheres \(S^{n-1}(t), t \in \mathcal{J}\), given by the equalities
\[
(X - x_0(t))^2 = r^2(t), \quad e(X - x_0(t)) = 0,
\]
where \(x_0(t)\) and \(r(t)\) are respectively centers and radii of the spheres. Further we assume that the rotational hypersurface \(M\) is also given by a vector-valued function \(X(u^1, ..., u^{n-1}, t)\) satisfying (10), where \(u^1, ..., u^{n-1}, t\) is a local coordinate system on \(M\).

Taking partial derivatives of (10) we find
\[
(X - x_0)X_\alpha = 0, \quad eX_\alpha = 0; \alpha = 1, ..., n-1,
\]
\[
(X - x_0)X_t = rr', \quad eX_t = 1.
\]
Then the vector \(X - x_0 - rr'e\) is normal to \(M\) at the point \(X\) and we can choose the unit normal to \(M\) by the equality
\[
N = -\frac{X - x_0 - rr'e}{r \sqrt{1 + r'^2}}.
\]
Denote by $\xi$ the unit vector field tangent to $M$ and perpendicular to the parallels $S^{n-1}(t)$. Up to a sign we have

$$\xi = \sqrt{1 + r'^2} e - r' N.$$ 

If $\nabla'$ is the standard flat connection in $\mathbb{R}^{n+1}$ we find the Weingarten formulas on $M$:

$$\nabla'_{x} N = \frac{1}{r \sqrt{1 + r'^2}} x, \quad x \perp \xi;$$

$$\nabla'_{\xi} N = \frac{r''}{(\sqrt{1 + r'^2})^3} \xi.$$ 

Hence, the second fundamental tensor $h$ of $M$ has the following structure

$$(11) \quad h = \frac{1}{r} \sqrt{1 + r'^2} g - \frac{1 + r'^2 + rr''}{r(1 + r'^2)^2} \eta \otimes \eta,$$

where $\eta$ is the dual 1-form of the unit vector field $\xi$.

Substituting $h$ from (11) into the Gauss equation we find the curvature tensor of any rotational hypersurface has the following form (see also [1]):

$$(12) \quad R = a \pi + b \Phi,$$

where $a$ and $b$ are the functions

$$(13) \quad a = \frac{1}{r^2(1 + r'^2)}, \quad b = -\frac{1 + r'^2 + rr''}{r^2(1 + r'^2)^2}$$

and $\Phi$ is the tensor

$$\Phi(X, Y, Z, U) = g(Y, Z) \eta(X) \eta(U) - g(X, Z) \eta(Y) \eta(U) + g(X, U) \eta(Y) \eta(Z) - g(Y, U) \eta(X) \eta(Z); \quad X, Y, Z, U \in \mathcal{X}M.$$ 

Let $\nabla$ be the Levi-Civita connection of the rotational hypersurface $(M, g)$. Applying the second Bianchi identity to (12) we obtain

$$(14) \quad \nabla_{x} \xi = \lambda x, \quad \lambda = \frac{\xi(a)}{2b}, \quad x \perp \xi;$$

$$(15) \quad (\nabla_{X} \eta)(Y) = \lambda [g(X, Y) - \eta(X) \eta(Y)], \quad X, Y \in \mathcal{X}M;$$

$$(16) \quad da = \xi(a) \eta = 2\lambda b \eta;$$

$$(17) \quad db = \xi(b) \eta;$$

Taking into account (12), (15), (16) and (17) we calculate with respect to local coordinates

$$(18) \quad \nabla_{i} R_{jkpq} = \lambda b (2\eta_{i} \pi_{jkpq} + \eta_{j} \pi_{ikpq} + \eta_{k} \pi_{ijpq} + \eta_{p} \pi_{jkqi} + \eta_{q} \pi_{jkpi} + (\xi(b) - 2b\lambda) \eta_{i} \Phi_{jkpq}.$$ 

If $E = \text{span}\{X, Y\}$ is an arbitrary 2-plane in $T_{p}M, p \in M$ with an orthonormal basis $\{X, Y\}$, we denote by $\gamma$ the angle between $\xi$ and $E$. Then we have

$$\cos^2 \gamma = \eta^2(X) + \eta^2(Y).$$

Taking into account the defining equality (4) from (18) we obtain

$$\varphi_{E} = [4\lambda b + (\xi(b) - 2b\lambda) \cos^2 \gamma] \eta.$$ 

Thus, we have

**Proposition 3.1.** Every rotational hypersurface is a directed Riemannian manifold.
Now we shall find the rotational hypersurfaces of pointwise constant relative sectional curvature.

As a consequence of (18), (16) and Theorem 2.4 we obtain

**Proposition 3.2.** A rotational hypersurface \((M, g)\) with curvature tensor (12) is of pointwise constant relative sectional curvature iff

\[ a - b = B = \text{const} \]

By use of the formulas (13) we find

\[ (19) \quad a - b = \frac{2(1 + r'^2) + rr''}{r^2(1 + r'^2)^2} = B. \]

Solving the differential equation (19) we obtain

**Proposition 3.3.** A rotational hypersurface \((M, g)\) with meridian \(t = t(r)\) is of pointwise constant relative sectional curvature iff

\[ (20) \quad t = \int \frac{r\sqrt{Ar^2 + B}}{\sqrt{1 - Ar^4 - Br^2}} dr, \quad 0 < (Ar^2 + B)r^2 < 1. \]

Putting \(u^2 = Ar^2 + B, m = \sqrt{B^2 + 4A - B}, m' = \frac{\sqrt{B^2 + 4A + B}}{2A}\), we obtain the meridian of the hypersurface has equations

\[ r = \sqrt{\frac{u^2 - B}{A}}, \quad t = \frac{1}{A} \int \frac{u^2}{\sqrt{(1 - mu^2)(1 + m'u^2)}} du. \]

Further we consider the cases:

I) \(A > 0\). Putting \(u = \sqrt{\frac{1-x^2}{m}}, x \in (0, 1)\) we find the meridian of the rotational hypersurface has the following equations:

\[ (21) \quad r = \sqrt{\frac{1-x^2}{m} - \frac{B}{A}}, \quad t = \frac{-1}{Am\sqrt{m + m'}}(J_1 - J_2), \]

where

\[ J_1 = \int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \quad J_2 = \int \frac{x^2dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \quad (k = \sqrt{\frac{m'}{m + m'}} < 1) \]

are the integrals of Legendre of first type and of second type, respectively.

II) \(A < 0\). Putting \(u = \sqrt{\frac{x}{-m'}}, x \in (0, 1)\) we find the equations

\[ (22) \quad r = \sqrt{- \frac{x^2}{m'A} - \frac{B}{A}}, \quad t = - \frac{1}{Am\sqrt{-m'}} J_2. \]

4. **The Case of a Totally Umbilical Distribution**

Let \((M, g)\) be a Riemannian manifold with a unit vector field \(\xi\). By \(\eta\) and \(\Delta\) we denote respectively the dual to \(\xi\) 1-form and the distribution perpendicular to \(\xi\).

The distribution \(\Delta\) is said to be **totally umbilical** if

\[ (23) \quad \nabla_x \xi = \lambda x, \]

where \(x \in \Delta\) and \(\lambda\) is a function on \(M\).

From (14) it follows that every rotational hypersurface has a totally umbilical distribution.

If we set \(\theta(X) = d\eta(\xi, X), \quad X \in \mathcal{X}M\), then from (23) it follows that

\[ (24) \quad (\nabla_X \eta)(Y) = \lambda [g(X, Y) - \eta(X)\eta(Y)] + \eta(X)\theta(Y). \]
and
\begin{equation}
    d\eta = \eta \wedge \theta.
\end{equation}

The last equality means that the distribution \( \Delta \) is involutive.

Taking into account (24) we find the Gauss formula for the distribution \( \Delta \):
\begin{equation}
    \nabla_x y = D_x y - \lambda g(x, y) \xi; \quad x, y \in \Delta,
\end{equation}
where \( D \) is the Levi-Civita connection of the distribution \( \Delta \).

Next we denote by \( K \) the curvature tensor of \( D \) and find the Gauss equation for the distribution \( \Delta \):
\begin{equation}
    R(x, y, z, u) = K(x, y, z, u) - \lambda^2 \pi(x, y, z, u); \quad x, y, z, u \in \Delta.
\end{equation}
Taking into account (26) and (27) we calculate
\begin{equation}
    (\nabla_w R)(x, y, z, u) = (D_w K)(x, y, z, u) + d\lambda^2(w) \pi(x, y, z, u); \quad w, x, y, z, u \in \Delta.
\end{equation}

When \( d\tau \neq 0 \) on the Riemannian manifold \((M, g)\) the distribution of the 1-form \( d\tau \) is said to be the scalar distribution.

Now we can prove

**Theorem 4.1.** Let \((M, g)\) be a connected directed Riemannian manifold of pointwise constant relative sectional curvature. If the scalar distribution of the manifold is totally umbilical, then \( M \) is a one-parameter family of locally symmetric submanifolds.

**Proof.** Because of (25) the distribution \( \Delta \) is involutive.

Let \( p \in M \) and \( S_p \) be the maximal integral submanifold of the distribution \( \Delta \) through the point \( p \). Since \( R \) and \( K \) satisfy the second Bianchi identity, then the equality (28) implies \( d\lambda^2 = 0 \) for all \( w \in \Delta \). This means \( d\lambda^2 = 0 \) on \( S_p \). Under the conditions of the theorem from (7) it follows that the restriction of \( \nabla R \) onto \( S_p \) is zero. Then the equality (28) implies \( DK = 0 \) on \( S_p \), i.e. \( S_p \) is a locally symmetric submanifold of \( M \).

The last question to consider is a theorem of Schur’s type for the pointwise constant relative sectional curvature (9).

**Theorem 4.2.** Let \((M, g)\) be a directed Riemannian manifold of pointwise constant relative sectional curvature (9) and totally umbilical scalar distribution. Then the curvature function \( k \) is constant on the integral submanifolds of the scalar distribution iff \( \eta \) is closed (\( \xi \) is geodesic).

**Proof.** Writing the equality (24) in local coordinates
\[
    \nabla_i \eta_j = \lambda(g_{ij} - \eta_i \eta_j) + \eta_i \theta_j
\]
we find
\[
    \nabla_i \tau_j = ||d\tau||_i \eta_j + ||d\tau||_j \lambda(g_{ij} - \eta_i \eta_j) - ||d\tau|| \eta_i \theta_j,
\]
\[
    (dk + k\theta) \wedge \eta = 0.
\]

The last equality shows that \( d\ln k + \theta = 0 \) on the integral submanifolds \( S_p \) of \( \Delta \). Hence, \( k \) is constant on \( S_p \) iff \( \theta = 0 \), i.e. \( d\eta = 0 \).

Finally the equalities
\[
    g(\nabla_\xi \xi, x) + \eta(\nabla_\xi x) = 0;
\]
\[
    [\xi, x] = \nabla_\xi x - \lambda x;
\]
\[
    \theta(x) = -\eta(\nabla_\xi x)
\]
for all \( x \in \Delta \) imply the condition \( \theta = 0 \) is equivalent to the condition \( \nabla_\xi \xi = 0 \), i.e. \( \xi \) being geodesic.
Remark 4.3. In the given examples in section 3 a simple calculation shows that

$$
    ||d\tau||^2 = -\frac{4(\tau - nB)^2(\tau + 2B)}{(n-1)(n+2)} + C(\tau - nB), \quad B, C = \text{const}.
$$

Hence, the curvature function $k$ is a constant on $S_p$, but it is not a global constant on $M$. Therefore Theorem 4.2 cannot be improved in this direction.

It is interesting to find examples of directed Riemannian manifolds of constant relative sectional curvature.

Remark 4.4. If $(M, g)$ is a surface with Gaussian curvature $K$ in the Euclidean space, then its sectional 1-form $\varphi$ satisfies the equality $\varphi = dK$ and consequently every surface is a directed Riemannian manifold of pointwise constant relative sectional curvature $k = ||dK||$.

Hence, the surfaces of constant relative sectional curvature are exactly the surfaces satisfying the condition $||\text{grad} K|| = \text{const}$.

The second author is partially supported by Sofia University Grant 99/2013.

References

[1] Ganchev G., V. Mihova. J. reine angew. Math. 522, 2000, 119-141.
[2] Gray A., L. Vanhecke. Decomposition of the space of covariant derivatives of curvature operators (not published)
[3] Nomizu K. Differential Geometry (in honor of K. Yano), Kinokuniya, Tokyo, 1972, 335-345.
[4] Singer I. M., J. A. Thorpe. Global Analysis (papers in honor of K. Kodaira), University of Tokyo Press, Tokyo, 1969, 355-365.
[5] Tricerri F., L. Vanhecke. Trans. Amer. Math. Soc., 267, 1981, 365-397.
[6] Weyl H. Princeton University Press, 1946.