A NOTE ON P-RESOLUTIONS OF CYCLIC QUOTIENT SINGULARITIES

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Abstract. P-resolutions of a cyclic quotient singularity are known to be in one-to-one correspondence with the components of the base space of its semi-universal deformation [KS]. Stevens [St] and Christophersen [Ch] has shown that P-resolutions are parametrized by so-called chains representing zero or, equivalently, certain subdivisions of polygons. I give here a purely combinatorial proof of the correspondence between subdivisions of polygons and P-resolutions.

1. Introduction

The discovery of Kollár and Shepherd–Barron in [KS] that the reduced components of the versal base space of a quotient singularity are in one-to-one correspondence with the P-resolutions of this singularity provided a more conceptual understanding of the deformation theory of cyclic quotient singularities. Christophersen [Ch] and Stevens [St] found a beautiful description of these components in terms of so-called chains representing zero or subdivisions of polygons. Nevertheless, their proofs were very involved and used algebro-geometric methods. Altmann [Al] gave a description of P-resolutions and their correspondence to chains representing zero in terms of toric varieties.

In this note, I will emphasize that P-resolutions on the one hand (cf. Lemma 6) and subdivisions of polygons on the other hand are purely combinatorial objects. Hence it is interesting to establish the correspondence between both classes by purely combinatorial methods. Therefore, I will refer to cyclic quotient singularities and P-resolutions only in so far as it is necessary to define the objects to be investigated here. The reader can find in [St] more information on cyclic quotient singularities and their deformation theory.

2. Sequences and subdivisions

Let \( W \) denote the monoid of sequences \((a_1, \ldots, a_k)\) with \( a_i \in \mathbb{N} \setminus \{0\} \) and \( W_2 \) the submonoid of such sequences with \( a_i \geq 2 \). The empty sequence \((\varepsilon)\) is denoted by \( \varepsilon \).

Furthermore, let \( M \) denote the free monoid with generators \( \alpha, \beta \) and \( M_2 \) the free monoid with generators \( \alpha, \gamma \). Both monoids act on \( W \) respectively \( W_2 \) in the following manner from the lefthand side or from the righthand side, respectively:

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\[ \alpha \varepsilon = \varepsilon \]
\[ \varepsilon \alpha = \varepsilon \]
\[ \alpha(a_1, \ldots, a_k) = (a_1, \ldots, a_k + 1) \quad (a_1, \ldots, a_k) \alpha = (a_1 + 1, \ldots, a_k) \]
\[ \beta a = a(1) \quad a \beta = (1)a \]
\[ \gamma a = \alpha \beta a = a(2) \quad a \gamma = a \beta \alpha = (2)a \]

Here \( a \) is an arbitrary element of \( W \) respectively \( W_2 \) and the composition in the respective monoids is denoted by juxtaposition. The following lemma is easy to observe:

**Lemma 1.** For each \( a \in W \) there exists a unique element \( \rho \in M \) such that \( a = \rho \varepsilon \) and \( \rho = 1 \) or \( \rho = \rho' \beta \) for a suitable \( \rho' \). An analogous statement holds for \( W_2 \) and \( M_2 \).

The inversion of a sequence in \( W \) is defined by
\[ \begin{align*}
\gamma a &= \gamma\alpha a \\
\beta a &= \beta \gamma a
\end{align*} \]

We define a map \( R : W \to W \) inductively as follows. On \( W_2 \) we have \( R(\varepsilon) = \varepsilon \) and
\[ R(\alpha a) = \gamma R(a) \quad \text{and} \quad R(\gamma a) = \alpha R(a) \]
for \( a \in W_2, a \neq \varepsilon \). For arbitrary elements \( a \in W \), the map \( R \) is defined by the rule
\[ R(a(1)a') = R(a)(1)R(a') \]

A straightforward induction shows:

**Lemma 2.** \( R \) is an involution, i.e. \( R^2 = id_W \), and
\[ R(\overline{a}) = \overline{R(a)} \]

**Lemma 3.** For \( a', a'' \in W \) and \( e, f \geq 2 \):
\[ R(a'(e + f - 1)a'') = R(a'(e))R((f)a''). \]

A nonempty sequence \((a_1, \ldots, a_k) \in W_2\) defines a continued fraction
\[ [a_1, \ldots, a_k] = a_1 - \frac{1}{a_2 - \frac{1}{\ldots - \frac{1}{a_k}}} \]

It is an easy consequence from Lemma 3 that \([R(a)] = \frac{n}{n-q}\) if \([a] = \frac{n}{q}\). Hence the operator \( R \) generalizes Riemenschneider’s point diagram rule to \( W \) (cf. [R], [S], 1.2).

For later use, we need a relation between the positions in the sequence \( a \) and the positions in the sequence \( R(a) \). This relation will be inductively defined. Let \( \ell \) be the length of \( a \) and \( r \) the length of \( R(a) \). The \((\ell + 1)\)-th position of \( \gamma a \) is associated with the \( r \)-th position of \( \gamma R(a) = \alpha R(a) \) and the \( \ell \)-th position of \( \alpha a \) is associated
with the \((r+1)\)-th position of \(R(\alpha a) = \gamma R(a)\). Furthermore, the \((\ell+1)\)-th position of \(\beta a\) is associated with the \((r+1)\)-th position of \(R(\beta a) = \beta R(a)\).

Obviously, the \(i\)-th position of \(a\) is associated with the \(j\)-th position of \(R(a)\) if and only if the \(j\)-th position of \(R(a)\) is associated with the \(i\)-th position of \(R(R(a)) = a\).

We fix a countable set \(\mathcal{V}\) together with a linear ordering \(\leq\) of \(V\). Elements of \(V\) will be called vertices and if \(V\) is a finite subset of \(r\) vertices then \(v_i\) with \(1 \leq i \leq r\) is defined by the equation \(V = \{v_0 < \cdots < v_r\}\).

A subdivision of an polygon with distinguished vertex, or shortly subdivision, is a pair \((V, \Delta)\) such that \(V\) is a nonempty finite subset of \(\mathcal{V}\) and \(\Delta\) is a set of subsets of \(V\) with the following property. There exists a subdivision of a plane polygon into triangles and a bijection between the vertices of the polygon and the elements of \(V\) such that \(v_i\) and \(v_{i+1}\) correspond to consecutive vertices and each set of vertices of a triangle is identified with an element of \(\Delta\) and vice versa. We accept a digon as plane polygon, hence \((V, \emptyset)\) is a subdivision if \(#V = 2\). An isomorphism between subdivisions \((V, \Delta)\) and \((V', \Delta')\) is a bijection from \(V\) to \(V'\) which maps \(\Delta\) bijectively onto \(\Delta'\). Hence this bijection either preserves or reverses the ordering \(\leq\). Obviously, the notion of subdivision can be defined in a purely combinatorial way without reference to plane polygons. But we do not bother the reader with an axiomatic approach since this would not enhance the understanding of this concept.

We define two laws of compositions on the set of subdivisions. Let \((V_1, \Delta_1)\) and \((V_2, \Delta_2)\) be subdivisions with

\[
    v^0 := \min V_1 = \min V_2 \\
    v^1 := \max V_1 < v^2 := \min V_2 \setminus \{\min V_2\}.
\]

Then we can define a new subdivision \((V_1, \Delta_1)(V_2, \Delta_2) := (V, \Delta)\) with

\[
    V := V_1 \cup V_2 \quad \text{and} \quad \Delta := \Delta_1 \cup \Delta_2 \cup \{v^0, v^1, v^2\}
\]

A subdivision is called irreducible if it is not the product of two subdivisions and it is called primary if there is at most one triangle with \(\min V\) as vertex. Obviously, a primary subdivision is also a irreducible one, but, in general, the converse is not true. Furthermore, each subdivision has a unique decomposition into irreducible factors.

Assume now \(v^1 = v^2\), then \((V_1, \Delta_1) * (V_2, \Delta_2) := (V, \Delta)\) with

\[
    V := V_1 \cup V_2 \quad \text{and} \quad \Delta := \Delta_1 \cup \Delta_2
\]

Each irreducible subdivision can by uniquely decomposed into primary factors with respect to the law of composition \(*\).

Let \((V, \Delta)\) be a subdivision of an \(n\)-gon and \(a = (a_1, \ldots, a_n) \in \mathcal{W}\). The sequence \(a\) can be understood as a labelling of the vertices \(v\) of \(V\) and we define the degree \(\deg v = \deg_v a\) of a vertex with respect to this labelling in such a way that it emphasizes the special role of vertices with label 1. Consider triangles of \(\Delta\) as equivalent if they have an edge in common and one of the triangles has a vertex with label 1. Then the degree of a vertex \(v\) is the number of equivalence classes of triangles having \(v\) as one of its vertices. In particular, a vertex with label 1 has always degree 1. We call \(a\) admissible for \((V, \Delta)\), if we always have

\[
    \deg_v v_i \leq a_i.
\]

The difference \(\text{def}_a v_i := a_i - \deg_v v_i\) is called the defect of the vertex \(v_i\). The set of triples \((V, \Delta, a)\) with \((V, \Delta)\) a subdivision and \(a\) admissible for \((V, \Delta)\) is the set of labelled subdivisions and is denoted by \(\mathcal{S}\).
A vertex \( v \) of \((V, \Delta)\) is called a splitting vertex if it is the maximal or the second vertex of an irreducible factor of \((V, \Delta)\). A vertex \( v \) is called saturated if its defect is 0, or if it is a splitting vertex and both neighbouring vertices have a label different from 1, or if this vertex together with its both neighbours forms a triangle. A nonempty sequence \( a \in W \) can be associated with a labelled graph \( \Gamma(a) \) as follows. If \( a = (a_1, \ldots, a_k) \) then \( \Gamma(a) \) is a chain with \( k \) nodes which are labelled consecutively with the numbers \(-a_1, \ldots, -a_k\). For \( a \in W_2 \) with \([a] = nq\), the graph \( \Gamma(R(a)) \) is the graph of the minimal resolution of the cyclic quotient singularity \( X(n,q) \) (cf. [Ri]).

### 3. P-resolutions and subdivisions

We recall the inductive definition of graphs of type T using our terminology (cf. [St]). The graph \( \Gamma(R(2,a,2)) \) for \( a \geq 2 \) is a graph of type T of the first kind. If \( \Gamma(b) \) is a graph of type T , then \( \Gamma(ab\gamma) \) and \( \Gamma(\gamma ba) \) are graphs of type T of the second kind. We define a subset \( T \) of subdivisions inductively as depicted in Fig.1. If \((V, \Delta) \in T \), then there exists exactly one vertex \( v_i = v_i \in V \) such that \( \{v_i-1, v_i, v_i+1\} \in \Delta \). This vertex is called the central vertex and \( i \) the central index of \((V, \Delta)\). By induction, one easily obtains:

**Lemma 4.** Let \((V, \Delta) \in T \) and \((V, \Delta, a) \in S \) such that all vertices different from the central one have defect 0. Then \( \Gamma(R(a)) \) is a graph of type T.

On the other hand, assume that \( \Gamma(R(a)) \) is a graph of type T. Then there exists a subdivision \((V, \Delta) \in T \) such that \((V, \Delta, a) \in S \) and all vertices different from the central one have defect 0. For each set \( V \) of vertices with the appropriate cardinality there exists exactly one \( \Delta \) with these properties.

**Lemma 5.** Let \((V, \Delta, a) \in S \) with \((V, \Delta) \in T \) such that all vertices \( v > v_0 \) are saturated. Then there exist unique numbers \( \ell, r \geq 0 \) and a unique \( a' \in W_2 \) such that \( \Gamma(R(a')) \) is a graph of type T and \( a = \alpha \ell a' \alpha^r \).

A P-resolution of a quotient singularity \( X \) is a partial resolution \( Y \rightarrow X \) such that \( K_Y \cdot E_i \) for all exceptional divisors, and all singularities of \( Y \) have a resolution graph of type T or \( A_k \) (cf. [St], 3.1). An easy calculation with the adjunction formula shows

**Lemma 6.** Let \( Y \) be a P-resolution of the cyclic quotient singularity \( X \) and \( G \) the graph of its minimal resolution \( \bar{Y} \). Let \( J \) be the subset of nodes which blow down to singularities of type T in \( Y \). Then each connected component of \( J \) is a subgraph
of $G$ being of type $T$. The $-1$-nodes in $G$ are adjacent to two nodes in $J$ and at least one of this node belongs to a component of $J$ which is of the second kind. The subset $K$ of nodes of $G$ which blow down to $A_k$-singularities can be characterized as follows: A $-2$-node belongs to $K$ if and only if none of its neighbours belongs to $J$.

On the other hand: If $G$ is a resolution graph of the quotient singularity $X$ and $J$ a subset of nodes of $G$ satisfying the above properties, then blowing down the nodes in $J$ and those in $K$ which is defined as above yields a $P$-resolution of $X$.

Assume $G = \Gamma(b)$ for a graph $G$ of the minimal resolution of a $P$-resolution. Let $I_1, \ldots, I_r$ be the maximal intervals of $J$, the set defined in lemma 6. Each interval $I_j$ is associated with a interval $H_j$ of positions of $a = R(b)$. These intervals need not longer to be disjoint, they may have extremal positions in common. Let \{ $M_1, \ldots, M_m$ \} be the set of all sets of positions of $R(b)$ which are either equal to one of the $H_j$ or a maximal interval in the complement of the union of all $H_j$. We define inductively a subdivision $(V, \Delta)$ with $(V, \Delta, a) \in S$ if $M_i$ is one of the $H_j$ then $(V_i, \Delta_i)$ is chosen according to Lemma 4 for the subgraph of $G$ determined by the interval $I_j$. For all other $M_i$ the subdivision $(V_i, \Delta_i)$ is chosen such that all triangles have the minimal vertex as vertex. Then we set

$$(V, \Delta) = (V_1, \Delta) \cdot \ldots \cdot (V_m, \Delta_m),$$

where $\cdot$ denotes $\cdot$ or $\ast$, respectively, depending on whether $M_i \cap M_{i+1} = \emptyset$ or not.

The set of all these labelled subdivisions is the set $P$. Obviously, we can reconstruct the resolution graph $G$ and the subset $J$ of nodes of $G$ from an element of $P$ which is associated with $G$ by our construction, i.e. there exists a canonical bijection between $P$-resolutions and isomorphism classes of $P$.

On the other hand, we have the set $M$ of all labelled subdivisions $(V, \Delta, a)$ with $a \in W_2$. The isomorphism classes of this set are in a canonical one-to-one correspondence with graphs of minimal resolutions of cyclic quotient singularities via the map $\Gamma \circ R$ (cf. [R]).

The remaining part of this paper will provide a proof of:

**Theorem 1.** There exists a canonical one-to-one correspondence between isomorphism classes of $M$ and isomorphism classes of $P$ with the following property. If $(V, \Delta, a) \in P$ and $(V', \Delta', a') \in M$ correspond to each other, then $\Gamma R(a')$ can be obtained from $\Gamma R(a)$ by successively blowing down $-1$-nodes.

**Proof.** Blowing down and blowing up yields operations on the labels of subdivisions via the operator $R$, which is an involution. In order to find the right correspondence between $M$ and $P$, it is necessary to find suitable positions for blowing up and to define suitable modifications of the underlying subdivision. As we will see later, the subdivision itself gives a hint where to blow up.

First of all, we prove a useful characterization of the set $P$:

**Lemma 7.** A subdivision $(V, \Delta, a)$ is an element of $P$ if and only if

1. There exist no interior triangles.
2. All vertices are saturated.
3. If an irreducible component has more than two vertices then its label is in $W_2$.
4. A vertex with label 1 has two neighbouring vertices which belong to primary factors being elements of $T$. One of this factor has more than four vertices.
Proof. The necessity of these conditions follows easily from Lemma 6. Let us assume that all these conditions hold.

Condition 1–3 imply with Lemma 5 that a primary component \((V', \Delta')\) with more than two vertices is an element of \(T\) and its label is \(a'a'\) with \(\Gamma R(a')\) a graph of type \(T\) and \(\ell, r \geq 0\). If the next vertex to the lefthand or the righthand side of this component has label 1 then \(\ell = 0\) or \(r = 0\), respectively, due to the definition of a saturated vertex.

The conditions 2 and 4 guarantee that the \(-1\)-nodes of \(\Gamma R(a')\) have the properties of lemma 6. Hence \(\Gamma R(a')\) is the graph of the minimal resolution of a \(P\)-resolution.

This lemma shows that one should blow up in such a way that interior triangles and non-saturated vertices disappear. In Fig. 2 and Fig. 3, one can see three operations on subdivisions. We call the first one an operation of type B1, and the second and third one operations of type B2a or B2b, respectively. All of them are operations of type B.

An operation of type B changes the label of the subdivision as follows. From \(a = a'(e + f - 1)a''\) we proceed to \(b = a'(e, 2, 1, 2, f)a''\). According to Lemma 3, we have \(R(a) = R(a'(e))R((f)a'')\). Therefore, \(R(b) = R(a'(e + 1))(1)R((f + 1)a'')\), i.e. \(\Gamma(R(b))\) is obtained by blowing up from \(\Gamma (R(a))\). An analogous computation shows that for operations of type B2a and B2b, the graph \(\Gamma (R(B))\) is obtained from the graph \(\Gamma (R(A))\) by blowing up between a \(-1\)-node and another node.

We will define a map \(F : S \rightarrow S\) which consists of applying a suitable operation of type \(B\), if appropriate, and show that \(F^k(V, \Delta, a) \in P\) for \((V, \Delta, a) \in M\) and \(k\) sufficiently large. On the other hand, a map \(G\) will be defined with the property \(F(G(V, \Delta, a)) = (V, \Delta, a) = G(F(V, \Delta, a))\) if \(F(V, \Delta, a) \neq (V, \Delta, a)\) and \(G(V, \Delta, a) \neq (V, \Delta, a)\). Furthermore, \(G^k(V, \Delta, a) \in M\) for \((V, \Delta, a) \in P\) and \(k\) sufficiently large.

Before we come to the definition of \(F\) we need some technical definitions. A configuration as depicted in Fig. 5 is called a hexagonal fan. The vertex which is incident to all four triangles is the root of the fan and the vertex opposite to the root is the apex of the fan. Furthermore, we require that the two triangles incident to the apex are not interior ones.

Assume that we are given a subdivision \((V, \Delta)\) and a subset \(W\) of \(V\) with \(v_0 \notin W\). The height of \(W\) is the number \(i+(k+1-j)\) where \(v_i\) is the minimal element of \(W\), \(v_j\) the maximal element of \(W\) and \(k + 1 = \#V\).
Let \((V, \Delta, a) \in \mathcal{S}\). If conditions 1-3 hold, then \(F(V, \Delta, a) := (V, \Delta, a)\). Otherwise, there is a vertex \(v_j\) such that one of the following conditions hold.

1. There exists an interior triangle \(\{v_i, v_j, v_k\} \in \Delta\) with \(i < j < k\).
2. The vertex \(v_j\) is not saturated.
3. We have \(a_j = 1\) and \(\{v_0, v_j\}\) is not an irreducible component of \((V, \Delta, a)\).

Choose \(j\) minimal with this property. Note that case 1 and case 2 may occur simultaneously. If condition 1 holds with \(a_j \geq 3\) and all involved vertices have a label different from 1, then we apply an operation of type B1 to the triangle \(\{v_i, v_j, v_k\}\), otherwise we set \(F(V, \Delta, a) = (V, \Delta, a)\). For this, we have to specify numbers \(e, f\) with \(a_j = e + f - 1\). Choose \(e\) such that it is the degree of the vertex whose label it is in the new subdivision. After having applied this operation either the number of interior triangles has become smaller or the height of the first such triangle has become smaller. Furthermore, we have obtained a hexagonal fan whose root is its minimal or maximal vertex, its apex has label 1 and all other vertices of this fan have label different from 1. Furthermore, no interior triangle has this apex as vertex.

In the remaining cases condition 2 holds in, we apply again a operation of type B1. First we specify the triangle the operation is applied to. Two cases can occur.
First, that there exists a triangle \( \{v_{j-1}, v_j, v_k\} \) with \( k > j \), secondly, that there exists a triangle \( \{v_j, v_{j+1}, v_k\} \) with \( k < j \). In the first case, the number \( f \) is chosen such that it is the degree of the vertex whose label it is in the new subdivision. In the second case, the number \( e \) is chosen in such manner. We observe, that the number of non-saturated vertices is reduced by this operation, since the vertex with label \( e \) or \( f \), respectively, will be a vertex of a triangle with three consecutive vertices, due to the construction.

We now deal with the case that condition 3 holds. We assume that there is a hexagonal fan with \( v_j \) as apex such that its root is the minimal or maximal vertex of this fan and all vertices of the fan being different from \( v_j \) have a label \( \geq 2 \). In all other cases, we define \( F(V, \Delta, a) = (V, \Delta, a) \). If the root is minimal, we can apply operation B2a, otherwise we apply operation B2b. Having applied this operation, we have again a hexagonal fan as above. If the root is not the minimal vertex \( v_0 \) then its height is smaller than the height of the fan above.

If \( (V, \Delta, a) \) is obtained from an element of \( M \) by successive application of \( F \), then proceeding by induction, one can easily verify the following facts:

- If \( v_j \) is as above, then all vertices with label 1 are smaller than this vertex.
- Each vertex with label 1 is apex of a hexagonal fan whose root is its minimal or maximal vertex. If \( v_0 \) is vertex of the hexagonal fan then it is its root.
- A vertex with label 1 has two neighbouring vertices which belong to primary factors being elements of \( T \). One of these factors has more than four vertices.

Let us call a labelled subdivision with these properties a good one. Hence, if \( F(V, \Delta, a) = (V, \Delta, a) \) and \( (V, \Delta, a) \) is good, then there exists no vertex \( v_j \) as above and hence \( (V, \Delta, a) \in \mathcal{P} \), as desired. The considerations above show that this happens after finitely many applications of \( F \), since in each step we diminish one of the following numbers: the number of interior triangles, the number of non-saturated vertices, the number of vertices \( v_j \) such that \( a_j = 1 \) and \( \{v_0, v_j\} \) is not an irreducible component, the height of the first interior triangle, the height of the first hexafan whose apex has label 1.

Furthermore, the inverse map \( G \) can be obtained as follows. Apply on good labelled subdivisions, if possible, the inverse of one of the operations of type \( B \) which yield a blowing down of the rightmost \(-1\)-node in \( \Gamma(R(a)) \). Otherwise, \( G \) maps a labelled subdivision onto itself. The desired properties of \( G \) are now easily to observe.

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\text{References}
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