Anomaly cancellation condition in an effective non-perturbative electroweak theory

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We establish the non-perturbative validity of the gauge anomaly cancellation condition in an effective electroweak theory of massless fermions with finite momentum cut-off and Fermi interaction. The requirement that the current is conserved up to terms smaller than the energy divided by the cut-off scale, which is the natural condition as gauge invariance is only emerging, produces the same constraint on charges as in the Standard Model. The result holds at a non-perturbative level as the functional integrals are expressed by convergent power series expansions and are analytic in a finite domain.

I. INTRODUCTION

In a chiral theory classical gauge invariance can be broken at the quantum level by anomalies, a fact producing a lack of renormalizability and of internal consistency. Therefore in the Standard Model the anomalies need to cancel out, and this produces an algebraic condition on the hypercharges (and therefore on the charges), see e.g. [1], [2] which gives a partial explanation to charge quantization without reference to grand unification. The values of hypercharges, that at a classical level can take any value, are constrained by a purely quantum effect.

In order to see how this condition arises the gauge fields can be decomposed in classical background and quantum part, see e.g. [3]. Neglecting the effect of the quantum gauge fields (but keeping quantum fermions) the conservation of current, following classically by Noether theorem, is broken by terms quadratic in the fields proportional to the three current correlations. They are expressed by sum of a small number of terms ("triangle graphs") and the condition on the charges comes from the request that such sum is vanishing. When the effect of quantum gauge fields is taken into account, the anomaly is instead sum of infinitely many graphs. Radiative corrections could produce extra conditions but the Adler-Bardeen theorem [4] is invoked to say that this is not the case. Such a property, which says that the anomaly is not renormalized by interactions, is a perturbative statement relying on cancellations valid assuming exact symmetries and removed cut-off. Non-perturbative versions of such property in a functional integral framework using the Jacobian method [5] are indeed valid only at one loop, see e.g. [6] or [7].

Natural questions are if the cancellation condition is valid at a non-perturbative level and if it still holds in an effective description when some symmetry is only approximate. The two questions are related; we can consider a theory with finite cut-off which can be possibly studied non-perturbatively, but some symmetry is necessarily broken. Such problems have been extensively analyzed over the years. Lattice gauge theory is a natural framework for a non-perturbative construction but the program of getting an anomaly free formulation of electroweak theory is still non complete, see for instance [8]-[12]. In [13],[14] the anomaly in a theory with finite momentum cut-off have been considered at a perturbative level. The role of symmetry breaking terms in the anomaly cancellation was considered in [15], finding that at one loop they do not break the cancellation condition.

In this paper we follow a different point of view. We keep a finite momentum ultraviolet cut-off and we still decompose the gauge field in a classical background and a quantum field; this second field is integrated out to produce an effective fermionic interaction. We consider therefore an effective electroweak theory of massless fermions with finite momentum cut-off and quartic Fermi interaction. It is not indeed restrictive to consider only a quartic interaction, as after the first Renormalization Group (RG) integration step, monomials of all orders are generated in the interaction. One can even look to such theory as the generic result of integrating out high energy degrees of freedom from some more fundamental theory, see e.g. [16]-[18]. We prove that the fermionic functional integrals defining such theory are mathematically well defined and the correlations are analytic in a finite domain of couplings of order of the cut-off divided by the gauge boson masses. Analyticity follows from determinant bounds for fermionic expectations. The results holds in the limit of removed infrared cut-off, so that the functional integrals are infinite-dimensional.

The Ward Identities in the effective theory have extra terms which would be formally vanishing removing the cut-off. Keeping only classical gauge fields not taking into account interactions the quadratic response is again sum of regularized triangle graphs which cancel out under the cancellation condition up to terms of the order of the

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energy/cut-off ratio. The non perfect cancellation is expected as finite momentum cut-off breaks gauge invariance already in the classical action. When the interaction is taken into account the response is sum of infinitely many graphs and the Adler-Bardeen argument does not hold, as it relies on symmetries which are broken by the cut-off. However by our exact RG analysis we show that the response can be decomposed in two terms; one is sum of triangle graphs with dressed vertices and propagators, which cancel out under the cancellation condition, and the other is given by a complicate series of renormalized graphs which again can be rigorously bounded by a power of the energy/cut-off ratio.

In conclusion, the requirement that the current is conserved up to terms of the energy divided by the cut-off scale, which is the natural condition in the effective QFT as gauge symmetry is emerging, provides the same constraint on charges as is found in the standard model at a perturbative level. It is therefore a robust condition holding at a non-perturbative level in an effective theory and even when gauge symmetry is classically not exact.

The results are obtained by Constructive RG methods, see e.g. [19] for an introduction, which have been previously used to construct chiral interacting theories and the chiral anomaly in $d = 1 + 1$ [20]-[22] in the limit of the removed ultraviolet cut-off. In the present case of effective electroweak in $d = 3 + 1$ analyticity is found only with a finite cut-off of the order of the gauge mass, and removed infrared cut-off, as Fermi interaction is not renormalizable. It would of course be interesting to get similar results for larger values of cut-off; this could be obtained avoiding of integrating out the bosons and considering the standard model electroweak theory with a momentum cut-off. Going beyond perturbation theory is in this case much more difficult as perturbation theory is expected to be non convergent and cluster expansions methods are needed. Another interesting question is considering lattice regularization such that chiral symmetry is non broken at a classical level. In the case of QED, the perfect validity of the Adler-Bardeen theorem has been recently rigorously established [23],[24] with a finite lattice (with emerging Lorentz symmetry), using the lattice regularization in [25], and it would be interesting to extend such result, if possible, to chiral theories.

The rest of the paper is organized in the following way. In §II we present the effective model and we state our main result. In §III we present the Renormalization Group analysis, in §IV we establish analyticity and in §V we derive the cancellation condition in the effective theory.

II. EFFECTIVE ELECTROWEAK THEORY

A. Grassmann integration and currents

We consider a single family of particles with two leptons, $\nu, e$ and two quarks $u, d$; the quarks have another color index which takes 3 values. We introduce therefore Grassmann variables $\psi^\pm_{i,L,x}, \psi^\pm_{i,R,x}$ with $L,R$ denoting chirality, $x \in [-L/2, L/2]^4$ with anti-periodic boundary conditions and

$$\{\psi^+_{i,s,x}, \psi^+_{i',s',x'}\} = \{\psi^-_{i,s,x}, \psi^-_{i',s',x'}\} = \{\psi^+_{i,s,x}, \psi^-_{i',s',x'}\} = 0$$

with $s = L,R$. One introduces the doublets $\Psi^\pm_{i,L,x} = (\psi^\pm_{i,L,x}, \psi^\pm_{i,x})$ and $\Psi^\pm_{q,x} = (\psi^\pm_{u,L,x}, \psi^\pm_{d,L,x})$. The index labeling the 2 components of $\psi^\pm_{x,i,s}$ and the color index for $\psi^\pm_{x,i,s}, \psi^\pm_{u,i}$ are omitted.

We define $\chi_{\mp i,s,x} = \frac{1}{\sqrt{2}} \sum_k e^{ikx} \psi^\pm_{i,s,x}$ with $k = \frac{2\pi}{L}n$, with $\chi^\pm_{i,s}$ another set of Grassmann variables. We introduce a smooth momentum cut-off $\chi_N(k)$ which is a infinitely differentiable compact support function (this is useful to get good decay properties in coordinate space) such that $\chi_N(k) = 0$ for $|k| \geq \gamma^{N+1}$ and $\chi_N(k) = 1$ for $|k| \leq \gamma^N$, $\gamma > 1$ a scaling parameter. Therefore $\gamma_N$ is the ultraviolet cut-off while the infrared cut-off is provided by $L$. The "fermionic gaussian measure" is defined as, $i = \nu, e, u, d, s = L, R$

$$P(d\psi) = \prod_{i,s,k} \delta^\ast_{i,s,k} (d\psi^\pm_{i,s,k}) e^{\frac{-1}{L^4} \sum_k \sigma_i^k \chi_N^\ast(k)(\sigma_i^k)}$$

(2)

where $\prod_k^\ast$ is a product over $k$ in the support of $\chi_N(k)$ and $\sigma_i^k = (\sigma_0, i\sigma)$ $e$ $\sigma_R^k = (\sigma_0, -i\sigma)$, $\sigma = \sigma_1, \sigma_2, \sigma_3$ with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(3)

The 2-point function is given by

$$\langle \psi^-_{i,s,x} \psi^+_{i',s',y} \rangle = \frac{\int P(d\psi) \psi^-_{i,s,x} \psi^+_{i',s',y}}{\int P(d\psi)} = \delta_{i,i'} \delta_{s,s'} g_8(x,y)$$

(4)
with
\[ g_{i,s}(x, y) = \frac{1}{L^d} \sum_k e^{i k(x - y)} \frac{\chi N(k)}{-i \sigma_k^p k} \]  

(5)

The n-point function \( \langle \psi^{i_1}_{1,s_1, x_1} \cdots \psi^{i_n}_{n,s_n, x_n} \rangle >_0 \) is given by the Wick rule. The cut-off function plays a very important role, as it makes the number of Grassmann variables finite, hence the Grassmann integral is well defined; at the end the limit \( L \to \infty \) is taken.

The currents relevant for electroweak interaction are the W and B ones

\[ j^k_{W,\mu, x} = \frac{1}{2} (j^k_{W,\mu, x} + j^k_{W,\mu, x}) j^k_{B,\mu, x} = \frac{1}{2} \sum_{i = e, \mu, \nu, \delta} (Y^i_{W,j} j^i_{j, \mu, x} + Y^i_{W,j} j^i_{j, \mu, x}) \]  

(6)

where \( Y^i_L, Y^i_R \) are the hypercharges, \( Y_L^e = Y_L^L, Y_L^u = Y_L^L \) and

\[ j^k_{j, \mu, x} = \psi^{i_1}_{i, s, x, \sigma} \psi^{-i_1}_{i, s, x} + j^k_{W,\mu, \xi, x} = \psi^{i_1}_{i, s, x} \sigma_\mu ^{i_1}_{i, s, x} j^k_{W,\mu, \xi, x} = \psi^{i_1}_{i, s, x, \sigma} \sigma_\mu ^{i_1}_{i, s, x} \]  

(7)

with \( \tau \) Pauli matrices. If \( \psi \) are classical fields verifying the Dirac equation, the currents \( j^k_{j, \mu, x}, j^k_{W,\mu, l, x}, j^k_{W,\mu, q, x} \) (7) are separately conserved.

The form of the interaction with classical fields is dictated by the requirement of invariance with respect to a gauge transformation; one gets that the average of the observable \( O \), a monomial in the Grassmann variables, is given by

\[ \langle O \rangle_{W, B} = \frac{\int P(d\psi) e^{i \int dx (g_{\mu, x} j^k_{W,\mu, x} + g'_{\mu, x} j^k_{B,\mu, x}) O} \int P(d\psi) e^{i \int dx (g_{\mu, x} j^k_{W,\mu, x} + g'_{\mu, x} j^k_{B,\mu, x})} \]  

(8)

Note that such invariance is true at a classical level (that is formally replacing the Grassmann variables with functions) only in the limit of removed cut-off \( N \to \infty \) but is violated at finite \( N \). Moreover, even when \( N \to \infty \) limit symmetry may be broken at a quantum level in the functional integral (39), what is exactly the anomaly phenomenon.

In order to see this, we can consider the average of the current with respect to (8). It is computed by expanding in series in the gauge fields; if \( A^0 = B, A^1 = W \) and the generating function is

\[ e^{W_0} = \int P(d\psi) e^{i \int dx (g_{\mu, x} j^k_{W,\mu, x} + g'_{\mu, x} j^k_{B,\mu, x})} \]  

(9)

then, if the sum over the choices of \( A \) and the combinatorial factor are understood, \( g^0 = g, g^1 = g', \varepsilon = 0, 1 \)

\[ \langle \hat{j}_{B, x} \rangle_{W, B} = \sum_n \frac{1}{n!} \int dp_1 \cdots dp_n \frac{\partial^{n+1} W_{W, B}}{\partial B_{\mu, \rho} \partial A^i_{\mu_1, p_1} \cdots \partial A_{\mu_n, p_n}} g^{i_1} A^i_{\mu_1, p_1} \cdots g^{i_n} A_{\mu_n, p_n} \delta(p + \sum_i p_i) \]  

(10)

Note that the coefficients are simply the truncated correlations of currents

\[ \frac{\partial^{n+1} W_{W, B}}{\partial B_{\mu, \rho} \partial A^i_{\mu_1, p_1} \cdots \partial A_{\mu_n, p_n}} |_0 = \langle \hat{j}_{B, \mu, p} \rangle_{W, B} \]  

(11)

so that the expansion can be represented as sum of simple Feynman graphs, see Fig.1. We are in particular interested

FIG. 1: Graphs contributing to (10)

on the conservation of the \( U(1) \) currents, which is expressed by \( \langle \partial_\mu j_{B, \mu, x} \rangle \).
B. Ward Identities

We introduce currents \( \tilde{j}_{\mu,1.p} = \tilde{j}^L_{\mu,v.p} + \tilde{j}^R_{\mu,c.p} \), \( \tilde{j}_{\mu,2.p} = \tilde{j}^L_{\mu,u.p} + \tilde{j}^R_{\mu,d.p} \), \( \tilde{j}_{3.\mu} = \tilde{R}^L_{\mu,1.p} \), \( \alpha = 1, 2, 3 \) and a generating function \( W_0(A, \phi) \) in which we add to the exponent of (9) a fermionic source term \( \sum_{\alpha=1,2,3} \int dx (\psi_{\alpha}^+ \phi_{\alpha}^- + \psi_{\alpha}^- \phi_{\alpha}^+ ) \), where \( \psi_1 = \psi_{L,v} + \psi_{U,u}, \psi_2 = \psi_{L,u} + \psi_{L,d}, \psi_3 = \psi_{R} \). Conservation laws are encoded in Ward Identities, which can be derived performing in the generating function \( W_0(A, \phi) \) the change of variables, \( i = \nu, \epsilon, u, d \)

\[
\begin{align*}
\psi_{1,L,x}^+ &\to e^{\pm i \alpha_{1,\nu} x} \psi_{1,L,x}^+, \\
\psi_{1,R,x}^- &\to e^{\pm i \alpha_{1,\epsilon} x} \psi_{1,R,x}^-, \\
\alpha_{L,v} &\to \alpha_{L,\nu}, \alpha_{L,c} &\to \alpha_{L,\epsilon}, \alpha_{L,u} &\to \alpha_{L,d}
\end{align*}
\]  

so obtaining, noting that the external currents are invariant \( c W_0(A, \phi) = \)

\[
\int P(d\psi) e^{-\int dx \psi_{\alpha}^+ (e^{i\alpha_{\epsilon} x} D e^{i\alpha_{\nu} x} - D) \psi_{\alpha}^-} \int dx (g W_{\mu}^\phi + g^* B_{\mu, x} B_{\mu, x}) + \int dx (\psi_{\alpha}^+ e^{i\alpha_{\epsilon} x} \phi_{\alpha, s, x} + \psi_{\alpha}^- e^{-i\alpha_{\nu} x} \phi_{\alpha, s, x}^+) 
\]

where

\[
D\psi_{\alpha,i} = \int dke^{-ikx} \chi_{-1}^- (k) \sigma_{\mu}^\epsilon \chi_{+1}^\epsilon \psi_{\alpha,i,k} \]

The fermionic source term acquire a phase but not the current source. The Jacobian of the transformation is unitary as a straightforward consequence of the fact that the number of Grassmann variables \( \psi_{\epsilon} \) is finite: this is an important difference with respect to what happens in (formal) functional integrals with infinitely many variables [5]. The exponent of the fermionic "measure" gets an extra term of the form

\[
\int dx \psi_{\alpha,i}^+ (e^{i\alpha_{\epsilon} x} D e^{-i\alpha_{\nu} x} - D) \psi_{\alpha,i}^- = \sum_{i,s} \int dx \delta T_{s,i} \alpha_{\epsilon,i,s} + O(\alpha^2)
\]

where

\[
\delta T_{s,i} = \frac{1}{L^4} \sum_{k,p} e^{-i p x} \chi_{+1}^\epsilon \sigma_{\mu}^\epsilon (\chi_{-1}^\mu k - \chi_{-1}^\mu (k + p)(k + p)) \psi_{\alpha,i,k+p}
\]

Replacing the cut-off function with 1 one gets that \( \chi^{+1}(k)p - \chi^{-1}(k + p)(k + p) = p \mu \) so that the r.h.s. of (15) reduces to \( \sum_{i,s} \int dx \psi_{\alpha,i,s} \delta_{\mu} \alpha_{\epsilon,i,s} \). Note that the expressions \( \chi^{-1} \) does not give any problem as identities have to be well understood between correlations. By performing derivatives of (13) with respect to \( \alpha_{\epsilon,i,s} \) and to the external fields we get

\[
p_\mu < \tilde{j}_{\mu,0.p}^-; \tilde{\psi}_{\alpha,i,k+p}^+ \tilde{\psi}_{\alpha,i,k+p}^- > + < \tilde{j}_{\mu,0.p}^+; \tilde{\psi}_{\alpha,i,k+p}^+ \tilde{\psi}_{\alpha,i,k+p}^- > = \delta_{\alpha,\alpha'} ( < \tilde{\psi}_{\alpha,i,k+p}^+ \tilde{\psi}_{\alpha',i,k+p}^- > - < \tilde{\psi}_{\alpha',i,k+p}^+ \tilde{\psi}_{\alpha,i,k+p}^- > )
\]

with \( \tilde{j}_{1.p} = \tilde{j}_{1.0.p} + \delta_{j_{1.p}} \), \( \tilde{j}_{2.p} = \tilde{j}_{2.0.p} + \delta_{j_{2.p}} \), \( \tilde{j}_{3.p} = \delta_{j_{3.p}} \) and

\[
\tilde{j}_{\mu,i,p}^\mu = \frac{1}{L^4} \sum_k C^\epsilon(k, p) \psi_{\epsilon,i,k+p}^+ \psi_{\epsilon,i,k+p}^- \quad C^\epsilon(k, p) = |(\chi^{-1}(k - 1)k - 1)k - (\chi^{-1}(k + p) - 1)(k + p)| \sigma_{\mu}^\epsilon
\]

The above identity has been written in a form closer to the formal WI writing \( \chi^{-1} \) as \( (\chi^{-1} - 1) + 1 \) in (16). With respect to the formal WI, (17) has an extra term dependent on the momentum cut-off. The origin of the term can also be understood from the equality between propagators with momentum cut-off

\[
g^\epsilon(k) = g^\epsilon(k + p) = g^\epsilon(k) \sigma_{\mu}^\epsilon p_\mu g^\epsilon(k + p) + g^\epsilon(k) C(k, p) g^\epsilon(k + p)
\]

which replace the identity \( g^\epsilon(k) = g^\epsilon(k + p) = g^\epsilon(k) \sigma_{\mu}^\epsilon p_\mu g^\epsilon(k + p) \) in presence of a cut-off.

We derive now the WI with respect to the currents; by performing derivatives in (13) with respect to \( \alpha \) and \( W, B \) we get

\[
p_\mu < \tilde{j}_{\mu,0.p}^-; \tilde{j}_{\nu,1,p_1}^-; \tilde{j}_{\nu,1,p_1}^-; \tilde{j}_{\nu,1,p_1}^-; \tilde{j}_{\nu,1,p_1}^-; \tilde{j}_{\nu,1,p_1}^- > = < \tilde{j}_{\mu,0.p}^+; \tilde{j}_{\nu,1,p_1}^+; \tilde{j}_{\nu,1,p_1}^+; \tilde{j}_{\nu,1,p_1}^+; \tilde{j}_{\nu,1,p_1}^+; \tilde{j}_{\nu,1,p_1}^+ >
\]

Again we get an extra term with respect to the formal WI proportional to \( \delta_{\mu,0}^\epsilon \); if such term would be vanishing then from (10) we get the current conservation \( p_\mu < \tilde{j}_{\mu,0.p}^- >_{W, B} = 0 \). Even more, the WI (20) with the l.h.s. vanishing is equivalent to the separate conservation of currents of different species. The l.h.s. of (20) is however in general non vanishing.
Let us consider now the r.h.s. of (20) with $n = 2$ which is given

$$p_{\mu} < \hat{j}_{s,i,\mu,p;\hat{j}_{s,i,\nu,p};\hat{j}_{s,i,\rho,p2}> = \int \frac{dk}{(2\pi)^4} \text{tr} \frac{\chi(k)}{-i\sigma^\mu k_\mu} C(k,p) \frac{\chi(k + p)}{-i\sigma^\nu(k_\mu + p_\mu)} \frac{\chi(k + p^2)}{-i\sigma^\rho(k_\mu + p_\mu)} (\nu, p_1) \rightarrow (\rho, p_2)$$

and, see [23]

$$p_{\mu} < \hat{j}_{s,p,\mu;\hat{j}_{s,p,\nu;\hat{j}_{s,p,\rho}\nu} >= \epsilon_{\mu, \nu, \rho} \frac{1}{12\pi^2} \epsilon_{\nu, \rho, \alpha} b_p p^a_p p^b_p + O(|p|^3 / \gamma^N)$$

where $|p| = \max(|p_1|, |p_2|)$ and $\epsilon_L = -\epsilon_R = 1$. From (23) we see that, in addition to terms proportional to the inverse cut-off, there are N-independent contributions which are the anomalies in the limit of removed cut-off.

The average of the $B$ current at second order, see (10), is given by

$$< \hat{j}_{B,\mu,p;\hat{j}_{W,\nu,p;\hat{j}_{W,\nu,\rho,p2}>> = \hat{L}_{B,\mu,\nu,p}(p_1, p_2)$$

The divergence follows by (23) and one finds

$$p_{\mu} \hat{L}_{B,\mu,\nu,p}(p_1, p_2) = \frac{1}{12\pi^2} \epsilon_{\nu, \rho, \alpha} \frac{1}{b_p} [\sum_i Y_i^L] + R_{B,\nu,\rho}(p_1, p_2)$$

$$p_{\mu} \hat{L}_{B,\mu,\nu,p}(p_1, p_2) = \frac{1}{12\pi^2} \epsilon_{\nu, \rho, \alpha} \frac{1}{b_p} [\sum_i Y_i^R^a] + R_{B,\nu,\rho}(p_1, p_2)$$

with

$$|R_{B,\nu,\rho}(p_1, p_2)|, |R_{B,\nu,\rho}(p_1, p_2)| \leq C \frac{|p|^3}{\gamma^N}$$

If we require that the current is conserved up to terms to the energy divided by the cut-off scale we get

$$\sum_i Y_i^L = 0, \quad \sum_i (Y_i^L)^3 - (Y_i^R)^3 = 0$$

These are of course the same conditions found in the standard electroweak theory with classical gauge fields; indeed the limit $N \rightarrow \infty$ can be taken safely and exact conservation is found. The condition is verified by elementary particles as $Y_i^L$ has value $(-1, -1, -1, -\frac{1}{3}, \frac{1}{3})$ and $Y_i^R$ has value $(0, 2, 2, -\frac{1}{3}, -\frac{1}{3})$ so that one gets $-2 + 6\frac{1}{3} = 0$ and $6(1/3)^3 + 2(-1)^3 - 3(4/3)^3 - 3(-2/3)^3 = 0$. The conservation of $W$ current does not give further constraint.

We want to investigate if the condition (27) still ensures the current conservation in the interacting case. In such a case the terms contributing to the divergence of the current are a series of infinitely many Feynman graphs and a direct verification is impossible. In addition, in order to get non perturbative results one needs to keep a finite ultraviolet cut-off, as the Standard Model is not asymptotically free. We ask therefore if also in the interacting case with finite cut-off the current is conserved up to terms proportional to the energy divided by the cut-off scale provided that the condition (27) is true.
D. Effective Fermi interaction

The Standard electroweak theory is obtained replacing in (9) the field $B_\mu, \mu$ with the sum of two fields $B_\mu + \tilde{B}_\mu$ and $W_\mu + \tilde{W}_\mu$, see e.g. [3], where $B_\mu, W_\mu$ are classical background fields and $\tilde{B}, \tilde{W}$ are quantum fields, with a gauge invariant action. The $Z$ and $e.m.$ currents are define by

\[
\int dx (g \tilde{W}_x^3 L I_{3,i} + g' \tilde{B}_x j_{B,\mu}) = \int dx \left( e A_{\mu,x} j_{\mu,x}^{em} + g' Z_{\mu,x} j_{\mu,x}^{Z} \right)
\]

where \( g' = \frac{g}{\cos \theta}, \) \( \tanh \theta = g'/g, \) \( g \sin \theta = g' \cos \theta = \epsilon \) and the charges are

\[
2Q_i^s = I_{3,i}^s + Y_i^s
\]

with $I_{3,i}^L = \pm 1$ and $I_{3,i}^R = 0$ (so that $Q_i$ is $(0, -1, 2/3, -1/3)$) and

\[
j_{\mu,x}^{c.m} = e \sum_i Q_i (j_{i,1,\mu,x} + j_{i,2,\mu,x}) \quad j_{Z,\mu,x} = \sum_{i,s} (I_{3,i}^s - \sin^2 \theta Q_i) j_{i,\mu,x}^s
\]

From (29) we see that the proof of charge quantization follows from the quantization of the hypercharges, as $I_{3,i}^s$ is quantized.

Due to the Higgs mechanism, the quantum $\tilde{Z}_\mu$ and $\tilde{W}$ gauge fields acquire a mass. The effective electroweak theory is obtained integrating the boson fields generating an effective quartic interaction; it is indeed not restrictive to consider only quartic interactions as monomials of any order in the fields are generated during the RG integrations, see §3. Neglecting for the moment the external gauge fields the correlations of the effective theory are given by

\[
\langle O \rangle = \frac{\int P(d\psi)e^{V(\psi)}O}{\int P(d\psi)e^{V(\psi)}}
\]

where $P(d\psi)$ is the fermionic integration with renormalized propagator

\[
g_{i,s}(x, y) = \frac{1}{Z_{N,i,s}} \frac{1}{L^4} \sum_k e^{ik(x-y)} \frac{\chi_N(k)}{-i\sigma^s k}\]

and

\[
V(\psi) = \int dx dy \lambda [W(x, y) (j_{W,\mu,x}^{1} j_{W,\mu,x}^{1} + j_{W,\mu,x}^{2} j_{W,\mu,x}^{2}) + w_z(x, y) j_{\mu,x} j_{\mu,x}]
\]

with $\lambda$ an effective coupling proportional to $g^2$ and

\[
v_W(x, y) = \int dk e^{ik(x-y)} \frac{\chi_N(k)}{|k|^2 + M_W^2} \quad v_Z(x, y) = A \int dk e^{ik(x-y)} \frac{\chi_N(k)}{|k|^2 + M_Z^2}
\]

with $M = M_Z > M_W$ and $A$ is a constant to take into account the difference in the effective couplings and masses.

At finite $N$ we can prove that the this effective theory has a well defined non-perturbative meaning, even if in the $L \to \infty$ limit the functional integrals are infinite dimensional. Indeed in §4 we prove the following result

**Theorem 1** The correlations corresponding to (31) are analytic in $\lambda$ for $|\lambda| \leq \left(\frac{M_Z}{M_W}\right)^6$ uniformly as $L \to \infty$

Analyticity in the coupling around the origin is a remarkable fact due to the purely fermionic nature of (31); indeed, in presence of bosons analyticity in zero cannot be true due to Dyson argument. The estimated radius of convergence is proportional to the gauge mass divided by the cut-off; this reflects the perturbative non-renormalizability of the theory, and implies that the cut-off must be chosen of the order of the gauge mass. The proof is based on non-pertubative methods and avoid the Feynman graph expansion, see §III.

E. Effective electroweak theory and main result

We include in the effective model the external gauge fields associate to the $B$ and $W$ currents. Due to the interaction, the charges are renormalized and one needs to introduce bare currents depending on parameters to be fixed so that the ir values correspond to the physical values at low momenta. We introduce therefore the bare background currents

\[
\tilde{j}_{W,\mu,x}^k = \sum_{a=L,q} Z_{N,a,k}^W j_{W,a,\mu,x}^k \quad \tilde{j}_{B,\mu,x}^s = \sum_{i,s, \mu, a = L,R} Y_i^s Z_{N,i,s}^B j_{i,\mu,x}^s
\]

(35)
with the parameters $Z_{N,a,k}^W$ and $Z_{N,i,s}^J$ to be chosen in order to fix the dressed parameters, which can be obtained by
the correlations. It is indeed an outcome of our RG analysis in §2 that in the analyticity domain $|\lambda| \leq \left( \frac{M}{\kappa g_{W,e}} \right)^6$ the
2-point function is

$$< \hat{\psi}_{k,i,s}^+ \hat{\psi}_{k,i,s}^- > = \frac{1}{Z_{-\infty,i,s}} \frac{1}{-i\sigma_\mu^p k_\mu} (1 + R(k))$$  \hspace{1cm} (36)
with $Z_{-\infty,i,s}$ is a non trivial analytic function of $\lambda$ representing the wave function renormalization and $|R(k)| \leq C|\lambda||k|^{-N}$. Similarly the 3-point functions are, $k \sim k + p \sim \kappa$

$$< \hat{J}_{\mu,x} \hat{\psi}_{i,s,k}^+ \hat{\psi}_{i,s,k+p}^- > = \frac{1}{\sigma_\mu^p k_\mu} \frac{1}{\sigma_\mu^p (k_\mu + p_\mu)} \left( \frac{Y_i^+ Z_i^+ Z_{-\infty}^+}{Z_{\infty,i,s} Z_{-\infty}^+} + R(k,k+p) \right)$$  \hspace{1cm} (37)
with $|R(k,k+p)| \leq C|\lambda||\kappa|^{-N}$ from which we see that the dressed hypercharge is $Y_i^+ Z_i^+ Z_{-\infty}^+$. A similar expression is
found for $< \hat{j}_{W,\mu,x} \hat{\psi}_{i,s,k}^+ \hat{\psi}_{i,s,k+p}^- >$ in which the dominant term is proportional to $\frac{Y_i^+ Z_i^+ Z_{-\infty}^+}{Z_{\infty,i,s} Z_{-\infty}^+}$.

The bare normalization are chosen in order to ensure the following conditions

$$Z_{i,s,-\infty} = 1 \quad Z_i^L_{i,s,-\infty} = 1 \quad Z_i^W_{\infty,-\infty} = 1$$  \hspace{1cm} (38)
The first condition ensures that the wave renormalization in the low-energy limit is the same for all particles; the second that the dressed hypercharge is equal to $Y_i^+$ and the third that the normalizations in the $W$ currents do not depend on the particle species in the low energy limit. The non-trivial renormalization of the charges is related to the extra terms with $\delta j$ in the WI for the 3-point function (17). Such WI holds also in the interacting case, as $V$ is invariant under the transformation (12). However the term depending on $\delta j$, which is proportional to the inverse of the cut-off in the non interacting case, is $N$ independent up to small corrections and $O(\lambda)$ in presence of interaction, see [21],[22] for a similar phenomenon in the $d = 1 + 1$ case.

The effective electroweak theory replacing (9) is therefore given by

$$< O >_{W,B} = \frac{\int P(d\psi) e^{V}\left( \int dx \left( g_{W,e} \hat{j}_{W,\mu,x} + g_{B,\mu,x} \hat{j}_{B,\mu,x} \right) \right) O \right)}{\int P(d\psi) e^{V} \left( \int dx \left( g_{W,e} \hat{j}_{W,\mu,x} + g_{B,\mu,x} \hat{j}_{B,\mu,x} \right) \right)}$$  \hspace{1cm} (39)
with $V$ given by (33) and $\hat{j}_{W,\mu,x}$ are given by (35) with the normalization condition (37) and (38). The response of the $U(1)$ current in the effective theory is given by

$$< \hat{j}_{B,\mu,p} >_{W,B} = \sum_{n} \frac{1}{n!} \int dp_1 ... dp_n \frac{\partial^{n+1} W_{W,B}}{\partial B_{\mu,p} \partial A_{\mu_1,p_1} \partial A_{\mu_n,p_n}} |g_{W,e} A_{\mu_1,p_1}^e ... g_{W,e} A_{\mu_n,p_n}^e| \delta(\sum_{i} p_i)$$  \hspace{1cm} (40)
with the derivative above given by $< j_{B,\mu,p} ; j_A^{\mu_1,p_1} ; ... ; j_A^{\mu_n,p_n} >$ and

![FIG. 3: Graphs contributing to the expansion at n = 2.](image)

$$e^{W(A)} = \int P(d\psi) e^{V} \left( \int dx \left( g_{W,e} \hat{j}_{W,\mu,x} + g_{B,\mu,x} \hat{j}_{B,\mu,x} \right) \right) \equiv \int P(d\psi) e^{V+B(A)}$$  \hspace{1cm} (41)
There are now radiative corrections, see Fig 3, which could produce extra conditions in order to impose that the current is conserved. This is however excluded by the following result.
**Theorem 2** For $|\lambda| \leq \frac{M}{(\epsilon N)^6}$ and choosing $Z_{i,s,N}, Z_{i,s,N}^T, Z_{n,N}$ as functions of $\lambda$ so that (38) holds, then the 3-point function can be written as

\[
\begin{align*}
\langle \tilde{J}_{B,\mu,x}; \tilde{J}_{B,\nu,x}; \tilde{J}_{B,\rho,x} \rangle &= L_{\mu,\nu,\rho} B(x,x_1,x_2) + R_{\mu,\nu,\rho}^1(x,x_1,x_2) \\
\langle \tilde{J}_{B,\mu,x}; \tilde{J}_{B,\nu,x}; \tilde{J}_{W,\rho,x} \rangle &= L_{\mu,\nu,\rho} W(x,x_1,x_2) + R_{\mu,\nu,\rho}^1 W(x,x_1,x_2)
\end{align*}
\]

with $\tilde{L}_{\mu,\nu,\rho}^B, \tilde{L}_{\mu,\nu,\rho}^W$ verifying (25) and

\[
|P_{\mu,\nu,\rho}^L(x,x_1,x_2)|, |R_{\mu,\nu,\rho}^1(x,x_1,x_2)| \leq C \left( \frac{1}{\gamma N \delta} \right)^\delta C
\]

with $\delta$ is the minimal distance between $x, x_1, x_2$.

We see from (42) that also in the interacting case the current is conserved up to terms proportional to the inverse of the cut-off scale, provided that the conditions $\sum Y_i^L = 0$ and $\sum (Y_i^R)^3 = 0$ hold; even if the average of the current is given by a complicated series of graphs, no new conditions arise. The crucial bound (43) is non-perturbative; graphs expansion is avoided and determinant bounds are used to implement cancellations due to Pauli principle and ensuring analyticity.

**III. RENORMALIZATION GROUP ANALYSIS**

The starting point for the analysis of $W_{B,W}(A)$ (41) is the following decomposition of the cut-off function

\[
\chi_N(k) = \sum_{h=-\infty}^{N} f_h(k) - f_h(k) = \chi(\gamma^{-h}k) - \chi(\gamma^{-h+1}k)
\]

so that $f_h(k)$ is a smooth cut-off function selecting momenta in $\gamma^{-h-1} \leq |k| \leq \gamma^{-h+1}$; we also call $\chi_h(k) = \sum_{j=-\infty}^h f_j(k)$ the cut-off function selecting momenta $|k| \leq \gamma^h$. We perform an exact RG integration. The starting point is the addition property

\[
P(d\psi) = P(d\psi^{(N)})P(d\psi^{(\leq N-1)})
\]

where $P(d\psi^{(N)})$ and $P(d\psi^{(\leq N-1)})$ is the gaussian grassmann measure with propagators

\[
g_{N,i,s}^{(N)}(x,y) = \frac{1}{Z_{N,i,s}} \frac{1}{L^4} \sum_k e^{ik(x-y)\chi_N(k)} = \frac{1}{Z_{N,i,s}} \frac{1}{L^4} \sum_k e^{ik(x-y)\chi_{N-1}(k)}
\]

Setting $V^{(N)} = V + B$ we can write

\[
\int P(d\psi) e^{V^{(N)}} = \int P(d\psi^{(N)})P(d\psi^{(\leq N-1)})e^{V^{(N)}} = \int P(d\psi^{(\leq N-1)})e^{V^{(N-1)}}
\]

where by definition

\[
e^{V^{(N-1)}} = \sum_{n=0}^\infty \frac{1}{n!} \mathcal{E}^{(N)}_n (V; n)
\]

where $\mathcal{E}^{(N)}_n (V; n)$ are the truncated expectations (that is the sum of connected Feynman graphs) with propagator $g^{(N)}$. After the integration of the first scale $\psi^{(N)}$ we get an effective potential which is sum of infinitely many monomials in the $\psi^{(\leq N-1)}, A$

\[
V^{(N-1)}(A, \psi^{(\leq N-1)}) = \sum_{l,m=0}^\infty \int d\mathcal{E}^{(N-1)}_{l,m} (\phi) \prod_{j=1}^l \psi_{\xi_j, s_j, x_j} \prod_{j=1}^m A_{\mu_j, x_j}
\]

The scaling dimension is $D = 4 - \frac{3}{2}l - m$ and we can separate the irrelevant terms $D < 0$ from the rest; moreover marginal and relevant terms $\psi^+ \psi^-$ or $A \psi^+ \psi^-$ are generally non-local (that is the fields have different coordinates),
and we can split them in a local plus an irrelevant part. In order to obtain this we define a localization operator \( \mathcal{L} \) such that \( \mathcal{L} \) gives a vanishing result on the irrelevant terms and

\[
\mathcal{L} \psi^+_{i,s,z} \psi^-_{i,s,z} = \psi^+_{i,s,z} \psi^-_{i,s,z} + (x - y)_\mu \psi^+_{i,s,z} \partial_\mu \psi^-_{i,s,z}.
\]

In conclusion we get the form

\[
\mathcal{L} \psi^+_{i,s,z} \psi^-_{i,s,z} = \mathcal{A}^e_{\mu,z} \psi^+_{i,s,z} \psi^-_{i,s,z}.
\]

In momentum space the above expression can be equivalently written as

\[
\mathcal{L} \int dk \bar{W}_{2,0}(k) \hat{\psi}^+_{i,s,k} \hat{\psi}^-_{i,s,k} = \int dk (\bar{W}_{2,0}(0) + k_\mu \partial_\mu \bar{W}_{2,0}(0)) \hat{\psi}^+_{i,s,k} \hat{\psi}^-_{i,s,k}
\]

\[
\mathcal{L} \int dk dp \bar{W}_{2,1,\mu}(k,p) A^e_{\mu,p} \hat{\psi}^+_{i,s,k+p} \hat{\psi}^-_{i,s,k+p} = \int dk dp \bar{W}_{2,1,\mu}(0,0) A^e_{\mu,p} \hat{\psi}^+_{i,s,k} \hat{\psi}^-_{i,s,k+p}
\]

The fields \( \psi \) have the same chirality, as the propagators are diagonal in the chiral index and the currents have the same chirality. By parity of the propagator \( \bar{W}_{2,0}(0) = 0 \), Lorentz symmetry, valid also in presence of cut-off, implies that \( \partial_\mu \bar{W}_{2,0}(0) \) and \( \bar{W}_{2,1,\mu}(0,0) \) are proportional to \( \sigma^\mu \). There are no contributions \( \psi^\mu \partial_\mu \psi^- \) with \( i \neq j \). Indeed if \( i \) and \( j \) belongs to different families then such term would violate the invariance under a global phase transformation \( \psi^\pm \rightarrow e^{\pm i \alpha} \psi^\pm \) with \( a = l, q \). If \( i, j \) belong to the same family then if the field \( i \) has \( s = R \) is impossible by a similar argument; if \( s = L \) we call \( n_1 \) the number of vertices containing only one field \( i \) (say \( e \)), \( n_2 \) or \( n_4 \) the number of vertices containing 2 or 4 fields; then \( (n_1 - 1 + 2n_2 + 4n_4)/2 \) must be integer; hence \( n_1 \) is odd but then there is an odd number of fields of the other family \( u \) or \( d \) and this is impossible. The marginal quadratic terms have therefore the form \( z_{N-1, i, s} \int dk \bar{k} \psi^+_{i,s,k} \psi^+_{i,s,k} \psi^-_{i,s,k} \psi^-_{i,s,k} \) which can be included in the wave function renormalization

\[
Z_{N-1, i, s} = Z_{N, i, s} + z_{N-1, i, s}
\]

In the same way there are no contribution to \( W_{2,1,\mu} \) with fields with different \( i \) index if the source is diagonal in the index, and the non vanishing terms can be included in the current renormalizations defining

\[
Z^J_{N-1, i, s} = Z^J_{N, i, s} + z^J_{N-1, i, s} \quad Z^W_{N-1, a, k} = Z^W_{N, a, k} + z^W_{N-1, a, k}
\]

In conclusion we get

\[
\int P(d\psi^{(\leq N-1)}) e^{\mathcal{L}V^{(N-1)}} + \mathcal{R}V^{(N-1)}
\]

with propagator

\[
g^{(\leq N-1)}(x, y) = \frac{1}{Z_{N-1, i, s}} \frac{1}{L^2} \sum_k \rho_k(x-y) \chi_{N-1}(k) \frac{1}{-i\sigma^\mu_k \bar{\psi}^\mu}
\]

where

\[
\mathcal{R}V^{(N-1)}(A, \psi^{(\leq N-1)}) = \sum_{l,m} \int dz W_{n,m}(z) \prod_{j=1}^l \partial_{\mu_s, j} \psi_{s_j, x_j} \prod_{j=1}^m A^e_{\mu_j, x_j}
\]

where \( \sum^* \) has the constraint that if \( l = 2, m = 0 \) then \( s_1 + s_2 = 2 \) and if \( l = 2, m = 1 \) then \( s_1 + s_2 = 1 \); that is, the effect of the \( \mathcal{R} \) operation is to produce a series with negative scaling dimension. Moreover

\[
\tilde{\mathcal{L}}V^{(N-1)} = \int dx (g W^W_{\mu,x} \gamma_k^{(N-1)} + g' B_{\mu,x} \gamma^{(N-1)})
\]

with

\[
\gamma_k^{(N-1)} = \sum_{a=L, q} Z^W_{N-1, a, k} \gamma_k^{(N-1)} \quad \gamma^{(N-1)} = \sum_{x=1} x \gamma_k^{(N-1)} Z^W_{N-1, i, s} \gamma_k^{(N-1)} (N-1)
\]

The expression is similar to the initial one, with the difference that the effective potential is sum over monomials of all orders and with derivatives in the fields, and the wave function renormalizations and the normalizations of the charges
is modified. The field $\psi(N-1)$ can be integrated and the procedure can be iterated. The generic RG integration step gives

$$e^{W(A)} = \int P(d\psi^{(\leq h)}) e^{\mathcal{L}V^h + \mathcal{R}V^h}$$

(59)

where $P(d\psi^{(\leq h)})$ is a Grassmann integration with propagator

$$g_{\lambda,s}^{(\leq h)} = \frac{1}{Z_{i,s,h} - i\sigma^\mu_k \delta_{\mu}}$$

(60)

and $\mathcal{L}V^h, \mathcal{R}V^h$ are similar with $N-1$ replaced by $h$.

As an outcome of the above construction the terms $W_{n,m}^{(h)}$ are expressed by convergent series in $\lambda$

$$W_{n,m}^{(h)} = \sum_{n=1}^{\infty} K_{n,l,m}^{h} \lambda^n$$

(61)

and in section §III it is proved that

$$|K_{n,l,m}^{h}| \leq C^{l+n+m} \gamma^{(4-3/2)(l-m)h^2 N(h-N)} \left[\frac{\gamma^6 N}{M^6}\right]^n$$

(62)

with $\delta_0 = 0$ and $\delta_n = \theta = 1/2$ for $n \neq 0$. The factor $\gamma^{\theta(h-N)}$ is a gain with respect to the "dimensional bound" in the term with at least a $\lambda$, and is due to the dimensional irrelevance of the quartic terms; such extra factor plays a crucial role in the following. Note that the estimated convergence radius is proportional to the cut-off and mass ratio, as a consequence of the perturbative non-renormalizability of the theory.

The effective renormalizations verify recursive equations, if $Z_h = (Z_{i,s,h}, Z_{i,s,h}, Z_{a,h})$

$$Z_{h-1} = Z_h + \beta_2^h \lambda; Z_{h,N}, Z_N) \quad |\beta_2^h| \leq \gamma^{\theta(h-N)} C\left[\frac{\gamma^6 N}{M^6}\right]$$

(63)

where the r.h.s. have an extra factor $\gamma^{\theta(h-N)}$ by (62), noting that there is no contribution to the $\beta$ function of order zero. The renormalizations are therefore finite

$$Z_{h-1} = Z_N + \sum_{k=h}^{N} \beta_2^h \lambda; Z_{k}, Z_N)$$

(64)

We impose the renormalization conditions; we can look to (64) as a self-consistence equation and by a contraction methods we find $Z_{i,s,N}$ as a function of $\lambda$ so that

$$Z_h = 1 + O(\gamma^{\theta(h-N)} \lambda \frac{\gamma^6 N}{M^6})$$

(65)

Analyticity stated in Theorem 1 is an immediate consequence of (62). Note that the denominator of the correlations (the partition function) at finite $L$ is analytic for any $\lambda$ in the whole complex plane at finite $L$ as it is a finite dimensional Grassmann integral; on the other hand the RG analysis above provides an expansion which coincides order by order and is analytic in a finite domain, so that it fully reconstructs the partition function. The correlation is also analytic, as the denominator is non vanishing in a finite disk for small $\lambda$ for any $L$ and the numerator is a finite dimensional integral; moreover it coincides order by order with the expansion found analyzing the generating function by RG which is also analytic in the same domain so that they coincide and analyticity as $L \to \infty$ follows.

IV. CONVERGENCE AND ANALYTICITY

We prove now the bound (62). The kernels of the effective potential generated in the Renormalization group analysis can be conveniently written as a sum of trees, defined in the following way, see e.g. [19].

Let us consider the family of all trees which can be constructed by joining a point $r$, the root, with an ordered set of $n \geq 1$ points, the endpoints of the unlabeled tree, so that $r$ is not a branching point. $n$ will be called the order of the unlabeled tree and the branching points will be called the non trivial vertices. The unlabeled trees are partially
ordered from the root to the endpoints in the natural way; we shall use the symbol \( < \) to denote the partial order. The number of unlabeled trees is \( 4^n \). The set of labeled trees \( \mathcal{T}_{h,n} \) is defined associating a label \( h \leq N - 1 \) with the root; moreover we introduce a family of vertical lines, labeled by an integer taking values in \([h, N + 1]\) intersecting all the non-trivial vertices, the endpoints and other points called trivial vertices. The set of the union of the endpoints, the trivial vertices and the non-trivial vertices. The scale label is \( v \) and in this case the scale is \( v < h \) vertices and \( v_1 < v_2 \), then \( h_{v_1} < h_{v_2} \). Moreover, there is only one vertex immediately following the root, which will be denoted \( v_0 \) and can not be an endpoint; its scale is \( h_0 = v_0 < h \). The effective potential can be written as

\[
\mathcal{V}^{(h)}(\psi^{(\leq h)}, A) = \sum_{n=1}^{\infty} \sum_{\tau \in \mathcal{T}_{h,n}} \mathcal{V}^{(h)}(\tau),
\]

where, if \( v_0 \) is the first vertex of \( \tau \) and \( \tau_1, \ldots, \tau_s \) \( (s = s_{\tau_0}) \) are the subtrees of \( \tau \) with root \( v_0 \), \( \mathcal{V}^{(h)} \) is defined inductively by the relation, \( h \leq N - 1 \)

\[
\mathcal{V}^{(h)}(\tau) = \frac{(-1)^{s+1}}{s!} \mathcal{E}^{h+1}_{h+1}[\mathcal{V}^{(h+1)}(\tau_1); \ldots; \mathcal{V}^{(h+1)}(\tau_s)]
\]

where \( \mathcal{E}^{h+1}_{h+1} \) is the truncated expectation and \( \mathcal{V}^{(h+1)}(\tau) = \mathcal{R}\mathcal{V}^{(h+1)}(\tau) \) if the subtree \( \tau_i \) contains more then one end-point, while if \( \tau_i \) contains only one end-point \( \mathcal{V}^{(h+1)}(\tau) = V(\psi^{(\leq h)}) \) if it is a normal end-point (and in such case \( h = N - 1 \)) or if is a special end-point \( \mathcal{L}\mathcal{V}^{h+1}(A, \psi^{(\leq h+1)}), \ h < N - 1 \) or \( \mathcal{B}(\psi^{(\leq h)}, A) \). We define \( P_v \) as the set of field labels of \( v \) representing the external fields and if \( v_1, \ldots, v_s \) are the \( s_\tau \) vertices immediately following \( v \), then we denote by \( Q_v \) the intersection of \( P_v \) and \( P_{v_i} \); this definition implies that \( P_v = \cup_i Q_v \). The union of the subsets \( P_{v_i} \setminus Q_{v_i} \) are the internal fields of \( v \). Therefore if \( \mathbf{P} \) is the family of all such choices and \( \mathbf{P} \) an element we can write

\[
\mathcal{V}^{(h)}(\tau) = \sum_{\mathbf{P} \in \mathbf{P}_v} \int dx v_0 W^{(h+1)}_{\mathbf{P},\mathbf{P}}(x v_0) \prod_{f \in P_{v_0}} \psi^{(f)}(x f)^{\leq h}[\prod_f A(x f)]
\]

where \( W^{(h+1)}_{\mathbf{P},\mathbf{P}}(x v_0) \) is defined inductively by the equation

\[
W^{(h+1)}_{\mathbf{P},\mathbf{P}}(x v_0) = \frac{1}{s_\tau} \prod_{i=1}^{s_\tau} W^{(h+1)}_{\mathbf{P},\mathbf{P}}(x v_0) \mathcal{E}^{h+1}_{h+1}[\psi^{(h)}(P_{v_i} / Q_{v_i}); \ldots; \psi^{(h)}(P_{v_{s_\tau}} / Q_{v_{s_\tau}})]
\]
where $\tilde{v}^{(h)}(P) = \prod_{f \in P} \tilde{v}^{(h)}_{v(f)}$ and $x_v$ are the coordinates associated to the vertex $v$. We use the well-known representation of the fermionic truncated expectation, if $P$ is a set of indices
\[
E^T_h(\tilde{v}^{(h)}(P_1); \tilde{v}^{(h)}(P_2); \ldots; \tilde{v}^{(h)}(P_s)) = \sum_{T} \prod_{t \in T} g^{(h)}(x_t - y_t) \int dP_T(t) \det G^{h,T}(t)
\] (69)
where $T$ is a set of lines forming an anchored tree graph between the clusters of points $x(f) \in P_i$, that is $T$ is a set of lines, which becomes a tree graph if one identifies all the points in the same cluster. Moreover, $dP_T(t)$ is a probability measure with support on a set of $t$ such that $t_{i,i'} = u_i \cdot u_{i'}$ for some family of vectors $u_i \in \mathbb{R}^4$ of unit norm. Finally $G^{h,T}(t)$ is a $(n-s+1) \times (n-s+1)$ matrix, whose elements are given by $G^{h,T}_{ij,i'j'} = t_{i,i'} g^{(h)}(x_{ij} - y_{ij})$.

By inserting the above representation we can write $W_{\tau P, T}^{(h+1)} = \sum_T W_{\tau P, T}^{(h+1)}$, where $T$ is the union of all the trees $T$.

The determinants are bounded by the Gram-Hadamard inequality, stating that, if $M$ is a square matrix with elements $M_{ij}$ of the form $M_{ij} = <A_i, B_j>$, where $A_i, B_j$ are vectors in a Hilbert space with scalar product $<\cdot , \cdot >$, then
\[
|\det M| \leq \prod_i \|A_i\| \cdot \|B_i\|
\]
where $\|\cdot \|$ is the norm induced by the scalar product. Let $\mathcal{H} = \mathbb{R}^s \otimes \mathcal{H}_0$, where $\mathcal{H}_0$ is the Hilbert space of complex two dimensional vectors with scalar product $<F,G> = \int dk F^*_K(k)G_k(k)$. It is easy to verify that
\[
G^{h_{ij},T}_{ij,i'j'} = t_{i,i'} g^{(h)}(x_{ij} - y_{ij}) = <u_i \otimes A^{(h)}_{x_{ij}}, u_{i'} \otimes B^{(h)}_{x_{ij}}>
\] (70)
where $u_i \in \mathbb{R}^s$, $i = 1, \ldots, s$, are the vectors such that $t_{i,i'} = u_i \cdot u_{i'}$ and $A, B$ suitable functions. The integrals over the coordinate are done integrating over the tree $T$ and the interactions, using that for any $K$
\[
|\psi(x)| \leq \frac{2^{4N}}{M^2} \frac{C_N}{1 + (|M||x|)^K} \int dx |\psi(x)| \leq \frac{2^{4N}}{M^6}
\] (71)
and
\[
|\tilde{g}^{(h)}(x)| \leq C\gamma^{3h} e^{-\gamma |x|^2} \int dx |\tilde{g}^{(h)}(x)| \leq C\gamma^{-h}
\] (72)
In conclusion we get
\[
\int dx_0 |W_{\tau P, T}(x_0)| \leq L^4 \prod_{v} \frac{1}{8 \nu_v!} \sum_{i=1}^{r_v} |P_{v_i}| - |P_v| \gamma^{-4h_v(s_v-1)} \gamma^{-2h_v(s_v-1)} \gamma^{3/2} \frac{2^{4N}}{M^6} \sum_i \gamma^{-z_v}\nu_v^{2^{4N}}
\]
where $z_v = 1$ if the external fields are $\psi \psi$ or $A \psi \psi$ and zero otherwise. By using that $h_v - h_{v'} = 1$
\[
\sum_v (h_v - h) (s_v - 1) = \sum_v (h_v - h_{v'}) (m_v^A + n_v^A - 1)
\]
\[
\sum_v (h_v - h) \sum_i |P_{v_i} - |P_v| = \sum_v (h_v - h_{v'}) (4m_v^A - |P_v| + 2n_v^A)
\] (73)
where $m_v^A$ is the number of end-points following $v, n_v^A$ is the number of external $A$ lines we get
\[
\int dx_0 |W_{\tau P, T}(x_0)| \leq L^4 \gamma^{-h} \frac{2^{4N}}{M^6} \sum_v \frac{1}{8 \nu_v!} \left( C^{r_v} \nu_v \gamma^{-2} \nu_v^{2^{4N}} \gamma^{-4h_v(s_v-1)} \gamma^{-2h_v(s_v-1)} \gamma^{3/2} \frac{2^{4N}}{M^6} \sum_i \gamma^{-z_v} \nu_v^{2^{4N}} \right)
\]
and finally
\[
\int dx_0 |W_{\tau P, T}(x_0)| \leq L^4 \gamma^{-bdw} C \left( \prod_v \frac{1}{8 \nu_v!} \gamma^{-d \psi(h_\psi - h_\psi')} \prod_v \gamma^{-2(N-h_v)(m_v^A)} \right) \frac{2^{4N}}{M^6} \sum_v \gamma^{-z_v} \nu_v^{2^{4N}}
\] (74)
where: \( \tilde{v} \in \tilde{V} \) are the vertices on the tree such that \( \sum_{\beta} |P_{\beta} v| - |P_{\beta} v| \neq 0 \), \( \tilde{v}^* \) is the vertex in \( \tilde{V} \) immediately preceding \( \tilde{v} \) or the root; \( m_{\tilde{v}} \) is the number of normal end-point following \( \tilde{v} \) and not any following vertex \( \tilde{v} \in \tilde{V} \); \( d_{\tilde{v}} = -4 + \frac{3|P_{\beta} v|}{2} + n_{\beta}^1 + z_{\beta} \). Finally the number of addenda in \( \sum_{\beta} |P_{\beta} v| \) is bounded by \( \prod_{\beta} s_{\beta} ! C^{\sum_{\beta} |P_{\beta} v| - |P_{\beta} v|} \). In order to bound the sums over the scale labels and \( \mathbf{P} \) we first use the inequality

\[
\prod_{\tilde{v}} \gamma^{-d_{\tilde{v}}(h_0 - h_{\tilde{v}})} \leq \left( \prod_{\tilde{v}} \gamma^{-\frac{1}{2}(h_0 - h_{\tilde{v}})} \right) \prod_{\tilde{v}} \gamma^{-\frac{3|P_{\beta} v|}{2}}
\]

(75)

where \( \tilde{v} \) are the non trivial vertices, and \( \tilde{v}^* \) is the non trivial vertex immediately preceding \( \tilde{v} \) or the root. The factors \( \gamma^{-\frac{1}{2}(h_0 - h_{\tilde{v}})} \) in the r.h.s. allow to bound the sums over the scale labels by \( C^n \). Finally if there if there is at least a normal end-point the bound improves by a factor \( \gamma^{\theta(N - N)} \) as \( m_{\tilde{v}} \geq 1 \) for some \( \tilde{v} \) so that, if

\[
\prod_{\tilde{v}} \gamma^{-d_{\tilde{v}}(h_0 - h_{\tilde{v}})} \prod_{\tilde{v}} \gamma^{-2(N - h_0) m_{\tilde{v}}}) \leq \gamma^{2(N - N)} \prod_{\tilde{v}} \gamma^{-d_{\tilde{v}}(h_0 - h_{\tilde{v}})}
\]

(76)

with \( \tilde{d}_{\tilde{v}} = d_{\tilde{v}} - \theta > 0 \) so that the sum over scales can be still done. This completes the proof of (62).

V. THE THREE CURRENT CORRELATION

We have now to prove Theorem 2. In order to compute \( S_3(x_1, x_2, x_3) = \langle \tilde{j}_{B, \mu, x_2}; \tilde{j}_{A, \mu, x_1}; \tilde{j}_{A, \mu, x_2, x_2} \rangle \) we perform the derivatives of \( \mathcal{W}(\mathbf{A}) \) given by (41) with respect to \( B_{\mu, x_2}; A_{\mu, x_1}; A_{\mu, x_2, x_2} \). The result can be written as \( S_3 = S_3^1 + S_3^2 \), where \( S_3^1 \) is given by trees with only special end-points, and \( S_3^2 = \sum_{\beta} \sum_{\gamma} S_3(x_1, x_2) \) where \( \sum_{\gamma} \) is sum over trees with at least one normal end-point, see Fig. 5. The renormalized triangle graphs have the form, in the case of three \( B \) currents

\[
\sum_{h_1, h_2, h_3} \left( \prod_{i=1}^3 Z_{h_i, x_i} \right) \int \frac{dk}{(2\pi)^4} \text{Tr} \left[ \frac{f^{h_1}(k)}{-i\sigma^\mu_k k_\mu} \right] \frac{f^{h_2}(k + p)}{-i\sigma^\mu_{k+p} (k + p_\mu)} (-i\sigma^\mu_p) (-i\sigma^\mu_{k+p} (-i\sigma^\mu_p) + [(\nu, p_1) \to (\sigma, p_2)])
\]

(77)

and a similar expression holds for the \( BWW \) currents. The main difference with respect to the triangle graphs seen in the non-interacting case is that the wave and the vertex are non trivial function of the momentum scale. We can use now (65) to further decompose the triangle graph as a sum of two terms, one in which \( Z_{h_1, x_1}, Z_{h_2, x_2}, Z_{h_3, x_3} \) are replaced by 1 and an extra term. The sum of such extra term plus \( S_3^1 \) constitute the term \( R_{h_1, \mu, \nu, \rho}; R_{h_2, \mu, \nu, \rho} \) in (42). The other term is exactly coinciding with the non-interacting case using that \( \sum_{h=-\infty}^N f^h = \chi_N \).

FIG. 5: Graphical representation to \( S_3 \). The first term in the r.h.s. represents \( S_3^1 \) and is a sum of triangle graphs; the dots represent the renormalization \( Z_{h_1}^\nu \) or \( Z_{h_1}^\nu \). The other term represents \( S_3^2 \), which is sum of terms with at least a \( \lambda \) vertex.

Let us consider \( S_3^2 \). With respect to the bound obtained in the previous section, we have to take into account that there is no contribution from the integrals over the coordinates. Given a tree \( \tau \), we can associate a tree \( \tau^* \), which is the tree obtained by \( \tau \) by erasing all the vertices not necessary to connect the special end-points; given a non-trivial vertex \( v \in \tau^* \), we call \( x_v^* \) the coordinates associated to end-points in \( \tau^* \) following \( v \), and \( \delta_v \) is the length of the shortest tree graph connecting the points \( x_v^* \). The number of non trivial vertices \( v \in \tau^* \) is \( \leq 3 \). The lack of integration over the external coordinates gives an extra factor \( \prod_{h} \gamma^{-4h_s(S_v^* - 1)h_s} \) where \( S_v^* \) is the number of branches in \( \tau^* \) following \( v \) (each integration contribute with a factor \( \gamma^{-4h_s(S_v^* - 1)h_s} \)); moreover we can write

\[
e^{-\sqrt{\gamma^2} |z|} \leq e^{-\frac{1}{2} \sqrt{\gamma^2} |z|} \prod_{k=-\infty}^{0} e^{-c \sqrt{\gamma^2} |z|} = e^{-\frac{1}{2} \sqrt{\gamma^2} |z|} \prod_{k=-\infty}^{\infty} e^{-c \sqrt{\gamma^2} |z|}
\]

(78)
The first factor is used to perform the integrations and from the second we get a factor \( e^{-(\gamma_h |\delta_v|)} \) for any non trivial vertex in \( v \) so that, if \( m = 3 \) and \( n \) is the number of normal end-points \( (\prod_v \gamma^{-d_v}(-h_h|\delta_v|) = \prod_v \gamma^{-d_v}) \)

\[
|S_r(x, x_1, x_2)| \leq C^m |\lambda|^n \gamma^{-h(4+2m)} \gamma^{\theta(h_n\tau - N)} \left[ \prod_v \gamma^{-d_v} \right] \prod_{n.t. v \in \tau} \gamma^{4h_n(S^* - 1)h_v e^{-(2h_n |\delta_v|)}}
\]  

(79)

with \( v^*_0 \) is the first non trivial vertex in \( \tau^* \). The factor \( \gamma^{-h(4+2m)} \) apparently forbids to sum over \( h \), and we need to use the decay factors associated to the propagators. In order to do that we can write

\[
\gamma^{-h(4+2m)} \prod_{v_0 \leq v \leq v^*_0} \gamma^{-d_v} = \gamma^{-h\tilde{d}_v(4+2m)} \prod_{v_0 \leq v \leq v^*_0} \gamma^{-d_v}
\]

(80)

where \( m_v = n^A_v \) for \( v_0 \leq v \leq v^*_0 \) and \( \tilde{d}_v = (-4 + n^A_v) = \tilde{d}_v = 3/2 |P_v| + z_v - \theta > 0 \). We call \( S^1_v \) the branches connecting to special \( A \) end-points. Using that \( S^1_v = S^1_v + S^2_v \), and \( n^A_v = S^1_v + \sum n^A_i, n^A_i \), the number of special end-points in \( \tau^*_v \) we have

\[
(-4 + 2n^A_v) + 4(S^1_v - 1) = -2S^1_v - \sum_{i=1}^{n^A_v} + 4S^1_v + 4S^2_v = 2S^1_v + S^2_v - \sum_{i=1}^{n^A_v} - (4 + 2n^A_v)
\]

(81)

Therefore to the vertex \( v^*_0 \) is associated \( \gamma^{\theta h\tilde{d}_v} \gamma^{2S^1_v h^*_{v^*_0}} \). We can repeat the same argument on each of the subtrees \( \tau_1^*, ..., \tau_{S^2_v}^* ; \) moreover \( \gamma^{2S^1_v h e^{-(2h_v |\delta_v|)}} \leq C_\delta \) as \( S^1_v \leq 3 \) so that

\[
|S_r(x, x_2, x_3)| \leq C^m |\lambda|^n \gamma^{-\theta N} |\gamma^{\theta h\tilde{d}_v} \gamma^{2S^1_v h^*_{v^*_0}} e^{-(2\delta |\delta|)}| \prod_v \gamma^{-d_v}
\]

(82)

where \( d_v = \tilde{d}_v \) if \( v \in \tau^* \), \( d_v = \tilde{d}_v \) otherwise. The sum over the scale difference is done using that \( \tilde{d}_v > 0 \); the remaining sum is done using that if \( \delta \equiv \gamma^{-h}\delta \), if \( \leq S^1 \leq 3 \)

\[
\sum_{h} \gamma^{(\theta + 2S^1)h e^{-(2\delta |\delta|)}} \leq \gamma^{(\theta + 2S^1)h e^{-(2\delta |\delta|)}} \sum_{h} \gamma^{(\theta + 2S^1)h e^{-(h - h\delta)}} \leq C_\delta
\]

(83)

uniformly in \( N \). In conclusion a bound \( |\lambda| (\frac{\pi}{\gamma})^\theta C_\delta \) is found. A similar bound is found for the corrections coming from the first term, as they have an extra factor \( \gamma^{\theta(h - N)} \) from (65). This concludes the proof of the bound (43).

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