Mass Gap in Weakly Coupled Abelian Higgs on a Unit Lattice

Abhishek Goswami

Abstract. The proof of the Higgs mechanism in a weakly coupled lattice gauge theory in $d \geq 2$ is revisited. A new power series cluster expansion is applied, and the mass gap is shown to exist for the observable $F_{\mu\nu}$.

Mathematics Subject Classification. Primary 81T08, 81T13, 81T25.

1. Introduction

We study Higgs field weakly coupled to an Abelian gauge field on a Euclidean unit lattice and establish mass gap. In standard model, the presence of Higgs field is responsible for the mass generation of $W^\pm$ and $Z$ gauge bosons as well as fermions. This principle is known as the Higgs mechanism. The recent discovery of a Higgs-like particle at LHC thus confirms this essential feature of the standard model. We demonstrate the Higgs mechanism via exponential decay of correlations between physical observable $F_{\mu\nu}$ (electromagnetic field strength). This decay implies that the gauge boson in the theory has acquired mass, and the theory is said to have a mass gap.

Mass generation is a long-distance problem, and thus, we want to study it without worrying about the ultraviolet (short distance) problem. We develop on the work of Balaban et al. [1]. We apply a new power series cluster expansion developed by Balaban et al. [2]. This is different from the decoupling expansion of [1].

Our results fit nicely in the program began by Balaban et al. [3,4] to develop a complete analysis of the ultraviolet problem. They employ a block averaging renormalization group technique which eventually takes us from an $\varepsilon = L^{-N}$ lattice, where $L$ is a positive number and $N$ is a positive integer, to a unit lattice, similar to our model.
The strong coupling regime of lattice gauge theory is widely studied. However, the weak coupling is more relevant at least in \( d = 3 \) to understand the continuum limit.

Before we proceed, we would like to mention some early significant contributions by Brydges et al. [5–7], toward the construction of two-dimensional scalar quantum electrodynamics without cutoffs. Regarding the development of cluster expansion methods to prove mass gap for simpler models such as \( P(\phi)^2 \), we refer reader to the fundamental papers of Glimm et al. [8, 9].

**Notation** Let \( \Lambda \subset \mathbb{Z}^d \) with \( d \geq 2 \) be a large finite unit lattice of dimension \( d \). The Higgs doublet \( \phi(x) = \phi_1(x) + i\phi_2(x) \) is defined on a site \( x \in \Lambda \). A bond \( b \in \Lambda \) is a pair of two adjacent sites \( (x, x + e_\mu) \). \( \Lambda^* \) denotes the set of all bonds \( b \).

The gauge field \( A_b \equiv A_\mu(x) \) is defined on the bond between \( (x, x + e_\mu) \). \( A_\mu(x) \) is the generator of the Abelian group \( U(1) \) such that \( U(x + e_\mu, x) = e^{ie_0 A_\mu(x)} \) is parallel translation operator from \( x \) to \( x + e_\mu \). \( e_0 \) is the gauge coupling and \( A_\mu(x) \in [\frac{-\pi}{e_0}, \frac{\pi}{e_0}] \) (compact or C), \( A_\mu(x) \in \mathbb{R} \) (non-compact or NC). A plaquette \( p \in \Lambda \) is a unit square formed by the bonds \( (x, x + e_\nu), (x + e_\mu, x + e_\mu + e_\nu), (x + e_\mu + e_\nu, x + e_\mu) \) and \( (x + e_\mu, x) \). \( \Lambda^{**} \) denotes the set of all plaquettes \( p \).

**Action** For our gauge field action, we use non-compact formalism,

\[
S_{NC}(A) = \frac{1}{2} \sum_{p \in \Lambda^{**}} (dA)^2(p). \tag{1.1}
\]

\((dA)(p) = \sum_{b \in \partial p} A_b \) is the field strength. The action for minimally coupled Higgs field is given by

\[
S_h(\phi, A) = \frac{1}{2} \sum_{\langle xy \rangle \in \Lambda} \left| e^{ie_0 A_{\langle xy \rangle}} \phi(y) - \phi(x) \right|^2 + \sum_{x \in \Lambda} \left( \lambda |\phi(x)|^4 - \frac{1}{4} \mu^2 |\phi(x)|^2 \right), \tag{1.2}
\]

where \( \langle xy \rangle \) denotes the bond connecting adjacent sites \( x, y \). The weak coupling regime is \((\lambda, e_0^2) \ll 1 \) with \( \frac{e_0^2}{\lambda} = \mathcal{O}(1) \). The total action for a non-compact weakly coupled Abelian Higgs theory is

\[
S(\phi, A) = S_{NC}(A) + S_h(\phi, A). \tag{1.3}
\]

For a theory defined by action \( S(\phi, A) \), the partition function is given by

\[
Z^{NC} = \int \mathcal{D}\phi \mathcal{D}A \ e^{-S(\phi, A)}, \tag{1.4}
\]

where \( \mathcal{D}\phi = \prod_{x \in \Lambda} d\phi_1(x)d\phi_2(x) \) and \( \mathcal{D}A = \prod_{b \in \Lambda^*} dA_b \) are product of Lebesgue measure on \( \mathbb{R} \).

**Proposition 1** [1] (Gauge fixing). Let \( \theta: \Lambda \rightarrow \mathbb{R} \) be the gauge function and \( \mathcal{D}\theta = \prod_{x \in \Lambda} d\theta_x \) denote product Lebesgue measure on \( \mathbb{R} \). Introduce a gauge fixing function \( G(A) \) satisfying

\[
\int e^{-G(A+\partial \theta)} \mathcal{D}\theta_{\Lambda \setminus x_0} = 1 \tag{1.5}
\]
with \( \theta(x_0) = 0 \) and \( d\theta = 0 \) on the boundary \( \partial \Lambda \). Let \( G(A) \) be given by

\[
G(A) = \sum_{x \in \Lambda} \frac{\alpha}{2} (\delta A)^2(x) + c. \tag{1.6}
\]

\( \delta \) is the adjoint of differential operator \( d \) with respect to \( l^2 \) inner product, and then \( c \) is a finite constant.

**Proof.** It is easy to see that substituting \( c = \ln \int e^{-\frac{\alpha}{2} \|\delta (A + d\theta)\|^2} d\theta \Lambda \setminus x_0 \) in (1.6), \( G(A) \) satisfies (1.5). To show that \( c \) is finite and independent of \( A \), let \( \Delta = \delta d \) on scalars. Then, \( \Delta \) is invertible on the orthogonal complement of the constants. Make a change of variable as \( \omega = \Delta \theta \) and rewrite

\[
c = \ln \int e^{-\frac{\alpha}{2} \|\delta A + \omega\|^2} D\omega + \text{const}. \tag{1.7}
\]

Then, the translation \( \omega \to \omega - \delta A \) completes the proof.

Gauge fixing makes the non-compact integral in Eq. 1.4 converge. Since addition of gauge fixing term with \( \alpha = 1 \) changes

\[
\int D\phi DAA e^{-\langle dA, J \rangle} \text{ to } \int D\phi DAA e^{-\langle A, \Delta A \rangle}
\]

where \( \Delta = \delta d + d\delta \). We take Dirichlet boundary condition, \( \Delta \) is positive definite, \( \langle A, \Delta A \rangle \geq (\text{const}) \|A\|^2 \), which makes the integral converge.

With the above gauge fixing function, the non-compact partition function is defined as

\[
Z_{NC} = \int D\phi DAA e^{-S(\phi, A) - G(A)}. \tag{1.8}
\]

As action \( S(\phi, A) \) and the Lebesgue measures are gauge invariant, we replace \( e^{-G(A)} \) in the partition function by its gauge average over \( \theta \). Define the compact gauge average of \( e^{-G(A)} \) as

\[
\langle \langle e^{-G(A)} \rangle \rangle = \left( \frac{2\pi}{e_0} \right)^{-|A|} \int_{-\frac{\pi}{e_0}}^{\frac{\pi}{e_0}} D\theta_A e^{-G(A + d\theta)}. \tag{1.9}
\]

Let \( J \) be a function defined on plaquettes which we allow to be complex and assume \( |J| < 1 \). Consider the observable \( e^{-\epsilon_0 \langle dA, J \rangle} \), where \( \langle dA, J \rangle \) denotes \( l^2 \) inner product. The non-compact expectation of an observable \( e^{-\epsilon_0 \langle dA, J \rangle} \) is defined as

\[
Z_{NC} \langle e^{-\epsilon_0 \langle dA, J \rangle} \rangle_{NC} = \int D\phi DAA e^{-S(\phi, A) - \epsilon_0 \langle dA, J \rangle} \langle \langle e^{-G(A)} \rangle \rangle. \tag{1.10}
\]

**Compact Formalism** Our non-compact formalism is equivalent to a compact formalism. For the compact, we do not use Wilson action which is

\[
S_C(A) = \frac{1}{2} \sum_{p \in \Lambda^{**}} \cos [(dA)(p)] + \text{const} \quad \text{with} |A_b| \leq \frac{\pi}{e_0}. \tag{1.11}
\]

Instead, we use Villain type which is

\[
e^{-S_C(A)} = \sum_{v: d\nu = 0} e^{-\frac{\alpha}{2} \sum_{p \in \Lambda^{**}} (dA(p) + \nu(p))^2} \quad \text{with} |A_b| \leq \frac{\pi}{e_0}, \tag{1.12}
\]
where \( v \) is an integer-valued two form, \( v: \Lambda^{**} \ni p \to \frac{2\pi}{e_0} \mathbb{Z} \). We explain the connection between this and non-compact formalism shortly in Theorem 1. The compact partition function for an Abelian Higgs theory is defined as

\[
Z^C = \sum_{v: dv = 0} \int_{-\frac{\pi}{e_0}}^{\frac{\pi}{e_0}} DA \int D\phi e^{-\frac{i}{2} \sum_{p \in \Lambda^{**}} (dA(p) + v(p))^2 - S_h(\phi, A)},
\]

where \( DA = \prod_{b \in \Lambda^*} dA_b \) is product Lebesgue measure on \( [-\frac{\pi}{e_0}, \frac{\pi}{e_0}] \). The compact expectation of an observable \( e^{-e_0(dA,J)} \) is defined as

\[
Z^C \langle e^{-e_0(dA,J)} \rangle^C = \sum_{v: dv = 0} \int_{-\frac{\pi}{e_0}}^{\frac{\pi}{e_0}} DA \int D\phi e^{-\frac{i}{2} \sum_{p \in \Lambda^{**}} (dA(p) + v(p))^2 - S_h(\phi, A) - e_0(dA + v, J)}.
\]

**Theorem 1.1** [1]. Given an observable \( e^{-e_0(dA,J)} \) and using the definition of expectation as above, the compact and non-compact formalism are equivalent in the following sense:

\[
Z^{NC} = \left(\frac{2\pi}{e_0}\right)^{-|\Lambda|+1} Z^C \quad \text{and} \quad \langle e^{-e_0(dA,J)} \rangle^{NC} = \langle e^{-e_0(dA+J)} \rangle^C.
\]

**Proof.** First break the integral in Eq. 1.10 into compact intervals \( [-\frac{\pi}{e_0}, \frac{\pi}{e_0}] \). Let \( n: \Lambda^* \to \left(\frac{2\pi}{e_0}\right) \mathbb{Z} \) be an integer-valued one form. Equation 1.10 becomes

\[
Z^{NC} \langle e^{-e_0(dA,J)} \rangle^{NC} = \sum_n \sum_v \int_{-\frac{\pi}{e_0}}^{\frac{\pi}{e_0}} DA \int D\phi e^{-S_h(\phi, A+n) - e_0(d(A+n), J)} \langle \langle e^{-G(A+n)} \rangle \rangle .
\]

Rewrite

\[
\sum_n = \sum_v \sum_n: dn = v .
\]

Note that in the Higgs action, \( e^{ie_0(A+n)} = e^{ie_0A} \) since \( n = \left(\frac{2\pi}{e_0}\right) \mathbb{Z} \). Replace the condition \( dn = v \) with the equivalent condition \( dv = 0 \) (since \( \Lambda \) has trivial topology). Rewrite the non-compact expectation as

\[
Z^{NC} \langle e^{-e_0(dA,J)} \rangle^{NC} = \sum_{v: dv = 0} \int_{-\frac{\pi}{e_0}}^{\frac{\pi}{e_0}} DA \int D\phi e^{-\frac{i}{2} \sum_{p} |dA + v|^2 - S_h(\phi, A) - e_0(dA + v, J)}
\]

\[
\sum_{n: dn = v} \langle \langle e^{-G(A+n)} \rangle \rangle .
\]

Let \( s: \Lambda \to \left(\frac{2\pi}{e_0}\right) \mathbb{Z} \) be a zero form with \( s(x_0) = 0 \). For a fixed \( v \) any two values, \( n \) and \( \tilde{n} \) satisfying \( dn = d\tilde{n} = v \) differ by \( ds \). The construction of such integer-valued forms and a choice of point \( x_0 \) is discussed in [10]. The gauge function \( \theta \) also satisfies \( \theta(x_0) = 0 \). Thus,
\[
\sum_{dn=\nu} \langle e^{-G(A+n)} \rangle \rangle = \sum_{s} \langle e^{-G(A+\tilde{n}+ds)} \rangle \\
= \sum_{s} \left( \frac{2\pi}{\epsilon_0} \right)^{-|A|} \int_{-\pi/\epsilon_0}^{\pi/\epsilon_0} D\theta_{\Lambda} e^{-G(A+\tilde{n}+d(\theta+s))} \\
= \left( \frac{2\pi}{\epsilon_0} \right)^{-|A|+1} \int_{\mathbb{R}} D\theta_{\Lambda}|_{x_0} e^{-G(A+\tilde{n}+d\theta)} \\
= \left( \frac{2\pi}{\epsilon_0} \right)^{-|A|+1}, \quad (1.19)
\]

where a factor of \( \frac{2\pi}{\epsilon_0} \) comes from evaluating the integral at \( x_0 \) and the final step is due to the gauge fixing condition of Eq. 1.5.

\[
Z^{NC} \langle e^{-e_0(dA,J)} \rangle_{NC} = \left( \frac{2\pi}{\epsilon_0} \right)^{-|A|+1} \sum_{v:dv=0} \int_{-\pi/\epsilon_0}^{\pi/\epsilon_0} DA \\
\int D\phi e^{-\frac{i}{2} \sum_{p} |dA+v|^2 - S_h(\phi,A) - e_0(dA+v,J)}. \quad (1.20)
\]

The gauge field \( A \) on the right-hand side takes value in the compact interval \( \left[ -\frac{\pi}{\epsilon_0}, \frac{\pi}{\epsilon_0} \right] \). Setting \( J = 0 \) (i.e., no external sources) gives \( Z^{NC} = \left( \frac{2\pi}{\epsilon_0} \right)^{-|A|+1} Z^{C} \) and hence \( \langle e^{-e_0(dA,J)} \rangle_{NC} = \langle e^{-e_0(dA+v,J)} \rangle^{C} \).

**Classical Higgs Mechanism** To show that the gauge field has acquired mass \( m_A \), we make standard change of variables as

\[
\phi(x) = \rho(x)e^{i\theta(x)}, \quad A \to A - \frac{1}{\epsilon_0} d\theta. \quad (1.21)
\]

\( \rho \) is the length of the Higgs field. Due to the absolute value of Higgs in the action (Eq. 1.2), there is no explicit \( \theta \) dependence since \( | e^{ie_0(A_{xy})} - \frac{\epsilon_0}{\epsilon_0} e^{i\theta(y)} \rho(y) - e^{i\theta(x)} \rho(x) | = | e^{i\theta(x)} || e^{ie_0A_{xy}} \rho(y) - \rho(x) | \). We thus take average over \( \theta(x) \) as \( \prod_{x \in \Lambda} \int_{-\pi}^{\pi} d\theta(x) = (2\pi)^{|A|} \) and drop this constant. The potential term in new variables is

\[
V(\rho) = \lambda \rho(x)^4 - \frac{1}{4} \mu^2 \rho(x)^2 \quad (1.22)
\]

with minimum at \( \rho_0 = \frac{\mu}{\sqrt{8\lambda}} \). Substituting \( \rho \to \rho_0 + \rho \), the total action is

\[
S(\rho, A) = \frac{1}{2} \sum_{p \in \Lambda^*} (dA+v)^2(p) + \frac{1}{2} m_A^2 \sum_{b \in \Lambda^*} A_b^2 + \frac{1}{2} \sum_{\langle xy \rangle \in \Lambda} (\rho(y) - \rho(x))^2 \\
+ \frac{1}{2} \mu^2 \sum_{x \in \Lambda} \rho^2(x) + \rho_0^2 \sum_{\langle xy \rangle \in \Lambda} (1 - \cos e_0 A_{xy}) - \frac{1}{2} e_0^2 A_{xy}^2 \\
+ \rho_0 \sum_{\langle xy \rangle \in \Lambda} (\rho(y) + \rho(x))(1 - \cos e_0 A_{xy}) \\
+ \sum_{\langle xy \rangle \in \Lambda} \rho(y)\rho(x)(1 - \cos e_0 A_{xy})
\]
Higgs theory is given by the generating functional for a non-compact Abelian Mass Gap
From the equivalence established in Theorem 1 and using the change of variables as above, the generating functional is given by
\[
Z[J] = \left(\frac{2\pi}{\epsilon_0}\right)^{-|\Lambda|+1} \sum_{v: d_v=0} \int_{-\epsilon_0}^{\epsilon_0} \mathcal{D}A \int_{-\rho_0}^{\infty} \mathcal{D}\rho ~ e^{-S(\rho,A)-\epsilon_0(dA+v,J)}. \tag{1.25}
\]
Let \( p_1 \) and \( p_2 \) be two plaquettes. The truncated correlation between \( dA(p_1) \) and \( dA(p_2) \) is given by
\[
\langle dA(p_1) dA(p_2) \rangle^T = \frac{\partial^2 \log Z[J]}{\partial J_{p_1} \partial J_{p_2}} \bigg|_{J=0}, \tag{1.26}
\]
where \( \langle dA(p_1) dA(p_2) \rangle^T = \langle dA(p_1) dA(p_2) \rangle - \langle dA(p_1) \rangle \langle dA(p_2) \rangle \).

**Theorem 1.2.** Given an Abelian Higgs theory on a unit lattice with mass \( m_A \) fixed and \( \min(\mu^2, m_A^2) \geq 2^4 + 1 \). Let the coupling constants \( \epsilon_0 \) and \( \lambda \) be sufficiently small depending on the masses. The correlation of \( dA(p) \) defined on two plaquettes \( p_1 \) and \( p_2 \) has an exponential decay as
\[
|\langle dA(p_1) dA(p_2) \rangle^T| \leq e^{-m d(p_1,p_2)} \tag{1.27}
\]
for some positive constant \( m \).

**Remark 1.3.** This says that the gauge field \( A_\mu \) has a mass at least as big as \( m \). In a continuum theory, the estimate implies that the spectrum of the Hamiltonian is contained in \( \{0\} \cup (m, \infty) \). This is the mass gap. Let \( F_{\mu\nu}(x) = dA(x, x+e_\mu, x+e_\mu+e_\nu, x+e_\nu) \). Alternatively with \( F_{\mu\nu}(x) \)
\[
|\langle F_{\mu\nu}(x_1) F_{\mu\nu}(x_2) \rangle - \langle F_{\mu\nu}(x_1) \rangle \langle F_{\mu\nu}(x_2) \rangle| \leq e^{-m d(x_1, x_2)}. \tag{1.28}
\]

**Remark 1.4.** Theorem 2 is new but similar to the results of Balaban et al. [1] who considered integer charge observables. An observable \( F(\phi, A) \) is integer charge observable if for every \( b \in \Lambda^* \), \( F(\phi, A + \frac{2\pi}{\epsilon_0} 1_b) = F(\phi, A) \), where \( 1_b \) is the characteristic function of the bond \( b \). Integer charge observables do not depend on \( v \). \( dA \) is not an integer charge observable since it acquires dependence on \( v \).

**Remark 1.5.** Let \( \gamma_1 \) and \( \gamma_2 \) be two closed curves composed of lattice bonds. A Wilson loop variable is given by \( W_{\gamma_i}(A) = \prod_{b \in \gamma_i} e^{i\epsilon_0 A_b} \). Note that \( W_{\gamma_i}(A + \frac{2\pi}{\epsilon_0} 1_b) = \prod_{b \in \gamma_i} e^{i\epsilon_0 (A_b + \frac{2\pi}{\epsilon_0} 1_b)} = \prod_{b \in \gamma_i} e^{2\pi i e^{i\epsilon_0 A_b}} = W_{\gamma_i}(A) \). So, Wilson loop is
an integer charge observable. Our methods also show that for some $m > 0$, the correlation of $W_{\gamma_1}(A)$ and $W_{\gamma_2}(A)$ has an exponential decay as
\begin{equation}
|\langle W_{\gamma_1}(A)W_{\gamma_2}(A) \rangle - \langle W_{\gamma_1}(A) \rangle \langle W_{\gamma_2}(A) \rangle| \leq e^{-md(\gamma_1, \gamma_2)}.
\end{equation}

Remark 1.6. In our model, $\langle \phi \rangle = 0$. In the physics literature, the expectation value of Higgs field $\langle \phi \rangle$ is taken to be nonzero to demonstrate the Higgs mechanism. While it is convenient to assume $\langle \phi \rangle \neq 0$ for computations, whether there exist such states in this simplified model is a difficult dynamical question mathematically. Our results show there is mass generation, whether or not $\langle \phi \rangle = 0$.

Outline of Proof We begin our analysis by splitting the lattice $\Lambda$ into small field (bounded) and large field (unbounded) regions. This is done by using characteristic functions that impose such a restriction on the fields. Our characteristic functions are exact and not just an approximation of characteristic function as in [1]. This simplifies the analysis at the very beginning of the proof. Then, we construct a localized small field integral having Gaussian measure with unit covariance and potential term $V$. Whereas [1] first defines a small field integral having Gaussian measure with covariance $C$ and then applies a decoupling cluster expansion due to Glimm–Jaffe–Spencer. Note that it is customary to treat such integrals via modern decoupling cluster expansions like [11,12]. Our construction is clean and completely bypasses all such expansions. Thus, it does not include the so-called weakening parameters ($s$ parameter). Weakening parameters are required to do interpolation between given covariance matrix and a more local version of it. This makes the analysis difficult as it further involves studying the differential aspects of all the functions with respect to $s$ parameter and other combinatorics.

Section 2 details our construction. This is a general, model-independent construction. In Sect. 3, we first write a power series of the potential $V$ and using the result of [2] generate a converging series expansion of the log of small field integral. In Theorem 3.1, we calculate the exact form of the series coefficients and show that they depend on the covariance matrix $C$. Thus, the decay properties of the covariance determine the decay nature of series coefficients. From Theorem 3.2, the series coefficients of the log of small field integral have the same form as of the power series of potential $V$. Since the covariance matrix $C$ is completely determined by the model system, our result provides a simple and clean alternative to decoupling cluster expansions. Then, in Sect. 4, we discuss the large field estimates. In Sect. 5, we combine various overlapping regions of the lattice into connected components and rewrite the generating functional as sum over those components. We establish mass gap and also show the decay of n-point truncated correlation functions in Sect. 6. We prove the existence of the infinite volume limit in Sect. 7.

To summarize what is new in the paper, there are four things: (i) a better cluster expansion, (ii) treatment of more observables like the field strength $F_{\mu\nu}$, (iii) decay of n-point truncated correlation functions and (iv) proof of infinite volume limit, $\Lambda \to \infty$. 
2. The Expansion

\( \Lambda \) consists of all the sites \( x \), \( \Lambda^\ast \) is the set of all the bonds \( b \), and \( \Lambda^{**} \) is the set of all the plaquettes \( p \). The Higgs field is \( \rho: \Lambda \to [\rho_0, \infty) \), the gauge field is \( A: \Lambda^\ast \to [\frac{\pi}{e_0}, \frac{\pi}{e_0}] \), and the vortex field is \( v: \Lambda^{**} \to \frac{2\pi}{e_0}\mathbb{Z} \). We represent all the fields on lattice as \( \Phi = (\rho, A): \Lambda \to \mathbb{R} \).

\[
\langle \Phi, T\Phi \rangle = \langle \rho, (-\Delta + \mu^2)\rho \rangle + \langle A, (\delta d + m^2 A)A \rangle \quad (2.1)
\]

\( \Delta \) and \( \delta d \) are operators on \( l^2(\Lambda) \) and \( l^2(\Lambda^\ast) \), respectively, with

\[
\langle \rho, -\Delta \rho \rangle = \sum_{(x,y) \subset \Lambda} (\rho(y) - \rho(x))^2 \quad \text{and} \quad \langle A, -\delta d A \rangle = \sum_{p \subset \Lambda^{**}} (dA)^2. \quad (2.2)
\]

Rewrite the action (1.23) as a sum of Gaussian part and higher-order interaction part,

\[
S(\Phi) = \frac{1}{2} \langle \Phi, T\Phi \rangle + V(\Phi). \quad (2.3)
\]

Define the covariance operator \( C \) as

\[
\langle \Phi, C\Phi \rangle = \langle \rho, (-\Delta + \mu^2)^{-1} \rho \rangle + \langle A, (\delta d + m^2 A)^{-1} A \rangle \quad (2.4)
\]

Thus, \( T = C^{-1} \). The generating functional is

\[
Z[J] = \left( \frac{2\pi}{e_0} \right)^{-|\Lambda|+1} \sum_{v: d_v=0} \int D\Phi e^{-\frac{1}{2} \langle \Phi, C^{-1}\Phi \rangle - V(\Phi) e^{-e_0(dA+v,J)}}. \quad (2.5)
\]

To construct our localized small field integral, we expand the generating functional into regions where the field \( \Phi \) is bounded and regions where it is not. The following four steps detail this expansion:

1. Let \( p_\lambda = |\log \lambda|^{2d+1} \) and \( r_\lambda = |\log \lambda|^2 \). Divide the lattice \( \Lambda \) into blocks \( \square \) of length \( [r_\lambda] \), where \( [\cdot] \) denotes the integer part. Define

\[
\Phi(\xi) = \begin{cases} 
\rho(x) & \text{if } \xi = x \in \Lambda, \\
A(b) & \text{if } \xi = b \in \Lambda^\ast.
\end{cases} \quad (2.6)
\]

Define the characteristic function

\[
\chi_{\square}(\Phi) = \begin{cases} 
1 & \sup_{\xi \in \square} |\Phi(\xi)| < p_\lambda, \\
0 & \text{otherwise}
\end{cases} \quad \zeta_{\square}(\Phi) = 1 - \chi_{\square}(\Phi). \quad (2.7)
\]

Then, the decomposition of unity is

\[
1 = \prod_{\square} (\chi_{\square} + \zeta_{\square}) = \sum_{Q \subset \Lambda} \prod_{\square \in \Lambda - Q} \chi_{\square} \prod_{\square \in Q} \zeta_{\square} = \sum_{Q \subset \Lambda} \chi_{\Lambda - Q} \zeta_Q. \quad (2.8)
\]

Thus, a \( \square \in Q \) if there is at least one \( \xi \in \square \) with \( |\Phi(\xi)| > p_\lambda \). Define

\[
\Lambda_0 = \Lambda - Q.
\]

Insert (decomposition of unity) 1 in the generating functional rewrite

\[
Z[J] = \left( \frac{2\pi}{e_0} \right)^{-|\Lambda|+1} \sum_{Q \subset \Lambda} \sum_{v: d_v=0} \int D\Phi e^{-\frac{1}{2} \langle \Phi, C^{-1}\Phi \rangle - V(\Phi) e^{-e_0(dA+v,J)}} \\
\chi_{\Lambda_0}(\Phi) \zeta_Q(\Phi). \quad (2.9)
\]
Set \( v = 0 \) in \( \Lambda_0 \).

2. Conditioning

Contract \( \Lambda_0 \) by \([r_\Lambda]\) to get \( \Lambda_1 \), such that, \( d(Q, \Lambda_1) = [r_\Lambda] \). Denote \( \tilde{Q} \equiv \Lambda_1^c = \Lambda - \Lambda_1 \).

Let \( 1_{\Lambda_1} \) and \( 1_{\tilde{Q}} \) be characteristic functions restricting operator to \( \Lambda_1 \) and \( \tilde{Q} \), respectively. Define \( T_{\Lambda_1} = 1_{\Lambda_1} T_1 \) and \( T_{\Lambda_1 , \tilde{Q}} = 1_{\Lambda_1} T_1 \). Then, rewrite

\[
\frac{1}{2} \langle \Phi, T \Phi \rangle = \frac{1}{2} \langle \Phi, T_{\Lambda_1} \Phi \rangle + \langle \Phi, T_{\Lambda_1 , \tilde{Q}} \Phi \rangle + \frac{1}{2} \langle \Phi, T_\tilde{Q} \Phi \rangle. \tag{2.10}
\]

The term \( \langle \Phi, T_{\Lambda_1} \Phi \rangle = \langle \Phi_{\Lambda_1}, T \Phi_{\Lambda_1} \rangle \) represents the interactions entirely in the small field region, the term \( \langle \Phi, T_\tilde{Q} \Phi \rangle = \langle \Phi_{\tilde{Q}}, T \Phi_{\tilde{Q}} \rangle \) represents the interactions in the large field region, and the term \( \langle \Phi, T_{\Lambda_1 , \tilde{Q}} \Phi \rangle = \langle \Phi_{\Lambda_1 , \tilde{Q}}, T \Phi_{\tilde{Q}} \rangle \) represents the interaction of the large field region \( \tilde{Q} \) over the boundary \( \partial \Lambda_1 \). Absorb the source term \( e_0 \langle dA + vJ \rangle \) into the potential \( V(\Phi) \) and rewrite

\[
Z[J] = \left( \frac{2\pi}{e_0} \right)^{-|A|+1} \sum_{Q \subseteq \Lambda \atop v \cdot \vec{d}e = 0} \int \mathcal{D} \Phi_{\tilde{Q}} e^{-\frac{1}{2} \langle \Phi, T \Phi \rangle - V(\Phi, \tilde{Q})} \zeta_{\tilde{Q}}(\Phi) \chi_{\Lambda_0 - \Lambda_1}(\Phi) \tag{2.11}
\]

\[
\int \mathcal{D} \Phi_{\Lambda_1} e^{-\frac{1}{2} \langle \Phi_{\Lambda_1}, T \Phi_{\Lambda_1} \rangle - V(\Lambda_1, \Phi_0)} \chi_{\Lambda_1}(\Phi)
\]

In the above equation, making the transformation

\[
\Phi_{\Lambda_1} \rightarrow \Phi_{\Lambda_1} - C_{\Lambda_1} T_{\Lambda_1} \Phi_{\tilde{Q}} \tag{2.12}
\]

the generating functional becomes

\[
Z[J] = \left( \frac{2\pi}{e_0} \right)^{-|A|+1} \sum_{Q \subseteq \Lambda \atop v \cdot \vec{d}e = 0} \int \mathcal{D} \Phi_{\tilde{Q}} e^{-\frac{1}{2} \langle \Phi, (T_{\tilde{Q}} - T_{\Lambda_1} C_{\Lambda_1} T_{\Lambda_1} \Phi_0) \rangle - V(\tilde{Q}, \Phi)} \chi_{\Lambda_0 - \Lambda_1}(\Phi) \int \mathcal{D} \Phi_{\Lambda_1} e^{-\frac{1}{2} \langle \Phi_{\Lambda_1}, (T_{\Lambda_1} \Phi_{\Lambda_1} - C_{\Lambda_1} T_{\Lambda_1} \Phi_{\tilde{Q}}) \rangle - V(\Lambda_1, \Phi_0)} \chi_{\Lambda_1}(\Phi)
\]

\[
\chi_{\Lambda_1}(\Phi) = C_{\Lambda_1} T_{\Lambda_1} \Phi_{\tilde{Q}} \tag{2.13}
\]

This step is equivalent to conditioning the small field integral on \( \Lambda_1 \) on values in the large field region \( \tilde{Q} \).

3. The integral over \( \Phi_{\Lambda_1} \) is Gaussian with covariance \( C_{\Lambda_1} = T_{\Lambda_1}^{-1} \). The operator \( C_{\Lambda_1} \) couples sites everywhere on lattice \( \Lambda \). Therefore, it is important to construct a localized operator. Define the kernel of operator \( C^{\text{loc}}_{\Lambda_1} \) as

\[
C^{\text{loc}}_{\Lambda_1}(x, y) = \begin{cases}
C_{\Lambda_1}(x, y) & \text{if } |x - y| < r_\Lambda, \\
0 & \text{otherwise}.
\end{cases} \tag{2.14}
\]

We adopt the representation for \( C^{\frac{1}{2}}_{\Lambda_1} \) and \( C^{\frac{1}{2}}_{\Lambda_1}^{\text{loc}} \) as discussed in [13]. Define

\[
C^{\frac{1}{2}}_{\Lambda_1} = \frac{1}{\pi} \int_0^\infty \frac{dr}{\sqrt{r}} C_{r, \Lambda_1}, \quad C^{\frac{1}{2}}_{\Lambda_1}^{\text{loc}} = \frac{1}{\pi} \int_0^\infty \frac{dr}{\sqrt{r}} C_{r, \Lambda_1}^{\text{loc}} \tag{2.15}
\]
with $C_{r,A_1} = (T_{A_1} + r)^{-1}$. Define $\delta C_{A_1}^{\frac{1}{2}}(x,y) = (C_{A_1}^{\frac{1}{2}} - C_{A_1}^{\frac{1}{2},\text{loc}})(x,y)$ as

$$\delta C_{A_1}^{\frac{1}{2}}(x,y) = \begin{cases} 0 & \text{if } |x - y| < r_\lambda, \\ C_{A_1}^{\frac{1}{2}}(x,y) & \text{otherwise}. \end{cases} \quad (2.16)$$

$C_{A_1}^{\frac{1}{2},\text{loc}}$ is a small perturbation of $C_{A_1}^{\frac{1}{2}}$ and hence invertible. To work in unit covariance, make the change of variables $\Phi = C_{A_1}^{\frac{1}{2},\text{loc}} \Phi'$

$$\langle \Phi, C_{A_1}^{-1} \Phi \rangle = \langle \Phi', C_{A_1}^{\frac{1}{2},\text{loc}} C_{A_1}^{-1} C_{A_1}^{\frac{1}{2},\text{loc}} \Phi' \rangle = ||\Phi'||_{A_1}^2 + V_\epsilon(\Lambda_1, \Phi') \quad (2.17)$$

This is the definition of $V_\epsilon(\Lambda_1, \Phi')$.

4. This change of variables introduces non-locality in characteristic function as $\chi_{A_1}(C_{A_1}^{\frac{1}{2},\text{loc}} \Phi' - C_{A_1} T_{A_1} Q \Phi_Q)$. To construct the localized small field integral, we therefore split the small field region $\Lambda_1$ into a new small field region and an intermediate field region. Let $p_{0,\lambda} = |\log \lambda|^a$ (with $2d < a < 2d + 1$) and $P \subset \Lambda_1$. We split the region $\Lambda_1$ into a region $P$ and $\Lambda_1 - P$ by using decomposition of unity as before. Define

$$\hat{\chi}_\Box(\Phi') = \begin{cases} 1 & \sup_{\xi \in \Box} |\Phi'(\xi)| < p_{0,\lambda}, \\ 0 & \text{otherwise} \end{cases} \quad \hat{\zeta}_\Box(\Phi') = 1 - \hat{\chi}_\Box(\Phi'). \quad (2.18)$$

Then, the decomposition of unity is

$$1 = \prod_{\Box \in \Lambda_1} (\hat{\chi}_\Box + \hat{\zeta}_\Box) = \sum_{P \subset \Lambda_1} \prod_{\Box \in \Lambda_1 - P} \hat{\chi}_\Box \prod_{\Box \in P} \hat{\zeta}_\Box = \sum_{P \subset \Lambda_1} \hat{\chi}_{\Lambda_1 - P} \hat{\zeta}_P. \quad (2.19)$$

Define

$$\Omega = \Lambda_1 - P.$$  

Due to unit covariance in $\Omega$, no conditioning is required along the boundary $\partial \Omega$. We contract $\Omega$ by $[r_\lambda]$ to get $\Omega_0$, such that $d(\Omega^c, \Omega_0) = [r_\lambda]$. In the potential, everything is analytic in the field, but the characteristic function $\chi_{A_1}$ is a mess which we have to clean. As a first step, we split the potential term as

$$V(\Lambda_1, C_{A_1}^{\frac{1}{2},\text{loc}} \Phi' - C_{A_1} T_{A_1} Q \Phi_Q) = V(\Omega_0, C_{A_1}^{\frac{1}{2},\text{loc}} \Phi' - C_{A_1} T_{A_1} Q \Phi_Q)$$

$$+ V(\Lambda_1 - \Omega_0, C_{A_1}^{\frac{1}{2},\text{loc}} \Phi' - C_{A_1} T_{A_1} Q \Phi_Q) \quad (2.20)$$

and factorize $\chi_{A_1}$ as

$$\chi_{A_1}(C_{A_1}^{\frac{1}{2},\text{loc}} \Phi' - C_{A_1} T_{A_1} Q \Phi_Q) = \chi_{A_1 - \Omega_0}(C_{A_1}^{\frac{1}{2},\text{loc}} \Phi' - C_{A_1} T_{A_1} Q \Phi_Q) \chi_{\Omega_0}(C_{A_1}^{\frac{1}{2},\text{loc}} \Phi' - C_{A_1} T_{A_1} Q \Phi_Q). \quad (2.21)$$

For $\xi \in \Omega_0$,

$$|C_{A_1}^{\frac{1}{2},\text{loc}} \Phi'(\xi)| \leq c ||\Phi'||_{\infty, \Omega} \leq c p_{0,\lambda} < \frac{p_\lambda}{2}$$

(see Lemma 2.3 to follow) and

$$|C_{A_1} T_{A_1} Q \Phi_Q| \leq c e^{-\gamma r_\lambda} p_\lambda < \frac{p_\lambda}{2}$$
where we have used estimate of covariance (Lemma 2.1) and \( |\Phi_Q| \leq p_\lambda \) since this is near \( \Lambda_1 \). Thus,

\[
\chi_{\Omega_0}(\det_{\Lambda_1} \Phi' - C_{\Lambda_1} T_{\Lambda_1} \Phi_Q) = 1. \tag{2.22}
\]

The non-locality in \( \chi_{\Lambda_1-\Omega_0}(\det_{\Lambda_1} \Phi' - C_{\Lambda_1} T_{\Lambda_1} \Phi_Q) \) prevents us to carry out integration over the region \( \Omega_0 \). As a second and final step, we contract the region \( \Omega_0 \) by \( [\alpha] \) to get a new small field region \( \Omega_1 \), that is, \( d(\Omega_0^c, \Omega_1) = [\alpha] \). Note that \( \chi_{\Lambda_1-\Omega_0}(\det_{\Lambda_1} \Phi' - C_{\Lambda_1} T_{\Lambda_1} \Phi_Q) \) and \( V(\Lambda_1 - \Omega_0, \det_{\Lambda_1} \Phi' - C_{\Lambda_1} T_{\Lambda_1} \Phi_Q) \) do not depend on \( \Phi'_{\Omega_1} \) since \( \det_{\Lambda_1} \) couples the fields only up to a distance of \([\alpha]\). After the above procedure, the generating functional becomes

\[
Z[J] = \left( \frac{2\pi}{\epsilon_0} \right)^{\frac{1}{2}} \sum_{Q \subseteq \Lambda} \sum_{P \subseteq \Lambda_1} \sum_{v: d = 0} \left( \det \det_{\Lambda_1} \right) \\
\int D\Phi_Q e^{\frac{1}{2}(\Phi, (T_{\Phi} - T_{\Phi_0} C_{\Lambda_1} T_{\Lambda_1} \Phi') - V(\Phi, \Phi))} \zeta_Q(\Phi) \chi_{\Omega_0-\Lambda_1}(\Phi) \\
\int D\Phi_{\Lambda_1-\Omega_1} e^{\frac{1}{2}||\Phi'||^2 - V(\Lambda_1 - \Omega_0, \det_{\Lambda_1} \Phi' - C_{\Lambda_1} T_{\Lambda_1} \Phi_Q)} \zeta_{\Lambda_1-\Omega_0}(\Phi) \chi_{\Lambda_1}(\Phi') \\
\int D\Phi_{\Omega_1} e^{\frac{1}{2}||\Phi'||^2 - V(\Omega_0, \det_{\Lambda_1} \Phi' - C_{\Lambda_1} T_{\Lambda_1} \Phi_Q) + V(\Lambda_1, \Phi')} \chi_{\Omega_1}(\Phi'). \tag{2.23}
\]

The above expression is the required expansion of the generating functional.

Consider the term \( \det \det_{\Lambda_1} \). As we will see in Lemma 2.5, for some \( W_1(\Lambda_1) = \sum_{\square \subseteq \Lambda_1} W_1(\square) \)

\[
\det \det_{\Lambda_1} = \left( \det \det_{\Lambda_1} \right) e^{W_1(\Lambda_1)}. \tag{2.24}
\]

As we will see in Lemma 2.8, for some \( W_2(\Lambda) = \sum_{\square \subseteq \Lambda} W_2(\square) \)

\[
\det \det_{\Lambda_1} = \left( \det \det_{\Lambda_1} \right) e^{W_2(\Lambda)} \tag{2.25}
\]

and we write \( W_2 = W_2(Q) + W_2(\Lambda_0 - \Omega_0) + W_2(\Omega_0) \). Consider the source term \( \epsilon_0(\delta A, J) \) with \( |J| < 1 \). Rewrite

\[
\langle dA, J \rangle = \sum_p dA(p) J(p) = \sum_p \sum_{b \in \partial p} A(b) J(b) \\
= \sum_b A(b) \sum_{p: b \in \partial p} J(b) = \langle A, \delta J \rangle \tag{2.26}
\]
and define
\[ V_1(\Omega_0, \Phi_{\Lambda_1}, \Phi_{\tilde{Q}}, J) = -W_1(\Lambda_1) - W_2(\Omega_0) \]
\[ + V(\Omega_0, C^{\frac{1}{2}, \text{loc}}_{\Lambda_1}\Phi' - C_{\Lambda_1}T_{\Lambda_1}Q\Phi_{\tilde{Q}}) \]
\[ + V_\varepsilon(\Lambda_1, \Phi') - \langle C^{\frac{1}{2}, \text{loc}}_{\Lambda_1}\Phi' - C_{\Lambda_1}T_{\Lambda_1}Q\Phi_{\tilde{Q}}, \delta J \rangle. \]  

Define the normalized Gaussian measure with unit covariance as
\[ d\mu_1(\Phi'_\Omega) = \frac{D\Phi'_\Omega e^{-\frac{1}{2}||\Phi'||^2_{\Omega_1}}}{Z_0(\Omega_1)}, \quad Z_0(\Omega_1) = \int D\Phi'_\Omega e^{-\frac{1}{2}||\Phi'||^2_{\Omega_1}}. \]

The localized small field integral is
\[ \Xi(\Omega_1, \Phi'_{\Lambda_1-\Omega_1}, \Phi_{\tilde{Q}}, J) = \int d\mu_1(\Phi'_\Omega) e^{-V_1(\Omega_0, \Phi'_{\Lambda_1}, \Phi_{\tilde{Q}}, J)} \chi(\Phi'_\Omega). \]

Rewrite the generating functional
\[ Z[J] = \left( \frac{2\pi}{\varepsilon_0} \right)^{|\Lambda|+1}(\det C^{\frac{1}{2}}) \sum_{Q \subseteq \Lambda} \sum_{P \subseteq \Lambda_1} \sum_{V: \delta V = 0} \int D\Phi e^{-\frac{1}{2}\langle \Phi, (T_Q - T_{\Lambda_1}Q, T_{\Lambda_1}\Phi) - V(\Phi_Q) \rangle \chi(\Phi)} e^{W_2(\Omega)} \]
\[ \int D\Phi'_\Lambda e^{-\frac{1}{2}||\Phi'||^2_{\Lambda_1 - \Omega_1} - V(\Lambda_1 - \Omega_0, C^{\frac{1}{2}, \text{loc}}_{\Lambda_1}\Phi' - C_{\Lambda_1}T_{\Lambda_1}Q\Phi_{\tilde{Q}})} \chi_{\Omega}(\Phi') \]
\[ \chi_{\Lambda_1 - \Omega_0}(C^{\frac{1}{2}, \text{loc}}_{\Lambda_1}\Phi' - C_{\Lambda_1}T_{\Lambda_1}Q\Phi_{\tilde{Q}}) e^{W_2(\Lambda_0 - \Omega_0)} Z_0(\Omega_1) \Xi(\Omega_1, \Phi'_{\Lambda_1 - \Omega_1}, \Phi_{\tilde{Q}}). \]

This is our basic expansion of the generating functional including the small field integral. Now, we prove some lemmas.

**Lemma 2.1** (Estimate on covariance). Let \( C \) be the positive, self-adjoint operator as defined in Eq. 2.4. Then, for some \( \gamma > 0 \), \( |C(x,y)| \leq ce^{-\gamma|x-y|} \).

**Proof.** Let \( x \) and \( y \) be two sites. For some vector \( a \) define an operator \( e_a(x) \) by its action on some function \( f \) as \( e_a(x)f = e^{ax}f \). Let \( \delta_x(y) = \delta_{xy} \) be the lattice delta function. Let \( |a| \) be small and \( || \cdot || \) be the \( l^2 \) norm. Then,

\[ e^{-a}\partial_x e_a = \partial_x + a \]
\[ e^{-a}(-\Delta + \mu^2)e_a = -\Delta - 2a \cdot \nabla + |a|^2 + \mu^2 \]
\[ -\Delta + \mu^2 = e_a(-\Delta - 2a \cdot \nabla + |a|^2 + \mu^2)e^{-a} \]
\[ (-\Delta + \mu^2)^{-1} = e_a(-\Delta - 2a \cdot \nabla + |a|^2 + \mu^2)^{-1}e^{-a} \]
\[ |(-\Delta + \mu^2)^{-1}(x,y)| = |e^{ax}(-\Delta - 2a \cdot \nabla + |a|^2 + \mu^2)^{-1}(x,y)e^{-ay}| \]
\[ = e^{a(x-y)}|\delta_x, (-\Delta - 2a \cdot \nabla + |a|^2 + \mu^2)^{-1}\delta_y|| \]
\[ \leq e^{a(x-y)}||\delta_x|| ||\delta_y|| |(-\Delta - 2a \cdot \nabla + |a|^2 + \mu^2)^{-1}|| \]
\[ \leq c e^{a(x-y)} \]  

(2.31)
setting $a = -\gamma \frac{(x-y)}{|x-y|}$ gives the result. Note that

$$(-\Delta + |a|^2 + \mu^2 - 2a \cdot \nabla)^{-1} = (-\Delta + |a|^2 + \mu^2)^{-1}$$

$$\sum_{n=0}^{\infty} [(-\Delta + |a|^2 + \mu^2)^{-1} 2a \cdot \nabla]^n$$

which converges if $|a| < \frac{\mu^2}{2 \pi^2 r}$. Note that $|a| = \gamma$. Similarly, we can prove the estimate for term $(-\delta d + m_A^2)^{-1}$ as long as $|a|$ is small to keep the expression $(-\delta d + |a|^2 + m_A^2 - 2a \cdot d)^{-1}$ positive, where $d$ denotes the exterior derivative.

**Remark 2.2.** Let $\xi, \xi' \in \Lambda \cup \Lambda^*$. Let $K$ be a positive, self-adjoint operator with $|K(\xi, \xi')| \leq e^{-\gamma d(\xi, \xi')}$ and $f$ be some function. Then,

$$(Kf)(\xi) = \sum_{\xi'} K(\xi, \xi') f(\xi')$$

$$||Kf||_\xi \leq \sum_{\xi'} |K(\xi, \xi')||f||_\infty$$

$$\leq \sum_{\xi'} e^{-\gamma d(\xi, \xi')}||f||_\infty \leq c||f||_\infty.$$  (2.33)

Note that $||Kf||_\infty \leq \sup_{\xi} ||Kf||(\xi)$. In the following lemmas, operator $K$ can be identified with $C, C^\frac{1}{2}, \delta C$.

**Lemma 2.3.** For $\Phi : \Lambda \rightarrow \mathbb{R}$,

1. Let $C^\frac{1}{2}, \text{loc}$ and $\delta C^\frac{1}{2}$ be the operators as defined in Eqs. 2.15 and 2.16, respectively. Then,

$$||C^\frac{1}{2}, \text{loc} \Phi(\xi)|| \leq c||\Phi||_\infty, \quad ||\delta C^\frac{1}{2} \Phi(\xi)|| \leq c e^{-\gamma r'||\Phi||_\infty}.\quad (2.34)$$

2. $C^\frac{1}{2}$ and $C^\frac{1}{2}, \text{loc}$ are invertible and

$$||C^{-\frac{1}{2}}(\xi)||, ||C^{-\frac{1}{2}}, \text{loc}(\xi)|| \leq c||\Phi||_\infty.$$  (2.35)

**Proof.** From definition,

$$\delta C^\frac{1}{2} = \frac{1}{\pi} \int_0^\infty \frac{dr}{\sqrt{r}} (C_r - C^\text{loc}_r) = \frac{1}{\pi} \int_0^\infty \frac{dr}{\sqrt{r}} \delta C_r.\quad (2.36)$$

Therefore,

$$|(\delta C^\frac{1}{2} \Phi(\xi)| \leq \sum_{\xi'} \frac{1}{\pi} \int_0^\infty \frac{dr}{\sqrt{r}} |\delta C_r(\xi, \xi')||\Phi||_\infty$$

$$\leq \sum_{\xi'} \frac{1}{\pi} \int_0^\infty \frac{dr}{\sqrt{r}} |(T + r)^{-1}(\xi, \xi')||\Phi||_\infty.\quad (2.37)$$

To estimate the integral, we use the estimate on covariance, $(-\Delta + \mu^2 + r)^{-1}(x, y) = (\delta_x, (-\Delta + \mu^2 + r)^{-1} \delta_y) \leq ce^{-\gamma|x-y|}$ and $(-\delta d + m_A^2 + r)^{-1}(b, b') =$
\[ \langle \delta_b, (-\delta d + m_A^2 + r)^{-1} \delta_{b'} \rangle \leq c e^{-\gamma d(b, b')} \], where \( d(b, b') \) is the infimum of distance between the sites containing bonds \( b \) and \( b' \). Note that (Eq. 2.1) \( T \) is a differential operator plus a mass term. Let \( m = \min(\mu^2, m_A^2) \) denote the mass term in \( T \). Since \( d(\xi, \xi') > r_\lambda \), rewrite

\[
| (T + r)^{-1}(\xi, \xi') | \leq \frac{1}{\pi} \int_0^\infty \frac{dr}{\sqrt{r}} | (T + r)^{-1}(\xi, \xi') | \leq \frac{1}{\pi} e^{-\frac{\gamma r_\lambda}{2}} \sum_{\xi'} e^{-\frac{\gamma d(\xi, \xi')}{2}} \int_0^\infty \frac{dr}{\sqrt{r} (m + r)^{\frac{5}{4}}} \leq c e^{-\gamma' r_\lambda}. \tag{2.39}
\]

The above inequality can be used to get the bound for \( C_{\frac{1}{2}, \text{loc}} \). For the second part, we note that operator \( C \) is invertible with \( C^{-1} = T \), it follows \( C^{-\frac{1}{2}} = TC^{\frac{1}{2}} \) and thus,

\[
| (C^{-\frac{1}{2}} \Phi)(\xi) | = | (TC^{\frac{1}{2}} \Phi)(\xi) | \leq O(m^2)||C^{\frac{1}{2}}\Phi||_\infty \leq c ||\Phi||_\infty. \tag{2.40}
\]

Writing \( C_{\frac{1}{2}, \text{loc}} = C^{\frac{1}{2}} - \delta C^{\frac{1}{2}} \), then

\[
C^{-\frac{1}{2}, \text{loc}} = C^{-\frac{1}{2}} \sum_{n=0}^\infty (\delta C^{\frac{1}{2}} C^{-\frac{1}{2}})^n, \tag{2.41}
\]

the result follows easily. \( \square \)

**Lemma 2.4.** For \( \Phi: \Lambda \to \mathbb{R} \), with change of variables, \( \Phi = C_{\Lambda_1}^{\frac{1}{2}, \text{loc}} \Phi' \), we have

\[
\langle \Phi, C_{\Lambda_1}^{-1} \Phi \rangle = \langle \Phi', C_{\Lambda_1}^{\frac{1}{2}, \text{loc}} C_{\Lambda_1}^{-1} C_{\Lambda_1}^{\frac{1}{2}, \text{loc}} \Phi' \rangle = ||\Phi'||_\Lambda^2 + V_\varepsilon(\Lambda_1, \Phi'). \tag{2.42}
\]

Then, \( V_\varepsilon(\Lambda_1, \Phi') \) has a local expansion in \( \Box \)

\[
V_\varepsilon = \sum_{\Box \subset \Lambda_1} V_\varepsilon(\Box) \text{ with } |V_\varepsilon(\Box)| \leq c [r_\lambda]^d e^{-\gamma' r_\lambda} ||\Phi'||^2_\infty. \tag{2.43}
\]

**Proof.** Using the identity \( 1 = \sum_{\Box} 1_\Box \), where

\[
1_\Box(\xi) = 1_\Box(x) \text{ if } \xi = x \text{ or } \xi = \langle x, x + e_k \rangle \tag{2.44}
\]

and \( C_{\frac{1}{2}, \text{loc}} = C^{\frac{1}{2}} - \delta C^{\frac{1}{2}} \), since \( C^{\frac{1}{2}} \) is self-adjoint, rewrite \( V_\varepsilon(\Box) \) as

\[
V_\varepsilon(\Box) = \langle \delta C^{\frac{1}{2}} \Phi', 1_\Box C^{-1} \delta C^{\frac{1}{2}} \Phi' \rangle - 2\langle C^{-\frac{1}{2}} \Phi', 1_\Box \delta C^{\frac{1}{2}} \Phi' \rangle \tag{2.45}
\]

Then,

\[
|V_\varepsilon(\Box)| \leq |\langle \delta C^{\frac{1}{2}} \Phi'(\xi), 1_\Box T \delta C^{\frac{1}{2}} \Phi'(\xi) \rangle| + 2|\langle C^{-\frac{1}{2}} \Phi'(\xi), 1_\Box \delta C^{\frac{1}{2}} \Phi'(\xi) \rangle| \leq \text{vol}(\Box) ||\delta C^{\frac{1}{2}} \Phi'(\xi)| |T \delta C^{\frac{1}{2}} \Phi'(\xi)| + 2 \text{ vol}(\Box) ||C^{-\frac{1}{2}} \Phi'(\xi)|| \delta C^{\frac{1}{2}} \Phi'(\xi)| \leq [r_\lambda]^d ||\delta C^{\frac{1}{2}} \Phi'||_\infty O(m^2)||C^{\frac{1}{2}} \Phi'||_\infty + 2 [r_\lambda]^d ||C^{-\frac{1}{2}} \Phi'||_\infty ||\delta C^{\frac{1}{2}} \Phi'||_\infty \leq c [r_\lambda]^d e^{-\gamma' r_\lambda} ||\Phi'||^2_\infty. \tag{2.46}
\]

\( \square \)
The change of variables carries with it the term $\det C_{\Lambda_1}^{\frac{1}{2}, \text{loc}}$ in partition function. In the region $\Lambda_1$, using $C_{\Lambda_1}^{\frac{1}{2}, \text{loc}} = C_{\Lambda_1}^{\frac{3}{2}} - \delta C_{\Lambda_1}^{\frac{3}{2}}$, rewrite

$$
\det C_{\Lambda_1}^{\frac{1}{2}, \text{loc}} = \det \left( C_{\Lambda_1}^{\frac{3}{2}} - \delta C_{\Lambda_1}^{\frac{3}{2}} \right)
= \det \left( C_{\Lambda_1}^{\frac{3}{2}} \right) \det \left( I - C_{\Lambda_1}^{\frac{3}{2}} \frac{1}{\gamma} \delta C_{\Lambda_1}^{\frac{3}{2}} \right)
= \det \left( C_{\Lambda_1}^{\frac{3}{2}} \right) e^{\frac{1}{\gamma} \mathrm{Tr} \log \left( I - C_{\Lambda_1}^{\frac{3}{2}} \frac{1}{\gamma} \delta C_{\Lambda_1}^{\frac{3}{2}} \right)}
= \det \left( C_{\Lambda_1}^{\frac{3}{2}} \right) e^{-\sum_{n=1}^{\infty} \frac{1}{\gamma} \mathrm{Tr} \left[ \left( C_{\Lambda_1}^{\frac{3}{2}} \frac{1}{\gamma} \delta C_{\Lambda_1}^{\frac{3}{2}} \right)^n \right]}. \tag{2.47}
$$

**Lemma 2.5.** Let $W_1(\Lambda_1) = -\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Tr} \left[ \left( C_{\Lambda_1}^{\frac{3}{2}} \frac{1}{\gamma} \delta C_{\Lambda_1}^{\frac{3}{2}} \right)^n \right]$. Then, $W_1$ has a local expansion in $\square$,

$$
W_1 = \sum_{\square \subset \Lambda_1} W_1(\square), \quad \text{with} \quad |W_1(\square)| \leq c [r_{\lambda}]^d e^{-\gamma r_{\lambda}} \tag{2.48}
$$

**Proof.** From the definition of $W_1$,

$$
W_1(\square) = -\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Tr} \left[ 1_{\square} \left( C_{\Lambda_1}^{\frac{3}{2}} \frac{1}{\gamma} \delta C_{\Lambda_1}^{\frac{3}{2}} \right)^n \right]
= -\sum_{n=1}^{\infty} \frac{1}{n} \sum_{x \in \square} \left[ \left( C_{\Lambda_1}^{\frac{3}{2}} \frac{1}{\gamma} \delta C_{\Lambda_1}^{\frac{3}{2}} \right)^n \delta_x \right](x). \tag{2.49}
$$

From the bounds on $C_{\Lambda_1}^{\frac{3}{2}}$ and $\delta C_{\Lambda_1}^{\frac{3}{2}}$, we have

$$
\left| \left[ \left( C_{\Lambda_1}^{\frac{3}{2}} \frac{1}{\gamma} \delta C_{\Lambda_1}^{\frac{3}{2}} \right)^n \delta_x \right](x) \right| \leq (c e^{-\gamma r_{\lambda}})^n \| \delta_x \| \leq (c e^{-\gamma r_{\lambda}})^n \tag{2.50}
$$

and so $|W_1(\square)| \leq \sum_{n=1}^{\infty} \frac{1}{n} \sum_{x} (c e^{-\gamma r_{\lambda}})^n = [r_{\lambda}]^d \sum_{n=1}^{\infty} \frac{1}{n} (c e^{-\gamma r_{\lambda}})^n$ and summing over $n$ gives the result. \hfill \Box

Next, we want to write $\det(C_{\Lambda_1}^{\frac{1}{2}})$ in terms of global term $\det(C_{\Lambda_1}^{\frac{1}{2}})$ as

$$
\det(C_{\Lambda_1}^{\frac{1}{2}}) = \det(C_{\Lambda_1}^{\frac{1}{2}}) e^{W_2}, \tag{2.51}
$$

where

$$
W_2 = \mathrm{Tr} \log \left( C_{\Lambda_1}^{\frac{1}{2}} \right) - \mathrm{Tr} \log \left( C_{\Lambda_1}^{\frac{1}{2}} \right)
= -\frac{1}{2} \mathrm{Tr} \log \left( T_{\Lambda_1} \right) + \frac{1}{2} \mathrm{Tr} \log \left( T \right). \tag{2.52}
$$

First, we prove a determinant identity which is due to Balaban [14].

**Lemma 2.6** (Determinant identity). Let $K$ be a positive self-adjoint operator. Then, $\det K = e^{\mathrm{Tr} \log K}$ where for any $R_0 > 0$

$$
\log K = K \int_{R_0}^{\infty} \frac{dx}{x} (K + x)^{-1} - \int_{0}^{R_0} dx (K + x)^{-1} + \log R_0. \tag{2.53}
$$
Proof. We start with resolvent equation for the function $\log K$,

$$\log K = \frac{1}{2\pi i} \int_{\Gamma} dz \log (z - K)^{-1}. \quad (2.54)$$

Construct $\Gamma$ as a simple closed curve traversed counterclockwise. Let $\theta = \arg z$. Then, $\Gamma$ consists of

- $\Gamma_R = \{z \in \mathbb{C}: |z| = R, -\pi + \epsilon \leq \theta \leq \pi - \epsilon\}.$
- $\Gamma_+ = \{z \in \mathbb{C}: r \leq |z| \leq R, \theta = \pi - \epsilon\}.$
- $\Gamma_r = \{z \in \mathbb{C}: |z| = r, -\pi + \epsilon \leq \theta \leq \pi - \epsilon\}.$
- $\Gamma_- = \{z \in \mathbb{C}: r \leq |z| \leq R, \theta = -\pi + \epsilon\}.$

For $R$ sufficiently large and $r$ sufficiently small, $\Gamma$ encloses the spectrum of operator $K$. The function $\log z$ is analytic inside $\Gamma$.

Now, we cut $\Gamma$ at $r < R_0 < R$ and write it as $\Gamma = \Gamma_- + \Gamma_+$ where $\Gamma_- = \Gamma \cap \{z: |z| \leq R_0\}$ and $\Gamma_+ = \Gamma \cap \{z: |z| \geq R_0\}$. In the integral over $\Gamma_+$, we use the following identity in resolvent equation

$$(z - K)^{-1} = z^{-1} + z^{-1}K(z - K)^{-1}. \quad (2.55)$$

The integral of the first term is

$$\lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{\Gamma_-} dz \frac{\log z}{z} = \frac{1}{2\pi i} \int_{|z|=R} \frac{dz}{z} \log z + \frac{1}{2\pi i} \int_R^{R_0} dx \frac{\log x}{x}, \quad (2.56)$$

where we have parametrized $z = -x$. Writing $\log x = \log |x| + i\pi$ above the real axis and $\log x = \log |x| - i\pi$ below the real axis and noting that $z = Re^{i\theta}$ implies $dz/z = i \, d\theta$, we get

$$\lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{\Gamma_-} dz \frac{\log z}{z} = \frac{1}{2\pi i} \int_{|z|=R} \frac{dz}{z} \log z - \int_{R_0}^{R} dx \frac{\log x}{x}$$

$$= \frac{1}{2\pi i} \int_{-\pi}^\pi i \, d\theta (\log R + i\theta) - \log R + \log R_0$$

$$= \log R(i2\pi) - \log R + \log R_0$$

$$= \log R_0. \quad (2.57)$$

Similarly, the integral of the second term is

$$\lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{\Gamma_-} dz \frac{\log z}{z} K(z - K)^{-1} = \frac{K}{2\pi i} \int_{|z|=R} \frac{dz}{z} \log z K(z - K)^{-1}$$

$$+ K \int_{R_0}^{R} dx \frac{1}{x} (x + K)^{-1}. \quad (2.58)$$

Take the limit $R \to \infty$. The first term is $O(R^{-1}\log R)$ which converges to zero. For the second term, the integrand is $O(x^{-2})$ which converges to the integral
over \([R_0, \infty)\). Next, we do the integral over \(\Gamma_<\) of the resolvent equation and get
\[
\lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{\Gamma_<} \mathrm{d}z \log (z - K)^{-1} = \frac{1}{2\pi} \int_{|z|=\epsilon} \frac{\mathrm{d}z}{z} \log (z - K)^{-1} - \int_{r}^{R_0} \mathrm{d}x (x + K)^{-1}.
\]
(2.59)

The minus sign is due to the fact that in \(\Gamma_<\) traversing from \(r\) to \(R_0\) is above the real axis, whereas in \(\Gamma_>\) it is from \(R\) to \(R_0\) and \(r < R_0 < R\). Take the limit \(r \to 0\). Since zero is not an eigenvalue, the first term is \(O(r \log r)\) and converges to zero. For the second term, the integrand is bounded and it converges to the integral over \([0, R_0]\). This completes the proof. \(\square\)

Use Lemma 2.6 with \(R_0 = 1\) to rewrite
\[
\text{Tr} \log T = - \sum_{\square \subset \Lambda} \int_{0}^{1} \mathrm{d}r \text{Tr} 1_{\square}(T + r)^{-1} 1_{\square} + \sum_{\square \subset \Lambda} \int_{1}^{\infty} \frac{\mathrm{d}r}{r} \text{Tr} 1_{\square} (T + r)^{-1} 1_{\square}
\]
\[
= - \sum_{\square \subset \Omega_1} \int_{0}^{1} \mathrm{d}r \text{Tr} 1_{\square}(T + r)^{-1} 1_{\square} + \sum_{\square \subset \Omega_1^c} \int_{1}^{\infty} \frac{\mathrm{d}r}{r} \text{Tr} 1_{\square} (T + r)^{-1} 1_{\square}
\]
\[
- \sum_{\square \subset \Omega_1} \int_{0}^{1} \mathrm{d}r \text{Tr} 1_{\square}(T + r)^{-1} 1_{\square} + \sum_{\square \subset \Omega_1^c} \int_{1}^{\infty} \frac{\mathrm{d}r}{r} \text{Tr} 1_{\square} (T + r)^{-1} 1_{\square}
\]
(2.60)

Consider the difference of Eqs. 2.60 and 2.61,
\[ \text{Tr} \log T - \text{Tr} \log T_{\Lambda_1} = \sum_{\square \subset \Omega_1} \int_0^1 \text{d}r \text{Tr} (1\square (T_{\Lambda_1} + r)^{-1} - (T + r)^{-1}1\square) \]

\[ - \sum_{\square \subset \Omega_1} \int_0^1 \text{d}r \text{Tr} 1\square (T + r)^{-1}1\square + \sum_{\square \subset \Lambda_1 - \Omega_1} \int_0^1 \text{d}r \text{Tr} 1\square (T_{\Lambda_1} + r)^{-1}1\square \]

\[ + \sum_{\square \subset \Omega_1} \int_1^\infty \frac{\text{d}r}{r} \text{Tr} 1\square (T + r)^{-1} - T_{\Lambda_1} (T_{\Lambda_1} + r)^{-1}1\square \]

\[ + \sum_{\square \subset \Omega_1} \int_1^\infty \frac{\text{d}r}{r} \text{Tr} 1\square T (T + r)^{-1}1\square \]

\[ - \sum_{\square \subset \Lambda_1 - \Omega_1} \int_1^\infty \frac{\text{d}r}{r} \text{Tr} 1\square T_{\Lambda_1} (T_{\Lambda_1} + r)^{-1}1\square. \quad (2.62) \]

**Lemma 2.7.** Let

\[ \mathcal{A}(\square) = \text{Tr}(1\square \mathcal{A} 1\square) = \text{Tr}(1\square ((T_{\Lambda_1} + r)^{-1} - (T + r)^{-1}1\square) \]

\[ \mathcal{B}(\square) = \text{Tr}(1\square \mathcal{B} 1\square) = \text{Tr}(1\square T (T + r)^{-1} - T_{\Lambda_1} (T_{\Lambda_1} + r)^{-1}1\square). \quad (2.63) \]

Let \( m = \min(\mu, m_A) \). For \( \square \subset \Omega_1 \) and \( m \) sufficiently large

\[ |\mathcal{A}(\square)| \leq O(m^{-2[r\lambda]}) \quad \text{and} \quad |\mathcal{B}(\square)| \leq O(m^{-2[r\lambda]}). \quad (2.64) \]

**Proof.** First consider the case \( T = -\Delta + \mu^2 \). Then,

\[ (-\Delta + \mu^2 + r)^{-1} = (\mu^2 + r)^{-1} \sum_{n=1}^\infty ((\mu^2 + r)^{-1}\Delta)^n. \quad (2.65) \]

Next, we construct a random walk expansion of the operator \((-\Delta + \mu^2 + r)^{-1},\)

\[ ((\mu^2 + r)^{-1}\Delta)^n(x, x') = (\mu^2 + r)^{-n} \sum_{x_1, x_2, \ldots, x_n} \Delta(x, x_1) \Delta(x_1, x_2) \ldots \Delta(x_n, x') \]

\[ (2.66) \]

where \( x_1, x_2, \ldots, x_n \) is a random walk \( \omega \) connecting sites \( x, x' \). Note that \( \Delta \) forces \( x_i \)'s \( \in \omega \) to be nearest neighbors. This random walk expansion exists everywhere on lattice \( \Lambda \) and converges for \( \mu \) sufficiently large.

Next, we note that in \( \Omega_1, T_{\Lambda_1} = T \). Thus, in \( \mathcal{A}(\square) \) and \( \mathcal{B}(\square) \) only those random walks contribute that join \( \Omega_1 \) with \( \tilde{Q} \) forcing \( n \geq 2[r\lambda] \). Since \( \Delta \) is bounded,

\[ \text{Tr}(1\square \mathcal{A} 1\square) = (\mu^2 + r)^{-1} \sum_{n=2[r\lambda]}^\infty \sum_{x \in \square} ((\mu^2 + r)^{-1}\Delta)^n(x, x) \]

\[ \leq [r\lambda]^d (\mu^2 + r)^{-2[r\lambda]} \]

\[ \text{Tr}(1\square \mathcal{B} 1\square) = (-\Delta + \mu^2)(\mu^2 + r)^{-1} \sum_{n=2[r\lambda]}^\infty \sum_{x \in \square} (\mu^2 + r)^{-n}\Delta^n(x, x) \]

\[ \leq [r\lambda]^d (\mu^2 + 2^d)(\mu^2 + r)^{-2[r\lambda]} \quad (2.67) \]
The case with $T = \delta d + m_\Lambda^2$ can be treated in the same manner, since $\delta d$ is also bounded.

**Lemma 2.8.** $W_2$ (as defined in 2.52) has a local expansion in $\Box$,

$$W_2 = \sum_{\Box \subset \Lambda} W_2(\Box), \quad \text{with} \quad |W_2(\Box)| \leq \begin{cases} \mathcal{O}(m^{-2[r_\lambda]}) & \text{if } \Box \subset \Omega_1, \\ \mathcal{O}([r_\lambda]^d) & \text{if } \Box \subset \Omega_1^c. \end{cases}$$

**Proof.** From definition of $W_2$ (2.52), for $\Box \subset \Omega_1$

$$W_2(\Box) = \int_0^1 dr \mathcal{A}(\Box) + \int_1^\infty \frac{dr}{r} \mathcal{B}(\Box)$$

using Lemma 2.7

$$\left| \int_0^1 dr \mathcal{A}(\Box) \right| \leq [r_\lambda]^d \mu^{-4[r_\lambda]-2} \int_0^1 dr [r_\lambda]^d \mu^{-4[r_\lambda]-2}$$

$$\left| \int_1^\infty \frac{dr}{r} \mathcal{B}(\Box) \right| \leq [r_\lambda]^d \left(2^d + \mu^2 \right) \int_1^\infty \frac{dr}{r} \mu^{-2[r_\lambda]+1} \leq [r_\lambda]^d \frac{1}{2} \left(2^d + \mu^2 \right) \mu^{-2[r_\lambda]+1}. \quad (2.70)$$

For $\Box \subset \Omega_1^c$,

$$W_2(\Box) = \int_0^1 dr \operatorname{Tr} l_\Box (T + r)^{-1} l_\Box + \int_1^\infty \frac{dr}{r} \operatorname{Tr} l_\Box T (T + r)^{-1} l_\Box \quad (2.71)$$

from 2.37,

$$\int_0^1 dr \operatorname{Tr} l_\Box (T + r)^{-1} l_\Box = \int_0^1 dr \sum_{\xi \in \Box} (T + r)^{-1} (\xi, \xi) \leq [r_\lambda]^d$$

$$\int_1^\infty \frac{dr}{r} \operatorname{Tr} l_\Box T (T + r)^{-1} l_\Box \leq \left(2^d + m^2 \right) \int_1^\infty \frac{dr}{r^{3/2}} \sum_{\xi \in \Box} (T + r)^{-\frac{1}{2}} (\xi, \xi) \leq 2 \left(2^d + m^2 \right)[r_\lambda]^d. \quad (2.72)$$

\[ \Box \]

### 3. Power Series Representation of $\Xi(\Omega_1, \Phi'_{\Lambda_1-\Omega_1}, \Phi_\tilde{Q}, J)$

Recall that $\Xi(\Omega_1, \Phi'_{\Lambda_1-\Omega_1}, \Phi_\tilde{Q}, J) = \int d\mu_1(\Phi'_\Omega) e^{-V_1(\Omega_0, \Phi'_\Lambda, \Phi_\tilde{Q}, J)} \hat{\chi}(\Phi'_\Omega)$, where $V_1(\Omega_0, \Phi'_\Lambda, \Phi_\tilde{Q}, J)$ is given by 2.27

$$1(\Omega_0, \Phi'_\Lambda, \Phi_\tilde{Q}, J) = -W_1(\Lambda_1) - W_2(\Omega_0) + V(\Omega_0, C_{\Lambda_1}^2 \Phi' - C_{\Lambda_1} T_{\Lambda_1} \Phi_\tilde{Q}) + V_\varepsilon(\Lambda_1, \Phi') - 0 \left( C_{\Lambda_1}^2 \Phi' - C_{\Lambda_1} T_{\Lambda_1} \Phi_\tilde{Q}, \delta J \right). \quad (3.1)$$
Note that \( W_1(\Lambda_1) \) and \( W_2(\Omega_0) \) are independent of \( \Phi' \). We want to construct a power series representation of \( \ln \Xi(\Omega_1, \Phi'_{\Lambda_1-\Omega_1}, \Phi_Q, J) \), where \( \ln \) denotes the natural logarithm.

**Notation** Let \( \xi \in \Lambda \cup \Lambda^* \). Then, for example,

\[
\sum_{x \in \Lambda} \rho(x)^4 = \sum_{\xi} \Phi'(\xi)^4 1_{\xi \in \Lambda},
\]

\[
\sum_{x \in \Lambda, x' \in \Lambda, b \in \Lambda^*} \rho(x)\rho(x') A(b) = \sum_{\xi, \xi', \xi''} \Phi'(\xi)\Phi'(\xi')\Phi''(\xi'') 1_{\xi \in \Lambda} 1_{\xi' \in \Lambda} 1_{\xi'' \in \Lambda^*},
\]

where \( 1_{\xi \in \Lambda} \) is the characteristic function that restricts \( \xi \in \Lambda \cup \Lambda^* \) to \( \xi \in \Lambda \).

**Power Series** Let \( X \subset \Lambda \). Let \( f(\Phi, \Psi) \) be smooth and analytic function of the fields \( \Phi, \Psi \) defined on \( X \). Then, a power series expansion of \( f \) is given by

\[
f(\Phi, \Psi) = \sum_{n, m \geq 0} \sum_{\xi_1, \ldots, \xi_n \in X} a(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m) \Phi(\xi_1) \cdots \Phi(\xi_n) \Psi(\eta_1) \cdots \Psi(\eta_m).
\]

Define a n-component vector \( \vec{\xi} \) as

\[
\vec{\xi} = \{\xi_1, \xi_2, \ldots, \xi_n\}.
\]

Then, for a n-component vector \( \vec{\xi} \), write

\[
\Phi(\vec{\xi}) = \Phi(\xi_1) \cdots \Phi(\xi_n).
\]

Rewrite power series expansion of \( f \) as

\[
f(\Phi, \Psi) = \sum_{n, m \geq 0} \sum_{\vec{\xi}=(\xi_1, \ldots, \xi_n) \in X} \sum_{\vec{\eta}=(\eta_1, \ldots, \eta_m) \in X} a(\vec{\xi}, \vec{\eta}) \Phi(\vec{\xi}) \Psi(\vec{\eta}).
\]

Let

\[
a_{\Psi}(\vec{\xi}) = \sum_{\vec{\eta} \in X} a(\vec{\xi}, \vec{\eta}) \Psi(\vec{\eta})
\]

and rewrite

\[
f(\Phi, \Psi) = \sum_{\vec{\xi} \in X} a_{\Psi}(\vec{\xi}) \Phi(\vec{\xi})
\]

such that

\[
e^{f(\Phi, \Psi)} = \sum_{l=0}^{\infty} \frac{1}{l!} f(\Phi, \Psi)^l
\]

\[
= 1 + \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{\vec{\xi}_1, \ldots, \vec{\xi}_l \in X} a_{\Psi}(\vec{\xi}_1) \cdots a_{\Psi}(\vec{\xi}_l) \Phi(\vec{\xi}_1) \cdots \Phi(\vec{\xi}_l).
\]

First set \( \Phi_Q = 0 \) and rewrite 3.1 as

\[
V_1(\Omega_0, \Phi'_{A_1}, J) = V(\Omega_0, C^{\frac{1}{2}, \text{loc}}_{A_1} \Phi') + V_\varepsilon(\Lambda_1, \Phi') - e_0(C^{\frac{1}{2}, \text{loc}}_{A_1} \Phi', \delta J).
\]
Next, note that
\[ C_{\Lambda_1}^{1,\text{loc}} \Phi' = C_{\Lambda_1}^{1,\text{loc}} (\Phi'_{\Lambda_1-\Omega_1}, \Phi'_{\Omega_1}). \]  
(3.11)

Let
\[ \Phi = \Phi'_{\Omega_1}, \quad \Psi = (\Psi_1, \Psi_2) = (\Phi'_{\Lambda_1-\Omega_1}, J). \]  
(3.12)

Thus, \( V_1(\Omega_0, \Phi'_{\Lambda_1}, J) = V_1(\Omega_0, \Phi, \Psi) \) and \( \Xi(\Omega_1, \Phi'_{\Lambda_1-\Omega_1}, J) = \Xi(\Omega_1, \Psi) \).

We are interested in the case \( f(\Phi, \Psi) = V_1(\Omega_0, \Phi, \Psi) \) and
\[ \Xi(\Omega_1, \Psi) = \int d\mu_1(\Phi) \tilde{\chi}(\Phi) e^{f(\Phi, \Psi)}. \]  
(3.13)

We want to write a power series of \( \ln \Xi(\Omega_1, \Psi) \) as
\[ \ln \Xi(\Omega_1, \Psi) = b_0 + \sum_{\alpha_1} b(\eta_1) \Psi_{\alpha_1}(\eta_1) + \sum_{\alpha_1, \alpha_2} b(\eta_1, \eta_2) \Psi_{\alpha_1}(\eta_1) \Psi_{\alpha_2}(\eta_2) + \cdots \]
(3.14)

\[ = \sum_{m \geq 0} \sum_{\eta = (\eta_1, \ldots, \eta_m)} \sum_{\alpha_1, \ldots, \alpha_m} b(\eta) \Psi_{\alpha}(\eta). \]

Note that \( \ln \Xi(\Omega_1, 0) = b_0 \). As a first step to construct a power series written above, we prove Theorem 3.1.

**Theorem 3.1.** There exists a coefficient system \( a(\tilde{\xi}, \tilde{\eta}) \) such that
\[ V_1(\Omega_0, \Phi, \Psi) = \sum_{n,m} a(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m) \]
(3.15)

\[ \Phi(\xi_1) \ldots \Phi(\xi_n) \Psi_{\alpha}(\eta_1) \ldots \Psi_{\alpha}(\eta_m) \]

with
\[ |a(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m)| \leq c(c_0 \rho_0)^{n+m-\gamma} e^{-\gamma t(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n)}, \]
(3.16)

where \( c \) and \( c_0 \) are constants and \( t(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m) \) is the length of minimal tree joining \( \xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m \).

**Proof.** Recall that \( V_1(\Omega_0, \Phi, \Psi) = V(\Omega_0, C_{\Lambda_1}^{1,\text{loc}} \Phi') + V_\varepsilon(\Lambda_1, \Phi') - e_0(C_{\Lambda_1}^{1,\text{loc}} \Phi', \delta J) \) and
\[ V(\Omega_0, C_{\Lambda_1}^{1,\text{loc}} \Phi') = \rho_0^2 \sum_{\xi} \left( 1 - \cos [e_0 C_{\Lambda_1}^{1,\text{loc}} \Phi'(\xi)]_{\xi \in \Omega_0} \right) \]

\[ - \frac{1}{2} e_0^2 \left( C_{\Lambda_1}^{1,\text{loc}} \Phi'(\xi) \right)^2_{\xi \in \Omega_0} + \rho_0 \sum_{\xi, \xi'} \left( C_{\Lambda_1}^{1,\text{loc}} \Phi'(\xi) \right)_{\xi \in \Omega_0} + C_{\Lambda_1}^{1,\text{loc}} \Phi'(\xi') \right)_{\xi' \in \Omega_0} \]

\[ (1 - \cos [e_0 C_{\Lambda_1}^{1,\text{loc}} \Phi'(\xi'')])_{|\xi - \xi'| = 1} \xi'' = (\xi, \xi') \]

\[ + \sum_{\xi, \xi'} C_{\Lambda_1}^{1,\text{loc}} \Phi'(\xi) C_{\Lambda_1}^{1,\text{loc}} \Phi'(\xi') \left( 1 - \cos [e_0 C_{\Lambda_1}^{1,\text{loc}} \Phi'(\xi'')] \right) \]

\[ 1 \xi \in \Omega_0, 1 \xi' \in \Omega_0, \xi - \xi' = 1 \xi'' = (\xi, \xi') \]

\[ \sum_{\xi} \left( \lambda (C_{\Lambda_1}^{1,\text{loc}} \Phi'(\xi))^4 + \sqrt{2} \lambda \mu (C_{\Lambda_1}^{1,\text{loc}} \Phi'(\xi))^3 - \log \left[ 1 + \frac{(C_{\Lambda_1}^{1,\text{loc}} \Phi'(\xi))^4}{\rho_0} \right] \right)_{\xi \in \Omega_0}. \]

(3.17)
Note that

\[ C_{\Lambda_1}^{\frac{1}{2}, \loc} \Phi'(\xi) = (C_{\Lambda_1}^{\frac{1}{2}, \loc} \Phi)(\xi) + (C_{\Lambda_1}^{\frac{1}{2}, \loc} \Psi_1)(\xi). \quad (3.18) \]

1. First rewrite

\[
\sum_{\xi} \left( \cos[e_0 C_{\Lambda_1}^{\frac{1}{2}, \loc} \Phi'(\xi)] - 1 \right) 1_{\xi \in \Omega_0^*} = \sum_{\xi} \sum_{l: \text{even}} \frac{(-1)^{l/2}}{l!} \frac{e_0 C_{\Lambda_1}^{\frac{1}{2}, \loc} \Phi'(\xi)}{l!} 1_{\xi \in \Omega_0^*} \\
= \sum_{\xi} \sum_{l: \text{even}} \frac{(-1)^{l/2}}{l!} \sum_{n,m} \frac{l!}{n!m!} \left( e_0 C_{\Lambda_1}^{\frac{1}{2}, \loc} \Phi(\xi) \right)^n \left( e_0 C_{\Lambda_1}^{\frac{1}{2}, \loc} \Psi_1(\xi) \right)^m 1_{\xi \in \Omega_0^*} \\
= \sum_{\xi} \sum_{n,m} e_0^{n+m} \frac{(-1)^{(n+m)/2}}{n!m!} \sum_{\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m} C_{\Lambda_1}^{\frac{1}{2}, \loc}(\xi, \xi_1) \Phi(\xi_1) \ldots C_{\Lambda_1}^{\frac{1}{2}, \loc}(\xi, \eta_1) \Psi(\eta_1) \ldots C_{\Lambda_1}^{\frac{1}{2}, \loc}(\xi, \eta_m) \Psi(\eta_m) \\
= \sum_{n,m} \sum_{\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m} a(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m) \Phi(\xi_1) \ldots \Phi(\xi_n) \Psi(\eta_1) \ldots \Psi(\eta_m),
\]

where

\[
a(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m) = e_0^{n+m} \frac{(-1)^{(n+m)/2}}{n!m!} \sum_{\xi} C_{\Lambda_1}^{\frac{1}{2}, \loc}(\xi, \xi_1) \ldots C_{\Lambda_1}^{\frac{1}{2}, \loc}(\xi, \xi_n) C_{\Lambda_1}^{\frac{1}{2}, \loc}(\xi, \eta_1) \ldots C_{\Lambda_1}^{\frac{1}{2}, \loc}(\xi, \eta_m) 1_{\xi \in \Omega_0^*} 1_{\xi_j \in \Lambda_1^*} 1_{\eta_j \in \Lambda_1^*}.
\]

Let \( \tilde{\xi}_j = \{\xi_j, \eta_j\} \). From Lemma 2.3, \( C_{\Lambda_1}^{\frac{1}{2}, \loc}(\xi, \tilde{\xi}_j) \leq ce^{-\frac{2}{3}d(\xi, \tilde{\xi}_j)} \), where \( d(\xi, \tilde{\xi}_j) \) denotes the infimum of the distance between the sites containing the bonds \( \xi, \tilde{\xi}_j \). Using

\[
\sum_{j} d(\xi, \tilde{\xi}_j) = \text{length of tree joining } (\xi, \xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m) \\
\geq t(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m)
\]

(3.21)
write
\[
|a(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m)|
\leq c \frac{e_0^{n+m}}{n!m!} \sum_{\xi} e^{-\frac{1}{8}d(\xi,\xi_1)-\cdots-\frac{1}{8}d(\xi,\xi_n)-\cdots-\frac{1}{8}d(\xi,\eta_m)}
\leq c \frac{e_0^{n+m}}{n!m!} e^{-\frac{1}{16}t(\xi_1,\ldots,\xi_n,\eta_1,\ldots,\eta_m)} \sum_{\xi} e^{-\frac{1}{16} \sum_j d(\xi,\tilde{\xi}_j)}
\leq c \frac{e_0^{n+m}}{n!m!} e^{-\frac{1}{16}t(\xi_1,\ldots,\xi_n,\eta_1,\ldots,\eta_m)}.
\] (3.22)

2. Similarly, rewrite
\[
\sum_\xi \lambda (C^{\frac{1}{2}}_{A_1})^n \Phi'(\xi) \Psi_1(\xi) 1_{\xi \in \Omega_0} = \sum_\xi \lambda [C^{\frac{1}{2}}_{A_1} \Phi(\xi) + C^{\frac{1}{2}}_{A_1} \Psi_1(\xi)]^n 1_{\xi \in \Omega_0}
\]
\[
= \sum_\xi \lambda \sum_{n,m=4}^{n+m} \frac{(n+m)!}{n!m!} (C^{\frac{1}{2}}_{A_1} \Phi(\xi))^n (C^{\frac{1}{2}}_{A_1} \Psi_1(\xi))^m 1_{\xi \in \Omega_0}
\]
\[
= \sum_\xi \lambda \sum_{n,m=4}^{n+m} \frac{(n+m)!}{n!m!} \sum_{\xi_1,\ldots,\xi_n,\eta_1,\ldots,\eta_m} C^{\frac{1}{2}}_{A_1}(\xi,\xi_1) \Phi(\xi_1) \cdots C^{\frac{1}{2}}_{A_1}(\xi,\xi_n) \Phi(\xi_n)
\]
\[
C^{\frac{1}{2}}_{A_1}(\xi,\eta_1) \Psi_1(\eta_1) \cdots C^{\frac{1}{2}}_{A_1}(\xi,\eta_{m}) \Psi_1(\eta_{m}) 1_{\xi \in \Omega_0} 1_{\xi_j \in A_1} 1_{\eta_j \in A_1}
\]
\[
= \sum_{n,m=4}^{n+m} \sum_{\xi_1,\ldots,\xi_n,\eta_1,\ldots,\eta_m} a(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m) \Phi(\xi_1) \cdots \Phi(\xi_n) \Psi_1(\eta_1) \cdots \Psi_1(\eta_{m}),
\]
\[
\text{where}
\]
\[
a(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m) = \lambda \frac{(n+m)!}{n!m!} \sum_\xi C^{\frac{1}{2}}_{A_1}(\xi,\xi_1) \cdots C^{\frac{1}{2}}_{A_1}(\xi,\xi_n)
\]
\[
C^{\frac{1}{2}}_{A_1}(\xi,\eta_1) \cdots C^{\frac{1}{2}}_{A_1}(\xi,\eta_{m}) 1_{\xi \in \Omega_0} 1_{\xi_j \in A_1} 1_{\eta_j \in A_1}.
\] (3.24)

Note that \( \lambda = O(e_0^2) \) by Eq. 1.24, using \( C^{\frac{1}{2}}_{A_1}(\xi,\tilde{\xi}_j) \leq ce^{-\frac{1}{8}d(\xi,\tilde{\xi}_j)} \) and \( \frac{(n+m)!}{n!m!} \leq 2^{n+m} = 16 \), from Eqs. 3.21 and 3.22,
\[
|a(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m)| \leq c (4e_0)^{\frac{n+m}{2}} e^{-\frac{1}{16}t(\xi_1,\ldots,\xi_n,\eta_1,\ldots,\eta_m)}.
\] (3.25)

The remaining terms in \( V_1(\Omega_0, \Phi, \Psi) \) can be treated similarly. This completes the proof of Theorem 3.1. \( \Box \)

**Norm of \( V_1 \)** Let \( w_{4t, \tilde{\kappa}}(\xi, \tilde{\eta}) = e^{nt(supp(\xi,\tilde{\eta}))}(4t)^n \tilde{\kappa}^m \) be the weight system with mass \( m \) giving weight at least \( 4t \) to \( \Phi \) and \( \tilde{\kappa} \) to \( \Psi_\alpha \). Let \( X \subset A_1 \) with \( X \cap \Omega_1 \neq \emptyset \). Define
Proof. Follows directly from Balaban et al. [2].

Then, the norm is defined as \( \|V_1\|_{w_{4r,\tilde{\kappa}}} = |a|_{w_{4r,\tilde{\kappa}}} \). Let \( 4r = p_0, \tilde{\kappa} = 1 \) and \( m < \gamma' \) (where \( \gamma' \) is the decay constant from Theorem 3.1). Then, from Theorem 3.1 \( \|V_1\|_{w_{4r,\tilde{\kappa}}} \).

\[
|a|_{w_{4r,\tilde{\kappa}}} = \sum_{n,m \geq 0} \max_{\xi \in X} \max_{1 \leq i \leq n} \sum_{\xi_i = \xi} e^{m(t(\text{supp}(\tilde{\xi},\tilde{\eta}))}(4r)^c_{\xi \gam\eta} |a(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m)|. \tag{3.26}
\]

Note that \( p_{0,\lambda} \leq e^{-n\epsilon}, (c_0 e_0) < 1 \) and \( (c_0 e_0)^{1-2\epsilon} < 1 \).

**Theorem 3.2.** Given a power series of \( V_1(\Omega_0, \Phi, \Psi) \) with coefficient system \( a(\xi, \tilde{\eta}) = e^{m(t(\text{supp}(\tilde{\eta})) \kappa^m(\tilde{\eta})} \) be the weight system of mass \( m \) giving weight \( \tilde{\kappa} \) to the field \( \Psi_\alpha \). Let \( Z = \text{supp}(\tilde{\eta}) \). Set \( \tilde{\kappa} = 1 \). Then, there exists a coefficient system \( b(\tilde{\eta}) \) having decay properties as \( a(\xi, \tilde{\eta}) \), with

\[
|b|_{w_{\tilde{\kappa}}} = \sum_{n \geq 0} \max_{\eta \in X} \max_{1 \leq i \leq n} \sum_{\eta_i = \eta} w_{\tilde{\kappa}}(\tilde{\eta})|b(\eta_1, \ldots, \eta_m)| \tag{3.28}
\]

and a function \( H(Z, \Psi) = \sum_{\text{supp}(\tilde{\eta}) = Z} \tilde{\eta} b(\tilde{\eta})\Psi_\alpha(\tilde{\eta}) \) with \( H(0) = b_0 \) and \( \ln \Xi(\Omega_1) \), \( \Psi = \sum_Z H(Z, \Psi) \). Define \( \|H\|_{w_{\tilde{\kappa}}} = |b|_{w_{\tilde{\kappa}}} \), then for \( e_0 \) sufficiently small

\[
\|H - H(0)\|_{w_{\tilde{\kappa}}} \leq \frac{\|V_1\|_{w_{4r,\tilde{\kappa}}} - 1}{16\|V_1\|_{w_{4r,\tilde{\kappa}}}} \leq c (c_0 e_0^{1-2\epsilon})^{\frac{1}{2}}. \tag{3.29}
\]

**Proof.** Follows directly from Balaban et al. [2].

Note that the series \( H(Z, \Psi) = \sum_{\text{supp}(\tilde{\eta}) = Z} \tilde{\eta} b(\tilde{\eta})\Psi_\alpha(\tilde{\eta}) \) converges in a unit disk in complex plane containing \( J \), and therefore, \( \sum_Z H(Z, \Psi) \) also converges. From Theorem 3.2,

\[
|H(Z)| \leq e^{-m(t(\text{supp}(\tilde{\eta}))} \sum_{n \geq 0} \max_{\eta \in X} \max_{1 \leq i \leq n} \sum_{\eta_i = \eta} w_{\tilde{\kappa}}(\tilde{\eta})|b(\eta_1, \ldots, \eta_m)| \leq \|H\|_{w_{\tilde{\kappa}}} e^{-m(Z)} \leq c (c_0 e_0^{1-2\epsilon})^{\frac{1}{2}} e^{-m(Z)}. \tag{3.30}
\]

So far we have assumed \( \Phi_Q = 0 \). Now, we drop this assumption.
**Proposition 3.3** (shift). Let $H$ be an analytic function. Let $\phi = -C_\Lambda T_{\Lambda_1} \tilde{\Phi} \tilde{Q}$ denote shift. Define $H^\#$ by

$$H^\#(\Psi, \phi) = H(\Psi + \phi) = \sum_{\vec{\eta} = (\eta_1, \ldots, \eta_{n+m}) \in X} b(\vec{\eta})(\Psi + \phi)(\vec{\eta}).$$

(3.31)

Let $w_{\vec{\kappa}, \vec{\kappa}_2}(\vec{\eta}) = e^{mt(supp(\vec{\eta}))} \tilde{\kappa}^n \tilde{\kappa}_2^m$ be the weight system of mass $m$ that gives weight $\tilde{\kappa}_2$ to $\phi$. Set $\tilde{\kappa} = 1$ and $\tilde{\kappa}_2 = p_\lambda$. Let $Z = supp(\vec{\eta})$. Then, $||H^\#||_{w_{\vec{\kappa}, \vec{\kappa}_2}} = ||H||_{w_{\vec{\kappa}, \vec{\kappa}_2}}$ and $|H(Z)| = |H^\#(Z)|$.

**Proof.** Follows directly from [2].

### 3.1. From $\vec{\eta}$ to Polymer

A union of $[r_\lambda]$ blocks $\square_i$ is called connected if any two blocks centered on nearest $[r_\lambda]$ neighbor sites have a common hypersurface, face, edge or a site. A polymer is a connected union of $[r_\lambda]$ blocks, $\square_i$. For a polymer $X$, let $|X|$ denote the number of $\square$ in $X$. Recall that $Z = supp(\vec{\eta})$. Define the set

$$X_Z = \{ \text{all } [r_\lambda] \text{ blocks, } \square \text{ containing } Z; X_Z \cap \Omega_1 \neq \emptyset \}.$$  

(3.32)

Let $X = \{X_i\} \subset \Omega_0$ be a collection of polymers. Rewrite,

$$\ln \Xi(\Omega_1, \Psi) = \sum_Z H^\#(Z, \Psi) = \sum_X \sum_{Z: X_Z = X} H^\#(Z, \Psi) = \sum_X H^\#(X, \Psi).$$

(3.33)

Rewrite $\Xi(\Omega_1, \Psi)$ using Mayer expansion as

$$\Xi(\Omega_1, \Psi) = e^{\sum_X H^\#(X, \Psi)} = \prod_X \left[ e^{H^\#(X, \Psi)} - 1 + 1 \right]$$

$$= \sum_{\{X_i\} \rightarrow Y} \prod_i \left[ e^{H^\#(X_i, \Psi)} - 1 \right]$$

$$= \sum_{\{Y_i\}, \text{disjoint}} \prod_i K(Y_i, \Psi),$$

(3.34)

where $\{Y_i\}$ are polymers and for a particular $Y_i$,

$$K(Y, \Psi) = \sum_{\cup X_i = Y} \prod_i \left[ e^{H^\#(X_i, \Psi)} - 1 \right].$$

(3.35)

**Lemma 3.4.** Given an analytic function $H^\#$ as defined earlier with $|H^\#(Z)| \leq c \left( c_0 e_0^{1-2\epsilon} \right)^{\frac{1}{2}} e^{-mt(Z)}$, for a polymer $X = X_Z$ containing $|X|$ blocks, there exists a constant $\kappa > 0$ such that $e^{-mt(Z)} \leq e^{-\kappa |X|}$ and thus

$$|H^\#(X)| \leq c \left( c_0 e_0^{1-2\epsilon} \right)^{\frac{1}{2}} e^{-\kappa |X|}.$$  

(3.36)

**Proof.** By definition, the polymer $X = X_Z$, therefore,

$$t(Z) \geq t(X) = (t(X) + 1) - 1$$

implies $e^{-mt(Z)} \leq e^{m(e^{-mt(X)} + 1)}$. Using the inequality (see for example [15]) $t(X) \leq |X| \leq c(t(X) + 1)$, it follows that $e^{-mt(Z)} \leq e^{m(e^{-\kappa |X|}}$, hence the result.
Lemma 3.5. Given a disjoint collection of polymers $Y = \{Y_i\}$ and $K(Y_i, \Psi)$ as defined above, for a new constant $\kappa'$, $|K(Y)| \leq c \left( c_0 \epsilon_0^{1-2\epsilon} \right)^{\frac{1}{2}} e^{-\kappa'|Y|}$.

Proof. First write the unordered collection $\{X_i\}$ as ordered sets $(X_1, X_2, \ldots, X_n)$. Then, for a particular $Y_i$,

$$K(Y, \Psi) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\cup X_i = Y} \prod_i e^{H^\#(X_i, \Psi)} - 1.$$  \hfill (3.37)

Now, using Lemma 3.4,

$$|e^{H^\#(X_i)} - 1| = \left| \int_0^1 \frac{d}{ds} e^{sH^\#(X_i)} ds \right| = \left| \int_0^1 H^\#(X_i) e^{sH^\#(X_i)} ds \right| \leq c \left( c_0 \epsilon_0^{1-2\epsilon} \right)^{\frac{1}{2}} e^{-\kappa|X_i|} \quad (3.38)$$

and

$$|K(Y)| \leq \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\cup X_i = Y} \prod_i c \left( c_0 \epsilon_0^{1-2\epsilon} \right)^{\frac{1}{2}} e^{-\kappa|X_i|} \leq c \left( c_0 \epsilon_0^{1-2\epsilon} \right)^{\frac{1}{2}} e^{-\frac{\kappa}{2}} |Y| \prod_{n=1}^{\infty} \left( \sum_{X_i \subseteq Y} \prod_{i=1}^{n} c \left( c_0 \epsilon_0^{1-2\epsilon} \right)^{\frac{1}{2}} e^{-\frac{\kappa}{2}} |X_i| \right)^n. \quad (3.39)$$

In second line, since $\{X_i\}$ overlap, $\sum_{X_i} |X_i| + \cdots + |X_n| \geq |Y|$ and we also extracted a factor of $c_0 \epsilon_0^{1-2\epsilon} \frac{1}{2}$ from $c_0 \epsilon_0^{1-2\epsilon} \frac{1}{2}$, leaving behind a factor of $c_0 \epsilon_0^{1-2\epsilon} \frac{1}{2}$. This inequality holds for $n \geq 2$. Next, we use [15, Lemma 25] (there exists a constant $b$, such that $\sum_{X \cap Y \neq \emptyset} e^{-a|X|} \leq b|Y|$) and rewrite the bracketed term as

$$\sum_{X \subseteq Y} c \left( c_0 \epsilon_0^{1-2\epsilon} \right)^{\frac{1}{2}} e^{-\frac{\kappa}{2}} |X| \leq c \left( c_0 \epsilon_0^{1-2\epsilon} \right)^{\frac{1}{2}} |Y|.$$

Thus,

$$|K(Y)| \leq c \left( c_0 \epsilon_0^{1-2\epsilon} \right)^{\frac{1}{2}} e^{-\frac{\kappa}{2}} |Y| \sum_{n=1}^{\infty} \frac{1}{n!} \left( c \left( c_0 \epsilon_0^{1-2\epsilon} \right)^{\frac{1}{2}} |Y| \right)^n \leq c \left( c_0 \epsilon_0^{1-2\epsilon} \right)^{\frac{1}{2}} e^{-\frac{\kappa}{2}} |Y| + c \left( c_0 \epsilon_0^{1-2\epsilon} \right)^{\frac{1}{2}} |Y| |Y| \leq c \left( c_0 \epsilon_0^{1-2\epsilon} \right)^{\frac{1}{2}} e^{-\kappa'|Y|}. \quad (3.40)$$

\hspace{1cm} $\square$

4. Large Field Region

There are two large field regions:

1. intermediate large field region $\mathcal{P}$ such that $\forall \square \in \mathcal{P}$, $\exists$ at least one $\xi \in \square$, with $p_{0,\lambda} < |\Phi(\xi)|$ and $\forall \xi \in \square, |\Phi(\xi)| < p_{\lambda}$,
2. large field region $Q$ such that $\forall \Box \in Q, \exists \xi \in \Box, \text{ with } |\Phi(\xi)| \geq p_\lambda$ and $\nu \neq 0$.

Recall that $\tilde{Q} = \Lambda_1^c$. Denote

$$\tilde{P} \equiv \Lambda_0 - \Omega_1.$$ 

Let $\{\tilde{P}_l\}$ and $\{\tilde{Q}_k\}$ denote connected components of $\tilde{P}$ and $\tilde{Q}$, respectively, that is, $\tilde{P}_l$ and $\tilde{Q}_k$ are polymers. Define $\cup_{l,i} P_{l,i} = P$ and $\cup_{k,i} Q_{k,i} = Q$ such that $\tilde{P}_l \supset P_l = \cup_{l,i} P_{l,i}$ and $\tilde{Q}_k \supset Q_k = \cup_{k,i} Q_{k,i}$. Here, $P_{l,i}$ and $Q_{k,i}$ are polymers in the regions $P$ and $Q$, respectively. Rewrite the large field quadratic part in the generating functional Eq. 2.30,

$$-\frac{1}{2} \langle \Phi, T_{\tilde{Q}} \Phi \rangle + \frac{1}{2} \langle \Phi, T_{\tilde{Q} \Lambda_1} C_{\Lambda_1} T_{\Lambda_1} \tilde{Q} \Phi \rangle$$

$$= -\frac{1}{2} \sum_i \langle \Phi, (T_{\tilde{Q}_i} - T_{\tilde{Q}_i} \Lambda_1 C_{\Lambda_1} T_{\Lambda_1} \tilde{Q}_i) \Phi \rangle + \frac{1}{2} \sum_{i,j} \langle \Phi, T_{\tilde{Q}_i \Lambda_1} C_{\Lambda_1} T_{\Lambda_1} \tilde{Q}_j \Phi \rangle.$$  

(4.1)

For the second term on the right-hand side, we construct polymers in the following manner. Consider two disjoint blocks $\Box$ and $\Box'$ such that $d(\Box, \Box') \geq r_\lambda$. Let $X$ be a polymer connecting $\Box$ and $\Box'$. For example, $X(\Box, \Box')$: all $d$-dimensional rectangular paths connecting $\Box$ and $\Box'$.

(By rectangular paths we mean starting at $\Box$ and selecting coordinate axis one at a time and traveling along it until the coordinate of $\Box'$ is reached.) Let $1_\Box$ and $1_{\Box'}$ denote the characteristic functions restricting operators to $\Box$ and $\Box'$, respectively. Then,

$$\frac{1}{2} \sum_{i,j} \langle \Phi, T_{\tilde{Q}_i \Lambda_1} C_{\Lambda_1} T_{\Lambda_1} \tilde{Q}_j \Phi \rangle = \frac{1}{2} \sum_\Box \sum_{i,j} \langle \Phi, 1_\Box T_{\tilde{Q}_i \Lambda_1} C_{\Lambda_1} T_{\Lambda_1} \tilde{Q}_j 1_{\Box'} \Phi \rangle$$

$$= \frac{1}{2} \sum_X \sum_{\Box, \Box' \rightarrow X} \sum_{i,j} \langle \Phi, 1_\Box T_{\tilde{Q}_i \Lambda_1} C_{\Lambda_1} T_{\Lambda_1} \tilde{Q}_j 1_{\Box'} \Phi \rangle$$

$$= \sum_X \sigma(X),$$  

(4.2)

where only disjoint $\Box, \Box'$ contribute and

$$\sigma(X) = \frac{1}{2} \sum_\Box \sum_{\Box' \rightarrow X} \sum_{i,j} \langle \Phi, 1_\Box T_{\tilde{Q}_i \Lambda_1} C_{\Lambda_1} T_{\Lambda_1} \tilde{Q}_j 1_{\Box'} \Phi \rangle.$$  

(4.3)

Thus, $e \sum_{i \neq j} \langle \Phi, T_{\tilde{Q}_i \Lambda_1} C_{\Lambda_1} T_{\Lambda_1} \tilde{Q}_j \Phi \rangle = e \sum_X \sigma(X)$. Using Mayer expansion, rewrite

$$e \sum_X \sigma(X) = \prod_X \left[ (e^{\sigma(X)} - 1) + 1 \right] = \sum_{\{X_i\} \text{ distinct}} \prod_i \left[ e^{\sigma(X_i)} - 1 \right]$$

$$= \sum_{\{X_i\} \text{ disjoint}} \prod_i f(X_i)$$  

(4.4)
where in the last step we group the \( \{X_i\} \) into connected sets and define for \( X \) connected,

\[
f(X) = \sum_{\cup X_i = X} \prod_i \left[ e^{\sigma(X_i)} - 1 \right]. \tag{4.5}
\]

Recall from Eq. 2.28 that \( Z_0(\Omega_1) = \int D\Phi' e^{-\|\Phi'\|^2_{P_1}}, \) and rewrite

\[
Z_0(\Omega_1) = Z_0(\Lambda) Z_0(\Omega_1^c)^{-1} = Z_0(\Lambda) Z_0(Q)^{-1} Z_0(\hat{P})^{-1}. \tag{4.6}
\]

Then, the generating functional is

\[
Z[J] = \left( \frac{2\pi}{e_0} \right)^{-|\Lambda|+1} (\det C^{\frac{1}{2}}) Z_0(\Lambda) \sum_{\nu : \nu \neq 0} \sum_{\{Q_k\}} \sum_{\{P_l\}} \sum_{\{Y_j\}} \prod_k e^{W_2(\tilde{Q}_k)} \\
\int \prod_k D\Phi_{\tilde{Q}_k} e^{-\frac{1}{2}(\Phi_{\tilde{Q}_k} - T_{\tilde{Q}_k} \Lambda_1 C_{\Lambda_1} T_{\Lambda_1} \tilde{Q}_k) \Phi) - V(\tilde{Q}_k, \Phi) \zeta_{\tilde{Q}_k} \chi_{\tilde{Q}_k} \chi_{Q_k \Lambda_1} (\Phi) \prod_k Z_0(\tilde{P}_l)^{-1} \prod_j K(Y_j, J). \tag{4.7}
\]

where \( \zeta_{P_l} = \prod_i \zeta_{P_l, i : \cup_i P_{l, i} \subset P_l} (\Phi') \) and \( \zeta_{Q_k} = \prod_i \zeta_{Q_k, i : \cup_i Q_{k, i} \subset Q_k} (\Phi) \). Denote

\[
\rho(\tilde{Q}_k, J) = e^{W_2(\tilde{Q}_k)} e^{-\frac{1}{2}(\Phi_{\tilde{Q}_k} - T_{\tilde{Q}_k} \Lambda_1 C_{\Lambda_1} T_{\Lambda_1} \tilde{Q}_k) \Phi) - V(\tilde{Q}_k, \Phi) \zeta_{\tilde{Q}_k} (\Phi) \chi_{\tilde{Q}_k} \chi_{Q_k \Lambda_1} (\Phi) Z_0(\tilde{Q}_k)^{-1} \tag{4.8}
\]

\[
\rho'(\tilde{P}_l, J) = e^{W_2(\tilde{P}_l)} e^{-\frac{1}{2}(\Phi'_{\tilde{P}_l} - T_{\tilde{P}_l} \Lambda_1 C_{\Lambda_1} \tilde{P}_l \Phi') - V(\tilde{P}_l, \Phi) \zeta_{\tilde{P}_l} (\Phi') \chi_{\tilde{P}_l} (\tilde{P}_l \Phi') Z_0(\tilde{P}_l)^{-1}
\]

and rewrite the generating functional as

\[
Z[J] = \left( \frac{2\pi}{e_0} \right)^{-|\Lambda|+1} (\det C^{\frac{1}{2}}) Z_0(\Lambda) \sum_{\nu : \nu \neq 0} \sum_{\{Q_k\}} \sum_{\{P_l\}} \sum_{\{Y_j\}} \prod_k \int D\Phi_{\tilde{Q}_k} \rho(\tilde{Q}_k, J) \prod_i f(X_i, J) \prod_l \int D\Phi'_{\tilde{P}_l} \rho'(\tilde{P}_l, J) \prod_j K(Y_j, J). \tag{4.9}
\]

**Lemma 4.1.** \(|f(X, J)| \leq c e_0 e^{-\gamma'|X|}, \) for some \( c, \gamma' > 0 \).

**Proof.** Let \( \xi \in \Box \) and \( \xi' \in \Box' \). Define a metric \( d(\xi, \xi') = \sup_\mu |\xi_\mu - \xi'_\mu| \) and \( d(\Box, \Box') = \inf_{\xi, \xi'} d(\xi, \xi') \). Then, \( |C(\xi, \xi')| \leq c e^{-\gamma d(\xi, \xi')} \leq c e^{-\gamma d(\Box, \Box')} \). Let \( X \) is a polymer joining \( \Box, \Box' \). First, we estimate \( \sigma(X) \) (Eq. 4.3). For a given \( X \), the number of \( (\Box, \Box') \) with \( \Box \in \tilde{Q}_i \) and \( \Box' \in \tilde{Q}_j \) is bounded by \( \mathcal{O}(1) \) depending only on the dimension. The number of \( (\tilde{Q}_i, \tilde{Q}_j) \) is bounded by \( \mathcal{O}(|X|) \). Now,
there are $d!$ ways to construct $X$. Thus, $d(\square, \square') \geq \frac{|X|}{d!}$. Also, as $\square, \square'$ are disjoint $d(\square, \square') \geq [r_\lambda]$. Note that $\Phi$ is at most $p_\lambda$ near the boundary $\partial \Lambda_1$ since $|\Phi(\xi)| \leq p_\lambda; \forall \xi \in \Lambda_0$. From the definition, Eq. 4.3 and using $e^{-\frac{x}{2}[r_\lambda]} = O(e_0^2)$, it follows
\[
|\sigma(X)| \leq \frac{1}{2} O(|X|) p_\lambda^2 e^{-\frac{x}{2}d(\square, \square')} e^{-\frac{x}{2}[r_\lambda]}
\leq \frac{1}{2} e_0^2 |X| p_\lambda^2 e^{-\frac{x}{2}[r_\lambda]} \tag{4.10}
\leq c e_0 e^{-\frac{x}{2}[r_\lambda]}|X|
\]
and following Eqs. 3.39 and 3.40 for some $\gamma' < \frac{x}{2d!}$,
\[
|f(X, J)| \leq c e_0 e^{-\gamma'|X|}. \tag{4.11}
\]

**Lemma 4.2.** Let $m_{\text{min}} = \min(\mu^2, m^2_\Lambda)$. Then, for $m_{\text{min}} \geq 2\frac{d}{2}+1$ and some constant $\gamma_1 > 0$, the large field action is bounded below as
\[
\langle \Phi, (T_{\bar{Q}_k} - T_{\bar{Q}_k \Lambda_1} C_{\Lambda_1} T_{\Lambda_1 \bar{Q}_k}) \Phi \rangle + V(\bar{Q}_k, \Phi) \geq \gamma_1 \langle \Phi, T\Phi \rangle_{\bar{Q}_k} + c \|\Phi\|^2_{\bar{Q}_k}. \tag{4.12}
\]

**Proof.** Rewrite
\[
\langle \Phi, (T_{\bar{Q}_k} - T_{\bar{Q}_k \Lambda_1} C_{\Lambda_1} T_{\Lambda_1 \bar{Q}_k}) \Phi \rangle + V(\bar{Q}_k, \Phi)
= \frac{1}{2} \langle \Phi, (T_{\bar{Q}_k} - 2 T_{\bar{Q}_k \Lambda_1} C_{\Lambda_1} T_{\Lambda_1 \bar{Q}_k}) \Phi \rangle + \frac{1}{2} \langle \Phi, T\Phi \rangle_{\bar{Q}_k} + V(\bar{Q}_k, \Phi). \tag{4.13}
\]
Using the positivity of $-\Delta$ and $\delta d$, $\langle \Phi, T\bar{Q}_k \Phi \rangle \geq m_{\text{min}} \|\Phi\|^2_{\bar{Q}_k}$ and from the definition of $T = C^{-1}$ (Eq. 2.8), note that $\|C_{\Lambda_1}\|_{\infty} \leq m_{\text{min}}^{-1}$. Due to locality in $T_{\bar{Q}_k \Lambda_1}$, there is no mass term in $T_{\bar{Q}_k \Lambda_1}$; therefore, $\langle \Phi, T_{\bar{Q}_k \Lambda_1} C_{\Lambda_1} T_{\Lambda_1 \bar{Q}_k} \Phi \rangle \leq m_{\text{min}}^{-1} \|T_{\bar{Q}_k \Lambda_1} \Phi\|^2 \leq m_{\text{min}}^{-1} (\sum_b |\partial \rho(b)|^2 + \sum_p |dA(p)|^2)$. Note that
\[
\sum_b |\partial \rho(b)|^2 = \sum_b \left| \sum_{x \in \partial b} \rho(x) \sum_{x' \in \partial b} \rho(x') \right| \leq \sum_b \sum_{x \in \partial b} \sum_{x' \in \partial b} \frac{1}{2} (\rho(x)^2 + \rho(x')^2) \tag{4.14}
\leq \sum_b \sum_{x \in \partial b} 2|\rho(x)|^2 \times 2 \leq 2 \sum_{x \in \partial b} |\rho(x)|^2 \leq 2^{d+1} \|\rho\|^2.
\]
Similarly,
\[
\sum_p |dA(p)|^2 \leq \sum_p \sum_{b \in \partial p} \sum_{b' \in \partial p} A(b) A(b') \leq \sum_p \sum_{b \in \partial p} 4|A(b)|^2 \times 2
\leq 4 \sum_{b \in \partial b} \sum_{p: \partial p \in b \text{ oriented}} |A(b)|^2 \leq 2^{d+1} \|A\|^2. \tag{4.15}
\]
Therefore,
\[
\langle \Phi, (T_{\tilde{Q}_k} - 2T_{\tilde{Q}_k} \lambda_1 C_{\lambda_1} T_{\lambda_1} \tilde{Q}_k) \Phi \rangle \geq \left( m_{\text{min}} - \frac{2(2d+1)}{m_{\text{min}}} \right) ||\Phi||^2_{\tilde{Q}_k} - 2(m_{\text{min}} - 2d+2) ||\Phi||^2_{\tilde{Q}_k}.
\]

(4.16)

The constant on the right is positive for \( m_{\text{min}} \geq 2^{\frac{d}{2}+1} \). And the result then follows from the stability Lemma 11.1 in [1]. □

**Proposition 4.3.** Let \( \tilde{Q} = \{ \tilde{Q}_k \} \) with \( \tilde{Q}_k \supset Q_k \) be polymers as defined and the function \( \rho(\tilde{Q}_k, J) \) be as defined in Eq. 4.8. Then,

\[
\left| \sum_{v \text{ on } \tilde{Q}_k, dv=0} \rho(\tilde{Q}_k, J) \right| \leq c e_0^2 e^{-\frac{3}{2}p_\lambda|Q_k|} e^{-(m_{\text{min}}\gamma_1 - p_\lambda^{-1} + c)} ||\Phi||^2_{\tilde{Q}_k}.
\]

(4.17)

**Proof.** Note that for \( \square \in Q_k \), \( \Phi(\xi) > p_\lambda \), for at least one \( \xi \in \square \) and so

\[
\zeta(\square) \leq e^{-p_\lambda + p_\lambda^{-1} ||\Phi||^2_{\tilde{Q}_k}}.
\]

(4.18)

Therefore, for \( \bigcup_i Q_k, i = Q_k \subset \tilde{Q}_k \),

\[
\zeta(Q_k) = \prod_i \prod_{\boxempty \in Q_k, i} \tilde{\zeta}(\square)
\]

\[
\leq e^{-p_\lambda \sum_i |Q_k, i| + p_\lambda^{-1} \sum_i ||\Phi||^2_{\tilde{Q}_k, i}} \leq e^{-p_\lambda |Q_k| + p_\lambda^{-1} ||\Phi||^2_{\tilde{Q}_k}}.
\]

(4.19)

From Lemma 2.7, \( W_2(\square) \leq O([r_\lambda]^d) \). Note that \( |Z_0(Q_k)^{-1}| = \pi^{-\frac{|Q_k|}{2}} \leq e^{-\frac{d}{2}|Q_k|} \). Thus,

\[
e^{W_2(Q_k)} Z_0(Q_k)^{-1} \zeta Q_k(\Phi) \chi_{\tilde{Q}_k \setminus Q_k}(\Phi)
\]

\[
\leq e^{O([r_\lambda]^d) |\tilde{Q}_k|} e^{-\frac{d}{2}|Q_k|} e^{-p_\lambda |Q_k| + p_\lambda^{-1} ||\Phi||^2_{\tilde{Q}_k}}
\]

\[
\leq e^{3d (O([r_\lambda]^d) - \frac{d}{2}) |Q_k|} e^{-p_\lambda |Q_k| + p_\lambda^{-1} ||\Phi||^2_{\tilde{Q}_k}}
\]

(4.20)

where we have used the fact that \( \Lambda_{\zeta}^\ast \leq 3^d |\Lambda_{\zeta}^\ast| \) and therefore \( |\tilde{Q}_k| \leq 3^d |Q_k| \) and \( ||\Phi||^2_{\tilde{Q}_k} \leq ||\Phi||^2_{Q_k} \). Now, using Lemma 4.2,

\[
\left| \sum_{v \text{ on } \tilde{Q}_k, dv=0} \rho(\tilde{Q}_k, J) \right| \leq e^{3d (O([r_\lambda]^d) - \frac{d}{2}) |Q_k|} e^{-p_\lambda |Q_k|} \sum_{v \text{ on } Q_k} e^{-\frac{d}{2}\gamma_1 (\langle \Phi, T\Phi \rangle_{Q_k} - (dA + v,J) + (p_\lambda^{-1} - c)) ||\Phi||^2_{\tilde{Q}_k}}.
\]

(4.21)
The bound of $|J| < \alpha$, where $\alpha < 1$, implies $e^{-\langle dA+v,J \rangle} \leq e^{\sum_p \alpha|dA+v(p)|}$. From the definition, $\langle \Phi, T\Phi \rangle = (dA + v)^2(p) + m_A^2|A(b)|^2 + O(\rho^2)$. Rewrite

$$
\sum_{p \in Q_k} \rho(\tilde{Q}_k, J) \leq e^{3d(O(r_\lambda)^d) - \frac{1}{2}|Q_k|} e^{-p_\lambda|Q_k|} e^{-\frac{1}{2}} \gamma_1 (\Phi, T\Phi) \tilde{Q}_k + \sum_{p} \alpha|dA+v(p)| + (p^{-1} - c)||\Phi||^2_{Q_k}
$$

$$
\leq e^{3d(O(r_\lambda)^d) - \frac{1}{2}|Q_k|} e^{-p_\lambda|Q_k|} e^{-\frac{1}{2}} \gamma_1 |dA(p) + v(p)|^2 + \alpha|dA(p) + v(p)|
$$

using $v(p) \in \frac{2\pi}{v_0} Z$ and $|dA| < \frac{4\pi}{v_0}$, only writing the terms containing $v$,

$$
\sum_{p \in \tilde{Q}_k} \sum_{v(p)} e^{\sum_p \in Q_k} - \frac{1}{2} \gamma_1 |dA(p) + v(p)|^2 + \alpha|dA(p) + v(p)|) = \prod_{p \in \tilde{Q}_k} \sum_{v(p)} e^{-\frac{1}{2}} \gamma_1 |dA(p) + v(p)|^2 + \alpha|dA(p) + v(p)|)
$$

$$
\leq \prod_{p \in \tilde{Q}_k} 5 + \sum_{|v| \geq \frac{\alpha}{\gamma_1}} e^{-\frac{1}{2}} \gamma_1 |dA(p) + v(p)|^2 + \alpha|dA(p) + v(p)|)
$$

$$
\leq \prod_{p \in \tilde{Q}_k} 5 + \sum_{|v| \geq \frac{\alpha}{\gamma_1}} e^{-\frac{1}{2}} \gamma_1 |dA(p) + v(p)|^2 + \alpha^2
$$

(4.23)

Since $|dA + v| > \frac{2\pi}{v_0} = \frac{v}{3}$,

$$
\sum_{v(p), p \in \tilde{Q}_k} e^{\sum_p \in Q_k} - \frac{1}{2} \gamma_1 |dA(p) + v(p)|^2 + \alpha|dA(p) + v(p)|) \leq \prod_{p \in \tilde{Q}_k} 5 + e^{\frac{\alpha^2}{\gamma_1}} \sum_{|v| \geq \frac{\alpha}{\gamma_1}} e^{-\frac{1}{4}} \gamma_1 v^2 \leq \prod_{p \in \tilde{Q}_k} c = e^{\ln c |\tilde{Q}_k|} \leq e^{3d|Q_k|}.
$$

(4.24)

Note that $\sum_{b \in \tilde{Q}_k} \gamma_1 m_A^2|A(b)|^2 + \gamma_1 O(\rho^2) \geq m_{\min} \gamma_1 ||\Phi||^2_{Q_k}$. Also note that $|\tilde{Q}_k|$ is getting canceled by $|Q_k|$ since we can assume that $3d(O(r_\lambda)^d) - \frac{1}{2} + 3d\ln c < \frac{\delta_\lambda}{8}$. Thus,
Lemma 4.4. In region \( \{ \tilde{P}_l \} \), the following relation holds
\[
e^{-V(\tilde{P}_l, C_{A_1}^{1, \text{loc}} \Phi') - C_{A_1} T_{A_1} \Phi_0)} \leq e^{e c_{0}^{1-4e} |\tilde{P}_l|}.
\] (4.26)

Proof. From Lemma 2.3, \( |C_{A_1}^{1, \text{loc}} \Phi'| \leq c \| \Phi' \|_{P_l, \infty} \leq c p_\lambda \). Also, near the boundary \( \partial A_1 \), \( |\Phi_0| \leq p_\lambda \); therefore, \( |C_{A_1} T_{A_1} \Phi_0| \leq \mathcal{O}(1) p_\lambda \). Next note that from Eq. 3.17, every term in \( V \) carries a factor of at least \( e_0 \). We can assume that for some \( 0 < \varepsilon \ll 1 \), \( p_\lambda \leq e_0^{-\varepsilon} \), and thus, \( |V(\tilde{P}_l, C_{A_1}^{1, \text{loc}} \Phi' - C_{A_1} T_{A_1} \Phi_0)| \leq c e_0^{-1-4\varepsilon} |\tilde{P}_l| \). The result follows using \( |e^{-V}| \leq |V| \).

\( \square \)

Proposition 4.5. Let \( \tilde{P} = \{ \tilde{P}_l \} \) be a collection of polymers and the function \( \rho'(\tilde{P}_l, J) \) be as defined in Eq. 4.8. Then,
\[
|\rho'(\tilde{P}_l, J)| \leq c e_0^{2} e^{-\frac{3}{2} p_0, \lambda |\tilde{P}_l| e^{-\frac{1}{2} - p_0, \lambda}} \| \Phi' \|^2_{\tilde{P}_l}.
\] (4.27)

Proof. From Lemma 4.4,
\[
e^{-\frac{1}{2} \| \Phi' \|^2_{\tilde{P}_l} - V(\tilde{P}_l, C_{A_1}^{1, \text{loc}} \Phi' - C_{A_1} T_{A_1} \Phi_0) \leq e^{e c_{0}^{1-4e} |\tilde{P}_l| e^{-\frac{1}{2} \| \Phi' \|^2_{\tilde{P}_l}}.\] (4.28)

Note that in \( P \), \( |\Phi(\xi)| \geq p_0, \lambda, \) for some \( \xi \in \square \) and so
\[
\hat{\zeta}(\square) \leq e^{-p_0, \lambda + p_0, \lambda} |\Phi'|_{\tilde{P}_l}^2.
\] (4.29)

Therefore,
\[
\hat{\zeta}(\tilde{P}_l) = \prod_{\square \in \tilde{P}_l} \hat{\zeta}(\square) \leq e^{-p_0, \lambda |\tilde{P}_l| + p_0, \lambda} |\Phi'|_{\tilde{P}_l}^2.
\] (4.30)

Note that \( |\tilde{P}_l| \leq 2 (3^d)|P| \) and \( |\Phi'|_{\tilde{P}_l}^2 \leq |\Phi'|_{\tilde{P}_l}^2 \). From Lemma 2.7, \( W_2(\square) \leq \mathcal{O}([r_{\lambda}]^d) \). We can assume by making \( p_0, \lambda \) big enough relative to \( r_\lambda \) that, \( [r_\lambda]^d + c e_0^{1-4\varepsilon} < \frac{p_0, \lambda}{8} \). Note that \( |Z_0(\tilde{P}_l)|^{-1} = \pi^{-\frac{|\tilde{P}_l|}{2}} \leq e^{-\frac{1}{2} |\tilde{P}_l|} \) and \( \lambda_{\tilde{P}_l} (C_{A_1}^{1, \text{loc}} \Phi' - C_{A_1} T_{A_1} \Phi_0) \leq 1 \). Thus,
\[
|\rho'(\tilde{P}_l, J)| \leq c e^{2 (3^d) (\mathcal{O}([r_{\lambda}]^d) - \frac{1}{2} + (c_0 e_0^{1-2\varepsilon}) \frac{1}{2}) |\tilde{P}_l| e^{-p_0, \lambda |\tilde{P}_l|} e^{-\frac{1}{2} |\Phi'|_{\tilde{P}_l}^2 + p_0, \lambda} |\Phi'|_{\tilde{P}_l}^2 \leq c e^{-\frac{3}{2} p_0, \lambda |\tilde{P}_l| e^{-\frac{1}{2} - p_0, \lambda}} |\Phi'|_{\tilde{P}_l}^2 \leq c e_0^{2} e^{-\frac{3}{2} p_0, \lambda |\tilde{P}_l| e^{-\frac{1}{2} - p_0, \lambda}} |\Phi'|_{\tilde{P}_l}^2,
\] (4.31)

where we have used \( e^{-\frac{p_0, \lambda}{2}} = \mathcal{O}(e_0^0) \).

\( \square \)
5. Convergence

Lattice Λ is composed of disjoint polymers \{Y_j\} of Ω_1, \{\tilde{P}_l\} of Λ_0 − Ω_1, \{X_i\} and \{\tilde{Q}_k\} of Λ_1. These connected components from various regions overlap near the boundaries of those regions. We combine these overlapping parts into connected components in two steps.

1. In small field region Λ_1, define connected components \{Z_m\}, such that any \(Z \in \{Z_m\}\),

\[ Z = \bigcup_j \{Y_j\} \cup_l \{\tilde{P}_l\} \tag{5.1} \]

where \(Y_j\) and \(\tilde{P}_l\) overlap and rewrite

\[ \sum \sum \prod_l \int \mathcal{D}\Phi'_{\tilde{P}_l} \rho'(\tilde{P}_l, J) \prod_j K(Y_j, J) \]

\[ = \sum \sum \prod_l \int \mathcal{D}\Phi'_{\tilde{P}_l} \rho'(\tilde{P}_l, J) \prod_j K(Y_j, J) \]

\[ = \sum \prod_m K'(Z_m, J) \tag{5.2} \]

where for connected \(Z\)

\[ K'(Z, J) = \sum_{\cup_j \{Y_j\} \cup_l \{\tilde{P}_l\} \rightarrow Z} \prod_l \int \mathcal{D}\Phi'_{\tilde{P}_l} \rho'(\tilde{P}_l, J) \prod_j K(Y_j, J). \tag{5.3} \]

Rewrite the generating functional

\[ Z[J] = \left(\frac{2\pi}{\epsilon_0}\right)^{-|\Lambda|+1} (\text{det} \ C^\dagger) Z_0(\Lambda) \sum \sum \sum \sum \]

\[ \prod_k \int \mathcal{D}\Phi_{\tilde{Q}_k} \rho(\tilde{Q}_k, J) \prod_i f(X_i, J) \prod_m K'(Z_m, J). \tag{5.4} \]

2. In entire lattice Λ, define connected components \{C_l\}, such that any \(C \in \{C_l\}\),

\[ C = \cup_i \{X_i\} \cup_k \{\tilde{Q}_k\} \cup_m \{Z_m\} \tag{5.5} \]

where \(X_i\), \(\tilde{Q}_k\) and \(Z_m\) overlap and rewrite

\[ \sum \sum \sum \prod_k \int \mathcal{D}\Phi_{\tilde{Q}_k} \sum v \rho(\tilde{Q}_k, J) \prod_i f(X_i, J) \prod_m K'(Z_m, J) \]

\[ = \sum \sum \prod_k \int \mathcal{D}\Phi_{\tilde{Q}_k} \sum v \rho(\tilde{Q}_k, J) \prod_i f(X_i, J) \prod_m K'(Z_m, J) \]

\[ = \sum \prod_l K'(C_l, J), \tag{5.6} \]
where for any connected $\mathcal{C}$,

\[
K^\#(\mathcal{C}, J) = \sum_{\cup_i \{X_i\} \cup_k \{\bar{Q}_k\} \cup_m \{Z_m\} \rightarrow \mathcal{C}} \prod_i f(X_i, J) \prod_m K'(Z_m, J). 
\]

(5.7)

Rewrite the generating functional

\[
Z[J] = \left( \frac{2\pi}{e_0} \right)^{-|\Lambda|+1} (\det C^\frac{1}{2}) Z_0(\Lambda) \sum_{\{\mathcal{C}_l\}} \prod_l K^\#(\mathcal{C}_l, J). 
\]

(5.8)

**Lemma 5.1.** There exists constant $\kappa_1 > 0$ such that

\[
|K'(Z)| \leq (c_0e_0^{5-2\epsilon})^{\frac{1}{2}} e^{-\kappa_1 |Z|}.
\]

**Proof.** From the definition,

\[
|K'(Z, J)| \leq \sum_{\cup_j \{Y_j\} \cup_l \{\bar{P}_l\} = Z} \prod_l \left| \int \mathcal{D}\Phi'_{\bar{P}_l} \rho'_{\bar{P}_l}(J) \right| \prod_j |K(Y_j, J)|
\]

\[
\leq \sum_{\cup_j \{Y_j\} \cup_l \{\bar{P}_l\} = Z} \prod_j \left( c_0e_0^{1-2\epsilon} \right)^{\frac{1}{2}} e^{-\kappa'|Y_j|} \prod_l e^{2\epsilon} e^{-\frac{\epsilon}{2} p_{0,\lambda} |P_l|} (5.9)
\]

\[
\int \mathcal{D}\Phi'_{\bar{P}_l} e^{-\left(\frac{1}{2} - p_{0,\lambda}^{-1}\right)||\Phi'||^2_{\bar{P}_l}}.
\]

Since $\{Y_j\}$ and $\{\bar{P}_l\}$ overlap, $|Y_j| + |\bar{P}_l| \geq |Z|$, we can extract a factor of $e^{-\min(\kappa', \frac{\epsilon}{2} p_{0,\lambda}^{-1}) |Z|}$. We also extract a factor of $(c_0e_0^{1-2\epsilon})^{\frac{1}{2}}$ from the first term, $e_0^2$ from the second term just like Eq. 3.39 (Lemma 3.5) and do the integration of the last term,

\[
|K'(Z, J)| \leq (c_0e_0^{5-2\epsilon})^{\frac{1}{2}} e^{-\frac{\epsilon}{2} |Z|} \sum_{\cup_j \{Y_j\} \cup_l \{\bar{P}_l\} \subset Z} \prod_j \left( c_0e_0^{1-2\epsilon} \right)^{\frac{1}{2}} e^{-\frac{\epsilon}{2} |Y_j|}
\]

\[
\prod_l e^{2\epsilon} e^{-\frac{\epsilon}{2} p_{0,\lambda} |P_l|} \int \mathcal{D}\Phi'_{\bar{P}_l} e^{-\left(\frac{1}{2} - p_{0,\lambda}^{-1}\right)||\Phi'||^2_{\bar{P}_l}}
\]

\[
\leq (c_0e_0^{5-2\epsilon})^{\frac{1}{2}} e^{-\frac{\epsilon}{2} |Z|} \left( \frac{\pi}{1 - p_{0,\lambda}^{-1}} \right)^{|\overline{Z}|} \frac{|Z|}{\pi^2}
\]

\[
\left[ \sum_{\cup_j \{Y_j\} \subset Z} \prod_j \left( c_0e_0^{1-2\epsilon} \right)^{\frac{1}{2}} e^{-\frac{\epsilon}{2} |Y_j|} \right] \left[ \sum_{\cup_l \{P_l\} \subset Z} \prod_l e^{2\epsilon} e^{-\frac{\epsilon}{2} p_{0,\lambda} |P_l|} \right].
\]

(5.10)
Rewrite \( \{Y_j\} \) and \( \{P_l\} \) as ordered collection \((Y_1, \ldots, Y_n)\) and \((P_1, \ldots, P_m)\),

\[
|K'(Z, J)| \leq (c_0 e_0^{5 - 2\varepsilon})^{\frac{1}{2}} e^{-(\frac{\kappa'}{2} - 1)|Z|} \\
\left[ \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(Y_1, \ldots, Y_n) \subset Z} \prod_{j=1}^{n} (c_0 e_0^{1 - 2\varepsilon})^{\frac{1}{2}} e^{-\frac{\kappa'}{2}|Y_j|} \right] \\
\left[ \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{(P_1, \ldots, P_m) \subset Z} \prod_{l=1}^{m} c e_0 e^{-\frac{2}{\pi} p_0, \lambda |P_l|} \right] \\
\leq (c_0 e_0^{5 - 2\varepsilon})^{\frac{1}{2}} e^{-(\frac{\kappa'}{2} - 1)|Z|} \\
\left[ \sum_{n=1}^{\infty} \frac{1}{n!} \left( (c_0 e_0^{1 - 2\varepsilon})^{\frac{1}{2}} \sum_{Y \subset Z} e^{-\frac{\kappa'}{2}|Y|} \right)^n \right] \\
\left[ \sum_{m=1}^{\infty} \frac{1}{m!} \left( c e_0 \sum_{P \subset Z} e^{-\frac{2}{\pi} p_0, \lambda |P|} \right)^m \right]. \tag{5.11}
\]

Following [15, Lemma 25] (there exists a constant \(b\), such that \(\sum_{X \cap Y \neq \emptyset} e^{-a|X|} \leq b|Y|\)), the first bracketed term is bounded by \(e^{c(c_0 e_0^{1 - 2\varepsilon})^{\frac{1}{2}}|Z|}\) and the second bracketed term is bounded by \(e^{c e_0|Z|}\). Then, for some \(\kappa_1 < \frac{\kappa'}{2} - 1\),

\[
|K'(Z, J)| \leq (c_0 e_0^{5 - 2\varepsilon})^{\frac{1}{2}} e^{-\kappa_1|Z|}. \tag{5.12}
\]

\[\square\]

**Lemma 5.2.** There exist constants \(\beta > 0\) and \(\kappa'_2 > 0\) such that

\[
|K^\#(C, J)| \leq c e_0^\beta e^{-\kappa'_2|C|}.
\]

**Proof.** From the definition,

\[
|K^\#(C, J)| \leq \sum_{\cup_i \{X_i\} \cup_k \{Q_k\} \cup_m \{Z_m\} = C} \prod_i |f(X_i, J)| \prod_m |K'(Z_m, J)| \\
\prod_k \left| \int \mathcal{D} \Phi_{Q_k} \sum_{\rho \text{ on } \tilde{Q}_k : d \nu = 0} \rho(\tilde{Q}_k, J) \right| \\
\leq \sum_{\cup_i \{X_i\} \cup_m \{Z_m\} \cup_k \{Q_k\} = C} \prod_i c e_0 e^{-\gamma|X_i|} \prod_m (c_0 e_0^{5 - 2\varepsilon})^{\frac{1}{2}} e^{-\kappa_1|Z_m|} \\
\prod_k c e_0^2 e^{-\frac{2}{\pi} p_0|Q_k|} \int \mathcal{D} \Phi_{Q_k} e^{-(m_{\min} \gamma_1 - p_\lambda^{-1} + c)|\Phi'|^2} \tilde{Q}_k. \tag{5.13}
\]

Since \(\{X_i\}, \{\tilde{Q}_k\}, \{Z_m\}\) overlap, \(|X_i| + |\tilde{Q}_k| + |Z_m| \geq |C|\), we can extract a factor of \(e^{-\min(\gamma, \kappa_1, \frac{2}{\pi} p_\lambda)|C|}\). We also extract a factor of \(e_0\) from the first term, \((c_0 e_0^{5 - 2\varepsilon})^{\frac{1}{2}}\) from the second term, \(c_0^2\) from the third term just like Eq. 3.39 (Lemma 3.5) and do the integration of the last term,
\[
|K^\#(\mathcal{C}, J)| \leq (c_0 e_0^{11-2\epsilon})^{\frac{1}{2}} e^{-\frac{5\epsilon}{2}|\mathcal{C}|} \sum_{\cup_i \{X_i\} \cup_m \{Z_m\} \cup_k \{\mathcal{Q}_k\} \subset \mathcal{C}} \prod_i c e_0^{\frac{1}{2}} e^{-\frac{\gamma_i}{2} |X_i|} \\
\prod_m (c_0 e_0^{5-2\epsilon})^{\frac{1}{2}} e^{-\frac{5\epsilon}{2} |Z_m|} \prod_k c e_0 e^{-\frac{\pi p_k}{\mathcal{C}}} |Q_k| \int \mathcal{D}\Phi_{\mathcal{Q}_k} e^{-(m_{\min} \gamma_1 - p_{\lambda}^{-1}) \|\Phi\|_{\mathcal{Q}_k}^2} \\
\leq (c_0 e_0^{11-2\epsilon})^{\frac{1}{2}} e^{-\frac{5\epsilon}{2}|\mathcal{C}|} \left( \frac{\pi}{m_{\min} \gamma_1 - p_{\lambda}^{-1}} \right)^{\frac{|\mathcal{C}|}{2}} \\
\left[ \sum_{\cup_i \{X_i\} \subset \mathcal{C}} \prod_i c e_0^{\frac{1}{2}} e^{-\frac{\gamma_i}{2} |X_i|} \right] \left[ \sum_{\cup_m \{Z_m\} \subset \mathcal{C}} \prod_m (c_0 e_0^{5-2\epsilon})^{\frac{1}{2}} e^{-\frac{5\epsilon}{2} |Z_m|} \right] \\
\left[ \sum_{\cup_k \{\mathcal{Q}_k\} \subset \mathcal{C}} \prod_k c e_0 e^{-\frac{\pi p_k}{\mathcal{C}}} |Q_k| \right]. \quad (5.14)
\]

Rewrite \(\{X_i\}, \{Z_m\}, \{\mathcal{Q}_k\}\) as an ordered collection, \((X_1, \ldots, X_{n_1}), (Z_1, \ldots, Z_{n_2})\) and \((Q_1, \ldots, Q_{n_3})\), then

\[
|K^\#(\mathcal{C}, J)| \leq (c_0 e_0^{11-2\epsilon})^{\frac{1}{2}} e^{-\frac{5\epsilon}{2}|\mathcal{C}|} \left( \frac{\pi}{m_{\min} \gamma_1 - p_{\lambda}^{-1}} \right)^{\frac{|\mathcal{C}|}{2}} \\
\left[ \sum_{n_1=1}^{\infty} \frac{1}{n_1!} \sum_{(X_1, \ldots, X_{n_1}) \subset \mathcal{C}} \prod_{i=1}^{n_1} c e_0^{\frac{1}{2}} e^{-\frac{\gamma_i}{2} |X_i|} \right] \\
\left[ \sum_{n_2=1}^{\infty} \frac{1}{n_2!} \sum_{(Z_1, \ldots, Z_{n_2}) \subset \mathcal{C}} \prod_{m=1}^{n_2} (c_0 e_0^{5-2\epsilon})^{\frac{1}{2}} e^{-\frac{5\epsilon}{2} |Z_m|} \right] \\
\left[ \sum_{n_3=1}^{\infty} \frac{1}{n_3!} \sum_{(Q_1, \ldots, Q_{n_3}) \subset \mathcal{C}} \prod_{k=1}^{n_3} c e_0 e^{-\frac{\pi p_k}{\mathcal{C}}} |Q_k| \right]. \quad (5.15)
\]

Following Lemma 5.1 and [15, Lemma 25] (there exists a constant \(b\), such that \(\sum_{X \cap Y \neq \emptyset} e^{-a|X|} \leq b|Y|\)), the first bracketed term is bounded by \(e^{c e_0^{\frac{3}{2}} |\mathcal{C}|}\), the second bracketed term is bounded by \(e^{c(c_0 e_0^{5-2\epsilon})^{\frac{1}{2}} |\mathcal{C}|}\) and the third bracketed term is bounded by \(e^{c e_0 |\mathcal{C}|}\). Then, for some \(\kappa_2 < \frac{\kappa_1}{2} - 1\) and \(\beta = \frac{11}{2} - \epsilon\),

\[
|K^\#(\mathcal{C}, J)| \leq c e_0^\beta e^{-\kappa_2 |\mathcal{C}|}. \quad (5.16)
\]

Since \(K^\#(\mathcal{C}, J)\) is small and has an exponential decay, we can use the standard procedure, see for example [15], and write

\[
\sum_{\{\mathcal{C}_l\}} \prod_l K^\#(\mathcal{C}_l, J) = e^{\sum_{\mathcal{C}} E(\mathcal{C}, J)}, \quad (5.17)
\]
where
\[ E(C, J) = \sum_{n=1}^{\infty} \sum_{\{C_1, \ldots, C_n\}; \cup_l C_l = C} \rho^T(C_1, \ldots, C_n) \prod_l K^#(C_l, J) \]  
(5.18)
and \( \rho^T(C_1, \ldots, C_n) = 0 \) if \( C_l \) can be divided into disjoint sets and from [15]
\[ |E(C, J)| \leq ce^3 e^{-\kappa z|C|}. \]  
(5.19)
Rewrite the generating functional
\[ Z[J] = \left( \frac{2\pi}{e_0} \right)^{-|\Lambda|+1} (\det C^{\frac{1}{2}}) Z_0(\Lambda) e^{\sum_c E(C, J)} \]  
(5.20)
and take the logarithm
\[ \log Z[J] = \sum_c E(C, J) + \log \left( \frac{2\pi}{e_0} \right)^{-|\Lambda|+1} (\det C^{\frac{1}{2}}) Z_0(\Lambda) \].  
(5.21)

6. Mass Gap

6.1. Proof of Theorem 1.2

Consider two plaquettes \( p_1 \) and \( p_2 \). From Eqs. 1.26 and 5.21,
\[ \langle (dA)(p_1)(dA)(p_2) \rangle - \langle (dA)(p_1) \rangle \langle (dA)(p_2) \rangle = \left. \frac{\partial}{\partial J_{p_1}} \frac{\partial}{\partial J_{p_2}} \log Z[J] \right|_{J=0} \]  
(6.1)
Now, \( E(J) = \sum_c E(C, J) \) is analytic in \( J \). First note that the potential term is linear in \( J \) due to which the functions \( \rho'(P, J), \rho(Q, J) \) and \( f(X, J) \) are analytic. Then, as stated in Theorem 3.2, the function \( H(Z, \Psi) \) converges in a unit disk in complex plane of \( J \), and therefore, the function \( K(Y, J) \) is also analytic. Then, from Eqs. 5.3 and 5.7 each \( E(C, J) \) is analytic in \( J \), so when we sum it up, the absolutely convergent series is also analytic in \( J \). The right-hand side only depends on \( J(p), p \in C \); we can take the derivative term by term but only those terms with nonzero support of \( J \) in \( C \) contribute. Thus,
\[ \left. \frac{\partial}{\partial J_{p_1}} \frac{\partial}{\partial J_{p_2}} E(J) \right|_{J=0} = \sum_{C \supset \{p_1, p_2\}} \left. \frac{\partial}{\partial J_{p_1}} \frac{\partial}{\partial J_{p_2}} E(C, J) \right|_{J=0}. \]  
(6.2)
Let \( \alpha_p = e^{3e_0^3} < 1 \) denote the bound of \( J_p \). Define a contour \( \Gamma_p \) to be a circle of radius \( O(e^{3e_0^3}) \) with center as origin in complex plane containing \( J_p \). By Cauchy integral formula,
\[ E(J) = \frac{1}{2\pi i} \oint_{\Gamma_p} \frac{E(J_{\neq p}, J'_p)}{(J_p - J'_p)} \, dJ'_p, \]  
(6.3)
where \( p \) denotes a plaquette \( p_1 \) or \( p_2 \) and \( J_{\sim p} = \{J_{p'}\}_{p' \neq p} \). By differentiating the above

\[
\frac{\partial}{\partial J_p} E(J) = -\frac{1}{2\pi i} \oint_{\Gamma_p} \frac{E(J_{\sim p}, J'_p)}{(J_p - J'_p)^2} \, dJ'_p
\]

and for \( J_p = 0 \), we get

\[
\frac{\partial}{\partial J_p} E(J) = -\frac{1}{2\pi i} \oint_{\Gamma_p} \frac{E(J_{\sim p}, J'_p)}{(J'_p)^2} \, dJ'_p.
\]

Therefore, for plaquettes \( p_1 \) and \( p_2 \), we have

\[
\frac{\partial}{\partial J_{p_1}} \frac{\partial}{\partial J_{p_2}} E(J) = \frac{1}{(2\pi i)^2} \oint_{\Gamma_{p_1}} \oint_{\Gamma_{p_2}} \frac{dJ'_{p_1}}{(J'_{p_1})^2} \frac{dJ'_{p_2}}{(J'_{p_2})^2} \, E(C, J)
\]

From Eqs. 6.1, 6.2 and 6.6,

\[
\langle (dA)(p_1)(dA)(p_2) \rangle - \langle (dA)(p_1) \rangle \langle (dA)(p_2) \rangle
\]

\[
= \sum_{C \supset \{p_1, p_2\}} \frac{1}{(2\pi i)^2} \oint_{\Gamma_{p_1}} \oint_{\Gamma_{p_2}} \frac{dJ'_{p_1}}{(J'_{p_1})^2} \frac{dJ'_{p_2}}{(J'_{p_2})^2} \, E(C, J)
\]

\[
\leq \sum_{C \supset \{p_1, p_2\}} \frac{1}{|\alpha_{p_1}| |\alpha_{p_2}|} |E(C, J)|
\]

\[
\leq \frac{1}{|\alpha_{p_1}| |\alpha_{p_2}|} e^{-\frac{\kappa_2}{2|\alpha_1|} d(p_1, p_2)} \sum_{C \supset \{p_1, p_2\}} c e_0^\beta e^{-\frac{\kappa_2}{2} |C|}
\]

\[
\leq c e_0^\beta e^{-6\epsilon} e^{-\frac{\kappa_2}{2|\alpha_1|} d(p_1, p_2)},
\]

where we have used Eq. 5.19 and Lemma 5.2. The second last step follows from, \( |C| \geq \frac{d(p_1, p_2)}{|\alpha_1|} \). The last step follows from [15, Lemma 25] (there exists a constant \( b \), such that \( \sum_{X \cap Y \neq \emptyset} e^{-a|X|} \leq b|Y| \)).

6.2. n-Point Truncated Correlation

Let \( P = \{p_1, p_2, \ldots, p_n\} \) be a set of plaquettes. Each plaquette \( p_i \in P \) carries a current \( J_{p_i} \). Denote \( (dA)(p_i) \equiv dA_{p_i} \).

**Theorem 6.1.** Define the \( n \)-point truncated correlation function as

\[
\frac{\partial^n}{\partial J_{p_1} \partial J_{p_2} \ldots \partial J_{p_n}} \log Z[J] \bigg|_{J_{p_i} = 0} = \langle dA_{p_1}, dA_{p_2}, \ldots, dA_{p_n} \rangle^T
\]

\[
= \langle dA_{p_1}, dA_{p_2}, \ldots, dA_{p_n} \rangle - \sum_{\text{partitions} P_i, i \in \{1, \ldots, n\}} \langle \prod_{P_i \in \pi_i} dA_{p_i} \rangle \cdots \langle \prod_{P_i \in \pi_k} dA_{p_i} \rangle
\]

Then,

\[
\langle dA_{p_1}, dA_{p_2}, \ldots, dA_{p_n} \rangle^T \leq c e_0^\beta - 3n \epsilon e^{-\frac{\kappa_2}{2|\alpha_1|} t(p_1, p_2, \ldots, p_n)},
\]
Recall that $\Lambda$ is a large finite unit lattice of dimension $d$. Infinite Volume Limit

$$\text{Vol. 20 (2019) Mass Gap in Weakly Coupled Abelian Higgs 3993}$$

Proof.

$$\frac{\partial^n}{\partial J_{p_1} \partial J_{p_2} \cdots \partial J_{p_n}} \log Z[J] \bigg|_{J_{p_i} = 0} = \sum C \frac{\partial^n}{\partial J_{p_1} \partial J_{p_2} \cdots \partial J_{p_n}} E(C, J) \bigg|_{J_{p_i} = 0}.$$ (6.10)

Set the bound of $J_{p_i}$ to be $\alpha_{p_i} = e^{3e} < 1$. Define a contour $\Gamma_{p_i}$ to be a circle of radius $e^{3e}$ with center as origin in complex plane containing $J_{p_i}$. Following Eqs. 6.2 and 6.6,

$$\langle dA_{p_1}, dA_{p_2}, \ldots, dA_{p_n} \rangle^T = \sum_{C > \{p_1, \ldots, p_n\}} \frac{1}{(2\pi i)^n} \int_{\Gamma_{p_1}} \cdots \int_{\Gamma_{p_n}} \frac{dJ'_{p_1}}{(J'_{p_1})^2} \cdots \frac{dJ'_{p_n}}{(J'_{p_n})^2} E(C, J)$$

$$\leq \sum_{C > \{p_1, \ldots, p_n\}} \frac{1}{|C|} \frac{1}{|\alpha_{p_1}|} \cdots \frac{1}{|\alpha_{p_n}|} |E(C, J)|$$

$$\leq \frac{1}{|\alpha_{p_1}|} \cdots \frac{1}{|\alpha_{p_n}|} e^{-\frac{\pi \alpha^2}{d} t(p_1, \ldots, p_n)} \sum_{C > \{p_1, \ldots, p_n\}} e^{\beta} e^{-\frac{\pi \alpha^2}{d} |C|}$$

$$\leq c \varepsilon^{\beta - 3n\epsilon} e^{-\frac{\pi \alpha^2}{d} t(p_1, \ldots, p_n)},$$ (6.11)

where we have used $|C| \geq t(p_1, \ldots, p_n)$. \qed

Remark 6.2. Let $\gamma_1$ and $\gamma_2$ be two closed curves composed of lattice bonds and $W_{\gamma_i}(A)$ be a Wilson loop. Let $t_i$ be a source function defined on bonds forming $\gamma_i$. Then, Theorem 2 is also true for the observable $e^{\sum_{i=1}^{N} t_i W_{\gamma_i}(A)}$.

7. Infinite Volume Limit

Recall that $\Lambda$ is a large finite unit lattice of dimension $d$ and is divided into $[r_\Lambda]$ blocks. Let $\Lambda(N) = [-r_\Lambda]N, [r_\Lambda]N]^d$ with side length $2[r_\Lambda]N$.

Theorem 7.1. For any $p_1, p_2 \in \Lambda(N)^*$, let $\langle dA(p_1)dA(p_2) \rangle^T_{\Lambda(N)}$ denote the two-point truncated correlation function of $dA(p_1)$ and $dA(p_2)$. Then, the infinite volume limit $\lim_{N \to \infty} \langle dA(p_1)dA(p_2) \rangle^T_{\Lambda(N)}$ exists.

Proof. The two-point truncated correlation function of $dA(p_1)$ and $dA(p_2)$ is given by

$$\langle dA(p_1)dA(p_2) \rangle^T_{\Lambda(N)} = \sum_{C \supset \{p_1, p_2\}} F_{\Lambda(N)}(C),$$ (7.1)

where $C$ is a polymer and

$$F_{\Lambda(N)}(C) = \frac{\partial}{\partial J_{p_1}} \frac{\partial}{\partial J_{p_2}} E_{\Lambda(N)}(J, C) \bigg|_{J=0}. \quad (7.2)$$
Let $N' > N$.

**Claim** $F_{\Lambda(N')}(\mathcal{C}) = F_{\Lambda(N)}(\mathcal{C})$ if $\mathcal{C}$ is well inside $\Lambda(N)$.

From 5.5, $\mathcal{C} = \cup_i \{X_i\} \cup_k \{\tilde{Q}_k\} \cup_m \{Z_m\}$, where $\{X_i\}$ and $\{\tilde{Q}_k\}$ are polymers in $\Lambda_i^c(N)$ and $\{Z_m\}$ are polymers in $\Lambda_1(N)$. From 5.1, $Z = \cup_j \{Y_j\} \cup_l \{P_l\}$, where, $\{Y_j\}$ and $\{P_l\}$ are polymers in $\Omega_1(N)$ and $\Lambda_1(N) - \Omega_1(N)$, respectively.

1. $V_1(\Omega_0, \Phi'_{\Lambda_1}, \Phi_{\Lambda_1^c}, J)$ only depends on the fields in the neighborhood of $\Lambda_1$ which is well separated from $\partial \Lambda(N)$. Since $T_{\Lambda_1^c\Lambda_1}$ only couples nearest neighbors, $J(p)$ is localized inside $\Lambda_1$ and $\Lambda_1 \subset \Lambda_0$ is a big separation of length $[r_\lambda]$. Hence, polymer $Y$ never touches the boundary $\partial \Lambda(N)$, and therefore, the function $K(Y)$ is independent of $N$. Similarly, the function $\rho'(P)$ is independent of $N$ (the big separation of $\Lambda_1$ from $\Lambda_0$ is critical). Hence, the function $K'(Z, J)$ defined in 5.3 is independent of $N$.

2. In the region $\Lambda_i^c(N)$, there is only nearest neighbor interaction due to operator $T_{\Lambda_i^c\Lambda_1}$ which is independent of $N$. Thus, the functions $f(X, J)$ and $\rho(\tilde{Q}, J)$ are independent of $N$.

From 5.7 and 5.18, $E_{\Lambda(N)}(J, C)$ is composed of functions $K'(Z, J), f(X, J)$ and $\rho(\tilde{Q}, J)$; therefore, $E_{\Lambda(N)}(J, C)$ does not depend on $N$. Moreover, if $\Lambda_0(N) = \Lambda(N)$, then the polymer $\mathcal{C} = Z$ and lies well inside $\Lambda(N)$. Therefore,

$$|\langle dA(p_1)dA(p_2)\rangle_{\Lambda(N')}^T - \langle dA(p_1)dA(p_2)\rangle_{\Lambda(N)}^T| = \sum_{\mathcal{C} \supset \{p_1,p_2\}} F_{\Lambda(N')}(\mathcal{C}) - \sum_{\mathcal{C} \supset \{p_1,p_2\}} F_{\Lambda(N)}(\mathcal{C})$$

$$\leq \sum_{\mathcal{C} \supset \{p_1,p_2\}} |F_{\Lambda(N')}(\mathcal{C})| \leq \mathcal{O}(e^{-K^2N}).$$

(7.3)

The last step follows from Lemma 5.2, and the fact that $\mathcal{C} \cap \Lambda(N)^c \neq \emptyset$ implies that $\mathcal{C}$ extends all the way to the boundary $\partial \Lambda(N)$; thus, $|\mathcal{C}| = \mathcal{O}(N)$. Since $N$ is arbitrary large, the limit $N \to \infty$ exists. This completes the proof of the theorem.

**Corollary 7.2.** The exponential decay of truncated correlations function $|\langle dA(p_1)dA(p_2)\rangle_{\Lambda(N)}^T| < e^{-m d(p_1,p_2)}$ holds for infinite volume limit.

**Remark 7.3.** Given a large finite unit lattice $\Lambda(N)$, let $\mathcal{P} = \{p_1, p_2, \ldots, p_n\}$ be a set of plaquettes. Then, the infinite volume limit of the $n$-point truncated correlation functions $\lim_{n \to \infty} \langle dA(p_1), dA(p_2), \ldots, dA(p_n)\rangle^T$ exists.

**Acknowledgements**

I would like to thank my advisor Jonathan Dimock for his guidance.

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.
References

[1] Balaban, T., Imbrie, J., Jaffe, A., Brydges, D.: The mass gap for Higgs models on a unit lattice. Ann. Phys. 158(2), 281–319, 0003-4916 (1984)

[2] Balaban, T., Feldman, J., Knörrer, H., Trubowitz, E.: Power series representations for bosonic effective actions. J. Stat. Phys. 134, 839 (2009)

[3] Balaban, T., Imbrie, J., Jaffe, A.: Renormalization of the Higgs model: minimizers, propagators, and the stability of mean field theory. Commun. Math. Phys. 97, 299–329 (1985)

[4] Balaban, T., Imbrie, J., Jaffe, A.: Effective action and cluster properties of the abelian Higgs model. Commun. Math. Phys. 114, 257–315 (1988)

[5] Brydges, D., Fröhlich, J., Seiler, E.: On the construction of quantized gauge fields. I. General results. Ann. Phys. 121, 227–284 (1979)

[6] Brydges, D., Fröhlich, J., Seiler, E.: On the construction of quantized gauge fields. II. Convergence of the lattice approximation. Commun. Math. Phys 71(2), 159–205 (1980)

[7] Brydges, D., Fröhlich, J., Seiler, E.: On the construction of quantized gauge fields. III. The two-dimensional abelian Higgs model without cutoffs. Commun. Math. Phys 79, 353–399 (1981)

[8] Glimm, J., Jaffe, A., Spencer, T.: The particle structure of the weakly coupled $P(\phi)^2$ model and other applications of high temperature expansions, part II: the cluster expansion. In: Wightman, A.S. (ed.) Constructive Quantum Field Theory. Springer Lecture Notes in Physics, vol. 25. Springer, Berlin (1973)

[9] Glimm, J., Jaffe, A., Spencer, T.: The Wightman axioms and particle structure in the weakly coupled $P(\phi)^2$ quantum field model. Ann. Math. 100, 585–632 (1974)

[10] Kennedy, T., King, C.: Spontaneous symmetry breakdown in the abelian Higgs model. Commun. Math. Phys. 104(2), 327–347 (1986), 1432-0916

[11] Brydges, D., Kennedy, T.: Mayer expansions and the Hamilton–Jacobi equation. J. Stat. Phys. 48, 19–49 (1987)

[12] Abdesselam, A., Rivasseau, V.: Trees, forests and jungles: a botanical garden for cluster expansions. In: Constructive Physics (Palaiseau, 1994), Lecture Notes in Physics, vol. 446, pp. 7–36. Springer, Berlin (1995)

[13] Dimock, J.: The renormalization group according to Balaban—II. Large fields. J. Math. Phys. 54, 092301 (2013)

[14] Balaban, T.: Localization expansions I. Function of the background configurations. Commun. Math. Phys. 182, 33–82 (1996)

[15] Dimock, J.: The renormalization group according to Balaban—I. Small fields. Rev. Math. Phys. 25, 1330010 (2013)
Abhishek Goswami
Department of Mathematics
SUNY at Buffalo
Buffalo NY 14260
USA
e-mail: goswami3@buffalo.edu

Communicated by Abdelmalek Abdesselam.
Received: January 4, 2019.
Accepted: September 4, 2019.