SCHEDULING PROBLEMS

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Abstract. We introduce the notion of a scheduling problem which is a boolean function $S$ over atomic formulas of the form $x_i \leq x_j$. Considering the $x_i$ as jobs to be performed, an integer assignment satisfying $S$ schedules the jobs subject to the constraints of the atomic formulas. The scheduling counting function counts the number of solutions to $S$. We prove that this counting function is a polynomial in the number of time slots allowed. Scheduling polynomials include the chromatic polynomial of a graph, the zeta polynomial of a lattice, the Billera-Jia-Reiner polynomial of a matroid and the newly defined arboricity polynomial of a matroid.

To any scheduling problem, we associate not only a counting function for solutions, but also a quasisymmetric function and a quasisymmetric function in non-commuting variables. These scheduling functions include the chromatic symmetric functions of Sagan, Gebhard, and Stanley, and a close variant of Ehrenborg’s quasisymmetric function for posets.

Geometrically, we consider the space of all solutions to a given scheduling problem. We extend a result of Steingrímsson by proving that the $h$-vector of the space of solutions is given by a shift of the scheduling polynomial. Furthermore, under certain niceness conditions on the defining boolean function, we prove partitionability of the space of solutions and positivity of fundamental expansions of the scheduling quasisymmetric functions and of the $h$-vector of the scheduling polynomial.

1. Introduction

A scheduling problem on $d$ items is given by a boolean formula $S$ over atomic formulas $x_i \leq x_j$ for $i, j \in [d]$. A $k$-schedule solving $S$ is a function $\omega : [d] \to [k]$ such that $S(\omega)$ is true. We consider the $d$ items as jobs to be scheduled into discrete time slots and the atomic formulas are interpreted as the constraints on jobs. A $k$-schedule satisfies all of the constraints using at most $k$ time slots. Consider three jobs and the following constraints: $x_1 \neq x_2$ if $x_3 \geq x_1$, but $x_1 = x_2$ is ok if $x_3 < x_1$. We interpret this as: tasks $x_1$ and $x_2$ can be performed at the same time only if $x_3$ happens first. The valid solutions to this scheduling problem are all integer points off of a cone in $\mathbb{R}^3$ as seen in Figure 1(a). Extending this example, another scheduling problem on three jobs might require that only one job may be started first but then the other two jobs can be started at any time. The valid schedules are then all integer points off of the shaded regions in Figure 1(b).

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We will be interested in the number of solutions to a given scheduling problem and define the scheduling counting function $\chi_S(k)$ to be the number of $k$ schedules solving $S$. Our first result shows that $\chi_S(k)$ is in fact a polynomial function in $k$. As special instances, scheduling polynomials include the chromatic polynomial of a graph, the zeta polynomial of a lattice, and the newly defined arboricity polynomial of a matroid. The arboricity polynomial counts the number of covers of a matroid with independent sets or, as a special case, the number of covers of a graph with disjoint spanning trees, naturally extending the notion of arboricity introduced by Nash-Williams [17], Tutte [24], and Edmonds [10], which is the size of the minimum possible independent cover. Many scheduling problems will be naturally associated to a graph or a matroid but do not generally satisfy a contraction/deletion recursion and are hence not Tutte evaluations. Instead, scheduling polynomials provide a large natural class of non-Tutte invariant polynomials.

Our approach to scheduling problems is both algebraic and geometric. Algebraically, to any scheduling problem, we associate not only a counting function for solutions, but also a quasisymmetric function and a quasisymmetric function in non-commuting variables which record successively more information about the solutions themselves. Geometrically, we consider the space of all solutions to a given scheduling problem via Ehrhart theory, hyperplanes arrangements, and the Coxeter complex of type A. As special instances, the varying scheduling structures include the chromatic functions of Sagan and Gebhard [12], and Stanley [22], the chromatic complex of Steingrímsson [23], the matroid invariant of Billera-Jia-Reiner [5], and a variant of Ehrenborg’s quasisymmetric function for posets [11].

We first use the interplay of geometry and algebra to prove a Hilbert series type result showing that the $h$-vector of the solution space is given by a shift of the scheduling polynomial. This includes and generalizes Steingrímsson’s result on the chromatic polynomial and coloring complex to all scheduling problems. Imposing certain niceness conditions on the space of solutions allows for stronger results. We focus on the case when the boolean function $S$ can be written as a particular kind of decision tree. Such decision trees provide a nested if-then-else structure for the scheduling problem. In this case we prove that the space of solutions is partitionable. This in turn implies positivity of the scheduling quasisymmetric functions in the fundamental bases and the $h$ coefficients of the scheduling polynomial.

Sections 2 and 3 review the necessary material from Ehrhart theory, hyperplane arrangements and the theory of quasisymmetric functions in non-commuting variables (NCQSym) and quasisymmetric functions (QSym). Section 4 connects the varying perspectives from which scheduling problems can naturally be viewed. The polynomiality of the scheduling counting function is established in Theorem 4.2 along with properties of its coefficients. Theorem 4.7 equates the scheduling polynomial to the $h$-polynomial of the allowed configuration of solutions. Section 5 introduced decision trees and establishes partitionability in Theorem 5.4. Theorem 5.8 proves positivity in the fundamental NCQSym basis under these conditions. Section 5 also includes
numerous examples of scheduling problems in connection to $P$-partitions, generalized permutahedra, and Bergman fans. Finally, section 6 introduces the arboricity polynomial as a scheduling counting function.

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2. Preliminaries: Ehrhart Theory

In this section we introduce the basic notions from Ehrhart theory that we need throughout the paper, however we assume some familiarity with polyhedral geometry. As references on these subjects we recommend [3, 25].

Consider a (bounded) set $X \subset \mathbb{R}^d$. The Ehrhart function

$$\text{ehr}_X : \mathbb{Z}_{>0} \rightarrow \mathbb{Z}_{\geq 0}$$

of $X$ counts the number of integer points in integer dilates of $X$, i.e.,

$$\text{ehr}_X(k) = \# \mathbb{Z}^d \cap k \cdot X$$

for any positive integer $k$. Ehrhart’s theorem states that if $X$ is a polytope whose vertices have integer coordinates, then the Ehrhart function $\text{ehr}_X$ of $X$ coincides with a polynomial, called the Ehrhart polynomial of the polytope. We call polytopes whose vertices have integer coordinates integral or lattice polytopes. Two polytopes in $\mathbb{R}^d$ are lattice equivalent if there is an affine automorphism $\phi$ of $\mathbb{R}^d$ with $\phi(P) = Q$ which induces a bijection on the integer lattice $\mathbb{Z}^d$. Lattice equivalent sets $X$ have the same Ehrhart function.

The most important example for our purposes are Ehrhart polynomials of simplices. The $d$-dimensional standard simplex $\Delta^d$ is the convex hull of the standard unit vector...
e_1, \ldots, e_d \in \mathbb{R}^{d+1}, \text{i.e.,}

\[ \Delta^d = \left\{ x \in \mathbb{R}^{d+1} \left| \sum_{i=1}^{d+1} x_i = 1, x_i \geq 0 \right. \right\} = \left\{ \sum_{i=1}^{d+1} \lambda_i e_i \in \mathbb{R}^{d+1} \left| \sum_{i=1}^{d+1} \lambda_i = 1, \lambda_i \geq 0 \right. \right\}. \]

Its Ehrhart polynomial is given by a binomial coefficient

\[ \text{ehr}_{\Delta^d}(k) = \binom{k+d}{d}, \]

which is a polynomial of degree \( d \) in the variable \( k \). An integral simplex is \textit{unimodular} if it is lattice equivalent to a standard simplex, whence all unimodular simplices of the same dimension have the same Ehrhart polynomial. Note that all faces of a unimodular simplex are themselves unimodular simplices.

Ehrhart polynomials of unimodular simplices arise naturally from the study of scheduling problems. Consider the following scheduling problem \( S \): “Schedule two jobs, 1 and 2, such that job 2 starts after or at the same time as job 1.” Suppose jobs can be run in \( k+1 \) discrete time slots, numbered 0 through \( k \). A feasible schedule is a pair of integer numbers \( 0 \leq x_1, x_2 \leq k \) such that \( x_1 \leq x_2 \). The feasible schedules are precisely the integer points \( x = (x_1, x_2) \) contained in the \( k \)-th dilate of the unimodular simplex \( \text{conv}(0, e_2, e_1 + e_2) \). Therefore, if we let the deadline \( k \) vary, the number of feasible \((k+1)\)-schedules is given by the Ehrhart polynomial of a 2-dimensional unimodular simplex, i.e., \( \chi_S(k+1) = \text{ehr}_{\Delta^2}(k) = \binom{k+2}{2} \).

If instead, we require job 2 to run strictly after job 1, feasible schedules are then integer points \( x \in \mathbb{Z}^2 \) such that \( 0 \leq x_1, x_2 \leq k \) and \( x_1 < x_2 \). Denote this scheduling problem by \( S' \). The strict inequality gives rise to a \textit{half-open simplex}. The \( d \)-dimensional standard simplex \( \Delta^d_i \) with \( i \) open faces is defined just as the standard simplex above, except that \( i \) of the inequalities are strict. More precisely

\[ \Delta^d_i = \left\{ x \in \mathbb{R}^{d+1} \left| \sum x_i = 1, x_1 \geq 0, \ldots, x_i > 0, x_{i+1} \geq 0, x_{d+1} \geq 0 \right. \right\}. \]

For every face of the standard simplex that we open, the Ehrhart polynomial is shifted by one:

\[ \text{ehr}_{\Delta^d_i}(k) = \binom{k+d-i}{d}. \quad (1) \]

In particular, the open simplex \( \Delta^d_{d+1} \) has Ehrhart polynomial \( \text{ehr}_{\Delta^d_{d+1}}(k) = \binom{k-1}{d} \).

In the example \( S' \), the feasible \((k+1)\)-schedules therefore correspond to the integer points in the \( k \)-th dilate of a 2-dimensional unimodular simplex with 1 open face, and their number is \( \chi_{S'}(k+1) = \text{ehr}_{\Delta^2_1}(k) = \binom{k+1}{2} \).\(^1\)

Half-open simplices do not suffice to capture all scheduling problems. Consider the following example \( S'' \) of a scheduling problem on three jobs: “Either run all three jobs at the same time, or first job 1, then job 2, and then job 3.” A feasible \((k-1)\)-schedule

\(^1\)In later sections, we will be using a slightly different setup than in the previous two examples: We will identify time slots with integers 1 through \( k \) by letting \( 0 < x_i < k+1 \). This alternate construction then yields \( \chi_S(k) = \text{ehr}_{\Delta^2_1}(k+1) = \binom{k+1}{2} \) and \( \chi_S(k) = \text{ehr}_{\Delta^3_2}(k+1) = \binom{k}{2} \).
is a triple \((x_1, x_2, x_3)\) with \(0 < x_1, x_2, x_3 < k\) such that \(x_1 < x_2 < x_3\) or \(x_1 = x_2 = x_3\). The resulting geometric region is the union of an open 3-dimensional simplex and (the relative interior of) the main diagonal of the cube \([0, k]^3\). This set is no longer a half-open simplex. In fact, it is not even the set-theoretic difference of two closed polytopes. In order to capture this type of geometric object, we work with partial polytopal complexes.

A polytopal complex \(K\) is a finite set of polytopes in some \(\mathbb{R}^d\) such that \(K\) is closed under taking faces and the intersection of any two polytopes in \(K\) is a face of both. A polytopal complex is integral if all vertices have integer coordinates. We define the relative interior \(\text{relint}(P)\) of a polytope \(P\) to be the interior of the polytope taken with respect to its affine hull, so that, for example, the interior of a 2-dimensional triangle in 3-space is the triangle minus its edges and vertices and, more generally, \(\text{relint} \Delta^d = \Delta^d_{d+1}\). The union of all faces of a polytopal complex is equal to the disjoint union of the relative interiors of all faces. For example, a triangle is equal to the open 2-dimensional region, plus the open line segments corresponding to the edges, plus the vertices (which are open with respect to their 0-dimensional affine hull). This observation implies that the Ehrhart function of a polytopal complex is just the sum of all Ehrhart functions of the relative interiors of all faces.

Now, returning to the example \(S''\), the set \(X\) of feasible points \(x \in \mathbb{R}^3\) for \(k = 1\) can be written as the disjoint union of the open 3-dimensional simplex \(\sigma\) which is the relative interior of the convex hull of the vectors \((0, 0, 0), (0, 0, 1), (0, 1, 1)\) and \((1, 1, 1)\) and the open line segment \(\lambda\) which is the relative interior of the convex hull of \((0, 0, 0)\) and \((1, 1, 1)\). Then the feasible \((k-1)\)-schedules are precisely the integer points in \(k \cdot X\), i.e.,

\[
\chi_{S''}(k-1) = \text{ehr}_X(k) = \text{ehr}_{\Delta^2_3}(k) + \text{ehr}_{\Delta^1_2}(k) = \binom{k-1}{2} + \binom{k-1}{1}.
\]

However, we note that the triangle \(\tau\) that is the relative interior of the convex hull of \((0, 0, 0), (0, 0, 1)\) and \((1, 1, 1)\) is not contained in \(X\), even though \(\tau\) is a face of \(\sigma\) and \(\lambda\) is a face of \(\tau\). This idea of taking the relative interiors of just some of the faces of a polytopal complex gives rise to the notion of a partial polytopal complex, which was introduced in [6]. A partial polytopal complex is a subset of the faces of a polytopal complex; in particular, a partial polytopal complex is not closed under taking faces. The support \(|K|\) of a partial polytopal complex \(K\) is the union of the relative interiors of all faces contained in \(K\). By abuse of terminology, we will often refer to both \(K\) and \(|K|\) as the polytopal complex \(K\) and let it be clear from context whether we are referring to the subset of Euclidean space or the set of polytopes ordered by the face relation.

Note that in the above example it is still convenient to speak of \((0, 0, 0), (0, 0, 1)\), etc. as vertices of \(\sigma\) and, consequently, as vertices of \(X\), even though they are neither contained in \(X\) nor in the relative interior of \(\sigma\). It will generally be clear from context, whether we are referring to vertices that are actually contained in \(X\), or merely to vertices of the closure of some face of the partial complex. However, when we need to carefully distinguish between these two cases, we adopt the following terminology:
We call the relative interior of a polytope $P$ an open polytope. The faces of an open polytope are the faces of its closure. By extension, we define the closure of any set of open polytopes as the set of all faces of these polytopes. A partial polytopal complex $\mathcal{K}$ is now any set of open polytopes whose closure $\overline{\mathcal{K}}$ is a polytopal complex. This allows us to distinguish between the faces of a partial polytopal complex $\mathcal{K}$ and the faces of its closure $\overline{\mathcal{K}}$. Note that indeed $|\mathcal{K}| = |\overline{\mathcal{K}}|$, i.e., the closure of the support is the support of the closure.

3. Preliminaries: Coxeter Complexes, QSym and NCQSym

Suppose we are given a scheduling problem with 3 jobs and three types of constraints. Any of the three jobs may be started first but different requirements are imposed depending on which starts first. If jobs 1 or 3 are started first, then the other must start at the same time as job 2. If job 2 starts first, then job 1 must occur next before job 3 can be started:

\[ (x_1 < x_2) \land (x_2 = x_3) \]
\[ (x_3 < x_2) \land (x_1 = x_2) \]
\[ (x_2 < x_1) \land (x_1 < x_3). \]

We can interpret the solutions as all those integer points such that $x_1 < x_2 = x_3$ or $x_3 < x_1 = x_2$ or $x_2 < x_1 < x_3$. Importantly, solutions only depend on the relative ordering of coordinates. It is this observation that allows solutions to be indexed by ordered set partitions.

An ordered set partition or set composition $\Phi \models [n]$ is a sequence of sets $(\Phi_1, \Phi_2, \ldots, \Phi_k)$ such that for all $i, j$, $(\Phi_i \cap \Phi_j = \emptyset)$ and $(\cup_i \Phi_i = [n])$. The $\Phi_i$ are the blocks of the ordered set partition and we will often use the notation $\Phi_1|\Phi_2|\cdots|\Phi_k$. Note that within each block, elements are not ordered, so the ordered set partition $13|4|2 \models [4]$ is the same as $31|4|2 \models [4]$. We will use ordered set partitions to represent integer points whose relative ordering of coordinates is given by the blocks of the partition. For example, $31|4|2$ represents all integer points such that $x_1 = x_3 < x_4 < x_2$.

As solutions to scheduling problems are naturally indexed by ordered set partitions they are closely related to the algebra of quasisymmetric functions in non-commuting variables, NCQSym, and the Coxeter complex of type $A$, Cox$_A$. In particular, the scheduling polynomial counts the number of solutions to a scheduling problem. We will interpret this polynomial as a specialization of the NCQSym scheduling function.

3.1. The Coxeter Complex of Type A. The braid arrangement $\mathcal{B}_n$ is the hyperplane arrangement in $\mathbb{R}^n$ consisting of hyperplanes $x_i = x_j$ for all $i, j \in [n]$. This arrangement is central, that is all hyperplanes contain the origin. It is also non-essential, that is the collection of normals to the hyperplanes do not span all of $\mathbb{R}^n$. The hyperplanes have a common intersection equal to the line $x_1 = x_2 = \cdots = x_n$.

Projecting the arrangement to the orthogonal complement of this line and intersecting with the unit sphere yields a spherical simplicial complex known as the Coxeter complex of type $A$, Cox$_A$. It can be realized combinatorially as the barycentric subdivision of the boundary of the simplex. In the Ehrhart setting of the previous section,
this complex can be obtained by (1) starting with the cube \([0, 1]^d\), (2) triangulating this with the braid arrangement, and (3) removing the two vertices consisting of only zeros and only ones, and all incident faces.

The faces of \(\text{Cox}_A\) are naturally labeled by ordered set partitions. Each face of the Coxeter complex is simply a normalization of a face of the cell decomposition induced by \(\mathcal{B}_n\) on \(\mathbb{R}^n\). A face of the cell decomposition of \(\mathcal{B}_n\) specifies for each pair \(i, j\) whether \(x_i < x_j\), \(x_i > x_j\), or \(x_i = x_j\), precisely the atomic formulas of scheduling problems. All points in a fixed face have the same relative ordering of coordinates. This relative ordering induces an ordered set partition on \([n]\). For example, if a face consists of all points such that \(x_2 = x_3 < x_1 = x_4 = x_6 < x_5\) then the induced ordered set partition is \(23\mid 146\mid 5\). Under this correspondence, we see that each maximal face corresponds to a partition into blocks of size one (i.e. a full permutation). Moreover, a face \(F\) is contained in a face \(G\) if and only if the ordered set partition corresponding to \(F\) coarsens the ordered set partition corresponding to \(G\); the face lattice is dual to the face lattice of the permutahedron.

3.2. NCQSym. Let \(\{x_1, x_2, \ldots\}\) be a collection of non-commuting variables. Given \(a \in \mathbb{N}^n\), let \(\Delta(a)\) be the ordered set partition \((\Delta_1|\Delta_2|\ldots|\Delta_k)\) such that \(a\) is constant on each set \(\Delta_i\) and satisfies \(a|_{\Delta_i} < a|_{\Delta_i+1}\) for all \(1 \leq i \leq k\). Define the order class of \(a\) to be the set of vectors \(b\) such that \(\Delta(b) = \Delta(a)\).

**Example 3.1.** For \(a = (3, 2, 2, 3, 1) \in \mathbb{N}^5\), \(\Delta(a) = (5|23|14)\). The order class of \(a\) consists of all vectors \(x \in \mathbb{N}^5\) such that \(x_5 < x_2 = x_3 < x_1 = x_4\).

The order class specifies the relative ordering of coordinates and contains all points in the relative interior of a cone of the braid arrangement. Hence an order class is naturally associated to (the relative interior) of a face of the Coxeter complex. The solutions to a satisfiable scheduling problem correspond to a union of these open faces.

**Definition 3.2.** A function in non-commuting variables is called quasisymmetric (an element of NCQSym) if \(\forall \gamma, \tau \in \mathbb{N}^n\) such that \(\gamma\) and \(\tau\) are in the same order class, \(\Delta(\gamma) = \Delta(\tau)\), the coefficient of \(x_{\gamma_1}x_{\gamma_2}\ldots x_{\gamma_n}\) is the same as the coefficient of \(x_{\tau_1}x_{\tau_2}\ldots x_{\tau_n}\).

Let \(\Phi\) be an ordered set partition. Define the monomial quasisymmetric function in non-commuting variables indexed by \(\Phi\) as follows:

\[
\mathcal{M}_\Phi = \sum_{a \in \mathbb{N}^n \mid \Delta(a) = \Phi} x_a.
\]

Alternatively, quasisymmetric functions in non-commuting variables can be defined as any function which can be expressed as a sum of monomial terms \(\mathcal{M}_\Phi\).

**Example 3.3.** Consider the order class of points that satisfy \(x_1 = x_3 < x_2 = x_4\). The corresponding ordered set partition \(\Phi\) is \((13|24)\).

\[
\mathcal{M}_{13|24} = x_1x_2x_1x_2 + x_1x_3x_1x_3 + x_2x_3x_2x_3 + x_3x_4x_3x_4 + \cdots
\]
One of the staples of geometric methods in combinatorics is to interpret a monomial as an integer point in space. The standard construction is to view a monomial in commuting variables $x_1^{a_1}x_2^{a_2} \cdots x_d^{a_d}$ as the point $(a_1, a_2, \ldots, a_d) \in \mathbb{Z}^d$. To translate quasisymmetric functions in non-commuting variables into a geometric setting, we need to use a different construction. Here we associate to a monomial $x_{i_1}x_{i_2} \cdots x_{i_d}$ the point $(i_1, \ldots, i_d) \in \mathbb{Z}^d$. For example $x_2x_1x_3$ becomes $(2, 1, 3) \in \mathbb{Z}^3$. That is, the entries of the vector are given by the *indices* of the monomial, as opposed to the exponents. This is well defined because we are working with non-commuting variables, so the factors $x_i$ appear in a fixed order.

Figure 3 provides examples of this construction. When viewed through the geometric lens explained above the infinite formal sum $\mathcal{M}_{3|1|2} = \sum_{0<i_3<i_1<i_2} x_{i_1}x_{i_2}x_{i_3}$
in the non-commuting variables \( \{x_1, x_2, x_3, \ldots \} \), corresponds to the set of integer points \( z \in \mathbb{Z}^3 \) with \( 0 < z_3 < z_1 < z_2 \), that is, the set of integer points in the relative interior of the cone generated by vectors \( e_2, e_1 + e_2 \) and \( e_1 + e_2 + e_3 \). If we turn one of the two inequalities into equalities, i.e., we pass to the sums \( M_{3|12} = \sum_{0<i_3<i_1=i_2} x_{i_1} x_{i_2} x_{i_3} \) and \( M_{13|2} = \sum_{0<i_3=i_1<i_2} x_{i_1} x_{i_2} x_{i_3} \), we obtain two facets of this cone. More precisely, we obtain the set of lattice points in the relative interiors of the cones \( \text{cone}_{\mathbb{Z}}(e_1 + e_2, e_1 + e_2 + e_3) \) and \( \text{cone}_{\mathbb{Z}}(e_2, e_1 + e_2 + e_3) \), respectively.

We will sometimes refer to quasisymmetric functions in non-commuting variables as nc-quasisymmetric functions. Informally, an nc-quasisymmetric function corresponds to a \( k \)-schedule where \( k \) has been taken to infinity, i.e., there is no deadline. Recall the example above on two jobs such that job 2 must start after job 1 unless they start at the same time. The solutions corresponds to the integer points satisfying \( \{x_1 < x_2\} \) or \( \{x_1 = x_2\} \) and are recorded by the nc-quasisymmetric function \( M_{1|2} + M_{12} \).

**Definition and Lemma 3.4.** Given a scheduling problem \( S \) on \( k \) items, the scheduling quasisymmetric function in noncommuting variables is the sum of monomial quasisymmetric functions in non-commuting variables \( M_{\Phi} \) over all ordered set partitions \( \Phi \) such that \( S \) is satisfied by any (and hence all) integer points \( a \in \mathbb{N}^k \) with relative ordering given by \( \Phi \),

\[
S_S := \sum_{a \in \mathbb{N}^k : \text{S}(a)} x_a = \sum_{\Phi \text{ such that } \exists a \in \mathbb{N}^k : (\Delta(a) = \Phi) \land \text{S}(a)} M_{\Phi}.
\]

### 3.3. From NCQSym to QSym to Polynomials

In order to impose a deadline, or \( k \) time slots, we restrict to a finite number of non-zero variables. Although not strictly necessary to arrive at a polynomial, we will also be interested in the intermediate specialization from an nc-quasisymmetric function to a quasisymmetric function.

A **composition** \( \alpha \) is a sequence \( (\alpha_1, \ldots, \alpha_k) \) of positive integers \( \alpha_i \). A **quasisymmetric function** is a formal power series in infinitely many variables \( \{x_1, x_2, \ldots\} \) which has
bounded degree and is shift invariant. Namely, for any composition \( \alpha \), the coefficients of all terms, \( x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k} \), running over all possible \( k \)-tuples, \( \{i_1 < i_2 < \ldots < i_k\} \), are the same.

The monomial quasisymmetric function indexed by the composition \( \alpha \) is

\[
M_\alpha := \sum_{1 \leq i_1 < i_2 < \ldots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}.
\]

One may equivalently define quasisymmetric functions as any series which can be written as a linear combination of monomial quasisymmetric functions.

Thus elements of QSym are naturally indexed by compositions and elements of NCQSym are naturally indexed by ordered set partitions. The type map maps elements of NCQSym to QSym by sending ordered set partitions to compositions. The type map of an ordered set partition simply records the size of each block:

\[
\text{type}(\Phi_1 | \Phi_2 | \cdots | \Phi_n) = (|\Phi_1|, |\Phi_2|, \ldots, |\Phi_n|).
\]

If \( S \in \text{NCQSym} \) is written as a sum of monomial terms, applying the type map to each index is equivalent to allowing the variables to commute. In this way we extend the type map to nc-quasisymmetric functions themselves.

**Example 3.5.** Let

\[
\mathcal{F} = M_{1|23} + M_{3|21} + M_{2|13}
\]

be an element of NCQSym written in terms of monomials. Then

\[
\text{type}(\mathcal{F}) = M_{\text{type}(1|23)} + M_{\text{type}(3|21)} + M_{\text{type}(2|13)}
\]

\[
= 2M_{(1,2)} + M_{(1,1,1)} \in \text{QSym}.
\]

Given a quasisymmetric function in non-commuting variables (or commuting), there is a naturally associated polynomial function defined by setting the first \( k \) variables \( x_1, x_2, \ldots, x_k \) equal to 1. For a single monomial term we have

\[
\mathcal{M}_\Phi(1^k) = \binom{k}{\ell(\Phi)},
\]

where \( \ell(\Phi) \) is equal to the length, i.e. the number of blocks, of the partition.

**Example 3.6.** Continuing the example above gives

\[
\mathcal{F} = M_{1|23} + M_{3|21} + M_{2|13}
\]

\[
\mathcal{F}(1^k) = 2 \binom{k}{2} + \binom{k}{3}.
\]

Just as in the study of Ehrhart polynomials of unimodular simplices, binomial coefficients make an appearance here and, as we will see in the next section, this is not an accident.
4. SCHEDULING PROBLEMS

In this section we bring the varying perspectives of Ehrhart theory, Coxeter complexes and quasisymmetric functions together to prove basic properties of scheduling problems. First, for any scheduling problem $S$, the number of $k$-schedules satisfying $S$ is seen to be a polynomial function in $k$. This scheduling polynomial is both an Ehrhart polynomial and a polynomial specialization of a quasisymmetric function. Next, we define the allowed and forbidden scheduling subconfigurations and connect the geometry of these spaces to the scheduling polynomial. Specifically, it is shown that the $h$-vectors of (shifted) scheduling polynomials are equal to the $h$-vectors of the allowed and forbidden subconfigurations.

**Definition 4.1.** A scheduling problem on $d$ items is given by a boolean formula $S$ over atomic formulas $\omega_i \leq \omega_j$ for $i, j \in [d]$. A $k$-schedule solving $S$ is a function $\omega : [d] \rightarrow [k]$ such that $S(\omega)$ is true.

For an ordered set partition $\Phi = (\Phi_1, \ldots, \Phi_m)$, we define the relatively open cone $\text{cone}(\Phi) \subset \mathbb{R}^n$ to be the set of all $a \in \mathbb{R}^n$ that satisfy the constraints

$$
0 < a_i \quad \text{for all } i \in [n],
$$

$$
a_i = a_j \quad \text{for all } l \in [m] \text{ and } i, j \in \Phi_l,
$$

$$
a_i < a_j \quad \text{for all } l_1 < l_2 \in [m] \text{ and all } i \in \Phi_{l_1}, j \in \Phi_{l_2}.
$$

Equivalently, $\text{cone}(\Phi)$ is the set of all real vectors $a$ in the positive orthant, $\mathbb{R}^n > 0$ such that $\Delta(a) = \Phi$:

$$
\text{cone}(\Phi) = \{ a \in \mathbb{R}^n > 0 \mid \Delta(a) = \Phi \}.
$$

If $\Phi$ ranges over all ordered set partitions of $[n]$, these cones form a triangulation $T^n_\Phi$ of the positive orthant $\mathbb{R}^n > 0$,

$$
\mathbb{R}^n > 0 = \bigcup_\Phi \text{cone}(\Phi).
$$

This triangulation of $\mathbb{R}^n > 0$ can also be obtained by subdividing the positive orthant according to the braid arrangement of all hyperplanes $x_i = x_j$.

As discussed in the previous section, we associate to every $a \in \mathbb{N}^n$ a monomial in non-commuting variables $x_{a_1} \cdots x_{a_n}$ which we abbreviate as $x_a$. In this way, we can associate to every set of lattice points, $A \subset \mathbb{N}^n$, a formal sum $N(A)$ of monomials in non-commuting variables

$$
N(A) = \sum_{a \in A} x_a.
$$

The function $N$ maps $\text{cone}(\Phi)$ to the monomial nc-quasisymmetric functions $\mathcal{M}_\Phi$,

$$
N(\text{cone}(\Phi) \cap \mathbb{Z}^n) = \mathcal{M}_\Phi.
$$

Conversely, we can think of any nc-quasisymmetric function $\mathcal{F}$ with non-negative coefficients in the monomial basis as a multiset of cones in the positive orthant, where the multiplicity of lattice points in $\text{cone}(\Phi)$ is given by the coefficient of $\mathcal{M}_\Phi$ in $\mathcal{F}$. 
Let $S$ be a scheduling problem on $n$ items. For any vector $a \in \mathbb{R}^n_{>0}$ we write $S(a)$ to denote the boolean value of whether or not $a$ satisfies the constraints imposed by $S$. $S(a)$ depends only on the relative order of the components $a_i$, and therefore is entirely determined by the ordered set partition $\Phi = \Delta(a)$. We use $S(\Phi)$ to denote the boolean value of whether or not $S(a)$ is satisfied for any (or, equivalently, all) $a$ with $\Delta(a) = \Phi$. Geometrically, this means that $S$, viewed as a function on $\mathbb{R}^n_{>0}$, is constant on each of the open cones, i.e., on the relative interior of, each face of the above triangulation $T^n$ of $\mathbb{R}^n_{>0}$.

These observations lead to the following theorem about scheduling counting functions.

**Theorem 4.2.** Let $S$ be a scheduling problem on $n$ items. The scheduling counting function,

$$\chi_S(k) = \text{the number of } k\text{-schedules satisfying } S,$$

is a polynomial in $k$. Its degree is at most $n$ and the coefficients $f_1, \ldots, f_n$ of $\chi_S$, given by

$$\chi_S(k) = \sum_{i=1}^{n} f_i \binom{k}{i} \quad (2)$$

are non-negative integers, counting the number of ordered set partitions $\Phi$ with $i$ non-empty parts such that $S(\Phi)$ holds. In particular, the $f_i$ are bounded above by $i! \cdot S(n,i)$, where the $S(n,i)$ are the Stirling numbers of the second kind.

As we have seen in Definition and Lemma 3.4, the coefficients of the scheduling nc-quasisymmetric function $\mathfrak{S}_S$ in terms of the monomial basis $\mathcal{M}_\Phi$ are 0 or 1 depending on whether or not $S(\Phi)$ holds. Theorem 4.2 can be viewed as a specialization of this results, obtained by collecting terms according to the number of parts of $\Phi$ or, equivalently, the dimension of the corresponding simplex. As we will see in the proof, Theorem 4.2 can be obtained both from the perspective of nc-quasisymmetric functions and the perspective of Ehrhart theory.

**Proof.** Let $S$ be a scheduling problem. Let $A$ denote the set of all vectors $a \in \mathbb{R}^n_{>0}$ such that $S(a)$ holds true. Recall that the scheduling nc-quasisymmetric function is

$$S_S = \sum_{a \in A \cap \mathbb{Z}^n} x_a = \sum_{\Phi : S(\Phi)} \mathcal{M}_\Phi,$$

where the latter sum ranges over all ordered set partitions $\Phi$ for which $S(\Phi)$ holds true. This means in particular that the coefficients of $S_S$ in the monomial basis are either 1 or 0, indicating which open cones $\text{cone}(\Phi)$ are contained in $A$ and which are disjoint from $A$.

Let $\Phi$ be an ordered set partition with $n$ non-empty parts, i.e., a permutation of the numbers $1, \ldots, n$. Assume without loss of generality that $\Phi = 1|2|\cdots|n$. Then

$$\text{cone}(\Phi) \cap [0, 1]^n = \{ a \in [0, 1]^n | a_1 < a_2 < \ldots < a_n \}$$

which is easily seen to be a unimodular simplex with vertices $(0, \ldots, 0)$, $(0, \ldots, 0, 1)$, $(0, \ldots, 0, 1, 1)$, $\ldots$, $(1, \ldots, 1)$. If, more generally, $\Phi = (\Phi_1, \ldots, \Phi_i)$ is any ordered
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partition with \( i \in [n] \) non-empty parts, then \( \text{cone}(\Phi) \cap [0,1]^n \) is an \( i \)-dimensional unimodular simplex with vertices \( v_0, \ldots, v_i \) where \( v_j \) for \( j = 0, \ldots, i \) has entries

\[
v_{j,k} = \begin{cases} 
1 & \text{if } k \in \Phi_{i-l} \text{ for some } 0 \leq l < j, \\
0 & \text{otherwise.}
\end{cases}
\]

In particular, \( v_0 = 0 \) and \( v_i = 1 \).

Passing to the open cube, we find that the collection of relatively open simplices \( \text{cone}(\Phi) \cap (0,1)^n \) where \( \Phi \) ranges over the ordered set partitions of \([n]\) is a partial simplicial complex that triangulates the open cube \((0,1)^n\). If \( \Phi \) has \( i \) parts, then \( \text{cone}(\Phi) \cap (0,1)^n \) is a relatively open \( i \)-dimensional simplex whose Ehrhart polynomial is

\[
ehr_{\text{cone}(\Phi) \cap (0,1)^n}(k) = \binom{k-1}{i} = \# \text{cone}(\Phi) \cap (0,k)^n \cap \mathbb{Z}^n.
\]

Given a nc-quasisymmetric function, substituting 1s for variables \( x_1, \ldots, x_k \) and 0s for variables \( x_{k+1}, \ldots \) sets all monomials \( x_a \) to 1 for which \( a_i \leq k \) for all \( i \), and all monomials \( x_a \) to 0 for which \( a_i > k \) for some \( i \). Thus substituting \( 1^k \) into \( M_\Phi \) corresponds to taking the intersection of \( \text{cone}(\Phi) \) with the open cube \((0,k+1)^n\), i.e.,

\[
N(\text{cone}(\Phi) \cap (0,k+1)^n \cap \mathbb{Z}^n) = M_\Phi(1^k).
\]

We observe that

\[
\text{cone}(\Phi) \cap (0,k+1)^n = (k+1) \cdot (\text{cone}(\Phi) \cap (0,1)^n),
\]

since the hyperplanes defining \( \text{cone}(\Phi) \) all go through the origin. We have thus found that

\[
M_\Phi(1^k) = ehr_{\text{cone}(\Phi) \cap (0,1)^n}(k+1), \tag{3}
\]

that is, \( M_\Phi(1^k) \) is the Ehrhart polynomial of the open polytope \((\text{cone}(\Phi) \cap (0,1)^n)\), up to a shift of 1.

It follows that

\[
\chi_S(k) = \# A \cap (0,k+1)^n \cap \mathbb{Z}^n = ehr_{A \cap (0,1)^n}(k+1) = \sum_{\Phi \in S(\Phi)} \text{ehr}_{\text{cone}(\Phi) \cap (0,1)^n}(k+1) = M_\Phi(1^k) = \sum_{i=1}^{n} f_i \binom{k}{i},
\]

where \( f_i \) is the number of ordered set partitions \( \Phi \) with \( i \) non-empty parts that satisfy \( S \). Note that the binomial coefficients \( \binom{k}{i} \) for \( i = 0, \ldots, n \) form a basis of the space of polynomials of degree at most \( n \), so the coefficients \( f_i \) are determined by the polynomial \( \chi_S(k) \). The vector \((0, f_1, \ldots, f_n)\) is the \( f^*-\)vector of the Ehrhart polynomial \( ehr_{A \cap (0,1)^n}(k) \) as defined in [6].
From this counting interpretation of the coefficients $f_i$ it follows immediately that the $f_i$ are non-negative integers. Moreover there are at most $i! \cdot S(n, i)$ ordered set partitions of $[n]$ with $i$ non-empty parts, which proves the upper bound on the $f_i$. □

Example 4.3. In our running example of two jobs in which job 1 may start before job 2 or both jobs can start simultaneously but job 2 can not start before job 1, the nc-quasisymmetric scheduling function was $M_{1|2} + M_{12}$. Thus the scheduling polynomial is

$$\chi_S(k) = S_S(1^k) = \binom{k}{2} + \binom{k}{1}.$$ 

Example 4.4. A particularly familiar example of a scheduling problem is graph coloring. Given a finite graph $G = (V, E)$, a $k$-coloring of $G$ is an assignment $\phi(G) : V \rightarrow [k]$ such that for all edges $\{i, j\} \in E$, $\phi(v_i) \neq \phi(v_j)$. Namely, a $k$-coloring colors the vertices of a graph with at most $k$-colors such that if two vertices are joined by an edge, they are given different colors. As a scheduling problem, the edges of the graph give strict atomic formulas: for all edges $\{i, j\} \in E$, $x_i \neq x_j$.

The scheduling quasisymmetric functions are in fact symmetric in the chromatic case. The chromatic symmetric function in non-commuting variables was introduced by Gebhard and Sagan [12] and the chromatic symmetric function was introduced by Stanley [22]. The scheduling counting function is the well studied chromatic polynomial. We point out that the chromatic counting function is usually established to be a polynomial using a contraction deletion argument on the edges of a graph. Our method instead establishes this function as a polynomial via a specialization of a symmetric function and as an Ehrhart polynomial. We will recover much of the well-known structure of these functions by considering the geometry of the scheduling solutions, as seen in the next section.

4.1. Triangulations and their Allowed and Forbidden configurations. Geometrically we have seen that the braid arrangement, consisting of all hyperplanes

$$H_{i,j} = \{ a \in \mathbb{R}^n \mid a_i = a_j \}$$

for $1 \leq i, j \leq n$, induces subdivisions $T_{\mathbb{R}^n}$, $T_{\mathbb{R}_{>0}^n}$, $T_{(0,1)^n}$, and $T_{S^{n-2}}$ of, respectively, the space $\mathbb{R}^n$, the positive orthant $\mathbb{R}_{>0}^n$, the open unit cube $(0,1)^n$ and the $n-2$ dimensional sphere

$$S^{n-2} = \left\{ x \in \mathbb{R}^n \mid \|x\|_2 = 1, \sum_{i=1}^{n} x_i = 0 \right\}.$$ 

The faces of these subdivisions are the regions of the respective ground set where the function $\Delta$, mapping points $a \in \mathbb{R}^n$ to ordered set partitions $\Phi$, is constant. The faces of $T_{\mathbb{R}^n}$ are relatively open polyhedral cones, each containing the lineality space $\{ a \in \mathbb{R}^n \mid a_i = a_j \forall i, j \}$. The faces of $T_{\mathbb{R}_{>0}^n}$ are pointed, relatively open simplicial cones. The faces of $T_{(0,1)^n}$ are relatively open unimodular simplices. The faces of $T_{S^{n-2}}$ are sections of the $n-2$ sphere. Combinatorially $T_{S^{n-2}}$ is equivalent to the simplicial complex known as the Coxeter complex of type $A_{n-1}$. 
All of these complexes are closely related. In fact, $T_{\mathbb{R}^n}$, $T_{\mathbb{R}^n_{\geq 0}}$, and $T_{(0,1)^n}$ are all combinatorially equivalent, as the function $\Delta$ defines a bijection between their sets of faces that preserves incidences. In particular, the faces of these three complexes are in one-to-one correspondence with the ordered set partitions $\Phi$ of $[n]$ into non-empty parts. In turn, $T_{S^{n-2}}$ can be obtained by intersecting $T_{\mathbb{R}^n}$ with the sphere $S^{n-2}$. The main difference is that in $T_{S^{n-2}}$ there is no face corresponding to the partition with only one part. Combinatorially, the other complexes are the cone over $T_{S^{n-2}}$. As these geometric objects are so closely related, we will frequently draw no clear distinction between them and refer loosely to the triangulation $T_n$. It will be clear from context which of these objects we are referring to in each case.

We write $\sigma \in T_n$ to denote that $\sigma$ is a face of the triangulation $T_n$. We use $\sigma_\Phi$ to mean the face of $T_n$ corresponding to the ordered set partition $\Phi$. Recall that for a scheduling problem $S$, $S(\Phi)$ denotes the truth value of whether $S(a)$ holds for any (or, equivalently, all) $a \in \mathbb{R}^n$ with $\Delta(a) = \Phi$. We define the *allowed configuration* $\Lambda(S)$ of the triangulation $T_n$ to be the set of faces

$$\Lambda(S) = \{ \sigma_\Phi \in T_n \mid S(\Phi) \text{ holds} \}.$$  

Correspondingly, define the *forbidden configuration* $\Gamma(S)$ to be the set of faces

$$\Gamma(S) = \{ \sigma_\Phi \in T_n \mid \neg S(\Phi) \text{ holds} \}.$$  

**Example 4.5.** As remarked above, a particularly familiar scheduling polynomial is the chromatic polynomial of a graph $G$. In this case, the graphical scheduling problem $S_G$ simply specifies which variables can not be equal to each other, $x_i \neq x_j$ for $\{i,j\}$ an edge of the graph, which simply means that no two jobs that are connected by an edge are allowed to run simultaneously. Steingrímsson’s coloring complex [23] can be described as the collection of ordered set partitions with at least one edge in at least one block, namely the forbidden configuration of the chromatic scheduling problem. In our framework, Steingrímsson’s coloring complex is the forbidden complex of the graphical scheduling problem $\Gamma_{S^{n-2}}(S_G)$ taken as a subcomplex of the sphere $S^{n-2}$. Much work has been done to understand the coloring complex in particular to better understand the chromatic polynomial. This avenue is possible because of the Hilbert series connection as shown in [23]. As we will see below this connection holds more generally for all scheduling problems.

Clearly, $\Lambda(\neg S) = \Gamma(S)$. This translates into basic observations about the counting function defined by $S$. Let

$$S_{\mathbb{N}^n} = \sum_{a \in \mathbb{N}^n} x_a$$

denote the nc-quasisymmetric function corresponding to all lattice points in the positive orthant. Furthermore, note that $(k - 1)^n$ is the Ehrhart polynomial of the open cube $(0, k + 1)^n$. 
Lemma 4.6. Let $S$ be a scheduling problem on $n$ items. Let $\neg S$ denote the scheduling problem given by the negative of the boolean formula $S$. Then

$$
\chi_S(k) + \chi_{\neg S}(k) = (k - 1)^n,
$$

$$
S_S + S_{\neg S} = S_n.
$$

Proof. The result on $S_S$ is immediate from the above observations. For the result on $\chi_S$, note that

$$
\chi_S(k) + \chi_{\neg S}(k) = \text{ehr}_{A \cap (0,1)^n}(k+1) + \text{ehr}_{0,1)^n \setminus A}(k+1) = \text{ehr}_{0,1)^n}(k+1) = (k - 1)^n.
$$

□

We have seen in the proof of Theorem 4.2, the scheduling polynomial $\chi_S(k)$ equals a shifted Ehrhart polynomial. In particular, the coefficients $f_i$ in the expansion (2) count the number of faces of the allowable configuration $\Lambda(S)$. Similarly, coefficients of $\chi_S$ in related binomial bases give rise to the $h$-vector of the partial simplicial complex $\Lambda(S)$ and its Ehrhart $h^*$-vector, as we will see in the next theorem.

As explained in the preliminaries that the Ehrhart polynomial of a $d$-dimensional unimodular simplex $\Delta_i^d$ with $i$ open faces is

$$
\text{ehr}_{\Delta_i^d}(k) = \binom{k + d - i}{d}.
$$

A $d$-dimensional simplex can have between 0 and $d+1$ open faces, and the polynomials $\binom{k+d-i}{d}$ for $i = 0, \ldots, d+1$ generate the space of all polynomials of degree at most $d$. That is, for every polynomial $p$ of degree at most $d$ there exist coefficients $h_0, \ldots, h_{d+1}$ such that

$$
p(k) = \sum_{i=0}^{d+1} h_i \binom{k+d-i}{d}.
$$

However, these coefficients $h_i$ are not uniquely determined as there is one degree of freedom too many. There are two different conventions for normalization, which determine the coefficient vector uniquely. The first convention is to fix $h_0 = 1$, which gives rise to the $h$-vector as commonly used in the study of simplicial complexes. This choice is motivated by the notion of a shelling, which induces a partition of a simplices into half-open simplices in which exactly one is closed [25]. The second convention is to fix $h_{d+1} = 0$, which gives rise to the $h^*$-vector (or Ehrhart $\delta$-vector) used in Ehrhart theory [3].

Formally, given numbers $f_0, \ldots, f_d$, the $h$-vector $h_0, \ldots, h_{d+1}$ and the $h^*$-vector $h_0^*, \ldots, h_d^*$ are defined, respectively, via $h_0 = 1$ and

$$
\sum_{i=0}^{d} f_i \binom{k-1}{i} = \sum_{i=0}^{d+1} h_i \binom{k+d-i}{d} = \sum_{i=0}^{d} h_i^* \binom{k+d-i}{d}.
$$

Typically, the numbers $f_0, \ldots, f_d$ are either the $f$-vector of a (partial) simplicial complex\(^2\) or the coefficients of a polynomial $p(k)$ given in the binomial basis.

\(^2\)That is, $f_i$ counts the number of $i$-dimensional faces of the complex.
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\( p(k) = \sum_{i=0}^{d} f_i \binom{k-1}{i} \). Given \( p(k) \) in this form, the \( h \)- and \( h^* \)-vectors can be defined, equivalently, by

\[
1 + \sum_{k=1}^{\infty} p(k) t^k = \frac{\sum_{i=0}^{d+1} h_i t^i}{(1 - t)^{d+1}} \quad \text{and} \quad \sum_{k=0}^{\infty} p(k) t^k = \frac{\sum_{i=0}^{d} h_i^* t^i}{(1 - t)^{d+1}}.
\]

Note that \( h_0^* = \sum_{i=0}^{d} (-1)^i f_i \), from which it follows that the \( h \)- and \( h^* \)-vectors of a complex \( X \) coincide if and only if the Euler characteristic of \( X \) is one.

**Theorem 4.7.** Let \( S \) be a scheduling problem. Then the \( h \)-vector of the shifted scheduling polynomial \( \chi_S(k - 1) \) is the \( h \)-vector of the allowed configuration \( \Lambda(S) \) and the \( h \)-vector of the polynomial \( (k - 2)^n - \chi_S(k - 1) \) is the \( h \)-vector of the forbidden configuration \( \Gamma(S) \), i.e.,

\[
h(\chi_S(k - 1)) = h(\Lambda(S)), \quad \text{and} \quad h((k - 2)^n - \chi_S(k - 1)) = h(\Gamma(S)),
\]

or equivalently,

\[
1 + t \sum_{k=0}^{\infty} \chi_S(k) t^k = \frac{h_{\Lambda(S)}(t)}{(1 - t)^{d+1}}, \quad \text{and} \quad 1 + t \sum_{k=0}^{\infty} ((k - 1)^n - \chi_S(k)) t^k = \frac{h_{\Gamma(S)}(t)}{(1 - t)^{d+1}},
\]

where \( h_{\Delta}(t) \) is the \( h \)-polynomial of \( \Delta, \sum_i h_i(\Delta) t^i \).

**Proof.** Let \( S \) be a scheduling problem, then

\[
\chi_S(k) = \text{ehr}_{\Lambda(S)}(k + 1) \quad \text{and} \quad (k - 1)^n - \chi_S(k) = \text{ehr}_{\Gamma(S)}(k + 1).
\]

Here, the allowed and forbidden configurations \( \Lambda(S) \) and \( \Gamma(S) \) are partial subcomplexes of \( T_{(0,1)^n} \) which is an integral unimodular triangulation of the open unit cube \( (0,1)^n \). In particular, as we know from Theorem 4.2, the coefficients \( f_i \) in (2) count the number of \( i \)-dimensional simplices in the respective complexes. Therefore, the \( h \)-vector of these complexes coincide with the \( h \)-vectors of their Ehrhart polynomials,

\[
1 + t \sum_{k=0}^{\infty} \chi_S(k) t^k = 1 + \sum_{k=1}^{\infty} \text{ehr}_{\Lambda(S)}(k) t^k = \frac{\sum_{i=0}^{d+1} h_i(\Delta(S)) t^i}{(1 - t)^{d+1}}
\]

and

\[
1 + t \sum_{k=0}^{\infty} ((k - 1)^n - \chi_S(k)) t^k = 1 + \sum_{k=1}^{\infty} \text{ehr}_{\Gamma(S)}(k) t^k = \frac{\sum_{i=0}^{d+1} h_i(\Gamma(S)) t^i}{(1 - t)^{d+1}},
\]

giving the stated results.

Analogous statements about the \( h^* \)-vector are straightforward to derive using the same technique.
Remark 4.8. The above theorem draws connections between the $h$- and $h^*$-vectors of the complexes $\Lambda_{(0,1)^n}(S)$ and $\Gamma_{(0,1)^n}(S)$ and the $h$- and $h^*$-vectors of the shifted scheduling polynomial $\chi_S$. This shift can be avoided by working with the half-open cube $(0,1]^n$ instead of the open cube $(0,1)^n$. In this case we have, for example,

$$\text{ehr}_{\Lambda_{(0,1)^n}(S)}(k) = \chi_S(k)$$

and hence

$$h^*(\Lambda_{(0,1)^n}(S)) = h^*(\chi_S(k))$$

with similar statements holding for the forbidden subcomplexes and/or the $h$-vector. The reason for our choice of working with the open cube throughout this paper, is that the faces of the partial simplicial complex $T_{(0,1)^n}$ are not in one-to-one correspondence with ordered set partitions: there are additional faces on the boundary of the half-open cube which correspond to the last part of an ordered set partition being maximal. To avoid this additional complication we work with the open cube at the expense of having to deal with the shift $\text{ehr}_{\Lambda_{(0,1)^n}(S)}(k + 1) = \chi_S(k)$.

Returning to the special case of graph coloring, Theorem 4.7 was established for $\chi_S$ equal to the chromatic polynomial and $\Gamma(S)$ equal to the coloring complex in [23]. Theorem 4.7 was also established for all Scheduling problems in which $\Gamma(S)$ forms a proper subcomplex of the Coxeter complex in [1]. Theorem 4.7 allows one to prove results on the scheduling polynomial by studying the geometry of the space of solutions to the scheduling problem. This approach has been used heavily to prove new bounds on the coefficients of the chromatic polynomial. Below we consider scheduling problems given by boolean functions of a particularly nice form. In this case, we will be able to prove partitionability of the allowed configuration.

5. Positivity and Partitionability

An important class of results in the literature on quasisymmetric functions, simplicial complexes and Ehrhart theory are theorems asserting the non-negativity of coefficient vectors in various bases. Here we consider the expansions of the scheduling nc-quasisymmetric, symmetric and polynomial functions in several bases and the relations of these expansions to the geometry of the allowable and forbidden configuration.

These results fall into two broad categories. In Section 5.1, we study non-negativity in the fundamental basis of (nc-)quasisymmetric functions, which is closely related to non-negativity of the $h^*$-vector of the Ehrhart polynomial. A geometric condition that guarantees non-negativity of this type is partitionability. In Section 5.2, we consider non-negativity in the co-fundamental basis of quasisymmetric functions, which, geometrically, is guaranteed if the allowed configuration is a subcomplex, i.e., closed under taking faces.
5.1. **Partitionability, the Ehrhart $h^*$-vector, and the Fundamental Basis.** A partial simplicial complex is partitionable if it can be written as a disjoint union of half-open simplices, see Figure 4. As we have seen in the preliminaries on Ehrhart theory (Section 2), unimodular half-open simplices have certain binomial coefficients as their Ehrhart polynomials. This ties the coefficients of Ehrhart polynomials in the corresponding binomial basis to the number of half-open simplices in a partition, and shows non-negativity of the $h^*$-vector if such a partition exists. On the level of nc-quasisymmetric functions, a partition corresponds to a decomposition into half-open simplicial cones which can be identified with elements of the fundamental basis.

![Figure 4](image-url)

**Figure 4.** a) A partial simplicial complex and a partition of it. Dashed edges are open, solid lines are closed. White vertices are not contained in the complex, but black vertices are. b) A triangle with one closed edge and one vertex is not partitionable. c) A non-partitionable partial simplicial complex. This complex has 0 vertices, 6 edges and 4 triangles. Thus, in a partition, one simplex would need to have at least two closed edges. But then it would also have to contain a vertex, which is impossible.

In this section we develop these connections. First, we introduce decision trees as a natural class of scheduling problems that give rise to partitionable allowable configurations. Then, we draw the connection between partitionability and non-negativity of coefficients on the level of polynomials and (nc-)quasisymmetric functions. Finally we discuss important special cases, including generalized permutahedra and $P$-partitions.

Let $\Delta$ be a simplicial complex and assume that $\Delta$ is pure, i.e., that all maximal simplices have the same dimension $k$. A *partition* of $\Delta$ is a map from maximal simplices $\sigma \in \Delta$ to half-open simplices $\hat{\sigma}$ such that $\text{relint}(\sigma) = \text{relint}(\hat{\sigma})$, any two half-open simplices $\hat{\sigma}_1, \hat{\sigma}_2$ are disjoint, and the union of all half-open simplices $\hat{\sigma}$ equals $\Delta$. A partial simplicial complex $\Delta$ is *partitionable* if there exists a partition
of $\Delta$. This generalizes the standard notion of partitionability to partial complexes. Examples of partitionable and non-partitionable complexes are given in Figure 4.

5.1.1. Decision Trees. Decision trees are a very commonly used form of boolean expression. Intuitively, they are simply nested if-then-else statements. We will work with decision trees where the conditions in the if-clauses are inequalities of the form $x_i \leq x_j$ and the conditions in the leaves of the tree are conjunctions of such inequalities (either strict or weak).

Figure 5 shows the example of a decision tree with only one if-then-else statement and the geometric region of points that satisfy the statement. The collection of points $x \in \mathbb{R}^n$ that satisfy the conditions of a tree form a union of faces of the cell decomposition induced by the braid arrangement or equivalently a collection of faces of the Coxeter complex $\text{Cox}_A$. In general, this union of faces can be non-convex (as seen in the Figure) and even non star-convex. Below we show that if a scheduling problem can be expressed as the union of decision trees which satisfy a certain property, then the allowable configuration $\Lambda(S)$ is partitionable, which in turn implies non-negativity of coefficients for the scheduling functions.

More formally, decision trees of boolean expressions are defined recursively as follows.

3For simplicial complexes, partitionability is often defined in terms of the face poset [15, 25], which is equivalent to the above definition in terms of half-open simplices. The face-poset-definition can also be carried over to partial simplicial complexes as follows. Let $\bar{\Delta}$ be the closure of $\Delta$ as defined in Section 2. Let $P$ be the face poset of $\bar{\Delta}$ and let $\bar{P}$ be the subposet consisting of those faces of $\bar{\Delta}$ whose relative interior is contained in $\Delta$. Then $\Delta$ is partitionable if $P$ can be written as the disjoint union of boolean lattices whose top elements are all at the same height, with respect to $\bar{\Delta}$. 

![Figure 5](image-url)
Definition 5.1. A leaf is a boolean expression $\psi$ that is a conjunction of inequalities of the form $x_i \leq x_j$ or $x_i < x_j$. A decision tree is either a leaf, or a boolean expression of the form

$$\text{if } \varphi \text{ then } \psi_t \text{ else } \psi_f$$

where $\varphi$ is an inequality of the form $x_i \leq x_j$ or $x_i < x_j$ and $\psi_t, \psi_f$ are decision trees.

As decision trees are binary trees, it will be convenient to distinguish notationally between a node $v$ of a tree $S$ and the boolean expression $S_v$ at that node.

Every leaf $v$ of a decision tree corresponds to a conjunction of inequalities which we call the cell at $v$. The cell at $v$ is the conjunction of $S_v$ (the inequalities given by the leaf $v$ itself) and the constraints given in the if-clauses of ancestors of $v$, negated according to whether $v$ resides in the “true” or “false” branch of the corresponding if-then-else clause. Let $v$ denote a leaf of the tree and let $v_0, \ldots, v_k$ denote its ancestors. Then $S_v$ is a boolean formula of the form “if $\varphi$ then $\psi_t$ else $\psi_f$” and we define $\varphi_v' := \varphi$ if $v$ resides in the branch $\psi_t$ and $\varphi_v' := \neg \varphi$ if $v$ resides in the branch $\psi_f$. We denote by $C_v$ the conjunction

$$C_v := \varphi_v' \land \cdots \land \varphi_{v_k}' \land S_v \land \bigwedge_{i=1}^{n}(0 < x_i < 1).$$

By a slight abuse of notation, we will also use $C_v$ to denote the polytope of all $x \in \mathbb{R}^n$ satisfying $C_v$. In the above formula, the purpose of $\bigwedge(0 < x_i)$ is to ensure that all solutions lie in the open cube $(0,1)^n$, as is usual in the Ehrhart theory setting. When working with quasisymmetric functions, this condition would be replaced, accordingly, with $\bigwedge(0 < x_i)$ so as to ensure that solutions are positive. In this case the solution sets $C_v$ are cones. Note that the set of formulas $\varphi_{v_0}' \land \cdots \land \varphi_{v_k}' \land S_v$ for all leaves $v$ of $S$ correspond exactly to the conjunctions in the disjunctive normal form of $S$.

To illustrate these definitions, we consider the following example, which is illustrated in Figure 6. Let $S$ denote the following decision tree.

if $x_1 \leq x_2$ then
  if $x_1 < x_4$ then
    if $x_2 < x_3$ then $x_4 < x_3$ else ($x_4 < x_2$ and $x_1 < x_3$)
  else
    $x_3 < x_2$ and $x_1 < x_3$
else
  if $x_1 \leq x_3$ then ($x_2 < x_4$ and $x_4 < x_3$) else ($x_1 < x_4$ and $x_2 < x_3$)

Of course, the same boolean function can be represented by different decision trees. The cells of this decision tree are the following (up to conjuncts of the form $\bigwedge(0 <$
Figure 6. The decision tree described in the text and its cells. a) The front faces of the Coxeter complex, with the faces contained in the allowable configuration of the decision tree shaded in gray. The face 4|1|3|2 of the allowable configuration lies on the back face of the Coxeter complex. b) A view from above onto the Coxeter complex, showing all faces in the allowable configuration at once. c) The faces of the allowable configuration as partitioned by the decision tree. The cells of the decision tree are the gray components shown in the figure. Dashed lines indicate open faces, while solid lines indicate closed faces. Note that $C_2$ satisfies both $x_1 \leq x_2$ and $x_3 \leq x_2$, however, $x_1 \leq x_2$ is redundant and therefore does not produce a closed facet.

$x_i < 1$) which have been omitted for brevity).

$$C_1 = (x_1 \leq x_2) \land (x_1 < x_4) \land (x_2 < x_3) \land (x_4 < x_3),$$

$$C_2 = (x_1 \leq x_2) \land (x_1 < x_4) \land (x_3 \leq x_2) \land (x_4 < x_2) \land (x_1 < x_3),$$

$$C_3 = (x_1 \leq x_2) \land (x_4 \leq x_1) \land (x_3 < x_2) \land (x_1 < x_3),$$

$$C_4 = (x_2 < x_1) \land (x_1 \leq x_3) \land (x_2 < x_4) \land (x_4 < x_3),$$

$$C_5 = (x_2 < x_1) \land (x_3 < x_1) \land (x_1 < x_4) \land (x_2 < x_3).$$
In the following, we are going to apply some geometric terminology to boolean formulas. A disjunction $\bigvee_i C_i(x)$ is called \textit{disjoint}, if the solution sets are disjoint, i.e., if for every $x$ at most one of the $C_i(x)$ is true. Moreover, the \textit{dimension} of a conjunction $C$ of inequalities is the dimension of the polyhedron of all solutions, i.e., it is the number of variables minus the maximal number of linearly independent equations implied by $C$.

We call a formula $C$ \textit{almost open}, if it is equivalent to a conjunction $\bigwedge_j I_j$ of inequalities $I_j$ such that at most one of the $I_j$ is weak and all of the $I_j$ are facet-defining. Note that the $I_j$ are facet-defining if the closure $\bar{C}$ of (the polytope defined by) $C$ has a non-redundant description $\bar{C} = \bigwedge_j \bar{I}_j$, where $\bar{I}_j$ denotes the weak inequality defined by $I_j$. Geometrically, almost open means that the polytope $C$ is open except for at most one face which may be closed.

In the above example, all cells of the decision tree $S$ are almost open. Note that in $C_2$, the weak inequality $x_1 \leq x_2$ is redundant, whence $C_2$ can equivalently be written as $C_2 = (x_1 < x_4) \land (x_3 \leq x_2) \land (x_4 < x_2) \land (x_1 < x_3)$. Similarly, $C_3$ can be rewritten using only one weak inequality (and two strict inequalities). Note that in Figure 6 all cells have at most one closed face.

As defined at the beginning of this section, a partial polytopal complex is partitionable, if it can be written as the disjoint union of half-open simplices of full dimension $n$. In our case, we are of course restricted to use simplices given by the braid triangulation.

**Theorem 5.2.** If a scheduling problem $S$ is equivalent to a formula of the form

$$ S \equiv \bigvee_i C_i \text{ where } C_i = \bigwedge_j I_{i,j} $$

where the disjunction is disjoint and the $C_i$ are almost open of dimension $d$, then $\Lambda(S)$ is partitionable.

Because the disjunction is disjoint, this theorem follows immediately, if we can prove it for almost open conjunctions.

**Lemma 5.3.** An almost open conjunction $C$ of dimension $d$ is partitionable.

It is well-known that boundary complexes of simplicial polytopes are partitionable, a fact that can for example be shown using line-shellings [8]. This method can also be used to construct shellings (and thus partitions) of regular triangulations of polytopes [25, Corollary 8.14]. These techniques extend naturally to almost open polytopes, as shown in the following proof. For the proof, we assume familiarity with regular triangulations, line-shellings and their connection to partitionability [9, 15, 25].

**Proof.** We first deal with the case where the polytope $C$ has exactly one closed face. Recall that having one closed face means that all but one of the facet-defining inequalities of $C$ are strict.

$C$ is a subconfiguration of the braid triangulation $T_{(0,1)^n}$ of the open cube, as defined in Section 4.1. Therefore, the induced triangulation $T$ of the almost open polytope $C$ is regular. Let $F$ be the facet of $C$ which is closed, and let $z$ denote a new point
close to $F$ but outside of $C$. We then create an open polytope $C'$ with $z$ as a new vertex by taking the open convex hull of $C$ and $z$:

$$C' = \text{oconv}(C, v) := \{ \lambda x + (1 - \lambda) v \mid x \in C, \lambda \in (0, 1) \}.$$ 

Note that $C'$ is open because $(1 - \lambda) < 1$ and $F$ is the only closed face of $C$. The extended polytope $C'$ then has a triangulation $T'$ which consists of all the simplices in $T$ and the open convex hulls $\text{oconv}(\sigma, z)$ of simplices $\sigma \in T$ such that $\sigma \subset F$ with the new vertex $z$.

The triangulation $T'$ is regular. Therefore, there exists a polytope $P$ that has $C'$ as its lower hull. We now construct a line shelling of the lower hull of $P$ which starts with a facet in $T' \setminus T$. The lifted polytope $P$ can be chosen such that all facets in $T' \setminus T$ are shelled first, again because $T$ is separated from $T' \setminus T$ by a single hyperplane, see also [7, Lemma 2].

Let $\sigma_1, \ldots, \sigma_N$ be the induced shelling order of the maximal-dimensional simplices in $C$. Let $\bar{\sigma}_i$ denote the closure of $\sigma_i$. Then, the sequence of half-open simplices

$$(\nu_i)_{i=1,\ldots,N} := \bar{\sigma}_1, \bar{\sigma}_2 \setminus \bar{\sigma}_1, \bar{\sigma}_3 \setminus (\bar{\sigma}_2 \cup \bar{\sigma}_1), \ldots, \sigma_N \setminus \bigcup_{i=1}^{N-1} \sigma_i$$

form a partition of the closed polytope $\bar{C}'$. By reversing which faces of these half-open simplices are open and which are closed, we obtain a partition

$$(\tau_i)_{i=1,\ldots,N} := \bar{\sigma}_N \setminus \partial C', \bar{\sigma}_{N-1} \setminus (\partial C' \cup \bar{\sigma}_N), \ldots, \bar{\sigma}_1 \setminus (\partial C' \cup \bigcup_{i=2}^N \bar{\sigma}_i)$$

of the open polytope $C'$. Let $\tau_i$ denote the $i$-th element in this sequence, that is,

$$\tau_i = \sigma_{N-i+1} \setminus (\partial C' \cup \bigcup_{i=N-i+2}^N \sigma_i)$$

for $i = 1, \ldots, N$. A face is open if it occurred in a previous simplex of the sequence or the boundary. Because we run through the sequence in reverse order, this flips the state of all faces versus the first partition defined above.

By construction, the simplices in $T' \setminus T$ are the last half-open simplexes in the sequence $(\tau_i)_i$ above, that is, there exists an $l$ such that $\sigma_i \subset C$ for all $i \leq l$ and $\sigma_i \subset C' \setminus C$ for all $i > l$. Then, the sequence $\tau_1, \ldots, \tau_l$ is a partition of $C$, as desired. Note in particular that $\bigcup_{i=1}^l \tau_i$ includes the closed face $F$, precisely because all faces of $C$ come first in the sequence $C$.

This concludes the proof for the case that $C$ has one closed face. The proof for the case that $C$ is an open polytope without closed faces is completely analogous, only simpler as there is no need to add the new vertex $z$. \[\square\]

---

4 The new simplices $\sigma \in T' \setminus T$ are separated from $C$ by the hyperplane defining $F$. Moreover $T$ induces a regular triangulation of $F$ and thus the triangulation of $C' \setminus C$ induced by $T'$ is regular as well.
Theorem 5.4. Let $S$ be a decision tree such that all cells of $S$ are almost open of dimension $d$. Then $\Lambda(S)$ is partitionable. Moreover, disjoint unions of such decision trees are partitionable.

Proof. A decision tree is the disjoint union of all its cells. By Lemma 5.3, all cells are partitionable because they are almost open. Therefore the decision tree is partitionable. \hfill \Box

Note that a cell can be in the "strict inequality" branch of many of its ancestors and still be almost open. The reason is that, in practice, many of these strict ancestral inequalities will be redundant.

Remark 5.5. In general, a partial complex of dimension $k$ is partitionable if it can be written as a disjoint union of half-open simplices of dimension $k$. In the above, we have defined partitionable with respect to the ambient dimension $n$, as this will be convenient when working with the fundamental bases of $\text{NCQSym}$ and $\text{QSym}$ below. However, this restriction is not essential. If $\Lambda(S)$ can be written as a disjoint union of almost open cells of a fixed dimension $k < d$, then the same proofs show partitionability of $\Lambda(S)$.

5.1.2. Fundamental Bases for $\text{NCQSym}$ and $\text{QSym}$. Above we established that if a scheduling problem can be written as a decision tree such that all cells are almost open then the allowable configuration is partitionable. Here we prove that partitionability of the allowable configuration implies positivity of the scheduling (nc-)quasisymmetric function in the (nc-)fundamental basis.

Let $\Phi_c$ and $\Phi_f$ (standing for coarse and fine) be two ordered set partitions such that $\Phi_f$ is a permutation, i.e., an ordered set partition of maximal length into blocks of size one, that refines $\Phi_c$. Then the poset of all ordered set partitions $\Phi$ between $\Phi_c$ and $\Phi_f$ under the refinement relation forms a boolean lattice of dimension $n - \ell(\Phi_c)$ where $\ell$ is the length of $\Phi_c$. Moreover, the type map gives an isomorphism from this boolean lattice to the boolean lattice of all compositions refining the composition type($\Phi_c$).

Thus,

$$L(\Phi_c; \Phi_f) := \sum_{\Phi_c \leq \Phi \leq \Phi_f} M_\Phi$$

is an nc-quasisymmetric function $L(\Phi_c; \Phi_f)$ that specializes to $L_{\text{type}(\Phi_c)}$ when the variables are allowed to commute.

As $(\Phi_c; \Phi_f)$ ranges over all such pairs of ordered set partitions, which are comparable under the refinement relation and where $\Phi_f$ has length $n$, the functions $L(\Phi_c; \Phi_f)$ form a generating system of the linear space of all nc-quasisymmetric functions. However, they do not form a basis as there are multiple representations of the same function, for example

$$L(1|2;3|1|2|3) + L(1|23;1|2|3|2) = L(1|23|1|2;3) + L(1|3|2;1|3|2)$$

$$= M_{1|2|3} + M_{1|23} + M_{1|3|2}.$$

We call this the fundamental generating system of the nc-quasisymmetric functions. To obtain a basis, we must fix a choice for $\Phi_f$ given $\Phi_c$. In particular, for any ordered
set partition \( \Phi \), let \( \hat{\Phi} \) denote the permutation refining \( \Phi \) with the property that the elements of each part of \( \Phi \) are listed in order. For example, if \( \Phi = 35|247|16 \) then \( \hat{\Phi} = 3|5|2|4|7|1|6 \). Given an ordered set partition, define

\[
\mathcal{L}_\Phi := \mathcal{L}_{(\Phi, \hat{\Phi})}.
\]

As \( \Phi \) ranges over all ordered set partitions, the functions \( \mathcal{L}_\Phi \) form a basis for the space of nc-quasisymmetric functions, which we call the nc-fundamental basis.

Alternatively, this fundamental basis for nc-quasisymmetric functions can be defined in terms of a directed refinement relation \( \preceq \) on ordered set partitions given by \((\Phi_i, \ldots, \Phi_{i-1}, \Phi_i \cup \Phi_{i+1}, \Phi_{i+2}, \ldots, \Phi_k) \prec \Phi \) where every element of the \( i \)th block is less than every element of the \( i+1 \)th block, see Figure 7. Then, the nc-fundamental basis for \( \text{NCQSym} \) can be defined by

\[
\mathcal{L}_\Phi := \sum_{\Psi: \Phi \preceq \Psi} M_\Psi,
\]

where \( \Phi \) ranges over all ordered set partitions. We note that this order is opposite to the order \( \leq \) used to define the basis \( Q_\Phi \) in [4]. Our choice of ordering is particularly motivated by its connection the the fundamental basis \( L \) of QSym; the \( \mathcal{L} \) basis of \( \text{NCQSym} \) restricts to the \( L \) basis of QSym when the variables are allowed to commute.

Namely, recall that the fundamental quasisymmetric functions of QSym are defined from the monomial quasisymmetric functions as follows. For any composition \( \alpha \),

\[
L_\alpha := \sum_{\beta: \beta \text{ refines } \alpha} M_\beta.
\]

The poset of all compositions that refine \( \alpha \), ordered by the refinement relation, forms a boolean lattice of dimension \( n - \ell(\alpha) \) where \( n \) is the total number of elements in the ground set and \( \ell \) is the length of \( \alpha \). For example, the composition \( 3+1 \) is refined by \( 1+2+1, 2+1+1 \) and \( 1+1+1+1 \), where \( 1+2+1 \) and \( 2+1+1 \) are incomparable under the refinement relation.

Applying the type map in the \( \mathcal{L} \) basis gives the corresponding quasisymmetric function in the \( L \) basis in such a way that directed refinement on the level of nc-quasisymmetric functions corresponds to refinement on the level of quasisymmetric functions:

\[
\text{NCQSym} \xrightarrow{\text{directed refinement}} \text{NCQSym} \xrightarrow{\text{type}} \text{QSym} \xrightarrow{\text{refinement}} \text{QSym} \xrightarrow{\text{type}} \text{QSym} \xrightarrow{L}
\]

**Example 5.6.** Consider the nc-quasisymmetric function

\[
\mathcal{F} = M_{12|3} + M_{1|23} + M_{1|2|3} + M_{2|1|3} + M_{1|3|2} = L_{12|3} + L_{1|23} - L_{1|2|3} + L_{2|1|3} + L_{1|3|2}.
\]
The corresponding quasisymmetric function; i.e. the quasisymmetric function obtained from $\mathcal{F}$ by allowing the variables to commute is given by

$$
F = M_{(2,1)} + M_{(1,2)} + 3M_{(1,1,1)}
= L_{(2,1)} + L_{(1,2)} + L_{(1,1,1)}
= L_{\text{type}(12|3)} + L_{\text{type}(1|23)} - L_{\text{type}(1|2|3)} + L_{\text{type}(2|1|3)} + L_{\text{type}(1|3|2)}.
$$

Let $\Delta$ be an $n$-dimensional half-open simplex with $i$ open faces, that, as a partial simplicial complex, is a subcomplex of $T_{(0,1)^n}$. As we have seen above, its faces are in one-to-one correspondence with ordered set partitions. Its minimal $i-1$-dimensional face corresponds to an ordered set partition $\Phi_c$ of length $i-1$. Its maximal $n$-dimensional face corresponds to an ordered set partition $\Phi_f$ of length $n$, i.e., a permutation of $n$. The interval of ordered set partitions between $\Phi_f$ and $\Phi_c$ corresponds precisely to the face poset of $\Delta$. Therefore, we now immediately have

$$
\mathcal{L}_{(\Phi_c;\Phi_f)}(1^k) = \text{ehr}_\Delta(k+1) = \binom{k + n - i + 1}{n}.
$$

In other words, if $\Phi_c$ is an ordered set partition of length $j = 1, \ldots, n$ and $\Phi_f$ is a permutation refining it, then

$$
\mathcal{L}_{(\Phi_c;\Phi_f)}(1^k) = \mathcal{L}_{\Phi_c}(1^k) = \binom{k + n - j}{n}.
$$

This observation extends directly to quasisymmetric functions. Writing $\text{type}(\mathcal{L})$ for the specialization that turns an nc-quasisymmetric function into a quasisymmetric function by allowing variables to commute, we have

$$
L_\alpha = \text{type}(\mathcal{L}_{\Phi_\alpha}) \quad \text{and} \quad L_\alpha(1^k) = \mathcal{L}_{\Phi_\alpha}(1^k) = \binom{k + n - \ell}{n}
$$

for any composition $\alpha$, where $\Phi_\alpha$ denotes any ordered set partition with $\text{type}(\Phi_\alpha) = \alpha$, and $\ell$ is the length of $\alpha$. We summarize the connections between the fundamental basis of (nc-)quasisymmetric functions and the $h^*$-vector of corresponding polynomials below.

**Proposition 5.7.** Let $\mathcal{F}$ denote an nc-quasisymmetric function, let $F$ denote a quasisymmetric function and let $p$ denote a polynomial such that

$$
\text{type}(\mathcal{F}) = F \quad \text{and} \quad \mathcal{F}(1^k) = F(1^k) = p(k).
$$

Namely, $\mathcal{F}$ specializes to $F$ when the variables are allowed to commute and both $\mathcal{F}$ and $F$ specialize to $p$ when the first $k$ variables are set equal to 1 and the rest to 0. Moreover, let $\mu_\Phi$, $\mu_{(\Phi_c;\Phi_f)}$ and $\lambda_\alpha$ denote the coefficient vectors of $\mathcal{F}$ and $F$ in terms
of the fundamental bases and let \((0, h^*_1, \ldots, h^*_n)\) denote the \(h^*\)-vector of \(p\),

\[
\mathcal{F} = \sum_{\Phi} \mu_\Phi \mathcal{L}_\Phi = \sum_{(\Phi_c; \Phi_f)} \mu_{(\Phi_c; \Phi_f)} \mathcal{L}_{(\Phi_c; \Phi_f)},
\]

\[
F = \sum_{\alpha} \lambda_\alpha L_\alpha,
\]

\[
p(k) = \sum_{i=1}^{n} h^*_i \binom{k+n-i}{n}.
\]

Then

\[
h^*_i = \sum_{\Phi \vdash (\Phi_f) \vdash (\Phi_c)} \mu_\Phi = \sum_{(\Phi_c; \Phi_f): \Phi_c \leq \Phi_f, \ell(\Phi_c) = i} \mu_{(\Phi_c; \Phi_f)} = \sum_{\alpha \vdash (\alpha) \vdash (\alpha)} \lambda_\alpha.
\]

The coefficients \(h^*_i\), \(\mu_\Phi\), \(\mu_{(\Phi_c; \Phi_f)}\) and \(\lambda_\alpha\) are integral but may be negative.

Non-negativity of the \(\mu_\Phi\) or the \(\mu_{(\Phi_c; \Phi_f)}\) implies non-negativity of the \(\lambda_\alpha\), and, in turn, non-negativity of the \(\lambda_\alpha\) implies non-negativity of the \(h^*_i\).

We now return to the geometry of the allowed and forbidden configurations and see that partitionability implies positivity expansions for the fundamental bases and the \(h^*\) coefficients.

**Theorem 5.8.** Let \(S\) be a scheduling problem such that \(\Lambda(S)\) is partitionable. Then there exists a representation

\[
\mathcal{S}_S = \sum_{(\Phi_c; \Phi_f)} \mu_{(\Phi_c; \Phi_f)} \mathcal{L}_{(\Phi_c; \Phi_f)}
\]

with non-negative coefficients \(\mu_{(\Phi_c; \Phi_f)} \in \{0, 1\}\). In particular, the scheduling quasisymmetric function is \(L\)-positive and the scheduling polynomial is \(h^*\)-positive.

Conversely, the existence of a representation of \(\mathcal{S}_S\) with \(0-1\) coefficients in terms of the fundamental generating system implies that \(\Lambda(S)\) is partitionable.

Note that partitionability is not implied when the coefficients of the scheduling polynomial or quasisymmetric function are non-negative. There are scheduling problems where \(h^*(\chi_S)\) is non-negative but \(\Lambda(S)\) is not partitionable.

**Proof.** As \(\Lambda((0,1)^n)(S)\) is partitionable, there exists a collection \(C\) of half-open \(n\)-dimensional simplices \(\sigma \in T_{(0,1)^n}\) such that \(\Lambda((0,1)^n)(S) = \bigcup_{\sigma \in C} \sigma\) and this union is disjoint. Each element \(\sigma \in C\) corresponds to a distinct pair \((\Phi_c; \Phi_f)\) of ordered set partitions, where \(\Phi_f\) refines \(\Phi_c\) and \(\Phi_f\) has length \(n\). Let \(P\) denote the collection of all pairs corresponding to half-open simplices in \(C\). Then

\[
\mathcal{S}_S = \sum_{(\Phi_c; \Phi_f) \in P} \mathcal{L}_{(\Phi_c; \Phi_f)}
\]

as desired. The non-negativity of the coefficients of the scheduling quasisymmetric function in the fundamental basis and the \(h^*\)-vector of \(\chi_S\) is implied by the existence of a non-negative representation of \(\mathcal{S}_S\) in the fundamental generating system. \(\square\)
Corollary 5.9. Let $S$ be a scheduling problem expressible as a union of decision trees with all cells almost open, then the scheduling quasisymmetric function is $L$-positive and the scheduling polynomial is $h^*$ positive.

Proof. This is an immediate consequence of Theorems 5.8 and 5.4. □

The next theorem guarantees positivity of the nc-quasisymmetric scheduling function $S$ in terms of the directed refinement relation. It is illustrated in Figure 7 below.

Theorem 5.10. Let $S$ be a scheduling problem such that $\Lambda(S)$ is closed under the directed refinement relation. If for every $\Phi \in \Lambda(S)$ there exists a unique coarsest allowed ordered set partition $\Phi_c \preceq \Phi$ such that $\Phi$ is a directed refinement of $\Phi_c$, then the coefficients of $S_S$ in terms of the fundamental basis as well as the $h^*$-vector of $\chi_S$ are non-negative.

Proof. For any ordered set partition $\Phi_f \in \Lambda(S)$ of length $n$, there exists by assumption a unique coarsest allowed ordered set partition $\Phi_c$ with $\Phi_c \preceq \Phi_f$. Let $P$ be the collection of all such pairs. Since $\Lambda(S)$ is closed under the directed refinement relation, the intervals $[\Phi_c, \Phi_f]$ are boolean lattices. Furthermore, for any $\Phi \in \Lambda(S)$ there exists a $\Phi_f$ of length $n$ that is a directed refinement of $\Phi$, hence $\Phi$ is contained in the interval with $\Phi_f$ as its maximal element. Therefore, $P$ forms a partition of $\Lambda(S)$ which completes the proof of the theorem. □

Note that the condition that $\Lambda(S)$ is closed under the directed refinement relation is equivalent to the requirement that the forbidden configuration $\Gamma(S)$ be a valid subcomplex of the Coxeter complex; i.e., a collection of faces closed under taking subsets.

5.1.3. Chromatic Complexes and Generalized Permutahedra. An important class of scheduling problems that satisfy the conditions of Theorem 5.10 are those scheduling problems that can be expressed as a disjunction of conjunctions of strict inequalities, i.e.,

$$S = \bigvee_i C_i$$

where $C_i = \bigwedge_j I_{i,j}$ such that the $I_{i,j}$ are strict inequalities of the form $x_a < x_b$ for some $a$ and $b$, and the conjunctions $C_i$ are satisfiable for all $i$. Such scheduling problems are a special case of decision trees with all cells fully open, whence all regions of the allowable configuration are convex. Therefore, Theorem 5.8 already provides $L$-positivity, but the conditions of Theorem 5.10 are easier to interpret in this case. The $C_i$ are conjunctions of strict inequalities. Geometrically, the allowable schedules given by a $C_i$ form a cone; the intersection of halfspaces defined by the inequalities $x_a < x_b$. On the Coxeter complex, such regions are known as posets of the complex. In particular, for a given $C_i = \bigwedge_j I_{i,j}$, the inequalities $I_{i,j}$ naturally induce a partial order on $[n]$. The collection

---

If only the halfspaces given by $x_a < x_b$ are intersected, the result is cone that will contain the line where all variables are equal. If we add the constraints $0 < x_a$ for all $a$ the resulting region becomes a pointed cone contained in $\mathbb{N}^n$. If furthermore we add the constraints $x_a < 1$, as in the Ehrhart construction, we obtain open polytopes contained in the cube $(0,1)^n$. 

of facets of the Coxeter complex contained in the cone corresponding to $C_i$, thought of as permutations, consist of all possible linear extensions of the partial order.

**Example 5.11.** Suppose $C = (x_1 < x_2) \land (x_1 < x_3) \land (x_3 < x_4)$. The corresponding poset $P$ is given in Figure 7(a). The linear extensions of $P$ are $\{1234, 1324, 1342\}$ and the corresponding cells of $\text{Cox}_A$ are shaded in Figure 7(b).

![Figure 7](image)

**Figure 7.** A poset corresponds to a convex region in the Coxeter complex $\text{Cox}_A$, given by a conjunction of inequalities. Part a) of this figure shows a poset $P$ and part b) the corresponding region. The directed refinement relation defines a partial order on the set of allowable ordered set partitions, shown in c). As this partial order has unique coarsest elements, this yields a partition of the allowable region as shown in d).

This case includes the coloring complex regarded as the forbidden subcomplex of the graph coloring problem. Again recall that the coloring complex can be interpreted as all those ordered set partitions of the vertices such that at least one block contains at least one edge. Let $G = (V, E)$ be a graph on $n$ vertices. The graphical arrangement $A_G$ associated to $G$ is the subarrangement of the braid arrangement consisting of the hyperplanes $\{x_i = x_j | i, j \in E\}$. The graphical zonotope $P_G$ is the zonotope dual to this arrangement formed by the sum of all normals to all planes in the arrangement. Geometrically, this leads to a perspective first noted explicitly by Hersch and Swartz:
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the coloring complex of $G$ is the codimension one skeleton of the normal fan of $P_G$ as subdivided by the Coxeter complex [14]. Equivalently, as a scheduling problem, the allowable configuration consists of all integer points in the interiors of maximal cones of the normal fan. These interiors are conjunctions of strict inequalities, one for each facet defining hyperplane of the cone. In this case, the disjunction is then over all cones in the fan.

**Example 5.12.** Consider the complete graph on 3 vertices, $K_3$. Any coloring of $K_3$ must give distinct colors to each vertex. Figure 8 shows the graphical zonotope associated to $K_3$ and it’s normal fan. The 6 maximal cones of the fan correspond to the 6 ordered set partitions which place the vertices in distinct blocks.

\[
\text{Chromatic NCQSym}(K_3) = M_{1|2|3} + M_{2|1|3} + M_{2|3|1} + M_{1|3|2} + M_{3|1|2} + M_{3|2|1},
\]

\[
\text{Chromatic QSym}(K_3) = 6M_{(1,1,1)},
\]

\[
\chi(K_3) = k(k - 1)(k - 2).
\]

**Figure 8.** a) The graphical zonotope of the complete graph $K_3$ and the corresponding graphical arrangement. b) A generalized permutahedron and its normal fan.

In [1], this construction was extended from graphical zonotopes to all generalized permutahedron. Generalized permutahedron are typically defined as the class of polytopes in $\mathbb{R}^n$ such that every edge is parallel to $e_i - e_j$ for some $i, j \in [n]$. An equivalent characterization of generalized permutahedron is the following.
Proposition 5.13. [18, 16] A polytope is a generalized permutahedron if and only if its normal fan is refined by the braid arrangement.

Hence the graphical zonotope, dual to a subarrangement of the braid arrangement, is a generalized permutahedron. Other examples include matroid polytopes, associahedra, and graph associahedra. A scheduling problem $S$ can be associated to any generalized permutahedron $GP$ by defining the forbidden configuration $\Gamma(S)$ to be the codimension one skeleton of the normal fan of $GP$ as subdivided by the Coxeter complex. Such a scheduling problem is then given as a disjunction of conjunctions. Namely, the scheduling problem of a generalized permutahedron is defined by taking, for each vertex $v$ of a generalized permutahedron $GP$, the conjunction of all defining hyperplanes meeting $v$. Hence the valid schedules correspond to all integer points in the interior of the normal fan. In [1], this allowable configuration and nc-quasisymmetric function were studied not as a scheduling problem but in connection to the Hopf monoid of generalized permutahedron. There it was shown that the generalized permutahedron nc-quasisymmetric function is $L$ positive and $\Gamma(S)$ is $h$-positive.

Returning to the graphical case, again in [14], this perspective of the normal fan is used to prove that the coloring complex has a convex ear decomposition which implies strong relations on the chromatic polynomial. The authors consider the generalization of their result from chromatic polynomials of graphs to characteristic polynomials of matroids. They note empirically however that their results do not seem to generalize. The perspective here suggests that the generalization should not be from chromatic polynomials to characteristic polynomials, but from the scheduling polynomials of graphic zonotopes to the scheduling polynomials of matroid polytopes. Namely, to scheduling problems whose forbidden configuration is the codimension one skeleton of the normal fan of a matroid polytope (and more generally to any generalized permutahedron). The corresponding scheduling polynomial for matroid polytopes is the polynomial restriction of the Billera-Jia-Reiner quasisymmetric function for matroids [5].

5.1.4. $P$-partitions. Scheduling problems that can be written as a disjunction of conjunctions can be described in terms of $P$-partitions for posets $P$ with all strict edges. Such a poset $P$ defines a conjunction $C$ of strict inequalities of the form $x_a < x_b$ and corresponds therefore to a disjunction of conjunction with only one $C_i$.

The order polytope of $P$, as by Stanley in [21], is the closure of the allowable configuration $\Lambda_{(0,1)}(C)$ given by the conjunction $C$. The order polynomial $\Omega_P$ of $P$ is the Ehrhart polynomial of the order polytope, and the strict order polynomial $\Omega_P^\circ$ is the Ehrhart polynomial of its relative interior. Thus the strict order polynomial of $P$ is the scheduling polynomial of $C$ and non-negativity of its $h^*$-vector follows from Stanley’s non-negativity theorem [20]. In this way, disjunctions of conjunctions can be viewed as disjoint unions of order polytopes. In this special case, however, the regions of allowable configuration defined by $P$ is always convex (even if some

\footnote{Due to the Ehrhart-MacDonald reciprocity theorem, see [3, Chapter 4], $\Omega_P$ is the reciprocal of $\Omega_P^\circ$.}
edges are weak). The benefit of decision trees is that they allow the construction of non-convex regions, while controlling partitionability through the use of almost open cells.

The scheduling quasisymmetric function restricted to a single $C_i$ is the generating function $K(P, \omega)$ where $\omega$ is a “fully un-natural” labeling of $P$. For schedules with a single $C_i$, Theorem 5.10 gives a special case of Stanley and Gessels [13, 19] expansion of $K(P, \omega)$ in the fundamental basis by descent sets. In this case, with all strict labels, the unique coarsest elements for each ordering can also be defined by the descent sets. Theorem 5.10 is neither stronger nor weaker than the $K(P, \omega)$ expansion. On the one hand, posets with non-strict edges may easily be non-positive in the $L$ basis, just as cones refined by the braid arrangement can be non-partitionable if they have both open and closed edges. On the other hand, many scheduling problems satisfy the conditions of Theorem 5.10 but do not arise from any poset or $P$-partition structure as seen in the example below.

**Example 5.14.** Consider the scheduling nc-quasisymmetric function,

$$S_S = L_{24|13} + L_{4|2|13} + L_{24|3|1} + L_{2|14|3} + L_{2|1|34} + L_{4|2|3}$$

$$= M_{24|13} + M_{2|4|13} + M_{24|1|3} + M_{2|4|1|3} + M_{4|2|1|3} + M_{24|3|1}$$

$$+ M_{2|4|3|1} + M_{2|4|3} + M_{2|1|4|3} + M_{2|1|34} + M_{2|1|3|4} + M_{4|2|3|1}.$$  

The corresponding scheduling problem $S$ does satisfy the conditions of Theorem 5.10, i.e. for every allowable ordered set partition $\Phi$ there is a unique coarsest ordered set partition $\Phi_c$ below $\Phi$ in the directed refinement ordering. On the other hand, $S$ cannot be written as the conjunction of disjunctions of strict inequalities. Geometrically, the allowable configuration is contractible but not convex. It can be realized by a decision tree in which all cells are almost open.

5.2. **Subcomplexes and the Co-fundamental basis.** We have seen how particular niceness conditions on the allowable configuration $\Lambda(S)$ and the forbidden configuration $\Gamma(S)$ translate into strong properties of the expansions of scheduling nc-quasisymmetric and quasisymmetric functions in the fundamental bases and the relationship between the $h$-vectors of these complexes and the polynomial restriction $\chi(S)$. In particular, many special cases above resulted in $\Gamma(S)$ being a valid subcomplex of the Coxeter complex (as opposed to simply a collection of faces). This was the case for example for the chromatic scheduling problem and more generally the generalized permutahedron scheduling problem. Here we consider scheduling problems such that $\Lambda(S)$ is a valid subcomplex of $\text{Cox}_4$, i.e., the ordered set partitions satisfying $S$ are closed under coarsening. In this case, expansion in the fundamental basis is not a natural choice. Expansion in the co-fundamental basis, however, is natural and does yield good behavior.

The co-fundamental basis for quasisymmetric functions is defined analogously to the fundamental basis with refinement replaced by coarsening. Namely, for a fixed composition $\alpha$, define $N_\alpha$ as the sum over all monomials indexed by compositions
that coarsen $\alpha$, 

$$N_\alpha := \sum_{\beta: \beta \text{ coarsens } \alpha} M_\beta.$$ 

The co-fundamental basis given by the $N_\alpha$ also results from allowing the variables to commute in an NCQSym basis defined analogously to the $L$ basis above. In this case a directed coarsening relation is used. This basis was first defined in [4], denoted $Q_\Phi$.

Recall that for an integer point $a \in \mathbb{N}^n$, $\Delta(a)$ is the ordered set partition recording the relative ordering of components of $a$; i.e., $a$ is constant on each $\Delta_i$ and satisfies $a|_{\Delta_i} < a|_{\Delta_{i+1}}$ for all $i$. Given an integer point $a$ or an ordered set partition $\Delta(a)$ we now associate a flag:

$$F_a = F_{\Delta(a)} := F_0 \subset F_1 \subset \ldots \subset F_k := [n]$$

such that $a$ is constant on each set $F_i - F_{i-1}$ and $a|_{F_i - F_{i-1}} < a|_{F_{i+1} - F_i}$.

**Example 5.15.** Suppose $a$ is an integer point satisfying $a_2 < a_1 = a_3 = a_4$ then $\Delta(a) = (2|134)$ and $\mathcal{F}_{\Delta(a)} = \emptyset \subset \{2\} \subset \{1, 2, 3, 4\}$.

Given a scheduling problem, if the allowable configuration corresponds to the collection of all flags of a graded poset, then clearly $\Lambda(S)$ is closed under coarsening. In this case, the scheduling quasisymmetric function has a particularly nice expansion.

**Proposition 5.16.** Let $S$ be a scheduling problem such that the collection of flags corresponding to the elements of $\Lambda(S)$,

$$\{\mathcal{F}_{\Delta(a)} \mid \Delta(a) \in \Lambda(S)\},$$

forms the collection of all flags of a graded poset. Then the coefficient of $N_\alpha$ in the expansion of scheduling quasisymmetric function in the co-fundamental basis is given by:

$$[N_\alpha] = (-1)^{|\alpha|} \sum_{\alpha \text{-flags}} \prod_i \mu(f_i, f_{i+1})$$

where an $\alpha$-flag is a flag such that $|f_{i+1}| - |f_i| = \alpha_i$ and $\mu$ is the Möbius function of the poset.

Proposition 5.16 associates a quasisymmetric function to a graded poset. The allowable flags are all chains in the poset. Thus, in the monomial basis, the scheduling quasisymmetric function is indexed by compositions recording the size of each step in the flag. This quasisymmetric function is very similar to but not the same as Ehrenborg’s quasisymmetric function associated to graded posets [11]. For any graded poset $P$, Ehrenborg defines a quasisymmetric function $F(P)$ by summing over all chains of the poset and recording the rank jump of the flag at each step. Although the quasisymmetric functions record different data from the poset, our Proposition 5.16 is equivalent to [11, Proposition 5.1], where our expansion in the co-fundamental basis is equivalent to his expansion of the image of the Malvenuto and Reutenauer involution of quasisymmetric functions. The equivalence is seen by noting that the coefficients are given by a manipulation of the Möbius function, and the compositions indexing the monomials could be any collection of compositions associated to a chain that is closed under coarsening.
5.2.1. The Bergman scheduling problem. As an explicit example of Proposition 5.16 we consider the Bergman scheduling problem, so named because the lattice points of the allowable configuration are precisely those contained in the Bergman fan of a matroid. In this case, the scheduling polynomial is the zeta-polynomial of the matroid.

Let \( M \) be a matroid on ground set \( E \) and let \( \omega \in \mathbb{N}^{|E|} \). Consider \( \omega \) as a weight function on the ground set of \( M \) and define the weight of a basis \( B \) to be: \( \omega(B) = \sum_{e \in B} \omega(e) \). The Bergman fan is defined to be all \( \omega \) such that every element of \( M \) is in some basis of minimal \( \omega \)-weight. Note that if \( \omega \) is in the Bergman fan, then so are all weight functions with the same relative ordering. Hence, we can define the Bergman scheduling problem to be the scheduling problem whose allowable configuration is given by the integer points of the Bergman fan. Considering \( \omega \) simply as an integer point, we associate a flag \( F_\omega \) as above. In [2] it is shown that a weight function \( \omega \) is in the Bergman fan if and only if \( F_\omega \) is a flag of flats of \( M \). Therefore, the Bergman scheduling problem is the scheduling problem with allowable configuration \( \Lambda(S) \) equal to all \( \Delta(\omega) \) such that \( F_\omega \) is a flag of flats of \( M \).

**Example 5.17.** Let \( U_{2,4} \) be the uniform matroid of rank 2 on a ground set of size 4.

\[
\text{Bergman NCQSym}(U_{2,4}) = M_{1234} + M_{1|234} + M_{2|134} + M_{3|124} + M_{4|123}.
\]

The presence of the term \( M_{1|234} \), for example, reflects that any \( \omega \) such that \( \omega_{i_1} < \omega_{i_2} = \omega_{i_3} = \omega_{i_4} \) yields the minimum weight bases: \( \{12, 13, 14\} \). Alternatively, \( \{1 \subset 1234\} \) is a flag of flats. The contribution to Bergman QSym(\( U_{2,4} \)) will be \( M_{(1,3)} \) since \( (1,3) \) is the image of the ordered set partition under the type map. Ehrenborg’s quasisymmetric function would instead record the rank jumps of this flag giving a contribution of \( M_{(1,2)} \).

The collection of ordered set partitions which index the Bergman nc-quasisymmetric function are closed under coarsening and moreover form the collection of flags of the lattice of flats of the matroid. Therefore Proposition 5.16 implies that the coefficients of the Bergman quasisymmetric function in the co-fundamental basis are given by

\[
[N_\alpha] = (-1)^{|\alpha|} \sum_{\alpha\text{-flags}} \prod_i \mu(f_i, f_{i+1}),
\]

where an \( \alpha \)-flag is a flag of flats such that \( |f_{i+1}| - |f_i| = \alpha_i \) and \( \mu \) is the Möbius function of the lattice of flats of \( M \). The lattice of flats of a matroid is a well studied object and allows us to further understand these coefficients. For example, the sign of \( [N_\alpha] \) alternates with the length of \( \alpha \), \( \sum_\alpha [N_\alpha] = 1 \), and up to sign, \( [N_{|E|}] \) is equal to the Möbius function of the lattice of flats \( \mu(0, 1) \) or equivalently the reduced Euler characteristic of the order complex of the proper part of the lattice of flats.

**Example 5.18.** The Schubert matroid \( M_{(135)} \) is the matroid consisting of all bases componentwise less than or equal to \( (1,3,5) \). The Bergman quasisymmetric function
is given by Bergman \( \text{QSym}(M_{(135)}) \)

\[
= M_5 + 3M_{2,3} + M_{3,2} + M_{4,1} + 3M_{1,4} + 4M_{1,1,3} + M_{1,2,2} + M_{2,1,2} + M_{2,2,1} + 2M_{1,3,1}
\]

\[
= 2N_{1,3,1} + N_{2,2,1} + N_{2,1,2} + N_{1,2,2} + 4N_{1,1,3} - 4N_{1,4} - 2N_{4,1} - N_{3,2} - 3N_{2,3} + 2N_5.
\]

The Bergman polynomial obtained by evaluating at \( 1^k \) counts the number of \( k \)-schedules satisfying the boolean formula prescribed by the closure operator.

**Proposition 5.19.** The Bergman polynomial of a matroid is equal to the zeta polynomial of the lattice of flats of the matroid, which counts multichains of length \( k \) in the lattice,

\[
\text{Bergman QSym}(M; 1^k) = Z(L(M), k) = |\{\hat{0} = y_0 \leq y_1 \leq \cdots \leq y_k = \hat{1} | y_i \in L(M)\}|.
\]

One could interpret the rank as a kind of cost function - once certain jobs are started, others of the same rank can be added without additional cost. To minimize cost, we require that in any scheduling of jobs, at each time step we have a closed subset of jobs.

**Example 5.20.** Returning to the example of \( U_{2,4} \), the Bergman polynomial is

\[
\text{Bergman QSym}(U_{2,4}; 1^k) = 2k^2 - k.
\]

The flats of this matroid are all singletons and the entire set \([4]\). In terms of scheduling, any job may be started first and then all others must be started simultaneously. Alternatively, all jobs may be started at the same time, but we do not allow for only two jobs to start at the same time.

6. **Arboricity**

In the last section we introduce the arboricity polynomial, a scheduling polynomial naturally associated to a graph or matroid. Given a simple graph \( G \), the arboricity of \( G \) is defined to be the minimum number of forests needed to decompose (cover) the edges of \( G \). The parameter was introduced by Nash-Williams and Tutte [17, 24]. This definition is easily extended to an arbitrary matroid \( M \) where the arboricity of \( M \) is then the minimum number of independent sets needed to cover the ground set of \( M \).

The constructive version of partitioning a matroid into as few independent subsets as possible is known as the matroid partitioning problem. The initial literature on arboricity is concerned with computing this minimum number. As shown in [17, 24] and extended by Edmonds [10], the arboricity \( a(M) \) is given by:

\[
a(M) = \max_{X \subseteq E} \left\lfloor \frac{|X|}{r_k(X)} \right\rfloor,
\]

where \( X \) ranges over all possible subsets of the ground set.

Here we will be concerned with the problem of enumerating independent set covers by size. In particular we will show that the number of covers of \( M \) with at most \( k \) independent sets is a polynomial in \( k \). Again, an appropriate analogy to keep in mind is the chromatic polynomial of a graph. The chromatic polynomial is the enumerate function which counts the number of proper coloring of a graph with at most \( k \) colors.
An important distinction for counting independent set coverings is that they do not satisfy a deletion contraction recursion hence the standard inductive proof technique for polynomiality does not apply in this case. Instead, independent set covers are seen as solutions to a scheduling problem and hence are enumerated by the corresponding scheduling polynomial.

6.1. The Arboricity polynomial.

**Definition 6.1.** Given a matroid \( M \) on a finite ground sets \( E \), define an independent cover of \( M \) with at most \( k \) parts to be a mapping \( f : E \to [k] \) such that \( f^{-1}(i) \) is an independent set for all \( i \).

The arboricity polynomial \( \chi_{A_M}(k) \) is the counting function equal to the number of independent covers of \( E \) with at most \( k \) parts.

As a scheduling problem, given a matroid \( M \) on a finite ground set \( E \), an independent cover is an ordered set partition of \( E \) such that no block contains a circuit of \( M \). Let \( C \) denote the set of circuits of \( M \), then the scheduling boolean function is

\[
S_M = \bigwedge_{C \in C} \neg(x_{i_1} = x_{i_2} = \cdots = x_{i_m}),
\]

where the conjunction runs over all circuits \( C \in C \) and \( \{i_1, i_2, \ldots, i_m\} = C \). The forbidden configuration consists of all ordered set partitions such that at least one block contains a circuit and the scheduling polynomial counts the number of covers with at most \( k \) parts. In terms of scheduling, the collection of jobs which may start at a fixed time can not be dependent.

**Example 6.2.** Let \( M \) be the free matroid on a ground set of size \( n \); i.e. all subsets of the ground set are independent. Then \( \chi_{A_M}(k) = k^n \).

**Example 6.3.** Let \( M(C_n) \) be the graphical matroid of the cycle graph on \( n \) vertices. Independent sets of this matroid correspond to acyclic subsets of edges (forests) of the graph. For \( k \leq 1 \) we find no independent covers of \( M \). For \( k \geq 2 \), independent covers correspond to ordered set partitions of \( [n] \) with at least 2 parts.

For \( n = 3 \), consider the braid arrangement in \( \mathbb{R}^3 \), the ground set space of the matroid, or edge set space of the graph. The allowable configuration consists of ordered set partitions corresponding to collections of integer points which lie off of the intersection: \( x_1 = x_2 = x_3 \). Hence the corresponding nc-quasisymmetric function element is

\[
\mathcal{A}(C_3) = \mathcal{M}_{(1|23)} + \mathcal{M}_{(2|13)} + \mathcal{M}_{(3|12)} + \mathcal{M}_{(23|1)} + \mathcal{M}_{(13|2)} + \mathcal{M}_{(12|3)} + \mathcal{M}_{(1|23)} + \mathcal{M}_{(1|32)} + \mathcal{M}_{(2|13)} + \mathcal{M}_{(2|31)} + \mathcal{M}_{(3|12)} + \mathcal{M}_{(3|21)}.
\]

Specializing under the type map gives the quasisymmetric function:

\[
A(C_3) = 3\mathcal{M}_{(1,2)} + 3\mathcal{M}_{(2,1)} + 6\mathcal{M}_{(1,1,1)}.
\]

Evaluating at \( 1^k \) gives the arboricity polynomial:

\[
\chi_{A(C_3)} = 3\binom{k}{2} + 3\binom{k}{2} + 6\binom{k}{3} = k^3 - k.
\]
In general,
\[ \chi_{A(C_n)}(k) = \sum_{m=2}^{n} S(n, m) \cdot k(k-1) \cdots (k-m+1), \]
where \( S(n, m) \) is the Stirling number of the second kind. Hence the number of covers of the cycle graph with at most \( k \) forests is
\[ \chi_{A(C_n)}(k) = k^n - k. \]

From this example, it can be seen that the arboricity polynomial does not satisfy a contraction deletion recursion analogous to the chromatic polynomial. Consider the graphical matroid of the 3-cycle, \( C_3 \). The number of independent covers of size at most 2 is \( 2^3 - 2 = 6 \). Deleting any edge gives a path of length 2 which has 1 independent cover of size 1 and 2 independent covers of size 2. The contraction of any edge gives a double edge which has no independent covers of size 1 and 2 independent covers of size 2,
\[
\begin{align*}
\chi_{A(C_3)}(2) &= 6 \\
\chi_{A(C_3)-e}(2) &= 3 \\
\chi_{A(C_3)/e}(2) &= 2.
\end{align*}
\]

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