Finite Temperature Correlators in the Schwinger Model

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ABSTRACT

We discuss the correlation function of hadronic currents in the Schwinger model at finite temperature $T$. We explicitly construct the retarded correlator in real time and obtain analytical results for the Euclidean correlator on a torus. Both constructions lead to the same finite temperature spectral function. The spatial screening lengths in the mesonic channels are related to the dynamical meson mass $m = e/\sqrt{\pi}$ and not $2\pi T$ even in the infinite temperature limit. The relevance of our results for the finite temperature problem in four dimensions is discussed.

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1. Introduction

With the possibility to use ultrarelativistic heavy ion colliders to probe hadronic matter at very high temperatures and/or densities it becomes imperative to get an improved understanding of finite temperature gauge field theories. A variety of approaches have been devised ranging from first principle lattice simulations \[\text{[1]}\] to approximate estimates in simplified models \[\text{[2]}\].

Just as in conventional plasma physics, a central problem in finite temperature QCD is to find the excitation spectrum as a function of temperature, density, external fields, \textit{etc.}. The spectral information in some particular channel $\alpha$ is contained in the spectral function $\sigma_\alpha$. Its knowledge would, at least in principle, allow us not only to calculate responses to external probes, but also production rates, multiplicities, \textit{etc.}.

While the spectral function at zero temperature contains poles corresponding to bound states and resonances and cuts corresponding to particle production thresholds, its interpretation at finite temperature is not always clear. It is usually assumed that the spectral function at finite temperature is dominated by hadronic quasiparticles in the low temperature phase and various collective excitations in the high temperature phase. However, the nature and quantum numbers of these collective excitations are not very well understood.

To determine the spectral function in a theory like QCD from first principles remains an unsolved problem. Besides the purely technical/numerical problems, the present lattice simulations of finite temperature systems can primarily provide information about space-like correlators only. The spectral function at finite temperature, which is related to time-like correlators, can only be obtained if one makes very specific, and unprovable, assumptions about its analytic structure.

Lacking suitable lattice techniques, other methods have been used to get spectral information. Examples are finite temperature QCD sum rule calculations and phenomenological models (usually inspired by models for the zero temperature vacuum), but here one is forced to \textit{assume} a special shape for the spectral function (shifted poles and cuts) and then proceed to estimate some unknown parameters like resonance masses and thresh-
olds (see [3] for a review). Another approach advocated by two of us [4], is to use the method of dimensional reduction to calculate certain static correlation functions at finite temperature.

The questions we would like to raise in this paper are twofold: 1. Is it meaningful to talk about spectral functions in interacting field theories at finite temperature? 2. To what extent can we learn about them from finite temperature Euclidean correlators? We will not attempt to answer these questions for QCD, but use a model theory simple enough to obtain answers.

We consider the Schwinger model at finite temperature. In section 2 we explicitly construct the real time correlation function for pseudoscalar and scalar sources (analogue of $e^+e^-$ production channel) at zero temperature and derive an explicit relation for the zero temperature spectral function. In section 3, we extend the analysis to finite temperature, by considering the retarded response function, and derive an explicit expression for the finite temperature spectral function for the Schwinger problem, using operator techniques. In section 4, we evaluate the scalar and pseudoscalar correlator for the Schwinger model on the torus. Exact analytical results are obtained with the help of the properties of the Jacobi theta-functions. The results are in full agreement with the naive bosonisation approach. Issues related to the temporal and spatial screening masses are discussed in section 5. In section 6, we show that the analytical continuation of the full nonperturbative Euclidean correlators yields temperature dependent spectral functions that are identical to the ones derived from the real time approach. Our conclusions are summarized in section 7. Helpful calculational details are given in two appendices.

2. Feynman Correlators

The Schwinger model is quantum electrodynamics [5] in one space and one time dimension defined by the Lagrangian density

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu}^2 - \bar{\psi} i \mathcal{D} \psi, \quad (2.1)$$
where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Abelian field strength and $\mathcal{D} = \gamma_\mu (\partial_\mu - ieA_\mu)$ the covariant derivative. It is well known that at zero temperature this model is equivalent to a theory with a single massive and free meson with mass $m = e/\sqrt{\pi}$ where $e$ is the electric charge of the electron [3]. The meson field is related to the polarization density induced by the electrons.

The mesonic Green’s functions in the vacuum can be constructed using the bosonisation techniques discussed by Coleman [7]. In this paper we consider the following scalar and pseudoscalar correlators,

\[
S_F(x) = \langle T(\bar{\psi}\psi(x)\bar{\psi}\psi(0)) \rangle, \\
P_F(x) = \langle T(\bar{\psi}\gamma_5\psi(x)\bar{\psi}\gamma_5\psi(0)) \rangle,
\]

(2.2)

which can be readily evaluated by using the operator identity [7]

\[
\frac{1}{2}(1 \pm \gamma_5)\psi(x) = -\left(\frac{meC}{4\pi}\right) : e^{\pm i2\sqrt{\pi}\phi(x)} :,
\]

(2.3)

where $\phi$ is a free massive meson and $C = 0.577$ is the Euler-Mascheroni constant. Normal ordering with respect to the vacuum is denoted by $: :$. The field $\phi(x)$ can be decomposed into positive and negative frequency components as

\[
\phi(x) = \int \frac{dp}{\sqrt{4\pi E_p}} \left( e^{-iE_p x_0 + ipx_1} a_p + e^{+iE_p x_0 - ipx_1} a_p^+ \right) = \phi^-(x) + \phi^+(x),
\]

(2.4)

with $E_p = \sqrt{p^2 + m^2}$. The product of normal ordered operators obtained by inserting (2.3) in (2.2) can be evaluated with the help of the following relation ($\epsilon_{x,0} = \pm 1$)

\[
:e^{\epsilon_{s,0}\phi(x)} : e^{\epsilon_{0}\phi(0)} := e^{\epsilon_{s,0}(\phi^+(0),\phi^-(x))} e^{\epsilon_{s,0}\phi^+(x)+i\epsilon_0\phi^+(0)} e^{i\epsilon_{s,0}\phi^-(x)+i\epsilon_0\phi^-(0)},
\]

(2.5)

which can be established using the Baker-Campbell-Hausdorff formula. Inserting (2.3) and (2.4) into (2.2) and using repeatedly (2.5) yield

\[
P_F(x) = (\bar{\psi}\psi)^2 \sinh(-4\pi i \Delta_F(x, m^2)),
\]

(2.6)

where the Feynman propagator is given by

\[
\Delta_F(x, m^2) = \theta(x^0)[\phi^-(x), \phi^+(0)] - \theta(-x^0)[\phi^+(x), \phi^-(0)],
\]

(2.7)
and the fermion condensate is
\[ \langle \overline{\psi} \psi \rangle = -\left( \frac{me^C}{2\pi} \right). \] (2.8)

Similarly,
\[ S_F(x) = (\overline{\psi} \psi)^2 \cosh(-4\pi i \Delta F(x, m^2)). \] (2.9)

These results were first obtained by Casher, Kogut and Susskind [8]. By expanding the hyperbolic functions in (2.6) and (2.9) we observe that the pseudoscalar correlator is a sum of exchanges of an odd number of massive mesons while the scalar correlator (2.9) is given by exchange of an even number of mesons. The scalar field carries odd parity. In addition, the scalar correlator is nonvanishing and equal to \( (\overline{\psi} \psi)^2 \) in the \( x \to \infty \) limit as expected from the cluster decomposition. The nonvanishing of \( (\overline{\psi} \psi)^2 \) in this model was first discussed by Loewenstein and Swieca [9].

The spectral function at zero temperature follows from (2.2) by inserting a complete set of physical states and making use of translational invariance,
\[ P_F(x) = -i \int_0^\infty dm^2 \sigma_P(m^2) \Delta_F(x, m^2), \] (2.10)

where we have defined \( (q^0 > 0 \text{ and } p_n^2 = m^2) \)
\[ \sigma_P(q^2) = 2\pi \sum_n \delta^2(q - p_n)|\langle 0|\overline{\psi} i\gamma_5 \psi(0)|n\rangle|^2. \] (2.11)

A comparison of (2.6) with (2.10) yields the following spectral function for the pseudoscalar channel
\[ \sigma_P(q^2) = (\overline{\psi} \psi)^2 \left( 4\pi \delta(q^2 - m^2) \right) + 2\pi \sum_{k=1}^\infty \frac{2^{2k+1}}{(2k+1)!} \prod_{i=1}^{2k+1} \int d^2 p_i \delta(p_i^2 - m^2) \theta(p_i^0) \delta^2(q - \sum_{j=1}^{2k+1} p_j), \] (2.12)

which was also obtained in [10]. The first term corresponds to the meson pole with a strength determined by the quark condensate. The sum is over the 3, 5, \( \cdots, \infty \)-particle
phase space with a coupling strength proportional to the pole strength. The spectral function (2.12) is just the sum over the cuts of the Feynman diagrams in the expansion of the correlator $P_F$ in powers of $\Delta_F(x)$. A similar expression can be derived for $S_F(x)$ involving an even exchange of pseudoscalar mesons.

3. Retarded Correlators

The finite temperature spectral function can be obtained from the retarded correlator. For that consider for instance the pseudoscalar correlator

$$P_R(x) = \theta(x^0)\langle \bar{\psi}i\gamma_5\psi(x), \bar{\psi}i\gamma_5\psi(0) \rangle_\beta,$$

(3.1)

where $\langle \cdots \rangle_\beta$ denotes the Gibbs average with respect to a heat bath of mesons of mass $m = e/\sqrt{\pi}$. The expectation value in (3.1) can be calculated using the operator identities (2.3) and (2.5). In a heat bath the expectation value $\langle \phi^+\phi^- \rangle_\beta$ does not vanish so that

$$\langle \exp \left( i\epsilon_x\phi^+(x) + i\epsilon_0\phi^+(0) \right) \exp \left( i\epsilon_x\phi^-(x) + i\epsilon_0\phi^-(0) \right) \rangle_\beta =$$

$$\exp \left( -\epsilon_0\epsilon_x \langle \phi^+(0)\phi^-(0) \rangle_\beta - \epsilon_x\epsilon_0 \langle \phi^+(x)\phi^-(x) \rangle_\beta - \epsilon_x\epsilon_x \langle \phi^+(0)\phi^-(x) \rangle_\beta \right).$$

(3.2)

With this in mind, the retarded pseudoscalar correlator becomes

$$P_R(x) = -\theta(x^0)\frac{m^2e^{2C}}{4\pi^2}e^{-8\pi\langle \phi^+(0)\phi^-(0) \rangle_\beta}$$

$$\times \left[ \text{sinh} \left( 4\pi \langle \phi^-(x), \phi^+(0) \rangle + 4\pi \langle \phi^+(x)\phi^-(0) \rangle_\beta + 4\pi \langle \phi^+(0)\phi^-(x) \rangle_\beta \right) \right]$$

$$- \left[ \text{sinh} \left( 4\pi \langle \phi^-(0), \phi^+(x) \rangle + 4\pi \langle \phi^-(x)\phi^+(0) \rangle_\beta + 4\pi \langle \phi^+(x)\phi^-(0) \rangle_\beta \right) \right].$$

(3.3)

In the heat bath

$$\langle \phi^-(x), \phi^+(0) \rangle + \langle \phi^+(x)\phi^-(0) \rangle_\beta + \langle \phi^+(0)\phi^-(x) \rangle_\beta = \int \frac{d^2p}{(2\pi)^2} \delta(p^2 - m^2)(\theta(p^0) + n_p)e^{-ip\cdot x},$$

(3.4)
where \( n_p = 1/(e^{\beta E_p} - 1) \) is the Bose-Einstein distributions for the massive bosons. Using (3.4) in (3.3) gives

\[
P_R(x) = -\theta(x^0)\langle \bar{\psi} \psi \rangle_\beta \left[ \sinh \left( 4\pi \int \frac{d^2p}{(2\pi)^2} e^{-ip \cdot x} \delta(p^2 - m^2)(\theta(p^0) + n_p) \right) \right. \\
- \sinh \left( 4\pi \int \frac{d^2p}{(2\pi)^2} e^{-ip \cdot x} \delta(p^2 - m^2)(\theta(-p^0) + n_p) \right) \right],
\]

(3.5)

where

\[
\langle \bar{\psi} \psi \rangle_\beta = -\frac{me^C}{2\pi} e^{-4\pi \langle \phi^+(0)\phi(0)^- \rangle_\beta} = -\frac{me^C}{2\pi} \exp \left( -\int dp \frac{n_p}{\sqrt{p^2 + m^2}} \right)
\]

(3.6)
is the temperature dependent chiral condensate, as calculated in [11, 12, 13]. A similar result holds for the retarded scalar correlator \( S_R(x) \) with the substitution \( \sinh \to \cosh \) in (3.3). Micro-causality can be exhibited explicitly in (3.5) by rewriting the result in the form

\[
P_R(x) = \langle \bar{\psi} \psi \rangle_\beta \sinh(2\pi \Delta_R(x, m^2)) \cosh \left( \int \frac{d^2p}{(2\pi)^2} (1 + 2n_p) \text{Im} \Delta_F(p, m^2) e^{ip \cdot x} \right),
\]

(3.7)

where \( \Delta_R(x, m^2) \) is the retarded propagator

\[
\Delta_R(x, m^2) = \theta(x^0) \left( [\phi^+(x), \phi^-(0)] + [\phi^-(x), \phi^+(0)] \right)
\]

(3.8)

and \( \Delta_F(p, m^2) \) the Fourier transform of the Feynman propagator (2.7). Again, the retarded pseudoscalar propagator in the heat bath corresponds to the exchange of an odd number of pseudoscalar mesons with temperature dependent strengths. The temperature dependent strengths follow from the bubble insertions at the vertices. Fig. 1a shows the temperature insertions on the vertices, and Fig. 1b the diagrammatic expansion of (3.7). A similar result can be derived for the scalar correlator. The answer is

\[
S_R(x) = \langle \bar{\psi} \psi \rangle_\beta \sinh(2\pi \Delta_R(x, m^2)) \sinh \left( \int \frac{d^2p}{(2\pi)^2} (1 + 2n_p) \text{Im} \Delta_F(p, m^2) e^{ip \cdot x} \right)
\]

(3.9)

and involves an exchange of an even number of pseudoscalar mesons, as expected from parity. Fig. 1c shows the corresponding diagrammatic expansion.

Using translational invariance, the retarded correlators can be related to the pertinent finite temperature spectral function. In the pseudoscalar case we have

\[
P_R(x) = \int d^2q \frac{\theta(x^0)}{2\pi} e^{iq \cdot x} \sigma_F(q^0, q^1)
\]

(3.10)
where the temperature dependent spectral function reads
\[
\sigma_P(q^0, q^1) = 2\pi \sum_{A,B} \delta^2(q - (q_A - q_B))(e^{-(E_A - F)/T} - e^{-(E_B - F)/T}) \mid < A|\overline{\psi}\gamma_5\psi(0)|B > |^2.
\]

(3.11)

The free energy of the system is denoted by \( F \) and \( q_A^2 = m^2 \). The spectral function for the retarded correlator is obtained by expanding the sinh in (3.5) in a power series. To proceed the following identity is useful
\[
\prod_{i=1}^{2k+1} \left( (\theta(p^0_i) + n_{p_i}) - \prod_{i=1}^{2k+1} (\theta(-p^0_i) + n_{p_i}) \right) = 2 \sinh(\beta p^0_0 / 2) \prod_{i=1}^{2k+1} e^{\beta |p^0_i|/2} n_{p_i},
\]

(3.12)

where \( p^0 = \sum_i p^0_i \). The proof of this identity follows immediately if one realizes that
\[
\prod_{i=1}^{2k+1} (1 + n_{p_i}) \prod_{p^0_i > 0} n_{p_i} - \prod_{p^0_i < 0} n_{p_i} \prod_{p^0_i > 0} (1 + n_{p_i}) = \frac{\prod_{p^0_i > 0} e^{\beta |p^0_i|} - \prod_{p^0_i < 0} e^{\beta |p^0_i|}}{\prod_{p^0_i} (e^{\beta |p^0_i|} - 1)}.
\]

(3.13)

By comparing the expansion of the correlator and (3.10) we can read the spectral function
\[
\sigma_P(q^0, q^1) = 4\pi \langle \overline{\psi}\psi \rangle \sinh(\beta q_0/2)
\times \sum_{k=0}^{\infty} \frac{2^{2k+1}}{(2k+1)!} \prod_{i=1}^{2k+1} \int d^2p_i \delta(p^2_i - m^2) e^{\beta |p^0_i|/2} n_{p_i} \delta^2(q - \sum_{j=1}^{2k+1} p_j).
\]

(3.14)

A rerun of the above argument for the retarded scalar correlator gives (following the substitution sinh \( \rightarrow \cosh \) in (3.3))
\[
\sigma_S(q^0, q^1) = 4\pi \langle \overline{\psi}\psi \rangle \sinh(\beta q_0/2)
\times \sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!} \prod_{i=1}^{2k} \int d^2p_i \delta(p^2_i - m^2) e^{\beta |p^0_i|/2} n_{p_i} \delta^2(q - \sum_{j=1}^{2k} p_j).
\]

(3.15)

The spectral functions at finite temperature are just the zero temperature spectral functions with temperature dependent strengths and thermally weighted phase space. The thresholds remain temperature independent. This result was expected, since the bosonised version of the model is a model with a free single massive meson.
The above results are consistent with the conventional cutting rules in real time. Specifically, we have for the scalar \((p_3^1 = q^1 - p_1^1)\) channel

\[
\sigma_S(q^0, q^1) = 4\pi\langle\bar{\psi}\psi\rangle^2 \int \frac{dp_1^3}{2E_1 2E_2} \times \\
(\delta(q^0 - E_1 - E_2)((1 + n_1)(1 + n_2) - n_1n_2) + \delta(q^0 + E_1 + E_2)(n_1n_2 - (1 + n_1)(1 + n_2)) + \delta(q^0 - E_1 + E_2)(n_2(1 + n_1) - n_1(1 + n_2)) + \delta(q^0 + E_1 - E_2)(n_1(1 + n_2) - n_2(1 + n_1)) + \cdots,
\]

(3.16)

where the dots refer to the processes with 4, 6, \cdots pseudoscalar meson exchanges. For the pseudoscalar channel the first two terms in the expansion of the sinh-functions in (3.5) result in \((p_3^1 = q^1 - p_1^1 - p_2^1)\)

\[
\sigma_P(q^0, q^1) = 4\pi\langle\bar{\psi}\psi\rangle^2 \delta(q^2 - m^2) + \frac{16\pi}{3}\langle\bar{\psi}\psi\rangle^2 \int \frac{dp_1^3 dp_2^3}{2E_1 2E_2 2E_3} \times \\
\left( (\delta(q^0 - E_1 - E_2 - E_3) - \delta(q^0 + E_1 + E_2 + E_3))((1 + n_1)(1 + n_2)(1 + n_3) - n_1n_2n_3) \\
+ \delta(q^0 - E_1 - E_2 + E_3)((1 + n_1)(1 + n_2)n_3 - n_1n_2(1 + n_3)) + (3 \leftrightarrow 1) + (3 \leftrightarrow 2) \\
+ \delta(q^0 + E_1 - E_2 + E_3)(n_1(1 + n_2)n_3 - (1 + n_1)n_2(1 + n_3)) + (2 \leftrightarrow 3) + (2 \leftrightarrow 1) \right) + \cdots,
\]

(3.17)

where the dots refer to the processes with 5, 7, \cdots pseudoscalar meson exchanges. Note that the pole contribution in the pseudoscalar channel remains temperature independent, only its strength is affected by temperature. Fig. 2a shows the various cuts contributing to (3.16) and Fig. 2b the various cuts contributing to (3.17). The thermal contributions to (3.16, 3.17) reflect on the exchanges of the sources with the heat bath. Incoming arrows are weighted by \(n\), outgoing arrows are weighted by \((n + 1)\). Lines on the left hand side of the vertex denote absorption into the heat bath, lines on the right hand side of the vertex denote emission from the heat bath. The lowest contribution to the scalar channel is consistent with the result discussed by Weldon [14] in four dimensions for scalar \(\phi^3\) theories. The next to leading order contribution to the pseudoscalar channel is consistent with the result discussed by Fujimoto et al. [15] in the context of scalar \(\phi^4\) theories. The results (3.14) and (3.15) summarize all the cuts for scalar \(\phi^n\) theories with \(n = 3, 4, \cdots, \infty\).
Their extension to four dimensions is immediate. The singularity structure of the spectral function at finite temperature is much more involved than the zero temperature one. Its structure is physically transparent when expressed in the energy variable $q^0$ (in the heat bath frame) as opposed to the invariant variable $q^2$.

4. Euclidean Correlators

Most calculations in finite temperature field theory are performed using the imaginary time or Euclidean formulation. At the perturbative level it has been well established how these results are related to similar calculations in real time. However, in the previous section we obtained the full nonperturbative correlation function based on operator techniques in real time and it is a nontrivial question whether it can be reproduced by an Euclidean calculation. In Euclidean space, finite temperature corresponds to imaginary time with a periodicity $\beta = 1/T$. The infrared divergences in the Schwinger model are regulated by putting the space axis on a circle with circumference $L$ that at the end of the calculation is taken to infinity $[12]$. Thus, we are led to study the Schwinger model on a torus.

We now outline the calculation of the scalar and pseudoscalar correlation functions on a torus. We will follow the methods developed in $[16, 17]$ and $[1]$ and use the notation of the latter reference when needed. In the following, all expectation values are defined on the torus except when stated otherwise. The starting point is the generating functional for (disconnected) fermionic Greens functions,

$$Z[\eta, \bar{\eta}] = \langle \prod_{p=1}^{k} (\bar{\eta}|\psi_p)(\bar{\psi}_p|\eta)e^{-\int d^2xd^2y \bar{\eta}(x)S(x-y)\eta(y)\det'(i\mathcal{D})} \rangle_A. \quad (4.1)$$

The fermionic Green’s function $S(x - y)$, with proper boundary conditions, is derived in Appendix A, and the gauge-field average $\langle \cdots \rangle_A$ is with respect to the usual Maxwell action $-\frac{1}{2}E^2$. The functions $\psi_p$ are the zero modes, and as explained in $[1]$, in the topological sector corresponding to a non zero electric flux $\Phi = 2\pi k/e$, there are precisely $|k|$ zero
modes, all with chirality $\Phi/|\Phi|$. In this sector the gauge potential can be decomposed as
\[
A_0 = \frac{2\pi}{\beta} h_0 + \partial_1 \phi + \partial_0 \lambda - \frac{\Phi}{V} x^1, \tag{4.2}
\]
\[
A_1 = \frac{2\pi}{L} h_1 - \partial_0 \phi + \partial_1 \lambda, \tag{4.3}
\]
where the $h_\mu$ are constant and $V = \beta L$ is the area of the torus. The explicit expression for the $p^{th}$ zero mode ($p = 1, 2, \cdots |k|$) with chirality $a = \pm$ can be found in [11]:
\[
\psi^a_p(x) = e^{-a\phi(x)} \left( \frac{2|k|}{\beta^2 V} \right)^{\frac{1}{2}} e^{2\pi i [h_1 x^1/\beta - k x^0 x^1/\gamma]} \Theta \left[ \frac{x^0/L + (p - \frac{1}{2} - h_1)/k}{k x^0/\beta + h_0} \right] (0, i|k|\tau) \tag{4.4}
\]
where $\Theta$ is the generalized theta function as defined in [18]. The determinant in (4.1) is over the states orthogonal to the zero modes and is given by [11]
\[
\det(iD) = \left| \frac{1}{\eta(i\tau)} \right|^{|k|/2} \exp \left( \frac{m^2}{2} \int d^2 x \phi(x) \square \phi(x) \right) \tag{4.5}
\]
for $k = 0$ and by
\[
\det'(iD) = \left( \frac{V}{2|k|} \right)^{|k|/2} e^{2\pi i \Phi(x)/\beta} \exp \left( \frac{m^2}{2} \int d^2 x \phi(x) \square \phi(x) + \frac{2\epsilon k}{V} \int d^2 x \phi(x) \right) \tag{4.6}
\]
for $k \neq 0$. Here $\eta(i\tau)$ is Dedekind’s eta function.

As in the previous section we study the connected scalar and pseudoscalar correlation functions defined by $\prod$
\[
S_E(x) = \langle \overline{\psi} \psi(x) \overline{\psi} \psi(0) \rangle, \tag{4.7}
\]
\[
P_E(x) = \langle \overline{\psi} i\gamma_5 \psi(x) \overline{\psi} i\gamma_5 \psi(0) \rangle, \tag{4.7}
\]
where the average is according to the partition function (4.1). It is convenient to use a chiral basis for the spinors and the fermionic Greens functions defined by
\[
\psi_{\epsilon}(x) = P_{\epsilon} \psi(x) = \frac{1}{2} (1 + \epsilon \gamma_5) \psi(x), \tag{4.8}
\]
where again $\epsilon = \pm 1$ and similarly for $S_{\epsilon\epsilon'}(x - y)$. The above correlation functions can be expressed as
\[
S_E(x) = 2C_{+-}(x) + 2C_{++},
\]
\[
P_E(x) = 2C_{+-}(x) - 2C_{++}, \tag{4.9}
\]
\footnote{Our conventions for the $\gamma$-matrices in Euclidean space are: $[\gamma_\mu, \gamma_\nu]_+ = 1_{\mu\nu}$, $\gamma_\mu^+ = \gamma_\mu$ and $\gamma_5 = i\gamma_1\gamma_2$.}
where the $C_{\alpha \epsilon'}(x)$ are defined by

$$C_{\alpha \epsilon'}(x) = \langle \bar{\psi} P_\alpha \psi(x) \bar{\psi} P_{\epsilon'} \psi(0) \rangle. \quad (4.10)$$

and we have simplified (4.7) by using the relations

$$C_{+-}(x) = C_{-+}(x), \quad C_{--}(x) = C_{++}(x). \quad (4.11)$$

From (4.1) and the chirality of the zero modes it follows that in fact only $k = 0$ and $k = \pm 2$ contribute to the correlators (4.7). In the trivial sector only $C_{+-}(x)$ and $C_{-+}(x)$ are non-vanishing whereas $C_{++}(x)$ and $C_{--}(x)$ are non-vanishing in the sectors $k = 2$ and $k = -2$, respectively.

We start with the trivial sector where the relevant non-vanishing correlation function reads

$$C_{+-}(x) = \langle \bar{\psi} P_+ \psi(x) \bar{\psi} P_+ \psi(0) \rangle = -\frac{1}{Z} \langle \det(iD) S_{+-}(x) S_{+-}(-x) \rangle_A, \quad (4.12)$$

where $Z = Z[0,0]$ is the partition function on the torus. To evaluate the functional integral over the gauge field we again follow (11) and write

$$\langle \cdot \cdot \cdot \rangle_A \equiv \int \mathcal{D}A e^{-\frac{1}{2} \int d^2x \mathcal{E}^2(x)} = J \sum_k \int_0^1 dh dh_1 \prod_{m \neq 0} d\phi_m d\lambda_m e^{-\frac{2\pi^2 k_s^2}{V \beta}} \frac{1}{2} \int d^2x \phi \Box \phi, \quad (4.13)$$

where $\phi_m$ and $\lambda_m$ are the Fourier components of the fields $\phi$ and $\lambda$ subject to the condition $\int d^2x \phi(x) = \int d^2x \lambda(x) = 0$. Furthermore, the Jacobian $J$ does not depend on the dynamical fields and cancels between numerator and denominator in (4.12), and the integration over the pure gauge $\lambda$ is trivial. Substituting the expression (A.11) for the fermionic Greens function, and the explicit Fourier expansion of the generalized theta function, it is now straightforward to perform the $h$-integrations to get

$$C_{+-}(x) = e^{2\pi^2 \frac{(x^1)^2}{V \beta}} \frac{1}{(2\pi \beta)^2} \left| \frac{\partial \vartheta_1'(0)}{\vartheta_1(z)} \right|^2 \langle e^{-2\phi(x) + 2\phi(0)} \rangle_S, \quad (4.14)$$

where $z = (x^0 + ix^1)/\beta$, and $\langle \cdots \rangle_S$ denotes the connected Greens functions with respect to the measure $\prod_{m \neq 0} d\phi_m \exp(-S_B)$ induced by the bosonic action

$$S_B = \frac{1}{2} \int d^2x \phi(x) \Box (\Box - m^2) \phi(x). \quad (4.15)$$
For simplicity of notation we have introduced the Jacobi theta function \( \vartheta_1(x) = -\vartheta_{11}(x, i\tau) \).

Finally using the identity [19]

\[
\vartheta_1'(0) = 2\pi \eta(i\tau)^3,
\]

we obtain

\[
C_{+-}(x) = \frac{1}{\beta^2} \eta(i\tau)^4 e^{2\pi i^2 / V} \left| \vartheta_1(z) \right|^2 \langle e^{-2\phi(x) + 2\phi(0)} \rangle_S.
\]

The integral over \( \phi \) is Gaussian and can be expressed immediately as

\[
\langle e^{-2\phi(x) + 2\phi(0)} \rangle_S = e^{4e^2(K(x,x) - K(x,0))},
\]

where the two-point function \( K(x, y) \) is defined by

\[
K(x, y) = \langle x \left[ \frac{1}{\Box - m^2} \right] y \rangle = \frac{1}{m^2} [\tilde{\Delta}_m(x - y) - \Delta(x - y)].
\]

Here \( \tilde{\Delta}_m(x) \) is the Greens function

\[
\tilde{\Delta}_m(x) = -\frac{1}{V} \sum_{(k,l)\neq(0,0)} \frac{e^{2\pi ikx^0 / \beta + 2\pi ilx^1 / L}}{\left( \frac{2\pi k}{\beta} \right)^2 + \left( \frac{2\pi l}{L} \right)^2 + m^2},
\]

and \( \Delta \) its massless counterpart. The full massive Greens function is obtained by adding the contribution from the constant mode, \textit{i.e.}

\[
\Delta_m(x - y) = \tilde{\Delta}_m(x - y) - \frac{1}{V m^2}.
\]

Thus we have

\[
\langle e^{-2\phi(x) + 2\phi(0)} \rangle_S = e^{4e^2K(x,x) - 4\pi \Delta_m(x) + 4\pi \Delta(x) - \frac{4\pi}{V m^2}}.
\]

Note that (4.17) apparently has a free massless singularity at \( x_\mu = 0 \) since \( \vartheta_1(z) \) is odd. However, the expression (4.22) for the massless propagator was studied by Kronecker [15] with the following result

\[
\Delta(x) = \frac{1}{4\pi} \log \left[ \frac{[\vartheta(z, i\tau)]^2}{\eta^2(i\tau)} e^{-\frac{2\pi i^2}{V}} \right],
\]
so this singularity cancels.\footnote{The cancellation of the singularity is expected because this model does not have massless excitations.}

Combining equations (4.17), (4.22) and (4.23) and using the result

\[
\langle \overline{\psi} \psi \rangle_\beta = -\frac{2}{\beta} e^{-\frac{2\pi}{\sqrt{\nu m^2}}} |\eta(i\tau)|^2 e^{2\pi^2 K(x,x)}
\]

from \cite{11} we finally get

\[
C_{+-}(x) = \frac{1}{\beta^2} |\eta(i\tau)|^4 e^{4\pi^2 K(x,x) - 4\pi \Delta m(x)} = \frac{1}{4} \langle \overline{\psi} \psi \rangle_\beta^2 e^{4\pi \Delta m(x)}.
\]

Note that this contribution to the scalar correlator does not saturate the cluster decomposition according to which

\[
\lim_{x \to \infty} \langle \overline{\psi} \psi(x) \overline{\psi} \psi(0) \rangle = \langle \overline{\psi} \psi \rangle^2.
\]

From this it is clear that the missing piece must come from the \(k = \pm 2\) sectors which we now evaluate.

According to the partition function (4.1), the correlation function \(C_{++}(x)\) reads

\[
C_{++}(x) = \frac{1}{Z} |\psi_1(x)\psi_2(0) - \psi_2(x)\psi_1(0)|^2 > A.
\]

The average over \(h_0\) and \(h_1\) is performed exactly in Appendix B resulting in

\[
C_{++}(x) = e^{-\frac{8\pi^2}{\beta^2}} \langle \overline{\psi}(x) \psi(0) \rangle = e^{-\frac{8\pi}{\sqrt{\nu m^2}}} |\phi_1(z)|^2 (e^{2\phi(x) + 2\phi(0)})_S,
\]

where we have used that the partition function is equal to \(Z = 1/|\eta(i\tau)|^2 \sqrt{2\tau}\). The integral over \(\phi\) can be carried out in exactly the same way as in the \(k = 0\) sector. The result is

\[
\langle e^{2\phi(x) + 2\phi(0)} \rangle_S = e^{4\pi^2 K(x,x) + 4\pi \Delta m(x) - 4\pi \Delta(x) + \frac{4\pi}{\sqrt{\nu m^2}}}.\]

Again we use the Kronecker-identity for the massless propagator which leads to a cancellation of the \(\theta\)-functions. Finally

\[
C_{++}(x) = \frac{1}{\beta^2} |\eta(i\tau)|^4 e^{4\pi^2 K(x,x) + 4\pi \Delta m(x)} = \frac{1}{4} \langle \overline{\psi} \psi \rangle_\beta^2 e^{-4\pi \Delta m(x)}.
\]
Combining (4.25) and (4.30) we get the remarkably simple results

\[ S_E(x) = \langle \overline{\psi} \psi \rangle_{\beta}^2 \cosh[4\pi \Delta_m(x)], \]
\[ P_E(x) = \langle \overline{\psi} \psi \rangle_{\beta}^2 \sinh[4\pi \Delta_m(x)], \]  

(4.31)

where \( S_E \) and \( P_E \) are the Euclidean counterparts of (2.9) and (2.6), respectively. These equations are derived for a finite circle, but as shown in [11] the \( L \to \infty \) limit of \( \langle \overline{\psi} \psi \rangle_{\beta} \) is well defined and exactly equal to the temperature dependent condensate \( \langle \overline{\psi} \psi \rangle_{\beta} \) appearing in section 2. These expressions were expected from the bosonisation approach described in the preceding sections. Indeed, (4.31) are the natural versions of the free massive boson correlators on the torus.

5. Screening Lengths

Lattice simulations in four dimensions in QCD have focused on the Euclidean correlation functions which particular emphasis on screening lengths \([20,1]\). It is interesting to note that the spatial screening lengths following from (4.31) are controlled by the meson mass \( m = e/\sqrt{\pi} \) and not \( 2\pi T \) whatever the temperature. Indeed, for large spatial separation and at low temperatures \((T \ll m/2\pi)\) (4.31) reduces to

\[ S_E(0, x^1) - \langle \overline{\psi} \psi \rangle_{\beta}^2 \sim 2\langle \overline{\psi} \psi \rangle_{\beta}^2 K_0^2(m|x^1|), \]
\[ P_E(0, x^1) \sim 2\langle \overline{\psi} \psi \rangle_{\beta}^2 K_0(m|x^1|), \]  

(5.1)

where \( K_0 \) is the McDonald function. We have used the approximation

\[ \Delta_m(0, x^1) = -\frac{1}{2} \int \frac{dp}{2\pi \sqrt{p^2 + m^2}} \left( 1 + \frac{2}{e^{\beta \sqrt{p^2 + m^2}} - 1} \right) \sim \frac{1}{2\pi} K_0(m|x^1|) \]  

(5.2)

valid for \( T \ll m/2\pi \), and dropped terms of order \( e^{-m/T} \). In (5.1) the first result reflects on a cut corresponding to the exchange of two massive pseudoscalars\(^3\), and the second on

\(^3\)Note that the pre-exponential factor is essential in determining the correct singularity structure of the spectral function. We believe this to be the case also in four dimensions and at low temperatures with \( K_0(m|x^1|) \to K_1(m|x_1|) \).
a pole. For temperatures such that $T \gg m/2\pi$, (4.3.1) reduce to

$$S_E(0, x^1) - \langle \overline{\psi} \psi \rangle_\beta^2 \sim 2 \langle \overline{\psi} \psi \rangle_\beta^2 \left( \frac{\pi T}{m} \right)^2 e^{-2m|x^1|},$$

$$P_E(0, x^1) \sim 2 \langle \overline{\psi} \psi \rangle_\beta^2 \left( \frac{\pi T}{m} \right) e^{-m|x^1|},$$

(5.3)

where we have used $\Delta_m(0, x^1) \sim -Te^{-m|x^1|/2m}$ instead of (5.2). We conclude that for any temperature, the screening length in the pseudoscalar channel is $m$ and in the scalar channel $2m$, whatever $T$. What is interesting to note is that for $T \gg m/2\pi$ the finite temperature Euclidean correlator along the spatial direction, no longer correctly reproduces the singularity structure of the real time spectral function.

If we were to dimensionally reduce the Schwinger model, we would expect the static correlators such as the ones described here to display screening lengths of the order of $2\pi T$, reflecting on the free field contribution to the correlators at high temperature. Indeed, for a free spatial massless fermion ($\omega_n = (2n + 1)\pi T$),

$$S(0, x^1) = -\frac{1}{\beta} \sum_n \int \frac{dp^1}{2\pi} e^{ip^1x^1} \overline{\gamma}^0_\omega \omega_n + \gamma^1 p^1 \overline{\omega_n}^2 + p^{12} = -\frac{i\gamma^1}{2\beta} \frac{\text{sgn}(x^1)}{\sinh(\frac{\pi |x^1|}{\beta})},$$

(5.4)

after contour integration. At large distances and/or large temperatures, only the lowest Matsubara modes $\pm \pi T$ contribute to (5.4), resulting in

$$S(0, x^1) \sim -iT \text{sgn}(x^1) \gamma^1 e^{-\pi T|x^1|},$$

(5.5)

in agreement with the leading behaviour from dimensional reduction. The free fermion contribution to the scalar and pseudoscalar Euclidean correlators are

$$S_{E}^{F}(0, x^1) = -\frac{1}{2\beta^2} \frac{1}{\sinh^2(\frac{\pi |x^1|}{\beta})} \sim -2T^2 e^{-2\pi T|x^1|},$$

$$P_{E}^{F}(0, x^1) = +\frac{1}{2\beta^2} \frac{1}{\sinh^2(\frac{\pi |x^1|}{\beta})} \sim +2T^2 e^{-2\pi T|x^1|},$$

(5.6)

in disagreement with (5.3). Why is that? The reason is due to the fact that the dimensionally reduced version is plagued with infrared divergences that makes the result (5.6) inapplicable. The model offers a counterexample to by now naive intuition.
The behaviour of (4.31) along the temporal direction at low temperature \( T \ll m/2\pi \), follows from the following form of \( \Delta_m(x^0, 0) \)

\[
\Delta_m(x^0, 0) = -\frac{1}{2} \int \frac{dp}{2\pi} \frac{1}{\sqrt{p^2 + m^2}} \frac{e^{x_0 \sqrt{p^2 + m^2}} + e^{(\beta - x_0) \sqrt{p^2 + m^2}}}{e^{\beta \sqrt{p^2 + m^2}} - 1} \sim -\frac{e^{-mx_0}}{\sqrt{2\pi m x_0}} - \frac{e^{-m(\beta - x_0)}}{\sqrt{2\pi m (\beta - x_0)}}.
\]

(5.7)

The singularity structure in the scalar and pseudoscalar correlator following from (5.7) at very low temperature agrees with the one derived from the spatial asymptotics, as expected from \( O(2) \) invariance in the zero temperature limit. The temperature corrections mostly affect the strengths of the singularities. These corrections are suppressed by \( e^{-m/T} \).

In the temperature range \( T \gg m/2\pi \), the behavior of (4.31) along the temporal direction is given by

\[
S_E(x^0, 0) \sim +\langle \bar{\psi} \psi \rangle_\beta \cosh \left( \frac{2\pi}{m\beta} - \ln(2 - 2\cos(\frac{2\pi}{\beta} x^0)) \right) \sim + \frac{1}{2\beta^2} \frac{1}{\sin^2(\frac{\pi x_0}{\beta})},
\]

\[
P_E(x^0, 0) \sim -\langle \bar{\psi} \psi \rangle_\beta \sinh \left( \frac{2\pi}{m\beta} - \ln(2 - 2\cos(\frac{2\pi}{\beta} x^0)) \right) \sim - \frac{1}{2\beta^2} \frac{1}{\sin^2(\frac{\pi x_0}{\beta})},
\]

(5.8)

We have used the fact that at high temperature \([11]\)

\[
\langle \bar{\psi} \psi \rangle_\beta \sim -2Te^{-\frac{\pi x_0}{\beta}}.
\]

(5.9)

The exponential growth in the hyperbolic functions in (5.8) is balanced by the exponential fall off in the condensate. To compare this result with the free massless fermion contribution at finite temperature, we recall that for a free temporal massless fermion \( (\omega_n = (2n + 1)\pi T) \)

\[
S(x^0, 0) = -\frac{1}{\beta} \sum_n \int \frac{dp}{2\pi} e^{i\omega_n x^0} \frac{\gamma^0 \omega_n + \gamma^1 p^1}{\omega_n^2 + p^2} = -\frac{i\gamma^0}{2\beta} \frac{1}{\sin(\frac{\pi x_0}{\beta})},
\]

(5.10)

where the last relation follows by contour integration with \( x^0 \neq 0 \) modulo \( \beta \). This immediately shows that at high temperature the scalar and pseudoscalar temporal correlators (5.8) reduce to the free fermion behavior \([11]\). By squeezing the temporal direction

\footnote{Note that the free results (5.4) and (5.10) in their exact form are analytical continuation of each other, even though the boundary conditions are not interchangeable for fermions.}
in Euclidean space (high temperature) the correlator in the temporal direction picks up exclusively the free field part. This is not the case along the spatial direction.
6. Analytical Continuation

We now show that the Euclidean correlators (4.31) give the same spectral functions as the ones derived from the retarded correlators. The derivation is fully nonperturbative. The Euclidean correlators at finite temperature are periodic. This can be readily seen by writing (4.7) in a Hamiltonian formalism. For instance 

\[ P_E(\tau, x^1) = \text{Tr} \left( e^{-\beta(H-F)}(\theta(\tau) \bar{\psi}i\gamma_5\psi(x) \bar{\psi}i\gamma_5\psi(0) + \theta(-\tau) \bar{\psi}i\gamma_5\psi(0) \bar{\psi}i\gamma_5\psi(x)) \right). \] (6.1)

Using \( \psi(\tau + \beta) = -e^{\beta H}\psi(\tau)e^{-\beta H} \), it follows immediately that \( P_E(\tau + \beta, x^1) = P_E(\tau, x^1) \). Similarly for the scalar correlator. Thus \( P_E(\tau, x^1) \) has a discrete Fourier representation

\[ \omega_n = \frac{2\pi n}{T} \]

\[ P_E(\tau, x^1) = \frac{1}{\beta} \sum_n e^{-i\omega_n\tau} P_E(\omega_n, x^1). \] (6.2)

For \( 0 < \tau < \beta \), the Euclidean correlator (6.1) can be rewritten in the form

\[ P_E(\tau, x^1) = \int dp^1 e^{ip^1x^1} \sum_{A, B} \delta(p^1 - (p_B^1 - p_A^1)) e^{-\beta(E_A - F)} e^{-(E_B - E_A)^T} | A | \bar{\psi}i\gamma_5\psi(0) | B > |^2, \] (6.3)

following the insertion of a complete set of physical states, \( p_B^2 = m^2 \). Inserting (6.3) and the identity in frequency space into (6.2), give

\[ P_E(\tau, x^1) = \frac{1}{\beta} \sum_n \int \frac{d\omega dp^1}{2\pi} e^{-i\omega_n\tau + ip^1x^1} \frac{\sigma_P(\omega, p^1)}{i\omega_n - \omega}. \] (6.4)

The sum over the Matsubara frequencies \( \omega_n \) can be carried out explicitly, using the relation

\[ \frac{1}{\beta} \sum_n e^{-i\omega_n\tau} \frac{1}{i\omega_n - \omega} = \oint_C \frac{dz}{2\pi i} \frac{e^{-iz\tau}}{iz - \omega} \frac{e^{i\beta z}}{e^{i\beta z} - 1}. \] (6.5)

The contour \( C \) encloses the real axis counterclockwise. Inserting (6.3) into (6.4) and unwinding the contour with \( 0 < \tau < \beta \) in mind give

\[ P_E(\tau, x^1) = -\int dp^1 d\omega \frac{e^{ip^1x^1 - \omega\tau}}{(2\pi)} \left( 1 + \frac{1}{e^{\beta\omega} - 1} \right) \sigma_P(\omega, p^1). \] (6.6)
We have deliberately disregarded the pinch singularity in (6.3) at \( \omega = 0 \), as it gives zero contribution in (6.6) since \( \sigma_p(0,p^1) = 0 \).

For \( L \to \infty \) the bosonic propagator \( \Delta_m(x) \) can be written as

\[
\Delta_m(\tau, x^1) = -\frac{1}{\beta} \sum_n \int \frac{dp_1}{2\pi} \frac{e^{ip_1 x^1 + i\omega_n \tau}}{\omega_n^2 + p_1^2 + m^2}.
\] (6.7)

The sum over \( n \) can be converted into an integral as in (6.5),

\[
\Delta_m(\tau, x^1) = -\oint_C \frac{dz}{2\pi} \int \frac{dp}{2\pi} \frac{e^{ip x^1}}{z^2 + p^2 + m^2} e^{i\beta z} - 1.
\] (6.8)

Again, the contour \( C \) encloses the real axis counterclockwise. \( C \) can be unwound to give contributions only from the poles \( \pm i\sqrt{p^2 + m^2} \) on the imaginary axis in the \( z \)-plane (with no pinch in this case). Thus

\[
\Delta_m(\tau, x^1) = -\int \frac{dp_0 dp_1}{2\pi} \delta(p^2 - m^2) \frac{1}{2\sinh[p^0/2]} e^{-p^0(\tau - \frac{\beta}{2}) + ip x^1}.
\] (6.9)

The lack of manifest periodicity in \( \tau \) in (6.9) is due to the fact that while unwinding the contour \( C \) above, we have explicitly restricted \( \tau \) in the range \([0, \beta]\) to allow convergence on the circles at infinity. The spectral function follows from (4.31) by expanding the \( \sinh \)-function in powers of \( \Delta_m \), using (6.9) and comparing with (6.6). The result is in complete agreement with (3.14) as obtained from the retarded correlator. This shows that the full real time spectral function can be reconstructed from the nonperturbative Euclidean correlator at finite temperature.

In general, the full analytical knowledge of the finite temperature Euclidean correlator is a "dream-stuff" in interacting quantum field theories in four dimensions. We are therefore left with approximate, and usually numerical, estimates of the these correlators. Given this fact of life, how do we go about reconstructing the true spectral function? More specifically, given some knowledge of the l.h.s. of (6.6), how do we go about the r.h.s. of (6.6)?

At zero temperature, the conventional wisdom is to look at the asymptotics of the Euclidean correlators, under the assumption that the leading singularity in the spectral function is well separated from possible background. At low temperature \( T \ll m/2\pi \),
this rationale seems to hold for the spatial asymptotics while most of the zero tempera-
ture structure is preserved. At temperatures $T \gg m/2\pi$ the spatial asymptotics of the
Euclidean correlator reflect solely on screening. The tails no longer encode properly the
real time singularity structure. This is more dramatic in the temporal direction, where
the free fermion behaviour is reached at high temperature, even in the presence of a mass
gap.

Besides the asymptotics of the Euclidean correlator at finite temperature, one can
probe its zero frequency (static) or zero momentum part, as first suggested by DeTar [21].
Indeed, from (6.4) we see that the large $x^1$-behavior of the static part of the Euclidean
correlator,

$$P_E(\omega_n = 0, x^1) = -\int \frac{d\omega dp^1}{2\pi} e^{ip^1x^1} \frac{\sigma_P(\omega, p^1)}{\omega}$$  \hfill (6.10)

reflects on the possible singularities of the spectral function in the pseudoscalar channel
and/or screening. In the Schwinger model, the covariant dispersion relation $\omega^2 = p^{12} + m^2$
holds even at finite temperature, tying the singularity in $p^1$ to the singularity in $\omega$. In
general, this is not true and we would expect the singularities (poles, resonances and
thresholds) to disperse in matter. The correlator (6.10) has been the object of interesting
lattice simulations in the context of QCD [20]. Note that the zero momentum part of
(6.10) relates directly to the pseudoscalar susceptibility

$$\chi_P = P_E(\omega_n = 0, p^1 = 0) = -2 \int_0^\infty d\omega \frac{\sigma_P(\omega, 0)}{\omega},$$  \hfill (6.11)

where we have used the fact that $\sigma_P(\omega, 0)$ is and odd function of $\omega$ at zero momentum\(^5\).
Susceptibilities of the type (6.11) reflect on the bulk properties of the theory at finite
temperature [22, 23]. They are important dynamical parameters in the theory of linear
response. For completeness, we remark that the zero momentum part of the Euclidean

\(^5\)The zero temperature limit (2.11) should be multiplied by sgn($q^0$) to allow for antiparticles, and is
consistent with this property.
correlator at finite temperature is given by

\[
P_E(\tau, q^1 = 0) = \int_0^\infty \frac{d\omega}{2\pi} \left( e^{\omega \tau} + e^{\omega(\beta - \tau)} \right) \frac{\sigma_P(\omega, 0)}{1 - e^{\beta \omega}} = -\int_0^\infty d\omega \frac{\cosh \left( \frac{\omega (\tau - \beta)}{2} \right)}{\sinh \left( \frac{\omega \beta}{2} \right)} \sigma_P(\omega, 0).
\]

(6.12)

For high temperatures, the l.h.s. is dominated by the free field behaviour which should reflect itself in \(\sigma_P(\omega \sim T, 0)\). One can think of a numerical inversion of (6.12) (inverse Laplace transforms) as a way to probe the time-like character of \(\sigma_P(\omega, 0)\) at zero momentum. In the Schwinger model, the validity of the inversion can be checked term by term. This inversion is worth pursuing on the lattice for the finite temperature Euclidean correlators in QCD.

7. Conclusions

We have explicitly constructed the zero and finite temperature correlation functions for the scalar and pseudoscalar channels in the context of the Schwinger model. The finite temperature spectral functions have been obtained directly from the retarded correlators using operator methods, and from the Euclidean correlators by analytical continuation. The two procedures are in agreement.

The temperature dependent spectral function in the Schwinger model reflects on thermalized, but free mesons of mass \(m = e/\sqrt{\pi}\). The poles and thresholds are temperature independent, with temperature dependent strengths. The absence of interactions in the bosonized version, makes the above results specific. In four dimensions and for theories such as QCD, we would expect the singularities to depend explicitly on the temperature (pions interact in the heat bath). However, we do not have a proof.

Our expressions for the temperature dependent spectral functions can be written as the sum of all possible cuttings of free scalar diagrams with an arbitrary exchange of pseudoscalar mesons, for both the scalar and pseudoscalar channel. In their phase space version they are directly amenable to four dimensions. They provide compact expressions for cutting rules in real time [14] in the context of \(\phi^n\) theories.
The exact expressions for the Euclidean correlators indicate that the screening lengths in the Schwinger model are amenable to the meson mass \( m = e/\sqrt{\pi} \) and independent of the temperature, whatever the temperature. This is so, even though the meson is a composite of two fermions. We strongly suspect that this result is due to the point like character of the meson in the massless Schwinger model, and not just the anomaly. Indeed, we conjecture that in the massive Schwinger model, the screening lengths asymptote \( 2\pi T \) at high temperature. We also conjecture that the heavy-light systems display screening lengths that asymptote \( \pi T \) in this model \[28\].

We note that in the Schwinger model the screening mass and the dynamical mass (leading singularity in the pseudoscalar channel at zero temperature) are temperature independent. The reason is that the process of mass generation in the model is through the \( U_A(1) \) anomaly, thus protected from matter corrections at all temperatures. This is not the case for physical excitations in four dimensions, to the exception of the \( \eta' \) mass. This raises an interesting thought for four dimensional QCD. Could it be that the \( \eta' \) mass survives matter effects? The standard scenario based on instanton-driven anomaly would not support this, as the instantons become suppressed at high temperature because of Debye screening. We remark, however, that the \( \eta' \) is not point like in nature. This point and the analogy are worth investigating, both in instanton and non-instanton approaches to the \( \eta' \) problem in matter.

The above considerations suggest that possible criteria for a weakly interacting gas of massless fermions at high temperature might be formulated in terms of the sums, \( S = S_E(x) + P_E(x) \), or the ratios, \( R = S_E(x)/P_E(x) \), taken along the spatial or temporal direction. For free massless fermions: \( S = 0 \) and \( R = -1 \). These criteria hold in the presence of isospin, for a variety of combinations with proper isospin weighting. They might still be evaded by the Coulomb bound state approach following from dimensional reduction as discussed by two of us \[4\]. This point is worth clarifying.

The massless Schwinger model through its severe infrared sensitivity, provides a counterexample to the general lore on the spatial screening lengths based on dimensional
reduction in three and four dimensions for both QED and QCD. Although the results are specific to the model, we think that they are still interesting. They provide a particular realization of field theoretical ideas at finite temperature, and show how naive arguments could be upset by infrared singularities.

In four dimensions, QCD is also expected to suffer from strong infrared singularities at finite temperature. However, lattice simulations [1] suggest that at high temperature the screening lengths are compatible with the dimensional reduction scheme [4, 24], to the exception of the pion and its chiral partner. To what extent these results reflect on a free spectral function is not clear though, since these results are still compatible with correlations in the heat bath [1, 24].

Our arguments in the Schwinger model indicate that at low temperature \( T \ll m/2\pi \), the large spatial asymptotics of the Euclidean correlators determines the leading isolated singularities of the spectral function, provided that the pre-exponents are properly identified. This identification is rapidly lost as the temperature is increased, poles and cuts are hardly discriminated for \( T \sim m/2\pi \). If the analogy with the Schwinger model holds, the break up temperature in the QCD context would be \( T \sim m_{\pi}/2\pi \sim 20 \) MeV. A similar remark holds for the behaviour of the Euclidean correlators along the temporal direction, except for the fact that the latter asymptote the free fermion behaviour at high temperature even in the presence of a mass gap. This point suggests that the lattice calculations of temporal correlators at high temperature [25] may be misleading on the interacting aspects of the theory, since they reflect primarily on the free field content of the correlators.

An interesting way to probe the dynamical part of the spectral function at zero momentum is to try a numerical inversion of (6.12) for QCD. Also, if the lattice results could be analyzed in frequency space, then perhaps one may think about analytical continuations a la Baym-Mermin [20], assuming that one can reconstruct singularities from a numerical output on a discrete mesh \( \omega_n = 2\pi nT \). In QCD at high temperature, it may be that the singularity is a simple cut throughout \( q^0 \) (analyticity aside) following the
change to an ionized (plasma) state. This simple fact is still elusive, however. Progress in these directions may come from real time alternatives to the present Euclidean time simulations. It should be emphasized, however, that the Euclidean correlators at finite temperature, are interesting in their own right. They reflect on spacelike physics in the form of screening and susceptibilities [24].

Finally, it would be interesting to see, whether finite temperature simulations of the massless Schwinger model, reproduce the exact results derived in this paper. Also, it would be amusing to test the ideas of finite temperature QCD sum rules [27] and finite temperature instanton (caloron) calculations [30] in this exactly solvable model. The extension of this work to higher correlation functions will be discussed elsewhere [29].

**Note Added**

After finishing up this work we became aware of a recent paper by A. Smilga ”Instantons in the Schwinger Model”, NSF-ITP-93-151, where the scalar correlator is evaluated at finite temperature using instanton techniques.
Appendix A: The Euclidean fermionic Green’s function

In this appendix we calculate the fermionic Green’s function defined by

\[ i\mathcal{D}S(x,y) = i\delta(x-y) - P(x,y), \tag{A.1} \]

where \( \mathcal{D} \) is the Dirac operator in the presence of a \( U(1) \) gauge field

\[ \mathcal{D} = \gamma_\mu (\partial_\mu - ieA_\mu), \tag{A.2} \]

and \( P \) is the projection operator onto the space of zero modes. We are interested in the case when there are no zero modes \( (P = 0) \). Since there are always zero modes present whenever the flux \( \Phi \neq 0 \) we set \( \Phi = 0 \)

\[ A_0 = \frac{2\pi h_0}{\beta} + \partial_1 \phi, \]
\[ A_1 = \frac{2\pi h_1}{L} - \partial_0 \phi, \tag{A.3} \]

where \( h_0, h_1 \) are defined modulo one. The Dirac operator has two zero modes (one of each chirality) at \((h_0, h_1) = (1/2, 0)\). However, these zero modes are cancelled by the determinant and do not give rise to singularities in the integration over \( h_0 \) and \( h_1 \). In this section we will stay away from this point. Notice that if we write

\[ S(x,y) = e^{-\gamma_5 \phi(x)}U(x)g(x,y)U^\dagger(y)e^{-\gamma_5 \phi(y)}, \tag{A.4} \]

where

\[ U(x) = \exp i2\pi \left( \frac{x^0 h_0}{\beta} + \frac{x^1 h_1}{L} \right), \tag{A.5} \]

then it follows from (A.1) that

\[ i\mathcal{D}g(x,0) = i\delta(x). \tag{A.6} \]

Because of translational invariance of the Green’s function we have set \( y = 0 \). The boundary conditions on the Green’s functions follow from the boundary conditions on the
fundamental fields. Specifically
\[
g \left( x^0 + \beta, x^1; 0 \right) = -e^{-i2\pi h_0} g \left( x; 0 \right),
g \left( x^0, x^1 + L; 0 \right) = e^{-i2\pi h_1} g \left( x; 0 \right). \tag{A.7}
\]

From \( \gamma_5 S \gamma_5 = -S \) it follows that \( S_{\pm,\pm} = g_{\pm,\pm} = 0 \). Equation (A.6) then reduces to
\[
+ 2\beta \partial_z g_+ (z) = \delta^{(2)}(z),
-2\beta \partial_{\bar{z}} g_+ (\bar{z}) = \delta^{(2)}(\bar{z}), \tag{A.8}
\]
where \( z = (x^0 + ix^1)/\beta \). The above equations tell us that \( g_- (g_+) \) is a holomorphic (anti-holomorphic) function of \( z \) with a single pole at \( z = 0 \) with residue \( 1/2\pi \beta (-1/2\pi \beta) \).

The solution that satisfies the boundary conditions (A.7) can be written down at once
\[
g_-(z) = \frac{1}{2\pi i \beta} \sum_{m,n} e^{2\pi i (h_0 + \frac{1}{2}) m + 2\pi i h_1 n} \frac{1}{z + m + n \tau} = \frac{1}{2\pi i \beta} \text{Ser}(h_0 + \frac{1}{2}, h_1, z, 1, \tau), \tag{A.9}
\]
where the function \( \text{Ser} \) was introduced and studied extensively by Kronecker. It can be expressed in terms of theta functions resulting in
\[
g_-(z) = \frac{1}{2\pi i \beta} \frac{\psi'_1(0) \psi'_1(z + h_1 - i(h_0 + \frac{1}{2})\tau)}{\psi'_1(1) \psi'_1(h_1 - i(h_0 + \frac{1}{2})\tau)} e^{-2\pi i z (h_0 + \frac{1}{2})}. \tag{A.10}
\]

The function \( g_-(z) \) is just the complex conjugate of \( g_+ \). The complete Green’s function follows immediately by combining this result with the gauge transformation
\[
S_{++} (x, y) = g_+(z) e^{-\phi(x) + \phi(0)} e^{\frac{2\pi i h_0 x^0}{y} + \frac{2\pi i h_1 x^1}{L}}. \tag{A.11}
\]

The function \( g_+(z) \) can also be constructed directly in terms of theta functions by writing down a ratio with the boundary conditions (A.7) and the pole structure as dictated by (A.10). The generalization of (A.11) to a nonvanishing flux sector can be found in [29].

\[6\text{Recall that } \Phi = 0 \text{ for the gauge fields, so that the gauge transformation one picks up under translation in the space direction is } 1.\]
Appendix B: Contribution from the $k = 2$ sector.

From (4.1), the correlation function $C_{++}(x)$ in the $k = 2$ sector reads

$$C_{++}(x) = \frac{1}{Z} < \det(iD)|\psi_1(x)\psi_2(0) - \psi_2(x)\psi_1(0)|^2 >_A$$  \hspace{1cm} (B.1)

where $\psi_1$ and $\psi_2$ are the zero modes (4.4) for $p = 0$ and $p = 1$, respectively. To evaluate (B.1), first, we re-express $\psi_1$ and $\psi_2$ in terms of the $\theta_{00}$ and $\theta_{10}$ functions, respectively, using the following relations

$$\Theta \left[ \begin{array}{c} a \\ b \end{array} \right] (0, 2i\tau) = \exp(-2\pi \tau a^2 + 2\pi iab)\theta_{00}(b + 2ia\tau, 2i\tau),$$

$$\Theta \left[ \begin{array}{c} a + \frac{1}{2} \\ b \end{array} \right] (0, 2i\tau) = \exp(-2\pi \tau a^2 + 2\pi iab)\theta_{10}(b + 2ia\tau, 2i\tau).$$  \hspace{1cm} (B.2)

We are using (10) for $(\frac{1}{2}0)$, (11) for $(\frac{1}{2}1)$, etc. Second, we apply the Caspari-Kronecker formula [31],

$$\theta_{00}(2v, 2i\tau)\theta_{10}(2u, 2i\tau) - \theta_{10}(2v, 2i\tau)\theta_{00}(2u, 2i\tau) = \theta_{11}(v + u, i\tau)\theta_{11}(v - u, i\tau)$$  \hspace{1cm} (B.3)

Note, that in our case the argument $(v - u)$ is $h$-independent, so the remaining integrand $|\theta_{11}(u + v, i\tau)|^2$ over the harmonic fields is Gaussian. Thus

$$C_{++}(x) = \frac{1}{\beta^2} \frac{1}{Z} |\theta_{11}(z, i\tau)|^2 e^{-\frac{8\pi^2}{\beta^2 T}} < e^{2\phi(x) + 2\phi(0)} >_S \times \int dh_0dh_1e^{-2\pi\tau[(\frac{x}{L} + 1 - h_0) + (\frac{1}{4} - \frac{h_0}{2})^2 + (\frac{1}{4} - \frac{h_0}{2})^2]}|\theta_{11}(t, i\tau)|^2,$$  \hspace{1cm} (B.4)

where $t = x_0/\beta + h_1 + i\tau(x_1/L + 1/2 - h_0)$. Since the integration over $h_1$ reduces the double sum in $|\theta_{11}(t, i\tau)|^2$ to a single one, the correlator reads

$$C_{++}(x) = \frac{1}{\sqrt{2\tau}} \frac{1}{\beta^2} \frac{1}{Z} e^{-\frac{8\pi^2}{\beta^2 T}}|\theta_{11}(z, i\tau)|^2 e^{-\frac{8\pi^2}{\beta^2 T}} < e^{2\phi(x) + 2\phi(0)} >_S \sum_n \int_0^1 dh_0e^{-2\pi\tau(n + 1 - h_0 + \frac{1}{4})^2}$$

$$= \frac{1}{\sqrt{2\tau}} \frac{1}{\beta^2} \frac{1}{Z} e^{-\frac{8\pi^2}{\beta^2 T}}|\theta_{11}(z, i\tau)|^2 e^{-\frac{8\pi^2}{\beta^2 T}} < e^{2\phi(x) + 2\phi(0)} >_S,$$  \hspace{1cm} (B.5)

where $z = (x_0 + ix_1)/\beta$ denotes the relative coordinate.
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Figure Captions

Fig. 1a. Temperature effects on the sources (a). The dashes are temperature insertions.

Fig. 1b. Diagrammatic expansion of the retarded pseudoscalar correlator $P_E$. The straight lines are retarded propagators $\Delta_R$, and the broken lines the imaginary part of the Feynman propagator $\text{Im} \Delta_F$.

Fig. 1c. The same as (b) but for the retarded scalar correlator $S_E$.

Fig. 2a. Leading processes contributing to the scalar spectral function $\sigma_S$ in the heat bath.

Fig. 2b. Leading processes contributing to the pseudoscalar spectral function $\sigma_P$ in the heat bath.
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