Trajectory and Attractor Convergence for a Nonlocal Kuramoto-Sivashinsky Equation

October 25, 1997

Jinqiao Duan¹, Vincent J. Ervin¹ and Hongjun Gao²
1. Department of Mathematical Sciences, Clemson University, Clemson, South Carolina 29634, USA.
2. Laboratory of Computational Physics, Institute of Applied Physics and Computational Mathematics, P. O. Box 8009, Beijing 100088, China.

Abstract

The nonlocal Kuramoto-Sivashinsky equation arises in the modeling of the flow of a thin film of viscous liquid falling down an inclined plane, subject to an applied electric field. In this paper, the authors show that, as the coefficient of the nonlocal integral term goes to zero, the solution trajectories and the maximal attractor of the nonlocal Kuramoto-Sivashinsky equation converge to those of the usual Kuramoto-Sivashinsky equation.

AMS subject classification: 35Q35, 58F39
Keywords. Nonlocal integral term, trajectory convergence, attractor convergence, infinite dimensional dynamical system

To Appear: Communications on Applied Nonlinear Analysis No. 4, Vol. 5, 1998.
1 Introduction

In this paper we investigate some dynamical issues of a Kuramoto-Sivashinsky equation with a nonlocal spatial integral term

\[ u_t + u_{xxxx} + u_{xx} + uu_x + \alpha H(u_{xxx}) = 0, \quad (1.1) \]

where \( \alpha \) is a non-negative constant coefficient, and

\[ H(f) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{x-\xi} d\xi, \quad (1.2) \]

is the Hilbert transform and the integral is understood in the sense of the Cauchy Principle Value.

The equation under consideration arises in the modeling of the flow of a thin film of viscous liquid falling down an inclined plane, subject to an applied electric field [9]. The application of a uniform electric field at infinity, perpendicular to the inclined plane, is to destabilize the liquid films on the surface of the plane. In an industrial setting it is hoped that this destabilization will lead to an enhancement of heat transfer. In this engineering context, \( \alpha We/\sqrt{W|\cot \beta - 4/5Re|} \), where \( We \) is the electrical Weber number, \( W \) the Weber number, \( Re \) the Reynolds number and \( \beta \) the angle between the plane and the horizontal. When the electrical field is absent, i.e. \( We = 0 \), the Hilbert transform term is gone and we obtain the usual (local) Kuramoto-Sivashinsky equation

\[ u_t + u_{xxxx} + u_{xx} + uu_x = 0. \quad (1.3) \]

Throughout this paper we restrict our attention to the case of \( u(x,t) \) periodic in \( x \) with period \( l > 0 \). We denote the interval \( I := (-l,l) \). Then, (1.2) is replaced by (see [1]) the “periodic” Hilbert transform

\[ H(f) = -\frac{1}{2l} \int_I \cot \frac{\pi(x-\xi)}{2l} f(\xi) d\xi. \]

Observe that \( u = C \), a constant, satisfies (1.1). Moreover, for

\[ u(x,0) = u_0(x) \quad \text{for} \quad x \in I, \quad \text{and} \quad \bar{u}_0 := \frac{1}{2l} \int_I u_0(x) dx, \quad (1.4) \]

integrating (1.1) over the interval \( I \) yields

\[ \frac{d}{dt} \int_I u(x,t) dx = 0, \quad \text{i.e.} \quad \frac{1}{2l} \int_I u(x,t) dx = \bar{u}_0. \]

Thus, we have that the mean of the solution is conserved with respect to time. This may be interpreted in the sense that the “dynamics” of \( u \) satisfying (1.1) are centered around the mean value of the initial data. Therefore,
without loss of generality, we will work in a periodic functional space with zero mean. That is, we will assume the following condition

\[ u(x,t) \text{ periodic on } I, \text{ with } u(x,0) = u_0(x), \text{ and } \int_I u_0(x) \, dx = 0. \quad (1.5) \]

In this paper, we will show that individual solution trajectories and the maximal attractor of the nonlocal Kuramoto-Sivashinsky equation approach the solution trajectories and the maximal attractor of the usual local Kuramoto-Sivashinsky equation, as the coefficient for the nonlocal term goes to zero.

2 Preliminaries

We denote by \( L^2_{\text{per}}(I) \), \( H^k_{\text{per}}(I) \), \( k = 1, 2, \cdots \), the usual Sobolev spaces of periodic functions on \( I \). Let \( \tilde{L}^2_{\text{per}}(I) \), \( \tilde{H}^k_{\text{per}}(I) \) denote the spaces of functions \( g \) in \( L^2_{\text{per}}(I) \), \( H^k_{\text{per}}(I) \), respectively, with mean zero, i.e. \( \tilde{g} := \frac{1}{2l} \int_I g(x) \, dx = 0. \)

In the following, \( \| \cdot \| \) denotes the usual \( L^2_{\text{per}}(I) \) norm. Due to the Poincare inequality, \( \| \cdots \| \) is an equivalent norm in \( \tilde{H}^k_{\text{per}}(I) \). All integrals are with respect to \( x \in I \), unless specified otherwise. We recall the following two inequalities:

Poincaré inequality ([23], p.49):

\[ \int_I g^2 \, dx \leq (2l)^2 \int_I g_x^2 \, dx, \quad (2.1) \]

for \( g \in \tilde{H}^1_{\text{per}}(I) \).

Agmon inequality ([23], p.50):

\[ \| g \|_\infty^2 \leq 2 \| g \| \| g_x \|, \]

(2.2)

for \( g \in \tilde{H}^1_{\text{per}}(I) \). This inequality also follows from \( g^2(x) = 2 \int_{x_0}^x gg_x \, dx \), where \( g(x_0) = 0 \) (as \( g \) is continuous and has zero mean).

The Hilbert transform \((\square)\) is a linear, invertible, bounded operator from \( L^2 \) to \( L^2 \), and from Sobolev space \( H^k \) to \( H^k \). Several noteworthy properties of the transform are (see [1], [2] or [24]):

\[
\begin{align*}
D_x H &= H(D_x), \\
H^{-1} &= -H, \\
\int vH(u) &= -\int uH(v), \\
\int H(u)H(v) &= \int uv, \\
\int uH(u) &= 0, \\
\|H(u)\| &= \|u\|, \\
(H(u))_x &= H(u_x).
\end{align*}
\]
These properties hold for the Hilbert transformation on both the real line and periodic intervals \((1\)). On the periodic interval \((-l, l)\), the Hilbert transformation has a simple representation

\[ H(f)(x) = i \sum_{k \in \mathbb{Z}} \text{sgn}(k) f_k e^{ik\pi x/l}, \]

for \(f(x) = \sum_{k \in \mathbb{Z}} f_k e^{ik\pi x/l}\), with \(f_k\) s the Fourier coefficients of \(f\).

Duan and Ervin \((6)\) (or, for similar arguments, see \((4), (5)\) or \((25)\)), have shown that the nonlocal dynamical system \((1.1)\) and \((1.5)\) has global solutions and a maximal attractor in \(\dot{L}^2_{\text{per}}(I)\). Moreover, the following time-uniform estimates hold for the nonlocal Kuramoto-Sivashinsky equation:

\[
\begin{align*}
\|u\|_{L^2_{\text{per}}} &\leq \rho_0, \\
\|u\|_{H^1_{\text{per}}} &\leq \rho_1, \\
\|u\|_{H^2_{\text{per}}} &\leq \rho_2,
\end{align*}
\]

where \(\rho_0, \rho_1, \rho_2\) are absolute positive constants for \(0 \leq \alpha \leq 1\). These estimates have also been proved for the usual Kuramoto-Sivashinsky equation \((3), (26)\) and references therein.

The usual Kuramoto-Sivashinsky equation \((1.3)\) together with \((1.5)\) also has global solutions and a maximal attractor in \(\dot{L}^2_{\text{per}}(I)\) as well as other dynamical properties; see, for example, \((20), (21), (16), (10), (17)\) and \((23)\).

### 3 Localization Limit

In this section, we will show that individual solution trajectories and the maximal attractor for the nonlocal Kuramoto-Sivashinsky equation \((1.1)\) converge to the solution trajectories and the maximal attractor of the usual Kuramoto-Sivashinsky equation \((1.3)\), as the coefficient \(\alpha\) of the nonlocal term goes to zero. We note that Duan et al. \((5)\) have studied this convergence for a Kuramoto-Sivashinsky equation with an extra dispersive term \(u_{xxx}\). Ercolani et al. \((8)\) have also studied the impact of \(u_{xxx}\) on the dynamics of the usual Kuramoto-Sivashinsky equation.

General convergence result for the maximal attractors is available in Hale et al. \((12)\), Temam \((25)\) and Hale-Lin-Raugel \((13)\). Similar applications can also be found in Hill and Sulin \((14)\).

Let us denote by \(T_\alpha(t)\) and \(T(t)\) the solution operators for the nonlocal Kuramoto-Sivashinsky equation \((1.1)\), and the usual Kuramoto-Sivashinsky equation \((1.3)\), respectively. We need to verify a few conditions that will ensure the convergence of the solution trajectories and attractors. In particular, we should show that operators \(T_\alpha(t)\) and \(T(t)\) satisfy

\[
\|T_\alpha(t)u_0 - T(t)u_0\| \leq \eta(\alpha, t, u_0),
\]
where \( \eta(\alpha, t, u_0) \to 0 \) as \( \alpha \to 0 \), and that the domain of attraction of \( T_\alpha(t) \) is independent of \( \alpha \). For more details of these conditions see Theorem I.1.2 of Temam [23], or Theorem 2.4 of Hale-Lin-Raugel [13].

Assume that \( u, v \) are the solutions of the nonlocal and usual Kuramoto-Sivashinsky equations, respectively, with the same initial data \( u_0(x) \), i.e., \( u = T_\alpha(t)u_0 \), \( v = T(t)u_0 \). We denote also \( w = u - v \), and hence \( \|w(0)\| \equiv 0 \).

Using the equations (1.1) and (1.3), we get

\[
\frac{1}{2} \frac{d}{dt} \|w\|^2 + \|w_{xx}\|^2 - \|w_x\|^2 + \int (uw_x + wv_x) \, dx + \alpha \int H(w_{xxx}) \, dx = 0. \tag{3.1}
\]

After integration by parts, and using the Cauchy-Schwarz inequality and Agmon inequality (2.2), we get

\[
\|w_x\|^2 \leq \|w\|^2 + \frac{1}{4} \|w_{xx}\|^2, \tag{3.2}
\]

\[
\int (uw_x + wv_x) \, dx = \int (-\frac{1}{2} u_x + v_x) w^2 \, dx \leq \frac{1}{2} \|u_x\|^2 \|u_{xx}\|^\frac{1}{2} + \|v_x\|^2 \|v_{xx}\|^\frac{1}{2} \|w\|^2 \leq 2 \rho_1^\frac{1}{2} \rho_2^\frac{3}{2} \|w\|^2, \tag{3.3}
\]

\[
\alpha \int H(w_{xxx}) \, dx = -\alpha \int H(w_{xx}) w_x \, dx \leq \alpha^2 \|w_x\|^2 + \frac{1}{4} \|w_{xx}\|^2 \leq \alpha^2 \rho_1^2 + \frac{1}{4} \|w_{xx}\|^2. \tag{3.4}
\]

Here we have used the estimates in (2.4) and (2.5). Substituting (3.2), (3.3) and (3.4) into (3.1), we obtain,

\[
\frac{d}{dt} \|w\|^2 + \|w_{xx}\|^2 \leq 2 \|w\|^2 + 4 \rho_1^\frac{1}{2} \rho_2^\frac{1}{2} \|w\|^2 + 2 \alpha^2 \rho_1^2. \tag{3.5}
\]

Let

\[
b_1 = 2 + 4 \rho_1^\frac{1}{2} \rho_2^\frac{1}{2}, \quad b_2 = 2 \rho_1^2.
\]

Thus we have

\[
\frac{d}{dt} \|w\|^2 \leq b_1 \|w\|^2 + b_2 \alpha^2, \tag{3.6}
\]

or

\[
\|w(t)\|^2 \leq \|w(0)\|^2 e^{b_1 t} + \frac{b_2}{b_1} (e^{b_1 t} - 1)\alpha^2, \tag{3.7}
\]

Since \( \|w(0)\| \equiv 0 \), we conclude that \( \|w(t)\| \to 0 \) as \( \alpha \to 0 \). This shows that the solution \( u \) of the nonlocal Kuramoto-Sivashinsky equation (1.1) approaches the solution \( v \) of the usual Kuramoto-Sivashinsky equation (1.3).
Moreover, solution operator $T_\alpha(t)$ has a uniform domain of attraction, i.e., $\mathcal{L}^2_{\text{per}}$, which is independent of $\alpha$.

Therefore, by using Theorem I.1.2 of Temam [23] or Theorem 2.4 of Hale-Lin-Raugel [13], we have the following theorem.

**Theorem 3.1** When the coefficient $\alpha$ of the nonlocal integral term goes to zero,

(i) A solution trajectory of the nonlocal Kuramoto-Sivashinsky equation (1.1) converges in $L^2$ to a solution trajectory of the usual Kuramoto-Sivashinsky equation (1.3), as long as both trajectories start at the same initial point.

(ii) The maximal attractor of the nonlocal Kuramoto-Sivashinsky equation (1.1) converges in $L^2$ to the maximal attractor of the usual Kuramoto-Sivashinsky equation (1.3).

### 4 Discussions

Nonlocal spatial integral terms appear in mathematical models in the form of partial integro-differential equations for various physical phenomena, for example, in beam theory [15], in electric systems [22], chemical diffusion [11], heat conduction processes [18], temperature evolution in an atmospheric system [19], in ferromagnetic system [7], and in chemical frontal development [27].

In this paper we have studied the limiting behavior of a prototypical partial-integral differential equation arising in fluid dynamics, as the coefficient of the nonlocal integral term goes to zero. This work is a part of our effort on the investigation of impact of nonlocal spatial interactions on the dynamics of (local) infinite dimensional dynamical systems.

**Acknowledgement.** This work was supported by the U.S. National Science Foundation grant DMS-9704345, and by the National Natural Science Foundation of China grant 19701023.

### References

[1] A. Abdelouhab, J. L. Bona, M. Felland and J.-C. Saut, “Nonlocal models for nonlinear dispersive waves,” Phys. D 40 (1989), 360-392.

[2] E. A. Alarcon, “Existence and finite dimensionality of the global attractor for a class of nonlinear dissipative equations,” Proc. R. Soc. Edinburgh 123A (1993), 893-916.

[3] P. Collet, J.-P. Eckmann, H. Epstein and J. Stubbe, “A global attracting set for the Kuramoto-Sivashinsky equation”, Comm. Math. Phys. 152 (1993), 203-214.
[4] J. Duan, H. V. Ly and E. S. Titi, “The effect of nonlocal interactions on the dynamics of Ginzburg-Landau equation”, Z. Angew. Math. Phys. 47 (1996), 433-455.

[5] J. Duan, H. V. Ly and E. S. Titi, “The effect of dispersion on the dynamics of the Kuramoto-Sivashinsky equation”, preprint, 1995.

[6] J. Duan and V. J. Ervin, Dynamics of a Nonlocal Kuramoto-Sivashinsky Equation, J. Diff. Eqns., 143 (1998), 243-266.

[7] F. J. Elmer, “Nonlinear and nonlocal dynamics of spatially extended systems: stationary states, bifurcations and stability”, Physica D 30 (1988), 321-341.

[8] N. M. Ercolani, D. W. McLaughlin and H. Poinier, “Attractors and transients for a perturbed periodic KdV equation: a nonlinear spectral analysis,” J. Nonlinear Sci. 3 (1993), 477-539.

[9] A. Gonzalez and A. Castellanos, “Nonlinear electrohydrodynamic waves on films falling down an inclined plane,” Phys. Rev. E 53 (1996), 3573-3578.

[10] J. Goodman, “Stability of the Kuramoto-Sivashinsky and related systems,” Comm. Pure Appl. Math. 47 (1994), 293-306.

[11] M. D. Graham, U. Middya and D. Luss, Pulses and global bifurcations in a nonlocal reaction-diffusion system, Phys. Rev. E 48 (1993), 2917-2923.

[12] J. K. Hale, “Asymptotic Behavior of Dissipative Systems”, American Math. Soc., 1988.

[13] J. K. Hale, X. B. Lin and G. Raugel, “Upper semicontinuity of attractors for approximations of semigroups and partial differential equations”, Math. Comput. 50 (1988), 89-123.

[14] A.T. Hill and E. Suli, “Set convergence for discretization of the attractor,” IMA Journal of Numerical Analysis 16 (1996), N2:289-296

[15] P. Holmes and J. Marsden, A partial differential equation with infinitely many periodic orbits: chaotic oscillations of a forced beam, Arch.Rat.Mech.Anal. 76 (1981), 135-166.

[16] J. S. Il’yashenko, “Global analysis of the phase portrait for the Kuramoto-Sivashinsky equation,” J. Dynamics Diff. Eqn. 4 (1992), 585-615.

[17] M. S. Jolly, I. G. Kevrekidis and E. S. Titi, “Approximate inertial manifolds for the Kuramoto-Sivashinsky equation: analysis and computations,” Physica D 44 (1990), 38-60.
[18] A. A. Lacey, Thermal runaway in a nonlocal problem modelling Ohmic heating: Part I: Model derivation and some special cases, Euro. J. Appl. Math. 6 (1995), 127-144.

[19] A. S. Murthy and J. G. Verwer, Solving parabolic integro-differential equations by an explicit integration method, J. Comp. Appl. Math. 39 (1992), 121-132.

[20] B. Nicolaenko, B. Scheurer and R. Temam, “Some global dynamical properties of the Kuramoto-Sivashinsky equation: nonlinear stability and attractors,” Physica D 16 (1985), 155-183.

[21] B. Nicolaenko, B. Scheurer and R. Temam, “Some global dynamical properties of a class of pattern formation equations,” Comm. in PDEs 14 (1989), 245-297.

[22] C. Radehaus, R. Dohmen, H. Willebrand and F.-J. Niedernostheide, Model for current patterns in physical systems with two charge carriers, Phys. Rev. A 42 (1990), 7426-7446.

[23] J. C. Robinson, “Inertial manifolds for the Kuramoto-Sivashinsky equation,” Phys. Lett. A 184 (1994), 190-193.

[24] E. M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Univ. Press, 1970.

[25] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Springer-Verlag, New York, 1988.

[26] R. Temam and X. Wang, “Estimates on the lowest dimension of inertial manifolds for the Kuramoto-Sivashinsky equation in the general case,” Diff. Integral Eqns 7 (1994), 1095-1108.

[27] J. W. Wilder, B. F. Edwards, D. A. Vasquez and G. I. Sivashinsky, “Derivation of a nonlinear front evolution equation for chemical waves involving convection”, Physica D 73 (1994), 217-226.