BIG RAMSEY DEGREES USING PARAMETER SPACES

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ABSTRACT. We show that the universal homogeneous partial order has finite big Ramsey degrees and discuss several corollaries. Our proof uses parameter spaces and the Carlson–Simpson theorem rather than (a strengthening of) the Halpern–Läuchli theorem and the Milliken tree theorem, which are the primary tools used to give bounds on big Ramsey degrees elsewhere (originating from work of Laver and Milliken). This new technique has many additional applications. To demonstrate this, we show that the homogeneous universal triangle-free graph has finite big Ramsey degrees, thus giving a short proof of a recent result of Dobrinen.

1. Introduction

We consider graphs, partial orders, (vertex)-ordered graphs and partial orders with linear extensions as special cases of model-theoretic relational structures (defined in Section 2). Given structures $A$ and $B$, we denote by $\mathcal{(B)}^A$ the set of all embeddings from $A$ to $B$. We write $C \rightarrow \mathcal{(B)}^A_{r,l}$ to denote the following statement:

For every colouring $\chi$ of $\mathcal{(C)}_A$ with $r$ colours, there exists an embedding $f : B \rightarrow C$ such that $\chi$ does not attain more than $l$ values on $\mathcal{(f(B))}^A$.

For a countably infinite structure $B$ and its finite substructure $A$, the big Ramsey degree of $A$ in $B$ is the least number $L \in \omega \cup \{\omega\}$ such that $B \rightarrow \mathcal{(B)}^A_{L}$ for every $r \in \omega$; see [KPT05]. A countably infinite structure $B$ has finite big Ramsey degrees if the big Ramsey degree of $A$ in $B$ is finite for every finite substructure $A$ of $B$.

A countable structure $A$ is called (ultra)homogeneous if every isomorphism between finite substructures extends to an automorphism of $A$. It is well known that there is an (up to isomorphism) unique homogeneous partial order $P$ with the property that every countable partial order has an embedding to $P$. We call $P$ the universal homogeneous partial order. Similarly, there is an up to isomorphism unique homogeneous triangle-free graph $H$ (called the universal homogeneous triangle-free graph, sometimes also triangle-free Henson graph) such that every countable triangle-free graph embeds to $H$. (See e.g. [Mac11] for more background on homogeneous structures.)

Our main result is the following.

Theorem 1.1. The universal homogeneous partial order has finite big Ramsey degrees.

Presently, there are just a few examples of structures with finite big Ramsey degrees known. As we show in Section 6 the universal homogeneous partial order represents an important new example of a structure in which many of the known examples can be interpreted and thus follow as a direct consequence.

Supported by project 18-13685Y of the Czech Science Foundation (GAČR), Center for Foundations of Modern Computer Science (Charles University project UNCE/SCI/004), by the PRIMUS/17/SCI/3 project of Charles University and by ERC Synergy grant DYNASNET 810115.
The study of big Ramsey degrees originates in work of Laver who, in 1969, showed that the big Ramsey degrees of the order of rationals are finite [Dev79, page 73], see also [EH74, Lav84]. In his argument, he re-invented the Halpern–Läuchli theorem [HL66]. His technique was later formulated more generally by means of the Milliken tree theorem [Mil79] and the notion of envelopes and embedding types [Tod10, Chapter 6]. The majority of existing results in the area continue to use the Milliken tree theorem as the primary proof technique. In particular, Devlin in 1979 [Dev79] refined Laver’s argument thereby giving a precise characterisation of the big Ramsey degrees of the order of rationals. In 2006, Sauer [Sau06] and Laflamme, Sauer, and Vuksanovic [LSV06] characterised the big Ramsey degrees of the Rado graph (with precise counts given by Larson [Lar08]). This was further generalised in several followup papers [LNVTS10, DLS16].

Our proof of Theorem 1.1, for the first time in the area, uses spaces described by parameter words. This leads to a finer control over the sub-trees compared to the aforementioned constructions. Our main Ramsey tool, formulated as Theorem 2.1, is an infinitary extension of the Graham–Rothschild theorem [GR71] and is a direct consequence of the Carlson–Simpson theorem [CS84]. While the connections of the Carlson–Simpson theorem, Halpern–Läuchli theorem for trees with bounded branching and the Milliken tree theorem are well known [CS84, DK16], the additional invariants parameter spaces can preserve have been not applied in this context so far.

The proof technique presented in this paper is flexible and can be used to obtain additional finite big Ramsey degrees results for restricted structures (that is, structures omitting given substructures or satisfying certain axioms). To demonstrate this, we give a new short proof of the following recent result of Dobrinen [Dob20a]:

**Theorem 1.2 (Dobrinen [Dob20a]).** The universal homogeneous triangle-free graph has finite big Ramsey degrees.

Both results have well known finitary counterparts. Given a class $\mathcal{K}$ of structures, the (small) Ramsey degree of $A$ in $\mathcal{K}$ is the least $l \in \mathbb{N} \cup \{\omega\}$ such that for every $B \in \mathcal{K}$ and $r \in \mathbb{N}$ there exists $C \in \mathcal{K}$ such that $C \rightarrow (B)^{A}_{r,l}$. A class $\mathcal{K}$ of finite structures is Ramsey (or has the Ramsey property) if the small Ramsey degree of every $A \in \mathcal{K}$ is one. The Ramsey property for finite partial orders with linear extensions was announced by Nešetřil and Rödl in 1984 [NR84] with the first proof published year later [PTW85]. The Ramsey property of finite ordered triangle free graphs is a direct consequence of the Nešetřil–Rödl theorem [NR77].

While there is a general framework which can be used to show that a given class $\mathcal{K}$ is Ramsey [HN19], the situation is very different in the context of big Ramsey degrees as there are still only a handful of structures where big Ramsey degrees are understood. The main difference is the lack of an infinite variant of the (Nešetřil and Rödl’s) partite construction [NR89] (see [NR18] for its adaptation to partial orders) which has proved to be a very versatile tool in the structural Ramsey theory. Partite construction separates the structural and Ramsey arguments. Ramsey objects are constructed by sequences of (structural) products and amalgamations which are derived by a combination of Ramsey and Hales–Jewett theorems. In this respect it differs from other proofs of these results which generally represents a structural object by means of tools provided by “unstructured” Ramsey theorems (see [Prö13, Theorem 12.13] for Ramsey property of ordered graphs and [Fou97] for Ramsey property of partial orders with linear extensions).

For several decades, it was not clear how to generalize Laver’s proof to (countable) restricted structures or structures in non-binary languages. Dobrinen’s recent proof of Theorem 1.2 started a significant progress. Her proof uses a new method
of bounding big Ramsey degrees inspired by Harrington’s proof of the Halpern–Läuchli theorem, which uses techniques from forcing and the Erdős–Rado theorem. The main pigeonhole argument is a technically challenging structured tree theorem, where the tree is built using a particular enumeration of the graph Henson \( \mathbf{H} \) in which certain tree levels are coding (and contain vertices of the graph being represented) while others are branching. This method was later generalized to (non-oriented) Henson graphs [Dob19]. Recently, Zucker simplified it and further generalized to finitely constrained free amalgamation classes of structures in binary languages [Zuc20]. Zucker’s proof is still based on a structured pigeonhole proved by forcing techniques, but it greatly simplifies the trees by eliminating distinction between coding and branching levels. This simplification comes at a cost; the upper bounds on big Ramsey degrees obtained from Zucker’s proof are bigger than ones obtained from the proof by Dobrinen (which are conjectured to be tight [Dob20a, Section 10]).

By unrelated techniques, free amalgamation classes with the property that the big Ramsey degree of a vertex is equal to one were recently characterised by Sauer [Sau20]. Bounds on big Ramsey degrees of unrestricted structures with arities greater than 2 were announced in [BCH+19] with a proof based on the vector (or product) form Milliken tree theorem [BCH+20b, BCH+20a]. Independently, similar results were obtained by Coulson, Dobrinen and Patel [CDP20] using Dobrinen’s method of strong coding trees [Dob20b].

We shall also remark that Theorem 2.1 has a direct proof based on Theorem 2 of [Kar13]. Consequently we obtain the first direct (and simpler) proof of Theorem 1.2.

The paper is organised as follows. In Section 2 we introduce parameter spaces. In Section 3 we introduce the corresponding notion of envelopes and embedding types. In Section 4 we prove the main results of this paper. In Section 5 we show that the construction is tight for determining small Ramsey degrees and thus give a new proof of a special case of the Nešetřil–Rödl theorem [NR77]. In Section 6 we discuss several corollaries. In Section 7 we briefly outline ongoing work and further directions to generalize techniques of this paper.

2. Preliminaries

We use the standard model-theoretic notion of relational structures. Let \( L \) be a language with relation symbols \( R \in L \) each having its arity. An \( L \)-structure \( A \) on \( A \) is a structure with vertex set \( A \) and relations \( R^A \subseteq A^r \) for every symbol \( R \in L \) of arity \( r \). If the set \( A \) is finite, we call \( A \) a finite structure. We consider only structures with finitely many or countably infinitely many vertices.

Given two \( L \)-structures \( A \) and \( B \), a function \( f: A \to B \) is an embedding \( f: A \to B \) if it is injective and for every \( R \in L \) of arity \( r \) we have that

\[
(v_1, v_2, \ldots, v_r) \in R^A \iff (f(v_1), f(v_2), \ldots, f(v_r)) \in R^B.
\]

We say that \( A \) and \( B \) are isomorphic if there is an embedding \( f: A \to B \) that is onto.

As usual in the structural Ramsey theory, given an embedding \( f: A \to B \) we will call the image of \( A \) in \( B \) (denoted by \( f(A) \)) a copy of \( A \) in \( B \). Structure \( A \) is rigid if the only automorphism of \( A \) (that is, isomorphism \( A \to A \)) is the identity. For rigid structures we will also slightly abuse the notation and write \( \tilde{A} \in \mathcal{B} \) for any structure for which there exists an embedding \( f \in \mathcal{B} \) such that \( f(A) = \tilde{A} \).

2.1. Parameter words and spaces. Given a finite alphabet \( \Sigma \) and \( k \in \omega \cup \{\omega\} \), a \( k \)-parameter word is a (possibly infinite) string \( W \) in alphabet \( \Sigma \cup \{\lambda_i : 0 \leq i < k\} \) containing each of \( \lambda_i, 0 \leq i < k \), such that for every \( 1 \leq j < k \), the first occurrence
of \( \lambda_j \) appears after the first occurrence of \( \lambda_{j-1} \). Given a parameter word \( W \), we denote by \(|W|\) its length and for every \( 0 \leq j < |W| \) by \( W_j \) the letter (or parameter) on index \( j \). (Note that the first letter of \( W \) has index 0). A 0-parameter word is simply a word. We will generally denote words by lowercase letters and parameter words by uppercase letters.

Let \( W \) be an \( n \)-parameter word and let \( U \) be a parameter word of length \( k \leq n \) (where \( k, n \in \omega \cup \{\omega\} \)). Then we denote by \( W(U) \) the parameter word created by substituting \( U \) to \( W \). More precisely, this is a parameter word created from \( W \) by replacing each occurrence of \( \lambda_i \), \( 0 \leq i < k \), by \( U_i \) and truncating it just before the first occurrence of \( \lambda_k \) (in \( W \)). Given an \( n \)-parameter word \( W \) and set \( S \) of parameter words of length at most \( n \), we denote by \( W(S) \) the set \( \{W(U): U \in S\} \).

We denote by \( [\Sigma]^n \) the set of all \( k \)-parameter words of length \( n \) (where \( k \leq n \in \omega \cup \{\omega\} \)). If \( k \) is finite we also denote by

\[
[\Sigma]^n_k = \bigcup_{i \leq n, i \in \omega} [\Sigma]^{i \choose k}
\]

the set of all finite \( k \)-parameter words of length at most \( n \). For brevity we put \( \Sigma^* = [\Sigma]^{|\omega|} \), the set of all words on the alphabet \( \Sigma \) with finite length and no parameters. Given an \( n \)-parameter word \( W \) and integer \( k < n \), we call \( W([\Sigma]^n_k) \) the \( k \)-dimensional subspace described by \( W \). We will denote by \( \emptyset \) the empty word.

We will make use of the following infinitary variant of the Graham–Rothschild Theorem [GR71] which is a direct consequence of the Carlson–Simpson theorem [CS84]. This theorem was also obtained by Voigt around 1983 in a manuscript which, to our knowledge, was never published (see, i.e., [PV85, Theorem A], [Car87]).

**Theorem 2.1.** Let \( \Sigma \) be a finite alphabet and \( k \geq 0 \) a finite integer. If the set \( [\Sigma]^n_k \) is coloured by finitely many colours, then there exists an infinite-parameter word \( W \) such that \( W ([\Sigma]^n_k) \) is monochromatic.

We will also make use of the finite version of Theorem 2.1 (which follows from the Graham–Rothschild theorem by assigning every word \( W \in [\Sigma]^{n - 1}_k \), \( k > 1 \), a word \( W' \in [\Sigma]^n_{k-1} \) created from \( W \) by truncating it just before the first occurrence of \( \lambda_{k-1} \).)

**Theorem 2.2.** Let \( \Sigma \) be a finite alphabet, \( 0 \leq k \leq n \) and \( r > 0 \) finite integers. Then there exists \( N = N([\Sigma], k, n, r) \) such that for every \( r \)-colouring of \( [\Sigma]^n_k \) there exists a word \( W \in [\Sigma]^n_N \) such that \( W ([\Sigma]^n_k) \) is monochromatic.

3. **Envelopes and Embedding Types**

Essentially all big Ramsey degree results are based on a notion of envelope and embedding type introduced by Laver and Milliken, see [Tod10, Section 6.2]. Precise definitions depend on the notion of a subspace (or a subtree). The following introduces these concepts for the context of parameter spaces.

**Definition 3.1.** Given a finite alphabet \( \Sigma \) and a set \( S \) of parameter words in alphabet \( \Sigma \), an envelope of \( S \) is a parameter word \( W \) in alphabet \( \Sigma \) such that for every \( U \in S \), there exists a parameter word \( U' \) such that \( W(U') = U \). We call the envelope \( W \) minimal if there is no envelope of \( S \) with fewer parameters than \( W \).

**Example 1.** Consider \( \Sigma = \{0\} \) The set \( S = \{0, 000\} \subseteq [\Sigma]^n_3 \) has two minimal envelopes: \( 0\lambda_0\lambda_0 \) and \( 0\lambda_0\lambda_0 \). Parameter word \( \lambda_0\lambda_1\lambda_2\lambda_3 \) is also an envelope of \( S \), but it is not a minimal envelope.

**Proposition 3.1.** Let \( \Sigma \) be a finite alphabet, let \( k \geq 0 \) be a finite integer, let \( S \) be a finite set of finite parameter words in alphabet \( \Sigma \) with at most \( k \) parameters and
let \( W \) be a minimal envelope of \( S \). Then \( W \) has at most \((|\Sigma| + k)^{|S|} + |S| - |\Sigma|\) parameters. Moreover, for every parameter \( \lambda_i \) of \( W \) and every minimal envelope \( W' \) of \( S \) it holds that the first occurrence of \( \lambda_i \) has the same position in \( W \) and \( W' \).

Proof. Fix \( \Sigma, k, \) and \( S \). If the set \( S \) is empty, then \( W \) can be chosen to be the empty word. We thus assume that \( |S| > 0 \) and put \( S = \{W^0, W^1, \ldots, W^{\ell-1}\} \). We show a method to construct an envelope \( W \).

Put \( m = \max_{0 \leq i < \ell} (|W^i|) \) and for every \( 0 \leq i < m \) we define the slice \( i \) as a sequence \( s^i = (W^0_0, W^1_1, \ldots, W^i_{i-1}) \) where we put \( W^i_0 = * \) if \( |W^i| \leq i \) (where * is a special symbol not in \( \Sigma \)). For every \( 0 \leq i < j < m \) we say that slice \( i \) is compatible with slice \( j \) if for every \( 0 \leq p < \ell \) it holds that either \( W^i_p = W^j_p \) or \( W^i_p = * \) and \( j \neq |W^p| \).

Now construct a word \( W \) of length \( m \) by putting for every \( 0 \leq j < m \)

\[
W_j = \begin{cases} 
  s & \text{if slice } j \text{ is } (s, s, \ldots, s) \text{ for some } s \in \Sigma, \\
  W_{j'} & \text{if there exists } 0 \leq j' < j \text{ such that slice } j' \text{ is compatible with slice } j \\
  \lambda_p & \text{otherwise, where } \lambda_p \text{ is the first so far unused parameter.}
\end{cases}
\]

It is easy to see that each new parameter must be introduced and thus the dimension of an envelope and first occurrences of parameters are uniquely determined by \( S \). Since there are at most \((|\Sigma| + k)^{\ell} + \ell\) mutually incompatible slices, we will use at most \((|\Sigma| + k)^{\ell} + \ell - |\Sigma|\) parameters. \( \square \)

**Definition 3.2.** Given a finite alphabet \( \Sigma \), a finite integer \( k \geq 0 \), a set \( S \) of parameter words in alphabet \( \Sigma \) and an envelope \( W \) of \( S \), an embedding type of \( S \) in \( W \), denoted by \( \tau_W(S) \), is the set of parameter words such that \( W(\tau_W(S)) = S \).

**Example 2.** The set \( S = \{0, 00\} \) has embedding type \( \{0, 0\} \) in both minimal envelopes given in Example 1.

**Corollary 3.2.** Let \( \Sigma \) be a finite alphabet and let \( k, \ell > 0 \) be finite integers. Then

1. the set

\[
\{ \tau_W(S) : S \subseteq [\Sigma]^*(\omega)^k, |S| = \ell, W \text{ is a minimal envelope of } S \}
\]

is finite, and,

2. for every finite set \( S \subseteq [\Sigma]^*(\omega) \) and its minimal envelopes \( W \) and \( W' \) it holds that \( \tau_W(S) = \tau_{W'}(S) \).

As a consequence of Corollary 3.2 we can also use \( \tau(S) \) for \( \tau_W(S) \) where \( W \) is some minimal envelope of \( S \).

**Remark 3.1.** Our Definitions 3.1 and 3.2 are closely related to the definition of envelopes and types used by Dodos, Kanellopoulos and Tyros [DKT14] and by Furstenberg and Katznelson [FK89], see also [DK16, Chapter 5]. The main difference is however the use of subspaces defined by variable words rather than parameter words. With respect to this notion subspaces the dimension of envelope and thus also the number of types is not bounded by the size of the set.

4. Big Ramsey degrees

In this section we prove Theorems 1.1 and 1.2. We start with Theorem 1.2 and later show that Theorem 1.1 follows by very similar arguments.
4.1. **Triangle-free graphs.** In this section we consider graphs to be structures in a language consisting of a single binary relation $E$. We fix alphabet $\Sigma = \{0\}$.

**Definition 4.1.** We define graph $G$ as follows:

1. The vertex set $G$ is $[\Sigma]^*(\omega^1)$ (that is, the set of all finite 1-parameter words).
2. Given two vertices $U$ and $V$ such that $|U| < |V|$, we put an edge between $U$ and $V$ if and only if
   
   (i) $V|U| = \lambda_0$ and
   
   (ii) for no $0 \leq j < |U|$ it holds that $U_j = V_j = \lambda_0$.

There are no other edges.

**Remark 4.1.** Condition (i) in Definition 4.1 is the passing number representation of the Rado graph used by Sauer [Sau06] (see also [Tod10, Theorem 6.25]). Condition (ii) is similar to Dobrinen’s parallel 1’s criterion [Dob20a, Definition 3.7]. The notion of subtree (or a subspace) used here is however different from [Sau06] and [Dob20a].

**Lemma 4.1.** Graph $G$ is triangle-free.

**Proof.** Assume to the contrary that $U$, $V$ and $W$ form a triangle. Without loss of generality we can assume that $|U| < |V| < |W|$. Because there is an edge between $U$ and $V$, we know that $V|U| = \lambda_0$. Because there is an edge between $U$ and $W$, we know that $W|U| = \lambda_0$. A contradiction with the existence of an edge between $V$ and $W$. □

The following follows directly from the definition of the substitution:

**Observation 4.2.** Let $W$ be an infinite-parameter word. Then for every $U, V \in G$ it holds that $U$ is adjacent to $V$ if and only if $W(U)$ is adjacent to $W(V)$.

Let $H$ with $H = \omega$ be (an enumeration of) the universal homogeneous triangle-free graph. We define the mapping $\varphi: \omega \to G$ by putting $\varphi(i) = U$ where $U$ is a 1-parameter word of length $i$ defined by putting for every $0 \leq j < i$

$$U_j = \begin{cases} 
\lambda_0 & \text{if and only if } \{j, i\} \text{ is an edge of } H, \\
0 & \text{otherwise.}
\end{cases}$$

It is easy to check:

**Observation 4.3.** Function $\varphi$ is an embedding $\varphi: H \to G$ and thus $G$ is a universal triangle-free graph.

Now we prove Theorem 1.2 in the following form:

**Theorem 4.4.** For every finite $k \geq 1$ and every finite colouring of induced subgraphs of $G$ with $k$ vertices there exists $f \in \binom{G}{k}$ such that the colour of every $k$-vertex subgraph $A$ of $f(G)$ depends only on $\tau([A]) = \tau(f^{-1}[A])$.

Observe that by Corollary 3.2, we obtain the desired finite upper bound on number of colours. The proof is again structured similarly to Milliken and Laver’s results, see [Tod10, Section 6.3]: by a repeated application of Theorem 2.1, we obtain the desired copy.

**Proof.** Fix $k$ and a finite colouring $\chi$ of subsets of $G$ of size $k$. Let $T^0, T^1, \ldots, T^{N-1}$ be all possible embedding types of subsets of $G$ of size $k$ in their minimal envelopes (given by Corollary 3.2). For every $0 \leq i \leq N - 1$, put $n_i = \max\{|U| : U \in T^i\}$.

Choose an infinite-parameter word $W^0 \in [\Sigma]^*(\omega^1)$ arbitrarily. We construct a sequence of infinite-parameter words $W^1, W^2, \ldots, W^N$ such that for every $0 < i \leq N$ the following is satisfied:
(1) \( W^i = W^{i-1}(Z') \) for some infinite-parameter word \( Z' \),
(2) There exists colour \( c' \) such that 
\[
\chi(W^i(U(T^{i-1}))) = c'
\]
for every \( U \in [\Sigma]^*_{(n_{i-1})} \).

Let \( f \) be defined by \( f(U) = W^N(U) \). By Observation 4.2 we know that this is an embedding with the desired properties.

It remains to show the construction of \( W^i \). Assume that \( W^{i-1} \) is constructed. Let \( \chi' : [\Sigma]^*_{(n_{i-1})} \) be a colouring given by \( \chi'(U) = \chi(W^{i-1}(U(T^{i-1}))) \). By Theorem 2.1 there exists an infinite-parameter word \( Z' \) and colour \( c' \) satisfying that 
\[
\chi'(Z'(U)) = c'
\]
for every \( U \in [\Sigma]^*_{(n_{i-1})} \). Put \( W^i = W^{i-1}(Z') \). \( \square \)

4.2. Partial orders. Throughout this section we fix a language with a single binary relation \( \leq \) and consider a partial order \( (A, \leq_A) \) to be a structure \( A \) with vertex set \( A \) and a binary relation \( \leq_A \). We also fix the alphabet \( \Sigma = \{ L, X, R \} \). We will use the lexicographic order of words that is based on the following order of the alphabet: \( L <_\text{lex} X <_\text{lex} R \). We define the following binary relation on \( \Sigma^* \):

**Definition 4.2.** For \( w, w' \in \Sigma^* \) we put \( w < w' \) if and only if there exists \( 0 \leq i < \min(|w|, |w'|) \) such that

(i) \((w_i, w'_i) = (L, R)\) and 
(ii) for every \( 0 \leq j < i \) it holds that \( w_j \leq \text{lex} w'_j \).

For \( w < w' \) we denote by \( i(w, w') \) the minimal \( i \) satisfying the condition (i) above. We put \( w \preceq w' \) if and only if either \( w = w' \) or \( w < w' \).

We denote by \( O \) the structure with vertex set \( O = \Sigma^* \) ordered by \( \preceq \). (Thus we put \( u \preceq v \) if and only if \( u \preceq v \).

**Proposition 4.5.** Structure \( O \) is a partial order.

The intuitive meaning of the definition above is that for every \( w \in \Sigma^* \) and every \( j \) the letter \( w_j \) describes a position of the vertex \( w \) with respect to an extension of the partial order by a new vertex. When extending a given partial order by a new vertex \( v \), we obtain a partitioning of its vertex set into three sets: \( L \) is the set of all vertices smaller than \( v \), \( X \) is the set of all vertices not comparable to \( v \) and \( R \) is the set of all vertices greater than \( v \). Because we aim to define partial order on the set of all words and because not every choice of \( L, X \) and \( R \) represent an extension, we simply disregard all the information which is in conflict with what has been decided earlier.

**Proof of Proposition 4.5.** It is easy to see that \( \preceq \) is reflexive and anti-symmetric. We verify transitivity. Let \( w < w' < w'' \) and put \( i = \min(i(w, w'), i(w', w'')) \).

First assume that \( i = i(w, w') \). Then we have \( w_i = L, w'_i = R \) which implies that \( w''_i = R \). For every \( 0 \leq j < i \) it holds that \( w_j \leq \text{lex} w'_j \leq \text{lex} w''_j \). It follows that \( w \preceq w'' \) and \( i(w, w'') \leq i \).

Now assume that \( i = i(w', w'') \). Then we have \( w'_i = L, w''_i = R \) and because \( w'_i = L \) then also \( w_i = L \). Again for every \( 0 \leq j < i \) it holds that \( w_j \leq \text{lex} w'_j \leq \text{lex} w''_j \). It also follows that \( w \preceq w'' \) and \( i(w, w'') \leq i \). \( \square \)

The key to our construction is the following:

**Lemma 4.6.** Let \( W \) be an infinite-parameter word. Then for every \( w, w' \in \Sigma^* \) it holds that \( w \preceq w' \) if and only if \( W(w) \preceq W(w') \).

**Proof.** This can be easily checked using the fact that for every \( i > 0, \lambda_i \) first occurs in \( W \) after the first occurrence of \( \lambda_{i-1} \). \( \square \)
Recall that by $P = (P, \leq_P)$ we denote the universal homogeneous partial order. Without loss of generality, we can assume that $P = \omega$ and thus fix an (arbitrary) enumeration of $P$. We define function $\varphi: \omega \to \Sigma^*$ by mapping $j \in P$ to a word $w$ of length $2j$ defined as:

$$(w_{2i}, w_{2i+1}) = \begin{cases} 
(L, L) & \text{for every } i < j, j \leq_P i, \\
(R, R) & \text{for every } i < j, i \leq_P j, \\
(X, X) & \text{for every } i < j, i \text{ is incomparable with } j \text{ by } \leq_P.
\end{cases}$$

Proposition 4.7. The function $\varphi$ is an embedding $\varphi: P \to O$. Consequently, $O$ is a universal partial order.

Proof. Given $i < j \in \omega$, put $u = \varphi(i)$ and $v = \varphi(j)$ and consider three cases:

1. $i \leq_P j \implies u \preceq v$: We have $u_{2i} = L$ and $v_{2i} = R$ and we check that for every $0 \leq k < i$ it holds that $u_{2k} \preceq_{\text{lex}} v_{2k}$ and thus also $u_{2k+1} \preceq_{\text{lex}} v_{2k+1}$. If $u_{2k} = L$ then this follows trivially. If $u_{2k} = X$ then we know that $k$ is incomparable with $i$ by $\leq_P$. It follows that $v_{2k} \neq L$ because $i \leq_P j$ and thus it can not hold that $j \leq_P k$. If $u_{2k} = R$ then we get $k \leq_P i \leq_P j$ and thus also $v_{2k} = R$.

2. $j \leq_P i \implies v \preceq i$: Here we have $u_{2i+1} = R$ and $v_{2i+1} = L$. Analogously as in the previous case we can check that for every $0 \leq k < i$ it holds that $v_{2k} \preceq_{\text{lex}} u_{2k}$.

3. If $i$ is incomparable with $j$ in $\leq_P$ then $u$ is incomparable with $v$ in $\preceq$: Assume the contrary and let $k \leq i$ be such that either $u_{2k} = L$ and $v_{2k} = R$ or $u_{2k+1} = L$ and $v_{2k+1} = R$. Clearly $k < i$ because $v_{2i} = v_{2i+1} = X$. We get that $i \leq_P k \leq_P j$. A contradiction.

Remark 4.2. Easy constructions of universal partial orders are interesting in their own right, see [Hed69, PT80, HN05b, HN05a, HN11]. Observe also that $\leq_{\text{lex}}$ is a linear extension of $\preceq$ and thus the construction can be seen as a direct refinement of the Laver–Devlin construction.

Now we are ready to prove Theorem 1.1 in the following form.

Theorem 4.8. For every finite $k \geq 1$ and every finite colouring of (induced) suborders of $O$ with $k$ elements, there exists $f \in (O^k)_O$ such that the colour of every suborder $A$ of $f(O)$ with $k$ vertices depends only on $\tau([A]) = \tau(f^{-1}[A])$.

Proof. This follows in an analogy to Theorem 4.4.

Fix $k$ and a finite colouring $\chi$ of subsets of $O$ of size $k$. Let $T^0, T^1, \ldots, T^{N-1}$ be all possible embedding types of subsets of $O$ of size $k$ in their minimal envelopes (given by Corollary 3.2). For every $0 \leq i \leq N - 1$ put $n_i = \max([U]: U \in T^i)$.

Choose infinite-parameter word $W^0 \in [\Sigma]^n_1$ arbitrarily. We construct a sequence of infinite-parameter words $W^1, W^2, \ldots, W^N$ such that for every $0 < i \leq N$ the following is satisfied:

1. $W^i = W^{i-1}(Z^i)$ for some infinite-parameter word $Z^i$,

2. There exists colour $c^i$ such that $\chi(W^i(U(T^{i-1}))) = c^i$ for every $U \in [\Sigma]^n_1(T^{i-1})$.

Let $f$ be defined by $f(U) = W^N(U)$. By Lemma 4.6 we know that this is an embedding with the desired properties.

Word $W^i$ is again constructed by an application of Theorem 2.1. \qed
5. Ramsey classes (of finite structures)

5.1. Ordered triangle-free graphs. An ordered graph is a relational structure $A$ in a language consisting of two binary relations $E$ and $\leq$ such that $(A, E_A)$ is a graph and $(A, \leq_A)$ is a linear order.

We prove a special case of the Neˇsetˇril–Rödl theorem. Our proof is based on the ideas developed in the previous sections and is arguably the most direct proof of this result known to date, giving a particularly simple description of the Ramsey graph $C$. We shall remark that similar constructions have been known for unrestricted classes, see [Pr613, Theorem 12.13] for a proof of the Ramsey property of the class of all finite ordered graphs. However, to our best knowledge, a similar strategy has been applied to a class of graphs with a forbidden subgraph in a special cases only (for coloring vertices and edges [NR75a, NR75b]).

**Theorem 5.1** (Neˇsetˇril–Rödl). For every integer $r > 0$ and every pair of finite ordered triangle-free graphs $A$ and $B$, there exists a finite ordered triangle-free graph $C$ such that $C \rightarrow (B)^A_{r,1}$.

**Proof.** We fix alphabet $\Sigma = \{0\}$. Recall the graph $G$ defined in Definition 4.1. By $G_N$, we denote the ordered graph created from $G$ by considering only vertices in $[\Sigma]^r(N)$ and adding a lexicographic ordering of the vertices (where we consider vertices to be strings in alphabet $\{0, \lambda_0\}$ ordered $0 \leq \text{lex} \lambda_0$).

We will show that for sufficiently large $N$ (to be specified at the end of the proof) it holds that $G_N \rightarrow (B)^A_{r,1}$. Towards this, we first define a more careful way to embed an ordered triangle-free graph $B$ to a graph $G_n$.

Let $B$ be an ordered triangle-free graph. For simplicity we can assume that $B = \{0, 1, \ldots, |B| - 1\}$ and that $\leq_B$ coincides with the order of integers. We define an embedding $\varphi: B \rightarrow G_n$ for some sufficiently large $n$ to be fixed later by the following procedure. We say that a function $f: B \rightarrow \{0, \lambda_0\}$ is a Katˇetov function (for $B$) if $B$ extended by a new vertex which is adjacent precisely to those vertices $v \in B$ satisfying $f(v) = \lambda_0$ is a triangle free graph. In other words, there are no two adjacent vertices $v, v' \in B$ such that $f(v) = f(v') = \lambda_0$.

Now enumerate all possible Katˇetov functions as $f_0, f_1, \ldots, f_{d-1}$ ordered lexicographically with respect to $\leq_B$. More precisely, we see every function $f_i$ as a word $w'_i$ of length $|B|$ with $w'_i = f_i(j)$ and order those words lexicographically.

Put $\varphi(v) = V$ where $|V| = d + v$ and

$$V_j = \begin{cases} f_j(v) & \text{for } j < d, \\ \lambda_0 & \text{for } d \leq j < d + v \text{ such that } v \text{ is adjacent to } j - d \text{ in } B, \\ 0 & \text{for } d \leq j < d + v \text{ such that } v \text{ is not adjacent to } j - d \text{ in } B. \end{cases}$$

Now put $n = d + |B|$. It is easy to see that $\varphi$ is an embedding of $B$ to $G_n$ (to see that the order is preserved, note that all extensions by a vertex connected to precisely one vertex of $B$ are triangle-free). An example of this representation is depicted in Figure 1.

Let $\varphi'$ be an embedding of $A \rightarrow G_k$ for some $k > 0$ constructed in the same way as above.

**Claim 5.2.** For every $\tilde{A} \in \mathcal{B}$ there exists a $k$-parameter word $W \in [\Sigma]^r(N)$ such that $W(\varphi'(A)) = \varphi(\tilde{A})$.

Let $f_0, f_1, \ldots, f_{d-1}$ be the enumeration of Katˇetov functions of $B$ in the lexicographic order and let $f'_1, f'_2, \ldots, f'_{d'-1}$ be the enumeration of Katˇetov functions of $\tilde{A}$ also ordered lexicographically. Let $h: \{0, 1, \ldots, d - 1\} \rightarrow \{0, 1, \ldots, d' - 1\}$ be the
mapping such that for every \( i \in \{0, 1, \ldots, d - 1\} \) function \( f_i \) restricted to \( \tilde{A} \) is \( f_{h(i)}' \).

Observe that every Katětov function \( f \) of \( \tilde{A} \) can be extended to a Katětov function \( f' \) of \( B \) by putting \( f' = f(v) \) for \( v \in \tilde{A} \) and \( f'(v) = 0 \) otherwise. It follows that \( h \) exists and is surjective.

Let \( \theta \) be the isomorphism \( A \to \tilde{A} \). For every \( v \in B \setminus \tilde{A} \) we put \( e(v) \) to be an integer such that \( f'_e(v) \) describes the neighbourhood of \( v \) in \( \tilde{A} \).

We now define a string \( W \) of length \( d + \max(\tilde{A}) \) as follows:

\[
W_j = \begin{cases} 
\lambda_{h(j)} & \text{for every } 0 \leq j < d, \\
\lambda_{h(j) + \theta^{-1}(j - d)} & \text{for every } d \leq j \text{ such that } j - d \in \tilde{A}, \\
\lambda_{c(j - d)} & \text{for every } d \leq j \text{ such that } j - d \notin \tilde{A}.
\end{cases}
\]

First observe that \( W_0 = \lambda_0 \). This is because \( f_0 \) and \( f_0' \) are both constant zero functions.

We verify that \( W \) is a \( k \)-parameter word, that is, for every \( 1 \leq j < k \) it holds that the first occurrence of \( \lambda_j \) comes after the first occurrence of \( \lambda_{j-1} \). We consider three cases:

1. \( j < d' \): Function \( f_j' \) can be extended to function \( f_j'': B \to \{0, \lambda_0\} \) by putting \( f_j''(v) = 0 \) for every \( v \notin \tilde{A} \). This is clearly a Katětov function of \( B \) and therefore there exists \( j' \) such that \( f_j'' = f_{j'}' \). From this it follows that \( W_j' = \lambda_j' \). Because zero is the minimal element of the alphabet we get that this is also the first occurrence of \( \lambda_j \) in \( W \). Finally because the first occurrence of \( \lambda_{j-1} \) can be found same way and the extension by zeros preserves the relative lexicographic order, we know that \( \lambda_j \) appears after \( \lambda_{j-1} \).

2. \( j = d' \): \( \lambda_j \) occurs once at position \( d + \theta(j - d') = d + \theta(0) \). We already checked that \( \lambda_{j-1} \) occurs before \( d \).

3. \( d' < j < k \): For every \( d' < j < k \) it holds that \( \lambda_j \) occurs precisely once at position \( d + \theta(j - d') \) so the desired ordering follows form the monotonicity of \( \theta \).

This finishes the proof that \( W \) is indeed \( k \)-parameter word. By substituting \( \varphi'(A) \) into \( W \) it can be also checked that \( W(\varphi'(A)) = \varphi(\tilde{A}) \). This finishes the proof of Claim 5.2.

Now let \( N = N(1, n, k, r) \) be given by Theorem 2.2. We claim that \( G_N \to (\mathcal{B}^n)^A_k \). Consider an \( r \)-colouring of \( G_N \). Observe that for every \( W \in [\Sigma]^* \binom{N}{k} \) we get a unique copy of \( A \) in \( G_N \) given by \( W(\varphi'(A)) \). We thus obtain an \( r \)-colouring of \([\Sigma]^* \binom{N}{k}\) and by an application of Theorem 2.2 a word \( \tilde{W} \in [\Sigma]^* \binom{N}{k} \) for which this colouring is constant. The monochromatic copy of \( B \) is now given by \( \tilde{W}(\varphi(B)) \). \( \square \)
5.2. Partial orders with linear extension. Now we will consider structures in
language with two binary relations ≤ and $\preceq$. A is a partial order with linear
extension if $(A, \leq_A)$ a partial order and $(A, \preceq_A)$ its linear extension.

We prove:

**Theorem 5.3** ([NR84, PTW85]). For every integer $r > 0$ and every pair of finite
partial orders with linear extensions $A$ and $B$ there exists a finite partial order with
linear extension $C$ such that $C \to (B)^A_{r+1}$.

**Remark 5.1.** The proof of Theorem 5.3 presented here is related to proofs of this re-
result based on the Graham–Rothschild theorem (by Fouché [Fou97], see also [Mas18,
Theorem 4.1]). We present it because our representation of the partial order by
finite words is different. This difference is necessary to show Theorem 1.1 (where
countably infinite partial orders need to be represented), but also perhaps makes
the proof of Theorem 5.3 a bit more systematic.

**Proof.** We fix alphabet $\Sigma = \{L, X, R\}$ and its ordering $L <_{\text{lex}} X <_{\text{lex}} R$. Denote
by $O_N$ the partial order induced on $[\Sigma]^* \binom{N}{A}$ by $O$ (given by Definition 4.2) with a
linear extension defined by the lexicographic order.

Fix $A$ and $B$ and proceed in analogy to the proof of Theorem 5.1. For simplicity
we can assume that $B = \{0, 1, \ldots, |B| - 1\}$ and that $\leq_B$ coincides with the order
of integers. We show that there exists $N$ such that $O_N \to (B)^A_{r+1}$.

We define an embedding $\varphi : B \to O_n$ for some sufficiently large $n$ (to be fixed
later) by the following procedure. We say that function $f : B \to \{L, X\}$ represents a
downset of $B$ if the set $\{v : f(v) = L\}$ is downwards closed with respect to $\leq_B$.

Now enumerate all possible functions representing a downset as $f_0, f_1, \ldots, f_{d-1}$
ordered lexicographically with respect to $\leq_B$. Put $\varphi(v) = w$ where is a word of
length $d + v + 1$ defined as follows:

$$w_j = \begin{cases} f_j(v) & \text{for } 0 \leq j < d, \\ R & \text{for } d \leq j < d + v, \\ L & \text{for } j = d + v. \end{cases}$$

An example of this representation is depicted in Figure 2.

Now put $n = d + |B| + 1$. It is easy to see that $\varphi$ is an embedding of $B$ to
$O_n$: levels $d$ to $d + |B|$ code the linear extensions given by $\leq_B$ while earlier levels
code all downsets. Every pair of vertices $u \leq_B v$ which are not comparable by
$\preceq$ have downsets witnessing this which makes sure that their images are also not
comparable by $\preceq$.

Let $\varphi'$ be an embedding of $A \to O_k$ for some $k > 0$ constructed the same way
as above.
Claim 5.4. For every \( \tilde{A} \in (B_A) \) there exists a k-parameter word \( W \in [\Sigma]^*(\tilde{A}) \) such that \( W(\varphi'(A)) = \varphi(\tilde{A}) \).

Let \( f_0, f_1, \ldots, f_{d-1} \) be the enumeration of functions representing downsets of \( B \) in the lexicographic order (with respect to \( \leq_B \)) and \( f'_1, f'_2, \ldots, f'_{d-1} \) be the enumeration of functions representing downsets of \( \tilde{A} \) also ordered lexicographically. Let \( h: \{0, 1, \ldots, d-1\} \to \{0, 1, \ldots, d' - 1\} \) be the mapping such that \( f_i \) restricted to \( \tilde{A} \) is \( f'_{h(i)} \). Observe that every downset \( f \) of \( \tilde{A} \) can be extended to a downset of \( f' \) and thus \( h \) is well defined and surjective.

Let \( \theta \) be the embedding \( A \to \tilde{A} \). We now define a string \( W \) of length \( d + \max(\tilde{A}) \) as follows:

\[
W_j = \begin{cases} 
\lambda_{h(j)} & \text{for every } 0 \leq j < d, \\
\lambda_{d + \theta^{-1}(j - d)} & \text{for every } d \leq j < |B| + 1 \text{ such that } j - d \in \tilde{A}, \\
R & \text{for every } d \leq j < |B| + 1 \text{ such that } j - d \notin \tilde{A}.
\end{cases}
\]

Next we verify that \( W \) is a k-parameter word. For this we need to find for every \( f'_j \) its lexicographically minimal extension \( f_{j'} \) and verify that the lexicographic order is preserved. Given \( f'_j \), we construct function \( f: B \to \{L, X, R\} \) by putting:

\[
f(v) = \begin{cases} 
f'_j(v) & \text{if } v \in \tilde{A}, \\
X & \text{if } v \notin \tilde{A} \text{ and there exists } u \in \tilde{A}, f'_j(u) = X \text{ and } u \leq_B v, \\
L & \text{otherwise.}
\end{cases}
\]

Observe that there is \( j' \) such that \( f = f_{j'} \) and that \( f_{j'} \) is lexicographically minimal among all functions \( f \) which represent a downset of \( B \) such that \( h(f) = f'_j \). This is due to fact that we put \( f(v) = X \) only when this was forced by a “witness” \( u \in \tilde{A} \) for which \( f'_j(u) = X \), and thus the value of \( v \) is \( X \) in every extension of \( f'_j \) which represents a downset. To see that this construction preserves the lexicographic order, it remains to observe that since \( u \leq_B v \), we also have \( u \leq_B v \) and thus while constructing the lexicographic order of extensions, \( f(u) \) will take a precedence over \( f(v) \).

Note that at this moment we make use of the fact that our representation uses downsets rather than all Katětov functions which would seem as more direct analogy of the proof of Theorem 5.1.

We thus conclude that \( W \) is indeed a parameter word. This finishes the proof of the claim.

Now let \( N = N(0, n, k, r) \) be given by Theorem 2.2. We claim that \( O_N \to (B_A)^{\infty} \). Consider an \( r \)-colouring of \( O_N \). Observe that for every \( W \in [\Sigma]^*(\tilde{N}) \) we get a unique copy of \( A \) in \( O_N \) given by \( W(\varphi'(A)) \). We thus obtain an \( r \)-colouring of \( [\Sigma]^*(\tilde{N}) \) and by application of Theorem 2.2 a word \( \tilde{W} \in [\Sigma]^*(\tilde{N}) \) for which this colouring is constant. The monochromatic copy of \( B \) is now given by \( \tilde{W}(\varphi(B)) \).

6. Applications

In this section we briefly discuss some examples of structures where finiteness of big Ramsey degrees follows as a direct consequence of Theorems 1.1 and 4.8. This includes some already known examples (linear orders, graphs, triangle-free graphs, ultrametric spaces) as well as a new example (\( S \)-metric spaces).

For each of the examples we will construct an interpretation in the universal homogeneous partial order \( P \) (or its fixed linear extension) which has the property that vertices of this interpretation are formed by \( (V)^{\infty} \) for some finite poset \( V \). By obtaining a common representation of these structures within partial orders we
also show that free superpositions of such structures have finite big Ramsey degrees, thereby giving a partial answer to a question asked by Zucker during the 2018 BIRS workshop “Unifying Themes in Ramsey Theory.”

We stress that the representations here generally only lead to very generous upper bounds on big Ramsey degrees.

6.1. Triangle-free graphs. It may be a bit of a surprise that Theorem 1.1 implies Theorem 1.2 in a particularly easy way. Given a homogeneous partial order \( P \), we denote by \( G_P \) the following graph:

1. Vertices of \( G_P \) are all triples of distinct vertices \((u_0, u_1, u_2)\) of \( P \) such that \( u_0 \prec_P u_2 \), while \((u_0, u_1)\) and \((u_1, u_2)\) are incomparable in \( P \).
2. Vertices \((u_0, u_1, u_2)\) and \((v_0, v_1, v_2)\) form an edge of \( G_P \) if and only if \( u_0 \prec_P v_1 \), \( v_0 \prec_P u_3 \prec_P v_2 \) and all other pairs \((u_i, v_j)\), \( i, j \in \{0, 1, 2\} \), are incomparable in \( P \).

By transitivity, \( G_P \) is triangle-free: if both \( \{(u_0, u_1, u_2), (v_0, v_1, v_2), (u_0, w_1, w_2)\} \) are edges of \( G_P \) then we have \( u_0 \leq_P w_2 \) which implies that \( \{(u_0, u_1, u_2), (w_0, w_1, w_2)\} \) is a non-edge.

It is not hard to check that there is an embedding \( \varphi \) form the homogeneous universal triangle free graph \( H \) to \( G_P \). Recall that the vertex set of \( H \) is \( \omega \) and construct the embedding \( \varphi \) inductively. For each vertex \( i \in \omega \) assume that \( \varphi(i') \) is constructed for every \( i' < i \) and apply the extension property of \( P \) to obtain three disjoint vertices \( i_0, i_1, i_2 \in P \), such \( (i_0, i_1, i_2) \) is a vertex of \( G_P \), adn for every \( j \leq i \) vertices \( \varphi(j) = (j_0, j_1, j_2) \) are disjoint from \( (i_0, i_1, i_2) \) and the following is satisfied:

1. If \( i, j \) forms an edge of \( H \) put \( i_0 \leq_P j_1 \leq_P i_2 \) and \( j_0 \leq_P i_1 \leq_P j_2 \) so \((i_0, i_1, i_2)\) and \((j_0, j_1, j_2)\) forms an edge of \( G_P \).
2. If \( i, j \) does not form an edge of \( H \) put \( i_0 \leq_P j_2 \) and \( j_0 \leq_P i_2 \) while keeping all other pairs \((i_k, j_k)\), \( k \in \{0, 1, 2\} \) incomparable in \( P \).

To finish the proof of Theorem 1.2, assume that we are given a finite colouring of \( \mathbb{R}^\omega_A \) for some finite triangle-free graph \( A \). Since \( H \) is universal, it contains a copy of \( G_P \) and hence it induces a colouring of \( \mathbb{R}^\omega_A \). This can be turned into a finite colouring of substructures of \( P \) on at most \( 3|A| \) vertices and hence, by Theorem 1.1, there is a copy of \( P \) with at most a bounded number of colours. This corresponds to a copy of \( G_P \) in \( G_P \) with at most a bounded number of colours and the rest follows by universality of \( G_P \).

6.2. Urysohn S-metric spaces. Let \( S \) be a set of non-negative reals such that \( 0 \in S \). A metric space \( M = (M, d) \) is an S-metric space if for every \( u, v \in M \) it holds that \( d(u, v) \in S \). We call a countable S-metric space \( M \) a Urysohn S-metric space if it is homogeneous (that is, every isometry of its finite subspaces extends to a bijective isometry from \( M \) to \( M \)) and every countable S-metric space embeds to it. In the following we will see S-metric spaces as relational structures in a language with a binary relation \( R_\ell \) for every \( \ell \in S \setminus \{0\} \).

Finite set of non-negative reals \( S = \{0 = s_0 < s_1 < \cdots < s_n\} \) is tight if if \( s_{i+j} \leq s_i + s_j \) for all \( 0 \leq i < j \leq i+j \leq n \) (see [Maš18]). It follows from a classification by Sauer [Sau13] that for every such \( S \) there exists an Urysohn S-metric space.

Mašulović in [Maš18, Theorem 4.4] shows a way to represent all S-metric cases with finitely many distances (for every tight set \( S \)) as a partial order. Using this construction we obtain: 

**Corollary 6.1.** Let \( S \) be a finite tight set of non-negative reals. Then the Urysohn S-metric space has finite big Ramsey degrees.
We will show a special case of Corollary 6.1 where \( S = \{0, 1, \ldots, d\} \). For other tight sets we refer the reader to [Maš18, Theorem 4.4].

**Proof (sketch).** Fix \( d \) and \( S = \{0, 1, \ldots, d\} \). Construct an \( S \)-metric space \( M_S \) as follows:

1. Vertices are chains of vertices of \( P \) of length \( d \).
2. Given two chains \( u_0 \leq_P \cdots \leq_P u_{d-1} \) and \( v_0 \leq_P \cdots \leq_P v_{d-1} \) their distance is the minimal \( \ell \in \{0, 1, \ldots, d\} \) such that for every \( i \in \{0, 1, \ldots, d-i\} \) it holds that \( u_i \leq_P v_{i+\ell} \) and \( v_i \leq_P u_{i+\ell} \).

Just as in the case of triangle-free graphs, triangle inequality follows from transitivity, one can embed the Urysohn \( S \)-metric space to \( M_S \) using an on-line algorithm and hence Corollary 6.1 follows. \( \Box \)

Note that not all finite sets \( S \) for which there exists a Urysohn \( S \)-metric space (these were characterised by Sauer [Sau13]) are tight and thus Corollary 6.1 is not a complete characterisation.

Just like Theorem 1.1, Corollary 6.1 has a known finite form. Ramsey property of the class of all finite ordered metric spaces was shown by Nešetřil [Nes07] (see also [DR12] for graph metric spaces). This result was later generalised to all \( S \)-metric spaces [HN19, HKN19a]. Big Ramsey degrees of vertices for all Urysohn \( S \)-metric spaces with \( S \) finite was shown to be one by Sauer [Sau12]. See also [DLPS07, DLPS08, NVT10] for more background on vertex partition theorems of Urysohn spaces.

### 6.3. Ultrametric spaces

Recall that metric space \( M = (M, d) \) is an ultrametric space if the triangle inequality can be strengthened to \( d(u, w) \leq \max\{d(u, v), d(v, w)\} \). The Urysohn ultrametric space of diameter \( d \) is the universal and homogeneous ultrametric space with distances \( \{0, 1, \ldots, d\} \). The following was shown by Nguyen Van Thé [NVT09] (along with a full characterisation of big Ramsey degrees of ultrametric spaces):

**Theorem 6.2.** For every \( d \geq 1 \) the Urysohn ultrametric space of diameter \( d \) has finite big Ramsey degrees.

**Proof.** We construct ultrametric space \( U_d \) as follows:

1. Vertices of \( U_d \) are \( d \)-tuples of vertices of \( P \).
2. The distance between vertices \( (u_0, u_1, \ldots, u_{d-1}) \) and \( (v_0, v_1, \ldots, v_{d-1}) \) is the minimal \( \ell \) such that for every \( 0 \leq i < d-\ell \) it holds that \( u_i = v_i \).

Again, it is easy to verify that this is a universal ultrametric space. The finiteness of big Ramsey degrees now follows by an application of Theorem 1.1. \( \Box \)

Observe that by replacing \( P \) by \( \omega \) above, the same result (with better bounds) follows by the infinite Ramsey theorem. Note that the construction above can be strengthened to the \( \Lambda \)-ultrametric spaces for a given finite lattice \( \Lambda \) [Bra17].

### 6.4. Linear orders

By fixing a linear extension of \( P \) one obtains an alternative proof of the Laver’s result:

**Corollary 6.3.** The order of rationals has finite big Ramsey degrees.

While this may not be very powerful observation at its own, we will discuss its consequences in Corollary 6.5. Observe also that \( P \) has a natural linear extension in the form of the lexicographic order.
6.5. **Structures with unary relations.** Another particularly simple consequence of Theorem 1.1 is:

**Corollary 6.4.** Let $L$ be a finite language consisting of unary relational symbols. Then the universal homogeneous $L$-structure has finite big Ramsey degrees.

**Proof.** For simplicity assume that $L$ consists of single unary relation $R$. Then the universal $L$-structure can be represented using $P$ as follows:

1. Vertices are all pairs of distinct vertices of $P$.
2. Put vertex $(u_0, u_1)$ to the relation $R$ if and only if $u_0 \leq_P u_1$.

\[ \square \]

6.6. **Free superpositions.** Recall that the age of a structure $M$ is the set of all finite structures having an embedding to $M$. Given a language $L$ and its sub-language $L^* \subseteq L$, an $L^*$-structure $M$ is the $L^*$-reduct of an $L$-structure $N$ if $M = N$ and it holds that $R^*_{M} = R^*_{N}$ for every $R \in L^*$.

Let $L$ and $L'$ be languages such that $L \cap L' = \emptyset$. Let $M$ be a homogeneous $L$-structure and $N$ a homogeneous $L'$-structure. Then the free superposition of $M$ and $N$, denoted by $M * N$, is the homogeneous $L \cup L'$-structure whose age consists precisely of those finite $(L \cup L')$-structures with the property that their $L$-reduct is in the age of $M$ and $L'$-reduct is in the age of $N$ (see e.g. [Bod15]).

It follows from the product Ramsey argument that the free interposition of finitely many Ramsey classes with strong amalgamation property and no algebraicity is also Ramsey [Bod15, Lemma 3.22], see also [HN19, Proposition 4.45]. Similar general result is not known for big Ramsey structures. However, we can combine the above observation to the following corollary (of Theorem 4.8) which heads in this direction by providing means to interpose many of the known structures with finite big Ramsey degrees:

**Corollary 6.5.** Let $M$ be a homogeneous structure that is a free superposition of finitely many copies of structures from the following list (each in a language disjoint from the others):

1. the homogeneous universal partial order,
2. the homogeneous universal triangle-free graph,
3. the Urysohn $S$-metric space for a finite thin set $S$ (for $S = \{0, 1, 2\}$ one obtains the Rado graph),
4. the Urysohn ultrametric space of a finite diameter $d$,
5. the order of rationals,
6. the homogeneous universal structure in a finite unary relational language,

then $M$ has finite big Ramsey degrees.

**Proof.** Let $M_1, M_2, \ldots, M_n$ be structures from the statement of the corollary, in mutually disjoint languages $L_1, L_2, \ldots, L_n$ such that for every $1 \leq i \leq n$ it holds that $M_i$ is $L_i$-structure. Put $M = M_1 * M_2 * \cdots * M_n$.

As shown above, for each structure $M_i$, $1 \leq i \leq n$, there exists a structure $N_i$ and an embedding $\varepsilon_i: M_i \to N_i$ such that $M_i = \{ (\varepsilon_i^\top) v_i \}$ for some finite structure $V_i$ and $M_i$ is represented using the partial order $P$ (or its linear extension).

Now consider a $(L_1 \cup L_2 \cup \cdots \cup L_n)$-structure $N$ defined as follows. The vertex set $N$ of $N$ consists of all $n$-tuples $(\vec{v}_1, \ldots, \vec{v}_n)$ with the property that for every $1 \leq i \leq n$ it holds that $\vec{v}_i$ is a vertex of $N_i$. Denote by $\pi_i$ the $i$-th projection $(\vec{v}_1, \ldots, \vec{v}_n) \mapsto \vec{v}_i$.

Now we define relations of $N$. For every $1 \leq i \leq n$, consider structure $N_i$:

1. If $N_i$ is a partial order, then the corresponding partial order of $N$ is created by putting $u \leq v$ if and only if either $u = v$ or $\pi_i(u) \neq \pi_i(v)$ and $\pi_i(u) \leq_{N_i} \pi_i(v)$.
(2) If $N_i$ is homogeneous universal triangle free graph then we put $u$ and $v$ adjacent if and only if $\pi_i(u)$ is adjacent to $\pi_i(v)$ in $N_i$.

(3) If $N_i$ is the order of rationals, then the corresponding linear order of $N$ is any linear order satisfying that $\pi_i$ is a monotone function.

(4) If $N_i$ is an $S$-metric space, then the corresponding metric space on $N$ is created by defining a distance of $u$ and $v$ to be 0 if $u = v$, $\min(S \setminus \{0\})$ if $\pi_i(u) = \pi_i(v)$ and the distance of $\pi_i(u)$ and $\pi_i(v)$ otherwise.

(5) If $N_i$ is an ultrametric space then the corresponding ultrametric space is created analogously, but by putting the distance to be 1 for every $u \neq v$, $\pi_i(u) = \pi_i(v)$.

(6) If $N_i$ is a structure with unary relations, then for every relation $R \in L_i$ we put $v$ to $\mathcal{R}_N$ if and only if $\pi_i(v) \in R$.

We say that substructure $A$ of $N$ is transversal if for every two vertices $(u_1, u_2, \ldots, u_n), (v_1, v_2, \ldots, v_n) \in A$ and every $1 \leq i \leq n$ it holds that $u_i \neq v_i$. Observe that embeddings $e_i : M_i \to N_i$, $1 \leq i \leq n$, can be combined to an embedding $e : M \to N$ defined by putting $e(v) = (e_1(v), e_2(v), \ldots, e_n(v))$, and that the image $e(M)$ is transversal. One can also verify that for every $1 \leq i \leq n$ it holds that the age of $N_i$ is the same as the age of the $L_i$-reduct of $N$. It follows that $N$ and $M$ have same ages. By universality of $M$ it follows that there is also an embedding $f : N \to M$.

7.1. Bigger forbidden substructures and bigger arities. The method presented in this paper can be used to strengthen Theorem 1.2 for free amalgamation classes in finite binary languages defined by finitely many forbidden irreducible substructures on at most 3 vertices.

For non-binary relations and bigger forbidden irreducible substructures it seems necessary to refine Theorem 2.1 for colouring multi-dimensional objects rather than
words in a similar manner as in [BCH+19, BCH+20a]. This seems to further develop the link between constructions in structural Ramsey theory and the extension property for partial automorphisms [HKN19b].

7.2. Optimality. Big Ramsey degree a vertex in the universal homogeneous triangle-free graph was shown to be one by Komjáth and Rödl [KR86] in 1986. Big Ramsey degree of an edge is four as shown by Sauer [Sau98] in 1998. Proof of Theorems 1.1 and 1.2 can be refined to precisely describe the big Ramsey degrees in a similar way as was done by Sauer [Sau06] for the random graph and Laflamme, Sauer and VukSANovic for free binary structures [LSV06]. This leads to the big Ramsey structure as defined by Zucker [Zuc19]. This will appear in [BCH+20c].

Acknowledgments. I am grateful to Martin Balko, David Chodounský, Natasha Dobrinen, Matěj Konečný, Jaroslav Nešetřil, Stevo Todoričević, Lluis Vena and Andy Zucker for remarks and corrections which improved presentation of this paper. This work was directly motivated by discussions with Jaroslav Nešetřil and also with Martin Balko, David Chodounský, Matěj Konečný and Lluís Vena during the meetings of project 18-13685Y. I was introduced to this interesting area by Claude Laflamme, Norbert Sauer and Robert Woodrow during my post-doc stay in Calgary.

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