Greenberger-Horne-Zeilinger States: Their Identifications and Robust Violations

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The $N$-qubit Greenberger-Horne-Zeilinger (GHZ) states are the maximally entangled states of $N$ qubits, which have had many important applications in quantum information processing, such as quantum key distribution and quantum secret sharing. Thus how to distinguish the GHZ states from other quantum states becomes a significant problem. In this work, by presenting a family of the generalized Clauser-Horne-Shimony-Holt (CHSH) inequality, we show that the $N$-qubit GHZ states can be indeed identified by the maximal violations of the generalized CHSH inequality under some specific measurement settings. The generalized CHSH inequality is simple and contains only four correlation functions for any $N$-qubit system, thus has the merit of facilitating experimental verification. Furthermore, we present a quantum phenomenon of robust violations of the generalized CHSH inequality, in which the maximal violation of Bell’s inequality can be robust under some specific noises adding to the $N$-qubit GHZ states.

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Introduction.—Quantum entanglement is the most surprising nonclassical property of composite quantum systems [1] that Schrödinger has singled out as “the characteristic trait of quantum mechanics” [2]. Nowadays quantum entangled states have become very powerful resources for quantum information processing, such as quantum cryptography, quantum teleportation, quantum communication protocols, quantum computation, and so on [3]. Among all entangled states, the maximally entangled ones are particularly important for quantum information tasks, for instance, Bell’s state is the indispensable state for realizing the perfect quantum teleportation [4].

As the maximally entangled state of two qubits, Bell’s state is usually given by $|\text{Bell}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Bell’s state has been extensively used to reveal quantum nonlocality through quantum violations of some types of Bell’s inequalities [5]. The typical two-qubit Bell’s inequality is the famous CHSH inequality [6], which is expressed as $T_{\text{CHSH}} = A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2 \leq 2$, where $A_i$, $B_j$, $(i,j = 1,2)$ being measurement settings. For any two-qubit pure state in the Schmidt-decomposition form $|\Psi\rangle = \cos \vartheta |00\rangle + \sin \vartheta |11\rangle$, the maximal violation of the CHSH inequality is $T_{\text{CHSH}}^{\text{max}}(\vartheta) = 2\sqrt{1 + \sin^2(2\vartheta)}$, which saturates the well-known Tsirelson’s bound $2\sqrt{2}$ [7] when Bell’s state is considered. Moreover, by exhausting all states of two qubits, one finds that Bell’s state is the only state can attain the bound. Therefore, the violations of CHSH inequality have two aspects of significance: (i) It may demonstrate the conflict between quantum mechanics and local-hidden-variable models [8, 9], and (ii) It can identify the two-qubit maximally entangled state due to its maximal quantum violation [10]. The latter merit plays a practical role in experimentally characterizing the entangled state.

From the viewpoint of maximally entangled state, the GHZ states [11] are the direct generalizations of Bell’s state from bipartite system to multipartite system. For the $N$-qubit system, the GHZ state is given by

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}( |00\cdots 0\rangle + |11\cdots 1\rangle )$$

and its other equivalent forms can be obtained under the local unitary transformations. The importance of the GHZ states is considered from two aspects: (i) Theoretically the GHZ states can be used to develop an all-verses-nothing proof of Bell’s nonlocality for $N$-qubit (i.e., the GHZ theorem), which is a significant improvement over Bell’s theorem as a way to disprove the concept of “elements of reality”, a concept introduced by the Einstein-Podolsky-Rosen problem in their attempt to prove that quantum theory is incomplete, and (ii) GHZ states have many important applications in quantum protocols, such as quantum secret sharing. Since about 1990, some researchers have attempted to generalize Bell’s inequalities from two-qubit to $N$-qubit, the well-known one is the Mermin-Ardehali-Belinski-Rikhspho (MABK) inequality [12–14], which is a tight Bell’s inequality for $N$-qubit and reduces to the CHSH inequality when $N = 2$. In 2004, Chen has explored a rigorous proof that the maximal violation of the MABK inequality can be applied to identify the $N$-qubit GHZ states [15]. Hence in principle, the MABK inequality offers a useful method for determining the $N$-qubit GHZ state in experiments,
not efficient due to the exponentially increasing number of correction-function terms contained in the inequality for $N$ qubits.

In this Letter, we shall advance the study of the connection between GHZ states and Bell’s inequality. Our purpose is two-fold: (i) The identification of GHZ states by the maximal violation of Bell’s inequality is a significant problem with practical interests since experimentally efficient methods are demanded for initializing multi-qubit entangled states required by different quantum protocols. Although the MABK inequality is a possible candidate of Bell’s inequality to identify GHZ states, its disadvantage in experimental implementation is also evident as mentioned earlier. The reason is that the number of correlation-function terms in the MABK inequality increases exponentially with the number of qubits ($2^N$ terms for even $N$, or $2^{N-1}$ terms for odd $N$), this in turn causes a scalable problem in practical experiments and thus the method based on the MABK inequality is time consuming and resource costly. In this work, we shall overcome such a problem by presenting a family of nontrivial Bell’s inequalities, which we call the generalized CHSH inequality.

**Identification of GHZ states.**—In order to identify the GHZ states, here we present a family of the generalized CHSH inequality for three qubits

$$\mathcal{I}_N = AB + AB' + A'B - A'B' \leq 2,$$ (2)

$$A = \bigotimes_{j=1}^{N-1} X_j, \ A' = \bigotimes_{j=1}^{N-1} X'_j, \ B = X_N, \ B' = X'_N,$$ (3)

and $X_j, \ X'_j$ are variables (valued $\pm 1$) for the $j$-th subsystem classically. Similar to the MABK inequality, the generalized CHSH inequality (2) is tight. When $N = 2$, the inequality (2) naturally reduces to the CHSH inequality, and when $N = 3$, it reduces to the third of 46 Sliwa’s inequalities for three qubits [16].

Quantum mechanically, we have the measurement operators as

$$X_k = \hat{n}_k \cdot \sigma_k, \ X'_k = \hat{n}'_k \cdot \sigma_k, \ (k = 1, 2, \cdots, N),$$ (4)

where $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices, $\hat{n}_k = (\sin \theta_k \cos \varphi_k, \sin \theta_k \sin \varphi_k, \cos \theta_k)$ and $\hat{n}'_k = (\sin \theta'_k \cos \varphi'_k, \sin \theta'_k \sin \varphi'_k, \cos \theta'_k)$ are the measurement settings for the $k$-th observer. Accordingly, we have

$$A = \bigotimes_{j=1}^{N-1} (\hat{n}_j \cdot \sigma_j), \ A' = \bigotimes_{j=1}^{N-1} (\hat{n}'_j \cdot \sigma_j),$$

$$B = \hat{n}_N \cdot \sigma_N, \ B' = \hat{n}'_N \cdot \sigma_N.$$. (5)

Now we come to determine the quantum upper bound for the inequality (2). To do this, let us define two new observables as

$$A'' = -\frac{i}{2}[A, A'] = -\frac{i}{2}(AA' - A'A),$$

$$B'' = -\frac{i}{2}[B, B'] = (\hat{n}_N \times \hat{n}'_N) \cdot \sigma_N.$$ (6)

Let $\alpha_k$ be the angle between two unit vectors $\hat{n}_k$ and $\hat{n}'_k$, then one can obtain

$$||A''|| \leq |\sin(\alpha_1 + \alpha_2 + \cdots + \alpha_{N-1})| \leq 1,$$

$$||B''|| = ||\hat{n}_N \times \hat{n}'_N|| = |\sin \alpha_N| \leq 1,$$ (7)

iff $\alpha_1 + \alpha_2 + \cdots + \alpha_{N-1} = \frac{(2\ell+1)\pi}{2}, \ \ell \in \mathbb{Z}$, and $\alpha_N = \frac{\pi}{2}$, the equality signs hold. Actually there exist some degrees of freedom to select the signs (either $+$ or $-$) before $\alpha_k$, however we can always restrict $\alpha_k$ in $[0, \pi]$ and remain the relation $|\sin(\alpha_1 + \alpha_2 + \cdots + \alpha_{N-1})| = 1$ unchanged in the case of $||A''|| = 1$. After some calculations, we may have the square operator for the Bell function $\mathcal{I}_N$ as

$$(\mathcal{I}_N)^2 = 4 - [A, A'] \otimes [B, B'] = 4([A'' \otimes B''] + \mathcal{I}_N),$$ (8)

here $\mathcal{I}$ is the identity operator. Since $||\mathcal{I}_N||^2 = 4||\mathcal{I}_N|| + 4||A''||||B''|| \leq 8$, one attains the quantum upper bound for the Bell function as $\mathcal{I}_N = \sqrt{||\mathcal{I}_N||^2} \leq 2\sqrt{2}$. In other words, the maximal violations of the generalized CHSH inequality cannot exceed the Tsirelson’s bound.

Next let us study the quantum violation of inequality (2) for the $N$-qubit generalized GHZ states ($\text{GHGHZ} = \cos \vartheta |00\cdots0\rangle + \sin \vartheta |11\cdots1\rangle$), whose density matrix are expressed as

$$\rho_N = \begin{bmatrix} \cos^2 \vartheta & \cdots & \cos \vartheta \sin \vartheta \\ \vdots & \ddots & \vdots \\ \cos \vartheta \sin \theta & \cdots & \sin^2 \vartheta \end{bmatrix}_{2^N \times 2^N},$$ (9)

and except the elements posed on four corners in $\rho_N$, all other elements vanish. For simplicity and without loss of generality [17], in this work we always choose the measurement setting as $\theta_k = \theta'_k = \pi/2$, which means that the Bloch vectors $\hat{n}_k$ and $\hat{n}'_k$ are located on the $xy$-plane. In this way, we have the measurement operators as

$$X_k = \begin{bmatrix} 0 & e^{-i\varphi_k} \\ e^{i\varphi_k} & 0 \end{bmatrix}, \ X'_k = \begin{bmatrix} 0 & e^{-i\varphi'_k} \\ e^{i\varphi'_k} & 0 \end{bmatrix}.$$ (10)

We then have

$$\langle \mathcal{I}_N \rangle = \text{tr}(\rho_N \mathcal{I}_N) = \sin 2\vartheta \left[ \cos(a + b) + \cos(a + b') + \cos(a' + b) - \cos(a' + b') \right],$$ (11)
with
\[ a = \sum_{j=1}^{N-1} \varphi_j, \quad b = \varphi_N, \quad a' = \sum_{j=1}^{N-1} \varphi'_j, \quad b' = \varphi'_N. \] (12)
It can be verified that
\[ \cos(a + b) + \cos(a' + b') = \cos(a + b') - \cos(a' + b') = 2F \cos \left( \frac{b - b'}{2} \right) - 2G \sin \left( \frac{b - b'}{2} \right) \]
\[ = 2\sqrt{F^2 + G^2} \cos \left( \frac{b - b'}{2} + \delta \right) \leq 2\sqrt{2}, \] (13)
with \( F = \cos \left( a + \frac{b+b'}{2} \right), \quad G = \sin \left( a' + \frac{b+b'}{2} \right), \) and \( \tan \delta = G/F. \) The equal sign in Eq. (13) occurs for \( F^2 = G^2 = 1 \) and \( \cos \left( \frac{b - b'}{2} + \delta \right) = 1, \) which leads to the following conditions:
\[ a + b + b' = p\pi, \quad a' + b + b' = (2q + 1)\pi. \]
\[ b - b' + \delta = 2r\pi, \quad \tan \delta = \pm 1. \] (14)
with \( p, q, r \in \mathbb{Z}. \) Note that the last two equations in Eq. (14) is same as the constraint of \( \|B''\| = 1. \) From above equations, we have
\[ \langle \mathcal{I}_N(\theta) \rangle_{\text{max}} = 2\sqrt{2}\sin(2\vartheta), \] (15)
which implies that the GHZ states can reach the Tsirelson’s bound when we take \( \vartheta = \pi/4. \)

By choosing some appropriate measurement settings, now we can identify the \( N \)-qubit GHZ states by maximal violations of the general CHSH inequality. We have the following theorem:

**Theorem 1.**—Under the conditions of measurement settings \( \varphi_j = 0, \varphi'_j = \frac{2\varphi_j}{\sqrt{N-1}}, \) \( j = 1, 2, ..., N - 1, \) and \( \varphi_N = -\frac{\varphi}{\sqrt{2}}, \varphi'_N = \frac{\varphi}{\sqrt{2}}, \) a \( N \)-qubit state \( |\psi\rangle \) satisfies
\[ \langle \mathcal{I}_N \rangle = \langle \psi | \mathcal{I}_N | \psi \rangle = 2\sqrt{2}, \] (16)
iff it is the \( N \)-qubit GHZ state in the form of Eq. (1).

**Proof.**—Firstly, we need to prove that if \( |\psi\rangle \) is the \( N \)-qubit GHZ state, then one has \( \langle \mathcal{I}_N \rangle = 2\sqrt{2}. \) Based on the conditions in **Theorem 1**, we have \( a = \sum_{j=1}^{N-1} \varphi_j = 0, \quad a' = \sum_{j=1}^{N-1} \varphi'_j = \frac{2\varphi}{\sqrt{N-1}}, \quad b = -\frac{\varphi}{\sqrt{2}}, \quad b' = \frac{\varphi}{\sqrt{2}}, \) this yields \( a + \frac{b+b'}{2} = 0, \quad a' + \frac{b+b'}{2} = \frac{\varphi}{\sqrt{2}}, \quad \delta = \frac{\varphi}{\sqrt{2}}, \quad b - b' + \delta = 0, \) which satisfy the conditions in Eq. (14). Therefore, it is easy to prove that if \( |\psi\rangle = \text{GHZ}, \) then \( \langle \mathcal{I}_N \rangle = 2\sqrt{2} \) holds.

Secondly, we need to prove that, under the conditions in **Theorem 1**, the GHZ state is the unique state that can satisfy \( \langle \mathcal{I}_N \rangle = 2\sqrt{2}. \) To reach this purpose, one need to show the largest eigenvalue of Bell-function matrix is not degenerate. Under the constraints in **Theorem 1**, one obtains
\[ X_j = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad X'_j = \begin{bmatrix} 0 & e^{i\varphi'_j} \\ e^{i\varphi'_j} & 0 \end{bmatrix}, \] (17)
with \( j = 1, 2, ..., N - 1, \) and
\[ B = X_N = \begin{bmatrix} 0 & e^{i\varphi} \\ e^{-i\varphi} & 0 \end{bmatrix}, \quad B' = X'_N = \begin{bmatrix} 0 & e^{-i\varphi'} \\ e^{i\varphi'} & 0 \end{bmatrix}. \] (18)
Then,
\[ A = \text{C-diag}(1, 1, ..., 1, 1), \quad A' = \text{C-diag} \left( -i, e^{-i\frac{(N-3)\pi}{2}}, ..., e^{-i\frac{(N-3)\pi}{2}}, i \right), \] (19)
where \( \text{C-diag} \left( e_1, e_2, ..., e_{N-2}, e_{N-1} \right) \) represents the counter-diagonal matrix with \( e_j \) the nonzero elements in the \( j \)-th row. After that, we deduce the Bell-function matrix as
\[ \mathcal{I}_N = \text{C-diag} \left( 2\sqrt{2}, 0, \sqrt{2} \left( 1 + e^{-i\varphi} \right), \sqrt{2} \left( 1 - e^{-i\varphi} \right), ..., \sqrt{2} \left( 1 + e^{-i\frac{(N-2)\pi}{2}} \right), \sqrt{2} \left( 1 - e^{-i\frac{(N-2)\pi}{2}} \right), 0, 2\sqrt{2} \right). \] (20)
Obviously, the largest eigenvalue \( 2\sqrt{2} \) of matrix \( \mathcal{I}_N \) is unique. This ends the proof.

**The Phenomenon of Robust Violations of Bell’s Inequality.**—Usually the maximal violations of Bell’s inequality are not robust. Let us take the CHSH inequality as an example. (i) Pure state case. As mentioned above, for Bell’s state, the maximal violation is \( T_{\text{max}}(\text{CHSH}(|\Phi\rangle)) = 2\sqrt{2}. \) It is well-known that for any two-qubit pure state \( |\psi\rangle, \) the maximal violation is given by \( T_{\text{max}}(\text{CHSH}(|\psi\rangle)) = 2\sqrt{1 + \epsilon^2}, \) where \( \epsilon \leq 1 \) is the entanglement degree of the state \( |\psi\rangle. \) For a pure state generated from the superposition of Bell’s state and an arbitrary two-qubit pure state \( |\Phi\rangle, \) i.e., \( |\psi\rangle = \sqrt{1 - \epsilon^2} |\Phi\rangle + \epsilon |\Phi\rangle, \) its maximal violation will be always less than \( 2\sqrt{2} \) because \( \epsilon(\langle \psi | \psi \rangle) < 1 \) for any \( \epsilon > 0. \) (ii) Mixed state case. Let us consider a two-qubit mixed states \( \rho = (1 - \epsilon)\rho_{\text{Bell}} + \epsilon \rho_{\text{noise}} \) (with \( 0 \leq \epsilon \leq 1 \)), which is a convex sum of density matrix of Bell’s state and a certain noise. In this case, one also finds that the maximal violation cannot reach \( 2\sqrt{2}. \) For instance, let us take \( \epsilon_{\text{noise}} = 1/4 \) that represents the white noise, and \( \rho \) becomes the well-known Werner state \([18]\), we then have the quantum violation as \( T_{\text{max}}(\text{CHSH}(\rho)) = (1 - \epsilon)2\sqrt{2} < 2\sqrt{2}. \) Therefore, any small disturbance adding to Bell’s state will let the maximal violation deviate from the Tsirelson’s bound.

In this work, we present for the first time the quantum phenomenon of robust violations of Bell’s inequality based on the generalized CHSH inequality (2). For \( N = 2, \) there is not the phenomenon of robust violations, as we have analyzed above. For \( N = 3, \) there are only trivial robust violations, because for at least one of observers, his two measurement settings are the same or opposite (i.e., \( \vec{n}_i = \vec{n}'_i \) or \( \vec{n}_i = -\vec{n}'_i \). For \( N \geq 4, \) there are nontrivial phenomena of robust violations of Bell’s inequality, in this case each observer performs two distinct measurements on his qubit. In the following, to address this topic, we will provide an explicit example for \( N = 4. \)
Example.—Let us take \( \varphi_1 = \varphi_2 = \varphi_4' = 0, \varphi_3 = \varphi_2 = \varphi_4 = \pi/2, \varphi_3 = -\pi/2, \) and \( \varphi_3' = \frac{\pi}{3} \), namely, in the \( xy \)-plane, the measurement directions are \( \mathbf{n_1} = \mathbf{n_2} = \hat{y}, \mathbf{n_3} = \hat{x}, \mathbf{n_1}' = \hat{y}, \mathbf{n_2}' = \hat{x} \). We have \( a = \sum_{j=1}^{3} \varphi_j = -\frac{\pi}{4}, a' = \sum_{j=1}^{3} \varphi_j' = \frac{\pi}{4}, b = \frac{\pi}{2}, b' = 0 \), this yields \( a + \frac{b + b'}{2} = 0, a' + \frac{b + b'}{2} = \frac{\pi}{4} \). \( \delta = -\pi/4, b - b' + \delta = 0 \), which satisfy the conditions in Eq. (14). Thus for the four-qubit GHZ state \( |\text{GHZ}_0\rangle = \frac{1}{\sqrt{2}}(|0000 \rangle + |1111 \rangle) \), one gets \( I_{N=4}^{\text{max}} = 2\sqrt{2} \).

Due to the above measurement settings, one then has
\[
\begin{align*}
A &= \text{C-diag}(\mu, \nu, \mu, \nu, \mu, \nu) \\
A' &= \text{C-diag}(-\nu, -\mu, \mu, \nu, -\nu, -\mu), \\
B &= \text{C-diag}(-i, i), \\
B' &= \text{C-diag}(1, 1).
\end{align*}
\]
with \( \mu = (1 + i)/\sqrt{2} \) and \( \nu = (1 - i)/\sqrt{2} \). This leads to
\[
I_4 = \text{C-diag}(2\sqrt{2}, 0, 0, 2\sqrt{2}, 0, i2\sqrt{2}, -i2\sqrt{2}, 0, \\
0, i2\sqrt{2}, -i2\sqrt{2}, 0, 2\sqrt{2}, 0, 0, 2\sqrt{2}).
\]

For the matrix of \( I_4 \), the largest eigenvalue \( 2\sqrt{2} \) is four-fold degeneracy, and the corresponding orthonormal eigenstates are all GHZ states in different forms:

\[
\begin{align*}
|\text{GHZ}_0\rangle &= \frac{1}{\sqrt{2}} \left( |0000 \rangle + |1111 \rangle \right), \\
|\text{GHZ}_1\rangle &= \frac{1}{\sqrt{2}} \left( |0011 \rangle + |1100 \rangle \right), \\
|\text{GHZ}_2\rangle &= \frac{1}{\sqrt{2}} \left( |i0101 \rangle + |1010 \rangle \right), \\
|\text{GHZ}_3\rangle &= \frac{1}{\sqrt{2}} \left( |-i0101 \rangle + |1010 \rangle \right).
\end{align*}
\]

It is easy to verify that for the following pure state
\[
|\psi\rangle = \sqrt{1 - \sum_{j=1}^{3} c_j^2} |\text{GHZ}_0\rangle + \sum_{j=1}^{3} c_j |\text{GHZ}_j\rangle,
\]
and mixed state
\[
\rho = (1 - \sum_{j=1}^{3} \epsilon_j) \rho_0 + \sum_{j=1}^{3} \epsilon_j \rho_j,
\]
with \( \rho_k = |\text{GHZ}_k\rangle \langle \text{GHZ}_k |, \) and for any \( 0 < \epsilon_j < 1, (j = 1, 2, 3) \), one always has the maximal violations as \( I_4^{\text{max}} = 2\sqrt{2} \), thus demonstrating the quantum phenomenon of robust violations of Bell’s inequality for the GHZ state.

Conclusion.—In summary, we have proposed a family of generalized CHSH inequality for \( N \) qubits, and based on which we have also addressed two significant topics related to the connection between the \( N \)-qubit GHZ states and Bell’s inequality. The first one is that we have successfully identified the \( N \)-qubit GHZ states by the maximal violation of the generalized CHSH inequality. The generalized CHSH inequalities are tight, indeed we have checked the tightness of the inequality for \( N \leq 7 \) and believe that it is also true for larger \( N \). Moreover, the generalized inequality is very simple in the sense that it contains only four correlation-function terms for arbitrary \( N \), thus providing a friendly way for experimental implementations with reduced complexity. In various quantum protocols with entangled states as computational resource, efficient characterization of the entangled states is of essential importance in the initial step of implementing the protocols. However for multi-qubit entangled states, it is still challenging to verify the effectiveness of the initialization of the states even though some efforts have been taken in the literature [19–23]. Our generalized inequality is a timely contribution to provide an efficient verification method for the \( N \)-qubit GHZ state in experiments. The second one is that for the GHZ states, we have present a quantum phenomenon of robust violations of Bell’s inequality. We find that for the GHZ states, the maximal violations of the generalized CHSH inequality are robust under some specific noises adding to the \( N \)-qubit GHZ states. We anticipate further experimental progresses in this direction in the near future.

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