1 Introduction

Our universe is a quantum one. This can be seen in experiments, like Tonomura’s 1989 electron double-slit experiment [18]. The setup of this experiment is an electron emitter, a detector screen, and a panel with two slits in between the emitter and the screen. A beam of electrons is then fired from the emitter to the detector screen after passing through the two slits. As expected, each electron leaves a point impression on the screen, but the resulting density pattern is that of a wave interfering with itself. This result cannot be explained by classical means.

Why we even study classical mechanics beyond historical or mathematical interest? For the physicist, the answer lies in the fact that classical mechanics provides models for producing quantum theories. For instance, the Schrödinger equation was “derived” in formal analogy with the Hamiltonian formulation of classical mechanics.

This brings us to the subject of this report: a method by which we can take a classical system and “quantize” it to obtain the corresponding quantum system. The key insight is due to Dirac [15], and is encoded in his famous Dirac axioms. In a nutshell, he noticed that the observables of a classical system and the observables of a quantum system have key formal similarities. Hence, a scheme for producing a quantum system from a classical one should respect these properties.

However, as shown by Groenwald and Van Hove, if the Dirac axioms are taken to be the definition of a quantization, then no quantization can exist. See Abraham and Marsden [14] for a proof of this fact. Rather than giving up, we weaken the axioms and study what’s called a prequantization, the first step towards a quantization scheme.

The prequantization scheme reviewed in this report is due to Kostant [1] and Souriau [2]. As a summary, if a symplectic manifold \((M, \omega)\) represents the state space of a classical system, then a Kostant-Souriau prequantization is a line bundle over \(M\) with some extra structure compatible with the symplectic form \(\omega\). It provides two things:

(i) a Hilbert space, the state space of a quantum system, and
(ii) a map from the classical observables $C^\infty(M)$ to the quantum observables on $\mathcal{H}$ which respects their formal properties.

The existence of such a line bundle turns out to be equivalent to an integrality condition on the symplectic form $\omega$, as the following theorem shows.

**Theorem 1. (Kostant-Souriau)**

A Kostant-Souriau prequantization of a symplectic manifold $(M, \omega)$ exists if and only if $[\omega]$ is integral.

As the name would suggest, Kostant-Souriau prequantization does not provide an actual quantization of a classical system. In some sense, the Hilbert space it constructs “depends on too many variables,” which the Heisenberg uncertainty principle explicitly forbids. Polarizations, which “cut down on half the variables,” are one solution, but they introduce their own technical difficulties. Metaplectic corrections and half-forms are then brought in to fix the issues arising from polarizations, but these half-forms don’t always exist. The interested reader should consult Bates and Weinstein [5] or Guillemin and Sternberg [6] for more details.

This report is divided into three sections. In the first, the basic ideas from classical and quantum physics will be discussed, which will then lead us to a description of Kostant and Souriau’s construction. The second section will delve into the theory of complex line bundles and the various structures associated to them, such as covariant derivatives, connections, and local systems. In the final section, a brief review of relevant ideas from Čech cohomology will be provided, paving the way to the proof of Theorem 1.
1.1 Notation

- \( C^\infty(M, N) \) - smooth maps between manifolds \( M \) and \( N \).
- \( C^\infty(M) = C^\infty(M, \mathbb{R}) \)
- \( \Gamma(U, A) \) - sections of a fibre bundle with total space \( A \) and \( U \) an open subset of base space.
- \( TM \) - tangent bundle of smooth manifold.
- \( \mathfrak{X}(M) \) - vector fields on a manifold.
- \( \mathfrak{X}_C(M) \) - complex vector fields on a manifold.
- \( \Omega^p(M) \) - differential \( p \)-forms on a manifold \( M \).
- \( H^p(M) \) - \( p \)-th De Rham cohomology group of \( M \).
- \( \check{H}^p(\mathcal{U}; \mathbb{R}) \) - \( p \)-th Čech cohomology group with respect to a cover \( \mathcal{U} \) with coefficients in \( \mathbb{R} \).
- \( S\mathcal{H} \) - essentially self-adjoint operators on a Hilbert space \( \mathcal{H} \).
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2 Prequantization

In this first section, we give a brief tour of the ideas from physics that are relevant to geometric prequantization. In particular, we review the symplectic formulation of classical mechanics, and the Schrödinger picture of non-relativistic quantum mechanics. We finish the review of physics with motivation for prequantization by example, then leap into the construction of Kostant and Souriau introduced in the previous section.

2.1 Geometric Mechanics

We begin with a review of the relevant formalism from geometric classical mechanics. The interested reader can consult Lee [11], Arnold [13], or da Silva [12] for a more detailed treatment.

If \((M, \omega)\) is a symplectic manifold, then since \(\omega\) is non-degenerate, we have the map
\[
\tilde{\omega} : TM \to T^*M; \quad v \mapsto \omega(v, -),
\]
where \(T^*M\) denotes the cotangent bundle, is an isomorphism of vector bundles. Using this fact, we can then define Hamiltonian vector fields.

**Definition 1.** Let \((M, \omega)\) be a symplectic manifold. If \(f \in C^\infty(M)\), then define the Hamiltonian vector field of \(f\), denoted \(X_f\) by
\[
df = \omega(X_f, -).
\]

**Definition 2.** A Hamiltonian system is a triple \((M, \omega, H)\), where \((M, \omega)\) is a symplectic manifold, called the phase space, and \(H \in C^\infty(M)\), called the Hamiltonian. We call integral curves of \(X_H\) trajectories and \(C^\infty(M)\) observables.

**Remark**

In this way of stating classical mechanics, we are thinking of \(M\) as representing all possible positions and momenta of a system. The level sets of \(H\) correspond to allowable energies of trajectories, and the observables correspond to the outcome of a measurement.

The observables of a Hamiltonian system \((M, \omega, H)\) can clearly be given a commutative algebra structure via pointwise multiplication. There is however another multiplication structure that we can give \(C^\infty(M)\), called the Poisson bracket.

**Definition 3.** Let \(f, g \in C^\infty(M)\). Define their Poisson bracket \(\{f, g\} \in C^\infty(M)\) by
\[
\{f, g\} = \omega(X_f, X_g).
\]

**Proposition 1.** Let \((M, \omega)\) be a symplectic manifold. Then the Poisson bracket
\[
\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M); \quad (f, g) \mapsto \{f, g\}
\]
is a Lie bracket. Furthermore, if \( f, g \in C^\infty(M) \), then

\[
[X_f, X_g] = X_{\{f, g\}}.
\]

**Proof.** Left to the reader. \(\square\)

The Poisson bracket is valuable since it allows us to compute "time derivatives" of observables along a trajectory. What I mean by that is if \( \Phi_t \) denotes the flow of \( X_H \) and \( f \in C^\infty(M) \), then \( f \circ \Phi_t \) is a measurement of whatever quantity \( f \) represents along the trajectory. Lie derivatives are used to show

\[
\frac{d}{dt} (f \circ \Phi_t) = \{f, H\} \circ \Phi_t.
\]

See [12] for a proof.

We get a very useful corollary from this result

**Corollary 1.** \( f \in C^\infty(M) \) is constant along every trajectory of the Hamiltonian system \((M, \omega, H) \iff \{f, H\} = 0\).

Let’s finish off this introduction to geometric mechanics with an example.

**Example 1.** One particle moving in \( \mathbb{R}^3 \) subject to a conservative force.

Let \( F : \mathbb{R}^3 \to \mathbb{R}^3 \) denote the force and \( m \) the mass. If \( q : \mathbb{R} \to \mathbb{R}^3 \) denotes the trajectory the particle takes when acted on by the force \( F \), then Newton tells us that \( q \) must satisfy

\[
F = m\frac{d^2q}{dt^2},
\]

a second order ODE. Thus, we are solving for both \( q \) and its first derivative. Hence, we choose the manifold which parameterizes this system to be \( M = \mathbb{R}^3 \times \mathbb{R}^3 \). Give \( M \) linear coordinates \((q, p) = (q_1, q_2, q_3, p_1, p_2, p_3)\). We think of the \( p_i \) as the coordinates for the momentum of the particle.

For the symplectic form, choose

\[
\omega = \sum_{j=1}^3 dq_i \wedge dp_i.
\]

It’s an easy exercise to show that \( \omega \) is indeed a symplectic form. Now we need our Hamiltonian.

A force \( F : \mathbb{R}^3 \to \mathbb{R}^3 \) is said to be conservative if there exists \( U : \mathbb{R}^3 \to \mathbb{R} \), called the potential energy function of \( F \), such that

\[
F = -\text{grad} \ U.
\]

Since \( U \) has units of energy and the Hamiltonian is supposed to represent the energy of a trajectory, we choose Hamiltonian

\[
H(q, p) = \frac{|p|^2}{2m} + U(q),
\]
where $\frac{1}{2m}p_\mu^2$ represents the kinetic energy of the trajectory. This is clearly smooth, hence $(M, \omega, H)$ is a Hamiltonian system. To see that trajectories of this Hamiltonian system correspond to solutions of Newton’s equation, we need to first compute $X_H$.

For any $f \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$, one can show that

$$X_f = \sum_{j=1}^3 \left( \frac{\partial f}{\partial p_j} \frac{\partial}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial}{\partial p_j} \right)$$

and hence a trajectory $\gamma(t) = (q(t), p(t))$ must satisfy

$$(X_H)_{\gamma(t)} = \sum_{j=1}^3 \left( \frac{\partial H}{\partial p_j} \frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q_j} \frac{\partial}{\partial p_j} \right)
= \gamma'(t)
= \sum_{j=1}^3 \left( \frac{dq_j}{dt} \frac{\partial}{\partial q_j} + \frac{dp_j}{dt} \frac{\partial}{\partial p_j} \right).$$

Comparing coefficients, we see that $\gamma$ satisfies the famous Hamilton’s canonical equations

$$\begin{cases}
\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j} \\
\frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j}
\end{cases}$$

In our case, this means the trajectory satisfies

$$\begin{cases}
\frac{dq_j}{dt} = \frac{p_j}{m} \\
\frac{dp_j}{dt} = -\frac{\partial U}{\partial q_j}
\end{cases}$$

Putting these equations together, we must have

$$m \frac{d^2 q}{dt^2} = -\nabla U(q) = F$$

so we’ve returned to Newton’s equation.

**Remark**

We see here why the classical picture cannot provide an adequate description of the double-slit experiment for an electron. If the electron is a particle, then we expect the screen to record a bunch of dots all focused around the path of the beam. We don’t see this. So if the electron is a wave, then we expect a continuous interference pattern. We also don’t see this. There is some wave-particle duality of the electron which this formulation is incapable of expressing.

### 2.2 Schrödinger Picture of Quantum Mechanics

To set up the Schrödinger picture of quantum mechanics, we need a list of data that is remarkably similar to that defining a classical mechanical system.
We need some object to represent all admissible states a system can be in and some observable that controls the evolution of the system. In classical mechanics, this role was played by the symplectic manifold and the Hamiltonian. In quantum mechanics, this role is played by a Hilbert space and an essentially self-adjoint operator, also called the Hamiltonian. As this review skips most of the important features of quantum mechanics, refer to Folland [16] or Bransden and Joachain [17] for more details.

Just for the sake of comparison, I introduce some nonstandard terminology.

**Definition 4.** A quantum Hamiltonian system is a pair $(\mathcal{H}, H)$, where $\mathcal{H}$ is a Hilbert space and $H$ is an essentially self-adjoint operator on $\mathcal{H}$. We call $\mathcal{H}$ the state space and $H$ the Hamiltonian operator.

We think of elements of $\mathcal{H}$ as representing all possible configurations that can occur in a given mechanical system. However, there is an arbitrariness to a state that can’t be measured called phase. So, in fact, we say two elements $v, w \in \mathcal{H}$ represent the same state if there exists $\lambda \in \mathbb{C}^*$ such that $v = \lambda w$.

We have the states of a quantum system, what are the observables? I.e. what objects represent the outcomes of experiments? It turns out [16] that observables on a quantum Hamiltonian system can be best interpreted as essentially self-adjoint operators. The set of all such operators is denoted $\mathcal{S}\mathcal{H}$. If $T \in \mathcal{S}\mathcal{H}$ represents some dynamical quantity we wish to measure, then we think of the eigenvalues of $T$ as the outcomes of said measurement.

The last element of this formalism we need to introduce is dynamics. We will make use of the Schrödinger picture to do this.

**Definition 5.** Let $(\mathcal{H}, H)$ be a quantum Hamiltonian system. A wave function is a map $\Psi : \mathbb{R} \rightarrow \mathcal{H}$ such that

(i) $\langle \Psi(t), \Psi(t) \rangle = 1$ for all $t \in \mathbb{R}$.

(ii) $\Psi$ satisfies the Schrödinger equation

$$i \frac{d}{dt} \Psi(t) = H \Psi(t).$$

A wavefunction $\Psi$ is supposed to represent all properties that an evolving mechanical system can possess for all times. Thus, (i) can be interpreted to mean that the state has probability 1 of existing for all times.

**Remark**

A quantum Hamiltonian system is analogous to a classical one. We still think of $H$ as representing the energy that a particular wave function carries with it. The only difference here is that we think of the eigenvalues $\{E_n\}_{n \in \mathbb{N}}$ of $H$ as the allowed energies, hence the discrete “quantum” nature of this formulation. Furthermore, the Schrödinger equation for a quantum Hamiltonian system is the quantum analog of the classical formula

$$\gamma'(t) = (X_H)_{\gamma(t)}$$

for integral curves of the Hamiltonian $H$ of a Hamiltonian system $(M, \omega, H)$.
We noted that the observables in the classical case had a Lie algebra structure given by the Poisson bracket. In the quantum case, the operators don’t even carry a vector space structure since their domains of definition may not be the same. But if $T : D_T \to \mathcal{H}$ and $S : D_S \to \mathcal{H}$ are two observables satisfying $D_S = D_T$ and $T(D_T) \subset D_T$, $S(D_S) \subset D_S$, then expressions of the form

$$T + S, \quad T \circ S - S \circ T = [T, S]$$

make sense. We say $T$ and $S$ are composable if these expressions are defined and essentially self-adjoint.

In the first formulations of quantum mechanics, the observables were always taken to be bounded self-adjoint operators. In this case, the set of quantum observables always carries a Lie algebra structure. However, as we shall see in the coming example, the Hamiltonian of a free particle in $\mathbb{R}^3$ is not even bounded!

**Example 2.** A particle in $\mathbb{R}^3$ with mass $m$ subject to a conservative force $F$.

Since our particle is moving through $\mathbb{R}^3$ and, as we said before, a wavefunction represents the states of the system through time, the natural Hilbert space is $L^2(\mathbb{R}^3)$ with the usual Lebesgue measure. Hence, a wave function $\Psi$ is of the form

$$\Psi(t) = \psi(q, t),$$

where $q = (q_1, q_2, q_3)$ is the coordinate on $\mathbb{R}^3$.

The Hamiltonian which experimentally works here is

$$H = -\frac{1}{2m} \nabla^2 + U(q),$$

where $\nabla^2$ denotes the Laplacian. Note that $H$ has domain of definition all $f \in L^2(\mathbb{R}^3)$ with square integrable first and second distributional derivatives.

Since $\Psi$ satisfies the Schrödinger equation, we have

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2m} \nabla^2 \psi + U(q) \psi.$$

Note that if we restrict to the case where no force is acting on the particle, then the Schrödinger equation reduces to

$$i \frac{\partial \psi}{\partial t} = \nabla^2 \psi.$$

It turns out, all solutions to this equation are of the form

$$\psi(r, t) = Ce^{i(q \cdot t)} E$$

where $C \in \mathbb{R}$ and $E \in \mathbb{R}^4$ are constants and $\cdot$ denotes the usual scalar product in $\mathbb{R}^4$. Notice that this is the equation of a standing wave evolving through time. I say here that this wave-function is not normalized in the sense of Definition 5. This isn’t a problem if instead of taking $\mathbb{R}^3$ as the domain of our wavefunction for each time, we take some box of side length $L$. 

9
Remark
The formula we obtained for a free particle moving through $\mathbb{R}^3$ goes a long way to explaining the outcome of the double-slit experiment discussed in the introduction. It explains the interference pattern on the screen and it explains why the interference pattern is composed of discrete dots.

In that experiment, there are four parts of the trajectory of the electron.

(i) The electron moves freely through space after being fired at the screen.
(ii) It interacts with the double-slit.
(iii) The particle moves freely through space once again.
(iv) Finally it hits the detector screen.

So if we model the electron as a free particle in $\mathbb{R}^3$ before and after it interacts with the screen, then indeed we will expect an interference pattern on the detector screen.

So now there’s the matter of the discrete dots. If the electron is a wave, then classically we expect a continuous interference pattern, not a discrete one. This is taken care of by what we said a quantum observable does: it returns its eigenvalues after an experiment. In our case, if we give $\mathbb{R}^3$ coordinates $(q_1, q_2, q_3)$ and suppose that the screen lies in the $(q_1, q_2)$-plane, then if $\psi(q, t)$ is the wavefunction of the electron, the numbers which the screen is recording are
\[
\int_{\mathbb{R}} q_1 |\psi(q, t_0)|^2 dq \quad \text{and} \quad \int_{\mathbb{R}} q_2 |\psi(q, t_0)|^2 dq
\]
where $t_0$ is the time when the electron hits the screen. Since $\psi(q, t)$ represents a wave which is interfering with itself, we get higher magnitude numbers when the wave constructively interferes with itself, and lower magnitude numbers when the wave destructively interferes with itself. These numbers represent the probability that the electron will have a given $q_1$ or $q_2$ coordinate at time $t_0$. Hence, since we are firing many electrons at a screen, we expect the pattern of measurements (the dots where the electrons hit the detector) to be discrete and mimic a wave interference pattern.

2.3 Dirac Quantization

A quantization is a procedure for taking a classical system and turning it into a quantum one. To get a feel for the Dirac axioms, let’s see what we need to do to turn example 1 into example 2.

In the classical case, we have a Hamiltonian system $(M, \omega, H)$ with

(i) manifold $M = \mathbb{R}^6$ with coordinates $(q, p) = (q_1, q_2, q_3, p_1, p_2, p_3)$,

(ii) symplectic form
\[
\omega = \sum_{j=1}^{3} dq_j \wedge dp_j,
\]
(iii) and Hamiltonian

\[ H_{\text{class}} = \frac{|p|^2}{2m} + U(q). \]

In the quantum case, we have a quantum Hamiltonian system \((\mathcal{H}, H)\), with

(i) Hilbert space \(\mathcal{H} = L^2(\mathbb{R}^3)\),

(ii) and Hamiltonian

\[ H_{\text{quant}} = -\frac{1}{2m} \nabla^2 + U(q) \]

We want a map \(Q : C^\infty(\mathbb{R}^3) \to S\mathcal{H}\) such that \(Q(H_{\text{class}}) = H_{\text{quant}}\). If we impose \(Q\) is linear, then

\[ Q\left(\frac{|p|^2}{2m} + U(q)\right) = \frac{1}{2m} \sum_{j=1}^{3} Q(p_j^2) + Q(U(q)) \]

In order for this to return the Schrödinger equation discussed in example 2 for any choice of \(U(q)\), we impose

\[
\begin{align*}
Q(1) &= 1 \\
Q(q_i) &= q_i \\
Q(p_j) &= i\frac{\partial}{\partial q_j}.
\end{align*}
\]

Note that in the classical case

\[ \{q_i, p_j\} = \delta_{ij} \]

and that in the quantum case we have

\[ \left[q_i, i\frac{\partial}{\partial q_j}\right] = -i\delta_{ij}. \]

Thus, by definition of \(Q\), we have

\[ Q([q_i, p_j]) = -i[Q(q_i), Q(p_j)]. \]

This motivates the axioms Dirac gave for a prequantization

**Definition 6.** Let \((M, \omega)\) be a symplectic manifold. A prequantization of \((M, \omega)\) is a pair \((Q, \mathcal{H})\), where \(\mathcal{H}\) is a Hilbert space and \(Q : C^\infty(M) \to S\mathcal{H}\) is a map such that

(i) \(Q(f)\) and \(Q(g)\) are composable for all \(f, g \in C^\infty(M)\).

(ii) \(Q\) is linear.

(iii) If \(1 \in C^\infty(M)\) is the constant function 1, then \(Q(1) = 2\pi I\), where \(I : \mathcal{H} \to \mathcal{H}\) is the identity.
\begin{equation}
Q(\{f, g\}) = -i [Q(f), Q(g)]
\end{equation}

**Remark**
One thing to note about the axioms as I’ve presented them is that I’ve introduced the factor of $2\pi$ in axiom (iii). This is to keep in line with the theory of prequantization as presented by Kostant \[1\]. It can be done away with, but we keep the $2\pi$ since it cleans up the statements of many of the theorems.

### 2.4 Kostant-Souriau Prequantization

Let $(M, \omega)$ be a symplectic manifold with $\dim M = 2n$. Our goal this section is to define a prequantum line bundle over $M$, then demonstrate how this induces a prequantization. The next section will then deal with when such a prequantum line bundle exists. We will mostly be following \[4\] and \[5\].

The first step is to define a Hilbert space from the manifold. For some motivation, consider the following example.

**Example 3.** Since $\omega$ is non-degenerate, $\omega^n$ is a volume form. So a natural choice for a Hilbert space would be the $L^2$ completion of

$$\{ f \in C^\infty(M, \mathbb{C}) \mid \text{supp } f \text{ is compact} \}$$

with respect to the inner product

$$(f, g) := \int_M f \overline{g} \omega^n.$$ 

Notice that $C^\infty(M, \mathbb{C})$ can be identified with the space of sections of the trivial complex line bundle $M \times \mathbb{C} \to M$.

So we see that we can obtain a Hilbert space from the trivial bundle on a symplectic manifold. This is due to the fact that the trivial bundle automatically comes equipped with a fibre-wise Hermitian metric. So more generally, we will want to consider all line bundles with Hermitian metrics over our manifold in order to have a large class of candidates for pre-quantization.

**Definition 7.** A line bundle over $M$ is fibre bundle $L \xrightarrow{\pi} M$ such that

(i) For all $x \in M$, $L_x := \pi^{-1}(x)$ has the structure of a 1-dimensional complex vector space.

(ii) For all $x \in M$, there exists open set $U \subset M$ about $x$ and a diffeomorphism

$$\psi : \pi^{-1}(U) \to U \times \mathbb{C}$$

such that for all $y \in U$, $\psi|_{L_y} : L_y \to \{ y \} \times \mathbb{C}$ is a linear isomorphism.

**Definition 8.** Let $U \subset M$ be an open subset. A local section of $L \to M$ is a smooth function $s : U \to L$ such that $\pi \circ s = \text{Id}_U$. Let $\Gamma(U, L)$ denote all such local section. Define $\Gamma(L) := \Gamma(M, L)$ and call such sections global. Let $\Gamma_c(L)$ denote compactly supported global sections.
As in the example, we want to use the fact that $\omega$ is non-degenerate to produce an inner product on compactly supported sections. To do this, we need a Hermitian form.

**Definition 9.** A Hermitian structure on $L \to M$ is a section $\langle , \rangle \in \Gamma(L^* \otimes L^*)$ such that for all $x \in M$, $\langle , \rangle_x : L_x \times L_x \to \mathbb{C}$ is a Hermitian inner product. Denote a line bundle with a Hermitian structure by the pair $(L, \langle , \rangle)$ and call it a Hermitian line bundle.

We can give $\Gamma_c(L)$ a pre-Hilbert space structure as follows. If $s_1, s_2 \in \Gamma_c(L)$, then $\langle s_1, s_2 \rangle : M \to \mathbb{C}$ is a smooth compactly supported function. Hence, is integrable. Define

$$\langle s_1, s_2 \rangle := \int_M \langle s_1, s_2 \rangle \omega^n.$$

It’s easy to show the following fact.

**Fact.** $(\Gamma_c(L), \langle \rangle)$ is a pre-Hilbert space.

Let $\mathcal{H}$ denote the $L^2$ completion of $\Gamma_c(L)$. Our goal now is to associate an unbounded self-adjoint operator on $\mathcal{H}$ to each smooth function on $M$. To motivate the construction, let’s return to the trivial bundle.

**Example 4.** Let $f \in C^\infty(M)$ and consider the trivial bundle $M \times \mathbb{C} \to M$. We want to make $f$ into a linear operator on $C^\infty(M, \mathbb{C})$. One way to do this is to define

$$m_f : C^\infty(M, \mathbb{C}) \to C^\infty(M, \mathbb{C}); \quad g \mapsto fg.$$

This is trivially linear.

The other way to make $f$ into a linear operator on $C^\infty(M, \mathbb{C})$ makes use of the symplectic structure on $M$. Let $X_f \in \mathfrak{X}(M)$ be the Hamiltonian vector field of $f$. Note that the map

$$C^\infty(M) \to C^\infty(M); \quad g \mapsto X_f g.$$

is a real linear map. Extending $\mathbb{C}$-linearly, $X_f$ is then a linear operator on $C^\infty(M, \mathbb{C})$.

Given a line bundle over $M$, we want to somehow lift the two operators defined in the previous example to act on arbitrary sections. To do this, we need the concept of the covariant derivative.

**Definition 10.** A covariant derivative $\nabla$ is a rule such that for all open $U \subset M$ there is a $\mathbb{C}$ bilinear map

$$\nabla_U : \mathfrak{X}_\mathbb{C}(U) \times \Gamma(U, L) \to \Gamma(U, L); \quad (\xi, s) \mapsto \nabla_\xi s,$$

where $\mathfrak{X}_\mathbb{C}(U)$ denotes local complex vector fields on $U$, such that

(i) If $U \subset V$, $\xi \in \mathfrak{X}_\mathbb{C}(V)$ and $s \in \Gamma(V, L)$, then

$$\left. \left( \nabla_\xi s \right) \right|_U = \nabla_{\xi|_U} (s|_U).$$
(ii) If \( f \in C^\infty(U, \mathbb{C}) \), \( \xi \in \mathfrak{X}_C(U) \), and \( s \in \Gamma(U, L) \), then
\[
\nabla f \xi s = f \nabla \xi s \\
\nabla \xi (fs) = (\xi f)s + f \nabla \xi s.
\]

Combining the maps from the example with a covariant derivative and adding some normalization factors, we can define the prequantization map.

**Definition 11.** Let \( f \in C^\infty(M) \) be a smooth function, \( X_f \in \mathfrak{X}(M) \) denote its Hamiltonian vector field. Define \( Q_f \in \text{Hom}(\Gamma(L), \Gamma(L)) \) by
\[
Q_f := i \nabla_{X_f} + 2\pi m_f,
\]
where \( m_f \) is the multiplication operator introduced in example 4. We call the map
\[
Q : C^\infty(M) \to \text{Hom}(\Gamma(L), \Gamma(L)); \quad f \mapsto Q_f
\]
the pre-quantization map.

**Remark**
If \( f \in C^\infty(M) \) is constant, then \( X_f = 0 \). Hence, \( \nabla X_f = 0 \). So, if \( f = 1 \), we then obtain \( Q_1 = 2\pi m_1 = 2\pi \text{Id} \). Thus, the map \( Q \) satisfies one of the axioms of a prequantization right away.

**Definition 12.** Let \((L, \langle \cdot, \rangle)\) be a Hermitian line bundle over \( M \), and \( \nabla \) a covariant derivative on \( L \). We say \( \nabla \) and \( \langle \cdot, \rangle \) are compatible if for all open \( U \subset M \), \( \xi \in \mathfrak{X}_C(U) \), and \( s_1, s_2 \in \Gamma(U, L) \)
\[
\xi \langle s_1, s_2 \rangle = \langle \nabla \xi s_1, s_2 \rangle + \langle s_1, \nabla \xi s_2 \rangle.
\]
Denote by \((L, \langle \cdot, \rangle, \nabla)\) a Hermitian line bundle with compatible covariant derivative.

**Lemma 1.** Let \((L, \langle \cdot, \rangle, \nabla)\) be a line bundle over \( M \) with compatible covariant derivative. Then for all \( s_1, s_2 \in \Gamma_c(L) \) and \( f \in C^\infty(M) \) we have
\[
(Q_f s_2, s_1) = (s_1, Q_f s_2).
\]

**Proof.** Trivially,
\[
(f s_1, s_2) = (s_1, f s_2).
\]
Next, since \( \nabla \) and \( \langle \cdot, \rangle \) are compatible, we have
\[
\int_M \langle \nabla X_f s_1, s_2 \rangle \omega^n = \int_M X_f (s_1, s_2) \omega^n - \int_M \langle s_1, \nabla X_f s_2 \rangle \omega^n.
\]
Now, since \( L_{X_f} \omega = 0 \), we have \( L_{X_f} \omega^n = 0 \). Hence,
\[
L_{X_f} (\langle s_1, s_2 \rangle \omega^n) = L_{X_f} (\langle s_1, s_2 \rangle) \omega^n + \langle s_1, s_2 \rangle L_{X_f} \omega^n = X_f (s_1, s_2) \omega^n.
\]
On the other hand, since \( (s_1, s_2) \omega^n \) is a top form, we have

\[
\mathcal{L}_{X_f}((s_1, s_2) \omega^n) = X_f \cdot d((s_1, s_2) \omega^n) + d(X_f \cdot (s_1, s_2) \omega^n) = d(X_f \cdot (s_1, s_2) \omega^n).
\]

Thus, by Stoke’s theorem

\[
\int_M X_f(s_1, s_2) \omega^n = \int_M d(X_f \cdot (s_1, s_2) \omega^n) = 0.
\]

Hence,

\[
(\nabla_{X_f} s_1, s_2) = -(s_1, \nabla_{X_f} s_2).
\]

Putting everything together,

\[
(Q_f s_1, s_2) = ([i \nabla_{X_f} + 2\pi f]s_1, s_2) \\
= i(\nabla_{X_f} s_1, s_2) + (2\pi f s_1, s_2) \\
= -i(s_1, \nabla_{X_f} s_2) + (s_1, 2\pi f s_2) \\
= (s_1, Q_f s_2).
\]

All that’s left for us is to obtain the relation

\[
Q_{\{f,g\}} = -i [Q_f, Q_g].
\]

This is a condition on the curvature of the covariant derivative on \( L \).

**Definition 13.** Let \( \nabla \) be a covariant derivative on \( L \rightarrow M \). Define the curvature of \( \nabla \), \( R \in \Omega^2(M) \otimes \text{End}(L) \) by

\[
R(\xi, \eta)s = [\nabla_\xi, \nabla_\eta]s - \nabla_{[\xi,\eta]}s,
\]

where \( \xi, \eta \) are local complex vector fields and \( s \) is a local section of \( L \).

We now impose the Bohr-Sommerfeld quantization condition on \((M, \omega)\). We demand that \( 2\pi i \omega \) is the curvature of \( \nabla \). That is, for all local complex vector fields \( \xi \) and \( \eta \) on \( M \):

\[
[\nabla_\xi, \nabla_\eta] - \nabla_{[\xi,\eta]} = 2\pi i \omega(\xi, \eta).
\]

**Lemma 2.** If equation (1) condition holds, then the map

\[
Q : C^\infty(M) \rightarrow \text{End}(\Gamma_c(L), \Gamma_c(L)); \quad f \mapsto Q_f
\]

satisfies

\[
Q_{\{f,g\}} = -i [Q_f, Q_g].
\]
Proof. Let \( f, g \in C^\infty(M) \). Then by equation (1),
\[
[\nabla_{X_f}, \nabla_{X_g}] = \nabla_{[X_f, X_g]} + 2\pi i \omega(X_f, X_g) = \nabla_{X_{\{f, g\}}} + 2\pi i \{f, g\}.
\]

Fix a local section \( s \). We compute
\[
Q_f(Q_g s) = Q_f(i\nabla_{X_g} s + 2\pi m s) = -\nabla_{X_f} \nabla_{X_g} s + 2\pi i \{f, g\} s - 4\pi^2 fgs = -\nabla_{X_f} \nabla_{X_g} s + 2\pi i \{f, g\} s + 2\pi i \{f, g\} s + 2\pi i \{f, g\} s + 2\pi i \{f, g\} s + 4\pi^2 fgs.
\]

Thus, swapping \( f, g \) we compute
\[
Q_f(Q_g s) - Q_g(Q_f s) = -[\nabla_{X_f}, \nabla_{X_g}] s + 4\pi i \{f, g\} s = -\nabla_{X_{\{f, g\}}} s - 2\pi i \{f, g\} s + 4\pi i \{f, g\} s = iQ_{\{f, g\}}.
\]

\( \square \)

**Definition 14.** Let \((M, \omega)\) be a symplectic manifold. A Kostant-Souriau prequantum line bundle over \((M, \omega)\) is a Hermitian line bundle \((L, \langle \cdot, \cdot \rangle, \nabla)\) over \(M\) with compatible covariant derivative with curvature \(R^\nabla = 2\pi i \omega\).

We have thus shown the following result.

**Theorem 2.** A Kostant-Souriau prequantum line bundle \((L, \langle \cdot, \cdot \rangle, \nabla)\) over a symplectic manifold \((M, \omega)\) induces a prequantization with Hilbert space the \(L^2\) completion of \(\Gamma_c(L)\) and prequantization map
\[
Q : C^\infty(M) \to Hom(\Gamma_c(L), \Gamma_c(L)); \quad f \mapsto i\nabla_{X_f} + 2\pi m_f.
\]

The savvy reader should now notice that we assumed a lot in order to get to this point. For the symplectic manifold \((M, \omega)\) and line bundle \(L \to M\) we assumed

(i) There exists a Hermitian structure on \(L\).

(ii) There exists a compatible covariant derivative.

(iii) The Kostant-Souriau prequantization condition holds.

Turns out, these are all conditions on the cohomology class of \([\omega]\). We will dedicate the rest of the report to showing this fact.

**Remark**

The Kostant-Souriau prequantization gives us other important facts for free. First, the prequantization map \(Q\) is actually injective. Second, this prequantization carries with it a projective unitary representation of the symplectomorphisms on \((M, \omega)\). To learn more, consult [5] or [4].
### 3 Line Bundles and Covariant Derivatives

#### 3.1 \(C^\ast\)-Principal Bundles And Connection 1-Forms

The first topic we will be discussing on our way to proving the main theorem will be \(C^\ast\)-principal bundles and connection 1-forms. We will use this material to obtain a convenient way of describing covariant derivatives on line bundles which we will use heavily in the final proof. For a more complete introduction to the topics discussed in this subsection in a broader context, consult [9] or [8].

\(C^\ast\)-principal bundles are a special kind of fibre bundle with a compatible action by \(C^\ast\) on the total space. Before we define \(C^\ast\)-principal bundles in full generality, let’s see the trivial example.

**Example 5.** Let \(M\) be a manifold and define \(P = M \times C^\ast\). With the natural projection \(pr_1 : P \to M\), we see that \(P \xrightarrow{pr_1} M\) is a trivial fibre bundle with typical fibre \(C^\ast\). This bundle has a smooth action by \(C^\ast\): Let \(x \in M\), \(\mu, \lambda \in C^\ast\). Define

\[(x, \mu) \cdot \lambda := (x, \mu\lambda).\]

This is trivially a smooth group action and is called the trivial action.

Note that the trivial action permutes elements of the fibre over \(x\).

Taking the above example as a local model, we define a \(C^\ast\)-principal bundle in full generality.

**Definition 15.** A \(C^\ast\)-principal bundle is a fibre bundle \(P \xrightarrow{\sigma} M\) with typical fibre \(C^\ast\) together with a smooth free action

\[P \times C^\ast \to P; \quad (p, \lambda) \mapsto p \cdot \lambda\]

such that

(i) If \(p \in P\), then \(q \in \sigma^{-1}(\sigma(p)) \iff\) there exists \(\lambda \in C^\ast\) such that \(q = p \cdot \lambda\).

(ii) For all \(x \in M\), there exists open neighbourhood \(U \subset M\) and a diffeomorphism

\[\psi : \pi^{-1}(U) \to U \times C^\ast\]

such that the diagram commutes

\[
\begin{array}{ccc}
\sigma^{-1}(U) & \xrightarrow{\psi} & U \times C^\ast \\
\downarrow{\sigma} & & \downarrow{pr_1} \\
U & \xrightarrow{\pi} & M
\end{array}
\]

And such that for all \(p \in \pi^{-1}(U)\) and \(\lambda \in C^\ast\)

\[\psi(p \cdot \lambda) = \psi(p) \cdot \lambda\]

where we give \(U \times C^\ast\) the trivial action. Call such a \((U, \psi)\) a local trivialization of \(P \to M\).
Remark
Let $\lambda \in \mathbb{C}^\ast$. For ease of notation, we will identify $\lambda$ with the smooth function

$$P \to P; \quad p \mapsto p \cdot \lambda.$$  

Example 6. As we’ve seen a trivial bundle $M \times \mathbb{C}^\ast \to M$ is a $\mathbb{C}^\ast$-principal bundle.

Example 7. For a less trivial example, let $L \xrightarrow{\pi} M$ be a line bundle. For all $x \in M$, define

$$L_x^+ := L_x \setminus \{0_x\}$$

where $0_x \in L_x$ denotes the zero element over $x$. We can then define the frame bundle of $L \to M$ to be

$$L^+ := \bigsqcup_{x \in M} L_x^+$$

and give it a projection $\tilde{\pi} := \pi|_{L^+}$.

Note that $L^+ \subset L$ is a submanifold. If we let $(U, \psi)$ be a local trivialization of $L \to M$, then since $\psi|_{L_x}$ is a linear isomorphism for all $x \in U$, we obtain that

$$\psi|_{\tilde{\pi}^{-1}(U)} : \tilde{\pi}^{-1}(U) \to U \times \mathbb{C}^\ast$$

is a diffeomorphism which commutes with the projections $\tilde{\pi}$ and $pr_1$.

Observe that $L^+$ carries a natural action by $\mathbb{C}^\ast$ which is smooth and free. Hence $L^+ \xrightarrow{\pi} M$ is a $\mathbb{C}^\ast$-principal bundle.

Turns out, the above example classifies all principal $\mathbb{C}^\ast$-principal bundles, but we will not need that fact in this report.

For the rest of this subsection, fix a principal $\mathbb{C}^\ast$-bundle $P \xrightarrow{\sigma} M$ over $M$. We will now see how the Lie algebra of $\mathbb{C}^\ast$, $\mathfrak{c}$, interacts with the fibre bundle structure.

Definition 16. For every $C \in \mathbb{C}$, define the fundamental vector field on $P$ associated to $C$, $X_C \in \mathfrak{X}(P)$, by

$$\langle X_C \rangle_p := \left. \frac{d}{dt} \right|_{t=0} p \cdot e^{2\pi i Ct}.$$  

Here’s a sequence of facts that we will be using, but will not be proving as they would constitute too large of a tangent from the core material.

Proposition 2. (i) The map

$$\mathbb{C} \to \mathfrak{X}(P); \quad C \mapsto X_C$$

is a Lie algebra homomorphism.

(ii) For all $C \in \mathbb{C}$, $X_C = CX_1$.  

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(iii) $X_1$ never vanishes and hence defines a subbundle $VP$ of $TP$, called the vertical bundle.

(iv) For all $p \in P$

$$d\sigma_p = V_p P.$$ 

Proof. See [8].

So a principal bundle always has a special subbundle of $TP$ specified by the action of $\mathbb{C}^*$. This attaches a copy of $\mathbb{C}$ to every point $p \in P$. Is there a way to attach a copy of $T_{\sigma(p)} M$ to $p$ as well in a transverse way? The answer is yes, and a formula on how to do this is provided by a connection 1-form.

**Definition 17.** A connection 1-form is a 1-form $\varphi \in \Omega^1(P) \otimes \mathbb{C}$ such that

(i) $X_C \hook C \varphi = C$ for all $C \in \mathbb{C}$.

(ii) For all $\lambda \in \mathbb{C}^*$ we have $\lambda^* \varphi = \varphi$.

**Example 8.** Let $P = M \times \mathbb{C}^*$. Give $\mathbb{C}^*$ coordinate $z$. Then for any $A \in \Omega^1(M) \otimes \mathbb{C}$, define

$$\varphi = \left( A, \frac{1}{2\pi i} \frac{dz}{z} \right).$$

First, let’s compute $X_1$ on $P$. For any $(x, \mu) \in M \times \mathbb{C}^*$, we have

$$(X_1)_{(x,\mu)} = \frac{d}{dt} \bigg|_{t=0} (x, \mu) \cdot e^{2\pi i t}$$

$$= \frac{d}{dt} \bigg|_{t=0} (x, \mu e^{2\pi i t}) = (0_x, 2\pi i \mu),$$

where $0_x \in T_x M$ denotes the zero element above $x$.

Now we have to show $\varphi(X_1) = 1$. We compute,

$$\varphi_{(x,\mu)}(X_1)_{(x,\mu)} = \left( A_x, \frac{1}{2\pi i} \frac{dz}{z} \right) (0_x, 2\pi i \mu)$$

$$= B_x(0_x) + \frac{1}{2\pi i} \frac{dz}{z} \bigg|_{\mu} (2\pi i \mu)$$

$$= \frac{1}{2\pi i \mu} (2\pi i \mu)$$

$$= 1.$$ 

Hence, $\varphi(X_C) = C \varphi(X_1) = C$.

Next, to show the equivariance, let $(v_x, \eta) \in T_x M \times \mathbb{C}$ and $\lambda \in \mathbb{C}^*$, then

$$(\lambda^* \varphi)_{(x,\mu)}(v_x, \eta) = \varphi_{(x,\lambda \mu)}(v_x, \lambda \eta)$$

$$= B_x(v_x) + \frac{1}{2\pi i} \frac{dz}{z} \bigg|_{\lambda \mu} (\lambda \eta)$$

$$= B_x(v_x) + \frac{1}{2\pi i} \frac{\eta}{\mu}$$

$$= \varphi_{(x,\mu)}(v_x, \eta).$$

Thus, $\varphi$ is a connection 1-form.
Given a connection 1-form, as was hinted above the definition, one can find a way of splitting the tangent bundle in a way which is transverse to the vertical bundle $VP$. This splitting due to the connection 1-form is called the Horizontal bundle. More formally.

**Definition 18.** Given a $C^*$-principal bundle $P \to M$, a connection 1-form $\varphi$, and $p \in P$, we say $v \in T_pP$ is horizontal if $\varphi(v) = 0$. Define the horizontal bundle $HP$ by

$$HP := \{v \in TP \mid v \text{ is horizontal}\}.$$ 

**Proposition 3.** $HP$ is a smooth subbundle of $TP$. Furthermore, there is a splitting $TP = VP \oplus HP$.

**Proof.** For each $p \in P$, $H_pP = \ker \varphi_p$. Now, since the map $\mathbb{C} \to \Gamma(VP); \ C \mapsto X_C$ is a Lie algebra isomorphism, this implies $\varphi_p|_{V_pP} : V_pP \to \mathbb{C}$ is the inverse and hence $H_pP = \dim P - 1$. Taking a trivializing neighbourhood in $M$ for $P$, it suffices to show that the Horizontal bundle is smooth on the trivial bundle. See [8] for a proof in this case.

It’s a fact from linear algebra that each tangent space $T_pP = V_pP \oplus H_pP$. And hence $TP = VP \oplus HP$.

**Corollary 2.** The projection maps

$$Ver : TP \to VP, \quad Hor : TP \to HP$$

are smooth.

Note that by dimensionality arguments alone, $d\sigma_p : H_pP \to T_{\sigma(p)}M$ is an isomorphism of vector spaces for all $p \in P$. This allows us to lift vector fields on $M$ to vector fields on $P$.

**Theorem 3.** Let $\xi \in \mathfrak{X}(U)$ be a local section of $M$. Then there exists a unique local vector field $\bar{\xi} \in \mathfrak{X}(\pi^{-1}(U))$ such that

(i) $Hor(\bar{\xi}) = \xi$

(ii) $d\pi_p(\bar{\xi}_p) = \xi_{\pi(p)}$

(iii) For all $\lambda \in \mathbb{C}^*$ we have

$$\lambda \bar{\xi} = \bar{\xi}.$$

**Proof.** See [9] \hfill $\square$

**Definition 19.** If $\xi \in \mathfrak{X}(U)$ and $\bar{\xi} \in \mathfrak{X}(\sigma^{-1}(U))$ is as in the above theorem, then we call $\bar{\xi}$ the horizontal lift of $\xi$.

**Corollary 3.** For each open subset $U \subset M$, the map

$$\mathfrak{X}(U) \to \mathfrak{X}(\pi^{-1}(U)); \quad \xi \mapsto \bar{\xi}$$

is linear.
3.2 Covariant Derivatives

The main result of this subsection is the correspondence between covariant derivatives and connection 1-forms. We will need $M$ to be paracompact for all these proofs to work. The layout of this subsection closely follows [7].

Throughout this section, let $L \xrightarrow{\pi} M$ be a line bundle over $M$ and $L^+ \xrightarrow{\tilde{\pi}} M$ be its frame bundle.

Let’s start with an example of a covariant derivative.

Example 9. Let $L = M \times \mathbb{C}$ be the trivial bundle over $M$. If $U \subset M$ is an open subset, then for all local sections $s \in \Gamma(U, L)$, there exists unique $f_s \in C^\infty(U, \mathbb{C})$ such that $s(x) = (x, f_s(x)), \ x \in M$.

For every $\xi \in \mathfrak{X}_C(U)$, define

$$\nabla_\xi s(x) := (x, \xi_x f)$$

Since tangent vectors are derivations, we immediately obtain that $\nabla$ is a covariant derivative.

Before we can continue, we need some ideas from point-set topology.

Definition 20. Let $X$ be a topological space.

(i) An open cover $\{U_i\}_{i \in I}$ is said to be locally finite if for all $x \in M$, there exists a neighbourhood $U$ of $x$ such that $U \cap U_i \neq \emptyset$ for only finitely many $i$.

(ii) $X$ is said to be paracompact if for every open cover $\{U_i\}_{i \in I}$ there exists a refinement $\{V_j\}_{j \in J}$ which is a locally finite open cover of $X$.

Proposition 4. If $M$ is paracompact, then any open cover admits a partition of unity.

Proof. See Lee [11].

We shall from now on assume that $M$ is paracompact. Note that since all smooth manifolds are paracompact, this will have no impact on the statement of the main theorem.

Using the paracompactness of $M$, we may now show a foundational result.

Proposition 5. If $\text{Co}(L)$ denotes the set of all covariant derivatives on $L \to M$, then $\text{Co}(L)$ is nonempty.

Proof. Let $\{U_i\}_{i \in I}$ be a trivializing cover of $L \to M$. Since $\pi^{-1}(U_i) \to U_i$ is isomorphic to the trivial bundle for all $i$, we have there exists a covariant derivative $\nabla_i$ on $\pi^{-1}(U) \to U_i$ for all $i$.

Take a partition of unity $\{h_i\}_{i \in I}$ subordinate to the cover. Define

$$\nabla := \sum_{i \in I} h_i \nabla_i.$$
That is, if $U \subset M$ is open, $\xi \in \mathfrak{X}(U)$ is a local vector field, and $s \in \Gamma(U, L)$ is a local section of $L$, then define

$$\nabla_\xi s := \sum_{i \in I} h_i(\nabla_i)\xi_i s_i$$

where $\xi_i = \xi|_{U_i}$ and $s_i = s|_{U_i}$.

Now, if $\nabla \in \text{Co}(L)$ and $\alpha \in \Omega^1(M) \otimes \mathbb{C}$, then we can define a new covariant derivative by

$$(\nabla + \alpha)_\xi s = \nabla_\xi s + \alpha(\xi)s.$$ 

Turns out, this procedure generates all covariant derivatives.

**Corollary 4.** $\text{Co}(L)$ is an affine space under the above action by the vector space $\Omega^1(M) \otimes \mathbb{C}$.

**Remark**

For notational convenience, if $U \subset M$ is open, $s \in \Gamma(U, L^+)$ and $t \in \Gamma(U, L)$, then define the smooth function

$$\frac{t}{s} : U \to \mathbb{C}$$

uniquely by the property

$$t = \frac{t}{s}s.$$

Note that if $t$ also doesn’t vanish (i.e. $t \in \Gamma(U, L^+)$), then $t/s$ doesn’t vanish as well.

**Proof.** (Of Corollary)

Let $\nabla, \nabla' \in \text{Co}(L)$. For any open subset $U \subset M$, $s \in \Gamma(U, L^+)$, and $\xi \in \mathfrak{X}(U)$, define

$$A_U(\xi) := \frac{\nabla_\xi s - \nabla'\xi s}{s}.$$ 

$A_U$ is linear in $\xi$ and smooth. Hence $A_U \in \Omega^1(U) \otimes \mathbb{C}$.

Choosing a different $t \in \Gamma(U, L^+)$, we obtain $g : U \to \mathbb{C}^*$ such that $t = gs$. Then we compute using the fact that both $\nabla$ and $\nabla'$ are derivations in the second entry:

$$\frac{\nabla_\xi t - \nabla'\xi t}{t} = \frac{\nabla_\xi gs - \nabla'\xi gs}{gs} = \frac{\nabla_\xi s - \nabla'\xi s}{s}.$$ 

Thus, $A_U$ doesn’t depend on choice of non-vanishing section. It’s then clear that if $U \cap V \neq \emptyset$, then $A_U|_{U \cap V} = A_V|_{U \cap V}$. Hence the $A_U$ form a globally defined 1-form $A \in \Omega^1(M) \otimes \mathbb{C}$. By construction, we have

$$\nabla - \nabla' = A.$$ 

$A$ is clearly the unique 1-form to have this property.
We will show a correspondence between covariant derivatives and connection 1-forms. First, we need a definition and a Lemma.

**Definition 21.** Let $V \subset L^+$ be an open subset and $f \in C^\infty(V, \mathbb{C})$. Say $f$ is $\mathbb{C}^*$-equivariant if for all $z \in V$ and $\lambda \in \mathbb{C}^*$
\[ f(\lambda z) = \lambda^{-1} f(z). \]

Denote the set of all such equivariant functions on $V$ by $C^*(V)$.

The following correspondence is from [3].

**Lemma 3.** Let $U \subset M$ be open.

(i) If $s \in \Gamma(U, L)$, then there exists a unique $\phi_s \in \mathbb{C}^*(\pi^{-1}(U))$ such that for all $z \in \pi^{-1}(U)$
\[ s(\pi(z)) = \phi_s(z)z. \]
(Recall that $L^+ \subset L$ so $\phi_s(z)z$ is defined as an element of $L$ for any value of $\phi_s$).

(ii) If $f \in \mathbb{C}^*(\pi^{-1}(U))$, then there exists a unique $\rho_f \in \Gamma(U, L)$ such that for all $z \in \pi^{-1}(U)$
\[ \rho_f(\pi(z)) = f(z)z. \]

(iii) The maps
\[ \phi : \Gamma(U, L) \to \mathbb{C}^*(\pi^{-1}(U)); \quad s \mapsto \phi_s \]
\[ \rho : \mathbb{C}^*(\pi^{-1}(U)) \to \Gamma(U, L); \quad f \mapsto \rho_f \]
are linear and inverse to one another.

**Proof.** Fix $s \in \Gamma(U, L)$ and $f \in \mathbb{C}^*(\pi^{-1}(U))$.

It’s clear that $\phi_s$ is well-defined. So we only show that $\rho_f$ is well-defined. Suppose $\lambda \in \mathbb{C}^*$ and $z \in \pi^{-1}(U)$. Then
\[ f(\lambda z)(\lambda z) = \lambda^{-1} \lambda f(z)z = f(z)z. \]

Hence, $\rho_f$ is well-defined.

We’ll show $\phi_s$ is smooth. It’s the exact same computation to show that $\rho_f$ is smooth.

Let $U \subset M$ be a trivializing neighbourhood and $\psi : \pi^{-1}(U) \to U \times \mathbb{C}$ a trivializing diffeomorphism. By definition, there exists $\beta : \pi^{-1}(U) \to \mathbb{C}$, which is a linear isomorphism over each fibre, such that
\[ \psi(z) = (\pi(z), \beta(z)). \]

Note that for any $z \in \pi^{-1}(U) \subset \pi^{-1}(U)$ we have $\beta(z) \neq 0$.

Fix $z \in \pi^{-1}(U)$. We have
\[ \psi \circ s(\pi(z)) = (\pi(z), F(\pi(z))) \]
for some $F : U \to \mathbb{C}$. On the other hand, we have
\[ \psi \circ s(\pi(z)) = \psi(\phi_s(z))z = (\pi(z), \phi_s(z)\beta(z)). \]
Hence, we have
\[ \phi_s(z) = \frac{f(\pi(z))}{\beta(z)}. \]
Both $F$ and $\beta$ are smooth and $\beta \neq 0$ on $\tilde{\pi}^{-1}(U)$, hence $\phi_s$ is smooth.

It's clear that both $\phi$ and $\rho$ are linear. Furthermore, their defining equations are inverse to one another. Hence, $\phi$ and $\rho$ are inverse. \qed

Now, we are ready for the main result of this section. The proof of (i) is due to Śniatycki [3] and the proof of (ii) is the argument given by Brylinski [7].

**Theorem 4.** (i) If $\varphi \in \Omega^1(L^+)$ is a connection 1-form, then there exists a unique covariant derivative $\nabla$ such that
\[ \nabla_{\xi}s = 2\pi i(s^*\varphi(\xi))s. \] (2)

(ii) If $\nabla$ is a covariant derivative, then there exists a unique connection 1-form $\varphi \in \Omega^1(L^+)$ satisfying equation (1).

**Proof.** (i) Let $\varphi \in \Omega^1(L^+) \otimes \mathbb{C}$ be a connection 1-form.

Let $U \subset M$ be an open subset, $\xi \in \mathfrak{X}_C(U)$ a local vector field, and $s \in \Gamma(U, L)$ a local section of $L$. If $\tilde{\xi} \in \mathfrak{X}_C(\pi^{-1}(U))$ is the horizontal lift of $\xi$ and $\phi_s \in C^\infty(\tilde{\pi}^{-1}(U), \mathbb{C})$ is the unique $\mathbb{C}^*$-equivariant smooth function such that
\[ s(\pi(z)) = \phi_s(z)z, \quad z \in \tilde{\pi}^{-1}(U) \]
then define
\[ \nabla_{\xi}s(\pi(z)) := (\tilde{\xi}_z\phi_s)z. \]

If $\nabla_{\xi}s$ is well-defined, then since it’s produced from the composition of smooth operations, $\nabla_{\xi}s \in \Gamma(U, L)$.

To show well-definedness, let $\lambda \in \mathbb{C}^*$, then
\[ \nabla_{\xi}s(\pi(\lambda z)) = \langle (\tilde{\xi}_z\phi_s(\lambda)) z \rangle = \langle (\lambda z \tilde{\xi}_z\phi_s) z \rangle \]
\[ = \langle (\tilde{\xi}_z(\phi_s(\lambda)) z \rangle \]
\[ = \langle \tilde{\xi}_z\phi_s z \rangle \]
\[ = \nabla_{\xi}s(\pi(z)). \]

Hence, $\nabla_{\xi}s$ is well-defined and smooth.

To see that $\nabla$ is a covariant derivative, first note that both the operations
\[ \mathfrak{X}_C(U) \to \mathfrak{X}_C(\tilde{\pi}^{-1}(U)); \quad \xi \mapsto \tilde{\xi} \]
\[ \Gamma(U, L) \to C^\infty(\tilde{\pi}^{-1}(U), \mathbb{C}); \quad s \mapsto \phi_s \]
are linear. Hence, $\nabla$ is linear in each entry. Second, since both the above maps respect restriction, so does $\nabla$.

To finish showing that $\nabla$ is a covariant derivative, we need to show that it is a derivation in the second entry. For this purpose, let $f \in C^\infty(U, \mathbb{C})$ and $z \in \pi^{-1}(U)$. Then we have $\nabla \xi (fs)(\pi(z)) = (\xi \phi_{fs})_z = (\xi f \circ \pi \phi_s)_z = (\xi \pi(z) f)(\pi(z)) + f(\pi(z)) \nabla \xi (\pi(z))$.

Since $z$ was arbitrary and $\pi : L^+ \to M$ is surjective, the result follows.

Finally, we need to demonstrate equation (1) holds. Let $s \in \Gamma(U, L^+)$ and $\xi \in \mathfrak{X}_C(M)$. Observe that for all $x \in M$, we have $ds_x(\xi_x) \in T_s(x)L^+$. Hence, we can decompose $ds_x(\xi_x) = \text{Hor}(ds_x(\xi_x)) + \text{Ver}(ds_x(\xi_x))$

Now, observe that since $\pi \circ s = \text{Id}$, we have $\text{Hor}(ds_x(\xi_x)) = \xi_{s(x)}$.

Recall the fundamental vector field $X_1$ on $L^+$ as in section 3.1. It’s easy to see that $\text{Ver}(ds_x(\xi_x)) = \varphi(ds_x(\xi_x))(X_1)_{s(x)}$.

Thus, $ds_x(\xi_x) = \xi_{s(x)} + \varphi(ds_x(\xi_x))(X_1)_{s(x)}$.

Using this, we may then compute

$$\xi_{s(x)} \phi_s = (dF_s)_{s(x)}(\xi_{s(x)})$$

$$= (d\phi_s)_{s(x)}(ds_x(\xi_x) - \varphi(ds_x(\xi_x))(X_1)_{s(x)})$$

$$= d(\phi_s \circ s)_x (\xi_x) - \varphi(ds_x(\xi_x))(d\phi_s)_{s(x)}((X_1)_{s(x)})$$

$$= 2\pi is^* \varphi(\xi_x).$$

Hence, since $\xi$ is invariant, one can compute for any $z \in L^+$

$$\nabla_z s(\pi(z)) = (\xi \phi_s)_z = [2\pi is^* \varphi(\xi_{s(x)}) \phi_s]_z = [2\pi is^* \varphi(\xi_{s(x)})]s(\pi(z)).$$

(ii) Let $\nabla$ be a covariant derivative on $L$. If the connection 1-form $\varphi$ exists, then it is uniquely characterized by equation (1). For if $\varphi'$ is another such connection 1-form, then we have for all local sections $s$ on $L^+$

$$s^* \varphi = s^* \varphi'.$$
Since every element of the tangent space of \(L^+\) can be obtained from the image of a local section, we thus conclude \(\varphi = \varphi'\). Hence, it suffices to show that \(\varphi\) exists locally, i.e. on the trivial bundle.

Let \(L = M \times \mathbb{C}\) be the trivial bundle. As in a previous example, we may identify the sections of \(L\) with complex valued functions on \(M\), i.e. a section of \(L\) is of the form \((\text{Id}_M, f)\), where \(f \in C^\infty(M, \mathbb{C})\). Furthermore, we showed that

\[
\mathcal{X}_\mathbb{C}(M) \times \Gamma(L) \to \Gamma(L); \quad (\xi, (\text{Id}_M, f)) \mapsto (\text{Id}, df(\xi))
\]

is a covariant derivative. Hence, since \(\text{Co}(L)\) is affine, if \(\nabla\) is a covariant derivative on \(L\), there exists \(B \in \Omega^1(M) \otimes \mathbb{C}\) such that for all \(f \in C^\infty(M, \mathbb{C})\)

\[
\nabla_\xi s = (\text{Id}_M, df(\xi) + 2\pi ifB(\xi)).
\]

If \(z\) is the coordinate on \(\mathbb{C}\), define

\[
\varphi = \left( B, \frac{1}{2\pi i} \frac{dz}{z} \right) \in \Omega^1(L) \otimes \mathbb{C}.
\]

But we showed in an even earlier example that \(\varphi\) is a connection 1-form. So all that’s left to do is to verify equation (2).

We then see that if \(s(x) = (x, f(x))\), then

\[
2\pi is^*\varphi(\xi)s = 2\pi if(\xi, df(\xi))s
\]

\[
= 2\pi i \left( B, \frac{1}{2\pi i} \frac{dz}{z} \right) (\xi, df(\xi))s
\]

\[
= 2\pi i \left[ B(\xi) + \frac{1}{2\pi i} \frac{df}{f}(\xi) \right] s
\]

\[
= (\text{Id}, df(\xi) + 2\pi ifB(\xi)).
\]

And this concludes the proof.

We can use this theorem to get a description of the curvature of the covariant derivative. Fix a line bundle \((L, \nabla)\) with covariant derivative over \(M\) and let \(\varphi\) the corresponding connection 1-form on \(L^+\).

**Corollary 5.** If \(R\) is the curvature of \(\nabla\), then for any open subset \(U \subset M\) and non-vanishing section \(s \in \Gamma(U, L^+)\), we have

\[
2\pi is^*d\varphi = R.
\]

**Proof.** Set \(\alpha = s^*\varphi\). We then have for any local vector field \(\xi\) on \(M\)

\[
\nabla_\xi s = 2\pi i\alpha(\xi)s.
\]
Now, if \( \eta \) is any other local vector field, we can compute
\[
\nabla_\xi (\nabla_\eta s) = 2\pi i \nabla_\xi (\alpha(\eta)s) \\
= 2\pi i [\xi(\alpha(\eta))s + \alpha(\eta) \nabla_\xi s] \\
= 2\pi i [\xi(\alpha(\eta))s - 4\pi^2 \alpha(\eta)\alpha(\xi)s].
\]

Using this, we can then compute,
\[
R(\xi, \eta)s = [\nabla_\xi, \nabla_\eta]s - \nabla_{[\xi, \eta]}s = 2\pi i \alpha(\xi, \eta)s.
\]

Hence,
\[
2\pi i \alpha = 2\pi i d\alpha = 2\pi is^*d\varphi = R.
\]

From here, one can deduce the following corollary.

**Corollary 6.** The curvature \( R \) of \( \nabla \) is a closed complex 2-form on \( M \) satisfying.
\[
\pi^* R = 2\pi i d\varphi. \tag{3}
\]

**Proof.** It’s clear by the above corollary that \( R \in \Omega^2(M) \otimes \mathbb{C} \). It’s also clear that \( R \) is closed since
\[
dR = 2\pi i d(s^*d\varphi) = 2\pi is^*d^2\varphi = 0.
\]

To show equation (3), let \( U \subset M \) be an open subset and \( s \in \Gamma(U, L^+) \). Note that since \( \pi \circ s = \text{Id}_U \), we have
\[
R = s^*\pi^* R = 2\pi is^*d\varphi.
\]

Hence, \( \pi^* R - 2\pi i d\varphi \in \ker s^* \). Since \( s \) was arbitrary, we conclude the corollary.

**3.3 Local Systems on a Line Bundle**

We finish this section with a technical discussion on the local nature of line bundles. This will be useful when we want to construct a line bundle from a closed 2-form on a manifold. This discussion closely follows [1] and [7].

**Definition 22.** A good cover of \( M \) is an open cover \( \{ U_i \}_{i \in I} \) such that for all \( i_0, \ldots, i_m \in I \) we have \( U_{i_0} \cap \cdots \cap U_{i_m} \) is either empty or connected and simply connected.

**Proposition 6.** On a paracompact manifold, every open cover admits a refinement which is a good cover.

**Proof.** See [7]

Let \( M \) be a paracompact manifold, \( L \xrightarrow{\pi} M \) a line bundle over \( M \), and \( L^+ \xrightarrow{\tilde{\pi}} M \) denote the frame bundle of \( L \) over \( M \), where
\[
\tilde{\pi} = \pi|_{L^+}.
\]
Definition 23. A local system of \( L \to M \) is a collection of pairs \( \{(U_i, s_i)\}_{i \in I} \) such that

(i) \( \{U_i\}_{i \in I} \) is a good cover of \( M \)

(ii) For all \( i \in I \), \( s_i \in \Gamma(U_i, L^+) \).

If \( U_i \cap U_j \neq \emptyset \), then define \( g_{ij} : U_i \cap U_j \to \mathbb{C}^* \) uniquely by

\[
s_i = g_{ij}s_j.
\]

Call the \( g_{ij} \) the transition functions of the local system.

In a local system, we can break our analysis of a line bundle down into the discussion of the trivial bundle, locally. We could always do this of course, but a local system then gives us a way of piecing the local analysis together again to give a global result. That’s great, but we first need them to exist.

Proposition 7. \( L \to M \) admits a local system.

Proof. Let \( \{U_i\}_{i \in I} \) be an open cover of \( M \) such that for all \( i \in I \) there exists a trivialization

\[
\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}.
\]

For all \( x \in U_i \), define

\[
s_i(x) := \psi_i^{-1}(x, 1).
\]

Since \( \psi_i \) is a diffeomorphism and a linear isomorphism on each fibre, we have \( s_i \in \Gamma(U_i, L^+) \).

Now, since \( M \) is paracompact, we may refine the cover \( \{U_i\}_{i \in I} \) into a good cover. Since refinement replaces our sets with even smaller sets, we will still have a trivializing cover. Hence, after some refinement, we may make \( \{(U_i, s_i)\}_{i \in I} \) into a local system.

A useful, but easy result is the following fact. We shall make use of it many times throughout this paper.

Fact. If \( \{(U_i, s_i)\}_{i \in I} \) is a local system and \( g_{ij} \) are the transition functions, then if \( U_i \cap U_j \cap U_k \neq \emptyset \), we have

\[
g_{ik} = g_{ij}g_{jk}.
\]

We call equation (4) the cocycle condition.

Just as we can piece together a manifold from local data, we can also piece a line bundle back together from a collection of transition functions. In a sense there is a one-to-one correspondence between transition functions and line bundles, see [7] for a more precise formulation of this result.

Proposition 8. Let \( \{U_i\}_{i \in I} \) be a good cover of \( M \) and suppose for each \( i, j \in I \) such that \( U_i \cap U_j \neq \emptyset \), there exists smooth \( g_{ij} : U_i \cap U_j \to \mathbb{C}^* \) satisfying the cocycle condition of equation (4). Then there exists a line bundle \( L \to M \) with local system \( \{(U_i, s_i)\}_{i \in I} \) such that \( g_{ij} \) are the transition functions.
Proof. We will use the good cover \( \{ U_i \}_{i \in I} \) and the \( g_{ij} \) to construct a \( \mathbb{C}^* \)-principal bundle, then use that to produce a line bundle which matches our requirements.

Let

\[
W = \bigsqcup_{i \in I} U_i \times \mathbb{C}^*.
\]

Give \( W \) the trivial action by \( \mathbb{C}^* \): if \((x, \mu) \in U_i \times \mathbb{C}^* \) and \( \lambda \in \mathbb{C}^* \), define

\[
(x, \mu) \cdot \lambda := (x, \mu \lambda).
\]

We impose equivalence relation on \( W \): if \((x, \mu) \in U_i \cap U_j \times \mathbb{C}^* \) then

\[
(x, \mu) \sim (x, g_{ij}(x) \mu).
\]

Define \( P = W/\sim \) and let \( Q : W \to P \) be the natural quotient map.

It’s easy to see that the action of \( \mathbb{C}^* \) descends to \( P \) and that this action is free. Further, if we restrict the quotient map \( Q : W \to P \) to \( Q_i : U_i \times \mathbb{C}^* \to P \), then this map is a homeomorphism. Hence, we may use the smooth structure on \( W \) to give \( P \) a smooth structure.

Finally, if we equip \( P \) with the projection

\[
\sigma : P \to M; \quad Q_i(x, \mu) \mapsto x, \quad x \in U_i.
\]

then \( P \) is a \( \mathbb{C}^* \)-principal bundle. See [7] for a more complete discussion of this construction.

Note that by construction, if \( x \in U_i \cap U_j \), then

\[
Q_j(x, \mu) \cdot g_{ij}(x) = Q_j(x, \mu g_{ij}(x)) = Q_i(x, \mu).
\]

Now, let’s construct our line bundle. Let

\[
L = (P \times \mathbb{C})/\sim
\]

where \((p, z) \sim (p \cdot \lambda, \lambda^{-1} z)\). Give it projection

\[
\pi([p, z]) := \sigma(p).
\]

It’s an easy exercise to show that \( \pi \) is well-defined. Further, if we fix \( x \in M \), we can give \( L_x \) a vector space structure as follows. If \([p, z], [q, w] \in \pi^{-1}(x)\), then there exists \( \lambda \in \mathbb{C}^* \) such that \( q = p \cdot \lambda \). Define

\[
[p, z] + [q, w] := [p, z + \lambda w].
\]

Again, another easy exercise to show that this addition is well-defined. For scalar multiplication, simply define

\[
\lambda[p, z] := [p, \lambda z], \quad \lambda \in \mathbb{C}.
\]

To show that \( L \) is a smooth manifold and that \( L \xrightarrow{\pi} M \) is locally trivial amounts to showing that the local trivialization which defines \( P \) induces a local
trivialization on $L$. See [8] for a more complete discussion of obtaining a line bundle from a $\mathbb{C}^*$-principal bundle.

Finally, we need our local sections. For each $i \in I$, define

$$s_i(x) = [Q_i(x, 1), 1] \in L.$$ 

By construction, $s_i$ is a smooth non-vanishing section for all $i$. To see that that $g_{ij}$ are the transition functions, let $x \in U_i \cap U_j$. Then

$$s_j(x) \cdot g_{ij}(x) = [Q_j(x, 1), g_{ij}(x)] = [Q_j(x, 1)g_{ij}(x), 1] = [Q_i(x, 1), 1] = s_i(x).$$

This completes the proof.

**Lemma 4.** Let $M$ be a manifold and $L = M \times \mathbb{C}$ the trivial bundle. If $s \in \Gamma(M, \mathbb{C}^*)$ and $\alpha \in \Omega^1(M) \otimes \mathbb{C}$, then there exists a unique covariant derivative $\nabla$ on $L$ such that for all local vector fields $\xi$ on $M$,

$$\alpha(\xi) = \frac{\nabla_\xi s}{2\pi is}.$$ 

**Proof.** Let $r = (\text{Id}_M, f)$ be a section of $L$. We’ve shown already in a previous example that

$$\nabla_\xi r(x) = (x, \xi_x f)$$

is a covariant derivative.

Thus, for our non-vanishing section $s \in \Gamma(M, L^+)$, written $s = (\text{Id}_M, f)$, if we define the complex 1-form

$$A = 2\pi i \alpha - \frac{df}{f},$$

then $\nabla = \nabla' + A$ is also a covariant derivative. We then simply compute for a local vector field $\xi$ on $M$:

$$\nabla_\xi s(x) = \left(x, \xi_x f + f(x) \left[2\pi i \alpha_x(\xi_x) - \frac{\xi_x f}{f}\right]\right) = (x, 2\pi i f(x) \alpha_x(\xi_x)).$$

Hence,

$$\nabla_\xi s = 2\pi i \alpha(\xi)s$$

and we’re done on existence. Uniqueness easily follows.

**Theorem 5.** Let $L \rightarrow M$ be a line bundle, $\{(U_i, s_i)\}_{i \in I}$ a local system with transition functions $g_{ij}$, and $\alpha_i \in \Omega^1(U_i) \otimes \mathbb{C}$ a collection of complex 1-forms so that

$$\alpha_i - \alpha_j = \frac{1}{2\pi i} \frac{dg_{ij}}{g_{ij}}.$$ 

Then there exists a unique covariant derivative $\nabla$ such that if $\varphi$ is the corresponding connection 1-form and $R$ the curvature, we have

(i) $\alpha_i = s_i^* \varphi$
(ii) $2\pi id\alpha_i = R|_{U_i}$.

Proof. On $U_i$, by the previous proposition, there exists a unique covariant derivative $\nabla_i$ on $U_i$ such that

$$\alpha_i = \frac{\nabla_i s_i}{2\pi i s_i}.$$ 

We will show that $\nabla_i = \nabla_j$ on $U_i \cap U_j$, which will then imply they piece together to form a globally defined covariant derivative. Suffices to show that

$$\nabla_j s_i = \nabla_i s_i.$$

Fix a local vector field $\xi$. Then we have

$$(\nabla_j)\xi s_i = (\nabla_j)(g_{ij}s_j)$$

$$= (\xi g_{ij})s_j + g_{ij}(\nabla_j)\xi s_j$$

$$= dg_{ij}(\xi)s_j + 2\pi ig_{ij} \alpha_j(\xi)s_j$$

$$= 2\pi i \left( \frac{1}{2\pi i} \frac{dg_{ij}(\xi)}{g_{ij}} + \alpha_j(\xi) \right) s_i$$

$$= 2\pi i \alpha_j(\xi) s_i$$

$$= (\nabla_i)\xi s_i.$$ 

Now, let $\varphi$ be the corresponding connection 1-form. Then we have by previous theorem

$$\nabla_\xi s = 2\pi i s^* \varphi(\xi)s$$

for every nonvanishing local section $s$ of $L$. But this implies on $U_i$

$$\alpha_i = \frac{\nabla s_i}{2\pi i s_i} = s^* \varphi.$$ 

It’s then a simple matter of applying Corollary 5 to obtain

$$R|_{U_i} = 2\pi is^* d\varphi = 2\pi id\alpha_i.$$ 

This completes the proof. \qed
4 Existence of a Prequantization

Before we prove the main theorem, there is the pesky matter of the definition of an integral cohomology class. We will need some Čech cohomology and a characterization of integrality given by Kostant [1]. This tangent will be supplied by [7] and [10].

4.1 A Little Čech Cohomology

The main uses of Čech cohomology for us will be the fact that we can compute cohomology with coefficients in any ring, so in particular \( \mathbb{Z} \). And, as we shall see, Čech cohomology is built into line bundles via local systems.

As with any cohomology theory, we need cochains.

**Definition 24.** Let \( \mathcal{U} = \{U_i\}_{i \in I} \) be a cover of a topological space \( X \) and \( R \) a ring. For all \( i_0, \ldots, i_p \in I \) with \( U_{i_0} \cap \cdots \cap U_{i_p} \neq \emptyset \), let

\[
U_{i_0 \cdots i_p} := U_{i_0} \cap \cdots \cap U_{i_p}.
\]

A Čech \( p \)-cochain \( \mu \) with respect to \( \mathcal{U} \) is a collection of constant functions \( \mu_{i_0 \cdots i_p} : U_{i_0 \cdots i_p} \to R \) for all \( i_0, \ldots, i_p \in I \) with \( U_{i_0 \cdots i_p} \neq \emptyset \). Let \( \check{C}^p(\mathcal{U}, R) \) denote the collection of all Čech \( p \)-cochains.

**Fact.** \( \check{C}^p(\mathcal{U}, R) \) is an abelian group for all \( p \).

To get a cohomology theory, we now need a differential.

**Definition 25.** Let \( \mu \in \check{C}^p(\mathcal{U}, R) \). Define

\[
(\delta \mu)_{i_0 \cdots i_{p+1}} := \sum_{j=0}^{p+1} (-1)^j \mu_{i_0 \cdots i_{j-1}i_{j+1} \cdots i_p}.
\]

In order \( \delta \) to make sense as a differential, we need it to be an abelian group homomorphism and it needs to square to 0. Thankfully, as the following fact shows, this is the case.

**Fact.**

(i) \( \delta : \check{C}^p(\mathcal{U}, R) \to \check{C}^{p+1}(\mathcal{U}, R) \) is an abelian group homomorphism.

(ii) \( \delta^2 = 0 \).

**Definition 26.** Define the Čech \( p \)-cocycles and \( p \)-coboundaries, respectively, by

\[
\check{Z}^p(\mathcal{U}, R) := \{ \mu \in \check{C}^p(\mathcal{U}, R) \mid \delta \mu = 0 \}.
\]

\[
\check{B}^p(\mathcal{U}, R) := \{ \mu \in \check{C}^p(\mathcal{U}, R) \mid \exists \eta \in \check{C}^{p-1}(\mathcal{U}, R) : \mu = \delta \eta \}.
\]

Define the \( p \)-th Čech cohomology group of \( X \) with respect to \( \mathcal{U} \) with coefficients in \( R \) by

\[
\check{H}^p(\mathcal{U}, R) = \check{Z}^p(\mathcal{U}, R) / \check{B}^p(\mathcal{U}, R).
\]
It might seem like the cohomology will depend on the choice of an open cover. Thankfully, in the case of a smooth manifold and coefficients in \( \mathbb{R} \), this is not the case.

**Theorem 6.** If \( M \) is a manifold and \( \mathcal{U} \) is a good cover of \( M \), then for all \( p \)

\[
\check{H}^p(\mathcal{U}; \mathbb{R}) \cong H^p(M).
\]

**Proof.** See Bott and Tu \[10\] \( \blacksquare \)

To finish off this section, we define what it means that a closed 2-form is integral.

**Fact.** Let \( \mathcal{U} \) be an open cover of \( X \), let \( R \) and \( S \) two rings, and \( \varphi : R \to S \) an abelian group homorphism. Then there is an induced group homomorphism

\[
\check{H}^p(\mathcal{U}, R) \to \check{H}^p(\mathcal{U}, S)
\]

for all \( p \).

**Proof.** Fix \( p \) and let \( \mu \in \check{C}^p(\mathcal{U}, R) \). Define \( \varphi(\mu) \) by

\[
\varphi(\mu)_{i_0 \ldots i_p} : U_{i_0 \ldots i_p} \to S; \quad x \mapsto \varphi(\mu_{i_0 \ldots i_p}(x)).
\]

This clearly induces a map of abelian groups

\[
\varphi : \check{C}^p(\mathcal{U}, R) \to \check{C}^p(\mathcal{U}, S).
\]

To show that this induces a homomorphism at the level of cohomology, we show \( \delta \varphi = \varphi \delta \). To do this, let \( \mu \in \check{C}^p(\mathcal{U}, R) \). Then we have

\[
(\varphi \delta \mu)_{i_0 \ldots i_{p+1}} = \varphi \left( \sum_{j=0}^{p+1} (-1)^j \mu_{i_0 \ldots i_{j-1} i_{j+1} \ldots i_p} \right)
\]

\[
= \sum_{j=0}^{p+1} (-1)^j \varphi(\mu)_{i_0 \ldots i_{j-1} i_{j+1} \ldots i_p}
\]

\[
= \delta \varphi(\mu).
\]

\( \blacksquare \)

**Definition 27.** Let \( \mathcal{U} \) be a good cover of the manifold \( M \) and \( \gamma_\mathcal{U} : \check{H}^2(\mathcal{U}; \mathbb{Z}) \to H^2(M) \) be the composition of the maps

\[
\check{H}^2(\mathcal{U}; \mathbb{Z}) \to \check{H}^2(\mathcal{U}; \mathbb{R}) \to \check{H}^2(M).
\]

Then we say \( [\omega] \in H^2(M) \) is integral if it lies in the image of \( \gamma_\mathcal{U} \).

This definition depended on the good cover chosen. However, as a corollary to the above theorem, we can see that it does not.
Corollary 7. If $\mathcal{U}$ and $\mathcal{V}$ are two good covers of $M$, then $\text{Im}(\gamma_{\mathcal{U}}) = \text{Im}(\gamma_{\mathcal{V}})$.

Proof. See Bott and Tu [10] \hfill \Box

Remark

Let $\omega \in \Omega^2(M)$ be a closed 2-form and $\{U_i\}_{i \in I}$ a good cover of $M$. We can construct a Čech 2-cocycle from $\omega$ as follows.

Since $\omega$ is closed and since $U_i$ is simply connected, there exists $\alpha_i \in \Omega^1(U_i)$ such that on $U_i$
$$d\alpha_i = \omega.$$ It’s easy to see then that if $U_i \cap U_j \neq \emptyset$, then $d(\alpha_i - \alpha_j) = 0$. Hence, there exists $f \in C^\infty(U_i \cap U_j)$ such that $\alpha_i - \alpha_j = df_{ij}$.

Doing a similar computation on $U_{ijk}$ can conclude
$$d(-f_{ij} + f_{ik} - f_{jk}) = 0$$ And hence there exists a constant function $\mu_{ijk}$ on $U_i \cap U_j \cap U_k$ such that
$$-f_{ij} + f_{ik} - f_{jk} = \mu_{ijk}.$$ It’s an easy computation to see that $\mu_{ijk}$ is indeed a Čech 2-cocycle.

Definition 28. Call the $\mu_{ijk}$ constructed in the above remark the Čech cocycle of $[\omega]$ with respect to the cover $\{U_i\}_{i \in I}$.

Theorem 7. $[\omega]$ is integral $\iff$ the Čech cocycles of $[\omega]$, $\mu_{ijk}$, are constant integer functions for all $i, j, k$ and any choice of good cover.

Proof. See Kostant [1] \hfill \Box

4.2 Main Theorem

At last, we have built up enough theory to prove the main result. The proof here is the one that can be found in Kostant's original paper [1].

Theorem 8. Let $M$ be a smooth manifold and $\omega \in \Omega^2(M)$ a closed 2-form. Then there exists a Hermitian line bundle with compatible covariant derivative $(L, \langle \cdot, \cdot \rangle, \nabla)$ over $M$ with curvature $2\pi i \omega \iff [\omega]$ is integral.

Proof. $\Rightarrow$: Suppose $(L, \langle \cdot, \cdot \rangle, \nabla)$ is Hermitian line bundle with compatible covariant derivative over $M$ with $2\pi i \omega$ the curvature.

If $\varphi$ is the corresponding connection 1-form, we have $d\varphi = \pi^* \omega$, where $\pi : L \to M$ is the line bundle projection.

Let $\{ (U_i, s_i) \}_{i \in I}$ be a local system, $g_{ij}$ the transition functions. Since we have a Hermitian structure, may pick the $s_i$ so that $\langle s_i, s_i \rangle = 1$ and hence $|g_{ij}| = 1$. 34
Define $\alpha_i = s_i^* \varphi$. Then since $\pi \circ s_i = \text{Id}_{U_i}$, we have on $U_i$

$$d\alpha_i = d(s_i^* \varphi) = s_i^* d\varphi = s_i^* \pi^* \omega = \omega.$$  

Furthermore, since $s_i = g_{ij} s_j$ and since for all local vector fields $\xi$ we have

$$\nabla_{\xi} s_i = 2\pi i \alpha_i(\xi)s_i, \quad \nabla_{\xi} s_j = 2\pi i \alpha_j(\xi)s_j,$$

one can easily show that

$$\alpha_i - \alpha_j = \frac{1}{2\pi i} \log g_{ij},$$

we obtain there is some integer valued $\mu_{ijk} : U_i \cap U_j \cap U_k \to \mathbb{Z}$ such that

$$-f_{ij} + f_{ik} - f_{jk} = \mu_{ijk}.$$  

Since $\mu_{ijk}$ is continuous and $U_i \cap U_j \cap U_k$ is connected, we must have that $\mu_{ijk}$ is constant.

Furthermore, since we now have

$$\alpha_i - \alpha_j = df_{ij}$$

we conclude that $\mu_{ijk}$ is a Cech cocycle of $[\omega]$ with respect to the cover $\{U_i\}_{i \in I}$. Since the $\mu_{ijk}$ are constant integer functions, we conclude by Theorem 7 that $[\omega]$ is integral.

$\Leftarrow$: Suppose $[\omega]$ is integral. Let $\{U_i\}_{i \in I}$ be a good cover. On $U_i$ there exists $\alpha_i \in \Omega^1(U_i)$ such that

$$\omega = d\alpha_i.$$  

On $U_i \cap U_j$, there exists $f_{ij} \in C^\infty(U_i \cap U_j)$ such that

$$\alpha_i - \alpha_j = df_{ij}.$$  

Define $g_{ij} = e^{2\pi i f_{ij}}$. Then the $g_{ij}$ satisfy the cocycle condition in equation (4) and

$$\alpha_i - \alpha_j = \frac{1}{2\pi i} \log g_{ij}.$$  

Hence by Theorem 5, there exists a line bundle with covariant derivative $(L, \nabla)$ over $M$ such that the $g_{ij}$ are the transition functions of the local
system \( \{(U_i, s_i)\}_{i \in I} \). Furthermore, if \( \varphi \) is the corresponding connection 1-form on \( L^+ \), we have
\[
\alpha_i = s_i^* \varphi
\]
Note that \( s_i^* d \varphi = d \alpha_i = \omega \). Hence, \( 2 \pi \iota \omega \) is the curvature form for \( \nabla \).
All that’s left is to demonstrate a Hermitian form that is compatible with \( \nabla \). To do this, let \( i \in I \) and \( z, w \in U_i \). Define
\[
\langle z, w \rangle_i := \frac{z \overline{w}}{s_i s_i^*}.
\]
We show the \( \langle , \rangle_i \) agree on overlaps and hence define a Hermitian form on \( L \).
Since \( |g_{ij}| = 1 \), we have if \( z, w \in \pi^{-1}(U_i \cap U_j) \)
\[
\langle z, w \rangle_i := \frac{z \overline{w}}{s_i s_i^*} = \frac{1}{|g_{ij}|^2} \frac{z \overline{w}}{s_j s_j^*} = \langle z, w \rangle_j.
\]
Hence the \( \langle , \rangle_i \) piece together to a globally defined Hermitian form \( \langle , \rangle \). We now just have to show that \( \nabla \) and \( \langle , \rangle \) are compatible. Suffices to show for all local vector fields \( \xi \)
\[
\langle \nabla_{\xi} s_i, s_i \rangle + \langle s_i, \nabla_{\xi} s_i \rangle = 0. \tag{5}
\]
To see this, let \( x \in U_i \) and let \( r, t \) be local sections of \( L \) in a neighbourhood of \( x \). Then there exists smooth complex valued functions \( F, G \) near \( x \) such that
\[
r = Fs_i \quad t = Gs_i.
\]
Then given that the equation above is true, we have for any local vector field \( \xi \) defined around \( x \)
\[
\xi_x \langle r, t \rangle = \xi_x (F \overline{G} \langle s_i, s_i \rangle) = \xi_x (FG).
\]
On the other hand,
\[
\langle \nabla_{\xi} r, t \rangle + \langle r, \nabla_{\xi} t \rangle = (\langle \xi F \rangle s_i + F \nabla_{\xi} s_i, Gs_i) + (Fs_i, \langle \xi G \rangle s_i + G \nabla_{\xi} s_i, t)
\]
\[
= (\xi F + \xi G) \langle s_i, s_i \rangle + F \overline{G} (\langle \nabla_{\xi} s_i, s_i \rangle + \langle s_i, \nabla_{\xi} s_i \rangle)
\]
\[
= \xi (F \overline{G}).
\]
Now all that’s left is to show equation (5) holds. Let \( i \in I \), then we compute,
\[
\langle \nabla_{\xi} s_i, s_i \rangle = \langle 2 \pi i \alpha_i (\xi) s_i, s_i \rangle = 2 \pi i \alpha_i (\xi).
\]
Hence, since \( \alpha_i \) is real valued
\[
\langle \nabla_{\xi} s_i, s_i \rangle + \langle s_i, \nabla_{\xi} s_i \rangle = 2 \pi i - 2 \pi i = 0.
\]
This completes the proof and this paper. \( \square \)
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