Remarks on BEC on Graphs

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Abstract: We consider Bose–Einstein condensation (BEC) on graphs with transient adjacency matrix. We prove the equivalence of BEC and non-factoriality of a quasi-free state. Moreover, quasi-free states exhibiting BEC decompose into generalized coherent states. We review necessary and sufficient conditions that a quasi-free state is faithful, factor, and pure and quasi-free states are quasi-equivalent, including the paper of H. Araki and M. Shiraishi (1971/72), H. Araki (1971/72), and H. Araki and S. Yamagami (1982). Using their formats and results, we prove necessary and sufficient conditions that a generalized coherent state is faithful, factor, and pure and generalized coherent states are quasi-equivalent as well.

Keywords: CCR algebra, generalized coherent state, quasi-equivalence, Bose–Einstein condensation.

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1 Introduction

In \cite{14}, T. Matsui studied the condition for Bose–Einstein condensation (BEC for short.) in terms of the random walk on a graph. In \cite{6}, F. Fidaleo, D. Guido, and T. Isola, in \cite{7} and \cite{8}, F. Fidaleo studied some spectral properties of the adjacency matrix of graphs and BEC. They obtained the criterion for BEC on graphs. In \cite{11}, J. T. Lewis and J. V. Pulé obtained the non-factoriality of a quasi-free state exhibiting BEC in $L^2(\mathbb{R}^3)$ case. However, in case of graphs, BEC implies non-factoriality of a quasi-free states exhibiting BEC is not clear, thus, we study a quasi-free state exhibiting BEC and prove the equivalence of the occurrence of BEC and non-factoriality of a quasi-free state. Moreover, we give factor decomposition of quasi-free states exhibiting BEC into generalized coherent states which are factor and mutually disjoint (Theorem 4.9.).

Generalized coherent states are generalization of coherent states in the following sense. Let $\mathfrak{h}$ be a Hilbert space and $\sigma$ be the symplectic form defined by $\sigma(f, g) := \text{Im} \langle f, g \rangle_\mathfrak{h}$.

In mathematics, a coherent state $\varphi$ on the Weyl CCR algebra $\mathcal{W}(\mathfrak{h}, \sigma)$ is given by

$$\varphi(W(f)) = \exp\{-\|f\|_\mathfrak{h}^2/4 + i\text{Re}\lambda(f)\} \quad (1.1)$$

for each $f \in \mathfrak{h}$, where $W(f)$, $f \in \mathfrak{h}$, are unitaries which generate $\mathcal{W}(\mathfrak{h}, \sigma)$ and $\lambda$ is a $\mathbb{C}$-linear functional on $\mathfrak{h}$. (See \cite{10} Theorem 3.1.) A state $\varphi$ on $\mathcal{W}(\mathfrak{h}, \sigma)$ is a generalized coherent state, if there exists a positive semi-definite sesquilinear form $S$ on $\mathfrak{h} \times \mathfrak{h}$ and an $\mathbb{R}$-linear functional $\lambda : \mathfrak{h} \to \mathbb{R}$ such that

$$\varphi(W(f)) = \exp\{-S(f, f)/4 + i\lambda(f)\}, \quad f \in \mathfrak{h}. \quad (1.2)$$

In Section 2, we review works of H. Araki and M. Shiraishi \cite{1}, H. Araki \cite{2}, and H. Araki and S. Yamagami \cite{3}. In \cite{1}, H. Araki and M. Shiraishi and in \cite{2}, H. Araki considered quasi-free states on the CCR algebra and obtained a condition that a quasi-free state is faithful, factor, and pure. In \cite{3}, H. Araki and S. Yamagami got necessary and sufficient conditions that quasi-free states are quasi-equivalent. In \cite{12}, J. Manuceau and A. Verbeure and in \cite{13}, J. Manuceau, F. Rocca, and D. Testard obtained a condition that a quasi-free state on the Weyl CCR algebra is pure and factor. In \cite{18}, A. van Daele obtained conditions of quasi-equivalence of quasi-free states on the Weyl CCR algebra as well. To consider conditions of factoriality, purity and faithfulness of a generalized coherent state and conditions of quasi-equivalence of generalized coherent states in a unified framework, we use formats in \cite{1}, \cite{2}, and \cite{3}.

In Section 3, we consider generalized coherent states on the Weyl CCR algebra. We prove necessary and sufficient conditions that a generalized coherent state is faithful, factor, and pure and necessary and sufficient conditions that generalized coherent states are quasi-equivalent. Moreover, we give an explicit form of factor decomposition of non-factor generalized coherent state. In \cite{9}, R. Honegger considered the decomposition of gauge-invariant quasi-free states. In the present paper, we only assume that a state on the Weyl CCR algebra is quasi-free or generalized coherent.

In Section 4, we review works of F. Fidaleo \cite{8} and consider the non-factoriality of quasi-free states with BEC. We show that a quasi-free state exhibiting BEC is non-factor and such state decomposes into generalized coherent states which are mutually disjoint. In \cite{15}, J. V. Pulé, A. F. Verbeure, and V. A. Zagrebnov considered inhomogeneous BEC on $L^2(\mathbb{R}^\nu)$, $\nu \geq 1$, and obtained that the
occurrence of BEC implies spontaneous symmetry breaking and an equilibrium state exhibiting BEC decompose into periodic states. In [14], T. Matsui obtained that the occurrence of BEC implies spontaneous symmetry breaking in case of graphs with some assumptions (See [14 Assumption 1.1.]) as well. In the present paper, generalized coherent states appeared in factor decomposition of a quasi-free state are not periodic. Thus, we give another decomposition of a quasi-free state exhibiting BEC.

2 Preliminaries

In this section, we review works of H. Araki and M. Shiraishi [1], H. Araki [2], and H. Araki and S. Yamagami [3]. In [1], H. Araki and M. Shiraishi and in [2], H. Araki considered quasi-free states on the CCR algebra and obtained necessary and sufficient conditions that a quasi-free state is factor, pure, and faithful. In [3], H. Araki and S. Yamagami obtained necessary and sufficient conditions that quasi-free states are quasi-equivalent. We use facts presented in this section to consider necessary and sufficient conditions that a generalized coherent state is factor, pure, and faithful and generalized coherent states are quasi-equivalent and to prove non-factoriality of quasi-free states exhibiting BEC.

2.1 Some Properties of a Quasi-free state

Let $\tilde{K}$ be a $\mathbb{C}$-linear space and $\gamma_{\tilde{K}} : \tilde{K} \times \tilde{K} \to \mathbb{C}$ be a sesquilinear form. Let $\Gamma_{\tilde{K}}$ be an anti-linear involution ($\Gamma_{\tilde{K}}^2 = 1$) satisfying $\gamma_{\tilde{K}}(\Gamma_{\tilde{K}} f, \Gamma_{\tilde{K}} g) = -\gamma_{\tilde{K}}(g, f)$.

A CCR algebra $A(\tilde{K}, \gamma_{\tilde{K}}, \Gamma_{\tilde{K}})$ over $(\tilde{K}, \gamma_{\tilde{K}}, \Gamma_{\tilde{K}})$ is the quotient of the complex $\ast$-algebra generated by $B(f), f \in \tilde{K}$, its adjoint $B(f)^\ast$, $f \in \tilde{K}$ and an identity over the following relations:

1. $B(f)$ is complex linear in $f$,
2. $B(f)^\ast B(g) - B(g)B(f)^\ast = \gamma_{\tilde{K}}(f, g)1$,
3. $B(\Gamma_{\tilde{K}} f)^\ast = B(f)$.

Any linear operator $P$ on $\tilde{K}$ satisfying

1. $P^2 = P$,
2. $\gamma_{\tilde{K}}(Pf, g) > 0$, if $Pf \neq 0$,
3. $\gamma_{\tilde{K}}(Pf, g) = \gamma_{\tilde{K}}(f, Pg)$,
4. $\Gamma_{\tilde{K}} P \Gamma_{\tilde{K}} = 1 - P$,

is called a basis projection.

Let $\mathfrak{h}$ be a complex pre-Hilbert space. A CCR (\ast)-algebra $A_{\text{CCR}}(\mathfrak{h})$ over $\mathfrak{h}$ is the quotient of the $\ast$-algebra generated by $a^\dagger(f)$ and $a(f)$, $f \in \mathfrak{h}$, and an identity by the following relations:

1. $a^\dagger(f)$ is complex linear in $f$,
2. $(a^\dagger(f))^\ast = a(f)$,
3. \([a(f), a^\dagger(g)] = (f, g)_h\) and \([a^\dagger(f), a(g)] = 0 = [a(f), a(g)]\).

Let \(P\) be a basis projection. Then the mapping \(\alpha(P)\) from \(A(\hat{K}, \gamma_{\hat{K}}, \Gamma_{\hat{K}})\) to \(\mathcal{A}_{\text{CCR}}(P\hat{K})\) defined by

\[
\alpha(P)(B(f_1)B(f_2)\cdots B(f_n)) = (\alpha(P)B(f_1))(\alpha(P)B(f_2))\cdots (\alpha(P)B(f_n))
\]

is a *-isomorphism of \(A(\hat{K}, \gamma_{\hat{K}}, \Gamma_{\hat{K}})\) onto \(\mathcal{A}_{\text{CCR}}(P\hat{K})\).

Let \(\mathcal{A}\) be a *-algebra with identity. A linear functional \(\varphi\) on \(\mathcal{A}\) is said to be state, if \(\varphi\) satisfies \(\varphi(A^*A) \geq 0, A \in \mathcal{A}\), and \(\varphi(1) = 1\). For a state \(\varphi\) on \(\mathcal{A}\), we have the GNS-representation space \((\mathcal{H}_\varphi, \pi_\varphi, \xi_\varphi)\) associated with \(\varphi\). We set \(\text{Re} \hat{K} := \{ f \in \hat{K} \mid \Gamma_{\hat{K}}f = f \}\). Then \(f \in \text{Re} \hat{K}\) if and only if \(B(f)^* = B(f)\).

On \(\text{Re} \hat{K}\), the operators \(B(f), f \in \text{Re} \hat{K}\), correspond to field operators. Moreover, \(a^\dagger(f)\) and \(a(f)\) correspond to the creation operators and the annihilation operators. We give examples of \(\hat{K}, \gamma_{\hat{K}}, \) and \(\Gamma_{\hat{K}}\) in Section 3 and 4.

Let \(\varphi\) be a state on \(A(\hat{K}, \gamma_{\hat{K}}, \Gamma_{\hat{K}})\) such that \(\pi_\varphi(B(f))\) is essentially self-adjoint for all \(f \in \text{Re} \hat{K}\). Then we put \(W_\varphi(f) = \exp(i\pi_\varphi(B(f)))\), \(f \in \text{Re} \hat{K}\). Such state \(\varphi\) is said to be regular if \(W_\varphi(f)\) satisfies the Weyl–Segal relations:

\[
W_\varphi(f)W_\varphi(g) = \exp(-\gamma_{\hat{K}}(f, g)/2)W_\varphi(f + g), \quad f, g \in \text{Re} \hat{K}.
\]

(2.2)

In general, the Weyl CCR algebra is the universal C*-algebra generated by unitaries \(W(f)\), \(f \in \text{Re} \hat{K}\), which satisfy (2.2) and we denote \(W(\text{Re} \hat{K}, \gamma_{\hat{K}})\) the Weyl CCR algebra. (See also [5], Theorem 5.2.8.)

A state \(\varphi\) on \(A(\hat{K}, \gamma_{\hat{K}}, \Gamma_{\hat{K}})\) is said to be quasi-free, if \(\varphi\) satisfies the following equations:

\[
\varphi(B(f_1)\cdots B(f_{2n-1})) = 0,
\]

\[
\varphi(B(f_1)\cdots B(f_{2n})) = \prod_{j=1}^n \varphi(B(f_{s(j)}))B(f_{s(j)+n})),
\]

where \(n \in \mathbb{N}\) and the sum is over all permutations \(s\) satisfying \(s(1) < s(2) < \cdots < s(n), s(j) < s(j+n), j = 1, 2, \ldots, n\). For any quasi-free state \(\varphi\) over \(A(\hat{K}, \gamma_{\hat{K}}, \Gamma_{\hat{K}})\), the sesquilinear form \(S_{\hat{K}} : \hat{K} \times \hat{K} \to \mathbb{C}\) defined by

\[
S_{\hat{K}}(f, g) = \varphi(B(f)^*B(g)), \quad f, g \in \hat{K}
\]

(2.4)

is positive semi-definite and satisfies

\[
\gamma_{\hat{K}}(f, g) = S_{\hat{K}}(f, g) - S_{\hat{K}}(\Gamma g, \Gamma f), \quad f, g \in \hat{K}.
\]

(2.5)

(See [1], Lemma 3.2.) Any quasi-free state on \(A(\hat{K}, \gamma_{\hat{K}}, \Gamma_{\hat{K}})\) determines the positive semi-definite sesquilinear form \(S_{\hat{K}}\), which satisfies the equation (2.5).

Conversely, for any positive semi-definite sesquilinear form \(S_{\hat{K}}\) on \(\hat{K} \times \hat{K}\) satisfying (2.5), there exists a unique quasi-free state \(\varphi\) satisfying (2.4) and \(\varphi\) is regular. (See [1], Lemma 3.5.) Thus, there exists a one-to-one correspondence between a positive semi-definite sesquilinear form \(S_{\hat{K}}\) on \(\hat{K} \times \hat{K}\) and a quasi-free state \(\varphi\) on \(A(\hat{K}, \gamma_{\hat{K}}, \Gamma_{\hat{K}})\). We denote the quasi-free state on \(A(\hat{K}, \gamma_{\hat{K}}, \Gamma_{\hat{K}})\) determined by a positive semi-definite sesquilinear form \(S_{\hat{K}}\) by \(\varphi_S\) defined in
We define the positive semi-definite form $(\cdot,\cdot)_{S}$ on $\tilde{K} \times \tilde{K}$ by the following equation:

$$(f,g)_{S} := S_{\tilde{K}}(f,g) + S_{\tilde{K}}(\Gamma_{\tilde{K}}g,\Gamma_{\tilde{K}}f), \quad f, g \in \tilde{K}. \quad (2.6)$$

We set $N_{S} := \{ f \in \tilde{K} \mid \|f\|_{S} = 0 \}$, where $\|f\|_{S} = (f,f)_{S}^{1/2}$. We denote the completion of $\tilde{K}/N_{S}$ with respect to the norm $\|\cdot\|_{S}$ by $K$. Since $S_{\tilde{K}}(f,f) \leq \|f\|_{S}^{2}, \|\Gamma_{\tilde{K}}(f,f)\| \leq \|f\|_{S}^{2}$, and $\|\Gamma_{\tilde{K}}f\|_{S} = \|f\|_{S}$ for any $f \in \tilde{K}$, we can extend the sesquilinear form $S_{\tilde{K}}$ and $\gamma_{\tilde{K}}$ to the sesquilinear form on $K \times K$ and the operator $\Gamma_{\tilde{K}}$ to the operator on $K$. We denote the extensions of $S_{\tilde{K}}, \gamma_{\tilde{K}}$, and $\Gamma_{\tilde{K}}$ by $S_{K}, \gamma_{K}$, and $\Gamma_{K}$, respectively. We define the bounded operators $S_{K}$ and $\gamma_{K}$ on $K$ by the following equations:

$$(\xi, S_{K}\eta)_{S} = S_{K}(\xi, \eta), \quad (\xi, \gamma_{K}\eta)_{S} = \gamma_{K}(\xi, \eta), \quad \xi, \eta \in K. \quad (2.7)$$

A quasi-free state $\varphi_{S}$ is said to be Fock type if $N_{S} = \{ 0 \}$ and the spectrum of the operator $S_{K}$ defined in (2.7) is contained in $(0,1/2,1]$. For any positive semi-definite sesquilinear form $S_{\tilde{K}}$ on $\tilde{K} \times \tilde{K}$, we can construct a Fock type state as follows. Let $L = K \oplus K$. For $\xi_{1}, \xi_{2}, \eta_{1}, \eta_{2} \in K$, we set

$$\gamma_{L}(\xi_{1} \oplus \xi_{2}, \eta_{1} \oplus \eta_{2}) = (\xi_{1}, \gamma_{K}\eta_{1}) - (\xi_{2}, \gamma_{K}\eta_{2})_{S}, \quad (2.9)$$

$$\Gamma_{L} = \Gamma_{K} \oplus \Gamma_{K}, \quad (2.10)$$

$$(\xi_{1} \oplus \xi_{2}, \eta_{1} \oplus \eta_{2})_{L} = (\xi_{1}, \eta_{1}) + (\xi_{2}, \eta_{2}) + 2(\xi_{1}, S_{K}^{1/2}(1 - S_{K})^{1/2}\eta_{2})_{S} + 2(\xi_{2}, S_{K}^{1/2}(1 - S_{K})^{1/2}\eta_{1})_{S}. \quad (2.11)$$

Let $N_{L} = \{ \xi \in \tilde{L} \mid (\xi, \xi)_{L} = 0 \}$. Then we denote the completion of $\tilde{L}/N_{L}$ with respect to the norm $\|\cdot\|_{L}$ by $L$. We define the bounded operators $\gamma_{L}$ and $\Pi_{L}$ on $L$ satisfying

$$(\xi, \gamma_{L}\eta)_{L} = \gamma_{L}(\xi, \eta), \quad \xi, \eta \in L. \quad (2.12)$$

$$\Pi_{L} = \frac{1}{2}(1 + \gamma_{L}). \quad (2.13)$$

Then the spectrum of $\Pi_{L}$ on $L$ is contained in $(0,1/2,1]$. (See Corollary 6.2.) Moreover the following three lemmas hold:

**Lemma 2.1.** [Corollary 6.2.] *The map $f \in \tilde{K} \mapsto [f] \in L$, where $[f] := (f \oplus 0)+N_{L}$, induces a $*$-homomorphism $\alpha_{L}$ of $A(\tilde{K}, \gamma_{\tilde{K}}, \Gamma_{\tilde{K}})$ into $A(L, \gamma_{L}, \Gamma_{L})$. The restriction of a Fock type state $\varphi_{L}$ of $A(L, \gamma_{L}, \Gamma_{L})$ to $\alpha_{L}(A(\tilde{K}, \gamma_{\tilde{K}}, \Gamma_{\tilde{K}}))$ gives a quasi-free state $\varphi_{S}$ of $A(\tilde{K}, \gamma_{\tilde{K}}, \Gamma_{\tilde{K}})$ through $\varphi_{L}(\alpha_{L}(A)) = \varphi_{S}(A)."*

**Lemma 2.2.** [Lemma 2.3.] *Let $R_{S}$ be the von Neumann algebra generated by spectral projections of all $\pi_{L}(B(f)), f \in \text{Re}\tilde{K}$, on the GNS representation space $(\Omega_{L}, \pi_{L}, \xi_{L})$ of $A(L, \gamma_{L}, \Gamma_{L})$ associated with $\varphi_{L}$. Then the following conditions are equivalent:

1. The GNS cyclic vector $\xi_{L}$ is cyclic for $R_{S}$.
2. The GNS cyclic vector $\xi_{L}$ is separating for $R_{S}$.
3. The operator $S_{K}$ on $K$ does not have an eigenvalue 0."*
4. The operator $S_K$ on $K$ does not have an eigenvalue 1.

**Lemma 2.3.** [2, Lemma 2.4.] The center of $R_S$ is generated by $\exp(i\pi_{PL}(B(h)))$, $h \in \Re(\mathbb{E}_0 K \odot 0)$, where $\mathbb{E}_0$ is the spectral projection of $S_K$ for $1/2$ and $(\mathbb{E}_0 K \odot 0)$ is the closure of $\mathbb{E}_0 K \odot 0$ with respect to the norm $\|\cdot\|_L$. In particular, $R_S$ is factor if and only if $\mathbb{K}_0 = \mathbb{E}_0 K = \{0\}$.

## 2.2 Quasi-equivalence of Quasi-free states

We recall the definitions of quasi-equivalence of representations and states.

**Definition 2.4.** [2, Definition 6.1.] Let $\pi_{S_1}$ and $\pi_{S_2}$ be representations associated with quasi-free states $\varphi_{S_1}$ and $\varphi_{S_2}$ on $A(\gamma_{\tilde{K}}, \gamma_{\tilde{K}}^*)$, respectively. The representations $\pi_{S_1}$ and $\pi_{S_2}$ are said to be quasi-equivalent, if there exists an isomorphism $\tau$ from $R_{S_1} = \{W_{S_1}(f) \mid f \in \Re\tilde{K}\}$ onto $R_{S_2} = \{W_{S_2}(f) \mid f \in \Re\tilde{K}\}$ such that

$$
\tau(W_{S_1}(f)) = W_{S_2}(f), \quad f \in \Re\tilde{K},
$$

(2.14)

where $W_{S_1}(f) = \exp(i\pi_{S_1}(B(f)))$ and $W_{S_2}(f) = \exp(i\pi_{S_2}(B(f)))$. Let $\varphi_{S_1}$ and $\varphi_{S_2}$ be quasi-free states on $A(\gamma_{\tilde{K}}, \gamma_{\tilde{K}}^*)$. The states $\varphi_{S_1}$ and $\varphi_{S_2}$ are said to be quasi-equivalent, if for each GNS-representations $(\mathcal{H}_{S_i}, \pi_{S_i}), i = 1, 2$ associated with $\varphi_{S_i}$, respectively, are quasi-equivalent.

This definition is equivalent to the definition of quasi-equivalence of states on a C*-algebra. (See [3, Definition 2.25.] and [4, Theorem 2.26].)

Let $\varphi_{S_1}$ and $\varphi_{S_2}$ be quasi-free states on $A(\gamma_{\tilde{K}}, \gamma_{\tilde{K}}^*)$. In [3], H. Araki and S. Yamagami showed the following theorem:

**Theorem 2.5.** [3, Theorem] Two quasi-free states $\varphi_{S_1}$ and $\varphi_{S_2}$ on $A(\gamma_{\tilde{K}}, \gamma_{\tilde{K}}^*)$ are quasi-equivalent if and only if the following conditions hold:

1. The topologies induced by $\|\cdot\|_{S_1}$ and $\|\cdot\|_{S_2}$ are equal.

2. Let $K$ be the completion of $\tilde{K}$ with respect to the topology $\|\cdot\|_{S_i}$ or $\|\cdot\|_{S_2}$. Then $S_1^{1/2} - S_2^{1/2}$ is in the Hilbert–Schmidt class on $K$, where the $S_1$ and $S_2$ are operators on $K$ defined in (2.7).

## 3 Generalized Coherent states

In this section, we consider generalized coherent states on the Weyl CCR algebra. Using facts in the previous section, we give necessary and sufficient conditions that a generalized coherent state is factor, pure, and faithful and generalized coherent states are quasi-equivalent as well.

### 3.1 The Weyl CCR algebra

Let $V$ be an $\mathbb{R}$-linear space with a symplectic form $\sigma : V \times V \to \mathbb{R}$, i.e., $\sigma$ is a bilinear form on $V$ and satisfy the following relations:

$$
\sigma(f, g) = -\sigma(g, f), \quad f, g \in V.
$$

(3.1)
We assume that there exists an operator $J$ on $V$ with the properties
\[ \sigma(Jf, g) = -\sigma(f, Jg), \quad J^2 = -1, \]
then $V$ is a $\mathbb{C}$-linear space with scalar multiplication defined by
\[ (c_1 + ic_2)f = c_1 f + c_2 Jf, \quad c_1, c_2 \in \mathbb{R}, \quad f \in V. \] (3.3)

Then we define the complexification $V^C$ of $V$ by $[3.3]$. We set $(f + ig)^* = f - ig$ for $f, g \in V$. We fix a symplectic space $(V, \sigma)$ with an operator $J$ satisfying (3.2). We puts $\tilde{K} = V^C$.

\[ \Gamma_{\tilde{K}} f = f^*, \quad f \in \tilde{K}, \]
\[ \gamma_{\tilde{K}}(f, g) = \frac{1}{2}\{\sigma(f, g^*) - \sigma(g, f^*)\}, \quad f, g \in \tilde{K}. \] (3.4)

Then on the GNS-representation space $(S, \lambda, \varphi)$ associated with a regular state $\varphi$ on $A(\tilde{K}, \gamma_{\tilde{K}})$, we define the annihilation operators $a(f)$ and creation operators $a^\dagger(f)$ on $\mathcal{H}_\varphi$ by the following equation:
\[ a_\varphi(f) := \{\pi_\varphi(B(f)) + i\pi_\varphi(B(\text{i}f))\}/\sqrt{2}, \quad a_\varphi^\dagger(f) := \{\pi_\varphi(B(f)) - i\pi_\varphi(B(\text{i}f))\}/\sqrt{2}, \]
for any $f \in \text{Re}\tilde{K}$. In this section, we identify the Weyl CCR algebra $W(V, \sigma)$ with a regular state $\varphi$ and $A(\tilde{K}, \gamma_{\tilde{K}}, \Gamma_{\tilde{K}})$ with $\varphi$, where $\tilde{K}, \gamma_{\tilde{K}}$ and $\Gamma_{\tilde{K}}$ defined in (3.4).

### 3.2 Generalized coherent states

For an $\mathbb{R}$-linear functional $\lambda : V \to \mathbb{R}$, there exists a $*$-automorphism $\tau_\lambda$ on $W(V, \sigma)$ defined by
\[ \tau_\lambda(W(f)) := e^{\text{i}\lambda(f)}W(f), \quad f \in V. \] (3.6)

Let $\varphi_S$ be a quasi-free state on $W(V, \sigma)$. Then we define the generalized coherent state $\varphi_{S, \lambda}$ by the following equation:
\[ \varphi_{S, \lambda}(W(f)) := \varphi_S \circ \tau_\lambda(W(f)) = e^{\text{i}\lambda(f)}\varphi_S(W(f)), \quad f \in V. \] (3.7)

We sets $N_S = \{ f \in V^C \mid \| f \|_S = 0 \}$, where $\| \cdot \|_S = (\cdot, \cdot)_S^{1/2}$ is the semi-norm defined in (2.6) and $V_S^C$ is the completion of $V^C/N_S$ by the norm $\| \cdot \|_S$. We denote the GNS-representation space with respect to $\varphi_S$ and $\varphi_{S, \lambda}$ by $(S_S, \pi_S, \xi_S)$ and $(S_{S, \lambda}, \pi_{S, \lambda}, \xi_{S, \lambda})$, respectively.

**Lemma 3.1.** Let $\varphi_S$ and $\varphi_{S, \lambda}$ be a quasi-free state and a generalized coherent state on $W(V, \sigma)$, respectively. Then
\[ R_S = R_{S, \lambda}, \] (3.8)
where $R_S$ and $R_{S, \lambda}$ is the von Neumann algebra generated by $\{ \pi_S(W(f)) \mid f \in V \}$ and $\{ \pi_{S, \lambda}(W(f)) \mid f \in V \}$, respectively.
Proof. Since $\varphi_S$ is regular, there exist self-adjoint operators $\Psi_S(f), f \in V$ such that $\pi_S(W(f)) = \exp(i\Psi_S(f))$. By definition of generalized coherent states, we have $\pi_{S,\lambda} = \exp(i\lambda f)\pi_S(W(f))$ and $(\delta_{S,\lambda}, \pi_{S,\lambda}, \xi_{S,\lambda}) = (\delta_S, \pi_S, \xi_S)$. On $H_S$, we have
\[
\{ \pi_S(W(f)) | f \in V \}'' = \{ e^{i\lambda f} \pi_S(W(f)) | f \in V \}'' = \{ \pi_{S,\lambda}(W(f)) | f \in V \}''.
\]
Thus, $R_S = R_{S,\lambda}$ by the double commutant theorem. ■

Theorem 3.2. Let $\varphi_{S,\lambda}$ be a generalized coherent state on $W(V, \sigma)$. Then $\varphi_{S,\lambda}$ is faithful if and only if $S$ does not have an eigenvalue 0 on $V^c_S$.

Proof. Note that $\varphi_S$ and $\varphi_{S,\lambda}$ has the same GNS cyclic vector space $\xi_{\Pi_L}$. By Lemma 2.2, $\varphi_{S,\lambda}$ is faithful if and only if $S$ does not have an eigenvalue 0 on $V^c_S$.

Theorem 3.3. Let $\varphi_{S,\lambda}$ be a generalized coherent state on $W(V, \sigma)$. Then $\varphi_{S,\lambda}$ is pure if and only if $S$ does not have an eigenvalue 1/2 on $V^c_S$.

Proof. By Lemma 2.3 and Lemma 3.1, we have the statement. ■

Theorem 3.4. Let $(V, \sigma)$ be a non-degenerate symplectic space and $\varphi_{S,\lambda}$ be a generalized coherent state on $W(V, \sigma)$. Then $\varphi_{S,\lambda}$ is pure if and only if $S$ is a basis projection.

Proof. If $S$ is a basis projection, then by Lemma 3.1 and [11, Lemma 5.5.] $\varphi_S$ is pure.

We use the notation in Section 2. Thus, $\tilde{K} = V^c$, $K = V^c_S$, and $L$ is the completion of $V^c_S \oplus V^c_S/N_L$ with respect to the norm $\| \cdot \|_L$ defined in (2.11). If $\varphi_{S,\lambda}$ is pure, then by Theorem 3.3, $S$ does not have an eigenvalue 1/2. Then $\Pi_L$ defined in (2.13) does not have an eigenvalue 1/2 because the eigenspace of $\Pi_L$ with 1/2 is the completion of the set $\{ f \oplus f | f \in E_0K \}$ with respect to the norm $\| \cdot \|_L$, where $E_0$ is the spectral projection of $S$ onto $\ker(S - 1/2)$. (See also the proof of (4) of [11, Lemma 6.1.].) Thus, $\Pi_L$ is a basis projection. Using the notation of [11, Lemma 5.5.], we have $R_S = R_{\Pi_L}(H_1)$, with $H_1 = [\Re \tilde{K}] \oplus 0 \subset L$ and $H_2 = [\Re \tilde{K}] \oplus L \oplus 0 \oplus 0$. If $\Pi_L \neq S$, then $K \neq L$. Thus, we have $R_{\Pi_L}(H_1)' = R_{\Pi_L}(H_2')$ by [11, Lemma 5.5.] and $H_1' \neq \{ 0 \}$, where $H_1'$ is the orthogonal complement with respect to the inner product $\langle \cdot, \cdot \rangle_L$ defined in (2.11). It leads $R_S' \neq C1$. It contradicts to the purity of $\varphi_S$. Thus, $S$ is a basis projection. ■

We have necessary and sufficient conditions that a generalized coherent state is faithful, factor, and pure. Next, we consider the quasi-equivalence of generalized coherent states.

Lemma 3.5. Let $\varphi_{S,\lambda}$ be a generalized coherent state on $W(V, \sigma)$. Then $f \in N_S$ if and only if $\pi_{S,\lambda}(W(f)) = e^{i\lambda f}1$.

Proof. If $f \in N_S$, then $\varphi_S(W(tf)) = 1$ for any $t \in \mathbb{R}$. Thus, by regularity of $\varphi_S$, $\pi_S(W(f)) = 1$. By definition of generalized coherent state, $\pi_{S,\lambda}(W(f)) = e^{i\lambda f}1$.

If $\pi_{S,\lambda}(W(f)) = e^{i\lambda f}1$, $f \in V$, then $\pi_S(W(f)) = 1$. Since $g^* = g$ for any $g \in V$, we have that $(f, f)_S = 0$. ■
Lemma 3.6. Let \( \varphi_{S_1, \lambda_1} \) and \( \varphi_{S_2, \lambda_2} \) be generalized coherent states on \( \mathcal{W}(V, \sigma) \). If \( \varphi_{S_1, \lambda_1} \) and \( \varphi_{S_2, \lambda_2} \) are quasi-equivalent, then \( N_{S_1} = N_{S_2} \).

Proof. Since \( \varphi_{S_1, \lambda_1} \) and \( \varphi_{S_2, \lambda_2} \) are quasi-equivalent, there exists \( \tau : \pi_{S_1, \lambda_1}(\mathcal{W}(V, \sigma))'' \to \pi_{S_2, \lambda_2}(\mathcal{W}(V, \sigma))'' \) such that

\[
\tau(\pi_{S_1, \lambda_1}(A)) = \pi_{S_2, \lambda_2}(A), \quad A \in \mathcal{W}(V, \sigma).
\]  

(3.10)

If \( N_{S_1} \neq N_{S_2} \), then there exists \( f \in V^\mathbb{C} \) such that \( f \in N_{S_1} \) and \( f \not\in N_{S_2} \). Put \( h = f + f^* \). Then \( h \in V = \text{Re}V^\mathbb{C} \) and \( h \in N_{S_1} \) and \( h \not\in N_{S_2} \). For such \( h \), we have

\[
\pi_{S_1, \lambda_1}(W(h)) = e^{i\lambda_1(h)} I
\]

by Lemma 3.5. However, we have

\[
\pi_{S_2, \lambda_2}(W(h)) = e^{i\lambda_2(h)} \pi_2(W(h)) = \tau(\pi_{S_1, \lambda_1}(W(h))) = e^{i\lambda_1(h)} I.
\]

(3.12)

It contradict to Lemma 3.5.

Theorem 3.7. Let \( \varphi_{S_1, \lambda_1} \) and \( \varphi_{S_2, \lambda_2} \) be generalized coherent states on \( \mathcal{W}(V, \sigma) \). Then \( \varphi_{S_1, \lambda_1} \) and \( \varphi_{S_2, \lambda_2} \) are quasi-equivalent if and only if the following conditions hold:

1. \( \|\cdot\|_{S_1} \) and \( \|\cdot\|_{S_2} \) induce the same topology,
2. \( S_1^{1/2} - S_2^{1/2} \) is a Hilbert–Schmidt class operator,
3. \( \lambda_1 = \lambda_2 \) on \( N_{S_1} = N_{S_2} \),
4. \( \lambda_1 - \lambda_2 \) is continuous with respect to the norm \( \|\cdot\|_{S_1} \) or \( \|\cdot\|_{S_2} \).

Proof. Assume that the topologies induced by \( \|\cdot\|_{S_1} \) and \( \|\cdot\|_{S_2} \) are equivalent, \( S_1^{1/2} - S_2^{1/2} \) is Hilbert–Schmidt class, \( \lambda_1 - \lambda_2 \) is continuous with respect to \( \|\cdot\|_{S_1} \) and \( \lambda_1 = \lambda_2 \) on \( N_{S_1} = N_{S_2} \). Then \( \varphi_{S_1} \) and \( \varphi_{S_2} \) are quasi-equivalent by [2] Theorem] and \( \varphi_{S_1, \lambda_1} \) and \( \varphi_{S_2, \lambda_2} \) are quasi-equivalent by continuity of \( \lambda_1 - \lambda_2 \) and \( \lambda_1 = \lambda_2 \) on \( N_{S_1} = N_{S_2} \).

Next, we assume that \( \varphi_{S_1, \lambda_1} \) and \( \varphi_{S_2, \lambda_2} \) are quasi-equivalent. The quasi-equivalence of \( \varphi_{S_1, \lambda_1} \) and \( \varphi_{S_2, \lambda_2} \) induces the quasi-equivalence of \( \varphi_{S_1, \lambda_1 - \lambda_2} \) and \( \varphi_{S_2} \). Put \( \lambda := \lambda_1 - \lambda_2 \). Then there exists a *-isomorphism \( \tau \) from \( \pi_{S_1, \lambda}(\mathcal{W}(V, \sigma))'' \) onto \( \pi_{S_2}(\mathcal{W}(V, \sigma))'' \) such that

\[
\tau(\pi_{S_1, \lambda}(A)) = \pi_{S_2}(A), \quad A \in \mathcal{W}(V, \sigma).
\]

(3.13)

For any \( f \in V \),

\[
\exp(i\lambda(f) - S_1(f, f)/2) = \langle \xi_{S_1}, \tau^{-1}(\pi_{S_2}(W(f)))\xi_{S_1} \rangle = \langle \xi_{S_1}, \tau^{-1}(\pi_{S_2}(W(f)))\xi_{S_1} \rangle
\]

(3.14)

is \( \|\cdot\|_{S_2} \)-continuous in \( f \in V \). Thus, \( \lambda \) and \( S_1 \) are \( \|\cdot\|_{S_2} \)-continuous. By symmetry, \( \lambda \) and \( S_2 \) are \( \|\cdot\|_{S_1} \)-continuous as well. By Lemma 3.5, \( N_S := N_{S_1} = N_{S_2} \). If \( \lambda \neq 0 \) on \( N_{S} \), then there exists \( f \in N_{S}\setminus\{0\} \) such that \( \lambda(f) \neq 0 \). If \( \lambda(f) = 2n\pi \) for some \( n \in \mathbb{Z} \), then we replace \( f \) by \( f/\pi \). For such \( f \), we have

\[
e^{i\lambda(f)} = \tau(\pi_{S_1, \lambda}(W(f))) = \pi_{S_2}(W(f)) = I
\]

(3.15)
by Lemma [3.5]. It contradicts to the quasi-equivalence of \( \varphi_{S_1, \lambda} \) and \( \varphi_{S_2} \). Thus, \( \lambda = 0 \) on \( N_S \). Let \( \tau' \) be the map from \( \pi_{S_1, \lambda}(W(V, \sigma)) \) to \( \pi_{S_1}(W(V, \sigma)) \) defined by
\[
\tau'(\pi_{S_1, \lambda}(A)) = \pi_{S_1}(A), \quad A \in W(V, \sigma).
\]
(3.16)
Since \( \lambda \) is continuous with respect to the norm \( \| \cdot \|_{S_1} \) and \( \lambda = 0 \) on \( N_S \), then we can extend \( \tau' \) to a map from \( \pi_{S_1, \lambda}(W(V, \sigma))' \) onto \( \pi_{S_1}(W(V, \sigma))'' \). Then \( \tau' \) induce the quasi-equivalence of \( \varphi_{S_1, \lambda} \) and \( \varphi_{S_1} \). Thus, \( \varphi_{S_1} \) and \( \varphi_{S_2} \) are quasi-equivalent and by Theorem [2.5] we have the statement. ■

**Remark 3.8.** In [20], S. Yamagami obtained quasi-equivalence conditions of (generalized) coherent states in terms of the transition amplitude. For applications to concrete models Hilbert-Schmidt conditions in Theorem [3.7] are easier to handle. Let \( \varphi_{S_1, \lambda_1} \) and \( \varphi_{S_2, \lambda_2} \) be generalized coherent states on the Weyl CCR algebra \( W(V, \sigma) \). Assume that \( \varphi_{S_1} \) and \( \varphi_{S_2} \) are quasi-equivalent. If \( \lambda_1 - \lambda_2 \) is not continuous in \( \| \cdot \|_{S_1} \) or \( \| \cdot \|_{S_2} \) or \( \lambda_1 \neq \lambda_2 \), then the transition amplitude \( \varphi_{S_1, \lambda_1}^{1/2} \varphi_{S_2, \lambda_2}^{1/2} = 0 \), where \( \varphi_{S_1}^{1/2} \) and \( \varphi_{S_2}^{1/2} \) is GNS-vector in the universal representation space \( L^2(W(V, \sigma))^{**} \). (See [20] Theorem 5.3.,)

Factor decompositions of quasi-free states are given in [9], [16] and [19], e.t.c.. For the convenience of the reader, we give an explicit form of factor decomposition of a non-factor generalized coherent state. We recall the definition of the disjointness of states. (See also [4] Definition 4.1.20, and [4] Lemma 4.2.8.)

**Definition 3.9.** Let \( \varphi_1 \) and \( \varphi_2 \) be positive linear functionals on a C*-algebra \( \mathcal{A} \). The positive linear functionals \( \varphi_1 \) and \( \varphi_2 \) are said to be disjoint, if for \( \omega = \varphi_1 + \varphi_2 \), there is a projection \( P \in \pi_{\omega}(\mathcal{A})'' \cap \pi_{\omega}(\mathcal{A})' \) such that
\[
\varphi_1(A) = (\xi_\omega, P \pi_{\omega}(A) \xi_\omega), \quad \varphi_2(A) = (\xi_\omega, (1 - P) \pi_{\omega}(A) \xi_\omega), \quad A \in \mathcal{A},
\]
(3.17)
where \( \pi_{\omega} \) is the GNS-representation and \( \xi_\omega \) is the GNS-cyclic vector associated with \( \omega \).

Note that factor representations are either quasi-equivalent or disjoint. (See e.g. [4] Proposition 2.4.22, [4] Theorem 2.4.26 (1), and [4] Proposition 2.4.27.)

**Theorem 3.10.** Let \( \varphi_{S, \lambda} \) be a generalized coherent state on \( W(V, \sigma) \). If \( \varphi_{S, \lambda} \) is non-factor, then there exists a probability measure \( \mu \) on \( \mathbb{R}^{2l} \) and \( \varphi_{S, \lambda} \) has factor decomposition of the form
\[
\varphi_{S, \lambda} = \int_{\mathbb{R}^{2l}} \varphi_{SE_0^{x, y}, \rho + \lambda} d\mu(x),
\]
(3.18)
where \( \varphi_{SE_0^{x, y}, \rho + \lambda}(W(f)) = \exp(-S(E_0^{x} f, E_0^{y} f)/4 + i x \cdot \rho(f) + i \lambda(f)) \) and \( \rho(f) = (\text{Re}(x, f), \text{Im}(x, f))_{k \in \mathbb{R}^{2l}} \). Moreover, \( \varphi_{SE_0^{x, y}, \rho + \lambda} \) and \( \varphi_{SE_0^{x, y}, \rho + \lambda} \) are disjoint unless \( x \neq y \), \( x, y \in \mathbb{R}^{2l} \).

**Proof.** If a generalized coherent state \( \varphi_{S, \lambda} \) on \( W(V, \sigma) \) is non-factor, then on \( V_S^C \), \( S \) has the spectral decomposition
\[
Sf = SE_0^{x} f + \frac{1}{2} \sum_{k \in \mathbb{R}^{2l}} (\epsilon_k, f) E_k, \quad f \in V_S^C,
\]
(3.19)
where $E_0$ is the spectral projection of $S$ with an eigenvalue $1/2$, $I$ is an index set such that $|I| = \dim \ker(S - 1/2)$, and $\{e_k\}_{k \in I}$ is an orthonormal basis for $\ker(S - 1/2)$. Thus, for any $W(f), f \in V$, we have

$$
\varphi_{S, \lambda}(W(f)) = \exp(- \frac{S(E_0^+ f, E_0^+ f)}{4} + i \lambda(f)) \exp(- \frac{\sum_{k \in I} |(e_k, f) S|^2}{8}).
$$

(3.20)

By a theorem of Bochner–Minlos (See e.g. [17, Theorem 2.2]), there exists a probability measure $\mu$ on $\mathbb{R}^{2I}$ such that

$$
\exp(- \frac{\sum_k |(e_k, f) S|^2}{8}) = \int_{\mathbb{R}^{2I}} \exp(i x \cdot \rho(f)) d\mu(x),
$$

(3.21)

where $\rho(f) = (\text{Re}(e_k, f) S, \text{Im}(e_k, f) S)_{k \in I} \in \mathbb{R}^{2I}$. For $\varphi_{SE_0^+, \lambda}$, we have $N_{SE_0^+} = E_0 V^C \neq \{0\}$. Since $E_0 V^C \neq \{0\}$, there exists a $f \in V$ such that $\text{Re}(e_k, f) S \neq 0$ or $\text{Im}(e_k, f) S \neq 0$. We put $f_n := E_0 f + 1/n E_0^+ f$. Then $\|f_n\|_{SE_0^+} \to 0$ and $\text{Re}(e_k, f_n) S \neq 0$ or $\text{Im}(e_k, f_n) S \neq 0$ as $n \to \infty$. Thus, the generalized coherent states $\varphi_{SE_0^+, x, \rho, \lambda}$ and $\varphi_{SE_0^+, y, \rho, \lambda}$, $x, y \in \mathbb{R}^{2I}$, are not quasi-equivalent unless $x = y$ by Theorem 3.7. Since $\|\cdot\|_S$ and $\|\cdot\|_{SE_0^+}$ induce the same topology on $V^C$ and $SE_0^+$ on $V^C$, $SE_0^+$ does not have an eigenvalue $1/2$, $\varphi_{SE_0^+, x, \rho, \lambda}$ is factor and $\varphi_{SE_0^+, y, \rho, \lambda}$ and $\varphi_{SE_0^+, z, \rho, \lambda}$ are disjoint unless $x \neq y$, $x, y \in \mathbb{R}^{2I}$.

4 BEC and Non-factor states

In this section, we consider quasi-free states on $W(\mathfrak{h}, \sigma)$, where $\mathfrak{h}$ is a pre-Hilbert space over $\mathbb{C}$ with an inner product $(\cdot, \cdot)_\mathfrak{h}$ and $\sigma(f, g) = \text{Im}(f, g)_\mathfrak{h}, f, g \in \mathfrak{h}$. We give the decomposition of quasi-free states on $W(\mathfrak{h}, \sigma)$ into generalized coherent states which are mutually disjoint.

4.1 General properties

In this subsection, we use the following notations. Let $\mathfrak{h}$ be a subspace of a Hilbert space over $\mathbb{C}$. We assume that $\mathfrak{h}$ is equipped with positive definite inner products $(\cdot, \cdot)_\mathfrak{h}$ and $(\cdot, \cdot)_0$. Let $q$ be a linear functional on $\mathfrak{h}$. We consider the quasi-free state $\varphi_{q, D}$, $D \geq 0$, on $W(\mathfrak{h}, \sigma)$ defined by

$$
\varphi_{q, D}(a^\dagger(f)a(g)) = \langle g, f \rangle_0 + Dq(g)q(f),
$$

(4.1)

where $a(f)$ and $a^\dagger(f), f \in \mathfrak{h}$, are the annihilation operators and the creation operators on the GNS representation space $\mathcal{B}_{\varphi_{q, D}}$, respectively. Note that the annihilation operators $a(f), f \in \mathfrak{h}$, and the creation operators $a^\dagger(f), f \in \mathfrak{h}$ satisfy the following equation:

$$
[a(f), a^\dagger(g)] = (f, g)_\mathfrak{h}, \quad [a(f), a(g)] = 0 = [a^\dagger(f), a^\dagger(g)], \quad f, g \in \mathfrak{h}.
$$

(4.2)

Our aim is to show that $\varphi_{q, D}$ is non-factor if $q$ is not continuous with respect to the norm $\|\cdot\|_{q, D}$ defined in (4.8) and $D > 0$, and to get factor decomposition
of $\varphi_{q,D}$, in this subsection. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis on a Hilbert space which is contained in $\mathfrak{h}$. Fix $\{e_n\}_{n\in\mathbb{N}}$. We set

$$
\mathcal{F} = \sum_{n\in\mathbb{N}} f_n e_n, \quad (4.3)
$$

for $f = \sum_{n\in\mathbb{N}} f_n e_n \in \mathfrak{h}$, where $f_n \in \mathbb{C}$, $n \in \mathbb{N}$ and $\overline{f_n}$ is the complex conjugate of $f_n$. For a linear functional $q$ and $D \geq 0$, we put $\tilde{K}_{q,D} = \mathfrak{h} \oplus \mathfrak{h}$. For $f_1, f_2, g_1, g_2 \in \mathfrak{h}$, we set

$$
\gamma_D(f_1 \oplus f_2, g_1 \oplus g_2) = \frac{1}{2} \left( \langle f_1, g_1 \rangle_{\mathfrak{h}} - \langle f_2, g_2 \rangle_{\mathfrak{h}} \right),
$$

$$
\Gamma(f_1 \oplus f_2) = \overline{f_2} \oplus \overline{f_1},
$$

$$
B(f_1 \oplus f_2) = \frac{1}{\sqrt{2}} (a^\dagger(f_1) + a(f_2)),
$$

$$
S_{q,D}(f_1 \oplus f_2, g_1 \oplus g_2) = \varphi_{q,D}(B(f_1 \oplus f_2) \ast B(g_1 \oplus g_2))
$$

$$
= \frac{1}{2} \varphi_{q,D}(a^\dagger(f_1) + a(f_2))\ast(a^\dagger(g_1) + a(g_2))
$$

$$
= \frac{1}{2} \langle f_1, g_1 \rangle_{\mathfrak{h}} + \frac{1}{2} \langle f_2, g_2 \rangle_{\mathfrak{h}} + \frac{1}{2} \langle f_1, g_2 \rangle_{\mathfrak{h}} + \frac{1}{2} \langle f_2, g_1 \rangle_{\mathfrak{h}} + D \frac{q(f_1)q(g_1)}{q(f_2)q(g_2)} + D \frac{q(f_2)q(g_2)}{q(f_2)q(g_2)} \quad (4.7)
$$

We define the inner product on $\tilde{K}_{q,D}$ by

$$
\langle f_1 \oplus f_2, g_1 \oplus g_2 \rangle_{\tilde{K}_{q,D}} = \frac{1}{2} \langle f_1, g_1 \rangle_{\mathfrak{h}} + \frac{1}{2} \langle f_2, g_2 \rangle_{\mathfrak{h}} + \langle f_1, g_1 \rangle_0 + \langle f_2, g_2 \rangle_0 + Dq(f_1)q(g_1) + Dq(f_2)q(g_2). \quad (4.8)
$$

Let $N_{K_{q,D}} = \{ f \in \tilde{K}_{q,D} \mid \|f\|_{q,D} = 0 \}$. Then we denote the completion of $\tilde{K}_{q,D}/N_{K_{q,D}}$ with respect to the norm $\|\cdot\|_{q,D}$ by $K_{q,D}$. In this case, $\|f_1 \oplus f_2\|_{q,D} = 0$ leads $f_1 = 0$ and $f_2 = 0$. Thus, $N_{K_{q,D}} = \{0\}$.

We put

$$
\langle f, g \rangle_{\mathfrak{h}} = \frac{1}{2} \langle f, g \rangle_{\mathfrak{h}} + \langle f, g \rangle_0, \quad f, g \in \mathfrak{h}, \quad (4.9)
$$

and $\|\cdot\|_{\mathfrak{h}} = \langle \cdot, \cdot \rangle_{\mathfrak{h}}^{1/2}$. We define the Hilbert space $\mathfrak{h}$ by the completion of $\mathfrak{h}$ with respect to the norm $\|\cdot\|_{\mathfrak{h}}$.

**Lemma 4.1.** The space $K_{q,D}$ has the following form:

1. If $D > 0$ and $q$ is not continuous with respect to the norm $\|\cdot\|_{\mathfrak{h}}$, then we have

$$
K_{q,D} = \mathbb{C} \oplus \mathfrak{F} \oplus \mathbb{C} \oplus \mathfrak{F}, \quad (4.10)
$$

2. If $D = 0$ or $q$ is continuous with respect to the norm $\|\cdot\|_{\mathfrak{h}}$, then we have

$$
K_{q,D} = \mathfrak{F} \oplus \mathfrak{F}. \quad (4.11)
$$

**Proof.** We consider the case of $D > 0$ and $q$ is not continuous with respect to the norm $\|\cdot\|_{\mathfrak{h}}$. It suffices to show that $\mathbb{C} \oplus \mathfrak{F} = \mathfrak{h}$, where $\mathfrak{h}$ is the completion of $\mathfrak{h}$ with respect to the norm $\|\cdot\|'$ defined by

$$
(\|f\|')^2 = \|f\|^2_{\mathfrak{h}} + D|q(f)|^2, \quad f \in \mathfrak{h} \quad (4.12)
$$

and $\|\cdot\|_{\mathfrak{h}}' = \langle \cdot, \cdot \rangle_{\mathfrak{h}}'^{1/2}$. We define the completion of $\mathfrak{h}$ with respect to the norm $\|\cdot\|'_{\mathfrak{h}}$.
We define \( \pi : \mathfrak{h} \rightarrow \mathbb{C} \oplus \mathfrak{r} \) by
\[
\pi(f) = q(f) \oplus f. \tag{4.13}
\]
Since \( q \) is not continuous, for any \( f \in \mathfrak{h} \), there exists a sequence \( f_n \) in \( \mathfrak{h} \) such that \( \lim_{n \rightarrow \infty} \|f_n - f\|_{\mathfrak{r}} = 0 \) and \( \lim_{n \rightarrow \infty} q(f_n) = 0 \). For such \( f_n \) and \( f \), we have
\[
\pi(f_n - f) \rightarrow q(f) \oplus 0, \quad \pi(f_n) \rightarrow 0 \oplus f. \tag{4.14}
\]
The case of \( D = 0 \) is clear. We assume that \( q \) is continuous with respect to the norm \( \|\cdot\|_{\mathfrak{r}} \). By continuity of \( q \), the norm \( \|\cdot\|' \), defined in (4.12), and \( \|\cdot\|_{\mathfrak{r}} \) induce the same topology. \( \blacksquare \)

**Theorem 4.2.** For a linear space \( \mathfrak{h} \) with positive definite inner products \( \langle \cdot, \cdot \rangle_{\mathfrak{h}} \) and \( \langle \cdot, \cdot \rangle_0 \), if \( D > 0 \) and \( q \) is not continuous with respect to the norm \( \|\cdot\|_{\mathfrak{r}} \), then the two-point function \( \varphi_{q,D} \) defined in (4.11) is a non-factor state on \( W(\mathfrak{h}, \sigma) \).

**Proof.** First, we consider the case of \( D > 0 \). By Lemma 2.1 and Lemma 2.2, it suffices to show that \( 1/2 \in \sigma_P(S_{q,D}) \). By Lemma 4.1, an element of \( K_{q,D} \) has the form \((a_1, f_1, a_2, f_2), a_1, a_2 \in \mathbb{C}, f_1, f_2 \in \mathfrak{r}\). For any \((a_1 \oplus f_1 \oplus a_2 \oplus f_2), (b \oplus 0 \oplus 0 \oplus 0) \in K_{q,D}, b \in \mathbb{C}, \) the operator \( S_{q,D} \) satisfies
\[
\{(a_1 \oplus f_1 \oplus a_2 \oplus f_2), S_{q,D}(b_1 \oplus 0 \oplus 0 \oplus 0)\}_{q,D} = \frac{D}{2} ab \tag{4.15}
\]
Thus, we have \( S_{q,D}(b \oplus 0 \oplus 0 \oplus 0) = 1/2(b \oplus 0 \oplus 0 \oplus 0) \) for any \( b \in \mathbb{C} \) and \( 1/2 \in \sigma_P(S_{q,D}) \). \( \blacksquare \)

**Proposition 4.3.** For a linear space \( \mathfrak{h} \) with positive definite inner products \( \langle \cdot, \cdot \rangle_{\mathfrak{h}} \) and \( \langle \cdot, \cdot \rangle_0 \), if \( D = 0 \) or \( q \) is continuous with respect to the norm \( \|\cdot\|_{\mathfrak{r}} \), the two-point function \( \varphi_{q,D} \) defined in (4.11) is a factor state on \( W(\mathfrak{h}, \sigma) \).

**Proof.** If \( q \) is continuous with respect to the norm \( \|\cdot\|_{\mathfrak{r}} \), then \( \varphi_{q,D} \) is quasi-equivalent to \( \varphi_{0,0} \) by Theorem 5.7. Thus, it suffices to show the case of \( D = 0 \). There exists the positive contraction operator \( A \) on \( \mathfrak{r} \) such that \( \langle \xi, A \eta \rangle_{\mathfrak{r}} = \langle \xi, \eta \rangle_{\mathfrak{r}}/2 \) and \( \langle \xi, (1 - A) \eta \rangle_{\mathfrak{r}} = \langle \xi, \eta \rangle_0 \), \( \xi, \eta \in \mathfrak{r} \). Then \( S_{0,0} \) has the following form:
\[
S_{0,0} = \eta_1 \eta_2 = (A + (1 - A)/2)\eta_1 \oplus \frac{1 - A}{2} \eta_2 = \frac{1 + A}{2} \eta_1 \oplus \frac{1 - A}{2} \eta_2. \tag{4.16}
\]
for \( \eta_1, \eta_2 \in \mathfrak{r} \). If \( 1/2 \in \sigma_P(S_{0,0}) \), then \( (1 + A)\eta_1 = \eta_1 \) and \( (1 - A)\eta_2 = \eta_2 \). Thus, \( \eta_1, \eta_2 \in \ker A \). Since the positive definiteness of \( \langle \cdot, \cdot \rangle_{\mathfrak{h}} \) and \( \langle \cdot, \cdot \rangle_0 \) on \( \mathfrak{h}, \mathfrak{h} \cap \ker A = \{0\} \). Thus, \( \ker A = \{0\} \) and \( \varphi_{0,0} \) is factor. \( \blacksquare \)

Next, we consider factor decomposition of \( \varphi_{q,D} \), if \( q \) is not continuous in \( \|\cdot\|_{\mathfrak{r}} \). Let \( (S_0, \pi_0, \xi_0) \) be the GNS-representation space with respect to \( \varphi_0 := \varphi_{q,0} = \varphi_{0,0} \). Since \( \varphi_0 \) is regular state on \( W(\mathfrak{h}, \sigma) \), there exist self-adjoint operators \( \Psi_0(f), f \in \mathfrak{h}, \) such that
\[
\pi_0(W(f)) = \exp(i\Psi_0(f)). \tag{4.17}
\]
Now we define the field operators \( \Psi_{s_1,s_2}(f), s_1, s_2 \in \mathbb{R}, f \in \mathfrak{h}, \) on \( \mathcal{S}_0 \) by
\[
\Psi_{s_1,s_2}(f) = \Psi_0(f) + s_1 D^{1/2} \Re q(f) \mathbbm{1} + s_2 D^{1/2} \Im q(f) \mathbbm{1}, \quad f \in \mathfrak{h}. \tag{4.18}
\]
Let $\pi_{s_1,s_2}$ be the representation of $W(\mathfrak{h},\sigma)$ on $\mathcal{H}_0$ defined by

$$\pi_{s_1,s_2}(W(f)) = \exp(i\Psi_{s_1,s_2}(f)), \quad f \in \mathfrak{h}. \quad (4.19)$$

Using the $\pi_{s_1,s_2}$, we define the state $\varphi_{s_1,s_2}$ on $W(\mathfrak{h},\sigma)$ by

$$\varphi_{s_1,s_2}(A) = \langle \xi_0, \pi_{s_1,s_2}(A)\xi_0 \rangle, \quad A \in W(\mathfrak{h},\sigma). \quad (4.20)$$

Then we have the following theorem.

**Theorem 4.4.** If $q$ is not continuous in $\|\cdot\|_2$, then for each $s_1, s_2, t_1, t_2 \in \mathbb{R}$, $\varphi_{s_1,s_2}$ and $\varphi_{t_1,t_2}$ are factor and disjoint unless $t_1 = s_1$ and $t_2 = s_2$.

**Proof.** By Lemma 3.1 and Proposition 4.3, $\varphi_{s_1,s_2}$ and $\varphi_{t_1,t_2}$ are factor. Since $q$ is not continuous with respect to the norm, $\varphi_{s_1,s_2}$ and $\varphi_{t_1,t_2}$ are disjoint unless $t_1 = s_1$ and $t_2 = s_2$ by Theorem 3.1.

Finally, we obtain factor decomposition of $\varphi_{q,D}$.

**Theorem 4.5.** If $q$ is not continuous in $\|\cdot\|_2$, then for any $D > 0$, factor decomposition of $\varphi_{q,D}$ defined in (4.1) is given by

$$\varphi_{q,D} = \frac{1}{2\pi} \int_{\mathbb{R}^2} \varphi_{s_1,s_2} e^{-\frac{s_1^2+s_2^2}{2}} ds_1 ds_2. \quad (4.21)$$

**Proof.** By Theorem 3.10, we are done.

### 4.2 On graphs

In this subsection, let $X = (VX, EX)$ be an undirected graph, where $VX$ is the set of all vertices in $X$ and $EX$ is the set of all edges in $X$. Two vertices $x, y \in VX$ are said to be adjacent if there exists an edge $(x, y) \in EX$ joining $x$ and $y$, and we write $x \sim y$. We denote the set of all the edges connecting $x$ with $y$ by $E_{x,y}$. Since the graph is undirected, $E_{x,y} = E_{y,x}$. Let $\ell^2(VX)$ be the set of all square summable sequence labeled by the vertices in $VX$. Let $A_X$ be the adjacency operator of $X$ defined by

$$\langle \delta_x, A_X \delta_y \rangle = |E_{x,y}|, \quad x, y \in VX. \quad (4.22)$$

In addition, for any $x \in VX$, we set the degree of $x$ by $\deg(x)$ and

$$\deg := \sup_{x \in VX} \deg(x). \quad (4.23)$$

We assume that $X$ is connected, countable and $\deg < \infty$. Then, the adjacency operator $A_X$ acting on $\ell^2(VX)$ is bounded. If for any $\delta_x, x \in VX$, $A_X$ satisfies the condition

$$\lim_{\lambda \to \|A_X\|} \langle \delta_x, (\lambda I - A_X)^{-1} \delta_x \rangle < \infty, \quad (4.24)$$

then $A_X$ is said to be transient. Let $H$ be the Hamiltonian on $\ell^2(VX)$ defined by $H := \|A_X\|I - A_X$.

A bounded operator $B$ on $\ell^2(VX)$ is called positivity preserving if $B_{x,y} := \langle \delta_x, B \delta_y \rangle \geq 0$ for any $x, y \in VX$. A sequence $\{v(x) | x \in VX\}$ is called a Perron–Frobenius weight for $B$ if it has positive entries and

$$\sum_{x \in VX} B_{x,y} v(y) = \|B\| v(x) \quad (4.25)$$
for any \( x \in VX \).

In [8], F. Fidaleo considered BEC on graphs and showed the following two results.

**Proposition 4.6.** [8 Proposition 4.1.] Let \( A_X \) be the adjacency operator of \( X \) on \( \ell^2(VX) \) and \( H \) be the Hamiltonian defined by \( H = \|A_X\|1 - A_X \). Let \( \mathfrak{h} \) be a subspace of \( \ell^2(VX) \) satisfying the following three conditions: For each \( \beta > 0 \),

1. \( e^{itH} \mathfrak{h} = \mathfrak{h} \), \( t \in \mathbb{R} \);
2. For each entire function \( f \), \( f(H)\mathfrak{h} \subseteq \mathcal{D}((e^{\beta H} - 1)^{-1/2}) \);
3. \( \sum_{x \in VX} |\langle (f(H)u)(x)|v(x)\rangle| < \infty \), and \( \langle f(H)u, v \rangle = \overline{f(0)}\langle u, v \rangle \), where \( v \) is a Perron–Frobenius weight for \( A_X \).

Then for \( D \geq 0 \), the two–point function

\[
\varphi_D(a^*(f_1)) = \langle (e^{H} - 1)^{-1/2} f_2, f_1 \rangle + D(f_2, v)\langle v, f_1 \rangle
\]

(4.26)
satisfies the KMS condition at inverse temperature \( \beta > 0 \) on the Weyl CCR algebra \( \mathcal{W}(\mathfrak{h}, \sigma) \) with respect to the dynamics generated by the Bogoliubov transformations

\[
f \in \mathfrak{h} \mapsto e^{itH} f, \quad t \in \mathbb{R}.
\]

(4.27)

By the above proposition and [14 Proposition 1.1.], we are said to be BEC occur if the case of \( D > 0 \) and BEC does not occur if the case of \( D = 0 \).

**Theorem 4.7.** [8 Theorem 4.5.] Suppose that \( A_X \) is transient. Let \( \mathfrak{h} \) be the subspace of \( \ell^2(VX) \) defined by

\[
\mathfrak{h}_1 = \{ e^{tH}\delta_x \mid t \in \mathbb{R}, x \in VX \}.
\]

Then \( \mathfrak{h}_1 \) satisfies the conditions 1, 2, and 3 in Proposition 4.6. Thus, for \( \mathfrak{h}_1 \) and any \( D \geq 0 \), the two-point function given in (4.26) defines KMS state on the Weyl CCR algebra \( \mathcal{W}(\mathfrak{h}_1, \sigma) \).

We give another example of \( \mathfrak{h} \). Let \( \mathcal{P}(\mathbb{C}) \) be the set of all polynomial functions on \( \mathbb{C} \). Let \( \mathfrak{h}_2 \) be the subspace defined by

\[
\mathfrak{h}_2 = \left\{ \int_{\mathbb{R}} p(t)e^{-\frac{b}{a}t}e^{itH}\delta_x dt \mid p \in \mathcal{P}(\mathbb{C}), a \in \mathbb{R}, b > 0, x \in VX \right\}.
\]

(4.29)

**Lemma 4.8.** The space \( \mathfrak{h}_2 \) satisfies the following conditions;

1'. \( e^{itH}\mathfrak{h}_2 = \mathfrak{h}_2 \), \( t \in \mathbb{R} \);
2'. \( e^{\beta H}\mathfrak{h}_2 \subseteq \mathcal{D}((e^{\beta H} - 1)^{-1/2}) \);
3'. \( \sum_{x \in VG} |\langle (e^{\beta H}u)(x)\rangle| < \infty \), and \( \langle e^{\beta H}u, v \rangle = \langle u, v \rangle \), \( u \in \mathfrak{h}_2 \).

**Proof.** The condition 1’, \( e^{itH}\mathfrak{h}_2 \subseteq \mathfrak{h}_2 \) is clear. Now we prove the condition 2’, \( e^{\beta H}\mathfrak{h}_2 \subseteq \mathcal{D}((e^{\beta H} - 1)^{-1/2}) \). Note that \( (e^{\beta x} - 1)^{-1} - (\beta x)^{-1} \) is continuous on \([0, \infty)\). Thus, it enough to show that \( e^{\beta H}\mathfrak{h}_2 \subseteq \mathcal{D}(H^{-1/2}) \). Since \( A_X \) is transient
and \( p(t)e^{-(t-a)^2/b} \) is a rapidly decreasing function on \( \mathbb{R} \), for a generator of \( \mathfrak{h}_2 \),
\[
\int_{\mathbb{R}} p(t)e^{-(t-a)^2/b} e^{itH} \delta_x dt,
\]
we have
\[
\left\langle (\lambda I - A_X)^{-1} e^{\beta H} \int_{\mathbb{R}} p(t)e^{-(t-a)^2/b} e^{itH} \delta_x dt, e^{\beta H} \int_{\mathbb{R}} p(t)e^{-(t-a)^2/b} e^{itH} \delta_x dt \right\rangle
\]
\[
= \left| \int_{\mathbb{R}} \int_{\mathbb{R}} p(t)p(s)e^{-(t-s)^2/s} e^{-(t-a)^2/b} \left\langle (\lambda I - A_X)^{-1} e^{\beta H} e^{itH} \delta_x, e^{\beta H} e^{isH} \delta_x \right\rangle dt ds \right|
\]
\[
\leq C_1 e^{4\beta \|A_X\|} \left\langle (\lambda I - A_X)^{-1} \delta_x, \delta_x \right\rangle \leq C_1 e^{2\beta \|A_X\|} \left\langle (\|A_X\| I - A_X)^{-1} \delta_x, \delta_x \right\rangle < \infty,
\]
(4.30)

where \( C_1 \) is a positive constant satisfying
\[
\int_{\mathbb{R}} \left| \int_{\mathbb{R}} p(t)p(s)e^{-(t-s)^2/s} e^{-(t-a)^2/b} \right| dt ds < C_1.
\]
(4.31)

Next, we show that \( \sup_{n \in \mathbb{N}} \sum_{x \in \Lambda_n} |(e^{\beta H} u)(x)| v(x) < \infty, \ u \in \mathfrak{h}_2 \), where \( \Lambda_n \) is a finite subgraph of \( X \) such that \( \Lambda_n \searrow X \). Let \( C_R \) be a circle centered at the origin with radius \( R > \|A_X\| \). We have
\[
\left| \left\langle e^{\beta H} \int_{\mathbb{R}} p(t)e^{-(t-a)^2/b} e^{itH} \delta_x dt, \delta_y \right\rangle \right| \leq \int_{\mathbb{R}} |p(t)|e^{-(t-a)^2/b} \left\langle e^{\beta H} e^{itH} \delta_x, \delta_y \right\rangle dt
\]
\[
= \int_{\mathbb{R}} |p(t)|e^{-(t-a)^2/b} \left| \frac{1}{2\pi i} \int_{C_R} e^{\beta z e^{itz}} \langle (z I - A_X)^{-1} \delta_x, \delta_y \rangle dz \right| dt
\]
\[
\leq R e^{\beta R} \int_{\mathbb{R}} |p(t)|e^{-(t-a)^2/b} e^{tr} dt \left\langle (R I - A_X)^{-1} \delta_x, \delta_y \right\rangle \leq C_2 \left\langle (R I - A_X)^{-1} \delta_x, \delta_y \right\rangle,
\]
(4.32)

for any \( x, y \in V_X \), where \( C_2 \) is a positive constant satisfying
\[
R e^{\beta R} \int_{\mathbb{R}} |p(t)|e^{-(t-a)^2/b} e^{tr} dt < C_2.
\]
(4.33)

By the above inequality (4.32), we get
\[
\sum_{y \in V_{\Lambda_n}} \left| \left\langle e^{\beta H} \int_{\mathbb{R}} p(t)e^{-(t-a)^2/b} e^{itH} \delta_x dt, \delta_y \right\rangle \right| v(y) \leq C_2 \sum_{y \in V_{\Lambda_n}} \left\langle (R I - A_X)^{-1} \delta_x, \delta_y \right\rangle v(y)
\]
\[
= C_2 \left\langle (R I - A_X)^{-1} \delta_x, v \right| _{V_{\Lambda_n}} \leq C_2 \sum_{k=0}^{\infty} \frac{\langle A_k \delta_x, v \rangle _{V_{\Lambda_n}}}{R^{k+1}}
\]
\[
\leq C_2 (R - \|A_X\|)^{-1} v(x).
\]
(4.34)

Finally, we show the latter part of the condition 3'. For any \( f \in \mathfrak{h}_2 \), by definition of \( v \),
\[
\langle e^{\beta H} f, v \rangle = (f, v).
\]
(4.35)

Thus, we are done. \( \blacksquare \)
Theorem 4.9. Suppose that the adjacency operator $A_X$ of a graph $X$ is transient. For $D > 0$, the two-point function $\varphi_D$ defined in (4.26) is a non-factor KMS state on $\mathcal{W}(h_1, \sigma)$ or $\mathcal{W}(h_2, \sigma)$. Moreover, we have factor decomposition of $\varphi_D$ into extremal KMS states

$$\varphi_D = \frac{1}{2\pi} \int_{\mathbb{R}^2} \varphi_{s_1, s_2} e^{-\frac{s_1^2 + s_2^2}{2}} ds_1 ds_2. \quad (4.36)$$

Proof. Since $\langle \cdot | (e^{\beta H} + 1)(e^{\beta H} - 1)^{-1} \cdot \rangle$ is positive definite inner product on $h_1$ and $h_2$, it suffice to show that $\langle v, f \rangle$, $f \in h_1$ or $f \in h_2$ is not continuous with respect to the norm $\langle \cdot | (e^{\beta H} + 1)(e^{\beta H} - 1)^{-1} \cdot \rangle$ by Theorem 4.4 and 4.5. Let $p_n$ be the polynomial defined by

$$p_n(x) = \sum_{k=0}^{n} \frac{(-nx)^k}{k!}. \quad (4.37)$$

For any $f \in h_1$, $(p_n(H) - 1)f \in h_1$. Put $f_n = (p_n(H) - 1)f$. Then

$$\langle f_n - f, (e^{\beta H} + 1)(e^{\beta H} - 1)^{-1}(f_n - f) \rangle \to 0, \quad (n \to \infty) \quad (4.38)$$

and

$$\langle v, f_n \rangle = 0 \quad (4.39)$$

for any $n \in \mathbb{N}$. Thus, we have that $\langle v, \cdot \rangle$ is not continuous.

For any $f \in h_2$, we put $f_n = p_n(H)f$. We can show $f_n \in h_2$. Similarly the case of $h_1$, we can prove the statement. ■

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