On the characterization of different synchronization stages by energy considerations

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Abstract. In this paper using an analogy with a mechanical dynamic system we have assigned to a Lorenz chaotic system an energy like measure that has allowed us to analyse the flows of energy that appear when two identical Lorenz systems are coupled via feedback, and their relationships with some salient features of the interaction between both systems. We study the whole range of values of the coupling strength ranging from no interaction at all to identical synchronization. Before the coupling strength reaches the level of identical synchronization it induces local changes in the stability of the synchronization manifold that have a reflect in the Lyapunov exponents and phase synchronization measures and that can be easily detected using quantitative measures of energy.

1. Introduction
Two chaotic systems conveniently coupled can be forced to reach different synchronization regimes. Although the synchronization phenomenon is a subject of intense research effort, there is no much work on the derivation of measures to evaluate the interdependences of the synchronizing systems. In [1], we developed a way to assign to a chaotic systems an energy-like measure that is used to illustrate how the maintenance of a synchronized regime between different chaotic systems requires a flow of energy between the guided system and an external source of energy. The energy approach can be used to explore some of the well-established phenomena in the synchronization of chaotic systems such as the emergence of phase synchronization or the collapse of the trajectory of the coupled system to some invariant subspace.

In this work we study two identical Lorenz systems. Studying the synchronization between identical systems helps to understand the synchronization process in general. To associate an energy like measure to the Lorenz system with a flavour of a mechanical energy we use the Kolmogorov-Lorenz formalism introduced in [2]. The reason for the Kolmogorov-Lorenz formalism is that it provides a unified framework for the analysis of the energy balance for unidirectional feedback coupled Lorenz systems (master-slave).Within this framework both systems, master and slave are analyzed with a Hamiltonian point of view.

We analyze the flows of energy that appear due to the coupling and its relationships with the appearance of some salient features of the interaction between both systems occurring in the transition from no interaction at all, corresponding to a zero coupling strength, to identical synchronization, at large enough values of the coupling strength. The transition to stability of the synchronization manifold is revealed by the transition to negative values of the conditional Lyapunov exponents. Nevertheless, before this global transition takes place, the coupling strength induces local transitions to stability which might be responsible for some of the observed interactions between the coupled...
systems. These regions of local stability could be detected by local Lyapunov exponents. We study their relationships with quantitative measures of phase synchronization and energy dissipation.

Section 2 contains the Kolmogorov-Lorenz formalism that enables us to deal with the synchronization phenomena of two unidirectional feedback coupled Lorenz systems on the basis of a unified framework. Within the framework of this approach, the energy balance is calculated to identify different domains of system behaviour as a function of the coupling parameter. In section 3 analytical expressions are given for the energy and dissipation of the master and slave systems when they are unidirectionally coupled. Section 4 presents simulation results that relate the energy balance with the synchronization error, the conditional Lyapunov exponents, with different measures of phase synchronization and with temporally intermittent stages of identical synchronization. Finally, in section 5 some conclusions are summarized.

2. Kolmogorov-Lorenz systems
The Lorenz system,

\[ \begin{align*}
\dot{x}_1 &= -\sigma x_1 + \sigma x_2, \\
\dot{x}_2 &= -x_1 x_3 + \rho x_1 - x_2, \\
\dot{x}_3 &= x_1 x_2 - \beta x_3,
\end{align*} \]

under the translation that preserves its divergence,

\[ \begin{align*}
&x_1 \rightarrow x_1, \\
&x_2 \rightarrow x_2, \\
&x_3 \rightarrow x_3 + \sigma + \rho,
\end{align*} \]

assumes the form,

\[ \begin{align*}
\dot{x}_1 &= -\sigma x_1 + \sigma x_2, \\
\dot{x}_2 &= -x_1 x_3 - \alpha x_1 - x_2, \\
\dot{x}_3 &= x_1 x_2 - \beta x_3 - \beta(\sigma + \rho).
\end{align*} \]

The normalized Lorenz system given by (3), Lorenz-nr, belongs to the general class of Kolmogorov-Lorenz models,[1, 2] described by,

\[ \begin{align*}
\dot{x}_i &= -\epsilon_{ijk} x^j (D^i_j x^i + w^i) - \Lambda_j^i x_j + f^i \quad i, j, k = 1, 2, 3
\end{align*} \]

where \( D \) is a diagonal matrix and,

\[ \begin{align*}
w = \begin{pmatrix} 0 \\ 0 \\ \sigma \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \beta \end{pmatrix}, \quad f = \begin{pmatrix} 0 \\ 0 \\ -\beta(\sigma + \rho) \end{pmatrix}.
\]

If \( w=0 \), equation (4) describes a Kolmogorov system with conservative quadratic term plus an added linear dissipation \( \Lambda \) and an added force \( f \). If we consider the Kolmogorov-Lorenz system without the forcing and dissipation, we have Hamiltonian system with a Hamiltonian function \( H \) that can be seen [3] as a mixture between that of a free rigid body and that of a spin system,

\[ H = \frac{x^2_i}{2I_j} + B_i x_i, \]

where \( B_i \) are constants, \( I_j \) are the constant in time principal components of the inertia tensor. The dynamics of this Hamiltonian system is consequently described by the set of differential equations:

\[ \dot{x}_i = -\epsilon_{ijk} x_k \frac{\partial H}{\partial x_j} = -\epsilon_{ijk} x_k \left(\frac{x_i}{I_j} + B_j\right). \]

In the system given by (6) and (7), \( H \) and Casimir \( C = x^2_i \) are invariants.
We can contemplate the Lorenz-nr system from this Hamiltonian perspective [4, 5]. In (3), if we suppress the dissipative term and the constant forcing, the subjacent system is that of (7), with a Hamiltonian function given by,

\[ H = \frac{1}{2} \left( dx_1^2 + (d+1)x_2^2 + (d+1)x_3^2 \right) + \alpha x_3, \tag{8} \]

where \(d\) is an arbitrary parameter. So, the full dynamics takes the simple form

\[ \dot{x}_i = -\epsilon_{ijk} x_k \frac{\partial H}{\partial x_j} - \Lambda_i x_j + f^i \quad i, j, k = 1, 2, 3. \tag{9} \]

It is clear that in this description of the Lorenz-nr system, the Hamiltonian \(H\) given by equation (8), and the Casimir \(C = x_i^2\) are not conserved.

2.1. Casimir of the Lorenz-nr system
Consider the time derivative of the Casimir \(C\) on the attractor,

\[ \dot{C} = x_1 x_1 + x_2 x_2 + x_3 x_3 \]
\[ = x_1 (-\alpha x_1 + \alpha x_2) + x_2 (-x_1 x_3 - \alpha x_1 - x_2) + x_3 (x_1 x_2 - \beta x_3 - \beta (\sigma + \rho) x_3) \]
\[ = -\alpha x_1^2 - x_2^2 - \beta x_3^2 - \beta (\sigma + \rho) x_3. \tag{10} \]

With the following definitions for the work \(W_f\) done by the force \(f\),

\[ W_f = -\beta (\sigma + \rho) x_3, \tag{11} \]

and the dissipation \(\text{Diss}\) as the work done by \(\Lambda\),

\[ \text{Diss} = \alpha x_1^2 + x_2^2 + \beta x_3^2, \tag{12} \]

the temporal derivative of the Casimir function can be expressed as

\[ \dot{C} = -\text{Diss} + W_f. \tag{13} \]

The decomposition of the temporal derivative of the Casimir function expressed by (13) supports the mechanical analogy of a definite positive dissipation term plus the work done by the external forced \(f\). The fact of the dissipation term being always positive proves to be advantageous as a measure of some features of the synchronization process. This decomposition does not apply to the actual energy of the system given by the Hamiltonian function \(H\) of equation (8). From now on we will consider the Casimir function \(C(x_1, x_2, x_3)\) as an energy-like measure of the condition of the states of the master and slave systems to which we can simply refer as energy.

2.2. Unidirectional coupled Lorenz system
Consider the Lorenz system coupled via feedback according to the equations,

\[ \dot{x}_1 = -\alpha x_1 + \alpha x_2 + K_1 (x_1^* - x_1), \]
\[ \dot{x}_2 = -x_1 x_3 + \rho x_1 - x_2 + K_2 (x_2^* - x_2), \]
\[ \dot{x}_3 = x_1 x_2 - \beta x_3 + K_3 (x_3^* - x_3), \tag{14} \]

where \(K=(K_1, K_2, K_3)\) is a vector parameter that measures the strength of the coupling and \((x_1^*, x_2^*, x_3^*)\) is the driving signal. Under the translation given by (2), the coupled Lorenz system adopts the normalized form,

\[ \dot{x}_1 = -\sigma x_1 + \alpha x_2 + K_1 x_1^*, \]
\[ \dot{x}_2 = -x_1 x_3 + \sigma x_1 - (1 + K_2) x_2 + K_2 x_2^*, \]
\[ \dot{x}_3 = x_1 x_2 - \beta x_3 - \beta (\sigma + \rho) + K_3 x_3^*, \tag{15} \]

that keeps the Kolmogorov-Lorenz structure (4), where
A convenient feature of this formalism is that the coupling feedback has not changed the conservative structure of the slave system, the strength of coupling has been added to $\Lambda$ and the forcing term depends on the driving signal and the strength of coupling. We note that transformation given by (2) has also preserved the diagonal elements of $\Lambda$.

3. Synchronization and energy balance of a Lorenz-Lorenz unidirectional coupling

Suppose we have two Lorenz-nr systems coupled via unidirectional feedback. In order to study the synchronization phenomena it is useful to have written both systems in the common formalism. With two identical Lorenz-nr systems coupled according (14), we will focus on the interaction energy between the two systems (master-slave) time-averaged on the attractor, specifically

- **Casimir**

  $<C_{\text{master}}>=\frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right)$,

  $<C_{\text{slave}}>=\frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right)$.

- **Work and dissipation for the master:** $f$ and $\Lambda$ given by (5)

  $<W_{\text{master}}>=-\beta(x_1^2 + x_2^2 + x_3^2)$,

  $<\text{Diss}_{\text{master}}> = \alpha x_1^2 + x_2^2 + \beta x_3^2$.

- **Work and dissipation for the slave:** $f$ and $\Lambda$ given by (16)

  $<W_{\text{slave}}>=K_1 x^*_1 x_1 + K_2 x^*_2 x_2 + (K_3 x_3 - \beta x_1^2 + (K_3 x_3 - \beta x_1^2 + \beta x_3^2)$,

  $<\text{Diss}_{\text{slave}}> = \alpha x_1^2 + x_2^2 + \beta x_3^2$.

Equations (17), (18) and (19) constitute our basis for self-consistent description of the energy balance in the synchronization phenomena.

If $K_1=K_2=K_3=K$, when the coupling strength is zero $K=0$ and, for all those values $K \geq K_{sc}$, where $K_{sc}$ is the coupling value at which identical synchronization happens $x_i = x_i^*$, so their averaged quantities of Casimir are equals

$<C_{\text{slave}}>=<C_{\text{master}}>$, \hspace{1cm} (20)

and

$<\text{Diss}_{\text{slave}}>=<\text{Diss}_{\text{master}}> + K <C_{\text{master}}>$, \hspace{1cm} (21)

this means a linear relation between $<\text{Diss}_{\text{slave}}>$ and the coupling parameter $K$.

For values of $K \in (0, K_{sc})$, the difference $<C_{\text{slave}}>-<C_{\text{master}}>$ and a dissipation term $<\text{Diss}_{\text{slave}}>$ that is not linear against the coupling strength, can help us to understand the synchronization process.

On the other hand, if we consider the time derivative of the slave Casimir averaged on the attractor we have

$<\dot{C}_{\text{slave}}>=x_1 \dot{x}_1 + x_2 \dot{x}_2 + x_3 \dot{x}_3$

$\quad - (\alpha + K_1) x_1^2 + (1 + K_2) x_2^2 + (\beta + K_3) x_3^2$

$\quad + <K_1 x^*_1 x_1 + K_2 x^*_2 x_2 + (K_3 x_3 - \beta x_1^2 + (K_3 x_3 - \beta x_1^2 + \beta x_3^2)$

$\quad = - <\text{Diss}_{\text{slave}}> + <W_{\text{slave}}>$, \hspace{1cm} (22)

that can be arranged as,
where the convenient feature of this arrangement is that the time derivative of the slave Casimir has split into two terms, one of which does not depend on the gain parameter $K$, and the other one given by

$$P(K) = \langle K_1 x_1 e_{x_1}^* + K_2 x_2 e_{x_2}^* + K_3 x_3 e_{x_3}^* \rangle,$$

is the amount of energy per unit time $P(K)$ [6] that is necessary to provide the slave system with in order to maintain the degree of synchronization attained with a coupling of gain parameter $K$.

4. Simulation results

In this section, with the exception of its last subsection, we will be considering aggregate data of the Casimir functions of the master and slave systems. This aggregation consists of averaging, for each value of the gain parameter $K$, the temporal values of the Casimir function of the master and slave systems along a trajectory long enough as to be licit to consider this temporal averaging as an average on the attractor. By energy balance we understand the use of this aggregate data of the slave system in comparison with the aggregate value that corresponds to the master system. In the last subsection the whole temporal series of the energy of the slave in its movement on the attractor will be used for each value of the gain parameter $K$.

All numerical results presented here are made with two identical Lorenz systems, $\sigma=16$, $\rho=45.92$, $\beta=4$, and $K_1=K_2=K_3=K$. Initial conditions are: $[20, 5, -5]$ for the master system, $[-2, 40, 11]$ for the slave system. Systems are integrated by Matlab6.5 using the Ode5 integration method and a fixed time step of 0.01s. For each value of $K$ the simulation is long enough, 5000s., as to consider the quantities averaged on the attractor. The synchronization error is defined by,

$$<\text{sync\_error}> = \langle <\sqrt{(x_1^* - x_1)^2 + (x_2^* - x_2)^2 + (x_3^* - x_3)^2}> \rangle,$$

where $e_{x_1}$, $e_{x_2}$, $e_{x_3}$ are the variable errors.

4.1. Energy balance and synchronization error

Figure 1 shows that complete synchronization is attained at $K_c=1.5$, and figure 2 the average value on their respective attractors of the energy of the master and slave systems for different values of the gain parameter $K$ ranging from $K=0$ to $K=1.6$. As the coupling only affects the slave system, the energy of the master system (dashed-line) is independent of $K$ and, consequently, remains constant in the whole range of values of the gain parameter. In contrast, the slave system (solid-line) experiences an oscillatory pattern of change of its average energy as a function of the gain parameter $K$, and reveals three singular points in the $[0, K_c]$ interval. Between the values $K=0$ and $K=0.55$ the average energy of the slave is greater than the average energy of the master system. That is to say, the average energy of the coupled system is in this range of values of the coupling strength greater than the energy that corresponds to its own autonomous chaotic oscillation. Its maximum value is attained for $K=0.35$. At this point the averaged energy starts to decrease with increasing values of the gain parameter. For $K=0.55$ the average energy of the slave system has fallen to its original value and continuous its fall from this value as far as $K=0.95$ where it has a minimum. In this range the average energy of the coupled system remains lower than the one corresponding to the free systems as far as $K=1.5$ where identical synchronization occurs and the energy recovers its original value at $K=0$. 

$$<\tilde{C_{\text{slave}}}>=\langle -\alpha x_1^2 + \beta x_3^2 - \beta(x_1 + \rho)x_3 > + \langle K_1 x_1 e_{x_1} + K_2 x_2 e_{x_2} + K_3 x_3 e_{x_3} \rangle,$$ (23)
Figure 1. Average synchronization error plotted against coupling parameter $K$. 

Figure 2. Average Casimir plotted against coupling parameter $K$; dashed-line: master system; solid-line slave system.

In figure 3, the average dissipation of the slave system $<\text{Diss}_{\text{slave}}>$ given by equation (19b) is shown for different values of the gain parameter. For $K > K_{sc}$, that is, for values of the gain greater than the corresponding to the onset of identical synchronization the average dissipated energy obeys the linear dependence given by (21). This linear relationship has been represented as a dotted line in figure 3. A characteristic fact of the dissipated average energy at different values of $K$ is that it deviates from linearity. Dissipation exceeds linearity in the whole range where the energy exceeds its reference value at $K=0$ and falls below linearity when the energy is lower than the level that corresponds to the free system. These different stages in the synchronization process can also be appreciated in figure 4. This figure shows the average energy per unit time required to maintain the degree of synchronization attained at each value of $K$. The required energy increases linearly with two different slopes, with limits in the linear regions at $K=K^-$ and $K=K^+$ respectively. Identical synchronization occurs at $K=1.5$ where no provision of energy is required to maintain the synchronized regime.

On the other hand, the average synchronization error, computed averaging the norm of the error vector at each value of $K$, fails to account for the singular points that we have observed with the energy measures as can be seen from figure 1. This synchronization error seems to be quite coarse a measure as to detect in detail the synchronization interactions responsible for the changes in the energy discussed before. Nevertheless, it can be appreciated in the figure a range of values of $K$, expanding from the beginning of the interaction at $K=0$ as far as to values of the gain in the order of $K^*=0.95$, where the synchronization error follows a linear pattern, followed by a much quicker than linear collapse to identical synchronization at $K=K_{sc}$. This behaviour clearly indicates at least two different stages. For values of the coupling strength in the region of high dissipation of energy the synchronization error simply reacts passively to the forcing. Nevertheless, if the forcing increases, the system enters a stage of lower dissipation of energy where there is an active cooperation with the forcing. Some kind of synergetic activity appears in the slave system that cooperates to reach synchrony.
4.2. Energy balance and Lyapunov exponents

Lyapunov exponents play an essential role in the analysis of the synchronization phenomena with coupled chaotic systems. For a Lorenz-Lorenz unidirectional coupling, the whole master-slave system is six-dimensional with two positive, two zero and two negative exponents. As $K$ is increased, three of them: one positive $\lambda_+$, one zero $\lambda_0$ and one negative $\lambda_-$, associate with the slave system and called conditional Lyapunov exponents, evolves with $K$ and their evolution reveals stages of synchronization. In reference [7] the authors show that for Lorenz coupled systems the originally null conditional Lyapunov exponent $\lambda_0$ does not behave in the way usually understood. Instead of becoming negative as the coupling parameter is increased, it starts to increase, reaches a maximum and then decreases as $K$ is increased further. For $K=K_0$, $\lambda_0(K)$ becomes negative.

Our measurements of the Lyapunov exponents for the coupled Lorenz-Lorenz system show that the singular points detected by the Casimir function are also related to changes in the conditional Lyapunov exponents. Figure 5a shows that $\lambda_0(K)$ reaches its maximum value at $K_0=0.35$, then decreases as $K$ is increased further, and finally becomes negative at $K=0.55$, so $K=K_0$. As $K$ is increased through $K^*$ the slave system has two negative conditional Lyapunov exponents, but for $K=K^*$, the $\lambda_-(K)$ exponent, that has been downhill, starts a smooth decrease. Referring to figure 5b, the $\lambda_-(K)$ exponent shows a linear decrease with the parameter $K$ being in the $[0, K^*]$ interval, levels off for $K^*<K<K^*$, then starts to decrease again with a small jump at about $K=1.4$. For $K>K_\infty=1.5$ all the conditional Lyapunov exponents are negative (complete synchronization) and they decrease against $K$ with the same slope.
4.3. Energy balance and phase synchronization

Phase synchronization in coupled chaotic systems is a weak form of synchronization, a situation where the systems adjust their time scales but their states remain uncorrelated. Non trivial phase synchronization, not as a consequence of state synchronization, occurs for weak coupling and it is generally related to the transition to a negative value in one of the originally null Lyapunov exponents. However, for Lorenz coupled systems, the authors [8] found that because of the double-scroll structure of the attractor phase synchronization in the above sense cannot occur. As the coupling is increased form zero, there is an interaction between the amplitude and the phase dynamics, so phase synchronization is impossible insofar as the state dynamics remains uncorrelated. Alternatively they observe a degree of state synchronization, intermittent synchronization, for $K > K_c = K^*$ when $\lambda_\parallel(K)$ has become negative. This phenomena of transition to intermittent synchronization at $K^*$ for coupled Lorenz systems had not been observed previously, as all previous works focus on the transition near $K_c = K_0$ for which the originally positive conditional Lyapunov exponent $\dot{\lambda}_\parallel$ becomes negative.

In order to look at the relationships between the characteristic features of our energy balance approach and the different stages of the synchronization process we now proceed to check the Lorenz-Lorenz coupling for phase synchronization. A common method to obtain the phase is to consider the analytical signal introduced by Gabor, and to decompose a time series into the instantaneous phase $\varphi(t)$ and instantaneous amplitude $A(t)$ by means of the Hilbert transform. For Lorenz systems, from amplitudes $\{x_i\}$ [9], we have

$$\ddot{x}_2(t) = x_2(t) + jH(t) = A(t)e^{j\varphi(t)},$$
$$H(t) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{x_2(t')}{t-t'} dt',$$
$$\varphi(t) = \arctan\left( \frac{H(t)}{\dot{x}_2(t)} \right),$$

where $PV$ refers to the Cauchy principal value.

To study phase synchronization we have computed the difference between the phase of the slave system and the phase of the master system $\Delta \varphi(t) = \varphi_{\text{slave}}(t) - \varphi_{\text{master}}(t)$, and also its velocity $\Delta \omega(t)$ which is the time derivative of the phase difference.

Phase difference portraits for various significant values of $K$ are displayed in figure 6, for $K=0$, no coupling; $K=0.25$, below the critical point $K^*$; $K'=0.35$; $K'=0.55$; $K'=0.95$; $K=1$, and finally at

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**Figure 5.** Lyapunov exponents plotted against coupling parameter $K$. (a) Originally positive and null exponents. (b) Originally negative exponents.
$K=1.45$, near the complete synchronization region. For $K=0$, we have two identical Lorenz systems and the temporal evolution of their phase difference is bound as both systems share a time scale. When the gain parameter $K$ is slowly increased the phase difference decreases being unbound, and its absolute slope increases with $K$ as the perturbed slave system temporarily gets caught inside stable orbits. At the point $K=K_m$, there is a change in the behaviour of the phase difference as its slope increases for larger values of the gain parameter $K>K_m$. The slave system is progressively drawn into the basin of attraction of the master system, and finally reaches, $K=K_{sc}$, the state of complete synchronization, so, a trivial phase synchronization.

![Figure 6](image-url)

**Figure 6.** Time evolution of the phase difference plotted for different coupling parameters $K$. Wider lines for singular values of $K$.

Figure 7a shows the velocity of phase difference, averaged on the attractor, plotted against the coupling parameter $K$. This figure shows another interesting feature: for $K=K_m$, at which $\lambda_p(K)$ has a maximum, the velocity starts to go downward significantly. On the other hand, it is interesting to analyse the standard deviation of the average velocity. Figure 7b reveals the average velocity has its maximal deviation at about $K^*$. 
4.4. Temporal energy measures and intermittent synchronization

We want to check for intermittent synchronization, so let us now focus on the relationships between the time dependencies $\Delta \phi(t)$, $C(t)$ and $\text{Diss}(t)$. In the next figures 8, 9 and 10, we present the temporal evolution of the above magnitudes for three values of the coupling parameter $K=1.45$, $K=1$ and $K=0.55$ respectively. These values of $K$ are in the interval $[K_c, K_s]$ where the slave system has only one positive conditional Lyapunov exponent so the appropriate region to look for a possible relationship between the phase synchronization and the amplitude synchronization.

For $K=1.45$, figure 8, the phase difference clearly shows staircase motions: $\Delta \phi(t)$ keeps constant in long-time segments and then jumps in shorter ones. Each stair corresponds to a time segment of phase synchronization and the jumps are desynchronization time periods. Inlet zooms into smaller time window to enlarge details of the phase synchronization. For these phase synchronization time intervals, the slave dissipation is high, and the slave Casimir keeps oscillating around of the master Casimir within a narrow range. As the Casimir is a function of the states, we can say that the slave system is in state vicinity of the master system. On the other hand, for the desynchronization periods the slave dissipation is at its lower level and the slave Casimir is far from the master Casimir, so the slave system is in its own basin of attraction. According to the plots presented in this figure 8, there is intermittent synchronization, that when happens the phase synchronization is associated with the state synchronization.

**Figure 7.** Velocity of the phase difference plotted against the coupling parameter $K$. (a) Average. (b) Standard deviation.
The analysis presented above can be viewed as a phenomenon of trivial phase synchronization as the value of the coupling parameter was near $K_{sc}$, so now we investigate the behaviour of the slave system for a coupling parameter of $K=1$, in between $K^*$ and $K_{sc}$. Figure 9 shows the results obtained. Phase difference staircases are not as clear a before, but when zoomed in they arise. Although time periods of phase synchronization are narrower, they also are associated with higher dissipation and instantaneous values of the slave Casimir around the line of the average master Casimir. The consequence is that, we have even far from the $K_{sc}$ point, intermittent synchronization and consequently trivial phase synchronization.

**Figure 8.** For $K=1.45$, temporal development of: (a) instantaneous phase difference; (b) instantaneous slave dissipation and average master dissipation (level line); (c) instantaneous slave Casimir and average master Casimir (level line).
Finally, in the course of our decrease of $K$, we arrived at the left boundary of the $[K_c, K_s]$ interval, $K=0.55$ where the originally null conditional Lyapunov exponent $\lambda_0(K)$ becomes negative. A vanishing Lyapunov exponent corresponds to a phase variable of an oscillator, so when it becomes negative the neutrality of the oscillator in the phase direction is broken and, as a well-accepted theory, we should look at phase synchronization. Figure 10 shows our results. A careful observation of the phase difference inset plot reveals a very narrow time intervals of synchronization then, we are at the beginning of the phase synchronization phenomena. The most important observation in figure 10 is that at the phase synchronization intervals the instantaneous slave Casimir is also around a narrow level of the average master Casimir, so it is a trivial phase synchronization. As before, these bursts of intermittent synchronization are accomplished by a high level of dissipation.
According to these observations, for two unidirectional feedback coupled Lorenz systems intermittent synchronization phenomena appears as soon as the coupling gain parameter goes beyond $K^*$, in agreement with the conclusions of reference [8].

5. Conclusions
Using an analogy with a mechanical dynamic system we have assigned to a Lorenz chaotic system an energy like measure. This measure can be used to study some of the interactions that appear when the system is coupled to another identical Lorenz system via feedback coupling. Before the coupling is established the chaotic slave system moves freely on its attractor with its own characteristic pattern of temporal energy and energy dissipation. When the coupling is established, its patterns of energy and energy dissipation change and so do their average values on the attractor. At each value of the gain parameter of the coupling, ranging from zero to a value large enough as to trigger identical synchronization, the energy and energy dissipation adopt different patterns and take different average
values that can be related with measures of the synchronization error, Lyapunov conditional exponents and measures of phase synchronization.

Our results confirm the relevant conclusion that in the case of two identical Lorenz systems, phase synchronization does not properly occur. The often attributed nontrivial phase synchronization, that is detected by the originally zero Lyapunov exponent, and by phase difference and difference derivative measures, is more likely to be intermittent identical synchronization. Our measures of energy show that from \( K = 0.55 \), where the null Lyapunov exponent begins to be negative, there are local regions where the energy and energy dissipation of the slave system are the ones that correspond to the free system. This fact is a clear indication that the synchronization manifold is locally stable and that not phase synchronization but intermittent identical synchronization is taking place. This fact can be corroborated by careful analysis of the phase synchronization measures.

The approach shown in this work is also applicable to homochaotic (different parameters) Lorenz systems. In this case, for \( K = 0 \) \( \langle C_{\text{free}} \rangle \) and \( \langle C_{\text{slave}}(0) \rangle \) are not equal but, in the limit, when \( K \to \infty \) the synchronization error tends to zero, so \( \langle C_{\text{free}}(K) \rangle \approx \langle C_{\text{free}} \rangle \) and we can think an energy balance of

\[
\text{Energ}_{-}\text{sc}(K) = (\langle C_{\text{slave}}(K) \rangle - \langle C_{\text{slave}}(0) \rangle) - (\langle C_{\text{slave}}(0) \rangle - \langle C_{\text{master}} \rangle).
\]  

(27)

On the other hand, there is a family of chaotic systems [10] that shares with the Lorenz system same characteristics within the Kolmogorov-Lorenz formalism, so these systems are also candidates for our work on the characterization of synchronization by energy considerations.

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