TYPE II₁ VON NEUMANN REPRESENTATIONS FOR HECKE OPERATORS ON MAASS FORMS AND RAMANUJAN-PETERSSON CONJECTURE

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Dedicated to Professor Dan Virgil Voiculescu on the occasion of his 60’th anniversary

ABSTRACT. We prove that classical Hecke operators on Maass forms are a special case of completely positive maps on II₁ factors, associated to a pair of isomorphic subfactors. This representation induces several matrix inequalities on the eigenvalues of the Hecke operators Maass forms. In particular the family of eigenvalues corresponding to an eigenvector is a completely bounded multiplier of the Hecke algebra. This representation of the Hecke algebra also implies that, for every prime $p$, the essential spectrum of the classical Hecke operator $T_p$ is contained in the interval $[-2\sqrt{p}, 2\sqrt{p}]$, predicted by the Ramanujan Petersson conjectures. In particular, given an open interval containing $[-2\sqrt{p}, 2\sqrt{p}]$, there are at most a finite number of possible exceptional eigenvalues lying outside this interval.

INTRODUCTION

In this paper we obtain an operator algebra representation for the classical Hecke operators. Using positivity properties, inherent to operator algebra structures, we are able to deduce various properties for the Hecke operators on Maass forms, e.g. we are able to compute the essential spectrum.

We start with a discrete group $G$ with an almost normal subgroup $\Gamma$, such that the set $S$ of subgroups $\Gamma_\sigma = \Gamma \cap \sigma \Gamma \sigma^{-1}$, $\sigma$ in $G$, is modular, with respect to inclusion.

The Hecke $\mathcal{H}_0 = \mathcal{H}_0(\Gamma, G)$ algebra of double cosets of $\Gamma$ in $G$ has a canonical representation, called left regular representation, on $l^2(\Gamma \backslash G)$, and our basic object will be $\mathcal{H}$, the closure of $\mathcal{H}_0$ in the weak topology on the bounded linear operators

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$B(\ell^2(\Gamma/G))$ acting on $\ell^2(\Gamma/G)$. Our main assumption will be that there exists a (projective) unitary representation $\pi$ of $G$ on $\ell^2(\Gamma)$, extending the left regular (projective, when a group 2-cocycle is present) representation of $\Gamma$ on $\ell^2(\Gamma)$.

This assumption (that also implies that $[\Gamma : \Gamma_{\sigma}] = [\Gamma : \Gamma_{\sigma^{-1}}]$ for all $\sigma \in G$) enables us to construct a canonical operator system structure on a more primitive structure, that dictates the Hecke algebra structure.

The operator system is $C(G, \Gamma)$ the linear space spanned by all sets of the form $[\sigma_1 \Gamma \sigma_2], \sigma_1, \sigma_2 \in G$, subject to the obvious relation that $\sum_i [\sigma_i^2 \Gamma \sigma_2^i] = \sum_j [\theta_j \Gamma \theta_j^2]$ if $\sigma_i, \theta_j, \varepsilon = 1, 2$ are elements in $G$ such that the sets $(\sigma_1^2 \Gamma \sigma_2^i)_i$ and respectively $(\theta_j \Gamma \theta_j^2)_j$ are disjoint of equal union. It is clear that the adjoint map on $C(G, \Gamma)$ is defined by mapping $[\sigma_1 \Gamma \sigma_2]$ into $[\sigma_2^{-1} \Gamma \sigma_1^{-1}]$ so that and the $*$ operation maps $[\sigma_1 \Gamma]$ into $[\Gamma \sigma_1^{-1}]$. Hence we also we get a canonical positive cone $C(\Gamma, G, \Gamma)$.

There exists a canonical pairing $C(\Gamma/G) \times C(\Gamma/G) \rightarrow C(\Gamma, G, \Gamma)$, mapping $[\sigma_1 \Gamma] \times [\Gamma \sigma_2]$ into $[\sigma_1 \Gamma \sigma_2]$.

It is obvious that the Hecke algebra structure is a particular case of the pairing.

A representation of $C(G, \Gamma)$ into a II$_1$ factor $\mathcal{M}$ is an isometric mapping of $\ell^2(\Gamma \setminus G), \ell^2(G/\Gamma)$ compatible with the $*$ operation, transforming the concatenation $[\sigma_1 \Gamma] \times [\Gamma \sigma_2] = [\sigma_1 \Gamma \sigma_2]$ into the algebra product.

We obtain such a representation of $C(G, \Gamma)$ into the von Neumann II$_1$ factor $\mathcal{L}(G, \varepsilon)$ associated with the group $G$ with cocycle $\varepsilon$.

In our representation $[\Gamma \sigma]$ are mapped into a family $t^{\Gamma \sigma}, \sigma \in G$ (constructed out of the matrix algebra coefficients) of the representation $\pi$ with respect to the unit vector 1 of $L(\Gamma)$. Let $\mathcal{L}(G, \varepsilon), \mathcal{L}(\Gamma, \varepsilon)$ be the finite von Neumann algebras, with cocycle $\varepsilon$ associated to the discrete groups $G, \Gamma$.

In addition, our construction shows that we may chose $(t^{\Gamma \sigma})_{\sigma \in G}$ so as to form a basis of $\mathcal{L}(G, \varepsilon)$ as a module over $\mathcal{L}(\Gamma, \varepsilon)$ (a Pimsner-Popa basis for $\mathcal{L}(\Gamma, \varepsilon) \subseteq \mathcal{L}(G, \varepsilon)$).

Moreover, $t^{\Gamma \sigma}$ is supported in $\ell^2(\Gamma \sigma)$.

This allows one to define an abstract Hecke operators (which as we explain bellow are unitarily equivalent to the classical Hecke operators for $G = \text{PSL}_2(\mathbb{Z}[\frac{1}{p}])$, modulo a positive phase operator, commuting to the Laplacian).

The formula for the abstract Hecke operator determined by such a representation $t$ of $C(G, \Gamma)$ is as follows:

$$\Psi_{[\Gamma \sigma \Gamma]} = [\Gamma : \Gamma_{\sigma}] E_{\mathcal{L}(\sigma \Gamma, \varepsilon)}^{\mathcal{L}(G, \varepsilon)}(t^{\Gamma \sigma} \cdot t^{\Gamma \sigma \Gamma}).$$

(1)

Here $t^{\Gamma \sigma \Gamma}$ is simply $\sum_{\Gamma \sigma \sigma \subseteq [\Gamma \sigma \Gamma]} t^{\Gamma \sigma \sigma}$ and $E_{\mathcal{L}(\Gamma, \varepsilon)}^{\mathcal{L}(G, \varepsilon)}$ is the canonical conditional expectation from $\mathcal{L}(G, \varepsilon) \rightarrow \mathcal{L}(\Gamma, \varepsilon)$ (that is the map killing all $g$ with $g$ not in $\Gamma$).

Thus the Hecke operators are canonically determined by the representation of $C(G, \Gamma)$ that we describe bellow. We will prove that for the modular groups this
abstract Hecke operators are unitarely equivalent (modulo a phase) to the classical Hecke operators on Maass forms.

The classical Hecke operators are acting on $L^2(F, \frac{dz}{(Im z)^2})$, where $F$ is a fundamental domain for the action of $\text{PSL}_2(\mathbb{Z})$. The classical Hecke operator (corresponding to the sum of double cosets in matrices of determinant $n$ are

$$ T(n) f(z) = \sum_{a,d = n \atop b = 0,1,\ldots,d-1} f \left( \frac{az+b}{d} \right) $$

and the normalized version

$$ \tilde{T}(n) = \frac{1}{\sqrt{n}} T_n. $$

The Ramanujan-Petersson conjecture states that if $c(n)$ are the eigenvalues for a common eigenvector $\xi \neq 0$, for all the $T_n$'s, then $c(p) \in [-2,2]$ for all primes $p$ (see [Hej]). This corresponds, when working with the non-normalized Hecke operator $T_p$, to the fact that the eigenvalues should be in the interval $[-2\sqrt{p}, 2\sqrt{p}]$.

Our setting, makes it natural to extend this conjecture to the more general setting of a group $G$ an almost normal subgroup $\Gamma$ and $\pi$ a projective unitary representation of $G$ on $l^2(\Gamma)$ extending the left regular representation (with cocycle) of $\Gamma$. The Hecke operators are replaced by the operators in formula (1), and the Ramanujan Petersson conjectured estimates become a conjecture about the continuity of the linear application which maps $[\Gamma \sigma \Gamma]$ into the completely positive map $\Psi_{[\Gamma \sigma \Gamma]}$. This is equivalent, by what we explained in the preceding paragraph to the classical case for $\text{PSL}_2(\mathbb{Z}[\frac{1}{p}]) \supset \text{PSL}_2(\mathbb{Z})$, as we will explain bellow. We prove that this continuity holds, when replacing the Hecke operators with their image in the Calkin algebra, and thus prove that the essential spectrum of the Hecke operator sits in the predicted interval $([-2\sqrt{p}, 2\sqrt{p}])$. We prove

**Theorem** For every prime $p$ the essential spectrum of the classical Hecke operator $T_p$ is contained in the interval $[-2\sqrt{p}, 2\sqrt{p}]$, predicted by the Ramanujan Petersson conjectures. In particular, given an open interval containing $[-2\sqrt{p}, 2\sqrt{p}]$, there are at most a finite number of possible exceptional eigenvalues lying outside this interval.

The representation $t$ of $\mathbb{C}(G, \Gamma)$ allows one to deduce that every eigenvector $\xi \in l^2(\Gamma)$ for the Hecke operators (defined by formula (1) above) determines a completely positive map on $\mathbb{C}(G, \Gamma)$, which when restricted to the Hecke algebra determines the eigenvalue. Such completely positive maps, in the case of the radial algebra ([Py]), of a free group with $N$ generators (which is isomorphic to the Hecke algebra for $\text{PSL}_2(\mathbb{Z}[\frac{1}{p}]) \supset \text{PSL}_2(\mathbb{Z})$, with $N = \frac{p+1}{2}$) have been studied in [DeCaHa]. They are derived out of Schur multipliers.

Our result shows that these completely positive maps have a canonical extension to $\mathbb{C}(G, \Gamma)$ and hence their knowledge is relevant for the determination of the eigenvalues.
The fact that the classical Hecke operators are unitarily equivalent to the abstract Hecke operators in formula (1) is outlined below.

First, we observe that given a family of Hilbert spaces acted by $G$, with a rich family of $\Gamma$ fixed vectors, then if we denote by $H^{\Gamma}\sigma$ the Hilbert space of vectors fixed by $\Gamma\sigma$, the Hecke operator $v \rightarrow T_\sigma(v) = \sum s_i\sigma v$ (where $\Gamma = \bigcup s_i\Gamma\sigma$ is the decomposition into right cosets) is obtained by composing the maps in the following diagram

$$H^{\Gamma\sigma^{-1}} \xleftarrow{\text{inc}} H^{\Gamma\sigma} \xrightarrow{\sigma} H^\Gamma$$

where $P$ is the orthogonal projection from $H^{\Gamma\sigma}$ onto $H^\Gamma$. Thus $T_\sigma v = P(\sigma v)$, $\sigma \in G$, $v \in H^\Gamma$. The Hilbert space structure is selected so that all the inclusions $H^{\Gamma\sigma_0} \subseteq H^{\Gamma\sigma_1}$ (where $\Gamma\sigma_0 \supseteq \Gamma\sigma_1$) are isometric.

In particular, if we let $\Gamma$ act on $\ell^2(\Gamma)$ (eventually with a cocycle $\varepsilon$) and $\pi$ a unitary representation of $G$ on $\ell^2(\Gamma)$ with cocycle $\varepsilon$ then the following diagram (with $E = E(\Gamma\sigma)^{\prime}$, the canonical conditional expectation from $\{\Gamma\sigma\}^{\prime}$ onto $\{\Gamma\}^{\prime}$)

$$\{\Gamma\sigma^{-1}\}^{\prime} \xleftarrow{\text{inc}} \{\Gamma\sigma\}^{\prime} \xrightarrow{\text{Ad}\pi(\sigma)} \{\Gamma\}^{\prime}$$

yields a Hecke operator

$$\Psi_\sigma(X) = E_{\{\Gamma\sigma\}^{\prime}}(\pi(\sigma)X\pi(\sigma)^*) = \sum_{i=1}^{n} \pi(s_i\sigma)(X)\pi(s_i\sigma)^*; \quad (2)$$

where $\Gamma = \bigcup s_i\Gamma_{\sigma}$.

Here the commutants are computed in the space $B(\ell^2(\Gamma))$, and hence all the algebras $\{\Gamma\}^{\prime}$, $\{\Gamma\sigma\}^{\prime}$, $\{\Gamma\sigma^{-1}\}^{\prime}$ are II$_1$ factors, so there is a canonical conditional expectation from $\{\Gamma\sigma\}^{\prime}$ onto $\{\Gamma\}^{\prime}$.

In the case of $G = \text{PSL}_2(\mathbb{Z}[1/2])$, $\Gamma = \text{PSL}_2(\mathbb{Z})$, the left regular representation of $\Gamma$ on $\text{PSL}_2(\mathbb{Z})$, with cocycle $\varepsilon$, is equivalent by [GHJ], with the restriction to $\Gamma$ of the 13-th element $\pi_{13}$ in the discrete series representation of $\text{PSL}_2(\mathbb{R})$. The Hilbert space $H_{13}$ of $\pi_{13}$ is the space $H^2(\mathbb{H}, d\nu_{13})$, where $d\nu_{13} = (\text{Im } z)^{13-2}dz$, and $\pi$ acts by left translations via M"obius transforms, corrected by the factor $J(g, z)^{13} = (cz + d)^{-13}$, $z \in \mathbb{H}$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\text{PSL}_2(\mathbb{R})$.

In this case the operators on $B(H_{13})$, by Berezin’s quantization [Be] are represented by reproducing kernels $k(\overline{z}, \eta)$, $z, \eta \in \mathbb{H}$, which are analytic functions on $\eta$ and antianalytic functions of $z$, subject to certain growth condition ([Ra1]). Then $\{\Gamma\}^{\prime}$ consists of kernels $k$ such that $k(\overline{\gamma z}, \gamma \eta) = k(\overline{z}, \eta)$, for all $\gamma \in \Gamma, z, \eta \in \mathbb{H}$.
The action of $T_{\sigma}$ on the operator $X$ with kernel $k_X$ gives a operator with kernel given by the kernel

$$z, \eta \mapsto \frac{1}{n} \sum k(s_i \sigma z, s_i \sigma \eta).$$

If we restrict to the diagonal we get the classical Hecke operators. By the theory of the Berezin transform ([Be]) (which is in fact the same as the Selberg transform) we know that the comparison between the kernel itself and its restriction to the diagonal is given by an invertible positive transformation (the Berezin transform). We will first find the formula

$$t_{\Gamma \sigma} = \sum_{\theta \in \Gamma} \langle \pi_{13}(\theta) I, I \rangle \theta,$$

where $I \in \ell^2(\Gamma) \cong H_{13}$ is the vector corresponding to the neutral element of $\Gamma$.

This allows to prove that the operators in (1) and (2) are equivalent.

Hence the analysis of the classical Hecke operators is reduced to the analysis of the operators in formula (1).

To analyze the essential spectrum of the operators in formula (1) means to switch to the Calkin algebra. The states are then generically averages, over points in $\Gamma$, equidistributed in cosets of modular subgroups. Thus when passing to the Calkin algebra, equalities of the type $g_1 \gamma_0 g_2 = \gamma$, become equalities on the averages (with respect Haar measure on the compact group $PSL_2(\mathbb{Z}_p)$, with $\mathbb{Z}_p$ the $p$-adic integers).

We compute this states by using the methods of counting lattice points in the paper by Gorodnik and Nevo ([GoNe], see also [EMS]). Thus we consider averages over sets of points of the form $\Gamma_t = \Gamma \cap B_t$, where $B_t = \{g|g \in PSL_2(\mathbb{R}), ||g||_2 \leq t\}$ be the hyperbolic ball in $PSL_2(\mathbb{R})$ of radius $t$. The computation then shows that such an essential state, coming from the Calkin algebra, in the representation of $G \times G$ as left and right convolution operators on $\ell^2(G)$ (modulo the compacts) is given by the formula, for all $g_1, g_2 \in G$:

$$F(g_1, g_2) = \lim_{t \to \infty} \frac{\text{vol}(B_t \cap g_1 B_t g_2)}{\text{vol} B_t}.$$ 

We prove that this is of the order of

$$\frac{\ln \|g_1\|_2 + \ln \|g_2\|_2}{\|g_1\|_2 \cdot \|g_2\|_2}.$$ 

In fact the computation for the essential states on the algebra generated by left and right convolutors is working for groups for which the hypothesis from ([GoNe]) By using this we find that the map $[\Gamma \sigma \Gamma] \to [\Psi_{\Gamma \sigma \Gamma}]_{Q(\ell^2(\Gamma))}$ is preserving the essential states, and hence is continuous with respect to the reduced Hecke algebra topology on $\mathcal{H}$, and hence it follows that the Ramanujan-Petterson condition holds true for the essential spectrum in the case $G = PSL_2(\mathbb{Z}[\frac{1}{p}]).$

Note that in fact we prove
Theorem. Let $\Gamma \subseteq G$ be an almost normal subgroup as above. Let $S$ be the set of finite index subgroups of $G$, of the form $\Gamma_{\sigma} = \sigma \Gamma \sigma^{-1} \cap \Gamma$ and assume that $S$ is directed downward, by inclusion. More precisely, we assume that for every $\Gamma_{\sigma_1}, \Gamma_{\sigma_2}$ in $S$ there exists a normal subgroup $N$ of $\Gamma$ and $\Gamma_{\sigma_3}$ such that $\Gamma_{\sigma_1} \supseteq \Gamma_{\sigma_2} \supseteq N$ $\supseteq \Gamma_{\sigma_3}$. We also assume that $G$ verifies the assumption in [GoNe]. Assume that for all $\sigma \in G$, the group indices $[\Gamma : \Gamma_{\sigma}], [\Gamma : \Gamma_{\sigma^{-1}}]$ are equal.

Let $X_\Gamma$ be the continuous functions on the profinite completion of $\Gamma$, that is the $C^*$-algebra in $L^\infty(\Gamma)$ generated by characteristic functions of cosets of elements in $\rho$ (in the projective case we have to add suitable sign functions to separate points).

Let $\tau$ be the functional on $X_\Gamma$ defined by
$$\tau(\chi_{s\Gamma_{\sigma}}) = \frac{1}{[\Gamma : \Gamma_{\sigma}]}$$
$s \in \Gamma, \sigma \in G$
which is a measure on the compact maximal space of $X_\Gamma$.

Because of the equality of the indices $[\Gamma : \Gamma_{\sigma}] = [\Gamma : \Gamma_{\sigma^{-1}}], \sigma \in G$, it follows that $\tau$ is invariant to the partial action of $G \times G^\text{op}$ on $X_\Gamma$.

Clearly, the maximal $C^*$ (groupoid) crossed product algebra $C^*((G \times G^\text{op}) \rtimes X_\Gamma)$ is represented in $B(\ell^2(\Gamma))$ by letting $G, G^\text{op}$ act by left and respectively right convolutors, and $X_\Gamma$ by multiplication operators. Then the states on $C^*((G \times G^\text{op}) \rtimes X_\Gamma)$ induced from the representation into $Q(\ell^2(\Gamma)) = B(\ell^2(\Gamma))/K(\ell^2(\Gamma))$ are of the form, for $T = \sum_{\tilde{g} \in G \times G^\text{op}} g f_\tilde{g}, f_\tilde{g} \in X_\Gamma$,
$$\Phi_\xi(T) = \Phi_0(\xi^* T \xi),$$
where $\Phi_0$ is the state on $C^*((G \times G^\text{op}) \rtimes X_\Gamma)$ defined by the formula
$$\Phi_0(T) = \sum_{(g_1, g_2) \in G \times G^\text{op}} \chi(g_1, g_2) \int_{\Gamma \Gamma^{-1} \cap \Gamma} f_\tilde{g}(k) d\mu,$$
where $d\mu$ is the (Haar) measure on $X_\Gamma$ induced by $\tau$ (the profinite completion of $\Gamma$) and for $g_1, g_2 \in G \times G^\text{op}$, $\chi(g_1, g_2)$ is the asymptotic displacement on a family of well rounded balls $B_t$ (as in [GoNe]), that is
$$\chi(g_1, g_2) = \lim_{t \to \infty} \frac{\text{vol}(B_t \cap g_1 B_t g_2)}{\text{vol}(B_t)}$$
(volumes computed with respect Haar measure on $G$).

As a consequence (by using the summability criteria in [DeCaHa]) we the following:

Theorem. The discrete group $\text{SL}_2(\mathbb{Z}[\frac{1}{p}])$ does not have the Akemann-Ostrand property.

The equivalence of the two representations of the Hecke operators is based on the following theorem of V.F.R Jones (see e.g. [GHJ]). Let $M$ be the factor generated
by the image of $\text{PSL}_2(\mathbb{Z})$ through the discrete series representation $\pi_{13}$ of $\text{PSL}_2(\mathbb{R})$, then as proven by Jones ([GHJ]), $M$ is unitarily equivalent to the factor $L(\text{PSL}_2(\mathbb{Z}))$ associated to the left regular representation of the discrete group $\text{PSL}_2(\mathbb{Z})$.

We can also derive some (matrix) inequalities on eigenvalues. This shows up, because of completely positive maps associated with eigenvectors. Assume the completely positive maps $\bar{\Psi}_\alpha$ in formula (1), where $\alpha$ runs over the space of cosets of $G$ have a joint eigenvector $\xi \neq 0$, and denote by $\bar{c}(\alpha)$ the corresponding eigenvalue.

The above description allows one to prove the following

**Theorem.** The map on the Hecke algebra that maps a coset $\alpha = [\Gamma \sigma \Gamma]$ into

$$\Phi_\bar{c}(\alpha) = [\Gamma \sigma \Gamma]$$

extends to a completely positive map on the reduced von Neumann algebra of the Hecke algebra.

*In particular, this proves that the sequence $(\bar{c}(\alpha))_{\alpha \in \Gamma \backslash G / \Gamma}$ is a completely positive multiplier for the Hecke $C^*$-algebra of $\Gamma$ in $G$.*

This information encodes positive definiteness for various matrices whose coefficients are linear combinations of the $\bar{c}(\alpha)$’s.

In fact, the representation of the $\bar{\Psi}_\alpha$ allows on even stronger positivity result, based on the complete positivity of the bilinear form of $H(\xi, a, b) \mapsto \tau_L(\Gamma)(\xi^* a \xi b^*)$.

This happens because the type II$_1$ representations encodes an action of $H \otimes H$. The Hecke operators on Maass form only the “diagonal” part of this action.

Another consequence is the following: let $A(G, \Gamma)$ the free $\mathbb{C}$-algebra generated by all the cosets $[\Gamma \sigma]$, $\sigma \in G$, and their adjoints ($[\Gamma \sigma]^* = [\sigma^{-1} \Gamma]$), subject to

$$\sum [\sigma_1^* \Gamma] [\Gamma \sigma_2] = \sum [\theta_1^* \Gamma] [\Gamma \theta_2]$$

if $\sigma_1$, $\theta_1$ are elements of $G$, and the disjoint union $\sigma_1^* \Gamma \sigma_2^*$ is equal to the disjoint union $\theta_1^* \Gamma \theta_2$. Note that the above relation corresponds exactly to the fact that the Hecke algebra of double cosets is a subalgebra of $A(G, \Gamma)$, with the trivial embedding mapping a double coset into the formal sum of its left or right cosets (using representatives). Then we have (see Appendix 2)

**Theorem.** The $\mathbb{C}$-algebra $A(G, \Gamma)$ admits at least one unital $C^*$-algebra representation.

Note that the Hecke algebra operator representation in formula (1) admits an extension to the algebra $A(G, \Gamma)$ ([Ra5]), and the content of the Ramanujan Petersson conjecture can be viewed as a conjecture on the representations of $A(G, \Gamma)$.

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1. Hecke Operators and Hilbert Spaces

In this chapter we present known facts about Hecke operators, from the point of view of Hecke operators as orthogonal projections composed with translation operators. This point of view is particularly relevant when dealing with finite von Neumann algebras, since in that case the projections are naturally conditional expectations. This representation of the Hecke operators as conditional expectations unveils an operators system structure on a more basic object, which in turn determines the structure of the Hecke algebra. The operator system is, as a vector space, the vector space having as a basis the sets $[\sigma_1 \Gamma \sigma_2], \sigma_1, \sigma_2$ in $G$.

Let $G$ be a discrete group and $\Gamma$ an almost normal subgroup. We assume that the set $S$ consisting of all finite index subgroups $\Gamma_\sigma$ of the form $\Gamma_\sigma = \sigma \Gamma \sigma^{-1} \cap \Gamma$, $\sigma \in \Gamma$ has the modular property, that is for any $\sigma_1, \sigma_2$ in $G$, there exists $\sigma_3$ in $G$, such that $\Gamma_\sigma_1 \cap \Gamma_\sigma_2 \supseteq \Gamma_\sigma_3$. Later we will also need the assumption that the indices $[\Gamma : \Gamma_\sigma]$ and $[\Gamma : \Gamma_{\sigma^{-1}}]$ are equal.

We will use the following type of unitary representations of the group $\Gamma$.

**Definition 1.** An adelic Hilbert space representation of the group $G$ described above consists of the following data. Let $V$ be a topological vector space, acted by $G$, $H \subseteq V$ be Hilbert space unitarily acted by $G$ (this is not the Hilbert space of the adelic Hilbert space representation). For $\Gamma_\sigma \in \rho$, we denote by $V_{\Gamma_\sigma}$ the set of vectors in $V$ fixed by $\Gamma_\sigma$. We assume that we are given a family of Hilbert space $H_{\Gamma_\sigma}$ for $\Gamma_\sigma$ in $S$ with the following properties:

1) $\Gamma_1 \subseteq \Gamma_0, \Gamma_1, \Gamma_0$ in $S$ then $H_{\Gamma_0} = H_{\Gamma_1} \cap V_{\Gamma_0}$.

2) The Hilbert space norm on $H_{\Gamma_\sigma}, \Gamma_\sigma$ in $S$ has the property that if $\Gamma_\sigma_1 \subseteq \Gamma_\sigma_0$ then the inclusion $H_{\Gamma_\sigma_0} \subseteq H_{\Gamma_\sigma_1}$ is isometric.

3) Note that if $v \in V_{\Gamma_\sigma}$ then $\sigma_1 v$ is invariant by the group $\sigma_1 \Gamma_\sigma \sigma_1^{-1}$ and thus by $\sigma_1 \Gamma_{\sigma_1} \sigma_1^{-1} \cap \Gamma = \Gamma_{\sigma_1} \cap \Gamma_{\sigma_1}$, which by modularity contains some subgroup $\Gamma_{\sigma_2} \in S$. Thus $\sigma_1 (V_{\Gamma_\sigma})$ is contained in $V_{\Gamma_{\sigma_2}}$ and consequently $\sigma (H_{\Gamma_\sigma})$ is contained in $H_{\Gamma_{\sigma_2}}$.

(In particular, $\sigma (H_{\Gamma_{\sigma^{-1}}}) = H_{\Gamma_\sigma}$.)

Thus $G$ acts on $H^{ad} = \bigcup_{\Gamma_\sigma \in \rho} H_{\Gamma_\sigma}$ and the inductive limit of Hilbert spaces (since all the inclusions are isometric) carries a natural inductive limit Hilbert space prenorm. Let $H^{ad}$ be Hilbert space completion of $H^{ad}$.

We assume that $G$ acts unitarily on $H^{ad}$, and this what we will call the adelic representation of $G$. 
The following axiom will not be used, although it is true in all examples. It relates the Hilbert space $H$ with the Hilbert spaces $H^{Γ_σ}$, $Γ_σ ∈ S$.

4) We assume that there exist $( , )$ a pairing between a dense subspace of $H$ and the Hilbert space $H^{Γ}$ such that if for $Γ_σ ∈ S$, and $v, w ∈ H^{Γ_σ}$ and there exists a vector $ξ$ in $V$, such that $v = \sum_{γ ∈ Γ_σ} γξ$, for the topology on $V$, then $(v, w)_{H^{Γ_σ}} = \frac{1}{|Γ : Γ_σ|} (ξ, w)$.

In the following we will describe the orthogonal projection from $H^{Γ_σ}$ onto $H^{Γ}$. This will then be used to define an abstract Hecke operator.

**Definition 2.** Fix $Γ_σ_0 ≥ Γ_σ_1$ two subgroups in $S$ and denote by $P_{H^{Γ_σ_0}}$ the orthogonal projection from $H^{ad}$ onto $H_{Γ_0}$ and by $P_{H^{Γ_σ_1}}$ the restriction of $P_{H^{Γ_σ_0}}$ to $H_{Γ_σ_1}$ (which is the same as the orthogonal projection from $H_{Γ_σ_1}$ onto $H_{Γ_0}$).

When $Γ_σ_0 = Γ$, we denote simply by $P_{H^{Γ_σ_1}}$ the projection.

The projection $P_{H^{Γ_σ_1}}$ has the following property

**Lemma 3.** For all $v$ in $H^{ad}$, $a$ in $Γ_σ$, $P_{H^{Γ_σ_1}}(av) = P_{H^{Γ_σ_1}}(v)$. To give a suggestive description of this property we will write $P_{H^{Γ_σ_1}}([Γ_σ]v) = P_{H^{Γ_σ_1}}(v)$.

**Proof.** Indeed for all $w ∈ H^{Γ_σ_1}$ we have

$$(P_{H^{Γ_σ_1}}(av), w)_{H^{Γ_σ_1}} = (av, w)_{H^{Γ_σ_1}} = (v, a^{-1}w)_{H^{Γ_σ_1}} = (v, w)_{H^{Γ_σ_1}}.$$  

The following proposition is almost contained in Sarnak [Sa].

**Proposition 4.** Let $Γ_σ$ in $S$ and let $(s_i)_{i=1}^n$ (where $n$ is the index $[Γ : Γ_σ]$) be a system of right coset, representatives for $Γ_σ$ in $Γ$ (that is $Γ = \bigcup_{i=1}^n s_iΓ_σ$). Define $Q_σ : V → V$ by the formula $Q_σ v = \frac{1}{n} (\sum_{i=1}^n s_i v)$, $v ∈ V$.

Then $Q_σ|_{H^{Γ_σ_1}}$ is the orthogonal projection from $H^{Γ_σ_1}$ onto $H^{Γ}$.

**Proof.** First, we note that indeed $Q_σ$ is a projection from $H^{Γ_σ_1}$ onto $H^{Γ}$. Indeed, for all $γ ∈ Γ$, and for every $i$ in $[1, 2, ..., n]$ there exists $θ_i(γ)$ an element in $Γ_σ$ and $π_γ$ a permutation of $[1, 2, ..., n]$ such that

$$γs_i = s_{π_γ(i)}θ_γ(i).$$

Hence for all $v$ in $V^{Γ_σ_1}$ (by the argument in [Sa]), for $v$ in $V^{Γ_σ_1}$

$$γ(Q_σ v) = \frac{1}{n} \sum_{i=1}^n γs_i v = \frac{1}{n} \sum_{i=1}^n s_{π_γ(i)}θ_γ(i) v = \frac{1}{n} \sum_{i=1}^n s_i v = Q_σ(v).$$
(This since \( \theta v = v \) for all \( \theta \) in \( \Gamma \)). Since \( Q_\sigma \) is obviously the identity when restricted to \( V^\Gamma \), it follows that \( Q_\sigma \) is a projection onto \( H^\Gamma \).

The complete the proof we have to show that \( Q_\sigma \) is indeed an orthogonal projection, i.e., that the adjoint of \( Q_\sigma \) is equal to \( Q_\sigma \).

For \( v, w \) in \( H^{\Gamma_\sigma} \) we have

\[
\langle Q_\sigma v, w \rangle_{H^{\Gamma_\sigma}} = \frac{1}{n} \sum_{i=1}^{n} \langle s_i v, w \rangle_{H^{\Gamma_\sigma}} = \frac{1}{n} \sum_{i=1}^{n} \langle v, s_i^{-1} w \rangle_{H^{\Gamma_\sigma}} = \frac{1}{n} \sum_{i=1}^{n} \langle v, P_{H^{\Gamma_\sigma}}(s_i^{-1} w) \rangle.
\]

Hence for \( w \) in \( H^{\Gamma_\sigma} \)

\[
Q_\sigma w = \frac{1}{n} \sum_{i=1}^{n} P_{H^{\Gamma_\sigma}}(s_i^{-1} w)
\]

and by using the notation in the previous lemma we have

\[
(Q_\sigma)^*(w) = \frac{1}{n} \sum_{i=1}^{n} P_{H^{\Gamma_\sigma}}([\Gamma]s_i^{-1} w).
\]

But \( \Gamma = \bigcup s_i \Gamma_\sigma \) and hence \( \Gamma = \bigcup \Gamma_\sigma(s_i)^{-1} \) and hence we can arrange by taking appropriate representatives for the right cosets of \( \Gamma_\sigma \) that

\[
(Q_\sigma)^*(w) = \frac{1}{n} \sum_{i=1}^{n} P_{H^{\Gamma_\sigma}}(s_i w) + \frac{1}{n} \sum_{i=1}^{n} s_i w.
\]

Since \( \sum_{i=1}^{n} s_i w \) is already in \( H^\Gamma \supseteq H^{\Gamma_\sigma} \) this is further equal to

\[
\sum_{i=1}^{n} s_i w = \theta_\sigma(w).
\]

Thus \( Q_\sigma \) is a selfadjoint projection. We note as a consequence of the previous proof that \( P_{H^\Gamma}(s\sigma v) = P(\sigma v) \) for all \( v \) in \( H^\Gamma \), \( s \) in \( \Gamma \), \( \sigma \) in \( G \). Indeed in this case \( \sigma v \) is in \( H^{\Gamma_\sigma} \) and hence

\[
P_{H^\Gamma}(\sigma v) = P_{H^\Gamma}([\Gamma] \sigma v). \quad \square
\]

As a corollary, we have the following equivalent description of the Hecke operator.

**Proposition 5.** Fix \( \sigma \) in \( G \). Let \( T_\sigma : H^\Gamma \to H^\Gamma \) be the abstract Hecke operator, defined by the formula

\[
T_\sigma v = \sum_{i=1}^{n} s_i \sigma v, \quad v \in H^\Gamma
\]

where \( s_i \) is a system of representatives for right cosets for \( \Gamma_\sigma \) in \( \Gamma \) (that is \( \Gamma = \bigcup s_i \Gamma_\sigma \) ).
Let \( P_{H_r}^{H_r^\sigma} \) be the orthogonal projection from \( H_r^\sigma \) onto \( H_r^\Gamma \) and note that \( \sigma \) belongs to \( H_r^\Gamma \). Then

\[
T_\sigma v = P_{H_r}^{H_r^\sigma}(\sigma v) = P_{H_r}^{H_r^\sigma}([\Gamma \sigma \Gamma]v),
\]

(where the last term of the equality is rather a notation to suggest that it doesn’t depend on which element in the coset we choose: that is \( P_{H_r}^{H_r^\sigma}(\sigma) = P_{H_r}^{H_r^\sigma}(\sigma_1 v) \) for all \( \sigma_1 \) in \( \Gamma \sigma \Gamma \); also \( \Gamma_{\sigma_1} = \Gamma_\sigma \) if \( \sigma_1 = \gamma_1 \sigma \gamma_2 \)).

**Proof.** This is a direct consequence of the last proposition and of the remark afterwards. \( \square \)

**Corollary 6.** The composition of the arrows in the following diagram gives the Hecke operator. Let \( \sigma \) in \( G \). The diagram is

\[
\begin{array}{cccc}
H_{\Gamma^{-1}} & \xrightarrow{\sigma} & H_{\Gamma} & \xrightarrow{\text{inc}} & V \\
& \circlearrowleft & & & \circlearrowright \quad P_{H_r}^{H_r^\sigma}
\end{array}
\]

Bellow, we present some basic examples of this construction. The first example corresponds to the induced \( C^* \)-Hecke algebra \((BC)\) which also assigns a canonical norm on the Hecke algebra (which is generated by the Hecke operator).

**Example 7.** Let \( V \) consists in function on the discrete group \( G \), and let \( G \) act by left translation. We let \( H_{\ell^2}(G) \) and define \( H_{\ell^2}(\Gamma/G) \subseteq V^I \) (since cosets of \( \Gamma \) are \( \Gamma \)-invariant functions).

We impose that the norm of \( [\Gamma \sigma] \) is 1, and then for the smaller cosets we renormalize that scalar product on \( \ell^2(\Gamma \sigma \backslash G) \) by the factor \( 1/[\Gamma : \Gamma \sigma] \). So that the canonical map \( \ell^2(\Gamma/G) \rightarrow \ell^2(\Gamma_\sigma/G) \) becomes an isometry.

In this setting \( s_i(\sigma \Gamma) \) is the coset \( s_i \sigma \Gamma \) and hence the \( (n = [\Gamma : \Gamma \sigma]) \) (non-normalized) Hecke operator \( nT_\sigma[\sigma_1 \Gamma] = \sum [s_i \sigma \sigma \Gamma] \).

This means that in this representation the Hecke operator \( nT_\sigma \) coincides with the multiplication by \( [\Gamma \sigma \Gamma] \) as defined in \((BC)\).

Thus the \( C^* \) algebra generated by the Hecke operators coincides with the Hecke algebra \( H_0 \) of double cosets. Recall \((BC)\) that if \( [\Gamma : \Gamma \sigma] = [\Gamma : \Gamma \sigma^{-1}] \) for all \( \Gamma \) in \( S \), then the vector state \( [\Gamma, [\Gamma]] \) is a trace on \( H_0 \) and the reduced \( C^* \) Hecke algebra is the closure of \( H_0 \) in the topology induced by this norm. (Here \( H \subseteq B(\ell^2(\Gamma/G)) \) is the closure of the \( * \)-algebra \( H_0 \)).

Recall \((Kr)\) that in the case \( G = \text{PSL}_2(\mathbb{Z}[\frac{1}{p}]) \), \( \Gamma = \text{PSL}_2(\mathbb{Z}) \) and \( \sigma_{p^n} = \begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix} \), then if \( \chi_n = [\Gamma_{p^n \Gamma}] \), the cosets \( \chi_n \) generate the Hecke algebra and are selfadjoint. The relation for the elements \( \chi_n \) are as follows

\[
\chi_1 \chi_n = \begin{cases} 
\chi_2 + (p + 1) \text{Id} & \text{if } n = 1, \\
\chi_{n+1} + p\chi_{n-1} & \text{if } n \geq 2.
\end{cases}
\]
and the value of the state $\langle \Gamma, \Gamma \rangle$ on $\chi_n$ is 0 unless $n = 0$, when the value is 1.

By comparing with [Py], we see that these are exactly the relations verified by the elements of the radial algebra of a free group with $N = \frac{p+1}{2}$ generators.

We can define polynomials $t_n(\lambda)$ by the recurrence relations above

$$t_1(\lambda)t_n(\lambda) = \begin{cases} t_2(\lambda) + 2N & \text{if } n = 1, \\ t_{n+1}(\lambda) + (2N - 1)t_{n-1}(\lambda) & \text{if } n \geq 2. \end{cases}$$

Let $\varphi_\lambda$ be the character of the $*$-algebra $\mathcal{H}_0$ define by requiring $\varphi_\lambda(\chi_1) = \lambda$ (and thus $\varphi_\lambda(\chi_n) = t_n(\lambda)$). It turns out ([PySw], see also [SwHa]) that $\varphi_\lambda$ is positive for $\lambda$ in $[-2N, 2N] = [-p+1, p+1]$ but if $\lambda$ is $[-2w, 2w]$, where $w = \sqrt{p}$, then $\varphi_\lambda$ is a state on the reduced $C^*$-algebra (it is actually a positive definite function on $F_N$, affiliated with the left regular representation). Thus the spectrum of $\chi_1$ in the reduced $C^*$-algebra is equal to $[-2w, 2w] = [-2\sqrt{p}, 2\sqrt{p}]$ and thus $\|\chi_1\| = 2\sqrt{p}$.

In particular, the norm of $[\Gamma\sigma_p\Gamma]$ in the reduced $C^*$-Hecke algebra is equal to $2\sqrt{p}$.

It is thus natural, in view of this example to formulate a generalized Ramanujan-Peterson conjecture as follows.

**Definition 8.** Generalized Ramanujan-Peterson conjecture for an adelic representation of a discrete group $G$, containing an almost normal subgroup $\Gamma$, such that the subgroups $\Gamma_\sigma = \sigma\Gamma\sigma^{-1} \cap \Gamma$ form a modular family and $[\Gamma : \Gamma_\sigma] = [\Gamma : \Gamma^{-1}_\sigma]$ (and thus $[\Gamma\sigma\Gamma] = [\Gamma\sigma^{-1}\Gamma]$) for all $\sigma$ in $G$. For all $\sigma$ in $G$, let $T_\sigma$ be the corresponding Hecke operator acting on $H^\Gamma$.

The claim of the conjecture is that $\|T_\sigma\| = \|[\Gamma\sigma\Gamma]||$, where the norm of $[\Gamma\sigma\Gamma]$ is calculated in the reduced $C^*$-Hecke algebra of double cosets of $\Gamma$ in $G$.

Equivalently, for any adelic representation of $G$ on $H^{ad}$ (as in the sense of Definition 1) the $\Gamma$-equivalent states of $G$ from this representation are weak limits of $\Gamma$-invariant states of $G$ derived from the left regular representation of the Hecke algebra.

**Proof.** (of the equivalence of the two statements). Indeed a $\Gamma$-equivariant state of $G$ is of the form $\varphi(g) = \langle gv, v \rangle$, where $v$ is $H^\Gamma$. On the other hand, the Hecke algebra decomposes the algebra generalized by $G$.

Indeed, if $v, w$ are two vectors in $H^\Gamma$ such that $\langle T_\sigma v, w \rangle$ for all $\sigma$ in $G$ then $\langle [\Gamma\sigma\Gamma]v, w \rangle$ for all $\sigma$ and thus $\langle gv, w \rangle = 0$ for all $g$ in $G$. \qed

**Remark 9.** In the case of $G = \text{PSL}_2(\mathbb{Z}[\frac{1}{p}])$, $\Gamma = \text{PSL}_2(\mathbb{Z})$ the positives states on $\mathcal{H}_0$, are $\varphi_\lambda$, $\lambda \in [-p+1, p+1]$.

In general, a positive state on $\mathcal{H}_0$ is necessary a positive state on $G$ (see [Ha]) but in the case of $\text{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ all such states are positive definite on $G$, and hence cannot he excluded a priori.

We now describe a second example, more related to operators algebra. The essential data here is a projective unitary representation $\pi$ (with cocycle $\varepsilon$) which
extends to \( G \) the left regular representation with cocycle \( \varepsilon \) of \( \Gamma \), on \( \ell^2(\Gamma) \), by keeping the same Hilbert space as the left regular representation.

**Example 10.** Let \( G, \Gamma \) as above; \( \pi \) a unitary representation of \( G \) on \( H = \ell^2(\Gamma) \) extending the left regular representation. Then, let \( \mathcal{V} = B(H) \), let \( G \) act on \( \mathcal{V} \) by \( \text{Ad}(\pi(g)) \). Then \( \mathcal{V}^\Gamma = \{ \pi(\Gamma) \}' \cong R(\Gamma) \). Let \( H = L_2(\mathcal{L}(G)) \cong \ell^2(G) \).

Hence \( H^\Gamma = \ell^2(\mathcal{L}(\Gamma)) \cong \ell^2(\Gamma) \) and naturally \( H^\Gamma \sigma = L^2(\mathcal{L}(\Gamma\sigma))' \), where \( \mathcal{L}(\Gamma\sigma)' \) is endowed with the normalized trace.

Then clearly, \( T_\sigma = \Psi_\sigma \) is a map from \( \ell^2(\Gamma) \) into \( \ell^2(\Gamma) \) induced by the map on \( (\mathcal{L}(\Gamma))' \) given as

\[
\Psi_\sigma(X) = E(\pi(\sigma)X\pi(\sigma)^{-1}) = \frac{1}{n} \sum \pi(s\sigma)X\pi(\sigma s)^{-1}
\]

for \( x \in \mathcal{L}(\Gamma) \). Note that \( \Phi_\sigma \) is a completely positive map.

The classical setting also fits into this pattern:

**Example 11.** Classical setting of Hecke operators acting on Maass forms. Let \( G = \text{PSL}_2(\mathbb{Z}[\frac{1}{p}]), \Gamma = \text{PSL}_2(\mathbb{Z}) \). The group \( G \) acts naturally on the upper halfplane \( \mathbb{H} \) by Moebius transforms. Here \( \mathcal{V} \) is the space of measurable functions on \( \mathcal{H} \), and \( G \) acts on a function \( f \) by mapping it into \( gf(z) = f(g^{-1}z), z \in \mathbb{H} \) and \( H^\Gamma = L^2(F_\Gamma, \nu_0), H^\Gamma \sigma = L^2(F_{\Gamma\sigma}, \frac{1}{|\varepsilon|} \nu_0) \), (where \( F_{\Gamma\sigma} \) is a fundamental domain for the action of the discrete group \( \Gamma_\sigma \) on the upper half plane \( \mathbb{H} \)). Here, \( T_\sigma f(z) = \sum f(s_i\sigma z) \), with \( s_i \) a system of representatives of left cosets of \( \Gamma_\sigma \) in \( \Gamma \). Let \( \sigma_p = \left( \begin{array}{cc} p^n & 0 \\ 0 & 1 \end{array} \right), n \in \mathbb{N} \). Then the Hecke operator, \( T_{\sigma_p} f(z) \), for \( n = 1 \) has the form.

\[
\sum_{d=0}^{p-1} f \left( \frac{zd + d}{p} \right) + f(pz), z \in \mathbb{H}.
\]

In the next chapter we explain why Example 11 is equivalent to Example 10 in the case \( G = \text{PSL}_2(\mathbb{Z}[\frac{1}{p}]), \Gamma = \text{PSL}_2(\mathbb{Z}) \).

Of course, the Hecke operators acting on automorphic forms are another example of this setting.

2. **Abstract Hecke Operators on \( \Pi_1 \) Factors**

In this section we introduce the abstract Hecke operators, associated with a pair of isomorphic subfactors, of equal indices, of a given factor \( M \).

In the case \( M = \mathcal{L}(\text{PSL}_2(\mathbb{Z})) \) we prove that with a suitable choice of the unitary implementing the isomorphism, one recovers the classical Hecke operators acting on Maass forms. This model will be based on the Berezin’s quantization of the upper half plane introduced in [Ra1], [Ra2].

First, we introduce the definition of an abstract Hecke operator.

**Definition 12.** Let \( M \) be a type \( \Pi_1 \) factor and let \( P_0, P_1 \) be two subfactors of finite equal indices.
Let $\theta : P_0 \to P_1$ be a von Neumann algebras isomorphism. Let $U$ be a unitary in $\mathcal{U}(L^2(M))$, that implements $\theta$, that is $U p U^* = \theta(p)$ for all $p$ in $P_0$. Since $P_0$, $P_1$ have equal indices there always exists such a unitary, which is unique up to left multiplication by a unitary in $P_1'$. Then $UP_0[U = P_1'$ and hence we can define $\Psi_U$ as the composition of the following diagram:

\[
\begin{array}{c}
E \\ \downarrow \\
\mathcal{P}'_1 \\
\uparrow \Ad U \\
\mathcal{P}'_0 \subseteq P_0'
\end{array}
\]

that is $\Psi_U(x) = E_{\mathcal{P}'_1}(U x U^*)$, $x \in \mathcal{P}'_0$.

**Remark 13.** If $\theta$ can be extended to an automorphism $\tilde{\theta}$ of $M$, then we can choose $U$ such that $U x U^* = \tilde{\theta}(x)$, for $x \in \mathcal{M}$ and hence in this case it follows that $UM'U^* = \mathcal{M}'$ and hence $\Psi_U(x)$ is simply $U x U^*$, $x \in \mathcal{M}'$, that is $\Psi_U$ is an automorphism of $\mathcal{M}'$.

To get a more exact description of $\Psi_U$ in the case of group von Neumann algebras, we need a more precise formula for the conditional expectation on the case of $\Gamma_1 \subseteq \Gamma$ a subgroup of a discrete group of finite index.

**Lemma 14.** Let $\Gamma$ be a discrete group and let $\Gamma_1$ be a discrete subgroup of finite index.

Let $\Gamma_1$ act on $\ell^2(\Gamma)$, and let $\mathcal{L}(\Gamma_1)'$ be the commutant of $\mathcal{L}(\Gamma_1)$ in $B(\ell^2(\Gamma))$. Then the conditional expectation $E_{\mathcal{L}(\Gamma_1)'}^{\mathcal{L}(\Gamma)'\Gamma}$ from $\mathcal{L}(\Gamma_1)'$ onto $\mathcal{L}(\Gamma)'$ is defined by following formula: choose $(s_i)_{i=1}^n$ be a system of representatives for right cosets for $\Gamma_1$ in $\Gamma$ (that is $\Gamma = \bigcup_{i=1}^n s_i \Gamma_1$ disjointly).

Denote by $L_{s_i}$ the operator of left convolution with $s_i$ acting on $\ell^2(\Gamma)$. Then

\[
E_{\mathcal{L}(\Gamma_1)'}^{\mathcal{L}(\Gamma)'\Gamma}(x) = \frac{1}{n} \sum_{i=1}^n L_{s_i} x L_{s_i}^*, \quad x \in \mathcal{L}(\Gamma_1)'
\]

This formula is reminiscent of the average formula in the definition of a double coset action on Maass forms.

**Proof.** The lemma is certainly well known for specialists in von Neumann algebras although we can not find a citation. For the sake of completeness we include the proof.

The proof is identical to the argument used for proving that Hecke operators, map $\text{PSL}_2(\mathbb{Z})$-invariant functions into $\text{PSL}_2(\mathbb{Z})$-invariant functions.

For every $\gamma$ in $\Gamma$ there exists a permutation $\pi_\gamma$ of $\{1, 2, \ldots, n\}$ such that

$\gamma s_i = s_{\pi_\gamma(i)} \theta_i(\gamma), \quad i = 1, 2, \ldots, n.$
Here $\pi_\gamma(i)$ is uniquely determined by the requirement that the element $\theta_i(\gamma) = s^{-1}_{\pi_\gamma(i)}\gamma s_i$ belongs to $\Gamma_1$.

Denoting for $x$ in $L(\Gamma_1)'$ by $E(x)$ the expression

$$E(x) = \frac{1}{n} \sum_{i=1}^{n} L_{s_i} x L_{s_i}^*,$$

we have that for all $\gamma$ in $\Gamma$

$$L_{\gamma}E(x)L_{\gamma} = \frac{1}{n} \sum_{i=1}^{n} L_{\gamma s_i} x L_{\gamma s_i}^* = \frac{1}{n} \sum_{i=1}^{n} L_{s_{\gamma s_i}(i)} L_{\theta_i(\gamma)} x L_{\theta_i(\gamma)} s_{\gamma s_i}(i),$$

Since $x$ belongs to $L(\Gamma_1)'$, and $\theta_i(\gamma)$ belongs to $\Gamma_1$, it follows that $L_{\theta_i(\gamma)} x L_{\theta_i(\gamma)} = x$ and hence that

$$L_{\gamma}E(x)L_{\gamma} = \frac{1}{n} \sum_{i=1}^{n} L_{s_{\gamma s_i}(i)} x L_{s_{\gamma s_i}(i)} = E(x).$$

Hence $E(x)$ belongs to $L(\Gamma_1)'$ for all $x$ in $L(\Gamma_1)'$. Moreover, it is obvious that $E$ is positive and $E(x) = x$ for $x$ in $L(\Gamma_1)'$. Hence $E$ is the conditional expectation $E_{L(\Gamma_1)'}$. The fact that $E$ is selfadjoint was proved in the previous chapter. This completes the proof. \(\square\)

Using this lemma we can conclude the unitary equivalence of the abstract Hecke operators (in the case of $\Gamma = PSL_2(\mathbb{Z})$) for a specific choice of the unitary $U$ implements the algebra morphism, with the classical Hecke operators on Maass forms. This has been observed in [Ra2], we recall the argument for the comfort of the reader.

The analytic discrete series $\pi_n$, $n \geq 2$ of representations of $PSL_2(\mathbb{R})$ is realized by considering the Hilbert space $H_n = H^2(\mathbb{H}, d\mu_n)$ of analytic square summable functions on the upper half plane $\mathbb{H} = \{ z \in \mathbb{C} | \text{Im} z > 0 \}$ with the measure $d\mu_n = (\text{Im} z)^{n-2} dz$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $PSL_2(\mathbb{R})$, with the standard action on $\mathbb{H}$, and automorphy factor $j(g, z) = (cz + d)$, $z \in \mathbb{H}$, the formula of the action is

$$\pi(g) f(z) = f(g^{-1} z) j(g, z)^{-n}, \quad f \in H_n, z \in \mathbb{H}.$$

For even $n$ this corresponds to a projective representation of $PSL_2(\mathbb{R})$ (thanks to the anonymous referee of a first submitted version of the paper, who reminded me this detail). We denote the 2-cocycle corresponding to the projective representation by $\varepsilon$, and note that it only takes the values $\pm 1$.

As a particular case of the results in [GJ], the space $H_{13}$ is unitarily equivalent to $\ell^2(P SL_2(\mathbb{Z}))$ by a unitary that transforms $\pi_{13}(\gamma)$ for $\gamma$ in $\Gamma = PSL_2(\mathbb{Z})$ into the unitary operator of left convolution with $\gamma$ on $\ell^2(\Gamma)$.
Another way to rephrase this is to say that the Hilbert space $H_{13}$ contains a vector $\xi$ such that $\langle \pi_{13}(\gamma) \xi, \xi \rangle = 0$ with the exception of the case $\gamma = e$.

In [Ra1] we proved that the commutant $A_{13} = \{ \pi_{13}(\Gamma) \}' \subseteq B(H_{13})$ (which is thus isomorphic to $L(P\text{SL}_2(\mathbb{Z}), \varepsilon)$-the $\varepsilon$ skewed $I_1$ associated to the discrete group $P\text{SL}_2(\mathbb{Z})$) can be described as the space of bivariant kernels $k : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ that are analytic in the first variable and anti-analytic in the second variable, and that are $\Gamma$-bivariant, that is $k(\gamma z, \gamma \eta) = k(z, \eta)$ for all $\gamma \in \Gamma$, $z, \eta \in \mathbb{H}$. One supplementary condition on $k$ is that it generates a bounded operator $X$ on $H_{13}$, via the reproducing kernel formula

$$(Xf)(z) = \int_{\mathbb{H}} k(z, \eta) f(\eta) d\mu_{13}(\eta)$$

for $z \in \mathbb{H}$, $f$ in $H_{13}$. It is obvious that $X$ commutes with $\{ \pi_{13}(\Gamma) \}$, and thus belongs to $A_{13}$, because of the $\Gamma$-invariance of the kernel.

The uniform norm of $X$ is difficult to compute, but the trace of $X$ ($X$ is an element in the type $I_1$ factor $A_{13}$) is given by the formula

$$\tau_{A_{13}}(X) = \frac{1}{\mu_0(F)} \int_{\mathbb{H}} k(z, z) d\mu_0(z).$$

Hence the $L^2$-norm of $X$, that is $\tau_{A_{13}}(X^*X)^{1/2}$, is given by the formula

$$\tau_{A_{13}}(X^*X)^{1/2} = \frac{1}{\mu(F)} \int_{\mathbb{H}} \int_{\mathbb{H}} |k(z, \eta)|^2 d(z, \eta)^{13} d\mu_0(z) d\mu_0(\eta).$$

Here $d(z, \eta) = \frac{\sqrt{1 - |z - \eta|^2}}{\text{Im} z \text{Im} \eta}$ for $z, \eta \in \mathbb{H}$ is the cosine of the hyperbolic distance from $z$ to $\eta$.

In [Ra1] it was proven that $L^2(A_{13}, \tau)$ is isomorphic to the Hilbert space of functions on $F$, with scalar product formula

$$\langle f, g \rangle_{13} = \langle f, B_{13}(\Delta) \rangle_{L^2(F)},$$

where $B_{13}(\Delta)$ is a positive, injective, selfadjoint operator, a well determined function of the $G$-invariant Laplacian $\Delta$, which therefore commutes with all the Hecke operators. (In fact $B_{13}(\Delta)$ is determined by the point pair invariant function $d(z, \eta)^{13}$.)

The unitary map $\Phi_{13}$ from $L^2(A_{13}, \tau)$ into functions on $F$, is simply the restriction of $k$ to the diagonal. If $g$ is an element in $P\text{SL}_2(\mathbb{R})$ and $X$ is an element in $A_{13}$ represented by the kernel $k$, then $\pi_{13}(\gamma) x \pi_{13}^{-1}(\gamma)$ is represented by the kernel $\sigma_g(k)$ defined by the formula

$$(1) \quad \sigma_g(k)(z, \eta) = k(g^{-1}z, g^{-1}\eta), \quad z, \eta \in \mathbb{H}.$$
Proposition 15. Let $\Gamma = \text{PSL}_2(\mathbb{Z})$. Let $\Gamma \sigma \Gamma$ in $\text{PGL}_2(\mathbb{Q})^+$ be a double coset of $\Gamma$ in $\text{PGL}_2(\mathbb{Q}_+)$, where $\sigma \in \text{PGL}_2(\mathbb{Q}_+)$. Then the classical Hecke operator associated to $\sigma$, is defined, by using a system of representatives $(s_i)^n_{i=1}$ for right cosets of $\Gamma_\sigma = \Gamma \cap \sigma \Gamma \sigma^{-1}$ in $\Gamma = \text{PSL}_2(\mathbb{Z})$ by the following formula, for a $\Gamma$-invariant function on $\mathbb{H}$,

$$(\tilde{T}_\sigma f)(z) = \frac{1}{n} \sum_{i=1}^n f((s_i \sigma)^{-1}z), \quad z \in \mathbb{H}.$$ 

Let $\tilde{\Psi}_\sigma(x) = E_{\{\pi_{13}(\Gamma_\sigma)^\prime\}} (\pi_{13}(\sigma) x \pi_{13}(\sigma)^*)$ be the abstract Hecke operator associated, to $L(\Gamma_{\sigma-1}), L(\Gamma_{\sigma})$, and the isomorphism $\theta_\sigma(x) = \sigma x \sigma^{-1}, x \in L(\Gamma_{\sigma-1})$, and unitary $U_\sigma = \pi_{13}(\sigma)$.

Then $\tilde{\Psi}_\sigma$ is unitarily equivalent to $\tilde{T}_\sigma$, up to a scalar phase, $B_{13}(\Delta)$, on $L^2(F, d\nu_0)$. Since $B_{13}(\Delta)$ commutes with all Hecke operators on $L^2(F, \nu_0)$, $\tilde{\Psi}_\sigma$ and $\tilde{T}_\sigma$ have the same eigenvalues, and the eigenvectors are the same correspondence.

Proof. For the sake of completeness we verify that $\tilde{T}_\sigma$ maps $\Gamma$-invariant functions into $\Gamma$-invariant functions.

Since $s_i$ was a system of representatives for right cosets of $\Gamma_\sigma$ in $\Gamma$, that is $\Gamma = \bigcup_{i=1}^n s_i \Gamma_\sigma$ as a disjoint union it follows that for every $\gamma$ in $\Gamma$, there exists a permutation $\pi_\gamma$ of $\{1, 2, \ldots, n\}$ such that

$$\gamma s_i = s_{\pi_\gamma(i)} \theta_i(\gamma),$$

with

$$\theta_i(\gamma) = s_{\pi_\gamma(i)}^{-1} \pi_\gamma s_i$$

belongs to $\Gamma_\sigma$.

Hence for all $i = 1, 2, \ldots, n$

$$\gamma s_i \sigma = s_{\pi_\gamma(i)} \theta_i(\gamma) \sigma = s_{\pi_\gamma(i)} \sigma^{-1} \theta_i(\gamma) \sigma.$$

Note that $\theta_i(\gamma)$ belongs to $\Gamma_\sigma = \Gamma \cap \sigma \Gamma \sigma^{-1}$ and hence that $\sigma^{-1} \theta_i(\gamma) \sigma$ belongs to $\Gamma_{\sigma-1} = \Gamma \cap \sigma^{-1} \Gamma \sigma \subseteq \Gamma$.

As a consequence, if $f$ is a $\Gamma$-invariant function on $\mathbb{H}$, then for $z \in \mathbb{H}$, we have

$$(\tilde{T}_\sigma f)(\gamma^{-1}z) = \frac{1}{n} \sum_{i=1}^n f((s_\gamma)^{-1} \gamma^{-1}z) = \frac{1}{n} \sum_{i=1}^n f((\gamma s_i \sigma)^{-1}z) =$$

$$= \frac{1}{n} \sum_{i=1}^n f((s_{\pi_\gamma(i)} \theta_i(\gamma) \sigma)^{-1}z) = \frac{1}{n} \sum_{i=1}^n f((s_{\pi_\gamma(i)} \sigma \cdot (\sigma^{-1} \theta_i(\gamma) \sigma))^{-1}z) =$$

$$= \frac{1}{n} \sum_{i=1}^n f((\sigma^{-1} \theta_i(\gamma) \sigma)^{-1} (s_{\pi_\gamma(i)} \sigma)^{-1}z)$$
but $f$ is $\Gamma$-invariant, $\sigma^{-1} \theta(\gamma) \sigma$ belongs to $\Gamma$ and hence this is equal to
\[
\frac{1}{n} \sum_{i=1}^{n} f((s_{\pi,\gamma}(i))^{-1} z) = \tilde{T}_\sigma f(z).
\]
Hence $\tilde{T}_\sigma f$ is a $\Gamma$-invariant function on $\mathbb{H}$.

The abstract Hecke operator associated to the unitary $U_\sigma = \pi_{13}(\sigma)$ is defined for $x \in \{ \pi_{13}(\Gamma) \}'$, by the formula
\[
\Psi_\sigma(x) = E_{\{ \pi_{13}(\Gamma) \}}' (U_\sigma x U_\sigma^*) = \sum_{i=1}^{n} \frac{1}{n} \pi_{13}(s_i) U_\sigma x U_\sigma^* \pi_{13}(s_i)^*,
\]
where $s_i$ are a system of right representatives for $\Gamma_\sigma$ in $\Gamma$ (that is $\Gamma = \bigcup s_i \Gamma_\sigma$). Because $\pi_{13}(s_i) U_\sigma = \pi_{13}(s_i) \sigma$, if $x$ is represented by a kernel $k$, then by formula (1), we get that $\Psi_\sigma(x)$ is represented by the kernel $\frac{1}{n} \sum_{i=1}^{n} k((s_i \sigma)^{-1} z, (s_i \sigma)^{-1} \eta)$, $z, \eta \in \mathbb{H}$. If we identify $L^2(A_{13}, \tau)$ with the Hilbert space $L^2(F, d\mu_0)$ with scalar product $\langle f, g \rangle = \langle f, B_{13}(\Delta) g \rangle_{L^2(F)}$ then in this identification $\Psi_\sigma$ will thus map a function $f$ into the function $\tilde{\Psi}_\sigma(f)(z) = \frac{1}{n} \sum_{i=1}^{n} f((s_i \sigma)^{-1} z)$. But this is exactly the Hecke operator $\tilde{T}_\sigma$, at least as a linear map. The structure of eigenvector, eigenvalues and the selfadjointness is unchanged by the new scalar product since $B_{13}(\Delta)$, as a function of the invariant Laplacian, commutes with all Hecke operators. \hfill $\Box$

3. **Explicit description of the abstract Hecke operator in the subgroup case**

In this section we assume that $\Gamma$ is a discrete subgroup and let $\Gamma_0, \Gamma_1$ be two isomorphic subgroups of equal finite index. Let $\theta$ be an isomorphism between $\Gamma_0, \Gamma_1$ and let $U$ be a unitary in $B(\ell^2(\Gamma))$ that implements $\theta$ (we can always find such a unitary since the subgroups have equal index). For $\gamma$ in $\Gamma$ we denote by $L_\gamma, R_\gamma$ the operators of left and respectively right convolution on $\ell^2(\Gamma)$ by $\Gamma$. All the statements in the chapters that follows are also valid in the presence of a group 2-cocycle $\epsilon$ on the group $G$, which restricts to the group $\Gamma$. We will assume that all the partial automorphisms of $\Gamma$, $\text{Ad} \sigma, \sigma \in G$ are $\epsilon$ preserving. This certainly happens in the case $G = \text{PSL}_2(\mathbb{Z}^{[1/p]})$, $\Gamma = \text{PSL}_2(\mathbb{Z})$, since $\epsilon$ is in this case canonical. All the statements below are obviously also valid in the presence of such a 2-cocycle. In the appendix 1 we will provide an alternative approach for the case with cocycle.

More generally, for $m$ in $\ell^2(\Gamma)$, we denote by $L_m, R_m$ the (eventually unbounded) operator of left (respectively right) convolution on $\ell^2(\Gamma)$ with $m$.

By $\mathcal{L}(\Gamma)$ and respectively $R(\Gamma)$, we denote the algebra of left (respectively right) bounded convolutions on $\Gamma$. $\mathcal{L}(\Gamma)$ is then the type $\text{II}_1$ factor associated with $\Gamma$. When $\epsilon$ is present we will use instead the notation $\mathcal{L}(\Gamma, \epsilon)$ and respectively $R(\Gamma, \epsilon)$

Recall that the antilinear involution operator $J : \ell^2(\Gamma) \to \ell^2(\Gamma)$, defined by $Jx = x^*, x \in \ell^2(\Gamma)$ has the property that $\mathcal{J} \mathcal{L}(\Gamma) J = R(\Gamma)$ and $J L_m J = R_m^*$. 
We have that $R_a R_b = R_{ba}$, $a, b \in \ell^2(\Gamma)$ and $\Phi(L_x) = R_{x^*} = JL_x J$ is a * isomorphism from $L(\Gamma)$ onto $R(\Gamma)$.

Moreover, for the von Neumann algebra of a group, the conjugation map $\gamma \mapsto \phi(\gamma)$ which maps $\sum_{\gamma \in \Gamma} a_\gamma \gamma$ into $\sum_{\gamma \in \Gamma} \phi(\gamma) a_\gamma \gamma$ is antilinear isomorphism of von Neumann algebras (from $L(\Gamma)$ onto $L(\Gamma)$).

Now, if $U$ is a unitary implementing $\phi$ that is, $UL_\gamma U^* = L_{\phi(\gamma)}$ for $\gamma$ in $\Gamma_0$, we obtain an expression for

\[ \Psi_U(R_x) = E_{R(\Gamma)}^R(U R_x U^*). \]

We will transfer, via the isomorphism $\Phi(L_x) = R_{x^*}$, this map to a completely positive map representation on $L(\Gamma)$. The ingredients for the explicit expression of $\Psi_U$ are the unit vector $t_i = U s_i$, where $s_i \in \Gamma \subseteq \ell^2(\Gamma)$ is a system of representatives for left cosets $\Gamma_0$ in $\Gamma$. Since $U$ maps $L(\Gamma_0)$ into $L(\Gamma_1)$ and since $\{s_i\}_i$ are a Pimsner–Popa basis [PP] for $L(\Gamma_0) \subseteq L(\Gamma)$ it follows that $(t_i)_i$ are Pimsner–Popa basis for $L(\Gamma_1) \subseteq L(\Gamma)$. More precisely that means $\ell^2(\Gamma)$ is the orthogonal sum of the subspaces $\ell^2(\Gamma_1) t_i$ and $\langle \gamma_1 t_i, \gamma_2 t_i \rangle_{L(\Gamma)}$ is equal to zero unless $\gamma_1 = \gamma_2$.

The properties of $t_i$ relative to $L(\Gamma_1)$ can be also expressed by saying that $\tau(\gamma t_i t_j^*)$ is zero unless $i = j$ and $\gamma$ is the identity, that is $E_{L(\Gamma_1)}(t_i t_j^*)$ is zero unless $i = j$ and in this case $E_{L(\Gamma_1)}(t_i t_i^*) = 1$.

To prove the result we need first a lemma, which gives a tool for calculating conditional expectations from elements in $L(\Gamma_1)'$ onto $R(\Gamma) = L(\Gamma)'$.

**Lemma 16.** Let $\Gamma$ be a discrete group and let $\Gamma_1$ be a subgroup of finite index.

Let $a, b$ two vectors in $\ell^2(\Gamma_1)$, that are left $\Gamma_1$ orthonormal, that is $E_{L(\Gamma_1)}(aa^*) = E_{L(\Gamma_1)}(bb^*) = 1$. Fix an element $m$ in $L(\Gamma_1)$ and consider the operator $V_{ab}^m$ acting on $\ell^2(\Gamma)$, with initial space $\ell^2(\Gamma_1) a$ and range contained in $\ell^2(\Gamma_1) b$ given by the formula

\[ V_{ab}^m(\gamma_1 a) = \gamma_1 mb. \]

Then $V_{ab}^m$ belongs to $L(\Gamma_1)'$ and $E_{L(\Gamma_1)'}(V_{ab}^m) = R^*_{a^*mb}$ (here the product $a^* mb$ is computed in $L(\Gamma)$).

**Proof.** Let $V_a$ (respectively $V_b$) be the partial isometries with initial space $\ell^2(\Gamma_1)$ and range $\ell^2(\Gamma_1) a$ and $\ell^2(\Gamma_1) b$ respectively.

Note that $V_a, V_b$ are partial isometries because $a, b$ are left orthonormal with respect to $L(\Gamma_1)$. Indeed, the relation $E_{L(\Gamma_1)}(aa^*) = 1$ implies that for $\gamma m \Gamma_1$, $\tau_{L(\Gamma_1)}(\gamma a a^*)$ is zero unless $\gamma$ is the identity and hence \( \langle \gamma_1 a, \gamma_2 a \rangle_{L(\Gamma)} = \tau(\gamma_2^{-1} \gamma_1 aa^*) \) is zero unless $\gamma_1 = \gamma_2$. Similarly for $V_b$.

If $e$ is the projection from $\ell^2(\Gamma)$ onto $\ell^2(\Gamma_1)$ then $e \in L(\Gamma_1)'$ and

\[ V_a = R_a e \quad \text{and} \quad V_b = R_b e. \]
Clearly, being an isometry $V^*_a$ is the partial isometry that maps $\gamma_1a$ into $\gamma_1$ for $\gamma_1$ in $\Gamma_1$. Consequently,

$$V^*_a = V_bR_mV^*_a = R_beR_mR^*_a$$

But if we use the map on $B(\ell^2(\Gamma))$ given by $x$ goes into $Jx^*J$ then $R(\Gamma)$ is mapped into $\mathcal{L}(\Gamma)$, $\mathcal{L}(\Gamma'_1)$ is mapped into $J\mathcal{L}(\Gamma'_1)'J$ and $JeJ = e$. The inclusion $R(\Gamma) \subseteq \mathcal{L}(\Gamma'_1)'$ is mapped into the first step of the Jones basic construction for $\mathcal{L}(\Gamma'_1) \subseteq \mathcal{L}(\Gamma)$.

Hence $e$ commutes with $R_m$ and $E_{\mathcal{L}(\Gamma'_1)'}(e) = \frac{1}{[\Gamma : \Gamma_1]}$.

Thus $V^*_a = R_bR_m^*eR_a^*$ and since $R_b, R_m, R^*_a$ belongs to $\mathcal{L}(\Gamma)'$, it follows that

$$E_{\mathcal{L}(\Gamma)'}(V^*_a) = \frac{1}{[\Gamma : \Gamma_1]}R_bR_mR^*_a$$

which is further equal to

$$\frac{1}{[\Gamma : \Gamma_1]}R_\alpha^*m_b.$$  \(\square\)

As an exemplification we note the following corollary, which is certainly known to specialists. We include it as an exemplification.

**Corollary.** Let $t$ in $\ell^2(\Gamma)$ be left orthonormal with respect to $\Gamma_1$ (that is $E_{\mathcal{L}(\Gamma'_1)}(tt^*) = 1$). Let $(s_i)_{i=1}^n$ be a system of right representatives for $\Gamma_1$ in $\Gamma$, that is $\Gamma$ is the disjoint union of $s_i\Gamma_1$.

Denote by $P_{[s_i,\Gamma]}$ the projection onto the space $S_{[t]}s_i\Gamma t$. Then

$$\sum_{t} P_{[s_i,\Gamma]} = R_{t^*t}.\$$

If we use the map $J \cdot J$ we get in $\mathcal{L}(\Gamma)$ that in $\mathcal{L}(\Gamma)$, if $r_\alpha$ is a system of representatives for left cosets of $\Gamma_1$ in $\Gamma$ (that is $\Gamma = \bigcup\Gamma_1r_\alpha$) then

$$\sum_{\alpha=1}^n P_{[t,\Gamma_1r_\alpha]} = t^*t.$$

**Proof.** The projection $P_{[\Gamma_1,t]}$ clearly belongs to $\mathcal{L}(\Gamma'_1)$ since it is invariant to left multiplication by $\Gamma_1$. In the terminology of the previous lemma we have that

$$p = P_{[\Gamma_1,t]} = v_{t^*t}^1$$

and hence

$$E_{\mathcal{L}(\Gamma)'}(p) = \frac{1}{[\Gamma : \Gamma_1]}R_{t^*t}.\$$

Now by Lemma 3, since $s_i$ is a system of right representatives for $\Gamma_1$ in $\Gamma$ it follows that

$$E_{\mathcal{L}(\Gamma)'}(p) = \frac{1}{[\Gamma : \Gamma_1]}\sum_{i=1}^n L_{s_i}pL_{s_i}'.$$
But $L_{s_i}P_{[\Gamma_1 t]}L_{s_i}^* = P_{[s_i \Gamma_1 t]}$. Hence
\[
\sum_{i=1}^n P_{[s_i \Gamma_1 t]} = R_{t^* t}.
\]
If we apply the conjugation map $J \cdot J$, the space $s_i \Gamma t$ gets mapped into $J(s_i \Gamma t) = t^* \Gamma s_i^{-1}$ and $J(R_{t^* t})J = L_{t^* t}$. But $(s_i^{-1})_{i=1}^n$ is there a system of left representatives for $\Gamma_1$ in $\Gamma$ and the result follows. \qed

We can now prove the main result of this section, which gives a concrete expression for the completely positive map $\Psi_\sigma(R_x) = E_{L(\Gamma_1)'}^L(uxu^*)$. We will also describe this map as an operator from $L(\Gamma)$ into $L(\Gamma)$.

**Theorem 17.** Let $\Gamma$ be a discrete subgroup and let $\Gamma_0, \Gamma_1$ be two isomorphic subgroups of equal finite index. Let $\theta$ be an isomorphism from $\Gamma_0$ onto $\Gamma_1$ and assume that $U$ is a unitary in $B(l^2(\Gamma))$ that implements $\theta$, that is $UL_{\gamma_0} = L\theta(\gamma_0)U$, for $\gamma_0$ in $\Gamma_0$.

Let $\Psi_0 : L(\Gamma)' \to L(\Gamma_1)'$ be the corresponding completely positive map, defined by the formula
\[
\Psi_U(x) = E_{L(\Gamma_1)'}(Uxu^*), \quad x \in L(\Gamma_1)'.
\]

Let $(s_i)_{i=1}^n$, with $n = [\Gamma : \Gamma_0] = [\Gamma : \Gamma_1]$ be a system of representatives for left cosets for $\Gamma_0$ in $\Gamma$, that is $\Gamma = \bigcup \Gamma_0 s_i$. Let $t_1 = U(s_i)$, $i = 1, 2, \ldots, n$, which as we observed before have the property that $E_{L(\Gamma_1)}(t_i^* t_j) = \delta_{ij}$. Then
\[
\Psi_U(R_x) = \sum_{i,j} R_{t_i^*}\theta(E_{L(\Gamma_0)}(s_is_j^*))t_j.
\]

Viewed as map from $L(\Gamma)$ onto $L(\Gamma)$ (via the identification of $L_x$ with $R_x^*$) the formula becomes
\[
\Psi_U(x) = \sum_{i,j=1}^n t_i^*\theta(E_{L(\Gamma_0)}(s_is_j^*))t_j, \quad x \in L(\Gamma).
\]

**Proof.** Fix $\gamma$ in $\Gamma$. We will first determine a formula for $URxU^*$. We use the fact $s_i$ are a system of representatives for right cosets for $\Gamma_0$ in $\Gamma$, so $\Gamma = \bigcup \Gamma_0 s_i$.

Hence for every $\gamma$ in $\Gamma$, and $i \in \{1, 2, \ldots, n\}$ there exists a permutation $\pi_\gamma$ of $1, 2, \ldots, n$ and an element $\theta_i(\gamma)$ in $\Gamma_0$ such that
\[
s_i\gamma = \theta_i(\gamma)s_{\pi_\gamma(i)}.
\]

More precisely, $\theta_i(\gamma) = s_i\gamma s_{\pi_\gamma(i)}^\gamma$. Clearly, one other possible way to write this expression is to say that
\[
\theta_i(\gamma) = \sum_j E_{L(\Gamma_0)}(s_i\gamma s_j^{-1}).
\]
Then for an arbitrary basis element $\gamma, t_i$ in $\ell^2(\Gamma_1) t_i$ we have 
\[(UR, U^*)(\gamma_1 t_i) = UR, \theta^{-1}(\gamma_1)s_i = U\theta^{-1}(\gamma_1)s_i\gamma = U\theta^{-1}(\gamma_1)\theta_i(\gamma)s_{\pi_i}(i).\]
Since $\theta^{-1}(\gamma_1)\theta_i(\gamma)$ belongs to $\Gamma_0$ this is further equal to 
\[\theta(\theta^{-1}(\gamma_1)\theta_i(\gamma))t_{\pi_i}(i) = \gamma_1\theta(\theta_i(\gamma))t_{\pi_i}(i).\]
Hence $(UR, U^*)(\gamma_1 t_i) = \gamma_1\theta(\theta_i(\gamma))t_{\pi_i}(i)$. With the terminology from Lemma 5, it follows that the restriction of $UR, U^*$ to $\ell^2(\Gamma_1) t_i$ is exactly $V\theta_i(\theta_i(\gamma))t_{\pi_i}(i)$. Since the space $\ell^2(\Gamma_1) t_i$ are pairwise orthogonal it follows that 
\[UR, U^* = \sum_{i=1}^{n} V\theta_i(\theta_i(\gamma))t_{\pi_i}(i).\]
Hence by Lemma 5 it follows that $E_{L(\Gamma)}^{\ell^2(\Gamma_1)'}(UR, U^*)$ is equal to the right convolutor by 
\[\frac{1}{[\Gamma : \Gamma_0]} \sum_i t_i^*\theta_i(\gamma)t_{\pi_i}(i).\]
By formula (2), this turns out to be 
\[\sum_{i,j} t_i^*\theta_i(E_{L(\Gamma_0)}(s_i x s_j^{-1}))t_j.\]
By linearity it then follows that 
\[\Psi_U(R_x) = \sum_{i,j} R_x^*\theta_i(E_{L(\Gamma_0)}(s_i x s_j^{-1}))t_j, \quad R_x \in R(\Gamma).\]
Passing from $R(\Gamma)$ to $L(\Gamma)$, $(R_x$ being mapped into $L_x^*)$ this is then the map $L_x^*$ goes into (after switching the indices $i$ and $j$)
\[\sum_{i,j} L_x^*\theta_i(E_{L(\Gamma_0)}(s_i x s_j^{-1}))t_j\]
and thus as a map on $L(\Gamma)$
\[\Psi_U(x) = \sum_{i,j} t_i^*\theta_i(E_{L(\Gamma_0)}(s_i x s_j^{-1}))t_j, \quad x \in L(\Gamma).\]
If we use the conjugation map $\overline{\cdot}$ on $L(\Gamma)$, this map becomes 
\[\overline{\Psi_U(x)} = T_j^* \theta_i(E_{L(\Gamma_0)}(s_i x s_j^{-1}))T_j\]
or 
\[\overline{\Psi_U(x)} = T_j^* \theta_i(E_{L(\Gamma_0)}(s_i x s_j^{-1}))\overline{T_j}\]
for $x$ in $L(\Gamma)$. \qed
4. THE TYPE II_1 REPRESENTATION FOR THE HECKE ALGEBRA OF A PAIR Γ ⊆ G, WHEN THE REGULAR REPRESENTATION OF Γ MAY BE EXTENDED TO G

In this section we consider the case of an almost normal subgroup Γ of G, where G has the property that there exists a unitary representation π : G → U(ℓ^2(Γ)) that extends the left regular representation of Γ. In this case the representation is extended to the Hecke algebra of the pair Γ ⊆ G. In this section we consider the case of an almost normal subgroup Γ ⊆ G, where G has the property that there exists a unitary representation π : G → U(ℓ^2(Γ)) that extends the left regular representation of Γ. In this case the representation is extended to the Hecke algebra of the pair Γ ⊆ G. All the results in this section remain valid in the presence of a group 2-cocycle on G, which restricts to the group Γ. We will assume that all the partial automorphisms of Γ, Ad σ, σ ∈ G are ε-preserving, (see also Appendix 1).

We recall from [Krieg], that H(Γ \ G/Γ) is simply the linearization of the algebra of double cosets of Γ in G. The product formula is as follows: let σ_1, σ_2 be elements of G

\[ [Γσ_1Γ][Γσ_2Γ] = \sum c(σ_1, σ_2, z)[ΓzΓ], \]

where [ΓzΓ] runs over the space of double cosets of Γ contained in Γσ_1Γσ_2Γ. The multiplicity c(σ_1, σ_2, z) is computed by the formula

\[ c(σ_1, σ_2, z) = \# \{ Γθ_2 | Γθ_2 ⊆ Γσ_2Γ \text{ s.t. } (∃)θ_1 \text{ in } Γσ_1Γ \text{ with } z = θ_1θ_2 \} \]

(see [Krieg], formula on page 15).

Moreover, H(Γ \ G/Γ) acts on the vector space of left cosets ℓ^2(Γ/G), which has as a basis the set \{Γs\} of left cosets representatives for Γ in G.

The formula of the action is for g, h ∈ G,

\[ [ΓgΓ][Γh] = \sum_{Γg_i \subseteq ΓgΓ} Γg_i h. \]

This is called ([BC], [CM], [Tz]) the left regular representation of the Hecke algebra on ℓ^2(Γ \ G) and is denoted by λ_{Γ\backslash G}.

Consequently, the above formula reads as

\[ λ_{Γ\backslash G}([ΓgΓ])([Γh]) = \sum_{Γg_i \subseteq ΓgΓ} Γg_i h, \]

where Γg_i are a system of representatives for left cosets of Γ that contained in ΓgΓ.

The Hecke algebra comes with a natural multiplicative homeomorphism ind : H(Γ \ G/Γ) → C which is defined by the requirement that

\[ \text{ind}[ΓgΓ] = \# \text{ right cosets of } Γ \text{ in } ΓgΓ = \text{card}[Γ : Γg]. \]

Moreover, there exists a natural reduced left reduced C* and von Neumann algebra associated with the action on ℓ^2(Γ/G). Namely, out of the space of cosets we make a Hilbert space by imposing the condition that the [Γg]’s form are orthonormal basis.
The reduced von Neumann algebra \( \mathcal{H}_{\text{red}}(\Gamma \setminus G/\Gamma) \) is the von Neumann subalgebra of \( B(\ell^2(\Gamma/G)) \) generated by the left multiplication with elements in \( \mathcal{H}(\Gamma \setminus G/\Gamma) \) (the weak closure). By \( \mathcal{H} \) we will denote the reduced C*-Hecke algebra which is the normic closure of \( \mathcal{H}(\Gamma \setminus G/\Gamma) \). These algebras are the weak (respectively the weak) closure of the algebra generated by the image of \( \lambda_{\Gamma/G} \). Note that this algebra comes with a natural state \( \omega_{\Gamma,G} \) which is simply

\[ \varphi(x) = \langle x[\Gamma], [\Gamma] \rangle. \]

In particular,

\[ \varphi([\Gamma g\Gamma]) = \text{ind}[\Gamma g\Gamma] \]

If for all \( g \) in \( G \), the subgroups \( \Gamma_g \) and \( \Gamma_{g^{-1}} \) have equal indices in \( \Gamma \) then \( \varphi \) is a trace, and another way to describe \( \mathcal{H}_{\text{red}}(\Gamma \setminus G/\Gamma) \) is to say that it is obtained via the GNS construction from the trace \( \varphi = \text{ind} \) on \( \mathcal{H}(\Gamma \setminus G/\Gamma) \). (Note that \( \mathcal{H}(\Gamma \setminus G/\Gamma) \) has a trivial involution \( \Gamma \sigma^{-1} \Gamma \) and hence the Hecke algebra is a \( * \)-algebra.)

**Remark 18.** The generators of the Hecke algebra of \( G = \text{PGL}_2(\mathbb{Q})^+ \) over \( \Gamma = \text{PSL}_2(\mathbb{Z}) \) are then the cosets of the form \( \alpha_{p,k} = \Gamma\sigma_{p,k} \Gamma \), with \( \sigma_{p,k} = \begin{pmatrix} 1 & 0 \\ 0 & p^k \end{pmatrix} \), where \( p \geq 2 \) runs over the prime numbers and \( k \) is a natural number. Fix \( p \geq 2 \) a prime number. Let \( N = (p-1)/2 \) and let \( F_N \) be the free group with \( N \) generators. Let \( \chi_k \in \mathcal{L}(F_N) \) be the sum of words of length \( k \), \( k \geq 1 \). It is proved in [Py] that the algebra generated by the selfadjoint elements \( \chi_k \), \( k \geq 1 \) is abelian, and that the spectrum of \( \chi_1 \) is exactly \( [-2\sqrt{p}, 2\sqrt{p}] \). Moreover the recurrence relations for \( \chi_k \) are the same as the one for \( \alpha_{p,k} \), and hence we have an algebra morphism mapping \( \alpha_{p,k} \) into \( \chi_k \). Since this morphism is trace preserving, we actually obtain an isomorphism of C*-algebras.

Consequently, the spectrum of \( \alpha_p \) in the reduced C*-Hecke algebra of \( G = \text{PGL}_2(\mathbb{Q})^+ \) over \( \Gamma = \text{PSL}_2(\mathbb{Z}) \) is exactly \( [-2\sqrt{p}, 2\sqrt{p}] \). In particular if \( \zeta \) is an eigenvector for the classical Hecke operator \( T_p \) with eigenvalue \( c_p \), let \( \mathcal{H}_p \) be the corresponding character induced by \( \zeta \) on the algebra generated by the double cosets \( \alpha_{p,k} \). It follows that \( c_p \) belongs to the interval \( [-2\sqrt{p}, 2\sqrt{p}] \) if and only if \( \mathcal{H}_p \) extends to a continuous character of the C*-algebra generated by the \( \alpha_{p,k}, k \geq 1 \) in the C*-Hecke algebra

We can now state the main result of this section. In particular, this proves that if \( G \) has a unitary representation on \( \ell^2(\Gamma) \) that extends the left regular representation, then \( \mathcal{H}_{\text{red}}(\Gamma \setminus G/\Gamma) \) embeds in a very natural way in \( \mathcal{L}(G) \).

**Theorem 19.** Let \( G \) be a discrete group with an almost normal subgroup \( \Gamma \). Assume that \( G \) admits a unitary representation \( \pi \) on \( \ell^2(\Gamma) \) that extends the left regular representation of \( \Gamma \) on \( \ell^2(\Gamma) \).

For \( \theta \in G \) define

\[ t(\theta) = \langle \pi(\theta)e, e \rangle \]
which is a matrix coefficient of the representation \( \pi \). Here \( e \) is the identity element of \( \Gamma \) viewed as a vector in the Hilbert space \( l^2(\Gamma) \).

For \( \alpha = [\Gamma \sigma \Gamma] \) a double coset in \( \mathcal{H}(\Gamma \setminus G/\Gamma) \) define
\[
t^\alpha = \sum_{\theta \in \alpha} t(\theta) \cdot \theta.
\]
Then \( t^\alpha \) is an element of \( l^2(\Gamma g \Gamma) \subseteq l^2(\Gamma) \) and the linear map \( S \)
\[
\alpha \mapsto t^\alpha, \quad \alpha = [\Gamma g \Gamma] \in \mathcal{H}(\Gamma \setminus G/\Gamma)
\]
extends to a unital \( \ast \)-normal isomorphism \( \rho \) from the von Neumann algebra \( \mathcal{H}_{\mathrm{red}}(\Gamma \setminus G/\Gamma) = \mathcal{H} \) into \( L(G) \). The restriction of \( \tau_{L(G)} \) to \( \mathcal{H} \) correspond to the state \( \omega_{\Gamma, \Gamma} \) on the Hecke algebra.

For \( c = [\Gamma s] \) a coset in \( l^2(\Gamma/G) \), define
\[
t^c = \sum_{\theta \in \Gamma s} t(\theta) \theta.
\]
Then \( t^c \in l^2(\Gamma s) \), and the family \( \{t^c\} \), where \( c \) runs over the space of left cosets is an orthonormal system generating a Hilbert space \( K \). Then \( K \) is a reducing space for the representation \( \rho \). The restriction of the representation \( \rho \) of \( \mathcal{H} \) to \( k \) is unitarily equivalent to the left representation \( \lambda_{\Gamma \setminus G} \) of \( \mathcal{H}_{\mathrm{red}}(\Gamma \setminus G/\Gamma) \) on \( l^2(\Gamma/G) \) by the unitary that maps \( t^c \) into the coset \( c \in l^2(\Gamma/G) \).

Remark 20. Note that, in particular, the theorem implies that the following properties hold true.

For all \( a_1 = [\Gamma \sigma_1 \Gamma], a_2 = [\Gamma \sigma_2 \Gamma] \) double cosets of \( \Gamma \) in \( G \)
a) \( t^{a_1} t^{a_2} = \sum_{\Gamma z \Gamma \subseteq \Gamma \sigma_1 \Gamma \sigma_2 \Gamma} \alpha(a_1, a_2, z) t^{\Gamma z \Gamma} \).
b) For all double cosets \( \Gamma \sigma \Gamma \) we have
\[
(t^{\Gamma \sigma \Gamma})^* = t^{\Gamma \sigma^{-1} \Gamma}.
\]
c) If \( a = [\Gamma \sigma \Gamma] \), and \( c = [\Gamma s] \) is a coset then
\[
t^{a} \cdot t^{c} = \sum_{\Gamma g_i \subseteq \Gamma \sigma \Gamma} t^{[\Gamma g_i s]},
\]
where \( \Gamma g_i \) runs over a set of representatives for left cosets of \( \Gamma \) that are contained in \( \Gamma \sigma \Gamma \).
d) For every coset \( c = \Gamma s, \|t^c\|_2^2 = 1 \) and \( \{t^c\} \) where \( c \) runs over cosets of \( \Gamma \) is an orthonormal forms.

Moreover, the following additional properties hold true.

1) \( \|t^{[\Gamma \sigma \Gamma]}\|_2^2 = \tau((t^{[\Gamma \sigma \Gamma]}^* t^{[\Gamma \sigma \Gamma]})) = \left\{ \begin{array}{ll} \text{ind}[\Gamma \sigma \Gamma] & \text{if } \sigma_1 = \sigma_2 \\
0 & \text{otherwise.} \end{array} \right. \)
2) If \( a_1 = [\Gamma \sigma_1 \Gamma], a_2 = [\Gamma \sigma_2 \Gamma] \) are two different double cosets then for all \( \gamma \) in \( \Gamma \)
\[
E_{L(G)}^{\mathcal{L}(G)}(t^{a_1} \gamma t^{a_2}) = 0.
\]
(In particular, $t^{a_1}, t^{a_2}$ are orthogonal.)

3) If $a = [\Gamma \sigma \Gamma]$ then

$$E_{L(\Gamma)}^{L(G)}(t^a(t^a)^*) = \text{ind} \, a.$$  

4) For all $\xi, \eta$ in $\ell^2(\Gamma)$ and $a = [\Gamma \sigma \Gamma]$ 

$$\eta t^a \xi^* = \sum_{\theta \in [\Gamma \sigma \Gamma]} \langle \pi(\theta) \xi, \eta \rangle \theta,$$

(where $\overline{\xi}, \overline{\eta}$ are the images of $\xi, \eta$ to the conjugation map: $\sum \xi_\gamma \gamma = \sum \xi_\gamma \gamma$).

5) If $s_i$ is a system of representatives for right cosets $\Gamma_{\sigma^{-1}}$ in $\Gamma$, so that $\Gamma \sigma \Gamma$ is as a set the disjoint union of $\Gamma \sigma s_i$ (since $\Gamma = \bigcup \Gamma_{\sigma^{-1}} s_i$) then 

$$t^{\Gamma \sigma \Gamma} = \sum_{i=1}^{\vert \Gamma : \Gamma_{\sigma^{-1}} \vert} t^{\Gamma \sigma s_i}.$$  

6) If $\sigma$ in $G$ commutes with $\Gamma$, then $t^{\Gamma \sigma \Gamma}$ is simply a multiple of $\sigma$ as an element of $L(G) \subseteq \ell^2(G)$.

7) The representation $\pi$ can be recovered from the coefficients $t(\theta)$, $\theta$ in $G$. Indeed, for all $\theta$ in $G$, $\gamma$ in $\Gamma$

$$\pi(\theta) \gamma = \sum_{\gamma_1} \gamma \gamma_1^{-1} \theta \gamma_1.$$  

In particular, $\pi(\sigma)e$ as an element of $\ell^2(\Gamma)$ is equal to $\sigma \cdot \pi^{-1} t$ and hence 

$$(\pi(\sigma)e)^* = \overline{t} \sigma^{-1}.$$  

Recall that if $x = \sum x_\gamma \gamma$ is an element of $L(\Gamma)$, then $x = \sum \overline{x_\gamma} \gamma$.

8) Let $\Gamma s_i, \Gamma t$ be two left cosets of $\Gamma$ in $G$. Let $A_{\Gamma s_i, \Gamma t}$ be the subset of $\Gamma$ defined by $A_{\Gamma s_i, \Gamma t} = \Gamma \cap s^{-1} \Gamma t.$

Let $\alpha_{\Gamma s_i, \Gamma t}$ be the projection from $\ell^2(\Gamma)$ onto the Hilbert space generated by the elements in $A_{\Gamma s_i, \Gamma t}$. In particular, $\gamma$ belongs to $A_{\Gamma s_i, \Gamma t}$ if and only if $\alpha_{\Gamma s_i, \Gamma t}(\gamma) \neq 0$ (and hence $\alpha_{\Gamma s_i, \Gamma t}(\gamma) = \gamma$) and this is further equivalent to the fact that there exist $\theta$ in $\Gamma$ such that 

$$s \gamma = \theta t \quad (\gamma = s^{-1} \theta t).$$  

Then, for $x$ in $L(\Gamma)$, 

$$E_{L(\Gamma)}^{L(G)}(t^{\Gamma s} x (t^{\Gamma t})^*) = t^{\Gamma s} \alpha_{\Gamma s_i, \Gamma t}(x) (t^{\Gamma t})^*.$$  

9) If $\Gamma \sigma \Gamma$ is a double coset in $G$, and if $(s_i)_{i=1}^{\vert \Gamma : \Gamma_{\sigma^{-1}} \vert}$ is a set of representatives for left $\Gamma_{\sigma^{-1}}$ cosets of $\Gamma_{\sigma^{-1}}$ in $\Gamma$ (that is $\Gamma = \bigcup \Gamma_{\sigma^{-1}} s_i$, so $\Gamma \sigma \Gamma = \bigcup \Gamma \sigma s_i$). For $r$ in $\Gamma$, let $\pi_r$ be the permutation of $\{1, 2, \ldots, [\Gamma : \Gamma_{\sigma^{-1}}] \}$ defined by the requirement that for $i$ in $\{1, 2, \ldots, [\Gamma : \Gamma_{\sigma} \Gamma_{\sigma^{-1}}] \}$, $\pi_r(i)$ is the unique element of $\{1, 2, \ldots, \pi_r(i) \}$ such that there exists $\theta$ in $\Gamma_{\sigma^{-1}}$ with $s_i \gamma = \theta s_{\pi_r(i)}$ (in particular, $\theta = s_1 \gamma s_{\pi_r(i)} \in \Gamma_{\sigma^{-1}}$).
Then
\[ E_{\mathcal{L}(\Gamma)}^{\mathcal{L}(\Gamma)}(t^{\sigma_s_i \gamma}(t^{\sigma_s_j})^*) = t^{\Gamma \sigma_s_i \alpha \Gamma \sigma_s_i \Gamma \sigma_s_j \gamma}(t^{\Gamma \sigma_s_j})^* \]
is different from 0, if and only if \( j = \pi \gamma(i) \), in which case it is equal to
\[ t^{\Gamma \sigma_s_i \gamma}(t^{\Gamma \sigma_s_j \gamma(i)})^*. \]
This is equivalent to fact that \( \gamma \) belongs to \( A_{\Gamma \sigma_s_i \Gamma \sigma_s_j} \) which is equivalent to the fact that there exists \( \theta \) in \( \Gamma \) such that \( (\sigma_s_i \gamma) = \theta(\sigma_s_j) \).

To prove the theorem we will first prove the following lemma, which gives a computational tool for all these equalities.

**Lemma 21.** For all \( \theta_1, \theta_2 \) in \( G \) the following equality holds:
1) \( t(\theta_1) = t(\theta_1^{-1}) \);
2) \( \sum_{\gamma \in \Gamma} t(\theta_1 \gamma) t(\gamma^{-1} \theta_2) = t(\theta_1 \theta_2) \).

**Proof.** Clearly
\[ t(\theta_1) = \langle \pi(\theta_1)e, e \rangle e \]
which by property 7) (that we will prove below) is
\[ \sum_{\gamma_1} t(\gamma_1^{-1} \theta_1) \gamma_1, \sum_{\gamma_2} t(\gamma_2^{-1} \theta_2) \gamma_2 \]
and hence this is equal to
\[ \sum_{\gamma} t(\gamma_1^{-1} \theta_1) t(\gamma_1^{-1} \theta_2^{-1}) = \sum_{\gamma} t(\theta_2 \gamma_1) t(\gamma_1^{-1} \theta_1). \]
This completes the proof of Lemma 1. \( \square \)

The proof of property 7) is
\[ \pi(\theta) \gamma = \sum_{\gamma_1} (\pi(\theta) \gamma, \gamma_1) \gamma_1 = \sum_{\gamma_1} (\pi(\gamma_1^{-1} \theta \gamma) e, e) \gamma_1 = \sum_{\gamma_1} t(\gamma_1^{-1} \theta \gamma) \gamma_1. \]

We now start the proof of Theorem 7.

The most relevant properties are a), c) that we will prove first.

To prove property a) let \( a_1 = [\Gamma \sigma_1 \Gamma] \), \( a_2 = [\Gamma \sigma_2 \Gamma] \) be two double cosets in \( \mathcal{H}(\Gamma \setminus G/\Gamma) \). Then
\[ t^{a_1} \cdot t^{a_2} = \sum_{\theta_1 \in \Gamma \sigma_1 \Gamma} t(\theta_1) t(\theta_2) \cdot \theta_1 \theta_2 \]
and hence this is equal to
\[ \sum_{z \in \Gamma \sigma_1 \Gamma \sigma_2 \Gamma} \left( \sum_{\theta_1 \in \Gamma \sigma_1 \Gamma \theta_2 \in \Gamma \sigma_2 \Gamma} t(\theta_1) t(\theta_2) \right). \]
To identify the coefficient

\[ (4) \sum_{\theta_1 \in \Gamma \sigma_1 \Gamma, \theta_2 \in \Gamma \sigma_2 \Gamma \atop \theta_1 \theta_2 = z} t(\theta_1)t(\theta_2) \]

for any \( z \in G \), that also belongs to \( \Gamma \sigma_1 \Gamma \sigma_2 \Gamma \), we consider

\[ A_z = \{ (\theta_1, \theta_2) \in \Gamma \sigma_1 \Gamma \times \Gamma \sigma_2 \Gamma \mid \theta_1 \theta_2 = z \} \].

Clearly, the group \( \Gamma \) acts on \( A_z \), the action of \( \gamma \) are an element \( (\theta_1, \theta_2) \) being

\[ \gamma(\theta_1, \theta_2) = (\theta_1 \gamma^{-1}, \gamma \theta_2). \]

It is obvious that this is a free action of \( \Gamma \). Let \( O \) be the space of orbits of \( \Gamma \). Each orbit is of the form \( \{ (\theta_1 \gamma^{-1}, \gamma \theta_2) \mid \gamma \in \Gamma \} \), with the action of \( \gamma \) being bijective. It follows by property 2) of Lemma 9 that for every orbit \( o \) in \( O \)

\[ \sum_{(\theta_1, \theta_2) \in o} t(\theta_1)t(\theta_2) = t(z). \]

Hence the coefficient in formula (4) is \( n(z)t(z) \), where \( n(z) \) is the number of orbits of \( \Gamma \) for the given action on \( A_z \).

We consider the following map \( \Phi \) from \( O \) into the space of cosets of \( \Gamma \) in \( G \). If \( o \in O \), is defined as \( o = \{ (\theta_1 \gamma, \gamma^{-1} \theta_2) \mid \gamma \in \Gamma \} \subseteq A_z \) for some \( \theta_1 \in \Gamma \sigma_1 \Gamma, \theta_2 \in \Gamma \sigma_2 \Gamma \), (with the necessary property that \( \theta_1 \theta_2 = z \)) then we define

\[ \Phi(o) = \Gamma \theta_2. \]

Clearly, this map is well defined.

Moreover, the image lies in the set \( M = M(\sigma_1, \sigma_2, z) \) of cosets of \( \Gamma y \) in \( G \) that verify that there exists \( x \) in \( \Gamma \sigma_1 \Gamma \) with \( xy = z \). (This is the set defining the coefficient \( c(\sigma_1, \sigma_2, z) \) in formula (3).

Now, clearly \( \Phi \) is injective since if \( o' = \{ (\theta'_1 \gamma, \gamma^{-1} \theta'_2) \mid \gamma \in \Gamma \} \) is another orbit in \( A_z \), such that

\[ \Phi(o') = \Phi(o) \]

then it follows that

\[ \Gamma \theta_2 = \Gamma \theta'_2. \]

But this implies \( \theta'_2 = \gamma_0 \theta_2 \) for some \( \gamma_0 \) in \( \Gamma \). Since \( \theta_1 \theta_2 = \theta'_1 \theta'_2 = z \), this implies that \( \theta'_1 = \theta'_2 \gamma_0^{-1} \) and hence that \( o \) and \( o' \) are the same orbit.

Thus the number \( u(z) \) in formula (4) is \( c(\sigma_1, \sigma_2, z) \), and since this only depends on the double coset of \( \Gamma z \Gamma \) and not of the individual value of \( z \), this proves that in the product \( t^x t^c \) the element \( t^x t^c \) shows up with coefficient \( c(\sigma_1, \sigma_2, z) \).

This completes the proof of property a).
We now prove property c). Let \( a = \Gamma \sigma \Gamma \) be a double coset and let \( c = \Gamma s \) be a left coset of \( \Gamma \) in \( G \). We want to determine \( t^a t^c \). Then

\[
(5) \quad t^a t^c = \sum_{\theta \in \Gamma \sigma \Gamma, g \in \Gamma s} t(\theta) t(g) \theta g = \sum_{z \in \Gamma \sigma \Gamma} z \left( \sum_{\theta \in \Gamma \sigma \Gamma, g \in \Gamma s} t(\theta) t(g) \right).
\]

Let \( (r_a)_{a=1}^n \), with \( n = [\Gamma : \Gamma_{\sigma^{-1}}] \) be a set of representatives for right cosets of \( \Gamma_{\sigma^{-1}} \) in \( \Gamma \). Then \( \Gamma = \bigcup_{a=1}^n \Gamma_{\sigma^{-1}} r_a \) as a disjoint union. Since \( \sigma \Gamma_{\sigma^{-1}} \sigma^{-1} = \Gamma \subseteq \Gamma \) it follows that

\[
\Gamma \sigma \Gamma = \bigcup_a \Gamma \sigma \Gamma_{\sigma^{-1}} r_a = \bigcup_{a=1}^n \Gamma \sigma r_a.
\]

Clearly, this is also a disjoint union since if \( \gamma_1 \sigma r_a = \gamma_2 \sigma r_b \) with \( \gamma_1, \gamma_2 \in \Gamma \), then it follows that

\[
r_b r_a^{-1} = \sigma^{-1}(\gamma_2^{-1} \gamma_1)\sigma
\]

and hence since \( r_a r_b^{-1} \) belongs to \( \Gamma \) it follows that \( \sigma^{-1}(\gamma_2^{-1} \gamma_1)\sigma \) belongs to \( \sigma^{-1} \Gamma \sigma \cap \Gamma \). Hence \( r_b r_a^{-1} \) belongs to \( \Gamma_{\sigma^{-1}} \) or \( r_b \) belongs to \( \Gamma_{\sigma^{-1}} r_a \). But this implies \( r_a = r_b \), since those were a set of representatives. We decompose the set \( \Gamma \theta \Gamma \times \Gamma s \) as the reunion \( \bigcup_{a=1,2,\ldots,n} \bigcup_{\gamma \in \Gamma} A_{\gamma_1, a} \), where \( A_{\gamma_1, a} \) is the set \( \{ (\gamma_1 \sigma r_a \gamma, \gamma^{-1} s) \mid \gamma \in \Gamma \} \). Note that the sets \( A_{\gamma_1, a} \) are disjoint.

Indeed, if \( A_{\gamma_1, a} \cap A_{\gamma_2, b} \neq \emptyset \), then there exists \( \gamma', \gamma'' \in \Gamma \) such that

\[
(\gamma_1 \sigma r_a \gamma', (\gamma')^{-1} s) = (\gamma_2 \sigma r_b \gamma'', (\gamma'')^{-1} s)
\]

but this implies that \( \gamma' = \gamma'' \) and hence this implies that

\[
\gamma_1 \sigma r_a = \gamma_2 \sigma r_b.
\]

Since as we have shown before the union \( \Gamma = \bigcup_{c=1}^n \Gamma \sigma r_c \) is disjoint it follows that \( r_a = r_b \) and hence that \( \gamma_1 = \gamma_2 \).

By formula (5) we thus have

\[
t^a t^c = \sum_{\gamma_1, a} \sum_{\gamma \in \Gamma} t(\gamma_1 \sigma r_a \gamma) t(\gamma^{-1} s) \gamma_1 \sigma r_a \gamma^{-1} s.
\]

By Lemma 9, this is further equal to

\[
\sum_{\gamma_1, a} t(\gamma_1 \sigma r_a s) \gamma_1 \sigma r_a s = \sum_a \left( \sum_{\gamma_1} t(\gamma_1 \sigma r_a s) \gamma_1 \sigma r_a s \right) = \sum_a t^{\Gamma \sigma r_a s}
\]

which is exactly

\[
\sum_{\Gamma \subseteq \Gamma \sigma} t^{\Gamma s},
\]

where the sum runs over right cosets of \( \Gamma \) contained in \( \Gamma \sigma \Gamma \). We now prove property d) in Remark 8. Let \( c = \Gamma s, d = \Gamma t \) be two cosets of \( \Gamma \) in \( G \).
Then \( \langle t^s, t^t \rangle \mathcal{L}(G) \) is equal to
\[
\sum_{\gamma_1 \in \Gamma} t(\gamma_1 s) \gamma_1 s \sum_{\gamma_2 \in \Gamma} t(\gamma_2 t) \gamma_2 t = \sum_{\gamma_1, \gamma_2 \in \Gamma} t(\gamma_1 s) t(\gamma_2 t) \langle \gamma_1 s, \gamma_2 t \rangle.
\]

If the cosets \( \Gamma s \) and \( \Gamma t \) are disjoint then this is clearly 0. Otherwise, if \( s = t \) then this is further equal to
\[
\sum_{\gamma_1 \in \Gamma} t(\gamma_1 s) t(\gamma_1 s) = \sum_{\gamma \in \Gamma} t(\gamma^{-1}_1) t(\gamma_1) = \sum_{\gamma \in \Gamma} t(s^{-1}_1 t) t(\gamma_1 s) = t(s^{-1}_1) t(s_1) = 1
\]
again by Lemma 9.

This completes the proof of properties a), b), c), d) from Remark 8.

We now proceed to the proof of Theorem 7.

By properties a), b) it is then obvious that the map \( \Phi \) from \( \mathcal{H}(\Gamma \setminus G / \Gamma) \) into \( \mathcal{L}(G) \) defined by \( \Phi([\Gamma \sigma]) = t^{\sigma \gamma} \) and then extended by linearity is * homeomorphism.

Because of properties c), d) the map \( v \) which maps \( t^{\gamma} \) into the coset \( \Gamma s \) in \( \ell^2(\Gamma \setminus G) \) is a unitary operator. Moreover, \( \Phi(\mathcal{H}(\Gamma \setminus G / \Gamma)) \) invariates \( K \), so the projection \( P_K \) from \( \ell^2(\Gamma \setminus G) \) onto \( K \) belongs to the commutant of the algebra \( \mathcal{H}_0 = \Phi(\mathcal{H}(\Gamma \setminus G / \Gamma)) \).

Moreover, by property d) and because of the definition of the left action \( \lambda_{\Gamma/G} \) of \( \mathcal{H}(\Gamma \setminus G / \Gamma) \) on \( \ell^2(\Gamma \setminus G) \) it follows that
\[
U(P\Phi(a)P)U^* = \lambda_{\Gamma/G}(a)
\]
for all \( a \) in \( \mathcal{H}(\Gamma \setminus G / \Gamma) \).

Moreover,
\[
\tau_{\mathcal{L}(G)}(\Phi(a)) = \varphi_{\Gamma,\Gamma}(\lambda_{\Gamma/G}(a)) = \langle \Phi(a)e, e \rangle = \langle P\Phi(a)P, e \rangle.
\]

Since \( e \) (the unit of \( \Gamma \subseteq G \)) belongs to \( k \) as \( t^\Gamma = e \).

To conclude the fact that \( \Phi \) is an isomorphism from \( \mathcal{H}_{red}(\Gamma \setminus G / \Gamma) \) into \( \mathcal{L}(G) \) we need the following lemmata that summarizes the properties we obtained so far

**Lemmata.** Let \( M \) be a finite von Neumann algebra with finite faithful trace \( \tau \). Let \( N_0 \) be a unital \(*\)-subalgebra of \( M \) that contains the unit. Assume that there exist a projection \( P \) onto a subspace \( K \) of \( L^2(M, \tau) \) that contains 1, and such that \( P \) commutes with \( N_0 \). Let \( B_0 = P_0N_0P_0 \), and let \( B \) be the von Neumann algebra generated by \( B_0 \) in \( B(K) \). Assume that \( \omega_{1,1} \) is a faithful special state on \( B \).

Then the reduction map, which maps \( \nu_0 \in N_0 \) into \( P_0\nu_0P_0 \) extends to a von Neumann algebra isomorphism from \( N = \{ N_0 \} \) onto \( B \).

**Proof.** Indeed \( \Phi \) becomes a unitary from \( L^2(N, \tau) \) onto \( L^2(B, \omega_{1,1}) \) which then implements the isomorphism from \( N \) onto \( B \).

This concludes the proof of the fact that \( \Phi : \mathcal{H}(\Gamma \setminus G / \Gamma) \) extends to a von Neumann algebras isomorphism, from \( \mathcal{H}_{red}(\Gamma \setminus G / \Gamma) \) into \( \mathcal{H} = \{ \Phi(\mathcal{H}(\Gamma \setminus G / \Gamma)) \} \)
because of the unitary \( U \) that intertwines the left regular representation of \( \mathcal{H} \) with the restriction of \( \Phi \) to \( \ell^2(\Gamma \backslash H) \) (which generates \( \mathcal{H}_{\text{red}}(\Gamma \backslash G / \Gamma) \) with the representation \( a \to P_K \Phi(a) P_K \).

We now proceed to the proof of the properties 1)–8) in Remark 8 (since 7) was already proven).

We start with property 2). Assume that \( a_1 = \Gamma \sigma_1 \Gamma, a_2 = \Gamma \sigma_2 \Gamma \) are two double cosets such that \( E_{\ell^2(G)}(a_1 \gamma (a_2)*) \) is different from 0 for some \( \gamma \) in \( \Gamma \).

The terms in \( t^a \gamma (t^{a_2})* \) are sums of multiples of elements of the form \((\gamma_1 \sigma_1 \gamma_2) \gamma (\gamma_3 \sigma_2^{-1} \gamma_4)\), with \( \gamma_1, \gamma_2, \gamma_3, \gamma_4 \neq 0 \) and hence if \( E_{\ell^2(G)}(a_1 \gamma (a_2)*) \) is different from 0, it follows that there exists \( \gamma_1, \gamma_2, \gamma_3, \gamma_4 \) and \( \theta \) in \( \Gamma \) such that
\[
(\gamma_1 \sigma_1 \gamma_2) \gamma (\gamma_3 \sigma_2^{-1} \gamma_4) = \theta.
\]
Hence \( \sigma_2 = (\gamma_4^{-1} \theta^{-1} \gamma_1) \sigma_1 (\gamma_2 \gamma_3) \) and hence \( \Gamma \sigma_2 \Gamma = \Gamma \sigma_1 \Gamma \) or \( a_1 = a_2 \).

This proves property 3) and also proves that
\[
\tau_{\ell^2(G)}((t^{\Gamma \sigma_1 \Gamma})^* (t^{\Gamma \sigma_2 \Gamma})) = 0
\]
if \( \Gamma \sigma_1 \Gamma \neq \Gamma \sigma_2 \Gamma \).

To prove the remaining part of property 1), note that by property 5) (which is obvious since the sets \( \Gamma \sigma_1 \) are disjoint) we have that
\[
t^{\Gamma \sigma \Gamma} = \sum_{\Gamma z \subseteq \Gamma \sigma \Gamma} t^{\Gamma z}.
\]
Since we know that different cosets \( \Gamma z_1, \Gamma z_2, t^{\Gamma z_1} \) and \( t^{\Gamma z_2} \) are orthogonal. It follows that
\[
\|t^{\Gamma \sigma \Gamma}\|^2 = \tau_{\ell^2(G)}(t^{\Gamma \sigma \Gamma} t^{\Gamma \sigma \Gamma}) = \sum_{\Gamma z \subseteq \Gamma \sigma \Gamma} \langle \Gamma z, \Gamma z \rangle = \sum_{\Gamma z \subseteq \Gamma \sigma \Gamma} 1,
\]
and this is exactly the number of left cosets in \( \Gamma \sigma \Gamma \), which is \( \text{ind}(\Gamma \sigma \Gamma) \).

Property 3) is now a consequence of property 1). Indeed, as we have proven in property 1), for every double coset \( a = [\Gamma \sigma] \), we have that \( t^a (t^a)^* \) is the sum
\[
\sum_{\Gamma z \subseteq \Gamma \sigma \sigma^{-1} \Gamma} c(\sigma, \sigma^{-1}, z) t^{\Gamma z}.
\]
Hence
\[
E_{\ell^2(G)}(t^a (t^a)^*) = c(\sigma, \sigma^{-1}, e) t^{\Gamma e}.
\]
But \( t^{\Gamma e} \) is just the identity.

On the other hand, if we apply the trace \( \tau \) into the previous relation, and since \( E \) preserves the trace it follows that
\[
\text{ind} a = \tau(t^a (t^a)^*) = \tau(E_{\ell^2(G)}(t^a (t^a)^*)) = c(\sigma, \sigma^{-1}, e).
\]
Hence $c(\sigma, \sigma^{-1}, e) = \text{ind } a$ where $a = [\Gamma \sigma \Gamma]$ and hence
\[
E_{\mathcal{L}(G)}^{\Gamma}(t^a(t^a)^*) = (\text{ind } a).
\]

We now proceed to the proof of property 4). By bilinearity it is sufficient to prove this property for $\xi = h_1, \eta = h_2$, where $h_1, h_2$ are two elements in $\Gamma$.

Hence we have to prove that for $a = [\Gamma \sigma \Gamma]$
\[
h_1 t^a h_2 = \sum_{\theta \in \Gamma \sigma \Gamma} \langle \pi(\theta) h_2, h_1 \rangle \theta,
\]
i.e.,
\[
t^a = \sum_{\theta \in \Gamma \sigma \Gamma} \langle \pi(h_1^{-1} \theta h_2) e, e \rangle h_1^{-1} \theta h_2.
\]
Doing a change of variable $\theta' = h_1^{-1} \theta h_2$ this is exactly the definition of $t^a$.

Finally, property 6) follows from the fact that in this case $\pi(\sigma)$ commutes with $\Gamma$ on $\ell^2(\Gamma)$ so it must be a scalar $\lambda$. Hence
\[
t(\gamma_1 \sigma \gamma_2) = \langle \pi(\gamma_1 \sigma \gamma_2) e, e \rangle = \lambda \langle \gamma_1 \gamma_2 e, e \rangle
\]
which is different from 0, if and only if $\gamma_1 \gamma_2 = e$.

But in this case $\Gamma \sigma \Gamma$ is simply $\Gamma \sigma$ and hence
\[
t^{\Gamma \sigma} = t^{\Gamma} = \sum_{\gamma \in \Gamma} t(\gamma) \gamma \sigma = t(\sigma) \sigma = \lambda \sigma.
\]

For property 7) note that
\[
\pi(\sigma)e = \sum_{\gamma \in \Gamma} \langle \pi(\sigma)e, \gamma \gamma \rangle = \sum_{\gamma} t(\gamma^{-1} \sigma) \gamma.
\]
Hence
\[
\sigma^{-1}(\pi(\sigma)e) = \sum_{\gamma \in \Gamma} t(\gamma^{-1} \sigma) \sigma^{-1} \gamma = \sum_{\gamma \in \Gamma} t(\sigma^{-1} \gamma) \sigma^{-1} \gamma = t^{\sigma^{-1} \Gamma}.
\]

Taking the adjoint we obtain
\[
(\pi(\sigma)e)^* \sigma = t^{\sigma^* \sigma}.
\]

We now prove property 8).

Let $\Gamma s, \Gamma t$ be two left cosets as in the statement. Let $\gamma$ be any element in $\Gamma$.
Then $E_{\mathcal{L}(G)}^{\Gamma}(t^{\Gamma s} \gamma (t^{\Gamma t})^*)$ is different from 0, if and only if there exists $\gamma_1, \gamma_2$ and $\theta$ in $\Gamma$ such that
\[
\gamma_1 s \gamma t^{-1} \gamma_2 = \theta
\]
which is equivalent to
\[
\gamma = s^{-1}(\gamma_1 \theta \gamma_2^{-1}) t = s^{-1}(\gamma') t,
\]
where $\gamma'$ belongs to $\Gamma$. 
Thus $E_{L(G)}^{\mathcal{L}(G)}(t^{\Gamma s}\gamma(t^{\Gamma t})^*)$ is different from 0, if and only if $\gamma$ belongs to $\Gamma \cap s^{-1}\Gamma t = A_{\Gamma s,\Gamma t}$. But this gives exactly that

$$E_{L(G)}^{\mathcal{L}(G)}(t^{\Gamma s}\gamma(t^{\Gamma t})^*) = \alpha_{\Gamma s,\Gamma t}(\gamma)$$

which by linearity proves the statement of property 8).

Note that $\alpha_{\Gamma s,\Gamma t}$ is the zero projection if $\Gamma \cap s^{-1}\Gamma t$ is void.

To prove property 9) we use property 8).

For $\psi_{s_i,\Gamma s_j}(\gamma)$ is different from 0, if and only if $\gamma$ belongs to $\Gamma \cap (s s_i)\Gamma (s s_j)$. For $\gamma$ in $\Gamma$.

So $\alpha_{\Gamma s_i,\Gamma s_j}(\gamma)$ is different from 0, if and only if there exists $\theta$ in $\Gamma$ such that

$$\gamma = s_i^{-1}(\sigma^{-1}\theta)s_j \quad \text{(or } \sigma s_i \gamma = \theta s_j)$$

or

$$s_i \gamma s_j^{-1} = \theta' = \sigma^{-1}\theta.$$

Hence $\theta$ belongs to $\Gamma_{\sigma^{-1}}$ and $s_i \gamma = \theta' s_j$ so $j$ must be equal to $\pi_\gamma(i)$. □

5. THE REPRESENTATION OF THE HECKE OPERATORS FOR $\Gamma \subseteq G$

This section contains the main result of the paper. In Section 2 we obtained an explicit formula for abstract Hecke operator.

In this section we prove that the algebra consisting of completely positive maps representing the Hecke operators has a lifting to $L(G)$. This lifting is similar to the dilation of a semigroup of completely positive maps. It relies on the representation for the Hecke algebra given in the previous section. The result is a formula that does not involve in its expression any choice of a system of representatives.

The main theorem of this paper is the following.

Note that all the result in this section remain valid in the presence of a group 2-cocycle on $G$, which restricts to the group $\Gamma$. We will assume that all the partial automorphisms of $\Gamma$, $\Ad \sigma \in G$ are $\varepsilon$ preserving, (see also Appendix 1).

**Theorem 22.** Let $G$ be a discrete group and $\Gamma \subseteq G$ an almost normal subgroup. Assume that $G$ admits a unitary representation on $\ell^2(\Gamma)$ that extends the left regular representation. For a coset $[\Gamma \sigma\Gamma]$ let $\Psi_\sigma$ be the abstract Hecke operator, associated with the unitary $\pi(\sigma)$,

$$\Psi_\sigma(x) = E_{L(G)}^{\mathcal{L}(G)}(\pi(\sigma)x\pi(\sigma)^*)$$

for $x$ in $R(\Gamma)$. We identify with $\mathcal{L}(G)$ via the canonical anti-isomorphism and consider $\Psi_\sigma$ as a map from $\mathcal{L}(\Gamma)$ into $\mathcal{L}(\Gamma)$.

Let $\rho : \mathcal{H}(\Gamma \setminus G/\Gamma) \to \mathcal{L}(G)$ be the representation of the Hecke algebra constructed in the previous section, so that

$$\rho([\Gamma \sigma\Gamma]) = \sum_{\theta \in [\Gamma \sigma\Gamma]} \langle \pi(\theta)e, e \rangle_{\ell^2(\Gamma)} = t^{\Gamma \sigma\Gamma}$$
for \( a = [\Gamma \sigma \Gamma] \) a double coset.

Then for \( x \) in \( \mathcal{L}(\Gamma) \),

\[
[\Gamma : \Gamma_{\sigma}] \Psi_\sigma(x) = E_{\mathcal{L}(\Gamma)}^{\mathcal{L}(G)}(\rho(a)x\rho(a)^*).
\]

Note that in particular \( \Psi_\sigma \) depends only on the coset \( \Gamma \sigma \Gamma \).

This formula is a dilation formula, for the "pseudo-semigroup" of completely positive maps \( \Psi_\sigma \), in the sense of the corresponding theory for semigroups of completely positive maps ([Ar]).

**Remark.** By using the antilinear isomorphism \( \overline{\cdot} : \mathcal{L}(G) \to \mathcal{L}(\Gamma) \) defined by \( \sum x_\gamma \gamma \to \sum \overline{x_\gamma} \gamma \), where \( x_\gamma \) are complex numbers, the formula for \( \Psi_\sigma(x) \) becomes

\[
[\Gamma : \Gamma_{\sigma}] \Psi_\sigma(x) = E_{\mathcal{L}(\Gamma)}^{\mathcal{L}(G)}(\rho(a)x\rho(a)^*), \quad x \in \mathcal{L}(G), \quad a = [\Gamma \sigma \Gamma], \sigma \in G.
\]

If \( \Gamma = \text{PSL}_2(\mathbb{Z}), G = \text{PSL}_2(\mathbb{Q})_+ \), by Proposition 4, \( \Psi_\sigma \) is unitary equivalent to the Hecke operator associated with \( \Gamma \sigma \Gamma \) on Maass form.

In the next proposition we will prove that, as in the classical case

\[
[\Gamma : \Gamma_{\sigma_1}] [\Gamma : \Gamma_{\sigma_2}] \Psi_{\sigma_1} \Psi_{\sigma_2} = \sum_{\Gamma z \Gamma \subseteq \Gamma_{\sigma_1} \Gamma_{\sigma_2} \Gamma} c(\sigma_1, \sigma_2, z) [\Gamma : \Gamma_{z}] \Psi_{z}.
\]

Recall that \( \Gamma_{\sigma^{-1}} = \Gamma \cap \sigma^{-1} \Gamma \sigma \), \( \Gamma_{\sigma} = \Gamma \cap \sigma \Gamma \sigma^{-1} \) and \( s_i \) is a system of left representatives for left cosets for \( \Gamma_{\sigma^{-1}} \) in \( \Gamma \), that is \( \Gamma = \bigcup_{i=1}^n \Gamma_{\sigma^{-1}} s_i \), where \( n = [\Gamma : \Gamma_{\sigma}] = [\Gamma : \Gamma_{\sigma^{-1}}] \). Let \( t_i = \pi(x)s_i \), which \( L(\Gamma_{\sigma}) \) an orthonormal family of vectors in \( L^2(\Gamma) \) (that is \( E_{L(\Gamma)}(t_i t_j^*) = \delta_{ij} \)).

Moreover, \( \theta_\sigma : \Gamma_{\sigma^{-1}} \to \Gamma_{\sigma} \) is the isomorphism implemented by \( \sigma \), that \( \theta_\sigma(\gamma) = \sigma \gamma \sigma^{-1} \) for \( \gamma \) in \( \Gamma_{\sigma^{-1}} \). In particular, for \( y \) in \( \mathcal{L}(\Gamma) \) we have that

\[
E_{\mathcal{L}(\Gamma)}^{\mathcal{L}(\Gamma_{\sigma^{-1}})}(\sigma x \sigma^{-1}) = \sigma E_{\mathcal{L}(\Gamma)}(x) \sigma^{-1}.
\]

In Proposition 6 we proved that \( \Psi_\sigma(x) \) is given by the formula, for \( x \) in \( \mathcal{L}(\Gamma) \),

\[
\Psi_\sigma(x) = \sum_{i,j=1}^{[\Gamma : \Gamma_{\sigma}]} t_i^* \theta_\sigma E_{\mathcal{L}(\Gamma)}^{\mathcal{L}(\Gamma)}(s_i x s_j^{-1}) t_j.
\]

By linearity we may assume that \( x \) is equal to \( \gamma = L_{\gamma} \) a group element in \( \mathcal{L}(\Gamma) \). Let \( \pi_\gamma \) be the permutation of the set \( \{1, 2, \ldots, [\Gamma : \Gamma_{\sigma^{-1}}]\} \), determined by the requirement that

\[
s_i \gamma = \theta_i(\gamma) s_{\pi_\gamma(i)},
\]

where \( \theta_i(\gamma) = s_i \gamma s_{\pi_\gamma(i)}^{-1} \) belongs to \( \Gamma_{\sigma^{-1}} \). Then

\[
\Psi_\sigma(\gamma) = \frac{1}{[\Gamma : \Gamma_{\sigma}]} \sum_{i=1}^{[\Gamma : \Gamma_{\sigma}]} t_i^* \theta_\sigma(s_i \gamma s_{\pi_\gamma(i)}^{-1}) h_{\pi_\gamma(i)}.
\]
Because $\theta_\sigma$ is conjugation by $\sigma$ this is further equal to

$$1 \left[ \Gamma : \Gamma_\sigma \right] \sum_{i=1}^{[\Gamma : \Gamma_\sigma]} t_i^\sigma \sigma s_i (\gamma) s_{\pi_\gamma(i)}^{-1} \sigma^{-1} t_{\pi_\gamma(i)}. \tag{7}$$

By property 7) in Remark 8

$$t_i^\sigma = (\pi(\sigma s_i)) = (\pi(\sigma s_i) e)^*$$

is equal to

$$t_i^\sigma \sigma s_i = \Gamma_{\sigma s_i}(\pi_{\sigma s_i})^{-1}$$

and hence

$$t_i^\sigma \sigma s_i = \Gamma_{\sigma s_i} \quad \text{for } i = 1, 2, \ldots, n.$$  

Consequently, combining this with formula 7) it follows that $\Psi_\sigma(\gamma)$ is further equal to

$$1 \left[ \Gamma : \Gamma_\sigma \right] \sum_{i,j=1}^{[\Gamma : \Gamma_\sigma]} E_L(\{I\sigma s_i \gamma (t_{\sigma s_j})^*\}. \tag{8}$$

Note that $E_L^G(\{I\sigma s_i \gamma (t_{\sigma s_j})^*\})$ is equal to $\Gamma_{\sigma s_i}(\pi_{\sigma s_j})^*$.  

Indeed, a term of the form $\Gamma_{\sigma s_i} t_{\sigma s_j}$ contains various terms of the form $a(\gamma_1 s_i \gamma s_j^{-1} \sigma^{-1} \gamma_2)$. Then $E_L^G(\{I\sigma s_i \gamma (t_{\sigma s_j})^*\})$ of such a term is different from 0 if and only if there exists a $\theta$ in $\Gamma$ such that

$$\gamma_1 \sigma s_i \gamma s_j^{-1} \sigma^{-1} \gamma_2 = \theta$$

which implies that

$$\sigma(s_i \gamma s_j^{-1}) \sigma^{-1} = \gamma_1^{-1} \theta \gamma_2$$

and hence

$$s_i \gamma s_j^{-1} = \sigma^{-1}(\gamma_1^{-1} \theta \gamma_2).$$

Thus, $s_i \gamma s_j^{-1} = \theta_1$ for some $\theta_1$ in $\Gamma_{\sigma^{-1}}$ and hence $s_i \gamma = \theta_1 s_j$. But this by the definition of the permutation $\pi_\gamma$ implies that $j = \pi_\gamma(i)$.  

Thus the equality (8) might be continued as

$$1 \left[ \Gamma : \Gamma_\sigma \right] \sum_{i,j=1}^{[\Gamma : \Gamma_\sigma]} E_L^G(\{I\sigma s_i \gamma t_{\sigma s_j}\}) \Gamma_{\sigma s_i}(\pi_{\sigma s_j})^*.$$  

But $t_{\Gamma \sigma} = \sum_{i=1}^{n} t_{\Gamma \sigma s_i}$ and hence this is further equal to

$$E_L^G(\{I\sigma \Gamma, \gamma t_{\Gamma \sigma}\})$$

By linearly this gives the required formula for $\Psi_\sigma$.  

It is well known that the Hecke operators on Maass forms (or map forms) give a representation for the Hecke algebra.
This is also true for the abstract Hecke operators, and we prove this, directly from the formula in the preceding theorem.

**Proposition.** The map \[ [\alpha] \rightarrow [\Gamma : \Gamma_\alpha] \Psi_\alpha \] described in the previous theorem is a \(*\) morphism from \( H(G \setminus \Gamma / G) \) into the algebra of bounded operators on \( \ell^2(\Gamma) \). If \( a_1 = \Gamma \sigma_1 \Gamma \), \( a_2 = \Gamma \sigma_2 \Gamma \) are two double cosets with multiplication rule

\[
a_1 a_2 = \sum_{\Gamma z \Gamma \subseteq \Gamma \sigma_1 \Gamma \sigma_2 \Gamma} \alpha(\sigma_1, \sigma_2, z) \Gamma z \Gamma.
\]

Then for all \( x \) in \( L(\Gamma) \)

\[
[\Gamma : \Gamma_\sigma_1][\Gamma : \Gamma_\sigma_2] \Psi_{\sigma_1} \Psi_{\sigma_2} = \sum_{\Gamma z \Gamma} \alpha(\sigma_1, \sigma_2, z) [\Gamma : \Gamma_z] \Psi_z
\]

and hence

\[
[\Gamma : \Gamma_\sigma][\Gamma : \Gamma_\sigma] E(t^{\bar{\sigma}_1} E(t^{\bar{\sigma}_2} x (t^{\bar{\sigma}_2})^* (t^{\bar{\sigma}_1})^*)) = \sum_{\Gamma z \Gamma \subseteq \Gamma \sigma_1 \Gamma \sigma_2 \Gamma} \alpha(\sigma_1, \sigma_2, z) [\Gamma : \Gamma_z] E^{\ell\ell}(G) (t^{\bar{\sigma}_1} x t^{\bar{\sigma}_2} \Gamma).
\]

**Proof.** To do this we need first to formulate another variant for the formula of \( \Psi^\Gamma_\sigma \Gamma \), for \( \Gamma \sigma \Gamma \) a double coset of \( \Gamma \) in \( G \).

Let \( (s_i)_{i=1}^{\Gamma : \Gamma_\sigma^{-1}} \) be a system of representatives for right cosets for \( \Gamma_\sigma^{-1} \) in \( \Gamma \), that is \( \Gamma \) is the disjoint union of \( \Gamma_s \sigma^{-1} s_i \), \( i = 1, 2, \ldots, [\Gamma : \Gamma_\sigma^{-1}] \). Then for each \( \gamma \) in \( \Gamma \) there exists a permutation \( \pi_\gamma \) of the set \( \{i = 1, 2, \ldots, [\Gamma : \Gamma_\sigma^{-1}]\} \) such that for each \( i \), there exists \( \theta_i(\gamma) \) in \( \Gamma_\sigma^{-1} \)

\[
s_i \gamma = \theta_i(\gamma) s_{\pi_i}(i).
\]

Then we proved that

\[
[\Gamma : \Gamma_\sigma] \Psi_\sigma(\gamma) = \sum_{i=1}^{[\Gamma : \Gamma_\sigma^{-1}]} t^{\bar{\sigma}_i} \alpha(\bar{\gamma}, \sigma_\bar{\gamma}(i)) \Psi_x.
\]

By property 9) in Remark 8, let \( \alpha_{ij} = \alpha_{\Gamma \sigma_i \Gamma, \Gamma \sigma_j} \) be the projection from \( \ell^2(\Gamma) \) onto the Hilbert space generated by

\[
\Gamma \cap (\sigma s_i)^{-1} \Gamma (\sigma s_j) = \Gamma \cap s_i (\sigma \Gamma \sigma^{-1}) s_j.
\]

Combining properties 8) and 9) it follows that

\[
[\Gamma : \Gamma_\sigma] \Psi_\sigma(x) = \sum_{i,j=1}^{n} t^{\bar{\sigma}_i} \alpha_{\Gamma \sigma_i \Gamma, \Gamma \sigma_j} (x) t^{\bar{\sigma}_j}.
\]

Let now \( a_1 = [\Gamma \sigma_1 \Gamma] \), \( a_2 = [\Gamma \sigma_2 \Gamma] \), be two double cosets in \( G \), for which we want to compute the composition

\[
[\Gamma : \Gamma_\sigma_1][\Gamma : \Gamma_\sigma_2] \Psi_{a_2 \circ \Psi_{a_1}}.
\]
Assume that $s_i$ are representatives for left $\Gamma^{-1}_{\sigma_1}$ cosets in $\Gamma$ (that is $\Gamma = \bigcup_{i=1}^{[\Gamma : \Gamma_{\sigma_1}]} s_i \Gamma_{\sigma_1}$) and similarly assume that $r_\alpha$, $\alpha = 1, 2, \ldots, [\Gamma : \Gamma_{\sigma_1}]$ are representatives for $\Gamma^{-1}_{\sigma_2}$ left cosets, that is $\Gamma = \bigcup_{\alpha=1}^{[\Gamma : \Gamma_{\sigma_2}]} r_\alpha$.

Recall that by property c) in Remark 8, we have that for all $i = 1, 2, \ldots, [\Gamma : \Gamma_{\sigma_1}]$

$$t^{\Gamma_{\sigma_2}}t^{\Gamma_{\sigma_1}s_i} = \sum_{a=1}^{[\Gamma : \Gamma_{\sigma_2}]} t^{\Gamma_{\sigma_2}r_\alpha \sigma_1 s_i}.$$  \hspace{1cm} (11)

Let $\pi_\gamma$ be the permutation associated to the cosets $\Gamma^{-1}_{\sigma_1}s_i, \Gamma^{-1}_{\sigma_1}s_2, \ldots$ as in Remark 8 and 9.

Then by using property 10) for $[\Gamma \sigma_1 \Gamma]$ we obtain that for every $\gamma$ in $\Gamma$, we have that

$$[\Gamma : \Gamma_{\sigma_1}] [\Gamma : \Gamma_{\sigma_1}] \Psi_{\sigma_2}(\Psi_{\sigma_1}(\gamma)) =$$

$$= \sum_{i,j=1}^{[\Gamma : \Gamma_{\sigma_1}]} E_{\mathcal{L}(\Gamma)}(t^\Gamma \sigma_2 \sigma_1 s_i) \alpha \Gamma \sigma_i, \Gamma \sigma_j(\gamma)(t^\Gamma \sigma_1 s_j)^* (t^\Gamma \sigma_2)^*$$

which by using the equality (11) is further equal to

$$\sum_{a,b=1}^{[\Gamma : \Gamma_{\sigma_2}]} \sum_{i,j=1}^{[\Gamma : \Gamma_{\sigma_1}]} E_{\mathcal{L}(\Gamma)}(t^\Gamma \sigma_2 r_\alpha \sigma_1 s_i) \alpha \Gamma \sigma_i, \Gamma \sigma_j(\gamma)(t^\Gamma \sigma_2 r_\alpha \sigma_1 s_j)^*.$$  \hspace{1cm} (12)

As noted in property 9) of Remark 8, $\alpha \Gamma \sigma_i, \Gamma \sigma_j(\gamma)$ is different from 0 if and only if $\gamma \in \Gamma \cap s_i^{-1} \sigma^{-1} \Gamma \sigma_j$ which is equivalent to the fact that there exists $\theta$ in $\Gamma$ such that $\sigma \gamma = \theta \sigma \sigma_j$.

Moreover, still as a consequence from property 8) in Remark 8 it follows that a term in the sum (12) is different from 0 if and only if there exists $\theta'$ in $\Gamma$ such that

$$\sigma_2 r_\alpha \sigma_1 s_i \gamma = \theta' (\sigma_1 r_\beta \sigma_1 s_j).$$  \hspace{1cm} (13)

But $j$ was determined by the fact that

$$\sigma_1 s_i = \theta \sigma_1 s_j \quad \text{for some } \theta \in \Gamma.$$  \hspace{1cm} (14)

From (14) we deduce that

$$\sigma_2 r_\alpha \sigma_1 s_i \gamma = \sigma_2 r_\alpha \theta \sigma_1 s_j$$

and using (13) we deduce that

$$\sigma_2 r_\alpha \theta \sigma_1 s_j = \theta' \sigma_1 r_\beta \sigma_1 s_j$$

and hence

$$\sigma_2 r_\alpha \theta = \theta' \sigma_1 r_\beta.$$

Hence $b$ and $\theta'$ are uniquely determined by $\theta$ and $a$ and hence by $a, i$ and $\gamma$. 
Thus there exists a bijection \( \alpha_\gamma = (\alpha_\gamma^1, \alpha_\gamma^2) \) of the set \( \{1, 2, \ldots, [\Gamma : \Gamma_{\sigma^{-1}}]\} \times \{1, 2, \ldots, [\Gamma : \Gamma_{\sigma^{-1}}]\} \) which to every pair \((a, i)\) associates the unique \( b = \alpha_\gamma^1(a, i)\), \( j = \alpha_\gamma^2(a, i) = \pi_\gamma(i) \) for which the term starting with \( \ell^{\Gamma_{\sigma^1 a \sigma_2 s_i}} \) in the sum (12) remains non zero after applying \( E^{L(\Gamma)}_{\ell} \). Moreover, this bijection has the property that for all \((a, i)\) in \( \{1, 2, \ldots, [\Gamma : \Gamma_{\sigma^{-1}}]\} \times \{1, 2, \ldots, [\Gamma : \Gamma_{\sigma^{-1}}]\} \) we have that there exists \( \theta' \) in \( \Gamma \) such that

\[
\sigma_2 r_a \sigma_1 s_i \gamma = \theta' \sigma_2 r_{\alpha_\gamma^1(a, i)} \sigma_1 s_{\pi_\gamma(i)}.
\]

Thus, \([\Gamma : \Gamma_{\sigma^{-1}}][\Gamma : \Gamma_{\sigma^{-1}}] \Psi_{\sigma_2} \Psi_{\sigma_1}(\gamma)\) is equal to

\[
(15) \quad \sum_{a=1}^{[\Gamma : \Gamma_{\sigma^{-1}}]} \sum_{i=1}^{[\Gamma : \Gamma_{\sigma^{-1}}]} \ell^{\Gamma_{\sigma_2 a \sigma_1 s_i \gamma}} \left( \ell^{\Gamma_{\sigma_2 r_{\alpha_\gamma^1(a, i)} \sigma_1 s_{\pi_\gamma(i)}}} \right)^*,
\]

where \( \alpha_\gamma = (\alpha_\gamma^1, \alpha_\gamma^2) \) is a bijection.

On the other hand, we know that

\[
\ell^{\Gamma_{\sigma_2 \Gamma} \ell^{\Gamma_{\sigma_1 \Gamma}}} = \sum_{[\Gamma_{\sigma_2 \Gamma} \subseteq \Gamma_{\sigma_1 \Gamma}] \sigma_2 \sigma_1 \Gamma} \alpha(\sigma_2, \sigma_1, z) \ell^{\Gamma_{\sigma_2 \Gamma}},
\]

where the multiplicities \( \alpha(\sigma_2, \sigma_1, z) \) are strictly positive integer numbers that come from the algebra structure of the Hecke algebra of double cosets.

Moreover, as we have seen above

\[
\ell^{\Gamma_{\sigma_2 \Gamma} \ell^{\Gamma_{\sigma_1 \Gamma}}} = \sum_{a=1}^{[\Gamma : \Gamma_{\sigma^{-1}}]} \sum_{i=1}^{[\Gamma : \Gamma_{\sigma^{-1}}]} \ell^{\Gamma_{\sigma_2 a \sigma_1 s_i}}.
\]

Hence the enumeration of left cosets in \([\Gamma_{\sigma_2 \Gamma}][\Gamma_{\sigma_1 \Gamma}]\) is \( \Gamma_{\sigma_2 r_a \sigma_1 s_i}, a = 1, 2, \ldots, [\Gamma : \Gamma_{\sigma^{-1}}], i = 1, 2, \ldots, [\Gamma : \Gamma_{\sigma^{-1}}] \).

This enumeration will contain for each coset \([\Gamma_{\sigma_2 \Gamma} \subseteq \Gamma_{\sigma_1 \Gamma}] \Gamma_{\sigma_2 \Gamma} \) exactly \( \alpha(\gamma_1, \gamma_2, z) \) groups of \([\Gamma : \Gamma_{\sigma^{-1}}] \) cosets, that together constitute of \( \Gamma_{\sigma_2 \Gamma} \).

The contribution of any such group in the sum (15), will be one copy of \( E^{L(\Gamma)}_{\ell}(t^z(\gamma(t^z)^x))^* \).

But this proves exactly that

\[
E^{L(\Gamma)}_{\ell}(t^{\Gamma_{\sigma_2 \Gamma}} \ell(t^{\Gamma_{\sigma_1 \Gamma}} \gamma(t^{\Gamma_{\sigma_1 \Gamma}})^x)(t^{\Gamma_{\sigma_2 \Gamma}})^x)
\]

is

\[
\sum_{[\Gamma_{\sigma_2 \Gamma} \subseteq \Gamma_{\sigma_1 \Gamma}] \sigma_2 \sigma_1 \Gamma} \alpha(\sigma_2, \sigma_1, z) E^{L(\Gamma)}_{\ell}(t^z(\gamma(t^z)^x))^*.
\]

By linearity this proves our result. \( \square \)
In concrete situation, it might happen that we have the unitary representation $\pi$ of $G$ on a Hilbert Space $H$, and that we know that $\pi|_{\Gamma}$ is unitarily equivalent to the left regular representation, but without knowing precisely the structure of these intertwiners. So, it would be useful to proceed with the construction of the elements $t^{\Gamma_\sigma \Gamma}$, but starting just with a cyclic vector $\eta$ (which automatically is separating) in the Hilbert space of the representation of $\pi$.

So, in this case we would start with

$$\tilde{t}^{\Gamma_\sigma \Gamma} = \sum_{\theta \in \Gamma_\sigma \Gamma} \langle \pi(\theta)\eta, \eta \rangle \theta.$$ 

For example, in the case of $\text{PSL}_2(\mathbb{Z})$ represented on the space $H_{13}([\text{GHJ}])$ by Perelomov ([Pe]) we know that evaluation vector at any given point in $\mathbb{H}$ is cyclic. Then the $\tilde{t}^{\Gamma_\sigma \Gamma}$ might have an easier expression. To exemplify we replace $\text{PSL}_2(\mathbb{R})$ by $SU(1,1)$, so the upper half plane gets replaced by the unit disk, and $\text{PSL}_2(\mathbb{Z})$ gets replaced by a discrete subgroup say $\Gamma_0$ of $SU(1,1)$. Let $\eta$ be the evaluation vector at 0, so $\eta$ becomes the constant function $|\cdot|$ and $\langle \pi(\gamma)\eta, \eta \rangle_{H_{13}}$ is clearly easy to compute (since $\pi(\gamma)$ is a multiple of the evaluation vector at $\gamma_0$).

In the next lemma we prove the family of “deformed” $\tilde{t}^{\Gamma_\sigma \Gamma}$ might be used to compute $\Psi_\sigma$.

**Proposition 23.** Let $\eta$ be a cyclic separating vector in $\ell^2(\Gamma)$ and let, for $\sigma$ in $G$,

$$\tilde{t}^{\Gamma_\sigma \Gamma} = \sum_{\theta \in \Gamma_\sigma \Gamma} \langle \pi(\theta)\eta, \eta \rangle \theta.$$ 

Let $x = (\eta^*\eta)^{1/2}$ which is invertible at least in the affiliated algebra of unbounded operators. Then $\xi = x^{-1/2}\eta$ is a cyclic trace vector, and hence by Remark 8, property 4),

$$t^{\Gamma_\sigma \Gamma} = x^{-1/2}\tilde{t}^{\Gamma_\sigma \Gamma}x^{-1/2}$$ 

and hence for $y$ in $L(\Gamma)$,

$$[\Gamma : \Gamma_\sigma] \Psi_\sigma(y) = E_{\mathcal{L}(\Gamma)}^{\mathcal{L}(G)}(x^{-1/2}\tilde{t}^{\Gamma_\sigma \Gamma}x^{-1/2}y)(\tilde{t}^{\Gamma_\sigma \Gamma})^*(x^{-1/2}).$$

**Proof.** This is now obvious. □

There is a very simple way to compute the element $x$ in the preceding lemma from the matrix coefficients $\langle \pi(\theta)\eta, \eta \rangle$, $\gamma \in \Gamma$. This is certainly well known to specialist, but for completeness we include the exact result here.

**Lemma.** Let $\eta$ in $\ell^2(\Gamma)$ be given. Assume we know the element $A = \sum \langle \gamma\eta, \eta \rangle \gamma^{-1}$. Then $\xi = (A^* A)^{-1/2}\eta$ is a cyclic trace vector in $\ell^2(\Gamma)$.

**Proof.** Indeed,

$$A = \sum_{\gamma} \tau(\eta^*\gamma)\gamma^{-1} = \sum_{\gamma} \tau((\eta\eta^*)\gamma^{-1})\gamma = \sum_{\gamma} \langle \eta\eta^*, \gamma \rangle \gamma = \eta\eta^*.$$
Hence \((\eta\eta^*)^{-1/2} = A^{-1/2}\) which is invertible since \(\eta\) is cyclic and separating. Then \((\eta\eta^*)^{-1/2}\eta\) is a unitary, that is (as a vector) a cyclic trace vector. \(\square\)

6. Complete positivity multipliers properties for eigenvalues for a joint eigenvector of the Hecke operators

In this section we derive further consequences, from the relations derived in the previous chapter, regarding the relative position in \(L(G)\) of the algebra \(L(\Gamma)\) and the \(H \subseteq L(G)\) generated by the \(T^\alpha\)'s \(\alpha\) running in the space of double cosets of \(\Gamma\) in \(G\). To avoid cumbersome notations we will use \(\rho(a)\) and \(t^a\) for \(\rho(a)\) and \(t^a\) for \(a = [\Gamma \sigma \Gamma]\) double cosets. This will continue to work with the image of \(H\) through the canonical conjugation anti-isomorphism in \(L(\Gamma)\).

Let \(D\) be the von Neumann subalgebra in \(B(\ell^2(G))\) generated by the operators of left and right multiplication with elements in \(H\), that is by \(L_\alpha = L_{t^\alpha}\), \(R_\alpha = R_{t^\alpha}\), the left and right convoluters by the elements \(t^\alpha \in H\) that are associated to double cosets \(\alpha = [\Gamma \sigma \Gamma]\), \(\sigma \in G\).

From an algebra point of view \(D\) is isomorphic to \(H \otimes H\), but when talking closures this might be false (e.g., see the action of the algebra \(D\) on the vector 1 (the unit of \(G\)), viewed as a vector \([Po]\)).

Let \(P\) be the projection from \(\ell^2(G)\) onto \(\ell^2(\Gamma)\). Then, by property 2) in Remark 8, it follows that

\[PL_\alpha R^*_{\beta} P = 0\]

unless \(\alpha = \beta\) in which case

\[PL_\alpha R^*_{\alpha} P = [\Gamma : \Gamma_\sigma]\Psi_\alpha, \quad \alpha = [\Gamma \sigma \Gamma].\]

Then \(D\) has the following remarkable property:

\[(PDP)(PDP) \subseteq (PDP)\]

and hence \(PDP\) is an algebra.

Moreover, the algebra \(t_\alpha \to P(L_\alpha R^*_{\alpha})P, \alpha = [\Gamma \sigma \Gamma], \sigma \in G\) extends to a \(*\)-algebra homeomorphism \(\Phi\) from the \(*\)-algebra generated by the \(t_\alpha\)'s, into \(PDP\).

Although we do not know the structure of the action of the algebra \(D\) on a vector \(\xi\) in \(\ell^2(\Gamma)\), that is different from 1, we can still derive some conclusion in the case when the unit vector \(\xi\) in \(\ell^2(\Gamma)\) is a joint eigenvector for all the \([\Gamma : \Gamma_\sigma^{-1}]\Psi_\alpha\)'s of eigenvalue \(\lambda(\alpha), \alpha = [\Gamma \sigma \Gamma]\) running over all double cosets of \(\Gamma\).

Let \(K\) be the Hilbert subspace of \(\ell^2(G)\) generated by \(H\xi\).

The fact that \(\xi\) is a norm 1 eigenvector for all the \(\Psi_\alpha\)'s, \(\alpha\) double coset implies

\[\tau(t_\alpha t^*_\beta)\xi = \delta_{\alpha\beta}\lambda(\alpha)\]

for all double cosets \(\alpha, \beta\) of \(\Gamma\) in \(G\), and hence

\[\tau(t_\alpha t^*_\beta)\xi^* = \delta_{\alpha\beta}\lambda(\alpha).\]
(Here $\xi$ is a norm 1 eigenvector for $[\Gamma : \Gamma_{\sigma}]\Psi_\alpha$, of eigenvalue $\lambda(\alpha)$, with $\alpha = \Gamma\sigma\Gamma$, $\sigma$ in $G$.)

We note the following consequence of these considerations.

**Lemma 24.** Let $\xi$ be a norm 1 joint eigenvector for the maps $[\Gamma : \Gamma_{\sigma-1}]\Psi_\alpha = E_{\mathcal{L}(\Gamma)}(t^\alpha \cdot (t^\alpha)^*)$ on $\ell_2(\Gamma)$, of eigenvalue $\lambda(\alpha)$ where $\alpha$ is the double coset $\Gamma\sigma\Gamma$, $\sigma$ in $G$.

Recall that $\mathcal{H}$ is the von Neumann algebra generated by all the $t_\alpha$’s. Then

$$E_{\mathcal{H}}(\xi^* t_\alpha \xi) = \frac{\lambda_\alpha}{[\Gamma : \Gamma_{\sigma-1}]} t_\alpha$$

for all double cosets $\alpha = \Gamma\sigma\Gamma$, $\sigma$ in $G$.

**Proof.** Let $\eta$ in $L^2(\mathcal{H}, \tau_G)$, be the vector $E_{\mathcal{H}}(\xi^* t_\alpha \xi)$.

Then for all cosets $\beta = \Gamma\sigma_1\Gamma$, $\sigma_1$ in $G$ we have that

$$\langle \eta, t_\beta \rangle_{\mathcal{L}(G)}(\Gamma) = \tau_{\mathcal{L}(G)}(\xi^* t_\alpha \xi^* t_\beta) = \tau_{\mathcal{L}(G)}(t_\alpha \xi t_\beta) = \tau_{\mathcal{L}(G)}(E_{\mathcal{L}(\Gamma)}(t_\alpha \xi t_\beta) \xi^*)$$

and this is 0 unless, $\alpha = \beta$, case in which the quantity above is further equal to

$$\tau_{\mathcal{L}(G)}(\lambda_\alpha \xi \xi^*) = \lambda_\alpha.$$

Thus $\eta$ is a vector in $L^2(\mathcal{H}, \tau_{\mathcal{L}(G)})$ which verifies that $\langle \eta, t_\alpha \rangle$ is 0 unless $\alpha = \beta$ case in which $\langle \eta, t_\alpha \rangle = \lambda_\alpha$.

Since as proven in Remark 8, $\{t^\alpha\}$ is an orthogonal basis for $L^2(\mathcal{H}, \tau_{\mathcal{L}(G)})$ implies that (again by Remark 8)

$$\eta = \frac{\lambda_\alpha}{\|t_\alpha\|^2} = \frac{\lambda_\alpha}{[\Gamma : \Gamma_{\sigma-1}]}$$

if $\alpha = [\Gamma \sigma\Gamma]$.

This observation has the following important corollary

**Corollary 25.** Let $G$ be a discrete group and $\Gamma \subset G$ an almost normal subgroup. Assume that $G$ admits a unitary representation $\pi$ that extends the left regular representation of $\Gamma$ on $\ell^2(\Gamma)$. For $\alpha = [\Gamma \sigma\Gamma]$ a double coset of $\Gamma$ in $G$, let $\Psi_\alpha$ be the completely positive map on $\mathcal{L}(\Gamma)'$ defined by the formula

$$\Psi_\alpha(x) = E_{\mathcal{L}(\Gamma)'}(\pi(\sigma)x\pi(\sigma^{-1})).$$

Let $\xi$ in $\ell^2(\Gamma)$ be a joint eigenvector of eigenvalue $\lambda(\alpha)$, for all the completely positive linear maps $[\Gamma : \Gamma_{\sigma-1}]\Psi_\alpha$, $\alpha = [\Gamma \sigma\Gamma]$, $\sigma$ in $G$.

Consider the linear map $\Phi_0$ on $\mathcal{H}(\Gamma \setminus G/\Gamma)$ (the linear span of double cosets) defined by

$$\Phi_0(\alpha) = \frac{\lambda(\alpha)}{\text{ind } \alpha} \alpha.$$

Here $\alpha = [\Gamma \sigma\Gamma]$, runs over all double cosets $\Gamma\sigma\Gamma$ of $\Gamma$ in $G$, and $\text{ind } \alpha [\Gamma : \Gamma_{\sigma-1}]$.

Then $\Phi_0$ extends to completely positive linear $\Phi_\lambda$ map on $\mathcal{H}$. 

□
In particular, the sequence \( \left( \frac{\lambda(\alpha)}{\text{ind} \alpha} \right)_{\alpha = [\Gamma \sigma \Gamma], \sigma \in G} \) is a completely positive bounded multiplier of the Hecke’s double cosets algebra.

**Proof.** The extension of the map \( \Phi_0 \) is the map \( \Phi \) on \( \mathcal{H} \) defined by \( \Phi(x) = E_\mathcal{H}(\xi^* x \xi) \). But this is clearly completely positive.

**Corollary 26.** Let \( \Gamma \subset G \) be an almost normal subgroup as above. Let \( \Delta \) be the map from the Hecke algebra \( \mathcal{H}_0 \) into \( \mathcal{H}_0 \otimes \mathcal{H}_0 \), defined by

\[
\Delta([\Gamma \sigma \Gamma]) = \frac{1}{[\Gamma : \Gamma_\sigma]} [\Gamma \sigma \Gamma] \otimes [\Gamma \sigma^{-1} \Gamma]
\]

for \( \sigma \in G \). (We may also extend \( \Delta \) to the reduced \( C^* \)-Hecke algebra \( \mathcal{H} \).)

Then \( \Delta \) is positive. In the terminology of Vershik ([Ve]) where this is proved for finite \( G \), the algebra \( \mathcal{H}_0 \) (with basis \( [\Gamma \sigma \Gamma] \)) is 2-positive.

**Proof.** We have to verify that if \( p \) is positive in \( \mathcal{H}_0 \) then \( \Delta(p) \) is positive. Since \( \mathcal{H}_0 \) is a commutative algebra, it is sufficient to prove that if \( \chi_\lambda \) is a character of \( \mathcal{H}_0 \), then \( (\text{Id} \otimes \chi_\lambda)(\Delta(p)) \in \mathcal{H}_0 \) is positive.

But obviously \( (\text{Id} \otimes \chi_\lambda) \) is the previous map \( \Phi_\lambda \), which is positive for all \( \lambda \) corresponding to values in the spectrum of \( [\Gamma \sigma \Gamma] \) in the reduced \( C^* \)-algebra.

**Remark 27.** In the case of \( G = \text{PSL}_2(\mathbb{Z}[\frac{1}{p}]) \), \( \Gamma = \text{PSL}_2(\mathbb{Z}) \), as we observed before the Hecke algebra \( \gamma \) is isomorphic to the radial algebra in the free group with \( N = \frac{2}{3} \) generators. The results of [Py], [DeCaHa], also prove that \( \Phi_\lambda \) is a completely positive map on \( \mathcal{H} \), for \( \lambda \) in the interval \([- (p + 1), (p + 1)]\). So, we cannot exclude values of \( \lambda \) by this method, in the case of \( \text{PSL}_2(\mathbb{Z}) \).

However, we have the following additional property of the map \( \Phi_\lambda \), that is derived from the representation of the primitive structure of the Hecke algebra.

**Proposition 28.** Let \( \tilde{C} \) be the vector space of sets of the form \([\sigma_1 \Gamma \sigma_2], \sigma_1, \sigma_2 \in G\). We let \( C(G, \Gamma) \) be the vector space obtained from \( \tilde{C} \) by factorizing at the linear relations of the form

\[
\sum [\sigma_1^1 \Gamma \sigma_2^1] = \sum [\theta_i^1 \Gamma \theta_i^2]
\]

if \( \sigma_i^1, \theta_i^1 \) are elements of \( G \), and the disjoint union \( \sigma_1^1 \Gamma \sigma_2^1 \) is equal to the disjoint union \( \theta_1^1 \Gamma \theta_2^1 \). Let \( \xi \) be an eigenvector (see Appendix 2). Then there exists a bilinear map \( \chi : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \) such that \( \chi \) recovers the value of the eigenvector, that is \( \chi|_{\mathcal{H}_0 \times \mathcal{H}_0} \) is defined by \( \chi([\Gamma \alpha \Gamma], [\Gamma \beta \Gamma]) = \delta_{\alpha \beta} \frac{\lambda([\Gamma \alpha \Gamma])}{\lambda([\Gamma \beta \Gamma])} \) and \( \chi \) is positive in the following sense

\[
\sum \lambda_{j_3 j_4} \lambda_{i_1 i_2} ([\sigma_1, \Gamma \sigma_2], ([\sigma_3 \Gamma \sigma_4])) \geq 0.
\]

**Proof.** Indeed we define

\[
\chi([\sigma_1 \Gamma \sigma_2], [\sigma_3 \Gamma \sigma_4]) = \tau(t^{\Gamma \sigma_1 \xi} \xi t^{\sigma_1 \Gamma} t^{\Gamma \sigma_2} t^{\sigma_2 \Gamma} t^{\Gamma \sigma_3} \xi^* t^{\sigma_3 \Gamma}).
\]
Note that $\chi$ has a second positivity property coming from the inequality
\[
\tau\left(\xi \left( \sum \eta_i \sigma_i \Gamma^{\sigma_i_1} \Gamma^{\sigma_i_2} \right) \right) \geq 0
\]
for all complex numbers $\eta_i, \theta_j$. □

**Remark 29.** It is not clear if the completely positive map, for values of $\lambda$ outside $[-2\sqrt{p}, 2\sqrt{p}]$, would have such an extension $\chi$.

7. THE STRUCTURE OF THE CROSSED PRODUCT ALGEBRA, MODULO THE COMPACT OPERATORS, OF LEFT AND RIGHT CONVULUTES IN $PSL_2(\mathbb{Z}[\frac{1}{p}])$

In this section $G$ will be the discrete group $PSL_2(\mathbb{Z}[\frac{1}{p}])$ and $\Gamma = PSL_2(\mathbb{Z})$. By $\varepsilon$ we denote the 2 group cocycle on $G$ with values in $\pm 1$ introduced in Chapter 2. We will prove an extension of the usual Akemann-Ostrand property, that asserts the $C^*$-algebra generated by left and right convolution of $\Gamma$ on $\ell^2(\Gamma)$, is isomorphic modulo the compact operators to the reduced $C^*$-algebra $C^*_\text{red}(\Gamma \times \Gamma^\text{op})$. (Here $\Gamma^\text{op}$ is the group $\Gamma$ considered with the opposite multiplication, so that we have a natural action of $\Gamma \times \Gamma^\text{op}$ on $\Gamma$.)

We will extend this result to the (partial) action of $G \times G^\text{op}$ on $\Gamma$ and identify the structure of the crossed product algebra obtained by taking the quotient modulo the compact operators.

As a consequence, since the representation of the Hecke algebra, giving unitarily equivalent operators to the classical ones, takes values into the $C^*$-algebra generated by left and right convolutions from $G \times G^\text{op}$ acting $\ell^2(\Gamma)$, and characteristic functions of cosets of modular subgroups, we can compute the essential spectrum of the classical Hecke operators.

Let $\mathbb{Z}_p$ be the $p$-adic integers and $K$ be the compact group $PSL_2(\mathbb{Z}_p)$. Note that $K$ is totally disconnected and that $\Gamma$ is dense in $K$. Let $\mu_p$ be the normalized Haar measure on $K$.

We will use the following embedding of the algebra of continuous functions on $K$ into $B(\ell^2(\Gamma))$. To each function $f$ in $C(K)$ we associate the diagonal multiplication operator on $\ell^2(\Gamma)$ with the restriction of $f$ to $\Gamma \subseteq K$.

In this way, $C(K)$ is identified with the commutative $C^*$-subalgebra $X_\Gamma$ of $\ell^\infty(\Gamma)$ generated by characteristic functions of left cosets (equivalently right) of modular subgroups.

The Haar measure $\mu_p$ on $K$ then correspond to the state (trace) on $X_\Gamma$ that associates to a coset $s\Gamma \sigma$ of a modular subgroup $\Gamma_\sigma$ of $\Gamma$ the value $\frac{1}{|\Gamma_\sigma|}$. Note the group $\tilde{G} = G \times G^\text{op}$ (where $G^\text{op}$ is the group $G$ with opposite multiplication) acts partially on $K$, by the formula
\[
(g_1 \times g_2^\text{op})(\gamma) = g_1 \gamma g_2^{-1}, \quad g_1 \times g_2 \in \tilde{G}, \quad \gamma \in \Gamma.
\]
If we take into account also the cocycle $\varepsilon$ (thus working with $\mathcal{L}(G, \varepsilon)$ instead of $\mathcal{L}(G)$), the formula of the action of $(g_1 \times g_2^{op})$ on $\gamma$ is modified by the factor $\varepsilon(g_1, \gamma)\varepsilon(\gamma, g_2)$.

The domain of $g_1 \times g_2^{op}$ is $\Gamma \cap g_1^{-1}\Gamma g_2$ and this shows that only elements of the form $g_1 \times g_2^{op}$ with $g_1, g_2$ belonging to the same double coset of $\Gamma$ in $G$ will have a nontrivial domain. Because $g_1, g_2$ are in the same double coset the action is measure preserving. Hence we can construct the reduced and maximal crossed product algebra

$$\mathcal{A} = C^*_{red}((G \times G^{op}) \rtimes K), \mathcal{A}_{\text{max}} = C^*((G \times G^{op}) \rtimes K)$$

To construct the reduced crossed product algebra we use the canonical trace $\tau_p$ on the algebraic crossed product $(G \times G^{op}) \rtimes C(K)$ induced by the $G \times G^{op}$ invariant measure $\mu_p$ on $K$.

We have a covariant representation of the crossed product $C^*((G \times G^{op}) \rtimes C(K))$ which comes from the embedding of $C(K)$ into $B(\ell^2(\Gamma))$ described above.

Indeed, let $\theta : G \times G^{op} \to B(\ell^2(\Gamma))$ be the representation (by partial isometries) of $G \times G^{op}$ by left and right convolutions on $\Gamma$. Then $\theta$ is compatible (equivalent) with respect to the action of $G \times G^{op}$ on $K$ and hence we get in this way a covariant representation of the $C^*$-algebra $C^*((G \times G^{op}) \rtimes K)$ into $B(\ell^2(\Gamma))$.

We denote this $C^*$-algebra by $B$ and note that $B$ is generated as a $C^*$-algebra by $\theta(G \times G^{op})$ and $C(K)$. By the results of Akemann-Ostrand ([AO]; see also [Co]) this algebra contains the compacts operators $\mathcal{K}(\ell^2(\Gamma))$.

Note that the algebras $B$ is in fact a corner (under the projection represented by the characteristic function of $\Gamma$, in larger crossed product algebra, of the group $G \times G^{op}$ acting on $\ell^2(G)$).

We formulate now our main result, which identifies the quotient algebra, as a crossed product algebra obtained by the GNS construction from a specific state on $\mathcal{A}_{\text{max}}$.

**Theorem 30.** Let $p$ be a prime number and let $G = \text{PSL}_2(\mathbb{Z}/p\mathbb{Z})$. Let $\mathcal{A}_0 = B/K(\ell^2(\Gamma))$ be the projection in the Calkin algebra of the algebra considered above. Then $\mathcal{A}_0$ is a $C^*$ algebra crossed product of $G \times G^{op}$ acting on $K$, obtained through the GNS construction from the state $\phi_p$ on $\mathcal{A}_{\text{max}}$, defined by the formula

$$\phi_p = \sum_{(g_1 \times g_2) \in G \times G^{op}} F(g_1, g_2) \chi_{\Gamma \cap g_1\Gamma(g_2)^{-1}} d\nu_p (g_1 \times g_2),$$

where $F$ is a numerical, positive definite function on $G \times G^{op}$, depending only on $\||g_1||_2, ||g_2||_2$, of the order of

$$\frac{\ln ||g_1||_2 + \ln ||g_2||_2}{||g_1||_2 \cdot ||g_2||_2}.$$
We will also use the notation $C^*_{\phi_p}(((G \times G^{op}) \rtimes K)$ for the algebra $A_0$ to distinguish the dependence on the state $\phi_p$. The exact formula for $F$ is implicit in the proof below.

**Proof.** As it was proved in the original article of Calkin on the Calkin algebra, if $\Omega$ is a non-trivial ultrafilter on $\mathbb{N}$, then every two sequences $(\xi_n), (\eta_n)$ of norm 1 vectors in $\ell^2(\Gamma)$, tending weakly to zero, define a norm 1 functional $\omega_{\xi,\eta}$ on the Calkin algebra $Q(\ell^2(\Gamma)) = B(\ell^2(\Gamma))/K(\ell^2(\Gamma))$ by the formula

$$\omega_{\xi,\eta} = \lim_{n \to \Omega} \langle X\xi_n, \eta_n \rangle, \quad X \in B(\ell^2(\Gamma)).$$

Moreover, all the states and functionals $B(\ell^2(\Gamma))$, vanishing on $K(\ell^2(\Gamma))$ may be obtained by such a construction.

We will analyse first the formula of these states and functionals on $C(K) \subseteq B(\ell^2(\Gamma))$ and then look at the behaviour of these states, when replacing $\xi$ by $g\xi$, $g \in \tilde{G} = G \times G^{op}$.

The goal is to prove that all such $\omega_{\xi,\eta}$ are continuous with respect to the norm induced by the GNS construction with respect to the specific state $\phi_p$. We may assume that all the vectors $\xi^n, \eta^n$ have finite supports (which might become arbitrarily large). By enlarging the supports we may assume that the finite supports $A_n, B_n \subseteq \Gamma$ of the vectors $\xi_n, \eta_n$ are respectively of the form $\{s_1^n, s_2^n, \ldots\}$, respectively $\{r_1^n, r_2^n, \ldots\}$, where $s_i^n$, respectively $r_i^n$ are coset representatives for $\Gamma \sigma_{p^kn}$, for large $k_n$. (Recall that $\sigma_{p^e} = \begin{pmatrix} 1 & 0 \\ 0 & p^e \end{pmatrix}$, for $e$ in $\mathbb{N}$.)

We may think of $s^n_1$ as an equidistribution of points in $K$, so that the average of the Dirac measure at these points converges weakly to the Haar measure on $K$. We may also assume that the sets $A_n, B_n$ are increasing.

We remark that $\omega_{\xi,\xi}$ induces a measure $\mu_\xi$ on $K$ defined by the formula

$$\int_K f(\omega) d\mu_\xi(\omega) = \lim_{n \to \Omega} \sum_{\omega \in A_n} f(\omega)|\xi^n(\omega)|^2.$$

The above formula establishes an isomorphism between measure on $K$, and weighted sequences of points in $\Gamma$, and hence by density it will be sufficient to work in the case $d\mu_\xi = d_\xi d\mu_p$, where $d_\xi$ is a distribution function on $K$ and $\mu_p$ is Haar measure on $K$.

Since the purpose of this analysis is to prove that the restriction of $\omega_{\xi,\eta}$ to $B/K$ is continuous with respect with respect to the norm induced by the GNS construction with respect to the specific state $\phi_p$, by linearity and density we may assume that $\xi^n, \eta^n$ have all positive entries and that the corresponding measures $\mu_\xi$ and $\mu_\eta$ are absolutely continuous, with respect to the Haar measure $\mu_p$ on $K$. Hence the formula
for the functional $\omega_{\xi,\eta}$ on $C(K) \subseteq Q(\ell^2(\Gamma))$ will be
$$\omega_{\xi,\eta}(f) = \int_K f(\omega) d_{\xi}^{1/2}(\omega) d_{\eta}^{1/2}(\omega) \theta_{A,B}(\omega) d_{\mu_p}(\omega)$$
for $f \in C(K)$. Here $\theta_{A,B}(\omega)$ is a density that only depends on the distribution of the intersection of the supports $A_n, B_n$.

More precisely, if $\Gamma_{\sigma_p,e}$ is a coset of a modular subgroup, and $K(\sigma_p,e, s)$ is the closure of the coset in $K$, then
$$\int_{K(\sigma_p,e, s)} \theta_{A,B}(\omega) d_{\mu_p}(\omega) = \lim_{n \to \Omega} \frac{\text{card } A_n \cap B_n \cap \Gamma_{\sigma_p,e,s}}{\text{card } A_n} \frac{1}{\text{card } B_n}. $$
(It will follow from our estimates below that this formula defines a measure absolutely continuous with respect $\mu_p$.)

In particular, if $X$ is a finite sum $\sum g f_g$ (where $f_g \in C(K)$ only a finite number of them non-zero), thus $X$ is an element in the algebraic crossed product, then
$$\omega_{\xi,\xi}(x) = \sum_{g \in \hat{G}} \int_K f_g(\omega) d_{\xi}^{1/2}(\omega) d_{\eta}^{1/2}(\omega) \theta_{A,g^{-1}A}(\omega) d_{\mu_p}(\omega).$$

Note that $d_{\eta}^{1/2}(\omega) = d_{\xi}^{1/2}(g\omega), g \in \hat{G}, \omega \in K$ and hence since states on the crossed product algebra with respect to the norm induced by the GNS construction with respect to the specific state $\phi_p$, are an ideal with respect Schur multiplication to prove the continuity for the states $\omega_{\xi,\xi}$ with respect to norm on the crossed product $C^*$-algebra, it will be sufficient to prove this for the case
$$\xi^n = \frac{1}{(\text{card } A_n)^{1/2}} \chi_{A_n},$$
where $\chi_{A_n}$ is the characteristic function of the sets described above.

We will denote the corresponding state by $\omega_{A,A}$ where by $A = (A_n)$ we mean the corresponding sequence of increasing finite sets.

Then for $X$ as above a finite sum
$$X = \sum_{g \in \hat{G}} g f_g, \quad f_g \in C(K)$$
we have the following formula for $\omega_{A,A}$
$$\omega_{A,A}(X) = \sum_{g \in \hat{G}} \int_K f_g(\omega) \theta_{A,g^{-1}A}(\omega) d_{\mu_p}(\omega)$$
and for $\Gamma_{\sigma_p,e}$ a coset, $e \in \mathbb{N}, s \in \Gamma$, and $K(p^e, s)$ the closure of the coset in $K$
$$\frac{1}{\mu_p(K(p^e, s))} \int_{K(p^e, s)} \theta_{A,g^{-1}A}(\omega) d_{\mu_p}(\omega) = \lim_{n \to \Omega} \frac{\text{card } A_n \cap g^{-1}A_n \cap \Gamma_{\sigma_p,e,s}}{\text{card } A_n \cap \Gamma_{\sigma_p,e,s}}.$$
Let $B_t = \{ g | g \in \text{PSL}_2(\mathbb{R}), ||g||_2 \leq t \}$ be the hyperbolic ball in $\text{PSL}_2(\mathbb{R})$ of radius $t$. Because of the work of Gorodnik and Nevo ([GoNe]), see also [EM], [DRS]), it follows that the sets $\Gamma_t = \Gamma \cap B_t$ are equidistributed in the cosets of modular subgroups in $\Gamma$, and hence we may use for the sets $A_n$ defined above the choice $A_n = \Gamma \cap B_n$. Since the sets $g_1 B_t g_2 \cap B_t, g_1, g_2 \in \text{PSL}_2(\mathbb{R})$, $t > 0$, ([GoNe]) are well rounded it follows that for $(g_1 \times g_2) \in G \times G^\text{op}$ the density $\theta_{A,g^{-1}A}$ defined above is equal to the limit, for $\Gamma_{\sigma, \rho}$ a modular subgroup coset,

$$\frac{1}{\mu_p(K(p^r, s))} \int_{K(p^r, s)} \theta_{A,g^{-1}A}(\omega) d\mu_p(\omega) = \lim_{t \to \infty} \frac{\text{card} (g_1 \Gamma g_2 \cap \Gamma \cap \Gamma_{\sigma, \rho})}{\text{card} (\Gamma_t \cap \Gamma_{\sigma, \rho})}.$$

Clearly this is equal to

$$\lim_{t \to \infty} \frac{\text{card} (g_1 \Gamma g_2 \cap \Gamma \cap (\Gamma_{\sigma, \rho} \cap (B_t \cap g_1 B_t g_2)))}{\text{card} (\Gamma \cap B_t \cap \Gamma_{\sigma, \rho})}.$$

For a large exponent $e$, the above quantity is non-zero, if and only if the coset $\Gamma_{\sigma, \rho}$ is contained in $g_1 \Gamma g_2 \cap \Gamma$, and hence it follows, by ([GoNe]), that $\theta_{A,g^{-1}A}$ is given by a constant density with respect to $\mu_p$, supported on the closure in $K$ of $g_1 \Gamma g_2 \cap \Gamma$, of weight

$$F(g_1, g_2) = \lim_{t \to \infty} \frac{\text{vol} (B_t \cap g_1 B_t g_2)}{\text{vol} B_t}.$$ 

Here $\text{vol}$ stands for volume computed with respect to Haar measure on $\text{PSL}_2(\mathbb{R})$. Note that $F$ is in itself a positive definite function on $\text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{R})^\text{op}$, but we are only interested in values of $F$ at $(g_1 \times g_2) \in G \times G^\text{op}$, whenever $g_1, g_2$ determine the same double coset of $\Gamma$ in $G$ (so that $g_1 \Gamma g_2 \cap \Gamma$ is nonvoid).

To finish the proof we have to find the order of growth of $F$. To do this we switch to $SU(1, 1)$ instead of $\text{PSL}_2(\mathbb{R})$. Assume $g_1, g_2 \in SU(1, 1)$ are given by:

$$g_1 = \begin{pmatrix} x & y \\ \eta & \overline{\eta} \end{pmatrix}, \quad g_2 = \begin{pmatrix} s & r \\ \overline{r} & \overline{s} \end{pmatrix}.$$

Since $B_t$ is invariant to left and right multiplication by unitaries, it follows that $F(g_1, g_2)$ only depends on $||g_1||_2, ||g_2||_2$ and hence we may assume that the numbers $x, y, s, t$ are all positive.

We have to compute the relative volume (with respect to the volume of $B_t$), as $t$ tends to infinity, of the intersection $g_1 B_t \cap B_t g_2^{-1}$. Using the $KAK$ decomposition of $SU(1, 1)$, and the corresponding Haar measure, we have to compute the volume of the set

$$\{ \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \in SU(1, 1) | xa + y \overline{b} \leq t, |as + br| \leq t \}.$$

We denote $|a| = \rho, a = \rho \exp i\theta_1, b = |b| \exp i\theta_2$. Since we are interested only in the asymptotic ratio of volumes as $t$ tends to infinity, we may substitute $|b| = \sqrt{|a|^2 - 1}$ with $|a|$ and we may replace the Haar measure on $SU(1, 1) = KAK$,
\[ d\mu_{\text{SU}(1,1)} = dk_1 d|a|dk_2 = dk_1 (\cosh^2 \alpha) d|a| dk_2 \] (where \(|a| = \cosh \alpha\)) with \( d\theta_1 \rho d\rho d\theta_2 \) (since we are interested only in asymptotic relative size of volumes).

Hence the formula for \( F(g_1, g_2) \) is
\[
\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{0}^{\min\left(\frac{1}{|\exp i\theta_1 + y|}, \frac{1}{|\exp i\theta_2 + x|}\right)} \rho d\rho d\theta_1 d\theta_2,
\]
which up to a constant is
\[
\frac{1}{x^2 s^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \min\left(\frac{1}{|\exp i\theta_1 + \frac{y}{x}|}, \frac{1}{|\exp i\theta_2 + \frac{x}{s}|}\right) d\theta_1 d\theta_2.
\]

Denote \( \alpha = \frac{y}{x} \) and \( \beta = \frac{r}{s} \) and note that these two quantities are of the order of respectively \( 1/x^2 \) and \( 1/s^2 \). Using arclength approximation it follows that the integral is of the order of
\[
\frac{1}{x^2 s^2} \int_{-1}^{1} \int_{-1}^{1} \min\left(\frac{1}{\alpha^2 + \theta_1^2}, \frac{1}{\beta^2 + \theta_2^2}\right) d\theta_1 d\theta_2.
\]

The result follows then by a straightforward computation.

In the course of the proof of the previous theorem, we also proved the following result:

**Theorem.** Let \( \Gamma \subseteq G \) be almost normal subgroup as above. Let \( S \) be the the set of finite index subgroups of \( G \), of the form \( \Gamma_\sigma = \sigma \Gamma \sigma^{-1} \cap \Gamma \) and assume that \( S \) is directed downward, by inclusion. More precisely, we assume that for every \( \Gamma_{\sigma_1}, \Gamma_{\sigma_2} \in S \) there exists a normal subgroup \( N \) of \( \Gamma \) and \( \Gamma_{\sigma_3} \) such that \( \Gamma_{\sigma_1} \supseteq \Gamma_{\sigma_2} \supseteq N \supseteq \Gamma_{\sigma_3} \). We also assume that \( G \) verifies the assumption in [GoNe]. Assume that for all \( \sigma \in G \), the group indices \( [\Gamma : \Gamma_{\sigma}] \), \( [\Gamma : \Gamma_{\sigma}^{-1}] \) are equal.

Let \( X_\Gamma \) be the continuous functions on the prefinite completion of \( \Gamma \), that is the \( \mathbb{C}^* \)-algebra in \( L^\infty(\Gamma) \) generated by characteristic functions of cosets of elements in \( \rho \) (in the projective case we have to add suitable sign functions to separate points).

Let \( \tau \) be the functional on \( X_\Gamma \) defined by
\[
\tau(\chi_{s\Gamma_\sigma}) = \frac{1}{[\Gamma : \Gamma_\sigma]}, \quad s \in \Gamma, \sigma \in G
\]
which is a measure on the compact maximal space of \( X_\Gamma \).

Because of the equality of the indices \([\Gamma : \Gamma_{\sigma}] = [\Gamma : \Gamma_{\sigma}^{-1}]\), \( \sigma \in G \), it follows that \( \tau \) is invariant to the partial action of \( G \times G^{\text{op}} \) on \( X_\Gamma \).

Clearly, the maximal \( C^* \) (groupoid) crossed product algebra \( C^*((G \times G^{\text{op}}) \rtimes X_\Gamma) \) is represented in \( B(l^2(\Gamma)) \) by letting \( G, G^{\text{op}} \) act by left and respectively right convolutors, and \( X_\Gamma \) by multiplication operators. Then the states on
\[
C^*((G \times G^{\text{op}}) \rtimes X_\Gamma)
\]
induced from the representation into $Q((\ell^2(\Gamma))) = \mathcal{B}(\ell^2(\Gamma))/K(\ell^2(\Gamma))$ are of the form, for $T = \sum_{g \in G \times G^{op}} g f_g$, $f_g \in X_\Gamma$.

$$\Phi_\xi(T) = \Phi_0(\xi^* T \xi),$$

where $\Phi_0$ is the state on $C^*((G \times G^{op}) \times X_\Gamma)$ defined by the formula

$$\Phi_0(T) = \sum_{\tilde{g} = (g_1, g_2) \in G \times G^{op}} \chi(g_1, g_2) \int_{g_1^{-1} \Gamma g_2^{-1} \cap \Gamma} f_g(k) dk,$$

where $dk$ is the (Haar) measure on $X_\Gamma$ induced by $\tau$ (the profinite completion of $\Gamma$) and for $g_1, g_2 \in G \times G^{op}$, $\chi(g_1, g_2)$ is the asymptotic displacement on a family of well rounded balls $B_t$ (as in [GoNe]), that is

$$\chi(g_1, g_2) = \lim_{t \to \infty} \frac{\text{vol}(B_t \cap g_1 B_t g_2)}{\text{vol}(B_t)}$$

(volumes computed with respect Haar measure on $G$).

**Theorem.** The discrete group SL$_2(\mathbb{Z}[\frac{1}{p}])$ does not have the Akemann-Ostrand property.

**Proof.** Indeed in this case the displacement function defining $\Phi_0$, is of the order $\frac{1}{\|g_1\| + \|g_2\|} (\ln \|g_1\| + \|g_2\|)$ (where for $g \in \text{PSL}_2(\mathbb{R})$, $\|g\|$ is the Hilbert-Schmidt norm). But this positive definite function doesn’t verify the $L^{2+\varepsilon}$ summability criteria in [DeCaHa].

In the following remark we describe a way to avoid the use of the cocycle $\varepsilon$.

**Remark** Assume that $\Gamma \subseteq G$ is an almost normal subgroup of $G$. We assume that $G$ is presented in the following way: (Here we assume $\mathbb{Z}_2$ is mapped into the center of $\widetilde{G}$.)

$$0 \to \mathbb{Z}_2 \to \widetilde{G} \to G \to 0$$

$$0 \to \mathbb{Z}_2 \to \tilde{\Gamma} \to \Gamma \to 0$$

Let $u$ be the image of the non-identity element of $\mathbb{Z}_2$ in $\widetilde{G}$ (or which is the same $\tilde{\Gamma}$). Then we assume that $u$ is central element in $\widetilde{G}$.

In the group algebra of $G$ let $P = 1 - u$, which is a projection (corresponding to the negative part of $u$). We consider the reduced algebra $A_P = PL(\widetilde{G})P \supset P\mathcal{C}(\widetilde{G})P$. ($P\mathcal{C}(\widetilde{G})P$ is like the group algebra of $\mathcal{C}(\widetilde{G})$ modulo the identity $u = -P$, $P$ being the identity of the reduced algebra.)

A similar construction the one in the preceding chapters, can be done in this setting, as follows.

Let $H_P = L^2(A_P, \tau_P)$, where $\tau_P$ is the reduced trace. The group $\tilde{\Gamma}$ acts by left and right convolutors $L_{\tilde{\gamma}}$, $R_{\tilde{\gamma}}$, $\tilde{\gamma} \in \tilde{\Gamma}$ on $H_P$ and we obviously have $L_u \tilde{\gamma} = -L_{\tilde{\gamma}}$, $R_u \tilde{\gamma} = -R_{\tilde{\gamma}}$. Assume $\tilde{\sigma} \in \tilde{G}$ and let $\tilde{\Gamma}_{\tilde{\sigma}} = \tilde{\sigma} \tilde{\Gamma} \tilde{\sigma}^{-1} \cap \tilde{\Gamma}$. Then every $\tilde{X}$ in
B(L^2(\mathcal{A}_P, \tau_P)), such that $\tilde{X}L_{\tilde{\gamma}_0} = L_{\tilde{\sigma}\tilde{\gamma}_0\tilde{\sigma}^{-1}}\tilde{X}$ for $\tilde{\gamma}_0 \in \tilde{\Gamma}_{\sigma}^{-1}$ will give raise to a completely positive map $\Psi_\sigma$ obtained from the following diagram

$$(PL(\tilde{\Gamma}_{\sigma}^{-1})P)' \xrightarrow{X^*,X} (PL(\tilde{\Gamma})P)' \supset \bigvee E.$$

Here the commutants are computed in the Hilbert space $H_P$ and $E$ is the canonical conditional expectation. By using the identification $PL(\tilde{\Gamma})P = PR(\tilde{\Gamma})P$ and the same construction as in the Appendix 1, we obtain that $\Psi_\sigma$ is unitarily equivalent to

$$\tilde{\Psi}_{[\tilde{\Gamma}\sigma\tilde{\Gamma}]}(x) = E_{P\tilde{\Gamma}P}(P\tilde{T}[\tilde{\Gamma}\sigma\tilde{\Gamma}]P)xP\tilde{T}[\tilde{\Gamma}\sigma\tilde{\Gamma}]P, \quad x \in P\tilde{\Gamma}P,$$

where $[\tilde{\Gamma}\sigma\tilde{\Gamma}] = [\tilde{\Gamma}\sigma\tilde{\Gamma}]$ is a double coset.

If we start with a representation of $\tilde{G}$ on $L^2(\mathcal{A}_P, \tau_P)$ extending the left regular representation of $\tilde{\Gamma}$ on $L^2(\mathcal{A}_P, \tau_P)$ (thus $\pi(u) = -1$), then we can construct as before

$$\tilde{T}[\tilde{\Gamma}\sigma\tilde{\Gamma}] = \frac{1}{2} \left( \sum_{\theta \in [\tilde{\Gamma}\sigma\tilde{\Gamma}]} \langle \pi(\theta)P, P \rangle \theta \right).$$

The factor $\frac{1}{2}$ is needed because when reducing by $P$ the terms $\langle \pi(\theta)P, P \rangle \theta$ and $\langle \pi(\theta u)P, P \rangle \theta u$ correspond to the same term.

Choosing a system of representatives for the elements $\tilde{\gamma}P, \tilde{\gamma} \in \tilde{\Gamma}$ amounts to give a cocycle $\varepsilon$, and working with $L(\Gamma, \varepsilon)$ instead of $\mathcal{A}_P$, and hence also the operators $\tilde{T}[\tilde{\Gamma}\sigma\tilde{\Gamma}]$ are unitarily equivalent to the classical Hecke operators where $\tilde{G} = \text{SL}_2(\mathbb{Z}[\frac{1}{\tilde{p}^1}]), \tilde{\Gamma} = \text{SL}_2(\mathbb{Z})$.

**Remark** In the context of Theorem 30 and the next theorem, the states from the Calkin algebra on $\ell^2(\mathcal{P}\tilde{\Gamma})$ will be of the form, for $X$ in $PC^\ast(\tilde{G} \times \tilde{G} \times K)$,

$$\Phi_0((\xi^s)^*X(\xi s)),$$

where $s$ is a sign function on $\tilde{K}$ that is $s(ku) = -s(k), k \in \tilde{K}$.

We may choose sign, one of sign function on the $p$-adic numbers $Q_p$ (see [GeGr]) and take $s(k) = \text{sign}_r(\text{Tr}(k)), k \in \tilde{K}$.

We now return to the context of Lemma 13. The previous Remark shows that we may always switch from the skewed algebra with cocycle to a reduced algebra of the cover group $\tilde{G}$.

In chapter 5, we proved that the Hecke algebra $\mathcal{H}_0$ of double cosets $[\Gamma\sigma\Gamma]$ of $\Gamma$ in $G$ admits a $*$-representation into $L(G)$, by mapping a double coset $[\Gamma\sigma\Gamma]$ into

$$t[\Gamma\sigma\Gamma] = \sum_{\theta \in \Gamma\sigma\Gamma} \langle \pi_{13}(\theta)1, 1 \rangle_{13} \theta \in \ell^2(\Gamma\sigma\Gamma) \cap L(G).$$
Here 1 is a vector in the Hilbert space $H_{13}$ that is a cyclic trace vector for the von Neumann algebra generated by $\pi_{13}(\Gamma)$ which is isomorphic to $L(G)$.

We will apply this theorem to the representation of the completely positive maps $\Psi_\sigma(x) = \Gamma : \Gamma_\sigma |E_{L(G)}(t^{13} \sigma t^{13} \Gamma), \; x \in L(\Gamma)$. We want to analyze $\Psi_\sigma$ modulo the compact operators.

The above theorem will apply to the $\Psi_\sigma$ only if the $t^{13} \sigma t^{13}$ are in the reduced $C^*$-algebra $C^*_r(G)$ and to do this we will prove that there exists a choice for the cyclic trace vector $\xi$ in $H_{13}$ such that $t^{13} \sigma t^{13}$ have the required property.

Note that changing $\xi$ into $u\xi$, where $u$ is a unitary changes $t^{13} \sigma t^{13}$ into $u^{*} t^{13} \sigma t^{13} u$ and in fact we are proving that the orbit $\{u^{*} t^{13} \sigma t^{13} u \mid u \in \mathcal{U}(L(\Gamma))\}$ intersects the $C^*$-algebra.

**Lemma 31.** With the notations from Proposition 4, there exists a choice of the cyclic trace vector $\xi$ in $H_{13}$ used in the construction of the elements $t^{13} \sigma t^{13}$, such that for all double cosets $[\Gamma \sigma \Gamma]$, the elements $t^{13} \sigma t^{13}$ belong to the reduced $C^*$-algebra $C^*_r(G)$.

**Proof.** Consider the space $H_{13}$ of positive functions on $\text{PSL}_2(\mathbb{R})$ that are obtained as matrix coefficients from elements $\eta$ in $H_{13}$ (that is $\varphi \colon G \to \mathbb{C}$ belongs to $H_{13}$ if there exists $\eta$ in $H_{13}$ such that $\varphi(g) = \langle \pi_{13}(g)\eta, \eta \rangle$, $g \in \text{PSL}_2(\mathbb{R})$).

Obviously, $H_{13}$ is a cone closed to infinite convergent sums. Indeed if $(\eta_i)$ is a family of vectors in $H_{13}$, $\sum \|\eta_i\|^2 < \infty$, each determining the positive functional $\varphi_i$. Consider the Hilbert subspace $L$ of $H_{13} \otimes \ell^2(I)$ generated by $\bigoplus_{i \in I} \pi(g)\eta_i$. This space is obviously invariant to the action of $G$. Since $\pi_{13}$ is irreducible $\pi(g)|_L$ is a multiple of the representation $\pi_{13}$ and because we have a cyclic vector, it is unitary equivalent to $\pi_{13}$. The vector $\eta = \bigoplus_{i \in I} \eta_i$ will then determine the positive definite function $\sum_i \varphi_i$.

In the sequel we denote $\pi_{13}$ simply by $\pi$.

As it was noted in the list of properties of $t^{13} \sigma t^{13}$, this is equal to $\sum_{g \in \Gamma \sigma \Gamma} \langle \pi(g)\xi, \xi \rangle g$.

If $\varphi_\eta(g) = \langle \pi(g)\eta, \eta \rangle$, $g \in \text{PSL}_2(\mathbb{R})$ is determined by the vector $\eta$, then for $a$ in $L^1(\mathcal{L}(\Gamma), \tau)$ the vector $\pi(a)\eta$ (note that $\pi|_\Gamma$ extends from $G$ to a representation of $\Gamma$ on $H_{13}$ to a representation of $\mathcal{L}(\Gamma)$) will determine a functional $\varphi_a$, that has the property that $\varphi_a|_{\text{PSL}_2(\mathbb{Q})} = a^* \varphi a$.

We are looking to find a positive functional in $H_{13}$ that has the property that $\varphi|_{\text{PSL}_2(\mathbb{Q})}$ belongs to the reduced $C^*$-algebra, and such that moreover $\varphi$ is implemented by a trace vector (as we have seen in Chapter 3, this is equivalent to the pseudo-multiplicative property

$$\varphi(g_1 g_2) = \sum_{\gamma \in \Gamma} \varphi(g_1 \gamma) \varphi(\gamma^{-1} g_2), \quad g_1, g_2 \in \text{PSL}_2(\mathbb{R}).$$
To find such a $\varphi = \varphi_\xi$ is therefore sufficient to find a vector $\xi$ such that the corresponding positive functional has the following properties:

1) the restriction of $\varphi_\xi$ to $\Gamma \sigma \Gamma$ determines an element in $C^*_\text{red}(\text{PGL}_2(\mathbb{Q}))$;

2) $\varphi_\xi|\Gamma$ is invertible in $C^*_\text{red}(\Gamma)$.

Indeed if we found such a vector $\xi$ then we are done because the vector $\xi_0 = \pi((\varphi_\xi|\Gamma)^{-1})\xi$ is a trace vector.

Moreover, let

$$\Psi(g) = (\pi(g)\xi_0, \xi_0)$$

and let $t^{\Gamma \sigma \Gamma}_0 = \sum_{g \in \Gamma \sigma \Gamma} \Psi(g)g$. Then $t^{\Gamma \sigma \Gamma}_0 = (\varphi_\xi|\Gamma)^{-1/2} t^{\Gamma \sigma \Gamma}(\varphi_\xi|\Gamma)^{-1/2}$, where $t^{\Gamma \sigma \Gamma}$ correspond to $\varphi_\xi|\Gamma \sigma \Gamma$ and hence are in $C^*_\text{red}(G)$ and thus belongs to $C^*_\text{red}(\Gamma)$.

Hence to conclude the proof it is sufficient to construct a vector with properties 1), 2). By Jolissaint estimates, it is sufficient to take a fast decreasing vector for the group $G$, such that $\varphi_\xi|\Gamma$ is invertible.

We now use a result by in [BH] (proof of Theorem A1) which says that given $x \geq 0, x \neq 0 \in C^*_\text{red}(\Gamma)$ there exists unitaries $\gamma_1, \ldots, \gamma_n$ in $\Gamma$ such that $\sum \gamma_i x \gamma_i^{-1}$ is invertible.

Consequently, take $\xi$ a fast decreasing vector in $H_{13}$ (e.g., the vector of evaluation at 0 in the model of the unit disk).

Then we construct the functional $\varphi_\xi$ and use the above mentioned result in [BH], to replace $\varphi_\xi$ by $\sum \gamma_i^{-1} \varphi_\xi \varphi_i = \Psi_0$.

Then $\Psi_0$ corresponds to the vector $\frac{1}{\sqrt{n}} (\oplus \pi(\gamma_i)\xi)$ which is still a fast decreasing vector and by construction $\Psi_0$ is invertible.

For another proof see [Ra5], where it is proved that the elements $t^{[\Gamma \sigma]}$, $\sigma \in G$ are a Pimsner Popa basis, and thus bounded.

The algebraic machinery that is inherent to the fact that we have a *-algebra morphism, constructed in Chapter 5, which maps a double coset $[\Gamma \sigma \Gamma]$ into the completely positive map $\Psi_\sigma$ on $\mathcal{L}(\Gamma)$ can be summarized as follows:

**Lemma 32.** The map $\Phi$ from $\mathcal{H}_0$ into $\mathcal{A}_0$, defined by

$$\Phi([\Gamma \sigma \Gamma]) = \chi_\Gamma(t^{[\Gamma \sigma \Gamma]} \otimes t^{[\Gamma \sigma \Gamma]}) \chi_\Gamma,$$

is a *-algebra morphism (see also [Ra5]).

**Proof.** This obtained by passing to the quotient, modulo the compacts the fact that the map taking the double coset $[\Gamma \sigma \Gamma]$ into the completely positive application $\Psi_\sigma$, defined by

$$\Psi_\sigma(x) = [\Gamma : \Gamma] E_{\mathcal{L}(\Gamma)}^{\mathcal{L}(G)}(t^{[\Gamma \sigma \Gamma]} x t^{[\Gamma \sigma \Gamma]}), \quad x \in \mathcal{L}(G),$$

is a *-algebra morphism.
In the next theorem, by taking advantage of the fact that in Theorem 30 we identified completely the structure of the algebra $A_0$ as the $C^*$ algebra crossed product of $G \times G^\op$ acting on $K$, obtained through the GNS construction from the state $\phi_p$ on $A_{\text{max}}$, we prove that the map $\Phi$ is an isomorphism.

**Theorem 33.** Let $G = \text{PSL}_2(\mathbb{Z}[\frac{1}{p}])$, $\Gamma = \text{PSL}_2(\mathbb{Z})$. Let $\pi_Q$ be the canonical projection from $B(\ell^2(\Gamma))$ onto the Calkin algebra $Q(\ell^2(\Gamma))$.

Then, the $*$-algebra morphism, constructed in Chapter 5, which maps a double coset $[\Gamma \sigma \Gamma]$ into the completely positive map $\Psi_\sigma$ on $L(\Gamma)$ given by the Stinespring dilation formula

$$\Psi_\sigma(x) = [\Gamma : \Gamma_\sigma]E^L(\Gamma) (t^{\Gamma \sigma \Gamma} xt^{\Gamma \sigma \Gamma}), \quad x \in L(\Gamma)$$

extends, when composing with the canonical projection $\pi_Q$ to an isomorphism from the reduced $C^*$-Hecke algebra $H$ into the the Calkin algebra $Q(\ell^2(\Gamma))$, mapping $\Gamma \sigma \Gamma$ into $\pi_Q(\Psi_\sigma)$.

Here we use the fact that for all $\sigma$ in $G$, $\Psi_\sigma$, which acts a priori on $L(\Gamma)$, extends to a bounded operator on $\ell^2(\Gamma)$.

**Note.** We showed in Chapter 4, (Remark 18, see also [Ra3]) that the validity of the estimates of the Ramanujan-Petersson conjecture is equivalent to the continuity with respect to the reduced $C^*$-algebra associated to the Hecke algebra of the map taking $\Gamma \sigma \Gamma$ into $\Psi_\sigma$. Hence we have proved that the Ramanujan-Petersson conjecture holds true, modulo the compact operators (that is for the essential spectrum of the Hecke operators) in the case $G = \text{SL}_2(\mathbb{Z}[\frac{1}{p}])$, $\Gamma = \text{PSL}_2(\mathbb{Z})$.

In fact, a trivial application of classical Fredholm theory gives the following corollary.

**Corollary 34.** For every prime $p$ the essential spectrum of the classical Hecke operator $T_p$ is contained in the interval $[-2\sqrt{p}, 2\sqrt{p}]$, predicted by the Ramanujan Petersson conjectures. In particular, given an open interval containing $[-2\sqrt{p}, 2\sqrt{p}]$, there are at most a finite number of possible exceptional eigenvalues lying outside this interval.

Note that as a corollary of the proof we reprove that the continuous part of the spectrum (corresponding to Eisenstein series) stays in the part given by Ramanujan-Petersson. See also the paper of P. Sarnak ([Sa]) where a distribution formula for the exceptional values is computed.

Also note that the existence of a spectral gap bellow the eigenvalue $[\Gamma : \Gamma_\sigma]$ (corresponding to the eigenvector 1) of $\Psi(\sigma)$ is equivalent to the existence to a spectral gap in the sense of [Po2] (that is to the fact that a sequence in $L(\Gamma)$ that asymptotically commutes with $t^{\Gamma \sigma \Gamma}$, $p$ a prime number, should be an asymptotically scalar sequence).
Proof of Corollary 34. The corollary follows from the Theorem 33. Indeed for every prime $p \geq 2$ let $\sigma_p = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ and let $\alpha_p = [\Gamma \sigma_p \Gamma]$ be the corresponding double coset. By Theorem 33 it follows that the essential spectrum of $\Psi_{\sigma_p}$ is equal to the spectrum of the double coset $\alpha_p$ as a selfadjoint convolutor in the reduced $C^*$-Hecke algebra. By the Remark 18, the spectrum of $\alpha_p$ is the interval $[-2\sqrt{p}, 2\sqrt{p}]$. Consequently the essential spectrum of $\Psi_{\sigma_p}$ is the interval $[-2\sqrt{p}, 2\sqrt{p}]$. By Proposition 15, the classical Hecke operators $T_p$ are unitarily equivalent (modulo a rescaling of the Hilbert space) to the completely positive map $\Psi_{\sigma_p}$ acting on $\ell^2(\Gamma)$. Hence the essential spectrum of $T_p$ is $[-2\sqrt{p}, 2\sqrt{p}]$ and hence by Fredholm theory the discrete spectrum can only accumulate at the endpoints of the interval.

Proof of Theorem 33. By Lemma 31, for $\sigma \in G$, the operator $\Psi_{\sigma}$ belongs to the algebra $B$ which is the $C^*$-algebra generated by $\epsilon_\Gamma L_{g1} R_{g2} \epsilon_\Gamma \subseteq B(\ell^2(\Gamma))$, $g_1, g_2 \in G$. Hence by Theorem 30, the only thing that remains to be proved is that the map

$$[\Gamma \sigma \Gamma] \to \chi_\Gamma(t^{\Gamma \sigma \Gamma} \otimes t^{\Gamma \sigma \Gamma}) \chi_\Gamma \in \mathcal{A}_0 = C^*_\phi_p((G \times G^{op}) \rtimes K).$$

is an isomorphism.

Since changing the vector 1 by another cyclic trace vector in the definition of $t^{\Gamma \sigma \Gamma}$ has the effect of conjugating $t^{\Gamma \sigma \Gamma}$ by a unitary, and hence the norm is unchanged, we may use a more convenient representative vector for 1. Thus we consider the vector 1 as the polar decomposition (in $\mathcal{L}(G)$ of the evaluation vector $E = \epsilon_0$ in $H_{13}$). (Here we have again switched from upper halfplane $\mathbb{H}$ to unit disk $\mathbb{D}$, and from $\text{PSL}(2, \mathbb{R})$ to $\text{SU}(1,1)$). Note that $E$ is defined by the relation $\langle E, f \rangle = f(0)$ for all $f \in H_{13}$. By ([KL], [Pe]) the vector $E$ is cyclic and separating for the von Neumann algebra generated by the image of $\pi_{13}(\Gamma)$ in $B(H_{13})$ and hence the isometry from the polar decomposition is a unitary.

Our choice of the vector 1, amounts by Proposition 23 to the following property:

$$t^{\Gamma \sigma \Gamma} = |E|^{-1/2} \tilde{t}^{\Gamma \sigma \Gamma} |E|^{-1/2},$$

where polar decomposition $|E|$ is computed by viewing $E$ as an element of $\mathcal{L}(G)$ and $\tilde{t}^{\Gamma \sigma \Gamma}$ is given by the easier formula

$$\tilde{t}^{\Gamma \sigma \Gamma} = \sum_{g \in [\Gamma \sigma \Gamma]} \frac{1}{\pi_{13}(g) E, E} g = \sum_{g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in [\Gamma \sigma \Gamma]} \left( \frac{1}{a} \right)^{13} g.$$

Note that because of Akeman-Ostrand property for $\Gamma$, since apriori $E$ belongs to the reduced algebra of $\Gamma$, it follows that $E \otimes E$ viewed as an element of the $C^*$-algebra $\mathcal{A}_0$ has zero kernel and hence the state on $\mathcal{A}_0$ defined by the formula

$$\psi_p(x) = \phi_p(|E|^{1/2} \otimes |E|^{1/2}) x (|E|^{1/2} \otimes |E|^{1/2}), x \in \mathcal{A}_0$$
is faithful on $A_0$. To show that the map $\Phi$ restricted to $H$ is an isomorphism we compute the moments of $t^{[\Gamma \sigma \Gamma]}$ with respect to the state $\psi_p$. We have for $\sigma \in G$

$$\psi_p(t^{[\Gamma \sigma \Gamma]}) = \phi_p(t^{[\Gamma \sigma \Gamma]}).$$

and the remark after Theorem 30 it follows that if $\tilde{\sigma} \in \tilde{G}$ projects onto $\sigma

$$\Psi_p([\Gamma \sigma \Gamma]) = \frac{1}{[\Gamma : \Gamma_{\sigma}]} \sum_{g_1, g_2 \in [\Gamma \sigma \Gamma] \times [\Gamma \sigma \Gamma]} \langle \pi_{13}(g_1)E, E \rangle \langle \pi_{13}(g_2)E, E \rangle \chi(\tilde{g}_1, \tilde{g}_2) \cdot \int_{g_1^{-1} \Gamma g_2^{-1} \cap \Gamma} s(\tilde{g}_1 \tilde{g}_2) s(\tilde{\gamma}) d\tilde{\gamma}.$$

(Here we integrate with respect Haar measure on $\mathrm{SL}_2(Z_p)$.) Recall that for $\tilde{k}$ in $\tilde{K}$, $s(k) = \text{sign}_c(\tilde{k})$ where $\text{sign}_c$ is one of the sign $p$-adic functions ([GeGr]). This sum might be reduced to a quarter of itself by choosing representatives in each class $(ug, \tilde{g}), \tilde{g} \in (\tilde{G} \times \tilde{G}^{op})$, which amounts to switching to a projective representation.

In the case $g = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ then $s(g)E, E \rangle = \frac{1}{\alpha_1}$. Switching back to the coordinates on $\mathrm{SL}_2(R)$ then if $g = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ corresponds to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$\frac{1}{\alpha_1} = \frac{a + d + i(b - c)}{a^2 + b^2 + c^2 + d^2}$$

(with $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$) and the terms corresponding to $(g_1, g_2), (sg_1 s^{-1}, g_2), (g_1, sg_2 s^{-1})$ and $(sg_1 s^{-1}, sg_2 s^{-1})$ will cancel out each other the corresponding terms in the sum for $\Psi_p([\Gamma \sigma \Gamma])$.

Hence the map $\Phi$ from $H_0$ into $A_0$ extends to a trace preserving isomorphism from $H$ endowed with the trace $\tau$ into $A_0$ endowed with the state $\psi_p$, which is faithful.

This completes the proof. Note that the above summation corresponds to a point-pair invariant as in variant B, of the Selberg trace Formula from ([He1]).

APPENDIX 1

A CONSTRUCTION OF ABSTRACT HECKE OPERATORS ON II$_1$ FACTORS

In this appendix we start with a pair of isomorphic subfactors of a type II$_1$ factor. We define the analogue of a first step of the Jones’ basic construction for such a data, which is a correspondence between spaces of intertwiners and von Neumann bimodules over the initial II$_1$ factor (see also [FV] for a related approach).

We then analyse the Connes’ fusion of this bimodules and prove a multiplicativity property for the associated completely positive maps.
Definition 35. Let $M$ be a type II$_1$ factors and $N_0, N_1 \subseteq M$ two subfactors of equal index and $\theta : N_0 \to N$ an isomorphism. We denote by $I_\sigma \subseteq M$ the linear space of all $X : L^2(M, \tau) \to L^2(M, \tau)$ such that

$$Xa = \theta(a)X \text{ for all } a \in N_0.$$  

Note that if $M = \mathcal{L}(\Gamma), N_0 = \mathcal{L}(\Gamma_{\sigma^{-1}})$ and $N_1 = \mathcal{L}(\Gamma_\sigma)$ then $\theta = \sigma \cdot \sigma^{-1}$ viewed as a map on $\Gamma_{\sigma^{-1}}$ into $\Gamma_\sigma$ and extended to the group algebra.

Also if $X$ belongs $I_\sigma$ then obviously $Y^*$ belongs to $I_{\sigma^{-1}}$. $I_\sigma$ plays the role of the commutant algebra of the subfactor.

The following construction can measure the distance of $\sigma$ being implemented by an automorphism of $M$.

Definition 36. Let $X, Y$ in $I_\sigma$. Then $X \cdot Y^*$ maps $N_0'$ into $N_1'$ (that is $XaY^*$ belongs to $N_1'$ for all $a$ in $N_0'$), and hence we have the following diagram

$$
\begin{array}{ccc}
N_0' & \xrightarrow{X \cdot Y^*} & N_1' \\
\text{inc} & \swarrow & \searrow E_{M'}^{N_1'} \\
M' & & \\
\end{array}
$$

where $E_{M'}^{N_1'}$ is the canonical expectation from $N_1'$ onto $M'$ (both $N_1', M'$ are II$_1$ factors, and the commutants are computed in the algebra $\mathcal{B}(L^2(M, \tau))$. Denote $\Psi_{X, Y^*}$ the composition, which is thus a linear map from $M'$ into $M'$.

Thus the formula for $\Psi_{X, Y^*}$ is

$$\Psi_{X, Y^*}(m') = E_{M'}^{N_1'}(Xm'Y^*), \quad m' \in M'.$$

Note that if $X = Y$, then $\Psi_{X, Y}$ is a completely positive map. As explained in chapter 2 $\Psi_{X, Y}$ is a generalization of the Hecke operators.

Considering the maps $\Psi_{X, Y}$ is indeed a way to measure how far is $\sigma$ from being implemented by an automorphism.

Indeed if $\sigma$ was the restriction of an automorphism of $M$, then $\sigma$ would implement an unitary $U$ on $L^2(M, \tau)$ which in turn would have the property that $UMU^* = M'$ and hence $\Psi_{U, U^*}$ would be simply an automorphism of $M'$.

We develop the analogy with the Jones basic construction, where for $B \subseteq A$, the first algebra in the basic construction is $Ae_B A$ (as an $A \times A'^{\text{op}}$ bimodule) and it isomorphic to $B' \subseteq \mathcal{B}(L^2(A, \tau))$.

In our sitation, we want to get an abstract definition of the $\Gamma \times \Gamma'^{\text{op}}$ bimodule $\ell^2(\Gamma \star \Gamma')$, starting from $\theta_\sigma : \Gamma_{\sigma^{-1}} \to \Gamma_\sigma$ defined by $\theta_\sigma(\gamma) = \sigma\gamma\sigma^{-1}$.

Definition 37. Let $N_0, N_1 \subseteq M$ and $\theta : N_0 \to N_1$ (which should correspond respectively to $\Gamma_{\sigma^{-1}}, \Gamma_\sigma \subseteq \Gamma$ and $\theta_\sigma(\gamma) = \sigma\gamma\sigma^{-1}$ in the group case). The bimodule generalizing the commutant in the Jones's construction case is the Hilbert space closure of $M\sigma M = M_\sigma M'^{\text{op}}$ where $\sigma$ is a virtual element with the property $\sigma n_0 \sigma^{-1} = \theta(n_0)$ or $\sigma^{-1} n_1 \sigma = \theta^{-1}(n_1)$ for $n_0$ in $N_0, n_1$ in $N_1$. 


Here the element $m\sigma m'$ is the tensor product $m \otimes m'$, where $m \otimes m'$ belongs to $M \otimes M^{\text{op}}$, and the scalar product is

$$\langle m \otimes m', a \otimes a' \rangle = \tau(a^*m\theta(E_{N_0}(m'(a')^*)))$$

for all $m, a$ in $M$, $m', a'$ in $M^{\text{op}}$. Here $\otimes$ stands for the product in $M^{\text{op}}$ that is $x \otimes y = yx$.

Thus the formula for the scalar product is

$$\langle m \otimes m', a \otimes a' \rangle = \tau(a^*m\theta(E_{N_0}(m'(a')^*)).$$

Proof of the consistency of the definition. We have to prove that the definition is consistent with the formal definition of $M\sigma M$, which is equal as a vector space to $M\sigma M^{\text{op}}$.

Thus we have to verify that

$$mn_1 \sigma m' = m\sigma(a^{-1}n_1\sigma)m' = m\sigma^{-1}(n_1)m' = m\sigma(m' \otimes \theta^{-1}(n_1))$$

for all $m, m'$ in $M$, $n_1$ in $N_1$.

Thus we have to verify that $\langle mn_1 \otimes m' - m \otimes \theta^{-1}(n_1)m', a \otimes a' \rangle$ is zero for all $m, m', a, a'$ in $M$, $n_1$ in $N_1$. But

$$\langle mn_1 \otimes m', a \otimes a' \rangle = \tau(a^*mn_1\theta(E_{N_0}(m'(a')^*)) =$$

$$= \tau(a^*m\theta(\theta^{-1}(n_1))\theta(E_{N_0}(m'(a')^*))) =$$

$$= \tau(a^*m\theta(\theta^{-1}(n_1))(E_{N_0}(m'(a')^*))) = \tau(a^*m\theta(E_{N_0}(\theta^{-1}(n_1)m'(a')^*)).$$

Here we use the fact that $E_{N_0}$ is a conditional expectation and that $\theta^{-1}(n_1)$ belongs to $N_0$.

Note that the scalar product corresponds exactly to the Stinespring dilation of the completely positive map $\theta(E_{N_0}(n_0))$ viewed as a map from $M$ with values into $N_1 \subseteq M^{\text{op}}$.

Remark 38. Without going into the complicated definition of $M^{\text{op}}$, which is only needed to have positivity of the scalar product, we could simply say that $M\sigma M$ is the Hilbert space completion, of the bimodule defined by the relation

$$mn_1 \sigma m' = m\sigma^{-1}(n_1)m'$$

for all $m, m'$ in $M$, $n_1$ in $N$ and $\theta$ is implemented formally by $\sigma$.

Then the scalar product $\langle m\sigma m', a\sigma a' \rangle$ is formally trace of $(a')^*a^{-1}a^*m\sigma m'$ which, by the trace property, is equal to the trace of $a^*m\sigma(m'(a')^*)\sigma^{-1}$ and is formally equal to $\tau(a^*m\theta(E_{N_0}(m'(a')^*)$).

We define an antilinear isomorphism between the intertwiner space and the bimodule as follows.

Definition 39. For $X$ in $I_\sigma$ (that is $Xn_0 = \theta(n_0)X$) we associate to $X$ a canonical element in $M\sigma M$, where as before $\sigma$ virtually implements $\theta$ (that is $mn_0 \sigma m' = m\sigma\theta(n_0)m'$, for all $m, m'$ in $M$, $n_0$ in $N$).
Then the antilinear map \( X \to \theta(X) \in L^2(M\sigma M) \) is defined by the relation
\[
\langle m\sigma m', \theta(X) \rangle = \tau(X(m)m')
\]
for all \( m, m' \) in \( M \).

**Proof** (of the consistency of the definition). We have to check that with this definition \( X(n_0m) = \theta(n_0)X(m) \) or by taking a trace against on element \( m' \) that \( \tau(X(n_0m)m') = \tau(X(m)m\theta(n_0)) \).

By using the above definition of \( \theta(X) \) this comes to
\[
\langle m'\sigma n_0m, \theta(X) \rangle = \langle m'\theta(n_0)\sigma m, \theta(X) \rangle
\]
which is obviously true from the definition of the bimodule property of \( M\sigma M \).

Out of this we get the following corollaries in the case \( M = L(\Gamma), N_0 = L(\Gamma_{\sigma^{-1}}), N_1 = L(\Gamma_{\sigma}) \) and \( \theta = \theta_{\sigma} \) is mapping \( \Gamma_{\sigma^{-1}} \) onto \( \Gamma_{\sigma} \) by mapping \( \gamma_0 \in \Gamma_{\sigma^{-1}} \) into \( \sigma\gamma_0\gamma^{-1} \).

**Corollary 40.** Assume that \( s_i \) an orthonormal basis for \( \Gamma_{\sigma^{-1}} \) in \( \Gamma \), that is \( \Gamma = \bigcup \Gamma_{\sigma^{-1}}s_i \) (and hence \( \Gamma = \bigcup \Gamma_{\sigma}s_i \)).

Then \( X\gamma_0s_i = \theta(\gamma_0)X(s_i) \). Denote \( t_i = X(s_i) \). Then \( t_i \) is \( \Gamma_{\sigma} \) Pimsner-Popa orthonormal basis for \( \Gamma_{\sigma} \) in \( \Gamma \).

Moreover, the formula for \( \theta(X) \) is in this case
\[
\theta(X) = \sum_i t_i^*\sigma s_i.
\]

**Proof.** Note that the decomposition \( \Gamma\sigma\Gamma = \bigcup [\Gamma\sigma s_i] \) is orthogonal. Hence assume \( \theta(X) = \sum_i X_1\sigma s_i \).

The relation between \( \theta(X) \) and \( X \) is
\[
\langle m_0\sigma m_1, \theta(X) \rangle = \tau(X(m_1), m_0)
\]
and hence
\[
\langle X(m_1), m_0 \rangle = \langle m_0^*\sigma m_1, \theta(X) \rangle.
\]
Hence taking \( m_1 = s_i \) we obtain
\[
\langle t_i, m_0 \rangle = \langle X(s_i), m_0 \rangle = \langle m_0^*\sigma s_i, \theta(X) \rangle = \langle m_0^*\sigma s_i, X_1\sigma s_i \rangle.
\]
Hence we get that for all \( m_0 \) in \( M = L(\Gamma) \) we have that
\[
\langle t_i, m_0 \rangle = \langle m_0^*, x_i \rangle
\]
or that \( \tau(t_i m_0^*) = \tau(x_i^* m_0^*) \) and hence that \( t_i = x_i^* \).

Hence
\[
\theta(X) = \sum_i (X(s_i))^*\sigma s_i.
\]
Another corollary is the explicit formula for \( \theta(X) \) in the case we have \( \gamma_1X\gamma_2 = X(\gamma_1\sigma\gamma_2) \)
Corollary 41. Let \( X \) be in \( I_{\sigma} \). Denote \( \gamma_1 X \gamma_2 \) by \( X(\gamma_1 \sigma \gamma_2) \). Then
\[
\theta(X) = \sum_{\theta \in \Gamma \sigma \Gamma} \langle X(\gamma_1 \sigma \gamma_2) I, I \rangle,
\]
where \( I \) is the unit element (or more generally a trace vector).

Proof. Again we use the formula \( \langle m_0 \sigma m_1, \theta(X) \rangle = \tau(X(m_1) m_0) \) and hence
\[
\langle X(a), b \rangle = \langle b^* \sigma a, \theta(X) \rangle
\]
or
\[
\langle \theta(X), b^* \sigma a \rangle = \langle b, X(a) \rangle.
\]
Thus
\[
\langle \theta(X), \gamma_0 \sigma \gamma_1 \rangle = \langle \gamma_0^{-1}, X(\gamma_1) \rangle = \langle I, \gamma_0 \gamma_1 I \rangle = \langle \gamma_0 X \gamma_1 I, I \rangle.
\]

The isometrical property of \( \theta \) is described in the next proposition.

First, we define an \( M \)-valued pairing \( \mathcal{P} \) from \( M\sigma M \times M\sigma^{-1} M \) into \( M \) as follows

Definition 42. There is a well defined projection \( \mathcal{P} : M\sigma M \times M\sigma^{-1} M \) into \( M \), defined by the formula
\[
\mathcal{P}((m_0 \sigma m_1), (m_2 \sigma^{-1} m_3)) = m_0 \theta(E_{N_0} (m_1 m_2)) m_3.
\]
Indeed, this is the \( M \) component of the Connes fusions \( (M\sigma M) \otimes (M\sigma^{-1} M) \).

Proposition 43. Let \( N_0, N_1 \subseteq M \) and \( \theta \) an isomorphism from \( N_0 \) into \( N_1 \), virtually implemented by \( \sigma \). Fix \( m' = R_m \in M' \), for \( m \) in \( M \) be the right convolutor by \( m \).

Then for all \( X, Y \) in \( I_\sigma \) we have that
\[
E_{N_0}(X m' Y^*) = \mathcal{P}(\theta(Y) m \theta(X^*))
\]
for all \( m' = R_m \) in \( M' \).

Proof. The proof is essentially that from Theorem 22. and won’t be repeated here. Note that by linearity here we can assume simply that \( \theta(X) = s \sigma r, \theta(Y) = s_1 \sigma_1 r_1 \), for unitaries \( s, s_1, r, r_1 \) in \( M \). \( \square \)

From here on we work (for simplicity) only in the case \( G, \Gamma \) and \( \sigma_1, \sigma_2 \) partial automorphism of \( \Gamma \) reduced by elements in \( G \), but we maintain the generality of the choice \( X, Y \). We assume that we are given \( \varepsilon \) a 2-cocycle on \( G \), preserved by all \( \sigma \)'s and all algebras are group algebras with cocycle.

Definition 44. Fix \( \sigma' \) in \( [\Gamma \sigma \Gamma] \) an element. Define two orthogonal projections \( \mathcal{P}_{\sigma' \Gamma} \) and \( \mathcal{P}_{\Gamma \sigma'} \) be the projections on \( \ell^2[\sigma' \Gamma] \) and \( \ell^2[\Gamma \sigma'] \) respectively. For \( \alpha \) in \( \ell^2(\Gamma \sigma \Gamma) \) we denote \( \alpha|_{\sigma' \Gamma} \) or \( \Gamma_{\sigma'}|_{\alpha} \) the projection \( \mathcal{P}_{\sigma' \Gamma}(\alpha) \) and \( \mathcal{P}_{\Gamma \sigma'}(\alpha) \).

We now prove various formulas of the multiplication of \( \theta(X), \theta(Y) \) where \( X, Y \) are in the intertwiners set \( I_{\sigma_1}, I_{\sigma_2} \) for various \( \sigma_1, \sigma_2 \).
Note that when working with a cocycle one should modify the definition of $I_\sigma$ into the set

$$X : \ell^2(\Gamma) \to \ell^2(\Gamma) \mid X \gamma_0 = \frac{\varepsilon(\sigma \gamma_0 \sigma^{-1}, \sigma)}{\varepsilon(\sigma, \gamma_0)} \theta(\gamma_0) X$$

for all $\gamma_0 \in \sigma^{-1} \Gamma \sigma \cap \Gamma = \Gamma_{\sigma^{-1}}$.

For simplicity, we will work under the assumption that the cocycle

$$\chi(\sigma, \gamma) = \frac{\varepsilon(\sigma \gamma_0 \sigma^{-1}, \sigma)}{\varepsilon(\sigma, \gamma_0)},$$

for all $\sigma \in G$, $\gamma \in \Gamma_{\sigma^{-1}}$, which is true in the case of $\text{PSL}_2(\mathbb{Z})$, where the canonical cocycle (taking values $\pm 1$) is preserved by $\sigma$.

The multiplicativity property for $\theta$ is then as follows:

**Proposition 45.** We assume that $G$ is a discrete group containing $\Gamma$ almost normal. For $\sigma$ in $G$ denote $\Gamma_{\sigma} = \Gamma \cap \sigma \Gamma \sigma^{-1}$.

Assume $\varepsilon$ is a cocycle on $G$, assume that $\varepsilon$ is invariant by the partial isomorphisms of $\sigma$ on $\Gamma$.

Let $\sigma_1, \sigma_2$ in $\Gamma$ and $X, Y$ in $I_{\sigma_1}$, $I_{\sigma_2}$ respectively.

We consider the algebra $M = \mathcal{L}(G, \varepsilon)$, $N = \mathcal{L}(\Gamma, \varepsilon)$, and by $N_{\sigma} = \mathcal{L}(\Gamma_{\sigma}, \varepsilon|_{\Gamma_{\sigma}})$ we denote the corresponding subalgebras for $\sigma \in G$. Denote the basis of $\mathcal{L}(G, \varepsilon)$ by $u_\sigma$, and note that $u_{\sigma_1} u_{\sigma_2} = \varepsilon(g_1, g_2) u_{\sigma_1 \sigma_2}$.

We assume further (which is always possible in our context) that by changing $\varepsilon$ by a coboundary, we have that $u_{\sigma_1 \sigma_2 \gamma} = u_{\sigma_1} u_{\sigma_2} u_{\gamma}$. We have:

1. The coefficient of $\alpha \in [\Gamma_{\sigma_1} \Gamma_{\sigma_2} \Gamma]$ in $\theta(X) \theta(Y)$ is given by the formula

$$\sum_{T} \langle (r_1 X r_2)(r_3 Y r_2', r_3' I), I \rangle \varepsilon(r_1 r_2 r_3, r_2' r_3') \varepsilon(r_1 r_2 r_3, r_2' r_3'),$$

where the sum runs over all $r_1, r_2, r_2', r_3, r_3'$ in $\Gamma$ such that $(r_1 \sigma_1 r_2)(r_2 \sigma_2 r_3) = \alpha$, with no repetitions of the type $[(r_1 \sigma_1 r_2), (r_2 \sigma_2 r_3)]$ allowed.

Note that if $\pi$ is a representation of $G$ extending the left regular representation and $X = \pi(\sigma_1)$, $Y = \pi(\sigma_2)$ then the summand becomes $\langle (\pi(X) I, I) \rangle$.

2. $\mathcal{P}_{\sigma_1 \Gamma} \mathcal{P}_{\sigma_2 \Gamma} \mathcal{P}_{\Gamma_{\sigma_1} \Gamma} \mathcal{P}_{\Gamma_{\sigma_2} \Gamma} \mathcal{P}_{\Gamma_{\sigma_1} \Gamma} \mathcal{P}_{\Gamma_{\sigma_2} \Gamma}$ the summand becomes $\langle (\pi(X) I, I) \rangle$.

3. For all $s$ in $\Gamma$, $\sigma_1, \sigma_2$ in $G$, $X$ in $I_{\sigma_1}$, $Y$ in $I_{\sigma_2}$

$$\theta(X) \mathcal{P}_{\sigma_1 \Gamma} \mathcal{P}_{\sigma_2 \Gamma} \mathcal{P}_{\Gamma_{\sigma_1} \Gamma} \mathcal{P}_{\Gamma_{\sigma_2} \Gamma} \mathcal{P}_{\Gamma_{\sigma_1} \Gamma} \mathcal{P}_{\Gamma_{\sigma_2} \Gamma} \mathcal{P}_{\Gamma_{\sigma_1} \Gamma} \mathcal{P}_{\Gamma_{\sigma_2} \Gamma} \mathcal{P}_{\Gamma_{\sigma_1} \Gamma} \mathcal{P}_{\Gamma_{\sigma_2} \Gamma} \mathcal{P}_{\Gamma_{\sigma_1} \Gamma} \mathcal{P}_{\Gamma_{\sigma_2} \Gamma} \mathcal{P}_{\Gamma_{\sigma_1} \Gamma} \mathcal{P}_{\Gamma_{\sigma_2} \Gamma} \mathcal{P}_{\Gamma_{\sigma_1} \Gamma} \mathcal{P}_{\Gamma_{\sigma_2} \Gamma} \mathcal{P}_{\Gamma_{\sigma_1} \Gamma} \mathcal{P}_{\Gamma_{\sigma_2} \Gamma} \mathcal{P}_{\Gamma_{\sigma_1} \Gamma} \mathcal{P}_{\Gamma_{\sigma_2} \Gamma} \mathcal{P}_{\Gamma_{\sigma_1} \Gamma} \mathcal{P}_{\Gamma_{\sigma_2} \Gamma} \mathcal{P}_{\Gamma_{\sigma_1} \Gamma} \mathcal{P}_{\Gamma_{\sigma_2} \Gamma} \mathcal{P}_{\Gamma_{\sigma_1} \Gamma} \mathcal{P}_{\Gamma_{\sigma_2} \Gamma} \mathcal{P}_{\Gamma_{\sigma_1} \Gamma} \mathcal{P}_{\Gamma_{\sigma_2} \Gamma} \mathcal{P}_{\Gamma_{\sigma_1} \Gamma} \mathcal{P}_{\Gamma_{\sigma_2} \Gamma} \mathcal{P}_{\Gamma_{\sigma_1} \Gamma} \mathcal{P}_{\Gamma_{\sigma_2} \Gamma} \mathcal{P}_{\Gamma_{\sigma_1} \Gamma} \mathcal{P}_{\Gamma_{\sigma_2} \Gamma} \mathcal{P}_{\Gamma_{\sigma_1} \Gamma} \mathcal{P}_{\Gamma_{\sigma_2} \Gamma} \mathcal{P}_{\Gamma_{\sigma_1} \Gamma} \mathcal{P}_{\Gamma_{\sigma_2} \Gamma} \mathcal{P}_{\Gamma_{\sigma_1} \Gamma} \mathcal{P}_{\Gamma_{\sigma_2} \Gamma}$

where $s_j$ is a system of coset representatives for $\Gamma_{\sigma_1}^{-1}$ in $\Gamma$.

**Proof.** It is clear that (2), (3) are consequences of formula (1); so, we will only prove (1).

First note that the following identity

$$\varepsilon(\sigma_1, \sigma_2) = \varepsilon(\sigma_1 \gamma, \sigma_2 \gamma^{-1}) \varepsilon(\sigma_1, \gamma) \varepsilon(\gamma^{-1} \sigma_2)$$
is a consequence of the projectivity property of a representation \( \pi \) having \( \varepsilon \) as a cocycle.

Indeed, just expand in two ways
\[
\pi(\sigma_1 \sigma_2) = \pi((\sigma_1 \gamma)(\gamma^{-1} \sigma_2)).
\]
Recall that
\[
\theta(X) = \sum_{\theta = \gamma_1 \sigma \gamma_2 \in \Gamma \sigma_1 \Gamma} \langle \gamma_1 \sigma_1 \gamma_2 I, I \rangle u_{\theta},
\]
\[
\theta(Y) = \sum_{\theta = \gamma_2 \sigma_2 \gamma_3 \in \Gamma \sigma_2 \Gamma} \langle \gamma_2 \sigma_2 \gamma_3 I, I \rangle u_{\theta}.
\]

We want to compute the coefficient of \( u_\alpha \), where \( \alpha = (\gamma_1 \sigma_1 \gamma_2)(\gamma_2' \sigma_2 \gamma_3) \), is in \( \theta(X) \theta(Y) \).

We will compute the sum of all the terms corresponding to non-allowable repetitions. Denote \( \sigma = \gamma_1 \sigma_1 \gamma_2 \), \( \sigma' = \gamma_2' \sigma_2 \gamma_3 \). Since \( u_{\sigma r} u_{r^{-1} \sigma^{-1}} = \varepsilon(\sigma r, r^{-1} \sigma^{-1}) \) this sum of coefficients will be
\[
\sum_{r} \varepsilon(\sigma r, r^{-1} \sigma^{-1}) \langle \gamma_1 X \gamma_2 r I, I \rangle \langle r^{-1} (\gamma_2' Y \gamma_3) I, I \rangle =
\]
\[
= \sum_{r} \varepsilon(\sigma r, r^{-1} \sigma^{-1}) \varepsilon(\sigma r, r) \langle r^{-1} \sigma^{-1} \rangle \langle \gamma_1 X \gamma_2 r I, I \rangle \langle \gamma_2' Y \gamma_3 I, r I \rangle =
\]
\[
= \sum_{r} \varepsilon(\sigma, \sigma') \langle \gamma_1 X \gamma_2 \rangle \langle r I, (\gamma_2' Y \gamma_3) I \rangle,
\]
which since \( r \) is an orthonormal basis is equal to
\[
\varepsilon(\sigma, \sigma') \langle \gamma_1 X \gamma_2 \rangle \langle \gamma_2' Y \gamma_3 I, I \rangle = \varepsilon(\sigma, \sigma') \langle I, (\gamma_1 X \gamma_2) \rangle \langle (\gamma_2' Y \gamma_3) I, I \rangle =
\]
\[
= \varepsilon(\sigma, \sigma') \langle (\gamma_1 X \gamma_2) \rangle \langle (\gamma_2' Y \gamma_3 I, I \rangle.
\]
This completes the proof of formula (1), and the other two are simple consequences. \( \square \)

Using formula (3) we obtain a generalization of the composition formula for the completely positive maps from Chapter 5.

**Proposition 46.** Let \( \sigma_1, \sigma_2 \) be elements in \( G \), and \( A, B \) in \( I_\sigma, C, D \) in \( I_{\sigma_2} \). Let \( N_{\sigma_j} = \mathcal{L}(\Gamma_{\sigma_j}), j = 1, 2 \).

Let \( I_{\sigma_1, \sigma_2} = \{ \sigma \in [\Gamma \sigma_1 \Gamma] \subseteq [\Gamma_1 \sigma_2 \Gamma] \} \) and let \( X_{\sigma_1}, Y_{\sigma_2} \) be the \( I_{\sigma_1} \) intertwiners that are obtained by taking products of the form \( \text{As}_i B, \text{Cs}_i D \), where \( s_i \) is a system of representatives for \( \Gamma_{\sigma_2} \).

Let \( \Psi_{AB} = E^{\mathcal{L}(G, \varepsilon)} G_{\sigma_1}(\theta(A) \cdot \theta(B)), \Psi_{CD} = E^{\mathcal{L}(G, \varepsilon)} G_{\sigma_2}(\theta(C) \cdot \theta(D)) \). Then
\[
\Psi_{CD} \Psi_{AB} = \sum N_{\sigma_1, \sigma_2}^{\sigma_3} \Psi_{X_{\sigma_3}, Y_{\sigma_3}},
\]
where \( \sigma_3 \) runs over \( I_{\sigma_1, \sigma_2} \), and \( N_{\sigma_1, \sigma_2}^{\sigma_3} \) are the multiplicities.
Proof. Fix $u$ a unitary in $M'$. Note that $u\theta(X)u^*$ that we denote by $\theta_u(X)$ has the same properties as $\theta$, as it is obtained by using the cyclic vector $u$ instead of the unit vector $1$ in the matrix coefficient computations for the map $\theta$.

Then $E(\theta(A)u\theta(B)) = E(\theta(A)\theta_u(B))u$ where $E = E_{L(G,\varepsilon)}$. Let $s_i$ be a system of representatives for $\Gamma_{\sigma_i^{-1}}$ in $\Gamma$.

Applying the condition expectation we obtain that $E(\theta(A)\theta_u(B))u = \sum_i \theta(A)\theta_u(B)$.

Apply $\theta(C), \theta(D)$ to the right and left, taking $r_j$ a system of representations for $\Gamma_{\sigma_j^{-1}}$ in $\Gamma$, we get by using formula (3) in the preceding statement that formula the following expression for $C(\Psi_{AB}(u))D^*$:

$$C(\Psi_{AB}(u))D^* = \sum_{j,k,i} \theta(C)\theta_u(B)u.$$

When applying $E_{L(G,\varepsilon)}$, only the terms with $j = k$ will remain in the above formula. The conclusion follows from the fact that the cosets $[\Gamma_{\sigma_j} \sigma_1 s_i]$ when grouped into double cosets will make a list of the double cosets in the product $[\Gamma_{\sigma_1} \Gamma][\Gamma_{\sigma_2} \Gamma]$ with taken into account.

□

APPENDIX 2

A MORE PRIMITIVE STRUCTURE OF THE HECKE ALGEBRA

Behind the structure of the Hecke algebras of double cosets of an almost normal subgroup $\Gamma$ of $G$ (discrete and countable) there exists in fact a more natural pairing operation between left and right cosets, which in fact gives contains into it all the information about the multiplication structure and embedding of the Hecke algebra. We call this a primitive structure of the Hecke algebra.

We prove here that our construction relies essentially on the representation of the primitive structure of the Hecke algebra.

First, we describe this primitive structure of the Hecke algebra.

Let $H_0 = C(\Gamma \setminus G/\Gamma)$ be the algebra of double cosets, which is represented either on $\ell^2(\Gamma \setminus G)$ or $\ell^2(G/\Gamma)$.

Definition 47. The primitive structure of the Hecke algebra. (This is an operator system in the sense of Pisier ([Pi]). Let $\tilde{C}$ be the vector space of sets of the form $[\sigma_1 \Gamma \sigma_2], \sigma_1, \sigma_2 \in G$. We let $C(G, \Gamma)$ be the vector space obtained from $\tilde{C}$ by factorizing at the linear relations of the form

$$\sum_{j,k} [\theta_1^j \Gamma \sigma_2^j] = \sum_{j,k} [\theta_1^j \Gamma \sigma_2^j]$$

if $\sigma_1, \sigma_2$ are elements of $G$, and the disjoint union $\sigma_1^1 \Gamma \sigma_2^1$ is equal to the disjoint union $\theta_1^j \Gamma \theta_2^j$. 


Then there exist a natural bilinear pairing $\mathbb{C}(\Gamma \setminus G) \times \mathbb{C}(G/\Gamma) \to \mathbb{C}(G, \Gamma)$ extending the usual product of the Hecke algebra. (Note that the Hecke algebra of double cosets is contained in $\mathbb{C}(G, \Gamma)$).

We prove that our construction, beyond proving a representation of the Hecke algebra (and of its subjacent left and right Hilbert space module) in $\mathcal{L}(G)$ it also gives a representation of the more primitive structure described above. The following theorem was entirely proved in Chapter 4.

**Theorem 48.** Let $\Gamma \subseteq G$ be almost normal where $G$ is discrete countable. Assume that there exists a projective representation $\pi$ with cocycle $\varepsilon$ of $G$, which, when restricted to $\Gamma$ is unitarily equivalent with the left regular representation $\lambda_{\Gamma,\varepsilon}$ of $\Gamma$ on $\ell^2(\Gamma)$. For $\sigma$ in $G$, let $t^{\Gamma\sigma} = (t^{\sigma^{-1}\Gamma})^*$ be the $\mathcal{L}(\Gamma_\sigma)$-unitary element (that is $E_{\mathcal{L}(\Gamma_\sigma)}((t^{\Gamma\sigma})(t^{\Gamma\sigma})^*) = 1$) constructed in Chapter 4.

Moreover we proved in Chapter 4 that $(t^{\Gamma\sigma})$ where $\Gamma \sigma$ runs over a system of representatives of right cosets of $\Gamma$ in $G$, is a Pimsner-Popa basis for $L(\Gamma) \subseteq L(G)$.

Then the map $\Phi: \ell^2(\Gamma/G) \to \mathcal{L}(G, \varepsilon)$ mapping $\Gamma \sigma$ into $t^{\Gamma\sigma}$ along with its dual $\tilde{\Phi}: \ell^2(G \setminus \Gamma) \to \mathcal{L}(G, \varepsilon)$ mapping (mapping $\sigma \Gamma$ into $t^\sigma \Gamma$) extends to a representation of the primitive structure by defining $\tilde{\Phi}^2(\sigma_1 \Gamma \sigma_2) = t^{\sigma_1 \Gamma \sigma_2} = \sum_{\theta \in [\sigma_1 \Gamma \sigma_2]} \langle \pi(\theta) \iota, \iota \rangle \theta$.

In particular, $t^{\Gamma\sigma}$ is determined by the following identity:

$$\sum (t^{\Gamma\sigma_1})^* (t^{\Gamma\sigma_2}) = \sum (t^{\Gamma\sigma_1})^* (t^{\Gamma\sigma_2})$$

if the disjoint union $\bigcup \sigma_1 \Gamma \sigma_2^j$ is equal to the disjoint union $\bigcup \theta_1 \Gamma \theta_2^j$. Moreover, $t^{\Gamma\sigma}$ is $L^2(\Gamma_\sigma) \cap \mathcal{L}(G)$.

**Remark 49.** There exists a remarkable pairing involving $\Phi^2$, which is defined by the following formula:

$$\chi([\sigma_1 \Gamma \sigma_2], [\sigma_3 \Gamma \sigma_4]) = \tau(\tilde{\Phi}([\sigma_1 \Gamma \sigma_2]) \tilde{\Phi}([\sigma_1 \Gamma \sigma_2])).$$

It is easy to compute that

$$\chi([\sigma_1 \Gamma \sigma_2], [\sigma_3 \Gamma \sigma_4]) = \sum_{\theta \in [\sigma_1 \Gamma \sigma_2] \cap \sigma_3 \Gamma \sigma_4} |t(\theta)|^2.$$

Moreover, $\chi$ has special positivity properties that derive from the fact that $\tau$ is a trace ($\chi$ is a cyclic Hilbert space product in the sense of [Ra4]).

$$\chi([\sigma_1 \Gamma \sigma_2], [\sigma_3 \Gamma \sigma_4]) = \tau(t^{\sigma_1 \Gamma} (t^{\sigma_2 \Gamma})^* t^{\sigma_3 \Gamma} (t^{\sigma_4 \Gamma})^*),$$

**Proof.** The proof of the representation $\tilde{\Phi}$ is a straight consequence of the identity (proved in Chapter 4, see also the preceding Appendix for the cocycle $\varepsilon$).

$$t(\theta_1 \theta_2) = \varepsilon(\theta_1, \theta_2) \sum_{\gamma \in \Gamma} t(\theta_1 \gamma) t(\gamma^{-1} \theta_2).$$
which implies

\[(t^{\Gamma \sigma_1})^* t^{\Gamma \sigma_2} = t^{\sigma_1 \Gamma \sigma_2}.\]

The existence of the representation \(\tilde{\Phi}\) is equivalent the existence of the representation.

**Proposition 50.** Assume that there exists such a representation \(\Phi^2, \Phi, \tilde{\Phi}\) of 
\(\mathbb{C}(G, \Gamma), \mathbb{C}(\Gamma \setminus G), \mathbb{C}(G/\Gamma),\) into the algebra \(\mathcal{L}(G, \varepsilon).\) Assume that \(a^{\Gamma \sigma} = \Phi(\Gamma \sigma)\)

is a Pimsner-Popa basis for \(L(\Gamma) \subseteq L(G)\) such that in addition \(\Phi([\Gamma \sigma])\) belongs to \(\ell^2([\Gamma \sigma]).\)

Then there exists a projective unitary representation \(\pi\) of \(G\) onto \(\ell^2(\Gamma),\) extending the left regular representation of \(\Gamma\) with cocycle \(\varepsilon,\) on \(\ell^2(\Gamma).\) Moreover, \(\pi\) is projective with cocycle \(\varepsilon.\)

**Proof.** Denote the basis of \(\mathcal{L}(G, \varepsilon)\) by \(u_\theta,\) and note that \(u_{g_1} u_{g_2} = \varepsilon(g_1, g_2) u_{g_1 g_2},\) \(g_1, g_2 \in G.\)

Let \(\sigma, s_i\) a set of representatives for \(\Gamma_{\sigma^{-1}}\) in \(\Gamma.\) Then we define

\[\pi(\sigma) s_i = \varepsilon(\sigma, s_i) [a^{\Gamma \sigma s_i}(\sigma s_i)^{-1}]^* \in \mathcal{L}(\Gamma, \varepsilon).\]

Then the fact that \(\pi(\sigma)\) is a representation follows form the identity \(a(\theta_1 \theta_2) = \sum a(\theta_1 \gamma) a(\gamma^{-1} \theta_2) \varepsilon(\sigma_1, \sigma_2).\) Here \(a(\theta)\) is the \(u_\theta\) coefficient of \(a^{[\Gamma \sigma]}\). The identity is a consequence of the fact that \(a^{\sigma_1 \Gamma} a^{\Gamma \sigma_2}\) depends only on the set \(\sigma_1 \Gamma \sigma_2\) and of the fact that \(a^{\Gamma \sigma_1 \Gamma} a^{\Gamma \sigma_2 \Gamma} = \sum \sigma_3 \varepsilon^{\sigma_3} a^{\Gamma \sigma_3 \Gamma},\) where \(\varepsilon^{\sigma_3}\) are the multiplicities from the Hecke algebra structure.

We also note that the free \(\mathbb{C}-\)algebra generated by left or right cosets, subject admits a canonical \(C^*\)-representation (in fact a representation into \(\mathcal{L}(G, \varepsilon),\) in the abov terms.

**Theorem 51.** Let \(\mathcal{A}(G, \Gamma)\) the free \(*-\mathbb{C}\)-algebra generated by all the cosets 
\([\Gamma \sigma], \sigma \in G,\) and their adjoints \(([\Gamma \sigma])^* = [\sigma^{-1} \Gamma],\) subject to the relation

\[\sum [\sigma_1 \Gamma] [\Gamma \sigma_2^\prime] = \sum [\theta_1^\prime \Gamma] [\Gamma \theta_2^\prime]\]

if \(\sigma_1^\prime, \theta_1^\prime\) are elements of \(G,\) and the disjoint union \(\sigma_1^\prime \Gamma \sigma_2^\prime\) is equal to the disjoint union \(\theta_1^\prime \Gamma \theta_2^\prime.\) Note that the above relation corresponds exactly to the fact that the Hecke algebra of double cosets is a canonical subalgebra of \(\mathcal{A}(G, \Gamma),\) with the trivial embedding mapping a double coset into the formal sum of its left or right cosets (using representatives).

Then we have that the \(*-\mathbb{C}\)-algebra \(\mathcal{A}(G, \Gamma)\) admits at least one unital \(C^*\)
algebra representation into \(\mathcal{L}(G, \varepsilon).\)

**Proof.** This is a trivial consequence of the relation described above, by mapping the coset \([\Gamma \sigma], \sigma \in G\) into \(\ell^2([\Gamma \sigma]).\)
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