A weighted Sobolev-Poincaré type trace inequality on Riemannian manifolds

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Abstract

Given \((M, g)\) a smooth compact \((n + 1)\)-dimensional Riemannian manifold with boundary \(\partial M\). Let \(\rho\) be a defining function of \(M\) and \(\sigma \in (0, 1)\). In this paper we study a weighted Sobolev-Poincaré type trace inequality corresponding to the embedding of \(W^{1,2}_\rho(M) \hookrightarrow L^p(\partial M)\), where \(p = \frac{2n}{n - 2\sigma}\). More precisely, under some assumptions on the manifold, we prove that there exists a constant \(B > 0\) such that, for all \(u \in W^{1,2}_\rho(M)\),

\[
\left( \int_{\partial M} |u|^p \, ds_g \right)^{2/p} \leq \mu^{-1} \int_M \rho^{1-2\sigma} |\nabla_g u|^2 \, dv_g + B \left( \int_{\partial M} |u|^{p-2} u \, ds_g \right)^{2/(p-1)}.
\]

This inequality is sharp in the sense that \(\mu^{-1}\) cannot be replaced by any smaller constant. Moreover, unlike the classical Sobolev inequality, \(\mu^{-1}\) does not depend on \(n\) and \(\sigma\) only, but depends on the manifold.

Key words: Sobolev-Poincaré trace inequality, Sharp constant, Manifold with boundary.

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1 Introduction

Let \((M, g)\) be a \((n + 1)\)-dimensional smooth compact Riemannian manifold with boundary \(\partial M\), \(n \geq 2\). There has been much work on the sharp Sobolev-type inequalities and sharp Sobolev-type trace inequalities on \((M, g)\) and their applications, see, for example, [21], [22], [16], [17], [26], and the references therein.

In [21], Li and Zhu established the sharp Sobolev trace inequalities corresponding to the embedding of \(H^1(M) \hookrightarrow L^{\frac{2n}{n-1}}(\partial M)\), they proved that there exists a constant \(A_1 = A_1(M, g) > 0\) such that, for any \(u \in H^1(M)\),

\[
\left( \int_{\partial M} |u|^{\frac{2n}{n-1}} \, ds_g \right)^{\frac{n-1}{n}} \leq S(n) \int_M |\nabla_g u|^2 \, dv_g + A_1 \int_{\partial M} u^2 \, ds_g,
\]  

(1.1)

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where \( S(n) = \frac{2}{n-1} \omega_n^{-1/n} \) is the best constant (see [8] or [1]), \( \omega_n \) denotes the volume of the unit sphere \( S^n \) of \( \mathbb{R}^{n+1} \), \( dv_g \) is the volume form of \((M, g)\) and \( ds_g \) is the induced volume form on \( \partial M \).

Later, Holeman and Humbert [13] studied the following Sobolev-Poincaré type inequality:

\[
\left( \int_{\partial M} |u|^{\frac{2n}{n-1}} \, ds_g \right)^{\frac{n-1}{n}} \leq A \int_M |\nabla_g u|^2 \, dv_g + B \int_{\partial M} |u|^{\frac{2}{n-1}} u \, ds_g \right)^{\frac{2(n-1)}{n+1}}. \tag{1.2}
\]

Contrary to (1.1), they showed that the best constant in (1.2) does not depend on the dimension only, but depends on the geometry of the boundary.

Considerable effort has also been devoted to the study of Sobolev-Poincaré type inequalities on compact Riemannian manifolds without boundary, we refer to [6, 12, 5, 27], among others.

Let \( \mathbb{R}^{n+1}_+ = \{(x, t) \mid x \in \mathbb{R}^n, t > 0\} \), \( \sigma \in (0, 1) \) and \( W^{1,2}(t^{1-2\sigma}, \mathbb{R}^{n+1}_+) \) be the homogeneous weighted Sobolev space with weight \( t^{1-2\sigma} \), i.e., the closure of \( C^\infty_c(\mathbb{R}^{n+1}_+) \) under the norm

\[
\|U\|_{W^{1,2}(t^{1-2\sigma}, \mathbb{R}^{n+1}_+)} = \left( \int_{\mathbb{R}^{n+1}_+} t^{1-2\sigma} (|U|^2 + |\nabla U|^2) \, dx \, dt \right)^{1/2}.
\]

The sharp weighted Sobolev trace inequality asserts that (see [23, 4])

\[
\|U(\cdot, 0)\|^2_{L^{t^{2\sigma}}(\mathbb{R}^n)} \leq S(n, \sigma) \int_{\mathbb{R}^{n+1}_+} t^{1-2\sigma} |\nabla U(x, t)|^2 \, dx \, dt \tag{1.3}
\]

for all \( U \in W^{1,2}(t^{1-2\sigma}, \mathbb{R}^{n+1}_+) \), where

\[
S(n, \sigma) = \frac{1}{2\pi} \frac{\Gamma(\sigma)}{\Gamma(1-\sigma)} \frac{\Gamma((n-2\sigma)/2)}{\Gamma((n+2\sigma)/2)} \frac{\Gamma(n/2)}{\Gamma(n/2)}^{2\sigma/n}. \tag{1.4}
\]

A function \( \rho \in C^\infty(\overline{M}) \) is called a defining function of \( M \) if

\[
\rho > 0 \quad \text{in} \ M, \quad \rho = 0 \quad \text{on} \ \partial M, \quad \nabla_g \rho \neq 0 \quad \text{on} \ \partial M.
\]

The weighted Sobolev space \( W^{1,2}(\rho^{1-2\sigma}, M) \) is defined as the closure of \( C^\infty(\overline{M}) \) under the norm

\[
\|U\|_{W^{1,2}(\rho^{1-2\sigma}, M)} = \left( \int_M \rho^{1-2\sigma} |\nabla U|^2 \, dv_g + \int_{\partial M} U^2 \, ds_g \right)^{1/2}. \tag{1.5}
\]

The weighted Sobolev space \( W^{1,2}(t^{1-2\sigma}, \mathbb{R}^{n+1}) \) and \( W^{1,2}(\rho^{1-2\sigma}, M) \) play important roles in the study of the fractional Nirenberg problem and the fractional Yamabe problem, respectively, see [14, 15, 10, 11, 18, 19, 20] and references therein.

In [16], Jin and Xiong established the sharp weighted Sobolev trace inequalities of type (1.3) on Riemannian manifolds with boundaries:
Theorem A. For $n \geq 2$, let $(M, g)$ be a $(n+1)$-dimensional smooth compact Riemannian manifold with boundary $\partial M$. Let $\sigma \in (0, 1/2]$, and $\rho$ be a defining function of $M$ satisfying $|\nabla g \rho| = 1$ on $\partial M$. Then there exists a constant $A_2 = A_2(M, g, \sigma, \rho) > 0$ such that, for all $u \in W^{1,2}(\rho^{1-2\sigma}, M)$,

$$
\left( \int_{\partial M} |u|^{2n/(n-2\sigma)} \, ds_g \right)^{n-2\sigma/n} \leq S(n, \sigma) \int_M \rho^{1-2\sigma} |\nabla_g u|^2 \, dv_g + A_2 \int_{\partial M} u^2 \, ds_g.
$$

(1.6)

If $\sigma \in (1/2, 1)$, $n \geq 4$ and $\partial M$ is totally geodesic. Let $\rho$ be a defining function of $M$ satisfying $\rho(x) = d(x) + O(d(x)^3)$ as $d(x) \to 0$, where $d(x)$ denotes the distance between $x$ and $\partial M$ with respect to the metric $g$. Then (1.6) still holds for all $u \in W^{1,2}(\rho^{1-2\sigma}, M)$.

Note that if $\sigma \in (1/2, 1)$, additional assumptions about the manifold are required in order to obtain sharp inequality (1.6). However, the following conclusion holds for any $\sigma \in (0, 1)$ without additional assumptions, see [16, Proposition 2.5].

Theorem B. For $n \geq 2$, let $(M, g)$ be a $(n+1)$-dimensional smooth compact Riemannian manifold with boundary $\partial M$. Let $\sigma \in (0, 1)$, and $\rho$ be a defining function of $M$ satisfying $|\nabla g \rho| = 1$ on $\partial M$. Then for any $\varepsilon > 0$, there exists a constant $A_\varepsilon > 0$ such that, for all $u \in W^{1,2}(\rho^{1-2\sigma}, M)$,

$$
\left( \int_{\partial M} |u|^{2n/(n-2\sigma)} \, ds_g \right)^{n-2\sigma/n} \leq (S(n, \sigma) + \varepsilon) \int_M \rho^{1-2\sigma} |\nabla_g u|^2 \, dv_g + A_\varepsilon \int_{\partial M} u^2 \, ds_g.
$$

From now on we denote $p = 2n/(n-2\sigma)$ for simplicity.

In this paper, we study a weighted Sobolev-Poincaré type trace inequality obtained by replacing the $L^2$ remainder term in (1.6) by another nonlinear term.

Before stating our main results, let us first denote

$$
\mu := \inf_{u \in \Lambda} \frac{\int_M \rho^{1-2\sigma} |\nabla_g u|^2 \, dv_g}{(\int_{\partial M} |u|^p \, ds_g)^{2/p}},
$$

(1.7)

where

$$
\Lambda := \{ u \in W^{1,2}(\rho^{1-2\sigma}, M) \mid \int_{\partial M} |u|^{p-2} u \, ds_g = 0, u \neq 0 \text{ on } \partial M \}.
$$

Our main result is as follows:

Theorem 1.1. Let $(M, g)$ be a $(n+1)$-dimensional smooth compact Riemannian manifold with boundary $\partial M$, $n \geq 2$. Let $\sigma \in (0, 1)$, and $\rho$ be a defining function of $M$ with $|\nabla g \rho| = 1$ on $\partial M$. Then there exists a constant $B > 0$ such that, for all $u \in W^{1,2}(\rho^{1-2\sigma}, M)$,

$$
\left( \int_{\partial M} |u|^p \, ds_g \right)^{2/p} \leq \mu^{-1} \int_M \rho^{1-2\sigma} |\nabla_g u|^2 \, dv_g + B \left| \int_{\partial M} |u|^{p-2} u \, ds_g \right|^{2/(p-1)},
$$

(1.8)

under the following conditions satisfied:
(1) \( n = 3, \sigma \in (0, 1/2), \) and \( H(P) > 0 \) at some point \( P \in \partial M, \)
(2) or \( n \geq 4, \sigma \in (0, 1), \) and \( H(P) > 0 \) at some point \( P \in \partial M, \)
(3) or \( n \geq 5, \sigma \in (0, 1), H(P) = 0 \) and
\[
\frac{3n^2 - 6n - 4\sigma^2 + 4}{12(1 - \sigma)(n - 1)(n - 2 - 2\sigma)} \bar{R}(P) + \|\pi\|^2(P) + \frac{3n - 2 - 2\sigma}{3n - 6 - 6\sigma} R_{tt}(P) > 0
\]
at some point \( P \in \partial M, \)
where \( H \) is the mean curvature of \( \partial M, \bar{R} \) denotes the scalar curvature for the metric induced by \( g \) on \( \partial M, \pi \) is the second fundamental form of \( \partial M, \) and \( R_{tt} \) is the component of the Ricci curvature tensor in \( M. \)

In fact, under the conditions of Theorem 1.1, we can prove that \( \mu < S(n, \sigma)^{-1}, \) where \( S(n, \sigma) \) is defined in (1.4). Then, by a standard variational arguments, we have the following existence results.

**Theorem 1.2.** Under the conditions of Theorem 1.1, there exists a function \( u \in \Lambda \) such that
\[
\mu = \frac{\int_M \rho^{1-2\sigma} |\nabla_g u|^2 \, dv_g}{\left( \int_{\partial M} |u|^p \, ds_g \right)^{2/p}} \quad \text{and} \quad \int_{\partial M} |u|^p \, ds_g = 1.
\]
Moreover, \( u \) satisfies
\[
\begin{aligned}
\text{div}_g (\rho^{1-2\sigma} \nabla_g u) &= 0 \quad \text{in } M, \\
-\lim_{\rho \to 0^+} \rho^{1-2\sigma} \partial_\rho u &= \mu |u|^{p-2} u \quad \text{on } \partial M.
\end{aligned}
\]

The proof of Theorem 1.2 is standard, see for example [3, 21, 22, 16], so we omit it.

**Remark 1.1.** Recall that the best constant in the sharp weighted Sobolev trace inequality (1.6) is \( S(n, \sigma). \) In (1.8), the best constant \( \mu^{-1} > S(n, \sigma). \) On the other hand, if \( n > 6\sigma, \) the exponent of the nonlinear term in the right hand side of (1.8) is less than 2.

The rest of this paper is organized as follows. In Section 2, we prove some preliminary results related to the main inequality (1.8). In Section 3, we prove Theorem 1.1 by constructing some test functions. In Appendix, we give some estimates required in the proof of Theorem 1.1.
2 Some preliminary results

In this section, we present some preliminary results. For all \( u \in W^{1,2}(\rho^{1-2\sigma}, M) \) and \( A, B > 0 \), consider the following inequality

\[
\left( \int_{\partial M} |u|^p \, ds_g \right)^{2/p} \leq A \int_M \rho^{1-2\sigma} |\nabla_g u|^2 \, dv_g + B \left| \int_{\partial M} |u|^{p-2} u \, ds_g \right|^{2/(p-1)}.
\] (2.1)

Define

\[
A_0 = \inf \{ A > 0 \mid \exists B > 0 \text{ s.t. } (2.1) \text{ is valid} \},
\]

where by “(2.1) is valid” we mean that (2.1) holds with \( A \) and \( B \), for all \( u \in W^{1,2}(\rho^{1-2\sigma}, M) \). By definition, \( A_0 \) is the best constant for (2.1) in the sense that (2.1) does not hold with some \( A' < A_0 \) in place of \( A_0 \).

Our first result in this section is about the validity of (2.1), which is:

**Proposition 2.1.** Let \((M, g)\) be a \((n+1)\)-dimensional smooth compact Riemannian manifold with boundary \( \partial M \), \( n \geq 2 \). Let \( \sigma \in (0,1) \), and \( \rho \) be a defining function of \( M \) with \( |\nabla_g \rho| = 1 \) on \( \partial M \). Then there exists \( A, B > 0 \) such that (2.1) is valid on \( M \).

Our second aim in this section is to study the best constant \( A_0 \) associated with inequality (2.1). We are able to prove that \( A_0 \) does not depend only on \( n \) and \( \sigma \), but on the geometry of the manifold. In fact we have the following result.

**Proposition 2.2.** Assume as in Proposition 2.1. Let \( \mu \) is defined as in (1.7), namely

\[
\mu = \inf_{u \in \Lambda} \frac{\int_M \rho^{1-2\sigma} |\nabla_g u|^2 \, dv_g}{\left( \int_{\partial M} |u|^p \, ds_g \right)^{2/p}},
\]

where

\[
\Lambda = \left\{ u \in W^{1,2}(\rho^{1-2\sigma}, M) \mid \int_{\partial M} |u|^{p-2} u \, ds_g = 0, \ u \neq 0 \text{ on } \partial M \right\}.
\]

Then \( A_0 = \mu^{-1} \).

Now a natural question is that in what conditions, the infimum \( A_0 = \mu^{-1} \) can be attained, namely the inequality (1.8) holds. In fact we have the following proposition.

**Proposition 2.3.** Assume as in Proposition 2.1. If \( \mu < S(n, \sigma)^{-1} \), where \( S(n, \sigma) \) is defined in (1.4). Then there exists a constant \( B > 0 \) such that, for all \( u \in W^{1,2}(\rho^{1-2\sigma}, M) \), the inequality (1.8) holds.

**Remark 2.1.** By Proposition 2.3, to prove our main result Theorem 1.1, we only need to show that under the assumptions about the mean curvature on the boundary \( \partial M \) of the manifold \( M \) in Theorem 1.1, it holds that \( \mu < S(n, \sigma)^{-1} \) and this will be done in the following Section 3.
Now, we are going to prove Propositions 2.1, 2.2, and 2.3.

**Proof of Proposition 2.1.** We begin to prove Proposition 2.1 through an argument by contradiction. Suppose the contrary of Proposition 2.1 is true, then we have, for all $\alpha \geq 1$, there exists $u_\alpha \in W^{1,2}(\rho^{1-2\sigma}, M)$ such that

$$
\left( \int_{\partial M} |u_\alpha|^p \, ds_g \right)^{2/p} > \alpha \left( \int_M \rho^{1-2\sigma} |\nabla_g u_\alpha|^2 \, dv_g + \left| \int_{\partial M} |u_\alpha|^{p-2} u_\alpha \, ds_g \right|^{2/(p-1)} \right). \tag{2.2}
$$

By homogeneity we may assume that

$$
\int_{\partial M} |u_\alpha|^p \, ds_g = 1. \tag{2.3}
$$

Firstly, it follows from (2.3) and Hölder inequality that $\{u_\alpha\}$ is bounded in $L^2(\partial M)$. Recall the definition of the norm $\| \cdot \|_{W^{1,2}(\rho^{1-2\sigma}, M)}$ given in (1.5), $\{u_\alpha\}$ is bounded in $W^{1,2}(\rho^{1-2\sigma}, M)$. Thus, by choosing a subsequence if necessary, there exists $u \in W^{1,2}(\rho^{1-2\sigma}, M)$ such that as $\alpha \to \infty$,

$$
u_\alpha \rightharpoonup u \quad \text{weakly in } W^{1,2}(\rho^{1-2\sigma}, M), $$

$$ u_\alpha \to u \quad \text{strongly in } L^2(\partial M). \tag{2.4}
$$

By weakly lower semi-continuous of the norm, we have

$$
\int_M \rho^{1-2\sigma} |\nabla_g u|^2 \, dv_g \leq \liminf_{\alpha \to \infty} \int_M \rho^{1-2\sigma} |\nabla_g u_\alpha|^2 \, dv_g.
$$

On the other hand, by (2.2) and (2.3),

$$
\liminf_{\alpha \to \infty} \int_M \rho^{1-2\sigma} |\nabla_g u_\alpha|^2 \, dv_g = 0.
$$

Therefore, $u$ is constant. This together with (2.4) implies that

$$
\lim_{\alpha \to \infty} \|u_\alpha\|_{W^{1,2}(\rho^{1-2\sigma}, M)} = \|u\|_{W^{1,2}(\rho^{1-2\sigma}, M)}.
$$

Furthermore, $u_\alpha \to u$ strongly in $W^{1,2}(\rho^{1-2\sigma}, M)$ as $\alpha \to \infty$. By Sobolev embeddings theorem, (2.2) and (2.3), we obtain that

$$
\int_{\partial M} |u|^{p-2} u \, ds_g = 0 \quad \text{and} \quad \int_{\partial M} |u|^p \, ds_g = 1,
$$

which contradicts to the fact that $u$ is constant. This finishes the proof of Proposition 2.1.

\[\square\]
Proof of Proposition 2.2. Choosing \( u \in \Lambda \), i.e., \( u \in W^{1,2}(\rho^{1-2\sigma}, M) \) satisfies
\[
\int_{\partial M} |u|^{p-2} u \, ds_g = 0 \quad \text{and} \quad u \not\equiv 0 \quad \text{on} \quad \partial M.
\]

For all \( \varepsilon > 0 \), by the definition of best constant \( A_0 \), we have
\[
\left( \int_{\partial M} |u|^p \, ds_g \right)^{2/p} \leq (A_0 + \varepsilon) \int_M \rho^{1-2\sigma} |\nabla_g u|^2 \, dv_g.
\]

Then one finds from the definition of \( \mu \) that \( A_0 \geq \mu^{-1} \). It remains to prove that \( A_0 \leq \mu^{-1} \). We argue by contradiction and assume that \( \mu^{-1} < A_0 \). Choosing \( S \in (\mu^{-1}, A_0) \).

For every \( \alpha \geq 1 \), define the functional
\[
I_\alpha(u) := \frac{\int_M \rho^{1-2\sigma} |\nabla_g u|^2 \, dv_g + \alpha \int_{\partial M} |u|^{p-2} u \, ds_g |^{2/(p-1)}}{\left( \int_{\partial M} |u|^p \, ds_g \right)^{2/p}},
\]

where \( u \in W^{1,2}(\rho^{1-2\sigma}, M) \) and \( u|_{\partial M} \not\equiv 0 \). Define
\[
\xi_\alpha := \inf_{u \in W^{1,2}(\rho^{1-2\sigma}, M), u|_{\partial M} \not\equiv 0} I_\alpha(u). \tag{2.5}
\]

Note that the proof of Theorem 1.1 implies that \( \mu \leq S(n, \sigma)^{-1} \) (See Section 3). Since \( S < A_0 \), by the definition of best constant \( A_0 \), for any \( \alpha \geq 1 \), there exists a function \( v_\alpha \in W^{1,2}(\rho^{1-2\sigma}, M) \) satisfies
\[
\left( \int_{\partial M} |v_\alpha|^p \, ds_g \right)^{2/p} > S \int_M \rho^{1-2\sigma} |\nabla_g v_\alpha|^2 \, dv_g + \alpha S \int_{\partial M} |v_\alpha|^{p-2} v_\alpha \, ds_g |^{2/(p-1)}.
\]

This implies that \( \xi_\alpha < S^{-1} \). Since \( S > \mu^{-1} \geq S(n, \sigma) \), we have \( \xi_\alpha < S(n, \sigma)^{-1} \). By standard calculus of variations (see for example [21, Proposition 1.2], [16, Proposition 2.4]), there exists a function \( u_\alpha \in W^{1,2}(\rho^{1-2\sigma}, M) \) satisfying
\[
\xi_\alpha = \int_M \rho^{1-2\sigma} |\nabla_g u_\alpha|^2 \, dv_g + \alpha \int_{\partial M} |u_\alpha|^{p-2} u_\alpha \, ds_g |^{2/(p-1)}
\]
and
\[
\int_{\partial M} |u_\alpha|^p \, ds_g = 1. \tag{2.6}
\]

Let
\[
C_\alpha := \left| \int_{\partial M} |u_\alpha|^{p-2} u_\alpha \, ds_g \right|.
\]

We claim that \( C_\alpha \neq 0 \). Indeed, if \( C_\alpha = 0 \), then \( u_\alpha \in \Lambda \). Thus, by the definition of \( \mu \), we have
\[
\xi_\alpha = \frac{\int_M \rho^{1-2\sigma} |\nabla_g u_\alpha|^2 \, dv_g}{\left( \int_{\partial M} |u_\alpha|^p \, ds_g \right)^{2/p}} \geq \inf_{u \in \Lambda} \frac{\int_M \rho^{1-2\sigma} |\nabla_g u|^2 \, dv_g}{\left( \int_{\partial M} |u|^p \, ds_g \right)^{2/p}} = \mu,
\]
which contradicts to the fact that $\xi_\alpha < S^{-1} < \mu$. Hence, $u_\alpha$ satisfies the Euler-Lagrange equation
\[
\begin{cases}
\text{div}_g(\rho^{1-2\sigma}\nabla_g u_\alpha) = 0 & \text{in } M, \\
-\lim_{\rho \to 0^+} \rho^{1-2\sigma} \partial_\rho u_\alpha = \xi_\alpha |u_\alpha|^{p-2} u_\alpha - \alpha C_{\alpha}^{p-1} |u_\alpha|^{p-2} & \text{on } \partial M.
\end{cases}
\]

Similarly to the proof of Proposition 2.1, $\{u_\alpha\}$ is bounded in $W^{1,2}(\rho^{1-2\sigma}, M)$. Therefore, there exists $u \in W^{1,2}(\rho^{1-2\sigma}, M)$ such that as $\alpha \to \infty$,
\[
\begin{align*}
&u_\alpha \rightharpoonup u \quad \text{weakly in } W^{1,2}(\rho^{1-2\sigma}, M), \\
&u_\alpha \to u \quad \text{strongly in } L^q(\partial M),
\end{align*}
\]
where $q < p$. It follows from Brézis-Lieb lemma that $u_\alpha$ and $u$ satisfy
\[
\int_{\partial M} |u_\alpha|^{p} \, ds_g - \int_{\partial M} |u_\alpha - u|^{p} \, ds_g - \int_{\partial M} |u|^{p} \, ds_g \to 0 \quad \text{as } \alpha \to \infty,
\]
and, in view of (2.6),
\[
\int_{\partial M} |u_\alpha - u|^{p} \, ds_g \leq 1 + o(1), \quad \int_{\partial M} |u|^{p} \, ds_g \leq 1,
\]
where $o(1) \to 0$ as $\alpha \to \infty$.

Since $\xi_\alpha = I_\alpha(u_\alpha) < S^{-1}$, it follows from the definition of $I_\alpha$ that
\[
\left| \int_{\partial M} |u|^{p-2} u \, ds_g \right| = \lim_{\alpha \to \infty} \left| \int_{\partial M} |u_\alpha|^{p-2} u_\alpha \, ds_g \right| = 0. \tag{2.9}
\]

We claim that $u \neq 0$. Indeed, by Theorem B, for any $\varepsilon > 0$, there exists a constant $A_\varepsilon > 0$ such that, for any $\alpha \geq 1$,
\[
1 = \left( \int_{\partial M} |u_\alpha|^{p} \, ds_g \right)^{2/p} \leq (S(n, \sigma) + \varepsilon) \int_{M} \rho^{1-2\sigma} |\nabla_g u_\alpha|^2 \, dv_g + A_{\varepsilon} \int_{\partial M} u_\alpha^2 \, ds_g.
\]

Since $\xi_\alpha = I_\alpha(u_\alpha) < S^{-1}$, using the definition of $I_\alpha$, we have
\[
\limsup_{\alpha \to \infty} \int_{M} \rho^{1-2\sigma} |\nabla_g u_\alpha|^2 \, dv_g \leq S^{-1}.
\]
In addition, since $S > \mu^{-1} \geq S(n, \sigma)$, choosing $\varepsilon > 0$ small enough, we get
\[
0 < 1 - (S(n, \sigma) + \varepsilon) S^{-1} \leq A_{\varepsilon} \int_{\partial M} u^2 \, ds_g,
\]
which concludes the proof of the claim.
By the compact embedding of $W^{1,2}(\rho^{1-2\sigma}, M)$ to $L^2(\partial M)$, Theorem B, (2.9), (2.5), (2.8), (2.7), and (2.6), we have

$$\xi_\alpha = \int_M \rho^{1-2\sigma} \left| \nabla_g u_\alpha \right|^2 \, dv_g + \alpha \left| \int_{\partial M} \left| u_\alpha \right|^{p-2} u_\alpha \, ds_g \right|^{2/(p-1)}$$

$$\geq \int_M \rho^{1-2\sigma} \left| \nabla_g (u_\alpha - u) \right|^2 \, dv_g + \int_M \rho^{1-2\sigma} \left| \nabla_g u \right|^2 \, dv_g$$

$$= \int_M \rho^{1-2\sigma} \left| \nabla_g (u_\alpha - u) \right|^2 \, dv_g + \frac{A_\varepsilon}{S(n, \sigma) + \varepsilon} \left( \int_{\partial M} \left| u_\alpha - u \right|^2 \, ds_g + \int_M \rho^{1-2\sigma} \left| \nabla_g u \right|^2 \, dv_g + o(1) \right)$$

$$\geq \frac{1}{S(n, \sigma) + \varepsilon} \left( \int_{\partial M} \left| u_\alpha - u \right|^p \, ds_g \right)^{2/p} + \frac{1}{S(n, \sigma) + \varepsilon} \left( \int_M \rho^{1-2\sigma} \left| \nabla_g u \right|^2 \, dv_g + \alpha \left| \int_{\partial M} \left| u \right|^{p-2} u \, ds_g \right|^{2/(p-1)} + o(1) \right)$$

$$\geq \frac{1}{S(n, \sigma) + \varepsilon} \left( \int_{\partial M} \left| u_\alpha - u \right|^p \, ds_g + \xi_\alpha \left( \int_{\partial M} \left| u \right|^p \, ds_g \right)^{2/p} + o(1) \right)$$

$$\geq \frac{1}{S(n, \sigma) + \varepsilon} \left( \int_{\partial M} \left| u_\alpha - u \right|^p \, ds_g + \xi_\alpha + o(1) \right)$$

$$= \left( \frac{1}{S(n, \sigma) + \varepsilon} - \xi_\alpha \right) \int_{\partial M} \left| u_\alpha - u \right|^p \, ds_g + \xi_\alpha + o(1).$$

Since $\xi_\alpha < S^{-1} < S(n, \sigma)^{-1}$, choosing $\varepsilon > 0$ small enough, we can derive that $\left\| u_\alpha - u \right\|_{L^p(\partial M)} \to 0$ as $\alpha \to \infty$. In particular,

$$\lim_{\alpha \to \infty} \int_{\partial M} \left| u_\alpha \right|^p \, ds_g = \int_{\partial M} \left| u \right|^p \, ds_g.$$

By weakly lower semi-continuous of the norm, we have

$$\int_M \rho^{1-2\sigma} \left| \nabla_g u \right|^2 \, dv_g \leq \liminf_{\alpha \to \infty} \int_M \rho^{1-2\sigma} \left| \nabla_g u_\alpha \right|^2 \, dv_g.$$

Therefore,

$$\frac{\int_M \rho^{1-2\sigma} \left| \nabla_g u \right|^2 \, dv_g}{(\int_{\partial M} \left| u \right|^p \, ds_g)^{2/p}} \leq \liminf_{\alpha \to \infty} \frac{\int_M \rho^{1-2\sigma} \left| \nabla_g u_\alpha \right|^2 \, dv_g}{(\int_{\partial M} \left| u_\alpha \right|^p \, ds_g)^{2/p}} \leq \liminf_{\alpha \to \infty} \xi_\alpha \leq S^{-1} < \mu.$$

Together with (2.9), this gives a contradiction to the definition of $\mu$, and hence completes the proof of Proposition 2.2.

Proof of Proposition 2.3. We prove this proposition using an argument by contradiction. Suppose the contrary of Proposition 2.3 is true, then we have, for all $\alpha \geq 1$,

$$\xi_\alpha < \mu,$$
where $\xi_\alpha$ is defined in (2.5). Since $\mu < S(n, \sigma)^{-1}$, we have $\xi_\alpha < S(n, \sigma)^{-1}$. As in the proof of Proposition 2.2, there exists $u_\alpha \in W^{1,2}(\rho^{1-2\sigma}, M)$ satisfying

$$\xi_\alpha = I_\alpha(u_\alpha) \quad \text{and} \quad \int_{\partial M} |u_\alpha|^p g \, ds = 1.$$ 

It follows that $u_\alpha$ satisfies

$$\begin{cases}
\text{div}_g (\rho^{1-2\sigma} \nabla_g u_\alpha) = 0 & \text{in } M, \\
-\lim_{\rho \to 0^+} \rho^{1-2\sigma} \partial_\rho u_\alpha = \xi_\alpha |u_\alpha|^{p-2} u_\alpha - \alpha C_\alpha^{-1} |u_\alpha|^{p-2} & \text{on } \partial M,
\end{cases} \quad (2.10)$$

where

$$C_\alpha = \left| \int_{\partial M} |u_\alpha|^{p-2} u_\alpha g \, ds \right|$$

satisfying $\lim_{\alpha \to \infty} C_\alpha = 0$.

Again proceeding as in the proof of Proposition 2.2, there exists $u \in W^{1,2}(\rho^{1-2\sigma}, M)$ such that as $\alpha \to \infty$,

$$u_\alpha \rightharpoonup u \quad \text{weakly in } W^{1,2}(\rho^{1-2\sigma}, M),$$

$$u_\alpha \to u \quad \text{strongly in } L^q(\partial M),$$

where $q < p$. Using the fact that $\xi_\alpha < \mu < S(n, \sigma)^{-1}$, similar to the proof of Proposition 2.2, we conclude that $u \not\equiv 0$ on $\partial M$. Now, integrating (2.10) over $M$ yields

$$\alpha C_\alpha^{\frac{4-2p}{p-1}} \int_{\partial M} |u_\alpha|^{p-2} g \, ds = \xi_\alpha < S(n, \sigma)^{-1}.$$ 

Note that $\frac{4-2p}{p-1} = -\frac{8\sigma}{n+2\sigma} < 0$, it follows from $\lim_{\alpha \to \infty} C_\alpha = 0$ that $\alpha C_\alpha^{\frac{4-2p}{p-1}} \to \infty$ as $\alpha \to \infty$. Hence, $\int_{\partial M} |u_\alpha|^{p-2} g \, ds \to 0$ as $\alpha \to \infty$. Since $u_\alpha \to u$ strongly in $L^{p-2}(\partial M)$, we get that

$$\int_{\partial M} |u|^{p-2} g \, ds = 0,$$

which is impossible since $u \not\equiv 0$ on $\partial M$. This ends the proof of Proposition 2.3.

\[\square\]

3 Proof of Theorem 1.1

In this section, we will prove Theorem 1.1. To this end, let us recall some known facts about the extremal functions of the sharp weighted Sobolev trace inequality (1.3). By definition, an extremal function $U_0 \not\equiv 0$ is such that it realizes the case of equality in (1.3).

Given any $\varepsilon > 0$ and $x_0 \in \mathbb{R}^n = \partial \mathbb{R}^{n+1}$, define

$$w_{\varepsilon, x_0}(x) := \alpha_{n, \sigma} \left( \frac{\varepsilon}{\varepsilon^2 + |x - x_0|^2} \right)^{\frac{n-2\sigma}{2}}, \quad x \in \mathbb{R}^n, \quad (3.1)$$
where
\[ \alpha_{n,\sigma} = 2^{n+2\sigma} \frac{\Gamma((n+2\sigma)/2)}{\Gamma((n-2\sigma)/2)} \frac{n-2\sigma}{4\sigma}. \]

For all \((x, t) \in \mathbb{R}^{n+1}_+\), define
\[ W_{\varepsilon,x_0}(x, t) := p_{n,\sigma} \int_{\mathbb{R}^n} \frac{t^{2\sigma}}{(|x-y|^2 + t^2)^{n+2\sigma/2}} w_{\varepsilon,x_0}(y) \, dy, \quad (3.2) \]
where
\[ p_{n,\sigma} = \frac{\Gamma((n+2\sigma)/2)}{\pi^{n/2} \Gamma(\sigma)}. \]

We say that \(W_{\varepsilon,x_0}\) is the \(\sigma\)-harmonic extension of \(w_{\varepsilon,x_0}\). The equality of (1.3) is attained by \(U = c W_{\varepsilon,x_0}\) for any \(c \in \mathbb{R}, \varepsilon > 0\) and \(x_0 \in \mathbb{R}^n\). If \(\sigma = 1/2\), the function \(W_{\varepsilon,x_0}\) can be explicitly written as
\[ W_{\varepsilon,x_0}(x, t) = \alpha_{n,1/2} \left( \frac{\varepsilon}{(\varepsilon + t)^2 + |x-x_0|^2} \right)^{n-1}. \]

Moreover, for any \(\sigma \in (0,1)\), we have
\[ \left( \int_{\mathbb{R}^n} w^p_{\varepsilon,x_0} \, dx \right)^{\frac{n}{2p}} = S(n,\sigma)^{-1} \kappa_\sigma, \quad (3.3) \]
where
\[ \kappa_\sigma = \frac{\Gamma(\sigma)}{2^{1-2\sigma} \Gamma(1-\sigma)}. \]

For future use, we denote \(w_\varepsilon = w_{\varepsilon,0}\) and \(W_\varepsilon = W_{\varepsilon,0}\). It can be easily checked that \(W_\varepsilon(\cdot,t)\) is radially symmetric for each \(t > 0\). Moreover,
\[ w_\varepsilon(x) = \varepsilon^{-(n-2\sigma)/2} w_1(\varepsilon^{-1} x) \quad \text{and} \quad W_\varepsilon(x,t) = \varepsilon^{-(n-2\sigma)/2} W_1(\varepsilon^{-1} x, \varepsilon^{-1} t). \]

Given \(P \in \partial M\), let \(x = (x^1, \ldots, x^n)\) be normal coordinates on \(\partial M\) at \(P\) and \(t = \rho\). In other words, let \((x,t)\) be Fermi coordinates at \(P\). Let us recall the expansion of the metric \(g\) on \(M\) near the boundary \(\partial M\). Its proof can be found in Escobar [7, Lemma 3.1 and Lemma 3.2].

**Lemma 3.1.** Suppose that \(P \in \partial M\). Then for \((x,t)\) small, it holds that
\[ \sqrt{|g|} = 1 - Ht + \frac{1}{2}(H^2 - ||\pi||^2 - R_{tt})t^2 - H_i x^i t - \frac{1}{6} R_{ij} x^i x^j + O((|x,t)|^3), \quad (3.4) \]
and
\[ g^{ij} = \delta^{ij} + 2 \pi^{ij} t - \frac{1}{3} R^{ij}_{kl} x^k x^l + g^{ij}_{lm} x^m t + (3 \pi^{im} \pi^{mj} + R^{ij}_{lm}) t^2 + O((|x,t)|^3), \quad (3.5) \]
where $H$ is the mean curvature of $\partial M$, $\pi^{ij}$ are the components of the second fundamental form $\pi$ of $\partial M$ and $\|\pi\|^2 = g^{ik}g^{jl}\pi_{ij}\pi_{kl}$, $\bar{R}^{ij}_{kl}$ are the components of the induced Riemannian curvature tensor on $\partial M$, $R_{ij}$ is that of the Riemannian curvature tensor in $M$, and $H^t_{ij}$, $g_{ij}$ denote the derivative of the mean curvature and metric tensor, respectively. Every tensor in the expansions is computed at $P$.

Let $\delta > 0$ be a fixed small number, define $B_\delta$ and $B_\delta^+$ be the $n$-dimensional ball and the $(n + 1)$-dimensional upper half-ball centered at 0 whose radius is $\delta$, respectively. Let $\eta \in C_\infty^\infty(\mathbb{R}^{n+1}_+)$ be a smooth radial cut-off function such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ in $B_\delta^+$, and $\eta \equiv 0$ in $\mathbb{R}^{n+1}_+ \backslash B_{2\delta}^+$. In the following, we will abuse notations by denoting $B_\delta$ and $B_\delta$ as the geodesic ball centered at $P$ with radius $\delta$.

Now, we define a family of test functions

$$
\phi_\varepsilon = \eta W_\varepsilon - \mu_\varepsilon = \begin{cases} 
W_\varepsilon - \mu_\varepsilon & \text{in } M \cap B_\delta^+, \\
\eta W_\varepsilon - \mu_\varepsilon & \text{in } M \cap (B_{2\delta}^+ \backslash B_\delta^+), \\
-\mu_\varepsilon & \text{in } M \backslash B_{2\delta}^+,
\end{cases}
$$

(3.6)

where $\mu_\varepsilon > 0$ is chosen such that

$$
\int_{\partial M} |\phi_\varepsilon|^{p-2}\phi_\varepsilon \, ds_g = 0,
$$

(3.7)

therefore, $\phi_\varepsilon \in \Lambda$.

In the remainder of this section, we will prove that, under the assumptions of Theorem 1.1, for $\varepsilon > 0$ small enough,

$$
\frac{\int_M \rho^{1-2\sigma} |\nabla g \phi_\varepsilon|^2 \, dv_g}{(\int_{\partial M} |\phi_\varepsilon|^p \, ds_g)^{2/p}} < S(n, \sigma)^{-1},
$$

thus by Proposition 2.3, Theorem 1.1 is proved.

We first estimate $\mu_\varepsilon$. Thanks to the definition of $\phi_\varepsilon$, we have that

$$
\int_{\partial M \backslash B_{2\delta}} |\phi_\varepsilon|^{p-2}\phi_\varepsilon \, ds_g \sim -\mu_\varepsilon^{\frac{n+2\sigma}{n-2\sigma}}.
$$

(3.8)
On the other hand, we get from (3.4) that, when \( \varepsilon \to 0 \),

\[
\int_{\partial M \cap B_{2\delta}} |\phi_{\varepsilon}|^{p-2} \phi_{\varepsilon} \, ds_g \\
= \int_0^{2\delta} \int_{S(r)} - \mu_{\varepsilon} \left( \frac{n-2}{n-2\delta} \left( \frac{\varepsilon}{\varepsilon^2 + r^2} \right)^{n-2\delta} - \mu_{\varepsilon} \right) \sqrt{|g|} \, d\sigma \, dr \\
\approx \int_0^{2\delta} \int_{S(r)} - \mu_{\varepsilon} \left( \frac{n-2}{n-2\delta} \left( \frac{\varepsilon}{\varepsilon^2 + r^2} \right)^{n-2\delta} - \mu_{\varepsilon} \right) \, d\sigma \, dr \\
\approx \int_0^{2\delta/\varepsilon} \int_{0}^{n-2\delta/\varepsilon} \left( \frac{\varepsilon}{1 + s^2} \right)^{n-2\delta} - \varepsilon \mu_{\varepsilon} \left( \frac{n-2\delta}{n-2} \right) - \varepsilon \mu_{\varepsilon} \left( \frac{n-2\delta}{n-2} \right) s^{n-1} \, ds,
\]

(3.9)

where \( d\sigma \) is the volume element of \( S(r) = \{ Q \in \partial M \mid \text{dist}_g(Q, P) = r \} \). Choosing a subsequence if necessary, we can suppose that \( \varepsilon \mu_{\varepsilon} \to 0 \) as \( \varepsilon \to 0 \).

Combining (3.7), (3.8), and (3.9), we obtain

\[
\mu_{\varepsilon} \sim \frac{\varepsilon^{(n-2\delta)^2}}{(n+2\delta)^3}
\]

(3.10)

as \( \varepsilon \to 0 \).

In order to prove Theorem 1.1, we distinguish two cases.

**Case 1.** \( n = 3, \sigma \in (0, 1/2), H(P) > 0 \) or \( n \geq 4, \sigma \in (0, 1), H(P) > 0 \).

**Step 1:** Computation of \( (\int_{\partial M} |\phi_{\varepsilon}|^p \, ds_g)^{-2/p} \).

By the definition of \( \phi_{\varepsilon} \), we have that

\[
\int_{\partial M} |\phi_{\varepsilon}|^p \, ds_g \\
= \int_{\partial M \setminus B_{2\delta}} |\phi_{\varepsilon}|^p \, ds_g + \int_{\partial M \cap B_{2\delta}} |\phi_{\varepsilon}|^p \, ds_g \\
= \mu_{\varepsilon}^p \text{Vol}(\partial M \setminus B_{2\delta}) + \int_0^{2\delta} \int_{S(r)} - \mu_{\varepsilon} \left( \frac{n-2}{n-2\delta} \left( \frac{\varepsilon}{\varepsilon^2 + r^2} \right)^{n-2\delta} - \mu_{\varepsilon} \right) \sqrt{|g|} \, d\sigma \, dr.
\]

It follows from (3.4) that

\[
\sqrt{|g|}(x, 0) = 1 - \frac{R_{ij}(P)}{6} x^i x^j + O(|x|^3).
\]

Therefore,

\[
\int_{S(r)} \sqrt{|g|} \, d\sigma = \omega_{n-1} r^{n-1} \left( 1 - \frac{R(P)}{6n} r^2 + o(r^2) \right),
\]

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where \( \bar{R} \) is the scalar curvature of \( \partial M \). Hence,

\[
\int_{\partial M} |\phi_\varepsilon|^p \, ds_g
\]

\[
= \mu_\varepsilon^p \text{Vol}(\partial M \setminus B_{2\delta}) + \int_0^{2\delta} \int_{S(\varepsilon r)} \left| \eta(r) \alpha_{n, \sigma} \left( \frac{\varepsilon}{\varepsilon^2 + r^2} \right)^{n-2\sigma} - \mu_\varepsilon \right|^{\frac{2n}{n-2\sigma}} \sqrt{|g|} \, d\sigma \, dr
\]

\[
= \mu_\varepsilon^p \text{Vol}(\partial M \setminus B_{2\delta})
\]

\[
+ \omega_{n-1} \int_0^{2\delta} \left| \eta(\varepsilon r) \alpha_{n, \sigma} \left( \frac{r}{1 + \varepsilon^2 r^2} \right)^{n-2\sigma} - \mu_\varepsilon \right|^{\frac{2n}{n-2\sigma}} r^{n-1} \left( 1 - \frac{\bar{R}(P)}{6n} \varepsilon^2 r^2 + o(r^2) \right) \, dr
\]

\[
= \mu_\varepsilon^p \text{Vol}(\partial M \setminus B_{2\delta}) + \omega_{n-1} \alpha_{n, \sigma}^{2(n-2\sigma)} \int_0^{\infty} \frac{s^{n-1}}{(1 + s^2)^n} \left( 1 - \frac{\bar{R}(P)}{6n} \varepsilon^2 s^2 + \varepsilon^2 o(s^2) \right) \, ds + O(\varepsilon^n)
\]

(3.11)

By the change of variables \( s = \varepsilon^{-1} r \), (3.1) and (3.3), we have

\[
\omega_{n-1} \alpha_{n, \sigma}^{2n/(n-2\sigma)} \int_0^{\infty} \frac{s^{n-1}}{(1 + s^2)^n} \, ds = \omega_{n-1} \alpha_{n, \sigma}^{2n/(n-2\sigma)} \int_0^{\infty} \frac{s^{n-1}}{(1 + s^2)^n} \, dr
\]

\[
= \omega_{n-1} \alpha_{n, \sigma}^{2n/(n-2\sigma)} \int_{\mathbb{R}^n} \frac{\varepsilon}{\varepsilon^2 + |x|^2}^n \, dx
\]

\[
= \int_{\mathbb{R}^n} w_\varepsilon^p \, dx
\]

\[
= (S(n, \sigma)^{-1} \kappa_\sigma)^{n/2\sigma}.
\]

Using integration by parts, we have

\[
\int_0^{\infty} \frac{s^{n+1}}{(1 + s^2)^n} \, ds = \int_0^{\infty} \frac{(1 + s^2)s^{n-1} - s^{n-1}}{(1 + s^2)^n} \, ds
\]

\[
= \int_0^{\infty} \frac{s^{n-1}}{(1 + s^2)^{n-1}} \, ds - \int_0^{\infty} \frac{s^{n-1}}{(1 + s^2)^n} \, ds
\]

\[
= \frac{2(n-1)}{n} \int_0^{\infty} \frac{s^{n+1}}{(1 + s^2)^n} \, ds - \int_0^{\infty} \frac{s^{n-1}}{(1 + s^2)^n} \, ds,
\]

therefore,

\[
\int_0^{\infty} \frac{s^{n+1}}{(1 + s^2)^n} \, ds = \frac{n}{n-2} \int_0^{\infty} \frac{s^{n-1}}{(1 + s^2)^n} \, ds
\]

\[
= \frac{n}{(n-2) \omega_{n-1} \alpha_{n, \sigma}^{2n/(n-2\sigma)} (S(n, \sigma)^{-1} \kappa_\sigma)^{n/2\sigma}}.
\]

(3.13)
Similarly, we have, if
\[ \int_{\partial M} |\phi_\varepsilon|^p \, ds_g \]
\[ = \mu_0^p \, \text{Vol}(\partial M \setminus B_{2\delta}) + \omega_{n-1} \alpha_{n, \sigma}^2 \int_0^\infty \varepsilon^{n-1} (1 - \frac{\tilde{R}(P)}{6n} \varepsilon^2) \, ds + O(\varepsilon^n) \]
\[ = O(\varepsilon^{\frac{n(n-2\sigma)}{n+2\sigma}}) + (S(n, \sigma)^{-1} \kappa_\sigma)^{n/2\sigma} - \frac{\tilde{R}(P)}{6(n-2)} (S(n, \sigma)^{-1} \kappa_\sigma)^{n/2\sigma} \varepsilon^2 + o(\varepsilon^2) + O(\varepsilon^n) \]
\[ = (S(n, \sigma)^{-1} \kappa_\sigma)^{n/2\sigma} \left( 1 - \frac{\tilde{R}(P)}{6(n-2)} \varepsilon^2 + o(\varepsilon^2) + O(\varepsilon^n) + O(\varepsilon^{\frac{n(n-2\sigma)}{n+2\sigma}}) \right). \]

If \( n = 3 \), we have
\[ \int_{\partial M} |\phi_\varepsilon|^p \, ds_g = \begin{cases} (S(n, \sigma)^{-1} \kappa_\sigma)^{n/2\sigma} \left( 1 - \frac{\tilde{R}(P)}{6(n-2)} \varepsilon^2 + o(\varepsilon^2) \right) & \text{if } 0 < \sigma < 3/10, \\
(S(n, \sigma)^{-1} \kappa_\sigma)^{n/2\sigma} \left( 1 + O(\varepsilon^{\frac{n(n-2\sigma)}{n+2\sigma}}) \right) & \text{if } 3/10 \leq \sigma < 1. \end{cases} \]

Hence,
\[ \left( \int_{\partial M} |\phi_\varepsilon|^p \, ds_g \right)^{-2/p} = \begin{cases} (S(n, \sigma)^{-1} \kappa_\sigma)^{-n/2\sigma} \left( 1 + \frac{(n-2) \tilde{R}(P)}{6(n-2)} \varepsilon^2 \right) & \text{if } 0 < \sigma < 3/10, \\
(S(n, \sigma)^{-1} \kappa_\sigma)^{-n/2\sigma} \left( 1 + O(\varepsilon^{\frac{3(n-2\sigma)}{n+2\sigma}}) \right) & \text{if } 3/10 \leq \sigma < 1. \end{cases} \]

Similarly, we have, if \( n = 4 \),
\[ \left( \int_{\partial M} |\phi_\varepsilon|^p \, ds_g \right)^{-2/p} = \begin{cases} (S(n, \sigma)^{-1} \kappa_\sigma)^{-(n-2\sigma)/2\sigma} \left( 1 + \frac{(n-2) \tilde{R}(P)}{6n(n-2)} \varepsilon^2 \right) & \text{if } 0 < \sigma < 2/3, \\
(S(n, \sigma)^{-1} \kappa_\sigma)^{-(n-2\sigma)/2\sigma} \left( 1 + O(\varepsilon^{\frac{1}{2+\sigma}}) \right) & \text{if } 2/3 \leq \sigma < 1. \end{cases} \]

If \( n \geq 5 \),
\[ \left( \int_{\partial M} |\phi_\varepsilon|^p \, ds_g \right)^{-2/p} = (S(n, \sigma)^{-1} \kappa_\sigma)^{-n/2\sigma} \left( 1 + \frac{(n-2) \tilde{R}(P)}{6n(n-2)} \varepsilon^2 \right). \]

By the definition \( (3.6) \) of the function \( \phi_\varepsilon \),
\[ \int_M \rho^{1-2\sigma} |\nabla_y \phi_\varepsilon|^2 \, dv_g = \int_{B_{2\delta}^+} \rho^{1-2\sigma} |\nabla_y \phi_\varepsilon|^2 \, dv_g \]
\[ = \int_{B_{\delta}^+} t^{1-2\sigma} |\nabla_y W_\varepsilon|^2 \, dv_g + \int_{B_{2\delta}^+ \setminus B_{\delta}^+} t^{1-2\sigma} |\nabla_y \phi_\varepsilon|^2 \, dv_g \]
\[ =: I_1 + I_2. \]
For the rest of this paper, we set
\[ |\nabla U|^2 = (\partial_{x_1} U)^2 + \cdots + (\partial_{x_n} U)^2 + (\partial_t U)^2, \quad |\nabla x U|^2 = (\partial_{x_1} U)^2 + \cdots + (\partial_{x_n} U)^2. \]

**Step 2:** Computation of \( I_1 \).
Before starting the computation, let us make one useful observation. For any \( k \in \mathbb{N} \), we get from Lemma A.1 that
\[
\int_{B_\delta^+} t^{1-2\sigma} (|\nabla W_\varepsilon|^2) \, dx \, dt = \begin{cases} O(\varepsilon^k) & \text{if } n > 2\sigma + k, \\ O(\varepsilon^k \log(\delta/\varepsilon)) & \text{if } n = 2\sigma + k, \\ O(\varepsilon^k(\delta/\varepsilon)^{2\sigma+n-k}) & \text{if } n < 2\sigma + k. \end{cases}
\] (3.18)

Using (3.5), we get
\[
I_1 = \int_{B_\delta^+} t^{1-2\sigma} |\nabla g W_\varepsilon|^2 \, dv_g = \int_{B_\delta^+} t^{1-2\sigma} (g^{ij} \partial_i W_\varepsilon(x,t) \partial_j W_\varepsilon(x,t) + (\partial_t W_\varepsilon(x,t))^2) \, dv_g \\
= \int_{B_\delta^+} t^{1-2\sigma} |\nabla W_\varepsilon|^2 \, dv_g + 2\pi^{ij}(P) \int_{B_\delta^+} t^{2-2\sigma} \partial_i W_\varepsilon \partial_j W_\varepsilon \, dv_g \\
+ \int_{B_\delta^+} t^{1-2\sigma} O(|(x,t)|^2) |\nabla x W_\varepsilon|^2 \, dv_g.
\] By (3.4), we have
\[
I_1 = \int_{B_\delta^+} t^{1-2\sigma} |\nabla W_\varepsilon|^2 \, dx \, dt - H(P) \int_{B_\delta^+} t^{2-2\sigma} |\nabla W_\varepsilon|^2 \, dx \, dt \\
+ 2\pi^{ij}(P) \int_{B_\delta^+} t^{2-2\sigma} \partial_i W_\varepsilon \partial_j W_\varepsilon \, dx \, dt + \int_{B_\delta^+} t^{1-2\sigma} O(|(x,t)|^2) |\nabla W_\varepsilon|^2 \, dx \, dt.
\] (3.19)

Since \( W_\varepsilon \) is the extremal function of (1.3), it follows from (3.3) that
\[
\int_{B_\delta^+} t^{1-2\sigma} |\nabla W_\varepsilon|^2 \, dx \, dt \leq \int_{\mathbb{R}^{n+1}_+} t^{1-2\sigma} |\nabla W_\varepsilon|^2 \, dx \, dt \\
= S(n, \sigma)^{-1} \left( \int_{\mathbb{R}^n} \frac{w_\varepsilon^{2n}}{2\sigma} \, dx \right)^{\frac{n-2\sigma}{n}} \\
= \kappa_\sigma^\frac{n-2\sigma}{2\sigma} S(n, \sigma)^{-\frac{n}{2\sigma}}.
\] (3.20)

On the other hand, since \( \partial_i W_\varepsilon \) is odd in \( x_i \) and \( \pi^{ij} \delta_{ij} = H \) at the point \( P \), using
(3.18) and the positivity of $H(P)$, it holds that

$$
-H(P) \int_{B_{\delta/\varepsilon}^+} t^{2-2\sigma} |\nabla W_\varepsilon|^2 \, dx \, dt + 2\pi i (P) \int_{B_{\delta/\varepsilon}^+} t^{2-2\sigma} \partial_i W_\varepsilon \partial_j W_\varepsilon \, dx \, dt
$$

$$
= -H(P) \int_{B_{\delta/\varepsilon}^+} t^{2-2\sigma} |\nabla W_\varepsilon|^2 \, dx \, dt + \frac{2}{n} H(P) \int_{B_{\delta/\varepsilon}^+} t^{2-2\sigma} |\nabla_x W_\varepsilon|^2 \, dx \, dt
$$

$$
\leq -H(P) \int_{B_{\delta/\varepsilon}^+} t^{2-2\sigma} |\nabla W_\varepsilon|^2 \, dx \, dt + \frac{2}{n} H(P) \int_{B_{\delta/\varepsilon}^+} t^{2-2\sigma} |\nabla W_\varepsilon|^2 \, dx \, dt
$$

$$
\sim \frac{2-n}{n} H(P) \varepsilon
$$

$$
\leq -CH(P) \varepsilon,
$$

where $C > 0$. Applying (3.18) again, we get

$$
\int_{B_{\delta/\varepsilon}^+} t^{1-2\sigma} O((|x,t|)^2)|\nabla W_\varepsilon|^2 \, dx \, dt = O(\varepsilon^2),
$$

(3.22)

if $n > 2\sigma + 2$.

Inserting (3.20), (3.21) and (3.22) into (3.19), we obtain

$$
I_1 \leq \frac{n-2\sigma}{n} S(n,\sigma) - \frac{\sigma}{\delta^2} - CH(P) \varepsilon + O(\varepsilon^2),
$$

(3.23)

where $C > 0$.

**Step 3:** Computation of $I_2$.

Note that

$$
|\nabla g \phi_\varepsilon|^2 \leq C |\nabla \phi_\varepsilon|^2 \leq C (\eta^2 |\nabla W_\varepsilon|^2 + W_\varepsilon^2 |\nabla \eta|^2),
$$

so that, because of the structure of the cut-off function $\eta$,

$$
|\nabla g \phi_\varepsilon|^2 \leq C |\nabla W_\varepsilon|^2 + \frac{C}{\delta^2} W_\varepsilon^2.
$$

Moreover, using the fact that $W_\varepsilon(\varepsilon x, \varepsilon t) = \varepsilon^{-(n-2\sigma)/2} W_1(x,t)$, we have, if $n > 2\sigma + 2$,

$$
\int_{B_{2\delta/\varepsilon}^+ \setminus B_{\delta/\varepsilon}^+} t^{1-2\sigma} W_\varepsilon^2 \, dx \, dt = \varepsilon^2 \int_{B_{2\delta/\varepsilon}^+ \setminus B_{\delta/\varepsilon}^+} t^{1-2\sigma} W_1^2 \, dx \, dt = \varepsilon^2 o(1),
$$

(3.25)

because by Lemma A.1, the integral $\int_{R_{n+1}^+} t^{1-2\sigma} W_1^2 \, dx \, dt$ is finite and $\delta/\varepsilon \rightarrow \infty$ as $\varepsilon \rightarrow 0$.

On the other hand, if $n > 2\sigma + 2$, then using Lemma A.1 we have

$$
\left( \frac{\delta}{\varepsilon} \right)^2 \int_{B_{2\delta/\varepsilon}^+ \setminus B_{\delta/\varepsilon}^+} t^{1-2\sigma} |\nabla W_1|^2 \, dx \, dt \leq \int_{B_{2\delta/\varepsilon}^+ \setminus B_{\delta/\varepsilon}^+} t^{1-2\sigma} (|x,t|)^2 |\nabla W_1|^2 \, dx \, dt < \infty.
$$
Hence,

\[
\int_{B_{2\varepsilon}^+ \setminus B_{3\varepsilon}^+} t^{1-2\sigma} |\nabla W_\varepsilon|^2 \, dx \, dt = \int_{B_{2\varepsilon/\varepsilon}^+ \setminus B_{3\varepsilon/\varepsilon}^+} t^{1-2\sigma} |\nabla W_1|^2 \, dx \, dt = O(\varepsilon^2). \tag{3.26}
\]

Putting together (3.24), (3.25), and (3.26), we obtain

\[
I_2 = \int_{B_{2\varepsilon}^+ \setminus B_{3\varepsilon}^+} t^{1-2\sigma} |\nabla_g \phi_\varepsilon|^2 \, dv_g = o(\varepsilon). \tag{3.27}
\]

**Step 4:** Conclusion.

Firstly, (3.17), (3.23), and (3.27) imply that

\[
\int_M \rho^{1-2\sigma} |\nabla_g \phi_\varepsilon|^2 \, dv_g \leq \kappa_{n-2\sigma} S(n, \sigma)^{-\frac{n}{2\sigma}} - CH(P)\varepsilon + o(\varepsilon).
\]

Moreover, if \( n = 3 \), using (3.14) we have

\[
\frac{\int_M \rho^{1-2\sigma} |\nabla_g \phi_\varepsilon|^2 \, dv_g}{(\int_M |\phi_\varepsilon|^p \, ds_g)^{2/p}} = S(n, \sigma)^{-1} - CH(P)\varepsilon + o(\varepsilon) \quad \text{if} \quad 0 < \sigma < 1/2.
\]

If \( n \geq 4 \), it follows from (3.15) and (3.16) that

\[
\frac{\int_M \rho^{1-2\sigma} |\nabla_g \phi_\varepsilon|^2 \, dv_g}{(\int_M |\phi_\varepsilon|^p \, ds_g)^{2/p}} = S(n, \sigma)^{-1} - CH(P)\varepsilon + o(\varepsilon) \quad \text{if} \quad 0 < \sigma < 1.
\]

Using the assumption of the mean curvature \( H(P) \), we obtain for \( \varepsilon \) small enough,

\[
\frac{\int_M \rho^{1-2\sigma} |\nabla_g \phi_\varepsilon|^2 \, dv_g}{(\int_M |\phi_\varepsilon|^p \, ds_g)^{2/p}} < S(n, \sigma)^{-1}.
\]

Using Proposition 2.3, this ends the proof of Theorem 1.1 in this case.

In the rest of this section we deal with the second case.

**Case 2.** \( n \geq 5, \sigma \in (0, 1), H(P) = 0 \) and

\[
\frac{3n^2 - 6n - 4\sigma^2 + 4}{12(1 - \sigma)(n - 1)(n - 2 - 2\sigma)} R(P) + ||\pi||^2(P) + \frac{3n - 2 - 2\sigma}{3n - 6 - 6\sigma} R_{tt}(P) > 0.
\]

Let us still use the notation of the first case.

**Step 1:** Computation of \( I_1 \).

Notice that we may assume that \( P \) is a maximum point of \( H \). Indeed, if there exists \( Q \) such that \( H(Q) > H(P) = 0 \), we can finish the proof using Case 1. Therefore, \( H_i(P) = 0 \) for \( 1 \leq i \leq n \). It follows from (3.4) that

\[
\sqrt{|g|}(x, t) = 1 - \frac{1}{2}(||\pi||^2 + R_{tt})t^2 - \frac{1}{6} \bar{R}_{ij} x^i x^j + O(\sqrt{(x, t)}^3). \tag{3.28}
\]
Similar to Case 1, using (3.5) we have

\[
I_1 = \int_{B_5^+} t^{1-2\sigma} |\nabla g W_\varepsilon|^2 \, dv_g
\]
\[
= \int_{B_5^+} t^{1-2\sigma} (g^{ij} \partial_i W_\varepsilon(x,t) \partial_j W_\varepsilon(x,t) + (\partial_t W_\varepsilon(x,t))^2) \, dv_g
\]
\[
=: J_1 + J_2 + J_3 + J_4 + J_5 + J_6,
\]

where

\[
J_1 = \int_{B_5^+} t^{1-2\sigma} |\nabla W_\varepsilon|^2 \, dv_g,
\]
\[
J_2 = 2\pi^{ij}(P) \int_{B_5^+} t^{2-2\sigma} \partial_i W_\varepsilon \partial_j W_\varepsilon \, dv_g,
\]
\[
J_3 = -\frac{1}{3} \bar{R}^{ij}_{kl}(P) \int_{B_5^+} t^{1-2\sigma} x^k x^l \partial_i W_\varepsilon \partial_j W_\varepsilon \, dv_g,
\]
\[
J_4 = g^{ij}_{,tm}(P) \int_{B_5^+} t^{2-2\sigma} x^m \partial_i W_\varepsilon \partial_j W_\varepsilon \, dv_g,
\]
\[
J_5 = (3\pi^{im}(P) \pi_m^j(P) + R^{ij}_{,t}(P)) \int_{B_5^+} t^{3-2\sigma} \partial_i W_\varepsilon \partial_j W_\varepsilon \, dv_g,
\]
\[
J_6 = \int_{B_5^+} t^{1-2\sigma} O(\|(x,t)^3\|) |\nabla x W_\varepsilon|^2 \, dv_g.
\]

In the following, we estimate these terms separately. Firstly, by (3.28) and (3.18), we have

\[
J_1 = \int_{B_5^+} t^{1-2\sigma} |\nabla W_\varepsilon|^2 \, dv_g
\]
\[
= \int_{B_5^+} t^{1-2\sigma} |\nabla W_\varepsilon|^2 \, dx \, dt - \frac{1}{2}(\|\pi\|_t^2 + R_{tt})(P) \int_{B_5^+} t^{3-2\sigma} |\nabla W_\varepsilon|^2 \, dx \, dt
\]
\[
- \frac{1}{6} \bar{R}_{ij}(P) \int_{B_5^+} t^{1-2\sigma} x^i x^j |\nabla W_\varepsilon|^2 \, dx \, dt + \int_{B_5^+} t^{1-2\sigma} O(\|(x,t)^3\|) |\nabla W_\varepsilon|^2 \, dx \, dt
\]
\[
= \int_{B_5^+} t^{1-2\sigma} |\nabla W_\varepsilon|^2 \, dx \, dt - \frac{1}{2}(\|\pi\|_t^2 + R_{tt})(P) \int_{B_5^+} t^{3-2\sigma} |\nabla W_\varepsilon|^2 \, dx \, dt
\]
\[
- \frac{1}{6n} \bar{R}(P) \int_{B_5^+} t^{1-2\sigma} |x|^2 |\nabla W_\varepsilon|^2 \, dx \, dt + O(\varepsilon^3)
\]
\[
= \int_{B_5^+} t^{1-2\sigma} |\nabla W_\varepsilon|^2 \, dx \, dt - \frac{1}{2}(\|\pi\|_t^2 + R_{tt})(P) \varepsilon^2 \int_{B_{5/\varepsilon}} t^{3-2\sigma} |\nabla W_1|^2 \, dx \, dt
\]
\[
- \frac{1}{6n} \bar{R}(P) \varepsilon^2 \int_{B_{5/\varepsilon}} t^{1-2\sigma} |x|^2 |\nabla W_1|^2 \, dx \, dt + O(\varepsilon^3)
\]
if $n > 2\sigma + 3$.

Now we try to estimate the second term $J_2$. By (3.28) and (3.18), we have

$$J_2 = 2\pi^{ij}(P) \int_{B_3^+} t^{2-2\sigma} \partial_i W_\varepsilon \partial_j W_\varepsilon \, dv_g$$

$$\leq 2\pi^{ij}(P) \int_{B_3^+} t^{2-2\sigma} \partial_i W_\varepsilon \partial_j W_\varepsilon \, dx \, dt + C \int_{B_3^+} t^{1-2\sigma} |(x, t)|^3 |\nabla W_\varepsilon|^2 \, dx \, dt$$

$$= 2\pi^{ij}(P) \int_{B_3^+} t^{2-2\sigma} \partial_i W_\varepsilon \partial_j W_\varepsilon \, dx \, dt + O(\varepsilon^3).$$

Since $\partial_i W_\varepsilon$ is odd in $x_i$ and $\pi^{ij}(P) \delta_{ij} = H(P) = 0$, it holds

$$2\pi^{ij}(P) \int_{B_3^+} t^{2-2\sigma} \partial_i W_\varepsilon \partial_j W_\varepsilon \, dx \, dt = 0.$$

Therefore,

$$J_2 = O(\varepsilon^3).$$

Again using (3.28) and (3.18), we have that

$$J_3 = -\frac{1}{3} \bar{R}^{ij}_{kl}(P) \int_{B_3^+} t^{1-2\sigma} x^k x^l \partial_i W_\varepsilon \partial_j W_\varepsilon \, dv_g$$

$$\leq -\frac{1}{3} \bar{R}^{ij}_{kl}(P) \int_{B_3^+} t^{1-2\sigma} x^k x^l \partial_i W_\varepsilon \partial_j W_\varepsilon \, dx \, dt + C \int_{B_3^+} t^{1-2\sigma} |(x, t)|^4 |\nabla W_\varepsilon|^2 \, dx \, dt$$

$$\leq -\frac{1}{3} \bar{R}^{ij}_{kl}(P) \int_{B_3^+} t^{1-2\sigma} x^k x^l \partial_i W_\varepsilon \partial_j W_\varepsilon \, dx \, dt + C \delta \int_{B_3^+} t^{1-2\sigma} |(x, t)|^3 |\nabla W_\varepsilon|^2 \, dx \, dt$$

$$= -\frac{1}{3} \bar{R}^{ij}_{kl}(P) \int_{B_3^+} t^{1-2\sigma} x^k x^l \partial_i W_\varepsilon \partial_j W_\varepsilon \, dx \, dt + O(\varepsilon^3).$$

It follows from the symmetries of the curvature tensor that

$$-\frac{1}{3} \bar{R}^{ij}_{kl}(P) \int_{B_3^+} t^{1-2\sigma} x^k x^l \partial_i W_\varepsilon \partial_j W_\varepsilon \, dx \, dt = 0.$$

Hence,

$$J_3 = O(\varepsilon^3).$$

Next, the calculations for $J_4$, $J_5$, and $J_6$ are very similar to the previous one. Indeed,

$$J_4 = g^{ij}_{tm}(P) \int_{B_3^+} t^{2-2\sigma} x^m \partial_i W_\varepsilon \partial_j W_\varepsilon \, dv_g$$

$$\leq g^{ij}_{tm}(P) \int_{B_3^+} t^{2-2\sigma} x^m \partial_i W_\varepsilon \partial_j W_\varepsilon \, dx \, dt + C \int_{B_3^+} t^{1-2\sigma} |(x, t)|^4 |\nabla W_\varepsilon|^2 \, dx \, dt$$

$$= O(\varepsilon^3),$$

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\[ J_5 = (3 \pi^{im}(P) \pi_m^j(P) + \bar{R}^{i}{}_{i}(P)) \int_{B_{3/0}^+} t^{3-2\sigma} \partial_i W_{\varepsilon} \partial_j W_{\varepsilon} \, dv_g \]

\[ \leq (3 \pi^{im}(P) \pi_m^j(P) + \bar{R}^{i}{}_{i}(P)) \int_{B_{3/0}^+} t^{3-2\sigma} \partial_i W_{\varepsilon} \partial_j W_{\varepsilon} \, dx \, dt + C \int_{B_{3/0}^+} t^{1-2\sigma} |(x, t)|^4 |\nabla W_{\varepsilon}|^2 \, dx \, dt \]

\[ = \frac{(3 \pi^2 + R_{ut})(P)}{n} \int_{B_{3/0}^+} t^{3-2\sigma} |\nabla_x W_{\varepsilon}|^2 \, dx \, dt + O(\varepsilon^3) \]

\[ = \frac{(3 \pi^2 + R_{ut})(P)}{n} \varepsilon^2 \int_{B_{3/0}^+} t^{3-2\sigma} |\nabla_x W_{1}|^2 \, dx \, dt \]

and

\[ J_6 = \int_{B_{3/0}^+} t^{1-2\sigma} O(|(x, t)|^3) |\nabla_x W_{\varepsilon}|^2 \, dv_g = O(\varepsilon^3). \]

By the above estimates and (3.20), we obtain

\[ I_1 = J_1 + J_2 + J_3 + J_4 + J_5 + J_6 \]

\[ \leq \kappa \sigma \frac{n^2 - 2\sigma}{n^2} S(n, \sigma) \frac{1}{2} (\pi^2 + R_{ut})(P) \varepsilon^2 (2(1 - \sigma)A_0 + o(1)) \]

\[ - \frac{1}{6n} \bar{R}(P) \varepsilon^2 \left( \frac{n(n^2 - 4n(1 - \sigma) + 4(1 - \sigma - \sigma^2))}{4\sigma(n-1)} A_0 + o(1) \right) \]

\[ + \frac{(3 \pi^2 + R_{ut})(P)}{n} \varepsilon^2 \left( \frac{2(1 - \sigma)^2}{3} A_0 + o(1) \right) + O(\varepsilon^3) \]

\[ = \kappa \sigma \frac{n^2 - 2\sigma}{n^2} S(n, \sigma) \frac{1}{2} - \frac{n^2 - 4n(1 - \sigma) + 4(1 - \sigma - \sigma^2)}{24\sigma(n-1)} \bar{R}(P) A_0 \varepsilon^2 \]

\[ - \frac{(1 - \sigma)(n - 2 - 2\sigma)}{n} ||\pi^2(P)|| A_0 \varepsilon^2 \]

\[ - \frac{(1 - \sigma)(3n - 2 - 2\sigma)}{3n} R_{ut}(P) A_0 \varepsilon^2 + o(\varepsilon^2). \]

**Step 2:** Computation of $I_2$. 

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By Lemma A.1, we have

\[
\left(\frac{\delta}{\varepsilon}\right)^3 \int_{E_{2s/e}^2 \setminus E_{3/e}^2} t^{1-2\sigma}|\nabla W_1|^2 \, dx \, dt \leq \int_{E_{2s/e}^2 \setminus E_{3/e}^2} t^{1-2\sigma}|(x,t)|^3 \nabla W_1|^2 \, dx \, dt < \infty
\]

if \( n > 2\sigma + 3 \). Therefore,

\[
\int_{E_{2s/e}^2 \setminus E_{3/e}^2} t^{1-2\sigma}|\nabla W_1|^2 \, dx \, dt = \int_{E_{2s/e}^2 \setminus E_{3/e}^2} t^{1-2\sigma}|\nabla W_1|^2 \, dx \, dt = O(\varepsilon^3). \tag{3.30}
\]

In view of (3.24), (3.25), and (3.30), we obtain

\[
I_2 = \int_{E_{2s/e}^2 \setminus E_{3/e}^2} t^{1-2\sigma}|\nabla g_\varepsilon|^2 \, dv = o(\varepsilon^2).
\]

**Step 3:** Conclusion.

Using (3.20) and Lemma A.2, we have

\[
\kappa_{\sigma, 2\sigma} S(n, \sigma)^{-\frac{n-2\sigma}{2\sigma}} = \int_{R_{n+1}^+} t^{1-2\sigma}|\nabla W_1|^2 \, dx \, dt = \frac{(n-2)(n-2+2\sigma)(n-2-2\sigma)}{4\sigma(n-1)} A_0,
\]

i.e.,

\[
A_0 = \left(\frac{4\sigma(n-1)}{(n-2)(n-2+2\sigma)(n-2-2\sigma)} \right)^{\frac{n-2\sigma}{2\sigma}} \kappa_{\sigma, 2\sigma} S(n, \sigma)^{-\frac{n-2\sigma}{2\sigma}}. \tag{3.31}
\]

Inserting (3.31) into (3.29), we obtain

\[
I_1 \leq \kappa_{\sigma, 2\sigma} S(n, \sigma)^{-\frac{n-2\sigma}{2\sigma}} - \frac{n^2 - 4n(1-\sigma) + 4(1-\sigma - \sigma^2)^2}{24\sigma(n-1)} \bar{R}(P) A_0 \varepsilon^2
\]

\[
- \frac{(1-\sigma)(n-2-2\sigma)}{n} \|\pi\|^2(P) A_0 \varepsilon^2
\]

\[
- \frac{(1-\sigma)(3n-2-2\sigma)}{3n} R_{tt}(P) A_0 \varepsilon^2 + o(\varepsilon^2)
\]

\[
= \kappa_{\sigma, 2\sigma} S(n, \sigma)^{-\frac{n-2\sigma}{2\sigma}} \left(1 - \frac{n^2 - 4n(1-\sigma) + 4(1-\sigma - \sigma^2)^2}{6(n-2)(n-2+2\sigma)(n-2-2\sigma)} \bar{R}(P) \varepsilon^2
\]

\[
- \frac{4\sigma(1-\sigma)(n-1)}{n(n-2)(n-2+2\sigma)} \|\pi\|^2(P) \varepsilon^2
\]

\[
- \frac{4\sigma(1-\sigma)(n-1)(3n-2-2\sigma)}{3n(n-2)(n-2+2\sigma)(n-2-2\sigma)} R_{tt}(P) \varepsilon^2 + o(\varepsilon^2) \right).
\]

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Therefore,
\[
\int_M \rho^{1-2\sigma} |\nabla_g \phi_\varepsilon|^2 \, dv_g \\
\leq \kappa_{\sigma}^{-\frac{n-2\sigma}{n-2}} S(n,\sigma)^{-\frac{n}{2n}} \left( 1 - \frac{n^2 - 4n(1-\sigma) + 4(1 - \sigma - \sigma^2)}{6(n-2)(n-2 + 2\sigma)(n-2 - 2\sigma)} \bar{R}(P) \varepsilon^2 \right)
\]

- \frac{4\sigma(1-\sigma)(n-1)}{n(n-2)(n-2 + 2\sigma)} \|\pi\|^2(\varepsilon) \\
- \frac{4\sigma(1-\sigma)(n-1)(3n-2 - 2\sigma)}{3n(n-2)(n-2 + 2\sigma)(n-2 - 2\sigma)} R_{tt}(P) \varepsilon^2 + o(\varepsilon^2) \bigg)
= \kappa_{\sigma}^{-\frac{n-2\sigma}{n-2}} S(n,\sigma)^{-\frac{n}{2n}} (1 + K_1 \bar{R}(P) \varepsilon^2 + K_2 \|\pi\|^2(\varepsilon) + K_3 R_{tt}(P) \varepsilon^2 + o(\varepsilon^2)),
\]

where
\[
K_1 = - \frac{n^2 - 4n(1-\sigma) + 4(1 - \sigma - \sigma^2)}{6(n-2)(n-2 + 2\sigma)(n-2 - 2\sigma)},
K_2 = \frac{4\sigma(1-\sigma)(n-1)}{n(n-2)(n-2 + 2\sigma)},
K_3 = \frac{4\sigma(1-\sigma)(n-1)(3n-2 - 2\sigma)}{3n(n-2)(n-2 + 2\sigma)(n-2 - 2\sigma)}.
\]

Note that (3.16) is always true, hence
\[
\int_M \rho^{1-2\sigma} |\nabla_g \phi_\varepsilon|^2 \, dv_g \\
= S(n,\sigma)^{-1}(1 + \bar{K}_1 \bar{R}(P) \varepsilon^2 + K_2 \|\pi\|^2(\varepsilon) + K_3 R_{tt}(P) \varepsilon^2 + o(\varepsilon^2)),
\]

where
\[
\bar{K}_1 = K_1 + \frac{n - 2\sigma}{6n(n-2)} = - \frac{\sigma(3n^2 - 6n - 4\sigma^2 + 4)}{3n(n-2)(n-2 + 2\sigma)(n-2 - 2\sigma)}.
\]

By direct calculations,
\[
\bar{K}_1 \bar{R}(P) + K_2 \|\pi\|^2(\varepsilon) + K_3 R_{tt}(P) < 0
\]
is equivalent to
\[
\frac{3n^2 - 6n - 4\sigma^2 + 4}{12(1-\sigma)(n-1)(n-2 - 2\sigma)} \bar{R}(P) + \|\pi\|^2(\varepsilon) + \frac{3n - 2 - 2\sigma}{3n - 6 - 6\sigma} R_{tt}(P) > 0.
\]

Under the hypothesis of this case, we obtain
\[
\int_M \rho^{1-2\sigma} |\nabla_g \phi_\varepsilon|^2 \, dv_g \\
= S(n,\sigma)^{-1}
\]
for \(\varepsilon\) small enough. By Proposition 2.3, this finishes the proof of Theorem 1.1.
Appendix

In this appendix, we will provide some lemmas used in the previous sections.

Lemma A.1. Let $\sigma \in (0, 1)$, $n > 2\sigma$, and $W_\varepsilon = W_{\varepsilon,0}$ is defined in (3.2). Then there holds

1. $W_\varepsilon(x, t) = O(\varepsilon^{(n-2\sigma)/2}(\varepsilon^2 + |(x, t)|^2)^{-n/2})$,
2. $\nabla_x W_\varepsilon(x, t) = O(\varepsilon^{(n-2\sigma)/2}(\varepsilon^2 + |(x, t)|^2)^{-n/2})$,
3. $\partial_t W_\varepsilon(x, t) = O(\varepsilon^{(n-2\sigma)/2}t^{2\sigma-1}(\varepsilon^2 + |(x, t)|^2)^{-n/2})$.

Proof. The proof can be found in [24, Corollary 3.2], see also [25, Lemma A.1].

Lemma A.2. Let $\sigma \in (0, 1)$, $n > 2\sigma + 2$, and $W_1 = W_{1,0}$ is defined in (3.2). Then there holds

\[
\int_{\mathbb{R}_+^{n+1}} t^{1-2\sigma} |x|^2 |\nabla W_1|^2 \, dx \, dt = \frac{n(n^2 - 4n(1-\sigma) + 4(1-\sigma - \sigma^2))}{4\sigma(n-1)} A_0, \quad (A.1)
\]

\[
\int_{\mathbb{R}_+^{n+1}} t^{1-2\sigma} |\nabla W_1|^2 \, dx \, dt = \frac{(n-2)(n-2+2\sigma)(n-2-2\sigma)}{4\sigma(n-1)} A_0,
\]

\[
\int_{\mathbb{R}_+^{n+1}} t^{3-2\sigma} |\nabla W_1|^2 \, dx \, dt = \frac{2(1-\sigma^2)}{3} A_0,
\]

\[
\int_{\mathbb{R}_+^{n+1}} t^{3-2\sigma} |\nabla W_1|^2 \, dx \, dt = 2(1-\sigma) A_0,
\]

where

\[
A_0 = \int_{\mathbb{R}_+^{n+1}} t^{1-2\sigma} W_1^2 \, dx \, dt < \infty.
\]

Proof. We only prove (A.1), the others are similar. The idea of the proof is using the Plancherel theorem and an explicit formula of the Fourier transform of $W_1(x, t)$ in $x$, which is motivated from [10], as well as [2, 11, 18, 19].

Firstly, by the Plancherel theorem, we obtain

\[
\int_{\mathbb{R}^n} |x|^2 |\nabla_x W_1(x, t)|^2 \, dx = \sum_{i=1}^n ||| \partial_i W_1(\cdot, t) |||_{L^2(\mathbb{R}^n)}^2
\]

\[
= \sum_{i=1}^n \int_{\mathbb{R}^n} \xi_i \hat{W}_1(\xi, t) \cdot (-\Delta) \xi_i (\xi, \hat{W}_1(\xi, t)) \, d\xi,
\]

where $\hat{W}_1(\xi, t)$ is the Fourier transform of $W_1(x, t)$ with respect to the variable $x \in \mathbb{R}^n$. It follows from [9] and [11] that

\[
\hat{W}_1(\xi, t) = \hat{w}_1(\xi) \varphi(|\xi| t) \quad \text{for all } \xi \in \mathbb{R}^n \text{ and } t > 0,
\]

where $\hat{w}_1(\xi)$ is supported by the ball $|\xi| < 1$ and $\varphi(|\xi| t)$ is a cut-off function.
where \( \hat{w}_1(r) \), \( r = |\xi| \) satisfies
\[
\hat{w}_1''(r) + \frac{1 + 2\sigma}{r} \hat{w}_1'(r) - \hat{w}_1(r) = 0, \tag{A.2}
\]
and \( \varphi(s) \) satisfies
\[
\varphi''(s) + \frac{1 - 2\sigma}{s} \varphi'(s) - \varphi(s) = 0. \tag{A.3}
\]
By some tedious calculations and the relation
\[
(-\Delta_\xi)(\xi_0 \hat{W}_1) = -2\partial_\xi \hat{W}_1 - \xi_0 \Delta_\xi \hat{W}_1
\]
and
\[
\Delta_\xi \hat{W}_1 = \hat{W}_1'' + (n - 1)r^{-1}\hat{W}_1,
\]
where ' represents the differentiation with respect to the radial variable \(|\xi|\), it holds that
\[
\int_{\mathbb{R}^{n+1}} t^{1-2\sigma} |x|^2 |\nabla_x W_1|^2 \, dx \, dt = - (n - 2\sigma) \omega_{n-1} \int_0^\infty \int_0^\infty t^{1-2\sigma} \hat{w}_1(r) \varphi^2(rt) \hat{w}_1'(r) r^n \, dr \, dt
\]
\[
- (n + 2\sigma) \omega_{n-1} \int_0^\infty \int_0^\infty t^{2-2\sigma} \hat{w}_1(r) \varphi(rt) \hat{w}_1'(r) r^{n+1} \, dr \, dt
\]
\[
- \omega_{n-1} \int_0^\infty \int_0^\infty t^{1-2\sigma} (1 + t^2) \hat{w}_1^2(r) \varphi^2(rt) r^{n+1} \, dr \, dt
\]
\[
- 2\omega_{n-1} \int_0^\infty \int_0^\infty t^{2-2\sigma} \hat{w}_1(r) \varphi(rt) \hat{w}_1'(r) \varphi'(r) r^{n+1} \, dr \, dt
\]
\[
= - (n - 2\sigma) \omega_{n-1} \left( \int_0^\infty r^{n-2+2\sigma} \hat{w}_1(r) \hat{w}_1'(r) \, dr \right) \left( \int_0^\infty s^{1-2\sigma} \varphi^2(s) \, ds \right)
\]
\[
- (n + 2\sigma) \omega_{n-1} \left( \int_0^\infty r^{n-3+2\sigma} \hat{w}_1^2(r) \, dr \right) \left( \int_0^\infty s^{2-2\sigma} \varphi(s) \varphi'(s) \, ds \right)
\]
\[
- \omega_{n-1} \left( \int_0^\infty r^{n-1+2\sigma} \hat{w}_1^2(r) \, dr \right) \left( \int_0^\infty s^{1-2\sigma} \varphi^2(s) \, ds \right)
\]
\[
- \omega_{n-1} \left( \int_0^\infty r^{n-3+2\sigma} \hat{w}_1^2(r) \, dr \right) \left( \int_0^\infty s^{3-2\sigma} \varphi^2(s) \, ds \right)
\]
\[
- 2\omega_{n-1} \left( \int_0^\infty r^{n-2+2\sigma} \hat{w}_1(r) \hat{w}_1'(r) \, dr \right) \left( \int_0^\infty s^{2-2\sigma} \varphi(s) \varphi'(s) \, ds \right).
\]

Using [19, Lemma B.2], we have
\[
\int_{\mathbb{R}^{n+1}} t^{1-2\sigma} |x|^2 |\nabla_x W_1|^2 \, dx \, dt
\]
\[
= \frac{(n + 2)(3n^2 - 6n - 4\sigma^2 + 4)}{8(n - 1)(1 - \sigma^2)} \omega_{n-1} \left( \int_0^\infty r^{n-3+2\sigma} \hat{w}_1^2(r) \, dr \right) \left( \int_0^\infty s^{3-2\sigma} \varphi^2(s) \, ds \right). \tag{A.4}
\]
Next, we will calculate \( \int_{\mathbb{R}^n} |x|^2 |\partial_t W_1(x, t)|^2 \, dx \). By the Plancherel theorem,

\[
\int_{\mathbb{R}^n} |x|^2 |\partial_t W_1(x, t)|^2 \, dx = \| \cdot |\partial_t W_1(\cdot, t)\|_{L^2(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} |\xi| |\hat{\partial_t W_1}(\xi)\varphi'(|\xi| t) \cdot (-\Delta)\xi(|\xi| \hat{\partial_t W_1}(\xi)\varphi'(|\xi| t)) \, d\xi.
\]

Then employing (A.2) and (A.3), we obtain

\[
\Delta_\xi(|\xi| \hat{\partial_t W_1}(\xi)\varphi'(|\xi| t)) = (r \hat{\partial_t W_1}(r))'' + \frac{n - 1}{r} (r \hat{\partial_t W_1}(r))'
\]

\[
= (n - 2 + 2\sigma) \hat{\partial_t W_1}(r) \varphi'(rt) + 2rt \hat{\partial_t W_1}(r) \varphi(rt)
\]

\[
+ \left( \frac{2\sigma(n - 2 + 2\sigma)}{r} + r(1 + t^2) \right) \hat{\partial_t W_1}(r) \varphi'(rt) + (n + 2\sigma)t \hat{\partial_t W_1}(r) \varphi(rt).
\]

Therefore, using [19, Lemma B.2],

\[
\int_{\mathbb{R}^n_+^{n+1}} t^{1-2\sigma} |x|^2 |\partial_t W_1|^2 \, dx \, dt
\]

\[
= -(n - 2 + 2\sigma) \omega_{n-1} \left( \int_0^\infty r^{n-2+2\sigma} \hat{\partial_t W_1}(r) \hat{\partial_t W_1}(r) \, dr \right) \left( \int_0^\infty s^{1-2\sigma} \varphi'^2(s) \, ds \right)
\]

\[
- 2\omega_{n-1} \left( \int_0^\infty r^{n-2+2\sigma} \hat{\partial_t W_1}(r) \hat{\partial_t W_1}(r) \, dr \right) \left( \int_0^\infty s^{2-2\sigma} \varphi(s) \varphi'(s) \, ds \right)
\]

\[
- 2\sigma(n - 2 + 2\sigma) \omega_{n-1} \left( \int_0^\infty r^{n-3+2\sigma} \hat{\partial_t W_1}(r) \, dr \right) \left( \int_0^\infty s^{1-2\sigma} \varphi'^2(s) \, ds \right)
\]

\[
- \omega_{n-1} \left( \int_0^\infty r^{n-1+2\sigma} \hat{\partial_t W_1}(r) \, dr \right) \left( \int_0^\infty s^{1-2\sigma} \varphi'^2(s) \, ds \right)
\]

\[
- \omega_{n-1} \left( \int_0^\infty r^{n-3+2\sigma} \hat{\partial_t W_1}(r) \, dr \right) \left( \int_0^\infty s^{3-2\sigma} \varphi'^2(s) \, ds \right)
\]

\[
- (n + 2\sigma) \omega_{n-1} \left( \int_0^\infty r^{n-3+2\sigma} \hat{\partial_t W_1}(r) \, dr \right) \left( \int_0^\infty s^{2-2\sigma} \varphi(s) \varphi'(s) \, ds \right)
\]

\[
= \frac{3n^3 - 12n^2 - 4n\sigma^2 + 8n\sigma + 12n - 8\sigma^2 - 8\sigma}{8\sigma(1 + \sigma)(n - 1)} \omega_{n-1} \left( \int_0^\infty r^{n-3+2\sigma} \hat{\partial_t W_1}(r) \, dr \right) \times \left( \int_0^\infty s^{3-2\sigma} \varphi'^2(s) \, ds \right).
\]

Similarly,

\[
A_0 = \int_{\mathbb{R}^n_+^{n+1}} t^{1-2\sigma} W_1^2 \, dx \, dt
\]

\[
= \frac{3}{2(1 - \sigma^2)} \omega_{n-1} \left( \int_0^\infty r^{n-3+2\sigma} \hat{\partial_t W_1}(r) \, dr \right) \left( \int_0^\infty s^{3-2\sigma} \varphi'^2(s) \, ds \right).
\]
Combining (A.4), (A.5) and (A.6) together, we see that,
\[ \int_{\mathbb{R}^{n+1}} t^{1-2\sigma}|x|^2|\nabla W_1|^2 \, dx \, dt = \frac{n(n^2 - 4n(1 - \sigma) + 4(1 - \sigma - \sigma^2))}{4\sigma(n-1)} A_0. \]
This finishes the proof. \qed

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