Antisymmetric paramodular forms of weight 3

V.A. Gritsenko and H. Wang

Abstract. The problem of the construction of antisymmetric paramodular forms of canonical weight 3 has been open since 1996. Any cusp form of this type determines a canonical differential form on any smooth compactification of the moduli space of Kummer surfaces associated to $(1, t)$-polarised abelian surfaces. In this paper, we construct the first infinite family of antisymmetric paramodular forms of weight 3 as automorphic Borcherds products whose first Fourier-Jacobi coefficient is a theta block.

Bibliography: 32 titles.

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§ 1. Introduction

Let $t$ be a positive integer. The paramodular group of level (or polarisation) $t$ is the integral symplectic group of the skew-symmetric form with elementary divisors $(1, t)$. This group is conjugate to a subgroup $\Gamma_t$ of the rational symplectic group $Sp_2(\mathbb{Q})$ (see §2). The Siegel modular threefold $\mathcal{A}_t = \Gamma_t \setminus \mathbb{H}_2$, where $\mathbb{H}_2$ is the Siegel upper half-space of genus 2, is isomorphic to the moduli space of abelian surfaces with polarisation of type $(1, t)$. This moduli space is not compact. If $F$ is a cusp form of weight 3 with respect to $\Gamma_t$, then $\omega_F = F(Z) dZ$ is a holomorphic 3-form on $\mathcal{A}_t$. According to Freitag’s criterion (see §6), $\omega_F$ can be extended to any smooth compactification of $\mathcal{A}_t$ of the moduli space. Therefore,

$$h^{3,0}(\mathcal{A}_t) = \text{dim}_\mathbb{C} S_3(\Gamma_t),$$

where $S_3(\Gamma_t)$ is the space of paramodular cusp forms of canonical weight 3. The lifting construction proposed by Gritsenko in [10] and [11] provides cusp forms of weight 3 with respect to the paramodular group $\Gamma_t$ for all $t$ except for the twenty polarisations

$$t = 1, \ldots, 12, 14, 15, 16, 18, 20, 24, 30, 36.$$

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In particular, $H^3(\Gamma_t, \mathbb{C})$ is nontrivial for all nonexceptional polarisations. We note that $\dim S_3(\Gamma_t) = 0$ for these twenty $t$ (see [5]). Due to the existence of canonical differential forms, the moduli space of $(1, t)$-polarised abelian surfaces might have trivial geometric genus only for the twenty exceptional polarisations. For $t \leq 20$ the rationality or unirationality of the moduli space is known (see [25]).

The paramodular group $\Gamma_t$ is not a maximal discrete group acting on $\mathbb{H}_2$ if $t \neq 1$. It has a normal extension $\Gamma^*_t$ such that $\Gamma^*_t / \Gamma_t \cong (\mathbb{Z}/2\mathbb{Z})^{\nu(t)}$, where $\nu(t)$ is the number of distinct prime divisors of $t$ (see [13]). In [13], Theorem 1.5, it was proved that the modular variety $\mathcal{H}_t = \Gamma^*_t \backslash \mathbb{H}_2$ can be considered as the moduli space of Kummer surfaces associated to $(1, t)$-polarised abelian surfaces. We note that the birational geometry of moduli spaces of Kummer surfaces is much more complicated than the geometry of moduli spaces of polarised abelian surfaces because the ramification divisor of the modular variety $\Gamma^*_t \backslash \mathbb{H}_2$ is much larger (see [16]). We expect a long list of the moduli spaces $\mathcal{H}_t$ for nonexceptional $t = 21$ was proved in [14].

If $t = p$ is a prime, then $\Gamma^*_t = \Gamma^+_t = \Gamma_t \cup \Gamma_t V_t$ contains only one additional involution $V_t$. A $\Gamma_t$-modular form $F$ of weight 3 will be modular with respect to the double extension $\Gamma^+_t$ if it satisfies an additional functional equation (see §2 for more details)

$$F\left(\begin{pmatrix} t \omega & z \\ z & \tau \end{pmatrix} \right) = -F\left(\begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \right).$$

(1.1)

We call such $\Gamma_t$-paramodular forms antisymmetric. We note that the modular forms obtained by Gritsenko’s lifting are symmetric, that is, they satisfy the equation of type (1.1) with plus sign.

The problem of the construction of antisymmetric paramodular forms of weight 3 has been open since 1996 (see [13]). For the Siegel modular group $\Gamma_1 = \text{Sp}_2(\mathbb{Z})$, there is essentially only one antisymmetric modular form. This is the Igusa modular form $\Delta_{35}$ of odd weight 35. The Borcherds product expansion for $\Delta_{35}$ was proposed in [17].

The theory of automorphic products gives a powerful instrument to construct antisymmetric cusp forms. The first six examples of weight 3 for $t = 122, 167, 173, 197, 213$ and 285 were constructed in [21] as automorphic Borcherds products determined by theta blocks. This sporadic construction was originally proposed for weight 2 as an answer to a question related to the Brumer-Kramer conjecture on modularity of abelian surfaces (see [6]).

In this paper we find the first infinite series of antisymmetric paramodular forms of weight 3 (see Theorem 2.2 in §2, and §5). The series starts with a non-cusp form for $t = 98$. Its first cusp form for $t = 122$ coincides with the example constructed in [21]. As an application (see §6) we prove that $H^{3,0}(\Gamma^+_t \backslash \mathbb{H}_2, \mathbb{C})$ and $H^3(\Gamma^+_t, \mathbb{C})$ are nontrivial for all square-free $t$ in the infinite series presented in Theorem 2.2.

The infinite series of antisymmetric paramodular forms is related to a very special reflective modular form in eight variables on an indefinite orthogonal group $O(2, 8)$. This modular form $\Phi^\text{Sch}_3$ is an automorphic Borcherds product (see §§4 and 5). It was discovered by Scheithauer in [29], §10, in the framework of his fundamental programme on the classification of reflective modular forms of singular weight.
The function $\Phi_{3}^{\text{Sch}}$ is similar to the Borcherds form $\Phi_{12}$ on $O^{+}(2, 26)$ which determines the Fake Monster Lie algebra and plays a crucial role in the Borcherds proof of the Moonshine Conjecture (see [2] and [3]).

Scheithauer’s original construction was given at a zero-dimensional cusp of the corresponding modular variety of orthogonal type as the Borcherds product of a certain nearly holomorphic modular form with respect to the Hecke congruence subgroup $\Gamma_{0}(7)$. In §5 we find another construction of the Scheithauer modular form at a one-dimensional cusp in a way proposed in [19] and [12]. It turns out that the first Fourier-Jacobi coefficient of the Borcherds product at this cusp is a holomorphic Jacobi form which coincides with the Kac-Weyl denominator function of the affine Lie algebra $\widehat{g}(A_{6})$. As a corollary we get that the corresponding Lorentzian Kac-Moody algebra is a hyperbolization of the affine Lie algebra $\widehat{g}(A_{6})$ (see §6).

In the last section, §7, we consider one more example of this type related to the root system $A_{4} \oplus A_{4}$ and construct an infinite family of antisymmetric paramodular forms of weight 4.

§2. Theta blocks and the main theorem

First we recall the definition of Siegel paramodular forms. Let

$$\mathbb{H}_{2} = \left\{ Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in M(2, \mathbb{C}) : \text{Im} Z > 0 \right\}$$

be the Siegel upper half-space of genus 2. The real symplectic group $\text{Sp}_{2}(\mathbb{R})$ acts on $\mathbb{H}_{2}$ via

$$M \langle Z \rangle = (AZ + B)(CZ + D)^{-1}, \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_{2}(\mathbb{R}).$$

Let $k \in \mathbb{Z}$. We define the slash operator on the space of holomorphic functions on $\mathbb{H}_{2}$ in the usual way:

$$(F|_{k}M)(Z) = \det(CZ + D)^{-k}F(M \langle Z \rangle). \quad (2.1)$$

Let $t$ be a positive integer. The paramodular group of level $t$ is a subgroup of $\text{Sp}_{2}(\mathbb{Q})$ defined by

$$\Gamma_{t} = \begin{pmatrix} * & t & * & * \\ * & * & * & * \end{pmatrix} \cap \text{Sp}_{2}(\mathbb{Q}), \quad \text{all } * \in \mathbb{Z}. \quad (2.2)$$

This group is conjugate to the integral symplectic group of the skew-symmetric form with elementary divisors $(1, t)$ (see [13] and [18]). As we mentioned in the introduction, the quotient $\mathcal{A}_{t} = \Gamma_{t} \backslash \mathbb{H}_{2}$ is isomorphic to the moduli space of abelian surfaces with a polarisation of type $(1, t)$.

For $t > 1$, we shall use the following double normal extension of $\Gamma_{t}$ in $\text{Sp}_{2}(\mathbb{R})$:

$$\Gamma_{t}^{+} = \Gamma_{t} \cup \Gamma_{t}V_{t}, \quad V_{t} = \frac{1}{\sqrt{t}} \begin{pmatrix} 0 & t & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -t & 0 \end{pmatrix}. \quad (2.3)$$
Definition 2.1. A holomorphic function \( F : \mathbb{H}_2 \to \mathbb{C} \) is called a Siegel paramodular form of weight \( k \) and level \( t \) if \( F|_k M = F \) for any \( M \in \Gamma_t \).

We denote the space of such modular forms by \( M_k(\Gamma_t) \). A paramodular form \( F \) is called a cusp form if \( \Phi(F|_k g) = 0 \) for all \( g \in \text{Sp}_2(\mathbb{Q}) \), where \( \Phi \) is the Siegel operator. The space of paramodular cusp forms is denoted by \( S_k(\Gamma_t) \).

Let \( \chi : \Gamma^+_t \to \{ \pm 1 \} \) be the nontrivial character with kernel \( \Gamma_t \). By virtue of this character, \( M_k(\Gamma_t) \) is decomposed into the direct sum of plus and minus \( V_t \)-eigenspaces, that is, \( M_k(\Gamma_t) = M_k(\Gamma^+_t) \oplus M_k(\Gamma^-_t, \chi) \). For \( F \in M_k(\Gamma^+_t, \chi^2) \) with \( \varepsilon = 0 \) or \( 1 \), we consider its Fourier and Fourier-Jacobi expansions

\[
F(Z) = \sum_{m \geq 0} \sum_{n \in \mathbb{N}, r \in \mathbb{Z}} c(n, r, m)q^n \zeta^r \xi^{mt} = \sum_{m \geq 0} \phi_{mt}(\tau, z)\xi^{mt},
\]

where \( q = \exp(2\pi i \tau), \zeta = \exp(2\pi i z) \) and \( \xi = \exp(2\pi i \omega) \). One can prove (see [11]) that \( F \) is a cusp form if \( c(n, r, m) \neq 0 \) implies that \( 4nmt - r^2 > 0 \). Then we see that each Fourier-Jacobi coefficient is a holomorphic Jacobi form of weight \( k \) and index \( mt \) in the sense of Eichler-Zagier [8], namely \( \phi_{mt} \in J_{k, mt} \) (see §3 for more details).

Moreover, according to the action of the involution \( V_t \), we obtain the equality

\[
(-1)^{k+\varepsilon} F(\tau, z, \omega) = F\left( \omega t, z, \frac{\tau}{t} \right),
\]

which yields \( c(n, r, m) = (-1)^{k+\varepsilon}c(m, r, n) \) (compare with (1.1)). When \( k + \varepsilon \) is even (odd), \( F \) is called symmetric (antisymmetric, respectively).

The paramodular forms constructed by additive Jacobi lifting due to Gritsenko [10], [11] are always symmetric. Thus the only regular way to construct antisymmetric paramodular forms is the method called the Borcherds automorphic product (see [2], [3], [12], [18], [19] and [29]–[32]). In the Gritsenko-Nikulin interpretation of the Borcherds product given in [18] one can control the action of the involution \( V_t \) in terms of the Fourier coefficients of weakly holomorphic Jacobi forms of weight 0. Unfortunately, one cannot produce any infinite series of such weakly holomorphic Jacobi forms because usually one gets meromorphic automorphic products. An attempt to overcome this difficulty was made in [20] and [21], using the theory of theta blocks (see [22]). This sporadic method gives natural candidates for the first Fourier-Jacobi coefficient of an antisymmetric paramodular form. As a result, an infinite series of antisymmetric paramodular forms with weights going to infinity was constructed in [21]. The first members of the constructed series (see Table 1 in [21]) are of weight 2 (three examples for \( t = 587, 713 \) and 893) and weight 3 (six examples for \( t = 122, 167, 173, 197, 213 \) and 285 mentioned in the introduction).

In this paper we construct antisymmetric paramodular forms using pull-backs of two special antisymmetric orthogonal modular forms of higher dimension. Like in [23] and [24] we use the construction of holomorphic theta blocks in many variables.

Let

\[
\eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n) \in S_{1/2}(\text{SL}_2(\mathbb{Z}), v_\eta)
\]
be the Dedekind \( \eta \)-function. This is a cusp form of weight \( 1/2 \) with the multiplier system \( v_\eta : \text{SL}_2(\mathbb{Z}) \to U_{24} \) of order 24. We consider the odd Jacobi theta-series
\[
\vartheta(\tau, z) = q^{1/8}(\zeta^{1/2} - \zeta^{-1/2}) \prod_{n \geq 1} (1 - q^n \zeta)(1 - q^n \zeta^{-1})(1 - q^n).
\]
(2.5)

It is known that \( \vartheta(\tau, -z) = -\vartheta(\tau, z) \) and \( \vartheta(\tau, z) \in J_{1/2,1/2}(v_\eta^3 \times v_H) \) is a holomorphic Jacobi form of weight \( 1/2 \) and index \( 1/2 \) (see [18]). We define a theta block
\[
\Theta_f = \eta^{f(0)} \prod_{a=1}^{\infty} \left( \frac{\vartheta_a}{\eta} \right)^{f(a)}.
\]
(2.6)

where \( f : \mathbb{N} \to \mathbb{N} \) is a sequence with finite support and \( \vartheta_a = \vartheta(\tau, az) \) (for details, see [20] and [22]).

The quotient \( \Theta_f \) is a weak Jacobi form of weight \( f(0)/2 \) with a character or a multiplier system. For some function \( f \) it is a holomorphic Jacobi form. The simplest example is the theta-quark
\[
\frac{\vartheta_a \vartheta_b \vartheta_{a+b}}{\eta} \in J_{1,a^2+ab+b^2}(v_\eta^6)
\]
(see [7] and [22]).

In this paper we prove the following theorem.

**Theorem 2.2.** For \( \mathbf{a} = (a_1, a_2, a_3, a_4, a_5, a_6) \in \mathbb{Z}^6 \), the theta block
\[
\Theta_a = \vartheta_{a_1} \vartheta_{a_2} \vartheta_{a_3} \vartheta_{a_4} \vartheta_{a_5} \vartheta_{a_6} \vartheta_{a_1+a_2} \vartheta_{a_2+a_3} \vartheta_{a_3+a_4} \vartheta_{a_4+a_5} \vartheta_{a_5+a_6} \vartheta_{a_1+a_2+a_3}
\times \vartheta_{a_2+a_3+a_4} \vartheta_{a_3+a_4+a_5} \vartheta_{a_4+a_5+a_6} \vartheta_{a_1+a_2+a_3+a_4+a_5+a_6}
\times \vartheta_{a_1+a_2+a_3+a_4+a_5+a_6} \frac{1}{\eta^{15}} = q^2(\ldots) \in J_{3,N(\mathbf{a})}
\]
(2.7)
of type \( (21 - \vartheta)/(15 - \eta) \) is a holomorphic Jacobi form of weight 3 and index \( N(\mathbf{a}) \), where
\[
N(\mathbf{a}) = 3a_1^2 + 5a_2a_1 + 4a_3a_1 + 3a_4a_1 + 2a_5a_1 + a_6a_1 + 5a_2^2
+ 8a_3a_2 + 6a_4a_2 + 4a_5a_2 + 2a_6a_2 + 6a_3^2 + 9a_4a_3 + 6a_5a_3
+ 3a_6a_3 + 6a_4^2 + 8a_5a_4 + 4a_6a_4 + 5a_5^2 + 5a_6a_5 + 3a_6^2.
\]
(2.8)

If this theta block is not identically zero, there exists an antisymmetric holomorphic paramodular form \( F_\mathbf{a} \in M_3(\Gamma_N(\mathbf{a})) \) of weight 3 and level \( N(\mathbf{a}) \) whose leading Fourier-Jacobi coefficient is the above theta block. Moreover, \( F_\mathbf{a} \) is a cusp form if \( N(\mathbf{a}) \) is square-free.

### § 3. Jacobi forms of lattice index and Borcherds products

In this section we introduce modular forms on orthogonal groups and Jacobi forms in many variables, which will be used in the proof of Theorem 2.2 (see [10], [7] or [12] for more details).
We consider an even integral lattice \( M = U \oplus U_1 \oplus L(-1) \) of signature \((2, n)\) with \( n \geq 3\), where \( U \) and \( U_1 \) are two hyperbolic planes and \( L \) is an even positive-definite integral lattice. We fix a basis of \( M \) of the form \((e, e_1, \ldots, f_1, f)\), where \( U = Ze + Zf\), \( U_1 = Ze_1 + Zf_1 \) and \( \ldots \) denotes a basis of the negative-definite lattice \( L(-1) \). Let

\[
\mathcal{D}(M) = \{[\omega] \in \mathbb{P}(M \otimes \mathbb{C}) : (\omega, \omega) = 0, (\omega, \overline{\omega}) > 0\}^+
\]

be the associated Hermitian symmetric domain of type IV (here \( + \) denotes one of its two connected components). We denote the index-2 subgroup of the orthogonal group \( O(M) \) preserving \( \mathcal{D}(M) \) by \( O^+(M) \).

Let \( \Gamma \) be a finite-index subgroup of \( O^+(M) \) and let \( k \in \mathbb{Z} \). A modular form of weight \( k \) and character \( \chi : \Gamma \to \mathbb{C}^* \) with respect to \( \Gamma \) is a holomorphic function \( F : \mathcal{D}(M)^\bullet \to \mathbb{C} \) on the affine cone \( \mathcal{D}(M)^\bullet \) satisfying

\[
F(tZ) = t^{-k}F(Z) \quad \forall t \in \mathbb{C}^*
\]

and

\[
F(gZ) = \chi(g)F(Z) \quad \forall g \in \Gamma.
\]

A modular form is called a \textit{cusp} form if it vanishes at every cusp (that is, boundary component) of the Baily-Borel compactification of the modular variety \( \Gamma \setminus \mathcal{D}(M) \).

Let \( D(M) = M^+/M \) be the discriminant group of \( M \). We denote the stable orthogonal group which is the subgroup of \( O^+(M) \) acting trivially on \( D(M) \) by \( \widetilde{O}^+(M) \). For any \( v \in M \otimes \mathbb{Q} \) satisfying \((v, v) < 0\), we define the rational quadratic divisor associated to \( v \) by

\[
\mathcal{D}_v = \{[Z] \in \mathcal{D}(M) : (Z, v) = 0\}.
\]

A reflective modular form is a modular form on \( \mathcal{D}(M) \) whose zero divisor is a union of rational quadratic divisors associated to primitive vectors determining reflections in \( O^+(M) \) (see, for example, [2], [3], [18], [19] or [12] for the exact definition).

We fix a tube realization of the homogeneous domain \( \mathcal{D}(M) \) related to the one-dimensional boundary component defined by the isotropic subspace \( P = \langle e, e_1 \rangle \)

\[
\mathcal{H}(L) = \{Z = (\tau, \delta, \omega) \in \mathbb{H} \times (L \otimes \mathbb{C}) \times \mathbb{H} : (\text{Im } Z, \text{Im } Z) > 0\},
\]

\[
(\text{Im } Z, \text{Im } Z) = 2 \text{Im } \tau \text{Im } \omega - (\text{Im } \delta, \text{Im } \delta).
\]

In this setting, Fourier-Jacobi coefficients and Jacobi forms can be viewed as modular forms with respect to the Jacobi group \( \Gamma^J(L) \) which is the parabolic subgroup \( \{g \in \text{SO}(M)^+ : gP = P, g|_L = \text{id}\} < \text{O}^+(M) \) (see [10] and [7]). The Jacobi group is the semidirect product of \( \text{SL}_2(\mathbb{Z}) \) with the Heisenberg group \( \text{H}(L) \) of the positive-definite integral lattice \( L \).

**Definition 3.1.** Let \( \varphi : \mathbb{H} \times (L \otimes \mathbb{C}) \to \mathbb{C} \) be a holomorphic function and let \( k \in \mathbb{Z} \) and \( t \in \mathbb{N} \). If \( \varphi \) satisfies the functional equations

\[
\varphi\left(\frac{a\tau + b}{c\tau + d}, \frac{3}{c\tau + d}\right) = (c\tau + d)^k e^{\pi t c(\delta, \delta) / (c\tau + d)} \varphi(\tau, \delta), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}),
\]
The notation class of and 1708 V.A. Gritsenko and H. Wang orthogonal group called (see [12], Theorem 4.2). Theorem 3.2 automorphic Borcherds product.

defines a meromorphic modular form of weight and assume that \( f(n, \ell) \neq 0 \Rightarrow 2n - (\ell, \ell) \geq 0 \), then \( \varphi \) is called a holomorphic Jacobi form. If \( \varphi \) also satisfies the stronger condition \( f(n, \ell) \neq 0 \Rightarrow 2n - (\ell, \ell) > 0 \), then \( \varphi \) is called a Jacobi cusp form. We denote by \( J_{k,L,t}^1 \) the vector space of weakly holomorphic (holomorphic or cusp, respectively) Jacobi forms of weight \( k \) and index \( t \) for \( L \).

We note that Jacobi modular forms in the sense of Eichler-Zagier [8] are identical to the Jacobi forms \( J_{k,A_1,t} \) for the lattice \( A_1 = (\mathbb{Z}, 2x^2) \) of rank 1.

The Fourier coefficient \( f(n, \ell) \) depends only on the number \( 2n - (\ell, \ell) \) and the class of \( \ell \) modulo \( tL \) (see [10]). The number \( 2n - (\ell, \ell) \) is called the hyperbolic norm of \( f(n, \ell) \). The Fourier coefficients \( f(n, \ell) \) with negative hyperbolic norm are called singular Fourier coefficients; they determine the divisor of the corresponding automorphic Borcherds product.

**Theorem 3.2** (see [12], Theorem 4.2). Let

\[
\varphi(\tau, \overline{z}) = \sum_{n \in \mathbb{Z}, \ell \in L^\vee} f(n, \ell) q^n \zeta^\ell \in J_{0,L,1}^1
\]

and assume that \( f(n, \ell) \in \mathbb{Z} \) for all \( 2n - (\ell, \ell) \leq 0 \). Fix an ordering in the vector system \( \{ \ell; f(0, \ell) \} \) which is analogous to positive and negative roots (see [12], §4). The notation \( (n, \ell, m) > 0 \) means that either \( m > 0 \), or \( m = 0 \) and \( n > 0 \), or \( m = n = 0 \) and \( \ell < 0 \). Set

\[
A = \frac{1}{24} \sum_{\ell \in L^\vee} f(0, \ell), \quad B = \frac{1}{2} \sum_{\ell > 0} f(0, \ell) \ell, \quad C = \frac{1}{2 \text{rank}(L)} \sum_{\ell \in L^\vee} f(0, \ell)(\ell, \ell).
\]

Then the product

\[
\text{Borch}(\varphi)(Z) = q^A r^B \xi^C \prod_{n,m \in \mathbb{Z}, \ell \in L^\vee (n, \ell, m) > 0} (1 - q^n \zeta^\ell \xi^m) f(nm, \ell),
\]

where \( Z = (\tau, \overline{z}, \omega) \in \mathcal{H}(L), q = \exp(2\pi i \tau), \zeta^\ell = \exp(2\pi i (\ell, \overline{z})) \) and \( \xi = \exp(2\pi i \omega) \), defines a meromorphic modular form of weight \( f(0, 0)/2 \) with respect to the stable orthogonal group \( \widetilde{O}^+(2U \oplus L(-1)) \) with the character \( \chi \) induced by

\[
\chi|_{SL_2(\mathbb{Z})} = v_2^{24} A, \quad \chi|_{H(L)}([\lambda, \mu; r]) = e^{\pi i C((\lambda, \lambda) + (\mu, \mu) - (\lambda, \mu) + 2r)}, \quad \chi(V) = (-1)^D,
\]
where \( V : (\tau, \mathfrak{z}, \omega) \rightarrow (\omega, \mathfrak{z}, \tau) \) and \( D = \sum_{n<0} \sigma_0(-n)f(n, 0) \). The poles and zeros of \( \text{Borch}(\varphi) \) lie on the rational quadratic divisors \( \mathcal{D}_v \), where \( v \in 2U \oplus L^\vee(-1) \) is a primitive vector with \((v, v) < 0\). The multiplicity of this divisor is given by
\[
\text{mult} \mathcal{D}_v = \sum_{d \in \mathbb{Z}, \ell > 0} f(d^2 n, d \ell),
\]
where \( n \in \mathbb{Z} \) and \( \ell \in L^\vee \) are such that \((v, v) = 2n - (\ell, \ell) \) and \( v \equiv \ell \mod 2U \oplus L(-1) \). Moreover, the first Fourier-Jacobi coefficient of \( \text{Borch}(\varphi) \) is given by
\[
\psi_{L,C}(\tau, \mathfrak{z}) = \eta(\tau)^{f(0,0)} \prod_{\ell > 0} \left( \frac{\vartheta(\tau, (\ell, \mathfrak{z}))}{\eta(\tau)} \right)^{f(0,\ell)}, \tag{3.1}
\]
which is a generalized theta block.

From the above theorem we see that the Borcherds product is antisymmetric if the integer \( D \) is odd.

\section*{§ 4. Lifting scalar-valued modular forms to Jacobi forms}

In [29] Scheithauer constructed a map which lifts scalar-valued modular forms on congruence subgroups to modular forms for the Weil representation. In view of the isomorphism between modular forms for the Weil representation and Jacobi forms (see [10]), we can easily build a lifting from scalar-valued modular forms on congruence subgroups to Jacobi forms. This lifting plays a crucial role in this paper.

For our purposes we focus on lattices of prime level. Let \( L \) be an even positive-definite lattice with bilinear form \( \langle \cdot, \cdot \rangle \). Denote the dual lattice of \( L \) by \( L^\vee \). The level of \( L \) is the smallest positive integer \( N \) such that \( N \langle x, x \rangle \in 2\mathbb{Z} \) for all \( x \in L^\vee \). We next assume that the level of \( L \) is a prime number \( p \). Let \( D(L) = L^\vee / L \) be the discriminant group of \( L \). Let \( \{ e_\gamma : \gamma \in D(L) \} \) be the formal basis of the group ring \( \mathbb{C}[D(L)] \). We denote the Weil representation of \( \text{SL}_2(\mathbb{Z}) \) on \( \mathbb{C}[D(L)] \) by \( \rho_{D(L)} \) and the (finite) orthogonal group of \( D(L) \) by \( \text{O}(D(L)) \) (for example, see [10], [3] and [30] for the Weil representation). Let \( M_k^{1,\text{inv}}(\rho_{D(L)}) \) be the space of nearly holomorphic modular forms for \( \rho_{D(L)} \) of weight \( k \) which are holomorphic except at infinity and invariant under the action of \( \text{O}(D(L)) \). By [29], Theorem 6.2, we have the following proposition.

\textbf{Proposition 4.1.} Let \( f \in M_k^1(\Gamma_0(p), \chi_{D(L)}) \) be a scalar-valued nearly holomorphic modular form on \( \Gamma_0(p) \) of weight \( k \) and character \( \chi_{D(L)} \) which is holomorphic except at cusps, where \( \chi_{D(L)} \) is the Dirichlet character defined by
\[
\chi_{D(L)}(A) = \left( \frac{a}{|D(L)|} \right); \quad A = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(p).
\]

Then
\[
F_{\Gamma_0(p),f,0}(\tau) = \sum_{M \in \Gamma_0(p) \setminus \text{SL}_2(\mathbb{Z})} f|M(\tau)\rho_{D(L)}(M^{-1})e_0 \in M_k^{1,\text{inv}}(\rho_{D(L)}). \tag{4.1}
\]
Set
\[ f|_S(\tau) = \sum_{t=0}^{p-1} g_t(\tau), \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \]
where
\[ g_t(\tau + 1) = \exp \left( \frac{2t\pi i}{p} \right) g_t(\tau), \quad 0 \leq t \leq p - 1. \]

Then
\[ F_{\Gamma_0(p), f, 0}(\tau) = f(\tau)e_0 + \xi_1 \frac{p}{\sqrt{|D(L)|}} \sum_{\gamma \in D(L)} g_{j_{\gamma}}(\tau)e_{\gamma}, \quad (4.2) \]
where \( j_{\gamma}/p = -\langle \gamma, \gamma \rangle/2 \mod 1 \) for \( \gamma \in D(L) \) and
\[ \xi_1 = \left( \frac{-1}{|D(L)|} \right) \exp \left( \frac{\text{rank}(L)\pi i}{4} \right). \]

We refer to [30] and [31] for more properties of the above lifting and some other similar constructions of this type.

Recall that the theta functions for the lattice \( L \) are defined by
\[ \Theta^L_\gamma(\tau, z) = \sum_{\ell \in \gamma + L} \exp(\pi i \langle \ell, \ell \rangle \tau + 2\pi i \langle \ell, z \rangle), \quad \gamma \in D(L). \quad (4.3) \]

By means of the isomorphism between vector-valued modular forms and Jacobi forms (see [10]), we obtain the following result.

**Proposition 4.2.** Under the assumptions of Proposition 4.1 set
\[ F_{\Gamma_0(p), f, 0}(\tau) = \sum_{\gamma \in D(L)} F_{\Gamma_0(p), f, 0; \gamma}(\tau)e_{\gamma}. \]

Then the function
\[ \Psi_{\Gamma_0(p), f, 0}(\tau, z) = \sum_{\gamma \in D(L)} F_{\Gamma_0(p), f, 0; \gamma}(\tau)\Theta^L_\gamma(\tau, z) \quad (4.4) \]
is a weakly holomorphic Jacobi form of weight \( k + \text{rank}(L)/2 \) and index 1 for \( L \) which is invariant under the action of the integral orthogonal group \( O(L) \).

§5. Antisymmetric paramodular forms of weight 3 on \( O(2, 8) \)

In this section we prove Theorem 2.2. The proof is based on Scheithauer’s work on the classification of reflective modular forms of singular (that is, minimal possible) weight. By [29], Theorem 10.3, there exists a holomorphic Borcherds product \( \Phi^\text{Sch}_3 \) of singular weight 3 with respect to the orthogonal group of the lattice
\[ U \oplus U(7) \oplus \text{Barnes-Craig lattice}, \quad (5.1) \]
whose genus is of type \( \Pi_{2,8}(7^{-5}) \). The modular form \( \Phi^\text{Sch}_3 \) is a reflective modular form with complete 2-divisor and 14-divisor whose multiplicities are all one.
Below, following ideas in [19], we give another model of the lattice (5.1) and a new construction of the reflective modular form $\Phi^\text{Sch}_3$.

Let $A_6$ be the classical root lattice

$$A_6 = \{(x_1, \ldots, x_7) \in \mathbb{Z}^7 : x_1 + \cdots + x_7 = 0\}.$$ 

Following [4] we fix the set of simple roots in $A_6$:

$$\begin{align*}
\alpha_1 &= (1, -1, 0, 0, 0, 0, 0), & \alpha_2 &= (0, 1, -1, 0, 0, 0, 0), & \alpha_3 &= (0, 0, 1, -1, 0, 0, 0), \\
\alpha_4 &= (0, 0, 1, -1, 0, 0, 0), & \alpha_5 &= (0, 0, 0, 0, 1, -1, 0), & \alpha_6 &= (0, 0, 0, 0, 0, 1, -1).
\end{align*}$$

Then the set of 21 positive roots in $A_6$ is

$$R^+_2(A_6) = \left\{ \sum_{s=i}^j \alpha_s : 1 \leq i \leq j \leq 6 \right\}.$$ 

Let $w_i$, $1 \leq i \leq 6$, be the fundamental weights of $A_6$. Then $(\alpha_i, w_j) = \delta_{ij}$ and $A_6^\vee/A_6 = \{0, w_1, w_2, w_3, w_4, w_5, w_6\}$. The level of $A_6$ is 7. Thus, the renormalization

$$A_6^\vee(7) = \mathbb{Z}w_1 + \mathbb{Z}w_2 + \mathbb{Z}w_3 + \mathbb{Z}w_4 + \mathbb{Z}w_5 + \mathbb{Z}w_6,$$

is an even integral lattice of determinant $7^5$ and its dual lattice is $(A_6^\vee(7))^\vee = A_6/7$. Throughout this section, $(\cdot, \cdot)$ denotes the standard scalar product on $\mathbb{R}^6$.

By [26], Corollary 1.13.3, we have

$$U \oplus U(7) \oplus \text{Barnes-Craig lattice} \cong 2U \oplus A_6^\vee(7),$$

because all of them are of level 7 and belong to the same genus, thus to the same class. We next use Proposition 4.2 to construct the reflective Borcherds product $\Phi^\text{Sch}_3$ at the one-dimensional cusp determined by the decomposition $2U \oplus A_6^\vee(-7)$. Scheithauer constructed a nearly holomorphic modular form of weight $-3$ for the Weil representation associated to the discriminant form of the lattice (5.3) by Proposition 4.1. The datum for it is a nearly holomorphic modular form $\eta^{-3}(\tau)\eta^{-3}(7\tau)$ of weight $-3$ and character $(\frac{7}{\tau})$ with respect to $\Gamma_0(7)$. By Proposition 4.2, we get a weakly holomorphic Jacobi form $\Psi_{A_6^\vee}(7)$ of weight 0 and index 1 for $A_6^\vee(7)$ which is invariant under the (finite) orthogonal group $O(A_6^\vee(7)) = O(A_6)$. As a result, we obtain the following.

**Theorem 5.1.** The Borcherds product $\Phi^\text{Sch}_3 = \text{Borch}(\Psi_{A_6^\vee}(7))$ is a reflective modular form of weight 3 and character $\det$ for the group $O^+(2U \oplus A_6^\vee(-7))$. Its zero divisors are all simple and represented as

$$\text{Div}(\Phi^\text{Sch}_3) = \sum_{r \in 2U \oplus A_6^\vee(-7)} \mathcal{D}_r + \sum_{s \in 2U \oplus (1/7)A_6(-1)} \mathcal{D}_s,$$

where $(\cdot, \cdot)_2$ is the bilinear form of the lattice $2U \oplus A_6^\vee(-7)$. 


Proof. From the construction of the Jacobi form $\Psi_{A_6^\vee}(7)$, we see that its singular Fourier coefficients are given by

$$\text{Sing}(\Psi_{A_6^\vee}(7)) = \sum_{n \in \mathbb{N}} \sum_{\substack{r \in A_6^\vee(7) \ (r,r)=2n}} q^{n-1} e^{2\pi i (r,3)} + \sum_{n \in \mathbb{N}} \sum_{\substack{s \in (1/7) \ A_6 \ (s,s)=2n+2/7}} q^n e^{2\pi i (s,3)}.$$ 

A Fourier coefficient depends only on the hyperbolic norm of its index and the class of $\ell$ in the discriminant group. In particular, all the Fourier coefficients in the $q^0$-term of $\Psi_{A_6^\vee}(7)$ are singular except the constant term $f(0,0) = 6$. Thus we have

$$\Psi_{A_6^\vee}(7)(\tau, \mathfrak{z}) = \Psi_{\Gamma_0(7), \eta^{-3}(\tau)} \eta^{-3}(7\tau), 0$$

$$= q^{-1} + \sum_{r \in A_6 \ (r,r)=2} e^{2\pi i (r,3)} + 6 + O(q) \in J_{0, A_6^\vee(7), 1}^{1, O(A_6)}$$

(5.5)

where $\mathfrak{z} = \sum_{i=1}^6 w_i z_i$, $z_i \in \mathbb{C}$. By [29], Proposition 3.2, there are 2352 classes of norm $2/7 \, (\text{mod} \, 2\mathbb{Z})$ in the discriminant group of $A_6^\vee(7)$. But we can only see 42 of them from the $q^0$-term in the Fourier expansion of $\Psi_{A_6^\vee}(7)$.

According to Theorem 3.2 and the Eichler criterion (see [15]) the automorphic products $\text{Borch}(\Psi_{A_6^\vee}(7))$ and $\Phi_{3}^{\text{Sch}}$ have the same divisor (5.4) with respect to the modular group $\hat{O}^+(2U \oplus A_6^\vee(-7))$. Therefore, these functions are equal up to a constant, due to the Köcher principle. To see that this constant is one, we can use the fact that both automorphic products are constructed from the same modular form $\eta^{-3}(\tau) \eta^{-3}(7\tau)$.

The lattice $2U \oplus A_6^\vee(-7)$ satisfies the Kneser condition (see [15]). Therefore the unique nontrivial character of $\hat{O}^+(2U \oplus A_6^\vee(-7))$ is $\text{det}$ (see [15], Corollary 1.8 and Proposition 3.4). Thus the modular form $\text{Borch}(\Psi_{A_6^\vee}(7))$ has character $\text{det}$ because it is antisymmetric.

The theorem is proved.

The advantage of our description of Scheithauer’s form $\Phi_{3}^{\text{Sch}}$ at the one-dimensional cusp related to $2U \oplus A_6^\vee(-7)$ is that we can give an explicit formula for its first Fourier-Jacobi coefficient.

**Corollary 5.2.** The first Fourier-Jacobi coefficient of $\Phi_{3}^{\text{Sch}}$ is a holomorphic Jacobi form defined by the following theta block:

$$\frac{1}{\eta^{15}(\tau)} \prod_{r \in R^+_3(A_6)} \vartheta(\tau, (r, \mathfrak{z}))$$

$$= \vartheta(z_1) \vartheta(z_2) \vartheta(z_3) \vartheta(z_4) \vartheta(z_5) \vartheta(z_6) \vartheta(z_1 + z_2) \vartheta(z_2 + z_3) \vartheta(z_3 + z_4)$$

$$\times \vartheta(z_4 + z_5) \vartheta(z_5 + z_6) \vartheta(z_1 + z_2 + z_3) \vartheta(z_2 + z_3 + z_4) \vartheta(z_3 + z_4 + z_5)$$

$$\times \vartheta(z_4 + z_5 + z_6) \vartheta(z_1 + z_2 + z_3 + z_4) \vartheta(z_2 + z_3 + z_4 + z_5)$$

$$\times \vartheta(z_3 + z_4 + z_5 + z_6) \vartheta(z_1 + z_2 + z_3 + z_4 + z_5)$$

$$\times \vartheta(z_2 + z_3 + z_4 + z_5 + z_6) \vartheta(z_1 + z_2 + z_3 + z_4 + z_5 + z_6) \frac{1}{\eta^{15}},$$

(5.6)
where $R^+_2(A_6)$ is the set of 21 positive roots of $A_6$ (see (5.2) and (6.1)) and $\vartheta(z) = \vartheta(\tau, z)$. It is a holomorphic Jacobi form of singular weight 3 and index 1 for $A_6^\vee(7)$ which is identical to the Kac-Weyl denominator function of the affine Lie algebra $\tilde{g}(A_6)$ (see [12], Corollary 2.7).

**Proof.** According to Theorem 3.2, to write the first Fourier-Jacobi coefficient of the Borcherds product $\text{Borcherds}(\Psi_{A_6^\vee}(7))$, we need to know only the $q^0$-part of the Fourier expansion of $\Psi_{A_6^\vee}(\tau)$. This implies (5.6). The corollary is proved.

**Remark 5.3.** In fact, $\Phi^\text{Sch}_3$ is a modular form for the full modular group $O^+(2U \oplus A_6^\vee(-7))$ because the vector-valued modular form $F_{l,0}(7), \eta^{-3}(\tau)\eta^{-3}(7\tau), 0$ is invariant under the orthogonal group of the discriminant form of $2U \oplus A_6^\vee(-7)$ (see [29]). It would be interesting to describe the character of $\Phi^\text{Sch}_3$ for $O^+(2U \oplus A_6^\vee(-7))$.

**Remark 5.4.** The theta function $\theta_{A_6^\vee}(7)(\tau) = \sum_{l \in A_6^\vee(7)} \exp(\pi i (l,l)\tau)$ is a scalar-valued nearly holomorphic modular form on $\Gamma_0(7)$ of weight 3 with the character $(\frac{7}{7})$. It can be expressed in terms of Dedekind $\eta$-functions:

$$\theta_{A_6^\vee}(7)(\tau) = \frac{\eta^7(\tau)}{\eta(7\tau)} + 7\eta^3(\tau)\eta^3(7\tau) + 7\frac{\eta^7(7\tau)}{\eta(\tau)} = 1 + 14q^3 + 42q^5 + 70q^6 + \cdots.$$ 

**Remark 5.5.** By Theorem 2.2 we get a holomorphic Borcherds product. Therefore, its first Fourier-Jacobi coefficient is also holomorphic. This gives a new ‘Borcherds-type’ proof of the holomorphicity of the theta blocks of type $(21 - \vartheta)/(15 - \eta)$.

We next consider the quasi-pull-back of the Borcherds product $\Phi^\text{Sch}_3$ (see [16] or [12]) to complete the proof of Theorem 2.2. In our case we can make this using pull-backs of the Jacobi modular form $\Psi_{A_6^\vee}(7)$ in six abelian variables. Given $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5, a_6) \in \mathbb{Z}^6$, we define a Jacobi form in one variable

$$\Psi_{A_6^\vee(7), \mathbf{a}}(\tau, z) = \Psi_{A_6^\vee(7)}(\tau, z \sum_{i=1}^{6} a_i w_i).$$ (5.7)

We denote by $n_0(\mathbf{a})$ the number of zeros among the following 21 integers

$$a_1, a_2, a_3, a_4, a_5, a_6, a_1 + a_2, a_2 + a_3, a_3 + a_4, a_4 + a_5, a_5 + a_6,$$

$$a_1 + a_2 + a_3, a_2 + a_3 + a_4, a_3 + a_4 + a_5, a_4 + a_5 + a_6,$$

$$a_1 + a_2 + a_3 + a_4, a_2 + a_3 + a_4 + a_5, a_3 + a_4 + a_5 + a_6,$$

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6,$$

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6.$$

The theta block (2.7) is not identically zero if and only if $n_0(\mathbf{a}) = 0$. We also set

$$N(\mathbf{a}) = \frac{7}{2} \left( \sum_{i=1}^{6} a_i w_i, \sum_{i=1}^{6} a_i w_i \right),$$ (5.8)
which equals half the sum of the squares of the above 21 integers. The explicit formula for \( N(\mathbf{a}) \) is given in (2.8). Then the function \( \text{Borch}(\Psi_{A_{\mathbf{a}}^{\eta}}(7), \mathbf{a}) \) is an anti-symmetric holomorphic Siegel modular form of weight \( 3 + n_0(\mathbf{a}) \) with respect to the paramodular group of level \( N(\mathbf{a}) \).

To finish the proof of Theorem 2.2, we have to apply the cuspidality test.

**Proposition 5.6** (see [27], Proposition 3.1). Let \( t \) be a square-free positive integer, and let \( k \) be a positive integer. If \( k = 2 \) or \( k \) is odd, then \( M_k(\Gamma_t) = S_k(\Gamma_t) \). If \( k = 4, 6, 8, 10, 14 \), then for all \( F \in M_k(\Gamma_t), F \in S_k(\Gamma_t) \) if and only if \( c(0, 0, 0) = 0 \) in (2.4).

Applying Theorem 2.2 to different \( \mathbf{a} \), we can construct an infinite series of anti-symmetric paramodular forms of weight 3. The first six values of \( N(\mathbf{a}) \) in Theorem 2.2 are

- \( 98: \mathbf{a} = (1, 1, 1, 1, 1, 1) \),
- \( 138: \mathbf{a} = (1, 2, 1, 1, 1, 1) \),
- \( 147: \mathbf{a} = (-1, 4, -6, 4, 1, 3) \),
- \( 167: \mathbf{a} = (1, 1, 1, 2, 2) \),
- \( 173: \mathbf{a} = (1, 1, 2, 1, 1, 2) \),
- \( 223: \mathbf{a} = (-2, 4, -7, 6, 3, -8) \),
- \( 227: \mathbf{a} = (-3, -2, 3, 4, 2, -8) \),
- \( 251: \mathbf{a} = (-6, 4, 1, 3, -5, -5) \),
- \( 257: \mathbf{a} = (8, -4, -1, 3, 4, -8) \),
- \( 269: \mathbf{a} = (6, 8, -3, -5, 6, 3) \),
- \( 271: \mathbf{a} = (4, -5, 3, 6, -3, -7) \),
- \( 283: \mathbf{a} = (-8, -2, 3, -4, 5, 2) \),
- \( 293: \mathbf{a} = (1, -6, 1, -5, 2, 6) \).

The paramodular form for \( t = 98 \) is not a cusp form because its first Fourier-Jacobi coefficient \( \vartheta^6 \vartheta^5 \vartheta^4 \vartheta^3 \vartheta^2 \vartheta \eta^{15} \) is not a Jacobi cusp form.

| \( N(\mathbf{a}) \) | \( \mathbf{a} = (a_1, \ldots, a_6) \) | Theta block |
|-----------------|-----------------|-------------|
| 167             | (1, 1, 1, 1, 2, 2) | \( \vartheta^4 \vartheta^3 \vartheta^2 \vartheta \eta^{15} \) |
| 173             | (1, 1, 2, 1, 1, 2) | \( \vartheta^3 \vartheta^2 \vartheta \eta^{15} \) |
| 223             | (-2, 4, -7, 6, 3, -8) | \( \vartheta^2 \vartheta \eta^{15} \) |
| 227             | (-3, -2, 3, 4, 2, -8) | \( \vartheta \eta^{15} \) |
| 251             | (-6, 4, 1, 3, -5, -5) | \( \vartheta \eta^{15} \) |
| 257             | (8, -4, -1, 3, 4, -8) | \( \vartheta \eta^{15} \) |
| 269             | (6, 8, -3, -5, 6, 3) | \( \vartheta \eta^{15} \) |
| 271             | (4, -5, 3, 6, -3, -7) | \( \vartheta \eta^{15} \) |
| 283             | (5, -6, -2, 3, 9, -2) | \( \vartheta \eta^{15} \) |
| 293             | (-8, -2, 3, -4, 5, 2) | \( \vartheta \eta^{15} \) |
The other levels are listed in Tables 1–3. We explain how to read the tables. For a fixed row, the corresponding paramodular form is constructed as $Borch(\Psi A^\vee_t(7), a)$. The number $N(a)$ is the level $t$ of the corresponding paramodular group. The ‘theta block’ is the first Fourier-Jacobi coefficient of $Borch(\Psi A^\vee_t(7), a)$.

### Table 2. Antisymmetric paramodular cusp forms of weight 3 and square-free (non-prime) level $t < 300$

| $N(a)$ | $a = (a_1, \ldots, a_6)$ | Theta block |
|--------|-----------------|-------------|
| 122    | $(2, 1, 1, 1, 1, 1)$ | $\vartheta_2^5 \vartheta_4^3 \vartheta_2^3 \vartheta_6 \vartheta_7 / \eta^{15} \vartheta$ |
| 138    | $(1, 2, 1, 1, 1)$ | $\vartheta_2^5 \vartheta_4^3 \vartheta_2^3 \vartheta_6 \vartheta_7 / \eta^{15}$ |
| 146    | $(1, 1, 2, 1, 1)$ | $\vartheta_2^5 \vartheta_4^3 \vartheta_2^3 \vartheta_6 \vartheta_7 / \eta^{15}$ |
| 174    | $(-1, 4, 1, -6, 3, 5)$ | $\vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_6 \vartheta_7 / \eta^{15}$ |
| 178    | $(1, 3, -2, -4, 9, -4)$ | $\vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_6 \vartheta_7 / \eta^{15}$ |
| 182    | $(7, -4, -1, 4, -2, -5)$ | $\vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_6 \vartheta_7 / \eta^{15}$ |
| 194    | $(2, 5, -2, -2, -5, 6)$ | $\vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_6 \vartheta_7 / \eta^{15}$ |
| 202    | $(5, -2, -2, -2, 3, 6)$ | $\vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_6 \vartheta_7 / \eta^{15}$ |
| 203    | $(2, 2, 4, -1, -1, -5)$ | $\vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_6 \vartheta_7 / \eta^{15}$ |
| 206    | $(-6, 2, 7, -8, 4, -1)$ | $\vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_6 \vartheta_7 / \eta^{15}$ |
| 210    | $(-3, -2, 7, -6, 3, -6)$ | $\vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_6 \vartheta_7 / \eta^{15}$ |
| 213    | $(9, -5, 2, 2, -3, -3)$ | $\vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_6 \vartheta_7 / \eta^{15}$ |
| 215    | $(-3, -7, 8, -3, 1, -2)$ | $\vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_6 \vartheta_7 / \eta^{15}$ |
| 222    | $(3, 7, -2, 1, -2, -6)$ | $\vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_6 \vartheta_7 / \eta^{15}$ |
| 224    | $(-2, 7, 1, 5, -2, -1)$ | $\vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_6 \vartheta_7 / \eta^{15}$ |
| 228    | $(7, -2, -2, 7, -6, -2)$ | $\vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_6 \vartheta_7 / \eta^{15}$ |
| 230    | $(1, -3, 1, -2, -1, -5)$ | $\vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_6 \vartheta_7 / \eta^{15}$ |
| 237    | $(2, 2, -6, 3, -4, -3)$ | $\vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_6 \vartheta_7 / \eta^{15}$ |
| 238    | $(7, 2, -1, 8, -3, -3)$ | $\vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_6 \vartheta_7 / \eta^{15}$ |
| 255    | $(-3, -3, 1, -3, 6, -8)$ | $\vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_6 \vartheta_7 / \eta^{15}$ |
| 258    | $(-7, 4, -6, -1, 6, -2)$ | $\vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_6 \vartheta_7 / \eta^{15}$ |
|        | $(-5, 1, 2, -4, 3, -8)$ | $\vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_6 \vartheta_7 / \eta^{15}$ |
|        | $(6, -9, 7, -3, -3, 7)$ | $\vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_6 \vartheta_7 / \eta^{15}$ |
|        | $(-2, -3, 1, 9, -4, 1)$ | $\vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_2^4 \vartheta_6 \vartheta_7 / \eta^{15}$ |

The first antisymmetric paramodular cusp form of weight 3 that we know of is in $S_3(\Gamma_{122}^+)$. We note that this function is not in $S_3(\Gamma_{122}^+)$ and it has Atkin-Lehner signs of $-1$ at both 2 and 61 (see [21]). These tables show that we can reconstruct all antisymmetric paramodular cusp forms of weight 3 and square-free level $t$ from [21] except for $t = 197$. 
Table 3. Antisymmetric paramodular cusp forms of weight 3 and square-free (non-prime) level \( t < 300 \)

| \( N(a) \) | \( a = (a_1, \ldots, a_6) \) | Theta block |
|----------|----------------|-------------|
| 262      | \((-5,3,-5,-1,2,7)\) | \( \vartheta^4 \vartheta_2^3\vartheta_3^3\vartheta_4\vartheta_5^2\vartheta_6^3\vartheta_7^2\vartheta_8^3\vartheta_9 \eta^{15} \) |
|          | \((-1,-1,7,3,-5,-1)\) | \( \vartheta^3 \vartheta_2^3\vartheta_3^3\vartheta_4\vartheta_5\vartheta_7\vartheta_8\vartheta_9 \eta^{15} \) |
|          | \((7,-6,-3,1,5,-7)\) | \( \vartheta^4 \vartheta_2^3\vartheta_3^3\vartheta_4\vartheta_5\vartheta_8^2\vartheta_9 \eta^{15} \) |
| 266      | \((-1,-7,3,6,-4,5)\) | \( \vartheta^3 \vartheta_2^3\vartheta_3^3\vartheta_4\vartheta_5^2\vartheta_6\vartheta_7^2\vartheta_8 \eta^{15} \) |
|          | \((4,-3,-6,7,-3,7)\) | \( \vartheta^3 \vartheta_2^3\vartheta_3^3\vartheta_4\vartheta_5^2\vartheta_6^2\vartheta_7 \eta^{15} \) |
|          | \((8,-1,-4,2,5,-8)\) | \( \vartheta^2 \vartheta_2^3\vartheta_3^3\vartheta_4\vartheta_5\vartheta_6^2\vartheta_8 \eta^{15} \) |
| 278      | \((4,1,2,3,-1,-7)\) | \( \vartheta^2 \vartheta_2^3\vartheta_3^3\vartheta_4\vartheta_5^2\vartheta_6^2\vartheta_7 \eta^{15} \) |
|          | \((5,-7,6,-5,8,-1)\) | \( \vartheta^5 \vartheta_2^3\vartheta_3^3\vartheta_4\vartheta_5^2\vartheta_6^2 \eta^{15} \) |
|          | \((-8,1,3,-5,4,-6)\) | \( \vartheta^3 \vartheta_2^3\vartheta_3^3\vartheta_4\vartheta_5^2\vartheta_6 \eta^{15} \) |
|          | \((-2,7,-8,7,-9,4)\) | \( \vartheta^4 \vartheta_2^3\vartheta_3^3\vartheta_4\vartheta_5^2\vartheta_6 \eta^{15} \) |
|          | \((1,-2,-9,7,1,-2)\) | \( \vartheta^5 \vartheta_2^3\vartheta_3^3\vartheta_4\vartheta_6^2 \eta^{15} \) |
|          | \((-6,-4,-1,4,2,-3)\) | \( \vartheta^4 \vartheta_2^3\vartheta_3^3\vartheta_4\vartheta_5^2 \eta^{15} \) |
| 285      | \((-4,-2,8,-5,-5,1)\) | \( \vartheta^3 \vartheta_2^3\vartheta_3^3\vartheta_4\vartheta_5^2 \eta^{15} \) |
| 286      | \((-1,-3,6,3,-4,-7)\) | \( \vartheta^3 \vartheta_2^3\vartheta_3^3\vartheta_4\vartheta_5 \eta^{15} \) |
| 287      | \((2,2,-3,-5,7,-9)\) | \( \vartheta^4 \vartheta_2^3\vartheta_3^3\vartheta_4\vartheta_5^2 \eta^{15} \) |
|          | \((8,4,-6,1,-2,-1)\) | \( \vartheta^4 \vartheta_2^3\vartheta_3^3\vartheta_4\vartheta_5 \eta^{15} \) |
| 290      | \((-2,-7,5,5,-8,6)\) | \( \vartheta^3 \vartheta_2^3\vartheta_3^3\vartheta_4\vartheta_5 \eta^{15} \) |
|          | \((-4,-2,1,3,1,6)\) | \( \vartheta^4 \vartheta_2^3\vartheta_3^3\vartheta_4\vartheta_5 \eta^{15} \) |
|          | \((1,5,5,-9,2,-1)\) | \( \vartheta^4 \vartheta_2^3\vartheta_3^3\vartheta_4\vartheta_5 \eta^{15} \) |
|          | \((-2,-5,1,7,2,-7)\) | \( \vartheta^3 \vartheta_2^3\vartheta_3^3\vartheta_4\vartheta_5 \eta^{15} \) |
|          | \((3,-4,7,-2,6,-5)\) | \( \vartheta^4 \vartheta_2^3\vartheta_3^3\vartheta_4\vartheta_5 \eta^{15} \) |
| 299      | \((5,1,4,-2,3,-7)\) | \( \vartheta^3 \vartheta_2^3\vartheta_3^3\vartheta_4\vartheta_5 \eta^{15} \) |
|          | \((5,4,-8,6,-5,-3)\) | \( \vartheta^3 \vartheta_2^3\vartheta_3^3\vartheta_4\vartheta_5 \eta^{15} \) |

Jerry Shurman has informed us that he can prove nonexistence of antisymmetric paramodular forms for many \( t < 300 \). For example, for square-free \( t \leq 220 \) the space \( S_3(\Gamma_t^+) \) may be nontrivial only for

\[ t = 122, 138, 146, 158, 167, 170, 173, 174, 178, 182, 183, 186, 194, 197, 202, 203, 206, 210, 213, 215, 218, 219. \]

In bold face, we write the polarisations for which we cannot construct an antisymmetric paramodular form of weight 3.

§ 6. Applications

6.1. Applications to the theory of moduli spaces and group cohomology. The paramodular group \( \Gamma_t \) and its normal extensions in \( \text{Sp}_2(\mathbb{R}) \) have realisations as
integral orthogonal groups of signature $(2, 3)$. This realisation describes the nature of the normal extensions $\Gamma_t^+$ and $\Gamma_t^*$ (see [13]).

Let $L_t = 2U \oplus \langle -2t \rangle$ be an even integral lattice of signature $(2, 3)$. The finite discriminant group $D_t = L_t^\vee / L_t = (2t)^{-1}Z/Z$ is a finite abelian group equipped with a quadratic form

$$q_t: D_t \times D_t \to (2t)^{-1}Z/2Z,$$

$$(l, l) \equiv (l, l)_{L_t} \mod 2Z$$

(see [26] for a general definition and properties). Any $g \in O(L_t)$ acts on the finite group $D_t$. By $\overline{O}(L_t)$ we denote the subgroup of the orthogonal group consisting of elements which act identically on the discriminant group.

The natural projection of $O^+(L_t)$ onto the finite orthogonal group $O(D_t)$ is surjective. The last group can be described as follows. For every $d || t$ (that is, $d | t$ and $(d, t/d) = 1$) there exists a unique (mod $2t$) integer $\xi_d$ satisfying

$$\xi_d = -1 \mod 2d \quad \text{and} \quad \xi_d = 1 \mod \frac{2t}{d}.$$  

All such $\xi_d$ form the group

$$\Xi(t) = \{ \xi \mod 2t \mid \xi^2 = 1 \mod 4t \} \cong (Z/2Z)^{\nu(t)},$$

where $\nu(t)$ is the number of prime divisors of $t$. It is evident that $O(D_t) \cong \Xi(t)$.

According to [11] and [13], Proposition 1.2 and Corollary 1.3, we have the following isomorphisms:

$$\Gamma_t^+ / \{ \pm E_4 \} \cong \overline{O}^+(L_t) / \{ \pm E_5 \} \quad \text{and} \quad \Gamma_t^* / \{ \pm E_4 \} \cong O^+(L_t) / \{ \pm E_5 \}.$$  

The coverings $\Gamma_t \backslash \mathbb{H}_2 \to \Gamma_t^+ \backslash \mathbb{H}_2$ and $\Gamma_t \backslash \mathbb{H}_2 \to \Gamma_t^* \backslash \mathbb{H}_2$ are Galois with a finite abelian Galois group. According to [13], Proposition 1.5, the modular variety $\mathscr{M}_t = \Gamma_t \backslash \mathbb{H}_2$ $(t$ is square-free) is isomorphic to the moduli space of polarized $K3$ surfaces with a polarisation of type $\langle 2t \rangle \oplus 2E_8(-1)$. According to [13], Theorem 1.5, the modular variety $\mathscr{M}_t = \Gamma_t^* \backslash \mathbb{H}_2$ is isomorphic to the moduli space of Kummer surfaces associated to abelian surfaces with a $(1, t)$-polarisation.

We mentioned in the introduction that weight-3 cusp forms are closely related to canonical differential forms on smooth models of the corresponding modular varieties. If $F$ is a cusp form of weight 3 with respect to an arithmetic group $\Gamma$, then $\omega_F = F(Z) dZ$ is a holomorphic 3-form over the open smooth part of the modular variety $(\Gamma \backslash \mathbb{H}_2)^0$ outside the ramification divisor and the boundary components. A very useful extension theorem due to Freitag implies that such a form can be extended to any smooth model of $\Gamma \backslash \mathbb{H}_2$. Let $\Gamma$ be an arbitrary subgroup of $Sp_2(\mathbb{R})$, which contains a principal congruence subgroup $\Gamma_1(N) \subset Sp_2(\mathbb{Z})$ of some level $N$. Then we have the following.

**Proposition 6.1** (see [9], Ch. 3, Hilfssatz 2.1). An element

$$\omega_F = F(Z) dZ \in H^0((\Gamma \backslash \mathbb{H}_2)^0, \Omega_3((\Gamma \backslash \mathbb{H}_2)^0))$$

can be extended to a canonical differential form on an arbitrary smooth compactification $\overline{\Gamma \backslash \mathbb{H}_2}$ if and only if the differential form $\omega_F$ is square integrable.
It is well known that a $\Gamma$-invariant differential form $\omega_F = F(Z) \, dZ$ is square integrable if and only if $F$ is a cusp form of weight 3. Thus we have the following identity for the geometric genus of the variety:

$$h^{3,0}(\Gamma \setminus \mathbb{H}_2) = \dim_{\mathbb{C}} S_3(\Gamma).$$

In particular, when $t$ is prime, we have $\Gamma_t^* = \Gamma_t^+$ and the space $S_3(\Gamma_t^*)$ is just the space of antisymmetric cusp forms of weight 3.

**Theorem 6.2.** The moduli space $\mathcal{X}_p = \Gamma_p^* \setminus \mathbb{H}_2$ of Kummer surfaces associated to $(1,p)$-polarised abelian surfaces has positive geometric genus for all primes $p = N(a)$ in Theorem 2.2. In particular, it is positive for $t = 167, 173, 223, 227, 251, 257, 269, 271, 283$ and 293. Moreover,

$$h^{3,0}(\Gamma_t^*, \mathbb{C}) \geq 2 \quad \text{for} \quad t = 227, 257, 269, 283 \quad \text{and} \quad h^{3,0}(\Gamma_{293}^*, \mathbb{C}) \geq 4.$$

For all square-free $t = N(a)$ in Theorem 2.2 the moduli space $\omega_t^+ = \Gamma_t^+ \setminus \mathbb{H}_2$ of polarized $K3$ surfaces with a polarisation of type $(2t) \oplus 2E_8(-1)$ has positive geometric genus. The smallest such $t$ equals 122 (see Tables 1–3).

It is known that $\dim S_3(\Gamma_t^*) = 0$ for $t \leq 40$ (see [5]). According to the calculation made by Shurman, the minimal level $t$ with $\dim S_3(\Gamma_t^*) \neq 0$ is 152 or 167. For $t = 152 = 8 \times 19$ the antisymmetric paramodular form of weight 3 and level 152 starts with the theta block $\vartheta_1^5 \vartheta_2^4 \vartheta_3^4 \vartheta_4^3 \vartheta_5^2 \vartheta_6 \vartheta_7 \vartheta_8 / \eta^{15}$. We leave to the readers two questions on this paramodular form. Does it belong to $M_3(\Gamma_{152}^*)$? Is it a cusp form?

We note that antisymmetric paramodular forms of weight 3 occur conjecturally as cohomology classes in $H^5(\Gamma_0(N), \mathbb{C})$, studied by Ash, Gunnells and McConnell in [1], where $\Gamma_0(N) \subseteq \text{SL}_4(\mathbb{Z})$ is defined by having a last row in $(N \mathbb{Z}, N \mathbb{Z}, N \mathbb{Z}, \mathbb{Z})$.

We hope to get more geometric applications of antisymmetric forms of weights 3 and 4 (see §7) in the near future.

**6.2. Automorphic $L$-functions.** For a prime polarisation $p$ such that the space $S_3(\Gamma_p^+)$ is one-dimensional, we get a new eigenfunction of all Hecke operators. (Cf. the Igusa modular form $\Delta_{35}$.) The first such prime is 167. For $t = 122$, the antisymmetric cusp form is an ‘old’ form, and comes from a ‘new’ form in $S_3(\Gamma_{61})$. Conjecturally, its Spin-$L$-function coincides with the motivic $L$-function of a non-rigid Calabi-Yau threefold.

Antisymmetric paramodular cusp forms of weight 2 are also very interesting. For a prime polarisation, such a form may exist starting from $p = 587$ (see [28]). For the moment, only three examples are known (see [21]), for $t = 587, 713$ and 893. The first supports the Paramodular Conjecture of Brumer and Kramer (see [6]), which is a generalization for two dimensions of the celebrated Shimura-Taniyama-Weil Conjecture on the modularity of elliptic curves. Unfortunately, there is no antisymmetric reflective modular form of singular weight 2 for a lattice of signature $(2,6)$ which splits into two orthogonal integral (renormalised) hyperbolic planes (see [29] and [32]). In addition, the leading Fourier-Jacobi coefficients of such antisymmetric paramodular forms are theta blocks of weight 2 with vanishing order $> 1$ in $q$. But so far, no such infinite family of theta blocks has been found (see [22]). Therefore, we cannot construct an infinite series of antisymmetric paramodular forms of weight 2 using the approach in our paper.
6.3. Hyperbolization of affine Lie algebras. One can set the following problem: to find all Lorentzian Kac-Moody algebras whose first term of the Kac-Weyl-Borcherds denominator function written at a one-dimensional cusp coincides with the Kac-Weyl denominator function of an affine Lie algebra. In such cases, one can study the Lorentzian Kac-Moody Lie algebra as a module over the corresponding affine Lie algebra.

The Kac-Weyl denominator function of an affine Lie algebra $\hat{g}(R)$ for a positive definite 2-root system $R$ of rank $n$ is the following theta block:

$$\psi_R(\tau, z) = \eta(\tau)^n \prod_{r \in R > 0} \frac{\vartheta(\tau, (r,z))}{\eta(\tau)}, \quad (6.1)$$

where the product is taken over all positive roots of the system $R$ and $z \in R \otimes \mathbb{C}$.

The possible list of the Lorentzian Kac-Moody algebras that are hyperbolizations of affine Lie algebras is rather short. They are the affine algebras for $A_1$ (see [18]), $2A_1$, $4A_1$, $A_2$ and $3A_2$, 23 root systems of Niemeier lattices of rank 24 (see [19], [12] and [22]) and the root system for $A_4$ (see [23] and [24]). The function $\Phi_{\text{Sch}}^3$ gives the case of $A_6$. We note that the Kac-Weyl-Borcherds denominator function of the Lorentzian Kac-Moody algebra for $R = A_1$, $2A_1$, $4A_1$, $A_2$, $3A_2$ and $A_4$ is the Gritsenko lifting of the corresponding Kac-Weyl denominator function of $\hat{g}(A_1)$.

At the end of this paper we consider another function of Scheithauer which gives a hyperbolization of the affine Lie algebra $\hat{g}(A_4 \oplus A_4)$. This interpretation gives antisymmetric paramodular forms of weight 4. We plan to apply them to algebraic geometry soon.

§ 7. Antisymmetric paramodular forms of weight 4

According to Scheithauer’s work (see [29], Theorem 10.3), there is a Borcherds product of singular weight 4 with respect to the lattice

$$U \oplus U(5) \oplus \text{Maass lattice},$$

whose genus is $\Pi_{10,2}(5^+6)$. We can check that

$$U \oplus U(5) \oplus \text{Maass lattice} \cong 2U \oplus A_4^\vee(5) \oplus A_4^\vee(5).$$

The explicit description and more properties of the lattice $A_4^\vee(5)$ can be found in our last preprint [24].

We can reconstruct the Borcherds product on $U \oplus U(5) \oplus \text{Maass lattice}$, at the one-dimensional cusp related to $2U \oplus A_4^\vee(5) \oplus A_4^\vee(5)$. By Proposition 4.2, we have

$$\Psi_{2A_4^\vee(5)}(\tau, z) = \Psi_{\Gamma_0(5), \eta^{-4}(\tau)\eta^{-4}(5\tau), 0}$$

$$= q^{-1} + \sum_{r \in A_4 \oplus A_4 \atop (r,r)=2} e^{2\pi i (r,z)} + 8 + O(q) \in J_{1,0}(2A_4)_{0,2A_4^\vee(5), 1}. \quad (7.1)$$
Therefore, the function $\text{Borch}(\Psi_{2A_4^\vee(5)})$ is a reflective modular form of weight 4 with respect to $O^+(2U \oplus 2A_4^\vee(-5))$ with divisor

$$\text{Div}(\text{Borch}(\Psi_{2A_4^\vee(5)})) = \sum_{r \in 2U \oplus 2A_4^\vee(5)} \mathcal{D}_r + \sum_{s \in 2U \oplus (1/5)2A_4(-1)} \mathcal{D}_s.$$  \hspace{1cm} (7.2)

Moreover, the character of $\text{Borch}(\Psi_{2A_4^\vee(5)})$ for the group $\widetilde{O}^+(2U \oplus 2A_4^\vee(-5))$ is det. Similarly to §5 we obtain the following theorem.

**Theorem 7.1.** Given $a = (a_1, a_2, a_3, a_4) \in \mathbb{Z}^4$ and $b = (b_1, b_2, b_3, b_4) \in \mathbb{Z}^4$, let $n_0(a, b)$ be the number of zeros in the following 20 integers

$$a_1, a_2, a_3, a_4, a_1 + a_2, a_2 + a_3, a_3 + a_4, a_1 + a_2 + a_3, a_2 + a_3 + a_4, a_1 + a_2 + a_3 + a_4,$$

$$b_1, b_2, b_3, b_4, b_1 + b_2, b_2 + b_3, b_3 + b_4, b_1 + b_2 + b_3, b_2 + b_3 + b_4, b_1 + b_2 + b_3 + b_4.$$  \hspace{1cm} (7.3)

Denote one-half of the sum of the squares of the above 20 integers by $N(a, b)$. We define a weakly holomorphic Jacobi form in one variable:

$$\Psi_{2A_4^\vee(5), a, b}(\tau, z) = \Psi_{2A_4^\vee(5)}\left(\tau, z \sum_{i=1}^{4} a_i u_i + z \sum_{j=1}^{4} b_j v_j\right),$$  \hspace{1cm} (7.4)

where the $u_i$ are the fundamental weights of the first copy of $A_4$ and the $v_j$ are the fundamental weights of the second copy of $A_4$.

Then $\text{Borch}(\Psi_{2A_4^\vee(5), a, b})$ is a holomorphic antisymmetric Siegel modular form of weight $4 + n_0(a, b)$ with respect to the paramodular group of level $N(a, b)$. Moreover, the first Fourier-Jacobi coefficient of $\text{Borch}(\Psi_{2A_4^\vee(5), a, b})$ is equal to

$$\eta^{3n_0(a, b) - 12} \prod_c \vartheta(\tau, (c, \frac{1}{c})),$$

where the product runs over all non-zero integers in the list (7.3).

We remark that all antisymmetric paramodular cusp forms of weights larger than 3 constructed in [21] can be reconstructed using our method. We list all of them and many new examples in Table 4.

**Remark 7.2.** We can also consider the pull-back to a lattice of signature $(2, 4)$. We define a weakly holomorphic Jacobi form for a lattice of rank 2:

$$\Psi_{2A_4^\vee(5), a + b}(\tau, z_1, z_2) = \Psi_{2A_4^\vee(5)}\left(\tau, z_1 \sum_{i=1}^{4} a_i u_i + z_2 \sum_{j=1}^{4} b_j v_j\right).$$

Assume that $n_0(a, b) = 0$. Denote one-half of the sum of the squares of the first ten integers related to $a$ by $N_0(a)$ and one-half of the sum of the squares of the last ten integers related to $b$ by $N_0(b)$. Then the Borcherds product $\text{Borch}(\Psi_{2A_4^\vee(5), a+b})$ will give an antisymmetric holomorphic modular form of canonical weight 4 for the stable orthogonal group of the lattice $2U \oplus (-2N_0(a)) \oplus (-2N_0(b))$. 
We hope that this type of modular forms will have applications to Hermitian modular forms and corresponding modular varieties. It will be interesting to seek a similar test to check, as in Proposition 5.6, the cuspidality of the constructed modular forms.

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Table 4. Antisymmetric paramodular cusp forms of weights larger than 3

| Weight | $N(a, b)$ | $a$, $b$ | Theta block |
|--------|-----------|----------|-------------|
| 4      | 62        | $(1, 1, 1, 1), (2, 1, 1, 1)$ | $\vartheta^7 \vartheta_2^6 \vartheta_3^2 \vartheta_5/\eta^{12}$ |
| 5      | 38        | $(1, 1, 1, 1), (1, 1, 1, 1)$ | $\vartheta^9 \vartheta_2^6 \vartheta_3^2 \vartheta_4/\eta^{9}$ |
| 6      | 42        | $(1, 1, 1, 1), (1, 1, 1, 1)$ | $\vartheta^8 \vartheta_2^4 \vartheta_3^2 \vartheta_4/\eta^{9}$ |
| 5      | 53        | $(1, 1, 1, 1), (1, 1, 1, 2)$ | $\vartheta_1^7 \vartheta_2^4 \vartheta_3^2 \vartheta_4^2/\eta^{9}$ |
| 5      | 65        | $(1, 1, 1, 1), (0, 1, 1, 2)$ | $\vartheta^6 \vartheta_2^2 \vartheta_3^2 \vartheta_4^2 \vartheta_5/\eta^{9}$ |
| 6      | 26        | $(-1, 1, 1, 1), (-1, 1, 1, 1)$ | $\vartheta^{10} \vartheta_2^2 \vartheta_3^2 \vartheta_4^2 \vartheta_5/\eta^{6}$ |
| 7      | 23        | $(-1, 1, 1, 1), (0, 1, 1, 0)$ | $\vartheta^9 \vartheta_2^7 \vartheta_3/\eta^{3}$ |
| 8      | 14        | $(1, -1, 1, 1), (1, -1, 1, 1)$ | $\vartheta^{12} \vartheta_2^7$ |
| 8      | 17        | $(0, 1, 1, 0), (1, -1, 1, 1)$ | $\vartheta^{10} \vartheta_2^6$ |
| 9      | 15        | $(0, 0, 1, 1), (1, -1, 1, 1)$ | $\eta^3 \vartheta^{10} \vartheta_2^5$ |
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Valery A. Gritsenko
Laboratoire Paul Painlevé, Université de Lille,
Villeneuve d’Ascq, France;
National Research University
Higher School of Economics,
Moscow, Russia
E-mail: valery.gritsenko@univ-lille.fr

Haowu Wang
Laboratoire Paul Painlevé, Université de Lille,
Villeneuve d’Ascq, France
E-mail: haowu.wangmath@gmail.com

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