Bianchi Type-III Dark Energy Cosmological Models in Brans-Dicke Theory of Gravitation

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Abstract: Dark energy cosmological models with variable equation of state (EoS) parameter in Brans-Dicke [1] scalar tensor theory of gravitation are obtained. The field equations have been solved by using the anisotropy feature of the universe in Bianchi type-III metric. Some important features of the models, thus obtained, have been discussed. We noticed that as a special case, it is always possible to obtain dark energy cosmological models in general relativity.

Key words
Bianchi type-III metric, Dark energy, Brans-Dicke theory, General relativity, EoS parameter.

1. Introduction

Brans-Dicke [1] theory of gravitation is a natural extension of general relativity which introduces an additional scalar field $\phi$ besides the metric tensor $g_{ij}$ and dimensionless coupling constant $\omega$. The Brans-Dicke field equations for a combined scalar and tensor fields (using geometrized units with $c = 1, G = 1$) are given by

$$G_{ij} = -8\pi\phi^{-1}T_{ij} - \omega\phi^{-2}(\phi,_{i,i,j} - \frac{1}{2}g_{ij}\phi^k\phi_k) - \phi^{-1}(\phi,_{i,j} - g_{ij}\phi^k)$$ (1.1)

and $\phi^k = 8\pi(3 + 2\omega)^{-1}T$ (1.2)

where $G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}$ is an Einstein tensor, $R$ is the scalar curvature, $\phi$ is the scalar field and $\omega$ is the coupling constant, $T_{ij}$ is the stress energy tensor of the matter and comma and semicolon denote partial and covariant differentiation respectively.

Also, we have energy – conservation equation

$$T^i_{;j} = 0.$$ (1.3)

Several aspects of Brans-Dicke cosmology have been extensively investigated by many authors. Reddy and Rao ([2], [3]) have obtained field of a charged particle and static plane symmetric solution in Brans-Dicke theory. Rao et al. [4] have obtained exact Bianchi type-V perfect fluid cosmological models in
Brans-Dicke theory of gravitation. Rao et al. [5] have obtained axially symmetric string cosmological model in Brans – Dicke theory of gravitation. Rao and Sireesha ([6], [7]) have discussed axially symmetric and Bianchi type-II, VIII and IX cosmological models with strange quark matter attached to string cloud in Brans-Dicke theory of gravitation respectively.

Based on Supernovae type Ia (Reiss et al.[8]; Perlmutter et al.[9]; and Tegmark et al. [10]) observations, cosmologists have accepted the idea of dark energy, which is a fluid with negative pressure making up around 70% of the present universe energy content to be responsible for this acceleration due to repulsive gravitation. Current studies to extract the properties of a dark energy component of the universe from observational data focus on the determination of its equation of state $w(t)$, which is the ratio of the dark energy’s pressure to its energy density $w(t) = P/\rho$, which is not necessarily constant. Recently, the parameter $w(t)$ has been calculated with some reasoning and the simplest dark energy candidate is the vacuum energy ($w = -1$), which is mathematically equivalent to the cosmological constant ($\Lambda$). The other conventional alternatives, which can be described by minimally coupled scalar fields, are quintessence ($w > -1$), phantom energy ($w < -1$) and quintom (that can across from phantom region to quintessence region as evolved) and have time dependent EoS parameter. Yadav and Yadav [11] has obtained Bianchi type-III anisotropic DE models with constant deceleration parameter and Pradhan and Amirhaschi [12] have investigated anisotropic Bianchi type-III DE model with variable $\omega$ without assuming constant deceleration parameter. Rao et al. [13] have studied Bianchi type- I dark energy model in Saez – Ballester [14] scalar tensor theory of gravitation. Rao et al. [15] have discussed LRS Bianchi type-I dark energy cosmological model in Brans-Dicke theory of gravitation. Naidu et al.[16] have obtained Bianchi type – III dark energy model in a scalar tensor theory of gravitation. Rao and Sreedevi Kumari [17] have discussed a cosmological model with negative constant deceleration parameter in general scalar tensor theory of gravitation. Shamir and Bhatti [18] have discussed anisotropic dark energy Bianchi type-III cosmological models in Brans Dicke theory of gravitation with the assumptions of constant deceleration parameter and power law relation between scalar field $\phi$ and scale factor. Rao et al. [19] have discussed Bianchi type-II, VIII & IX dark energy cosmological models in Saez - Ballester theory of gravitation. Rao et al. [20] have obtained perfect fluid dark energy cosmological models in Saez - Ballester and general theory of gravitation. Recently Rao et al.[21] have obtained Kantowski - Sachs dark energy cosmological models in general scalar tensor theory of gravitation.
In continuation to the above investigations, we once again study Bianchi type-III dark energy cosmological models in Brans-Dicke theory of gravitation without using the assumptions of either constant deceleration parameter or power law relation between scalar field $\phi$ and scale factor.

2. Metric And Field Equations

We consider the Bianchi type-III metric in the form

$$ds^2 = dt^2 - A^2 dx^2 - B^2 e^{-2x} dy^2 - C^2 dz^2$$  \hspace{1cm} (2.1)

where $A, B & C$ are functions of cosmic time $t$ only.

The energy momentum tensor components of the fluid can be written in anisotropic diagonal form as

$$T^i_j = diag[T^0_0, T^1_1, T^2_2, T^3_3]$$  \hspace{1cm} (2.2)

We can parameterize the components of the energy momentum tensor as follows:

$$T^i_j = diag[\rho, -p_x, -p_y, -p_z]$$

$$= diag[1, -w_x, -w_y, -w_z] \rho$$

$$= diag[1, -w, -(w+\gamma), -(w+\delta)] \rho$$  \hspace{1cm} (2.3)

where $\rho$ is the energy density of the fluid. $p_x, p_y$ and $p_z$ are the pressures. $w_x, w_y$ and $w_z$ are the directional equation of state (EoS) parameters of the fluid along x, y and z axes respectively and $w(t) = \frac{p}{\rho}$ is the deviation free EoS parameter of the fluid.

Here we have parameterized the deviation from isotropy by setting $w_x = w$. Also $\gamma$ & $\delta$ are the skewness parameters, which are the deviations from $w$ along y and z axes. The parameters $w, \gamma$ & $\delta$ are not necessarily constants and can be functions of the cosmic time $t$.

Using commoving coordinates, the field equations (1.1) to (1.3) for the metric (2.1) with the help of (2.3) can be written as

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} + \omega \left( \frac{\ddot{\phi}}{\dot{\phi}} \right)^2 + \left( \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) = -8\pi \phi^{-3} \rho \omega$$  \hspace{1cm} (2.4)
\[ \frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A} \ddot{C}}{AC} + \frac{\omega}{2} \left( \frac{\dot{\phi}}{\phi} \right)^2 + \frac{\phi}{\phi} \left( \frac{\dot{A}}{A} + \frac{\dot{C}}{C} \right) = -8\pi \phi^{-1} (w + \gamma) \rho \] (2.5)

\[ \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A} \ddot{B}}{AB} - \frac{1}{A^2} + \frac{\omega}{2} \left( \frac{\dot{\phi}}{\phi} \right)^2 + \frac{\phi}{\phi} \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) = -8\pi \phi^{-1} (w + \delta) \rho \] (2.6)

\[ \frac{\dot{A} \dot{B}}{AB} + \frac{\dot{B} C}{BC} + \frac{\dot{C} A}{CA} - \frac{\omega}{2} \left( \frac{\dot{\phi}}{\phi} \right)^2 + \frac{\phi}{\phi} \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) = 8\pi \phi^{-1} \rho \] (2.7)

\[ \frac{\dot{A}}{A} - \frac{\dot{B}}{B} = 0 \] (2.8)

\[ \dot{\phi} + \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) = 8\pi \rho \left( 1 - \gamma - \delta - 3w \right) \] (3 + 2\omega) \] (2.9)

\[ \dot{\rho} + (w + 1) \rho \left( 2 \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) + \delta \rho \frac{\dot{C}}{C} = 0 \] (2.10)

where the overhead dot (\( \cdot \)) denotes derivative with respect to the cosmic time \( t \).

From (2.8), we get

\[ A = B \] (2.11)

Using (2.11), the field equations (2.4) to (2.10) reduce to

\[ \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B} \ddot{C}}{BC} + \frac{\omega}{2} \left( \frac{\dot{\phi}}{\phi} \right)^2 + \frac{\phi}{\phi} \left( \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) = -8\pi \phi^{-1} w \rho \] (2.12)

\[ \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B} \ddot{C}}{BC} + \frac{\omega}{2} \left( \frac{\dot{\phi}}{\phi} \right)^2 + \frac{\phi}{\phi} \left( \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) = -8\pi \phi^{-1} (w + \gamma) \rho \] (2.13)

\[ 2 \frac{\ddot{B}}{B} + \frac{\ddot{B}^2}{B^2} - \frac{1}{B} + \frac{\omega}{2} \left( \frac{\dot{\phi}}{\phi} \right)^2 + \frac{\phi}{\phi} \left( \frac{\ddot{B}}{B} \right) = -8\pi \phi^{-1} (w + \delta) \rho \] (2.14)

\[ 2 \frac{\dot{B} \dot{C}}{BC} + \frac{\dot{B}^2}{B^2} - \frac{1}{B^2} + \frac{\omega}{2} \left( \frac{\dot{\phi}}{\phi} \right)^2 + \frac{\phi}{\phi} \left( \frac{2 \dot{B} + \dot{C}}{B} \right) = 8\pi \phi^{-1} \rho \] (2.15)
\[
\ddot{\phi} + \phi \left( 2 \frac{\ddot{B}}{B} + \frac{\dot{C}}{C} \right) = \frac{8\pi \rho (1 - \gamma - \delta - 3w)}{(3 + 2\omega)} 
\tag{2.16}
\]

\[
\dot{\rho} + (w + 1)\rho \left( 2 \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) + \delta \rho \frac{\dot{C}}{C} = 0 
\tag{2.17}
\]

From (2.12) & (2.13), we get

\[
\gamma = 0 
\tag{2.18}
\]

Using (2.18), and by using the transformation \( dt = B^2 CdT \), the field equations (2.12) to (2.18) can be written as

\[
\frac{B''}{B} - 2 \frac{B'^2}{B^2} + \frac{C''}{C} - \frac{C'^2}{C^2} - 2 \frac{B'C'}{BC} + \frac{\omega \phi'^2}{2 \phi^2} + \frac{\phi''}{\phi} - \frac{\phi'B'}{\phi B} = -\frac{8\pi}{\phi} w \rho (B^2 C^2) 
\tag{2.19}
\]

\[
2 \frac{B''}{B} - 2 \frac{B'C'}{BC} - 3 \frac{B'^2}{B^2} - B^2 C^2 + \frac{\omega \phi'^2}{2 \phi^2} + \frac{\phi''}{\phi} - \frac{\phi'C'}{\phi C} = -\frac{8\pi}{\phi} (w + \delta) \rho (B^2 C^2) 
\tag{2.20}
\]

\[
\frac{B'^2}{B^2} + 2 \frac{B'C'}{BC} - B^2 C^2 - \frac{\omega \phi'^2}{2 \phi^2} + \phi' \left( 2 \frac{B'}{B} + \frac{C'}{C} \right) = \frac{8\pi}{\phi} \rho (B^2 C^2) 
\tag{2.21}
\]

\[
\phi'' = \frac{8\pi}{(3 + 2\omega)} (1 - \delta - 3w) \rho (B^2 C^2) 
\tag{2.22}
\]

\[
\rho' + (w + 1)\rho \left( 2 \frac{B'}{B} + \frac{C'}{C} \right) + \delta \rho \frac{C'}{C} = 0 
\tag{2.23}
\]

where the overhead dash denotes derivative with respect to \( T \).

From (2.19) to (2.22), we get

\[
\omega \left( \frac{\phi''}{\phi} - \frac{\phi'^2}{2 \phi^2} \right) = 2 \frac{B''}{B} - 3 \frac{B'^2}{B^2} + \frac{C''}{C} - \frac{C'^2}{C^2} - 2 \frac{B'C'}{BC} - B^2 C^2 
\tag{2.24}
\]
The field equations (2.19) to (2.22) are four independent equations with six unknowns $B, C, \phi, \rho, w & \delta$. In order to get a deterministic solution we take the following plausible physical condition, the shear scalar $\sigma$ is proportional to scalar expansion $\theta$, which leads to the linear relationship between the metric potentials $B$ and $C$, i.e.,

$$B = C^n \quad (n \neq 0) \quad (2.25)$$

From (2.24) & (2.25), we get

$$\omega \left( \phi^* - \frac{1}{2} \phi'^2 \right) = (2n + 1) \frac{C^n}{C} - (n^2 + 4n + 1) \frac{C'^2}{C^2} - C^{2n+2} \quad (2.26)$$

From (2.26), we get

$$\phi = k_2 e^{\frac{k_1}{\omega}}. \quad (2.27)$$

$$C = \left[ \left( \frac{k_1}{k_3} \right) \text{sec}(n+1)k_4T \right]^{\frac{1}{n+1}}, n \neq -1 \quad (2.28)$$

where $k_1 & k_2$ are arbitrary constants and without loss of generality we can assign unity for $k_2$, i.e. $k_2 = 1$ and $k_3^2 = \frac{1}{n^2 + n + 1} \quad \& \quad k_4^2 = \frac{k_1}{2n(n+2)}$.

From (2.25) & (2.11), we get

$$B = A = \left[ \left( \frac{k_1}{k_3} \right) \text{sec}(n+1)k_4T \right]^{\frac{n}{n+1}} \quad (2.29)$$
Then the metric (2.1) can now be written in the form
\[
ds^2 = \left[ \frac{k_4}{k_5} \right] \frac{2^{(n+1)}}{n+1} \frac{dx^2}{dT^2} - \left[ \frac{k_4}{k_5} \right] \frac{2^{n}}{n+1} \frac{dy^2}{dz^2} - \left[ \frac{k_4}{k_5} \right] \frac{2^{n}}{n+1} \frac{dz^2}{dx^2} - \left[ \frac{k_4}{k_5} \right] \frac{2^{n}}{n+1} \frac{dz^2}{dy^2} - \left[ \frac{k_4}{k_5} \right] \frac{2^{n}}{n+1} \frac{dy^2}{dz^2}
\]

From (2.21), we get the Energy density
\[
\rho = \frac{1}{8\pi k_5^2} e^\omega \left\{ nk_4^2 \left( n + 1 \right) k_4 T - k_4^2 \sec^2 \left( n + 1 \right) k_4 T \right\} + \left( 2n + 1 \right) k_4 \left( \frac{k_4}{\omega} \right) \tan \left( n + 1 \right) k_4 T - \frac{k_4 \left( 3 + 2n \right)}{2 \left( n + 2 \right)} \left( \cos \left( n + 1 \right) k_4 T \right)^{\frac{2 \left( n + 1 \right)}{n + 1}}
\]
where \( k_5 = \left( \frac{k_4}{k_3} \right) \frac{2 \left( n + 1 \right)}{n + 1} \)

From (2.19), we get the EoS parameter
\[
\omega = \frac{\left\{ nk_4 \left( \frac{k_4}{\omega} \right) \tan \left( n + 1 \right) k_4 T - \frac{k_4^2 \sec^2 \left( n + 1 \right) k_4 T - \frac{k_4 \left( 1 + \omega \right)}{\omega} \right\}}{nk_4^2 \tan^2 \left( n + 1 \right) k_4 T - \frac{k_4^2 \sec^2 \left( n + 1 \right) k_4 T + \left( 2n + 1 \right) k_4 \left( \frac{k_4}{\omega} \right) \tan \left( n + 1 \right) k_4 T - \frac{k_4 \left( 3 + 2n \right)}{2 \left( n + 2 \right)} \right\}}
\]

From (2.19) & (2.20), we get the skewness parameter
\[
\delta = \frac{\left\{ (n + 2) k_4^2 \sec^2 \left( n + 1 \right) k_4 T + \left( 1 - n \right) k_4 \left( \frac{k_4}{\omega} \right) \tan \left( n + 1 \right) k_4 T \right\}}{nk_4^2 \sec^2 \left( n + 1 \right) k_4 T - \frac{k_4^2 \sec^2 \left( n + 1 \right) k_4 T + \left( 2n + 1 \right) k_4 \left( \frac{k_4}{\omega} \right) \tan \left( n + 1 \right) k_4 T - \frac{k_4 \left( 3 + 2n \right)}{2 \left( n + 2 \right)} \right\}}
\]

Thus the metric (2.30) together with (2.27) & (2.31) to (2.33) constitutes Bianchi type-III dark energy cosmological model in Brans-Dicke theory of gravitation which is entirely different from the model obtained by Shamir and Bhatti [18].
Bianchi Type-III Dark Energy Cosmological Model in General Relativity

It is interesting to mention here that the dark energy cosmological model in general relativity is recovered as $\omega \rightarrow \infty$.

From (2.31), we get the Energy density

$$\rho = \frac{1}{8\pi k_5} \left[ nk_4^2 \tan^2(n+1) k_4 T - k_4^2 \sec^2(n+1) k_4 T \right] \left[ k_4 (3 + 2n) \right] \left[ \cos(n+1) k_4 T \right]^2 \left( \frac{2n+1}{n+1} \right)$$

(2.34)

From (2.32), we get the EoS parameter

$$w = \frac{\left( k_4^2 \sec^2(n+1) k_4 T + k_4 \right)}{nk_4^2 \tan^2(n+1) k_4 T - k_4^2 \sec^2(n+1) k_4 T - k_4 (3 + 2n) \left( \frac{2n+1}{n+1} \right)}$$

(2.35)

From (2.33), we get the skewness parameter

$$\delta = \frac{(n+2)k_4^2 \sec^2(n+1) k_4 T}{nk_4^2 \tan^2(n+1) k_4 T - k_4^2 \sec^2(n+1) k_4 T - k_4 (3 + 2n) \left( \frac{2n+1}{n+1} \right)}$$

(2.36)

Thus the metric (2.30) together with (2.34) to (2.36) constitutes Bianchi type-III dark energy cosmological model in Einstein theory of gravitation and which is entirely different from the model obtained by Pradhan and Amirhaschi [12].

3. Some important features of the models

The volume element of the model (2.30) is given by

$$V = (-g)^{\frac{1}{2}} = \sqrt{k_5} \left[ \sec(n+1) k_4 T \right]^{2n+1} e^{-X}$$

(3.1)

$$\frac{2n+1}{n+1}$$
The expansion scalar $\theta$ is given by

$$ \theta = u^i_j = (2n+1)k_i Tan(n+1)k_i T $$

(3.2)

The shear scalar $\sigma^2$ is given by

$$ \sigma^2 = \frac{7}{18} (2n+1)^2 k_i^2 Tan^2(n+1)k_i T $$

(3.3)

Mean Hubble parameter $H$ is given by

$$ H = \frac{1}{3} (H_1 + 2H_2) = \frac{(2n+1)k_i Tan(n+1)k_i T}{3} $$

(3.4)

The deceleration parameter $q$ is given by

$$ q = -3\theta^2 (\theta, u^i_j + \frac{1}{3} \theta^2) = -\left[ \frac{3(n+1)}{(2n+1)} \text{Cosec}^2(n+1)k_i T + 1 \right] $$

(3.5)

The tensor of rotation $W_{ij} = u_{i,j} - u_{j,i}$ is identically zero and hence this universe is non-rotational.

4. Conclusions

In this paper, we have presented spatially homogeneous and anisotropic Bianchi type - III dark energy cosmological models in Brans-Dicke and Einstein theory of gravitation without taking the assumption of negative constant deceleration parameter proposed by Bermann [22]. Recent experiments show that there is a certain amount of anisotropy in the universe. In standard cosmology, data tells us that the universe is homogeneous and isotropic. However, after the discovery of temperature anisotropies of the Cosmic Microwave background (CMB) radiation, it is conjectured that there are unobservable small amount of anisotropies are present in the early stages of evolution of the universe. The presence of this feature seems to be inconsistent with isotropic FRW model. The current CMB data supports an inflationary big bang model of cosmic origin for our universe. Hence in our model there is an unobservable small amount of anisotropy present. Hence anisotropic space-times are important. Dark energy models with variable equation of state (EoS) parameter are significant since dark energy is supposed to be the best candidate to explain cosmic acceleration in general relativity and in modified theories of gravitation. It is observed that
the model (2.30) has no singularities except at $T = \frac{(2r + 1)\pi}{2(n+1)k_4}, r = 0, \pm 1, \pm 2, \ldots$ for $n > 0$ and the spatial volume varies with time $T$. We observe that at the initial epoch $T = 0$, the expansion scalar $\theta$, shear scalar $\sigma$ and the Hubble parameter $H$ vanish whereas $T \to \infty$, they vary with a constant rate. We can also observe that the energy density $\rho$ diverges as $T \to \infty$. The EoS parameter $W$, skewness parameter $\delta$ approach constant value as $T \to \infty$. Also, the deceleration parameter appears with negative sign for $-\infty < n < -1 \& \frac{-1}{2} < n < \infty$, which implies accelerating expansion of the universe, which is consistent with the present day observations. Also for $n = -\frac{4}{5}, q = 0$ and hence the universe expands at a constant rate. But for $n = 1$, the model (2.30) represents isotropic dark energy cosmological model in this theory. These anisotropic as well as isotropic exact models are new, more general and represent not only the early stage of evolution but also the present universe. Finally we noticed that as a special case, it is always possible to obtain dark energy cosmological models in general relativity.

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