Conformal Field Theory Techniques
in Large $N$ Yang-Mills Theory

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Following some motivating comments on large $N$ two-dimensional Yang-Mills theory, we discuss techniques for large $N$ group representation theory, using quantum mechanics on the group manifold $U(N)$, its equivalence to a quasirelativistic two-dimensional free fermion theory, and bosonization. As applications, we compute the free energy for two-dimensional Yang-Mills theory on the torus to $O(1/N^2)$, and an interesting approximation to the leading answer for the sphere. We discuss the question of whether the free energy for the torus has $R \to 1/R$ invariance.

1 Introduction

The first part of this article is an introduction to what might be called “large $N$ representation theory,” Lie group representation theory with the focus on the limit $N \to \infty$ of $SU(N)$ and the other classical groups. This has many applications in physics and mathematics, and good mathematical introductions exist, which tie it to its applications in group theory, soliton theory, combinatorics, and so forth. (See [1], 5.4 for a treatment very much like the one here; see also [2].) Now although it might seem that this theory would be invaluable for studying the large $N$ limit of models with $SU(N)$ symmetry, and some examples in the physics literature are in [3], it does not get as much use as one might expect. Whether this is simply because the language is taking time to standardize,

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or because the physics really is too diverse to capture in one formalism, I leave for the reader to judge. However a model which seems made to order as an application is large $N$ two-dimensional Yang-Mills theory ($YM_2$ in the following).

I would like to make some comments about the potential physical relevance of this model, which is certainly a long way from realistic four-dimensional models. The starting point is the old idea that QCD could be reformulated as a string theory, which for many reasons can only be a free string in the large $N$ limit. [4] Although our understanding of such things is primitive, it seems clear that a “QCD string” if it exists is a very different type of string than those studied as theories of quantum gravity, and that a low dimensional solvable model might provide a context in which we could discover and understand such a different type of string theory. This line of thought has led to a revival of the study of two-dimensional Yang-Mills theory and QCD with the objective of developing a string representation which would generalize to higher dimensions. The earliest work in this direction concentrated on reproducing Wilson loop expectation values; even in two dimensions these are non-trivial for self-intersecting loops and evidence was found that their structure could be described by a sum of surfaces each with weight $\exp -\text{area}$, modified by local factors associated with features on the surface such as branch points. [5] More recently the partition function on a closed Riemann surface has been studied and a complete set of rules derived which reproduces it as a sum over surfaces, again with additional features. [6] The idea is not restricted to two dimensions and a $D$-dimensional lattice formulation exists. [7]

Besides explicit string constructions of this sort, much can be learned from comparing precise results from a field theory and candidate equivalent string theories. Of course if we can get exact results for a theory we do not really need a string formulation but even crude results in higher dimensions are valuable to show whether a string formulation could work at all. The most important issue for the constructions of [6, 7] is that they are essentially strong coupling expansions. They improve on Wilson’s original expansion by replacing the expansion parameter $1/g^2$ by $\exp -g^2$ ($g$ is a dimensionless bare lattice coupling) but still face the essential problem of the strong coupling expansion for gauge theory – the continuum limit requires the limit of weak bare coupling. As it turns out there is an interesting two-dimensional case which illustrates the problems, namely $YM_2$ on the sphere.

Assuming the usefulness of these string representations, the next step would be to take a world-sheet continuum limit and hope that this still reproduces the Yang-Mills continuum limit. The results of [6] are similar to topological field theory results, and [3, 4] pointed out that for torus target space there is a natural candidate for comparison, the topological sigma model with torus target space (possibly with coupling to gravity), and that this worked for the torus world-sheet. One consequence of this on arbitrary genus world-sheet would be $R \rightarrow 1/R$ duality invariance, and we will look for this at genus two in the $YM_2$ results.

This relation to topological theory seems very special to $D = 2$, and in higher dimensions we expect a string with less trivial dynamics. However the generic string the-
ory seems to have a trivial world-sheet continuum limit; this is the famous “c > 1” or “branched polymer” problem which has been argued to be inevitable in a theory defined as a sum over world-sheets with positive weights. A sensible QCD string must escape this problem, and the additional world-sheet features have to play an essential role in this, leading to the question of which of the many features are essential and which are irrelevant. One would like to understand this before trying to reproduce the string theory with a continuum world-sheet action. It seems to me that this can only be properly understood in $D > 2$; nevertheless the $D = 2$ results do suggest that some features are more important than others.

2 Quantum Mechanics and Group Representations

The prototypical system we study is quantum mechanics on the group manifold $U(N)$. This allows us to quickly classify representations and derive the Weyl character formula. Physically this system is already interesting, since it describes the (global) degrees of freedom of two-dimensional Yang-Mills theory. We go on to discuss calculations of the YM$_2$ free energy on a Riemann surface.

The natural Hamiltonian is

$$H = \text{tr} \left( U \frac{\partial}{\partial U} \right)^2 = \sum_a E^a E^a. \quad (2.1)$$

Here $E^a = \text{tr} t^a dU/dU$ generates left rotations of $U$ and represents the Lie algebra $u(N)$. Thus acting on a wave function which could be any matrix element of an irreducible representation $R$, $\psi(U) = D^{(R)}(U)$, $H = C_2(R)$, the second Casimir (normalized so that $C_2(\mathbf{1}) = N$). In fact it is the unique invariant and purely second order linear differential operator on the group manifold, the Laplacian.

To classify representations we should find their characters $\chi_R(U) = \text{tr} D^{(R)}(U)$. These will be wave functions invariant under $\psi(U) \rightarrow \psi(gUg^{-1})$, so we should make the change of variables $U_{ij} = g_{ik} z_k g_{kj}^{-1}$. (This is familiar from the quantum mechanics of a hermitian matrix and for the group manifold case is much older, going back to Harish-Chandra.

$$\sqrt{h} = |\Delta(z)|^2 = \Delta(z)^2 \quad (2.2)$$

where $\Delta(z) = \prod_{i<j} (z_i - z_j)$ and $\tilde{\Delta}(z) = \prod_{i<j} \sin \frac{\theta_i - \theta_j}{2} = \Delta(z)/\prod_i z_i^{(N-1)/2}$. The “radial” components of the metric are simply $h_{ij} = \delta_{ij}$. Thus on wave functions independent of $g$

$$H = -\sum_i \frac{1}{\Delta^2} \frac{d}{d\theta_i} \tilde{\Delta} \frac{d}{d\theta_i}. \quad (2.3)$$

We can rewrite this as

$$H = -\sum_i \left[ \frac{1}{\Delta} \frac{d^2}{d\theta_i^2} \tilde{\Delta} - \frac{1}{\Delta} \left( \frac{d^2 \tilde{\Delta}}{d\theta_i^2} \right) \right]. \quad (2.4)$$
For hermitian matrix quantum mechanics Δ was a Vandermonde and the second term, thanks to a non-trivial identity, gave zero. Here, after a similar calculation, the second term is found to equal \( -N(N^2 - 1)/12 \). [13]

Thus, after redefining the wave functions by \( \psi \to \tilde{\Delta} \psi \), we have a theory of \( N \) free fermions on the circle. The boundary conditions are also determined by this redefinition; they become periodic (antiperiodic, respectively) for \( N \) odd (even). An orthonormal basis for wave functions is Slater determinants

\[
\psi_{\vec{n}} = \det_{i,j} z_i^n
\]

with energy \( E = \sum_i n_i^2 - N(N^2 - 1)/12 \). The ground state has fermions distributed symmetrically about \( n = 0 \), and energy zero, so the Fermi level \( n_F = (N - 1)/2 \).

Going back to the original wave functions, we have rederived the Weyl character formula (for \( U(N) \) actually due to Schur):

\[
\chi_{\vec{n}}(\vec{z}) = \frac{\det_{1 \leq i,j \leq N} z_i^{n_j}}{\det_{1 \leq i,j \leq N} z_i^{n_j - 1 - n_F}}.
\]

In terms of roots and weights, the indices \( n_i \) with \( n_1 > n_2 > \ldots > n_N \) are the components of the highest weight vector shifted by half the sum of the positive roots (usually denoted \( \mu + \rho \)) where the basis of the Cartan subalgebra is just \( (H_i)_{jk} = \delta_{ij} \delta_{jk} \). In the language of Young tableaux, if \( h_i \) is the number of boxes in the \( i \)’th row, \( n_i = (N - 1)/2 + 1 - i + h_i \).

The \( U(1) \) charge is \( Q = \sum_i n_i \). We can change this by a multiple of \( N \) by shifting all the fermions \( n_i \to n_i + a \), but \( Q \) mod \( N \) is correlated with the conjugacy class of the \( SU(N) \) representation (in other words the action of the center) reflecting the identification \( U(N) \cong SU(N) \times U(1)/\mathbb{Z}_N \).

Interesting observables in this quantum mechanics, invariant under the adjoint action, are the invariant “position” operators

\[
W_n = \text{tr } U^n = \sum_i z_i^n
\]

and “generalized Hamiltonians”

\[
H_m = (-i)^m \sum_i \frac{\partial^m}{\partial \theta_i^m}.
\]

For \( m > 2 \) these are not the higher Casimirs \( \text{tr } E^m \) but are polynomial in them (see [14, p. 163] for an explicit expression). We will not discuss \( m > 2 \) further here.

We next go to a second quantized formalism with operators \( B^+_n \) and \( B_n \) creating and destroying the fermion mode \( z^n \), and \( \psi(\theta) = \sum_n e^{in\theta} B_n \). Then \( H = \int d\theta \partial \psi^+ \partial \psi - E_0 \). The operators \( W_n \) and \( H_m \) will become fermion bilinears.

The first simplification of the large \( N \) limit now appears. If we never consider operators \( W_n \) with \( n \sim N \), then fermions near the positive and negative Fermi surfaces completely
decouple. We can then speak of a quasi-relativistic fermi system, with complex chiral left- and right-moving fermions. We should also speak of $U$ raising the left-moving (upper) fermions while lowering the right-movers, and $U^{-1}$ doing the opposite. This suggests that we refer to representations contained in tensor products of $O(N^0)$ fundamentals as “chiral,” and their complex conjugates as “anti-chiral.” The full representation theory is a product of chiral and anti-chiral sectors. So, let $b^+_n = B^+_{nF-\epsilon+n}$, $b_n = B_{nF+\epsilon+n}$, $\bar{b}^+_n = B^+_{n\epsilon-n}$, $\bar{b}_n = B_{n\epsilon-n}$, where $\epsilon = \frac{1}{2}$ is an choice of definition, introduced to give antiperiodic ($n \in \mathbb{Z} + \frac{1}{2}$) moding for all $N$. The local operators $\psi(z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} z^{-n}b_n$, $\bar{\psi}(z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \bar{z}^{-n}\bar{b}_n$, and $\bar{\psi}^+(z)$ now satisfy standard 2d field theory commutation relations. The standard Fock vacuum ($b_n|0\rangle = b^+_n|0\rangle = 0$ for $n > 0$) corresponds to the identity representation, and higher representations can be built by acting with the bilinears $b^+_n b_{-m}$ and $\bar{b}^+_n \bar{b}_{-m}$. Of these, clearly the simplest are the $W_n$’s which become \[ W_n = \text{tr} U^n \]

\begin{align*}
W_n &= \int dz_{-1-n} \psi^+(z) \psi(z) + \int d\bar{z} \bar{z}^{-1+n} \bar{\psi}^+(z) \bar{\psi}(z) \\
&= \sum_m b^+_m b_{-m} + \bar{b}^+_m \bar{b}_{-m}.
\end{align*}

We recognize the operators here as the left- and right-moving conformal field theory $U(1)$ currents, and the construction of the $W_n$’s as bosonization:

\[ W_n \equiv \alpha_n + \bar{\alpha}_n = \int d\theta e^{i\theta} \partial_\tau \phi(z = e^{n(\tau+i\theta)}, \bar{z} = e^{n(\tau-i\theta)}) \]

(2.12)

defining the standard free boson oscillator expansion with

\[ \partial_\tau \phi(z) = i \sum_{m \in \mathbb{Z}} \alpha_m z^{m-1} \]

\[ [\alpha_m, \alpha_n] = [\bar{\alpha}_m, \alpha_n] = m \delta_{m+n,0} \]

(2.13)

\[ [\alpha_m, \bar{\alpha}_n] = 0. \]

Notice that the $W_n$ commute as operators, as they should.

The charges $\alpha_0$ and $\bar{\alpha}_0$ count fermion numbers. Their sum is constant in our application. There is a normal ordering ambiguity in the definition (2.11) which we use to define it to be zero. As for the difference $\alpha_0 - \bar{\alpha}_0$, clearly it can be changed by operators like

\[ \sum_n B^+_{n-nF-\epsilon+n} B_{nF+n} = \sum_{n \in \mathbb{Z} + \frac{1}{2}} b^+_n \bar{b}_n = \int dz \frac{dz}{z} \psi^+(z) \bar{\psi}(z^*). \]

(2.14)

In the bosonic language it is winding number;

\[ w = \alpha_0 - \bar{\alpha}_0 = -i \int dz \partial \phi + i \int d\bar{z} \bar{\partial} \phi = -\frac{1}{2\pi}(\phi(2\pi) - \phi(0)). \]

(2.15)

\[^2\text{Our CFT conventions are generally as in [13], except that our boson } \phi = \sqrt{2} \text{ times theirs; in other words } S_{\text{free}} = \int d^2x (\partial \phi)^2 / 2\pi. \text{ Our contour integrals always have an implicit } 1/2\pi i, \text{ and } \bar{z} \text{ integrals go counterclockwise.} \]
A better way to change the winding number is to turn it on continuously from zero; taking this back to the fermi picture we are continuously changing the fermion boundary conditions, or equivalently multiplying the wave function by

\[ \psi(\vec{z}) \rightarrow \left( \prod_{i=1}^{N} z_i \right)^{s} \psi(\vec{z}). \]  

(2.16)

Taking \( s \) from zero to one gives a new state with the same \( SU(N) \) quantum numbers but \( U(1) \) charge increased by \( N \). We conclude that the original quantum mechanics on \( U(N) \) is equivalent to a free bosonic field theory whose zero modes are treated rather asymmetrically: we sum over integer winding numbers, but not over momenta.

We should keep in mind that although the formalism so far suggests a close relationship with two-dimensional conformal field theory, there is no a priori guarantee that the Hamiltonian or observables in a specific problem will be local in the two dimensions. Of course the positions \( \theta_i \) parameterize the maximal torus of our original group, so local time evolution on the group manifold will reduce to local two dimensional evolution. On the other hand group multiplication is an example of a natural operation with no locality properties. So not all problems will have simple conformal field theory translations.

We could summarize by saying that the bose-fermi correspondence is the large \( N \) limit of the Frobenius relation between characters and symmetric polynomials. A character \( \chi_{\vec{n}} \) corresponds to a fermion Fock basis state in a simple way; if one takes a “chiral” representation \( R \), with Young tableau with \( h_i \) boxes in the \( i \)'th row, \( 1 \leq i \leq r \), this corresponds to the state

\[ |\vec{h} \rangle = \chi_{\vec{h}}(U)|0 \rangle = b_{\epsilon-h_1}^+ b_{\epsilon-h_2}^+ \cdots b_{\epsilon-h_r}^+, b_{-\epsilon-1} b_{-\epsilon-2} \cdots b_{-\epsilon-r+1}|0 \rangle. \]  

(2.17)

An alternate basis for class functions is

\[ \prod_{i}(\text{tr } U^i)^{\sigma_i} \]  

(2.18)

(in terms of the \( z_i \) these functions give a basis for the symmetric polynomials) which will correspond to a state built with bosonic operators

\[ |\sigma \rangle = \prod_{i} W_i^{\sigma_i}|0 \rangle. \]  

(2.19)

Orthonormality of the characters gives us

\[ \chi_{\vec{n}}(U)|0 \rangle = |\vec{n} \rangle = \sum_{\sigma} <\vec{n}|\sigma > W_i^{\sigma_i}|0 \rangle. \]  

(2.20)

Another application of this is the integration of class functions, which is just expectation values of products of operators. For example,

\[ \int dU(TrU^{-2})(TrU)^2 = <0| (\alpha_2 + \bar{\alpha}_{-2})(\alpha_{-1} + \bar{\alpha}_1)^2|0 \rangle = 0 \]  

(2.21)
to all orders in $1/N$. (and for finite $N > 2$, by going back to the non-relativistic fermions.)

The above was all for $U(N)$; the $U(1)$ generator is $Q = H_1 = \int d\theta \psi^+ \partial \psi = \sum_n n b^+_n b_n$ which in the large $N$ limit becomes

$$Q = \sum_n n b^+_n b_n - \sum_n n \bar{b}^+_n \bar{b}_n = L_0 - \bar{L}_0$$

(2.22)

(but see below.) Constraining this to zero (keeping integer winding numbers) gives representations of the quotient $SU(N)/\mathbb{Z}_N$. If we are interested in $SU(N)$ we have two options. We can choose a representation of $U(N) \cong SU(N) \times U(1)/\mathbb{Z}_N$ for each representation of $SU(N)$, and subtract the $U(1)$ part of the second Casimir, $H_{U(1)} = Q^2/N$ from the Hamiltonian. Or, we can extend our sum over states to get $SU(N) \times U(1)$. A $U(1)$ character of charge 1 is $\chi_1 = \prod_i z_i^{1/N}$, so the appropriate modification is simply to sum over winding numbers $k/N$, or equivalently fermion sectors with twisted boundary conditions $\psi(e^{2\pi i z}) = e^{2\pi i (1/2+k/N)} \psi(z)$ and $\bar{\psi}(e^{-2\pi i \bar{z}}) = e^{2\pi i (1/2+k/N)} \bar{\psi}(\bar{z})$.

Our quantum mechanical Hamiltonian is the second Casimir, which is not the relativistic Hamiltonian $L_0 + \bar{L}_0$. In terms of relativistic fermions it is

$$H_{U(N)} = \sum_{n \in \mathbb{Z} + \frac{1}{2}} (n_F + \epsilon - n)^2 (b^+_n b_n + \bar{b}^+_n \bar{b}_n) - E_0$$

$$= NL_0 + N\bar{L}_0 + \sum_{n \in \mathbb{Z} + \frac{1}{2}} n^2 : b^+_n b_n + \bar{b}^+_n \bar{b}_n :$$

$$= NL_0 + N\bar{L}_0 + \oint dz \ z^2 : \partial \psi^+ \partial \psi : + \oint d\bar{z} \ \bar{z}^2 : \partial \bar{\psi}^+ \partial \bar{\psi} :$$

(2.23)

(we know that the vacuum energy in this ground state is zero).

The bosonization of this Hamiltonian is very well known in the context of matrix models, as it is just the Das-Jevicki-Sakita Hamiltonian governing the dynamics of the eigenvalue density in hermitian matrix quantum mechanics. We are retracing the steps of Gross and Klebanov and of Wadia and Sengupta [10] to arrive at it. The difference here is that there is no potential, and the fermions live on a circle. We have a left and right moving decomposition of the boson $\phi(z, \bar{z}) = \phi_L(z) + \phi_R(\bar{z})$, and the standard formulas: $e^{i\phi_L(z)} := \psi(z), e^{-i\phi_R(z)} := \psi^+(z)$, etc... Substituting into (2.23), we can define the second derivative term by point-splitting the two operators and taking the limit. The result must be a sum of operators of charge zero and dimension $(3,0)$ and $(0,3)$. In fact the operators $\partial^3 \phi$ or $\partial \phi \partial^2 \phi$ would be unimportant here because they are total derivatives (and there are enough derivatives to kill the winding mode), so the only possibility is (the coefficients are easily checked on low lying states)

$$H = -\frac{N}{2} \oint dz \ z : (\partial \phi)^2 : -\frac{N}{2} \oint d\bar{z} \ \bar{z} : (\partial \phi)^2 :$$

$$+ \frac{i}{3} \oint dz \ z^2 : (\partial \phi)^3 : + \frac{i}{3} \oint d\bar{z} \ \bar{z}^2 : (\partial \phi)^3 :$$

$$= NL_0 + N\bar{L}_0 + H_I.$$
The cubic interaction term in this Hamiltonian is quite natural, as we could see by considering the action of our original (2.1) on states (2.19) – it would contain terms preserving the “string number” (number of traces), as well as terms joining or splitting strings in higher order in $1/N$. $H_I$ is conserved under free time evolution (as are all the $H_m$’s).

Actually there is a slight awkwardness in the bosonic formalism at this point: with our present definitions, (2.22) is not quite correct. The contribution of $w$ to the $U(1)$ charge is $Nw$ and is correctly reproduced by (2.24) only if we take the momentum $p = N$. Although this might sound like a more natural choice, it obscures the large $N$ limit: we will constantly need to expand $\partial \phi = N/2z + O(1)$ to calculate. Rather we take instead $p = 0$ and

$$Q = Nw + L_0 - L_0.$$  \hspace{1cm} (2.25)

This point is important only if we are interested in the operator $Q$; in particular (2.24) is correct with $p = 0$.

All this could be done for a general group manifold. Computing singlet wave functions again leads to the Weyl character formula. For the groups $Sp(2N)$ the maximal torus can be taken to be diagonal matrices diag$(z_i, z_i^{-1})$ and the Weyl group includes both permutations and the reflections $z_i \rightarrow z_i^{-1}$. Although we will not try to develop it here, in the large $N$ limit this should produce a free fermion theory on a surface with boundary. For $SO(N)$ at finite $N$ we would need an additional global degree of freedom to incorporate the spinor representations; however these have $C_2 \sim N^2$ so would drop out of our large $N$ considerations.

3 YM$_2$ on the cylinder and torus

This is really the same quantum mechanics on a group manifold under a different name. Let us do canonical quantization with our space being a circle of radius 1; time evolution will generate a cylinder of area $A = 2\pi t$. The Hamiltonian is $g^2 \int dxtr E^2$, with $E(x)^a = -i\partial/\partial A^a(x)$, and we must impose Gauss’ law $D_xE = 0$, which is solved by gauge invariant wave functions, i.e. satisfying $\psi[g^{-1}(x)(\partial_x + A(x))g(x)] = \psi[A(x)]$. A wave function is determined by its value on configurations of constant $A(x)$, and gauge orbits are in one-to-one correspondence with values of the holonomy $U = P \exp i \int_0^{2\pi} A(x)dx$ modulo the adjoint action $U \rightarrow g^{-1}Ug$, completing the reduction to the singlet sector of quantum mechanics. The standard large $N$ limit is taken with gauge coupling $g^2 \sim 1/N$ and in two dimensions we can set $g^2 = 1/N$, defining our unit of length. Then time evolution is generated by an $O(N^0)$ free Hamiltonian with an $O(1/N)$ interaction term. The ground state energy $E_0$ is freely adjustable, say by adding $E_0 \int d^2x \sqrt{g}$ to our original Lagrangian.

The simplest physical quantity is the partition function on the torus,

$$\text{Tr} \ e^{-2\pi iH} = Z_{1-1} + O(N^{-2}).$$  \hspace{1cm} (3.1)

\footnote{See \cite{18} for a complete elaboration of this.}
The leading term is $O(N^0)$ and the notation “$1 \to 1$” indicates that the string interpretation of this is a sum of (disconnected) maps from genus one world-sheets (at $N^0$) to a genus one target space.

Since the gauge invariant states correspond directly to our conformal field theory states, and the interaction is subleading, $Z_{1\to1}$ is almost the standard torus partition function of free $c=1$ conformal field theory. The “almost” is there because the total charge of our fermi theory or momentum zero mode $\alpha_0 + \bar{\alpha}_0$ of our bose theory is conserved; thus we have a partition function with this constraint. This is particularly easy to implement in the bose description; clearly

$$Z_{1\to1} = q^{E_0} \prod_{n \geq 1} \frac{1}{(1 - q^n)^2} \sum_{w \in \mathbb{Z}} q^{w^2}.$$  \hfill (3.2)

where $q \equiv e^{-2\pi t}$. The sum over winding modes is a rather uninteresting side effect of the $U(1)$ factor. The nicest way to eliminate it is to decouple the $SU(N)$ and $U(1)$ in the way described above, by summing over winding numbers $w = k/N$. In the large $N$ limit we clearly want to interpret such a sum as an integral; it is Gaussian, giving

$$Z'_{1\to1} = \frac{N^2}{\sqrt{2\pi t}} q^{E_0} \prod_{n \geq 1} \frac{1}{(1 - q^n)^2}.$$  \hfill (3.3)

Another way of saying this is, we have an $SU(N) \times U(1)$ gauge theory with the same coupling constant in both sectors. Using the coupling constant which gives a nice large $N$ limit, $g^2/N$, gives the extreme weak coupling limit in the $U(1)$ sector. In this limit we cannot see the compact nature of the group $U(1)$.

Subleading corrections to this will have a string interpretation in terms of maps from higher genus world-sheets. We can write an all-orders expression quite explicitly from the free fermion formalism: for $U(N)$,

$$Z_{\text{all} \to 1} = \sum_R e^{-AC_2(R)/N}$$  \hfill (3.4)

$$= q^{E_0} \oint \frac{dz}{z} \left[ \prod_{m \geq 1} (1 + z q^{m-1/2+(m-1)/2}/N) \prod_{n \geq 1} (1 + z^{-1} q^{n-1/2-(n-1)/2}/N) \right]^2.$$  \hfill (3.6)

The contour integral is there to implement the constraint of zero total charge. It complicates the interpretation so again it is useful to do a bosonic calculation. From both the CFT and string points of view, the free energy is a sum over connected diagrams. Expanding $\exp \left( -\frac{A}{N} H_f \right)$ we have the series

$$F_1(A) = F_{1\to1} + \sum_{g \geq 2} N^{2-2g} F_{g\to1}$$  \hfill (3.5)

with

$$F_{g\to1}(A) = \frac{1}{(2g-2)!} \langle \left( \frac{iA}{3} \oint dz \ z^2 : \partial \phi(z)^3 : + \text{c.c.} \right)^{2g-2} \rangle_c.$$  \hfill (3.6)
This is a connected correlation function on the torus (an annulus with \( z \) and \( qz \) identified). The integrals are taken over contours of constant \(|z|\), and since \( H_I \) is conserved, we can take \(|z|\) to be slightly different for each contour, avoiding any possible singularities. The Green’s function (defined by the original oscillator expansion) will be the usual one [15, p. 571] if we use the same prescription as in (3.3) of integrating the boson winding mode:

\[
G(\nu_1, \nu_2) = \langle \partial_\nu \phi(\nu_1) \partial_\nu \phi(\nu_2) \rangle = \partial_1^2 \log \theta_1(\nu_1 - \nu_2|\tau) + \frac{\pi}{t} \tag{3.7}
\]

\[
= -\phi(\nu_1 - \nu_2|\tau) - \frac{\pi^2}{3} E_2(\tau) + \frac{\pi}{t},
\]

\[ z = e^{2\pi i\nu}, \tau = it \] and the Weierstrass function and Eisenstein series are defined in [13]. The \( \langle \partial \phi \bar{\partial} \phi \rangle \) propagator is the same with \( \nu \rightarrow \bar{\nu} \). Also

\[
\langle \partial \phi \bar{\partial} \phi \rangle = -\frac{\pi}{t}. \tag{3.8}
\]

If we want a group other than \( SU(N) \times U(1) \), the zero mode contribution \( \pi/t \) will be modified.

The first correction will be at \( 1/N^2 \) from two insertions of our interaction Hamiltonian. In changing variables from \( z \) to \( \nu \) we should remember that the normal ordering of (2.24) was defined with respect to the \( z \) coordinate. Taking this into account however gives a contribution proportional to the momentum, in other words zero. The sum of terms involving contractions of a pair of operators from the same appearance of \( H_I \) (in [3], contributions which can be disconnected by cutting a “tube”) vanish (for \( \text{Re} \ \tau = 0 \)). So,

\[
F_{2 \rightarrow 1} = \frac{2A^2}{3(2\pi)^6} \int_0^1 d\nu \ G(\nu, 0)^3. \tag{3.9}
\]

The contour integrals are in the appendix, giving

\[
F_{2 \rightarrow 1} = \frac{A^2}{2^5 \cdot 3^4 \cdot 5} \Bigl( 10E_2^3 - 6E_2E_4 - 4E_6 \Bigr) + \frac{A}{2^4 \cdot 9} \Bigl( E_4 - E_2^2 \Bigr) \tag{3.10}
\]

\[
= A^2(8q^2 + 64q^3 + \ldots) + A(2q + 12q^2 + \ldots).
\]

In the interpretation of [3] this is the generating function counting maps without folds from a genus two surface to a torus. These can have two branch points (the \( A^2 \) term) or a “handle” (the \( O(A) \) term). Since the \( U(1) \) piece does not contribute at subleading orders in \( 1/N \), this is exactly the \( SU(N) \) result. One can also see this by expanding \( \partial \phi = w/z + \ldots \) and integrating out \( w \), which produces the correction to \( H_I \) appropriate for \( SU(N) \).

As for \( F_{1 \rightarrow 1} \), the most striking thing about this answer is how close it is to being a modular form (here of weight 6) in the variable \( \tau \). The Eisenstein series \( E_k \) for \( k \geq 4 \)
are forms of weight $k$, and while $E_2$ is not a form it has a very simple anomaly in its transformation law:

$$E_k(-1/τ) = \tau^k E_k(τ), \quad k ≥ 4 \quad (3.11)$$
$$E_2(-1/τ) = \tau^2(E_2(τ) + 12/2πiτ). \quad (3.12)$$

There is no analog of this at finite $N$; it is a non-trivial consequence of the quasi-relativistic nature of the degrees of freedom in the large $N$ limit. Indeed, in terms of the original (unrescaled) couplings this is the transformation $g^2 A/N \rightarrow N/g^2 A$, very different from familiar strong-weak coupling duality. It is a general property of the torus answers; from (3.6) we see that $F \sim 1/τ^2 (g − 1)$ will “almost” be a modular form of weight $6(g − 1)$.

It is very tempting to look for a string theory interpretation of this. It is a target space duality invariance, like the $R \rightarrow 1/R$ of the free compact boson CFT. In fact if the world-sheet embedding was described by a free (complex) boson we would expect precisely this symmetry. Furthermore, if we found that the (world-sheet) genus $g$ free energy was precisely a modular form with weight proportional to $g − 1$, we could define a combined transformation on area and string coupling which left the total free energy invariant, just as was the case for the compactified $c = 1$ fundamental string.

We should ask whether by changing definitions or modifying YM_2 slightly we could get a truly modular covariant answer. From (3.3), it seems most promising to consider the $SU(N) \times U(1)$ case, though we have no deep understanding of why this choice of group should be better than $SU(N)$, say. Adding a sum over momenta to complement the sum over integer windings appropriate for $U(N)$ does not seem promising, because the Hamiltonian has a term $p^2$, which would be unbounded below.

We would then like to make two changes: first, extend $τ$ to a complex parameter; second, extend the contour integrals in (3.11) to integrals $∫ d^2 ν$. Ways to accomplish the first have been proposed by several physicists. Since the area controls $L_0 + \bar{L}_0$, we need to combine it with a parameter which controls $L_0 - \bar{L}_0$. Now in $D = 2$ there is a theta term for $U(1)$ but not for $SU(N)$. For the $U(N)$ theory we could just add the $U(1)$ theta term, which would weigh the contribution of charge $q$ in the sum over representations by $e^{iqθ}$. Modding out by $Z_N$ correlates this with the conjugacy class in $SU(N)$, which in the large $N$ limit is just $L_0 - \bar{L}_0$. However this does not work for $SU(N) \times U(1)$, where there is no correlation. Instead we could consider twisted boundary conditions for the gauge field on the original torus. A twist by a group element $C$ identifies holonomy $U$ at time $t$ with holonomy $CU$ at time 0; gauge invariance $ψ(U) = ψ(gUg^{-1})$ requires that $C$ be in the center, so there are $N$ possible twists $θ = 0, 1/N, \ldots$ which act as $exp 2πiθ(L_0 - \bar{L}_0)$. In the large $N$ limit we can consider $τ = it + θ$ to be a continuous variable.

The other modification seems necessary if we want a modular covariant answer, but so far we have no real justification for doing this from the YM_2 point of view. Naively we would say that since $H_I$ is conserved we have $A∫ dνH_I = ∫ d^2 νH_I$ but since (3.10) we

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\(^4\) The following points were developed in discussions with D. Gross and C. Vafa.

\(^5\) See [8] for a more precise version of this.
is not modular covariant there must be a subtlety. It is that we ignored short distance
singularities. Integrating a meromorphic function \( \int d^2 \nu f(\nu) \) will not produce divergences
but we might get a finite piece which fixes things up. In fact, for \( f(\nu) \) with a singularity
only at zero, we can write

\[
\frac{1}{2i} \int d\nu f(\nu) \wedge d(\bar{\nu} - \nu) = \text{Im} \int_0^1 d\nu f(\nu) + \pi \text{Res} f(\nu) \big|_{\nu=0}
\]

(3.13)

and with this term

\[
F^{\text{inv}}_{2\to 1} = \frac{A^2}{2^5 \cdot 3^4 \cdot 5} \text{Re} \left( 10\hat{E}_2^3 - 6\hat{E}_2 E_4 - 4E_6 \right)
\]

(3.14)

where \( \hat{E}_2(\tau) = E_2(\tau) - 3/\pi \text{Im} \tau \) is modular covariant. The diagrams mentioned just above
(3.9) now vanish. This is different from (3.10) at \( O(A) \) and now there are also terms at
\( O(A^0) \) and \( O(A^{-1}) \) whose string interpretation is unclear. Furthermore, while the “chiral”
and “antichiral” parts are separately modular covariant, the combination appearing here
is a bit strange.

We could certainly imagine other modifications of (3.10) to get modular covariance,
but this modification does generalize to all genus and seems relatively natural. A modular
covariant answer is a prerequisite for comparison with topological string theory. There is
a striking similarity between the form of the collective field theory (2.24) and the Kodaira-
Spencer field theory developed in [21], describing topological string theory on a Calabi-Yau
target space, which may be an important clue to the continuum string interpretation. [22]

4 YM\(_2\) on other Riemann surfaces

Before discussing the formalism let us review some known results for the large \( N \) partition
function on other topologies. The qualitative structure is quite clear from the expression
as a sum over representations,

\[
\exp F_{\text{all} \to G} = \sum_R (\dim R)^{2-2G} \exp -\frac{A}{N} C_2(R),
\]

(4.1)

and the leading order behavior \( \dim R \sim N^{n+\bar{n}} \) for a representation made from \( n \) and \( \bar{n} \)
(anti)fundamentals.

For \( G > 1 \), at order \( 1/N^{2s} \) only the finite set of representations with \( n + \bar{n} \leq s \) can contribute. Although there are interesting things to say about a string theory which
reproduces these answers, since no observable exhibits a sum over an infinite number of
degrees of freedom, there is no evidence for any non-topological field theory description.
In the present framework we will see that although formally we can build higher genus
surfaces by sewing cylinders, the “vertex” which accomplishes this is highly non-local.

For \( G = 0 \), there is an \( O(N^2) \) contribution to \( F \). This is in many ways the most
interesting case, because the leading behavior of the free energy for higher dimensional
Yang-Mills theory is $O(N^2)$. It is quite possible that the qualitative behavior of the two-dimensional problem is of direct relevance for higher dimensions, because any higher dimensional space (and certainly a lattice as well) will contain embedded two-spheres.

Computing an $O(N^2)$ free energy in large $N$ is quite different from the problems we treated above, because we can hope to find a saddle point which dominates the path integral. Of course this is usually the reason we think that a large $N$ limit will simplify a problem; however it brings with it a complication: there can be more than one saddle point, and a phase transition where their actions are equal.

In terms of our present formalism we would expect to describe the saddle point in terms of an expectation value for the boson $\phi$, determined by minimizing the Jevicki-Sakita action (2.24). Although this makes sense it is much easier to describe the saddle point in terms of the conjugate variables $n_i$, since these are time-independent. A new feature of the problem is that since these are discrete, there is an upper bound on their density, and an associated phase transition when the bound is saturated.

To return to our Hamiltonian formalism, the problem in this section is to find the wave functions for the disk and the three-holed sphere. We might expect to be able to represent them as simple conformal field theory states, as is done in string field theory. Let us start with the disk, or its zero-area limit, in other words the wave function $\psi = \delta(U)$. Clearly acting on this state $\text{Tr} U^n = N \ \forall n \neq 0$, or

$$ (\alpha_n + \bar{\alpha}_{-n} - N) |D_0> = 0. \quad (4.2) $$

These constraints are easily solved:

$$ |D_0> = \exp \sum_{n \geq 1} \frac{1}{n} (\alpha_{-n} \bar{\alpha}_{-n} - N\alpha_{-n} - N\bar{\alpha}_{-n}) |0>. \quad (4.3) $$

This state can be used to calculate the dimension of a representation:

$$ \text{dim } R_\vec{n} = <D_0|\vec{n}>. \quad (4.4) $$

(try for example the characters $\frac{1}{2}((\text{Tr } U)^2 + \text{Tr } U^2)$.)

Another characterization of this state is through boundary conditions of the fields – we put a boundary $\tau = 0$ with $\partial \phi / \partial \tau = N\delta(\theta)$. This is equivalent to taking Neumann boundary conditions, and inserting the operator $: \exp iN\phi(0) :$. In the fermi language this is a rather singular state, which is defined by the boundary condition that all the fermions (eigenvalues) are at $z = 1$. In the non-relativistic formalism, we can produce it by taking the o.p.e. of $N$ fermions, resulting in the state $\prod_{i=0}^{N-1} \partial^i \psi |0>$. Taking the inner product of (for example) a character with this state will reproduce the calculation of the dimension of a representation by taking the limit of all $z_i \to 1$ in the Weyl character formula using l’Hôpital’s rule.

The $YM_2$ sphere free energy is then

$$ \exp F_{all-0} = <D_0|e^{-AH}|D_0> = <:e^{-iN\phi(\tau)} : e^{iN\phi(0)} :>_{\text{cyl}} \quad (4.5) $$
a correlation function on a cylinder of length \( \tau \). We see that we cannot expand \( \exp \frac{i}{N} H_I \); the \( 1/N \) is compensated in correlation functions by the explicit \( N \) in the vertex operators. On the other hand the problem is classical; we can rescale \( \phi \rightarrow N\phi \) and get an overall \( N^2 \) in the exponential.

The qualitative behavior of the dominant classical solution is clear. We have a fluid with a conserved particle number; the boundary condition is that at time \( t = 0 \) it is concentrated at \( \theta = 0 \); as it evolves it spreads out but must recontract from \( t = \tau/2 \) on to meet the boundary condition at \( t = \tau \). This is not the easiest way to solve the problem, which is treated more simply in \([23]\), but it does make the nature of the large \( N \) transition clear: at a critical \( \tau_c \), the two edges of the particle distribution meet at \( t = \tau_c/2 \) and \( \theta = \pi \). For \( \tau \leq \tau_c \) they do not know they live on a circle; beyond this value it is crucial, and the classical solution is a non-analytic function of \( \tau \) at this point. (See also \([24]\).)

The expansion of \([6]\) reproduces the \( \tau \geq \tau_c \) behavior but has no apparent relation with the \( \tau < \tau_c \) behavior. Putting back the \( g^2 \) dependence shows that it is valid at strong coupling. Now the main reason for the transition was simply the compactness of the gauge group, just as for the Gross-Witten transition \([25]\), which suggests that higher dimensional theories will have the same problem. This is not just a property of an approximation but of the true continuum result, and the dynamical picture above suggests the strange possibility that in higher dimensions we could have non-analyticity in the local observables at large \( N \), for example in the dependence of a Wilson loop expectation value on its length. (This is not the case in \( D = 2 \), see \([26]\).)

To try to understand the effects of the various world-sheet features of \([4]\), we could ask how modifying them or removing them affects the answers. Comparison with the derivation there shows that the \( \Omega \)-points of \([6]\) are represented here by the states \( |D_0> \) while the “movable branch points” (those which come with a weight proportional to the area of the world-sheet) correspond to insertions of the interaction \( H_I \) (as we saw above for the torus target space). On the sphere we get an interesting result even after dropping the movable branch points. \([6]\) Since we could accomplish this by multiplying \( H_I \) by a new coupling constant and taking it to zero, the resulting theory is in a sense continuously connected with the standard \( \text{YM}_2 \). In fact this modification corresponds to a local modification of the original \( \text{YM}_2 \) action. \([27]\) Thus it is not inconceivable that the corresponding modification of a \( D > 2 \)-dimensional string theory would be in the same universality class as \( \text{YM}_D \). We might take this as a working hypothesis in interpreting string rules like those of \([1, 2]\): features we cannot change by a local modification of the original Yang-Mills action (like the \( \Omega \) points) are more likely to be relevant for physics than features (like the movable branch points) that can be so changed.

What makes it all the more interesting is that in \( D = 2 \) this modification pushes the transition to zero coupling! For simplicity drop the winding modes of the boson, so this result exactly enumerates covers of the sphere branched only at the \( \Omega \) points. Correlation functions on the cylinder with Neumann boundary conditions are equal to those of a chiral

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\*6This modification has also been considered by D. Gross.
boson on the doubled surface, here a torus of modulus $2\tau$. Then
\[
\exp F_{\text{all}}^{\text{modified}} = \langle e^{-iN\phi(\tau)} : e^{iN\phi(0)} : \rangle_{\text{cyl}}
\]
\[
= \frac{1}{\eta(2\tau)} \exp N^2 \left[ \langle \phi(\tau)\phi(0) \rangle - \lim_{z \to 0} \langle \phi(z)\phi(0) \rangle + \log z \right]
\]
\[
= \frac{1}{\eta(2\tau)} \exp -N^2 \log \frac{\theta_1(\tau|2\tau)}{\theta_1'(0|2\tau)}
\]
\[
= \prod_{m \geq 1} \frac{1}{(1 - q^{2m})} \left( \frac{1 - q^{2m}}{1 - q^{2m-1}} \right)^{2N^2}.
\]

This is an analytic function for all real $\tau > 0$, with an expansion in $\exp -\tau$ whose terms have a string representation, and whose $\tau \to 0$ behavior is remarkably similar to the "correct" (heat kernel) weak coupling behavior $F \sim -\frac{1}{2} N^2 \log \tau$. Whether this is of any relevance to $D > 2$ is unclear, and the action which defines this theory is quite peculiar, but this suggests that the heat kernel action might not be the last chapter in the story.

We briefly return to $G > 1$. We can build higher dimensional surfaces by sewing, and the appropriate "pants" vertex is well known in the character language:

\[
\langle V_3 | \psi_1 > | \psi_2 > | \psi_3 >= \int dU dV dW \psi_1(UVU^{-1}W)\psi_2(V^{-1})\psi_3(W^{-1})
\]

\[
|V_3 >= \sum_R \dim R \langle \chi_R > 1 | \chi_R > 2 | \chi_R > 3.
\]

Since this expression involves group multiplication, a Ward identity analogous to (4.2) would be highly non-local in our two-dimensional auxiliary space. The result is clearly non-local at each order in $1/N$ as well; it is a sum over finitely many states which separately have no local definition. All this and consideration of what sort of vertex could satisfy the equation $\langle D_0 | V_3 >= 1$ leads to the conclusion that there is not likely to be an expression for the vertex much simpler than (4.8).

A final comment, which we will not apply here (but see [28]). Clearly there are many other group theoretic ideas we could try to fit into this formalism. One interesting one is multiplication of characters, which allows computing tensor product decompositions. Evidently it is much easier to multiply symmetric functions expressed in the basis of power sums (2.19), so this will be a simpler operation in the bosonic language. Since we are multiplying wave functions at the same point on the group manifold, we expect this to be a local operation in our two dimensional CFT language. In fact we can write multiplication of $n$ wave functions as a vertex $|V_n>$, determined by the condition that

\[
\langle \psi_1 | \langle \psi_2 | \ldots \text{Tr } U^k_i | V_n > = \int dU \text{ Tr } U^k \prod_i \psi_i(U)
\]

is the same no matter which wave function we multiply by $\text{Tr } U^k$. In conformal field theory terms this means that we have a boundary on which $n$ cylinders meet, and the local boundary condition

\[
\partial_\tau \phi_i(z) = \partial_\tau \phi_j(z) \quad \forall i, j.
\]
This condition only couples mode $n$ to $-n$ and is easy to translate into an oscillator expression for the vertex

$$|V_n \rangle = \exp \sum_{i \neq j} \sum_{n \geq 1} \frac{1}{n} \alpha^{(i)}_n \alpha^{(j)}_{-n} |0 \rangle .$$

(4.11)

In a certain sense this is a trivial result; however it would be quite amusing to translate it into the fermionic language, because it would amount to a new proof of the Littlewood-Richardson rule for tensor product decompositions, at least in the large $N$ limit.

Some contour integrals used in section 3: [19]

$$\int_0^1 d\nu \frac{\partial^2}{\partial \nu} \log \theta_1(\nu) = 0$$

$$\int_0^1 d\nu \varphi(\nu)^2 = \frac{\pi^4}{9} E_4$$

$$\int_0^1 d\nu \varphi(\nu)^3 = \frac{4\pi^6}{5 \cdot 27} E_6 - \frac{\pi^6}{15} E_4 E_2$$

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