ON CERTAIN INTEGRAL TENSOR CATEGORIES AND INTEGRAL TQFTS

QI CHEN

Abstract. We construct certain tensor categories that are dominated by finitely many simple objects. Objects in these categories are modules over rings of algebra integers. We show how to obtain TQFTs defined over algebra integers from these categories.

1. Introduction

The goal of this note is twofold: (1) to give certain tensor categories dominated by finitely many simple objects (cf. section 2.1), (2) to construct topological quantum field theories (TQFTs) whose ground rings are algebraic integers (or integral TQFTs for short) (cf. section 2.4). We use representation theory of quantum groups in the construction. Results in this note are related to those in [A], [GK] and [G].

We fix a complex simple Lie algebra \( \mathfrak{g} \), an integer \( r \) and a primitive \( r \)-th root of unity \( \xi \) throughout this note.

Let \( U_v = U_v(\mathfrak{g}) \) be the quantum group associated to \( \mathfrak{g} \). It is well known that \( U_v \) is a Hopf algebra over the ring \( \mathbb{Q}(v) \) where \( v \) is a formal parameter. Hence the category \( V_v \) of finite-dimensional \( U_v \)-modules of type 1 is a tensor category (cf. section 2.3). It turns out that \( V_v \) is equivalent to the category of finite-dimensional \( \mathfrak{g} \)-modules (cf. 6.3 of [Lu]).

Let \( A = \mathbb{Z}[v, v^{-1}] \). Lusztig showed that \( U_v \) has an \( A \)-subalgebra \( U_A \) which inherits the Hopf algebra structure from \( U_v \). Let \( U_C = U_A \otimes_A \mathbb{C} \) (resp. \( U_\xi = U_A \otimes_A \mathbb{Z}[\xi] \)) where \( \mathbb{C} \) (resp. \( \mathbb{Z}[\xi] \)) is considered as an \( A \)-algebra by sending \( v \) to \( \xi \). The category \( V_C \) (resp. \( V_\xi \)) of finite-dimensional \( U_C \)-modules (resp. finite-ranked \( U_\xi \)-modules) of type 1 with the ground ring \( \mathbb{C} \) (resp. \( \mathbb{Z}[\xi] \)) behaves very differently from \( V_v \).

If \( r \) is prime to the non-zero entries of the Cartan matrix for \( \mathfrak{g} \) and is bigger than the Coxeter number, then Andersen proved in [A] that (in the language of [K]) there is a full subcategory \( \mathcal{V}_C \) of \( \mathcal{V}_v \) which is a quotient category dominated by finitely many simple objects. (This is proved in [GK] when \( r \) is prime and big enough.) Objects in these categories are vector spaces over \( \mathbb{C} \). The following proposition is our first main result.

Proposition 1.1. If \( \mathfrak{g} \) is not of type \( E_8, F_4 \) or \( G_2 \) and \( r \) is a prime bigger than \( m(\mathfrak{g}) \) then there exists a subcategory \( \mathcal{V}_C \) of \( \mathcal{V}_v \) which has a quotient category \( \mathcal{V}_C' \) dominated by finitely many simple objects.

See section 4 for \( m(\mathfrak{g}) \) and the construction for the categories. Note that objects in \( \mathcal{V}_C' \) are \( \mathbb{Z}[\xi] \)-modules of finite rank. This is essential to our construction of integral TQFTs. In the construction we have to restrict the allowed 3-cobordisms (cf. section 6.3). TQFTs for such restricted 3-cobordisms will be denoted by \( (\mathcal{S}, \tau) \) (as opposed to \( (\mathcal{F}, \tau) \) for the non-restricted ones).
Proposition 1.2. If \( r \) and \( g \) are as in proposition 1.1 then there exists a TQFT \((S_+, \tau)\) such that the images of \( S_+ \) are free \( \mathbb{Z}[\xi] \)-modules of finite rank.

This proposition is contained in proposition 6.4. It provides us infinitely many mapping class group representations over \( \mathbb{Z}[\xi] \), which may shed lights on the structure of mapping class groups.

The note is organized as follows. Basic knowledge is recalled in section 2. Sections 3 and 4 deal with representations of quantum groups at generic parameter and root of unity respectively. Proof of proposition 1.4 occupies section 5. Integral TQFTs are constructed in section 6.

2. Preliminaries

The material in this section is quite standard and may be skipped.

2.1. Tensor categories. A good reference for this subsection is [Ka]. A category \( \mathcal{C} \) is a tensor category if

(a) each hom-set \( \text{Hom}_\mathcal{C}(U, V) \) is an additive abelian group and composition of morphisms is bilinear relative to this addition,
(b) \( \mathcal{C} \) has biproduct (or direct sum) \( U \oplus V \) for any two objects \( U \) and \( V \),
(c) there is a bifunctor \( \otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C} \) which is associative up to a natural isomorphism, called the associativity constraint,
\[ \alpha_{U,V,W}: U \otimes (V \otimes W) \to (U \otimes V) \otimes W, \]
(d) \( \mathcal{C} \) has an object \( 1_\mathcal{C} \), unique up to natural isomorphism, and two natural isomorphisms, called the left and the right unit constraint,
\[ \lambda_U: 1_\mathcal{C} \otimes U \to U, \quad \rho_U: U \otimes 1_\mathcal{C} \to U, \]
(e) associativity, left unit and right unit constraints satisfy the pentagon and the triangle axioms.

Our definition of a tensor category is different from the usual one, which assumes neither (a) nor (b). Let \( \mathcal{C} \) be a tensor category. We say \( \mathcal{C} \) is dominated by a set of objects \( A \) if every object is a direct sum of objects from \( A \). Such a set is called a dominant set of \( \mathcal{C} \). The ground ring of \( \mathcal{C} \) is \( k_\mathcal{C} = \text{Hom}_\mathcal{C}(1_\mathcal{C}, 1_\mathcal{C}) \). An object \( U \) in \( \mathcal{C} \) is simple if \( \text{Hom}_\mathcal{C}(U, U) = k_\mathcal{C} \). If \( \mathcal{C} \) is dominated by the set of simple objects then it is said to be semisimple.

A tensor category is said to be strict if its associativity, left unit and right unit constraints are all identities. By Mac Lane’s coherence theorem, every tensor category is equivalent to a strict one. Therefore we assume all tensor categories in this note to be strict. A tensor category \( \mathcal{C} \) is said to have duality if

(f) there is a contravariant functor \( *: \mathcal{C} \to \mathcal{C} \). (For any object \( U \) and any morphism \( f \) we denote \( *U \) and \( *f \) by \( U^* \) and \( f^* \) respectively.)
(g) for every object \( U \) there are two morphisms, called the coevaluation and the evaluation respectively,
\[ b_U: 1_\mathcal{C} \to U \otimes U^*, \quad d_U: U^* \otimes U \to 1_\mathcal{C}, \]
such that
\[ (\text{Id}_U \otimes d_U)(b_U \otimes \text{Id}_U) = \text{Id}_U, \quad (d_U \otimes \text{Id}_{U^*})(\text{Id}_{U^*} \otimes b_U) = \text{Id}_{U^*}. \]
Let $\mathcal{C}$ be a tensor category and $P : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$ be the flip functor defined by $P(U,V) = (V,U)$. Then $\mathcal{C}$ is said to be braided if there is a natural isomorphism $c$ between $\otimes$ and $\otimes P$ which satisfies the hexagon axiom. Denote $c_{U,V} : U \otimes V \rightarrow V \otimes U$ for the braiding between $U$ and $V$.

Let $\mathcal{C}$ be a braided tensor category with duality. A twist is a natural transformation $\theta$ from the identity functor of $\mathcal{C}$ to itself such that $\theta_{V \otimes W} = (\theta_V \otimes \theta_W)c_{U,V}c_{V,U}$ and $\theta_{V^*} = \theta_V^*$, for all objects $V,W$ in $\mathcal{C}$. A braided tensor category with duality is called a ribbon category if it has a twist.

Let $\mathcal{C}$ be a ribbon category with a finite dominant set $A$. Set $s_{U,V} = \text{tr}_q(c_{U,V}c_{V,U})$ for every pair $U,V \in A$. Here

$$\text{tr}_q(f) = \text{tr}(d_Wc_{W,W^*}(f \theta_W \otimes \text{Id})b_W) \in k_{\mathcal{C}}$$

is the quantum trace of $f : W \rightarrow W$. The matrix $(s_{U,V})_{U,V \in A}$ is called the $S$-matrix of $\mathcal{C}$ with respect to the dominant set $A$ and is denoted by $S(\mathcal{C},A)$ or simply $S(\mathcal{C})$ if $A$ is clear from the context.

A ribbon category $\mathcal{C}$ is said to be a modular category if

- it has a finite dominant set $A$,
- $A$ contains the identity object $1_\mathcal{C}$ and is closed under duality,
- $S(\mathcal{C},A)$ is invertible over $k_{\mathcal{C}}$.

### 2.2. Lie algebras

Let $(a_{ij})$, $i,j = 1, \ldots, \ell$, be the Cartan matrix and $\Phi$ be the root system of $\mathfrak{g}$. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a set of simple roots $\Pi_\mathfrak{h} = \{\alpha_1, \ldots, \alpha_\ell\}$ in its dual space $\mathfrak{h}^*$. We consider $\Pi_\mathfrak{h}$ as a subset of $\Phi$. Let $X$ and $Y$ be the weight lattice and the root lattice of $\mathfrak{g}$. The order of the group $X/Y$ is $\det(a_{ij})$. Let $X_+$ be the set of dominant weights. For any $\alpha, \beta \in X$, we say $\alpha > \beta$ if $\alpha - \beta \in \mathbb{Z}_+\Pi_\mathfrak{h}$. This defines a partial ordering on $X$. Let $\Phi_+ = \{\alpha \in \Phi \mid \alpha > 0\}$ be the set of positive roots.

One can define a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}^*$ in the following way. Multiply the $i$-th row of $(a_{ij})$ by $d_i \in \{1, 2, 3\}$ such that $(d_ia_{ij})$ is a symmetric matrix with $d_i = 1, i = 1, \ldots, \ell$ or $|\{d_i\}| = 2$. Set $(\alpha_i|\alpha_j) = d_ia_{ij}$. This bilinear form is non-degenerate and is proportional to the dual of the Killing form restricted on $\mathfrak{h}$. Set $d = \max_{1 \leq i \leq \ell}(d_i)$. Let $\alpha_0$ be the highest short root with respect to $\Pi_\mathfrak{h}$. The Coxeter number $h = (\rho|\alpha_0) + 1$ where $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$. Then we have $h = \ell + 1, 2\ell, 2\ell, 2\ell - 2, 12, 18, 30, 12, 6$ when $\mathfrak{g} = A_{\ell}, B_{\ell}, C_{\ell}, D_{\ell}, E_6, E_7, E_8, F_4, G_2$ respectively.

The fundamental dominant weights $\lambda_i$, $i = 1, \ldots, \ell$, are defined by $2(\lambda_i|\alpha_j)/(\alpha_j|\alpha_j) = \delta_{ij}$. Here $\delta_{ij}$ is the Kronecker symbol. Then $X$ and $X_+$ are $\mathbb{Z}$-span and $\mathbb{Z}_+$-span of $\lambda_i^\prime$s.

Let $\mathcal{V}_\mathfrak{g}$ be the category of finite-dimensional $\mathfrak{g}$-modules over $\mathbb{C}$. Then $\mathcal{V}_\mathfrak{g}$ is a semisimple tensor category whose simple objects are parameterized by the dominant weights, i.e., for each dominant weight $\lambda$ there exists a unique simple $\mathfrak{g}$-module $V_\lambda$ of highest weight $\lambda$ and all simple objects in $\mathcal{V}_\mathfrak{g}$ are of this form.

The Weyl group $W$ acts on $X$ and $Y$ naturally. For each integer $m > 0$, the affine Weyl group $W_m$ is a group of linear transformations of $\mathfrak{h}^*$ generated by the Weyl group $W$ and the reflection $s_m$ along the hyperplane $F_m = \{x \in \mathfrak{h}^* \mid (x|\alpha_0) = m\}$. There is another $W_m$ action on $\mathfrak{h}^*$, called the dot action, defined by $w.x = w(x + \rho) - \rho$ where the action on the right-hand side is the natural one. Set

$$C_m = \{x \in \mathfrak{h}^* \mid (x + \rho|\alpha_0) < m, 0 < (x + \rho|\alpha_i) \text{ for } i = 1, \ldots, \ell\}.$$
A fundamental domain of the dot action of $W_m$ on $\mathfrak{h}^*$ is $\bar{C}_m$, the topological closure of $C_m$. It is well known that $W_m = W \ltimes mY$.

2.3. Quantum groups. Notation in this subsection is consistent with that of [J] except that we substitute $q$ there by $v$. Recall that the quantum integer

$$[n]_i = \frac{v^n_i - v^{-n}_i}{v_i - v_i^{-1}}$$

for $n \in \mathbb{Z}$ where $v_i = v^{a_i}$, the quantum factorial $[n]_i^j = \prod_{k=1}^{n} [k]_i$ for $n \geq 0$, and the quantum binomial coefficient

$$\begin{bmatrix} a \\ b \end{bmatrix}_i = [a]_i[a - 1]_i \cdots [a - b + 1]_i/[b]_i$$

with $a, b \in \mathbb{Z}$ and $b \geq 0$ (with the convention that $[a]_0 = 1$).

The quantum group $U_v = U_v(\mathfrak{g})$ associated to $\mathfrak{g}$ is a non-commutative $\mathbb{Q}(v)$-algebra generated by $E_i, F_i, K_i$ and $K_i^{-1}$ (for $i = 1, \ldots, \ell$, $K_i K_i^{-1} = 1 = K_i^{-1} K_i$, $K_i E_i K_i^{-1} = v^{(\alpha_i | \alpha_i)} E_i$, $K_i F_i K_i^{-1} = v^{-(\alpha_i | \alpha_i)} F_i$, $E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{v_i - v_i^{-1}}$ and if $i \neq j$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_i E_i^{1-a_{ij}-k} E_j E_i^k = 0,$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_i F_i^{1-a_{ij}-k} F_j F_i^k = 0.$$}

For $\alpha = \sum k_i \alpha_i \in Y$, we denote $\prod_i K_i^{k_i}$ by $K_\alpha$.

A $U_v$-module $M$ is said to have type 1 if it admits a weight space decomposition

$$M = \bigoplus_{\lambda \in X} M^\lambda$$

where $M^\lambda = \{x \in M \mid K_\alpha(x) = v^{(\alpha | \lambda)} x\}$. If $M^\lambda \neq 0$ then $\lambda$ is called a weight of $M$. A type 1 $U_v$-module is said to have highest weight $\lambda$ if all of its weights are less than $\lambda$, with respect to the partial ordering on $X$ defined in section 2.2.

One can put a Hopf algebra structure on $U_v$ with coproduct $\Delta_{U_v}$, antipode $S_{U_v}$, and counit $\epsilon_{U_v}$ defined by

$$\Delta_{U_v}(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \epsilon_{U_v}(E_i) = 0, \quad S_{U_v}(E_i) = -K_i^{-1} E_i,$$

$$\Delta_{U_v}(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \epsilon_{U_v}(F_i) = 0, \quad S_{U_v}(F_i) = -F_i K_i,$$

$$\Delta_{U_v}(K_i) = K_i \otimes K_i, \quad \epsilon_{U_v}(K_i) = 1, \quad S_{U_v}(K_i) = K_i^{-1}.$$}

Therefore the category $\mathcal{V}_v$ of finite-dimensional $U_v$-modules of type 1 is a tensor category with duality. The tensor product and the duality are the usual ones. The unit object $1 = 1_{\mathcal{V}_v}$.
is the one dimensional representation \( \mathcal{V}_0 \). The coevaluation and the evaluation are

\[
\begin{align*}
b_V : 1 & \to V \otimes V^* \quad \text{defined by} \quad b_V(1) = \sum_i e_i \otimes e^i, \\
d_V : V^* \otimes V & \to 1 \quad \text{defined by} \quad d_V(f \otimes a) = f(a),
\end{align*}
\]

where \( \{e_i\} \) and \( \{e^i\} \) are bases of \( V \) and \( V^* \) with \( e^i(e_j) = \delta_{ij} \). Let \( \mathcal{V}_\lambda, \lambda \in X_+ \), be the module of highest weight \( \lambda \) in \( \mathcal{V}_v \) with \( \dim M^\lambda = 1 \). Recall that \( w_0 \) is the longest element of \( W \). We have

**Proposition 2.1** (cf. [Lu] proposition 6.3.6). \( \mathcal{V}_v \) is semisimple and \( \mathcal{V}_\lambda, \lambda \in X_+ \) deplete all simple objects in \( \mathcal{V}_v \). Furthermore \( v\mathcal{V}_\lambda^* \cong v\mathcal{V}_{-w_0(\lambda)} \).

Recall that \( \mathcal{A} = \mathbb{Z}[v, v^{-1}] \). Let \( \mathcal{U}_\mathcal{A} \) be the \( \mathcal{A} \)-subalgebra of \( \mathcal{U}_v \) generated by elements \( E_i^{(n)}, F_i^{(n)}, K_i, K_i^{-1} \) for \( i = 1, \ldots, \ell \) and \( n \in \mathbb{Z}, n > 0 \). The Hopf algebra structure on \( \mathcal{U}_v \) can be restricted to \( \mathcal{U}_\mathcal{A} \). Let \( \mathcal{U}_\mathcal{A}^+, \mathcal{U}_\mathcal{A}^- \) and \( \mathcal{U}_\mathcal{A}^0 \) be the subalgebras of \( \mathcal{U}_\mathcal{A} \) generated by \( \{E_i^{(n)}\}, \{F_i^{(n)}\} \) and \( \{K_i, K_i^{-1}, [K_i; c]_t \mid c, t \in \mathbb{Z}, t > 0\} \) respectively. Here

\[
[K_i; c]_t = \prod_{s=1}^t \frac{K_i v_i^{c+1-s} - K_i^{-1} v_i^{-c-1+s}}{v_i^s - v_i^{-s}}.
\]

Then \( \mathcal{U}_\mathcal{A} = \mathcal{U}_\mathcal{A}^- \mathcal{U}_\mathcal{A}^0 \mathcal{U}_\mathcal{A}^+ \).

The subalgebra \( \mathcal{U}_\mathcal{A} \) is a free \( \mathcal{A} \)-module. It has two bases, the PBW basis \( \mathcal{P} \) and the canonical basis \( \mathcal{B} \). The canonical basis is unique while the PBW basis is unique up to the choice of a reduced expression of \( w_0 \). The definitions of these bases are quite involved and we refer the readers to [L] for details.

2.4. TQFTs based on a ribbon category. This subsection is a recap of section 2.2 of [CL]. See also IV of [L] and section 3.3 of [BK]. Manifolds are always orientable and smooth. Maps between manifolds are always smooth. Non-zero (tangent or normal) vectors are equivalent up to scalar multiple. We fix a ribbon category \( \mathcal{R} \) in this subsection.

2.4.1. Ribbon graphs. A band is a homeomorphic image of \([0, 1]\) with a non-zero normal field. The images of 0 and 1 are called the bases of the band. An annulus is a homeomorphic image of \( S^1 \) with a non-zero normal field. A band or an annulus is oriented if it is equipped with a non-zero tangent field. A coupon is a homeomorphic image of \([0, 1] \times 0 \) and \([0, 1] \times 1 \) are called the bottom and the top of the coupon respectively. Coupons are always oriented.

Let \( M \) be a 3-manifold. A ribbon graph \( \Omega \) in \( M \) is a union of oriented bands, annuli, and coupons embedded in \( M \) such that the bases of bands are on \( \partial M \) or the bottoms/tops of coupons.

Let \( \Omega \) be a ribbon graph. An \( \mathcal{R} \)-coloring of \( \Omega \) is an assignment to each band and annulus of \( \Omega \) an arbitrary object of \( \mathcal{R} \), and to each coupon of \( \Omega \) a morphism of \( \mathcal{R} \). A ribbon graph \( \Omega \) together with a \( \mathcal{R} \)-coloring \( \mu \) is called a \( \mathcal{R} \)-colored ribbon graph, denoted \( \Omega(\mu) \). A ribbon graph is partially \( \mathcal{R} \)-colored if some bands and/or annuli and/or coupons are \( \mathcal{R} \)-colored.
2.4.2. Extended surfaces. An $\mathcal{R}$-mark on a closed surface $\Gamma$ is a point $p$ on $\Gamma$ associated with a triple $(t, V, \nu)$, where $t$ (direction of the mark) is a non-zero tangent vector at $p$, $V$ (label of the mark) is an arbitrary object from $\mathcal{R}$ and $\nu$ (sign of the mark) is $+$ or $\sim$. An $\mathcal{R}$-extended surface (or $e$-surface for short) is a closed oriented surface $\Gamma$ together with a finite set of $\mathcal{R}$-marks on it and a decomposable Lagrangian subspace of $H_1(\Gamma, \mathbb{Q})$. If $\Gamma$ is an $e$-surface we denote by $-\Gamma$ another $e$-surface obtained from $\Gamma$ by reversing the orientation of $\Gamma$, keeping the Lagrangian subspace and, for every $\mathcal{R}$-mark on $\Gamma$, changing its sign while keeping its label and direction unchanged. The empty surface is considered an $e$-surface with no $\mathcal{R}$-marks.

An $e$-homeomorphism between two $e$-surfaces is a homeomorphism between the underlying surfaces respecting the extended structure.

2.4.3. Extended cobordisms. An $\mathcal{R}$-extended 3-manifold is a triple $(M, \Omega, w)$ that consists of an oriented 3-manifold $M$, an integer $w$ (weight of $M$) and an $\mathcal{R}$-colored ribbon graph $\Omega$ sitting in it. The boundary of $M$ is an $e$-surface that is compatible with $\Omega$.

An $\mathcal{R}$-extended cobordism (or $e$-cobordism for short) is a triple $(M, \Gamma, \Lambda)$ where $M$ is an $\mathcal{R}$-extended 3-manifold and $\partial M = (-\Gamma) \sqcup \Lambda$ (as an $e$-surface). An $e$-homeomorphism between two $e$-cobordisms is a homeomorphism between the underlying 3-manifolds respecting the extended structures on 3-manifolds and their boundaries.

If $(M, \Gamma, \Lambda)$ and $(M', \Gamma', \Lambda')$ are two $e$-cobordisms and there is an $e$-homeomorphism $f : \Lambda \to \Gamma'$. One can glue these two $e$-cobordisms along $f$ to get a new $e$-cobordism $(M \cup_f M', \Gamma, \Lambda')$. Here $M \cup_f M'$ is an $\mathcal{R}$-extended 3-manifold with an $\mathcal{R}$-colored ribbon graph (obtained by gluing ribbon graphs in $M$ and $M'$) sitting inside and weight computed as in IV.9.1 of [T].

2.4.4. TQFTs based on $\mathcal{R}$. Let $K$ be a commutative ring with unit and $\text{Mod}(K)$ be the category of projective $K$-modules. Let $\mathcal{C}(\mathcal{R})$ be the category of $e$-surfaces and $e$-homeomorphisms. A topological quantum field theory (TQFT) based on $\mathcal{R}$ with ground ring $K$ is a pair $(\mathcal{T}, \tau) = (\mathcal{T}_\mathcal{R}, \tau_\mathcal{R})$. Here $\mathcal{T} : \mathcal{C}(\mathcal{R}) \to \text{Mod}(K)$ is a modular functor, cf. III.1.2 of [T], based on $\mathcal{R}$ with ground ring $K$ and $\tau$ assigns to every $e$-cobordism $(M, \Gamma, \Lambda)$ a $K$-homomorphism

$$\tau(M) = \tau(\sigma(M, \Gamma, \Lambda)) : \mathcal{T}(\Gamma) \to \mathcal{T}(\Lambda)$$

that satisfies the naturality, multiplicativity, functorality and normalization axioms.

Let $(\mathcal{T}, \tau)$ be a TQFT based on $\mathcal{R}$ with ground ring $K$. For an $e$-cobordism $(M, \emptyset, \emptyset)$ one has $\tau(M) : K \to \mathcal{T}(\Gamma)$. Denote $\tau(M)(1) \in \mathcal{T}(\Gamma)$ by $[M]$, called a vacuum state in $\mathcal{T}(\Gamma)$. The TQFT is called non-degenerate if for every $e$-surface $\Gamma$, the module $\mathcal{T}(\Gamma)$ is generated over $K$ by vacuum states.

An $e$-cobordism with closed underlying 3-manifold is called closed. The image of a closed $e$-cobordism $(M, \emptyset, \emptyset)$ under a TQFT is a linear map from $K$ to $K$. Therefore $[M] \in K$. The following definition is due to Gilmer [G].

**Definition 2.2.** Let $D$ be a Dedekind domain contained in $K$ and $\mathcal{E}$ be an element in $K$. A TQFT $(\mathcal{T}, \tau)$ is called almost $(D)$-integral if $\mathcal{E}[M]$ is in $D$ for every closed connected $e$-cobordism $(M, \emptyset, \emptyset)$.

2.5. Twisted modules. Let $K$ be a commutative ring and $A$ be a $K$-algebra. Let $V$ be an $A$-module and $f : A \to A$ be a $K$-algebra endomorphism. One can define another $A$-module
structure on $V$ induced by $f$. We denote $V$ with this induced $A$-module structure by $fV$. Let $\cdot$ and $\cdot_f$ denote these two actions of $A$ on $V$ then
\[ x \cdot_f u = f(x) \cdot u, \quad \forall x \in A, \; u \in V. \]

Let $g : A \to A$ be a $K$-algebra antiendomorphism. Then one can define an $A$-module structure on $V^* = \text{Hom}_K(V, K)$ through $g$. We denote $V^*$ with this induced $A$-module structure by $g^*V^*$. The action of $A$ on it is denoted by $*_g$. (Dot is used for actions on $V$ and asterisk is used for actions on $V^*$.) One has
\[ (x *_g \alpha)(u) = \alpha(g(x) \cdot u), \quad \forall x \in A, \; \alpha \in V^*, \; u \in V. \]

Proof of the following lemma is left for the readers.

**Lemma 2.3.** Suppose $V, W$ are $A$-modules and $f : A \to A$ is a $K$-algebra automorphism. If $g : A \to A$ is a $K$-algebra endomorphism (resp. antiendomorphism) and $\phi : fV \to gW$ (resp. $\phi : fV \to g^*W$) is an $A$-module homomorphism then $\phi$ can be considered as an $A$-module homomorphism $V \to g f^{-1} W$ (resp. $V \to g^* f^{-1} W$).

We adopt the following convention: if $A$ is a Hopf algebra with antipode $S_A$ and $V$ is an $A$-module then we will denote $S_A V^*$ simply by $V^*$.

## 3. Representations of Quantum Groups for Generic $v$

### 3.1. Finite-ranked $U_A$-modules.

Let $P$ and $B$ be a PBW basis and the canonical basis of $U^-_A$ respectively (cf. section 2.3). Recall that $\lambda V, \lambda \in X_+$, is the simple object in $V_v$ of highest weight $\lambda$.

Fix an element $v_\lambda \in \lambda V$. The $U_A$-submodule $\lambda V_\lambda$ of $\lambda V$ generated by $v_\lambda$ is free as an $A$-module. The canonical basis (up to scalar) $B_\lambda$ of $\lambda V_\lambda$, and $\lambda V_\lambda$ of course, is the set $\{b v_\lambda \neq 0 \mid b \in B\}$. A PBW basis $P_\lambda$ of $\lambda V_\lambda$ is $\{b v_\lambda \neq 0 \mid b \in P\}$, which is not unique up to scalar.

Let
\[ \lambda V_\lambda^* = \text{Hom}_A(\lambda V_\lambda, A), \]

which is also a $U_A$-module because the antipode $S_{U_A}$ restricts to $U_A$. The following remark demonstrates that proposition 2.1 is no longer true for $\lambda V_\lambda$.

**Remark.** The quantum group $U_v(\mathfrak{sl}_2)$ is an algebra over $\mathbb{Q}(v)$ generated by $E, F, K, K^{-1}$ with the relations
\[ KK^{-1} = K^{-1}K = 1, \quad EF - FE = \frac{K - K^{-1}}{v - v^{-1}}, \]
\[ KE = v^2EK, \quad KF = v^{-2}FK. \]

It is a Hopf algebra with antipode $S_{U_v(\mathfrak{sl}_2)} : U_v(\mathfrak{sl}_2) \to U_v(\mathfrak{sl}_2)$ defined by
\[ S_{U_v(\mathfrak{sl}_2)}(E) = -K^{-1}E, \quad S_{U_v(\mathfrak{sl}_2)}(F) = -FK, \quad S_{U_v(\mathfrak{sl}_2)}(K) = K^{-1}. \]

For any integer $n \geq 0$ there exists a unique $(n + 1)$-dimensional $U_v(\mathfrak{sl}_2)$-module $V_n$ over $\mathbb{Q}(v)$ with basis $b_n = \{e_i\}_{i=0, \ldots, n}$ such that the actions of $E$ and $F$ on $V_n$ are depicted by the diagram in figure 1 where the action of $F$ is indicated by solid lines and the action of $E$ by dashed lines. Let $b'_n = \{e'_i\}_{i=0, \ldots, n} \subset V_n$ such that $e'_i(e'_j) = \delta_{ij}$. Using the antipode $S_{U_v(\mathfrak{sl}_2)}$, one has the diagram in figure 2 depicting the actions of $E$ and $F$ on $V_n^*$ where $\alpha_k = -v^{a_k - 2k + 2}$. From these two diagram it is easily seen that $V_n \cong V_n^*$. It turns out that $b_n$ is the canonical
basis of $V_n$. Hence $b_n$ is a basis for $AV_n$ and $b_n^*$ is a basis for $AV_n^*$. These two diagrams also show that $AV_n$ and $AV_n^*$ are not isomorphic as $U_A(\mathfrak{sl}_2)$-modules. Proposition 3.2 below says that in order to have isomorphism one only needs to invert quantum integers.

3.2. Inverting quantum integers. The $U_A$-module $AV_\lambda, \lambda \in X_+$, carries a unique bilinear form $H_\lambda$ (cf. [DCK]) such that:

\begin{align*}
H_\lambda(v_\lambda, v_\lambda) &= 1, \\
H_\lambda(x, ay) &= aH_\lambda(x, y) \text{ for } a \in A \text{ and } x, y \in AV_\lambda, \\
H_\lambda(g \cdot x, y) &= H_\lambda(x, \varphi(g) \cdot y) \text{ for } g \in U_A \text{ and } x, y \in AV_\lambda.
\end{align*}

Here $\varphi$ is the unique $\mathbb{Z}$-algebra antiautomorphism of $U_A$ such that:

\begin{align*}
\varphi(E_i^{(n)}) &= F_i^{(n)}, & \varphi(F_i^{(n)}) &= E_i^{(n)}, & \varphi(K_i) &= K_i^{-1}, & \varphi(v) &= v^{-1}.
\end{align*}

$H_\lambda$ is often referred to as the quantum Shapovalov’s form. We have $H_\lambda(AV_\lambda^\mu, AV_\lambda^\nu) = 0$ if $\mu \neq \nu$ and $\text{Ker}(H_\lambda) = 0$. Let $\det(\lambda)$ denote the determinant of the matrix of $H_\lambda$ in the basis $P_\lambda$.

Proposition 3.1 ([DCK] proposition 1.9). For any $\lambda \in X_+$, $\det(\lambda)$ is a product of quantum integers.

Let $A_n = A[1/[n]_i^{\lambda};i=1,\ldots,\ell]$ and $\mathcal{A}V_\lambda = AV_\lambda \otimes \mathcal{A}n$. For $\lambda \in X_+$, let

\[ \mathcal{A}V^*_\lambda = \text{Hom}_{A_n}(\mathcal{A}V_\lambda, A_n). \]

Since $AV_\lambda$ and $AV^*_\lambda$ are free modules of same rank $AV^*_\lambda = AV^*_\lambda \otimes \mathcal{A}A_n$. We have

Proposition 3.2. For $\lambda \in X_+$, the $U_A$-modules $AV_\lambda^*$ and $AV^*_\lambda$ are isomorphic for large enough $n$ where $\lambda' = -w_0(\lambda)$. 
Proof. The bilinear form \( H_\lambda : \mathcal{A}_\lambda \otimes Z \mathcal{A}_\lambda \to \mathcal{A} \) induces a \( \mathcal{U}_A \)-linear map over ground ring \( Z \) (cf. section 2).\(^{25,3}\)

\[
H_\lambda : \mathcal{A}_\lambda \to \mathcal{A}_\lambda^*. 
\]

Let \( \bar{\cdot} : \mathcal{U}_A \to \mathcal{U}_A \) be the unique \( Z \)-algebra automorphism such that

\[
\bar{E}_i^{(s)} = E_i^{(s)}, \quad \bar{F}_i^{(s)} = F_i^{(s)}, \quad \bar{K}_i = K_i^{-1}, \quad \bar{v} = v^{-1}.
\]

Then \( \bar{\varphi} = \varphi \circ \bar{\cdot} \) is an antiautomorphism for \( \mathcal{U}_A \) (cf. equation (2)) over \( \mathcal{A} \). By lemma 2.3, equation (3) can be considered as a \( \mathcal{U}_A \)-module homomorphism

\[
H_\lambda : \mathcal{A}_\lambda \to \mathcal{A}_\lambda^*
\]

over the ground ring \( Z \). By proposition \( 3.1 \) \( H_\lambda \) is actually an isomorphism.

There is a unique \( \mathcal{U}_A \)-module isomorphism \( \bar{\cdot} : \mathcal{A}_\lambda \to \mathcal{A}_\lambda \), over ground ring \( Z \), that is identity on the canonical basis \( \mathcal{B}_\lambda \) of \( \mathcal{A}_\lambda \) (cf. \( \text{Lu} \) page 171 where \( \mathcal{A}_\lambda \) and \( \mathcal{B}_\lambda \) are denoted by \( \Lambda_\lambda \) and \( \mathcal{B}(\Lambda_\lambda) \) respectively). Then

\[
\bar{H}_\lambda = H_\lambda \circ \bar{\cdot} : \mathcal{A}_\lambda \to \mathcal{A}_\lambda^*
\]

is a \( \mathcal{U}_A \)-module isomorphism over \( Z \). Easy computation shows that \( \bar{H}_\lambda \) is an honest \( \mathcal{U}_A \)-module isomorphism, i.e. over ground ring \( \mathcal{A} \). There is a unique \( \mathcal{U}_A \)-module isomorphism \( \bar{\omega} \) of \( \mathcal{U}_A \) such that

\[
\bar{\omega}(E_i^{(s)}) = F_i^{(s)}, \quad \bar{\omega}(F_i^{(s)}) = E_i^{(s)}, \quad \bar{\omega}(K_i) = K_i^{-1}.
\]

Lusztig proved that there exists a unique \( \mathcal{U}_A \)-module isomorphism

\[
\chi : \mathcal{A}_\lambda \to \mathcal{A}_\lambda^*
\]

that maps \( \mathcal{B}_\lambda \) to \( \mathcal{B}_\lambda^* \) (cf. \( \text{Lu} \) page 177). Hence

\[
\bar{H}_\lambda \circ \chi^{-1} : \mathcal{A}_\lambda^* \to \mathcal{A}_\lambda^*
\]

is a \( \mathcal{U}_A \)-module isomorphism. By lemma \( 3.3 \) below \( \mathcal{S}_{\mathcal{U}_A} \mathcal{A}_\lambda^* \mathcal{A}_\lambda^* \) is isomorphic to \( \mathcal{A}_\lambda^* \), which in turn is isomorphic to \( \mathcal{A}_\lambda^* \) by equation (4). Hence \( \mathcal{S}_{\mathcal{U}_A} \mathcal{A}_\lambda^* \mathcal{A}_\lambda^* \) are isomorphic. This isomorphism can be considered as an isomorphism between \( \mathcal{A}_\lambda \) and \( \mathcal{S}_{\mathcal{U}_A} \mathcal{A}_\lambda^* \mathcal{A}_\lambda^* \).

Lemma 3.3. For any \( \mu \in X_+ \), \( \mathcal{S}_{\mathcal{U}_A} \mathcal{A}_\lambda \mathcal{A}_\lambda^* \mathcal{A}_\mu \cong \mathcal{A}_\lambda^* \).

Proof. By lemma \( 2.3 \) it’s enough to prove \( \mathcal{V}_1 = \mathcal{S}_{\mathcal{U}_A} \mathcal{A}_\lambda \mathcal{A}_\lambda \mathcal{A}_\mu \cong \mathcal{V}_2 = \mathcal{A}_\lambda^* \) (because \( \omega^2 = \text{Id} \)). Let \( z : \mathcal{V}_1 \to \mathcal{V}_2 \) and \( z' : \mathcal{V}_2 \to \mathcal{V}_1 \) be \( \mathcal{U}_A \)-homomorphisms such that \( z(v_\mu) = v_\mu \) and \( z'(v_\mu) = v_\mu \). They are well defined because \( \mathcal{S}_{\mathcal{U}_A} \mathcal{A}_\lambda \mathcal{A}_\mu \) is a \( \mathcal{U}_A \)-automorphism sending \( K_i \) to \( K_i \).

Set \( \mathcal{A}_\infty = \bigcup_{n=1}^\infty \mathcal{A}_n \) and \( \mathcal{A}_\lambda \mathcal{A}_\lambda \mathcal{A}_\mu \mathcal{A}_\lambda \mathcal{A}_\mu \) because the coproduct \( \Delta_{\mathcal{U}_A} \) of \( \mathcal{U}_v \) restricts to \( \mathcal{U}_A \), tensor product of \( \mathcal{U}_A \)-modules is also a \( \mathcal{U}_A \)-module. Using proposition \( 3.2 \) and the canonical basis we have

Lemma 3.4. \( \mathcal{A}_\lambda \mathcal{A}_\mu \) is a direct summand of \( \mathcal{A}_\lambda \mathcal{A}_\mu \mathcal{A}_\lambda \mathcal{A}_\mu \mathcal{A}_\lambda \mathcal{A}_\mu \) for \( \lambda, \mu \in X_+ \).
Proof. Let $M_1 = A\mathcal{V}_\lambda$, $M_2 = A\mathcal{V}_\mu$, $M_3 = M_1 \otimes M_2$, $M_4 = A\mathcal{V}_{\lambda + \mu}$, $M_5 = A\mathcal{V}_{-w_0(\lambda + \mu)}$, and $M_6 = M_5 \otimes M_3$. Also let $M_0 = A\mathcal{V}_0$ be the rank 1 $U_A$-module. Following [Lu], one can define the canonical basis $B$ for $M_i$. Note that $B_3 = B_1 \otimes B_2$ (resp. $B_6 = B_5 \otimes B_3$) is not the tensor product basis $B_1 \otimes B_2$ (resp. $B_5 \otimes B_3$).

There is a $U_A$-module monomorphism

$$\phi : M_4 \to M_3$$

that carries $B_4$ into $B_3$ (cf. [Lu] proposition 27.1.7). To prove the lemma we need to find a $U_A$-module epimorphism $\phi' : M_3 \to M_4$ such that $\phi' \circ \phi = \text{Id}_{M_4}$.

It is easy to see that $M_6 \otimes \mathbb{Q}(v)$ has a unique quotient isomorphic to $M_0 \otimes \mathbb{Q}(v)$ considered as objects in $\mathcal{V}_\nu$. (Use weight argument and proposition 2.1.) By proposition 27.2.6 in [Lu] one has a $U_A$-epimorphism

$$\pi : M_6 \to M_0$$

carrying $B_6$ onto $B_0 \cup \{0\}$. There is a $(U_A$-linear) right coevaluation $\vartheta : A_\infty \to M'_5 \otimes M_5$ such that

$$(5) \quad \vartheta(1) = \sum_{i=1}^m b_i^* \otimes K_{-2\rho}(b_i).$$

Here $B_5 = \{b_i\}$ and $b_i^*(b_j) = \delta_{ij}$. Suppose $u_4$ is of highest weight in $B_4$ and $b_m$ is of lowest weight in $B_5$. By proposition 3.2 there is a $U_A$-isomorphism $\psi : M'_5 \to M_4$ such that

$$(6) \quad \psi(b_m^*) = cu_4$$

for some invertible $c \in A_\infty$.

Let

$$\phi'' = \psi \circ (\text{Id}_{M'_5} \otimes \pi) \circ (\vartheta \otimes \text{Id}_{M_3}) : M_3 \to M_4.$$ 

Since $M_i \otimes \mathbb{Q}(v)$ is simple in the category $\mathcal{V}_\nu$, $(\phi'' \circ \phi) \otimes \text{Id}_{\mathbb{Q}(v)} = a \text{Id}_{M_i \otimes \mathbb{Q}(v)}$ for some $a \in \mathbb{Q}(v)$. But $\phi''$ and $\phi$ are defined over $A_\infty$ we know that $a \in A_\infty$.

We prove that $a$ is invertible in $A_\infty$ and this will complete the proof of the lemma. We only have to look at the image of $u_4$ under $\phi'' \circ \phi$. Let $u_3$ be the unique element of weight $\lambda + \mu$ in $B_3$. Because $\phi$ takes canonical basis to canonical basis one has $\phi(u_4) = u_3$ and

$$(\vartheta \otimes \text{Id}_{M_3}) \circ \phi(u_4) = \sum_{i=1}^m b_i^* \otimes K_{-2\rho}(b_i) \otimes u_3$$

(cf. equation 3). By theorem 24.3.3.(b) of [Lu] $b_m \otimes u_3 = b_m \otimes u_3 \in B_6$. By proposition 27.3.8 of [Lu] $\pi(b_m \otimes u_3) = u_0$ where $u_0$ is the unique element in $B_0$. Since $b_m$ has weight $-\lambda - \mu$ one has $\pi(K_{-2\rho}(b_m) \otimes u_3) = v^{(2\rho(\lambda + \mu))}u_0$. For $i \neq m$, $\pi(K_{-2\rho}(b_i) \otimes u_3) = 0$ because $b_i \otimes u_3$ is of nonzero weight. Therefore $(\text{Id}_{M'_5} \otimes \pi) \circ (\vartheta \otimes \text{Id}_{M_3}) \circ \phi(u_4) = v^{(2\rho(\lambda + \mu))}b_m^*$. By equation 3, we have $a = v^{(2\rho(\lambda + \mu))}c$, which is invertible.

The finite-dimensional $U_A$-modules of type 1 form a ribbon category whose morphisms have ribbon graph presentations (cf. [T]). Figure 3 presents the morphisms used in the proof.

Furthermore we have
Figure 3. Morphisms used in the proof of lemma 3.4

Proposition 3.5. For $\lambda, \mu \in X_+$, there exist $\mu_1, \ldots, \mu_m \in X_+$ such that

$$A^\infty V_\lambda \otimes A^\infty V_\mu = \bigoplus_{i=1}^m A^\infty V_{\mu_i}.$$  

Proof. As we saw above, $A^\infty V_{\lambda+\mu}$ is a direct summand of $A^\infty V_\lambda \otimes A^\infty V_\mu$. Let $M$ be the direct summand of $A^\infty V_\lambda \otimes A^\infty V_\mu$ that is complementary to $A^\infty V_{\lambda+\mu}$. We must show that $M$ contains a direct summand isomorphic to $A^\infty V_\nu$ for some $\nu \in X_+$. Note that $M$ is a based module (cf. [Lu]). Let $\nu$ be a highest weight of $M$. By proposition 27.1.7 in [Lu] $M$ contains a submodule isomorphic to $A^\infty V_\nu$. This submodule is actually a direct summand. This can be proved similarly as the previous lemma. The only difference is that $M$ may contain more than one copy of $A^\infty V_\nu$ but it is easy to get around. One only has to be a little more careful when constructing the projection map. We leave the details for the readers. □

Let $V_\infty$ be a category of $U_A$-modules and $U_A$-morphisms over ground ring $A_\infty$. Objects in $V_\infty$ are direct summands of $A^\infty V_\mu_1 \star \cdots \star A^\infty V_\mu_m$ where $\star$ is either $\otimes$ or $\oplus$ and $\mu_i \in X_+$. For any $\lambda \in X_+$, $A^\infty V_\lambda$ is simple because even its extension to $\mathbb{Q}(v)$ is simple in $V_v$. An easy consequence of the previous proposition is the following.

Corollary 3.6. The tensor category $V_\infty$ is semisimple and $A^\infty V_\lambda$'s, $\lambda \in X_+$, deplete all simple objects. Furthermore $A^\infty V_*^\ast \cong A^\infty V_{\lambda'}$ where $\lambda' = -w_0(\lambda)$.

4. REPRESENTATIONS AT ROOTS OF 1

From now on $r$ is assumed to be an odd prime greater than the Coxeter number and $\xi$ is a root of unity of order $r$.

4.1. $U_\xi$-modules. Let $U_\xi = U_A \otimes_A \mathbb{Z}[\xi]$ where $\mathbb{Z}[\xi]$ is considered as an $A$-algebra by sending $v$ to $\xi$. Similarly we define $U_\xi^-$, $U_\xi^+$ and $U_\xi^0$. One has $U_\xi = U_\xi^- U_\xi^0 U_\xi^+$. We study the representation theory for $U_\xi$ in this section using results from section 3.2 and [A]. Let $V_\lambda^\xi = A^\lambda V_\lambda \otimes_A \mathbb{Z}[\xi]$ and $V^\ast_\lambda = \text{Hom}_{\mathbb{Z}[\xi]}(V_\lambda, \mathbb{Z}[\xi])$ for $\lambda \in X_+$. They are $U_\xi$-modules and are free of finite rank as $\mathbb{Z}[\xi]$-modules. They will be the building blocks of our construction. The next two lemmas concern their duality and tensor product.
Lemma 4.1. For \( \lambda \in \bar{C}_r \cap X_+ \) and \( \lambda' = -w_0(\lambda) \) the \( \mathfrak{U}_\xi \)-modules \( \mathcal{V}_\lambda \) and \( \mathcal{V}_{\lambda'}^* \) are isomorphic.

Note that \( \lambda' \in \bar{C}_r \cap X_+ \) if and only if \( \lambda \in \bar{C}_r \cap X_+ \) where \( \bar{C}_r \) is defined in section 2.2. Let \( \mathcal{V}_\lambda = \mathcal{V}_\lambda \otimes \mathbb{C}, \lambda \in X_+ \). It is well known that \( \mathcal{V}_\lambda \) is a simple \( \mathfrak{U}_\mathbb{C} \)-module for \( \lambda \in \bar{C}_r \cap X_+ \). Note that the image of the quantum integer \( [n]_i \) in \( \mathbb{Z}[\xi] \) is invertible if \( r \) is not a factor of \( n \) and the image is 0 otherwise.

Proof. The quantum Shapovalov's form \( H_\lambda \) on \( \mathcal{A}_\mathcal{V}_\lambda \) (cf. section 3.1) induces a bilinear form on \( \mathcal{V}_\lambda \). Reading the proof of proposition 3.2 one realizes that it's enough to show that \( \det(\lambda) \) contains no factor of \( [sr] \) for some integer \( s \). Assume there was one. Then \( \det(\lambda) = 0 \). Since \( H_\lambda \) can also induce a bilinear form on \( \mathcal{V}_\lambda \) this would imply that \( \mathcal{V}_\lambda \) has a nontrivial submodule, namely the kernel of \( H_\lambda \). We get a contradiction. \( \Box \)

Lemma 4.2. If \( \lambda, \mu, \lambda + \mu \in X_+ \cap \bar{C}_r \) then \( \mathcal{V}_\lambda \otimes \mathcal{V}_\mu = \oplus_{i=1}^s \mathcal{V}_{\mu_i} \) with \( \mu_i \in X_+ \cap \bar{C}_r \).

Proof. The proof goes like those of lemma 3.4 and proposition 3.5. Remember we had to prove that the element \( a = v^{(2p(\lambda + \mu))} \) is invertible, which follows from the fact that \( c \) is invertible. Situation here is similar. We have to prove one element is invertible, which can be proved by using the proof of lemma 4.1 and the discussion before it. \( \Box \)

4.2. Negligible objects and morphisms. Let \( \mathcal{V}_\xi' \) be a full subcategory of \( \mathcal{V}_\xi \), the category of finite-ranked \( \mathfrak{U}_\xi \)-modules of type 1. Every object in \( \mathcal{V}_\xi' \) is a direct summand of \( \mathcal{V}_{\mu_1} \ast \cdots \ast \mathcal{V}_{\mu_m} \) where \( \ast \) is either \( \otimes \) or \( \oplus \) and \( \mu_i \in Y_+ \cap C_r \) where \( Y_+ = Y \cap X_+ \). Morphisms in \( \mathcal{V}_\xi' \) are \( \mathfrak{U}_\xi \)-morphisms. (We use the root lattice \( Y \) instead of \( X \) because (1) it’s required in the case \( g = B_\xi \) and (2) it gives modular category.)

An object \( V \) in \( \mathcal{V}_\xi' \) is called negligible if the quantum trace \( \text{tr}_q(f) = \text{tr}(K_{2p},f) = 0 \) for all \( f \in \text{End}_{\mathcal{V}_\xi}(V) \). A morphism \( f : V \rightarrow W \) in \( \mathcal{V}_\xi' \) is called negligible if it factors through a negligible object. (Compare \( \text{Ki} \) section 3.)

We record some basic facts about negligible objects and morphisms.

Lemma 4.3. In \( \mathcal{V}_\xi' \) we have

1. \( M \otimes N \) is negligible if \( M \) is,
2. \( \mathcal{V}_\lambda \) is negligible for every \( \lambda \in F_r \cap Y_+ \) (cf. section 2.2),
3. \( \mathcal{V}_\lambda, \lambda \in C_r \cap Y_+ \), is not a direct summand of a negligible module,
4. direct summand of a negligible object is negligible,
5. direct sum of negligible objects is negligible,
6. \( f, f^*, fg, gf, f \otimes g \) and \( g \otimes f \) are negligible if \( f \) is,
7. \( f + g \) is negligible if \( f \) and \( g \) are.

Proof. One can prove (1) and (6) using graphical calculus. We prove (2). Any \( f \in \text{End}_{\mathcal{V}_\xi}(\mathcal{V}_\lambda) \) induces \( f \otimes 1 \in \text{End}_{\mathcal{V}_\xi}(\mathcal{V}_\lambda) \). We see that \( \text{tr}_q(f) = 0 \) because \( \text{tr}_q(f \otimes 1) = 0 \) (cf. \( \text{Ki} \)). We prove (3). If \( \mathcal{V}_\lambda, \lambda \in C_r \cap Y_+ \), was a direct summand of a negligible module \( M \) then \( \mathcal{V}_\lambda \otimes \mathbb{C} \) would be a direct summand of \( M \otimes \mathbb{C} \). But Andersen proved this can not be true (cf. \( \text{Ki} \) again). Assertion (4) is obvious. Assertion (5) follows from \( \text{tr}_q(f) = \text{tr}_q(\pi_1 \circ f \circ \iota_1) + \text{tr}_q(\pi_2 \circ f \circ \iota_2) \) for \( f \in \text{End}_{\mathcal{V}_\xi}(V_1 \otimes V_2) \) where \( \pi_i \) and \( \iota_i, i = 1, 2 \), are standard projection and injection. Assertion (7) follows from the universal property of the direct sum and (5). \( \Box \)
Let
\[
m(A_\ell) = \ell + 1, \quad m(B_\ell) = 2\ell, \quad m(C_\ell) = 3\ell - 1,
\]
\[
m(D_\ell) = 3\ell - 6, \quad m(E_6) = 14, \quad m(E_7) = 21.
\]
Note that \(m(\mathfrak{g}) \geq h\). The following proposition is the key to prove proposition 1.1.

**Proposition 4.4.** If \(\mathfrak{g}\) is not of type \(E_8, F_4\) or \(G_2\) and \(r\) is a prime bigger than \(m(\mathfrak{g})\) then for any object \(V\) in \(\mathcal{V}_\xi\) we have
\[
V = \bigoplus_i \xi V_{\mu_i} \oplus Z
\]
where \(\mu_i \in Y_+ \cap C_r\) and \(Z\) is negligible.

Under the assumption \(r\) is, in particular, prime to the non-zero entries of the Cartan matrix for \(\mathfrak{g}\) and is bigger than the Coxeter number. This proposition will be proved in section 5.

### 4.3. Reduction to semisimple tensor categories.

Let \(\mathcal{V}'_\xi\) be the category of \(\mathcal{V}_\xi\) quotient by negligible morphisms. Objects in \(\mathcal{V}'_\xi\) are the same as those in \(\mathcal{V}_\xi\) and
\[
\text{Hom}_{\mathcal{V}'_\xi}(V,W) = \text{Hom}_{\mathcal{V}_\xi}(V,W)/\text{negligible morphisms}.
\]
Negligible objects in \(\mathcal{V}'_\xi\) become 0 in \(\bar{\mathcal{V}}'_\xi\). By assertions (6) and (7) of lemma 4.3, \(\bar{\mathcal{V}}'_\xi\) is a tensor category. According to proposition 4.4, \(\bar{\mathcal{V}}'_\xi\) is semisimple and is dominated by \(\xi V_\lambda\), \(\lambda \in Y_+ \cap C_r\). This proves proposition 1.1.

### 5. Proof of proposition 4.4

We will make use of some techniques from [GK]. The proof is done by checking different cases. In all the cases except \(B_\ell\) we need to enlarge \(\mathcal{V}_\xi\) a little bit. Let \(\mathcal{V}_\xi''\) be a subcategory of \(\mathcal{V}_\xi\). Every object in \(\mathcal{V}_\xi''\) is a direct summand of \(\xi V_{\mu_1} \ast \cdots \ast \xi V_{\mu_m}\) where \(\ast\) is either \(\oplus\) or \(\otimes\) and \(\mu_i \in X_+ \cap C_r\). Morphisms in \(\mathcal{V}_\xi''\) are \(U_\xi\)-morphisms. Obviously \(\mathcal{V}_\xi''\) is a full subcategory of \(\mathcal{V}_\xi''\). We can define negligible objects and morphisms in \(\mathcal{V}_\xi''\) similarly as in \(\mathcal{V}_\xi''\). If we prove proposition 4.4 in \(\mathcal{V}_\xi''\) (i.e. changing \(Y\) to \(X\)) then it is true in \(\mathcal{V}_\xi''\) also. The Dynkin diagrams in figure 4 help us identify fundamental representations.

#### 5.1. Type \(A_\ell\)

Recall that \(\lambda_i, i = 1, \ldots, \ell\), are the fundamental weights. In this case, the highest (short) root \(\alpha_0 = \alpha_1 + \cdots + \alpha_\ell\) (cf. [11]). We have \((\lambda_i + \rho|\alpha_0) = 1 + \ell\). Hence for any \(r > m(\mathfrak{g}) = \ell + 1\), \(\xi V_{\lambda_i}\) is an object in \(\mathcal{V}_\xi''\).

Let \(\lambda = \sum_i a_i \lambda_i\) be a dominant weight in \(C_r \cap X_+\). By lemma 4.2, \(\xi V_\lambda\) is a direct summand of \(\bigotimes_i (\xi V_{\lambda_i})^{a_i}\). Since \(\xi V_{\mu_i}, \mu_i \in X_+ \cap C_r\), is simple and direct summand of negligible object is negligible (lemma 4.3(4)) it is enough to prove proposition 4.4 when \(V = \xi V_\lambda \otimes \xi V_{\lambda_i}\) with \(\lambda \in X_+ \cap C_r\). In this case we have \((\lambda + \lambda_i + \rho|\alpha_i) = (\lambda + \rho|\alpha_0) + 1 \leq r\). The proposition now follows from lemma 4.2 and lemma 4.3(2).

#### 5.2. Type \(C_\ell\)

It’s known that \(\xi V_{\lambda_i} = \xi V_{\lambda_i^1}, i = 1, \ldots, \ell\), which is a direct summand of \(\xi V_{\lambda_i^1}\) if the highest weight \(i\lambda_1\) is in \(C_r \cap X_+\) (cf. theorem 10.5.7 of [W] and lemma 4.2). We have \(\alpha_0 = \alpha_1 + 2(\alpha_2 + \cdots + \alpha_{\ell-1}) + \alpha_\ell\) and \((\lambda_1|\alpha_0) = 1\). So \((i\lambda_1 + \rho|\alpha_0) \leq (\ell\lambda_1 + \rho|\alpha_0) = 3\ell - 1 = m(\mathfrak{g}) < r\) and \(i\lambda_1\) is in \(C_r \cap X_+, i = 1, \ldots, \ell\). Therefore we only have to prove the proposition when \(V = \xi V_{\mu} \otimes \xi V_{\lambda_i}\) for \(\mu \in C_r \cap X_+\). This can be proved as in the case \(A_\ell\).
5.3. **Type** $D_\ell$. We have $\alpha_0 = \alpha_1 + 2(\alpha_2 + \cdots + \alpha_{\ell-2}) + \alpha_{\ell-1} + \alpha_\ell$. For $i = 1, \ell - 1$ and $\ell$ and $n = 1, 2, \ldots, \ell - 3$, we have

\[(n\lambda_i + \rho|\alpha_0) = n + 2\ell - 3 \leq 3\ell - 6 = m(\mathfrak{g}) < r.\]

According to theorem 10.5.7 of [W] and lemma 4.2, all fundamental representations are direct summands of $\xi V_{\lambda_i}^{\otimes j}, i = 1, \ell - 1, \ell$ and $j = 1, 2, \ldots, \ell - 3$. We can prove the proposition just like $A_\ell$ because $(\lambda_i|\alpha_0) = 1$, $i = 1, \ell - 1, \ell$.

5.4. **Type** $E_6$. We have $\alpha_0 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + \alpha_6$ and $(n\lambda_i + \rho|\alpha_0) \leq 14 = m(\mathfrak{g}) < r$, $i = 1, 6$ and $n = 1, 2, 3$. So $\xi V_{\lambda_i}, i = 1, 6$ belong to $C_r \cap X_+$. Using [S] and proposition 4.2 we see that all fundamental representations $\xi V_{\lambda_i}, 1 \leq i \leq 6$ are direct summands of $\xi V_{\lambda_1}^{\otimes j}$ or $\xi V_{\lambda_6}^{\otimes j}, j = 1, 2, 3$. By lemma 4.3 (2) we only have to prove the proposition when $V = \xi V_{\mu} \otimes \xi V_{\lambda_i}, i = 1, 6$. This can be proved as $A_\ell$ since $(\lambda_i|\alpha_0) = 1$ for $i = 1, 6$.

5.5. **Type** $E_7$. We have $\alpha_0 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$ and $(n\lambda_7 + \rho|\alpha_0) \leq 21 = m(\mathfrak{g}) < r$, $n = 1, 2, 3, 4$. So $\xi V_{\lambda_7}$ belongs to $C_r \cap X_+$. Using [S] and proposition 4.2 we see that all fundamental representations $\xi V_{\lambda_i}, 1 \leq i \leq 7$ are direct summands of $\xi V_{\lambda_7}^{\otimes j}$,
j = 1, 2, 3, 4. By lemma 4.3 we only have to prove the proposition when \( V = \xi V_\mu \otimes \xi V_\lambda \). This can be proved as \( A_\xi \) since \( (\lambda_7|\alpha_0) = 1 \).

5.6. Type \( B_\ell \). We work in \( \mathcal{V}_\xi' \). Let \( b_\ell = \{ \lambda_1, \ldots, \lambda_{\ell-1}, 2\ell \} \subset Y \). Because \( |X/Y| = 2 \) and \( b_\ell \subset Y \) we see that \( b_\ell \) is a basis of the root lattice \( Y \) over \( \mathbb{Z} \). For any \( \mu \in b_\ell, (\mu|\alpha_0) = 2 \) where \( \alpha_0 = \sum \alpha_i \), cf. [H]. Therefore \( \xi V_\mu, \mu \in b_\ell \), is an object in \( \mathcal{V}_\xi' \). For any \( \lambda \in C_r \cap Y_+ \), \( \xi V_\lambda \) is a direct summand of \( \xi V_{\mu_1} \otimes \cdots \otimes \xi V_{\mu_m} \) for \( \mu_i \in b_\ell \) by lemma 4.2. Therefore we only have to prove the proposition when \( V = \xi V_\lambda \otimes \xi V_\mu \) for \( \lambda \in Y_+ \cap C_r \) and \( \mu \in b_\ell \).

From the Cartan matrix we see that \( (\alpha_i|\alpha_j) \) is always an even number. For any \( \lambda \in Y \), there is an integer \( a \) such that

\[
(\lambda + \rho|\alpha_0) = 2a + (\rho|\alpha_0) = 2a + \sum_{i,j=1}^{\ell}\langle \alpha_i| \lambda_j \rangle
= 2a + \sum_{i=1}^{\ell-1} \sum_{j=1}^{\ell} \langle \alpha_i| \lambda_j \rangle + \sum_{j=1}^{\ell} \langle \alpha_\ell| \lambda_j \rangle
= 2a + \sum_{i=1}^{\ell-1} \sum_{j=1}^{\ell} 2\delta_{ij} + \sum_{j=1}^{\ell} \delta_{j\ell}
= 2a + 2(\ell - 1) + 1 = 2(\ell + 1) - 1.
\]

If \( \lambda \in Y_+ \cap C_r \) then \( (\mu + \rho|\alpha_0) \leq r - 2 \) because \( r \) is odd. Hence \( \lambda + \mu \in \bar{C}_r \cap Y_+ \) for any \( \mu \in b_\ell \) and we are done by proposition 4.2 and lemma 4.3 (2).

6. INTEGRAL TQFTs

We will follow [1] to construct a modular category from the braided tensor category \( \bar{\mathcal{V}}_\xi \). Then we will follow [3] to get integral TQFTs. The construction of the TQFT is similar to the one in [CL].

Let \( g \) be a complex simple Lie algebra not of type \( E_8, F_4, G_2 \) and \( r \) be an odd prime greater than \( m(g) \). We fix a set of dominant simple objects

\[
A = A(\bar{\mathcal{V}}_\xi) = \{ \xi V_\lambda \mid \lambda \in Y_+ \cap C_r \}.
\]

We identify \( A \) with the set \( \{ \lambda \in Y_+ \cap C_r \} \).

6.1. Modular categories. It is well known that \( \bar{\mathcal{V}}_\xi \) is a ribbon category, cf. [CL, Le1]. In order to make it into a modular category, we only have to make the S-matrix

\[
S = S(\bar{\mathcal{V}}_\xi) = (s_{\lambda,\mu})_{\lambda,\mu \in A}
\]

invertible.

Let \( \mathbb{C}[X] \) be the algebra over \( \mathbb{C} \) generated by \( v^\lambda, \lambda \in X \). The addition in \( \mathbb{C}[X] \) is formal and the multiplication is given by \( v^\lambda v^\mu = v^{\lambda+\mu} \). One can map \( \mathbb{C}[X] \) onto a subalgebra of \( \text{Hom}_\mathbb{C}(U^0_{\xi}, \mathbb{C}) \) through

\[
v^\lambda(K_i) = \psi_r \left( v^{(\alpha_i|\lambda)} \right), \quad v^\lambda \left( \left[ K_i; \frac{c}{t} \right] \right) = \psi_r \left( \left[ (\alpha_i|\lambda)/d_i + c \right]/t \right),
\]

where \( \psi_r : \mathcal{A} \to \mathbb{Z}[\xi] \) is a ring homomorphism sending \( v \) to \( \xi \). Let \( \delta \in \mathbb{C}[X] \) be the Weyl denominator

\[
\delta = \prod_{\alpha \in \Phi_+} (v^{\alpha/2} - v^{-\alpha/2}) = \sum_{w \in W} \text{sn}(w)v^{w(\rho)}, \quad (8)
\]
where \(\text{sn}(w) = (-1)^{\text{length of } w}\). Let

\[
\mathcal{D}^2 = \mathcal{D}^2_{\mathcal{V}_{\xi}} = \sum_{\lambda \in Y_+ \cap C_r} \dim_q(\mathcal{V}_\lambda)^2,
\]

where \(\dim_q(\mathcal{V}_\lambda) = \text{tr}_q(\text{Id}_{\mathcal{V}_\lambda})\) is the quantum dimension of \(\mathcal{V}_\lambda\). Recall that \(w_0\) is the longest element in \(W\) and \(\ell\) is the dimension of \(\mathfrak{h}\).

**Lemma 6.1.** Let \(r > m(\mathfrak{g})\) be a prime and \(\mathfrak{g}\) be a simple Lie algebra not of type \(E_8, F_4, G_2\). Then \(\mathcal{D}^2 = \frac{\text{sn}(w_0)r^2}{\delta(K_{2p})^2}\) and \(\det(S)^2 = \pm \mathcal{D}^2_{\mathcal{V}_+ \cap C_r}\).

Note that \(\delta(K_{2p}) \neq 0\) because \(r\) is odd and greater than \(h\). This lemma is proved in the proof of theorem 3.3 in [CL] and section 2.4. In the rest of this paper \(\mathcal{D}\) will mean \(\mathcal{D}^2\) in the ground ring. Let \(\zeta\) be a root of unity of order \(r\geq -1\) is an integer, \(\mathcal{D}\) is in the ground ring. Let \(\mathcal{D} = \mathbb{Z} \in \mathbb{Z}[\zeta, r^{-1}]\).

**Lemma 6.2.** \(\mathcal{D} \in \mathbb{Z}[\zeta, r^{-1}]\).

**Proof.** By equation (8), \(\delta(K_{2p}) \sim (\xi - 1)^{\Phi_q}\). Since \(r \sim (\xi - 1)^{r^{-1}}\), \(\frac{1}{\delta(K_{2p})}\) is in \(\mathbb{Z}[\zeta, r^{-1}]\). It remains to notice that \(\sqrt{\text{sn}(w_0)r^2} \in \mathbb{Z}[\zeta]\). \(\square\)

Let \(\mathcal{V}_\lambda = \mathcal{V}_\lambda \otimes \mathbb{Z}[\zeta, r^{-1}]\). Let \(\mathcal{V}'\) be a category of \(U_\zeta\)-modules and \(U_\zeta\)-morphisms. Objects in \(\mathcal{V}'\) are direct summands of \(\mathcal{V}_{\mu_1} \ast \cdots \ast \mathcal{V}_{\mu_m}\) where \(\ast\) is either \(\otimes\) or \(\oplus\) and \(\mu_i \in Y_+ \cap C_r\). Let \(\mathcal{V}'_{\mathcal{V}}\) be the category \(\mathcal{V}_{\mathcal{V}}\) quotient by negligible morphisms. Clearly \(\mathcal{V}'_{\mathcal{V}}\) is a modular category dominated by \(A = \{\mathcal{V}_\lambda | \lambda \in Y_+ \cap C_r\}\) and \(\mathcal{D}\) is in the ground ring.

There is a unique functor \(e_r : \mathcal{V}'_{\mathcal{V}} \to \mathcal{V}'_{\mathcal{V}}\) respecting the ribbon structure with \(e_r(\mathcal{V}_\lambda) = \mathcal{V}_\lambda\) and \(e_r(f) = f \otimes \text{Id}_{\mathbb{Z}[\zeta, r^{-1}]}\).

**6.2. A TQFT based on \(\mathcal{V}'_{\mathcal{V}}\).** We define a TQFT based on the ribbon category \(\mathcal{V}'_{\mathcal{V}}\) (cf. section 3.2 of [CL] and section 2.4). In the rest of this paper \(e\)-surfaces and \(e\)-cobordisms will mean \(\mathcal{V}'_{\mathcal{V}}\)-extended surfaces and \(\mathcal{V}'_{\mathcal{V}}\)-extended cobordisms unless otherwise specified.

**6.2.1. Parametrization of \(e\)-surfaces.** A \(\mathcal{V}'_{\mathcal{V}}\)-extended type (or type for short) is a tuple \((g; (V_1, \nu_1), \ldots, (V_m, \nu_m))\) where \(g \geq -1\) is an integer, \(V_i\)'s are arbitrary objects from \(\mathcal{V}'_{\mathcal{V}}\) and \(\nu_i\)'s belong to \(\{+, -\}\). If \(g = -1\) then \(m = 0\). We construct a standard handlebody \(H_t\) for every type \(t = (g; (V_1, \nu_1), \ldots, (V_m, \nu_m))\) shown in figure 5. It is a genus \(g\) handlebody standardly embedded in \(\mathbb{R}^3\) with a partially \(\mathcal{V}'_{\mathcal{V}}\)-colored ribbon graph \(R_t\) sitting in it (cf. section 2.4.1). Here we assume the empty space is a handlebody of genus \(-1\). The ribbon graph \(R_t\) consists of a coupon (the narrow rectangle near the bottom), \(m\) vertical bands (with \(m\) bases on the coupon and \(m\) bases \(p_i, i = 1, \ldots, m\) on \(\partial H_t\)) and \(g\) half-circled bands (oriented to the left with bases on the coupon). The \(m\) vertical bands are oriented and colored according to \(t\) such that the \(i\)-th
vertical band is colored by \( V_i \) and oriented up (resp. down) if \( \nu_i \) is \( - \) (resp. \( + \)). The normal vectors on the bands point toward the reader. Recall that the normal vector \( n_i \) at \( p_i \) is tangent to \( \partial \mathcal{H}_t \). The boundary of \( \mathcal{H}_t \) is an \( e \)-surface, called the standard \( e \)-surface of type \( t \) and denoted \( \Sigma_t \). The \( \mathcal{V}_t' \)-marks \( p_i, i = 1, \ldots, m \) are associated with \( (n_i, V_i, \nu_i) \). The Lagrangian subspace is the kernel of the map \( H_1(\partial \mathcal{H}_t, \mathbb{Q}) \to H_1(\mathcal{H}_t, \mathbb{Q}) \) induced by the inclusion. Note that \( \mathcal{H}_t \) is generally not an \( e \)-cobordism because \( R_t \) is only partially colored.

For any connected \( e \)-surface \( \Gamma \) let

\[
\pi(\Gamma) = \{ p \mid p : \Sigma_t \to \Gamma \text{ is an } e\text{-homeomorphism} \}
\]

be the set of all parametrizations of \( \Gamma \) up to \( e \)-isotopy (in the obvious sense). Note that \( \Gamma \) may be parametrized by more than one standard surfaces because the \( \mathcal{V}_t' \)-marks on \( \Gamma \) are not ordered. For any \( e \)-surface \( \Gamma \), \( \pi(\Gamma) \) is not empty (IV.6.4.2 of [1]).

**6.2.2. The modular functor.** For any integer \( a \) let \( A^a \) be the set \((C_\tau \cap Y_\tau)^a\) if \( g > 0 \) and the one-element set \( \{ \bullet \} \) otherwise. Let \( \Lambda_\bullet = \mathbb{Z}[\zeta, r^{-1}] \) and for \( \lambda \in A^g \) with \( g > 0 \) let

\[
\Lambda_\lambda = \bigotimes_{i=1}^g (\mathcal{V}_{\lambda_i} \otimes \mathcal{V}_{\lambda_i}^*)
\]

where \( \lambda_i \) is the \( i \)-th coordinate of \( \lambda \) and the tensor products are taken over \( \mathbb{Z}[\zeta, r^{-1}] \). Note that \( \Lambda_\lambda \) is an object in \( \mathcal{V}_\zeta' \).

For any \((\mathcal{V}_\zeta'\text{-extended})\) type \( t = (g; (V_1, \nu_1), \ldots, (V_m, \nu_m)) \) and \( \lambda \in A^g \) let

\[
T_{t,\lambda} = \text{Hom}_{\mathcal{V}_\zeta} (\mathbb{Z}[\zeta, r^{-1}], (\bigotimes_{i=1}^m e_r(V_i)^{\nu_i}) \otimes \Lambda_\lambda)
\]

where the tensor products are taken over \( \mathbb{Z}[\zeta, r^{-1}] \). Here \( e_r(V_i)^+=e_r(V_i) \) and \( e_r(V_i)^-=e_r(V_i)^* \) for \( i = 1, \ldots, m \).

Let \( T_t = \oplus_{\lambda \in A^g} T_{t,\lambda} \). Suppose \( \Gamma \) is a connected \( e \)-surface of type \( t \). For a parametrization \( p \in \pi(\Gamma) \), set \( \mathcal{T}_p(\Gamma) = T_t \). For two parametrizations \( p, p' \in \pi(\Gamma) \) define an isomorphism

\[
\varphi(p, p') : \mathcal{T}_p(\Gamma) \to \mathcal{T}_{p'}(\Gamma) \text{ of } \mathbb{Z}[\zeta, r^{-1}]\text{-modules}.
\]

Identify the vector spaces \( \{ \mathcal{T}_p(\Gamma) \}_{p \in \pi(\Gamma)} \) along the isomorphisms \( \{ \varphi(p, p') \}_{(p, p')} \). The resulting vector spaces \( \mathcal{T}(\Gamma) \) depends only on \( \Gamma \). For any parametrization \( p : \Sigma_t \to \Gamma \), \( \mathcal{T}(\Gamma) \) is canonically isomorphic to \( \mathcal{T}_p(\Gamma) \). Denote this isomorphism by \( p_2 \).

\[\text{1The actual formulas of } \varphi(p, p') \text{ will not be used in the sequel. Interested readers can find them in IV.6.3 and IV.6.4.2 of [1].}\]
Let $\Gamma$ and $\Gamma'$ be connected $e$-surfaces. For an $e$-homeomorphism $f : \Gamma \to \Gamma'$ we define $\mathcal{T}(f) : \mathcal{T}(\Gamma) \to \mathcal{T}(\Gamma')$ as follows. Pick any parametrization $p : \Sigma \to \Gamma$. Then $fp$ is a parametrization of $\Gamma'$. Set $\mathcal{T}(f) = (fp)_*(p_\ast)^{-1}$ which does not depend on the parametrization we choose (cf. IV.6.3.1 of [1]).

For non-connected $e$-surfaces we can do the above componentwisely and then form tensor product. Recall that $\mathcal{E}(\bar{\mathcal{V}}_\zeta')$ is the category of $e$-surfaces and $e$-homeomorphisms. We have a modular functor $\mathcal{T} : \mathcal{E}(\bar{\mathcal{V}}_\zeta') \to \text{Mod}(\mathbb{Z}[\zeta, r^{-1}])$ (cf. IV.6.3.3 of [1]).

Remark. Unlike [1], which uses only a modular category in the construction, we start with closed (cf. [G]). We say $\mathcal{T}(\emptyset)$ if $\mathcal{T}(\emptyset)$ is an even vacuum state if $(\mathcal{T}(\emptyset), \tau(\emptyset)) = (\emptyset, \emptyset)$.

6.2.3. An almost integral TQFT. Because $\bar{\mathcal{V}}_\zeta'$ is a modular category one can follow IV.9 in [1] to get a TQFT $(\mathcal{T}^e, \tau^e)$ based on $\bar{\mathcal{V}}_\zeta'$ with ground ring $\mathbb{Z}[\zeta, r^{-1}]$. Any $\bar{\mathcal{V}}_\zeta'$-extended cobordism $(M, \Gamma, \Lambda)$ induces a $\bar{\mathcal{V}}_\zeta'$-extended cobordism $(M, \bar{\Gamma}, \bar{\Lambda})$. One just needs to replace any object $V \in \bar{\mathcal{V}}_\zeta'$ by $e_r(V)$ and any morphism $f \in \bar{\mathcal{V}}_\zeta'$ by $e_r(f)$. Hence one has a $\mathbb{Z}[\zeta, r^{-1}]$-linear map $\tau^e(M) : \mathcal{T}^e(\emptyset) \to \mathcal{T}^e(\emptyset)$.

From the definition of the modular functor $\mathcal{T}$ given in section 6.2.2 it is easy to see that $\mathcal{T}(Z) = \mathcal{T}^e(\emptyset)$ for any $\bar{\mathcal{V}}_\zeta'$-extended surface $Z$. Set $\tau(M) = \tau^e(M)$.

Proposition 6.3. Suppose $\mathfrak{g}$ is not of type $E_8, F_4, G_2$ and $r \geq m(\mathfrak{g})$ is an odd prime. The pair $(\mathcal{T}, \tau)$ is a non-degenerate TQFT based on $\bar{\mathcal{V}}_\zeta'$ with ground ring $\mathbb{Z}[\zeta, r^{-1}]$. Furthermore, $(\mathcal{T}, \tau)$ is almost $\mathbb{Z}[\zeta]$-integral.

This proposition can be proved as lemma 3.1 and theorem 3.2 in [CL]. We will sketch the proof for the almost integrality in section 6.3.2.

6.3. Integral TQFTs. Gilmer showed in [G] that when $\mathfrak{g} = \mathfrak{sl}_2$ the TQFT constructed in [BHMV] can be restricted to a sub-cobordism such that the modules of surfaces are free over some ring of algebraic integers. We prove that the same can be done for the TQFT $(\mathcal{T}, \tau)$ constructed in section 6.2. We again fix a simple Lie algebra $\mathfrak{g}$ not of type $E_8, F_4, G_2$ and an odd prime $r > m(\mathfrak{g})$.

6.3.1. Restricted cobordisms. A cobordism $(M, \Sigma', \Sigma)$ is said to be targeted if the 0-th relative Betti number $\beta_0(M, \Sigma)$ is zero. We say $[M] = \tau(M)(1) \in \mathcal{T}(M)$ is a targeted vacuum state if $(M, \emptyset, \Sigma)$ is targeted. A cobordism $(M, \Sigma', \Sigma)$ is said to be even if

$$w(M) \equiv \beta_0(\Sigma) + \beta_1(\Sigma)/2 + \beta_0(K(M)) + \beta_1(K(M)) \mod 2,$$

where $K(M)$ is a homologically canonical closed 3-manifold, which is equal to $M$ if $M$ is closed (cf. [G]). We say $[M]$ is an even vacuum state if $(M, \emptyset, \Sigma)$ is even.
For any \(e\)-surface \(\Sigma\), let \(S(\Sigma)\) (resp. \(T_+(\Sigma), S_+(\Sigma)\)) be the submodule of \(\mathcal{F}(\Sigma)\) generated by targeted (resp. even, even targeted) vacuum states over \(\mathbb{Z}[\zeta]\) (resp. \(\mathbb{Z}[\xi, r^{-1}], \mathbb{Z}[\xi]\)).

**Proposition 6.4.** Given the data above, for any targeted (resp. even, even targeted) cobordism \((M, \Sigma', \Sigma)\),

\[
\tau(M) : S(\Sigma') \rightarrow S(\Sigma), \quad \text{(resp.} \ T_+(\Sigma') \rightarrow T_+(\Sigma), \ S_+(\Sigma') \rightarrow S_+(\Sigma)\).
\]

For any \(e\)-surface \(\Sigma\), \(\mathcal{T}(\Sigma)\) (resp. \(T_+(\Sigma), S(\Sigma), S_+(\Sigma)\)) is a free \(\mathbb{Z}[\zeta, r^{-1}], \mathbb{Z}[\xi, r^{-1}], \mathbb{Z}[\zeta]^{-}\)module of finite rank.

We will prove this proposition in section 6.3.3. It implies proposition 1.2 and contains theorem 5 of [G] because \(\text{sn}(w_0) = -1\) and \(\ell = 1\) when \(g = \mathfrak{sl}_2\).

6.3.2. Surgery formula. Let \(J = J^\#\) be the functor from \(\mathcal{V}_\xi\)-colored ribbon graphs to \(\mathcal{V}_{\xi'}\) that respects the ribbon structure, cf. I.2.5 of [I]. The functor \(J^\#\) is usually called the colored Jones polynomial. Let \(\Omega \sqcup L\) be a ribbon graph in \(S^3\) where \(\Omega\) is \(\mathcal{V}_{\xi'}\)-colored and \(L\) is an uncolored framed link of \(m\) components. Recall that for any integer \(a \geq 0\), \(A^a = (C_r \cap Y_+)^a\). For \(\mu = (\mu_1, \ldots, \mu_m) \in A^m\), denote by \(L(\mu)\) the \(\mathcal{V}_{\xi'}\)-extended framed link with the \(i\)-th component colored by \(\xi_{\mu_i}\). Let \(U^m\) be the trivial link of \(m\) components. Define

\[
Q_{\Omega \sqcup L(\mu)} = J_{U^m(\mu)J_{\Omega \sqcup L(\mu)}},
\]

and

\[
F_{(L, \Omega)} = \sum_{\mu \in A^m} Q_{\Omega \sqcup L(\mu)}.
\]

(11)

Let \((M, \Omega, \omega)\) be a closed connected \(\mathcal{V}_{\xi'}\)-extended 3-manifold. We first recall how to compute \([M]\) from a surgery description of \(M\). Suppose that \(M\) is the result of surgery on a framed link \(L \subset S^3\). Then there exists a \(\mathcal{V}_{\xi'}\)-colored ribbon graph \(\Omega'\) in \(S^3\) disjoint from \(L\) such that \((M, \Omega)\) is the result of \((S^3, \Omega' \sqcup \bar{L})\) surgery on \(L\). By slight abuse of notation we will write \(\Omega\) for \(\Omega'\). Let \(U_+\) and \(U_-\) be the trivial knots with +1 and \(-1\) framing respectively. It’s known that \(F_- F_+ = D^2\) where \(F_\pm = F_{(U_\pm, \emptyset)}\) as in equation (11), cf. [K]. Let \(\kappa\) be the square root of \(F_- / F_+\) such that \(\kappa D = F_-\).

According to IV.9.2 and II.2.2 of [I]

\[
[M] = \tau(M, \Omega, \omega)(1) = F^\sigma_{(L, \Omega)} D^{-(\sigma + m + 1)} \kappa^w
\]

\[
= F_{(L, \Omega)} D^{-(m + 1)} \kappa^{w + \sigma}
\]

\[
= \frac{F_{(L, \Omega)}}{F^\sigma_{-} + \beta_1 F^\sigma_{+}} F^{-1} \kappa^{\beta_1 + w + 1}
\]

where \(\beta_1\) is the first Betti number of \(M\) and \(\sigma, \sigma_+\) and \(\sigma_-\) are the signature, the number of positive eigenvalues and the number of negative eigenvalues of the linking matrix of \(L\) respectively. Recall that \(\sigma = \sigma_+ - \sigma_-\) and \(m = \sigma_+ + \sigma_- + \beta_1\).

**Lemma 6.5.** If \((M, \Omega, \omega)\) is a closed connected \(\mathcal{V}_{\xi'}\)-extended 3-manifold then \(F_-[M]\) is in \(\mathbb{Z}[\zeta]\) and it is in \(\mathbb{Z}[\xi]\) if \(M\) is furthermore even.
Proof. It is proved in [CL] that $\frac{F(L, \Omega)}{F(-, \Omega)}$ is in $\mathbb{Z}[\xi]$. Therefore we only need to prove $\kappa^{\beta_1 + w + 1}$ belongs to an appropriate ring. By lemma 6.2, $\kappa = F_- / D$ is in the fractional field of $\mathbb{Z}[\xi]$. Since $\kappa$ is always a root of 1 (cf. [BK]), $\kappa$ is an element of $\mathbb{Z}[\xi]$. By equation (10) $\beta_1 + w + 1$ is an even number if $M$ is even. □

This lemma implies that $(\mathcal{T}, \tau)$ is $\mathbb{Z}[\xi]$-almost integral and $(\mathcal{T}_+, \tau)$ is $\mathbb{Z}[\xi]$-almost integral.

**Corollary 6.6.** For any $e$-surface $\Sigma$, $S(\Sigma)$ (resp. $S_+(\Sigma)$) is a finitely generated projective $\mathbb{Z}[\xi]$- (resp. $\mathbb{Z}[\xi]$-) module and $S(\Sigma) \otimes_{\mathbb{Z}[\xi]} \mathbb{Z}[\xi, r^{-1}] = \mathcal{T}(\Sigma)$ (resp. $S_+(\Sigma) \otimes_{\mathbb{Z}[\xi]} \mathbb{Z}[\xi, r^{-1}] = \mathcal{T}_+(\Sigma)$).

This can be proved using theorem 1 of [G] and lemma 6.5.

**6.3.3. Proof of proposition 6.4** By the definition of $\mathcal{T}$ in section 6.2.2 and the fact that $\tilde{V}_\xi'$ is dominated by free $\mathbb{Z}[\xi, r^{-1}]$-modules of finite rank, we see that $\mathcal{T}(\Sigma)$ is free of finite rank.

The freeness of $\mathcal{T}_+(\Sigma)$ can be proved similarly as proposition 7 in [G] using the freeness of $\mathcal{T}(\Sigma)$ and proposition 6.5. Note that $\tau$, $\xi$, and $\zeta$ are denoted $p$, $A_1^\alpha$, and $\alpha_p$ respectively in [G]. (Note that $\mathbb{Z}[A_p] = Z[A_1^2]$ if $p$ is odd.) Since $S_+(\Sigma) \otimes_{\mathbb{Z}[\xi]} \mathbb{Z}[\xi, r^{-1}]$ is isomorphic to $\mathcal{T}_+(\Sigma)$ one can show that $S_+(\Sigma)$ is a free $\mathbb{Z}[\xi]$-module using lemma 1 (and the discussion before it) of [G]. Hence $S(\Sigma) = S_+(\Sigma) \otimes_{\mathbb{Z}[\xi]} \mathbb{Z}[\xi]$ is also free.

**6.3.4. Bases.** Gilmer and Masbaum announced that explicit bases for the integral TQFT in [G] can be obtained using the Kauffman bracket skein module. It will be interesting to know what these bases stand for in the representation theory of quantum groups. Are they related to the canonical bases?

**References**

[A] Henning Haahr Andersen. Tensor products of quantized tilting modules. *Comm. Math. Phys.*, 149(1):149–159, 1992.

[BHMV] C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel. Topological quantum field theories derived from the Kauffman bracket. *Topology*, 34(4):883–927, 1995.

[BK] Bojko Bakalov and Alexander Kirillov, Jr. *Lectures on tensor categories and modular functors*, volume 21 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2001.

[CL] Qi Chen and Thang Le. Almost integral TQFTs from simple Lie algebras. *Submitted for publication*.

[DCK] Corrado De Concini and Victor G. Kac. Representations of quantum groups at roots of 1. In Operator algebras, unitary representations, enveloping algebras, and invariant theory (Paris, 1989), volume 92 of *Progr. Math.*, pages 471–506. Birkhäuser Boston, Boston, MA, 1990.

[G] Patrick Gilmer. Integrality for TQFTs. *arXiv:math.QA/0105059*.

[GK] Sergei Gelfand and David Kazhdan. Examples of tensor categories. *Invent. Math.*, 109(3):595–617, 1992.

[H] James E. Humphreys. *Introduction to Lie algebras and representation theory*. Springer-Verlag, New York, 1978. Second printing, revised.

[J] Jens Carsten Jantzen. *Lectures on quantum groups*. American Mathematical Society, Providence, RI, 1996.

[Ka] Christian Kassel. *Quantum groups*, volume 155 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[Ki] Alexander A. Kirillov, Jr. On an inner product in modular tensor categories. *J. Amer. Math. Soc.*, 9(4):1135–1169, 1996.

[Le1] Thang T. Q. Le. Integrality and symmetry of quantum link invariants. *Duke Math. J.*, 102(2):273–306, 2000.
[Le2] Thang T. Q. Le. Quantum invariants of 3-manifolds: integrality, splitting, and perturbative expansion. In Proceedings of the Pacific Institute for the Mathematical Sciences Workshop “Invariants of Three-Manifolds” (Calgary, AB, 1999), volume 127, no. 1-2, pages 125–152, 2003.

[Lu] George Lusztig. Introduction to quantum groups. Birkhäuser Boston Inc., Boston, MA, 1993.

[S] John Stembridge. The coxeter and weyl Packages. www.math.lsa.umich.edu/ jrs/maple.html.

[T] V. G. Turaev. Quantum invariants of knots and 3-manifolds. Walter de Gruyter & Co., Berlin, 1994.

[W] Zhe Xian Wan. Lie algebras. Pergamon Press, Oxford, 1975. Translated from the Chinese by Che Young Lee, International Series of Monographs in Pure and Applied Mathematics, Vol. 104.

E-mail address: qichen@buffalo.edu