Topological fundamental groups can distinguish spaces with isomorphic homotopy groups

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Abstract

We exhibit path connected metric spaces and a map $f : X \to Y$ such that $f$ induces an isomorphism on homotopy groups but such that, with natural topologies, $X$ and $Y$ have nonhomeomorphic fundamental groups. Consequently we can conclude $X$ and $Y$ have distinct homotopy types despite the failure of the Whitehead theorem in this context.

1 Introduction

Given CW complexes $X$ and $Y$, the Whitehead theorem ([9]) asserts that a map $f : X \to Y$ is a homotopy equivalence provided $f$ induces an isomorphism on homotopy groups. However the result can fail in the context of path connected metric spaces. For example the standard Warsaw circle has trivial homotopy groups but fails to have the homotopy type of a point. This note aims to show the topological fundamental group can help counterbalance the general failure of the Whitehead theorem.

For a general space $X$ work of Biss [1] initiates the development of a theory whose fundamental notion is the following. Endowed with the quotient topology inherited from the path components of based loops in $X$, the familiar based fundamental group $\pi_1(X, p)$ of a topological space $X$ becomes a topological group. For example if $X$ is locally contractible then loops in $X$
are homotopically invariant under small perturbation, and consequently the fundamental group $\pi_1(X, p)$ has the discrete topology. For spaces that are complicated both locally and globally, the topology of $\pi_1(X, p)$ can be more interesting ([3] [4] [7]). An important feature of the theory is that if $X$ and $Y$ have the same homotopy type then $\pi_1(X, p)$ and $\pi_1(Y, p)$ are isomorphic and homeomorphic (Proposition 3.3 [4]).

These facts motivate the question of whether the added topological structure on $\pi_1(X, p)$ can ever succeed in distinguishing the homotopy type of spaces $X$ and $Y$ in instances when the hypotheses of the Whitehead theorem are satisfied.

In fact this is so and in this note we exhibit aspherical spaces $X$ and $Y$ such that inclusion $j : X \to Y$ induces an isomorphism on homotopy groups. However $\pi_1(X)$ and $\pi_1(Y)$ fail to be homeomorphic, and thus we can conclude that $X$ and $Y$ do not have the same homotopy type despite the failure of the Whitehead theorem for this pair of examples.

The theory of topological fundamental groups is still in the early stages of development ([5] [6] [8]) and it is hoped this note will be seen as promoting its utility and helping to motivate its continued investigation. For example the space $Y$ constructed in this paper is not locally path connected. This suggests the following.

**Question.** Suppose $Y$ is an aspherical (metric) Peano continuum and $X \subset Y$ is aspherical and path connected. Suppose inclusion $j : X \hookrightarrow Y$ induces an isomorphism $j^* : \pi_1(X, p) \to \pi_1(Y, p)$. Must $j^*$ be a homeomorphism? If $j^*$ is a homeomorphism must $j$ be a homotopy equivalence?

## 2 Definitions and Preliminaries

All definitions are compatible with those found in Munkres [10]. If $X$ is a metrizable space and $p \in X$ let $C_p(X) = \{ f : [0, 1] \to X \text{ such that } f \text{ is continuous and } f(0) = f(1) = p \}$. Endow $C_p(X)$ with the topology of uniform convergence.

The **topological fundamental group** $\pi_1(X, p)$ is the set of path components of $C_p(X)$ endowed with the quotient topology under the canonical surjection $q : C_p(X) \to \pi_1(X, p)$ satisfying $q(f) = q(g)$ if and only if $f$ and $g$ belong to the same path component of $C_p(X)$. Thus a set $U \subset \pi_1(X)$ is open if and only if $q^{-1}(U)$ is open in $C_p(Y)$.

**Remark 1** The topological fundamental group $\pi_1(X, p)$ is a topological group
under concatenation of paths. (Proposition 3.1[1]). A map $f : X \to Y$ determines a continuous homomorphism $f^* : \pi_1(X, p) \to \pi_1(Y, f(p))$ via $f^*([\alpha]) = [f(\alpha)]$ (Proposition 3.3[1]). If $X$ and $Y$ have the same homotopy type then $\pi_1(X)$ is homeomorphic and isomorphic to $\pi_1(Y)$ (Corollary 3.4[1]). For the remainder of this paper all fundamental groups will be considered topological groups.

The space $X$ is **semilocally simply connected** at $p$ if there exists an open set $U \subset X$ such that inclusion $j : U \hookrightarrow X$ induces the trivial homomorphism $j^* : \pi_1(U, p) \to \pi_1(X, p)$. The space $Z$ is **discrete** if each one point subset of $Z$ is open.

**Remark 2** The main result of [3] shows that if $X$ is locally path connected then $\pi_1(X, p)$ is discrete if and only if $\pi_1(X, p)$ is semilocally simply connected.

### 3 Main result

**Theorem 3** There exist path connected aspherical separable metric spaces $X$ and $Y$ such that $X \subset Y$ and inclusion $j : X \hookrightarrow Y$ induces an isomorphism $j^* : \pi_1(X, p) \to \pi_1(Y, p)$. Thus $(X, Y, j)$ satisfies the hypothesis of the Whitehead theorem. However the topological fundamental groups $\pi_1(X, p)$ and $\pi_1(Y, p)$ are not homeomorphic. Hence the **topology** of fundamental groups has the capacity to distinguish the homotopy type of $X$ and $Y$ when the algebra fails to do so.

**Proof.** The basic idea is to let $X$ denote the countable union of a sequence of large simple closed curves $C_1 \cup C_2 \ldots$ joined at a common point $p$. Such a space is sometimes called a bouquet of infinitely many loops. In particular $X$ is locally contractible and should not be mistaken for the Hawaiian earring. The space $Y$ is a compactification of $X$ obtained by attaching a line segment $\alpha$ based at $p$ such that the curves $C_n$ converge to $\alpha$ in the Hausdorff metric.

Since each of $X$ and $Y$ is path connected and 1 dimensional, if $n \neq 1$ then $\pi_n(X, p) = \pi_n(Y, p) = 1$. Thus, to show that $(X, Y, j)$ satisfies the hypothesis of the Whitehead theorem it suffices to show that $j^* : \pi_1(X, p) \to \pi_1(Y, p)$ is an isomorphism.

Formally for $n \geq 2$ let $C_n \subset \mathbb{R}^2$ denote boundary of the convex hull of the following 3 point set: $\{(0,0), (\frac{1}{n}, 1), (\frac{1}{n}, 1) + \frac{1}{10(n+1)}(n, -1)\}$. Then for each
$n \geq 2$ $C_n$ is the boundary of a triangle and in particular $C_n$ is a simple closed curve. Let $p = (0, 0)$. Note $C_n \cap C_m = p$ if $n \neq m$. Let $\alpha$ denote the line segment $[(0, 0), (0, 1)] \subset R^2$. Let $X = \cup_{n=2}^{\infty} C_n$ and let $Y = \overline{X}$. Note $X \cup \alpha$. Note the path connected spaces $X$ and $Y$ are 1 dimensional and hence aspherical ([2]). We will show inclusion $j : X \hookrightarrow Y$ induces an isomorphism $j^* : \pi_1(X, p) \rightarrow \pi_1(Y, p)$.

To prove $j^*$ is one to one suppose $f : \partial D^2 \rightarrow X$ is inessential in $Y$ and suppose $f(1) = p$. Let $F : D^2 \rightarrow Y$ satisfy $F_{\partial D^2} = f$. Let $U = F^{-1}(\alpha \setminus p)$. Since $D^2$ is locally path connected, and since $\alpha \setminus p$ is a component of $X \setminus p$ the set $U$ is open. Suppose $x \in \overline{U} \setminus U$. Then $F(x) = p$ since $\alpha = \overline{\alpha \setminus p}$. Thus, we may redefine $F$ to be $p$ on the set $U$ and obtain a continuous function $G : D^2 \rightarrow X$ such that $G_{\partial D^2} = f$. This proves $j^*$ is one to one.

To prove $j^*$ is a surjection suppose $\beta \in C_p(Y)$. We must show there exists $\gamma \in C_p(X)$ such that $\gamma$ and $\beta$ are path homotopic in $Y$. Since $\text{im}(\beta)$ is a Peano continuum $\text{im}(\beta)$ is locally path connected. Thus we may choose $N$ such that $\text{im}(\beta) \cap (\{\frac{1}{N}, \frac{1}{N+1}, \ldots\} \times \{1\}) = \emptyset$. Let $A = \text{im}(\beta) \cap (\alpha \cup C_N \cup C_{N+1} \ldots)$. Let $B = C_1 \cup C_2 \ldots \cup C_{N-1}$. Note $A$ is a contractible Peano continuum such that $p \in A$. Moreover $B$ is a strong deformation retract of $B \cup A$. Thus there exists a homotopy $h_t : A \cup B \rightarrow B$ such that $h_0 = \text{id}_{A \cup B}$ and $h_t$ fixes $B$ pointwise. Thus the homotopy $h_t(\beta)$ determines that $\beta$ is path homotopic in $Y$ to $\gamma = h_1(\beta)$. Note $\text{im}(\gamma) \subset X$. Hence $j^*$ is a surjection and therefore $j^*$ is an isomorphism.

Since the space $X$ is locally contractible $\pi_1(X, p)$ has the discrete topology (Remark [2]). On the other hand $\pi_1(Y, p)$ does not have the discrete topology, since there exists an inessential loop $f \in C_p(Y)$ which is the uniform limit of inessential loops. (Let the (inessential) map $f$ go up and down once on $\alpha$ and let $f_n$ be an (essential) loop going once around $C_n$). Thus the path component of the constant map is not open in $C_p(Y)$ and thus $\pi_1(Y, p)$ cannot have the discrete topology. Thus $\pi_1(X, p)$ and $\pi_1(Y, p)$ are not homeomorphic and hence $X$ and $Y$ do not have the same homotopy type. ■

**Remark 4** The space $Y$ constructed is semilocally simply connected. However $\pi_1(Y, p)$ does not have the discrete topology. Consequently $Y$ is a counterexample to the (false) Theorem 5.1 [2] which asserts that $\pi_1(Y, p)$ is discrete if and only if $\pi_1(Y, p)$ is semilocally simply connected.
References

[1] Biss, Daniel K. *The topological fundamental group and generalized covering spaces*. Topology Appl. 124 (2002), no. 3, 355–371.

[2] Curtis, M. L.; Fort, M. K., Jr. *Homotopy groups of one-dimensional spaces*. Proc. Amer. Math. Soc. 8 (1957), 577–579.

[3] Fabel, Paul. *A characterization of spaces with discrete topological fundamental group*. Preprint. http://front.math.ucdavis.edu/math.GN/0502249

[4] Fabel, Paul. *The fundamental group of the harmonic archipelago*. Preprint. http://front.math.ucdavis.edu/math.AT/0501426

[5] Fabel, Paul. *The topological Hawaiian earring group does not embed in the inverse limit of free groups*. Preprint. http://front.math.ucdavis.edu/math.GN/0501482

[6] Fabel, Paul *A retraction theorem for topological fundamental groups with applications to the Hawaiian earring*. Preprint. http://front.math.ucdavis.edu/math.AT/0502218

[7] Fabel, Paul *The Hawaiian earring group is topologically incomplete*. Preprint. http://front.math.ucdavis.edu/math.GN/0502148

[8] Fabel, Paul *A monomorphism theorem for the inverse limit of nested retracts*. Preprint. http://front.math.ucdavis.edu/math.AT/0502275

[9] Hatcher, Allen. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[10] Munkres, James R., *Topology: a first course*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1975.