THE CONSTRAINED MKP HIERARCHY AND THE GENERALIZED KUPERSHMIKT-WILSON THEOREM

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ABSTRACT The constrained Modified KP hierarchy is considered from the viewpoint of modification. It is shown that its second Poisson bracket, which has a rather complicated form, is associated to a vastly simpler bracket via Miura-type map. The similar results are established for a natural reduction of MKP.

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The aim of the paper is to construct the generalized Miura map and the modification for the constrained modified KP hierarchy (cMKP). We shall show that the rather complicated second Hamiltonian structure for cMKP is transformed to a vastly simpler structure (essentially $\partial^{\pm}_x$).

As a transformation between KdV and MKdV systems, Miura map played important role in the development of the Soliton theory. Due to its importance, Miura map has been generalized to various integrable systems and we here just cite the well-known Kupershmidt-Wilson theorem [13] for the Gelfand-Dickey (GD) hierarchy, Drinfeld-Sokolov’s results for the integrable systems related to Kac-Moody-Lie algebras [7], Miura maps for the hierarchies from the energy-dependent Schrödinger operator [4] and matrix GD hierarchy [8], etc.. Apart from its mathematical interest, Miura type map is relevant from the viewpoint of physics. Indeed, in many cases, Miura maps serve as free field realizations for the corresponding W algebras.

More recently, so-called the constrained KP hierarchy attracts much attention (see [4] [18] and the references there). It is known that this hierarchy plays a role in theory of matrix models [2]. Its modifications and Miura transformation were considered initially in [14] for the concrete cases and in [4] for the general case. The same sort of problem is considered for the \((n, m)^{th}\) -KdV hierarchy [3] (see also [8] [16]).

We are interested in the constrained modified KP hierarchy. It is to be remarked that the MKP is introduced in [11] and studied extensively in [12] [10]. The Lax operator for cMKP is

\[ \mathcal{L} = \partial^n + v_{n-1} \partial^{n-1} + \cdots + v_0 + \partial^{-1} v_{-1}, \]

and the Lax equation reads

\[ \frac{d}{dt} \mathcal{L} = [(L^q_{\leq 1}, \mathcal{L}], \quad (1) \]

where $\partial = \partial / \partial x$ and for any pseudo-differential operator $A$, we assume $A = \sum_{i \geq 0} a_i \partial^i + \sum_{i \leq -1} \partial^i a_i =: \sum_{i \geq 0} A_i + \sum_{i \leq -1} A_i$. As shown by Oevel and Strampp [18], the cMKP is a bi-Hamiltonian system

\[ \frac{d}{dt} \mathcal{L} = P_1 \delta H_{q+1} = P_2 \delta H_q, \quad (3) \]
with the Poisson tensors or Hamiltonian operators $\mathcal{P}_i$ are defined by

$$\mathcal{P}_1 \frac{\delta H}{\delta \mathcal{L}} = \left( \left( \frac{\delta H}{\delta \mathcal{L}} \right)_{\geq 1}, \mathcal{L} \right) - \left( \left( \frac{\delta H}{\delta \mathcal{L}} \right)_{\geq -1}, \mathcal{L} \right),$$

(4)

$$\mathcal{P}_2 \frac{\delta H}{\delta \mathcal{L}} = \left( \mathcal{L} \frac{\delta H}{\delta \mathcal{L}} \right)_{\geq 0} \mathcal{L} - \mathcal{L} \left( \frac{\delta H}{\delta \mathcal{L}} \mathcal{L} \right)_{\geq 0} - \left( \left( \frac{\delta H}{\delta \mathcal{L}} \mathcal{L} \right)_{\geq 0}, \mathcal{L} \right) + \mathcal{L} \delta^{-1} \left( \left[ \mathcal{L}, \delta \frac{\delta H}{\delta \mathcal{L}} \right] \right),$$

(5)

$$H_q = \frac{n}{q} \int \text{res} \mathcal{L}^2 \delta \mathcal{L} dx,$$

where res stands for the usual residue: for any operator $A = \sum a_i \partial^i$, $\text{res} A = a_{-1}$; also the notion $A_0$ means taking the projection to $a_0$; the gradient $\frac{\delta H}{\delta \mathcal{L}}$ is defined as $\frac{\delta H}{\delta \mathcal{L}} = \sum_{i=-1}^{n-1} \partial^{-i-1} \frac{\delta H}{\delta \mathcal{L}}$.

While the first bracket induced by $\mathcal{P}_1$ is the Lie-Poisson bracket associated to the relevant algebra, the second one has a rather complicated form. Our aim is to get a better understanding of its structure. We will see that this complicated Poisson tensor is Miura-related to a very simple one. Indeed, the initial steps along this direction were taken in a previous paper[15] and there we made a conjecture which now is formulated as the following:

**Theorem 1** Define a Miura transformation between the coordinates $(v_{n-1}, \ldots, v_0, v_{-1})$ and $(w_n, \ldots, w_0)$ by the following factorization of the Lax operator $\mathcal{L}$

$$\mathcal{L} = \partial^{-1} (\partial - w_n)(\partial - w_{n-1}) \cdots (\partial - w_0),$$

(6)

then, the second bracket induced by $\mathcal{P}_2$

$$\{F, G\} = \int \text{res} \left( \frac{\delta F}{\delta \mathcal{L}} (\mathcal{P}_2 \frac{\delta G}{\delta \mathcal{L}}) \right) dx,$$

(7)

is transformed to

$$\{F, G\} = - \int \sum_{i=0}^{n} \frac{\delta F}{\delta w_i} \frac{\delta G}{\delta w_i}^t dx + \int \sum_{i=0}^{n} \frac{\delta F}{\delta w_i} \sum_{j=0}^{n} \frac{\delta G}{\delta w_j}^t dx.$$

(8)

where $t = \partial$ for short.
To make the things easier, we first convert the bracket(7) to more familiar Gelfand-Dickey brackets as in [9]. This is the context of the following Lemma.

**Lemma 1** Let us define a new operator by

$$ L \equiv \partial L = \partial^{n+1} + u_{n}\partial^n + \cdots + u_0, $$

then, the following relation between brackets is satisfied

$$ \{F, G\} = \{F, G\}_2 + \{F, G\}_3, $$

where

$$ \{F, G\}_2 = \int \text{res} \frac{\delta F}{\delta L} \left( (L \frac{\delta G}{\delta L})_{\geq 0} L - L (\frac{\delta G}{\delta L} L)_{\geq 0} \right) dx, $$

and

$$ \{F, G\}_3 = \int \text{res} \frac{\delta F}{\delta L} \left[ L, D^{-1} \text{res} \left[ L, \frac{\delta G}{\delta L} \right] \right] dx, $$

are the so-called second and third Gelfand-Dickey brackets associated to $L$ respectively. $D^{-1}$ denotes an integration: $D^{-1} = \int x \, dz$; $\{F, G\}$ is given by (7).

**Proof.** Since $\delta F = \int \text{res}(\frac{\delta F}{\delta L} \delta L) dx = \int \text{res}(\frac{\delta F}{\delta L} \partial \delta L) dx = \int \text{res}(\frac{\delta F}{\delta L} \delta L) dx$, one has

$$ \frac{\delta F}{\delta L} = \frac{\delta F}{\delta \partial} \partial, $$

Then,

$$ \left( L \frac{\delta F}{\delta L} \right)_{\geq 0} = \left( \partial L \frac{\delta F}{\delta L} \partial^{-1} \right)_{\geq 0} = \left( L \frac{\delta F}{\delta L} \right)_{\geq 0} + \left( L \frac{\delta F}{\delta L} \right)_{\geq 1} \partial^{-1}, $$

$$ \left( L \frac{\delta F}{\delta L} \right)_{\geq 0} L = \left( L \frac{\delta F}{\delta L} \right)_{\geq 0} \partial L + \left( L \frac{\delta F}{\delta L} \right)_{\geq 1} \partial L $$

$$ = \partial \left( L \frac{\delta F}{\delta L} \right)_{\geq 0} L - \left( L \frac{\delta F}{\delta L} \right)_{\geq 1} L. $$
\[ L(\frac{\delta F}{\delta L}L) \geq 0 = \partial L(\frac{\delta F}{\delta L}L) \geq 0. \]  

(15)

\[ \text{res} \left[ L, \frac{\delta F}{\delta L} \right] = \text{res} \left( \partial L \frac{\delta F}{\delta L} \partial^{-1} - \frac{\delta F}{\delta L} L \right) \]

\[ = \text{res} \left( \left[ L, \frac{\delta F}{\delta L} \right] - \frac{\delta F}{\delta L} + \partial L \frac{\delta F}{\delta L} \partial^{-1} \right) \]

\[ = \text{res} \left[ L, \frac{\delta F}{\delta L} \right] + \left( \frac{\delta F}{\delta L} \right)'_0. \]

(16)

Thus, using the above equations(14-16), we obtain

\[ \{F, G\}_2 + \{F, G\}_3 = \]

\[ = \int \text{res} \left( \frac{\delta F}{\delta L} \partial^{-1} \left( \partial \left( L \frac{\delta G}{\delta L} \right) \right)_{\geq 0} L - \left( L \frac{\delta G}{\delta L} \right)'_0 \right) \]

\[ - \partial L \left( \frac{\delta G}{\delta L} \right)_{\geq 0} + \left[ \partial L, D^{-1} \text{res} \left[ L, \frac{\delta G}{\delta L} \right] \right] + \left[ \partial L, \left( L \frac{\delta G}{\delta L} \right)'_0 \right] \right) \, dx \]

\[ = \int \text{res} \left( \frac{\delta F}{\delta L} L \partial^{-1} \left( \partial \left( L \frac{\delta G}{\delta L} \right) \right)_{\geq 0} L - \partial L \left( \frac{\delta G}{\delta L} \right)_{\geq 0} - \partial \left[ \left( L \frac{\delta G}{\delta L} \right)'_0, L \right] \right) \]

\[ + \partial \left[ L, D^{-1} \text{res} \left[ L, \frac{\delta G}{\delta L} \right] \right] + \left[ \left[ L, \frac{\delta G}{\delta L} \right], L \right] \right) \right) \, dx \]

\[ = \{F, G\}. \]

This completes the proof the lemma 1. □

With lemma 1 in hand, we may use the standard Kupershmidit-Wilson theorem for the second GD bracket. It states that

\[ \{F, G\}_2 = - \int \sum_{i=0}^n \left( \frac{\delta F}{\delta w_i} \right) \left( \frac{\delta G}{\delta w_i} \right)' \, dx, \]  

(17)

thus, to prove the theorem 1, we need only to prove the following lemma

**Lemma 2**

\[ \{F, G\}_3 = \int \sum_{i=0}^n \frac{\delta F}{\delta w_i} \sum_{j=0}^n \left( \frac{\delta G}{\delta w_j} \right)' \, dx. \]
Proof. Using $\delta F = \int \text{res} \frac{\delta F}{\delta L} \delta L dx = \int \sum_{i=0}^{n} \frac{\delta F}{\delta w_i} \delta w_i dx$, we obtain
\[ \frac{\delta F}{\delta w_i} = -\text{res} \left( \partial_{i-1} \cdots \partial_0 \frac{\delta F}{\delta L} \partial_n \cdots \partial_{i+1} \right), \quad (18) \]
here we used the notion $\partial_i = \partial - w_i$ for short.

Now,
\[ \int \sum_{i=0}^{n} \frac{\delta F}{\delta w_i} \sum_{j=0}^{n} \left( \frac{\delta G}{\delta w_j} \right)' dx = \int \text{res} \left( \sum_{i=0}^{n} \left( \partial_{i-1} \cdots \partial_0 \frac{\delta F}{\delta L} \partial_n \cdots \partial_{i+1} \right) \left( \text{res} \sum_{j=0}^{n} \left( \partial_{j-1} \cdots \partial_0 \frac{\delta G}{\delta L} \partial_n \cdots \partial_{j+1} \right)' \right) \right) dx \]
\[ = \int \text{res} \left( \sum_{i=0}^{n} \left( \partial_{i-1} \cdots \partial_0 \frac{\delta F}{\delta L} \partial_n \cdots \partial_{i+1} \right) \left[ \partial, \text{res} \sum_{j=0}^{n} \left( \partial_{j-1} \cdots \partial_0 \frac{\delta G}{\delta L} \partial_n \cdots \partial_{j+1} \right) \right] \right) dx \]
\[ = \int \text{res} \left( \sum_{i=0}^{n} \left( \partial_{i-1} \cdots \partial_0 \frac{\delta F}{\delta L} \partial_n \cdots \partial_{i+1} \right) \left( \partial \left( \text{res} \sum_{j=0}^{n} \left( \partial_{j-1} \cdots \partial_0 \frac{\delta G}{\delta L} \partial_n \cdots \partial_{j+1} \right) \right) \right) \right) dx. \]

We replace $\partial$ by $\partial_i$ in the last expression, which changes nothing, then we find
\[ \int \sum_{i=0}^{n} \frac{\delta F}{\delta w_i} \sum_{j=0}^{n} \frac{\delta G}{\delta w_j} dx = \int \text{res} \left( \left[ \frac{\delta F}{\delta L}, L \right] \right. \left. \text{res} \sum_{i=0}^{n} \left( \partial_{i-1} \cdots \partial_0 \frac{\delta G}{\delta L} \partial_n \cdots \partial_{i+1} \right) \right), \quad (19) \]
however,
\[ \left( \text{res} \sum_{i=0}^{n} \left( \partial_{i-1} \cdots \partial_0 \frac{\delta G}{\delta L} \partial_n \cdots \partial_{i+1} \right) \right)' = \]
\[ = \text{res} \left[ \partial, \sum_{i=0}^{n} \left( \partial_{i-1} \cdots \partial_0 \frac{\delta G}{\delta L} \partial_n \cdots \partial_{i+1} \partial \right) \right] = \text{res} \left[ L, \frac{\delta G}{\delta L} \right], \quad (20) \]
which implies
\[ \text{res} \sum_{i=0}^{n} \left( \partial_{i-1} \cdots \partial_0 \frac{\delta G}{\delta L} \partial_n \cdots \partial_{i+1} \right) = D^{-1} \text{res} \left[ L, \frac{\delta G}{\delta L} \right], \quad (21) \]
now substituting the above expression into the equation (19), we complete the proof. □

Thus, the combination of the last two Lemmas provides us a completed proof for our theorem 1.

Next let us turn our attention to a special reduction of the general case. That is, we consider the following Lax operator

\[
\hat{L} = \partial^n + v_{n-1} \partial^{n-1} + \cdots + v_1 \partial,
\]

(22)

It is easy to see that this is a consistent reduction. The Hamiltonian structures for the associated hierarchy are inherited from the one (4) by Dirac reduction [17]. Since we are only interested in the second one, let us write it down here

\[
\hat{P}(\delta H / \delta \hat{L}) = \left( \hat{L} \frac{\delta H}{\delta \hat{L}} \right)_{\geq 0} \hat{L} - \hat{L} \left( \frac{\delta H}{\delta \hat{L}} \hat{L} \right)_{\geq 1} - \left( \frac{\delta H}{\delta \hat{L}} \hat{L} \right)_{0} \hat{L}
- \hat{L} \partial^{-1} \text{res} \left( \left[ \frac{\delta H}{\delta \hat{L}}, \hat{L} \right] \right) - \left[ D^{-1} \left( \text{res} \left[ \frac{\delta H}{\delta \hat{L}}, \hat{L} \right] \right), \hat{L} \right],
\]

(23)

where \( \frac{\delta H}{\delta \hat{L}} = \sum_{i=1}^{n-1} \partial^{-(i+1)} \frac{\delta H}{\delta v_i} \).

We define a bracket with \( \hat{P} \) as

\[
\{ F, G \} = \int \text{res} \left( \frac{\delta F}{\delta \hat{L}} \hat{P} \frac{\delta G}{\delta \hat{L}} \right) dx.
\]

(24)

This bracket is also complicated and its structure is revealed by the following lemma:

**Lemma 3**

\[
\{ F, G \} = \{ F, G \}_2 - \{ F, G \}_3,
\]

(25)

where the left side of the equation is the second and third GD brackets associated with the operator \( \hat{L} = \hat{L} \partial^{-1} \).

*Proof.* Using the identity \( \partial \frac{\delta H}{\delta \hat{L}} = \frac{\delta H}{\delta \hat{L}} \) and the rest is similar to the proof of lemma 1. □

At this point, we have the following theorem
Theorem 2. Let the Miura map be given by the following factorization of the Lax operator \( \hat{L} \):
\[
\hat{L} = (\partial - \hat{w}_{n-1}) \cdots (\partial - \hat{w}_1).
\] (26)

Then in the coordinates \( \{\hat{w}_i\} \), the Poisson tensor \( \hat{P} \) defined by (23) is the following simple \((n - 1) \times (n - 1)\) matrix operator:
\[
\begin{pmatrix}
2 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & 2
\end{pmatrix}
\partial.
\] (27)

Remark: We know that a Poisson tensor will define a Poisson algebra. So, our theorem 1 and theorem 2 provide free field realizations for the corresponding Poisson algebras. This may be interesting for the theory of \( W \) algebra.

To conclude the paper, we give here one example.

Example. \( n = 3 \) case: \( \hat{\mathcal{L}} = \partial^3 + v_2 \partial^2 + v_1 \partial \). The Lax equation
\[
\hat{\mathcal{L}}_t = [(\hat{\mathcal{L}}^2)^{\geq 1}, \hat{\mathcal{L}}],
\]
gives
\[
v_2_t = \frac{1}{3}(-3v_{2xx} + 6v_1x - 2v_2v_x),
\]
\[
v_1_t = \frac{1}{3}(-2v_{2xxx} + 3v_{1xx} + 2v_2v_1x - 2v_1v_2x - 2v_2v_{2xx}).
\]

The second Poisson tensor is
\[
\hat{\mathcal{P}} = \begin{pmatrix}
-6\partial & 3\partial(\partial - v_2) \\
-3(\partial + v_2)\partial & 2\partial^2 + v_1\partial + \partial v_1 + 2v_2\partial^2 - 2\partial^2v_2 - 2v_2\partial v_2
\end{pmatrix}.
\]

According to the theorem 2, the modified Poisson tensor and the transformation between the fields are respectively
\[
\begin{pmatrix}
2\partial \\
\partial \\
2\partial
\end{pmatrix},
\]
\( v_2 = -(w_2 + w_1), \quad v_1 = w_1w_2 - w_1x \)
this of course can be verified by a simple calculation.

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