Can one see the number of colors in $\eta, \eta' \rightarrow \pi^+ \pi^- \gamma$?

B. Borasoy$^{a,b}$ and E. Lipartia$^a$

$^a$ Physik Department, Technische Universität München, 85747 Garching, Germany

$^b$ Helmholtz-Institut für Strahlen- und Kernphysik (Theorie), Universität Bonn, Nußallee 14-16, 53115 Bonn, Germany

Abstract

We investigate the decays $\eta, \eta' \rightarrow \pi^+ \pi^- \gamma$ up to next-to-leading order in the framework of the combined $1/N_c$ and chiral expansions. Counter terms of unnatural parity at next-to-leading order with unknown couplings are important to accommodate the results both to the experimental decay width and the photon spectrum. The presence of these coefficients does not allow for a determination of the number of colors from these decays.

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1email: borasoy@itkp.uni-bonn.de
2email: lipartia@ph.tum.de
The anomalous decay $\pi^0 \to \gamma\gamma$ is presented as a textbook example to confirm from low-energy hadron dynamics the number of colors to be $N_c = 3$, see e.g. [4], since this decay originates at tree level from the Wess-Zumino-Witten (WZW) term \([2, 3]\) with a quantized prefactor $N_c$. The decay width $\Gamma_{\pi^0 \to \gamma\gamma}$ is thus proportional to $N_c^2$, being quite sensitive to the number of colors, and in fact the result for $N_c = 3$ is in perfect agreement with experiment.

Recently, however, it was shown in \([4, 5]\) that the cancellation of triangle anomalies in the standard model with an arbitrary number of colors leads to $N_c$ dependent values of the quark charges, such that the vertex with one pion and two photons is completely canceled by the $N_c$ dependent part of a Goldstone-Wilczek term \([5, 6]\). Within this scenario the decay $\pi^0 \to \gamma\gamma$ cannot be utilized to support $N_c = 3$. A similar cancellation also occurs for the decay $\eta \to \gamma\gamma$, if one neglects $\eta$-$\eta'$ mixing. The $N_c$ independence is maintained at one-loop order, i.e. at next-to-next-to-leading order in the combined chiral/large $N_c$ expansion, for both the $\pi^0$ and the $\eta$ decay, but the strong $N_c$ dependence of the singlet decay $\eta_0 \to \gamma\gamma$ induces also a strong $N_c$ dependence for $\eta \to \gamma\gamma$ due to $\eta$-$\eta'$ mixing \([7]\). One concludes then that both the $\eta$ and the $\eta'$ decay show clear evidence that we live in a world with three colors.

On the other hand, it has been pointed out in \([5]\) that at tree level the decay width of the process $\eta \to \pi^+\pi^-\gamma$ is proportional to $N_c^2$ and should replace the textbook process $\pi^0 \to \gamma\gamma$ lending support to $N_c = 3$. In analogy to the two-photon decays, the effects of $\eta$-$\eta'$ mixing along with the inclusion of subleading contributions must be treated systematically, in order to make a rigorous statement on a possible determination of the number of colors from this process. In the present work we will therefore investigate the decays $\eta, \eta' \to \pi^+\pi^-\gamma$ up to next-to-leading order within the framework of large $N_c$ chiral perturbation theory (ChPT) \([8]\).

At leading order in the combined chiral and $1/N_c$ expansions the decays $\eta, \eta' \to \pi^+\pi^-\gamma$ originate from a piece in the WZW Lagrangian

$$S_{wzw}(U, v) = -\frac{N_c}{48\pi^2} \int \langle \Sigma_L^2 v - \Sigma_R^3 v \rangle, \quad (1)$$

where $\Sigma_L = U^\dagger dU$, $\Sigma_R = UdU^\dagger$, and we adopted the differential form notation of \([8]\),

$$v = dx^\mu v_\mu, \quad d = dx^\mu \partial_\mu \quad (2)$$

with the Grassmann variables $dx^\mu$ which yield the volume element $dx^\mu dx^\nu dx^\alpha dx^\beta = \epsilon^{\mu\nu\alpha\beta} d^4x$. The brackets $\langle \ldots \rangle$ denote the trace in flavor space, while the unitary matrix $U = e^{i\phi}$ collects the pseudoscalar meson nonet $(\pi, K, \eta_8, \eta_0)$. The external vector field $v = -eQA$ contains the photon field $A = A_\mu dx^\mu$ and the quark charge matrix $Q$ of the $u$- $d$- and $s$-quarks which is usually assumed to be independent of the number of colors with $Q = \frac{2}{3} \text{diag}(2, -1, -1)$. However, the cancellation of triangle anomalies requires $Q$ to depend on $N_c$ \([4, 5]\).

$$Q = \frac{1}{3} \text{diag}(1, 1, 1) + \frac{1}{N_c} - 1, \frac{1}{N_c} - 1 \quad (3)$$

with $\hat{Q} = \frac{1}{3} \text{diag}(2, -1, -1)$ being the conventional charge matrix, while the second term is proportional to the baryon number and gives rise to the Goldstone-Wilczek term. The anomalous Lagrangian of Eq. \([1]\) decomposes into the conventional WZW Lagrangian of the $U(3)$ theory.
with the charge matrix $\hat{Q}$ and a Goldstone-Wilczek term which vanishes for $N_c = 3$

$$S_{WZW}(U, v) = S_{WZW}(U, \hat{v}) + \left(1 - \frac{N_c}{3}\right)S_{GW}(U, A)$$

with $\hat{v} = -e\hat{Q}A$ and

$$S_{WZW}(U, \hat{v}) = \frac{N_c e}{48\pi^2} \int \left\langle \left(\Sigma_L^3 - \Sigma_R^3\right)\hat{Q}\right\rangle A,$$

$$S_{GW}(U, A) = \frac{e}{48\pi^2} \int \left\langle \Sigma_L^3\right\rangle A.$$ (4)

However, this presentation is not convenient to perform calculations within the framework of large $N_c$ ChPT. To this end, one rather expands the quark charge matrix $Q$ in powers of $1/N_c$

$$Q = \frac{1}{2} \text{diag}(1, -1, -1) + \frac{1}{2N_c} I \equiv Q^{(0)} + Q^{(1)},$$ (6)

where the superscript denotes the order in the combined large $N_c$ and chiral counting scheme, i.e. $Q^{(0)}$ ($Q^{(1)}$) is of order $O(1)$ ($O(\delta)$). From $S_{WZW}$ one obtains the tree level contributions

$$S_{WZW}(U, v) = \int d^4x \mathcal{L}_{WZW} = -\frac{iN_c e}{24\pi^2} \int \left\langle d\phi d\phi d\phi \hat{Q}\right\rangle A$$

$$= -\frac{iN_c e}{24\pi^2} \int \left\langle d\phi d\phi d\phi Q^{(0)}\right\rangle A,$$ (7)

since for the processes $\eta_8, \eta_0 \to \pi^+\pi^-\gamma$ the trace with $Q^{(1)}$ in Eq. (4) vanishes. The pertinent amplitudes have the structure

$$A^{(\text{tree})}(\phi \to \pi^+\pi^-\gamma) = -\frac{N_c e}{\sqrt{3}12\pi^2 f^3} k_\mu \epsilon_\nu p^{+ \alpha} p^{- \beta} \epsilon^{\mu\nu\alpha\beta} \alpha^{(\text{tree})}_\phi,$$ (8)

where $p^{+}(p^{-})$ is the momentum of the outgoing $\pi^+(\pi^-)$ and $k(\epsilon)$ is the momentum (polarization) of the outgoing photon. Next we replace $f^3$ by $F_\phi F_\pi^2$ in Eq. (8) with the decay constants $F_\phi$ defined via

$$\langle 0|\bar{q}_\gamma \gamma_\mu \gamma_5 q|^\phi\rangle = i\sqrt{2} p_\mu F_\phi^i$$ (9)

which is consistent at leading order. Neglecting $\eta-\eta'$ mixing for the moment, the coefficients $\alpha^{(\text{tree})}_\phi$ read

$$\alpha^{(\text{tree})}_\eta = 1, \quad \alpha^{(\text{tree})}_{\eta'} = \sqrt{2}.$$ (10)

These are the expressions which were suggested to be utilized for a determination of $N_c$ \cite{5}. Employing the experimental values \cite{9}

$$\Gamma_{\eta \to \pi^+\pi^-\gamma} = 56.1 \pm 5.4 \text{ eV},$$

$$\Gamma_{\eta' \to \pi^+\pi^-\gamma} = 59.6 \pm 5.2 \text{ keV},$$ (11)

we extract from the $\eta$ decay $N_c = 7$ and $N_c = 10$ from the $\eta'$ decay which is clearly in contradiction to the well-established value $N_c = 3.$

\footnote{Note, however, that a factor of 1/3 is missing in the amplitudes given in \cite{5}.}
At next-to-leading order we replace the charge matrix $Q$ with the mass matrix $\chi$, where additional derivatives \[11, 12\]lation. First, there is a term of fourth chiral order which is suppres sed by one order in and the QCD vacuum angle $\theta$ with the coefficients \[8, 10, 11\]situation changes by including next-to-leading order corrections. At next-to-leading order gauge invariant counter terms of unnat ural parity enter the calcu-
lation of large $N_c$ ChPT counter terms of sixth chiral order contribute which can be decomposed into explicitly symmetry breaking terms and terms with additional derivatives \[11, 12\]

\[
\sin 2\vartheta^{(0)} = -\frac{4\sqrt{2}m_K^2 - m_\pi^2}{3m_{\eta'}^2 - m_\eta^2}
\]

the experimental values given in Eq. \[11\] allow either for $N_c = 4$ or $N_c = 5$, but $N_c = 3$ is clearly ruled out. We can therefore conclude that the decays $\eta, \eta' \rightarrow \pi^+\pi^-\gamma$ at leading order are not suited to confirm the number of colors. In the following we investigate whether the situation changes by including next-to-leading order corrections.

At the same order in the $\delta$ expansion of large $N_c$ ChPT counter terms of unnatural parity enter the calculation. First, there is a term of fourth chiral order which is suppressed by one order in $N_c$ with respect to the leading order result \[8, 10, 11\]

\[
d^4x \hat{L}_{\rho^4} = i\hat{L}_1 \psi (dv dU dU^\dagger + dv dU^\dagger dU)
\]

with $\psi = -i \ln \det U$ and we have neglected for brevity both the external axial-vector fields and the QCD vacuum angle $\theta$.

At the same order in the $\delta$ expansion of large $N_c$ ChPT counter terms of sixth chiral order contribute which can be decomposed into explicitly symmetry breaking terms and terms with additional derivatives \[11, 12\]

\[
\hat{L}_{\rho^6} = \hat{L}_\chi + \hat{L}_{\partial^2},
\]

where

\[
d^4x \hat{L}_\chi = K_1 \langle (U^\dagger \chi - \chi^\dagger U) [ (U^\dagger dvU + dv) U^\dagger dUU^\dagger dU + U^\dagger dUU^\dagger dU (U^\dagger dvU + dv)] \rangle
\]

\[
+ K_2 \langle (U^\dagger \chi - \chi^\dagger U) U^\dagger dU \left( U^\dagger dvU + dv \right) U^\dagger dU \rangle
\]

with the mass matrix $\chi = \text{diag}(m_\pi^2, m_\eta^2, 2m_K^2 - m_\pi^2)$ and

\[
d^4x \hat{L}_{\partial^2} = K_3 \langle (U^\dagger dvU + dv) [(U^\dagger \partial^\lambda dU - (\partial^\lambda dU)^\dagger U] [U^\dagger dU \ U^\dagger \partial_\lambda U + U^\dagger \partial_\lambda U \ U^\dagger dU [U^\dagger \partial^\lambda dU - (\partial^\lambda dU)^\dagger U]] \rangle
\]

\[
+ K_4 \langle (U^\dagger dvU + dv) [(U^\dagger \partial^\lambda dU - (\partial^\lambda dU)^\dagger U] U^\dagger dU \ U^\dagger \partial_\lambda U + U^\dagger \partial_\lambda U \ U^\dagger dU [U^\dagger \partial^\lambda dU - (\partial^\lambda dU)^\dagger U]] \rangle.
\]

At next-to-leading order we replace the charge matrix $Q$ by $Q^{(0)}$, since $Q^{(1)}$ contributes beyond our working precision. Without mixing the counter terms yield the amplitudes

\[
A^{(ct)}(\phi \rightarrow \pi^+\pi^-\gamma) = \frac{8e}{\sqrt{3}f_3} k_\mu e_\nu p_\alpha^+ p_\beta^- \epsilon^{\mu\nu\alpha\beta} \beta_\phi
\]

with the coefficients

\[
\beta_{\eta_8} = m_\pi^2 \left[ 2\tilde{K}_1 + \tilde{K}_2 \right] - [m_\eta^2 + 2s_{+/-} - 2m_\pi^2] \tilde{K}_3 - [s_{+/-} - 2m_\pi^2] \tilde{K}_4
\]

\[
\beta_{\eta_0} = \frac{3}{\sqrt{2}} L_1 + \sqrt{2} m_\pi^2 \left[ 2\tilde{K}_1 + \tilde{K}_2 \right] - \sqrt{2}[m_\eta^2 + 2s_{+/-} - 2m_\pi^2] \tilde{K}_3 - \sqrt{2}[s_{+/-} - 2m_\pi^2] \tilde{K}_4
\]
and $s_{+-} = (p^+ + p^-)^2$. One must furthermore account for the $Z$-factors of the mesons and $\eta$-$\eta'$ mixing up to next-to-leading order. For each pion leg the pertinent $Z$-factor

$$\sqrt{Z_\pi} = 1 - \frac{4}{f^2} m^2_\pi L_{5}^{(r)}$$  \hspace{1cm} (20)

can be completely absorbed by replacing one factor of $f$ by the physical decay constant $F_\pi$ in the denominator of the amplitude, Eq. (18),

$$F_\pi = f \left( 1 + \frac{4}{f^2} m^2_\pi L_{5}^{(r)} \right).$$  \hspace{1cm} (21)

The coupling constant $L_{5}^{(r)}$ originates from the effective Lagrangian of natural parity

$$\mathcal{L}_{\text{eff}} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \ldots$$  \hspace{1cm} (22)

which reads at lowest order $\delta^0$

$$\mathcal{L}^{(0)} = \frac{f^2}{4} (\partial_\mu U^\dagger \partial^\mu U) + \frac{f^2}{4} (\chi U^\dagger + U \chi^\dagger) - \frac{1}{2} \tau \psi^2$$  \hspace{1cm} (23)

and at next-to-leading order $\mathcal{O}(\delta)$

$$\mathcal{L}^{(1)} = L_5 (\partial_\mu U^\dagger \partial^\mu U (\chi^\dagger U + U^\dagger \chi)) + L_8 (\chi^\dagger U \chi^\dagger U + U^\dagger \chi U^\dagger \chi)$$

$$+ \frac{f^2}{12} \Lambda_1 \partial_\mu \psi \partial^\mu \psi + i \frac{f^2}{12} \Lambda_2 \psi (\chi^\dagger U - U^\dagger \chi).$$  \hspace{1cm} (24)

Note that both $L_5$ and $L_8$ contain divergent pieces which compensate divergencies from loop integrals at order $\mathcal{O}(\delta^2)$ and are thus suppressed by one order in $N_c$ with respect to the finite parts $L_{5}^{(r)}$, $L_{8}^{(r)}$. To the order we are working, we omit the divergent portions.

In the tree level expression for the decay amplitude, Eq. (8), the states $\eta_8$ and $\eta_0$ are replaced by the physical states $\eta$ and $\eta'$ via [7]

$$\frac{1}{f} \eta_8 = \frac{1}{F_{\eta_8}} \left[ \cos\vartheta^{(1)} - \sin\vartheta^{(0)} A^{(1)} \right] \eta + \frac{1}{F_{\eta}} \left[ \sin\vartheta^{(1)} + \cos\vartheta^{(0)} A^{(1)} \right] \eta'$$

$$\frac{1}{f} \eta_0 = \frac{1}{F_{\eta_0}} \left[ \cos\vartheta^{(0)} A^{(1)} - \sin\vartheta^{(1)} \right] \eta + \frac{1}{F_{\eta'}} \left[ \sin\vartheta^{(0)} A^{(1)} + \cos\vartheta^{(1)} \right] \eta'$$  \hspace{1cm} (25)

where

$$A^{(1)} = \frac{8\sqrt{2}}{3F_{\pi}} L_{5}^{(r)} [m_K - m^2_\pi],$$

$$\sin 2\vartheta^{(1)} = \sin 2\vartheta^{(0)} \left( \frac{1 + \Lambda_2}{\sqrt{1 + \Lambda_1}} + \frac{8}{F_{\pi}} [2L_{8}^{(r)} - L_{5}^{(r)}] (m^2_K - m^2_\pi) - \frac{24}{F_{\pi}} L_{5}^{(r)} \tau \right).$$  \hspace{1cm} (26)

The numerical discussion of these expressions is presented in [7]. For the counter term contributions in Eq. (18), on the other hand, we keep only the leading order pieces in Eq. (23)

$$\frac{1}{f} \eta_8 = \frac{1}{F_{\eta_8}} \cos\vartheta^{(0)} \eta + \frac{1}{F_{\eta}} \sin\vartheta^{(0)} \eta'$$

$$\frac{1}{f} \eta_0 = - \frac{1}{F_{\eta'}} \sin\vartheta^{(0)} \eta + \frac{1}{F_{\eta}} \cos\vartheta^{(0)} \eta'$$  \hspace{1cm} (27)
Figure 1: Photon spectrum for $N_C = 3$

which was already employed in the discussion of the leading order decay amplitude, cf. Eq. (12).

From our results it is easy to see that the $\eta'$ decay does not depend on the QCD renormalization scale. Due to the anomalous dimension of the singlet axial current, the decay constant $F_{\eta'}^0$ scales as, cf. Eq. (9),

$$ F_{\eta'}^0 \to Z_A F_{\eta'}^0, $$

(28)

where $Z_A$ is the multiplicative renormalization constant of the singlet axial current. Furthermore, the $\tilde{K}_i$ are scale independent, whereas $\tilde{L}_1$ transforms according to

$$ \tilde{L}_1 \to \tilde{L}_1^{\text{ren}} = Z_A \tilde{L}_1 - \frac{N_C}{144\pi^2} [Z_A - 1]. $$

(29)

Since $\tilde{L}_1$ appears in the $\eta'$ decay amplitude in the combination

$$ \left( \frac{N_C}{12\pi^2} - 12\tilde{L}_1 \right) \to \frac{N_C}{12\pi^2} - 12\tilde{L}_1^{\text{ren}} = Z_A \left( \frac{N_C}{12\pi^2} - 12\tilde{L}_1 \right), $$

(30)

the amplitude remains renormalization group invariant.

We now determine the unknown coefficients $\tilde{K}_i$ by fitting them to both the decay width $\Gamma_{\eta \to \pi^+ \pi^- \gamma}$ and the corresponding photon spectrum. To this end, we rewrite the coefficient $\beta_{\eta s}$ in terms of effectively two parameters

$$ \beta_{\eta s} \equiv \beta_{\eta s}^{(1)} + \beta_{\eta s}^{(0)} s_{+-}. $$

(31)

Setting $N_c = 3$ we obtain a perfect fit to both the experimental decay width $\Gamma_{\eta \to \pi^+ \pi^- \gamma} = 56.1 \pm 5.4$ eV and the photon spectrum, see Fig. 1, with $\beta_{\eta s}^{(1)} = 1.3 \times 10^{-3}$ and $\beta_{\eta s}^{(0)} = 28.4 \times 10^{-3}$ GeV$^{-2}$ which shows that the subleading contributions from the counter terms are important and not suppressed with respect to the leading order originating from the WZW term. However, for $N_c = 2$ an equally good fit to the experimental data, see Fig. 2, is achieved by setting $\beta_{\eta s}^{(1)} = -3.2 \times 10^{-3}$ and $\beta_{\eta s}^{(0)} = 22.0 \times 10^{-3}$ GeV$^{-2}$. Although a fit for $N_c = 1$ would be possible as well, we do not present the results here, as a world with $N_c = 1$ has no strong interactions. Note that in the present work we do not explore the possibility of estimating the values of the unknown couplings by means of model-dependent assumptions such as resonance saturation.

It thus does not seem to be possible to strictly determine the number of colors at next-to-leading order in large $N_c$ ChPT, unless one imposes in addition the cancellation of Witten’s global $SU(2)_L$ anomaly which requires $N_c$ to be odd [13]. In that case $N_c = 2$ is ruled out.
and for $N_c = 5$ it turns out that – to the order we are working – one cannot bring the results into agreement with experiment by varying the couplings. In particular, the photon spectrum can only be reproduced with a larger decay width. One may be inclined to argue that the restriction to odd $N_c$ enables a determination of $N_c$, but it is well-known from the one-loop calculation of this decay in conventional ChPT that the loop contributions reduce the decay width [11, 14]. It is therefore possible that a next-to-next-to-leading order calculation in large $N_c$ ChPT including one-loop corrections can be brought to agreement with experiment also for $N_c = 5$. However, such an investigation is beyond the scope of the present work. In any case, a rigorous statement on the number of colors cannot be made due to the failure of the anomalous contribution from the WZW term to accommodate the decay width for $N_c = 3$ and the presence of unknown couplings.

In the case of the $\eta'$ decay unitarity effects via final state interactions are dominating [15, 16]. Therefore, a perturbative approach is insufficient to describe the $\eta'$ decay, and we will refrain from presenting numerical results here.

We conclude that a clean derivation of the number of colors cannot be achieved by investigating the decays $\eta, \eta' \rightarrow \pi^+\pi^- \gamma$. In particular, $\eta \rightarrow \pi^+\pi^- \gamma$ should not be utilized as a textbook example to confirm the number of colors to be $N_c = 3$.

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