Solutions of the Euler equations and stationary structures in an inviscid fluid.

O. V. Kaptsov
Institute of Computational Modeling, Siberian Branch, Russian Academy of Sciences, Krasnoyarsk, Russia
E-mail: kaptsov@icm.krasn.ru

Abstract.
The Euler equations describing two-dimensional steady flows of an inviscid fluid are studied. These equations are reduced to one equation for the stream function and then, using the Hirota function, solutions of three nonlinear elliptic equations are found. The solutions found are interpreted as sources in a rotating fluid, jets, chains of sources and sinks, vortex structures. We propose a new simple method for constructing solutions in the form of rational expressions of elliptic functions. It is shown that the flux of fluid across a closed curve is quantized in the case of the elliptic Sin-Gordon equation.

Keywords: Euler equations, Hirota function, elliptic functions.

1 Introduction
It is well known that the equations
\[ uu_x + vu_y + p_x = 0, \quad uv_x + vv_y + p_y = 0, \quad u_x + v_y = 0, \]
describe steady two-dimensional flows of inviscid fluid, where \( u, v \) are the components of velocity and \( p \) is the pressure. This system is reduced to one equation
\[ \Delta \psi = \omega(\psi) \]  
(1.1)
for the stream function \( \psi \). Here, and below, \( \Delta \) represents the two-dimensional Laplacian operator, \( \omega \) is vorticity and \( u = \psi_y, \ v = -\psi_x \). Equation (1.1) also occurs in various applications such as plasma physics and condensed matter physics [2, 3, 4].

Equation (1.1) is well investigated in the linear case. In addition, the elliptic Liouville equation
\[ \Delta \psi = \exp(\psi) \]
connected by transformation
\[ \psi = \log(-2\Delta(\log \tau)) \]
with the Laplace equation $\Delta \tau = 0$. This is a direct consequence of the classical formula for the general solution of the hyperbolic Liouville equation [5].

At the end of the nineteenth and beginning of the twentieth century, the Bäcklund and Tzitzeica [6] found transformations that allow generating solutions to the equations

$$u_{xy} = \sin(u), \quad u_{xy} = \exp(u) - \exp(-2u).$$

Using the Bäcklund transformation, in [4] some vortex-type singular solutions of the elliptic Sine-Gordon equation

$$\Delta \psi = \sin(\psi) \quad (1.2)$$

were found. Multiparameter solution formula for the Tzitzeica equation

$$\Delta \psi = \exp(\psi) - \exp(-2\psi)$$

was presented in [8]. Note the works [9, 10], which used separation of variables to construct solutions of the equation (1.1) with other functions $\omega(\psi)$.

This paper is divided into two parts. The first deals with solutions of the Sine-Gordon equation (1.2) and Sinh-Gordon one. So the solutions of the Sine-Gordon equation are represented as

$$\psi = 4 \tan^{-1} \frac{G}{F},$$

where $F$ and $G$ are smooth functions in the plane $\mathbb{R}^2(x, y)$. It is shown that the flux of fluid volume crossing the simple closed curve $\gamma \subset \mathbb{R}^2(x, y)$ is equal to

$$Q \equiv \oint_{\gamma} d\psi = 8\pi I, \quad I \in \mathbb{Z}.$$ 

The integer $I$ is equal to the sum of the Poincaré indices of zero points of the vector field $V = (F, G)$ lying inside the bounded curve $\gamma$. We have found exact solutions of the Sine-Gordon (1.2) and Sinh-Gordon equations expressed in terms of elementary functions. These solutions can be interpreted as sources and sinks, jet streams, periodic chains of sources and sinks, vortexes, and combinations thereof.

In the second part, a new method for constructing elliptic solutions of the Sin-Gordon, Sinh-Gordon and Tzitzéica equations is proposed. These classes of solutions are represented as rational expressions of elliptic functions. To find them, the Maple computer algebra system is used. The calculations are similar to those performed in [11].

### 2 Elementary solutions

In this section we will find some elementary solutions of equation (1.1), that is solutions which can be expressed in terms of algebraic operations, logarithms and exponentials. We begin with the Sine-Gordon and Sinh-Gordon equations

$$\Delta \psi = \sin(\psi), \quad (2.1)$$

$$\Delta \psi = \sinh(\psi). \quad (2.2)$$
Let us reduce these equations to a same form using complex and double numbers \[12\]. The field of complex numbers will be denoted by \( \mathbb{C} \), and the algebra of double numbers by \( \mathbb{D} \). Every complex and double number has the form \( z = a + \delta b \), where \( a, b \in \mathbb{R} \), \( \delta \notin \mathbb{R} \). So if \( \delta = i \in \mathbb{C} \), then \( \delta^2 \) is equal to \(-1\), and if \( \delta \in \mathbb{D} \), then \( \delta^2 \) is equal to \(1\). Multiplication of double numbers is given by:

\[
(a_1 + \delta b_1)(a_2 + \delta b_2) = a_1a_2 + b_1b_2 + \delta(a_1b_2 + a_2b_1).
\]

Thus, the equations (2.1), (2.2) can be written as

\[
\Delta \psi = \exp(\delta \psi) - \exp(-\delta \psi) = 2\delta.
\]

Let \( v = \exp(\delta \psi) \) be a new function, then the previous equation is of the form

\[
v(v_{xx} + v_{yy}) - v_x^2 - v_y^2 - v^3/2 + v/2 = 0.
\]

Suppose \( F \) and \( G \) are smooth functions on an open set \( \Omega \subset \mathbb{R}^2 \) and \( H = F + \delta G \). Then we say that the function \( \bar{H} = F - \delta G \) is conjugate to \( H \). Next we look for solutions of (2.1), (2.2) in the form

\[
v = \frac{H^2}{\bar{H}^2}.
\]

Obviously, the functions \( v \) and \( \psi \) can be written as

\[
v = \left( \frac{1 - \delta G}{1 + \delta G} \right)^2, \quad \psi = \frac{2}{\delta} \log \left( \frac{1 - \delta G}{1 + \delta G} \right).
\]

Therefore the function \( \psi \) is

\[
\psi = 4 \tan^{-1} \frac{G}{F}, \quad \text{when } \delta = i \in \mathbb{C}; \text{ but if } \delta \in \mathbb{D}, \text{ then }
\]

\[
\psi = 4 \tanh^{-1} \frac{G}{F}.
\]

We note the useful statement about the amount of fluid crossing a closed curve. Suppose the stream function \( \psi \) satisfies the Sine-Gordon equation and has the form (2.5), where \( F \) and \( G \) generate a vector field \( V = (F, G) \) in the domain \( \Omega \subset \mathbb{R}^2 \). Let \( a \in \Omega \) be an zero of \( V \) and \( c \) is a small circle around the zero. Then the line integral

\[
\text{ind}(a) \equiv \frac{1}{2\pi} \int_c d \tan^{-1}(G/F)
\]

is integer called Poincaré's index of the point \( a \) \[13\]. Thus, the source strength

\[
\int_c d\psi,
\]
is equal to $8\pi \text{ind}(a) \in 8\pi \mathbb{Z}$, i.e., the source strength is quantized in this case. We say that a line integral
$$q = \frac{1}{2\pi} \oint_c d\psi$$
is the topological charge of the point $a$.

Let us suppose that a simple closed curve $\gamma$ is the boundary of $\Omega$ and the vector field $V$ has several zeros $a_1, \ldots, a_n$ with topological charges $q_1, \ldots, q_n$. It follows from Poincaré’s theorem [14] that
$$\oint_{\gamma} d\psi = 8\pi \sum_{j=1}^{n} \text{ind}(a_j) = 2\pi \sum_{j=1}^{n} q_j.$$Thus, the flux of fluid volume across the closed curve $\gamma$ is
$$Q \equiv \oint_{\gamma} d\psi = 2\pi \sum_{j=1}^{n} q_j \in 8\pi \mathbb{Z},$$i.e., this means that $Q$ is also quantized. A useful way to think of singular solutions with topological charges is as point defects in the fluid. Other topological quantum numbers are discussed in [15].

We now look for solutions of the equations (2.1) and (2.2) by using the function $H$ of the form
$$1 + \delta \exp(kx + ny + \eta),$$with $k, n, \eta \in \mathbb{R}$. It is easy to see that $v$ satisfies the equation (2.3), if $k^2 + n^2 = 1$. In the case of the Sine-Gordon equation, the function $\psi$ is smooth; its graph is a two-dimensional kink. The streamlines $\psi = \text{const}$ corresponding to this solution are obviously straight lines. For simplicity we assume that $k = 1, n = \eta = 0$. The corresponding stream function and velocity components are
$$\psi = 4 \tanh^{-1}(\exp(x)), \quad v_x = 0, \quad v_y = -\frac{2}{\cosh^2(x/2)}.$$Then the solution may be interpreted as a steady jet that is parallel to the $y$-axis. When $kn \neq 0$ we also have a jet flow. We remark that in the case of the Sinh-Gordon equation, the solution $\psi$ is discontinuous.

Next, we call any linear combination of the functions $\exp(kx + ny + \eta)$, with $k, n, \eta \in \mathbb{C}$, a Hirota function. Consider a Hirota function
$$H = 1 + \delta(f_1 + f_2) + \delta^2 s_{12} f_1 f_2,$$(2.6)where $f_i = \exp(k_i x + n_i y + \eta_i), i = 1, 2$. Substituting the function (2.4) into left side of (2.3), we obtain a rational function whose numerator is a polynomial $P$ in $k_i, n_i, s_{12}$. Equating the coefficients of $P$ to zero, we have a system NAS of nonlinear algebraic equations. The system has a non-trivial solution
$$s_{12} = \frac{n_1 n_2 + k_1 k_2 - 1}{n_1 n_2 + k_1 k_2 + 1}, \quad n_i^2 + k_i^2 = 1 \quad i = 1, 2.$$
Hence solutions of the equations (2.1), (2.2) are given by

$$\psi_1 = 4 \tan^{-1} \left( \frac{f_1 + f_2}{1 - s_{12} f_1 f_2} \right), \quad \psi_2 = 4 \tanh^{-1} \left( \frac{f_1 + f_2}{1 + s_{12} f_1 f_2} \right).$$

We further consider only the solution $\psi_1$, since the function $\psi_2$ is discontinuous. Suppose that $\eta_1 = \eta_2 = 0$, $n_i, k_i \in \mathbb{R}$. In this case, the graph of $\psi_1$ looks like two kinks and has one saddle point. Streamlines are shown in Figure 1. It may be interpreted as an interaction of two jets.

Now we set $\eta_1 = 0$, $\eta_2 = i\pi$, $n_i, k_i \in \mathbb{R}$. It gives a singularity of the velocity distribution at some point $A$, where $G = f_1 + f_2 = 0$ and $F = 1 - s_{12} f_1 f_2 = 0$. As we said above, this solution may be interpreted as a point source or sink in the rotational flow. Because of $A$ is a simple point of vector field $V = (F, G)$ then the topological charge of the point $A$ is equal to $\pm 4$ and the source strength is $\pm 8\pi$. Since equations (2.1) and (2.2) are invariant under the transformation $\psi \mapsto -\psi$ we can obtain a source or a sink. Figure 2 shows the pattern of streamlines in the $(x, y)$-plane for the flow associated with a source or a sink.

Let us set $\eta_1 = \eta_2 = 0$, $n_1 = a + ib$, $n_2 = a - ib \ (b \neq 0)$. Then the relation $f_1 + f_2 = 0$ gives countable number of straight and parallel streamlines. The perpendicular line defined by the equation $1 - s_{12} f_1 f_2 = 0$ is also a streamline. The points of intersection of the streamlines are sources or sinks. Each semi-infinite strip between the nearest parallel streamlines is filled with streamlines connecting a source and a sink.

Let us consider a Hirota function

$$\tau = 1 + \delta (f_1 + f_2 + f_3) + \delta^2 (s_{12} f_1 f_2 + s_{13} f_1 f_3 + s_{23} f_2 f_3) + \delta^3 s_{123} f_1 f_2 f_3. \quad (2.7)$$
Here, as above, the functions $f_i$ have the form $\exp(k_ix+n_iy+\eta_i)$, with $k_i, n_i, \eta_i \in \mathbb{C}$, and $s_{ij}$ is given by the following formula

$$s_{ij} = \frac{n_in_j + \eta_i\eta_j - 1}{n_in_j + \eta_i\eta_j + 1}, \quad n_i^2 + k_i^2 = 1 \quad (i = 1, 2, 3). \quad (2.8)$$

Substituting the Hirota function (2.7) into the equation (2.3) and (2.4), we find

$$s_{123} = s_{12}s_{13}s_{23}.$$  

To obtain flow patterns, one must again choose constants $n_i, \eta_i \in \mathbb{C}$ ($i = 1, 2, 3$) and signs of $k_i = \pm \sqrt{1 - n_i^2}$. We have the simplest case when $\eta_i = 0$ and $n_i, k_i \in \mathbb{R}$ ($i = 1, 2, 3$). If $n_1 = -0.7, n_2 = 0.4, n_3 = 0.1, k_1 = \sqrt{1 - k_1^2}, k_2 = -\sqrt{1 - k_2^2}, k_3 = \sqrt{1 - k_3^2}$, then we get the flow pattern shown in Figure 3. There we see three jets and a vortex.

If we set $\eta_1 = i\pi, \eta_2 = \eta_3 = 0, n_1 = 0.1, n_2 = 0.4, n_3 = 0.3$ and $k_1 = \sqrt{1 - k_1^2}, k_2 = -\sqrt{1 - k_2^2}, k_3 = \sqrt{1 - k_3^2}$, then we have the flow pattern, including a sink, a source and jets (see Fig. 3). It is easy to construct other solutions by choosing, for example, $n_1$ and $n_2$ to be complex conjugate and $n_3$ to be real.

Formulas (2.6) and (2.7) can be written as

$$(1 + \delta f_1) \ast (1 + \delta f_2), \quad (1 + \delta f_1) \ast (1 + \delta f_2) \ast (1 + \delta f_3).$$

Here addition is defined in the usual way, and the operation $\ast$ is given by the formulas

$$f_i \ast f_j = s_{ij}f_if_j, \quad f_i \ast f_j \ast f_k = s_{ij}s_{ik}s_{jk}f_if_jf_k.$$
Рис. 3: three jets and a vortex.

Рис. 4: a sink, a source and jets.
where $s_{ij}, s_{ik}, s_{jk}$ are calculated according to (2.8). For arbitrary $n$ the Hirota function has the form of the product

$$H_n = (1 + \delta f_1) \ast (1 + \delta f_2) \ast \cdots \ast (1 + \delta f_{n-1}) \ast (1 + \delta f_n).$$

In this case, the following relations must be fulfilled

$$f_{i_1} \cdots f_{i_m} = s_{i_1 \cdots i_m} f_{i_1} \cdots f_{i_m},$$

where $s_{i_1 \cdots i_m}$ is the product of all $s_{jk}$ such that $j, k \in \{i_1, \ldots i_m\}$ and $j < k$.

Let us consider only some solutions to the Sine-Gordon equation for the case $n = 4$. The corresponding Hirota function is

$$H_4 = (1 + \delta f_1) \ast (1 + \delta f_2) \ast (1 + \delta f_3) \ast (1 + \delta f_4).$$

We set $\eta_i = 0$ ($1 \leq i \leq 4$) and $n_1 = 0.5, n_2 = 0.4, n_3 = 0.3, n_4 = 0.2$, when $k_i < 0$ ($1 \leq i \leq 3$), $k_4 > 0$. This gives a flow pattern with four jets and two vortices (see Fig. 5). Assuming one of $\eta_i$ is imaginary, we can get a pair of sources and a pair of sinks or a single sink. Choosing $n_1 = a + ib, n_2 = a - ib, n_3 = a + ib$ and $n_4 = a - ib$ we obtain intersecting chains of sources and sinks, or the resonance of these chains resembling the resonance of solitons for the Kadomtsev-Petviashvili equation.

At the end of this section a few words should be said about the solutions of the Tzitzéica equation

$$\Delta \psi = \exp(\psi) - \exp(-2\psi).$$  

(2.9)
We have proposed in [8] the following representation
\[ \psi = \log (1 - 2(\Delta \log H)) \]
of solutions of the Tzitzeica equation. The function \( \psi \) satisfies the equation (2.9) if \( H \) is the Hirota function
\[ H = (1 + f_1) \ast (1 + f_2) \ast \cdots \ast (1 + f_n), \]
where \( f_i = \exp(k_i x + n_i y + \eta_i), n_i^2 + k_i^2 = 3 \), \( f_i \ast f_j = s_{ij} f_i f_j \),
\[ s_{ij} = \frac{4k_i k_j n_i n_j + 4k_i^2 k_j^2 - 9n_i n_j - 6k_i^2 - 9k_i k_j - 6k_j^2 + 27}{4k_i k_j n_i n_j + 4k_i^2 k_j^2 + 9n_i n_j - 6k_i^2 + 9k_i k_j - 6k_j^2 + 27}. \]

3 Solutions expressed in elliptic functions

The Sine-Gordon equation (2.1) has exact solutions that are expressed in terms of elliptic functions. Some such solutions were found using the Bäcklund transformation [4] and the Steuerwald ansatz [16, 7], often called the Lamb ansatz. Unfortunately, the monograph [10] contains some errors in constructing solutions in elliptic functions. In this section we will find a number of new ansatz for the equations (2.1) and (2.9). It should be noted that in the case of hyperbolic Sine-Gordon equation, solutions expressed in terms of elliptic functions were obtained in [17] using reductions from algebro-geometric solutions.

Let us now introduce a new function \( w = \tan(\psi/4) \). Then the equation (2.1) is rewritten as
\[ (1 + w^2)(w_{xx} + w_{yy}) - 2w(w_x^2 + w_y^2) + w^3 - w = 0. \]
We look for solutions of the last equation in the form
\[ w = \frac{s_0 + s_1 F + s_2 G}{p_0 + p_1 F + p_2 G}, \]
with \( s_i, p_i \in \mathbb{R} \) (\( i = 0, 1, 2 \)). Assume that \( F(x), G(y) \) are functions satisfying ordinary differential equations
\[ (F'_x)^2 = a_4 F^4 + a_3 F^3 + a_2 F^2 + a_1 F + a_0, \quad a_i \in \mathbb{R}, \]
\[ (G'_y)^2 = b_4 G^4 + b_3 G^3 + b_2 G^2 + b_1 G + b_0, \quad b_i \in \mathbb{R}. \]
Substitute (3.2) into the left side of the equation (3.1) and express all derivatives using (3.3) and (3.4). As a result, we obtain a rational function of \( F \) and \( G \). Equating the coefficients of the numerator to zero, we obtain a NAS system of nonlinear algebraic equations with respect to \( a_i, b_i \) (\( 0 \leq i \leq 4 \)) and \( s_j, p_j \) (\( 0 \leq j \leq 2 \)). NAS system solutions are quite cumbersome and therefore we consider only two cases.

The first case is an analogue of the Steuerwald ansatz
\[ w = \frac{G}{F}, \]
where the functions $F, G$ satisfy the equations

$$(F')^2 = b_4 F^4 + (1 - b_2) F^2 + b_0, \quad (G')^2 = b_4 G^4 + b_2 G^2 + b_0, \quad b_0, b_2, b_4 \in \mathbb{R}.$$ 

If $b_4 < 0$, then the functions are expressed in terms of the Jacobi function $dn$ (the delta amplitude). Thus the functions $F$ and $G$ are periodic and vanish twice on the period. As a result, we have a partition of $xy$-plane into equal rectangles. There are sources (or sinks, respectively) at two opposite vertices of the rectangle; inside there is one saddle point, and streamlines connect sources and sinks.

The second solution has the form

$$w = \frac{F - G}{F + G}. \quad (3.6)$$

Moreover, the functions $F(x), G(y)$ satisfy the equations

$$(F')^2 = b_4 F^4 - (1 + b_2) F^2 + b_0, \quad (G')^2 = b_4 G^4 + b_2 G^2 + b_0.$$ 

The flow pattern is qualitatively the same as in the previous case.

Now we look for solutions to equation (3.1) in the form

$$w = \frac{s_0 + s_1 F + s_2 G + s_3 FG}{p_0 + p_1 F + p_2 G + p_3 FG} \quad (3.7)$$

Assume that the functions $F, G$ satisfy equations (3.3), (3.4). Substitute the $w$ given by the formula (3.7) into the left side of (3.1) and express all derivatives using (3.3), (3.4). Then we have a rational function of $F, G$, and equating the numerator coefficients to zero, we obtain a nonlinear algebraic system with respect to $a_i, b_i (0 \leq i \leq 4)$ and $s_j, p_j (0 \leq j \leq 3)$. Its solutions can be found using computer algebra systems. Here are some representations for the function $w$:

$$w = \frac{s_0 + s_1 F + s_3 FG}{p_0 + p_1 F + p_3 FG}, \quad w = \frac{s_1 F + s_3 FG}{p_2 G + p_3 FG},$$

$$w = 4 \frac{a_0 s_1 F + s_3 FG}{p_0 (4 a_0 + a_1 F)}, \quad w = \frac{s_1 s_2 / s_3 + s_1 F + s_2 G + s_3 FG}{p_0}.$$ 

Coefficients of ordinary differential equations for the functions $F, G$ are cumbersome and we do not present them.

Back to the Tzitzeica equation (2.9) and introduce a new function

$$v = \exp(\psi).$$

Then the equation (2.9) is rewritten as follows

$$v(v_{xx} + v_{yy}) - v_x^2 - v_y^2 - v^3/2 + 1 = 0. \quad (3.8)$$

First, we look for solutions to this equation in the form

$$v = \frac{s_0 + s_1 F + s_2 G}{p_0 + p_1 F + p_2 G} \quad (3.9)$$
where $F$ and $G$ satisfy the equations (3.3), (3.4). Substitute $v$ again into the right side (3.8) and repeating the reasoning above, we get the following representation

$$v = s_0 + F + G, \quad s_0 \in \mathbb{R}.$$  

The equations for the functions $F$ and $G$ have the form

$$(F_x')^2 = 2F^3 + \left(3s_0 + \frac{a_1 - b_1}{2s_0}\right) F^2 + a_1 F - s_0^3 + \frac{s_0(a_1 + b_1)}{2} + 1 - b_0,$$

$$(G_y')^2 = 2G^3 + \left(3s_0 + \frac{-a_1 + b_1}{2s_0}\right) G^2 + b_1 G + b_0,$$

with $s_0 \neq 0$. The solutions of the previous two equations are expressed in terms of the Weierstrass elliptic functions. To get a specific solution, let’s set $s_0 = 1, b_0 = -0.1, a_1 = b_1 = -0.2$. As the initial data, we choose the values $F(0) = G(0) = -0.2$. A two-dimensional contour graph of the function $v$ is shown in Fig. 6. On one of the contour lines, the function $v$ is equal to zero, and therefore the stream function $\psi$ is not defined on it.

![Contour plot of a solution of the equation (3.8).](image)

We are now looking for a function $v$ in the form

$$v = \frac{s_0 + s_1 F + s_2 G + s_3 FG}{p_0 + p_1 F + p_2 G + p_3 FG}.$$  

By repeating the previous reasoning, we can obtain several different representations for $v$. The simplest of them is

$$v = \frac{p_0 + p_3 FG}{s_2 G}.$$
In this case, the functions $F$ and $G$ must satisfy the following equations

\[ (F')^2 = 2p_3 F^3 - b_2 F^2 + b_3 p_0 F/p_3 + a_0, \]  
\[(3.11)\]

\[ (G')^2 = b_4 G^4 + b_3 F^3 + b_2 G^2 + 2p_0 G/s_2. \]  
\[(3.12)\]

In conclusion, we give a few more representations for the function of the function $v$:

\[ v = \frac{p_0 + p_1 F + p_3 FG}{s_1 F}, \]

\[ v = \frac{p_0 + p_1 F + p_0 s_3 G/s_1 + p_3 FG}{s_1 F + s_3 FG}, \]

\[ v = \frac{p_0 + p_1 F + p_2 G + p_3 FG}{p_0 s_3 G/p_1 + s_3 FG}. \]

We do not present the form of the corresponding equations (3.3), (3.4) due to their cumbersomeness.

4 Conclusions

In this paper, we found new classes of solutions of the two-dimensional Euler equations for an inviscid fluid. They describe various smooth and singular vortex flows. A new method for constructing solutions in elliptic functions is proposed. It would be useful to try to classify all such solutions. It is interesting to construct similar solutions in elliptic functions for other mathematical models. The question of finding similar solutions for non-stationary Euler equations remains open. It would be very important to prove the quantization of the flux of fluid volume across the closed curve without the assumption of representing solutions in the form (2.5).

Список литературы

[1] G. Batchelor. An introduction to fluid dynamics. Cambridge University Press, 1970.

[2] G. Bateman, MHD instability. Cambridge, Mass., MIT Press, 1978.

[3] Yu. B. Movsesyants. Solitons in collisionless cold plasma // Physica A 140 (1987) pp. 554-566

[4] A.B. Borisov and V.V. Kiseliev. Topological defects in incommensurate magnetic and crystal structures quasi-periodic solutions of elliptic Sine-Gordon equation // Physica D (1988) V.31, pp. 49-64

[5] N.H. Ibragimov. Transformation groups applied to mathematical physics, Reidel, Boston, 1985

[6] C. Rogers, W. K. Schief, Bäcklund and Darboux Transformations: Geometry and Modern Applications in Soliton Theory. (2002) Cambridge University Press.
[7] O. Kaptsov. Some classes of plane vortex flows of an ideal fluid // Journal of Applied Mechanics and Technical Physics, 1989, V.30, No 1. pp. 109-117

[8] O. Kaptsov. Some classes of two-dimensional stationary vortex structures in an ideal fluid // Journal of Applied Mechanics and Technical Physics, 1998, V. 39, No 4, pp. 50-53.

[9] J. Shercliff. Simple rotational flows // J. Fluid Mech. (1977), vol. 82, part 4, pp. 687-703

[10] V. Andreev, O. Kaptsov, V. Pukhnachov, A. Rodionov. Applications of group-theoretical methods in hydrodynamics. Kluwer Academic Publishers, 1998.

[11] Kaptsov, O.V., Kaptsov, D.O. Exact solution of Boussinesq equations for propagation of nonlinear waves // Eur. Phys. J. Plus (2020) 135: 723

[12] I.L. Kantor, A.S. Solodovnikov, Hypercomplex Numbers: An Elementary Introduction to Algebras .Springer New York, 1989

[13] A. A. Andronov, A. A. Vitt, and S. E. Khaikin, Theory of Oscillators. Oxford/London/New York. Pergamon Press. 1966.

[14] V. I. Arnold. Ordinary differential equations. Springer Berlin Heidelberg. 1992

[15] D. Thouless. Topological Quantum Numbers in Nonrelativistic Physics. World Scientific Publishing Company. 1998

[16] R. Steuerwald. Über Enneper’sche Flächen und Bäcklund’sche Transformation // Abh. Bayer. Akad. Wiss.— 1936.— Heft 40, pp. 1—105

[17] Belokolos, E.D.; Bobenko, A. I.; Enol’skii, V. Z.; Its, A. R.; Matveev, V. B. Algebro-Geometric Approach to Nonlinear Integrable Equations. Berlin etc., Springer-Verlag 1994.
