A robust numerical scheme for transcritical flow simulation in a Braided river

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Abstract: Flow simulation over a braided channel is a challenging task as in the monsoon season flow moves over an undulating bed while as in non-monsoon season or in lean period it moves through several sub-channel. Therefore a more robust scheme is necessary for simulating such kind flow particularly when transcritical flow occurs in river reach. Shallow water equations can be derived by depth integration of the Navier Stokes equation and assuming hydrostatic pressure distribution. These equations are basically based upon the conservation of mass and momentum approach. Shallow water equations are a set of a first-order nonlinear hyperbolic partial differential equation and do not have closed form of analytical solution without considering a large number of approximation because of which these equation needs to be solved by different numerical techniques like finite difference, finite element, and finite volume. The hyperbolic nature of the equations results in a generation of numerical dispersion in the solution domain because of which proper shock capturing methods need to be implemented along with the scheme. The presence of bed slope term in the governing equations causes some difficulty to apply them in complex bathymetry. Researchers have modified the original form of governing equations so that these equations can be implemented in a complex topography with proper mass conservation. In this paper, the modified form of the 2D governing equation is combined with the proper shock capturing technique to investigate the applicability of the scheme in complex bathymetry. This scheme is used as an integral part of BRAHMA model (Braided River Aid: Hydro-Morphological Analyser) developed by IIT-Guwahati in collaboration with Brahmaputra Board. Governing equations solved by Mac Cormack predictor-corrector scheme and the scheme has been tested with some classical problem including dam break, quiescent water above an irregular bed, steady flow over an irregular bed, transcritical flow over a bump etc. For each test case numerical results are compared with the analytical result and found satisfactory.

Keywords: Complex Bathymetry, Transcritical Flow, Shallow water equation

1 Introduction

Due to the advancement of computer solutions shallow water equations have been used widely nowadays for the simulation of river flow. Shallow water equations are derived from Navier-Stokes equations by integrating over the depth and assuming the pressure distribution as hydrostatic. Shallow water equations are nothing but the sets of non-linear first order hyperbolic partial differential equations which contains continuity and momentum equation and used in the simulation of unsteady flow. It is difficult to get the analytical solution for these equations without considering a large number of approximation. Because of this difficulty several numerical methods like finite difference, finite volume and finite element method have been adopted for simulation of shallow water equations (Tseng & Chu, Chaudhry[10], Liang et al. [14], Sanyal “et al”, Hriday Mani Kalita, Kalita H.M, Sarma A.K[13]). The hyperbolic nature of the equations leads to dissipative and dispersion errors in the solution domain and it is necessary to adopt a proper shock capturing technique to suppress these oscillations (Anderson).

As in case of natural channels such as river flow, open channel flow usually transits from one flow condition to another due to bed slope changes and variation in channel width. These conditions are
classified into subcritical where the flow velocity is less than the wave celerity and supercritical in which the flow velocity is greater than the wave speed (celerity) and critical flow condition where the flow velocity and celerity are the same. When there is a change from one flow condition to another, the condition is regarded as transcritical flow. Among the family of finite difference schemes MacCormack Predictor corrector (MacCormack) scheme is most widely used for the solution of governing equation. This scheme is shock capturing and suitable to apply in the transcritical region where hydraulic jump formed. However, this scheme is second order accurate in nature which leads to dispersion error and further propagation of the error in the solution creates spurious oscillation and ultimately leads to an unstable solution. During 80s different investigator (Chaudhry, 1986; Jamson “et al”, 1981) have used a method termed as artificial viscosity to suppress the oscillations but the disadvantage of this method is that modeller has to choose the coefficient by trial and error procedure which is tedious. In order to overcome this trial and error approach Garcia-Navarro “et al”. used a theory named, TVD technique which can capture the sharp discontinuities without generating the spurious oscillations. This methodology was originally put forwarded by Harten (1984) and the advantage with it is it can automatically take the required amount of dissipation on different places in the computational domain. Liang et al [14] used a TVD technique proposed by Davis (1984) in MacCormack predictor-corrector scheme to simulate unsteady flow. Later different investigator have used this TVD technique in the simulation in hydrodynamic models. In their models they first spilt the 2D equation into two 1D equation and 1D equations are solved by traditional predictor corrector step in a sequence of four times.

In this paper transcritical flow occurred over complex bathymetry is simulated using MacCormack predictor corrector scheme and TVD technique is used for shock capturing.

2 Hydrodynamic Model

Assuming the hydrostatic pressure distribution following form of the governing equation is used

\[ U_t + E_x + F_y + S = 0 \]  
where \[ U = \begin{bmatrix} \eta \\ hu \\ hv \end{bmatrix}, \quad E = \begin{bmatrix} hu \\ u^2 h \\ uhv \end{bmatrix}, \quad F = \begin{bmatrix} hv \\ huv \\ v^2 \end{bmatrix}, \quad S = \begin{bmatrix} 0 \\ gh \frac{d \eta}{dx} - S_{fx} \\ gh \frac{d \eta}{dy} - S_{fy} \end{bmatrix} \] (1)

Where \( \eta \)=water surface elevation, \( h \)=depth of flow \( u \)=velocity in x direction, \( v \)=velocity in y direction, \( S_{fx} \) 
& \( S_{fy} \) are the friction slope in x and y direction. Governing equation presented above are solved by using TVD Mac-Cormack predictor corrector scheme.

\[ U_{i,j}^p = U_{i,j} - \Delta t \left( E_{i,j}^k - E_{i,j-1}^k \right) - \frac{\Delta t}{\Delta y} \left( F_{i,j}^k - F_{i-1,j}^k \right) - \Delta t S_{i,j} \] (2)

\[ U_{i,j}^c = U_{i,j} - \Delta t \left( E_{i,j+1}^p - E_{i,j}^p \right) - \frac{\Delta t}{\Delta y} \left( F_{i+1,j}^p - F_{i,j}^p \right) - \Delta t S_{i,j}^p \] (3)

\[ U_{i,j}^{k+1} = \frac{1}{2} \left( U_{i,j}^p + U_{i,j}^c \right) + TVD_{i,j} \] (4)
where $\Delta x$ is the grid spacing in x direction designated by the subscript i, $\Delta y$ is the grid spacing in y direction designated by the subscript j, $\Delta t$ is the grid spacing in the time axis (time step) designated by the superscript k, superscripts p and c stands for predicted and corrected values. BRAHMA2D model has been developed using the above mentioned numerical scheme and has been tested for complex bathymetry with different flow conditions.

3 Application of the BRAHMA2D model

BRAHMA2D model has applied in the following case

3.1 Transcritical flow over a bump without shock:

A classical test problem is considered for a channel of length 25m, width 1m with the bump. The bathymetry of the bump is given by the following equation

$$z_B = \begin{cases} 
2-0.05(x-10)^2 & \text{if } 8<x<12 \\
0 & \text{otherwise} 
\end{cases} \quad (5)$$

Transcitical flow is the region in which both subcritical and supercritical flow is exist. To simulate this the computational domain is divided into 100 number of cells by taking $\Delta x = 0.25m$. A discharge per unit width of $q=1.53 \text{ m}^3/\text{sec}$ is considered at upstream and a water level of $\eta=0.66m$ is considered at downstream. Near the side walls free slip boundary condition is used. Courant number is found to be 0.85. Simulated water surface profile is plotted in fig.1

![Figure 1-Transcritical flow without shock](image)

After the application of the BRAHMA2D in above mentioned flow condition it is found that model is able to simulate the transcritical flow region and also able to capture the discontinuities.
3.2 Transcritical flow over a bump with shock:
The second case considered here is the simulation of steady flow for a channel of length 25m, width 1m over the hump with shock. The bathymetry of the hump is given by the following equation:

\[
z_b = \begin{cases} 
2^{-0.05(x - 10)^2} & \text{if } 8 < x < 12 \\
1 & \text{otherwise}
\end{cases}
\] (6)

To simulate this, the computational domain is divided into 250 number of cells by taking \( \Delta x = 0.1 \)m. A discharge per unit width of \( q = 0.18 \) m\(^2\)/sec is considered at upstream and a water level of \( \eta = 0.33 \) m is considered at downstream. The side walls free slip boundary condition is used. Courant number is found to be 0.77. Simulated water surface profile is plotted in fig. 2.

![Figure 2 - Transcritical flow with shock](image)

After the application of the BRAHMA2D in above mentioned flow condition it is found that model is able to simulate the complex flow condition as shown in figure 2. This can be considered as complex case because here hydraulic jump is formed due to the variation of flow parameters. Because of the Shock capturing technique employed in the model it is able to smooth the oscillations and maintain the stability.

3.3 Steady flow over an irregular bed:
Third case considered here is the simulation of steady flow over an irregular bed to check the Well balanced property of the numerical scheme. The test channel is 1500 m long and manning’s roughness is considered as 0.1. To simulate this the computational domain is divided into 300x5 number of cells by taking \( \Delta x = \Delta y = 5 \). Time step value is considered as
0.3 sec which leads to a courant number of 0.68. Near the side walls no slip boundary condition is used. For initial condition water surface elevation is considered as 15m over the whole domain.

4 Conclusion

BRAHMA2D(Braided River Aid: Hydro-Morphological Analyser ) model used in this study to simulate the transcritical flow region with and without shock flow over irregular bed topography is found to be stable and conservative.

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