Complete bifurcation analysis of DC-DC converters under current mode control

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Abstract. The purpose of this research is to investigate to what extent application of novel method of complete bifurcation groups to the analysis of global dynamics of piecewise-smooth hybrid systems enables one to highlight new nonlinear effects before periodic and chaotic regimes. Results include the construction of complete one and two-parameter bifurcation diagrams, detection of various types of bifurcation groups and investigation of their interactions, localization of rare attractors, and the investigation of different principles of birth of chaotic attractors. Effectiveness of the approach is illustrated in respect to one of the most widely used switching systems – boost converter under current mode control operating in continuous current mode.

1. Introduction
Bifurcations of piecewise-smooth hybrid dynamical systems have received much attention in recent years due to some unexpected, and even unique, phenomena observed in this kind of systems [1-6, 9]. As a matter of fact, the numerical and analytical tools that can be used to study these bifurcations are still very incomplete [1, 2, 9] and a need for an efficient methods and software tools that can handle a large variety of nonlinear problems associated with piecewise-smooth systems therefore remains. The brute-force approach is a general method used for calculating bifurcation diagrams of stable steady-state solutions of arbitrary systems. In contrast, this paper presents the improvement of innovative approach, called Method of Complete Bifurcation Groups (MCBG) [8, 10], allowing the complete bifurcation analysis of highly nonlinear systems, considering stable and unstable periodic orbits. The purpose of this research is to study the possibilities of application of MCBG to the analysis of dynamics of piecewise-smooth hybrid systems, constructing the complete one-parametric bifurcation diagrams and two-parametric diagram (bifurcation map) for simple switching converter – boost one.

2. The discrete model of the current mode controlled boost converter with compensating ramp
The schematic diagram of the conventional current controlled boost converter operating in continuous conduction mode (CCM – when the inductor current never falls to zero) with a control circuitry including a comparator and a RS flip-flop is shown in the figure 1.

The comparator likens the inductor current to a reference current $I_{ref}$ in order to control the state of switching element $S$. Thus the circuit could be described by two different structures, depending on whether the switch is open (OFF) or closed (ON). When the switch $S$ is in the ON position, the diode $D$ is reverse biased and non-conducting. Assuming that the resistance of inductor $L$ is negligible, the current $i_L$ rises almost linearly (see figure 1(b)) and the energy is accumulated in the magnetic field of the inductor. The switch $S$ is opened when the inductor current reaches the reference value $I_{ref}$, then the voltage $Ldi_L/dt$ is induced in the inductor in order to maintain the current flow. This forward biases the...
diode and the current decays almost linearly, transferring the energy from the inductor to the capacitor \( C \). The switch is closed each time the clock pulse with period \( T \) arrives [1].

\[
\begin{align*}
\text{Figure 1. (a) The current controlled boost converter; (b)-(d) typical waveforms of inductor current and the control signal for the switch.}
\end{align*}
\]

It is well known and documented fact that almost all current mode controlled switching converters are capable of exhibiting subharmonic modes of operation [1]. Thus, in order to avoid the occurrence of undesirable operating regimes, special compensating ramp is added to the reference current. This compensating signal is widely integrated in the DC-DC converter control chips and is taken into account while defining the model of the boost converter under study.

The operation of boost converter could be described by two systems of differential equations, accordingly to the two possible switching structures. However, it is now a common practice to use the discrete-time models, which provide the simple tool for the direct construction of bifurcation diagrams as well as analytical investigation of various types of nonlinear phenomena. The discrete-time model of the boost converter under study is the function that relates the voltage and the current vector at one instant \((i_{n+1}, v_{n+1})\) to the vector at previous instant \((i_n, v_n)\); the sequence of time instants here is clearly defined by the arrival of clock pulses. The boost converter is unusual in that the discrete-time model can be derived in closed form without approximations [1, 6]. It could be deduced that there are two ways in which the state can move from one clock instant to the next. A clock pulse may arrive before the inductor current reaches the \( I_{ref} \) (figure 1(b)). In that case, the obtained discrete-time model is the following:

\[
\begin{align*}
\text{(1) } v_{n+1} &= v_n e^{-T/RC} \\
\text{(2) } i_{n+1} &= i_n + ET/L
\end{align*}
\]
On the other hand, if the inductor current reaches $I_{\text{ref}}$ before the arrival of the next clock pulse (figure 1(c)) the map would include the ON and OFF intervals. In such case, the discrete time mapping is as follows:

\[
v_{n+1} = e^{-mt_1^2} [K_1 \cos(\mu t_2) + K_2 \sin(\mu t_2)] + E \tag{3}
\]

\[
i_{n+1} = e^{-mt_1^2} \left[ C[-m(K_1 \cos(\mu t_2) + K_2 \sin(\mu t_2))] + \mu(-K_1 \sin(\mu t_2) + K_2 \cos(\mu t_2))] + (K_1 \cos(\mu t_2) + K_2 \sin(\mu t_2))/R \right] + E/R, \tag{4}
\]

where $K_1 = v_n e^{-2mt_1^2} - E$, $K_2 = [I_{\text{ref}}C-m(v_n e^{-2mt_1^2} + E)]/\mu$, $t_1 = (I_{\text{ref}} - i_n)/(E/L + S_c)$, $t_2 = T - t_1$, $S_c = V_{p-p}/T$, $\mu = \sqrt{(p^2 - m^2)}$, $m = 1/(2RC)$ and $p = 1/\sqrt{(LC)}$.

The borderline between the two regimes described by maps (1)-(2) and (3)-(4) is given by the value $I_{\text{border}}$ for which the inductor current reaches the reference value exactly at the arrival of the next clock pulse (figure 1(d)):

\[
I_{\text{border}} = I_{\text{ref}} - T(E/L + S_c) \tag{5}
\]

The model presented in this section is used to obtain the complete bifurcation diagrams and provide all necessary investigation of phenomena observed in the current controlled DC-DC boost converter.

3. The concepts of the Method of Complete Bifurcation Groups (MCBG)

The Method of Complete Bifurcation Groups, developed in Riga Technical University in a group under guidance of Professor M.V. Zakrzhevsky [10], includes the complex of approaches used for the analysis of highly nonlinear dynamical systems. Precisely, it implies the following procedures:

- for fixed parameters of the system: detection of all stable and unstable periodic regimes and bifurcation subgroups with so called unstable periodic infinitiums (UPI – the area in which only unstable periodic orbits of definite bifurcation group exist [8,10]) on plane of states, construction of regimes’ basins of attraction;
- for varied system parameters: construction of complete one parameter bifurcation diagrams and bifurcation maps.

The continuation along a solution branch of selected regime (not parameter) is of special importance in the MCBG. This approach allows finding new unknown before stable regimes in widely used dynamical models of strongly nonlinear oscillatory systems.

One-parameter bifurcation analysis within the MCBG includes following procedures:

- the construction of periodic skeleton: all stable and unstable periodic solutions at fixed value of bifurcation parameter are found, the number and types of bifurcation groups is identified;
- on the basis of obtained periodic skeleton, using the numerical continuation, the branches of bifurcation diagram, corresponding to each found group are constructed. In many cases, on the branches of diagram corresponding to unstable regimes, the greater or smaller stable periodic solutions are found (rare attractors – RA);
- the complete bifurcation diagram is composed, including stability estimation procedure, defining (analytically or numerically) appropriate monodromy matrixes for all found regimes and determining their multipliers;
- the revision of the diagram allows detection of different types of RA and UPI.

In the forthcoming Sections, the described procedure will be applied to the model (1)-(4) and appropriate complete bifurcation diagrams for boost DC-DC converter under study will be constructed.

4. The complete bifurcation analysis of the boost converter

At the beginning of the investigation, the two-parameter bifurcation diagram (map) is constructed, dividing the two parameter space to different regions of periodic and chaotic behavior. The reference current and the output capacitance are selected accordingly as primary and secondary bifurcation parameters. The constructed bifurcation map was obtained for the boost converters with parameters
defined in Table 1. The map shows that for small values of output capacitance, the boost converter
exhibits period doubling (PD) bifurcation cascade, chaotic mode of operation and a great variety of
subharmonic windows. However, the increment of the secondary bifurcation parameter leads to the
contraction of area of the first period doubling cascade with the subsequent transition to robust chaotic
operation (without periodic windows and coexisting attractors).

Table 1. Parameters of the boost converter.

| Parameter              | Value       |
|------------------------|-------------|
| Switching period, T    | 100 μs      |
| Inductance, L          | 1.5 mH      |
| Load resistance, R     | 40 Ω        |
| Output capacitance, C  | 2-20 μF     |
| Input voltage, E       | 5 V         |
| Reference current, I<sub>ref</sub> | 0.2…1.5 A |
| Compensating ramp, S<sub>c</sub> | 0           |

Figure 2. Two parameter bifurcation diagram of the boost converter.

To provide the detailed analysis of the dynamics of boost converter, the complete bifurcation
diagrams are constructed as the cross-sections of the bifurcation map. The first complete bifurcation
diagram (for C=5 μF) depicted in Figure 3 (where dark lines represent stable and light lines represent
unstable periodic regimes), shows that for the small values of reference current the system operates in
the stable period-1 (P1) regime (within the 1T bifurcation group), that loses its stability through
smooth period doubling bifurcation at I<sub>ref</sub>≈0.42 A. Increasing the primary bifurcation parameter (I<sub>ref</sub>),
the classical period doubling bifurcation cascade and the subsequent chaotization is observed (which is
connected to the formation of the UPI-1). Further increment of I<sub>ref</sub> leads to the occurrence of the 6T
bifurcation group that also includes the smooth period doubling cascade ending with the chaotic mode
of operation. Figure 4 shows the amplified fragment of the complete bifurcation diagram, exhibiting the sudden disappearance and the subsequent appearance of the chaotic attractor defined by 6T bifurcation group. The described phenomenon could not be interpreted in terms of the loss of local stability of definite periodic regime as it is connected to the appearance of different types of global crises. The type of the crisis could be simply detected, studying the unstable branches of bifurcation diagram, constructed within the MCBG.

![Bifurcation Diagram](image)

**Figure 3.** The complete bifurcation diagram for the boost converter $C = 5 \mu F$.

Figure 4 shows that, at the point of “disappearance” of the chaotic attractor, the unstable branch of P6 regime touches the attractor that becomes unstable through the boundary crisis. It is important to note, that the unstable chaotic attractor of 6T bifurcation group still exists, and becomes stable through the internal crisis when the unstable branch of P6 touches the 1T chaotic attractor (at $I_{ref} \approx 0.73 A$), forming the global chaotic dynamics of the system.
The dynamics of each bifurcation group could be studied separately; however, the global dynamics of the boost converter is defined by the interaction of various bifurcation groups. The bifurcation diagram, depicted in the figure 3, shows that for certain parameter values the 1T chaotic attractor coexists with the P6 regime of 6T bifurcation group, defining the region of multistability, in which each regime has its own basin of attraction. The mentioned attractors along with the appropriate basins of attraction are shown in the figure 5 (black area represents the basin of attraction of P6 regime, and the white area – that of 1T chaotic regime). It could be observed that the border of the basin of attraction is rather complex and the occurrence of external noise -or even small deviation in system parameters- could lead to skipping between chaotic and subharmonic modes of operation.
Figure 6. Coexistence of P6 and P9 regimes, both born from single P3 regime for $C = 7 \, \mu F$.

The complete bifurcation diagrams, obtained for the greater values of the output capacitance in general have structure similar to one shown in the figure 3, so for the sake of simplicity only the most atypical bifurcation groups will be shown.

The increment of the output capacitance leads to the growth of the influence of the borderline, defined in (5), on the global dynamics of the converter. For example figure 6 shows the 3T bifurcation group within which the first non-smooth period doubling bifurcation caused by the collision with $I_{\text{border}}$ leads to the appearance of two stable coexisting P6 and P9 regimes with their own chaotic attractors (UPI$_{1}$-3T and UPI$_{2}$-3T). In fact the observed bifurcation is simultaneous period doubling and period tripling, leading to very complicated dynamics of the boost converter.

Figure 7. The complete bifurcation diagram for the boost converter for $C = 17 \, \mu F$.  

Another example of the effect of the pronounced borderline is shown in the figure 7, where after the non-smooth period doubling bifurcation no stable orbits occur (see the region of robust chaos in figure 2). All bifurcation groups (see for example $6T_1$ and $5T_1$) are formed by uncommon border collision, which leads to the jump of the characteristic multiplier across the whole unity circle, causing the simultaneous occurrence of period doubling and saddle-node (SN) bifurcation (see figure 8). The only analogous of this “strange bifurcation” up to authors knowledge is described in [7], within the study of complex behavior of mechanical systems.

![Figure 8. The dependence of the maximal multiplier of P5 regime within 5T1 bifurcation group on parameter for $C=17 \mu F$.](image)

Thus it has been shown, that for the small values of output capacitance the smooth bifurcation dominate in the boost converter. However, for greater values of $C$ the collision with the borderline (5) may lead to the manifestation of sudden “strange” bifurcations and the appearance of rather uncommon types of coexisting bifurcation groups.

5. Conclusions
Nowadays the typical design objective in majority of engineering systems stable operation with wide and predictable stability margins. Switching mode DC–DC converters often do not fit cleanly into such a framework. The use of switching elements makes this kind of systems highly nonlinear and the prediction of their instabilities is a challenging task.

It has been demonstrated in this communication that the MCBG could be efficiently applied to investigation of the dynamics of switching DC–DC converters in order to provide the researcher and designer with necessary information about the domains of existence and stability of period-1 and subharmonic operation, as well as the appearance of smooth and various types of unordinary non-smooth bifurcations in this kind of circuits.

The application of MCBG to the analysis of complex behavior of switching converters demonstrates potential for general applicability of pronounced technique to exploration of dynamics of switching (hybrid) systems described by set of difference or differential equations. The development in this direction may bring us to a more detailed understanding of the complex dynamics of even more complicated systems with several control loops and sophisticated switching borders, using the concepts of complete bifurcation groups and rare attractors.

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References

[1] Banerjee S, Verghese G C 2001 *Nonlinear Phenomena in Power Electronics . Attractors, Bifurcations, Chaos, and Nonlinear Control* (IEEE Press)

[2] di Bernardo M, Budd C J, Champneys A R and Kowalczyk P 2008 *Piecewise-Smooth Dynamical Systems: Theory and Application* (Berlin: Springer)

[3] di.Bernardo M, Fosas E, Olivar G, Vasca F 1997 Secondary bifurcations and high-periodic orbits in voltage controlled buck converter *International Journal of Bifurcation and Chaos* 12 pp 2755-2771

[4] Chan W C Y, Tse C K 1996 A universal bifurcation path conjectured from current-programmed switching converters *Proceedings of International Symposium on Nonlinear Theory and Its Applications* (Japan) pp 121-124

[5] Fosas E and Olivar G 1996 Study of chaos in buck converter *IEEE Transactions on Circuits and Systems Part I* 43 pp 13-25

[6] Hammil D C, Deane J H B and Jefferies D J 1992 Modeling of chaotic dc/de converters by iterative nonlinear mappings *IEEE Transactions on Circuits and Systems Part I* 35 pp 25-36.

[7] Leine R I 2000 *Bifurcations in Discontinuous Mechanical Systems of Filippov-Type* Ph.D. thesis (Eindhoven University of Technology)

[8] Schukin I, Zakrzhevsky M 2008 Application of Software SPRING and Method of Complete Bifurcation Groups for the Bifurcation Analysis of Nonlinear Dynamical System *Journal of Vibroengineering* 10 pp. 510-518

[9] Tse C K 2003 *Complex Behavior of Switching Power Converters* (Boca Raton: CRC Press)

[10] Zakrzhevsky M 2008 New Concepts of Nonlinear Dynamics: Complete Bifurcation Groups, Protuberances, Unstable Periodic Infinitiums and Rare Attractors *Journal of Vibroengineering* 10 pp 421-441