Counting $\text{SO}(9) \times \text{SU}(2)$ representations in coordinate independent state space of SU(2) Matrix Theory

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Abstract

We consider decomposition of coordinate independent states into $\text{SO}(9) \times \text{SU}(2)$ representations in SU(2) Matrix theory. To see what and how many representations appear in the decomposition, we compute the character, which is given by a trace over the coordinate independent states, and decompose it into the sum of products of SO(9) and SU(2) characters.

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1 Introduction

Matrix theory, which is expected to be a correct description of M-theory, is a quantum mechanics with two sets of operators: bosonic coordinate matrices $X^a_i$ and fermionic matrices $\theta^a_{\alpha}$. This quantum mechanics has SO(9) symmetry of space rotation, and gauge symmetry SU($N$). To investigate the structure of wavefunctions in this theory, it is necessary to know the structure of the space spanned by coordinate independent states i.e. states constructed only by $\theta^a_{\alpha}$. Especially we want to know what representations of SO(9) and SU($N$) those states form. However the number of states are enormous even if we only take coordinate independent states, and it makes explicit construction of representations difficult.

In this paper we count numbers of representations appearing in the space of coordinate independent states in the case of SU(2) gauge group, avoiding explicit construction of representations. To do it efficiently we employ the notion of characters in group theory: We introduce $\chi$, a trace of a group element of SO(9)$\times$SU(2) over the coordinate independent states. If we take an appropriate basis of the states, $\chi$ can be calculated explicitly, and by decomposing it into sum of products of SO(9) and SU(2) characters, we can immediately read off what and how many representations appear in the space of coordinate independent states. Similar analyses have been made in [1] for SO(7)$\times$SU(2) singlets, and in [3] for SU($N$) gauge group singlets. As a byproduct of our calculation we can give another proof of the uniqueness of SO(9)$\times$SU(2) singlet proven in [1, 2].

In the next section we compute $\chi$ and perform the decomposition. In Appendix A and B we collect information on group theory necessary for the analysis. Calculations are made with the help of symbolic manipulation program Mathematica.

2 SO(9)$\times$SU(2) character

Matrix theory has real Grassmann odd operators $\theta^a_{\alpha}$, where $\alpha = 1, 2, \ldots, 16$ is an SO(9) spinor index and $a = 1, 2, \ldots, N^2 - 1$ is an adjoint index of the gauge group SU($N$). Their anticommutation relation is
\[
\{\theta^a_{\alpha}, \theta^b_{\beta}\} = \delta_{\alpha\beta}\delta^{ab}.
\] (2.1)

For SU(2) gauge group, we have $16 \times 3 = 48$ operators, and half of those are regarded as creation operators and the other half as annihilation operators. Then we can construct $2^{48/2} = 2^{24}$ states. If we fix the adjoint index $a$ then the 8 creation operators give 256 states,
which are classified into 44-dimensional symmetric traceless representation, 84-dimensional 3-
rank antisymmetric representation, and 128-dimensional vector-spinor representation of SO(9).
If we take the adjoint index into account, the decomposition of $2^{24}$ states into $\text{SO}(9) \times \text{SU}(2)$
representations is not immediately clear. To construct gauge invariant wavefunctions it is
important to know it. To this end, we introduce the character $\chi$, which is given by the trace
over the $2^{24}$-dimensional space and is a function of parameters $x_1, x_2, x_3, x_4$ and $y$:

$$\chi = \text{tr}[\exp(i x_1 J_{12} + i x_2 J_{34} + i x_3 J_{56} + i x_4 J_{78} + iyg^1)],$$

where $J_{ij} = -i \theta^a(\gamma_{ij})_{\alpha\beta} \theta^b$ are SO(9) generators and $g^a = \frac{i}{2} \epsilon_{abc} \theta^b \theta^c$ are SU(2) generators. In
addition we define $\tilde{\chi}$ by the following, with fermion number operator insertion $(-1)^F$:

$$\tilde{\chi} = \text{tr}[(-1)^F \exp(i x_1 J_{12} + i x_2 J_{34} + i x_3 J_{56} + i x_4 J_{78} + iyg^1)].$$

Here we define traces of states as the sum of contributions from boson states and fermion
states. Therefore $\tilde{\chi}$ gives the difference of contributions from boson and fermion states.

Since the trace does not depend on choice of orthogonal basis of states, we will take one
which makes calculation of the characters easier. Although $\theta^a$ is real, let us temporarily
consider the case where $\theta^a$ is complex and their anticommutation relation is

$$\{\theta^a, (\theta^b) \dagger\} = \delta_{ab} \delta^{\alpha\beta}.$$ (In other words, $\theta^a$ is given by two copies of the real one

$\theta^{(1)} \theta^{(2)}$: $\theta^a = (\theta^{(1)a} + i \theta^{(2)a})/\sqrt{2}$.) Then we could regard $(\theta^a) \dagger$ as creation operators. Since these operators are
covariant under both of SO(9) and SU(2) transformations, the character were given just by sum
of characters of antisymmetric tensor product representations $\text{Alt}_n[(\text{SO}(9) \text{ spinor}) \times (\text{SU}(2) \text{ adjoint})]$, which can be calculated by Frobenius formula (See Appendix A). (Such calculation
has been done in [3].) However in our case where $\theta^a$ are real, creation and annihilation
operators cannot be separated without losing manifest covariance. Therefore we will take a
different way and calculation will be harder than the complex case. First we define $\theta^\pm$ as

$$\theta^\pm = \frac{1}{\sqrt{2}} \theta^2 \pm i \theta^3.$$ (2.4)

Note that $(\theta^a) \dagger = \theta^\dagger_\mp$. Then nontrivial anticommutation relations are given by

$$\{\theta_\alpha^+, \theta_\beta^1\} = \delta_{\alpha\beta}, \quad \{\theta_\alpha^-, \theta_\beta^+\} = \delta_{\alpha\beta}.$$ (2.5)

A vacuum $|0\rangle$ for these operators is defined as follows:

$$\theta^- |0\rangle = 0.$$ (2.6)
and $\theta^{\dagger}_\alpha$ work as creation operators on this vacuum. Since it is not necessary in the following, we do not specify the action of $\theta^{\dagger}_\alpha$ on $|0\rangle$. Of course we can make different choices of vacuum and creation operators. All of them give the same coordinate independent state space, and make part of $\text{SO}(9)\times\text{SU}(2)$ symmetry not manifest. As a result those vacua cannot be $\text{SO}(9)\times\text{SU}(2)$ singlet. Our choice makes $\text{SO}(9)$ symmetry manifest at the expense of $\text{SU}(2)$ symmetry, and each step of computation of the characters in the following has manifest $\text{SO}(9)$ symmetry. At the end of the computation we will recover manifest $\text{SU}(2)$ symmetry.

We can immediately see that actions of $\theta^{\dagger}_\alpha$ and $\theta^{\pm}_\alpha$ can be considered separately, and the characters are decomposed into two parts corresponding to them:

$$
\chi = \chi^{\theta^1}_\alpha \chi^{\theta^z}_\alpha, \quad \tilde{\chi} = \tilde{\chi}^{\theta^1}_\alpha \tilde{\chi}^{\theta^z}_\alpha.
$$

(2.7)

Since $[g^1, \theta^1_\alpha] = 0$, $\chi^{\theta^1}_\alpha$ and $\tilde{\chi}^{\theta^1}_\alpha$ can be readily computed. Indeed $\chi^{\theta^1}_\alpha$ is just the sum of $\text{SO}(9)$ characters of 2-rank symmetric traceless, 3-rank antisymmetric, and vector-spinor representation:

$$
\chi^{\theta^1}_\alpha = \chi^{[2000]} + \chi^{[0010]} + \chi^{[1001]}, \quad \tilde{\chi}^{\theta^1}_\alpha = \chi^{[2000]} + \chi^{[0010]} - \chi^{[1001]},
$$

(2.8)

where $\text{SO}(9)$ representations are indicated by Dynkin labels $[q_1q_2q_3q_4]$. See Appendix B for more information and notation for $\text{SO}(9)$ characters.

Next we compute $\chi^{\theta^\pm}_\alpha$ and $\tilde{\chi}^{\theta^\pm}_\alpha$. States are classified by the number of $\theta^{\dagger}_\alpha$ on $|0\rangle$:

$$
|0\rangle, \quad \theta^{\dagger}_\alpha |0\rangle, \quad \theta^{\dagger}_\alpha \theta^{\dagger}_\alpha |0\rangle, \quad \ldots, \quad \theta^{\dagger}_\alpha \ldots \theta^{\dagger}_{\alpha_1} |0\rangle.
$$

(2.9)

Since $g^1 = 8 - \theta^{\dagger}_\alpha \theta^\alpha$ and $\theta^{\dagger}_\alpha \theta^\alpha$ works as the number operator for $\theta^{\dagger}_\alpha$, in the sector of $n \theta^{\dagger}_\alpha$, the factor $\exp(ig^1)$ in the characters gives $e^{i(8-n)y}$. Obviously this sector forms $n$-rank antisymmetric product representation of $\text{SO}(9)$ spinor, and therefore contribution to $\chi^{\theta^\pm}_\alpha$ from this sector is given by

$$
e^{i(8-n)y} \chi(\text{Alt}_n(\text{spinor})).
$$

(2.10)

$\chi(\text{Alt}_n(\text{spinor}))$ can be calculated by Frobenius formula. In fact, sectors of $n \theta^{\dagger}_\alpha$ and of $16 - n \theta^{\dagger}_\alpha$ are in the same representation of $\text{SO}(9)$, because states $\theta^{\dagger}_\alpha \ldots \theta^{\dagger}_{\alpha_1} |0\rangle$ are also expressed as $\epsilon_{\alpha_1 \ldots \alpha_n \alpha_{n+1} \ldots \alpha_{16}} \theta^{\dagger}_{\alpha_1} \ldots \theta^{\dagger}_{\alpha_n} |0\rangle$. Indeed straightforward calculation shows $\chi(\text{Alt}_{16-n}(\text{spinor})) = \chi(\text{Alt}_n(\text{spinor})).$

*In [1] the coordinate independent state space for fixed gauge indices is constructed, and its natural extension to the case with gauge indices is given by $\lambda^a_\alpha \equiv \frac{1}{\sqrt{2}} (\theta^{\dagger}_\alpha + i \theta^{\dagger}_{\alpha + 8}) (\alpha = 1, \ldots, 8), (\lambda^a_\alpha)^\dagger |0'\rangle = 0$. This retains manifest gauge symmetry at the expense of $\text{SO}(9)$ symmetry. We retain $\text{SO}(9)$ instead of $\text{SU}(2)$ because the structure of characters of $\text{SU}(2)$ is simpler than that of $\text{SO}(9)$ and it is easier to recover $\text{SU}(2)$ than $\text{SO}(9)$. The vacuum $|0'\rangle$ is related to our $|0\rangle$ by $|0\rangle = \prod_{\alpha=1}^8 \lambda^{-\alpha} |0'\rangle$. 

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Then the total contributions are

\[
\chi_{\theta^{\pm}} = \sum_{n=0}^{7} \left[ e^{i(8-n)y} + e^{-i(8-n)y} \right] \chi(\text{Alt}_n(\text{spinor})) + \chi(\text{Alt}_8(\text{spinor})),
\]

(2.11)

\[
\tilde{\chi}_{\theta^{\pm}} = \sum_{n=0}^{7} \left[ e^{i(8-n)y} + e^{-i(8-n)y} \right] (-1)^n \chi(\text{Alt}_n(\text{spinor})) + \chi(\text{Alt}_8(\text{spinor})).
\]

(2.12)

As is well-known, representations of SU(2) are labeled by nonnegative half integers (spins), and for spin \( n \) representation eigenvalues of \( g^1 \) are \(-n, -n + 1, \ldots, n - 1, n\) and the character \( \chi_n^{SU(2)} \) for this representation is given by

\[
\chi_n^{SU(2)} = \text{tr}_{\text{spin}} e^{iyg^1} = e^{i(-n)y} + e^{i(-n+1)y} + \cdots + e^{i(n-1)y} + e^{iny}.
\]

(2.13)

Note that \( e^{iny} + e^{-iny} = \chi_n^{SU(2)} - \chi_{n-1}^{SU(2)} \). Using this we can rewrite \( y \) dependent part of (2.11) and (2.12). Then the total characters are given by the following, in the forms which make decomposition into SU(2) representations manifest:

\[
\chi = \sum_{n=0}^{8} \chi_n^{SU(2)} \chi_n^{SO(9)}, \quad \tilde{\chi} = \sum_{n=0}^{8} \chi_n^{SU(2)} \tilde{\chi}_n^{SO(9)},
\]

(2.14)

where

\[
\chi_n^{SO(9)} = \begin{cases} 
\chi_{\theta^1} \left[ \chi(\text{Alt}_{8-n}(\text{spinor})) - \chi(\text{Alt}_{7-n}(\text{spinor})) \right] & (n = 0, \ldots, 7), \\
\chi_{\theta^1} & (n = 8),
\end{cases}
\]

(2.15)

\[
\tilde{\chi}_n^{SO(9)} = \begin{cases} 
(-1)^n \tilde{\chi}_{\theta^1} \left[ \chi(\text{Alt}_{8-n}(\text{spinor})) + \chi(\text{Alt}_{7-n}(\text{spinor})) \right] & (n = 0, \ldots, 7), \\
\tilde{\chi}_{\theta^1} & (n = 8).
\end{cases}
\]

(2.16)

\( \chi_n^{SO(9)} \) and \( \tilde{\chi}_n^{SO(9)} \) can be decomposed further, into contributions from boson states and fermion states, denoted by \( \chi_n^{SO(9),B} \) and \( \chi_n^{SO(9),F} \) respectively:

\[
\chi_n^{SO(9),B} = \frac{1}{2} (\chi_n^{SO(9)} + \tilde{\chi}_n^{SO(9)}), \quad \chi_n^{SO(9),F} = \frac{1}{2} (\chi_n^{SO(9)} - \tilde{\chi}_n^{SO(9)}).
\]

(2.17)

Thus we have obtained explicit expressions for the characters, because we know explicit expressions of \( \chi_{\theta^1}, \tilde{\chi}_{\theta^1} \) and \( \chi(\text{Alt}_n(\text{spinor})) \). However the expressions (2.15) and (2.16) do not tell us what SO(9) irreducible representations they contain. So our next task is to decompose (2.15) and (2.16) into the sums of SO(9) characters. Since \( \chi_n^{SO(9),B} \) and \( \chi_n^{SO(9),F} \) are given in the form of products of SO(9) characters, the decomposition can be performed by decomposing tensor product representations of corresponding representations one by one, or by using the following orthogonality relation:

\[
\frac{1}{2^4 \cdot 4!} \prod_{i=1}^{4} \left( \int_{0}^{2\pi} \frac{dx_i}{2\pi} \right) [D_{\rho}]^2 \chi[q_{11},q_{12},q_{13},q_{14}] \chi[q_{21},q_{22},q_{23},q_{24}] = \delta_{q_{11},q_{21}} \delta_{q_{12},q_{22}} \delta_{q_{13},q_{23}} \delta_{q_{14},q_{24}}.
\]

(2.18)
where $D_\rho$ is defined in Appendix B. We take the latter method. We just computed integrals of products of $SO(9)$ characters and $\chi_{n}^{SO(9),B}$ or $\chi_{n}^{SO(9),F}$ using Mathematica and determined the decompositions completely. Then from them we can immediately read off what $SO(9) \times SU(2)$ representations our $2^{24}$-dimensional space are decomposed into, and the multiplicities of those representations. The result is given in Table I.

As a check of our result, let us compute the numbers of states contributing to $\chi_{n}^{SO(9),B}$ and $\chi_{n}^{SO(9),F}$. Those can be counted by reading each column of Table I or by setting $x_i = 0$ in (2.15) and (2.16). We see that these two ways give the same values and the numbers of boson states and fermion states are equal:

\[
\begin{align*}
\chi_0^{SO(9),B}, \chi_0^{SO(9),F} & \to 183040 \text{ states}, \quad (2.19) \\
\chi_1^{SO(9),B}, \chi_1^{SO(9),F} & \to 439296 \text{ states}, \quad (2.20) \\
\chi_2^{SO(9),B}, \chi_2^{SO(9),F} & \to 465920 \text{ states}, \quad (2.21) \\
\chi_3^{SO(9),B}, \chi_3^{SO(9),F} & \to 326144 \text{ states}, \quad (2.22) \\
\chi_4^{SO(9),B}, \chi_4^{SO(9),F} & \to 161280 \text{ states}, \quad (2.23) \\
\chi_5^{SO(9),B}, \chi_5^{SO(9),F} & \to 56320 \text{ states}, \quad (2.24) \\
\chi_6^{SO(9),B}, \chi_6^{SO(9),F} & \to 13312 \text{ states}, \quad (2.25) \\
\chi_7^{SO(9),B}, \chi_7^{SO(9),F} & \to 1920 \text{ states}, \quad (2.26) \\
\chi_8^{SO(9),B}, \chi_8^{SO(9),F} & \to 128 \text{ states}. \quad (2.27)
\end{align*}
\]

Then we can confirm that the total number of states is equal to $2^{24}$:

\[
2^{24} = 16777216 = (183040 \times 2) \times 1 + (439296 \times 2) \times 3 + (465920 \times 2) \times 5 \\
+ (326144 \times 2) \times 7 + (161280 \times 2) \times 9 + (56320 \times 2) \times 11 \\
+ (13312 \times 2) \times 13 + (1920 \times 2) \times 15 + (128 \times 2) \times 17. \quad (2.28)
\]

From the first row of Table I, we see that $SO(9)$ singlet states are decomposed into one singlet and one 13-dimensional representation of $SU(2)$. This is consistent with the result of [1, 2, 4], and gives another proof of the uniqueness of $SO(9) \times SU(2)$ singlet. The second row of Table I tells us that $SO(9)$ vector states are decomposed into one 3-dimensional, one 7-dimensional, one 11-dimensional and one 15-dimensional representation, which is consistent with the result of [4]. From the fifth row we see that there is no gauge invariant $SO(9)$ spinor, which means that the condition of full supersymmetry for the linear term in $X_i^a$ in the expansion of zero energy wavefunction is always satisfied, because the condition is in the form that a gauge invariant $SO(9)$ spinor made of the linear term is equal to zero[4].
| SO(9) | representation | dimension | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-------|----------------|-----------|---|---|---|---|---|---|---|---|---|
| 0, 0, 0 | 9 | 1 | 1 | 1 | 1 | 1 | | | | | |
| 0, 1, 0 | 36 | 1 | 1 | 1 | 1 | 1 | | | | | |
| 0, 0, 1 | 84 | 2 | 1 | 2 | 1 | 2 | 1 | | | | |
| 0, 0, 2 | 16 | 1 | 1 | 1 | 1 | 1 | 1 | | | | |
| 0, 0, 3 | 44 | 2 | 1 | 2 | 1 | 1 | | | | | |
| 0, 0, 4 | 90 | 1 | 1 | 2 | 2 | 1 | 1 | | | | |
| 0, 0, 5 | 256 | 1 | 1 | 2 | 2 | 1 | 1 | | | | |
| 0, 0, 6 | 672 | 1 | 1 | 2 | 1 | 1 | | | | | |
| 0, 0, 7 | 900 | 1 | 1 | 2 | 1 | 1 | | | | | |
| 0, 0, 8 | 2560 | 1 | 1 | 1 | 1 | 1 | | | | | |
| 0, 0, 9 | 5040 | 1 | 1 | 3 | 1 | 1 | | | | | |
| 0, 0, 10 | 12672 | 1 | 1 | 1 | 1 | 1 | | | | | |
| 0, 1, 0 | 2772 | 1 | 2 | 3 | 2 | 1 | | | | | |
| 0, 1, 1 | 2772 | 1 | 2 | 3 | 2 | 1 | | | | | |
| 0, 1, 2 | 450 | 1 | 2 | 1 | 1 | | | | | | |
| 0, 1, 3 | 2772 | 1 | 1 | 1 | 1 | | | | | | |
| 0, 1, 4 | 7700 | 2 | 1 | 2 | | | | | | | |
| 0, 1, 5 | 1920 | 2 | 2 | 2 | 2 | | | | | | |
| 0, 1, 6 | 4608 | 1 | 2 | 3 | 2 | | | | | | |
| 0, 1, 7 | 12672 | 1 | 1 | 2 | 1 | | | | | | |
| 0, 1, 8 | 16968 | 1 | | | | | | | | | |
| 0, 1, 9 | 8748 | 1 | 1 | 1 | | | | | | | |
| 0, 2, 0 | 2772 | 1 | 1 | 1 | | | | | | | |
| 0, 2, 1 | 4452 | 1 | 1 | | | | | | | | |
| 0, 2, 2 | 3900 | 1 | 2 | 4 | 2 | | | | | | |
| 0, 2, 3 | 2756 | 1 | 1 | 1 | | | | | | | |
| 0, 2, 4 | 31500 | 1 | 1 | 1 | | | | | | | |
| 0, 2, 5 | 9504 | 1 | 2 | 3 | 2 | | | | | | |
| 0, 2, 6 | 19712 | 1 | 2 | 3 | 2 | | | | | | |
| 0, 2, 7 | 24192 | 1 | 1 | 1 | | | | | | | |
| 0, 2, 8 | 15444 | 1 | 2 | 3 | 2 | | | | | | |
| 0, 2, 9 | 76632 | 1 | | | | | | | | | |
| 0, 3, 0 | 25740 | 1 | 1 | 2 | 1 | | | | | | |
| 0, 3, 1 | 60690 | 1 | 1 | 1 | | | | | | | |
| 0, 3, 2 | 65536 | 1 | 1 | 1 | | | | | | | |
| 0, 3, 3 | 1122 | 1 | | | | | | | | | |
| 0, 3, 4 | 7140 | 1 | | | | | | | | | |
| 0, 3, 5 | 20196 | 1 | | | | | | | | | |
| 0, 3, 6 | 5280 | 1 | | | | | | | | | |
| 0, 3, 7 | 18018 | 1 | | | | | | | | | |
| 0, 3, 8 | 12375 | 1 | 1 | 2 | 1 | | | | | | |
| 0, 3, 9 | 18480 | 1 | 1 | 1 | | | | | | | |
| 0, 4, 0 | 27648 | 1 | 1 | 1 | | | | | | | |
| 0, 4, 1 | 59136 | 1 | | | | | | | | | |
| 0, 4, 2 | 67200 | 1 | | | | | | | | | |
| 0, 4, 3 | 54675 | 1 | | | | | | | | | |
| 0, 4, 4 | 96228 | 1 | | | | | | | | | |
| 0, 4, 5 | 2558 | 1 | | | | | | | | | |
| 0, 4, 6 | 12672 | 1 | | | | | | | | | |
| 0, 4, 7 | 32725 | 1 | | | | | | | | | |
| 0, 4, 8 | 56320 | 1 | | | | | | | | | |

Table 1: Multiplicities of SO(9)×SU(2) representations in the 2^24-dimensional space. Shaded rows indicate contributions from fermion states.
3 Discussion

We have computed the SO(9) × SU(2) character for coordinate independent states in SU(2) Matrix theory and have decomposed it into the sum of products of SO(9) and SU(2) characters. It immediately gives the decomposition of those states into SO(9) × SU(2) representations, and gives another proof of the uniqueness of the coordinate independent SO(9) × SU(2) singlet state.

A next natural question is if similar calculation can be done in the case of SU(N) gauge group\[5\]. Especially it is an interesting question if there are two or more SO(9) × SU(N) singlet states, or it is unique as in the SU(2) case.

Another question is if all the states can be constructed by acting $\theta^a_\alpha$ on the unique SO(9) × SU(2) singlet state. We can give a hint for it if we can count numbers of such states and compare the result with Table 1.

Appendix

A Frobenius formula

The character $\chi(R)$ of a representation $R$ is given by a trace of a group element $g$ over states in $R$: $\chi(R) = \text{tr}_R(g)$, and we define $\chi(R^k)$ by $\chi(R^k) = \text{tr}_R(g^k)$. Then the character of $n$-rank antisymmetric tensor product $\text{Alt}_n(R)$ of a representation $R$ can be computed by the following Frobenius formula:

$$
\chi(\text{Alt}_n(R)) = \sum_{\sum_{k=1}^n k_i = n \atop i_k: \text{nonnegative integer}} (-1)^{n+\sum_{k=1}^n i_k} \prod_{k=1}^n \frac{[\chi(R^k)]^{i_k}}{i_k! \cdot k_i^{i_k}}.
$$

(A.1)

Here we count contributions from boson states and fermions states additively. If states in representation $R$ are fermionic and we count them with minus sign, then the sign factor in the above formula must be changed from $(-1)^{n+\sum_{k=1}^n i_k}$ to $(-1)\sum_{k=1}^n i_k$. Since we need explicit expressions in the text, we show some of them for reader’s convenience:

$$
\begin{align*}
\chi(\text{Alt}_0(R)) & = 1, \quad \quad (A.2) \\
\chi(\text{Alt}_1(R)) & = \chi_1, \quad \quad (A.3) \\
\chi(\text{Alt}_2(R)) & = \frac{1}{2}(\chi_1^2 - \chi_2), \quad \quad (A.4) \\
\chi(\text{Alt}_3(R)) & = \frac{1}{6}(\chi_1^3 - 3\chi_1\chi_2 + 2\chi_3), \quad \quad (A.5)
\end{align*}
$$
\[ \chi(\text{Alt}_4(R)) = \frac{1}{24}(\chi^4 - 6\chi^2 + 3\chi^2 + 8\chi_3 - 6\chi_4), \]  
\[ \chi(\text{Alt}_5(R)) = \frac{1}{120}(\chi^5 - 10\chi^3\chi_2 + 15\chi_1\chi^2 + 20\chi^2_1\chi_3 
- 20\chi_2\chi_3 - 30\chi_1\chi_4 + 24\chi_5), \]  
\[ \chi(\text{Alt}_6(R)) = \frac{1}{720}(\chi^6 - 15\chi^4\chi_2 + 45\chi^2\chi^2 - 15\chi_3^3 + 40\chi_3^3\chi_3 
- 120\chi_1\chi_2\chi_3 + 40\chi_1^2 - 90\chi_1\chi_4 + 90\chi_2\chi_4 + 144\chi_1\chi_5 - 120\chi_6), \]  
\[ \chi(\text{Alt}_7(R)) = \frac{1}{5040}(\chi^7 - 21\chi^5\chi_2 + 105\chi_3\chi^2 - 105\chi_1\chi^2 + 70\chi_1^3\chi_3 
- 420\chi_1\chi_2\chi_3 + 210\chi_1^2\chi_3 + 280\chi_1\chi_3^2 - 210\chi_1^3\chi_4 + 630\chi_1\chi_2\chi_4 
- 420\chi_3\chi_4 + 504\chi_1\chi_5 - 504\chi_2\chi_5 - 840\chi_1\chi_6 + 720\chi_7), \]  
\[ \chi(\text{Alt}_8(R)) = \frac{1}{40320}(\chi^8 - 28\chi^6\chi_2 + 210\chi^4\chi^2 - 420\chi^2\chi^3 + 105\chi^4_2 
+ 112\chi_3^5\chi_3 - 1120\chi_3\chi_2\chi_3 + 1680\chi_1\chi_2^2\chi_3 + 1120\chi_1^2\chi_3^2 
- 1120\chi_2\chi_3^3 - 420\chi_1^4\chi_4 + 2520\chi_1^2\chi_2\chi_4 - 1260\chi_2^2\chi_4 
- 3360\chi_1\chi_3\chi_4 + 1260\chi_4^2 + 1344\chi_1^3\chi_5 - 4032\chi_1\chi_2\chi_5 
+ 2688\chi_3\chi_5 - 3360\chi_1^2\chi_6 + 3360\chi_2\chi_6 + 5760\chi_1\chi_7 - 5040\chi_8), \]  

where \( \chi_k = \chi(R^k) \).

**B SO(9) representations and characters**

Representations of SO(9) are uniquely specified by the Dynkin label \( [q_1, q_2, q_3, q_4] \), where \( q_1, q_2, q_3, \) and \( q_4 \) are nonnegative integers. In the context of Matrix theory, even and odd \( q_4 \) correspond to bosonic and fermionic states respectively. The highest weight \( \mu \) of the representation \([q_1, q_2, q_3, q_4] \) is given by the linear combination of fundamental weights \( \mu_1, \mu_2, \mu_3, \) and \( \mu_4 \):  
\[ \mu = q_1\mu_1 + q_2\mu_2 + q_3\mu_3 + q_4\mu_4, \]  
where  
\( \mu_1 = (1,0,0,0), \quad \mu_2 = (1,1,0,0), \quad \mu_3 = (1,1,1,0), \quad \mu_4 = (1/2,1/2,1/2,1/2). \)  

Dimension of \([q_1, q_2, q_3, q_4] \) can be computed by the following expression obtained from Weyl dimension formula:  
\[ \text{dim}[q_1, q_2, q_3, q_4] = \prod_{i=1}^{4} \left( 1 + \frac{2(q_i + \cdots + q_3 + q_4)}{9 - 2i} \right) \times \prod_{1 \leq i < j \leq 4} \left( 1 + \frac{q_i + \cdots + q_{j-1} + 2(q_j + \cdots + q_3 + q_4)}{9 - 2i - 2j} \right) \]
\[ \times \prod_{1 \leq i < j \leq 4} \left( 1 + \frac{q_i + \cdots + q_{j-1}}{j-i} \right), \]  
where, for \( i = 4 \), expressions as \( q_i + \cdots + q_3 \) should be ignored.

Since an element \( w \) of Weyl group of SO(9) acts on a weight \( \lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \) as sign flip and permutation \( \sigma \) of components:

\[ w \cdot \lambda = (\pm \lambda_\sigma(1), \pm \lambda_\sigma(2), \pm \lambda_\sigma(3), \pm \lambda_\sigma(4)), \]  
the character for \([q_1, q_2, q_3, q_4]\), denoted by \( \chi_{[q_1, q_2, q_3, q_4]} \), is given by the following expression obtained from Weyl character formula:

\[ \chi_{[q_1, q_2, q_3, q_4]} \equiv \text{tr}_{[q_1, q_2, q_3, q_4]} \left[ \exp(iJ_{12}x_1 + iJ_{34}x_2 + iJ_{56}x_3 + iJ_{78}x_4) \right] = \frac{D_{\rho+\mu}}{D_\rho}, \]  
where \( \rho = \mu_1 + \mu_2 + \mu_3 + \mu_4 \) is the Weyl vector, and

\[ D_\lambda = 16 \sum_\sigma \text{sgn}(\sigma) \prod_{i=1}^{4} \sin(\lambda_\sigma(i)x_i) = 16 \det[\sin(\lambda_jx_i)]. \]  
Table 2 shows some correspondences between Dynkin labels and representations which we usually construct by taking tensor products of vector and spinor representations, and characters of some of them are given by

\[
\begin{align*}
\chi[0000] &= 1, \\
\chi[1000] &= 1 + (c_1^2 + c_2^2 + c_3^2 + c_4^2), \\
\chi[0010] &= 4 + 3(c_1^2 + c_2^2 + c_3^2 + c_4^2) + (c_1^2c_2^2 + c_1^2c_3^2 + c_1^2c_4^2 + c_2^2c_3^2 + c_2^2c_4^2 + c_3^2c_4^2) \\
& \quad + (c_1^2c_2c_3^2 + c_1^2c_2c_4^2 + c_1^2c_3c_4^2 + c_2^2c_3c_4^2), \\
\chi[0001] &= c_1^4c_2c_3c_4, \\
\chi[0010] &= 4 + (c_1^2 + c_2^2 + c_3^2 + c_4^2) \\
& \quad + (c_1^2c_2^2 + c_1^2c_3^2 + c_1^2c_4^2 + c_2^2c_3^2 + c_2^2c_4^2 + c_3^2c_4^2) + (c_1^4 + c_2^4 + c_3^4 + c_4^4), \\
\chi[1000] &= 4c_1^3c_2c_3c_4 + (c_1^3c_2c_3c_4 + c_1^3c_2c_3c_4 + c_1^3c_2c_3c_4 + c_1^3c_2c_3c_4),
\end{align*}
\]  
where \( c_i^n = 2\cos(nx_i/2) \).

References

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### Table 2: Correspondences between Dynkin labels and representations

| Dynkin label | dimension | representation |
|--------------|-----------|----------------|
| [0,0,0,0]    | 1         | singlet        |
| [1,0,0,0]    | 9         | vector         |
| [0,1,0,0]    | 36        | 2-rank antisymmetric |
| [0,0,1,0]    | 84        | 3-rank antisymmetric |
| [0,0,0,1]    | 16        | spinor         |
| [0,0,0,2]    | 126       | 4-rank antisymmetric |
| [1,0,0,1]    | 128       | vector-spinor  |
| [n,0,0,0]    | $(2n+7)(n+6)!/(7! \cdot n!)$ | n-rank symmetric traceless |

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