Subharmonic Almost Periodic Functions of Slowly Growth

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Abstract

We obtain a complete description of the Riesz measures of almost periodic subharmonic functions with at most of linear growth on \( \mathbb{C} \); as a consequence we get a complete description of zero sets for the class of entire functions of exponential type with almost periodic modulus.

2000 Mathematics Subject Classification: Primary 31A05, Secondary 42A75, 30D15
Keywords: subharmonic function, almost periodic function, Riesz measure, zero set, entire function of exponential type

Bohr’s Theorem (see [2], or [10], Ch.6, §1) implies that each almost periodic function on real axis \( \mathbb{R} \) with the bounded spectrum is just the restriction to \( \mathbb{R} \) of an entire almost periodic function \( f \) of exponential type. Moreover, \( f \) has no zeros outside of some strip \( |\text{Im}z| \leq H \) if and only if supremum and infimum of the spectrum \( f \) also belongs to the spectrum. In [9] (see also [10], Appendix VI) M.G.Krein and B.Ya.Levin obtained a complete description of zeros of functions from the last class. Namely, a set \( \{a_k\}_{k \in \mathbb{Z}} \) in a horizontal strip of a finite width is just the zero set of an entire almost periodic function \( f \) of exponential type if and only if the set is almost periodic and has the representation

\[
a_k = dk + \psi(k), \quad k \in \mathbb{Z},
\]

where \( d \) is a constant, the function \( \psi(k) \) is bounded, and the values

\[
S_n = \lim_{r \to \infty} \sum_{|k| < r} \left[ \psi(k + n) - \psi(k) \right] \frac{k}{k^2 + 1} (2)
\]

are bounded uniformly in \( n \in \mathbb{Z} \).

It can be proved that almost periodicity of \( \{a_k\} \) yields representation (1) and a finite limit in (2) for every fixed \( n \in \mathbb{Z} \). Also, one can obtained a complete description of zero sets for the class of entire functions of exponential type with almost periodic modulus and zeros in horizontal strip of finite width: we should only replace the \( S_n \) by \( \text{Re}S_n \).

Observe that every entire function of exponential type bounded on \( \mathbb{R} \) has the form

\[
f(z) = C e^{i\nu z} \lim_{r \to \infty} \prod_{|a_k| < r} \left(1 - \frac{z}{a_k}\right), \quad \nu \in \mathbb{R}
\]

([10], Ch. 5; for simplicity we suppose \( 0 \notin \{a_k\} \)); so we have an explicit representation for functions from the classes above-mentioned.
Note that one of the authors of the present paper obtained in [5] a complete description of zero sets for holomorphic almost periodic functions on a strip and on the plane without any growth conditions. An implicit representation for a special case of almost periodic holomorphic functions was obtained earlier in [3]. Besides, it was proved in [3] that zero sets of holomorphic functions with the almost periodic modulus on a strip (or on the plane) are just almost periodic discrete sets. This result is a consequence of more general one: every almost periodic measure on a strip is just the Riesz measure of some subharmonic almost periodic function on the strip.

In §3 of our paper we obtain a complete description of the Riesz measures for almost periodic subharmonic functions of the normal type with respect to the order 1 (note that it is the smallest growth for bounded on \( \mathbb{R} \) subharmonic function). In particular, we consider the case of periodic subharmonic functions. As a consequence, we get a complete description of zero sets for the class of entire functions of exponential type with the almost periodic modulus without any additional requirements on distributions of zeros. Note that representation (1) with a bounded function \( \psi(k) \) is incorrect here, therefore methods of paper [9] do not work in our case. The integral representation from [3] creates an almost periodic subharmonic function with a given almost periodic Riesz measure, but does not allow to control the growth of the function, therefore it isn’t fit for our problem as well.

We make use of a subharmonic analogue of representation

\[
\log |f(z)| = \int_0^\infty \frac{n(0,t) - n(z,t)}{t} dt - \nu y + \log |C|,
\]

for functions of the form (3) (see [6] or review [7], p.45); here \( n(c,t) \) is a number of zeros in the disc \( \{ z : |z - c| \leq t \} \). We obtain this analogue in §2 of our paper. Also, we get a complete description of the Riesz measures for bounded on \( \mathbb{R} \) subharmonic functions with at most of linear growth on \( \mathbb{C} \).

Here we base on a subharmonic analogue of (3) as well. Of course, this analogue can be obtained by repeating all the steps of the proof (3) for entire functions in [10], nevertheless we prefer to give a short proof in §1, using Azarin’s theory of limit sets for subharmonic functions [1]. The idea of the proof belongs to prof. A.F.Grishin, and the authors is very grateful to him.

\[\text{§1}\]

In this section we prove the following theorem:

**Theorem 1.** Let \( v(z) \) be a subharmonic function on \( \mathbb{C} \) such that

\[
v^+(z) = O(|z|) \quad \text{as} \quad z \to \infty
\]

and

\[
\sup_{x \in \mathbb{R}} v(x) < \infty.
\]

Then

\[
v(z) = \lim_{R \to \infty} \int_{|w| < R} (\log |z - w| - \log^+ |w|) d\mu(w) + A_1 y + A_2.
\]
Here \( z = x + iy, \mu = \frac{1}{2\pi} \Delta v \) is the Riesz measure of \( v, A_1, A_2 \in \mathbb{R} \), and the limit exists uniformly on compact subsets in \( \mathbb{C} \).

Note that the condition (5) means just \( v \) is at most of normal type with respect to the order 1.

Our proof of Theorem 1 based on Azarin’s theory of limit sets \([1]\). Thus, if a subharmonic function \( v \) satisfies (5), then

\[ a) \text{ the family } v_t(z) = t^{-1}v(tz), t > 1 \text{ is a relatively compact set in the space of distributions } \mathcal{D}'(\mathbb{C}); \text{ in other words, for every sequence of functions from this family there is a subsequence converging to a subharmonic function.} \]

Note that the convergence in \( \mathcal{D}'(\mathbb{C}) \) is the weak convergence on all functions from the class of infinitely smooth compactly supported functions \( \mathcal{D}(\mathbb{C}); \text{ moreover, the class of subharmonic functions is closed with respect to this convergence.} \]

\[ b) \text{ If } v_\infty = \lim_{t \to \infty} v_t(z), \text{ then the Riesz measure } \mu_\infty \text{ of the function } v_\infty \text{ satisfies the equality} \]

\[ \mu_\infty = \lim_{t \to \infty} \mu_t, \]

where \( \mu_t(E) = t^{-1}\mu(tE) \) for Borel subsets of \( \mathbb{C}; \text{ limits } (9) \text{ exists in the sense of weak convergence on continuous compactly supported functions on } \mathbb{C}. \text{ Moreover, in this case there exists} \]

\[ \lim_{R \to \infty} \int_{1 \leq |w| < R} \frac{d\mu(w)}{w} \neq \infty. \] (10)

If a subharmonic function satisfies (5) and (8), then it is called a completely regular growth (with respect to the order 1).

In what follows we need the simple criterion of compactness for family of subharmonic functions (see\([1]\)).

**Lemma A.** A family \( \{u_\alpha\} \) of subharmonic functions on \( \mathbb{C} \) is a relatively compact set in the space of distributions \( \mathcal{D}'(\mathbb{C}) \) if and only if

\[ a) \sup_\alpha \sup_{z \in K} u_\alpha(z) < \infty \text{ for all compacta } K \subset \mathbb{C}, \]

\[ b) \inf_\alpha \sup_{z \in K_0} u_\alpha(z) > -\infty \text{ for some compact set } K_0 \subset \mathbb{C}. \]

Also, we need the following variant of Fragment-Lindelof Theorem.

**Theorem FL.** If a function \( v \) is subharmonic in a neighborhood of the closure of upper half-plane \( \mathbb{C}^+ = \{z = x + iy : y > 0\} \) and satisfies conditions (5), (6), then for all \( z \in \mathbb{C}^+ \)

\[ v(z) \leq \sup_{x \in \mathbb{R}} v(x) + \sigma^+ y, \]

with \( \sigma^+ = \limsup_{y \to +\infty} y^{-1}v(iy). \)

The proof of this statement is the same as for holomorphic on \( \mathbb{C}^+ \) and continuous on \( \overline{\mathbb{C}}^+ \) functions (see, for example, \([8]\), p.28).

First let us prove a subharmonic analogue of Cartwright Theorem (the holomorphic case see, for example, \([10]\), Ch. V).
Theorem 2. Let a subharmonic function $v$ on $\mathbb{C}$ be satisfied (5) and (6). By definition, put
\[
\sigma_{\pm} = \limsup_{y \to \pm\infty} \frac{v(iy)}{|y|}.
\] (11)
Then $v(z)$ is a completely regular growth; moreover, the function $v_\infty$ from (8) has the form
\[
v_\infty(z) = \begin{cases} \sigma^+ y, & y \geq 0, \\ \sigma^- |y|, & y < 0, \end{cases}
\] (12)

Remark. From Theorem FL it follows that if a subharmonic function $v$ on $\mathbb{C}$ satisfies the conditions of Theorem 2 with $\sigma^+ \leq 0$ and $\sigma^- \leq 0$, then $v$ is a constant.

The proof is based on the following lemma

Lemma 1. Let $u < 0$ be a subharmonic function on $\mathbb{C}^+$. Then for every $R < \infty$ and $r \in (0, R/2)$
\[
\frac{u(ir)}{r} \leq \frac{C}{R^3} \int_{|z-iR|<R} u(z) dm_2(z),
\] (13)
where $C$ is an absolute constant and $m_2$ is the plain Lebesque measure.

Proof
Since Poisson formula for the disc $B(iR/2, R/2) = \{z : |z - iR/2| < R/2\}$, we have
\[
u(ir) \leq \frac{1}{2\pi} \int_0^{2\pi} u(ir/2 + e^{i\theta} R/2) \frac{(R/2)^2 - (R/2 - r)^2}{(R/2)^2 + (R/2 - r)^2 - R(R/2 - r) \cos(\pi/2 + \theta)} d\theta.
\]
Using the inequality $u < 0$, replace the interval of integration by $[\pi/4, 3\pi/4]$. We obtain
\[
u(ir) \leq \frac{r}{(R - r)} \frac{1}{4} \sup_{\theta \in [\pi/4, 3\pi/4]} u \left( \frac{iR}{2} + \frac{R}{2} e^{i\theta} \right)
\] (14)
Since for all $\theta \in [\pi/4, 3\pi/4]$
\[B(iR, R/2) \subset B(iR/2 + e^{i\theta} R/2, (1 + \sqrt{2})R/2) \subset \mathbb{C}^+,
\]
we have
\[
u\left( \frac{iR}{2} + \frac{R}{2} e^{i\theta} \right) \leq \frac{4}{\pi R^2 (1 + \sqrt{2})^2} \int_{|z - iR/2 - e^{i\theta} R/2| \leq (1 + \sqrt{2})R/2} u(z) dm_2(z) \leq
\]
\[
\frac{8}{\pi R^2 (3 + 2\sqrt{2})} \int_{|z - iR| < R/2} u(z) dm_2(z)
\]
Then replace the average over the disc $B(iR, R/2)$ by the average over the disc $B(iR, R)$. So the assertion of the Lemma follows from (14).
Proof of theorem 2.
Without loss of generality it can be assumed that \( \sup_{x \in \mathbb{R}} v(x) = 0 \). Put \( u(z) = v(z) - \sigma_+ y \).
From Theorem FL it follows that \( u(z) < 0 \) on \( \mathbb{C}^+ \), then
\[
\limsup_{y \to +\infty} \frac{u(iy)}{y} = 0 \tag{15}
\]
Fix \( z_0 \in \mathbb{C}^+ \). Let \( \varphi, 0 \leq \varphi \leq 1 \), be an infinite differentiable and compactly supported function on \( \mathbb{C}^+ \), depending only on \( |z - z_0| \). Apply Lemma 1 for the function \( u_t(z) = u(tz)t^{-1} \). If \( \text{supp} \varphi \subset B(iR, R) \), then we get
\[
\int_{\mathbb{C}^+} u_t(z) \varphi(z) dm_2(z) \leq -\varepsilon R^3 \tag{16}
\]
Since (15), we see that for each \( \varepsilon > 0 \) and \( t > t(\varepsilon) \) there is \( r \in (0, R/2) \) such that \( u(itr) \geq -\varepsilon tr \). We obtain
\[
\int_{\mathbb{C}^+} u_t(z) \varphi(z) dm_2(z) \to 0 \quad \text{as} \quad t \to \infty.
\]
Hence,
\[
\int_{\mathbb{C}^+} u_t(z) \varphi(z) dm_2(z) \to 0 \quad \text{as} \quad t \to \infty.
\]
Therefore, \( v_t(z) = u_t(z) + \sigma_+ y \to \sigma_+ y \) in the space \( \mathcal{D}'(\mathbb{C}^+) \). Similarly, \( v_t(z) \to -\sigma_- y \) in the space \( \mathcal{D}'(\mathbb{C}^-) \), where \( \mathbb{C}^- = \{ z = x + iy : y < 0 \} \). Each limit function for \( v_t(z) \) is always subharmonic, therefore we get (12). So limit (8) exists. Theorem is proved.

Consequence. The Riesz measure of the limit function \( v_\infty(z) \) equals
\[
\frac{\sigma_+ + \sigma_-}{2\pi} m_1(x),
\]
where \( m_1 \) being the Lebesgue measure on \( \mathbb{R} \).

Proof of theorem 1. From Jensen-Privalov formula for subharmonic function, we get the estimate for \( r \geq 1 \)
\[
\mu(B(0, r)) \leq C_1 r \tag{16}
\]
where constant \( C_1 \) depends only on \( v \). Using Brelot-Hadamard Theorem for subharmonic function (see, for example, [12]), we obtain that there is a harmonic polynomial \( H(z) \) of degree 1 such that
\[
v(z) = \int_{|w|<1} \log |z-w| d\mu(w) + \int_{|w|\geq1} \log \left( \left| 1 - \frac{z}{w} \right| + \text{Re} \frac{z}{w} \right) d\mu(w) + H(z). \tag{17}
\]
Denote by \( v^0(z) \) the first integral in (17). Since (10), we get
\[
v(z) = v^0(z) + \lim_{R \to \infty} \int_{1 \leq |w| < R} \log \left| 1 - \frac{z}{w} \right| d\mu(w) + A_0 x + A_1 y + A_2, \tag{18}
\]
The application of Theorem 2 yields that the function $v$ is a completely regular growth, hence the measures $\mu_t$ converge weakly to the measure

$$\mu_\infty = \frac{\sigma_+ + \sigma_-}{2\pi}m_1.$$  \hspace{1cm} (19)

Let $\mu'$ be the restriction of the measure $\mu$ to $\mathbb{C} \setminus B(0, 1)$. Obviously, measures $\mu'_t$ weakly converge to the measure $\mu_\infty$ as well. Therefore,

$$v_t(z) = v^0_t(z) + \lim_{R \to \infty} \int_{|w| < R} \log \left| 1 - \frac{z}{w} \right| d\mu'_t(w) + A_0x + A_1y + \frac{A_2}{t}. \hspace{1cm} (20)$$

Pass to a limit in (20) as $t \to \infty$ in the space $D'(\mathbb{C})$. First, by Theorem 2, the functions $v_t(z)$ converge to the function $v_\infty(z)$ from (12). Since $v^0_t(z) = O(|\log |z||)$ as $|z| \to \infty$, we see that $v^0_t(z) \to 0$. By (16), we obtain $\mu'_t(B(0, R)) \leq C_1 R$ for all $t \geq 1$ and $R > 0$. Therefore, we get uniformly in $t \geq 1$,

$$\int_{|w| \geq R} \frac{d\mu'_t(w)}{|w|^2} = 2 \int_{R}^{\infty} \frac{\mu'_t(B(0, s))}{s^3} ds - \frac{\mu'_t(R)}{R^2} \to 0. \hspace{1cm} (21)$$

as $R \to \infty$. Also, by (10), uniformly in $t \geq 1$, $R' \geq R$

$$\int_{R \leq |w| \leq R'} \frac{d\mu'_t(w)}{w} = \int_{R \leq |w| \leq R'} \frac{d\mu(w)}{w} \to 0, \hspace{1cm} (22)$$

as $R \to \infty$. For $|z| < C$ and a sufficiently large $|w|$,

$$\left| \log \left| 1 - \frac{z}{w} \right| + \text{Re} \frac{z}{w} \right| \leq \frac{|z|^2}{|w|^2}.$$

Therefore, taking into account (21) and (22), we obtain for all $\varphi \in D(\mathbb{C})$ uniformly in $t \geq 1$

$$\int_{\mathbb{C}} \left( \lim_{R \to \infty} \int_{|w| \leq R} \log \left| 1 - \frac{z}{w} \right| d\mu'_t(w) \right) \varphi(z) dm_2(z) =$$

$$= \lim_{R \to \infty} \int_{|w| \leq R} \left( \int_{\mathbb{C}} \log \left| 1 - \frac{z}{w} \right| \varphi(z) dm_2(z) \right) d\mu'_t(w). \hspace{1cm} (23)$$

Note that measure $\mu_\infty$ does not charge any circle $|w| = R$, therefore the restrictions of measures $\mu'_t(w)$ to any disc $B(0, R)$ weakly converge to the restriction of the measure $\mu_\infty(w)$. The function $\int \log |z - w| \varphi(z) dm_2(z)$ is continuous in the variable $w$, so we have

$$\lim_{t \to \infty} \int_{|w| \leq R} \int \log |z - w| \varphi(z) dm_2(z) d\mu_t(w) = \int_{|w| \leq R} \int \log |z - w| \varphi(z) dm_2(z) d\mu_\infty(w). \hspace{1cm} (24)$$
By the same reason for each \( \delta > 0 \)

\[
\lim_{t \to \infty} \int_{\delta \leq |w| \leq R} \log |w| \mu_t(w) = \int_{\delta \leq |w| \leq R} \log |w| \mu_\infty(w).
\]  

(25)

Furthermore,

\[
\int_{|w| \leq \delta} \log |w| \mu'_t(w) = \log \delta \int_0^{\delta} \frac{\mu'(B(0, \delta t))}{\delta} \, ds - \int_0^\delta \frac{\mu'(B(0, st))}{st} \, ds
\]  

(26)

Since \( \mu'(B(0, r)) \leq C_1 r \) for all \( r > 0 \), we see that (26) tends to zero as \( \delta \to 0 \) uniformly in \( t \geq 1 \). Combining (23), (24), (25), and (26), we get the equality

\[
v_\infty(z) = \lim_{R \to \infty} \int_{|w| \leq R} \log |1 - \frac{z}{w}| \, d\mu_\infty(w) + A_0 x + A_1 y.
\]

Take \( y = 0 \). Since \( v_\infty(x) = 0 \) and

\[
\lim_{R \to \infty} \int_{|u| \leq R} \log |1 - \frac{x}{u}| \, du = 0
\]

for all \( x \in \mathbb{R} \), we obtain \( A_0 = 0 \). Now the assertion of Theorem 1 follows from (18).

\section*{2}

Here we get a complete description of Riesz measures for subharmonic functions with at most of linear growth on \( \mathbb{C} \) (i.e., do not exceed \( C(|z| + 1) \) with \( C < \infty \)) and with some additional conditions (bounded on \( \mathbb{R} \) or with the compact family of translations along \( \mathbb{R} \)). Holomorphic analogues of the corresponding theorems were obtained earlier one of the author in [6].

First prove some lemmas.

**Lemma 2.** *If a measure \( \mu \) on \( \mathbb{C} \) satisfies the condition

\[
\mu(B(0, R + 1)) - \mu(B(0, R)) = \bar{o}(R) \quad \text{as} \quad R \to \infty,
\]

and the limit

\[
\lim_{R \to \infty} \int_{|w| < R} (\log^+ |z - w| - \log^+ |w|) \, d\mu(w)
\]

exists at some point \( z \in \mathbb{C} \), then the limit equals

\[
\int_1^\infty \frac{\mu(B(0, t)) - \mu(B(z, t))}{t} \, dt.
\]

*
Proof

For all $z \in \mathbb{C}$ and $R \in (|z| + 1, \infty)$ we have
\[
\int_{|w| < R} \log^+ |z - w| d\mu(w) - \int_{|w| < R} \log^+ |w| d\mu(w) = (\log R) \mu(B(z, R)) - \int_{1}^{R} \frac{\mu(B(z, t))}{t} dt - \log R \mu(B(0, R)) + \int_{1}^{R} \frac{\mu(B(0, t))}{t} dt + \int_{|w| < R, |w-z| \geq R} \log^+ |z - w| d\mu(w) - \int_{|w| \geq R, |w-z| < R} \log^+ |z - w| d\mu(w) = \int_{1}^{R} \frac{\mu(B(0, t)) - \mu(B(z, t))}{t} dt + \int_{|w| < R, |w-z| \geq R} \log \left| \frac{z-w}{R} \right| d\mu(w) - \int_{|w| \geq R, |w-z| < R} \log \left| \frac{z-w}{R} \right| d\mu(w). \tag{28}
\]

If $|w| < R$ and $|z-w| \geq R$ or $|w| \geq R$ and $|z-w| < R$, then we have
\[
1 - \frac{|z|}{R} \leq \left| \frac{z-w}{R} \right| \leq 1 + \frac{|z|}{R}.
\]

Therefore the integrand functions of last two integrals in (28) are $O(1/R)$ as $R \to \infty$. The domains of integrations are subsets of the ring $R - |z| \leq |w| \leq R + |z|$, hence, by (27), these integrals tends to 0 as $R \to \infty$. Lemma is proved.

Lemma 3. Let a measure $\mu$ be satisfied (10), (16), and (27), and let
\[
V(z) = \lim_{R \to \infty} \int_{|w| < R} (\log |z - w| - \log^+ |w|) d\mu(w). \tag{29}
\]

Then $V$ is a subharmonic function with Riesz measure $\mu$ and
\[
V(z) = \int_{1}^{\infty} \frac{\mu(B(0, t)) - \mu(B(z, t))}{t} dt + \int_{|w-z| < 1} \log |z - w| d\mu(w) \tag{30}
\]

for all $z \in \mathbb{C}$. Furthermore, the function
\[
\tilde{V}(z) = \frac{1}{2\pi} \int_{0}^{2\pi} V(z + e^{i\theta}) d\theta \tag{31}
\]

satisfies the equality
\[
\tilde{V}(z) = \int_{1}^{\infty} \frac{\mu(B(0, t)) - \mu(B(z, t))}{t} dt. \tag{32}
\]
Proof

It follows from (16) that the integral in (17) is a subharmonic function on \( \mathbb{C} \) with the Riesz measure \( \mu \); besides, it satisfies \((5)\) (see, for example, [12], Ch.1). If, in addition, \( \mu \) satisfies \((10)\), then the limit in (29) exists uniformly on bounded sets, and the function \( V \) coincides with the integral in (17) up to a linear term; so it has the same properties as well.

Using the equality \( \int_0^{2\pi} \log |a + e^{i\theta}| d\theta = 2\pi \log^+ |a| \), we get

\[
\tilde{V}(z) = \lim_{R \to \infty} \int_{|w|<R} (\log^+ |z - w| - \log^+ |w|) d\mu(w).
\]

and \( V(z) = \tilde{V}(z) + \int_{|w-z|<1} \log |z - w| d\mu(w) \). Then Lemma 2 implies \((32)\) and \((30)\). Lemma 3 is proved.

Lemma 4. Let a subharmonic on \( \mathbb{C} \) function \( v \) be satisfied \((5)\). Then the family of translations \( \{v(z+h)\}_{h \in \mathbb{R}} \) is a relatively compact subset in \( \mathcal{D}'(\mathbb{C}) \) if and only if the function

\[
\hat{v}(z) = \frac{1}{2\pi} \int_0^{2\pi} v(z + e^{i\theta}) d\theta
\]  

is bounded on \( \mathbb{R} \); this function is bounded simultaneously with the function

\[
\hat{\nu}(z) = \frac{1}{\pi} \int_{|w|<1} v(z + w) dm_2(w)
\]  

Proof. If the family \( \{v(z + h)\}_{h \in \mathbb{R}} \) is a relatively compact subset, then it is uniformly bounded from above on compacta in \( \mathbb{C} \) and the function \( v \) is uniformly bounded from above on every strip \( |y| < H \). Then by Lemma A there is a compact subset \( K_0 \) of \( \mathbb{C} \) such that

\[
\sup_{K_0} v(z + h) \geq C_2, \quad \forall h \in \mathbb{R}.
\]

Take \( d > \sup_{K_0} |z| \). Then for each \( h \in \mathbb{R} \) there is a point \( z(h), |z(h)| < d \), such that

\[
\frac{1}{(d+1)^2\pi} \int_{|w-z(h)| \leq d+1} v(w + h) dm_2(w) \geq v(z(h) + h) > C_2 - 1.
\]

Further, we have

\[
\int_{|w-z(h)| \leq d+1} v(w + h) dm_2(w) \leq \int_{|w| \leq 1} v(w + h) dm_2(w) + (d^2 + 2d)\pi \sup_{|y|<d+1} v(z).
\]

Therefore, \( \sup_{h \in \mathbb{R}} \hat{v}(h) > -\infty \). Since \( \hat{v}(h) \geq \hat{v}(h) \), we see that the functions \( \hat{v} \) and \( \hat{\nu} \) are bounded uniformly from below on \( \mathbb{R} \). It is clear that these functions are bounded uniformly from above on \( \mathbb{R} \) as well.

On the other hand, if \( \hat{v}(z) \) is bounded from below on \( \mathbb{R} \), then \( \inf_{h \in \mathbb{R}} \sup_{|w|=1} v(h + w) > -\infty \); if \( \hat{\nu}(h) \) is bounded from above on \( \mathbb{R} \), then \( v(h) \) is bounded from above on \( \mathbb{R} \) as well. Since Theorem FL, we see that \( v(z) \) is bounded from above on every strip \( |y| < H \). It follows from Lemma A that \( \{v(z+h)\}_{h \in \mathbb{R}} \) is a relatively compact set. Hence \( \hat{v}(x) \) is bounded on \( \mathbb{R} \).

Now we can prove the theorems mentioned above.
Theorem 3. For a measure \( \mu \) on \( \mathbb{C} \) to be the Riesz measure for some subharmonic function satisfying conditions (5) and (6) it is necessary and sufficient that the conditions (10), (16), (27), and

\[
\sup_{x \in \mathbb{R}} \int_1^\infty \frac{\mu(B(0,t)) - \mu(B(x,t))}{t} \, dt < \infty. \tag{35}
\]

be fulfilled.

Proof. If a subharmonic function \( v \) satisfies (5) and (6), then, by Theorem 2, its Riesz measure \( \mu \) satisfies (10) and the measures \( \mu_t \) converge to the measure \( \mu_\infty = (\sigma_+ + \sigma_-)(2\pi)^{-1}m_1 \).

The last measure does not charge any circle \( |w| = R \), therefore (9) implies that \( \mu(B(0,R)) = CR + \sigma(R) \) as \( R \to \infty \). Hence we get (16) and (27). By theorem 1, \( v(z) = V(z) + A_1y + A_2 \). So the function \( V \) is also bounded from above on \( \mathbb{R} \). Since Theorem FL, we see that the same is true for the function \( \tilde{V} \) from (31). Now Lemma 3 implies (35).

Conversely, if a measure \( \mu \) satisfies (10), (16), and (27), then, by Lemma 3, the subharmonic function \( V \) has the Riesz measure \( \mu \) and satisfies (5) and (6). Theorem is proved.

Theorem 4. For a measure \( \mu \) on \( \mathbb{C} \) to be the Riesz measure for some subharmonic function \( v \) with the property (5) such that the family of translations \( \{v(z+h)\}_{h \in \mathbb{R}} \) is a relatively compact subset in \( D'(\mathbb{C}) \), it is necessary and sufficient that the conditions (10), (16), (27), and

\[
\sup_{x \in \mathbb{R}} \left| \int_1^\infty \frac{\mu(B(0,t)) - \mu(B(x,t))}{t} \, dt \right| < \infty \tag{36}
\]

be fulfilled.

Proof. If a function \( v \) satisfies (5), and the function \( \tilde{v} \) from (33) is bounded on \( \mathbb{R} \), then \( v \) is bounded from above on \( \mathbb{R} \). Theorem 3 implies conditions (10), (16), and (27) for the Riesz measure \( \mu \) of the function \( v \). By Theorem 1, \( \tilde{v} \) equals \( V \) from (31) on \( \mathbb{R} \) up to a constant term, therefore Lemmas 3 and 4 imply (36).

Conversely, if a measure \( \mu \) satisfies (10), (16), (27), and (36), then \( V \) from (29) satisfies (5), and the function \( \tilde{V} \) is bounded on \( \mathbb{R} \).

Hence the assertion of Theorem 4 follows from Lemma 4.

§3

A continuous function \( F(z) \) on a closed strip \( \{z = x + iy : x \in \mathbb{R}, |y| \leq H\} \) with \( H \geq 0 \) is almost periodic if the family of translations \( \{F(z+h)\}_{h \in \mathbb{R}} \) is a relatively compact set with respect to the topology of uniform convergence on the strip; a function is almost periodic on an open strip (in particular, on \( \mathbb{C} \)), if it is almost periodic on every closed substrip of a finite width.

A measure (maybe complex) \( \mu \) on \( \mathbb{C} \) is called almost periodic if for any test-function \( \varphi \in D(\mathbb{C}) \) the convolution \( \int \varphi(w+t)d\mu(w) \) is an almost periodic function in \( t \in \mathbb{R} \) (11).
The following statements are valid:

**Theorem R** (Theorem 1.8 [11]). For a measure \( \mu \) to be almost periodic it is necessary and sufficient that the following condition be fulfilled: for each sequence \( \{h_n\} \subset \mathbb{R} \) there exists a subsequence \( \{h'_n\} \) such that the convolutions \( \int \varphi(w + x + h'_n)d\mu(w) \) converge uniformly with respect to \( x \in \mathbb{R} \) and functions \( \varphi \in L \), where \( L \) being a compact subset in \( D(\mathbb{C}) \). Moreover, for a measure to be almost periodic it is sufficient to check this condition only for all single-point sets \( L \subset D(\mathbb{C}) \).

If we take \( L = \{\varphi(z + iy)\}_{|y|\leq H} \) for some \( \varphi \in D(\mathbb{C}) \), we obtain that the convolutions \( \int \varphi(w + z + h'_n)d\mu(w) \) actually converge uniformly on any strip \( |y| \leq H \), hence the function \( \int \varphi(w + z)d\mu(w) \) is almost periodic on \( \mathbb{C} \).

Further, a subharmonic function \( v \) on \( \mathbb{C} \) is called *almost periodic*, if the measure \( v(z)dm_2(z) \) is almost periodic (see [3]; equivalent definition see [4]).

It follows from definition that the Riesz measure of an almost periodic subharmonic function is also almost periodic. Conversely, each almost periodic measure is the Riesz measure of some almost periodic subharmonic function (see [3], where to be investigated the case of a strip as well). Note that the family of translations \( \{v(z + h)\}_{h \in \mathbb{R}} \) is a relatively compact subset of \( D'(\mathbb{C}) \) for every almost periodic subharmonic function \( v \) on \( \mathbb{C} \). Note also that each almost periodic subharmonic function is bounded from above on every horizontal strip of a finite width (see [3]).

Here we obtain the following result:

**Theorem 5**. A necessary and sufficient conditions for a measure \( \mu \) on \( \mathbb{C} \) to be the Riesz measure of some almost periodic subharmonic function at most of linear growth is that the measure be almost periodic and satisfied (10), (27), (10), and (36).

The proof of the theorem bases on the following lemmas.

**Lemma 5**. Suppose subharmonic functions \( v_1(z) \) and \( v_2(z) \) on \( \mathbb{C} \) with the common Riesz measure satisfy (7) and (8); then \( v_1(z) = v_2(z) + p_1 + p_2y \). Further, if

\[
\sup_{x \in \mathbb{R}} v_1(x) = \sup_{x \in \mathbb{R}} v_2(x) \quad \text{and} \quad \sigma_+(v_1) = \sigma_+(v_2),
\]

where \( \sigma_+ \) be defined in (11), then \( v_1 \equiv v_2 \).

**Proof** The first part follows from Theorem 1, the second one is evident.

**Lemma 6**. Let a subharmonic on \( \mathbb{C} \) function \( v \) be satisfied (5) and a family \( \{v(z + h)\}_{h \in \mathbb{R}} \) be a relatively compact subset of \( D'(\mathbb{C}) \); if \( v(z + h_n) \to v^*(z) \) in the space \( D'(\mathbb{C}) \), then the family \( \{v^*(z + h)\}_{h \in \mathbb{R}} \) is a relatively compact subset as well and

\[
\sup_{x \in \mathbb{R}} v^*(x) \leq \sup_{x \in \mathbb{R}} v(x), \tag{37}
\]

\[
\inf_{t \in \mathbb{R}} \int_{|z-t|<1} v^*(z)dm_2(z) \geq \inf_{t \in \mathbb{R}} \int_{|z-t|<1} v(z)dm_2(z), \tag{38}
\]

\[
\sigma_+(v^*) \leq \sigma_+(v), \quad \sigma_-(v^*) \leq \sigma_-(v). \tag{39}
\]
Proof Put \( M = \sup_{x \in \mathbb{R}} v(x) \). By Theorem FL, we get for any \( \varepsilon > 0 \)
\[
v(z) \leq M + 2\varepsilon \max\{\sigma_+(v), \sigma_-(v)\}, \quad |y| < 2\varepsilon.
\]
Let \( \varphi \) be a function from \( \mathcal{D}(\mathbb{C}) \) such that \( \varphi \) depends only on \(|z|\), \( \varphi \geq 0 \), \( \varphi(z) = 0 \) for \(|z| \geq \varepsilon \), \( \int \varphi(z)dm_2(z) = 1 \). Then
\[
(v \ast \varphi)(z) \leq M + 2\varepsilon \max\{\sigma_+(v), \sigma_-(v)\}, \quad |y| < \varepsilon.
\]
Therefore,
\[
(v^* \ast \varphi)(z) \leq M + 2\varepsilon \max\{\sigma_+(v), \sigma_-(v)\}, \quad |y| < \varepsilon.
\]
Note that \( v^* \) is a subharmonic function, hence \((v^* \ast \varphi)(z) \geq v^*(z)\). Since \( \varepsilon \) is arbitrary, we obtain (37). By the same argument, for all \( y \in \mathbb{R} \)
\[
\sup_{x \in \mathbb{R}} v^*(x + iy) \leq \sup_{x \in \mathbb{R}} v(x + iy).
\]
Therefore we obtain (5) and (39).

Further, the functions \( v(z + h_n) \) are integrable on every disc and uniformly bounded from above, therefore we can replace the convergence of measures \( v(z + h_n)dm_2(z) \) in the sense of distributions by the weak convergence of measures. Since the limit measure \( v^*(z)dm_2(z) \) does not charge any circle, we have
\[
\lim_{n \to \infty} \int_{|z-t|<1} v(z + h_n)dm_2(z) = \int_{|z-t|<1} v^*(z)dm_2(z)
\]
for each \( t \in \mathbb{C} \). Hence we get (38). Taking into account Lemma 4, we see that the family \( \{v^*(z + h)\}_{h \in \mathbb{R}} \) is a compact subset of \( \mathcal{D}'(\mathbb{C}) \). Lemma is proved.

Lemma 7. Under the conditions of the previous lemma, suppose that the Riesz measure \( \mu \) of the function \( v(z) \) is almost periodic. Then inequalities (37) - (39) turn into equalities, the Riesz measure \( \mu^* \) of the function \( v^*(z) \) becomes almost periodic, and there is a subsequence \( \{h_{n'}\} \) such that for every \( \varphi \in \mathcal{D}(\mathbb{C}) \)
\[
\lim_{n' \to \infty} \sup_{t \in \mathbb{R}} \left| \int \varphi(w - t - h_{n'})d\mu(w) - \int \varphi(w - t)d\mu^*(w) \right| = 0. \tag{40}
\]

Proof

For all \( \varphi \in \mathcal{D}(\mathbb{C}) \) we have
\[
\lim_{n \to \infty} \int \varphi(z - h_n)v(z)dm_2(z) = \lim_{n \to \infty} \int \varphi(z)v(z + h_n)dm_2(z) = \int \varphi(z)v^*(z)dm_2(z).
\]
Since \( \mu = (2\pi)^{-1}\Delta v \), we obtain
\[
\lim_{n \to \infty} \int \varphi(z - h_n)d\mu(z) = \int \varphi(z)d\mu^*(z). \tag{41}
\]

From Theorem R it follows that there is a subsequence \( \{h_{n'}\} \) such that for any \( \varphi \in \mathcal{D}(\mathbb{C}) \) the almost periodic functions \( \int \varphi(z - t - h_{n'})d\mu(z) \) converge to an almost periodic function uniformly in \( t \in \mathbb{R} \). If we replace \( z \) by \( z - t \) in (11), then we get (40). Consequently,
the function $\int \varphi(w - t)d\mu^*(w)$ is almost periodic in $t \in \mathbb{R}$, and $\mu^*$ is an almost periodic measure.

Pass to a subsequence again if necessary, we may assume that the functions $v^*(z - h_n)$ converge in the space $\mathcal{D}'(\mathbb{C})$ to some subharmonic function $v^{**}(z)$ with the Riesz measure $\mu^{**}$. Therefore,

$$\lim_{n' \to \infty} \int \varphi(z + h_{n'})d\mu^*(z) = \int \varphi(z)d\mu^{**}(z).$$

On the other hand, it follows from (40) that

$$\lim_{n \to \infty} \left| \int \varphi(w)d\mu(w) - \int \varphi(w + h_n)d\mu^*(w) \right| = 0.$$

Here $\varphi$ is an arbitrary function from $\mathcal{D}(\mathbb{C})$, hence, $\mu^{**} = \mu$. By Lemma 5, we get $v^{**}(z) = v(z) + D_1 + D_2y$.

Since (37) is valid for pairs $v, v^*$ and $v^*, v^{**}$, we get $D_1 \leq 0$. Then (38) for pairs $v, v^*$ and $v^*, v^{**}$ implies $D_1 \geq 0$, and we obtain $D_1 = 0$ and the equality in (37) and (38). By the same way, we obtain the equalities in (39). Lemma is proved.

**Proof of Theorem 5.** The necessity follows immediately from Theorem 4. Let us prove a sufficiency. Suppose $\mu$ satisfies the conditions of Theorem 5. Let $V$ be the function from (29), and let $\{h_n\} \subset \mathbb{R}$ be an arbitrary sequence. It follows from Theorem 4 that the family $\{V(z + h_n)\}$ is a relatively compact subset of $\mathcal{D}'(\mathbb{C})$. Therefore we can assume without loss of generality that $V(z + h_n) \to V^*(z)$ in $\mathcal{D}'(\mathbb{C})$. To prove the Theorem, we need to check that

$$\int \varphi(z - t - h_n)V(z)dm_2(z) \to \int \varphi(z - t)V^*(z)dm_2(z).$$

uniformly in $t \in \mathbb{R}$ for any $\varphi \in \mathcal{D}(\mathbb{C})$.

Assume the contrary. Then there is $\varphi_0 \in \mathcal{D}(\mathbb{C}), \varepsilon_0 > 0$, and $\{t_n\} \subset \mathbb{R}$ such that

$$\left| \int \varphi_0(w)V(w + h_n + t_n)dm_2(w) - \int \varphi_0(w)V^*(w + t_n)dm_2(w) \right| \geq \varepsilon_0 \quad (42)$$

(if necessary we can replace the sequence $\{h_n\}$ by a subsequence).

We may assume also that $V(z + h_n + t_n) \to V^{**}(z), V^*(z + t_n) \to V^{**}(z)$ in $\mathcal{D}'(\mathbb{C})$ as $n \to \infty$. By $\mu^*, \mu^{**}, \mu^{***}$ denote the Riesz measures of the functions $V^*, V^{**}, V^{***}$, respectively. Then we have

$$\lim_{n \to \infty} \int \varphi(w - h_n - t_n)d\mu(w) = \int \varphi(w)d\mu^{**}(w), \quad (43)$$

$$\lim_{n \to \infty} \int \varphi(w - t_n)d\mu^*(w) = \int \varphi(w)d\mu^{***}(w) \quad (44)$$

for any $\varphi \in \mathcal{D}(\mathbb{C})$.

On the other hand, the measure $\mu$ satisfies (40). Hence the integrals in left-hand sides of (43) and (44) have the same limit, and $\mu^{**} = \mu^{***}$.

By Lemma 7 we obtain

$$\sup_{\mathbb{R}} V^{**}(x) = \sup_{\mathbb{R}} V(x) = \sup_{\mathbb{R}} V^*(x) = \sup_{\mathbb{R}} V^{***}(x)$$

13
and

\[ \sigma_+(V^{**}) = \sigma_+(V) = \sigma_+(V^*) = \sigma_+(V^{**}) \]

Using Lemma 5, we get \( V^{**} \equiv V^{***} \). This contradicts (12). Theorem 5 is proved.

Now let \( d \) be a divisor in \( \mathbb{C} \), i.e., a sequence \( \{a_k\} \subset \mathbb{C} \) without finite limit points such that each value may appear with a finite multiplicity. A divisor is called almost periodic if the discrete measure supported at the points \( a_k \) with mass at each point equals the multiplicity of the point in the sequence is almost periodic (see [12], [3]; in [3] there is an equivalent geometric definition). Moreover, almost periodic divisors are just the divisors of entire functions with almost periodic modulus in every substrip \( \{ z = x + iy : |y| < H \} \).

**Theorem 6.** For a divisor \( \{a_k\} \) to be the divisor of an entire function of exponential type with the almost periodic modulus, it is necessary and sufficient the following conditions be fulfilled:

a) The divisor is almost periodic,

b) there is a finite limit

\[ \lim_{R \to \infty} \sum_{|a_k| < R} \frac{1}{a_k}, \]

c) \( n(0, t) = O(t) \),

d) \( n(0, t + 1) - n(0, t) = \sigma(t) \),

e) \( \sup_{x \in \mathbb{R}} \left| \int_1^\infty \frac{n(0, t) - n(x, t)}{t} dt \right| < \infty; \)

here \( n(c, t) = \text{card}\{ k : |a_k - c| \leq t \} \).

**Proof** By Theorem 6, conditions a) – e) mean just the existence of an almost periodic subharmonic function \( v \) at most of linear growth with the Riesz measure supported at the points \( \{a_k\} \) with mass equals a multiplicity of the point in the sequence. Then \( v(z) = \log |f(z)| \) for an entire function \( f \) of exponential type such that the divisor of \( f \) is \( \{a_k\} \). Since \( v \) is an almost periodic, we get that \( |f| \) is almost periodic (see [3]). Theorem is proved.

Now we consider the periodic case.

**Theorem 7.** A necessary and sufficient condition for a measure \( \mu \) on \( \mathbb{C} \) to be the Riesz measure of some periodic subharmonic function with period 1 at most of linear growth is the measure be stable with respect to the translation on 1 and

\[ \mu\{ z = x + iy : 0 \leq x < 1, y \in \mathbb{R} \} < \infty. \]  \hspace{1cm} (45)

A divisor \( \{a_k\} \) is the divisor of a holomorphic function \( f \) when zeros of \( f \) coincide with the values \( \{a_k\} \) and a multiplicity of every zero equals the multiplicity of corresponding \( a_k \).
Proof Let $v$ be a subharmonic function such that $v(z + 1) = v(z)$. It is clear that its Riesz measure is stable with respect to the translation on 1. By Theorem 1, it follows that $\mu$ satisfies (16). Using the equality

$$
\mu\{z = x + iy: 0 \leq x < 1, |y| < n\} = \frac{1}{2n} \mu\{z = x + iy: -n \leq x < n, |y| < n\} , \tag{46}
$$

we get (45).

Conversely, let $\mu$ be stable with respect to the translation on 1 and be satisfied (45). Using (16), we obtain (16). Then for any $r > (48)$ is less than $r\varepsilon$

$$
\mu\{z: r \leq |z| < r + 1\} \leq \sum_{n \in \mathbb{Z}, |n| \leq r+1} \mu\{z: x \in [0, 1), r \leq |z + n| < r + 1\} \tag{47}
$$

Fix $\delta \in (0, 1)$. For $|n| < r(1 - \delta) - 1$ we have

$$
\{z: x \in [0, 1), r \leq |z + n| < r + 1\} \subset \{z: x \in [0, 1), |y| > r\delta\}
$$

Hence (47) is majorized by

$$
2r(1 - \delta)\mu\{z: x \in [0, 1), |y| > r\delta\} + (2\delta r + 5)\mu\{z: x \in [0, 1), y \in \mathbb{R}\}. \tag{48}
$$

It follows from (45) that for any $\varepsilon > 0$ there exist $\delta > 0$ and $r_0 < \infty$ such that for $r \geq r_0$ (48) is less than $r\varepsilon$. This yields (27).

Further, take $R > r > 1$. We have

$$
\left| \int_{r < |z| < R} \frac{d\mu(z)}{z} - \int_{r < |x + iy| < R} \frac{d\mu(z)}{z} \right| \leq \frac{1}{r - 1} \mu\{z: r - 1 < |z| < r + 1\} + \frac{1}{R - 1} \mu\{z: R - 1 < |z| < R + 1\}; \tag{49}
$$

where $[x]$ being the integral part of a real $x$. By (27), the right-hand side of (49) tends to zero as $r \to \infty$. Then we obtain

$$
\int_{r < |x + iy| < R} \frac{d\mu(z)}{z} = \sum_{n \in \mathbb{Z}, |n| \leq R, x \in [0, 1), r < |n + iy| < R} \int \frac{d\mu(z)}{z + n} = \sum_{x \in [0, 1), y \in \mathbb{R}} \sum_{n \in \mathbb{Z}, r < |n + iy| < R} \frac{z + n}{|z + n|^2} d\mu(z) \tag{50}
$$

Now for any $x \in [0, 1), y \in \mathbb{R}$ we have

$$
\sum_{n \in \mathbb{Z}, |n + iy| > r} \frac{|z|}{|z + n|^2} \leq \sum_{n \in \mathbb{N} \cup \{0\}, n^2 > r^2 - y^2} \frac{1 + |y|}{n^2 + y^2} + \sum_{n \in \mathbb{N}, n^2 > r^2 - y^2} \frac{1 + |y|}{(n - 1)^2 + y^2} \leq 2 \int_{\sqrt{\max\{0, r^2 - y^2\}}}^{\infty} \frac{1 + |y|}{t^2 + y^2} dt + \frac{2(1 + |y|)}{y^2} \chi_r(y); \tag{51}
$$

where $\chi_r(y)$ is the characteristic function of $[0, r]$.
here \( \chi_r(y) \) is a characteristic function of interval \((\sqrt{r^2-1}, \infty)\). Besides,
\[
- \sum_{n \in \mathbb{Z}: r < |n+iy| < R} \frac{n}{|z+n|^2} = \sum_{n \in \mathbb{N}: r < |n+iy| < R} n \left( \frac{1}{|z-n|^2} - \frac{1}{|z+n|^2} \right) = \\
= \sum_{n \in \mathbb{N}: r^2 < n^2+y^2 < R^2} \frac{2n^2x}{((n-x)^2+y^2)((n+x)^2+y^2)}
\]

It is easy to see that the right-hand side of (52) is also majorized by (51). Both terms monotonically decrease to 0 as \( r \to \infty \), hence (50) tends to 0 as \( r \to \infty \) uniformly in \( R \). It follows from (49) and (50) that (10) is valid. Finally, by (31) and (32), the integral
\[
\int_1^\infty \frac{\mu(B(0,t)) - \mu(R(z,t))}{t} \, dt
\]
is bounded for \( z = x \in [0,1] \). Since \( \mu \) is stable with respect to the translation on 1, we get (36). Now the assertion of Theorem 7 follows from Theorem 5.

**Consequence.** A necessary and sufficient conditions for a divisor \( \{a_k\} \) to be the divisor of an entire periodic (with period 1) function of exponential type bounded on real axis is that the divisor be periodic with period 1 and its restriction to the strip \( \{z : 0 \leq x < 1, y \in \mathbb{R}\} \) be finite.

In this case the corresponding function is a finite product of elementary functions \( \sqrt{1 - \cos 2\pi(z - \gamma_k)} \) with \( \text{Re}\gamma_k \in [0,1] \); it is unique up to a multiplier \( Ce^{i\nu z}, \nu \in \mathbb{R} \).

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