RING THEORETIC ASPECTS OF QUANGLES

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ABSTRACT. We associate to every quandle $X$ and an associative ring with unity $k$, a nonassociative ring $k[X]$ following [3]. The basic properties of such rings are investigated. In particular, under the assumption that the inner automorphism group $\text{Inn}(X)$ acts orbit 2-transitively on $X$, a complete description of right (or left) ideals is provided. The complete description of right ideals for the dihedral quandles $R_n$ is given. It is also shown that if for two quandles $X$ and $Y$ the inner automorphism groups act 2-transitively and $k[X]$ is isomorphic to $k[Y]$, then the quandles are of the same partition type. However, we provide examples when the quandle rings $k[X]$ and $k[Y]$ are isomorphic, but the quandles $X$ and $Y$ are not isomorphic. These examples answer some open problems in [3].

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1. INTRODUCTION

Quandles are generally non-associative algebraic structures (the exception being the trivial quandles). They were introduced independently in the 1980’s by Joyce [12] and Matveev [17] with the purpose of constructing invariants of knots in the three space and knotted surfaces in four space. However, the notion of a quandle can be traced back to the 1940’s in the work of Mituhisa Takasaki [24]. The three axioms of a quandle algebraically encode the three Reidemeister moves in classical knot theory. For a recent treatment of quandles (see [8]). Joyce and Matveev introduced the notion of the fundamental quandle of a knot and gave a theorem that brings the problem of equivalence of knots to the problem of the quandle isomorphism of their fundamental quandles. Precisely, two knots $K_1$ and $K_2$ are equivalent (up to reverse and mirror image) if and only if the fundamental quandles $Q(K_1)$ and $Q(K_2)$ are isomorphic. But determining isomorphism classes of quandles is a difficult task in general. Thus the need of restricting oneself to some specific families of quandles
such as connected quandles (called also indecomposable), medial and Alexander quandles. Recall that the fundamental quandles of knots are connected. Recently, there has been investigations of quandles from algebraic point of views and their relations to other algebraic structures such as Lie algebras [4, 5], Leibniz algebras [13, 14], Frobenius algebras and Yang-Baxter equation [6], Hopf algebras [2, 5], transitive groups [26], quasigroups and Moufang loops [9], ring theory [3] etc. This article will add to this list since we introduce new concepts motivated by ring theory to the theory of quandles. We follow [3] and we associate to every quandle \((X, \triangleright)\) and an associative ring \(k\) with unity, a nonassociative ring \(k[X]\). Precisely, Let \(k[X]\) be the set of elements that are uniquely expressible in the form \(\sum_{x \in X} a_x x\), where \(x \in X\) and \(a_x = 0\) for almost all \(x\). Then the set \(k[X]\) becomes a ring with the natural addition and the multiplication given by the following, where \(x, y \in X\) and \(a_x, a_y \in k\),

\[
(\sum_{x \in X} a_x x) \cdot (\sum_{y \in X} b_y y) = \sum_{x, y \in X} a_x b_y (x \triangleright y).
\]

Linearization of quandles appeared in the work on categorical groups and other notions of categorification in [4] and [5], where self-distributive structures in the categories of coalgebras, cocommutative coalgebras and Hopf algebras were studied. Precisely, in studying self-distributivity maps in coalgebras, the authors of [4] gave a broad examples with a focus on the case of \(k \oplus k[X]\) with the multiplication

\[
(a + \sum_{x \in X} a_x x) \cdot (b + \sum_{y \in X} b_y y) = \sum_{y \in X} a b_y + \sum_{x, y \in X} a_x b_y (x \triangleright y).
\]

In [3], the authors showed that the ring \(k[X]\) gives interesting information on the quandle \(X\). In this article we investigate the basic properties of quandle rings and also solve some of the open problems stated in [3]. In particular, under the assumption that the inner automorphism group \(\text{Inn}(X)\) acts orbit 2-transitively on the quandle \(X\) and the ring \(k\) is a field of characteristic zero (or a certain semigroup \(H_x\) acts 2-transitively on \(X\)) a complete description of right (or left) ideals is provided. The corresponding results for fields of positive characteristic are given in Corollary 4.10. The complete description of right ideals of \(k[R_n]\), where \(R_n\) is the dihedral quandle of order \(n\), is given. It is also shown that the rings \(k[X]\) are Noetherian, when the quandle \(X\) is finite and the ring \(k\) is Noetherian. We also give an example of a quandle \(X\) with \(k[X]\) not Noetherian. These rings are, in general, not domains and neither every right nor left ideal is principal. It is also shown that if for two quandles \(X\) and \(Y\) the inner automorphism groups act 2-transitively and \(k[X]\) is isomorphic to \(k[Y]\) (here \(k\) is a field of \(\text{char}(k) = 0\)), then the quandles are of the same partition type. However, we provide examples when the quandle rings \(k[X]\) and \(k[Y]\) are isomorphic, but the quandles \(X\) and \(Y\) are not isomorphic when \(\text{char}(k) = p\) for any prime \(p\).

The following is the organization of the article. In Section 2, we recall the basics of quandles with examples. In Section 3, we investigate an open question raised in [3] concerning the power associativity of non-trivial quandles. Precisely, we prove that quandle rings are never power associative when the quandle is non-trivial and the ring has characteristic zero.

Section 4 deals with various properties of quandle rings. We show that for a Noetherian ring \(k\) and a finite quandle \(X\), the quandle ring \(k[X]\) is both left and right Noetherian ring. We also give, for any positive integer \(n\) and \(k = \mathbb{R}\) or \(\mathbb{C}\), the complete list of simple right ideals of the quandle ring \(k[R_n]\). In section 5, we investigate the problem of isomorphisms of quandle rings. We introduce the notion of partition-type of quandles and show that if the
quandle rings $k[X]$ and $k[Y]$ are isomorphic and the quandles $X$ and $Y$ are orbit 2-transitive, then $X$ and $Y$ are of the same partition type. Section 6 deals with the augmentation ideal of a quandle rings. Precisely we give a solution to conjecture 6.5 in [3].

Throughout the paper, $k$ always denotes a ring unless specified otherwise.

2. Review of Quandles

We start this section by giving the basics of quandles with examples.

**Definition 2.1.** A quandle, $X$, is a set with a binary operation $(a, b) \mapsto a \triangleright b$ such that

(I) For any $a \in X$, $a \triangleright a = a$.

(II) For any $a, b \in X$, there is a unique $c \in X$ such that $a = c \triangleright b$.

(III) For any $a, b, c \in X$, we have $(a \triangleright b) \triangleright c = (a \triangleright c) \triangleright (b \triangleright c)$.

A rack is a set with a binary operation that satisfies (II) and (III). Racks and quandles have been studied extensively in, for example, [12, 17]. For more details on racks and quandles see the book [8].

The following are typical examples of quandles:

- A group $G$ with conjugation as the quandle operation: $a \triangleright b = b^{-1}ab$, denoted by $X = \text{Conj}(G)$, is a quandle.
- Any subset of $G$ that is closed under such conjugation is also a quandle. More generally if $G$ is a group, $H$ is a subgroup, and $\sigma$ is an automorphism that fixes the elements of $H$ (i.e. $\sigma(h) = h \forall h \in H$), then $G/H$ is a quandle with $\triangleright$ defined by $Ha \triangleright Hb = H\sigma(ab^{-1})b$.
- Any $\mathbb{Z}[t, t^{-1}]$-module $M$ is a quandle with $a \triangleright b = ta + (1-t)b$, for $a, b \in M$, and is called an Alexander quandle.
- Let $n$ be a positive integer, and for elements $i, j \in \mathbb{Z}_n$, define $i \triangleright j = 2j - i \pmod{n}$. Then $\triangleright$ defines a quandle structure called the dihedral quandle, and denoted by $R_n$, that coincides with the set of reflections in the dihedral group with composition given by conjugation.
- Any group $G$ with the quandle operation: $a \triangleright b = b^{-1}ab$ is a quandle called Core($G$).

The notions of quandle homomorphisms and automorphisms are clear. Let $X$ be a quandle, thus the second axiom of Definition 2.1 makes any right multiplication by any element $x$, $R_x : y \mapsto y\triangleright x$, into a bijection. The third axiom of Definition 2.1 makes $R_x$ into a homomorphism and thus an automorphism. Let $\text{Aut}(X)$ denotes the group of all automorphisms of $X$ and let $\text{Inn}(X) := \langle R_x : x \in X \rangle$ denotes the subgroup generated by right multiplications. The quandle $X$ is called connected quandle if the group $\text{Inn}(X)$ acts transitively on $X$, that is, there is only one orbit. Later in the paper, in Section 4, we will use the left multiplication in a quandle denoted $L_x : y \mapsto x \triangleright y$. In general these maps need not to be bijective. Quandles in which left multiplications $L_x$ are bijections are called Latin quandles.

3. Power associativity of quandle rings

In [3], power associativity of dihedral quandles was investigated and the question of determining the conditions under which the quandle ring $R[X]$ is power associative was raised. In this section we give a complete solution to this question. Precisely, we prove that quandle rings are never power associative when the quandle is non-trivial and the ring has characteristic zero.
But first let's recall the following definition from [1]

**Definition 3.1.** A ring $k$ in which every element generates an associative subring is called a power-associative ring.

**Example 3.2.** Any alternative algebra is power associative. Recall that an algebra $A$ is called alternative if $x \cdot (x \cdot y) = (x \cdot x) \cdot y$ and $x \cdot (y \cdot y) = (x \cdot y) \cdot y$, $\forall x, y \in A$, (for more details see [10]).

It is well known [1] that a ring $k$ of characteristic zero is power-associative if and only if $(x \cdot x) \cdot x = x \cdot (x \cdot x)$ and $(x \cdot x) \cdot (x \cdot x) = [(x \cdot x) \cdot x] \cdot x$, for all $x \in k$.

**Theorem 3.3.** Let $k$ be a ring of characteristic zero and let $(X, \triangleright)$ be a non-trivial quandle. Then the quandle ring $k[X]$ is not power associative.

**Proof.** Let $u = \sum_{x \in X} a_x x$. Then

$$u \cdot u = \left( \sum_{x \in X} a_x x \right) \cdot \left( \sum_{x \in X} a_x x \right) = \sum_{x \in X} a_x^2 x + \sum_{x, y \in X, x \neq y} a_x a_y (x \triangleright y).$$

$$(u \cdot u) \cdot u = \left( \sum_{x \in X} a_x^2 x + \sum_{x, y \in X, x \neq y} a_x a_y (x \triangleright y) \right) \cdot \sum_{x \in X} a_x x$$

$$= \sum_{x \in X} a_x^3 x + \sum_{x, y \in X, x \neq y} a_x^2 a_y (x \triangleright y) + \sum_{x, y \in X} a_x^2 a_y (x \triangleright y)$$

$$+ \sum_{x, y \in X} a_y^2 a_x (x \triangleright y) \triangleright x + \sum_{x, y \in X} a_y^2 a_x (x \triangleright y) \triangleright y$$

$$+ \sum_{x, y, z \in X} a_x a_y a_z (x \triangleright y) \triangleright z.$$

and

$$u \cdot (u \cdot u) = \left( \sum_{x \in X} a_x x \right) \cdot \left( \sum_{x \in X} a_x^2 x + \sum_{x, y \in X, x \neq y} a_x a_y (x \triangleright y) \right)$$

$$= \sum_{x \in X} a_x^3 x + \sum_{x, y \in X, x \neq y} a_x^2 a_y (y \triangleright x) + \sum_{x, y \in X} a_y^2 a_x (x \triangleright y)$$

$$+ \sum_{x, y \in X} a_x^2 a_y x \triangleright (x \triangleright y) + \sum_{x, y \in X} a_x^2 a_y y \triangleright (x \triangleright y)$$

$$+ \sum_{x, y, z \in X} a_x a_y a_z x \triangleright (y \triangleright z).$$
The formulas above show that the equality \((u \cdot u) \cdot u = u \cdot (u \cdot u)\) for every \(u \in k[X]\) implies \((x \triangleright y) \triangleright z = x \triangleright (y \triangleright z)\), i.e. \((X, \triangleright)\) is associative, which contradicts the fact that \((X, \triangleright)\) is non-trivial.

\[ \square \]

4. **Various properties of the quandle ring \(k[X]\)**

In this section, we investigate different properties of quandle rings.

4.1. **Basic properties.** The following proposition shows that if the quandle \(X\) is a union of a finite orbit \(X_1\) and any quandle \(X_2\), then the quandle ring \(k[X]\) is not an integral domain.

**Proposition 4.1.** Let \(X = X_1 \amalg X_2\) be a quandle with \(|X_1| < \infty\). Then \(k[X]\) is not an integral domain.

**Proof.** Indeed, it follows from property (II) of Definition 2.1 that

\[
(\sum_{z \in X_1} z) \triangleright (x - y) = 0,
\]

where \(x\) and \(y\) are any two distinct elements of \(X\).

\[ \square \]

**Definition 4.2.** Let \(X = \{e_1, \ldots, e_n\}\) be a quandle of finite cardinality. We define the *order* of an element \(x \in k[X]\) to be the largest index \(i\) that occurs in the expression \(x = \sum a_i e_i\).

We then have the following proposition that gives the conditions for the quandle ring \(k[X]\) to be both left and right Noetherian.

**Proposition 4.3.** Let \(X\) be a quandle of finite cardinality and \(k\) be a Noetherian ring. Then \(k[X]\) is both left and right Noetherian ring.

**Proof.** Let \(I \subset k[X]\) be an ideal, \(I_m\) the subset of elements of order \(m\) in \(I\) and \(\tilde{I}_m \subset k\) the ideal of leading coefficients of elements in \(I_m\). Clearly, \(I = \bigcup_m I_m\), moreover, each \(\tilde{I}_m\) is finitely generated, since the ring \(k\) is Noetherian. Now it is sufficient to verify that \(I_m\) is generated by any \((f_{m_1}, \ldots, f_{m_k}) \subset \bigcup_{s=1}^m I_s\), whose leading coefficients \((a_{m_1}, \ldots, a_{m_k})\) generate \(\bigcup_{s=1}^m \tilde{I}_s\). This is checked via induction on \(m\). Indeed, let \(g = \alpha_m e_m + \sum \alpha_i e_i \in I_m\) be an element of order \(m\). Then \(\alpha_m = \sum_{i=1}^k \beta_i a_{m_i}\) and the element \(g - \sum_{i=1}^k \beta_i f_{m_i}\) has order strictly less than \(m\).

\[ \square \]

The next example shows that \(k[X]\) is neither necessarily a right nor a left Noetherian ring, if the quandle \(X\) is not of finite cardinality.

**Example 4.4.** Consider the trivial quandle \(X = X_1 \amalg X_2 \amalg X_3 \amalg \ldots\) with each \(X_i = \{e_i\}\) consisting of a single element. Take \(I \subset k[X]\) to be the augmentation ideal. Assume that \(I\) is generated by the elements \((a_1, \ldots, a_m)\), where each \(a_i = \sum_{j} \alpha_{ij} e_j\) with \(\sum_{i} \alpha_{ij} = 0\) and the sum is finite. Then for any \(\gamma \in k[X]\) and any \(a_i \in \{a_1, \ldots, a_m\}\), we have \(\gamma \triangleright a_i = 0\) and \(a_i \triangleright \gamma = c a_i\) for some \(c \in k\). Now let \(n := \max(\text{ord}(a_1), \ldots, \text{ord}(a_m))\). Then \(e_1 - e_N\) is not generated by \(a_i\)'s for any \(N > n\). Hence \(I\) is not generated by \((a_1, \ldots, a_m)\).
A similar example shows that $k[X]$ is not necessarily a PID, even if the quandle $X$ is of finite cardinality.

**Example 4.5.** Consider the trivial quandle $X = X_1 \sqcup X_2 \sqcup \ldots \sqcup X_k$ consisting of a single element and let $I \subset k[X]$ be the augmentation ideal. For the same reasons as in the example above, the ideal $I$ can not be generated by a single element if $n \geq 3$.

### 4.2. Study of ideals in $k[X]$ via groups and semigroups.

Let $T_X$ stand for the semigroup of all maps from a finite set $X$ to itself (the full transformation semigroup of $X$). Representations of $T_X$ were extensively studied by A. H. Clifford (see for example [11]) and are intrinsically connected to the study of left ideals in $k[X]$ as shown below.

Let $X = X_1 \sqcup X_2 \sqcup \ldots \sqcup X_k$ be the decomposition of a quandle $X$ into orbits. Recall that any element $x \in X$ gives rise to two functions from $X$ to itself given by $L_x : X \to X$ and $R_x : X \to X$ (the latter function is a bijection). The left multiplication produces a map $\psi : X \to T_X$. The function $R_x : X \to X$ restricts to functions $X_i \to X_i$ for any orbit $X_i$ by definition. This gives rise to the map $\varphi : X \to S_n$, which restricts to the maps $\varphi_i : X \to S_{n_i}$, where $n_i = |X_i|$ and $n = \sum_{i=1}^{k} n_i$. The composition of functions $L_y \circ L_x$ and $R_y \circ R_x : X \to X$ correspond to the products of the (semi)group elements in the image.

We denote the underlying $k$-vector space of $k[X]$ by $V$. By remarks of the preceding paragraph to study the right ideals in $k[X]$ is equivalent to viewing $V$ as a representation of the group $\text{Inn}(X)$. Clearly, $V = \bigoplus_{i=1}^{k} V^i$, where $V^i = k[X_i]$, and therefore, one needs to understand the decompositions of the $V^i$'s into irreducibles. First, notice that each $V^i$ contains a one-dimensional subspace $V^i_{\text{triv}} := k v^i_{\text{triv}}$ invariant under the action of $G_{X_i}$, where $v^i_{\text{triv}} = \sum_{x \in X_i} x$.

The following notion of 2-transitivity (and also higher $k$-transitivity) of quandles was introduced in [15, 16]. Two-transitive quandles are called two-point homogeneous quandles in [25].

**Definition 4.6.** The action of a group (semigroup) $G$ on a set $X$ is called 2-transitive if for any two pairs $(x_1, y_1) \in X \times X$ and $(x_2, y_2) \in X \times X$, there exists an element $g \in G$, such that $g \cdot x_1 = x_2$ and $g \cdot y_1 = y_2$.

**Definition 4.7.** Let $X = X_1 \sqcup X_2 \sqcup \ldots \sqcup X_k$ be a finite quandle. $X$ is said to be left (right) 2-transitive if the semigroup $H_X$ (the group $\text{Inn}(X)$) acts 2-transitively on $X$. $X$ is said to be left (right) orbit 2-transitive if the semigroup $H_X$ (each of the groups $G_{X_i}$) acts 2-transitively on $X$ (orbit $X_i$).

Let $V_{st} \subset V$ be the subspace orthogonal to the vector $v_{\text{triv}} = \sum_{x \in X_i} x$. 
The following theorem gives the list of subgroups $G \leq S_n$ for which the representation $V_{st}$ is irreducible. The first assertion can be found as Theorem 1(b) in [19] and second as (ii) in the Main Theorem of [20], and the proof and description of groups (i) – (v) can be found in [21]. The groups (i) – (v) are respectively the affine and projective general semilinear groups, projective semilinear unitary groups, Suzuki and Ree groups.

**Theorem 4.8.**
(a) If $\text{char}(k) = 0$ then $V_{st}$ is an irreducible representation for any 2-transitive subgroup of $S_n$.
(b) In case $\text{char}(k) = p > 3$, the representation $V_{st}$ is irreducible for any 2-transitive subgroup $G$ of $S_n$ except
(i) $G \leq A\Pi L(m, q)$, and $p$ divides $q$;
(ii) $G \leq P\Pi L(m, q)$, $m \geq 3$ and $p$ divides $q$;
(iii) $G \leq P\Pi L(3, q)$, and $p$ divides $q + 1$;
(iv) $G \leq S\Pi (q)$, and $p$ divides $q + 1 + m$, where $m^2 = 2q$;
(v) $G \leq R\Pi (q)$, and $p$ divides $(q + 1)(q + 1 + m)$, where $m^2 = 3q$.

**Definition 4.9.** Let $S$ be a finite semigroup. If $e$ is an idempotent, then $eSe$ is a monoid with identity $e$; its group of units $G_e$ is called the maximal subgroup of $S$ at $e$.

**Corollary 4.10.**
(1) Let $X = X_1 \sqcup X_2 \sqcup \ldots \sqcup X_k$ be a finite quandle with a 2-transitive action of each $G_{X_i}$ on the corresponding orbit $X_i$ and $\text{char}(k) = 0$. Then $k[X] \simeq \bigoplus_{i=1}^{k} (V_{st}^{i} \oplus V_{triv}^{i})$ where the r.h.s. consists of simple right ideals.
(2) Let $X = X_1 \sqcup X_2 \sqcup \ldots \sqcup X_k$ be a finite quandle with a 2-transitive action of each $G_{X_i}$ on the corresponding orbit $X_i$ and $G_{X_i}$’s not among the groups from (i) – (v) in Theorem 4.8, $\text{char}(k) = p > 3$. Then $k[X] \simeq \bigoplus_{i=1}^{k} (V_{st}^{i} \oplus V_{triv}^{i})$ where the r.h.s. consists of simple right ideals.
(3) Let $X$ be a finite quandle such that $H_X \subset T_X$ contains a maximal subgroup with a 2-transitive action on $X$ and $\text{char}(k) = 0$. Then $k[X] \simeq V_{st} \oplus V_{triv}$ where the r.h.s. consists of simple left ideals.
(4) Let $X$ be a finite quandle such that $H_X \subset T_X$ contains a maximal subgroup with a 2-transitive action on $X$ not among the groups from (i) – (v) in Theorem 4.8 and $\text{char}(k) = p > 3$. Then $k[X] \simeq V_{st} \oplus V_{triv}$ where the r.h.s. consists of simple left ideals.

**Definition 4.11.** A quandle $X$ is of right (left) cyclic type (or cyclic) if for each $x \in X$ the permutation $\varphi(x)$ (or the semigroup element $\psi(x)$) acts on $X \setminus \{x\}$ as a cycle of length $|X| - 1$, where $|X|$ denotes the cardinality of $X$.

The above discussion motivates to find the conditions on $X$ to be 2-transitive and orbit 2-transitive. The following theorem was obtained in [26] (see Corollary 4 and references therein).

**Theorem 4.12.** Every finite right 2-transitive quandle is of right cyclic type.

**Remark 4.13.** It is also easy to see that if $X$ is of left cyclic type then it is a left 2-transitive quandle.

The above observations allow to strengthen Corollary 4.10.
Corollary 4.14. (1) Let $X = X_1 \sqcup X_2 \sqcup \ldots \sqcup X_k$ be a finite quandle with each subquandle $X_i$ of right cyclic type. Then $k[X] \simeq \bigoplus_{i=1}^{k} (V_{st}^i \oplus V_{triv}^i)$, where the r.h.s. consists of simple right ideals.

(2) Let $X$ be a finite quandle of left cyclic type. Then $k[X] \simeq V_{st} \oplus V_{triv}$ (an equality of left ideals with the r.h.s. consisting of simple left ideals).

For all quandles with order up to eight, the following chart gives the number of quandles (up to isomorphism) and their corresponding number of right 2-transitive quandles.

| Order | # of quandles | # of right 2-transitive quandles | # of left 2-transitive quandles |
|-------|---------------|----------------------------------|-------------------------------|
| 3     | 3             | 3                                | 2                             |
| 4     | 7             | 6                                | 3                             |
| 5     | 22            | 16                               | 7                             |
| 6     | 73            | 42                               | 14                            |
| 7     | 298           | 151                              | 39                            |
| 8     | 1581          | 656                              | 105                           |

Table 1. Right 2-transitive quandles

Example 4.15. Consider the dihedral quandle $R_n$ with odd $n > 3$. This quandle is connected and the action of $\text{Inn}(R_n)$ is not 2-transitive.

The next proposition provides the decomposition of $k[R_n]$ into the sum of simple right ideals.

Proposition 4.16. Fix the ground field $k = \mathbb{C}, \mathbb{R}$.

(1) Let $n$ be an odd number. Then $k[R_n] \simeq V_{triv} \oplus \bigoplus_{\xi \in \mu_n} V_{\xi, \zeta}$ where the r.h.s. consists of simple right ideals and $V_{\xi, \zeta}$ are the two-dimensional irreducible representations of $D_n$ spanned by the eigenvectors for the normal subgroup $\mathbb{Z}_n \triangleleft D_n$ with eigenvalues $\xi, \zeta$.

(2) Let $n = 2k$ be an even number. Then $k[R_n] \simeq V_{triv,even} \oplus V_{triv,odd} \oplus \bigoplus_{\xi \in \mu_k} V_{\xi, \zeta}^{\oplus 2}$ where the r.h.s. consists of simple right ideals and $V_{\xi, \zeta}$ are the two-dimensional irreducible representations of $D_k$ spanned by the eigenvectors for the normal subgroup $\mathbb{Z}_k \triangleleft D_k$ with eigenvalues $\xi, \zeta$. The representations $V_{triv,even}$ and $V_{triv,odd}$ are the trivial one-dimensional representations corresponding to the orbits $X_{even}$ and $X_{odd}$ defined in the proof below.

Proof. We start with the case of odd $n$. Notice that $\text{Inn}(R_n)$ is a subgroup of $S_n$ isomorphic to the dihedral group $D_n$. Moreover, from multiplication table of $R_n$ we notice that multiplication by $e_i$ on the right correspond to reflection with respect to the line through the vertex $i$ and the midpoint of the opposite edge. Thus finding the decomposition of $k[R_n]$ into the sum of where the r.h.s. consists of simple right ideals is equivalent to decomposing the representation $V = \{v_1, \ldots, v_n\}$ given by $g \cdot v_i = v_{g \cdot i}$, where the action on the r.h.s. corresponds to the action of $\text{Inn}(R_n)$ on the $i$th vertex of a regular $n$-gon.

Now we consider the case $n = 2k$. Let $X_{odd} := \{e_i \in X \mid i \ is \ odd\}$ and $X_{even} := \{e_i \in X \mid i \ is \ even\}$. Note that $X = X_{even} \amalg X_{odd}$ with $|X_{even}| = |X_{odd}| = k$ and $e_i \triangleright e_j = e_i \triangleright e_{j+k}$.
for any $i, j \in \{1, \ldots, n\}$. The statement for $n = 2k$ follows from the observation that the dihedral group of order $k$ (generated by $\varphi(e_1), \ldots, \varphi(e_k) \in S_n$) acts on each orbit $X_{\text{even}}$ and $X_{\text{odd}}$ the same way as described in the case of odd $n$. □

Remark 4.17. Let $X = \text{Conj}(G)$ be the conjugation quandle on a group $G$, then the problem of decomposition of $k[X]$ into indecomposable right $k[X]$-modules is equivalent to decomposing $k[G]$ with conjugation action into indecomposable representations. This was studied, see for example [18, 22, 23].

5. On isomorphisms of quandle rings.

In this section, we investigate the problem of isomorphisms of quandle rings. We introduce the notion of partition-type of quandles and show that if the quandle rings $k[X]$ and $k[Y]$ are isomorphic and the quandles $X$ and $Y$ are orbit 2-transitive, then $X$ and $Y$ are of the same partition type. First, we start with the following definition.

**Definition 5.1.** Let $X$ be a finite quandle of cardinality $n$. The *partition type* of $X$ is $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_j$ being the number of orbits of cardinality $j$ in $X$.

**Example 5.2.** The partition type of the quandle $X = \{e_1, e_2\} \amalg \{e_3, e_4, e_5\} \amalg \{e_6, e_7\} \amalg \{e_8\}$ is $\lambda = 1, 2, 1, 0, 0, \ldots$.

**Theorem 5.3.** Assume $\text{char}(k) \neq 2, 3$. If the quandle rings $k[X]$ and $k[Y]$ are isomorphic and the quandles $X$ and $Y$ are orbit 2-transitive ($G_X$'s are not among the groups from (i) — (v) in Theorem 4.8 in case $\text{char}(k) = p > 3$), then $X$ and $Y$ are of the same partition type.

**Proof.** Let $s$ stand for the number of orbits in $X$, $d$ for the number of orbits in $Y$ and $n$ be the number of elements in $X$ (and $Y$). The partition types of $X$ and $Y$ will be denoted by $\lambda$ and $\mu$. The number of elements in $X_j$ will be denoted by $n_j$. Corollary 4.10 implies $k[X] \simeq \bigoplus_{i=1}^{s} (V_{st}^i \oplus V_{\text{triv}}^i)$ as the sum of right simple ideals, similarly, $k[Y] \simeq \bigoplus_{j=1}^{d} (W_{st}^j \oplus W_{\text{triv}}^j)$. Notice that $V_{st}^i = 0$, if $|X_i| = 0$. The isomorphism $\varphi : k[X] \to k[Y]$ induces another decomposition $k[Y] = \bigoplus_{j=1}^{d} \varphi(V_{st}^i) \oplus \varphi(V_{\text{triv}}^i)$ with each summand being a simple ideal in $k[Y]$. The Krull-Schmidt theorem asserts that the decomposition $k[Y] \simeq \bigoplus_{j=1}^{d} (W_{st}^j \oplus W_{\text{triv}}^j)$ is unique up to permutation of summands, from which we conclude that $\lambda = \mu$. □

**Corollary 5.4.** Let $X$ be a quandle of order 3. Then the three quandle rings arising from $X$ are not pairwise isomorphic.

**Corollary 5.5.** The ring $k[X]$ with $X$ the trivial quandle is not isomorphic to any other quandle ring.

**Remark 5.6.** Another way to see that the assertion of the corollary holds is as follows. If $\varphi : k[X] \to k[Y]$ is an isomorphism of quandle rings, it induces an isomorphism of augmentation ideals $\bar{\varphi} : I_X \to I_Y$. In particular, this implies that the trivial quandle $X$ is not isomorphic to any other quandle $Y$ as $\alpha \triangleright I_X = 0$ for any $\alpha \in k[X]$, but this property does not hold in $k[Y]$.

**Proposition 5.7.** Let $k = \mathbb{Z}_p$, where $p$ is prime, and $X$ a quandle of order 3. Then the three quandle rings arising from $X$ are not pairwise isomorphic.
Proof. We count the number of zeros in the multiplication table of each quandle ring and show that they are different.

**Case 1.** Let $X$ be $T_3$, the trivial quandle.

Let $\alpha_j e_1 + \beta_j e_2 + \gamma_j e_3$ be an element of $\mathbb{Z}_p[T_3]$ and consider $a_{11} e_1 + a_{12} e_2 + a_{13} e_3 \in \mathbb{Z}_p[T_3]$.

Then

$$(\alpha_j e_1 + \beta_j e_2 + \gamma_j e_3) \cdot (a_{11} e_1 + a_{12} e_2 + a_{13} e_3) = 0$$

implies

$$a_{11} + a_{12} + a_{13} = 0.$$

So $a_{11} = -a_{12} - a_{13}$. Hence there are $p^2$ zero columns.

**Case 2.** Now let $X = \{1, 2\} \sqcup \{3\}$, the quandle with two orbits.

Let $\alpha_j e_1 + \beta_j e_2 + \gamma_j e_3$ be an element of $\mathbb{Z}_p[X]$ and consider $a_{11} e_1 + a_{12} e_2 + a_{13} e_3 \in \mathbb{Z}_p[X]$.

Then

$$(\alpha_j e_1 + \beta_j e_2 + \gamma_j e_3) \cdot (a_{11} e_1 + a_{12} e_2 + a_{13} e_3) = 0$$

implies

$$\gamma_j (a_{11} + a_{12} + a_{13}) = 0,$$

$$\alpha_j (a_{11} + a_{12}) e_1 + \alpha_j a_{13} e_2 = 0,$$

and

$$\beta_j (a_{11} + a_{12}) e_2 + \beta_j a_{13} e_1 = 0.$$

Hence we have $a_{13} = 0$ and $a_{11} = -a_{12}$ which implies there are $p$ number of zero rows.

**Case 3.** Now $X = R_3$, the Takasaki quandle.

Let $\alpha_j e_1 + \beta_j e_2 + \gamma_j e_3$ be an element of $\mathbb{Z}_p[R_3]$ and consider $a_{11} e_1 + a_{12} e_2 + a_{13} e_3 \in \mathbb{Z}_p[R_3]$.

Then

$$(\alpha_j e_1 + \beta_j e_2 + \gamma_j e_3) \cdot (a_{11} e_1 + a_{12} e_2 + a_{13} e_3) = 0$$

implies

$$\alpha_j (a_{11} e_1 + a_{12} e_3 + a_{13} e_2) = 0$$

$$\beta_j (a_{11} e_3 + a_{12} e_2 + a_{13} e_1) = 0$$

$$\gamma_j (a_{11} e_2 + a_{12} e_1 + a_{13} e_3) = 0$$

for all $j$.

Since the system

$$a_{11} e_1 + a_{12} e_3 + a_{13} e_2 = 0$$

$$a_{11} e_3 + a_{12} e_2 + a_{13} e_1 = 0$$

$$a_{11} e_2 + a_{12} e_1 + a_{13} e_3 = 0$$

is consistent, there is only one solution which is $a_{11} = a_{12} = a_{13} = 0$. Thus there is only one zero column.

Next we provide two examples, when the quandle rings $k[X]$ and $k[Y]$ are isomorphic, but the quandles $X$ and $Y$ are not, answering question 7.4 of [3].
Example 5.8. Let \( k \) be a field with \( \text{char}(k) = 3 \) and quandles \( X \) and \( Y \) be of cardinality 4 with multiplication tables as below.

\[
\begin{array}{c|cccc}
\triangleright & e_1 & e_2 & e_3 & e_4 \\
\hline
e_1 & e_1 & e_1 & e_2 & e_2 \\
e_2 & e_2 & e_2 & e_1 & e_1 \\
e_3 & e_3 & e_3 & e_3 & e_3 \\
e_4 & e_4 & e_4 & e_4 & e_4 \\
\end{array}
\quad \text{and} \quad
\begin{array}{c|cccc}
\triangleright & e'_1 & e'_2 & e'_3 & e'_4 \\
\hline
e'_1 & e'_1 & e'_1 & e'_2 & e'_2 \\
e'_2 & e'_2 & e'_2 & e'_1 & e'_1 \\
e'_3 & e'_3 & e'_3 & e'_3 & e'_3 \\
e'_4 & e'_4 & e'_4 & e'_4 & e'_4 \\
\end{array}
\]

The isomorphism is given by \( \varphi : k[X] \to k[Y] \), where \( \varphi = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \).

Generalizing Example 5.8, we consider \( \tilde{X} = X \amalg \{ e_3 \} \amalg \{ e_6 \} \amalg \ldots \amalg \{ e_n \} \) and \( \tilde{Y} = Y \amalg \{ e_5 \} \amalg \{ e_6 \} \amalg \ldots \amalg \{ e_n \} \). Clearly the quandles \( \tilde{X} \) and \( \tilde{Y} \) are not isomorphic. Let \( k \) be a field with \( \text{char}(k) = p \) with \( p \mid n - 1 \). Then \( \varphi : k[X] \to k[Y] \) given by \( \varphi(e_i) = e'_i \) for \( i \neq 4 \) and \( \varphi(e_4) = \sum_{j=1}^{n} e'_j \) is a ring isomorphism.

Example 5.9. Let \( k \) be a field with \( \text{char}(k) = 5 \) and quandles \( X \) and \( Y \) be of cardinality 6 with multiplication tables as below.

\[
\begin{array}{c|cccccc}
\triangleright & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
\hline
e_1 & e_1 & e_1 & e_1 & e_1 & e_1 & e_2 \\
e_2 & e_2 & e_2 & e_2 & e_2 & e_2 & e_1 \\
e_3 & e_3 & e_3 & e_3 & e_3 & e_3 & e_4 \\
e_4 & e_4 & e_4 & e_4 & e_4 & e_4 & e_3 \\
e_5 & e_5 & e_5 & e_5 & e_5 & e_5 & e_5 \\
e_6 & e_6 & e_6 & e_6 & e_6 & e_6 & e_6 \\
\end{array}
\quad \text{and} \quad
\begin{array}{c|cccccc}
\triangleright & e'_1 & e'_2 & e'_3 & e'_4 & e'_5 & e'_6 \\
\hline
e'_1 & e'_1 & e'_1 & e'_1 & e'_1 & e'_1 & e'_2 \\
e'_2 & e'_2 & e'_2 & e'_2 & e'_2 & e'_2 & e'_1 \\
e'_3 & e'_3 & e'_3 & e'_3 & e'_3 & e'_3 & e'_4 \\
e'_4 & e'_4 & e'_4 & e'_4 & e'_4 & e'_4 & e'_3 \\
e'_5 & e'_5 & e'_5 & e'_5 & e'_5 & e'_5 & e'_5 \\
e'_6 & e'_6 & e'_6 & e'_6 & e'_6 & e'_6 & e'_6 \\
\end{array}
\]

Define \( \phi(e_i) = e'_i \) for \( 1 \leq i \leq 5 \) and \( \phi(e_6) = \sum_{i=1}^{6} e'_i \). It is easy to see that \( \phi \) is an isomorphism from \( k[X] \) to \( k[Y] \).

Now we present another family of examples. Consider \( \mathbb{Z}_{3n-1} \) and let \( k = \frac{3n-2}{2} \). Consider two quandles \( X \) and \( Y \) of order \( 3n \), where \( n \) is a power of 2.

\( X = X_1 \amalg X_2 \amalg \ldots \amalg X_{k} \amalg X_{k+1} \amalg X_{k+2} \) and \( X_{k+2} = \{ e_{3n} \} \) and

\( Y = Y_1 \amalg Y_2 \amalg \ldots \amalg Y_{k} \amalg Y_{k+1} \amalg Y_{k+2} \) and \( Y_{k+2} = \{ e'_{3n} \} \).
5.8 provides a negative answer to Example 5.10 if the corresponding matrix is of full rank. We provide one possible application.

Example 5.10. Let \( X_1, X_2 \) and \( X_3 \) be quandles of cardinality one. We show that the rings \( k[X_1] \oplus k[X_2] \oplus k[X_3] \) and \( k[X_1] \amalg X_2 \amalg X_3 \) are not isomorphic. Indeed the conditions

|\( \triangleright \) | \( e_1 \) | \( e_2 \) | \( \ldots \) | \( e_{3n-1} \) | \( e_{3n} \) |
|---|---|---|---|---|---|
|\( e_1 \) | \( \ldots \) | \( \ldots \) | \( X_1 \) | \( \ldots \) | \( \ldots \) |
|\( e_2 \) | \( \ldots \) | \( \ldots \) | \( X_1 \) | \( \ldots \) | \( \ldots \) |
|\( e_3 \) | \( \ldots \) | \( \ldots \) | \( X_2 \) | \( \ldots \) | \( \ldots \) |
|\( e_4 \) | \( \ldots \) | \( \ldots \) | \( X_2 \) | \( \ldots \) | \( \ldots \) |
|\( \ldots \) | \( \ldots \) | \( \ldots \) | \( \ldots \) | \( \ldots \) | \( \ldots \) |
|\( e_{3n-3} \) | \( e_{3n-3} \) | \( e_{3n-3} \) | \( e_{3n-3} \) | \( e_{3n-3} \) | \( e_{3n-2} \) |
|\( e_{3n-2} \) | \( e_{3n-2} \) | \( e_{3n-2} \) | \( e_{3n-2} \) | \( e_{3n-2} \) | \( e_{3n-3} \) |
|\( e_{3n-1} \) | \( e_{3n-1} \) | \( e_{3n-1} \) | \( e_{3n-1} \) | \( e_{3n-1} \) | \( e_{3n-1} \) |
|\( e_{3n} \) | \( e_{3n} \) | \( e_{3n} \) | \( e_{3n} \) | \( e_{3n} \) | \( e_{3n} \) |

Now define \( \phi(e_i) = e'_i \) for \( 1 \leq i \leq 3n - 1 \) and \( \phi(e_{3n}) = \sum_{i=1}^{3n} e'_i \). Then \( \phi \) is an isomorphism from \( k[X] \) to \( k[Y] \).

These examples give negative answers to both Questions 7.4 and 7.5 of Bardakov et al. in case \( R = \mathbb{Z}_n \) and it is not hard to see that Example 5.8 provides a negative answer to Question 7.5 of Bardakov et al. for \( \text{char}(k) = 0 \). Indeed, let \( I \subseteq X \) and \( \overline{I} \subseteq Y \) be the augmentation ideals. Then \( I^{\geq 2} = \overline{I}^{\geq 2} = (e_1 - e_2) \), hence, \( I^k/I^{k+1} = \overline{I}^k/\overline{I}^{k+1} = 0 \) for \( k \geq 2 \). Also, \( I^0/I^1 \cong I^0/I^1 \) and \( I^1/I^2 \cong I^1/I^2 \) are 1-dimensional and 2-dimensional vector spaces over \( k \).

### 5.1. Parameter spaces for quandle ring morphisms

Let \( k[X] \) and \( k[Y] \) be two quandle rings with \( |X| = |Y| = n \). Then \( \varphi = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{1n} & \ldots & a_{nn} \end{pmatrix} \) defines a map \( k[X] \to k[Y] \) if \( \varphi(e_i e_j) = \varphi(e_i) \varphi(e_j) \) for all \( i, j \in \{1, \ldots, n\} \). This in turn produces \( n^3 \) quadratic equations in the \( n^2 \)-dimensional vector space of parameters \((a_{ij})\). Furthermore, \( \varphi \) is an isomorphism if the corresponding matrix is of full rank. We provide one possible application.

**Example 5.10.** Let \( X_1, X_2 \) and \( X_3 \) be quandles of cardinality one. We show that the rings \( k[X_1] \oplus k[X_2] \oplus k[X_3] \) and \( k[X_1] \amalg X_2 \amalg X_3 \) are not isomorphic. Indeed the conditions
\( \varphi^2(e_i) = \varphi(e_i) \) and \( \varphi(e_i e_j) = \varphi(e_i) \varphi(e_j) = 0 \) give rise to the equations

\[
\begin{align*}
\begin{cases}
a_{11} &= a_{11}(a_{11} + a_{12} + a_{13}) \\
a_{12} &= a_{12}(a_{11} + a_{12} + a_{13}) \\
a_{13} &= a_{13}(a_{11} + a_{12} + a_{13}) \\
a_{21} &= a_{21}(a_{21} + a_{22} + a_{23}) \\
a_{22} &= a_{22}(a_{21} + a_{22} + a_{23}) \\
a_{23} &= a_{23}(a_{21} + a_{22} + a_{23}) \\
a_{31} &= a_{31}(a_{31} + a_{32} + a_{33}) \\
a_{32} &= a_{32}(a_{31} + a_{32} + a_{33}) \\
a_{33} &= a_{33}(a_{31} + a_{32} + a_{33}) \\
0 &= a_{11}(a_{21} + a_{22} + a_{23}) \\
0 &= a_{12}(a_{21} + a_{22} + a_{23}) \\
0 &= a_{13}(a_{21} + a_{22} + a_{23}) \\
\end{cases}
\end{align*}
\] (1)

We show that the system is already inconsistent. As \( \text{rk}(\varphi) = 3 \), to satisfy the first nine equations, we must have \( a_{11} + a_{12} + a_{13} = a_{21} + a_{22} + a_{23} = a_{31} + a_{32} + a_{33} = 1 \), however, this leaves the only possibility \( a_{11} = a_{12} = a_{13} = 0 \) for the remaining three equations, which contradicts the assumption on the rank of \( \varphi \).

6. A PROOF OF CONJECTURE 6.5 IN [3]

The goal of this section is to give a solution to conjecture 6.5 in [3] concerning the quotient of the powers of the augmentation ideal of the quandle ring of dihedral quandles.

In this Section, we confirm one of the conjectures suggested in [3] and present the progress of another conjecture. First we state the conjecture as presented in [3].

**Conjecture 6.1.** Let \( R_n \) be the dihedral quandle.

1. If \( n > 1 \) is an odd integer, then \( \Delta^k(R_n)/\Delta^{k+1}(R_n) \cong \mathbb{Z}_n \) for all \( k \geq 1 \).
2. If \( n > 2 \) is an even integer, then \( |\Delta^k(R_n)/\Delta^{k+1}(R_n)| = n \) for all \( k \geq 2 \).

We prove the first part of the above conjecture. We slightly abuse the notation and write \( e_i \) for \([e_i]\).

**Theorem 6.2.** Let \( n \) be odd. Then \( \Delta^k(R_n)/\Delta^{k+1}(R_n) \cong \mathbb{Z}_n \) for all \( k \geq 1 \).

**Proof.** We prove the result by induction on \( k \). First we show that \( \Delta(R_n)/\Delta^2(R_n) \) is generated as an abelian group by \( e_1 \) and \( \Delta(R_n)/\Delta^2(R_n) \cong \mathbb{Z}_n \). Consider the integral quandle ring of the dihedral quandle \( R_n = \{a_2, a_1, a_2, \ldots, a_{n-1}\} \). Let \( e_1 = a_1 - a_0, e_2 = a_2 - a_0, \ldots, e_{n-1} = a_{n-1} - a_0 \). Then \( \Delta(R_n) = \langle e_1, e_2, \ldots, e_{n-1} \rangle \). The abelian group \( \Delta^2(R_n) \) is generated by the products \( e_i \cdot e_j \). It is clear that \( e_{2i} = -e_{n-2i} \), where \( 1 \leq i \leq \frac{n-1}{2} \). In particular, when \( i = \frac{n-1}{2} \) we have \( e_1 = -e_{n-1} \). Since \( e_1 = -e_{n-1} \), we have the following.
$e_1 \cdot e_1 = 0$ implies $e_2 = 2e_1$, and $e_{n-1} \cdot e_1 = 0$ implies $e_3 = 3e_1$.

Continuing in this manner in the first column from bottom to top, we get $ne_1 = 0$. Hence $\Delta/\Delta^2(R_n)$ is generated as an abelian group by $e_1$, and $\Delta(R_n)/\Delta^2(R_n) \cong \mathbb{Z}_n$.

Now assume that $\Delta^k(R_n)/\Delta^{k+1}(R_n) \cong \mathbb{Z}_n$. We show that $\Delta^{k+1}(R_n)/\Delta^{k+2}(R_n) \cong \mathbb{Z}_n$.

Let $[\alpha]$ be a generator of $\Delta^k(R_n)/\Delta^{k+1}(R_n)$, i.e., $\Delta^k(R_n)/\Delta^{k+1}(R_n) = \{\alpha, 2\alpha, 3\alpha, \ldots, (n-1)\alpha, n\alpha\}$. Note that the elements $e_1\alpha$ where $1 \leq i \leq n-1$, generate $\Delta^{k+1}(R_n)/\Delta^{k+2}(R_n)$.

Since $e_k = ke_1$ for $1 \leq k \leq n - 1$, the above set is generated by $e_1\alpha$. Recall that $ne_1 = 0$.

Thus we have
\[
\Delta^{k+1}(R_n)/\Delta^{k+2}(R_n) \cong \mathbb{Z}_n.
\]

Now we present a generalization of [3, Proposition 6.3 (1)] where $n > 2$ is even. We will use the following equation (see (4) in Case 2 in appendix) in the proof of Theorem 6.3.

(2) $e_i e_1 = -e_2 - e_{n-i} + e_{n-i+2}$, for $3 \leq i \leq n - 3$.

**Theorem 6.3.** Let $n = 2k$ for some positive integer $k$. Then $\Delta(R_n)/\Delta^2(R_n) \cong \mathbb{Z} \oplus \mathbb{Z}_k$.

**Proof.** We adhere to the following strategy:

1. Show that $\Delta(R_n)/\Delta^2(R_n)$ is generated by the classes of $e_1$ and $e_2$
2. Verify that $e_{2s} = se_2$ for $2 \leq s \leq k - 1$. It follows that the abelian subgroup generated by $e_2$ is $\mathbb{Z}_k$.
3. Check that the abelian subgroup generated by $e_1$ has no torsion.

First we show that (1) holds. For this we claim that

(3) $e_1 = \begin{cases} \frac{1}{2}e_2 & \text{if } l \text{ is even}, \\ \left[\frac{1}{2}\right] e_2 + e_1 & \text{if } l \text{ is odd}. \end{cases}$

Let $l > 3$ be even. We prove by induction on $l$.

Let $l = 4$. Then we have $e_4 = 2e_2$ which is true since $0 = e_{n-2}e_1 = -2e_2 + e_4$. Assume that it is true for $l$ and consider $l + 1$. Let $i = n - l$ in (2) (consider the first column in table similarly in the case $n \equiv 0 \pmod{4}$).

Then we have
\[
e_{n-i}e_1 = -e_2 - e_1 + e_{i+2}.
\]

$e_{n-1}e_1 = 0$ implies $e_{i+2} = e_1 + e_2$. From inductive assumption we have
\[
e_{i+2} = \left(\frac{l + 2}{2}\right) e_2.
\]

Let $l$ be odd. The by Division Algorithm we have
\[
l = \left\lfloor\frac{l}{2}\right\rfloor 2 + 1,
\]

which implies $l - 1$ is even. Then we have $e_{l-1} = e_{\left\lfloor\frac{l}{2}\right\rfloor 2} = \left\lfloor\frac{l}{2}\right\rfloor e_2$.

We show that for $3 \leq l \leq n - 1$ we have
\[
e_1 = e_{l-1} + e_1
\]
by induction on $l$.
When $l = 3$, we have $e_3 = e_2 + e_1$ which is true since $0 = e_{n-1}e_1 = -e_1 - e_2 + e_3 = 0$.
Assume that it is true for $l$. When $l + 2$, let $i = n - l$ in (2) (consider the first column in table similarly in the case $n \equiv 0 \pmod{4}$).
Then we have

$$e_{n-1}e_1 = -e_2 - e_1 + e_{l+2}.$$

$e_{n-1}e_1 = 0$ implies $e_{l+2} = e_1 + e_2$. From inductive assumption we have

$$e_{l+2} = e_{l-1} + e_1 + e_2 = \left(\frac{l-1}{2}\right)e_2 + e_1 + e_2 = \left(\frac{l+1}{2}\right)e_2 + e_1 = \left\lfloor \frac{l+2}{2} \right\rfloor e_2 + e_1.$$

This completes the claim. Now consider

$$e_i e_j = (a_i - a_0)(a_j - a_0) = a_{2j-i} - a_{n-i} - a_{2j} + a_0 = -e_{2j-i} + e_{n-i} + e_{2j}.$$

Using the above claim we show that $\Delta^2(R_n)$, which consists of $e_i e_j$, for all $1 \leq i, j \leq n-1$, is generated by relations from (3). This proves that the abelian group generated by $e_1$ is torsion free.

Let $i$ be even.

$$e_i e_j = -e_{2j-i} + e_{n-i} + e_{2j}$$

$$= -\left(\frac{2j-i}{2}\right)e_2 + \left(\frac{n-i}{2}\right)e_2 + \left(\frac{2j}{2}\right)e_2$$

$$= \frac{n}{2}e_2$$

$$= ke_2$$

$$= 0.$$

Now let $i$ be odd.

$$e_i e_j = -e_{2j-i} + e_{n-i} + e_{2j}$$

$$= -\left[\frac{2j-i}{2}\right]e_2 - e_1 + \left[\frac{n-i}{2}\right]e_2 + e_1 + \left(\frac{2j}{2}\right)e_2$$

$$= -\left[\frac{2j-i}{2}\right]e_2 + \left[\frac{n-i}{2}\right]e_2 + je_2$$

$$= -\left(\frac{2j-i-1}{2}\right)e_2 + \left(\frac{n-i-1}{2}\right)e_2 + je_2$$

$$= \frac{n}{2}e_2$$

$$= ke_2$$

$$= 0.$$

Therefore we have

$$\Delta(R_n)/\Delta^2(R_n) \cong \mathbb{Z} \bigoplus \mathbb{Z}_k.$$

\[\square\]

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Here we present some patterns in multiplication tables for $\Delta(R_n)$ considering two cases: $n \equiv 0 \pmod{4}$ and $n \equiv 2 \pmod{4}$. We also give two examples to elaborate following equations.

**Case 1.** $n \equiv 0 \pmod{4}$.

Consider the integral quandle ring of the dihedral quandle $R_n = \{a_0, a_1, a_2, \ldots, a_{n-1}\}$. Let $e_1 = a_1 - a_0$, $e_2 = a_2 - a_0$, $\ldots$, $e_{n-1} = a_{n-1} - a_0$. Then $\Delta(R_n) = \langle e_1, e_2, \ldots, e_{n-1} \rangle$. To determine $\Delta^2(R_n)$, we compute the products $e_i \cdot e_j$. Then we have the following.

\[
e_{2i}e_i = -e_{2i} - e_{n-2i}, \quad \text{for } 1 \leq i \leq \frac{n}{4}
\]
\[
e_{n-2i}e_i = -2e_{2i} + e_{4i}, \quad \text{for } 1 \leq i \leq \frac{n}{4} - 1
\]
\[
e_{2i}e_{\frac{n}{2}-i} = e_{n-4i} - 2e_{n-2i}, \quad \text{for } 1 \leq i \leq \frac{n}{4} - 1
\]
\[
e_i e_{\frac{n}{2}} = -e_{n-i} - e_{\frac{n}{2}} + e_{\frac{n}{2}+n-i}, \quad \text{for } \frac{n}{2} + 1 \leq i \leq n - 1
\]
\[
e_i e_{\frac{n}{2}} = e_{\frac{n}{2}-i} - e_{\frac{n}{2}} - e_{n-i}, \quad \text{for } 1 \leq i \leq \frac{n}{2} - 1
\]
\[
e_i e_{\frac{n}{2}} = 0, \quad \text{for all } i.
\]

We give an example when $n = 8$.

| $e_i$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|-------|-------|-------|-------|-------|-------|-------|-------|
| $e_1$ | $e_1 - e_4 - e_7$ | $0$ | $-e_2 - e_6$ | $e_4 - 2e_6$ | $e_4$ |
| $e_2$ | $-e_2 - e_6$ | $e_2 - e_4 - e_6$ | $e_4 - 2e_6$ | $0$ | $0$ |
| $e_3$ | $e_1 - e_4 - e_5$ | $0$ | $0$ | $0$ | $0$ |
| $e_4$ | $0$ | $0$ | $-2e_4$ | $0$ | $0$ |
| $e_5$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $e_6$ | $0$ | $-e_2 - e_4 - e_6$ | $-e_2 - e_6$ | $0$ | $0$ |
| $e_7$ | $0$ | $-e_1 - e_4 - e_5$ | $0$ | $0$ | $0$ |

Note that $i$-th column = $(\frac{n}{2} + i)$-th column.

**Case 2.** $n \equiv 2 \pmod{4}$.

Similar computations to Case 1 yield the following.

\[
e_{2i}e_i = -e_{2i} - e_{n-2i}, \quad \text{for } 1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor
\]
\[
e_{n-2i}e_i = -2e_{2i} + e_{4i}, \quad \text{for } 1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor
\]
\[
e_{2i}e_{\frac{n}{2}-i} = e_{n-4i} - 2e_{n-2i}, \quad \text{for } 1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor
\]
\[
e_i e_{\frac{n}{2}} = 0, \quad \text{for all } i
\]
\[
e_1e_1 = e_1 - e_2 - e_{n-1}
\]
\[
e_{n-1}e_1 = -e_1 - e_2 + e_3
\]

(4) \[e_1e_1 = -e_2 - e_{n-1} + e_{n-1+2}, \quad \text{for } 3 \leq i \leq n - 3.\]

We give an example when $n = 10$. 

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**APPENDIX**

Here we present some patterns in multiplication tables for $\Delta(R_n)$ considering two cases: $n \equiv 0 \pmod{4}$ and $n \equiv 2 \pmod{4}$. We also give two examples to elaborate following equations.
RINGLE THERETIC ASPECTS OF QUANDLES

|   | \( e_1 \) | \( e_2 \) | \( e_3 \) | \( e_4 \) | \( e_5 \) | \( e_6 \) | \( e_7 \) | \( e_8 \) | \( e_9 \) |
|---|---|---|---|---|---|---|---|---|---|
| \( e_1 \) | \( e_1 - e_2 - e_9 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( e_2 \) | \( -e_2 - e_8 \) | \( e_6 - 2e_8 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( e_3 \) | \( -e_2 - e_7 + e_9 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( e_4 \) | \( -e_2 - e_6 + e_9 \) | \( -e_4 - e_6 \) | \( e_2 - 2e_6 \) | 0 | 0 | 0 | 0 | 0 | 0 |
| \( e_5 \) | \( -e_2 - e_5 + e_7 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( e_6 \) | \( -e_2 - e_4 + e_6 \) | \( -2e_4 + e_8 \) | \( -e_4 - e_6 \) | 0 | 0 | 0 | 0 | 0 | 0 |
| \( e_7 \) | \( -e_2 - e_3 + e_5 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( e_8 \) | \( -2e_2 + e_4 \) | \( -e_2 - e_4 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( e_9 \) | \( -e_1 - e_2 + e_3 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Note that \( i \)-th column = \( \left( \frac{n}{2} + i \right) \)-th column.

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