An explicit solution to a random walk game

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Abstract. We study a two player zero sum game where a stone is initially placed on an integer point \((i,j)\) with \(0 < i < m\) and \(0 < j < n\) on the Cartesian plane. At each step, player A makes the choice between +1 and −1 and player B makes the choice between x-axis and y-axis and they move the stone accordingly. Player A’s objective is to let the stone first hit the x-axis or the line \(x = m\) while player B’s objective is to let the stone first hit the y-axis or the line \(y = n\). We present an explicit formula for the Nash equilibrium of this game.

Key words. Game theory, Mixed-strategy, Trigonometry, Value.

1. Introduction

In this paper, we consider a game where a stone is placed on an integer point \((i,j)\) with \(0 < i < m\) and \(0 < j < n\) on the Cartesian plane. At each turn, the stone will move to one of its neighbouring integer point. As illustrated in Figure 1, Player B chooses in which axis it moves and player A chooses whether it moves in the +1 direction or −1 direction. Player A wins if the stone hits the blue boundary lines \(y = 0\) or \(x = m\) before it hits the red boundary lines \(x = 0\) or \(y = n\) and Player B wins otherwise. Both players want to maximise their winning probability. We assume the payoff for player A to be 1 if player A wins and -1 if player B wins. Let \(V_{i,j}\) be the expected payoff for player A of the game starting at \([i,j]\) provided both players play according to the Nash equilibrium solution. At each step, it can thus be viewed as a two player zero sum game with payoff matrix shown in Table 1. In [1], it has been shown that this type of two person

Figure 1: An illustration of the game: a stone is placed at \([2,1]\), it moves according to player A and player B’s choices and player A wants it to hit the blue lines while player B wants it to hit the red lines.
zero-sum games has a Nash equilibrium point. Intuitively, we would expect $V_{i,j-1}, V_{i+1,j} > V_{i,j} > V_{i,j+1}, V_{i-1,j}$ and therefore the players should play mixed strategies in the Nash equilibrium solution. In this paper, we investigate the Nash equilibrium solution of this game and obtain an explicit expression for the solution $V_{i,j}$.

2. Solution of the random walk game

In this section, we will present the Nash equilibrium solution of the game described in the Introduction. As discussed earlier, at each step of the game, we assume the two players to play the Nash equilibrium of the game shown in Table 1. Therefore, we can express $V_{i,j}$ as a function of $V_{i+1,j}$, $V_{i,j+1}$, $V_{i-1,j}$ and $V_{i,j-1}$. In addition, we know the boundary conditions $V_{i,0} = 1$, $V_{i,n} = -1$, $V_{0,j} = -1$, $V_{m,j} = 1$ for $0 < i < m$ and $0 < j < n$. We now present a explicit solution of this system of equations and show that it is indeed the Nash equilibrium solution of this game.

**Lemma 2.1.** For a game with payoff matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, if $\min(a,d) > \max(b,c)$, then the value of this game is $\frac{ad-bc}{a+d-b-c}$.

**Proof.** As $\min(a,d) > \max(b,c)$, we know that both players must play mixed strategies in the Nash equilibrium solution. Let $p$ be the probability that player A chooses the first row and $q$ the probability that player A chooses the second row. At equilibrium, we must have

$$p + q = 1,$$

$$ap + cq = bp + dq.$$

Solving this, we deduce

$$p = \frac{d - c}{a + d - b - c},$$

$$q = \frac{a - b}{a + d - b - c},$$

And the value of this game is therefore

$$ap + cq = \frac{a - b}{a + d - b - c} \left( \frac{d - c}{a + d - b - c} \right) + \frac{a - b}{a + d - b - c} \left( \frac{a - b}{a + d - b - c} \right)$$

$$= \frac{ad - bc}{a + d - b - c}.$$ 

So, we now expect that the Nash equilibrium solution satisfies

$$V_{i,j} = \frac{V_{i+1,j}V_{i,j-1} - V_{i,j+1}V_{i-1,j}}{V_{i+1,j} + V_{i,j-1} - V_{i-1,j} - V_{i,j+1}}.$$ 

Solving this type of systems of equations in general is usually difficult. However, we have found a particular solution of this system of equations by making an educated guess.
Theorem 2.2. Let \( z \in \mathbb{R} \) be a constant and for \( 0 \leq x, 0 \leq y, x + y < \pi \) and \( x, y \) not both zero define the function

\[
f_z(x, y) = \begin{cases} 
1 & \text{if } y = 0, \\
-1 & \text{if } x = 0, \\
\frac{\cot(y) - \cot(x) + z}{\cot(y) + \cot(x)} & \text{otherwise}.
\end{cases}
\]

Now, for \( 0 < \theta < \frac{\pi}{2} \) and \( \theta \leq x + y < \pi - \theta \), let

\[
a = f_z(x + \theta, y), \\
b = f_z(x, y + \theta), \\
c = f_z(x - \theta, y), \\
d = f_z(x, y - \theta),
\]

then

\[
\frac{ac - bd}{a + d - b - c} = f_z(x, y).
\]

Proof. This can be shown using standard trigonometric identities. Instead, we have used Mathematica and verified these results. \( \square \)

Theorem 2.3. Define \( f_z \) as in Theorem 3.2, let \( \theta = \frac{\pi}{m+n} \) let and \( z = 2\cot(m\theta) \), then \( V_{i,j} = f_z(i\theta, j\theta) \).

Proof. Let \( \tilde{V}_{i,j} = f_z(i\theta, j\theta) \). By Theorem 3.2, we have

\[
\tilde{V}_{i,j} = \frac{\tilde{V}_{i+1,j} - \tilde{V}_{i,j-1} - \tilde{V}_{i,j+1} + \tilde{V}_{i-1,j}}{\tilde{V}_{i+1,j} + \tilde{V}_{i,j-1} - \tilde{V}_{i,j} - \tilde{V}_{i,j+1}}.
\]

Now, note that for \( 0 < x, y, x < m\theta \) and \( x + y < \pi \)

\[
\frac{\partial f_z(x, y)}{\partial x} = \frac{1}{\sin^2(x)} \times \frac{2 \cot(m\theta) + 2 \cot(y)}{(\cot(y) + \cot(x))^2} > 0.
\]

So, \( f_z \) is strictly increasing in \( x \). Further, for \( 0 < i \leq m, 0 < j < n \), we have
\[ f_z(i\theta, j\theta) = \frac{\cot(j\theta) - \cot(i\theta) + z}{\cot(j\theta) + \cot(i\theta)} = -1 + \frac{2\cot(m\theta) + 2\cot(j\theta)}{\cot(i\theta) + \cot(j\theta)} > -1, \]

as \( i\theta + j\theta < \pi \) and \( m\theta + j\theta < \pi \). Therefore, we have \( f_z(i\theta, j\theta) > f_z((i−1)\theta, j\theta) \). Similarly, we have

\[
\frac{\partial f_z(x, y)}{\partial y} = \frac{1}{\sin^2(y)} \times \frac{2\cot(m\theta) - 2\cot(x)}{(\cot(y) + \cot(x))^2} < 0.
\]

And, for \( 0 < i < m, 0 < j \leq n \), we have

\[
f_z(i\theta, j\theta) = \frac{\cot(j\theta) - \cot(i\theta) + z}{\cot(j\theta) + \cot(i\theta)} = 1 + \frac{2\cot(m\theta) - 2\cot(i\theta)}{\cot(i\theta) + \cot(j\theta)} < 1.
\]

Hence, we have \( f_z(i\theta, j\theta) < f_z(i\theta, (j - 1)\theta) \). Recall \( \theta = \frac{\pi}{m+n} \) when \( j = n, 0 < i < m \), we have

\[
f_z(i\theta, j\theta) = \frac{\cot(n\theta) - \cot(i\theta) + 2\cot(m\theta)}{\cot(n\theta) + \cot(i\theta)} = \frac{\cot(n\theta) - \cot(i\theta) - 2\cot(n\theta)}{\cot(n\theta) + \cot(i\theta)} = -\cot(i\theta) - \cot(n\theta) < 0
\]

and when \( i = m, 0 < j < n \)

\[
f_z(i\theta, j\theta) = \frac{\cot(j\theta) - \cot(m\theta) + 2\cot(m\theta)}{\cot(j\theta) + \cot(m\theta)} = \frac{\cot(j\theta) + \cot(m\theta)}{\cot(j\theta) + \cot(m\theta)} = 1.
\]

So, all the boundary conditions are satisfied by \( \tilde{V}_{i,j} \). Moreover, for \( 0 < i < m, 0 < j < n \),

\[
\min(V^{−i+1,j},V^{−i,j−1}) > \max(V^{−i−1,j},V^{−i,j+1}).
\]

So, the Nash equilibrium of the game with payoff matrix

\[
\begin{pmatrix}
V_{i+1,j} & \tilde{V}_{i,j+1} \\
V_{i−1,j} & \tilde{V}_{i,j−1}
\end{pmatrix}
\]

has mixed strategies and the value of this game is \( \tilde{V}_{i,j} \). Let \( (it,jt) \) be the position of the stone at step \( t \) of the game. Now, suppose player A plays the Nash equilibrium strategy of the payoff matrix \( \begin{pmatrix} V_{it+1,jt} & \tilde{V}_{it,jt+1} \\ V_{it−1,jt} & \tilde{V}_{it,jt−1} \end{pmatrix} \) at each step, then no matter what player B does, \( \tilde{V}_{it,jt} \) will be
a martingale. So, suppose the stone starts at position i,j. By optional stopping theorem, player A can guarantee an expected payoff $\tilde{V}_{i,j}$. Similarly, if player B plays the Nash equilibrium strategy of the payoff matrix $\begin{pmatrix} V_{i+1,j} & V_{i+1,j+1} \\ V_{i-1,j} & V_{i,j-1} \end{pmatrix}$, then according to [2], no matter what player A does, $\tilde{V}_{i,j}$ will be a martingale and thus player B can guarantee that the expected payoff for player A in this game is $\tilde{V}_{i,j}$. So, $V_{i,j} = \tilde{V}_{i,j}$ is the expected payoff of player A in the Nash equilibrium solution.

We now look at some properties of this solution. We now write $V_{m,n}^{i,j}$ be the solution of the expected payoff for player A if the stone starts at i,j on a m × n board.

**Corollary 2.4.** Let $s > 0$ be an integer, then

$$V_{i,j}^{m,s,n} = V_{i,j}^{m,n}.$$ 

Proof. By Theorem 3.3,

$$V_{i,j}^{m,s,n} = f_2 \cot \left( \frac{\pi}{m+n} \right) \left( \frac{is \pi}{ms + ns} \right) \left( \frac{js \pi}{ms + ns} \right)$$

$$= f_2 \cot \left( \frac{\pi}{m+n} \right) \left( \frac{1}{m+n} \right)$$

$$= V_{i,j}^{m,n},$$

as desired.

**Corollary 2.5.**

$$\lim_{s \to \infty} V_{i,j}^{m,s,n} = \frac{i-j}{i+j}.$$ 

Proof. The result is trivial when i = 0 or j = 0. When i,j ≠ 0, we have

$$\lim_{s \to \infty} V_{i,j}^{m,s,n} = \lim_{s \to \infty} \frac{\cot \left( \frac{\pi}{ms + ns} \right) - \cot \left( \frac{\pi}{ms + ns} \right)}{\cot \left( \frac{\pi}{ms + ns} \right) + \cot \left( \frac{\pi}{ms + ns} \right)}$$

$$= \lim_{s \to \infty} \frac{\cot \left( \frac{\pi}{ms + ns} \right) - \cot \left( \frac{\pi}{ms + ns} \right)}{\cot \left( \frac{\pi}{ms + ns} \right) + \cot \left( \frac{\pi}{ms + ns} \right)}$$

$$= \lim_{s \to \infty} \frac{\sin \left( \frac{\pi}{ms + ns} \right) \sin \left( \frac{\pi}{ms + ns} \right) - \cos \left( \frac{\pi}{ms + ns} \right) \sin \left( \frac{\pi}{ms + ns} \right)}{\cos \left( \frac{\pi}{ms + ns} \right) \sin \left( \frac{\pi}{ms + ns} \right) + \cos \left( \frac{\pi}{ms + ns} \right) \sin \left( \frac{\pi}{ms + ns} \right)}$$

$$= \frac{i-j}{i+j} \quad \text{(because} \sin(x) \sim x \text{ when} x \text{ tends to zero.)}$$

Now, we simplify the solution in the case of a square board.

**Corollary 2.6.** In the case when $m = n$, $V_{i,j}^{m,n} = \frac{\sin \left( \frac{\pi i}{2m} \right)}{\sin \left( \frac{\pi j}{2m} \right)}$. 

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Proof. Apply Theorem 3.3, \( z \) will equal to zero in this case and we have

\[
V_{i,j}^{m,n} = \frac{\cot(\frac{j\pi}{2m}) - \cot(\frac{i\pi}{2m})}{\cot(\frac{j\pi}{2m}) + \cot(\frac{i\pi}{2m})} = \frac{\cos(\frac{j\pi}{2m})\sin(\frac{i\pi}{2m}) - \sin(\frac{j\pi}{2m})\cos(\frac{i\pi}{2m})}{\cos(\frac{j\pi}{2m})\sin(\frac{i\pi}{2m}) + \sin(\frac{j\pi}{2m})\cos(\frac{i\pi}{2m})} = \frac{\sin(\frac{i-j}{2m})}{\sin(\frac{i+j}{2m})} \]

Intuitively, when the starting point is in the midpoint of the board, the expected payoff of the game will be zero. We show this result in the following corollary.

**Corollary 2.7.** When \( m = 2i \) and \( n = 2j \), then

\[
V_{i,j}^{m,n} = 0.
\]

**Proof.** Using the notations in Theorem 3.3, denote \( x = \frac{i\theta}{m+n}, \ y = \frac{j\theta}{m+n} \). Recall \( \theta = \frac{m+n}{m+n}, \ z = 2\cot(m\theta) \) we have \( \theta + x = \frac{\pi}{2} \) and \( z = 2\cot(2x) \). Hence,

\[
V_{i,j}^{m,n} = \frac{\cot(y) - \cot(x) + z}{\cot(x) + \cot(y)} = \frac{\tan(x) - \cot(x) + 2\cot(2x)}{\cot(x) + \tan(x)} = \frac{\tan(x) - \frac{1}{\tan(x)} + \frac{2}{\tan(2x)}}{\cot(x) + \tan(x)} = \frac{\tan(x) - \frac{1}{\tan(x)} + \frac{1 - \tan^2(x)}{\tan(x)}}{\cot(x) + \tan(x)} = 0,
\]

as desired. \( \square \)

### 3. Conclusion

As a result, the given solution can be used to show the Nash equilibrium solution of the game. We have shown this with the examples as written above. We have also shown that the given solution satisfies the boundary conditions. Therefore, we have proven that the given solution is an explicit solution of the game.

### 4. Appendix

We have discussed that when \( m = n \), \( V_{i,j}^{m,n} = \frac{\sin(\frac{i-j}{2m})}{\sin(\frac{i+j}{2m})} \). In this section, we will present the proof for this statement.

**Proof.** Let \( V_{i,j}^{m,n} = \frac{\sin(\frac{i-j}{2m})}{\sin(\frac{i+j}{2m})} \). By applying Theorem 3.2, we have \( \theta = \frac{\pi}{2m} \) and
Now, for $0 < \theta < \frac{\pi}{2}$ and $\theta \leq x + y < \pi - \theta$, let $(i - j)\theta$ be A and $(i + j)\theta$ be B, we then have

$$a = \frac{\sin(A + \theta)}{\sin(B + \theta)},$$

$$b = \frac{\sin(A - \theta)}{\sin(B + \theta)},$$

$$c = \frac{\sin(A - \theta)}{\sin(B - \theta)},$$

$$d = \frac{\sin(A + \theta)}{\sin(B - \theta)},$$

$$ad - bc = \frac{\sin^2(A + \theta)}{\sin(B + \theta)\sin(B - \theta)} - \frac{\sin^2(A - \theta)}{\sin(B + \theta)\sin(B - \theta)}$$

$$= \frac{\sin^2(A + \theta) - \sin^2(A - \theta)}{\sin(B + \theta)\sin(B - \theta)}$$

$$a + d - b - c = \frac{\sin(A + \theta)\sin(B - \theta) + \sin(A + \theta)\sin(B + \theta) - \sin(A - \theta)\sin(B + \theta) - \sin(A - \theta)\sin(B - \theta)}{\sin(B + \theta)\sin(B - \theta)}$$

$$= \frac{\sin(A + \theta)(\sin(B - \theta) + \sin(B + \theta)) - \sin(A - \theta)(\sin(B + \theta) + \sin(B - \theta))}{\sin(B + \theta)\sin(B - \theta)}$$

We substitute these results to

$$\bar{V}_{i,j} = \frac{ad - bc}{a + d - b - c}$$

and obtain

$$\bar{V}_{i,j} = \frac{\sin^2(A + \theta) - \sin^2(A - \theta)}{\sin(A + \theta)\sin(B - \theta) + \sin(B + \theta) - \sin(A - \theta)\sin(B + \theta) - \sin(A - \theta)\sin(B - \theta)}$$

$$= \frac{\sin(A + \theta) + \sin(A - \theta)}{\sin(B - \theta) + \sin(B + \theta)}$$

$$= \frac{2\sin(A)\cos(\theta)}{2\sin(B)\cos(\theta)}$$

$$= \frac{\sin((i - j)\theta)}{\sin((i + j)\theta)}$$
as desired.

5. References
[1] J. von Neumann and O. Morgenstern Theory of Games and Economic Behavior. Princeton University Press

[2] P.Hall/C.C Heyde Martingale Limit Theorem and Its Application. Academic Press

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