Eisenstein series of weight one, \( q \)-averages of the 0-logarithm and periods of elliptic curves

by

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To Kumar Murty: Hyapi aravai

Abstract

For any elliptic curve \( E \) over \( k \subset \mathbb{R} \) with \( E(\mathbb{C}) = \mathbb{C}^\times/q^\mathbb{Z} \), \( q = e^{2\pi i z}, \text{Im}(z) > 0 \), we study the \( q \)-average \( D_{0,q} \), defined on \( E(\mathbb{C}) \), of the function \( D_0(z) = \text{Im}(z/(1 - z)) \). Let \( \Omega^+(E) \) denote the real period of \( E \). We show that there is a rational function \( R \in \mathbb{Q}(X_1(N)) \) such that for any non-cuspidal real point \( s \in X_1(N) \) (which defines an elliptic curve \( E(s) \) over \( \mathbb{R} \) together with a point \( P(s) \) of order \( N \)), \( \pi D_{0,q}(P(s)) \) equals \( \Omega^+(E(s)) R(s) \). In particular, if \( s \) is \( \mathbb{Q} \)-rational point of \( X_1(N) \), a rare occurrence according to Mazur, \( R(s) \) is a rational number.

1. Introduction

The relationship between modular forms of weight one and periods of elliptic curves is well-known, certainly to experts in the field. In this paper we study in some detail those modular forms of weight one which arise from \( q \)-averages of the 0-logarithm function, \( \ell_0(z) = z/(1 - z) \). We are led to consider these rather special forms to point out the analogy with the situation in algebraic K-theory, where the other polylogarithm functions have already played an important role in connection with special values of \( L \)-functions.

Let \( E \) be an elliptic curve over \( \mathbb{Q} \) and let \( P_0 \in E(\mathbb{Q}) \) be a rational point of order \( N > 1 \). Writing \( E(\mathbb{C}) = \mathbb{C}/L \) for some complex lattice \( L = \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \tau \), with \( \text{Im} \tau > 0 \), we let \( q = e^{2\pi i \tau} \); the exponential map \( u \mapsto e^{2\pi i u} \) gives an isomorphism \( \Phi : E(\mathbb{C}) \cong \mathbb{C}^\times/q^\mathbb{Z} \). Let \( z_0 \) be a representative for the coset \( \Phi(P_0) \), so that \( \Phi(P_0) = z_0 \cdot q^\mathbb{Z} \).

For \( k \geq 0 \) define the polylogarithm functions,

\[
\ell_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k},
\]

these functions can be regarded as single-valued functions on the disk \( \{ z : |z| < 1 \} \), or they can be regarded, by analytic continuation, as multi-valued functions on \( \{ z : z \neq 1, z \neq 0 \} \). The Bloch-Wigner dilogarithm function

\[
D_2(z) = \log |z| \arg(1 - z) + \text{Im} \ell_2(z) = \log |z| \arg(1 - z) - \text{Im} \int_0^z \log(1 - t) \frac{dt}{t}
\]

is a single-valued real function of the complex variable \( z \). Summing over a coset of \( q^\mathbb{Z} \) yields \( D_{2,q}(z) = \sum_{n \in \mathbb{Z}} D_2(zq^n) \), which is a real function on \( E(\mathbb{C}) \). Its value at \( P_0 \) plays an important role in conjectures of Spencer Bloch about values of the \( L \)-series \( L(E, s) \) at \( s = 2 \) in connection with values of the higher regulator map \( K_2 E \to \mathbb{R} \) from algebraic K-theory ([4], [2]) and certain Eisenstein-Kronecker-Lerch series ([26]). In particular, it often happens that

\[
\frac{\pi D_{2,q}(z_0)}{L(E, 2)}
\]

is a rational number to great numerical accuracy.

In this paper we explore the analogous situation at \( s = 1 \).

To do this, we replace the dilogarithm by the first member of the sequence of polylogarithm functions, namely, the 0-logarithm. We let

\[
D_0(z) = \text{Im} \ell_0(z) = \text{Im} (z/(1 - z)) = \text{Im} (1/(1 - z))
\]

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for \( z \neq 1 \), and let \( D_0(1) = 0 \). Summing over a coset gives a function

\[
D_{0,q}(z) = \sum_{n \in \mathbb{Z}} D_0(zq^n)
\]

which amounts to a real function on \( E(\mathbb{C}) \). Something similar works for the higher polylogarithm functions \( \ell_m(z) \), but other terms must be added to arrive at a function which is single-valued and real-analytic on the complex plane, generalizing the \( m = 2 \) case of [3], cf. [21], [28], where the latter presents Zagier’s conjecture relating to the values of Dedekind zeta functions of arbitrary number fields \( F \) at positive integers \( m \geq 2 \) and \( K_{2m-1}(F) \).

The conjecture of Birch and Swinnerton-Dyer [24, p. 362] states, among other things, that the coefficient of the leading term in the Taylor series of \( L(E,s) \) at \( s = 1 \) is a rational number times the product of the real period of \( E \) and the determinant of the height pairing on the Mordell-Weil group \( E(\mathbb{Q}) \), with the rank of the Mordell-Weil group and the order of vanishing of \( L(E,s) \) at \( s = 1 \) predicted to be the same. In particular, \( L(E,1) \) should always be a rational multiple (possibly zero) of the real period of \( E \), which has long been known in the case where \( E \) is a modular elliptic curve (via the theory of modular symbols and the Manin–Drinfeld theorem [9]). Now every elliptic curve over \( \mathbb{Q} \) is known to be modular by Wiles, Taylor, Diamond, Conrad and Bruehl ([5], [25], [27]). Thus, in the search for an analogue of the rationality of (1.1), we may as well use the real period instead of \( L(E,1) \); this even simplifies the analysis, for ignoring the arithmetic data that underlies \( L(E,1) \) allows us to study the variation as \( E \) moves in a family. (It is not important to what follows, but it may be worthwhile to remark that by Kolyvagin [16], \( L(E) \) is finite when \( L(E,1) = 0 \), and one of the ingredients of his theorem was a non-vanishing hypothesis, established later by two sets of authors, one of them being Kumar and Ram Murty, and the other being Bump, Friedberg and Hoffstein.)

We write the equation for \( E \) as \( y^2 = f(x) \) with \( f \) a monic polynomial of degree 3. Let \( \omega = dx/(2y) \) be the standard holomorphic differential on \( E(\mathbb{C}) \); it depends on the choice of equation. Set the real period to be

\[
\Omega^+ = \int_{E(\mathbb{R})} \omega = \int_{\gamma} \frac{dx}{\sqrt{f(x)}},
\]

where \( \gamma \) is the largest real root of \( f(x) \).

We introduce the group

\[
\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N \right\}.
\]

Let \( X_1(N) \) denote the modular curve corresponding to the group \( \Gamma_1(N) \). It is a projective curve defined over \( \mathbb{Q} \), so that \( X_1(N) \otimes \mathbb{C} \) contains the quotient \( \Gamma_1(N) \backslash H \) as an affine open subset. For each \( s \in X_1(N) \), not a cusp, there is a corresponding elliptic curve \( E(s) \) equipped with a point \( P(s) \) of order \( N \); the curve and point are both defined over \( \mathbb{Q}(s) \). It is possible to pick a differential \( \omega(s) \neq 0 \) on \( E(s) \) that varies algebraically in \( s \); this enables us to interpret the various quantities above, such as \( q \) and \( \Omega^+ \), as functions of \( s \).

**Theorem 1.2.** Let \( N > 0 \). There is a rational function \( R \in \mathbb{Q}(X_1(N)) \) such that for each real point \( s \in X_1(N) \), not a cusp, we have

\[
R(s) = 2\pi D_{0,q(s)}(P(s))/\Omega^+(s).
\]

In particular, if \( s \) is a rational point, then the quantity \( R(s) \) is a rational number.

This theorem was suggested by two sequences of numerical experiments. Our first sequence of experiments showed that

\[
\frac{\pi D_{0,q}(P_0)}{\Omega^+}
\]

appears to be a rational number of small height to great accuracy, for example, 100 digits.

Our second sequence of numerical experiments was this. The possible numbers \( N \) that occur are \( 1 \leq N \leq 10 \) or \( N = 12 \) (according to a theorem of Mazur, [19]), and for each of these there is a one-parameter family (defined over \( \mathbb{Q} \)) of pairs \((E_t,P_t)\) where \( P_t \) is a point of order \( N \) on the elliptic curve \( E_t \). We found that the ratio above is a simple function of the modular parameter \( t \in \mathbb{C} \). The results are summarized in
Table 1.

| N | k | \(2NR = 4N\pi D_{0,q}(kP)/\Omega^+\) |
|---|---|---------------------------------|
| 3 | 1 | \(-a_1\) (with \(a_1 = a_3\)) |
| 4 | 1 | \(-2\) |
| 5 | 1 | \(c - 3\) |
|   | 2 | \(-3c - 1\) |
| 6 | 1 | \(-4\) |
| 7 | 1 | \(d^2 - 3d - 3\) |
|   | 2 | \(-5d^2 + d + 1\) |
|   | 3 | \(3d^2 - 9d + 5\) |
| 8 | 1 | \((-8d + 2)/d\) |
|   | 3 | \((-8d + 6)/d\) |
| 9 | 1 | \(f^3 - 3f^2 - 5\) |
|   | 2 | \(-7f^3 + 3f^2 - 1\) |
|   | 4 | \(-5f^3 + 15f^2 - 18f + 7\) |
| 10 | 1 | \(-4f - 4\) |
|   | 3 | \(-12f + 8\) |
| 12 | 1 | \((24\tau^3 - 48\tau^2 + 36\tau - 10)/(1 - \tau)^3\) |
|   | 5 | \((24\tau^3 - 24\tau^2 + 12\tau - 2)/(1 - \tau)^3\) |

Table 1, which presents, for each case, a rational function which is a good fit to numerical results with many digits of accuracy, for a finite number of values of \(t\). Refer to [17, p.217], for the parametrizations used; we use Kubert’s names for the parameters. In each case, the family of elliptic curves is represented by an equation of the form \(y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6\); we choose \(w = dy/(2y + a_1x + a_3)\), as is customary.

We expected that the ratio \(R\) would be a rational function of the parameter, i.e., would be a modular function. We were surprised that its coefficients were always rational, but we shouldn’t have been, for it turns out that the well-known Hecke-Eisenstein modular forms of weight one have sufficient rationality properties to explain the observations. These forms are presented in Lang’s book [18] in chapter XV, section 1, in a concise form. The best explanation of their rationality properties is presented by Nicholas Katz in [14, Appendix C], where he presents a purely algebraic construction of the Hecke-Eisenstein modular forms; we hadn’t been aware of his work during our initial investigations. His construction shows the forms are defined over the rational numbers, and are computable exactly and algebraically, so we were able to verify that the entries in Table 1 are correct by means of a second computer program.

In this paper we explain how to deduce the rationality of the ratios \(R\) presented above from the rationality of the Hecke-Eisenstein forms. This is done by using \(q\)-expansions to identify our ratio with the modular form presented in Katz’s paper. This identification is easy in the case where \(P_0\) is on the identity component of \(E(\mathbb{R})\), but harder otherwise, essentially because Katz provides the \(q\)-expansion just at certain cusps. We then explain the relationship with the Hecke-Eisenstein forms as presented in Lang’s book. We present an exposition of the \(q\)-expansion principle which is sufficient to deduce the desired rationality of the ratios \(R\) directly from the rationality of the \(q\)-expansion coefficients, allowing one to bypass, if desired, the elegant algebraic construction of Katz.

We end with a general question. Let \(M\) be a motive over \(\mathbb{Q}\), of weight \(k \geq 1\), defined by an irreducible direct factor of the cohomology of an abelian variety \(A\). If \(r\) is an integer with \(k/2 \leq r \leq k + 1\), critical or not, is there an analog of the function \(D_{0,q}\) (or \(D_{2,q}\)) on \(A(\mathbb{C})\), whose values at torsion points have (at times)
rational relations with $L(M,k)$? For $M = \text{Sym}^2(E)$, with $E$ an elliptic curve over $\mathbb{Q}$, $k=2$, and $r=3$, one has such a phenomenon; see the papers [8] and [20].

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2. Interpreting $D_{0,q}$ as a modular form of weight one for $\Gamma_1(N)$

We use the same notation as in the introduction. Since $E$ is real, we may choose the isomorphism $E \cong \mathbb{C}/L$ in such a way that the real structure on $E$ corresponds to complex conjugation on $\mathbb{C}$. Then since $L$ is invariant under complex conjugation we may choose $\tau$ so that $\Re \tau$ is either 0 or 1/2; in the first case, $E(\mathbb{R})$ has two components and $q > 0$, and in the second case, $E(\mathbb{R})$ has one component and $q < 0$. In both cases $|q| < 1$. Since $P_0$ is a point of order $N$, we may write $z_0$ in the form $z_0 = \zeta_N^k q^{1/N}$ where $\zeta_N = \exp(2\pi i/N)$. Since $P_0$ is real, we may pick $z_0$ so that either

case A) $z_0 = \zeta_N^k$

or

case B) $z_0 = \zeta_N^k q^{1/2}$.

Case B occurs when $2 \mid N$, $q > 0$, and $P_0$ is not on the identity component of $E(\mathbb{R})$.

We occupy ourselves with case A first. Making use of the easy identities $D_0(\bar{z}) = -D_0(z) = D_0(1/z)$ and remembering that $q$ is real we compute

$$
\frac{1}{i} D_{0,q}(\zeta_N^k) = \frac{1}{i} \sum_{j \in \mathbb{Z}} D_0(\zeta_N^k q^j) = \frac{1}{i} D_0(\zeta_N^k) + \frac{2}{i} \sum_{j=1}^{\infty} D_0(\zeta_N^k q^j) = \frac{1}{i} D_0(\zeta_N^k) + \frac{2}{i} \Im \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} (\zeta_N^k q^j)^m
$$

(2.1)

$$
= \frac{1}{i} D_0(\zeta_N^k) + \frac{2}{i} \Im \sum_{n=1}^{\infty} \left( \sum_{m|n \ m > 0} \zeta_N^{km} \right) q^n = \frac{1}{i} D_0(\zeta_N^k) - \sum_{n=1}^{\infty} \left( \sum_{m|n \ m > 0} \zeta_N^{km} - \zeta_N^{-km} \right) q^n
$$

The double sums above are absolutely convergent because $|q| < 1$, and

$$
\sum_{j=1}^{\infty} \sum_{m=1}^{\infty} |q|^{jm} \leq \frac{|q|}{(1-|q|)^2}.
$$

We set

$$
g_k(\tau) = \frac{1}{i} D_0(\zeta_N^k) - \sum_{n=1}^{\infty} \left( \sum_{m|n \ m > 0} \zeta_N^{km} - \zeta_N^{-km} \right) q^n
$$

so that $g_k(\tau) = (1/i) D_{0,q}(\zeta_N^k)$ when $q$ is real. We will use this $q$-expansion for values of $q$ which are not necessarily real, as $D_{0,q}(z_0)$ is inappropriate when $q$ is not real.
Now we consider case B. We have

\[
\frac{1}{i} D_{0,q}(\zeta_N^k q^{1/2}) = \frac{1}{i} D_{0,q}(\zeta_N^k q^{\frac{k}{2}}) = \frac{1}{i} \sum_{j \in \mathbb{Z}} D_0(\zeta_N^k q^{\frac{k}{2}j})
\]

\[
= \frac{2}{i} \sum_{j=1}^{\infty} D_0(\zeta_N^k q^{-\frac{j}{2}}) = \frac{2}{i} \sum_{j=1}^{\infty} \text{Im} \sum_{m=1}^{\infty} \zeta_N^{km} q^{(j-\frac{1}{2})m}
\]

\[
= \frac{2}{i} \sum_{n=1}^{\infty} \left( \sum_{\substack{m|n \\text{odd} \\ \ \ m > 0}} \text{Im} \zeta_N^{km} \right) q^{n/2}
\]

(2.2)

\[
= \sum_{n=1}^{\infty} \left( \frac{2}{i} \sum_{\substack{m|n \\text{odd} \\ \ m > 0}} \left( \text{Im} \zeta_N^{kn/m} \right) \right) q^{n/2}
\]

\[
= \sum_{n=1}^{\infty} \left( - \sum_{\substack{m|n \\text{odd} \\ \ m > 0}} \left( \zeta_N^{kn/m} - \zeta_N^{-kn/m} \right) \right) q^{n/2}.
\]

We set

\[
h_k(\tau) = \sum_{n=1}^{\infty} \left( - \sum_{\substack{m|n \\text{odd} \\ \ m > 0}} \left( \zeta_N^{kn/m} - \zeta_N^{-kn/m} \right) \right) q^{n/2}
\]

so that \( h_k(\tau) = (1/i) D_{0,q}(\zeta_N^k q^{1/2}) \) when \( q \) is real.

Let \( \mathcal{L}_N \) denote the set of all lattices \( L \) in \( \mathbb{C} \). Let \( \mathcal{N} \) denote the set of all pairs \( (L, u) \) where \( L \) is a lattice in \( \mathbb{C} \) and \( u \) is an element of \( N^{-1}L \) of order \( N \) in \( N^{-1}L/L \) (see [18, p. 101]). Consider a function \( F : \mathcal{L}_N \to \mathbb{C} \) satisfying the identity \( F(\alpha L, \alpha u) = \alpha^{-\ell} F(L, u) \) for all \( \alpha \in \mathbb{C} \). Call such a function homogeneous of degree \( -\ell \). Since \( k \) is relatively prime to \( N \), we have a map \( \phi_k : \mathbb{H} \to \mathcal{L}_N \) from the upper half plane \( \mathbb{H} \), defined by

\[
\phi_k(\tau) = (Z \cdot 2\pi i + Z \cdot 2\pi i \tau, 2\pi k/N).
\]

Notice the factor of \( 2\pi i \) used here. The composite \( f_k = F \circ \phi_k \) is a modular form of weight \( \ell \) for the group \( \Gamma_1(N) \) if it is meromorphic on the upper half plane and at the cusps; the main import of being a modular form is the identity

\[
f_k(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^\ell f_k(\tau)
\]

for all \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N) \), which follows from the homogeneity of \( F \).

For \( f_k \) to be meromorphic at the cusps means that

\[
(c\tau + d)^{-\ell} f_k \left( \frac{a\tau + b}{c\tau + d} \right)
\]

is a meromorphic function of \( e^{2\pi i \tau/N} \), for each \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \) (this condition is formulated incorrectly in [18, p. 103]). This condition is independent of \( k \), and in terms of \( F \) it means that

\[
F \left( \mathbb{Z} \cdot 2\pi i + \mathbb{Z} \cdot 2\pi i \tau, 2\pi i \left( \frac{c\tau + d}{N} \right) \right)
\]

is a meromorphic function of \( e^{2\pi i \tau/N} \) for each pair of relatively prime integers \( c, d \).
The field of definition of a modular form is properly understood as described in [14]. Given a scheme $S$ on which the integer $N$ is invertible, we let $E_N(S)$ denote the set of triples $(E, \omega, P)$, where $E$ is a family of elliptic curves over $S$, $\omega$ is a holomorphic differential form on $E$ relative to $S$ which is nonzero on each fiber, and $P$ is a section of $E$ over $S$ which has order $N$. There is a bijection $L \rightarrow E_N(C)$ which sends $(L, u)$ to $(C/L, dz, u + L)$, where $z$ is the coordinate function on $C$.

If $R$ is a ring, then a holomorphic modular form for $\Gamma_1(N)$ of weight $\ell$ over $R$ is a collection of maps $F : E_N(S) \rightarrow H^0(S, O_S)$ defined for any $R$-scheme $S$, which is natural in $S$, and which is homogeneous of weight $\ell$ in the sense that $F(E, \alpha \omega, P) = \alpha^{-\ell}F(E, \omega, P)$ for each $\alpha \in H^0(S, O_S)$. When $R$ is a field, then we say that $R$ is a field of definition of $F$.

When $C$ is an $R$-algebra, we will make use of the bijection mentioned previously implicitly, regarding such an $F$ also as a function $F : E_N(C) \rightarrow C$, and writing $F(C/L, dz, u + L) = F(L, u)$.

The $q$-expansions at the cusps are obtained by choosing $k$ and $\ell$ so $\gcd(k, \ell, N) = 1$ and expanding

$$F\left(\mathbb{Z} \cdot 2\pi i + \mathbb{Z} \cdot 2\pi i\tau, 2\pi i\left(\frac{k + \ell \tau}{N}\right)\right)$$

as a Laurent series in $q^{d/N}$, where $d = \gcd(\ell, N)$.

In [14, p. 260] is described a modular form, $A_1$, of weight 1. Some of its $q$-expansions are given in [14, (2.7.10)] as follows.

$$A_1(\mathbb{Z} \cdot 2\pi i + \mathbb{Z} \cdot 2\pi i\tau, 2\pi i k/N) = \frac{1}{2i} \cot(\pi k/N) - \sum_{n=1}^{\infty} \left(\sum_{m|n \text{ and } m > 0} (\zeta_N^m - \zeta_N^{-m})\right) q^n$$

One sees easily that

$$\frac{1}{i} D_0(\zeta_N^k) = \frac{1}{i} \text{Im} \left(\frac{\zeta_N^k}{1 - \zeta_N^k}\right) = \frac{1}{2} \left(\frac{\zeta_N^k + 1}{\zeta_N^k - 1}\right) = \frac{1}{2i} \cot(\pi k/N).$$

Thus from the identity of the $q$-expansions (2.1) and (2.3), we see that for case A

$$g_k(\tau) = A_1(\mathbb{Z} \cdot 2\pi i + \mathbb{Z} \cdot 2\pi i\tau, 2\pi i k/N).$$

In case B, we expect the analogous identity

$$h_k(\tau) = A_1(\mathbb{Z} \cdot 2\pi i + \mathbb{Z} \cdot 2\pi i\tau, 2\pi i\left(\frac{k}{N} + \frac{1}{\tau}\right)).$$

to hold (it doesn’t seem to follow immediately from [14, 2.7.8]); proving it amounts to computing the $q$-expansion for $A_1$ at cusps other than the ones Katz considered, and we do this later in (3.6).

In [14, Appendix C] is an amazing algebraic construction of $A_1$ which shows that it is defined over $\mathbb{Z}[\frac{1}{N}]$. We will use this result to get the rationality.

Let $E$ be an elliptic curve, as in the introduction, defined over a subfield $K$ of $C$, with a point $P_0$ of order $N$ and a differential $\omega = dx/(2y)$. We can find a lattice $L$, a number $u_0 \in \mathbb{C}$, and an isomorphism $(E, \omega, P_0) \cong (C/L, dz, u_0 + L)$. Since $A_1$ is defined over $\mathbb{Q}$, we see that $r := A_1(L, u_0) \in K$. Now write $L = \mathbb{Z} \cdot v_1 + \mathbb{Z} \cdot v_2$ with $\text{Im}(v_2/v_1) > 0$, and let $\tau := v_2/v_1$. Thus

$$\frac{2\pi i}{v_1} A_1(\mathbb{Z} \cdot 2\pi i + \mathbb{Z} \cdot 2\pi i\tau, 2\pi i u_0/v_1) = \frac{1}{v_1} A_1(\mathbb{Z} \cdot \tau, u_0/v_1) = A_1(\mathbb{Z} \cdot v_1 + \mathbb{Z} \cdot u_0) = A_1(L, u_0) = r \in K.$$

Now suppose we are in case A, so that $K \subset \mathbb{R}$, we take $\Omega^+ = v_1$ to be real and positive, and we pick $k \in \mathbb{Z}$ so $k/N = u_0/v_1$. Then from (2.4) we deduce:

$$\frac{2\pi i D_{0,q}(\zeta_N^k)}{\Omega^+} = \left(\frac{1}{i} D_{0,q}(\zeta_N^k)\right) / \left(\frac{1}{2\pi i} \Omega^+\right) = (g_k(\tau)) / \left(\frac{1}{2\pi i} \Omega^+\right) = \frac{2\pi i}{v_1} A_1(\mathbb{Z} \cdot 2\pi i + \mathbb{Z} \cdot 2\pi i\tau, 2\pi i k/N) = r \in K.$$
This shows the desired rationality of the values of $R$ presented in the introduction. In case B, we have $\frac{k}{N} + \frac{1}{2} \tau = u_0/v_1$, so by (2.5) we have:

$$
\frac{2\pi D_{0,q}(\zeta_N q^{1/2})}{\Omega^+} = \left( \frac{1}{4} D_{0,q}(\zeta_N q^{1/2}) \right) / \left( \frac{1}{2\pi i} \Omega^+ \right)
= h_k(\tau) / \left( \frac{1}{2\pi i} \Omega^+ \right)
= \frac{2\pi i}{v_1} A_1(\zeta \cdot 2\pi i + \zeta \cdot 2\pi i \tau, 2\pi i (k/N + \tau/2)) = r \in K,
$$

yielding the desired rationality in this case.

The sense in which $R$ is a rational function of the parameter is this. Each of the families of elliptic curves in Table 1 can each be viewed as an element $(E, \omega, P_0) \in \mathcal{E}_N(S)$, where $S$ is an open subset of $\mathbb{P}_Q^1$. Thus $R = A_1(E, \omega, P_0) \in \mathbb{Q}(\mathbb{P}^1)$ is a rational function. Alternatively, for $N \geq 3$, one may take $S = X_1(N) - \{\text{cusps}\}$ and $(E, \omega, P)$ the universal family of elliptic curves, thereby proving theorem 1.2.

### 3. Hecke-Eisenstein modular forms

Now we recall work of Hecke ([10], [11]). (It also occurs in [12, Chapter 3].) We always use an unadorned $\equiv$ to denote congruence modulo $N$.

For $(L, u) \in \mathcal{L}_N$ and $s \in \mathbb{C}$ with $\text{Re } s > 1$ we define

$$
\Phi(L, u, s) = \sum_{\omega \equiv u \mod L} \omega^{-1}|\omega|^{-s},
$$

where the prime means that the term with $\omega = 0$ is to be omitted if it occurs. We define

$$
G(L, u) = \Phi(L, u, 0)
$$

by analytic continuation as Hecke does. It is evident that $G$ is a homogeneous function of degree $-1$. For $a, b \in \mathbb{Z}$ we let $G_{a,b}(\tau) = G(\tau + Z, (a\tau + b)/N)$, in accordance with the notation in [22]. If it happens that $a = 0$ then $G_{a,b}(\tau)$ is a modular function for $\Gamma_1(N)$, but in any case, it is a modular function for $\Gamma(N)$.

Providing a $q$-expansion at $\tau = \infty$ for each function $G_{a,b}(\tau)$ in terms of $e^{2\pi i \tau/N}$ is the same as providing a $q$-expansion for $G$ at each of the cusps of $X(N)$. Conversely, finding a $q$-expansion for $G_{a,b}(\tau)$ at $\tau = \infty$ for all $a, b$ is the same as finding a $q$-expansion for one of the functions $G_{a,b}(\tau)$ at all of the cusps. Now we present some notation needed to write down the $q$-expansion found by Hecke.

Introduce the Hurwitz zeta function

$$
\zeta(s, \alpha) = \sum_{n > -\alpha} (n + \alpha)^{-s}
$$

and the notation

$$
\delta(v) = \begin{cases} 1, & \text{if } v \in \mathbb{Z}; \\ 0, & \text{otherwise.} \end{cases}
$$

Define

$$
\alpha_n(a, b) = \begin{cases} \frac{1}{N} \delta\left(\frac{a}{N}\right) \lim_{s \to 1} \left[ \zeta(s, b) - \zeta(s, -b) \right] - \frac{\pi i}{N} \left[ \zeta(0, \frac{a}{N}) - \zeta(0, -\frac{a}{N}) \right] & \text{if } n = 0 \\ \frac{2\pi i}{N} \sum_{m|n} (\text{sgn } m) \zeta_N^{km} & \text{if } n > 0 \end{cases}
$$
Here is Hecke’s result.

**Proposition 3.1.** [22, p. 168], or [11, p. 203]. \( G_{a,b}(\tau) = \sum_{n=0}^{\infty} \alpha_n(a, b)q^{n/N} \)

From this we can deduce the following.

**Proposition 3.2.** \( \frac{1}{2\pi i} \sum_{a \mod N} \sum_{b \mod N} \zeta_N^{ka} G_{a,b}(\tau) = g_k(\tau) \)

**Proof.** Part of this proof is just like the proof of [14, (2.7.12)].

We expand the left-hand side using Proposition 3.1, obtaining

\[
\sum_{n=0}^{\infty} \beta_n q^{n/N},
\]

where

\[
\beta_n = \frac{1}{2\pi i} \sum_{a \mod N} \sum_{b \mod N} \zeta_N^{ka} \alpha_n(a, b).
\]

We take the case \( n = 0 \) first. We have

\[
\beta_0 = \frac{1}{2\pi i} \sum_{a \mod N} \sum_{b \mod N} \zeta_N^{ka} \alpha_0(a, b)
= -\frac{1}{2} \left( \sum_{a \mod N} \zeta_N^{ka} \left( \zeta(0, \frac{a}{N}) - \zeta(0, \frac{-a}{N}) \right) \right)
= -\frac{1}{2} \sum_{a \mod N} (\zeta_N^{ka} - \zeta_N^{-ka}) \zeta(0, \frac{a}{N})
\]

We see from [26, p.59], that \( \zeta(0, \alpha) = \frac{1}{2} - \alpha \) for \( 0 < \alpha < 1 \), and it is known that \( \zeta(0, 1) = \zeta(0) = -\frac{1}{2} = \frac{1}{2} - 1 \). Thus (compare with [14, p. 263]) we have

\[
\beta_0 = -\frac{1}{2} \sum_{a=1}^{N} \left( \zeta_N^{ka} - \zeta_N^{-ka} \right) \left( \frac{1}{2} - \frac{a}{N} \right)
= \frac{1}{2N} \sum_{a=1}^{N} (\zeta_N^{ka} - \zeta_N^{-ka}) a
= -\frac{1}{N} \Im \sum_{a=1}^{N} a \zeta_N^{-ka} = \frac{1}{i} \Im \lim_{T \to \zeta_N^k} \frac{\sum_{a=1}^{N} T^a}{1 - T^N}
= \frac{1}{i} \Im \lim_{T \to \zeta_N^k} \frac{T}{1 - T} = \frac{1}{i} \Im \frac{\zeta_N^k}{1 - \zeta_N^k} = \frac{1}{i} D_0(\zeta_N^k).
\]

(The middle equality above is an application of l’Hôpital’s rule.) This agrees with the coefficient of \( q^0 \) in \( g_k(\tau) \).

Now we take up the case \( n > 0 \). We have

\[
\beta_n = -\frac{1}{N} \sum_{a \mod N} \sum_{b \mod N} \zeta_N^{ka} \sum_{m \mid n} (\text{sgn } m) \zeta_N^b m
= -\frac{1}{N} \sum_{m \mid n} \sum_{b \mod N} (\text{sgn } m) \zeta_N^b m
= -\sum_{m \mid n} (\text{sgn } m) \zeta_N^b m
\]
and so $\beta_n = 0$ unless $n \equiv 0$, for else the summation has no terms. We compute, for $n > 0$,

$$
\beta_{NN} = - \sum_{m|n} (\text{sgn } n) \zeta_{n/m} = - \sum_{m|n} (\text{sgn } m) \zeta_{m/n}
$$

$$
= - \sum_{m|n} \left( \zeta_{m/n} - \zeta_{-m/n} \right)
$$

This is the coefficient of $q^n$ in the expansion for $g_k(\tau)$. Q.E.D.

Now we show how to use Proposition 3.2 to interpret $g_k(\tau)$ as a holomorphic modular form of weight one for the group $\Gamma_1(N)$.

As standard basis for the exterior power $\bigwedge^2 \mathbb{C}$ we will use $1 \wedge i$. We let $\alpha_L$ denote the generator of the group $\bigwedge^2 L$ which is a positive multiple of $1 \wedge i$. For any $u, v \in N^{-1}L$ we have

$$
N \frac{u \wedge v}{\alpha_L} = N^{-1} N \frac{u \wedge Nv}{\alpha_L} \in N^{-1} \mathbb{Z}
$$

and thus

$$
\chi^L_u(v) = \exp \left( 2\pi i N \frac{u \wedge v}{\alpha_L} \right)
$$

is an $N^{th}$ root of 1. We remark that $\chi^L_u(v) = e_N(u, v)$ in terms of the $e_N$-pairing (see [12, p. 477]). One checks the following formulas.

$$
\chi^L_{u+u'}(v) = \chi^L_u(v) \chi^L_{u'}(v)
$$

$$
\chi^L_{\alpha u}(\alpha v) = \chi^L_u(v) \quad \text{for } \alpha \in \mathbb{C}
$$

$$
\chi^L_u(v + v') = \chi^L_u(v) \chi^L_{u'}(v')
$$

$$
\chi^L_u(v) = 1 \quad \text{if } u \in L
$$

$$
\chi^{Z+Z\tau}_{(k+i\ell)N}(\frac{a\tau + b}{N}) = \zeta_N^{ka-\ell b} \quad \text{for } a, b, k \in \mathbb{Z}
$$

It follows that $\chi^L_u$ depends only on the class of $u$ in $N^{-1}L/L$, and is a homomorphism $N^{-1}L \rightarrow \mathbb{C}^\times$.

For $(L, u) \in \mathcal{L}_N$ we define

$$
G'(L, u) = \sum_{w \in N^{-1}L/L} \chi^L_u(w)G(L, w);
$$

the function $G'$ is a meromorphic modular form for $\Gamma_1(N)$. From (3.2) we see that

$$
g_k(\tau) = G'(\mathbb{Z} \cdot 2\pi i + \mathbb{Z} \cdot 2\pi i \tau, 2\pi ik/N),
$$

which shows that $g_k(\tau)$ is a holomorphic modular form for $\Gamma_1(N)$. Combining this with (2.4) we see that

$$
A_1 = G'
$$
We can also deduce the following proposition.

**Proposition 3.5.** If \(2 \mid N\), then
\[
\frac{1}{2\pi i} \sum_{a \mod N \atop b \mod N} \zeta_N^{ka} (-1)^b G_{a,b}(\tau) = h_k(\tau)
\]

**Proof.** We expand the left hand side, using Proposition 3.1, obtaining
\[
\sum_{n=0}^{\infty} \gamma_n q^{n/N},
\]
where
\[
\gamma_n = \frac{1}{2\pi i} \sum_{a \mod N \atop b \mod N} \zeta_N^{ka} (-1)^b \alpha_n(a, b).
\]
We compute (keeping in mind that \(N\) is even)
\[
\gamma_0 = \frac{1}{2\pi i} \sum_{a \mod N \atop b \mod N} \zeta_N^{ka} (-1)^b \alpha_0(a, b)
= \frac{1}{2\pi i} \left[ \sum_{b \mod N} (-1)^b \frac{1}{N} \lim_{s \to 1} \left\{ \zeta(s, \frac{b}{N}) - \zeta(s, \frac{-b}{N}) \right\} \right]
= 0
\]
Now for \(n > 0\) we find
\[
\gamma_n = -\frac{1}{N} \sum_{a \mod N \atop b \mod N} \zeta_N^{ka} (-1)^b \sum_{m \mid n \atop m \equiv a} \text{sgn}(m) \zeta_N^{km}
= -\frac{1}{N} \sum_{m \mid n} \text{sgn}(m) \zeta_N^{kn/m} \sum_{b \mod N} (-\zeta_N^m)^b
= -\sum_{m \mid n \atop m \equiv N/2} \text{sgn}(m) \zeta_N^{kn/m}
\]
We see that \(\gamma_n = 0\) unless \(\frac{N}{2} \mid n\), and we find that
\[
\gamma_{nN/2} = -\sum_{m \mid n \atop m \text{ odd}} \text{sgn}(m) \zeta_N^{kn/m}
= -\sum_{m \mid n \atop m \text{ odd}} \zeta_N^{-kn/m} - \zeta_N^{-kn/m},
\]
which is, indeed, the coefficient of \(q^{n/2}\) in \(h_k(\tau)\). **Q.E.D.**

We make the connection with modular forms by observing that
\[
(3.6) \quad h_k(\tau) = G'(\mathbb{Z} \cdot 2\pi i + \mathbb{Z} \cdot 2\pi i \tau, 2\pi i (\frac{k}{N} + \frac{1}{2} \tau)).
\]
which together with (3.4) justifies (2.5).
4. Expansion at the cusp $\tau = 0$.

In order to test the rationality of $G'$ we will examine the $q$-expansion at a cusp which happens to be a rational point on the curve $X_1(N)$. We may, for example, expand

$$G'(Z \cdot 2\pi i + Z \cdot 2\pi i\tau, 2\pi i\frac{\ell\tau}{N})$$

in terms of $q^{1/N} = e^{2\pi i\tau/N}$, where $\ell$ is any integer relatively prime to $N$. Only one such cusp is needed to apply the $q$-expansion principle, so we could set $\ell = 1$, but the computation is no harder if we refrain from doing that.

Proceeding as in Proposition 3.2, we may write down the expansions we desire in the following proposition; they don’t seem to follow easily from [14, 2.7.8].

**Proposition 4.1.**

(a) $G'(Z \cdot 2\pi i + Z \cdot 2\pi i\tau, 2\pi i\frac{\ell\tau}{N}) = \left(\frac{\ell}{N} - \frac{1}{2}\right) - \sum_{n=1}^{\infty} \left( \sum_{m|n, m \equiv \ell} \text{sgn } m \right) q^{n/N}$

(b) $G'(Z \cdot 2\pi i + Z \cdot 2\pi i\tau, 2\pi i\left(\frac{1}{2} + \frac{\ell\tau}{N}\right)) = \left(\frac{\ell}{N} - \frac{1}{2}\right) - \sum_{n=1}^{\infty} \left( \sum_{m|n, m \equiv \ell} (-1)^{n/m} \text{sgn } m \right) q^{n/N}$

**Proof.** We have

$$G'(Z \cdot 2\pi i + Z \cdot 2\pi i\tau, 2\pi i\frac{\ell\tau}{N}) = \frac{1}{2\pi i} G'(Z + Z\tau, \frac{\ell\tau}{N})$$

$$= \frac{1}{2\pi i} \sum_{a \mod N} \sum_{b \mod N} \chi_{N}(a\tau + b) G(Z + Z\tau, \frac{a\tau + b}{N})$$

$$= \frac{1}{2\pi i} \sum_{a \mod N} \sum_{b \mod N} \zeta_{N}^{-\ell b} G_{a,b}(\tau),$$

so applying (3.1) we can write

$$G'(Z \cdot 2\pi i + Z \cdot 2\pi i\tau, 2\pi i\frac{\ell\tau}{N}) = \sum_{n=0}^{\infty} \beta_{n}' q^{n/N}$$

where

$$\beta_{n}' = \frac{1}{2\pi i} \sum_{a \mod N} \sum_{b \mod N} \zeta_{N}^{-\ell b} \alpha_{n}(a, b)$$

and

$$G'(Z \cdot 2\pi i + Z \cdot 2\pi i\tau, 2\pi i\left(\frac{1}{2} + \frac{\ell\tau}{N}\right)) = \sum_{n=0}^{\infty} \gamma_{n}' q^{n/N},$$

where

$$\gamma_{n}' = \frac{1}{2\pi i} \sum_{a \mod N} \sum_{b \mod N} (-1)^{a} \zeta_{N}^{-\ell b} \alpha_{n}(a, b).$$
We compute

$$\beta'_0 = \frac{1}{2\pi i} \sum_{a \mod N} \sum_{b \mod N} \frac{\zeta_N^{-\ell b}}{N} \left[ \frac{1}{N} \delta \left( \frac{a}{N} \right) \lim_{s \to 1} [\zeta(s, b/N) - \zeta(s, -b/N)] \right]$$

$$= \frac{1}{2\pi i N} \lim_{s \to 1} \sum_{b \mod N} \zeta_N^{-\ell b} [\zeta(s, b/N) - \zeta(s, -b/N)]$$

$$= \frac{-1}{2\pi i N} \lim_{s \to 1} \sum_{b \mod N} \left( \zeta_N^{\ell b} - \zeta_N^{-\ell b} \right) \zeta(s, b/N)$$

$$= \frac{-1}{2\pi i N} \lim_{s \to 1} \sum_{b=1}^{N} \left( \zeta_N^{\ell b} - \zeta_N^{-\ell b} \right) \sum_{n=0}^{\infty} \left( \frac{n + b}{N} \right)^{-s}$$

$$= \frac{-1}{\pi N} \lim_{s \to 1} N^s \text{Im} \sum_{n=0}^{\infty} \sum_{b=1}^{N} \zeta_N^{\ell b} (Nn + b)^{-s}$$

$$= \frac{-1}{\pi} \lim_{s \to 1} \text{Im} \sum_{m=1}^{\infty} \zeta_N^{\ell m} m^{-s}$$

$$= \frac{-1}{\pi} \lim_{s \to 1} \sum_{m=1}^{\infty} \zeta_N^{\ell m} m^{-1}$$

The latter series converges because \( \sum_{b \mod N} \zeta_N^{\ell b} = 0 \), which tells us that its terms, when taken \( N \) at a time, are of the order of \( m^{-2} \). The last equality used above is justified by the continuity of Dirichlet series up to the line of convergence, \([6, p. 87]\), provided we let \( s \) approach 1 from the right through real values. The last series above is summed in formula (7) of \([26, p. 54]\); alternatively, we may apply the continuity of power series up to the circle of convergence (analogous to the continuity of Dirichlet series up to the line of convergence), \([1, \text{Lehrsatz IV}]\), letting \( z \) approach \( \zeta_N^{\ell} \) radially from the origin, we obtain

$$\beta'_0 = \frac{-1}{\pi} \lim_{z \to \zeta_N^{\ell}} \text{Im} \sum_{m=1}^{\infty} z^m m^{-1}$$

$$= \frac{1}{\pi} \lim_{z \to \zeta_N^{\ell}} \text{Im} \log(1 - z)$$

$$= \frac{1}{\pi} \arg(1 - \zeta_N^{\ell})$$

$$= \frac{\ell}{N} - \frac{1}{2}$$

Now for \( n \geq 1 \) we compute

$$\beta'_n = \frac{1}{2\pi i} \sum_{a \mod N} \sum_{b \mod N} \frac{\zeta_N^{-\ell b}}{N} \left[ \frac{2\pi i}{N} \sum_{m|n} (\text{sgn } m) \zeta_N^{bm} \right]$$

$$= \frac{-1}{N} \sum_{m|n} (\text{sgn } m) \sum_{b \mod N} \zeta_N^{b(m-\ell)}$$

$$= - \sum_{m|n} \text{sgn } m$$

This proves (b). The proof for (c) is similar. \( \text{Q.E.D.} \)

**Alternate proof of 4.1.** We offer another proof based on formula **H6** of \([18, p. 250]\); it uses the Weierstrass \( \zeta \)-function. For the purpose of reconciling the notation in \([18, pp. 247–250]\) with that in \([14, \text{Appendix C}]\),
we imagine that each $\eta$ in [18] has been replaced by $-\eta$. We use the notations $G_{1,\ell}$, $G^{1,\ell}$, $V_N$, $F$, $h_{a_1,a_2}$, and $H_{a_1,a_2}$ from [18] without repeating the definitions, and we assume $0 < \ell < N$.

From [18, H6, p. 250] and the definition of $G_{1,\ell}$ we have

$$G_{1,\ell}(\tau) = \frac{\ell}{N} - \frac{1}{2} - \sum_{j=0}^{\infty} \frac{q^{(jN+\ell)/N}}{1 - q^{(jN+\ell)/N}} + \sum_{j=1}^{\infty} \frac{q^{(jN-\ell)/N}}{1 - q^{(jN-\ell)/N}}.$$ 

By collecting similar powers of $q$ we obtain

$$G_{1,\ell}(\tau) = \frac{\ell}{N} - \frac{1}{2} - \sum_{n=1}^{\infty} \left( \sum_{m|n \atop m \equiv \ell} \text{sgn } m \right) q^{n/N}.$$ 

On the other hand, we may simply trace the definitions in [18] and [14].

$$G_{1,\ell}(\tau) = \frac{1}{2\pi i} h_{\ell/N,0}(\tau)$$
$$= \frac{1}{2\pi i} H_{\ell/N,0}(\tau,1)$$
$$= \frac{1}{2\pi i} \left( \zeta(\ell\tau/N, Z + Z \cdot \tau) + \eta(\ell\tau/N, Z + Z \cdot \tau) \right)$$
$$= \frac{1}{2\pi i} \left( \zeta(\ell\tau/N, Z + Z \cdot \tau) + \frac{1}{N} \eta(\ell\tau, Z + Z \cdot \tau) \right)$$
$$= \frac{1}{2\pi i} A_1(Z + Z \cdot \tau, \ell\tau/N)$$
$$= A_1(Z \cdot 2\pi i + Z \cdot 2\pi i \tau, 2\pi i \ell\tau/N)$$
$$= G'(Z \cdot 2\pi i + Z \cdot 2\pi i \tau, 2\pi i \ell\tau/N).$$

Combining the previous two formulas gives (a).

As for (b), we see as above that

$$G'(Z \cdot 2\pi i + Z \cdot 2\pi i \tau, 2\pi i(\frac{1}{2} + \frac{\ell\tau}{N})) = \frac{1}{2\pi i} h_{\ell/N,1/2}(\tau).$$

Then we apply H4 of [18]:

$$\frac{1}{2\pi i} h_{\ell/N,1/2}(\tau) = \frac{\ell}{N} + F(q, -q^{\ell/N})$$
$$= \frac{\ell}{N} - \frac{1}{2} - \sum_{j=0}^{\infty} \frac{-q^{(jN+\ell)/N}}{1 + q^{(jN+\ell)/N}} + \sum_{j=1}^{\infty} \frac{-q^{(jN-\ell)/N}}{1 + q^{(jN-\ell)/N}}$$
$$= \frac{\ell}{N} - \frac{1}{2} - \sum_{n=1}^{\infty} \left( \sum_{m|n \atop m \equiv \ell} (-1)^{n/m} \text{sgn } m \right) q^{n/N}$$

Q.E.D.

**Remark 4.2.** One may also follow Lang [18, H4] as in the alternate proof above to get the $q$-expansions
in (3.3) and (3.6). One uses

\[ A_1(\mathbb{Z} \cdot 2\pi i + \mathbb{Z} \cdot 2\pi i \tau, 2\pi i k/N) \]
\[ = \frac{1}{2\pi i} h_{0, k/N}(\tau) \]
\[ = F(q, \zeta_N^k) \]
\[ = -\frac{1}{2} - \frac{\zeta_N^k}{1 - \zeta_N^k} - \sum_{j=1}^{\infty} \frac{q^j \zeta_N^k}{1 - q^j \zeta_N^k} + \sum_{j=0}^{\infty} \frac{q^{-j \zeta_N^k}}{1 - q^{-j \zeta_N^k}} \]
\[ = \frac{1}{2} \left( \frac{\zeta_N^k + 1}{\zeta_N^k - 1} \right) - \sum_{n=1}^{\infty} \left( \sum_{m|n} \left( \zeta_N^{kn} - \zeta_N^{-kn} \right) q^n \right) \]
\[ = g_k(\tau) \]

and

\[ A_1(\mathbb{Z} \cdot 2\pi i + \mathbb{Z} \cdot 2\pi i \tau, 2\pi i(\frac{k}{N} + \frac{1}{2} \tau)) \]
\[ = \frac{1}{2\pi i} h_{1/2, k/N}(\tau) \]
\[ = \frac{1}{2} + F(q, q^{1/2} \zeta_N^k) \]
\[ = \frac{1}{2} - \frac{1}{2} \sum_{j=0}^{\infty} \frac{q^{j+1/2} \zeta_N^k}{1 - q^{j+1/2} \zeta_N^k} + \sum_{j=1}^{\infty} \frac{q^{-j-1/2} \zeta_N^{-k}}{1 - q^{-j-1/2} \zeta_N^{-k}} \]
\[ = -\sum_{n=1}^{\infty} \left( \sum_{0 \leq j | n, r \text{ odd}} \left( \zeta_N^{kn/r} - \zeta_N^{-kn/r} \right) q^{n/2} \right) \]
\[ = h_k(\tau) \]

5. The \( q \)-expansion principle.

The \( q \)-expansion principle is part of the classical theory of modular forms. When the modular curve in question has a model over a subfield \( k \subseteq \mathbb{C} \), a modular function tends to be defined over \( k \) if and only if the coefficients of its \( q \)-expansion are in \( k \). In this section we state the purely algebraic portion of the \( q \)-expansion principle.

Let \( K/k \) be a field extension. In our application, it will be \( \mathbb{C}/\mathbb{Q} \).

Suppose \( X \) is a curve over \( K \), with \( x \) a nonsingular point on \( X \) defined over \( K \), and let \( q \) be an element of valuation 1 in the complete local ring \( \mathcal{O}_x \). We have a \( K \)-algebra isomorphism \( \mathcal{O}_x \cong K[[q]] \) with the ring \( K[[q]] \) of formal power series in the variable \( q \), and an inclusion of fields \( K(X) \to K((q)) \) into the field of formal Laurent series. Pick generators \( f_1, \ldots, f_r \) over \( K \) for the field \( K(X) \), so that \( K(X) = K(f_1, \ldots, f_r) \).

**Proposition 5.1.** With the notation above, if \( f_1, \ldots, f_r \in k((q)) \), then
(a) \( k((q)) \) and \( K \) are linearly disjoint over \( k \).
(b) \( K(X) \cap k((q)) = k(f_1, \ldots, f_r) \).
(c) If \( X \) has a model \( Y \) over \( k \) with respect to which the functions \( f_1, \ldots, f_r \) are defined over \( k \) (i.e., \( f_i \in k(Y) \)), then \( k(Y) = k(f_1, \ldots, f_r) \).

When we say that \( X \) has a model \( Y \) over \( k \), we mean \( Y \) is an irreducible curve over \( k \) with an isomorphism \( g : Y \otimes_k K \cong X \).

**Proof.** First notice that \( k(f_1, \ldots, f_r) \subseteq k((q)) \). To prove (a) we proceed as in [23, p. 141]. Given \( a_1, \ldots, a_m \in K \) linearly independent over \( k \), we suppose that \( g_1, \ldots, g_m \in k((q)) \) satisfy \( \sum a_i g_i = 0 \). We write each \( g_i \) as a Laurent series \( g_i = \sum_j b_{ij} q^j \) with coefficients \( b_{ij} \in k \). Then we deduce that
\[
\sum_j a_j b_{ij} q^j = 0,
\]
and thus, for each \( j \), we have \( \sum_i a_i b_{ij} = 0 \), whence each \( b_{ij} = 0 \), and thus each \( g_i = 0 \).

We claim that when we have intermediate fields \( k \subseteq L \subseteq M \subseteq k((q)) \), then \( K \cdot L \subseteq K \cdot M \), with equality holding iff \( L = M \). We need only prove that \( K \cdot L = K \cdot M \) implies \( L = M \). We know \( K \) and \( k((q)) \) are linearly disjoint subfields of \( K((q)) \), so it follows that so are \( K \) and \( L \), since \( L \) is a subfield of \( k((q)) \). It follows that \( K \otimes L \) maps isomorphically onto a subring \( K \cdot L \) of the field \( K \cdot L \) and generates that field, so \( K \cdot L \) is the fraction field of \( K \cdot L \). Similarly, \( K \cdot M \) is the fraction field of \( K \cdot M \). But \( M \) is a field extension of \( L \), hence a free \( L \)-module, a property which is preserved by tensor product. Hence \( K \cdot M \) is a free \( K \cdot L \)-module, and since \( K \cdot L = K \cdot M \), we can view \( K \cdot M \) as a submodule of the fraction field of \( K \cdot L \). But any free submodule of the fraction field of a ring has rank at most 1, so \( K \cdot L = K \cdot M \). Thus \( K \otimes L \cong K \otimes M \), implying \( L = M \), proving the claim.

To prove (b), we apply this claim with \( L = k(f_1, \ldots, f_r) \) and \( M = K(f_1, \ldots, f_r) \cap k((q)) \).

The same fact proves (c). Indeed, the isomorphism \( g \) above induces an injective map from \( k(Y) \otimes_k K \) into \( K(X) \) whose image generates the field \( K(X) \). This implies that \( k(Y) \) and \( K \) are linearly disjoint as subfields of \( K(X) \). As \( k(f_1, \ldots, f_r) \) is a submodule of \( k(Y) \), it is also linearly disjoint from \( K \). Now we are done as before.

Q.E.D.

We refer to [7], [12], [13] and [15] for discussions of the \( q \)-expansion.

6. The function field of the rational model of \( \Gamma_1(N) \setminus H \).

We recall the details of the standard examples of modular forms and functions for \( \Gamma = \text{SL}_2(\mathbb{Z}) \). For each even integer \( \ell \geq 4 \) define a homogeneous function \( E_\ell : \mathcal{L} \to \mathbb{C} \) of degree \( -\ell \) by the formula

\[
E_\ell(L) = \sum_{w \in L} w^{-\ell}.
\]

Define \( G_2 = 60 E_4 \) and \( G_3 = 140 E_6 \). Let \( g_2 = (2\pi i)^4 G_2 \circ \phi_k \) and \( g_3 = (2\pi i)^6 G_3 \circ \phi_k \); these are modular forms of weight 4 and 6 respectively. Take \( \Delta = g_2^3 - 27 g_3^2 \) to be the standard cusp form of weight 12, and let \( j = g_2^3/\Delta \) be the \( j \)-invariant, a modular function.

It is known that the \( q \)-expansion of \( j \) has rational coefficients; since \( j \) is a modular function for the full modular group \( \Gamma \), the same is true for its \( q \)-expansion at the other cusps.

The Weierstrass \( \wp \)-function gives a modular form of weight 2 for \( \Gamma_1(N) \) as follows. We define a function \( \wp : \mathcal{L}_N \to \mathbb{C} \) by the formula \( \wp(L, u) = \sum_{w \in L} ((w - u)^2 - w^2) \); it is a homogeneous function of degree \(-2\). We let \( p_k = \wp \circ \phi_k \) be the corresponding modular form. According to [23, 6.2.1, p. 141] the \( q \)-expansion of \( \wp \) is

\[
\wp(Z, \tau + Z, \frac{r \tau + s}{N}) = (2\pi i)^2 \left\{ \frac{1}{12} - 2 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} \right. \\
+ \zeta_N^s q^{r/N} \left( 1 - \zeta_N^{s} q^{r/N} \right)^{-2} + \sum_{n=1}^{\infty} \left( \zeta_N^{ns} q^{nr/N} + \zeta_N^{-ns} q^{-nr/N} \right) \frac{nq^n}{1-q^n} \right\},
\]

for \( 0 \leq r < N \), \( (r, s) \in \mathbb{Z}^2 \), and \( (r, s) \not\in N\mathbb{Z}^2 \). Taking \( r = k \) and \( s = 0 \) we find that the \( q \)-expansion of

\[
\wp(Z, 2\pi i + Z, 2\pi i, 2\pi i k \tau / N),
\]

which is the \( q \)-expansion for \( p_k(\tau) \) at \( \tau = 0 \), has rational coefficients.

We define \( f_k = g_2 g_3 p_k / \Delta \); it is a modular function for \( \Gamma_1(N) \). It is known that \( (2\pi i)^{-4} g_2, (2\pi i)^{-6} g_3, \) and \( (2\pi i)^{-12} \Delta \) have \( q \)-expansions with rational coefficients (see [23, 2.2]). It follows that the \( q \)-expansion at the cusp \( \tau = 0 \) of the modular function \( f_k \) has rational coefficients.

Proposition 6.1. The field \( \mathbb{C}(j, f_k) \) is the field of all modular functions for \( \Gamma_1(N) \).

Proof. Let \( M \) be the field of all modular functions for the group

\[
\Gamma(N) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z}) \mid \begin{array}{cc} a & b \\ c & d \end{array} \equiv \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \mod N \right\}.
\]
(This field $M$, called the modular function field of level $N$, is the function field of the modular curve $X(N)$ over $\mathbb{C}$.) We know that $C(j)$ is the field of all modular functions for the group $\Gamma = \text{SL}_2(\mathbb{Z})$, that $M$ is a Galois extension of $\mathbb{C}(j)$, and that $\text{Gal}(M/C(j)) = \Gamma/\Gamma(N) \cdot \{ \pm 1 \} \cong \text{SL}_2(\mathbb{Z}/N)/\{ \pm 1 \}$.

Let $L$ be the field of all modular functions for $\Gamma_1(N)$. We know that $K = C(j, f_k) \subseteq L$, and that $\text{Gal}(M/L) = \Gamma_1(N) \cdot \{ \pm 1 \}/\Gamma(N) \cdot \{ \pm 1 \}$. To determine the group corresponding to the intermediate field $K$ we consider an element $\gamma \in \Gamma$ which leaves every element of $K$ fixed. Then, as in [23, 6.1-A], we see that the equation $f_k \circ \gamma = f_k$ implies that

$$\left( \begin{array}{c} 0 \\ \frac{k}{N} \end{array} \right) \cdot \gamma \equiv \left( \begin{array}{c} 0 \\ \frac{k}{N} \end{array} \right) \mod \mathbb{Z}^2,$$

and thus that

$$\gamma \equiv \left( \begin{array}{cc} * & * \\ 0 & \epsilon \end{array} \right) \mod N,$$

where $\epsilon = \pm 1$. The matrix $\gamma' = \epsilon \gamma$ has determinant 1, so

$$\gamma' \equiv \left( \begin{array}{cc} 1 & * \\ 0 & 1 \end{array} \right) \mod N,$$

i.e., $\gamma' \in \Gamma_1(N)$, showing that $\gamma \in \Gamma_1(N) \cdot \{ \pm 1 \}$. Thus $L = K$. Q.E.D.

Applying (5.1) yields the following corollary.

**Corollary 6.2.** The field $\mathbb{Q}(j, f_k)$ is the field of modular functions for $\Gamma_1(N)$ whose $q$-expansions at the cusp $\tau = 0$ have rational coefficients, and is the function field of the canonical rational model of $X_1(N)$.

**REMARKS**

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