**QUANTUM ALGEBRAIC TORI**

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**Introduction.** The notion of quantum torus appears in mathematics due to the following physical consideration. Let the operators $u, v$ satisfy the Heisenberg commutation relation $[u, v] = q$, where $q = i\hbar$. Then the operators $x = e^u$, $y = e^v$ satisfy the relation $xy = qyx$. The algebra, generated by $x$ and $y$, is called quantum torus.

In the preprint the author gives the definition of quantum algebraic torus over the arbitrary field. Quantum algebraic tori can be characterized in terms of exact sequences (Theorem 1). The description of the center is given in Proposition 1. In the case of roots of 1 QAT produces the fibering of central simple algebras on its center (Proposition 2). QAT can be used for the construction of central simple algebras and skew fields. It is unknown whether these central simple algebras be crossed products or not. The description of QAT of dimension 2 is given in terms of generating elements and there relations. The case of roots of 1 is considered (Proposition 3).

**Quantum algebraic tori and exact sequences**

Let $L$ be a field and $Q = (q_{ij})_{ij=1}^n$ be a matrix with the entries $q_{ij} \in L^*$, $q_{ij}q_{ji} = q_{ii} = 1$. An algebra of twisted Laurent polynomials is the algebra $L_Q[x_1^{\pm 1}, \ldots , x_n^{\pm 1}]$, generated by the elements $x_1^{\pm 1}, \ldots , x_n^{\pm 1}$ with the relations $x_i x_j = q_{ij} x_j x_i$. The group of diagonal matrices $D_n(L) = L^* \times \cdots \times L^*$ acts by automorphisms $x_i \mapsto t_i x_i$ on $L_Q[x_1^{\pm 1}, \ldots , x_n^{\pm 1}]$. We shall call this algebra a quantum form of diagonal torus $D_n$.

Let $T$ be an algebraic $k$-torus splited over the Galois extension $L/k$. Denote by $\Gamma = Gal(L/k)$, $M = T$ the group of characters of the torus $T$, defined over the algebraic closure of the field $k$. The group $\Gamma$ acts on the group $M$. The map $T \mapsto M$ is an 1-1 correspondence of the set of algebraic tori, splited over $L$, to the set of finitely generated $Z$-free $Z[\Gamma]$-modules. The both sets is defined up to equivalence[1,2].

Denote by $^a M$ the group $M$ in additive form. We shall denote elements of the group $M$ by $t^m$, $m \in ^a M$. Choose the $Z$-basis $e_1, \ldots , e_n$ in $^a M$. The characters $t_i = t^{e_i}$ free generate $M$. For $m = m_1 e_1 + \cdots + m_n e_n$ the element $t^m$ is equal to $t_1^{m_1} \cdots t_n^{m_n}$.

Now we shall give the main definition.

**Definition 1.** A quantum form of the $k$-torus $T$ is the $k$-algebra $A_Q$ with
following properties: 1) \( A_{Q,L} = A_Q \otimes_k L = L_Q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \), 2) there exists the map of coaction \( R : A_Q \mapsto k[T] \otimes_k A_Q \) of the Hopf algebra \( k[T] \) on \( A_Q \), such that \( R_L = R \otimes 1 : A_{Q,L} \mapsto L[T] \otimes L A_{Q,L} \) coinside with the coaction of diagonal torus \( R_L(x_i) = t_i \otimes x_i \). We shall also call \( A_Q \) quantum algebraic torus.

The group \( T(k) \) actes by automorphisms on \( A_Q \).

**Definition 2.** The homomorphism of \( k \)-algebras \( \phi : A_Q \mapsto A'_{Q'} \) is called a homomorphism of quantum \( k \)-forms of algebraic torus \( T \) is the following diagram commutes:

\[
\begin{array}{ccc}
A_Q & \xrightarrow{R} & k[T] \otimes A_Q \\
\downarrow \phi & & \downarrow 1 \otimes \phi \\
A'_{Q'} & \xrightarrow{R'} & k[T] \otimes A'_{Q'}
\end{array}
\]

Denote by \( Quan_L(T) \) the category of quantum \( k \)-forms of algebraic torus \( T \), splited over \( L \) and defined up to an isomorphism. Denote by \( Ext^c_\Gamma(M, L^*) \) the category of central extensions \( \Gamma \)-\( 1 \mapsto L^* \mapsto E \mapsto M \mapsto 1 \).

Here \( L^* \) is in the center of \( E \). The elements of this category are also defined up to an isomorphism.

**Theorem 1.** The categories \( Quan_L(T) \) and \( Ext^c_\Gamma(M, L^*) \) are isomorphic.

**Proof.**

1) Let \( A_Q \) be a quantum form of the \( k \)-torus \( T \). Denote by \( E \) the group \( A_{Q,L}^* \) of invertible elements of the algebra \( A_{Q,L} = L_Q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \). The group \( E \) is generated by \( x_1^{\pm 1}, \ldots, x_n^{\pm 1}, L^* \). The group \( E \) is a \( \Gamma \)-group with respect to the natural action of the Galois group \( \Gamma \) in \( A_Q \otimes L = A_{Q,L} \). We restrict \( R_L \) on \( E \). From \( R_L(x_i) = t_i \otimes x_i \) we have \( R_L : E \mapsto M \times E \). The map \( R_L \) s a homomorphism of \( \Gamma \)-groups. Consider the projection \( \pi_1 : M \times E \mapsto M \). The superposition \( \pi = \pi_1 R_L : E \mapsto M \) is a homomorphism of \( \Gamma \)-groups with the kernel \( L^* \). There exists the exact sequence of \( \Gamma \)-groups \( 1 \mapsto L^* \mapsto E \mapsto M \mapsto 1 \).

The algebra \( A_Q \) is uniquely reconstructed from the \( \Gamma \)-group \( E \) as the \( k \)-algebra of \( \Gamma \)-invariant elements in \( L_Q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \).

2) The homomorphism of \( k \)-algebras \( \phi : A_Q \mapsto A'_{Q'} \) is extended to the \( \Gamma \)-invariant \( L \)-homomorphism \( \phi \otimes 1 : A_{Q,L} \mapsto A'_{Q',L} \). We restrict it on \( E \) and have the \( \Gamma \)-homomorphism of groups \( \phi : E = (A_{Q,L})^* \mapsto E' = (A'_{Q',L})^* \). The
commutativity of the following diagrams one can be derived from (1):

\[
\begin{array}{ccc}
E & \overset{R}{\longrightarrow} & M \times E \\
\downarrow \phi & & \downarrow 1 \otimes \phi \\
E' & \overset{R}{\longrightarrow} & M \times E'
\end{array}
\] (3)

\[
\begin{array}{ccc}
1 \quad \longrightarrow \quad L^* \quad \longrightarrow \quad E \quad \overset{\pi}{\longrightarrow} \quad M \quad \longrightarrow \quad 1 \\
\downarrow \text{id} & & \downarrow \phi & & \downarrow \text{id} \\
1 \quad \longrightarrow \quad L^* \quad \longrightarrow \quad E' \quad \overset{\pi'}{\longrightarrow} \quad M \quad \longrightarrow \quad 1
\end{array}
\] (4)

Isomorphic quantum forms correspond to isomorphic exact sequences.

3) The aim of this section is to reconstruct a quantum form from the exact sequence. According to the section 1, this reconstruction is unique.

Let $E$ be a central extension of the $\Gamma$-groups $1 \rightarrow L^* \rightarrow E \overset{\pi}{\rightarrow} M \rightarrow 1$. The group $L^*$ embedes in $E$. Consider the $L$-algebra $L_Q[M] = E \otimes_L L$. The algebra $L_Q[M]$ is an algebra of twisted Laurent polynomials. The group $\Gamma$ acts in $L_Q[M]$ by automorphisms. The group $E$ is a group of invertible elements in $L_Q[M]$.

Consider the $k$-algebra $A_Q = L_Q[M]^\Gamma$. We shall show that $A_Q$ is a quantum form of $T$. This will finish the proof of the theorem.

3.1) The aim of this section is to proof $A_Q \otimes_k L = L_Q[M]$. Choose the elements $x_1, \ldots, x_n \in E$ such, that $\pi(x_i) = t_i$. Denote $x_m = x_1^{m_1} \cdots x_n^{m_n}$. The elements $x_m, m \in ^a M$ form the basis of $L_Q[M]$ over $L$. The algebra $A_Q$ is $k$-spanned by the elements $a = \alpha x_m + \alpha_1 x_m^{(1)} + \cdots + \alpha_n x_m^{(n)}$ such, that the set of degrees $m, m(1), \ldots, m(n)$ is a $\Gamma$-orbit in $^a M$. Let $H = St(m, \Gamma)$. Then for $h \in H$ we have $h x_m = \gamma_h x_m$, where $\gamma \in L^*$. The following equality holds $\gamma_{h_1 h_2} = \gamma_{h_1} \gamma_{h_2}$. Therefore $\gamma_h$ is a cocycle on $H$ with values in $L^*$. Apply the Hilbert Theorem 90 to the extension $L$ over $F = L^H$. There exists the element $\gamma \in L^*$ such, that $\gamma_h = h \gamma \gamma^{-1}$. Then $h(\gamma^{-1} x^m) = \gamma^{-1} x^m$.

Let $e, g_1, \ldots, g_n$ be the representatives of cosets $G/H$ such, that $g_i(m) = m(i)$. The element $g_i(x^m)$ is up to constant equals to $x^m(i)$. Denote $x^{<m>} = \gamma^{-1} x^m$, $x^{<m(i)>} = g_i x^{<m>}$. The element $a$ equals to $\beta x^{<m>} + \beta_1 x^{<m(1)>} + \cdots + \beta_n x^{<m(n)>}$, where $\beta = \alpha \gamma \in F$. $\beta_i = g_i \beta$. Since the extension $L/k$ is separable, then the extension $F/k$ is also separable. According to the Primitive Element Theorem there exists the element $\delta \in F$ such, that $F = k(\delta)$. If $f(x)$ is minimal $k$-polynomial of $\delta$, then $dim_k F = \deg f(x) = s + 1$. Let $\delta_0 = \delta, \delta_1 = g_1 \delta, \ldots, \delta_s = g_s \delta$ be all roots of the polynomial $f(x)$. Since $L/k$ is the Galois extension, then all roots are contained in $L$. The algebra $A_Q$ is spanned over the field $k$ by the elements $a_r = \delta_0^r x^{<m>} + \delta_1^r x^{<m(1)>} +$
\[ \ldots + \delta^r_x x^{<m>} = \delta_{ij}^r, r = 0, 1, \ldots, s. \]

Denote by $W$ the matrix $\left( \delta_{ij}^r \right)_{i,j=0}^s$. The determinant of $W$ is equals to $\prod_{i>j}^s (\delta_i - \delta_j) \neq 0$. There exists the $s + 1$-tuple $\nu = (\nu_0, \nu_1, \ldots, \nu_s)$ such that $\nu W = (1, 0, \ldots, 0)$. Then $\sum_{r=0}^s \nu_r a_r = x^{<m>}$ and $x^m \in A_Q \otimes_k L$ for all $m \in \mathbb{a} M$. This proves $A_Q \otimes_k L = L_Q[M]$. 

3.2) The aim of this section is to construct the map of coaction on $A_Q$. Recall, that the map $E \mapsto M$ is a homomorphism of $\Gamma$-groups. Consider the $\Gamma$-homomorphism $R_L = (\pi, id) : E \mapsto M \times E$. Extent it to the $\Gamma$-invariant homomorphism $R_L : L_Q[M] \mapsto L_Q[T] \otimes_L L_Q[M]$. The homomorphism $R_L$ coincides with the coaction of diagonal torus $R_L(x_i) = t_i \otimes x_i$. Show, that the restriction $R_L$ on $A_Q$ be a coaction of $k[T]$. The image $R_L(A_Q) = R_L(L_Q[M]^\Gamma)$ contains in $(L[T] \otimes_L L_Q[M])^\Gamma$. Recall that $k[T] = L[T]^\Gamma$. Let us show

\[ (L[T] \otimes_L L_Q[M])^\Gamma = L[T]^\Gamma \otimes_k L_Q[M]^\Gamma \]

Let $c \in (L[T] \otimes_L L_Q[M])^\Gamma$. Consider the normal basis $a_1, \ldots, a_p$ in the extension $L/k$. The Galois group $\Gamma$ acts on the normal basis by permutations. The element $c$ can be represented in the form $c = \sum_{i=1}^p (f_i \otimes \alpha_i) a_i$, where $f_i, \alpha_i$ are $\Gamma$-invariant elements in $L[T], L_Q[M]$. For all $\sigma \in \Gamma$ we have $c = \sigma c$. Hence

\[ \sum_{i=1}^p (f_i \otimes \alpha_i) a_i = \sum_{i=1}^p (f_i \otimes \alpha_i)(\sigma a_i) = \sum_{i=1}^p (f_i \otimes \alpha_i) a_{\sigma(i)} \]

Then $f_{\sigma(i)} \otimes \alpha_{\sigma(i)} = f_i \otimes a_i$ for all $i$. The group $\Gamma$ transitively acts on the normal basis. We have $c = f_1 \otimes \alpha_1(a_1 + \cdots + a_p) \in L[T]^\Gamma \otimes_k L_Q[M]^\Gamma$. \(\square\)

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**On construction of extensions**

Let $Q = (q_{ij})_{i,j=1}^n$ be a matrix with the entries

\[ q_{ij} \in L^*, q_{ij} q_{ji} = q_{ii} = 1 \]  \(5\)

Let $L_Q[M] = L_Q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be an algebra of twisted Laurent polynomials over the field $L$. Denote by $k_Q[T]$ or $A_Q$ the quantum form of torus $T$, constructed by the extension (2). This algebra coincides with $L_Q[M]^\Gamma$. Consider bihomomorphism $Q : \mathbb{a} M \times \mathbb{a} M \mapsto L^*$, where $Q(e_i, e_j) = q_{ij}$. For $m = n \in e_1 + \cdots + m_n e_n$ and $k = k_1 e_1 + \cdots + k_n e_n$ we have

\[ Q(m, k) = \prod_{i,j=1}^n q_{ij}^{m_i k_j} \]
The bihomomorphism \( Q \) satisfies the condition \( Q(m, k) = Q(k, m)^{-1} \). Denote \( x^m = x_1^{m_1} \cdots x_n^{m_n} \). We have

\[
x^m x^k = Q(m, k)x^k x^m
\]

(6)

For all \( \sigma \in \Gamma \) the element \( \sigma(x^m) \) is equals to \( x^\sigma m \) up to constant:

\[
\sigma(x^m) = \gamma(\sigma(m)) x^\sigma m, \gamma(\sigma(m)) \in L^*
\]

(7)

Acting by \( \sigma \) on the left and right sides of the equation (6), we have

\[
Q(\sigma m, \sigma k) = \sigma Q(m, k)
\]

(8)

As a result by the exact sequence (2) there constructed the matrix \( Q \) with the conditions (5) and (8).

**Question.** Let \( Q \) be a matrix with the conditions (5) and (8), Can we reconstruct the exact sequence (2)? Or (the same in different terms), can we define the action of Galois group on \( L_Q[M] \) by automorphisms such that (7) holds?

The answer is positive in the following two cases.

1) Let \( M \) is a permutation module. The group \( \Gamma \) acts by permutations on the basis \( ^\sigma(e_i) = e_{\sigma(i)}, i = 1, n \). The equality (8) for the matrix \( Q \) is equivalent to

\[
q_{\sigma(i), \sigma(j)} = ^\sigma q_{ij}
\]

(9)

Let \( L\{M\} = L\{x_1^{\pm 1}, \ldots, x_n^{\pm 1}\} \) be a free algebra generated by \( x_1^{\pm 1}, \ldots, x_n^{\pm 1} \).

The action \( ^\sigma(x^m) = x_{\sigma(i)}^{\sigma(m)} \) extents to the automorphism of \( L\{M\} \). Denote by \( I \) the ideal, generated by the elements \( x_ix_j - q_{ij}x_jx_i \) in \( L\{M\} \). It follows from (9), that the ideal \( I \) is invariant under the action of the group \( \Gamma \). The group \( \Gamma \) acts by autimorphisms in the factor-algebra \( L_Q[M] = L\{M\}/I \).

2) Let \( M \) be an arbitrary module over the cyclic group of the second order \( \Gamma = \{e, \sigma : \sigma^2 = 1\} \). The module \( M \) is a direct sum indecomposable modules. Indecomposable modules have rank 1 or 2. In the first case the action \( \sigma e_1 = \pm e_1 \). In the second case \( \sigma \) acts by permutations of basis elements \( \sigma e_1 = e_2, \sigma e_2 = e_1 \). the action \( \Gamma \) extents to the action in \( L\{M\} \) by automorphisms: \( ^\sigma x_1 = x_{\sigma_1}^{\pm 1} \) for rank 1 and \( ^\sigma x_1 = x_2, ^\sigma x_2 = x_1 \) for rank 2. Further the \( \Gamma \)-action in \( L_Q[M] \) is constructed similar to 1).

**Center of** \( A_Q \)
Denote by $Z_Q[M]$ the center of $L_Q[M]$. The Galois group $\Gamma$ acts in $L_Q[M]$ by automorphisms. The center $Z_Q[M]$ is invariant with respect to this action. Denote $aM_c = \{m \in aM : Q(m,k) = 1, \forall k \in aM\}$. It follows from (5) and (8) that the subgroup $aM_c$ is invariant with respect to $\Gamma$-action. Denote by $M_c$ the group $aM_c$ in multiplicative form. The center $Z_Q[M]$ is a linear space over the field $L$, spanned by the elements $x^m, m \in aM_c$. We have $Z_Q[M] = L[M_c]$. Denote by $Z_Q$ the center of $A_Q$.

**Proposition 1.** The center $Z_Q$ coincides with $Z_Q[M]^\Gamma$.

Proof. The subalgebra $Z_Q[M]^\Gamma$ consists of central elements and is contained in $A_Q = L_Q[M]^\Gamma$. Therefore $Z_Q[M]^\Gamma$ is contained in $Z_Q$. From the other side, $Z_Q \subset Z_Q \otimes L \subset Z_Q[M]$. Therefore $Z_Q = Z_Q \subset Z_Q[M]^\Gamma$. Denote by $T_c$ the $k$-torus corresponding to $\Gamma$-module $M_c$. By the embedding of groups $i : M_c \hookrightarrow M$ we can construct the embedding of $k$-algebras $i : k[T_c] \hookrightarrow k[T]$ and the homomorphism $i_* : T \mapsto T_c$.

**Corollary 1.1.** $Z_Q = k[T_c]$. □

Recall that the coaction $R : A_Q \hookrightarrow k[T] \otimes_k A_Q$ lifts to $R_L = R \otimes 1 : A_Q,L \mapsto L[T] \otimes L A_Q,L$ and coincides over $L$ with coaction of diagonal torus $R_L(x_i) = t_i \otimes x_i$. Hence $R_L : Z_Q[M] \hookrightarrow L[T_c] \otimes Z_Q[M]$. The algebra $Z_Q[M]$ is spanned over the field $L$ by the elements $x^m, m \in aM_c$. The algebra $L[T_c]$ is spanned over the field $L$ by the elements $t^m, m \in aM_c$. The algebras $Z_Q[M]$ and $L[T_c]$ are isomorphic. The coaction $R_L : L[T_c] \hookrightarrow L[T_c] \otimes L[T_c]$ coincides with comultiplication on the diagonal torus $T_c \otimes L$. The restriction $R_L$ on $A_Q$ is equal to $R$. Similar to the proof of Theorem 1, one can show, that $R : L[T_c]^\Gamma \hookrightarrow L[T_c]^\Gamma \otimes_k L[T_c]^\Gamma$. The coaction $R$ coincides with comultiplication $k[T_c] \hookrightarrow k[T_c] \otimes_k k[T_c]$.

**Corollary 1.2.** Coaction $R$ on the center $Z_Q = k[T_c]$ coincides with comultiplication in $k[T_c]$. The action of $T(k)$ on the center is a superposition of the homomorphism $i_*$ and the regular action of the torus $T_c$. □

Consider the case when $q_{ij}$ are all roots of degree $l$ of unity. Then $q_{ij} = \epsilon^{s_{ij}}$, where $s_{ij} \in \mathbb{Z}$, $\epsilon \in L$ is a primitive root of degree $l$ of unity. The center $Z_Q[M]$ of the algebra $L_Q[M]$ contains the subalgebra $L[x_1^{\pm l}, \ldots, x_n^{\pm l}]$. The center $Z_Q$ of the algebra $A_Q$ contains the subalgebra $Z_Q^{(l)} = L[x_1^{\pm l}, \ldots, x_n^{\pm l}]^\Gamma$.

We shall call this algebra a $l$-center. The algebra $Z_Q^{(l)}$ is an algebra of regular functions on the $k$-torus $T^{(l)}$, which corresponds to the $\Gamma$-module $aM^{(l)} = l(aM) = \mathbb{Z}l_1 + \cdots + \mathbb{Z}l_n$. By the embedding of groups $i_l : M^{(l)} \hookrightarrow M$ we can construct the embedding of $k$-algebras $i_l : k[T^{(l)}] \hookrightarrow k[T]$ and the ho-
momorphism $i_{t,*} : T \mapsto T^{(l)}$. The homomorphisms $i_*$ and $i_{t,*}$ are isogenies. Similar to the case of center the coaction on the $l$-center coincides with the comultiplication on $T^{(l)}$. The action of $T$ on the $l$-center is a superposition of the homomorphism $i_{t,*}$ and the regular action of the $T^{(l)}$.

Spectrum of quantum torus at roots of 1

We shall consider the case $q_{ij} = e^{a_{ij}}$, where $s_{ij} \in \mathbb{Z}_l$, $e \in L$ is a primitive root degree $l$ of unity.

Let $\chi \in T_c(k) = Hom_{k-alg}(Z_Q,k)$. Denote by $\chi_l$ the restriction of $\chi$ on the $Z_Q^{(l)}$. Extent $\chi$ to the $\Gamma$-invariant $L$-homomorphism $\chi_L$ of the algebra $Z_Q[M] = Z_Q \otimes_k L$ into $L$. Similar extent $\chi_l$ to the $\Gamma$-invariant $L$-homomorphism $\chi_l,L$ of the algebra $Z_Q^{(l)}[M] = Z_Q^{(l)} \otimes_k L$ into $L$.

**Notations 1.**

$P_{\chi,L} = L_Q[M]/\text{Ker}\chi_L$, $P_{\chi,l}^{(l)} = L_Q[M]/\text{Ker}\chi_L,l$

$P_{\chi} = L_Q[M]/\text{Ker}\chi$, $P_{\chi,l}^{(l)} = L_Q[M]/\text{Ker}\chi_l$

**Notations 2.**

1) $C_l(a,b,\omega)$ is an algebra over the field $L$ generated by the elements $x, y$ with the relations $x^l = a, y^l = b, xy = \omega yx, a, b \in L^*, \omega = e^k, k$ divide $l$;

2) $C_l(a, b) := C_l(a, b, \omega)$, if $\omega$ is a primitive root of degree $l$ of 1; This algebra is called a cyclic algebra ;

3) $C_l(a) := L[x]/(x^l - a), a \in L^*$;

**Definition.** The tensor product of some copies of cyclic algebras is called a polycyclic algebra.

**Proposition 2.**

1) $P_{\chi}^{(l)}$ and $P_{\chi,l}^{(l)}$ are semisimple algebras, $P_{\chi}$ and $P_{\chi,L}$ are central simple algebras , $P_{\chi,L}$ is a polycyclic algebra;

2) $P_{\chi}^{(l)} \otimes_k L = P_{\chi,l}^{(l)}$, $P_{\chi} \otimes_k L = P_{\chi,L}$

**Proof.** One can choose the set of generators in the algebra $L_Q[M]$ of twisted Laurent polynomials as follows: $x_i, y_j, z_i$, $i = \overline{1,p}$, $j = \overline{1,n-2p}$ with the relations $x_i y_j = e^{k_i} y_j x_i, k_1 | k_2, \ldots, k_p | l[3]$. The else pairs of generators commutes. Denote $d_i = \frac{l}{k_i}$.

The center $Z_Q[M]$ of the algebra $L_Q[M]$ is generated by $x_i^{d_i}, y_j^{d_i}, z_j$; the $l$-center $Z_Q^{(l)}[M]$ is generated by $x_i^l, y_j^l, z_j^l$. We have

$$P_{\chi,l}^{(l)} = \otimes_{i=1}^p C_l(a_i, b_i, e^{k_i}) \otimes_{j=1}^{n-2p} C_l(c_j),$$
The algebra $A$ action of $\Gamma$ on $a$ algebra $L$.

In the case $k$ coinsides with the algebra of twisted Laurent polynomials $\sigma$ if the following matrices:

$$
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix},
\begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix},
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
$$

Note, that $(\ker \chi L)^\Gamma = \ker \chi$ and $\ker \chi \otimes L = \ker \chi L$. Hence we have

$$
P_\chi \otimes L = (A_Q / \ker \chi) \otimes_k L = A_Q \otimes L / \ker \chi \otimes L = L_Q[M] / \ker \chi L = P_{\chi L}
$$

Similar one can get $P^{(l)}_\chi \otimes_k L = P^{(l)}_{\chi L}$. □

**Quantum algebraic tori of small dimension**

Let $L = k(\alpha)$, $\alpha^2 = D$, $D \in k$ be a quadratic extension of the field $k$. The Galois group of $L/k$ is $\Gamma = Gal(L/k) = \{1, \sigma : \sigma^2 = 1\}$ In every $\Gamma$-module $^aM$ of rank 2 there exists a basis such, that the matrix of $\sigma$ is equal to one of the following matrices: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The algebra $L_Q[M]$ is generated by $x_1^{\pm 1}, x_2^{\pm 1}$ with the relations $x_1x_2 = qx_2x_1$. The action of $\Gamma$ on $^aM$ extents to the action on $L_Q[M]$, if $q \in k$ in 1,3 cases and if $^aq = q^{-1}$ (i.e. $N(q) = 1$) in 2,4 cases. Our aim to find the generators of $A_Q = L_Q[M]^\Gamma$ and there relations.

1) $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The group $\Gamma$ trivially acts on $^aM$, $q \in k$. The algebra $A_Q$ coinsides with the algebra of twisted Laurent polynomials $k_Q[M]$.

In the case $q = 1$, we have $A_Q = A = k[x_1^{\pm 1}, x_2^{\pm 1}]$ and $T(k) = k^* \times k^*$

2) $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Denote $q = \lambda + \alpha \mu$. In this section $N(q) = \lambda^2 - D\mu^2 = 1$.

The algebra $A_Q$ is generated by $x = x_1^{\pm 1}$, $z_1 = \frac{x_2 + x_2^{-1}}{2}$, $z_2 = \frac{x_2 - x_2^{-1}}{2\alpha}$ with the relations

$$
x \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \lambda & D\mu \\ \mu & \lambda \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} x, \quad z_1^2 - Dz_2^2 = 1, \quad z_1z_2 = z_2z_1
$$

If $q = 1$, then $A_Q = A$ coinsides with quotient algebra $k[x_1^{\pm 1}, z_1, z_2]$ with respect to the ideal generated by $z_1^2 - Dz_2^2 = 1$. The group $T(k) = k^* \times T_{L/k}^{(1)}$. 

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where $T^{(1)}_{L/k} = \{ x \in L^* : N(x) = 1 \}$.  

3) $\sigma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Denote $y_1 = \frac{x_1 + x_2^{-1}}{2}$, $y_2 = \frac{x_1 - x_2^{-1}}{2}$, $z_1 = \frac{x_2^2 - x_2^{-1}}{2}$, $z_2 = \frac{x_2 - x_2^{-1}}{2}$. The algebra $A_Q$ is generated by $y_1, y_2, z_1, z_2$ with the relations $y_1 z_1 + D y_2 z_2 = q^\pm (z_1 y_1 \pm D z_2 y_2)$, $y_2 z_1 \pm y_1 z_2 = q^\pm (z_2 y_1 \pm z_1 y_2)$, $y_1^2 - D y_2^2 = z_1^2 - D z_2^2 = 1$, $[y_1, y_2] = [z_1, z_2] = 0$. In the case $q = 1$ the algebra $A_Q = A$ coincides with the quotient algebra $k[y_1, y_2, z_1, z_2]$ with respect to ideal generated by $y_1^2 - D y_2^2 - 1$, $z_1^2 - D z_2^2 - 1$. The group $T(k)$ coincides with $T^{(1)}_{L/k} \times T^{(1)}_{L/k}$.

4) $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $q = \lambda + \alpha \mu$.

Let $q \neq -1$. Denote $u = \frac{x_1 + x_2}{2}$, $v = \frac{x_1 - x_2}{2\alpha}$, $w = \frac{1 + \sigma a}{2} x_1 x_2$. One can proof the following relations:

\begin{align*}
\text{a)} & \ u^2 - D v^2 = w, \\
\text{b)} & \ uv - vu = -\frac{\mu}{1 + \lambda} w, \\
\text{c)} & \ w \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \lambda & -D \mu \\ -\mu & \lambda \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} w
\end{align*}

The formula c) is derived from a) and b).

The algebra $A_Q$ is generated by $u, v, (u^2 - D v^2)^{-1}$ with the unique relation $(1 + \lambda)(uv - vu) + \mu(u^2 - D v^2) = 0$.

In the case $q = -1$ denote $u = \frac{x_1 + x_2}{2}$, $v = \frac{x_1 - x_2}{2\alpha}$, $w = \alpha x_1 x_2$. One can proof the following relations:

\begin{align*}
\text{a)} & \ u^2 - D v^2 = 0, \\
\text{b)} & \ uv - vu = -\frac{w}{D}, \\
\text{c)} & \ w \begin{pmatrix} u \\ v \end{pmatrix} = -\begin{pmatrix} u \\ v \end{pmatrix} w
\end{align*}

The formula c) is derived from a) and b).

The algebra $A_{-1}$ is generated by $u, v, (uv - vu)^{-1}$ with the unique relation $u^2 - D v^2 = 0$.

In the case $q = 1$ the algebra $A_Q = A = k[u, v, (u^2 - D v^2)^{-1}]$. The group $T(k)$ coincides with $L^*$.

**Case of roots of 1**

Let $q$ be a primitive root of odd degree $l$ of unit. In all of above cases 1)-4) the center $Z_Q[M]$ coincides with the $l$-center. $Z_Q^{(l)}[M] = L_Q[x_1^{\pm 1}, x_2^{\pm 1}]$. The
center $Z_Q$ is a $k$-form of the center $Z_Q[M]$ (Proposition 1). Our aim is to study the structure of the algebras $P(\chi)$ in common point.

Denote by $K = \text{Fract}(Z_Q)$ the field of fractions of $Z_Q$ and $K_L = K \otimes_k L = \text{Fract}(Z_Q[M])$ is the field of fractions of $Z_Q[M]$. Let $\chi : Z_Q \mapsto K$ be an embedding $Z_Q$ into $K$. Extend $\chi$ to $\Gamma$-invariant homomorphism $Z_Q[M]$ into $K_L$.

Denote $\chi(x_1) = a_1$, $\chi(x_2) = a_2$. The algebra $P_{\chi,L}$ coincides with $C_1(a_1, a_2)$ and $P_{\chi}$ is its $K$-form.

**Proposition 3.** In the cases 1), 2), 4) the skew field $P_{\chi}$ is a cyclic crossed product over the field $K$. (The case 3) is unknown to the author).

**Proof.** It is sufficient to prove that $P_{\chi}$ contains the maximal subfield, which is a Galois field with cyclic Galois group. It is trivial in the case 1). In the cases 2) and 4) there exists the element $y \in P_{\chi,L}$ such that $^s y = y^{-1}$ (in the case 2) put $y = x_2$; in the case 3) put $y = x_1^{-1} x_2$). In every case $y^l \in K$. As $q \in L$, then $K_L(y)$ is a cyclic extension of the field $K_L$. Denote $H = \text{Gal}(K_L(y)/K_L) = \{1, s, \ldots, s^{l-1} : s^l = 1\}$, $^s y = q y$. Denote by $G$ the Galois group of the extension $K_L(y)/K$. The group $G$ contains two subgroups $\Gamma$ and $H$. Note that $^s y = ^s y^{-1} = q^{-1} y^{-1} = ^s (q y) = ^s s y$. The subgroups $\Gamma$ and $H$ commutes. As $[K_L(y) : K] = 2l$, then $G = H \times \Gamma$. The factor group $G/\Gamma$ is isomorphic to $H$. The field $K_L(y)^\Gamma$ is a cyclic extension of degree $l$ over $K$. □

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