A relativistic bouncer on a vibrating surface

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Abstract. We consider a special relativity scenario in order to study the dynamical behavior of a particle flying under the influence of a gravitational field, colliding successive times against a rigid vibrating surface, via a restitution coefficient. We define two re-scaled dimensionless dynamical variables, namely: the relative particle velocity $W$ with respect to the surface’s velocity, and the real parameter $\tau$ accounting for the temporal evolution of the system. In order to analyze the system’s nonlinear dynamical behavior, we construct the mapping at the particle-surface contact point, described for the $k^{th}$ collision, by the couple of variables $(\tau_k W_k)$. Then, from the dynamical mapping, we compute the fixed point trajectory and analyze its stability. We find the dynamical behavior of the fixed point trajectory to be stable or unstable, according to the values of the re-scaled vibrating surface amplitude, the restitution coefficient and the auxiliary variable $\beta \in (0, 1)$, sweeping from the non relativistic to the ultra relativistic regime. Other important dynamical aspects such as the phase space volume and the one cycle vibrating surface (decomposed into absorbing and transmitting regions) are also discussed. Furthermore, the model rescues well known results in the non relativistic limit.

1. Introduction

The bouncer was introduced as an alternative to the Fermi-Ulam model for the cosmic ray Fermi’s acceleration mechanism [1, 2]. This model considers the dynamics of successive collisions, via a restitution coefficient, between a bouncing particle flying under the influence of a gravitational field against a rigid vibrating surface; within the special relativity scenario. This dynamical system is exactly iterated in time, under the assumption that the surface is unaffected by repeated impacts of the ball. Different values of the restitution coefficient yield different motion regimes, such as periodic, non-periodic and chaotic motion. The model has been extensively studied, both theoretically and experimentally, within a nonrelativistic [3, 4, 5] as well as a relativistic framework [6, 7]. Following [8, 9], the model we study considers explicit dynamical equations for the system. By doing this, we are able to define a parameter space where the various dynamical regimes can be easily characterized. In section 2 are presented the dynamical equations governing the system on the special relativity scheme, and the mapping is presented in section 3. Fixed points are computed and its stability are discussed in section 4. Some general aspect related to the dynamics of the system are presented in section 5 and the conclusion follows in section 6.

2. The relativistic bouncer

We consider a particle with unitary rest mass flying under the influence of a gravitational field $(g)$. The particle collides repeated times with a vibrating rigid platform. The vibration is sinusoidal with period $T$ and strength $\Gamma/2\pi$. As usual, the nature of the collision is defined
by a restitution coefficient $\alpha$. Convenient dimensionless variables are $\tau$ for time, $X$ for the trajectory particle and $S$ for the vibrating amplitude, defined by $t_{old} = T\tau$, $X_{old} = 0.5gT^2X$ and $S_{old} = 0.5gT^2S$, respectively. These equations of motion for the nonrelativistic case, with dissipation were presented in [9]. Here, we present the relativistic case (with no dissipation), the equations of motion being [10],

$$\frac{d}{d\tau} \left[ \frac{V}{\sqrt{1-\beta^2V^2}} \right] = -2\beta, \quad \frac{d}{d\tau} \left[ \frac{\dot{\beta}}{\sqrt{1-\beta^2S^2}} \right] = -(2\pi)^2 S; \quad (1)$$

where $V = \dot{X}$ and $\beta = 0.5gT/c$, with $c$ the speed of light. Integrating these equations with initial conditions $X_0$ and $V_0$ at $\tau_0$, we have

$$V(\tau) = \frac{\sqrt{1-\beta^2V_0^2}}{\sqrt{1+\beta^2|\tau_0-2(\tau-\tau_0)|^2}} \dot{X}, \quad X(\tau) = X_0 + \frac{1}{2\beta^2} \left[ \sqrt{1+\beta^2v_0^2} - \sqrt{1+\beta^2[\tau_0-2(\tau-\tau_0)]^2} \right], \quad (2)$$

with $v_0 = V_0/\sqrt{1-\beta^2V_0^2}$ and the approximated solution (in the harmonic balance scheme [11])

$$\dot{S}(\tau) = \frac{4\sqrt{1+(\frac{\dot{\beta}}{2})^2}}{\sqrt{1+\beta^2}\cos(2\pi\tau)} \Gamma \cos(2\pi\tau), \quad S(\tau) = \frac{\sqrt{1+(\frac{\dot{\beta}}{2})^2}}{2\pi\beta} \arcsin \left[ \frac{\beta S}{\sqrt{1+\beta^2}} \right] \sin(2\pi\tau). \quad (3)$$

All the above expressions yield the Newtonian limit as $\beta \to 0$.

3. The bouncer mapping

The temporal variables $\tau_0$ and $\tau_1$ are respectively, the time at which the system initiate its dynamics and the time at which the first collision occurs. With $X_0 = S(\tau_0)$, we define the relative distance particle-surface as $D(\tau)$ and the respective relative velocity $W(\tau)$ (according to [10]). Then for $\tau_0 \leq \tau \leq \tau_1$,

$$D(\tau) = \{S(\tau_0) - S(\tau)\} + \frac{1}{2\beta^2} \left[ \sqrt{1+\beta^2v_0^2} - \sqrt{1+\beta^2[\tau_0-2(\tau-\tau_0)]^2} \right],$$

$$W(\tau) = \frac{V(\tau) - \dot{S}(\tau)}{1-\beta^2V(\tau)S(\tau)}. \quad (4)$$

We denote $W_1$ as the relative velocity particle-surface just after the first collision have occurred, and we construct the dynamical mapping $\mathcal{M}((\tau_0, W_0)) \to (\tau_1, W_1)$ by imposing at $\tau_1$ both the relative distance and the relative velocity, to satisfy respectively

$$D(\tau_1) = 0,$$

$$W_1(\tau_1) = -\alpha W_1(\tau_1), \quad (5)$$

Figure 1. The dynamic of the system represented in the $(\tau, W)$ space.
where $\alpha$ is the restitution coefficient.

Figure 1 schematically depicts the system’s dynamics in the $(\tau, W)$-space, for parameters $\alpha = 0.5$, $\beta = 0.125$ and $\Gamma = 0.70$. The successive collisions occurred at $\tau_0 = 0.36$ and $\tau_1 = 1.68$, with relative velocities particle-system $W_0 = 1.665$ and $W_1 = 2.974$; respectively. Notice that $\beta W_{0,1} < 1$, as it should be.

4. Fixed points and their stability

The mapping fixed points are defined by $\tau_4$. Fixed points and their stability

The mapping fixed points are defined by $(\tau_*, W_*)$, satisfying the condition $\tau_0 = \tau_*$, $\tau_1 = \tau_* + m$ (the integer $m$ represents a platform cycle) and $W_0 = W_1 = W_*$, thus we have

$$D(\tau_* + m) = \frac{1}{2\alpha^2} \left[ \sqrt{1 + \beta^2 v_*^2} - \sqrt{1 + \beta^2 (v_* - 2m)^2} \right] = 0,$$

by inspection we find $v_* = m$. Note that at the fixed point: $V_1 \rightarrow -V_*$ and $V_* = \frac{m}{\sqrt{1 + \beta^2 m^2}}$.

In order to compute $\dot{S}_*$ and $W_*$, we take into account the second condition in Eqn.(5) and the initial condition

$$W_0 = \frac{v_0 - \dot{S}_0}{1 - 3\beta^2 S_0 v_0}.$$

We obtain approximately (to lowest order in $\beta^2$)

$$\dot{S}_* = \frac{(1 - \alpha)}{(1 + \alpha)} \frac{V_*}{(1 - \beta^2 V_*^2)} \quad , \quad W_* = \frac{2\alpha}{(1 + \alpha)} \frac{V_*}{(1 + \beta^2 (1 + \alpha)^2 V_*^2)}.$$

Now, we discuss the stability of the fixed points by allowing small displacements in the neighborhood of $\tau_0 = \tau_* + \delta \tau_0$, $\tau_1 = \tau_* + \delta \tau_1$, $W_0 = W_* + \delta W_0$ and $W_1 = W_* + \delta W_1$. From Eqn.(5), we have

$$\Delta D = \dot{S}_* (\delta \tau_0 - \delta \tau_1) + \frac{1}{2} V_* (\Delta v_0 + \Delta v_1),$$

$$\delta W_1 = -\alpha \frac{[(1 - \beta^2 \dot{S}_*)^\Delta V_1 - (1 - \beta^2 V_*^2) \Delta \dot{S}_1]}{(1 + \beta^2 V_* S_*).}$$

We define the convenient parameter $\sigma = -\frac{1}{2} \frac{\Delta \dot{S}_*}{\Delta V_1}$, thus the linear variations become:

$$\Delta \dot{S}_1 = -2\sigma \delta \tau_1, \quad \Delta V_1 = \frac{\Delta v_0}{(1 + \beta^2 V_*^2)}$$

and $\Delta v_1 = \Delta v_0 - 2(\delta \tau_1 - \delta \tau_0)$. Then, with

$$V_0 = \frac{v_0}{\sqrt{1 + \beta^2 v_*^2}} = \frac{W_0 + \dot{S}_0}{1 + \beta^2 W_0 S_0},$$

we obtain

$$\Delta v_0 = \frac{(1 + \beta^2 V_*^2)}{(1 + \beta^2 W_0 S_*)} \left[ -2\sigma (1 - \beta^2 V_*^2) \delta \tau_0 + (1 - \beta^2 V_*^2) \delta W_0 \right].$$

Finally, the linearized form of Eqn.(5) is given by

$$\left[ \begin{array}{c} \delta \tau_1 \\ \delta W_1 \end{array} \right] = \left[ \begin{array}{c} 1 - \sigma \theta q \mathcal{P} \\ 2\alpha \sigma t^2 \{ [p (1 - \theta) + \sigma \theta q r] \mathcal{P} - r \} \end{array} \right], -\alpha t^2 \{ p (1 - \theta) + \sigma \theta q r \} \mathcal{Q} \left[ \begin{array}{c} \delta \tau_0 \\ \delta W_0 \end{array} \right].$$

where

$$\mathcal{P} = 1 - \beta^2 V_*^2, \quad \mathcal{Q} = 1 - \beta^2 V_*^2 \dot{S}_*, \quad \mathcal{R} = 1 - \beta^2 \dot{S}_*^2, \quad t = (2 - \mathcal{Q})^{-1},$$

$$p = \frac{(1 - \mathcal{Q})}{2 - (p + \mathcal{Q} + \mathcal{R} + p R)}, \quad q = \frac{(1 - \mathcal{Q}) r^{-3/2}}{2 - (p + \mathcal{Q} + \mathcal{R} + p R)}, \quad r = (1 + \beta^2 m^2)^{-1}, \quad \theta = \frac{(1 - \mathcal{Q})}{2(1 - \mathcal{Q})}.\]
We compute the trace $\Sigma$ and the determinant $\Pi$ of the monodromy matrix $\mathcal{M}$ Eqn. (12)

$$
\Sigma = 1 + \alpha (\theta - 1) t^2 p Q - \sigma \theta q \left[ (1 + \alpha t^2 r) - (1 - P) - \alpha t^2 r (1 - Q) \right],
$$

$$
\Pi = \alpha (\theta - 1) t^2 p Q.
$$

The $\mathcal{M}$ eigenvalues are given by $\chi_{1,2} = \frac{\Sigma}{2} \pm \sqrt{\left(\frac{\Sigma}{2}\right)^2 - \Pi}$, and we highlight some features, namely:

A) Complex eigenvalues $\left(\frac{\Sigma}{2}\right)^2 - \Pi < 0$ when $\sigma^- < \sigma < \sigma^+$,

B) Real eigenvalues $\left(\frac{\Sigma}{2}\right)^2 - \Pi > 0$ when $\sigma < \sigma^-$ or $\sigma > \sigma^+$,

where $\sigma^\pm = \left[ \frac{1 \pm \sqrt{\Pi}}{\sqrt{\theta (1 + \alpha t^2 r q)}} \right]^2 \left[ 1 - \frac{(1 - P)^2 + \alpha t^2 r (1 - Q)}{(1 + \alpha t^2 r)} \right]$.

The stability regions of the system are as presented below:

| Complex $\chi$ | $|\chi_{1,2}| = \sqrt{\Pi} < 1$, stable/focus | Depending on $\alpha, m$ and $\beta$ |
|----------------|------------------------------------------|---------------------------|
| $|\chi_{1,2}| = \sqrt{\Pi} > 1$, unstable/source |

| Real $\chi$ | $|\chi_{1,2}| < 1$, stable/node | Depending on $\alpha, m, \beta$ and $\Gamma$ |
|--------------|-------------------------------|----------------------------------|
| $|\chi_{1,2}| > 1$, unstable/node |

Notice that for a given value of $\sigma$, we have to compute the corresponding $\Gamma$ value from the solution of

$$
(\frac{\sigma}{2})^2 q^4 + (q^2 - \beta^2 \zeta^2)^2 \left[ (1 + \beta^2 \Gamma^2) \zeta^2 - q^2 \Gamma^2 \right] = 0,
$$

where $q = \sqrt{1 + \left(\frac{\beta \Gamma}{2}\right)^2}$ and $\zeta_* = \frac{(1 - \alpha)}{(1 + \alpha)} \sqrt{1 + \beta^2 m^2 / \left[ 1 + \beta^2 \left(\frac{1 - \alpha}{1 + \alpha}\right)^2 m^2 \right]}$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{We plot $\sigma$ and $\Gamma$ versus $\alpha$, for both $m = 2$ and $m = 8$; with $\beta = 0.125$, respectively. Values of $\sigma$ above the continuous red line and below the continuous black line define real eigenvalues. The region between the continuous-black-line and the continuous-red-line define the complex eigenvalues.}
\end{figure}
Figure 2 displays $\sigma$ and $\Gamma$ as functions of the restitution coefficient $\alpha$, for $\beta m = 0.25$ and $\beta m = 1.0$. For the $\sigma$’s values above the continuous red line and below the continuous black lines the eigenvalues are real. For values in between the red and the black lines the eigenvalues are complex.

5. The global dynamics

5.1. The phase space volume

The system will remain bounded as long as the parameters $\alpha$ and $\beta$ are chosen appropriately, even both dynamical variables $W$ and $\Delta \tau$. We consider the reduced phase space (from eqn.5) in order to estimate the upper bound $W_{\text{max}}$ and $\Delta \tau_{\text{max}}$. A rough upper bound can be derived from Eqn.(4),

$$\frac{\rho \Gamma}{\pi \sqrt{1 + \beta^2 \Gamma^2}} + \frac{\Delta \tau (\tau_0 - \Delta \tau)}{\sqrt{1 + \beta^2 \nu_0^2}} , \quad W(\tau) \leq \alpha \left\{ \frac{\rho \Gamma}{\pi \sqrt{1 + \beta^2 \Gamma^2}} - \nu \right\} . \quad (16)$$

From the above relations by minimizing $D(\tau)$ we obtain $\rho \Gamma / \sqrt{1 + \beta^2 \Gamma^2} = \zeta_\ast$. With $\pi \Lambda = \zeta_\ast \sqrt{1 + \beta^2 m^2}$ and $\kappa = \sqrt{m^2 + 4 \Lambda / [1 + \beta^2 (m^2 + 4 \Lambda)]}$, we finally obtain

$$\Delta \tau_{\text{max}} = \frac{m}{\pi} + \sqrt{\left( \frac{m}{\pi} \right)^2 + \Lambda} , \quad W_{\text{max}} = \alpha \left\{ \frac{\zeta_\ast / \pi + \kappa}{1 + \beta^2 \kappa \zeta_\ast / \pi} \right\} . \quad (17)$$

5.2. Absorbing and transmitting regions

Here we consider a one cycle vibrating surface that can be decomposed into a transmitting plus an absorbing region [8], according to whether or not long flights are allowed. Then, for any take-off time $0 < \tau_0 < 1$, we define $(\vartheta, W_A)$ as the point where the collision happens with relative zero velocity. From Eqn.(4), we compute $\vartheta$ and then $W_A$ as

$$\sqrt{1 - \beta^2 \dot{S}^2(\vartheta)} \left[ S(\tau_0) - S(\vartheta) \right] + (\vartheta - \tau_0) \left[ \dot{S}(\vartheta) - (\vartheta - \tau_0) \right] = 0 , \quad W_A = \frac{\dot{S}(\vartheta) - \dot{S}(\tau_0)}{1 - \beta^2 S(\vartheta) S(\tau_0)} . \quad (18)$$

If $W_0 < W_A$ (absorbing region) the particle ride on the membrane allowing some small bouncing (chattering) or simply sticking to the membrane (locking) until the velocity is reversed and $W_0 > W_A$. In the latter case, called transmitting region, the ball is launched with a strong impulse.

6. Conclusions

We introduced a model derived from the strict equation of motion for a ball flying under the influence of a gravitational field colliding against an oscillating surface, in the special relativity scenario. The associated dynamical mapping governing the system’s motion was presented. We computed the mapping fixed points and discussed its stability, thus explicit values for the system’s parameters for different dynamical regimes are found. The bounded motion regime and the complexity of the absorbing-transmitting regions depending on the parameter $\beta$ were also computed. We emphasize that our model rescue well known nonrelativistic results as in [8].

The unbounded velocity limit in Newtonian theory, implies that some results are to be taken with a “grain of salt”, for example when considering “runaway” solutions and chaos suppression at high velocities. This work is a necessary step in order to test the well established chaotic nonrelativistic scenarios.

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