Variation on the similar-size disk tower of hanoi puzzle

S Zuchri
SMP Negeri 1 Batujajar Jl. SMP No 12 Batujajar, Bandung Barat 40561, Indonesia
Email: saefudinzuchri.71@gmail.com

ABSTRACT. The famous Tower of Hanoi puzzle was invented by Édouard Lucas in 1883. This puzzle proposed three pegs, and the number of disks with different size. The puzzle starts with the disks in a neat stack in ascending order of size on one peg, the smallest at the top. The objective of the puzzle is to move the entire stack to another peg, by following these simple rules: (1) only one disk can be moved at a time; (2) Each move consists of taking the upper disk from one of the stacks and placing it on top of another stack; (2) No disk is placed on the top of a smaller disk and the minimum number of move is the goal of this puzzle. Many variations have been proposed as exercises and challenges. Some have more than three pegs and some with colours. This paper poses a new variation. There are two or more disks with similar size. The disks are labelled from 1 to \( n \) in increasing order of size so the disk with similar size has the same label. If \( m \) is the label of the similar disks, so \( \mathcal{M}_p(n; m) \) is the minimum number moves needed to move all the disks in original peg to destination peg. We have,

\[
\mathcal{M}_2(n; m) = 2^{n-1} + 2^{n-m-1} - 1 \\
\mathcal{M}_3(n; m) = 2^{n-2} + 2^{n-m-1} - 1 \\
\mathcal{M}_p(n; m) = 2^{n-p+2} + (p - 1)2^{n-m-p+1} - 1
\]

The number moves needed to move if there are \( p_1 \) similar size disks \( m_1 \) and \( p_2 \) similar size disks \( m_2 \) is

\[
\mathcal{M}_{p_1,p_2}(n; m_1, m_2) = 2^{-p_1-p_2+4}(p_12^{-m_1} + p_22^{-m_2}) - (2^{-m_1} + 2^{-m_2}) + 1 - 1
\]

1. Introduction
The tower of Hanoi Puzzle was invented by French Mathematician Édouard Lucas in 1883. He is a famous French number theorist and recreational mathematician. The tower of Hanoi Puzzle has been around for at least a hundred years and is a favorite subject of recreational mathematician.

There are three vertical pegs and some number \( n \) of disk, all with a hole in the center that allows them to be stacked on the pegs. The disks are all different size in increasing order. They are stacked in order on one of the pegs-original peg, with the biggest on the bottom and the smallest on the top. The objective of the puzzle is to move entire stack from original peg to destination peg, by following these simple rules: (1) only one disk can be moved at a time; (2) Each move consists of taking the upper disk from one of the stack and put it on top of another stack; (3) No disk is placed on the top of smaller disk. The third peg is used to temporarily hold disk during the process. The objective of the puzzle is to find the minimum move to transfer the stack of disk from origin peg to destination peg.

We know that \( 2^n - 1 \) is a unique minimum-move solution. It is easily described recursively: for \( n > 0 \), recursively move a tower of \( n - 1 \) disks from original peg to temporary peg; move disk from original peg to destination peg; and then recursively move the tower of \( n - 1 \) disk from temporary peg to destination peg. If \( \mathcal{M}_0(n) \) is the number of moves made for \( n \) disks,
we have

\[ M_0(n) = M_0(n - 1) + 1 + M_0(n - 1) \]
\[ = 2 \times M_0(n - 1) + 1 \]
\[ = 2 \times [2 \times M_0(n - 2) + 1] + 1 \] 

or

\[ M_0(n) = 4 \times M_0(n - 2) + 2 + 1 \]
\[ = 8 \times M_0(n - 3) + 4 + 2 + 1 \]
\[ = 2^{n-1} + \ldots + 8 + 4 + 2 + 1 \]
\[ = 2^n - 1 \] 

(1)

We can see that disk \( n \) need one move, disk \( n - 1 \) need two move, disk \( n - 2 \) need four move, and disk \( m \) need \( 2^{n-m} \) move.

Many variations have been proposed, with feature such as colored disk, multiple pegs, restricred moves. Lukas himself invented a version with five pegs and a stack of disks of four different color. Indeed Paul invented variation with four pegs. This paper will discuss two, three, and \( m \) similar size disks but still with three pegs.

2. Main Result

In this section we present the formula of minimum number move \( n \) disks from original peg to destination peg. It will be easier if we divided it by some cases. First, there are two similar size disks. There are at least three disk on the stack. The second is three similar size disks, there are more than three disks on the stack. From case 1 and case 2 we can see the pattern and make the formula for \( m \) similar size disks.

2.1 Case 1:

Let \( n \) be a number of disks, two of the disks have similar size. The disks are labelled from 1 to \( n - 1 \) in increasing order of size. The similar size disks have the same label. Tabel 1 is the number of move from the stack with two similar size that have label 2.

It's clear that if there are two similar size disks, the number of move that are needed for the disk is twice that it should be. Let \( n \) be a number of disks, \( m \) is the label of the the similar size disks for \( 1 < m < n \). The number of move needed for a disk with labelled \( m \) is \( 2^{n-m-1} \). Because there are two disks with labelled \( m \), so the number of move that needed for the disks with labelled \( m \) is twice than \( 2^{n-m-1} \).

| The Number of Move at Disk \( D_i \) | The Number of The Disk |
|-----------------------------------|------------------------|
| \( D_1 \)                         | 3 4 5 6 ...           |
| \( D_2 \)                         | 2 \times 1 2 \times 2 2 \times 4 2 \times 8 ... | \( 2 \times 2^{n-2} \) |
| \( D_3 \)                         | - 1 2 4 ...            | \( 2^{n-4} \)            |
| ...                              | - - - - ...           | ...                     |
| \( D_m \)                         | - - - - ...            | \( 2^{n-m-1} \)         |
| ...                              | - - - - ...           | ...                     |
| Total Steps                      | 4 9 18 36 ...         | ...                     |
\( D_i \) is the number of moves that have label \( i \).

The total number of moves for label \( m \) is,

\[
M_2(n; m) = 1 + 2 + 4 + \cdots + 2(2^{n-m-1}) + \cdots + 2^{n-2} \\
= 1 + 2 + 4 + \cdots + 2^{n-2} + 2^{n-m-1} \\
= 2^{n-1} + 2^{n-m-1} - 1
\]  

\[ (3) \]

2.2 Case 2:

Let \( n \) is the number of disks and there are three similar size disks, the disks on a stack have label from 1 to \( n - 2 \), in increasing order. The similar size of disk have the same label. If \( m \) is the label of the similar disks then the number of move of each disk can we see at Table 2.

| The Number of Move at Disk \( D_i \) | The Number of The Disk |  \\
|-------------------------------------|-------------------------|
| \( D_1 \) | 2 | 4 | 8 | \cdots | \( 2^{n-3} \) |
| \( D_2 \) | \( 3 \times 1 \) | \( 3 \times 2 \) | \( 3 \times 4 \) | \cdots | \( 3 \times 2^{n-4} \) |
| \( D_3 \) | - | 1 | 2 | \cdots | \( 2^{n-5} \) |
| \( \cdots \) | - | - | \cdots | \cdots | \cdots |
| \( D_m \) | - | - | \cdots | \cdots | \( 2^{n-m-2} \) |
| \( \cdots \) | - | - | \cdots | \cdots | \cdots |
| **Total Steps** | 5 | 11 | 23 | \cdots | \cdots |

\( D_i \) is the number of moves that have label \( i \).

Table 2 shows that if there are three similar size disks, the number of moves for the similar disk is three times that it should be. The number of moves that are needed for disk \( m \) is \( 2^{n-m-2} \). So the number of moves that needed for the disks with label \( m \) is three times than \( 2^{n-m-2} \).

The total number of move is,

\[
M_3(n; m) = 1 + 2 + 4 + \cdots + 3(2^{n-m-1}) + \cdots + 2^{n-3} \\
= 1 + 2 + 4 + \cdots + 2^{n-3} + 2 \cdot 2^{n-m-2} \\
= 2^{n-2} + 2^{n-m-1} - 1
\]  

\[ (4) \]

Without Loss of Generality, we can find the total number of move \( m \) similar size disks. Let say \( n \) as a number of disks, and \( p \) as a number of similar size disks, \( 1 < p < n \). The disks are labelled from 1 to \( n - p + 1 \) in increasing order of size. If \( m \) is a label of similar size disks, the number of moves needed to move disk \( m \) is \( 2^{n-m-p+1} \). Because there are \( p \) disks label \( m \), so the number of moves needed to move all the disk to destination peg are,

\[
M_p(n; m) = 1 + 2 + 4 + \cdots + p(2^{n-m-p+1}) + \cdots + 2^{n-p+1} \\
= 2^{n-p+2} + (p - 1) \cdot 2^{n-m-p+1}
\]  

\[ (5) \]
We can find if there are \( p_1, p_2 \) similar size disks where their disks have different label. Let \( n \) be a number of disks, the stack of disks have \( p_1 \) disks label \( m_1 \) and \( p_2 \) disks label \( m_2 \), for \( p_1 + p_2 < n \).

The number of move that needed for a disk whom have label \( m_1 \) is \( 2^{n-p_1-p_2-m_1+2} \) and label \( m_2 \) is \( 2^{n-p_1-p_2-m_2+2} \). There are \( p_1 \) disks have label \( m_1 \) and \( p_2 \) disks have label \( m_2 \), so the number of move that needed for the disks label \( m_1 \) and \( m_2 \) consecutively are \( p_1 \times 2^{n-p_1-p_2-m_1+2} \) and \( p_2 \times 2^{n-p_1-p_2-m_2+2} \).

| The Number of Move at Disk \( D_i \) | The Number of The Disk |
|-------------------------------------|------------------------|
| \( D_1 \)                           | \( 8 \)  \( 16 \)  \( 32 \) ... \( 2^{n-p_1-p_2+1} \) |
| \( D_2 \)                           | \( 2 \times 4 \) \( 2 \times 8 \) \( 2 \times 16 \) ... \( 2 \times 2^{n-p_1-p_2} \) |
| \( D_3 \)                           | \( 2 \) \( 4 \) \( 8 \) ... \( 2^{n-p_1-p_2-1} \) |
| \( D_4 \)                           | \( 3 \times 1 \) \( 3 \times 2 \) \( 3 \times 4 \) ... \( 3 \times 2^{n-p_1-p_2-2} \) |
| ...                                | ... ... ... ... ... ... |
| \( D_m \)                           | \( - \) \( - \) ... ... ... ... ... |
| \( D_m \)                           | \( - \) \( - \) ... ... ... ... ... |
| \( D_m \)                           | \( - \) \( - \) ... ... ... ... ... |
| ...                                | ... ... ... ... ... ... |
| **Total Steps**                    | \( 21 \) \( 42 \) \( 84 \) ... ... ... |

\( D_i \) is the number of move that have label \( i \).

The total number moves needed to move all disk are,

\[
M_{p_1,p_2}(n; m_1, m_2) = 1 + 2 + 4 + \cdots + p_1(2^{n-p_1-p_2-m_1+2}) + \cdots + p_2(2^{n-p_1-p_2-m_2+2}) + \cdots + 2^{n-p_1-p_2+1} \\
= 2^{n-p_1-p_2+2} + (p_1 - 1)(2^{n-p_1-p_2-m_1+2}) + (p_2 - 1)(2^{n-p_1-p_2-m_2+2}) - 1 \\
= 2^{n-p_1-p_2+2}[1 + (p_1 - 1)2^{-m_1} + (p_2 - 1)2^{-m_2}] - 1 \\
= 2^{n-p_1-p_2+2}[p_12^{-m_1} + p_22^{-m_2} - (2^{-m_1} + 2^{-m_2}) + 1] - 1
\]

(6)

3. References

[1] Hinz A M, et al 2013 *The Tower of Hanoi—Myths and Maths* (Switzerland: Birkhäuser Basel)

[2] Allouche J A, Astoorian D, Randall J and Shallit J 1994 *Mathematical Association of America* **101** 651-658

[3] Stockmeyer P K and Lunnon F 2008 *Proc. of International Conference on Fibonacci Numbers and Their Applications* (Patras) (Dodrecht: Kluwer Academic Publisher)

[4] Stockmeyer PK 1994 *Congressus Numerantium* **102** 3-12
