The Hirzebruch-Mumford covolume of some hermitian lattices

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Abstract

Let \( L = \text{diag}(1, 1, \ldots, 1, -1) \) and \( M = \text{diag}(1, 1, \ldots, 1, -2) \) be the lattices of signature \((n, 1)\). We consider the groups \( \Gamma = SU(L, \mathcal{O}_K) \) and \( \Gamma' = SU(M, \mathcal{O}_K) \) for an imaginary quadratic field \( K = \mathbb{Q}(\sqrt{-d}) \) of discriminant \( D \) and its ring of integers \( \mathcal{O}_K \), \( d \) odd and square free. We compute the Hirzebruch-Mumford volume of the factor spaces \( \mathbb{B}^n/\Gamma \) and \( \mathbb{B}^n/\Gamma' \). The result for the factor space \( \mathbb{B}^n/\Gamma \) is due to Zeltinger [9], but as we’re using it to prove the result for \( \mathbb{B}^n/\Gamma' \) and it is hard to find his article, we prove the first result here as well.

Introduction

Let \( L \) be the lattice \( \text{diag}(1, 1, \ldots, 1, -1) \) of signature \((n, 1)\). We consider the group \( \Gamma = SU(L, \mathcal{O}_K) \) for the ring of integers \( \mathcal{O}_K \) of an imaginary quadratic field \( K = \mathbb{Q}(\sqrt{-d}) \) of discriminant \( D \), \( d \) odd and square free. Denote by \( \text{Vol}_{HM}(\mathbb{B}^n/\Gamma) \) the Hirzebruch-Mumford volume of the factor space \( \mathbb{B}^n/\Gamma \). We denote by \( L(k) \) the L-function of the quadratic field \( K \) with character \( \chi_D(p) = \left( \frac{D}{p} \right) \) (Kronecker symbol). We prove the following theorem:

**Theorem 1.** The Hirzebruch-Mumford volume of the factor space \( \mathbb{B}^n/\Gamma \) equals

| \( n \) | \( D \) | \( \text{Vol}_{HM}(\mathbb{B}^n/\Gamma) \) |
|-------|--------|----------------------------------|
| even  | -4d    | \( D \frac{n^2 + 3n}{4} \cdot \prod_{j=1}^{n} \frac{1}{(2\pi)^{j+1}} \cdot \zeta(2) \cdot L(3) \cdot \zeta(4) \cdot L(5) \cdot \ldots \cdot L(n+1) \) |
| even  | -d     | \( D \frac{n^2 + 3n}{4} \cdot \prod_{j=1}^{n} \frac{1}{(2\pi)^{j+1}} \cdot \zeta(2) \cdot L(3) \cdot \zeta(4) \cdot L(5) \cdot \ldots \cdot L(n+1) \) |
| odd   | -4d    | \( D \frac{n^2 + 3n}{4} \cdot (1 - 2^{-(n+1)}) \cdot \prod_{p|d} \left( 1 + \left( \frac{-1}{p} \right)^{\frac{n+3}{2}} \right) \cdot p^{-\frac{n+1}{2}} \cdot \prod_{j=1}^{n} \frac{1}{(2\pi)^{j+1}} \cdot \zeta(2) \cdot \ldots \cdot \zeta(n+1) \) |
| odd   | -d     | \( D \frac{n^2 + 3n}{4} \cdot \prod_{p|d} \left( 1 + \left( \frac{-1}{p} \right)^{\frac{n+3}{2}} \right) \cdot p^{-\frac{n+1}{2}} \cdot \prod_{j=1}^{n} \frac{1}{(2\pi)^{j+1}} \cdot \zeta(2) \cdot L(3) \cdot \ldots \cdot \zeta(n+1) \) |

Using theorem[8] we compute the \( \text{Vol}_{HM}(\mathbb{B}^n/\Gamma') \) for the group \( \Gamma' = SU(M, \mathcal{O}_K) \), where the lattice \( M = \text{diag}(1, 1, \ldots, 1, -2) \), and prove the following theorem:

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Theorem 2. The Hirzebruch-Mumford volume of the factor space $\mathbb{B}^n/\Gamma'$ equals

\[
Vol_{HM}(\mathbb{B}^n/\Gamma') =
\begin{array}{ll}
\text{even} & D \frac{n^2-3n}{4} \cdot \prod_{j=1}^{n} \frac{\zeta(2) \cdot L(3) \cdot \zeta(4) \cdot L(5) \cdot \ldots \cdot L(n+1) \cdot (2^n - 1)}{\pi^j (2\pi)^{2j-1}} \\
\text{even} & D \frac{n^2-3n}{4} \cdot \prod_{j=1}^{n} \frac{\zeta(2) \cdot L(3) \cdot \zeta(4) \cdot L(5) \ldots \cdot L(n+1) \cdot 2^n \cdot \frac{1-(\frac{2}{\pi})^{2n+2(n+1)}}{1-(\frac{2}{\pi})^{2-1}}}{\pi^j (2\pi)^{2j-1}} \\
\text{odd} & D \frac{n^2-3n}{4} \cdot \prod_{j=1}^{n} \frac{\zeta(2) \cdot L(3) \cdot \zeta(4) \cdot L(5) \ldots \cdot L(n+1) \cdot 2^n \cdot \frac{1-(\frac{2}{\pi})^{2n+2(n+1)}}{1-(\frac{2}{\pi})^{2-1}}}{\pi^j (2\pi)^{2j-1}} \\
\text{odd} & D \frac{n^2-3n}{4} \cdot \prod_{j=1}^{n} \frac{\zeta(2) \cdot L(3) \cdot \zeta(4) \cdot L(5) \ldots \cdot L(n+1) \cdot 2^n \cdot \frac{1-(\frac{2}{\pi})^{2n+2(n+1)}}{1-(\frac{2}{\pi})^{2-1}}}{\pi^j (2\pi)^{2j-1}}
\end{array}
\]

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Local measures $\tau_p$

In this section we remind the notion of a local measure and some well-known facts about local measures. Consider the group $G = SU(L,K)$. More explicitly, $G$ is the group of matrices $A$ such that

\[
\begin{cases}
AL\Lambda' = L, \\
\det(A) = 1.
\end{cases}
\]

The Lie algebra $\mathcal{L}$ of the group $G$ is defined by the system

\[
\begin{align*}
BL + LB' &= 0, \\
Tr(B) &= 0,
\end{align*}
\]

Let $\mathcal{L}_\mathbb{Z}$ be the elements of $\mathcal{L}$ with coefficients in $O_K$. Consider the matrix norm $||A|| = \max|a_{ij}|_p$ in the Lie algebra $\mathcal{L}_p = \mathcal{L}_\mathbb{Q} \otimes \mathbb{Q}_p$. Let $B(k) = \{A, ||A|| < \frac{1}{p^k}\}$ be a ball in $\mathcal{L}_p$ of a sufficiently small radius. Denote by $G_{\mathbb{Z}}^{(k)}$ the image of $B(k)$ under the exponential map. The following facts are well-known and can be found, for example, in [2].

Proposition 1. a) The group $G_{\mathbb{Z}}^{(k)}$ is a congruence subgroup of $G_{\mathbb{Z}}$ modulo $p^k$ for $k \geq 1$ and $p \neq 2$ or $k \geq 2$ and $p = 2$.

b) The manifold $G_{\mathbb{Z}}$ splits into a disjoint union of $[G_{\mathbb{Z}} : G_{\mathbb{Z}}^{(k)}]$ balls of the same volume.

c) The volume of $G_{\mathbb{Z}}^{(k)}$ is equal to the volume of $B(k)$ starting from $k = 1$ for $p \neq 2$ and from $k = 2$ for $p = 2$.

It follows that

\[
Vol_{\tau_p}(G_{\mathbb{Z}}) =
\begin{cases}
\frac{N_p}{p^{(n+1)^2-1}}, & p \neq 2; \\
\frac{N_2}{4^{(n+1)^2-1}}, & p = 2,
\end{cases}
\]

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Then it follows from the corollary that \( N \) is a solution of \( \phi \). Let 

**Corollary 1.** \( Z \) be the Cartan decomposition. Denote by \( | \) and \( M \) the volume form on \( \omega \). Consider the bilinear form \( \lambda \equiv SU(L, O/2^3) \). \( p \neq 2 \). Then it follows from the corollary that \( N_p = \# SU(L, O/pO) \) and \( N_2 = \# ker(\phi_1) = \# SU(L, O/2^3)O \).

So, 

\[
Vol_{\tau_p}(G_{Z_p}) = \begin{cases} 
\frac{\#SU(L, O/pO)}{p^{(n+1)/2}1}, & p \neq 2; \\
\frac{\#SU(L, O/2^3O)}{4^{(n+1)/2}1}, & p = 2,
\end{cases}
\]

The following theorem is well-known ([10]):

**Theorem 4.** The Tamagawa number of the group \( SU \) equals 1.

It means that \( \tau_\infty(G_{Z}/G_{Z}) = \frac{1}{\prod \tau_p(G_{Z_p})} \).

**Hirzebruch-Mumford volume computation**

Consider the bilinear form \( B = Tr(XY) \) on the Lie algebra \( \mathcal{L} \). Let \( \mathcal{L} = k \oplus m \) be the Cartan decomposition. Denote by \( \omega_1 \) the volume form on \( k \) and by \( \omega_2 \) the volume form on \( m \) constructed by the bilinear form \( B \). Let \( \omega^{(d)} = \omega_1 \cdot \omega_2 \).

**Lemma 1.** \( |Vol_{\omega(\phi)}(G_{Z}/G_{Z})| = d^{\frac{n(n+1)}{2}}1|Vol_{\tau_\infty}(G_{Z}/G_{Z})| \) for \( d \equiv 3 \) (mod 4) and \( |Vol_{\omega(\phi)}(G_{Z}/G_{Z})| = d^{\frac{n(n+1)}{4}}1|Vol_{\tau_\infty}(G_{Z}/G_{Z})| \) for \( d \equiv 1 \) (mod 4).

**Proof.** Let \( O_K = \mathbb{Z}1+\mathbb{Z}e \). We fix the basis of \( \mathcal{L} \) over \( \mathbb{Z} \): \( e_k = \begin{pmatrix} 0 & \ldots & 0 & \ldots & 0 \\
\vdots & & \vdots & & \vdots \\
0 & \ldots & 0 & \ldots & \varepsilon \\
\vdots & & \vdots & & \vdots \\
0 & \ldots & \varepsilon & \ldots & 0 \end{pmatrix} \) (non-zero elements are in the \( k \)-th row of \( n+1 \)-th column and \( k \)-th column of
the intersection of \( \{ \ldots, n-1,n, f_{n-1,n}, e_1, f_1, \ldots, e_n, f_n \} \) is:

\[
B_L = \begin{pmatrix}
-2d & d & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & \ldots \\
-2d & -2d & d & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & d & -2d & d & \ldots & 0 & 0 & 0 & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & 0 & 0 & \ldots & -2\epsilon & -\epsilon - \epsilon & \ldots & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & \ldots & -\epsilon - \epsilon & -2 & \ldots & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 2\epsilon & \epsilon + \epsilon & \ldots \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & \epsilon + \epsilon & 2 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots 
\end{pmatrix}
\]

So \( \sqrt{\det(B)} = d^{7}(\epsilon - \bar{\epsilon})\frac{n(n+1)}{2}\sqrt{n+1} \). Clearly, \( \sqrt{\det(B)} = d^{\frac{n(n+1)}{4}}\sqrt{n+1} \).
for \( \epsilon = \frac{1 + \sqrt{d}}{2} \) and \( \sqrt{|\det(B)|} = d^{\frac{n(n+3)}{2}} \cdot 2^{-\frac{n(n+1)}{2}} \sqrt{n+1} \) for \( \epsilon = \sqrt{-d} \). We get

the statement of the lemma.

Consider a closed Riemannian manifold \( M^{2s} \) of dimension \( 2s \). Let \( \Omega \) be the Euler-Poincaré measure on it. The following theorem is well-known:

**Theorem 5.** [3] Let \( \chi(M^{2s}) \) be the Euler characteristic of a closed Riemannian manifold \( M^{2s} \). Then \( \int_{M^{2s}} \Omega = \chi(M^{2s}) \). Moreover, if \( M^{2s} \) is a homogeneous manifold of constant holomorphic curvature \( k \), then \( \Omega = \frac{(s+1)!k^s}{(4\pi)^s} \right d\nu \), where \( d\nu \) is a volume element corresponding to the Riemannian metric.

**Lemma 2.** The curvature \( k \) of space \( G_\mathbb{R}/G_\mathbb{Z} \) in metrics \( B|_n \) equals \(-2\).

**Proof.** The curvature \( k \) can be computed by the formula:

\[
k = \frac{B([X, JX], X], JX)}{B(X, X) \cdot B(JX, JX)},
\]

where \( J \) is a complex structure on \( \mathbb{B}^n \), \( J = 
\[
\begin{pmatrix}
0 & \ldots & 0 & a_1 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & a_n \\
-a_1 & \ldots & -a_n & 0
\end{pmatrix}
\]

Substituting, for example, basis vector \( f_n \) as \( X \) in the above formula, we get

the statement of the lemma.

**Definition 1.** The quotient \( \text{Vol}_{HM}(\mathbb{B}^n/\Gamma) = \frac{\chi(\mathbb{B}^n/\Gamma)}{\chi(\mathbb{B}^n/\mathbb{C})} \) is called the Hirzebruch-Mumford volume of the quotient space \( \mathbb{B}^n/\Gamma \) or the Hirzebruch-Mumford covolume of the group \( \Gamma \).

It means that \( \text{Vol}_{HM}(\mathbb{B}^n/\Gamma) = \frac{\text{Vol}_G(\mathbb{B}^n/\Gamma)}{n+1} \), since \( \chi(\mathbb{B}^n/\mathbb{C}) = n+1 \).

So,

\[
\text{Vol}_{HM}(\mathbb{B}^n/\Gamma) = \frac{\text{Vol}_G(\mathbb{B}^n/\Gamma)}{n+1} = \text{Vol}_{\omega_1}(\mathbb{B}^n/\Gamma) \cdot \frac{n!}{(2\pi)^n}.
\]

Let \( K \) be the maximal compact subgroup \( S(U(n) \times U(1)) \) of \( G_\mathbb{R} \).

We note that

\[
\text{Vol}_{\omega_1}(G_\mathbb{R}/G_\mathbb{Z}) = \text{Vol}_{\omega_1}(K) \cdot \text{Vol}_{\omega_2}(K/G/G_\mathbb{Z}) = \\
= \text{Vol}_{\omega_1}(K) \cdot \text{Vol}_{\omega_2}(\mathbb{B}^n/G_\mathbb{Z}) = \text{Vol}_{\omega_1}(K) \cdot \text{Vol}_{HM}(\mathbb{B}^n/\Gamma) \cdot \frac{(2\pi)^n}{n!}.
\]

So,

\[
\text{Vol}_{HM}(\mathbb{B}^n/\Gamma) = \frac{n! \cdot \text{Vol}_{\omega_1}(G_\mathbb{R}/G_\mathbb{Z})}{(2\pi)^n \cdot \text{Vol}_{\omega_1}(K)}.
\]

Using lemma [1] we obtain:

**Lemma 3.**

\[
\text{Vol}_{HM}(\mathbb{B}^n/\Gamma) = \frac{n! \cdot D^{\frac{2}{3} - \frac{1}{n}} \sqrt{n+1} \cdot \text{Vol}_{\omega_2}(G_\mathbb{R}/G_\mathbb{Z})}{(2\pi)^n \cdot \text{Vol}_{\omega_1}(K)}, \text{ for } d \equiv 3 \pmod{4}
\]

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and

\[ \text{Vol}_{HM}(\mathbb{R}^n / \Gamma) = \frac{n! \cdot D \cdot \sqrt{n+1} \cdot \text{Vol}_{\tau_\omega}(G_{\mathbb{Z}}/G_{\mathbb{Z}})}{(4\pi)^n \cdot \text{Vol}_{\omega_1}(K)}, \text{ for } d \equiv 1 \pmod{4}. \]

So it suffices to compute \( \text{Vol}_{\omega_1}(K) \) and \( \text{Vol}_{\tau_\omega}(G_{\mathbb{R}}/G_{\mathbb{Z}}) \).

**Computation of \( \tau_\infty \) for \( L \) using local measures \( \tau_p \)**

**Lemma 4.** The index \( |U(L, \mathcal{O}_K/p^k \mathcal{O}_K) : SU(L, \mathcal{O}_K/p^k \mathcal{O}_K)| \) equals \( 2p^k \) if \( p \) is ramified and \( p^k(1 - \left( \frac{D}{p} \right) p^{-1}) \) if \( p \) is unramified.

**Proof.** We consider the exact sequence

\[ 1 \rightarrow SU(L, \mathcal{O}_K/p^k \mathcal{O}_K) \rightarrow U(L, \mathcal{O}_K/p^k \mathcal{O}_K) \rightarrow U(1, \mathcal{O}_K/p^k \mathcal{O}_K) \rightarrow 1. \]

Now it suffices to compute \( \#U(1, \mathcal{O}_K/p^k \mathcal{O}_K) \) which has been done, for example, in [7].

**Lemma 5.** The local volume \( \tau_p(G_{Z_p}) \) for \( p \nmid D \) equals

\[ \tau_p(G_{Z_p}) = \frac{n+1}{\beta_p} \left( 1 - \left( \frac{D}{p} \right)^{-1} \right). \]

**Proof.** We compute \( \beta_p \) using the article [1]. By definition, \( \beta_p = \lim_{N \to \infty} p^{-N(n+1)^2} \#U(L, \mathcal{O}_K/p^N \mathcal{O}_K). \) This limit stabilizes for \( N \geq 2\nu_p(\det(L)) + 1 \), so in our case \( N = 1 \). By lemma [3] \( U(L, \mathcal{O}_K/p^k \mathcal{O}_K) : SU(L, \mathcal{O}_K/p^k \mathcal{O}_K) = p(1 - \left( \frac{D}{p} \right) p^{-1}) \), so \( \#SU(L, \mathcal{O}_K/p^k \mathcal{O}_K) = \beta_p \cdot p^{(n+1)^2-1} \cdot \left( 1 - \left( \frac{D}{p} \right) p^{-1} \right) \). So it suffices to compute \( \beta_p \). The Jordan decomposition of the lattice \( L \) contains only one component \( L_0, d_0 = 0 \) (using the notation from [1]), so \( \beta_p = p^{-\dim G_0} \#G_0(F_p) \) (\( F_p \) - finite field of order \( p \)). Here \( G_0 = U(n+1) \) if \( \chi_D(p) = -1 \) and \( G_0 = GL(n+1) \) if \( \chi_D(p) = 1 \). So, \( \dim G_0 = (n+1)^2 \). It is well-known [1] that \( \#GL(n+1, F_p) = p^{(n+1)^2} \prod_{i=1}^{n+1} (1 - p^{-i}) \) and \( \#U(n+1, F_p) = p^{(n+1)^2} \prod_{i=1}^{n+1} (1 - (-1)^i p^{-i}) \). So

\[ \tau_p(G_{Z_p}) = \frac{\#SU(L, \mathcal{O}_K/p^k \mathcal{O}_K)}{p^{(n+1)^2-1}} = \frac{\beta_p}{\left( 1 - \left( \frac{D}{p} \right) p^{-1} \right) \prod_{i=1}^{n+1} \left( 1 - \left( \frac{D}{p} \right) p^{-(n+1)} \right)}. \]

**Lemma 6.** The local volume \( \tau_p(G_{Z_p}) \) for odd \( p \) such that \( p \nmid D \) equals

\[ \tau_p(G_{Z_p}) = \left( 1 - \left( \frac{D}{p} \right)^{n+1} \right) \prod_{i=1}^{(n-1)/2} \left( 1 - (-1)^i p^{-2i} \right) \text{ for odd } n, \]

\[ \tau_p(G_{Z_p}) = \prod_{i=1}^{n/2} \left( 1 - p^{-2i} \right) \text{ for even } n. \]
Proof. We compute $\beta_p$ using the article \cite{9}. By definition, $\beta_p = \lim_{N \to \infty} p^{-N(n+1)^2} \#U(L, \mathcal{O}_K/p^N \mathcal{O}_K)$. This limit stabilizes for $N \geq 2\nu_p(\det(L)) + 3$, so in our case $N = 1$. By lemma \cite{9} \[ U(L, \mathcal{O}_K/p\mathcal{O}_K) : SU(L, \mathcal{O}_K/p\mathcal{O}_K) = 2p, \] so $\#SU(L, \mathcal{O}_K/p\mathcal{O}_K) = \frac{\beta_p \cdot p^{(n+1)^2}}{2p}$. So it suffices to compute $\beta_p$. The Jordan decomposition of the lattice $L$ contains only one component $L_0$, $d_0 = 0$ (using the notation from \cite{9}), so $\beta_p = p^{-\dim_{\mathbb{F}_p} \#G_0(\mathbb{F}_p)} (\mathbb{F}_p$ - finite field of order $p$). Here $G_0 = O(n + 1)$, so $\dim G_0 = \frac{n(n+1)}{2}$. It is well-known \cite{9} that $\#O(n+1, \mathbb{F}_p) = 2p^{\frac{n(n+1)}{2}} \prod_{i=1}^{\frac{n}{2}} (1 - p^{-2i})$ if $n$ is even and $\#O(n + 1, \mathbb{F}_p) = 2p^{\frac{(n+1)(n+2)}{2}} \prod_{i=1}^{\frac{n-1}{2}} (1 - p^{-2i})$ if $n$ is odd. Here $\epsilon = 1$ if the discriminant of $L$ is $(-1)^{\frac{n+1}{2}}$ and $\epsilon = -1$ otherwise, so $\epsilon = \left(\frac{-1}{p}\right)$. We get

$$\tau_p(G_{zp}) = \frac{\#SU(L, \mathcal{O}_K/p\mathcal{O}_K)}{p^{(n+1)^2-1}} = \frac{\beta_p}{2} \left( 1 - \left(\frac{-1}{p}\right)^{\frac{n+1}{2}} \right) \prod_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} (1 - p^{-2i})$$

for odd $n$ and

$$\tau_p(G_{zp}) = \prod_{n=1}^{\frac{n}{2}} (1 - p^{-2i})$$

for even $n$.

Lemma 7. The local volume $\tau_2(G_{z_2})$ equals $\tau_2(G_{z_2}) = \prod_{i=2}^{n+1} (1 - \left(\frac{-2}{p}\right)^i 2^{-i})$ if $2 \nmid D$, $\tau_2(G_{z_2}) = 2^{-n} \prod_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} (1 - 2^{-2i})$ if $2 \mid D$.

Proof. The computations in case $2 \nmid D$ have already been done in lemma \cite{5} so we need only to prove the lemma for $2 \mid D$. We compute $\beta_2$ using the article \cite{2}. By definition, $\beta_2 = \lim_{N \to \infty} p^{-N(n+1)^2} \#U(L, \mathcal{O}_K/2^N \mathcal{O}_K)$. This limit stabilizes for $N \geq 2\nu_p(\det(L)) + 3$ (because $p = 2$ is ramified in our case), so $N = 3$. By lemma \cite{9} the index $[U(L, \mathcal{O}_K/2^3 \mathcal{O}_K) : SU(L, \mathcal{O}_K/2^3 \mathcal{O}_K)] = 2^4$, so $\#SU(L, \mathcal{O}_K/2^3 \mathcal{O}_K) = \beta_2 \cdot 2^3(p^{n+1})^3$. So it suffices to compute $\beta_2$. The Jordan decomposition of the lattice $L$ contains only one component $L_0$ of type $I^0$ when $n$ is even and of type $I^c$ when $n$ is odd. Moreover, $d_0 = 0$, $N = 0$, $\beta = 1$, $f = 2$ (using the notation from \cite{2}). So $\beta_2 = 2^{-\frac{(n+1)^2}{2}} \cdot 2^{m+1} \cdot \#Sp(n, 2)$ for even $n$ and $\beta_2 = 2^{-\frac{(n+1)^2}{2}} \cdot 2^{m+1} \cdot \#Sp(n-1, 2)$ for odd $n$. Considering that $m = (n + 1)^2 - \frac{n(n+1)}{2}$ for even $n$ and $m = (n + 1)^2 - \frac{n(n-1)}{2}$ for odd $n$ and that $\#Sp(n, 2) = 2^{\frac{n(n+1)}{2}} \prod_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} (1 - 2^{-2i})$, we get $\beta_2 = 2 \prod_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} (1 - 2^{-2i})$. Now we need to compute $[ker(\phi)]$, where $\phi : SU(L, \mathcal{O}_K/2^3 \mathcal{O}_K) \to SU(L, \mathcal{O}_K/2^2 \mathcal{O}_K)$. If $A \in ker(\phi)$ then $A = E + 4B$, where $B$ is a matrix with coefficients in $\mathcal{O}/2\mathcal{O}$. The matrix $B$ must satisfy the system:

$$\begin{cases}
-2BL = LB' \\
Tr(B) = 0.
\end{cases}$$

If $B = (b_{ij})$, we get
the following equations: 
\[-b_{ij} = b_{ii}, \ i \in \{1; n\}\] 
\[-b_{ij} = b_{ji}, \ i, j \in \{1; n\}, i \neq j\] 
Each equation for 
\[b_{i,n+1} = b_{n+1,i}, \ i \in \{1; n\}.\]

\(b_{ii}\) has 4 solutions, and in the remaining equations one summand is uniquely determined by the other one, so it gives 4 options for each pair \((b_{ij}, b_{ji})\). So 
\[|\ker(\phi)| = 4^n \cdot 4^{\binom{n+1}{2} - \binom{n+1}{2}} = 2^{n^2 + 3n}.\]

We get 
\[
\tau_2(G_{\mathbb{Z}^n}) = \frac{\#SU(L_n \cdot O_K/2^3 \cdot O_K)}{4^{(n+1)^2 - 1} \cdot |\ker(\phi)|} = \frac{2^{3(n+1)^2-3} \prod_{i=1}^{[n/2]} (1 - 2^{-2i})}{2^{2n^2+4n} \cdot 2^{n(n+3)}} = 2^{-n} \prod_{i=1}^{[n/2]} (1 - 2^{-2i}).
\]

The values of \(\tau_\infty(G_{\mathbb{R}}/G_{\mathbb{Z}})\) are written in the table below. To compute them we used the results of lemma \(\Box\) lemma \(\Box\) and lemma \(\Box\) and the fact that 
\[\tau_\infty(G_{\mathbb{R}}/G_{\mathbb{Z}}) = \prod_{p \neq \infty} \tau_p(G_{\mathbb{Z}_p}).\]

| n   | D    | \(\tau_\infty(G_{\mathbb{R}}/G_{\mathbb{Z}})\)                           |
|-----|------|-------------------------------------------------------------------------|
| even| -4d  | \(2^n \zeta(2) \cdot L(3) \cdot \zeta(4) \cdot L(5) \cdot \ldots \cdot L(n+1)\) |
| even| -d   | \(\zeta(2) \cdot L(3) \cdot \zeta(4) \cdot L(5) \cdot \ldots \cdot L(n+1)\) |
| odd | -4d  | \(2^{n} (1 - 2^{-\binom{n+1}{2}}) \cdot \prod_{p|d} (1 + \left(\frac{(-1)^{\binom{n+1}{2}}}{p}\right) \cdot 2p^{-n+1}) \cdot \zeta(2) \cdot \ldots \cdot \zeta(n+1)\) |
| odd | -d   | \(\prod_{p|d} (1 + \left(\frac{(-1)^{\binom{n+1}{2}}}{p}\right) p^{-n+1}) \cdot \zeta(2) \cdot L(3) \cdot \ldots \cdot \zeta(n+1)\) |

The volume \(Vol_{\omega_1}(K)\) of the maximal compact group 
\(K = SU(n) \times G_{\mathbb{Z}}\)

We will need the following result.

**Lemma 8.** \(Vol_{\omega_1}(SU(n)) = \sqrt{n(2\pi)^{\frac{n^2+n-2}{2}}} \prod_{i=1}^{n-1} i!\)

**Proof.** It is well-known (\[\Box\]) that if we take the measure to be \(tr(ad_X ad_Y)\), then the volume of the group \(SU(n)\) equals 
\[\frac{2^{n^2-1} \pi^2 (2\pi)^{\frac{n^2+n-2}{2}}}{1!2!\ldots(n-1)!}.\]

In the Lie algebra \(su(n)\) the form \(tr(ad_X ad_Y)\) corresponds to the form \(2n \cdot tr(XY) = 2n \cdot B(X,Y)\), so the proportionality coefficient is \((2n)^{\frac{n^2+n-2}{2}}\). So,

\[
Vol_{\omega_1}(SU(n)) = \frac{2^{n^2-1} \pi^2 (2\pi)^{\frac{n^2+n-2}{2}}}{1!2!\ldots(n-1)! (2n)^{\frac{n^2+n-2}{2}}} = \sqrt{n(2\pi)^{\frac{n^2+n-2}{2}}} \prod_{i=1}^{n-1} i!\]

\(\Box\)
Lemma 9. \(Vol_{\omega_1}(SU(n) \times U(1)) = \frac{\sqrt{n + 1}(2\pi)^{n^2+n}}{n!} \).

Proof. Consider the vector \(e_1 = \begin{pmatrix} i & 0 & \ldots & 0 & 0 \\ 0 & i & \ldots & 0 & 0 \\ 0 & 0 & \ldots & i & 0 \\ 0 & 0 & \ldots & 0 & -ni \\ \end{pmatrix} \) as the first basis vector in the Lie algebra of the group \(SU(n) \times U(1)\). Then the Lie algebra is a direct sum of \(\mathbb{C}e_1\) and \(su(n)\). The direct product of \(SU(n)\) on a one-parameter subgroup, generated by \(e_1\) (a circle), is an \(n\)-sheet cover of \(SU(n) \times U(1)\). The radius of the circle, corresponding to \(e_1\), is \(\sqrt{n^2+n}\). So we get:

\[
Vol_{\omega_1}(SU(n) \times U(1)) = \frac{2\pi\sqrt{n^2+n} \cdot Vol_{\omega_1}(SU(n))}{n} = \frac{\sqrt{n + 1}(2\pi)^{n^2+n}}{n!}.
\]

Now if we substitute \(Vol_{\omega_1}(SU(n) \times U(1))\) and \(\tau_{\infty}(G_\mathbb{R}/G_\mathbb{Z})\) into the formulas in lemma 3, we get the statement of the theorem 1.

The computation of \(Vol_{HM}(\mathbb{B}^n/\Gamma')\)

Consider the group \(G' = SU(M, K)\). More explicitly, \(G'\) is the group of matrices \(A\) such that \(AMA^t = L\), \(det(A) = 1\). The Lie algebra \(\mathcal{L}'\) of the group \(G'\) is defined by the system \(\begin{cases} BM + MB' = 0, \\ Tr(B) = 0. \end{cases}\) We need to compute new local densities \(\tau_2(G'_\mathbb{R}/G'_\mathbb{Z})\), \(\tau_p(G'_\mathbb{R}/G'_\mathbb{Z})\) for odd \(p\) such that \(p|D\) and the new determinant of the Killing form. Everything else remains the same as for the lattice \(L\) and group \(G\).

Lemma 10. \(|Vol_{\omega_1}(G'_\mathbb{R}/G'_\mathbb{Z})| = 2^n \cdot d^{\frac{n(n+3)}{2}} \cdot \sqrt{n+1}|Vol_{\tau_{\omega_1}}(G'_\mathbb{R}/G'_\mathbb{Z})|\) for \(d \equiv 3 \pmod{4}\) and \(|Vol_{\omega_1}(G'_\mathbb{R}/G'_\mathbb{Z})| = d^{\frac{n(n+3)}{2}} \cdot 2^{n+1} \cdot \sqrt{n+1}|Vol_{\tau_{\omega_1}}(G'_\mathbb{R}/G'_\mathbb{Z})|\) for \(d \equiv 1 \pmod{4}\).

Proof. We fix the basis of \(\mathcal{L}'_\mathbb{Z}\) over \(\mathbb{Z}\). It consists of the elements \(g_k, e_{i,j}, f_{i,j}\)

\[
\begin{pmatrix}
0 & \ldots & 0 & \ldots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & \ldots & \epsilon \\
0 & \ldots & 2\epsilon & \ldots & 0
\end{pmatrix}
\]

\(e_k\), non-zero elements are in the \(k\)-th row of \(n+1\)-th column and \(k\)-th column of
where $U$ is the set of elements $x$, such that $x$ is the norm of the determinant of matrix in $U(M, O/2^k O)$. Each element $X \in U(M, O/2^k O)$ satisfies the equation $XMX' = M$, which means that $2\text{Norm}(\det(X)) = 2$. It means that $\text{Norm}(\det(X)) = 1$ or $\text{Norm}(\det(X)) = 1 + 2^{k-1}$. We note that there exists a matrix $X \in U(M, O/2^k O)$ such that $\text{Norm}(\det(X)) = 1 + 2^{k-1}$. For example, we can take $X = \text{diag}(1, 1, \ldots, 1, 1 + 2^{k-2})$. Clearly, $|U| = 2|U(1, O/2^k O)|$.

Lemma 12. The values of $\tau_2(G_{22}^\prime)$ are listed in the table below:
\[ \tau_2(G'_{Z_2}) \]

**Proof.** We first consider the case \( D = -d \). We compute \( \beta_2 \) using the article [3]. By definition, \( \beta_2 = \lim_{N \to \infty} p^{-N(n+1)^2} \#U(M, \mathcal{O}_K/2^N \mathcal{O}_K) \). This limit stabilizes for \( N \geq 2 \nu_p(\det(M)) + 1 \), so \( N = 3 \). We need to do the following steps:

1. Compute \( \beta_2 \) and \( |U(M, \mathcal{O}/8 \mathcal{O})| \) using the article [3];

2. Compute \( |SU(M, \mathcal{O}/8 \mathcal{O})| \) using lemma [11];

3. Divide \( |SU(M, \mathcal{O}/8 \mathcal{O})| \) by \( 2^{3(n+1)^2-1} \) (it follows from the proposition [1]).

The Jordan decomposition of \( M \) is \( M_0 \oplus M_1 \), \( rk(M_0) = n \), \( rk(M_1) = 1 \) (using the notation from [3]). We get \( d_0 = 0 \), \( d_1 = 1 \), \( d = 1 \), \( \beta_2 = 2 \cdot 2^{-n^2} \cdot 2^{-1} \cdot |G_0| \cdot |G_1| \).

The group \( G_0 = U(n) \) when \( d \equiv 3 \pmod{8} \) and \( GL(n) \) when \( d \equiv -1 \pmod{8} \). Similarly, \( G_1 = U(1) \) when \( d \equiv 3 \pmod{8} \) and \( GL(1) \) when \( d \equiv -1 \pmod{8} \). So, \( |G_0| = 2^n \prod_{i=1}^{n} (1 - \left( \frac{-d}{2} \right)^i 2^{-i}) \), \( |G_1| = \begin{cases} 3, & d \equiv 3 \pmod{8} \\ 1, & d \equiv -1 \pmod{8} \end{cases} \). On the other hand, by definition \( \beta_2 = 2^{-3(n+1)^2} |U(M, \mathcal{O}/8 \mathcal{O})| \). Using that \( |U(M, \mathcal{O}/8 \mathcal{O}) : SU(M, \mathcal{O}/8 \mathcal{O})| = 2^4(1 - \left( \frac{-d}{2} \right)^i 2^{-i}) \), we get

\[
|SU(M, \mathcal{O}/8 \mathcal{O})| = 2^{3(n+1)^2-1} \prod_{i=1}^{n} (1 - \left( \frac{-d}{2} \right)^i 2^{-i})
\]

so

\[ \tau_2(G'_{Z_2}) = \prod_{i=1}^{n} (1 - \left( \frac{-d}{2} \right)^i 2^{-i}) \]

We note that this result doesn’t depend on the parity of \( n \).

Now consider the case \( D = -4d \). We compute \( \beta_2 \) using the article [2]. By definition, \( \beta_2 = \lim_{N \to \infty} p^{-N(n+1)^2} \#U(M, \mathcal{O}_K/2^N \mathcal{O}_K) \). This limit stabilizes for \( N \geq 2 \nu_p(\det(M)) + 3 \) (because \( p = 2 \) is ramified), so \( N = 5 \). We need to do the following steps:

1. Compute \( \beta_2 \) and \( |U(M, \mathcal{O}/32 \mathcal{O})| \) using the article [2];

2. Compute \( |SU(M, \mathcal{O}/32 \mathcal{O})| \) using lemma [11];

3. Consider \( \phi : SU(M, \mathcal{O}/32 \mathcal{O}) \to SU(M, \mathcal{O}/8 \mathcal{O}) \). It follows from lemma [8] and corollary [1] that \( im(\phi) \) coincides with the order of \( SU(M, \mathbb{Z}_2)/SU(M, \mathbb{Z}_2) \) (using the notations from proposition [1]). So we need to compute the \( \#ker(\phi) \) in order to compute \( im(\phi) \).

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4. Divide $\text{im}(\phi)$ by $2^{3((n+1)^2-1)}$ (it follows from the proposition $[1]$).

The Jordan decomposition of $M$ is $M_0 \oplus M_2$, $\text{rk}(M_0) = n$, $\text{rk}(M_2) = 1$.

We first consider the case when $n$ is even. Using the notation from $[2]$ we see that the lattice $M_0$ has type $I^n$, the lattice $M_2$ has type $I^0$, $\dim(B_0/Y_0) = n - 2$, $\dim(B_2/Y_2) = 0$. Besides, $\beta = 1$, $d_0 = 0$, $d_2 = 2 \cdot 1 \cdot \frac{d_0}{d_2} = 0$, $N = 1$. So,

$$\beta_2 = 2 \cdot 2^{-(n+1)^2} |\tilde{G}|,$$

where $|\tilde{G}| = 2^m \cdot 2^\beta |\text{Sp}(n-2)|$, $m = (n+1)^2 - \dim(\text{Sp}(n-2))$;

$$m = (n + 1)^2 - \frac{(n - 1)(n - 2)}{2}, \quad |\text{Sp}(n - 2)| = 2^{(2n-2)^2} \prod_{i=1}^{n-2} (2^{2i} - 1),$$

$$\beta_2 = 2^2 \prod_{i=1}^{n-2} (1 - 2^{-2i}).$$

By definition $\beta_2 = \lim_{N \to \infty} 2^{-N \dim(G)} |G'(A/\pi N A)|$. Since $N = 5$ we get

$$2^{-5(n+1)^2} |U(M, O/32O)| = 2^2 \prod_{i=1}^{n-2} (1 - 2^{-2i}),$$

so

$$|U(M, O/32O)| = 2^{5(n+1)^2+2} \prod_{i=1}^{n-2} (1 - 2^{-2i}).$$

Using that $|U(M, O/32O) : SU(M, O/32O)| = 2^7$, we obtain

$$|SU(M, O/32O)| = 2^{5(n+1)^2-5} \prod_{i=1}^{n-2} (1 - 2^{-2i}).$$

Now we need to compute $|\ker(\phi)|$. It means we need to find the number of solutions $B$ of the system

\[
\begin{align*}
-BM &= M\bar{B}^t \mod O/4O, \\
TrB &= 0
\end{align*}
\]

Taking $B = (b_{ij})$,

$$-b_{ii} = \bar{b}_{ii}, \quad i \in \{1; n\}$$

$$-b_{ij} = \bar{b}_{ji}, \quad i, j \in \{1; n\}$$

$$2b_{n+1} = \bar{b}_{n+1}, \quad i \in \{1; n\}.$$}

and in the remaining equations one summand is uniquely determined by the other one, so it gives 16 options for each pair $(b_{ij}, b_{ji})$. So $|\ker(\phi)| = 8^n - 16^{\frac{n(n+1)}{2}} = 2n^2 + 5n$.

So we get

$$\tau_2 = \frac{|SU(M, O/32O)|}{|\ker(\phi)|} \cdot 2^{3((n+1)^2-1)} = \prod_{i=1}^{n-2} (1 - 2^{-2i}) \cdot \frac{2n}{2n}.$$

When $n$ is odd the proof is similar and the difference occurs only in computation of $\beta_2$. The Jordan decomposition of $M$ is $M_0 \oplus M_2$, $\text{rk}(M_0) = n$, $\text{rk}(M_2) = 1$. Using the notation from the article $[2]$ the lattice $M_0$ has type $I^0$. 

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the lattice $M_2$ has type $I^0$, $\dim(B_0/Y_0) = n - 1$, $\dim(B_2/Y_2) = 0$. Besides, $\beta = 1$, $d_0 = 0$, $d_2 = 2 \cdot \frac{1}{2} = 0$, $N = 1$. We get:

$$\beta_2 = 2 \cdot 2^{-(n+1)}|\tilde{G}|,$$

where $|\tilde{G}| = 2^m \cdot 2^\beta|Sp(n-2)|$, $m = (n+1)^2 - \dim(Sp(n-1))$;

$$n = (n+1)^2 - \frac{(n-1)(n)}{2}, \quad |Sp(n-1)| = 2^{\left(\frac{n+1}{2}\right)^2} \prod_{i=1}^{\frac{n-1}{2}} (2^{2i} - 1),$$

$$\beta_2 = 2^2 \prod_{i=1}^{\frac{n-1}{2}} (1 - 2^{-2i}).$$

Everything else is the same as in the case of even $n$, so we get:

$$\tau_2 = \frac{|SU(M, \mathcal{O}/32\mathcal{O})|}{|\ker(\phi)| \cdot 2^{3(n+1)^2 - 1}} = \frac{n+1}{\prod_{i=1}^{n+1} (1 - 2^{-2i})}. \quad \square$$

**Lemma 13.** Let $p$ be odd such that $p|D$. Then $\tau_p(G_{Z_p}) = \prod_{i=1}^{\frac{n}{2}} (1 - p^{-2i}) = \tau_p(G_{Z_p})$ for even $n$ and $\tau_p(G_{Z_p}) = (1 - \left(\frac{n+1}{2p}\right)^2) \prod_{i=1}^{\frac{n-1}{2}} (1 - p^{-2i}) = \tau_p(G_{Z_p}) \cdot \frac{1 - \left(\frac{n+1}{p}\right)^2}{1 - \left(\frac{n+1}{p}\right)}$ for odd $n$.

**Proof.** As well as in the proof of lemma we notice that $\tau_p(G_{Z_p}) = \frac{1}{2} \beta_p$ (using the notation from [1]). It follows from this article that $\beta_p = p^{-(n+1)^2}|G_0(\mathcal{O}/p\mathcal{O})|$. If $n$ is even then $G_0$ is an orthogonal group of type I or II, $|G_0(\mathcal{O}/p\mathcal{O})| = 2p^{\frac{n^2}{2}} \prod_{i=1}^{\frac{n}{2}} (p^{2i} - 1)$, and $\tau_p(G_{Z_p})$ is the same as for the lattice $L$ so the value of $\tau_p(G_{Z_p})$ is also the same. If $n$ is odd then $G_0$ is an orthogonal group of type III or IV, it’s order is $|G_0(\mathcal{O}/p\mathcal{O})| = 2p^{\frac{n^2}{2}} |(p^{\frac{n+1}{2}} - 1)|$ where $\epsilon = -1$ if $G_0$ is of type III and $\epsilon = 1$ if $G_0$ is of type IV. Type III means that the determinant of $M$ is $(-1)^{\frac{n+1}{2}}$, and type IV means that the determinant of $L$ is $\omega(-1)^{\frac{n+1}{2}}$, where $\omega$ is a quadratic nonresidue modulo $p$. Since the determinant of $M$ is $-2$, we get $\epsilon = \left(\frac{-1}{p}\right)^{\frac{n+1}{2}}$. For the lattice $L$ the value of corresponding $\epsilon$ was $\left(\frac{-1}{p}\right)^{\frac{n+1}{2}}$, so we get the statement of the lemma. $\square$

Now we notice that

$$Vol_{HM}(\mathbb{B}^n/I^\nu) = Vol_{HM}(\mathbb{B}^n/I) \cdot \frac{\tau_2(G_{Z_p})}{\tau_2(G_{Z_p})} \cdot \prod_{p|d} \frac{\tau_p(G_{Z_p})}{\tau_p(G_{Z_p})} \cdot \frac{\det(B_M)}{\det(B_L)}$$

The proportionality coefficient is listed in the table below.
Substituting it into the above formula we get the statement of theorem 2.

\[
\begin{array}{|c|c|c|}
\hline
n & D & \frac{\tau_d(G_2)}{\tau_d(G'_2)} \cdot \prod_{p | d} \frac{\tau_p(G_p)}{\tau_p(G'_p)} \cdot \frac{\det(B_M)}{\det(B_L)} \\
\hline
\text{even} & -4d & 2^n \cdot (1 - 2^{-n}) \\
\text{even} & -d & 2^n \cdot \frac{1 - \left(\frac{-d}{p}\right) 2^{-(n+1)}}{1 - \left(\frac{-d}{p}\right)^2} \\
\text{odd} & -4d & 2^n \cdot \prod_{p | d} \frac{1 - \left(\frac{-1}{p}\right)\left(\frac{-d}{p}\right)}{1 - \left(\frac{-1}{p}\right)^2} p^{\frac{2n+1}{2}} \\
\text{odd} & -d & 2^n \cdot \frac{1 - \left(\frac{-d}{p}\right)^{n+1} 2^{-(n+1)}}{1 - \left(\frac{-d}{p}\right)^2} \cdot \prod_{p | d} \frac{1 - \left(\frac{-1}{p}\right)\left(\frac{-d}{p}\right)^2}{1 - \left(\frac{-1}{p}\right)^2} p^{\frac{n+1}{2}} \\
\hline
\end{array}
\]

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