HOMOLOGICAL MIRROR SYMMETRY OF $\mathbb{CP}^N$ AND THEIR PRODUCTS VIA MORSE HOMOTOPY

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Abstract. We propose a way of understanding homological mirror symmetry when a complex manifold is a smooth compact toric manifold. So far, in many example, the derived category $D^b(\text{coh}(X))$ of coherent sheaves on a toric manifold $X$ is compared with the Fukaya-Seidel category of the Milnor fiber of the corresponding Landau-Ginzburg potential. We instead consider the dual torus fibration $\pi : M \to B$ of the complement of the toric divisors in $X$, where $\bar{B}$ is the dual polytope of the toric manifold $X$. A natural formulation of homological mirror symmetry in this set-up is to define $\text{Fuk}(\bar{M})$ a variant of the Fukaya category and show the equivalence $D^b(\text{coh}(X)) \simeq D^b(\text{Fuk}(\bar{M}))$. As an intermediate step, we construct the category $\text{Mo}(P)$ of weighted Morse homotopy on $P := \bar{B}$ as a natural generalization of the weighted Fukaya-Oh category proposed by [M. Kontsevich and Y. Soibelman, In Symplectic geometry and mirror symmetry, page 203 (2001)]. We then show a full subcategory $\text{Mo}_E(P)$ of $\text{Mo}(P)$ generates $D^b(\text{coh}(X))$ for the cases $X$ is a complex projective space and their products.

1. Introduction

In this paper, we propose a way of understanding homological mirror symmetry for the case of smooth compact toric manifolds. So far, in many studies, the derived category $D^b(\text{coh}(X))$ of coherent sheaves on a toric manifold $X$ is compared with the Fukaya- Seidel category of the Milnor fiber of the corresponding Landau-Ginzburg potential. In this paper, we consider the dual torus fibration $\pi : M \to B$, in the sense of Strominger-Yau-Zaslow construction [20], of the complement of the toric divisors in $X$, where $P = \bar{B}$ is the dual polytope of the toric manifold $X$. Fukaya discussed the Calabi-Yau cases in [8] where the Kähler metric degenerates at singular fibers of the torus fibration. In our set-up, the Kähler metrics go to infinity at the boundaries $\partial(P)$. In [6], Fang discusses homological mirror symmetry of $\mathbb{CP}^n$ along this line. There, he starts with considering line bundles on $\mathbb{CP}^n$ and the corresponding Lagrangians in the mirror dual side. His idea of discussing the homological mirror symmetry is to consider the category of constructible sheaves as an intermediate step. We instead apply Kontsevich-Soibelman’s approach [16] to our case and consider a category $\text{Mo}(P)$ of Morse homotopy. Namely, a natural formulation of homological mirror symmetry in our situation is to define a variant of the

Date: April 20, 2022.
Fukaya category $Fuk(\tilde{M})$ and show the equivalence $D^b(\text{coh}(X)) \simeq D^b(Fuk(M))$. As an intermediate step, we construct the category $Mo(P)$ of weighted Morse homotopy on $P$ as a natural generalization of the weighted Fukaya-Oh category proposed in [16]. We then show a full subcategory $Mo_\mathcal{E}(P)$ of $Mo(P)$ generates $D^b(\text{coh}(X))$ for the case $X$ is a complex projective space and their products. For more general $X$, we may consider the Lagrangian sections discussed in [5], where Chan discusses the correspondence between holomorphic line bundles over projective toric manifolds and Lagrangian sections in the mirror dual. The relation of such an approach with Abouzaid’s one [1] is also mentioned there.

While Abouzaid [1] employed the geometric perturbation technique to establish transversality and used the notion of $A_\infty$ pre-categories to avoid the self-intersection problem, we pick up finitely many Lagrangians explicitly and allowed clean intersections. This simplifies the comparison of the symplectic and complex sides and makes the mirror functor more word-by-word.

This paper is organized as follows. In section 2, we recall the SYZ torus fibration set-up [20] following [18, 17]. There, a pair of dual torus fibrations $M \to B$ and $\tilde{M} \to B$ is defined. We use this set-up by identifying $\tilde{M}$ with the complement of the toric divisors in a toric manifold $X$. In section 3, we recall the correspondence of Lagrangian sections of $M \to B$ and holomorphic line bundles on $\tilde{M}$, again, following [18, 17]. In the last subsection, we demonstrate a Lagrangian section to be derived from the line bundle $\mathcal{O}(k)$ restricted on the complement of the toric divisors for $X = \mathbb{C}P^n$ by the correspondence above. In section 4, we first recall DG-categories $\mathcal{F}(M)$ and $\mathcal{V}(\tilde{M})$ associated to $M$ and $\tilde{M}$, respectively, in [12]. Kontsevich-Soibelman’s approach for the homological mirror symmetry [16] proposes an intermediate category $Mo(B)$ and the existence of an $A_\infty$-equivalence

$$Fuk(M) \simeq Mo(B) \to \mathcal{F}(M).$$

This $Mo(B)$ is called the weighted Fukaya-Oh category or the category of weighted Morse homotopy on $B$. In subsection 4.3, we propose a modification $Mo(P)$ of $Mo(B)$ where $P = \tilde{B}$ is the dual polytope of a smooth compact toric manifold $X$. In the last section, we discuss the correspondence between $D^b(\text{coh}(X))$ and $Mo(P)$ when $X$ is a complex projective space or their products. In particular, we see that we can take strongly exceptional collections $\mathcal{E}$ of $D^b(\text{coh}(X))$ consisting of line bundles and the corresponding full subcategory $Mo_\mathcal{E}(P)$ of $Mo(P)$ so that

$$Tr(Mo_\mathcal{E}(P)) \simeq D^b(\text{coh}(X))$$

where $Tr$ is the Bondal-Kapranov-Kontsevich construction [4, 15] of triangulated categories from $A_\infty$-categories.
2. Toric manifolds and $T^n$-invariant manifolds

2.1. Dual torus fibrations. In this subsection, we briefly review the SYZ torus fibration set-up [20]. For more details see [18, 17]. We follow the convention of [12].

Throughout this section, we consider an $n$-dimensional tropical Hessian manifold $B$ which we will define shortly below. Our goal of this subsection is then to construct $B$.

A smooth manifold $B$ is called affine if $B$ has an open covering $\{U_\lambda\}_{\lambda \in \Lambda}$ such that the coordinate transformation is affine. This means that, for any $U_\lambda$ and $U_\mu$ such that $U_\lambda \cap U_\mu \neq \emptyset$, the coordinate systems $x_\lambda := (x_1(\lambda), \ldots, x_n(\lambda))^t$ and $x_\mu := (x_1(\mu), \ldots, x_n(\mu))^t$ are related to each other by

$$x_\mu = \varphi_{\lambda\mu} x_\lambda + \psi_{\lambda\mu}$$

with some $\varphi_{\lambda\mu} \in GL(n; \mathbb{R})$ and $\psi_{\lambda\mu} \in \mathbb{R}^n$. If in particular $\varphi_{\lambda\mu} \in GL(n, \mathbb{Z})$ for any $U_\lambda \cap U_\mu$, then $B$ is called tropical affine. (If in addition $\psi_{\lambda\mu} \in \mathbb{Z}^n$, $B$ is called integral affine.)

For simplicity, we take such an open covering $\{U_\lambda\}_{\lambda \in \Lambda}$ so that the open sets $U_\lambda$ and their intersections are all contractible. It is known that $B$ is an affine manifold iff the tangent bundle $TB$ is equipped with a torsion free flat connection. When $B$ is affine, then its tangent bundle $TB$ forms a complex manifold. This fact is clear as follows. For each open set $U = U_\lambda$, let us denote by $(x, y) = (x^1, \ldots, x^n; y^1, \ldots, y^n)$ the coordinates of $U \times \mathbb{R}^n \simeq TB|_U$ so that a point $\sum_{i=1}^n y_i^i \frac{\partial}{\partial x^i} x \in T_x B \subset TB$ corresponds to $(x^1, \ldots, x^n; y^1, \ldots, y^n) \in U \times \mathbb{R}^n$.

We locally define the complex coordinate system by $z := (z^1, \ldots, z^n)^t$, where $z^i := x^i + iy^i$ with $i = 1, \ldots, n$. By the coordinate transformation [11], the bases are transformed by

$$\frac{\partial}{\partial x_\mu} = (\varphi_{\lambda\mu})^{-1} \frac{\partial}{\partial x_\lambda}, \quad \frac{\partial}{\partial x} := (\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n})^t,$$

and hence the corresponding coordinates are transformed by

$$y_\mu = \varphi_{\lambda\mu} y_\lambda, \quad y := (y^1, \ldots, y^n)^t$$

so that the combination $\sum_i y_i \frac{\partial}{\partial x^i}$ is independent of the coordinate systems. This shows that the transition functions for the manifold $TB$ are given by

$$\begin{pmatrix} x_\mu \\ y_\mu \end{pmatrix} = \begin{pmatrix} \varphi_{\lambda\mu} & 0 \\ 0 & \varphi_{\lambda\mu} \end{pmatrix} \begin{pmatrix} x_\lambda \\ y_\lambda \end{pmatrix} + \begin{pmatrix} \psi_{\lambda\mu} \\ 0 \end{pmatrix},$$

and hence the complex coordinate systems are transformed holomorphically:

$$z_\mu = \varphi_{\lambda\mu} z_\lambda + \psi_{\lambda\mu}.$$

On the other hand, for any smooth manifold $B$, the cotangent bundle $T^*B$ has a (canonical) symplectic form $\omega_{T^*B}$. For each $U_\lambda = U$, when we denote the coordinates of
This is actually defined globally since the coordinate transformations on $T^*B$ are induced from the coordinate transformations of $\{U_\lambda\}_{\lambda \in \Lambda}$. Actually, one has

$$dx(\lambda) = \varphi_{\lambda \mu} dx(\mu)$$

and the corresponding coordinates are transformed by

$$\hat{y}(\lambda) = (\varphi_{\lambda \mu})^{-1} \hat{y}(\mu), \quad \hat{y} := (y_1, \ldots, y_n)^t$$

so that the combination $\sum_{i=1}^n y_i dx^i \in T^*B$ is independent of the coordinates. From this, it follows that the symplectic form $\omega_{T^*B} = -d(\sum_{i=1}^n y_i dx^i)$ is defined globally.

By choosing a metric $g$ on a smooth manifold $B$, one obtains a bundle isomorphism between $TB$ and $T^*B$. For each $b \in B$, this isomorphism $TB \to T^*B$ is defined by $\xi \mapsto g(\xi, -)$ for $\xi \in T_bB$. This actually defines a bundle isomorphism since $g$ is nondegenerate at each point $b \in B$. This bundle isomorphism also induces a diffeomorphism from $TB$ to $T^*B$. In this sense, hereafter we sometimes identify $TB$ and $T^*B$. By this identification, $y^i$ and $y_i$ is related by

$$y_i = \sum_{j=1}^n g_{ij} y^j, \quad g_{ij} := g \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right).$$

When an affine manifold $B$ is equipped with a metric $g$ which is expressed locally as

$$g_{ij} = \frac{\partial^2 \phi}{\partial x^i \partial x^j}$$

for some local smooth function $\phi$, then $(B, g)$ is called a Hessian manifold. When $B$ is a Hessian manifold, then $TB \simeq T^*B$ is equipped with the structure of Kähler manifold as we explain below. In this sense, a Hessian manifold is also called an affine Kähler manifold.

First, when $B$ is affine, then $TB$ is already equipped with the complex structure $J_{TB}$. We fix a metric $g$ and set a two-form $\omega_{TB}$ on $TB$ as

$$\omega_{TB} := \sum_{i,j=1}^n g_{ij} dx^i \wedge dy^j.$$  

This $\omega_{TB}$ is nondegenerate since $g$ is nondegenerate. Furthermore, $\omega_{TB}$ is closed iff $(B, g)$ is Hessian, where $\omega_{TB}$ coincides with the pullback of $\omega_{T^*B}$ by the diffeomorphism $TB \to$
Thus, a Hessian manifold \((B, g)\) is equipped with the complex structure \(J_{TB}\) and the symplectic structure \(\omega_{TB}\). A metric \(g_{TB}\) on \(TB\) is then given by

\[
g_{TB}(X, Y) := \omega_{TB}(X, J_{TB}(Y))
\]

for \(X, Y \in \Gamma(T(TB))\). This is locally expressed as

\[
g_{TB} = \sum_{i,j=1}^{n} (g_{ij} dx^i dx^j + g_{ij} dy^i dy^j).
\]

This shows that \(g_{TB}\) is positive definite. To summarize, for a Hessian manifold \((B, g)\), \((TB, J_{TB}, \omega_{TB})\) forms a Kähler manifold, where \(g_{TB}\) is the Kähler metric.

In order to define a Kähler structure on \(T^*B\), we employ the dual affine local coordinate system

\[
\tilde{x} = (x_1, \ldots, x_n)^t
\]

of \(x = (x^1, \ldots, x^n)\) on \(B\), that is, the coordinate system \(\tilde{x}\) satisfying

\[
dx_i = \sum_{j=1}^{n} g_{ij} dx^j.
\]

Such an \(\tilde{x}\) actually exists since \((B, g)\) is Hessian; we may set \(x_i(x) := (\partial \phi/\partial x^i)(x)\). We thus obtain the dual coordinate system \(\tilde{x}^{(\lambda)} := (x_1^{(\lambda)}, \ldots, x_n^{(\lambda)})^t\) for each \(\lambda\). The dual coordinates then define another affine structure on \(B\). Actually, the local description of the metric is changed by

\[
g^{(\lambda)} = \{((g_{ij})^{(\lambda)})_{i,j=1}^{n}\} = (\phi_{\lambda}^t)^{-1} g_{(\mu)} \phi_{\lambda}^{-1},
\]

so one has \(d\tilde{x}^{(\lambda)} = (\phi_{\lambda}^t)^{-1} d\tilde{x}^{(\mu)} \) and then

\[
\tilde{x}^{(\lambda)} = (\phi_{\lambda}^t)^{-1} \tilde{x}^{(\mu)} + \tilde{\psi}_{\lambda}^{\mu}, \tag{3}
\]

for some \(\tilde{\psi}_{\lambda}^{\mu} \in \mathbb{R}^n\). Thus, the combinations \(z_i := x_i + iy_i, i = 1, \ldots, n\), form a complex coordinate system on \(T^*B\), and \(T^*B\) forms a complex manifold. Actually, by eq.(2) and (3), one has the holomorphic coordinate transformation

\[
\tilde{z}^{(\mu)} = (\phi_{\lambda}^t)^{-1} \tilde{z}^{(\lambda)} + \tilde{\psi}_{\lambda}^{\mu}, \quad \tilde{z} := (z_1, \ldots, z_n)^t.
\]

Using this dual coordinates, the symplectic form \(\omega_{T^*B}\) is expressed locally as

\[
\omega_{T^*B} = \sum_{i,j=1}^{n} g^{ij} dx_i \wedge dy_j,
\]

where \(g^{ij}\) is the \((i, j)\) element of the inverse matrix of \(\{g_{ij}\}\). Then, we set a metric on \(T^*B\) by

\[
g_{T^*B}(X, Y) := \omega_{T^*B}(X, J_{T^*B}(Y))
\]
for $X, Y \in \Gamma(T(T^*B))$, which is locally expressed as

$$g_{T^*B} = \sum_{i,j=1}^{n} (g^{ij} dx_i dx_j + g^{ij} dy_i dy_j).$$

These structures define a Kähler structure on $T^*B$.

For a tropical Hessian manifold $B$, we consider two $T^n$-fibrations over $B$ obtained by a quotient $M$ of $TB$ and a quotient $\tilde{M}$ of $T^*B$ by fiberwise $\mathbb{Z}^n$ action as follows.

For $TB$, we locally consider $TB|_U$ and define a $\mathbb{Z}^n$-action generated by $y^i \mapsto y^i + 2\pi$ for each $i = 1, \ldots, n$. For $T^*B$, we again locally consider $T^*B|_U$ and define a $\mathbb{Z}^n$-action generated by $y_i \mapsto y_i + 2\pi$ for each $i = 1, \ldots, n$. Both $\mathbb{Z}^n$-actions are well-defined globally since $B$ is tropical affine, i.e., the transition functions of $n$-dimensional vector bundles $TB$ and $T^*B$ belong to $GL(n; \mathbb{Z})$.

Then, $M := TB/\mathbb{Z}^n$ is a Kähler manifold whose symplectic structure $\omega_M$ and complex structure $J_M$ are those naturally induced from $\omega_{TB}$ and $J_{TB}$ on $TB$. Similarly, $\tilde{M} := T^*B/\mathbb{Z}^n$ is a Kähler manifold whose symplectic structure $\omega_{\tilde{M}}$ and complex structure $J_{\tilde{M}}$ are those induced from $\omega_{T^*B}$ and $J_{T^*B}$, respectively. In particular, $z = x+iy$ and $\tilde{z} = \tilde{x}+i\tilde{y}$ turn out to be local complex coordinates of the complex manifolds $M$ and $\tilde{M}$, respectively. The fibrations $\pi : M \to B$ and $\tilde{\pi} : \tilde{M} \to B$ are often called semi-flat torus fibrations or $T^n$-invariant manifolds. See [18] [17] and also [8]. Since $M$ and $\tilde{M}$ are dual to each other, we can construct them in the opposite way. That is, if we consider the coordinate systems $\tilde{x}^{(\lambda)}$ for $B$, then the tangent bundle over $B$ is $T^*B$ above, and the cotangent bundle is $TB$. Following [18] [17], we treat $M$ as a symplectic manifold and $\tilde{M}$ as a complex manifold and discuss the homological mirror symmetry.

2.2. Toric manifolds and $T^n$-invariant manifolds. The set-up in the previous subsection is originally applied to the mirror symmetry of compact Calabi-Yau manifolds $M, \tilde{M}$. We would like to extend this set-up to the case $\tilde{M}$ is the complement of the toric divisors of a smooth compact toric manifold $X$. The complement $\tilde{M}$ is actually a trivial torus fibration $\tilde{\pi} : \tilde{M} \to B$ where the base $B$ is identified with the interior of the dual polytope $P$ of $X$.

What may be more interesting is that $B$ is actually tropical affine in this situation. Of course, since $B$ is a contractible open set, $B = \text{Int}(P)$ has an open covering by itself, which means that $B$ is tropical affine. However, what we meant is something stronger in the following sense. A smooth compact toric manifold $X$ has a natural open covering $\{\tilde{U}_\lambda\}_{\lambda \in \Lambda}$, where each $\tilde{U}_\lambda$ is associated to each cone in the fan of $X$ of maximal dimension, which induces the open covering $\{U_\lambda := \pi(\tilde{U}_\lambda)\}_{\lambda \in \Lambda}$ of $P$ (where the origin of each $\tilde{U}_\lambda$ corresponds by $\pi$ to each vertex of $P$). Then we see that the coordinate transformations are tropical affine (though $\tilde{U}_\lambda \cap B = B$ for any $\lambda$.) This seems important since we need
to include some information from the boundary $\partial(B)$ of $B$ when we discuss homological mirror symmetry of $X$ and its mirror dual.

Let us see the above construction explicitly for $X = \mathbb{C}P^n$. For

$$\mathbb{C}P^n = \{[t_0: \cdots : t_n]\},$$

the natural open covering is $\{\tilde{U}_\lambda\}_{\lambda=0,1,\ldots,n}$ where

$$\tilde{U}_\lambda = \{[t_0: \cdots : t_n] \mid t_\lambda \neq 0\}.$$

The corresponding local coordinates are $(w_1^{(\lambda)}, \ldots, w_n^{(\lambda)})$ where

$$w_1^{(\lambda)} = t_0/t_\lambda, \ldots, w_\lambda^{(\lambda)} = t_{\lambda-1}/t_\lambda, w_{\lambda+1}^{(\lambda)} = t_{\lambda+1}/t_\lambda, \ldots, w_n^{(\lambda)} = t_n/t_\lambda. \quad (4)$$

We identify $\tilde{M}$ with the complement of the toric divisors of $\mathbb{C}P^n$:

$$\tilde{M} = \{[t_0: \cdots : t_n] \mid t_0 \cdot t_1 \cdots t_n \neq 0\},$$

where $\tilde{\pi}: \tilde{M} \to B$ is given by

$$\tilde{\pi}([t_0: \cdots : t_n]) := [\lvert t_0 \rvert: \cdots: \lvert t_n \rvert]$$

(though we express this in a different coordinate system below). So we have $U_\lambda := \tilde{U}_\lambda \cap \tilde{M} = \tilde{M}$ for any $\lambda$. We further denote $U_\lambda := \pi(U_\lambda)$. For each $U_\lambda$, we express

$$\tilde{w}_i^{(\lambda)} = e^{e_i^{(\lambda)}} = e^{e_i^{(\lambda)} + b_i^{(\lambda)}}.$$

Since the coordinate transformation between $\tilde{z}^{(\lambda)}$ and $\tilde{z}^{(\mu)}$ is tropical affine by (4), so is the coordinate transformation between $\tilde{x}^{(\lambda)}$ and $\tilde{x}^{(\mu)}$.

Hereafter we consider $U := U_0$ (since $U_0 = U_1 = \cdots = U_n = B$) and drop the upper index $(0)$; for instance $w_i^{(0)} =: w_i$ and $x_i^{(0)} =: x_i$. The Fubini-Study Kähler form is then expressed in $\tilde{M} = (\tilde{\pi})^{-1}(U)$ as

$$\omega_{\tilde{M}} = -2i d\left(\tilde{w}_1dw_1 + \cdots + \tilde{w}_ndw_n\over 1 + \tilde{w}_1w_1 + \cdots + \tilde{w}_nw_n\right).$$

When we express this as $\omega_{\tilde{M}} = \sum_{i,j=1}^n g^{ij}dx_i \wedge dy_j$, we have

$$g^{ij} = \frac{\partial^2 \tilde{\phi}}{\partial x_i \partial x_j},$$

$$\tilde{\phi} = \log(1 + e^{2x_1} + \cdots + e^{2x_n}).$$

Thus, $B$ is a Hessian manifold. The dual coordinates $(x^1, \ldots, x^n)$ is obtained by

$$dx^i = \sum_{j=1}^n \frac{\partial^2 \tilde{\phi}}{\partial x_i \partial x_j} dx^j = d \left(\frac{\partial \tilde{\phi}}{\partial x_i}\right),$$
so

\[ x^i = \frac{\partial \tilde{\phi}}{\partial x_i} = \frac{2e^{2x_i}}{1 + e^{2x_1} + \cdots + e^{2x_n}}. \]

(5)

By this \((x^1, \ldots, x^n)\), \(B\) is expressed as

\[ B = \{(x^1, \ldots, x^n) | x^1 > 0, \ldots, x^n > 0, x^1 + \cdots + x^n < 2\}. \]

In particular, in this coordinate system, \(\pi: \tilde{M} \to B\) is expressed as

\[ \pi(\tilde{x}, \tilde{y}) = (x^1(\tilde{x}), \ldots, x^n(\tilde{x})) = \left( \begin{array}{c} 2\bar{w}_1w_1 \\
1 + \bar{w}_1w_1 + \cdots + \bar{w}_nw_n \\
1 + \bar{w}_1w_1 + \cdots + \bar{w}_nw_n \end{array} \right). \]

In this way, \(\pi\) is regarded as the restriction to \(\tilde{M} \subset X\) of the moment map \(X \to \mathbb{R}^n\) for the \(T^n\) action on \(X = \mathbb{C}P^n\).

Note also that we can regard \(M\), the dual torus fibration of \(\tilde{M}\), as an open complex submanifold of \((\mathbb{C}^\times)^n\), where \((e^{-(x^1+iy^1)}, \ldots, e^{-(x^n+iy^n)})\) is the coordinate system of \((\mathbb{C}^\times)^n\). However, the symplectic form \(\omega_M = \sum_{i,j=1}^n g_{ij} dx^i \land dy^j\) diverges at the boundary \(\partial(M) = \pi^{-1}(\partial(B))\).

3. Lagrangian submanifolds and holomorphic vector bundles

In the first two subsections, we first recall the construction of line bundles on \(\tilde{M}\) associated to Lagrangian sections of \(M \to B\) discussed in [18, 17]. Then, in subsection 3.3 we apply this construction to the case \(\tilde{M}\) is the complement of the toric divisors of \(\mathbb{C}P^n\).

This construction gives an objectwise correspondence of the corresponding homological mirror symmetry. More generally, on \(M\) we can consider Lagrangian sections equipped with local systems as objects of the Fukaya category \(\text{Fuk}(M)\). However, we do not discuss this generalized set-up since we need only Lagrangian sections equipped with trivial local system for our purpose. See also Remark 3.1 at the end of subsection 3.2.

3.1. Lagrangian submanifolds in \(M\). We fix a tropical affine open covering \(\{U_\lambda\}_{\lambda \in \Lambda}\). Let \(s : B \to M\) be a section of \(M \to B\). Locally, we may regard \(s\) as a section of \(TB \simeq T^*B\). Then, \(s\) is locally described by a collection of functions as

\[ y^i_\lambda = s^i_\lambda(x) \]

on each \(U_\lambda\).

On \(U_\lambda \cap U_\mu\), these local expressions are related to each other by

\[ s^i_\mu(x) = s^i_\lambda(x) + 2\pi I_{\lambda \mu} \]

(6)
for some $I_{\lambda \mu} \in \mathbb{Z}^n$. Here, $x$ may be identified with either $x(\lambda)$ or $x(\mu)$. Also, $s(\lambda)(x)$ and $s(\mu)(x)$ are expressed by the common coordinates $y(\lambda)$ or $y(\mu)$. This transformation rule automatically satisfies the cocycle condition
\[ I_{\lambda \mu} + I_{\mu \nu} + I_{\nu \lambda} = 0 \] (7)
for $U_\lambda \cap U_\mu \cap U_\nu \neq \emptyset$. We denote by $s$ such a collection $\{s(\lambda) : U_\lambda \to TB|_{U_\lambda}\}_{\lambda \in \Lambda}$ which is equipped with the transformation rule (6) satisfying the cocycle condition (7).

Now we discuss when the graph of $s$ forms a Lagrangian submanifold in $M$. By definition, an $n$-dimensional submanifold $L$ in a $2n$-dimensional symplectic manifold $(M, \omega_M)$ is Lagrangian iff $\omega_M|_L = 0$. This is a local condition. Thus, in order to discuss whether the graph of a section $s : B \to M$ is Lagrangian or not, we may check the condition locally and in particular in $T^*B$.

It is known (as shown easily by taking the basis) that the graph of $\sum_{i=1}^n y_i dx^i$ with local functions $y_i$ is Lagrangian in $T^*B$ iff there exists a local function $f$ such that $\sum_{i=1}^n y_i dx^i = df$. Now, a section $s : B \to M$ is locally regarded as a section of $T^*B$ by setting $y_i = \sum_{j=1}^n g_{ij} y^j = \sum_{j=1}^n g_{ij} s^j$, from which one has
\[ \sum_{i=1}^n y_i dx^i = \sum_{i=1}^n \left( \sum_{j=1}^n g_{ij} s^j \right) dx^i = \sum_{j=1}^n s^j dx_j. \]

Thus, the graph of the section $s : B \to M$ is Lagrangian iff there exists a local function $f$ such that $\sum_{j=1}^n s^j dx_j = df$.

Note that $y = s(x)$ defines a special Lagrangian submanifold if $s$ is affine with respect to $x^i$. (Thus, the zero section of $M \to B$ is a special Lagrangian submanifold.)

The gradient vector field is of the form:
\[ \text{grad}(f) := \sum_{i,j} \partial f/\partial x^j g^{ji} \partial/\partial x^i = \sum_{i} \partial f/\partial x^i \partial/\partial x^i. \] (8)

3.2. Holomorphic vector bundles on $\tilde{M}$. Consider a section $s : B \to M$ and express it as a collection $s = \{s(\lambda)\}_{\lambda \in \Lambda}$ of local functions. We define a line bundle $V$ with a $U(1)$-connection on the mirror manifold $\tilde{M}$ associated to $s$. We set the covariant derivative locally as\(^\dagger\)
\[ D := d - \frac{i}{2\pi} \sum_{i=1}^n s^i(x) dy_i, \] (9)
\(^\dagger\)We switch the sign of the connection one form compared to that in [12] so that the mirror correspondence of objects fits with the one in homological mirror symmetry of tori as in [14] and references therein. We also include $2\pi$ in various places which are missing in [12].
whose curvature is
\[ D^2 = \frac{i}{2\pi} \sum_{i,j=1}^{n} \frac{\partial s^i}{\partial x_j} dx_j \wedge dy_i. \]

The \((0,2)\)-part vanishes iff the matrix \(\frac{\partial s^i}{\partial x_j}\) is symmetric, which is the case when there exists a function \(f\) locally such that \(df = \sum_{i=1}^{n} s^i dx_i\). Thus, the condition that \(D\) defines a holomorphic line bundle on \(\tilde{M}\) is equivalent to that the graph of \(s\) is Lagrangian in \(M\).

This covariant derivative \(D\) is in fact defined globally. Suppose that \(D\) is given locally on each \(\tilde{M}|_{U_\lambda}\) of the \(T^n\)-fibration \(\tilde{M} \to B\) with a fixed tropical affine open covering \(\{U_\lambda\}_{\lambda \in \Lambda}\). Namely, we continue to employ \(\{U_\lambda\}_{\lambda \in \Lambda}\) for local trivializations of the line bundle associated to a section \(s : B \to M\). The transition functions for \((V,D)\) are defined as follows. Recall that the section \(s : B \to M\) is expressed locally as
\[ y^i_{(\lambda)} = s^i_{(\lambda)}(x) \]
on each \(U_\lambda\), where, on \(U_\lambda \cap U_\mu\), the local expression is related to each other by
\[ s_{(\mu)}(x) = s_{(\lambda)}(x) + 2\pi I_{\lambda\mu} \]
for some \(I_{\lambda\mu} \in \mathbb{Z}^n\) (see eq. (6)). Correspondingly, the transition function for the line bundle \(V\) with the connection \(D\) is given by
\[ \psi_{(\mu)} = e^{I_{\lambda\mu} \cdot \hat{y}} \psi_{(\lambda)} \]
for local expressions \(\psi_{(\lambda)}, \psi_{(\mu)}\) of a smooth section \(\psi\) of \(V\), where \(I_{\lambda\mu} \cdot \hat{y} := \sum_{j=1}^{n} i_j y_j\) for \(I_{\lambda\mu} = (i_1, \ldots, i_n)\). We see the compatibility
\[ (D\psi_{(\lambda)})_{(\mu)} = D(\psi_{(\mu)}) \]
holds true since the left hand side turns out to be
\[ e^{I_{\lambda\mu} \cdot \hat{y}} \left( \left( d - \frac{i}{2\pi} s_{(\lambda)}(x) \cdot dy \right) e^{-I_{\lambda\mu} \cdot \hat{y}} \psi_{(\mu)} \right) = e^{I_{\lambda\mu} \cdot \hat{y}} e^{-I_{\lambda\mu} \cdot \hat{y}} \left( \left( d - \frac{i}{2\pi} (s_{(\lambda)}(x) + 2\pi I_{\lambda\mu}) \cdot dy \right) \psi_{(\mu)} \right) \]
\[ = \left( d - \frac{i}{2\pi} s_{(\mu)}(x) \cdot dy \right) \psi_{(\mu)}. \]

Since \((V,D)\) is locally-trivialized by \(\{\tilde{M}|_{U_\lambda}\}_{\lambda \in \Lambda}\), for each \(x \in B\), \(\psi(x, \cdot)\) gives a smooth function on the fiber \(T^n\). Thus, on each \(U_\lambda\), \(\psi(x, y)\) can be Fourier-expanded as
\[ \psi(x, y)|_{U_\lambda} = \sum_{I \in \mathbb{Z}^n} \psi_{I,\lambda}(x) e^{I \cdot \hat{y}}, \]
where \( I \cdot \check{y} := \sum_{j=1}^{n} i_j y_j \) for \( I = (i_1, \ldots, i_n) \). Note that each coefficient \( \psi_{\lambda, I} \) is a smooth function on \( U_\lambda \). In this expression, the transition function acts to each \( \psi_{\lambda, I} \) as
\[
\sum_{I \in \mathbb{Z}^n} \psi_{\mu, I} e^{i(I+I_{\lambda_{\mu}}) \cdot \check{y}} = \sum_{I \in \mathbb{Z}^n} \psi_{\lambda, I} e^{i(I_{\lambda_{\mu}}) \cdot \check{y}}
\]
and hence \( \psi_{\mu, I} = \psi_{\lambda, I - I_{\lambda_{\mu}}} \).

**Remark 3.1.** We can also associate a line bundle to a Lagrangian section equipped with a local system, where the holonomy turns out to be included as the real coefficient of each \( dy_i \) in the covariant derivation (9). We do not discuss this generalized set-up since the real coefficients are trivial for any line bundle we need in this paper and hence the corresponding object is a Lagrangian section with the trivial local system.

### 3.3. Holomorphic line bundles on \( \mathbb{C}P^n \) and the corresponding Lagrangians.

In the previous subsections, we assign a line bundle on \( \check{M} \) to each Lagrangian section in \( M \to B \). In this subsection, we start from a line bundle on \( \mathbb{C}P^n \). We identify \( \check{M} \) with the complement of the toric divisors of \( \mathbb{C}P^n \), and restrict the line bundle to \( \check{M} \). We will see that, by twisting it with an appropriate isomorphism, the result actually comes from a Lagrangian section in \( M \to B \). In this way, we construct a Lagrangian section in \( M \to B \) corresponding to \( \mathcal{O}(a) \) on \( \mathbb{C}P^n \) for any \( a \in \mathbb{Z} \).

We continue the convention in subsection 2.2. The complement of the toric divisors of \( \mathbb{C}P^n \) is
\[
\check{M} = \{ [t_0 : \cdots : t_n] \mid t_0 \cdot t_1 \cdots t_n \neq 0 \},
\]
where
\[
e^{x_i + iy_i} = w_i = t_i / t_0.
\]
A connection one-form of \( \mathcal{O}(a) \) is given by the one which is expressed locally on \( \check{M} \) as
\[
A_a = -a \frac{\check{w}_1 w_1 + \cdots + \check{w}_n w_n}{1 + \check{w}_1 w_1 + \cdots + \check{w}_n w_n} = \frac{e^{2x_1}(dx_1 + i dy_1) + \cdots + e^{2x_n}(dx_n + i dy_n)}{1 + e^{2x_1} + \cdots + e^{2x_n}}.
\]

We twist this by
\[
\Psi_a := (1 + e^{2x_1} + \cdots + e^{2x_n})^{a/2},
\]
and then obtain
\[
\Psi_a^{-1}(d + A_a) \Psi_a = d - ia \frac{e^{2x_1} dy_1 + \cdots + e^{2x_n} dy_n}{1 + e^{2x_1} + \cdots + e^{2x_n}}.
\]
By the previous subsections, this is the line bundle on $\tilde{M}$ which corresponds to the Lagrangian section $L_a$ in $M \to B$ expressed as

\[
\begin{pmatrix}
  y^1 \\
  \vdots \\
  y^n
\end{pmatrix}
= \begin{pmatrix}
  s^1_a \\
  \vdots \\
  s^n_a
\end{pmatrix}
= 2\pi a
\begin{pmatrix}
  \frac{e^{2x_1}}{1 + e^{2x_1} + \cdots + e^{x_n}} \\
  \vdots \\
  \frac{e^{2x_n}}{1 + e^{2x_1} + \cdots + e^{x_n}}
\end{pmatrix}
= 2\pi a
\begin{pmatrix}
  x^1 \\
  \vdots \\
  x^n
\end{pmatrix}
\]

where $x^i > 0$ for $i = 1, \ldots, n$ and $x^1 + \cdots + x^n < 2$.

We see that the local function

\[
f_a = 2\pi a \left( \log(1 + e^{2x_1} + \cdots + e^{2x_n}) - \log 2 \right)
= -2\pi a \log(2 - x^1 - x^2 - \cdots - x^n)
\]

satisfies $df_a = \sum_{i=1}^n s^i_a \, dx_i$. The corresponding gradient vector field is

\[
\grad(f_a) = \sum_{i=1}^n \frac{\partial f_a}{\partial x_i} \frac{\partial}{\partial x^i} = 2\pi a \left( x^1 \frac{\partial}{\partial x^1} + \cdots + x^n \frac{\partial}{\partial x^n} \right)
\]

by (8).

**Remark 3.2.** This Lagrangian section $L_a$ is a special Lagrangian since it is expressed locally as the graph of linear functions $y^i(x)$ of $(x^1, \ldots, x^n)$.

Furthermore, we see that $L_a$ includes a critical point of the corresponding Landau-Ginzburg potential. In fact, the Landau-Ginzburg potential $W$ is

\[
W(w^1, w^2, \ldots, w^n) := w^1 + \cdots + w^n + \frac{e^{-2}}{w^1 w^2 \cdots w^n},
\]

where $w^i := e^{-(x^i + iy^i)}$. The critical points are given by

\[
(w^1, \ldots, w^n) = (e^{-\frac{2}{a+1}}\zeta^a, \ldots, e^{-\frac{2}{a+1}}\zeta^a) =: c_a, \quad a = 0, 1, \ldots, n
\]

where $\zeta = (1)^{\frac{1}{n+1}}$ is the $(n+1)$-th root of unity. Thus, we see that each critical point $c_a \in (\mathbb{C}^\times)^n$ is included in $L_a$.

### 4. Homological mirror symmetry set-up

In this section, we first recall DG-categories $\mathcal{F}(M)$ and $\mathcal{V}(\tilde{M})$ associated to $M$ and $\tilde{M}$ respectively, following [12]. Kontsevich-Soibelman’s approach for the homological mirror symmetry [16] introduces an intermediate category $\mathcal{M}_o(B)$ and the existence of an $A_\infty$-equivalence

\[
\Fuk(M) \simeq \mathcal{M}_o(B) \xrightarrow{\sim} \mathcal{F}(M).
\]
This $Mo(B)$ is called the weighted Fukaya-Oh category or the category of weighted Morse homotopy on $B$. In subsection 4.5, we propose a modification $Mo(P)$ of $Mo(B)$ where $P = \bar{B}$ is the dual polytope of a smooth compact toric manifold $X$.

4.1. DG-category $\mathcal{V}$ associated to $\bar{M}$. We define a DG-category $\mathcal{V} = \mathcal{V}(\bar{M})$ of holomorphic line bundles over $\bar{M}$ as follows. The objects are holomorphic line bundles $\mathcal{V}$ with $U(1)$-connections $D$ associated to lifts $s$ of sections as we defined in subsection 3.2. We sometimes label these objects as $s$ instead of $(\mathcal{V}, D)$. For any two objects $s_a = (\mathcal{V}_a, D_a), s_b = (\mathcal{V}_b, D_b) \in \mathcal{V}$, the space $\mathcal{V}(s_a, s_b)$ of morphisms is defined by

$$\mathcal{V}(s_a, s_b) := \Gamma(\mathcal{V}_a, \mathcal{V}_b) \otimes_{C^\infty(\bar{M})} \Omega^{0,*}(\bar{M}),$$

where $\Omega^{0,*}(\bar{M})$ is the space of anti-holomorphic differential forms and $\Gamma(\mathcal{V}_a, \mathcal{V}_b)$ is the space of homomorphisms from $\mathcal{V}_a$ to $\mathcal{V}_b$. The space $\mathcal{V}(s_a, s_b)$ is a $\mathbb{Z}$-graded vector space, where the grading is defined as the degree of the anti-holomorphic differential forms. The degree $r$ part is denoted by $\mathcal{V}_r(s_a, s_b)$. We define a linear map $d_{ab} : \mathcal{V}_r(s_a, s_b) \to \mathcal{V}^{r+1}(s_a, s_b)$ as follows. We decompose $D_a$ into its holomorphic part and anti-holomorphic part $D_a = D_a^{(1,0)} + D_a^{(0,1)}$, and set $2D_a^{(0,1)} =: d_a$. Then, for $\psi \in \mathcal{V}_r(s_a, s_b)$, we set

$$d_{ab}(\psi) := d_b\psi - (-1)^r\psi d_a \in \mathcal{V}^{r+1}(s_a, s_b).$$

Note that $d_{ab}^2 = 0$ since each $(\mathcal{V}_a, D_a)$ is holomorphic, i.e., $(d_a)^2 = 0$.

The product structure $m : \mathcal{V}(s_a, s_b) \otimes \mathcal{V}(s_b, s_c) \to \mathcal{V}(s_a, s_c)$ is defined by the composition of homomorphisms of line bundles together with the wedge product for the anti-holomorphic differential forms. More precisely, for $\psi_{ab} \in \mathcal{V}_{r_{ab}}(s_a, s_b)$ and $\psi_{bc} \in \mathcal{V}_{r_{bc}}(s_b, s_c)$, we set

$$m(\psi_{ab}, \psi_{bc}) := (-1)^{r_{ab}r_{bc}}(\psi_{ab} \wedge \psi_{bc} (= \psi_{ab} \wedge \psi_{bc}),$$

where $\wedge$ denotes the operation consisting of the composition and the wedge product.

In order to construct another equivalent curved DG-category, we rewrite this DG-category $\mathcal{V}$ more explicitly. For an element $\psi \in \mathcal{V}_r(s_a, s_b)$, we Fourier-expand this locally as

$$\psi(\tilde{x}, \tilde{y}) = \sum_{I \in \mathbb{Z}^n} \psi_I(\tilde{x})e^{H\tilde{y}},$$

Here we again make a minor change of the formulation of the DG category compared to [12] due to the change of sign in (9).

In [12], we construct a curved DG category $DG_{\bar{M}}$ where the objects are not necessarily holomorphic. The relation is given by $\mathcal{V} = DG_{\bar{M}}(0)$. 

Note that $d_{ab}^2 = 0$ since each $(\mathcal{V}_a, D_a)$ is holomorphic, i.e., $(d_a)^2 = 0$.

The product structure $m : \mathcal{V}(s_a, s_b) \otimes \mathcal{V}(s_b, s_c) \to \mathcal{V}(s_a, s_c)$ is defined by the composition of homomorphisms of line bundles together with the wedge product for the anti-holomorphic differential forms. More precisely, for $\psi_{ab} \in \mathcal{V}_{r_{ab}}(s_a, s_b)$ and $\psi_{bc} \in \mathcal{V}_{r_{bc}}(s_b, s_c)$, we set

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$$\psi(\tilde{x}, \tilde{y}) = \sum_{I \in \mathbb{Z}^n} \psi_I(\tilde{x})e^{H\tilde{y}},$$

In [12], we construct a curved DG category $DG_{\bar{M}}$ where the objects are not necessarily holomorphic. The relation is given by $\mathcal{V} = DG_{\bar{M}}(0)$. 

Note that $d_{ab}^2 = 0$ since each $(\mathcal{V}_a, D_a)$ is holomorphic, i.e., $(d_a)^2 = 0$.
where \( \psi_I \) is a smooth anti-holomorphic differential form of degree \( r \). Namely, it is expressed as

\[
\psi_I = \sum_{i_1, \ldots, i_r} \psi_{I;i_1 \cdots i_r} d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_r}
\]

with smooth functions \( \psi_{I;i_1 \cdots i_r} \). Let us express the transformation rule for \( s_a \) as

\[
(s_a)_{(\mu)} = (s_a)_{(\lambda)} + 2\pi I_a
\]

with \( I_a = I_{a;\lambda\mu} \in \mathbb{Z}^n \). The transition function is then given by

\[
\psi(\mu)_I = e^{i(\mathfrak{q}_b - \mathfrak{q}_a)} \frac{\mathfrak{q}_a}{\mathfrak{q}_b} \psi(\lambda)_I
\]

and hence

\[
\psi(\mu)_I, I = \psi(\lambda)_I + I_a - I_b.
\]

The differential \( d_{ab} \) is expressed locally as follows. Since

\[
D_a = d - \frac{i}{2\pi} \sum_{j=1}^{n} s_a^j(x)dy_j
\]

\[
= \sum_{j=1}^{n} \left( \frac{\partial}{\partial x_j} dx_j + \left( \frac{\partial}{\partial y_j} - \frac{i}{2\pi} s_a^j \right) dy_j \right)
\]

\[
= \frac{1}{2} \sum_{j=1}^{n} \left( \frac{\partial}{\partial x_j} - i \left( \frac{\partial}{\partial y_j} - \frac{i}{2\pi} s_a^j \right) \right) d\bar{z}_j + \frac{1}{2} \sum_{j=1}^{n} \left( \frac{\partial}{\partial x_j} + i \left( \frac{\partial}{\partial y_j} - \frac{i}{2\pi} s_a^j \right) \right) d\bar{z}_j,
\]

one has

\[
d_a = 2D_a^{(0,1)} = \sum_{j=1}^{n} \left( \frac{\partial}{\partial x_j} + \frac{s_a^j}{2\pi} + i \frac{\partial}{\partial y_j} \right) d\bar{z}_j
\]

and then

\[
d_{ab}(\psi) = 2\partial(\psi) - \frac{1}{2\pi} \sum_{i=1}^{n} (s_a - s_b)^i d\bar{z}_i \wedge \psi.
\]

4.2. **DG-category \( \mathcal{F} \) associated to \( M \).** We define a DG-category \( \mathcal{F} = \mathcal{F}(M) \) consisting of Lagrangian sections in \( M \) as follows\(^4\). As we shall see, we construct it so that it is canonically isomorphic to the previous DG-category \( \mathcal{V} \). We fix a tropical affine open covering \( \{U_{\lambda}\}_{\lambda \in \Lambda} \) of \( B \).

After the identification of \( (V, D) \) with \( s \) made in the previous subsection\(^1\), the objects are the same as those in \( \mathcal{V} \), that is, lifts \( s \) of sections of \( M \rightarrow B \). Under this identification the object \( s_a \in \mathcal{F} \), satisfies the transformation rule \((\ref{transformation_rule})\) as above. For each \( \lambda \in \Lambda \) and \( I \in \mathbb{Z}^n \), let \( \Omega_{\lambda,I}(s_a, s_b) \) be the space of complex valued smooth differential forms on \( U_{\lambda} \).

The space \( \mathcal{F}(s_a, s_b) \) is then the subspace of

\[
\prod_{\lambda \in \Lambda} \prod_{I \in \mathbb{Z}^n} \Omega_{\lambda,I}(s_a, s_b)
\]

\(^4\)This \( \mathcal{F} \) corresponds to \( DG_M(0) \) in \(\ref{previous_works}\).
such that

- \( \phi_{\lambda,I} \in \Omega_{\lambda,I}(s_a, s_b) \) satisfies
  \[ \phi_{\mu,I}|_{U_{\lambda} \cap U_{\mu}} = \phi_{\lambda,I+I_a-I_b}|_{U_{\lambda} \cap U_{\mu}} \]
  for any \( U_{\lambda} \cap U_{\mu} \neq \emptyset \) and

- the sum \( \sum_{I \in \mathbb{Z}^n} \phi_{\lambda,I} e^{iI \cdot \bar{y}} \) converges as smooth differential forms on each \( M|_{U_{\lambda}} \).

The space \( \mathcal{F}(s_a, s_b) \) is a \( \mathbb{Z} \)-graded vector space, where the grading is defined as the degree of the differential forms. The degree \( r \) part is denoted \( \mathcal{F}^r(s_a, s_b) \). We define a linear map \( d_{ab} : \mathcal{F}^r(s_a, s_b) \to \mathcal{F}^{r+1}(s_a, s_b) \) which is expressed locally as

\[
d_{ab}(\phi_{\lambda,I}) := d(\phi_{\lambda,I}) - \frac{1}{2\pi} \sum_{j=1}^{n} (s_j^a - s_j^b + 2\pi i_j) dx_j \wedge \phi_{\lambda,I}
\]

for \( \phi_{\lambda,I} \in \Omega_{\lambda,I}(s_a, s_b) \) with \( I := (i_1, \ldots, i_n) \in \mathbb{Z}^n \), where \( d \) is the exterior differential on \( B \).

We have \( d_{ab}^2 = 0 \).

The composition of morphisms \( m : \mathcal{F}(s_a, s_b) \otimes \mathcal{F}(s_b, s_c) \to \mathcal{F}(s_a, s_c) \) is defined by

\[
m(\phi_{ab,\lambda,I}, \phi_{bc,\mu,J}) := \phi_{ab,\lambda,I} \wedge \phi_{bc,\mu,J} \in \Omega_{\lambda,I+J}(s_a, s_c)
\]

for \( \phi_{ab,\lambda,I} \in \Omega_{\lambda,I}(s_a, s_b) \) and \( \phi_{bc,\mu,J} \in \Omega_{\mu,J}(s_b, s_c) \). These structures define a DG-category \( \mathcal{F} \). Note that this \( \mathcal{F} \) is believed to be \( A_\infty \)-equivalent to the corresponding full subcategory of the Fukaya category \( \text{Fuk}(M) \). (Compare this \( \mathcal{F} \) with what is called the deRham model for the Fukaya category in Kontsevich-Soibelman [16], in particular a construction in the Appendix (Section 9.2). ) In subsection 4.4 we shall explain the outline of how to compare \( \mathcal{F} \) with the Fukaya category.

### 4.3. Equivalence between \( \mathcal{F} \) and \( \mathcal{V} \)

The DG-category \( \mathcal{F} \) is canonically isomorphic to the DG-category \( \mathcal{V} \). In fact, we see that the objects in \( \mathcal{F} \) are the same as those in \( \mathcal{V} \). The spaces of morphisms in \( \mathcal{F} \) and in \( \mathcal{V} \) are also identified canonically as follows. For a morphism \( \phi_{ab} = \{\phi_{ab,\lambda,I}\} \in \mathcal{F}^r(s_a, s_b) \), each \( \phi_{ab,\lambda,I} \) is expressed as

\[
\phi_{ab,\lambda,I} = \sum_{i_1, \ldots, i_r} \phi_{ab,\lambda,I; i_1 \ldots i_r} dx_{i_1} \wedge \cdots \wedge dx_{i_r}.
\]

To this, we correspond an element in \( \mathcal{V}^r(s_a, s_b) \) which is locally given as

\[
\sum_{i_1, \ldots, i_r} (\phi_{ab,\lambda,I; i_1 \ldots i_r} e^{iI \cdot \bar{y}}) d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_r}
\]

on \( U_{\lambda} \). We denote this correspondence by \( \mathfrak{f} : \mathcal{F} \to \mathcal{V} \). It is easily seen that our construction guarantees the following fact.

**Proposition 4.1 ([12 Proposition 4.1.]).** The functor \( \mathfrak{f} : \mathcal{F} \to \mathcal{V} \) is a DG-isomorphism.
4.4. The DG-category $\mathcal{F}$ and the Fukaya category $Fuk(M)$. The DG-category $\mathcal{F}$ is expected to be $A_\infty$-equivalent to the Fukaya category $Fuk(M)$ \cite{7} of Lagrangian sections. The idea discussed in \cite{16} to relate them is to apply homological perturbation theory to the DG-category $\mathcal{F}$ (as an $A_\infty$-category) in an appropriate way so that the induced $A_\infty$-category coincides with (the full subcategory of) the Fukaya category $Fuk(M)$. More precisely, what should be induced directly from $\mathcal{F}$ is the category $Mo(B)$ of weighted Morse homotopy or the Fukaya-Oh category for the torus fibration $M \to B$ introduced in section 5.2 of \cite{16}. Here, the Fukaya-Oh category means the $A_\infty$-category of Morse homotopy on $B$ introduced in \cite{7}. It is shown in \cite{9} that the Fukaya-Oh category is equivalent to the Fukaya category $Fuk(T^*B)$ consisting of the corresponding objects. The Fukaya-Oh category for the torus fibration $M \to B$ is a generalization of the Fukaya-Oh category on $B$ so that it corresponds to the Fukaya category $Fuk(M)$ instead of $Fuk(T^*B)$. Thus, a natural way of obtaining the $A_\infty$-equivalence $\mathcal{F} \simeq Fuk(M)$ is to interpolate the category $Mo(B)$ so that

$$Fuk(M) \simeq Mo(B) \xrightarrow{\sim} \mathcal{F},$$

where an $A_\infty$-equivalence $Mo(B) \to \mathcal{F}$ is expected to be obtained by the homological perturbation theory and then $Mo(B)$ is identified with $Fuk(M)$ by a linear $A_\infty$-isomorphism. There are some technical difficulties in proceeding this story precisely. See \cite{12, subsection 4E.}.

4.5. The category $Mo(P)$ of weighted Morse homotopy. If we start with a toric manifold $X$ and set $\tilde{M}$ as the complement of the toric divisors, we obtain $M$ as a torus fibration over the interior $B$ of the dual polytope $P$. As we discuss in the next section, from the homological mirror symmetry viewpoint, what we should discuss is not $Fuk(M)$ but a kind of Fukaya category $Fuk(\tilde{M})$ of a torus fibration over $P = \tilde{B}$. As an intermediate step, we consider the category $Mo(P)$ of weighted Morse homotopy for the dual polytope $P$. This $Mo(P)$ is a generalization of the weighted Fukaya-Ohe category given in \cite{16} to the case where the base manifold has boundaries and critical points may be degenerate.

Although we need only Lagrangians of constant slopes (which we call affine Lagrangians) in this paper, we formulate $Mo(P)$ as general as possible in the following description, as the framework works in more general toric cases. In a subsequent paper \cite{10}, we use this formulation again to extend the result to the case of the Hirzebruch surface $\mathbb{F}_1$, where Lagrangians defined by rational functions appear.

**Definition of $Mo(P)$** The definition is as follows. The objects of $Mo(P)$ are Lagrangian sections of $\pi : M \to B$ satisfying certain conditions (see for instance \cite{5, Definition 3.1]). We extend each Lagrangian section on $B$ to that on $\tilde{B}$ smoothly. We say that two objects $L, L'$ intersect *cleanly* if there exists an open set $\tilde{B} \subset \mathbb{R}^n$ such that $\tilde{B} \subset \tilde{B}$ and $L, L'$ over
$B$ can be extended to graphs of smooth sections over $\tilde{B}$ so that they intersect cleanly. We assume that any two objects $L, L'$ intersect cleanly.

For each $L$, we take a function $f_L$ on $\tilde{B}$ so that $L$ is the graph of $df_L$. For a given ordered pair $(L, L')$, we assign a grading $|V|$ for each connected component $V$ of the intersection $\pi(L \cap L')$ in $P = \tilde{B}$ as the dimension of the stable manifold $S_v \subset \tilde{B}$ of the gradient vector field $-\text{grad}(f_L - f_{L'})$ with a point $v \in V$. This does not depend on the choice of the point $v \in V$. The space $\text{Mo}(P)(L, L')$ of morphisms is then set to be the $\mathbb{Z}$-graded vector space spanned by the connected components $V$ of $\pi(L \cap L') \in P$ such that there exists a point $v \in V$ which is an interior point of $S_v \cap P \subset S_v$.

Now let us consider an $(l + 1)$-tuple $(L_1, \ldots, L_{l+1})$, $l \geq 2$, and take a generator $V_i(l+1) \in \text{Mo}(P)(L_i, L_{l+1})$ for each $i$ and $V_1(l+1) \in \text{Mo}(P)(L_1, L_{l+1})$. We denote by

$$\mathcal{GT}(v_{12}, \ldots, v_{l(l+1)}; v_{1(l+1)})$$

the set of gradient trees starting at $v_{12}, \ldots, v_{l(l+1)}$, where $v_i(l+1) \in V_i(l+1)$, and ending at $v_{1(l+1)} \in V_1(l+1)$. Here, a gradient tree $\gamma \in \mathcal{GT}(v_{12}, \ldots, v_{l(l+1)}; v_{1(l+1)})$ is a continuous map $\gamma : T \rightarrow P$ with a rooted trivalent $l$-tree $T$. Regarding $T$ as a planar tree, the leaf external vertices from the left to the right are mapped to $v_{12}, \ldots, v_{l(l+1)}$, and the root external vertex is mapped to $v_{1(l+1)}$ by $\gamma$. Furthermore, for each edge $e$ of $T$, the restriction $\gamma|_e$ is a gradient trajectory of the corresponding gradient vector field. See [10].

We then denote

$$\mathcal{GT}(V_{12}, \ldots, V_{l(l+1)}; V_{1(l+1)}) := \bigcup_{(v_{12}, \ldots, v_{l(l+1)}; v_{1(l+1)}) \in V_{12} \times \cdots \times V_{l(l+1)} \times V_{1(l+1)}} \mathcal{GT}(v_{12}, \ldots, v_{l(l+1)}; v_{1(l+1)}).$$

We say that two gradient trees $\gamma, \gamma' \in \mathcal{GT}(V_{12}, \ldots, V_{l(l+1)}; V_{1(l+1)})$ are $C^\infty$-homotopic to each other if $\gamma : T \rightarrow P$ is homotopic to $\gamma' : T \rightarrow P$ so that $\gamma|_e$ is $C^\infty$-homotopic to $\gamma'|_e$ for each edge of $T$. We further denote

$$\mathcal{HT}(V_{12}, \ldots, V_{l(l+1)}; V_{1(l+1)}) := \{[\gamma] \mid \gamma \in \mathcal{GT}(V_{12}, \ldots, V_{l(l+1)}; V_{1(l+1)})\},$$

where $[\gamma]$ is the $C^\infty$-homotopy class of $\gamma$.

We in particular consider the case where $|V_{1(l+1)}| = |V_{12}| + \cdots + |V_{l(l+1)}| + 2 - l$, then assume that $\mathcal{HT}(V_{12}, \ldots, V_{l(l+1)}; V_{1(l+1)})$ is a finite set. For this we need that, in the case of the trivalent tree with one interior vertex for example, i) the functions assigned to each edge, which are of the form $f_L - f_{L'}$, are (Bott-)Morse-Smale, and ii) the (un)stable manifolds of the critical points of the above mentioned functions intersect transversely. It is well-known that such transversality can always be achieved by a small perturbation of the functions. See [9] for the precise formulation. As we shall see in the next section,
this transversality condition is satisfied for our specific choices of Lagrangians and we do not further discuss this here.

For each element in \( \gamma \in \mathcal{GT}(V_{i_2}, \ldots, V_{i_{t+1}}; V_{v_1(i+1)}) \), we can assign the weight \( e^{-A(\gamma)} \) where \( A(\gamma) \in [0, \infty] \) is the symplectic area of the piecewise smooth disk in \( \pi^{-1}(\gamma(T)) \) as is done in Kontsevich-Soibelman [16]. This weight is invariant with respect to a \( C^\infty \)-homotopy. Then, we define a multilinear product

\[
m_l : Mo(P)(L_1, L_2) \otimes Mo(P)(L_2, L_3) \otimes \cdots \otimes Mo(P)(L_t, L_{t+1}) \rightarrow Mo(P)(L_1, L_{t+1})
\]

of degree \( 2 - l \) by

\[
m_l(V_{i_2}, \ldots, V_{i_{t+1}}) = \sum_{V_{i_{t+1}}} \sum_{[\gamma] \in \mathcal{GT}(V_{i_2}, \ldots, V_{i_{t+1}}; V_{v_1(i+1)})} \pm e^{-A(\gamma)}V_{i(t+1)}, \quad (14)
\]

where \( V_{i_{t+1}} \) are the bases of \( Mo(P)(L_1, L_{n+1}) \) of degree \( |V_{i_2}| + \cdots + |V_{i_{t+1}}| + 2 - l \). We expect that \( \{m_l\}_{l \geq 2} \) forms a minimal \( A_\infty \)-structure with the sign \( \pm \) given by the formula of the homological perturbation lemma. See for example [13] subsection 2.2).

We do not prove the \( A_\infty \)-relations of \( m_l \)'s in general for the following reason. For a given set \( \mathcal{E} \) of objects of \( Mo(P) \), we denote by \( Mo_{\mathcal{E}}(P) \) the full subcategory of \( Mo(P) \) consisting of objects in \( \mathcal{E} \), and consider only \( Mo_{\mathcal{E}}(P) \) instead of the whole \( Mo(P) \). In our examples, we can find a strongly exceptional collection \( \mathcal{E} \) in \( Tr(Mo_{\mathcal{E}}(P)) \), which implies that all higher \( m_l \)'s with \( l \geq 3 \) vanish in \( Mo_{\mathcal{E}}(P) \) for the degree reason. As we shall explain later, we take \( \mathcal{E} \) to be the set of Lagrangian sections \( L_a \) corresponding to \( O(a) \) for the \( X = \mathbb{C}P^n \) case, and take \( \mathcal{E} \) to be their products for the \( X = \mathbb{C}P^m \times \mathbb{C}P^m \) case.

Establishing the whole \( Mo(P) \) for general \( P \) requires much more work (see [9] for the case of closed manifolds). We shall carry this out in a forthcoming paper.

**A strong minimality assumption** For each pair \( (L, L') \), the differentials of a Morse-Bott version of the Floer complex on \( Mo(P)(L, L') \) should be trivial in order that \( \{m_l\}_{l \geq 2} \) forms a minimal \( A_\infty \)-structure. In our construction, we rather impose a stronger assumption for the class of objects as follows. For any pair \( (L, L') \) and any two distinct elements of the basis \( V, W \in Mo(P)(L, L') \), there does not exist any gradient flow starting at a point in \( V \) and ending at a point in \( W \).

**The identity morphism** For each \( L \in Mo(P) \), the space \( Mo(P)(L, L) \) of morphisms is generated by \( P \) itself which is of degree zero. When the above \( Mo(P) \) is well-defined and forms a minimal \( A_\infty \)-category, we believe that \( P \) is the strict unit. We see that \( P \in Mo(P)(L, L) \) forms at least the identity morphism with respect to \( m_2 \) under the strong minimality assumption above. In order to show that it is the strict unit, we need to show that \( Mo(P) \) is obtained by applying homological perturbation theory to a DG
category. Thus, that $Mo(P)$ forms a minimal $A_\infty$-category and that $Mo(P)$ is strictly unital should be shown at the same time.

**More explicit expression** For each $L$, let us choose a local expression $s : B \to TB$ as we did for $\mathcal{F}$ or $\mathcal{V}$. (A different choice leads to an isomorphic object.) This enables us to assign each generator $V$ of a morphism space a $\mathbb{Z}^n$-grading (which is different from the grading $|V|$ above). For instance, for lifts $s_a : B \to TB$ and $s_b : B \to TB$ of $L_a$ and $L_b$, consider $s_{b,I} : B \to TB$ defined by

$$y_j = s_{b,I} = s_{b}^j - 2\pi i_j,$$

where $I = (i_1, \ldots, i_n) \in \mathbb{Z}^n$. Denote by $Mo_I(P)(s_a, s_b)$ the space generated by the generators of $Mo(P)(s_a, s_b)$ which are included in the image of the intersection graph$(s_a) \cap$ graph$(s_{b,I})$ by $TB \to B$. Then, we have the decomposition

$$Mo(P)(s_a, s_b) = \prod_{I \in \mathbb{Z}^n} Mo_I(P)(s_a, s_b).$$

In this way, each generator of $Mo(P)(s_a, s_b)$ is assigned a $\mathbb{Z}^n$-grading. The multilinear product $[14]$ preserves these gradings in order for the corresponding gradient trees to be well-defined. (See $[12]$ subsection 4E.)

**Connection to $DG(X)$** We end with this subsection by explaining why we expect this $Mo(P)$ to be a candidate of the category on the mirror dual of $X$. We start with the DG category $DG(X)$ of holomorphic line bundles on $X$, as is constructed explicitly in subsection 5.1 and remove the toric divisors of $X$ to obtain $\hat{M}$. Then, $DG(X)$ should be regarded as a subcategory, which we denote by $\mathcal{V}(\hat{M})$, of $\mathcal{V}(\hat{M})$. In particular, $\mathcal{V}(\hat{M})$ is not full since the smoothness condition at the removed toric divisors is imposed in $\mathcal{V}(\hat{M})$. The cohomologies of the morphism spaces in $\mathcal{V}(\hat{M}) \simeq DG(X)$ then differs from those in $\mathcal{V}(\hat{M})$. They can be larger than those in $\mathcal{V}(\hat{M})$ though the morphism spaces in $DG(X)$ are smaller at the cochain level.

In the original set-up where $M$ and $\hat{M}$ are supposed to be compact Calabi-Yau manifolds, the cohomologies of the morphism spaces in $\mathcal{V}(\hat{M})$ are in one-to-one correspondence with those in $\mathcal{F}(M)$ since $\mathcal{F}(M)$ and $\mathcal{V}(\hat{M})$ are isomorphic DG-categories (subsection 4.3). Furthermore, the cohomologies of the morphism spaces $\mathcal{F}(M)(s_a, s_b)$ are isomorphic to $Fuk(M)(s_a, s_b)$ at least when $s_a$ and $s_b$ define Lagrangian sections $L_a$ and $L_b$ which are transversal to each other. Namely, the cohomologies $H(\mathcal{V}(s_a, s_b))$ are spanned by bases which are associated with connected components of $L_a \cap L_b$. We would like to keep this relation even when $\hat{M}$ is noncompact. Then, if the smoothness condition at the removed toric divisors produces additional generators in $H(DG(X)(s_a, s_b))$ from $H(\mathcal{V}(\hat{M})(s_a, s_b))$, we would like to enlarge $M$ so that there exist the corresponding additional connected
components of the intersections of the Lagrangians. Our feeling is that it seems to go well if we add the boundary of $M$ and consider a kind of Fukaya category $Fuk(\bar{M})$ or the corresponding category $Mo(P = \bar{B})$ of weighted Morse homotopy. A candidate is the category $Mo(P)$ we defined in this subsection. Actually, we can consider a DG category $\mathcal{F}'(M)$ which is canonically isomorphic to $\mathcal{V}'(\bar{M})$ just in a similar way as in subsection 4.2. (We may just replace $d\bar{z}_i$ by $dx_i$.) Then, we expect to have a sequence of $A_\infty$-equivalences

$$Mo(P) \simeq \mathcal{F}'(M) \simeq \mathcal{V}'(\bar{M}) \simeq DG(X)$$

or the corresponding derived equivalence $Tr(Mo(P)) \simeq Tr(DG(X)) \simeq D^b(coh(X))$. However, as mentioned in subsection 4.4, there are some technical difficulties, even in the original setting, to show the $A_\infty$-equivalence $Mo(B) \to \mathcal{F}$. Therefor, before discussing the above equivalences in a general set-up, in this paper we explicitly proceed with this story successfully for $X$ the projective spaces and their products in the next section. More precisely, we consider full subcategories $Mo_\varepsilon(P) \subset Mo(P)$ and $\mathcal{V}_\varepsilon(\bar{M}) \subset \mathcal{V}(\bar{M})$ and an $A_\infty$-equivalence $Mo_\varepsilon(P) \simeq \mathcal{V}_\varepsilon(\bar{M})$, which induces the derived equivalence $Tr(Mo(P)) \simeq D^b(coh(X))$. Note that we show the above $A_\infty$-equivalence directly and skip the intermediate category $\mathcal{F}_\varepsilon(M)$ there. Once we are convinced that $Mo(P)$ is the correct notion, we would like to construct the corresponding Fukaya category $Fuk(\bar{M})$ in the future.

5. Homological mirror symmetry of $\mathbb{C}P^n$

In this section, we discuss a version of homological mirror symmetry of $\mathbb{C}P^n$ as the complex side by explicitly proceeding with the story described in the last subsection. In subsection 5.1, we construct the DG category $DG(\mathbb{C}P^n)$ of holomorphic line bundles on $\mathbb{C}P^n$, and recall the structure of its cohomologies. Then, we discuss the homological mirror symmetry for $\mathbb{C}P^n$ in subsection 5.2. We extend the story to $\mathbb{C}P^m \times \mathbb{C}P^n$ in subsection 5.3.

5.1. DG category $DG(\mathbb{C}P^n)$ of line bundles over $\mathbb{C}P^n$. We first construct the DG category $DG(\mathbb{C}P^n)$ consisting of holomorphic line bundles $\mathcal{O}(a)$, $a \in \mathbb{Z}$. The space $DG(\mathbb{C}P^n)(\mathcal{O}(a), \mathcal{O}(b))$ of morphisms is defined as the Dolbeault resolution of $\Gamma(\mathcal{O}(a), \mathcal{O}(b))$. Namely, it is the graded vector space, each graded piece of which is given by

$$DG^r(\mathbb{C}P^n)(\mathcal{O}(a), \mathcal{O}(b)) := \Gamma(\mathcal{O}(a), \mathcal{O}(b)) \otimes \Omega^{0,r}(\mathbb{C}P^n)$$

with $\Gamma(\mathcal{O}(a), \mathcal{O}(b))$ being the space of smooth bundle morphisms from $\mathcal{O}(a)$ to $\mathcal{O}(b)$. The composition of morphisms is defined in a similar way as that in $\mathcal{V}(\bar{M})$ in subsection 4.1. Each $\mathcal{O}(a)$ is associated with the connection $D_a$, which is expressed locally as

$$D_a = d - a \frac{\bar{w}_1 dw_1 + \cdots + \bar{w}_n dw_n}{1 + \bar{w}_1 w_1 + \cdots + \bar{w}_n w_n}$$
on $U = U_0$ (eq.\ref{eq:U_0}), and the differential
\[
d_{ab} : DG^r(\mathbb{C}P^n)(\mathcal{O}(a), \mathcal{O}(b)) \to DG^{r+1}(\mathbb{C}P^n)(\mathcal{O}(a), \mathcal{O}(b))
\]
is defined by
\[
d_{ab}(\hat{\psi}) := 2 \left( D_b^{0,1} \hat{\psi} - (-1)^r \hat{\psi} D_a^{0,1} \right).
\]
This differential satisfies the Leibniz rule with respect to the composition. Thus, $DG(\mathbb{C}P^n)$ is a DG category.

The generators of $H^0(DG(\mathbb{C}P^n))(\mathcal{O}(a), \mathcal{O}(a+1))$ are given by
\[
1, w_1, w_2, \ldots, w_n
\]
locally on $U$. These generate $H^0(DG(\mathbb{C}P^n))(\mathcal{O}(a), \mathcal{O}(b))$ as products of functions, so $H^0(DG(\mathbb{C}P^n))(\mathcal{O}(a), \mathcal{O}(b))$ is represented by polynomials in $(w_1, \ldots, w_n)$ of degree equal to or less than $b - a$. In particular, $H^0(DG(\mathbb{C}P^n))(\mathcal{O}(a), \mathcal{O}(b)) = 0$ for $a > b$. It is known by \cite{2} that $\mathcal{E} := (\mathcal{O}(q), \ldots, \mathcal{O}(q+n))$ forms a full strongly exceptional collection of $D^b(\text{coh}(\mathbb{C}P^n))$ for each $q \in \mathbb{Z}$. That $\mathcal{E}$ forms a strongly exceptional collection means
\[
H^0(DG(\mathbb{C}P^n))(\mathcal{O}(a), \mathcal{O}(a)) \simeq \mathbb{C},
\]
\[
H^0(DG(\mathbb{C}P^n))(\mathcal{O}(a), \mathcal{O}(b)) = 0, \quad a > b
\]
\[
H^r(DG(\mathbb{C}P^n))(\mathcal{O}(a), \mathcal{O}(b)) = 0, \quad r \neq 0
\]
for any $a, b = \{q, q+1, \ldots, q+n\}$. Let $DG_\mathcal{E}(\mathbb{C}P^n)$ be the full DG subcategory of $DG(\mathbb{C}P^n)$ consisting of $\mathcal{E}$. Then the strongly exceptional collection $\mathcal{E}$ is full means that it generates $D^b(\text{coh}(\mathbb{C}P^n))$ in the sense that
\[
\text{Tr}(DG_\mathcal{E}(\mathbb{C}P^n)) \simeq D^b(\text{coh}(\mathbb{C}P^n)),
\]
where $\text{Tr}$ is the Bondal-Kapranov construction \cite{3}.

Note that $H^n(DG(\mathbb{C}P^n))(\mathcal{O}(q+n+1), \mathcal{O}(q)) \neq 0$; it includes an element represented by
\[
\frac{dw_1 \cdots dw_n}{(1 + w_1 \bar{w}_1 + \cdots + w_n \bar{w}_n)^2}.
\]

5.2. Homological mirror symmetry of $\mathbb{C}P^n$. First, we identify the DG category $DG(\mathbb{C}P^n)$ with a (non-full) subcategory $\mathcal{V}'$ of the DG category $\mathcal{V} = \mathcal{V}(\hat{M})$ consisting of the same objects $\mathcal{O}(a), a \in \mathbb{Z}$, where
\[
\hat{M} = \mathbb{C}P^n \setminus \{[t_0 : t_1 : \cdots : t_n] | t_0 \cdot t_1 \cdots t_n = 0\}.
\]
For a given morphism $\hat{\psi} \in DG^0(\mathbb{C}P^n)(\mathcal{O}(a), \mathcal{O}(b))$, we express it locally on $U$ (see subsection \underline{22}), and remove the origin (corresponding to $t_0 = 0$). We send this to $\mathcal{V}'(\mathcal{O}(a), \mathcal{O}(b))$ using \underline{111}:
\[
\hat{\psi} \mapsto \psi := \Psi_b^{-1} \circ \hat{\psi} \circ \Psi_a.
\]
Clearly, this map is compatible with the differentials and the compositions in both sides. In this way, we obtain a functor
\[ \mathcal{I} : DG(\mathbb{C}P^n) \to \mathcal{V} \]
of DG-categories. We see that \( \mathcal{I} \) is faithful. However, \( \mathcal{I} \) is not full since \( \tilde{\psi} \) is smooth at the points \( \{ [t_0 : t_1 : \cdots : t_n] \mid t_0 \cdot t_1 \cdots t_n = 0 \} \). Thus, the image
\[ \mathcal{V}' := \mathcal{I}(DG(\mathbb{C}P^n)) \]
is a non-full DG subcategory of \( \mathcal{V} \).

The local expression for morphisms are transformed by \( \mathcal{I} \) as follows. By \( w_i = e^{x_i + iy_i} = r_i e^{iy_i} \), we Fourier-expand \( \tilde{\psi} \) as
\[ \tilde{\psi} = \sum_{I \in \mathbb{Z}^n} \tilde{\psi}_I(r)e^{i\hat{y}I}, \quad r := (r_1, \ldots, r_n). \]
Now, recall (5) and then
\[
x^1 + \cdots + x^n = \frac{2(e^{x_1} + \cdots + e^{2x_n})}{1 + e^{2x_1} + \cdots + e^{2x_n}} = 2 - \frac{2}{1 + e^{2x_1} + \cdots + e^{2x_n}},
\]
so we have
\[ \Psi_a = \left( 1 + e^{2x_1} + \cdots + e^{2x_n} \right)^{\frac{a}{2}} = \left( \frac{2}{2 - x^1 - x^2 - \cdots - x^n} \right)^{\frac{a}{2}}, \]
and
\[ r_i = e^{x_i} = \left( \frac{x_i^1 + \cdots + e^{2x_n}}{2} \right)^{\frac{1}{2}} = \left( \frac{x_i}{2 - x^1 - x^2 - \cdots - x^n} \right)^{\frac{1}{2}}. \]
Then, \( \psi = \mathcal{I}(\tilde{\psi}) \) turns out to be
\[
\sum_{I \in \mathbb{Z}^n} \tilde{\psi}_I(r) \left( \frac{2 - x^1 - x^2 - \cdots - x^n}{2} \right)^{\frac{b-a}{2}} e^{i\hat{y}I} = \sum_{I \in \mathbb{Z}^n} \psi(x)e^{i\hat{y}I},
\]
so Fourier-componentwisely we have the transformation
\[ \psi_I(x) = \tilde{\psi}_I(r(x)) \left( \frac{2 - x^1 - x^2 - \cdots - x^n}{2} \right)^{\frac{b-a}{2}}. \]
We can bring the generators (15) of \( H^0(DG(\mathbb{C}P^n)(\mathcal{O}(a), \mathcal{O}(a+1))) \) over \( \mathbb{C} \) to those of \( H^0(\mathcal{V})(\mathcal{O}(a), \mathcal{O}(a+1)) \), which are given by
\[ \sqrt[4]{\frac{2 - x^1 - x^2 - \cdots - x^n}{2}} \] (16)
and

\[
\left[ \sqrt{\frac{x^1}{2} e^{iy_1}} \right], \left[ \sqrt{\frac{x^2}{2} e^{iy_2}} \right], \ldots, \left[ \sqrt{\frac{x^n}{2} e^{iy_n}} \right].
\] (17)

The above bases (16) and (17) generate the whole space \( H^0(V')(O(a), O(b)) \) as products of these functions. Explicitly, the bases \( e_{ab,I} = (i_1, \ldots, i_n) \), of the vector space \( H^0(V')(O(a), O(b)) \) are

\[
e_{ab,I} = c_{ab,I} \cdot \left( \sqrt{\frac{2 - x^1 - x^2 - \cdots - x^n}{2} b-a-|I|} \right)^{i_1} \left( \sqrt{\frac{x^1}{2} e^{iy_1}} \right)^{i_1} \cdots \left( \sqrt{\frac{x^n}{2} e^{iy_n}} \right)^{i_n},
\] (18)

where \( i_1 \geq 0, \ldots, i_n \geq 0 \) and \( |I| := i_1 + \cdots + i_n \leq b - a \), and we attach \( c_{ab,I} \) so that \( \max_{x \in P} |e_{ab,I}(x)| = 1 \). Note that this is valid for \( a = b \), too, where we only have \( I = (0, \ldots, 0) =: 0 \) and \( e_{aa,0} \) is the identity element; \( e_{aa,0}(x) = 1 \) for any \( x \in P \).

Since all the exponents in (18) are non-negative, we see that each \( e_{ab,I} \) extends to a continuous function on \( P \). By direct calculations, we have the following lemma.

**Lemma 5.1.** For a fixed \( a < b \) and \( e_{ab,I} \in H^0(V')(O(a), O(b)) \), the set

\[
\{ x \in P \mid |e_{ab,I}(x)| = 1 \}
\]

consists of a point

\[
v_{ab,I} := \left( \frac{2i_1}{b-a}, \ldots, \frac{2i_n}{b-a} \right),
\]

which is the intersection \( V_{ab,I} \subset \pi(L_a \cap L_b) \) with label \( I \). This correspondence then gives a quasi-isomorphism

\[
\iota : Mo(P)(L_a, L_b) \to V'(O(a), O(b))
\]
of cochain complexes. □

For each \( a < b \) and \( I \), we later employ a function \( f_{ab,I} \) on \( P \) defined uniquely by

\[
\sum_{j=1}^{n} (s_a^j - s_b^j + 2\pi i_j) dx_j = df_{ab,I}, \quad f_{ab,I}(v_{ab,I}) = 0.
\] (19)

**Remark 5.2.** In the original set-up where \( B \) is compact, this \( f_{ab,I} \) is a Morse function, where \( v_{ab,I} \) is the critical point of degree zero. However, now in our case, \( v_{ab,I} \) may be at the boundary \( \partial(P) \). Even if we extend \( P \) to \( \tilde{B} \) naturally, \( v_{ab,I} \in \partial(P) \) may not be a critical point since the symplectic form on \( M \) diverges at the boundary.

For each \( a \), the space \( Mo(P)(L_a, L_a) \) is generated by \( P \). The two conditions

\[
\max_{x \in P} |e_{aa,0}(x)| = 1, \quad \{ x \in P \mid |e_{aa,0}(x)| = 1 \} = P
\]
are clearly satisfied. We define a quasi-isomorphism \( \iota : Mo(P)(L_a, L_a) \to V'(\mathcal{O}(a), \mathcal{O}(a)) \) by \( \iota(P) = e_{ax0} \).

For \( a > b \), both the space \( Mo(P)(L_a, L_b) \) and the cohomology \( H(V'(\mathcal{O}(a), \mathcal{O}(b))) \) are trivial. Thus, the zero map \( \iota : Mo(P)(L_a, L_b) \to V'(\mathcal{O}(a), \mathcal{O}(b)) \) is a quasi-isomorphism.

Now, let us fix \( q \in \mathbb{Z} \) and consider \( \mathcal{E} := (\mathcal{O}(q), \mathcal{O}(q + 1), \ldots, \mathcal{O}(q + n)) \). We denote the corresponding full subcategories by \( DG_{\mathcal{E}}(\mathbb{C}P^n) \subset DG(\mathbb{C}P^n), V'_{\mathcal{E}} \subset V' \) and \( Mo_{\mathcal{E}}(P) \subset Mo(P) \). It is known that \( \mathcal{E} \) (with any \( q \)) forms a full strongly exceptional collection in \( Tr(DG_{\mathcal{E}}(\mathbb{C}P^n)) \sim D^b(\text{coh}(\mathbb{C}P^n)) \) \( \mathbb{R} \). Recall that an \( A_\infty \)-equivalence is an \( A_\infty \)-functor which induces a category equivalence on the corresponding cohomology categories.

**Theorem 5.3.** For each \( q \in \mathbb{Z} \), the quasi-isomorphisms

\[
\iota : Mo(P)(L_a, L_b) \to V'(\mathcal{O}(a), \mathcal{O}(b))
\]

with \( a, b \in \{q, \ldots, q + n\} \) extend to a linear \( A_\infty \)-equivalence

\[
\iota : Mo_{\mathcal{E}}(P) \xrightarrow{\sim} V'_{\mathcal{E}}.
\]

Now, we have the DG isomorphism \( DG_{\mathcal{E}}(\mathbb{C}P^n) \sim I(DG_{\mathcal{E}}(\mathbb{C}P^n)) = V'_{\mathcal{E}} \). Since a DG functor is a linear \( A_\infty \)-functor, we immediately obtain the following.

**Corollary 5.4.** One has a linear \( A_\infty \)-equivalence

\[
Mo_{\mathcal{E}}(P) \to DG_{\mathcal{E}}(\mathbb{C}P^n).
\]

**Corollary 5.5.** One has an equivalence of triangulated categories

\[
Tr(Mo_{\mathcal{E}}(P)) \sim D^b(\text{coh}(\mathbb{C}P^n)).
\]

In the rest of this subsection, we show Theorem 5.3 by computing the structure of \( Mo_{\mathcal{E}}(P) \). We already see that nontrivial morphisms in \( Mo(P) \) are of degree zero only. This implies, by degree counting, that the higher \( A_\infty \)-products of \( Mo(P) \) are trivial. Thus, what remains to show the theorem is to construct the product \( m_2 \) and show the compatibility of the products with respect to \( \iota \).

**Lemma 5.6.** For \( a < b < c \) and bases \( V_{ab}:L_{ab} \in Mo(P)(L_a, L_b), V_{bc}:L_{bc} \in Mo(P)(L_b, L_c) \), we have

\[
\iota m_2(V_{ab}:L_{ab}, V_{bc}:L_{bc}) = e_{ab}:L_{ab} \cdot e_{bc}:L_{bc}.
\]

**proof.** Recall that each base consists of a point; \( V_{ab}:L_{ab} = \{v_{ab}:L_{ab}\} \) and so on. We take the function \( f_{ab}:L_{ab} \) defined by \( (5.2) \). Since its gradient vector field is of the form

\[
-\text{grad}(f_{ab}:L_{ab}) = 2\pi \frac{(b - a)}{2} \left( \left( x^1 - i_{ab1} \right) \frac{\partial}{\partial x^1} + \cdots + \left( x^n - i_{abn} \right) \frac{\partial}{\partial x^n} \right),
\]
its gradient trajectories starting from \( v_{ab;I_{ab}} \) go straight. Similarly, gradient trajectories of \(-\text{grad}(f_b - f_c)\) starting from \( v_{bc;I_{bc}} \) go straight. On the other hand, the only gradient trajectory of \(-\text{grad}(f_a - f_c)\) ending at \( v_{ac;I_{ac}} \), \( I_{ac} := I_{ab} + I_{bc} \) is the one staying at \( v_{ac;I_{ac}} \) since it is of degree zero. This means that these three gradient trajectories should meet at \( v_{ac;I_{ac}} \). Thus we obtained the gradient tree \( \gamma \) defining the product \( m_2(V_{ab;I_{ab}}, V_{bc;I_{bc}}) \) explicitly. (The result is that \( v_{ac;I_{ac}} \) sits on the straight line segment \( v_{ab;I_{ab}}, v_{bc;I_{bc}} \) in all cases.) Now, \( A(\gamma) \) turns out to be

\[
A(\gamma) = \frac{1}{2\pi} f_{ab;I_{ab}}(v_{ac;I_{ac}}) + \frac{1}{2\pi} f_{bc;I_{bc}}(v_{ac;I_{ac}}).
\]

Here, \( f_{ab;I_{ab}}(v_{ac;I_{ac}})/2\pi \) is the symplectic area of the triangle disk whose edges belong to \( s_a(\gamma(T)), s_b(\gamma(T)) \) and \( \pi^{-1}(v_{ac;I_{ac}}) \). Similarly, \( f_{bc;I_{bc}}(v_{ac;I_{ac}})/2\pi \) is the symplectic area of the corresponding triangle disk. We thus obtain the weight \( +e^{-A(\gamma)} \).

Next, we look at the product in \( \mathcal{V}' \) side. We can express the bases as \( e_{ab;I_{ab}} = e^{(1/2\pi)f_{ab;I_{ab}}} \cdot e^{H_{ab}\gamma} \) and \( e_{bc;I_{bc}} = e^{(1/2\pi)f_{bc;I_{bc}}} \cdot e^{H_{bc}\gamma} \). We have

\[
e_{ab;I_{ab}} \cdot e_{bc;I_{bc}} = e^{\frac{1}{2\pi}(f_{ab;I_{ab}} + f_{bc;I_{bc}})} \cdot e^{H_{ac}\gamma}.
\]

Since this is the product of the zero-th cohomologies, the result is also a closed morphism in \( \mathcal{V}' \). Hence, the right hand side is proportional to \( e_{ac;I_{ac}} \), whose absolute value takes the maximal value at \( v_{ac;I_{ac}} \). Namely, we have

\[
e_{ab;I_{ab}} \cdot e_{bc;I_{bc}} = e^{\frac{1}{2\pi}(f_{ab;I_{ab}}(v_{ac;I_{ac}}) + f_{bc;I_{bc}}(v_{ac;I_{ac}}))} \cdot e^{H_{ac}\gamma}.
\]

This shows that the compatibility \[20\] holds true.

We need to show the compatibility \[20\] for any \( a \leq b \leq c \). If \( a = b \), then \( V_{ab;I_{ab}} = P \). If \( b = c \), then \( V_{bc;I_{bc}} = P \). Now, we see that \( Mo_\mathcal{E}(P) \) satisfies the strong minimality assumption in subsection \[15\] which implies that \( P \) forms the identity morphism in \( Mo_\mathcal{E}(P) \). Since we already know that \( \iota(P) \) is the identify morphism in \( \mathcal{V}' \), the compatibility \[20\] follows and the proof of Theorem \[5.3\] is completed.

**Remark 5.7.** The product \( m_2 \) and the linear \( A_\infty \)-equivalence \( \iota \) can be induced by applying homological perturbation theory to \( DG(\mathbb{C}P^n) \) in a suitable way. As the higher \( A_\infty \)-products of \( Mo(P) \) are trivial, the induced \( A_\infty \)-equivalence turns out to be linear by degree counting since nontrivial cohomologies of morphisms are of degree zero only.

As a biproduct of the proof, we see that \( Mo_\mathcal{E}(P) \) has the following properties.

**Proposition 5.8.** For any \( L_a, L_b \in Mo_\mathcal{E}(P) \) such that \( L_a \neq L_b \), \( V_{ab} = \pi(L_a \cap L_b) \) belongs to the boundary \( \partial(P) \).

For given bases \( V_{ab} \in Mo_\mathcal{E}(L_a, L_b) \) and \( V_{bc} \in Mo_\mathcal{E}(L_b, L_c) \), the image \( \gamma(T) \) by any gradient tree \( \gamma \in \mathcal{GT}(V_{ab}, V_{bc}; V_{ac}) \) belongs to the boundary \( \partial(P) \) unless \( L_a = L_b = L_c \).
Remark 5.9. If \( L_a = L_b = L_c \), then \( V_{ab} = P, V_{bc} = P \) and \( V_{ac} = P \). Then \( \gamma \in \mathcal{GT}(V_{ab}, V_{bc}; V_{ac}) \) is a constant map to a point in \( P \). If \( L_a = L_b \neq L_c \), then \( V_{ab} = P \) and \( V_{bc} = \{ v_{bc} \} = V_{ac} \). Then \( \gamma \in \mathcal{GT}(V_{ab}, V_{bc}; V_{ac}) \) is the constant map to the point \( v_{bc} \in \partial(P) \).

Similarly, if \( L_a \neq L_b = L_c \), then \( \gamma \in \mathcal{GT}(V_{ab}, V_{bc}; V_{ac}) \) is the constant map to the point \( v_{ab} \in \partial(P) \).

We expect that for many other toric Fano manifolds \( X \) and (strongly) exceptional collections \( \mathcal{E} \), \( Mo_\mathcal{E}(P) \) may satisfy these properties.

We also believe that there exists an \( A_\infty \)-equivalence

\[
Mo(P) \rightarrow DG(\mathbb{C}P^n)
\]

between the whole categories. However, it is not easy to show directly that the whole category \( Mo(P) \) is well-defined as an \( A_\infty \)-category since there are infinitely many gradient trees for which we should check whether our assumption holds or not. In particular, if \( Mo(P) \) is well-defined, it should have nontrivial higher \( A_\infty \)-products.

5.3. Homological mirror symmetry of \( \mathbb{C}P^m \times \mathbb{C}P^n \). In this subsection we shall see how the framework presented in the last subsections works for the case of the product of projective spaces. The point here is that we need not only transversal but clean intersections of Lagrangians in the symplectic side. That’s why we included clean intersections in the definition of \( Mo(P) \) in subsection 4.5. We still do not need higher products \( m_3, m_4, \ldots \) because we can pick up full strongly exceptional collections on both sides (see remark at the end of subsection 4.5).

Let \( X = \mathbb{C}P^m \times \mathbb{C}P^n \), and \( \check{M} \) be the complement of the toric divisors. For

\[
\begin{array}{ccc}
\mathbb{C}P^m & \rightarrow & X \\
p_1 & & \downarrow p_2 \\
& \mathbb{C}P^n & \leftarrow \\
\end{array}
\]

we denote \( \mathcal{O}(a, b) := p_1^* \mathcal{O}(a) \otimes p_2^* \mathcal{O}(b) \). Then \( \mathcal{E} := \{ \mathcal{O}(a, b) \}_{a=0,1,\ldots,m, b=0,1,\ldots,n} \) with the lexicographic order forms a strongly exceptional collection. According to Orlov [19] the semi orthogonal ordered set of admissible subcategories \( (\mathcal{D}_0, \ldots, \mathcal{D}_n) \), where \( \mathcal{D}_0 \) is the image of \( D^b(\text{coh}(\mathbb{C}P^m)) \) in \( D^b(\text{coh}(\mathbb{C}P^m \times \mathbb{C}P^n)) \) under the pull-back functor \( p_1^* \) and \( \mathcal{D}_i \)'s are its twists along \( \mathbb{C}P^n \), generates \( D^b(\text{coh}(\mathbb{C}P^m \times \mathbb{C}P^n)) \), which means that the collection of \( \mathcal{O}(a, b) \)'s above is full. We consider the DG category \( DG(X) \) of these line bundles, and the corresponding DG-category \( \mathcal{V}(\check{M}) \). Just in a similar way as in the previous subsection, we have a DG subcategory \( \mathcal{I}(DG(X)) = \mathcal{V}' \subset \mathcal{V} = \mathcal{V}(\check{M}) \) so that \( DG(X) \simeq \mathcal{V}' \). Their full subcategories consisting of \( \mathcal{E} \) are denoted \( DG_\mathcal{E}(X) \) and \( \mathcal{V}'_\mathcal{E} \).

Then, the parallel statements to the case \( X = \mathbb{C}P^n \) hold.
**Theorem 5.10.** There exists a linear $A_\infty$-equivalence

$$\iota : \text{Mo}_\varepsilon(P) \xrightarrow{\sim} \mathcal{V}_\varepsilon'$$

such that for each generator $V \in \text{Mo}_\varepsilon(P)(L, L')$ we have $\max_{x \in P} |\iota(V)(x)| = 1$ and

$$V = \{ x \in P \mid |\iota(V)(x)| = 1 \}.$$

**Corollary 5.11.** We have a linear $A_\infty$-equivalence

$$\text{Mo}_\varepsilon(P) \simeq D\text{G}_\varepsilon(\mathbb{C}P^m \times \mathbb{C}P^n).$$

**Corollary 5.12.** We have an equivalence of triangulated categories

$$\text{Tr}(\text{Mo}_\varepsilon(P)) \simeq D^b(\text{coh}(\mathbb{C}P^m \times \mathbb{C}P^n)).$$

**Proposition 5.13.** If $L \neq L'$, any generator $V \in \text{Mo}_\varepsilon(P)(L, L')$ belongs to the boundary $\partial(P)$.

For bases $V \in \text{Mo}_\varepsilon(L, L')$ and $V' \in \text{Mo}_\varepsilon(L', L'')$ such that $L \neq L'$ and $L' \neq L''$, any gradient tree $\gamma \in \mathcal{G}T(V, V', V'')$ with $V'' \in \text{Mo}_\varepsilon(P)(L, L'')$ belongs to the boundary $\partial(P)$.

**proof of Theorem 5.10** The bases of the space $H^0(V')(\mathcal{O}(a_1, a_2), \mathcal{O}(b_1, b_2))$ are

$$e_{a_1;b_1;I} \otimes e_{a_2;b_2;J},$$

where $e_{a_1;b_1;I}$ and $e_{a_2;b_2;J}$ are the bases of the corresponding zero-cohomology spaces of morphisms defined in [18] for $\mathbb{C}P^m$ and $\mathbb{C}P^n$, respectively, so $I = (i_1, \ldots, i_m)$ and $J = (j_1, \ldots, j_n)$ run over

$$i_1 \geq 0, \ldots, i_m \geq 0, \quad |I| \leq b_1 - a_1,$$

$$j_1 \geq 0, \ldots, j_n \geq 0, \quad |J| \leq b_2 - a_2.$$
for \( a_1 = b_1 \) and \( a_2 = b_2 \), where \( P_1 \) and \( P_2 \) are the dual polytope of \( \mathbb{C}P^m \) and \( \mathbb{C}P^n \), respectively.

For \( V_{a_1:b_1} \times V_{a_2:b_2} \in \text{Mo}(P)(L(a_1,a_2), L(b_1,b_2)) \) and \( V_{b_1,c_1} \times V_{b_2,c_2} \in \text{Mo}(P)(L(b_1,b_2), L(c_1,c_2)) \), the equation

\[
\gamma_{m_2(V_{a_1:b_1} \times V_{a_2:b_2}; b_1,c_1 \times V_{b_2,c_2}; L)} = (e_{a_1:b_1} \otimes e_{a_2:b_2}; J) \cdot (e_{b_1,c_1} \otimes e_{b_2,c_2}; L)
\]

follows immediately from looking at the structure of the gradient tree \( \gamma \) defining the product \( m_2(V_{a_1:b_1} \times V_{a_2:b_2}; b_1,c_1 \times V_{b_2,c_2}; L) \). Actually, let us denote by \( \gamma_1 \) the gradient tree obtained as the composition of \( \gamma \) with the projection \( P \rightarrow P_1 \). We see that \( \gamma_1 \) is the gradient tree defining the product \( m_2(V_{a_1:b_1}; b_1,c_1; J) \). Similarly, we consider \( \gamma_2 \). Then, we have \( A(\gamma) = A(\gamma_1) + A(\gamma_2) \), and we see that this is compatible with the product

\[
(e_{a_1:b_1} \otimes e_{a_2:b_2}; J) \cdot (e_{b_1,c_1} \otimes e_{b_2,c_2}; L) = (e_{a_1:b_1} \cdot e_{b_1,c_1}; K) \otimes (e_{a_2:b_2} \cdot e_{b_2,c_2}; L)
= e^{-A(\gamma_1)} e_{a_1:b_1} \cdot e_{b_1,c_1} \otimes e^{-A(\gamma_2)} e_{a_2:b_2} \cdot e_{b_2,c_2}.
\]

This completes the proof of Theorem 5.10. \( \square \)

**Proof of Proposition 5.13** Each \( \gamma \) is obtained from the pair \( (\gamma_1, \gamma_2) \) in the proof of Theorem 5.10. In particular, the image \( \gamma(T) \) of a trivalent tree \( T \) by \( \gamma \) is obtained as \( (\gamma_1, \gamma_2)(T) \subset P_1 \times P_2 \). By Proposition 5.13, \( \gamma_1(T) \subset \partial(P_1) \) if \( V_{a_1:b_1} \neq P_1 \) or \( V_{b_1,c_1} \neq P_1 \), i.e., if \( a_1, b_1, c_1 \) do not satisfy \( a_1 = b_1 = c_1 \). Similarly, \( \gamma_2(T) \subset \partial(P_2) \) unless \( a_2 = b_2 = c_2 \).

On the other hand, when at least either \( L(a_1,a_2) \neq L(b_1,b_2) \) or \( L(b_1,b_2) \neq L(c_1,c_2) \) is satisfied, at least one of the inequalities \( a_1 < b_1, b_1 < c_1, a_2 < b_2, b_2 < c_2 \) is satisfied. Thus, at least neither \( a_1 = b_2 = c_1 \) nor \( a_2 = b_2 = c_2 \) is satisfied. This means that at least either \( \gamma_1(T) \subset \partial(P_1) \) or \( \gamma_2(T) \subset \partial(P_2) \) holds, which implies that \( \gamma(T) \subset \partial(P) \). \( \square \)

Lastly, we show more explicitly the gradient trees \( \gamma \) defining the products

\[
m_2(V_{a_1:b_1} \times V_{a_2:b_2}; b_1,c_1 \times V_{b_2,c_2}; L).
\]

A different point from the previous subsection is that \( V_{a_1:b_1} \times V_{a_2:b_2}; b_1,c_1 \times V_{b_2,c_2}; L \) and \( V_{a_1,c_1; K} \times V_{a_2,c_2}; J \) may not be points even if \( V_{a_1:b_1} \times V_{a_2:b_2} \neq P \) and \( V_{b_1,c_1} \times V_{b_2,c_2} \neq P \). There are \( 2^4 = 16 \) types of the products depending on whether each " \( \leq \)" in \( (a_1 \leq b_1 \leq c_1; a_2 \leq b_2 \leq c_2) \) is " \( = \)" or " \( < \)". When " \( = \)" is included, then the corresponding morphism consists of a clean intersection (with nonzero dimension). We further divide them by the number of " \( = \)" as follows.

1. the case \( a_1 < b_1 < c_1, a_2 < b_2 < c_2 \).
2. only one of the equations \( a_1 = b_1, b_1 = c_1, a_2 = b_2, b_2 = c_2 \) is satisfied.
3. \( a_1 = b_1 < c_1 \) and \( a_2 = b_2 < c_2 \), or \( a_1 < b_1 = c_1 \) and \( a_2 < b_2 = c_2 \).
4. \( a_1 = b_1 < c_1 \) and \( a_2 < b_2 = c_2 \), or \( a_1 < b_1 = c_1 \) and \( a_2 = b_2 < c_2 \).
5. \( a_1 = b_1 < c_1 \) and \( a_2 < b_2 < c_2 \), or \( a_1 < b_1 < c_1 \) and \( a_2 = b_2 < c_2 \).
The case (4) corresponds to \( P \cdot P = P \) in \( \text{Mo}(P) \). The cases (3) and (2) correspond to the products \( P \cdot V = V \) or \( V \cdot P = P \) for some \( V \). The case (2”) corresponds to the product \((P_1 \times V) \cdot (P_1 \times W) = P_1 \times (V \cdot W) \) or \((V \times P_2) \cdot (W \times P_2) = (V \cdot W) \times P_2 \), where \( V \cdot W \) is a product in \( \text{Mo}(P_1) \) or \( \text{Mo}(P_2) \). Thus, the argument reduces to the one in the previous subsection. In the case (0), all \( V_{a_1b_1;I} \times V_{a_2b_2;J}, V_{b_1c_1;K} \times V_{b_2c_2;L} \) and \( V_{a_1c_1;I+K} \times V_{a_2c_2;J+K} \) consist of points. Thus, the situation is just the product of the ones in Lemma \[5.6\]

Now we discuss more carefully the cases (1) and (2”). In the case (1), if \( a_1 = b_1 \), then

\[
V_{a_1b_1;I} \times V_{a_2b_2;J} = P_1 \times \{v_{a_2b_2;J}\}
\]

which is not a point. In this case, a gradient tree \( \gamma \in \mathcal{GT}(P_1 \times \{v_{a_2b_2;J}\}, \{v_{a_1c_1;K}\} \times \{v_{b_2c_2;L}\}; \{v_{a_1c_1;K}\} \times \{v_{a_2c_2;J+L}\}) \) belongs to \( \mathcal{GT}((v_{a_1c_1;K}, v_{a_2b_2;J}); (v_{a_1c_1;K}, v_{b_2c_2;L}); (v_{a_1c_1;K}, v_{a_2c_2;J+L})) \) and the image \( \gamma(T) \) is a straight segment connecting \( (v_{a_1c_1;K}, v_{a_2b_2;J}) \) and \( (v_{a_1c_1;K}, v_{b_2c_2;L}) \) on which \( (v_{a_1c_1;K}, v_{a_2c_2;J+L}) \) sits. Similarly, in the case (2”), if \( a_1 = b_1 \) and \( b_2 = c_2 \), then we have

\[
V_{a_1b_1;I} \times V_{a_2b_2;J} = P_1 \times \{v_{a_2b_2;J}\}, \quad V_{b_1c_1;K} \times V_{b_2c_2;L} = \{v_{b_1c_1;K}\} \times P_2.
\]

A gradient tree \( \gamma \in \mathcal{GT}(P_1 \times \{v_{a_2b_2;J}\}, \{v_{b_1c_1;K}\} \times P_2; \{v_{b_1c_1;K}\} \times \{v_{a_2b_2;J}\}) \) belongs to \( \mathcal{GT}((v_{b_1c_1;K}, v_{a_2b_2;J}); (v_{b_1c_1;K}, v_{a_2b_2;J}); (v_{b_1c_1;K}, v_{a_2b_2;J})) \), and then the image \( \gamma(T) \) is just the point \( (v_{b_1c_1;K}, v_{a_2b_2;J}) \) which is the intersection \( (P_1 \times \{v_{a_2b_2;J}\}) \cap (\{v_{b_1c_1;K}\} \times P_2) \). Though this example is still too simple in this sense, this is actually an example of products of Morse homotopy where the gradient trees start from (non-transverse) clean intersections instead of intersection points, generalizing the original set-up of Morse homotopy \[7, 9\] and Kontsevich-Soibelman \[16\]. More examples of clean intersections appear in the case of \( \mathbb{P}_1 \) \[10\].

ACKNOWLEDGMENTS

We are grateful to Akira Ishii for introducing Orlov’s paper \[19\] to us. We also thank the referee for valuable suggestions which improve expositions in this paper very much.

M. F. is supported by Grant-in-Aid for Scientific Research (C) (18K03269) of the Japan Society for the Promotion of Science. H. K. is supported by Grant-in-Aid for Scientific Research (C) (18K03293) of the Japan Society for the Promotion of Science.
References

[1] M. Abouzaid. Morse homology, tropical geometry, and homological mirror symmetry for toric varieties. *Selecta Mathematica*, 15(2):189–270, 2009.

[2] A.A. Beilinson. Coherent sheaves on $\mathbb{P}^n$ and problems of linear algebra. *Functional Analysis and Its Applications*, 12(3):214–216, 1978.

[3] A. Bondal, Representations of associative algebras and coherent sheaves. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* 53:25–44, 1989; translation in *Math. USSR-Izv.* 34:23–42, 1990.

[4] A.I. Bondal and M.M. Kapranov. Enhanced triangulated categories. *Math. USSR-Sb.*, 70:93–107, 1991.

[5] K. Chan. Holomorphic line bundles on projective toric manifolds from Lagrangian sections of their mirrors by SYZ transformations. *International Mathematics Research Notices*, 2009(24):4686–4708, 2009.

[6] B. Fang. Homological mirror symmetry is T-duality for $\mathbb{P}^n$. *Commun. Number Theory Phys.* 2:719–742, 2008.

[7] K. Fukaya. Morse homotopy, $A^\infty$-category, and Floer homologies. In *Proceedings of GARC Workshop on Geometry and Topology ’93 (Seoul, 1993)*, volume 18 of *Lecture Notes Ser.*, pages 1–102. Seoul Nat. Univ., Seoul, 1993.

[8] K. Fukaya. Asymptotic analysis, multivalued morse theory, and mirror symmetry. In *Graphs and patterns in mathematics and theoretical physics*, volume 73 of *Proc. Sympos. Pure Math.*, pages 205–278, Providence, RI, 2005. Amer. Math. Soc.

[9] K. Fukaya and Y.-G. Oh. Zero-loop open strings in the cotangent bundle and morse homotopy. *Asian J. Math.*, 1:96–180, 1997.

[10] M. Futaki and H. Kajiura, Homological mirror symmetry of $F_1$ via Morse homotopy. preprint arXiv:2012.06801.

[11] H. Kajiura. An $A^\infty$-structure for lines in a plane. *Internat. Math. Res. Notices*, 2009(20):3913–3955, 2009.

[12] H. Kajiura. On some deformations of Fukaya categories. In *Symplectic, Poisson, and Noncommutative Geometry*, volume 62, page 93. Cambridge University Press, 2014.

[13] H. Kajiura. Homological perturbation theory and homological mirror symmetry. In *Higher Structures in Geometry and Physics*, pages 201–226. Springer, 2011.

[14] H. Kajiura. Fukaya categories of two-tori revisited. *J. Geom. Phys.* 160:103965, 18 pp, 2021.

[15] M. Kontsevich. Homological algebra of mirror symmetry. In *Proceedings of the International Congress of Mathematicians (Zürich, 1994)*, pages 120–139, Birkhäuser, Basel, 1995.

[16] M. Kontsevich and Y. Soibelman. Homological mirror symmetry and torus fibrations. In *Symplectic geometry and mirror symmetry (Seoul, 2000)*, pages 203–263. World Sci. Publishing, River Edge, NJ, 2001.

[17] N.C. Leung. Mirror symmetry without corrections. *Communications in Analysis and Geometry*, 13(2):287–331, 2005.

[18] N.C. Leung, S.-T. Yau, and E. Zaslow. From special Lagrangian to hermitian-Yang-Mills via Fourier-Mukai transform. *Adv. Theor. Math. Phys.* 4:1319–1341, 2000.

[19] D.O. Orlov. Projective bundles, monoidal transformations, and derived categories of coherent sheaves. *Izv. Ross. Akad. Nauk Ser. Mat.* 56 852–862, 1992; translation in *Russian Acad. Sci. Izv. Math.* 41 133–141, 1993.
[20] A. Strominger, S.-T. Yau, and E. Zaslow. Mirror symmetry is T-duality. *Nucl. Phys. B*, 479:243–259, 1996.

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