SELF-RESTRICTING NOISE IN QUANTUM DYNAMICS

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Abstract. States of open quantum systems usually decay continuously under environmental interactions. Quantum Markov semigroups model such processes in dissipative environments. It is known that a finite-dimensional quantum Markov semigroup with detailed balance induces exponential decay toward a subspace of invariant or fully decayed states, under what are called modified logarithmic Sobolev inequalities. We analyze continuous processes that combine coherent and stochastic processes, breaking detailed balance. We find counterexamples to analogous decay bounds for these processes. Through analogs of the quantum Zeno effect, noise can suppress interactions that would spread it. Faster decay of a subsystem may thereby slow overall decay. Hence the relationship between the strength of noise on a part and induced decay on the whole system is often non-monotonic. We observe this interplay numerically and its discrete analog experimentally on IBM Q systems. Our main results then explain and generalize the phenomenon theoretically. In contrast, we also lower bound decay rates above any given timescale by combining estimates for simpler, effective processes across times.

A quantum state exposed to its environment usually decays toward a fixed point that is invariant under environmental interactions. This open system time-evolution is responsible for decoherence, the equilibration of thermal systems, noise in quantum transmission, and many other important processes in quantum information. A common expectation is that a quantum state decays exponentially unless actively protected, though well-isolated systems may lengthen the decay time. An important job of quantum information theory is to rigorously characterize such decay and its exceptions, quantitatively estimate rates, and invent strategies to forestall unwanted decay or enable desired equilibration.

Existing theory shows exponential decay for processes involving just noise or thermal equilibration. Much of quantum science, however, is concerned with processes that involve coherent time-evolution driven by a Hamiltonian, such as gates in quantum computation, interactions in many-body physics, etc. The theory of decay induced by such processes has been less clear. As suggested by quantum error correction, active intervention may suppress decoherence. More surprisingly, we find that sometimes noise suppresses its own spread, yielding a non-monotonic relationship between noise strength and overall decay rate. Exceedingly strong noise applied to a subsystem often slows the decay it induces on other parts of a system, effectively isolating itself through analogs of the quantum Zeno effect.

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1.1. Background and Summary of Theoretical Findings. In general, dynamics of open systems are given by quantum channels. A quantum channel is (mathematically) a completely positive, trace-preserving map on densities, and it (physically) models a transformation on a quantum system given by unitary dynamics including both the original system and its environment. When that environment is stationary and dissipative, these dynamics take the form of a quantum Markov semigroup, a family of quantum channels $\Phi^t$ parameterized by time $t \geq 0$.

Just as a Hamiltonian $H$ generates a family of unitaries $U^t = \exp(-itH)$, a Lindbladian $\mathcal{L}$ generates a quantum Markov semigroup $\Phi^t = \exp(-t\mathcal{L})$ [1, 2]. The Lindbladian is an open system generalization of the Hamiltonian to non-unitary dynamics. Quantum Markov semigroups can induce non-invertible processes, as information is lost to the environment.

A series of results [3, 4, 5] have shown that all Lindbladians having detailed balance with respect to a GNS inner product [6] obey a complete, modified logarithmic-Sobolev inequality (CMLSI) based on the modified logarithmic Sobolev inequality defined in [7, 8, 9]. For a semigroup $(\Phi^t : t \in \mathbb{R}^+)$ in terms of the quantum relative entropy $D(\cdot \| \cdot)$, $\lambda$-CMLSI states that

$$D(\Phi^t(\rho)\|\mathcal{E}(\rho)) \leq e^{-\lambda t} D(\rho\|\mathcal{E}(\rho))$$

for all input densities $\rho$, where $\lambda > 0$, and $\mathcal{E}$ is a projection to the invariant or fixed point subspace of $\Phi^t$. Furthermore, the “completeness” of the inequality refers to its stability under tensor extensions and products: we may extend $\Phi^t$ to $\Phi^t \otimes \text{Id}$ for an auxiliary system of any size and the CMLSI constant $\lambda$ remains the same. Though the quantum relative entropy has many information-theoretic interpretations, we use it primarily as a distance-like notion. Decay of relative entropy implies loss of information or distinguishability from a fully decohered or equilibrated fixed point state. CMLSI implies that the diamond norm distance $\|\Phi^t - \mathcal{E}\|_\diamond$ and trace distance $d_{tr}$ of $\Phi^t(\rho)$ to a fixed point decay exponentially:

$$\|\Phi^t - \mathcal{E}\|_\diamond = 2 \max_{\rho} d_{tr}(\Phi^t(\rho), \mathcal{E}(\rho)) \leq 2e^{-\lambda t/2} \sqrt{\ln d}$$

in dimension $d$, where the maximum includes extensions by an untouched auxiliary system. Diamond norm distance is complementary to process fidelity, which quantifies similarity of channels. CMLSI also implies bounds on decoherence times, capacities, a variety of resource measures, mixing times, and even quantum advantage in near-term algorithms [10, 11, 12]. Properties of relative entropy as discussed in Section 5.2 have made general CMLSI more tractable than directly obtaining analogous trace distance bounds.

Nonetheless, important questions have remained open. In particular, detailed balance is often broken when a process includes coherent rotations in addition to noise. Should exponential decay still hold? We consider Lindbladian generators of the form

$$\mathcal{L}(\rho) = i[H, \rho] + \mathcal{S}(\rho)$$

1In mathematics, the Lindbladian would conventionally be a Heisenberg picture superoperator with pre-adjoint $\mathcal{L}^*$ acting on densities. Since we work with the Schrödinger picture superoperator on densities, we denote this by $\mathcal{L}$ instead of $\mathcal{L}^*$. We nonetheless use the mathematical sign convention that $-\mathcal{L}$ is the generator, rather than $+\mathcal{L}$. 


on an input state $\rho$, where $\mathcal{S}$ is a Lindbladian with detailed balance, and $H$ is a Hamiltonian generator of unitary time evolution. We often $E_0 := \lim_{t \to \infty} \exp(-tS)$ as the fixed point projection of $\mathcal{S}$, and $E := \lim_{t \to \infty} \exp(-tL)$ as that of $\mathcal{L}$ when one exists. We refer to $H$ as generating coherent dynamics, rotation, or drift, and to $\mathcal{S}$ as generating stochastic dynamics, noise, or decay. We refer to this combined form as a decay+drift Lindbladian.

Based on results for Lindbladans with detailed balance, one might expect exponential decay to hold generally in the absence of error correction. Analogous to CMLSI, we define a notion of rotated CMLSI with a time-varying, decay-invariant subspace. While this notion extends the concept of CMLSI to include coherent subprocesses, we find it does not hold in general. Some systems decay sub-exponentially at short times as demonstrated in Counterexamples 3.1 and 3.2. Information may hide in a subsystem or basis that is far in some sense from the noise. More broadly, strong noise induces a competition between Zeno-like effects and long-time decay. In the limit of overwhelming noise, the overall decay rate of a decay+drift Lindbladian often decreases as the stochastic part’s decay rate increases. Nonetheless, exponential decay appears after finite time for decay+drift Lindbladians.

2. Experimental and Numerical Observations

2.1. Simulated Spin Chain with Depolarizing End. Though the main results of this paper are analytical, we set the stage by simulating a simple, commonly studied example: a spin chain in one spatial dimension with open boundary conditions. Using Qiskit Dynamics, we simulate

![Figure 1](image)

**Figure 1.** Relative entropy of a 4-qubit spin chain to (fully decayed) complete mixture: (1) Spin chain illustration. The noised qubit is on top and shaded yellow. The 3 qubits below are shaded green with nearest-neighbor interactions. Under overwhelmingly strong noise, the interaction between the noised qubit and the rest of the chain is suppressed by Zeno-like effects. (2) Relative entropy vs. time with input $\hat{1}/2^{\otimes 3} \otimes |0\rangle \langle 0|$, where the legend notes $\gamma$ in Equation (3); (3) Relative entropy vs. $\gamma$ at time $t = 0.5$ with input $|0000\rangle$. 

XX + YY nearest neighbor interactions (see [13] for a simple example of a similar system) on 4 qubits. We add depolarizing noise to the first qubit in the chain, which continuously randomizes the state of that subsystem concurrently with the aforementioned interactions. The simulated
Lindbladian has the form
\[
\mathcal{L} = 2\pi i \sum_{j=1}^{3} (X_j X_{j+1} + Y_j Y_{j+1}) - \gamma (\hat{1}/2 \otimes \rho^{(2\rightarrow 4)} - \rho),
\]
(3)
where $\rho^{(2\rightarrow 4)}$ denotes the marginal on the 2nd-4th qubits, and $X, Y, Z$ denote unnormalized Pauli matrices. We compute relative entropy of the 4-qubit chain with respect to its fixed point of complete mixture, starting from initial state $|0000\rangle$ or $(\hat{1}/2) \otimes \rho(2 \rightarrow 4) \otimes |0\rangle\langle 0|$. Results are plotted in Figure 1.(1). We denote by $\gamma$ a parameter multiplying the noise terms, which controls the strength of noise relative to time and interaction terms.

In Figure 1.(2), we examine the model defined by Equation (3) with initial state $(\hat{1}/2) \otimes \rho \otimes |0\rangle\langle 0|$. Here we see almost no initial decay, as we examine in Counterexample C.5. More surprising is the inversion in the relationship between noise strength and entropy decay when going from 10.0 to 100.0. Figure 1.(3) further illuminates this observation by showing relative entropy at fixed time as a function of $\gamma$ for input state $|0000\rangle$. While decay rate expectedly correlates with noise strength for weak noise, the relationship soon inverts. The explanation of this odd phenomenon is a Zeno-like effect known as the generalized adiabatic theorem [14]. Extremely strong noise supresses the interaction between the noised qubit and others, slowing its own spread.

This simple example serves primarily to motivate the further studies in this paper. First, however, we turn to an even simpler system to study experimentally.

2.2. **Self-Restricting Noise in a 2-Qubit Interaction.** Consider the following scenario: two qubits $A$ and $B$ undergo coherent time-evolution under an interaction Hamiltonian $H = Z \otimes X/2$, while $A$ undergoes depolarizing noise. Hence the system’s evolution is described by the (adjoint) Lindbladian
\[
\mathcal{L}(\rho) := i[Z \otimes X/2, \rho] - \gamma (\hat{1}/2 \otimes \rho^B - \rho).
\]
(4)
The fixed point conditional expectation of the stochastic part, $S(\rho) = -\gamma (\hat{1}/2 \otimes \rho^B - \rho)$, is $E_0(\rho) = \hat{1}/2 \otimes \rho^B$. That of $\mathcal{L}$ is $E(\rho) = 1/4$, complete mixture. Here $E_0(H) = 0$, generating the identity. Let $\Phi_{ZX(t)}$ denote the unitary generated by $H$ in time $t$.

2.2.1. *Simulations Approaching Continuum.* We simulate time-evolution under $\mathcal{L}$ using Qiskit. We first choose $t = 4\pi$ so that a fully coherent rotation results in perfect fidelity with the original state. We choose an input state of $|00\rangle\langle 00|$, which decays to complete mixture. We vary $\gamma$ such that $(1 - \exp(-\gamma t))$ (the depolarizing likelihood) falls between $3.35 \times 10^{-8}$ and 1.0, scaled logarithmically. Since qubit $A$ is assumed fully depolarized, we study how qubit $B$ decays toward the expected fixed point of complete mixture. Results appear in Figure 2.

We observe counter-intuitively non-monotonic decay with noise strength. For very small values of $\gamma$, the state expectedly does not decay noticeably. As we tune $\gamma$ up, we see a region of strong decay, where a time of $4\pi$ is long enough to mostly dephase qubit $B$ under almost continuous interactions with a regularly noised qubit $A$. With large $\gamma$, however, the relationship inverts. As the noise channel approaches completely depolarizing, we see the decay slow with increasing $\gamma$. We vary Trotter step number to rule this out as an underlying mechanism.
2.2.2. Discrete Zeno-Like Effect in Experiment. To better understand and confirm observations from Subsubsection 2.2.1, we recall the analogy between the generalized Zeno effect [15, 16], which describes continuous processes frequently interrupted by discrete channels, and the adiabatic Theorem [14], in which a fast, continuous process suppresses some aspects of a simultaneous, slower, continuous process. In this Section, we experimentally observe how the generalized Zeno effect causes depolarizing noise to self-restrict.

We consider the channel given by
\[ \Phi_{(k)} = (\mathcal{E}_0 \circ \Phi_{ZX(\pi/(2k))})^k \circ \mathcal{E}_0. \] (5)

Since a rotation of \( \pi/2 \) corresponds to a fully entangling gate, we take a total possible rotation of \( \pi/2 \). The number of interruptions is \( k \). Combining Equation (4) with a Zeno-like bound derived
from Proposition B.5 and Lemma B.7 in the Supplementary Information, a tightened special case of Theorem 5.1 we calculate theoretically that

\[ \| \Phi_k - \mathcal{E}_0 \|_\infty \leq \min \left\{ \frac{\pi^2}{4k}, e^{\pi/2k}, 1.0 \right\}. \tag{6} \]

We implement \( \Phi_k \) experimentally on an IBM Q system as described in Subsection 5.1. Overall, we observe the expected generalized Zeno dynamics: as \( k \) increases, qubit \( B \) is protected from interactions with the frequently depolarized qubit \( A \). The real experiment appears to converge to Zeno dynamics more quickly than its simulated, noiseless counterpart, as seen in Figures 3.(1) and 3.(3). The accelerated convergence is potentially explainable by a small but \( k \)-independent under-rotation in each instance of \( \Phi_{ZX}(t) \), which compounds with increasing \( k \).

Comparing the experimental results shown in Figure 3 to theory and simulation, we see qualitative agreement. Since Equation (6) is intended as an upper bound, not a prediction of the experimental value, we do not necessarily expect quantitative agreement. We do see that the experimental value exceeds the ideal upper bound at large step number, which probably reflects the prevalence of real hardware noise that is not part of our model. Primarily, the experimented illustrated in Figure 3 demonstrates how what appears to be more depolarizing noise applied can reduce ultimate mixture in an interacting system. This example is analogous to the preceding, continuous scenarios, in which directly tuning up the noise rate protected neighboring qubits. Though intended to support theoretical and numerical results, this experiment may be of independent interest in demonstrating a counter-intuitive version of the generalized Zeno effect.

3. Theoretical Results and Explanation of Observed Phenomena

Rather than a universal generalization of (C)MLSI, we first discover and prove barriers. Hamiltonians introduce a technical complication: unitary components in a semigroup may continue to rotate states within protected subspaces indefinitely, so they need not approach an invariant subspace. Purely Hamiltonian time-evolution is the simplest example of a non-trivial semigroup that never decays to a fixed point. To accommodate this and more sophisticated examples, we introduce a rotated analog of (C)MLSI denoted (C)RMLSI (for rotated MLSI). We say that a semigroup \( \Phi_t \) has RMLSI if it converges in relative entropy to \( R_t \mathcal{E} \), where \( \mathcal{E} \) is a projector and \( R_t \) a time-dependent unitary rotation commuting with that projector:

\[ D(\Phi_t(\rho)\| R_t \mathcal{E}(\rho)) \leq e^{-\lambda t} D(\rho\| \mathcal{E}(\rho)) \]

Complete RMLSI (CRMLSI) is defined analogously on arbitrary extensions by auxiliary systems.

GNS detailed balance states that for any pair of operators \( x, y \), \( \text{tr}(\omega x \dagger \mathcal{L}(y)) = \text{tr}(\omega \mathcal{L}(x) \dagger y) \), where \( \dagger \) denotes Hermitian conjugation. Detailed balance is self-adjointness with respect to the \( \omega \)-weighted GNS inner product, as described in Subsection 5.1 and fully developed in [6].

3.1. Barriers to Decay. RMLSI does not hold for all finite-dimensional semigroups, as illustrated by the following counterexamples.

Counterexample 3.1 (Nearest-Neighbor Interactions with Endpoint Noise). In this counterexample, we consider an \( n \)-qubit Hamiltonian of the form \( H = \sum_{j=1}^n H_{j,j+1} + H_j \). This form
represents nearest-neighbor interactions on a one-dimensional chain with open boundary conditions. Notable examples include the Heisenberg and Ising models. We also consider a stochastic generator $S = S_1 \otimes \mathbf{1}^{\otimes n-1}$, which acts only on the leftmost qubit. Physically, we may think of such a system as well-isolated from its noisy environment except for the left end of the chain. Until the $(n-1)$th term in a Taylor expansion of the semigroup around $t = 0$, no term contains more than $n - 2$ qubit swaps.

Via continuity of relative entropy to a subalgebra restriction (see [17, Lemma 7] and [18, Proposition 3.7]),
$$D(\Phi^t(\rho)\|\mathcal{E}(\rho)) \geq (1 - O(t^{n-1} \log t)) D(\rho\|\mathcal{E}(\rho)).$$

Since $\Phi^t$ cannot have RMLSI with any $\lambda > 0$, it does not have RMLSI.

To be more specific, consider a swap chain in which $H = \sum_{j=1}^{n-1} X_j X_{j+1} + Y_j Y_{j+1}$ as in Subsubsection 2.1. Let $A_1, ..., A_n$ denote qubit subsystems. We add a stochastic generator of noise on the 1st qubit, $S(\rho) = \rho - \frac{\mathbf{1}}{2} \otimes \text{tr}_{A_1}(\rho)$. The equilibrium state of the swap chain is an overall complete mixture. Now consider the input state $\rho = (\frac{\mathbf{1}}{2})^{\otimes (n-1)} \otimes |0\rangle\langle 0|$, which is in equilibrium everywhere except the rightmost qubit. Relative entropy decay at small $t$ proceeds as $O(t^{n-1} \log t)$, representing sublinear tunneling amplitude for the state at one end to undergo noise at the other. The fixed point accounts for propagation of noise throughout the entire system, but noise takes time to propagate along the chain. This counterexample is illustrated numerically in Subsubsection 2.1.

**Counterexample 3.2 (Dephasing + Basis Drift).** Consider the Lindbladian
$$\mathcal{L}(\rho) = i[X, \rho] - \gamma(Z\rho Z - \rho),$$
where $X, Z$ are the usual qubit Pauli matrices. This Lindbladian combines rotation via the Pauli $X$ matrix with dephasing in the $Z$ basis. The long-term behavior of this Lindbladian is depolarizing, as any state not in the $Z$ basis becomes more mixed, and a state diagonal in the $Z$ basis rotates to another basis. We apply $\mathcal{L}$ to the input state $|0\rangle\langle 0|$. Again using continuity of relative entropy to a subalgebra restriction,
$$D(\Phi^t(|0\rangle\langle 0|)\|\frac{\mathbf{1}}{2}) \geq (1 - O(t^2 \log t)) D(|0\rangle\langle 0|\|\frac{\mathbf{1}}{2}).$$

Analogously to how in Counterexample C.5 noise takes time to propagate between qubits, here it takes time to propagate between bases. This counterexample to RMLSI also hints at another feature: the quantum Zeno effect [19].

For a Lindbladian in the form of Equation (2), the fixed point of the stochastic generator $S$ will not coincide with the overall fixed point of the process unless its projector $\mathcal{E}_0$ commutes with $H$. When these fixed point projectors differ, a system initially in the fixed point subspace of $S$ takes time to see decay. Furthermore, strong noise can actually suppress its own spread:

**Theorem 3.3.** Let $\mathcal{L}$ given by $\mathcal{L}(\rho) = i[H, \rho] + S(\rho)$ be a decay+drift Lindbladian such that for any input density $\rho$, $\exp(-tS)(\rho)$ decays exponentially in trace distance or relative entropy to $S$’s invariant subspace with rate at least $\lambda_0$. Assume the Hamiltonian $H$ does not commute with the invariant subspace projection of $S$. If the relative entropy or trace distance of $\exp(-t\mathcal{L})(\rho)$ to a
fixed point decays exponentially for every input $\rho$ with rate $\lambda$ at sufficiently long times, and $\lambda_0$ is sufficiently large, then $\lambda \leq O(1/\sqrt{\lambda_0})$.

The surprising aspect of Theorem 3.3 is that the decay rate of $L$, $\lambda$, is bounded inversely to that of $S$, $\lambda_0$. Theorem 3.3 starts to explain the observations in Section 2. In the limit of infinite noise strength, the system approaches Zeno dynamics generated by $E_0(H)$ acting on $E_0(\rho)$ - see Theorem 5.1. While Zeno dynamics immediately apply the maximal decay induced by $S$, they also constrain how $H$ can effectively change $E_0$, maintaining protected subspaces from $S$ that would otherwise be exposed via interplay with $H$.

A technical version of Theorem 3.3 with calculable constants appears as Theorem C.6 in the Supplementary Information. Furthermore, Theorem C.6 generalizes $H$ to another Lindbladian $L_1$ under some assumptions about invariant subspace projections.

In general, if a Lindbladian’s Hamiltonian does not commute with the decay part’s invariant projection, overall CRMLSI will fail at short times, and Zeno-like self-restriction will appear for strong noise. In practice, one might expect noise to affect different bases or subsystems heterogeneously, but not so much that it only applies to one part. Analogous results apply: weakly noised subsystems’ initial decay reflects their decay rates at short times rather than those of interacting, noisier subsystems. Even at long times, a system is protected from its extremely noisy components.

3.2. Lower Bounds on Decay Rates. Despite counterexamples to universal MLSI and the Zeno effect, we still expect exponential decay with estimable rate to emerge after finite time. When $S$ induces slow decay, the overall decay should obviously be correspondingly slow. As one continuously increases the strength of $S$ such as by taking $S \rightarrow \gamma S$ for $\gamma \geq 0$, the overall decay rate for fixed $t$ should at first ascend before again declining due to Zeno-like effects. These extremes are distinct, as $\gamma = 0$ corresponds to fully coherent time-evolution, while $\gamma \rightarrow \infty$ converges to a potentially rotating invariant subspace of $S$. Nonetheless, we predict slower decay at extremely large and small $\gamma$ than in an intermediate regime.

**Theorem 3.4.** Let $L$ be a decay+drift Lindbladian as in Equation (2). Let $E$ be the invariant subspace projection of $L$ up to a possible persistent rotation $R_t$. For any finite timescale $\tau > 0$, there is some $\lambda_\tau > 0$ such that $\Phi^t = \exp(-tL)$ has $D(\Phi^t(\rho)\|R_tE(\rho)) \leq \exp(-\lambda_\tau [t/\tau])D(\rho\|E(\rho))$ for any $t > 0$, where $[\cdot]$ denotes the floor function.

A technical version of Theorem 3.4 with calculable constants appears as Theorem C.10 in the Supplementary Information. For a given value of $t$, one may optimize the choice of $\tau$ given the technical constants of Theorem C.10. To understand the assumptions and constants in this Theorem, we examine how distinct regimes compare qualitatively to observations in Subsection 2.2.1. We consider a Lindbladian given by $L(\rho) = i[H,\rho] + \gamma S(\rho)$, attaching a scalar factor $\gamma > 0$ to $S$ in the decay+drift form of Equation (2). This $\gamma$ factor allows us to tune the strength of the stochastic part and was used in the numerical demonstrations of Subsection 2.2.1. Under a channel ordering condition that is assured to hold for some constants in finite dimension, we
obtain
\[
\lambda_\tau = a\gamma e^{-b\gamma},
\]
where \(\lambda_0\) is the assumed CMLSI constant of \(S\), and \(a, b\) are constants depending on \(S, \tau,\) and \(\epsilon\).

Analyzing distinct regimes:

- For small \(\gamma\), \(e^{-b\gamma} \approx 1\). We may expand Equation (8) as \(\lambda_\tau = a\gamma - O(\gamma^2)\), observing here that \(\lambda_\tau\) increases proportionally to \(\gamma\). In Figure 2, this regime corresponds roughly to 0-1% depolarizing noise.
- The regime of strongest overall decay appears for intermediate values of \(\gamma\), such as with 1-10% depolarizing noise in Figure 2.
- When \(\gamma\) is large, the exponential decay in \(\gamma\) dominates over the linear \(\gamma\) factor in Equation (8). This regime corresponds to 10-100% depolarizing noise in Figure 2. Zeno-like effects as in Theorem 3.3 protect information in the invariant subspace of \(S\), which might be larger than the invariant subspace of \(L\).

Though Theorem 3.4 agrees qualitatively with simulation, we expect the quantitative correspondence to be loose. The simulations of Subsection 2.2.1 use input states chosen to show the largest possible effects in low dimension. Theorem 3.4 trades away this optimality for generality.

4. Conclusions and Outlook

While the breakdown of CRMLSI complicates decay estimation, it seems optimistic for near-term quantum technology. For example, [11] shows how exponential decay leads to classical simulatability of quantum optimization algorithms. Though the current work shows that CRMLSI may fail for decoherence co-occurring with gate executions, exponential decay with circuit depth is still expected. Additional avenues to slow that decay, however, seem plausible. For example, real quantum processors often have heterogeneous qubit decoherence rates. Bad qubits might contribute less overall mixture and more gate miscalibration than previously expected. Also, frequent syndrome measurements can suppress errors via the Zeno effect without feed-forward correction [19, 20]. Short-time effects observed herein may enhance such schemes by restricting effective errors to initial, more corrigible forms, e.g. dephasing does not immediately become depolarizing under concurrent basis rotation.

A natural follow-up considers a reversal of Hamiltonian suppression: do fast, coherent dynamics suppress noise? First-order analysis does not show this conjecture. The QMS formulation of dissipation, however, assumes fast environmental dynamics that may preclude faster coherent dynamics. Explicitly modeling system-environment interactions as a larger Hamiltonian reveals that coherent interactions can suppress noise, unifying the Zeno effect with dynamical decoupling, a noise reduction technique [21, 22]. Future studies may examine strategies combining self-restricting noise, dynamical decoupling, and measurement-induced protection.

5. Methods

5.1. Experimental and Numerical Methods. Simulations in Subsection 2.1 were run in Qiskit dynamics using the Lindbladian solver. Von Neumann entropies were calculated using
Qiskit’s ‘quantum_info.entropy’ subroutine and subtracted from 4.0 to obtain the relative entropy with respect to the completely mixed fixed point state. We denote the (unnormalized) Pauli matrices

\[
X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

The simulations in Subsubsection 2.2.1 were conducted using Qiskit and Qiskit Dynamics. To approximate continuous dynamics on the circuit-based Qiskit, we simulate the Trotter expanded version

\[
\left(\exp\left(-\frac{itH}{k}\right) \exp\left(-\frac{t\gamma S}{k}\right)\right)^k.
\]

The unitary rotation was implemented as a parameterized RZX gate, and the depolarizing noise using a Qiskit NoiseModel in the AerSimulator. These simulations were carried out using the ‘density_matrix’ method with 8192 shots and channels inferred using Qiskit’s ‘process_tomography_circuits’ and ‘process_tomography_fitter’ subroutine and class for the single ‘B’ qubit. The ‘cts’ simulation was run using Qiskit Dynamics with the Lindbladian solver using a Bell state input to directly compute the Choi matrix. Different simulators and Trotter step values were used to corroborate that counter-intuitive observations reflect a quantum phenomenon, rather than a quirk of discretization or finite stepsize effect.

In simulation, one may program the channel \( \Phi_{(k)} \) almost exactly. In experiment, there are several challenges. First, IBMQ devices do not natively implement depolarizing channels, since they are typically undesirable in computation. Second, unintended noise on qubit \( B \) may induce mixture independently from any process on \( A \), confounding intended effects. Third, small, two-qubit rotations also differ from the typical use case of gate-based quantum computers. For the experiment, we apply the following procedure:

1. The device is initialized in the computational basis \( |0\ldots0\rangle \) state.
2. Channel tomography preparation gates are applied to qubit \( B \).
3. The following sequence of steps is repeated for \( k \) rounds:
   a. Apply \( E_0 \): \( S_Z \) gates are applied to each of two auxiliary qubits. One \( CX \) gate is applied from the first auxiliary to \( A \). One Hadamard is applied to \( A \). One \( CX \) is applied from the second auxiliary to \( A \). Reset operations begin on both auxiliaries.
   b. Apply \( \Phi_{ZX}(\pi/(2k)) \): using the Qiskit class RZXCalibrationBuilder and OpenPulse access, a pulsed implementation of \( \Phi_{ZX}(\pi/(2k)) \) is applied from qubit \( A \) to qubit \( B \).
   c. Wait for the next cycle: between each application of \( \Phi_{ZX}(\pi/(2k)) \) or the beginning or end of the circuit, dynamical decoupling is applied to \( B \) via a pair of \( X \) gates inserted via Qiskit’s dynamical decoupling routine.
4. \( E \) is applied again without the reset operations.
5. Channel tomography is applied to qubit \( B \). Qubit \( A \) is assumed to be fully depolarized, as the state of it and the auxiliaries is discarded.

To bypass unintended noise, we apply a principle inspired by error mitigation. In our scenario, an analogous procedure is greatly simplified by the fact that most forms of device noise do not resemble dephasing in the \( X \) basis. Given a \( 4 \times 4 \) Choi matrix \( M \) from process tomography on
Figure 4. The circuit to apply the channel $\Phi_{(2)}$. Qubit $q_6$ on the `ibmq_lagos` serves as $B$, $q_5$ as $A$, and $q_3$ and $q_4$ as auxiliaries. Figure created using IBM Quantum, generated using Qiskit and the Matplotlib backend. Gates within the two barriers apply $\Phi_{(k)}$, while gates surrounding the barriers are inserted by Qiskit’s channel tomography.

a single qubit, one may model the effect of incoherently applying the $X$ operator by

$$\frac{|M_{14}| + |M_{23}|}{|M_{14}| - |M_{23}|} = \exp(\chi t),$$

where $\chi \in \mathbb{R}^+$ is a parameter controlling the strength of $X$-basis dephasing noise. One may see via the “$D^4$” model introduced in [23] that a combination of $Z$-basis dephasing, amplitude damping, depolarizing, and coherent phase drift do not directly affect the ratio given in Equation (9). Hence we may extract the expected effect of $\Phi_{(k)}$ independently from common forms of noise in superconducting qubits. To extract the Zeno effect from the noise induced by real hardware, we solve for $\chi$ as in Equation (9) via the Choi matrix from process tomography, then use $\chi$ to construct a purely dephasing channel. See Section D in the Supplementary Information for more details on inferring different contributions to noise.

At $k = 16$, $|M_{14}| - |M_{23}| < 0$, resulting in a negative argument to the logarithms used to solve Equation (9) for $\chi$. This over-rotation in $X$ may result from passive drift, from overrotation in the application of $\Phi_{ZX(\pi/(2k))}$, or even from random fluctuations when both $|M_{14}|$ and $|M_{23}|$ are small. Based on the unexpectedly fast convergence observed in Figure 3(1) and Figure 3(3), it appears that a buildup in calibration errors is likely. This explanation is consistent with the difficulty in calibrating $\Phi_{ZX(\pi/(2k))}$ for large values of $k$. Since experiments with $k > 15$ may fail to reflect intended parameters, we truncate the results presented in Figure 3 to earlier points.

5.2. Theoretical Methods. By $\mathcal{B}(\mathcal{H})$ we denote the space of bounded operators on Hilbert space $\mathcal{H}$, and by $\mathcal{D}(\mathcal{H})$ we denote the space of densities. By $\hat{1}$ we denote the identity matrix. For a unitary matrix $U$, we denote by $R_U$ the channel that applies that unitary matrix via conjugation such that $R_U(X) = UXU^\dagger$ for any $X \in \mathcal{B}(\mathcal{H})$. If there is a family of unitaries ($U_j$), we may denote $R_j := R_{U_j}$ when it is clear from context. A quantum channel is a completely positive, trace-preserving map, usually denoted by $\Phi$, $\Psi$, or $\Theta$. For products of quantum channels, we use left multiplication to denote composition or the “$\circ$” symbol to optimize readability, e.g. $\Phi\Psi(\rho) = \Phi(\Psi(\rho)) = \Phi \circ \Psi(\rho)$. We often use the diamond norm to compare quantum channels,
denoted $\| \cdot \|_\Diamond$. We also characterize similarity of quantum densities using the fidelity given by $F(\rho, \omega) = \text{tr}(\sqrt{\sqrt{\rho} \omega \sqrt{\rho}})^2$. By process fidelity we refer to the fidelity between Choi matrices. See Appendix A for more information on norms used in this paper.

Relative entropy inequalities herein are restricted to finite dimension, though they should in principle extend to infinite dimension with re-derivation of some referenced inequalities. The Zeno-like norm bounds derived in Appendix B in principle generalize to some infinite-dimensional settings, but that is not the focus of this paper.

A quantum Markov semigroup (QMS) is a family of channels $(\Phi^t)$ for $t \in \mathbb{R}^+$ (continuous case) or $t \in \mathbb{N}$ (discrete case) with the essential property that $\Phi^t \Phi^s = \Phi^{t+s}$. For any quantum channel $\Phi$, $(\Phi^t : t \in \mathbb{N})$ is a discrete semigroup of powers of the channel. Any continuous QMS has an adjoint Lindbladian generator $\mathcal{L}$ such that $\Phi^t = \exp(-t\mathcal{L})$ for any $t \in \mathbb{R}^+$. For a Hamiltonian $H$, we denote by $i[H, \cdot]$ the transformation given by $i[H, \cdot](x) := i[H, x]$ for an operator $x$. Hence $R_{\exp(-iHt)} = \exp(-i[H, \cdot])$. A Lindbladian generalizes Hamiltonian time-evolution to open systems.

For a normal, faithful density $\omega \in \mathcal{D}(\mathcal{H})$, the GNS inner-product with respect to $\omega$ is defined for $x, y \in \mathcal{B}(\mathcal{H})$ as $\langle x, y \rangle_\omega = \text{tr}(\omega x^* y)$ in a tracial setting. We say that a Lindbladian has GNS detailed balance or that $\mathcal{L}$ is GNS self-adjoint when $\mathcal{L}$ is self-adjoint with respect to the GNS inner product for an implied (or explicitly written) invariant density $\omega$. Formally, the standard Lindbladian is $\mathcal{L}^*$, the adjoint of $\mathcal{L}$ with respect to the trace.

Any Lindbladian with GNS detailed balance has a fixed point subspace $\mathcal{N}_\omega$ related to its fixed point von Neumann algebra $\mathcal{N}$ [6]. There is a fixed point projector $E_\omega$ to $\mathcal{N}_\omega$, the pre-adjoint of a conditional expectation $E_\omega$ with respect to the trace [2]. We may denote by $E$ a conditional expectation or related channel, at times suppressing the explicit subscript. Some additional properties of $E$ are described in Appendix A. We also recall the Pimsner-Popa indices, which we denote for subspace projections,

$$C(E) = \inf\{c > 0|\rho \leq cE(\rho)\forall \rho \in \mathcal{M}_\omega\}, \quad C_{ub}(E) = \sup_{n \in \mathbb{N}} C(E \otimes \text{Id}^n).$$

as considered in [18, 5] and originally by Pimsner and Popa [24] as a finite-dimensional analog of the Jones index [25]. When $E$ has is GNS self-adjoint with respect to density $\omega$, $C_{ub}(E) \leq d^2 \omega_{\min}^{-1}$, where $d$ is the dimension of the space and $\omega_{\min}^{-1}$ the minimum eigenvalue of $\omega$.

A starting point for this work is the quantum version of the modified logarithmic Sobolev inequality (MLSI) introduced by Kastoryano and Temme [9]. Stochastic versions of this inequality appear in earlier literature [7, 8]. MLSI differs from but was inspired by the earlier notion of logarithmic Sobolev inequalities [26, 27]. An important, more recent observation is that while the canonical log-Sobolev inequality fails for semigroups that lack a unique fixed point state [28], the modified version may remain valid [29]. This observation motivated the notion of a complete modified logarithmic-Sobolev inequality (CMLSI) [30]. A finite-dimensional semigroup

\footnote{As with the Lindbladian, we denote by $E$ the completely positive, trace-preserving or Schrödinger picture quantum channel, and by $E^*$ its adjoint. We may refer to $E$ as a conditional expectation.}
generated by $\mathcal{L}$ has $\lambda$-CMLSI if and only if
\[ D\left( (e^{-\mathcal{L}t} \otimes \mathbb{1}_B)(\rho) \parallel (\mathcal{E} \otimes \mathbb{1}_B)(\rho) \right) \leq e^{-\lambda t} D(\rho \parallel (\mathcal{E} \otimes \mathbb{1}_B)(\rho)) \] (10)
for all extensions by a finite-dimensional auxiliary subsystem $B$. CMLSI with some constant is known for all finite-dimensional quantum Markov semigroups having GNS detailed balance with respect to a faithful state. This result was derived in a self-contained way in [5]. It was simultaneously derived in [4] for semigroups that are self-adjoint with respect to the trace, which via [3] also extends to all finite-dimensional semigroups with detailed balance.

An important tool is the multiplicative relative entropy comparison from earlier work. Let $\rho$ be a density and $\mathcal{E}, \Phi$ be quantum channels such that $\Phi(\rho) \leq c \mathcal{E}(\rho)$ for constant $c > 1$, and let $\zeta \in (0, 1)$. Furthermore, let $\Phi \mathcal{E} = \mathcal{E}$. Then
\[ D(\rho \parallel (1 - \zeta \mathcal{E}(\rho) + \zeta \Phi(\rho)) \geq \beta_{c, \zeta} D(\rho \parallel \mathcal{E}(\rho)) \] for $\beta_{c, \zeta} = 1 - O(c\zeta)$.

A concrete value of $\beta_{c, \zeta}$ is determined in [31], from which we see that
\[ \beta_{c, \zeta} \geq \frac{1}{1 + \zeta c + 2\zeta^2 (c - 1) \left( 1 - 2\zeta (1 + \zeta) \right)} \left( \frac{(c - 1)^2}{c(\ln c - 1) + 1} - 4\zeta - \zeta^2 \right) . \]
One may also use the some of the approximate tensorization estimates from [5]. A common form of Equation (11) with one parameter states that if a channel $\Psi$ has that $\Psi \mathcal{E} = \mathcal{E} \Psi = \mathcal{E}$ and $(1 - \epsilon) \mathcal{E} \leq \Psi \leq (1 + \epsilon) \mathcal{E}$, then $D(\rho \parallel \Psi(\rho)) \geq \beta_{2, \epsilon} D(\rho \parallel \mathcal{E}(\rho))$, and $\beta_{2, \epsilon} = 1 - O(\epsilon)$.

Because CMLSI yields tensor-stable decay rate bounds on processes such as decoherence [10] and thermal equilibration [32], concrete estimates of involved constants are valuable and have practical relevance. A method of particular relevance noted in [31, 33] allows one to build up MLSI estimates for a complicated system from its constituents. In particular, let $\mathcal{L}_1, ..., \mathcal{L}_m$ be Lindbladian generators with detailed balance and respective fixed point projector $\mathcal{E}_1, ..., \mathcal{E}_m$, and $\mathcal{L} = \alpha_1 \mathcal{L}_1 + ... + \alpha_m \mathcal{L}_m$ with fixed point projector $\mathcal{E}$ and constants $\alpha_1, ..., \alpha_m > 0$. If each $\mathcal{L}_j : j \in 1...m$ has (C)MLSI with respective constant at least $\lambda$ and if
\[ \sum_{j=1}^{m} \alpha_j D(\rho \parallel \mathcal{E}_j(\rho)) \geq D(\rho \parallel \mathcal{E}(\rho)) , \] (12)
then $\mathcal{L}$ has $\lambda$-(C)MLSI. We call this trick decay merging. The condition of Equation (12) is known as quasi-factorization or approximate tensorization. Quasi-factorization was first shown for the classical case in [34], generally proven for the quantum relative entropy in [5], and refined in [31, 33]. We use versions of (12) from [31].

Though the quantum Zeno effect is historically stated in terms of measurements [19], a number of results that include versions of the generalized Zeno effect [35, 15, 16, 36, 37], dynamical decoupling [21, 22], and adiabatic theorems [38, 14] show that many kinds of fast quantum processes can modify the effective dynamics of a simultaneous, slower process. Here we show a Zeno-like bound in terms of MLSI constants:

**Theorem 5.1.** Let $\mathcal{L}$ be a bounded Lindbladian and $t > 0$. Let $\Phi$ be any quantum channel with invariant subspace projection $\mathcal{E}_0$. If $D(\Phi(\rho) \parallel \mathcal{E}(\rho)) \leq e^{-\lambda t} D(\rho \parallel \mathcal{E}(\rho))$ for all $\rho$ including auxiliary
extensions or if \(\|\Phi^k - \mathcal{E}_0\|_\diamond \leq e^{-\lambda k b}\) for constant \(b > 0\) and all \(k \in \mathbb{N}\), then

\[
\| (\Phi \circ e^{-L t/k})^k - e^{-\mathcal{E}_0 L^t E_0} \mathcal{E}_0 \|_\diamond \leq \begin{cases} O(t^2/k) & \text{for } k \text{ large}, \\ O(t^2/\lambda k) & \text{for } \lambda \text{ small, } k \text{ large.} \end{cases}
\]

For any Lindbladian \(\mathcal{S}\) such that \(\|e^{-\mathcal{S} t} - \mathcal{E}_0\|_\diamond \leq e^{-\mathcal{S}_0 b}\) (as implied by \(\lambda\)-CMLSI),

\[
\|e^{-(\mathcal{S}+\mathcal{L}) t} - e^{-\mathcal{E}_0 \mathcal{L} E_0} \mathcal{E}_0 \|_\diamond \leq O\left(\frac{t^2}{\lambda}\right).
\]

If \(\mathcal{L} = i[H, \cdot]\) for some Hamiltonian \(H\), then \(\exp(-\mathcal{E}_0 \mathcal{L} E_0 t) \circ \mathcal{E}_0(\rho)\) is equivalent to unitary evolution generated by \(\mathcal{E}_0(H)\) applied to \(\mathcal{E}_0(\rho)\) for any input \(\rho\).

Theorem 5.1 is a version of the generalized quantum Zeno effect. Theorem 5.1 is a shortened version of Theorem B.14, Remark B.16 and Remark B.6 shown in the Supplementary Information, which derives constants, can account for time-varying interruption channels in the discrete case, and may for some cases substitute alternate norm conditions. The semigroup generated by \(\mathcal{E}_0 \mathcal{L} E_0\) is conventionally referred to as the Zeno dynamics. For discrete interruption by a fixed point projection as in Subsection 2, Proposition B.5 yields a more specific bound with tighter constants.

Though Theorem 5.1 is similar to results of [14, 16, 37], it bounds convergence in terms of CMLSI and decay constants. This distinction may seem subtle, but it is essential to Theorem 3.3, which compares decay constants of a Lindbladian and its rotation-free constituent. Furthermore, the asymptotic dependence on \(t\) and \(\mathcal{L}\) is explicitly shown and of polynomial order. As seen in Example C.8 in the Supplementary Information, there are cases in which stronger decay does not arise from multiplicative comparability. The primary argument of the proof is that the first order terms in Taylor series for the original dynamics and Zeno limit are equivalent after expanding in both the number of discrete steps \((k\) in Theorem 5.1) and a matrix order comparison parameter determined by the decay constant. The result then follows from analytically and combinatorially bounding the higher-order terms in both of these parameters. To derive the continuous case, we use a Kato-Suzuki-Trotter formula in the \(k \to \infty\) limit. Because Theorem 5.1 is similar to known inequalities, we state it as a method rather than a main result of this paper. Still, it is possible that Theorem 5.1 is of independent interest.

The relative entropy decay Theorem 3.4 follows from variations on the decay merging trick as described in [31] and in [33]. Theorem 3.4 is more subtle because of the possibility for Zeno-like effects to block decay to the long-time fixed point subspace. A counter-intuitive consequence of Theorem 5.1 and the broader Zeno effect is that a chain of projections may approach the action of a unitary on a particular subspace. Let \(\mathcal{E}_t := R_{\exp(-i H t)} \circ \mathcal{E}_0 \circ R_{\exp(i H t)}\) for Hamiltonian \(H\) and any \(t \in \mathbb{R}\). Note that for any \(k \in \mathbb{N}\), \(\mathcal{E}_t \mathcal{E}_{t-1/k} \cdots \mathcal{E}_1/k = R_{\exp(-i H t)} \circ (\mathcal{E}_0 R_{\exp(i H t/k)})^k\). As a direct consequence,

\[
\lim_{k \to \infty} \mathcal{E}_t \mathcal{E}_{t-1/k} \cdots \mathcal{E}_1/k \mathcal{E}_0 = R_{\exp(i(H_0(H)-H)t)} \circ \mathcal{E}_0
\]

Though each \(\mathcal{E}_t, \ldots, \mathcal{E}_1/k\) is a projection that we might interpret as rotation-free, in the continuum limit, the chain of composed projections approaches unitary rotation following \(\mathcal{E}_0\).
To counter the breakdown in decay merging as composed projections approach effective unitarity, we show a converse bound to CMLSI:

**Theorem 5.2.** Let \((\Phi^t : t \in \mathbb{R}^+)\) be a continuous quantum Markov semigroup with decay+drift Lindbladian \(\mathcal{L} : \mathcal{L}(\rho) = i[H, \rho] + \mathcal{S}(\rho)\), such that \(\mathcal{S}\) has detailed balance and invariant subspace projection \(\mathcal{E}_0\). Then

\[
D(\Phi^t(\rho)\|\mathcal{E}_0 \Phi^t(\rho)) \geq \exp(-C_{cb}(\mathcal{E}_0)\|\mathcal{S}\|_2 t/2)D(\rho\|\mathcal{E}_t(\rho)),
\]

where \(\mathcal{E}_t = R_{\exp(-iHt)}\mathcal{E}_0 R_{\exp(iHt)}\), and \(C_{cb}(\mathcal{E}_0) \leq d^2\) in dimension \(d\) is a Pimsner-Popa index constant as described in Supplementary Information Section 5.2.

Theorem 5.2 is paradoxically the key to showing that despite the restricting impact of Zeno-like effects, continuous quantum Markov semigroups with drift show exponential decay after finite time. Since fast convergence toward the Zeno limit hinders decay toward the long-time fixed point, a speed limit on decay due to \(\mathcal{S}\) allows drift to spread noise across subspaces. Building on Theorem 5.2, we may use Equation (12) to show that combined entropy subtractions from \(\lambda\)-CMLSI at different times do not vanish. Theorem 5.1 implies that there must be some penalty in terms of the stochastic part’s decay strength, because the process cannot be arbitrarily close to its Zeno limit while also decaying quickly toward its long-time fixed point unless they are compatible (such as when \(H\) commutes with \(\mathcal{E}_0\)).

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7. Data Availability

All data that support the plots within this paper and other findings of this study are available from the corresponding author upon reasonable request for uses compatible with the IBM Quantum End User Agreement.

8. Code Availability

Code used for the experimental and numerical portions of this paper is available from the corresponding author upon reasonable request.
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Supplementary Information

Appendix A. Mathematical Background

A.1. Distance and Entropy Measures. We recall the quantum relative entropy given by
\[ D(\rho \parallel \sigma) := \text{tr}(\rho \log \rho - \rho \log \sigma) \]
for two densities \( \rho, \sigma \in \mathcal{D}(\mathcal{H}) \) and introduced by Umegaki [39]. Beyond tracial settings, the relative entropy has a more general definition via Tomita-Takesaki modular theory. The logarithm base does not matter when comparing ratios or asymptotic orders, so many inequalities herein need not explicitly specify a base. We will by default denote “log” with base 2 for compatibility with the experimental conventions in Section 2, and “ln” for the natural logarithm.

For a quantum channel (completely positive, trace-preserving map) \( \Phi \), we denote by \( \Phi^* \) the adjoint of \( \Phi \) with respect to the trace (a completely positive, unital map).

By \( \rho \geq \sigma \), we mean that \( \rho \) is greater than \( \sigma \) in the Loewner order: \( \rho - \sigma \) is a non-negative matrix. We say for a pair of quantum channels \( \Phi, \Psi : \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H}) \) that \( \Psi \geq_{cp} \Phi \) iff
\[ (\Psi \otimes \hat{1}_B)(\rho) \geq (\Psi \otimes \hat{1}_B)(\rho) \]
for all input densities \( \rho \) on finite-dimensional auxiliary systems \( B \cong M_n \) such that \( n \in \mathbb{N} \). We call this and the associated symbols \( <_{cp}, \leq_{cp}, >_{cp} \) cp-order relations. Via the Choi-Jamiolkowski isomorphism, a finite-dimensional quantum channel is fully defined by its action on one side of a maximally entangled state between its input space and an auxiliary space of the same dimension. Hence if \( \Phi \geq_{cp} (1 - \epsilon)\Phi \), then \( \Phi = (1 - \epsilon)\Psi + \epsilon\Theta \) for some \( \epsilon \in (0, 1) \) and other channel \( \Theta \).

We denote the Schatten norms \( \| \cdot \|_p \) for \( p \in [0, \infty] \). The trace distance is given by
\[ d_{tr}(\rho, \sigma) := \frac{1}{2} \| \rho - \sigma \|_1 . \]
In general, for a pair of Banach spaces \( \mathcal{A} \) and \( \mathcal{B} \) with respective norms \( \| \cdot \|_\mathcal{A} \) and \( \| \cdot \|_\mathcal{B} \), the \( \mathcal{A} \to \mathcal{B} \) norm on maps from \( \mathcal{A} \) to \( \mathcal{B} \) is given by
\[ \| \Phi \|_{\mathcal{A} \to \mathcal{B}} := \sup_{\rho \in \mathcal{A}} \frac{\| \Phi(\rho) \|_\mathcal{B} \| \rho \|_\mathcal{A} - }{\| \rho \|_\mathcal{A} } . \]
For von Neumann algebras of the same type \( \mathcal{A} \) and \( \mathcal{B} \), we denote \( \| \Phi \|_{\mathcal{A} \to \mathcal{B}, cb} := \sup_{\rho \in \mathcal{A}} \| \Phi \otimes \hat{1}_C \|_{\mathcal{A} \otimes \mathcal{C} \to \mathcal{B} \otimes \mathcal{C}} \), where \( \mathcal{C} \) is an extension over von Neumann algebras of the same type as \( \mathcal{A} \) and \( \mathcal{B} \) with a compatible norm in each tensor product. We denote \( \| \Phi \|_{p \to q, cb} = \| \Phi \|_{\mathcal{A} \to \mathcal{B}, cb} \) in which \( \| \cdot \|_\mathcal{A} = \| \cdot \|_p \) and \( \| \cdot \|_\mathcal{B} = \| \cdot \|_q \). The diamond norm on a map \( \Phi \) is defined as \( \| \Phi \|_\diamond := \| \Phi \|_{1 \to 1, cb} \). We call a map \( \Phi \) an \( \mathcal{A} \to \mathcal{B} \) contraction if \( \| \Phi \|_{\mathcal{A} \to \mathcal{B}} \leq 1 \). When \( \Phi \) is a map from a normed Banach space \( \mathcal{A} \) to itself, we denote \( \| \Phi \| := \| \Phi \|_{\mathcal{A} \to \mathcal{A}} \).

When \( \mathcal{E} \) is the conditional expectation to the invariant subspace of a Lindbladian with detailed balance, we note the following properties:

(1) \( \mathcal{E} \) is idempotent (hence a projection).
(2) \( \mathcal{E} \) is self-adjoint with respect to the GNS inner product for \( \omega \).
(3) $\mathcal{E}$ has the following bimodule property: for any $a, b \in \mathcal{N}$ and $x \in \mathcal{B}(\mathcal{H})$, $\mathcal{E}^*(axb) = a\mathcal{E}^*(x)b$. Following, for any density $\rho \in \mathcal{D}(\mathcal{H})$, $\mathcal{E}(a\rho b) = a\mathcal{E}(\rho)b$.

**APPENDIX B. REANALYSIS OF THE GENERALIZED ZENO EFFECT**

This Subsection is devoted to a technical reanalysis of the generalized Zeno effect. Rather than spectral properties of the channels involved, we base our estimates on cp-order inequalities and seek comparability with CMLSI and $\lambda$-decay.

The bounds derived herein are nonetheless in terms of norms as described in Appendix A. These bounds are in principle very general, requiring only sub-multiplicativity of $\|\cdot\|_{A \rightarrow B}$ in addition to its being a norm. A restriction, however, is that many of the results must assume contractivity of most or all maps involved. The diamond norm is especially convenient in this sense, as channels automatically satisfy this assumption. The diamond norm is however only clearly usable in algebras with a finite trace, which extends it to some but not all infinite-dimensional settings. In principle, one could apply results herein using the Kosaki norms [40] in all von Neumann algebras, but more care would be needed to ensure contractivity of involved maps and in some cases boundedness and analyticity. Since the purpose of this paper is however to understand the relationship between the Zeno effect and mostly finite-dimensional semigroup theory, we will not be too concerned with technical barriers in non-tracial settings. For infinite-dimensional versions of the generalized Zeno effect, see [36, 37].

To simplify notation, let

$$F^{(m)}_a := \frac{a^m \exp(a)}{m!}$$

for any scalar $a > 0, k \in \mathbb{N}$.

**Remark B.1.** For any $a > 0$ such that $\exp(a)$ equals its Taylor series,

$$\exp(a) - \sum_{n=0}^{k} \frac{a^n}{n!} = \sum_{n=k+1}^{\infty} \frac{a^n}{n!(n+k+1)} \leq \frac{a^{k+1} \exp(a)}{(k+1)!} = F^{(k+1)}_a.$$  

**Lemma B.2.** Let $\mathcal{L}$, and $\mathcal{E}$ be respectively a Lindbladian and a map on the same von Neumann algebra. Let $\mathcal{A}$ be a normed input subspace that is preserved by $\mathcal{L}$, and $\mathcal{B}$ be the normed output space of $\mathcal{E}$. Then for any $t \in \mathbb{R}$

$$\left\| \mathcal{E} \circ \sum_{m=k}^{\infty} \frac{(it)^m}{m!} \mathcal{L}^m \right\|_{A \rightarrow B,(cb)} \leq F^{(k)}_{\|\mathcal{L}\|_{A,(cb)}} \|\mathcal{E}\|_{A \rightarrow B,(cb)}.$$

**Proof.** First, we name a given input $\rho$ and use the triangle inequality to separate terms.

$$\left\| \mathcal{E} \circ \sum_{m=k}^{\infty} \frac{(it)^m}{m!} \mathcal{L}^m \right\|_{A \rightarrow B,(cb)} \leq \sum_{m=k}^{\infty} \frac{\|\mathcal{E} \circ \mathcal{L}^m\|_{A \rightarrow B,(cb)}}{m!} \leq \sum_{m=k}^{\infty} \frac{\|\mathcal{L}^m\|_{A \rightarrow B,(cb)}}{m!}.$$

We then consider each term.

$$\|\mathcal{E} \circ \mathcal{L}^m(\rho)\|_{A \rightarrow B,(cb)} \leq \|\mathcal{E}\|_{A \rightarrow B,cb} \|\mathcal{L}^m(\rho)\|_{A,(cb)}.$$  

The proof then follows from Remark B.1. \qed
Lemma B.3. Let \((f_m)_{m=1}^k, (g_m)_{m=1}^k\) be families of maps for \(k \in \mathbb{N}\) such that \(f_m \circ f_{m-1}\) and \(g_m \circ g_{m-1}\) are valid compositions. Let 
\[ \omega_l = \prod_{m=1}^l f_m(\rho) \text{ for input } \rho \text{ and each } l \in 1..k. \]
Then
\[
\left( \prod_{m=1}^k f_m - \prod_{m=1}^k g_m \right)(\rho) = \sum_{l=1}^k \left( \prod_{n=l+1}^k g_n \right) (f_l - g_l)(\omega_{l-1}).
\]

Proof. For each value of \(l\),
\[
\left( \prod_{n=l+1}^k g_n \right)(\omega_l) - \left( \prod_{n=l}^k g_n \right)(\omega_{l-1}) = \left( \prod_{n=l+1}^k g_n \right)(f_l - g_l)(\omega_{l-1}).
\]
The Lemma then follows from induction. \(\square\)

Corollary B.4. Let \((f_m)_{m=1}^k, (g_m)_{m=1}^k\) be families of maps as in Lemma B.3, and further assume that these maps are submultiplicative and defined from a normed Banach space to itself. Then
\[
\left\| \prod_{m=1}^k g_m - \prod_{m=1}^k f_m \right\| \leq \sum_{l=0}^{k-1} \left\| \prod_{n=l+1}^k g_n \right\| \left\| \prod_{n=1}^{l-1} f_n \right\| \left\| f_l - g_l \right\|.
\]

Proof. We apply Lemma B.3 to the normed quantity in the left hand side, obtaining that
\[
\left\| \prod_{m=1}^k g_m - \prod_{m=1}^k f_m \right\| = \sup_\rho \frac{1}{\|\rho\|} \sum_{l=1}^k \left( \prod_{n=l+1}^k g_n \right) (f_l - g_l) \left( \prod_{m=1}^{l-1} f_m \right)(\rho).
\]
Via the triangle inequality, we may separate the terms in the sum. We then split the product via submultiplicativity. The overall supremum over \(\rho\) then underestimates the per-term and per-factor suprema, completing the Corollary. \(\square\)

B.1. Results for Interruptions by Conditional Expectations. Here we consider a continuous process interrupted by a conditional expectation. The results of this Subsection underpin the the theory bound in Subsection 2.2.2 recalled as Equation (1). Furthermore, they show in a relatively simple calculation how Zeno-like bounds arise from Taylor expansion and norm comparison. These calculations may guide the intuition for the more complicated results of Subsection B.2.

Proposition B.5. Let \(\mathcal{L}\) and \(\mathcal{E}\) be respectively a Lindbladian and projection to the subspace \(\mathcal{N} \subseteq \mathcal{A}\) defined on Banach space \(\mathcal{A}\) such that \(\exp(-\mathcal{L})\) is equal to its Taylor series. Let \((t_m)_{m=1}^k\) be a family of values in \(\mathbb{R}^+\) such that \(t_m = O(1/k)\). Let \(t = \sum_{m=1}^k t_m\). Then
\[
\left\| \prod_{m=1}^k (\mathcal{E}\exp(-t \mathcal{L})\mathcal{E}) - \exp(-t \mathcal{E} \circ \mathcal{L} \circ \mathcal{E}) \right\|_{\mathcal{N}\rightarrow\mathcal{N},(cb)} = O(1/k)
\]
\[
\leq \sum_{m=1}^k \left( \|\mathcal{E}\|_{\mathcal{A}\rightarrow\mathcal{N},(cb)} F_{t_m}^{(2)} t_m \|\mathcal{L}\|_{\mathcal{A}\rightarrow\mathcal{A},(cb)} + F_{t_m}^{(2)} t_m \|\mathcal{E}\|_{\mathcal{N}\rightarrow\mathcal{N},(cb)} \right) \left\| \exp \left( -\mathcal{E} \mathcal{L} \mathcal{E} \sum_{n=m+1}^k t_n \right) \right\|_{\mathcal{N}\rightarrow\mathcal{N},(cb)}.
\]
Proof. First, we show for one value of \( t \) that
\[
\| \mathcal{E} \exp(-\mathcal{L}t)\mathcal{E}(\rho) - \exp(-t\mathcal{E} \circ \mathcal{L} \circ \mathcal{E})\|_N \leq \left( F^{(2)}_{\mathcal{L},N} + F^{(2)}_{\mathcal{E} \circ \mathcal{L} \circ \mathcal{E},N} \right).
\] (16)

For any input \( \rho \), one may Taylor expand
\[
(\mathcal{E} \circ \exp(-\mathcal{L}t) \circ \mathcal{E})(\rho) = \mathcal{E}(\rho) - t\mathcal{E}(\mathcal{L}(\mathcal{E}(\rho))) + \mathcal{E}\left( \sum_{m=2}^{\infty} \frac{(-t)^m}{m!} \mathcal{L}^m(\mathcal{E}(\rho)) \right).
\] (17)

The terms up to first order in \( t \) match those of \( \exp(-t\mathcal{E} \circ \mathcal{L} \circ \mathcal{E}) \). Via the triangle inequality and idempotence of \( \mathcal{L} \), what remains is to bound the distance of higher order terms,
\[
\| \mathcal{E}\left( \sum_{m=2}^{\infty} \frac{(-t)^m}{m!} \mathcal{L}^m(\mathcal{E}(\rho)) \right) - \sum_{m=2}^{\infty} \frac{(-t)^m}{m!} (\mathcal{E} \circ \mathcal{L} \circ \mathcal{E})^m(\mathcal{E}(\rho)) \|_N.
\]

We apply Lemma B.7 to each higher-order sequence individually, using the triangle inequality to separate them. This step completes the proof of Equation (16).

We then apply Corollary B.4. The fact that \( \exp(a) \exp(b) = \exp(a+b) \) whenever \( [a,b] = 0 \) completes the Proposition. \( \square \)

Proposition B.5 simplifies when bounding the diamond norm, because quantum channels are contractions. Hence
\[
\| \prod_{m=1}^{k} (\mathcal{E} \exp(-\mathcal{L}t_m/k)\mathcal{E}) - \exp(-t\mathcal{E} \circ \mathcal{L} \circ \mathcal{E}) \|_0 \leq \sum_{m=1}^{k} \left( F^{(2)}_{\mathcal{L},0} + F^{(2)}_{\mathcal{E} \circ \mathcal{L} \circ \mathcal{E},0} \right).
\] (18)

Proposition B.5 yields additional simplifications when \( \mathcal{L} \) has the form of a Hamiltonian:

Remark B.6. When \( \mathcal{E} \) is a conditional expectation and \( \mathcal{L} = i[H,\cdot] \) for some Hamiltonian \( H \), the bimodule property of conditional expectations implies that for any input \( \rho \),
\[
\mathcal{E}(i[H,\mathcal{E}(\rho)]) = i[H,\mathcal{E}(\rho)] - i\mathcal{E}(i[H,\rho]) = i[\mathcal{E}(H),\mathcal{E}(\rho)] \cdot
\] (19)

Lemma B.7. For any \( p \) such that \( \|\cdot\|_p \) is a norm and \( \|\cdot\|_p \) obeys Hölder’s inequality,
\[
\|[H,\cdot]^m\|_{p,q,(\rho)} \leq 2^m\|H\|_\infty \sup_{\rho} \|\rho\|_q/\|\rho\|_p
\]

Proof. For Hamiltonians, we use the fact that \( [H,\cdot]^m(\rho) \) generates \( 2^m \) terms on any density \( \rho \), each of which contains \( m \) powers of \( H \) and one of \( \rho \). Using Hölder’s inequality and its inductive generalization,
\[
\|H^k\rho H^{m-k}\|_p \leq \|H^k\|_\infty \|\rho H^{m-k}\|_p \leq \|H^k\|_\infty \|\rho\|_p \|H^{m-k}\|_\infty \leq \|\rho\|_p \|H\|_\infty^m
\]
for any integer \( k \) such that \( 0 \leq k \leq m \). Hence
\[
\|[H,\cdot]^m\|_{p,q} = \sup_{\rho} \|[H,\cdot]^m(\rho)\|_p/\|\rho\|_p \leq 2^m\|H\|_\infty^m.
\]

To conclude the Lemma, we return to Equation (15) and re-assemble the exponential series, using Remark B.1. \( \square \)
B.2. Results for Maps Converging to Fixed Points. Here we show Zeno-like effects for both discrete channels compositions and Lindbladian-generated semigroups that converge toward a fixed point projection \( \mathcal{E} \). Generalizing the results of the previous Section, those of this Section no longer assume that the interruption is itself a projection.

**Lemma B.8.** Let \((\Phi_m)_{m=1}^k\) be a family of bounded maps on Banach space \( \mathcal{A} \). Let \((\mathcal{L}_m)_{m=1}^k\) be a family of bounded Lindbladians. Let \( t_1, ..., t_k \in \mathbb{R}^+ \). Then

\[
\left\| \prod_{m=1}^k (\Phi_m \circ e^{-\mathcal{L}_m t_m}) - \prod_{m=1}^k (\Phi_m \circ (1 - \mathcal{L}_m)) \right\| \leq \sum_{r=1}^k \left\| \prod_{m=r+1}^k \Phi_m e^{-\mathcal{L}_m t_m/k} \right\| F^{(2)}_{\|\mathcal{L}_m\| t_r}.
\]

**Proof.** The Lemma follows from noting that \((1 - \mathcal{L} t)\) is the 1st order Taylor expansion of \( e^{-\mathcal{L} t} \) for any \( t \in \mathbb{R}^+ \), so

\[
\|e^{-\mathcal{L}_m t_m} - (1 - \mathcal{L}_m)\| \leq F^{(2)}_{\|\mathcal{L}_m\| t_m}
\]

for each \( m \in 1...k \). Corollary B.4 completes the Lemma. \(\Box\)

While it is often intuitive to think of a Lindbladian as having units of inverse time and appearing alongside a time parameter in the expression \( \exp(-\mathcal{L} t) \), \( t \) is formally redundant in many of the mathematical expressions we will use. When \( t \) is an overall parameter (not changing by interval as in Lemma B.8), we may instead write \( \exp(-\mathcal{L}) \), implicitly absorbing \( t \) as a multiplying factor in \( \mathcal{L} \). Doing so simplifies notation, and one may trivially re-extract the parameter \( t \) by substituting \( \mathcal{L} \rightarrow t\mathcal{L} \) in resulting expressions.

Subsequent Lemmas require some combinatoric notation. For \( m < k \in \mathbb{N} \), let

\[
WS(m, k) \subset \{[l_1, ..., r_1], ..., [l_m, ..., r_m] : 1 = l_1 \leq r_1 \leq l_2 \leq r_2 ... < l_m \leq r_m = k\}
\]

denote the set of partitions of \( k \) into \( m \) contiguous, ordered, non-overlapping intervals. For given \( W \in WS(m, k) \), let \( W(j) \) denotes a contiguous sequence of indices for \( j \in 1...m \). Let \( |W(j)| \) denote the number of indices in \( W(j) \), which we will refer to as its length. By \( W(j)[l] \) we denote the \( l \)th index in \( W(j) \) for \( l \in 1...|W(j)| \). As an example, we might take \( W = (1 \mapsto [1,2,3], 2 \mapsto [4,5]) \in WS(2,5) \), in which case \( W(1) = [1,2,3], W(2) = [4,5] \), and \( W(2)[1] = 4 \). In this example, we would have \( l_1 = 1, r_1 = 3, l_2 = 4, r_2 = 5 \).

For any \( n \leq k \), let \( WS(m,k,n) \subseteq WS(m,k) \) denote the subset of partitions such that \( |W(j)| \geq n \) for all \( j \in 1...m \). Note that \( WS(m,k,n) \) is the empty set whenever \( n > k/m \).

**Lemma B.9.** Let \((\Phi_m)_{m=1}^k\) be a family contractions on a Banach space for any \( k \in \mathbb{N} \). Let \((\mathcal{L}_m)_{m=1}^k\) be bounded Lindbladians such that \( e^{-\mathcal{L}_m} \) is also contractive for each \( m \). Let \( \|\mathcal{L}\| := \max_m \{\|\mathcal{L}_m\|\} \). Then for any \( \alpha : \mathbb{N} \rightarrow (1/k, 1) \) and \( n \in 1...k \),

\[
\left\| \prod_{m=1}^k \left( \Phi_m \left(1 - \frac{\mathcal{L}_m}{k}\right) \right) - \sum_{m=0}^n \frac{(-1)^m}{k^m} \sum_W \prod_j (\Phi_{W(j)} \mathcal{L}_{W,j}) \Phi_{W(m+1)} \right\|
\leq \sum_{m=1}^n \frac{\alpha(m)m\|\mathcal{L}\|^m}{(1 - m)!} + \sum_{m=n+1}^k \frac{\|\mathcal{L}\|^m}{m!},
\]

(20)
where $\Phi_{W(j)} = \Phi_{W(j)||W(j)||,k} \circ \ldots \circ \Phi_{W(j)||,k}$, and the sum over $W$ is within the set $WS(m + 1,k,\alpha(m)k)$, and each $L_{W,j} \in (L_m)$ is the Lindbladian appearing between the partitions $W(j)$ and $W(j + 1)$.

**Proof.** For convenience of notation, let the norm distance in this Lemma be denoted $\Delta$.

The first step is the binomial expansion, where we substitute the index $r$ for $m$ on the left hand side,

$$\prod_{r=1}^{k}(\Phi_{r}(1 - L_{m}/k)) = \sum_{m=0}^{k} \frac{(-1)^{m}}{k^{m}} \sum_{W \in WS(m+1,k)} \left( \prod_{j=1}^{m} (\Phi_{W(j)}L_{W,j}) \right) \Phi_{W(m+1)}.$$  \hspace{1cm} (21)

We see that the right hand side of Equation (21) is the same as that in the compared quantity from the desired inequality, except that the latter sums over $WS(m + 1, k, \alpha(l)k)$ instead of over $WS(m + 1, k)$. Hence we must bound the number and magnitude of terms containing short partitions.

Assume we are given some function $\alpha : \mathbb{N} \rightarrow \mathbb{R}^{+}$ and consider a particular value of $m$ as in Equation (21). The number of partitions containing at least one segment of length at most $\alpha(m)k$ is upper-bounded by $(k - 1)!/2$, since we can consider first placing $m - 1$ partition boundaries within $k$ locations, then choose a final partition boundary that is no more than $\alpha(m)k$ indices away from one of the $m - 1$ original boundaries or from first or last index. The divisor of $2$ arises from the invariance under exchange between the final boundary and its close neighbor. This bound is an overcount, since the first $m - 1$ placements might already contain one or more partitions that are too small. We will ignore this overcounting, since for $k/m >> \alpha(m)k$, it is not expected to contribute much. Using the triangle inequality to recombine the sum in Equation (21),

$$\Delta \leq \sum_{m=0}^{k} \left\| L \right\| \left\| \left( \sum_{m=0}^{k} \frac{(-1)^{m}}{k^{m}} \sum_{W \in WS(m+1,k)} \left( \prod_{j=1}^{m} (\Phi_{W(j)}L_{W,j}) \right) \Phi_{W(m+1)} \right) \right\| \leq \sum_{m=0}^{k} \left\| L \right\| \left\| \left( \sum_{m=0}^{k} \frac{(-1)^{m}}{k^{m}} \sum_{W \in WS(m+1,k)} \left( \prod_{j=1}^{m} (\Phi_{W(j)}L_{W,j}) \right) \Phi_{W(m+1)} \right) \right\|.$$

It is easy to see that $(k - 1)!(k - 1)^{m-2}/(m - 1)!$. We then observe that $(k - 1)/k \leq k/(k + 1)$ and that $[\alpha(m)mk] \leq \alpha(m)m(k + 1)$. Hence

$$\frac{k(k - 1)^{m-2}[\alpha(m)mk]}{k^m(m - 1)!} \leq \frac{\alpha(m)m}{(1 - m)!}!.$$

We then separately handle the terms with $m > n$. Returning to Equation (21), we apply the coarse bound that the cardinality $|WS(m + 1, k)| \leq (k choose m) \leq k^m/m!$. Hence

$$\left\| \left( \sum_{m=n+1}^{k} \frac{(-1)^{m}}{k^{m}} \sum_{W \in WS(m+1,k)} \left( \prod_{j=1}^{m} (\Phi_{W(j)}L_{W,j}) \right) \Phi_{W(m+1)} \right) \right\| \leq \sum_{m=n+1}^{k} \left\| L \right\| \left\| \left( \sum_{m=n+1}^{k} \frac{(-1)^{m}}{k^{m}} \sum_{W \in WS(m+1,k)} \left( \prod_{j=1}^{m} (\Phi_{W(j)}L_{W,j}) \right) \Phi_{W(m+1)} \right) \right\|.$$

£
Lemma B.10. Let \((\Phi_m)_{m=1}^{k}\) and \((\mathcal{L}_m)_{m=1}^{k}\) be as in Lemma B.9 with the additional assumption that for given \(r \in \mathbb{N}\) and \(\gamma \in \mathbb{R}^+\), \(\Phi_{W(j)} \succeq_{cp} (1 - \epsilon)\mathcal{E}\) whenever \(|W(j)| \geq k/\gamma\). Then

\[
\left\| \sum_{m=0}^{k} \frac{(-1)^m}{k^m} \sum_{W \in WS_k} \prod_{j=1}^{m} (\Phi_{W(j)} \mathcal{L}_{W,j}) \Phi_{W(m+1)} - \sum_{m=0}^{k} \frac{(-1)^m}{k^m} \prod_{j=1}^{m} (\mathcal{E} \mathcal{L}_{W,j}) \right\|
\]

\[
\leq \epsilon^{\gamma/r} \|\mathcal{L}\| \exp(\|\mathcal{L}\|) + \sum_{m=r}^{k} \frac{\|\mathcal{L}\|^m}{(m-1)!},
\]

where \(WS_k := WS(m + 1, k, [k/m])\) in the sums.

Proof. We first consider the terms for individual values of \(m \leq n, k\), rewriting

\[
\sum_{W} \prod_{j=1}^{m} (\Phi_{W(j)} \mathcal{L}_{W,j}) \Phi_{W(m+1)} = \sum_{W} \prod_{j=1}^{m} ((1 - \epsilon_m)\mathcal{E} + \epsilon_m \Psi_{W(j)})(1 - \epsilon_m)\mathcal{E} + \epsilon_m \Psi_{W(m+1)})
\]

for some maps \(\{\Psi_{W(j)} : j = 1...m + 1\}\) such that \(\mathcal{E} \Psi_{W(j)} \mathcal{E} = \mathcal{E}\). Here \(W \in WS(m + 1, k, [k/m])\), and \(\epsilon_m = \epsilon^{\gamma/m}\). We begin by estimating the distance

\[
\left\| \prod_{j=1}^{m} (1 - \epsilon_m)\mathcal{E} + \epsilon_m \Psi_{W(j)}(1 - \epsilon_m)\mathcal{E} + \epsilon_m \Psi_{W(m+1)}) - \prod_{j=1}^{m} (\mathcal{E} \mathcal{L}_{W,j}) \right\|
\]

\[
\leq \|\mathcal{L}\|^m (1 - (1 - \epsilon^{\gamma/r})^m)
\]

for a single value of \(W\). This bound follows from the number of \(\mathcal{E}\) vs. \(\Psi_{W(j)}\) being binomially distributed in the first term after expanding, since the compared expressions match when the former contains only \(\mathcal{E}\) and \(\mathcal{L}\) factors. When \(m = 0\), the inequality is trivial. Via Bernoulli’s inequality, we simplify the expression for \(m > 0\) to \((1 - (1 - \epsilon^{\gamma/m})^m) \leq m\epsilon^{\gamma/m}\). Hence

\[
\left\| \sum_{m=0}^{k} \frac{1}{k^m} \sum_{W} \prod_{j=1}^{m} (1 - \epsilon_m)\mathcal{E} + \epsilon_m \Psi_{W(j)}(1 - \epsilon_m)\mathcal{E} + \epsilon_m \Psi_{W(m+1)}) - \prod_{j=1}^{m} (\mathcal{E} \mathcal{L}_{W,j}) \right\|
\]

\[
\leq \sum_{m=0}^{k} \frac{1}{k^m} \binom{k}{m} \|\mathcal{L}\|^m m\epsilon^{\gamma/m} \leq \sum_{m=0}^{k} \frac{\|\mathcal{L}\|^m}{m!} m\epsilon^{\gamma/m},
\]

where the final inequality follows from recalling that \((k \text{ choose } m) \leq k^m/m!\). Then

\[
\sum_{m=0}^{k} \frac{\|\mathcal{L}\|^m}{m!} m\epsilon^{\gamma/m} \leq \sum_{m=0}^{r} \frac{\|\mathcal{L}\|^m}{m!} m\epsilon^{\gamma/m} + \sum_{m=r}^{k} \frac{\|\mathcal{L}\|^m}{(m-1)!}.
\]

For an overestimate, we note that \(m\epsilon^{\gamma/m} \leq r\epsilon^{\gamma/r}\) for all \(m \in 1...r\). Hence

\[
\sum_{m=0}^{k} \frac{\|\mathcal{L}\|^m}{m!} m\epsilon^{\gamma/m} \leq \epsilon^{\gamma/r} \|\mathcal{L}\| \exp(\|\mathcal{L}\|) + \sum_{m=r}^{k} \frac{\|\mathcal{L}\|^m}{(m-1)!}.
\]
Theorem B.11. Let \((\Phi_m)_{m=1}^k\) be a family of norm contractions for any \(k \in \mathbb{N}\) all having fixed point projector \(E\). Let \((\mathcal{L}_m)_{m=1}^k\) be bounded Lindbladians such that \(e^{-\mathcal{L}_m s}\) is contractive for all \(s \in \mathbb{R}^+\) and \(m\). Let \(\|\mathcal{L}\| := \max_m \|\mathcal{L}_m\|\). Let \(\gamma \in \mathbb{R}^+\) such that for any consecutive sequence \(\Phi_W = \Phi_{j_1}, \ldots, \Phi_{j_{k/\gamma}}\), \(\Phi_W \geq c_p (1 - \epsilon) E\). For any \(q \in 1 \ldots k\), if \(\gamma/q (\log \epsilon (\gamma/q) + 2) \geq (1 + 1/2 \pi) e\|\mathcal{L}\|/q + 2\), then

\[
\left\| \prod_{m=1}^k (\Phi_m \circ e^{-\mathcal{L}_m/k}) - \prod_{m=1}^k e^{-\mathcal{L}_m E} \right\| \leq (k + k\%q) \left( F^{(2)}_{\|\mathcal{L}\|/k} + F^{(2)}_{\|\mathcal{L}E\|/k} \right) + q \left( \frac{1}{\gamma} F^{(1)}_{\|\mathcal{L}\|/q} + \epsilon \right) + \frac{1}{(1 + 1/2 \pi)(\gamma/q) / \log (\gamma/q) - 2},
\]

where \(\%\) denotes the modulus operator.

Proof. The Theorem follows from Lemmas B.8, B.9, and B.10 with appropriate parameters. For general convenience, note that

\[
\sum_{m=r}^k \|\mathcal{L}\|/(m - 1)! = \|\mathcal{L}\| \sum_{m=r-1}^{k-1} \|\mathcal{L}\|^m/m! \tag{23}
\]

for any \(r \in 1 \ldots k\).

Let \(\Delta_1\) denote the contribution from Lemma B.8. Under the assumptions of this Theorem,

\[
\Delta_1 \leq k \left( F^{(2)}_{\|\mathcal{L}\|/k} + F^{(2)}_{\|\mathcal{L}E\|/k} \right).
\]

Let \(\Delta_2\) denote the contribution from Lemma B.9 with \(\alpha(m) = 1/\gamma m\) and \(n = k\). Using Equation (23),

\[
\Delta_2 \leq \sum_{m=1}^k \|\mathcal{L}\|^m \frac{m\alpha(m)}{(m - 1)!} = \|\mathcal{L}\| \sum_{m=0}^{k-1} \frac{\|\mathcal{L}\|^m}{\gamma m!} = \frac{1}{\gamma} F^{(1)}_{\|\mathcal{L}\|}.
\]

Consider Lemma B.10 for \(r \in \mathbb{N}\). Via Equation (23) and Robbins’s precise form of Stirling’s approximation [41], \(m! \geq \sqrt{2\pi m (m/e)^m}\). Hence

\[
\sum_{m=r-1}^{k-1} \frac{\|\mathcal{L}\|^m}{m!} \leq \sum_{m=r-1}^{k-1} \frac{1/\gamma}{\sqrt{2\pi m}} \left( \frac{\|\mathcal{L}\|}{m} \right)^m \leq \frac{1}{2\pi} \left( \frac{\|\mathcal{L}\|}{r - 1} \right)^{r-1} \frac{r - 1}{r - 1 - e\|\mathcal{L}\|},
\]

since we can overestimate the sum by replacing \(k\) by \(\infty\). If we assume that \(r > a\epsilon\|\mathcal{L}\| + 1\) for \(a > 1\), then the bound is non-trivial as overestimated by \(1/2\pi (a - 1)^a r^{-2}\). For large \(r\), this function decays faster than exponentially in \(r\). Letting the contribution from Lemma B.10 be denoted \(\Delta_3\),

\[
\Delta_3 \leq e^{\lceil \gamma/r \rceil} + \frac{1}{2\pi (a - 1)^a r^{-2}}.
\]

To optimize this expression, it will be convenient to choose \(r = \gamma / (\log \epsilon \gamma + 2)\), and \(a = 1 + 1/2 \pi\). We then find

\[
\Delta_3 \leq \frac{\epsilon}{\gamma} + \frac{1}{(1 + 1/2 \pi) \gamma / \log \epsilon \gamma - 2},
\]
as long as \( \gamma/(\log_\gamma \gamma + 2) \geq (1 + 1/2\pi)e\|E\| + 2 \).

Let \( \Delta_4 \) denote the contribution from again applying Lemma B.8 this time to relate the term \( \sum_{m=0}^{k}(E\mathcal{L}\mathcal{E})^m/k^m = (1 - E\mathcal{L}\mathcal{E}/k)^k \) to the desired \( \exp(-E\mathcal{L}\mathcal{E}/k) \). We find \( \Delta_4 \leq kF_{\|E\|}^{(2)} \).

Via the triangle inequality, \( \sum_{r=1}^{4} \Delta_r \) yields that
\[
\left\| \prod_{m=1}^{k} \left( \Phi_m \circ e^{-E\mathcal{L}/k} \right) - e^{-E\mathcal{L}\mathcal{E}} \right\| \leq k(F_{\|E\|/k}^{(2)} + F_{\|E\mathcal{L}\mathcal{E}\|/k}^{(2)}) + \frac{1}{\gamma}(F_{\|E\|/q}^{(1)} + \epsilon) + \frac{1}{(1 + 1/2\pi)(\gamma/q)\log_\gamma \gamma - 2} \cdot (24)
\]
Finally, we introduce \( q \). We may trivially re-express
\[
\prod_{m=1}^{k} \left( \Phi_m \circ e^{-E\mathcal{L}/k} \right) - e^{-E\mathcal{L}\mathcal{E}} = \prod_{m=0}^{q-2} \left( \prod_{n=1}^{k/q} \left( \Phi_{m(k/q) + n} \circ e^{-(L/q)/(k/q)} \right) \right) \prod_{n=1}^{k\%q} \left( \Phi_{m(k/q) + n} \circ e^{-(L/q)/(k/q)} \right) - (e^{-E\mathcal{L}\mathcal{E}/q})^q \mathcal{E}.
\]
If \( q \) does not divide \( k \), then we may greatly simplify and slightly loosen the bound by effectively extending the product until it does, for instance letting \( \Phi_m = \Phi_m \circ \mathcal{L} \) and \( \mathcal{L}_m = \mathcal{L}_m \circ \mathcal{L} \). Hence letting \( \tilde{k} = k + k\%q \),
\[
\left\| \prod_{m=1}^{k} \left( \Phi_m \circ e^{-L_m/k} \right) - \prod_{m=1}^{k} e^{-E\mathcal{L}m\mathcal{E}} \right\| \leq \left\| \prod_{m=0}^{q-1} \left( \prod_{n=1}^{\tilde{k}/q} \left( \Phi_{m(k/q) + n} \circ e^{-(L/q)/(k/q)} \right) \right) \right\| - (e^{-E\mathcal{L}\mathcal{E}/q})^q \mathcal{E}.
\]
Via Corollary B.4 and the norm-contractiveness assumption on the maps involved,
\[
\ldots \leq \sum_{m=0}^{q-1} \frac{\tilde{k}}{q} (F_{\|E\|/k}^{(2)} + F_{\|E\mathcal{L}\mathcal{E}\|/k}^{(2)}) + \frac{q}{\gamma}(F_{\|E\|/q}^{(1)} + \epsilon) + \frac{1}{(1 + 1/2\pi)(\gamma/q)\log_\gamma \gamma - 2} \cdot (24).
\]
The final expression results from the sum. \( \square \)

Theorem B.11 applies straightforwardly in the continuous case, in which we will take \( k \to \infty \) and show that the corresponding conditions on \( \gamma \) scale appropriately. In the discrete case, however, even a large value of \( k \) does not immediately appear to eliminate dependence on \( \gamma \). This is however arguably an inefficiency: for larger \( k \), we should be able to obtain the same \( \epsilon \) with correspondingly scaled \( \gamma \). Here we do so explicitly as a Corollary:

**Corollary B.12.** Let \( (\Phi_m)_{m=1}^{k} \) be a family of contractive channels with shared fixed point projector \( \mathcal{E} \). Let \( q \in \mathbb{N} \) such that for any consecutive sequence \( \Phi_W = \Phi_{j_1}, ..., \Phi_{j_q}, \Phi_W \leq \epsilon \). Let \( \mathcal{L} \) be a bounded Lindbladian such that \( e^{-\mathcal{L}t} \) is contractive for all \( t \in \mathbb{R}^+ \), \( w \in \mathbb{N} \), and \( k/gw(\log_e(k/gw) + 2) \geq (1 + 1/2\pi)e\|E\|/w + 2 \). Then
\[
\left\| \prod_{m=1}^{k} \left( \Phi_m \circ e^{-L/k} \right) - e^{-E\mathcal{L}\mathcal{E}} \right\| \leq (k + k\%w)(F_{\|E\|/k}^{(2)} + F_{\|E\mathcal{L}\mathcal{E}\|/k}^{(2)}) + w\left( \frac{gw}{k} (F_{\|E\|/w}^{(1)} + \epsilon) + \frac{1}{(1 + 1/2\pi)(k/gw)\log_\gamma (k/gw) - 2} \right).
\]
Proof. Let \( \gamma := (k + k\%w)/g \), so that \( g = \lceil (k + k\%w)/\gamma \rceil \). Then Theorem B.11 yields that
\[
\left\| \prod_{m=1}^{k} (\Phi_m \circ e^{-L/k}) - e^{-\varepsilon \mathcal{E}} \right\| \leq (k + k\%w)(F_{\mathcal{L}/k}^{(2)} + F_{\mathcal{E} \mathcal{L}/k}^{(2)}) + w\left( \frac{g w}{F_{\mathcal{L}/w}} (F_{\mathcal{L}/k}^{(1)} + \epsilon) + \frac{1}{(1 + 1/2\pi)(k/g)\log(\gamma/k)/log_{\gamma}(k/g) - 2} \right).
\]

Corollary B.13. Let \( \mathcal{S} \) be a bounded Lindbladian with fixed point projector \( \mathcal{E} \) on a Banach space such that \( \exp(-\mathcal{S}/\gamma) \geq_{cp} (1 - \varepsilon)\mathcal{E} \), \( \gamma \in \mathbb{R}^+ \). Let \( \mathcal{L} \) be a bounded Lindbladian such that \( e^{-L} \) is also contractive for all \( t \in \mathbb{R}^+ \), \( w \in \mathbb{N} \), and \( \gamma / (\log \gamma + 2) \geq (1 + 1/2\pi)e\|\mathcal{L}\|/w + 2 \). Then
\[
\left\| e^{-(\mathcal{S}+\mathcal{L})} - e^{-\varepsilon \mathcal{E}} \right\| \leq w\left( \frac{w}{\gamma} (F_{\mathcal{L}/w}^{(1)} + \epsilon) + \frac{1}{(1 + 1/2\pi)(\gamma/w)\log_{\gamma}(\gamma/w) - 2} \right).
\]

Proof. The Corollary follows from the Kato-Suzuki-Trotter formula [42] as \( k \to \infty \) and from Theorem B.11.

A subtle but ultimately crucial distinction between Corollary B.13 and the main Theorems of [14] is that instead of an explicit weighting factor in the exponential, \( \gamma \) describes a potentially more implicit decay rate. Hence we may relate these Zeno-like bounds to semigroup decay. First, we consider decay in the operator norm.

Theorem B.14. Let \( \mathcal{L} \) be a bounded Lindbladian in dimension \( d \) such that \( e^{-L} \) is a contractive quantum Markov semigroup for all \( t \in \mathbb{R}^+ \). Let \( \mathcal{E}_0 \) be a projection. Given \( \varepsilon \in (0,1) \), let \( c_\varepsilon \leq d^2 \) be the minimum constant such that if any channel \( \Psi \) has \( \Psi \mathcal{E}_0 = \mathcal{E}_0 \Psi = \mathcal{E}_0 \) and \( \sup_{\rho} \| \Psi - \mathcal{E}_0 \| \leq \varepsilon/c_\varepsilon \), then \( \Psi \geq_{cp} (1 - \varepsilon)\mathcal{E}_0 \). Such a \( c_\varepsilon \) exists as long as \( \| \cdot \| \) bounds the infinity norm of outputs. Furthermore, let
\[
a(\gamma) := (1 + 1/2\pi)^{\gamma/\log_{\gamma}\gamma - 2}
\]
for \( \gamma > 0 \), set \( \gamma := \lambda/\ln(c_\varepsilon b/\varepsilon w) \) and \( w \in 1..k \). Assume \( \gamma \log_{\gamma} \gamma \geq (1 + 1/2\pi)e\|\mathcal{L}\|/w + 2 \).

1. Let \( \mathcal{S} \) be a Lindbladian in dimension \( d \) generating contractive semigroup \( \Phi^t \) with fixed point projection \( \mathcal{E}_0 \) such that for all \( t > 0 \),
\[
\|\Phi^t - \mathcal{E}_0\| \leq e^{-\lambda t b}
\]
for some \( b > 0 \). Then
\[
\left\| e^{-(\mathcal{S}+\mathcal{L})} - e^{-\mathcal{E}} \right\| \leq w\left( \frac{1}{\gamma} (F_{\mathcal{L}/w}^{(1)} + \epsilon) + \frac{1}{a(\gamma)} \right).
\]

2. Let \( (\Phi_m)_{m=1}^n \) be a family of contractive quantum channels in dimension \( d \) with shared fixed point projection \( \mathcal{E}_0 \) such that
\[
\|\Phi_{m1}...\Phi_{mk} - \mathcal{E}_0\| \leq e^{-\lambda k b}
\]
for \( k \in 1...n \), any consecutive, increasing subsequence \( (m_j \in 1...n)_{j=1}^k \), and \( b > 0 \). Then for sufficiently large \( k \),

\[
\left\| \prod_{m=1}^{k} (\Phi_m \circ e^{-\mathcal{L}/k}) - e^{-\mathcal{E}_E \mathcal{E}} \right\| \leq (k + k\%w)(F^{(2)}_\|\mathcal{L}\|/k + F^{(2)}_\|\mathcal{E}_E \mathcal{E}\|/k)
\]

\[
+ w\left( \frac{1}{k} \left( F^{(1)}_\|\mathcal{L}\|/w + \epsilon \right) + \frac{1}{a(k^\left\lfloor 1/\gamma \right\rfloor)} \right).
\]

**Proof.** Existence and an explicit value of \( c_0 \) is given by [31, Proposition 2.16] or [10, Lemma 41]. For (1), we apply Corollary B.13. For (2), we use Corollary B.12 with \( g = \left\lfloor 1/\gamma \right\rfloor /w \). □

Though the conditions of Theorem B.14 might not always be satisfied, one can always multiply \( S \) by a constant factor until they are reached. This multiplication is analogous to the formulation in [14], where the bound is in terms of an explicit such factor. In contrast, our bound also depends explicitly on other aspects of \( S \), which might include such components as the connectivity of an underlying model, effective temperature, etc.

**Remark B.15.** In principle, we could extend Theorem B.14 to unequally spaced times, replacing \( t/k \) by \( t_m \) for \( m \in 1...k \). Using Theorem B.11 it would be easy to do so in terms of \( k \max_m t_m \).

A more sophisticated approach might obtain a bound in terms of \( \sum_m t_m \) or other moments.

Similarly, we could also in principle allow the semigroups \( \mathcal{L} \) and \( \mathcal{S} \) to vary in time, as \( \mathcal{S}(t) \) and \( \mathcal{L}(t) \). As long as all constants involved were appropriately bounded over every time interval, one might again obtain an overall bound in terms of their maxima/minima or sums.

Either of these generalizations would however complicate the Theorem and its proof to obtain some technical enhancements we do not need for the primary results of this paper. Hence we leave the option to future work should it be desired. A potential follow-up paper may attempt to more fully address the issue of time-varying Lindbladians, including the possibility that the invariant subalgebra varies in ways not captured by a unitarily rotating fixed point projection.

**Remark B.16.** If a \( d \)-dimensional semigroup \( \Phi^t \) generated by \( S \) has \( \lambda \)-CMLSI to fixed point projection \( \mathcal{E}_0 \), then via Pinsker’s inequality,

\[
\|\Phi^t - \mathcal{E}_0\|_\diamond \leq e^{-\lambda t/2} \sqrt{2 \ln C_{cb}(\mathcal{E}_0)}.
\]

Similarly, if a family of channels \( (\Phi_m)_{m=1}^n \) all have a complete strong data processing inequality to a shared fixed point projections \( \mathcal{E}_0 \) in that

\[
D((\Phi_m(\rho) \otimes \mathcal{I}^B)(\mathcal{E}_0 \otimes \mathcal{I}^B)(\rho)) \leq e^{-\lambda/2} D(\rho\|\mathcal{E}_0) \mathcal{I}^B(\rho)
\]

for any \( \rho^A \mathcal{B} \) with \( |A| = d \), then

\[
\|\Phi_{m_1}...\Phi_{m_k} - \mathcal{E}_0\|_\diamond \leq e^{-\lambda k} \sqrt{2 \ln C_{cb}(\mathcal{E}_0)}
\]

for any \( k \in \mathbb{N} \) and \( (\Phi_{m_j})_{j=1}^k \). Hence diamond norm bounds imply the conditions of Theorem B.14. Via Pinsker’s inequality, relative entropy decay bounds imply norm bounds used in Theorem B.14 with \( b = \sqrt{2 \ln C_{cb}(\mathcal{E}_0)} \) and exponential decay rate \( \lambda/2 \), noting that \( \ln C_{cb}(\mathcal{E}_0) \leq 2 \ln d \).
To prove Theorem 5.1, we invoke Theorem B.14 with Remark B.16. We choose $w = \lceil t \|L\| \rceil$, which extracts the second term’s dependence on $t \|L\|$ to a quadratic factor instead of the potentially exponentially dependence in $F^{(1)}_t$. The choice of $w$ in Theorem 5.1 is ultimately why the dependence of $\lambda$ on $\lambda_0$ in Theorem 3.3 is inverse square root, rather than e.g. inverse logarithmic.

APPENDIX C. PROOFS OF ENTROPY DECAY BOUNDS

CMLSI may technically fail if the fixed point subspace contains persistent rotations. To account for drifting fixed points of discrete channels $\Phi_1,...,\Phi_m$, we modify the assumed decay to fixed point subalgebras such that for each $j \in 1...m$, there is a unitary $U_j$ such that (denoting $R_j := R_{U_j}$)

$$\Phi_j E = E \Phi_j = R_j E = E R_j .$$

for some common $E$. We may further assume that each $\Phi_j$ has its own fixed point expectation $E_j$ and corresponding $U_j$ such that

$$\Phi_j E_j = E_j \Phi_j = R_j E_j = E_j R_j ,$$

and $\Phi_j^r \to R_j^r E_j$ as $r \to \infty$. Under these assumptions, $U_j$ accounts for the invertible drift that is not eliminated by stochastic decay. We may deduce from Equation (25) that

$$\Phi_m...\Phi_1 E = U_m...U_1 E$$

and from Equation (26) that $E_j E = E$ for each $j$.

Definition C.1. A channel $\Phi$ has rotated decay with constant lambda (\(\lambda\)-decay) if $\Phi^r \to R_{U} E$ for some unitary $U$ and fixed point projection $E$ such that

$$\Phi E = E \Phi = R_{U} E = E R_{U} ,$$

and

$$D(\Phi(\rho)\|R_{U} E(\rho)) \leq e^{-\lambda D(\rho\|E(\rho))} .$$

We say $\Phi$ has complete $\lambda$-decay when $\lambda$-decay holds for all extensions by an auxiliary system.

Analogously to Definition C.1, we define a continuous analog of MLSI for Lindbladians with a rotating protected subspace.

Definition C.2 (RMLSI). A quantum Markov semigroup $\Phi^t$ and associated Lindbladian generator obey rotated MLSI ($\lambda$-RMLSI) if for all $t \in [0, \infty)$,

$$D(\Phi^t(\rho)\|E(\Phi^t(\rho))) \leq e^{-\lambda t D(\rho\|E(\rho)))}$$

for some continuously parameterized unitary $R^t$ such that $\Phi^t E = E \Phi^t = R^t E = E R^t$. It has $\lambda$-CRMLSI if $\lambda$-RMLSI holds under extension by an arbitrary auxiliary system.

If the stochastic generator $S$ in Equation (2) commutes with the Hamiltonian $H$, then CMLSI of $L$ is at least that of $S$ (even if $S$ may not commute with $H$).
Proposition C.3. Let $\mathcal{L}$ be a Lindbladian with $\sigma$-detailed balance and fixed point projector $\mathcal{E}$. Let $\tilde{\mathcal{L}}$ be a Lindbladian that commutes with $\mathcal{E}$. If $\mathcal{L}$ has $\lambda$-(C)MLSI, then $\mathcal{L} + \tilde{\mathcal{L}}$ decays states to the projection given by $\mathcal{E}$ as though having $\lambda$-(C)MLSI. If $\Psi_1, ..., \Psi_m$ are quantum channels that commute with $\mathcal{E}$, then

$$D(\Psi_1 \Phi^{t_1} \Psi_2 \Phi^{t_2} ... \Phi^{t_{m-1}} \Psi_m(\rho)\|\Psi_1 ... \Psi_m \mathcal{E}(\rho)) \leq e^{-\lambda t} D(\rho\|\mathcal{E}(\rho))$$

for any $t_1, ..., t_{m-1} > 0$ such that $t_1 + ... + t_{m-1} = t$.

As a technical subtlety, $\tilde{\mathcal{L}}$ may decay to a smaller subspace than that given by $\mathcal{E}$ at a slower rate, such as by applying noise to a subsystem untouched by $\mathcal{L}$. For this reason we say that the semigroup induces relative entropy to $\mathcal{L}$’s fixed point subspace as though it has $\lambda$-(C)MLSI. The semigroup given by $\mathcal{L} + \tilde{\mathcal{L}}$ may not have $\lambda$-(C)MLSI to its ultimate fixed point subspace, though it necessarily does under the conditions of Proposition C.3 if $\tilde{\mathcal{L}}$ is a Hamiltonian.

Proof. First, we prove the discrete case, in which channels $\Psi_1, ..., \Psi_m$ surround and intersperse with $\exp(-\mathcal{L}t)$. Via the data processing inequality,

$$D(\Psi_1 \Phi^{t_1} \Psi_2 \Phi^{t_2} ... \Phi^{t_{m-1}} \Psi_m(\rho)\|\Psi_1 ... \Psi_m \mathcal{E}(\rho)) \leq D(\Phi^{t_1} \Psi_2 ... \Phi^{t_{m-1}} \Psi_m(\rho)\|\Psi_2 ... \Psi_m \mathcal{E}(\rho)).$$

Then using assumed (C)MLSI,

$$D(\Phi^{t_1} \Psi_2 ... \Phi^{t_{m-1}} \Psi_m(\rho)\|\Psi_2 ... \Psi_m \mathcal{E}(\rho)) = D(\Phi^{t_1} \Psi_2 ... \Phi^{t_{m-1}} \Psi_m(\rho)\|\mathcal{E}(\Psi_2 \Phi^{t_1} ... \Phi^{t_{m-1}} \Psi_m(\rho)))$$

$$\leq (1 - \lambda(t - 1)) D(\Psi_2 ... \Phi^{t_{m-1}} \Psi_m(\rho)\|\mathcal{E}(\Psi_2 \Phi^{t_1} ... \Phi^{t_{m-1}} \Psi_m)) \cdot$$

Iterating the inequality completes the discrete case. For the continuous case, replacing $\Psi_1, ..., \Psi_m$ by a Lindbladian $\tilde{\mathcal{L}}$, we apply the same argument with the Kato-Suzuki-Trotter expansion, stating for small time $\tau$ and bounded Lindbladians of the form in Equation (2) that

$$\Phi^\tau(\rho) = \Phi^\tau_{\tilde{\mathcal{L}}} \Phi^\tau(\rho) + O(\tau^2),$$

where $\Phi^\tau$ is generated by $\mathcal{L}$ and $\Phi^\tau_{\tilde{\mathcal{L}}}$ by $\tilde{\mathcal{L}}$. We then have

$$D(\Phi^\tau(\rho)\|\mathcal{E}(\Phi^\tau(\rho))) = D(\Phi^\tau \tilde{\Phi}^\tau \Phi^{\tau-\tau}(\rho)\|\mathcal{E} \tilde{\Phi}^\tau \tilde{\Phi}^{\tau-\tau}(\rho)) + O(\tau^2 \log \tau),$$

where the correction term follows from the continuity of relative entropy with respect to a subalgebraic restriction, [17, Lemma 7] and [18, Proposition 3.7]. Via assumed (C)MLSI and the data processing inequality for relative entropy, the above Equation leads to the conclusion that

$$D(\Phi^\tau(\rho)\|\mathcal{E}(\Phi^\tau(\rho))) \leq e^{-\lambda \tau} D(\Phi^{\tau-\tau}(\rho)\|\tilde{\Phi}^{\tau-\tau}(\rho)) + O(\tau^2 \log \tau).$$

Iterating completes the Remark as we take the limit $\tau \to \infty$, using the fact that CMLSI holds for tracially self-adjoint Lindbladians as shown in [5] and [4].

In general, CMLSI will not hold for semigroups of the form in Equation (2). We will however prove decay as in Theorem 3.4. Our first step is to use an iterated chain rule of relative entropy to derive a sequential analog to the decay merging results of [31].
Lemma C.4. Let \((\Phi_j)^m_{j=1}\) and respective \((\mathcal{E}_j)^m_{j=1}\), \((U_j)^m_{j=1}\) be families quantum channels satisfying equation (26), (complete) \(\lambda_j\)-decay for each \(j\), and equation (25) for intersection fixed point projector \(\mathcal{E}\). Then
\[
D(\Phi_m...\Phi_1(\rho)\|R_m...R_1\mathcal{E}(\rho)) \leq D(\rho|\mathcal{E}(\rho)) - \sum_j \lambda_j D(\Phi_j...\Phi_1(\rho)|\mathcal{E}_{j+1}\Phi_j...\Phi_1(\rho))
\]

Proof. For any \(m \in \mathbb{N}\), using the chain rule and assumed complete decay for \(\Phi_m\),
\[
D(\Phi_m...\Phi_1(\rho)\|R_m...R_1\mathcal{E}(\rho)) = D(\Phi_m\Phi_{m-1}...\Phi_1(\rho)\|E_mR_m\mathcal{E}R_{m-1}...R_1(\rho))
\]
\[
= D(\Phi_m\Phi_{m-1}...\Phi_1(\rho)\|E_mR_m\Phi_{m-1}...\Phi_1(\rho)) + D(\mathcal{E}_mR_m\Phi_{m-1}...\Phi_1(\rho)\|R_m\mathcal{E}R_{m-1}...R_1(\rho))
\]
\[
\leq (1 - \lambda_m)D(\Phi_{m-1}...\Phi_1(\rho)\|E_m\Phi_{m-1}...\Phi_1(\rho)) + D(\mathcal{E}_m\Phi_{m-1}...\Phi_1(\rho)\|\mathcal{E}(R_{m-1}...R_1(\rho)))
\]
\[
= D(\Phi_{m-1}...\Phi_1(\rho)\|R_{m-1}...R_1\mathcal{E}(\rho)) - \lambda_mD(\Phi_{m-1}...\Phi_1(\rho)\|E_m\Phi_{m-1}...\Phi_1(\rho))
\]
(27)

Iterating, we arrive at
\[
D(\Phi_m...\Phi_1(\rho)\|R_m...R_1\mathcal{E}(\rho)) \leq D(\rho|\mathcal{E}(\rho)) - \sum_j \lambda_j D(\Phi_j...\Phi_1(\rho)|\mathcal{E}_{j+1}\Phi_j...\Phi_1(\rho)),
\]
completing the Lemma.

We may extend Lemma C.4 to the continuous limit, replacing \(\mathcal{E}_j\) by \(\mathcal{E}_t = \text{Rot}_{-iHt} \circ \mathcal{E}_0 \circ \text{Rot}_{iHt}\) and \(\Phi_j\) by \(\Phi^t\) for a semigroup \((\Phi^t)\) with Hamiltonian part \(H\). Then
\[
\lim_{\tau \to 0} \tau \sum_t \lambda D(\Phi_{t+\tau}\Phi_{t}\Phi_0(\rho)|\mathcal{E}_{t+\tau}\Phi_{t}...\Phi_0(\rho)) = \lambda \int_0^t D(\Phi^s(\rho)|\mathcal{E}_0\Phi^s(\rho))ds .
\]
(28)

At this point, we encounter a barrier due to Zeno-like effects. A major technical underpinning of decay merging [31],
\[
D(\rho|\mathcal{E}(\rho)) + D(\rho|\hat{\mathcal{E}}(\omega)) \geq D(\rho|\mathcal{E}\hat{\mathcal{E}}(\omega)) ,
\]
approaches triviality when \(\mathcal{E} \approx \hat{\mathcal{E}}\). The first step in deriving this inequality is to apply data processing, taking \(D(\rho|\mathcal{E}(\omega)) \to D(\mathcal{E}(\rho)||\mathcal{E}\hat{\mathcal{E}}(\omega))\). One may expand via the chain rule, noting that \(D(\rho|\mathcal{E}(\omega)) = D(\rho|\hat{\mathcal{E}}(\omega)) + D(\hat{\mathcal{E}}(\rho)||\mathcal{E}(\omega))\), where the first term is nearly eliminated in the data processing step when \(\mathcal{E} \approx \hat{\mathcal{E}}\). There is however no way to tighten this inequality for general densities even when \(\mathcal{E} \approx \hat{\mathcal{E}}\):

Counterexample C.5. Assume
\[
D(\rho|\mathcal{E}(\rho)) + D(\rho|\hat{\mathcal{E}}(\rho)) \geq aD(\rho|\mathcal{E}\hat{\mathcal{E}}(\rho)) .
\]

Via the chain rule,
\[
D(\rho|\hat{\mathcal{E}}(\rho)) \geq (a - 1)D(\rho|\mathcal{E}(\rho)) + aD(\mathcal{E}(\rho)||\mathcal{E}\hat{\mathcal{E}}(\rho)) .
\]
If \(\rho = \check{\mathcal{E}}(\rho)\), but \(\rho \neq \mathcal{E}(\rho)\), then
\[
0 \geq (a - 1)D(\rho|\mathcal{E}(\rho)) + aD(\mathcal{E}(\rho)||\mathcal{E}\hat{\mathcal{E}}(\rho)) ,
\]
which can hold only if \( a \leq 1 \). Since we know that \( a \geq 1 \) by the original Lemma, \( a = 1 \). To make this counterexample more concrete, we may take \( \mathcal{E} \) to be a pinch on the qubit Bloch sphere and \( \tilde{\mathcal{E}} = \text{ad}_{U(\tau)} \circ \mathcal{E} \circ \text{ad}_{U(\tau)} \) for some qubit unitary family \( U(t) \).

As per Theorem 5.1 and Equation (13), \( \lim_{\tau \to \infty} \mathcal{E}_t \) is entropy-preserving, so there is no way for it to converge in general to a convex combination including \( \mathcal{E} \). More broadly, Zeno dynamics imply that if we tune up the strength of the decay component relative to the Hamiltonian, there are at least some instances in which the overall decay rate begins to decrease toward zero. Counterexample C.5 highlights that if we take a pair of conditional expectations that are infinitesimally rotated with respect to each other, their composition looks more like the first-applied conditional expectation followed by a rotation than like a broader mixing process.

For a GNS self-adjoint Lindbladian, the strategy of \([31]\) is to reduce the problem of combining constituent Lindbladians to one of quasi-factorization, which estimates the relative entropy to an intersection fixed point conditional expectation in terms of the relative entropy with respect to constituents. As illustrated in Counterexamples 3.1 and 3.2, this approach often fails with time dependence, as early dynamics may not sufficiently represent later mixing processes.

**Theorem C.6** (Technical Version of Theorem 3.3). Let \( \mathcal{L} = \mathcal{L}_1 + \mathcal{S} \) generate \( \Phi_t \), where \( \mathcal{L}_1 \) is a bounded Lindbladian and \( \mathcal{S} \) a Lindbladian with GNS detailed balance such that

\[
\|e^{-t\mathcal{S}} - \mathcal{E}_0\|_\diamond \leq e^{-\lambda_0 t} b_0
\]

for constant \( b > 0 \). For given \( \epsilon \), let \( e_\epsilon \) and \( a : \mathbb{R}^+ \to \mathbb{R}^+ \) be defined as in Theorem B.14. If there exists a \( t_0 > 0 \) for which

\[
\|\Phi_t - \mathcal{E}\|_\diamond \leq e^{-\lambda t} b
\]

when \( t \geq t_0 \), then letting \( \alpha = \inf_{t > t_0} \{\|R_t \mathcal{E} - e^{-\epsilon \mathcal{E}_0 \mathcal{L}_1 \mathcal{E}_0}\|_\diamond\} \),

\[
\lambda \leq \frac{2 \|\mathcal{L}\|(e + \epsilon)}{\alpha} \sqrt{\frac{2 \ln(c_b e_\epsilon / \epsilon)}{\lambda_0}} \ln \left( \frac{2b}{\alpha} \right)
\]

for sufficiently large \( \lambda_0 \).

**Proof.** Via the triangle inequality,

\[
\|\Phi_t - R_t \mathcal{E}\|_\diamond \geq \alpha - \|\Phi_t - e^{-t\mathcal{E}_0 \mathcal{L}_1 \mathcal{E}_0}\|_\diamond.
\]

Let \( \gamma := \lambda_0 / 2 \ln(c_b e_\epsilon / \epsilon) \) for arbitrary \( t > t_0 \). Via Theorem B.14 with \( \lambda_0 \) sufficiently large and \( q = \lceil t \|\mathcal{L}_1\| \rceil \),

\[
\|\Phi_t - e^{-t\mathcal{E}_0 \mathcal{L}_1 \mathcal{E}_0}\|_\diamond \leq \lceil t \|\mathcal{L}_1\| \rceil \left( \frac{\lceil t \|\mathcal{L}_1\| \rceil}{\gamma} (e + \epsilon) + \frac{1}{a(\gamma / \lceil t \|\mathcal{L}_1\| \rceil)} \right).
\]

Combining the above with Equation (30) and assumed decay of \( \Phi_t \),

\[
e^{-\lambda t} b \geq \|R_t \mathcal{E} - e^{-t\mathcal{E}_0 \mathcal{L}_1 \mathcal{E}_0}\|_\diamond - \lceil t \|\mathcal{L}_1\| \rceil \left( \frac{\lceil t \|\mathcal{L}_1\| \rceil}{\gamma} (e + \epsilon) + \frac{1}{a(\gamma / \lceil t \|\mathcal{L}_1\| \rceil)} \right).
\]
Re-arranging the inequality, we obtain that
\[
\lambda \leq -\frac{1}{t} \ln \left( \frac{1}{b} \left( \| R_tE - e^{-tE_0L_1E_0}E_0 \|_\diamond - \| tL_1 \| \left( \frac{\| tL_1 \|}{\gamma} (e + \epsilon) + \frac{1}{a(\gamma/\| tL_1 \|)} \right) \right) \right).
\] (31)

To obtain a concrete bound, we choose a value of \( t \). This choice is constrained by two aspects: first, Theorem B.14 requires that
\[
\frac{(\gamma/\| tL_1 \|)}{\log (\epsilon / (\gamma/\| tL_1 \|))} \geq (1 + 1/2\pi) e^{\| L_1 \| t / \| tL_1 \|} + 2 \geq (1 + 1/2\pi) + 2.
\]
Second, the argument of the logarithm in Equation (31) must remain positive. As long as \( E_0 \neq E \), and \( E_0 L_1 E_0 \) has a fixed point also differing from \( E \), one may easily see that \( \alpha > 0 \) via some input states, such as \( \rho \) such that \( E_0(\rho) = \rho \). For large \( \gamma \), the second constraint dominates. We choose \( t \) to be the largest such value such that
\[
\| tL_1 \| = \sqrt{\frac{\alpha \gamma}{2(e + \epsilon)}}.
\]
We again note that for sufficiently large \( \gamma \), the ceiling function is effectively absorbed by the non-tight factor of \( 1/\sqrt{2} \). The \( 1/a(\ldots) \) term in Equation (31) is subleading, so we absorb it by appending the above factor of \( 1/\sqrt{2} \). Equation (31) thereby implies that
\[
\lambda \leq -\frac{2\| L_1 \|(e + \epsilon)}{a\sqrt{\gamma}} \ln \left( \frac{\alpha}{2b} \right).
\]

This Equation completes the Theorem, which we finish by substituting the defined value of \( \gamma \). \( \square \)

**Remark C.7.** As with Remark B.16, we may easily extend Theorem 3.3 to use CMLSI constants. In particular, we would use Pinsker’s inequality to make the substitutions \( \lambda \to \lambda/2 \), \( \lambda_0 \to \lambda_0/2 \), \( b = \sqrt{2 \ln C_{cb}(E)} \), and \( b_0 = \sqrt{2 \ln C_{cb}(E)} \).

Though our Zeno-like bound, Theorem 5.1, is analogous to the primary result of [14], it does not immediately follow. An essential difference is that the main Theorems of [14] control the relative strength of processes through an explicit multiplier, “\( \gamma \).” In contrast, Theorem 5.1 uses the CMLSI constant of the stochastic part of the process. The following example illustrates a scenario in which growth of the CMLSI constant emerges not from an explicit multiplier but from the internal structure of the process:

**Example C.8.** Let \( G \) be a finite, undirected graph on \( n \) vertices, defined as a set of pairs \( \{i, j\} : i, j \in 1...n \). Let
\[
\Phi_{i,j}(\rho) = |i\rangle \langle j| \rho |i\rangle \otimes \rho_j^B + |j\rangle \langle i| \rho |j\rangle \otimes \rho_l^B + \left( \sum_{s \neq i,j} |s\rangle \langle s| \otimes 1^B \right) \rho \left( \sum_{r \neq i,j} |r\rangle \langle r| \otimes 1^B \right)
\]
represent a single edge on Hilbert space of dimension \( n \), with the possibility of extension by an arbitrary auxiliary system with an interaction Hamiltonian \( H \). As noted in [31], the complete graph Lindbladian given by
\[
S_n(\rho) = \rho - \sum_{i \in 1...n} |i\rangle \langle i|
\]
Using the data processing inequality for relative entropy, for any $\tau > 0$ the complete graph is $O(\ln n/n)$ even though for $\alpha > 1$, $\mathcal{S}_{\alpha n}$ is not a straightforward extension of $a\mathcal{S}_n$ for any scalar $a > 1$. Accelerated convergence to the Zeno limit arises because the structure of all-to-all interactions, which in the absence of a $1/n$ normalization factor cause the degree of the graph and hence the mixing time of a random walk to decrease with size. Hence this family of Lindbladians exhibits growing growing time rate that is not captured by an overall factor multiplying the Lindbladian.

The continuous structure of Lindbladian-driven decay nonetheless lets us to transform Equation (28) into a quasi-factorization inequality. We show a general converse to CMLSI:

**Theorem C.9** (Restatement of Theorem 5.2). Let $(\Phi^t : t \in \mathbb{R}^+)$ be a continuous quantum Markov semigroup in the form of Equation (2) with stochastic generator $S$ having fixed point conditional expectation $E_0$. Then

$$D(\Phi^t(\rho)||E_0\Phi^t(\rho)) \geq \exp(-C_{cb}(E_0)||S||\tau t/2)D(\rho||E_t(\rho)),$$

where $E_t = R_{\exp(-iHt)}E_0R_{\exp(iHt)}$.

**Proof.** We may write $S(\rho)$ as a sum of positive and negative parts for each input density $\rho$. Here $S(\rho) = S_+(\rho) - S_-(\rho)$, where $S_+(\rho), S_-(\rho) \geq 0$. Since $\text{tr}(S(\rho)) = 0$, $\text{tr}(S_+(\rho)) = \text{tr}(S_-(\rho))$ for all $\rho$. Hence $\text{tr}(S_+(\rho)) = (\text{tr}(S_-(\rho)) + \text{tr}(S_+(\rho)))/2 \leq ||S||\tau/2$, and $||S_+||_{\alpha} \leq ||S||\tau/2$. Since $S_+(\rho) \geq 0$, $S_+(\rho)/\alpha$ is a normalized density matrix for any $\rho$ with some $0 \leq \alpha \leq ||S_+||_{\alpha}$.

For $r > 0$ and any density $\omega$, let $\tilde{S}_\rho(\omega) := c(||S||\tau/2)(\omega - \mathcal{E}_r(S_+(\rho)/\alpha))$, where $c := C_{cb}(E_0)$ as shorthand. We know that $\tilde{S}_\rho$ is a valid Lindbladian, because its generated semigroup,

$$\exp(-\tilde{S}_\rho t)(\omega) = (1 - \exp(-tc||S||\tau/2))\mathcal{E}_r(S_+(\rho)/\alpha) + \exp(-tc||S||\tau/2)\omega,$$

has the form of a convex state replacement. Then

$$-(S + \tilde{S}_\rho)(\rho) = S^- - S^+ + c(||S||\tau/2)\mathcal{E}_r(S_+(\rho)/\alpha) - c(||S||\tau/2)\rho \geq -(c||S||\tau/2)\rho.$$

For asymptotically small $\tau > 0$,

$$\exp(-\tau(S + \tilde{S}_\rho))(\omega) \geq (1 - \tau c||S||\tau/2)\omega - O(\tau^2).$$

Using the data processing inequality for relative entropy, for any $t, r > 0$,

$$D(\Phi^t(\rho)||E_t\Phi^t(\rho)) \geq D(\exp(-\tau\tilde{S}_{\Phi^{t-r}(\rho)}\phi^{t-r}(\rho))\exp(-\tau\tilde{S}_{\Phi^{t-r}(\rho)})\mathcal{E}_r\phi^{t-r}(\rho)\).

One can easily check that $[\tilde{S}_\rho, \mathcal{E}_r] = 0$, so we can move $\exp(-\tau\tilde{S}_{\Phi^{t-r}(\rho)}\phi^{t-r}(\rho))$ past $\mathcal{E}_r$ in the second argument to relative entropy. Using the Kato-Suzuki-Trotter formula,

$$\exp(-\tau\tilde{S}_{\Phi^{t-r}(\rho)}\phi^{t-r}(\rho)) = \exp(-\tau(S + \tilde{S}_{\Phi^{t-r}(\rho)}))\mathcal{F}_r\phi^{t-r}(\rho) + O(\tau^2) \geq (1 - \tau c||S||\tau/2)\phi^{t-r}(\rho) + O(\tau^2).$$

Hence using the convexity of relative entropy and continuity of subalgebra-relative entropy,

$$D(\Phi^{t}(\rho)||E_0\Phi^{t}(\rho)) \geq (1 - \tau c||S||\tau/2)D(\mathcal{F}_r\phi^{t-r}(\rho)||E_r\mathcal{F}_r\phi^{t-r}(\rho)) + O(\tau^2\ln \tau)$$

$$= (1 - \tau c||S||\tau/2)D(\Phi^{t-r}(\rho)||\mathcal{E}_r\phi^{t-r}(\rho)) + O(\tau^2\ln \tau).$$

To complete the Lemma, we iterate $t/\tau$ times starting from $r = 0$ and take the limit as $\tau \to 0$. □
Returning to Equation (28),

$$\lambda \int_0^t D(\Phi_s(\rho) | \mathcal{E}_0 \Phi_s(\rho)) ds \geq \lambda \int_0^t e^{-r\|S\|/2} D(\rho | \mathcal{E}_s(\rho)) dr. \tag{34}$$

Since the right hand side is an integral of relative entropies with respect to distinct conditional expectations, we may apply a quasi-factorization as in [31].

**Theorem C.10.** [Technical Restatement of Theorem 3.4] Let $\mathcal{E}$ be a Lindbladian in the form of Equation (2) with stochastic generator $S$ and Hamiltonian $H$. Let $E_t := R_{\exp(-iHt)} \mathcal{E}_0 R_{\exp(iHt)}$. Let $\mathcal{E}$ denote the invariant subspace under decay at all values of $t$. Assume that $S$ has $\lambda$-CMLSI and for given $\tau \in \mathbb{R}^+$ that

$$(1 + \zeta(c - 1)) \mathcal{E} \geq_{cp} \int_0^\tau \ldots \int_0^\tau \mathcal{E}_{t_m} \ldots \mathcal{E}_{t_1} d\mu_1(t_1) \ldots d\mu_m(t_m) \geq_{cp} (1 - \zeta) \mathcal{E}$$

where $\mu_1, \ldots, \mu_m : [0, \tau] \rightarrow [0, 1]$ are probability measures and $\alpha > 0$ such that

$$\sum_{j=1}^m \int_0^\tau d\mu_j(t) |t\rangle \langle t| \leq \alpha \int_0^\tau e^{-E_{\mathcal{E}_g}(\mathcal{E}_g)(S)(t)/2} |t\rangle \langle t| dt,$$

where $|t\rangle \langle t|$ denotes a basis vector on the compact set $[0, \tau]$. Then $\exp(-\mathcal{E}\tau)$ has $\alpha\beta\zeta\lambda$-decay.

The conditions are satisfied for any $\tau > 0$ with some $\alpha > 0$, $\zeta \in (0, 1)$, and $c > 1$.

The overall approach to proving Theorem 3.4 is to relate the problem to a kind of decay merging: given a set of Lindbladians $S_t$ parameterized by time $t$ and each having $\lambda$-CMLSI, the composite Lindbladian

$$S_{tot} = \int_0^\tau S_t dt \tag{35}$$

would have CMLSI with a constant depending on $\lambda$ and $\tau$.

**Lemma C.11** (Continuous Quasi-factorization). Let $\mathcal{G}$ be a compact set and $\mu_1, \ldots, \mu_m : \mathcal{G} \rightarrow [0, 1]$ a sequence of probability measures on $\mathcal{G}$. Let $\mathcal{E}_g$ denote a conditional expectation parameterized by $g$ to subalgebra $\mathcal{N}_g$, such that $\cap_{g \in \mathcal{G}} \mathcal{N}_g = \mathcal{N}$ with conditional expectation $\mathcal{N}$. Assume Riemann integrability and that

$$(1 - \zeta) \mathcal{E} \leq_{cp} \ldots \int \mathcal{E}_{g_m} \ldots \mathcal{E}_{g_1} d\mu_1(g_1) \ldots d\mu_m(g_m) \leq_{cp} (1 + \zeta(c - 1)) \mathcal{E}.$$ 

Then

$$\sum_{j=1}^m \int D(\rho | \mathcal{E}_g) d\mu_j(g) \geq \beta_{c,\zeta} D(\rho | \mathcal{E}(\rho)),$$

Proof of Lemma C.11. The main trick is similar to the proof of quasi-factorization in [31], using Equation (29) to obtain

$$D(\rho | \mathcal{E}_g) d\mu_j(g) \geq \ldots \int \mathcal{E}_{g_m} \ldots \mathcal{E}_{g_1} d\mu_1(g_1) \ldots d\mu_m(g_m) D(\rho | \mathcal{E}_{g_m} \ldots \mathcal{E}_{g_1}(\rho)).$$

This Lemma then follows from convexity of relative entropy and Equation (11).
\( \mathcal{L} \) as in Theorem 3.4 however does not have the exact form of \( \mathcal{S}_{\text{tot}} \) as in (35): rather than apply effectively rotated versions of \( \mathcal{S} \) simultaneously, \( \mathcal{L} \) applies them sequentially. Theorem 5.2 shows that at time \( t \), there is still some remnant of the original state at least for relative entropy. Hence we may apply decay merging as though different \( \mathcal{S}_t \) occurred simultaneously.

**Proof of Theorem 3.4.** The first part of the proof is given by the extension of Lemma C.4 via Equation (28), from which we obtain that

\[
D(\Phi^t(\rho) \| R_{\exp(-iHt)} \mathcal{E}(\rho)) \leq D(\rho \| \mathcal{E}(\rho)) - \gamma \lambda \int_0^t D(\Phi^r(\rho) \| \mathcal{E}_0 \Phi^r(\rho)) dr .
\]

From Theorem 5.2

\[
\int_0^t D(\Phi^r(\rho) \| \mathcal{E}_0 \Phi^r(\rho)) dr \geq \int_0^t e^{-c\|S\|\phi r} D(\rho \| \mathcal{E}_r(\rho)) dr .
\]

Lemma C.11 for \( m \) copies of the above completes the quantitative part of the Theorem.

The existence of constants satisfying the conditions of the Theorem follows [31, Remark 1.10], in which it was shown that, within finite dimensions, a composition of conditional expectations to high enough power eventually reaches a convex combination involving the overall fixed point. Via the semigroup property, we know that the fixed point of \( \exp(-\tau \mathcal{L}) \) is the same fixed point as for the entire semigroup. \( \square \)

The above proof and Theorem 3.4 imply that for any \( \tau > 0 \), \( \exp(-\tau \mathcal{L}) \) shows some relative entropy decay with respect to the fixed point of the overall semigroup. Examining the primary argument in terms of continuous quasi-factorization, the set of conditional expectations \( \{ \mathcal{E}_t : 0 \leq t \leq \tau \} \) suffices to obtain quasi-factorization, and a sequence of these conditional expectations ultimately decays to the overall fixed point. One may wonder if it is possible to extrapolate to \( \tau = 0 \), finding CMLSI after all. This fails not because there is some finite time lacking decay, but because the constant may approach zero too quickly in this limit. While \( \{ \mathcal{E}_t : 0 \leq t \leq \tau \} \) contains enough conditional restrictions to reach the overall fixed point of the semigroup, it is not necessarily efficient. Indeed, at small \( \tau \), the projectors are extremely close together and generally have a highly unfavorable quasi-factorization constant. Nonetheless, they contain enough information about the semigroup to fully distinguish its ultimate fixed point assuming one exists.

**Appendix D. Experimental Details**

Experiments were run on the *ibmq_lagos* through Qiskit. To minimize shot noise, 32000 shots were used per circuit. A single-qubit process tomography uses 12 circuits, each with a distinct combination of preparation and post-processing gates. Tomography circuits were generated automatically using Qiskit’s “process_tomography_circuits” subroutine and fit using Qiskit’s “ProcessTomographyFitter.” At the time of running, the two auxiliary qubits respectively had reset times of 0.99\( \mu s \) and 1.00\( \mu s \). The auxiliary qubit had \( T_1 \) of 156\( \mu s \) and \( T_2 \) of 158\( \mu s \). The \( \text{CX} \) gate from \( A \) to \( B \) had reported error 0.0076, which one may interpret as one minus the fidelity as determined by IBM’s randomized benchmarking [43]. Though the full \( \text{CX} \) gate would take
256ns, the pulsed $\Phi_{ZX(\pi/(2k))}$ was slightly shorter, taking 214ns for its fully entangling version and 155ns when $k > 1$. For larger values of $k$, smaller $XZ$ rotations were applied by reducing the pulse amplitude using Qiskit’s RZXCalibrationBuilder based on techniques of [44]. The RZX form of interaction was chosen because of its relation to commonly used gates on this computing platform.

![Figure 5.](image)

Figure 5. Plots of raw metrics for qubit $B$ undergoing the channel described by Equation (5). (1) Fidelity of induced channel’s Choi matrix with the identity process. (2) Fidelity with the long-time fixed point.

Raw metrics are shown in Figure 5. To better represent and understand the actual channels observed, we use a similar method as the $D^4$ model in [23]. While the intended decoherence is $X$-basis dephasing as inferred in Subsection 5.1, unintended decoherence commonly appears as a combination of depolarizing, amplitude damping, dephasing, and coherent phase drift. Unlike in [23], here we study single channels rather than repeated composition of the same channel, so we do not use the “$t$” parameter considered therein. We make the simplifying assumption that noise is applied simultaneously as in a continuous semigroup - this does not constrain the parameter range of the model but resolves the ambiguity due to non-commutativity of amplitude damping with depolarizing noise and $X$-basis decoherence. We add continuous $X$-basis dephasing to the original model. We denote by $\epsilon$ a depolarizing parameter, by $\eta$ a $Z$-basis amplitude damping parameter, by $\delta$ a $Z$-basis dephasing parameter, by $\theta$ a phase drift angle, and by $\chi$ the same $X$-basis dephasing parameter as in Equation (9).

For small values of $k$, there is an initial increase in fidelity of $\Phi(k)$ with the identity channel on $B$ shown in Figure 5. Nonetheless, fidelity quickly begins to drop. Experimental results then begin to diverge from noiseless simulation. Though the pulse-based $\Phi_{ZX(\pi/(2k))}$ yields improved performance compared with the default mapping to native gates, it does not decline as $1/k$ but enters a regime of diminishing returns (see [44] for details). Furthermore, each application of $\mathcal{E}_0$ requires approximately $1\mu s$ waiting for the auxiliary qubits to reset. Based on the reported $T_1$ and $T_2$ for qubit $B$, one may reasonably estimate that each application of $\mathcal{E}_0 \circ \Phi_{ZX(\pi/(2k))}$ induces on the order of $1 - 2\%$ infidelity via passive noise. The effects of unintended hardware noise increase with $k$, contrasting the protective scaling of the Zeno effect. For $k > 5$, unintended
noise appears to dominate. Not only does fidelity of $\Phi(k)$ with the identity channel quickly begin to decrease, but fidelity with the long-time fixed point (in which $B$ is fully dephased in the $X$ basis) also drops quickly.

A channel is fully characterized by its Choi matrix, the result of applying the channel to one half of a maximally entangled pair. The Choi matrix of an identity channel is given by the density matrix of $(|00\rangle + |11\rangle)/\sqrt{2}$, a maximally entangled state in which the output and reference mirror each other. To infer noise parameters from a Choi matrix, we solve for specific elements of the Choi matrix under modeled noise in the computational basis. In particular, letting $M_{j,l}$ denote the $j,l$th entry,

$$M_{11} = \left( \frac{1}{2} - \frac{\epsilon + 2\eta + \chi}{4(\epsilon + \eta + \chi)} \right) e^{-(\epsilon + \eta + \chi)} + \frac{\epsilon + 2\eta + \chi}{4(\epsilon + \eta + \chi)},$$

$$M_{14} = \left( \frac{1}{2} - \frac{\epsilon + \chi}{4(\epsilon + \eta + \chi)} \right) e^{-(\epsilon + \eta + \chi)} + \frac{\epsilon + \chi}{4(\epsilon + \eta + \chi)}.$$

Because the channel only touches one half of the Bell pair, we may assume that $M_{22} = 1/2 - M_{11}$ and that $M_{33} = 1/2 - M_{44}$. For off-diagonal elements,

$$|M_{14}| + |M_{23}| = \frac{1}{2} e^{-2(\eta/2 + \epsilon + \delta)}$$

$$|M_{14}| - |M_{23}| = \frac{1}{2} e^{-2(\eta/2 + \epsilon + \delta + \chi)} .$$

These 4 matrix elements suffice to fully define the 4 parameters $\epsilon, \eta, \delta$, and $\chi$. In practice, we find simple a simple formula for $\chi$ as in (9), which allows us to extract this parameter immediately. We then use Scipy’s “scipy.optimize.minimize” subroutine to solve for $\epsilon$ and $\eta$, after which we can easily solve for $\delta$ in terms of $|M_{14}| + |M_{23}|$. We find $\theta$ independently as the phase of $M_{14}$. These inferred parameters uniquely determine $M_{11}, M_{22}, M_{12}, M_{14}, M_{23}, M_{41}, M_{32}, M_{33},$ and $M_{44}$. In this model, we assume that other elements are zero.

Figure 6 shows unintended noise parameters over time. While coherent phase drift is non-trivial, this should not have a substantial effect on inferred $X$-basis dephasing. Otherwise, the dominant noise contribution is from $Z$-basis dephasing, which by reducing the magnitude of off-diagonal elements may reduce the precision of the inferred $\chi$ in Equation (9). Dominance of $Z$-basis dephasing as unintended noise is consistent with passive decoherence during resets.
but this explanation is not consistent with the similarity in reported $T_1$ and $T_2$ noise and lack of substantial amplitude damping contribution. Dephasing noise appears to peak at 16 steps, the same point where $\chi$ becomes negative (and is set to 0 in further parameter inference), while other parameters show spikes at this point. This observation is consistent with the explanation that $\chi$ becomes negative due to uncertainty in the ratio of the sum and difference in Equation (36) when both have small values. It also appear that the pulsed interactions might drive dephasing noise or suppress other kinds of noise.

A qubit undergoing dephasing noise in two bases also is effectively depolarized. Though complete depolarization is indistinguishable from complete dephasing in both of two mutually unbiased bases, the partial versions of these channels do allow one to distinguish noise contributions via the ratio of each dephased contribution to the depolarized portion. In Figure 6, the depolarizing parameter corresponds to that left over after accounting for both kinds of dephasing. Here we see evidence that in this case, depolarizing noise arises more as a consequence of dephasing in two bases than via direct replacement of the state by complete mixture.

Finally, we arrive at the culmination of this analysis in Figure 7. First, Figure 7(1) shows the process fidelity of the inferred model with the channel tomography. Since observed fidelities are at least 95% with a mean of 99%, the noise model does not lose much information about the state. Figure 7(2) shows the inferred $\chi$ parameter, which is used to reconstruct the cleaned data for Figure 3.

Figure 7. (1) Process fidelity of inferred model’s reconstructed channel with the observed channel. (2) Inferred dephasing parameter $\chi$ as in Equation (9).