THE EXISTENCE OF A SMOOTH INTERFACE IN THE EVOLUTIONARY ELLIPTIC MUSKAT–VERIGIN PROBLEM WITH NONLINEAR SOURCE.

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(This is an English translation of the article: S.P. Degtyarev, "The existence of a smooth interface in the Muskat-Verigin elliptic evolution problem with a nonlinear source", Ukr. Mat. Visn. 7 (2010), no. 3, 301–330, http://www.ams.org/mathscinet-getitem?mr=2809065)

Translated from Russian by V. V. Kukhtin

Abstract

We study the two-phase Muskat–Verigin free-boundary problem for elliptic equations with nonlinear sources. The existence of a smooth solution and a smooth free boundary is proved locally in time by applying the parabolic regularization of a condition on the free boundary.

Key words: Free boundary, Muskat–Verigin problem, classical solution, smooth interface.

MSC: 35R35, 35K65, 35R37, 35K60

1 Statement of the problem and the main result

The present work is devoted to the study of a boundary-value problem with unknown boundary describing the process of multidimensional nonstationary filtration of two fluids in a porous medium [18, 24, 4, 5, 16, 8] under assumption that the densities of these fluids are practically independent of the pressure and, hence, are constant values. Namely such a situation happens, for example, in the process of filtration of two unmixed noncompressible fluids (oil, water) at the displacement of a fluid by a silicate solution, etc.

A mathematical model of such a problem is the evolutionary problem with free boundary for elliptic equations.
A three-dimensional problem of such a kind without bulk sources was considered in [8]. The goal of the present work is to generalize results in [8] to the case of a space with any dimension and to take the effect of nonlinear bulk sources into account. As distinct from [8], we will study the corresponding linearized problem within the method of parabolic regularization of the condition on a free boundary. This method taken by us from [17] was applied to the given problem in [6, 7].

We note that the Muskat–Verigin problem для parabolic equations was studied in [3] in details and was analyzed in [1]. Moreover, with regard for the Gibbs–Thomson condition, the Muskat–Verigin problem for parabolic equations was considered in [25, 26, 14].

We note also that, in the case of two spatial variables (i.e., at \(N = 2\)), the problem under study was considered in [6, 7] with the use of the same regularization as in [17] and in the present work. Therefore, we will consider that the dimension of the space \(\mathbb{R}^N\) satisfies the condition \(N \geq 3\) and use the method of work [2].

Let \(\Omega\) be a doubly connected domain in \(\mathbb{R}^N\) with the boundary \(\partial \Omega = \Gamma^+ \cup \Gamma^-\), where \(\Gamma^\pm\) are smooth closed surfaces without self-intersections, \(\Gamma(\tau)\) and \(\tau \in [0, T]\) are smooth closed surfaces without self-intersections that lie between \(\Gamma^\pm\) and divide the domain \(\Omega\) into two doubly connected domains \(\Omega^\pm\). Moreover, \(\partial \Omega^\pm = \Gamma(\tau) \cup \Gamma^\pm\), and the surface \(\Gamma(0) \equiv \Gamma\) is given.

In the domains \(\Omega^\pm\), we consider the following boundary-value problem for the unknown functions \(u^\pm(y, \tau)\) and the unknown surfaces \(\Gamma(\tau)\) with the conditions

\[
L_0 u^\pm \equiv \triangle u^\pm(y, \tau) = f^\pm(u^\pm), \quad y \in \Omega^\pm, \quad (1.1)
\]

\[
u^+|_{\Gamma(\tau)} = u^-|_{\Gamma(\tau)}, \quad (1.2)
\]

\[
a^+ (\nabla u^+, \vec{N})|_{\Sigma_T} = a^- (\nabla u^-, \vec{N})|_{\Sigma_T} = m \cos(\vec{N}, \tau), \quad (1.3)
\]

\[
u^\pm|_{\Gamma_T} = g^\pm(y, \tau), \quad (1.4)
\]

\[
\Gamma(0) = \Sigma_T \cap \{t = 0\} = \Gamma, \quad (1.5)
\]

\[
u^\pm(y, 0) = u_0^\pm(y), \quad y \in \Omega^\pm. \quad (1.6)
\]

Here, \(\Omega^\pm\) are the domains into those \(\Omega\) is divided by the initial surface \(\Gamma = \Gamma(0)\), \(\Gamma_T^\pm = \Gamma^\pm \times [0, T]\), \(\Gamma_T = \Gamma \times [0, T]\), \(f^\pm(u), g^\pm(y, \tau), \text{ and } u_0^\pm(y)\) are the given functions, \(a^\pm\) and \(m\) are positive constants, \(\Sigma_T = \{(y, \tau) : \tau \in [0, T], \ y \in \Gamma(\tau)\}\) is a surface in \(\mathbb{R}^N \times [0, T]\), \(\vec{N}\) is a normal to \(\Sigma_T\) in the space \(\mathbb{R}^N \times [0, T]\) that is directed so that its projection on \(\{t = \text{const}\}\) is directed toward \(\Omega_T^+\).

We note that condition (1.3) can be presented also in the form

\[
a^+ \frac{\partial u^+}{\partial n}(\tau) = a^- \frac{\partial u^-}{\partial n}(\tau) = -mV_n, \quad (1.7)
\]
where $V_n$ is the velocity of motion of the surface $\Gamma(\tau)$ along the normal $\vec{n}(\tau)$ to $\Gamma(\tau)$ in the space $x \in \mathbb{R}^N$ directed inward $\Omega^+$. We note that $\vec{n}(0)$ is a normal to the initial surface $\Gamma$ denoted below simply by $\vec{n}$.

We denote $\Omega_T = \Omega \times (0, T)$ and $\Omega^+_T = \Omega^+ \times (0, T)$. We will use the standard Hölder spaces $H^l(\Omega)$ and $H^{l,1/2}(\Omega_T)$ with a noninteger $l > 0$ that were introduced in \[11\] with the norm

$$|u|_{\Omega_T}^{(l)} = \sum_{j=0}^{\lfloor l \rfloor} \sum_{2r+s=j} |D^r \xi^s u|^{(0)} + \sum_{2r+s=[l]} \langle D^r \xi^s u \rangle_{x, \Omega_T}^{(l-[l])} + \sum_{0<l-2r-s<2} \langle D^r \xi^s u \rangle_{x, \Omega_T}^{(\frac{l-2r-s}{2})},$$

where $|D^r \xi^s u|^{(0)} = \max_{\Omega_T} |D^r \xi^s u|$, $\langle u \rangle_{x, \Omega_T}$, and $\langle u \rangle_{t, \Omega_T}$ are the Hölder constants of the function $u(x, t)$ in $x$ and in $t$, respectively.

We also use the following Hölder spaces. Let $\alpha, \beta \in (0, 1)$. We define a semi-norm (see \[19\])

$$[u]_{\Omega_T}^{(\alpha, \beta)} = \sup_{(x,t), (y,\tau) \in \Omega_T} \frac{|u(x, t) - u(y, t) - u(x, \tau) + u(y, \tau)|}{|x - y|^\alpha |t - \tau|^\beta}, \quad \alpha, \beta \in (0, 1).$$

Then we define the spaces $E^{k+\alpha}(\Omega_T)$, $k = 0, 1, 2, 3$, with a bounded norm

$$|u|_{\Omega_T}^{(k+\alpha, \alpha)} = \max_{t \in [0, T]} |u(\cdot, t)|_{\Omega}^{(k+\alpha)} + \sum_{|\alpha|=0}^k \langle D^r_x u \rangle_{t, \Omega_T}^{(\alpha)} + \sum_{|\alpha|=0}^k \langle D^r_x u \rangle_{t, \Omega_T}^{(\alpha, \alpha)}.$$

We note that a part of terms in definition \[1.8\] can be interpolated in terms of lower and higher orders. Therefore,

$$|u|_{\Omega_T}^{(k+\alpha, \alpha)} \leq C \left( \max_{t \in [0, T]} |u(\cdot, t)|_{\Omega}^{(0)} + \langle u \rangle_{t, \Omega_T}^{(\alpha)} + \sum_{|\alpha|=0}^k [D^r_x u]_{t, \Omega_T}^{(\alpha, \alpha)} \right). \quad \text{(1.9)}$$

We note also that norm \[1.8\] is equivalent to the norm

$$|u|_{\Omega_T}^{(k+\alpha, \alpha)} \sim \max_{t \in [0, T]} |u(\cdot, t)|_{\Omega}^{(0)} + \langle u \rangle_{t, \Omega_T}^{(\alpha)} + \sum_{|\alpha|=0}^k \sup_{0<h<1} \left( \frac{D^r_x(x, t+h) - D^r_x(x, t)}{h^\alpha} \right)_{x, \Omega_T}^{(\alpha)}.$$

With the use of a local parametrization, we can define smooth surfaces from the above-indicated classes and the corresponding spaces of smooth functions on these surfaces in the standard way.

We also define a space $P^{k+\alpha}(\Gamma_T)$, $k = 2, 3$, with the norm

$$|u|_{P^{k+\alpha}(\Gamma_T)} = |u|_{E^{k+\alpha}(\Gamma_T)} + |u|_{E^{1+\alpha}(\Gamma_T)}.$$

We made the following assumptions about data of problem \[1.1\]–\[1.6\]:

$$\Gamma^\pm, \Gamma \in H^{4+\alpha}, \quad g^\pm \in H^{4+\alpha, \frac{1+\alpha}{2}}(\Gamma^\pm_T), \quad u_0^\pm(x) \in H^{4+\alpha}(\Omega^\pm), \quad \text{(1.12)}$$
\[ f^\pm(u) \in C^2_{\text{loc}}(\mathbb{R}^1), \quad (f^\pm)'(u) \geq \nu > 0. \quad (1.13) \]

In addition, we suppose that the conditions of consistency hold for problem (1.1)–(1.6) that mean that

\[ u_0^+|_{\Gamma^+} = g^+(y,0), \quad u_0^-|_{\Gamma} = u_0^-|_{\Gamma}, \quad a^+ \frac{\partial u_0^+}{\partial \mathbf{n}}|_{\Gamma} = a^- \frac{\partial u_0^-}{\partial \mathbf{n}}|_{\Gamma}, \quad \Delta u_0^\pm(y) = f^\pm(u_0^\pm(y)), \quad y \in \Omega^\pm, \quad (1.14) \]

\[ \nabla u_0^\pm(y) = f^\pm(u_0^\pm(y)), \quad y \in \Omega^\pm, \quad (1.15) \]

where \( \mathbf{n} \) is a normal to \( \Gamma \) directed toward \( \Omega^+ \). We note that condition (1.15) is, in essence, the assumption for the right-hand sides \( f^\pm(u) \) that means that the corresponding boundary-value problem for Eq. (1.15) is solvable. We also assume that the problem is nondegenerate, namely,

\[ \frac{\partial u_0^\pm}{\partial \mathbf{n}} \geq \nu > 0, \quad \frac{\partial u_0^+}{\partial \mathbf{n}} - \frac{\partial u_0^-}{\partial \mathbf{n}} \geq \nu > 0, \quad y \in \Gamma, \quad (1.16) \]

where \( \nu \) is some positive constant.

In what follows, by \( C, b, \nu, \) and \( \gamma \), we denote all absolute constants or constants depending only on once and for all fixed data of the problem.

In order to formulate the main result, we introduce a parametrization of the unknown boundary with the help of some unknown function and reduce the initial problem to a problem in a fixed domain. In so doing, in a sufficiently small neighborhood \( \mathcal{N} \) of the surface \( \Gamma \), we introduce the coordinates \( (\omega, \lambda) \), where \( \omega \) are the coordinates on the surface \( \Gamma \), \( \lambda \in \mathbb{R} \), and \( |\lambda| \leq \lambda_0 \) so that if \( x \in \mathcal{N} \), then

\[ x = x_\Gamma(\omega) + \lambda \mathbf{n}(\omega) = x(\omega, \lambda), \quad |\lambda| \leq \lambda_0 \quad (1.17) \]

in a unique way, where \( x_\Gamma(\omega) \in \Gamma \), and \( \lambda \) is a deviation of the point \( x \) from the surface \( \Gamma \) along the normal \( \mathbf{n} \) to \( \Gamma \) directed, we recall, inward \( \Omega^+ \).

Let \( \rho(\omega, t) \) be a sufficiently small-value function defined on \( \Gamma_T = \Gamma \times [0, T] \), \( \rho(\omega, 0) \equiv 0 \). Then the parametrization

\[ x = x_\Gamma(\omega) + \mathbf{n}(\omega)\rho(\omega, t) \]

at every \( t \in [0, T] \) sets some surface \( \Gamma_{\rho(t)} \) dividing the domain \( \Omega \) into two subdomains \( \Omega^+_{\rho} \) and \( \Omega^-_{\rho} \). We denote a surface in \( \Omega_T = \Omega \times [0, T] \) by \( \Gamma_{\rho,T} \equiv \bigcup_{t \in [0, T]} \Gamma_{\rho}(t) \times \{ t \} \). By \( \Omega^\pm_{\rho,T} \), we denote those domains, into those the surface \( \Gamma_{\rho,T} \) divides the domain \( \Omega_T \). We assume (and will prove it below) that the unknown surface \( S_T = \Gamma_{\rho,T} \) with some unknown function \( \rho \).

Let a function \( \chi(\lambda) \in C^\infty \) be such that \( \chi(0) = 1, 0 \leq \chi(\lambda) \leq 1, \chi(\lambda) \equiv 0 \) at \( |\lambda| \geq \lambda_0, |\lambda'| \leq 2/\lambda_0 \), where \( \lambda_0 \) is a number from relation (1.17). Let also \( \rho(\omega, t) \) be a function of the class \( S^{2+\alpha}(\Gamma_T) \) and such that \( |\rho| \leq \lambda_0/4 \). We define the mapping \((x, t) \rightarrow (y, \tau)\) of the domain \( \Omega_T \) onto itself by the formula

\[ e_\rho: \begin{cases} y = \begin{cases} x_\Gamma(\omega(x)) + \mathbf{n}(\omega(x))\lambda(x) + \chi(\lambda(x))\rho(\omega(x), t) = x + \mathbf{n}(\omega(x))\chi(\lambda(x))\rho(\omega, t), & x \in \mathcal{N}, \\ x, & x \notin \mathcal{N}, \end{cases} \\ \tau = t. \end{cases} \quad (1.18) \]
Thus, the spatial coordinates \((\omega, \lambda)\) of points \(y = y(x,t)\) and \(x\) in a neighborhood \(\mathcal{N}\), where the mapping \(e_\rho\) differs from the identity one, are connected by the relations

\[
\omega(y) = \omega(x), \quad \lambda(y) = \lambda(x) + \chi(\lambda(x))\rho(\omega(x), t).
\]

It is easy to see that the mapping \(e_\rho\) transfers bijectively the domains \(\Omega_\pm^T\) onto the domains \(\Omega_{\rho,T}^\pm\). Moreover, since \(\rho(\omega,0) \equiv 0, e_\rho(x,0) \equiv (x,0)\). For simplicity, we will denote the functions \(u^\pm\) after the change of variables by the same symbol, i.e.,

\[
u^\pm(x,t) \equiv u^\pm(y,\tau) \circ e_\rho(x,t),
\]

and \(u^\pm(x,t)\) are already defined in the known fixed domains \(\Omega_\pm^\pm\).

Let us change the variables \((y,\tau) = e_\rho(x,t)\) in problem \((1.1)-(1.6)\). We arrive at the equivalent formulation of the problem, but already in fixed domains:

\[
L_\rho u^\pm \equiv \nabla_\rho^2 u^\pm(x,t) = f^\pm(u^\pm), \quad (x,t) \in \Omega_\pm^T, \tag{1.19}
\]

\[
u^+(x,t) - u^-(x,t) = 0, \quad (x,t) \in \Gamma_T, \tag{1.20}
\]

\[
\rho_t(\omega,t) + a^\pm S(\omega,\rho,\nabla_\omega\rho) \frac{\partial u^\pm}{\partial \eta} + a^\pm \sum_{i=1}^N S_i(\omega,\rho,\nabla_\omega\rho) \frac{\partial u^\pm}{\partial \omega_i} = 0, \quad (x,t) \in \Gamma_T, \tag{1.21}
\]

\[
u^\pm(x,t) = g^\pm(x,t), \quad (x,t) \in \Gamma^\pm, \tag{1.22}
\]

\[
u^\pm(x,0) = u^\pm_0(x), \quad x \in \Omega^\pm, \tag{1.23}
\]

where \(\nabla_\rho = J_\rho \nabla\), and \(J_\rho\) is the matrix inverse and conjugate to the Jacobi matrix \(\partial y/\partial x\). Here, \(S(\omega,\rho,\nabla_\omega\rho)\) and \(S_i(\omega,\rho,\nabla_\omega\rho)\) are smooth functions of the arguments. Moreover,

\[
S(\omega,0,0) \equiv 1, \quad \frac{\partial S}{\partial \rho_{\omega_i}}(\omega,0,0) \equiv 0, \tag{1.24}
\]

and

\[
S_i(\omega,0,0) \equiv 0, \quad i = 1,2,\ldots,N. \tag{1.25}
\]

We now briefly explain the derivation of relations \((1.21)\) and properties \((1.24)\) and \((1.25)\). In the variables \((y,\tau)\) in a neighborhood \(\Gamma_T\), let

\[
\Phi_\rho(y,\tau) \equiv \lambda(y) - \rho(\omega(y), \tau), \tag{1.26}
\]

where \(\lambda(y), \omega(y)\) are the \((\omega, \lambda)\) coordinates of a point \(y\) in a neighborhood of the surface \(\Gamma\). We note that the vector \(\nabla_{(y,\tau)}\Phi_\rho\) is directed toward the domain \(\Omega_{\rho,T}^+\). Thus, the normal \(\vec{N}\) in condition \((1.3)\) is

\[
\vec{N} = \nabla_{(y,\tau)}\Phi_\rho / |\nabla_{(y,\tau)}\Phi_\rho|. \tag{1.27}
\]
Thus, condition (1.3) can be rewritten in the form

\[ a^\pm (\nabla_y \Phi_\rho, \nabla_y u^\pm) = m \frac{\partial \Phi_\rho}{\partial \tau} = -m \rho_x (\omega(y), \tau). \quad (1.28) \]

In relation (1.28), we pass to the variables \((x, t)\) in correspondence with the change \((x, t) = e_\rho(x, t)\), by taking into account that \(\Phi_\rho(x, t) \equiv \lambda(x)\) in the variables \((x, t)\) in a neighborhood of \(\Gamma\), since \(\omega(y) = \omega(x)\) and \(\lambda(y) = \lambda(x) + \rho(\omega(x), t)\). Thus, relation (1.28) takes the form

\[ a^\pm (J_\rho \nabla_x \lambda(x), J_\rho \nabla_x u^\pm) = -m \rho_t (\omega(x), t), \quad (1.29) \]

or

\[ a^\pm (J_\rho^* J_\rho \nabla_x \lambda(x), \nabla_x u^\pm) + m \rho_t (\omega(x), t) = 0. \quad (1.30) \]

Let else \(\nabla_x = R \nabla_{(\omega, \lambda)}\), where \(R\) is a Jacobi orthogonal matrix of the transition \(x \to (\omega, \lambda)\) between the coordinates in a neighborhood of \(\Gamma\). Since \(\nabla_x \lambda(x) = \overrightarrow{n}(\omega(x))\), relation (1.30) can be written in the form

\[ a^\pm (R^* J_\rho^* J_\rho R \overrightarrow{n}, \nabla_{(\omega, \lambda)} u^\pm) + \rho_t (\omega, t) = 0. \quad (1.31) \]

In other words, due to the smoothness of the surface \(\Gamma\) and, respectively, the smoothness of the change \(x \to (\omega, \lambda)\), relation (1.31) is a relation of the form (1.21) with some smooth functions \(S\) and \(S_i\), because

\[ \frac{\partial u^\pm}{\partial \lambda} = \frac{\partial u^\pm}{\partial \overrightarrow{n}}. \]

In this case, at \(\rho = 0\) and \(\rho_\omega = 0\), the matrix \(J_\rho \equiv I\), i.e., relation (1.31) takes the form (since \(R^* R = R^{-1} R = I\))

\[ a^\pm \frac{\partial u^\pm}{\partial \overrightarrow{n}} + \rho_t (\omega, t) = 0, \quad (1.32) \]

which yields properties (1.24) and (1.25).

Below, we formulate the main result.

**Theorem 1.1** Let conditions (1.12)–(1.15) and (1.16) be satisfied. Then there exists \(T > 0\) such that problem (1.19)–(1.23) (and, hence, problem (1.1)–(1.6)) has the unique smooth solution at \(t \in [0, T]\), and

\[ |u^\pm|_{L^{2+\alpha}(\Omega_T^\pm)} + |\rho|_{L^{2+\alpha}(\Gamma_T)} \leq C(u_0^\pm, g^\pm, \Gamma, \Gamma^\pm). \quad (1.33) \]

The subsequent sections are devoted to the proof of Theorem 1.1.
2 Linearization of problem (1.19)–(1.23)

To prove Theorem 1.1, we use the method developed in \cite{8,2} that presents, in essence, a version of the Newton method for the solution of nonlinear equations.

First, we construct the “initial approximation” to the solution of the nonlinear problem (1.19)–(1.23). By \( \sigma(\omega, t) \), we denote a function of the class \( H_{4+\alpha, 4+\alpha/2}(\Gamma_T) \) such that

\[
\sigma(\omega, 0) = \rho(\omega, 0) = 0, \quad \frac{\partial \sigma}{\partial t}(\omega, 0) = \rho_t(\omega, 0) \equiv \rho^{(1)}(\omega) = a^+ \frac{\partial u^+_0}{\partial n}.
\] (2.1)

We now continue the functions \( u^+_0 \) through the surface \( \Gamma \) onto the whole domain \( \Omega \) and construct the functions \( w^\pm(x, t) \in H_{4+\alpha, 4+\alpha/2}(\Omega_T) \) such that

\[
w^\pm(x, 0) = u^+_0(x), \quad x \in \Omega,
\] (2.2)

\[
\frac{\partial w^\pm}{\partial t}(x, 0)|_\Gamma = - \frac{\partial u^+_0}{\partial n}(x)|_\Gamma \rho_t(\omega(x), 0) = - \frac{\partial u^+_0}{\partial n}(x)|_\Gamma \rho^{(1)}(\omega(x)).
\] (2.3)

The procedure of construction of such functions is described in \cite{11}. We note that condition (2.3) implies that the complicated functions \( w^\pm \sigma \) satisfy the relation

\[
\frac{\partial (w^\pm \circ e_\sigma)}{\partial t}(x, 0)|_\Gamma = \frac{\partial w^\pm}{\partial t}(x, 0) + \frac{\partial w^\pm}{\partial n}(x, 0)|_\Gamma \sigma_t(\omega, 0) = 0.
\] (2.4)

It will be used in what follows and yields, for example, \( \partial F_3/\partial t(x, 0) = 0 \) in (2.8) below. (We note that we could require that a single condition,

\[
\frac{\partial w^+}{\partial t}(x, 0)|_\Gamma - \frac{\partial w^-}{\partial t}(x, 0)|_\Gamma = - \left( \frac{\partial u^+_0}{\partial n}(x)|_\Gamma - \frac{\partial u^-_0}{\partial n}(x)|_\Gamma \right) \rho^{(1)},
\]

be satisfied instead of conditions (2.3), which would also give the necessary result.)

We note that the principal linear part of the mapping \( \delta \in P^{2+\alpha}(\Gamma_T) \rightarrow w^\pm \circ e_{\sigma+\delta} \in P^{2+\alpha}(\Omega_T) \) is

\[
\lim_{\varepsilon \to 0} \frac{w^\pm \circ e_{\sigma+\delta} - w^\pm \circ e_\sigma}{\varepsilon} = \left( \frac{\partial w^\pm}{\partial \lambda} \circ e_\sigma \right) \chi(\lambda(x)) \delta(\omega(x), t) \equiv b^\pm(x, t) \delta.
\]

We denote

\[
\delta(\omega, t) = \rho(\omega, t) - \sigma(\omega, t), \quad v^\pm(x, t) = u^\pm(x, t) - w^\pm \circ e_\sigma - b^\pm \delta.
\] (2.5)

Since any function \( f(x, t) \) satisfies the relation

\[
(L_0 f) \circ e_\rho = L_\rho(f \circ e_\rho),
\] (2.6)
we present relations (1.19)—(1.23) in the form

\[ \Delta v^\pm(x, t) - (f^\pm)'(w^\pm \circ e_\sigma) v^\pm \]

\[ = \left\{ \left( L^\pm_0 - L^\pm_\sigma \right) v^\pm(x, t) + \left[ f^\pm(w^\pm \circ e_\sigma + v^\pm + b^\pm \delta) - (f^\pm)'(w^\pm \circ e_\sigma) \right] v^\pm \right\} 
\]

\[ + (f^\pm)'(w^\pm \circ e_\sigma) b^\pm \delta + [f^\pm(w^\pm \circ e_\sigma) - (L^\pm_0 w^\pm) \circ e_{\sigma+\delta}] \]

\[ + \left\{ \left( L^\pm_\sigma \left( w^\pm \circ e_{\sigma+\delta} - w^\pm \circ e_\sigma \right) - \left( \frac{\partial w^\pm}{\partial \lambda} \circ e_\sigma \right) \chi(\lambda) \delta \right) \right. \]

\[ - (L^\pm_{\sigma+\delta} - L^\pm_\sigma) \left( v^\pm + \left( \frac{\partial w^\pm}{\partial \lambda} \circ e_\sigma \right) \chi(\lambda) \delta \right) \]

\[ + (L^\pm_{\sigma+\delta} - L^\pm_\sigma)(w^\pm \circ e_{\sigma+\delta} - w^\pm \circ e_\sigma) \right\} \]

\[ \equiv F^+_3(x, t; v^\pm, \delta) + F^+_2(x, t; v^\pm, \delta), \quad (x, t) \in \Omega_T^\pm, \quad (2.7) \]

\[ v^+ - v^- + \left( \frac{\partial w^+}{\partial n} - \frac{\partial w^-}{\partial n} \right) \circ e_\sigma \delta = w^+ \circ e_\sigma - w^- \circ e_\sigma \equiv F_3(x, t), \quad (x, t) \in \Gamma_T, \quad (2.8) \]

\[ \delta_t + a^+ \frac{\partial v^+}{\partial n} + \sum_{i=1}^{N-1} \left( a^\pm \frac{\partial S_i}{\partial \rho \omega_i} (\omega, \sigma, \nabla_\omega \sigma) \frac{\partial w^\pm \circ e_\sigma}{\partial \omega_i} \right) \delta_{\omega_i} = \left\{ a^\pm \left[ 1 - S(\omega, \sigma, \nabla_\omega \sigma) \right] \frac{\partial v^\pm}{\partial n} \right\} \]

\[ - \left[ \sigma_t + a^\pm S(\omega, \sigma, \nabla_\omega \sigma) \frac{\partial w^\pm \circ e_\sigma}{\partial \omega_i} + \sum_{i=1}^{N-1} a^\pm S_i(\omega, \sigma, \nabla_\omega \sigma) \frac{\partial w^\pm \circ e_\sigma}{\partial \omega_i} \right] \]

\[ - S(\omega, \sigma + \delta, \nabla_\omega \sigma + \nabla_\omega \delta) \frac{\partial^2 w^\pm}{\partial n \partial \omega_i} - \sum_{i=1}^{N-1} \left( a^\pm S_i(\omega, \sigma, \nabla_\omega \sigma) \frac{\partial v^\pm}{\partial \omega_i} - a^\pm \sum_{i=1}^{N-1} S_i(\omega, \sigma, \nabla_\omega \sigma) \frac{\partial^2 w^\pm}{\partial n \partial \omega_i} \right) \]

\[ + \left\{ - a^\pm S(\omega, \sigma + \delta, \nabla_\omega \sigma + \nabla_\omega \delta) - S(\omega, \sigma, \nabla_\omega \sigma) \right\} \left[ \frac{\partial v^\pm}{\partial n} + \frac{\partial^2 w^\pm \circ e_\sigma}{\partial n^2 \delta} \right] \]

\[ - a^\pm \sum_{i=1}^{N-1} \left[ S_i(\omega, \sigma + \delta, \nabla_\omega \sigma + \nabla_\omega \delta) \right. \]

\[ - \sum_{j=1}^{N-1} \frac{\partial S_i}{\partial \rho \omega_j} (\omega, \sigma, \nabla_\omega \sigma) \delta_{\omega_j} - S_i(\omega, \sigma, \nabla_\omega \sigma) \left( \frac{\partial v^\pm}{\partial \omega_i} + \frac{\partial w^\pm \circ e_\sigma}{\partial \omega_i} + \frac{\partial^2 w^\pm \circ e_\sigma}{\partial n \partial \omega_i} \right) \]

\[ - \sum_{i,j=1}^{N-1} \frac{\partial S_i}{\partial \rho \omega_j} (\omega, \sigma, \nabla_\omega \sigma) \delta_{\omega_j} \left( \frac{\partial v^\pm}{\partial \omega_i} + \frac{\partial^2 w^\pm \circ e_\sigma}{\partial n \partial \omega_i} \right) \right\} \]

\[ \equiv F^+_4(x, t, v^\pm, \delta) + F^+_5(x, t, v^\pm, \delta), \quad (2.9) \]

\[ v^\pm(x, t) = g^\pm(x, t) - w^\pm \circ e_\sigma \equiv F^+_6(x, t), \quad (x, t) \in \Gamma_T^\pm, \quad (2.10) \]

\[ v^\pm(x, 0) = 0, \quad x \in \overline{\Omega_T^\pm}, \quad \delta(\omega, 0) = 0, \quad (2.11) \]
where we took into account that $\chi(\lambda) \equiv 1$ in a neighborhood of $\Gamma$, and the mappings $e_\rho$ and $e_\sigma$ are the identity ones outside of some neighborhood of $\Gamma$. Moreover, the procedure of construction of the functions $w^\pm$ and $\sigma$ implies that we search for actually the functions $v^\pm$ and $\delta$ such that, additionally to (2.11),

$$v^\pm \in \dot{E}^{2+\alpha}(\Omega^\pm_T), \quad \delta \in \dot{P}^{2+\alpha}(\Gamma_T),$$

where the dot above the symbol of a space means a subspace consisting of functions vanishing at $t = 0$ together with all derivatives with respect to $t$ that are admitted by the class. Of basic importance is the circumstance that such classes with a dot satisfy relations analogous to those for the classes $\dot{H}^{1/2}$. Namely, if $u, v \in \dot{H}^{1/2}$, then

$$|u|_{H^{l',l'/2}((\Omega_T^+ \cup \Omega_T^-) \times \Gamma_T)} \leq CT^{\frac{l-l'}{2}} |u|_{H^{l,l/2}(\Omega_T^+ \cup \Omega_T^-) \times \Gamma_T},$$

$$|u|_{S^{l'}}(\Omega_T^+ \cup \Omega_T^-) \leq CT^{\frac{l-l'}{2}} |u|_{S^l(\Omega_T^+ \cup \Omega_T^-) \times \Gamma_T},$$

$$|u|_{E^{2+\alpha'}}(\Omega_T^+ \cup \Omega_T^-) \leq CT^{\frac{\alpha-\alpha'}{2}} |u|_{E^{2+\alpha}(\Omega_T^+ \cup \Omega_T^-) \times \Gamma_T},$$

$$|u|_{P^{2+\alpha'}}(\Gamma_T) \leq CT^{\frac{\alpha-\alpha'}{2}} |u|_{P^{2+\alpha}(\Gamma_T)}$$

with $\alpha' < \alpha$.

$$|uv|_{H^{l,l/2}((\Omega_T^+ \cup \Omega_T^-) \times \Gamma_T)} \leq CT^{\frac{l}{2}} |u|_{H^{l,l/2}(\Omega_T^+ \cup \Omega_T^-) \times \Gamma_T} |v|_{H^{l,l/2}(\Omega_T^+ \cup \Omega_T^-) \times \Gamma_T}. \tag{2.12}$$

It is easy to see that the right-hand sides $F_1 - F_6$ of relations (2.7)–(2.11) also vanish at $t = 0$ and, thus, belong to the classes with a dot.

The sense of relations (2.7)–(2.11) consists in the separation of the principal part in the nonlinear relations (2.23)–(2.27) that is linear in $v^\pm$ and $\delta$. In this case, all “free terms” (possessing a high smoothness) and “quadratic” terms are transferred to the right-hand side. Then, using directly the definition of the functions $F_i$ on right-hand sides of (2.7)–(2.11) and considering separately each term, it is easy to verify the validity of the following proposition. We denote

$$\mathcal{H} = \dot{E}^{2+\alpha}(\Omega_T^+ \cup \Omega_T^-) \times \dot{E}^{2+\alpha}(\Omega_T^+ \cup \Omega_T^-) \times \dot{P}^{2+\alpha}(\Gamma_T), \quad \psi = (v^+, v^-, \delta) \in \mathcal{H},$$

$$\|\psi\| = |v^+|_{E^{2+\alpha}(\Omega_T^+ \cup \Omega_T^-)} + |v^-|_{E^{2+\alpha}(\Omega_T^+ \cup \Omega_T^-)} + |\delta|_{P^{2+\alpha}(\Gamma_T)} \tag{2.13}$$

and will consider the functions $F_i$ as functions of $\psi$.

**Lemma 2.1** Let $\psi, \psi_1, \psi_2 \in \mathcal{H}$. Then

$$\left| F_i^\pm(x, t; \psi) \right|_{E^\alpha(\Omega_T^\pm)} \leq C(1 + \|\psi\|)T^{\alpha/2}, \tag{2.14}$$
where the constants to conditions on the free boundary are supplemented by the regularizing term \( \varepsilon \) with a small factor 3.

We now consider the problem of the determination of the unknown functions linear problem that is set by the left-hand sides of relations (2.7)–(2.11). Such a

\[
\begin{align*}
|F_1^\pm(x, t; \psi_2) - F_1^\pm(x, t; \psi_1)|_{E^0(\mathbb{P}^2_+)} C(||\psi_1||) T^{\alpha/2} ||\psi_2 - \psi_1||, \\
|F_2^\pm(x, t; \psi)|_{E^0(\mathbb{P}^2_+)} \leq C(||\psi||) ||\psi||^2, \\
|F_3^\pm(x, t; \psi_1)|_{E^{2\alpha}(\mathbb{P}^2_+)} \leq CT^{\alpha/2}, \\
|F_4^\pm(x, t)|_{E^{1+\alpha}(\Gamma_T)} \leq CT^{\alpha/2}, \\
|F_5^\pm(x, t; \psi_1)|_{E^{1+\alpha}(\Gamma_T)} \leq C(||\psi||) ||\psi||^2, \\
|F_6^\pm(x, t; \psi_1)|_{E^{1+\alpha}(\Gamma_T)} \leq C(||\psi||) ||\psi||^2, \\
|F_7^\pm(x, t; \psi_1)|_{E^{2\alpha}(\Gamma_T)} \leq CT^{\alpha/2},
\end{align*}
\]

where the constants \( C(||\psi||) \) remain bounded at bounded \(||\psi||\).

We note that, while verifying inequality (2.14)–(2.23), it is necessary to consider also relations (1.24) and (1.25).

3 Model problem corresponding to problem (2.7)–(2.11)

In this section, we consider a simple problem corresponding to the essence of the linear problem that is set by the left-hand sides of relations (2.7)–(2.11). Such a problem follows from problem (2.7)–(2.11) by fixing the coefficients on the left-hand sides of (2.7)–(2.11) at some point on the boundary \( \Gamma \) at \( t = 0 \) and by a local straightening of the surface \( \Gamma \). In addition, the boundary conditions corresponding to conditions on the free boundary are supplemented by the regularizing term with a small factor \( \varepsilon > 0 \) so as it was made in [17, 6, 7].

Let \( \mathbb{R}^N_+ = \{ x \in \mathbb{R}^N : +x_N \geq 0 \} \), \( \mathbb{R}^N_{\pm,T} = \mathbb{R}^N_+ \times [0, T] \), \( x' = (x_1, x_2, \ldots, x_{N-1}) \).

We now consider the problem of the determination of the unknown functions \( u^\pm(x, t) \) that are set on \( \mathbb{R}^N_{\pm,\infty} = \mathbb{R}^N_+ \times [0, \infty) \), respectively, and the unknown function \( \rho(x', t) \) set on \( \mathbb{R}^{N-1}_N = (\mathbb{R}^N \times [0, \infty)) \cap \{ x_N = 0 \} \) by the conditions

\[
\begin{align*}
- \Delta u^\pm = f^\pm_1(x, t) = f^\pm_1(x, t), \quad (x, t) \in \mathbb{R}^N_{\pm,\infty}, \\
u^+(x, t) - u^-(x, t) + A\rho(x', t) = f^\pm_2(x', t), \quad x_N = 0, \quad t \geq 0, \\
\rho_\ast(x', t) - \varepsilon \Delta x' \rho + a^\pm \frac{\partial u^\pm}{\partial x_N} + \sum_{i=1}^{N-1} h^\pm_i \rho_{x_i} = f^\pm_3(x', t), \quad x_N = 0, \quad t \geq 0,
\end{align*}
\]

\[
\begin{align*}
u^\pm(x, 0) = 0, \quad x \in \mathbb{R}^N_{\pm}, \quad \rho(x', 0) = 0.
\end{align*}
\]
Here, \( f_1^\pm, f_2, \) and \( f_3^\pm \) are finite functions,
\[
f_1^\pm \in \dot{E}^\alpha(\mathbb{R}^N_{+,\infty}), \quad f_2 \in \dot{E}^{2+\alpha}(\mathbb{R}^{N-1}_\infty), \quad f_3^\pm \in \dot{E}^{1+\alpha}(\mathbb{R}^{N-1}_\infty),
\] (3.5)
\( \varepsilon > 0 \) is a small fixed positive constant, and \( A, a^\pm, \) and \( h^\pm \) are given positive constants.

The following proposition is valid.

**Theorem 3.1** Let \( \varepsilon \in (0, 1] \), and let condition (3.5) be satisfied. Then, for any finite solution \((u^+, u^-, \rho)\) of problem (3.1)–(3.4), the estimates
\[
|u^+|_{E^{2+\alpha}(\mathbb{R}^N_{+,T})} + |u^-|_{E^{2+\alpha}(\mathbb{R}^N_{+,T})} + |\rho|_{p^{2+\alpha}(\mathbb{R}^{N}_T)} + \varepsilon |\rho|_{p^{1+\alpha}(\mathbb{R}^{N-1}_T)} \\
\leq C( |f_1^+|_{\dot{E}^\alpha(\mathbb{R}^N_{+,T})} + |f_1^-|_{\dot{E}^\alpha(\mathbb{R}^N_{+,T})} + |f_2|_{E^{2+\alpha}(\mathbb{R}^{N-1}_T)} \\
+ |f_3^+|_{\dot{E}^{1+\alpha}(\mathbb{R}^{N-1}_T)} + |f_3^-|_{\dot{E}^{1+\alpha}(\mathbb{R}^{N-1}_T)}),
\] (3.6)
are true, and the constant \( C \) in (3.6) is independent of \( \varepsilon \).

The subsequent content of this section is devoted to the proof of Theorem 3.1. In this case, we consider the continuation of all functions into the region \( t < 0 \) to be zero since all these functions belong to classes with a dot (i.e., they vanish at \( t = 0 \) together with their derivatives).

The proof of Theorem 3.1 is preceded by the following lemma. We now consider the boundary-value problem in the half-space \( \mathbb{R}^N_{+,T} \) with a parameter \( \varepsilon > 0 \):
\[
- \Delta u = f(z, t), \quad (z, t) \in \mathbb{R}^N_{+,T},
\] (3.7)
\[
\frac{\partial u}{\partial z_N}(z', 0, t) = F(z', t), \quad (z', t) \in \mathbb{R}^{N-1}_T,
\] (3.8)
\[
u(z, 0) = 0, \quad u(z, t) \in \dot{E}^{2+\alpha}(\mathbb{R}^N_{+,T});
\] (3.9)
where \( f \in \dot{E}^\alpha(\mathbb{R}^N_{+,T}), \) \( F \in \dot{E}^{1+\alpha}(\mathbb{R}^{N-1}_T). \) Moreover, \( f \) and \( F \) are finite in \( z \), or they decrease sufficiently rapidly at infinity.

**Lemma 3.2** Problem (3.7)–(3.9) has the unique smooth solution bounded at infinity that satisfies the estimate
\[
|u|_{E^{2+\alpha}(\mathbb{R}^N_{+,T})} \leq C( |f|_{\dot{E}^\alpha(\mathbb{R}^N_{+,T})} + |F|_{\dot{E}^{1+\alpha}(\mathbb{R}^{N-1}_T)}).
\] (3.10)

**Proof.** We omit a detailed proof of this lemma, since it well known in the case where the spaces \( H^{k+\alpha} \) are used instead of the spaces \( E^{k+\alpha} \) and consists in the well-known estimates of the potential of a simple layer (see, e.g., [12] and references therein). In order to prove the lemma for the spaces \( E^{k+\alpha} \), it is sufficient to note that the function
\[
u_h(x, t) = \frac{u(x, t) - u(x, t - h)}{h^\alpha}, \quad h \in (0, 1),
\]
satisfies problem (3.7)–(3.9) with the change of the appropriate functions on the right-hand side of the equation and in the boundary condition by the functions

\[
f_h(x, t) = f(x, t) - f(x, t - h), \quad F_h(x, t) = F(x, t) - F(x, t - h) .
\]

In this case, by virtue of the assumption about data of the problem, \( f_h \in H^\alpha(R^N_T), F_h \in H^{1+\alpha}(R^N_T) \) uniformly in \( t \) and \( h \). Therefore,

\[
\sup_{h} \max_{t} |u_h|^{(2+\alpha)} \leq C \left( |f|_{E^\alpha(R^N_T)} + |F|_{E^{1+\alpha}(R^N_T)} \right).
\]

This yields, with regard for the definition of the spaces \( E^{k+\alpha} \), the assertion of the lemma.

An analogous proposition is valid also for problem (3.7)–(3.9) if the Neumann condition (3.8) is replaced by the Dirichlet condition

\[
u(x', 0, t) = F(x', t), \quad (x', t) \in R^N_T .
\]

Lemma 3.3 In (3.11), let \( F \in E^{2+\alpha}(R^N_T) \), and let \( F \) be finite in \( z \) or sufficiently rapidly decrease at infinity. Then problem (3.7), (3.11), (3.9) has the unique smooth solution bounded at infinity, for which the following estimate is true:

\[
|u|_{E^{2+\alpha}(R^N_T)} \leq C \left( |f|_{E^\alpha(R^N_T)} + |F|_{E^{2+\alpha}(R^N_T)} \right).
\]

The proof of this lemma is identical to that of the previous lemma.

Lemmas 3.2 and 3.3 allow us, without any loss of generality, to consider \( f_1 \equiv 0, f_2 \equiv 0 \), and \( f_3 \equiv 0 \) in problem (3.1)–(3.4), so that only the function \( f_3 \) is nonzero.

To problem (3.1)–(3.4), we now apply the Laplace transformation with respect to the variable \( t \) and the Fourier transformation with respect to the variables \( x' \). We denote the result of such a transformation of the function \( f(x', t) \) by \( \tilde{f}(\xi, p) \), i.e.,

\[
\tilde{f}(\xi, p) = C \int_0^\infty e^{-pt} dt \int e^{-ix' \xi} f(x, t) dx.
\]

As a result, problem (3.1)–(3.4) is reduced to a boundary-value problem for ordinary differential equations and takes the form (we denote \( h^\pm = (h^1_1, h^2_2, \ldots, h^N_{N-1}) \))

\[
\begin{align*}
\frac{d^2 \bar{u}^\pm}{dx_N^2} - \xi^2 \bar{u}^\pm &= 0, \quad x_N > 0 \ (x_N < 0), \\
\bar{u}^+ - \bar{u}^- + A \bar{\rho} &= 0, \quad x_N = 0, \\
\bar{\rho}(p + \varepsilon \xi^2 - ih^+ \xi) + a^+ \frac{d \bar{u}^+}{dx_N} &= 0, \quad x_N = 0,
\end{align*}
\]

(3.14) (3.15) (3.16)
\[ \tilde{\rho}(p + \varepsilon\xi^2 - ih^- \xi) + a^- \frac{d\tilde{u}^-}{dx_N} = \tilde{f}_3^- , \quad x_N = 0. \] (3.17)

The condition additional to relations (3.14)–(3.17) is the condition of boundedness at infinity, i.e.,

\[ |\tilde{u}^\pm| \leq C, \quad x_N \to \pm\infty. \] (3.18)

With regard for condition (3.18), Eq. (3.14) yields

\[ \tilde{u}^+(\xi, p, x_N) = \tilde{g}^+(\xi, p)e^{-x_N|x|}, \] (3.19)

\[ \tilde{u}^-(\xi, p, x_N) = \tilde{g}^-(\xi, p)e^{x_N|x|}, \] (3.20)

where \( \tilde{g}^\pm(\xi, p) = \tilde{u}^\pm|_{x_N=0} \) are some functions.

Substituting these formulas in (3.16) and (3.17), we obtain

\[ \tilde{\rho}(p + \varepsilon\xi^2 - ih^+ \xi) - a^+ \tilde{g}^+|\xi| = 0, \] (3.21)

\[ \tilde{\rho}(p + \varepsilon\xi^2 - ih^- \xi) + a^- \tilde{g}^-|\xi| = \tilde{f}_3^- . \] (3.22)

Let us divide relations (3.21) and (3.22) by \( a^+ \) and \( a^- \), respectively. Summing these equalities and taking relations (3.15) into account, we obtain

\[ \tilde{\rho} \left[ \left( \frac{1}{a^+} + \frac{1}{a^-} \right) p - i \left( \frac{h^+}{a^+} + \frac{h^-}{a^-} \right) \xi + \varepsilon \left( \frac{1}{a^+} + \frac{1}{a^-} \right) \xi^2 + A|\xi| \right] = \frac{\tilde{f}_3^-}{a^-} , \]

or

\[ \tilde{\rho} = \frac{\tilde{f}}{p - iH\xi + \varepsilon\xi^2 + B|\xi|} , \] (3.23)

where

\[ \tilde{f} = \frac{\tilde{f}_3^-}{a^- \left( \frac{1}{a^+} + \frac{1}{a^-} \right)} , \quad H = \left( \frac{h^+}{a^+} + \frac{h^-}{a^-} \right) / \left( \frac{1}{a^+} + \frac{1}{a^-} \right) , \]

\[ B = A / \left( \frac{1}{a^+} + \frac{1}{a^-} \right) . \]

It is obvious that

\[ |f|_{E^{1+\alpha}(R^{N-1})} = C|\tilde{f}_3^-|_{E^{1+\alpha}(R^{N-1})} . \] (3.24)

Thus, (3.23) gives the formula for the unknown function \( \tilde{\rho} \) in terms of the Fourier–Laplace transform.

We note that the denominator in (3.23) does not become zero at \( \Re(p) > 0 \).

By performing the inverse Laplace–Fourier transformation in (3.23), we get

\[ \rho(x', t) = \int_{0}^{t} \int_{\mathbb{R}^{N-1}} K_\varepsilon(x' - \xi, t - \tau) f(\xi, \tau) d\xi d\tau , \] (3.25)
where \( K_\varepsilon(x, t) \) is the inverse Laplace–Fourier transform of the function \((p - iH\xi + \varepsilon\xi^2 + B|\xi|)^{-1}\), i.e., \((a > 0)\)

\[
K_\varepsilon(x', t) = C \int_{\mathbb{R}^{N-1}} e^{ix'\xi} \frac{dp}{p - iH\xi + \varepsilon\xi^2 + B|\xi|},
\]

(3.26)

The inverse Laplace transform of the function \((p - iH\xi + \varepsilon\xi^2 + B|\xi|)^{-1}\) can be easily calculated as

\[
\hat{K}_\varepsilon(\xi, t) = e^{iH\xi t - \varepsilon\xi^2 t - B|\xi| t},
\]

(3.27)
where \( \varphi(z') = V(z',0) \), \( z' = (z_1, \ldots, z_{N-1}) \). On the other hand, such Dirichlet problem in a half-space can be solved with the use of the Fourier transformation with respect to the variables \( z' \) that gives, as is easily verified,

\[
\hat{V}(\xi, z_N) = C \hat{\varphi}(\xi) e^{-z_N|\xi|}. \tag{3.33}
\]

Comparing (3.33) and (3.32), we may conclude that the inverse Fourier transform of the function \( e^{-z_N|\xi|} \) is \( C z_N/(z^2 + z_N^2)^{N/2} \). In other words, by replacing \( z_N \) by \( Bt \), we have

\[
G(x', t) = C \int_{\mathbb{R}^{N-1}} e^{ix'\xi} e^{-B|\xi|t} d\xi = C \frac{t}{(x'^2 + B^2t^2)^{N/2}}, \quad x' \in \mathbb{R}^{N-1}. \tag{3.34}
\]

With regard for the well-known properties of the functions \( \Gamma_\varepsilon(x', t) \) and the explicitly given \( G(x', t) \) in (3.34), we can verify that the kernel \( K_\varepsilon(x', t) \) possesses the properties (analogous to those of \( \Gamma_\varepsilon(x', t) \)):

\[
\int_{\mathbb{R}^{N-1}} D^r_x D^s_t K_\varepsilon(y, t) dy = \begin{cases} 
1, & |r| + s = 0, \\
0, & |r| + s > 0.
\end{cases} \tag{3.35}
\]

In addition, at any \( \varepsilon \in (0, 1] \), the derivatives of the kernel \( K_\varepsilon(x', t) \) have properties that inherit those of the kernel \( G(x', t) \) in (3.34). Namely, the following lemma is valid.

**Lemma 3.4** The function \( K_\varepsilon(x', t) \) and its derivatives with respect to \( x \) satisfy the estimates

\[
|D^r_x K_\varepsilon(x', t)| \leq C (x'^2 + t^2)^{-\frac{N-1}{2} - |r|}, \quad |r| = 0, 1, 2, \tag{3.36}
\]

where the constant \( C \) is independent of \( \varepsilon \).

**Proof.** It is easily seen that the function \( G(x', t) \) satisfies the estimates

\[
|D^r_x G(x', t)| \leq Ct((x')^2 + t^2)^{-\frac{N}{2} - |r|} \leq C((x')^2 + t^2)^{-\frac{N-1}{2} - |r|}, \quad |r| = 0, 1, 2. \tag{3.37}
\]

For simplicity, we consider only the case \( |r| = 1 \), since the remaining cases can be studied quite analogously. Let \( i = 1, N - 1 \). Then

\[
K_{ex_i}(x', t) = \int_{\mathbb{R}^{N-1}} \Gamma_\varepsilon(y, t) G_{x_i}(x' - y, t) dy = \int_{|x' - y| \geq |x'|/2} \Gamma_\varepsilon(y, t) G_{x_i}(x' - y, t) dy + \int_{|x' - y| < |x'|/2} \Gamma_\varepsilon(y, t) G_{x_i}(x' - y, t) dy \equiv A_1 + A_2. \tag{3.38}
\]
In view of (3.34), the estimate
\[ |G_{x_i}(x' - y, t)| \leq C(x'^2 + t^2)^{-\frac{N}{2}} \]
is valid for the function \( G_{x_i}(x' - y, t) \) on the set \( |x' - y| \geq |x'|/2 \). Therefore,
\[ |A_1| \leq C(x'^2 + t^2)^{-\frac{N}{2}} \int_{\mathbb{R}^{N-1}} \Gamma_{x}(y, t) \, dy = C(x'^2 + t^2)^{-\frac{N}{2}}. \tag{3.39} \]

Passing to the estimate of \( A_2 \), we note that the quantities \( |y| \) and \( |x'| \) are equivalent on the set \( |x' - y| < |x'|/2 \). Therefore, on this set with some \( \gamma > 0 \), we have
\[ e^{-\gamma \frac{t^2}{|x'|}} \leq e^{-\gamma \frac{t^2}{|x'|}}. \tag{3.40} \]

Then we consider two cases. First, let \( t \geq |x'| \). Then, by estimating \( |G_{x_i}| \leq C t^{-N} \), we have
\[ |A_2| \leq C(\varepsilon t) \frac{N-1}{2} e^{-\gamma \frac{t^2}{|x'|}} \cdot \text{Measure}\{ |x' - y| < |x'|/2 \} t^{-N} = C \left( \frac{x'^2}{\varepsilon t} \right)^{\frac{N-1}{2}} e^{-\gamma \frac{t^2}{|x'|}} t^{-N} \leq C t^{-N} \leq C(x'^2 + t^2)^{-\frac{N}{2}}, \tag{3.41} \]
since \( t \geq |x'| \).

But if \( t < |x'| \), then we apply the integration by parts and represent \( A_2 \) as follows:
\[ A_2 = - \int_{|x' - y| < |x'|/2} \Gamma_{x}(y, t) G_{y_i}(x' - y, t) \, dy \]
\[ = - \int_{|y - x'| = \frac{1}{2} |x'|} \Gamma_{x}(y, t) G(x' - y, t) \, dS_y \]
\[ + \int_{|y - x'| < \frac{1}{2} |x'|} \Gamma_{x}(y, t) G(x' - y, t) \equiv I_1 + I_2. \]

Taking the relation \( |x' - y| = |x'|/2 \) into account, by virtue of (3.37) and (3.30), we have
\[ |I_1| \leq C(\varepsilon t)^{-\frac{N-1}{2}} e^{-\gamma \frac{t^2}{|x'|}} |x'|^{N-2} (x'^2 + t^2)^{-\frac{N-1}{2}} \leq C |x'|^{-1} (x'^2 + t^2)^{-\frac{N-1}{2}} \leq C(x'^2 + t^2)^{-\frac{N}{2}}, \tag{3.42} \]
since \( t \leq |x'| \).

Analogously, in view of (3.40) and properties of the function \( \Gamma_{y_i} \), we have
\[ |I_2| \leq C(\varepsilon t)^{-\frac{N}{2}} e^{-\gamma \frac{t^2}{|x'|}} \int_{|y - x'| < \frac{1}{2} |x'|} \frac{t \, dy}{[(x' - y)^2 + t^2]^\frac{N}{2}}. \]
Performing the change \( x' - y = tz \) with \( dy = t^{N-1} dz \) in the last integral, we obtain

\[
|I_2| \leq C(\varepsilon t)^{-\frac{N}{2}} e^{-\gamma \frac{t^2}{2\varepsilon}} t^{N-1} \int_{\mathbb{R}^{N-1}} \frac{t \, dy}{(z^2 + 1)^{\frac{N}{2}}}
\]

\[
\leq C \left( \frac{x'^2}{\varepsilon t} \right)^N e^{-\gamma \frac{t^2}{2\varepsilon}} |x'|^{-N} \leq C |x'|^{-N} \leq C(x'^2 + t^2)^{-\frac{N}{2}}. \tag{3.43}
\]

Thus, estimates (3.33), (3.42), (3.41), and (3.39) yield estimate (3.36) of the lemma in the case \(|r| = 1\).

The remaining estimates are proved analogously.

By virtue of properties (3.35) and (3.36) in the full analogy to [12, Chap. III], we obtain the estimate of the smoothness of potential (3.28) by the variable \( x' \):

\[
\max_t |\rho(\cdot, t)|_{(2+\alpha)_{\mathbb{R}^N_+}} \leq C \max_t |f|_{1+\alpha(\mathbb{R}^N_+)} \tag{3.44}
\]

Analogously to the proof of Lemma 3.2, let us consider the functions

\[
\rho_h = \rho(x', t) - \rho(x', t - h), \quad u^\pm_h = \frac{u^\pm(x', t) - u^\pm(x', t - h)}{h^\alpha},
\]

These functions satisfy the same problem (3.1)–(3.4) with the appropriate right-hand sides. Therefore, we see, by completely repeating the previous reasoning, that the function \( \rho_h(x', t) \) is potential (3.28) with the density

\[
f_h = \frac{f(x', t) - f(x', t - h)}{h^\alpha}
\]

instead of \( f \). Thus, (3.44) yields

\[
|\rho|_{E^{1+\alpha}(\mathbb{R}^N_+)} \leq C \sup_{t, h} |f_h|_{1+\alpha(\mathbb{R}^N_+)} \leq C |f|_{E^{1+\alpha}(\mathbb{R}^N_+)} \tag{3.45}
\]

We now represent relation (3.23) in the form

\[
\rho_t - \varepsilon \Delta x' \rho - B \Delta x'(\Lambda \rho) + \sum_{i=1}^N H_i \rho_{x_i} = f(x', t)
\]

or

\[
\rho_t - \varepsilon \Delta x' \rho = F(x', t) \equiv \frac{f + B \Delta x'(\Lambda \rho) - \sum_{i=1}^N H_i \rho_{x_i}}{h^\alpha}, \tag{3.46}
\]

where \( \Lambda \rho \) is the operator with the symbol \(|\xi|^{-1} \), i.e., \( \Lambda : \tilde{\rho} \rightarrow \tilde{\rho}/|\xi| \). As is well known (see, e.g., [23]),

\[
\Lambda \rho(x', t) = C \int_{\mathbb{R}^{N-1}} \rho(y, t) \frac{dy}{|x' - y|^{N-2}}
\]
Moreover, analogously to the standard Hölder spaces with respect to \( x' \),
\[
|\Lambda \rho|_{E^{k+r}(\mathbb{R}^N_+)} \leq C|\rho|_{E^{k-1,1}(\mathbb{R}^N_+)} , \quad k = 1, 2, 3 .
\] (3.47)
Thus, (3.47), (3.46), and (3.45) yield
\[
|F|_{E^{1+r}(\mathbb{R}^N_+)} \leq C(|\rho|_{E^{2+r}(\mathbb{R}^N_+)} + |f|_{E^{1+r}(\mathbb{R}^N_+)}) \leq C|f|_{E^{1+r}(\mathbb{R}^N_+)} .
\] (3.48)
Since \( f(x', 0) = 0 \) and \( \rho(x', 0) = \rho_t(x', 0) = 0 \), Eq. (3.46) yields, completely analogously to [11] Chap. IV,
\[
\varepsilon \langle D^3_\alpha \rho \rangle_{x, \mathbb{R}^N_+} \leq C \langle D_\alpha F \rangle_{x, \mathbb{R}^N_+} .
\] (3.49)
In addition, with regard for the function \( \rho_h = (\rho(x', t) - \rho(x', t - h))/h^\alpha \), we have
\[
\varepsilon \left[ D^3_\alpha \rho \right]_{x, \mathbb{R}^N_+} \leq C \left[ D_\alpha F \right]_{x, \mathbb{R}^N_+} .
\] (3.50)
Then it follows from (3.46) that
\[
\langle D_x \rho \rangle_{x, \mathbb{R}^N_+} + [D_x \rho]_{x, \mathbb{R}^N_+} \leq C \left( \langle D_x F \rangle_{x, \mathbb{R}^N_+} + [D_x F]_{x, \mathbb{R}^N_+} \right) .
\] (3.51)
By virtue of the finiteness of the function \( \rho(x', t) \) relations (3.49)–(3.51) and (3.48) yield
\[
|\rho|_{E^{1+r}(\mathbb{R}^N_+)} + |\rho_t|_{E^{1+r}(\mathbb{R}^N_+)} \leq C \left| f \right|_{E^{1+r}(\mathbb{R}^N_+)} ,
\] (3.52)
where the constant \( C \) is independent of \( \varepsilon \), i.e.,
\[
|\rho|_{E^{2+r}(\mathbb{R}^N_+)} + |\rho_t|_{E^{1+r}(\mathbb{R}^N_+)} \leq C \left| f \right|_{E^{1+r}(\mathbb{R}^N_+)} ,
\] (3.53)
which gives the required estimate for the function \( \rho(x', t) \).

Possessing the estimate for the function \( \rho(x', t) \), we can consider the functions \( u^\pm(x, t) \) as solutions of the Neumann problems in the appropriate domains with the condition
\[
u^\pm \frac{\partial u^\pm}{\partial \nu_{x_N}}|_{x_N=0} = F_1^\pm = f_3^\pm - \rho_t + \varepsilon \Delta x' \rho - h^\pm \nabla \rho .
\] (3.54)
In this case, \( F_1^\pm \) are finite, and, by virtue of (3.53), we have
\[
\left| F^\pm_1 \right|_{E^{1+r}(\mathbb{R}^N_+)} \leq C \left| f \right|_{E^{1+r}(\mathbb{R}^N_+)} .
\] (3.55)
Then Lemma 3.2 yields
\[
\left| u^\pm \right|_{E^{1+r}(\mathbb{R}^N_+)} \leq C \left| f \right|_{E^{1+r}(\mathbb{R}^N_+)} .
\] (3.56)
Estimate (3.56) together with estimate (3.53) complete the proof of Theorem 3.1 □
4 A linear problem in the domain $\Omega_T$

In this section, we prove the solvability of a linear problem corresponding to problem (2.7)–(2.11) with the given right-hand sides from the appropriate classes. In this case, we regularize the boundary condition on the surface $\Gamma$ in the same manner, as it was made in [17, 6, 7].

We now consider the problem of the determination of the unknown functions $u^\pm(x, t)$ defined in the domains $\overline{\Omega_T^\pm}$, respectively, and the unknown function $\rho(x, t)$ defined on the surface $\Gamma_T$ by the conditions

$$- \Delta u^\pm + b^\pm(x, t)u^\pm = f_1^\pm, \quad (x, t) \in \Omega_T^\pm,$$

$$u^+ - u^- + A(x, t)\rho = f_2, \quad (x, t) \in \Gamma_T,$$

$$\rho_t - \varepsilon \Delta_{\Gamma} \rho + a^\pm \frac{\partial u^\pm}{\partial n} + \sum_{i=1}^{N-1} h_i^\pm(x, t)\rho_{x_i} = f_3^\pm, \quad (x, t) \in \Gamma_T,$$

$$u^\pm(x, 0) = 0, \quad \rho(x, 0) = 0,$$

$$u^\pm(x, t) = f_4^\pm, \quad (x, t) \in \Gamma_T^\pm,$$

where $\Delta_{\Gamma}$ is the Laplace operator on the surface $\Gamma$ (see, e.g., [17]), $a^\pm = \text{const} > 0$, $b^\pm(x, t) \in E^\alpha(\overline{\Omega_T^\pm})$, $A(x, t) \in E^{2+\alpha}(\Gamma_T)$, $h_i^\pm(x, t) \in E^{1+\alpha}(\Gamma_T)$, and the conditions $\nu \leq A(x, t) \leq C$ and $b^\pm(x, t) \geq \nu > 0$ are satisfied. The right-hand sides $f_i$ in relations (4.1)–(4.5) are assumed to be such that the quantities

$$\mathcal{M}_T^\pm \equiv |f_1^\pm|_{E^\alpha(\overline{\Omega_T^\pm})} + |f_2|_{E^{2+\alpha}(\Gamma_T)} + |f_3^\pm|_{E^{1+\alpha}(\Gamma_T)} + |f_4^\pm|_{E^{2+\alpha}(\Gamma_T)} < \infty,$$

$$\mathcal{M}_T \equiv \mathcal{M}_T^+ + \mathcal{M}_T^-$$

are finite, and

$$f_1^\pm(x, 0) = 0, \quad f_2(x, 0) = 0, \quad f_3^\pm(x, 0) = 0, \quad f_4^\pm(x, 0) = 0,$$

i.e., all functions $f_i$ belong to spaces marked with a dot.

The following theorem is valid.

**Theorem 4.1** If conditions (4.6) and (4.7) are satisfied, problem (4.1)–(4.5) has the unique solution at any $\varepsilon > 0$ from the space $u^\pm \in \dot{E}^{2+\alpha}(\overline{\Omega_T^\pm})$, $\rho \in \dot{P}^{3+\alpha}(\Gamma_T)$, and the estimate

$$|u^+|_{E^{2+\alpha}(\overline{\Omega_T^\pm})} + |u^-|_{E^{2+\alpha}(\overline{\Omega_T^\pm})} + |\rho|_{P^{2+\alpha}(\Gamma_T)} + \varepsilon|\rho|_{P^{3+\alpha}(\Gamma_T)} \leq C_T \mathcal{M}_T,$$

(4.8)

where the constant $C_T$ from (4.8) is independent of $\varepsilon \in (0, 1]$, is true.

At $\varepsilon = 0$, problem (4.1)–(4.5) has the unique solution from the spaces $u^\pm \in \dot{E}^{2+\alpha}(\overline{\Omega_T^\pm})$, $\rho \in \dot{P}^{2+\alpha}(\Gamma_T)$, and estimate (4.8) with $\varepsilon = 0$ is valid.
\textbf{Proof.} First, we prove estimate (4.8), by assuming the availability of a solution of problem (4.1)–(4.5) from the appropriate class. The following lemma is valid.

\textbf{Lemma 4.2} For any solution of problem (4.1)–(4.5) from the class \( u^\pm \in \dot{E}^{2+\alpha}(\Omega_T^\pm) \), \( \rho \in \dot{P}^{3+\alpha}(\Gamma_T) \), estimate (4.8) is valid at \( \varepsilon > 0 \).

\textbf{Proof.} By using the standard Schauder technique for estimates and by considering the results of Section 3 on properties of the model problems corresponding to points of the boundary \( \Gamma \), we obtain, in the ordinary manner, the following a priori estimate of the solution of problem (4.1)–(4.5):

\[ |u^+_h|_{E^{2+\alpha}(\Omega_T^+)} + |u^-_h|_{E^{2+\alpha}(\Omega_T^-)} + |\rho|_{P^{2+\alpha}(\Gamma_T)} + \varepsilon |\rho|_{P^{2+\alpha}(\Gamma_T)} \leq C(M_T + C \langle u^+_t \rangle_{\alpha,T} + \langle u^-_t \rangle_{\alpha,T}), \tag{4.9} \]

Whereas \( |u^+_h(0)_{\Omega_T^+} \leq C \langle u^+_t \rangle_{\alpha,T} \), the Hölder constants \( \langle u^+_t \rangle_{\alpha,T} \) and \( \langle u^-_t \rangle_{\alpha,T} \) for the functions \( u^\pm(x, t) \) with respect to the variable \( t \) cannot be estimated by the interpolation in the space \( \dot{E}^{2+\alpha}(\Omega_T^\pm) \). In order to estimate these Hölder constants, we consider the functions \( (h \in (0, 1)) \)

\[ u^\pm_h = \frac{u^\pm(x, t) - u^\pm(x, t - h)}{h^\alpha}. \]

and estimate their modulus maximum \( |u^+_h(0)_{\Omega_T^+} \) uniformly in \( h \). We take into account that \( u^+_h \in H^{2+\alpha}(\Omega_T^+) \), \( t \in [0, T] \), and the space \( H^{2+\alpha}(\Omega^+) \) is compactly embedded in the space \( L_\infty(\Omega^+) \subset L_2(\Omega^+) \). Therefore, for any \( \delta > 0 \) and \( t \in [0, T] \), we have the inequality (see [13])

\[ |u^+_h(0)_{\Omega_T^+} \leq \delta |u^+_h(2+\alpha)_{\Omega_T^+} + C_\delta \| u^+_h \|_{2,\Omega_T^+}, \tag{4.10} \]

where \( \| u^+_h \|_{2,\Omega_T^+} \) is the \( L_2 \)-norm of the functions \( u^+_h \). Relation (4.10) yields

\[ |u^+_h(0)_{\Omega_T^+} \leq \delta |u^+_h|_{E^{2+\alpha}(\Omega_T^+)} + C_\delta \max_{t \in [0, T]} \| u^+_h \|_{2,\Omega_T^+}. \tag{4.11} \]

Thus, we need to estimate the quantity \( \max_{t \in [0, T]} \| u^+_h \|_{2,\Omega_T^+} \).

Without any loss of generality, we can consider that \( f^+_4 \equiv 0 \), since these functions can be extended inward \( \Omega_T^+ \) with the preservation of a class. Then we can consider new unknown functions \( v^\pm = u^\pm - f^+_4 \) that satisfy the same problem with the same estimate of the right-hand sides. Thus, by taking \( f^+_4 \equiv 0 \) without any loss of generality and by subtracting relation (4.3) for the sign \( "-" \) from the same relation for the sign \( "+" \), we get a problem for the functions \( u^\pm_h \):

\[ -\Delta u^+_h + b^+_h u^+_h = F^+_1 \equiv f^+_1 - b^+_h u^+_h, \quad (x, t) \in \Omega_T^+, \tag{4.12} \]

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\[ u^+_h - u^-_h = F_2 \equiv -A\rho_h - A_h\tilde{\rho} + f_{2h}, \quad (x, t) \in \Gamma_T, \]  
\[ a^+ \frac{\partial u^+_h}{\partial n} - a^- \frac{\partial u^-_h}{\partial n} = F_3 \equiv f^{\pm}_{3h} - f^{-}_{3h} + \bar{H} \nabla \omega \rho_h + \bar{H}_h \nabla \tilde{\omega} \rho, \quad (x, t) \in \Gamma_T, \]  
\[ u^+_h = 0, \quad (x, t) \in \Gamma^+_T, \]  
\[ u^+_h(x, 0) = 0, \quad x \in \Omega^+_h, \]  
where the lower index \( h \) of the designation of functions means the corresponding difference relation, \( \tilde{a}^\pm(x, t) = u^\pm(x, t - h), \rho(x, t) = \rho(x, t - h), \bar{H} = \{ h^+_h - h^-_h \}. \)

Let us multiply Eqs. \((4.12)\) by the functions \( a^\pm u^\pm_h \), respectively, and integrate by parts over the domains \( \Omega^\pm \). With regard for the direction of the normal \( \vec{n} \) to the boundary \( \Gamma \), we obtain

\[ a^\pm \int_{\Omega^\pm} (\nabla u^\pm_h)^2 \, dx + a^\pm \int_{\Omega^\pm} b^\pm (u^\pm_h)^2 \, dx \pm a^\pm \int_{\Gamma} u^+_h \left( a^\pm \frac{\partial u^+_h}{\partial n} \right) \, dS = \pm a^\pm \int_{\Omega^\pm} u^+_h F^\pm_1 \, dx. \]

Since the relation \( u^+_h = u^-_h + F_2 \) is satisfied on the surface \( \Gamma \) relation \((4.17)\) for the sign \( '+' \) can be presented in the form

\[ a^+ \int_{\Omega^+} (\nabla u^+_h)^2 \, dx + a^- \int_{\Omega^-} (\nabla u^-_h)^2 \, dx + a^+ \int_{\Gamma} u^+_h \left( a^+ \frac{\partial u^+_h}{\partial n} \right) \, dS - a^- \int_{\Gamma} F_2 \left( a^- \frac{\partial u^-_h}{\partial n} \right) \, dS = a^+ \int_{\Omega^+} u^+_h F^+_1 \, dx - a^- \int_{\Omega^-} u^-_h F^-_1 \, dx. \]

Adding \((4.18)\) and relation \((4.17)\) for the sign \( '-' \) and taking conditions \((4.14)\) into account, we get

\[ a^+ \int_{\Omega^+} (\nabla u^+_h)^2 \, dx + a^- \int_{\Omega^-} (\nabla u^-_h)^2 \, dx + a^+ \int_{\Gamma} u^+_h \left( a^+ \frac{\partial u^+_h}{\partial n} \right) \, dS - a^- \int_{\Gamma} F_2 \left( a^- \frac{\partial u^-_h}{\partial n} \right) \, dS = a^+ \int_{\Omega^+} u^+_h F^+_1 \, dx - a^- \int_{\Omega^-} u^-_h F^-_1 \, dx. \]

We now estimate the terms on the right-hand side of \((4.19)\), by using the Cauchy inequality with small parameter \( \mu > 0 \):

\[ \int_{\Omega^\pm} u^\pm_h F^\pm_1 \, dx \leq \mu^2 \left\| u^\pm_h \right\|^2_{2,\Omega^\pm} + C_\mu \left\| F^\pm_1 \right\|^2_{2,\Omega^\pm} \leq C \mu^2 \left\| u^\pm \right\|^2_{E^{2+\alpha}(\mathbb{T}^2)} + C_\mu \left( \left\| F^\pm_1 \right\|_{\Omega^\pm} \right)^2, \]

\[ \int_{\Gamma} F_2 \left( a^+ \frac{\partial u^+_h}{\partial n} \right) \, dS \leq \mu^2 \left\| \frac{\partial u^+_h}{\partial n} \right\|^2_{2,\Gamma} + C_\mu \left\| F_2 \right\|^2_{2,\Gamma} \leq C \mu^2 \left\| u^+ \right\|^2_{E^{2+\alpha}(\mathbb{T}^2)} + C_\mu \left( \left\| F_2 \right\|_{\Gamma} \right)^2, \]
\[
\left| \int \nabla u^+ \cdot F_3 \, dS \right| \leq C \mu^2 \left| u^- \right|^2_{E^{2+\alpha}(\Gamma_T)} + C_\mu \left( \left| F_3 \right|^2_{\Gamma_T} \right).
\]  
(4.22)

We note also that, at \( T < 1 \),

\[
|\tilde{u}^+|_{\Omega_T^+} \leq CT^\alpha \left\langle \tilde{u}^+ \right\rangle_{t,\Omega_T^+} \leq CT^\alpha \left| u^+ \right|_{E^{2+\alpha}(\Gamma_T)} ,
\]
\[
|\tilde{\rho}|_{\Gamma_T} + |\rho_h|_{\Gamma_T} \leq CT^{1-\alpha} \left| \frac{\partial \rho}{\partial t} \right|_{\Gamma_T} \leq CT^{1-\alpha} |\rho|_{L^2(\Gamma_T)} ,
\]
\[
|\nabla \rho_h|_{\Gamma_T^+} \leq C \left\langle \nabla \rho \right\rangle_{t,\Gamma_T^+} \leq CT^\frac{1}{2} \left( \nabla \rho \right)_{t,\Gamma_T^+} \leq CT^\frac{1}{2} |\rho_{2+\alpha}(\Gamma_T) .
\]

Thus, the functions \( F_{1}^\pm, F_{2}, \) and \( F_{3} \) in (4.12)–(4.16) satisfy the estimate

\[
\left| F_{1}^+ \right|_{\Gamma_T^+} + \left| F_{1}^- \right|_{\Gamma_T^-} + \left| F_{2} \right|_{\Gamma_T^+} + \left| F_{3} \right|_{\Gamma_T^-} \leq CT^\lambda \left( |u^+|_{E^{2+\alpha}(\Gamma_T)} + |u^-|_{E^{2+\alpha}(\Gamma_T)} + |\rho|_{L^2(\Gamma_T)} \right) + CM_T
\]
(4.23)

with some \( \lambda > 0 \).

We note that, by virtue of conditions (4.15), the inequality

\[
\int_{\Omega^\pm} \left( u^\pm \right)^2 \, dx \leq C \int_{\Omega^\pm} \left( \nabla u^\pm \right)^2 \, dx
\]

is valid. Then relations (4.19)–(4.23) yield

\[
\max_{t \in [0,T]} \| u^\pm \|_{L^2(\Omega^\pm)} \leq C(\mu + C_\mu T^\lambda) \left( |u^+|_{E^{2+\alpha}(\Gamma_T)} + |u^-|_{E^{2+\alpha}(\Gamma_T)} + |\rho|_{L^2(\Gamma_T)} \right) + C_\mu M_T .
\]
(4.24)

By choosing firstly \( \mu \) and then \( T \) to be sufficiently small and by joining estimates (4.21), (4.11), and (4.9), we obtain that estimate (4.8) is satisfied on some interval \([0,T]\) independent of the values of the right-hand sides of the problem.

By moving now upward along the axis \( t \) step-by-step, as it was made in [11], Chap. IV, we prove estimate (4.8) on any finite time interval \([0,T]\).

Thus, Lemma 4.2 and estimate (4.8) are proved.

Let us continue the proof of Theorem 4.1. We write problem (4.1)–(4.5) in the form

\[
- \Delta u^\pm + b^+(x,t)u^\pm = f_1^\pm , \quad (x,t) \in \Omega_T^\pm ,
\]
(4.25)

\[
u^+ - u^- = -A(x,t)\rho + f_2 , \quad (x,t) \in \Gamma_T ,
\]
(4.26)

\[
a^+ \frac{\partial u^+}{\partial n} - a^- \frac{\partial u^-}{\partial n} = f_3^- - f_3^+ + \vec{H} \nabla \rho , \quad (x,t) \in \Gamma_T ,
\]
(4.27)

\[
u^\pm = f_4^\pm , \quad (x,t) \in \Gamma_T^\pm ,
\]
(4.28)

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\[ u^+(x,0) = 0, \quad x \in \Omega^+ \quad (4.29) \]

\[ \rho_1 - \varepsilon \Delta \rho = f^+_3 - a^+ \varepsilon \frac{\partial u^+}{\partial n} - \sum h^+_i \rho_{\omega_i}, \quad \rho(\omega,0) = 0. \quad (4.30) \]

It follows from results in \cite{10, 20} that, for the given function \( \rho \in E^{2+\alpha}(\Gamma_T) \) on the right-hand sides of relations (4.25)–(4.28), the problem of conjugation (4.25)–(4.28) has the unique solution that satisfies the estimate

\[ |u^+|_{E^{2+\alpha}(\Gamma_T)} + |u^-|_{E^{2+\alpha}(\Gamma_T)} \leq CM_T + C |\rho|_{E^{2+\alpha}(\Gamma_T)}, \quad (4.31) \]

so that

\[ |\nabla u^+|_{E^{1+\alpha}(\Gamma_T)} \leq CM_T + C |\rho|_{E^{2+\alpha}(\Gamma_T)}. \quad (4.32) \]

We note that the results in \cite{10, 20} concern the spaces \( H^{2+\alpha}(\Omega) \), but the transition to the spaces \( E^{2+\alpha}(\Omega) \) is realized simply by the consideration of the appropriate problem for the functions \( u^+_h = (u^+_h(x,t) - u^+_h(x,t-h))/h^\alpha \).

Thus, we have correctly defined the operator \( L_\varepsilon: \rho \rightarrow L_\varepsilon \rho \) that puts each function \( \rho \in E^{2+\alpha}(\Gamma_T) \) given on the right-hand sides of relations (4.25)–(4.28) and (4.30) in correspondence with the function \( L_\varepsilon \rho \) that is the solution of the Cauchy problem (4.30) with the function \( \rho \) and the function \( \partial u^+ / \partial n \) determined by \( \rho \) that are given on the right-hand side of (4.30).

Relations (4.31) and (4.32) and results in \cite{17} yield

\[ |L_\varepsilon \rho|_{P^{3+\alpha}(\Gamma_T)} \leq C_\varepsilon \left( |\nabla u^+|_{E^{1+\alpha}(\Gamma_T)} + |\nabla \rho|_{E^{1+\alpha}(\Gamma_T)} + |f^+_3|_{E^{1+\alpha}(\Gamma_T)} \right) \]

\[ \leq C_\varepsilon \left( |f^+_3|_{E^{1+\alpha}(\Gamma_T)} + |\rho|_{E^{2+\alpha}(\Gamma_T)} \right), \quad (4.33) \]

and, in addition, for \( \rho_1, \rho_2 \in E^{2+\alpha}(\Gamma_T) \),

\[ |L_\varepsilon \rho_2 - L_\varepsilon \rho_1|_{P^{3+\alpha}(\Gamma_T)} \leq C_\varepsilon |\rho_2 - \rho_1|_{E^{2+\alpha}(\Gamma_T)}. \quad (4.34) \]

It follows from results in \cite{15, 21, 22} about the interpolation in Hölder spaces that the quantity

\[ [D^2_\rho]^{(\alpha,\frac{1}{2}+\alpha)}_{x,t,\Gamma_T} \leq C |\rho|_{P^{3+\alpha}(\Gamma_T)} \quad (4.35) \]

is finite. Hence, since \( \rho(x,0) = 0, \rho_t(x,0) = 0 \), we have

\[ [D^2_\rho]^{(\alpha,\frac{1}{2}+\alpha)}_{x,t,\Gamma_T} \leq CT^{\frac{1}{2}} [D^2_\rho]^{(\alpha,\frac{1}{2}+\alpha)}_{x,t,\Gamma_T} \leq CT^{\frac{1}{2}} |\rho|_{P^{3+\alpha}(\Gamma_T)}. \quad (4.36) \]

Analogous inequalities with the factor \( T^\lambda \) are valid also for other terms in the definition of the norm \( \rho \) in the space \( E^{2+\alpha}(\Gamma_T) \). For \( \rho_1, \rho_2 \in E^{2+\alpha}(\Gamma_T) \), this result and relation (4.34) yield

\[ |L_\varepsilon \rho_2 - L_\varepsilon \rho_1|_{E^{2+\alpha}(\Gamma_T)} \leq C_\varepsilon T^\lambda |\rho_2 - \rho_1|_{E^{2+\alpha}(\Gamma_T)}. \quad (4.37) \]
Thus, by choosing $T$ to be sufficiently small, we obtain that the operator $L_\varepsilon$ is a contraction one on $E^{2+\alpha}(\Gamma_T)$ and, hence, has the single fixed point. Together with (4.33), this yields a solution of problem (4.1)–(4.5) on some interval $[0, T]$ independent of the values of the right-hand sides of the problem. By moving upward along the axis $t$, as it was made in [11, Chap. IV], we obtain the solution of problem (4.1)–(4.5) on some interval $[0, T]$ in the required class an any finite time interval. The estimate of the solution was proved above in Lemma 4.2.

Thus, we prove the assertion of the theorem for $\varepsilon > 0$. We now transit to the limit as $\varepsilon \to 0$. We note that, by virtue of estimate (4.8), the sequence $u^{\pm}_n, \rho_n$ is compact in the spaces $E^{2+\beta}(\Omega^\pm_T)$ and $P^{2+\beta}(\Gamma_T)$ with any $\beta < \alpha$. Hence, we can separate a subsequence $u^{\pm}_n \to u^{\pm}, \rho_n \to \rho$ that converges in these spaces, and the functions $u^{\pm}$ and $\rho$ present the solution of problem (4.1)–(4.5) at $\varepsilon = 0$. Indeed, in view of the available estimate, the limit transition is possible in each of the relations. In addition, the limiting functions will belong to the same spaces $E^{2+\alpha}(\Omega^\pm_T)$ and $P^{2+\alpha}(\Gamma_T)$, because, by virtue of the estimate uniform, for example, in $\varepsilon$,

$$\left| \frac{D^2_x u^+_n(x + T, t) - D^2_x u^+_n(x, t)}{|T|^\alpha} \right| \leq CMT,$$

we can transit to the limit in this inequality as $\varepsilon_n \to 0$ due to the uniform convergence of the functions $D^2_x u^+$ on $\Omega^+_T$. This yields

$$\langle D^2_x u^+_n(x, t) \rangle^{(\alpha)}_{x, \Omega^+_T} \leq CMT.$$

The remaining estimates are analogous.

Eventually, the uniqueness of the solution that was obtained by the limit transition follows directly from estimate (4.8).

Thus, Theorem 4.1 is proved. $\blacksquare$

5 A nonlinear problem: the proof of Theorem 1.1

The proof of Theorem 1.1 is based on Theorem 4.1 and a representation of the problem under consideration in the form (2.7)–(2.11). We now define a nonlinear operator $F(\psi), \psi = (v^+, v^-, \delta)$ in (2.7)–(2.11) that puts every given $\psi$ on the nonlinear right-hand sides of relations (2.7)–(2.11) in correspondence with the solution of the linear problem determined by the left-hand sides of these relations. In this case, Theorem 4.1 and Lemma 2.1 imply that the operator $F(\psi)$ possesses the following properties on a ball $B_r = \{ \psi : \| \psi \| \leq r \} \subset H$:

$$\| F(\psi) \|_H \leq C(T^{\alpha/2} + r)\| \psi \|_H, \quad (5.1)$$

$$\| F(\psi_1) - F(\psi_2) \|_H \leq C(T^{\alpha/2} + r)\| \psi_1 - \psi_2 \|_H. \quad (5.2)$$
It is easy to see that relations (5.1) and (5.2) imply that, at sufficiently small $T$ and $r$, the operator $\mathcal{F}(\psi)$ maps the closed ball $B_r$ into itself and is a contraction operator there. The single fixed point of this operator gives the solution of the initial nonlinear problem with free boundary that is related to Theorem 1.1. Thus, Theorem 1.1 is proved.

References

[1] B. V. Bazaliy, I. I. Danilyuk, and S. P. Degtyarev, “Classical solvability of a multidimensional nonstationary filtration problem with free boundary,” *Dokl. AN UkrSSR, Ser. A*, No. 2, 9–15 (1987).

[2] B. V. Bazaliy and S. P. Degtyarev, “Classical solvability of a multidimensional Stefan problem at the convective motion of a viscous noncompressible fluid,” *Mat. Sb.*, **132**(174), No. 1, 3–19 (1987).

[3] G. I. Bizhanova and V. A. Solonnikov, “On problems with free boundaries for parabolic second-order equations,” *Alg. Analiz*, **12**, Iss. 6, 98–139 (2000).

[4] I. I. Danilyuk, “Nonstationary filtration in a dispersion system of two barotropic media,” *Dokl. AN UkrSSR, Ser. A*, No. 1, 14–18 (1985).

[5] I. I. Danilyuk, “On the joint nonstationary filtration of a gas and a dispersion system of barotropic media,” *Dokl. AN UkrSSR, Ser. A*, No. 11, 9–13 (1985).

[6] Fahuai Yi, “Local classical solution of Muskat free boundary problem,” *J. Partial Differ. Eq.*, **9**, No. 1, 84–96 (1996).

[7] Fahuai Yi, “Global classical solution of Muskat free boundary problem,” *J. Math. Anal. Appl.*, **288**, No. 2, 442–461 (2003).

[8] V. N. Gusakov and S. P. Degtyarev, “Existence of a smooth solution of the filtration problem,” *Ukr. Mat. Zh.*, **41**, No. 9, 1192–1198 (1989).

[9] E.-I. Hanzawa, “Classical solutions of the Stefan problem,” *Tohoku Math. J.*, **33**, 297–335 (1981).

[10] O. A. Ladyzhenskaya, V. Ya. Rivkind, and N. N. Uralt’seva, “On the classical solvability of the diffraction problem,” *Trudy Mat. Inst. V. A. Steklova*, **92**, 116–146 (1966).

[11] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Uralt’seva, *Linear and Quasilinear Equations of Parabolic Type*, Amer. Math. Soc., Providence, RI, 1968.
[12] O. A. Ladyzhenskaya and N. N. Ural’tseva, *Linear and Quasilinear Equations of Elliptic Type*, Academic Press, New York, 1968.

[13] J.-L. Lions, *Quelques Methodes de Resolution des Problemes aux Limites Non Lineaires*, Gauthier–Villars, Paris, 1969.

[14] Longfeng Xu, “A Verigin problem with kinetic condition,” *Appl. Math. Mech.*, 18, No. 2, 191–199 (1997).

[15] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Birkhäuser, Basel, 1995.

[16] A. M. Meiermanov, “The problem of motion of the contact rupture surface at the filtration of unmixed compressible fluids,” *Sib. Mat. Zh.*, 23, No. 1, 85–103 (1982).

[17] A. M. Meiermanov, “On the classical solution of a multidimensional Stefan problem for quasilinear parabolic equations,” *Mat. Sb.*, 112(154), No. 2(6), 170–192 (1980).

[18] M. Muskat, *The Flow of Homogeneous Fluids through Porous Media*, McGraw-Hill, New York, 1937.

[19] V. A. Solonnikov, “Solvability of the problem of motion of a viscous non-compressible fluid bounded by a free surface,” *Izv. AN SSSR, Ser. Mat.*, 41, No. 6, 1388–1424 (1977).

[20] V. A. Solonnikov, “On general boundary-value problems for the systems elliptic in the Douglis–Nirenberg meaning,” *Trudy Mat. Inst. V. A. Steklova*, 92, 233–297 (1966).

[21] V. A. Solonnikov, “A priori estimates for second-order equations of the parabolic type,” *Trudy Mat. Inst. V. A. Steklova*, 70, 133–212 (1964).

[22] V. A. Solonnikov, “Estimates of the solutions of a nonstationary linearized system of the Navier–Stokes equations,” *Trudy Mat. Inst. V.A. Steklova*, 70, 213–317 (1964).

[23] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, 1970.

[24] N. N. Verigin, “On a class of hydromechanical problems for domains with movable boundaries,” in *Dynamics of a Fluid with Free Boundaries* [in Russian], Institute of Hydrodynamics of SD RAS, Novosibirsk, 1980, 23–33.

[25] Youshun Tao and Fahuai Yi, “Classical Verigin problem as a limit case of Verigin problem with surface tension at free boundary,” *Appl. Math.-JCU*, 11B, 307–322 (1996).
[26] Youshan Tao, “Classical solution of Verigin problem with surface tension,” 
Chin. Ann. Math., Ser. B, 18, No. 3, 393–404 (1997).