q-analogs of group divisible designs

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October 15, 2018

Abstract

A well known class of objects in combinatorial design theory are group divisible designs. Here, we introduce the q-analogs of group divisible designs. It turns out that there are interesting connections to scattered subspaces, q-Steiner systems, design packings and qr-divisible projective sets.

We give necessary conditions for the existence of q-analogs of group divisible designs, construct an infinite series of examples, and provide further existence results with the help of a computer search.

One example is a (6, 3, 2, 2)2 group divisible design over GF(2) which is a design packing consisting of 180 blocks that such every 2-dimensional subspace in GF(2)6 is covered at most twice.

1 Introduction

The classical theory of q-analogs of mathematical objects and functions has its beginnings as early as in the work of Euler [Eul53]. In 1957, Tits [Tit57] further suggested that combinatorics of sets could be regarded as the limiting case q → 1 of combinatorics of vector spaces over the finite field GF(q). Recently, there has been an increased interest in studying q-analogs of combinatorial designs from an applications’ view. These q-analog structures can be useful in network coding and distributed storage, see e.g. [GP18].

It is therefore natural to ask which combinatorial structures can be generalized from sets to vector spaces over GF(q). For combinatorial designs, this question was first studied by Ray-Chaudhuri [BRC74], Cameron [Cam74a, Cam74b] and Delsarte [Del76] in the early 1970s.

Specifically, let GF(q)v be a vector space of dimension v over the finite field GF(q). Then a t-(v, k, λ)q subspace design is defined as a collection of

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$k$-dimensional subspaces of $\text{GF}(q)^v$, called blocks, such that each $t$-dimensional subspace of $\text{GF}(q)^v$ is contained in exactly $\lambda$ blocks. Such $t$-designs over $\text{GF}(q)$ are the $q$-analogs of conventional designs. By analogy with the $q \to 1$ case, a $t$-(v,k,1)$_q$ subspace design is said to be a $q$-Steiner system, and denoted $S(t, k, v)_q$.

Another well-known class of objects in combinatorial design theory are group divisible designs [MG07]. Considering the above, it therefore seems natural to ask for $q$-analogs of group divisible designs.

At first glance, this seems like a somewhat artificial task without much justification. But quite surprisingly, it turns out that $q$-analogs of group divisible designs have interesting connections to scattered subspaces which are central objects in finite geometry, as well as to coding theory via $q^r$-divisible projective sets. We will also discuss the connection to $q$-Steiner systems [BEO+16] and to design packings [EZ18].

Let $K$ and $G$ be sets of positive integers and let $\lambda$ be a positive integer. A $(v, K, \lambda, G)$-group divisible design of index $\lambda$ and order $v$ — denoted as $(v, K, \lambda, G)_q$-GDD — is a triple $(V, G, B)$, where $V$ is a finite set of cardinality $v$, $G$, where $\# G > 1$, is a partition of $V$ into parts (groups) whose sizes lie in $G$, and $B$ is a family of subspaces (blocks) of $V$ (with $\# B \in K$ for $B \in B$) such that every pair of distinct elements of $V$ occurs in exactly $\lambda$ blocks or one group, but not both. See—for example— [MG07, Han75] for details. We note that the “groups” in group divisible designs have nothing to do with group theory.

The $q$-analog of a combinatorial structure over sets is defined by replacing subsets by subspaces and cardinalities by dimensions. Thus, the $q$-analog of a group divisible design can be defined as follows.

**Definition 1** Let $K$ and $G$ be sets of positive integers and let $\lambda$ be a positive integer. A $q$-analog of a group divisible design of index $\lambda$ and order $v$ — denoted as $(v, K, \lambda, G)_q$-GDD — is a triple $(V, G, B)$, where

- $V$ is a vector space over $\text{GF}(q)$ of dimension $v$,
- $G$ is a vector space partition of $V$ into subspaces (groups) whose dimensions lie in $G$, and
- $B$ is a family of subspaces (blocks) of $V$,

that satisfies

1. $\# G > 1$,
2. if $B \in B$ then $\dim B \in K$,
3. every 2-dimensional subspace of $V$ occurs in exactly $\lambda$ blocks or one group, but not both.

A $(v, K, \lambda, \{g\})_q$-GDD is called $g$-uniform. Subsequently, if $K$ or $G$ are one-element sets, we denote it by small letters, e.g. $(v, k, \lambda, g)_q$-GDD for $K = \{k\}$ and $G = \{g\}$. 

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1 A set of subspaces of $V$ such that every 1-dimensional subspace is covered exactly once is called vector space partition.
In the rest of the paper we study the case $K = \{k\}$ and $G = \{g\}$. The latter implies that the vector space partition $G$ is a partition of the 1-dimensional subspaces of $V$ in subspaces of dimension $g$. In finite geometry such a structure is known as $(g-1)$-spread. Additionally, we will only consider so called simple group divisible designs, i.e. designs without multiple appearances of blocks.

A possible generalization would be to require the last condition in Definition 1 for every $t$-dimensional subspace of $V$, where $t \geq 2$. For $t = 1$ such a definition would make no sense.

An equivalent formulation of the last condition in Definition 1 would be that every block in $B$ intersects the spread elements in dimension of at most one. The $q$-analog of concept of a transversal design would be that every block in $B$ intersects the spread elements exactly in dimension one. But for $q$-analogs this is only possible in the trivial case $g = 1$, $k = v$. However, a related concept was defined in [ES13].

Among all 2-subspaces of $V$, only a small fraction is covered by the elements of $G$. Thus, a $(v, k, \lambda, g)_q$-GDD is “almost” a 2-$(v, k, \lambda)_q$ subspace design, in the sense that the vast majority of the 2-subspaces is covered by $\lambda$ elements of $G$. From a slightly different point of view, a $(v, k, \lambda, g)_q$-GDD is a 2-$(v, k, \lambda, g)_q$ packing design of fairly large size, which are designs where the condition “each $t$-subspace is covered by exactly $\lambda$ blocks” is relaxed to “each $t$-subspace is covered by at most $\lambda$ blocks” [BKW18a]. In Section 6 we give an example of a (6, 3, 2, 2)$_2$-GDD consisting of 180 blocks. This is the largest known 2-$(6, 3, 2)_2$ packing design.

We note that a $q$-analog of a group divisible design can be also seen as a special graph decomposition over a finite field, a concept recently introduced in [BNW]. It is indeed equivalent to a decomposition of a complete $m$-partite graph into cliques where: the vertices are the points of a projective space $PG(n, q)$; the parts are the members of a spread of $PG(n, q)$ into subspaces of a suitable dimension; the vertex-set of each clique is a subspace of $PG(n, q)$ of a suitable dimension.

2 Preliminaries

For $1 \leq m \leq v$ we denote the set of $m$-dimensional subspaces of $V$, also called Grassmannian, by $[V]_m$. It is well known that its cardinality can be expressed by the Gaussian coefficient

$$\# \left[ V \atop m \right]_q = \binom{v}{m}_q = \frac{(q^v - 1)(q^{v-1} - 1) \cdots (q^{v-m+1} - 1)}{(q^m - 1)(q^{m-1} - 1) \cdots (q - 1)}.$$ 

**Definition 2** Given a spread in dimension $v$, let $[V]_k'$ be the set of all blocks that contain no 2-dimensional subspace which is already covered by the spread.

The intersection between a $k$-dimensional subspace $B \in [V]_k'$ and all elements of the spread is at most one-dimensional. In finite geometry such a subspace $B \in [V]_k'$ is called scattered subspace with respect to $G$ [BBL00, BL00].

In case $g = 1$, i.e. $G = [V]_q$, no 2-dimensional subspace is covered by this trivial spread. Then, $(V, B)$ is a 2-$(v, k, \lambda)_q$ subspace design. See [BKW18a].

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for surveys about subspace designs and computer methods for their construction.

Let \( q \cdot s = v \) and \( V = \text{GF}(q)^v \). Then, the set of 1-dimensional subspaces of \( \text{GF}(q^s)^v \) regarded as \( s \)-dimensional subspaces in the \( q \)-linear vector space \( \text{GF}(q)^v \), i.e.

\[
\mathcal{G} = \left[ \text{GF}(q^s)^v \right]_{q^s}^1,
\]

is called Desarguesian spread.

A \( t \)-spread \( \mathcal{G} \) is called normal or geometric, if \( U, V \in \mathcal{G} \) then any element \( W \in \mathcal{G} \) is either disjoint to the subspace \( \langle U, V \rangle \) or contained in it, see e.g. \( \text{Lun99} \).

Since all normal spreads are isomorphic to the Desarguesian spread \( \text{Lun99} \), we will follow \( \text{Lav16} \) and denote normal spreads as Desarguesian spreads.

If \( s \in \{1, 2\} \), then all spreads are normal and therefore Desarguesian. The automorphism group of a Desarguesian spread \( \mathcal{G} \) is \( \text{PGL}(s, q^s) \).

“Trivial” \( q \)-analogs of group divisible designs. For subspace designs, the empty set as well as the the set of all \( k \)-dimensional subspaces in \( \text{GF}(q)^v \) always are a designs, called trivial designs. Here, it turns out that the question if trivial \( q \)-analogs of group divisible designs exist is rather non-trivial.

Of course, there exists always the trivial \( (v, k, 0, g) \)-GDD \( (V, G, \{\} ) \). But it is not clear if the set of all scattered \( k \)-dimensional subspaces, i.e. \( (V, G, \left[ V \right]_{q}^k) \), is always a \( q \)-GDD. This would require that every subspace \( L \in \left[ V \right]_{q}^k \) that is not covered by the spread, is contained in the same number \( \lambda_{\text{max}} \) of blocks of \( \left[ V \right]_{q}^k \).

If this is the case, we call \((V, \left[ V \right]_{q}^k, \mathcal{G}) \) the complete \((v, k, \lambda_{\text{max}}, g) \)-GDD.

If the complete \((v, k, \lambda_{\text{max}}, g) \)-GDD exists, then for any \((v, k, \lambda, g) \)-GDD \((V, G, B) \) the triple \((V, G, \left[ V \right]_{q}^k \setminus B) \) is a \((v, k, \lambda_{\text{max}} - \lambda, g) \)-GDD, called the supplementary \( q \)-GDD.

For a few cases we can answer the question if the complete \( q \)-GDD exists, or in other words, if there is a \( \lambda_{\text{max}} \). In general, the answer depends on the choice of the spread. In the smallest case, \( k = 3 \), however, \( \lambda_{\text{max}} \) exists for all spreads.

**Lemma 1** Let \( \mathcal{G} \) be a \((g - 1)\)-spread in \( V \) and let \( L \) be a 2-dimensional subspace which is not contained in any element of \( \mathcal{G} \). Then, \( L \) is contained in

\[
\lambda_{\text{max}} = \left( \frac{v - 2}{3 - 2} \right) \left( \frac{g - 1}{1} \right)
\]

blocks of \( \left[ V \right]_{q}^k \).

**Proof.** Every 2-dimensional subspace \( L \) is contained in \( \left[ v - 2 \right]_{3 - 2} \) 3-dimensional subspaces of \( \left[ V \right]_{q}^k \). If \( L \) is not contained in any spread element, this means that \( L \) intersects \( \left[ V \right]_{q}^k \) different spread elements and the intersections are 1-dimensional.

Let \( S \) be one such spread element. Now, there are \( \left[ g - 1 \right]_{1} \) choices among the 3-dimensional subspaces in \( \left[ V \right]_{q}^k \) which contain \( L \) to intersect \( S \) in dimension two. Therefore, \( L \) is contained in

\[
\lambda_{\text{max}} = \left( \frac{v - 2}{3 - 2} \right) \left( \frac{g - 1}{1} \right)
\]
blocks of $\left[\begin{array}{c}v \\ 1 \end{array}\right]_q$.

In general, the existence of $\lambda_{\max}$ may depend on the spread. This can be seen from the fact that the maximum dimension of a scattered subspace depends on the spread, see [BL00]. However, for a Desarguesian spread and $g = 2$, $k = 4$, we can determine $\lambda_{\max}$.

**Lemma 2** Let $\mathcal{G}$ be a Desarguesian $(g - 1)$-spread in $V$ and let $L$ be a 2-dimensional subspace which is not contained in any element of $\mathcal{G}$. Then, $L$ is contained in

$$\lambda_{\max} = \left[\frac{v - 2}{4 - 2}\right]_q - 1 - q \left[\frac{2}{1}\right]_q \left[\frac{v - 4}{1}\right]_q - \left[\frac{v}{1}\right]_q / \left[\frac{2}{1}\right]_q + \left[\frac{4}{1}\right]_q / \left[\frac{2}{1}\right]_q$$

blocks of $\left[\begin{array}{c}v \\ 1 \end{array}\right]_q$.

**Proof.** Every 2-dimensional subspace $L$ is contained in $\left[\frac{v - 2}{4 - 2}\right]_q$ 4-dimensional subspaces. If $L$ is not covered by the spread this means that $L$ intersects $\left[\frac{2}{1}\right]_q$ spread elements $S_1, \ldots, S_{q+1}$, which span a 4-dimensional space $F$. All other spread elements are disjoint to $L$. Since $L \subseteq F$, we have to subtract one possibility. For each $1 \leq i \leq q + 1$, $(S_i, L)$ is contained in $q^{\left[\frac{v - 4}{1}\right]}_q$ 4-dimensional subspaces with a 3-dimensional intersection with $F$. All other spread elements $S'$ of $F$ satisfy $(S', L) = F$. If $S''$ is one of the $q^{\left[\frac{v - 4}{1}\right]}_q, q^{\left[\frac{4}{1}\right]}_q, q^{\left[\frac{2}{1}\right]}_q$ spread elements disjoint from $F$, then $F'' := (S'', L)$ intersects $F$ in dimension 2. Moreover, $F''$ does not contain any further spread element, since otherwise $F''$ would be partitioned into $q^2 + 1$ spread elements, where $q + 1$ of them have to intersect $L$. Thus, $L$ is contained in exactly $\lambda_{\max}$ elements from $\left[\begin{array}{c}v \\ 1 \end{array}\right]_q$. □

3 Necessary conditions on $(v, k, \lambda, g)_q$

The necessary conditions for a $(v, k, \lambda, g)_q$-GDD over sets are $g \mid v$, $k \leq v/g$, $\lambda(\frac{v}{g} - 1)g \equiv 0 \pmod{k - 1}$, and $\lambda(\frac{v}{g} - 1)g^2 \equiv 0 \pmod{k(k - 1)}$, see [Han75].

For $q$-analogues of GDDs it is well known that $(g - 1)$-spreads exist if and only if $g$ divides $v$. A $(g - 1)$-spread consists of $\left[\begin{array}{c}v \\ 1 \end{array}\right]_q / \left[\begin{array}{c}g \\ 1 \end{array}\right]_q$ blocks and contains

$$\left[\begin{array}{c}g \\ 2 \end{array}\right]_q, \left[\begin{array}{c}v \\ 1 \end{array}\right]_q / \left[\begin{array}{c}g \\ 1 \end{array}\right]_q$$

2-dimensional subspaces.

Based on the pigeonhole principle we can argue that if $B$ is a block of a $(v, k, \lambda, g)_q$-GDD then there can not be more points in $B$ than the number of spread elements, i.e. if $\left[\begin{array}{c}k \\ 1 \end{array}\right]_q \leq \left[\begin{array}{c}v \\ 1 \end{array}\right]_q / \left[\begin{array}{c}g \\ 1 \end{array}\right]_q$. It follows that (see [BL00], Theorem 3.1)

$$k \leq v - g. \quad (1)$$

This is the $q$-analog of the restriction $k \leq v/g$ for the set case.

If $\mathcal{G}$ is a Desarguesian spread, it follows from [BL00], Theorem 4.3] for the parameters $(v, k, \lambda, g)_q$ to be admissible that

$$k \leq v/2.$$
By looking at the numbers of 2-dimensional subspaces which are covered by spread elements we can conclude that the cardinality of $B$ has to be

$$
\#B = \lambda \frac{\binom{v-1}{1}_q - \binom{g-1}{1}_q \cdot \binom{k}{1}_q / \binom{v}{1}_q}{\binom{2}{1}_q}.
$$

A necessary condition on the parameters of a $g$-uniform $q$-$(k, \lambda)$ GDD is that the cardinality in (2) is an integer number.

Any fixed 1-dimensional subspace $P$ is contained in $\binom{v-1}{1}_q$ 2-dimensional subspaces. Further, $P$ lies in exactly one block of the spread and this block covers $\binom{g-1}{1}_q$ 2-dimensional subspaces through $P$. Those 2-dimensional subspaces are not covered by blocks in $B$. All other 2-dimensional subspaces containing $P$ are covered by exactly $\lambda$ $k$-dimensional blocks. Such a block contains $P$ and there are $\binom{k-1}{1}_q$ 2-dimensional subspaces through $P$ in this block. It follows that $P$ is contained in exactly

$$
\lambda \frac{\binom{v-1}{1}_q - \binom{g-1}{1}_q \cdot \binom{k}{1}_q / \binom{v}{1}_q}{\binom{k-1}{1}_q}
$$

$k$-dimensional blocks and this number must be an integer. The number (3) is the replication number of the point $P$ in the $q$-GDD.

Up to now, the restrictions (1), (2), (3), as well as $g$ divides $v$, on the parameters of a $(v, k, \lambda, g)_q$-GDD are the $q$-analogs of restrictions for the set case. But for $q$-GDDs there is a further necessary condition whose analog in the set case is trivial.

Given a multiset of subspaces of $V$, we obtain a corresponding multiset $P$ of points by replacing each subspace by its set of points. A multiset $P \subseteq \binom{v}{1}_q$ of points in $V$ can be expressed by its weight function $w_P$: For each point $P \in V$ we denote its multiplicity in $P$ by $w_P(P)$. We write

$$
\#P = \sum_{P \in V} w_P(P) \quad \text{and} \quad \#(P \cap H) = \sum_{P \in H} w_P(P)
$$

where $H$ is an arbitrary hyperplane in $V$.

Let $1 \leq r < v$ be an integer. If $\#P \equiv \#(P \cap H) \pmod{q^r}$ for every hyperplane $H$, then $P$ is called $q^r$-divisible. In [KK17, Lemma 1] it is shown that the multiset $P$ of points corresponding to a multiset of subspaces with dimension at least $k$ is $q^{k-1}$-divisible.

**Lemma 3** ([KK17, Lemma 1]) For a non-empty multiset of subspaces of $V$ with $m_i$ subspaces of dimension $i$ let $P$ be the corresponding multiset of points. If $m_i = 0$ for all $0 \leq i < k$, where $k \geq 2$, then

$$
\#P \equiv \#(P \cap H) \pmod{q^{k-1}}
$$

for every hyperplane $H \leq V$.

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2Taking the elements of $P$ as columns of a generator matrix gives a linear code of length $\#P$ and dimension $k$ whose codewords have weights being divisible by $q^r$. 


Proof. We have \( \#\mathcal{P} = \sum_{i=0}^{v} m_i \binom{v}{i} \). The intersection of an \( i \)-subspace \( U \leq V \) with an arbitrary hyperplane \( H \leq V \) has either dimension \( i \) or \( i - 1 \). Therefore, for the set \( \mathcal{P}' \) of points corresponding to \( U \), we get that \( \#\mathcal{P}' = \binom{v}{i} \) and that \( \#(\mathcal{P}' \cap H) \) is equal to \( \binom{v}{i} \) or \( \binom{v-1}{i-1} \). In either case, it follows from \( \binom{v}{i} = \binom{v-1}{i-1} \) (mod \( q^{i-1} \)) that
\[
\#(\mathcal{P}' \cap H) \equiv \binom{i}{1} \text{ (mod } q^{i-1}).
\]
Summing up yields the proposed result. \( \square \)

If there is a suitable integer \( \lambda \) such that \( w_{\mathcal{P}}(P) \leq \lambda \) for all \( P \in V \), then we can define for \( \mathcal{P} \) the complementary weight function
\[
\bar{w}(P) = \lambda - w(P)
\]
which in turn gives rise to the complementary multiset of points \( \bar{\mathcal{P}} \). In [KK17, Lemma 2] it is shown that a \( q^r \)-divisible multiset \( \mathcal{P} \) leads to a multiset \( \bar{\mathcal{P}} \) that is also \( q^r \)-divisible.

**Lemma 4 ([KK17, Lemma 2])** If a multiset \( \mathcal{P} \) in \( V \) is \( q^r \)-divisible with \( r < v \) and satisfies \( w_{\mathcal{P}}(P) \leq \lambda \) for all \( P \in V \) then the complementary multiset \( \bar{\mathcal{P}} \) is also \( q^r \)-divisible.

**Proof.** We have
\[
\#\bar{\mathcal{P}} = \binom{v}{1} \lambda - \#\mathcal{P} \quad \text{and} \quad \#(\bar{\mathcal{P}} \cap H) = \binom{v-1}{1} \lambda - \#(\mathcal{P} \cap H)
\]
for every hyperplane \( H \leq V \). Thus, the result follows from \( \binom{v}{1} \equiv \binom{v-1}{1} \) (mod \( q^r \)) which holds for \( r < v \). \( \square \)

These easy but rather generally applicable facts about \( q^r \)-divisible multiset of points are enough to conclude:

**Lemma 5** Let \( (V, G, B) \) be a \( (v, k, \lambda, g)_q \)-GDD and \( 2 \leq g \leq k \), then \( q^{k-g} \) divides \( \lambda \).

**Proof.** Let \( P \in \binom{[v]}{1} \) be an arbitrary point. Then there exists exactly one spread element \( S \in G \) that contains \( P \). By \( B_P \) we denote the elements of \( B \) that contain \( P \). Let \( S' \) and \( B'_P \) denote the corresponding subspaces in the factor space \( V/P \).

We observe that every point of \( \binom{[S']}{} \) is disjoint to the elements of \( B'_P \) and that every point in \( \binom{[v]}{1} \setminus \binom{[S']}{} \) is met by exactly \( \lambda \) elements of \( B'_P \) (all having dimension \( k - 1 \)). We note that \( B'_P \) gives rise to a \( q^{k-2} \)-divisible multiset \( \mathcal{P} \) of points. So, its complement \( \bar{\mathcal{P}} \), which is the \( \lambda \)-fold copy of \( S' \), also has to be \( q^{k-2} \)-divisible. For every hyperplane \( H \) not containing \( S' \), we have \( \#(\mathcal{P} \cap H) = \lambda \binom{g-2}{1} \) and \( \#(\mathcal{P} \cap H) = \lambda \binom{g-2}{1} \). Thus, \( \lambda q^{k-2} = \#\mathcal{P} - \#(\mathcal{P} \cap H) \equiv 0 \) (mod \( q^{k-2} \)), so that \( q^{k-g} \) divides \( \lambda \). \( \square \)

We remark that the criterion in Lemma 5 is independent of the dimension \( v \) of the ambient space. Summarizing the above we arrive at the following restrictions.

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Theorem 1  Necessary conditions for a $(v,k,\lambda,g)_q$-GDD are

1. $g$ divides $v$,
2. $k \leq v - g$,
3. the cardinalities in (2), (3) are integer numbers,
4. if $2 \leq g \leq k$ then $q^{k-g}$ divides $\lambda$.

If these conditions are fulfilled, the parameters $(v,k,\lambda,g)_q$ are called admissible.

Table 1 contains the admissible parameters for $q = 2$ up to dimension $v = 14$. Column $\lambda_\Delta$ gives the minimum value of $\lambda$ which fulfills the above necessary conditions. All admissible values of $\lambda$ are integer multiples of $\lambda_\Delta$. In column $\#B$ the cardinality of $B$ is given for $\lambda = \lambda_\Delta$. Those values of $\lambda_{\max}$ that are valid for the Desarguesian spread only are given in italics, where the values for $(v,g,k) = (8,4,4)$ and $(9,3,4)$ have been checked by a computer enumeration.

For the case $\lambda = 1$, the online tables [HKKW16] 
http://subspacecodes.uni-bayreuth.de
may give further restrictions, since $B$ is a constant dimension subspace code of minimum distance $2(k-1)$ and therefore

$$\#B \leq A_q(v,2(k-1);k).$$

The currently best known upper bounds for $A_q(v,d;k)$ are given by [HHK+17] Equation (2) referring back to partial spreads and $A_2(6,4;3) = 77$ [HKK15], $A_2(8,6;4) = 257$ [HHK+17] both obtained by exhaustive integer linear programming computations, see also [KK17].

4  $q$-GDDs and $q$-Steiner systems

In the set case the connection between Steiner systems $2-(v,k,1)$ and group divisible designs is well understood.

Theorem 2 ([Han75, Lemma 2.12]) A $2-(v+1,k,1)$ design exists if and only if a $(v,k,1,k-1)$-GDD exists.

There is a partial $q$-analog of Theorem 2.

Theorem 3 If there exists a $2-(v+1,k,1)_q$ subspace design, then a $(v,k,q^2,k-1)_q$-GDD exists.

Proof. Let $V'$ be a vector space of dimension $v+1$ over $\text{GF}(q)$. We fix a point $P \in \left[ V' \right]_q$ and define the projection

$$\pi : \text{PG}(V') \to \text{PG}(V'/P), \ U \mapsto (U + P)/P.$$ 

For any subspace $U \leq V'$ we have

$$\dim(\pi(U)) = \begin{cases} 
\dim(U) - 1 & \text{if } P \leq U, \\
\dim(U) & \text{otherwise}.
\end{cases}$$
Table 1: Admissible parameters for $(v, k, g)_2$-GDDs with $v \leq 14$.

| $v$ | $g$ | $k$ | $\lambda_\Delta$ | $\lambda_{\text{max}}$ | #B | #G |
|-----|-----|-----|-------------------|------------------------|----|----|
| 6   | 2   | 3   | 2                 | 12                     | 180| 21 |
| 6   | 3   | 3   | 3                 | 6                      | 252| 9  |
| 8   | 2   | 3   | 2                 | 60                     | 3060| 85 |
| 8   | 2   | 4   | 4                 | $480$                  | 1224| 85 |
| 8   | 4   | 3   | 7                 | 42                     | 10200| 17 |
| 8   | 4   | 4   | 7                 | $14$                  | 2040| 17 |
| 9   | 3   | 3   | 1                 | 118                    | 6132| 73 |
| 9   | 3   | 4   | 10                | $1680$                 | 12264| 73 |
| 10  | 2   | 3   | 14                | 252                    | 347820| 341 |
| 10  | 2   | 4   | 28                | $10080$                | 139128| 341 |
| 10  | 2   | 5   | 8                 |                        | 8976| 341 |
| 10  | 5   | 3   | 21                | 210                    | 507408| 33 |
| 10  | 5   | 4   | 35                |                        | 169136| 33 |
| 10  | 5   | 5   | 15                |                        | 16368| 33 |
| 12  | 2   | 3   | 2                 | 1020                   | 797940| 1365 |
| 12  | 2   | 4   | 28                | $171360$               | 2234232| 1365 |
| 12  | 2   | 5   | 40                |                        | 720720| 1365 |
| 12  | 2   | 6   | 16                |                        | 68640| 1365 |
| 12  | 3   | 3   | 3                 | 1014                   | 1195740| 585 |
| 12  | 3   | 4   | 2                 |                        | 159432| 585 |
| 12  | 3   | 5   | 1860              |                        | 33480720| 585 |
| 12  | 3   | 6   | 248               |                        | 1062880| 585 |
| 12  | 4   | 3   | 1                 | 1002                   | 397800| 273 |
| 12  | 4   | 4   | 7                 |                        | 556920| 273 |
| 12  | 4   | 5   | 62                |                        | 1113840| 273 |
| 12  | 4   | 6   | 124               |                        | 530400| 273 |
| 12  | 6   | 3   | 1                 | 930                    | 393120| 65 |
| 12  | 6   | 4   | 1                 |                        | 78624| 65 |
| 12  | 6   | 5   | 155               |                        | 2751840| 65 |
| 12  | 6   | 6   | 31                |                        | 131040| 65 |
| 14  | 2   | 3   | 2                 | 4092                   | 12778740| 5461 |
| 14  | 2   | 4   | 4                 | $2782560$             | 5111496| 5461 |
| 14  | 2   | 5   | 248               |                        | 71560944| 5461 |
| 14  | 2   | 6   | 496               |                        | 34076640| 5461 |
| 14  | 2   | 7   | 32                |                        | 536640| 5461 |
| 14  | 7   | 3   | 21                | 3906                   | 133161024| 129 |
| 14  | 7   | 4   | 35                |                        | 44387008| 129 |
| 14  | 7   | 5   | 465               |                        | 133161024| 129 |
| 14  | 7   | 6   | 651               |                        | 44387008| 129 |
| 14  | 7   | 7   | 63                |                        | 1048512| 129 |
Let $\mathcal{D} = (V', B')$ be a 2-$(v + 1, k, 1)_q$ subspace design. The set
\[ \mathcal{G} = \{ \pi(B) \mid B \in B', P \in B \} \]
is the derived design of $\mathcal{D}$ with respect to $P$. It is shown that if a group $G$ acts transitively on the subsets of cardinality $A$. A very successful approach to construct designs, since in [CK79, Prop. 8.4] it is shown that if a group $G$ acts transitively on the subsets of cardinality $A$, in other words, it is a $(k - 2)$-spread in $V'/P$. Now define
\[ B = \{ \pi(B) \mid B \in B', P \notin B \} \]
and $V = V'/P$.

We claim that $(V, \mathcal{G}, B)$ is a $(v, k, q^2, k - 1)_q$-GDD.

In order to prove this, let $L \in [V']_{(2)}$ be a line not covered by any element in $\mathcal{G}$. Then $L = E/P$, where $E \in [V']_{q}$, $P \leq E$ and $E$ is not contained in a block of the design $\mathcal{D}$. The blocks of $\mathcal{B}$ covering $L$ have the form $\pi(B)$ with $B \in B'$ such that $B \cap E$ is a line in $E$ not passing through $P$. There are $q^2$ such lines and each line is covered in a unique block in $B'$. Since these $q^2$ blocks $B$ have to be pairwise distinct and do not contain the point $P$, we get that there are $q^2$ blocks $\pi(B) \in B$ containing $L$.

Since there are 2-(13, 3, 1)$_2$ subspace designs, by Theorem 3 there are also 2-(13, 3, 4, 2)$_2$-GDDs. The smallest admissible case of a 2-$(v, 3, 1)_q$ subspace design is $v = 7$, which is known as a $q$-analog of the Fano plane. Its existence is a notorious open question for any value of $q$. By Theorem 3 the existence would imply the existence of a $(6, 3, q^2, 2)_q$-GDD, which has been shown to be true in [EH17] for any value of $q$, in the terminology of a “residual construction for the $q$-Fano plane”. In Theorem 4 we will give a general construction of $q$-GDDs covering these parameters. The crucial question is if a $(6, 3, q^2, 2)_q$-GDD can be “lifted” to a 2-$(7, 3, 1)_q$ subspace design. While the GDDs with these parameters constructed in Theorem 3 have a large automorphism group, for the binary case $q = 2$ we know from [BKN18] that the order of the automorphism group of a putative 2-$(7, 3, 1)_2$ subspace design is at most two. So if the lifting construction is at all possible for the binary $(6, 3, 4, 2)_2$-GDD from Theorem 4 necessarily many automorphisms have to “get destroyed”.

In Table 2 we can see that there exists a $(8, 3, 4, 2)_2$-GDD. This might lead in the same way to a 2-$(9, 3, 1)_2$ subspace design, which is not known to exist.

5 A general construction

A very successful approach to construct $t$-$(v, k, \lambda)$ designs over sets is to prescribe an automorphism group which acts transitively on the subsets of cardinality $t$. However for $q$-analogues of designs with $t \geq 2$ this approach yields only trivial designs, since in [CK79, Prop. 8.4] it is shown that if a group $G \leq \text{PGL}(v, q)$ acts transitively on the $t$-dimensional subspaces of $V$, $2 \leq t \leq v - 2$, then $G$ acts transitively also on the $k$-dimensional subspaces of $V$ for all $1 \leq k \leq v - 1$.

The following lemma provides the counterpart of the construction idea for $q$-analogues of group divisible designs. Unlike the situation of $q$-analogues of designs, in this slightly different setting there are indeed suitable groups admitting the general construction of non-trivial $q$-GDDs, which will be described in the sequel. Itoh’s construction of infinite families of subspace designs is based on a similar idea [Itoh].
Lemma 6 Let \( \mathcal{G} \) be a \((q - 1)\)-spread in \( \text{PG}(V) \) and let \( G \) be a subgroup of the stabilizer \( \text{PGL}(v, q)_{\mathcal{G}} \) of \( \mathcal{G} \) in \( \text{PGL}(v, q) \). If the action of \( G \) on \( \bigcup_{S \in \mathcal{G}} \binom{S}{2} \) is transitive, then any union \( \mathcal{B} \) of \( G \)-orbits on the set of \( k \)-subspaces which are scattered with respect to \( \mathcal{G} \) yields a \((v, k, \lambda, g)_{\mathcal{G}}\)-\( \text{GDD} \) \((V, \mathcal{G}, \mathcal{B})\) for a suitable value \( \lambda \).

Proof. By transitivity, the number \( \lambda \) of blocks in \( \mathcal{B} \) passing through a line \( L \in \binom{V}{2} \setminus \bigcup_{S \in \mathcal{G}} \binom{S}{2} \) does not depend on the choice of \( L \). \( \square \)

In the following, let \( V = \text{GF}(q^2)^s \), which is a vector space over \( \text{GF}(q) \) of dimension \( v = gs \). Furthermore, let \( \mathcal{G} = \binom{V}{1}_{q^g} \) be the Desarguesian \((g - 1)\)-spread in \( \text{PG}(V) \). For every \( \text{GF}(q) \)-subspace \( U \leq V \) we have that

\[
\dim_{\text{GF}(q)}((U)_{\text{GF}(q^2)}) \leq \dim_{\text{GF}(q)}(U).
\]

In the case of equality, \( U \) will be called fat. Equivalently, \( U \) is fat if and only if one (and then any) \( \text{GF}(q) \)-basis of \( U \) is \( \text{GF}(q^2) \)-linearly independent. The set of fat \( k \)-subspaces of \( V \) will be denoted by \( \mathcal{F}_k \).

We remark that for a fat subspace \( U \), the set of points \( \{x_{\text{GF}(q^s)} : x \in U\} \) is a Baer subspace of \( V \) as a \( \text{GF}(q^s) \)-vector space.

Lemma 7

\[
\#\mathcal{F}_k = q^{g(q-1)}\binom{q}{2} \prod_{i=0}^{k-1} \frac{q^{g(s-i)} - 1}{q^{k-i} - 1}.
\]

Proof. A sequence of \( k \) vectors in \( V \) is the \( \text{GF}(q) \)-basis of a fat \( k \)-subspace if and only if it is linearly independent over \( \text{GF}(q^2) \). Counting the set of those sequences in two ways yields

\[
\#\mathcal{F}_k \cdot \prod_{i=0}^{k-1} (q^q - q^i) = \prod_{i=0}^{k-1} ((q^q)^s - (q^q)^i),
\]

which leads to the stated formula. \( \square \)

We will identify the unit group \( \text{GF}(q)^* \) with the corresponding group of \( s \times s \) scalar matrices over \( \text{GF}(q^2) \).

Lemma 8 Consider the action of \( \text{SL}(s, q^2)/\text{GF}(q)^* \) on the set of the fat \( k \)-subspaces of \( V \). For \( k < s \), the action is transitive. For \( k = s \), \( \mathcal{F}_k \) splits into \( \frac{q^s-1}{q-1} \) orbits of equal length.

Proof. Let \( U \) be a fat \( k \)-subspace of \( V \) and let \( B \) be an ordered \( \text{GF}(q) \)-basis of \( U \). Then \( B \) is an ordered \( \text{GF}(q^2) \)-basis of \( (U)_{\text{GF}(q^2)} \).

For \( k < s \), \( B \) can be extended to an ordered \( \text{GF}(q^2) \)-basis \( B' \) of \( V \). Let \( A \) be the \((s \times s)\)-matrix over \( \text{GF}(q^2) \) whose columns are given by \( B' \). By scaling one of the vectors in \( B' \setminus B \), we may assume \( \det(A) = 1 \). Now the mapping \( V \rightarrow V, x \mapsto Ax \) is in \( \text{SL}(s, q^2) \) and maps the fat \( k \)-subspace \( \langle e_1, \ldots, e_k \rangle \) to \( U \langle e_i \rangle \) denoting the \( i \)-th standard vector of \( V \). Thus, the action of \( \text{SL}(s, q^2)/\text{GF}(q)^* \) is transitive on \( \mathcal{F}_k \).

It remains to consider the case \( k = s \). Let \( A \) be the \((s \times s)\)-matrix over \( \text{GF}(q^2) \) whose columns are given by \( B \). As any two \( \text{GF}(q) \)-bases of \( U \) can be
mapped to each other by a GF($q$)-linear map, we see that up to a factor in GF($q^*$), det($A$) does not depend on the choice of $B$. Thus,

$$\det(U) := \det(A) \cdot \text{GF}(q^*) \in \text{GF}(q^*)^*/\text{GF}(q)^*$$

is invariant under the action of $\text{SL}(s,q^*)$ on $\mathcal{F}_k$. It is readily checked that every value in $\text{GF}(q^*)^*/\text{GF}(q)^*$ appears as the invariant $\det(U)$ for some fat $s$-subspace $U$, and that two fat $s$-subspaces having the same invariant can be mapped to each other within $\text{SL}(s,q^*)$. Thus, the number of orbits of the action of $\text{SL}(s,q^*)$ on $\mathcal{F}_s$ is given by the number $\#(\text{GF}(q^*)^*/\text{GF}(q)^*) = \frac{q^s - 1}{q^i - 1}$ of invariants. As $\text{SL}(s,q^*)$ is normal in $\text{GL}(s,q^*)$ which acts transitively on $\mathcal{F}_s$, all orbits have the same size. Modding out the kernel $\text{GF}(q)^*$ of the action yields the statement in the lemma. 

\[\square\]

**Theorem 4** Let $V$ be a vector space over $\text{GF}(q)$ of dimension $gs$ with $g \geq 2$ and $s \geq 3$. Let $\mathcal{G}$ be a Desarguesian $(g-1)$-spread in $\text{PG}(V)$. For $k \in \{3, \ldots, s-1\}$, $(V, \mathcal{G}, \mathcal{F}_k)$ is a $(gs, k, \lambda, g)_q$-GDD with

$$\lambda = q^{(g-1)((s^2)-1)} \prod_{i=2}^{k-1} \frac{q^{(s-i)} - 1}{q^{k-i} - 1}.$$

Moreover, for each $\alpha \in \{1, \ldots, \frac{q^s - 1}{q^i - 1}\}$, the union $\mathcal{B}$ of any $\alpha$ orbits of the action of $\text{SL}(s,q^*)/\text{GF}(q)^*$ on $\mathcal{F}_s$ gives a $(gs, s, \lambda, g)_q$-GDD $(V, \mathcal{G}, \mathcal{B})$ with

$$\lambda = \alpha q^{(g-1)((s^2)-1)} \prod_{i=2}^{s-2} \frac{q^{gi} - 1}{q^i - 1}.$$

**Proof.** We may assume $V = \text{GF}(q^*)^*$ and $\mathcal{G} = \{V_{\mathcal{H}}\}_{i=2}^{s}$. The lines covered by the elements of $\mathcal{G}$ are exactly the non-fat $\text{GF}(q)$-subspaces of $V$ of dimension 2. By Lemma 8 and Lemma 9 $(V, \mathcal{G}, \mathcal{F}_k)$ is a GDD. Double counting yields $\#\mathcal{F}_2 \cdot \lambda = \#\mathcal{F}_k \cdot \frac{V}{\mathcal{H}}$. Using Lemma 7 this equation transforms into the given formula for $\lambda$.

In the case $k = s$, by Lemma 8 each union $\mathcal{B}$ of $\alpha \in \{1, \ldots, \frac{q^s - 1}{q^i - 1}\}$ orbits under the action of $\text{SL}(s,q^*)/\text{GF}(q)^*$ on $\mathcal{F}_s$ yields a GDD with

$$\lambda = \alpha q^{(g-1)((s^2)-1)} \frac{q - 1}{q^i - 1} \prod_{i=2}^{s-1} \frac{q^{(s-i)} - 1}{q^{i-1} - 1} = \alpha q^{(g-1)((s^2)-1)} \prod_{i=2}^{s-2} \frac{q^{gi} - 1}{q^i - 1}.$$

\[\square\]

**Remark 1** In the special case $q = 2$, $s = 3$ and $\alpha = 1$ the second case of Theorem 4 yields $(6,3,q^2,2)_q$-GDDs. These parameters match the “residual construction for the $q$-Fano plane” in [EH17].

**Remark 2** A fat $k$-subspace ($k \in \{3, \ldots, s\}$) is always scattered with respect to the Desarguesian spread $\{V_{\mathcal{H}}\}_{i=2}^{s}$. The converse is only true for $g = 2$. Thus, Theorem 4 implies that the set of all scattered $k$-subspaces with respect to the Desarguesian line spread of $\text{GF}(q)^{2s}$ is a $(2s, k, \lambda_{\text{max}}, 2)_q$-GDD.
Table 2: Existence results for \((v, k, \lambda, g)_{q}\)-GDD for \(q = 2\).

| \(v\) | \(g\) | \(k\) | \(\lambda\) | \(\lambda_{\text{max}}\) | \(\lambda\) | comments |
|---|---|---|---|---|---|---|
| 6 | 2 | 3 | 2 | 12 | 4 | EHL7 |
| 6 | 3 | 3 | 6 | 3, 6 | \(\langle \sigma^{7} \rangle\) |
| 8 | 2 | 3 | 2 | 60 | 2, 58 | \(\langle \sigma, \phi^{4} \rangle\) |
| 8 | 2 | 4 | 4 | 480 | 20, 40, \ldots, 480 | \(\langle \sigma, \phi \rangle\) |
| 8 | 4 | 3 | 7 | 42 | 7, 21, 35 | Thm. 4 |
| 8 | 4 | 4 | 7 | 14 | 14, 28, 42 | \(\langle \sigma, \phi \rangle\) |
| 9 | 3 | 3 | 1 | 118 | 2, 3, \ldots, 115, 116, 118 | \(\langle \sigma, \phi \rangle\) |
| 9 | 3 | 4 | 10 | 1680 | 30, 60, \ldots, 1680 | \(\langle \sigma \rangle\) |
| 10 | 2 | 3 | 14 | 252 | 14, 28, \ldots, 252 | \(\langle \sigma \rangle\) |
| 10 | 5 | 3 | 21 | 210 | 105, 210 | \(\langle \sigma, \phi^{2} \rangle\) |
| 12 | 2 | 3 | 2 | 1020 | 4 | BEÖ+16 |
| 12 | 2 | 6 | 16 | 12533760 | \(\alpha = 1, \ldots, 3\) | Thm. 4 |
| 12 | 3 | 4 | 2 | 21504 | \(\alpha = 1, \ldots, 7\) | Thm. 4 |
| 12 | 4 | 3 | 1 | 1002 | 64 \(\alpha = 1, \ldots, 15\) | Thm. 4 |

6 Computer constructions

An element \(\pi \in \text{PGL}(v, q)\) is an automorphism of a \((v, k, \lambda, g)_{q}\)-GDD if \(\pi(G) = G\) and \(\pi(B) = B\).

Taking the Desarguesian \((g - 1)\)-spread and applying the Kramer-Mesner method [KM76] with the tools described in [BKL05, BKW18b, BKW18a] to the remaining blocks, we have found \((v, k, \lambda, g)_{q}\)-GDDs for the parameters listed in Tables 2. In all cases, the prescribed automorphism groups are subgroups of the normalizer \(\langle \sigma, \phi \rangle\) of a Singer cycle group generated by an element \(\sigma\) of order \(q^{v} - 1\) and by the Frobenius automorphism \(\phi\), see [BKW18a]. Note that the presented necessary conditions for \(\lambda_{\Delta}\) turn out to be tight in several cases.

Example. We take the primitive polynomial \(1 + x + x^{3} + x^{4} + x^{6}\), together with the canonical Singer cycle group generated by

\[
\sigma = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

For a compact representation we will write all \(\alpha \times \beta\) matrices \(X\) over \(\text{GF}(q)\).
Table 3: Existence results for \((v, k, \lambda, g)_q\)-GDD for \(q = 3\).

| \(v\) | \(g\) | \(k\) | \(\lambda_\Delta\) | \(\lambda_{\text{max}}\) | \(\lambda\) | comments |
|---|---|---|---|---|---|---|
| 6 | 2 | 3 | 3 | 36 | 9 | \([\text{EH}17]\) |
| | | | | | | \(9\alpha, \alpha = 1, \ldots, 4\) Thm. 4 |
| 6 | 3 | 3 | 4 | 24 | 12, 24 | \(\langle \sigma^{13}, \phi \rangle\) |
| 8 | 2 | 4 | 9 | 9720 | 2430\(\alpha\), \(\alpha = 1, \ldots, 4\) Thm. 4 |
| 8 | 4 | 3 | 13 | 312 | 52, 104, 156, 208, 260, 312 | \(\langle \sigma, \phi \rangle\) |
| 9 | 3 | 3 | 1 | 1077 | 81\(\alpha\), \(\alpha = 1, \ldots, 13\) Thm. 4 |
| 10 | 2 | 5 | 27 | 22044960 | 5511240\(\alpha\), \(\alpha = 1, \ldots, 4\) Thm. 4 |
| 12 | 2 | 6 | 81 | 439267872960 | 109816968240\(\alpha\), \(\alpha = 1, \ldots, 4\) Thm. 4 |
| 12 | 3 | 4 | 3 | 5373459\(\alpha\), \(\alpha = 1, \ldots, 13\) Thm. 4 |
| 12 | 4 | 3 | 1 | 29472 | 729\(\alpha\), \(\alpha = 1, \ldots, 40\) Thm. 4 |

with entries \(x_{i,j}\), whose indices are numbered from 0, as vectors of integers

\[ \sum_j x_{0,j}q^j, \ldots, \sum_j x_{\alpha-1,j}q^j, \]

i.e. \(\sigma = [2, 4, 8, 16, 32, 27]\).

The block representatives of a \((6, 3, 2, 2)_2\)-GDD can be constructed by pre-
scribing the subgroup \(G = \langle \sigma^7 \rangle\) of the Singer cycle group. The order of \(G\) is
9, a generator is \([54, 55, 53, 49, 57, 41]\). The spread is generated by \([1, 14]\), under
the action of \(G\) the 21 spread elements are partitioned into 7 orbits. The blocks
of the GDD consist of the \(G\)-orbits of the following 20 generators.

\[ [3, 16, 32], [15, 16, 32], [4, 8, 32], [5, 8, 32], [19, 24, 32], [7, 24, 32], [10, 4, 32],
\[ [18, 28, 32], [17, 20, 32], [1, 28, 32], [17, 10, 32], [25, 2, 32], [13, 6, 32], [29, 30, 32],
\[ [33, 12, 16], [38, 40, 16], [2, 36, 16], [1, 36, 16], [11, 12, 16], [19, 20, 8]\]

Acknowledgements

The authors are grateful to Anton Betten who pointed out the connection to
scattered subspaces.

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