Rough Center Manifolds

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Abstract

Since the breakthrough in rough paths theory for stochastic ordinary differential equations (SDEs), there has been a strong interest in investigating the rough differential equation (RDE) approach and its numerous applications. Rough path techniques can stay closer to deterministic analytical methods and have the potential to transfer many pathwise ordinary differential equation (ODE) techniques more directly to a stochastic setting. However, there are few works that analyze dynamical properties of RDEs and connect the rough path / regularity structures, ODE and random dynamical systems approaches. Here we contribute to this aspect and analyze invariant manifolds for RDEs. By means of a suitably discretized Lyapunov-Perron-type method we prove the existence and regularity of local center manifolds for such systems. Our method directly works with the RDE and we exploit rough paths estimates to obtain the relevant contraction properties of the Lyapunov-Perron map.

1 Introduction

Classically, the theory of center manifolds is well-studied for ODEs [12, 51] of the form

\[
\frac{dU}{dt} := U' = AU + F(U), \quad U = U_t \in \mathbb{R}^n,
\]

where \(A \in \mathbb{R}^{n \times n}\) is a matrix, \(F(0) = 0\), and \(F(U) = O(\|U\|^2)\) as \(\|U\| \to 0\). Suppose \(A\) has spectrum on the imaginary axis, then the equilibrium \(U \equiv 0\) is non-hyperbolic, yet there usually also eigenvalues with positive and negative real parts. Center manifolds help us to reduce the dynamics to the dimension of the number of eigenvalues with zero real part [12, 51]. The theory also naturally extends to several classes of infinite-dimensional partial differential equations (PDEs) [52], e.g., thinking of \(A\) in (1) as a differential operator and viewing (1) as an evolution equation on a Banach space [4, 35].

Our aim here is to develop a theory for center manifolds for SDEs driven by multiplicative noise, which goes far beyond the case of a Brownian motion. We are going to develop center manifolds within the theory of rough paths [43, 26, 22]. Of course, the first step is to establish existence of center manifolds. Therefore, this work is entirely devoted to this aspect. Our proof shows that rough path theory is ideally suited to carry out the Lyapunov-Perron method for existence of center manifolds for stochastic systems.

Based on the existence theory, and motivated by numerous physical applications, we are going to analyze approximations of center manifolds [8, 9, 15, 18, 19] and bifurcations [11, 38] in future work. Regarding this, a challenging question that naturally arises is whether the expansions used in the rough paths/regularity structures theory can help us gain dynamical insight.

There is already considerable interest in analyzing stable/unstable or center manifolds for stochastic (partial) differential equations (SDEs/SPDEs), see e.g. [11, 17, 8, 9, 15, 19, 18]. However, in previous
works the standard approach to derive invariant manifold results is a transformation argument, often carried out for SDEs for $U = U_t$ of the form
\[
dU = (AU + F(U)) \, dt + U \circ d\tilde{B}, \quad U_0 =: \xi \in \mathbb{R}^n. \tag{2}
\]
Here, $\circ$ stands for Stratonovich differential, $\tilde{B} = \tilde{B}_t$ is a two-sided real-valued Brownian motion, $A \in \mathbb{R}^{n \times n}$ is a matrix, $n \geq 1$, and $F : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous with $F(0) = 0$. As for the ODE/PDE case, some results for (2) can be extended to certain unbounded operators $A$ generating strongly-continuous semigroups on separable Banach spaces, which is relevant for SPDEs. The classical transformation of (2) relies on the Ornstein-Uhlenbeck (OU) process as follows: Consider the unique solution $z = z_t$ of the one-dimensional OU process given by the SDE
\[
dz = -z \, dt + d\tilde{B}. \tag{3}
\]
Performing the Doss-Sussmann-type transformation $\tilde{U} = U e^{-z}$, one obtains that (2) can be transformed to a non-autonomous random differential equation
\[
\tilde{U}' = A\tilde{U} + \tilde{F}(\tilde{U}; z), \tag{4}
\]
where the map $\tilde{F}$ depends upon the time-dependent process $z$ and hence on $\tilde{B}$; see [1]. Now, the random differential equation (4) can be analyzed using the properties of the OU process [19, Lem. 2.1] and suitable assumptions on the coefficients. Of course, the same methods apply if one takes in (2) linear multiplicative noise, namely $G(U) \circ d\tilde{B}$, where $G$ is a linear operator generating a strongly-continuous group commuting with the strongly-continuous semigroup generated by $A$.

Under suitable spectral assumptions on $A$, which imply the existence of an exponential trichotomy that entails invariant splittings of the phase space in stable/unstable/center subspaces, one can set up a classical Lyapunov-Perron method and derive under certain gap conditions invariant manifolds for (4). Since the random dynamical systems generated by (2) and (4) are conjugated, one can transfer all these results to (2).

Instead, the goal of this paper is to use a direct pathwise approach [25] to investigate center manifolds for RDEs of the form (5). More precisely, we consider (2) driven by a quite general nonlinear multiplicative noise. Moreover, the random input goes far beyond the case of a Brownian motion and it can be a certain Gaussian process. The only restriction comes from the Hölder regularity of its trajectories. We are going to prove our results using rough paths, which have not been employed for invariant manifolds so far. Since the breakthrough in the rough paths theory introduced by T. Lyons [43], there has been a growing interest in analyzing flows driven by rough paths [2] or random dynamical systems for rough differential equations [3]. The main techniques used in this work rely on Gubinelli’s controlled rough paths as described in [26, 22]. This theory gives a pathwise meaning to (2) without reducing it to an equation with random coefficients. Since the solution of an RDE driven by a Stratonovich rough path lift yields a strong solution of a classical Stratonovich SDE [22, Thm. 9.1(ii)], many classical center manifold results for (2) can be recovered as special cases.

This work is structured as follows. In Section 2 we provide background on controlled rough paths and RDEs. Section 3 is devoted to dynamics of rough differential equations [3]. The existence of center manifolds is based on a discrete-time Lyapunov-Perron method. This technically challenging step is necessary since we work with pathwise integrals and therefore need to control at each step the norms of the random input on a fixed time-interval. After deriving suitable estimates of the controlled rough integrals, the center manifold theory follows using a random dynamical systems approach. The results obtained for the discrete Lyapunov-Perron map can then be extended to the time-continuous one [41, 25]. Finally, we point out in Section 5 the main arguments, which lead to the smoothness of the manifolds obtained. This requires technical transfer of several existing ideas [19, 15, 30] to the RDE
bounded derivatives, and as a major direction for future work. Eventually applicable for quite large classes of singular SPDEs [31, 27, 32, 33, 5]. We leave this aspect to dichotomies to trichotomies.

see [10]. Hence, the techniques we developed here for RDEs/SDEs may have the potential to be possible as controlled rough paths can be viewed as one particular instance of a regularity structure; see [10].

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Notation: We use $\mathbb{E}$ to denote the expectation, $\otimes$ for the tensor product, $\oplus$ for the direct sum, $C^m$ for the space of $m$-times differentiable maps, $C^m_0$ for the space of $m$-times differentiable maps with bounded derivatives, and $C^\alpha$ for $\alpha$-Hölder maps.

The theory of controlled rough paths gives a meaning to (6) even if the paths of the driving process are not smooth, but only $\alpha$-Hölder for $\alpha \in (1/3, 1/2)$ and a finite-dimensional vector space $\mathcal{V}$ (the reader may think of $\mathcal{V} = \mathbb{R}^d$ as one concrete example we shall often use for $\mathcal{V}$). Then we define $\mathcal{W} := (W, \mathcal{W})$ as an $\alpha$-Hölder rough-path, more precisely

$$W \in C^\alpha([0, 1]; \mathcal{V}) \quad \text{and} \quad \mathcal{W} \in C^{2\alpha}([0, 1]^2; \mathcal{V} \otimes \mathcal{V}),$$

and the connection between $W$ and $\mathcal{W}$ is given by Chen’s relation. This means that

$$\mathcal{W}_{s,t} - \mathcal{W}_{s,u} - \mathcal{W}_{u,t} = W_{s,u} \otimes W_{u,t}, \quad (7)$$

where $W_{s,t} := W_t - W_s$. For a smooth path $W$, $\mathcal{W}$ could be constructed using iterated integrals of $W$. The theory of controlled rough paths gives a meaning to (6) even if the paths of the driving process are not smooth, but only $\alpha$-Hölder regular for $\alpha \in (1/3, 1/2)$. This includes Brownian motion and fractional Brownian motion with Hurst parameter $H \in (1/3, 1/2)$. The second order process $\mathcal{W}$ can be thought of as

$$\int_0^t W_{s,u} \otimes dW_u.$$
Remark 2.1. Results regarding global existence and uniqueness of a solution of (5) for smooth $F$ and $G$ can be found in [22, Ch. 8]. These are stated without using Duhamel’s formula and incorporating $dt$ in a space-time rough path. Results without incorporating $dt$ in a space-time rough path can be found in [10, 23]. Since we are going to investigate center manifolds for (5), we prefer the mild formulation specified in [4]. In fact, mild solutions to SPDEs [40] and their variants [44, 47], are used implicitly in regularity structures [27] and form a cornerstone of deterministic PDE dynamics [35] so it is also natural for our future goals to use this notion of solution.

Now we consider preliminary results, which are necessary to establish the existence of a solution for (5), cf. [22, Sec. 8.5]. These results, and the techniques by which they are proven, are going to also play a key role for our center manifold problem. Throughout this section (5), $C$ can be found in [22, Ch. 8]. These are stated without using Duhamel’s formula and incorporating $dt$. We proceed with some elementary rough path estimates. Here we emphasize that

$$
\|Y\|_{\infty} = \|Y\|_{\infty} |Y|_0 + \|Y\|_\alpha + \|R^Y\|_{2\alpha},
$$

where the remainder $R^Y$ has $2\alpha$-Hölder regularity. The space of controlled rough paths $(Y, Y')$ is denoted by $D^\alpha_W([0,1];\mathcal{X}) = D^\alpha_W$. This space is endowed with the semi-norm

$$
\|Y, Y'\|_{D^\alpha_W} := \|Y'\|_\alpha + \|R^Y\|_{2\alpha}.
$$

We could also endow $D^\alpha_W$ with a norm $|Y_0| + |Y'_0| + \|Y'\|_\alpha + \|R^Y\|_{2\alpha}$ but note that this reduces to the semi-norm if $Y_0 = 0$ and $Y'_0 = 0$, which is a fact we shall in Section 4.2. $Y'$ is referred to as the Gubinelli derivative of $Y$; see [20] and [22, Ch. 4]. We shall always use $Y, Y', \mathcal{V}, \mathcal{X}$ in the context of an abstract controlled rough path. Yet, there are obviously more concrete cases.

Example 2.3. Consider integrating the RDE (5) directly over time, then one mayo give meaning to the integral $\int G(U) \, dW_r$ by using

$$
Y = G(u), \quad Y' = DG(u)G(u), \quad \mathcal{V} = \mathbb{R}^d, \quad \mathcal{X} = \mathbb{R}^{n\times d},
$$

DG denotes the total derivative of $G$. In this case, we also note that

$$
\mathcal{L}(\mathcal{V}, \mathcal{X}) = \mathcal{L}(\mathbb{R}^d, \mathbb{R}^{n\times d}) = \mathcal{L}(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n)) = \mathcal{L}(\mathbb{R}^d \otimes \mathbb{R}^d, \mathbb{R}^n).
$$

Yet, we shall see that this example does not suffice for our case as we also would like to study solution formulas involving a (semi-)group.

We proceed with some elementary rough path estimates. Here we emphasize that $C$ stands for a universal constant, which may vary from line to line. The dependence of this constant

$$
C = C[\cdot, \cdot\ldots];
$$

on certain parameters and/or problem input will be explicitly stated in square brackets. $C$ can always be uniformly chosen over the time interval, i.e., for $T \in [0,1]$.

Re-writing (5) entails $Y_{s,t} = Y'_s W_{s,t} + R^Y_{s,t}$, so one obtains

$$
|Y_{s,t}| \leq \|Y'\|_\infty \|W\|_\alpha (t-s)^\alpha + \|R^Y\|_{2\alpha} (t-s)^{2\alpha}, \quad \text{for } s, t \in [0,1].
$$

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This immediately entails the following estimates for the path \( Y \):
\[
\|Y\|_\infty \leq \|Y'\|_\infty \|W\|_\alpha + \|R_Y\|_{2\alpha}
\]
as well as
\[
\|Y\|_\alpha \leq \|Y'\|_\infty \|W\|_\alpha + \|R_Y\|_{2\alpha}.
\]  
(10)
Since
\[
\|Y\|_\infty \leq |Y_0| + \|Y\|_\alpha,
\]
the previous estimate leads to
\[
\|Y\|_\alpha \leq C(1 + \|W\|_\alpha)(|Y_0| + \|Y, Y'\|_{D_W^{2\alpha}}).
\]  
(11)
The second estimate immediately follows from (10) combined with the definition of the semi-norm on \( D_W^{2\alpha} \). The next step is to explain the concept of the integral we use in (10). 

**Theorem 2.4.** (Prop. 1) Let \((Y, Y') \in D_W^{2\alpha}([0,1]; \mathbb{R}^{d \times n})\) and \(W = (W, \mathbb{W})\) be a \(\mathbb{R}^d\)-valued \(\alpha\)-Hölder rough path for some \(\alpha \in (\frac{1}{3}, \frac{1}{2})\). Furthermore, \(P\) stands for a partition of \([0,1]\). Then, the integral of \(Y\) against \(W\) defined as
\[
\int_s^t Y_r \, dW_r := \lim_{|P| \to 0} \sum_{[u,v] \in P} (Y_u W_{u,v} + Y'_u \mathbb{W}_{u,v})
\]  
(12)
extists for every pair \(s, t \in [0,1]\). Moreover, the estimate
\[
\left| \int_s^t Y_r \, dW_r - Y_s W_{s,t} - Y'_s \mathbb{W}_{s,t} \right| \leq C(\|W\|_\alpha \|R_Y\|_{2\alpha} + \|\mathbb{W}\|_{2\alpha} \|Y'\|_\alpha) |t - s|^{3\alpha}
\]  
(13)
holds true for all \(s, t \in [0,1]\). The map from \(D_W^{2\alpha}([0,1]; \mathbb{R}^{d \times n})\) to \(D_W^{2\alpha}([0,1]; \mathbb{R}^n)\) given by
\[
(Y, Y') \mapsto (P, P') := \left( \int_0^t Y_r \, dW_r, Y_t \right)
\]
is linear and continuous. Furthermore, the estimate
\[
\|P, P'\|_{D_W^{2\alpha}} \leq \|Y\|_\alpha + \|Y'\|_\infty \|\mathbb{W}\|_{2\alpha} + C(\|W\|_\alpha \|R_Y\|_{2\alpha} + \|\mathbb{W}\|_{2\alpha} \|Y'\|_\alpha)
\]  
(14)
is valid.

Next, we have to consider the semigroup \(S = S(t)\) and provide estimates for certain constants depending on \(S\), which are helpful throughout this section, see also Section 4. We recall the following simple result:

**Lemma 2.5.** Let \(\mathcal{X}\) be a finite-dimensional vector space and \(A \in \mathcal{L}(\mathcal{X}, \mathcal{X}) =: \mathcal{L}(\mathcal{X})\). Then there exists a constant \(\overline{M} \geq 1\) such that
\[
|S(t)| \leq \overline{M} e^{\|A\|t} \quad \text{and} \quad |S(t) - \text{Id}| \leq \overline{M} |A| te^{\|A\|t}, \quad t \geq 0.
\]
For simplicity, we can set \(\overline{M} = 1\) throughout this section, since we only need to derive appropriate time-regularity results.

In order to construct (12) we need that \((Y, Y') \in D_W^{2\alpha}([0,1]; \mathcal{L}(\mathcal{V}, \mathcal{X}))\). Therefore, to justify that
\[
\int_0^t S(\cdot - r)Y_r \, dW_r,
\]  
(15)
can be defined by (12) we need to establish that \((S(t - \cdot)Y, (S(t - \cdot)Y)' \in D_W^{2\alpha}([0,1]; \mathcal{L}(\mathcal{V}, \mathcal{X}))\). This is contained in the next result, which contains the choice (13) as a special case.
Lemma 2.6. Let \((Y, Y') \in D^{2\alpha}_W([0, 1]; L(V, \mathcal{X})).\) For every \(t \in [0, 1]\) we set \(Z^t := S(t - \cdot)Y.\) Then we have \((Z^t, (Z^t)' : D^{2\alpha}_W([0, 1]; L(V, \mathcal{X})),\) where \((Z^t)' = S(t - \cdot)Y'.\)

Proof. Regarding the definition of \(D^{2\alpha}_W\) we have to show that \(Z^t \in C^\alpha, (Z^t)' \in C^\alpha\) and \(R^{Z^t} \in C^{2\alpha}\). We fix \(0 < u < v \leq 1\). Then it follows that

\[
|Z^t_{u,v}| = |S(t - v)Y_v - S(t - u)Y_u| \\
\leq |S(t - v)Y_v - S(t - v)Y_u| + |(S(t - v) - S(t - u))Y_u| \\
\leq |S(t - v)||Y_{u,v}| + |S(t - v)||S(v - u) - \text{Id}||Y_u| \\
\leq e^{t|A||Y|_\alpha|v - u|_\alpha} + |A|e^{2t|A||Y||_\infty|v - u|} \\
\leq e^{t|A||Y|_\alpha|v - u|_\alpha} + |A|e^{2t|A|(|Y_0| + ||Y||_\alpha)|v - u|}.
\]

Using the previous computation, we derive

\[
|Z^t|_{\alpha} \leq C|A|e^{2t|A|(|Y_0| + ||Y||_\alpha)}. \tag{16}
\]

Replacing \(Y\) with \(Y'\) in the previous computation, one obtains the same estimate for the Gubinelli derivative \((Z^t)'\) from which the \(\alpha\)-Hölder regularity immediately follows. Next, we now focus on the remainder \(R^{Z^t}\) and aim to show that it is \(2\alpha\)-Hölder continuous. We have:

\[
|R^{Z^t}_{u,v}| = |Z^t_{u,v} - (Z^t)'W_{u,v}| \\
= |S(t - v)Y_v - S(t - u)Y_u - S(t - u)Y_u'W_{u,v}| \\
= |S(t - v)Y_v - S(t - u)Y_u - S(t - u)Y_v + S(t - u)R_{u,v}'| \\
\leq |S(t - v)||S(v - u) - \text{Id}||Y_v| + |S(t - u)||R_{u,v}'Y| \\
\leq |A|e^{2t|A||Y||_\infty|v - u|} + e^{t|A||R_{u,v}'Y|_{2\alpha}|v - u|^{2\alpha}}.
\]

From this we infer that \(R^{Z^t} \in C^{2\alpha}\) and

\[
||R^{Z^t}|_{2\alpha} \leq C|A|e^{2t|A|(|Y_0| + ||Y||_\alpha + ||R_{u,v}'Y|_{2\alpha})}. 
\]

This completes the proof. \(\square\)

Up to now, we have shown that we can define the integral \((15)\) by \((12)\). Now, we compute its Gubinelli derivative and prove that

\[
\left(\int_0^t S(\cdot - r)Y_r \ dW_r, \left(\int_0^t S(\cdot - r)Y_r \ dW_r\right)\right)'
\]

forms a controlled rough path.

Lemma 2.7. Let \((Y, Y') \in D^{2\alpha}_W([0, 1]; L(V, \mathcal{X})).\) Then

\[
\left(\int_0^t S(\cdot - r)Y_r \ dW_r, Y\right) \in D^{2\alpha}_W([0, 1]; \mathcal{X}). \tag{18}
\]

Proof. Due to Lemma 2.6 we can define according to Theorem 2.4 the integral

\[
I_t := \int_0^t S(t - r)Y_r \ dW_r.
\]
To prove (18) we only show that

\[ I_{s,t} = (S(t-s) - \text{Id}) \int_0^s S(s-r)Y_r \, dW_r + \int_s^t S(t-r)Y_r \, dW_r. \]

Consequently, the first term gives us

\[ \left| (S(t-s) - \text{Id}) \int_0^s S(s-r)Y_r \, dW_r \right| \leq |A| |t-s| e^{(t-s)|A|} \left| \int_0^s S(s-r)Y_r \, dW_r \right| \]

and the second one

\[ \left| \int_s^t S(t-r)Y_r \, dW_r \right| \leq |S(t-s)||Y||W|_\alpha (t-s)^\alpha \]

\[ + |S(t-s)||Y'||W|_{2\alpha} (t-s)^{2\alpha} + C(t-s)^{3\alpha}. \]

We now prove the $2\alpha$-Hölder regularity of the remainder. To this aim we compute for $0 \leq s \leq t \leq 1$

\[ |R_{s,t}^I| = |I_{s,t} - I_{s,t}'W_{s,t}| = \int_0^t S(t-r)Y_r \, dW_r - \int_0^s S(s-r)Y_r \, dW_r - Y_s W_{s,t} \]

\[ \leq \left| \int_0^s (S(t-r) - S(s-r))Y_r \, dW_r \right| + \left| \int_s^t S(t-r)Y_r \, dW_r - Y_s W_{s,t} \right|. \]

We start estimating the first term in the previous inequality as

\[ \left| \int_0^s (S(t-r) - S(s-r))Y_r \, dW_r \right| \leq |S(t-s) - \text{Id}| \left| \int_0^s S(s-r)Y_r \, dW_r \right| \]

\[ \leq |A| |t-s| e^{(t-s)|A|} \left| \int_0^s S(s-r)Y_r \, dW_r \right|. \]

To estimate the rough integral we apply (13) and obtain

\[ \left| \int_0^s (S(t-r) - S(s-r))Y_r \, dW_r \right| \]

\[ \leq |A| |t-s| e^{(t-s)|A|} \left( |S(s)Y_0W_0| + |S(s)Y_0'W_{0,s}| + C(\|W\|_\alpha \|R^{Z^s}\|_{2\alpha} + \|W\|_{2\alpha} \|(Z^s)'\|_\alpha s^{3\alpha}) \right) \]

\[ \leq C[|A|] |t-s| (|Y_0| \|W\|_\alpha s^\alpha + |Y_0'| \|W\|_{2\alpha} s^{2\alpha} + C(\|W\|_\alpha \|R^{Z^s}\|_{2\alpha} + \|W\|_{2\alpha} \|(Z^s)'\|_\alpha s^{3\alpha})) \].

Plugging in the estimates obtained for $\|R^{Z^s}\|_{2\alpha}$ and $\|(Z^s)'\|_\alpha$ in Lemma 2.6 we conclude that we can find a constant $C = C[|A|]$ such that

\[ \left| \int_0^s (S(t-r) - S(s-r))Y_r \, dW_r \right| \]

\[ \leq C[|A|] (|Y_0| + |Y_0'| + \|Y\|_\alpha + \|Y'\|_\alpha + \|R^Y\|_{2\alpha}) (\|W\|_\alpha + \|W\|_{2\alpha}) |t-s|. \]
We now estimate the second term as follows. Using again (13) we infer that
\[
\left| \int_s^t S(t-r)Y_r \, dW_r - Y_s W_{s,t} \right|
\]
\[
\leq |S(t-s)Y_s W_{s,t} - Y_s W_{s,t}| + |S(t-s)Y_s' W_{s,t}| + C(\|W\|_\alpha \|R^{Z^t}\|_{2\alpha} + \|W\|_{2\alpha}) \|Z^t\|_\alpha |t-s|^{3\alpha}
\]
\[
\leq C[\|A\|] \left( \|Y\|_\infty \|W\|_\alpha |t-s|^{1+\alpha} + \|Y'\|_\infty \|W\|_{2\alpha} |t-s|^{2\alpha} \right)
\]
\[
+ C \left( \|W\|_\alpha \|R^{Z^t}\|_{2\alpha} + \|W\|_{2\alpha} \|(Z^t)'\|_\alpha \right) |t-s|^{3\alpha}.
\]
Consequently, we have
\[
\left| \int_s^t S(t-r)Y_r \, dW_r - Y_s W_{s,t} \right|
\]
\[
\leq C[\|A\|] \left( |Y_0| + \|Y'_0\| + \|Y\|_\alpha + \|Y'\|_\alpha + \|R^Y\|_{2\alpha} \right) \left( \|W\|_\alpha + \|W\|_{2\alpha} \right) |t-s|^{2\alpha},
\]
which proves the required regularity of the remainder of the controlled rough integral. Putting all these estimates together and recalling that
\[
\|Y\|_\alpha \leq C(1 + \|W\|_\alpha)(\|Y'_0\| + \|Y, Y'\|_{D^{R^2}_W}),
\]
we finally infer that
\[
\left| \int_0^t S(\cdot - r)Y_r \, dW_r, Y \right|_{D^{R^2}_W}
\]
\[
\leq \|Y\|_\alpha + C[\|A\|] \left( |Y_0| + \|Y'_0\| + \|Y, Y'\|_{D^{R^2}_W} \right) \left( 1 + \|W\|_\alpha \right) \left( \|W\|_\alpha + \|W\|_{2\alpha} \right).
\]
This finishes the proof. \qed

Keeping the previous computation in mind, one can easily derive that for \((Y, Y') \in D^{2\alpha}_W([0,1]; \mathcal{X})\) and \(G \in C^3_b(\mathcal{X}; \mathcal{L}(\mathcal{Y}, \mathcal{X}))\) the following estimate holds true
\[
\left| \int_0^t S(\cdot - r)G(Y_r) \, dW_r, G(Y) \right|_{D^{R^2}_W} \leq \|G(Y)\|_\alpha + C[\|A\|] \left( |G(Y_0)| + \|G'(Y_0)\| + \|G(Y), (G(Y))'\|_{D^{2\alpha}_W} \right) \left( 1 + \|W\|_\alpha \right) \left( \|W\|_\alpha + \|W\|_{2\alpha} \right).
\]
\]
Note that
\[
\|G(Y)\|_\alpha \leq \|DG\|_\infty \|Y\|_\alpha \leq \|G\|_{C^1_b} \|Y\|_\alpha
\]
and
\[
\|G'(Y_0)\| = \|DG(Y_0)Y'_0\| \leq \|G\|_{C^3_b}.
\]
Now, one uses [22] Lem. 7.3 which gives a meaning of the operation of composition of a controlled rough path with a smooth function together with the estimate
\[
\|G(Y), (G(Y))'\|_{D^{R^2}_W} = \|G(Y), DG(Y)Y'\|_{D^{R^2}_W}
\]
\[
\leq C\|G\|_{C^2_b} M \left( \|Y'_0\| + \|Y, Y'\|_{D^{2\alpha}_W} \right) \left( 1 + \|W\|_\alpha \right)^2,
\]
\]
\]
\]
\]
\]
\]
\]
\]
\]
for \((Y, Y') \in D_{W}^{2\alpha}([0, 1]; \mathcal{X})\) with \(|Y_0'_{\mathcal{X}}| + ||Y, Y'||_{D_{W}^{2\alpha}} \leq M\).

We point out following result which is essential for fixed-point arguments, since it provides explicit estimates for the difference of two controlled rough paths.

**Lemma 2.8.** Let \(Y, \bar{Y} \in D_{W}^{2\alpha}([0, 1]; \mathcal{X})\) with \(|Y_0| + ||Y, Y'||_{D_{W}^{2\alpha}} \leq M\) and \(|\bar{Y}_0| + ||\bar{Y}, \bar{Y}'||_{D_{W}^{2\alpha}} \leq M\). Then, there exists a constant \(C = C[\alpha, ||W||_\alpha]\), such that the estimate

\[
\|G(Y) - G(\bar{Y}), (G(Y))' - (G(\bar{Y}))'||_{D_{W}^{2\alpha}}
\leq CM^2\|G\|_{C^3_B}(|Y_0 - \bar{Y}_0| + |Y_0' - \bar{Y}_0'| + ||Y - \bar{Y}, Y' - \bar{Y}'||_{D_{W}^{2\alpha}})
\]

is valid.

**Proof.** Fix \(x, y \in \mathcal{X}\). Then it holds

\[
G(x) - G(y) = (x - y) \int_0^1 DG(tx + (1 - t)y) \, dt.
\]

Therefore, we define for \(Y, \bar{Y} \in D_{W}^{2\alpha}\) and \(s \in [0, 1]\)

\[
\mathcal{R}_s := g(Y_s, \bar{Y}_s) \quad \text{and} \quad \mathcal{S}_s := Y_s - \bar{Y}_s,
\]

where we set

\[
g(x, y) := \int_0^1 DG(tx + (1 - t)y) \, dt.
\]

Consequently, we have that \(G(Y) - G(\bar{Y}) = \mathcal{R}\mathcal{S}\). First of all, we infer from [22, Lem. 7.3] that \((\mathcal{R}, \mathcal{R}') \in D_{W}^{2\alpha}\), where

\[
\mathcal{R}' = D_x g(Y, \bar{Y})Y' + D_y g(Y, \bar{Y})\bar{Y}'.
\]

Again the result [22, Lem. 7.3] entails

\[
||\mathcal{R}, \mathcal{R}'||_{D_{W}^{2\alpha}} \leq C\|g\|_{C^2_B} M(|Y_0| + |ar{Y}_0| + ||Y, Y'||_{D_{W}^{2\alpha}} + ||\bar{Y}, \bar{Y}'||_{D_{W}^{2\alpha}})(1 + ||W||_\alpha)^2
\leq C\|g\|_{C^3_B} M(|Y_0| + |ar{Y}_0| + ||Y, Y'||_{D_{W}^{2\alpha}} + ||\bar{Y}, \bar{Y}'||_{D_{W}^{2\alpha}})(1 + ||W||_\alpha)^2.
\]

Furthermore, [22, Cor. 7.4] gives us that \((\mathcal{R}\mathcal{S}, (\mathcal{R}\mathcal{S})') \in D_{W}^{2\alpha}\) and \((\mathcal{R}\mathcal{S})' = \mathcal{R}'\mathcal{S} + \mathcal{R}\mathcal{S}'\). Moreover, we obtain

\[
||\mathcal{R}\mathcal{S}, (\mathcal{R}\mathcal{S})'||_{D_{W}^{2\alpha}} \leq C(||\mathcal{R}_0|| + |\mathcal{R}'_0| + ||\mathcal{R}, \mathcal{R}'||_{D_{W}^{2\alpha}})(||\mathcal{S}_0|| + |\mathcal{S}'_0| + ||\mathcal{S}, \mathcal{S}'||_{D_{W}^{2\alpha}}).
\]

Using [22] in [23] we derive

\[
\|G(Y) - G(\bar{Y}), (G(Y) - G(\bar{Y}))'||_{D_{W}^{2\alpha}}
\leq C\|g\|_{C^4_B} M^2(|Y_0 - \bar{Y}_0| + |Y_0' - \bar{Y}_0'| + ||Y - \bar{Y}, Y' - \bar{Y}'||_{D_{W}^{2\alpha}}),
\]

where again \(C = C[\alpha, ||W||_\alpha]\). \(\square\)

Putting [20] and Lemma 2.8 together, we get the following result:
Lemma 2.9. Let \( Y, \tilde{Y} \in D^2_{W}(0,1; \mathcal{X}) \), with \( |Y_0| + \|Y,Y'\|_{D^2_{W}} \leq M \), \( |\tilde{Y}_0| + \|\tilde{Y},\tilde{Y}'\|_{D^2_{W}} \leq M \) and \( G \in C^3_b(\mathcal{X}; \mathcal{L}(\mathcal{V}, \mathcal{X})) \). Then the following estimate holds true

\[
\left\| \int_0^t S(\cdot - r)(G(Y_r) - G(\tilde{Y}_r)) \, dW_r, G(Y) - G(\tilde{Y}) \right\|_{D^2_{W}} \\
\leq \|G(Y) - G(\tilde{Y})\|_\alpha + C|A| \|G\|_{C^3_b} M^2 \|Y - \tilde{Y}, Y' - \tilde{Y}'\|_{D^2_{W}} (1 + \|W\|_\alpha)(\|W\|_\alpha + \|\mathcal{W}\|_{2\alpha}).
\]

Recall that according to \( (11) \) one has

\[
\|G(Y) - G(\tilde{Y})\|_\alpha \leq C(1 + \|W\|_\alpha)\|G(Y) - G(\tilde{Y}), (G(Y) - G(\tilde{Y}))'\|_{D^2_{W}}.
\]

From this one can further infer that

\[
\left\| \int_0^t S(\cdot - r)(G(Y_r) - G(\tilde{Y}_r)) \, dW_r, G(Y) - G(\tilde{Y}) \right\|_{D^2_{W}} \\
\leq C|A| \|G\|_{C^3_b} M^2 (1 + \|W\|_\alpha)(\|W\|_\alpha + \|\mathcal{W}\|_{2\alpha})\|Y - \tilde{Y}, Y' - \tilde{Y}'\|_{D^2_{W}}.
\]

In addition to the last estimate, we also need another result to be employed later on in order to estimate the terms containing the initial condition and the drift of a rough differential equation in \( D^2_{W} \). In this case, the computation simplifies since the Gubinelli derivative no longer plays a role. We have:

Lemma 2.10. Let \( \xi \in \mathcal{X} \) and let \( F : \mathcal{X} \to \mathcal{X} \) be Lipschitz continuous. Then, for \((Y,Y') \in D^2_{W}(0,1; \mathcal{X})\), we have

\[
\left\| S(\cdot)\xi + \int_0^t S(\cdot - r)F(Y_r) \, dr, 0 \right\|_{D^2_{W}} \leq C|A|\|\xi\| + |F(Y_0)| + L_F \|Y\|_\alpha,
\]

where \( L_F \) denotes the Lipschitz constant of \( F \).

Proof. Since the Gubinelli derivative of the expression in \( (25) \) is zero we only need to estimate the \( 2\alpha \)-Hölder norm of the remainder in order to prove the assertion. In our case this is constituted by

\[
\left\| S(\cdot)\xi + \int_0^t S(\cdot - r)F(Y_r) \, dr \right\|_{2\alpha} \leq \|S(\cdot)\xi\|_{2\alpha} + \left\| \int_0^t S(\cdot - r)F(Y_r) \, dr \right\|_{2\alpha}.
\]

Letting \( 0 \leq s < t \leq 1 \) the first term easily results in

\[
|S(t) - S(s)\xi| \leq |S(s)| \|S(t - s) - \text{Id}\| \|\xi\| \leq e^{s|A|} |A| e^{(t-s)|A|} (t-s) \|\xi\|,
\]

which gives us

\[
\|S(\cdot)\xi\|_{2\alpha} \leq C|A| \|\xi\|.
\]

Furthermore, we get

\[
\left\| \int_0^t S(t - r)F(Y_r) \, dr - \int_0^s S(s - r)F(Y_r) \, dr \right\| \\
\leq \left\| \int_0^s (S(t - r) - S(s - r))F(Y_r) \, dr \right\| + \left\| \int_s^t S(t - r)F(Y_r) \, dr \right\|.
\]
Moreover, one can obtain that

\[
\left| \int_0^s (S(t-r) - S(s-r)) F(Y_r) \, dr \right| \leq |S(t-s) - \text{Id}| \int_0^s |S(s-r)| |F(Y_r)| \, dr
\]

\[
\leq |A| (t-s) e^{(t-s)|A|} \|F(Y)\|_\infty \int_0^s e^{(s-r)|A|} \, dr
\]

\[
\leq C[|A|] (|F(Y_0)| + L_F \|Y\|_\alpha)(t-s).
\]

It is then straightforward to observe that

\[
\left| \int_s^t S(t-r) F(Y_r) \, dr \right| \leq C[|A|](|F(Y_0)| + L_F \|Y\|_\alpha)(t-s),
\]

which implies

\[
\left\| S(\cdot)\xi + \int_0^\cdot S(\cdot-r)F(Y_r) \, dr \right\|_{2\alpha} \leq C[|A|](|\xi| + |F(Y_0)| + L_F \|Y\|_\alpha).
\]

Moreover, one can obtain that

\[
\left\| S(\cdot)(\xi - \tilde{\xi}) + \int_0^\cdot S(\cdot-r)(F(Y_r) - F(\tilde{Y}_r)) \, dr \right\|_{2\alpha} \leq C[|A|](|\xi - \tilde{\xi}| + L_F \|Y - \tilde{Y}\|_\infty),
\]

where \((Y, Y')\) and \((\tilde{Y}, \tilde{Y}')\) are two controlled rough paths with \(Y_0 = \xi\) and \(\tilde{Y}_0 = \tilde{\xi}\).

Regarding all these preliminary results one can then prove easily, in analogy to \cite[Thm. 8.4]{22}, existence and uniqueness result for \cite{5}. A local solution is derived by a fixed-point argument, which can be extended to a global one by a standard concatenation argument \cite[Ch. 8]{22}.

**Theorem 2.11.** The RDE \cite{3} has a unique global solution represented by a rough path \((U,U') \in D_W^2([0,1]; \mathbb{R}^n)\) given by

\[
(U,U') = \left( S(\cdot)\xi + \int_0^\cdot S(\cdot-r)F(U_r) \, dr + \int_0^\cdot S(\cdot-r)G(U_r) \, dW_r, G(U) \right).
\]

The solution \cite{27} is a rough path, consequently the two components are connected via \cite{8}. The first one is also referred to as the path component. The rough path involves the group \(S = S(t)\) explicitly, which is going to turn out to be crucial to construct center manifolds. The results stated here for \(\alpha\)-Hölder continuous paths, carry over to \(p\)-variation paths, see \cite[Sec. 5.3]{23}, \cite[Ch. 12]{23} or \cite{16}.

We conclude this subsection pointing out some results which are required for the computation of invariant manifolds for \cite{5}. As commonly met within this framework (see \cite{7,11,23}), since the gap conditions may be too restrictive and require a certain smallness of the Lipschitz constants of the nonlinearities, the first step is to introduce an appropriate cut-off technique in order to truncate these nonlinearities. As a consequence the Lipschitz constants of \(F\) and \(G\) will be made small, as required to derive the contraction property of the Lyapunov-Perron map in Section 4.2 in a suitable Banach space. This technique will be extended here to the setting of rough paths.
As seen above, we intensively used the operation of composition of a controlled rough path with a smooth function, recall \((21)\). In order to introduce the cut-off procedure explained above, we now compose \((Y, Y') \in D^2_W([0, 1]; \mathcal{X})\) with a smooth cut-off function. To this aim, let \(\chi\) be a smooth cut-off function, such that

\[
\begin{cases}
\chi(Y) = Y, & \|Y, Y'\|_{D^2_W} \leq \frac{1}{2} \\
\chi(Y) = 0, & \|Y, Y'\|_{D^2_W} \geq 1.
\end{cases}
\] (28)

Opposite to the deterministic setting, we apply here the cut-off technique as follows. For a positive random variable \(R(W)\) we set

\[\chi_R(Y) := \chi_{R(W)}(Y) := R(W)\chi\left(\frac{Y}{R(W)}\right)\]

and \(F_R := F_{R(W)} := F \circ \chi_R\) and \(G(R) := G_{R(W)} := G \circ \chi_R\). Obviously \((\chi_R(Y), D\chi_R(Y)Y') \in D^2_W\) for \((Y, Y') \in D^2_W\). Furthermore, note that and \(G(R) = G(Y), F_R(Y) = F(Y)\) if \(\|Y, Y'\|_{D^2_W} \leq \frac{R(W)}{2}\).

Recalling Lemma \(2.12\), we have

\[
\left\| \int_0^T S(\cdot - r)(F_R(Y_r) - F_R(Y_r)) \, dr \right\|_{D^2_W} \leq C\|A\|L_F(R(W))\|Y - \tilde{Y}, Y' - \tilde{Y}'\|_{D^2_W},
\] (29)

where \(L_F(R)(W)\) denotes the Lipschitz constant of \(F_R\). Furthermore, Lemma \(2.9\) yields for \((Y, Y')\) and \((\tilde{Y}, \tilde{Y}') \in D^2_W([0, 1]; \mathcal{X})\) that

\[
\left\| \int_0^T S(\cdot - r)(G_R(Y_r) - G_R(Y_r)) \, dW_r, G_R(Y) - G_R(\tilde{Y}) \right\|_{D^2_W} \leq C\|A\|L_G(R(W))(1 + \|W\|_\alpha)(\|W\|_\alpha + \|W\|_\alpha)\|Y - \tilde{Y}, Y' - \tilde{Y}'\|_{D^2_W},
\] (30)

where \(L_G(R(W))\) stands for \(\|G_R\|_{C^b}\). Next, we fix \(K > 0\). Keeping \((2.10)\) and \((30)\) in mind we let \(\tilde{R}(W)\) be the unique solution of

\[
C\|A\|L_F(\tilde{R}(W)) + C\|A\|(1 + \|W\|_\alpha)(\|W\|_\alpha + \|W\|_\alpha)L_G(\tilde{R}(W)) = K.
\] (31)

and set \(R(W) := \min\{\tilde{R}(W), 1\}\). Combining \((29), (30)\) and \((31)\) leads to the following result. We consider \(T_R := T_{R(W)}\) such that \(T : D^2_W([0, 1]; \mathcal{X}) \rightarrow D^2_W([0, 1]; \mathcal{X})\) and

\[
T_R(W, Y, Y') := \left( \int_0^T S(\cdot - r)F_R(Y_r) \, dr + \int_0^T S(\cdot - r)G_R(Y_r) \, dW_r, G_R(Y) \right).
\] (32)

**Lemma 2.12.** Let \((Y, Y'), (\tilde{Y}, \tilde{Y}') \in D^2_W([0, 1]; \mathcal{X})\). We have

\[
\|T_R(W, Y, Y') - T_R(W, \tilde{Y}, \tilde{Y}')\|_{D^2_W([0, 1]; \mathcal{X})} \leq K\|Y - \tilde{Y}, Y' - \tilde{Y}'\|_{D^2_W([0, 1]; \mathcal{X})}.
\] (33)

This immediately yields suitable estimates for the solution of \((5)\) when one replaces \(F\) by \(F_R\) respectively \(G\) by \(G_R\).
3 Random Dynamics

The main techniques and results established in the previous section using controlled rough paths are necessary in order to provide pathwise estimates for the solutions of [3]. In this section, we provide some concepts from the random dynamical systems theory [1], which allow us to give a definition of an invariant manifold for [3]; for even further information regarding random dynamical systems generated by RDEs, see [3, Sec. 3].

The next concept is fundamental in the theory of random dynamical systems, since it describes a model of the driving noise.

**Definition 3.1.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) stand for a probability space and \(\theta : \mathbb{R} \times \Omega \to \Omega\) be a family of \(\mathbb{P}\)-preserving transformations (i.e., \(\theta_t \mathbb{P} = \mathbb{P}\) for \(t \in \mathbb{R}\)) having following properties:

(i) the mapping \((t, \omega) \mapsto \theta_t \omega\) is \((\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})\)-measurable, where \(\mathcal{B}(\cdot)\) denotes the Borel sigma-algebra;

(ii) \(\theta_0 = \text{Id}_\Omega\);

(iii) \(\theta_{t+s} = \theta_t \circ \theta_s\) for all \(t, s \in \mathbb{R}\).

Then the quadruple \((\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})\) is called a metric dynamical system.

In our context, constructing a metric dynamical system is going to rely on constructing \(\theta\) as a shift map. We start to check that shifts act quite naturally on rough paths. For an \(\alpha\)-Hölder rough path \(W = (W, \mathbb{W})\) and \(\tau \in \mathbb{R}\) let us define the time-shift \(\Theta_\tau W := (\Theta_\tau W, \Theta_\tau \mathbb{W})\) by

\[
\Theta_\tau W_t := W_{t+\tau} - W_\tau \\
\Theta_\tau W_{s,t} := \mathbb{W}_{s+\tau, t+\tau}
\]

Note that the time shift naturally extends linearly to sums of rough paths, e.g., \(\Theta_\tau W_{s,t} = W_{t+\tau} - W_{s+\tau}\). Furthermore, the shift leaves the path space invariant:

**Lemma 3.2.** Let \(T_1, T_2, \tau \in \mathbb{R}\), and \(W = (W, \mathbb{W})\) be an \(\alpha\)-Hölder rough path on \([T_1, T_2]\) for \(\alpha \in (1/3, 1/2)\). Then the time-shift \(\Theta_\tau W = (\Theta_\tau W, \Theta_\tau \mathbb{W})\) is also an \(\alpha\)-Hölder rough path on \([T_1 - \tau, T_2 - \tau]\).

**Proof.** Let \(s, u, t \in [T_1 - \tau, T_2 - \tau]\). The \(\alpha\)-Hölder-continuity of \(\theta_t W\) and the \(2\alpha\)-Hölder continuity of \(\theta_t \mathbb{W}\) are obvious. We only prove that Chen’s relation (7). We have

\[
\Theta_\tau W_{s,t} - \Theta_\tau W_{s,u} - \Theta_\tau W_{u,t} = W_{s+\tau, t+\tau} - W_{s+\tau, u+\tau} - W_{u+\tau, t+\tau} = W_{s+\tau, u+\tau} \otimes W_{u+\tau, t+\tau} = (W_{u+\tau} - W_\tau - W_{s+\tau} + W_\tau) \otimes (W_{t+\tau} - W_\tau - W_{u+\tau} + W_\tau) = \Theta_\tau W_{s,u} \otimes \Theta_\tau W_{u,t}.
\]

where in [34] we use Chen’s relation (7).

Based upon [3] we consider the following concept:

**Definition 3.3.** Let \((\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})\) be a metric dynamical system. We call \(W = (W, \mathbb{W})\) a rough path cocycle if the identity

\[
W_{s,s+t}(\omega) = W_{0,t}(\theta_s \omega)
\]

holds true for every \(\omega \in \Omega\), \(s \in \mathbb{R}\) and \(t \geq 0\).
The previous definitions hint already at the fact that one may be able to just use as a probability space \( \Omega \) a space of paths. One classical case, where we get via this construction a metric dynamical system and a rough cocycle is fractional Brownian motion.

**Example 3.4.** As a concrete example for \( W \) consider the fractional Brownian motion \( B^H \) restricted to any compact interval \([-L,L]\) with \( L \geq 1 \) and for \( H \in (1/3,1/2] \). This includes classical Brownian motion as the case when \( H = 1/2 \). \( B^H \) can be lifted to an \( \alpha \)-Hölder rough-path \( B^H = (B^H, \mathcal{B}^H) \) as discussed in \([22, Ex. 10.11]\), where

\[
\mathcal{B}^H_{s,t} := \int_{s}^{t} B^H_{s,u} \otimes dB^H_u.
\]

Gluing together lifts on compact time intervals, one may extend \( B^H \) to the whole real line. Furthermore, we may consider the canonical probability space \((C_0(\mathbb{R}; \mathbb{R}^d), \mathcal{B}(C_0(\mathbb{R}; \mathbb{R}^d)), \mathbb{P})\), where \( C_0(\mathbb{R}; \mathbb{R}^d) \) denotes the space of all \( \mathbb{R}^d \)-valued continuous functions, which are 0 in 0, endowed with the compact open topology. The shift on the sample path space is given by

\[
(\Theta_t f)(\cdot) := f(\tau + \cdot) - f(\tau), \quad \tau \in \mathbb{R}, \quad f \in C_0(\mathbb{R}, \mathbb{R}^d).
\]  

Using Kolmogorov’s Theorem or the Garsia-Rodemich-Rumsey inequality \([22, A.2]\) one can conclude that maps in \( C^\alpha_0(\mathbb{R}; \mathbb{R}^d) \) have a finite \( \alpha \)-Hölder semi-norm on every compact interval \( \mathbb{P} \)-almost surely. Hence, we can restrict this metric dynamical system to the set \( C^\alpha_0(\mathbb{R}, \mathbb{R}^d) \). For the metric dynamical system

\[
(C^\alpha_0(\mathbb{R}, \mathbb{R}^d), \mathcal{B}(C^\alpha_0(\mathbb{R}, \mathbb{R}^d)), \mathbb{P}, (\Theta_t)_{t \in \mathbb{R}}) =: (\Omega_B, \mathcal{F}_B, \mathbb{P}, (\Theta_t)_{t \in \mathbb{R}})
\]

one may check that \( B^H = (B^H, \mathcal{B}^H) \) represents a rough path cocycle as introduced in Definition 3.3.

Of course, the same construction of a path-space \((\Omega_W, \mathcal{F}_W, \mathbb{P})\), also referred to as canonical probability space, can be carried out for more general \( \alpha \)-Hölder rough paths \( W = (W, \mathcal{W}) \) constructed from a (stochastic) process \( W_t \), not just fractional Brownian motion, where the definition of a shift map is still as above, i.e.,

\[
(\Theta_t W)(t) := W_{t+\tau} - W_{\tau}.
\]

We now have the abstract definition of, as well as concrete examples for, metric dynamical systems for our problem modelling the underlying rough driving process (or “noise”). Now we have to also define the dynamical systems structure of the solution operator of our RDE. As a first step we recall the classical definition of a random dynamical system \([1]\).

**Definition 3.5.** A random dynamical system on \( \mathcal{X} \) over a metric dynamical system \((\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})\) is a mapping

\[
\varphi : [0, \infty) \times \Omega \times \mathcal{X} \rightarrow \mathcal{X}, \quad (t, \omega, x) \mapsto \varphi(t, \omega, x),
\]

which is \((\mathcal{B}([0, \infty)) \times \mathcal{F} \times \mathcal{B}(\mathcal{X}), \mathcal{B}(\mathcal{X}))\)-measurable and satisfies:

(i) \( \varphi(0, \omega, \cdot) = \text{Id}_{\mathcal{X}} \) for all \( \omega \in \Omega \);

(ii) \( \varphi(t + \tau, \omega, x) = \varphi(t, \theta_{\tau} \omega, \varphi(\tau, \omega, x)) \), for all \( x \in \mathcal{X} \), \( t, \tau \in [0, \infty) \), \( \omega \in \Omega \);

(iii) \( \varphi(t, \omega, \cdot) : \mathcal{X} \rightarrow \mathcal{X} \) is continuous for all \( t \in [0, \infty) \) and all \( \omega \in \Omega \).

The second property in Definition 3.5 is referred to as the cocycle property. One can now expect that the solution operator of \([5]\) generates a random dynamical system. Indeed, working with a pathwise interpretation of the stochastic integral as given in \([12]\), no exceptional sets can occur. For completeness, we indicate a proof of this fact, see also \([3]\).
Lemma 3.6. Let $W$ be a rough path cocycle. Then the solution operator
\[ t \mapsto \varphi(t, W, \xi) = U_t = S(t)\xi + \int_0^t S(t - r) F(U_r) \, dr + \int_0^t S(t - r) G(U_r) \, dW_r, \]
for any $t \in [0, \infty)$ of the RDE \([5]\) generates a random dynamical system over the metric dynamical system \((\Omega_W, F_W, \mathbb{P}, (\Theta_t)_{t \in \mathbb{R}})\).

Proof. The relevant properties to define the metric dynamical system we need have been discussed in Example (3.1). The only difficulty is checking cocycle property for the solution operator. We calculate
\[
U_{t+\tau} = S(t + \tau)\xi + \int_0^{t+\tau} S(t + \tau - r) F(U_r) \, dr + \int_0^{t+\tau} S(t + \tau - r) G(U_r) \, dW_r
\]
\[
= S(t)S(\tau)\xi + \int_0^\tau S(\tau - r) F(U_r) \, dr + \int_0^\tau S(\tau - r) G(U_r) \, dW_r
\]
\[
+ \int_0^\tau S(t + \tau - r) G(U_r) \, dW_r + \int_\tau^{t+\tau} S(t + \tau - r) G(U_r) \, dW_r
\]
\[
= S(t) \left( S(\tau)\xi + \int_0^\tau S(\tau - r) F(U_r) \, dr + \int_0^\tau S(\tau - r) dW_r \right)
\]
\[
+ \int_0^t S(t - r) F(U_{t+\tau}) \, dr + \int_0^t S(t - r) G(U_{t+\tau}) \, d\Theta_r W_r
\]
\[
= S(t)U_\tau + \int_0^t S(t - r) F(U_{t+\tau}) \, dr + \int_0^t S(t - r) G(U_{t+\tau}) \, d\Theta_r W_r.
\]
The above computation are rigorously justified, since one can check that if \((U, U') \in D^{2\alpha}_W([T_1 + \tau, T_2 + \tau]; \mathcal{A})\) then \((U_{s+\tau}, U'_{s+\tau}) \in D^{2\alpha}_{\Theta W}([T_1, T_2]; \mathcal{A})\). Here \(T_1, T_2 \in \mathbb{R}\) with \(T_1 < T_2\). The \(\alpha\)-Hölder continuity of \(U_{s+\tau}\) and \(U'_{s+\tau}\) is obvious. For the remainder we have
\[
|R^{U}_{s+\tau} - U'_{s+\tau} \Theta_r W_s, t| = |U_{s+\tau,t+\tau} - U'_{s+\tau} W_{s+\tau,t+\tau}| = |R^{U}_{s+\tau,t+\tau}| \leq \|R_U\|_{2\alpha} (t - s)^{2\alpha}.
\]
Furthermore, to show the shift property of the rough integral, we take a partition \(\mathcal{P}\) of \([T_1, T_2]\) and have
\[
\int_{T_1}^{T_2} U_{t+\tau} \, d\Theta_r W_r = \lim_{|\mathcal{P}| \to 0} \sum_{[s,t] \in \mathcal{P}} (U_{s+\tau} \Theta_r W_{s,t} + U'_{s+\tau} \Theta_r W_{s,t})
\]
\[
= \lim_{|\mathcal{P}| \to 0} \sum_{[s,t] \in \mathcal{P}} U_{s+\tau} W_{s+\tau,t+\tau} + U'_{s+\tau} W_{s+\tau,t+\tau}
\]
\[
= \lim_{|\mathcal{P}| \to 0} \sum_{[s,t] \in \mathcal{P}} U_{s} W_{s+\tau,t+\tau} + U'_{s} W_{s+\tau,t+\tau} = \int_{T_1+\tau}^{T_2+\tau} U_r \, dW_r. \tag{36}
\]
Here \(\mathcal{P}\) is a partition of \([T_1 + \tau, T_2 + \tau]\) given by \(\mathcal{P} := \{[s + \tau, t + \tau] : [s, t] \in \mathcal{P}\}\).
The \((\mathcal{B}([0, \infty)) \times \mathcal{F}_W \times \mathcal{B}(\mathcal{X}), \mathcal{B}(\mathcal{X}))\)-measurability of \(\varphi\) follows be well-known arguments. One considers a sequence of (classical) solutions \((U^n, (U^n)' )_{n \in \mathbb{N}}\) of \((3)\) corresponding to smooth approximations \((W^n, W^n)_{n \in \mathbb{N}}\) of \((W, W)\). Obviously, the mapping \((t, W, \xi) \mapsto U^n_t\) is \((\mathcal{B}([0, T]) \times \mathcal{F}_W \times \mathcal{B}(\mathcal{X}), \mathcal{B}(\mathcal{X}))\)-measurable for any \(T > 0\). Since \(U\) continuously depends on the rough input \(W\), according to Thm. 8.5 in [22], one immediately concludes that \(\lim_{n \to \infty} U^n_t = U_t\). This gives the measurability of \(U\) with respect to \(\mathcal{F}_W \times \mathcal{B}(\mathcal{X})\). Due to the time-continuity of \(U\), we obtain by Ch. 3 in [13] the \((\mathcal{B}([0, T]) \times \mathcal{F}_W \times \mathcal{B}(\mathcal{X}), \mathcal{B}(\mathcal{X}))\)-measurability of the mapping \((t, \omega, \xi) \mapsto U_t\) for any \(T > 0\).

Note that the role of the random elements in \(\Omega_W\) is played by the paths \(W\) as we one uses the canonical probability space of paths. So we directly denote these elements by \(W\) and do not employ the additional notation \(W_t(\omega) := \omega(t)\). The random dynamical system \(\varphi : \mathbb{R}^+ \times \Omega \times \mathbb{R}^n \to \mathbb{R}^n\) obviously depends upon the \(t, \xi, W\), and \(\mathbb{W}\) although we do not directly display the dependence upon \(\mathbb{W}\) in the notation.

To construct local random invariant manifolds, which can be characterized by the graph of a smooth function in a ball with a certain radius [11, 19, 25] one requires the concept of tempered random variables [1] Chapter 4], which we recall next:

**Definition 3.7.** A random variable \(R : \Omega \to (0, \infty)\) is called tempered from above, with respect to a metric dynamical system \((\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})\), if

\[
\limsup_{t \to \pm \infty} \frac{\ln^+ R(\theta_t \omega)}{t} = 0, \quad \text{for all } \omega \in \Omega,
\]

where \(\ln^+ a := \max\{\ln a, 0\}\). A random variable is called tempered from below if \(1/R\) is tempered from above. A random variable is tempered if and only if is tempered from above and from below.

Note that the set of all \(\omega \in \Omega\) satisfying \((37)\) is invariant with respect to any shift map \((\theta_t)_{t \in \mathbb{R}}\), which is an observation applicable to our case when \(\theta_t = \Theta_t\). A sufficient condition for temperedness is according to [1] Prop. 4.1.3] that

\[
\mathbb{E} \sup_{t \in [0,1]} \ln^+ R(\theta_t \omega) < \infty.
\]

Moreover, if the random variable \(R\) is tempered from below with \(t \mapsto R(\theta_t \omega)\) continuous for all \(\omega \in \Omega\), then for every \(\varepsilon > 0\) there exists a constant \(C[\varepsilon, \omega] > 0\) such that

\[
R(\theta_t \omega) \geq C[\varepsilon, \omega]e^{-\varepsilon|t|},
\]

for any \(\omega \in \Omega\). Again, for our concrete example when \(\Omega = \Omega_B\) one can easily check that norms are tempered.

**Lemma 3.8.** Let \(B^H = (B^H, \mathbb{B}^H)\) be the rough path cocycle associated to a fractional Brownian motion \(B^H\) with Hurst parameter \(H \in (1/3, 1/2]\). Then the random variables

\[
R_1(B^H) = \|B^H\|_\alpha \quad \text{and} \quad R_2(B^H) = \|\mathbb{B}^H\|_{2\alpha}
\]

are tempered from above.

**Proof.** The first assertion is valid due to the fact that \(\mathbb{E}\|B^H\|_m^2 < \infty\) and the second one follows regarding that \(\mathbb{E}\|\mathbb{B}^H\|_{2m}^{2m} < \infty\), for \(m \in \mathbb{N}\) as contained in [22] Thm. 10.4]. This shows the temperedness of both random variables. \(\square\)
As before, the last result holds more generally for broader classes of rough paths, e.g., by applying the results bounded norms of rough paths found in [22, Sec. 10]. From now, we shall simply assume that $\mathbf{W} = (W, \mathbb{W})$ is a rough path cocycle such that the random variables

$$R_1(W) = \|W\|_\alpha \quad \text{and} \quad R_2(\mathbb{W}) = \|\mathbb{W}\|_{2\alpha}$$

are tempered from above. This will be necessary in the existence proof of a local center manifold. One wants to ensure [42, 41, 11] that for initial conditions belonging to a ball with a sufficiently small tempered from below radius, the corresponding trajectories remain within such a ball (see the proof of Lemma 4.13 below). To this aim we emphasize.

**Lemma 3.9.** The constant $K$ in (31) is tempered from below.

**Proof.** This immediately follows by Lemma 3.8 regarding (31).

## 4 Local center manifolds for RDEs

In this section we prove the existence of a local center manifold for (5). The approach is similar to the one employed in [25] in order to compute unstable manifolds for SPDEs driven by a fractional Brownian motion with Hurst parameter $H > 1/2$ using elements from fractional calculus [45]. However, here we want to connect the theory of random invariant manifolds for S(P)DEs as in [19, 41, 25] to rough paths theory. This allows us to consider SDEs driven by general $\alpha$-Hölder continuous processes, as described in Section 3.

**Assumptions 4.1.** We assume that we are in a center-stable situation, namely there are eigenvalues $\{\lambda_j^c\}_{j=1}^{n_c}$ of the linear operator $A$ on the imaginary axis $i\mathbb{R}$ as well as eigenvalues $\{\lambda_j^s\}_{j=1}^{n_s}$ in the left-half plane $\{z \in \mathbb{C} : \text{Re}(z) < 0\}$. Upon counting multiplicities we have $n_c + n_s = n$. Hence, there exists a decomposition of the phase space $\mathbb{R}^n = X = X^c \oplus X^s$, where the linear spaces $X^c$ and $X^s$ are spanned by the (generalized) eigenvectors with eigenvalues $\lambda_j^c$ and $\lambda_j^s$ respectively. Moreover, there exist two bounded projections $P^c$ and $P^s$ associated to this splitting such that

1) $\text{Id} = P^s + P^c$;

2) $P^cS(t) = S(t)P^c$ and $P^sS(t) = S(t)P^s$ for $t \geq 0$.

Additionally, we assume that there exist two exponents $\gamma$ and $\beta$ with $-\beta < 0 \leq \gamma < \beta$ and constants $M_c, M_s \geq 1$, such that

$$|S(t)P^c x| \leq M_c e^{\gamma t} |P^c x|, \quad \text{for } t \leq 0 \text{ and } x \in X;$$

$$|S(t)P^s x| \leq M_s e^{-\beta t} |P^s x|, \quad \text{for } t \geq 0 \text{ and } x \in X.$$  \hspace{1cm} (40)

(41)

For further details and similar assumptions, see [20, Sec. 6.1.1] and [50, Sec. 7.1.2]. According to our restrictions we have $\gamma \geq 0$ and $-\beta < 0$ which gives us the spectral gap $\gamma + \beta > 0$. We also use the notations $\xi^c := P^c \xi$, $S^c(\cdot) := P^cS(\cdot)$, respectively $S^s(\cdot) := P^sS(\cdot)$ and refer to $X^c$ and $X^s$ as center, respectively stable, subspace.

For the nonlinear terms, additionally to the regularity assumptions made in Section 2 we assume the usual conditions

**Assumptions 4.2.**

$$F(0) = G(0) = 0, \quad DF(0) = 0.$$
Remark 4.3. One can easily extend the techniques presented below if one additionally has an unstable subspace, namely if there exist eigenvalues of $A$ with real part greater than zero. In this case the classical exponential trichotomy condition is satisfied, see for instance [15, 50] and Appendix [13]. For simplicity and from the point of view of applications we assume that we are in a center-stable situation and work with Assumptions [4.1] similar to [20, Sec. 6].

Definition 4.4. We call a random set $\mathcal{M}^c(W)$, which is invariant with respect to $\varphi$ (i.e. $\varphi(t, W, \mathcal{M}^c(W)) \subset \mathcal{M}^c(\Theta_t W)$ for $t \in \mathbb{R}$ and $W \in \Omega_W$), a center manifold if this can be represented as

$$\mathcal{M}^c(W) = \{ \xi + h^c(\xi, W) : \xi \in \mathcal{X}^c \},$$

(42)

where $h^c(\cdot, W) : \mathcal{X}^c \to \mathcal{X}^s$. Moreover, $h^c(0, W) = 0$ and $\mathcal{M}^c(W)$ is tangent to $\mathcal{X}^c$ at the origin, meaning that the tangency condition $Dh^c(0, W) = 0$ is satisfied.

We prove the existence of a local center manifold $\mathcal{M}^c_{loc}(W)$ for (5), namely (42) holds true when $\xi$ belongs to a random ball of $\mathcal{X}^c$ with a tempered radius. The Lipschitz continuity of $h^c$ with respect to $\xi$ will also be justified.

Remark 4.5. For a better comprehension Appendix [A] summarizes basic methods used to establish the invariance of random manifolds. In the theory of random dynamical systems the suitable concept for invariance of a random set (see [1, 19]) is that each orbit starting inside this random set, evolves and remains there omega-wise modulo the changes that occur due to the noise. These changes can be characterized by a suitable shift of the fiber of the noise, as argued in the proof of Lemma [4.13].

One of the proof technique for the existence of (local) center manifolds for deterministic and stochastic ODEs/PDEs is based on the Lyapunov-Perron method. We employ here the Lyapunov-Perron method in conjunction with rough path estimates. Note that the continuous-time Lyapunov-Perron map for (5) is constituted by (compare [20, Sec. 6.2] or [53])

$$J(W, U)[\tau] := S^c(\tau)\xi^c + \int_0^\tau S^c(\tau - r)F(U_r) \, dr + \int_0^\tau S^c(\tau - r)G(U_r) \, dW_r$$

(43)

$$+ \int_{-\infty}^\tau S^s(\tau - r)F(U_r) \, dr + \int_{-\infty}^\tau S^s(\tau - r)G(U_r) \, dW_r.$$

Due to the presence of the rough stochastic integrals we cannot directly work with (43), since we have to keep track of $\|W\|_\alpha$ and $\|\mathcal{W}\|_{2\alpha}$ occurring in (30) on a finite-time horizon. Similar to [25] we derive an appropriate discretized version of the Lyapunov-Perron map and show that this possesses a fixed-point in a suitable function space. We provide the explicit derivation of the discrete Lyapunov-Perron map in Subsection [4.1].

In the following sequel we will consider the solution of (5) at discrete times and obtain a sequence of mild solutions. The local center manifold theory will be developed for the discrete-time random dynamical system and will be shown to hold true for the original one, as in [25, 41].

4.1 Derivation of a discrete Lyapunov-Perron transform

The strategy is to rewrite (43) such that we only have to deal with stochastic integrals on the time-interval $[0,1]$, as considered in Section [2]. To this aim we let $W \in \Omega_W$, $t \in [0,1]$ and $i \in \mathbb{Z}^-$. Replacing
In order to simplify the expressions above we perform the following substitutions and use (36). More precisely, replacing \( r \) by \( t + i - 1 \) in (43) we obtain

\[
J(W,U)[t + i - 1] = \sum_{\tau = 0}^{t+i-1} S^c(\tau + i - 1) \xi^{\tau} + \int_0^{i-1+t} S^c(t + i - 1 - r) F(U_r) \, dr + \int_0^{i-1+t} S^c(t + i - 1 - r) G(U_r) \, dW_r \tag{44}
\]

\[
+ \int_{t+i-1}^{i-1} S^c(t + i - 1 - r) F(U_r) \, dr + \int_{t+i-1}^{i-1} S^c(t + i - 1 - r) G(U_r) \, dW_r \tag{45}
\]

\[
= \sum_{\tau = 0}^{i+1} \int_{k=0}^{k-1} S^c(t + i - 1 - r) F(U_r) \, dr + \int_{k=0}^{k-1} S^c(t + i - 1 - r) G(U_r) \, dW_r \tag{46}
\]

\[
+ \int_{i-1}^{i+1+t} S^c(t + i - 1 - r) F(U_r) \, dr + \int_{i-1}^{i+1+t} S^c(t + i - 1 - r) G(U_r) \, dW_r \tag{47}
\]

\[
+ \sum_{k=-\infty}^{k-1} \int_{t+i-1}^{k} S^c(t + i - 1 - r) F(U_r) \, dr + \sum_{k=-\infty}^{k-1} \int_{t+i-1}^{k} S^c(t + i - 1 - r) G(U_r) \, dW_r \tag{48}
\]

\[
+ \int_{t+i-1}^{i-1} S^c(t + i - 1 - r) F(U_r) \, dr + \int_{t+i-1}^{i-1} S^c(t + i - 1 - r) G(U_r) \, dW_r. \tag{49}
\]

In order to simplify the expressions above we perform the following substitutions and use (36). More precisely, replacing \( r \) by \( r - k + 1 \), the sum in (46) yields

\[
- \sum_{k=0}^{i+1} S^c(t + i - 1 - k) \int_0^1 S^c(1 - r) F(U_{r+k-1}) \, dr
\]

\[
- \sum_{k=0}^{i+1} S^c(t + i - 1 - k) \int_0^1 S^c(1 - r) G(U_{r+k-1}) \, d\Theta_{k-1} W_r.
\]
Substituting \( r \) with \( r - i + 1 \), we may re-write (47) as

\[
- \int_0^1 S^s(t - r) F(U_{r+i-1}) \, dr - \int_0^1 S^s(t - r) G(U_{r+i-1}) \, d\Theta_{i-1} W_r.
\]

Using again the substitution \( r \to r - k + 1 \), we may re-formulate (48) as

\[
\sum_{k=-\infty}^{i-1} S^s(t + i - 1 - k) \int_0^1 S^s(1 - r) F(U_{r+k-1}) \, dr
\]

\[
+ \sum_{k=\infty}^{i-1} S^s(t + i - 1 - k) \int_0^1 S^s(1 - r) G(U_{r+k-1}) \, d\Theta_{k-1} W_r.
\]

Finally, replacing \( r \) by \( r - i + 1 \) in (49) entails

\[
\int_0^t S^s(t - r) F(U_{r+i-1}) \, dr + \int_0^t S^s(t - r) G(U_{r+i-1}) \, d\Theta_{i-1} W_r.
\]

Summarizing, we have for \( W \in \Omega_W, t \in [0,1] \) and \( i \in \mathbb{Z}_- \) that

\[
J(W, U)[t + i - 1] = S^c(t + i - 1) \xi^c
\]

\[
- \sum_{k=0}^{i+1} S^c(t + i - 1 - k) \left( \int_0^1 S^c(1 - r) F(U_{r+i-1}) \, dr + \int_0^1 S^c(1 - r) G(U_r) \, d\Theta_{i-1} W_r \right) \tag{50}
\]

\[
- \int_0^1 S^c(1 - r) F(U_{r+i-1}) \, dr - \int_0^1 S^c(1 - r) G(U_{r+i-1}) \, d\Theta_{i-1} W_r \tag{51}
\]

\[
+ \sum_{k=-\infty}^{i-1} S^s(t + i - 1 - k) \left( \int_0^1 S^s(1 - r) F(U_{r+k-1}) \, dr + \int_0^1 S^s(1 - r) G(U_{r+k-1}) \, d\Theta_{k-1} W_r \right) \tag{52}
\]

\[
+ \int_0^t S^s(t - r) F(U_{r+i-1}) \, dr + \int_0^t S^s(t - r) G(U_{r+i-1}) \, d\Theta_{i-1} W_r, \tag{53}
\]

This will lead us to the structure of the \textit{discrete} Lyapunov-Perron map, as defined below (51). To simplify the notation, motivated by (50), (52) and (53), we write for \((Y, Y') \in D_{\Omega_W}^c([0,1]; X)

\[
T^c(W, Y, Y')[: \ldots := \left( \int_0^1 S^c/\xi (\cdot - r) F(Y) \, dr + \int_0^1 S^c/\xi (\cdot - r) G(Y_r) \, dW_r, G(Y) \right), \tag{54}
\]

and

\[
\hat{T}^c(W, Y, Y')[: \ldots := \left( \int_0^1 S^c(\cdot - r) F(Y_r) \, dr + \int_0^1 S^c(\cdot - r) G(Y_r) \, dW_r, G(Y) \right). \tag{55}
\]

\textbf{Remark 4.6.} \textit{When we work with the cut-off nonlinearities \( F_R \) and \( G_R \) instead of \( F \) and \( G \), we indicate this fact using the notation \( T^c_{R^c} \), respectively \( \hat{T}^c_{R^c} \), see also (52).}

The main goal now is to find an appropriate framework, in which we can formulate a meaningful fixed-point problem for \( J \).
4.2 The fixed-point argument

Before we proceed with the existence proof of local center manifolds for (5) we point out the main differences between our approach and a known approach for random center manifold theory for SDEs driven by linear multiplicative Stratonovich noise, e.g.

\[ du = (Au + f(u)) \, dt + u \circ d\tilde{B}_t. \]  

(56)

Here \( \tilde{B} \) stands for a two-sided real-valued Brownian motion. In this case, using the transformation \( \tilde{u} := ue^{-z(\tilde{B})} \), where \( (t, W) \mapsto z(\theta_t \tilde{B}) \) is the Ornstein-Uhlenbeck process (recall (3)), one obtains the non-autonomous random differential equation

\[ du = (Au + z(\theta_t \tilde{B})u + g(\theta_t \tilde{B}, u)) \, dt, \]

(57)

where \( g(\tilde{B}, u) := e^{-z(\tilde{B})}f(e^{z(\tilde{B})}u) \). Regarding Assumptions 4.1 one immediately infers that the continuous-time Lyapunov-Perron transform for (57) is given by

\[
J(\tilde{B}, u)[t] := S^c(t)e^{\int_0^t z(\theta_r \tilde{B}) \, dr} P^c x + \int_0^t S^c(t-r)e^{\int_r^t z(\theta_r \tilde{B}) \, dr} P^c g(\theta_r \tilde{B}, u(r)) \, dr \\
+ \int_{-\infty}^t S^a(t-r)e^{\int_r^t z(\theta_r \tilde{B}) \, dr} P^a g(\theta_r \tilde{B}, u(r)) \, dr.
\]

(58)

Further details on the derivation/setting of this operator can be found in [53], [20, Sec. 6.2.2], [14, Ch.4] and the references specified therein. The next natural step is to show that (58) possesses a fixed-point in a certain function space. One possible choice turns out to be \( BC^{\eta,z}(\mathbb{R}^-; \mathcal{X}) \), see [20, p. 156]. This space is defined as

\[
BC^{\eta,z}(\mathbb{R}^-; \mathcal{X}) := \left\{ u : \mathbb{R}^- \to \mathcal{X}, \ u \text{ is continuous and } \sup_{t \leq 0} e^{-\eta t - \int_0^t z(\theta_r \tilde{B}) \, dr} \|u(t)\| < \infty \right\}
\]

(59)

and is endowed with the norm

\[ \|u\|_{BC^{\eta,z}} := \sup_{t \leq 0} e^{-\eta t - \int_0^t z(\theta_r \tilde{B}) \, dr} |u(t)|. \]

Here \( \eta \) is determined from (40) and (41), namely one has \( -\beta < \eta < 0 \). Note that the previous expressions are well-defined since

\[ \lim_{t \to \pm \infty} \frac{|z(\theta_t \tilde{B})|}{|t|} = 0, \]

according to [19, Lem. 2.1] and the references specified therein. Yet, note that the Lyapunov-Perron map (58) always works with an implicitly transformed equation and not directly on the space of solutions of the original problem.

In our context, we directly work with solutions to RDEs. However, since our Lyapunov-Perron transform (58) contains stochastic integrals, the entire machinery applicable to (58) breaks down. Therefore, we have to find an appropriate setting for the fixed-point argument. To this aim we introduce now a function space which helps us incorporate the discretized version of (13) derived in the previous subsection. Namely, we work with the space of sequences \( BC^\eta(D^{2a}_W) \), whose elements are constituted by \( \mathcal{X} \)-valued controlled rough paths on \([0,1]\).
Furthermore, \( J \) of \( U \in \mathbb{t} \) specified in (28), i.e., we replace \( F \) and \( G \) by \( F_R \) respectively \( G_R \). Obviously, the following result holds true.

**Lemma 4.8.** The solution operator of the truncated RDE \([5]\) generates a random dynamical system \( \varphi_R \).

**Proof.** This follows analogously to Lemma 3.6

Motivated by Subsection 4.1 we are justified to introduce the discrete Lyapunov-Perron transform \( J_d(W, U, \xi) \) for a sequence of controlled rough paths \( U \in BC^n(D_{W}^{2\alpha}) \) and \( \xi \in \mathcal{X} \) as the pair \( J_d(W, U, \xi) := (J_d^{1}(W, U, \xi), J_d^{2}(W, U, \xi)) \), where the precise structure is given below. For \( t \in [0, 1] \), \( W \in \Omega_W \) and \( i \in \mathbb{Z}_- \) we define

\[
J_d^{1}(W, U, \xi)[i - 1, t] := S^c(t + i - 1)\xi^c
\]

\[
- \sum_{k=0}^{i+1} S^c(t + i - 1 - k) \left( \int_0^1 S^c(1 - r)F_R(U_r^{k-1}) \, dr + \int_0^1 S^c(1 - r)G_R(U_r^{k-1}) \, d\Theta_{k-1}W_r \right)
\]

\[
- \int_t^1 S^c(t - r)F_R(U_r^{i-1}) \, dr - \int_t^1 S^c(t - r)G_R(U_r^{i-1}) \, d\Theta_{i-1}W_r
\]

\[
+ \sum_{k=-\infty}^{i-1} S^s(t + i - 1 - k) \left( \int_0^1 S^s(1 - r)F_R(U_r^{k-1}) \, dr + \int_0^1 S^s(1 - r)G_R(U_r^{k-1}) \, d\Theta_{k-1}W_r \right)
\]

\[
+ \int_0^t S^s(t - r)F_R(U_r^{i-1}) \, dr + \int_0^t S^s(t - r)G_R(U_r^{i-1}) \, d\Theta_{i-1}W_r.
\]

Furthermore, \( J_d^{2}(W, U, \xi) \) stands for the Gubinelli derivative of \( J_d^{1}(W, U, \xi) \), i.e. \( J_d^{2}(W, U, \xi)[i - 1, \cdot] := \langle J_d^{1}(W, U, \xi)[i - 1, \cdot] \rangle^{s} \). Note that \( \xi^c \) can be recovered setting \( i = 0 \) and \( t = 1 \) in the definition of \( J_d^{1}(W, U, \xi) \), i.e., \( J_d^{1}(W, U, \xi)[-1, 1] = \xi^c \).

We emphasize that for a sequence \( U \in BC^n(D_{W}^{2\alpha}) \) the first index \( i \in \mathbb{Z}_- \) in the definition of \( J_d(W, U, \xi)[\cdot, \cdot] \) gives the position within the sequence and the second one refers to the time variable \( t \in [0, 1] \). Not to overburden the notation in \([61]\) for the elements of \( U \) we simply write \( U_t \) instead of \( U[i, t] \) for \( i \in \mathbb{Z}_- \) and \( t \in [0, 1] \). We are going to show that \([61]\) maps \( BC^n(D_{W}^{2\alpha}) \) into itself and is a contraction if the constant \( K \) specified in \([31]\) is chosen small enough, as justified by the following computation.

Now we can state our first main result. In the following, \( C_S \) stands for a constant which exclusively depends on the group \( S \).
Theorem 4.9. Let Assumptions 4.1 and 4.2 hold true and let $K$ satisfy the gap condition
\[ K \left( \frac{e^{\beta+y}(C SM_\gamma e^{-y} + 1)}{1 - e^{-(\beta+y)}} + \frac{e^{\gamma-y}(C SM_\gamma e^{-y} + 1)}{1 - e^{-(\beta+y)}} \right) < \frac{1}{4}. \] (62)

Then, the map $J_d : \Omega \times BC^\eta(D_\gamma W) \to BC^\eta(D_\gamma W)$ possesses a unique fixed-point $\Gamma \in BC^\eta(D_\gamma W)$.

Remark 4.10. Note that (62) can be obtained for instance by choosing the constant appearing in (61) as
\[ K^{-1} := 4e^{(\beta+y)/2} \left( \frac{e^{(\beta+y)/2}C S(M_\alpha + M_\epsilon) + 1}{1 - e^{-(\beta+y)/2}} \right), \] (63)

which follows by setting $\eta := \frac{\beta+y}{2} < 0$.

Proof. Let two sequences $\mathbb{U} = ((U_{i-1}, (U_{i-1}'))_{i \in \mathbb{Z}^+}$ and $\tilde{\mathbb{U}} = ((\tilde{U}_{i-1}, (\tilde{U}_{i-1}'))_{i \in \mathbb{Z}^+}$ belong to $BC^\eta(D_\gamma W)$ and satisfy $P^c U_{i-1} = P^c \tilde{U}_{i-1} = \xi_c$. We want to verify the contraction property. The fact that $J_d(\cdot)$ maps $BC^\eta(D_\gamma W)$ into itself can be derived by setting $\tilde{\mathbb{U}} = 0$ in the next computation and using that $F_R(0) = G_R(0) = 0$. According to (60) we have
\[ \|S^c(t + i - 1)\xi_c, 0\|_{BC^\eta(D_\gamma W)} = \|S^c(\cdot + i - 1)\xi_c\|_{2\alpha e^{-\eta(i-1)}} = \|S^c(i - 1)\xi_c\| \|S(\cdot)\|_{2\alpha e^{-\eta(i-1)}} \leq C S M_c e^{(\gamma-y)(i-1)} \|\xi_c\|. \]

The previous expression remains bounded for $i \in \mathbb{Z}^+$ since we assumed that $-\beta < \eta < 0 \leq \gamma < \beta$. Next, we are going to estimate the difference
\[ \|J_d(W, \mathbb{U}, \xi) - J_d(W, \tilde{\mathbb{U}}, \xi)\|_{BC^\eta(D_\gamma W)} \]
in several intermediate steps. Verifying the contraction property on the stable part of (61), one has to compute two terms. First of all, due to (63)
\[ \sum_{k=-\infty}^{i-1} e^{-\eta(i-1)}\|S^c(\cdot + i - 1 - k)\|_{2\alpha} \|T^c_R(\Theta_{k-1} W, \tilde{U}_{k-1}, (\tilde{U}_{k-1}'))[1] - T^c_R(\Theta_{k-1} W, \tilde{U}_{k-1}, (\tilde{U}_{k-1}'))[1]\|_{D_\gamma W} \]
\[ \leq \sum_{k=-\infty}^{i-1} C S M_c e^{-\eta(i-1)} e^{-\beta(i-1-k)}K \|U_{k-1} - \tilde{U}_{k-1}, (U_{k-1} - \tilde{U}_{k-1})'\|_{D_\gamma W} \]
\[ \leq \sum_{k=-\infty}^{i-1} C S M_c e^{-\eta(i-1)} e^{-\beta(i-1-k)}e^{\eta(k-1)}Ke^{-\eta(k-1)} \|U_{k-1} - \tilde{U}_{k-1}, (Y_{k-1} - \tilde{U}_{k-1})'\|_{D_\gamma W} \]
\[ \leq \sum_{k=-\infty}^{i-1} e^{-(\eta+y)(i-1-k)}C S M_c e^{-\eta}Ke^{-\eta(k-1)} \|U_{k-1} - \tilde{U}_{k-1}, (U_{k-1} - \tilde{U}_{k-1})'\|_{D_\gamma W}. \]

Combining this with the last term of (61) entails the final estimate on the stable part
\[ \sum_{k=-\infty}^{i-1} e^{-\eta(i-1)}\|S^c(\cdot + i - 1 - k)\|_{2\alpha} \|T^c_R(\Theta_{k-1} W, \tilde{U}_{k-1}, (\tilde{U}_{k-1}'))[1] - T^c_R(\Theta_{k-1} W, \tilde{U}_{k-1}, (\tilde{U}_{k-1}'))[1]\|_{D_\gamma W} + e^{-\eta(i-1)}\|T^c_R(\Theta_{i-1} W, U_{i-1}, (U_{i-1}'))[\cdot] - T^c_R(\Theta_{i-1} W, \tilde{U}_{i-1}, (\tilde{U}_{i-1}'))[\cdot]\|_{D_\gamma W} \]
\[ \leq \sum_{k=-\infty}^{i} e^{-(\eta+y)(i-k)}K(C S M_c e^{-\eta} + 1)e^{-\eta(k-1)} \|U_{k-1} - \tilde{U}_{k-1}, (U_{k-1} - \tilde{U}_{k-1})'\|_{D_\gamma W} \]
\[ \leq K e^{\eta+y}(C S M_c e^{-\eta} + 1)\|U - \tilde{U}, U' - \tilde{U}'\|_{BC^\eta(D_\gamma W)}. \]
We focus now on the center part. Here we obtain

\[ \sum_{k=0}^{i+1} e^{-\eta(i-1)} \|S^c(\cdot + i - 1 - k)\|_{2\alpha} \|T_R(\Theta_{k-1}W, U^{k-1}, (U^{k-1})')[1] - T_R(\Theta_{k-1}W, \tilde{U}^{k-1}, (\tilde{U}^{k-1})')[1]\|_{D_W^{2\alpha}} \]

\[ \leq \sum_{k=0}^{i+1} C_S M e^{-\eta(i-1)} e^{\gamma(i-1-k)} K \|U^{k-1} - \tilde{U}^{k-1}, (U^{k-1} - \tilde{U}^{k-1})'\|_{D_W^{2\alpha}} \]

\[ \leq \sum_{k=0}^{i+1} C_S M e^{-\eta(i-1)} e^{\gamma(i-1-k)} e^{\eta(k-1)} K \|U^{k-1} - \tilde{U}^{k-1}, (U^{k-1} - \tilde{U}^{k-1})'\|_{D_W^{2\alpha}} \]

\[ \leq \sum_{k=0}^{i+1} C_S M e^{(\gamma - \eta)(i-1-k)} e^{-\eta(k-1)} K \|U^{k-1} - \tilde{U}^{k-1}, (U^{k-1} - \tilde{U}^{k-1})'\|_{D_W^{2\alpha}}. \]

Combining this and estimating the third summand in (61) yields

\[ \sum_{k=0}^{i+1} \sum_{k=0}^{i+1} e^{-\eta(i-1)} \|S^c(\cdot + i - 1 - k)\|_{2\alpha} \|T_R(\Theta_{k-1}W, U^{k-1}, (U^{k-1})')[1] - T_R(\Theta_{k-1}W, \tilde{U}^{k-1}, (\tilde{U}^{k-1})')[1]\|_{D_W^{2\alpha}} \]

\[ + e^{-\eta(i-1)} \|T_R(\Theta_{i-1}W, U^{i-1}, (U^{i-1})')[1] - T_R(\Theta_{i-1}W, \tilde{U}^{i-1}, (\tilde{U}^{i-1})')[1]\|_{D_W^{2\alpha}} \]

\[ \leq \sum_{k=0}^{i} e^{(\gamma - \eta)(i-1-k)} K (C_S M e^{-\eta + 1}) e^{-\eta(k-1)} \|U^{k-1} - \tilde{U}^{k-1}, (U^{k-1} - \tilde{U}^{k-1})'\|_{D_W^{2\alpha}} \]

\[ \leq K e^{\gamma - \eta(\frac{C_S M e^{-\eta + 1}}{1 - e^{-(\gamma - \eta)}})} \|U - \tilde{U}, U' - \tilde{U}'\|_{BC^n(D_W^{2\alpha})}. \]

Due to (62) we have that

\[ \|J_d(W, U, \xi) - J_d(W, \tilde{U}, \xi')\|_{BC^n(D_W^{2\alpha})} \leq \frac{1}{4} \|U - \tilde{U}, U' - \tilde{U}'\|_{BC^n(D_W^{2\alpha})}. \]

Applying Banach’s fixed-point theorem, we infer that \(J_d(W, U, \xi)\) possesses a unique fixed point \(\Gamma(\xi, W) \in BC^n(D_W^{2\alpha})\) for each fixed \(\xi \in X^c\).

The fixed-point will further help us to characterize the local center manifold.

**Lemma 4.11.** The mapping \(\xi \mapsto \Gamma(\xi, W) \in BC^n(D_W^{2\alpha})\) is Lipschitz continuous.

**Proof.** One easily obtains for \(\xi_1, \xi_2 \in X^c\) that

\[ \|\Gamma(\xi_1, W) - \Gamma(\xi_2, W)\|_{BC^n(D_W^{2\alpha})} = \|J_d(W, \Gamma(\xi_1, W), \xi_1) - J_d(W, \Gamma(\xi_2, W), \xi_2)\|_{BC^n(D_W^{2\alpha})} \]

\[ \leq \|J_d(W, \Gamma(\xi_1, W), \xi_1) - J_d(W, \Gamma(\xi_1, W), \xi_2)\|_{BC^n(D_W^{2\alpha})} \]

\[ + \|J_d(W, \Gamma(\xi_1, W), \xi_2) - J_d(W, \Gamma(\xi_2, W), \xi_2)\|_{BC^n(D_W^{2\alpha})} \]

\[ \leq \|S^c(\cdot + i - 1)(\xi_i - \xi_2), 0\|_{BC^n(D_W^{2\alpha})} + \frac{1}{4} \|\Gamma(\xi_1, W) - \Gamma(\xi_2, W)\|_{BC^n(D_W^{2\alpha})} \]

\[ \leq C_S M e^{\gamma|\xi_1 - \xi_2|} + \frac{1}{4} \|\Gamma(\xi_1, W) - \Gamma(\xi_2, W)\|_{BC^n(D_W^{2\alpha})}, \]

which proves the statement. \(\square\)

Before stating the existence result of local center manifolds for (5), we must fix further notations. In the following we write \(U(\xi)\) to emphasize the dependence of the path \(U\) on the initial condition \(\xi\) of the RDE (5). Furthermore, consider \(\Gamma(\xi, W)\), which is the fixed-point of \(J_d(W, U, \xi)\) belonging to
In order to justify the invariance of the center manifold as required in Definition 4.4, we must show that there exists a tempered from below random variable \( \rho(W) \) such that the local center manifold of (5) is given by the graph of a Lipschitz function, namely

\[
\mathcal{M}^c_{\text{loc}}(W) = \{ \xi + h^c(\xi, W) : \xi \in B_{\mathcal{X}^c}(0, \rho(W)) \},
\]

where we define

\[
h^c(\xi, W) := P^s\Gamma(\xi, W)[1, 1] |_{B_{\mathcal{X}^c}(0, \rho(W))},
\]

and consequently

\[
h^c(\xi, W) = \sum_{k=-\infty}^{0} S^s(-k) \int_{0}^{1} S^s(1-r)F(\Gamma(\xi, W)[k-1, r]) \, dr + \sum_{k=-\infty}^{0} S^s(-k) \int_{0}^{1} S^s(1-r)G(\Gamma(\xi, W)[k-1, r]) \, d\Theta_{k-1} \cdot W_r.
\]

Proof. First of all, since \( F(0) = G(0) = 0 \) we have that \( h^c(0, W) = 0 \), consequently \( 0 \in \mathcal{M}^c_{\text{loc}}(W) \). Regarding this, the tangency condition will be clear in Section 4 when we investigate the smoothness of \( \mathcal{M}^c_{\text{loc}}(W) \). We now show that \( \mathcal{M}^c_{\text{loc}}(W) \) is a local center manifold with discrete time for the random dynamical system \( \varphi(\cdot) \) associated to (5). Namely, for initial conditions \( \xi \in B_{\mathcal{X}^c}(0, \rho(W)) \), where \( \rho(W) \) will be appropriately chosen, we are going to show that

\[
\Gamma(\xi, W)[-1, 1] \subseteq \mathcal{M}^c_{\text{loc}}(W) \quad \text{and} \quad \Gamma(\xi, W)[i-1, 1] \subseteq \mathcal{M}^c_{\text{loc}}(\Theta_i W)
\]

for \( i \in \mathbb{Z}^- \). Furthermore, the corresponding trajectories starting in \( \Gamma(\xi, W)[-1, 1] \) and \( \Gamma(\xi, W)[i-1, 1] \) remain within a ball with a tempered radius. We set \( \tilde{\gamma}^i(\xi) := \Gamma(\xi, W)[i-1, \cdot] \). We index \( \tilde{\gamma} \) by \( i \) and not \( i-1 \) because we want to derive expressions for \( \mathcal{M}^c_{\text{loc}}(W) \) respectively \( \mathcal{M}^c_{\text{loc}}(\Theta_i W) \) instead of \( \mathcal{M}^c_{\text{loc}}(\Theta_{i-1} W) \). When we compute the \( D^{\psi_\alpha}_{\mathcal{W}} \)-norm we use for simplicity the notation \( \tilde{\gamma}^i \). However, when we analyze the shift with respect to \( W \) of \( \Gamma(\xi, W)[\cdot, \cdot] \), i.e., \( \Gamma(\xi, \Theta_i W)[\cdot, \cdot] \) we explicitly write all the arguments.
In order to justify the claimed assertions we firstly set
\[
\rho(W) := \frac{R(\Theta_{i-1}W)}{2L_\Gamma e^{-\eta}},
\] (66)
where \(L_\Gamma\) denotes the Lipschitz constant of the mapping \(\xi \mapsto \Gamma(\xi, W) \in BC^n(D_W^{2\alpha})\) as obtained in Lemma 4.11. Recall that \(R\) stands for the tempered from below radius used in the cut-off procedure (28). Therefore, for \(\xi \in \mathcal{X}\) we have
\[
\|\Gamma(P^c\xi, W)\|_{BC^n(D_W^{2\alpha})} \leq L_\Gamma|P^c\xi|,
\]
so letting \(\xi \in B_{X^c}(0, \rho(W))\) immediately entails
\[
\|\Gamma(\xi, W)\|_{BC^n(D_W^{2\alpha})} \leq L_\Gamma \rho(W).
\] (67)
Using the definition of the \(\| \cdot \|_{BC^n(D_W^{2\alpha})}\)-norm, the previous inequality rewrites as
\[
\sup_{i \in \mathbb{Z}_-} e^{-\eta(i-1)}\|\tilde{y}^i(\xi), (\tilde{y}^i(\xi))'\|_{D_W^{2\alpha}} \leq L_\Gamma \frac{R(\Theta_{i-1}W)}{2L_\Gamma e^{-\eta}}.
\] (68)
Setting \(i = 0\) in (68) we infer that for \(|\xi| \leq \rho(W)\), the norm of the trajectory \(\tilde{y}^{-1}(\xi) = \Gamma(\xi, W)[{-1, 1}]\) can be estimated by
\[
\|\tilde{y}^0(\xi), (\tilde{y}^0(\xi))'\|_{D_W^{2\alpha}} \leq \frac{R(\Theta_{i-1}W)}{2}.
\]
The next step is to derive that \(\Gamma(\xi, W)[i-1, 1] = \tilde{y}^i_1(\xi) \in \mathcal{M}_{loc}(\Theta_iW)\) and to show that for the corresponding trajectory, the relation
\[
\|\tilde{y}^i(\xi), (\tilde{y}^i(\xi))'\|_{D_W^{2\alpha}} \leq \frac{R(\Theta_{i-1}W)}{2}
\]
holds true. To this aim, we firstly employ (69). Note that this is also valid in the discrete-time setting, according to [1, Sec. 4.1.1]. This further yields that there exists a positive random variable \(\hat{\rho}(W)\) and a constant (which we choose \(L_\Gamma\)) such that
\[
\rho(\Theta_iW) \geq \hat{\rho}(W)L_\Gamma e^{\eta(i-1)}.
\] (69)
Now, taking \(\xi \in B_{X^c}(0, \hat{\rho}(W))\) we have according to (67) that
\[
\sup_{i \in \mathbb{Z}_-} e^{-\eta(i-1)}\|\tilde{y}^i(\xi), (\tilde{y}^i(\xi))'\|_{D_W^{2\alpha}} \leq L_\Gamma \hat{\rho}(W).
\] (70)
Combining this with (69) entails
\[
\rho(\Theta_iW) \geq \hat{\rho}(W)L_\Gamma e^{\eta(i-1)} \geq |P^c\tilde{y}^i_1(\xi)| = |P^c\Gamma(\xi, W)[i-1, 1]|.
\] (71)
Summarizing we obtained for \(\xi \in B_{X^c}(0, \hat{\rho}(W))\) that
\[
\Gamma(\xi, W)[{-1, 1}] \in \mathcal{M}_{loc}^c(W)
\]
\[
\Gamma(\xi, W)[i-1, 1] \in \mathcal{M}_{loc}(\Theta_iW).
\]
Secondly, in order to derive the invariance property of \(\mathcal{M}_{loc}^c(W)\), analogously to [25, Lem. 4.7] or [12, Lem. 5.5] one can establish a connection between the fixed-points of \(J_d(W, \cdot, \cdot)\) and \(J_d(\Theta_iW, \cdot, \cdot)\) for
Recalling Remark 4.12, this further leads to
\[ \Gamma(\xi, W)[i - 1, 0] = \Gamma(\xi, W)[i - 2, 1] \]
and infers using (85)
\[ \Gamma(\xi, W)[i - 1, \cdot] = \Gamma(P^\varepsilon \Gamma(\xi, W)[i - 1, 1], \Theta_i W)[\cdot, \cdot], \]
which tells us that \( \Gamma(\xi, W)[i - 1, \cdot] \) can be obtained from \( \Gamma(\cdot, \Theta_i W)[-1, \cdot] \) on the \( \Theta_i W \)-fiber starting with the initial condition \( P^\varepsilon (\Gamma(\xi, W)[i - 1, 1]) \). For more information and a detailed computation compare (85) and consult Appendix A. Regarding this, we derive using (70) and (71) we get
\[ \|\tilde{y}_i^j(\xi), (\tilde{y}_i^j)(\xi)\|_{D^2_W} \leq L \Gamma e^{-\eta \rho(\Theta_i W)}, \]
and consequently we have
\[ \|\tilde{y}_i^j(\xi), (\tilde{y}_i^j)(\xi)\|_{D^2_W} \leq \frac{R(\Theta_i W)}{2}. \] (73)
Recalling Remark 4.12, this further leads to
\[ (\Gamma(\xi, W)[i - 1, t], (\Gamma(\xi, W)[i - 1, t])') = (\tilde{y}_i^j(\xi), (\tilde{y}_i^j)(\xi))' \]
\[ = \left( S(t)\Gamma(\xi, W)[i - 1, 0] + \int_0^t S(t - r)F(\tilde{y}_i^j(\xi)) \, dr + \int S(t - r)G(\tilde{y}_i^j(\xi)) \, d\Theta_{i - 1} W, G(\tilde{y}_i^j(\xi)) \right) \]
\[ = \left( S(t)\Gamma(\xi, W)[i - 2, 1] + \int_0^t S(t - r)F(\tilde{y}_i^j(\xi)) \, dr + \int S(t - r)G(\tilde{y}_i^j(\xi)) \, d\Theta_{i - 1} W, G(\tilde{y}_i^j(\xi)) \right), \]
so setting \( t = 1 \) in (73) one obtains
\[ \varphi(1, \Theta_{i - 1} W, (\Gamma(\xi, W)[i - 2, 1]) = (\Gamma(\xi, W)[i - 1, 1] \in M^s_{loc}(\Theta_i W). \]
Now, the cocycle property established in Lemma 3.6 implies that
\[ M^s_{loc}(W) \ni \Gamma(\xi, W)[-1, 1] = \varphi(1, \Theta_{-1} W, (\Gamma(\xi, W)[-1, 0]) = \varphi(-i + 1, \Theta_{i - 1} W, (\Gamma(\xi, W)[i - 1, 0]). \]
Letting \( j := -i + 1 \) in the previous relation yields
\[ \varphi(j, \Theta_{-j} W, (\Gamma(\xi, W)[-j - 1, 1]) \in M^s_{loc}(W). \]
Replacing \( \Theta_{-j} W \) by \( W \), we finally conclude that
\[ \varphi(j, W, (\Gamma(\xi, \Theta_j W)[-j - 1, 1]), \in M^s_{loc}(\Theta_j W). \]
One can extend these results to the continuous-time setting, namely one follows the steps presented in the previous proof replacing \( i - 1 \) by \( i - 1 + t \), where \( i \in \mathbb{Z}^- \) and \( t \in (0, 1) \). This can easily be achieved regarding that
\[ \varphi(-i + 1, \Theta_{i - 1} W, (\Gamma(\xi, W)[i - 1, 0]) = \varphi(-i + 1 - t, \Theta_{i - 1 + t} W, (\Gamma(\xi, W)[i - 1, t]), \]
according to the cocycle property. Consequently, for a sufficiently small initial condition \( \xi \), i.e. \( \xi \in B_{\chi_0}(0, \hat{\rho}(W)) \), one can show that \( (\Gamma(\xi, W)[i - 1, t] \in M^s_{loc}(\Theta_{i + 1} W) \). Indeed, as argued before one constructs as in (71) a random tempered radius \( \hat{\rho}(W) \) such that \( \|P^\varepsilon \Gamma(\xi, W)[i - 1, t]\| \leq \hat{\rho}(\Theta_{i + 1} W); \) see also Appendix A and [25, 11].
Putting all these insights together we infer:

**Lemma 4.14.** The local center manifold $\mathcal{M}_{loc}^c(W)$ for (5) is given by

$$\mathcal{M}_{loc}^c(W) = \{\xi + h^c(\xi, W) : \xi \in B_{X^c}(0, \hat{\rho}(W))\},$$

where

$$h^c(\xi, W) := \int_{-\infty}^{0} S^a(-r)F(U_r(\xi)) \, dr + \int_{-\infty}^{0} S^a(-r)G(U_r(\xi)) \, dW_r.$$

Note that if one takes $\alpha \in (1/2, 1)$, i.e., this corresponds to a fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$, the above computations are applicable to this case as well, yet could even be simplified further in this special case. Random center manifolds for Itô/Stratonovich for certain classes of SDEs have also been investigated in [18, Sec. 4] and [42, Sec. 5], yet these works always using the transformation of the corresponding SDE in a random ODE as shortly indicated at the beginning of Section 4.2. In fact, our results turn out to be compatible to previous works if the same assumptions are made, yet are obtained by very different techniques and in the broader class of RDEs.

## 5 Smoothness of center manifolds

We first point out the main arguments, which guarantee the smoothness of random invariant manifolds. These have been employed in [19, Sec. 4] and [12, Sec. 5] for random stable/unstable manifolds and in [15] for center manifolds. From the rough path point of view it is essential to investigate the differentiability of the Itô-Lyons map, i.e., the map that associates a controlled rough path to the solution of the RDE driven by this path; see also [16] or [22, Sec. 8.9]. For our goals, it suffices to show the continuous-differentiability of the mapping $\xi \mapsto U(\xi)$, where $\xi \in X^c$ and $U(\xi)$ is the solution of (5) as discussed in Section 2.

**Remark 5.1.** Note that $h^c(\cdot, W)$ is Lipschitz due to Lemma 4.14. In this section we establish that additional smoothness assumptions on $F$ and $G$ (i.e., $F : \mathbb{R}^n \to \mathbb{R}^n$ is $C^m$ and $G : \mathbb{R}^n \to \mathbb{R}^{n \times d}$ is $C^{m+3}_b$ for $m \geq 1$) lead to better regularity of $h^c(\cdot, W)$.

We now indicate the main ideas in the classical proof of smoothness of invariant manifolds for S(P)DEs. In order to show that $\mathcal{M}_{loc}^c(W)$ obtained in Lemma 4.13 is $C^1$ one needs to verify that $h^c(\xi, W)$ is continuously differentiable in $\xi \in X^c$. Therefore, one has to establish the differentiability of the solution of the Lyapunov-Perron fixed-point problem in $\xi$. For notational simplicity, we firstly describe the main ideas without using the controlled rough path notation. This will be used only when we state the main result of this section, namely Theorem 5.2.

We consider the continuous-time Lyapunov-Perron transform associated to (5) and do a formal computation to illustrate the main idea. For $\xi \in X^c$ we have

$$U_t(\xi) = S^c(t)\xi + \int_{0}^{t} S^c(t-r)F(U_r(\xi)) \, dr + \int_{0}^{t} S^c(t-r)G(U_r(\xi)) \, dW_r$$

$$+ \int_{-\infty}^{t} S^a(-r)F(U_r(\xi)) \, dr + \int_{-\infty}^{t} S^a(-r)G(U_r(\xi)) \, dW_r.$$

Since we analyze the differentiability of $\xi \mapsto U_t(\xi)$ we have to investigate the difference $U_t(\xi) - U_t(\xi_0)$, where $\xi_0 \in X^c$. We consider

$$U_t(\xi) - U_t(\xi_0) - T(U_t(\xi) - U_t(\xi_0)) = S^c(t)(\xi - \xi_0) + I,$$  \hspace{1cm} (75)
where

\[ T(Z) := \int_0^t S^c(t-r)DF(U_r(\xi_0))Z \, dr + \int_0^t S^c(t-r)DG(U_r(\xi_0))Z \, dW_r \]  
\[ + \int_{-\infty}^t S^s(t-r)DF(U_r(\xi_0))Z \, dr + \int_{-\infty}^t S^s(t-r)DG(U_r(\xi_0))Z \, dW_r \]  

and

\[ I := \int_0^t S^c(t-r)(F(U_r(\xi))-F(U_r(\xi_0))-DF(U_r(\xi_0))(U_r(\xi)-U_r(\xi_0))) \, dr \]  
\[ + \int_{-\infty}^t S^s(t-r)(F(U_r(\xi))-F(U_r(\xi_0))-DF(U_r(\xi_0))(U_r(\xi)-U_r(\xi_0))) \, dr \]  
\[ + \int_0^t S^c(t-r)(G(U_r(\xi))-G(U_r(\xi_0))-DG(U_r(\xi_0))(U_r(\xi)-U_r(\xi_0))) \, dW_r \]  
\[ + \int_{-\infty}^t S^s(t-r)(G(U_r(\xi))-G(U_r(\xi_0))-DG(U_r(\xi_0))(U_r(\xi)-U_r(\xi_0))) \, dW_r. \]

Now, the goal is to derive conditions which ensure that \( ||T|| < 1 \), in order for \((\text{Id} - T)\) to be invertible together with \(|I| = o(|\xi - \xi_0|)\) as \( \xi \to \xi_0 \). Then, due to (75) one concludes that

\[ U_t(\xi) - U_t(\xi_0) = (\text{Id} - T)^{-1}S^c(t)(\xi - \xi_0) + o(|\xi - \xi_0|), \]  
as \( \xi \to \xi_0 \),

which implies that \( U_t(\xi) \) is differentiable in \( \xi \). Its derivative is constituted by

\[ D_\xi U_t(\xi) = S^c(t) + \int_0^t S^c(t-r)DF(U_r(\xi))D_\xi U_r(\xi) \, dr + \int_0^t S^c(t-r)DG(U_r(\xi))D_\xi U_r(\xi) \, dW_r \]  
\[ + \int_{-\infty}^t S^s(t-r)DF(U_r(\xi))D_\xi U_r(\xi) \, dr + \int_{-\infty}^t S^s(t-r)DG(U_r(\xi))D_\xi U_r(\xi) \, dW_r. \]

The fact that such a formula is valid for the controlled rough integral introduced in Theorem 2.4 follows according to [16, Cor. 2], see also [22, Thm. 8.10].
To prove the continuity of the mapping $\xi \mapsto D_\xi U_t(\xi)$ one computes for $\xi$ and $\xi_0 \in \mathcal{X}^c$

$$D_\xi U_t(\xi) - D_\xi U_t(\xi_0) = \int_0^t S^c(t - r)(DF(U_r(\xi))D_\xi U_r(\xi) - DF(U_r(\xi_0))D_\xi U_r(\xi_0)) \, dr$$

$$+ \int_{-\infty}^t S^s(t - r)(DG(U_r(\xi))D_\xi U_r(\xi) - DG(U_r(\xi_0))D_\xi U_r(\xi_0)) \, dW_r$$

$$= \int_0^t S^c(t - r)(DF(U_r(\xi)))(D_\xi U_r(\xi) - D_\xi(U_r(\xi_0))) \, dr$$

$$+ \int_{-\infty}^t S^s(t - r)(DG(U_r(\xi)))(D_\xi U_r(\xi) - D_\xi(U_r(\xi_0))) \, dW_r + \mathcal{T},$$

where

$$\mathcal{T} = \int_0^t S^c(t - r)(DF(U_r(\xi)) - DF(U_r(\xi_0))D_\xi U(\xi_0)) \, dr$$

$$+ \int_{-\infty}^t S^s(t - r)(DG(U_r(\xi)) - DG(U_r(\xi_0))D_\xi U(\xi_0)) \, dW_r.$$

For further details and properties of flows associated to rough differential equations, see [2] [10] and the references specified therein. Regarding all these preliminary considerations, we now proceed with the proof of the smoothness of the local center manifold $M^c_{loc}(W)$ for (5) obtained in Lemma 4.14.

**Theorem 5.2.** Assume that $F$ is $C^m$ and $G$ is $C^{m+3}_b$ for $m \geq 1$. If $-\beta < m\eta < \gamma$ and

$$K \left( \frac{e^{\beta + \eta}(CS_M e^{-\eta}) + 1}{1 - e^{-(\beta + \eta)}} \right) + \frac{e^{\gamma - \eta}(CS_M e^{-\eta}) + 1}{1 - e^{-(\gamma - \eta)}} < 1 \quad \text{for } 1 \leq j \leq m,$$

then $M^c_{loc}(W)$ is a local $C^m$-center manifold.

**Proof.** In order to prove the assertion we have to justify the formal computation presented above. The first step is pass to the discrete-time setting, as in Section 4.2.

Let $\xi, \xi_0 \in \mathcal{X}^c$ be fixed. We first consider $j = 1$, the statement follows thereafter by induction. Since (69) holds true we can find a small number $\delta > 0$ such that $-\beta < \eta + 2\delta < 0$ and that

$$\text{gap}(\eta') := K \left( \frac{e^{\beta + \eta'}(CS_M e^{-(\eta')}) + 1}{1 - e^{-(\beta + \eta')}} \right) + \frac{e^{\gamma - \eta'}(CS_M e^{-(\eta')}) + 1}{1 - e^{-(\gamma - \eta')}} < 1 \quad \text{for } 0 \leq \eta' \leq 2\delta. \quad (80)$$

As in Section 4.2 one can define the mapping $J_d(W, U, \xi)$ for a sequence $U := ((U^{i-1}, (U^i)'))_{i \in \mathbb{Z}^-} \in BC^{\eta + \eta'}(D^2_W)$ with $0 \leq \eta' \leq 2\delta$. Following the steps of the proof of Theorem 4.9 we infer using (80) that $J_d(W, U, \xi)$ is a contraction and possesses therefore a fixed-point $\tilde{\Gamma}(\xi, W) \in BC^{\eta + \eta'}(D^2_W)$ for $0 \leq \eta' \leq 2\delta$. Furthermore, we set $\tilde{u}^{-1}_t(\xi) := \tilde{\Gamma}(\xi, W)[i - 1, t], \text{ for } i \in \mathbb{Z}^-, \ t \in [0, 1]$ and $\tilde{U}(\xi) := (\tilde{u}^{-1}_t(\xi), (\tilde{u}^{-1}_t(\xi))')_{i \in \mathbb{Z}^-}$. 

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Now we define the discrete version of the operator \( T \) specified in (78). Namely, for a sequence \( Z := (z^i, (z^i)^\gamma)_{i \in \mathbb{Z}} \in BC^{\eta+\delta}(D^{\alpha\gamma}_W) \) we introduce \( T_d(Z) := (T_d^1(Z), T_d^2(Z)) \), where

\[
T_d^1(Z)[i-1, t] := \sum_{k=-\infty}^{i-1} S^c(t + i - 1 - k) z_t^{k-1} \left( \int_0^t S^c(1-r) D F_R(\tilde{u}_r^{k-1}(\xi_0)) \, dr \right) + \int_0^t S^c(1-r) D G_R(\tilde{u}_r^{k-1}(\xi_0)) \, d\Theta_{k-1} W_r
\]

\[
+ \int_0^t S^c(1-r) D G_R(\tilde{u}_r^{k-1}(\xi_0)) \, d\Theta_{i-1} W_r
\]

\[
- \sum_{k=0}^{i+1} S^c(t + i - 1 - k) z_t^{k-1} \left( \int_0^t S^c(1-r) D F_R(\tilde{u}_r^{k-1}(\xi_0)) \, dr \right) + \int_0^t S^c(1-r) D G_R(\tilde{u}_r^{k-1}(\xi_0)) \, d\Theta_{k-1} W_r
\]

\[
- \int_0^t S^c(t-r) D F_R(\tilde{u}_r^{k-1}(\xi_0)) \, dr - \int_0^t S^c(t-r) D G_R(\tilde{u}_r^{k-1}(\xi_0)) \, d\Theta_{i-1} W_r
\]

and \( T_d^2(Z) = (T_d^1(Z))' \). By the same computation made in Theorem 4.9 we immediately conclude that \( T_d \) maps \( BC^{\eta+\delta}(D^{\alpha\gamma}_W) \) into itself and

\[
\|T_d\|_{BC^{\eta+\delta}(D^{\alpha\gamma}_W)} \leq K \left( \frac{e^{\beta+(\eta+\delta)}(C_S M_k e^{-(\eta+\delta)} + 1)}{1 - e^{-(\beta+(\eta+\delta))}} + \frac{e^{\gamma-(\eta+\delta)}(C_S M_k e^{-(\eta+\delta)} + 1)}{1 - e^{-(\gamma-(\eta+\delta))}} \right) < 1.
\]

Regarding (78) we have to take into account the discrete analogue \( I_d \) of \( I \) given in (77) and show that \( |I_d|_{BC^{\eta+\delta}(D^{\alpha\gamma}_W)} \approx o(|\xi - \xi_0|) \) as \( \xi \to \xi_0 \). As in the standard proofs of such assertions, e.g. Thm. 4.1 in [19] we have to split \( I_d \) into more small parts and analyze them separately. Let \( N \) and \( \bar{N} \) be two positive numbers which will be determined later and \( t \in [0, 1] \). For \( i+1 \leq -N \) we start with

\[
I_1 := e^{-(\eta+\delta)(i-1)} \left( \sum_{k=i+1}^{-N} \|S^c(\cdot + i - 1 - k) \tilde{\xi}(\tilde{u}_r^{k-1}(1) + \tilde{\xi}(\tilde{u}_r^{i-1}(1)) \| \right),
\]

where \( \| \cdot \| \) stands for \( \| \cdot \|_{D^{\alpha\gamma}_W} \). For notational simplicity we use the symbols

\[
\tilde{\xi}^{c/s}(\tilde{u}_r^{k-1}(\cdot)) := (\tilde{\xi}^{c/s}(\tilde{u}_r^{k-1}(\cdot)), (\tilde{\xi}^{c/s}(\tilde{u}_r^{k-1}(\cdot)))'),
\]

where

\[
\tilde{\xi}^{c/s}(\tilde{u}_r^{k-1}(t)) := \int_0^t S^c(s) (t-r) (F_R(\tilde{u}_r^{k-1}(\xi_0)) - F_R(\tilde{u}_r^{k-1}(\xi_0)) - DF_R(\tilde{u}_r^{k-1}(\xi_0))(\tilde{u}_r^{k-1}(\xi) - \tilde{u}_r^{k-1}(\xi_0))) \, dr
\]

\[
+ \int_0^t S^c(s) (t-r) (G_R(\tilde{u}_r^{k-1}(\xi_0)) - G_R(\tilde{u}_r^{k-1}(\xi_0)) - DG_R(\tilde{u}_r^{k-1}(\xi_0))(\tilde{u}_r^{k-1}(\xi) - \tilde{u}_r^{k-1}(\xi_0))) \, d\Theta_{k-1} W_r,
\]

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and \( \hat{I} := (\hat{r}^{c/s}(\hat{u}^{i-1}(\cdot)), (\hat{r}^{c/s}(\hat{u}^{i-1}(\cdot)))' \) for
\[
\hat{I}(\hat{u}^{i-1})(t) := \int_t^1 S^{c/s}(t - r)(F_R(\hat{u}^{i-1}(\xi)) - F_R(\hat{u}^{i-1}(\xi_0)) - DF_R(\hat{u}^{i-1}(\xi))(\hat{u}^{i-1}(\xi) - \hat{u}^{i-1}(\xi_0))) \, dr
\]
\[
+ \int_t^1 S^{c/s}(t - r)(G_R(\hat{u}^{i-1}(\xi)) - G_R(\hat{u}^{i-1}(\xi_0)) - DG_R(\hat{u}^{i-1}(\xi))(\hat{u}^{i-1}(\xi) - \hat{u}^{i-1}(\xi_0))) \, d\Theta_{t-}W_r.
\]
For \( i + 1 \geq -N \) we set \( I_1 := 0 \). We continue analyzing the center component and introduce
\[
I_2 := e^{-(\eta + \delta)(i-1)} \left( \sum_{k=-N}^{-1} \|S^{c}(\cdot + i - 1 - k)\hat{r}^{c}(\hat{u}^{k-1})(1) + \hat{r}^{c}(\hat{u}^{i-1}(\cdot))\| \right).
\]
Furthermore, for the stable part we need the following expressions. For \( |i - 1| \leq \overline{N} \) we define
\[
I_3 := e^{-(\eta + \delta)(i-1)} \sum_{k=-\infty}^{-\overline{N}} \|S^{s}(\cdot + i - 1 - k)\hat{r}^{s}(\hat{u}^{k-1})(1) + \hat{r}^{s}(\hat{u}^{i-1}(\cdot))\|,
\]
and respectively
\[
I_4 := e^{-(\eta + \delta)(i-1)} \sum_{k=-\infty}^{i-1} \|S^{s}(\cdot + i - 1 - k)\hat{r}^{s}(\hat{u}^{k-1})(1) + \hat{r}^{s}(\hat{u}^{i-1}(\cdot))\|.
\]
Finally, for \( |i - 1| \geq \overline{N} \)
\[
I_5 := e^{-(\eta + \delta)(i-1)} \sum_{k=-\infty}^{i-1} \|S^{s}(\cdot + i - 1 - k)\hat{r}^{s}(\hat{u}^{k-1})(1) + \hat{r}^{s}(\hat{u}^{i-1}(\cdot))\|.
\]
Note that
\[
|I_d|_{BC^{\eta + \delta}(D_{\overline{2}\nu})} \leq \sup_{i \in \mathbb{Z}^-} I_1 + \sup_{i \in \mathbb{Z}^-} I_2 + \sup_{|i-1| \leq \overline{N}} I_3 + \sup_{|i-1| \leq \overline{N}} I_4 + \sup_{|i-1| \geq \overline{N}} I_5.
\]
It is sufficient to show that for any \( \varepsilon > 0 \) there is a \( \overline{v} > 0 \) such that if \( |\xi - \xi_0| \leq \overline{v} \) then we have \( |I_d|_{BC^{\eta + \delta}(D_{\overline{2}\nu})} \leq \varepsilon |\xi - \xi_0| \). To this aim, we provide the corresponding estimates for \( I_1 \) and \( I_2 \), since the rest can be obtained by analogous computations; see [19] Sec. 4] and [30]. First of all we recall that according to Lemma 4.14 the mapping \( \xi \mapsto \hat{U}(\xi) \) is Lipschitz continuous from \( \mathcal{X}^c \) to \( BC^{\eta + 2\delta}(D_{\overline{2}\nu}) \) with
\[
||\hat{U}(\xi) - \hat{U}(\xi_0)||_{BC^{\eta + 2\delta}(D_{\overline{2}\nu})} \leq \frac{C_S M_e e^\gamma}{1 - gap(2\delta)} |\xi - \xi_0|.
\]
Consequently, we find
\[
I_1 \leq C \sum_{k=i}^{-N} e^{-(\eta + \delta)(i-1)} C_S M_e e^\gamma(i-1-k) e^{(\eta + 2\delta)(k-1)}
\]
\[
e^{-\eta + 2\delta)(k-1)} \|\hat{u}^{k-1}(\xi) - \hat{u}^{k-1}(\xi_0), (\hat{u}^{k-1}(\xi) - \hat{u}^{k-1}(\xi_0))'\|_{D_{\overline{2}\nu}}
\]
\[
\leq C_S M_c \sum_{k=i}^{N} e^{(\gamma-(\eta+\delta))(i-1-k)} e^{-(\eta+\delta)+\delta(k-1)} ||\tilde{U}(\xi) - \tilde{U}(\xi_0)||_{BC^{\eta+2\delta}(D^2_W)}
\]
\[
\leq (C_S M_c)^2 \frac{e^{\gamma-2(\eta+\delta)}}{(1 - gap(2\delta))(1 - e^{-(\gamma-(\eta+\delta))})} e^{-\delta(N+1)} |\xi - \xi_0|.
\]

We fix \(\varepsilon > 0\) and choose \(N\) sufficiently large such that the previous expression becomes small, more precisely
\[
|I_1| \leq \tilde{C}\varepsilon|\xi - \xi_0|,
\]
where \(\tilde{C} < 1\). With this determined \(N\) one can further estimate \(I_2\). We employ arguments used in Lemmas 2.8 and 2.9. Keeping the structure of \(I_2\) in mind we introduce
\[
H_k := \int_0^1 (DF_R(\tau\tilde{u}^{k-1}_r(\xi) + (1 - \tau)\tilde{u}^{k-1}_r(\xi_0)) - DF_R(\tilde{u}^{k-1}_r(\xi_0))) \, d\tau
\]
\[
+ \int_0^1 (DG_R(\tau\tilde{u}^{k-1}_r(\xi) + (1 - \tau)\tilde{u}^{k-1}_r(\xi_0)) - DG_R(\tilde{u}^{k-1}_r(\xi_0))) \, d\tau
\]
and set
\[
G_k := \tilde{u}^{k-1}_r(\xi) - \tilde{u}^{k-1}_r(\xi_0).
\]

According to [22, Cor. 7.4] the product of two controlled rough paths \(G\) and \(\tilde{G}\) is again a controlled rough path and
\[
||G\tilde{G}, (G\tilde{G})'||_{D^2_W} \leq C ||G, G'||_{D^0_W} ||\tilde{G}, \tilde{G}'||_{D^2_W},
\]
recall also Section 2. In our case we apply (82) for \((H_k, H'_k)\) and \((G_k, G'_k)\). Therefore we infer that
\[
I_2 \leq M_c \sum_{k=-N}^{0} e^{-(\eta+\delta)(i-1-k)} e^{(\gamma-(\eta+\delta))(k-1)} e^{-(\eta+\delta)(k-1)} ||H_k, H'_k||_{D^2_W} ||\tilde{u}^{k-1}_r(\xi) - \tilde{u}^{k-1}_r(\xi_0) - (\tilde{u}^{k-1}_r(\xi) - \tilde{u}^{k-1}_r(\xi_0))'||_{D^2_W}.
\]
\[
\leq M_c \sum_{k=-N}^{0} e^{(\gamma-(\eta+\delta))(i-1-k)} ||H_k, H'_k||_{D^2_W} ||\tilde{U}(\xi) - \tilde{U}(\xi_0)||_{BC^{\eta+\delta}(D^2_W)}
\]
\[
\leq \frac{e^{-(\delta+\eta)} M_e^2 |\xi - \xi_0|}{1 - gap(\delta)} \sum_{k=-N}^{0} e^{(\gamma-(\eta+\delta))(i-1-k)} ||H_k, H'_k||_{D^2_W}.
\]

Now, taking \(\xi\) and \(\xi_0\) close to each other we can control (83) and therefore \(I_2\). Namely, we can find \(\sigma > 0\) such that if \(|\xi - \xi_0| \leq \sigma\), then
\[
I_2 \leq \tilde{C}\varepsilon|\xi - \xi_0|,
\]
where \(\tilde{C} < 1\). Applying the same computation for the terms \(I_3, I_4, I_5\) and letting \(N\) and \(\overline{N}\) be large enough we finally obtain that if \(|\xi - \xi_0| \leq \overline{\sigma}\) for some \(\overline{\sigma} > 0\), then
\[
|I_d|_{BC^{\eta+\delta}(D^2_W)} \leq \epsilon|\xi - \xi_0|,
\]
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which proves the desired statement. To prove the continuity of the mapping $\xi \mapsto D\xi U_t(\xi)$ as indicated above, one passes again to the discrete setting in (79) and estimates

$$
\|D\xi \tilde{u}_t(\xi) - D\xi \tilde{u}_t(\xi_0)\|_{BC^0(D^2_W)} \leq \frac{\|T\|_{C^0(X^c, BC^0(D^2_W))}}{1 - \text{gap}(\eta)}.
$$

Now, using the same arguments, one can show that $\|T\|_{C^0(X^c, BC^0(D^2_W))} = o(1)$ as $\xi \to \xi_0$. \hfill \Box

A Properties of the discrete Lyapunov-Perron map

We start by pointing out following general technique, which is required in order to prove the invariance of a manifold $\tilde{M}^c$ for a continuous-time random dynamical system $\tilde{\varphi}$, i.e.

$$
\tilde{\varphi}(T, W, \tilde{M}^c(W)) \subset \tilde{M}^c(T, W).
$$

Again, $\tilde{M}^c(W)$ is given by the graph of a function $\tilde{h}(\cdot, W) : X^c \to X^s$, i.e., $\tilde{M}^c(W) := \{\xi^c + \tilde{h}(\xi^c, W) : \xi^c \in X^c\}$, where $\tilde{h}(\xi^c, W) := P^s \tilde{\Gamma}(\xi^c, W)[0]$. Here we denote with $\tilde{\Gamma}(\xi^c, B)[\cdot]$ the fixed-point of the corresponding Lyapunov-Perron map, where $\cdot$ stands only for the time-variable. For further details, see [12] Lem. 5.5 or [25] Lem. 4.7, where similar computations are performed.

Typically, one shows that for $\tilde{\xi}^c \in X^c$ small enough it holds that $P^s \tilde{\Gamma}(\xi^c, W)[0] \in \tilde{M}^c(W)$. To justify the invariance property one has to infer that $\tilde{\varphi}(T, W, \tilde{\Gamma}(\xi^c, W)[0]) \in \tilde{M}^c(T, W)$. To this aim, regarding the structure of the manifold, one needs to analyze $\tilde{h}$ on the $\Theta_t W$-fiber replacing $\tilde{\xi}^c$ with $P^c \tilde{\varphi}(T, W, \tilde{\Gamma}(\xi^c, W)[0])$, more precisely $\tilde{h}(P^c \tilde{\varphi}(T, W, \tilde{\Gamma}(\xi^c, W)[0]), \Theta_t W)$. Therefore one needs to derive an expression for the fixed-point of the Lyapunov-Perron transform on the $\Theta_t W$-fiber replacing $\xi^c$ by $P^c \tilde{\varphi}(T, W, \tilde{\xi}^c)$, i.e., $\tilde{\Gamma}(P^c \tilde{\varphi}(T, W, \tilde{\xi}^c), \Theta_t W)[\cdot]$, compare [72]. Hence, it is enough to prove that

$$
\tilde{\Xi}_T(\sigma, W) = \begin{cases} 
\tilde{\Gamma}(\tilde{\xi}^c, W)[\sigma + T] & : \sigma < T \\
\tilde{\varphi}(\sigma + T, W, \tilde{\Gamma}(\tilde{\xi}^c, W)[0]) & : \sigma \in [-T, 0]
\end{cases}
$$

is the fixed-point of $\tilde{\Gamma}(P^c \tilde{\varphi}(T, W, \tilde{\xi}^c), \Theta_t W)[\sigma]$, see [12] Lem. 5.5. This implies

$$
P^s \tilde{\Xi}_T(0, W) = \tilde{h}(P^c \tilde{\varphi}(T, W, \tilde{\Gamma}(\tilde{\xi}^c, W)[0]), \Theta_t W),
$$

which yields that $\tilde{\varphi}(T, W, \tilde{\Gamma}(\tilde{\xi}^c, W)[0]) \in \tilde{M}^c(\Theta_t W)$. For completeness we now derive the analogue of (83) for discrete-time random dynamical systems, compare (72). We indicate here the computation, which allows us to express the fixed-point of $J_d$ as introduced in Section 4.2 when shifting the fiber of the noise. More precisely, we investigate a connection between the fixed-points of $J_d(\cdot, \cdot)$ and $J_d(\Theta_i W, \cdot)$ for $i \in \mathbb{Z}_-$.\hfill \Box

Lemma A.1. Let $\xi_{-1} := P^c \Gamma(\xi^c, W)[-1, 0]$. The fixed-point $\Gamma(\xi^c, W)$ of $J_d(\cdot, \cdot, \xi^c)$ can be expressed by

$$
\Xi(\xi^c, W)[i, \cdot] := \begin{cases} 
\Gamma(\xi_{-1}, \Theta_{-1} W)[i + 1, \cdot], & \text{if } i = -2, -3, \ldots \\
\varphi_R(\cdot, \Theta_{-1} W, \Gamma(\xi_{-1}, \Theta_{-1} W)[-1, 1]), & \text{if } i = -1
\end{cases}
$$

Proof. To prove the statement we are going to show that $\Xi(\xi^c, W)$ is a fixed-point of $J_d(\cdot, \cdot, \xi^c)$. Due to the uniqueness of the fixed-point we may then infer that

$$
\Xi(\xi^c, W) = \Gamma(\xi^c, W).
$$
Since \( \Gamma(\xi_{-1}, \Theta_{-1}W) \) is the fixed-point of \( J_d(\Theta_{-1}W, \cdot, \xi_{-1}^c) \) we explore (61). For notational simplicity we suppress the Gubinelli derivative in the following computation, i.e. we only use the expression for \( J_d^1(\cdot, \cdot, \cdot) \). On the stable part this entails

\[
\Gamma^s(\xi_{-1}, \Theta_{-1}W)[-1, 1] = \sum_{k=-\infty}^{-1} S^s(-k) \left( \int_0^1 S^s(1-r)F_R(\Gamma(\xi_{-1}^c, \Theta_{-1}W)[k-1, r]) \, dr \right.
\]

\[
+ \int_0^1 S^s(1-r)G_R(\Gamma(\xi_{-1}^c, \Theta_{-1}W)[k-1, r]) \, d\Theta_{k-1} \Theta_{-1}W_r \biggr)
\]

\[
+ \int_0^1 S^s(1-r)F_R(\Gamma(\xi_{-1}^c, \Theta_{-1}W)[-1, r])) \, dr
\]

\[
+ \int_0^1 S^s(1-r)G_R(\Gamma(\xi_{-1}^c, \Theta_{-1}W)[-1, r]) \, d\Theta_{-1} \Theta_{-1}W
\]

\[
= \sum_{k=-\infty}^{-1} S^s(-k) \left( \int_0^1 S^s(1-r)F_R(\Gamma(\xi_{-1}^c, \Theta_{-1}W)[k-1, r]) \, dr \right.
\]

\[
+ \int_0^1 S^s(1-r)G_R(\Gamma(\xi_{-1}^c, \Theta_{-1}W)[k-1, r]) \, d\Theta_{k-1} \Theta_{-1}W_r \biggr).
\]

Using (60) for \( i = -1 \) we have on the stable component

\[
\Xi^s(\xi_{-1}, W)[-1, t] = \varphi^s_R(t, \Theta_{-1}W, \Gamma(\xi_{-1}^c, \Theta_{-1}W)[-1, 1])
\]

\[
= \sum_{k=-\infty}^{0} S^s(t-k) \left( \int_0^1 F_R(\Gamma(\xi_{-1}^c, \Theta_{-1}W)[k-1, r]) \, dr \right.
\]

\[
+ \int_0^t G_R(\Gamma(\xi_{-1}^c, \Theta_{-1}W)[k-1, r]) \, d\Theta_{k-1} \Theta_{-1}W_r \biggr)
\]

\[
+ \int_0^t S^s(t-r)F_R(\Xi(\xi_{-1}^c, \Theta_{-1}W)[-1, r)) \, dr + \int_0^t S^s(t-r)G_R(\Xi(\xi_{-1}^c, \Theta_{-1}W)[-1, r]) \, d\Theta_{-1} \Theta_{-1}W_r
\]

\[
= \sum_{k=-\infty}^{-1} S^s(t-k-1) \left( \int_0^1 F_R(\Gamma(\xi_{-1}^c, \Theta_{-1}W)[k, r]) \, dr + \int_0^t G_R(\Gamma(\xi_{-1}^c, \Theta_{-1}W)[k, r]) \, d\Theta_{k-1} \Theta_{-1}W_r \right)
\]

\[
+ \int_0^t S^s(t-r)F_R(\Xi(\xi_{-1}^c, \Theta_{-1}W)[-1, r)) \, dr + \int_0^t S^s(t-r)G_R(\Xi(\xi_{-1}^c, \Theta_{-1}W)[-1, r]) \, d\Theta_{-1} \Theta_{-1}W_r
\]

\[
= \sum_{k=-\infty}^{-1} S^s(t-k-1) \left( \int_0^1 S^s(1-r)F_R(\Xi(\xi_{-1}^c, \Theta_{-1}W)[k-1, r]) \, dr \right.
\]

\[
+ \int_0^1 S^s(1-r)G_R(\Xi(\xi_{-1}^c, \Theta_{-1}W)[k-1, r]) \, d\Theta_{k-1} \Theta_{-1}W_r \biggr).
\]
\[ + \int_0^t S^s(t-r)F_R(\Xi(\xi_{-1}^c, \Theta_{-1}W)[-1, r]) \, dr + \int_0^t S^s(t-r)G_R(\Xi(\xi_{-1}^c, \Theta_{-1}W)[-1, r]) \, d\Theta_{-1}W_r, \]

where in the last step we use again (86). Furthermore, regarding that \( \Gamma(\xi_{-1}^c, \Theta_{-1}W) \) is the fixed-point of \( J_d(\Theta_{-1}W, \cdot, \xi_{-1}^c) \) we now compute analogously to the first step from (61) for \( i = -2, -3, \ldots \)

\[ \Gamma^s(\xi_{-1}^c, \Theta_{-1}W)[i+1, t] = \sum_{k=-\infty}^{i+1} S^s(t+i+1-k) \left( \int_0^1 S^s(1-r)F_R(\Gamma(\xi_{1}^c, \Theta_{-1}W)[k-1, r]) \, dr \right. \]

\[ \left. + \int_0^1 S^s(1-r)G_R(\Gamma(\xi_{1}^c, \Theta_{-1}W)[k-1, r]) \, d\Theta_{k-1}W_r \right) \]

\[ + \int_0^t S(t-r)F_R(\Gamma(\xi_{1}^c, \Theta_{-1}W)[i+1, r]) \, dr \]

\[ + \int_0^t S(t-r)F_R(\Gamma(\xi_{1}^c, \Theta_{-1}W)[i+1, r]) \, d\Theta_{i+1} \Theta_{-1}W_r \]

\[ = \sum_{k=-\infty}^{i} S^s(t+i-k) \left( \int_0^1 S^s(1-r)F_R(\Gamma(\xi_{1}^c, \Theta_{-1}W)[k, r]) \, dr \right. \]

\[ \left. + \int_0^1 S^s(1-r)G_R(\Gamma(\xi_{1}^c, \Theta_{-1}W)[k, r]) \, d\Theta_{k-1}W_r \right) \]

\[ + \int_0^t S^s(t-r)F_R(\Gamma(\xi_{1}^c, \Theta_{-1}W)[i+1, r]) \, dr \]

\[ + \int_0^t S^s(t-r)G_R(\Gamma(\xi_{1}^c, \Theta_{-1}W)[i+1, r]) \, d\Theta_{i} \Theta_{-1}W_r. \]

On the other hand (86) gives us on the stable part

\[ \Xi^c(\xi^c, W)[i, t] = \sum_{k=-\infty}^{i} S^s(t+i-k) \left( \int_0^1 S^s(1-r)F_R(\Xi(\xi^c, W)[k-1, r]) \, dr \right. \]

\[ \left. + \int_0^1 S^s(1-r)G_R(\Xi(\xi^c, W)[k-1, r]) \, d\Theta_{k-1}W_r \right) \]

\[ + \int_0^t S^s(t-r)F_R(\Xi(\xi^c, W)[i, r]) \, dr \]

\[ + \int_0^t S^s(t-r)G_R(\Xi(\xi^c, W)[i, r]) \, d\Theta_{i} \Theta_{-1}W_r. \]
In order to prove the assertion one carries out a similar computation for the center component. One infers that

\[ \Gamma(\xi^c, W)[i, t] := S^c(t + i)\xi^c \]

\[ -\sum_{k=0}^{i+2} S^c(t + i - k) \left( \int_0^1 S^c(1 - r) F_R(\Gamma(\xi^c, W)[k - 1, r]) \, dr \right. \]

\[ + \int_0^1 S^c(1 - r) G_R(\Gamma(\xi^c, W)[k - 1, r]) \, d\Theta_{k-1}W_r \]

\[ \left. - \int_t^1 S^c(t - r) F_R(\Gamma(\xi^c, W)[i, r]) \, dr - \int_t^1 S^c(t - r) G_R(\Gamma(\xi^c, W)[i, r]) \, d\Theta_lW_r \right. \]

\[ + \sum_{k=-\infty}^i S^s(t + i - k) \left( \int_0^1 S^s(1 - r) F_R(\Gamma(\xi^c, W)[k - 1, r]) \, dr \right. \]

\[ + \int_0^1 S^s(1 - r) G_R(\Gamma(\xi^c, W)[k - 1, r]) \, d\Theta_{k-1}W_r \]

\[ + \int_0^t S^s(t - r) F_R(\Gamma(\xi^c, W)[i, r]) \, dr + \int_0^t S^s(t - r) G_R(\Gamma(\xi^c, W)[i, r]) \, d\Theta_lW_r. \]

This shows that \( \Xi(\xi^c, W) \) is the fixed-point of \( J_d(W, \cdot, \xi^c) \). Due to uniqueness we have that \( \Xi(\xi^c, W) = \Gamma(\xi^c, W) \). This proves the statement.

Hence, our discrete Lyapunov-Perron map also naturally leads to invariance of the center manifold.

\section*{B \ Exponential trichotomy}

We shortly indicate how the results obtained in Section \( \S \) can be applied if there additionally exists an unstable subspace. In that case one assumes that there exist constants \( \rho_1 > \rho_2 \geq 0 \geq -\rho_2 > -\rho_3 \) such that

\[ |S(t)P^c x| \leq M_c e^{\rho_2 |t|} |P^c x|, \quad \text{for } t \in \mathbb{R} \text{ and } x \in X; \]

\[ |S(t)P^u x| \leq M_u e^{\rho_1 |t|} |P^u x|, \quad \text{for } t \leq 0 \text{ and } x \in X; \]

\[ |S(t)P^s x| \leq M_s e^{-\rho_3 |t|} |P^s x|, \quad \text{for } t \geq 0 \text{ and } x \in X, \]
holds true, compare [50] Sec. 7.1.2. Then, the continuous-time Lyapunov-Perron map given by

\[ J(W, U)[\tau] := S^c(\tau)\xi^c + \int_0^\tau S^c(\tau - r)F(U_r) \, dr + \int_0^\tau S^c(\tau - r)G(U_r) \, dW_r \]

\[ + \int_{-\infty}^\tau S^s(\tau - r)F(U_r) \, dr + \int_{-\infty}^\tau S^s(\tau - r)G(U_r) \, dW_r \]

\[ - \int_\tau^\tau S^u(\tau - r)F(U_r) \, dr - \int_\tau^\tau S^u(\tau - r)G(U_r) \, dW_r, \]

has to be discretized for \( \tau \in \mathbb{Z} \). In this case, we introduce for \( \eta > 0 \) satisfying \( \rho_2 < \eta < \min\{\rho_1, \rho_3\} \) the space

\[ ||U||_{BC^\eta(D^2_W)} := \sup_{i \in \mathbb{Z}} e^{-\eta|i|}||U^{i-1}, (U^{i-1})'||_{D^2_W}. \]

Proceeding exactly as in Subsection 4.1 leads to the following definition of a discrete Lyapunov-Perron transform \( J_d(W, U, \xi) := (J_d^1(W, Y, \xi), J_d^2(W, U, \xi)) \) for a sequence \( U \in BC^\eta(D^2_W), t \in [0, 1], W \in \Omega_W \) and \( i \in \mathbb{Z} \):

\[ J_d^1(W, U, \xi)[i - 1, t] := S^c(t + i - 1)\xi^c \]

\[ - \sum_{k=0}^{i+1} S^c(t + i - 1 - k) \left( \int_0^1 S^c(1 - r)F(U_r^{i-1}) \, dr + \int_0^1 S^c(1 - r)G(U_r^{i-1}) \, d\Theta_{k-1}W_r \right) \]

\[ - \int_0^1 S^c(t - r)F(U_r^{i-1}) \, dr - \int_0^1 S^c(t - r)G(U_r^{i-1}) \, d\Theta_{i-1}W_r \]

\[ + \sum_{k=-\infty}^{i-1} S^s(t + i - 1 - k) \left( \int_0^1 S^s(1 - r)F(U_r^{i-1}) \, dr + \int_0^1 S^s(1 - r)G(U_r^{i-1}) \, d\Theta_{k-1}W_r \right) \]

\[ + \int_0^t S^s(t - r)F(U_r^{i-1}) \, dr + \int_0^t S^s(t - r)G(U_r^{i-1}) \, d\Theta_{i-1}W_r \]

\[ + \int_0^t S^u(t - r)F(U_r^{i-1}) \, dr + \int_0^t S^u(t - r)G(U_r^{i-1}) \, d\Theta_{i-1}W_r \]

\[ + \sum_{k=i-1}^{\infty} S^u(t + i - 1 - k) \left( \int_0^1 S^u(1 - r)F(U_r^{i-1}) \, dr + \int_0^1 S^u(1 - r)G(U_r^{i-1}) \, d\Theta_{k-1}W_r \right). \]

Of course, \( J_d^2(W, U, \xi) := (J_d^1(W, Y, \xi))' \). By the same arguments employed in Theorem 4.9 one can infer that the gap condition is now given by

\[ K \left( \frac{e^{\eta-\rho_2}(C_S M_e e^{-\eta} + 1)}{1 - e^{-(\eta-\rho_2)}} + \frac{e^{\rho_1-\eta}(C_S M_e e^{-\eta} + 1)}{1 - e^{-(\rho_1-\eta)}} + \frac{e^{\rho_2-\eta}(C_S M_e e^{-\eta} + 1)}{1 - e^{-(\rho_2-\eta)}} \right) < \frac{1}{4}. \]

Hence, using our techniques one can also prove the existence of center-unstable, respectively center-stable manifolds for rough differential equations in exactly the same steps as demonstrated above.
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