Learning Lyapunov Functions for Piecewise Affine Systems with Neural Network Controllers

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Abstract

We propose an iterative method for Lyapunov-based stability analysis of piecewise affine dynamical systems in feedback with piecewise affine neural network controllers. In each iteration, a learner uses a collection of samples of the closed-loop system to propose a Lyapunov function candidate by solving a convex program. The learner then queries a verifier, which then solves a mixed-integer program to either validate the proposed Lyapunov function candidate or reject it with a counterexample, i.e., a state where the stability condition fails. We design the learner and the verifier based on the analytic center cutting-plane method (ACCPM), in which the verifier acts as the cutting-plane oracle to refine the set of Lyapunov function candidates. We show that the overall algorithm terminates in a finite number of iterations. We demonstrate the utility of the proposed method in searching for quadratic and piecewise quadratic Lyapunov functions.

1 Introduction

Deep neural networks (DNNs) have achieved remarkable success in various challenging tasks across different fields such as image classification [1], machine translation [2], speech recognition [3], reinforcement learning, and control [4]. The high performance of DNNs, however, does not come with guarantees as they are highly complex and therefore, hard to reason about. Moreover, it is relatively well-known that neural networks can be sensitive to input perturbations and adversarial attacks. These drawbacks limit the application of DNNs in safety-critical systems, in which standard control strategies, although potentially inferior to DNNs in performance, do have performance guarantees. Therefore, it is critical to develop tools that can provide useful certificates of stability, safety, and robustness for DNN-driven systems.

Stability analysis aims to show that a set of initial states of a dynamical system would stay around and potentially converge to an equilibrium point. Stability analysis of dynamical systems often relies on constructing Lyapunov functions that can find positive invariant sets, certify the stability of equilibrium points, and estimate their region of attraction. Finding Lyapunov functions for generic nonlinear systems, however, is not an easy task and often requires substantial expertise and manual effort. This task becomes even more challenging when neural networks (NNs) are used as stabilizing controllers for nonlinear systems. In this context, it is of great interest to automate the construction of Lyapunov functions for nonlinear systems [5] and in particular, neural-network-controlled feedback systems.

In this paper, our particular focus is on stability analysis of neural-network-controlled feedback systems, in which the system being controlled is piecewise affine (PWA) and the controller is a neural network. Piecewise affine functions are a powerful class for identifying or approximating...
nonlinear systems via multiple linearizations at different operating points [6]. Control design for these systems, however, is computationally challenging. For example, controlling PWA systems with a Model Predictive Control (MPC) strategy would require solving a mixed-integer program, which is NP-complete [7]. Therefore, for PWA systems it is desirable to design alternate control strategies such as neural networks to reduce the computational cost. However, since neural networks are a composition of nonlinear functions, providing useful certificates of stability or safety remains a significant challenge.

Contributions: We propose an algorithmic approach to synthesize Lyapunov functions for discrete-time piecewise affine systems in feedback with ReLU neural network controllers. The proposed algorithm relies on a “learner” to propose a Lyapunov function candidate, and a “verifier” to validate the Lyapunov function candidate or reject it with a counterexample, i.e., a state where the stability condition fails. We design the learner and the verifier according to the analytic center cutting-plane method (ACCPM) from convex optimization, with which the algorithm can efficiently and exhaustively search over the considered Lyapunov function candidate class. Notably, the algorithm is guaranteed to terminate in finite steps and upon termination, we either find a Lyapunov function that certifies the closed-loop stability or verify that no Lyapunov function exists in the entire class of Lyapunov functions under consideration. We demonstrate the application of our algorithm to search for quadratic and piecewise quadratic Lyapunov functions.

1.1 Related work

Output range analysis of NNs: Driven by the need to verify the robustness of deep neural networks against adversarial attacks [8], a large body of work has been reported on bounding the output of DNNs for a given range of inputs [9, 10, 11, 12, 13, 14, 15]. These methods can be categorized into exact and approximate depending on how they evaluate/approximate the nonlinear activation functions in the DNN. Of particular interest to this paper is the tight certification of ReLU NNs through formal verification techniques such as Satisfiability Modulo Theories (SMT) solvers [13, 15, 16] or mixed-integer programming (MIP) [10, 17]. These approaches can verify the linear properties of the ReLU NN output given linear constraints on the network inputs. More scalable but less accurate methods that apply linear programming (LP) relaxation [11] or semidefinite programming (SDP) relaxation [12, 14, 18, 19] of NNs to over approximate the output range exist.

Stability and reachability analysis of NN-controlled systems: Despite their high performance, NN-controlled systems lack safety and stability guarantees. Motivated by this, several works have studied the challenging task of verifying NN-controlled systems. In [20], built upon the framework of [12], the authors over approximate the reachable set of NN-controlled LTI systems by bounding the nonlinear activation functions of the NN through quadratic constraints (QCs). [21, 22] approximate NN functions through polynomial approximation to conduct closed-loop reachability analysis. Ivanov et al. [23] transform a sigmoid-based NN into a hybrid system and propose verification of the NN-controlled system through standard hybrid system verification tools.

For stability analysis, linear matrix inequality (LMI)-based sufficient stability conditions are derived by abstracting the nonlinear activation function in NNs through QCs [21] or by linear difference inclusions [25, 26]. Yin et al. [27] use a QC abstraction of NNs to analyze uncertain plants with perturbations described by integral quadratic constraints (IQC). The input-output stability of an NN-controlled system is considered in [25] where QCs are constructed from the bounds of partial gradients of NN controllers. For ReLU NN controllers, the authors in [20] exploit the fact

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1 Rectified Linear Unit
that a ReLU NN is a PWA function and analyze the closed-loop stability by identifying the linear dynamics around the equilibrium. However, the number of modes generated by the ReLU NN is exponential in the number of neurons, and can easily make general PWA system stability analysis tools intractable [30, 31, 32, 33].

Sample-based Lyapunov function synthesis: Synthesizing Lyapunov functions for nonlinear dynamical systems is very difficult in general and often involves solving large, nonconvex optimization problems. To avoid solving hard optimization problems directly, several sample-based Lyapunov function synthesis methods have been proposed. Topeu et al. [34] use simulation data to help postulate the region of attraction of continuous-time polynomial systems and find Lyapunov function candidates for solving bilinear matrix inequalities (BMIs). Closely related to our work is [35] where an iterative approach formulates Lyapunov function candidates for continuous-time nonlinear systems from simulation traces and improves the result by the counterexample trace generated by a falsifier at each iteration. Our method differs from theirs in that we provide a finite-step termination guarantee for our iterative algorithm and show that our algorithm terminates with either a valid Lyapunov function or a certification that Lyapunov functions of a specific form do not exist.

Counterexample guided inductive synthesis: The iterative approach alternating between a learning module and a verification module to synthesize a certificate for control systems is known as the Counter-Example Guided Inductive Synthesis (CEGIS) framework proposed by [36, 37] in the verification community. The application of CEGIS on Lyapunov function synthesis for continuous-time nonlinear autonomous systems can be found in [38, 5] using SMT solvers for verification. However, the termination of the iterative procedures in these works is not guaranteed. Notably, Ravanbakhsh et al. [39] apply the CEGIS framework to synthesize control Lyapunov functions for nonlinear continuous-time systems and provide finite-step termination guarantees for the iterative algorithm through careful design of the learner, which essentially implements the maximum volume ellipsoid cutting-plane method [40]. Although our method also provides termination guarantees through the (analytic center) cutting-plane method, the considered problem setting (stability verification of NN-controlled PWA system), and the construction of the learner and the verifier are highly different.

The rest of the paper is organized as follows. After introducing the preliminaries on Lyapunov functions in Section 2, we state our problem formulation as finding an estimate of region of attraction of a NN-controlled PWA system in Section 3. We develop our method in Section 4 and discuss its extensions and variations in Section 5. Section 6 demonstrates the application of the proposed method through numerical examples and Section 7 concludes the paper.

1.2 Notations

We denote the set of real numbers by \( \mathbb{R} \), the set of integers by \( \mathbb{Z} \), the \( n \)-dimensional real vector space by \( \mathbb{R}^n \), and the set of \( n \times m \)-dimensional real matrices by \( \mathbb{R}^{n \times m} \). The standard inner product between two matrices \( A, B \in \mathbb{R}^{n \times m} \) is given by \( \langle A, B \rangle = \text{tr}(A^\top B) \) and the Frobenius norm of a matrix \( A \in \mathbb{R}^{n \times m} \) is given by \( \| A \|_F = (\text{tr}(A^\top A))^{1/2} \). Denote \( S^n \) the set of \( n \times n \)-dimensional symmetric matrices, and \( S^n_+ \) (\( S^n_{++} \)) the set of \( n \times n \)-dimensional positive semidefinite (definite) matrices. For two vectors \( x \in \mathbb{R}^{n_x} \) and \( y \in \mathbb{R}^{n_y} \), \( (x, y) \in \mathbb{R}^{n_x+n_y} \) represents their concatenation. Given a set \( S \subseteq \mathbb{R}^{n_x+n_y} \), \( \text{Proj}_S(x) = \{ x \in \mathbb{R}^{n_x} \mid \exists y \in \mathbb{R}^{n_y} \text{ s.t. } (x, y) \in S \} \) denotes the orthogonal projection of \( S \) onto the subspace \( \mathbb{R}^{n_x} \). Given a point \( x \in \mathbb{R}^{n_x} \) and a compact set \( \mathcal{R} \subseteq \mathbb{R}^{n_x} \), define the projection of \( x \) to the set \( \mathcal{R} \) as \( \text{Proj}_\mathcal{R}(x) = \arg\min_{y \in \mathcal{R}} \text{dist}(x, y) \) where \( \text{dist}(x, y) = \| x - y \|_2 \) is the Euclidean norm of \( x - y \). For a set \( S \subseteq \mathbb{R}^n \), we denote \( \text{cl}(S) \) the closure of \( S \), \( \text{int}(S) \) the set of
all interior points in $S$, and $S_\infty = \{ d \in \mathbb{R}^n | x + \alpha d \in S, \forall x \in S, \forall \alpha \in \mathbb{R}_+ \}$ the recession cone of $S$.

2 Preliminaries

2.1 Stability of nonlinear autonomous systems

Consider a discrete-time autonomous system

$$x_{k+1} = f(x_k),$$

where $x_k \in \mathbb{R}^{n_x}$ is the state at time $k \geq 0$ and $f : \mathcal{R} \to \mathbb{R}^{n_x}$ is a nonlinear, continuous function with domain $\mathcal{R} \subseteq \mathbb{R}^{n_x}$. Without loss of generality, we assume $0 \in \text{int}(\mathcal{R})$ is an equilibrium of the system, i.e., $f(0) = 0$. To study the stability properties of autonomous systems at their equilibrium points, Lyapunov functions may be used.

**Definition 1.** (Lyapunov stability [41, Chapter 13]) The equilibrium point $x = 0$ of the autonomous system (1) is

- **Lyapunov stable** if for each $\epsilon > 0$, there exists $\delta = \delta(\epsilon)$ such that if $\|x_0\| < \delta$, then $\|x_k\| < \epsilon, \forall k \geq 0$.

- **asymptotically stable** if it is Lyapunov stable and there exists $\delta > 0$ such that if $\|x_0\| < \delta$, then $\lim_{k \to \infty} \|x_k\| = 0$.

- **geometrically stable** if there exists $\delta > 0$, $\alpha > 1$, $0 < \rho < 1$ such that if $\|x_0\| < \delta$, then $\|x_k\| \leq \alpha \rho^k \|x_0\| \quad \forall k \geq 0$.

Lyapunov stability describes the local behavior of autonomous systems around their equilibrium. For an asymptotically stable equilibrium point, the region of attraction (ROA) is the set of initial states from which the trajectories of the autonomous system converge to the equilibrium.

**Definition 2** (Region of attraction). The region of attraction of the autonomous system (1) is defined as

$$\mathcal{O} = \{ x_0 \in \mathcal{R} | \lim_{k \to \infty} \|x_k\| = 0 \}.$$

**Definition 3** (Successor set). For the closed-loop system (8), we denote the successor set from the set $X$ as $\text{Suc}(X) = \{ y \in \mathbb{R}^{n_x} | \exists x \in X \text{ s.t. } y = f(x) \}$.

**Definition 4** (Positive invariant set). A set $\mathcal{X} \subseteq \mathcal{R}$ is said to be positive invariant for the autonomous system (1) if for all $x_0 \in \mathcal{X}$, we have $x_k \in \mathcal{X}, \forall k \geq 0$.

**Theorem 1.** ([41, Chapter 13]) Consider the discrete-time nonlinear system (1) and assume there is a continuous function $V(x) : \mathcal{R} \to \mathbb{R}$ with domain $\mathcal{R}$ such that

$$V(0) = 0 \text{ and } V(x) > 0, \forall x \in \mathcal{X} \setminus \{0\}$$  \hspace{1cm} (2a)

$$V(f(x)) - V(x) \leq 0, \forall x \in \mathcal{X},$$ \hspace{1cm} (2b)

where the set $\mathcal{X}$ is the region of interest (ROI) satisfying $\mathcal{X} \subseteq \mathcal{R}$, $\text{Suc}(\mathcal{X}) \subseteq \mathcal{R}$, and $0 \in \text{int}(\mathcal{X})$. Then the origin is Lyapunov stable. If, in addition,

$$V(f(x)) - V(x) < 0, \forall x \in \mathcal{X} \setminus \{0\},$$ \hspace{1cm} (3)

then the origin is asymptotically stable.
We denote the closed-loop dynamics of the PWA system (5) in feedback with the ReLU network as
\[ x_+ = \psi(x,u) = A_i x + B_i u + c_i, \quad \forall x \in R_i = \{ x \in \mathbb{R}^n x | F_i x \leq h_i \}, \] (5)
where \( x \in \mathbb{R}^n x \) is the state, \( u \in \mathbb{R}^n u \) is the control input, \( x_+ \in \mathbb{R}^n x \) denotes the state at the next time instant, \( i \in I = \{ 1, 2, \ldots, N_m \} \) is the mode of the system, and \( \{ R_i \}_{i \in I} \) are polytopic partitions (i.e., \( R_i \) is a bounded polyhedron) of the state space. We assume that the partitions \( R_i \) satisfy \( \text{int}(R_i) \cap \text{int}(R_j) = \emptyset \) and the PWA dynamics (5) is well-posed, i.e., \( \psi(x,u) = \psi_j(x,u), \forall x \in R_i \cap R_j, \forall (i,j) \in I^2 \). We denote the PWA system (5) compactly as \( x_+ = \psi(x,u) \) and define its domain as \( R = \bigcup_{i=1}^{N_m} R_i \). Without loss of generality, we assume the origin \((x,u) = 0\) is an equilibrium point of the PWA system, i.e., \( \psi(0,0) = 0 \).

As a special case, the PWA system (5) reduces to a linear time-invariant (LTI) system when there is only one mode. In this case, we denote the LTI system as
\[ x_+ = Ax + Bu, \quad x \in R = \{ x \in \mathbb{R}^n x | F x \leq h \}. \] (6)

3 Problem Statement

3.1 Plant model

In this paper, we consider discrete-time piecewise affine (PWA) dynamical systems of the form
\[ x_+ = \psi_i(x,u) = A_i x + B_i u + c_i, \quad \forall x \in R_i = \{ x \in \mathbb{R}^n x | F_i x \leq h_i \}, \] (5)
where \( x \in \mathbb{R}^n x \) is the state, \( u \in \mathbb{R}^n u \) is the control input, \( x_+ \in \mathbb{R}^n x \) denotes the state at the next time instant, \( i \in I = \{ 1, 2, \ldots, N_m \} \) is the mode of the system, and \( \{ R_i \}_{i \in I} \) are polytopic partitions (i.e., \( R_i \) is a bounded polyhedron) of the state space. We assume that the partitions \( R_i \) satisfy \( \text{int}(R_i) \cap \text{int}(R_j) = \emptyset \) and the PWA dynamics (5) is well-posed, i.e., \( \psi_i(x,u) = \psi_j(x,u), \forall x \in R_i \cap R_j, \forall (i,j) \in I^2 \). We denote the PWA system (5) compactly as \( x_+ = \psi(x,u) \) and define its domain as \( R = \bigcup_{i=1}^{N_m} R_i \). Without loss of generality, we assume the origin \((x,u) = 0\) is an equilibrium point of the PWA system, i.e., \( \psi(0,0) = 0 \).

As a special case, the PWA system (5) reduces to a linear time-invariant (LTI) system when there is only one mode. In this case, we denote the LTI system as
\[ x_+ = Ax + Bu, \quad x \in R = \{ x \in \mathbb{R}^n x | F x \leq h \}. \] (6)

3.2 Neural network controller model

In this paper we assume the PWA dynamical system (5) is driven by a multi-layer piecewise linear neural network controller \( u = \pi(x) \), where \( \pi : \mathbb{R}^n x \to \mathbb{R}^n u \) is described by
\[ z_0 = x \]
\[ z_{\ell+1} = \max(W_{\ell} z_{\ell} + b_{\ell}, 0), \quad \ell = 0, \ldots, L - 1 \]
\[ \pi(x) = W_L z_L + b_L. \] (7)

Here \( z_0 = x \in \mathbb{R}^n_0 \) (\( n_0 = n_x \)) is the input to the neural network, \( z_{\ell+1} \in \mathbb{R}^{n_{\ell+1}} \) is the vector representing the output of the \((\ell + 1)\)-th hidden layer with \( n_{\ell+1} \) neurons, \( \pi(x) \in \mathbb{R}^{n_{L+1}} \) (\( n_{L+1} = n_u \)) is the output of the neural network, and \( W_{\ell} \in \mathbb{R}^{n_{\ell+1} \times n_{\ell}}, b_{\ell} \in \mathbb{R}^{n_{\ell+1}} \) are the weight matrix and the bias vector of the \((\ell + 1)\)-th hidden layer.

3.3 Closed-loop stability analysis

We denote the closed-loop dynamics of the PWA system (5) in feedback with the ReLU network controller (7) as
\[ x_+ = f_{cl}(x) := \psi(x, \pi(x)). \] (8)
Under the assumption $\pi(0) = 0$, the origin is an equilibrium of the closed-loop system, i.e., $f_{cl}(0) = 0$. We define the region of interest as a polytopic set given by

$$\mathcal{X} = \{x \in \mathbb{R}^n_x | F_{\mathcal{X}} x \leq h_{\mathcal{X}}\}. \quad (9)$$

As the first problem, we are interested in verifying the stability of the equilibrium point and finding an inner estimate of its region of attraction contained in $\mathcal{X}$.

**Problem 1.** Find an inner estimate of the ROA $\tilde{\mathcal{O}} \subseteq \mathcal{X}$ for the closed-loop system (8).

As the second problem, suppose the ROI set $\mathcal{X}$ is a positive invariant set for the closed-loop system. We are interested in verifying whether $\mathcal{X}$ is an estimate of ROA.

**Problem 2.** Assume the region of interest $\mathcal{X} \subseteq \mathcal{R}$ defined by (9) is positive invariant for the closed-loop system (8). Verify if $\lim_{k \to \infty} x_k = 0$ for all $x_0 \in \mathcal{X}$.

The ROI $\mathcal{X}$ can be interpreted as a user-defined set that guides the search for an estimate of ROA. With the assumption on the positive invariance of $\mathcal{X}$, Problem 2 allows us to find a larger estimate of ROA (i.e., $\mathcal{X}$ itself) as shown in Fig. 1.

![Illustration of Problem 1](image1)

![Illustration of Problem 2](image2)

Figure 1: In Problem 1, the aim is to find an estimate of ROA $\tilde{\mathcal{O}}$ inside the ROI $\mathcal{X}$. In Problem 2, under the assumption that $\mathcal{X}$ is a positive invariant set, we want to verify if $\mathcal{X} \subset \mathcal{O}$. Note that the ROA is an unknown, possibly nonconvex set and $\mathcal{X}$ is chosen agnostic to $\mathcal{O}$.

### 4 Learning Lyapunov functions for the closed-loop system

To solve Problems 1 and 2, we propose an iterative learning-based method to construct a valid Lyapunov function for the closed-loop system (8). In each iteration of this algorithm, a “learner” proposes a Lyapunov function candidate using a collection of samples of the closed-loop system. The learner then queries a “verifier” that either certifies that the proposed function is a valid Lyapunov function or rejects it with a counterexample, i.e., a state where the stability condition fails. This counterexample is then added to the training set of the learner. In the next iteration, the learner uses the new counterexample and a cutting-plane strategy to refine the set of Lyapunov function candidates. The learner repeats this process until termination when either it finds a valid Lyapunov function, or it verifies its non-existence in the given parameterized function space. In our framework, the learner solves a convex optimization problem to synthesize Lyapunov function candidates, and
the verifier solves a mixed-integer program (MIP) to generate the counterexamples. We provide a finite-step termination guarantee for our algorithm through the analysis of cutting-plane methods from convex optimization. The proposed method is illustrated in Fig. 2.

![Fig. 2: Learning-based method for Lyapunov function synthesis.](image)

### 4.1 Quadratic Lyapunov functions

We begin by considering a quadratic Lyapunov function candidate of the form

\[
V(x; P) = x^\top Px
\]

with the matrix parameter \( P \in \mathbb{S}^n_{++} \) and domain \( \mathcal{R} \) the same as that of the PWA system. \(^5\) We aim to find a \( P \) such that \( V(x; P) \) would satisfy the constraint. \(^3\) Since we can always scale \( P \) while satisfying constraint \(^3\), without loss of generality, we describe the set of valid Lyapunov function candidate parameters as

\[
\mathcal{F} = \{P \in \mathbb{S}^n \mid 0 \preceq P \leq I, f_{cl}(x)^\top P f_{cl}(x) - x^\top Px < 0, \forall x \in \mathcal{X} \setminus \{0\}\}. \tag{11}
\]

We observe that \( \mathcal{F} \) is a convex set defined by semidefinite constraints and infinitely many linear constraints on \( P \). Although, theoretically, a feasible point in \( \mathcal{F} \) can be obtained by solving a convex semi-infinite feasibility problem, \(^42\) such an approach is numerically intractable. This motivates sample-based approaches which instead require the constraints in \(^1\) to only hold for a finite set of samples \( S = \{x^1, x^2, \ldots, x^N\} \subset \mathcal{X} \), resulting in an over-approximation of \( \mathcal{F} \)

\[
\mathcal{F} \subset \tilde{\mathcal{F}} = \{P \in \mathbb{S}^n \mid 0 \preceq P \leq I, f_{cl}(x)^\top P f_{cl}(x) - x^\top Px \leq 0, \forall x \in S\}. \tag{12}
\]

Finding a feasible point in \( \tilde{\mathcal{F}} \) is now a tractable convex feasibility program. However, there is no guarantee that the resulting solution \( P \) would correspond to a valid Lyapunov function, even if the number of samples approaches infinity. In this paper, we propose an efficient sampling strategy based on the analytic center cutting-plane method (ACCPM) to iteratively grow the sample set and refine the set \( \tilde{\mathcal{F}} \) until we find a feasible point in \( \mathcal{F} \). In the next subsections, we describe the proposed approach and provide finite-step termination and validity guarantees.

\(^2\)For numerical computation, we formulate the over-approximation of \( \mathcal{F} \) as compact sets. As will be shown in the following sections, our proposed method will find a feasible point in the interior of the closure of \( \mathcal{F} \) using barrier functions. Therefore, if \( \mathcal{F} \) is non-empty, our method will find a feasible point in \( \mathcal{F} \) satisfying all the strict inequalities.
4.2 The analytic center cutting-plane method

Cutting-plane methods \([43, 44, 45]\) are iterative algorithms to find a point in a target convex set \(\mathcal{F}\) or to determine whether \(\mathcal{F}\) is empty. In these methods, we have no information of \(\mathcal{F}\) except for a “cutting-plane oracle”. At each iteration \(i\), we have a “localization” set \(\tilde{\mathcal{F}}_i\) defined by a finite set of inequalities that overapproximates the target set, i.e., \(\mathcal{F} \subseteq \tilde{\mathcal{F}}_i\). If \(\tilde{\mathcal{F}}_i\) is empty, then we have a proof that the target set \(\mathcal{F}\) is also empty. Otherwise, we query the oracle at a point \(P(i) \in \tilde{\mathcal{F}}_i\). If \(P(i) \in \mathcal{F}\), the oracle returns ‘yes’ and the algorithm terminates; if \(P(i) \notin \mathcal{F}\), it returns ‘no’ together with a separating hyperplane that separates \(P(i)\) and \(\mathcal{F}\). In the latter case, the cutting-plane method updates the localization set by \(\tilde{\mathcal{F}}_{i+1} = \tilde{\mathcal{F}}_i \setminus \{\text{half space defined by the separating hyperplane}\}\). This process continues until either a point in the target set is found or the target set is certified to be empty.

Based on how the query point is chosen, different cutting-plane methods have been proposed including the center of gravity method \([46]\), the maximum volume ellipsoid (MVE) cutting-plane method \([40]\), the Chebyshev center cutting-plane method \([44]\), the ellipsoid method \([17, 48]\), the volumetric center cutting-plane method \([39]\), and the analytic center cutting-plane method \([50, 51, 43]\). In this paper, we use the analytic center cutting-plane method since it allows the localization set \(\tilde{\mathcal{F}}_i\) to be described by linear matrix inequalities. In the ACCPM, the query point \(P(i)\) is chosen as the analytic center of the localization set \(\tilde{\mathcal{F}}_i\).

**Definition 5** (Analytic center \([52]\)). The analytic center \(x_{ac}\) of a set of convex inequalities and linear equalities \(f_i(x) \leq 0, i = 1, \ldots, m, \mathcal{F}x = g\), is defined as the optimal solution of the convex problem

\[
\begin{align*}
\text{minimize} \quad & -\sum_{i=1}^{m} \log(-f_i(x)) \\
\text{subject to} \quad & \mathcal{F}x = g.
\end{align*}
\]  

(13)

To implement the ACCPM, we introduce a learner, which proposes Lyapunov function candidates based on a set of samples, and a verifier, which serves as a cutting-plane oracle.
4.3 The learner

Let $\mathcal{S} = \{x^1, x^2, \ldots, x^n\} \subset \mathcal{X}$ be the collection samples from $\mathcal{X}$. The Lyapunov difference as a function of the current state and the matrix $P$ is

$$\Delta V(x, P) := V(x_+; P) - V(x; P) = f_d(x)^\top P f_d(x) - x^\top P x,$$

which is linear in $P$. Based on the sample set $\mathcal{S}$, we construct an outer-approximation of $\mathcal{F}$ as

$$\tilde{\mathcal{F}} = \{P \in \mathbb{S}^n | 0 \preceq P \preceq I, \Delta V(x, P) \leq 0, \forall x \in \mathcal{S}\}. \quad (14)$$

The localization set $\tilde{\mathcal{F}}$ represents the learner’s knowledge about $\mathcal{F}$ by observing the one-step state transitions of the dynamical system from the sample set $\mathcal{S}$. Utilizing the ACCPM, the learner proposes a Lyapunov function candidate $V(x; P_{ac})$ with $P_{ac}$ as the analytic center of $\tilde{\mathcal{F}}$:

$$P_{ac} := \text{argmin}_P -\sum_{x \in \mathcal{S}} \log(-\Delta V(x, P)) - \log \det(I - P) - \log \det(P). \quad (15)$$

We denote the objective function in (15) as $\phi(\tilde{\mathcal{F}})$ and call it the potential function on the set $\tilde{\mathcal{F}}$. If (15) is infeasible, then we have a proof that no quadratic Lyapunov function exists for the closed-loop system. Otherwise, the learner proposes $V(x; P_{ac}) = x^\top P_{ac} x$ as a Lyapunov function candidate. Due to the log-barrier function in (15), $P_{ac}$ is in the interior of $\tilde{\mathcal{F}}$. At initialization, we have an empty sample set, i.e., $\mathcal{S} = \emptyset$ and the potential function simply becomes $\phi(\tilde{\mathcal{F}}) = -\log \det(I - P) - \log \det(P)$ with $P_{ac} = \frac{1}{2} I$ as the analytic center.

4.4 The verifier

Given a Lyapunov function candidate $V(x; P^{(i)})$ proposed by the learner at iteration $i$, the verifier either ensures that this function satisfies the constraints in (2a) and (3), or returns a state where constraint (2a) or (3) is violated as a counterexample. Since the log-barrier function in (15) guarantees $P^{(i)} \succ 0$, constraint (2a) is readily satisfied and the verifier must check the violation of constraint (3). This can be done by solving the optimization problem

$$\text{maximize}_{x \in \mathcal{X} \setminus \{0\}} \Delta V(x, P^{(i)}). \quad (16)$$

Next, we will show that for PWA systems and ReLU NN controllers, the optimization problem (16) can be formulated as a mixed-integer quadratic program (MIQP). This is based on the fact that both the PWA dynamics $\psi(x, u)$ and the ReLU NN controller can be described by a set of mixed-integer linear constraints.

4.4.1 Mixed-integer formulation of PWA dynamics

Recall that the PWA dynamics (5) is given by $x_+ = \psi_i(x, u) = A_i x + B_i u + c_i$ if $x \in \mathcal{R}_i$ in the state space. Equivalently, we can describe the PWA dynamics on $\mathcal{R}_i$ in the joint space of $(x, u, x_+) \in \mathbb{R}^{2n_x + n_u}$ as $(x, u, x_+) \in \text{gr}(\psi_i)$ where $\text{gr}(\psi_i)$ is the graph of $\psi_i$ and is defined as a polytope in $\mathbb{R}^{2n_x + n_u}$:

$$\text{gr}(\psi_i) = \{(x, u, x_+) | P_i(x, u, x_+) \leq q_i\}, \text{ with } P_i = \begin{bmatrix} A_i & B_i & -I \\ -A_i & -B_i & I \\ F_i & 0 & 0 \\ 0 & G_u & 0 \end{bmatrix}, \quad q_i = \begin{bmatrix} -c_i \\ c_i \\ h_i \\ h_{iu} \end{bmatrix}. \quad (17)$$
The first two block rows of $P_i$ and $q_i$ define the dynamics $x_+ = A_i x + B_i u + c_i$ and the third block row constrains $x$ to lie in $\mathcal{R}_i = \{ x | F_i x \leq h_i \}$. In the last block row, we add an artificial constraint $u \in \bar{U} = \{ u | G_u u \leq h_u \}$ on the control input where $\bar{U}$ is a bounded set in $\mathbb{R}^{n_u}$. This guarantees that $\text{gr}(\psi_i)$ have a common recession cone $\text{gr}(\psi_i)_\infty = \{ (0, 0, 0) \}$ for all $i \in \mathcal{I}$. Details of choosing $\bar{U}$ will be discussed in Section 4.5. Following from the definition of $\text{gr}(\psi_i)$, we have

$$(x, u, x_+) \in \text{gr}(\psi_i) \iff x_+ = \psi_i(x, u), \quad x \in \mathcal{R}_i.$$  

By denoting the PWA dynamics collectively as $x_+ = \psi(x, u)$, we define the graph of $\psi(x, u)$ as $\text{gr}(\psi) := \bigcup_{i \in \mathcal{I}} \text{gr}(\psi_i)$. Then we have

$$(x, u, x_+) \in \text{gr}(\psi) \iff x_+ = \psi(x, u),$$

which indicates that the evolution of the PWA dynamics is constrained in a union of disjunctive sets $\text{gr}(\psi)$. As a result, optimization over $x, u, x_+$ subject to the PWA system dynamics is equivalent to optimizing over the set $\text{gr}(\psi)$. Next, we will show that $\text{gr}(\psi)$ lends itself to a mixed-integer formulation.

**Definition 6 (Mixed-integer formulation of a set [53]).** For a set $Q \subset \mathbb{R}^{n_z}$, consider the set $L_Q \subseteq \mathbb{R}^{n_z+n_\lambda} \times \mathbb{Z}^{n_\mu}$ in a lifted space given by

$$L_Q = \{ (z \in \mathbb{R}^{n_z}, \lambda \in \mathbb{R}^{n_\lambda}, \mu \in \mathbb{Z}^{n_\mu}) | \ell(z, \lambda, \mu) \leq v \},$$  

with a function $\ell : \mathbb{R}^{n_z+n_\lambda} \times \mathbb{Z}^{n_\mu} \rightarrow \mathbb{R}^n$ and a vector $v \in \mathbb{R}^n$. The set $L_Q$ is a mixed-integer formulation of $Q$ if $\text{Proj}_z(L_Q) = Q$. If the function $\ell$ is linear (convex), we call the related formulation MIL (MIC).

In Definition 6 if $Q$ has a mixed-integer formulation $L_Q$, then $Q$ can be represented by the inequalities $\ell(z, \lambda, \mu) \leq v$ which means

$$z \in Q \iff \exists \lambda, z \text{ s.t. } \ell(z, \lambda, \mu) \leq v.$$  

For the PWA dynamics $x_+ = \psi(x, u)$, we construct an MIL formulation of $\text{gr}(\psi)$ which is known as the convex-hull formulation [53] in the following lemma.

**Lemma 1.** For PWA system $x_+ = \psi(x, u)$ defined in [5], let $x_i \in \mathbb{R}^{n_z}, u_i \in \mathbb{R}^{n_u}, \mu_i \in \{ 0, 1 \}$ for $i \in \mathcal{I}$ be the additional variables. Then the set $L_{\text{gr}(\psi)} = \{ (x, u, x_+), \{ x_i \}, \{ u_i \}, \{ \mu_i \} | (19) \}$ with the constraints

$$F_i x_i \leq \mu_i h_i, \quad G_i u_i \leq \mu_i h_u, \quad \mu_i \in \{ 0, 1 \}, \quad \forall i \in \mathcal{I}$$

$$(1, x, u, x_+) = \sum_{i \in \mathcal{I}} (\mu_i, x_i, u_i, A_i x_i + B_i u_i + \mu_i c_i),$$  

is an MIL formulation of $\text{gr}(\psi)$.

**Proof.** Since $\bar{U}$ and $\mathcal{R}_i$ are bounded, $\text{gr}(\psi_i)$ share the common recession cone $\text{gr}(\psi_i)_\infty = \{ (0, 0, 0) \}$ for all $i \in \mathcal{I}$. By [53] Corollary 3.6, the set $L_{\text{gr}(\psi)}$ in the lemma is an MIL formulation of $\text{gr}(\psi)$. □

In constraints (19), the binary variable $\mu_i$ can be interpreted as an indicator of the current mode of the state $x$. For instance, let $\mu_i = 1$ and consequently $\mu_j = 0, \forall j \neq i$. Since $F_j x_j \leq \mu_j h_j = 0 \Rightarrow x_j = 0, \forall j \neq i$, we have $x = \sum_{k \in \mathcal{I}} x_k = x_i$. Then the state $x$ satisfies $F_i x \leq h_i$, i.e., $x \in \mathcal{R}_i$. Similarly, $G_i u_i \leq \mu_j h_u = 0 \Rightarrow u_j = 0, \forall j \neq i$ and $u = \sum_{k \in \mathcal{I}} u_k = u_i$. It then follows $x_+ = \sum_{k \in \mathcal{I}} A_k x_k + B_k u_k + \mu_k c_k = A_i x_i + B_i u_i + c_i = \psi_i(x, u)$. The MIL formulation of $\text{gr}(\psi)$ in (19) allows us to represent the PWA dynamical constraint $x_+ = \psi(x, u)$ through finite mixed-integer linear constraints. It consists a key component in our construction of the verifier.
4.4.2 Mixed-integer formulation of ReLU NN

Consider a scalar ReLU function \( y = \max(0, x) \) where \( x \leq x \leq \bar{x} \). Then it can be shown that the ReLU function admits the following mixed-integer representation [17],

\[
y = \max(0, x), \quad x \leq x \leq \bar{x} \iff \{ y \geq 0, \ y \geq x, y \leq x - x(1 - t), \ y \leq \bar{x}t, \ t \in \{0, 1\}\}, \quad (20)
\]

where the binary variable \( t \in \{0, 1\} \) is an indicator of the activation function being active (\( y = x \)) or inactive (\( y = 0 \)). Now consider a ReLU network described by the equations in [7]. Suppose \( m^\ell \) and \( \bar{m}^\ell \) are known element-wise lower and upper bounds on the input to the \( \ell \)-th activation layer, i.e., \( m^\ell \leq W_\ell z^\ell + b_\ell \leq \bar{m}^\ell \). Then the neural network equations are equivalent to a set of mixed-integer constraints:

\[
z_{\ell+1} = \max(W_\ell z^\ell + b_\ell, 0) \iff \begin{cases} z_{\ell+1} \geq W_\ell z^\ell + b_\ell \\ z_{\ell+1} \leq W_\ell z^\ell + b_\ell - \text{diag}(m^\ell)(1 - t_\ell) \\ z_{\ell+1} \geq 0 \\ z_{\ell+1} \leq \text{diag}(\bar{m}^\ell)t_\ell, \end{cases} \quad (21)
\]

where \( t_\ell \in \{0, 1\}^{n_{\ell+1}} \) is a vector of binary variables for the \( (\ell + 1) \)-th activation layer and \( \mathbf{1} \) denotes vector of all 1’s. We note that the element-wise pre-activation bounds \( \{m^\ell, \bar{m}^\ell\} \) can be found by, for example, interval bound propagation or linear programming assuming known bounds on the input of the neural network [55, 57, 56, 57, 58]. Since state-of-the-art solvers for mixed-integer programming are based on the Branch & Bound algorithm [58], tight upper/lower bounds \( \{\bar{m}^\ell, m^\ell\} \) will allow the Branch & Bound algorithm to prune branches more efficiently and reduce the total running time.

4.4.3 Verifier construction through MIQP

Having the mixed-integer formulation of the PWA dynamics and the ReLU network at our disposal, we can formulate the verifier’s optimization problem (16) as the following MIQP:

\[
\begin{array}{ll}
\text{maximize} & y^TP^{(i)}y - x^TP^{(i)}x \\
\text{subject to} & F_{\mathcal{X}} x \leq h_{\mathcal{X}} \\
 & \|x\|_\infty \geq \epsilon \\
 & z_0 = x \\
 & \text{for } \ell = 0, \ldots, L - 1: \\
 & z_{\ell+1} \geq 0, \ z_{\ell+1} \geq W_\ell z^\ell + b_\ell \\
 & z_{\ell+1} \leq W_\ell z^\ell + b_\ell - \text{diag}(m^\ell)(1 - t_\ell) \\
 & z_{\ell+1} \leq \text{diag}(\bar{m}^\ell)t_\ell, \ t_\ell \in \{0, 1\}^{n_{\ell}} \\
 & u = W_L z_L + b_L \\
 & F_{\mathcal{I}} x_i \leq \mu_i h_i, \ \forall i \in \mathcal{I} \\
 & G_{\mathcal{I}} u_i \leq \mu_i h_u, \ \forall i \in \mathcal{I} \\
 & (1, x, u, y) = \sum_{i \in \mathcal{I}} (\mu_i, x_i, u_i, A_i x_i + B_i u_i + \mu_i c_i) \\
 & \mu_i \in \{0, 1\}, \ \forall i \in \mathcal{I}.
\end{array} \quad (22)
\]
Constraint \((22b)\) restricts the search space to \(\mathcal{X}\), constraint \((22c)\) with \(\epsilon > 0\) excludes a neighborhood around the origin\(^3\) to verify constraint \((3)\) with strict inequalities, constraints \((22d)\) to \((22i)\) exactly model the ReLU network controller \(u = \pi(x)\) with known layer-wise upper and lower bounds \(\{\bar{m}_l, m_\ell\}\) (see Section 4.4.2), and constraints \((22j)\) to \((22m)\) describe the PWA dynamics \(y = \psi(x,u)\) (see Section 4.4.1). The objective function \((22a)\) aims to find a counterexample where the Lyapunov condition \((3)\) is violated with the largest margin.

Denote by \(x^{(i+1)}_*\) the optimal solution and \(p^*\) the optimal value of problem \((22)\). If \(p^* < 0\), then we have a proof that \(V(x;P^{(i)})\) is a Lyapunov function since \(\Delta V(x, P^{(i)}) < 0\) for all \(x \in \mathcal{X} \setminus B_\epsilon\) where \(B_\epsilon = \{x ||x||_\infty < \epsilon\}\) and we terminate the search. As will be shown in Section 4.5, the choice of \(\epsilon\) guarantees \(p^* < 0\) is sufficient to prove the asymptotic stability of the origin. If \(p^* \geq 0\), then we reject the Lyapunov function candidate \(V(x;P^{(i)})\) with \(x^{(i+1)}_*\) as the counterexample since constraint \((3)\) is violated at a non-zero state \(x^{(i+1)}_* \in \mathcal{X}\) with \(\Delta V(x^{(i+1)}_*, P^{(i)}) = p^* \geq 0\). Meanwhile, we obtain a separating hyperplane \(\Delta V(x^{(i+1)}_*, P) = 0\) in \(P\) such that \(\Delta V(x^{(i+1)}_*, P) \leq 0\) for all \(P \in F\). Therefore, the MIQP \((22)\) serves as a cutting-plane oracle.

### 4.5 Implementation of the ACCPM

With a learner that solves the convex program in \((15)\) and a verifier that solves the MIQP in \((22)\), we summarize the overall procedure in Algorithm 1. In Algorithm 1, we iteratively grow the sample set \(S_i\) through the interaction between the learner and the verifier to guide the search for Lyapunov functions. Several practical issues of implementing Algorithm 1 are addressed below.

**Algorithm 1 Analytic center cutting-plane method (ACCPM)**

**Input:** initial sample set \(S_0 \subset \mathcal{X}\)  
**Output:** \(P_*, \text{Status}\)  
1: procedure ACCPM  
2: \(i = 0\)  
3: while True do  
4:     Generate an outer-approximation \(\tilde{F}_i\) from sample set \(S_i\) by \((14)\).  
5:     if \(\tilde{F}_i = \emptyset\) then  
6:         Return: Status = Infeasible, \(P_* = \emptyset\).  
7:     end if  
8:     \(P^{(i)} = \text{AnalyticCenter}(\tilde{F}_i)\) \(\triangleright \) The learner proposes \(P^{(i)} = \arg \min (15)\)  
9:     Query the verifier (cutting-plane oracle) at \(P^{(i)}\) \(\triangleright \) The verifier solves \((22)\) with \(P^{(i)}\)  
10:    if Verifier returns ‘yes’ then  
11:        Return: Status = Feasible, \(P_* = P^{(i)}\). \(\triangleright \) if \(\text{min}(22) < 0\)  
12:    else  
13:        Extract the counterexample \(x^{(i+1)}_* = \arg \min_x (22)\) \(\triangleright \) if \(\text{min}(22) \geq 0\)  
14:        \(S_{i+1} = S_i \cup \{x^{(i+1)}_*\}\)  
15:    end if  
16:    \(i = i + 1\)  
17: function AnalyticCenter(\(\tilde{F}_i\))  
18:    \(P_{ac} = \text{Optimal solution to } (15)\)  
19: return \(P_{ac}\)

**Exclusion of the origin:** In the MIQP \((22)\), we exclude the \(\ell_\infty\)-norm ball \(B_\epsilon = \{x ||x||_\infty < \epsilon\}\) in order to verify asymptotic stability of the origin. The choice of \(\epsilon\) is based on the observation that the

\(^3\)For choice of \(\epsilon\) see the discussion in Section 4.5.
closed-loop system $x_+ = f_d(x)$ is actually a PWA autonomous system and we can identify the local linear dynamics $x_+ = A_d x$ in the polytopic partition $D_0$ that contains the origin [29]. It follows from linear system theory that the origin is asymptotically stable if and only if all eigenvalues of $A_d$ have magnitudes less than 1 [59]. As a result, we choose $\epsilon$ small enough such that $B_\epsilon \subseteq D_0$ and divide the asymptotic stability verification into two steps: use MIQP \cite{69} to show the convergence of closed-loop trajectories to $B_\epsilon$ and use $A_d$ to prove the convergence to the origin. Since the stability of $A_d$ can be easily verified, our goal is to find a Lyapunov function that certifies the convergence of closed-loop trajectories to $B_\epsilon$.

**Solvability of nonconvex MIQP:** The optimization problem \cite{69} is a nonconvex MIQP since the quadratic objective function \cite{69} is indefinite. Therefore, the relaxation of the problem after removing the integer constraints would result in a nonconvex quadratic program. Nonconvex MIQP can be solved to global optimality through Gurobi v9.0 [60] by transforming the nonconvex quadratic expression into a bilinear form and applying spatial branching [61]. More information on solving nonconvex mixed-integer nonlinear programming can be found in [62, 63, 64]. In this paper, we rely on Gurobi to solve the nonconvex MIQP \cite{69} automatically.

**Choice of upper/lower bounds** $\{\bar{m}_i, \underline{m}_i\}$: As discussed in Section 4.4.2, tight element-wise upper and lower bounds on the pre-activation values would lead to a more efficient pruning in the Branch&Bound algorithm and consequently reduce the overall running time. Finding tighter bounds, however, would require more computation. This trade-off has been explored in the context of neural network verification in [17].

**Choice of artificial control input constraint:** In the definition of $\text{gr}(\psi_i)$, we introduced the artificial control input constraint $\mathcal{U} = \{u | G_u u \leq h_u\}$. First, $\mathcal{U}$ should be a bounded set to guarantee the validity of \cite{19}. Second, $\mathcal{U}$ should be an over-approximation of the output range of $\pi(x)$ over $\mathcal{X}$ to guarantee that the MIQP \cite{69} searches over all $x \in \mathcal{X}$. This can be done by choosing $\mathcal{U}$ as a box constraint and applying interval arithmetic [65] to over-approximate the output range of $\pi(x)$. Other output range analysis tools based on LP [11], SDP [12, 14], or MILP [17] are also applicable.

**Normalization of the separating hyperplane:** Let $\mathcal{S} = \{x^1, \cdots, x^N\}$. When generating the localization set $\bar{F}_i$, we apply the linear constraint $\Delta \bar{V}(x^j_1, P) = f_d(x^j_1)^T P f_d(x^j_1) - x^j_1 x^j_1^T P = (f_d(x^j_1) f_d(x^j_1)^T - x^j_1 x^j_1^T, P) = (D_{j, j}, P) \leq 0$ for all $1 \leq j \leq N$ where $D_i = f_d(x^j_1) f_d(x^j_1)^T - x^j_1 x^j_1^T$.

We normalize the linear constraint $\langle D_i, P \rangle \leq 0$ by $\langle D_i/\|D_i\|_F, P \rangle \leq 0$ and use the normalized constraint in the analytic center optimization problem \cite{15}.

Next, we provide finite-step termination guarantees for the proposed algorithm based on the analysis of the ACCPM from convex optimization.

### 4.6 Termination of ACCPM

The convergence and complexity of the ACCPM have been studied in \cite{43, 51, 66, 67, 68, 69} under various assumptions on the localization set, the form of the separating hyperplane, whether multiple cuts or multiple constraints drop is applied, etc. Directly related to Algorithm \cite{1} and the search for quadratic Lyapunov functions is [69] which analyzes the complexity of the ACCPM with a matrix variable and semidefiniteness constraints. Notably, it provides an upper bound on the number of iterations that Algorithm \cite{1} can run before termination. In [69], it is assumed that

1. A1: $\mathcal{F}$ is a convex subset of $\mathbb{S}^{n_x}$.
2. A2: $\mathcal{F}$ contains a non-degenerate ball of radius $\epsilon > 0$, i.e., there exists $P_{center} \in \mathbb{S}^{n_x}$ such that $\{P \in \mathbb{S}^{n_x} | \|P - P_{center}\|_F \leq \epsilon\} \subseteq \mathcal{F}$ where $\|\cdot\|_F$ is the Frobenius norm.
• A3: \( \mathcal{F} \subset \{ P \in \mathbb{S}^{n_x} | 0 \preceq P \preceq I \} \).

Let the ACCPM start with the localization set \( \mathcal{F}_0 = \{ P | 0 \preceq P \preceq I \} \) (or empty sample set \( \mathcal{S}_0 \) in (15)) and initialize the first query point \( P(0) = \frac{1}{2}I \) correspondingly. If at iteration \( i \), a query point \( P(i) \) is rejected by the oracle, a separating hyperplane of the form \( \langle D_i, P - P(i) \rangle = 0 \) is given and \( D_i \) is normalized to satisfy \( \|D_i\|_F = 1 \). Then each localization set \( \tilde{\mathcal{F}}_i \) is given by

\[
\tilde{\mathcal{F}}_i = \{ P | 0 \preceq P \preceq I, \langle D_i, P \rangle \leq c_i, i = 1, \ldots, N \},
\]

with \( D_i \) defining the separating hyperplane at iteration \( i \) and \( c_i = \langle D_i, P(i) \rangle \). The analytic center of \( \tilde{\mathcal{F}}_i \) is found by minimizing its potential function

\[
\phi(\tilde{\mathcal{F}}_i) = -\sum_{i=1}^N \log(c_i - \langle D_i, P \rangle) - \log \det(P) - \log \det(I - P),
\]

i.e., the query point \( P(i) = \arg\min \phi(\tilde{\mathcal{F}}_i) \) at iteration \( i \). Then the ACCPM following the above description achieves the following finite-step termination guarantee.

**Theorem 2** (Theorem 6.5 [69]). The analytic center cutting-plane method under assumptions A1 to A3 terminates in at most \( O(n_x^3/\epsilon^2) \) iterations.

**Proof.** Sketch of proof: For the sequence of localization sets \( \tilde{\mathcal{F}}_k \), [69] computes an upper bound on the potential function \( \phi(\tilde{\mathcal{F}}_k) \) which is approximately \( k \log(\frac{k}{\epsilon}) \), and a lower bound on \( \phi(\tilde{\mathcal{F}}_k) \) which is proportional to \( \frac{k}{2} \log(\frac{k}{n_x^2}) \). Since the ACCPM must terminate before the lower bound exceeds the upper bound, we obtain the finite-step termination guarantee in Theorem 2.

The application of Theorem 2 to Algorithm 1 is direct. In Algorithm 1 we choose \( D_i = f_d(x^{(i)}_*) f_d(x^{(i)}_*)^\top - x^{(i)}_* x^{(i)}_*^\top \) and normalize it to be of unit norm. In order to be consistent with the ACCPM described in [69], we relax the separating hyperplane from \( \langle D_i, P \rangle \leq 0 \) to \( \langle D_i, P \rangle \leq \langle D_i, P(i) \rangle \) (note \( \langle D_i, P(i) \rangle \geq 0 \) by definition of \( D_i \)) with which we can directly apply Theorem 2 to guarantee that Algorithm 1 terminates in at most \( O(n_x^3/\epsilon^2) \) iterations.

## 5 Extensions and variations

### 5.1 Piecewise quadratic Lyapunov functions

In the previous section, we used quadratic Lyapunov functions to verify the stability of the closed-loop system. When no globally quadratic Lyapunov function can be found or the results are too conservative, a natural extension is to consider *piecewise quadratic* (PWQ) functions. In this section, we consider this class. Specifically, consider the following piecewise quadratic Lyapunov function candidate

\[
V(x; P) = \left[ \begin{array}{c} x \\ f_d(x) \end{array} \right]^\top P \left[ \begin{array}{c} x \\ f_d(x) \end{array} \right],
\]

with \( P \in \mathbb{S}^{2n_x} \). This function is a composition of a quadratic function and the PWA closed-loop dynamics \( f_d(x) \), and hence is a PWQ function. By the formulation in (24), we parameterize a class of continuous PWQ functions with modes as many as those of the closed-loop system [8] but with only a small-size parameter \( P \in \mathbb{S}^{2n_x} \) in order to keep the complexity of running the ACCPM at a manageable level.
Remark 1. The standard parameterization of PWQ functions requires defining a quadratic function on a prefixed partition, which will introduce quadratic equality constraints in the MIP-based verifier. Considering the complexity of nonconvex mixed-integer quadratically constrained programming, we apply the parameterization in (24) instead to reduce the number of quadratic constraints in the MIP-based verifier.

Similar to Section 4, let $\mathcal{F}$ be the set of parameter $P$ such that $V(x;P)$ satisfies the conditions (2a) and (3):

$$\mathcal{F} = \{ P \in \mathbb{S}^{2n_x} | 0 < P \preceq I, V(f_{cl}(x);P) - V(x;P) < 0, \forall x \in X \setminus \{0\} \}.$$ 

Since $\mathcal{F}$ is a convex set in $P$, we can apply the ACCPM to find a feasible point in $\mathcal{F}$.

5.1.1 Design of the learner

Consider the sample set with $N$ samples $S = \{x^1, \cdots, x^N\}$. The Lyapunov difference for the closed-loop system is

$$\Delta V(x,P) = V(f_{cl}(x);P) - V(x;P) = \begin{bmatrix} f_{cl}(x) \\ f^{(2)}_{cl}(x) \end{bmatrix}^\top P \begin{bmatrix} f_{cl}(x) \\ f^{(2)}_{cl}(x) \end{bmatrix} - \begin{bmatrix} x \\ f_{cl}(x) \end{bmatrix}^\top P \begin{bmatrix} x \\ f_{cl}(x) \end{bmatrix}$$

where $f^{(2)}_{cl}(x) = f_{cl}(f_{cl}(x))$. The localization set $\bar{\mathcal{F}}$ is thus given by

$$\bar{\mathcal{F}} = \{ P | 0 \preceq P \preceq I, \Delta V(x,P) \leq 0, \forall x \in S \},$$

which contains $\mathcal{F}$. When $S = \emptyset$, we have $\bar{\mathcal{F}} = \{ P | 0 \preceq P \preceq I \}$. The learner finds the analytic center $P_{ac}$ of the localization set $\bar{\mathcal{F}}$ by solving

$$\min_P - \sum_{x \in S} \log(-\Delta V(x,P)) - \log \det(I - P) - \log \det(P)$$

If problem (27) is infeasible, there is no valid Lyapunov function in the parameterized function class (24); otherwise, the solution $P_{ac} = \arg \min P_{ac}$ constructs the Lyapunov function candidate $V(x;P_{ac})$ proposed by the learner. Similarly, we can relax the separating hyperplane as shown in (23) to apply the finite-step termination guarantee.

5.1.2 Design of the verifier

With the PWQ Lyapunov function candidate $V(x;P^{(i)})$ proposed by the learner at iteration $i$, the verifier solves the following MIQP:

$$\max_{\{x^j, u^j, \mu^j, z_i^j, t_{ij}\}} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}^\top P^{(i)} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} - \begin{bmatrix} x^0 \\ x^1 \end{bmatrix}^\top P^{(i)} \begin{bmatrix} x^0 \\ x^1 \end{bmatrix}$$

subject to

$$F_X x^0 \leq h_X$$

$$\|x^0\|_\infty \geq \epsilon$$

for $j = 0, 1$:

$$z_j^0 = x^j$$

for $\ell = 0, \cdots, L - 1$:
\[
    z^i_j \geq 0, \quad z^i_j \geq W_i z^i_{j-1} + b_i \\
    z^i_j \leq W_i z^i_{j-1} + b_i - \text{diag}(m^i_j)(1 - t^i_j) \\
    z^i_j \leq \tilde{m}^i_j t^i_j, \quad t^i_j \in \{0, 1\}^{n_x} \\
    u^i_j = W_L z^i_j + b_L \\
    F_i x^i_j \leq \mu^i_j h_i, \quad G_u u^i_j \leq \mu^i_j h_u, \quad \forall i \in I \\
    (1, x^i_j, u^i_j, x^{i+1}) = \sum_{i \in I} (\mu^i_j, x^i_j, u^i_j, A_i x^i_j + B_i u^i_j + \mu^i_j g_i) \\
    \mu^i_j \in \{0, 1\}, \forall i \in I
\]

where we evolve the closed-loop system dynamics for two steps. For each state \( x^0 \in \mathcal{X} \setminus B_\epsilon \) (constraint (28b) and (28c)), constraints (28d) to (28i) model the PWA dynamics \( x^1 = \psi(x^0, u^0), x^2 = \psi(x^1, u^1) \) while constraints (28j) to (28l) describe the neural network controller \( u^0 = \pi(x^0), u^1 = \pi(x^1) \). The interpretation of the solution of problem (28) is the same as that in Section 4.4.3.

It is straightforward to parameterize more complex PWQ Lyapunov function candidate classes by concatenating more ‘future’ states \( f^{(i)}_d(x) \), \( i = 1, 2, \cdots, k \) in (24). This only requires modification of the verifier (28) by cloning the MIL constraints for \( k + 1 \) times and choosing the ROI \( \mathcal{X} \) accordingly to guarantee that \( f^{(i)}_d(x) \) is well-defined.

\[\text{\textbf{5.2 Neural network controllers with projection}}\]

In Section 4 we studied the closed-loop stability of the PWA system (5) with a neural network controller \( u = \pi(x) \). In practical applications, we often encounter hard constraints on the state and on the control input (e.g., actuator saturation). In this section, we consider neural networks with a projection layer in feedback interconnection with the LTI system (6) with dynamic matrices \( (A, B) \). Specifically, we assume the projected neural network satisfies the following constraints:

- **Control input constraint:** the control input \( u \) has to satisfy the polytopic constraint
  \[
  u \in \mathcal{U} = \{ u \in \mathbb{R}^{n_u} | C_u u \leq d_u \}. 
  \]

- **Positive invariance constraint:** the region of interest \( \mathcal{X} = \{ x \in \mathbb{R}^{n_x} | F_X x \leq h_X \} \subseteq \mathcal{R} \) is positive invariant for the closed-loop system.

While the control input constraint can always be satisfied by projecting \( \pi(x) \) onto \( \mathcal{U} \), to make the ROI \( \mathcal{X} \) positive invariant we require \( \mathcal{X} \) to be a control invariant set under the control input constraint \( \mathcal{U} \).

**Definition 7** (Control invariant set). A set \( \mathcal{C} \subseteq \mathcal{R} \) is said to be a control invariant set for the LTI system \( x_+ = Ax + Bu \) subject to the control input constraint (29) if

\[
\exists u \in \mathcal{U} \text{ such that } x_+ = Ax + Bu \in \mathcal{C}, \quad \forall x \in \mathcal{C}.
\]

**Assumption 1.** The region of interest \( \mathcal{X} \) is a control invariant set for the LTI system (6) with control input constraints (29).

**Remark 2.** While identifying a positively invariant set for the nonlinear closed-loop autonomous system (8) is challenging, finding a control invariant set for the LTI system (6) with a polytopic control input constraint \( \mathcal{U} \) can be done through an iterative algorithm [7].
To satisfy the positive invariance constraint, we project the neural network output
\[ u = \pi(x), \quad x \in \mathcal{X} \]
on to the state-dependent polyhedron
\[ \Omega(x) = \{ u \in \mathbb{R}^n | Ax + Bu \in \mathcal{X}, u \in \mathcal{U} \}. \] (30)
As a result, the projected control input \( u^p = \pi_{proj}(x) := \text{Proj}_{\Omega(x)}(\pi(x)) \) is given by the optimal solution to the following convex quadratic program,
\[
\begin{align*}
    u^p &= \arg \min_u \frac{1}{2} \| u - \pi(x) \|^2_2 \\
    \text{subject to} \quad &F_{\mathcal{X}}Bu \leq h_{\mathcal{X}} - F_{\mathcal{X}}Ax \\
    &C_u u \leq d_u.
\end{align*}
\] (31)
From the definition of \( \Omega(x) \), the projected NN controller \( \pi_{proj}(x) \) ensures that \( \pi_{proj}(x) \in \mathcal{U} \) and \( x_+ = Ax + B\pi_{proj}(x) \in \mathcal{X} \) for all \( x \in \mathcal{X} \). Hence, the ROI \( \mathcal{X} \) is rendered positive invariant under \( \pi_{proj}(x) \) and satisfies the invariance assumption in Problem 2. The application of the projected NN controller to guarantee constraint satisfaction or set invariance can be found in [70]. However, the authors do not verify the closed-loop stability under the projected NN controller. In the following, we modify the ACCPM to account for control input projection.

### 5.2.1 ACCPM for the projected controller
Since \( \pi_{proj}(x) \) guarantees \( \mathcal{X} \) is positive invariant for the closed-loop system, we can verify \( \mathcal{X} \subseteq \mathcal{O} \) by searching over quadratic Lyapunov functions through Algorithm 1. For the learner, the analytic center optimization problem (15) remains unchanged except that the new closed-loop dynamics are applied. For the verifier, we need to construct a new MIQP that reflects the projection operation in (31).

We observe that the projected control input \( u^p \) is the optimal solution to the convex quadratic program (31). By the KKT conditions, \( u^p \) is the solution to the following system of inequalities and equalities:
\[
\begin{align*}
    u^p - \pi(x) + (F_{\mathcal{X}}B)^\top \lambda + C_u^\top \nu &= 0 \\
    F_{\mathcal{X}}Bu^p + F_{\mathcal{X}}Ax &\leq h_{\mathcal{X}} \\
    C_u u^p &\leq d_u \\
    \lambda &\succeq 0, \nu \succeq 0 \\
    \lambda^\top (F_{\mathcal{X}}Bu^p + F_{\mathcal{X}}Ax - h_{\mathcal{X}}) &= 0 \\
    \nu^\top (C_u u^p - d_u) &= 0
\end{align*}
\] (32)
Let \( n_f \) and \( n_c \) be the number of rows of \( F_{\mathcal{X}} \) and \( C_u \), respectively. Then \( \lambda \in \mathbb{R}^{n_f}, \nu \in \mathbb{R}^{n_c} \) are the Lagrangian dual variables. Constraints (32c) and (32d) are bilinear but they can be modeled as mixed-integer linear constraints through the big-\( M \) method [71]:
\[
\begin{align*}
    0 \leq \lambda_i &\leq M t_{\lambda,i}, \quad t_{\lambda,i} \in \{0, 1\} \\
    0 \leq (h_{\mathcal{X}} - F_{\mathcal{X}}Bu^p - F_{\mathcal{X}}Ax)_i &\leq M(1 - t_{\lambda,i}), \quad i = 1, \ldots, n_f \\
    0 \leq \nu_j &\leq M t_{\nu,j}, \quad t_{\nu,j} \in \{0, 1\} \\
    0 \leq (d_u - C_u u^p)_j &\leq M(1 - t_{\nu,j}), \quad j = 1, \ldots, n_c
\end{align*}
\] (33)
where the subscript \( i \) denotes the \( i \)-th entry of a vector. In (33), a single \( M \) is chosen large enough for all relevant constraints for simplicity of exposition while tighter element-wise lower and upper
We first consider model predictive control of the LTI system. We demonstrate the application of the ACCPM through two numerical examples: one is a double integrator LTI system and the other is a PWA system modeling an inverted pendulum in contact with an elastic wall. Following from approximate model predictive control (MPC) [70, 72], we first synthesize an MPC controller $u = \pi_{\text{MPC}}(x)$ for the underlying system, and then train a ReLU NN $\pi(x)$ through supervised learning to approximate the MPC controller, i.e., $\pi(x) \approx \pi_{\text{MPC}}(x)$. The closed-loop stability of the NN-controlled system is analyzed by synthesizing a Lyapunov function through Algorithm 1. All the simulation is implemented in Python 3.7 with Gurobi v9.0 [60] on an Intel i7-6700K CPU.

### 6.1 A double integrator example

We first consider model predictive control of the LTI system $x_+ = Ax + Bu, x \in \mathbb{R}^2$ with state and control input constraints $X$ and $U$:

$$A = \begin{bmatrix} 1.1 & 1.1 \\ 0 & 1.1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \quad X = \{x | -5 \leq x \leq 5\}, \quad U = \{u | -1 \leq u \leq 1\}.$$  

(35)

The MPC horizon is given by $T = 20$ together with the stage cost $q(x, u) = x^\top Q x + u^\top R u$, $Q = \text{diag}(1, 1)$, $R = 1$ and terminal cost $p(x) = x^\top P_\infty x$, $P_\infty = \text{DARE}(A, B, Q, R)$ which means $P_\infty$ is the solution to the discrete algebraic Riccati equation defined by $(A, B, Q, R)$. The terminal set $X_T$ is chosen as the maximum positive invariant set [7] Chapter 10] of the closed-loop system $x_+ = (A + BK_\infty)x$ where $K_\infty = -(B^\top P_\infty B + R)^{-1}B^\top P_\infty A$. The choice of $Q, R, P_\infty, X_T$ guarantees the asymptotic stability of the closed-loop system $x_+ = Ax + B\pi_{\text{mpc}}(x)$ with a polytopic ROA shown in Fig. 4b [7] Chapter 12].
6.1.1 Synthesis of a NN controller

By the explicit MPC approach \cite{73, 74}, the MPC controller, denoted by $\pi_{mpc}(x)$, is a PWA function with polyhedral partitions as shown in Fig. 4a. By construction of the MPC controller, the ROA of the closed-loop system is a polytopic set shown in Fig. 4b. We denote the ROA as $\mathcal{X}_0 := \{x | F_{\mathcal{X}_0}x \leq h_{\mathcal{X}_0}\}$ and use it as a reference region of interest.

To obtain a neural network approximate of $\pi_{mpc}(x)$, we sample 650 states uniformly from $\mathcal{X}_0$ and compute $\pi_{mpc}(x)$ at these samples as the training data. Through supervised learning with Keras \cite{75}, we obtain a ReLU NN controller $\pi(x)$ with 3 hidden layers and 10 neurons in each layer to approximate $\pi_{mpc}(x)$. The bias term in the output layer of the NN is modified such that $\pi(0) = 0$.

The ReLU NN is visualized in Fig. 4c.

Next, we want to verify the stability of double integrator system under the NN controller $\pi(x)$.

![Figure 4: The explicit MPC controller (left) is approximated by a neural network controller (right). The ROA of the MPC controller (middle) is used as the ROI reference to guide the search for Lyapunov functions.](image)

6.1.2 Estimate ROA through Quadratic/PWQ Lyapunov functions

For the closed-loop system of the double integrator interconnected with $\pi(x)$, we first apply the ACCPM to search for a quadratic Lyapunov function $V(x; P) = x^TPx$ with $P \in \mathbb{S}^2_{++}$. Since the set $\mathcal{X}_0$ (Fig. 4b) is the ROA of the MPC controller, the ROI is chosen as $\mathcal{X} = \gamma\mathcal{X}_0 := \{x | F_{\mathcal{X}_0}x \leq \gamma h_{\mathcal{X}_0}\}$ with a scaling variable $0 < \gamma \leq 1$ to guide our search of Lyapunov functions. Since the estimate of ROA $\tilde{\mathcal{O}}$ is always contained in the ROI, we want $\mathcal{X}$ to be as large as possible as long as the ACCPM can find a Lyapunov function.

In the ACCPM implementation, the learner solves the convex program (15) through CVXPY \cite{76} with MOSEK \cite{77} as the solver, and the verifier solves the MIQP (22) through Gurobi \cite{60}. We set $\epsilon = 0.0172$ in the MIQP (22) and denote $B_\epsilon$ the $\ell_\infty$ norm ball centered at the origin with radius $\epsilon$. Inside $B_\epsilon$, the closed-loop dynamics is given by

$$x_+ = A_{cl}x = \begin{bmatrix} 0.50355752 & 0.02697626 \\ -0.29822124 & 0.56348813 \end{bmatrix} x$$

Since $A_{cl}$ has eigenvalues 0.53352282 $\pm$ 0.08453977i with norm less than 1, the closed-loop system is asymptotically stable inside $B_\epsilon$. The ACCPM starts with an empty initial sample set $S_0 = \emptyset$. Through bisection, the largest scaling variable is given by $\gamma = 0.89$, i.e., we can find a valid quadratic Lyapunov function through the ACCPM with $\mathcal{X} = 0.89\mathcal{X}_0$. After 10 iterations, the ACCPM

\footnote{Fig. 4a and Fig. 4b are generated through the MPT3 toolbox \cite{74} in MATLAB R2019b.}
terminated with total solver time 9.788 seconds and found a valid quadratic Lyapunov function
\[ V(x) = x^\top P x \]
with
\[ P = \begin{bmatrix} 0.14240707 & 0.02797589 \\ 0.02797589 & 0.78732241 \end{bmatrix} \] (36)
together with an estimate of ROA shown in Fig. 5a. In running the ACCPM, we show the optimal values of the MIQP (22) and the counterexamples found by the verifier in Fig. 5. At termination, the optimal solution of the MIQP (22) is \( x^* = (-0.0172, 0.0048) \) (red star in Fig. 5b) which lies on the boundary of \( B_\epsilon \) with optimal value \( p^* = -7.083 \times 10^{-7} < 0 \). This certifies that the terminating parameter \( P \) in (36) generates a Lyapunov function.

Figure 5: For the double integrator system with ROI \( \mathcal{X} = 0.89\mathcal{X}_0 \) and quadratic Lyapunov function candidates, the ACCPM terminates in 10 iterations. Fig. 5a plots the optimal value of the MIQP (22) solved by the verifier. Fig. 5b shows the counterexamples found in each iteration with the optimal solution of (22) in the last iteration marked by the red star.

The same procedure is applied to searching PWQ Lyapunov functions with \( P \in S^4 \). The ACCPM is initialized with \( S_0 = \emptyset \) and through bisection on \( \gamma \) we set \( \mathcal{X} = 1.0\mathcal{X}_0 \). In this case, the ACCPM terminates in 5 iterations with total solver time 51.398 seconds, and generates a Lyapunov function with
\[ P = \begin{bmatrix} 0.27336075 & 0.04068795 & -0.1110683 & 0.09147914 \\ 0.04068795 & 0.79551997 & 0.02651292 & 0.15342152 \\ -0.1110683 & 0.02651292 & 0.13499793 & -0.10682033 \\ 0.09147914 & 0.15342152 & -0.10682033 & 0.48064250 \end{bmatrix}. \] (37)
The corresponding estimate of ROA given by the PWQ Lyapunov function is shown in Fig. 6b. Compared with the quadratic Lyapunov function example in Fig. 6a, the application of PWQ Lyapunov functions enlarges the estimate ROA found by the ACCPM.

### 6.1.3 NN controller with projection

We project the synthesized NN controller \( \pi(x) \) to the state-dependent polytope \( \Omega(x) \) shown in (30) with \( \mathcal{X} = \mathcal{X}_0, \mathcal{U} = U \) since \( \mathcal{X}_0 \) is a control invariant set under the control input constraint \( U \) in (35) by construction. By applying the MIQP (34) as the verifier, the ACCPM found a PWQ Lyapunov function \( V(x; P) \) with the initial sample set \( S_0 = \emptyset \) in 5 iterations and with total solver time 64.470
(a) Estimate of ROA by a quadratic Lyapunov function.  (b) Estimate of ROA by a PWQ Lyapunov function.

Figure 6: Left: with $\mathcal{X} = 0.89\mathcal{X}_0$, the ACCPM found a quadratic Lyapunov function with an ellipsoidal estimate of ROA (yellow) for $\pi(x)$. Right: with the PWQ Lyapunov function candidate parameterized in [24], the ACCPM found a valid candidate with $\mathcal{X} = 1.0\mathcal{X}_0$ and an estimate ROA (yellow). Simulated closed-loop trajectories with the NN controller are plotted for a grid of initial conditions.

seconds. Thus, we certify that $\mathcal{X}_0 \subseteq \mathcal{O}$ when the NN controller with projection is applied. As shown in Fig. 7 the corresponding estimate of ROA now becomes a polytope and preserves the asymptotic stability guarantee of the MPC controller constructed in Section 6.1.

Figure 7: For the NN controller with projection, the polytopic ROI $\mathcal{X} = \mathcal{X}_0$ is an estimate of ROA.

6.2 Inverted pendulum with an elastic wall

We use a hybrid system example from [78] and consider the inverted pendulum shown in Fig. 8a with parameters $m = 1, \ell = 1, g = 10, k = 100, d = 0.1, h = 0.01$. The state is $x = (q, \dot{q})$ which represents the angle $q$ and angular velocity $\dot{q}$ of the pendulum. By linearizing the dynamics of the inverted pendulum around $x = 0$, we obtain a hybrid system which has two modes: not in contact with the elastic wall (mode 1) and in contact with the elastic wall (mode 2). After discretizing the model using the explicit Euler scheme with a sampling time $h = 0.01$, a PWA model $x_+ = \psi(x, u)$
in the form (5) is obtained with the following parameters

\[
A_1 = \begin{bmatrix} 1 & 0.01 \\ 0.1 & 1 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}, c_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathcal{R}_1 = \{ x | (-0.2, -1.5) \leq x \leq (0.1, 1.5) \} \\
A_2 = \begin{bmatrix} 1 & 0.01 \\ -0.9 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}, c_2 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \mathcal{R}_2 = \{ x | (0.1, -1.5) \leq x \leq (0.2, 1.5) \}
\]  

(38)

Figure 8: A ReLU neural network (right) that approximates a hybrid MPC controller is applied on the inverted pendulum system with an elastic wall (left).

We then synthesize a hybrid MPC controller \( \pi_{\text{MPC}}(x) \) for the PWA system through disjunctive programming \[53\] where the control input constraints are given by \( u \in U = \{ u \in \mathbb{R} | -4 \leq u \leq 4 \} \) and the horizon of MPC is set as \( T = 10 \). The stage and terminal costs are given by

\[
q(x_t, u_t) = x_t^\top Q x_t + u_t^\top R u_t, \quad p(x_T) = x_T^\top Q_T x_T
\]

where \( Q = I, R = 1 \), and \( Q_T = \text{DARE}(A_1, B_1, Q, R) \) is the solution to the discrete algebraic Riccati equation defined by \( (A_1, B_1, Q, R) \). We evaluate \( \pi_{\text{MPC}}(x) \) on a uniform \( 40 \times 40 \) grid samples from the state space and let the reference ROI \( X_0 \) be the convex hull of all the feasible state samples. Then the ROI \( \mathcal{X} = \gamma X_0 \) with \( 0 < \gamma \leq 1 \) is applied to guide the search for an estimate of ROA.

A total number of 1354 feasible samples of state and control input pairs \((x, \pi_{\text{MPC}}(x))\) are generated to train a ReLU neural network \( \pi(x) \) in Keras to approximate the MPC controller. The NN has 2 hidden layers with 20 neurons in each layer and its output layer bias term is modified after training to guarantee \( \pi(0) = 0 \). The NN controller is shown in Fig. 8b. For the NN controller, we set \( \epsilon = 0.0158 \) and verify that the linear closed-loop dynamics inside \( B_\epsilon \) is asymptotically stable.

With empty initial sample set \( S_0 \), the ACCPM is run with both quadratic and PWQ Lyapunov function candidates in order to find a large estimate of ROA. With quadratic Lyapunov function candidates, the largest ROI is given by \( \mathcal{X} = 0.81 X_0 \) through bisection on \( \gamma \). The ACCPM terminates in 10 iterations with total solver time 9.928 seconds. The corresponding estimate of ROA is shown in Fig. 9a. With PWQ Lyapunov function candidates, the largest ROI is given by \( \mathcal{X} = 1.0 X_0 \) in which case the ACCPM terminates in 9 iterations with total solver time 124.210 seconds. The estimate of ROA obtained by the found PWQ Lyapunov function is shown in Fig. 9b. It is observed that the PWQ Lyapunov function is less conservative compared with the quadratic one. In addition
to the theoretical guarantees provided by the ACCPM, the validity of the estimate of ROA is also shown by the simulated closed-loop trajectories in Fig. 9.

7 Conclusion

In this paper, we proposed an iterative algorithm that learns quadratic and piecewise quadratic Lyapunov functions for piecewise affine systems with ReLU neural network controllers. The proposed algorithm is composed of a learner and a verifier, in which the learner uses a cutting-plane strategy to propose Lyapunov function candidates from a set of sampled states of the closed-loop system, and the verifier either certifies the validity of the proposed Lyapunov function or rejects it with a counterexample to be accounted for by the learner in the next round. We provide finite-step termination guarantee for the overall algorithm. Future work includes extending the proposed algorithm to stability analysis of neural-network-controlled uncertain systems.

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