Glueball spectrum in a (1+1)-dimensional model for QCD

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ABSTRACT

We consider (1+1)-dimensional QCD coupled to scalars in the adjoint representation of the gauge group SU(N). This model results from dimensional reduction of the (2+1)-dimensional pure glue theory. In the large-N limit we study the spectrum of glueballs numerically, using the discretized light-cone quantization. We find a discrete spectrum of bound states, with the density of levels growing approximately exponentially with the mass. A few low-lying states are very close to being eigenstates of the parton number, and their masses can be accurately calculated by truncated diagonalizations.

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1. INTRODUCTION

The light-cone quantization supplemented with a regulator in the form of discretized longitudinal momenta is a promising approach to nonperturbative QCD [1,2,3], which may provide tools for calculating the hadron spectrum as well as the interaction cross-sections starting from first principles. It has been successfully used in studies of various field theories in 1+1 dimensions, and some progress has been made in generalizing it to (3+1)-dimensional theories [2,4]. Recently, Dalley and one of the authors applied this method to a new type of models, namely, two-dimensional large-\(N\) QCD coupled to matter in the adjoint representation of SU(\(N\)) [5]. These models are far more complex than the large-\(N\) QCD coupled to matter in the fundamental representation [6–9], but they are the simplest models where one can study some genuine QCD effects. There are two color flux tubes attached to each quantum and in that sense the quanta of the adjoint matter resemble physical gluons. The glueball-like bound states may contain any number of quanta connected into a closed string by the color flux tubes. Pair creation inside the string is not suppressed in the large-\(N\) limit, leading to eigenstates which are mixtures of strings with different numbers of partons. (In the 't Hooft model, due to the absence of transverse gluons, one finds only the quark–antiquark bound states [6].) Therefore, the adjoint matter imitates some transverse gluon effects.

Another way to see this is to dimensionally reduce (2+1)-dimensional pure glue theory, and to note that the zero mode of the transverse gluon field acts as the adjoint matter field coupled to (1+1)-dimensional QCD. Consider the pure glue SU(\(N\)) theory in 2+1 dimensions,

\[
S = -\frac{1}{4g^2} \int \, d^3x \, \text{Tr} \, F_{\mu\nu} F^{\mu\nu}.
\]

(1)

Compactifying one of the spatial dimensions, \(y \sim y + L\), we may ignore the dependence of fields on \(y\) as \(L \to 0\), i.e. \(\partial A^\mu / \partial y = 0\). The action then reduces to

\[
S_{sc} = \int dx^0 dx^1 \, \text{Tr} \left[ \frac{1}{2} D_\alpha \phi D^\alpha \phi - \frac{1}{4g^2} F_{\alpha\beta} F^{\alpha\beta} \right],
\]

(2)

where \(g^2 = g_3^2 / L\), and \(\phi(x^0, x^1) = A_y / g\) is a traceless \(N \times N\) hermitian matrix field, whose covariant derivative is given by \(D_\alpha \phi = \partial_\alpha \phi + i [A_\alpha, \phi]\). Therefore, \(\phi\) represents the remnants
of the transverse gluon degrees of freedom. In order to absorb the logarithmically divergent mass renormalization it is necessary to add a mass term for $\phi$

$$ S_{sc} = \int dx^0 dx^1 \text{Tr} \left[ \frac{1}{2} D_\alpha \phi D^{\alpha} \phi + \frac{1}{2} m_0^2 \phi^2 - \frac{1}{4 g^2} F_{\alpha \beta} F^{\alpha \beta} \right]. \quad (3) $$

The light-cone quantization and the spectrum of the theory (3) were considered in ref. [5]. In this paper we present results of a new numerical diagonalization. We find that holding the renormalized mass fixed results in a convergent and physically reasonable spectrum of glueballs. We compute this spectrum by solving the linear light-cone Schrödinger equation describing the bound states. The density of states grows roughly exponentially with their mass, as expected in a physically interesting gauge theory. We also find that a few low-lying states are very close to being eigenstates of the parton number, while a typical excited state is a complex mixture of states with different parton numbers. These results are similar to the results of our previous work [10] where we performed a detailed numerical study of a similar model, with the adjoint scalar replaced by an adjoint Majorana fermion. For a different approach to the model with the adjoint fermions see ref. [11].

2. LIGHT-CONE QUANTIZATION

In this section we summarize the light-cone quantization of $(1+1)$-dimensional SU($N$) gauge theory coupled to a scalar field in the adjoint representation [5]. Consider the action (3) and choose the light-cone gauge $A_- = 0$. Introducing the light-cone coordinates $x^\pm = (x^0 \pm x^1)/\sqrt{2}$, we obtain

$$ S_{sc} = \int dx^+ dx^- \text{Tr} \left[ \partial_+ \phi \partial_- \phi - \frac{1}{2} m_0^2 \phi^2 + \frac{1}{2 g^2} (\partial_- A_+)^2 + A_+ J_+^i \right], \quad (4) $$

where $J_+^i = i [\phi, \partial_+ \phi]_{ij}$ is the longitudinal momentum current. A similar model, with $\phi$ taken to be a general complex matrix, was considered in ref. [12] in an attempt to solve higher-dimensional pure glue theory.
In the light-cone quantization $x^+$ is treated as the time and the canonical commutation relations are imposed at equal $x^+$:

$$[\phi_{ij}(x^-), \partial_- \phi_{kl} (\tilde{x}^-)] = \frac{1}{2} \delta(x^- - \tilde{x}^-) \delta_{il} \delta_{jk}. \quad (5)$$

The action does not depend on the time derivatives of the gauge potential $A_+$ which, therefore, can be eliminated by its constraint equations. The light-cone momentum and energy, $P^\pm = \int dx^- T^\pm$, are found to be [5]:

$$P^+ = \int dx^- \text{Tr}[(\partial_- \phi)^2],$$
$$P^- = \int dx^- \text{Tr} \left[ \frac{1}{2} m_0^2 \phi^2 - \frac{1}{2} g^2 J^+ \frac{1}{\partial_-^2} J^+ \right]. \quad (6)$$

Our goal is to solve the eigenvalue problem

$$2P^+ P^- |\Phi\rangle = M^2 |\Phi\rangle, \quad (7)$$

where the physical states must satisfy the zero-charge constraint

$$\int dx^- J^+ |\Phi\rangle = 0, \quad (8)$$

arising from integration over the zero-mode of $A_+$. Since $[P^+, P^-] = 0$, $|\Phi\rangle$ is a simultaneous eigenstate of $P^+$ and $P^-$. In practice it is easy to ensure that $|\Phi\rangle$ carries a definite $P^+$, but the subsequent solution of eq. (7) is highly non-trivial.

Introducing the mode expansion

$$\phi_{ij} = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{dk^+}{\sqrt{2k^+}} \left( a_{ij}(k^+) e^{-ik^+ x^-} + a_{ij}^\dagger(k^+) e^{ik^+ x^-} \right), \quad (9)$$

with

$$[a_{ij}(k^+), a_{lk}^\dagger(\tilde{k}^+)] = \delta(k^+ - \tilde{k}^+) \delta_{il} \delta_{jk}, \quad (10)$$
eq. (6) can be written in terms of the oscillators as follows:

\[ P^+ = \int_{0}^{\infty} dk \, k a_{ij}^\dagger(k) a_{ij}(k), \]  

\[ P^- = \frac{1}{2} m^2 \int_{0}^{\infty} \frac{dk}{k} a_{ij}^\dagger(k) a_{ij}(k) \]

\[ + \frac{g^2}{8\pi} \int_{0}^{\infty} dk_1 dk_2 dk_3 dk_4 \frac{dk}{k_1 k_2 k_3 k_4} \left\{ A \delta(k_1 + k_2 - k_3 - k_4) a_{k_j}^\dagger(k_3) a_{j_i}^\dagger(k_4) a_{k_l}^\dagger(k_1) a_{l_i}(k_2) \right. \]

\[ + B \delta(k_1 + k_2 + k_3 - k_4) \left( a_{k_j}^\dagger(k_1) a_{j_i}^\dagger(k_2) a_{k_l}^\dagger(k_3) a_{l_i}(k_4) + a_{k_j}^\dagger(k_4) a_{k_l}^\dagger(k_1) a_{l_i}(k_2) a_{i_j}(k_3) \right) \}, \]

where

\[ A = \frac{(k_2 - k_1)(k_4 - k_3)}{(k_1 + k_2)^2} - \frac{(k_3 + k_1)(k_4 + k_2)}{(k_4 - k_2)^2}, \]

\[ B = \frac{(k_1 + k_4)(k_3 - k_2)}{(k_3 + k_2)^2} + \frac{(k_3 + k_4)(k_1 - k_2)}{(k_1 + k_2)^2}, \]

and we have dropped the superscript plus on \( k_i \) for brevity. The renormalized mass \( m \) is defined by

\[ m^2 = m_0^2 + \frac{g^2 N}{4\pi} \int_{0}^{\infty} \frac{dp}{p} + \frac{g^2 N}{\pi} \int_{0}^{k} \frac{d(k - p)^2}{p}. \]

Note that the second term in the mass renormalization is finite in the principal value sense, while the first term is logarithmically divergent. We will see that the spectrum is finite if the renormalized mass \( m \) is held fixed.

An important simplification in the light-cone quantization is that the oscillator vacuum satisfies

\[ P^+ |0\rangle = 0; \quad P^- |0\rangle = 0. \]

Other Fock states are then constructed by acting with creation operators \( a_{ij}^\dagger \) on the vacuum. The zero-charge condition (8) requires that all the color indices be contracted. Therefore,
we look for eigenstates of eq. (7) in the form

\[ |\Phi(P^+)\rangle = \sum_{j=1}^{\infty} \int_0^{P^+} dk_1 \ldots dk_j \delta \left( \sum_{i=1}^{j} k_i - P^+ \right) \]

\[ f_j(k_1, k_2, \ldots, k_j) N^{-j/2} \text{Tr} \left[ a^\dagger(k_1) \ldots a^\dagger(k_j) \right] |0\rangle , \]

where the wave functions have cyclic symmetry

\[ f_i(k_2, k_3, \ldots, k_i, k_1) = f_i(k_1, k_2, k_3, \ldots, k_i) . \]

This state is trivially an eigenstate of \( P^+ \), and the problem is to ensure that it is an eigenstate of \( P^- \).

The complexity of the coupling to adjoint matter arises mainly from the presence of pair production and pair annihilation terms in \( P^- \). These terms appear in the leading order of the \( 1/N \) expansion, provided that \( g^2N \) is kept fixed in the large-\( N \) limit. On the other hand, the terms in \( P^- \) that take one color singlet into two are suppressed by \( 1/N \), which implies the stability of bound states in the large-\( N \) limit. The glueball wave functions satisfy linear eigenvalue equations [13, 5], which, upon introducing longitudinal momentum fractions \( x_i = k^+_i / P^+ \), can be written as

\[
\frac{M^2 \pi}{g^2 N} f_i(x_1, x_2, \ldots, x_i) = \frac{m^2 \pi}{g^2 N} f_i(x_1, x_2, \ldots, x_i) + \frac{\pi}{4 \sqrt{x_1 x_2}} f_i(x_1, x_2, \ldots, x_i)
\]

\[ + \int \frac{dy}{0} f_i(y, x_1 + x_2 - y, x_3, \ldots, x_i) \frac{(x_2 - x_1)(x_1 + x_2 - 2y)}{(x_1 + x_2)^2} \]

\[ + \int \frac{dy}{0} \frac{(x_1 + y)(x_1 + 2x_2 - y)}{(x_1 - y)^2} \frac{f_i(x_1, x_2, \ldots, x_i) - f_i(y, x_1 + x_2 - y, x_3, \ldots, x_i)}{4 \sqrt{x_1 x_2 y(x_1 + x_2 - y)}} \]

\[ + \int \frac{dy}{0} \frac{f_{i+2}(y, z, x_1 - y - z, x_2, \ldots, x_i)}{4 \sqrt{x_1 y z(x_1 - y - z)}} \left[ \frac{(x_1 + y)(x_1 - y - 2z)}{(x_1 - y)^2} + \frac{(2x_1 - y - z)(y - z)}{(y + z)^2} \right] \]

\[ + \int \frac{dy}{0} \frac{f_{i-2}(x_1 + x_2 + x_3, x_4, \ldots, x_i)}{4 \sqrt{x_1 x_2 x_3(x_1 + x_2 + x_3)}} \left[ \frac{(2x_1 + x_2 + x_3)(x_3 - x_2)}{(x_3 + x_2)^2} + \frac{(x_1 + x_2 + 2x_3)(x_1 - x_3)}{(x_1 + x_2)^2} \right] \]

\( \pm \) cyclic permutations of \( (x_1, x_2, \ldots, x_i) \)
where \( M^2 = 2P^+P^- \) is the bound state (mass)\(^2\). In writing this equation we have used a principal value integral \([8]\)

\[
\int_0^{x_1+x_2} \frac{dy}{(x_1-y)^2} \frac{(x_1+y)(x_1+2x_2-y)}{\sqrt{y(x_1+x_2-y)}} = -\pi. \tag{19}
\]

Note that the only singular term in eq. (18) is well-defined in the principal value sense.

By truncating eq. (18) to the two-body sector [see eq. (23)], we can show using a constant variational wave function that \( M^2 \) is unbounded from below for \( m^2 < 0 \). For \( m^2 \geq 0 \), on the other hand, the \( M^2 \) of every state is positive. Thus, the critical value of \( m^2 \) is zero. We believe that this is the correct value to consider if we want to regard our model as the proper dimensional reduction of the (2+1)–dimensional pure glue QCD. Therefore, we will be interested primarily in the \( m^2 = 0 \) case.

The form of eq. (18) indicates that the interactions conserve the number of partons modulo 2. Therefore, there are two kinds of bound states: those involving mixtures of states with even parton numbers, and those involving mixtures of states with odd parton numbers. From here on, we will call them even and odd glueballs respectively. Furthermore, eq. (18) possesses another \( \mathbb{Z}_2 \) symmetry, \( T \) \([11]\):

\[
f_i(x_1, x_2, \ldots, x_i) = T f_i(x_i, \ldots, x_2, x_1). \tag{20}
\]

The \( \mathbb{Z}_2 \) quantum number \( T \) has two possible values, 1 and \(-1\). In terms of the original field, \( T : \phi_{ij} \rightarrow \phi_{ji} \), which obviously leaves \( P^\pm \) invariant \([11]\). Physically, every bound state can be thought of as a superposition of oriented closed strings, and the quantum number \( T \) describes the transformation property under a reversal of orientation.

3. THE DISCRETIZED APPROXIMATION AND NUMERICAL RESULTS

The system of equations (18) involves an infinite number of multivariable functions. It seems very difficult, if not impossible, to solve this system of equations analytically. Therefore, here we follow the same strategy as in \([10]\) and replace the continuum equations (18) by a sequence of discretized approximations, such that the eigenvalues of the discretized
problems eventually converge to the eigenvalues of (18). In the light-cone quantization, a simple discretized approximation is obtained by replacing the continuous momentum fractions $x$ by a discrete set $n/K$, where $n$ are odd positive integers, and the positive integer $K$ is sent to infinity as the cut-off is removed [3,2]. The restriction to odd integers corresponds to the choice of antiperiodic boundary conditions in the discretized light-cone quantization, and leads to the best convergence towards continuum limit. Thus, the functions $f_i(x_1, x_2, \ldots, x_i)$ are replaced by

$$g_i(n_1, n_2, \ldots, n_i) = K^{(1-i)/2} f_i \left( \frac{n_1}{K}, \frac{n_2}{K}, \ldots, \frac{n_i}{K} \right)$$

with $\sum_{j=1}^i n_j = K$, and

$$\int_0^1 dx \rightarrow \frac{2}{K} \sum_{\text{odd } n>0}^K .$$

For the even glueballs, $g_i(n_1, n_2, \ldots, n_i)$ are normalized as

$$\sum_{\text{even } i} \sum_{\text{odd } n_1>0} \cdots \sum_{\text{odd } n_{i-1}>0} \left| g_i(n_1, n_2, \ldots, n_i-1, K - \sum_{j=1}^{i-1} n_j) \right|^2 = 1 ,$$

(21)

and similarly for the odd glueballs.

The number of independent components of $g_i(n_1, n_2, \ldots, n_i)$ is equal to the number of partitions of $K$ into positive odd integers, modulo cyclic permutations. Finding all such partitions is a combinatorial problem that is easily solved with a computer program. If we regard all the independent $g_i(n_1, n_2, \ldots, n_i)$ as components of a vector, then the discretized eigenvalue problem (18) is equivalent to diagonalizing a matrix

$$M^2 \pi \frac{\pi^2 N}{g^2} = K(xV + T) ,$$

(22)

where $x = \frac{\pi m^2}{g^2 N}$ is the dimensionless parameter. The main difficulty in this approach is that the number of states increases rapidly with $K$. For example, there are 765, 1169, 1810, 2786, 4340, 6712 states for $K = 20, 21, 22, 23, 24, 25$, respectively. In actual calculations it is advantageous to consider separately the even and odd sectors under $T$, and introduce the
linear combinations

\[ g_i^{\pm}(n_1, n_2, \ldots, n_i) = \frac{1}{\sqrt{2}} \left( g_i(n_1, n_2, \ldots, n_i) \pm g_i(n_i, n_{i-1}, \ldots, n_1) \right). \]

The entries of \( V \) and \( T \) can be read off from eqs. (18). For example, for \( K = 6 \) there are 4 partitions which correspond to the \( \mathbb{Z}_2 \)-even sector: \( \{(1, 1, 1, 1, 1), (3, 1, 1, 1), (5, 1), (3, 3)\} \).

The matrix to be diagonalized in this case reads:

\[
K(xV + T) = \begin{pmatrix}
36x + 9\pi & 0 & 0 & 0 \\
0 & 20x + (3 + \sqrt{3})\pi & 0 & 0 \\
0 & 0 & \frac{36x}{5} + \frac{16 + 3\pi}{\sqrt{5}} & -\frac{\sqrt{512}}{5} \\
0 & 0 & -\frac{\sqrt{512}}{5} & 4x + \frac{32}{\sqrt{5}} + \pi
\end{pmatrix}.
\]

For this simple example there are no partitions corresponding to the \( \mathbb{Z}_2 \)-odd sector. For larger values of \( K \), however, the total number of partitions splits approximately equally between the even and odd sectors, allowing us to reach higher values of \( K \). For \( K = 24 \) there are 2358 components in the \( \mathbb{Z}_2 \)-even sector and 1982 components in the \( \mathbb{Z}_2 \)-odd sector. For \( K = 25 \) there are 3544 and 3168 even and odd components, respectively.

A good numerical procedure is to calculate the spectrum for a fixed \( x \) and a range of values of \( K \), and then to extrapolate the results to infinite \( K \), the continuum limit. We will also assume that some bulk properties of the spectrum can be estimated from the results at a fixed large \( K \). We will be most interested in \( x = 0 \) which corresponds to the limit of massless quanta. Note that, with our choice of the regulator, we observe only even (odd) glueballs for an even (odd) \( K \). We will present here the results for our biggest diagonalization in each sector separately: for even \( K = 24 \) and for odd \( K = 25 \). Our analysis is very similar to the one performed in ref. [10] for the model with the adjoint fermions.

In fig. 1 we show the spectrum of even glueballs for \( x = 0 \) and \( K = 24 \), with the mass plotted vs. the expectation value of the number of partons, \( n \). The density of states obviously increases rapidly with the mass, and almost all the states lie within a band bounded by two \( \langle n \rangle \sim M \) lines. One interesting feature of our results, already noted in ref. [5] for smaller \( K \), is that for a few low-lying eigenstates the wave functions are strongly peaked on states with a definite number of partons. For example, for \( K = 24 \) the ground state has probability 0.99899 to consist of two partons, and the first excited state has probability 0.99734 to consist
of four partons. For the ground state, it is a good approximation, therefore, to truncate eq. (18) to the two-body sector described by the wave function $\phi(x) = f_2(x, 1 - x)$. From eq. (17) we have $\phi(x) = \phi(1 - x)$. Thus, we obtain

$$M^2 \phi(x) = m^2 \frac{\phi(x)}{x(1 - x)} + \frac{g^2 N}{2} \frac{\phi(x)}{\sqrt{x(1 - x)}} + \frac{g^2 N}{2\pi} \int_0^1 dy \frac{\phi(x) - \phi(y)}{(x - y)^2} \frac{(x + y)(2 - x - y)}{\sqrt{x(1 - x)y(1 - y)}}.$$

This is simply the bound state equation for the theory with scalar quarks in the fundamental representation of SU($N$) [8,9], with $g^2 \to 2g^2$. The interaction strength is doubled because now there are two flux tubes connecting the pair of partons. The lowest eigenvalue of eq. (23), $M^2 = 4.76g^2N/\pi$, provides a good upper bound on the lowest eigenvalue of eq. (18).

As the excitation number increases, however, the wave functions typically become quantum superpositions of states with different parton numbers. The physical picture is that a typical excited state contains some number of virtual pairs, and our data supports this expectation. In order to quantify this effect, we will call a state pure if it has probability $> 0.9$ to be in one of the number sectors. Table 1 shows the total number of states and the number of pure states in each mass interval of fig. 1. We also show the expectation value of the number of partons averaged over all states in each mass interval. Evidently, a few low-lying states are pure, while there are almost no pure states among the high excitations. In fig. 2 we plot the spectrum of the odd glueballs for $x = 0$ and $K = 25$, and in table 2 we quantify their purity. The lightest odd glueball contains, to a good approximation, only three partons, and the first excited state is almost purely a five-parton state. If we increase the mass of the quantum, the pair creation becomes somewhat suppressed. We have performed the diagonalization for $x = 1$ and found, indeed, that the number of pure states increases. While the distributions of states for $x = 0$ shown in figs. 1 and 2 are almost uniform within a band, for $x = 1$ there is an increase in the number of states with the average parton number near 2, 4, 6, 8, . . . for $K = 24$, and a similar increase in the number of states with the average parton number near 3, 5, 7, 9, . . . for $K = 25$.

A striking property of figs. 1 and 2 is the rapid growth of the density of states with increasing mass. In fig. 3 we plot the logarithm of the number of odd glueballs vs. the mass for the data in table 2. For a certain range of masses the graph is approximately linear. The deviation from linearity for large enough mass is clearly due to the effects of the cut-off.
Our results indicate that the density of states grows roughly exponentially with the mass, exhibiting the Hagedorn behavior

$$\rho(m) \sim m^\alpha e^{\beta m}.$$  

Thus, although the mass spectrum is discrete, it rapidly becomes virtually indistinguishable from a continuum. From our data we estimate that the inverse Hagedorn temperature is

$$\beta \approx (0.65 - 0.7) \sqrt{\pi/(g^2N)}.$$  

Another physical effect that is pronounced in our results (tables 1 and 2) is that the mass increases roughly linearly with the average number of partons. In fig. 4 we plot these results for $x = 0$ and $K = 24$ (table 1). A heuristic explanation of this effect was suggested in [10] using the result of ref. [14] that the ground state energy of a system of $n$ non-relativistic particles connected into a closed string by harmonic springs, which should be identified with $M^2$, behaves as $\sim n^2$ for sufficiently large $n$.

Since the low-lying states are very pure, they can be well approximated by truncating the diagonalization to a single parton number sector. For instance, for $x = 0$ and $K = 24$ the ground state has probability 0.999999 to consist of 2, 4 or 6 partons. Extrapolating to infinite $K$, we find 0.999996, and therefore this truncation is highly reliable. In table 3 we compare the full and truncated calculations. We have performed the truncated diagonalizations up to $K = 40$, and extrapolating these results to infinite $K$, we find the upper bounds $M^2 \approx 4.33 g^2N/\pi$ for $x = 0$ and $M^2 \approx 13.43 g^2N/\pi$ for $x = 1$. These are extremely close to the extrapolations from exact diagonalizations. This shows that, by judiciously truncating the space of states, certain eigenvalues can be determined to a good accuracy with relatively small diagonalizations. Similar approximations work well for the odd glueballs. For $x = 0$ and $K = 25$ the lightest state has probability 0.999992 to consist of three or five partons. Extrapolating it to infinite $K$, we find 0.999994. The advantage of this truncation is that we can access higher value of $K$ (up to 51) than in the full diagonalization, and extrapolate more reliably. We find that the lowest eigenvalue extrapolated in this fashion is $M^2 \approx 11.95 g^2N/\pi$ for $x = 0$, and $M^2 \approx 30.87 g^2N/\pi$ for $x = 1$. Good agreement of these values with extrapolations from exact diagonalizations gives us some confidence that our methods are consistent. We have used the Bulirsch-Stoer algorithm which has proved to be particularly efficient for extrapolating short series [15]. In fig. 5 we show the $M^2$ of the
ground state for $x = 0$ as a function of $1/K$. For $K \leq 24$ we show the results of the full diagonalization, while for $26 \leq K \leq 40$ we use the results of truncated diagonalization in the $2-$, $4-$ and $6-$bit sector.

4. DISCUSSION

Our calculations suggest that (1+1)-dimensional large-$N$ QCD coupled to adjoint scalar matter captures some of the physical features of the higher-dimensional pure glue gauge theories. This model may also serve as a good test of the light-cone quantization methods that are promising to become a useful tool for studying the non-perturbative structure of the strong interactions. We mention below some interesting questions for further study.

1. Even though the model is (1+1)-dimensional, we have observed a rich structure of glueball states. In fact, it is much richer than in the models coupled to matter in the fundamental representation of SU($N$), which have only one state per unit mass-squared [7, 8]. It would be interesting to gain further insight into the Hagedorn behavior, eq. (24). Could there be exact formulae for the constants $\alpha$ and $\beta$? Their determination is relevant to the properties of the deconfining phase transition. Such a transition has been recently studied in the model with adjoint fermions [11, 16].

2. We found that some low-lying states are exceedingly close to being eigenstates of the parton number. However, they fail to be exact eigenstates. We feel that this deserves a deeper understanding. If this property holds true in higher-dimensional gauge theories, it could lead to an enormously simplified picture of the low-lying states.

3. The glueball states can be simply pictured as closed strings that cannot split or join in the large-$N$ limit. Is there a continuum string description of our model? A study of this question seems to be a logical next step in the program of ref. [17], where a string representation of the pure glue theory was found.

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REFERENCES

1. For a review, see S. J. Brodsky and H.-C. Pauli, “Light-Cone Quantization of Quantum Chromodynamics,” Lectures at the 30th Schladming Winter School in Particle Physics, SLAC-PUB-5558 (1991).

2. H.-C. Pauli and S. Brodsky, Phys. Rev. D32 (1985) 1993, 2001; K. Hornbostel, S. Brodsky and H.-C. Pauli, Phys. Rev. D41 (1990) 3814; for a good review, see K. Hornbostel, Ph.D. thesis, SLAC report No. 333 (1988).

3. C. B. Thorn, Phys. Lett. 70B (1977) 85; Phys. Rev. D17 (1978) 1073; D19 (1979) 639; D20 (1979) 1435.

4. A. C. Tang, S. J. Brodsky, and H. C. Pauli, Phys. Rev. D44 (1991) 1842; D. Mustaki, S. Pinsky, J. Shigemitsu, and K. Wilson, Phys. Rev. D43 (1991) 3411; A. Harindranath and R. J. Perry, Phys. Rev. D43 (1991) 492; A. Harindranath, R. J. Perry and J. Shigemitsu, Phys. Rev. D46 (1992) 4580.

5. S. Dalley and I. R. Klebanov, Phys. Rev. D47 (1993) 2517.

6. G. ’t Hooft, Nucl. Phys. B72 (1974) 461.

7. G. ’t Hooft, Nucl. Phys. B75 (1974) 461.

8. S.-S. Shei and H.-S. Tsao, Nucl. Phys. B141 (1978) 445.

9. W. Bardeen and R. Pearson, Phys. Rev. D14 (1976) 547; M. B. Halpern and P. Senjanović, Phys. Rev. D15 (1977) 1655; T. N. Tomaras, Nucl. Phys. B163 (1980) 79.

10. G. Bhanot, K. Demeterfi and I. R. Klebanov, “(1+1)-dimensional large- N QCD coupled to adjoint fermions,” PUPT-1413, IASSNS-HEP-93/42, hep-th/9307111 (to appear in Phys. Rev. D).

11. D. Kutasov, “Two Dimensional QCD coupled to Adjoint Matter and String Theory,” EFI-93-30, hep-th/9306013.
12. W. A. Bardeen, R. B. Pearson and E. Rabinovici, Phys. Rev. D21 (1980) 1037.

13. I. R. Klebanov and L. Susskind, Nucl. Phys. B309 (1988) 175.

14. M. Karliner, I. R. Klebanov and L. Susskind, Int. Jour. Mod. Phys. A3 (1988) 1981.

15. R. Bulirsch and J. Stoer, Numer. Math. 6 (1964) 413;
   M. Henkel and G. Schütz, J. Phys. A 21 (1988) 2617.

16. I. Kogan, “Hot Gauge Theories and $\mathbb{Z}_N$ Phases,” PUPT-1415.

17. D. J. Gross, Nucl. Phys. B400 (1993) 161;
   D. J. Gross and W. Taylor, Nucl. Phys. B400 (1993) 181; B403 (1993) 395.
Table 1. Numerical data for $K = 24$, $x = 0$ shown in fig. 1; the 3.0 bin includes all states whose masses are $1.5 - 3.0$; etc.

| $M$     | 3.0 | 4.5 | 6.0 | 7.5 | 9.0 | 10.5 | 12.0 | 13.5 | 15.0 | 16.5 |
|---------|-----|-----|-----|-----|-----|------|------|------|------|------|
| number of states | 1   | 1   | 1   | 2   | 7   | 26   | 103  | 262  | 598  | 1013 |
| number of pure states | 1   | 1   | 1   | 1   | 2   | 3    | 4    | 3    | 1    | 1    |
| average parton number | 2.00 | 4.00 | 2.17 | 4.98 | 4.64 | 4.96 | 6.01 | 7.36 | 8.54 | 9.91 |

Table 2. Numerical data for $K = 25$, $x = 0$ shown in fig. 2.

| $M$     | 4.5 | 6.0 | 7.5 | 9.0 | 10.5 | 12.0 | 13.5 | 15.0 | 16.5 | 18.0 |
|---------|-----|-----|-----|-----|------|------|------|------|------|------|
| number of states | 1   | 1   | 1   | 7   | 26   | 99   | 277  | 651  | 1270 | 1757 |
| number of pure states | 1   | 1   | 1   | 3   | 5    | 5    | 1    | 0    | 0    | 0    |
| average parton number | 3.00 | 4.99 | 3.09 | 4.52 | 5.12 | 5.96 | 7.18 | 8.39 | 9.69 | 11.12 |
Table 3. Numerical values of the ground state mass for \( x = 0 \) and \( x = 1 \) for even \( K \).

| \( K \) | \( M^2 \) (full) | \( M^2 \) (2+4+6-bit) | \( M^2 \) (full) | \( M^2 \) (2+4+6-bit) |
|-------|----------------|-----------------------|----------------|-----------------------|
| 12    | 4.1064         | 4.1064                | 10.8712        | 10.8712               |
| 14    | 4.1476         | 4.1476                | 11.0652        | 11.0652               |
| 16    | 4.1789         | 4.1789                | 11.2219        | 11.2219               |
| 18    | 4.2034         | 4.2034                | 11.3520        | 11.3520               |
| 20    | 4.2230         | 4.2230                | 11.4624        | 11.4624               |
| 22    | 4.2388         | 4.2388                | 11.5577        | 11.5577               |
| 24    | 4.2518         | 4.2518                | 11.6411        | 11.6411               |
| \( \infty \) | 4.3            | 4.33                  | 13.4           | 13.43                 |

Table 4. Numerical values of the ground state mass for \( x = 0 \) and \( x = 1 \) for odd \( K \).

| \( K \) | \( M^2 \) (full) | \( M^2 \) (3+5-bit) | \( M^2 \) (full) | \( M^2 \) (3+5-bit) |
|-------|----------------|---------------------|----------------|---------------------|
| 15    | 10.5001        | 10.5003              | 26.0496        | 26.0497             |
| 17    | 10.6569        | 10.6572              | 26.5323        | 26.5326             |
| 19    | 10.7853        | 10.7858              | 26.9358        | 26.9362             |
| 21    | 10.8925        | 10.8932              | 27.2797        | 27.2802             |
| 23    | 10.9835        | 10.9843              | 27.5775        | 27.5781             |
| 25    | 11.0616        | 11.0626              | 27.8387        | 27.8393             |
| \( \infty \) | 11.9           | 11.95                | 30.8           | 30.87                |
FIGURE CAPTIONS

Fig.1. The spectrum of even glueballs for $K = 24$, $x = 0$, $M < 18$; mass $M$ is measured in units of $\sqrt{g^2N/\pi}$ and plotted vs. the expectation value of the parton number.

Fig.2. The spectrum of odd glueballs for $K = 25$, $x = 0$, $M < 18$.

Fig.3. Logarithm of the density of states and a linear fit for $K = 25$ and $x = 0$.

Fig.4. Average number of partons as a function of mass for $K = 24$ and $x = 0$.

Fig.5. Extrapolation towards infinite $K$ of the ground state at $x = 0$; $M^2\pi/g^2N$ is plotted versus $1/K$. 
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