Exact Kink Solitons in the Presence of Diffusion, Dispersion, and Polynomial Nonlinearity

E.P. Raposo and D. Bazeia

Lyman Laboratory of Physics
Harvard University, Cambridge, Massachusetts 02138

Center for Theoretical Physics
Laboratory for Nuclear Science and Department of Physics
Massachusetts Institute of Technology, Cambridge, Massachusetts 02139-4307

(MIT-CTP-2742, May 1998)

Abstract

We describe exact kink soliton solutions to nonlinear partial differential equations in the generic form $u_t + P(u)u_x + \nu u_{xx} + \delta u_{xxx} = A(u)$, with polynomial functions $P(u)$ and $A(u)$ of $u = u(x, t)$, whose generality allows the identification with a number of relevant equations in physics. We emphasize the study of chirality of the solutions, and its relation with diffusion, dispersion, and nonlinear effects, as well as its dependence on the parity of the polynomials $P(u)$ and $A(u)$ with respect to the discrete symmetry $u \rightarrow -u$. We analyze two types of kink soliton solutions, which are also solutions to $1 + 1$ dimensional $\phi^4$ and $\phi^6$ field theories.

PACS numbers: 03.40.-t, 52.35.Fp, 63.20.Ry

Keywords: Nonlinear partial differential equations, solitons

---

1This work is supported in part by funds provided by the U. S. Department of Energy (D.O.E.) under cooperative research agreement DE-FC02-94ER40818, and by Conselho Nacional de Desenvolvimento Científico e Tecnológico, CNPq, Brazil.

2Corresponding author. Present address: Departamento de Física, Universidade Federal de Pernambuco, 50670-901 Recife, Pernambuco, Brazil. Telephone: 55 81 271.0111. Fax: 55 81 271.0359. E-mail: raposo@cmt.harvard.edu or ernesto@lftc.ufpe.br.

3On leave from Departamento de Física, Universidade Federal da Paraíba, Caixa Postal 5008, 58051-970 João Pessoa, Paraíba, Brazil.
I Introduction

Nonlinear differential equations are known [1] to describe a wide variety of phenomena not only in physics, in which applications extend over magnetofluid dynamics, water surface gravity waves, electromagnetic radiation reactions, and ion acoustic waves in plasmas, just to name a few, but also in biology and chemistry, among several other fields. In spite of the increasing development of mathematical techniques and concepts to solve nonlinear equations, exact solutions seem not to be the rule and numerical methods have been the most common approach to study their properties [2].

Since the seminal work of Korteweg and de Vries (KdV) [3], in which a third-order nonlinear equation was studied to explain the shallow-water solitary wave experiments by Russel [4], solitary-wave or soliton solutions have been found in a number of nonlinear differential equations [1], with phenomenologies related to liquid crystals, dynamics of growing interfaces and domain walls, electromagnetism, nonlinear optics, acoustics, and elasticity, among others. These localized non-dispersive travelling waves can be of several distinct types [5], such as kink, pulse, breather, envelope, and dark solitons, all of them presenting the property that their shapes and velocities are preserved along propagation, and even upon collision with other solitary waves. In fact, they result from a precise balance among the competing elements that define the equation, namely the tendency of spreading due to the presence of a dispersive term and the action of nonlinear terms, which in general favour large amplitude disturbances and velocities that assure the stability of the travelling waves with respect to small distortions in form. In some special cases, solitary waves do not support arbitrary speeds and can propagate only with definite velocities determined from the parameters of the nonlinear equation. These so-called chiral solitons have received much attention lately due to their recent associations with the nonlinear Schrödinger equation and the fractional quantum Hall effect [6, 7, 8]. Moreover, we have also recently pointed [9] to the presence of chiral kink soliton solutions in generalized KdV-Burgers-Huxley (gKdVBH) equations in the form

\[ u_t + f_x + g_{xx} - \delta u_{xxx} = h(u), \]  

where \( f, g, \) and \( h \) are smooth functions in \( u \), and \( u_t \) (\( u_x \)) stands for the partial derivative of \( u(x, t) \) with respect to time (position). The above equation combines dispersion, controlled by the real parameter \( \delta \), nonlinearity, described by the functions \( f(u), g(u), \) and \( h(u) \), and diffusion related to the second-order space derivative.

This work is inspired in our previous study [9], although here we follow a different approach. We consider exact kink solutions to nonlinear differential equations of the generic form

\[ u_t + P(u) u_x + \nu u_{xx} + \delta u_{xxx} = A(u), \]  

with polynomial functions defined as

\[ P(u) = \sum_{i=0}^{N_p} p_i u^i, \]
and
\[ A(u) = \sum_{i=0}^{N_a} a_i u^i . \]  

The general form of Eq. (2) allows the identification of several interesting cases. For instance, by considering propagating waves \( u(x,t) = u(x-ct) = u(y) \), with velocity \( c \), it is easy to see that the gKdVBH equation is recovered from Eq. (2) for \( df/du = P(u) \), \( g(u) = g_0 + \nu u \), \( \delta = -\tilde{\delta} \), and \( h(u) = A(u) \). Furthermore, the standard KdVB equation corresponds to identifying \( P(u) = \lambda u \), \( \delta = -\bar{\delta} \), and \( A(u) = 0 \), with the particular case \( \nu = 0 \) accounting respectively for the standard and modified KdV equations. On the other hand, the Burgers-Huxley equation represents the situation in which \( P(u) = \bar{\lambda} u \), \( \delta = 0 \), and \( A(u) = h(u) \), with the case \( h(u) = 0 \) corresponding to the standard Burgers equation, whose applications include nonlinear heat diffusion, shock waves, viscous effects in gas dynamics, and an important connection with the deterministic KPZ equation in one spatial dimension, known to provide the evolution of the profile of a growing interface or a domain wall of general nature.

Other examples include the generalized Boussinesq equation,
\[ u_{tt} - c_0^2 u_{xx} - p(u^2)_{xx} - q(u^3)_{xx} - \bar{h} u_{xxxx} = 0 , \]  
which by a trivial integration in \( y \) for travelling solutions \( u(x,t) = u(x-ct) = u(y) \) is related to Eq. (2) through the identifications \( P(u) = -c_0^2 - 2pu - 3qu^2 \), \( \nu = 0 \), \( \delta = -\bar{h} \), \( A(u) = a_0 \), where \( c_0 \) is the speed of the sound. The standard and modified Boussinesq equations correspond respectively to the cases \( q = 0 \) and \( p = 0 \). They were first described in the context of the theory of long water waves, but also present innumerable applications in electromagnetism, plasma physics, elasticity, and other fields. Another generic equation commonly found in physics is
\[ u_{tt} - \alpha^2 u_{xx} + \frac{dV}{du} = 0 , \]  
which can also be compared to Eq. (2) by considering travelling waves with \( y = x - \tilde{c} t \) and identifying \( P(u) = c \), \( \nu = \tilde{c}^2 - \alpha^2 \), \( \delta = 0 \), and \( A(u) = -(dV/du) \). As examples of Eq. (6), we recall the Klein-Gordon equation of field theory in which \( V(u) = \beta^2 u^2 \), also applied to describe standard vibrations for a displacement \( u \) with presence of an additional restoring force proportional to \( u \), and the sine-Gordon equation corresponding to \( V(u) = \beta^2 (1 - \cos u) \), which embodies a wide range of applications including Josephson junctions in superconductors, dislocations in crystals, waves in ferromagnetic materials, laser pulses in two state media, and geometry of surfaces, just to name a few. Also, the standard nonlinear Schrödinger equation,
\[ iu_t + \mu u_{xx} = \frac{dV}{d\rho} u , \]  

3
corresponds to formally assigning for the travelling wave \( u(x - \bar{c} t) \) the values \( P(u) = c - i\bar{c}, \nu = \mu, \delta = 0, \) and \( A(u) = (dV/d\rho) u, \) where \( V = V(u^* u) = V(\rho) \) is commonly expressed as a quadratic or cubic potential in the charge density \( \rho. \)

Regarding the polynomial \( A(u) \) in Eq. (2), it first appeared as a generalization [5] of the study of monoatomic chains of equal masses interacting via a number of nearest-neighbor realistic potentials such as the Toda, Morse, and Lennard-Jones potentials [11]. In the continuum spatial limit, the longitudinal displacement of the particle \( n \) from its equilibrium position, \( u_n(t), \) is replaced by an analytic function \( u(x, t), \) and the series expansion of the interaction terms gives rise to the spatial derivatives of high orders. In this context, \( A(u) \) thus represents an on-site local interaction potential in the differential equation of motion that governs the system [5].

In the following we generalize the procedure applied in Ref. [12] to some nonlinear equations, in order to investigate Eq. (2). The extension to equations with higher-order derivative terms, such as the fifth-order KdV-type equations [4], is straightforward and will not be considered here. Our emphasis will be on the study of chirality of exact kink solutions to the general equation, and analysis of its relation with combined diffusion, dispersion, and nonlinear effects, as well as its dependence on the parity of the polynomials \( P(u) \) and \( A(u) \) with respect to the discrete symmetry \( u \rightarrow -u. \) In Sec. II we analyze the case in which kink solutions are of the type \( u(y) = a \tanh(\lambda ay), \) whereas in Sec. III solutions in the form \( u(y) = \{(a^2/2)[1 + \tanh(\lambda a^2 y)]\}^{1/2} \) are considered. Finally, comments and conclusions are presented in Sec. IV.

### II Exact Kink Solutions of Type \( \phi^4 \)

We start by considering travelling solutions \( u(x, t) = u(x - ct) = u(y) \) to the general third-order nonlinear differential equation in the presence of a local polynomial potential \( A(u), \) Eq. (2),

\[
[-c + P(u)] \frac{du}{dy} + \nu \frac{d^2u}{dy^2} + \delta \frac{d^3u}{dy^3} = A(u). \tag{8}
\]

From Eqs. (3) and (4), the orders of the polynomials, \( N_p \) and \( N_a, \) are not arbitrary, but are instead strongly related to the characteristics of the solutions. For instance, if \( N_p = N_a - 2 \) one can search for exact kink soliton solutions in the form,

\[
u(y) = a \tanh(\lambda ay), \tag{9}\]

since the property,

\[
\frac{du}{dy} = \lambda (a^2 - u^2) , \tag{10}\]

leads to the general result,

\[
O(\frac{d^n u}{dy^n}) = u^{n+1} . \tag{11}\]

4
Indeed, from Eq. (11) one sees that if \( N_a > N_p + 2 \) no solutions of the form presented in Eq. (9) can be found to Eq. (8). We have shown [4, 13] that Eq. (9) also corresponds to kink solutions \( \phi = \phi(y) \) in the context of the relativistic \( \phi^4 \) field theory in 1+1 dimensions, which presents potential

\[
V(\phi) = \frac{1}{2} \lambda^2 (\phi^2 - a^2)^2 ,
\]

that develops spontaneous symmetry breaking of the discrete \( Z_2 \) symmetry \( \phi \rightarrow -\phi \). Here we mention that the kink solutions resembles the two asymmetric and degenerate vacua of this \( \phi^4 \) system.

The coefficients \( \{p_i\} \) and \( \{a_i\} \) in Eqs. (3) and (4) are not arbitrary, but instead must obey a series of relations in order to allow Eq. (9) to be solution of Eq. (8). In fact, this is actually expected to occur to solitary-wave solutions, which can only exist for a precise balance of the diffusion, dispersion, and nonlinear effects. One can see that by explicitly substituting Eqs. (3), (4), and (9) in Eq. (8), to obtain

\[
a_0 = -c + p_0 - 2\delta \lambda^2 a^2 , \quad (13)
\]

\[
a_1 = p_1 - 2\nu \lambda , \quad (14)
\]

\[
a_2 = p_2 + 8\delta \lambda^2 + \frac{(c-p_0)}{a^2} , \quad (15)
\]

\[
a_3 = -p_1 + 2\nu \lambda , \quad (16)
\]

\[
a_4 = -p_2 - 6\delta \lambda^2 . \quad (17)
\]

Although this technique can be easily generalized to any polynomial orders \( N_p \) and \( N_a \), provided that in the case of solution given by Eq. (9) they obey the relation \( N_p = N_a - 2 \), we have restricted ourselves in Eqs. (13)-(17) to situations of physical interest, representing quadratic nonlinearity in the first-order derivative term, \( N_p = 2 \), as pointed from the examples listed in Sec. I. Indeed, considering the general \( N_p > 2 \), \( N_a > 4 \) case only generates relations among highest-order coefficients \( \{p_i\} \) and \( \{a_i\} \) not involving the parameters of interest \( \nu, \delta, \) and \( c \). For instance, in this case Eqs. (16) and (17) would respectively read

\[
a_3 = -p_1 + a^2 p_3 + 2\nu \lambda , \quad (18)
\]

\[
a_4 = -p_2 + a^2 p_4 - 6\delta \lambda^2 , \quad (19)
\]

and new relations involving higher-order coefficients would emerge:

\[
a_i = -p_{i-2} + a^2 p_i , \quad n > 4 . \quad (20)
\]

Notice that Eq. (20) represents matches exclusively between coefficients of the polynomials \( P(u) \) and \( A(u) \) necessary in order to allow Eq. (9) as solutions to Eq. (8). No extra physical
information concerning diffusion or dispersion effects, or even chirality (see below) is gained from Eq. (20), contrarily to what happens to Eqs. (13)-(17), as we discuss below. Moreover, Eqs. (13)-(20) also indicate that the coefficients are not totally independent, but instead must obey a set of relations, which in the particular case of solutions given by Eq. (9) are given by, in the case of $N_a \text{ even}$,

$$\sum_{i=0}^{N_a/2-1} a^{2i}a_{2i+1} = 0 ,$$

and

$$\sum_{i=0}^{N_a/2} a^{2i}a_{2i} = 0 .$$

Consequently, setting the values of the $N_a/2$ ($N_a/2 - 1$) even (odd) $a_i$’s automatically determines the values of the remaining even (odd) coefficients.

We notice from Eqs. (13)-(20) that the diffusion and dispersion parameters, respectively $\nu$ and $\delta$, are related to definite and distinct parities of the coefficients. For instance, $\nu$ is associated only with odd coefficients, $p_1, p_3, a_1, a_3$, whereas $\delta$ is related to the even ones, $p_0, p_2, a_0, a_2, a_4$. No higher-order coefficients depend on $\nu$ or $\delta$. Indeed, this is consequence of properties of the solution considered, Eqs. (9) and (10), namely that each derivative multiplies the original function by a factor of order $u$, thus changing its parity with respect to the symmetry $u \to -u$, as one can also see from Eq. (11). Another important feature is that the velocity $c$ of the kink solitons also depends only on even coefficients up to quadratic order, i.e., $p_0, p_2, a_0, a_2$. The fact that $c$ is not arbitrary, being instead determined by the parameters that define the nonlinear differential equation, indicates that solutions given in terms of Eq. (9) are actually chiral solitons. For instance, Eq. (13) can be used to determine the velocity of the localized travelling solutions,

$$c = p_0 - \frac{1}{\lambda a^2} (a_0 + 2\delta\lambda^3 a^4) .$$

We observe from the above equation that $p_0$ plays the role of the velocity of the reference frame, as expected since it represents the $u$-independent term in the polynomial $P(u)$. Furthermore, Eq. (23) also tells that the diffusion term associated with the parameter $\nu$ does not play any role in fixing the velocity of the solitons, thus not contributing to their chirality. This is a consequence of the distinct parities of the coefficients associated with $c$ and $\nu$, as commented above. Indeed, by considering the simplest case in which $p_0 = a_0 = 0$, we point to the relevance of the presence of a dispersive media ($\delta \neq 0$) in order to support chiral kink solitons of the type described by Eq. (9). Nevertheless, this does not seem to be a property shared by all types of kink solitons, as we comment on Sec. III.

From the above discussions we also notice the remarkable fact that distinct choices of the parameters can lead to solitary-wave solutions with the same shape, Eq. (9), even if
they do not necessarily have the same velocity. Indeed, this feature was first verified [11] with the finding that Toda, Morse, and Lennard-Jones solitons arising from distinct nonlinear differential equations are very nearly alike, a result then attributed to the similarity in the shapes of their respective potential walls.

Let us now consider some specific examples. The connection with the gKdVH equation, Eq. (1) with the ansatz \( u(x, t) = u(x - ct) = u(y) \), is easily established from the following identifications (see also Introduction):

\[
P(u) = \frac{df(u)}{du}, \quad \nu u = g(u) - g_0, \quad \delta = -\bar{\delta}, \quad A(u) = h(u),
\]

where \( g_0 \) is some unimportant constant. To illustrate with a previously reported [9] example of soliton solutions related to Eq. (1) and with form given by Eq. (9), let us consider the case in which

\[
P(u) = p_0 + p_2 u^2, \quad \nu = 0, \quad A(u) = a_0 + a_2 u^2,
\]

that corresponds to the gKdVH equation with functions

\[
f(u) = f_0 + p_0 u + \frac{p_2 u^3}{3}, \quad g(u) = -g_0, \quad h(u) = a_0 + a_2 u^2.
\]

Solving Eqs. (13)-(17) along with the functions expressed by Eqs. (28)-(30) leads to chiral solitons, Eq. (9), with velocity

\[
c = p_0 - \frac{a_0 p_2}{3 a_2} + \text{sgn}(\lambda) a_2 \sqrt{-\frac{6\delta}{p_2}}.
\]

The result reported in Ref. [9] is restored by considering \( p_0 = -2\alpha^2\beta^2, p_2 = 6\beta^2, a_0 = \alpha^2\beta, \) and \( a_2 = -\beta \). In this case the velocity of the chiral solitons are simply given by \( (\delta < 0) \)

\[
c = \text{sgn}(\lambda) \sqrt{-\delta}.
\]
III Exact Kink Solutions of Type $\phi^6$

We now search for exact kink soliton solutions to Eq. (2) in the form

$$u(y) = \left\{ \frac{a_0^2}{2} \left[ 1 + \tanh(\lambda a^2 y) \right] \right\}^{1/2}.$$  \hspace{1cm} (36)

As we have reported in Refs. [9, 13], differently from the solutions investigated in Sec. II, Eq. (35) corresponds to kink solitons of the $\phi^6$ field theoretical system in the 1 + 1 dimensional spacetime. In this case the potential is

$$V(\phi) = \frac{1}{2} \lambda^2 \phi^2 (\phi^2 - a^2)^2,$$  \hspace{1cm} (37)

which also develops spontaneous symmetry breaking of the discrete $Z_2$ symmetry $\phi \to -\phi$. Here, however, the kink solutions resembles both the symmetric and asymmetric phases this $\phi^6$ system engenders.

In the present case we get

$$\frac{du}{dy} = \lambda u(a^2 - u^2),$$  \hspace{1cm} (38)

so that, by contrasting with the solutions studied in Sec. II,

$$\mathcal{O}\left(\frac{d^n u}{dy^n}\right) = u^{2n+1},$$  \hspace{1cm} (39)

thus implying that each derivative operated preserves parity with respect to $u \to -u$ by multiplying the original function by an even factor on $u$. As a consequence, one must now have $N_p = Na - 3$ to allow Eq. (36) to be solutions to Eq. (8). In this case, however, since the third-order derivative of Eq. (36) gives rise to a polynomial of order $u^7$, then the matching procedure involving physical ingredients such as diffusion and/or dispersion requires the presence of higher-order nonlinearity in $A(u)$, $Na = 7$. By substituting Eq. (36) in Eq. (8), we obtain the following relations

$$a_0 = 0,$$  \hspace{1cm} (40)

$$\frac{a_1}{\lambda a^2} = -c + p_0 + \lambda a^2 \nu + \lambda^2 a^4 \delta,$$  \hspace{1cm} (41)

$$a_2 = \lambda a^2 p_1,$$  \hspace{1cm} (42)

$$\frac{a_3}{\lambda} = c - p_0 + a^2 p_2 - 4\lambda a^2 \nu - 13\lambda^2 a^4 \delta,$$  \hspace{1cm} (43)

$$\frac{a_4}{\lambda} = -p_1 + a^2 p_3,$$  \hspace{1cm} (44)

$$\frac{a_5}{\lambda} = -p_2 + a^2 p_4 + 3\lambda \nu + 27\lambda^2 a^2 \delta,$$  \hspace{1cm} (45)

$$a_6 = -\lambda p_3,$$  \hspace{1cm} (46)

$$\frac{a_7}{\lambda} = -p_4 - 15\lambda^2 \delta.$$  \hspace{1cm} (47)
First we notice that due to Eq. (39) all terms in the left-hand side of Eq. (8) depend on \( u \), thus leading to the result of Eq. (40). Furthermore, as another consequence of Eq. (39), the diffusion and dispersion parameters, respectively \( \nu \) and \( \delta \), are both related only to even coefficients \( \{p_i\} \) and odd coefficients \( \{a_i\} \), differently from the case considered in Sec. II in which they were associated with coefficients of definite but distinct parities (even for \( \delta \), odd for \( \nu \)). Consequently, the physics related to diffusion and dispersion effects in Eq. (8) with solutions provided by Eq. (36) is irrespective to the presence of odd parity terms in the polynomial \( P(u) \), and even terms in \( A(u) \). Indeed, in this case the nonlinear differential equation, Eq. (8), presents odd symmetry with respect to \( u \rightarrow -u \). The difference between the odd-symmetric and non-symmetric cases (even symmetry is not allowed due to the presence of the constant \( c \)) appears from the way nonlinearity enters the equation through \( P(u) \) and \( A(u) \). It is also interesting to observe that the velocity of the solitons \( c \) only depends on the even coefficients \( p_0 \) and \( p_2 \), and odd coefficients \( a_1 \) and \( a_3 \). This implies that, contrarily to the solutions discussed in Sec. II, both diffusion and dispersion effects are actually relevant to determine \( c \):

\[
c = p_0 - \frac{1}{\lambda a^2}(a_1 - \nu \lambda^2 a^4 - \delta \lambda^3 a^6) .
\] (48)

We also recall that, as for the previous case, the coefficients are not totally independent and relations given by Eqs. (21) and (22) still hold for soliton solutions in the form of Eq. (36).

Finally, as an illustrative example we consider the case in which

\[
P(u) = p_0 ,
\] (49)

and

\[
A(u) = -9\lambda^3 a^4 \delta u^3 + 24\lambda^3 a^2 \delta u^5 - 15\lambda^3 \delta u^7 .
\] (50)

If we set

\[
\nu = -\lambda a^2 \delta ,
\] (51)

then Eq. (8) with solutions given by Eq. (36) is equivalent to a gKdVBH equation, Eq. (1), defined by the functions

\[
f(u) = f_0 + p_0 u ,
\] (52)

\[
g(u) = g_0 + \nu u ,
\] (53)

\[
h(u) = A(u) ,
\] (54)

and with \( \tilde{\delta} = -\delta \), if the velocity of the chiral kink soliton solution, Eq. (36), is

\[
c = p_0 ,
\] (55)

which is also in agreement with Eq. (48) and the results reported in Ref. [9].
IV Conclusion

In this work we have investigated the presence of travelling kink soliton solutions $u(x, t) = u(x - ct)$ to nonlinear partial differential equations of the generic form

$$u_t + P(u)u_x + \nu u_{xx} + \delta u_{xxx} = A(u),$$

with polynomial functions defined as

$$P(u) = \sum_{i=0}^{N_p} p_i u^i, \quad A(u) = \sum_{i=0}^{N_a} a_i u^i.$$

These equations combine diffusion, dispersion, and nonlinearity in distinct ways, and include a number of examples of relevant differential equations in physics and other fields of nonlinear science.

We have analyzed two types of kink soliton solutions, namely $u(y) = a \tanh(\lambda ay)$, which is also related to solutions of the relativistic $1 + 1$ dimensional $\phi^4$ field-theoretical system, and $u(y) = \{(a^2/2)[1 + \tanh(\lambda a^2 y)]\}^{1/2}$, which is associated with kinks of the $\phi^6$ system. These kinks present distinct properties. For instance, while in the $\phi^4$ system they connect the two asymmetric vacua, in the $\phi^6$ model they relate the symmetric vacuum $\phi = 0$ to the asymmetric ones.

The emphasis of our study was on the chirality of the localized soliton solutions, and its relation with diffusion, dispersion, and nonlinear effects, as well as its dependence on the parity of the polynomials $P(u)$ and $A(u)$ with respect to the discrete symmetry $u \rightarrow -u$.

Acknowledgments

EPR and DB would respectively like to thank the Condensed Matter Theory group at Harvard University and the Center for Theoretical Physics at Massachusetts Institute of Technology for hospitality.
References

[1] See for instance G.B. Whitham, Linear and Nonlinear Waves (Wiley, New York, 1974); L. Lam and J. Prost, eds., Soliton in Liquid Crystals (Springer-Verlag, New York, 1992).

[2] V.I. Karpman and J.-M. Vanden-Broeck, Phys. Lett. A 200 (1995) 423 and references therein.

[3] D.J. Korteweg and G. de Vries, Phil. Mag. 39 (1895) 422.

[4] G.S. Emmerson, John Scott Russell (John Murray, London, 1977).

[5] N. Flytzanis, S. Pnevmatikos and M. Remoissenet, J. Phys. C 18 (1985) 4603.

[6] S.J. Beneton Rabello, Phys. Rev. Lett. 76 (1996) 4007; (E) 77 (1996) 4851.

[7] U. Aglietti, L. Griguolo, R. Jackiw, S-Y. Pi and D. Seminara, Phys. Rev. Lett. 77 (1996) 4406; R. Jackiw, J. Nonlin. Math. Phys. 4 (1997) 241.

[8] L. Griguolo and D. Seminara, preprint MIT-CTP 2578, hep-th/9709075.

[9] D. Bazeia and E.P. Raposo, preprint MIT-CTP 2734, solv-int/9804017; see also D. Bazeia and F. Moraes, preprint MIT-CTP 2713, solv-int/9802002; D. Bazeia, preprint MIT-CTP 2714, solv-int/9802007.

[10] M. Kardar, G. Parisi and Y-C. Zhang, Phys. Rev. Lett. 56 (1986) 889.

[11] T.J. Rolfe, S.A. Rice and J. Dancz, J. Chem. Phys. 70 (1979) 26.

[12] R. Han, J. Yang, L. Huibin and W. Kelin, Phys. Lett. A 236 (1997) 319.

[13] D. Bazeia, M.J. dos Santo and R.F. Ribeiro, Phys. Lett. A 208 (1995) 84; D. Bazeia, R.F. Ribeiro and M.M. Santos, Phys. Rev. E 54 (1996) 2934.