Ramp-up of Hawking radiation in BEC analogue black holes

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Inspired by a recent experiment by J. Steinhauer and his group, we present a simple model which describes the formation of an acoustic black hole in a BEC, allowing an analytical computation of the evolution in time of the corresponding density-density correlator. The ramp-up of analog Hawking radiation is quantitatively studied both at $T = 0$ and, most interestingly, even in the presence of an initial temperature $T$, as is always the case experimentally.

Hawking black hole (BH) evaporation \[1\] can be understood as a pairs creation process in which a member of the pair is created outside the horizon constituting the thermal radiation emitted by the BH. The other member, called partner, is created inside the horizon and has negative (Killing) energy. The members of the pair are entangled and this leads to correlations. The presence of a causal horizon in a BH prevents a direct measurement of these correlations. This kind of particle-partner creation mechanism is however not peculiar to BHs \[2\]. The same process is at work in various others physical settings (called analogue BHs), in particular in flows that from subsonic turn to supersonic (for a review, see \[3\]).

Sonic BHs constructed from Bose-Einstein condensates (BECs) are the most studied examples \[4\]. In this case these correlations have not only been predicted \[5\] but indeed have been experimentally observed by Steinhauer in 2016 and 2019 \[6, 7\] and this is the most stringent evidence of Hawking like (phonons in this case) radiation in a BEC.

The striking feature of the correlation pattern is a stationary peak which appears at late time after the formation of the sonic horizon in the equal time density-density correlator \[8\] when one point is taken outside the horizon and the other inside. Two secondary peaks are also predicted \[9, 10\], they are caused by the backscattering of the modes, but so far one has not been able to see them given the much weaker signature of these compared to the former.

Recently, Steinhauer and his group have reported on an experiment which was able to follow through correlations measurements the time evolution of a BEC BH from the formation of the sonic horizon and the subsequent rump up of Hawking radiation to its stationary regime \[11\].

The purpose of this paper is to provide a simple toy model based on QFT in curved space which, in the spirit of the gravitational analogy \[12\], reproduces quite nicely in an analytical manner the time evolution of the relevant correlator peak towards the stationary configuration (a preliminary study in this direction can be found in Appendix C of \[13\]). We will also be able to see the effects of a non vanishing ambience temperature of the BEC (always present) on this evolution. One should say that this stationary configuration does not last forever. According to Steinhauer after some time an inner horizon forms and lasing and stimulated emission from the inner horizon dominate. These effects are however completely dispersive and cannot be reproduced by the gravitational analogy.

In the spirit of the gravitational analogy, phonons in a spatially one dimensional BEC can by described by a simple model consisting of a (3+1) dimensional massless scalar field $\delta \vartheta$, representing the phase fluctuation of the condensate, propagating in an acoustic metric associated to the one dimensional (along the $x$ axis) BEC flow:

$$ds^2 = \frac{n}{mc}[-(c^2-V^2)dt^2+2Vdtdx+dx^2+dy^2+dz^2] \tag{1}$$

where $V(x)$ is the velocity of the flow and $c(x)$ the speed of sound, $n(x)$ is the condensate density and $m$ the mass of the single atom. The transverse size $l_{\perp}$ of the condensate is assumed to be much smaller than the healing length $\xi = \frac{n}{mc}$, as it happens in the experimental realization of Steinhauer. This allows to treat the system as effectively 1D.

The field $\delta \vartheta$ satisfies

$$\Box \delta \vartheta = 0 \ , \tag{2}$$

where the covariant D’Alembertian is constructed from the metric \[1\]. We will be interested in the correlator of the 1D density fluctuations $\delta \hat{n}^{(1)}$ which is constructed from $\delta \vartheta$ as

$$\delta \hat{n}^{(1)} = -\frac{n^{(1)}}{mc^2}(\partial_t + V \partial_x)\delta \vartheta \ , \tag{3}$$

where $n^{(1)} = nl_{\perp}^2$ is the 1D density of the condensate. Relation \[3\] holds in the so called hydrodynamical approximation which is valid on longitudinal scales much bigger than the healing length $\xi$. This approximation is the core of the gravitational analogy in BECs.
We shall consider the flow directed from right to left at a constant velocity $V$ ($< 0$) and whose density $n$ is also constant. The profile for the speed of sound is assumed to be the following:

$$
\begin{cases}
  c = c_{in}, & t < 0 \\
  c = |V| \left( 1 + \frac{2}{3} \tanh \frac{3x}{2V} \right), & t > 0
\end{cases}
$$

where $c_{in}$ ($>|V|$) is a constant and $\kappa$ is also a constant. One can vary the speed of sound $c$ (= $\sqrt{\frac{mc}{A}}$) for example by varying the atom-atom interaction coupling $g$ using Feshbach resonances [14], or as described in Ref. [7]. We can generalize the model by letting both $c$ and $V$ vary as it is the case in the set-up of the experiment described in [11]. We believe however that our simple toy model is sufficient to reproduce at least qualitatively the relevant features recently observed by Steinhauer and his group.

Our choice describes a uniform subsonic flowing condensate for $t < 0$. Instantaneously at $t=0$ a sonic BH forms: the flow remains subsonic for $x > 0$, while it becomes supersonic for $x < 0$. The horizon is at $x = 0$ and $\kappa$ is its surface gravity.

For the reasons given before we shall be interested in the equal time density-density correlator $G_2^{(1)} (t; x, x') = \lim_{t \to t'} \langle \delta n(x, t) \delta n(x', t') \rangle$ which following [5] we approximate as:

$$G_2^{(1)} = \frac{n^{(1)}(x)n^{(1)}(x')}{m^2 c^2(x)c(x')} \lim_{t \to t'} \frac{m^2 c(x)c(x')}{n^{(1)}(x)n^{(1)}(x')} D \langle \delta \tilde{\theta}^{(2)}(t) \delta \tilde{\theta}^{(2)}(t') \rangle$$

(5)

where $D$ is the differential operator $D = (\partial_t + V \partial_x)(\partial_{t'} + V \partial_{x'})$ and $\langle \delta \tilde{\theta}^{(2)}(t, x) \delta \tilde{\theta}^{(2)}(t', x') \rangle$ is the two-point function of a 2D massless scalar field $\delta \tilde{\theta}^{(2)} = \sqrt{n^{(1)} \ln \delta \tilde{\theta}}$ propagating in the 1+1D metric

$$ds^2 = -(c^2 - V^2)dt^2 + 2V dt dx$$

(6)

which satisfies

$$\Box \delta \tilde{\theta}^{(2)} = 0,$$

(7)

where the 2D $\Box$ is calculated from [6]. The approximation used is familiar in QFT in curved space-time when dealing with Hawking BH evaporation in spherically symmetric spacetimes (Schwarzschild for example): the Unruh vacuum (the quantum state that describes Hawking radiation [12]) 4D stress tensor $T_{ab}$ is approximated by $\tfrac{4\pi}{V} t_{ab}$. $t_{ab}$ corresponds to the stress tensor of a 2D massless scalar field. See for instance [16]-[18]. The conformal factor $\tfrac{1}{t^{1/2}}$ of the transverse $(y, z)$ space plays the role of $v^2$. The approximation introduced in eq. (5) makes the model analytically soluble and although one neglects the backscattering of the modes (studied in [19]), the presence and basic features of the main correlation peak can be nicely reproduced.

For zero temperature (we shall discuss later the case $T \neq 0$) we have (up to an, irrelevant in our case, diverging constant related to the infrared divergence of our 2D theory)

$$\langle \delta \tilde{\theta}^{(2)}(t, x) \delta \tilde{\theta}^{(2)}(t', x') \rangle = - \frac{\hbar}{4\pi} \ln(u_{in} - u_{in}')(v_{in} - v_{in}'),$$

(8)

where

$$u_{in} = t - \frac{x}{c_{in} - |V|}, \quad v_{in} = t + \frac{x}{c_{in} + |V|}$$

(9)

are the null coordinates associated to the metric (1) for $t < 0$. This two point function characterizes our “in” quantum state for the field $\delta \tilde{\theta}$ which corresponds to an initial vacuum (i.e. no incoming phonons from both left and right past null infinity).

This expression can be extended for $t > 0$ simply by matching the null coordinates along the spacelike shell at $t = 0$ (see Fig. [1]).

Note that our theory is conformal invariant and so there is no scattering both inside the condensate and at the transition layer. For $t > 0$ the corresponding null coordinates are

$$
\begin{align*}
  u &= t - \frac{1}{\kappa} \ln \sinh \frac{3 \kappa |x|}{2|V|}, \\
  v &= t + \frac{1}{8 \kappa} \left( \frac{9 \kappa x}{2|V|} - \ln \cosh \left( \frac{3 \kappa x}{2|V|} + \tanh^{-1} \frac{1}{3} \right) \right).
\end{align*}
$$

(10)

Matching $u$ and $u_{in}$ at $t = 0$ we get

$$- \kappa u = \ln \sinh |Au_{in}|,$$

(11)

where $A = \frac{3 \kappa}{2|V|} (|V| - c_{in}) < 0$, which can be inverted giving

$$|u_{in} A| = \ln(\sqrt{1 + e^{-2\kappa u}} + e^{-\kappa u}).$$

(12)
FIG. 2. Ramp-up of the density correlator (14) for \( t_1 = \frac{1}{\kappa} \) (orange), \( t_2 = \frac{2}{\kappa} \) (blue) and \( t_3 = \frac{3}{\kappa} \) (green). Here and in all the figures that follow we have plotted the correlator up to the overall factor \( \frac{\hbar n(1)}{4\pi m} \) and chosen the values \( \kappa = \frac{1}{2}, |V| = 1, c_n = \frac{3}{2} \), with \( 5 < x < 20 \) and \( -20 < x' < -5 \). This plot, as well as that in Fig. (3), has been reversed to better appreciate the differences at different times.

Similarly for the advanced null coordinates we obtain

\[
v_{in} = \frac{4\kappa v}{B} + \frac{1}{2B} \ln \left( 1 + \sqrt{1 + 2\sqrt{2e^{-8\kappa v}}} \right), \tag{13}\]

where \( B = \frac{3\kappa(c_n + |V|)}{2|V|} \). Given these relations, the correlator (5) for \( t > 0 \) can be analytically evaluated starting from the expression

\[
G^{(1)}_2(t, x, x') = -\frac{\hbar n(1)}{4\pi mc(x)^{1/2}c(x')^{1/2}} \times \tag{14}
\frac{1}{(c(x) - |V|)(c(x') - |V|)} \frac{1}{|u_n - u_{n'}|} \left( \frac{uc_{in} du_{in}}{du} \frac{1}{du'} (u_{in} - u_{in'})^2 + \frac{1}{dv} \frac{1}{dv'} (v_{in} - v_{in'})^2 \right)^{1/2}.
\]

The resulting expression in terms of \((t, x)\) coordinates is rather long and will be given elsewhere \cite{20}. For \( t < 0 \) a similar expression holds, just replace \( c(x) \) by \( c_n \) and \((u, v)\) by \((u_{in}, v_{in})\). In Fig. 2 we have represented the correlator for points \( x > 0 \) (outside the horizon) and \( x' < 0 \) (inside the horizon) at three increasing times \((t_1 = \frac{1}{\kappa}, t_2 = \frac{2}{\kappa}, t_3 = \frac{3}{\kappa})\).

One sees very nicely the ramp-up and the formation at late time of the deep valley (the correlator is negative) located at \( x = -x' \). This can be confirmed analytically by taking the late time limit, \( u_{in} \to 0, u \to +\infty \) of eq. (12) yielding \( |u_{in}| \sim e^{-\kappa u} \) and, from (13), \( v_{in} \sim \frac{4\kappa v}{B} \).

FIG. 3. Plot of the density correlator (14) for \( x' = -5 \) and \( t_1 = \frac{1}{\kappa} \) (orange), \( t_2 = \frac{2}{\kappa} \) (blue) and \( t_3 = \frac{3}{\kappa} \) (green).

The correlator in terms of \((t, x)\) coordinates is

\[
\frac{-\hbar n(1)}{4\pi m |V|^2 \sqrt{(1 + \frac{3}{2} \tanh \frac{3\kappa x}{2|V|})(1 + \frac{3}{2} \tanh \frac{3\kappa x'}{2|V|})}} \times \left[ -\frac{9\kappa^2}{16 \tanh \frac{3\kappa x}{2|V|} \tanh \frac{3\kappa x'}{2|V|} \cosh^2 \left( \frac{1}{2} \ln \frac{\sinh \frac{3\kappa x}{2|V|}}{\sinh \frac{3\kappa x'}{2|V|}} \right) \right] + \tag{15}
\frac{1}{\sqrt{(2 + \frac{3}{2} \tanh \frac{3\kappa x}{2|V|})(2 + \frac{3}{2} \tanh \frac{3\kappa x'}{2|V|})}} \left( v(t, x) - v'(t, x') \right)^2 \right],
\]

where in the second term (see the second of (10))

\[
v(t, x) - v'(t, x') = \frac{9}{16|V|}(x - x') - \frac{1}{8\kappa} \ln \frac{2e^{-\frac{3\kappa x}{2|V|}} + e^{-\frac{3\kappa x'}{2|V|}}}{2e^{-\frac{3\kappa x}{2|V|}} + e^{-\frac{3\kappa x'}{2|V|}}} \tag{16}\]

Note, however, that this is true only for \( x = -x' \) sufficiently far away from the horizon, as remarked in Ref. \cite{21}. In this region the \tanh terms appearing in (15) can be well approximated by their asymptotic values, while the remaining \( \frac{1}{\sinh \kappa} \) term has indeed a minimum for \( x = -x' \). In Fig. (3) we plot the value of the correlator (14) as a function of \( x \) for \( x' = -5 \) and, in Fig. (4), for \( x' = -10 \) for the previous time slots. One clearly sees that the peak is located at \( x = -x' \) only for \( x' = -10 \),

FIG. 4. Plot of the density correlator (14) for \( x' = -10 \) and \( t_1 = \frac{1}{\kappa} \) (orange), \( t_2 = \frac{2}{\kappa} \) (blue) and \( t_3 = \frac{3}{\kappa} \) (green).
while for the the point \( x' = -5 \) closer to the horizon the peak does not appear at \( x = -x' \). This is due to the fact that as one approaches the horizon \( (x = -x' \rightarrow 0) \) the singularity of the two point function at coincidence points starts dominating. Pairs production appears not to be located near the horizon. We know in gravity that near the singularity of the two point function at coincidence points we have, for \( b \) corresponding to right moving phonons \( (\omega > 0) \), the initial population of phonons in thermal equilibrium in the comoving frame is \( N = 1 \). Such \( t = t(x) \) governs the formation of the peak at \( x = -x' \) and also its length. Away enough from the horizon, the length of the peak grows linearly in \( t \), as noticed also in [8].

Let us now consider the case in which the condensate has an initial temperature \( T \), so instead of the vacuum we have, for \( t \), a thermal distribution of phonons. This analysis is rather important since experimentally a condensate has always a non vanishing temperature which may be comparable or even bigger than the Hawking one \( T_H = \frac{\hbar c}{2 \pi k_B} \) associated to the thermal emission of phonons by the sonic horizon. The initial population of phonons in thermal equilibrium in the comoving frame is characterized by an occupation number

\[
N\omega_u(v) = \frac{1}{e^{\frac{\hbar c}{\pi k_B T} - 1}}
\]

where \( \omega_u \) and \( \omega_v \) are the Doppler rescaled frequencies corresponding to right moving phonons \( (u) \) and left moving ones \( (v) \)

\[
\omega_u = \frac{\omega c_{in}}{c_{in} - |V|}, \quad \omega_v = \frac{\omega c_{in}}{c_{in} + |V|}.
\]

The corresponding two-point function for the field \( \delta \hat{\theta}^{(2)} \) reads \[22\]

\[
\langle \delta \hat{\theta}^{(2)}(t, x) \delta \hat{\theta}^{(2)}(t', x') \rangle = -\frac{\hbar}{4\pi} \ln \left| \frac{A_u \Delta u_{in} \sinh \left( \frac{3\omega |x|}{2V} \right)}{A_v} \right|
\]

\[
(19)
\]

where \( A_u = \frac{\pi k_B T}{\hbar c_{in}} \), \( \Delta u_{in} = (u_{in} - u_{in}') \), and similarly for \( v_{in} \). The time evolution of the corresponding density-density correlator

\[
G_T^{(1)}(t; x, x') = -\frac{\hbar n^{(1)}}{4\pi mc \sqrt{c(x')^2}} \times
\]

\[
\left[ \frac{1}{c(x) - |V|} \frac{1}{c(x') - |V|} \right] \left[ \frac{A_u^2}{\left( c(x) - |V| \right) \left( c(x') - |V| \right)} \frac{dU_{in}}{du} \frac{dU_{in}'}{du'} \sinh^2 A_u \left( u_{in} - u_{in}' \right) + \right.
\]

\[
\left. \frac{1}{c(x) + |V|} \frac{1}{c(x') + |V|} \right] \left[ \frac{A_v^2}{\left( c(x) + |V| \right) \left( c(x') + |V| \right)} \frac{dV_{in}}{dv} \frac{dV_{in}'}{dv'} \sinh^2 A_v \left( v_{in} - v_{in}' \right) \right]_{t=t'}
\]

is shown in Fig. 6 for \( T = 10T_H \). We do not see noticeable differences in the time evolution when the BEC temperature equals the Hawking temperature \( T = T_H \) with respect to the \( T = 0 \) case.

One sees that the stationary configuration with the valley located at \( x = -x' \) \( (x \gg 0) \) appears even in this case but at a time that is bigger than the one required at \( T = 0 \) and this time can increase with \( T \). This can be seen by noticing that at finite temperature the late time limit we got at \( T = 0 \) has to be supplemented (for \( T > \frac{3\omega}{2\pi} T_H \)) by the more stringent requirement

\[
\frac{\pi k_B T}{\hbar c_{in}} e^{-\frac{\hbar c}{\pi k_B T}} \sinh \left( \frac{3\omega |x|}{2V} \right) = \text{cst} \ll 1.
\]

At late times the first term in eq. (20) coming from the \( u \) modes contribution reduces to the corresponding one at \( T = 0 \) (the first term in eq. (15)). This because the late time contribution comes only from the modes propagating very close to the horizon (i.e. \( u \rightarrow +\infty \) and these are highly redshifted, so any information of the initial state is washed out (no hair theorem for Hawking radiation). An initial population causes stimulated emission of \( u \) phonons which is however just a transient effect [23]. This does not hold for the \( v \) modes because they are not redshifted and hence the second term in eq. (20) shows a temperature dependence also in the stationary regime at late time. The
thermal $v$ contribution to the correlator is positive, but, as the $T_H=0$ one (which is negative), smaller than the $u$ contribution. For instance, for $x=-x'=10$ such contributions are one order of magnitude smaller than the $u$ contribution. So the temperature corrections, within the hydrodynamical approximation (gravitational analogy), are small.

To sum up, in this paper we have presented an analytical computation of the density-density correlation in a BEC acoustic black hole formed at some instant of laboratory time $t$, for both $T=0$ and, most interestingly for experimental purposes, when a non vanishing initial temperature $T$ of the BEC is present. The time-dependent formation of the deep valley at $x=-x'$ sufficiently far away from horizon, which signals the ramp-up of analog Hawking radiation, is first studied at $T=0$. A close inspection of our results shows that this happens starting from $x=-x'\sim 8$ (i.e., in our units, at a distance $\frac{2|V|}{\kappa}$ from the horizon) and time $\frac{4}{\kappa}$. When $T \neq 0$, at $x=-x'\sim 8$ and $T=10T_H$ the ramp-up process is slower and lasts up to a time $t$ of roughly $\frac{2}{8}$.

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