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Some Fejér-Type Inequalities for Generalized Interval-Valued Convex Functions

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Abstract: The goal of this study is to create new variations of the well-known Hermite–Hadamard inequality (HH-inequality) for preinvex interval-valued functions (preinvex I-V-Fs). We develop several additional inequalities for the class of functions whose product is preinvex I-V-Fs. The findings described here would be generalizations of those found in previous studies. Finally, we obtain the Hermite–Hadamard–Fejér inequality with the support of preinvex interval-valued functions. Some new and classical special cases are also obtained. Moreover, some nontrivial examples are given to check the validity of our main results.

Keywords: preinvex interval valued functions; interval riemann integrals; hermite–hadamard inequalities; hermite–hadamard–fejér inequality

MSC: 26A33; 26A51; 26D10

1. Introduction

Let \( \mathbb{R} \) be the set of real numbers and \( \mathcal{J} : K \subseteq \mathbb{R} \to \mathbb{R} \) be a convex function. If \( o, q \in K \) with \( o \leq q \), then

\[
\mathcal{J}\left(\frac{o + q}{2}\right) \leq \frac{1}{q - o} \int_o^q \mathcal{J}(\omega) d\omega \leq \frac{\mathcal{J}(o) + \mathcal{J}(q)}{2},
\]

where \( K \) is a convex set, which is named Jensen’s inequality [1]. The famous HH-inequality is then created by Hermite and Hadamard by adding the integral mean value of the convex function \( \mathcal{J} \) to inequality (1).

Let \( \mathcal{J} : K \subseteq \mathbb{R} \to \mathbb{R} \) be a convex function. If \( o, q \in K \) with \( o \leq q \), then

\[
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\]

If \( \mathcal{J} \) is concave, the inequality (2) holds in the inverse fashion, see [2,3]. This inequality has several applications in numerical integration since it may be used to give a rough approximation of the integral mean on \([o, q]\). For details on the use and rising popularity of the HH-inequality, readers might refer to [4–14].

The results point to the prevalence of fractional-order phenomena and show that fractional calculus is more accurate and reliable than classical calculus. The fractional calculus technique, as a result, has become one of the most well-liked study topics in academia. Numerous fractional-order research findings on the HH-inequality have recently been published. Examples include the Riemann–Liouville fractional integral inequalities [15–24].
conformable fractional integral inequalities [25], k-Riemann–Liouville fractional integral inequalities [26], and local fractional integral inequalities [27–39].

Research on fractional operator-type integral inequalities is becoming more and more popular, since fractional integral operators have several applications in a number of fields. Set et al. [40] used Raina’s fractional integral operators to create new Hermite–Hadamard–Mercer inequalities. With a modified Mittag-Leffler kernel, Srivastava et al. [41] created the generalized left-side and right-side fractional integral operators, and they then used this large family of fractional integral operators to study the fascinating Chebyshev inequality. Sun established certain Hermite–Hadamard-type inequalities for extended h-convex functions and modified preinvex functions in refs. [28,42] using two local fractional integral operators with a Mittag-Leffler kernel. The two local fractional integral operators were then used by Xu et al. [43] to examine Hermite–Hadamard–Mercer for extended h-convex functions. For more information, see [44–53] and the references therein.

Ahmad et al. established various inequalities pertaining to the right side of the HH-inequality in [54], as well as two new fractional integral operators with exponential kernels. The constraint on the left side of the HH-inequality was then studied by Wu et al. [55] using these integral operators. In order to create various inequalities of the HH- and Ostrowski types, Budak et al. [56] combined exponential kernels with these integral operators. The use of the novel integral operators with exponential kernels in interval-valued and interval-valued coordinated HH-inequalities was expanded by Du and Zhou et al. (see [57–62]).

Furthermore, Khan et al. introduced the different classes of convex functions such as $(h_1, h_2)$-convex fuzzy I-V-Fs [63], $(h_1, h_2)$-preinvex fuzzy I-V-Fs [64], log-s-convex fuzzy I-V-Fs in the second sense [65], harmonically convex fuzzy I-V-Fs [66], generalized p-convex fuzzy I-V-Fs [67] and introduced HH-type inequalities of these functions. For more information, see [68–78].

The goal of this study is to find certain HH-inclusions that are more generic. The study’s general format consists of five sections, including an introduction. The remainder of the paper is organized as follows: Section 2 introduces certain types of integrals of real-valued functions and their accompanying HH-inequalities. In Section 2, we briefly summarize the idea of interval-valued functions. We discuss generalized integrals of interval-valued functions in Section 2, and provide some examples of these integrals. In Section 3, we use defined generalized integrals to show many HH-inclusions for interval-valued convex functions and to validate the main results; we have provided some nontrivial examples. Finally, in Section 4, some findings and future study areas are explored.

2. Preliminaries

We offer some fundamental arithmetic regarding interval analysis in this paragraph, which will be quite useful throughout the article.

\[
a = [a_*, a^*], b = [b_*, b^*] \quad (a_* \leq x \leq a^* \text{ and } b_* \leq y \leq b^* \quad \forall x, y \in \mathbb{R})
\]

\[
a + b = [a_*, a^*] + [b_*, b^*] = [a_* + b_*, a^* + b^*],
\]

\[
a - b = [a_*, a^*] - [b_*, b^*] = [a_* - b_*, a^* - b^*],
\]

\[
a \times b = [a_*, a^*] \times [b_*, b^*] = [\min \lambda', \max \lambda']
\]

\[
\min \lambda' = \min \{a_* b_*, a_* b^*, a^* b_*, a^* b^*\}, \max \lambda' = \max \{a_* b_*, a_* b^*, a^* b_*, a^* b^*\}
\]

\[
\nu, [a_*, a^*] = \begin{cases} [\nu a_*, \nu a^*] & \text{if } \nu > 0, \\ \{0\} & \text{if } \nu = 0, \\ [\nu a^*, \nu a_*] & \text{if } \nu < 0. \end{cases}
\]
Let \( \mathcal{K}_C, \mathcal{K}_C^+, \mathcal{K}_C^- \) be the set of all closed intervals of \( \mathbb{R} \), the set of all closed positive intervals of \( \mathbb{R} \) and the set of all closed negative intervals of \( \mathbb{R} \). Then, \( \mathcal{K}_C, \mathcal{K}_C^+, \mathcal{K}_C^- \) are defined as

\[
\mathcal{K}_C = \{ [a_-, a_+] : a_-, a_+ \in \mathbb{R} \text{ and } a_- \leq a_+ \}, \\
\mathcal{K}_C^+ = \{ [a_-, a_+] : a_-, a_+ \in \mathcal{K}_C \text{ and } a_- > 0 \}, \\
\mathcal{K}_C^- = \{ [a_-, a_+] : a_-, a_+ \in \mathcal{K}_C \text{ and } a_- < 0 \}.
\]

(8)

(9)

(10)

For \([a_-, a_+], [b_-, b_+] \in \mathcal{K}_C\), the inclusion “\( \subseteq \)” is defined by \([a_-, a_+] \subseteq [b_-, b_+]\), if and only if \(b_- \leq a_-, a_+ \leq b_+\).

**Theorem 1** ([68]). If \( \mathcal{J} : [a, q] \subset \mathbb{R} \to \mathcal{K}_C \) is an I-V-F such that \( \mathcal{J} (\omega) = [\mathcal{J}_s (\omega), \mathcal{J}_t (\omega)] \), then \( \mathcal{J} \) is interval Riemann integrable (IR-integrable) over \([a, q]\) if and only if \( \mathcal{J}_s (\omega) \) and \( \mathcal{J}_t (\omega) \) are both Riemann-integrable (R-integrable) over \([a, q]\) such that

\[
(\text{IR}) \int_a^q \mathcal{J} (\omega) d\omega = \left[ (\text{R}) \int_a^q \mathcal{J}_s (\omega) d\omega, (\text{R}) \int_a^q \mathcal{J}_t (\omega) d\omega \right],
\]

(11)

where \( \mathcal{J}_s, \mathcal{J}_t : [a, q] \to \mathbb{R} \).

The collection of all Riemann-integrable real-valued functions and Riemann-integrable I-V-Fs is denoted by \( \mathcal{R}_{[a,q]} \) and \( \mathcal{I} \mathcal{R}_{[a,q]} \), respectively.

**Definition 1** ([72]). Let \( K \) be a convex set. Then I-V-F \( \mathcal{J} : K \to \mathcal{K}_C^+ \) is named as convex on \( K \) if

\[
\mathcal{J} (\xi \omega + (1 - \xi) \omega') \supseteq \xi \mathcal{J} (\omega) + (1 - \xi) \mathcal{J} (\omega'),
\]

(12)

for all \( \omega, \omega' \in K, \xi \in [0, 1] \). \( \mathcal{J} \) is named as concave on \( K \) if inequality (12) is reversed.

**Definition 2** ([78]). Let \( K \) be an invex set. Then, I-V-F \( \mathcal{J} : K \to \mathcal{K}_C^- \) is named as preinvex on \( K \) with respect to \( \mathfrak{w} \) if

\[
\mathcal{J} (\omega + (1 - \xi) \mathfrak{w} (\omega, \omega')) \supseteq \xi \mathcal{J} (\omega) + (1 - \xi) \mathcal{J} (\omega'),
\]

(13)

for all \( \omega, \omega' \in K, \xi \in [0, 1], \) where \( \mathfrak{w} : K \times K \to \mathbb{R} \). \( \mathcal{J} \) is named as preconcave on \( K \) with respect to \( \mathfrak{w} \) if inequality (13) is reversed. \( \mathcal{J} \) is named as affine if \( \mathcal{J} \) is both convex and concave.

**Remark 1.** The preinvex I-V-Fs have some very nice properties similar to convex I-V-F:

1. if \( \mathcal{J} \) is preinvex I-V-F, then \( \beta \mathcal{J} \) is also preinvex for \( \beta \geq 0 \).
2. if \( \mathcal{J} \) and \( \mathcal{T} \) both are preinvex I-V-Fs, then max(\( \mathcal{J}(\omega), \mathcal{T}(\omega) \)) is also preinvex I-V-F.

In the case of \( \mathfrak{w}(\omega, \omega') = \omega - \omega' \), we obtain the Definition 2 of convex I-V-F.

**Theorem 2.** Let \( K \) be an invex set and \( \mathcal{J} : K \to \mathcal{K}_C^+ \) be a I-V-F such that

\[
\mathcal{J} (\omega) = [\mathcal{J}_s (\omega), \mathcal{J}_t (\omega)], \ \forall \ \omega \in K.
\]

(14)

for all \( \omega \in K \). Then, \( \mathcal{J} \) is preinvex I-V-F on \( K \), if and only if, \( \mathcal{J}_s (\omega) \) and \( \mathcal{J}_t (\omega) \) are preinvex and preconcave functions, respectively.

**Proof.** The proof of this result is similar to Theorem 6, see [63]. \( \square \)

**Example 1.** We consider the I-V-F \( \mathcal{J} : [0, 1] \to \mathcal{K}_C^+ \) defined by

\[
\mathcal{J} (\omega) = [2\omega^2, 4\omega].
\]

(15)
Hence, end point functions $\mathcal{J}_*(\omega)$, $\mathcal{J}^*(\omega)$ are preinvex functions with respect to $w(\omega, \omega) = \omega - \omega$. Hence, $\mathcal{J}(\omega)$ is preinvex $I-V-F$.

3. Main Results

In this section, we propose interval $HH$-inequalities for preinvex $I-V-F$s. Moreover, some examples are presented that verify the applicability of theory developed in this study.

**Theorem 3.** (The interval $HH$-inequality for preinvex $I-V-F$). Let $\mathcal{J} : [a, b] \rightarrow \mathcal{K}_C^+$ be a preinvex $I-V-F$ such that $\mathcal{J}(\omega) \subseteq [\mathcal{J}_*(\omega), \mathcal{J}^*(\omega)]$ for all $\omega \in [a, b]$. If $\mathcal{J} \in \tau R_{(C, w(q, q))]$, then

$$3\left(\frac{2a + w(q, q)}{2}\right) \geq \frac{1}{w(q, q)} (IR) \int_a^{a + w(q, q)} \mathcal{J}(\omega) d\omega \geq \frac{3(a) + \mathcal{J}(q)}{2}. \quad (16)$$

**Proof.** Let $\mathcal{J} : [a, b] \rightarrow \mathcal{K}_C^+$ be a preinvex $I-V-F$. Then, by hypothesis, we have

$$2\mathcal{J}\left(\frac{2a + w(q, q)}{2}\right) \geq \mathcal{J}(a + (1 - \xi)w(q, q)) + \mathcal{J}(a + \xi w(q, q)).$$

Therefore, we have

$$2\mathcal{J}_*(\frac{2a + w(q, q)}{2}) \leq \mathcal{J}_*(a + (1 - \xi)w(q, q)) + \mathcal{J}_*(a + \xi w(q, q))$$

$$2\mathcal{J}^*(\frac{2a + w(q, q)}{2}) \geq \mathcal{J}^*(a + (1 - \xi)w(q, q)) + \mathcal{J}^*(a + \xi w(q, q)).$$

Then,

$$2 \int_0^1 \mathcal{J}_*\left(\frac{2a + w(q, q)}{2}\right) d\xi \leq \int_0^1 \mathcal{J}_*(a + (1 - \xi)w(q, q)) d\xi + \int_0^1 \mathcal{J}_*(a + \xi w(q, q)) d\xi,$$

$$2 \int_0^1 \mathcal{J}^*\left(\frac{2a + w(q, q)}{2}\right) d\xi \geq \int_0^1 \mathcal{J}^*(a + (1 - \xi)w(q, q)) d\xi + \int_0^1 \mathcal{J}^*(a + \xi w(q, q)) d\xi.$$

It follows that

$$\mathcal{J}_*\left(\frac{2a + w(q, q)}{2}\right) \leq \frac{1}{w(q, q)} \int_a^{a + w(q, q)} \mathcal{J}_*(\omega) d\omega,$$

$$\mathcal{J}^*\left(\frac{2a + w(q, q)}{2}\right) \geq \frac{1}{w(q, q)} \int_a^{a + w(q, q)} \mathcal{J}^*(\omega) d\omega.$$

That is,

$$\left[\mathcal{J}_*\left(\frac{2a + w(q, q)}{2}\right), \mathcal{J}^*\left(\frac{2a + w(q, q)}{2}\right)\right] \supseteq \frac{1}{w(q, q)} \left[\int_a^{a + w(q, q)} \mathcal{J}_*(\omega) d\omega, \int_a^{a + w(q, q)} \mathcal{J}^*(\omega) d\omega\right].$$

Thus,

$$\mathcal{J}\left(\frac{2a + w(q, q)}{2}\right) \supseteq \frac{1}{w(q, q)} (IR) \int_a^{a + w(q, q)} \mathcal{J}(\omega) d\omega \geq \frac{3(a) + \mathcal{J}(q)}{2}. \quad (17)$$

In a similar way as above, we have

$$\frac{1}{w(q, q)} (IR) \int_a^{a + w(q, q)} \mathcal{J}(\omega) d\omega \geq \frac{3(a) + \mathcal{J}(q)}{2}. \quad (18)$$

Combining (17) and (18), we have

$$\mathcal{J}\left(\frac{2a + w(q, q)}{2}\right) \supseteq \frac{1}{w(q, q)} (IR) \int_a^{a + w(q, q)} \mathcal{J}(\omega) d\omega \geq \frac{3(a) + \mathcal{J}(q)}{2}.$$
Remark 2. If \( w(q, o) = q - o \), then Theorem 3 reduces to the result for convex I-V-F, see [78]:

\[
\mathcal{J}\left(\frac{o + q}{2}\right) \geq \frac{1}{q - o} \int_q^o \mathcal{J}(\omega) d\omega \geq \frac{\mathcal{J}(o) + \mathcal{J}(q)}{2}.
\]  

(19)

If \( \mathcal{J}_*(\omega) = \mathcal{J}^*(\omega) \), then Theorem 3 reduces to the result for preinvex function, see [69]:

\[
\mathcal{J}\left(\frac{2o + w(q, o)}{2}\right) \leq \frac{1}{w(q, o)} \int_{o}^{o+w(q,o)} \mathcal{J}(\omega) d\omega \leq \left[\mathcal{J}(o) + \mathcal{J}(q)\right] \int_0^1 \xi d\xi.
\]  

(20)

If \( \mathcal{J}_*(\omega) = \mathcal{J}^*(\omega) \) with \( w(q, o) = q - o \), then Theorem 3 reduces to inequality (2).

Example 2. We consider the I-V-F \( \mathcal{J} : [o, o + w(q, o)] \rightarrow \mathbb{R}^+_C \) defined by \( \mathcal{J}(\omega) = [2\omega^2, 4\omega] \). Hence, end point functions \( \mathcal{J}_*(\omega) = 2\omega^2 \), \( \mathcal{J}^*(\omega) = 4\omega \) are preinvex functions with respect to \( w(q, o) = q - o \). Hence, \( \mathcal{J}(\omega) \) is preinvex I-V-F with respect to \( w(q, o) = q - o \). We now compute the following:

\[
\mathcal{J}\left(\frac{2o + w(q, o)}{2}\right) \geq \frac{1}{w(q, o)} \int_{o}^{o+w(q,o)} \mathcal{J}(\omega) d\omega \geq \frac{\mathcal{J}(o) + \mathcal{J}(q)}{2}.
\]

\[
\mathcal{J}_*\left(\frac{2o + w(q, o)}{2}\right) = \mathcal{J}_*(1) = 2,
\]

\[
\frac{1}{w(q, o)} \int_{o}^{o+w(q,o)} \mathcal{J}_*(\omega) d\omega = \frac{1}{2} \int_0^2 2\omega^2 d\omega = \frac{8}{3},
\]

\[
\frac{\mathcal{J}_*(o) + \mathcal{J}_*(q)}{2} = 4,
\]

which means

\[
2 \leq \frac{8}{3} \leq 4.
\]

Similarly, it can be easily shown that

\[
\mathcal{J}^*\left(\frac{2o + w(q, o)}{2}\right) \geq \frac{1}{w(q, o)} \int_{o}^{o+w(q,o)} \mathcal{J}^*(\omega) d\omega \geq \frac{\mathcal{J}^*(o) + \mathcal{J}^*(q)}{2}.
\]

such that

\[
\mathcal{J}^*\left(\frac{2o + w(q, o)}{2}\right) = \mathcal{J}^*(1) = 4,
\]

\[
\frac{1}{w(q, o)} \int_{o}^{o+w(q,o)} \mathcal{J}^*(\omega) d\omega = \frac{1}{2} \int_0^2 4\omega d\omega = 4,
\]

\[
\frac{\mathcal{J}^*(o) + \mathcal{J}^*(q)}{2} = 4.
\]

From which, it follows that

\[
4 = 4 = 4,
\]

that is,

\[
[2, 4] \supseteq \left[\frac{8}{3}, 4\right] \supseteq [4, 4].
\]

Hence,

\[
\mathcal{J}\left(\frac{2o + w(q, o)}{2}\right) \geq \frac{1}{w(q, o)} \int_{o}^{o+w(q,o)} \mathcal{J}(\omega) d\omega \geq \frac{\mathcal{J}(o) + \mathcal{J}(q)}{2}.
\]
Theorem 4. Let $\mathcal{J}, \mathfrak{A} : [q,o + w(q,o)] \rightarrow \mathcal{K}_{+}^{\mathbb{C}}$ be two preinvex I-V-Fs such that $\mathcal{J}(\omega) = [\mathcal{J}_{+}(\omega), \mathcal{J}_{+}^{*}(\omega)]$ and $\mathfrak{A}(\omega) = [\mathfrak{A}_{+}(\omega), \mathfrak{A}_{+}^{*}(\omega)]$ for all $\omega \in [q,o + w(q,o)]$. If $\mathcal{J}, \mathfrak{A}$ and $\mathcal{J} \times \mathfrak{A} \in \mathcal{TR}([q,o + w(q,o)])$, then

$$\frac{1}{w(q,o)} (IR) \int_{0}^{q + w(q,o)} \mathcal{J}(\omega) \times \mathfrak{A}(\omega) d\omega \geq \frac{\omega(0,q)}{3} + \frac{\eta(0,q)}{6},$$

where $\omega(0,q) = \mathcal{J}(q) \times \mathfrak{A}(q) + \mathcal{J}(q) \times \mathfrak{A}(q)$, $\eta(0,q) = \mathcal{J}(q) \times \mathfrak{A}(q) + \mathcal{J}(q) \times \mathfrak{A}(q)$, and

$$\omega(0,q) = [\omega(0,q), \omega^{*}(0,q)]$$

and $\eta(0,q) = [\eta(0,q), \eta^{*}(0,q)]$.

Proof. Since $\mathcal{J}, \mathfrak{A} \in \mathcal{TR}([q,q + w(q,o)])$, then we have

$$\mathcal{J}_{+}(q + (1 - \xi)w(q,o)) \leq \xi \mathcal{J}_{+}(q) + (1 - \xi)\mathcal{J}_{+}(0),$$

$$\mathcal{J}_{+}^{*}(q + (1 - \xi)w(q,o)) \geq \xi \mathcal{J}_{+}^{*}(q) + (1 - \xi)\mathcal{J}_{+}^{*}(0).$$

Moreover,

$$\mathfrak{A}_{+}(q + (1 - \xi)w(q,o)) \leq \xi \mathfrak{A}_{+}(q) + (1 - \xi)\mathfrak{A}_{+}(0),$$

$$\mathfrak{A}_{+}^{*}(q + (1 - \xi)w(q,o)) \geq \xi \mathfrak{A}_{+}^{*}(q) + (1 - \xi)\mathfrak{A}_{+}^{*}(0).$$

From the definition of left and right preinvex IV-F, it follows that $0 \leq \mathcal{J}(\omega)$ and $0 \leq \mathfrak{A}(\omega)$, so

$$\mathcal{J}_{+}(q + (1 - \xi)w(q,o)) \times \mathfrak{A}_{+}(q + (1 - \xi)w(q,o))$$

$$\leq (\xi \mathcal{J}_{+}(q) + (1 - \xi)\mathcal{J}_{+}(0)) (\xi \mathfrak{A}_{+}(q) + (1 - \xi)\mathfrak{A}_{+}(0))$$

$$= \mathcal{J}_{+}(q) \times \mathfrak{A}_{+}(q) \xi^{2} + \mathcal{J}_{+}(q) \times \mathfrak{A}_{+}(0) \xi^{2} + \mathcal{J}_{+}(0) \times \mathfrak{A}_{+}(q) \xi(1 - \xi)$$

$$+ \mathcal{J}_{+}(0) \times \mathfrak{A}_{+}(0) \xi(1 - \xi);$$

$$\mathcal{J}_{+}^{*}(q + (1 - \xi)w(q,o)) \times \mathfrak{A}_{+}^{*}(q + (1 - \xi)w(q,o))$$

$$\geq (\xi \mathcal{J}_{+}^{*}(q) + (1 - \xi)\mathcal{J}_{+}^{*}(0)) (\xi \mathfrak{A}_{+}^{*}(q) + (1 - \xi)\mathfrak{A}_{+}^{*}(0))$$

$$= \mathcal{J}_{+}^{*}(q) \times \mathfrak{A}_{+}^{*}(q) \xi^{2} + \mathcal{J}_{+}^{*}(q) \times \mathfrak{A}_{+}^{*}(0) \xi^{2} + \mathcal{J}_{+}^{*}(0) \times \mathfrak{A}_{+}^{*}(q) \xi(1 - \xi)$$

$$+ \mathcal{J}_{+}^{*}(0) \times \mathfrak{A}_{+}^{*}(0) \xi(1 - \xi).$$

Integrating both sides of the above inequality over $[0,1]$, we obtain

$$\int_{0}^{1} \mathcal{J}_{+}(q + (1 - \xi)w(q,o)) \mathfrak{A}_{+}(q + (1 - \xi)w(q,o)) d\omega$$

$$= \frac{1}{w(q,o)} \int_{0}^{q + w(q,o)} \mathcal{J}_{+}(\omega) \mathfrak{A}_{+}(\omega) d\omega$$

$$\leq (\mathcal{J}_{+}(q) \mathfrak{A}_{+}(q) + \mathcal{J}_{+}(0) \mathfrak{A}_{+}(0)) \int_{0}^{1} \xi^{2} d\xi$$

$$+ (\mathcal{J}_{+}(q) \mathfrak{A}_{+}(0) + \mathcal{J}_{+}(0) \mathfrak{A}_{+}(q)) \int_{0}^{1} \xi(1 - \xi) d\xi,$$

$$\int_{0}^{1} \mathcal{J}_{+}^{*}(q + (1 - \xi)w(q,o)) \mathfrak{A}_{+}^{*}(q + (1 - \xi)w(q,o)) d\omega$$

$$= \frac{1}{w(q,o)} \int_{0}^{q + w(q,o)} \mathcal{J}_{+}^{*}(\omega) \mathfrak{A}_{+}^{*}(\omega) d\omega$$

$$\geq (\mathcal{J}_{+}^{*}(q) \mathfrak{A}_{+}^{*}(q) + \mathcal{J}_{+}^{*}(0) \mathfrak{A}_{+}^{*}(0)) \int_{0}^{1} \xi^{2} d\xi$$

$$+ (\mathcal{J}_{+}^{*}(q) \mathfrak{A}_{+}^{*}(0) + \mathcal{J}_{+}^{*}(0) \mathfrak{A}_{+}^{*}(q)) \int_{0}^{1} \xi(1 - \xi) d\xi.$$

It follows that

$$\frac{1}{w(q,o)} \int_{0}^{q + w(q,o)} \mathcal{J}_{+}(\omega) \mathfrak{A}_{+}(\omega) d\omega \leq \omega_{+}([q,o]) \int_{0}^{1} \xi^{2} d\xi + \eta_{+}([q,o]) \int_{0}^{1} \xi(1 - \xi) d\xi,$$

$$\frac{1}{w(q,o)} \int_{0}^{q + w(q,o)} \mathcal{J}_{+}^{*}(\omega) \mathfrak{A}_{+}^{*}(\omega) d\omega \geq \omega^{*}([q,o]) \int_{0}^{1} \xi^{2} d\xi + \eta^{*}([q,o]) \int_{0}^{1} \xi(1 - \xi) d\xi,$$
that is,
\[
\frac{1}{w(0, q)} \left[ \int_{q}^{1+w(q, o)} \mathcal{J}(\omega) \mathcal{A}(\omega) d\omega - \int_{0}^{w(q, o)} \mathcal{J}(\omega) \mathcal{A}(\omega) d\omega \right] \geq \left[ \frac{w^*(q, o)}{3}, \frac{\omega^*(q, o)}{3} \right] + \left[ \frac{\eta^*(q, o)}{6}, \frac{\eta^*(q, o)}{6} \right].
\]

Thus,
\[
\frac{1}{w(0, q)} \left( IR \right) \int_{q}^{1+w(q, o)} \mathcal{J}(\omega) \mathcal{A}(\omega) d\omega \geq \frac{\omega(0, q)}{3} + \frac{\eta(0, q)}{6},
\]
and the theorem has been established. □

**Example 3.** We consider the I-V-Fs \( \mathcal{J}, \mathcal{A} : [0, \omega + w(q, o)] \to K_C^+ \) defined by \( \mathcal{J}(\omega) = [2\omega^2, 4\omega] \) and \( \mathcal{A}(\omega) = [\omega, 2\omega] \). Since end point functions \( \mathcal{J}_s(\omega) = 2\omega^2, \mathcal{J}^*(\omega) = 4\omega \)
and \( \mathcal{A}_s(\omega) = \omega, \mathcal{A}^*(\omega) = 2\omega \) preinvex functions with respect to \( w(q, o) = q - o \). Hence, \( \mathcal{J}, \mathcal{A} \) both are preinvex I-V-Fs. We now compute the following:

\[
\frac{1}{w(q, o)} \int_{0}^{\mathcal{J}(\omega) \times \mathcal{A}(\omega) d\omega = \frac{1}{2},
\]

\[
\frac{1}{w(q, o)} \int_{0}^{\mathcal{J}^*(\omega) \times \mathcal{A}(\omega) d\omega = \frac{8}{3},
\]

\[
\frac{\omega_s(q, o)}{3} = \frac{2}{3},
\]

\[
\frac{\omega^*(q, o)}{3} = \frac{8}{3},
\]

\[
\frac{\eta_s(q, o)}{6} = 0,
\]

that means
\[
\left[ \begin{array}{c}
1 \\
\frac{2}{3}
\end{array} \right] \supseteq \left[ \begin{array}{c}
8 \\
\frac{8}{3}
\end{array} \right].
\]

Hence, Theorem 4 is verified.

The following assumption is required to prove the next result regarding the bi-function \( w : K \times K \to R \) which is known as:

**Condition C [70].** Let \( K \) be an invex set with respect to \( w \). For any \( o, q \in K \) and \( \xi \in [0, 1] \),

\[
w(q, o + \xi w(q, o)) = (1 - \xi)w(q, o),
\]

\[
w(o, o + \xi w(q, o)) = -\xi w(q, o).
\]

Clearly for \( \xi = 0 \), we have \( w(q, o) = 0 \) if and only if, \( q = o \), for all \( o, q \in K \). For the applications of Condition C, see [5,8,42,69,70].

**Theorem 5.** Let \( \mathcal{J}, \mathcal{A} : [0, o + w(q, o)] \to K_C^+ \), be two preinvex I-V-Fs such that \( \mathcal{J}(\omega) = [\mathcal{J}_s(\omega), \mathcal{J}^*(\omega)] \) and \( \mathcal{A}(\omega) = [\mathcal{A}_s(\omega), \mathcal{A}^*(\omega)] \) for all \( \omega \in [0, o + w(q, o)] \). If \( \mathcal{J}, \mathcal{A} \) and \( \mathcal{J} \times \mathcal{A} \in \mathcal{TR}(w(q, o)) \) and condition C hold for \( w \), then

\[
2 \mathcal{J} \left( \frac{2 + w(q, o)}{2} \right) \times \mathcal{A} \left( \frac{2 + w(q, o)}{2} \right) \geq \frac{1}{w(q, o)} \left( IR \right) \int_{0}^{\mathcal{J}(\omega) \times \mathcal{A}(\omega) d\omega + \frac{\omega(0, q)}{6} + \frac{\eta(0, q)}{3},
\]

where \( \omega(o, q) = \mathcal{J}(o) \times \mathcal{A}(o) + \mathcal{J}(q) \times \mathcal{A}(q), \eta(o, q) = \mathcal{J}(o) \times \mathcal{A}(q) + \mathcal{J}(q) \times \mathcal{A}(o), \) and \( \omega(o, q) = [\omega_s(o, q), \omega^*(o, q)] \) and \( \eta(o, q) = [\eta_s(o, q), \eta^*(o, q)] \).
Proof. Using condition C, we can write

\[
\omega + \frac{1}{2} \omega(q, o) = \omega + \tilde{\omega}(q, o) + \frac{1}{2} \omega(o + (1 - \tilde{\omega})\omega(q, o), o + \tilde{\omega}(q, o)).
\]

By hypothesis, we have

\[
\begin{align*}
\mathcal{J}_s\left(\frac{2 \omega + \omega(q, o)}{2}\right) \times \mathcal{A}_s\left(\frac{2 \omega + \omega(q, o)}{2}\right) & 
= \mathcal{J}_s\left(o + \tilde{\omega}(q, o) + \frac{1}{2} \omega(o + (1 - \tilde{\omega})\omega(q, o), o + \tilde{\omega}(q, o))\right) \\
& \times \mathcal{A}_s\left(o + \tilde{\omega}(q, o) + \frac{1}{2} \omega(o + (1 - \tilde{\omega})\omega(q, o), o + \tilde{\omega}(q, o))\right) \\
& = \mathcal{J}_s\left(o + \tilde{\omega}(q, o) + \frac{1}{2} \omega(o + (1 - \tilde{\omega})\omega(q, o), o + \tilde{\omega}(q, o))\right) \\
& \times \mathcal{A}_s\left(o + \tilde{\omega}(q, o) + \frac{1}{2} \omega(o + (1 - \tilde{\omega})\omega(q, o), o + \tilde{\omega}(q, o))\right)
\end{align*}
\]

Integrating over \([0, 1]\), we have

\[
\begin{align*}
2 \mathcal{J}_s\left(\frac{2 \omega + \omega(q, o)}{2}\right) \times \mathcal{A}_s\left(\frac{2 \omega + \omega(q, o)}{2}\right) & \leq \frac{1}{\omega(q, o)} \int_{o}^{\omega+\omega(q, o)} \mathcal{J}_s(\omega) \times \mathcal{A}_s(\omega) d\omega + \frac{\omega_s((o, q))}{6} + \frac{\eta_s((o, q))}{3}, \\
2 \mathcal{J}_s\left(\frac{2 \omega + \omega(q, o)}{2}\right) \times \mathcal{A}_s\left(\frac{2 \omega + \omega(q, o)}{2}\right) & \geq \frac{1}{\omega(q, o)} \int_{o}^{\omega+\omega(q, o)} \mathcal{J}_s(\omega) \times \mathcal{A}_s(\omega) d\omega + \frac{\omega_s((o, q))}{6} + \frac{\eta_s((o, q))}{3},
\end{align*}
\]

from which, we have

\[
\begin{align*}
2 \mathcal{J}_s\left(\frac{2 \omega + \omega(q, o)}{2}\right) \times \mathcal{A}_s\left(\frac{2 \omega + \omega(q, o)}{2}\right) & \leq \mathcal{J}_s\left(\frac{2 \omega + \omega(q, o)}{2}\right) \times \mathcal{A}_s\left(\frac{2 \omega + \omega(q, o)}{2}\right) \\
\mathcal{J}_s\left(\frac{2 \omega + \omega(q, o)}{2}\right) & \leq \frac{1}{\omega(q, o)} \int_{o}^{\omega+\omega(q, o)} \mathcal{J}_s(\omega) \times \mathcal{A}_s(\omega) d\omega + \frac{\omega_s((o, q))}{6} + \frac{\eta_s((o, q))}{3},
\end{align*}
\]
that is,

\[ 2 \mathcal{J} \left( \frac{2\omega + \mathfrak{w}(q, o)}{2} \right) \times \mathcal{A} \left( \frac{2\omega + \mathfrak{w}(q, o)}{2} \right) \]

\[ \geq \frac{1}{\mathfrak{w}(q, o)} (IR) \int_{0}^{\omega + \mathfrak{w}(q, o)} \mathcal{J}(\omega) \times \mathcal{A}(\omega) d\omega + \frac{\omega(q, o)}{6} + \eta(q, o) \]

\[ \geq 32 \tau \text{signed functions ions with} \]

this completes the proof. \( \square \)

**Example 4.** We consider the I-V-Fs \( \mathfrak{J}, \mathfrak{A} : [0, \omega + \mathfrak{w}(q, o)] \rightarrow \mathcal{K}^{+}_{C} \) defined by, \( \mathfrak{J}(\omega) = \{2\omega^{2}, 4\omega\} \) and \( \mathfrak{A}(\omega) = \{\omega, 2\omega\} \), then \( \mathfrak{J}(\omega), \mathfrak{A}(\omega) \) both are preinvex I-V-Fs with respect to \( \mathfrak{w}(q, o) = q - o \). We have \( \mathfrak{J}^{*}(\omega) = 2\omega^{2}, \mathfrak{A}^{*}(\omega) = 4\omega \) and \( \mathfrak{A}^{*}(\omega) = \omega, \mathfrak{A}^{*}(\omega) = 2\omega \).

We now compute the following:

\[ \frac{2 \mathfrak{J} \left( \frac{2\omega + \mathfrak{w}(q, o)}{2} \right)}{\mathfrak{w}(q, o) \times \mathfrak{A} \left( \frac{2\omega + \mathfrak{w}(q, o)}{2} \right)} \times \mathcal{A} \left( \frac{2\omega + \mathfrak{w}(q, o)}{2} \right) = \frac{1}{2'} \]

\[ \frac{1}{\mathfrak{w}(q, o)} (IR) \int_{0}^{\omega + \mathfrak{w}(q, o)} \mathcal{J}(\omega) \times \mathcal{A}(\omega) d\omega = \frac{1}{2} \]

\[ \frac{1}{\mathfrak{w}(q, o)} \omega_{j}(o, q) \]

\[ \frac{1}{\mathfrak{w}(q, o)} \omega^{+}(o, q) \]

\[ \mathfrak{J}^{*}(o, q) \]

\[ \mathfrak{A}^{*}(o, q) \]

\[ \eta^{*}(o, q) \]

\[ \eta^{*}(o, q) \]

that means

\[ \left[ \frac{1}{2'}, \frac{4}{3} \right] \supset \left[ \frac{5}{6}, \frac{4}{3} \right] \]

Hence, Theorem 5 is verified.

We now give HH-Fejér inequalities for preinvex I-V-Fs. Firstly, we obtain the second HH-Fejér inequality for preinvex I-V-F.

**Theorem 6.** Let \( \mathfrak{J} : [0, \omega + \mathfrak{w}(q, o)] \rightarrow \mathcal{K}^{+}_{C} \) be a preinvex I-V-F with \( \omega < o + \mathfrak{w}(q, o) \) such that \( \mathfrak{J}(\omega) = \{\mathfrak{J}^{*}(\omega), \mathfrak{J}^{*}(\omega)\} \) for all \( \omega \in [0, \omega + \mathfrak{w}(q, o)] \). If \( \mathfrak{J} \in \mathcal{IR}_{[0, \omega + \mathfrak{w}(q, o)]} \) and \( \mathcal{C} : [0, \omega + \mathfrak{w}(q, o)] \rightarrow \mathbb{R}, \mathcal{C}(\omega) \geq 0 \), symmetric with respect to \( \omega + \frac{1}{2} \mathfrak{w}(q, o) \), then

\[ \frac{1}{\mathfrak{w}(q, o)} (IR) \int_{0}^{\omega + \mathfrak{w}(q, o)} \mathfrak{J}(\omega)\mathcal{C}(\omega) d\omega \geq \left[ \mathfrak{J}(o) + \mathfrak{J}(q) \right] \int_{0}^{1} \xi \mathcal{C}(o + \xi \mathfrak{w}(q, o)) d\xi. \quad (21) \]

**Proof.** Let \( \mathfrak{J} \) be a preinvex I-V-F. Then, we have

\[ \mathfrak{J}(o + (1 - \xi)\mathfrak{w}(q, o))\mathcal{C}(o + (1 - \xi)\mathfrak{w}(q, o)) \leq (\xi \mathfrak{J}(o) + (1 - \xi)\mathfrak{J}(q))\mathcal{C}(o + (1 - \xi)\mathfrak{w}(q, o)), \]

\[ \mathfrak{J}^{*}(o + (1 - \xi)\mathfrak{w}(q, o))\mathcal{C}(o + (1 - \xi)\mathfrak{w}(q, o)) \geq (\xi \mathfrak{J}^{*}(o) + (1 - \xi)\mathfrak{J}^{*}(q))\mathcal{C}(o + (1 - \xi)\mathfrak{w}(q, o)). \quad (22) \]
Moreover,
\[ J_*(o + \xi w(q, o))C(o + \xi w(q, o)) \leq (1 - \xi)J_*(o) + \xi J_*(q))C(o + \xi w(q, o)), \]
\[ J^*(o + \xi w(q, o))C(o + \xi w(q, o)) \geq (1 - \xi)J^*(o) + \xi J^*(q))C(o + \xi w(q, o)). \] (23)

After adding (23) and (24), and integrating over [0, 1], we get
\[
\begin{align*}
\int_0^1 J_*(o + (1 - \xi)w(q, o))C(o + (1 - \xi)w(q, o))d\xi & \leq \int_0^1 J_*(o + \xi w(q, o))C(o + \xi w(q, o))d\xi \\
& + \int_0^1 J_*(o + \xi w(q, o))C(o + \xi w(q, o))d\xi \\
& + \int_0^1 J^*(o + \xi w(q, o))C(o + \xi w(q, o))d\xi \\
& + \int_0^1 J^*(o + \xi w(q, o))C(o + \xi w(q, o))d\xi \\
& = 2J_*(o)\int_0^1 \xi C(o + (1 - \xi)w(q, o))d\xi + 2J^*(q)\int_0^1 \xi C(o + \xi w(q, o))d\xi.
\end{align*}
\] (24)

Since \( C \) is symmetric, then
\[
\begin{align*}
\int_0^1 J_*(o + (1 - \xi)w(q, o))C(o + (1 - \xi)w(q, o))d\xi & \leq 2\int_0^1 J_*(o + \xi w(q, o))C(o + \xi w(q, o))d\xi \\
& + 2\int_0^1 J_*(o + \xi w(q, o))C(o + \xi w(q, o))d\xi \\
& = 2J_*(o)\int_0^1 \xi C(o + (1 - \xi)w(q, o))d\xi + 2J^*(q)\int_0^1 \xi C(o + \xi w(q, o))d\xi.
\end{align*}
\] (25)

We have
\[
\begin{align*}
\int_0^1 J_*(o + (1 - \xi)w(q, o))C(o + (1 - \xi)w(q, o))d\xi & = \int_0^1 J_*(o + \xi w(q, o))C(o + \xi w(q, o))d\xi \\
& = \int_0^1 \xi C(o + \xi w(q, o))d\xi \\
& = \int_0^1 J^*(o + (1 - \xi)w(q, o))C(o + (1 - \xi)w(q, o))d\xi \\
& = \int_0^1 \xi C(o + \xi w(q, o))d\xi \\
& = \int_0^1 \xi C(o + \xi w(q, o))d\xi.
\end{align*}
\] (26)

From (26), we have
\[
\frac{1}{\omega(q, o)} \int_0^{\omega(q, o)} J_*(\omega)C(\omega)d\omega \leq [J_*(o) + J_*(q)] \int_0^1 \xi C(o + \xi w(q, o))d\xi,
\]
\[
\frac{1}{\omega(q, o)} \int_0^{\omega(q, o)} J^*(\omega)C(\omega)d\omega \geq [J^*(o) + J^*(q)] \int_0^1 \xi C(o + \xi w(q, o))d\xi,
\]
that is,
\[
\left[ \frac{1}{\omega(q, o)} \int_0^{\omega(q, o)} J_*(\omega)C(\omega)d\omega, \frac{1}{\omega(q, o)} \int_0^{\omega(q, o)} J^*(\omega)C(\omega)d\omega \right] \supseteq [J_*(o) + J_*(q), J^*(o) + J^*(q)] \int_0^1 \xi C(o + \xi w(q, o))d\xi.
\]
Hence,
\[
\frac{1}{m(q,o)} \left( \int_q^{a+m(q,o)} \tilde{J}(\omega) C(\omega) d\omega \right) \geq [\tilde{J}(\omega) + \tilde{J}(q)] \int_q^1 \xi C(\omega + \xi m(q,o)) d\xi.
\]

Next, we construct first HH-Fejér inequality for preinvex I-V-F, which generalizes first HH-Fejér inequalities for preinvex function, see \[69,70\]. □

**Theorem 7.** Let \( \tilde{J} : [0, o + m(q,o)] \to K_C^+ \) be a preinvex I-V-F with \( o < o + m(q,o) \) such that \( \tilde{J}(\omega) = [\tilde{J}_-(\omega), \tilde{J}_+(\omega)] \) for all \( \omega \in [0, o + m(q,o)] \). If \( \tilde{J} \in \mathcal{F} \mathcal{R}_{(o, o + m(q,o))} \) and \( C : [o, o + m(q,o)] \to \mathbb{R}, C(\omega) \geq 0, \) symmetric with respect to \( o + \frac{1}{2}m(q,o) \), and \( \int_q^{a+m(q,o)} C(\omega) d\omega > 0 \), and Condition C for \( m, \) then
\[
\tilde{J} \left( o + \frac{1}{2}m(q,o) \right) \geq \frac{1}{\int_q^{a+m(q,o)} C(\omega) d\omega} \left( \int_q^{a+m(q,o)} \tilde{J}(\omega) C(\omega) d\omega \right).
\]

**Proof.** Using condition C, we can write
\[
o + \frac{1}{2}m(q,o) = o + \tilde{\xi} m(q,o) + \frac{1}{2}m(o + (1 - \tilde{\xi})m(q,o), o + \tilde{\xi} m(q,o)).
\]

Since \( \tilde{J} \) is a preinvex, we have
\[
\tilde{J}_-(o + \frac{1}{2}m(q,o)) = \tilde{J}_-(o + \tilde{\xi} m(q,o) + \frac{1}{2}m(o + (1 - \tilde{\xi})m(q,o), o + \tilde{\xi} m(q,o)) \leq \frac{1}{2} (\tilde{J}_+(o + (1 - \tilde{\xi})m(q,o)) + \tilde{J}_+(o + \tilde{\xi} m(q,o))),
\]
\[
\tilde{J}_+(o + \frac{1}{2}m(q,o)) = \tilde{J}_+(o + \tilde{\xi} m(q,o) + \frac{1}{2}m(o + (1 - \tilde{\xi})m(q,o), o + \tilde{\xi} m(q,o)) \geq (\tilde{J}_+(o + (1 - \tilde{\xi})m(q,o)) + \tilde{J}_+(o + \tilde{\xi} m(q,o))).
\]

By multiplying (28) by \( C(o + (1 - \tilde{\xi})m(q,o)) = C(o + \tilde{\xi} m(q,o)) \) and integrating it by \( \tilde{\xi} \) over \([0,1]\), we obtain
\[
\begin{align*}
\tilde{J}_+(o + \frac{1}{2}m(q,o)) & \int_0^1 C(o + \tilde{\xi} m(q,o)) d\tilde{\xi} \\
& \leq \frac{1}{2} \left( \int_0^1 \tilde{J}_+(o + (1 - \tilde{\xi})m(q,o)) C(o + (1 - \tilde{\xi})m(q,o)) d\tilde{\xi} + \int_0^1 \tilde{J}_+(o + \tilde{\xi} m(q,o)) d\tilde{\xi} \right) C(o + \tilde{\xi} m(q,o)) \\
& \geq \frac{1}{2} \left( \int_0^1 \tilde{J}_+(o + (1 - \tilde{\xi})m(q,o)) C(o + (1 - \tilde{\xi})m(q,o)) d\tilde{\xi} + \int_0^1 \tilde{J}_+(o + \tilde{\xi} m(q,o)) C(o + \tilde{\xi} m(q,o)) d\tilde{\xi} \right).
\end{align*}
\]

Since
\[
\int_0^1 \tilde{J}_+(o + (1 - \tilde{\xi})m(q,o)) C(o + (1 - \tilde{\xi})m(q,o)) d\tilde{\xi} = \int_0^1 \tilde{J}_+(o + \tilde{\xi} m(q,o)) C(o + \tilde{\xi} m(q,o)) d\tilde{\xi} = \frac{1}{m(q,o)} \int_q^{a+m(q,o)} \tilde{J}_+(\omega) C(\omega) d\omega
\]
\[
\int_0^1 \tilde{J}_+(o + (1 - \tilde{\xi})m(q,o)) C(o + (1 - \tilde{\xi})m(q,o)) d\tilde{\xi} = \frac{1}{m(q,o)} \int_q^{a+m(q,o)} \tilde{J}_+(\omega) C(\omega) d\omega.
\]

From (30), we have
\[
\begin{align*}
\tilde{J}_+(o + \frac{1}{2}m(q,o)) & \leq \frac{1}{\int_q^{a+m(q,o)} C(\omega) d\omega} \int_q^{a+m(q,o)} \tilde{J}_+(\omega) C(\omega) d\omega, \\
\tilde{J}_+(o + \frac{1}{2}m(q,o)) & \geq \frac{1}{\int_q^{a+m(q,o)} C(\omega) d\omega} \int_q^{a+m(q,o)} \tilde{J}_+(\omega) C(\omega) d\omega.
\end{align*}
\]
From which, we have
\[
\left[ \mathcal{J}_n \left( \sigma + \frac{1}{2} w(q, o) \right), \mathcal{J}_n^* \left( \sigma + \frac{1}{2} w(q, o) \right) \right] \supseteq \left\{ \int_a^{\sigma + w(q, o)} \mathcal{J}_n(\omega)C(\omega) d\omega, \int_a^{\sigma + w(q, o)} \mathcal{J}_n^*(\omega)C(\omega) d\omega \right\},
\]
that is,
\[
\mathcal{J} \left( \sigma + \frac{1}{2} w(q, o) \right) \supseteq \frac{1}{\int_a^{\sigma + w(q, o)} C(\omega) d\omega} (IR) \int_a^{\sigma + w(q, o)} \mathcal{J}(\omega)C(\omega) d\omega.
\]
This completes the proof. \(\square\)

**Remark 3.**

(i) If \(w(q, o) = q - o\), then inequalities in Theorems 6 and 7 reduce for convex I-V-Fs, see [78].

(ii) If \(\mathcal{J}_n(\sigma) = \mathcal{J}_n^*(\sigma)\), then Theorems 6 and 7 reduce to classical first and second HH-Fejér inequality for preinvex function, see [69].

(iii) If \(\mathcal{J}_n(\sigma) = \mathcal{J}_n^*(\sigma)\) and \(w(q, o) = q - o\) then Theorems 6 and 7 reduce to classical first and second HH-Fejér inequality for convex function, see [71].

**Example 5.** We consider the I-V-F \(\mathcal{J} : [1, 1 + w(4, 1)] \to K^+_C\) defined by \(\mathcal{J}(\omega) = \left[ \frac{1}{\omega}, \omega \right]\). Since end point functions \(\mathcal{J}_n(\omega), \mathcal{J}_n^*(\omega)\) are preinvex functions \(w(\sigma, \omega) = \sigma - \omega\), then \(\mathcal{J}(\omega)\) is preinvex I-V-F. If

\[
C(\omega) = \begin{cases} \omega - 1, & \sigma \in [1, \frac{5}{2}], \\ 4 - \omega, & \sigma \in \left(\frac{5}{2}, 4\right). \end{cases}
\]

Then, we have
\[
\begin{align*}
\frac{1}{w(4, 1)} & \int_1^{1 + w(4, 1)} \mathcal{J}_n(\omega)C(\omega) d\omega \\
\frac{1}{w(4, 1)} & \int_1^{1 + w(4, 1)} \mathcal{J}_n^*(\omega)C(\omega) d\omega \\
& = \frac{1}{2} \int_1^4 \mathcal{J}_n(\omega)C(\omega) d\omega + \frac{1}{2} \int_4^5 \mathcal{J}_n(\omega)C(\omega) d\omega + \frac{1}{2} \int_2^4 \mathcal{J}_n(\omega)C(\omega) d\omega,
\end{align*}
\]
\[
= \frac{1}{2} \int_1^4 \mathcal{J}_n^*(\omega)C(\omega) d\omega + \frac{1}{2} \int_4^5 \mathcal{J}_n^*(\omega)C(\omega) d\omega + \frac{1}{2} \int_2^4 \mathcal{J}_n^*(\omega)C(\omega) d\omega,
\]
\[
= \frac{1}{2} \int_1^4 \frac{1}{\omega} (\omega - 1) d\omega + \frac{1}{2} \int_4^5 \frac{1}{\omega} (4 - \omega) d\omega = \frac{1}{2} (4 \log \left(\frac{8}{5}\right) + \log \left(\frac{5}{2}\right)),
\]
\[
= \frac{1}{2} \int_1^4 \omega (\omega - 1) d\omega + \frac{1}{2} \int_2^4 \omega (4 - \omega) d\omega = \frac{15}{8}.
\]

Moreover,
\[
\begin{align*}
[\mathcal{J}_n(\sigma) + \mathcal{J}_n(q)] & \int_0^1 \xi C(\sigma + \xi w(q, o)) d\xi \\
[\mathcal{J}_n^*(\sigma) + \mathcal{J}_n^*(q)] & \int_0^1 \xi C(\sigma + \xi w(q, o)) d\xi \\
& = \frac{5}{4} \left[ \int_0^1 \frac{3}{2} \xi^2 d\xi + \frac{1}{2} \xi (3 - 3\xi) d\xi \right] = \frac{15}{32},
\end{align*}
\]
\[
= \frac{5}{4} \left[ \int_0^1 \frac{3}{2} \xi^2 d\xi + \frac{1}{2} \xi (3 - 3\xi) d\xi \right] = \frac{15}{8}.
\]

From (31) and (32), we have
\[
\left[ \frac{1}{2} \left( 4 \log \left(\frac{8}{5}\right) + \log \left(\frac{5}{2}\right) \right), \frac{15}{8} \right] \supseteq \left[ \frac{15}{32}, \frac{15}{8} \right].
\]

Hence, Theorem 6 is verified.
For Theorem 7, we have
\[
J^* \left( o + \frac{1}{2} w(q, o) \right) = \frac{2}{5}, \\
J^* \left( o + \frac{1}{2} w(q, o) \right) = \frac{5}{2},
\]
(32)
\[
\int_o^{o + \frac{1}{2} w(q, o)} C(\omega) d\omega = \int_{\frac{5}{2}}^1 (\omega - 1) d\omega + \int_{\frac{5}{2}}^2 (4 - \omega) d\omega = \frac{9}{4},
\]
(33)
\[
\int_o^{o + \frac{1}{2} w(q, o)} C(\omega) d\omega = \int_1^4 J^*(\omega) C(\omega) d\omega = \frac{4}{3} \left( 4\log \left( \frac{8}{5} \right) + \log \left( \frac{5}{2} \right) \right)
\]
\[
\int_o^{o + \frac{1}{2} w(q, o)} C(\omega) d\omega = \int_1^4 J^*(\omega) C(\omega) d\omega = \frac{5}{2}
\]
From (33) and (34), we have
\[
\left[ \frac{2}{5}, \frac{5}{2} \right] \supseteq \left[ \frac{4}{3} \left( 4\log \left( \frac{8}{5} \right) + \log \left( \frac{5}{2} \right) \right), \frac{5}{2} \right]
\]
Hence, Theorem 7 is verified.

4. Conclusions
We constructed the new Hermite–integral Hadamard’s inequality for preinvex interval-valued functions in this study employing the interval integral operators with exponential kernel supplied by Moore in ref. [68]. In order to show the size relationship of the function values of the inequalities and to confirm the veracity of the findings, we offered four numerical examples. Our study of interval integral operator-type integral inequalities will broaden the practical application of Hermite–Hadamard-type inequalities, because integral operators are frequently used in engineering technology, such as mathematical models, and because different integral operators are suitable for different types of practical problems. We will study these inequalities further using various types of integral operators, because we are aware that integral operators are employed in many other fields. This will also provide a direction for our future research.

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