Sharp interface limit in a phase field model of cell motility

Leonid Berlyand · Volodymyr Rybalko · Mykhailo Potomkin

Abstract We consider a system of two PDEs arising in modeling of motility of eukaryotic cells on substrates. This system consists of the Allen-Cahn equation for the scalar phase field function coupled with another vectorial parabolic equation for the orientation of the actin filament network.

The two key properties of this system are (i) presence of gradients in the coupling terms (gradient coupling) and (ii) mass (volume) preservation constraints. We first prove that the sharp interface property of initial conditions is preserved in time. Next we formally derive the equation of the motion of the interface, which is the mean curvature motion perturbed by a nonlinear term that appears due to the properties (i)-(ii). This novel term leads to surprising features of the motion of the interface.

Because of these properties maximum principle and classical comparison techniques do not apply to this system. Furthermore, the system can not be written in a form of gradient flow, which is why recently developed $\Gamma$-convergence techniques also can not be used for the justification of the formal derivation. Such justification is presented in a one-dimensional model problem and it leads to a stability result in a class of “sharp interface” initial data.

The work of LB was supported by NSF grant DMS-1106666. The work of VR and MP was partially supported by NSF grant DMS-1106666. The authors are grateful to I. Aronson and F. Ziebert for useful discussions on the phase field model of cell motility introduced in their paper.

L. Berlyand
Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA
E-mail: berlyand@math.psu.edu

V. Rybalko
Mathematical Division, B. Verkin Institute for Low Temperature, Physics and Engineering of National Academy of Sciences of Ukraine, 47 Lenin Ave., 61103 Kharkiv, Ukraine
E-mail: vrybalko@ilt.kharkov.ua

M. Potomkin
The Pennsylvania State University, University Park, PA 16802, USA,
E-mail: potomkin@math.psu.edu
1 Introduction

The problem of cell motility has been a classical subject in biology for several centuries. It dates back to the celebrated discovery by van Leeuwenhoek in 17th century who drastically improved microscope to the extent that he was able to observe motion of single celled organisms that moved due contraction and extension. Three centuries later this problem continues to attract the attention of biologists, biophysicists and, more recently, applied mathematicians. A comprehensive review of the mathematical modeling of cell motility can be found in [22].

This work is motivated by the problem of motility (crawling motion) of eukariotic cells on substrates. The network of actin (protein) filaments (which is a part of cytoskeleton in such cells) plays an important role in cell motility. We are concerned with the cell motility caused by extension of front of the cell due to polymerization of the actin filaments and contraction of the back of the cell due to detachment of these filaments. Modeling of this process in full generality is at present a formidable challenge because several important biological ingredients (e.g., regulatory pathways [22]) are not yet well understood.

However, in recent biophysical studies several simplified phase field models have been proposed. Simulations performed for these models demonstrated good agreement with experiments (e.g., [10,13] and references their in). Recall that phase field models are typically used to describe the evolution of an interface between two phases (e.g., solidification or viscous fingering). The key ingredient of such models is an auxiliary scalar field, which takes two different values in domains describing the two phases (e.g., 1 and 0) with a diffuse interface of a small (non zero) width. The advantage of such an approach is that it allows us to consider one set of PDEs in the whole domain occupied by both phases and therefore avoids an issue of coupling two different sets of PDEs in each phase, which is typically quite complicated in both simulations and analysis.

In this work we present rigorous mathematical analysis of the 2D phase field model proposed in [13] that consists of a system of two PDEs for the phase field function and orientation vector with an integral mass conservation constraint. This model can be rewritten in a simplified form suitable for asymptotical analysis, so that all key features of the qualitative behavior are preserved, which can be seen from a comparison of simulations from [13] with our analytical results. First, in [13] the integral mass conservation constraint is introduced in the PDE system by adding a penalization parameter into the double-well potential (formulas (2.2), (2.5)-(2.6) in [13]). We introduce this constraint via a dynamic Lagrange multiplier $\lambda(t)$ defined below, which provides the same qualitative behavior of solutions. Second, for technical simplicity we drop two terms in the second equation (for polarization) in the phase field system [13], since our analysis shows that these terms can be incorporated with minor changes in both the results and the techniques. Thirdly, in order to study the long term behavior of the system, we perform the diffusive scaling ($t \mapsto \epsilon^2 t$, $x \mapsto \epsilon x$).
Indeed, the crawling motion is very slow and time variable needs to be “accelerated”. Thus, we arrived at the following system of parabolic PDEs for a scalar phase field function $\rho_{\varepsilon}$ and the orientation vector $P_{\varepsilon}$:

\[
\frac{\partial \rho_{\varepsilon}}{\partial t} = \Delta \rho_{\varepsilon} - \frac{1}{\varepsilon^2} W'(\rho_{\varepsilon}) - P_{\varepsilon} \cdot \nabla \rho_{\varepsilon} + \lambda_{\varepsilon}(t) \quad \text{in} \quad \Omega, \tag{1}
\]

\[
\frac{\partial P_{\varepsilon}}{\partial t} = \varepsilon \Delta P_{\varepsilon} - \frac{1}{\varepsilon} P_{\varepsilon} - \beta \nabla \rho_{\varepsilon} \quad \text{in} \quad \Omega. \tag{2}
\]

On the boundary $\partial \Omega$ we impose the Neumann and the Dirichlet boundary conditions respectively:

\[
\frac{\partial \rho_{\varepsilon}}{\partial n} = 0 \quad \text{and} \quad P_{\varepsilon} = 0, \quad \text{where} \quad \Omega \subset \mathbb{R}^2 \text{ is a bounded smooth domain.}
\]

Equation (1) is a perturbation of the following Allen-Cahn equation:

\[
\frac{\partial \rho_{\varepsilon}}{\partial t} = \Delta \rho_{\varepsilon} - \frac{1}{\varepsilon^2} W'(\rho_{\varepsilon}). \tag{3}
\]

The latter equation is a scalar version of the celebrated Ginzburg-Landau equation and it plays a fundamental role in mathematical modeling of phase transitions. It consists of the standard linear parabolic equation and a nonlinear lower order term, which is the derivative of a smooth double well potential

\[
W(\rho) = \frac{1}{4}\rho^2(1-\rho)^2. \tag{4}
\]

Equation (3) was introduced to model the motion of phase-antiphase boundary (interface) between two grains in a solid material. Analysis of (3) as $\varepsilon \to 0$ led to the asymptotic solution that takes values 0 and 1 in the domains corresponding to two phases separated by an interface of the width of order $\varepsilon$, the so-called sharp interface. Furthermore, it was shown that this sharp interface exhibits the mean curvature motion. Recall that in this motion the normal component of the velocity of each point of the surface is equal to the mean curvature of the surface at this point. This motion has been extensively studied in the geometrical community (e.g., [17,18,16,4] and references therein). It also received significant attention in PDE literature. Specifically [9] and [11] established existence of global viscosity solutions (weak solutions) for the mean curvature flow. The mean curvature motion of the interface in the limit $\varepsilon \to 0$ was formally derived in [19], [14] and then justified in [12] by using the viscosity solutions techniques.

Note that equation (3) is closely related to another well-known model of phase separation, the so-called Cahn-Hilliard equation (4), which is a forth order reaction diffusion equation that models how two components of a binary fluid spontaneously separate and form domains of two pure fluids.

There are two distinguishing features in the problem (1)-(2): coupling and a non-local mass conservation constraint. We first comment on the coupling. Note that another prominent biological FitzHugh-Nagumo model has similar coupling feature but in (1)-(2) the coupling occurs via spatial gradients (gradient coupling) of the unknown functions where as in FitzHugh-Nagumo [23], [27] the two equations are coupled via lower order terms (unknown functions rather than their derivatives). There are several phase field models for Allen-Cahn (also Cahn-Hilliard) equation coupled
with another parabolic equation via lower order terms \[12,7\]. Our analysis shows that the gradient coupling results in novel mathematical features such as the following nonlinear nonlocal equation for the velocity of the interface curve \( \Gamma(t) \) derived below:

\[
V = \kappa + \frac{\beta}{c_0} \Phi(V) - \frac{1}{|\Gamma(t)|} \int_{\Gamma(t)} \left( \kappa + \frac{\beta}{c_0} \Phi(V) \right) ds.
\]  

(5)

Here \( V \) stands for the normal velocity of \( \Gamma(t) \) with respect to the inward normal, and \( \kappa \) denotes the curvature of \( \Gamma(t) \), \( c_0 \) is a constant determined by the potential \( W \) (\( c_0 = \sqrt{3/2} \) for the specific choice \( \Phi \) of the double-well potential), \(|\Gamma(t)|\) is the curve length, and function \( \Phi(V) \) is given by (91).

Next note that the term \( \lambda_\varepsilon (t) \) in (1) is a Lagrange multiplier responsible for the volume constraint (conservation mass in the original physical problem [13]) and it has the following form

\[
\lambda_\varepsilon (t) = \frac{1}{|\Omega|} \int_\Omega \left( \frac{1}{\varepsilon^2} W'(\rho_\varepsilon) + P_\varepsilon \cdot \nabla \rho_\varepsilon \right) dx
\]  

(6)

Solutions of stationary Allen-Cahn equation with the volume constraint were studied in [21] by \( \Gamma \)-convergence techniques applied to the stationary variational problem corresponding to (3). It was established that the \( \Gamma \)-limiting functional is the interface perimeter (curve length in 2D or surface area in higher dimensions). Subsequently in the work [25] an evolutionary reaction-diffusion equation with double-well potential and nonlocal term that describes the volume constraint was studied. The following asymptotic formula for evolution of interface \( \Gamma \) in the form of volume preserving mean curvature flow was formally derived in [25]

\[
V = \kappa - \frac{1}{|\Gamma(t)|} \int_{\Gamma(t)} \kappa ds
\]  

(7)

Formula (7) was rigorously justified in the radially symmetric case in [5] and in general case in [8].

There are three main approaches to the study of asymptotic behavior (sharp interface limit) of solutions of phase field equations and systems.

When comparison principle for solutions applies, a PDE approach based on viscosity solutions techniques was successfully used in [12,3,28,20,2,15]. This approach can not be applied to the system (1)-(2), because of the gradient coupling and nonlocal multiplier \( \lambda_\varepsilon(t) \). It is an open issue to introduce weak (e.g., viscosity or Brakke type) solutions in problems with constraints. Furthermore, since there is no comparison principle, the only technique available for the justification of the sharp interface limit is energy bounds which become quite difficult due to the coupling and the volume preservation.

Another technique used in such problems is \( \Gamma \)-convergence (see [25] and references therein). It also does not work for the system (1)-(2). Standard Allen-Cahn equation (3) is a gradient flow (in \( L^2 \) metric) with GL energy functional, which is why one can use the \( \Gamma \)-convergence approach. However, there is no energy functional such that problem (1)-(2) can be written as a gradient flow.
As explained above the gradient coupling and the volume constraint are the key features of the problem (1), (2), (6) and they led to both novel results and analysis techniques. Specifically, the objectives of our study are three fold:

(i) To show that there is no finite time blow up and the sharp interface property of the initial data propagates in time.
(ii) To investigate how the gradient coupling combined with the nonlocal volume constraints changes the limiting equation of the interface motion.
(iii) To develop novel techniques for the justification of the limiting equation of the interface motion that, in particular, includes rigorous derivation of asymptotic expansion for the solution of the problem (1)-(2).

The paper is organized as follows. In section 2 is devoted to the objective (i). Section 3 the objectives (ii) and (iii) are addressed in the context of a model one-dimensional problem. In Section 4 the equation for the interface motion (5) is formally derived.

2 Existence of the sharp interface solutions that do not blow up in finite time

In this section we consider the boundary value problem (1), (2) with \( \lambda_\varepsilon \) given by (6).

Introduce the following functionals

\[
E_\varepsilon(t) := \frac{\varepsilon}{2} \int_\Omega |\nabla \rho_\varepsilon(x,t)|^2 \, dx + \frac{1}{\varepsilon} \int_\Omega W(\rho_\varepsilon(x,t)) \, dx, \\
F_\varepsilon(t) := \int_\Omega \left( |P_\varepsilon(x,t)|^2 + |P_\varepsilon(x,t)|^4 \right) \, dx.
\]

The system (1)-(2) is supplied with “well prepared” initial data, which means that two conditions hold:

\[-\varepsilon^{1/4} \leq \rho_\varepsilon(x,0) \leq 1 + \varepsilon^{1/4}, \]

and

\[E_\varepsilon(0) + F_\varepsilon(0) \leq C, \]

Theorem 1 If the initial data \( \rho_\varepsilon^0 := \rho_\varepsilon(x,0), P_\varepsilon^0 := P_\varepsilon(x,0) \) satisfy (9) and (10), then for any \( T > 0 \) the solution \( \rho_\varepsilon, P_\varepsilon \) exists on the time interval \( (0, T) \) for sufficiently small \( \varepsilon > 0, \varepsilon < \varepsilon_0(T) \). Moreover, it satisfies \(-\varepsilon^{1/4} \leq \rho_\varepsilon(x,t) \leq 1 + \varepsilon^{1/4} \) and

\[\varepsilon \int_0^T \int_\Omega \left( \frac{\partial \rho_\varepsilon}{\partial t} \right)^2 \, dx \, dt \leq C, \quad E_\varepsilon(t) + F_\varepsilon(t) \leq C \quad \forall t \in (0, T), \]

where \( C \) is independent of \( t \) and \( \varepsilon \).
Remark 1 This theorem implies that if the initial data are well-prepared in the sense of (9)–(10) then for $0 < t < T$ the solution exists and has the structure of $\varepsilon$-transition layer. Moreover, the bound on initial data (5) remains true for $t > 0$, while it relies on maximum principle argument, it also requires additional estimates on $\lambda_\varepsilon$ as seen from (13) below.

Proof
STEP 1. First multiply (1) by $\partial_t \rho_\varepsilon$ and integrate over $\Omega$:

$$
\int_{\Omega} |\partial_t \rho_\varepsilon|^2 \, dx + \frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} |\nabla \rho_\varepsilon|^2 + \frac{1}{\varepsilon^2} W(\rho_\varepsilon) \right) \, dx = - \int_{\Omega} P_\varepsilon \cdot \nabla \rho_\varepsilon \partial_t \rho_\varepsilon \, dx
$$

$$
\leq \frac{1}{2} \int_{\Omega} |\partial_t \rho_\varepsilon|^2 \, dx + \frac{1}{2} \int_{\Omega} |P_\varepsilon|^2 |\nabla \rho_\varepsilon|^2 \, dx.
$$

(12)

Here we used that due to (6) the integral of $\partial_t \rho_\varepsilon$ over $\Omega$ is zero and thus

$$
\int \lambda_\varepsilon(t) \partial_t \rho_\varepsilon \, dx = 0.
$$

Next, using the maximum principle in (1) we get

$$
-2\varepsilon^2 \sup_{\tau \in [0,t]} |\lambda_\varepsilon(\tau)| \leq \rho_\varepsilon \leq 1 + 2\varepsilon^2 \sup_{\tau \in [0,t]} |\lambda_\varepsilon(\tau)|.
$$

(13)

Let $T_\varepsilon > 0$ be the maximal time such that

$$
-\varepsilon^{1/4} \leq \rho_\varepsilon \leq 1 + \varepsilon^{1/4}, \quad \text{when } t \leq T_\varepsilon,
$$

(14)

and from now on we assume that $t \leq T_\varepsilon$.

Using (12), (14) and integration by parts we obtain

$$
\frac{d}{dt} E_\varepsilon + \frac{\varepsilon}{4} \int_{\Omega} |\partial_t \rho_\varepsilon|^2 \, dx \leq \varepsilon \int \left( |P_\varepsilon|^2 |\Delta \rho_\varepsilon| + |\nabla |P_\varepsilon|^2| |\nabla \rho_\varepsilon| \right) \, dx
$$

(15)

We proceed by deriving an upper bound for the integral in the right hand side of (15). By (1) we have

$$
\int_{\Omega} |\Delta \rho_\varepsilon| |P_\varepsilon|^2 + |\nabla |P_\varepsilon|^2|| |\nabla \rho_\varepsilon| \right) \, dx \leq \int_{\Omega} |\partial_t \rho_\varepsilon||P_\varepsilon|^2 \, dx + \int_{\Omega} |P_\varepsilon| \cdot |\nabla P_\varepsilon||P_\varepsilon|^2 \, dx + \int_{\Omega} |\nabla \rho_\varepsilon| \left( |P_\varepsilon|^2 \right) \, dx
$$

$$
+ \frac{1}{\varepsilon^2} \int_{\Omega} |W'(\rho_\varepsilon)| |P_\varepsilon|^2 \, dx + |\lambda_\varepsilon| \int_{\Omega} |P_\varepsilon|^2 \, dx =: \sum_{i=1}^{5} I_i.
$$

(16)

The following bounds are obtained by routine applying the Cauchy-Schwarz and Young’s inequalities. For the the sum of the first three terms in (16) we get,

$$
\sum_{i} I_i \leq \varepsilon \int_{\Omega} |\partial_t \rho_\varepsilon|^2 \, dx + \varepsilon \int_{\Omega} |P_\varepsilon|^2 |\nabla P_\varepsilon|^2 \, dx + \frac{1}{2\varepsilon} \int_{\Omega} |P_\varepsilon|^4 \, dx + \frac{1}{\varepsilon} \int_{\Omega} |\nabla |P_\varepsilon|^2||^2 \, dx + \frac{1}{\varepsilon} E_\varepsilon.
$$

Since $(W'(\rho_\varepsilon))^2 \leq 6W(\rho_\varepsilon)$ we also have

$$
\frac{1}{\varepsilon^2} \int_{\Omega} |W'(\rho_\varepsilon)||P_\varepsilon|^2 \, dx \leq \frac{3}{\varepsilon^2} \int_{\Omega} W(\rho) \, dx + \frac{1}{2\varepsilon} \int_{\Omega} |P_\varepsilon|^4 \, dx \leq \frac{3}{\varepsilon} E_\varepsilon + \frac{1}{2\varepsilon} \int_{\Omega} |P_\varepsilon|^4 \, dx.
$$
Finally, in order to bound $I_5$ we first derive,

$$|\lambda_{e}(t)| \leq \frac{C}{\varepsilon^2} \left( \int_{\Omega} W(\rho_e) dx \right)^{1/2} + \left( \int_{\Omega} |\nabla \rho_e|^2 dx \right)^{1/2} \left( \int_{\Omega} |P_e|^2 dx \right)^{1/2}$$

$$\leq \frac{C}{\varepsilon} \left( \frac{E_e}{\varepsilon} \right)^{1/2} + \left( \frac{2E_e}{\varepsilon} \right)^{1/2} \left( \int_{\Omega} |P_e|^2 dx \right)^{1/2},$$

(17)

then

$$I_5 \leq \frac{C}{\varepsilon} \left( \frac{E_e}{\varepsilon} \right)^{1/2} \int_{\Omega} |P_e|^2 dx + \left( \frac{2E_e}{\varepsilon} \right)^{1/2} \left( \int_{\Omega} |P_e|^2 dx \right)^{3/2} \leq \frac{C}{\varepsilon} E_e + \frac{1}{2\varepsilon^2} \int_{\Omega} |P_e|^4 dx + E_e^2 + \frac{C}{\varepsilon^{2/3}} \int_{\Omega} |P_e|^4 dx.$$

Thus,

$$\sum_{1}^{5} I_i \leq \frac{C}{\varepsilon^2} E_e + E_e^2 + \frac{1}{\varepsilon^2} \int_{\Omega} |\partial_t \rho_e|^2 dx + E_e^2 + \frac{1}{\varepsilon^2} \int_{\Omega} |P_e|^4 dx + \int_{\Omega} |\nabla |P_e|^2|^2 dx,$$

and using this inequality, (16) and (14) in (15), then substituting the resulting bound in (12) we obtain, for sufficiently small $\varepsilon$,

$$\frac{1}{4} \int_{\Omega} |\partial_t \rho_e|^2 dx + \frac{1}{\varepsilon^2} \int_{\Omega} \|\rho_e\|^2 dx + \frac{1}{\varepsilon^2} \int_{\Omega} |P_e|^4 dx + \int_{\Omega} |\nabla |P_e|^2|^2 dx \tag{18}$$

STEP 2. Now we obtain a bound for the last two terms in (18). Taking the scalar product of (2) with $2kP_e + 4|P_e|^2 P_e$, $k > 0$, and integrating over $\Omega$ we get

$$\frac{d}{dt} \int_{\Omega} (k|P_e|^2 + |P_e|^4) dx + \varepsilon \int_{\Omega} (2k|\nabla P_e|^2 + 4|\nabla P_e|^2 |P_e|^2 + 2|\nabla |P_e|^2|^2 |P_e|^2 + 2 \int_{\Omega} (k|P_e|^2 + 2|P_e|^4) dx$$

$$\leq -2k \beta \int_{\Omega} P_e \cdot \nabla P_e dx + 4\beta \int_{\Omega} P_e \text{div}(P_e|P_e|^2) dx$$

$$\leq kC \int_{\Omega} |\nabla P_e|^2 dx + \frac{k}{\varepsilon} \int_{\Omega} |P_e|^2 dx + \frac{C}{\varepsilon} \int_{\Omega} |P_e|^2 dx.$$

We chose $k := C_1 + 1$ to obtain

$$\varepsilon \int_{\Omega} |\nabla |P_e|^2|^2 dx + \frac{1}{\varepsilon} \int_{\Omega} |P_e|^4 dx \leq CE_e - \frac{d}{dt} \int_{\Omega} (k|P_e|^2 + |P_e|^4) dx. \tag{19}$$

STEP 3. Finally, introducing $G_e = E_e + \int_{\Omega} (4k|P_e|^2 + |P_e|^4) dx$, by (18) and (19) we have the differential inequality,

$$\frac{dG_e}{dt} \leq CG_e + \varepsilon G_e^2, \tag{20}$$

with a constant $C > 0$ independent of $\varepsilon$. Considering the bounds on the initial data and assuming that $\varepsilon$ is sufficiently small, one can easily construct a bounded supersolution $G$ of (20) on $[0,T]$ such that $G(0) \geq G_e$. We now have, $G_e \leq C$ on $[0,T_e]$ for sufficiently small $\varepsilon$. By (13) and (17) we then conclude that $T_e$ in (14) coincides actually with $T$ when $\varepsilon$ is small. Theorem is proved.
Remark 2 We briefly summarize the key points of the proof. Equation (1) has the form of Allen-Cahn PDE perturbed by a quadratic term $P_\varepsilon \cdot \nabla \rho_\varepsilon$ and term $\lambda_\varepsilon$ which is constant in $x$. Therefore the blow up issue is determined by the competition between these two parts. The first part is a gradient flow corresponding to scalar Ginzburg-Landau energy. This energy admits well known estimates that rule out the blow up (by nonnegativity of energy). On the other hand even ODEs with quadratic nonlinearity are known to exhibit a finite time blow up. In presence of the second term blow up is ruled out by establishing differential inequality (20) (cf. standard Gronwall). The key difficulty here is due to the “fourth power” term $\frac{1}{2} \int_\Omega |P_\varepsilon|^2 |\nabla \rho_\varepsilon|^2 \, dx$ in (12). Here a straightforward way would be to get $L^\infty$ estimates on $P_\varepsilon$ directly, which is, however, not possible. Thus we first lower the power using integration by parts and maximum principle (14) to obtain a “third power term” in the RHS of (15). A subtle point here is that maximum principle gives (13) which contains $\lambda_\varepsilon$ and one needs estimate (17) on $\lambda_\varepsilon$ to get bound (14) on an extended interval. Combining (15) with (1)-(2) leads to (20), which in turn rules out the blow up.

3 1D model problem: rigorous derivation of the sharp interface limit and remarks on stability

In Section 4 we present derivation of the formal asymptotic expansion of the solution of (1)-(2) and use it to obtain the equation of motion (5), which is the principal object of interest in the study of cell motility. There are two main sources of difficulties in rigorous justification of this derivation (i) the possible non-smoothness of limiting velocity field $V$ and (ii) dimension greater than one is much harder to handle technically. That is why in this Section we consider a simplified one-dimensional model and impose smallness assumption on $\beta$ that guarantees regularity of asymptotic solutions as well as plays an important technical role in our proof.

Specifically, we study the limiting behavior as $\varepsilon \to 0$ of the solution of the system

\[
\begin{align*}
\frac{\partial \rho_\varepsilon}{\partial t} &= \rho_\varepsilon^2 \rho_\varepsilon - \frac{W'(\rho_\varepsilon)}{\varepsilon^2} + P_\varepsilon \partial_x \rho_\varepsilon + \frac{F(t)}{\varepsilon}, \quad x \in \mathbb{R}^1 \quad (21) \\
\frac{\partial P_\varepsilon}{\partial t} &= \varepsilon \partial_x^2 P_\varepsilon - \frac{1}{\varepsilon} P_\varepsilon + \beta \partial_x \rho_\varepsilon, \quad (22)
\end{align*}
\]

assuming that the initial data $\rho_\varepsilon(x, 0), P_\varepsilon(x, 0)$ has the following (“very well-prepared”) form

\[
\rho_\varepsilon(x, 0) = \theta_0 \left( \frac{x}{\varepsilon} \right) + \sum_{i=1}^{N} \varepsilon^i \theta_i \left( \frac{x}{\varepsilon} \right) + \varepsilon^\alpha u_\varepsilon \left( \frac{x}{\varepsilon}, 0 \right)
\]

and

\[
P_\varepsilon(x, 0) = \sum_{i=1}^{N} \varepsilon^i \Psi_i \left( \frac{x}{\varepsilon} \right) + \varepsilon^\alpha Q_\varepsilon \left( \frac{x}{\varepsilon}, 0 \right)
\]

where $\alpha < N + 1$. Here functions $\theta_i$ are solutions of (14) for $i = 0$ and (38) for $i \geq 1$, and $\Psi_i$ are solutions of (37) for $i = 0$ and (39) for $i \geq 1$. The functions $V_i$ which are
involved in the definition of $\Psi_i$ are defined by (40) and (42) ($V = V_0 + \varepsilon V_1 + \ldots$ is the expansion the velocity of the cell’s sharp interface). We also assume that there exists a constant $C$, independent of $\varepsilon$, such that

$$\|u_\varepsilon(y,0)\|_{L^2} + \|Q_\varepsilon(y,0)\|_{L^2} \leq C.$$  

(23)

We emphasize that $F(t)$ in the RHS of (21) is a given function rather than an unknown Lagrange multiplier in (1). The main distinction of 1D case is because in 1D there is no motion due to curvature of the interface since the interface is a point. Therefore if we take initial data such that the domain $\rho = 1$ is a finite interval, then such a “one dimensional mathematical cell” will not move, which corresponds to a well known fact that the motion in the one-dimensional Allen-Cahn problem is exponentially slow (that is very different from AC in higher dimensions). Thus, we choose initial data to be a step like function that is a transition from an unbounded left interval $\rho = 0$ to an unbounded right interval $\rho = 1$ (the “cell”).

Hereafter $\theta_0$ denotes the classical standing wave solution of

$$\theta_0''(x) = W'(\theta_0(x)) \quad x \in \mathbb{R}^1.$$  

(24)

with step-like conditions at infinity

$$\theta_0(x) \to 0, \text{ as } x \to -\infty; \quad \theta_0 \to 1, \text{ as } x \to +\infty.$$  

(25)

Since the solution of (24)-(25) is uniquely defined up to translations (shifts), we impose the following normalization

$$\theta_0(0) = 1/2.$$  

(26)

In the particular case of the double-well potential $W$ having the form (4) the solution $\theta_0$ is explicitly given by $\theta_0 = (1 - \tanh(x/\sqrt{8}))/2$. Note that $\rho_\varepsilon(x,t) = \theta_0(x/\varepsilon)$ solves (21) if the coupling term and $F(t)$ are both identically zero.

The main result of this sections is the following theorem:

**Theorem 2** Let $\rho_\varepsilon$ and $P_\varepsilon$ solve (21) and (22) on $[0, T]$ with initial conditions satisfying (23). Assume also that $F(t)$ is a given smooth function and $|\beta| < \beta_0$, where $\beta_0 > 0$ depends on the potential $W$ only. Then we have, for sufficiently small $\varepsilon$,

$$\rho_\varepsilon(x,t) = \theta_0 \left( \frac{x - x_\varepsilon(t)}{\varepsilon} \right) + \varepsilon \rho_\varepsilon^{(1)}(t,x),$$  

(27)

where $x_\varepsilon(t)$ denotes the location of the interface between 0 and 1 phases and

$$\int (\rho_\varepsilon^{(1)}(t,x))^2 \, dx \leq C \text{ for all } t \in [0, T].$$  

(28)
Moreover, \( x_\varepsilon(t) \) converges to \( x_0(t) \) which solves the interface motion equation similar to (5):

\[
-c_0 \dot{x}_0(t) = \Phi(x_0(t)) + F(t), \quad \Phi(V) := \frac{1}{\beta} \int \Psi_0(y, -V)(\theta'_0)^2 dy,
\]

where \( c_0 = \int (\theta'_0)^2 dy \), and \( \Psi_0(y; V) \) is defined as solution of (28).

**Remark 3** From the definition of function \( \Psi_0 \) it follows that it depends linearly on the parameter \( \beta \) so that function \( \Phi(V) \) does not depend on \( \beta \).

**Remark 4** While Theorem 2 describes the leading term of the asymptotic expansion for \( \rho_\varepsilon \), in the course of the proof we also construct the leading term of the asymptotic expansion of \( P_\varepsilon \) in the form \( \Psi_0 \left( \frac{\varepsilon \theta(x)}{x_\varepsilon(t)}, \dot{x}_0(t) \right) \).

**Remark 5** (on stability) Equation (29) is rigorously derived when \(|\beta| < \beta_0\) but it could be formally derived for any real \( \beta \). This equation has the unique smooth solution \( x_0(t) \) when \(|\beta| < \beta^*\), for some \( \beta^* > \beta_0 > 0 \). Roughly speaking, if \( \beta < \beta^* \), then \( x_0(t) \) is determined by (29) due to the implicit function theorem otherwise multiple solutions \( x_0(t) \) may appear. Thus, assumption on smallness of \( \beta \) can be viewed as a stability condition. By contrast, for large enough \( \beta \) one can observe instability due to the fact that the limiting equation (29) has multiple solutions and, therefore, perturbation of initial data may result in switching between multiple solutions of equation (29). Indeed, to explain we rewrite equation (29)

\[
c_0 V - \beta \Phi(V) = F_0(t)
\]

where the left hand side of (30) can be resolved in \( V \), but not uniquely.

**Remark 6** The estimate (28) justifies the asymptotic expansion (27) and this estimate is the principal claim of this Theorem. However, in the course of the proof we actually obtain and justify a more precise asymptotic expansion of the form \( \rho_\varepsilon = \theta_0(\frac{\varepsilon \theta}{\varepsilon \theta}) + \varepsilon \theta_1(\frac{\varepsilon \theta}{\varepsilon \theta}, t) + \varepsilon^2 \theta_2(\frac{\varepsilon \theta}{\varepsilon \theta}, t) + O(\varepsilon^3) \), which corresponds to \( N = 3 \) and \( \alpha = 3 \) in the expansions (32) below.

The proof of Theorem 2 is divided into two steps, presented in the following two subsections.

In the first step (Subsection 3.1) we formally construct approximate solution of the order \( N \) and obtain equations for the residuals

\[
u_\varepsilon = \frac{1}{\varepsilon^\alpha}(\rho_\varepsilon - \bar{\rho}_\varepsilon) \quad \text{and} \quad Q_\varepsilon = \frac{1}{\varepsilon^\alpha}(P_\varepsilon - \bar{P}_\varepsilon),
\]

where \( \bar{\rho}_\varepsilon = \theta_0 + \varepsilon \theta_1 + \ldots + \varepsilon^N \theta_N \) and \( \bar{P}_\varepsilon = \Psi_0 + \varepsilon \Psi_1 + \ldots + \varepsilon^N \Psi_N \),

(32)

for some \( \alpha \geq 1 \). It would be natural to expect that \( \alpha = N + 1 \), however, it turned out that due to the gradient coupling and nonlinearity of the problem, we can only prove boundedness of \( u_\varepsilon \) and \( Q_\varepsilon \) for some \( 1 < \alpha < N + 1 \).
The second step (Subsection 3.2) is the central mathematical part of this paper, and we briefly outline its main ideas. The goal there is to obtain bounds on residuals $u_\varepsilon$ and $Q_\varepsilon$ for appropriate $\alpha$ and $N$. The bounds on $u_\varepsilon$ play central role and they imply bounds on $Q_\varepsilon$ though these bounds are coupled. Therefore the bound (28) is the main claim of the Theorem.

The techniques of asymptotic expansions that include bounds on residuals were first developed for Allen-Cahn PDE in [23]. The proofs in [23] are based on the lower bound of the spectrum of linearized self-adjoint stationary Allen-Cahn operator in an unbounded domain. The techniques of this type were subsequently developed and applied in Alikakos, Bates, Chen [1] for the Cahn-Hilliard equation, Caginalp and Chen [6] for the phase field system, and [8] for volume preserving Allen-Cahn PDE.

In the system (1)-(2) or its one-dimensional analog (21)-(22) the corresponding linearized operator is not self-adjoint and the previously developed techniques can not be directly applied.

The results of this Section are based on the analysis of a time-dependent linearized problem that corresponds to the entire system. We represent (split) the residual not be directly applied.

First, we seek formal approximations for $\rho_\varepsilon$ and $P_\varepsilon$ in the form:

$$
\rho_\varepsilon(x,t) \approx \theta_0 \left( \frac{x-x_\varepsilon(t)}{\varepsilon} \right) + \sum_i e^i \theta_i \left( \frac{x-x_\varepsilon(t)}{\varepsilon}, t \right) \\
P_\varepsilon(x,t) \approx \sum_i e^i P_i \left( \frac{x-x_\varepsilon(t)}{\varepsilon}, t \right).
$$

We also assume $x_\varepsilon(t)$ admits a power series expansion,

$$
x_\varepsilon(t) = x_0(t) + \varepsilon x_1(t) + \ldots + \varepsilon^N x_N(t) + \ldots,
$$

so that we also have expansion for the velocity $V = -\dot{x}_\varepsilon$,

$$
V(t) = V_0(t) + \varepsilon V_1(t) + \ldots + \varepsilon^N V_N(t) + \ldots, \quad V_i(t) = -\dot{x}_i(t), \; i = 0,1,\ldots
$$

3.1 Construction of asymptotic expansions.

First, we seek formal approximations for $\rho_\varepsilon$ and $P_\varepsilon$ in the form:

$$
\rho_\varepsilon(x,t) \approx \theta_0 \left( \frac{x-x_\varepsilon(t)}{\varepsilon} \right) + \sum_i e^i \theta_i \left( \frac{x-x_\varepsilon(t)}{\varepsilon}, t \right) \\
P_\varepsilon(x,t) \approx \sum_i e^i P_i \left( \frac{x-x_\varepsilon(t)}{\varepsilon}, t \right).
$$

We also assume $x_\varepsilon(t)$ admits a power series expansion,

$$
x_\varepsilon(t) = x_0(t) + \varepsilon x_1(t) + \ldots + \varepsilon^N x_N(t) + \ldots,
$$

so that we also have expansion for the velocity $V = -\dot{x}_\varepsilon$,

$$
V(t) = V_0(t) + \varepsilon V_1(t) + \ldots + \varepsilon^N V_N(t) + \ldots, \quad V_i(t) = -\dot{x}_i(t), \; i = 0,1,\ldots
$$
Next we expand $W'(\rho_\varepsilon)$.

$$W'(\rho_\varepsilon) = W'(\theta_0) + \varepsilon W''(\theta_0)\theta_1 + \varepsilon^2 \left[ W'''(\theta_0)\theta_2 + \frac{W''(\theta_0)}{2}\theta_1^2 \right] + ...$$

$$+ \varepsilon^i \left[ W''(\theta_0)\theta_i + (dW)^{(i)} \right] + ...$$

where

$$(dW)^{(i)} = \sum_{i_1 + i_2 = i, i_1, i_2 \geq 1} \frac{W'''(\theta_0)}{2} \theta_{i_1} \theta_{i_2} + \sum_{i_1 + i_2 + i_3 = i, i_1, i_2, i_3 \geq 1} \frac{W^{(iv)}(\theta_0)}{6} \theta_{i_1} \theta_{i_2} \theta_{i_3}.$$ 

Substitute the series (44) and (33) into (21)-(22), and equate terms of like powers of $\varepsilon$ to obtain that $\theta_i$ and $\Psi_i$ for $i = 0, 1, 2$ satisfy

$$\theta_0'' = W'(\theta_0),$$

$$-\theta_1'' + W''(\theta_0)\theta_1 = -V_0\theta_0' + \Psi_0\theta_0' - V_0\theta_0' - \frac{W'''(\theta_0)}{2}\theta_1^2 + \Psi_0\theta_1' + \Psi_1\theta_0',$$

$$-\theta_2'' + W''(\theta_0)\theta_2 = -V_1\theta_0' - V_0\theta_1' - \frac{W'''(\theta_0)}{2}\theta_2^2 + \Psi_0\theta_2' + \Psi_1\theta_0' + \Psi_1\theta_1',$$

and

$$\Psi_0'' - V_0\Psi_0' - \Psi_0 = -\beta \theta_0',$$

$$\Psi_1'' - V_0\Psi_1' - \Psi_1 = -\beta \theta_1' + V_1\Psi_0' + \Psi_0,'$$

$$\Psi_2'' - V_0\Psi_2' - \Psi_2 = -\beta \theta_2' + V_1\Psi_1' + V_2\Psi_0' + \Psi_1.'$$

The equations for $i > 2$ have the following form

$$-\theta_i'' + W''(\theta_0)\theta_i = -\theta_{i-2} - \sum_{j=0}^{i-1} V_j\theta_{i-1-j} - (dW)^{(i)} + \sum_{j=0}^{i-1} \Psi_j\theta_{i-1-j}.'$$

$$\Psi_i'' - V_0\Psi_i' - \Psi_i = -\beta \theta_i' + \sum_{j=1}^{i} V_j\Psi_{i-j} + \Psi_{i-1}.'$$

Remark 7 Due to the fact that $\theta_i$ is an eigenfunction of the linearized Allen-Cahn operator corresponding to the zero eigenvalue, the following solvability conditions for (35), (36) and (38) arise

$$\int \left\{ -V_0\theta_0' + \Psi_0\theta_0' + F(\varepsilon) \right\} \theta_0' dy = 0,$$

$$\int \left\{ -V_1\theta_0' - V_0\theta_1' - \frac{W'''(\theta_0)}{2}\theta_1^2 + \Psi_0\theta_1' + \Psi_1\theta_0' \right\} \theta_0' dy = 0,$$

$$\int \left\{ -\theta_{i-2} - \sum_{j=0}^{i-1} V_j\theta_{i-1-j} - (dW)^{(i)} + \sum_{j=0}^{i-1} \Psi_j\theta_{i-1-j} \right\} \theta_0' dy = 0.$$

and uniquely define the functions $V_i(t), i = 0, ..., N - 1$ such that the solvability conditions (42) are satisfied. Equations (40), (41) and (42) are solvable for $V_0, V_1$ and
The functions $\psi_i$, respectively, for sufficiently small $\beta$. Also we note that once the solvability conditions are satisfied then equations (35),(36) and (38) have a family of solutions: $\psi_i = \theta_i + \gamma \theta'_0, \gamma \in \mathbb{R}$, where $\theta_i$ is a particular solution. We choose $\gamma$ s.t.

$$\int \theta'_0 \theta_i dy = 0.$$ 

Define functions $u_\epsilon(y,t)$ and $Q_\epsilon(y,t)$ by

$$\rho_\epsilon = \tilde{\rho}_\epsilon(y,t) + \epsilon^\alpha u_\epsilon(y,t), \quad \text{and} \quad P_\epsilon = \tilde{P}_\epsilon(y,t) + \epsilon^\alpha Q_\epsilon(y,t) \quad \text{for} \quad y = \frac{x-x_\epsilon(t)}{\epsilon}, \quad (43)$$

where

$$\tilde{\rho}_\epsilon(y,t) = \theta_0(y) + \sum_{i=1}^{N} \epsilon^i \theta_i(y,t) \quad \text{and} \quad \tilde{P}_\epsilon(y,t) = \sum_{i=0}^{N} \epsilon^i \Psi_i(y,t). \quad (44)$$

Substituting the representation for $\rho_\epsilon$ from (43) into (21) we derive the PDE for $u_\epsilon$ (note that the differentiation in time and new spatial variable $y = \frac{x-x_\epsilon(t)}{\epsilon}$ are no longer independent)

$$\frac{\partial u_\epsilon}{\partial t} = \frac{u''_\epsilon}{\epsilon^2} - \frac{V_0 u'_\epsilon}{\epsilon} - \frac{W''(\theta_0) u_\epsilon}{\epsilon} - \frac{W'''(\theta_0) u_\epsilon \theta_1}{\epsilon} + \frac{U'_0 u'_\epsilon}{\epsilon} + \frac{Q_\epsilon \theta'_0}{\epsilon} + R_\epsilon(t,y), \quad (45)$$

where $R_\epsilon$ is of the form

$$R_\epsilon(t,y) = \epsilon^{N-1} a_\epsilon(t,y)$$

$$+ \epsilon^N b_{0,\epsilon}(t,y) + b_{1,\epsilon}(t,y) u_\epsilon + \epsilon^{\alpha-2} b_{2,\epsilon}(t,y) u^2_\epsilon + \epsilon^{2\alpha-2} b_{3,\epsilon}(t,y) u^3_\epsilon$$

$$+ e_\epsilon(t,y) u'_\epsilon + g_\epsilon(t,y) Q_\epsilon + \epsilon^{\alpha-1} Q_\epsilon u'_\epsilon. \quad (46)$$

where $a_\epsilon(t,y), b_{k,\epsilon}(t,y), k = 1, 2, 3, e_\epsilon(t,y), g_\epsilon(t,y)$ are bounded functions in $y, t$ and $\epsilon$ and square integrable with respect to $y$ (except $e_\epsilon$). Moreover, the function $a_\epsilon$ is orthogonal to $\theta'_0$:

$$\int \theta'_0(y) a_\epsilon(t,y) dy = 0.$$ 

The functions $a_\epsilon, b_{k,\epsilon}, e_\epsilon$ and $g_\epsilon$ are expressed in terms of $\theta_i, V_i$ and $\Psi_i$, their exact form, which is not important for the proof, is given in Appendix.

Substituting the representation for $P_\epsilon$ from (43) into (22) we derive also the PDE for $Q_\epsilon$:

$$\frac{\partial Q_\epsilon}{\partial t} = \frac{Q''_\epsilon}{\epsilon} - \frac{V Q'_\epsilon}{\epsilon} - \frac{Q_\epsilon}{\epsilon} + \frac{\beta u'_\epsilon}{\epsilon} + \epsilon^{N-\alpha} m_\epsilon(t,y). \quad (47)$$

For more details on derivation of (45) and (47) we refer to Appendix.
3.2 Bounds for residuals \( u_e \) and \( Q_e \)

In this section we obtain bounds for the coupled system of PDEs:

\[
\begin{align*}
\frac{\partial u_e}{\partial t} &= \frac{u''_0 + V_0 u'_{e}}{\varepsilon} - \frac{W''(\theta_0) u_e}{\varepsilon^2} \frac{\Psi(u'_{e})}{\varepsilon} + \frac{Q_{e} u''_{e}}{\varepsilon} + Q_{e} \frac{Q_{e}}{\varepsilon} + R_{e}(t, y) \\
\frac{\partial Q_{e}}{\partial t} &= \frac{Q''_{e}}{\varepsilon} - \frac{V Q'_{e}}{\varepsilon} + \frac{\Psi(u'_{e})}{\varepsilon} + \frac{Q_{e} Q_{e}}{\varepsilon} + \varepsilon^{N-a} m_{e}(t, y).
\end{align*}
\]

To this end we write the unknown function \( u_e \) in the following form,

\[
u_e(t, y) = \theta'_0(y) [v_e(t, y) + \xi_e(t)], \quad \text{where} \quad \int (\theta'_0(y))^2 v_e(t, y) dy = 0. \quad (48)
\]

Then \((45)\) becomes

\[
\frac{\partial}{\partial t} (\theta'_0 (v_e + \xi_e)) = -\frac{V_0}{\varepsilon} (\theta'_0 (v_e + \xi_e)')' - \frac{W''(\theta_0)}{\varepsilon^2} \theta'_0 (v_e + \xi_e) - \frac{W''(\theta_0)}{\varepsilon} (v_e + \xi_e) \theta_0 \theta_1 \\
+ \frac{\Psi(\theta'_0 (v_e + \xi_e))'}{\varepsilon} + \frac{Q_{e} \theta''_{e}}{\varepsilon} + R_{e}(t, y). \quad (49)
\]

**Lemma 1** The following inequality holds

\[
\frac{d}{dt} \int (\theta'_0)^2 v_{e}^2 dy + \frac{1}{2} \varepsilon^2 \int (\theta'_0)^2 (v_e)^2 dy \\
\leq C \xi_e^2 + \frac{1}{\varepsilon} \left[ \int Q_{e} (\theta'_0)^2 (v_e + \xi_e) dy - \xi_e^2 \int \Psi(\theta'_0)^2 dy \right] + \int R_{e} \theta'_0 (v_e + \xi_e) dy. \quad (50)
\]

**Proof:**

Multiply \((49)\) by \( \theta_e = \theta'_0 (v_e + \xi_e) \) and integrate to obtain

\[
\frac{d}{dt} \int (\theta'_0)^2 (v_e + \xi_e)^2 dy + \frac{1}{\varepsilon^2} \int (\theta'_0 (v_e + \xi_e)')^2 dy \\
= -\frac{V_0}{\varepsilon} \int (\theta'_0 (v_e + \xi_e)')' \theta'_0 (v_e + \xi_e) dy \\
- \int \frac{W''(\theta_0)}{\varepsilon^2} (\theta'_0)^2 (v_e + \xi_e) dy - \int \frac{W''(\theta_0)}{\varepsilon} (\theta'_0)^2 \theta_1 (v_e + \xi_e) dy \\
+ \int \frac{\Psi(\theta'_0 (v_e + \xi_e))'}{\varepsilon} \theta_0 (v_e + \xi_e) dy \\
+ \int \frac{Q_{e}}{\varepsilon} (\theta'_0)^2 (v_e + \xi_e) dy + \int R_{e} \theta'_0 (v_e + \xi_e) dy. \quad (51)
\]

In order to derive \((50)\) we simplify equality \((51)\). First, we notice that the integral in the first term can be rewritten as follows

\[
\int (\theta'_0)^2 (v_e + \xi_e)^2 dy = \int (\theta'_0)^2 v_e^2 dy + \xi_e^2 \int (\theta'_0)^2 dy. \quad (52)
\]
Thus, the first term in the right hand side of (51) vanishes. Indeed,

\[ \int (\theta'_0 (v_e + \xi_y))' \theta'_0 (v_e + \xi_y) dy = \frac{1}{2} (\theta'_0)^2 (v_e + \xi_y)^2 \bigg|_{y=-\infty}^{+\infty} = 0. \]

The second term in the left hand side can be split into three terms:

\[ \frac{1}{\epsilon^2} \int \{ (\theta'_0 (v_e + \xi_y))' \}^2 dy = \frac{1}{\epsilon^2} \int (\theta'_0 (v_e))' dy \]

\[ + \frac{2 \xi_y}{\epsilon^2} \int (\theta'_0 (v_e))' dy + \frac{\xi_y^2}{\epsilon^2} \int (\theta''_0) dy. \tag{53} \]

Rewrite the first term in (53), using (34) and integrating by parts,

\[ \int (\theta'_0 (v_e))' dy = \int (\theta''_0 (v_e))' v_e^2 dy + 2 \int \theta'_0 \theta''_0 v_e dy + \int (\theta'_0 (v_e))' 2 dy 
= \left\{ - \int \theta''_0 \theta'''_0 v_e dy - 2 \int \theta''_0 \theta''_0 v_e dy \right\} + 2 \int \theta''_0 \theta'_0 v_e dy 
+ \int (\theta'_0)^2 (v_e)^2 dy 
= - \int \theta''_0 \theta''_0 v_e dy + \int (\theta'_0 (v_e))' 2 dy 
= - \int W'' (\theta) (\theta'_0)^2 v_e dy + \int (\theta'_0 (v_e))' 2 dy. \]

Thus,

\[ - \frac{1}{\epsilon^2} \int ((\theta'_0 (v_e))')^2 dy - \frac{W'' (\theta)}{\epsilon^2} (\theta'_0)^2 v_e^2 dy = - \frac{1}{\epsilon^2} \int (\theta'_0 (v_e))' dy. \tag{54} \]

Next we rewrite the second and the third terms in (52) using the fact that \( \theta''_0 = W'' (\theta) \theta'_0 \) (this latter equality is obtained by differentiating (34) with respect to \( y \)),

\[ - \frac{1}{\epsilon^2} \int (\theta'_0)^2 dy \xi_y^2 - \frac{W'' (\theta)}{\epsilon^2} (\theta'_0)^2 dy \xi_y^2 = 0. \tag{55} \]

\[ - \frac{2}{\epsilon^2} \int (\theta'_0 (v_e))' \theta'_0 dy \xi_y - 2 \int \frac{W'' (\theta)}{\epsilon^2} (\theta'_0)^2 v_e dy \xi_y = 0. \tag{56} \]

Also, we make use the following equality which follows from (36),

\[ - \frac{W''' (\theta)}{\epsilon} \theta_1 (\theta'_0)^2 dy + \frac{\Psi_0}{2 \epsilon} (\theta'_0)^2 dy = \int [-V_0 \theta'_0 + \Psi_0 \theta'_0 + \theta''_0] \theta'_0 dy 
+ \int W'' (\theta) \theta'_0 \theta'_0 dy + \int \Psi_0 \theta''_0 \theta'_0 dy 
= - \int \Psi_0 (\theta'_0)^2 dy. \tag{57} \]
Finally, we use equalities \( (52), (54), (57), (55) \) and \( (56) \) to rewrite \( (51) \) as follows
\[
\frac{d}{dt} \left[ \int (\theta_0')^2 v_\varepsilon^2 dy + \int (\theta_0')^2 dy \xi_\varepsilon^2 \right] + \frac{1}{\varepsilon} \int (\theta_0')^2 (v_\varepsilon')^2 dy = \\
-2 \xi_\varepsilon \frac{W''(\theta)}{\varepsilon} \theta_1 (\theta_0')^2 v_\varepsilon dy - \int \frac{W''(\theta)}{\varepsilon} \theta_1 (\theta_0')^2 v_\varepsilon^2 dy \\
-\xi_\varepsilon \frac{\Psi''}{\varepsilon} (\theta_0')^2 v_\varepsilon dy - \int \frac{\Psi''}{2\varepsilon} (\theta_0')^2 v_\varepsilon^2 dy \\
+ \frac{1}{\varepsilon} \int Q_\varepsilon (\theta_0')^2 (v_\varepsilon + \xi_\varepsilon) dy - \frac{\xi_\varepsilon^2}{\varepsilon} \int \Psi''(\theta_0')^2 dy \\
+ \int R_\varepsilon \theta'_0 (v_\varepsilon + \xi_\varepsilon) dy.
\]
Then \( (50) \) is obtained by applying the standard Cauchy-Schwarz inequality and the Poincare inequality \( (98) \) from Appendix.

It follows from the previous lemma that in order show the boundedness of \( u_\varepsilon \) we need to find an appropriate upper bound on the term
\[
\frac{1}{\varepsilon} \left[ \int Q_\varepsilon (\theta_0')^2 (v_\varepsilon + \xi_\varepsilon) dy - \xi_\varepsilon^2 \int \Psi''(\theta_0')^2 dy \right].
\]
To this end, we use the equation \( (47) \) for \( Q_\varepsilon \).

Note that \( \Psi_0 = \Psi_0(y; V_0) \) depends on time \( t \) through \( V_0(t) \). For simplicity of the presentation we suppress the second argument \( \Psi_0(y) := \Psi_0(y; V_0) \), if it equals to \( V_0 = -x_0(t) \).

It follows from \( (37) \) functions \( \Psi_0' \) and \( \Psi_0, V_0 := \partial \Psi_0 / \partial V_0 \) solve the following equations
\[
(\Psi_0'')'' - V_0(\Psi_0')' - (\Psi_0')'' = -\beta \theta_0'' \text{ and } \Psi_0'' - V_0 \Psi_0' - V_0 \Psi_0 = \Psi_0'.
\]

Rewrite \( (47) \) substituting \( u_\varepsilon = \theta_0'(v_\varepsilon + \xi_\varepsilon) \),
\[
Q_\varepsilon'' - V Q_\varepsilon' - Q_\varepsilon - \varepsilon \frac{\partial Q_\varepsilon}{\partial t} = -\beta \theta_0'' \xi_\varepsilon - \beta (\theta_0' v_\varepsilon)' - \varepsilon^{N+1-\alpha} m_\varepsilon.
\]
Thus, \( Q_\varepsilon \) can be written as
\[
Q_\varepsilon = A_\varepsilon + B_\varepsilon,
\]
where \( A_\varepsilon \) and \( B_\varepsilon \) are solutions of the following problems,
\[
A_\varepsilon'' - V A_\varepsilon' - A_\varepsilon - \varepsilon \frac{\partial A_\varepsilon}{\partial t} = -\beta \theta_0'' \xi_\varepsilon, \quad A_\varepsilon(0) = Q_\varepsilon(0) \quad (60)
\]
\[
B_\varepsilon'' - V B_\varepsilon' - B_\varepsilon - \varepsilon \frac{\partial B_\varepsilon}{\partial t} = -\beta (\theta_0' v_\varepsilon)', \quad B_\varepsilon(0) = 0.
\]
Next we note that the function \( A_\varepsilon \) can be written as \( A_\varepsilon = \xi_\varepsilon \Psi_0' + D_\varepsilon (\Psi_0 = \Psi_0(y; V)) \), where \( D_\varepsilon \) solves the following problem,
\[
D_\varepsilon'' - V D_\varepsilon' - D_\varepsilon - \varepsilon \frac{\partial D_\varepsilon}{\partial t} = \xi_\varepsilon \Psi_0' + \xi_\varepsilon V \xi_\varepsilon \Psi_0, \quad D_\varepsilon(0) = Q_\varepsilon(0) - \xi_\varepsilon \Psi_0', \quad \xi_\varepsilon = 0. \quad (61)
\]
Thus,

\[ Q_\varepsilon = \xi_\varepsilon \Psi_0'(y; V) + D_\varepsilon + B_\varepsilon. \]  

(62)

This representation (62) allows us to rewrite the term (58) as follows,

\[ \frac{\varepsilon}{\xi_\varepsilon} \int Q_\varepsilon (\theta_0')^2 v_\varepsilon dy + \frac{1}{\varepsilon} \int [D_\varepsilon + B_\varepsilon] (\theta_0')^2 dy. \]  

(63)

The bounds for these terms are collected in the next lemma.

**Lemma 2** The following inequalities hold

(i)

\[ \varepsilon d\frac{d}{dt} \left[ \int B_\varepsilon^2 dy \right] + \int B_\varepsilon^2 dy + \int (B_\varepsilon')^2 dy \leq c \int (\theta_0')^2 (v_\varepsilon')^2 dy + c\varepsilon^{2(N+1-a)}, \]  

(64)

(ii)

\[ \varepsilon d\frac{d}{dt} \left[ \int D_\varepsilon^2 dy \right] + \int D_\varepsilon^2 dy + \int (D_\varepsilon')^2 dy \leq c\beta^2 \varepsilon^2 \frac{\varepsilon_\varepsilon^2}{\varepsilon_\varepsilon} + c\varepsilon^{2}\varepsilon_\varepsilon^2 \]  

(65)

(iii)

\[ \int Q_\varepsilon^2 dy \leq C\varepsilon_\varepsilon \frac{\varepsilon_\varepsilon^2}{\varepsilon_\varepsilon} + \int D_\varepsilon^2 dy; \]  

(66)

and

\[ \int (Q_\varepsilon')^2 dy \leq C\varepsilon_\varepsilon^2 + \int (D_\varepsilon')^2 dy + \int (B_\varepsilon')^2 dy. \]  

(67)

**Proof.**

Items (i) and (ii) are proved by means of energy relations that are obtained after multiplying (60) and (61) by \( B_\varepsilon \) and \( A_\varepsilon \), respectively, and integrating in \( y \). The resulting energy relation for the function \( B_\varepsilon \) is

\[ \frac{\varepsilon}{2} \frac{d}{dt} \left[ \int B_\varepsilon^2 dy \right] + \int B_\varepsilon^2 dy + \int (B_\varepsilon')^2 dy = -\beta \int \theta_0' v_\varepsilon B_\varepsilon dy + \varepsilon^{N+1-a} \int m_\varepsilon B_\varepsilon dy, \]

and the energy relation for \( D_\varepsilon \) reads

\[ \frac{\varepsilon}{2} \frac{d}{dt} \left[ \int D_\varepsilon^2 dy \right] + \int D_\varepsilon^2 dy + \int (D_\varepsilon')^2 dy = -\varepsilon_\varepsilon \Psi_\varepsilon' D_\varepsilon dy - \varepsilon V \xi_\varepsilon \Psi_0' D_\varepsilon dy. \]  

(68)

Then (64) and (65) are obtained by applying the Cauchy-Schwarz inequality. Note that \( \Psi_0' \) depends linearly on \( \beta \), this allows us to bound the right hand side of (68) by \( c\beta^2 \varepsilon^2 \frac{\varepsilon_\varepsilon}{\varepsilon_\varepsilon} + c\varepsilon^2 \frac{\varepsilon_\varepsilon^2}{\varepsilon_\varepsilon} + \frac{1}{\varepsilon} \int D_\varepsilon^2 dy \) and this eventually leads to (65).

Item (iii) easily follows from representation (62).

□

Using (50), representation (63) for (58) and the previous Lemma we derive the following corrolary.
Corollary 1 The following inequality holds

\[
\frac{d}{dt} \left[ \int (\theta_0')^2 v_e^2 dy + c_0 \xi_e^2 + \frac{1}{\varepsilon} \int D_e^2 dy + \frac{1}{\varepsilon} \int B_e^2 dy \right] + \frac{1}{2\varepsilon^2} \left[ \int (\theta_0')^2 (v_e')^2 dy + \int D_e^2 dy + \int (D_e')^2 dy + \int B_e^2 dy + \int (B_e')^2 dy \right] \leq c \xi_e^2 + cB^2 \xi_e^2 + \int R_e \theta_0^2 (v_e + \xi_e) dy + c \varepsilon^{2(N-1-\alpha)}. \tag{69}
\]

Thus, we reduced the estimation of (59) to estimation of \( \xi_e^2 \).

The key observation leading to the desired bound on \( \xi_e \) can be explained now as follows. Observe that the presence of \( \xi_e \) in the RHS of (50) might result in the exponential growth of \( \xi_e \) (the best one can guarantee from \( \xi_e \) type bound). Fortunately, the term \( \int \Psi''_e (\theta_0')^2 dy \xi_e^2 \) in (50) cancels with the leading term appearing after substitution of expansion (62) for \( Q_e \). However, this results in the appearance of lower order terms depending on \( \xi_e \). Lemma 3 below provides the control of \( |\xi_e| \).

Lemma 3 The following inequality holds

\[
\xi_e^2 \leq \frac{C}{\varepsilon^2} \int (\theta_0')^2 (v_e')^2 dy + \frac{C}{\varepsilon} \int \left\{ B_e^2 + D_e^2 \right\} dy + C \xi_e^2 + \int R_e \theta_0^2 dy \xi_e. \tag{70}
\]

Proof:
Multiply equation (49) by \( \theta_0' \xi_e \) and integrate in \( y \) to obtain

\[
\int (\theta_0')^2 (v_e + \xi_e) \xi_e dy = -\frac{V_0}{\varepsilon} \int (\theta_0' (v_e + \xi_e)' \theta_0' dy + \frac{\xi_e}{\varepsilon^2} \int (\theta_0' (v_e + \xi_e)')' \theta_0' dy + \frac{\xi_e}{\varepsilon} \int \frac{W''(\theta_0)}{\varepsilon} (v_e + \xi_e) (\theta_0')^2 dy - \frac{\xi_e}{\varepsilon} \int \frac{W''(\theta_0)}{\varepsilon} (v_e + \xi_e) (\theta_0')^2 dy + \xi_e \int \frac{Q_e}{\varepsilon} (\theta_0')^2 dy + \frac{\xi_e}{\varepsilon} \int R_e (t, y) \theta_0' dy. \tag{71}
\]

Using (48) we simplify the left hand side of (71).

\[
\int (\theta_0')^2 (v_e + \xi_e) \xi_e dy = \xi_e \frac{d}{dt} \left\{ \int (\theta_0')^2 v_e \right\} + \xi_e^2 \int (\theta_0')^2 dy = \xi_e^2 \int (\theta_0')^2 dy. \tag{72}
\]

In the next four steps inequality (70) will be derived by estimating the right hand side of (71) term by term.
STEP 1. The first term in the right hand side of (71) is estimated by using integration by parts and the Cauchy-Schwarz inequality,

$$-\frac{V_0}{\varepsilon} \int \left( \theta_0'(v_e + \xi_e) \right)' \theta_0' \xi_e \, dy = -\frac{V_0}{\varepsilon} \int \theta_0' \left( v_0 + \xi_e \right) \xi_e \, dy = -\frac{V_0 \xi_e}{2\varepsilon} \int \left( \theta_0' \right)^2 \, dy \leq \frac{C}{\delta \varepsilon^2} \int \left( \theta_0' \right)^2 \, dy + \delta \xi_e^2. \quad (73)$$

Here we introduced small parameter $\delta$ which does not depend on $\varepsilon$ and will be chosen later.

STEP 2. In this step we show that the sum of the second and the third terms in the right hand side of (71) gives zero. Indeed, use integration by parts and the definition of $\theta_0'$ ($\theta'' = W'(\theta_0)$) to obtain

$$\frac{1}{\varepsilon^2} \left\{ \int \left( \theta_0'(v_e + \xi_e) \right)'' \theta_0' \xi \, dy - \int W''(\theta_0)(v_e + \xi_e) \left( \theta_0' \right)^2 \, dy \right\}$$

$$= \frac{\xi_e}{\varepsilon^2} \left\{ - \int \left( \theta_0'(v_e + \xi_e) \right)' \theta_0'' \, dy + \int W'(\theta_0)(v_e + \xi_e)' \, dy \right\} = 0. \quad (74)$$

STEP 3. In this step we estimate the sum of the fourth, the fifth and the sixth terms in the right hand side of (71),

$$-\frac{\xi_e}{\varepsilon} \int \frac{W''(\theta_0)}{\varepsilon} (v_e + \xi_e)(\theta_0')^2 \, dy + \xi_e \int \frac{W_0}{\varepsilon} (v_0 + \xi_e)(\theta_0') \, dy + \xi_e \int \frac{Q_0}{\varepsilon} (\theta_0')^2 \, dy$$

To this end we first show that

$$\xi_e \frac{\xi_e}{\varepsilon} \int \frac{W''(\theta_0)}{\varepsilon} (\theta_0')^2 \, dy + \xi_e \frac{\xi_e}{\varepsilon} \int \frac{W_0}{\varepsilon} \theta_0' \, dy = 0. \quad (75)$$

Indeed, differentiating (35) in y one obtains the equation $\theta''' - W''(\theta_0) \theta_0' = W''(\theta_0) \theta_0' \theta_1 + V_0 \theta_0' - \Psi_0 \theta_0' - \Psi_1 \theta_0' + F'$. whose solvability condition reads

$$\int \left\{ W''(\theta_0) \theta_0' \theta_1 + V_0 \theta_0' - \Psi_0 \theta_0' - \Psi_1 \theta_0' + F' \right\} \theta_0' \, dy = 0.$$

The latter equality contains five terms. The second and the fifth term vanish since they are integrals of derivatives ($\left( \frac{W_0}{\varepsilon} (\theta_0')^2 \right)'$ and $\left( F' \theta_0' \right)'$, respectively). Integrating by parts the third term we get,

$$\int W''(\theta_0)(\theta_0')^2 \, dy = 0.$$

This immediately implies (75). Now, taking into account the equality (75) and representation (62) for $Q_0$ written in the form

$$Q_0 = \xi_e \Psi_0' + \xi_e \left( \Psi_0'(y; V) - \Psi_0(y; V_0) \right) + D_\varepsilon + B_\varepsilon,$$
we get
\[-\dot{\xi} \int \frac{W'''(\theta_0)}{\varepsilon} (v_\varepsilon + \xi_\varepsilon)(\theta_0')^2 \theta_1 \, dy + \dot{\theta}_1 \int \frac{\Psi_0'}{\varepsilon} (\theta_0' (v_\varepsilon + \xi_\varepsilon))' \theta_0' \, dy + \dot{\xi} \int \frac{Q_0}{\varepsilon} (\theta_0')^2 \, dy \]
\[= -\dot{\xi} \int \frac{W'''(\theta_0)}{\varepsilon} v_\varepsilon (\theta_0')^2 \theta_1 \, dy + \dot{\xi} \int \frac{\Psi_0'}{\varepsilon} (\theta_0' v_\varepsilon) \theta_0' \, dy \]
\[+ \dot{\xi} \int (\Psi''_0(y,V) - \Psi'_0(y,V_0))(\theta_0')^2 \, dy + \dot{\xi} \int \frac{B_0}{\varepsilon} (\theta_0')^2 \, dy + \dot{\xi} \int \frac{D_0}{\varepsilon} (\theta_0')^2 \, dy, \]
where we have also used integration by parts.

Applying the Poincare inequality (78) we get the following estimate,
\[-\dot{\xi} \int \frac{W'''(\theta_0)}{\varepsilon} (v_\varepsilon + \xi_\varepsilon)(\theta_0')^2 \theta_1 \, dy + \dot{\xi} \int \frac{\Psi_0'}{\varepsilon} (\theta_0' (v_\varepsilon + \xi_\varepsilon))' \theta_0' \, dy + \dot{\xi} \int \frac{Q_0}{\varepsilon} (\theta_0')^2 \, dy \]
\[\leq \frac{C}{\varepsilon \delta^2} \int (\theta_0')^2 (v_\varepsilon')^2 \, dy + \frac{C}{\varepsilon \delta^2} \left\{ B_0^2 + D_0^2 \right\} \delta + \frac{c \varepsilon^2}{\delta^2} + \delta \varepsilon^2 \xi_\varepsilon^2. \quad (76) \]

STEP 4. We use equalities (72), (73) and estimates (74), (76) in (71), and take \(\delta > 0\) such that \(\int (\theta_0')^2 \, dy > 4 \delta\) to derive (70).

Now, multiplying (70) by \(|\beta|\) and adding to (69) we obtain the following corollary.

**Corollary 2** The following inequality holds for sufficiently small \(|\beta|\)

\[
\frac{d}{dt} \left[ \int (\theta_0')^2 v_\varepsilon^2 \, dy + c_0 \xi_\varepsilon^2 + \frac{1}{\varepsilon} \int D_0^2 \, dy + \frac{1}{\varepsilon} \int B_0^2 \, dy \right] \\
+ \frac{1}{4 \varepsilon^2} \left[ \int (\theta_0')^2 (v_\varepsilon')^2 \, dy + \int D_0^2 \, dy + \int (D_0')^2 \, dy + \int B_0^2 \, dy + \int (B_0')^2 \, dy \right] \\
\leq c \xi_\varepsilon^2 + (c \beta^2 - |\beta|) \dot{\xi}^2 + \varepsilon^{2(N-\alpha - 1)} \\
+ \int R_\varepsilon \theta_0' (v_\varepsilon + \xi_\varepsilon) \, dy + |\dot{\xi}| |\beta| \int R_\varepsilon \theta_0' \, dy. \quad (77) \]

In the following lemma we obtain appropriate bounds for the last two terms in (77), i.e. terms containing \(R_\varepsilon\).

**Lemma 4.** The following inequalities hold true

(i) for all \(\alpha > 2\) and \(N \geq \alpha + 1\)
\[
\int R_\varepsilon \theta_0' (v_\varepsilon + \xi_\varepsilon) \, dy \leq \frac{c}{\varepsilon} \int (\theta_0')^2 (v_\varepsilon')^2 \, dy + c \xi_\varepsilon^2 + c \\
+ c \varepsilon^{2 \alpha - 2} \left( \int (\theta_0')^2 v_\varepsilon^2 \, dy \right)^2 + c \varepsilon^{2 \alpha - 2} \left( \int (\theta_0')^2 v_\varepsilon^2 \, dy \right)^3 \\
+ c \varepsilon^{2 \alpha - 2} \xi_\varepsilon^2 + c \varepsilon^{2 \alpha - 2} \xi_\varepsilon^2 \\
+ c \int B_0^2 \, dy + c \int (B_0')^2 \, dy \\
+ c \int D_0^2 \, dy + c \int (D_0')^2 \, dy. \quad (78) \]
(ii) for all $\alpha > 2$ and $N \geq \alpha + 1$

$$
|\xi| \int R_\varepsilon \theta_0'(v_\varepsilon + \xi_\varepsilon) dy \leq \frac{1}{8} \frac{\xi_\varepsilon^2}{\varepsilon} + c e^{4(\alpha - 1)} \frac{\xi_\varepsilon^6}{\varepsilon} + c \left[ \int (\theta_0')^2 v_\varepsilon^2 dy + \frac{\xi_\varepsilon^2}{\varepsilon} \right] \\
+ c e^{2(\alpha - 2)} \left[ \int (\theta_0')^2 v_\varepsilon^2 dy + \frac{\xi_\varepsilon^2}{\varepsilon} \right]^2 + c \int B_\varepsilon^2 dy + c \int D_\varepsilon^2 dy \\
+ c e^{2\alpha - 2} \left[ \int (\theta_0')^2 v_\varepsilon^2 dy \right]^{3/2} \left[ \int (\theta_0')^2 (v'_\varepsilon)^2 dy \right]^{1/2} |\xi_\varepsilon| \\
+ c e^{\alpha - 1} \left[ \int Q_\varepsilon^2 + (Q'_\varepsilon)^2 dy \right]^{1/2} |\xi_\varepsilon|.
$$

(79)

**Proof.**

To show (i) we estimate $\int R_\varepsilon \theta_0'(v_\varepsilon + \xi_\varepsilon) dy$ using the definition of $R_\varepsilon$ (46), the Cauchy-Schwarz inequality, Poincare inequalities (98), (101), (103), and estimates (66) and (67) from Lemma 4.

$$
\int R_\varepsilon \theta_0'(v_\varepsilon + \xi_\varepsilon) dy \leq c \left[ 1 + e^{2(N-1-\alpha)} + e^{\alpha} + e^{2\alpha - 2} \right] \int (\theta_0')^2 (v'_\varepsilon)^2 dy \\
+ c \left[ 1 + e^{\alpha - 1} + e^{2(N-\alpha)} \right] \frac{\xi_\varepsilon^2}{\varepsilon} + c e^{2\alpha - 2} \frac{\xi_\varepsilon^3}{\varepsilon} + c e^{2\alpha - 2} \frac{\xi_\varepsilon^4}{\varepsilon} \\
+ c e^{\alpha - 2} \left[ \int (\theta_0')^2 v_\varepsilon^2 dy \right]^2 + c e^{2\alpha - 2} \left[ \int (\theta_0')^2 (v'_\varepsilon)^2 dy \right]^3 \\
+ c \int B_\varepsilon^2 dy + c \int B'_\varepsilon^2 dy + c e^{\alpha - 1} \left[ \int Q_\varepsilon^2 + (Q'_\varepsilon)^2 dy \right].
$$

Taking $\alpha > 2$ and $N \geq \alpha + 1$ we derive (78).

To prove (ii) we note that definition (46) of $R_\varepsilon$, the Cauchy-Schwarz inequality, Poincare inequalities (98) and (101) yield

$$
|\xi| \int R_\varepsilon \theta_0' dy \leq \frac{1}{8} \frac{\xi^2}{\varepsilon} + c e^{4(\alpha - 1)} \frac{\xi^6}{\varepsilon} + c \left[ \int (\theta_0')^2 (v'_\varepsilon)^2 dy \right] \\
+ c e^{2(\alpha - 2)} \left[ \int (\theta_0')^2 v_\varepsilon^2 dy + \frac{\xi^2}{\varepsilon} \right]^2 \\
+ c e^{2\alpha - 2} \left[ \int (\theta_0')^2 v_\varepsilon^2 dy \right]^{3/2} \left[ \int (\theta_0')^2 (v'_\varepsilon)^2 dy \right]^{1/2} |\xi| \\
+ c \int B_\varepsilon^2 dy + c \int D_\varepsilon^2 dy \\
+ c e^{\alpha - 1} \left[ \int (\theta_0')^2 v_\varepsilon^2 dy + \frac{\xi^2}{\varepsilon} \right]^{1/2} \left[ \int Q_\varepsilon^2 dy + \int (Q'_\varepsilon)^2 dy \right]^{1/2} |\xi|.
$$

□
Now it is convenient to introduce the following notation,

\[ \mathcal{E}_\varepsilon(t) = \int (\theta_0')^2 \nu^2 dy + c_0 \varepsilon^2 + \frac{1}{\varepsilon} \int B_t^2 dy + \frac{1}{\varepsilon} \int D_t^2 dy \]

\[ \mathcal{G}_\varepsilon(t) = \frac{1}{8 \varepsilon^2} \left\{ \int (\theta_0')^2 (v')^2 dy + \int B_t^2 dy + \int (B_t')^2 dy + \int D_t^2 dy + \int (D_t')^2 dy \right\} + \left( \frac{\beta^*}{8} - \varepsilon \beta^2 \right) \varepsilon^2. \]

In terms of \( \mathcal{E}_\varepsilon \) and \( \mathcal{G}_\varepsilon \) we can rewrite (79) for \( \alpha = 3 \) and \( N = 4 \) in the following form,

\[ \int R_\varepsilon \theta dy \mathcal{E}_\varepsilon \leq c \mathcal{E}_\varepsilon + c \varepsilon \mathcal{E}^2 + c \varepsilon^3 \mathcal{G}_\varepsilon + c (c + \varepsilon^2 \mathcal{G}^3/2 + \varepsilon^4 \mathcal{G}^3/2) \mathcal{G}_\varepsilon \]

Substituting the above inequality and (79) into (77) we obtain

\[ \dot{\mathcal{E}}_\varepsilon + \frac{1}{2} \mathcal{G}_\varepsilon \leq c \mathcal{E}_\varepsilon + c \varepsilon \mathcal{E}^3/2 + c \varepsilon^3 \mathcal{G}_\varepsilon + c (c + \varepsilon^2 \mathcal{G}^3/2 + \varepsilon^4 \mathcal{G}^3/2) \mathcal{G}_\varepsilon. \]

Assume that

\[ \mathcal{E}_\varepsilon(t) \leq c \text{ for all } t \in [0, t^*]. \]

Then for \( t \in [0, t^*] \) we have

\[ \dot{\mathcal{E}}_\varepsilon \leq c \mathcal{E}_\varepsilon \tag{80} \]

Thus, \( [0, T] \subset [0, t^*] \) sufficiently small \( \varepsilon \). This concludes the proof of Theorem 2.

4 Formal derivation of (5)

In this section we formally derive equation (5) for 2D system (112) with gradient coupling. Analogous derivation for the single Allen-Cahn equation has been done in [19] and [8].

Consider a subdomain \( \omega_0 \subset \Omega \subset \mathbb{R}^2 \) (\( \omega_0 \) is occupied by the cell) so that \( \Gamma(0) = \partial \omega_0 \) (boundary of the cell at \( t = 0 \)). For all \( t \in [0, T] \) consider a closed curve \( \Gamma(t) \) s.t. \( \partial \omega_0 = \Gamma(t) \) and \( \omega_0 \subset \Omega \). Let \( \omega_0(s, t) \) be a parametrization of \( \Gamma(t) \). In a vicinity of \( \Gamma(t) \) the parameters \( s \) and the signed distance \( r \) to \( \Gamma(t) \) will be used as local coordinates, so that

\[ x = X_0(s, t) + r n(s, t) = X(r, s, t), \quad \text{where } n \text{ is an inward normal.} \]

The inverse mapping to \( x = X(r, s, t) \) is given by

\[ r = \pm \text{dist}(x, \Gamma(t)), \quad s = S(x, t), \]

where in the formula for \( r \) we choose \( - \) if \( x \in \omega_0 \) and \( + \), if \( x \notin \omega_0 \). Recall that \( \Gamma(t) \) is the limiting location of interface as \( \varepsilon \to 0 \). For fixed \( \varepsilon \) we are looking for interface in the form of \( \varepsilon \)-perturbation of \( \Gamma(t) \):

\[ \Gamma_\varepsilon(t) = \Gamma(t) + \varepsilon h_\varepsilon(s, t). \]
Introduce the limiting velocity $V_0 := -\partial_r r$ and the distance to $\Gamma_0(t)$ rescaled by $\varepsilon$:

$$z = z^\varepsilon(x,t) = \frac{r - \varepsilon h_\varepsilon(S(x,t),t)}{\varepsilon}.$$  \hspace{1cm} (81)

Next we define a rule that for all positive $t$ transforms any given function $w$ of original variable $x$ into the corresponding function $\hat{w}$ in local coordinates $(z,s)$:

$$w(x,t) = \hat{w} \left( \frac{r(x,t) - \varepsilon h_\varepsilon(S(x,t),t)}{\varepsilon}, S(x,t), t \right).$$

By applying this rule for the functions $\rho_\varepsilon$ and $P_\varepsilon$ we define $\hat{\rho}_\varepsilon$ and $\hat{P}_\varepsilon$:

$$\hat{\rho}_\varepsilon(z,s,t) = \rho_\varepsilon(x,t) \text{ and } \hat{P}_\varepsilon(z,s,t) = P_\varepsilon(x,t).$$

We now introduce asymptotic expansions in local coordinates:

$$\hat{\rho}_\varepsilon(z,s,t) = \Theta_0(z,s,t) + \varepsilon \Theta_1(z,s,t) + \ldots$$ \hspace{1cm} (82)

$$\hat{P}_\varepsilon(z,s,t) = \Psi_0(z,s,t) + \varepsilon \Psi_1(z,s,t) + \ldots$$ \hspace{1cm} (83)

$$h_\varepsilon(s,t) = h_1(s,t) + \varepsilon h_2(s,t) + \ldots$$ \hspace{1cm} (84)

$$\lambda_\varepsilon(t) = \frac{\lambda_0(t)}{\varepsilon} + \lambda_1(t) + \varepsilon \lambda_2(t) + \ldots$$ \hspace{1cm} (85)

Now, substitute (82-85) into (1) and (2). Equating coefficients at $\varepsilon^{-2}, \varepsilon^{-1}$ and $\varepsilon^0$, we get:

$$\frac{\partial^2 \Theta_0}{\partial z^2} = W''(\Theta_0),$$ \hspace{1cm} (86)

and

$$- V_0 \frac{\partial \Theta_0}{\partial z} = \frac{\partial^2 \Theta_1}{\partial z^2} - W''(\Theta_0) \Theta_1 + \frac{\partial \Theta_0}{\partial z} \kappa(s,t) + (\Psi_0 \cdot n) \frac{\partial \Theta_0}{\partial z} + \lambda_0(t),$$ \hspace{1cm} (87)

$$- V_0 \frac{\partial \Psi_0}{\partial z} = \frac{\partial^2 \Psi_1}{\partial z^2} - \Psi_0 + \beta \frac{\partial \Theta_0}{\partial z} n.$$ \hspace{1cm} (88)

where $\kappa(s,t)$ is the curvature of $\Gamma_0(t)$. The curvature $\kappa$ appears in the equation when one rewrites the Laplace operator in (1) in local coordinates $(z,s)$. The solvability condition for the equation for $\Theta_1$ (87) yields

$$c_0 V_0 = c_0 \kappa(s,t) + \int (\Psi_0 \cdot n) \left( \frac{\partial \Theta_0}{\partial z} \right)^2 \, dz + \lambda_0(t).$$ \hspace{1cm} (89)

From the definition (6) it follows that $\int_{\Omega} \partial_s \rho_\varepsilon = 0$. Substitute expansions for $\rho_\varepsilon$ into $\int_{\Omega} \partial_s \rho_\varepsilon = 0$, take into account that $\Theta_0$ does not explicitly depend on $s$ and $t$ (which follows from the equation (86); note that $\Theta_0$ still depends on $t$ implicitly, through variable $z$ which by (81) is a function of $t$). By integrating RHS of (89) we get

$$\lambda_0(t) = - \int \left\{ c_0 \kappa(s,t) + \int (\Psi_0 \cdot n) \left( \frac{\partial \Theta_0}{\partial z} \right)^2 \, dz \right\} \, ds.$$ \hspace{1cm} (90)
Introduce $\Psi_0 := \Psi_0/\beta$. Since equation (88) is linear with respect to $\Psi_0$ and the in-homogeneity is linearly proportional to $\beta$, function $\Psi_0$ does not depend on $\beta$. Finally, define

$$
\Phi(V) := \int (\Psi_0 \cdot n) \left( \frac{\partial \theta_0}{\partial z} \right)^2 dz. \tag{91}
$$

By substituting (91) and (90) into equation (89) we derive (5).

A Appendix

A.1 Derivation of equations for $u\epsilon$ and $Q\epsilon$.

First, rewrite the linear part of the equation:

$$\frac{\partial p\epsilon}{\partial t} - \vec{\alpha}^2 \epsilon^2 p\epsilon - F(t) = \epsilon h \left( \frac{\partial u\epsilon}{\partial t} + \frac{V_0 u\epsilon}{\epsilon} - \frac{u\epsilon^2}{\epsilon^2} \right) + \epsilon^2 \frac{V^2 - V_0 u\epsilon}{\epsilon}$$

$$\frac{\theta_\epsilon^2}{\epsilon^2} = \frac{-\theta_\epsilon^2 + V_0 \theta_\epsilon^2 - F_0 \epsilon + \cdots + \epsilon^{N-2} \left( \theta_{N-2} - \theta_\epsilon^2 + \sum_{j=0}^{N-1} V_j \theta_{N-2-j} \right) + \epsilon^{N-1} \left( -\theta_{N-1} + \sum_{j=0}^{N} V_j \theta_{N-j} \right) + \epsilon^N r\epsilon (\epsilon, y, t)}{\epsilon} \tag{92}
$$

Here

$$\epsilon^N r\epsilon (\epsilon, y, t) = \epsilon^N \left[ \theta_N + \sum_{k=0}^{N} \left( \sum_{j=1}^{k+1} V_j \theta_{N-k-j} \right) \right]$$

To analyze nonlinear part denote

$$h\epsilon(t, y) = \sum_{i=1}^{N} \epsilon^{i-1} \theta_i(t, y) \text{ and } r\epsilon(t, y) = \sum_{i=1}^{N} \epsilon^{i-1} \Psi_i(t, y). \tag{93}$$

Thus, $p\epsilon = \theta_0 + \epsilon h\epsilon + \epsilon^d u\epsilon$ and $P\epsilon = \Psi_0 + \epsilon r\epsilon + \epsilon^d Q\epsilon$. Rewrite nonlinear terms:

$$P\epsilon \vec{\alpha}^2 \epsilon^2 p\epsilon = \epsilon^{-1} (\Psi_0 + \epsilon r\epsilon + \epsilon^d Q\epsilon) (\theta_0^2 + \epsilon h\epsilon^2 + \epsilon^d u\epsilon^2)$$

$$= \epsilon^{2a-1} Q\epsilon \epsilon^2 u\epsilon^2 + \epsilon^{a-1} \Psi_0 u\epsilon^2 + \epsilon^{a-1} \theta_\epsilon^2 Q\epsilon + \epsilon^d r\epsilon u\epsilon^2 + \epsilon^d h\epsilon Q\epsilon + \epsilon^{N-1} \sum_{i=0}^{N} \Psi_i \theta_{N-i}^2 + \epsilon^{N-1} \sum_{i=0}^{N} \theta_{N-i} \theta_{N-j} + \epsilon^N r\epsilon^2 (\epsilon, y, t). \tag{94}$$

Here

$$\epsilon^{N-2} r\epsilon^2 (\epsilon, y, t) = \epsilon^N \sum_{i=N+1}^{2N} \epsilon^{i-N-1} \sum_{j=0}^{N} \Psi_i \theta_{N-j}^2$$

$$\frac{W'(p\epsilon)}{\epsilon^2} = \left\{ \frac{W'(\theta_0)}{2} + \epsilon \frac{W'(\theta_0)}{2} \frac{\theta_0}{\epsilon} + \epsilon^2 \frac{W'(\theta_0)}{2} \frac{\theta_0^2}{\epsilon^2} \right\} \epsilon^{N-2} u\epsilon$$

$$+ \left\{ \frac{W'(\theta_0)}{2} + \epsilon \frac{W'(\theta_0)}{2} \frac{\theta_0}{\epsilon} \right\} \epsilon^{2a-2} u\epsilon^2 + \frac{W'(\theta_0)}{6} \epsilon^{3a-2} u\epsilon^3$$

$$+ \epsilon^{N-2} (\epsilon^d W^N) \epsilon^2 + \epsilon^{N-1} \sum_{i=N+1}^{2N} \epsilon^{i-N} \sum_{j=0}^{N} \Psi_i \theta_{N-j} + \epsilon^N r\epsilon^3 (\epsilon, y, t), \tag{95}$$

$$\epsilon^{N-1} (\epsilon^d W^N) \epsilon^2 + \epsilon^N r\epsilon^3 (\epsilon, y, t),$$
Here

\[
\epsilon^N r_3(x,y,t) = \epsilon^N \sum_{k=0}^{2N} \epsilon^{k-2} \left\{ \sum_{i_1+i_2 = N+k \atop 1 \leq i_1,i_2 \leq N} \frac{W''(\theta_k)}{2} \theta_{i_1} \theta_{i_2} + \sum_{i_1+i_2+i_3 = N+k \atop 1 \leq i_1,i_2,i_3 \leq N} \frac{W^{(i)}(\theta_k)}{6} \theta_{i_1} \theta_{i_2} \theta_{i_3} \right\}
\]

Summing together (92), (94), (95) and dividing by \( \epsilon^N \) we get (45). Derivation (47) is simple since (22) is linear.

In the end of this subsection we write expression for functions in (46).

\[
a_i(t,y) = -\theta_{N-1} - \sum_{j=0}^{N} V_j \theta_{N-j} - F_N + \sum_{j=1}^{N} \Psi_j \theta_{N-j} - (dW)^{(N+1)},
\]

\[
b_{i,i}(t,y) = -r_1(x,y,t) - r_3(x,y,t) + \varepsilon \theta_1 + \varepsilon \theta_3 + \varepsilon^{N-2} \theta_N + \frac{W^{(i)}(\theta_1)}{2} h_x^2,
\]

\[
b_{2,i}(t,y) = \frac{W''(\theta_1)}{2} + \frac{W^{(i)}(\theta_1)}{2} h_x,
\]

\[
b_{i,i}(t,y) = \frac{W^{(i)}(\theta_1)}{6},
\]

\[
e_i(t,y) = \frac{V-V_0}{\varepsilon} + r_e,
\]

\[
g_e(t,y) = h_e.
\]

and the function \( m_e \) from (47).

\[
m_e(t,y) = \sum_{k=0}^{N} \left[ \sum_{j=1}^{N-k+1} V_j \Psi_{N-k+1-j} \right]
\]

### A.2 Auxiliary Inequalities.

**Assumption on \( \theta_1^2 \).** There exist \( \kappa > 0 \) and \( c_0 > 1 \) such that

\[
e_0 \epsilon^{-|y|} < (\theta_1^2(y))^2 \leq c_0 \epsilon e^{-|y|}, \quad y \in \mathbb{R}
\]

**Remarks.** 1. All inequalities below will be proven for particular case \( (\theta_1^2(y))^2 = e^{-|y|} \). The result is obviously extended for all \( \theta_1^2 \) satisfying (96).

2. It is easy to make sure that \( \theta_1^2 \) from the cell movement problem satisfies (96). Indeed,

\[
\theta_1(y) = \frac{1}{2} (1 - \tanh \frac{y}{2\sqrt{2}}) \quad \text{and} \quad (\theta_1^2(y))^2 = \frac{1}{32} \cosh^2 \frac{y}{2\sqrt{2}}
\]

and \( 2e^{-|y|} \leq \cosh x \leq 4e^{-|y|} \) for \( x \in \mathbb{R} \).

**Theorem.** *(Poincare inequality)*

\[
\int (\theta_1^2)^2 (v - <v>)^2 dy \leq C_P \int (\theta_1^2)^2 (v')^2 dy, \quad v \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}),
\]

where

\[
<v> = \frac{\int (\theta_1^2)^2 v dy}{\int (\theta_1^2)^2 dy}
\]
**Proof.**

**Step 1. (Friedrich’s inequality.)** Take any \( u \in C^4(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) s.t. \( u(0) = 0 \) and let us prove the Friedrich’s inequality:

\[
\int_0^\infty e^{-\kappa y} u^2 \, dy \leq c_F \int_0^\infty (\theta_0^2(u')^2) \, dy
\]  

(100)

where \( c_F \) does not depend on \( u \). Indeed,

\[
\int_0^\infty e^{-\kappa y} u^2 \, dy = \int_0^\infty \left( \int_0^\infty e^{-\kappa y} \, dt \right) u^2 \, dy
\]

\[
\leq \int_0^\infty \left( \int_0^\infty e^{-\kappa y} \, dt \right) |u'| |u| \, dy
\]

\[
= \frac{1}{\kappa} \int_0^\infty e^{-\kappa y} |u'| |u| \, dy
\]

\[
\leq \frac{1}{\kappa} \left( \int_0^\infty e^{-\kappa y} (u')^2 \, dy \right)^{1/2} \left( \int_0^\infty e^{-\kappa y} u^2 \, dy \right)^{1/2}.
\]

Thus,

\[
\int_0^\infty (\theta_0^2(u')^2) \, dy \leq \frac{c_F^2}{\kappa} \int_0^\infty (\theta_0^2(u')^2) \, dy
\]

(101)

Dividing the latter inequality by \( \left( \int_0^\infty (\theta_0^2(u')^2) \, dy \right)^{1/2} \), and then taking square of both sides we have

\[
\int_0^\infty (\theta_0^2(u')^2) \, dy \leq \frac{c_F^2}{\kappa} \int_0^\infty (\theta_0^2(u')^2) \, dy.
\]

Of course, similar inequality is valid on \((-\infty,0)\):

\[
\int_{-\infty}^0 (\theta_0^2(u')^2) \, dy \leq \frac{c_F^2}{\kappa} \int_{-\infty}^0 (\theta_0^2(u')^2) \, dy.
\]

Hence we proven the Friedrich’s inequality (100) with the constant \( c_F = c_F^1 / \kappa^2 \).

**Step 2.** We prove the Poincare inequality (98) by contradiction: assume that there exists a sequence \( v_n \in C^4(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) such that

\[
\int_0^\infty (\theta_0^2 v_n^2) \, dy = 1, \quad \int_0^\infty (\theta_0^2 v_n^2) \, dy = 0 \quad \text{and} \quad \int_0^\infty (\theta_0^2 v_n^2) \, dy \to 0.
\]

Apply Friedrich’s inequality (100) for functions \( v_n(y) - v_n(0) \):

\[
\int_0^\infty (\theta_0^2 (v_n(y) - v_n(0))^2) \, dy \leq c_F \int_0^\infty (\theta_0^2 v_n^2) \, dy \to 0.
\]

On the other hand,

\[
\int_0^\infty (\theta_0^2 (v_n(y) - v_n(0))^2) \, dy = \int_0^\infty (\theta_0^2 v_n^2) \, dy + \int_0^\infty (\theta_0^2 v_n^2) \, dy - \int_0^\infty (\theta_0^2 v_n^2) \, dy \geq \int_0^\infty (\theta_0^2 v_n^2) \, dy.
\]

Hence,

\[
\int_0^\infty (\theta_0^2 v_n^2) \, dy \to 0
\]

which contradicts to \( \int_0^\infty (\theta_0^2 v_n^2) \, dy = 1 \). The contradiction proves the theorem.

\[\square\]

**Theorem. (Interpolation inequality)**

\[
\int_0^\infty (\theta_0^2 y^2) \, dy \leq c_I \left( \int_0^\infty (\theta_0^2 (y^2 + (y')^2)) \, dy \right)^{1/2} \left( \int_0^\infty (\theta_0^2 y^2) \, dy \right)
\]

(101)
In this appendix we prove uniqueness of the solution to the problem (1)-(2) in the following class:

A.3 Uniqueness of original problem (1)-(2)

Proof. Consider first

$$e^{-x^2}|y(y)| = |\mathbf{e}^\mathbf{d} e^{-x^2}v(x)dx| = \int_0^\infty \int_0^\infty e^{-x^2}v(x)dx$$

Thus, we have

$$e^{-x^2}|y| \leq c_1 \left( \int_0^\infty e^{-x^2} \left\{ v^2 + (v')^2 \right\} dt \right)^{1/2} e^{-x^2/2}$$

Multiply by $e^{-x^2/2}|v|^2$ and integrate over $(0,\infty)$:

$$\int_0^\infty e^{-x^2/2}|v|^2dy \leq c_2 \left( \int \int_0^\infty e^{-x^2} \left\{ v^2 + (v')^2 \right\} dt \right)^{1/2} \int e^{-x^2}v^2dy.$$ Rederiving the same estimate for $(-\infty,0)$ we prove the theorem.

\textbf{Theorem. (Interpolation inequality for $n = 4$):}

$$\int |\mathbf{e}^\mathbf{d}^n|v|^4dy \leq c_4 \left( \int \int_0^\infty \left\{ v^2 + (v')^2 \right\} dt \right)^{1/2} \left( \int \int_0^\infty e^{-x^2}v^2dy \right)^{3/2}.$$ 

Proof. Take $y > 0$

$$e^{-x^2}v^2(y) = 2x \int_0^\infty e^{-x^2}v^2dt - 2 \int_0^\infty e^{-x^2}vv' \cdot$$

$$\leq 2x \int_0^\infty e^{-x^2}v^2dt + 2 \left( \int_0^\infty e^{-x^2}v^2dt \right)^{1/2} \left( \int_0^\infty e^{-x^2}v^2dt \right)^{1/2}$$

$$\leq c \left( \int \int_0^\infty \left\{ v^2 + (v')^2 \right\} dt \right)^{1/2} \left( \int \int_0^\infty e^{-x^2}v^2dt \right)^{1/2}.$$ Multiply by $e^{-x^2}v^2(y)$ and integrate over positive $y$:

$$\int_0^\infty e^{-x^2}v^2(y)dy \leq c \left( \int \int_0^\infty \left\{ v^2 + (v')^2 \right\} dt \right)^{1/2} \left( \int \int_0^\infty e^{-x^2}v^2dt \right)^{3/2}.$$ Use (55) and the same inequality for $(-\infty,0).$
Assume that $p_1^{(1)}$, $p_1^{(2)}$ and $p_2^{(1)}$, $p_2^{(2)}$ are solutions satisfying (102) and (103) for some positive $T > 0$.

Take $\mathbf{p}_c = p_1^{(1)} - p_2^{(1)}$ and $\mathbf{p}_e = p_1^{(2)} - p_2^{(2)}$. Then equation for $\mathbf{p}_c$:

$$\frac{d\mathbf{p}_c}{dt} = \Delta \mathbf{p}_c + a(x,t)\mathbf{p}_c - b(x,t) \cdot \nabla \mathbf{p}_c + c(x,t) \cdot \mathbf{p}_e - \overline{\lambda}(t),$$

where

$$a(x,t) = \int_0^1 W''(p(x) + s\mathbf{p_c})ds, \quad b(x,t) = p(x), \quad c(x,t) = \nabla p \cdot \mathbf{p_c}, \quad \overline{\lambda}(t) = \lambda_1(t) - \lambda_2(t).$$

We know that $|a(x,t)| < c$, $|b(\cdot, t)|_{L^2(\Omega)} + ||b(\cdot, t)||_{L^2(\Omega)} < c$, $||c(\cdot, t)||_{L^2(\Omega)} \leq c$ and

$$\overline{\lambda}(t) = \frac{1}{|\Omega|} \int_{\partial\Omega} \{a(x, t)\mathbf{p}_c - b(x, t) \cdot \nabla \mathbf{p}_c + c(x, t) \cdot \mathbf{p}_e\} dx$$

and thus,

$$|\overline{\lambda}(t)| \leq c||\mathbf{p}_c|| + c||\nabla \mathbf{p}_c|| + c||\mathbf{p}_e||$$

And energy estimate is (if multiply by $\mathbf{p}_c$):

$$\frac{d}{dt} \left[ \int \mathbf{p}_c^2 \right] + 2 \int \nabla \mathbf{p}_c \cdot \nabla \mathbf{p}_c \leq c||\mathbf{p}_c||^2 + c||\nabla \mathbf{p}_c||^2 + c||\mathbf{p}_e||^2.$$ 

Thus, using interpolation inequality

$$||\mathbf{p}_c||_{L^2} \leq c||\mathbf{p}_c||^{1/2}||\mathbf{p}_c||^{1/2} \leq \sqrt{c||\mathbf{p}_c||^2 + c||\mathbf{p}_c||^2},$$

and

$$||\nabla \mathbf{p}_c||_{L^2} \leq c||\nabla \mathbf{p}_c||^{1/2}||\nabla \mathbf{p}_c||^{1/2} \leq \sqrt{c||\nabla \mathbf{p}_c||^2 + c||\mathbf{p}_c||^2},$$

we get

$$\frac{d}{dt} \left[ \int \mathbf{p}_c^2 \right] + \int \nabla \mathbf{p}_c \cdot \nabla \mathbf{p}_c \leq c||\mathbf{p}_c||^2 + c||\mathbf{p}_e||^2.$$ 

Equation for $\mathbf{p}_e$:

$$\frac{d\mathbf{p}_e}{dt} = \Delta \mathbf{p}_e - \mathbf{p}_c - \nabla \mathbf{p}_c.$$

Energy estimate for $\mathbf{p}_e$ (if multiply by $\mathbf{p}_e$):

$$\frac{d}{dt} \left[ \int \mathbf{p}_e^2 \right] + 12 \int \nabla \mathbf{p}_e \cdot \nabla \mathbf{p}_e + 4 \int \mathbf{p}_e \cdot \mathbf{p}_c \leq \int (\nabla \mathbf{p}_e \cdot \mathbf{p}_e) \mathbf{p}_e.$$

Estimate the right hand side using integration by parts:

$$\int (\nabla \mathbf{p}_e \cdot \mathbf{p}_e) \mathbf{p}_e \leq c||\nabla \mathbf{p}_e||^2 + c||\mathbf{p}_e||^2$$

Thus, using

$$||\nabla \mathbf{p}_e||^2 + \mathbf{p}_e ||\mathbf{p}_e||_{L^2} \leq \nu ||\nabla \mathbf{p}_e||^2 + \frac{1}{\nu} ||\mathbf{p}_e||^2 \leq \nu ||\nabla \mathbf{p}_e||^2 + c||\mathbf{p}_e||^2,$$

we have

$$\frac{d}{dt} \left[ \int \mathbf{p}_e^2 \right] \leq c \int \mathbf{p}_e^2 + c \int \mathbf{p}_c^2.$$ 

Adding (106) to (107) we get:

$$\frac{d}{dt} \left[ \int \mathbf{p}_c^2 + \int \mathbf{p}_e^2 \right] \leq c \left[ \int \mathbf{p}_c^2 + \int \mathbf{p}_e^2 \right]$$

which proves uniqueness.
A.4 Maximum principle

Consider the equation
\[ \frac{\partial \rho}{\partial t} = \Delta \rho - \frac{W'(\rho)}{\varepsilon^2} - P \cdot \nabla \rho + \lambda(t), \quad \text{in} \quad \Omega \tag{108} \]
with Neumann boundary conditions:
\[ \nu \cdot \nabla \rho = 0. \]

Functions $P$ and $\lambda$ are assumed to be given, $W'(\rho) = \frac{1}{2} \rho (1 - \rho)(1 - 2\rho)$.

**Theorem.** Assume
\[ 0 \leq \rho(x, 0) \leq 1 \tag{109} \]
Then for all $t > 0$
\[ -2\varepsilon^2 \sup_{\tau \in (0,t]} |\lambda(\tau)| \leq \rho(x,t) \leq 1 + 2\varepsilon^2 \sup_{\tau \in (0,t]} |\lambda(\tau)|. \tag{110} \]

**Proof:** Denote $M := \max_{x \in \Omega, \tau \in [0,t]} \rho(x, \tau)$ and assume that the maximum is attained in $x_0 \in \Omega$ and $s_0 > 0$. For such $x_0$ and $s_0$ we have:
\[ \partial_{t} \rho \geq 0, \quad \Delta \rho \leq 0, \quad P \cdot \nabla \rho = 0. \]
Thus,
\[ W'(M) \leq \varepsilon^2 \sup_{s \in (0,t]} |\lambda(s)|. \]
Assume that $M > 1$, then $W'(M) = \frac{1}{2} M(1 - M)(1 - 2M) \geq \frac{1}{2} (M - 1)$, so
\[ M \leq 1 + 2\varepsilon^2 \sup_{s \in (0,t]} |\lambda(s)|. \]

Denote $m := \min_{x \in \Omega, s \in (0,t]} \rho(x, s)$ and assume that the maximum is attained in $x_0 \in \Omega$ and $s_0 > 0$. For such $x_0$ and $s_0$ we have:
\[ \partial_{t} \rho \leq 0, \quad \Delta \rho \geq 0, \quad P \cdot \nabla \rho = 0. \]
Thus,
\[ W'(m) \geq -\varepsilon^2 \sup_{s \in (0,t]} |\lambda(s)|. \]
Assume that $m < 0$, then $W'(m) = \frac{1}{2} m(1 - m)(1 - 2m) \leq \frac{1}{4} m$, so
\[ -m \geq 2\varepsilon^2 \sup_{s \in (0,t]} |\lambda(s)|. \]

Acknowledgements We would like to thank referees for careful reading of the manuscript.

References

1. Allen, S., Cahn, J.: A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening. Acta. Metall (27), 1084–1095 (1979)
2. Barles, G., Lio, F.D.: A geometrical approach to front propagation problems in bounded domains with neumann-type boundary conditions. INTERFACES AND FREE BOUNDARIES 5(3), 239–274 (2003)
3. Barles, G., Soner, H.M., Souganidis, P.E.: Front propagation and phase field theory. SIAM J. Control Optim. 31(2), 439–469 (1993)
4. Brakke, K.A.: The Motion of a Surface by Its Mean Curvature. Princeton University Press (1978)
5. Bronsard, L., Stoth, B.: Volume-preserving mean curvature flow as a limit of a nonlocal ginzburg-landau equation. SIAM J. Math. Anal. 28(4), 769–807 (1997)
6. Cahn, J.W., Hilliard, J.E.: Free energy of a nonuniform system. i. interfacial free energy. J. Chem. Phys. 28(2), 258 (1958)
7. Chen, X.: Spectrums for the allen-cahn, cahn-hilliard, and phase field equations for generic interface. Comm. P.D.E. 19, 1371–1395 (1994)
8. Chen, X., Hilhorst, D., Logak, E.: Mass conserving Allen-Cahn equation and volume preserving mean curvature flow. Interfaces Free Bound. 12(4), 527–549 (2010). DOI 10.4171/IFB/244. URL http://dx.doi.org/10.4171/IFB/244
9. Chen, Y.G., Giga, Y., Goto, S.: Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. J. Differential Geom. 33(3) (1991)
10. D. Shao, W.R., Levine, H.: Computation model for cell morphodynamics. Physical Review Letters 105(10)
11. Evans, L., Spruck, J.: Motion by mean curvature. I. J. Diff. Geom (33), 635–681 (1991)
12. Evans, L.C., Soner, H.M., Souganidis, P.E.: Phase transitions and generalized motion by mean curvature. Comm. Pure Appl. Math. 45, 1097–1123 (1991)
13. F. Ziebert, S.S., Aranson, I.: Model for self-polarization and motility of keratocyte fragments. Journal of the Royal Society (2011). Published online
14. Fife, P.C.: Dynamics of internal layers and diffusive interfaces, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 53. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA (1988). DOI 10.1137/1.9781611970180. URL http://dx.doi.org/10.1137/1.9781611970180
15. Golovaty, D.: The volume preserving motion by mean curvature as an asymptotic limit of reaction-diffusion equations. Q. of Appl. Math. 55, 243–298 (1997)
16. Grayson, M.: The heat equation shrinks embedded plane curves to points. J. Diff. Geom. (26), 285–314 (1987)
17. Hamilton, R.S.: Three-manifolds with positive Ricci curvature. J. Differential Geometry 17, 255–306 (1982)
18. Huisken, G.: Flow by mean curvature of convex surface into point. J. Diff. Geom (20), 237–266 (1984)
19. J. Rubinstein, P.S., Keller, J.: Fast reaction, slow diffusion, and curve shorting. Siam J. Appl. Math (49), 116–133 (1989)
20. Lio, F.D., Kim, C.I., Slepcev, D.: Nonlocal front propagation problems in bounded domains with neumann-type boundary conditions and applications. Journal Asymptotic Analysis 37(3-4), 257–292 (2004)
21. Modica, L.: Gradient theory of phase transition and singular perturbation. Arch. Rat. Mech. Anal. (98), 123–142 (1986)
22. Mogilner, A.: Mathematics of cell motility: have we got its number? J. Math. Biol. 58, 105–134 (2009)
23. Mottoni, P., Schatzman, M.: Geometrical evolution of developed interfaces. Trans. Amer. Math. Soc. 347, 1533–1589 (1995)
24. Roccaoreau, C., Georgescu, A., Giurgiteanu, N.: The FitzHugh-Nagumo model: bifurcation and dynamics, vol. 10. Springer (2000)
25. Rubinstein, J., Sternberg, P.: Nonlocal reaction-diffusion equations and nucleation. IMA J. Appl. Math. 48(3), 249–264 (1992). DOI 10.1093/imamat/48.3.249. URL http://dx.doi.org/10.1093/imamat/48.3.249
26. Serfaty, S.: Gamma-convergence of gradient flows on hilbert and metric spaces and applications. Disc. Cont. Dyn. Systems, A 31, No 4, 1427–1451 (2011)
27. Soravia, P., Souganidis, P.E.: Phase field theory for fitzhugh-nagumo-type systems. SIAM J. Math. Anal. 27(5), 1341–1359 (1996)
28. Souganidis, P.E.: Recent developments in the theory of front propagation and its applications. Modern Methods in Scientific Computing and Applications 75, 397–449 (2002)