Anomalies and non-log-normal tails in one-dimensional localization with power-law disorder

M. Titov and H. Schomerus

Max-Planck-Institut für Physik komplexer Systeme, Nöthnitzer Str. 38, 01187 Dresden, Germany

(Dated: February 2003)

Within a general framework, we discuss the wave function statistics in the Lloyd model of Anderson localization on a one-dimensional lattice with a Cauchy distribution for the random on-site potential. We demonstrate that already in leading order in the disorder strength, there exists a hierarchy of anomalies in the probability distributions of the wave function, the conductance, and the local density of states, for every energy which corresponds to a rational ratio of wave length to the lattice constant. We also show that these distribution functions do have power-law rather than log-normal tails and do not display universal single-parameter scaling. These peculiarities persist in any model with power-law tails of the disorder distribution function.

PACS numbers: 72.15.Rn, 05.45.Ac

Wave function localization in one spatial dimension has attracted enormous attention since the pioneering work by Anderson. Much of our present understanding of this phenomenon is based on the original Anderson model taken on a one-dimensional lattice

\[-t(\Psi_{n+1} + \Psi_{n-1}) + V_n \Psi_n = E\Psi_n,\]  

(1)

with a white-noise disorder \((V_n) \sim \delta_{nm}, \langle V_n \rangle = 0,\) and fixed hopping element \(t.\) The potential \(V_n\) at each site takes real values according to a probability density \(P(V).\) Most of the theoretical investigations of Anderson localization assume a finite variance \(\text{var} V \equiv 2D < \infty,\) and then consider the weak-disorder limit \(D \ll t^2.\) This condition is a prerequisite for single-parameter scaling \(\delta \gg 1\) in which the product \(L \xi^{-1}\) of system length \(L\) and inverse localization length \(\xi^{-1} = -\lim_{n \to \infty} \frac{1}{n} \ln |\Psi_n|\) is the only free parameter in the universal distribution function of the Lyapunov exponents \(\alpha = -\frac{1}{t} \ln |\Psi_n|\) for finite \(n.\) This carries over to universal distribution functions of the dimensionless conductance \(g\) and the local density of states \(\nu.\) For \(L/\xi \gg 1\) these distribution functions follow log-normal laws — the paradigm for large fluctuations. On the other hand, in many realistic applications the distribution function \(P(V)\) displays power-law tails, with \(D = \infty,\) such that in a sense disorder never really is weak. The most prominent example is the localization of wave functions in the momentum space of the kicked rotator. This dynamical problem has been mapped onto the Anderson model in the seminal works, with an effectively random Cauchy-distributed potential

\[P_{\text{Cauchy}}(V) = \frac{1}{\pi} \Im \left\{ \frac{1}{V - i\delta} \right\}, \quad \delta > 0.\]  

(2)

Equation (2) with the disorder distribution function given by Eq. (1) has been proposed for the first time by Lloyd. In this model the localization length can be computed analytically for arbitrary \(\delta \gg 1\) and the variance of the Lyapunov exponents \(\alpha\) has been analyzed very recently in Ref. 8.

A beautiful experimental realization of the kicked rotator is the dynamics of atoms driven by a regular train of laser pulses. In these experiments, the probability distribution function in momentum space is seen to relax from an initial Gaussian into an exponential profile, demonstrating the absence of diffusion in momentum direction. However, since the wave function statistics in localization is a prototypical example of large fluctuations, the localization length of the eigenstates can be quantitatively inferred from moments of the wave function (like the measured mean probability) only if the full shape of the distribution function is known, including its tails. The impact of the tails of the probability density \(P(V)\) on the statistics of the wave functions has been already mentioned by Halperin but has not been analyzed, let alone sufficiently appreciated, in the literature.

In this paper we address the fluctuations of the wave function \(\Psi\) and related quantities for distributions \(P(V)\) with power-law tails, by going beyond the mean \(\xi^{-1}\) of \(\alpha\) and its variance, studied so far.\(^{2,8}\) \(^{2,8}\) We first set out a general framework for arbitrary \(P(V)\), which then is applied to the Lloyd model with \(P(V)\) given by Eq. (2). The fluctuations turn out to be highly non-universal, with an anomalous energy dependence \(\exp(\alpha n)\) reflecting also the spatial discreetness of the Anderson model \(\delta\) and non-log-normal tails that strongly affect the behavior of the moments of \(\Psi, g,\) and \(\nu\) even in the weak-disorder limit \(\delta \ll t,\) and even for low fractional orders of the moments. These characteristics of the wave function statistics are in striking contrast to the universality for models with \(D \ll t^2\) and single-parameter scaling.

The central quantity of interest in our calculation is the generating function \(\mu(\lambda)\) of the cumulants of \(\ln \Psi.\) As pointed out by Borland and Thouless instead of solving Eq. (1) as a boundary-value problem it suffices to investigate the specific solution \(\Phi_n\) of the initial-value problem \(\Phi_0 = a, \Phi_1 = b.\) At large distances \(n \gg 1\) this solution exponentially increases as \(\Phi_n \sim \exp(\alpha n)\) for almost all values of \(a, b,\) and \(\alpha\) statistically is equivalent to the inverse wave-function decay \(\Psi_n \sim \Phi_n^{-1}\) in the original problem. The cumulant-generating function

\[\mu(\lambda) = \lim_{n \to \infty} \frac{1}{n} \ln \langle |\Phi_n|^\lambda \rangle = \sum_{k=1}^{\infty} c_k \lambda^k\]  

(3)

accounts for the details of convergence of the Lyapunov exponent \(\alpha\) to its mean value \(c_1 = \xi^{-1}\). The coefficients \(c_k\) with \(k \geq 2\) are numerical constants which characterize the deviations of \(\alpha\) from \(c_1.\) As follows from Eq. (3) the cumulants of \(\alpha\) vanish according to the law dictated by the generalized
Each coefficient may be used to define a length scale $\xi_k = e^{-c_k^{-1}}$, and the question for single-parameter scaling can be posed as whether these length scales are independent quantities or not.

We now set out a general approach to calculate the generating function $\mu(\lambda)$ and the coefficients $c_k$ for arbitrary form of $P_V(V)$. We built up on the formalism previously used to calculate the inverse localization length $\xi^{-1} = c_1^{16,17,18}$ The Anderson model (11) can be written in terms of new variables $z_n = \Psi_n/\Psi_n, r_n = \ln |\Psi_n|$ in the simple way

$$z_n = v_n - 1/z_{n-1}, \quad r_n = r_{n-1} + \ln |z_{n-1}|,$$

where $v_n \equiv (V_n - E)/t$. We seek the specific solution of the initial-value problem $z_0 = b/a, r_0 = \ln |a|$. Iterating the map (6) we observe that $z_n$ and $r_n$ take real values $z$ and $r$ with a probability density $P_n(z, r)$, which obeys

$$P_n(z, r) = \int_{-\infty}^{\infty} F(v) dv \int_{-\infty}^{\infty} P_{n-1}(z', r') dz' dr' \times \delta(z - v + 1/z') \delta(r - r' - \ln |z'|),$$

with $F(v) = tP_V(vt + E)$ the probability density of $v$.

It is convenient to introduce the function

$$h_n(z, \lambda) = |z|^\lambda \int_{-\infty}^{\infty} e^{r\lambda} P_n(z, r) dr$$

and rewrite Eq. (7) as

$$h_n(z, \lambda) = |z|^\lambda \int_{-\infty}^{\infty} F(z + 1/z') h_{n-1}(z', \lambda) dz'.$$

According to Eqs. (5) and (8) we have

$$\mu(\lambda) = \lim_{n \to \infty} \frac{1}{n} \ln \left[ \int_{-\infty}^{\infty} dz h_{n-1}(z, \lambda) \right].$$

If the function $\mu(\lambda)$ exists, the solution to Eq. (9) at large $n$ must fulfill the relation $h_n(z, \lambda) = e^{\mu(\lambda)} h_{n-1}(z, \lambda)$. Thus Eq. (9) is transformed into the functional eigenvalue problem

$$e^{\mu(\lambda) - \lambda \ln |z|} h(z, \lambda) = F[h](z, \lambda),$$

$$F[h](z, \lambda) \equiv \int_{-\infty}^{\infty} F(z + 1/z') h(z', \lambda) dz'.$$

This is the central general equation of this paper. In any practical case it has to be solved perturbatively in $\lambda$. We expand the function $h(z, \lambda)$ in a series

$$h(z, \lambda) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} h_k(z)$$

and introduce the notation $\xi_1(z) \equiv c_1 - \ln |z|$. Equation (11) is transformed into the following set of equations

$$F[h_0] - h_0 = 0,$$

$$F[h_1] - h_1 = c_1 h_0,$$

$$F[h_2] - h_2 = (c_1^2 + c_2) h_0 + 2 c_3 h_1,$$

$$F[h_3] - h_3 = (c_1^3 + 3 c_1 c_2 + c_3) h_0 + 3(c_1^2 + c_2) h_1 + 3 c_1 c_2 h_2,$$

Equation (15a) delivers the stationary distribution function of $z$ and has been used before$^{16,17,18}$ to calculate the localization length $\xi = c_1^{-1}$ from

$$c_1 = \int_{-\infty}^{\infty} h_0(z) \ln |z| dz.$$

So far we have shown that Eq. (15a) can be considered as just the first member of a hierarchy of equations that determine the complete wave function statistics for finite $n \gg \xi$.

The integrals $\int_{-\infty}^{\infty} dz$ of the left-hand sides of Eqs. (13) equal zero. Equation (14) indeed can be derived by integrating both sides of Eq. (13b) along the real axis. Once the distribution function $h_0(z)$ and the mean Lyapunov exponent $c_1$ are known (as is analytically the case in the Lloyd model), one can construct the solution to Eq. (13b) iteratively by

$$h_1(z) = K[c_1h_0](z),$$

$$K[f](z) \equiv -f(z) - \int_{-\infty}^{\infty} K(z, z') f(z') dz'.$$

The second coefficient $c_2$ in the cumulant expansion is readily found by integrating Eq. (15c) along the real axis,

$$c_2 = \int_{-\infty}^{\infty} [h_0(z) (\ln |z| - c_1) + 2 h_1(z)] (\ln |z| - c_1) dz.$$
FIG. 1: The ratio \( c_3/c_1 \) is plotted according to the analytical result (21) for the Lloyd model at the disorder strength \( \delta = 0.01t \). The ratio is never small inside the band and reveals anomalies at energies \( E = -2t \cos(\pi p/q) \) with \( p \) and \( q \) integers. The corresponding rational number \( p/q \) is indicated in the figure. The size of the anomaly only depends on the value of the denominator \( q \).

The kernel function \( K(z, z') \) can be obtained by iterative application of the operator \( \mathcal{F}[\hbar] \),

\[
K(z, z') = \frac{1}{\pi} \text{Im} \sum_{n=1}^{\infty} \left( \frac{1}{z-p_n(z') - \frac{1}{z-r_n}} \right), \tag{19a}
\]

\[
p_n(z) = \frac{(s^n - s^{-n}) - z(s^{n+1} - s^{-(n+1)})}{(s^n - s^{-(n-1)}) - z(s^n - s^{-n})}, \tag{19b}
\]

\[
r_n = p_n(\pm \infty), \tag{19c}
\]

where the second term in the parenthesis on the right-hand side of Eq. (19a) is added to provide a better convergence of the intermediate expressions.

Applying the result (19) to Eqs. (15) and (16) and performing the summation one recovers the result of Ref. 8.

\[
c_2 = \text{Re} \left[ \text{Li}_2(s^{-2}) - \text{Li}_2(1/s^{-2}) \right] + \text{arg}(s)(\pi - \text{arg}(s)) + \ln |s|^2 \left[ \ln((|s|^2 - 1) - \ln |s^2 - 1| \right], \tag{20}
\]

where \( \text{Li}_n(z) = \sum_{k=1}^{\infty} z^k/k^n \) is the polylogarithmic function. As has been shown in Ref. 8, in the limit \( \delta \to 0 \) the ratio \( c_2/c_1 \) equals 2 (not 1 as for conventional weak disorder) inside the band (it vanishes outside the band). This energy-insensitivity has encouraged the authors of Ref. 8 to conclude that single-parameter scaling is fulfilled, and to introduce a novel criterion for single-parameter scaling. As we will discuss now, these findings do not carry over to the fluctuations beyond the variance, characterized by \( c_k \) with \( k \geq 3 \).

The coefficient \( c_3 \) can be found from Eq. (15d) as

\[
c_3 = 3 \int_{-\infty}^{\infty} \frac{dz}{z} \left[ c_2 - \ln^2 \left( \frac{s}{z} \right) \right] h_0(z),
\]

\[
+ 3 \int_{-\infty}^{\infty} \frac{dz}{z} \left[ \ln \left( \frac{s}{z} \right) - 2 \ln \left( \frac{s}{z-1/s} \right) \right] \Sigma(z), \tag{21a}
\]

\[
\Sigma(z) = \frac{1}{2\pi} \text{Im} \sum_{n=1}^{\infty} \left[ \frac{1}{z - s} - \frac{1}{z - p_n(s)} \right] \ln \left( \frac{s}{p_n^{-1}(z)} \right), \tag{21b}
\]

where the function \( p_n^{-1}(z) \) stands for the inverse of \( p_n(z) \). The ratio \( c_3/c_1 \) is plotted at Fig. 1 versus the energy. The plot clearly displays a sequence of sharp dips, which appear exactly at energies \( E = -2t \cos(\pi p/q) \) where \( p \) and \( q \) are integer, and become more narrow in the limit \( \delta \to 0 \). The anomaly in the band center is the biggest one and reaches about 3% of the absolute value of the ratio \( c_3/c_1 \) in the limit \( \delta \to 0 \). The existence of such anomalies for the inverse localization length \( \xi^{-1} = c_1 \) has been pointed out by Lambert 30, but for this quantity they only show up in higher orders of the expansion in the disorder strength, with exception of the band edge \( |E| = 2 \) and the band center \( E = 0 \). For conventional weak disorder with \( D << t^2 \) the other anomalies should be seen in the higher coefficients \( c_k \) with \( k \geq 3 \). However, those cumulants are themselves suppressed by orders of \( D/t^2 \) (see Table I), again with the exception of the band edge and the band center, where they are of the same order as \( c_3 \) and \( c_4 \).

In striking contrast, in the Lloyd model the coefficients \( c_k \) increase very rapidly with increasing index \( k \). In the limit \( \delta \to 0 \) we indeed observe \( c_2/c_1 = 2 \), \( c_3/c_1 = 5 \,

| \( D^{2/3} t^{-1/3} \approx \varepsilon \ll t \) | \( D/t \ll |E| \ll t \) | \( \varepsilon, |E| \sim t \) |
|---|---|---|
| \( c_1, c_2 \) | \( D/(4\varepsilon) \) | \( D/(4t^2 - E^2) \) |
| \( c_3, c_4 \) | \( 33D^3/(128 \varepsilon^4 t^3) \) | \( 9D^3/(32 E^2 t^4) \) |
| \( c_5, c_6 \) | \( 5175D^5/(2048 \varepsilon^7 t^5) \) | \( 135D^5/(128 E^4 t^6) \) |

The last column represents the generic values inside the band.
$c_4/c_1 \approx 20$, $c_5/c_1 \approx 100$ for $p/q$ irrational. The analysis of Eqs. (14) for the Lloyd model demonstrates that the generating function $\mu(\lambda)$ exists only for $\lambda < \lambda_c$, where the convergence radius $\lambda_c < 1$, which implies a factorial growth of the ratios $c_k/c_1$ for large $k$. Such a behavior is consistent with a power-law tail in the conductance distribution function $P_g(g) \sim g^{-(2-\lambda_c)/2}$ for $g \to 0$. For a general power law $P_V(V) \propto |V|^{-\beta}$ for $|V| \to \infty$, $\lambda_c < \beta - 1$ must be expected to depend on $\beta$, implying that the precise form of the tail in $P_g(g)$ is not universal. The power-law tail in $P_g(g)$ is certified by the numerical result for the conductance distribution function for Cauchy disorder, shown for $L/\xi = 6$ in Fig. 2. The probability to find a vanishing conductance in the Lloyd model is strongly enhanced as compared to the case of conventional weak disorder, which displays log-normal tails. In this case, the generating function $\mu(\lambda)$ is well-defined for all $\lambda$ in the whole energy range. Moreover, far from the band edges ($\varepsilon \equiv 2t - |E| \gg D^{2/3}t^{-1/3}$) and far from the band center ($E \gg D/t$), it acquires a universal parabolic form $\mu(\lambda) = \xi^{-1}((\lambda + \lambda_c^2/2)$, since $c_1 = c_2 = D/(4t^2 - E^2)$, while all other coefficients can be disregarded in the limit $D \ll t^2$ (see Table I).

In conclusion, we have studied analytically the statistics of localized wave functions in the Lloyd model, which is frequently used to analyze dynamical localization. We have found that the distribution functions of the conductance $g$ and of the local density of states $\nu$ do not have a log-normal form. Moreover, even in the limit of vanishing disorder these distribution functions reveal sharp anomalies at energies $E = -2t \cos(p\pi/q)$, with $p/q$ a rational number. These specific features can be attributed to the power-law decay of the disorder distribution function. They sensitively affect the moments (including fractional moments) of $g$ and $\nu$ and demonstrate that for such distribution functions the wave-function statistics are highly non-universal. It would be striking to see similar effects for the prelocalized states in the diffusive regime $L \ll \xi$ of multichannel systems.

We thank B. L. Altshuler and S. Fishman for illuminating discussions.

1. P. W. Anderson, Phys. Rev. 109, 1492 (1958).
2. P. W. Anderson, D. J. Thouless, E. Abrahams, and D. S. Fisher, Phys. Rev. B 22, 3519 (1980).
3. S. Fishman, D. R. Grempel, and R. E. Prange, Phys. Rev. Lett. 49, 509 (1982).
4. D. L. Shepelyansky, Phys. Rev. Lett. 56, 677 (1986); Physica D 28, 103 (1987).
5. P. Lloyd, J. Phys. C 2, 1717 (1969).
6. D. J. Thouless, J. Phys. C 5, 77 (1972).
7. K. Ishii, Prog. Th. Phys. Suppl. 53, 77 (1973).
8. L. I. Deych, A. A. Lisyansky, and B. L. Altshuler, Phys. Rev. Lett. 84, 2678 (2000); Phys. Rev. B 64, 224202 (2002).
9. F. L. Moore, J. C. Robinson, C. Bharucha, P. E. Williams, and M. G. Raizen, Phys. Rev. Lett. 73, 2974 (1994).
10. C. F. Bharucha, J. C. Robinson, F. L. Moore, B. Sundaram, Q. Niu, and M. G. Raizen, Phys. Rev. E 60, 3881 (1999).
11. B. I. Halperin, Adv. Chem. Phys. 13, 123 (1967).
12. R. E. Borland, Proc. R. Soc. London A 274, 529 (1963).
13. D. J. Thouless, in Ill-Condensed Matter, edited by R. Balian, R. Maynard, and G. Toulouse (North-Holland, Amsterdam, 1979).
14. We refer here to the local density of states which is averaged over a range $\ell_{m\alpha\alpha\alpha}$ such that $\lambda_{F} \ll \ell_{m\alpha\alpha\alpha} \ll \xi$, where $\lambda_{F}$ is the Fermi wave length, see H. Schomerus, M. Titov, P. W. Brouwer, and C. W. J. Beenakker, Phys. Rev. B 65, 121101(R) (2002).
15. H. Schomerus and M. Titov, cond-mat/0302148 (2003).
16. I. M. Lifshitz, S. A. Gredeskul, and L. A. Pastur, Introduction to the Theory of Disordered Systems (Wiley, New York, 1988).
17. B. Derrida and E. Gardner, J. Phys. (Paris) 45, 1283 (1984).
18. A. Bovier and A. Klein, J. Stat. Phys. 51, 501 (1988); A. Bovier, J. Stat. Phys. 56, 669 (1989).
19. The kernel function $K(z, z')$ is not uniquely defined, since one can always add to it an arbitrary function depending only on $z$. The fact that such transformations do not affect the results ensures the independence of the coefficients $c_k$ on the conditions of the initial-value problem.
20. C. J. Lambert, Phys. Rev. B 29, 1091 (1984).
21. M. Kappus and F. Wegner, Z. Phys. B: Condens. Matter 45, 15 (1981).
22. H. Schomerus and M. Titov, Phys. Rev. E 66, 066207 (2002).
23. H. Schomerus and M. Titov, cond-mat/0208457 [in press Phys. Rev. B (R) March (2003)].
24. L. I. Deych, M. V. Erementchouk, A. A. Lisyansky, to appear in Phys. Rev. Lett. cond-mat/0207169.
25. B. L. Altshuler, V. E. Kravtsov, and I. V. Lerner, in Mesoscopic Phenomena in Solids, edited by B. L. Altshuler, P. A. Lee, R. A. Webb (North-Holland, Amsterdam, 1991).