FORM-TYPE CALABI-YAU EQUATIONS ON KÄHLER MANIFOLDS OF NONNEGATIVE ORTHOGONAL BISECTIONAL CURVATURE

JIXIANG FU, ZHIZHANG WANG, AND DAMIN WU

Abstract. In this paper we prove the existence and uniqueness of the form-type Calabi-Yau equation on Kähler manifolds of nonnegative orthogonal bisectional curvature.

1. Introduction

In the previous paper [1], we introduced the form–type Calabi–Yau equation on a compact complex n-dimensional manifold with a balanced metric and with a non-vanishing holomorphic n-form Ω. A balanced metric ω on X is a hermitian metric such that $dω^{n−1} = 0$. Given a balanced metric $ω_0$ on $X$, let us denote by $P(ω_0)$ the set of all smooth real $(n−2,n−2)$–forms $ψ$ such that $ω_0^{n−1} + \sqrt{−1} \partial \bar{∂}ψ > 0$ on $X$. Then, for each $φ \in P(ω_0)$, there exists a balanced metric, which we denote by $ω_φ$, such that $ω_φ^{n−1} = ω_0^{n−1} + \sqrt{−1} \partial \bar{∂}φ$. We say that such a metric $ω_φ$ is in the balanced class of $ω_0$. Our aim is to find a balanced metric $ω_φ$ in the balanced class of $ω_0$ such that

$$||Ω||_{ω_φ} = \text{a constant } C_0 > 0.$$  

The geometric meaning of such a metric is that its Ricci curvatures of the hermitian connection and the spin connection are zero. On the other hand, the direct non-Kähler analogue of the Calabi conjecture has recently been solved by Tosatti–Weinkove [7] (see also [4], and the references in [7, 4]). In general their solutions provides hermitian Ricci-flat metrics which are not balanced.

As in the Kähler case, equation (1.1) can be reformulated in the following form

$$\frac{ω_φ^n}{ω_0^n} = e^f \frac{\int_X ω_φ^n}{\int_X ω_0^n},$$

where $f \in C^∞(X)$ is given and satisfies the compatibility condition:

$$\int_X e^f ω_0^n = \int_X ω_0^n.$$  

We would like to find a solution $φ \in P(ω_0)$. The equation (1.2) is called a form-type Calabi–Yau equation, a reminiscent of the classic function type Calabi–Yau equation. We have constructed solutions for (1.1) when $X$ is a complex torus [1]. A natural approach to solve (1.2) is to use the continuity method. The openness and
uniqueness were discussed in the previous work [1]. We do not know whether there
is a geometric obstruction for solving (1.2) in general.

Equation (1.2) is still meaningful on a compact complex manifold with a balanced
metric, whose canonical line bundle is not holomorphically trivial. Geometrically,
solving (1.2) allows us to solve the problem of prescribed volume form on $X$, in the
balanced class of each balanced metric on $X$. Namely, given any positive $(n, n)$–form $W$ on $X$ and a balanced metric $\omega_0$, we let

$$e^f = \left( \frac{W}{\omega_0^n} \right) \int_X \frac{\omega_0^n}{W},$$

then by solving (1.2) we are able to find a metric $\omega_\varphi$ in the balanced class of $\omega_0$ such that $\omega_\varphi^n$ is equal to $W$, up to a constant rescaling.

It seems to us very hard to understand equation (1.2) in general. In this paper, we
want to look for solutions in a subset within the balanced class of a given balanced
metric. The idea is, in some sense, to transfer the form-type Calabi–Yau equation
to a function type equation.

So in the following we let $(X, \eta)$ be an $n$-dimensional Kähler manifold, $n \geq 2$, and $\omega_0$ be a balanced metric on $X$. We let

$$\mathcal{P}_\eta(\omega_0) = \{ v \in C^\infty(X) \mid \omega_0^{n-1} + (\sqrt{-1}/2)\bar{\partial}\partial v \wedge \eta^{n-2} > 0 \text{ on } X \}.$$ 

For each $v \in \mathcal{P}_\eta(\omega_0)$, we denote by $\omega_v$ the positive $(1, 1)$–form on $X$ such that $\omega_v^{n-1} = \omega_0^{n-1} + (\sqrt{-1}/2)\bar{\partial}\partial v \wedge \eta^{n-2}$ on $X$.

Then we consider the equation

$$(1.4) \quad \frac{\omega_{u}^n}{\omega_0^n} = e^f \frac{\int_X \omega_{u}^n}{\int_X \omega_0^n},$$

where $f \in C^\infty(X)$ is given and satisfies the compatibility condition (1.3).

Obviously, the function-type equation (1.4) is a special case of the form-type
equation (1.2). However, an important observation is that, solving (1.4) will enable
one to find all the solutions to (1.2), in the balanced class of a given balanced metric
(see Remark 2).

In this paper, we are able to solve (1.4), under the assumption that the Kähler
metric $\eta$ has nonnegative orthogonal bisectional curvature; that is, for any orthonor-
mal tangent frame $\{e_1, \ldots, e_n\}$ at any $x \in M$, the curvature tensor of $\eta$ satisfies that

$$(1.5) \quad R_{i\bar{j}j} \equiv R(e_i, \bar{e}_i, e_j, \bar{e}_j) \geq 0, \quad \text{for all } 1 \leq i, j \leq n \text{ and } i \neq j.$$ 

We remark that nonnegativity of the orthogonal bisectional curvature is weaker than
nonnegativity of the bisectional curvature. In fact, the former condition are satisfied
by not only complex projective spaces and the Hermitian symmetric spaces, but
also some compact Kähler manifolds of dimension $\geq 2$ whose holomorphic sectional
curvature is strictly negative somewhere. We refer the reader to the recent work
Gu–Zhang [3] for the study of nonnegative orthogonal bisectional curvature, which
generalizes the earlier work of Mok [5] and Siu–Yau [6].

Our main result is as follows:
Theorem 1. Let \((X, \eta)\) be a compact Kähler manifold of nonnegative orthogonal bisectional curvature, and \(\omega_0\) be a balanced metric on \(X\). Then, for any smooth function \(f\) on \(X\), equation \((1.4)\) admits a solution \(u \in P_\eta(\omega_0)\), which is unique up to a constant.

Remark 2. Recall that our aim is to find all solutions of equation \((1.2)\) in the balanced class of a given balanced metric \(\omega_0\) for a given \(f \in C^\infty(X)\) satisfying the compatibility condition \((1.3)\). By Theorem 1, we achieve this goal on a Kähler manifold \(X\) of nonnegative orthogonal bisectional curvature. (In particular, the form-type equation on a complex torus is completely settled.)

Indeed, for any \(\varphi \in P(\omega_0)\) and any \(f \in C^\infty(X)\) satisfying \((1.3)\), we claim that there exists a unique smooth function \(u_\varphi\) up to a constant such that \((\varphi + u_\varphi \wedge \eta^{n-2})\) is in \(P(\omega_0)\) and solves \((1.2)\); namely, if we denote by \(\omega_{u_\varphi}\) the positive \((1,1)\)-form such that

\[
\omega_{u_\varphi}^{n-1} = \omega_0^{n-1} + \frac{\sqrt{-1}}{2} \partial\bar{\partial} \varphi + \frac{\sqrt{-1}}{2} \partial\bar{\partial} u_\varphi \wedge \eta^{n-2} > 0,
\]

then,

\[
(1.6) \quad \frac{\omega_{u_\varphi}^n}{\omega_0^n} = e^{f_\varphi} \frac{\int_X \omega_{u_\varphi}^n}{\int_X \omega_0^n}.
\]

To see this, we define a function \(f_\varphi \in C^\infty(X)\) by

\[
e^{f_\varphi} = e^{f} \frac{\omega_0^n}{\omega_\varphi^n} \frac{\int_X \omega_\varphi^n}{\int_X \omega_0^n}.
\]

Then, \(f_\varphi\) satisfies

\[
\int_X e^{f_\varphi} \omega_\varphi^n = \frac{\int_X e^{f} \omega_0^n}{\int_X \omega_0^n} \frac{\int_X \omega_\varphi^n}{\int_X \omega_\varphi^n} = \int_X \omega_\varphi^n.
\]

Applying Theorem 2 to \(\varphi\) and \(f_\varphi\) yields that, there exists a unique solution \(u_\varphi \in P_\eta(\omega_\varphi)\) satisfying that

\[
\frac{\omega_{u_\varphi}^n}{\omega_\varphi^n} = e^{f_\varphi} \frac{\int_X \omega_{u_\varphi}^n}{\int_X \omega_\varphi^n}.
\]

Clearly, this is equivalent to \((1.6)\). The claim is proved. Therefore, in this way, we can find all solutions (which are infinitely many) to equation \((1.2)\) in the balanced class of a given balanced metric on \(X\) of nonnegative orthogonal bisectional curvature.

So the idea used in this paper, which is to transfer from the form-type Calabi–Yau equation to a function-type equation, may be useful. Later we will establish the theorem 1 on any compact Kähler manifold.

We employ the continuity method to prove Theorem 1. In Section 2, we establish an \textit{a priori} \(C^2\) estimate for the solution \(u\). This is the place where we need the curvature condition. The \(C^2\) estimate enables us to obtain a general \textit{a priori} \(C^0\) estimate, via the classic Moser’s iteration. This is the content of Section 3. We then adapt the Evans–Krylov theory to our form–type equation, and obtain in Section 4 the Hölder estimates for second derivatives. The openness is covered by Theorem 3.
in our previous paper [1]. For readers’ convenience, we briefly indicate the argument in the last section, Section 5. The uniqueness is also proved in Section 5.

Acknowledgment. The authors would like to thank Professor S.-T. Yau for helpful discussion. Part of the work was done while the third named author was visiting Fudan University, he would like to thank their warm hospitality. Fu is supported in part by NSFC grants 10831008 and 11025103.

2. $C^2$ estimates for Form–type equations

In this section, we would like to establish the following estimate:

**Lemma 3.** Given $F \in C^2(X)$, let $u \in C^4(X)$ satisfy that

$$\omega_{0}^{n-1} + (\sqrt{-1}/2)\partial\bar{\partial}u \wedge \eta^{n-2} > 0 \quad \text{on } X,$$

and that

(2.1) $$\det[\omega_{0}^{n-1} + (\sqrt{-1}/2)\partial\bar{\partial}u \wedge \eta^{n-2}] = e^F \det \omega_{0}^{n-1}.$$ 

Then, we have

(2.2) $$\Delta_\eta u \leq C + C(u - \inf_X u) \quad \text{on } X,$$

and

$$\sup_X |\omega_{0}^{n-1} + \partial\bar{\partial}u \wedge \eta^{n-2}| \leq C + \left(\sup_X u - \inf_X u\right).$$

Here $\Delta_\eta v = \sum \eta^j v_{ij}$ denotes the Laplacian of a function $v$ with respect to $\eta$, and $C > 0$ is a constant depending only on $\inf_X (\Delta_\eta F)$, $\sup_X F$, $\eta$, $n$, and $\omega_0$.

Here are some conventions: For an $(n-1,n-1)$–form $\Theta$, we denote

$$\Theta = \left(\frac{-1}{2}\right)^{(n-1)!} \sum_{p,q} s(p,q)\Theta_{pq} dz^1 \wedge d\bar{z}^1 \cdots \wedge dz^p \wedge d\bar{z}^p \wedge \cdots \wedge dz^q \wedge d\bar{z}^q \wedge \cdots \wedge dz^n \wedge d\bar{z}^n,$$

in which

(2.3) $$s(p,q) = \begin{cases} -1, & \text{if } p > q; \\ 1, & \text{if } p \leq q. \end{cases}$$

Here we introduce the sign function $s$ so that,

$$dz^p \wedge d\bar{z}^q \wedge s(p,q)dz^1 \wedge d\bar{z}^1 \cdots \wedge dz^p \wedge d\bar{z}^p \wedge \cdots \wedge dz^q \wedge d\bar{z}^q \wedge \cdots \wedge dz^n \wedge d\bar{z}^n$$

$$= dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n,$$

for all $1 \leq p, q \leq n$.

We denote

$$\det \Theta = \det(\Theta_{pq}).$$

If the matrix $(\Theta_{pq})$ is invertible, we denote by $(\Theta^{pq})$ the transposed inverse of $(\Theta_{pq})$, i.e.,

$$\sum_l \Theta_{il} \Theta^{lj} = \delta_{ij}.$$
Note that, for a positive \((1,1)\)-form \(\omega\) given by
\[
\omega = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^{n} g_{ij} dz_i \wedge d\bar{z}_j,
\]
we have
\[
\omega^n = \left(\frac{\sqrt{-1}}{2}\right)^n n! \det(g_{ij}) dz^1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n,
\]
and by our convention,
\[
(\omega^{n-1})_{ij} = \det(g_{ij}) g_{ij}.
\]
It follows that
\[
(\omega^{n-1})_{ij} = \det(g_{ij})^{n-1},
\]
and
\[
(\omega^{n-1})_{ij} = \frac{g_{ij}}{\det(g_{ij})}.
\]

In the following, the subscripts such as "\(,p\)" stand for the ordinary local derivatives; for example,
\[
\eta_{ij,k} = \frac{\partial \eta_{ij}}{\partial z^k}, \quad \eta_{ij,lm} = \frac{\partial^2 \eta_{ij}}{\partial z^l \partial \bar{z}^m}.
\]
For a function \(h\) we can omit the comma: \(h_{l} = h_{,l}, h_{lm} = h_{,lm}, \) etc. Unless otherwise indicated, all the summations below range from 1 to \(n\). We remark that, under the convention, equation (1.4) can be rewritten as
\[
det[\omega_{n-1}^n + (\sqrt{-1}/2) \partial \bar{\partial} u \wedge \eta^{n-2}] = e^{(n-1)f} \left(\int_X \omega_n^0 \right)^{n-1},
\]
which is convenient for deriving the estimates.

**Proof of Lemma 3** Let
\[
\Psi_u = \Psi + (\sqrt{-1}/2) \partial \bar{\partial} u \wedge \eta^{n-2}, \quad \text{where} \quad \Psi = \omega_{0}^{n-1}.
\]
Let
\[
\phi = \frac{\sum_{i,j} \eta_{ij}(\Psi_u)_{i\bar{j}}}{\det \eta} - Au,
\]
where \(A > 0\) is a large constant to be determined. Using wedge products, the function \(\phi\) can also be written as
\[
\phi = \frac{n \eta \wedge \Psi_u}{\eta^n} - Au
\]
(2.5)
\[
= (h + \Delta_{\eta} u) - Au, \quad \text{where} \quad h = \frac{n \eta \wedge \omega_{0}^{n-1}}{\eta^n}.
\]
Consider the operator
\[
L\phi = (n - 1) \sum_{k,l} \Psi_{u}^{kl} \left(\frac{\sqrt{-1}}{2} \partial \bar{\partial} \phi \wedge \eta^{n-2}\right)_{k\bar{l}}.
\]
Suppose that $\phi$ attains its maximum at some point $P$ in $X$. We choose a normal coordinates, such that at $P$, $\eta_{ij} = \delta_{ij}$ and $d\eta_{ij} = 0$. Then, we rotate the axes so that at $P$ we have $(\Psi_u)_{pq} = \delta_{pq}(\Psi_u)_{p\bar{p}}$. Thus, for any smooth function $v$ on $X$, we have at $P$ that

$$
(n - 1)(\frac{\sqrt{-1}}{2}\partial \bar{\partial} v \wedge \eta^{n-2})_{ij} = \delta_{ij} \sum_{p \neq i} v_{p\bar{p}} + (1 - \delta_{ij})v_{j\bar{j}}.
$$

By (2.7) we obtain that

$$
(\Psi_u)_{ii} = \Psi_{ii} + \frac{1}{n - 1} \sum_{q \neq i} u_{q\bar{q}},
$$

(2.8)

$$
(\Psi_u)_{ij} = \Psi_{ij} + \frac{u_{j\bar{j}}}{n - 1} = 0, \quad \text{for all } i \neq j.
$$

(2.9)

It follows that

$$
\sum_{i=1}^{n} (\Psi_u)_{ii} = \sum_{i=1}^{n} \Psi_{ii} + \sum_{i=1}^{n} u_{ii} = h + \Delta \eta u.
$$

(2.10)

Furthermore, we have

$$
(\Psi_u)_{ij,p} = \Psi_{ij,p} + \frac{\delta_{ij}}{n - 1} \sum_{q \neq i} u_{q\bar{q}p} + \frac{1 - \delta_{ij}}{n - 1} u_{j\bar{j}p},
$$

(2.11)

and

$$
(\Psi_u)_{i\bar{i},p\bar{p}} = \Psi_{i\bar{i},p\bar{p}} + \frac{1}{n - 1} \sum_{k \neq i} u_{k\bar{k}p\bar{p}} + \frac{1}{n - 1} \sum_{k \neq i} u_{k\bar{k}} \left( \sum_{j \neq k} \sum_{j \neq i} \eta_{j\bar{j},pp} \right)
$$

$$
- \frac{1}{n - 1} \sum_{a \neq i, b \neq i, a \neq b} u_{ab} \eta_{a\bar{a},p\bar{p}}.
$$

Note that under the normal coordinate system, the curvature $(R_{ij\bar{k}\bar{l}})$ of $\eta$ reads

$$
R_{ij\bar{k}\bar{l}} = -\eta_{ij,k\bar{l}} + \sum_{a,b} \eta^{ab} \eta_{a\bar{k},b\bar{l}} = -\eta_{ij,k\bar{l}}, \quad \text{at } P.
$$

This together with (2.9) imply that

$$
(\Psi_u)_{i\bar{i},p\bar{p}} = \Psi_{i\bar{i},p\bar{p}} + \frac{1}{n - 1} \sum_{k \neq i} u_{k\bar{k}p\bar{p}} - \frac{1}{n - 1} \sum_{k \neq i} u_{k\bar{k}} \left( \sum_{j \neq k} \sum_{j \neq i} R_{j\bar{j},pp} \right)
$$

$$
- \sum_{a \neq i, b \neq i, a \neq b} \Psi_{ab} R_{a\bar{b}p\bar{p}}.
$$

(2.12)

We compute at $P$ that

$$
L_{\phi} = (n - 1) \sum_{i} (\Psi_u)^{il} \left( \frac{1}{2} \partial \bar{\partial} \phi \wedge \eta^{n-2} \right)_{il} = \sum_{i} \sum_{p \neq i} (\Psi_u)^{il} \phi_{p\bar{p}}.
$$

Note that

$$
0 = \phi_{p}(P) = h_{p} + (\Delta \eta u)_{p} - Au_{p}.
$$

(2.13)
Apply (2.13) to obtain that
\[ 0 \geq \phi_{p\bar{p}}(P) = h_{p\bar{p}} + (\Delta_\eta u)_{p\bar{p}} - A u_{p\bar{p}}. \]
It follows that
\[ 0 \geq L\phi = \sum_l \sum_{p \neq l} (\Psi_u)^{\bar{l}l} \phi_{p\bar{p}} \]
(2.14)
\[ = \sum_l \sum_{p \neq l} (\Psi_u)^{\bar{l}l} [h_{p\bar{p}} + (\Delta_\eta u)_{p\bar{p}}] - A \sum_l \sum_{p \neq l} (\Psi_u)^{\bar{l}l} u_{p\bar{p}}. \]
Notice that
\[ \sum_l \sum_{p \neq l} (\Psi_u)^{\bar{l}l} [h_{p\bar{p}} + (\Delta_\eta u)_{p\bar{p}}] \]
\[ = \sum_l \sum_{p \neq l} (\Psi_u)^{\bar{l}l} h_{p\bar{p}} + \sum_{l,a} \sum_{p \neq l} (\Psi_u)^{\bar{l}l} u_{a\bar{p}p\bar{p}} + \sum_l \sum_{p \neq l} (\Psi_u)^{\bar{l}l} \sum_{a,b} \eta_{a\bar{p}p\bar{p}} u_{a\bar{b}} \]
(2.15)
\[ = \sum_l \sum_{p \neq l} (\Psi_u)^{\bar{l}l} h_{p\bar{p}} + \sum_{l,a} \sum_{p \neq l} (\Psi_u)^{\bar{l}l} u_{a\bar{p}p\bar{p}} + \sum_{l,a} \sum_{p \neq l} (\Psi_u)^{\bar{l}l} R_{a\bar{p}p\bar{p}} u_{a\bar{a}} \]
\[ - (n-1) \sum_{l,a} \sum_{p \neq l} (\Psi_u)^{\bar{l}l} R_{b\bar{p}p\bar{b}} \Psi_{b\bar{a}}, \text{ by (2.21)}. \]
Here the fourth derivative term can be handled by the equation (2.1): We rewrite (2.1) as
\[ \log \det \Psi_u = F + \log \det \Psi. \]
Differentiating this in the direction of \( \partial / \partial z^a \) yields
\[ \sum_{k,l} (\Psi_u)^{kl} (\Psi_u)^{kl,a} = (F + \log \det \Psi)_a. \]
and then,
\[ \sum_{k,l} (\Psi_u)^{kl} (\Psi_u)^{kl,a\bar{b}} = (F + \log \det \Psi)_{a\bar{b}} + \sum_{k,l,p,q} (\Psi_u)^{kl} (\Psi_u)^{pq} (\Psi_u)^{kl,a} (\Psi_u)^{p\bar{q}a\bar{b}}. \]
Contracting this with \( \eta^{a\bar{b}} \) and applying the normal coordinates yield that
\[ \sum_{l,a} (\Psi_u)^{\bar{l}l} (\Psi_u)^{\bar{l}l,a\bar{a}} = \sum_a (F + \log \det \Psi)_{a\bar{a}} + \sum_{k,l,a} \frac{|(\Psi_u)^{kl,a}|^2}{(\Psi_u)^{kl}(\Psi_u)_{kk}}. \]
This together with (2.12) imply that
\[ \sum_{l,a} (\Psi_u)^{\bar{l}l} \Psi_{l,a\bar{a}} + \frac{1}{n-1} \sum_{l,a} \sum_{p \neq l} (\Psi_u)^{\bar{l}l} u_{p\bar{a}a\bar{a}} \]
\[ \geq \sum_{k,l,a} \frac{|(\Psi_u)^{kl,a}|^2}{(\Psi_u)^{kl}(\Psi_u)_{kk}} + \frac{1}{n-1} \sum_{l,a} (\Psi_u)^{\bar{l}l} \sum_{p \neq l} u_{p\bar{p}} \left( \sum_{m \neq p, m \neq l} R_{m\bar{m}a\bar{a}} \right) \]
\[ + \sum_{l,a} (\Psi_u)^{\bar{l}l} \left( \sum_{p \neq l, q \neq l, p \neq q} \Psi_{p\bar{q}a} R_{p\bar{q}a\bar{a}} \right) + \Delta_\eta F + \Delta_\eta (\log \det \Psi). \]
Combining this with (2.15) yields

\[
\sum_{l} \sum_{p \neq l} (\Psi_{u})^{il} (h_{p\bar{p}} + (\Delta_{\eta} u)_{p\bar{p}}) \geq \sum_{l,a} \sum_{p \neq l} (\Psi_{u})^{il} R_{a\bar{p}p} u_{a\bar{a}} + \sum_{l,a} (\Psi_{u})^{il} u_{p\bar{p}} \left( \sum_{m \neq p, m \neq l} R_{a\bar{a}m\bar{m}} \right) \\
+ (n - 1) \sum_{k,l,a} \left| (\Psi_{u})_{kl,\bar{a}} \right|^2 + (n - 1) \Delta_{\eta} F + (n - 1) \Delta_{\eta} (\log \det \Psi) \\
+ \sum_{l} \sum_{p \neq l} (\Psi_{u})^{il} h_{p\bar{p}} - (n - 1) \sum_{l,a} (\Psi_{u})^{il} \Psi_{il,a\bar{a}} \\
+ (n - 1) \sum_{l,a} (\Psi_{u})^{il} \left( \sum_{p \neq l, q \neq l, p \neq q} \Psi_{pq} R_{p\bar{q}a\bar{a}} \right) \\
- (n - 1) \sum_{l} \sum_{p \neq l} \sum_{a \neq b} (\Psi_{u})^{il} R_{ab\bar{p}p} \Psi_{ab}.
\]  

(2.16)

The first two terms on the right hand side of above inequality can be handled as follows.

\[
\sum_{l,a} \sum_{p \neq l} (\Psi_{u})^{il} R_{a\bar{p}p} u_{a\bar{a}} + \sum_{l,a} (\Psi_{u})^{il} u_{p\bar{p}} \left( \sum_{m \neq p, m \neq l} R_{a\bar{a}m\bar{m}} \right) \\
= \sum_{l,a} (\Psi_{u})^{il} R_{l\bar{a}a} u_{l\bar{l}} - \sum_{l,a} (\Psi_{u})^{il} R_{a\bar{a}l} u_{a\bar{a}} + \sum_{l,a} (\Psi_{u})^{il} u_{a\bar{a}} R_{a\bar{a}p\bar{p}} \\
+ \sum_{l,a} \sum_{p \neq l} (\Psi_{u})^{il} u_{p\bar{p}} \left( \sum_{m \neq p, m \neq l} R_{a\bar{a}m\bar{m}} \right) \\
= \frac{1}{2} \sum_{l,a} (\Psi_{u})^{il} R_{l\bar{a}a} (u_{l\bar{l}} - u_{a\bar{a}}) + \frac{1}{2} \sum_{l,a} (\Psi_{u})^{il} \left[ (\Psi_{u})_{il} - \Psi_{il} \right] \\
+ (n - 1) \sum_{l} \left( \sum_{m \neq l} R_{m\bar{m}} \right) (\Psi_{u})^{il} [(\Psi_{u})_{il} - \Psi_{il}] \\
+ (n - 1) \sum_{l} \left( \sum_{m \neq l} R_{m\bar{m}} \right) \frac{(\Psi_{u})^{il} [(\Psi_{u})_{il} - \Psi_{il}]}{(\Psi_{u})^{il}(\Psi_{u})_{il}} \\
+ (n - 1)^2 \sum_{l} R_{l\bar{l}a} - (n - 1) \sum_{l} (\Psi_{u})^{il} \Psi_{il} \left( \sum_{m \neq l} R_{m\bar{m}} \right) + \text{(by 2.8)}
\]  

(2.17)
Apply (2.8) to estimate the first term of last equality
\[
\frac{1}{2} \sum_{l,a} R_{l\bar{a}a} (u_{l\bar{l}} - u_{a\bar{a}}) \frac{((\Psi_u)_{a\bar{a}} - (\Psi_u)_{l\bar{l}})}{(\Psi_u)_{l\bar{l}}(\Psi_u)_{a\bar{a}}}
\]
\[
= \frac{n-1}{2} \sum_{l,a} R_{l\bar{a}a} \frac{[(\Psi_u)_{a\bar{a}} - (\Psi_u)_{l\bar{l}}]^2}{(\Psi_u)_{l\bar{l}}(\Psi_u)_{a\bar{a}}}
\]
(2.18)
\[
+ \frac{n-1}{2} \sum_{l,a} R_{l\bar{a}a} \frac{(\Psi_u - \Psi_{a\bar{a}})((\Psi_u)_{a\bar{a}} - (\Psi_u)_{l\bar{l}})}{(\Psi_u)_{l\bar{l}}(\Psi_u)_{a\bar{a}}}
\]
\[
\geq (n-1) \sum_{l,a} R_{l\bar{a}a} \frac{\Psi_{l\bar{l}} - \Psi_{a\bar{a}}}{(\Psi_u)_{l\bar{l}}}, \quad \text{by (1.5).}
\]
Combining (2.16) with (2.17) and then with (2.18), we obtain
\[
\sum_{l} \sum_{p \neq l} (\Psi_u)^{l\bar{l}} (h_{pp} + (\Delta_\eta u)^{p\bar{p}})
\]
(2.19)
\[
\geq -C_1(n-1) \sum_l (\Psi_u)^{l\bar{l}} - (n-1)^2 C_1 + (n-1) \inf \Delta_\eta F.
\]
Here and throughout this section, we denote by $C_1 > 0$ a generic constant depending only on $\Psi$ and the curvature of $\eta$.

Substituting (2.19) into (2.14) yields
\[
0 \geq L\phi \geq -A \sum_l \sum_{p \neq l} (\Psi_u)^{l\bar{l}} u_{p\bar{p}} - C_1(n-1) \sum_l (\Psi_u)^{l\bar{l}}
\]
\[
- (n-1)^2 C_1 - \inf \Delta_\eta F
\]
\[
= -n(n-1)A + (n-1)A \sum_l (\Psi_u)^{l\bar{l}} \Psi_{l\bar{l}} - C_1(n-1) \sum_l (\Psi_u)^{l\bar{l}}
\]
\[
- (n-1)^2 C_1 + (n-1) \inf \Delta_\eta F.
\]
Now we choose $A > 0$ sufficiently large so that
\[
A \inf \left( \min_l (\Psi_u)^{l\bar{l}} \right) \geq 2C_1.
\]
It follows that
\[
\frac{nA}{C_1} + (n-1) - \frac{\inf \Delta_\eta F}{C_1} \geq \sum_{l=1}^n (\Psi_u)^{l\bar{l}}
\]
\[
\geq \left[ \sum_{i=1}^n (\Psi_u)^{i\bar{i}} \right]^{\frac{1}{n}} \left\{ \det \left[ (\Psi_u)^{i\bar{j}} \right] \right\}^{\frac{1}{n-1}}
\]
\[
= \left[ \sum_{i=1}^n (\Psi_u)^{i\bar{i}} \right]^{\frac{1}{n-1}} e^\frac{F}{n-1} \left( \det \Psi \right)^{\frac{1}{n-1}}.
\]


Hence,
\[ h + \Delta_\eta u = \sum_{i=1}^{n} (\Psi u)_{ij} \leq C_2 \quad \text{at } P. \]

Here and throughout this section, we denote by \( C_2 \) a generic positive constant depending only on \( n, \Psi, \eta, \sup \Delta_\eta F, \) and \( \sup F. \) Therefore, at any point in \( X, \)
\[ (h + \Delta_\eta u) \leq (h + \Delta_\eta u)(P) + A u - A u(P) \leq C_2 + C_2(u - \inf u). \]

Since \( [(\Psi u)_{ij}] \) is positive definite everywhere, we have
\[ |(\Psi u)_{ij}| \leq C_2 + C_2(u - \inf u), \quad \text{for all } 1 \leq i, j \leq n. \]

This completes the proof. □

Lemma 3 enables us to establish the \( C^2 \) estimate for equation (1.4):

**Corollary 4.** For any \( f \in C^\infty(X), \) let \( u \in C^\infty(X) \) be a solution of
\[ (2.20) \quad \det(\omega_u^{n-1}) = e^{(n-1)f} \left( \frac{\int_X \omega_u^n}{\int_X e^f \omega_0^n} \right)^{n-1}, \]
where \( \omega_u \) is a positive \((1,1)\)-form on \( X \) such that
\[ \omega_u^{n-1} = \omega_0^{n-1} + (\sqrt{-1}/2) \partial \bar{\partial} u \wedge \eta^{n-2}. \]

Then, we have
\[ (2.21) \quad \Delta_\eta u \leq C + C(u - \inf u) \quad \text{on } X, \]
and
\[ \sup_X |\omega_u^{n-1}| \leq C + C(u - \inf u), \]
where \( C > 0 \) is a constant depending only on \( f, \eta, n, \) and \( \omega_0. \)

**Proof.** Let
\[ F = (n-1) \left( f + \log \int_X \omega_u^n - \log \int_X e^f \omega_0^n \right). \]

To apply Lemma 3, it suffices to estimate \( \inf(\Delta_\eta F) \) and \( \sup F. \) Note that
\[ \Delta_\eta F = (n-1)\Delta_\eta f. \]
Applying the maximum principle to (2.20) at the points where \( u \) attain its maximum and minimum, respectively, yields a uniform bound for the constant:
\[ -\sup f \leq \log \int_X \omega_u^n - \log \int_X e^f \omega_0^n \leq -\inf f. \]
This implies that \( \sup |F| \leq (n-1)(\sup f - \inf f). \) □
3. $C^0$ ESTIMATES

In this section, we first would like to derive the following general $C^0$ estimate. This then combining Corollary 4 will settle the $C^0$ estimate for manifolds of nonnegative orthogonal bisectional curvature.

Lemma 5. Let $(X, \eta)$ be an arbitrary Kähler manifold with complex dimension $n \geq 2$. Suppose that $u \in C^2(X)$ satisfies

$$\Delta u \leq C_1 + C_1(u - \inf_X u),$$
$$\Delta u > -C_2,$$

where $\Delta$ stands for the Laplacian with respect to $\eta$, and $C_1, C_2$ are two positive constants. Then,

$$\sup_X u - \inf_X u \leq C,$$

in which $C > 0$ is a constant depending only on $\eta, n, C_1, \text{ and } C_2$.

The proof uses Moser’s iteration, consisting of the following two propositions. For simplicity, we denote, throughout this section, that

$$\int h = \int_X h\eta^n, \quad \text{for all } h \in L^1(X, \eta),$$

and for $p > 0$,

$$\|h\|_p = \left(\int h^p\right)^{1/p}, \quad \text{for all } h \in L^p(X, \eta).$$

And we abbreviate $\Delta = \Delta_\eta$ in this section.

Proposition 6. Let $v \in C^2(X), v > 0$ on $X$, satisfy that

(3.1) \quad $$\Delta v + cv \geq d$$ \quad \text{on } X,

where $c$ and $d$ are constants. Then, for any real number $p > 0$,

$$\sup_X v \leq C^{1/p}(1 + |c|)^{n/p}(|v|_p + |d|),$$

where $C > 0$ is a constant depending only on $\eta$ and $n$.

Proof. Let

$$\tilde{v} = v + |d|.$$

Then,

$$\tilde{v} \geq v > 0.$$

Multiplying both sides of (3.1) by $-\tilde{v}^p$, $p \geq 1$, and then integrating by parts yield that

$$p \int \nabla \tilde{v}^2 \tilde{v}^{p-1} \leq c \int \tilde{v}^p v - d \int \tilde{v}^p.$$
Then,
\[
\int_X \left| \nabla \tilde{v}^{p+1} \right|^2 \leq p|c| \int \tilde{v}^{p+1} + p|d\right| \int \tilde{v}^p \\
\leq p(|c| + 1) \int \tilde{v}^{p+1}, \quad p \geq 1.
\]

Now invoke the Sobolev inequality
\[
\|h\|_{2n/(n-1)}^2 \leq C(\|\nabla h\|_2^2 + \|h\|_2^2), \quad \text{for all } h \in C^1(X).
\]
Here and below, we denote by \(C > 0\) a generic constant depending only on \(\eta\) and \(n\). Substituting \(h = \tilde{v}^{p+1}\) into the Sobolev inequality gives that
\[
\|\tilde{v}\|_{(p+1)\kappa} \leq [C(1 + |c|)(p + 1)]^{\frac{1}{p+1}} \|\tilde{v}\|_{p+1}, \quad \text{for all } p \geq 1.
\]
Here
\[
\kappa = \frac{n}{n-1}.
\]

Now we fix a real number \(p \geq 2\), and define a sequence \(\{p_i\}\) as follows:
\[
p_0 = p, \quad p_i = p_{i-1} \kappa = p\kappa^i, \quad \text{for all } i = 1, 2, \ldots
\]
Iterating (3.2) with respect to \(\{p_i\}\) yields that
\[
\|\tilde{v}\|_{p_{k+1}} \leq \exp \left( \log \left[ C(1 + |c|) \right] \sum_{i=0}^{k} \frac{1}{p_i} + \sum_{i=0}^{k} \frac{\log p_i}{p_i} \right) \|\tilde{v}\|_{p_0} \\
\leq [C(1 + |c|)]^{n/p} \|\tilde{v}\|_p, \quad \text{for any } k \geq 0,
\]
where we use the fact that
\[
\sum_{i=0}^{\infty} \frac{1}{\kappa^i} = n.
\]

Letting \(k\) tend to infinity gives that for \(p \geq 2\),
\[
(3.3) \quad \sup_X \tilde{v} \leq [C(1 + |c|)]^{n/p} \|\tilde{v}\|_p.
\]
For \(0 < p < 2\), it follows from above inequality that
\[
\sup_X \tilde{v} \leq [C(1 + |c|)]^{n/2} \left( \int \tilde{v}^2 \right)^{1/2} \\
\leq [C(1 + |c|)]^{n/2} \left( \int \tilde{v}^p \right)^{1/2} (\sup \tilde{v})^{(1-p)/2}.
\]
Then we still have (3.3). So for any real number \(p > 0\),
\[
\sup_X v \leq \sup_X \tilde{v} \leq C^{1/p}(1 + |c|)^{n/p}(\|v\|_p + |d|).
\]
\(\square\)
Proposition 7. Let \( v \in C^2(X) \), \( v > 0 \) on \( X \) and satisfy
\[
\Delta v - cv \leq 0 \quad \text{on } X,
\]
where \( c \) is a constant. Then, there exists a real number \( p_0 > 0 \), depending on \( \eta, n \), and \( c \), such that
\[
\inf_X v \geq C^{-1/p_0} (1 + |c|)^{-n/p_0} \|v\|_{p_0},
\]
where \( C > 0 \) depends only on \( \eta \) and \( n \).

Proof. By (3.4) we have
\[
\Delta(v^{-1}) + cv^{-1} \geq -v^{-2}(\Delta v - cv) \geq 0.
\]
Applying Proposition 6 to \( v^{-1} \) yields that, for any \( p > 0 \),
\[
\sup(v^{-1}) \leq C^{1/p}(1 + |c|)^{n/p} \|v^{-1}\|_p.
\]
It follows that
\[
\inf v \geq C^{-1/p}(1 + |c|)^{-n/p} \|v\|_{p} \left( \int v^{-p} \cdot \int v^p \right)^{-1/p}.
\]
Then, it suffices to show that, there exists some \( p_0 > 0 \) such that
\[
\int v^{-p_0} \cdot \int v^{p_0} \leq C.
\]
Here and below, we always denote by \( C > 0 \) a generic constant depends only on \( \eta \) and \( n \), unless otherwise indicated. Denote
\[
w = \log v - \frac{\int \log v}{\int \eta^n}.
\]
To show (3.5), it is sufficient to establish
\[
\int e^{p_0|w|} \leq C,
\]
for, (3.6) implies both
\[
\int e^{p_0w} \leq C, \quad \text{and} \quad \int e^{-p_0w} \leq C;
\]
and multiplying these two inequalities gives (3.5).

Note that
\[
e^{p_0|w|} = \sum_{m=0}^{\infty} \frac{p_0^m |w|^m}{m!}.
\]
Let us estimate \( \|w\|_m \) for each \( m \geq 1 \). Multiplying both sides of (3.4) by \( \phi v^{-1} \), where \( \phi \in C^1(X) \) and \( \phi \geq 0 \), and then integrating by parts yield that
\[
\int \phi |\nabla w|^2 \leq c \int \phi + \int \nabla \phi \cdot \nabla w.
\]
We first set \( \phi \equiv 1 \) in (3.7) to obtain that
\[
\int |\nabla w|^2 \leq |c| \int \eta^n.
\]
We can apply Poincaré inequality to get
\begin{equation}
\int w^2 \leq C|c|.
\end{equation}

Then, by Hölder inequality,
\begin{equation}
\int |w| \leq C|c|^{1/2}.
\end{equation}

It remains to estimate \(\|w\|_m\) for \(m \geq 3\). We now set \(\phi = w^{2p}\), \(p \geq 1\), in (3.7). Then,
\[
\int |w|^{2p} |\nabla w|^2 \leq |c| \int |w|^{2p} + 2p \int |w|^{2p-1} |\nabla w|^2.
\]
By Young’s inequality,
\[
2p|w|^{2p-1} \leq \frac{2p-1}{2p} |w|^{2p} + (2p)^{2p-1}.
\]
It follows that
\[
\int |w|^{2p} |\nabla w|^2 \leq 2p|c| \int |w|^{2p} + (2p)^{2p} \int |\nabla w|^2.
\]
Observe that
\[
|\nabla w|^p = p^2 w^{2p-2} |\nabla w|^2 \leq w^{2p} |\nabla w|^2 + p^2 |\nabla w|^2.
\]
We then have
\[
\int |\nabla w|^p \leq 2p|c| \int |w|^{2p} + 2(2p)^{2p} \int |\nabla w|^2, \quad \text{for all } p \geq 1.
\]
Apply the Sobolev inequality to obtain that
\[
\left( \int w^{2p\kappa} \right)^\kappa \leq C p(1 + |c|) \int |w|^{2p} + C(2p)^{2p} \int |\nabla w|^2 \leq C p(1 + |c|) \left( \int |w|^{2p} + (2p)^{2p} \right), \quad \text{by (3.8)}.
\]
Here we denote
\[
\kappa = \frac{n}{n-1}.
\]
Hence, we have for all \(p \geq 1\) that
\begin{equation}
\|w\|_{2p\kappa} \leq \left[ C(1 + |c|) \right]^\frac{1}{2p} (2p)^{\frac{1}{2p}} \left( \|w\|_{2p} + 2p \right),
\end{equation}
in view of the inequality
\[(a + b)^\epsilon \leq a^\epsilon + b^\epsilon, \quad \text{for all } 0 < \epsilon < 1, \ a \geq 0, \ \text{and} \ b \geq 0.
\]
We shall iterate (3.11) with respect to the sequence \(\{p_i\}_{i=0}^\infty\) given below:
\[
p_0 = 2, \quad p_i = p_{i-1}\kappa = 2\kappa^i \quad \text{for all } i \geq 1.
\]
Thus, we obtain for each \( k \geq 0 \) that
\[
\|w\|_{p_{k+1}} \leq C \sum_{i=0}^{k} p_i + \exp \left( \log |C(1 + |c|)] \sum_{i=0}^{k} \frac{1}{p_i} + \frac{\log p_i}{p_i} \right) \|w\|_{p_0}
\]
\[
\leq C p_k + C(1 + |c|)^{n/2}\|w\|_2,
\]
in which we use the fact that
\[
\sum_{i=0}^{k} \kappa^i \leq n \kappa^k.
\]
Now note that for any integer \( m \geq 2 \), there exists an integer \( i \geq 0 \) such that
\[
2\kappa^i \leq m < 2\kappa^{i+1}.
\]
Then,
\[
\|w\|_m \leq \|w\|_{p_{i+1}} \leq \|w\|_{p_{i+1}} \leq C m + C(1 + |c|)^{n/2}\|w\|_2
\]
\[
\leq C m + C(1 + |c|)^{n/2} \quad \text{by (3.3)}
\]
\[
\leq C(1 + |c|)^{n/2} m.
\]
Hence,
\[
\int \frac{|w|^m}{m!} \leq C^m (1 + |c|)^{\frac{nm}{m!}} \leq C^m (1 + |c|)^{\frac{nm}{m!}} e^m.
\]
Let
\[
p_0 = \frac{1}{2C(1 + |c|)^{n/2}e};
\]
and then
\[
\int \frac{p_0^m |w|^m}{m!} \leq \frac{1}{2^m}, \quad \text{for all } m \geq 2.
\]
This together with (3.10) yields (3.6). This completes the proof. \( \square \)

We are in a position to prove Lemma 5.

**Proof of Lemma 5** Let
\[
v = u - \inf_X u + 1.
\]
Then,
\[
v \geq 1, \quad \text{and} \quad \inf_X v = 1,
\]
since \( X \) is compact and so \( u \) attains its minimum. On the other hand, we have
\[
\Delta v - C_1 v \leq 0, \quad \text{by (3.12)}
\]
and
\[
\Delta v > -C_2, \quad \text{by (3.13)}
\]
Applying Proposition 7 to (3.12) obtains that
\[
\inf_X v \geq C^{-1/p_0} (1 + |C_1|)^{-n/p_0} \|v\|_{p_0}.
\]
Here $p_0 > 0$ is a number depending only on $\eta$, $n$, and $C_1$; and $C > 0$ is a constant depending only on $\eta$ and $n$. Applying Proposition 6 to (3.13) with $p = p_0$ yields that

$$\sup_X v \leq (C')^{1/p_0} (\|v\|_{p_0} + C_2),$$

where $C' > 0$ depends only on $\eta$ and $n$. Combining these two inequalities we have

$$\sup_X v \leq (C')^{1/p_0} \left( C_{1/p_0} (1 + |C_1|)^{n/p_0} \inf_X v + C_2 \right) \leq (C')^{1/p_0} \left( C_{1/p_0} (1 + |C_1|)^{n/p_0} + C_2 \right).$$

It follows that

$$\sup_X u - \inf_X u \leq \sup_X v \leq C,$$

where $C > 0$ depends only on $\eta$, $n$, $C_1$, and $C_2$.

Let us now return to equation (1.4). We let $(X, \eta)$ be the complex $n$-dimensional Kähler manifold of nonnegative quadratic bisectional curvature, and $\omega_0$ be a Hermitian metric on $X$.

**Corollary 8.** Given any $f \in C^\infty(X)$, let $u \in C^\infty(X)$ be a solution of

$$\det(\omega_u^{n-1}) = e^{(n-1)f} \left( \frac{\int_X \omega_u^n}{\int_X e^f \omega_0^n} \right)^{n-1},$$

where $\omega_u$ is a positive $(1,1)$-form such that

$$\omega_u^{n-1} = \omega_0^{n-1} + (\sqrt{-1}/2) \partial \bar{\partial} u \wedge \eta^{n-2} > 0 \text{ on } X.$$

Then,

$$\sup_X |\omega_u^{n-1}|_\eta \leq C,$$

where $C > 0$ is a constant depending only on $f$, $\eta$, $n$, and $\omega_0$.

**Proof.** By Corollary 4 it suffices to estimate $(\sup u - \inf u)$. Contracting

$$\omega_0^{n-1} + (\sqrt{-1}/2) \partial \bar{\partial} u \wedge \eta^{n-2} > 0$$

with $\eta$ yields that

$$\Delta_\eta u > \frac{n\eta \wedge \omega_0^{n-1}}{\eta^n} > -C_2 \text{ on } X.$$

Here the constant $C_2 > 0$ depends only on $\eta$, $n$, and $\omega_0$. We have (2221), on the other hand. Therefore, the result is an immediate consequence of Lemma 5. □
4. Hölder estimates for second derivatives

Let $X$ be a $n$-dimensional Kähler manifold, $\eta$ be a Kähler metric on $X$, and $\omega_0$ be a balanced metric on $X$. We will establish the following estimate.

**Lemma 9.** Given $F \in C^2(X)$, let $u \in C^4(X)$ satisfy that

$$\omega_0^{n-1} + (\sqrt{-1}/2)\partial\bar{\partial}u \wedge \eta^{n-2} > 0$$

on $X$, and that

$$\det[\omega_0^{n-1} + (\sqrt{-1}/2)\partial\bar{\partial}u \wedge \eta^{n-2}] = e^F \det \omega_0^{n-1}. \tag{4.1}$$

Suppose that

$$\sup_X |\omega_0^{n-1} + \sqrt{-1}/2\partial\bar{\partial}u \wedge \eta^{n-2}| \eta \leq C_3 \tag{4.2}$$

for some constant $C_3 > 0$. Then,

$$\|u\|_{C^2,\alpha} \leq C,$$

where $0 < \alpha < 1$ and $C > 0$ are constants depending only on $C_3$, $n$, $\omega_0$, and $\eta$.

We shall apply the Evans–Krylov theory (see, for example, Gilbarg–Trudinger [2, p. 461, Theorem 17.14]), which is on the real fully nonlinear elliptic equation. Note that Evans–Krylov theory is based on the weak Harnack estimate (see, for example, [2, p. 246, Theorem 9.22]), which, in turn, makes uses of the Aleksandrov’s maximum principle (see, for example, [2, p. 222, Lemma 9.3]).

We first adapt the Aleksandrov’s maximum principle to the complex setting. To see this, we start from the following result (see, for example, Lemma 9.2 in [2]): Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with smooth boundary.

**Lemma (Aleksandrov).** For $v \in C^2(\overline{\Omega})$ with $v \leq 0$ on $\partial\Omega$, we have

$$\sup_{\Omega} v \leq \frac{\text{diam}(\Omega)}{\sigma_{2n}} \left( \int_{\Gamma_v^+} |\det D^2v| \right)^{\frac{1}{2n}}. \tag{4.3}$$

Here $\sigma_{2n}$ is the volume of unit ball in $\mathbb{C}^n$, $D^2v$ denotes the real Hessian matrix of $v$, and $\Gamma_v^+$ is the upper contact set of $v$, i.e.,

$$\Gamma_v^+ = \{ y \in \Omega; v(x) \leq v(y) + Dv(y) \cdot (x - y) \text{ for all } x \in \Omega \}. $$

Then, it suffices to control the real Hessian $D^2v$ by the complex Hessian $(w_{ij})$ of $v$, over $\Gamma_v^+$. Note that $\Gamma_v^+ \subset \{ y \in \Omega; (D^2v)(y) \leq 0 \}$. We shall make use of the following inequality:

**Proposition 10.** Let $w$ be a real $C^2$ function in $\Omega$. For $P \in \Omega$ such that $D^2w \geq 0$,

$$\det(D^2w) \leq 8^n |\det w_{ij}|^2$$

at $P$.

**Proof.** Recall that

$$\frac{\partial}{\partial z^i} = \frac{1}{2} \left( \frac{\partial}{\partial x^i} - \sqrt{-1} \frac{\partial}{\partial y^i} \right), \quad 1 \leq i \leq n.$$
We denote
\[ w_{x^i} = \frac{\partial w}{\partial x^i}, \quad w_{x^i y^j} = \frac{\partial^2 w}{\partial x^i \partial y^j}, \quad \ldots \]
Then,
\[ w_{ij} = \frac{1}{4} (w_{x^i x^j} + w_{y^i y^j}) + \sqrt{\frac{-1}{4}} (w_{x^i y^j} - w_{x^j y^i}), \quad 1 \leq i, j \leq n. \]
Since \( D^2w \geq 0 \) at \( P \), we can choose a coordinate system \((x^1, y^1, \ldots, x^n, y^n)\) near \( P \) such that \( D^2w \) is diagonalized at \( P \); and hence,
\[ w_{x^i x^i} \geq 0, \quad w_{y^i y^i} \geq 0, \quad \text{for all } 1 \leq i \leq n. \]
Then, under this coordinate system, the complex Hessian of \( w \) is also diagonalized, i.e.,
\[ w_{ij} = \delta_{ij} \frac{1}{4} (w_{x^i x^i} + w_{y^i y^i}). \]
It follows that, at \( P \),
\[ 16^n |\text{det} w_{ij}|^2 = \prod_{i=1}^{n} (w_{x^i x^i} + w_{y^i y^i})^2 \geq 2^n \prod_{i=1}^{n} w_{x^i x^i} \prod_{i=1}^{n} w_{y^i y^i} = 2^n \text{det}(D^2w). \]

Moreover, for any Hermitian matrix \((a^{ij}) > 0\) on \( \Gamma_v^+ \), we have by the elementary inequality that
\[ \det(a^{ij}) \det(-v_{ij}) \leq \left( \frac{-\sum_{i,j} a^{ij} v_{ij}}{n} \right)^n. \]
Now apply Proposition [10] and (4.4) to (4.3) to obtain the following complex version Aleksandrov’s maximum principle (compare with [2, p. 222, Lemma 9.3]):

**Lemma 11.** Let \((a^{ij})\) be a positive definite Hermitian matrix in \( \Omega \). For \( v \in C^2(\overline{\Omega}) \) with \( v \leq 0 \) on \( \partial \Omega \),
\[ \sup \Omega v \leq \frac{2^n \text{diam}(\Omega)}{n \sigma_2^{1/(2n)}} \left[ \int_{\Gamma_v^+} \left| \frac{-\sum a^{ij} v_{ij}}{\text{det}(a^{ij})} \right|^{2n} \right]^\frac{1}{2n}. \]
Then, the weak Harnack inequality below (compare with [2, p. 246, Theorem 9.22]) follows from Lemma [11] and the cube decomposition procedure.

**Theorem (Krylov–Safonov).** Let \( v \in W^{2, 2n}(\Omega) \) satisfy \( \sum a^{ij} v_{ij} \leq g \) in \( \Omega \), where \( g \in L^{2n}(\Omega) \), and \((a^{ij})\) satisfies that
\[ 0 < \lambda |\zeta|^2 \leq \sum_{i,j} a^{ij}(z) \zeta_i \zeta_j \leq \Lambda |\zeta|^2, \quad \text{for all } z \in \Omega \text{ and } \zeta \in \mathbb{C}^n, \]
in which \( \lambda \) and \( \Lambda \) are two constants. Suppose that \( v \geq 0 \) in an open ball \( B_{2R}(y) \subset \Omega \) centered at \( y \) of radius \( 2R \). Then,
\[
\left( \frac{1}{|B_R|} \int_{B_R} v^p \right)^{1/p} \leq C \left[ \inf_{B_R} v + \frac{R}{\lambda} \|g\|_{L^2(B_{2R})} \right],
\]
where \( |B_R| \) denotes the measure of \( B_R \), and \( p > 0 \) and \( C > 0 \) are constants depending only on \( n \), \( \lambda \), and \( \Lambda \).

Let us denote by
\[
E[(u_{ij})] = \log \det \left[ \omega_0^{n-1} + (\sqrt{-1}/2) \bar{\partial} \partial u \wedge \eta^{n-2} \right].
\]

To apply Evans–Krylov theory, it remains to check the following two conditions ([2, p. 456]):

1. \( E \) is uniformly elliptic with respect to \((u_{ij})\),
2. \( E \) is concave on the range of \((u_{ij})\).

As in Section 2, we denote \( \Psi = \omega_0^{n-1} \) and
\[
(4.5) \quad \Psi_u = \Psi + (\sqrt{-1}/2) \bar{\partial} \partial u \wedge \eta^{n-2}.
\]

We use the index convention (2) for an \((n-1, n-1)\)–form. Then,
\[
E[(u_{ij})] = \log \det[(\Psi_u)_{ij}];
\]
and thus,
\[
\frac{\partial E}{\partial (\Psi_u)_{ij}} = (\Psi_u)^{ij}, \quad \frac{\partial^2 E}{\partial (\Psi_u)_{ij} \partial (\Psi_u)_{kl}} = - (\Psi_u)^{ij} (\Psi_u)^{kl}.
\]

Clearly, \( E \) is concave on \([ (\Psi_u)_{ij} ] \). By (4.1) and (4.2), we know that the eigenvalues of \([ (\Psi_u)_{ij} ] \) with respect to \((\eta_{ij})\), have uniform bounds which depend only on \( F, \omega_0 \), and \( C_3 \). Therefore, \( E \) is uniformly elliptic with respect to \([ (\Psi_u)_{ij} ] \). Observe that by (4.5), \([ (\Psi_u)_{ij} ] \) depends linearly on \((u_{pq})\). Since \((\eta_{ki}) > 0 \) on \( X \), the conditions (1) and (2) follows immediately from the chain rule.

Now we can apply the procedure in [2, p. 457–461], and this proves Lemma 9. As a corollary, we obtain the Hölder estimate of \( C^2 \) for equation (1.4).

**Corollary 12.** Let \((X, \eta)\) an \( n \)-dimensional Kähler of nonnegative quadratic bisectional curvature, and \( \omega_0 \) be a Hermitian metric on \( X \). Given any \( f \in C^\infty(X) \), let \( u \in C^\infty(X) \) be a solution of
\[
\frac{\det(\omega_u^{n-1})}{\det(\omega_0^{n-1})} = e^{(n-1)f} \left( \frac{\int_X \omega_u^n}{\int_X e^f \omega_0^n} \right)^{n-1},
\]
where \( \omega_u \) is a positive \((1, 1)\)–form such that
\[
\omega_u^{n-1} = \omega_0^{n-1} + (\sqrt{-1}/2) \bar{\partial} \partial u \wedge \eta^{n-2} > 0 \quad \text{on} \; X.
\]

Then,
\[
\|u\|_{C^{2, \alpha}(X)} \leq C,
\]
where \( 0 < \alpha < 1 \) and \( C > 0 \) are constants depending only on \( f, \eta, n, \) and \( \omega_0 \).
5. Openness and uniqueness

Throughout this section, we let $\omega_0$ be a balanced metric, and let $\eta$ be an arbitrary Kähler metric, unless otherwise indicated. We fix $k \geq n + 4$, $0 < \alpha < 1$, and a function $f \in C^{k,\alpha}(X)$ satisfying

$$\int_X e^{tf} \omega_0^n = V \equiv \int_X \omega_0^n.$$ 

Here $C^{k,\alpha}(X)$ is the usual Hölder space on $X$. Consider for $0 \leq t \leq 1$,

$$\det(\omega_{u_t}^{n-1}) \det(\omega_0^{n-1}) = e^{(n-1)tf} \left( \frac{\int_X \omega_{u_t}^n}{\int_X e^{tf} \omega_0^n} \right)^{n-1},$$

(5.1)

where $u_t \in P_\eta(\omega_0)$. By abuse of notation, in this section we denote $P_\eta(\omega_0) = \{ v \in C^{k+2,\alpha}(X); \omega_{u_t}^{n-1} + (\sqrt{-1}/2) \partial \bar{\partial} v \land \eta^{n-2} > 0 \}.$

Let

$$T = \{ t \in [0, 1]; \text{the equation (5.1) has a solution } u_t \in C^{k+2,\alpha}(X) \text{ such that } u_t \in P_\eta(\omega_0). \}. \tag{5.2}$$

Clearly, we have $0 \in T$.

**Lemma 13.** Let $T$ be the set given as above. Then $T$ is open in $[0,1]$.

**Proof.** Notice that (5.1) is the same as

$$\frac{\omega_{u_t}^n}{\omega_0^n} = e^{tf} \frac{\int_X \omega_{u_t}^n}{\int_X e^{tf} \omega_0^n}.$$ 

As in Section 3 of [1], we define

$$M(w) \equiv \log \frac{\omega_0^n}{\omega_w^n} - \log \left( \frac{1}{V} \int_X \omega_w^n \right),$$

for any $w \in P_\eta(\omega_0)$. Then, $M(w) \in F^{k,\alpha}(X)$, where $F^{k,\alpha}(X)$ is the hypersurface in $C^{k,\alpha}(X)$ given by

$$F^{k,\alpha}(X) = \left\{ g \in C^{k,\alpha}(X); \int_X e^g \omega_0^n = V \right\}.$$ 

Now suppose that $t \in T$. Then, the corresponding $u_t$ defines a positive $(1,1)$–form $\omega_{u_t}$ such that

$$\omega_{u_t}^{n-1} = \omega_0^{n-1} + (\sqrt{-1}/2) \partial \bar{\partial} u \land \eta^{n-2} > 0 \quad \text{on } X;$$

furthermore, $u_t$ satisfies that

$$M(u_t) = tf + \log V - \log \left( \int_X e^{tf} \omega_0^n \right) \in F^{k,\alpha}(X).$$

The tangent space of $F^{k,\alpha}(X)$ at $M(u_t)$ is identically the same as the Banach space $E^{k,\alpha}(X)$, which consists of all $h \in C^{k,\alpha}(X)$ such that

$$\int_X h \omega_{u_t}^n = 0.$$
In view of the Implicit Function Theorem, it suffices to show that the linearization operator \( L_t = M_{\nu t} \), given by

\[
L_t(v) = \frac{n(\sqrt{-1}/2)\partial\bar{\partial} v \wedge \eta^{n-2} \wedge \omega_{ut}}{(n-1)\omega_{ut}^n} - \frac{n\int_X (\sqrt{-1}/2)\partial\bar{\partial} v \wedge \eta^{n-2} \wedge \omega_{ut}}{(n-1)\int_X \omega_{ut}^n},
\]

is a linear isomorphism from \( \mathcal{E}_t^{k+2,\alpha}(X) \) to \( \mathcal{E}_t^{k,\alpha}(X) \). This is guaranteed by Lemma 13 in [1]. The proof is thus finished.

\[\square\]

**Remark 14.** We thank John Loftin for pointing out that the openness argument in [1] also works for \( \eta \) being a astheno-Kähler metric, i.e., \( \eta \) is a hermitian metric such that \( \partial\bar{\partial}\eta^{n-2} = 0 \).

By the results in the previous section, we know that \( T \) is also closed, provided that the orthogonal bisectional curvature of \( \eta \) is nonnegative. Therefore, the existence part in Theorem [1] is proved. The uniqueness follows immediately from the following proposition.

**Proposition 15.** Let \( v \in \mathcal{P}_\eta(\omega_0) \) satisfying

\[
(5.3) \quad \det \left[ \omega_0^{n-1} + (\sqrt{-1}/2)\partial\bar{\partial} v \wedge \eta^{n-2} \right] = \delta \det \omega_0^{n-1},
\]

where \( \delta > 0 \) is a constant. Then, \( v \) must be a constant function and \( \delta = 1 \).

**Proof.** Applying the maximum principle to equation (5.3) at the maximum points of \( v \) yields that \( \delta \leq 1 \). Similarly, we get \( \delta \geq 1 \) by considering (5.3) at the minimum points of \( v \). Thus, \( \delta = 1 \). Then, we apply the arithmetic–geometric mean inequality to obtain

\[
1 = \left[ \frac{\det \omega_0^{n-1}}{\det \omega_0^{n-1}} \right]^{1/n} \leq 1 + \frac{1}{n} \sum_{i,j=1}^{n} (\omega_0^{n-1})^{-i} j \left( (\sqrt{-1}/2)\partial\bar{\partial} v \wedge \eta^{n-2} \right)_{ij} \leq 1 + \frac{\omega_0 \wedge \eta^{n-2} \wedge (\sqrt{-1}/2)\partial\bar{\partial} v}{\omega_0^n} \equiv 1 + K v.
\]

Note that the linear operator \( K \) so defined is uniformly elliptic, by the metric equivalence of \( \eta \) and \( \omega_0 \) on the compact manifold \( X \). Applying the strong maximum principle to \( K v \geq 0 \) yields that \( v \) is a constant function. \[\square\]

Therefore, the proof of Theorem [1] is completed.

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Institute of Mathematics, Fudan University, Shanghai 200433, China
E-mail address: majxfu@fudan.edu.cn

Institute of Mathematics, Fudan University, Shanghai 200433, China
E-mail address: youxiang163wang@163.com

Department of Mathematics, The Ohio State University, 1179 University Drive, Newark, OH 43055, U.S.A.
E-mail address: dwu@math.ohio-state.edu