The Minimal Dimension of a Sphere with an Equivariant Embedding of the Bouquet of $g$ Circles is $2g - 1$

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Abstract
To embed the bouquet of $g$ circles $B_g$ into the $n$-sphere $S^n$ so that its full symmetry group action extends to an orthogonal actions on $S^n$, the minimal $n$ is $2g - 1$. This answers a question raised by Zimmermann.

Keywords Groups acting on finite graphs · Equivariant embeddings into spheres

Mathematics Subject Classification 57M25 · 57S17 · 57S25 · 05C10

In this note graphs are finite and connected, and group actions are faithful. Denote the $n$-dimensional Euclidean space and orthogonal group by $\mathbb{R}^n$ and $O(n)$, and the unit sphere in $\mathbb{R}^{n+1}$ by $S^n$. For a given graph $\Gamma$ with an action of a finite group $G$, an embedding $e: \Gamma \to S^n$ is $G$-equivariant, if there is a subgroup $G \subset O(n+1)$ such that $G$ acts on the pair $(S^n, e(\Gamma))$ and satisfies $g(e(x)) = e(g(x))$ for any $x \in \Gamma$ and $g \in G$. In the definition of $G$-equivariant embedding, one may replace $(S^n, O(n+1))$ by $(\mathbb{R}^n, O(n))$. See Remark 5 for relations between the two definitions.

The existence of $G$-equivariant embedding for any pair $(G, \Gamma)$ is known, see [3,4]. On the other hand, for given $\Gamma$ and $G$, usually it is difficult to find the minimal $n$ such that there is a $G$-equivariant embedding $\Gamma \to \mathbb{R}^n$.

The most significant finite group associated to a graph $\Gamma$ is its full symmetry group $\text{Sym} \Gamma$. The simplest and also most symmetric graph is the bouquet of $g$ circles (the graph with one vertex and $g$ closed edges, see Fig. 1), denoted as $B_g$. $B_g$ is the simplest in the sense that $B_g$ has only one vertex. $B_g$ is the most symmetric in the sense that: for $g \geq 3$, the maximum order of finite group action on a hyperbolic graph of rank $g$ (defined as its fundamental group rank) is $2^g g!$, which is realized uniquely by the
action of Sym $B_g$ on $B_g$ [7]. A graph $\Gamma$ is hyperbolic if it has rank $> 1$ and it has no free edge. See Example 6 for the rank 2 case.

For simplicity, (Sym $\Gamma$)-equivariant embeddings of $\Gamma$ will be called just equivariant embeddings of $\Gamma$. Recently, Zimmermann [8] asked a very interesting question (as commented by the MR review of the paper).

**Question 1** What is the minimal value of $n$ such that there is an equivariant embedding $B_g \to S^n$?

Zimmermann pointed out that $n \geq g - 1$, which follows from a group theoretical argument. He also described an equivariant embedding of $B_g$ into $S^{2g}$. So, he concluded, the minimal $n$ is between $g - 1$ and $2g$ [8]. We answer Zimmerman’s question with

**Theorem 2** The minimal $n$ such that there is an equivariant embedding $B_g \to S^n$ is $2g - 1$.

Theorem 2 follows from the next two propositions.

**Proposition 3** Suppose there is an equivariant embedding of $B_g$ into $S^n$, then $n \geq 2g - 1$.

**Proposition 4** There are equivariant embeddings of $B_g$ into $S^{2g-1}$ for each $g > 1$.

**Proof of Proposition 3** First, we recall carefully a description of the action of Sym $B_g$ on $B_g$. Denote the vertex of $B_g$ by $v$ and the $g$ circles by $C_1, \ldots, C_g$, see Fig. 2. From
now on we consider each circle $C_i$ as oriented. It is known that $\text{Sym} \ B_g$ is the semidirect product $(\mathbb{Z}_2)^g \rtimes S_g$, where $S_g$ is the permutation group of $g$ elements, which permutes those $g$ oriented circles, and the normal subgroup $(\mathbb{Z}_2)^g$ acts by orientation reversing involutions of those $g$ oriented circles.

More precisely, for each $\sigma \in S_g$, denote the corresponding action on $B_g$ by $\tau_\sigma$, which sends $C_i$ to $C_{\sigma(i)}$ and preserves the orientations. Denote the element $\rho_i$ to be the element in $(\mathbb{Z}_2)^g \subset \text{Sym} \ B_g$, which is the orientation reversing involution on $C_i$ and is the identity on the remaining $C_j$, for $j \neq i$. Now $\rho_i$ has two fixed points, one is $v$, another we denote as $p_i$. It is clear that

$$\tau_\sigma \rho_i \tau_\sigma^{-1} = \rho_{\sigma(i)}. \quad (1)$$

By (1) we have

$$\rho_{\sigma(i)} \tau_\sigma (p_i) = \tau_\sigma \rho_i \tau_\sigma^{-1} \tau_\sigma (p_i) = \tau_\sigma \rho_i (p_i) = \tau_\sigma (p_i).$$

So, we have $\rho_{\sigma(i)} \tau_\sigma (p_i) = \tau_\sigma (p_i)$. That is, $\tau_\sigma(p_i)$ is the fixed point of $\rho_{\sigma(i)}$ on $C_{\sigma(i)} - \{v\}$. Since the fixed point of $\rho_{\sigma(i)}$ on $C_{\sigma(i)} - v$ is unique, we have

$$\tau_\sigma(p_i) = p_{\sigma(i)}. \quad (2)$$

Suppose now we have an equivariant embedding $B_g \subset S^n$. That is, we have embedded $\text{Sym} \ B_g$ into $O(n+1)$ which acts on the pair $(S^n, B_g)$ and the restriction on $B_g$ is the action of $\text{Sym} \ B_g$ described as above. Since $S^n$ is the unit sphere of $\mathbb{R}^{n+1}$ and $O(n+1)$ is the orthogonal transformation group of $\mathbb{R}^{n+1}$, the action of $\text{Sym} \ B_g$ on $S^n$ extends to a $(\text{Sym} \ B_g)$-action on $\mathbb{R}^{n+1}$ as an orthogonal transformation group.

Consider each point in $\mathbb{R}^{n+1}$ as a vector in $\mathbb{R}^{n+1}$. Since $v, p_1, p_2, \ldots, p_g$ are in the unit sphere, each of them is a non-zero vector. Let

$$V = \langle v, p_1, p_2, \ldots, p_g \rangle$$

be the subspace of $\mathbb{R}^{n+1}$ spanned by the vectors $v, p_1, p_2, \ldots, p_g$. By (2), the $S_g$-action on $B_g$ permutes those vectors $p_1, p_2, \ldots, p_g$ and fixes $v$. Hence $S_g$ acts faithfully on $V$, that is, $\tau_\sigma|_V = \text{id}|_V$ if and only if $\sigma$ is the identity on $S_g$.

Now we show that the vectors $v, p_1, \ldots, p_{g-1}$ are linearly independent: Suppose $a, a_1, \ldots, a_{g-1}$ are real numbers such that

$$av + \sum_{i=1}^{g-1} a_i p_i = 0. \quad (3)$$

For each $j \in \{1, 2, \ldots, g-1\}$, recall that $(j, g) \in S_g$ exchanges $j$ and $g$, fixes all the remaining $i$, and $\tau_{(j,g)}$ corresponds to the element $(j, g) \in S_g$. Since $\tau_{(j,g)}$ is an orthogonal transformation, $\tau_{(j,g)}$ is linear. Applying $\tau_{(j,g)}$ to (3), we have
\[ 0 = \tau(j, g) \left( a v + \sum_{i=1}^{g-1} a_i p_i \right) = a \tau(j, g)(v) + \sum_{i=1}^{g-1} a_i \tau(j, g)(p_i). \]

Since \( v \) is the common fixed point of \( S_g \), and \( \tau(j, g)(p_i) = p_i \) if \( j \neq i \) and \( \tau(j, g)(p_j) = p_g \), we have further

\[ av + \sum_{i=1 \atop i \neq j}^{g-1} a_i p_i + ajp_g = 0. \] (4)

On the other hand, by (3) and some elementary transformations,

\[ av + \sum_{i=1 \atop i \neq j}^{g-1} a_i p_i + ajp_g = av + \sum_{i=1}^{g-1} a_i p_i - ajp_j + ajp_g = aj(p_g - p_j). \] (5)

Combining (4) and (5) we get \( aj(p_g - p_j) = 0 \). Since \( B_g \subset \mathbb{R}^{n+1} \) is an embedding, and \( j \neq g \),

\[ p_g - p_j \neq 0. \]

Hence \( a_j = 0 \). Since \( j \) can be any element in \( \{1, 2, \ldots, g-1\} \), \( a_j = 0 \) for each \( j \in \{1, 2, \ldots, g-1\} \). By (3), \( av = 0 \). Since \( v \in S^n, v \neq 0 \), we have \( a = 0 \). That is, (3) implies that

\[ a = a_1 = a_2 = \ldots = a_{g-1} = 0, \]

so \( v, p_1, \ldots, p_{g-1} \) are linearly independent. As a conclusion we have

\[ \dim V \geq g. \] (6)

Let \( V^\perp \) be the orthogonal complement of \( V \) in \( \mathbb{R}^{n+1} \). We have

\[ \mathbb{R}^{n+1} = V \oplus V^\perp. \] (7)

For any \( i \in \{1, 2, \ldots, g\} \), \( \rho_i \) is orthogonal, so is linear. Moreover, each point in \( \{v, p_1, p_2, \ldots, p_g\} \) is a fixed point of \( \rho_i \). Therefore \( \rho_i \) is the identity on \( V \), the subspace spanned by \( \{v, p_1, p_2, \ldots, p_g\} \). Furthermore, we conclude that the action of the subgroup \( (\mathbb{Z}_2)^g \) on \( V \) is trivial. In particular, \( V \) is an invariant subspace of the \( (\mathbb{Z}_2)^g \)-action.

Since the \( (\mathbb{Z}_2)^g \)-action on \( \mathbb{R}^{n+1} \) is orthogonal and \( V \) is invariant under the \( (\mathbb{Z}_2)^g \)-action, \( (\mathbb{Z}_2)^g \) acts orthogonally on the orthogonal complement \( V^\perp \) of \( V \). Since the \( (\mathbb{Z}_2)^g \)-action is trivial on \( V \) and is faithful on \( \mathbb{R}^{n+1} \), the \( (\mathbb{Z}_2)^g \)-action must be faithful on \( V^\perp \) by (7).
Let \( q \) be the dimension of \( V^\perp \), and consider \( O(q) \) as the \( q \) by \( q \) orthogonal matrix group. Then the \( (\mathbb{Z}_2)^g \)-action on \( V^\perp \) is by a matrix group in \( O(q) \). Let \( \Omega_q \) be the set of all \( q \) by \( q \) diagonal matrices whose entries on the diagonal are either 1 or \(-1\). Then there are exactly \( 2^q \) matrices in \( \Omega_q \).

Each element in \( (\mathbb{Z}_2)^g \) is of order 1 or 2, therefore it can be diagonalized, in fact, so that it is in \( \Omega_q \). By linear algebra, a finite number of commuting diagonalizable matrices can be diagonalized simultaneously. Since \( (\mathbb{Z}_2)^g \) is a finite abelian group, we may assume that under a basis of \( V^\perp \), all matrices in \( (\mathbb{Z}_2)^g \) are diagonalized, therefore are elements of \( \Omega_g \). Since \( (\mathbb{Z}_2)^g \) has \( 2^g \) elements, \( (\mathbb{Z}_2)^g \) acts faithfully on \( V^\perp \), the image of \( (\mathbb{Z}_2)^g \) in \( \Omega_g \) must be also \( 2^g \) pairwise different elements. Therefore \( 2^g \leq 2^q \), that is,

\[
g \leq q = \dim V^\perp. \tag{8}
\]

By (6), (7), and (8) we have

\[
n + 1 = \dim V + \dim V^\perp \geq 2g.
\]

That is, \( n \geq 2g - 1 \). We finished the proof.

**Proof of Proposition 4** We will construct two different equivariant embeddings. The first one is an equivariant embedding \( B_g \rightarrow \mathbb{R}^{2^g-1} \); here the required \( G \)-equivariant embedding \( B_g \rightarrow S^{2^g-1} \) can be obtained by one point compactification via the inverse of stereographic projection \( p: S^{2^g-1} \rightarrow \mathbb{R}^{2^g-1} \). The second one is an equivariant embedding of \( B_g \rightarrow \mathbb{R}^{2^g} \), where \( B_g \) stays equivariantly in the unit sphere \( S^{2^g-1} \).

**The first construction** Before we give a general construction, we would like to provide a visible equivariant embedding of \( B_2 \subset \mathbb{R}^3 \) in the familiar dimension in which we live. Fix a standard \( xyz \)-coordinate system of \( \mathbb{R}^3 \). Let \( C_1 \) be the unit circle in the lower-half \( zx \)-plane that is tangent to the \( x \)-axis at the origin, and \( C_2 \) be the
unit circle in the upper-half $yz$-plane that is tangent to the $y$-axis at the origin, as shown in Fig. 3. Then $C_1 \cup C_2$ provides an embedding $B_2 \subset \mathbb{R}^3$. The action of $\text{Sym} \ B_2 = (\mathbb{Z}_2)^2 \ltimes S_2$ is also realized by a subgroup of $O(3)$: where the inversion of $C_1$ is given by the reflection about the $yz$-plane, the inversion of $C_2$ is given by the reflection about the $zx$-plane, and the permutation of $C_1$ and $C_2$ is given by the $\pi$-rotation around a diagonal $L$ in the $xy$-plane. This provides an equivariant embedding.

Now we provide the equivariant embedding of $B_g \subset \mathbb{R}^{2g-1}$ for $g > 2$. Let $V_1 = \mathbb{R}^{g-1}$ and $v_1, v_2, \ldots, v_g$ be the vertices of a regular $(g - 1)$-dimensional simplex $\Delta_{g-1}$ centered at the origin $O_1$ of $V_1$ with the length $|O_1v_i| = 1$. Then $S_g$, the full symmetry group of $\Delta_{g-1}$, is a subgroup of $O(g - 1)$. Let $V_2 = \mathbb{R}^g$ and $e_1, e_2, \ldots, e_g$ be the standard orthogonal basis of $V_2$. Let $V = V_1 \times V_2$, then $V = \mathbb{R}^{2g-1}$ and $(v_i, e_i)$ is a standard orthogonal basis of the Euclidean plane $E_i$ spanned by $v_i$ and $e_i$, $i \in \{1, \ldots, g\}$.

Define the isometrical embedding $i_i : S^1 \rightarrow V$ by

$$i_i(\cos \theta, \sin \theta) = (1 + \cos \theta) \cdot v_i + \sin \theta \cdot e_i, \quad (9)$$

where $0 \leq \theta < 2\pi$ is the parameter on the unit circle. Let $i_i(S^1) = C_i, i \in \{1, \ldots, g\}$. Clearly, the origin $O$ of $V$ belongs to each $C_i$. Moreover, $C_i$ belongs to $E_i$ and $E_i \cap E_j = O$ for $i \neq j$, so $C_i \cap C_j = O$ for $i \neq j$. Hence $\bigcup_{i=1}^g C_i$ is a bouquet of $g$ circles embedded in $V$ with the origin $O$ as the vertex, still denoted as $B_g$.

For each $i \in \{1, 2, \ldots, g\}$ and each $\sigma \in S_g$, we define $i_i, \tau_\sigma \in O(2g - 1)$ as follows:

$$i_i(v_j) = v_j \quad \text{for all } j, \quad i_i(e_j) = e_j \quad \text{for } j \neq i, \quad i_i(e_i) = -e_i, \quad \tau_\sigma(1) = C_{i(i)}, \quad \tau_\sigma(v_i) = v_{i(i)}. \quad (11)$$

By (9), (10), and (11), one can check directly that

(i) $i_i$ is the identity on $C_j$ for $j \neq i$ and is an orientation reversing involution on $C_i$;

(ii) $\tau_\sigma(C_i) = C_{i(i)}$ and $\tau_\sigma^{-1}(C_i) = C_{i^{-1}(i)}$.

Let

$$H = \langle i_i, \tau_\sigma : 1 \leq i \leq g, \sigma \in S_g \rangle.$$

Then $H$ is a subgroup of $\text{Sym} \ B_g$ and each $h \in H$ has the form

$$h = \tau_\sigma \prod_{i=1}^g i_i^{a_i},$$

where $a_i = 0$ or $1$, since $\langle i_i : 1 \leq i \leq g \rangle$ is a normal subgroup of $H$. Clearly $H$ acts faithfully on $B_g$. Moreover, if $h \in H$ is the identity on $B_g$, $h$ must fix all $v_i$ and $e_i$, which implies that $h$ is the identity on $V$. So $H$ also acts faithfully on $V$. The restriction of $H$ on $B_g$ is $\text{Sym} \ B_g$, since it exhausts all symmetries of $\text{Sym} \ B_g$. Therefore we get an equivariant embedding $B_g \rightarrow \mathbb{R}^{2g-1}$.
The second construction

View $\mathbb{R}^{2g}$ as the product

$$\mathbb{R}^{2g} = \mathbb{R}^2_1 \times \mathbb{R}^2_2 \times \ldots \times \mathbb{R}^2_g,$$

where each $\mathbb{R}^2_i$ is a Euclidean plane with standard coordinate system $(x_i, y_i)$. Then $(x_1, y_1, x_2, y_2, \ldots, x_g, y_g)$ provides a standard coordinate system of the Euclidean space $\mathbb{R}^{2g}$. Let $C_i$ be the circle in $\mathbb{R}^{2i}_i \subset \mathbb{R}^{2g}$ given by

$$x^2_i + y^2_i = \frac{1}{g}, \quad x_j = \frac{1}{\sqrt{g}} \quad \text{and} \quad y_j = 0 \quad \text{for} \quad j \neq i,$$

with anti-clockwise orientation in the $x_i y_i$-plane. Now we make three observations:

(a) Clearly, the point $v = \left(\frac{1}{\sqrt{g}}, 0, \frac{1}{\sqrt{g}}, 0, \ldots, \frac{1}{\sqrt{g}}, 0\right) \in \mathbb{R}^{2g}$ belongs to each $C_i$, and each pair $C_i$ and $C_j$ intersect only at $v$ for $i \neq j$. Therefore the union $\bigcup_{i=1}^g C_i$ provides an embedding of $B_g \rightarrow \mathbb{R}^{2g}$.

(b) For each $i \in \{1, 2, \ldots, g\}$ and each $\sigma \in S_g$, we define $\iota_i, \tau_{\sigma} : \mathbb{R}^{2g} \rightarrow \mathbb{R}^{2g}$ as below:

$$\iota_i(x_1, y_1, \ldots, x_{i-1}, y_{i-1}, x_i, y_i, x_{i+1}, y_{i+1}, \ldots, x_g, y_g) = (x_1, y_1, \ldots, x_{i-1}, y_{i-1}, x_i, -y_i, x_{i+1}, y_{i+1}, \ldots, x_g, y_g),$$

$$\tau_{\sigma}(x_1, y_1, \ldots, x_i, y_i, \ldots, x_g, y_g) = (x_{\sigma(1)}, y_{\sigma(1)}, \ldots, x_{\sigma(i)}, y_{\sigma(i)}, \ldots, x_{\sigma(g)}, y_{\sigma(g)}).$$

Clearly $\iota_i, \tau_{\sigma} \in O(2g)$, $\iota_i$ is the identity on $C_j$ for $j \neq i$ and is an orientation reversing involution on $C_i$, and $\tau_{\sigma}(C_i) = C_{\sigma(i)}$. Moreover, the group

$$H = \langle \iota_i, \tau_{\sigma} : 1 \leq i \leq g, \sigma \in S_g \rangle$$

is isomorphic to $(\mathbb{Z}_2)^g \rtimes S_g \cong \text{Sym } B_g$ and acts faithfully on the pair $(\mathbb{R}^{2g}, B_g)$.

(c) For each $w \in B_g$, $w \in C_i$ for some $i$. By (11) we have

$$|w|^2 = \sum_{j=1}^g (x_j^2 + y_j^2) = \sum_{j=1}^g (x_j^2 + y_j^2) + \sum_{j \neq i}^g (x_i^2 + y_i^2)$$

$$= \sum_{j=1}^g \left(\left(\frac{1}{\sqrt{g}}\right)^2 + 0^2\right) + \frac{1}{g} = 1.$$

That is, $B_g$ stays in the unit sphere $S^{2g-1} \subset \mathbb{R}^{2g}$.
By the conclusion of (b), the given $B_\ell \subset S^{2\ell-1}$ is an equivariant embedding. \qed

**Remark 5** For a given pair $(G, \Gamma)$, as we explained in the proof of Proposition 4, each $G$-equivariant embedding $\Gamma \to \mathbb{R}^n$ provides a $G$-equivariant embedding $\Gamma \to S^n$. However a $G$-equivariant embedding $\Gamma \to S^n$ does not guarantee a $G$-equivariant embedding $\Gamma \to \mathbb{R}^n$, see the example below.

**Example 6** The maximum order of finite group action on a hyperbolic graph of rank 2 is 12, which is realized uniquely by the action of $\text{Sym} M_3 = S_3 \ltimes \mathbb{Z}_2$ on $M_3$ [7], where $M_3$ is the graph with two vertices joined by three edges, $S_3$ permutes three edges and fixes each vertex, and $\mathbb{Z}_2$ is an orientation-reversing involution on each edge. There is an equivariant embedding $M_3 \to S^2$, which is indicated by the left side of Fig. 4.

There is no equivariant embedding $M_3 \subset \mathbb{R}^2$. A quick proof uses Jordan curve theorem: For any embedding $M_3 \to \mathbb{R}^2$, $M_3$ divides $\mathbb{R}^2$ into two bounded regions $R_1$, $R_2$, and one unbounded region $R_3$, see the right side of Fig. 4. Then $C_2$, the unique edge neighboring two bounded regions, must be invariant under any $\tau \in O(2)$, therefore the embedding is not equivariant.

**Remark 7** A related question is when a finite group action $G$ on a graph $\Gamma$ can be $G$-equivariantly embedded into $\mathbb{R}^3 (S^3)$, see [2,6]. Similar question for surfaces has also been raised, see [1,5].

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