Asymptotic Fermionic Symmetry From Soft Gravitino Theorem

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Abstract

We discuss the semiclassical scattering problem for massless matter coupled to Rarita-Schwinger field in four dimensional Minkowski space. We rewrite the soft gravitino theorem as a Ward identity for the $S$-matrix and discuss the relationship of corresponding symmetry charges to new asymptotic fermonic symmetries of massless Rarita-Schwinger field.
1 Introduction

Our work was inspired by the recent progress in understanding soft theorems as Ward identities for certain asymptotic symmetries. The possibility for such correspondence can be guessed from an observation that the points on null infinity are causally disconnected. We can define conserved charges for each generator of null infinity, which is essentially the same as introducing continuous symmetry parameters (functions, vector fields or spinors on two dimensional sphere at infinity) for global symmetries in four dimensional Minkowski. Symmetries act on massless scattering data at asymptotic infinity, which may include gravitational data. Gravitational scattering data is the same as geometry, so the symmetry transform one asymptotically Minkowski space into another asymptotically Minkowski space, which makes this symmetry \textit{asymptotical} in contrast with ordinary symmetry that preserves particular
metric. In pure gravity such symmetry was discovered by Bondi, Van der Berg, Metzner and Sachs [1–3] and is often denoted as the BMS symmetry.

It is not surprising that the asymptotic symmetry is a symmetry of the massless $S$-matrix, however the precise relation requires careful description of the scattering data and matching across spatial infinity. Furthermore, the Ward identity for an asymptotic symmetry is nothing but a soft theorem, discovered in context of scattering theory by Weinberg [4]. The relationship between the asymptotic symmetry of abelian gauge theory and Weinberg’s soft photon theorem was described by Strominger [5] and was later generalized in [6–26]. Moreover it has been shown [27–29] that the changes in asymptotic data can be measured in a form of various memory effects.

Most of the asymptotic symmetries, that were successfully related to soft theorems, were bosonic, except the recent work [30], where authors discovered an infinite-dimensional sermonic symmetry in supersymmetric QED coupled to massless charged matter.

Supersymmetry algebra of Minkowski space includes Poincare transformations as bosonic generators, which asymptotically are elevated to the infinite-dimensional BMS algebra [1–3, 31], so it in natural to address a question whether such an infinite-dimensional generalization exists for global supersymmetries. Apparently such generalization was discussed by Awada, Gibbons and Shaw [32], while the corresponding soft theorems were studied by Grisaru and Pendleton [33] for pure $\mathcal{N} = 1$ supergravity case and further expanded using modern spinor-helicity formalism to $\mathcal{N} = 8$ case by [34].

In this article, we will use the simplest $\mathcal{N} = 1$ soft gravitino theorem, recast it into a Ward, identity and analyze the symmetry generated by this procedure. It turns out that this symmetry, parametrized by a spinor on two sphere, generates a local supersymmetry transformation on massless supermatter and changes gravitino by a zero mode, so it can be identified with a sermonic asymptotic symmetry.

The paper is organized as follows: in section 2 we introduce a Rarita-Shwinger field coupled to supersymmetric matter. We describe gauge fixing, coordinate system in vicinity of asymptotic null infinity and define 4d spinor decomposition in terms of the 2d spinors. In section 3 we describe free scattering data on null infinity and relate it to the plane wave expansion and zero modes. In section 4 we derive a Ward identity from the gravitino’s soft theorem and identify symmetry charges. We discuss symmetry transformations generated by these charges in section 5 and propose their relation to asymptotic symmetries.
2 Rarita-Schwinger field

Massless gravitino field $\psi_\mu$ coupled to supersymmetric matter is described by Rarita-Schwinger (RS) equation

$$\gamma^{\mu\nu\rho} \nabla_\nu \psi_\rho = J^\mu, \quad (2.1)$$

where $\gamma^\mu$ are 4d gamma matrices, $J^\mu$ is a supercurrent for the matter fields. Note that covariant derivative $\nabla_\mu$ may have a nontrivial spin-connection part. Equation (2.1) is invariant under gauge transformations

$$\delta \psi_\mu = \nabla_\mu \epsilon \quad (2.2)$$

where $\epsilon$ is a four dimensional spinor. Gauge transformations (2.2) are local supersymmetries in purely bosonic background. In order to describe the radiation data for RS equation we need to fix a gauge. We are going to analyze the (2.1) equation in four dimensional Minkowski so will use the most common covariant gauge

$$\nabla^\mu \psi_\mu = 0. \quad (2.3)$$

These gauge conditions require that the matter supercurrent $J^\mu$ is $\gamma$-traceless

$$\gamma^\mu \cdot J_\mu = 0. \quad (2.4)$$

Superconformal matter automatically satisfy this condition since it related to the trace of stress tensor by supersymmetry. For general supersymmetric matter in flat space we often can an the improvement term $\delta J^\mu = \partial_\nu B^{[\mu\nu]}$ to make $\gamma$-traceless supercurrent.

Using $\nabla_\mu \gamma_\nu = 0$ and gauge fixing conditions (2.3) we can simplify RS equation

$$\gamma^{\mu\nu\rho} \nabla_\nu \psi_\rho = \gamma^\nu \nabla_\nu \psi_\mu = J^\mu. \quad (2.5)$$

Radiation data for massless fields lives at null asymptotic infinity $I$ so we are going to introduce retarded coordinates $(u, r, z^a)$ related to Cartesian coordinates $(t = x^0, x^i)$ by

$$u = t - r, \quad r^2 = x^i x_i, \quad x^i = r \hat{x}^i(z), \quad (2.6)$$

where $\hat{x}^i(z^a)$ defines an embedding of unit round $S^2$ with coordinates $z^1, z^2$ in $\mathbb{R}^3$ with
coordinates \(x^1, x^2, x^3\). Minkowski metric in this coordinates

\[
ds^2 = -du^2 - 2dudr + r^2 \gamma_{ab} dz^a dz^b.
\]  

(2.7)

Here \(\gamma_{ab}\) is a unit metric on the round \(S^2\). Future null infinity \(I^+\) is given by the null hypersurface \((r = \infty, u, z^a)\), with future \((u = \infty)\) and past \((u = -\infty)\) boundaries denoted by \(I^+_+\) and \(I^+_\), respectively.

In the vicinity of the past null infinity \(I^-\), we are going to introduce advanced coordinates \((v, r, z^a)\) related to Cartesian coordinates \((t = x^0, x^i)\) by

\[
v = t + r, \quad r^2 = x^i x_i, \quad x^i = -r \hat{x}^i(z),
\]

(2.8)

where \(\hat{x}^i(z^a)\) is the same embedding of unit round \(S^2\) in \(R^3\) as before. Note in particular that the angular coordinate \(z^a\) on \(I^-\) is antipodally related to the angular coordinate on \(I^+\), so that null generators of \(I^\) passing through spatial infinity \((r^0)\) are labeled by the same numerical value of \(z^a\). \(I^-\) is the \((r = \infty, v, z^a)\) null hypersurface, with future \((v = \infty)\) and past \((v = -\infty)\) boundaries denoted by \(I^-_+\) and \(I^-\), respectively. Minkowski metric in advanced coordinates takes the form

\[
ds^2 = -dv^2 + 2dvdz^a dz^b.
\]

(2.9)

Note that most equations in advanced coordinates can be obtained from the ones in retarded coordinates by a simple redefinition \(u \rightarrow -v\), which serves as a good reason to use retarded coordinates in our detailed considerations, while presenting final statements for advanced coordinates when needed.

In order to define a covariant derivative \(\nabla_\mu\) for spinor fields in retarded coordinates we need to specify a spinor frame. We will use Minkowski space in Cartesian coordinates \((x^0, x^i)\) as a flat spinor frame

\[
e^0_\mu dx^\mu \equiv dx^0 = du + dr, \quad e^i_\mu dx^\mu \equiv dx^i = \hat{x}^i dr + r \partial_a \hat{x}^i dz^a.
\]

(2.10)

Spin connection is trivial for such frame since one-forms \(e^0_\mu dx^\mu, e^i_\mu dx^\mu\) are exact. Gamma matrices in retarded coordinates are of the form

\[
\gamma_t \equiv e^i_t \gamma_i + e^0_t \gamma_0 = \gamma_0 + \hat{x}^i \gamma_i, \\
\gamma_u \equiv e^i_u \gamma_i + e^0_u \gamma_0 = \gamma_0, \quad \gamma_a \equiv e^i_a \gamma_i + e^0_a \gamma_0 = r \partial_a \hat{x}^i \gamma_i.
\]

(2.11)
In retarded coordinates the RS equation (2.1) takes the form
\[
\begin{align*}
\gamma^r \partial_r \psi_u - \gamma_r \partial_a \psi_u + r^{-1} \gamma^a \partial_a \psi_u &= J_u, \\
\gamma^r \partial_r \psi_r - \gamma_r \partial_a \psi_r + r^{-1} \gamma^a \partial_a \psi_r - r^{-2} \gamma^a \psi_a &= J_r, \\
\gamma^r \partial_r \psi_a - \gamma_r \partial_a \psi_a + r^{-1} \hat{\gamma}^b D_b \psi_a + \hat{\gamma}_a (\psi_r - \psi_a) &= J_a,
\end{align*}
\] (2.12)

where we introduced matrices \( \hat{\gamma}^a \equiv r^{-1} \gamma^a \) which are indeed gamma matrices on \( S^2 \). Here \( D_a \) is an ordinary covariant (with respect to \( \gamma_{ab} \) metric) derivative \( D_a \) on the sphere with spin connection
\[
D_a = D_a + \frac{1}{2} \gamma^r \hat{\gamma}_a,
\] (2.13)

We can further simplify our analysis of the RS equations (2.12) using decomposition of 4d spinors into pair of 2d spinors. Such decomposition is realized using pair of projectors
\[
P_0 = -\frac{1}{2} \gamma_r \gamma_u = -\frac{1}{2} \gamma^u \gamma^r, \quad P_1 = -\frac{1}{2} \gamma_u \gamma_r = -\frac{1}{2} \gamma^r \gamma^u,
\]
\[
P_0 + P_1 = 1, \quad P_0 P_1 = P_1 P_0 = 0, \quad P_0 P_0 = P_0, \quad P_1 P_1 = P_1,
\] (2.14)

which obey the following useful properties
\[
D_a P_0 = P_0 D_a, \quad P_0 \gamma^r = \gamma^r P_1, \quad P_1 \gamma^r = \gamma^r P_0, \quad P_0 \gamma_r = \gamma_r P_1, \quad P_1 \gamma_r = \gamma_r P_0.
\] (2.15)

In order to analyze RS equations (2.12) near \( \mathcal{I}^+ \) we assume an asymptotic expansion for gravitino field and supercurrent
\[
\psi_\mu (r, u, z) = \sum_{n=0}^{\infty} \frac{1}{r^n} \psi_\mu^{(n)} (u, z), \quad J_\mu (r, u, z) = \sum_{n=0}^{\infty} \frac{1}{r^n} J_\mu^{(n)} (u, z),
\] (2.16)

while our equations (2.12) assume the following form
\[
\begin{align*}
(1 - n) \gamma^r \psi_{1u}^{(n)} - \gamma_r \partial_a \psi_{1u}^{(n+1)} + \hat{\gamma}^a D_a \psi_{0u}^{(n)} &= J_{0u}^{(n+1)}, \\
(1 - n) \gamma^r \psi_{0u}^{(n)} - \hat{\gamma}^a D_a \psi_{1u}^{(n)} &= J_{1a}^{(n+1)}, \\
(1 - n) \gamma^r \psi_{1r}^{(n)} - \gamma_r \partial_a \psi_{1r}^{(n+1)} + \hat{\gamma}^a D_a \psi_{0r}^{(n)} - \hat{\gamma}_a \psi_{0a}^{(n-1)} &= J_{0r}^{(n+1)}, \\
(1 - n) \gamma^r \psi_{0r}^{(n)} + \hat{\gamma}_a D_a \psi_{1r}^{(n)} - \hat{\gamma}^a \psi_{1a}^{(n+1)} &= J_{1r}^{(n+1)}, \\
-n \gamma^r \psi_{1a}^{(n)} - \gamma_r \partial_a \psi_{1a}^{(n+1)} + \hat{\gamma}^b D_b \psi_{0a}^{(n)} + \hat{\gamma}_a (\psi_{0r}^{(n+1)} - \psi_{0u}^{(n+1)}) &= J_{0a}^{(n+1)}, \\
-n \gamma^r \psi_{0a}^{(n)} + \hat{\gamma}^b D_b \psi_{1a}^{(n)} + \hat{\gamma}_a (\psi_{1r}^{(n+1)} - \psi_{1u}^{(n+1)}) &= J_{1a}^{(n+1)}.
\end{align*}
\] (2.17)

The large-r falloff for supercurrent components is \( J_a, J_u = \mathcal{O}(r^{-2}) \), \( J_r = \mathcal{O}(r^{-3}) \), while trace

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free condition (2.4) further restricts \( J_{1u}^{(2)} = 0 \). The gauge fixing conditions (2.3)

\[
\begin{align*}
\gamma^r \psi_{1r}^{(n+1)} - \gamma^r \psi_{1u}^{(n+1)} + \tilde{\gamma}^a \psi_{0a}^{(n)} &= 0, \\
\gamma^r \psi_{0r}^{(n+1)} + \tilde{\gamma}^a \psi_{1a}^{(n)} &= 0, \\
(1 - n)(\psi_{0r}^{(n+1)} - \psi_{0u}^{(n+1)}) - \partial_a \psi_{0a}^{(n+2)} + D_a \psi_{0a}^{(n)} - \frac{1}{2} \gamma^r \tilde{\gamma}^a \psi_{1a}^{(n)} &= 0, \\
(1 - n)(\psi_{1r}^{(n+1)} - \psi_{1u}^{(n+1)}) - \partial_a \psi_{1r}^{(n+2)} + D_a \psi_{1a}^{(n)} - \frac{1}{2} \gamma^r \tilde{\gamma}^a \psi_{0a}^{(n)} &= 0.
\end{align*}
\]  

Let us note that our gauge fixing (2.3) allows for a residual gauge symmetry in the form of free solution to Dirac equation. In four dimensional Minkowski free solution to Dirac equation \( \epsilon = r^{-1} \epsilon_0^{(1)} (u, z) \), so we can further set

\[
\psi_{0r}^{(2)} (u, z) = 0. \tag{2.19}
\]

### 3 The semiclassical scattering problem

#### 3.1 Scattering data for gravitino at \( \mathcal{I}^+ \)

In this subsection we will present some details on the analysis of RS equations (2.17) and gauge fixing conditions (2.18). The \( \mathcal{O}(r^0) \) trace free condition (2.18)

\[
\gamma^r \psi_{1r}^{(0)} - \gamma^r \psi_{1u}^{(0)} = 0, \quad \gamma^r \psi_{0r}^{(0)} = 0. \tag{3.1}
\]

Since \( \gamma^r \) is invertible due to the relation \( \gamma^r \gamma^r = 1 \) we can immediately write

\[
\psi_{0r}^{(0)} = 0. \tag{3.2}
\]

Now let us look at \( \mathcal{O}(r^{-1}, r^{-2}) \) orders of r-component of RS equation (2.17)

\[
\begin{align*}
-\gamma^r \partial_u \psi_{1r}^{(0)} &= J_{0r}^{(0)} = 0, \\
\gamma^r \psi_{1r}^{(0)} - \gamma^r \partial_a \psi_{1r}^{(1)} + \tilde{\gamma}^a D_a \psi_{1r}^{(0)} &= J_{0r}^{(1)} = 0, \\
\gamma^r \psi_{0r}^{(0)} + \tilde{\gamma}^a D_a \psi_{0r}^{(1)} &= J_{1r}^{(1)} = 0.
\end{align*} \tag{3.3}
\]

The last equation combined with \( \psi_{0r}^{(0)} = 0 \) leads to

\[
\tilde{\gamma}^a D_a \psi_{1r}^{(0)} = 0. \tag{3.4}
\]
There are no nontrivial solution to the Dirac equation on $S^2$

$$\hat{\gamma}^a D_a \psi = 0, \quad \Rightarrow \psi = 0. \quad (3.5)$$

In order to prove (3.5) let us consider

$$(\hat{\gamma}^a D_a)^2 \psi = \hat{\gamma}^a \hat{\gamma}^b D_a D_b \psi = (\hat{\gamma}^{[ab]} + \gamma^{ab}) D_a D_b \psi = (D^2 + \frac{1}{4} \hat{\gamma}^{[ab]} \hat{\gamma}^{[ab]}) \psi = \left(D^2 - \frac{1}{2}\right) \psi, \quad (3.6)$$

where we used

$$[D_a, D_b] \psi = \frac{1}{2} \hat{\gamma}_{[ab]} \psi = \frac{1}{4} (\hat{\gamma}_a \hat{\gamma}_b - \hat{\gamma}_b \hat{\gamma}_a) \psi. \quad (3.7)$$

Eigenvalues of $D^2$ are negative on $S^2$ since it is a compact manifold, therefore all nontrivial eigenmodes of $(\hat{\gamma}^a D_a)^2$ have negative eigenvalues, what concludes our proof of (3.5).

Using (3.5) and the first equation from (3.1) we can further evaluate

$$\psi_0^{(0)} = \psi_1^{(0)} = \psi_{1u}^{(0)} = 0. \quad (3.8)$$

Now using $O(r^0, r^{-1}, r^{-2})$ orders of the $u$-component RS equation (2.17)

$$-\gamma_r \partial_u \psi_{1u}^{(0)} = J_{0u}^{(0)} = 0,$$
$$\gamma_r \psi_{1u}^{(0)} - \gamma_r \partial_u \psi_{1u}^{(1)} + \hat{\gamma}^a D_a \psi_{0u}^{(0)} = J_{1u}^{(1)} = 0,$$
$$\gamma_r \psi_{0u}^{(0)} - \hat{\gamma}^a D_a \psi_{0u}^{(1)} = J_{1u}^{(1)} = 0,$$
$$-\hat{\gamma}^a D_a \psi_{1u}^{(1)} = J_{1u}^{(2)} = 0, \quad (3.9)$$

we can solve for

$$\psi_0^{(0)} = \psi_1^{(0)} = \psi_{1u}^{(0)} = \psi_{0u}^{(0)} = \psi_{1u}^{(1)} = 0. \quad (3.10)$$

Similar analysis of the system (2.17, 2.18) at orders where supercurrent components $J_{\mu}^{(n)}$ are trivial leads to

$$\psi_0^{(1)} = \psi_1^{(1)} = \psi_{1a}^{(1)} = \hat{\gamma}_a \psi_{0a}^{(0)} = 0. \quad (3.11)$$

The trace-free part of $\psi_{0a}^{(0)}(u, z)$ is unconstrained and represents free scattering data for gravitino in the form of two independent function on $T^+$, which correspond to the two independent physical polarizations. The subleading orders of $\psi_{\mu}$ are expressible in terms of this free data, up to a boundary data. In particular, the $O(r^{-2})$ order of $\nabla^\mu \psi_\mu = 0$ and residual gauge fixing condition (2.19) determine

$$\psi_{0a}^{(1)} = D_a \psi_{0a}^{(0)}, \quad (3.12)$$

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while the u-component of RS equation (2.17)

\[- \gamma_r \partial_u \psi_{1u}^{(2)} + \hat{\gamma}^a D_a \psi_{0u}^{(1)} = J_{0u}^{(2)}\]

(3.13)
determines \(\psi_{1u}^{(2)}(u, z)\) up to its boundary value at \(\mathcal{I}_+\) in terms of the scattering data for gravitino \(\psi_{0a}^{(0)}\) and matter supercurrent \(J_{0u}^{(2)}\).

### 3.2 Scattering data for gravitino at \(\mathcal{I}^-\)

Similar analysis can be applied to gravitino field \(\psi_{-\mu}\) in advanced coordinates (2.9) near \(\mathcal{I}^-\) using the same gauge (2.3). Free scattering data at \(\mathcal{I}^-\) is represented in terms of the trace free part of the \(\psi_{0a}^{(0)}(u, z)\). Equations (3.12, 3.13) in advanced coordinates take the form

\[\psi_{0v}^{(1)-} = -D^a \psi_{0a}^{(0)-}, \quad \gamma_r \partial_v \psi_{1v}^{(2)-} + \hat{\gamma}^a D_a \psi_{0v}^{(1)-} = J_{0v}^{(2)}.\]

(3.14)

### 3.3 Mode expansions

The mode expansion for the RS field is given by the following formula

\[\psi_\mu = \sum_{\alpha=\pm} \int \frac{d^3q}{(2\pi)^3} \left[ u^{\mu}_\alpha e^{iq \cdot x} b_\alpha(\vec{q}) + u^{\mu}_\alpha e^{-iq \cdot x} b_\alpha(\vec{q})^\dagger \right],\]

(3.15)

where \(b_\alpha(\vec{q})^\dagger\) is a creation operator for gravitino with spatial momenta \(\vec{q}\) and polarization \(\alpha\), that satisfies the commutation relations

\[\{b_\alpha(\vec{p}), b_\beta(\vec{q})^\dagger\} = \delta_{\alpha\beta} (2\pi)^3 \delta^3(\vec{p} - \vec{q}).\]

(3.16)

We can evaluate the free scattering data \(\psi_{0a}^{(0)}\) for RS field in terms of the mode sum using

\[\psi_{0a}^{(0)}(u, z) = P_0 \lim_{r \to \infty} \partial_a x^\mu \psi_{\mu}(u + r, r x^j(z)) = \partial_a \hat{x}^i P_0 \lim_{r \to \infty} r \psi_i(r, u, z).\]

(3.17)

At \(r \to \infty\) phases in the mode expansion (3.15) become large. In particular

\[i q \cdot x = -i \omega_q (u + r) + i r q^j \hat{x}_j(z) = -i \omega_q u - i r (1 - \hat{x}^j(z) q_j/\omega_q),\]

(3.18)
where $\omega_q$ is the energy for gravitino with spatial momentum $q_i$. The saddle point approximation of (3.15) leads to

$$
\psi_0^{(0)}(u, z) = -\frac{i}{4\pi^2} \partial_a \hat{x}^i(z) \sum_{\alpha=\pm} \int_0^\infty \omega_q d\omega_q [u_i^{*\alpha} e^{-i\omega_q u_i} b_\alpha (\omega_q \hat{x}(z)) + h.c].
$$

(3.19)

The positive and negative frequency modes are given by

$$
\psi^{\omega(0)}_a(z) = -\frac{i\omega}{2\pi} \partial_a \hat{x}^i(z) \sum_{\alpha} u_i^{*\alpha} b_\alpha (\omega \hat{x}(z)),
$$

$$
\psi^{-\omega(0)}_{0a}(z) = -\frac{i\omega}{2\pi} \partial_a \hat{x}^i(z) \sum_{\alpha} u_i^{\alpha} b_\alpha (\omega \hat{x}(z))^\dagger,
$$

(3.20)

where $\omega > 0$ in both formulas. The $\omega \rightarrow 0$ limit of these expressions defines a zero mode operator

$$
\psi_0^{0(0)} = \frac{1}{2} \lim_{\omega \rightarrow 0} (\psi^{\omega(0)}_0 + \psi^{-\omega(0)}_0).
$$

(3.21)

The asymptotic data at $I^-$ is given by

$$
\psi_{0a}^{(0)}(v, z) = P_0 \lim_{r \rightarrow \infty} \partial_a \epsilon^\mu \psi_\mu (v - r, r \hat{x}(z)),
$$

(3.22)

which may be decomposed into the positive and negative frequency modes

$$
\psi^{\omega(0)}_{0a}(z) = -\frac{i\omega}{2\pi} \partial_a \hat{x}^i(z) \sum_{\alpha} u_i^{*\alpha} b_\alpha (-\omega \hat{x}(z)),
$$

$$
\psi^{-\omega(0)}_{0a}(z) = -\frac{i\omega}{2\pi} \partial_a \hat{x}^i(z) \sum_{\alpha} u_i^{\alpha} b_\alpha (-\omega \hat{x}(z))^\dagger.
$$

(3.23)

The associated zero mode creation operator is given by

$$
\psi_{0a}^{0(0)} = \frac{1}{2} \lim_{\omega \rightarrow 0} (\psi^{\omega(0)}_{0a} + \psi^{-\omega(0)}_{0a}).
$$

(3.24)

4 Ward identity from soft theorem

Soft gravitino theorem relates an $n + 1$-point on-shell amplitude with a single soft gravitino insertion to a soft operator acting on $n$-point amplitude. Since gravitino is a fermion the soft operator is fermionic as well. We can add an extra fermionic label to the external states in addition to physical polarizations and external momenta so that we can consider both bosonic and fermionic states simultaneously. For massless particles external momenta $p_k^\mu$ can
be labeled in terms of the energies $E_k$ and points on a sphere $z_k$, so that
\[
p_k^\mu = E_k(1, \hat{x}^i(z_k)). \tag{4.1}
\]
For notational simplicity we will suppress all labels except $z_k$ for external incoming and outgoing states
\[
|z_1, \ldots, z_n\rangle, \quad \langle z_{n+1}, \ldots, z_{n+m}|. \tag{4.2}
\]
Using an $S$-matrix presentation for an $n$-point on-shell amplitude we can write the soft theorem for an outgoing gravitino
\[
\lim_{\omega \to 0} \omega \langle z_{n+1} \ldots | b_\alpha(\vec{q}) S | z_1 \ldots \rangle = i \left[ \sum_{k=n+1}^{n+m} \frac{p_k^\mu(u_{\alpha\mu}Q_k)}{p_k \cdot q} - \sum_{k=1}^{n} \frac{p_k^\mu(u_{\alpha\mu}Q_k)}{p_k \cdot q} \right] \langle z_{n+1} \ldots | S | z_1 \ldots \rangle, \tag{4.3}
\]
where $u_{\alpha\mu}$ is a polarization tensor for the gravitino with helicity $\alpha$; $p_k^\mu$, $Q_k$ - momenta and supercharges of matter fields.

Using our results for zero modes of radiation data (3.22) we can write the soft theorem (4.3) in the following form
\[
\langle z_{n+1} \ldots | \psi_0^{(0)}(z) S | z_1 \ldots \rangle = - \frac{1}{4\pi} F_{a}(z; z_k) \langle z_{n+1} \ldots | S | z_1 \ldots \rangle, \tag{4.4}
\]
where functions $F_a^{out}(z; z_k)$
\[
F_a^{out}(z; z_k) = \omega D_a \hat{x}^i(z) \sum_{\alpha} u_i^{\alpha*} \left[ \sum_{k=1}^{n} \frac{p_k^\mu(u_{\alpha\mu}Q_k)}{p_k \cdot q} - \sum_{k=n+1}^{n+m} \frac{p_k^\mu(u_{\alpha\mu}Q_k)}{p_k \cdot q} \right] =
\]
\[
= \frac{1}{2} \hat{\gamma}_b \gamma_a P_0(z) \left[ \sum_{k=1}^{n} Q_k \partial^b \log(1 - P(z, z_k)) - \sum_{k=n+1}^{n+m} Q_k \partial^b \log(1 - P(z, z_k)) \right]. \tag{4.5}
\]
Here we used the completeness relation for polarization tensors
\[
\sum_{\alpha} u^{\alpha i} u^{\alpha j} = (\delta^{ij} - \hat{x}^i \hat{x}^j) P_0 - \frac{1}{2} \partial_a \hat{x}^i \partial_b \hat{x}^j \gamma^{a} \gamma^{b} P_0, \tag{4.6}
\]
\footnote{Note the that the definition of the $F_a^{out}(z; z_k)$ is such that it is proportional to expectation value for gravitino’s field zero mode at $I^+$. It would be interesting to design a memory, that can measure this zero mode. The fermionic nature of zero mode suggests that we may need to couple it to a superparticle to create a bosonic observable.}
supercharge conservation
\[
\sum_{k=1}^{n} Q_k - \sum_{k=n+1}^{n+m} Q_k = 0, \quad (4.7)
\]
and defined an invariant distance on \( S^2 \)
\[
P(z, z_k) \equiv \hat{x}^i(z) \hat{x}_i(z_k). \quad (4.8)
\]
Note that \( F_a(z; z_k) \) obeys the differential equation
\[
\sqrt{\gamma} D^a F^\text{out}_a(z; z_k) = 2\pi \left[ n \sum_{k=1}^{n} Q_k \delta^2(z - z_k) - \sum_{k=n+1}^{n+m} Q_k \delta^2(z - z_k) \right]. \quad (4.9)
\]
We can consider a soft theorem for an incoming soft gravitino
\[
\lim_{\omega \to 0} \omega \langle z_{n+1} \ldots | S b_a(q)^\dagger | z_1 \ldots \rangle = i \left[ \sum_{k=n+1}^{n+m} \frac{p_k^\mu(u_{\alpha\mu} Q_k)}{p_k \cdot q} - \sum_{k=1}^{n} \frac{p_k^\mu(u_{\alpha\mu} Q_k)}{p_k \cdot q} \right] \langle z_{n+1} \ldots | S | z_1 \ldots \rangle, \quad (4.10)
\]
and rewrite it using incoming zero mode (3.24)
\[
\langle z_{n+1} \ldots | S \psi_{0a}^{(0)-(0)}(z) | z_1 \ldots \rangle = \frac{1}{4\pi} F^\text{in}_a(z; z_k) \langle z_{n+1} \ldots | S | z_1 \ldots \rangle, \quad (4.11)
\]
where
\[
F^\text{in}_a(z; z_k) = \frac{1}{2} \gamma_b \gamma_a P_0(z) \left[ n \sum_{k=1}^{n} Q_k \partial^b \log(1 + P(z, z_k)) - \sum_{k=n+1}^{n+m} Q_k \partial^b \log(1 + P(z, z_k)) \right]. \quad (4.12)
\]
After applying (4.9) to equations (4.4) and (4.11), we may integrate against an arbitrary spinor \( \bar{\epsilon}(z) \) on the sphere to obtain
\[
- \int d^2 z \sqrt{\gamma} \bar{\epsilon}^\dagger(z) \gamma^r D^a \langle z_{n+1} \ldots | \psi_{0a}^{(0)(0)}(z) S | z_1 \ldots \rangle + \int d^2 z \sqrt{\gamma} \bar{\epsilon}^\dagger(z) \gamma^r D^a \langle z_{n+1} \ldots | S \psi_{0a}^{(0)-(0)}(z) | z_1 \ldots \rangle
\]
\[
= \left[ \sum_{k=n+1}^{n+m} \epsilon^\dagger(z_k) \cdot Q_k - \sum_{k=1}^{n} \epsilon^\dagger(z_k) \cdot Q_k \right] \langle z_{n+1} \ldots | S | z_1 \ldots \rangle, \quad (4.13)
\]
with \( \bar{\epsilon} = \epsilon^\dagger \gamma_u \) and \( \bar{\epsilon}^\dagger(z) \) being antipodally identified with \( \bar{\epsilon}(z) \) what can be written as Ward identity
\[
\langle z_{n+1} \ldots | (Q^+ S - SQ^-) | z_1 \ldots \rangle = 0. \quad (4.14)
\]
The charges $Q^\pm = Q^\pm_H + Q^\pm_S$ commute with the $S$-matrix and induce infinitesimal symmetry transformations on $\mathcal{I}^\pm$ states. $Q^\pm_H$ is defined by its action on the asymptotic states:

$$Q^+_H|z_1, \ldots \rangle = \sum_{k=1}^n \epsilon^\dagger(z_k) \cdot Q_k|z_1, \ldots \rangle, \quad \langle z_{n+1}, \ldots |Q^-_H = \langle z_{n+1}, \ldots | \sum_{k=n+1}^{n+m} \epsilon^\dagger(z_k) \cdot Q_k.$$  \hspace{1cm} (4.15)

The soft charges are given by

$$Q^+_S = \int d^2z \bar{\epsilon} \gamma^r D^a \psi^{0(0)}_{0a}, \hspace{1cm} (4.16)$$

$$Q^-_S = \int d^2z \bar{\epsilon}^- \gamma^r D^a \psi^{0(0)-}_{0a}. \hspace{1cm} (4.17)$$

5 From Ward identity to asymptotic symmetry

5.1 Action of matter charges

The hard charges (4.15) defined by their action on the in- and out-states can be written using super current

$$Q^+_H = \lim_{r \to \infty} r^2 \int_{\mathcal{I}^+} \sqrt{\gamma} d^2zd\epsilon J_{0u}(u, r, z) \hspace{1cm} (5.1)$$

$$Q^-_H = \lim_{r \to \infty} r^2 \int_{\mathcal{I}^-} \sqrt{\gamma} d^2zd\epsilon^- J_{0v}(v, r, z) \hspace{1cm} (5.2)$$

These expressions can be further rewritten in the form

$$Q^+_H = \lim_{\Sigma \to \mathcal{I}^+} \int_\Sigma d\Sigma \ n^\mu_{\Sigma} \epsilon J_\mu, \quad Q^-_H = \lim_{\Sigma \to \mathcal{I}^-} \int_\Sigma d\Sigma \ n^\mu_{\Sigma} \epsilon^- J_\mu. \hspace{1cm} (5.3)$$

Here $\Sigma$ is a space-like Cauchy surface, $n_{\Sigma}$ is a unit normal to $\Sigma$. Written in this form, it is clear that the hard charges generate supersymmetry transformations with parameter $\epsilon$ on the asymptotic states if we assume standard commutation relations for the matter fields.

5.2 Vanishing soft charge

Symmetry charges (4.15, 4.17, 4.16) apply a supersymmetry transformation to matter fields and create a zero mode for gravitino. However global supersymmetries are just a symmetries of the matter sector of the coupled theory. Thus the spinors $\epsilon$ for which the soft charge (4.16) vanishes should represent global supersymmetries of Minkowski space. Since $\psi^{0(0)}_{0a}$ is trace-
free the most general such $\epsilon$ is a solution to

$$\mathcal{D}_a \epsilon = \hat{\gamma}_a \omega \quad (5.4)$$

for some spinor $\omega$. Using (3.7) we can show that

$$\mathcal{D}_a \omega = -\frac{1}{4} \hat{\gamma}_a \epsilon. \quad (5.5)$$

The linear combinations

$$\epsilon_{\pm} = \epsilon \pm 2\gamma^r \omega \quad (5.6)$$

solve the canonical Killing spinor equation on the two-sphere

$$\mathcal{D}_a \epsilon_{\pm} = \pm \frac{1}{2} \gamma^r \hat{\gamma}_a \epsilon_{\pm}. \quad (5.7)$$

Furthermore $\epsilon_{\pm}(z)$ trivially satisfies

$$\partial_u \epsilon_{\pm} = \partial_r \epsilon_{\pm} = 0, \quad (5.8)$$

what allows us to use $\epsilon_{\pm}(z)$ to parametrize solutions to covariantly constant spinor in Minkowski.

**5.3 Action of soft charges**

Let us use our prediction for the symplectic structure in conjugation with our proposal for the soft charge. From the mode expansion (3.19) we can read off the following bracket

$$\{\psi^{(0)}_a (u, z), \psi^{(0)*}_b (v, w)\} = \left( \gamma_{ab} - \frac{1}{2} \hat{\gamma}_a \hat{\gamma}_b \right) P_0 \delta^{(2)}(z - w) \delta(u - v), \quad (5.9)$$

and use it to evaluate the soft charge action

$$\delta \psi^{(0)}_a = \left( \gamma_{ab} - \frac{1}{2} \hat{\gamma}_a \hat{\gamma}_b \right) \mathcal{D}^b \epsilon_0 = \mathcal{D}_a \epsilon_0 - \frac{1}{2} \hat{\gamma}_a \hat{\gamma}^b \mathcal{D}_b \epsilon_0. \quad (5.10)$$

The local gauge transformation (2.2) of $\psi^{(0)}_{0a}$

$$\delta \psi^{(0)}_{0a} = \mathcal{D}_a \epsilon_0 - \frac{1}{2} \gamma^r \hat{\gamma}_a \epsilon_1 \quad (5.11)$$
contains residual gauge transformation that preserves (2.18), (2.19) for
\[ \epsilon_1 = -\gamma^r \hat{\gamma}^a D_a \epsilon_0. \] (5.12)

Therefore we conclude that \(Q^+_S\) acts linearly on zero mode of gravitino
\[ \psi^{(0)}_{0a}(z, u) = D_a \sigma_0(z) - \frac{1}{2} \hat{\gamma}^a \hat{\gamma}^b D_b \sigma_0(z), \quad \delta \epsilon_0(z) = \epsilon_0(z). \] (5.13)

Similarly for the scattering data on \(\mathcal{I}^-\) we have
\[ \xi^{(0)}_{0a}(z, v) = D_a \sigma_0^-(z) - \frac{1}{2} \hat{\gamma}^a \hat{\gamma}^b D_b \sigma_0^-(z), \quad \delta \epsilon_0^-(z) = \epsilon_0^-(z). \] (5.14)

## 5.4 Scattering

So far we treated \(\mathcal{I}^+\) and \(\mathcal{I}^-\) separately and symmetry charges \(Q^\pm\) acted on two separate scattering datasets. In order to connect \(\mathcal{I}^-\) to \(\mathcal{I}^+\) we must match the scattering data at \(\imath^0\). Following the analysis in [6], all fields and functions are taken to be continuous along the null generators of \(\mathcal{I}\) passing through \(\imath^0\). Due to the antipodal identification of the angular coordinates on \(\mathcal{I}^+\) and \(\mathcal{I}^-\), the zero modes (5.13) for \(\psi^{(0)}_{0a}\) and (5.14) for \(\psi^{(0)-}_{0a}\) are matched according to
\[ \sigma_0(z) = \sigma_0^-(z). \] (5.15)

This identification also allows for a canonical identification of transformation parameters according to the rule
\[ \epsilon(z) = \epsilon^-(z), \] (5.16)

yielding a diagonal subgroup that may be identified as a symmetry of the \(S\)-matrix.

## 5.5 Total charges

Typical total charge in theory with a gravity is total derivative on corresponding part of null infinity. In our case we have the following expression for the total charge
\[ Q^+ = Q^+_S + Q^+_H = \int dud^2z \sqrt{\gamma} \epsilon (\gamma^r D^a \psi^{(0)}_{0a} + J^{(2)}_{0u}). \] (5.17)

Using (5.12) and (3.13) we can write total charge in the form
\[ Q^+ = -\int_{\mathcal{I}^+} \bar{\epsilon} \gamma_r \partial_u \psi^{(2)}_{u1}. \] (5.18)
Furthermore we can impose an additional boundary condition

\[ \psi^{(2)} |_{\mathcal{I}^+} = 0, \quad (5.19) \]

so that the total charge is an integral over \( \mathcal{I}^+ \) surface, which we use to match the scattering data. This charge is similar to the charge introduced for global super symmetries in [35].

6 Concluding remarks

At the last stages of the work we were contacted by the Steven Avery and Burkhard U.W. Schwab [36], who work on a similar problem. Authors take a different approach to the analysis of asymptotic supersymmetries in \( \mathcal{N} = 1, d = 4 \) supergravity, while the final conclusion is similar to ours: There is an infinite dimensional fermonic symmetry for the gravitino coupled to matter in four dimensional Minkowski space.

It is worth mentioning that we used notations which can be adapted to the higher dimensional cases, in a similar way as we did in [11,20]. The whole construction of the new symmetries from soft gravitino theorem might as well work in higher dimensions.

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