New solvable problems in the dynamics of a rigid body about a fixed point in a potential field.

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Abstract
We determine the general form of the potential of the problem of motion of a rigid body about a fixed point, which allows the angular velocity to remain permanently in a principal plane of inertia of the body. Explicit solution of the problem of motion is reduced to inversion of a single integral. A several-parameter generalization of the classical case due to Bobylev and Steklov is found. Special cases solvable in elliptic and ultraelliptic functions of time are discussed.

Key words: Rigid body dynamics. Integrable cases. Solvable cases. Particular solutions. Bobylev-Steklov case.

1 Introduction
Integrable systems constitute a rare exception in mechanics. This is most clearly manifested in the field of rigid body dynamics, where all known integrable cases constitute few small tables, see e.g. [1], [2] and [3]. Second comes particular solutions of equations of motion of a rigid body in various settings. Those are solutions valid only for certain particular sets of the initial position and angular velocity. In the classical problem of motion of a rigid body about a fixed point in a uniform gravity field there are eleven solutions of this type known after authors of the 19th and the 20th centuries. All of them are collected in table 1 below (in chronological order):
Table 1: Known particular solvable cases of the classical problem.

For a detailed account of those cases see [16] or [17]. Some of them were generalized through the addition of a gyrostatic moment [16] and other potential and gyroscopic forces [16], [18].

In the present article we aim at exploring the possibility of particular solutions of the Bobylev-Steklov type for the problem of motion of a rigid body about a fixed point in a field that generalizes the classical setting. We assume that the body is acted upon by certain potential forces, which admit symmetry axis fixed in space. The equations of motion for this problem can be written in the Euler-Poisson form (e.g. [1]):

\[
\begin{align*}
A\dot{p} + (C - B)qr &= \gamma_2 \frac{\partial V}{\partial \gamma_3} - \gamma_3 \frac{\partial V}{\partial \gamma_2}, \\
B\dot{q} + (A - C)pr &= \gamma_3 \frac{\partial V}{\partial \gamma_1} - \gamma_1 \frac{\partial V}{\partial \gamma_3}, \\
C\dot{r} + (B - A)pq &= \gamma_1 \frac{\partial V}{\partial \gamma_2} - \gamma_2 \frac{\partial V}{\partial \gamma_1}, \quad (1)
\end{align*}
\]

\[
\begin{align*}
\dot{\gamma}_1 + q\gamma_3 - r\gamma_2 &= 0, \\
\dot{\gamma}_2 + r\gamma_1 - p\gamma_3 &= 0, \\
\dot{\gamma}_3 + p\gamma_2 - q\gamma_1 &= 0, \quad (2)
\end{align*}
\]

where \(A, B, C\) are the principal moments of inertia, \(p, q, r\) are the components of the angular velocity of the body and \(\gamma_1, \gamma_2, \gamma_3\) are the components of the unit vector \(\gamma\) along the axis of symmetry of the force field, all being referred to the principal axes of inertia at the fixed point. The potential \(V\) depends only on the Poisson variables \(\gamma_1, \gamma_2, \gamma_3\). In the classical problem of a heavy body \(V = a\gamma_1 + b\gamma_2 + c\gamma_3\).
Equations (1) and (2) admit three general first integrals:

\[ I_1 = \frac{1}{2}Ap^2 + \frac{1}{2}Bq^2 + \frac{1}{2}Cr^2 + V, \text{ the energy integral} \quad (3) \]

\[ I_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1, \text{ the geometric integral} \quad (4) \]

\[ I_3 = Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3, \text{ the areas integral} \quad (5) \]

2 A new solvable case

The aim of this paper is to look for potentials \( V \) which allow full solution of the system (1-2) under the condition

\[ q = 0 \quad (6) \]

on the zero level of the areas integral, i.e.

\[ I_3 = 0 \quad (7) \]

We assume that \( A \neq C \) without restriction on the third moment of inertia \( B \). This choice of the problem is motivated by the classical Bobylev-Steklov solution in the classical problem, which is characterized by the same condition (6) but with the potential \( V = a\gamma_1 \) and the additional restriction on the moments of inertia \( A = 2C \).

From (6) and the middle equation of (1) we get

\[ (A - C)p\gamma_3 - \gamma_3 \frac{\partial V}{\partial \gamma_1} + \gamma_1 \frac{\partial V}{\partial \gamma_3} = 0 \quad (8) \]

Differentiating this equality and using equations (1)-(7) we arrive at the expressions for the remaining angular velocities

\[ p = -\sqrt{\frac{C\gamma_3}{A(A - C)\gamma_1}}(\gamma_1 \frac{\partial V}{\partial \gamma_3} - \gamma_3 \frac{\partial V}{\partial \gamma_1}) \]

\[ r = \sqrt{\frac{A\gamma_1}{C(A - C)\gamma_3}}(\gamma_1 \frac{\partial V}{\partial \gamma_3} - \gamma_3 \frac{\partial V}{\partial \gamma_1}) \quad (9) \]

where \( V \) satisfies the linear partial differential equation
\[ \gamma_1 \gamma_2 \gamma_3 (A \frac{\partial^2 V}{\partial \gamma_1^2} - C \frac{\partial^2 V}{\partial \gamma_3^2}) + (A \gamma_1^2 + C \gamma_2^2)(\gamma_1 \frac{\partial^2 V}{\partial \gamma_2 \partial \gamma_3} - \gamma_3 \frac{\partial^2 V}{\partial \gamma_1 \partial \gamma_2}) \\
- \gamma_2 (A \gamma_1^2 - C \gamma_2^2) \frac{\partial^2 V}{\partial \gamma_1 \partial \gamma_3} \\
- (A - 2C) \gamma_2 \gamma_3 \frac{\partial V}{\partial \gamma_1} + 2(A - C) \gamma_3 \gamma_1 \frac{\partial V}{\partial \gamma_2} - (2A - C) \gamma_1 \gamma_2 \frac{\partial V}{\partial \gamma_3} = 0 \quad (10) \]

On the other hand, from the first and third equations of (2), in virtue of (6), we have
\[ \frac{d\gamma_3}{d\gamma_1} = \frac{\dot{\gamma}_3}{\dot{\gamma}_1} = \frac{C \gamma_3}{A \gamma_1} \quad (11) \]
This can be readily integrated to give
\[ \gamma_3 = \lambda \gamma_1^{C/A}, \quad \lambda = \text{const.} \quad (12) \]
This means the trace of the vertical unit vector on the unit sphere attached to the body lies on a cylindrical surface whose generators are parallel to the \( y \)-axis. Also, from (11) one can express \( \gamma_2 \) as
\[ \gamma_2 = \sqrt{1 - \gamma_1^2 - \lambda^2 \gamma_1^{2C/A}} \quad (13) \]
and Thus, five of the six Euler-Poisson variables are expressed in terms of the last one: \( \gamma_1 \). The relation with time can be determined by separation of variables in the first Poisson equation. This finally gives
\[ t = \int \frac{d\gamma_1}{\sqrt{g(\gamma_1)}} \quad (14) \]
where
\[ g(\gamma_1) = \frac{A \gamma_1^{\frac{A-C}{C(A-C)}} (1 - \gamma_1^2 - \lambda^2 \gamma_1^{\frac{2C}{A}})(\gamma_1 \frac{\partial V}{\partial \gamma_3} - \gamma_3 \frac{\partial V}{\partial \gamma_1})_0} \]
and \( _0 \) in the right hand side means the value of the expression in virtue of relations (12, 13), so that \( g \) is a function of the single variable \( \gamma_1 \).

Summing up, we formulate the following

**Theorem 1** For an arbitrary rigid body moving about a fixed point while acted upon by forces with potential \( V \) satisfying (10), the Euler-Poisson equations (1-2) are solvable on the zero level of the areas integral under the condition \( q = 0 \). The solution is parameterized in terms of \( \gamma_1 \) by expressions (9, 12, 13) and relation with time is given by (14).
3 The general form of the solution

It is not hard to construct the general solution of the linear PDE (10), which may be written in the form

\[ V = V_1 + V_2, \]  

(15)

where

\[ V_1 = (A\gamma_1^2 + C\gamma_2^2) \int F\left( \frac{[A(\gamma_1^2 + \gamma_2^2) - u]^{A/C}}{C(\gamma_1^2 + \gamma_2^2) - u} \right) \frac{du}{u^2} \]  

(16)

\[ V_2 = (A\gamma_1^2 + C\gamma_2^2)G(\gamma_2) \]  

(17)

This form involves two arbitrary functions \( F \) and \( G \), which we assume well behaved on the poisson sphere (11), so that all subsequent operations on the potential \( V \) are justified. With this form of the potential we can rewrite the expressions for the Euler-Poisson variables in their final form parametrized by \( \gamma_1 \)

\[ (p, q, r) = (-\sqrt{\frac{2C}{A}}\lambda^{C/A} \gamma_1, \sqrt{\frac{2A}{C}} \gamma_1) \sqrt{\varpi} \]

(18)

\[ (\gamma_1, \gamma_2, \gamma_3) = (\gamma_1, \sqrt{1 - \gamma_2^2 - \lambda^2 \gamma_1}, \lambda^{C/A}) \]

where

\[ \varpi = \frac{F(-\lambda^{2A}(A - C))^{A-1}}{A\gamma_1^2 + C\lambda^2 \gamma_1^{2C/A}} + \int^{A\gamma_1^2 + C\lambda^2 \gamma_1^{2C/A}} F\left( \frac{[A(\gamma_1^2 + \gamma_2^2) - u]^{A/C}}{C(\gamma_1^2 + \gamma_2^2) - u} \right) \frac{du}{u^2} \]

(19)

\[ + G(\sqrt{1 - \gamma_1^2 - \lambda^2 \gamma_1}) \]

and \( \gamma_1 \) is determined in terms of time by inverting the integral

\[ t = \int \frac{d\gamma_1}{g(\gamma_1)}, \quad g(\gamma_1) = -2A\gamma_1^2(C(1 - \gamma_2^2 - \lambda^2 \gamma_1^{2C/A}))) \]

(20)

It may now be verified that the solution given by the expressions (15,20) satisfies the Euler-Poisson equations (11,12). Also, one can check that the total energy of the motion is in fact preserved and has the value

\[ h = -F(-\lambda^{2A}(A - C))^{A-1} \]

(21)
4 Generalization of the classical Bobylev-Steklov, Chaplygin and Goriatchev cases

It is difficult to foresee the explicit form of the part (16) of the potential with a given choice of the function $F$ in the integrand. It may also be impractical or even impossible to evaluate the integral in (16) for arbitrary $A$ and $C$ in closed form. In the classical Bobylev-Steklov the moments of inertia are subject to a single condition $A = 2C$. We now explore the form of the potential corresponding to certain simple forms of the function $F$ under this same condition, to obtain a generalization of the classical Bobylev-Steklov case. It turns out that the sequences of functions \{u^{n+\frac{1}{2}}\}, \{u^{-n}\} lead to rational potentials. As an example we take the first three terms of each, so that

$$F(u) = \sqrt{u}(A_1 + A_2 u + a_3 u^2) + \frac{B_1}{u} + \frac{B_2}{u^2} + \frac{B_3}{u^3}$$

(22)

gives the potential

$$V_1 = a_1 \gamma_1 + a_2 [8 \gamma_1 (\gamma_1^2 + \gamma_3^2) + \frac{\gamma_2^2}{\gamma_1}] + a_3 [16 \gamma_1 (8 \gamma_1^2 + 16 \gamma_1^2 \gamma_3^2 + 9 \gamma_3^4) + \frac{16 \gamma_3^6}{\gamma_1} + \frac{\gamma_3^8}{\gamma_1^2}]$$

$$+ b_1 \gamma_3 + b_2 \frac{\gamma_1^2 - \gamma_3^2}{\gamma_3^6} + b_3 \frac{2 \gamma_1^4 - 2 \gamma_1^2 \gamma_3^2 + \gamma_3^4}{\gamma_3^6}$$

(23)

$a_i, b_i$ being arbitrary parameters. If also we choose $G(\gamma_2) = C_1 + C_2 \gamma_2^2$ and use the geometric relation (41), we may write the full potential (15) as

$$V = V_1 + c_1 (\gamma_1^2 - \gamma_3^2) + c_2 \gamma_2^2 (2 \gamma_1^2 + \gamma_3^2)$$

(24)

The solution of the equations of motion corresponding to this potential may be expressed by inserting it into (9) and then substituting expressions (12), (13) for $\gamma_3, \gamma_2$. One can obtain $\gamma_1$ as a function of time by inverting the hyperelliptic integral

$$t = \lambda^2 \sqrt{\frac{C}{2}} \int \frac{dx}{\sqrt{g(x)}},$$

(25)

g(x) = \frac{1}{x}(1 - x^2 - \lambda x)$$

$$\times \{\lambda^4 [2c_2 x^5 - 2(4a_2 - c_2 \lambda)x^4 - 2(2a_2 \lambda + c_1 + c_2)x^3 - (a_1 - a_2 \lambda^2)x^2]$$

$$- 2(b_1 \lambda^2 + 3b_2)x + 2b_2 \lambda}\}$$

(26)
This integral becomes ultraelliptic when \( c_2 = 0 \) and elliptic if, moreover, \( a_2 = b_2 = 0 \). In the last case the whole solution may be written as follows:

\[
V = a_1 \gamma_1 + c_1 (\gamma_1^2 - \gamma_2^2) + \frac{b_1}{\gamma_3^3} \quad (27)
\]

\[
\gamma_3 = \sqrt{\nu \gamma_1}, \quad \gamma_2 = \sqrt{1 - \nu x - x^2}
\]

\[
p = -\sqrt{-\frac{1}{2C \nu \gamma_1}}(2b_1 + a_1 \nu^2 \gamma_1 + 2c_1 \nu^2 \gamma_1^2), \quad q = 0,
\]

\[
r = \frac{1}{\nu} \sqrt{-\frac{2}{2}} \frac{b_1 + a_1 \nu^2 \gamma_1 + 2c_1 \nu^2 \gamma_1^2}{C} \quad (28)
\]

\[
t = \sqrt{\frac{C}{2}} \int_{\gamma_1}^{\gamma_1} \frac{dx}{\sqrt{(x^2 + \nu x - 1)(2b_1 + \nu^2 a_1 x + 2\nu^2 c_1 x^2)}} \quad (29)
\]

Thus, the solution of the Euler-Poisson system \((1-2)\) with the three-terms potential \((27)\) can be expressed explicitly, for the body satisfying \( A = 2C \) and on the zero level of \( I_3 \), in terms of elliptic functions of time.

It should be noted here that the the solution \((27, 29)\) does not depend on the second moment of inertia \( B \), which remains arbitrary. If in this solution we add the condition \( B = A \), so that the the body has the Kovalevskaya configuration \( A = B = 2C \), we obtain an intersection with three well-known integrable problems of rigid body dynamics:

1. The full potential \((27)\) with three arbitrary parameters \( a_1, c_1, b_1 \) was pointed out first by Goriachev in \([10]\), but only under the condition \( A = B = 2C \).

2. When \( b_1 = 0 \), we get a problem of motion of a rigid body by inertia in an ideal incompressible fluid (see e.g. \([19]\)).

3. \( b_1 = c_1 = 0 \), Kovalevskaya’s case of motion about a fixed point in the uniform gravity field. This case is general integrable, i.e. on arbitrary level of \( I_3 \).

Formulas \((29)\) give the explicit solutions of the first two cases under the additional restriction \( q = 0 \) and the third under the two restrictions \( q = I_3 = 0 \), in terms of elliptic functions of time.
5 Some closed-form and polynomial potentials

Most problems of physical importance and all known completely integrable and solvable cases in rigid body dynamics are characterized exactly or approximately by potentials of simple forms that are mainly polynomial or algebraic in the Poisson variables $\gamma_1, \gamma_2, \gamma_3$. It is quite interesting to isolate possible solutions of polynomial or finite form of the linear PDE (10) satisfied by $V$ for arbitrary moments of inertia. The part $V_2$ of (15) is polynomial for any polynomial choice of $G(\gamma_2)$. The part $V_1$ as expressed by (16) depends only on $\gamma_1, \gamma_3$ and hence satisfies the following particular version of equation (10):

$$
\gamma_1\gamma_3(A \frac{\partial^2 V}{\partial \gamma_1^2} - C \frac{\partial^2 V}{\partial \gamma_3^2}) - (A\gamma_1^2 - C\gamma_3^2) \frac{\partial^2 V}{\partial \gamma_1 \partial \gamma_3} - (A - 2C) \frac{\partial V}{\partial \gamma_1} - (2A - C) \frac{\partial V}{\partial \gamma_3} = 0 \quad (30)
$$

We now assume a solution of this linear PDE as a combination of homogeneous terms of the two variables $\gamma_1, \gamma_3$. After some trials to normalize the singularities of the resulting differential equations we arrive at the following form

$$
V = \sum_{\nu} a_{\nu} \gamma_1^\nu g_{\nu}(v), \quad v = 1 + \frac{\gamma_2^2}{\gamma_1^2} \quad (31)
$$

where the summation, with arbitrary constant coefficients $a_{\nu}$, extends over the set of possible values of $\nu$ that will be determined later. This leads to the differential equations for each of the functions $g_{\nu}$

$$
v(1 - v) \frac{d^2 g_{\nu}}{dv^2} + [(\nu - 2 + \frac{\nu}{2(\alpha - 1)})v - \nu + 2] \frac{dg_{\nu}}{dv} + (\frac{\nu}{\alpha - 1} - 2)g_{\nu} = 0 \quad (32)
$$

in which $\alpha = \frac{A}{C}$. This is a hypergeometric equation whose solutions for generic values of the parameters are

$$
g_{\nu 1} = F(\frac{\nu}{2}, 1 - \frac{\alpha \nu}{2(\alpha - 1)}; \frac{\nu}{2} + 2; v), \quad (33)
g_{\nu 2} = v^{\nu/2 - 1} F(-1, -\frac{\nu}{2(\alpha - 1)}; \frac{\nu}{2}; v)
$$

$$
= v^{\nu/2 - 1} \frac{\alpha - 1 + v}{\alpha - 1} \quad (34)
$$

The second solution $g_{\nu 2}$ contributes to (31) a term

$$
A_{\nu}(\gamma_1^2 + \gamma_3^2)^{\nu/2 - 1}(\alpha \gamma_1^2 + \gamma_3^2) = A_{\nu}(1 - \gamma_2^2)^{\nu/2 - 1}(\alpha \gamma_1^2 + \gamma_3^2) \quad (35)
$$
which is of the type \([17]\), and can be better considered as included in that form. It thus remains to consider the decomposition of the potential part \(V_1\) in the form implied by \([15]\) and \([33]\). The typical term will now read

\[
V_{1\nu} = \gamma_1^\nu F\left(-\frac{\nu}{2}, 1 - \frac{\alpha \nu}{2(\alpha - 1)}, -\frac{\nu}{2} + 2; \nu\right) \tag{36}
\]

As we are looking for closed-form solutions we try to isolate cases when the hypergeometric series terminates, giving a polynomial expression. To that end, we note that the hypergeometric function satisfies the relation

\[
F\left(a, b; c; v\right) = (1 - v)^{-c + a + b} F\left(c - a, c - b, c; 1 + \frac{\alpha}{\nu}\right) \tag{37}
\]

There are 3 obvious possible types of such closed-form expressions, in which the hypergeometric series truncates:

1) \(\alpha\) is a negative integer, while \(c\) is not a negative integer greater than \(-n\). This gives only one non-trivial possibility \(\nu = 2\), which leads to potential of the type \([17]\) for a constant \(G(\gamma_2)\).

2) \(b\) is a negative integer \(-n\), while \(c\) is not one of the numbers \(\frac{n + 1 - k}{n - k}, k = 0, 1, \ldots, n\). This leads to the sequence \(\nu = -2(\alpha - 1)(n + 1), n = 0, 1, 2, \ldots\). The third index for this choice becomes \(\alpha + 1 + (\alpha - 1)n\), which is positive for all positive \(n\). This choice invokes an infinite sequence of terms in the potential. The typical term in this case has the form

\[
\frac{\gamma_1^{2n}}{\gamma_3} F\left(2, -n ; \alpha + 1 + (\alpha - 1)n; 1 + \frac{\gamma_3^2}{\gamma_1}\right) \tag{38}
\]

The numerator of this term is a homogeneous polynomial of degree \(2n\) in the two variables \(\gamma_1\) and \(\gamma_3\). The power \(2(\alpha + \alpha - 1)\) in the denominator is always positive. This sequence of potential terms is undefined at \(\gamma_3 = 0\) for all \(n\).

3) \(c - b\) is a negative integer \(-n\), while \(\alpha\) is not one of the \(n + 1\) rational numbers \(\frac{n + 1 - k}{n - k}, k = 0, 1, \ldots, n\). This leads to the sequence \(\nu = -2(\alpha - 1)(n + 1), n = 0, 1, 2, \ldots\) and to potentials of the type

\[
\frac{\gamma_1^{2(\alpha - 1 - n)}}{\alpha} \cdot \frac{\gamma_2^{2n}}{\gamma_1^{\alpha}} F\left(-n, -\frac{\alpha - 1}{\alpha}(n + 1); 1 - n + \frac{n + 1}{\alpha}; 1 + \frac{\gamma_3^2}{\gamma_1}\right) \tag{38}
\]
The product of the second and third factors in this expression is a polynomial of degree $2n$. The behaviour of the first term differs depends on the two numbers $\alpha$ and $n$.

The final form of the potential becomes

$$V = (A\gamma_1^2 + C\gamma_3^2) \sum_n A_n \gamma_2^n$$

$$+ \gamma_1^{2(\alpha-1)} \sum_n B_n \gamma_1^{2n(1-1/\alpha)} F(-n, -\frac{\alpha - 1}{\alpha}(n + 1); 1 - n + \frac{n + 1}{\alpha}; 1 + \frac{\gamma_2^2}{\gamma_1^2})$$

$$+ \frac{1}{\gamma_3^{2(\alpha-1)}} \sum_n C_n \gamma_3^{2n} \gamma_1^{-2n} F(2, -n; \alpha + 1 + (\alpha - 1)n; 1 + \frac{\gamma_2^2}{\gamma_1^2})$$

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