The type problem for Riemann surfaces via Fenchel–Nielsen parameters

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Abstract
A Riemann surface \( X \) is said to be of \textit{parabolic type} if it does not support a Green’s function. Equivalently, the geodesic flow on the unit tangent bundle of \( X \) (equipped with the hyperbolic metric) is ergodic. Given a Riemann surface \( X \) of arbitrary topological type and a hyperbolic pants decomposition of \( X \), we obtain sufficient conditions for parabolicity of \( X \) in terms of the Fenchel–Nielsen parameters of the decomposition. In particular, we initiate the study of the effect of twist parameters on parabolicity. A key ingredient in our work is the notion of \textit{nonstandard half-collar} about a hyperbolic geodesic. We show that the modulus of such a half-collar is much larger than the modulus of a standard half-collar as the hyperbolic length of the core geodesic tends to infinity. Moreover, the modulus of the annulus obtained by gluing two nonstandard half-collars depends on the twist parameter, unlike in the case of standard collars. Our results are sharp in many cases. For instance, for zero-twist flute surfaces as well as for half-twist flute surfaces with concave sequences of lengths our results provide a complete characterization of parabolicity in terms of the length parameters. It follows that parabolicity is equivalent to completeness in these cases. Applications to other topological types such as surfaces with infinite genus and one end (also known as the infinite Loch–Ness monster), the ladder surface, and abelian covers of compact surfaces are also studied.
1. INTRODUCTION AND RESULTS

1.1 The type problem

A fundamental question in the classification theory of Riemann surfaces, also known as the type problem, is whether a Riemann surface $X$ supports a Green’s function. A Riemann surface is said to be of parabolic type if it does not support a Green’s function (equivalently, Brownian motion on $X$ is recurrent). Classically the class of parabolic surfaces has been denoted by $O_G$ [6].

There are numerous known characterizations of parabolic surfaces coming from function theory, dynamics, and geometry. Specifically, if the Riemann surface $X$ is the quotient of the hyperbolic plane by a Fuchsian group, that is, $X = \mathbb{H} / \Gamma$ then $X$ is parabolic if and only if one of the following conditions holds, see, for example, [1, 6, 8, 14, 16, 36, 38, 45, 46]:

1. harmonic measure of the ideal boundary $\partial_\infty X$ vanishes;
2. geodesic flow on the unit tangent bundle of $X$ (equipped with the hyperbolic metric) is ergodic;
3. Poincaré series of $\Gamma$ diverges;
4. $\Gamma$ has the Mostow rigidity property;
5. $X$ has the Bowen property;
6. almost every geodesic ray is recurrent. Equivalently, the set of escaping geodesic rays from a point $p \in X$ has zero (visual) measure.

Various sufficient conditions for being of parabolic type in terms of explicit constructions were classically studied by Myrberg, Ahlfors, Nakai, Mori, Ohtsuka, Sario, Nevanlinna, and many others (for reference, see [6, 43]).

The main goal of the present work is to make transparent the relationship between the hyperbolic geometry of a Riemann surface and its type. Our main results give sufficient conditions on
the Fenchel–Nielsen parameters of a surface (length and twist parameters on a pants decomposition) to guarantee that it is of parabolic type, see Theorems 1.1 and 1.2. Some of the important aspects of these sufficient conditions are described next.

(1) **Twists.** For the first time in the literature we explicitly identify the effect of the twist parameters on parabolicity. For instance, we show that the intuitive heuristic ‘increasing twists preserves parabolicity’ holds in wide generality.

(2) **Sharpness.** Our sufficient conditions are often sharp. This allows us to obtain a characterization of parabolicity in geometric terms in many cases. For instance, we prove that $X$ is parabolic if and only if it is complete, provided $X$ is a zero-twist flute surface, or a half-twist flute surface with a concave sequence of lengths of the pants decomposition, see Theorems 1.5 and 1.7.

(3) **Generality.** We do not impose any restrictions on the topology of the Riemann surface and thus our results are valid in the general context.

The study of the relationship between the geometry of a Riemann surface and the type problem has a long history. Besides the works mentioned above, Nicholls [37] and Fernández-Rodríguez [17] obtained sufficient conditions for parabolicity in terms of the growth of the fundamental domains of the corresponding Fuchsian groups. However, a complete characterization of parabolicity in terms of the growth of the fundamental domain is impossible, see [37]. More recently, Matsuzaki–Rodríguez [31] considered the type problem for tight flute surfaces with uniformly distributed cusps. The use of twists/shears has also played a crucial role in the celebrated work of Kahn–Markovic [23, 24] on the surface subgroup and Ehrenpreis conjectures.

### 1.2 General results

Let $X$ be an infinite type Riemann surface, and $\{X_n\}$ an exhaustion of $X$ by finite area geodesic surfaces so that no boundary component of $X_n$ is a boundary component of $X_{n+1}$. All exhaustions in this paper are assumed to be of this type.

We denote by $\partial_0 X_n$ the collection of boundary components of $X_n$. Thus, the elements of $\partial_0 X_n$ are pairwise disjoint simple closed geodesics. By adding additional simple closed geodesics we complete $\partial_0 X_n$ to a pants decomposition of $X_n$. Hence, the Riemann surface $X$ endowed with a conformal hyperbolic metric can be viewed as infinitely many geodesic pairs of pants glued along their boundary geodesics. In this paper, we are not concerned with marked hyperbolic structures. Thus, the choice of geodesic pairs of pants is given by the lengths of the boundary geodesics, while the choice in the gluing is given by an angular parameter in the interval $(-\frac{1}{2}, \frac{1}{2})$, called the twist, see Sections 2 and 6. The lengths and twists, called the Fenchel–Nielsen parameters (relative to the pants decomposition), determine the conformal hyperbolic metric on $X$ (see Section 2 for details).

Let $P$ be a pair of pants in the pants decomposition as above that is contained in $X_{n+1} - X_n$. We will denote by $\alpha$ one of the pants curves of $P$, and by $\gamma$ a simple orthogeodesic in $P$ from $\alpha$ to one of the other pants curves on the boundary of $P$, see Figure 1.3. The twist along $\alpha$ will be denoted by $t(\alpha)$. We denote by $\ell(\gamma)$ the length of a geodesic $\gamma$ on a hyperbolic Riemann surface.

For each topological type, we give a sufficient condition for parabolicity.

**Theorem 1.1.** Let $X$ be an infinite type hyperbolic surface with an exhaustion $\{X_n\}$. Suppose there are constants $\alpha_0, \gamma_0 > 0$ such that for every pair of pants $P$, and curves $\alpha$ and $\gamma$ in $P$ as above we have
\( \ell(y) \geq \gamma_0 \) and \( \ell(\alpha) \geq \alpha_0 \). If

\[
\sum_{n=1}^{\infty} \frac{1}{\sum_{\alpha \in \partial_0 X_n} e^{\ell(\alpha)/2}} = \infty, \tag{1}
\]

then \( X \) is parabolic.

Theorem 1.1 is a consequence of a more general result, cf. Theorem 8.3, which is valid without the lower bound assumptions for \( \ell(y) \) and \( \ell(\alpha) \).

Theorem 1.1 is a twist-free result in the sense that it holds for any choice of twist parameters. We obtain a stronger result by bringing into the twist parameters into the sufficient condition for parabolicity. To do this, we make the mild technical assumption that the boundaries of \( X_{n+1} \) and \( X_n \) are not too close and the connected components of \( X_{n+1} - X_n \) are not too small (that is, not a pair of pants), see Theorem 8.5 for the precise hypotheses.

**Theorem 1.2.** Let \( X \) be an infinite type hyperbolic surface with exhaustion \( \{X_n\} \) as in Theorem 1.1 with the assumptions mentioned above. If

\[
\sum_{n=1}^{\infty} \frac{1}{\sum_{\alpha \in \partial_0 X_n} e^{(1-|t(\alpha)|) \ell(\alpha)/2}} = \infty, \tag{2}
\]

then \( X \) is of parabolic type (see Theorem 8.5 for the precise formulation).

In the above theorem, the twist parameter, \( t(\alpha) \), is measured with respect to a pants decomposition which includes the boundary components of the \( X_n \). Clearly, condition (1) implies (2). Therefore, if \( X \) satisfies (1) then not only \( X \) is parabolic but so are all the hyperbolic surfaces obtained by deforming \( X \) by twisting along the boundary curves of the exhaustion \( \{X_n\} \).

**Remark 1.3 (Increasing twists preserves parabolicity).** Since \( e^{(1-|t|)\ell/2} \) is decreasing in \( |t| \), Theorem 1.2 implies that if \( X \) satisfies (2) and \( X' \) is a surface obtained from \( X \) by increasing the absolute value of twist parameters \( t(\alpha) \) then \( X' \) is also parabolic.

### 1.3 Tight flute surfaces

Arguably, the simplest infinite type (hyperbolic) Riemann surface \( X \) is a tight flute surface, see [9]. It is obtained by starting with a geodesic pair of pants \( P_0 \) with two punctures and then consecutively gluing geodesic pairs of pants \( P_n, n \geq 1 \), with one puncture and two boundary geodesics in an infinite chain. Let \( \ell_n \) and \( t_n \) be the length and twist parameters of the closed geodesic \( \alpha_n \) on the boundary after gluing \( n \) pairs of pants. We denote the resulting surface by \( X = X(\{\ell_n, t_n\}) \), see Figure 1.1. It is relatively simple to see that if an infinite subsequence of \( \{\ell_n\} \) is bounded above by a positive constant then \( X \) is of parabolic type. When \( \ell_n \to \infty \), applying Theorem 1.2 we obtain the following.
Theorem 1.4. Let \( X = X(\{\ell_n, t_n\}) \) be a tight flute surface such that \( \ell_n \to \infty \). Then \( X \) is of parabolic type if
\[
\sum_{n=1}^{\infty} e^{-(1-|t_n|)\frac{\ell_n}{2}} = \infty.
\] (3)

If we set the twists \( t_n \) equal to zero in (3) we have that a flute surface \( X = X(\{\ell_n, 0\}) \) is parabolic if \( \sum e^{-\ell_n/2} = \infty \). It turns out that this condition is not only sufficient but also necessary. Moreover, we prove the following, see Theorem 9.4.

Theorem 1.5 (Parabolicity of zero-twist flutes). A zero-twist tight flute surface \( X = X(\{\ell_n, 0\}) \) is parabolic if and only if one of the following holds:

1. \( X \) is complete,
2. \( \sum_{n=1}^{\infty} e^{-\ell_n/2} = \infty \).

If \( t_n = 1/2 \) for all \( n \in \mathbb{N} \) then we obtain a half-twist tight flute \( X(\{\ell_n, 1/2\}) \). In this case, Equation (3) becomes \( \sum_n e^{-\ell_n/4} = \infty \). Unlike the zero-twist case we do not know if this condition is necessary and sufficient for parabolicity for an arbitrary sequence \( \{\ell_n\} \). However, we show that the condition is sharp in many cases. To do this, we first obtain a sufficient condition for non-completeness and hence non-parabolicity for half-twist tight flutes.

Theorem 1.6. A half-twist tight flute surface \( X(\{\ell_n, \frac{1}{2}\}) \) is incomplete if
\[
\sum_n e^{-\sigma_n/2} < \infty,
\] (4)

where \( \sigma_n = \ell_n - \ell_{n-1} + \cdots + (-1)^{n-1} \ell_1 \).
Using Theorem 1.6, we identify a class of half-twist tight flute surfaces for which we have a characterization of parabolicity.

We say that \( \{\ell_n\} \) is a concave sequence if there is a non-decreasing concave function \( f : [0, \infty) \to [0, \infty) \) such that \( \ell_n = f(n) \) for \( n \geq 0 \). Equivalently, \( \ell_n \) is concave if it is non-decreasing and for \( n \geq 1 \) the following holds:

\[
2\ell_n \geq \ell_{n+1} + \ell_{n-1}. \tag{5}
\]

For half-twist surfaces corresponding to concave sequences we show that \( e^{-\sigma_n/2} \asymp e^{-\ell_n/4} \). Theorems 1.4 and 1.6 then give the following characterization, see Theorem 9.7.

**Theorem 1.7** (Parabolicity of half-twist flutes). Let \( X = X(\{\ell_n, 1/2\}) \), where \( \{\ell_n\} \) is a concave sequence. Then \( X \) is parabolic if and only if one of the following conditions holds

1. \( X \) is complete,
2. \( \sum_n e^{-\ell_n/4} = \infty \).

Given Theorems 1.5 and 1.7, one may think that a tight flute surface is parabolic if and only if it is complete. This is not the case. Indeed, let \( X \) be obtained by taking out a sequence of points from the unit disk that converge to every point of the unit circle. Then \( X \) is the tight flute surface that is the union of countably many pairs of pants, complete and not of parabolic type (see also [20, 25]). It is not known if there are such examples among half-twist tight flute surfaces. In Section 9, we construct examples of half-twist tight flutes (necessarily with not concave \( \{\ell_n\} \)) for which Theorems 1.7 and 1.6 do not apply, see Example 9.9. A particular case of that is the following.

**Example 1.8.** Let \( X_s = X(\{s\ell_n, 1/2\}) \) be a tight flute surface, where for \( n \geq 1 \) we have

\[
\ell_{2n} = \ln(n + 1) + 2\ln n,
\]

\[
\ell_{2n+1} = 3\ln(n + 1).
\]

Applying the above results, we obtain that \( X_s \) is parabolic if \( s \in (0, 4/3] \), and \( X_s \) is incomplete if \( s > 2 \) (see Example 9.9 for the proofs). For \( s \in (4/3, 2] \), the results of this paper are inconclusive. It would be interesting to know whether \( X_s \) is complete and non-parabolic if \( s \in (4/3, 2] \).

Motivated by the discussion above, we ask the following.

**Question 1.9.** Suppose \( \ell_n \to \infty \).

1. Is \( X = X(\{\ell_n, 1/2\}) \) incomplete if and only if (4) holds?
2. Is \( X' = X(\{\ell_n', t_n'\}) \) parabolic if \( X = X(\{\ell_n, t_n\}) \) is parabolic and \( t_n < t_n', \forall n \geq 1 \)?
3. Given \( \{\ell_n\} \) is there a sequence of twists \( \{t_n\} \) such that \( X = X(\{\ell_n, t_n\}) \) is parabolic.

1.4 Applications to various surfaces and regular covers

Besides considering flute surfaces in this paper, we apply the sufficient conditions for parabolicity (for example, Theorems 1.1 and 1.2) to other topological types as well. Here we mention three such
examples: (1) the Loch–Ness monster (surface of infinite genus and one non-planar topological end) first studied in [41]; (2) the complement of the Cantor set (uncountably many ends); (3) topological abelian covers of compact surfaces.

### 1.4.1 Loch–Ness monster

Let $X_1^\infty$ be a hyperbolic Loch–Ness monster as in Figure 1.2. Suppose that the lengths of geodesics which cut off the genus, denoted by $\beta_n$, are uniformly bounded above. We show, see Theorem 10.1, that $X_1^\infty$ is of parabolic type if

$$\sum_{n=1}^{\infty} e^{-(1-t(\alpha_n))/2} \ell(\alpha_n) = \infty.$$  \hfill (5)

In the above theorem, the twist parameter, $t(\alpha_n)$, is measured relative to the end points in $\alpha_n$ of the orthogeodesic from $\beta_{n-1}$ to $\alpha_n$, and the orthogeodesic from $\alpha_n$ to $\beta_n$.

### 1.4.2 Complement of a Cantor set

Let $X_\infty$ be a genus zero surface whose space of topological ends is a Cantor set as in Figure 10.2. The surface $X_\infty$ is homeomorphic to the complement of a Cantor set on the Riemann sphere and has an exhaustion $\{X_n\}$, where $X_n$ is a genus zero surface with $2^n$ geodesic boundary curves for every $n \geq 1$, see the discussion before Theorem 10.3. As before we denote by $\partial_0 X_n$ the collection of boundary components of $X_n$.

It is well-known that if the lengths of the boundary geodesics of $X_n$’s are uniformly bounded from below then the surface $X_\infty$ is not of parabolic type, [32]. In the opposite direction, we show (see Theorem 10.3) that if there is a constant $C \geq 1$ such that for every $n \geq 1$ and all $\alpha \in \partial_0 X_n$ we have

$$\ell(\alpha) \leq Cn/2^n, \hfill (6)$$

then $X_\infty$ is of parabolic type.
It is an open problem whether $X_{\infty}$ can be parabolic if the lengths of the boundary geodesics decay slower than in (6) (for example, if there is a constant $k > 0$ such that $\ell(\alpha) \lesssim n^{-k}$ for $\alpha \in \partial_0 X_n$).

1.4.3 Abelian covers of closed surfaces

In [33] and [42], it was shown that a hyperbolic Riemann surface $X$ is of parabolic type if it is a $\mathbb{Z}$ or $\mathbb{Z}^2$ geometric cover $\pi: X \to Y$ over a closed Riemann surface. Our methods give an alternative proof of this result along with a generalization to hyperbolic Riemann surfaces $X$ which are topological covers of a closed Riemann surface, see Theorem 10.5. In fact, the hyperbolic structure on $X$ can be chosen, so that it is quasiconformally distinct from the hyperbolic structure on the geometric cover but in a suitable sense has Fenchel–Nielsen parameters that agree with the parameters of the regular cover for almost all pants curves. See Example 10.7 for details.

1.5 Tools of the trade: Extremal distance in nonstandard and standard collars

A key ingredient in the proofs of our results is the characterization of parabolicity in terms of the extremal distance. See Section 3 for the definition and properties of extremal distance. The method of extremal length (or the length–area principle) was initiated by Ahlfors in 1935 for the study of the type problem for simply connected Riemann surfaces, see [3]. He showed that a simply connected Riemann surface $X$ is parabolic if and only if there is a conformal metric $\rho(z)|dz|$ on $X$ and $r_0 > 0$ such that

$$\int_{r_0}^{\infty} \frac{dr}{L(r)} = \infty,$$

where $L(r)$ is the $\rho$-length of the circle of radius $r$ centered at some point $z_0 \in X$. Ahlfors’ criterion (7) was generalized and reformulated by several authors and was later often referred to as the modular test.

Let $\{X_n\}$ be an exhaustion of $X$ by a family of relatively compact regions with piecewise analytic boundary such that $\overline{X}_n \subset X_{n+1}$. Denote by $\beta_n$ the boundary of $X_n$. Let $\lambda_{X_n-X_1}(\beta_1, \beta_n)$ be the extremal length of the family of curves contained in $X_n - X_1$ which connect $\beta_1$ and $\beta_n$. The following characterization of parabolicity is due to Nevanlinna [36], see also [44, p. 328].

Modular test. The Riemann surface $X$ is parabolic if and only if

$$\lambda_{X_n-X_1}(\beta_1, \beta_n) \to \infty \quad \text{as} \quad n \to \infty. \quad (8)$$

Informally, $X$ is parabolic if the extremal distance between any compact subset of $X$ and its ideal boundary is infinite, that is, $\partial_\infty X$ cannot be reached in finite time. Equivalently, $X$ is parabolic if and only if the capacity of $\partial_\infty X$ vanishes, see [43].

Since $\lambda_{X_n-X_1}(\beta_1, \beta_n)$ is difficult to compute or estimate, one usually uses condition (8) in conjunction with the so-called serial rule for the extremal length. For that we suppose that the connected components $\beta_{k,j}$ of $\beta_k$ are contained in pairwise disjoint collars (topological annuli), denoted by $A_{k,j}$. Let $\lambda_{k,j}$ be the extremal length of the path family in $A_{k,j}$ connecting its boundary
components, and denote $\lambda_k = \sum_j \lambda_{k,j}$. From the serial rule and the fact that $A_{k,j}$’s are disjoint, it follows that $\lambda_{X_{n-1}} (\beta_1, \beta_n) \geq \sum_{k=1}^{n-1} \lambda_k$. Therefore, by (8) $X$ is parabolic, provided

$$\sum_{k=1}^{\infty} \lambda_k = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \lambda_{k,j} = \infty,$$

where $|\partial_0(X_n)|$ denotes the number of boundary components of $X_n$.

Thus, if $X$ is a Riemann surface with an exhaustion $\{X_n\}$, where the boundary components of $X_n$ are geodesics then we would like to construct disjoint collars around these boundary geodesics and calculate or estimate the extremal distance between their boundaries.

The well-known collar lemma (see [15]) tells us that a simple closed geodesic of length $\ell$ on a hyperbolic Riemann surface is guaranteed to have a collar (annular neighborhood) of width $\text{arcsinh}(\frac{1}{\sinh \frac{\ell}{2}})$, see Figure 1.3 for the picture of a half-collar. We call this a standard collar. The important point is that the width only depends on the length of the geodesic and not the ambient hyperbolic structure of the surface. The extremal distance between the boundary components of the one-sided standard collar up to a constant multiple is bounded below by $e^{-\ell/2}$ (see [29] and Lemma 5.5). This is a good asymptotic estimate when $\ell \to 0$ (or in the thin parts of the surface) but not for large $\ell$.

To deal with large $\ell$ (or thick parts of the surface), we introduce what we call a nonstandard half-collar about a geodesic in a pair of pants. This half-collar will depend on local data of the pair of pants as opposed to the standard half-collar which depends only on the length of the closed geodesic. Most of the sufficient conditions for parabolicity we obtain follow from the extremal distance bounds on the nonstandard half-collars and collars, which we describe next.

Let $\alpha, \alpha_1,$ and $\alpha_2$ be the boundary geodesics (we allow $\alpha_1$ or $\alpha_2$ to be a puncture) of a pair of pants $P$, and $\gamma$ the unique simple orthogeodesic between $\alpha$ and $\alpha_1$ (see Figure 1.3). Letting $b$ be the end point of $\gamma$ on $\alpha_1$, there exist exactly two shortest geodesic segments from $b$ to the simple orthogeodesic from $\alpha$ to $\alpha_2$. These segments have equal length and the union of the two connect to make a geodesic loop $\hat{\beta}$ with non-smooth point $b$. The nonstandard half-collar around $\alpha$ is the region in $P$ between $\alpha$ and the geodesic loop $\hat{\beta}$. It is topologically an annulus which we denote by $R_{\alpha, \gamma}$ (see Figure 1.3). A more general type of collar has been considered by Parlier in a different context [39, 40].
FIGURE 1.4  Pink and blue regions above are the standard and nonstandard half-collars about the hyperbolic geodesic $\alpha$, respectively. When the length of the orthogeodesic $\gamma$ from $\alpha$ to another boundary component of the pants is infinite (on the right) then the standard collar is completely contained in the nonstandard collar. As $\ell = \ell(\alpha) \to \infty$, the modulus of the standard collar is comparable to $e^{-\ell/2}/\ell'$, while for the nonstandard collar the lower bound is of the order $e^{-\ell/2}$

To simplify the notation, we say that two positive quantities $a$ and $b$ satisfy $a \gtrsim b$ if $a/b$ is greater than or equal to a positive constant; $a \lesssim b$ if $a/b$ is less than or equal to a positive constant; and $a \asymp b$ if $a/b$ is between two positive constants or equivalently, if $a \gtrsim b$ and $a \lesssim b$.

Neither type of half-collar (standard or nonstandard) contains the other except when $\alpha_1$ is a puncture (that is, $\ell(\gamma) = \infty$) in which case the nonstandard half-collar contains the standard half-collar (see Figure 1.4). Nevertheless, for $\ell' = \ell'(\alpha)$ large, the nonstandard half-collar produces a larger extremal distance between the boundary components than the standard half-collar, see Theorem 5.6 and Corollary 5.10. For example, when $\ell(\gamma) \geq \gamma_0 > 0$, we have the following asymptotic behavior for $\lambda(R,\gamma)$ as $\ell(\alpha) \to \infty$:

$$\lambda(R,\gamma) \gtrsim e^{-\ell(\alpha)/2}, \quad (9)$$

while the extremal distance between the boundary components of the standard half-collar, as noted above, is comparable to $e^{-\ell(\alpha)/2}/\ell(\alpha)$.

When two standard half-collars $R$ and $R'$ around geodesics of the same length are glued by an isometry along the geodesics, the obtained surface $\hat{R}$ is invariant under the isometric reflection in the geodesic regardless of the twist. Consequently, the extremal distance between the boundary curves of the collar $\hat{R}$ equals the sum of the extremal distances between the boundary components of the two half-collars, that is,

$$\lambda(\hat{R}) = \lambda(R) + \lambda(R') \asymp e^{-\ell'/2}/\ell'.$$  

(10)

On the other hand, when two nonstandard half-collars are glued, see Figure 1.5, the extremal distance between the boundary components of the glued surface may significantly increase.
Let $R'_{\alpha,\gamma,\gamma'}$ be the nonstandard collar (as above) obtained by gluing nonstandard half-collars $R_{\alpha,\gamma}$ and $R_{\alpha,\gamma'}$ along the common geodesic $\alpha$ of length $\ell$ with a twist $t$. We write $R'_{\alpha}$ if $\ell(\gamma) = \ell(\gamma') = \infty$. Denoting by $\hat{R}_{\alpha}$ the standard collar around $\alpha$, we have $\lambda(R'_{\alpha}) \geq \ell(e^{\ell/2}\lambda(\hat{R}_{\alpha}))$ as $\ell \to \infty$, see (12).

depending on the twist of the gluing (Theorem 6.1). For instance, we show that if $R'_{\alpha,\gamma,\gamma'}$ is the annular region obtained by gluing two nonstandard half-collars $R_{\alpha,\gamma}$ and $R_{\alpha,\gamma'}$ with twist $t \in (-\frac{1}{2},\frac{1}{2}]$ and $\ell(\gamma) = \ell(\gamma') = \infty$ then

$$\lambda(R'_{\alpha,\gamma,\gamma'}) \geq e^{-(1-|t|)\frac{\ell(\alpha)}{2}},$$

provided $\ell(\alpha) \geq \ell_0 \geq 2$. Therefore, comparing the nonstandard and standard two-sided collars we have from (10) and (11) the following asymptotic inequality, as $\ell(\alpha) \to \infty$,

$$\frac{\lambda(R'_{\alpha,\gamma,\gamma'})}{\lambda(\hat{R})} \geq e^{\ell(\alpha)e^{1/2}},$$

In particular, as $\ell(\alpha) \to \infty$ the ratio between the nonstandard and standard collars grows linearly if $t = 0$ and exponentially as soon as there is a non-trivial twist $t \neq 0$ involved.

The extremal distance estimates for nonstandard collars follow from our technical tool on general collars about a simple closed geodesic, see Corollary 5.3. We show that the extremal distance between the boundary components of a general collar is comparable to the extremal length of the curve family of geodesic orthorays based at the core curve $\alpha$, see Section 3 for the definition of the extremal length of curve families. To achieve this, we use logarithmic coordinates and express the universal cover of the collar as a region bounded by two graphs in the plane. Then the extremal distance between the boundary components of the collar in the Riemann surface is related to the extremal length of curves connecting the top graph to the bottom graph in the universal cover. These curve families degenerate as the length of the core curve $\ell(\alpha) \to \infty$ and our key result is an estimate of the extremal length of such degenerating families of curves. A similar approach was also used in the context of the Teichmüller theory, see [21, 26]. However, the degeneration of the families in our setting are much more involved and the estimates do not follow from any previous work.
1.6 Outline of the paper

The rest of this paper is organized as follows. In Section 2, we introduce geodesic pairs of pants, the Fenchel–Nielsen parameters, and the construction of infinite type Riemann surfaces from the geodesic pairs of pants. In Section 3, we recall the definition and basic properties of modulus of curve families. Sections 4–6 are the technical core of the paper. In Section 4, we obtain estimates for the moduli of degenerating curve families connecting two graphs of real functions over a compact interval. In Sections 5 and 6, we apply the results of Section 4 and prove the main modulus bounds for the collars around geodesics, in particular here we prove the estimates (9) and (11). In Section 7, we recall Nevanlinna’s modular test of parabolicity (8) and prove a slight generalization which is used in our applications. In Section 8, we combine the previous results obtaining our most general sufficient conditions on the Fenchel–Nielsen parameters which guarantee parabolicity of an arbitrary infinite type Riemann surface. In particular, Theorems 1.1 and 1.2 follow from the results in Section 8. In Section 9, we consider the applications to flute surfaces which imply Theorems 1.4–1.7. Section 10 describes sufficient conditions for parabolicity for various topological types of infinite type surfaces including the infinite Loch–Ness monster, surfaces with finitely many ends, surfaces with a Cantor set of ends, and topological $\mathbb{Z}^r$ covers of compact surfaces. In particular, we recover the results of Mori and Reeson on conformal $\mathbb{Z}$ and $\mathbb{Z}^2$ covers of compact surfaces.

We conclude the introduction by listing some of the notation used in the text together with the sections where the corresponding quantities are defined.

| Definition | Section | Notation |
|------------|---------|----------|
| Length and twist parameters | 2 | $l(\alpha), t(\alpha)$ |
| Modulus and extremal length | 3 | $\text{mod} \Gamma$ |
| Extremal distance | 3 and 7 | $\lambda(R), \lambda_{X_n-X_1}(\beta_1, \beta_n)$ |
| Simply degenerating families of functions | 4.1 | $\{ (f_\ell, g_\ell) \}_{\ell \geq \ell_0}$ |
| Standard half-collar | 5 | $R_{\ell}$ |
| nonstandard half-collar | 5 | $R_{\ell, y}$ |
| nonstandard collar | 6 | $R_{x, y, \gamma, \gamma'}^{(\alpha)}$ |
| Geodesic subsurface | 7 | $X_\alpha$ |
| Boundary components of $X$ | 7 | $\delta_0 X$ |
| Tight flute surface | 9 | $X(\{ l_n, t_n \})$ |

2 Riemann Surfaces of Infinite Topological Type

Every Riemann surface $X$ in this paper is assumed to admit a hyperbolic metric, that is a conformal metric of constant curvature equal to $-1$. Thus, $X$ is not conformal to the Riemann sphere $\hat{\mathbb{C}}$, the complex plane $\mathbb{C}$, the punctured complex plane $\mathbb{C} \setminus \{0\}$ or the torus. See [15] for background on hyperbolic geometry.

We will interchangeably use the terms Riemann surface and hyperbolic surface for the same object. A Riemann surface $X$ is of infinite topological type if its fundamental group $\pi_1(X)$ is infinitely generated.

A geodesic pair of pants is a complete hyperbolic surface (homeomorphic to a sphere minus three disks) whose boundary components are either closed geodesics or punctures with at least
one boundary a closed geodesic. A *tight* pair of pants is a geodesic pair of pants that has at least one puncture. In Figure 1.4, we illustrated two geodesic pairs of pants, on the left with three boundary geodesics and on the right with two boundary geodesics and a puncture. The geodesic pair of pants on the right is a tight pair of pants.

Consider a geodesic pair of pants $P$ which is not tight and fix a boundary geodesic $\alpha$ of $P$. Let $\alpha_1$ be another closed geodesic on the boundary of $P$. Let $\gamma$ be the orthogeodesic from $\alpha$ to $\alpha_1$. The foot $x \in \alpha$ of $\gamma$ on $\alpha$ is called a marked point. If $P$ is a tight pair of pants with one boundary puncture, we choose $\gamma$ to be the simple orthogeodesic in $P$ from $\alpha$ to the puncture. If $P$ has two punctures then we choose one puncture and repeat the construction above.

Let $P'$ be another geodesic pair of pants with boundary geodesic $\alpha'$. Assume $\ell(\alpha) = \ell(\alpha')$. We identify $\alpha$ and $\alpha'$ by an isometry to obtain a bordered hyperbolic surface from the two pairs of pants. The isometric identification $\alpha \equiv \alpha'$ is determined by the relative position of the marked points $x \in \alpha$ and $x' \in \alpha'$ which is recorded by the twist parameter $t(\alpha) \in (-\frac{1}{2}, \frac{1}{2})$. Namely, if $x = x'$, then $t(\alpha) = 0$. If $x \neq x'$, then $\alpha - \{x, x'\}$ consists of two arcs and $|t(\alpha)|$ is the length of the shorter arc divided by $\ell(\alpha)$. If $|t(\alpha)| = \frac{1}{2}$, then we have $t(\alpha) = \frac{1}{2}$. If $|t(\alpha)| < \frac{1}{2}$, then we orient $\alpha$ as a part of the boundary of $P$. If the shorter of the two arcs of $\alpha - \{x, x'\}$ is $(x, x')$ for the orientation of $\alpha$ then $t(\alpha) = |t(\alpha)|$; otherwise $t(\alpha) = -|t(\alpha)|$.

By glueing countably many geodesic pairs of pants in this manner, we obtain a not necessarily complete surface $X$ with hyperbolic metric induced by the hyperbolic metric on the geodesic pairs of pants. The choices in the gluings are given by the twist parameters and the geodesic pairs of pants are uniquely determined by the lengths of the boundary geodesics called the length parameters. When the boundary geodesic is a puncture then by convention the length is zero. Therefore, the hyperbolic metric on $X$ is uniquely determined by the length and twist parameters on the boundary geodesics of the pairs of pants called the Fenchel–Nielsen parameters. Since we do not consider the space of Riemann surfaces to have a base point surface and need not consider marked Riemann surfaces, we are content to use the twist parameters in $(-\frac{1}{2}, \frac{1}{2})$ in order to describe all hyperbolic metrics.

Finally, the surface $X$ obtained by gluing countably many geodesic pairs of pants might not be complete in the induced hyperbolic metric. The boundary of the metric completion of $X$ consists of simple closed geodesics and bi-infinite simple geodesics (see [7, 9, 11]). By attaching funnels to the closed geodesics and attaching geodesic half-planes to the bi-infinite geodesics of the boundary of the metric completion of $X$, we obtain a hyperbolic surface $\hat{X}$ homeomorphic to $X$ with a geodesically complete hyperbolic metric such that the inclusion $X \to \hat{X}$ is an isometric embedding. Any infinite type hyperbolic surface can be obtained as the above by gluing of countably many geodesic pairs of pants and by attaching funnels and half-planes (see [7, 11]). The hyperbolic surface structure is completely determined by the length and twist parameters called Fenchel–Nielsen parameters.

We are mainly interested in determining whether a hyperbolic surface is or is not of parabolic type. The geodesic flow on the unit tangent bundle of a hyperbolic surface with a funnel preserves two disjoint open subsets and hence cannot be ergodic. Therefore, a hyperbolic surface with a funnel supports a Green’s function and thus it is not of parabolic type. In our constructive approach to hyperbolic surfaces, a funnel appears only if a boundary geodesic of a pair of pants is not glued to another boundary geodesic. For this reason, we always assume that a boundary component of a pair of pants which is not glued to another boundary component is a puncture. Thus, we are not considering surfaces with funnels because they are known to not be of parabolic type. Under our assumption, a hyperbolic surface obtained by gluing countably many geodesic pairs of pants could still be incomplete due to a possible accumulations of boundary geodesics of the pairs of pants [9].
However, determining for which Fenchel–Nielsen parameters precisely $X$ is incomplete appears to be a difficult problem.

## 3 MODULUS OF A CURVE FAMILY

Let $X$ be an arbitrary Riemann surface which supports a conformal hyperbolic metric. Denote by $\Gamma$ a family of curves in $X$ that are locally rectifiable in the charts. A metric $\rho$ on $X$ is an assignment in each local chart $z = x + iy$ of a metric $\rho(z)|dz|$ invariant under transition maps. We require that $\rho$ is non-negative and Borel measurable.

A metric $\rho$ on $X$ is allowable for $\Gamma$ if the $\rho$-length satisfies

$$\ell_\rho(\gamma) = \int_\gamma \rho(z)|dz| \geq 1$$

for each $\gamma \in \Gamma$. If a curve $\gamma$ is not rectifiable then we set $\ell_\rho(\gamma) = \infty$.

**Definition 3.1.** The modulus $\text{mod}\Gamma$ of the family $\Gamma$ is defined by

$$\text{mod}\Gamma = \inf_\rho \iint_X \rho^2(z)dx\,dy,$$

where the infimum is over all allowable metrics $\rho$ for $\Gamma$ (see [5, 18]).

The extremal length $\lambda(\Gamma)$ of the curve family $\Gamma$ is defined by

$$\lambda(\Gamma) = \frac{1}{\text{mod}\Gamma}.$$

It is clear that any information about the modulus gives equivalent information about the extremal distance. We slightly favor the modulus for the simplicity of the subadditivity formula (compare inequality (14) and Lemma 3.3, Property 2). Additionally, the definition of an allowable metric, as being a metric where all curves have length at least one, makes geometric arguments more streamlined.

The modulus and the extremal length of a family of curves is invariant under conformal mappings and quasi-invariant under quasiconformal mappings (for example, see [5]).

Let $R = \{z : r_1 < |z| < r_2\}$ be an annulus with inner radius $r_1 \geq 0$ and outer radius $r_2 \leq \infty$. Let $\Gamma$ be the family of all curves in $R$ with one end point on $|z| = r_1$ and the other on $|z| = r_2$. It is well-known that the modulus of $\Gamma$ is (see [27])

$$\text{mod}\Gamma = \frac{1}{\lambda(\Gamma)} = \frac{2\pi}{\log \frac{r_2}{r_1}}.
$$

Consider a radial segment $\{z = re^{i\theta} : r_1 < r < r_2\}$ and a conformal map $z \mapsto \log z$ defined on $R - \{z = re^{i\theta} : r_1 < r < r_2\}$. The image of $R - \{z = re^{i\theta} : r_1 < r < r_2\}$ under $z \mapsto \log z$ is the rectangle $Q = \{z = x + iy : \log r_1 < x < \log r_2, \theta < y < \theta + 2\pi\}$. Let $\Gamma_Q$ be the family of all curves connecting the left and right sides of $Q$. A direct computation shows that (see [27])

$$\text{mod}\Gamma = \text{mod}\Gamma_Q. \quad (13)$$
Our goal is to recognize the conditions under which the equality (13) holds in a general doubly connected region $R$. Any doubly connected region $R$ on a Riemann surface is conformally equivalent to an annulus $\{r_1 < |z| < r_2\}$ in the complex plane $\mathbb{C}$. Let $\Gamma$ be the family of curves connecting one components $\partial_1 R$ to the other component $\partial_2 R$ of the boundary of $R$. Since the modulus of a family of curves is a conformal invariant, it follows that $\text{mod} \Gamma = \frac{2\pi}{\log r_2/r_1}$.

Fix a Jordan arc $\tau$ connecting two boundary components of $R$ with end points $z_i \in \partial_i R$ for $i = 1, 2$. Let $\Gamma_{\tau}$ be the family of curves in $R - \tau$ connecting $\partial_1 R$ to $\partial_2 R$. Then $\text{mod} \Gamma \geq \text{mod} \Gamma_{\tau}$ and the strict inequality is possible. However, we observe the following.

**Lemma 3.2.** Let $R$ be a doubly connected domain and $\tau$ a Jordan arc connecting the two boundary components of $R$. If there exists an anticonformal map $c : R \to R$ which pointwise fixes $\tau$ then $\text{mod} \Gamma = \text{mod} \Gamma_{\tau}$.

**Proof.** Upon conformally mapping $R$ onto an annulus, the image of $\tau$ is pointwise fixed by an anticonformal map of the annulus. Thus, the image of $\tau$ is a radial segment and we obtain $\text{mod} \Gamma = \text{mod} \Gamma_{\tau}$. \hfill $\square$

Next, we list some important properties of the modulus, which will be used repeatedly throughout the paper, see [27] for the proofs of these results.

**Lemma 3.3.** Let $\Gamma_1, \Gamma_2, \ldots$ be curve families in $X$.

1. **Monotonicity:** If $\Gamma_1 \subset \Gamma_2$, then $\text{mod} (\Gamma_1) \leq \text{mod} (\Gamma_2)$.
2. **Subadditivity:** $\text{mod} (\bigcup_{i=1}^{\infty} \Gamma_i) \leq \sum_{i=1}^{\infty} \text{mod} (\Gamma_i)$.
3. **Overflowing:** If $\Gamma_1 < \Gamma_2$, then $\text{mod} (\Gamma_1) \geq \text{mod} (\Gamma_2)$.

The notation $\Gamma_1 < \Gamma_2$ above denotes the fact that for every curve $\gamma_2 \in \Gamma_2$ there is a curve $\gamma_1 \in \Gamma_1$ such that $\gamma_1 \subset \gamma_2$. If this is the case we say $\Gamma_1$ minorizes $\Gamma_2$.

The subadditivity property for the extremal length is given by

$$\lambda \left( \bigcup_{i=1}^{\infty} \Gamma_i \right) \geq \frac{1}{\sum_{i=1}^{\infty} \frac{1}{\lambda(\Gamma_i)}}. \quad (14)$$

When the curve families $\Gamma_1, \Gamma_2$, and so on have disjoint supports (that is, are contained in disjoint domains) the inequality in the subadditivity property turns into equality.

**Lemma 3.4.** Let $\Gamma_n$, for $n = 1, 2, \ldots$, be at most a countable set of families of curves such that the support of any two families are disjoint. If $\Gamma = \bigcup_{n} \Gamma_n$, then

$$\text{mod} \Gamma = \sum_{n} \text{mod} \Gamma_n.$$ 

The following property is most conveniently expressed in terms of extremal length (see [19, p. 135, section IV.3]).

**Lemma 3.5 (Serial rule).** Assume that $\{R_n\}_{n=1}^{\infty}$ are mutually disjoint doubly connected domains separating boundaries of a doubly connected domain $R$. Let $\Gamma$ be the curve family connecting the two
boundary components of $R$ and let $\Gamma_n$ be the curve family that connects the two boundary components of $R_n$. Then

$$\lambda(\Gamma) \geq \sum_n \lambda(\Gamma_n).$$

An allowable metric $\rho_1$ for a family of Jordan curves $\Gamma$ is extremal if

$$\text{mod} \Gamma = \int_X \rho(z)^2 dx dy.$$

The following sufficient condition for extremality of a metric is known as Beurling’s criterion [4].

**Lemma 3.6** (Beurling’s criterion). The metric $\rho_1$ is extremal for $\Gamma$ if there is a subfamily $\Gamma_1 \subset \Gamma$ such that

1. $\int_\gamma \rho_1|dz| = 1$, for all $\gamma \in \Gamma_1$;
2. for any real valued $h$ on $X$ satisfying $\int_\gamma h|dz| \geq 0$, for all $\gamma \in \Gamma_1$ the following holds

$$\iint_X h \rho_1 dx dy \geq 0.$$

Beurling’s criterion can be applied to a family of curves consisting of vertical segments in the complex plane to find an explicit expression for the modulus of this family (note the similarity to Ahlfors’ integral (7)).

**Lemma 3.7** see [21, Lemma 4.1]. Given a measurable set $E \subset \mathbb{R}$, let $\Gamma = \{\gamma_x\}_{x \in E}$ be a family of curves such that $\gamma_x$ is contained in a vertical line through $x$. Then

$$\text{mod} \Gamma = \int_E \frac{dx}{\ell_E(x)},$$

where $\ell_E(x)$ is the Euclidean length of $\gamma_x$.

We will also need to use the notion of extremal distance between two boundary components of an annulus.

**Definition 3.8.** Let $R$ be a doubly connected region in a Riemann surface $X$. The extremal distance between boundary components of $R$ is

$$\lambda(R) := \lambda(\Gamma_R) = \frac{1}{\text{mod} \Gamma_R},$$

where $\Gamma_R$ is the curve family in $R$ connecting the boundary components.

## 4 Modulus of Curve Families between Graphs

Let $x_1 < x_2$ and $f, g : \mathbb{R} \to \mathbb{R}$ be two continuous periodic functions with period $x_2 - x_1$. We estimate the modulus of curves connecting the graph of $f$ to the graph of $g$. For simplicity, we assume
that \( f(x) > g(x) \) for all \( x \in \mathbb{R} \). Define \( \Pi(x + iy) = x \), and let \( R \) be the region bounded by \( f \) and \( g \).

Let \( \Gamma \) be the family of curves in \( R \) connecting the graphs of \( f \) and \( g \) such that \( \Pi(\gamma(0)) \in [x_1, x_2] \) for every \( \gamma \in \Gamma \). Let \( 0 < \delta < x_2 - x_1 \) be fixed and denote by \( \Gamma^{\leq \delta} \) the family of all \( \gamma : [0, 1] \rightarrow R \) in \( \Gamma \) such that \( \{\Pi(\gamma(t)); t \in [0, 1]\} \) is an interval of length at most \( \delta \). Let \( \Gamma^{> \delta} \) be the family of all \( \gamma \in \Gamma \) such that \( \{\Pi(\gamma(t)); t \in [0, 1]\} \) is an interval of length greater than \( \delta \). Then by subadditivity (see Lemma 3.3, Property 2), we have

\[
\text{mod} \Gamma \leq \text{mod} \Gamma^{\leq \delta} + \text{mod} \Gamma^{> \delta}.
\]

We first observe that \( \text{mod} \Gamma^{> \delta} \) can be easily estimated in terms of \( \delta \) and is bounded even if the curves in \( \Gamma \) degenerate.

**Lemma 4.1.** Under the above assumptions,

\[
\text{mod} \Gamma^{> \delta} \leq A/\delta^2,
\]

where \( A \) is the Euclidean area between the graphs and above \( [x_1 - \delta, x_2 + \delta] \).

**Proof.** Let \( D_\delta \) be the region bounded by the graphs of \( f \) and \( g \) such that \( \Pi(z) \in [x_1 - \delta, x_2 + \delta] \). Let \( \rho(z) = 1/\delta \) for all \( z \in D_\delta \), and set \( \rho(z) = 0 \) for \( z \notin D_\delta \). Then \( \rho \) is an allowable metric for \( \Gamma^{> \delta} \) and the lemma follows. \( \square \)

We next estimate \( \text{mod} \Gamma^{\leq \delta} \). In [21], an estimate for \( \text{mod} \Gamma^{\leq \delta} \) is given when the degeneration of the domain is done by vertical shrinking. We need an estimate for more general degeneration of the domain where not only the vertical direction is shrinking but also the shape of \( f(x) \) and \( g(x) \) is changing in the process.

Note that each \( \gamma \in \Gamma^{\leq \delta} \) lies inside the region \( D_\delta \), used in the proof of Lemma 4.1. Define

\[
m_\delta(x) = m(x) := \min_{a,b \in [x-\delta,x+\delta]} [f(a) - g(b)]
\]

for \( x \in [x_1, x_2] \). Equivalently, \( m_\delta(x) = \min_{a \in [x-\delta,x+\delta]} f(a) - \max_{b \in [x-\delta,x+\delta]} g(b) \). The quantity \( m_\delta(x) \) is the height of the tallest rectangle between the graphs of \( f \) and \( g \) whose vertical sides are contained in \( x - \delta \) and \( x + \delta \). For fixed \( \delta > 0, m_\delta(x) \) is a continuous function.

**Lemma 4.2.** The metric \( \rho \) defined by \( \rho(z) = 1/m(\Pi(z)) \) for all \( z \) between the graphs of \( f(x) \) and \( g(x) \) with \( \Pi(z) \in [x_1 - \delta, x_2 + \delta] \), and by \( \rho(z) = 0 \) elsewhere is allowable for the curve family \( \Gamma^{\leq \delta} \).

**Proof.** Let \( \gamma \in \Gamma^{\leq \delta} \). Fix \( z = \gamma(t) \) for some \( t \in [0, 1] \) and denote by \( I_\delta = [\Pi(z) - \delta, \Pi(z) + \delta] \) the closed interval centered at \( \Pi(z) \). Then \( \gamma \) connects the top and bottom of the rectangle \( I_\delta \times [\max_{b \in I_\delta} g(b), \min_{a \in I_\delta} f(a)] \). Thus, the Euclidean length of \( \gamma \) is at least \( m(\Pi(z)) \). \( \square \)

We use the above lemma to find an effective estimate for \( \text{mod} \Gamma^{\leq \delta} \). It turns out that the estimate is, up to a positive multiplicative constant, equal to the modulus of the vertical arcs connecting the two graphs.
For each $\delta > 0$ and each pair $(f, g)$, we set

$$c_\delta := \inf_x \frac{m_\delta(x)}{|f(x) - g(x)|}. \quad (15)$$

Since $\frac{m_\delta(x)}{|f(x) - g(x)|}$ is continuous on $[x_1, x_2]$, it is easy to see that $0 < c_\delta \leq 1$. Moreover, $c_\delta \to 1$ as $\delta \to 0$. Geometrically, $\frac{m_\delta(x)}{|f(x) - g(x)|}$ measures how far the region above $[x - \delta, x + \delta]$ and between the graphs of $f$ and $g$ is from being a rectangle. Thus, $c_\delta$ is the largest deviation of an inscribed rectangle of width $2\delta$ with sides parallel to the coordinate axes is from having height $(f(x) - g(x))$ for any $x \in \mathbb{R}$. We call $c_\delta$ the $\delta$-rectangle deviation between $f$ and $g$.

**Lemma 4.3.** For any $\delta > 0$,

$$\int_{x_1}^{x_2} \frac{1}{f(x) - g(x)} \, dx \leq \text{mod } \Gamma \leq \int_{x_1}^{x_2+\delta} \frac{1}{m_\delta(x)} \, dx \leq \int_{x_1-\delta}^{x_1} \frac{1}{f(x) - g(x)} \, dx. \quad (16)$$

*Proof.* Recall that $\rho(z) = 1/m_\delta(\Pi(z))$ is allowable and we have

$$\text{mod } \Gamma_{\leq \delta} \leq \int_{x_1-\delta}^{x_1} \int_{g(x)}^{f(x)} \frac{1}{m_\delta(x)^2} \, dy \, dx = \int_{x_1-\delta}^{x_1} \frac{f(x) - g(x)}{m_\delta(x)^2} \, dx \leq \frac{1}{c_\delta^2} \int_{x_1-\delta}^{x_1} \frac{1}{f(x) - g(x)} \, dx.$$

The family $\Gamma^u$ of vertical segments connecting the graph of $f(x)$ to graph of $g(x)$ above the interval $[x_1, x_2]$ is a subfamily of $\Gamma_{\leq \delta}$ so that $\text{mod } \Gamma^u \leq \text{mod } \Gamma_{\leq \delta}$. Lemma 3.7 gives the left-hand inequality.

**Theorem 4.4.** For any $\delta > 0$,

$$\text{mod } \Gamma^u \leq \text{mod } \Gamma \leq \frac{3}{c_\delta^2} \text{mod } \Gamma^u + \frac{A}{\delta^2}, \quad (16)$$

where $A$ is equal to the Euclidean area between the graphs of $f$ and $g$ above the interval $[x_1, x_2]$.

*Proof.* The left-hand side of (16) follows from $\Gamma^u \subseteq \Gamma$ and the monotonicity of modulus.

By Lemma 4.1, we have $\text{mod } \Gamma^{>\delta} \leq \frac{A}{\delta^2}$. Next note that by Lemma 4.3,

$$\text{mod } \Gamma_{\leq \delta} \leq \frac{1}{c_\delta^2} \int_{x_1-\delta}^{x_1} \frac{1}{f(x) - g(x)} \, dx \leq \frac{3}{c_\delta^2} \int_{x_1}^{x_2} \frac{1}{f(x) - g(x)} \, dx, \quad (17)$$

where the second inequality above follows from breaking the integral over the intervals, $[x_1 - \delta, x_1]$, $[x_1, x_2]$, $[x_1, x_1]$ and using the periodicity of $f$ and $g$. Now by the Beurling’s criterion, Lemma 3.7, the last integral is $\text{mod } \Gamma^u$.

Finally, using the fact that $\text{mod } \Gamma \leq \text{mod } \Gamma_{\leq \delta} + \text{mod } \Gamma^{>\delta}$ yields the right-hand side of (16).
4.1 Modulus of degenerating families of curves

Fix $\ell_0 \geq 0$ and $[x_1, x_2] \subset \mathbb{R}$. Consider a setting where we have a family of continuous periodic function pairs $\{ (f_\ell, g_\ell) \}$ all having the same period $x_2 - x_1$ and depending on a positive real parameter $\ell \geq \ell_0$. In later applications of the results in this section, the family of periodic functions depends on the lengths of simple closed geodesics and as such we use the subscript $\ell$ as a common notation for the length.

**Definition 4.5.** A family of continuous periodic function pairs $\{ (f_\ell, g_\ell) \}_{\ell \geq \ell_0}$ as above is simply degenerate if

1. $f_\ell(x) > g_\ell(x)$ for all $x \in \mathbb{R}$ and $\ell \geq \ell_0$,
2. the graphs of $(f_\ell, g_\ell)$ pointwise go to 0 as $\ell \to \infty$, and
3. the area bounded by their graphs above the interval $[x_1, x_2]$ is at most 1, for all $\ell$.

The choice of 1 in condition (3) is not crucial, and serves merely as a matter of convenience, as long as it is finite.

**Remark 4.6.** Given a simply degenerate family $\{ (f_\ell, g_\ell) \}$, let $\Gamma_\ell$ be the curve family connecting the part of the graph of $f_\ell$ over $[x_1, x_2]$ to the graph of $g_\ell$, that is, $\Pi(\gamma(0)) \in [x_1, x_2]$, and let $\Gamma^v_\ell$ be the vertical subfamily of $\Gamma_\ell$. We observe that as $\ell \to \infty$ we have

$$\text{mod} \Gamma_\ell \geq \text{mod} \Gamma^v_\ell \to \infty.$$  

Since the first inequality follows from the monotonicity of modulus it is enough to show that $\text{mod} \Gamma^v_\ell \to \infty$. For that let $M_\ell = \max \{|f_\ell(x) - g_\ell(x)| : x \in [x_1, x_2]|$, and note that $M_\ell \to 0$. Therefore, by Lemma 3.7 we have $\text{mod} \Gamma^v_\ell \geq M_\ell^{-1}|x_2 - x_1| \to \infty$, as desired.

Next, we formulate a condition that implies that the modulus of the vertical curve family $\Gamma^v_\ell$ is comparable to the modulus of the full family $\Gamma_\ell$, as $\ell$ goes to infinity.

Recall, that $c_{\delta, \ell}$ is the $\delta$-rectangle deviation between $f_\ell$ and $g_\ell$ as in (15).

**Corollary 4.7.** Suppose $\{ (f_\ell, g_\ell) \}_{\ell \geq \ell_0}$ is a simply degenerating family. If there exists a positive real-valued function $\delta = \delta(\ell)$ bounded above by the period $x_2 - x_1$ so that

1. $d = \inf_{\ell \geq \ell_0}[[\delta(\ell)]^2\text{mod} \Gamma^v_\ell] > 0$,
2. $c = \inf_{\ell \geq \ell_0}(c_{\delta, \ell}) > 0$,

then for all $\ell \geq \ell_0$

$$1 \leq \frac{\text{mod} \Gamma_\ell}{\text{mod} \Gamma^v_\ell} \leq \frac{3}{c^2} + \frac{3}{d}.$$  

**Proof.** For $\ell \geq \ell_0$, plugging $\delta = \delta(\ell)$ into inequality (16) of Theorem 4.4, noting that the area between the graphs and above the interval $[x_1 - \delta, x_2 + \delta]$ is at most 3, and dividing by $\text{mod} \Gamma^v_\ell$ we obtain

$$1 \leq \frac{\text{mod} \Gamma_\ell}{\text{mod} \Gamma^v_\ell} \leq \frac{3}{c_{\delta(\ell), \ell}^2} + \frac{3}{[\delta(\ell)]^2\text{mod} \Gamma^v_\ell}.$$  

The right-hand side, by our assumptions, is bounded above by $\frac{3}{c^2} + \frac{3}{d}$ and we are done. □
Remark 4.8. There are simply degenerate families of functions where we must allow that $\delta(\ell) \to 0$ as $\ell \to \infty$ in order to be able to apply Corollary 4.7.

Example 4.9. We next give an example to show that item (2) in Corollary 4.7 is necessary for (18) to hold. That is, a judicious choice of $\delta(\ell)$ satisfying the hypotheses of Corollary 4.7 does not always exist. Namely, we give a simply degenerating family for which the ratio of $\text{mod}(\Gamma_v)\,$ to $\text{mod}(\Gamma)\,$ goes to infinity as $\ell \to \infty$.

We first start with a computation involving a degenerating family of generalized quadrilaterals as $\varepsilon \to 0$. Given $\varepsilon \in (0, 1)$, let $\Omega_\varepsilon$ be the domain obtained from the rectangle $[0, 1] \times [0, \varepsilon]$ in $\mathbb{C}$ by removing the closed segments $L_k = \{ k \frac{N_\varepsilon}{N_\varepsilon - 1} + t_i : \varepsilon^2 \leq t \leq \varepsilon \}$ for $k = 2, \ldots, N_\varepsilon - 1$, where $N_\varepsilon > \frac{1}{2(\varepsilon - \varepsilon^2)}$ (see the gray region in Figure 4.1).

The domain $\Omega_\varepsilon$ has two vertical sides and the complement of the vertical sides of the boundary of $\Omega_\varepsilon$ has two components: the top and the bottom. The bottom component is the interval $[0, 1]$ on the real axis, and the top component is the union of $[0, 1] \times \{ \varepsilon \}$ and the segments $L_k$.

Let $\Gamma_\varepsilon$ be the family of arcs in $\Omega_\varepsilon$ connecting top to bottom. Let $\Gamma_v^\prime \varepsilon$ be the curves in $\Gamma_\varepsilon$ that are vertical.

Let $G_\varepsilon$ be the family of curves in $\Omega_\varepsilon$ connecting the vertical sides. Let $\Omega'_\varepsilon = \Omega_\varepsilon \cap [\frac{1}{N_\varepsilon}, \frac{N_\varepsilon - 1}{N_\varepsilon}] \times [0, \varepsilon]$. Let $G'_\varepsilon$ be the family of curves in $\Omega'_\varepsilon$ that connect $\{ \frac{1}{N_\varepsilon} + t_i : 0 \leq t \leq \varepsilon^2 \}$ and $\{ \frac{N_\varepsilon - 1}{N_\varepsilon} + t_i : 0 \leq t \leq \varepsilon^2 \}$ as in Figure 4.1.

Note that $G'_\varepsilon < G_\varepsilon$ hence $\text{mod}G'_\varepsilon \geq \text{mod}G_\varepsilon = \frac{1}{\text{mod}\Gamma_\varepsilon}$. By [26, Lemma 5.4] if $N_\varepsilon = \lceil \frac{1}{\varepsilon^2} \rceil$ we have $\text{mod}G'_\varepsilon \leq C\varepsilon^2$ for some $C > 0$. Therefore, $\frac{\text{mod}\Gamma_v^\prime \varepsilon}{\text{mod}\Gamma_\varepsilon} \leq C\varepsilon \to 0$ as $\varepsilon \to 0$.

The top side of the domain $\Omega_\varepsilon$ can be approximated by the graph of a continuous function such that $\frac{\text{mod}\Gamma_v^\prime \varepsilon}{\text{mod}\Gamma_\varepsilon} \leq 2C\varepsilon$ and we have the same conclusion for a simply degenerate family of functions.

Remark 4.10. The construction in Example 4.9 gives regions not bounded by graphs of continuous functions which do not satisfy item (2) in Corollary 4.7. We point out that similarly constructed regions which are not bounded by graphs of continuous functions may satisfy the assumption of...
Corollary 4.7 and one can prove that Corollary 4.7 applies to these more general regions although we do not pursue this here.

5 | MODULUS OF HALF-COLLARS

Let $X$ be a Riemann surface endowed with its conformal hyperbolic metric and $\alpha$ a simple closed geodesic on $X$. A collar about $\alpha$ is an annular (doubly connected) open neighborhood of $\alpha$. A half-collar about $\alpha$ is an annular neighborhood with $\alpha$ being one of its boundary components. In this section, we will compute the extremal distances between the two boundary components of standard and nonstandard collars around simple closed geodesics using the results from Section 4.

5.1 | General collars

The most direct way for computing the extremal distance between the two boundary components of a collar is to use Lemma 3.2. The fixed point set of the reflection symmetry of a general collar is not easily identified. For this reason we lift the collar to the universal covering and make use of a curve family in the universal covering which is related to the curve family connecting the two boundaries of the collar on the surface $X$.

Let $\alpha$ be a simple closed geodesic on a conformally hyperbolic Riemann surface $X$ of length $\ell$ and let $R$ be a collar or half-collar about $\alpha$. Fix the universal covering of $X$ to be the upper half-plane $\mathbb{H}^2$ such that the positive $y$-axis covers $\alpha$. Let $\tilde{R}$ be the image under $z(w) = \frac{1}{\ell} \log w$ of the component of the pre-image of $R$ in $\mathbb{H}^2$ that contains the positive $y$-axis. Note that $\tilde{R}$ is a universal cover of $R$ with a covering translation $h(z) = z + 1$ such that $\tilde{R}/<h> = R$, where $<h>$ is the cyclic group generated by $h$.

Definition 5.1. We will call $\tilde{R}$ the universal cover of $R$ in logarithmic coordinates.

The domain $\tilde{R}$ lies between graphs of two functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(x + 1) = f(x)$, $g(x + 1) = g(x)$ and $f(x) > g(x) > 0$ for all $x \in \mathbb{R}$ (see Figure 6.2).

Lemma 5.2. Given a collar or half-collar $R$ about a simple closed geodesic $\alpha$ on a Riemann surface $X$, let $\tilde{R}$ be universal cover of $R$ in logarithmic coordinates and $I$ a fundamental interval for the action of $<h>$ on one boundary component of $\tilde{R}$. Denote by $\Gamma_R$ the curve family in $R$ that connects the two boundary components of $R$. Consider the curve family $\Gamma$ in $\tilde{R}$ starting in $I$ and ending at the other boundary component. Then

$$\frac{1}{3} \text{mod} \Gamma - \frac{2}{3} A \leq \text{mod} \Gamma_R \leq \text{mod} \Gamma,$$

where $A$ is equal to the Euclidean area of the part of $\tilde{R}$ over the interval $I$.

Proof. The domain $\tilde{R}$ lies between graphs of two functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(x + 1) = f(x)$, $g(x + 1) = g(x)$ and $f(x) > g(x)$ for all $x \in \mathbb{R}$. Without loss of generality we assume that $I$ lies on the graph of $f$. Let $\Omega$ be the set of points in $\tilde{R}$ below $I$. Then $\Omega$ is a fundamental set for the action of $<h>$. 
Assume that $\rho(z)|dz|$ is an allowable metric for the family $\Gamma$. We define a metric $\bar{\rho}(z)|dz|$ for $z \in \Omega$ by

$$\bar{\rho}(z) = \sqrt[\infty]{\sum_{k=-\infty}^{\infty} [\rho(z + k)]^2}$$

and denote its projection to $R$ by $\bar{\rho}$ again. Since $\iint_{R} \rho^2(z)dxdy = \iint_{\Omega} \rho^2(z)dxdy$ we have that the series defining $\bar{\rho}$ converges almost everywhere when $\iint_{R} \rho^2(z)dxdy < \infty$.

Let $\gamma \in \Gamma_R$ and we compute its $\bar{\rho}$-length. Consider the lift $\tilde{\gamma}$ of $\gamma$ that starts on $I$ and ends on the graph of $g$. Then $\tilde{\gamma} \in \Gamma$. Lift $\rho$ to $\tilde{\rho}$. Then the $\tilde{\rho}$-length of $\gamma$ is equal to $\bar{\rho}$-length of $\tilde{\gamma}$.

We divide $\gamma$ into arcs $\{\gamma_k\}_k$ that lie in different translates $\Omega_k := h^k(\Omega)$ of $\Omega$ by elements of the group $<h>$. On each $\Omega_k$, we have

$$\bar{\rho}(z) \geq \rho(z).$$

Therefore, $l_{\bar{\rho}}(\gamma) = l_{\tilde{\rho}}(\tilde{\gamma}) \geq l_{\rho}(\tilde{\gamma}) \geq 1$ and the $\bar{\rho}$ metric is allowable for $\Gamma_R$. Since

$$\iint_{R} \bar{\rho}^2(z)dxdy = \sum_{k=-\infty}^{\infty} \iint_{\Omega_k} \rho^2(z)dxdy = \iint_{R} \rho^2(z)dxdy,$$

we have that

$$\text{mod}\Gamma_R \leq \text{mod}\Gamma.'$$

Assume that $\tilde{\rho}(z)|dz|$ is an allowable metric for the family $\Gamma_R$. Let $\Omega_1$ be the subdomain of $\tilde{R}$ which contains $\Omega$ and points whose $x$-coordinate differ by at most 1 from the $x$-coordinate of a point in $\Omega$. We take a lift $\rho(z)|dz|$ of $\tilde{\rho}(z)|dz|$ to the region $\Omega_1$. Then we define a metric $\rho_1(z)|dz|$ on the universal covering by

$$\rho_1(z) = \begin{cases} 
\rho(z), & \text{for } z \in \Omega \\
\max\{\rho(z), 1\}, & \text{for } z \in \Omega_1 \setminus \Omega \\
0, & \text{otherwise.}
\end{cases}$$

The metric $\rho_1(z)|dz|$ is allowable for the family $\Gamma$. This is because any curve $\gamma \in \Gamma$ is either completely contained in $\Omega_1$ or connects the vertical sides of one of the components in $\Omega_1 - \Omega$. In the first case, we have $\ell_{\rho_1}(\gamma) = \ell_{\rho}(\gamma) = \ell_{\bar{\rho}}(\pi(\gamma)) \geq 1$, where $\pi : \tilde{R} \to R$ is the covering map. In the second case, $\ell_{\rho_1}(\gamma) \geq \int_{\gamma \cap [\Omega_1 - \Omega]} |dz| \geq 1$.

We have

$$\iint_{R} \bar{\rho}^2(z)dxdy = \iint_{\Omega} \rho^2(z)dxdy \geq \frac{1}{3} \iint_{\Omega_1} \rho^2(z)dxdy \geq \frac{1}{3} \iint_{\Omega_1} \rho_1^2(z)dxdy - \frac{2}{3}A,$$

where the last inequality follows by $\rho_1^2 \leq \rho^2 + 1$ on $\Omega_1 - \Omega$ and $A$ is the area of $\Omega$. Taking the infimum over all allowable $\tilde{\rho}$, we obtain

$$\text{mod}\Gamma_R \geq \frac{1}{3} \text{mod}\Gamma - \frac{2}{3}A.$$

\[\square\]
From now on, we use subscript $\ell$ to emphasize the dependence on the length $\ell$ of the closed geodesic $\alpha$. Let $f_{\ell} : \mathbb{R} \to \mathbb{R}$ and $g_{\ell} : \mathbb{R} \to \mathbb{R}$ be the functions whose graphs are the upper and lower boundaries of the universal covering $\tilde{R}$. Note that $f_{\ell}(x + 1) = f_{\ell}(x)$ and $g_{\ell}(x + 1) = g_{\ell}(x)$.

The above lemma compares the modulus of the curve family $\Gamma_R$ connecting the two boundary components of $R$ to the modulus of a curve family $\Gamma_{\ell}$ in the universal covering $\tilde{R}$ in the logarithmic coordinates which connects a fundamental interval on the graph of $f_{\ell}$ to the graph of $g_{\ell}$ inside $\tilde{R}$. Since the modulus of $\Gamma_{\ell}$ cannot be directly computed, we compare it to the modulus of a family of vertical curves in the universal covering. We give the estimate under the assumption that the length $\ell$ of a closed geodesic $\alpha$ is bounded below by some fixed $\ell_0 > 0$.

Denote by $\Gamma^v_{\ell}$ the vertical family of curves in $\tilde{R}$ between these graphs and above $[-\frac{1}{2}, \frac{1}{2}]$. Recall that $c_{\delta(\ell), \ell}$ measures, up to scale $\delta(\ell)$, how far the area between the graphs are from being a rectangle. See Section 4 above Corollary 4.7 for the precise definition. In the following corollary, we put together Corollary 4.7 and Lemma 5.2 to derive a criterion for the $\text{mod} \Gamma_R$ to be comparable to $\text{mod} \Gamma^v_{\ell}$.

Here and in what follows the notation

$$a \asymp b$$

means that there is a constant $1 \leq C < \infty$ such that $C^{-1} \leq a/b \leq C$.

**Corollary 5.3.** For each $\ell \geq \ell_0$, let $R$ be a collar or half-collar about a geodesic of length $\ell$ and let $\Gamma_R$ be the curve family connecting the two boundary components of $R$. Let $\{(f_{\ell}, g_{\ell})\}_{\ell \geq \ell_0}$ be the lifts of the boundary components of $R$ to the universal covering $\tilde{R}$ in logarithmic coordinates such that $\{(f_{\ell}, g_{\ell})\}_{\ell \geq \ell_0}$ is a simply degenerate family. If there exists a positive real-valued function $\delta = \delta(\ell)$ so that

1. $d = \inf_{\ell \geq \ell_0} \delta(\ell)^2 \text{mod} \Gamma^v_{\ell} > 0$,
2. $c = \inf_{\ell \geq \ell_0} (c_{\delta(\ell), \ell}) > 0$,

then

$$\text{mod} \Gamma_R \asymp \text{mod} \Gamma^v_{\ell},$$

when $\ell \to \infty$. The bound on the constant of the above comparison depends on $c$ and $d$ and it goes to infinity when either of them goes to zero.

**Proof.** Since $\{(f_{\ell}, g_{\ell})\}_{\ell \geq \ell_0}$ is a simply degenerating family, Remark 4.6 shows that $\text{mod} \Gamma^v_{\ell} \to \infty$ as $\ell \to \infty$.

Since $A \leq 1$, Lemma 5.2 gives

$$\frac{1}{3} \left[ \text{mod} \Gamma_{\ell} - 2 \right] \leq \text{mod} \Gamma_R \leq \text{mod} \Gamma_{\ell}.$$ (19)

Corollary 4.7 applied to both sides of Equation (19) gives us

$$\frac{1}{3} \left[ \text{mod} \Gamma^v_{\ell} - 2 \right] \leq \text{mod} \Gamma_R \leq \left( \frac{3}{c^2} + \frac{3}{d} \right) \text{mod} \Gamma^v_{\ell}.$$ (20)
and hence
\[
\frac{1}{3} \left[ 1 - \frac{2}{\text{mod} \Gamma_{\ell'}} \right] \leq \frac{\text{mod} \Gamma_R}{\text{mod} \Gamma_{\ell'}} \leq \frac{3}{c^2} + \frac{3}{d}.
\]

Since \(\text{mod} \Gamma_{\ell'} \to \infty\) as \(\ell' \to \infty\), we have \(\frac{1}{3} \left[ 1 - \frac{2}{\text{mod} \Gamma_{\ell'}} \right] \geq \frac{1}{6}\) for all \(\ell' \geq \ell_1 > 0\) with \(\ell_1\) large enough and the result follows for \(\ell' \geq \ell_1\). The result follows for \(\ell_0 \leq \ell' \leq \ell_1\) by continuity.

**Remark 5.4.** The family \(\Gamma_{\ell'}\) corresponds to the set of geodesics between boundary components of \(R\) which are orthogonal to the core geodesic of \(R\). In particular, the modulus of these orthogonals is the same as the modulus of \(\Gamma_{\ell'}\). Hence, Corollary 5.3 could be rephrased purely in hyperbolic terms on the collar.

### 5.2 Standard collars

The **standard collar** about \(\alpha\) is the set of points a distance less than \(r\left(\frac{\ell'(\alpha)}{2}\right)\) from \(\alpha\), where
\[
r(x) := \sinh^{-1}\left(\frac{1}{\sinh x}\right). \tag{21}
\]

The standard collar is bounded by two equidistant curves. It is well-known that a standard collar always exists and disjoint simple closed geodesics have disjoint standard collars, see [15]. The **standard half-collar** consists of the points on one side of the standard collar. Note that by (21) for large \(x\), we have the following asymptotics
\[
r(x) \asymp \frac{1}{e^x}, \quad \text{as } x \to \infty. \tag{22}
\]

On the other hand, since \(\sinh^{-1}(t) = \ln(t + \sqrt{t^2 + 1})\), from (21) we have
\[
r(x) = \ln \left(\frac{1 + \cosh x}{\sinh x}\right) = \ln \left(\frac{2}{x}\right) + o(x), \quad \text{as } x \to 0. \tag{23}
\]

The extremal length of the curve family \(\Gamma_R\) connecting the two boundary components of the standard full collar neighborhood \(R\) about \(\alpha\) was computed by Maskit, see [29]. For the convenience of the reader, we give the computation below.

**Lemma 5.5.** Let \(R\) be the standard half-collar about a simple closed geodesic of length \(\ell'\). Then
\[
\lambda(\Gamma_R) = \frac{1}{\ell'} \arctan \left[ \frac{1}{\sinh \frac{\ell'}{2}} \right]. \tag{24}
\]

**Proof.** By the collar lemma, the geodesic \(\alpha\) of length \(\ell\) has a one-sided collar of length \(\rho = \sinh^{-1}\frac{1}{\sinh \frac{\ell}{2}}\), see, for instance, [15, p. 94]. We lift the collar \(R\) to the upper half-plane so that \(\alpha\) is lifted to the geodesic with end points 0 and \(\infty\). One lift \(\tilde{R}\) of \(R\) is between two hyperbolic geodesics.
orthogonal to the \(y\)-axis, intersecting the imaginary axis at points \(iy_1\) and \(iy_2\), with \(0 < y_1 < y_2\), and between a Euclidean ray from 0 subtending angle \(\theta\) with the \(y\)-axis. Thus, \(\tilde{R}\) is an annular sector

\[
\tilde{R} = \{ re^{i\phi} : \phi \in (\pi/2 - \theta, \pi/2), r \in (y_1, y_2) \}.
\]

Therefore, \(\lambda(\Gamma_R) = \lambda(\tilde{\Gamma}_R)\), where \(\tilde{\Gamma}_R\) is the family of curves connecting the rays \(\{re^{i\phi} : \phi = \pi/2, r > 0\}\) and \(\{re^{i\phi} : \phi = \pi/2 - \theta, r > 0\}\) in the quadrilateral \(\tilde{R}\). The standard extremal length formula for curves in an annular sector, cf. [47, Remark 7.7], then gives us

\[
\lambda(\tilde{\Gamma}_R) = \frac{\theta}{\ln \frac{y_2}{y_1}} = \frac{\theta}{\ell}.
\]

Finally, the last equality implies (24), since \(\tan \theta = \sinh \rho = 1/\sinh \ell/2\), cf. [12, p. 162].

We note here that from (24), we have the following asymptotic behavior for the extremal distance \(\lambda(R) := \lambda(\Gamma_R)\) (see Definition 3.8) between the boundary components of the standard collar \(R\) about a simple closed geodesic of length \(\ell\):

\[
\lambda(R) \approx \frac{1}{\ell e^{\ell/2}}, \quad \text{as } \ell \to \infty,
\]

\[
\lambda(R) \approx \frac{1}{\ell}, \quad \text{as } \ell \to 0.
\]

5.3 | Nonstandard half-collars

One of the main objects in our study is what we call a nonstandard half-collar. Let \(P\) be a geodesic pair of pants with three boundary geodesics \(\alpha, \alpha_1\) and \(\alpha_2\). Fix a boundary geodesic \(\alpha\) of \(P\) and an orthogeodesic \(\gamma\) from \(\alpha\) to one of the other boundary geodesics \(\alpha_1\) of \(P\). Let \(b\) be the end point of \(\gamma\) located on the other geodesic \(\alpha_2\). Since every geodesic pair of pants has a unique decomposition into two right angled hexagons, it follows that \(b\) lies on two such identical right angled hexagons. On each such hexagon, we drop a perpendicular from \(b\) to the other simple orthogeodesic emanating from \(\alpha\) (see Figure 1.3). Then the union of these two perpendiculars is a piecewise geodesic loop \(\beta\) with a non-smooth point at \(b\), and the annular domain bounded by \(\alpha\) and \(\beta\) is what we call the nonstandard half-collar \(R_{\alpha, \gamma}\) about \(\alpha\).

When a geodesic pair of pants \(P\) has a puncture \(\alpha_1\), then \(\gamma\) is a geodesic ray orthogonal to \(\alpha\) that converges to the puncture. In this case, the geodesic loop \(\beta\) becomes a bi-infinite geodesic which converges to the puncture in both directions and is orthogonal to the orthogeodesic between \(\alpha\) and the third boundary component of \(P\) (see the right side of Figure 1.4). Again \(R_{\alpha, \gamma}\) is the annular domain between \(\alpha\) and \(\beta\).

There are three parameters associated with a nonstandard half-collar: the length \(\ell(\alpha)\) of \(\alpha\), the length \(\ell(\gamma)\) of the orthogeodesic \(\gamma\), and the length \(\ell(\eta)\) of the geodesic segment \(\eta\) from the boundary geodesic \(\alpha\) to the geodesic loop \(\beta\). Using basic hyperbolic geometry, the three quantities are related by (see [15, Theorem 2.3.1(iv)])

\[
\tanh \ell(\eta) = \frac{\tanh \ell(\gamma)}{\cosh \ell(\alpha)/2}.
\]
So, the nonstandard half-collar can be parameterized (determined) by the length of the boundary geodesic \( \ell(\alpha) \) and the length \( \ell(\gamma) \) with the constraint \( \ell(\gamma) > r(\frac{\ell(\alpha)}{2}) \) coming from the collar lemma.

When \( \ell(\gamma) = \infty \), that is when the orthogeodesic goes out a cusp, the nonstandard half-collar contains the standard half-collar. On the other hand, for \( \ell(\gamma) < \infty \), using the quadrilateral formula in hyperbolic geometry, one can show that

\[
\ell(\alpha) \leq 2r(\ell(\eta)).
\] (27)

Therefore, in this case, it is not hard to see that neither collar contains the other, see Figure 1.4. Nevertheless, using the above notation we have the following result.

**Theorem 5.6.** Let \( R_{\alpha,\gamma} \) be the nonstandard half-collar about a boundary geodesic \( \alpha \) of length \( \ell(\alpha) > 1 \) on a pair of pants. Then

\[
\lambda(R_{\alpha,\gamma}) \asymp \ell(\eta),
\] (28)

where \( \ell(\eta) \) is given by Equation (26).

**Remark 5.7.** Using (26), the estimate (28) can be rephrased in terms of the lengths \( \ell(\alpha) \) and \( \ell(\gamma) \) rather than \( \ell(\eta) \), see Corollary 5.10. In particular, in contrast to (25) we have

\[
\lambda(R_{\alpha,\gamma}) \asymp \frac{1}{e^{\ell(\gamma)/2}},
\] (29)

as \( \ell(\alpha) \to \infty \), for \( \ell(\gamma) > 1 \).

**Proof.** Consider a lift of \( R_{\alpha,\gamma} \) to \( \mathbb{H} \) such that the lift of \( \alpha \) is on the imaginary axis, and the lift of \( \beta \) lies on the semicircle \( C \), as indicated in the right top part of Figure 6.1. The semicircle \( C \) lies on the circle

\[
|w - \cosh r(\ell(\eta))| = \sinh r(\ell(\eta)).
\] (30)

The shaded region in the right top part of Figure 6.1 is the fundamental domain for the action of the covering transformations.

The conformal map \( z = \frac{1}{\ell(\alpha)} \log w \) sends the shaded region in Figure 6.1 onto a region bounded by the graphs of functions

\[
f_{\ell(\alpha)}(x) = \frac{\pi}{2\ell(\alpha)},
\]

\[
g_{\ell(\alpha)}(x) = \frac{1}{\ell(\alpha)} \cos^{-1} \left( \frac{\cosh x\ell(\alpha)}{\cosh r(\ell(\eta))} \right)
\]

and vertical lines \( x = -1/2 \) and \( x = 1/2 \).
The inequality $\frac{\pi}{2} - \cos^{-1} t \geq t$ gives

$$g_{\ell(\alpha)}(x) \leq \frac{\pi}{2\ell(\alpha)} - \frac{1}{\ell(\alpha)} \cosh x\ell(\alpha) \leq \frac{\pi}{2\ell(\alpha)} - \frac{1}{2\ell(\alpha)} e^{\frac{|x|\ell(\alpha)}} =: h_{\ell(\alpha)}(x).$$

Set $\delta(\ell(\alpha)) = \frac{1}{\ell(\alpha)}$. Let $R'_{\alpha, \gamma}$ be the sub-annulus of $R_{\alpha, \gamma}$ whose lifted boundary components are the graphs of the functions $f_{\ell(\alpha)}(x)$ and $h_{\ell(\alpha)}(x)$. We first estimate $\lambda(R'_{\alpha, \gamma})$ using Corollary 5.3. For $\delta_2 \in [-\delta, \delta]$, we have

$$m(x) = \frac{\pi}{2\ell(\alpha)} - h_{\ell(\alpha)}(x + \delta_2) = \frac{e^{\frac{r(\ell(\eta))}{2\ell(\alpha)}}}{2\ell(\alpha)} e^{\frac{|x+\delta_2|\ell(\alpha)}} \geq \frac{e^{\frac{r(\ell(\eta))}{2\ell(\alpha)}}}{2\ell(\alpha)} e^{\frac{|x|\ell(\alpha)}} e^{-\frac{|\delta_2|\ell(\alpha)}} =: f_{\ell(\alpha)}(x) - h_{\ell(\alpha)}(x) e^{-1}.$$ (31)

This gives $c \geq e^{-1}$. Let $\Gamma'_{\alpha, \gamma}$ be the vertical family above $[-\frac{1}{2}, \frac{1}{2}]$ between the graphs of $f_{\ell(\alpha)}(x)$ and $h_{\ell(\alpha)}(x)$. Then we have

$$\text{mod} \Gamma'_{\alpha, \gamma} = \int_{-\frac{1}{2}}^{\frac{1}{2}} f_{\ell(\alpha)}(x) - h_{\ell(\alpha)}(x) dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} 2\ell(\alpha) e^{-r(\ell(\eta))\frac{|x|\ell(\alpha)}} dx = 4(1 - e^{-\frac{1}{2}r(\ell(\alpha))}) > 4(1 - e^{-\frac{1}{2}r(\ell(\eta))}) e^{\frac{r(\alpha)}{2}}.$$ (32)

This implies that $b = \inf_{\ell(\alpha) \geq 1} \delta(\ell(\alpha))^2 \text{mod} \Gamma'_{\alpha, \gamma} > 0$ and the conditions of Corollary 5.3 are met for the family $\Gamma'_{\alpha, \gamma}$. Therefore,

$$\lambda(R'_{\alpha, \gamma}) \asymp \frac{1}{\text{mod} \Gamma'_{\ell(\alpha)}} \asymp e^{-r(\ell(\eta))}.$$ Since $\lambda(R_{\alpha, \gamma}) \geq \lambda(R'_{\alpha, \gamma})$, we have $\lambda(R_{\alpha, \gamma}) \geq e^{-r(\ell(\eta))}$.

For the upper bound for $\lambda(R_{\alpha, \gamma})$ we need to show that $\text{mod} \Gamma_{\alpha, \gamma} \geq c_1 e^{r(\ell(\eta))}$ for some constant $c_1 > 0$ where $\Gamma_{\alpha, \gamma}$ is the family of curves connecting the boundary components of $R_{\alpha, \gamma}$. Let $\Gamma_{\alpha, \gamma}^\perp$ be the geodesic arcs connecting two boundary components of $R_{\alpha, \gamma}$ that start orthogonal to the boundary geodesic $\alpha$. The family $\Gamma_{\alpha, \gamma}^\perp$ is in a one to one correspondence with the family $\Gamma_{\alpha, \gamma}^v$ of the vertical segments above the interval $[-\frac{1}{2}, \frac{1}{2}]$ connecting the graphs of $f_{\ell(\alpha)}(x)$ and $g_{\ell(\alpha)}(x)$. Then we have

$$\text{mod} \Gamma_{\alpha, \gamma} \geq \text{mod} \Gamma_{\alpha, \gamma}^\perp = \text{mod} \Gamma_{\alpha, \gamma}^v.$$
Note that there exists $c_0 > 0$ such that $\pi/2 - \cos^{-1} t \leq c_0 t$ for $0 \leq t \leq 1$. Then for $-1/2 \leq x \leq 1/2$, we have

$$g_{\ell(\alpha)}(x) = \frac{1}{\ell(\alpha)} \cos^{-1} \frac{\cosh x}{\cosh r(\ell(\eta))} \geq \frac{\pi}{2\ell(\alpha)} - \frac{c_0}{\ell(\alpha)} \frac{\cosh |x|}{\cosh r(\ell(\eta))}$$

and

$$f_{\ell(\alpha)}(x) - g_{\ell(\alpha)}(x) \leq \frac{c_0}{\ell(\alpha)} \frac{\cosh |x|}{\cosh r(\ell(\eta))} \cdot$$

By Lemma 3.7, we have

$$\text{mod} \Gamma_v^{\psi} = \int_{-\frac{1}{2}}^{\frac{1}{2}} dx f_{\ell(\alpha)}(x) - g_{\ell(\alpha)}(x) \geq e^{r(\ell(\eta))} \int_{-\frac{1}{2}}^{\frac{1}{2}} \ell(\alpha) e^{-\frac{x}{\ell(\alpha)}} dx \geq c_1 e^{r(\ell(\eta))}$$

for some $c_1 > 0$. This finishes the proof. \qed

**Remark 5.8.** A key point in Theorem 5.6 is the fact that the asymptotics of $\lambda(R_{x,y})$ is sharp, up to a bounded multiplicative constant. This is a consequence of comparing the extremal length of all curves connecting the graphs of $f_{\ell(\alpha)}(x)$ and $g_{\ell(\alpha)}(x)$ over $[-1/2, 1/2]$ to the modulus of the vertical subfamily, using the theory developed in Section 4. Much easier (and not sharp) estimates can be obtained by comparing to families of curves in rectangles. For instance, $\lambda(R_{x,y})$ can be bounded below by considering the family of curves connecting $f_{\ell(\alpha)}(x) = \pi/(2\ell(\alpha))$ and

$$y = \max_{(-\frac{1}{2}, \frac{1}{2})} g_{\ell(\alpha)}(x) = g_{\ell(0)} = \frac{1}{\ell(\alpha)} \cos^{-1} \frac{1}{\cosh r(\ell(\eta))}.$$ 

The extremal length of the latter family is estimated by (if say $\ell(\gamma) > 1$)

$$f_{\ell}(x) - g_{\ell}(x) \geq \frac{\pi}{2\ell(\alpha)} - \frac{1}{\ell(\alpha)} \left[ \frac{\pi}{2} - \frac{1}{\cosh r(\ell(\eta))} \right] \geq e^{-r(\ell(\eta))} \frac{\ell(\alpha)}{\ell(\alpha)} > e^{-\frac{\ell(\alpha)}{\ell(\alpha)}},$$

which is much weaker than (28) or (34).

**Remark 5.9.** The universal covering $\tilde{R}$ in the logarithmic coordinates agrees with the lift of the nonstandard collar to the strip model of the hyperbolic plane (see [13, Example 7.9]). In fact, the expression for $g_{\ell}(x)$ is an explicit formula for a geodesic in the strip model up to a linear map.

**Corollary 5.10.** Let $R_{x,y}$ be the nonstandard half-collar as in Figure 1.4. Then, for $\ell(\alpha) \to \infty$, we have

$$\lambda(R_{x,y}) \simeq \begin{cases} \ell(\gamma)e^{-\ell(\alpha)/2}, & \text{if } 0 < \ell(\gamma) < 1, \\ e^{-\ell(\alpha)/2}, & \text{if } \ell(\gamma) \geq 1. \end{cases}$$

(34)

For $\ell(\alpha) \leq 1$, we have

$$\lambda(R_{x,y}) \gtrsim \frac{1}{\ell(\alpha)}. \quad (35)$$
In particular, $\ell(\alpha)$ stays bounded between two positive constants if and only if $\lambda(R_{\alpha,\gamma})$ stays bounded between two other positive constants.

**Proof.** By (26), for fixed $\ell(\gamma)$, we have that $\ell(\alpha) \to \infty$ if and only if $\ell(\eta) \to 0$. Therefore, combining (28) with (23) and (26) we obtain, respectively,

$$\lambda(R_{\alpha,\gamma}) \approx \frac{\tanh(\ell(\eta))}{e^{\ell(\alpha)/2}},$$

as $\ell(\alpha) \to \infty$, which implies (34) by using standard estimates for $\tanh(\ell(\eta))$.

The estimate (35) follows from (33) since $\ell(\alpha) < 1$. Finally, the last statement follows immediately from (34) and (35).

**Corollary 5.11.** Let $P$ be a pair of pants with boundary lengths $\ell(\alpha), \ell(\alpha_1), \ell(\alpha_2)$ with $\ell(\alpha) > 0$ and $\ell(\alpha_1) < 2 \coth^{-1}(\cosh 1)$. Then the nonstandard half-collar $R_{\alpha,\gamma}$ about the geodesic $\alpha$ satisfies

$$\lambda(R_{\alpha,\gamma}) \approx e^{-\ell(\alpha)/2},$$

as $\ell(\alpha) \to \infty$.

**Proof.** The hexagon formula, cf. [12], applied to a half of the pair of pants with boundary lengths $\ell(\alpha), \ell(\alpha_1), \ell(\alpha_2)$ gives

$$\cosh \ell(\gamma) = \coth \frac{\ell(\alpha_1)}{2} \coth \frac{\ell(\alpha)}{2} + \frac{\cosh(\ell(\alpha_2)/2)}{\sinh(\ell(\alpha_1)/2) \sinh(\ell(\alpha)/2)} \geq \coth \frac{\ell(\alpha_1)}{2} \coth \frac{\ell(\alpha)}{2} \geq \cosh 1,$$

where $\gamma$ is the orthogeodesic between $\alpha$ and $\alpha_1$.

Then we get that $\ell(\gamma) \geq 1$ which extends the same conclusion to this case by Corollary 5.10.

6 | GLUING NONSTANDARD HALF-COLLARS WITH A TWIST

Let $P$ be a geodesic pair of pants with boundary geodesic curve $\alpha$ and an orthogeodesic $\gamma$ from $\alpha$ to another boundary geodesic or a puncture $\alpha_1$ of $P$. Let $P'$ be another geodesic pair of pants with boundary geodesic $\alpha'$ and an orthogeodesic $\gamma'$ from $\alpha'$ to another boundary geodesic or a puncture $\alpha'_1$ of $P$. If $\ell(\alpha) = \ell(\alpha')$ then we can glue $\alpha$ to $\alpha'$ by an isometry such that the relative position of the marked points is given by the twist parameter $t(\alpha) \in (-\frac{1}{2}, \frac{1}{2})$ (see Section 2).

Let $R_{\alpha,\gamma,\gamma'}^{t(\alpha)}$ be the union of the two nonstandard half-collars $R_{\alpha,\gamma}$ and $R_{\alpha',\gamma'}$ around $\alpha \equiv \alpha'$ with the twist parameter $t(\alpha)$. We give an estimate of the extremal distance between the two boundary components of the annulus $R_{\alpha,\gamma,\gamma'}^{t(\alpha)}$. When the twist is not zero the extremal distance between the two boundary components of $R_{\alpha,\gamma,\gamma'}^{t(\alpha)}$ is larger than the sum of the extremal distances between the boundary components of the two nonstandard half-collars $R_{\alpha,\gamma}$ and $R_{\alpha',\gamma'}$. 

Theorem 6.1. Let \( R_{\alpha,\gamma,\gamma'}^{(\alpha)} \) be an annulus obtained by isometrically gluing two nonstandard half-collars along their boundary geodesics \( \alpha \) and \( \alpha' \) of equal length with twist \(-\frac{1}{2} \leq t(\alpha) < \frac{1}{2}\). Given \( \ell_0 \geq 2 \), there exists \( C > 0 \) such that for all \( \ell(\alpha) \geq \ell_0 \), we have

\[
\lambda(R_{\alpha,\gamma,\gamma'}^{(\alpha)}) \geq \frac{C}{\max\{e^\ell(\eta) - |t(\alpha)| \frac{\ell(\alpha)}{2}, e^\ell(\eta') - |t(\alpha)| \frac{\ell(\alpha)}{2}\}},
\]

where \( \eta \) and \( \eta' \) are orthogonals on the respective nonstandard half-collars as in Subsection 5.3.

Remark 6.2. By Equation (26), we have \( r(\ell(\eta)) \approx \frac{\ell(\eta)}{2} - \ln - \frac{\ell(\eta)}{2} \), where \( \ln - x = \min\{\ln x, 0\} \).

Then for \( \min\{\ell(\eta), \ell(\eta')\} \leq 1 \), the above estimate is equivalent to

\[
\lambda(R_{\alpha,\gamma,\gamma'}^{(\alpha)}) \geq C \min\{\ell(\gamma), \ell(\gamma')\} e^{-\left(\frac{1}{2} - \frac{|t(\alpha)|}{2}\right)\frac{\ell(\alpha)}{2}}, \tag{37}
\]

and for \( \min\{\ell(\gamma), \ell(\gamma')\} \geq 1 \),

\[
\lambda(R_{\alpha,\gamma,\gamma'}^{(\alpha)}) \geq Ce^{-\left(\frac{1}{2} - \frac{|t(\alpha)|}{2}\right)\frac{\ell(\alpha)}{2}}. \tag{38}
\]

If the two nonstandard half-collars come from tight pairs of pants, we have \( r(\ell(\eta)) = r(\ell(\eta')) = \ell(\alpha)/2 \) and \( \ell(\gamma) = \ell(\gamma') = \infty \), and we immediately obtain the following.

Corollary 6.3. Let \( R_{\alpha,\gamma,\gamma'}^{(\alpha)} \) be the annular region obtained by gluing two nonstandard half-collars \( R_{\alpha,\gamma} \) and \( R_{\alpha',\gamma'} \) with twist \( t(\alpha) \) and \( \ell(\gamma) = \ell(\gamma') = \infty \). If \( \ell(\alpha) \geq \ell_0 \geq 2 \) and \(-\frac{1}{2} \leq t(\alpha) < \frac{1}{2}\) then, for the constant \( C \) from Theorem 6.1,

\[
\lambda(R_{\alpha,\gamma,\gamma'}^{(\alpha)}) \geq Ce^{-\left(\frac{1}{2} - \frac{|t(\alpha)|}{2}\right)\frac{\ell(\alpha)}{2}}. \tag{39}
\]

It is interesting to note that if two standard half-collars are glued together then the extremal distance between the boundary components does not increase with twist. In fact, the extremal distance between the boundary components of the glued standard collars is equal to twice the extremal distance between the boundary components of the half standard collar due to the fact that the glued collar is symmetric about the closed core geodesic.

Proof of Theorem 6.1. The two nonstandard half-collars have corresponding parameters \( \ell(\alpha), \ell(\eta), \ell(\gamma), \) and \( \ell(\alpha') = \ell(\alpha), \ell(\eta'), \) and \( \ell(\gamma') \), respectively. Note that the orthogeodesics \( \gamma \) and \( \gamma' \) do not meet at the geodesic \( \alpha \) unless \( t(\alpha) = 0 \). In fact, the distance between \( \gamma \) and \( \gamma' \) along the geodesic \( \alpha \) is \(|t(\alpha)|\ell(\alpha)\) (see Figure 1.5). Denote by \( \Gamma_{\beta,\beta'} \) the family of curves connecting \( \beta \) to \( \beta' \) in \( R_{\alpha,\gamma,\gamma'}^{(\alpha)} \), where \( \beta \) and \( \beta' \) are the piecewise geodesic loops on the respective half-collars. Note that

\[
\lambda(R_{\alpha,\gamma,\gamma'}^{(\alpha)}) = 1/\text{mod } \Gamma_{\beta,\beta'}.
\]

We extend \( \gamma \) until it hits \( \beta' \) and continue to call the extension \( \gamma \). Cut \( R_{\alpha,\gamma,\gamma'}^{(\alpha)} \) along \( \gamma \) and choose a lift \( Q_{\ell(\alpha)} \) of the simply connected region \( R_{\alpha,\gamma,\gamma'}^{(\alpha)} - \gamma \) to the upper half-plane so that \( \alpha \) is lifted to the segment \([e^{-\ell(\alpha)/2}i, e^{\ell(\alpha)/2}i]\) of the \( y \)-axis and the two lifts of \( \gamma \) are geodesic arcs orthogonal to the \( y \)-axis that pass through the points \( e^{-\ell(\alpha)/2}i \) and \( e^{\ell(\alpha)/2}i \) as in Figure 6.1.
The lift of the geodesic loop $\beta$ of the left half-collar lies on a geodesic with end points $-e^{r(\epsilon(\eta))}$ and $-e^{-r(\epsilon(\eta))}$. Let $\tilde{\gamma}$ be a lift of $\gamma$ and denote by $\tilde{b}$ the lift of $b$, where $\tilde{\gamma}$ meets the geodesic with end points $-e^{r(\eta)}(-e^{r(\eta)}))$ and $-e^{r(\eta)}$). The lift of the geodesic loop $\beta'$ of the right nonstandard half-collar that intersects the lifts of $\gamma$ lies on two geodesics with end points $e^{t(\alpha)}e^{\alpha r(\eta')}(-e^{\alpha r(\eta')})$, $e^{t(\alpha)}e^{\alpha r(\eta')}(-e^{\alpha r(\eta')}+r(\eta'))$ and $e^{t(\alpha)}e^{\alpha r(\eta')}(-e^{\alpha r(\eta')}+r(\eta'))$. The lift of $b'$ is the intersection of the two geodesics above (see Figure 6.1).

Let $\tau_{\ell(\alpha)}(w) = e^{\ell(\alpha)}w$ be the hyperbolic translation corresponding to the geodesic $\alpha$. Then $\tilde{\mathcal{A}}_{t(\alpha)} = \bigcup_{k \in \mathbb{Z}} \tau_{\ell(\alpha)}(Q_{t(\alpha)})$ is a universal covering of $R_{t(\alpha)}^{\alpha,\gamma,\gamma'}$. We apply the map $z = \frac{1}{\ell(\alpha)} \log w$ to $\tilde{\mathcal{A}}_{t(\alpha)}$. The image of the boundary component of $R_{t(\alpha)}^{\alpha,\gamma,\gamma'}$ that covers the geodesic loop $\beta'$ is the graph of the function $f_{t(\alpha)}(x)$ obtained by taking the $y$-coordinate of the image of the geodesic with end points $-e^{r(\epsilon(\eta))}$ and $-e^{e(\epsilon(\eta))}$. From the equation of the circle containing the geodesic

$$|w + \cosh r(\epsilon(\eta))| = \sinh r(\epsilon(\eta))$$

and by $w = e^{\ell(\alpha)x}$ with $z = x + iy$, we obtain

$$f_{t(\alpha)}(x) = \frac{1}{\ell(\alpha)} \cos^{-1} \left( - \frac{\cosh \ell(\alpha)x}{\cosh r(\epsilon(\eta))} \right) = \frac{1}{\ell(\alpha)} \left( \pi - \cos^{-1} \frac{\cosh \ell(\alpha)x}{\cosh r(\epsilon(\eta))} \right)$$

$$\geq \frac{1}{\ell(\alpha)} \left( \frac{\pi}{2} + \frac{e^{\ell(\alpha)\epsilon(\eta)}}{2} \right) = h_1(x)$$

for $-1/2 \leq x \leq 1/2$, where we used the inequality $\frac{\pi}{2} - \cos^{-1} s \geq s$ for $0 \leq s \leq 1$. Note that $\frac{\cosh \ell(\alpha)x}{\cosh r(\epsilon(\eta))} \leq 1$ for $x \in [-1/2, 1/2]$ by (27). The functions $f_{t(\alpha)}$ and $h_1(x)$ extend to $\mathbb{R}$ by invariance under $x \mapsto x + 1$.

Similarly, the image in the $z$-plane of the boundary component of $\tilde{\mathcal{A}}_{t(\alpha)}$ that covers the geodesic loop $\beta'$ is the graph of the function $g_{t(\alpha)}(x)$ obtained by taking the $y$-coordinate of the image of the geodesic with end points $\cosh r(\epsilon(\eta') = t(\alpha)\gamma(\alpha) + r(\epsilon(\eta'))$ and $e^{t(\alpha)\gamma(\alpha) + r(\epsilon(\eta'))}$ over the interval $[-1/2 +
Fig. 6.2  The lift in log coordinates of the glued nonstandard collars

\[ t(\alpha), 1/2 + t(\alpha) \] and extending it by \( g_{t(\alpha)}(x + k) = g_{t(\alpha)}(x) \) for all \( k \in \mathbb{Z} \). We obtain

\[ g_{t(\alpha)}(x) = \frac{1}{\ell(\alpha)} \cos^{-1} \left( \frac{\cosh \ell(\alpha)(x - t(\alpha))}{\cosh r(\eta')} \right) \leq \frac{1}{\ell(\alpha)} \left( \frac{\pi}{2} - \frac{e^{\ell(\alpha)|x-t(\alpha)|}}{2e^{r(\eta')}} \right) =: h_2(x) \]

for \(-1/2 + t(\alpha) \leq x \leq 1/2 + t(\alpha)\) and extend \( g_{t(\alpha)}(x) \) and \( h_2(x) \) to \( \mathbb{R} \) by invariance under \( x \mapsto x + 1 \). The region \( Q_{t(\alpha)} \) is mapped to the region above the segment \([-1/2, 1/2]\) and between the graphs of \( f_{t(\alpha)}(x) \) and \( g_{t(\alpha)}(x) \) for \( x \in [-1/2, 1/2] \). The graphs of \( h_1(x) \) and \( h_2(x) \) bound a subregion between the graphs of \( f_{t(\alpha)}(x) \) and \( g_{t(\alpha)}(x) \) that is invariant under the translation \( x \mapsto x + 1 \) and projects to a subring \( R' \) of \( R_{\alpha,\gamma,\gamma'} \), see Figure 6.2. Let \( \Gamma_{R'} \) be the family of curves that connects the two boundary components of \( R' \). Since each \( \gamma \in \Gamma_{R'} \) contains a subarc in \( \Gamma_{R'} \), it follows that

\[ \text{mod} \Gamma_{\beta,\beta'} \leq \text{mod} \Gamma_{R'}. \]

Therefore, we only need to estimate \( \text{mod} \Gamma_{R'} \) from above.

Let \( \Gamma_{R'}^v \) be the family of vertical lines connecting the graphs of \( h_1(x) \) and \( h_2(x) \) above \(-1/2 \leq x \leq 1/2\). We show that the conditions of Corollary 5.3 are satisfied. Let \( \delta = \delta(\ell(\alpha)) = 1/\ell(\alpha) \) and define

\[ m_\delta(x) = \inf_{|\delta_1|,|\delta_2| \leq \delta} [h_1(x + \delta_1) - h_2(x + \delta_2)] \]

for \( x \in [-1/2, 1/2] \).

We estimate \( m_\delta(x) \) from below similar to the proof of Theorem 5.6. Define

\[ k_1(x) = \frac{e^{\ell(\alpha)|x|}}{2e^{r(\eta')}} \text{ for } x \in \left[-\frac{1}{2}, \frac{1}{2}\right] \text{ and } \]

\[ k_2(x) = \frac{e^{\ell(\alpha)|x-t(\alpha)|}}{2e^{r(\eta')}} \text{ for } x \in \left[-\frac{1}{2} + t(\alpha), \frac{1}{2} + t(\alpha)\right] \]

and extend both functions to \( \mathbb{R} \) with period one. Note that

\[ h_1(x) - h_2(x) = k_1(x) + k_2(x). \]
Assume that \(|\delta_1| \leq \delta\) and \(x, x + \delta_1 \in [-1/2, 1/2]\). Then we have
\[
k_1(x + \delta_1) = \frac{e^{\ell(\alpha)|x + \delta_1|}}{2e^{\ell(\alpha)}e^{\ell'(\eta)}} \geq e^{-\ell(\alpha)|\delta_1|} \frac{e^{\ell(\alpha)|x|}}{2e^{\ell(\alpha)}e^{\ell'(\eta)}} \geq e^{-1}k_1(x)
\]
because \(|\delta_1| \ell(\alpha) \leq \delta(\ell(\alpha))\ell(\alpha) = 1\).

Assume that \(\ell(\alpha) > 2\), \(x \in [-\frac{1}{2}, \frac{1}{2}]\) and \(x + \delta_1 < -\frac{1}{2}\). We refer the reader to Figure 6.3 in order to easily follow the arguments in this case.

Then \(|\delta_1| < \frac{1}{2}\) and \(-\frac{1}{2} \leq x < 0\). Using the facts that \(k_1(x) = k_1(-x)\) and \(k_1(x + 1) = k_1(x)\), we obtain
\[
k_1(x + \delta_1) = k_1(-(1 + x + \delta_1)).
\]
Let \(x_1 = -(1 + x + \delta_1)\). By \(x + \delta_1 < -\frac{1}{2}\), we immediately get \(x_1 + 1/2 > 0\) which in turn gives
\[
|x - x_1| \leq |x + \frac{1}{2}| + |x_1 + \frac{1}{2}| = x + \frac{1}{2} - \frac{1}{2} - x - \delta_1 = -\delta_1 \leq \delta.
\]
Then, by \(x, x_1 \in [-\frac{1}{2}, \frac{1}{2}]\) and the above, we get
\[
k_1(x + \delta_1) = k_1(x_1) \geq e^{-1}k_1(x).
\]
For \(x \in [-1/2, 1/2]\) and \(x + \delta_1 > 1/2\), we similarly obtain \(k_1(x + \delta_1) \geq e^{-1}k_1(x)\).

Finally, since \(k_2(x) = k_1(x - t)\) we have that
\[
k_2(x + \delta_2) \geq e^{-1}k_2(x)
\]
for all \(x\) and \(|\delta_2| \leq \delta\).

Thus, for \(|\delta_1|, |\delta_2| \leq \delta\), and all \(x\) we have
\[
h_1(x + \delta_1) - h_2(x + \delta_2) \geq e^{-1}[h_1(x) - h_2(x)],
\]
which implies \(\inf_{\ell(\alpha) \geq \ell_0} c_{\delta(\ell(\alpha)), \ell'(\alpha)} \geq e^{-1} > 0\). Thus, condition (2) of Corollary 5.3 is satisfied.

To verify condition (1) in Corollary 5.3, we find a lower bound on \(\text{mod} \Gamma_{K'}^0\). Assume first that \(0 \leq t \leq \frac{1}{2}\). Then \(-\frac{1}{2} + t \leq 0\) and thus \([-\frac{1}{2} + t, \frac{1}{2}] \supset [0, \frac{1}{4}]\). For \(x \in [\frac{1}{8}, \frac{1}{4}]\), we have that
\[
k_1(x) \leq \frac{e^{\ell(\alpha)}}{2\ell(\alpha)e^{\ell'(\eta)}} \quad \text{and} \quad k_2(x) \leq \frac{e^{\ell(\alpha)}}{2\ell(\alpha)e^{\ell'(\eta')}}.
\]
By setting \(r^* := \min\{r(\ell(\eta)), r(\ell'(\eta'))\} \geq \frac{\ell(\alpha)}{2}\), we obtain, for \(x \in [\frac{1}{8}, \frac{1}{4}]\),
\[
k_1(x) + k_2(x) \leq \frac{e^{\ell(\alpha)}}{\ell(\alpha)e^{r^*}},
\]
which gives
\[
\text{mod}^{\gamma_2}_{\Gamma'} = \int_{-1/2}^{1/2} \frac{dx}{k_1(x) + k_2(x)} \geq \int_{-1/4}^{1/8} \frac{dx}{k_1(x) + k_2(x)} \geq \frac{1}{8} \ell(\alpha) e^{\frac{3\ell(\gamma)}{8}}.
\]

Thus, \( \inf_{\ell(\alpha) \geq \ell_0} \frac{\delta(\ell(\alpha))^2}{\text{mod}^{\gamma_2}_{\Gamma'}} \geq \inf_{\ell(\alpha) \geq \ell_0} \frac{\frac{3\ell(\gamma)}{8}}{8\ell(\alpha)} > 0 \) and the condition (1) in Corollary 5.3 is satisfied when \( 0 \leq t(\alpha) \leq 1/2 \).

We prove the condition (1) in Corollary 5.3 under the assumption that \( 1/2 \leq t(\alpha) \leq 1 \) (which is equivalent to \( -1/2 \leq t(\alpha) \leq 0 \)). Note that \( [-1/4, -1/2 + t(\alpha)] \supset [-1/4, -1/8] \). For \( x \in [-1/4, -1/2] \) we have that \( k_1(x) \leq \frac{e^{\ell(\alpha)}}{2\ell(\alpha)e^{\ell(\gamma)}} \) as before. On the other hand, we have \( k_2(x) = \frac{e^{\ell(\alpha)|x-t(\alpha)|}}{2\ell(\alpha)e^{\ell(\gamma)}} \) for \( x \in [-1/4, -1/2 + t(\alpha)] \). Since \( -1/4 \leq 1 + x - t(\alpha) \leq 3/8 \), we obtain \( k_2(x) \leq \frac{e^{3\ell(\gamma)/8}}{2\ell(\alpha)e^{\ell(\gamma)}} \). Then condition (1) in Corollary 5.3 is satisfied as in the above paragraph.

The lower estimate for \( \text{mod}^{\gamma_2}_{\Gamma'} \) obtained above is crude. The only purpose was to show that the condition (1) in Corollary 5.3 is satisfied. We now proceed to obtain a finer upper estimate in order to prove the desired inequality in the theorem.

Let \( \Gamma_{\Gamma'} \) be the vertical family above \([-1/2, 1/2]\) connecting the graphs of \( h_1 \) and \( h_2 \). We proceed to compute \( \text{mod}^{\gamma_2}_{\Gamma'} = \int_{-1/2}^{1/2} \frac{dx}{h_1(x) - h_2(x)} \). To facilitate the integration, we will assume that \( 0 \leq t(\alpha) \leq 1 \) with the understanding that the twist \( t(\alpha) \in [1/2, 1] \) gives the same surface as \( t(\alpha) - 1 \in [-1/2, 0] \). The integration over \( x \in [-1/2, 1/2] \) is divided into four intervals. For \(-1/2 \leq x \leq -1/2 + t(\alpha)/2 \) we have the inequality \( h_1(x) - h_2(x) \geq \frac{1}{2\ell(\alpha)} e^{\ell(\gamma)/2} \) which gives
\[
\left( -\frac{1}{2} + \frac{t(\alpha)}{2} \right) \frac{dx}{h_1(x) - h_2(x)} \leq 2e^{\ell(\gamma)/2} \int_{-1/2}^{-1/2 + t(\alpha)/2} \ell(\alpha) e^{\ell(\alpha)x} dx \leq 2e^{\ell(\gamma)/2} - t(\alpha) e^{\ell(\gamma)/2}.
\]

For \(-1/2 + t(\alpha)/2 \leq x \leq -1/2 + t(\alpha) \), we have the inequality \( h_1(x) - h_2(x) \geq \frac{1}{2\ell(\alpha)} e^{\ell(\gamma)/2} \) which gives
\[
\left( -\frac{1}{2} + \frac{t(\alpha)}{2} \right) \frac{dx}{h_1(x) - h_2(x)} \leq 2e^{\ell(\gamma)/2} \int_{-1/2}^{-1/2 + t(\alpha)/2} \ell(\alpha) e^{-\ell(\alpha)x} dx \leq 2e^{\ell(\gamma)/2} - t(\alpha) e^{\ell(\gamma)/2}.
\]

For \(-1/2 + t(\alpha) \leq x \leq t(\alpha)/2 \), we have the inequality \( h_1(x) - h_2(x) \geq \frac{1}{2\ell(\alpha)} e^{\ell(\gamma)/2} \) which gives
\[
\left( -\frac{1}{2} + t(\alpha) \right) \frac{dx}{h_1(x) - h_2(x)} \leq 2e^{\ell(\gamma)/2} \int_{-1/2}^{-1/2 + t(\alpha)/2} \ell(\alpha) e^{\ell(\alpha)x} dx \leq 2e^{\ell(\gamma)/2} - t(\alpha) e^{\ell(\gamma)/2}.
\]
For $t(\alpha)/2 \leq x \leq 1/2$, we have the inequality $h_1(x) - h_2(x) \geq \frac{1}{2t(\alpha)} e^{r(\alpha)\frac{x}{2}}$ which gives

$$\int_{t(\alpha)/2}^{1/2} \frac{dx}{h_1(x) - h_2(x)} \leq 2e^{r(\eta)} \int_{t(\alpha)/2}^{1/2} e^{-r(\alpha)\frac{x}{2}} dx \leq 2e^{r(\eta)} - t(\alpha)\frac{r(\alpha)}{2}. \tag{42}$$

Putting together (39), (40), (41), and (42), we obtain

mod$_R^V \leq c \max\{e^{r(\eta) - t(\alpha)\frac{r(\alpha)}{2}}, e^{r(\eta) - t(\alpha)\frac{r(\alpha)}{2} + t(\alpha)\frac{r(\alpha)}{2}}, e^{r(\eta') - t(\alpha)\frac{r(\alpha)}{2}}, e^{r(\eta') - t(\alpha)\frac{r(\alpha)}{2} + t(\alpha)\frac{r(\alpha)}{2}}\}$

for a fixed $c > 0$ and all $t(\alpha) \in [0, 1]$.

For $t(\alpha) \in [0, 1/2]$, we have

$$\text{mod}_{R'} \leq c \max\{e^{r(\eta) - t(\alpha)\frac{r(\alpha)}{2}}, e^{r(\eta') - t(\alpha)\frac{r(\alpha)}{2}}\}$$

and for $t(\alpha) \in [1/2, 1]$, we have

$$\text{mod}_{R'} \leq c \max\{e^{r(\eta) - t(\alpha)\frac{r(\alpha)}{2}(1-t(\alpha))}, e^{r(\eta') - t(\alpha)\frac{r(\alpha)}{2}(1-t(\alpha))}\}$$

and hence the theorem follows. \qed

Suppose $\alpha$ and $\beta$ are disjoint simple closed geodesics on a hyperbolic surface. By the collar lemma, the standard collars about $\alpha$ and $\beta$ are disjoint. On the other hand, it is possible that nonstandard collars overlap. Indeed, even nonstandard half-collars can overlap as can be seen on a pair of pants. In fact, a pair of pants can have at most two disjoint nonstandard half-collars. Given any two components of a pair of pants one can construct nonstandard half-collars about each one by using the unique simple orthogeodesic from the boundary component to the third component. These half-collars are disjoint. This is the key point in the following topological lemma.

**Lemma 6.4.** Let $Y$ be a geodesic subsurface of finite area which is not a pair of pants and has non-empty boundary. Then for any choice of pants decomposition of $Y$ there are choices of nonstandard half-collars about each boundary component that are pairwise disjoint.

**Proof.** Consider a pair of pants $P$ in the given decomposition having at least one geodesic boundary in common with $Y$. Since there is at least one boundary component of $P$, say $\beta$, interior to $Y$ (otherwise $Y$ would be a pair of pants) one can run a simple orthogeodesic from the boundary component of $Y$ to $\beta$ thus constructing a nonstandard half-collar. As was observed above, if $P$ contains two nonstandard half-collars they are disjoint in $P$, hence in $Y$. \qed

### 7 MODULAR TESTS FOR PARABOLICITY

Recall, from Subsection 1.5 that a characterization of parabolic type Riemann surfaces can be given in terms of extremal distance. Let $\{X_n\}$ be an exhaustion of $X$ by a family of relatively compact regions with piecewise analytic boundary such that $\bar{X}_n \subset X_{n+1}$. Denote by $\beta_n$ the boundary of
Let $\lambda_{X_n - X_1}(\beta_1, \beta_n)$ be the extremal distance between $\beta_1$ and $\beta_n$ in $X_n - X_1$. We will use the following criterion for parabolicity, cf. [6, p. 229].

**Proposition 7.1** (Modular test). The Riemann surface $X$ is parabolic if and only if

$$\lambda_{X_n - X_1}(\beta_1, \beta_n) \to \infty \quad \text{as} \quad n \to \infty.$$

To simplify the application of the Modular test, given an exhaustion $\{X_n\}$ we choose an open set $B_n \subset X_{n+1} - X_{n-1}$ that contains the boundary $\beta_n$ of $X_n$. The boundary of $B_n$ is divided into two sets $a_n$ and $b_n$ such that $a_n \subset X_n$ and $b_n \subset X_{n+1} - X_n$. The interiors of $B_{n-1}$ and $B_n$ are disjoint by the assumption. Denote by $\lambda_n$ the extremal distance between $a_n$ and $b_n$ in $B_n$. Then by the serial rule for extremal length (see [6, p. 222])

$$\lambda_{X_n - X_1}(\beta_1, \beta_n) \geq \sum_{k=1}^{n} \lambda_k$$

and we obtain the following sufficient condition for parabolicity

**Proposition 7.2.** [6] If

$$\sum_{n=1}^{\infty} \lambda_n = \infty,$$

then the Riemann surface $X$ is of parabolic type.

Assume that the Riemann surface $X$ has punctures. Since punctures are points at infinity, any such exhaustion would have boundary components bounding punctured discs which would add terms corresponding to the punctures. We remove this difficulty by showing that an exhaustion that contains full neighborhoods of punctures can be used to replace the compact exhaustion in Propositions 7.1 and 7.2. To this end, since our applications are hyperbolic geometric in nature, we will work in the hyperbolic category.

A geodesic subsurface in $X$ is a subsurface with geodesic boundary. We are interested in finite area geodesic subsurfaces, namely, ones that have finitely many cusps and finitely many closed geodesics on the boundary. The next proposition allows us to relax the criteria from a compact exhaustion to an exhaustion by finite area geodesic subsurfaces.

**Proposition 7.3.** Let $X$ be an infinite type hyperbolic surface with $\{X_n\}$ an exhaustion by finite area geodesic subsurfaces. Then $X$ is parabolic if and only if

$$\lambda_{X_n - X_1}(\partial X_1, \partial X_n) \to \infty \quad \text{as} \quad n \to \infty.$$

A puncture in a hyperbolic surface has a standard cusp neighborhood of hyperbolic area one which we denote by $\mathcal{C}_0$. The set of paths in $C_0$ from $\partial C_0$ that go out the cusp end have infinite extremal length. This leads to the fact that the standard cusp contains a decreasing sequence of cusp neighborhoods, denoted $\{C_R\}$, so that the extremal distance $\lambda_{C_R \setminus C_0}(\partial C_0, \partial C_R) = R$. We start with a lemma that builds on this fact. It is a straightforward application of basic extremal length properties, we leave the proof to the reader.
Lemma 7.4. Suppose $Y$ is a geodesic subsurface with $p \geq 1$ punctures, and $K \subset Y$ a connected compact subset disjoint from the standard cusp neighborhoods of each puncture. Given $R > 0$, denote by $Y'$ the subsurface $Y$ with deleted cusp neighborhoods about each puncture where each of these neighborhoods is isomorphic to $C_R$. Let $\Gamma'$ be the curve family in $Y' \setminus K$ which connects $\partial K$ to the boundary curves of the deleted neighborhoods of the punctures. Then

$$\lambda(\Gamma') \geq \frac{R}{p}.$$  

Proof. Since $K$ is disjoint from the standard cusp neighborhood of each puncture of $Y$, the family of curves $\Gamma'$ overflows the family of curves $\Gamma'_R$ that connect the boundary of the union $U_0$ of the standard cusp neighborhoods of the punctures to the boundary of the union $U_R$ of the cusp neighborhoods isomorphic to $C_R$ inside $U_0 \setminus U_R$. Then we have (see Lemma 3.3, Property 1)

$$\mod\Gamma' \leq \mod\Gamma'_R.$$  

Note that $\Gamma'_R$ is a disjoint union of $p$ families of curves connecting the two boundaries of the two neighborhoods of each puncture and the supports of the $p$ families are disjoint. Since the modulus of each family is $1/R$, the additivity property gives (see Lemma 3.4)

$$\mod\Gamma'_R = \frac{p}{R}.$$  

Then

$$\lambda(\Gamma') = \frac{1}{\mod\Gamma'} \geq \frac{1}{\mod\Gamma'_R} = \frac{R}{p}. \hfill \square$$

We are now ready to prove Proposition 7.3.

Proof. Clearly the proposition holds if there are no punctures, and it is easy to check that it holds in the presence of funnels in $X$. For this reason, we henceforth assume that $X$ has no funnels and at least one puncture.

Let $p(n)$ be the number of punctures in $X_n$. For each puncture in $X_n$, delete the cusp neighborhood isomorphic to $C_{p(n)n}$ and denote the excised domain by $X'_n$ (see Figure 7.1). If there is no puncture in $X_n$, set $X'_n = X_n$. Clearly $\{X'_n\}$ is a compact exhaustion of $X$. As a matter of notational convenience we assume $X_1 = X'_1$. We let

$$\Gamma_n = \Gamma(\partial X_1, \partial X_n; X_n), \quad \Gamma'_n = \Gamma(\partial X_1, \partial X'_n; X'_n),$$

and would like to prove that $\lambda(\Gamma_n)$ and $\lambda(\Gamma'_n)$ diverge simultaneously.

Let $\Gamma^{cusp}_n$ and $\Gamma^{3}_n$ denote the subfamilies of curves in $\Gamma_n$ that go through the cusp boundary of $X'_n$ and those that do not, respectively. Similarly, we define $\Gamma'^{cusp}_n$ and $\Gamma'^{3}_n$. More precisely,

$$\Gamma^{cusp}_n = \{ \gamma \in \Gamma_n : \gamma \cap (\partial X'_n \setminus \partial X_n) \neq \emptyset \}, \quad \Gamma^{3}_n = \Gamma_n \setminus \Gamma^{cusp}_n$$

$$\Gamma'^{cusp}_n = \{ \gamma \in \Gamma'_n : \gamma \cap (\partial X'_n \setminus \partial X_n) \neq \emptyset \}, \quad \Gamma'^{3}_n = \Gamma'_n \setminus \Gamma'^{cusp}_n.$$
By Lemma 7.4, with $Y = X_n$, $K = X_1$, and $R = p(n)n$, we have

$$\lambda(\Gamma_n^{\text{cusp}}) \geq n. \quad (43)$$

Noting that $\Gamma_\partial^\beta = \Gamma_n^{\text{cusp}}$, and using monotonicity and subadditivity in Lemma 3.3, we obtain

$$\frac{1}{\lambda(\Gamma_n^{\text{cusp}})} \leq \frac{1}{\lambda(\Gamma_n^{\partial})} \leq \frac{1}{\lambda(\Gamma_n^{\partial, \beta})} + \frac{1}{\lambda(\Gamma_n^{\text{cusp}})}, \quad (44)$$

Since every curve in $\Gamma_n^{\text{cusp}}$ contains a subcurve in $\Gamma_n^{\text{cusp}}$, by the overflowing property of modulus we have $\lambda(\Gamma_n^{\text{cusp}}) \leq \lambda(\Gamma_n^{\text{cusp}})$. By (43) and (44), we obtain

$$\left| \frac{1}{\lambda(\Gamma_n)} - \frac{1}{\lambda(\Gamma_n')} \right| \leq \frac{1}{\lambda(\Gamma_n^{\text{cusp}})} \leq \frac{1}{n}.$$

Therefore, $\lambda(\Gamma_n) \to \infty$ if and only if $\lambda(\Gamma_n') \to \infty$, thus completing the proof.

We note that the specific choice of the constant $R = p(n)n$ in the proof is not important.

**Remark 7.5.** Proposition 7.3 is part of a much more general phenomena in which a subsurface of the original hyperbolic surface has a compact exhaustion for which the extremal distance goes to infinity.

### 8 APPLICATIONS TO GENERAL INFINITE TYPE SURFACES

Let $X$ be an infinite type Riemann surface equipped with its conformal hyperbolic metric. Recall that a collar is an annular neighborhood about a simple closed geodesic, and that a half-collar has the simple closed geodesic as a boundary component.
We fix an exhaustion \( \{X_n\}_n \) of \( X \) by finite area geodesic subsurfaces (that is, hyperbolic surfaces of finite area with finitely many cusps and finitely many closed geodesics on their boundaries) such that \( X_{n+1} \) contains the closure of \( X_n \) in its interior. We denote by \( \partial_0 X_n \) the collection of geodesic boundary components of \( X_n \) and recall that \( \ell'(\alpha) \) is the length of \( \alpha \in \partial_0 X_n \). We first state an abstract theorem that holds for any set of disjoint collars about the \( \alpha \in \partial_0 X_n \). For a collar \( R_\alpha \), the extremal distance between its two boundary components is denoted by \( \lambda(R_\alpha) \). We rephrase Proposition 7.2 in terms of collars.

**Proposition 8.1.** Let \( X \) be an infinite type hyperbolic surface and \( \{X_n\}_n \) an exhaustion by finite area geodesic subsurfaces as above. Given a collection of disjoint collars \( \{R_\alpha\}_{\alpha \in \partial_0 X_n} \) if

\[
\sum_{n=1}^{\infty} \frac{1}{\sum_{\alpha \in \partial_0 X_n} \frac{1}{\lambda(R_\alpha)}} = \infty,
\]

then \( X \) is of parabolic type.

Since standard collars about disjoint closed geodesics are disjoint, Proposition 7.2 and Lemma 5.5 together with \( \frac{1}{\ell} \arctan\left(\frac{1}{\sinh \frac{\ell}{2}}\right) \approx 1/(e^{\ell/2}) \) give us the following.

**Proposition 8.2.** Let \( X \) be an infinite type hyperbolic surface and \( \{X_n\}_n \) an exhaustion by finite area geodesic subsurfaces as above. If

\[
\sum_{n=1}^{\infty} \frac{1}{\sum_{\alpha \in \partial_0 X_n} \ell'(\alpha)e^{\ell'(\alpha)/2}} = \infty,
\]

then \( X \) is of parabolic type.

The above proposition can be strengthened by using Corollary 5.10 when \( \ell'(\alpha) \) is large. Let \( P \) be a single geodesic pair of pants with boundary geodesics \( \alpha, \beta \) and \( \beta' \), where \( \beta' \) can degenerate to a puncture. Then the nonstandard half-collars around \( \alpha \) and \( \beta \) with the other boundaries having non-smooth point on \( \beta' \) are disjoint.

Assume that \( X_1 \) contains at least two pairs of pants. It follows that each boundary geodesic of \( X_1 \) belongs to a pair of pants in \( X_1 \) such that at least one other boundary geodesic is in the interior of \( X_1 \). Thus, the set of boundary geodesics of \( X_1 \) has a corresponding set of one-sided nonstandard collars in \( X_1 \) that are mutually disjoint by Lemma 6.4. The same property is true for each boundary geodesic of \( X_n \) because it belongs to a pair of pants in \( X_n - X_{n-1} \) that has at least one boundary geodesic in the interior of \( X_n \). By Lemma 6.4 all nonstandard half-collars around boundary geodesics of the exhaustion \( \{X_n\}_n \) are mutually disjoint.

Let \( \alpha \in \partial_0 X_n \) and let \( \lambda(R_{\alpha,\gamma}) \) be the extremal distance between the boundary components of the nonstandard half-collar \( R_{\alpha,\gamma} \) around \( \alpha \) corresponding to orthogeodesic \( \gamma \). Fix \( \varepsilon_0 > 0 \) and \( \gamma_0 > 0 \). For \( \ell'(\gamma) < \gamma_0 \) and \( \ell'(\alpha) > \varepsilon_0 \) we have \( 1/\lambda(R_\alpha) \leq e^{\ell'(\gamma)/2}/\ell'(\alpha) \); for \( \ell'(\gamma) \geq \gamma_0 \) and \( \ell'(\alpha) > \varepsilon_0 \) we have \( 1/\lambda(R_\alpha) \leq e^{\ell'(\gamma)/2} \). For \( \ell'(\alpha) < \varepsilon_0 \) we have \( 1/\lambda(R_\alpha) \leq 1/\ell'(\alpha) \) by Lemma 5.5. By setting

\[
\sigma(R_{\alpha,\gamma}) := \max \left\{ \frac{e^{\ell'(\alpha)/2}}{\ell'(\gamma)}, \frac{e^{\ell'(\alpha)/2}}{\ell'(\alpha)}, \frac{1}{\ell'(\alpha)} \right\},
\]

we obtain the following.
**Theorem 8.3.** Let $X$ be an infinite type hyperbolic surface with an exhaustion $\{X_n\}$. If

$$\sum_{n=1}^{\infty} \frac{1}{\sum_{\alpha \in \delta_0 X_n} \sigma(R_{\alpha, \gamma})} = \infty,$$

then $X$ is of parabolic type.

We use $|\delta_0 X_n|$ to denote the number of components on the boundary of $X_n$.

**Corollary 8.4.** Let $X$ be an infinite type hyperbolic surface with an exhaustion $\{X_n\}$. Let $\sigma_n = \sup_{\alpha \in \delta_0 X_n} \sigma(\lambda(R_{\alpha, \gamma}))$ where $R_{\alpha, \gamma}$ is the nonstandard half-collar around $\alpha$ in $X_n$. If

$$\sum_{n=1}^{\infty} \frac{1}{|\delta_0 X_n| \sigma_n} = \infty,$$  \hspace{1cm} (45)

then $X$ is of parabolic type.

Theorem 8.3 and its corollary are twist-free results. To include twists to achieve sufficient conditions for parabolicity we use nonstandard collars. When the nonstandard collar is a collar (that is, not a half-collar) one has to make some topological restrictions (either on the topology of the surface or on the exhaustion) to ensure the nonstandard collars are disjoint.

Suppose $\{X_n\}$ is an exhaustion of $X$ as discussed at the beginning of this section, and fix a pants decomposition of $X$ whose pants curves include the boundary curves of $\{X_n\}$. Each boundary geodesic $\alpha$ of $X_n$ is an interior geodesic of $X$ and hence it makes sense to talk about the twist $t(\alpha)$ of $\alpha$.

**Theorem 8.5.** Let $X$ be an infinite type hyperbolic surface with an exhaustion $\{X_n\}$. Assume each connected component $Y$ of $X_{n+1} - X_n$ is not a pair of pants and there is a pants decomposition of $Y$ so that the distance from any boundary geodesic of $Y$ to the boundary component of the pair of pants that is interior is uniformly bounded from below. If

$$\sum_{n=1}^{\infty} \frac{1}{\sum_{\alpha \in \delta_0 X_n} e^{-(1-t(\alpha)) \frac{\ell(\alpha)}{2}}} = \infty,$$  \hspace{1cm} (46)

then $X$ is of parabolic type.

**Proof.** By using Lemma 6.4 on the given pants decomposition, we can conclude that the boundary components of $X_n$ have disjoint nonstandard half-collars. Since the boundary components of $X_{n+1} - X_n$ also have disjoint nonstandard half-collars, they glue together to allow us to conclude that the boundary geodesics of each $X_n$ have nonstandard collars that are pairwise disjoint. The assumption on each connected component $Y$ implies that the nonstandard collars around the boundaries of $X_n$ are simultaneously disjoint for all $n$. Using the estimates for the extremal length of nonstandard collars coming from Theorem 6.1 and Remark 6.2, formula (38) in terms of length and twist allows us to conclude the result. \qed
If we set
\[ L_n = \max \{ \ell'(\alpha) : \alpha \in \partial_0 X_n \}, \]
\[ \tau_n = \min \{|t_n(\alpha)| : \alpha \in \partial_0 X_n \}, \]
we obtain the following corollary.

**Corollary 8.6.** Let \( X \) be as in Theorem 8.5. Then \( X \) is parabolic if
\[ \sum_{n=1}^{\infty} \frac{1}{|\partial_0 X_n| e^{(1-\tau_n)\frac{L_n}{2}}} = \infty. \] (47)

If we assume that \( |\partial_0 X_n| \approx n^p \) for some \( p \geq 0 \) and \( \tau_n \geq \tau > 0 \) then by Equation (47) \( X \) would be parabolic if
\[ L_n \leq \frac{2}{1-\tau}[(1-p) \log n + \log \log n]. \]

In particular, we have the following two cases which will be useful in the discussion of abelian covers in Subsection 10.5. Cases (1) and (2) correspond to abelian \( \mathbb{Z} \) and \( \mathbb{Z}^2 \) covers.

**Example 8.7.** We have the following sufficient conditions for parabolicity:
(1) \( |\partial_0 X_n| = O(1) \), \( \tau_n \geq \tau > 0 \), and \( L_n \leq \frac{2}{1-\tau} [\log n + \log \log n] \);
(2) \( |\partial_0 X_n| = O(n) \), \( \tau_n \geq \tau > 0 \), and \( L_n \leq \frac{2}{1-\tau} [\log \log n] \).

**Remark 8.8.** We note that increasing the twist in Equation (46) preserves divergence. More specifically, if \( |t'(\alpha)| \geq |t(\alpha)| \) for all \( \alpha \), then parabolicity persists.

The hypotheses of the next theorem are somewhat different from the others in that we make an assumption about the existence of half-collars with no relation to the length of the core geodesic.

**Theorem 8.9.** Let \( X \) be an infinite type hyperbolic surface with an exhaustion \( \{X_n\} \) so that \( \partial X_n \) has one boundary component and is contained in a half-collar of width \( \varepsilon_n \). Set \( \ell'_n = \ell'(\partial X_n) \). If
\[ \sum \frac{\arctan(\sinh \varepsilon_n)}{\ell'_n} = \infty, \] (48)
then \( X \) is parabolic.

**Remark 8.10.** The main application here is to situations where the length of the core geodesic is going to infinity but the collar width is larger than the standard collar width. For example, this happens if there is a lower bound on the width for all \( n \) (see Theorem 10.5).

**Proof.** In the proof of Lemma 5.5, using \( \varepsilon_n \) for the collar width instead of the standard collar width \( r(\ell'_n/2) \) yields that the curve family joining the boundary components of the half-collar...
has extremal length \(\frac{\arctan(\sinh \varepsilon_n)}{\varepsilon_n}\). The theorem now follows by applying Proposition 8.1 to this case. The details are left to the reader.

\[ \square \]

## 9 | FLUTE SURFACES AND PANTS DECOMPOSITION

In this section, we apply results of previous sections to obtain sufficient conditions for parabolicity for tight flute surfaces. We say that \(X\) is a \textit{tight flute surface} if it can be constructed by consecutively gluing a sequence of tight pairs of pants \(P_n\) (see Figure 1.1). More specifically, given a sequence \(\ell_n > 0\) of lengths and a sequence \(t_n \in [-1/2, 1/2)\) of twists, \(n \geq 1\), we define the corresponding tight flute surface

\[
X = X(\{\ell_n, t_n\})
\]

as follows.

Let \(P_0\) be a tight pair of geodesic pants with two punctures and one boundary geodesic \(\alpha_1\) of length \(\ell(\alpha_1) = \ell_1\). For \(n \geq 1\), let \(P_n\) be a tight pair of geodesic pants with one puncture and two boundary geodesics \(\alpha_n\) and \(\alpha_{n+1}\) of lengths \(\ell(\alpha_n) = \ell_n\) and \(\ell(\alpha_{n+1}) = \ell_{n+1}\). In particular, \(P_{n-1}\) and \(P_n\) both have boundary geodesics of length \(\ell_n\) which (abusing the notation slightly) will both be denoted by \(\alpha_n\). Then we glue by an isometry \(P_{n-1}\) to \(P_n\) along \(\alpha_n\) for \(n \geq 1\).

The choice in the gluing of \(P_{n-1}\) and \(P_n\) is given by a (relative or angle) \textit{twist parameter} \(t_n \in [-1/2, 1/2)\) (see Section 2). The \textit{absolute twist} \(|t_n|\ell_n\) is the shortest distance between the feet of the orthogeodesics \(\gamma_{n-1}\) and \(\gamma_n\) to \(\alpha_n\) (see Section 2 for the definition and Figure 1.1 for the notation). The surface \(X\) obtained by these consecutive gluings is called a tight flute surface [9]. We denote by \(X(\{\ell_n, t_n\})\) the resulting hyperbolic surface. The surface \(X(\{\ell_n, t_n\})\) may not be complete in which case its completion would contain geodesic half-planes and would not be of parabolic type.

One of the main applications of the general results of this paper is a sufficient condition for parabolicity for flute surfaces. Specifically, from Corollary 6.3 and Proposition 7.2, we obtain the following.

**Theorem 9.1.** A tight flute surface \(X(\{\ell_n, t_n\})\), with \(\ell_n \geq \ell_0 > 0\) is of parabolic type if

\[
\sum_{n=1}^{\infty} e^{-(1-|t_n|)\frac{\ell_n}{2}} = \infty. \tag{49}
\]

**Proof.** Let \(R_{\alpha_n, \gamma_n'}\) and \(R_{\alpha_{n+1}, \gamma_{n+1}'}\) be the nonstandard half-collars in \(P_n\) about \(\alpha_n\) and \(\alpha_{n+1}\), where \(\gamma_n'\) and \(\gamma_{n+1}'\) are as in Figure 1.1. The bi-infinite simple geodesic \(\beta_n \subset P_n\) from the puncture to itself is the common boundary of these half-collars. Then \(R_{\alpha_n, \gamma_n'}\) and \(R_{\alpha_{n+1}, \gamma_{n+1}'}\) are disjoint and \(R_{\alpha_n, \gamma_n'} \cup R_{\alpha_{n+1}, \gamma_{n+1}'} = P_n\). Let \(R_{\alpha_n, \gamma_n''}^{(\alpha_n)}\) be the nonstandard collar around \(\alpha_n\) in \(P_{n-1} \cup P_n\). Thus, \(R_{\alpha_n, \gamma_n''}^{(\alpha_n)} = R_{\alpha_n, \gamma_{n-1}'} \cup \alpha_n \cup R_{\alpha_{n+1}, \gamma_n'}\). If \(X_n = \bigcup_{i=0}^{n-1} P_i\) then \(R_{\alpha_n, \gamma_n''}^{(\alpha_n)} \subset X_{n+1} - X_{n-1}\). The boundary components of \(R_{\alpha_n, \gamma_n''}^{(\alpha_n)}\) are \(\beta_{n-1} \subset X_n\) and \(\beta_n \subset X_{n+1} - X_n\). The interiors of \(R_{\alpha_{n-1}, \gamma_{n-1}'}^{(\alpha_{n-1})}\) and \(R_{\alpha_{n-2}, \gamma_{n-2}'}^{(\alpha_{n-1})}\) are disjoint and Proposition 7.2 applies.
If $\lambda_n$ is the extremal distance between the boundary components $\beta_{n-1}$ and $\beta_n$ of $R_{\alpha_n,\gamma''_{n-1},\gamma'_n}$, then by Corollary 6.3 we have
\[
\sum_n \lambda_n = \sum_n \lambda(R_{\alpha_n,\gamma''_{n-1},\gamma'_n}) \geq C \sum_n e^{-(1-|t_n|)\varepsilon_n/2}.
\]
Thus, (49) implies that $\sum_n \lambda_n = \infty$ and $X$ is of parabolic type by Proposition 7.2.

**Remark 9.2.** Note that in the case of a flute surface $X$ even though $X_{n+1} - X_n$ is a pair of pants the nonstandard collars around the geodesics $\alpha_n$ are disjoint. This is a consequence of the fact that pairs of pants $\{P_n\}$ are glued in a chain by attaching one boundary component of $P_{n+1}$ to the boundary component of $X_n$. Thus, the nonstandard collars have disjoint interior. Therefore, we do not need to apply Lemma 6.4 to flute surfaces.

Since $|t_n| \in [0, 1/2]$, we obtain the following ‘twist-free’ corollary of Theorem 9.1.

**Corollary 9.3.** A tight flute surface $X(\{\ell'_n, t_n\})$ is of parabolic type, independent of twists, if
\[
\sum_n e^{-\ell'_n/2} = \infty.
\] (50)

In the case of zero-twist flute surfaces, Theorem 9.1 gives a complete characterization of parabolicity. We denote by $\delta_n$ the orthogeodesic between $\alpha_n$ and $\alpha_{n+1}$ and by $\ell(\delta_n)$ its length.

**Theorem 9.4 (Zero-twist flutes).** Let $X = X(\{\ell'_n, 0\})$ be a zero-twist tight flute surface. The following are equivalent:

1. $X$ is of parabolic type,
2. $X$ is complete,
3. $\sum_n \ell(\delta_n) = \infty$,
4. $\sum_{n=1}^\infty e^{-\ell'_n/2} = \infty$.

**Proof.** Since every incomplete surface carries a Green’s function it is not parabolic, and therefore (1) $\Rightarrow$ (2). To show that (2) $\Rightarrow$ (3), note that if $t_n = 0$ then the orthogeodesics $\delta_n$ between $\alpha_n$ and $\alpha_{n+1}$ connect to each other at their end points to form a geodesic ray of length $\sum_{n=1}^\infty \ell(\delta_n)$ that leaves every compact subset of $X$. Therefore, if $\sum_{n=1}^\infty \ell(\delta_n) < \infty$ then $X$ is incomplete.

By (26) in Section 5, we have $\ell(\delta_n) \approx e^{-\ell'_n/2}$ if $\ell(\gamma) = \infty$ and $\ell'_n \to \infty$ and hence (3) $\Leftrightarrow$ (4).

Finally, by Theorem 9.1 we have (4) $\Rightarrow$ (1) since $t_n = 0$.

**Example 9.5.** Let $X = X(\{\ell'_n, 0\})$ be a zero-twist surface.

1. Suppose $\lim_{n \to \infty} \ell'_n ln n = c > 0$. Then $X$ is parabolic if $c \in [0, 2)$, not parabolic if $c \in (2, \infty)$, and $X$ could be either parabolic or not parabolic if $c = 2$, see (3) below.
2. Let $\ell'_n = c ln n$. Then $X$ is parabolic if and only if $c \in (0, 2]$.
3. Let $\ell'_n = 2 \ln n + c \ln(\ln n)$. Then $X$ is parabolic if and only if $c \in [0, 2]$. 

9.1 Incomplete half-twist tight flutes

A tight flute surface all of whose twists are 1/2 is called a half-twist tight flute. By Theorem 9.1, we have that a half-twist surface $X(\{\ell_n, 1/2\})$ is parabolic if

$$\sum_n e^{-\frac{\ell_n}{2}} = \infty. \quad (51)$$

Even though we do not know if (51) is equivalent to parabolicity of $X$, in general we will show that this is the case under some mild assumptions on the lengths $\ell_n$, see Theorem 9.7. We also provide examples illustrating how our sufficient conditions for parabolicity and non-parabolicity (in fact incompleteness) complement each other, see Example 9.9. To do this, we first obtain a sufficient condition for $X$ to be incomplete and hence not of parabolic type.

As mentioned above, every parabolic surface is necessarily complete. However, there are examples of complete flute surfaces which are not parabolic, cf. [25]. Also, in [11] it was shown that for any choice of lengths $\{\ell_n\}$ there is a choice of twists $\{t_n\}$ such that $X(\{\ell_n, t_n\})$ is complete. This choice is not constructive and given an explicit choice of twists $\{t_n\}$ it is difficult to decide whether a surface is complete or incomplete. Thus, it is natural to look for conditions implying completeness or incompleteness in terms of the Fenchel–Nielsen parameters, $\{\ell_n\}$ and $\{t_n\}$.

One might expect that given a sequence of lengths $\{\ell_n\}$ then the choice of largest possible twists, that is, half-twists $|t_n| = 1/2$ is ‘the best possible’ choice of twists for $X = X(\{\ell_n, t_n\})$ implying completeness (see Matsuzaki [30] for a related choice $t_n = 1/4$). In a sense, our next result identifies a class of surfaces for which this is not true.

**Theorem 9.6.** A half-twist tight flute surface $X(\{\ell_n, \frac{1}{2}\})$ is incomplete if

$$\sum_n e^{-\frac{\sigma_n}{2}} < \infty, \quad (52)$$

where $\sigma_n = \ell_n - \ell_{n-1} + \cdots + (-1)^{n-1}\ell_1$.

Before proving Theorem 9.6, we discuss some of its consequences. An immediate corollary of Theorem 9.6 is that if a half-twist tight flute surface $X(\{\ell_n, \frac{1}{2}\})$ is parabolic then

$$\sum_n e^{-\frac{\sigma_n}{2}} = \infty. \quad (53)$$

In view of the above, it is natural to seek conditions on $\{\ell_n\}$ so that (51) and (53) are equivalent (or not). First, note that if $\ell_n$ has a bounded subsequence then (51) holds and $X$ is parabolic. Therefore, to obtain a non-trivial condition for parabolicity, we will always assume that $\ell_n \to \infty$.

We say that $\{\ell_n\}$ is a concave sequence if and there is non-decreasing concave function $f : [0, \infty) \to [0, \infty)$ such that $\ell_n = f(n)$ for $n \geq 0$. Equivalently, $\{\ell_n\}$ is concave if it is non-decreasing and for $n \geq 1$ the following holds:

$$2\ell_n \geq \ell_{n+1} + \ell_{n-1}. \quad (54)$$
For half-twist surfaces corresponding to concave sequences Theorem 9.6 gives the following characterization of parabolicity (analogous to Theorem 9.4 in the case of zero-twists).

**Theorem 9.7.** Let \( X = X(\{\ell_n, 1/2\}) \) such that \( \{\ell_n\} \) is a concave sequence. Then the following are equivalent:

1. \( X \) is parabolic,
2. \( X \) is complete,
3. \( \sum_n \sqrt{\ell(\delta_n)} = \infty \),
4. \( \sum_n e^{-\ell_n/4} = \infty \).

**Proof of Theorem 9.7.** Just like in the proof of Theorem 9.4, the implication (1) \( \Rightarrow \) (2) is true in general, (3) \( \Leftrightarrow \) (4) holds because of \( \ell(\delta_n) \simeq e^{-\ell_n/2} \), and (4) \( \Rightarrow \) (1) follows from Theorem 9.1. Thus, we only need to show the implication (2) \( \Rightarrow \) (4). Equivalently, we will show that \( X \) is incomplete if

\[
\sum_n e^{-\ell_n/4} < \infty. \tag{55}
\]

Observe that from the definition of \( \sigma_n \) we have for every \( n \geq 1 \)

\[
\sigma_n + \sigma_{n-1} = \ell_n, \\
\sigma_{n+2} - \sigma_n = \ell_{n+2} - \ell_{n-1}. \tag{56}
\]

Let \( \varepsilon_n = \ell_{2n} - \ell_{2n-1} \), and \( \eta_n = \ell_{2n+1} - \ell_{2n} \) for \( n \geq 1 \). Since \( \{\ell_n\} \) is non-decreasing we have that \( \varepsilon_n \geq 0 \) and \( \eta_n \geq 0 \). Moreover, since \( \{\ell_n\} \) is concave, rewriting (54) we have \( \ell_{n+1} - \ell_n \leq \ell_n - \ell_{n-1} \). In particular, for all \( n \geq 1 \) we have

\[
\varepsilon_{n+1} \leq \eta_n \leq \varepsilon_n. \tag{57}
\]

Using (57), we obtain

\[
\sigma_{2k+1} = \sum_{i=1}^k \eta_i + \ell_1 \leq \sum_{i=1}^k \varepsilon_i + \ell_1 = \sigma_{2k} + \ell_1, \\
\sigma_{2k+1} \geq \sum_{i=1}^{k-1} \eta_i + \ell_1 \geq \sum_{i=1}^k \varepsilon_i - \varepsilon_1 + \ell_1 \geq \sigma_{2k} - \ell_2.
\]

Thus, \( |\sigma_{2k+1} - \sigma_{2k}| \leq \max\{\ell_1, \ell_2\} \leq \ell_2 \). Therefore, using the second line in (56) and the fact that \( \ell_{2k+1} - \ell_{2k} \) is non-increasing, we obtain

\[
|\sigma_{2k} - \sigma_{2k-1}| \leq |\sigma_{2k+1} - \sigma_{2k}| + |\sigma_{2k+1} - \sigma_{2k-1}|
\leq \ell_2 + \ell_{2k+1} - \ell_{2k-1}
\leq 2\ell_2.
\]
In particular, \(|\sigma_n - \sigma_{n-1}| \leq 2 \epsilon_2\) for every \(n \geq 1\). Since \(\epsilon_n = \sigma_n + \sigma_{n-1}\) we obtain

\[
|2\sigma_n - \epsilon_n| = |\sigma_n - \sigma_{n-1}| \leq 2 \epsilon_2.
\]

From (58), it follows that \(e^{-\sigma_n/2} \leq e^{\epsilon_2/2} e^{-\epsilon_n/4}\). Thus, by Theorem 9.6 (55) implies (52) and \(X\) is incomplete, which completes the proof. \(\square\)

**Example 9.8.** Let \(X = X(\{\epsilon_n, 1/2\})\) be a half-twist tight flute surface such that \(\{\epsilon_n\}\) is concave.

1. Suppose \(\lim_{n \to \infty} \epsilon_n = c \geq 0\). Then \(X\) is parabolic if \(c \in [0, 4)\), not parabolic if \(c \in (4, \infty)\), and \(X\) could be either parabolic or not parabolic if \(c = 4\), see (3) below.

2. Let \(\epsilon_n = c \ln n\). Then \(X\) is parabolic if and only if \(c \in (0, 4)\).

3. Let \(\epsilon_n = 4 \ln n + c \ln(\ln n)\). Then \(X\) is parabiliic and only if \(c \in [0, 4]\).

**Proof of Theorem 9.6.** Let \(P_n\), for \(n \geq 1\), be a tight pair of pants with two boundary geodesics \(\alpha_n\) and \(\alpha_{n+1}\) of lengths \(\epsilon_n\) and \(\epsilon_{n+1}\) and third boundary a puncture. Let \(P_0\) be a tight pair of pants with one geodesic boundary \(\alpha_1\) of length \(\epsilon_1\) and two other boundaries being punctures. Denote by \(\delta_n\) the orthogeodesic connecting \(\alpha_n\) and \(\alpha_{n+1}\) in \(P_n\). Denote by \(\gamma'\) the simple geodesic ray orthogonal to \(\alpha_n\) ending in the puncture of \(P_n\), and denote by \(\gamma''\) the simple geodesic ray orthogonal to \(\alpha_{n+1}\) ending in the puncture of \(P_n\). Then \(P_n\) has front to back hyperbolic symmetry with \(\delta_n\) and \(\gamma' \cup \gamma''\) as two arcs of fixed points of the symmetry.

We glue \(\{P_n\}_n=0^\infty\) along boundary geodesics of equal lengths with half-twists \(t_n = \frac{1}{2}\) to obtain a half-twist tight flute surface \(X\). The geodesic arc \(\delta_n\) is continued by \(\gamma'_n+1\) and they make a geodesic ray. Similarly, \(\gamma''_n \cup \delta_{n+1}\) is a geodesic ray. The surface \(X\) has front to back orientation reversing hyperbolic symmetry with one arc of fixed point being \(\gamma'_1 \cup \gamma''_1 \cup \delta_2 \cup \gamma'_3 \cup \gamma''_3 \cup \delta_4 \cup \cdots\) and the other arc of fixed points being \(\delta_1 \cup \gamma'_2 \cup \gamma''_2 \cup \delta_3 \cup \gamma'_4 \cup \gamma''_4 \cup \cdots\) (see Figure 9.1).

We consider the front side of \(X\) and construct a path \(p\) (in the front side of \(X\)) of finite length starting from \(\alpha_1\) and intersecting each \(\alpha_n\). Such a path \(p\) leaves every compact subset of \(X\) and hence it will follow that \(X\) is incomplete. The front side of \(X\) consists of pentagons \(\Sigma_n\) which are the front sides of the pants \(P_n\). In the pentagon \(\Sigma_1\), we make a Saccheri quadrilateral with the base \(\delta_1\), one side being half of \(\alpha_1\) (which has length \(\frac{1}{2} \epsilon_1\)), the other side being a part of \(\alpha_2\) of length \(\frac{1}{2} \epsilon_2\) and denote the length of the summit by \(s_1\). We construct a Saccheri quadrilateral in \(\Sigma_2\) which has base \(\delta_2\), one sides lying on \(\alpha_2\) of length \(\frac{1}{2}(\epsilon_2 - \epsilon_1)\) and the other side on \(\alpha_3\), and the summit whose length is \(s_2\) shares a point in common with the summit in \(\Sigma_1\). We continue this construction through all pentagons \(\Sigma_n\) and the path \(p\) is the concatenation of the segments of the Saccheri quadrilaterals (see Figure 9.2).

We first estimate the length of the base \(\delta_n\) of the pentagon \(\Sigma_n\). Draw an orthogonal to \(\delta_n\) from the ideal vertex of \(\Sigma_n\). Let \(\delta^1_n\) and \(\delta^2_n\) be the two arcs that \(\delta_n\) is divided into by the orthogonal. Then by taking \(\varphi = 0\) in [15, p. 38, Theorem 2.3.1(i)], we have

\[
sinh \frac{\epsilon(\delta^1_n)}{2} = 1, \quad \sinh \frac{\epsilon(\delta^2_n)}{2} = 1.
\]

Since \(\epsilon_n\) is large, we obtain \(\epsilon(\delta^1_n) + \epsilon(\delta^2_n) = \epsilon(\delta_n) \approx e^{-\epsilon_n/2} + e^{-\epsilon_{n+1}/2}\).

Next, we estimate the lengths \(s_n\). Note that the two lengths of the sides of the Saccheri quadrilateral in \(\Sigma_n\) are \(\frac{1}{2}(\epsilon_n - \epsilon_{n-1} + \cdots + (-1)^{n-1} \epsilon_1)\) for \(n \geq 2\). We divide each Saccheri quadrilateral into two quadrilaterals by the common orthogonal to \(s_n\) and \(\delta_n\). The two new quadrilaterals have
A half-twist tight flute surface $X = X(\{\ell_n, 1/2\})$. Note that $\ell_n$ and $d_n$ denote the lengths of $\alpha_n$ and the orthogeodesic $\delta_n$, respectively.

The dotted path $p$ consisting of summits of Saccheri quadrilaterals.
FIGURE 9.3 The parameter space \((a, b)\) for the family of half-twist tight flute surfaces \(X_{a,b}\). In the closure of the blue triangle surfaces are parabolic. In the interior of the pink unbounded region the surfaces are incomplete and hence not parabolic. In the white regions, our results are inconclusive. If \(m > 1\) the surfaces \(X_{a,ma}\) are parabolic for \(a \in \left[0, 4/(m+1)\right]\), hyperbolic for \(a > 2\), and are of unknown type for \(a \in (4/(m+1), 2]\)

three right angles and we have (see [15, p. 38, Theorem 2.3.1(v)])

\[
\sinh \frac{1}{2}s_n = \sinh \frac{\ell'(\delta_n)}{2} \cosh \frac{\ell_n - \ell_{n-1} + \cdots + (-1)^{n-1}\ell_1}{2}.
\]

By \(\cosh \frac{\ell_n - \ell_{n-1} + \cdots + (-1)^{n-1}\ell_1}{2} \approx e^{\ell_n - \ell_{n-1} + \cdots + (-1)^{n-1}\ell_1/2}\) and \(\ell'(\delta_n) \approx e^{-\ell_n/2} + e^{-\ell_{n+1}/2}\), the above formula implies that

\[
s_n \approx e^{-\ell_{n+1} + \ell_n - \ell_{n-1} + \cdots + (-1)^{n+1}\ell_1/2} + e^{-\ell_{n-1} + \cdots + (-1)^{n}\ell_1/2}.
\]

Therefore, the length of the path \(p\), that is, the sum \(\sum_n s_n\), is finite and \(X\) is incomplete. □

**Example 9.9.** To illustrate the range of applicability of Theorems 9.1 and 9.6 we consider a two-parameter family of half-twist tight flute surfaces \(X_{a,b}\) with \(a, b > 0\) (see Figure 9.3).

For every pair of points \(0 < a, b < \infty\) we will define a sequence \(\ell_{a,b} = \{\ell_n\}\) and consider the corresponding half-twist tight flute surface \(X_{a,b} = X(\ell_{a,b}, 1/2)\). The sequence \(\ell_{a,b}\) will be chosen in such a way that

\[
\sigma_{2n} \approx a \ln n, \\
\sigma_{2n+1} \approx b \ln n, \tag{59}
\]

where as before \(\sigma_n = \ell_n - \ell_{n-1} + \cdots + (-1)^{n-1}\ell_1\). For instance, if we let \(\ell_1 \in (0, a \ln 2)\) and for \(n \geq 1\) define

\[
\ell_{2n} = a \ln (n+1) + b \ln n, \\
\ell_{2n+1} = (a + b) \ln (n + 1), \tag{60}
\]
then a simple calculation shows that 
\[ \sigma_{2n+1} = b \ln(n+1) + \ell_1 \] and \[ \sigma_{2n} = a \ln(n+1) - \ell_1, \] and therefore (59) is satisfied.

From (60), we have that \[ \ell_n \simeq (a + b) \ln n. \] Therefore, \[ e^{-\ell_n/4} \asymp n^{-(a+b)/4}, \] and by Theorem 9.1 \( X_{a,b} \) is parabolic if the series \( \sum_n n^{-(a+b)/4} \) diverges, or equivalently if \( a + b \leq 4 \), see the blue triangle in Figure 9.3.

On the other hand, from (59) we have
\[
\sum_n e^{-\sigma_n/2} = \sum_k \left( \frac{1}{e^{\sigma_{2k}/2}} + \frac{1}{e^{\sigma_{2k+1}/2}} \right) \asymp \sum_k \left( \frac{1}{k^{a/2}} + \frac{1}{k^{b/2}} \right)
\]
\[
\asymp \sum_k \frac{1}{k^{\min(a,b)/2}}.
\]

Thus, by Theorem 9.6, \( X_{a,b} \) is incomplete if \( \min\{a, b\} > 2 \), see the interior of the unbounded shifted quadrant in Figure 9.3.

In particular, for every \( m > 1 \) the surface \( X_{a,ma} \) is parabolic for \( a \in [0, 4/(m+1)] \), incomplete hence not parabolic for \( a > 2 \), and the type is not known for \( a \in (4/(m+1), 2] \) (this corresponds to the line through the origin in Figure 9.3).

**9.2 Bi-infinite tight flute surface**

We refer to Figure 9.4 which we call the bi-infinite flute surface. All of its ends are planar and its space of ends is homeomorphic to the subset of the real line, \( \{0\} \cup \{1\} \cup \{1 - \frac{1}{n}\}_{n \in \mathbb{N}} \cup \{0\} \cup \{1 - \frac{1}{n}\}_{n \in \mathbb{N}}. \) The isolated ends are all cusps and by Theorem 8.5, we have the following.

**Theorem 9.10.** Let \( X \) be a bi-infinite tight flute as in Figure 9.4. If
\[
\sum_{n=1}^{\infty} \frac{1}{e^{(1-|r_n|)/2}} e^{(1-|r_{-n}|)/2} = \infty,
\]
then \( X \) is of parabolic type.
10 | A TRIP TO THE MENAGERIE: APPLICATIONS TO VARIOUS TOPOLOGICAL TYPES

To give the reader a sense of the scope of the applications of Theorem 8.3, we give a sampling of our sufficient conditions for the parabolicity (ergodicity of the geodesic flow) theorem applied to various topological settings. With the exception of Subsection 10.5, each subsection is devoted to a particular topological type (illustrated by a figure in the subsection) of an infinite type hyperbolic surface. The geodesic pants decomposition of a surface is given by the family of closed geodesics \( \{\alpha_n\} \). The relative twist around \( \alpha_n \) is denoted \( t_n = t(\alpha_n) \), and the length of \( \alpha_n \) is denoted by \( \ell_n = \ell(\alpha_n) \). Any other notation in the figure should be self-explanatory. Finally, in Subsection 10.5 we address the question of when a topological abelian cover of a compact surface is of parabolic type.

10.1 | Loch–Ness monster (infinite genus and one non-planar end)

Let \( X_1^\infty \) be as in Figure 1.2. The surface \( X_1^\infty \) is obtained from the tight flute surface by replacing each puncture with a closed geodesic and attaching a finite genus to each closed geodesic (genus of the attached surfaces may vary and the supremum might be equal to infinity). The surface has one topological non-planar end and infinite genus. We note that if an infinite subsequence of \( \{\alpha_n\}_n \) have lengths bounded from above then the surface \( X_1^\infty \) is of parabolic type using the estimate for standard collars in Lemma 5.5. It is therefore of interest to assume that the lengths of \( \alpha_n \) converge to infinity as \( n \) increases. Using Corollaries 5.11 and 6.3, we obtain the following.

**Theorem 10.1.** Let \( X_1^\infty \) be a hyperbolic Loch–Ness monster as in Figure 1.2. Suppose there exists \( M > 0 \) so that \( \ell(\beta_n) \leq M \), for all \( n \). Then \( X_1^\infty \) is of parabolic type if

\[
\sum_{n=1}^{\infty} e^{-(1-|t(\alpha_n)|)} \frac{\ell(\alpha_n)}{2} = \infty.
\]

10.2 | Ladder surface (Infinite genus and two non-planar ends)

We denote by \( X_2^\infty \) the infinite genus surface with two non-planar ends. Denote by \( \alpha_n \), for \( n \in \mathbb{Z} \), the geodesics which together with \( \beta_n \) make a geodesic pants decomposition of the flute part of \( X_2^\infty \). Using Corollaries 5.11 and 6.3 and Theorem 8.3, we obtain the following.

**Theorem 10.2.** Let \( X_2^\infty \) be as in Figure 10.1. Assume there exists \( M > 0 \) such that \( \ell(\beta_n) \leq M \), for all \( n \). Then \( X_2^\infty \) is of parabolic type if

\[
\sum_{n=1}^{\infty} \frac{1}{e^{(1-|t(\alpha_n)|)} \frac{\ell(\alpha_n)}{2} + e^{(1-|t(\alpha_n)|)} \frac{\ell(\alpha_n)}{2}} = \infty.
\] (61)

10.3 | The complement of the Cantor set

Let \( X_\infty \) be a genus zero surface whose space of topological ends is a Cantor set as in Figure 10.2. The surface \( X_\infty \) is homeomorphic to the complement of a Cantor set on the Riemann sphere. The
surface $X_\infty$ is obtained by gluing pairs of pants with boundary geodesics $\alpha^n_j$, for $n = 1, 2, \ldots$ and $j = 1, 2, \ldots, 2^{n+1}$, as follows. In step $n = 1$, we glue two pairs of pants along a geodesic boundary $\alpha_1^1$. The obtained surface is a sphere minus four disks with four geodesic boundary curves $\alpha_j^1$, for $j = 1, 2, 3, 4$. In step $n$ the obtained surface has genus zero and $2^{n+1}$ boundary geodesics $\alpha_k^n$, for $n = 1, 2, \ldots$ and $k = 1, 2, \ldots, 2^{n+1}$. In the next step we glue a pairs of pants to each $\alpha_j^n$ and obtain genus zero surface with $2^{n+2}$ boundary geodesics $\alpha_j^{n+1}$ for $j = 1, 2, \ldots, 2^{n+2}$. It is well-known that if the lengths of $\alpha_j^n$ are bounded from below, then the surface $X_\infty$ is not of parabolic type [32].
Theorem 10.3. Consider the compact exhaustion of the complement of the Cantor set $X_\infty$ as in Figure 10.2. Then $X_\infty$ is of parabolic type if for every $n \geq 1$ and all $\alpha \in \partial_0 X_n$, we have

$$\ell(\alpha) \leq C \frac{n}{2^n}.$$  

Proof. Denote by $R$ the standard one-sided collar around the geodesic $\alpha \in \partial_0 X_n$. Since $\ell(\alpha) \leq C \frac{n}{2^n}$, Corollary 5.10 implies that $1/\lambda(R) \leq C' \frac{n}{2^n}$. Thus,

$$\sum_{n=1}^{\infty} \frac{1}{\sum_{j=1}^{2^n} 1/\lambda(R)} \geq \frac{1}{C'} \sum_{n=1}^{\infty} \frac{1}{2^n n/2^{n+1}} = \infty$$

and the surface $X_\infty$ is of parabolic type by Theorem 8.3. □

We remark that the same theorem holds if we replace $X_\infty$ with the blooming Cantor tree, that is, a Riemann surface whose space of ends is a Cantor set and each end is accumulated by genus.

10.4 Surfaces with $|\partial_0 X_n|$ bounded

Let $\{X_n\}$ be an exhaustion of $X$ so that the number of boundary components of $X_n$ is constant as for the surface in Figure 10.3. Note that this implies that $X$ has finitely many non-isolated ends as well as finitely many non-planar ends. Setting $L_n = \max\{\ell(\alpha) : \alpha \in \partial_0 X_n\}$ and $\tau_n = \min\{|t(\alpha)| : \alpha \in \partial_0 X_n\}$, an application of Corollary 8.6 yields the following.
Theorem 10.4. Let \( \{X_n\} \) be an exhaustion of \( X \) with a constant number of components and assume for each \( n \) that the boundary geodesics in \( \partial_0X_n \) have the same twist \( t_n \). If

\[
\sum_{n=1}^{\infty} \frac{1}{e^{(1-|t_n|)^{3/2}}} = \infty,
\]

then \( X \) is of parabolic type.

10.5 Abelian covers of compact surfaces

All covers discussed in this section are regular covers. We consider a topological abelian cover \( X \) of a compact Riemann surface \( Y \), that is, a regular cover with a properly discontinuous action by a torsion-free abelian group \( G \) for which \( X/G \) is topologically \( Y \). Such a cover induces a compact exhaustion of \( X \) in the following way: choose a fundamental domain, \( P \), for the action of \( G \) on \( X \) so that its closure \( \overline{P} \) is a compact subsurface whose boundary projects to the simple closed geodesics on \( Y \) which induce the abelian covering. The translates of \( \overline{P} \) by \( G \) tile the surface \( X \). Given \( P \), define \( |g| \) to be the least number of translates of \( \overline{P} \) that a path traverses from \( \overline{P} \) to \( g\overline{P} \).

Let

\[
X_n = \bigcup \{g\overline{P} : g \in G, |g| \leq n\} \subset X,
\]

The \( \{X_n\} \) form a compact exhaustion of \( X \). We say that this exhaustion is induced by the cover. We consider a fixed pants decomposition of \( Y \) whose lifted curves in \( X \) together with the curves in \( \partial X_n \) form a pants decomposition of \( X \). We call this a pants decomposition induced by the cover. If the cover is given by an isometric action of \( G \), we say it is a geometric cover. When \( G \) is either \( \mathbb{Z} \) or \( \mathbb{Z}^2 \), Mori [33] showed that a geometric cover is parabolic. Rees [42] extended these results in various directions as well as to higher dimensions and both authors (Mori and Rees) independently showed that if \( G = \mathbb{Z}^r \), for \( r \geq 3 \), a geometric cover is not parabolic. In particular, these results show that in the case of \( \mathbb{Z} \) and \( \mathbb{Z}^2 \) covers, to achieve parabolicity it is sufficient for the cover to be given by an isometric action. In this subsection, we use our methods to generalize this sufficiency condition for parabolicity. Let \( P \) be a fundamental domain as above.

- \( G = \mathbb{Z} \), lifting a single curve: In this case, \( \partial P \subset X \) is composed of two simple closed curves that are lifts of a non-separating simple closed curve on \( Y \). The compact subsurfaces \( \{X_n\} \) of the exhaustion induced by the cover have two boundary curves for each \( n \). \( X \) is topologically a ladder surface.

- \( G = \mathbb{Z}^2 \), lifting a pair of disjoint curves: In this case, \( \partial P \subset X \) is the union of four simple closed curves and the covering is given by two disjoint non-separating simple closed curves on \( Y \) and the compact subsurfaces \( \{X_n\} \) of the exhaustion induced by the cover have at most \( 4n \) boundary curves for each \( n \). \( X \) is topologically a Loch–Ness monster.

- \( G = \mathbb{Z}^2 \), lifting a pair of intersecting curves: In this case, \( \partial P \subset X \) is a topological rectangle and the cover is given by the lift of two intersecting non-separating simple closed curves on \( Y \). A schematic picture of the cover is provided in Figure 10.4, where each square represents a copy of \( P \), \( X_n \) is the \( n \) by \( n \) square centered at \( P \), and \( \partial X_n \) consists of the closed curve \( \alpha_n \). \( X \) is topologically a Loch–Ness monster.
We have the following theorem for topological abelian covers. Recall that $L_n = \max\{\ell(\alpha) : \alpha \in \partial_0 X_n\}$ and $\tau_n = \min\{|t_n(\alpha)| : \alpha \in \partial_0 X_n\}$.

**Theorem 10.5.** Let $X$ be an infinite type hyperbolic surface that topologically covers a compact Riemann surface where the covering group $G$ is either $\mathbb{Z}$ or $\mathbb{Z}^2$. Let $\{X_n\}$ be the compact exhaustion of $X$ induced by the cover. Any of the following items are sufficient to imply that $X$ is of parabolic type.

1. If $G = \mathbb{Z}$ and the cover is given by a single non-separating simple closed curve, assume

   \[
   \sum_{n=1}^{\infty} \frac{1}{e^{(1-\tau_n)L_n/2}} = \infty.
   \]

2. If $G = \mathbb{Z}^2$ and the cover is given by two disjoint non-separating simple closed curves, assume

   \[
   \sum_{n=1}^{\infty} \frac{1}{n e^{(1-\tau_n)L_n/2}} = \infty.
   \]

3. If $G = \mathbb{Z}^2$ and the cover is given by two non-separating simple closed curve that intersect, for each $n$ assume that the geodesic in the homotopy class of $\partial_0 X_n$ has length $\ell_n$ and a half-collar of width $\varepsilon_n$ satisfying,

   \[
   \sum_{n=1}^{\infty} \frac{\arctan(\sinh \varepsilon_n)}{\ell_n} = \infty.
   \]

**Proof.** Note that on the boundary of $X_n$, there are two components for each $n$ in the case of the cover corresponding to a single simple closed curve, at most $4n$ components if the cover corresponds to two disjoint curves, and one boundary component for each $n$ if it corresponds to two intersecting curves. Items (1) and (2) now follow from Theorem 8.5, and the fact that $X_n = \bigcup\{g\bar{P} : g \in G, |g| \leq n\}$ where the translate $g\bar{P}$ is homeomorphic to $\bar{P}$, Item (3) follows from Theorem 8.9. □

As a special case we recover the results of Mori and Rees for $G = \mathbb{Z}$ or $\mathbb{Z}^2$.

**Corollary 10.6** [33, 42]. Let $X$ be an infinite type hyperbolic surface that geometrically covers a compact Riemann surface $Y$ where the covering group is either $\mathbb{Z}$ or $\mathbb{Z}^2$. Then $X$ is of parabolic type.
Proof. If the cover is associated to a single curve or two disjoint curves then the maximal length, \( L_n \), of the boundary geodesics of \( X_n \) is constant for all \( n \), and hence the corollary follows from items (1) and (2) of Theorem 10.5.

We next take up the case that \( G = \mathbb{Z}^2 \) and the cover is given by the intersection of two simple closed geodesics on \( Y \), say \( \beta_1 \) and \( \beta_2 \). Cutting \( Y \) open along these geodesics gives us \( P \subset X \) along with its \( \mathbb{Z}^2 \)-translates depicted in Figure 10.4 where the horizontal, respectively, vertical, geodesics project to \( \beta_1 \), respectively, \( \beta_2 \). The fact that \( \beta_1 \) and \( \beta_2 \) have standard collars guarantees that there is an embedded \( \epsilon \)-neighborhood about the vertical and horizontal geodesics. In fact, the constant \( \epsilon \) can be taken to be the smaller of the standard collar widths of \( \beta_1 \) and \( \beta_2 \).

Note that in this case \( \partial X_n \) consists of one component, whose geodesic representative we call \( \alpha_n \). The geodesic \( \alpha_n \) for geometrical and topological reasons lies in the \( n \) by \( n \) square determined by \( X_n \), and together with \( \partial X_n \) bounds a topological annulus. Using the fact that \( \partial X_n \) has an embedded \( \epsilon \)-neighborhood, we may conclude that \( \alpha_n \) has a half-collar of width \( \epsilon \). Finally, noting that \( \ell'(\alpha_n) < 2n(\ell'(\beta_1) + \ell'(\beta_2)) \), in particular \( \ell'(\alpha_n) \) grows at most linearly in \( n \), we apply Theorem 10.5, item (3) to conclude parabolicity. \( \square \)

Example 10.7. Suppose \( X \) is an infinite type hyperbolic Riemann surface, and \( \pi : X \to Y \) is a geometric cover over the compact Riemann surface \( Y \) where the covering group is either \( \mathbb{Z} \) or \( \mathbb{Z}^2 \). Let \( \{X_n\} \) be a compact exhaustion of \( X \) induced by the cover and fix a pants decomposition of \( X \) that includes the simple closed curves in \( \bigcup \partial_0 X_n \). As we saw in Corollary 10.6, \( X \) is of parabolic type. On the other hand, by varying the parameters along just the curves in the boundary of the exhaustion and applying Theorem 10.5 we can obtain a large class \( \{X^\alpha\}_{\alpha \in A} \) of hyperbolic Riemann surfaces of parabolic type with the following properties:

- each \( X^\alpha \) has the same topology as \( X \),
- \( X^\alpha \) is quasiconformally distinct from \( X^\beta \) for any pair \( \alpha \neq \beta \) (see [10]),
- for each \( \alpha \), the Fenchel–Nielsen parameters of \( X^\alpha \) agree with the parameters of the geometric cover, \( X \), along all the pants curves except the ones on the boundary of the compact exhaustion.

Remark 10.8. For rank \( r \geq 3 \), the analogue of the sufficient conditions in items (2) and (3) of Theorem 10.5 must have the boundary lengths of the induced cover going to zero. This is ostensibly because the \( n \)th term of the series contains \( \frac{1}{n^{r-1}} \). This is underscored by the result mentioned earlier ([33] or [42]) that if \( G = \mathbb{Z}^r \), for \( r \geq 3 \), then a geometric cover of the closed Riemann surface is not parabolic. On the other hand, one does not have to change the hyperbolic structure much to regain parabolicity in the rank \( r \geq 3 \) case. This point is illustrated by the following corollary which follows from Theorem 8.5.

Corollary 10.9. Suppose \( \pi : X \to Y \) is a \( \mathbb{Z}^r \)-covering of the compact Riemann surface \( Y \) by the infinite type hyperbolic surface \( X \), where the covering is given by \( r \geq 3 \) disjoint non-separating simple closed curves on \( Y \). Assume that the compact exhaustion \( \{X_n\} \) is induced by the cover and recall that \( L_n = \max \{\ell(\alpha) : \alpha \in \partial_0 X_n\} \). If

\[
\sum_{n=1}^{\infty} \frac{1}{n^{r-1} L_n e^{\frac{L_n}{2}}} = \infty,
\]

then \( X \) is of parabolic type.
**Example 10.10.** Consider the hypothesis of the above corollary and lift a pants decomposition of $Y$ whose pants curves include the $r$ non-separating curves associated to the cover. Lifting these curves we obtain a pants decomposition of $X$; it is not difficult to see that for each $n$, $\partial X_n$ is a pants curve in this decomposition. Save the curves in $\partial X_n$ which we assume satisfy expression (63), the Fenchel–Nielsen parameters of all the other pants curves can be equal to the Fenchel–Nielsen parameters of the corresponding curve in $Y$. In this fashion, we provide examples of parabolic surfaces that are $\mathbb{Z}^r$-topological coverings for $r \geq 3$ of a compact Riemann surface which are geometric coverings away from the $r$ non-separating curves associated to the cover. This stands in contrast to the results of Mori and Rees which show that geometric covers are not parabolic.

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