IBN-varieties of algebras.

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Abstract

The concept of a variety with IBN (invariant basic number) propriety first appeared in the ring theory. But we can define this concept for an arbitrary variety $\Theta$ of universal algebras with an arbitrary signature $\Omega$; see Definition 1.4.

The proving of the IBN propriety of some variety is very important in universal algebraic geometry. This is a milestone in the study of the relation between geometric and automorphic equivalences of algebras of this variety.

In this paper we prove very simple but very useful for studying of IBN proprieties of different varieties Theorems 2.1 and 3.2. We will consider some applications of this theorem.

We will consider many-sorted universal algebras as well as one-sorted. So all concepts and all results will by generalized for the many-sorted case.

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1 Motivation and introduction

In this paper we denote a cardinality of a set $A$ by $|A|$. The cardinality $\aleph_0$ we denote by $\infty$. We will denote a disjoint union of sets by symbol $\uplus$. 
We will consider all problems and prove all results in this paper in the general situation of many-sorted universal algebras. Also the results of this paper are more important in the studying of varieties of many-sorted universal algebras.

We will define the many-sorted universal algebras as in [13]. We suppose that there is a set \( \Gamma \) of names of sorts. A many-sorted algebra is a set \( H \) which has "sorting", that is, the mapping \( \eta_H : H \to \Gamma \). The set \( \eta_H^{-1}(i) \) is the set of elements of the sort \( i \) of the algebra \( H \), where \( i \in \Gamma \). We denote \( \eta_H^{-1}(i) = H^{(i)} \).

An element from \( H^{(i)} \) we denote by \( h^{(i)} \), in order to emphasize that it is an element of the sort \( i \). The situation, when exists \( i \in \Gamma \) such that \( H^{(i)} = \emptyset \), is possible. For every subset \( Y \subseteq H \) we denote the set \( \eta_H(Y) \subseteq \Gamma \) by \( \Gamma_Y \). In particular we denote \( \text{im} \eta_H = \{ i \in \Gamma \mid H^{(i)} \neq \emptyset \} \) by \( \Gamma_H \).

We denote by \( \Omega \) a signature (set of operations) of our algebras. In many-sorted case every operation \( \omega \in \Omega \) has the type \( \tau_\omega = (i_1, \ldots, i_n; j) \), where \( n \in \mathbb{N}, i_1, \ldots, i_n, j \in \Gamma \). Operation \( \omega \in \Omega \) of the type \( (i_1, \ldots, i_n; j) \) is a mapping \( \omega : H^{(i_1)} \times \ldots \times H^{(i_n)} \to H^{(j)} \).

**Definition 1.1** We suppose that many-sorted algebras \( A \) and \( B \) have a same set \( \Gamma \) of names of sorts and a same signature \( \Omega \). We say that the mapping \( \varphi : A \to B \) is a **homomorphism** from algebra \( A \) to algebra \( B \) if the equation

\[
\eta_A = \eta_B \varphi
\]

fulfills and for every \( \omega \in \Omega \), the equation

\[
\varphi(\omega(a_1, \ldots, a_n)) = \omega(\varphi(a_1), \ldots, \varphi(a_n))
\]

holds.

We suppose in (1.2) that \( \tau_\omega = (i_1, \ldots, i_n; j), a_k \in A^{(i_k)}, 1 \leq k \leq n \).

The equation (1.1) means that the mapping \( \varphi \) transform all elements of every sort \( i \in \Gamma \) of algebra \( A \) to elements of same sort of algebra \( B \). Also we can conclude from (1.1) that no exist homomorphisms from \( A \) to \( B \) if \( \Gamma_A \not\subseteq \Gamma_B \).

We denote by \( \text{Hom}(A, B) \) the set of homomorphisms from algebra \( A \) to algebra \( B \). As we can see, the situation when \( \text{Hom}(A, B) = \emptyset \) is possible. We denote \( \varphi|_{A^{(i)}} \) by \( \varphi^{(i)}, i \in \Gamma \). By (1.1) we have that \( \varphi^{(i)} \) is a mapping from \( A^{(i)} \) to \( B^{(i)} \).

A congruence \( U \) in an universal algebra \( H \) with a set \( \Gamma \) of names of sorts and a signature \( \Omega \) is a subset \( U \subseteq \bigcup_{i \in \Gamma} (H^{(i)} \times H^{(i)}) \), such that for every \( \omega \in \Omega \) with \( \tau_\omega = (i_1, \ldots, i_n; j) \) and for every \( (h_k, g_k) \in U \cap (H^{(i_k)} \times H^{(i_k)}) \), where \( 1 \leq k \leq n \), holds \( \omega(h_1, \ldots, h_n), \omega(g_1, \ldots, g_n) \in U \). Every congruence \( U \) in an algebra \( H \) gives a natural epimorphism \( \delta_U : H \to H/U \) from the algebra \( H \) to the quotient algebra \( H/U \). In our consideration we will use without special reminder the

**Lemma 1.1** If \( H, G \) two algebras with same set of names of sorts and same signature, \( U \) is a congruence in algebra \( H \), \( V \) is a congruence in algebra \( G \), \( \varphi : H \to G \) is a homomorphism and \( \varphi(U) = \{(\varphi(h'), \varphi(h'')) \mid (h', h'') \in U\} \subseteq V \) then there exists a homomorphism \( \psi : H/U \to G/V \), such that \( \delta_V \varphi = \psi \delta_U \).
which can be proved very easy.

The definition of a kernel of a homomorphism of many-sorted algebras should be formulated as follows. A kernel of a homomorphism \( \varphi : A \to B \) of algebras with a set \( \Gamma \) of names of sorts is a subset \( \ker \varphi \subseteq \bigcup_{i \in \Gamma} (A^{(i)} \times A^{(i)}) \) such that for every \((a, a') \in \ker \varphi\) the equality \( \varphi(a) = \varphi(a') \) holds. A kernel of every homomorphism is a congruence.

For a little bit different approach to the concept of a many-sorted algebra see [10].

We fix the set of names of sorts \( \Gamma \) and the signature \( \Omega \). We take for every \( \eta \in \Gamma \) the countable set \( X_0^{(i)} = \{x_1^{(i)}, \ldots, x_k^{(i)}, \ldots\} \). The elements of sets \( X_0^{(i)} \), \( i \in \Gamma \), we can call symbols, or letters, or variables, or generators of the sort \( i \).

After this we consider the disjoint union \( X_0 = \bigcup_{i \in \Gamma} X_0^{(i)} \). We choose the subset \( X = \bigcup_{i \in \Gamma} X^{(i)} \subset X_0 \), where \( X^{(i)} = X_0^{(i)} \cap X \). It is possible that there exists \( i \in \Gamma \) such that \( X^{(i)} = \emptyset \). There exists a mapping \( \eta_X : X \to \Gamma \), such that \( \eta_X(x) = i \) if \( x \in X^{(i)} \subset X \). Similar to above we can denote \( \text{im} \eta_X = \Gamma_X \).

We consider the algebra of terms with a set of names of sorts \( \Gamma \) and a signature \( \Omega \) over the set (alphabet) \( X \). We denote this algebra by \( \mathfrak{T}(\Gamma, \Omega, X) \) or by \( \mathfrak{T}(X) \), if it cannot cause errors. We can define by induction by construction in algebra \( \mathfrak{T}(X) \) sets of terms of the sort \( i \), where \( i \in \Gamma \), i.e., sets \( \mathfrak{T}(X)^{(i)} \).

Of course, the decomposition \( \mathfrak{T}(X) = \bigcup_{i \in \Gamma} \mathfrak{T}(X)^{(i)} \) fulfills. Hence, exists a mapping \( \eta_{\mathfrak{T}(X)} : \mathfrak{T}(X) \to \Gamma \), such that \( \eta_{\mathfrak{T}(X)}(t) = i \) if \( t \in \mathfrak{T}(X)^{(i)} \). The equality \( \eta_{\mathfrak{T}(X)|X} = \eta_X \) holds.

It is easy to prove by induction by construction the following

**Proposition 1.1** For every \( X \subset X_0 \), every algebra \( H \) with signature \( \Omega \) such that \( \Gamma_X \subseteq \Gamma_H \) and every mapping \( \varphi^* : X \to H \) such that \( \eta_X = \eta_H \varphi^* \), there exists unique homomorphism \( \varphi : \mathfrak{T}(X) \to H \) such that \( \varphi|_X = \varphi^* \).

From this proposition we can easy prove the projective property of the algebra of terms

**Proposition 1.2** Let \( \mathfrak{T}(\Gamma, \Omega, X) = \mathfrak{T}(X) \) the algebra of terms with a set of names of sorts \( \Gamma \) and a signature \( \Omega \) over the set \( X \) and \( A, B \) two algebras with same set of names of sorts and same signature. If there exist a homomorphism \( \varphi : \mathfrak{T}(X) \to B \) and an epimorphism \( \alpha : A \to B \) then there exists a homomorphism \( \chi : \mathfrak{T}(X) \to A \), such that \( \alpha \chi = \varphi \).

We suppose that \( X = \{x_1, \ldots, x_r\} \) and \( f_1(x_1, \ldots, x_r) = f_1, f_2(x_1, \ldots, x_r) = f_2 \in \mathfrak{T}(X)^{(i)} \) for certain \( i \in \Gamma_{\mathfrak{T}(X)} \). We say that identity \( f_1 = f_2 \) holds in algebra \( H \) with the set of names of sorts \( \Gamma \) and the signature \( \Omega \) or algebra \( H \) satisfies the identity \( f_1 = f_2 \) if for every \( h_1, \ldots, h_r \in H \) such that \( \eta_X(x_j) = \eta_H(h_j), 1 \leq j \leq r \), the equality \( f_1(h_1, \ldots, h_r) = f_2(h_1, \ldots, h_r) \) holds, or, in other words, if for every \( \varphi \in \text{Hom}(\mathfrak{T}(X), H) \) the equality \( \varphi(f_1) = \varphi(f_2) \)
holds. We denote this fact by $H \models (f_1 = f_2)$. In particular $H \models (f_1 = f_2)$ if $\text{Hom} (\mathcal{F}(X), H) = \emptyset$. Sometimes we will denote identities briefly: without brackets.

We consider a set of identities $I$. This set can by infinite. But for every $(f_1 = f_2) \in I$ exists finite $X \subset X_0$ such that $f_1, f_2 \in \mathcal{F}(X)$. We say that an algebra $H$ satisfies the set of identities $I$ if for every $(f_1 = f_2) \in I$ the $H \models (f_1 = f_2)$ fulfills. This fact we denote by $H \models I$. The class of all algebras $H$ with set of names of sorts $\Gamma$ and signature $\Omega$ such that $H \models I$ we call a variety defined by the set of identities $I$ and denote by $\text{Var}(I)$. Therefore, if we consider some variety of algebras, we suppose that set of names of sorts $\Gamma$ and signature $\Omega$ are known.

On the other hand we can consider a class of algebras $\Theta$ and the set of all identities $I$ of some variety $\mathcal{V}$ with set of names of sorts $\Gamma$ and signature $\Omega$ such that $\mathcal{V} \models I$ and denote by $\text{Var}(I)$. We consider a set of identities $I$ which contain only variables from a subset $X \subset X_0$ we denote by $\mathcal{F}(X)$. If we consider the identity $(f_1 = f_2)$ as a pair $(f_1, f_2)$ then for every $X \subset X_0$ we have that $\mathcal{F}(X)$ is a congruence in $\mathcal{F}(X)$.

The proof of the following facts is an easy exercise in logic rather than in algebra.

**Claim 1.1** If $\Lambda, \Lambda_1, \Lambda_2$ some classes of algebras with set of names of sorts $\Gamma$ and signature $\Omega$ and $I, I_1, I_2$ some sets of identities, i.e., $I, I_1, I_2 \subseteq \bigsqcup_{i \in I} (\mathcal{F}(\Gamma, \Omega, X)^{(i)} \times \mathcal{F}(\Gamma, \Omega, X)^{(i)})$, then

1. $I_1 \subseteq I_2 \implies \text{Var}(I_1) \supseteq \text{Var}(I_2)$,
2. $\Lambda_1 \subseteq \Lambda_2 \implies \mathcal{J}(\Lambda_1) \supseteq \mathcal{J}(\Lambda_2)$,
3. $\text{Var}(\Lambda) = \text{Var}(\mathcal{J}(\Lambda)) \supseteq \Lambda$,
4. $\mathcal{J}(\text{Var}(I)) \supseteq I$,
5. $\mathcal{J}(\text{Var}(\mathcal{J}(\Lambda))) = \mathcal{J}(\Lambda)$,
6. $\text{Var}(\mathcal{J}(\text{Var}(I))) = \text{Var}(I)$, in particular, if $\Theta$ is a variety of algebras, then $\text{Var}(\mathcal{J}(\Theta)) = \Theta$,
7. if $(f_1 = f_2) \in \mathcal{J}(\Lambda, X)$ and $\varphi \in \text{Hom}(\mathcal{F}(X), \mathcal{F}(Y))$, then $(\varphi(f_1) = \varphi(f_2)) \in \mathcal{J}(\Lambda, Y)$,
8. if $\Theta_1, \Theta_2$ are varieties of algebras with same set of names of sorts and same signature then $\Theta_1 \cap \Theta_2$ is also a variety and for every $X \subset X_0$ the equality $\mathcal{J}(\Theta_1 \cap \Theta_2, X) = (\mathcal{J}(\Theta_1, X), \mathcal{J}(\Theta_2, X))$ holds, where $(\mathcal{J}(\Theta_1, X), \mathcal{J}(\Theta_2, X))$ is a congruence generated by congruences $\mathcal{J}(\Theta_1, X)$ and $\mathcal{J}(\Theta_2, X)$, i.e., the minimal congruence which contains $\mathcal{J}(\Theta_1, X)$ and $\mathcal{J}(\Theta_2, X)$.

**Definition 1.2** We say that algebra $F_\Theta(Y)$, is a free algebra of some variety $\Theta$, generated by the set of free generators $Y$ if
1. $Y \subseteq F_{\Theta}(Y)$,
2. $F_{\Theta}(Y) \in \Theta$,
3. for every $H \in \Theta$ such that $\Gamma_H \supseteq \Gamma_Y$ and for every mapping $\varphi^* : Y \to H$ such that $\eta_Y = \eta_H \varphi^*$, there exists unique homomorphism $\varphi : F_{\Theta}(Y) \to H$ such that $\varphi|_Y = \varphi^*$.

By Proposition 1.1, we have that algebra $\mathfrak{T}(X)$ is a free algebra of the variety, which defined by empty set of identities. This free algebra is generated by the set of free generators $X$.

The natural epimorphism $\mathfrak{T}(X) \to \mathfrak{T}(X)/\mathfrak{J}(\Theta, X)$ we denote by $\delta_{\Theta, X}$. From Proposition 1.1 and from definition of $\mathfrak{J}(\Theta, X)$ we can conclude that

$$\mathfrak{T}(X)/\mathfrak{J}(\Theta, X) \cong F_{\Theta}(Y),$$

where $Y = \delta_{\Theta, X}(X)$.

**Definition 1.3** We say that a variety $\Theta$ is $i$-degenerate, where $i \in \Gamma$, if $(x_1^{(i)} = x_2^{(i)}) \in \mathfrak{J}(\Theta)$. If for every $i \in \Gamma$ the inclusion $(x_1^{(i)} = x_2^{(i)}) \in \mathfrak{J}(\Theta)$ holds, then we call the variety $\Theta$ degenerate. We say that a variety $\Theta$ is $i$-nondegenerate, where $i \in \Gamma$, if $(x_1^{(i)} = x_2^{(i)}) \notin \mathfrak{J}(\Theta)$. A variety $\Theta$ is called nondegenerate if it is $i$-nondegenerate for every $i \in \Gamma$.

**Proposition 1.3** If a variety $\Theta$ is $i$-degenerate, then for every $H \in \Theta$ the inclusion $|H^{(i)}| \in \{0, 1\}$ holds.

**Proof.** We suppose that $|H^{(i)}| > 1$. It means that there are $h_1^{(i)}, h_2^{(i)} \in |H^{(i)}|$. We consider the algebra $\mathfrak{T}\left(x_1^{(i)}, x_2^{(i)}\right)$ and the mapping $\varphi^* : \left\{x_1^{(i)}, x_2^{(i)}\right\} \to H$, such that $\varphi^*\left(x_j^{(i)}\right) = h_j^{(i)}$, $j = 1, 2$. By Proposition 1.1 there exists a homomorphism $\varphi : \mathfrak{T}\left(x_1^{(i)}, x_2^{(i)}\right) \to H$, such that $\varphi\left(x_1^{(i)}, x_2^{(i)}\right) = \varphi^*\left(x_1^{(i)}\right) = \varphi\left(x_2^{(i)}\right)$. It means that $h_1^{(i)} = h_2^{(i)}$ and gives a contradiction with $|H^{(i)}| > 1$. ■

**Proposition 1.4** Let $X \subseteq X_0$. A variety $\Theta$ is $i$-nondegenerate, if and only if the natural epimorphism $\delta_{\Theta, X} : \mathfrak{T}(X) \to \mathfrak{T}(X)/\mathfrak{J}(\Theta, X) = F_{\Theta}(\delta_{\Theta, X}(X))$ is an injection on $X^{(i)}$.

**Proof.** We suppose that the variety $\Theta$ is $i$-nondegenerate, i.e., $(x_1^{(i)} = x_2^{(i)}) \notin \mathfrak{J}(\Theta)$, but there are $x_1^{(i)}, x_2^{(i)} \in X^{(i)}$ such that $x_1^{(i)} \neq x_2^{(i)}$ and $\delta_{\Theta, X}\left(x_1^{(i)}\right) = \delta_{\Theta, X}\left(x_2^{(i)}\right)$. It means that $(x_1^{(i)} = x_2^{(i)}) \in \mathfrak{J}(\Theta, X)$. There exists an endomorphism $\alpha$ of $\mathfrak{T}(X)$ such that $\alpha(x_1^{(i)}) = x_1^{(i)}$ and $\alpha(x_2^{(i)}) = x_2^{(i)}$. Hence $(x_1^{(i)} = x_2^{(i)}) \in \mathfrak{J}(\Theta)$.  

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We suppose that the variety $\Theta$ is $i$-degenerate. $F_{E_{\Theta}}(\delta_{\Theta,X}(X)) \in \Theta$, so, by Proposition 1.3 \((F_{E_{\Theta}}(\delta_{\Theta,X}(X)))^{(i)} \leq 2\). Hence, for every $x_{j}^{(i)},x_{j}^{(i)} \in X^{(i)}$ the equality $\delta_{\Theta,X}(x_{j}^{(i)}) = \delta_{\Theta,X}(x_{j}^{(i)})$ holds. □

We can prove from definition of a free algebra of variety, that if $F_{E_{\Theta}}(Y), F_{E_{\Theta}}(Z)$ are free algebras of the variety $\Theta$ with the sets of free generators $Y = \bigsqcup_{i \in \Gamma} Y^{(i)}$, $Z = \bigsqcup_{i \in \Gamma} Z^{(i)}$ correspondingly, and for every $i \in \Gamma$ the equality $|Y^{(i)}| = |Z^{(i)}|$ holds, then $F_{E_{\Theta}}(Y) \cong F_{E_{\Theta}}(Z)$.

**Proposition 1.5** Up to isomorphism all free algebras of a variety $\Theta$ have a form (1.3).

**Proof.** Let $F_{E_{\Theta}}(Z)$ is a free algebra of the variety $\Theta$, where $Z = \bigsqcup_{i \in \Gamma} Z^{(i)}$. We take the set $X = \bigsqcup_{i \in \Gamma} X^{(i)}$, such that for every $i \in \Gamma$ the equality $|X^{(i)}| = |Z^{(i)}|$ holds. We consider the algebra of terms $\mathfrak{T}(X)$ and the free algebra $F_{E_{\Theta}}(Y)$ of the variety $\Theta$ where $Y = \delta_{\Theta,X}(X)$. We have that $Y = \delta_{\Theta,X}(X) = \bigsqcup_{i \in \Gamma} \delta_{\Theta,X}(X^{(i)})$. We denote $\delta_{\Theta,X}(X^{(i)})$ by $Y^{(i)}$. If for any $i \in \Gamma$ we have that $\Theta$ is $i$-nondegenerate variety, then by Proposition 1.4 we have that $|X^{(i)}| = |Y^{(i)}| = |Z^{(i)}|$. If for $i \in \Gamma$ we have that $\Theta$ is $i$-degenerate variety then by Proposition 1.3 $|Z^{(i)}| = 0$ or $|Z^{(i)}| = 1$. We have in the first case that $|X^{(i)}| = 0$ and $|Y^{(i)}| = 0$. We have in the second case that $|X^{(i)}| = 1$ and $|Y^{(i)}| = 1$. Therefore in all these cases we have that $|Y^{(i)}| = |Z^{(i)}|$. Hence $F_{E_{\Theta}}(Z) \cong F_{E_{\Theta}}(Y)$. □

Now we consider two varieties $\Theta$ and $\Delta$ such that $\Theta \supseteq \Delta$. Similar to Claim 1.3 item 2 we obtain the inclusion $\mathfrak{I}(\Theta,X) \subseteq \mathfrak{I}(\Delta,X)$. We have by the Third Theorem of Isomorphism that

$$\mathfrak{I}(X)/\mathfrak{I}(\Theta,X) / \mathfrak{I}(\Delta,X) / \mathfrak{I}(\Theta,X) \cong \mathfrak{I}(X)/\mathfrak{I}(\Delta,X) \cong F_{\Delta}(\delta_{\Delta,X}(X)).$$

For every $X = \bigsqcup_{i \in \Gamma} X^{(i)} \subset X_{0}$ we consider all identities of the form $\delta_{\Theta,X}(f_{1}) = \delta_{\Theta,X}(f_{2})$, where $f_{1},f_{2} \in \mathfrak{T}(X)$, which hold in all algebras of the variety $\Delta$. As above the set of all these identities can be considered as congruence in $F_{E_{\Theta}}(\delta_{\Theta,X}(X))$ and we denote it by $\mathfrak{I}_{\Theta}(\Delta,X)$. For every algebra $H \in \Theta$ and every homomorphism $\varphi \in \text{Hom}(\mathfrak{T}(X),H)$ there exists unique homomorphism $\psi \in \text{Hom}(F_{E_{\Theta}}(\delta_{\Theta,X}(X)),H)$ such that $\varphi = \psi \delta_{\Theta,X}$. From this fact we immediately conclude that $(\mathfrak{I}(\Delta,X)/\mathfrak{I}(\Theta,X)) = \mathfrak{I}_{\Theta}(\Delta,X)$ and

$$F_{\Delta}(\delta_{\Delta,X}(X)) \cong F_{E_{\Theta}}(\delta_{\Theta,X}(X))/\mathfrak{I}_{\Theta}(\Delta,X). \quad \text{(1.4)}$$

Now we will prove one proposition which is a generalization of item 7 from Claim 1.1.
Proposition 1.6 If Θ is some variety of algebras, Δ ⊆ Θ its subvariety, (g_1 = g_2) ∈ \( \mathcal{J}_\Theta(\Delta, X) \) and \( \varphi \in \text{Hom}(F_\Theta(\delta_{\Theta,X}(X)), F_\Theta(\delta_{\Theta,Y}(Y))) \), then \( (\varphi(g_1) = \varphi(g_2)) \in \mathcal{J}_\Theta(\Delta, Y) \).

Proof. If \( (g_1 = g_2) \in \mathcal{J}_\Theta(\Delta, X) \) then \( g_i = \delta_{\Theta,X}(f_i) \), where \( f_i \in \mathcal{I}(X) \), \( i = 1, 2 \), and for every \( H \in \Delta \) the \( H \vdash (f_1 = f_2) \) holds. By Proposition 1.2 for every \( \varphi \in \text{Mor}_{\Theta}(F_\Theta(\delta_{\Theta,X}(X)), F_\Theta(\delta_{\Theta,Y}(Y))) \) there exists a homomorphism \( \bar{\varphi} : \mathcal{I}(X) \rightarrow \mathcal{I}(Y) \), such that \( \varphi\delta_{\Theta,X} = \delta_{\Theta,Y}\bar{\varphi} \). By Claim 1.1 item 4 the \( H \vdash (\bar{\varphi}(f_1) = \bar{\varphi}(f_2)) \) holds for every \( H \in \Delta \). It means that \( (\delta_{\Theta,Y}\bar{\varphi}(f_1) = \delta_{\Theta,Y}\bar{\varphi}(f_2)) \in \mathcal{J}_\Theta(\Delta, Y) \). But \( \delta_{\Theta,Y}\bar{\varphi}(f_i) = \varphi\delta_{\Theta,X}(f_i) = \varphi(g_i) \), where \( i = 1, 2 \). So \( (\varphi(g_1) = \varphi(g_2)) \in \mathcal{J}_\Theta(\Delta, Y) \). ■

We will denote \( \mathcal{J}_\Theta(\Delta, X) \) by \( \mathcal{J}(\Delta, X) \), if it cannot cause errors.

This research is motivated by universal algebraic geometry. All definitions of the basic notions of the universal algebraic geometry can be found, for example, in [11], [12], [13] and [14]. Also, there are fundamental papers [2], [9] and [3], [4].

The relation between geometric and automorphic equivalences of universal algebras of some variety \( \Theta \) is a one important question of universal algebraic geometry (see [13] and [14]). We need for the studying of this relation, at first, consider a category \( \Theta^0 \) of the finitely generated free algebras of a variety \( \Theta \). Objects of this category are the finitely generated free algebras of the variety \( \Theta \) and morphisms of this category are their homomorphisms. After this we consider the quotient group \( \mathfrak{A}/\mathfrak{Q} \), where \( \mathfrak{A} \) is the group of all automorphisms of the category \( \Theta^0 \) and \( \mathfrak{Q} \) is the subgroup of all inner automorphisms of this category. If this quotient group is trivial, then the geometric and automorphic equivalences are coincides in the variety \( \Theta \) (see [13]). If the group \( \mathfrak{A}/\mathfrak{Q} \) is not trivial, then often, but not always, we can give example of two algebras of the variety \( \Theta \) which are automorphic equivalent but not geometric equivalent. From this place onwards we consider only finitely generated free algebras of some variety \( \Theta \), i.e., algebras \( F_\Theta(X) \), where \( X \subset X_0 \) and \( |X| < \infty \).

For the computing of the group \( \mathfrak{A}/\mathfrak{Q} \) was elaborated on the simple but very strong method of the verbal operations (see [15] and [18]). This method can be applied only when the

Condition 1.1 For every \( \Phi \in \mathfrak{A} \), every \( i \in \Gamma \) and every \( x^{(i)} \in X_0^{(i)} \) the isomorphism \( \Phi(F_\Theta(x^{(i)})) \cong F_\Theta(x^{(i)}) \) holds.

fulfills in the variety \( \Theta \).

The IBN (invariant basis number) property or invariant dimension property was defined initially in the theory of rings and modules, see, for example, [6] Definition 2.8. But we can generalize this property for an arbitrary variety of universal algebras.

Definition 1.4 We say that a variety \( \Theta \) is an \( i \)-IBN-variety, where \( i \in \Gamma \), if from \( F_\Theta(Y) \cong F_\Theta(Z) \), where \( Y = \bigcup_{j \in \Gamma} Y^{(j)} \), \( Z = \bigcup_{j \in \Gamma} Z^{(j)} \) are sets of free generators of corresponding algebras, we can conclude that \( |Y^{(i)}| = |Z^{(i)}| \). A
variety Θ is called **IBN-variety** or variety which has an **IBN propriety** if it is an i-IBN-variety for every i ∈ Γ.

In the case of one-sorted algebras the Condition [1,1] is weaker than the IBN propriety. From the IBN propriety of a variety Θ we can conclude by the method of [16, Section 5] that in this variety the Condition [1,1] holds. But exist varieties in which the Condition [1,1] holds though these varieties have no the IBN propriety. The next example is a folklore of the theory of rings and modules.

**Example 1** We consider some field k and a vector space V over this field such that \( \dim_k V = \infty \). We denote by A the ring of all linear operators over the vector space V: \( A = \text{End}_k V \). It is known that variety \( A\mathfrak{M} \) of all lefts modules over the ring A has not the IBN propriety.

In this variety hold these isomorphisms: \( A \cong A \oplus A \cong A \oplus A \cong \ldots \cong \bigoplus_{i=1}^{n} A \cong \ldots \). But (E. Aladova) in this variety the Condition [1,1] holds, because in this variety all finitely generated free algebras are isomorphic.

In the case of many-sorted algebras we can not conclude Condition [1,1] directly from the IBN propriety.

**Example 2** \( \mathcal{SET} – \text{COUP} \) is the variety of the couples of sets.

This is a variety of two-sorted algebras, i.e., \( \Gamma = \{1, 2\} \): elements of the first set of a couple are elements of the first sort, elements of the second set of a couple are elements of the second sort. Algebras of this variety has the empty signature and this variety is defined by the empty set of identities. We denote this variety by \( \mathcal{SET} – \text{COUP} \). Section 2 we will prove that this variety has an IBN propriety. We consider the functor \( \Phi : \mathcal{SET} – \text{COUP}^0 \to \mathcal{SET} – \text{COUP}^0 \) which "switch" the sets in every couple, or, more formal, the functor \( \Phi \) such that

\[
\Phi (F_{\mathcal{SET} – \text{COUP}} (X)) = F_{\mathcal{SET} – \text{COUP}} (Y),
\]

where \( X = \{x_{a_1}^{(1)}, \ldots, x_{a_m}^{(1)}\} \cup \{x_{\beta_1}^{(2)}, \ldots, x_{\beta_n}^{(2)}\} \), \( Y = \{x_{\beta_1}^{(1)}, \ldots, x_{\beta_n}^{(1)}\} \cup \{x_{a_1}^{(2)}, \ldots, x_{a_m}^{(2)}\} \), and

\[
\Phi (f : F_{\mathcal{SET} – \text{COUP}} (A) \to F_{\mathcal{SET} – \text{COUP}} (C)) =
\]

\[
g : F_{\mathcal{SET} – \text{COUP}} (\Phi (A)) \to F_{\mathcal{SET} – \text{COUP}} (\Phi (C)),
\]

where \( A = \{x_{a_1}^{(1)}, \ldots, x_{a_k}^{(1)}\} \cup \{x_{\beta_1}^{(2)}, \ldots, x_{\beta_l}^{(2)}\} \), \( C = \{x_{\gamma_1}^{(1)}, \ldots, x_{\gamma_p}^{(1)}\} \cup \{x_{\beta_1}^{(2)}, \ldots, x_{\beta_l}^{(2)}\} \), \( \Phi (A) = \{x_{\beta_1}^{(1)}, \ldots, x_{\beta_l}^{(1)}\} \cup \{x_{\alpha_1}^{(2)}, \ldots, x_{\alpha_k}^{(2)}\} \), \( \Phi (C) = \{x_{\alpha_1}^{(1)}, \ldots, x_{\alpha_k}^{(1)}\} \cup \{x_{\beta_1}^{(2)}, \ldots, x_{\beta_l}^{(2)}\} \), and if \( f (x_{a_i}^{(1)}) = x_{\gamma_j}^{(1)} \), \( f (x_{\beta_j}^{(2)}) = x_{\alpha_i}^{(2)} \), then \( g (x_{\alpha_i}^{(1)}) = x_{\beta_j}^{(1)} \), \( g (x_{\beta_j}^{(2)}) = x_{\alpha_i}^{(2)} \).

This functor is the inverse of itself, but

\[
\Phi (F_{\mathcal{SET} – \text{COUP}} (x_{a_1}^{(1)})) = F_{\mathcal{SET} – \text{COUP}} (x_{\beta_1}^{(2)}) \neq F_{\mathcal{SET} – \text{COUP}} (x_{a_1}^{(1)}).
\]
It means that Condition 1.1 holds not in this variety.

But in many cases, we can use some additional considerations to deduce Condition 1.1 from the IBN propriety of some variety. See, for example, [18, Section 5] about a variety of all actions of semigroups over sets, variety of all automats, variety of all representations of groups, and [19, Section 4] about a very wide class of subvarieties of the variety of all representation of Lie algebras.

Therefore, the proving of the IBN propriety of some variety is a milestone in the study of the relation between geometric and automorphic equivalences of algebras of this variety.

2 Simple cases

In this section we consider such varieties for which the IBN property can be proved directly.

Example 3 The variety of all vector spaces over a fixed field \( k \).

We consider vector spaces over a fixed field \( k \) as one-sorted algebras. For every scalar \( \lambda \in k \) the multiplication of vectors by this scalar we consider as one unary operation. A linear map from one vector space to other can be uniquely defined by images of basic vectors. Therefore all vector spaces over a fixed field \( k \) are free algebras of this variety and the vectors of basis are free generators of these algebras. It is known that isomorphic vector spaces have same cardinality of bases, therefore this variety has the IBN property.

We will prove this

**Theorem 2.1** If \( \mathcal{F}(X) \cong \mathcal{F}(Y) \) then for every \( i \in \Gamma \) the equality \( |X^{(i)}| = |Y^{(i)}| \) holds.

**Proof.** We will define by induction by construction the length of terms, i.e., elements of algebra \( \mathcal{F}(X) \). We will denote a length of \( t \in \mathcal{F}(X) \) by \( l(t) \). We define that length 0 have only generators, i.e., elements of the set \( X \), and constants of the signature \( \Omega \). We suppose that we just defined what terms have a length \( j \) for every \( j < n \), where \( j, n \in \mathbb{N} \). Now we we will defined what terms have a length \( n \). We consider an operation \( \omega \in \Omega \) such that the arity \( r \) of this operation is positive, i.e., \( r > 0 \). If \( t_1, \ldots, t_r \in \mathcal{F}(X) \), such that \( l(t_1), \ldots, l(t_r) < n \) and \( \max \{l(t_1), \ldots, l(t_r)\} = n - 1 \), then \( l(\omega(t_1, \ldots, t_r)) = n \).

Now we consider a situation when there exists a homomorphism \( \varphi : \mathcal{F}(X) \to \mathcal{F}(Y) \). The length of terms is defined both in \( \mathcal{F}(X) \) and \( \mathcal{F}(Y) \). We will prove by induction by length of term that for every \( t \in \mathcal{F}(X) \) the inequality \( l(t) \leq l(\varphi(t)) \) holds. If \( l(t) = 0 \), then this inequality is trivial. We suppose that we proved this inequality when \( l(t) < n \), where \( n \in \mathbb{N} \). Let \( l(t) = n \). It means by our definition that \( t = \omega(t_1, \ldots, t_r) \), where \( \omega \in \Omega \), the arity of the operation \( \omega \) is equal to \( r \), \( t_1, \ldots, t_r \in \mathcal{F}(X) \) and \( \max \{l(t_1), \ldots, l(t_r)\} = n - 1 \). So, \( \varphi(t) = \omega(\varphi(t_1), \ldots, \varphi(t_r)) \). By induction hypothesis we have that \( l(t_i) \leq \ldots \)
\( l(\varphi(t_i)), \) where \( 1 \leq i \leq r. \) Therefore \( \max\{l(\varphi(t_1)),\ldots,l(\varphi(t_r))\} \geq n - 1 \) and \( l(\varphi(t)) = \max\{l(\varphi(t_1)),\ldots,l(\varphi(t_r))\} + 1 \geq n = \ell(t) = n. \)

Now we suppose that there exists an isomorphism \( \varphi : \mathcal{T}(X) \to \mathcal{T}(Y). \) We will prove that for every \( i \in \Gamma \) there exists a bijection \( \psi_i : X^{(i)} \to Y^{(i)}. \) We consider \( x^{(i)} \in X^{(i)} \) and will prove that \( \varphi(x^{(i)}) \in Y^{(i)}. \) If \( \varphi(x^{(i)}) = t \in \mathcal{T}(Y), \) then \( \varphi^{-1}(t) = x^{(i)}. \) \( \varphi^{-1} : \mathcal{T}(Y) \to \mathcal{T}(X) \) is also a homomorphism. It was proved that \( \ell(t) \leq \ell(\varphi^{-1}(t)) = \ell(x^{(i)}) = 0. \) Therefore \( t \in Y \) or \( t = c \) is a constant. We have in the second case that \( t = c \) is also an element of \( \mathcal{T}(X) \) and \( \varphi(x^{(i)}) = t = c = \varphi(c), \) which contradicts the injectivity of the mapping \( \varphi. \) In the first case we have by (1.1) that \( \varphi(x^{(i)}) \in Y^{(i)}. \) It means that there exists a mapping \( \psi_i = \varphi|_{X^{(i)}} : X^{(i)} \to Y^{(i)}. \) We have by symmetry that also there exists a mapping \( \chi_i = (\varphi^{-1})|_{Y^{(i)}} : Y^{(i)} \to X^{(i)}. \) It is clear that \( \chi_i \psi_i = id_{X^{(i)}} \) and \( \psi_i \chi_i = id_{Y^{(i)}}. \) Therefore \( \psi_i \) is a bijection. This completes the proof.

We conclude from this theorem that

**Example 4** Varieties defined by empty set of identities

are IBN-varieties.

In particular

**Example 5** The variety of all sets

is an IBN-variety.

**Example 6** The variety of all graphs.

We consider graphs as 2-sorted algebras. The first sort is the sort of edges of a graph, the second sort is the sort of vertices of a graph. The signature of graphs contain 2 unary operations: \( h \) and \( t. \) The operation \( h \) define a head of an edge and the operation \( t \) define a tail of an edge. These operations have a same type: \( \tau_h = \tau_t = (1; 2). \) The variety of all graphs defined by empty set of identities, so this variety is an IBN-variety.

**Example 7** The variety of all automatons.

This variety was considered in [18]. We consider automatons as 2-sorted algebras. The first sort is a sort of input signals, the second sort is a sort of statements of an automaton, the third sort is a sort of output signals. \( \Omega = \{\ast, \circ\}. \) The operation \( \ast \) gives as a new statement of an automaton according to an input signal and a previous statement of automaton: \( \tau_\ast = (1, 2; 2). \) The operation \( \circ \) gives an output signal according to an input signal and a statement of automaton. \( \tau_\circ = (1, 2; 3). \) The variety of all automatons defined by empty set of identities, so this variety is an IBN-variety.
3 Functor $\mathcal{D}$. General results.

In this section we study IBM property of a variety by properties of its subvarieties. This method seems to us rather strong. We can choose such subvarieties that an observation of their free algebras is very simple. The first result in this direction was obtained by [5]. Were considered the one-sorted algebras and was proved the

**Theorem 3.1** The variety $\Theta$ is an IBM-variety if there exists a subvariety $\Delta \subseteq \Theta$ such that $\Delta$ is a nondegenerate and for every $F \in \mathrm{Ob}\Delta^0$ the inequality $|F| < \infty$ holds.

We will consider one example as an application of this theorem. We will use this example below.

**Example 8** A nondegenerate variety of groups.

We will prove that every nondegenerate variety of groups is an IBN-variety. We denote some nondegenerate variety of groups, which we will consider, by $\Theta$. By [7, Theorem 15.1.10], $\Theta$ defined by set of identities

$$W = \left\{ x_1^d = 1, u_1 = 1, u_2 = 1, \ldots \right\},$$

where $d \in \mathbb{N}, d \neq 1$, $u_i$ are elements of commutators of free groups $\mathbb{F}(X)$, such that $|X| < \infty$. We will consider two cases:

**Case 1** $2 \mid d$, in particular $d = 0$.

In this case we consider the subvariety $\Delta \subseteq \Theta$ defined by identity

$$x_1^2 = 1.$$

This is a nondegenerate variety, because this variety contains the group $\mathbb{Z}_2$. By theorem about finitely generated abelian group every group of this variety generated by $n$ generators contains $2^n$ elements. Therefore $\Delta$ fulfills conditions of Theorem 3.1 and $\Theta$ is an IBN-variety.

**Case 2** $2 \nmid d$.

In this case there exists a prime number $p > 2$ such that $p \mid d$. We consider the subvariety $\Delta \subseteq \Theta$ defined by identities

$$x_1^p = 1, x_1x_2 = x_2x_1.$$

As above, this is a nondegenerate variety, because this variety contains the group $\mathbb{Z}_p$, and every group of this variety generated by $n$ generators contains $p^n$ elements. And as above, we conclude from Theorem 3.1 that $\Theta$ is an IBN-variety.

In this section we generalize the Theorem 3.1.
We consider only free algebras of varieties which have form \([1,3]\), because by Proposition \([1,3]\) all free algebras of varieties are isomorphic to the free algebras of this form.

We consider a variety of universal algebras \(\Theta\) with a set \(\Gamma\) of names of sorts. If \(\Delta \subseteq \Theta\) then by \([1,4]\) \(F_\Delta (\delta_{\Delta,X} (X)) \cong F_\Theta (\delta_{\Theta,X} (X)) / J_\Theta (\Delta,X)\). We will denote by \(\delta_{\Delta,X} : F_\Theta (\delta_{\Theta,X} (X)) \rightarrow F_\Theta (\delta_{\Theta,X} (X)) / J_\Theta (\Delta,X)\) the natural epimorphism.

**Lemma 3.1** If \(\Delta \subseteq \Theta\) then for every \(\varphi \in \text{Mor}_\Theta (F_\Theta (\delta_{\Theta,X} (X)), F_\Theta (\delta_{\Theta,Y} (Y)))\) there exists unique \(\varphi^* \in \text{Mor}_\Delta (F_\Delta (\delta_{\Delta,X} (X)), F_\Delta (\delta_{\Delta,Y} (Y)))\) such that \(\delta_{\Delta,Y} \varphi = \varphi^* \delta_{\Delta,X}\).

**Proof.** By Proposition \([1,4]\) \(\varphi (J_\Theta (\Delta, X)) \subseteq J_\Theta (\Delta, Y)\). So, there exists a homomorphism

\[
\varphi^* : F_\Theta (\delta_{\Theta,X} (X)) / J_\Theta (\Delta, X) \cong F_\Delta (\delta_{\Delta,X} (X))
\]

such that \(\delta_{\Delta,Y} \varphi = \varphi^* \delta_{\Delta,X}\).

If there are \(\varphi_1, \varphi_2 \in \text{Mor}_\Theta (F_\Theta (\delta_{\Theta,X} (X)), F_\Theta (\delta_{\Theta,Y} (Y)))\) such that \(\delta_{\Delta,Y} \varphi = \varphi_1 \delta_{\Delta,X}\) and \(\delta_{\Delta,Y} \varphi = \varphi_2 \delta_{\Delta,X}\), then \(\varphi_1 \delta_{\Delta,X} = \varphi_2 \delta_{\Delta,X}\), and, because \(\delta_{\Delta,X}\) is an epimorphism, \(\varphi_1 = \varphi_2\).

Now we will define the functor \(D_\Delta^\Theta : \Theta^0 \rightarrow \Delta^0\). We define the mapping \(D_\Delta^\Theta\) from \(\text{Ob} \Theta^0\) to \(\text{Ob} \Delta^0\) as follows:

1. \(D_\Delta^\Theta (F_\Theta (\delta_{\Theta,X} (X))) = F_\Delta (\delta_{\Delta,X} (X)) \cong F_\Theta (\delta_{\Theta,X} (X)) / J_\Theta (\Delta, X)\),
2. if \(\varphi \in \text{Mor}_\Theta (F_\Theta (\delta_{\Theta,X} (X)), F_\Theta (\delta_{\Theta,Y} (Y)))\) then

\[
D_\Delta^\Theta (\varphi) = \varphi^* \in \text{Mor}_\Delta (F_\Delta (\delta_{\Delta,X} (X)), F_\Delta (\delta_{\Delta,Y} (Y)))
\]

such that \(\delta_{\Delta,Y} \varphi = \varphi^* \delta_{\Delta,X}\).

By Lemma \([3,1]\) the \(D_\Delta^\Theta (\varphi)\) exists and is well-defined.

**Corollary 1** The mapping \(D_\Delta^\Theta\) is a functor.

**Proof.** For every \(F_\Theta (\delta_{\Theta,X} (X)) \in \text{Ob} \Theta^0\) we have that \(\delta_{\Delta,X} id_{F_\Theta (\delta_{\Theta,X} (X))} = id_{F_\Delta (\delta_{\Delta,X} (X))} \delta_{\Delta,X}\), so \(D_\Delta^\Theta (id_{F_\Theta (\delta_{\Theta,X} (X))}) = id_{D_\Delta^\Theta (F_\Theta (\delta_{\Theta,X} (X)))}\).

If \(F_\Theta (\delta_{\Theta,X_1} (X_1)), F_\Theta (\delta_{\Theta,X_2} (X_2)), F_\Theta (\delta_{\Theta,X_3} (X_3)) \in \text{Ob} \Theta^0\) and

\[
\varphi_1 \in \text{Mor}_\Theta (F_\Theta (\delta_{\Theta,X_1} (X_1)), F_\Theta (\delta_{\Theta,X_2} (X_2))),
\]

\[
\varphi_2 \in \text{Mor}_\Theta (F_\Theta (\delta_{\Theta,X_2} (X_2)), F_\Theta (\delta_{\Theta,X_3} (X_3))),
\]

then by Lemma \([3,1]\) we have that \(\delta_{\Delta,X_3} \varphi_1 = D_\Delta^\Theta (\varphi_1) \delta_{\Delta,X_1}\) and \(\delta_{\Delta,X_3} \varphi_2 = D_\Delta^\Theta (\varphi_2) \delta_{\Delta,X_2}\). So

\[
\delta_{\Delta,X_3} \varphi_2 \varphi_1 = D_\Delta^\Theta (\varphi_2) \delta_{\Delta,X_2} \varphi_1 = D_\Delta^\Theta (\varphi_2) D_\Delta^\Theta (\varphi_1) \delta_{\Delta,X_1}.
\]

Therefore \(D_\Delta^\Theta (\varphi_2 \varphi_1) = D_\Delta^\Theta (\varphi_2) D_\Delta^\Theta (\varphi_1)\).
Theorem 3.2 The variety $\Theta$ is an IBM-variety if for every $i \in \Gamma$ there exists a subvariety $\Delta_i \subseteq \Theta$ such that $\Delta_i$ is an $i$-nondegenerate $i$-IBM-variety.

Proof. The variety $\Theta$, which fulfills the condition of this theorem, is a nondegenerate variety by Claim [1.1] item 2.

We consider $F_\Theta (\delta_{\Theta, X} (X))$, $F_\Theta (\delta_{\Theta, Y} (Y)) \in \text{Ob}\Theta^0$ such that $F_\Theta (\delta_{\Theta, X} (X)) \cong F_\Theta (\delta_{\Theta, Y} (Y))$. The functor $D^\Theta_{\Delta_i}$ transforms this isomorphism to the isomorphism $F_\Delta, (\delta_{\Delta_i, X} (X)) \cong F_\Delta, (\delta_{\Delta_i, Y} (Y))$. $\Delta_i$ is an $i$-IBM-variety, so $\big( (\delta_{\Delta_i, X} (X))^{(i)} \big) \cong \big( (\delta_{\Delta_i, Y} (Y))^{(i)} \big)$. $\delta_{\Delta_i, X}$ and $\delta_{\Delta_i, Y}$ are homomorphisms, hence, we have by [1.1] that $\big( (\delta_{\Delta_i, X} (X))^{(i)} \big) = \big( (\delta_{\Delta_i, X} (X))^{(i)} \big)$, $\big( (\delta_{\Delta_i, Y} (Y))^{(i)} \big) = \big( (\delta_{\Delta_i, Y} (Y))^{(i)} \big)$. $\Delta_i$ is an $i$-nondegenerate, therefore, by Proposition [1.3] $\big| X^{(i)} \big| = \big| Y^{(i)} \big|$. $\Theta$ is a nondegenerate variety, so $\big| \delta_{\Theta, X} (X) \big| = \big| \delta_{\Theta, Y} (Y) \big|$ and, because $\delta_{\Theta, X}$ and $\delta_{\Theta, Y}$ are homomorphisms $\big| (\delta_{\Theta, X} (X))^{(i)} \big| = \big| (\delta_{\Theta, Y} (Y))^{(i)} \big|$. This equality holds for every $i \in \Gamma$, hence $\Theta$ is an IBM-variety.

Now we will demonstrate the application of this theorem to some varieties of one-sorted and many-sorted algebras. In the following examples we will consider wide enough classes of varieties of algebras. For arbitrary variety $\Theta$ from theses wide classes and for every name of sort $i \in \Gamma$ of elements of theses algebras we will define by some identity a variety $\Delta_i$ which will by an $i$-IBM-variety. If for every $i \in \Gamma$ the varieties will be $i$-nondegenerate we can conclude that the variety $\Theta$ is an IBM variety.

Example 9 A nondegenerate variety of linear algebras over a fixed field $k$.

Linear algebras are the vector spaces with an additional binary operation called multiplication, which is a bilinear. We consider a nondegenerate variety $\Theta$ of linear algebras over a fixed field $k$. We take a subvariety $\Delta \subseteq \Theta$ defined in $\Theta$ by identity

$$x_1 x_2 = 0,$$

i.e., subvariety of linear algebras with null multiplication. We will prove

Proposition 3.1 The variety $\Delta$ is a nondegenerate variety.

Proof. The variety of all linear algebras we denote by $\Lambda$. We consider some free algebra $F_\Theta (\delta_{\Theta, X} (X)) = F$ of the variety $\Theta$. This algebra is isomorphic to the quotient algebra $F (Y) / I (Y)$, where $Y = \delta_{\Lambda, X} (X)$, $F (Y) = F_\Lambda (\delta_{\Lambda, X} (X))$ is an absolutely free linear algebra generate by set of generators $Y$, $I (Y) = \mathcal{J}_\Lambda (\Theta, X)$ is a multihomogeneous ideal of $F (Y)$. The proof of this fact see, for example, in [1] Ch. VII, 3.1, Proposition 2. We suppose that $I (Y)$ contains generators of degree 1, i.e., $\lambda_i y_i + \ldots + \lambda_{i_n} y_{i_n} \in I (Y)$, where $y_1, \ldots, y_n \in Y$, $\lambda_1, \ldots, \lambda_n \in k \setminus \{ 0 \}$. There exists an endomorphism $\alpha$ of $F (Y)$ such that $\alpha (y_1) = y_1$, $\alpha (y_2) = \ldots = \alpha (y_{i_n}) = 0$. By Proposition [1.6] we have that $\alpha (\lambda_1 y_1 + \ldots + \lambda_{i_n} y_{i_n}) = \lambda_i y_i \in I (Y)$, so $y_i \in I (Y)$ and $\Theta$ is a degenerate variety. From this contradiction we conclude that $I (Y) \subseteq (F (Y))^2$. By Claim [1.4] item 3 we have that $F_\Delta (\delta_{\Theta, X} (X)) \cong F (Y) / (F (Y))^2$, i.e., this is a vector
space with the basis $Y$ equipped by null multiplication. From Proposition 1.3 we conclude that $\Delta$ is a nondegenerate variety.

Every algebra of the subvariety $\Delta$ is a vector space equipped by null multiplication. Every homomorphism of algebras of this subvariety are linear maps. So, as in Example 3 every algebra of this subvariety is a free algebra and the vectors of the basis of this vector space are free generators. Hence, as in Example 3 the subvariety $\Delta$ has the IBN property. Therefore by Theorem 3.2 the variety $\Theta$ has the IBN property.

**Example 10** (A. Sivatski) A nondegenerate variety of semigroups.

Let $\Theta$ is a nondegenerate variety of semigroups. We consider a subvariety $\Delta \subseteq \Theta$ defined in $\Theta$ by identity

$$x_1 x_2 = x_1. \quad (3.2)$$

We suppose that the variety $\Delta$ is a nondegenerate variety. Then the free semigroup $F_\Delta (X)$ of the subvariety $\Delta$ is a set $X$ with the multiplication defined by identity (3.2) and every mapping of sets from $F_\Delta (X) = X$ to $F_\Delta (Y) = Y$ is a homomorphism of these semigroups. Therefore, by Example 5 if $F_\Delta (X) \cong F_\Delta (Y)$ then $|X| = |Y|$ and the variety $\Theta$ has the IBN property.

If $\Theta$ is a nondegenerate variety of commutative semigroups then the subvariety defined in $\Theta$ by identity (3.2) is a degenerate variety, because from the identities $x_1 x_2 = x_2 x_1$ and $x_1 x_2 = x_1$ we conclude the identity $x_1 = x_2$. So we need use another approach for the proving of IBN property of varieties of commutative semigroups.

**Example 11** A nondegenerate variety of commutative semigroups.

Let $\Theta$ is a nondegenerate variety of commutative semigroups. We consider a subvariety $\Delta \subseteq \Theta$ defined in $\Theta$ by identity

$$x^2 = x. \quad (3.2)$$

We suppose that the variety $\Delta$ is a nondegenerate variety. We consider a finite set $X = \{x_1, \ldots, x_n\}$ and a semigroup $F_\Delta (X)$ which is free semigroup in the subvariety $\Delta$ and generated by the set $X$ of free generators. Every element of $F_\Delta (X)$ can be presented in the form $x_{i_1} x_{i_2} \ldots x_{i_m}$, where $1 \leq m \leq n$, $i_1 < i_2 < \ldots < i_m$. Hence $|F_\Delta (X)| < \infty$ and by Theorem 3.1 the variety $\Theta$ is an IBM-variety.

### 4 Examples of varieties of 2-sorted algebras

Now we will consider three examples of varieties of 2-sorted algebras i.e., such that $\Gamma = \{1, 2\}$. The studying of the IBM properties of these varieties was started in [8]. Here we present an approach that is more general and uniform. So the considerations in all of these examples will be very similar.

A signature $\Omega$ of every algebra which we will study in these examples fulfills the
Condition 4.1 A signature $\Omega$ is separated into three classes of algebraic operations: $\Omega = \Omega^{(1)} \uplus \Omega^{(2)} \uplus \Omega_\omega$. The operations from the first class $\Omega^{(1)}$ give results in the first sort of algebras and have arguments also only from the first sort. The operations from the second class $\Omega^{(2)}$ give results in the second sort of algebras and have arguments also only from the second sort. And the third class $\Omega_\omega$ contain only one binary operation which we call "action" and denote by $\circ$, this operation has type $(1, 2; 2)$.

Now we will prove some proprieties of algebras which signatures have such separation.

We consider some signature which fulfills Condition 4.1, some set $X = X^{(1)} \uplus X^{(2)} \subset X_0$ such that $|X| < \infty$ and algebra of terms $\mathcal{T} (\{1\}, \Omega, X) = \mathcal{T} (X)$.

Proposition 4.1 The equality $\mathcal{T} (X)^{(1)} = \mathcal{T} (\{1\}, \Omega^{(1)}, X^{(1)})$ holds.

Proof. We will prove the inclusion $\mathcal{T} (X)^{(1)} \subseteq \mathcal{T} (\{1\}, \Omega^{(1)}, X^{(1)})$ by induction by construction. The elements of the set $X^{(1)}$ and all constants of the first sort are elements of $\mathcal{T} (\{1\}, \Omega^{(1)}, X^{(1)})$. This is the induction basis. Now we will make the induction step. We consider a term $\omega (t_1, \ldots, t_n) \in \mathcal{T} (X)^{(1)}$, where $t_1, \ldots, t_n \in \mathcal{T} (X)$. So $\tau_\omega = (i_1, \ldots, i_n; 1)$, where $i_1, \ldots, i_n \in \{1, 2\}$. Our signature fulfills Condition 4.1 hence $i_1 = \ldots = i_n = 1$. Therefore $\omega \in \Omega^{(1)}$ and $t_1, \ldots, t_n \in \mathcal{T} (X)^{(1)}$. By the induction hypothesis $t_1, \ldots, t_n \in \mathcal{T} (\{1\}, \Omega^{(1)}, X^{(1)})$, so $\omega (t_1, \ldots, t_n) \in \mathcal{T} (\{1\}, \Omega^{(1)}, X^{(1)})$.

We can prove the inclusion $\mathcal{T} (\{1\}, \Omega^{(1)}, X^{(1)}) \subseteq \mathcal{T} (X)^{(1)}$ also by induction by construction even simpler than previous inclusion. 

Now we consider some variety $\Theta$ of algebras which fulfill Condition 4.1. The inclusion

$$\mathcal{I} (\Theta, X) \subseteq \left( \mathcal{T} (X)^{(1)} \times \mathcal{T} (X)^{(1)} \right) \cup \left( \mathcal{T} (X)^{(2)} \times \mathcal{T} (X)^{(2)} \right)$$

holds, because $\mathcal{I} (\Theta, X)$ is a congruence in $\mathcal{T} (X)$. The intersection $\mathcal{I} (\Theta, X) \cap \left( \mathcal{T} (X)^{(1)} \times \mathcal{T} (X)^{(1)} \right)$ which we denote by $(I (\Theta, X))^{(1)}$ is a congruence in $\mathcal{T} (X)^{(1)} = \mathcal{T} (\{1\}, \Omega^{(1)}, X^{(1)})$, so elements of $(I (\Theta, X))^{(1)}$ can be considered as identities in the class of one-sorted algebras with signature $\Omega^{(1)}$. We denote an union

$$\bigcup_{X \subset X_0, |X| < \infty} (I (\Theta, X))^{(1)}$$

by $(I (\Theta))^{(1)}$. If we considered elements of $(I (\Theta))^{(1)}$ as identities of one-sorted algebras with the signature $\Omega^{(1)}$, then they define the variety of one-sorted algebras with signature $\Omega^{(1)}$, which we denote by $\Theta^{(1)}$.

Proposition 4.2 If $A = A^{(1)} \uplus A^{(2)}$ is an arbitrary algebra of the variety $\Theta$, then $A^{(1)} \in \Theta^{(1)}$.

Proof. By Condition 4.1 the set $A^{(1)}$ is a one-sorted algebra with signature $\Omega^{(1)}$. We consider some identity

$$(f_1 (x_1, \ldots, x_r) = f_2 (x_1, \ldots, x_r)) \in (I (\Theta))^{(1)}.$$

(4.1)
By Proposition 4.1, all elements of the set \( X = \{ x_1, \ldots, x_r \} \) are elements of the first sort, so \( X = X^{(1)} = \{ x_1^{(1)}, \ldots, x_r^{(1)} \} \). We consider some homomorphism \( \psi : \mathfrak{I} (\{ 1 \}, \Omega^{(1)}, X^{(1)}) \to A^{(1)} \). By Proposition 4.1 there exists a homomorphism \( \varphi : \mathfrak{I} (X) \to A \), such that \( \varphi (x_j^{(1)}) = \psi (x_j^{(1)}) \), \( 1 \leq j \leq r \). \( A \in \Theta \), so \( A \) fulfills the identity (4.1) as identity of 2-sorted algebras with signature \( \Omega \). Therefore these equalities fulfill:

\[
\varphi \left( f_1 \left( x_1^{(1)}, \ldots, x_r^{(1)} \right) \right) = \varphi \left( f_2 \left( x_1^{(1)}, \ldots, x_r^{(1)} \right) \right),
\]
\[
f_1 \left( \varphi \left( x_1^{(1)} \right), \ldots, \varphi \left( x_r^{(1)} \right) \right) = f_2 \left( \varphi \left( x_1^{(1)} \right), \ldots, \varphi \left( x_r^{(1)} \right) \right),
\]
\[
f_1 \left( \psi \left( x_1^{(1)} \right), \ldots, \psi \left( x_r^{(1)} \right) \right) = f_2 \left( \psi \left( x_1^{(1)} \right), \ldots, \psi \left( x_r^{(1)} \right) \right).
\]

Thus, \( f_1 (x_1, \ldots, x_r), f_2 (x_1, \ldots, x_r) \in \mathfrak{I} (\{ 1 \}, \Omega^{(1)}, X^{(1)}) \), hence we obtain an equality

\[
\psi \left( f_1 \left( x_1^{(1)}, \ldots, x_r^{(1)} \right) \right) = \psi \left( f_2 \left( x_1^{(1)}, \ldots, x_r^{(1)} \right) \right).
\]

It means that \( A^{(1)} \) fulfills the identity (4.1) as identity of one-sorted algebras with signature \( \Omega^{(1)} \). Therefore \( A^{(1)} \in \Theta^{(1)} \). ■

**Proposition 4.3** If \( \Theta \) is nondegenerate variety then \( \Theta^{(1)} \) is nondegenerate variety.

**Proof.** We consider the set \( X = X^{(1)} = \{ x_1^{(1)}, x_2^{(1)} \} \). The variety \( \Theta \) is nondegenerate, so by Proposition 4.1 the free algebra \( F_{\Theta} (\delta_{\Theta,X} (X)) \) contains no less than 2 elements of the first sort. By Proposition 4.2 \( (F_{\Theta} (\delta_{\Theta,X} (X)))^{(1)} ) \in \Theta^{(1)} \). Hence, by Proposition 4.3 the variety \( \Theta^{(1)} \) is nondegenerate. ■

**Proposition 4.4** If \( \Theta^{(1)} \) is an IBN-variety, then \( \Theta \) is an 1-IBN-variety.

**Proof.** In the beginning we consider a set \( X = X^{(1)} \cup X^{(2)} \). There exists a natural epimorphism \( \delta_{\Theta,X} : \mathfrak{I} (X) \to F_{\Theta} (\delta_{\Theta,X} (X)) \), where \( F_{\Theta} (\delta_{\Theta,X} (X)) \) is a free algebra of the variety \( \Theta \) with the set of free generators \( \delta_{\Theta,X} (X) \). We will prove that \( (F_{\Theta} (\delta_{\Theta,X} (X)))^{(1)} \) is a free algebra of the variety \( \Theta^{(1)} \) with the set of free generators \( \delta_{\Theta,X} (X^{(1)}) \). By Condition 4.4 \( \delta_{\Theta,X} ^{(1)} \) is a homomorphism of one-sorted algebras with signature \( \Omega^{(1)} \). By definition \( \ker \delta_{\Theta,X} = \mathfrak{J} (\Theta, X) \), so \( \ker \delta_{\Theta,X} ^{(1)} = \mathfrak{J} (\Theta, X) \cap \left( \mathfrak{I} (X)^{(1)} \times \mathfrak{I} (X)^{(1)} \right) = (I (\Theta, X))^{(1)} \). \( \delta_{\Theta,X} \) is an epimorphism, so \( \operatorname{im} \delta_{\Theta,X} ^{(1)} = (F_{\Theta} (\delta_{\Theta,X} (X)))^{(1)} \). By Proposition 4.1 the equality \( \mathfrak{I} (X)^{(1)} = I \left( \{ 1 \}, \Omega^{(1)}, X^{(1)} \right) \) holds. So \( \operatorname{im} \delta_{\Theta,X} ^{(1)} = (F_{\Theta} (\delta_{\Theta,X} (X)))^{(1)} \cong \mathfrak{I} (X)^{(1)} \). So, by (4.4), \( (F_{\Theta} (\delta_{\Theta,X} (X)))^{(1)} \) is a free algebra of the variety \( \Theta^{(1)} \) with the set of free generators \( \delta_{\Theta,X} ^{(1)} (X^{(1)}) = \delta_{\Theta,X} (X^{(1)}) \).
Now we suppose that there exists an isomorphism $\varphi : F_\Theta (\delta_{\Theta,X} (X)) \to F_\Theta (\delta_{\Theta,Y} (Y))$ of two free algebras of the the variety $\Theta$. $\varphi^{(1)} : (F_\Theta (\delta_{\Theta,X} (X)))^{(1)} \to (F_\Theta (\delta_{\Theta,Y} (Y)))^{(1)}$ is a bijection and a homomorphism of one-sorted algebras with signature $\Omega^{(1)}$, so $\varphi^{(1)}$ is an isomorphism. As we proved above $(F_\Theta (\delta_{\Theta,X} (X)))^{(1)}$ and $(F_\Theta (\delta_{\Theta,Y} (Y)))^{(1)}$ are free algebras of the variety $\Theta^{(1)}$ with sets of free generators $\delta_{\Theta,X} (X^{(1)})$ and $\delta_{\Theta,Y} (Y^{(1)})$ respectively. If $\Theta^{(1)}$ is an IBN-variety, then $|\delta_{\Theta,X} (X^{(1)})| = |\delta_{\Theta,Y} (Y^{(1)})|$. The equalities $\delta_{\Theta,X} (X^{(1)}) = (\delta_{\Theta,X} (X))^{(1)}$ and $\delta_{\Theta,Y} (Y^{(1)}) = (\delta_{\Theta,Y} (Y))^{(1)}$ hold, so $\Theta$ is an 1-IBN-variety. □

Now we consider the variety $\Lambda$ defined by identity

$$x^{(1)} \circ x^{(2)} = s \left( x^{(2)} \right),$$

where $s (x^{(2)})$ is some term from element $x^{(2)}$ constructed by operations of $\Omega^{(2)}$, i.e., element of $\mathfrak{T} (\{2\}, \Omega^{(2)}, x^{(2)})$. Further, in particular examples, we will choose the term $s (x^{(2)})$ in different ways.

**Proposition 4.5** If some variety $\Delta$ is a subvariety of $\Lambda$, then for every $X \subseteq X_0$ and every $f \in \mathfrak{T} (X)^{(2)}$ there exists $f' \in \mathfrak{T} (\{2\}, \Omega^{(2)}, X^{(2)})$ such that $\delta^{(2)}_{\Delta,X} (f) = \delta^{(2)}_{\Delta,X} (f')$, where $\delta^{(2)}_{\Delta,X} : \mathfrak{T} (X) \to F_{\Delta} (\delta^{(2)}_{\Delta,X} (X))$ is a natural epimorphism.

**Proof.** We will prove this fact by induction by construction. For elements of the set $X^{(2)}$ and for constant of the second sort this fact is trivial. This is the induction basis. Now we will make the induction step.

In the first case we suppose that $f = \omega (t_1, \ldots, t_n)$, where $\omega \in \Omega^{(2)}$ and $t_1, \ldots, t_n \in \mathfrak{T} (X)^{(2)}$. By induction hypothesis there exist $t'_1, \ldots, t'_n \in \mathfrak{T} (\{2\}, \Omega^{(2)}, X^{(2)})$, such that $\delta^{(2)}_{\Delta,X} (t_i) = \delta^{(2)}_{\Delta,X} (t'_i)$, $1 \leq i \leq n$. Therefore

$$\delta^{(2)}_{\Delta,X} (f) = \delta^{(2)}_{\Delta,X} (\omega (t_1, \ldots, t_n)) = \omega \left( \delta^{(2)}_{\Delta,X} (t_1), \ldots, \delta^{(2)}_{\Delta,X} (t_n) \right) = \omega \left( \delta^{(2)}_{\Delta,X} (t'_1), \ldots, \delta^{(2)}_{\Delta,X} (t'_n) \right) = \delta^{(2)}_{\Delta,X} (\omega (t'_1, \ldots, t'_n)).$$

In the second case we suppose that $f = t^{(1)} \circ t^{(2)}$, where $t^{(1)} \in \mathfrak{T} (X)^{(1)}$, $t^{(2)} \in \mathfrak{T} (X)^{(2)}$. By the induction hypothesis there exists $t' \in \mathfrak{T} (\{2\}, \Omega^{(2)}, X^{(2)})$, such that $\delta^{(2)}_{\Delta,X} (t^{(2)}) = \delta^{(2)}_{\Delta,X} (t')$. $F_{\Delta} (\delta^{(2)}_{\Delta,X} (X))$ is an algebra of the variety $\Delta$, so, by item 2 of Claim 1.1 it fulfills the identity 1.2. Therefore $\delta^{(2)}_{\Delta,X} (f) = \delta^{(2)}_{\Delta,X} (t^{(1)}) \circ \delta^{(2)}_{\Delta,X} (t^{(2)}) = s \left( \delta^{(2)}_{\Delta,X} (t^{(2)}) \right) = s \left( \delta^{(2)}_{\Delta,X} (t') \right) = \delta^{(2)}_{\Delta,X} (s (t')).$ $s (t') = f' \in \mathfrak{T} (\{2\}, \Omega^{(2)}, X^{(2)}).$ □

**Corollary 1** Under the conditions of this proposition the equality

$$\delta^{(2)}_{\Delta,X} \left( \mathfrak{T} (X)^{(2)} \right) = (F_{\Delta} (\delta^{(2)}_{\Delta,X} (X)))^{(2)} = \delta^{(2)}_{\Delta,X} \left( \mathfrak{T} (\{2\}, \Omega^{(2)}, X^{(2)}) \right)$$

holds.

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Now we denote by $\Delta$ a subvariety of the variety $\Theta$, defined by the identity \[\text{Claim 1.1.}\] Therefore $\Delta$ is a subject of Proposition 4.5.

For every $X \subset X_0$ we denote $\mathcal{I}(\Delta_X) \cap (\mathcal{I}(\{2\}, \Omega(2), X(2)) \times \mathcal{I}(\{2\}, \Omega(2), X(2)))$ by $(I(\Delta_X))(2)$. An union $\bigcup_{X \subset X_0, |X| < \infty} (I(\Delta_X))(2)$ we denote by $(I(\Delta))(2)$.

$(I(\Delta))(2)$ is a congruence in $\mathcal{I}(\{2\}, \Omega(2), X(2))$, so elements of $(I(\Delta))(2)$ can be considered as identities of one-sorted algebras with the signature $\Omega(2)$. The variety defined by these identities we denote by $\Delta(2)$. We define the variety $\Delta(1)$ in the same way as we previously defined the variety $\Theta(1)$.

Now we consider an one-sorted algebra $H(1)$ with signature $\Omega(1)$, such that $H(1) \in \Delta(1)$, and an one-sorted algebra $H(2)$ with signature $\Omega(2)$, such that $H(2) \in \Delta(2)$. We construct a 2-sorted algebra $H = H(1) \uplus H(2)$, such that $H(1)$ is a set of all elements of the first sort of algebra $H$ and $H(2)$ is a set of all elements of the second sort of algebra $H$. We define the action elements of $H(1)$ over elements of $H(2)$ this way

$$\forall h(1) \in H(1), h(2) \in H(2), h(1) \circ h(2) = s(h(2)), \quad (4.3)$$

where $s$ is a term from (4.2). By Condition 4.1 the algebra $H$ is an algebra of signature $\Omega$.

**Proposition 4.6** $H$ is an algebra of the variety $\Delta$.

**Proof.** We have from (4.3) that $H$ fulfills the identity (4.2), so $H \in \Delta$. We will prove that algebra $H$ fulfills all identities of variety $\Delta$.

We consider a set $X = X(1) \uplus X(2) \subset X_0$ and algebra of terms $\mathcal{I}(X)$. We suppose that $(f, g) \in \mathcal{I}(\Delta, X) \cap (\mathcal{I}(X)(2) \times \mathcal{I}(X)(2)) \subset \mathcal{I}(\Delta)$. It means that $(f, g) \in \ker \delta_{\Delta,X}$, i.e., $\delta_{\Delta,X}(f) = \delta_{\Delta,X}(g)$. By Proposition 4.5 there exist $f', g' \in \mathcal{I}(\{2\}, \Omega(2), X(2))$ such that $\delta_{\Delta,X}(f) = \delta_{\Delta,X}(f') = \delta_{\Delta,X}(g') = \delta_{\Delta,X}(g')$. Therefore

$$(f, f'), (g, g') \in \ker \delta_{\Delta,X} = \mathcal{I}(\Lambda, X). \quad (4.4)$$

Now we consider an arbitrary homomorphism $\varphi: \mathcal{I}(X) \to H$. $H \in \Lambda$, so

$$\varphi(f) = \varphi(f'), \varphi(g) = \varphi(g'). \quad (4.5)$$

$\Delta \subseteq \Lambda$, so, similar to item 2 of Claim 1.1, $\ker \delta_{\Delta,X} = \mathcal{I}(\Delta, X) \supseteq \ker \delta_{\Delta,X} = \mathcal{I}(\Lambda, X)$. Hence, by (4.4), $\delta_{\Delta,X}(f) = \delta_{\Delta,X}(f') = \delta_{\Delta,X}(g) = \delta_{\Delta,X}(g')$. Hence $(f', g') \in \ker \delta_{\Delta,X} = \mathcal{I}(\Delta, X)$. So $(f', g') \in (I(\Delta, X))(2) \subset (I(\Delta))(2)$. Here we consider $(f' = g')$ as an identity of one-sorted algebras with the signature $\Omega(2)$. $\varphi|_{\mathcal{I}(\{2\}, \Omega(2), X(2))}: \mathcal{I}(\{2\}, \Omega(2), X(2)) \to H(2)$ is a homomorphism from the one-sorted algebra of
morphism, we conclude that $\varphi(f') = \varphi(g')$. By Proposition 1.5 we conclude that $\varphi(f) = \varphi(g)$, i.e., $H \models (f = g)$.

Now we consider $(f, g) \in \mathcal{J}(\Delta, X) \cap \left(\mathfrak{T}(X)^{(1)} \times \mathfrak{T}(X)^{(1)}\right)$. By Proposition 4.4 we can consider $(f = g)$ as an identity of one-sorted algebras with the signature $\Omega^{(1)}$. We consider an arbitrary homomorphism $\varphi : \mathfrak{T}(X) \to H$. By Proposition 4.1 $\varphi^{(1)} : \mathfrak{T}\left(\{1\}, \Omega^{(1)}\right) \to H^{(1)}$ is a homomorphism from the one-sorted algebra of terms with signature $\Omega^{(1)}$ to the one-sorted algebra $H^{(1)}$ with same signature, $(f, g) \in (I(\Delta))^{(1)}$, $H^{(1)} \in \Delta^{(1)}$, so $\varphi(f) = \varphi^{(1)}(f) = \varphi^{(1)}(g) = \varphi(g)$, i.e., $H \models (f = g)$.

**Proposition 4.7** For every set $X = X^{(1)} \cup X^{(2)} \subset X_0$ the algebra $(F_\Delta(\delta_{\Delta,X}(X)))^{(2)} = \delta_{\Delta,X}^{(2)}\left(\mathfrak{T}\left(\{2\}, \Omega^{(2)}\right), X^{(2)}\right)$ is a free algebra of the variety $\Delta^{(2)}$ with the set of generators $\delta_{\Delta,X}^{(2)}(X^{(2)})$.

**Proof.** We consider an arbitrary algebra of the variety $\Delta^{(2)}$. This algebra we denote by $H^{(2)}$. We consider an arbitrary mapping $\varphi^* : \delta_{\Delta,X}^{(2)}(X^{(2)}) \to H^{(2)}$. $\Delta$ is a subject of the Proposition 4.2, so the algebra $(F_\Delta(\delta_{\Delta,X}(X)))^{(1)}$ which we denote by $H^{(1)}$ is an algebra of the variety $\Delta^{(1)}$. We consider the set $H = H^{(1)} \cup H^{(2)}$ with operations of the signature $\Omega^{(1)}$ defined in $H^{(1)}$ and operations of the signature $\Omega^{(2)}$ defined in $H^{(2)}$. We define the action elements of $H^{(1)}$ over elements of $H^{(2)}$ by formula (4.3). For every $x^{(1)} \in X^{(1)}$ we define $\varphi^*\left(\delta_{\Delta,X}^{(1)}(x^{(1)})\right) = \delta_{\Delta,X}^{(1)}(x^{(1)}) \in (F_\Delta(\delta_{\Delta,X}(X)))^{(1)} = H^{(1)}$. By Proposition 4.6 we have that $H \in \Delta$. So there exists a homomorphism $\varphi : (F_\Delta(\delta_{\Delta,X}(X))) \to H$, such that $\varphi|_{\delta_{\Delta,X}(X)} = (\varphi^*)|_{\delta_{\Delta,X}(X)}$.

$$\varphi^{(2)} : (F_\Delta(\delta_{\Delta,X}(X)))^{(2)} = \delta_{\Delta,X}^{(2)}\left(\mathfrak{T}\left(\{2\}, \Omega^{(2)}\right), X^{(2)}\right) \to H^{(2)}$$

is a homomorphism of one-sorted algebras with signature $\Omega^{(2)}$, such that $(\varphi^{(2)})|_{\delta_{\Delta,X}^{(2)}(X^{(2)})} = \varphi^*$, i.e., the equality $\left(\varphi^{(2)}\delta_{\Delta,X}^{(2)}\right)(x^{(2)}) = \left(\varphi^*\delta_{\Delta,X}^{(2)}\right)(x^{(2)})$ holds for every $x^{(2)} \in X^{(2)}$.

We suppose that there is an other homomorphism $\chi : \delta_{\Delta,X}^{(2)}\left(\mathfrak{T}\left(\{2\}, \Omega^{(2)}\right), X^{(2)}\right) \to H^{(2)}$ of one-sorted algebras with signature $\Omega^{(2)}$ such that $\chi|_{\delta_{\Delta,X}^{(2)}(X^{(2)})} = \varphi^*$, i.e., the equality $\left(\chi\delta_{\Delta,X}^{(2)}\right)(x^{(2)}) = \left(\varphi^*\delta_{\Delta,X}^{(2)}\right)(x^{(2)})$ holds for every $x^{(2)} \in X^{(2)}$. By Proposition 1.1 we have that $\varphi^{(2)}\delta_{\Delta,X}^{(2)} = \chi\delta_{\Delta,X}^{(2)}$ and, because $\delta_{\Delta,X}^{(2)}$ is an epimorphism, we conclude that $\varphi^{(2)} = \chi$.

**Corollary 1** If $\Delta^{(2)}$ is a nondegenerate variety, then $\Delta$ is 2-nondegenerate variety.

**Proof.** $\left(x_1^{(2)}, x_2^{(2)}\right) \in \mathfrak{T}\left(\{2\}, \Omega^{(2)}, \left\{\left.x_1^{(2)}, x_2^{(2)}\right\}\right) \times \mathfrak{T}\left(\{2\}, \Omega^{(2)}, \left\{\left.x_1^{(2)}, x_2^{(2)}\right\}\right)$. So, if $\left(x_1^{(2)}, x_2^{(2)}\right) \in \mathcal{J}(\Delta)$, then $\left(x_1^{(2)}, x_2^{(2)}\right) \in (I(\Delta))^{(2)}$. ■
Corollary 2 If $\Delta^{(2)}$ is an IBN-variety, then $\Delta$ is a 2-IBN variety.

Proof. We suppose that there exists an isomorphism $\varphi: (F_\Delta(\delta, X(X))) \to (F_\Delta(\delta, Y(Y)))$, where $X = X^{(1)} \cup X^{(2)}, Y = Y^{(1)} \cup Y^{(2)} \subset X_0$. Therefore $\varphi^{(2)}: (F_\Delta(\delta, X(X))^{(2)}) \to (F_\Delta(\delta, Y(Y))^{(2)})$ is an isomorphism. $(F_\Delta(\delta, X(X)))^{(2)}$ is a free algebra of the variety $\Delta^{(2)}$ with the set of generators $\delta^{(2)}, X$, $\delta^{(2)}, Y$, $\delta^{(2)}$ is an IBN-variety, so $\delta^{(2)} X = \delta^{(2)} Y$. 

Now we will prove two propositions, which we will use in the consideration of the following three examples.

**Proposition 4.8** The variety of all sets has no nondegenerate subvarieties.

Proof. Nontrivial identities in the variety of all sets have form $x_i = x_j$, where $i \neq j$. But, if $\Theta$ is a subvariety of the variety of all sets and $(x_i, x_j) \in I(\Theta)$, where $i \neq j$, then $\Theta$ is a degenerate variety.

**Proposition 4.9** The variety of all vector spaces over some fixed field $k$ has no nondegenerate subvarieties.

Proof. A vector space $F(Y)$ with basis $Y$ is a free algebra of the variety of all vector spaces. Nontrivial identities in this variety have form $\lambda_1 y_1 + \ldots + \lambda_n y_n = 0$, where $y_1, \ldots, y_n \in Y$, $\lambda_1, \ldots, \lambda_n \in k \setminus \{0\}$. We conclude as in the Proof of the Proposition 3.1 that the subvariety of the variety of all vector spaces defined by this identity is a degenerate variety.

The following three examples satisfy Condition 4.1.

**Example 12** Varieties of actions of semigroups over sets.

In this example we consider varieties of actions of semigroups over sets. Actions of semigroups over sets are 2-sorted algebras, i.e., $\Gamma = \{1, 2\}$. The first sort is a sort of elements of semigroups, the second sort is a sort of elements of sets. $\Omega = \{\cdot, \circ\}$: $\cdot$ is a multiplication in the semigroup, $\circ$ is an action of the elements of the semigroup over the elements of the set. $\tau_\cdot = (1, 1; 1)$, $\tau_\circ = (1, 2; 2)$.

We denote by $\Theta$ some nondegenerate variety of actions of semigroups over sets.

If $\Theta^{(1)}$ is an IBN-variety, in particular, one from varieties considered in Examples 10 and 11 then, by Proposition 4.3 $\Theta$ is an 1-IBN-variety. As a variety $\Lambda$ we consider the variety of trivial action, i.e., the variety defined by identity $x^{(1)} \circ x^{(2)} = x^{(2)}$. This is identity 4.2 with $s(x^{(2)}) = x^{(2)}$.

If $\Delta^{(2)}$ is a nondegenerate variety, then, by Proposition 4.8 $\Delta^{(2)}$ is a variety of all sets. Hence, by Example 5 $\Delta^{(2)}$ is an IBN-variety. By Corollaries 1 and 2 from Proposition 4.7 we have that $\Delta$ is a 2-nondegenerate 2-IBN-variety. By Theorem 3.2 we can conclude in this case that $\Theta$ is an IBN-variety.
Example 13 Varieties of representations of Lie algebras over a fixed field $k$.

In this example we consider varieties of representations of Lie algebras over some fixed field $k$. A representation of Lie algebras we consider as 2-sorted algebra. The first sort is a sort of elements of some Lie algebra over a fixed field $k$, the second sort is a sort of vectors of some vector space over same field $k$. As in Example 3, a multiplication of elements of the Lie algebra and vectors by any scalar $\lambda \in k$ we consider as two different unary operations. All operations over vectors of the vector space have all arguments of the sort 2 and give a result of the sort 2. All operations over elements of the Lie algebra have all arguments of the sort 1 and give a result of the sort 1. An operation of an action of elements of the Lie algebra over vectors of the vector space, which we denote by $\circ$, has a type $\tau_\circ = (1, 2; 2)$.

We denote by $\Theta$ some nondegenerate variety of representations Lie algebras over the field $k$. By Proposition 4.3 the variety $\Theta^{(1)}$ is nondegenerate. So, $\Theta^{(1)}$ is a nondegenerate variety of Lie algebras over the field $k$ and, by Example 9, $\Theta^{(1)}$ is an IBN-variety. Hence, by Proposition 4.4, $\Theta$ is an 1-IBN-variety.

As a variety $\Lambda$ we consider the variety of null action, i.e., the variety defined by identity $x^{(1)} \circ x^{(2)} = 0^{(2)}$.

This is identity (4.2) with $s (x^{(2)}) = 0^{(2)}$.

**Proposition 4.10** $\Delta^{(2)}$ is the variety of all vector spaces over the field $k$.

**Proof.** In the proof of this proposition we denote elements $\delta_{\Xi, X} (x)$, $\delta_{\Lambda, X} (x)$, $\delta_{\Theta, X} (x)$, $\delta_{\Delta, X} (x)$ and $x \in X \subset X_0$ by same symbol $x$. This cannot cause confusion here.

$\Lambda$ is a nondegenerate variety because for every Lie algebras over the field $k$ we can define a null action over an arbitrary vector space over the field $k$.

In [19, Theorem 3.1.] (see also [17]) was proved that for every variety $\Xi$ of representations Lie algebras over the field $k$ the free algebra $F_{\Xi} (X)$ of this variety generated by set $X = X^{(1)} \uplus X^{(2)} \subset X_0$ has this form:

$$(F_{\Xi} (X))^{(1)} = L_{\Xi} \left( X^{(1)} \right) = L \left( X^{(1)} \right) / I_{\Xi} \left( X^{(1)} \right),$$

$$(F_{\Xi} (X))^{(2)} = \bigoplus_{x^{(2)} \in X^{(2)}} \left( A_{\Xi} \left( X^{(1)} \right) x^{(2)} \right),$$

where $L \left( X^{(1)} \right)$ is a free Lie algebra, generated by the set $X^{(1)}$, $I_{\Xi} \left( X^{(1)} \right)$ is a multihomogeneous two-sided ideal of this algebra, $\bigoplus_{x^{(2)} \in X^{(2)}} \left( A_{\Xi} \left( X^{(1)} \right) x^{(2)} \right)$ is a direct sum of $|X^{(2)}|$ copies of $A \left( X^{(1)} \right)$ cyclic module $A_{\Xi} \left( X^{(1)} \right) = A \left( X^{(1)} \right) / B_{\Xi} \left( X^{(1)} \right)$, where $A \left( X^{(1)} \right)$ is a free associative algebra with unit, generated by the set $X^{(1)}$, $B_{\Xi} \left( X^{(1)} \right)$ is a multihomogeneous two-sided ideal of this algebra.
It is clear that $B_\Lambda (X^{(1)}) \supseteq \langle X^{(1)} \rangle$, where $\langle X^{(1)} \rangle$ is a two-sided ideal of algebra $A (X^{(1)})$, generated by all elements of set $X^{(1)}$. $\langle X^{(1)} \rangle$ is a maximal two-sided ideal of $A (X^{(1)})$, because $A (X^{(1)}) / \langle X^{(1)} \rangle \cong k$. $\Lambda$ is a nondegenerate variety, so $B_\Lambda (X^{(1)}) = \langle X^{(1)} \rangle$.

Now we consider $B_\Theta (X^{(1)})$. If there exists $\lambda \in k$ such that $\lambda \neq 0$ and $\lambda \in B_\Theta (X^{(1)})$, then the $B_\Theta (X^{(1)}) = A (X^{(1)})$ holds. But this contradicts the fact that $\Theta$ is a nondegenerate variety. Therefore $B_\Theta (X^{(1)}) = \langle X^{(1)} \rangle = B_\Lambda (X^{(1)})$.

By item 8 of Claim 1.1 we have that $B_\Delta (X^{(1)}) = \langle B_\Theta (X^{(1)}), B_\Lambda (X^{(1)}) \rangle = B_\Lambda (X^{(1)}) = \langle X^{(1)} \rangle$. Hence

$$(F_\Delta (X))^{(2)} = \bigoplus_{x^{(2)} \in X^{(2)}} \left( \left( A \left( X^{(1)} \right) / \langle X^{(1)} \rangle \right) x^{(2)} \right) \cong \bigoplus_{x^{(2)} \in X^{(2)}} k x^{(2)}$$

is a vector space over $k$ which has dimension $|X^{(2)}|$. By Corollary II from Proposition 1.3 $\delta^{(2)}_{\Delta,X} (\mathfrak{T} (X))^{(2)} = (F_\Delta (X))^{(2)} = \delta^{(2)}_{\Delta,X} (\mathfrak{T} (\{2\}, \Omega^{(2)}, X^{(2)}))$ and by Proposition 4.7 $\delta^{(2)}_{\Delta,X} (\mathfrak{T} (\{2\}, \Omega^{(2)}, X^{(2)}))$ is a free algebra of the variety $\Delta^{(2)}$. Now Propositions 1.3 and 4.9 complete the proof. ■

**Corollary 1** $\Delta^{(2)}$ is an IBN-variety.

**Proof.** See Example 3. ■

So $\Delta$ is a 2-nondegenerate 2-IBN-variety by Corollaries II and I from Proposition 4.7. Now we can conclude from Theorem 3.2 that $\Theta$ is an IBN-variety.

**Example 14** Varieties of representations of groups over a fixed field $k$.

In this example we consider varieties of representations of groups over a fixed field $k$. A representation of a group we consider as a 2-sorted algebra. The first sort is a sort of elements of a group, the second sort is a sort of vectors of some vector space over a fixed field $k$. As in Example 3 a multiplication of vectors by any scalar $\lambda \in k$ we consider as an unary operation. All operations over vectors of the vector space have all arguments of the sort 2 and give a result of the sort 2. All operations over elements of the group: the multiplication; the operation that gives to each element its inverse and the constant 1 - have all arguments of the sort 1 and give a result of the sort 1. An operation of action of elements of the group over vectors of the vector space, which we denote by $\circ$, has a type $\tau_\circ = (1, 2; 2)$.

We denote by $\Theta$ some nondegenerate variety of representations of groups over a fixed field $k$. By Proposition 4.3 the variety $\Theta^{(1)}$ is nondegenerate. So, $\Theta^{(1)}$ is a nondegenerate variety of groups and, by Example 8 $\Theta^{(1)}$ is an IBN-variety. Hence, by Proposition 4.4 $\Theta$ is an 1-IBN-variety.

As a variety $\Lambda$ we consider the variety of trivial action, i.e., the variety defined by identity

$$x^{(1)} \circ x^{(2)} = x^{(2)}.$$ 

This is identity 4.2 with $s (x^{(2)}) = x^{(2)}$.

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Proposition 4.11 $\Delta^{(2)}$ is the variety of all vector spaces over the field $k$.

Proof. In the proof of this proposition we denote elements $\delta_{\Xi, X}(x)$, $\delta_{\Lambda, X}(x)$, $\delta_{\Theta, X}(x)$, $\delta_{\Delta, X}(x)$ and $x \in X \subseteq X_0$ by same symbol $x$. This cannot cause confusion here.

We can prove by method of [19, Section 3] that for every variety $\Xi$ of representations Lie algebras over the field $k$ the free algebra $F_{\Xi}(X)$ of this variety generated by set $X = X^{(1)} \uplus X^{(2)} \subseteq X_0$ has this form:

$$(F_{\Xi}(X))^{(1)} = G_{\Xi}(X^{(1)}) = G(X^{(1)}) / H_{\Xi}(X^{(1)}),$$

$$(F_{\Xi}(X))^{(2)} = \bigoplus_{x^{(2)} \in X^{(2)}} (kG(X^{(1)}) / I_{\Xi}(X^{(1)})) x^{(2)},$$

where $G(X^{(1)})$ is a free group, generated by the set $X^{(1)}$, $H_{\Xi}(X^{(1)})$ is a fully invariant subgroup of this group, $kG(X^{(1)})$ is a $k$-group algebra over the group $G(X^{(1)})$, $\bigoplus_{x^{(2)} \in X^{(2)}} (kG(X^{(1)}) / I_{\Xi}(X^{(1)})) x^{(2)}$ is a direct sum of $|X^{(2)}|$ copies of $kG(X^{(1)})$ cyclic module, $I_{\Xi}(X^{(1)})$ is a two-sided ideal of algebra $kG(X^{(1)})$, which is invariant under all endomorphisms of $kG(X^{(1)})$, defined by endomorphisms of $G(X^{(1)})$. See also [20, Section 0.2].

By [20, Section 0.2, Example 1], $I_{\Lambda}(X^{(1)}) = \text{Aug}(X^{(1)})$ an augmentation ideal of $kG(X^{(1)})$, i.e., kernel of the homomorphism $\varepsilon : kG(X^{(1)}) \rightarrow k$, defined by homomorphism $G(X^{(1)}) \rightarrow \{1_k\}$. $I_{\Theta}(X^{(1)}) \subseteq \text{Aug}(X^{(1)})$ by [20, Proposition 0.2.3]. Therefore, as in Example [13] $I_{\Delta}(X^{(1)}) = \langle I_{\Theta}(X^{(1)}), I_{\Lambda}(X^{(1))} \rangle = I_{\Lambda}(X^{(1)}) = \text{Aug}(X^{(1)})$. Hence

$$(F_{\Delta}(X))^{(2)} = \bigoplus_{x^{(2)} \in X^{(2)}} (kG(X^{(1)}) / \text{Aug}(X^{(1)})) x^{(2)} \cong \bigoplus_{x^{(2)} \in X^{(2)}} kx^{(2)}$$

is a vector space over $k$ which has a dimension $|X^{(2)}|$. Now we complete the proof as in Example [13].

We have, as in Example [13] that $\Delta$ is a 2-nondegenerate 2-IBN-variety and $\Theta$ is an IBN-variety.

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