The asymptotic behaviour of recurrence coefficients for orthogonal polynomials with varying exponential weights

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Abstract

We consider orthogonal polynomials \( \{ p_{n,N}(x) \} \) on the real line with respect to a weight \( w(x) = e^{-NV(x)} \) and in particular the asymptotic behaviour of the coefficients \( a_{n,N} \) and \( b_{n,N} \) in the three term recurrence \( x\pi_{n,N}(x) = \pi_{n+1,N}(x) + b_{n,N}\pi_{n,N}(x) + a_{n,N}\pi_{n-1,N}(x) \). For one-cut regular \( V \) we show, using the Deift-Zhou method of steepest descent for Riemann-Hilbert problems, that the diagonal recurrence coefficients \( a_{n,n} \) and \( b_{n,n} \) have asymptotic expansions as \( n \to \infty \) in powers of \( 1/n^2 \) and powers of \( 1/n \), respectively.

1 Introduction

We consider the asymptotic behavior of the recurrence coefficients \( a_{n,N} \) and \( b_{n,N} \) in the three-term recurrence relation

\[
x\pi_{n,N}(x) = \pi_{n+1,N}(x) + b_{n,N}\pi_{n,N}(x) + a_{n,N}\pi_{n-1,N}(x)
\]

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for orthogonal polynomials with respect to varying exponential weights. Here $\pi_{n,N}$ is the $n$-th degree monic orthogonal polynomial with respect to a varying weight

$$w_N(x) = e^{-NV(x)}$$

where $V$ is real analytic on $\mathbb{R}$ with $\lim_{x \to \pm \infty} \frac{V(x)}{\log(1+x^2)} = +\infty$. Moreover, $V$ is assumed to be one-cut regular, which means that the equilibrium measure $d\mu_V = \psi_V(x)dx$ associated with $V$ is supported on one interval $[a, b]$ where it has the form

$$\psi_V(x) dx = \sqrt{(b-x)(x-a)} h(x) \chi_{[a,b]}(x) dx$$

(1.1)

where $h$ is real analytic, strictly positive on $[a, b]$, and in addition the inequality (3.1) is strict for $x \in \mathbb{R} \setminus [a, b]$. See e.g. [1, 2, 5, 11, 17] for the definition of the equilibrium measure and for more information on the one-cut regular case.

Under these assumptions Deift et al. [7] proved that $a_{n,n}$ and $b_{n,n}$ have asymptotic expansions in powers of $1/n$. Their approach is based on the Deift-Zhou method of steepest descent applied to the Riemann-Hilbert problem for orthogonal polynomials of Fokas, Its, and Kitaev [12]. This method was first introduced in [9] and further developed in [6, 7, 8] and many papers since then.

The asymptotic result on the recurrence coefficients was considerably refined by Bleher and Its [2, Theorem 5.2] who showed for polynomial $V$ that there exists $\varepsilon > 0$ and real analytic functions $f_{2k}(s)$, $g_{2k}(s)$, $k = 0, 1, \ldots$, on $[1 - \varepsilon, 1 + \varepsilon]$ such that the asymptotic expansions

$$a_{n,N} \sim f_0 \left( \frac{n}{N} \right) + \sum_{m=1}^\infty N^{-2m} f_{2m} \left( \frac{n}{N} \right)$$

(1.2)

$$b_{n,N} \sim g_0 \left( \frac{n + 1/2}{N} \right) + \sum_{m=1}^\infty N^{-2m} g_{2m} \left( \frac{n + 1/2}{N} \right)$$

(1.3)

hold uniformly as $n, N \to \infty$ with $1 - \varepsilon \leq n/N \leq 1 + \varepsilon$. These $1/N^2$ expansions are used in [2] to prove the $1/N^2$ expansion of the free energy (a.k.a. logarithm of the partition function or Hankel determinant) of the associated random matrix ensemble in the one-cut regular case, see also [11].

The proof of (1.2) and (1.3) in [2] is based on the Deift et al. result referred to above, in combination with so-called string equations. It is of some interest to find a proof that is based on the Riemann-Hilbert steepest
descent analysis only. Here we do this for the diagonal case \( n = N \), and we obtain the following.

**Theorem 1.1.** Let \( V \) be real analytic and one-cut regular. Then there exist constants \( \alpha_{2m} \) and \( \beta_m, m = 1, 2, \ldots \) (depending on \( V \)) such that \( a_{n,n} \) and \( b_{n,n} \) have the following asymptotic expansions as \( n \to \infty \):

\[
a_{n,n} \sim \frac{(b-a)^2}{16} + \sum_{m=1}^{\infty} \frac{\alpha_{2m}}{n^{2m}}, \quad b_{n,n} \sim \frac{b+a}{2} + \sum_{m=1}^{\infty} \frac{\beta_m}{n^m},
\]

where \( a \) and \( b \) are the endpoints of the support of \( \psi_V \). The first coefficient \( \beta_1 \) in the expansion for \( b_{n,n} \) is given explicitly by

\[
\beta_1 = \frac{1}{2\pi(b-a)} \left( \frac{1}{h(b)} - \frac{1}{h(a)} \right)
\]

where \( h \) is the function appearing in the expression (1.1) for the equilibrium measure \( \psi_V \) associated with \( V \).

In our proof of Theorem 1.1 we follow the main lines of the steepest descent analysis of [7]. We will deduce that the odd powers in the expansion of \( a_{n,n} \) vanish from the structure of the local Airy parametrices around the endpoints. The expression (1.5) for \( \beta_1 \) is new, although it is likely that it can be deduced from the approach of [2] as well. The explicit formula (1.5) shows that \( \beta_1 = 0 \) if and only if \( h(a) = h(b) \). It is very easy to construct examples of one-cut regular \( V \) such that \( h(a) \neq h(b) \) and so \( \beta_1 \neq 0 \). We have thus corrected an error in a paper of Albeverio, Pastur, and Shcherbina [1, Theorem 1, formula (1.34)] who claim that \( \beta_1 = 0 \) always in the one-cut regular case.

**Example 1.2.** We may explicitly check Theorem 1.1 using Jacobi polynomials with varying parameters \( \alpha = AN, \beta = BN, A, B > 0 \). These polynomials are orthogonal with weight \((1-x)^A(1+x)^B\) on \([-1, 1]\). The equilibrium measure takes the form (1.1) with

\[
a, b = \frac{B^2 - A^2 \pm 4 \sqrt{(1+A+B)(1+A)(1+B)}}{(2+A+B)^2}
\]

and

\[
h(x) = \frac{2 + A + B}{2\pi(1-x^2)}.
\]

see [16 [15]. We are in the one-cut regular case, but for weights restricted to \([-1, 1]\). An analysis of the proof of Theorem 1.1, however, will show that the results (1.4)-(1.5) remain valid in this case as well.
From the explicit form of the recurrence coefficients for Jacobi polynomials, see e.g. [4, 15],

\[ a_{n,n} = \frac{4(1 + A + B)(1 + A)(1 + B)}{(2 + A + B)^2 - \frac{1}{n})(2 + A + B)^2} \]
\[ b_{n,n} = \frac{B^2 - A^2}{(2 + A + B)(2 + A + B + \frac{2}{n})} \]

it is easy to see that (1.4) holds. Using (1.6)-(1.7) we can also ascertain the validity of (1.5).

2 The Riemann-Hilbert Problem

The Riemann-Hilbert problem for orthogonal polynomials was introduced by Fokas, Its, and Kitaev [12]. It asks for a \(2 \times 2\) matrix valued function \(Y(z)\) satisfying

\[
\begin{cases}
    Y(z) \text{ is analytic in } \mathbb{C} \setminus \mathbb{R} \\
    Y_+(x) = Y_-(x) \begin{pmatrix} 1 & e^{-NV(x)} \\ 0 & 1 \end{pmatrix} \text{ for } x \in \mathbb{R} \\
    Y(z) = (I + \mathcal{O}(\frac{1}{z})) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix} \text{ as } z \to \infty.
\end{cases}
\] (2.1)

The unique solution of (2.1) is (see e.g. [5])

\[
Y(z) = \left( \begin{array}{cc}
\kappa_{n,N}^{-1} p_{n,N}(z) & \frac{1}{2\pi i \kappa_{n,N}} \int_{\mathbb{R}} \frac{p_{n,N}(t)}{t-z} dt \\
-2\pi i \kappa_{n-1,N} p_{n-1,N}(z) & -\kappa_{n-1,N} \int_{\mathbb{R}} \frac{p_{n-1,N}(t)}{t-z} dt
\end{array} \right) (2.2)
\]

where \(p_{n,N}(x) = \kappa_{n,N} x^n N(x)\) is the \(n\)th degree orthonormal polynomial. The recurrence coefficients are expressed as follows in terms of the solution of the Riemann-Hilbert problem (2.1), see [5, 10].

**Proposition 2.1.** Let

\[
Y(z) = \left( I + \frac{1}{z} Y_1 + \frac{1}{z^2} Y_2 + \mathcal{O} \left( \frac{1}{z^3} \right) \right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix} (2.3)
\]

Then

\[
a_{n,N} = (Y_1)_{12} (Y_1)_{21} (2.4)
\]
and
\[ b_{n,N} = \frac{(Y_2)_{12}}{(Y_1)_{12}} - (Y_1)_{22} \quad (2.5) \]

For the remainder of this paper we will take \( N = n \). We closely follow [5, 7] in applying the Deift-Zhou method of steepest descent for Riemann-Hilbert problems to (2.1).

### 3 The Deift-Zhou method of steepest descent

The goal of the Deift-Zhou method of steepest descent for Riemann-Hilbert problems is to change the original problem into a problem for which the asymptotics for \( z \to \infty \) are normalised and for which all matrices, jump matrices and solutions alike, are asymptotically close to the identity matrix for large \( n \) which can be solved iteratively. The specific details and steps needed to achieve this goal shall be explained below.

#### 3.1 The First Step: Transformation \( Y \mapsto T \)

The key aspect of the first step of the analysis is the equilibrium measure \( \mu_V \) corresponding to \( V \). This equilibrium measure \( \mu_V \) is the unique probability measure that satisfies for some \( l \),

\[ 2 \int \log |x - y|^{-1} \, d\mu_V(y) + V(x) \geq l, \quad \text{for all } x \in \mathbb{R}, \quad (3.1) \]

\[ 2 \int \log |x - y|^{-1} \, d\mu_V(y) + V(x) = l, \quad \text{for all } x \in \text{supp } \mu_V. \quad (3.2) \]

For the one-cut regular case that we are considering we have that the support is one interval \([a, b]\) and \( d\mu_V(x) = \psi_V(x) \, dx \) as in (1.1). In addition the inequality (3.1) is strict for \( x \in \mathbb{R} \setminus [a, b] \).

Define
\[ g(z) = \int \log(z - s) \, d\mu_V(s) = \int \log(z - s) \psi_V(s) \, ds \quad (3.3) \]

and
\[ \phi(z) = \pi \int_b^z ((s - b)(s - a))^{\frac{1}{2}} h(s) \, ds, \quad z \in \mathbb{C} \setminus (-\infty, b] \quad (3.4) \]

\[ \tilde{\phi}(z) = \pi \int_a^z ((s - b)(s - a))^{\frac{1}{2}} h(s) \, ds, \quad z \in \mathbb{C} \setminus [a, +\infty). \quad (3.5) \]
If we now put
\[ T(z) = e^{-n(l/2)}\sigma_3 Y(z)e^{-ng(z)}\sigma_3 e^{n(l/2)}\sigma_3, \] (3.6)
where \( \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) is the third Pauli matrix, then \( T \) satisfies the Riemann-Hilbert problem
\[
\begin{align*}
T(z) & \text{ is analytic in } \mathbb{C} \setminus \mathbb{R}, \\
T_+(x) &= T_-(x)J_T(x) \text{ for } x \in \mathbb{R}, \\
T(z) &= I + O\left(\frac{1}{z}\right) \text{ as } z \to \infty,
\end{align*}
\] (3.7)
where
\[
J_T(x) = \begin{cases}
\begin{pmatrix} 1 & e^{-2n\tilde{\phi}(x)} \\ 0 & 1 \end{pmatrix} & \text{for } x < a, \\
\begin{pmatrix} e^{2n\phi_+(x)} & 0 \\ 0 & e^{2n\phi_-(x)} \end{pmatrix} & \text{for } x \in (a, b), \\
\begin{pmatrix} 1 & e^{-2n\phi(x)} \\ 0 & 1 \end{pmatrix} & \text{for } x > b.
\end{cases}
\] (3.8)

Since the inequality in (3.1) is strict for \( x < a \) and \( x > b \) we have that \( \tilde{\phi}(x) > 0 \) for \( x < a \) and \( \phi(x) > 0 \) for \( x > b \). Thus the jump matrices for \( T \) on \( (-\infty, a) \) and \( (b, \infty) \) tend to the identity matrix as \( n \to \infty \).

### 3.2 The Second Step: Transformation \( T \mapsto S \)

The second transformation is the so-called *opening of the lens* and it is based on the factorisation
\[
\begin{pmatrix} e^{2n\phi_+(x)} & 0 \\ 0 & e^{2n\phi_-(x)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \] (3.9)
of the jump matrix \( J_T \) on \( (a, b) \). The factorisation (3.9) allows us to split the jump on \( (a, b) \) as shown in Figure 1.

We use \( \Sigma_1 \) and \( \Sigma_2 \) to denote the upper and lower lips of the lens, respectively. We define \( S \) as follows:

- For \( z \) outside the lens, we put \( S = T \).
- For \( z \) within the region enclosed by \( \Sigma_1 \) and \( (a, b) \),
\[ S = T \begin{pmatrix} 1 & 0 \\ e^{-2n\phi} & 1 \end{pmatrix}. \] (3.10)
For $z$ within the region enclosed by $\Sigma_1$ and $(a,b)$,

$$S = T \begin{pmatrix} 1 & 0 \\ e^{2n\phi} & 1 \end{pmatrix}. \quad (3.11)$$

Then $S$ satisfies the following Riemann-Hilbert problem:

$$\begin{cases} 
S(z) \text{ is analytic in } \mathbb{C} \setminus (\mathbb{R} \cup \Sigma_1 \cup \Sigma_2) \\
S_+(z) = S_-(z)J_S(z) \text{ for } z \in \mathbb{R} \cup \Sigma_1 \cup \Sigma_2 \\
S(z) = I + O \left( \frac{1}{z} \right) \text{ for } z \to \infty
\end{cases} \quad (3.12)$$

where

$$J_S(z) = \begin{cases} 
\begin{pmatrix} 1 & 0 \\ e^{2n\phi(z)} & 1 \end{pmatrix} & \text{ for } z \in \Sigma_1 \cup \Sigma_2, \\
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{ for } z \in (a, b), \\
\begin{pmatrix} 1 & e^{-2n\phi(z)} \\ 0 & 1 \end{pmatrix} & \text{ for } z < a, \\
\begin{pmatrix} 1 & e^{-2n\phi(z)} \\ 0 & 1 \end{pmatrix} & \text{ for } z > b.
\end{cases} \quad (3.13)$$

We may (and do) assume that the lips of the lens are in the region where $\text{Re } \phi < 0$, so that the jump matrices for $S$ on $\Sigma_1$ and $\Sigma_2$ tend to the identity matrix as $n \to \infty$.

\begin{figure}[h]
\centering
\begin{tikzpicture}[scale=1.5]
\draw[thick] (0,0) -- (3,0);
\draw[thick] (0,0) -- (0,2);
\draw[thick] (3,0) -- (3,2);
\draw[thick] (0,0) arc (180:0:1);
\node at (1.5,0) {$a$};
\node at (0,1.5) {$b$};
\node at (0,0) {$\begin{pmatrix} 1 & e^{-2n\phi} \\ 0 & 1 \end{pmatrix}$};
\node at (0,1) {$\begin{pmatrix} 1 & 0 \\ e^{2n\phi} & 1 \end{pmatrix}$};
\node at (1,0) {$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$};
\node at (1,1) {$\begin{pmatrix} 1 & e^{-2n\phi} \\ 0 & 1 \end{pmatrix}$};
\node at (2,0) {$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$};
\end{tikzpicture}
\caption{Jump matrices for $S$ after opening of the lens}
\end{figure}
3.3 The Third Step: Parametrix Away From Endpoints

The parametrix away from the branch points is a 'global solution' $N(z)$ satisfying the Riemann-Hilbert problem
\[
\begin{align*}
N(z) & \text{ is analytic in } \mathbb{C} \setminus [a, b] \\
N_+(x) &= N_-(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ for } x \in (a, b) \\
N(z) &= I + \mathcal{O} \left( \frac{1}{z} \right) \text{ for } z \to \infty
\end{align*}
\] (3.14)
which has solution (see [5])
\[
N(z) = \begin{pmatrix} \beta(z) + \beta^{-1}(z) & \beta(z) - \beta^{-1}(z) \\ \beta(z) - \beta^{-1}(z) & \beta(z) + \beta^{-1}(z) \end{pmatrix}^{2i} \] (3.15)
where $\beta(z) = \left( \frac{z-b}{z-a} \right)^{\frac{1}{2}}$.

3.4 The Fourth Step: Parametrices Near Endpoints

Having constructed the 'global solution', the next step is finding 'local solutions' close to the endpoints $a$ and $b$. Near $b$, the local situation is described as in the left picture of Figure 2 with jump matrix
\[
J_P(z) = J_S(z) = \begin{cases} 
\begin{pmatrix} 1 & 0 \\ e^{2n\phi(z)} & 1 \end{pmatrix} & \text{on } \Sigma_1 \cap U \text{ and } \Sigma_2 \cap U \\
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{on } (a, b) \cap U \\
\begin{pmatrix} 1 & e^{-2n\phi(z)} \\ 0 & 1 \end{pmatrix} & \text{on } (b, \infty) \cap U
\end{cases}
\]
where $U$ is a (small) disk around $b$.

We therefore want to find a matrix function $P$, that solves
\[
\begin{align*}
P(z) & \text{ is analytic on } U \setminus (\Sigma_1 \cup \Sigma_2 \cup \mathbb{R}) \\
P_+(z) &= P_-(z)J_P(z) \text{ on } (\Sigma_1 \cup \Sigma_2 \cup \mathbb{R}) \cap U \\
P(z) &= N(z) (I + \mathcal{O} \left( \frac{1}{n} \right)) \text{ as } n \to \infty \text{ uniformly for } z \in \partial U
\end{align*}
\]
Then $P(z)e^{n\phi(z)\sigma_3}$ should have constant jumps on $(\Sigma_1 \cup \Sigma_2 \cup \mathbb{R}) \cap U$, namely
\[
\begin{pmatrix} P(z)e^{n\phi(z)\sigma_3} \end{pmatrix}_+ = \begin{pmatrix} P(z)e^{n\phi(z)\sigma_3} \end{pmatrix}_- \times \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ for } z \in (\Sigma_1 \cup \Sigma_2) \cap U \\
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ for } z \in (a, b) \cap U \\
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ for } z \in (b, \infty) \cap U
\end{pmatrix}
\]
Figure 2: Mapping of neighbourhood of \( b \) onto a neighbourhood of \( f(b) = 0 \)

Shrinking \( U \) if necessary, we have that

\[
\zeta = f(z) = \left( \frac{3}{2} \phi(z) \right)^{2/3}
\]

defines a conformal map from \( U \) to a convex neighborhood of \( \zeta = 0 \). We may and do assume that the lips of the lens are taken so that \( \Sigma_1 \cap U \) is mapped into \( \arg \zeta = 2\pi/3 \), and \( \Sigma_2 \cap U \) is mapped into \( \arg \zeta = 2\pi/3 \), see Figure 2.

Denoting the sectors in the \( \zeta \)-plane by I, II, III, IV as in Figure 2, and using the usual Airy function \( \text{Ai}(\zeta) \), we construct the Airy model solution \( \Phi \) by

\[
\Phi(\zeta) = \begin{cases} 
\begin{pmatrix} \text{Ai}(\zeta) & \omega \text{Ai}(\omega \zeta) \\ \text{Ai}'(\zeta) & \omega^2 \text{Ai}'(\omega \zeta) \end{pmatrix} & \text{for } \zeta \text{ in sector IV} \\
\begin{pmatrix} \text{Ai}(\zeta) & -\omega \text{Ai}(\omega^2 \zeta) \\ \text{Ai}'(\zeta) & -\omega \text{Ai}'(\omega^2 \zeta) \end{pmatrix} & \text{for } \zeta \text{ in sector I} \\
\begin{pmatrix} -\omega \text{Ai}(\omega \zeta) & -\omega^2 \text{Ai}(\omega^2 \zeta) \\ -\omega^2 \text{Ai}'(\omega \zeta) & -\omega \text{Ai}'(\omega^2 \zeta) \end{pmatrix} & \text{for } \zeta \text{ in sector II} \\
\begin{pmatrix} -\omega^2 \text{Ai}(\omega^2 \zeta) & \omega \text{Ai}(\omega \zeta) \\ -\omega \text{Ai}'(\omega^2 \zeta) & \omega^2 \text{Ai}'(\omega \zeta) \end{pmatrix} & \text{for } \zeta \text{ in sector III}
\end{cases}
\]

which has the jump matrices in the \( \zeta \)-plane indicated in the right side of Figure 2.

Then for any analytic prefactor \( E_n(z) \) we have that

\[
P(z) = E_n(z) \Phi(n^{2/3} f(z)) e^{n\phi(z)} \sigma_3
\]

has the required jump matrices \( J_P \). If we choose

\[
E_n = \sqrt{\pi} N(z) \left( \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} \right) \left( n^{2/3} f(z) \right)^{\sigma_3/4}
\]

then the matching condition \( P(z) = N(z)(1 + O(1/n)) \) as \( n \to \infty \) for \( z \in \partial U \), is satisfied as well, see e.g. \([3, 5, 7]\) for further detail.
A similar construction yields a parametrix $\hat{P}$ in a small disc $\hat{U}$ around $a$. One can see that $\hat{P}$ can be obtained by taking $P$ and interchanging $a$ and $b$ and conjugating with $\sigma_3$.

### 3.5 The Fifth Step: Transformation $S \mapsto R$

Using the parametrices $N$, $P$, and $\hat{P}$, we define the third transformation $S \mapsto R$ as follows

$$R(z) = \begin{cases} 
S(z)N(z)^{-1} & \text{for } z \in \mathbb{C} \setminus (U \cup \hat{U}) \\
S(z)P(z)^{-1} & \text{for } z \in U \\
S(z)\hat{P}(z)^{-1} & \text{for } z \in \hat{U}
\end{cases}$$

Then $R$ has no jump on $[a, b] \setminus (U \cup \hat{U})$, as the jumps of $S$ and $N^{-1}$ cancel out. In $U$ and $\hat{U}$ the jumps of $S$ cancel out with the jumps of $P$ and $\hat{P}$, leaving only jumps for $R$ on the contour $\Sigma_R$ shown in Figure 3.

![Figure 3: Contour $\Sigma_R$ for the Riemann-Hilbert problem for $R$](image)

The Riemann-Hilbert problem for $R$ is

$$\begin{cases} 
R(z) \text{ is analytic on } \mathbb{C} \setminus \Sigma_R \\
R_+(z) = R_-(z)J_R(z) & \text{for } z \in \Sigma_R \\
R(z) = I + O\left(\frac{1}{z}\right) & \text{for } z \to \infty
\end{cases}$$

where

$$J_R(z) = \begin{cases} 
N(z)J_S(z)N(z)^{-1} & \text{for } z \in \Sigma_R \setminus (\partial U \cup \partial \hat{U}) \\
P(z)N(z)^{-1} & \text{for } z \in \partial U \\
\hat{P}(z)N(z)^{-1} & \text{for } z \in \partial \hat{U}
\end{cases}$$

The jump matrices $J_R(z) = N(z)J_S(z)N(z)^{-1}$ tend to the identity matrix at an exponential rate as $n \to \infty$. The jump matrices on $\partial U$ and $\partial \hat{U}$ tend to the identity matrix but at a slower rate of $1/n$ as $n \to \infty$. The
precise form is obtained from the asymptotic expansion of the Airy function
as $z \to \infty$, $-\pi < \arg z < \pi$, (see \[13\])

$$\text{Ai}(z) \sim \frac{e^{-\frac{2}{3}z^{\frac{3}{2}}}}{2\sqrt{\pi}z^{\frac{1}{4}}} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma\left(3k + \frac{1}{2}\right)}{9^k (2k)! \Gamma\left(\frac{1}{2}\right)} \frac{1}{z^{\frac{3}{2}k}}$$ (3.19)

and the corresponding asymptotic expansion for $\text{Ai}'(z)$. Using these facts
in the parametrix $P$ we find an asymptotic expansion for the jump of $R$ on $\partial U$

$$J_R(z) = P(z) N(z)^{-1} \sim I + \sum_{k=1}^{\infty} \frac{1}{n^k} \Delta_k(z)$$ (3.20)

where

$$\Delta_k(z) = \frac{1}{\sqrt{\pi}} \left( \frac{\Gamma\left(3k + \frac{1}{2}\right)}{9^k (2k)!} - \frac{\Gamma\left(3k - \frac{3}{2}\right)}{4 \cdot 9^{k-1} (2(k-1))!} \right) \frac{1}{\left(\frac{3}{2} \phi(z)\right)^k}$$

$$- \frac{1}{4\sqrt{\pi}} \frac{\Gamma\left(3k - \frac{3}{2}\right)}{9^{k-1} (2(k-1))! \left(\frac{3}{2} \phi(z)\right)^k \sigma_2} \text{ for } k \text{ even}$$ (3.21)

and

$$\Delta_k(z) = -\frac{\beta(z)^2}{\left(\frac{3}{2} \phi(z)\right)^k} \frac{1}{2\sqrt{\pi}} \left( \frac{\Gamma\left(3k + \frac{1}{2}\right)}{9^k (2k)!} - \frac{\Gamma\left(3k - \frac{3}{2}\right)}{2 \cdot 9^{k-1} (2(k-1))!} \right) (\sigma_3 + i \sigma_1)$$

$$- \frac{\beta(z)^{-2}}{\left(\frac{3}{2} \phi(z)\right)^k} \frac{1}{2\sqrt{\pi}} \frac{\Gamma\left(3k + \frac{1}{2}\right)}{9^k (2k)!} (\sigma_3 - i \sigma_1) \text{ for } k \text{ odd}$$ (3.22)

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ (3.23)

are the Pauli matrices.

A similar expansion

$$J_R(z) = \tilde{P}(z) N(z)^{-1} \sim I + \sum_{k=1}^{\infty} \frac{1}{n^k} \tilde{\Delta}_k(z)$$ (3.24)

holds for the jump matrix on $\partial \tilde{U}$.

As a result we find by the methods of [7], see also [14, Lemma 8.3],
Lemma 3.1. There exist matrix valued functions $R_k(z)$ with the property that for every $l \in \mathbb{N}$, there exist constants $C > 0$ and $r > 0$ such that for every $z$ with $|z| \geq r$,

$$
\left\| R(z) - I - \sum_{k=1}^{l} \frac{R_k(z)}{n^k} \right\| \leq \frac{C}{|z|^{l+1}}
$$

(3.25)

So we write

$$
R(z) \sim I + \sum_{k=1}^{\infty} \frac{1}{n^k} R_k(z)
$$

(3.26)

From (3.26), (3.20) and (3.24) and the Riemann-Hilbert problem for $R$, we find an additive Riemann-Hilbert problem for $R_k(z)$,

$$
\begin{cases}
R_k(z) \text{ is analytic on } \mathbb{C} \setminus (\partial U \cup \partial \tilde{U}) \\
R_{k+}(z) = R_{k-}(z) + \sum_{l=0}^{k-1} R_{l-}(z) \Delta_{k-l}(z) \text{ for } z \in \partial U \\
R_{k+}(z) = R_{k-}(z) + \sum_{l=0}^{k-1} R_{l-}(z) \tilde{\Delta}_{k-l}(z) \text{ for } z \in \partial \tilde{U} \\
R_k(z) = O \left( \frac{1}{z} \right) \text{ as } z \to \infty
\end{cases}
$$

(3.27)

where $R_0(z) = I$. These Riemann-Hilbert problems can be successively solved using the Sokhotskii-Plemelj formula, or using a technique based on Laurent series expansion as in [14].

4 Proof of Theorem 1.1

For the proof of (1.4) we do not need to compute the explicit forms of the $R_k$’s. However, we need to know that they have the following structure. Recall that the Pauli matrices are given in (3.23).

Lemma 4.1. For $k$ odd, $R_k(z)$ is a linear combination of $\sigma_1$ and $\sigma_3$ and for $k$ even, $R_k(z)$ is a linear combination of $I$ and $\sigma_2$.

Proof. For $k = 1$, we know because of (3.27) that $R_{1+} = R_{1-} + \Delta_1$ on $\partial U$ and $R_{1+} = R_{1-} + \tilde{\Delta}_1$ on $\partial \tilde{U}$. As $\Delta_1$, $\Delta_1 \in \text{span}\{\sigma_1, \sigma_3\}$ on account of (3.22), $R_1(z)$ must be a linear combination of $\sigma_1$ and $\sigma_3$ as well.

Let $k \geq 1$ and once more observe (3.27). If $k$ is odd, then again by (3.22) $\Delta_k$, $\tilde{\Delta}_k \in \text{span}\{\sigma_1, \sigma_3\}$ and using induction on $k$, for every $l < k$, $R_{l-}(z) \Delta_{k-l}(z)$ and $R_{l-}(z) \tilde{\Delta}_{k-l}(z)$ are products of a linear combination of
\( \sigma_1 \) and \( \sigma_3 \) and a linear combination of \( I \) and \( \sigma_2 \) (see also (3.21)–(3.22)), which results in a linear combination of \( \sigma_1 \) and \( \sigma_3 \). Thus all terms in the (additive) jump for \( R_k \) on \( \partial U \) and on \( \partial \tilde{U} \) are in the span of \( \sigma_1 \) and \( \sigma_3 \), and it follows that \( R_k \in \text{span} \{ \sigma_1, \sigma_3 \} \) if \( k \) is odd.

If \( k \) is even, then by induction, where we use again ((3.21)–(3.22)), we have that \( R_{l-}(z) \Delta_{k-l}(z) \) and \( R_{l-}(z) \tilde{\Delta}_{k-l}(z) \) are either products of two linear combinations of \( I \) and \( \sigma_2 \) (in case \( l \) is even), or products of two linear combinations of \( \sigma_1 \) and \( \sigma_3 \) (in case \( l \) is odd). In both cases we find that \( R_{l-}(z) \Delta_{k-l}(z) \) and \( R_{l-}(z) \tilde{\Delta}_{k-l}(z) \) are linear combinations of \( I \) and \( \sigma_2 \), which implies that \( R_k \in \text{span} \{ I, \sigma_2 \} \) if \( k \) is even.

Now we can finally prove our main result.

**Proof of Theorem 1.1.** We start from the expressions (2.4) and (2.5) for \( a_{n,n} \) and \( b_{n,n} \) in terms of the solution of the Riemann-Hilbert problem for \( Y \). Following the transformations \( Y \mapsto T \mapsto S \), we find that

\[
a_{n,n} = (S_1)_{12} (S_1)_{21} \tag{4.1}
\]

and

\[
b_{n,n} = \frac{(S_2)_{12}}{(S_1)_{12}} - (S_1)_{22} \tag{4.2}
\]

where \( S_1 \) and \( S_2 \) are the terms in the expansion of \( S(z) \) as \( z \to \infty \),

\[
S(z) = I + \frac{1}{z} S_1 + \frac{1}{z^2} S_2 + O \left( \frac{1}{z^3} \right).
\]

To obtain (4.2) we use that \( g(z) = \log z + O(1/z) \), see also [10].

By (3.18), we know that \( S(z) = R(z) N(z) \) for \( |z| \) large enough, so we need the first terms in the expansions of \( N(z) \) and \( R(z) \) as \( z \to \infty \). From (3.15) we have

\[
N(z) = \frac{\beta(z) + \beta(z)^{-1}}{2} I + \frac{\beta(z) - \beta(z)^{-1}}{2 \sigma_2}
\]

\[
= I - \frac{(b - a)}{4} \sigma_2 \frac{1}{z} + \left( \frac{(b - a)^2}{32} I - \frac{b^2 - a^2}{8} \sigma_2 \right) \frac{1}{z^2} + O \left( \frac{1}{z^3} \right) \tag{4.3}
\]
and from Lemma 4.1

\[
R(z) = I + \frac{1}{z}\left(\sum_{m \text{ odd}} \frac{1}{n^m} (R_{m1\sigma_1} + R_{m1\sigma_3}\sigma_3) + \sum_{m \text{ even}} \frac{1}{n^m} (R_{m1I} + R_{m1\sigma_2}\sigma_2)\right) \\
+ \frac{1}{z^2}\left(\sum_{m \text{ odd}} \frac{1}{n^m} (R_{m2\sigma_1} + R_{m2\sigma_3}\sigma_3) + \sum_{m \text{ even}} \frac{1}{n^m} (R_{m2I} + R_{m2\sigma_2}\sigma_2)\right) \\
+ O\left(\frac{1}{z^3}\right)
\]  

(4.4)

where the constants \(R_{mjI}, R_{mj\sigma_k}\), for \(m \in \mathbb{N}\), \(j = 1, 2\), and \(k = 1, 2, 3\) are such that \(R_{mjI} + \sum_{k=1}^{3} R_{mj\sigma_k}\sigma_k\) is the coefficient of \(z^{-j}\) in the Laurent expansion of \(R_m(z)\) around \(z = \infty\).

Therefore, by (4.3) and (4.4),

\[
S(z) = R(z)N(z) \sim I + \frac{1}{z}\left(\frac{(b-a)}{4}\sigma_2 + \sum_{m \text{ odd}} \frac{1}{n^m} (R_{m1\sigma_1} + R_{m1\sigma_3}\sigma_3) \\
+ \sum_{m \text{ even}} \frac{1}{n^m} (R_{m1I} + R_{m1\sigma_2}\sigma_2)\right) \\
+ \frac{1}{z^2}\left(\frac{(b-a)^2}{32}I - \frac{b^2 - a^2}{8}\sigma_2 + \sum_{m \text{ odd}} \frac{1}{n^m} \left(\left(R_{m2\sigma_1} + \frac{b-a}{4} R_{m1\sigma_3}\right)\sigma_1 \\
+ \left(R_{m2\sigma_3} - \frac{b-a}{4} R_{m1\sigma_1}\right)\sigma_3\right) + \sum_{m \text{ even}} \frac{1}{n^m} \left(\left(R_{m2I} - \frac{b-a}{4} R_{m1\sigma_2}\right)I \\
+ \left(R_{m2\sigma_2} - \frac{b-a}{4} R_{m1I}\right)\sigma_2\right)\right) + O\left(\frac{1}{z^3}\right)
\]  

(4.5)

which implies that

\[
(S_1)_{12} \sim \frac{b-a}{4}i + \sum_{m \text{ odd}} \frac{1}{n^m} R_{m1\sigma_1} - i \sum_{m \text{ even}} \frac{1}{n^m} R_{m1\sigma_2} 
\]  

(4.6)

and

\[
(S_1)_{21} \sim -\frac{b-a}{4}i + \sum_{m \text{ odd}} \frac{1}{n^m} R_{m1\sigma_1} + i \sum_{m \text{ even}} \frac{1}{n^m} R_{m1\sigma_2} 
\]  

(4.7)

Inserting (4.6) and (4.7) into (1.1) then finally gives

\[
a_{n,n} \sim \frac{(b-a)^2}{16} + \sum_{m=1}^{\infty} \frac{a_{2m}}{n^{2m}}
\]
for certain constants $\alpha_{2m}$.

Similar to (4.6) and (4.7) we have that $(S_2)_{12}$ and $(S_1)_{22}$ have asymptotic expansions in powers of $1/n$. From the expansion (4.5) for $S$, we see

$$(S_2)_{12} \sim \frac{b^2 - a^2}{8} i + \sum_{m \text{ odd}} \frac{1}{n^m} \left( \frac{b-a}{4} iR_{m1\sigma_3} + R_{m2\sigma_1} \right)$$

and

$$(S_1)_{22} \sim - \sum_{m \text{ odd}} \frac{1}{n^m} iR_{m1\sigma_3} + \sum_{m \text{ even}} \frac{1}{n^m} R_{m1\sigma_1}$$

From (4.2) it then follows that

$$b_{n,n} \sim \sum_{m=0}^{\infty} \frac{\beta_m}{n^m}$$

where $\beta_0 = \frac{b+a}{2}$ and

$$\beta_1 = 2R_{11\sigma_3} - \frac{4}{b-a} iR_{12\sigma_1} + \frac{2(b+a)}{b-a} iR_{11\sigma_1}.$$  (4.9)

Our final task is to further evaluate the right-hand side of (4.9). As in [14], we have that $\Delta_1$ is meromorphic in a neighborhood of $b$ with a pole in $b$. Indeed, if we write

$$\frac{\beta(z)^{-2}}{\phi(z)} = (z-b)^{-2} \sum_{m=0}^{\infty} B_m (z-b)^m, \quad B_0 = \frac{3}{2\pi h(b)},$$

and use (3.22), then we find for $z$ in a neighborhood of $b$,

$$\Delta_1(z) = \left( -\frac{5B_1}{144} (\sigma_3 - i\sigma_1) + \frac{7B_0}{144(b-a)} (\sigma_3 + i\sigma_1) \right) \frac{1}{z-b} - \frac{5B_0}{144} (\sigma_3 - i\sigma_1) \frac{1}{(z-b)^2} + O(1).$$

Similarly, for $z$ in a neighborhood of $a$, we have

$$\frac{\beta(z)^2}{\phi(z)} = (z-a)^{-2} \sum_{m=0}^{\infty} A_m (z-a)^m, \quad A_0 = \frac{3}{2\pi h(a)},$$
and
\[
\tilde{\Delta}_1(z) = \left( -\frac{5A_1}{144} (\sigma_3 + i\sigma_1) - \frac{7A_0}{144(b-a)} (\sigma_3 - i\sigma_1) \right) \frac{1}{z-a} \\
- \frac{5A_0}{144} (\sigma_3 + i\sigma_1) \frac{1}{(z-a)^2} + O(1). 
\] (4.13)

As in [14] we have that \( R_1(z) \) for \( z \in \mathbb{C} \setminus \overline{U \cup U} \) is equal to the sum of the Laurent parts of (4.11) and (4.13). Expanding \( R_1(z) \) as \( z \to \infty \), we then get
\[
R_1(z) = R_{11} \frac{1}{z} + R_{12} \frac{1}{z^2} + O \left( \frac{1}{z^3} \right) \quad \text{as} \quad z \to \infty,
\]
where
\[
R_{11} = -\frac{5A_1}{144} (\sigma_3 + i\sigma_1) - \frac{7A_0}{144(b-a)} (\sigma_3 - i\sigma_1) \\
- \frac{5B_1}{144} (\sigma_3 - i\sigma_1) + \frac{7B_0}{144(b-a)} (\sigma_3 + i\sigma_1) \\
R_{12} = -\frac{5aA_1}{144} (\sigma_3 + i\sigma_1) - \frac{7aA_0}{144(b-a)} (\sigma_3 - i\sigma_1) \\
- \frac{5bB_1}{144} (\sigma_3 - i\sigma_1) + \frac{7bB_0}{144(b-a)} (\sigma_3 + i\sigma_1) \\
- \frac{5A_0}{144} (\sigma_3 + i\sigma_1) - \frac{5B_0}{144} (\sigma_3 - i\sigma_1).
\]

Thus
\[
R_{11\sigma_3} = -\frac{5(A_1 + B_1)}{144} - \frac{7(A_0 - B_0)}{144(b-a)}, 
\]
(4.14)
\[
R_{11\sigma_1} = -i\frac{5(A_1 - B_1)}{144} + i\frac{7(A_0 + B_0)}{144(b-a)}. 
\]
(4.15)
\[
R_{12\sigma_1} = -i\frac{5(aA_1 - bB_1)}{144} + i\frac{7(aA_0 + bB_0)}{144(b-a)} - i\frac{5(A_0 - B_0)}{144}. 
\]
(4.16)

Inserting (4.14)–(4.16) into (4.9), we find after straightforward calculations that \( A_1 \) and \( B_1 \) fully disappear and that (4.9) reduces to
\[
\beta_1 = \frac{B_0 - A_0}{3(b-a)}. 
\]

Using the explicit formulas for \( A_0 \) and \( B_0 \) given in (4.10) and (4.12), we arrive at (1.5), which completes the proof of Theorem 1.1.
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