A Solution for a Fundamental Problem of 3D Inference based on 2D Representations

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Abstract—3D inference from monocular vision using neural networks is an important research area of computer vision. Applications of the research area are various with many proposed solutions and have shown remarkable performance. Although many efforts have been invested, there are still unanswered questions, some of which are fundamental. In this paper, I discuss a problem that I hope will come to be known as a generalization of the Blind Perspective-n-Point (Blind PnP) problem for object-driven 3D inference based on 2D representations. The vital difference between the fundamental problem and the Blind PnP problem is that 3D inference parameters in the fundamental problem are attached directly to 3D points and the camera concept will be represented through the sharing of the parameters of these points. By providing an explainable and robust gradient-decent solution based on 2D representations for an important special case of the problem, the paper opens up a new approach for using available information-based learning methods to solve problems related to 3D object pose estimation from 2D images.

Index Terms—Machine Learning, Computer Vision, Analysis-by-Synthesis, 3D Inference, Differentiable Rendering

I. INTRODUCTION

Learning properties of 3D objects from 2D images is a typical module of 3D inference from monocular vision and is also a common task in computer vision applications. Based on the success of 2D convolutional neural networks in understanding semantic information of 2D images [1], these neural networks were quickly adapted to extract 3D information from 2D images [2], [3]. Among approaches of using 2D convolutional neural networks for extracting 3D information from 2D images, differentiable rendering approaches are diverse and have many advantages in improving performance [4], [5]. A key point of differentiable rendering techniques is a differentiable rendering pipeline that allows flowing gradients from rendered pixels to parameters of the neural network. However, the variety of graphical representations make differentiation of rendering process is not uniquely defined [4]. Inspired by regular grids of convolutional layers, as well as the desire for making contributions in this paper benefit as many differentiable rendering techniques as possible, I introduce a novel graphical representation based on the result of translating a certain graphics code [6] to a grid of pixel-level 3D information under the support of a linear perspective projection, called 2D representation of 3D information.

Instead of constructing a complete solution of extracting 3D information from 2D images, I focus on a fundamental problem of inverse rendering. Thus, I will construct a dataset that represents an important special case of the problem, propose a solution for the special case and analyze the solution on the dataset. The dataset, Sky [7] was constructed on the ApolloCar3D dataset [7] but was modified to balance practicality and complexity to make it simple enough for analysis. Because of the nature of the ApolloCar3D dataset, lights and material appearance are not taken into account. However, they are not the purpose of this paper.

In summary, my contributions are:

• Describe a fundamental problem that can be applied for differentiable rendering techniques.
• Introduce a dataset that represents an important special case of the problem.
• Propose a solution for the special case, demonstrate and analyze the performance of the solution on the dataset.

II. A FUNDAMENTAL PROBLEM

A. 2D Representation

Define a fragment \( g \) is a tuple of coordinates and rendering information of a 3D point has form \( g = (\text{coord}(g), \text{info}(g)) = ((x, y, z), r) \in \mathbb{R}^3 \times \mathbb{R}^+ \), where \( \text{coord}(\cdot) \) and \( \text{info}(\cdot) \) are, respectively, functions permit accessing the coordinates and the rendering information value of a fragment. Given a set of \( K \) fragments \( S = \{g_i\}_{i=1}^K \), a 2D representation of \( S \) on a screen has size \( H \times W \) using a linear perspective projection (without extrinsic parameters) \( P : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) is a function \( L_{S,P,\text{AGG}}^{H \times W} : [1..H] \times [1..W] \rightarrow \mathbb{R} \),

\[
L_{S,P,\text{AGG}}^{H \times W}(i,j) = \begin{cases} \text{AGG}(T_{S,P}(i,j)), & \text{if } T_{S,P}(i,j) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}
\]

where \( \text{AGG}(\cdot) \) is an aggregation function of fragments’ rendering information, and \( T_{S,P}(i,j) \) returns a set of fragments in \( S \) which have \((i,j)\) as the result of applying the projection \( P \) to their coordinates \((i,j)\) is called screen position of fragments (in the set). Precisely, \( T_{S,P}(i,j) = \{ g \in S | [P(\text{coord}(g))] = (i,j) \} \) in that the floor function \([\cdot]\) was expanded for tuples,

\[
[[x_1, x_2, \ldots, x_n]] \triangleq ([x_1], [x_2], \ldots, [x_n]).
\]

For the simplification of notations, the matrix notation \( L_{ij} \) usually used to substitute \( L_{S,P,\text{AGG}}^{H \times W}(i,j) \).

Proper 2D representation. Considering a function \( f : \mathbb{R}^+ \rightarrow \mathbb{R}^3 \) that decodes coordinates of a 3D point from a value of rendering information. We say a 2D representation \( L \) is proper under the function \( f \) if \((i,j) = [P(f(L_{ij}))] \), for every \((i,j)\) satisfies \( L_{ij} \neq 0 \).

1https://github.com/thienannguyen-cv/Sky-dataset
B. An aggregation function

Consider a set of $N$ fragments $S = \{(x_i, y_i, z_i, r_i)\}_{i=1}^N$. Let $z_{\text{min}} = \min\{z_i | z_i > 0\}_{i=1}^N$ is the minimum depth value of visible points of $S$, define $\text{rmin}(\cdot)$ aggregation function:

$$\text{rmin}(S) \triangleq \min\{|r| | (x, y, z, r) \in S_{z_{\text{min}}}\},$$

where $S_{z_{\text{min}}} = \{(x, y, z, r) \in S | z = z_{\text{min}}\}$. Thus, $\text{rmin}(S)$ returns the minimum render information value of closest-to-camera visible points of $S$. A convention that $\text{rmin}(\emptyset) = 0$.

C. Problem Description

Given a set of $N$ 3D points $Q = \{x_i \in \mathbb{R}^3\}_{i=1}^N$, a screen has size $H \times W$ and a linear perspective projection (without extrinsic parameters) $P : \mathbb{R}^3 \rightarrow \mathbb{R}^2$, $Q' = \{(x'_i = (x_i^1 + a, x_i^2 + b, x_i^3 + c) | x_i \in Q\}_{i=1}^N$ is a set of $N$ 3D points generated by adding unknown amounts $a$, $b$, and $c$ to coordinates of each point in $Q$. Let $F' = \{(x'_i^{(1)}, x'_i^{(2)}, x'_i^{(3)}) | x'_i \in Q'\}_{i=1}^N$ is a set of fragments constructed on $Q'$. Let $Q$, $P$, and the value of the 2D representation $I_{F', P, r_{\text{min}}}$ is known, find the values of $a$, $b$, and $c$.

III. SKY DATASET

The dataset is constructed on the ApolloCar3D dataset, a dataset of urban street views with car-object 3D pose annotations. Fig. [1] is a sample in the Sky dataset. While the ApolloCar3D dataset contains 5,277 driving images of $3384 \times 2710$ resolution and over 60K car instances of 79 car models [7], the Sky dataset is built only upon 3D annotations of their 4,248 images which are chosen randomly. With $K$ is the projection was used in the ApolloCar3D dataset, construct configurations of the problem (names of variables are still remain) for each image in 4,248 chosen images:

- Let $O = \{c_i = (x_i, y_i, z_i)\}_{i=1}^M$ is the set of locations of car objects in the 3D annotation of the image, $G = \{((x_i, y_i, z_i) \in O, z_i)\}_{i=1}^M$ is a set of fragments constructed on $O$.
- Let $P(x) = K(x) \odot \frac{[256 \ 256 \ 3384]}{2710} - [128 \ 0]$ is a projection constructed on the projection $K$. Let $E_{G, P, r_{\text{min}}}$ is a 2D representation of $G$ on a screen has size $128 \times 256$.
- A 2D representation $E_{128 \times 256}$ is constructed on $E$ by drawing uniform circles (described in the Appendix) at non-zero points of active points in $E$. Thus, $Q$ is the set of $N$ 3D points are reconstructed from $E'$ so that projection of a 3D point in $Q$ is also screen position of an active point on $E'$. Also, a point in $E'$ corresponds to only one point in $Q$, ignored obscured points.
- According to the fundamental problem, I construct a 2D representation $E_{128 \times 256}$ from $Q$, $P$ is called target 2D representation of the configuration. With $a = 0.517$, $b = 0.303$ and $c = 0$, $F' = \{(x'_i^{(1)}, x'_i^{(2)}, x'_i^{(3)}) | x'_i \in Q'\}_{i=1}^N$ is a set of $N$ fragments created from $Q'$, where $Q' = \{x'_i = (x_i^1 + a, x_i^2 + b, x_i^3 + c) | x_i \in Q\}_{i=1}^N$.

In summary, the dataset is 4,248 configurations of a special case of the fundamental problem where $c$ is ignored, the projection $P$, the unknowns $a$ and $b$ is unique.

IV. A GRADIENT DESCENT SOLUTION

Consider the special case of the problem with its 4,248 instances in the Sky dataset. According to the description of the dataset, we have a rendering algorithm $R$ that takes a set of $N$ 3D points of a configuration, $Q$, and returns an image $A$ that is also a 2D representation $E_{128 \times 256}$ is an image in the Sky dataset. And, $F = \{(x_i = (x_i^{(1)}, x_i^{(2)}, x_i^{(3)}), r)x_i \in Q, r_{\text{inv}}(r_i) = (x_i^{(1)}, x_i^{(2)}, x_i^{(3)})\}_{i=1}^N$, where $r_{\text{inv}}$ is a differentiable function. Via the function $r_{\text{inv}}$, we imply that spatial information in values of a point in $E'$ is linked to its screen position on $E'$. In other words, $A$ is a proper 2D representation under the function $r_{\text{inv}}$ (proper 2D representation for short). Let $\theta^* = (a, b)$ is the ground truth parameter and $\theta = (\bar{a}, \bar{b})$ is its estimation, $\bar{I}$ is an image constructed on $A$,

$$\bar{I}_{ij} = \begin{cases} \text{add}(A_{ij}, \tilde{\theta}_{3D}), & \text{if } A_{ij} \neq 0, \\ 0, & \text{otherwise}, \end{cases}$$

where $\tilde{\theta}_{3D} = (\bar{a}, \bar{b}, 0)$ is represented $\theta$ in 3D space and the differentiable function $\text{add}(\cdot)$ adds $\theta_{3D}$ to the coordinates of 3D points encoded in the rendering information $A_{ij}$,

$$r_{\text{inv}}(\tilde{r} = \text{add}(r, \theta_{3D})) \triangleq r_{\text{inv}}(r) + (\bar{a}, \bar{b}, 0).$$
As a part of the solution, I propose a differentiable mapping algorithm which permits the gradient propagates to \( \theta \), called \textit{Spatial Domain Mapping} (SDM), for generating a \textit{proper} 2D representation \( L \) from \( \tilde{I} \). After the forward pass of SDM, spatial information of a point in \( \tilde{I} \) is linked to its screen position (called \textit{proper screen position}, at this point). So, we have

\[
\tilde{I}_{\text{p}, \text{p}, \text{rmin}}^{128 \times 256}(i, j) = \text{rmin}(q(\tilde{I}, (i, j))),
\]

where \( q(\tilde{I}, (i, j)) \) returns a set of \textit{fragments} constructed from 3D points encoded in \( \tilde{I} \) which have \((i, j)\) as the result of applying the projection \( P \) on them,

\[
q(\tilde{I}, (i, j)) = \{(\text{rinv}(\tilde{I}_{kl}), \tilde{I}_{kl})|(i, j) = [P(\text{rinv}(\tilde{I}_{kl}))]_{k,l}.
\]

Note that, for a configuration in the Sky dataset, the value \( D_{22} \) is linked to its corresponding screen position instead of “bubbling” a point at \( \text{rinv}(\tilde{I}_{ij}) \). As a part of the solution, I propose a differentiable mapping \( I \) to its \textit{XY-adjustments} (other circles). In this case, \( D_{22} \) is the maximum element of \( D_{22} \), (d) the output of the algorithm.

\[ \text{\text{otherwise}, } \tilde{I}_{ij} = 0, \text{ I set a safe value for } \text{XY-adjustments, } \tilde{T}_{ijkl} = (0, 0, 0). \text{ Precisely, } D_{ij} \text{ is the result of applying a soft-max-ish function to distances from the 3D point represented by } \tilde{I}_{ij} \text{ to its \textit{XY-adjustments}, } \{\tilde{T}_{ijkl}\}_{k,l}.
\]

1) \text{Forward pass}: The forward pass of the algorithm generates the \textit{proper} 2D representation \( L \) from the parameter \( \theta \) in a differentiable way. This is illustrated in Fig. 2. Thus, for each point \( \tilde{I}_{ij} \), define a kernel \( D_{ij} \) has size \( h \times w \) (\( h, w \) are odds) to indicate the offset of its \textit{proper screen position} relative to its current screen position by a real value in the interval \([0, 1]\). Let \((c_h \triangleq \lfloor h/2 \rfloor, c_w \triangleq \lfloor w/2 \rfloor)\) is the position of the center element of the kernel. If \( \tilde{I}_{ij} \neq 0 \), each point \( D_{ijkl} \) in the kernel corresponds to a 3D point \( T_{ijkl} = (x_{ijkl}, y_{ijkl}, z_{ijkl}) \) has the same depth value as \( \text{rinv}(\tilde{I}_{ij}) \) and its screen position via applying the linear perspective projection \( P \) is \((i + k - c_h, j + l - c_w + 0.5, i + k - c_h + 0.5, j + l - c_w + 0.5) \) \( T_{ijkl} \) is called the \textit{XY-adjustment} of \( \tilde{I}_{ij} \) at the screen position \((i + k - c_h, j + l - c_w)\).

\[
\text{for } k \in [1..c_h], l \in [1..c_w] : z_{ijkl} = \text{zinv}(\tilde{I}_{ij}), \quad (i + k - c_h + 0.5, j + l - c_w + 0.5) = P(T_{ijkl}).
\]

I use the gradient descent method \cite{LDR21} and a technique called \textit{dynamic gradient flow} to solve the optimization problem:

\[
\theta^* = \arg \min_{\theta} J(L, L).
\]

A. Spatial Domain Mapping

For the convenience of explanation, I define functions \( \text{xinv}(-), \text{yinv}(-), \text{zinv}(-) \) through the below equation:

\[
\text{rinv}(r) \triangleq \{(\text{xinv}(r), \text{yinv}(r), \text{zinv}(r))\}.
\]

2) \text{Comment 1}: Let \( B_{ijkl} = D_{ijkl} \cdot \tilde{I}_{ij} \),

\[
C_{ij} \triangleq \text{rmin}(\{(\text{rinv}(B_{(i-k+c_h),(j-l+c_w)kl}), B_{(i-k+c_h),(j-l+c_w)kl})\}_{1 \leq k \leq c_h, 1 \leq l \leq c_w} \quad (13)
\]

is a \( H \times W \) matrix. We have \( \lim_{c \to \infty} C_{ij} = \tilde{L}_{\text{p}, \text{p}, \text{rmin}}^{128 \times 256}(i, j) \).

Proofs of all Comments and Lemmas can be found in the Appendix. Finally, through applying the result of Comment \cite{Tie22} the matrix \( C \) is the output of the forward pass.
2) Backward pass: The backward pass refers to the gradient flow from the loss function $J$ to the parameter $\theta$ in computing the gradient $\nabla_{\theta} J(L, L)$. This is illustrated in Fig. 3. To understand the solution, let’s consider a basic case and then the general case:

**Single-point.** Assume the set $Q$ has only one element. Consider the corresponding point of that element in $\bar{I}$, $\bar{I}_{uv0} \neq 0$, and its kernel $\bar{D}_{uv0}$. Determine $k_0$ and $l_0$ so that the max value of $D_{uv0}$ is $D_{uv0(k_0,l_0)} \approx 1$. Let $\Upsilon_0$ be a set of positions of $N_0$ points in $\bar{L}$ which depend on $L_{uv0}$. $\Upsilon_0 \triangleq \{(\mu_0, \nu_0) \mid -c_k < \mu_0 < u_0 + c_h, \nu_0 - c_w < \nu_0 < v_0 + c_w\}_{i=1}$. As the gradient from these points will be passed to $\bar{I}_{uv0}$, defining a loss function based on these points will help with optimizing $\theta$. Under the assumption of a single point, I create a proper 2D representation from the binary mask $L$ by assigning to the active point of $L$ the property of $\bar{I}_{uv0}$. Let $L_{y0}' = \theta$ be the target active point of $\bar{I}_{uv0}$, define a proper 2D representation $L^*$,

$$L^*_{ij} \triangleq \begin{cases} r_{uv0(i-u_0+c_h)(j-v_0+c_w)}(u_0, v_0), & \text{if } (i, j) = (u_0', v_0'), \\ 0, & \text{otherwise}, \end{cases}$$

where $rinv(r_{ijkl}) = T_{ijkl}$. I assume that $(u_0', v_0') \in \Upsilon_0$.

**Comment 2:** Let

$$\theta_{opt} \triangleq \arg \min_{\theta} \sum_{i,j} |rinv(\lim_{c \to \infty} L_{ij}) - rinv(L^*_{ij})|.$$  

(15)

If $\tilde{\theta} = \theta_{opt}$ then the binary mask of $\bar{L}$ is equal to $L$ ($\mu^\ast + (\bar{L}) = L$).

Apply this result to (3), we have

$$J(\bar{L}, L) = \sum_{i,j} |rinv(\bar{L}_{ij}) - rinv(L^*_{ij})|,$$

as the suitable loss function.

Consider points in $\bar{L}$ and $L^*$ which have screen positions in $\Upsilon_0$, as only these points can affect the value of the gradient flow at $\bar{I}_{uv0}$. I realize that, according to (13), there will be at most two active points are $L_{(u_0+k_0-c_h,l_0-v_0-c_w)}$ and $L_{y0}'$, as shown in Fig. 3. Thus, let $(\bar{u}_0, \bar{v}_0) \triangleq (u_0 + k_0 - c_h, v_0 + l_0 - c_w)$ and by the chain rule, we have

$$\frac{dJ}{d\theta} = \frac{dJ}{d\bar{I}_{uv0}} \cdot \frac{d\bar{I}_{uv0}}{d\theta}$$

(17)

$$= \frac{d}{d\theta} \left| rinv(\bar{L}_{\bar{u}_0, \bar{v}_0}) - rinv(L^*_{\bar{u}_0, \bar{v}_0}) \right| \frac{d\bar{I}_{uv0}}{d\theta} + \frac{d}{d\theta} \left| rinv(\bar{L}_{u_0', v_0'}) - rinv(L^*_{u_0', v_0'}) \right| \frac{d\bar{I}_{uv0}}{d\theta},$$

(18)

when $(\bar{u}_0, \bar{v}_0) \neq (u_0', v_0')$. In case $(\bar{u}_0, \bar{v}_0) = (u_0', v_0')$, the first term is decayed. However, my purpose is matching the binary mask of $\bar{L}$ and $L$ so, if nothing is specified, we imply that

$$(\bar{u}_0, \bar{v}_0) \neq (u_0', v_0').$$

In this case, because of the assumption of a single point, we have $L^*_{\bar{u}_0, \bar{v}_0} = 0$. Thus, the first term of (13) supports moving the proper screen position of $\bar{I}_{uv0}$ away the $(\bar{u}_0, \bar{v}_0)$ position. And so, the second term supports moving it toward the $(u_0', v_0')$ position. Now, I can simplify the gradient harmlessly by canceling the first term and removing the role of the kernel $D$ in computing the gradient:

$$\frac{dJ}{d\theta} = \frac{d}{d\bar{I}_{uv0}} \left| rinv(\bar{L}_{\bar{u}_0, \bar{v}_0}) - rinv(\bar{L}_{\bar{u}_0, \bar{v}_0} + \Delta u, \Delta v) \right| \frac{d\bar{I}_{uv0}}{d\theta},$$

(20)

where $(\Delta u, \Delta v) \triangleq (u_0' - u_0 + c_h, v_0' - v_0 + c_w)$. This is equivalent to reducing the distance between two 2D points encoded in the rendering information of $\bar{I}_{uv0}$ and of $L^*_{u_0', v_0'}$.

**Multiple points**. This case brings us back to the problem described in the Sky dataset. Let $W = \{(u_i, v_i) | L_{u_i, v_i} \neq 0\}_{i=1}^M$ is the set of positions of $M'$ active points in $\bar{I}$. Thus, to be able to apply the same solution as the single-point case for each element in $W$, these two problems must be solved:

- (18) is only valid for the single-point case.
- Let $U = \{J_i\}_{i=1}^{M'}$ is the set of loss functions for active points in $I$. How will these loss functions be aggregated?

Consider a set $\{\Upsilon_i\}_{i=1}^{M'}$, $\Upsilon_i$ is constructed from $(u_i, v_i)$ in the same way as constructing $\Upsilon_0$ from $(u_0, v_0)$ in the single-point case. From $\{\Upsilon_i\}_{i=1}^{M'}$, construct a set $\{\Upsilon_i\}_{i=1}^{M'}$. Thus, for each element $(u_i, v_i)$ in $W$, $\Upsilon_i$ is a set of screen positions of $N_i$ active points in $\bar{L}$ which depend on $I_{u_i, v_i}$ except the proper screen position $(\bar{u}_i, \bar{v}_i)$ of $I_{u_i, v_i}$ on $L$.

$$\Upsilon_i \triangleq \{(u_i^{r(i)}, v_i^{r(i)}) \in \Upsilon_i \setminus \{u_i, v_i\} | L^{r(i)}_{u_i^{r(i)}, v_i^{r(i)}) \neq 0\}_{i=1}^{N_i}.$$  

(21)

In that, $(u_i^{0}, v_i^{0})$ is always the target active points of $(u_i, v_i)$. So, the valid expansion for the gradient at (17) is

$$\frac{dJ_i}{d\theta} = \frac{d}{d\bar{L}_{u_i, v_i}} \left| rinv(\bar{L}_{u_i, v_i}) - rinv(L^*_{u_i, v_i}) \right| \frac{d\bar{I}_{u_i, v_i}}{d\theta} + \frac{d}{d\bar{I}_{u_i, v_i}} \left| rinv(\bar{L}_{u_i, v_i}) - rinv(L^*_{u_i, v_i}) \right| \frac{d\bar{I}_{u_i, v_i}}{d\theta}$$

(22)

For the second problem, because loss functions in $U$ are independent of each other, all loss functions can be optimized by optimizing their summation

$$J(\bar{L}, L) \triangleq \sum_{i} J_i(\bar{L}, L).$$

(23)

**B. Optimization and Convergence**

In a sense, the Sky dataset was dividing a problem into small configurations. This helps with preventing the binary 2D representation $L$ from being full of active points, which means it doesn’t contain any useful information for estimation. On the other hand, as a result, an optimization method for the problem must be able to estimate $\theta^*$ through configurations. This makes the stochastic gradient descent (SGD) (8) is the most natural and suitable choice for optimization.
Comment 3: Assume that active points in the binary mask $L$ are distributed independently, $H$ and $W$ are much bigger than $h$ and $w$, respectively, and the size of the kernels (and the size of the screen) are big enough to be considered, in the scope of this comment, as boundless. We have:

- The first terms of (22) are independent random variables.
- The expectation of the first term is zero.
- The variance of the first term is less than a constant.

The assumption of distributing points independently in Comment 3 is only reasonable for the positions of circles’ centers. However, through zooming 2D representations out as will be mentioned in the implementation details section, we can ignore the shape of circles. This helps retain the result of Comment 3 for the Sky dataset.

Lemma 1: Consider a set of 3D points $Q = \{q_i \in \mathbb{R}^3\}_{i=1}^{N}$, where $N > 0$ is the number of elements of $Q$ and a tuple $\theta = (a,b)$. Let $\theta = (a,b)$ be a tuple of unknowns. $Q' = \{x'_i = (q_i(1) + a, q_i(2) + b, q_i(3)) | q_i \in Q\}_{i=1}^{N}$ and $Q = \{\tilde{x}_i = (q_i(1) + \alpha, q_i(2) + \beta, q_i(3)) | q_i \in Q\}_{i=1}^{M}$ ($0 < M \leq N$) are constructed by adding $\alpha$ and $\beta$ to the coordinates of points in $Q$. Let $F' = \{(x_i, r_i) | x_i \in Q', r_i \in \mathbb{R}^+\}_{i=1}^{N}$, and $\tilde{F} = \{\tilde{x}_i, \tilde{r}_i) | \tilde{x}_i \in \tilde{Q}, \tilde{r}_i \in \mathbb{R}^+\}_{i=1}^{M}$ be two sets of fragments constructed on $Q'$ and $Q$. Let $\tilde{L}^{H \times W}_{F', P, r_{\text{min}}}$ and $\tilde{L}^{H \times W}_{F, P, r_{\text{min}}}$ be 2D representations constructed on $F'$ and $\tilde{F}$, where $P$ is a linear perspective projection (without extrinsic parameters). Assume that there exists points $p \in Q$, $p' \in Q'$, $\tilde{p} \in Q$, and a screen position $(u, v)$. So that, $p = (X, Y, z)$, $p' = (X + a, Y + b, z)$, and $\tilde{p} = (X + \alpha, Y + \beta, z)$, where $X$ and $Y$ are independent random variables. And, $(u, v)$ is not the projection on $\tilde{L}$ of any 3D point in $\tilde{Q}$ other than $\tilde{p}$. Assume the probability that $(u, v)$ is the projection on $L'$ of $p'$ is $c \in (0, 1) \subset \mathbb{Q}$. Thus, if $\tilde{L}$ always has the same binary mask as $\tilde{L} \odot L'$,

$$P(\mathcal{B}_{\tilde{L}}(\tilde{L}) = \mathcal{B}_{\tilde{L}}(\tilde{L} \odot L')) = 1,$$

then $\lim_{z \to 0} \tilde{\theta} = \theta$.

Although the result of Lemma 1 is for ideal data, there is a more practical result that can apply to almost all kinds of data. Thus, if the screen has an ideal density (unlimited fined) the result can be applied for any $z$ (without the limit notation).

Known that the SGD algorithm accumulates gradients calculated from independent samples to update the parameter $\tilde{\theta}$, if the learning rate $r$ is small enough then the variance of the contribution of the first term of (22) in the optimization will decay to zero. Thus, based on Comment 3 the loss function was defined in (23) is still suitable for the optimization problem. Based on Comment 3 and Lemma 1 I construct a convergence criterion. Thus, I estimate the probability in (24) through supervising the empirical expectation over $n$ consecutive times with $n$ is large enough for applying the law of large numbers in the estimation, and stop the SGD algorithm if it is equal to 0. While the empirical convergence is still assured, the learning rate $r$ will be decided empirically through hyperparameter tuning 9.
V. EXPERIMENTS

I split randomly 768 configurations of the Sky dataset, which were chosen randomly, into two datasets are train set and dev set. Thus, the train set has 512 configurations for training and the dev set has 256 configurations for validation.

A. Implementation Details

I used the Pytorch deep learning framework for implementation. There are many ways for implementing the algorithm but my criteria was utilizing built-in functions of the framework and simplifying implementation. I chose the size of each mini-batch as large as possible and it was 16. The convergence criterion was substituted by using the number of epochs concept [10], [8]. Thus, the algorithm will pass over the entire dataset many times until it is almost sure that the algorithm was converged. The number of epochs $n_{\text{epoch}}$ and the learning rate $r$ were determined through hyperparameter tuning, as shown in Table I.

Additionally, instead of tuning the hyperparameters $h, w$, I chose $(h, w) = (3, 3)$ and used a training algorithm as a valid substitution for expanding the kernels’ size. Precisely, I optimized the parameter $\theta$ with smaller 2D representations through an algorithm that reduces the size of 2D representations by half in all dimensions and generates four new zoomed-out 2D representations. Because the distance between two points in 3D space is directly proportional to their distance in the projection plane, I propose a divide-and-conquer algorithm for reshaping. Thus, a point in a new 2D representation is mapped into a set of points in the original 2D representation which are nearest to the center point of these points on the original screen. And, to be still able to use the result of Comment 2 screen positions used to construct XY-adjustments must reflect the positions of these center points on the original screen.

For each training batch, the size of the 2D representation $\tilde{T}$ is reduced by $2^s$ in all two dimensions to generate $4^s$ new 2D representations, where $s$ is a positive integer. This can be done by applying the reshape algorithm multiple times to 2D representations. Then, the SDM algorithm is performed iteratively along with increasing the size of 2D representations until these 2D representations reach the desired size. Precisely, after performing the SDM algorithm, the size of 2D representations is increased (doubled in all dimensions) by inverting the reshape algorithm. This implementation helps with a valid forward pass as the original SDM algorithm. However, it makes the backward pass different from the original algorithm (now, the 2D representation $\tilde{T}$ for the backward pass is the output of the forward pass instead of its original version). Nonetheless, this difference is needed to guarantee the results of Comment 2 for $3 \times 3$ kernels.

A zoomed-out target 2D representation is used for optimization. The idea is a correct zoom-out version of the target 2D representation will help the algorithm still intact the results of Comment 2 and Lemma 7. Finally, after convergence of the algorithm for zoomed-out 2D representations, the size of these 2D representations is increased (again, by inverting the reshape algorithm) for optimizing the loss function with nearer target active points until convergence of the algorithm for the original size. Additionally, the learning rate $r$ should be reduced by half as the distances between XY-adjustments of a point are decreased along with the increase of the size of the 2D representation.

The pseudocode of the SDM algorithm can be found in the Appendix.

B. Evaluation Method

I evaluate the algorithm on criteria are accuracy and robustness.

1) Accuracy: Instead of creating a test set for considering the performance of the algorithm, I evaluate directly based on the Euclidean distance between the parameter $\theta^*$ and its estimation, $\hat{\theta}$.

2) Robustness: To evaluate the robustness, I add noise into the train set to understand the behavior of the algorithm against noise, which is a common characteristic of real data especially in computer vision. The noise will be added to the target 2D representations. Precisely, considering a non-negative integer $n$, I randomly change screen positions of active points in the target 2D representations. Thus, for each target 2D representation $T$, I construct a new target 2D representation $T'$ to substitute for $T$, so that the set of screen positions of active points in the new target 2D representation is $\{(u_i + X_i, v_i + Y_i)\}_{i=1}^N$, where $X_i, \ldots, X_N, Y_1, \ldots, Y_N$ are identically distributed independent random variables uniformly distributed in $[-n..n]$. So that, $n$ is called noise level. I also use the term $n$-pixels to mention $n$ because it affects screen positions. The effect of the noise on the data is illustrated in Fig. 4.

C. Result Analysis

I chose $s = 4$ which helps the algorithm considers transitions at a range of up to 16 pixels on the screen. Via hyperparameter tuning, I decided $n_{\text{epoch}} = 1$ and $r = 3e^{-4}$ (the best performance configuration in Table I). I considered noise levels are 0-pixels for evaluating the accuracy criteria and 1-pixel, 2-pixels, 3-pixels, and 4-pixels for evaluating the robustness criteria. For each noise level, I performed the experiment 60 times. The result is summarized in Table II.

| Loss value | Hyperparameters |
|------------|-----------------|
| $6.703^*$  | number of epochs ($n_{\text{epoch}}$) |
| $0.930^*$  | 1 |
| $1.004$    | 1 $3e^{-4}$ |
| $1.355$    | 2 $1e^{-4}$ |
| $0.565^*$  | 2 $3e^{-4}$ |
| $1.227$    | 2 $1e^{-3}$ |

* The best performance.
For comparison, I exchange the noise levels to Euclidean distance (the second column in Table II) through interpolating the transition of XY coordinates of a point based on its screen position and Z coordinate. The first row of Table II describes the accuracy of the algorithm in the noise-free environment is $0.000682 \pm 0.00068404$, with a 95% confidence level. Compared with the average deviation (0.2834) in the case of the smallest level of noise (1-pixel noise-level), the accuracy is usually less than one hundred times smaller than the average deviation. As also shown in Table II the average performance of the algorithm (0.040056) even in the worst case of noise (4-pixels noise-level) is much less than the average deviation in the case of the 1-pixel noise-level. Thus, 75% of outcomes are less than a sixth of the average deviation. Also, the value of the mean plus triple the standard deviation is less than a fourth of the average deviation. Although distributions are slightly right-skewed, the evidence is still strong enough to demonstrate that the algorithm is accurate and stable against noise.

### VI. Conclusion

The problem mentioned in this paper is basic and directly related to more complex and practical problems in 3D inference from synthesis 2D views. Despite the number of parameters is small (only two parameters), compared to the size of the train set, being there are no explicit links between a 3D point and its corresponding representation on synthesis 2D views is the real challenge in solving the problem. Fortunately, with some practical assumptions, I could come up with an optimization solution for a major special case of the problem. The solution is stable against noise and has many potentials for further improvements in the future.

Being constructed on the 2D representation concept brings the ability to embed the solution into current differentiable rendering techniques, and thus helps reduce efforts in developing ideas in this paper for other research and applications. Although ideas need to be improved, I hope these contributions can motivate and benefit others’ works.

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APPENDIX A
DETAILS ABOUT CIRCLES IN THE SKY DATASET

Let $E'$ be an $128 \times 256$ matrix of zero values. For each active point $E_{uv}$ in the 2D representation $E$, consider points in $E'$ which have positions in $S = \{(u_i, v_i) \in \mathbb{N}^2 \mid \exp(-\frac{d(u_i-u_i)^2+(v_i-v_i)^2}{\sigma_i^2}) \geq 0.25, 1 \leq u_i \leq 128, 1 \leq v_i \leq 256\}$. $N'$ is the number of elements of $S$. A circle on $E'$ is drawn by following these steps for each $(u', v')$ in $S$:

- If $E_{u'v'} = 0$, assign the value of $E_{uv}$ to $E_{u'v'}$.
- Else, if $E_{u'v'} > E_{uv}$, assign the value of $E_{uv}$ to $E_{u'v'}$.

APPENDIX B
PROOF FOR COMMENT 1

According to (6), we consider two cases. For the first case, consider every screen position $(u', v')$ on $\tilde{L}$ that has $q(I, (u', v')) \neq 0$. Recall that in (7),

$$q(I, (u', v')) = \{(\text{lin}(I)_{ij}), I_{ij}) \mid \tilde{I}_{ij} \neq 0, (u', v') = \{P(\text{lin}(I)_{ij})\}_{i,j}.$$}

Thus, for each $\tilde{I}_{uv}$ satisfies the conditions in (7), we consider the corresponding kernel of this active point, $D_{uv}$. Let $(\Delta' u', \Delta' v') = (u' - u + c_h, v' - v + c_w)$, we have $\lim_{c \to \infty} D_{uv} \Delta' u' \Delta' v' = 1$. Indeed, because $I_{uv} \neq 0$ and according to (12), we have $D_{uv} \Delta' u' \Delta' v' = \lim_{c \to \infty} D_{uv} \Delta' u' \Delta' v' = 1$. If the 3D point represented by $\tilde{I}_{uv}$ is nearer to the XY-adjustment $T_{uv} \Delta' u' \Delta' v'$ than other XY-adjustments of $\tilde{I}_{uv}$, we consider points that have the same depth value as indicated in (10), let's consider them in a Euclidean plane $\Pi : z = z_{\text{lin}}(I_{uv})$. Consider the set of XY-adjustments of $I_{uv}$, $\{T_{uvij} \mid 1 \leq i \leq c_h, 1 \leq j \leq c_w\}$, we have the set of points $V_{ij} = \{(x, y, z) \mid z = z_{\text{lin}}(I_{uv}), \{P((x, y, z))\} = \{P(T_{uvij})\}\}$ in the plane is the Voronoi cell associated with $T_{uvij}$ (based on the nature of $\lfloor \cdot \rfloor$ function and its relationship with $\lfloor \cdot \rfloor$ function). According to (7), we have $(u', v') = \{P(\text{lin}(I_{uv}))\}$ and thus, according to (11), the 3D point represented by $I_{uv}$ is in the Voronoi cell of $T_{uvij}$. Because points in the Voronoi cell associated with $T_{uvij}$ is nearer to the XY-adjustment $T_{uvij}$ than other XY-adjustments, we have the 3D point represented by $I_{uv}$ is nearer to the XY-adjustment $T_{uv} \Delta' u' \Delta' v'$ than other XY-adjustments of $\tilde{I}_{uv}$. Therefore, as said above, we have $\lim_{c \to \infty} D_{uv} \Delta' u' \Delta' v' = 1$. Let $(r, s) = (u' - i + c_h, v' - j + c_w)$ and recall that $B_{ijkl} = D_{ijkl} \cdot I_{ij}$, so, according to (13), we can rewrite (7) as

$$q(\tilde{I}, (u', v')) = \{(\text{lin}(\lim_{c \to \infty} B_{r_{s_{ij}}}), \lim_{c \to \infty} B_{r_{s_{ij}}}) \mid I_{ij} \neq 0, \lim_{c \to \infty} D_{r_{s_{ij}}} = 1\}_{1 \leq i \leq c_h, 1 \leq j \leq c_w}. $$

Consider the second case, screen positions $(u', v')$ on $\tilde{L}$ that have $q(I, (u', v')) = 0$. Thus, we have $B_{r_{s_{ij}}} = 0$. In general,

$$\tilde{L}(u', v') = \begin{cases} \text{rmin}(\{(\text{lin}(\lim_{c \to \infty} B_{r_{s_{ij}}}), \lim_{c \to \infty} B_{r_{s_{ij}}})\}_{k,i}), \quad \text{if } \lim_{c \to \infty} B_{r_{s_{ij}}} \neq 0, \\ 0, \quad \text{otherwise.} \end{cases}$$\(26\)

Finally, based on the definition of the function $\text{rmin}$, we have $\lim_{c \to \infty} C_{u'v'} = \tilde{E}_{\tilde{F}, P, \text{rmin}}(u', v')$.

APPENDIX C
PROOF FOR COMMENT 2

Recall that the max value of the kernel $D_{uvij}$ is $D_{uvij} \approx 1$, while its other values are 0. On the other hand, we have the expression $|\text{rmin}(\lim_{c \to \infty} I_{ij}) - \text{rmin}(L_{ij})|$ has a possible minimum value is 0. Assume that $|\text{rmin}(\lim_{c \to \infty} I_{ij}) - \text{rmin}(L_{ij})| = 0$, we have $\text{rmin}(\lim_{c \to \infty} I_{uvij}) = \text{rmin}(L_{uvij})$. Thus, according to the definition of $\tilde{L}$ in Comment 1, the definition of $L^*$ in (14), and the assumption of a single point, these equations must be respectively satisfied:

$$\tilde{L}_{uvij} = D_{uvij}(u_{0i} - \bar{k}_0 + c_h, v_{0j} - \bar{k}_0 + c_w) \cdot \tilde{I}(u_{0i} - \bar{k}_0 + c_h, v_{0j} - \bar{k}_0 + c_w),$$

$$\lim_{c \to \infty} \text{rmin}(\tilde{L}_{uvij}) = \text{rmin}(L_{uvij}) = \{T_{uvij}(u_{0i} - u_0 + c_h, v_{0j} - v_0 + c_w), (u_{0i} - \bar{k}_0 + c_h, v_{0j} - \bar{k}_0 + c_w) = (u_0, v_0)\}. $$

The equation system is solved if $\text{rmin}(\tilde{I}_{uvij}) = \text{rmin}(L_{uvij}) = \text{rmin}(L_{uvij}) = 0$. Inversely, if the equation system is solved, we also have $|\text{rmin}(\lim_{c \to \infty} I_{ij}) - \text{rmin}(L_{ij})| = 0$. Thus, based on the definition of $\theta_{\text{opt}}$, we have that if $\theta - \theta_{\text{opt}}$ then $\theta_{\text{opt}}(\tilde{L}) = L$.

APPENDIX D
PROOF FOR COMMENT 3

Consider an active point in $I_{uvij}$, where $(u_i, v_i)$ is in the set of active points in $I (W = \{(u_i, v_i) \mid I_{uvij} \neq 0\})$, and its corresponding active point $\tilde{L}_{uvij}$, where $(u_{0i} , v_{0j})$ is the proper screen position of $I_{uvij}$. Let $(r, s)$ is a screen position in $\hat{T}_{ij}$ as illustrated in Fig. 5. So, $(r, s)$ corresponds to a term of the first term in (22). Based on the assumptions of the Comment 3, we have that the distribution of the value of the first term is independent of $(u_{0i}, v_{0j})$ and, obviously, the first terms are independent random variables.

Because of symmetry and the independent distribution of active points in $L$, we have that the absolute value of the expectation of the gradient of active points at the position $(r, s)$ is equal to the absolute value of the expectation of the gradient of active points at the reflection of $(r', s')$ in $(u_{0i}, v_{0j})$. Moreover, we have gradients at these two screen position represent for two opposite transition directions of the 3D point represented by $I_{uvij}$. Thus sum of the expectation of the gradient of active points at $(r, s)$ and $(r', s')$ is zero. Apply
the same fact for others points in \( \mathcal{T}_l \) we have the expectation of the first term is zero.

From (20), we have that the absolute values of terms in (22) are less than a constant. Thus, along with the fact that the expectation of the first term is zero, we have its variance is less than a constant.

**APPENDIX E**

**PROOF FOR LEMMA 1**

First, we will prove that if \( (u, v) \) is the projection of \( \tilde{p} \) on \( \tilde{L} \) then it is also the projection of \( p' \) on \( L' \). According to (24), we have that if \( (u, v) \) is the projection of \( \tilde{p} \) on \( \tilde{L} \) then \( (u, v) \) must also be the projection of a point \( q' \in Q' \). Assume that \( q' \) is not \( p' \). Let \( A \) be the event of \( (u, v) \) is the projection of a point in \( Q' \) and \( B \) be the event of \( (u, v) \) is the projection of a point in \( \bar{Q} \). We have \( P(A) = 1 \) as \( q' \) is deterministic and \( P(B) = c \) (the assumption of the lemma). Because \( P(A) = 1 \), we have the event \( A \cap B \) is a necessary condition of the event of \( \tilde{L} \) has the same binary mask as \( L' \). So, we have

\[
P(\tilde{p}_R'(L') = \tilde{p}_R(\tilde{L} \odot L')) < P(A \cap B),
\]

\[
< P(B),
\]

\[
< c. \tag{27}
\]

Because \( c \in (0, 1) \), we have \( P(\tilde{p}_R'(L') = \tilde{p}_R(\tilde{L} \odot L')) < 1 \) (contrary to the assumption of the lemma in (24)). So, \( q' \) must be the same point as \( p' \), this leads to \( (u, v) \) is also the projection of \( p' \) on \( L' \).

Without loss of generality, let \( p = (x, y, z) \), \( p' = (x + a, y + b, z) \) and \( \tilde{p} = (x + \tilde{a}, y + \tilde{b}, z) \). Because \( (u, v) = [P(p')] = [P(\tilde{p})] \), we have

\[
|P(p') - P(\tilde{p})| < C, \tag{28}
\]

where \( C \) is a constant vector. According to the definition of linear perspective projection (without extrinsic parameters), we have

\[
|P(p') - P(\tilde{p})| = \frac{|A p' - A \tilde{p}|}{z}, \tag{29}
\]

where \( A \) is a constant matrices. From (28) and (29), we have \( A \frac{p' - \tilde{p}}{z} < C \). Along with the fact that \( A \) is constant matrices, we know that there exist a constant vector \( C' \), so that \( \lim_{z \to 0} \frac{p' - \tilde{p}}{z} = \lim_{z \to 0} C' z = (0, 0, 0) \). Because \( p' - \tilde{p} = (a - \tilde{a}, b - \tilde{b}, 0) \) and \( \theta - \tilde{\theta} = (a - \tilde{a}, b - \tilde{b}) \), we have \( \lim_{z \to 0} (\theta - \tilde{\theta}) = (0, 0) \). Therefore, \( \lim_{z \to 0} \theta = \tilde{\theta} \).

**APPENDIX F**

**PSEUDO-CODE OF THE SPATIAL DOMAIN MAPPING ALGORITHM**

**Algorithm 1** Spatial Domain Mapping algorithm

**Require:** Computational graph of 2D representation \( \tilde{I} \), target binary mask \( L \), original size of 2D representation \( (H, W) = (128, 256) \), order of kernel size \( s \in \mathbb{N}^+ \), zoom-out index \( 0 \leq \text{zoom} \leq s \)

**Ensure:** Computational graph of proper 2D representation \( \tilde{L} \)

1: **procedure** reshape(\( As \)) \( \triangleright \) Reshape algorithm to reduce size of 2D representations by half in all spatial dimensions
2: Create an empty list \( B \)
3: for \( A \in As \) do
4: Use row-major order reshape method of Pytorch to reshape \( A \) to shape \( (h/2) \times (w/2) \times 2 \)
5: Use row-major order permute method of Pytorch to transpose \( A \) to shape \( 4 \times (h/2) \times (w/2) \) while keeping the first and the second dimensions
6: Decouple the first order of \( A \) to have a list of 4 new 2D representations
7: Store the list to \( B \)
8: end for
9: return \( B \)
10: end procedure
11: **procedure** SDM3X3(\( Is \)) \( \triangleright \) The SDM algorithm used 3 \( \times \) 3 kernels
12: Set an empty list \( Ls \)
13: for \( I \in Is \) do
14: Construct XY-adjustments of active points of \( I \)
15: Apply the result of Comment 7 to generate computational graph of proper 2D representation \( \tilde{L} \) from \( I \)
16: Modify the computational graph of the proper 2D representation to reflect (22)
17: Store \( L' \) to \( Ls \)
18: end for
19: return \( Ls \)
20: end procedure
21: for \( i \leftarrow 1, s - \text{zoom} \) do
22: Construct a list of size-reduced 2D representation, \( \tilde{I} \leftarrow \text{reshape}(\tilde{I}) \)
23: \( L \leftarrow \text{reshape}(L) \)
24: end for
25: Perform SDM algorithm used 3 \( \times \) 3 kernels, \( \tilde{L} \leftarrow \text{SDM3X3}(\tilde{I}) \)
26: for \( i \leftarrow 1, s - \text{zoom} \) do
27: \( \tilde{I} \leftarrow \tilde{I} \)
28: Increase (doubled in all dimensions) the size of 2D representations \( \tilde{I} \) and \( L \) by inverting the reshape algorithm
29: \( \tilde{L} \leftarrow \text{SDM3X3}(\tilde{I}) \)
30: end for
31: Cancel the gradient flow at screen positions of \( \tilde{I} \) which satisfy (19)
32: Cancel the gradient flow at inactive screen positions of \( \tilde{L} \) as mentioned at (21)
33: Construct zoomed-out target 2D representation, \( L \leftarrow \tilde{p}_R'(\sum L_s) \)
34: return \( \tilde{L} \) \( \triangleright \) proper 2D representation \( \tilde{L} \) from \( \tilde{I} \)