Finding Small Weight Isomorphisms with Additional Constraints is Fixed-Parameter Tractable*

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Abstract. Lubiw showed that several variants of Graph Isomorphism are NP-complete, where the solutions are required to satisfy certain additional constraints [14]. One of these, called Isomorphism With Restrictions, is to decide for two given graphs $X_1 = (V_1, E_1)$ and $X_2 = (V_2, E_2)$ and a subset $R \subseteq V \times V$ of forbidden pairs whether there is an isomorphism $\pi$ from $X_1$ to $X_2$ such that $\pi(i) \neq \pi(j)$ for all $(i, j) \in R$.

We prove that this problem and several of its generalizations are in fact in FPT:

• The problem of deciding whether there is an isomorphism between two graphs that moves $k$ vertices and satisfies Lubiw-style constraints is in FPT, with $k$ and $|R|$ as parameters. The problem remains in FPT even if a CNF of such constraints is allowed. As a consequence of the main result it follows that the problem to decide whether there is an isomorphism that moves exactly $k$ vertices is in FPT. This solves a question left open in [2].

• When the weight and complexity are unrestricted, finding isomorphisms that satisfy a CNF of Lubiw-style constraints is in FPTGI.

• Checking if there is an isomorphism between two graphs that has complexity $t$ is also in FPT with $t$ as parameter, where the complexity of a permutation $\pi$ is the Cayley measure defined as the minimum number $t$ such that $\pi$ can be expressed as a product of $t$ transpositions.

• We consider a more general problem in which the vertex set of a graph $X$ is partitioned into Red and Blue, and we are interested in an automorphism that stabilizes Red and Blue and moves exactly $k$ vertices in Blue, where $k$ is the parameter. This problem was introduced in [6], and in [2] we showed that it is W[1]-hard even with color classes of size 4 inside Red. Now, for color classes of size at most 3 inside Red, we show the problem is in FPT.

In the non-parameterized setting, all these problems are NP-complete. Also, they all generalize in several ways the problem to decide whether there is an isomorphism between two graphs that moves at most $k$ vertices, shown to be in FPT by Schweitzer [15].

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1 Introduction

The Graph Isomorphism problem (GI) consists in deciding whether two given input graphs are isomorphic, i.e., whether there is a bijection between the vertex sets of the two graphs that preserves the adjacency relation. It is an intensively researched algorithmic problem for over four decades, culminating in Babai’s recent quasi-polynomial time algorithm [3].

There is also considerable work on the parameterized complexity of GI. For example, already in 1980 it was shown [9] that GI, parameterized by color class size, is fixed-parameter tractable (FPT). It is also known that GI, parameterized by the eigenvalue multiplicity of the input graph, is in FPT [4]. More recently, GI, parameterized by the treewidth of the input graph, is shown to be in FPT [13].

In a different line of research, Lubiw [14] has considered the complexity of GI with additional constraints on the isomorphism. Exploring the connections between GI and the NP-complete problems, Lubiw defined the following version of GI.

**Isomorphism With Restrictions:** Given two graphs $X_1 = (V_1, E_1)$ and $X_2 = (V_2, E_2)$ and a set of forbidden pairs $R \subseteq V_1 \times V_2$, decide whether there is an isomorphism $\pi$ from $X_1$ to $X_2$ such that $i \pi \neq j$ for all $(i, j) \in R$.

When $X_1 = X_2$, the problem is to check if there is an automorphism that satisfies these restrictions. Lubiw showed that the special case of testing for fixed-point-free automorphisms is NP-complete. Klavík et al. recently reexamined Isomorphism With Restrictions [12]. They show that it remains NP-complete when restricted to graph classes for which GI is as hard as for general graphs. Conversely, they show that it can be solved in polynomial time for several graph classes for which the isomorphism problem is known to be solvable in polynomial time by combinatorial algorithms, e.g. planar graphs and bounded treewidth graphs. However, they also show that the problem remains NP-complete for bounded color class graphs, where an efficient group theoretic isomorphism algorithm is known.

A different kind of constrained isomorphism problem was introduced by Schweitzer [15]. The weight (or support size) of a permutation $\pi \in \text{Sym}(V)$ is $|\{ i \in V \mid i \pi \neq i \}|$. Schweitzer showed that the problem of testing if there is an isomorphism $\pi$ of weight at most $k$ between two $n$-vertex input graphs in the same vertex set can be solved in time $k^{O(k)} \text{poly}(n)$. Hence, the problem is in FPT with $k$ as parameter. Schweitzer’s algorithm exploits interesting properties of the structure of an isomorphism $\pi$. Based on Lubiw’s reductions [14], it is not hard to see that the problem is NP-complete when $k$ is not treated as parameter.

In this paper we consider the problem of finding isomorphisms with additional constraints in the parameterized setting. In our main result we formulate a graph isomorphism/automorphism problem with additional constraints that generalizes Lubiw’s setting as follows. For a graph $X = (V, E)$, let $\pi \in \text{Aut}(X)$ be an automorphism of $X$. We say that a permutation $\pi \in \text{Sym}(V)$ satisfies a formula $F$ over the variables in $\text{Var}(V) = \{ x_{u,v} \mid u, v \in V \}$ if $F$ is satisfied by the assignment that has $x_{u,v} = 1$ if and only if $u \pi = v$. For example, the conjunction $\bigwedge_{u \in V} \neg x_{uu}$ expresses the condition that $\pi$ is fixed-point-free. We define:

**Exact-CNF-GI:** Given two graphs $X_1 = (V, E_1)$ and $X_2 = (V, E_2)$, a CNF formula $F$ over $\text{Var}(V)$, and $k \in \mathbb{N}$, decide whether there is an isomorphism from $X_1$ to $X_2$ that has weight exactly $k$ and satisfies $F$. The parameter is $|F| + k$, where $|F|$ is the number of variables used in $F$. 

2
In Section 4, we first give an FPT algorithm for EXACT-CNFGA, the automorphism version of this problem. The algorithm uses an orbit shrinking technique that allows us to transform the input graph into a graph with bounded color classes, preserving the existence of an exact weight $k$ automorphism that satisfies the formula $F$. The bounded color class version is easy to solve using color coding; see Section 3 for details. Building on this, we show that EXACT-CNFGI is also in FPT. In particular, this allows us to efficiently find isomorphisms of weight exactly $k$, a problem left open in [2], and extends Schweitzer’s result mentioned above to the exact case. In our earlier paper [2] we have shown that the problem of exact weight $k$ automorphism is in FPT using a simpler orbit shrinking technique which does not work for exact weight $k$ isomorphisms. In this paper, we use some extra group-theoretic machinery to obtain a more versatile orbit shrinking.

In Section 5, we turn from restrictions on weight and complexity to restrictions given only by a CNF formula over Lubiw-style constraints. We show that hypergraph isomorphism constrained by a CNF formula is in FPT GI. Note that the problem remains GI-hard even when the formula is constantly true, so an FPT algorithm without GI oracle would imply GI $\in$ P.

In Section 6, we consider the problem of computing graph isomorphisms of complexity exactly $t$: The complexity of a permutation $\pi \in \text{Sym}(V)$ is the minimum number of transpositions whose product is $\pi$. Checking for automorphisms or isomorphisms of complexity exactly $t$ is NP-complete in the non-parameterized setting. We show that the problem is in FPT with $t$ as parameter. Again, the “at most $t$” version of this problem was already shown to be in FPT by Schweitzer [15] as part of his algorithmic strategy to solve the weight at most $k$ problem. Our results in Sections 4 and 6 also hold for hypergraphs when the maximum hyperedge size is taken as additional parameter.

In Section 7, we examine a different restriction on the automorphisms being searched for. Consider graphs $X = (V,E)$ with vertex set partitioned into Red and Blue. The Colored Graph Automorphism problem (defined in [6]; we denote it Col-GA), is to check if $X$ has an automorphism that respects the partition and moves exactly $k$ Blue vertices. We showed in [2] that this problem is W[1]-hard. In our hardness proof the orbits of the vertices in the Red part of the graph have size at most 4, while the ones for the Blue vertices have size 2. We show here that this cannot be restricted any further. If we force the size of the orbits of $\text{Aut}(X)$ in the Red part to be bounded by 3 (i.e., the input graph has Red further partitioned into color classes of size at most 3 each), then the problem to test whether there is an automorphism moving exactly $k$ Blue vertices can be solved in FPT (with parameter $k$). The Blue part of the graph remains unconstrained. Observe that Schweitzer’s problem [15] coincides with the special case of this problem where there are no Red vertices. This implies that the non-parameterized version of Col-GA is NP-complete (even when $X$ has only Blue vertices). Similarly, finding weight $k$ automorphisms of a hypergraph reduces to Col-GA by taking the incidence graph, where the original vertices become Blue and the vertices for hyperedges are Red; note that this yields another special case, where both Red and Blue induce the empty graph, respectively.

2 Preliminaries

We use standard permutation group terminology, see e.g. [5]. Given a permutation $\sigma \in \text{Sym}(V)$, its support is $\text{supp}(\sigma) = \{ u \in V \mid u^\sigma \neq u \}$ and its (Hamming) weight is $|\text{supp}(\sigma)|$. The complexity of $\sigma$ (sometimes called its Cayley weight) is the minimum number $t$ such that $\sigma$ can be written as the product of $t$ transpositions.
Let $G \leq \text{Sym}(V)$ and $\pi \in \text{Sym}(V)$; this includes the case $\pi = \text{id}$. A permutation $\sigma \in G\pi \setminus \{\text{id}\}$ has \textit{minimal complexity in} $G\pi$ if for every way to express $\sigma$ as the product of a minimum number of transpositions $\sigma = \tau_1 \cdot \cdots \cdot \tau_{\text{compl}(\sigma)}$ and every $i \in \{2, \ldots, \text{compl}(\sigma)\}$ it holds that $\tau_i \cdots \tau_{\text{compl}(\sigma)} \notin G\pi$. The following lemma observes that every element of $G\pi$ can be decomposed into minimal-complexity factors.

**Lemma 2.1** [2, Lemma 2.2]. Let $G\pi$ be a coset of a permutation group $G$ and let $\sigma \in G\pi \setminus \{\text{id}\}$. Then for some $\ell \geq 1$ there are $\sigma_1, \ldots, \sigma_{\ell-1} \in G$ with minimal complexity in $G$ and $\sigma_\ell \in G\pi$ with minimal complexity in $G\pi$ such that $\sigma = \sigma_1 \cdots \sigma_\ell$ and $\text{supp}(\sigma_i) \subseteq \text{supp}(\sigma)$ for each $i \in \{1, \ldots, \ell\}$.

An action of a permutation group $G \leq \text{Sym}(V)$ on a set $V'$ is a group homomorphism $h: G \to \text{Sym}(V')$; we denote the image of $G$ under $h$ by $G(V')$. For $u \in V$, we denote its stabilizer by $G_u = \{ \pi \in G \mid u^\pi = u \}$. For $U \subseteq V$, we denote its pointwise stabilizer by $G[U] = \{ \pi \in G \mid \forall u \in U : u^\pi = u \}$ and its setwise stabilizer by $G\{U\} = \{ \pi \in G \mid U^\pi = U \}$.

For $S \subseteq \mathcal{P}(V)$, we let $G_S = \{ \pi \in G \mid \forall U \in S : U^\pi = U \}$.

A \textit{hypergraph} $X = (V, E)$ consists of a vertex set $V$ and a hyperedge set $E \subseteq \mathcal{P}(V)$. Graphs are the special case where $|e| = 2$ for all $e \in E$. The \textit{degree} of a vertex $v \in V$ is $|\{ e \in E \mid v \in e \}|.$ A (vertex) \textit{coloring} of $X$ is a partition of $V$ into color classes $C = (C_1, \ldots, C_m)$. The color classes $C$ are $b$-\textit{bounded} if $|C_i| \leq b$ for all $i \in [m]$. An \textit{isomorphism} between two hypergraphs $X = (V, E)$ and $X' = (V', E')$ (with color classes $C = (C_1, \ldots, C_m)$ and $C' = (C'_1, \ldots, C'_m)$) is a bijection $\pi: V \to V'$ such that $E' = \{ \{ \pi(v) \mid v \in e \} \mid e \in E \}$ (and $C'_i = \{ \pi(v) \mid v \in C_i \}$). The isomorphisms from $X$ to $X'$ form a coset that we denote by $\text{Iso}(X, X')$. The automorphisms of a hypergraph $X$ are the isomorphisms from $X$ to itself; they form a group which we denote by $\text{Aut}(X)$.

## 3 Bounded color class size

To show that \textsc{exact-cnf-ga} for hypergraphs with $b$-bounded color classes can be solved in \textsc{fpt}, we recall our algorithm for exact weight $k$ automorphisms of bounded color class hypergraphs [2] and show how it can be adapted to the additional constraints given by the input formula.

**Definition 3.1.** Let $X = (V, E)$ be a hypergraph with color class set $C = (C_1, \ldots, C_m)$.

(a) For a subset $C' \subseteq C$, we say that a color-preserving permutation $\pi \in \text{Sym}(V)$ \textit{$C'$-satisfies a CNF formula $F$ over $\text{Var}(V)$} if every clause of $F$ contains a literal $x_{u,v}$ or $\neg x_{u,v}$ with $u \in \bigcup C'$ that is satisfied by $\pi$.

(b) For a color-preserving permutation $\pi \in \text{Sym}(V)$, let $C[\pi] = \{ C_i \in C \mid \exists v \in C_i : v^\pi \neq v \}$ be the subset of color classes that intersect $\text{supp}(\pi)$. For a subset $C' \subseteq C[\pi]$, we define the permutation $\pi_{C'} \in \text{Sym}(V)$ as

$$
\pi_{C'}(v) = \begin{cases} 
v^\pi, & \text{if } v \in \bigcup C', \\
v, & \text{if } v \notin \bigcup C'.
\end{cases}
$$

Note that $\pi_{C[\pi]} = \pi$.

(c) A color-preserving automorphism $\sigma \neq \text{id}$ of $X$ is said to be \textit{color-class-minimal}, if for every set $C'$ with $\emptyset \subsetneq C' \subsetneq C[\sigma]$, the permutation $\sigma_{C'}$ is not in $\text{Aut}(X)$.
Lemma 3.2. Let $X = (V, E)$ be a hypergraph with color class set $C = \{C_1, C_2, \ldots, C_m\}$. For $\emptyset \neq C' \subseteq C$ and a CNF formula $F$ over $\text{Var}(V)$, the following statements are equivalent:

- There is a nontrivial automorphism $\sigma$ of $X$ with $C[\sigma] = C'$ that satisfies $F$.
- $C'$ can be partitioned into $C_1, \ldots, C_{\ell}$ and $F$ (seen as a set of clauses) can be partitioned into CNF formulas $F_0, \ldots, F_{\ell}$ such that $F_0 = (C \setminus C')$-satisfied by id and for each $i \in \{1, \ldots, \ell\}$ there is a color-class-minimal automorphism $\sigma_i$ of $X$ with $C[\sigma_i] = C_i$ that $C_i$-satisfies $F_i$. Moreover, the automorphisms $\sigma$ and $\sigma_i$ can be chosen to satisfy $\sigma_i = \sigma_{C_i}$ for $1 \leq i \leq \ell$, respectively.

Proof. To show the forward direction, let $\sigma$ be a nontrivial automorphism of $X$ with $C[\sigma] = C'$ that satisfies $F$. We put those clauses of $F$ into $F_0$ that are $(C \setminus C')$-satisfied by id, and the remaining clauses of $F$ into the CNF formula $F'$. Note that $F'$ is $C'$-satisfied by $\sigma$: Every clause of $F$ must contain a literal $x_{u,v}$ or $\neg x_{u,v}$ that is satisfied by $\sigma$. If this literal is not $C'$-satisfied by $\sigma$, we have $u \notin \bigcup C'$ and thus $u^\sigma = u$, so this clause is contained in $F_0$.

We show by induction on $|C[\sigma]|$ that for an automorphism $\sigma$ of $X$ which $C[\sigma]$-satisfies a CNF formula $F''$ over $\text{Var}(V)$, we can partition $C[\sigma]$ into $C_1, \ldots, C_{\ell}$ and the clauses of $F''$ into $F_1, \ldots, F_\ell$ such that $\sigma_{C_i}$ is a color-class-minimal automorphism of $X$ that $C_i$-satisfies $F_i$, for $1 \leq i \leq \ell$. If $\sigma$ itself is color-class-minimal, which always happens if $|C[\sigma]| = 1$, we are done: We can set $\ell = 1$, $C_1 = C[\sigma]$, and $F_1 = F''$. Otherwise there is a non-empty $D \subsetneq C[\sigma]$ such that $\varphi = \sigma_D \in \text{Aut}(X)$. This implies $\varphi' = \sigma \varphi^{-1} \in \text{Aut}(X)$. Note that $\varphi' = \sigma_{C[\sigma \setminus D]}$ and thus $C[\varphi'] = C[\sigma] \setminus D$. Next, we partition the clauses of $F''$ into two CNF formulas $F$ and $F'$: If a clause of $F''$ is $D$-satisfied by $\sigma$, we include it in $F$; otherwise we include it in $F'$. In the former case, this implies that this clause is also $D$-satisfied by $\varphi$, as $u^\sigma = u^\varphi$ for all $u \in \bigcup D$. In the latter case, the clause must be $(C[\sigma] \setminus D)$-satisfied by $\sigma$, and consequently also by $\varphi'$, as $u^\sigma = u'^\varphi$ for all $u \in \bigcup (C[\sigma] \setminus D)$. Thus $F$ is $C[\varphi]$-satisfied by $\varphi$, and $F'$ is $C[\varphi']$-satisfied by $\varphi'$; so we can apply the inductive hypothesis to both $\varphi$ and $\varphi'$. This yields partitions of $D$ and $F$ as well as $C[\sigma] \setminus D$ and $F'$, which we can combine to obtain the desired partitions of $C[\sigma]$ and $F''$.

To show the backward direction, let $C_1, \ldots, C_{\ell}$ be a partition of $C'$, let $F_0, \ldots, F_{\ell}$ be a partition of the clauses of $F$, and let $\sigma_1, \ldots, \sigma_\ell$ be color-preserving automorphisms of $X$ with $C[\sigma_i] = C_i$ such that $F_i$ is $C_i$-satisfied by $\sigma_i$ for $1 \leq i \leq \ell$, and $F_0$ is $(C \setminus C')$-satisfied by id. Consider the automorphism $\sigma = \sigma_1 \cdots \sigma_\ell$. As $C_i \cap C_j = \emptyset$ for $i \neq j$, the following definition of $\sigma$ is equivalent and well-defined:

$$u^\sigma = \begin{cases} u^{\sigma_i} & \text{if } \exists i \in \{1, \ldots, \ell\} : v \in \bigcup C_i \\ v & \text{otherwise} \end{cases}$$

Thus we have $\sigma_i = \sigma_{C_i}$ and $C[\sigma] = C'$. Moreover, any clause of $F$ is contained in some $F_i$. If $i > 0$, this clause is $C_i$-satisfied by $\sigma_i$, and thus also by $\sigma$, as $u^{\sigma_i} = u^\sigma$ for all $u \in \bigcup C_i$. It remains to consider the case $i = 0$. Then the clause is $(C \setminus C')$-satisfied by id and thus also by $\sigma$, as $u^\sigma = u$ for all $u \in \bigcup (C \setminus C')$. 

In [2] an algorithm is presented that, given a hypergraph $X$ on vertex set $V$ with $b$-bounded color classes and $k \in \mathbb{N}$, computes all color-class-minimal automorphisms of $X$ that have weight exactly $k$ in $O((kh)^{O(k^2)} \text{poly}(N))$ time. We use it as a building block for the following algorithm (see line 5).
Algorithm 1: ColorExactCNFGA\(_b(X,C,k,F)\)

1. **Input:** A hypergraph \(X = (V,E)\) with \(b\)-bounded color classes \(C = \{C_1,\ldots,C_m\}\), a parameter \(k \in \mathbb{N}\), and a CNF formula \(F\) over \(\text{Var}(V)\)
2. **Output:** A color-preserving automorphism \(\sigma\) of \(X\) with \(|\text{supp}(\sigma)| = k\) that satisfies \(F\), or \(\bot\) if none exists

\[
A_0 = \{\text{id}\}
\]

for \(i \in \{1,\ldots,k\}\) do

\[
A_i \leftarrow \{\sigma \in \text{Aut}(X) \mid \text{\(\sigma\) is color-class-minimal and has weight \(i\)\)}\} \quad // \text{see [2]}
\]

if there is an automorphism \(\sigma\) of \(X\) that has weight \(k\) and satisfies \(F\), or \(\bot\) if none exists

\[
\text{return } \sigma = \sigma_1 \cdots \sigma_\ell
\]

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**Theorem 3.3.** Given a hypergraph \(X = (V,E)\) with \(b\)-bounded color classes \(C\), a CNF formula \(F\) over \(\text{Var}(V)\), and \(k \in \mathbb{N}\), the algorithm ColorExactCNFGA\(_b(X,C,k,F)\) computes a color-preserving automorphism \(\sigma\) of \(X\) with weight \(k\) that satisfies \(F\) in \((kbl)^{O(k^2)k^{O(|F|)}}\) poly\((N)\) time (where \(N\) is the size of \(X\)), or determines that none exists.

**Proof.** If the algorithm returns \(\sigma = \sigma_1 \cdots \sigma_\ell\), we know \(\sigma_i \in A_{k_i}\) and \(\text{supp}(\sigma_i) \subseteq \bigcup(h' \circ h)^{-1}(i)\) and \(F_i\) is \(C[\sigma_i]\)-satisfied by \(\sigma_i\), and \(F_0\) is \((C \setminus \bigcup_{i=1}^\ell C[\sigma_i])\)-satisfied by id then

\[
\text{return } \sigma = \sigma_1 \cdots \sigma_\ell
\]

We next show that the algorithm does not return \(\bot\) if there is an automorphism \(\pi\) of \(X\) that has weight \(k\) and satisfies \(F\). By Lemma 3.2, we can partition \(C[\pi]\) into \(C_1,\ldots,C_\ell\) and the clauses of \(F\) into \(F_0,\ldots,F_\ell\) such that \(F_0\) is \((C \setminus C[\pi])\)-satisfied by id and, for \(1 \leq i \leq \ell\), the permutation \(\pi_i = \pi_{C_i}\) is a color-class-minimal automorphism of \(X\) that \(C[\pi_i]\)-satisfies \(F\). Now consider the iteration of the loop where \(h\) is injective on \(C[\pi]\); such an \(h\) must exist as it is chosen from a perfect hash family. Now let \(h': [k] \rightarrow [\ell]\) be a function with \(h'(h(C)) = i\) if \(C \in C[\pi_i]\); such an \(h'\) exists because \(h\) is injective on \(C[\pi]\). In the loop iterations where \(h'\) and the partition of \(F\) into \(F_0,\ldots,F_\ell\) is considered, the condition on line 10 is true (at least) with \(\sigma_i = \pi_i\), so the algorithm does not return \(\bot\).

Line 5 can be implemented by using the algorithm ColoredAut\(_{k,b}(X)\) from [2] which runs in \(O((kbl)^{O(k^2)}\text{poly}(N))\) time, and this also bounds \(|A_i|\). As \(|C| \leq n\), the perfect hash family \(\mathcal{H}_{C,b}\) has size \(2^{O(k)}\log n\), and can also be computed in this time. The inner loops take at most \(k^b\), \(k^k\) and \((k + 1)^{|F|}\) iterations, respectively. Together, this yields a runtime of \((kbl)^{O(k^2)k^{O(|F|)}}\text{poly}(N)\).

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4 Exact weight

In this section, we show that finding isomorphisms that have an exactly prescribed weight and satisfy a CNF formula is fixed parameter tractable. In fact, we show that this is true even for hypergraphs, when the maximum hyperedge size \(d\) is taken as additional parameter.
Exact-CNF-HGI: Given two hypergraphs $X_1 = (V, E_1)$ and $X_2 = (V, E_2)$ with hyperedge size bounded by $d$, a CNF formula $F$ over $\text{Var}(V)$, and $k \in \mathbb{N}$, decide whether there is an isomorphism from $X_1$ to $X_2$ of weight $k$ that satisfies $F$. The parameter is $|F| + k + d$.

Our approach is to reduce Exact-CNF-HGI to Exact-CNF-HGA (the analogous problem for automorphisms), which we solve first.

We require some permutation group theory definitions. Let $G \leq \text{Sym}(V)$ be a permutation group. The group $G$ partitions $V$ into orbits: $V = \Omega_1 \cup \Omega_2 \cdots \cup \Omega_r$. On each orbit $\Omega_i$, the group $G$ acts transitively. A subset $\Delta \subseteq \Omega_i$ is a block of the group $G$ if for all $\pi \in G$ either $\Delta^\pi = \Delta$ or $\Delta^\pi \cap \Delta = \emptyset$. Clearly, $\Omega_i$ is itself a block, and so are all singleton sets. These are trivial blocks. Other blocks are nontrivial. If $G$ has no nontrivial blocks it is primitive. If $G$ is not primitive, we can partition $\Omega_i$ into blocks $\Omega_i = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_r$, where each $\Delta_i$ is a maximal nontrivial block. Then the group $G$ acts primitively on the block system $\{\Delta_1, \Delta_2, \ldots, \Delta_r\}$. In this action, a permutation $\pi \in G$ maps $\Delta_i$ to $\Delta^\pi_i = \{u^\pi \mid u \in \Delta_i\}$.

The following two theorems are the main group-theoretic ingredients to our algorithms; they imply that every primitive group on a sufficiently large set contains the alternating group.

**Theorem 4.1** [5, Theorem 3.3A]. Suppose $G \leq \text{Sym}(V)$ is a primitive subgroup of $\text{Sym}(V)$. If $G$ contains an element $\pi$ such that $|\text{supp}(\pi)| = 3$ then $G$ contains the alternating group $\text{Alt}(V)$. If $G$ contains an element $\pi$ such that $|\text{supp}(\pi)| = 2$ then $G = \text{Sym}(V)$.

**Theorem 4.2** [5, Theorem 3.3D]. If $G \leq \text{Sym}(V)$ is primitive with $G \notin \{\text{Alt}(V), \text{Sym}(V)\}$ and contains an element $\pi$ such that $|\text{supp}(\pi)| = m$ (for some $m \geq 4$) then $|V| \leq (m - 1)^{2m}$.

The following lemma implies that the alternating group in a large orbit survives fixing vertices in a smaller orbit.

**Lemma 4.3.** Let $G \leq \text{Sym}(\Omega_1 \cup \Omega_2)$ be a permutation group such that $\Omega_1$ is an orbit of $G$, and $|\Omega_1| \geq 5$. Recall that $G(\Omega_1)$ denotes the image of $G$ under its action on $\Omega_1$. Suppose $G(\Omega_1) \in \{\text{Alt}(\Omega_1), \text{Sym}(\Omega_1)\}$ and $|G(\Omega_1)| > |G(\Omega_2)|$. Then for some subgroup $H$ of $G(\Omega_2)$, the group $G$ contains the product group $\text{Alt}(\Omega_1) \times H$. In particular, the pointwise stabilizer $G_{\Omega_2}$ contains the subgroup $\text{Alt}(\Omega_1) \times \{\text{id}\}$.

**Proof.** Let $p_2 : G \to G(\Omega_2)$ denote the surjective projection homomorphism. Then

$$\text{Ker}(p_2) = \{(x, 1) \mid (x, 1) \in G\}.$$ 

Let $K = \{x \mid (x, 1) \in \text{Ker}(p_2)\}$. It is easily checked that $K$ is a normal subgroup of $G(\Omega_1)$. As $G(\Omega_1) \in \{\text{Alt}(\Omega_1), \text{Sym}(\Omega_1)\}$, $\text{Alt}(\Omega_1)$ is simple, and the only nontrivial normal subgroup of $\text{Sym}(\Omega_1)$ is $\text{Alt}(\Omega_1)$, it follows that either $K = \{1\}$ or $\text{Alt}(\Omega_1) \leq K$.

**Case 1.** Suppose $K = \{1\}$. In this case $\text{Ker}(p_2)$ is trivial. Thus, $p_2$ is an isomorphism from $G$ to $G(\Omega_2)$ implying that $|G| = |G(\Omega_2)|$. As $|G| \geq |G(\Omega_1)|$, this contradicts the assumption that $|G(\Omega_2)| < |G(\Omega_1)|$.

**Case 2.** Suppose $\text{Alt}(\Omega_1) \leq K$. Consider the other surjective projection homomorphism $p_1 : G \to G(\Omega_1)$. Then $\text{Ker}(p_1) = \{(1, y) \mid (1, y) \in G\}$, and $H = \{y \mid (1, y) \in \text{Ker}(p_1)\}$ is a normal subgroup of $G(\Omega_2)$. We show that $G$ contains the product group $\text{Alt}(\Omega_1) \times H$ as claimed by the lemma.

Consider any pair $(x, y) \in \text{Alt}(\Omega_1) \times H$. We can write it as $(x, y) = (x, 1) \cdot (1, y)$, and note that by definition $(x, 1) \in \text{Ker}(p_2)$ and $(1, y) \in \text{Ker}(p_1)$. As both $\text{Ker}(p_1)$ and $\text{Ker}(p_2)$ are subgroups of $G$, it follows that $(x, y) \in G$. 

\[\square\]
Remark 4.4. In a special case of Lemma 4.3, suppose \( G \leq \text{Sym}(\Omega_1 \cup \Omega_2) \) such that \( \Omega_1 \) is an orbit of \( G \), \( |\Omega_1| \geq \max\{5, |\Omega_2| + 1\} \), and \( \text{Alt}(\Omega_1) \leq G(\Omega_1) \). As \( |G(\Omega_1)| > |G(\Omega_2)| \) is implied by this assumption, the consequence of the lemma follows.

The effect of fixing vertices of some orbit on other orbits of the same size depends on how the group relates these orbits to each other.

Definition 4.5. Two orbits \( \Omega_1 \) and \( \Omega_2 \) of a permutation group \( G \leq \text{Sym}(\Omega) \) are linked if there is a group isomorphism \( \sigma: G(\Omega_1) \to G(\Omega_2) \) with \( G(\Omega_1 \cup \Omega_2) = \{ (\varphi, \sigma(\varphi)) \mid \varphi \in G(\Omega_1) \} \). (This happens if and only if both \( G(\Omega_1) \) and \( G(\Omega_2) \) are isomorphic to \( G(\Omega_1 \cup \Omega_2) \).)

We next show that two large orbits where the group action includes the alternating group are (nearly) independent unless they are linked.

Lemma 4.6. Suppose \( G \leq \text{Sym}(\Omega) \) where \( \Omega = \Omega_1 \cup \Omega_2 \) is its orbit partition such that \( |\Omega_i| \geq 5 \) and \( G(\Omega_i) \in \{ \text{Alt}(\Omega_i), \text{Sym}(\Omega_i) \} \) for \( i = 1, 2 \). Then either \( \Omega_1 \) and \( \Omega_2 \) are linked in \( G \), or \( G \) contains \( \text{Alt}(\Omega_1) \times \text{Alt}(\Omega_2) \).

Proof. For \( i = 1, 2 \), let \( p_i: G \to G(\Omega_i) \) denote the surjective projection homomorphisms. Further, let \( H = \{ x \mid (1, x) \in \ker(p_1) \} \) and \( K = \{ x \mid (x, 1) \in \ker(p_2) \} \). It is easily checked that \( H \) is a normal subgroup of \( G(\Omega_2) \). Therefore, \( H \) is either \( G(\Omega_2) \) or \{id\} or \( \text{Alt}(\Omega_2) \) (note: the last case coincides with the first if \( G(\Omega_2) = \text{Alt}(\Omega_2) \)). Similarly, \( K \) is a normal subgroup of \( G(\Omega_1) \) and thus either \( G(\Omega_1) \) or \{id\} or \( \text{Alt}(\Omega_1) \).

Case 1: \( H = \{\text{id}\} \) (the case \( K = \{\text{id}\} \) is symmetric). Then \( \ker(p_1) \) is trivial, and \( p_1 \) is an isomorphism from \( G \) to \( G(\Omega_1) \), implying that \( |G| = |G(\Omega_1)| \). By the basic isomorphism theorem, we have \( \frac{G}{\ker(p_2)} \cong G(\Omega_2) \), and hence \( |G| = |\ker(p_2)| \cdot |G(\Omega_2)| \). But \( G \) is isomorphic to \( G(\Omega_1) \) and hence has only three possible normal subgroups: isomorphic to \( G(\Omega_1) \), isomorphic to \( \text{Alt}(\Omega_1) \), or isomorphic to \{id\}. In the first two cases, \( |\ker(p_2)| \geq |\text{Alt}(\Omega_2)| \geq |G(\Omega_1)|/2 \). Hence, \( |G| > |G(\Omega_1)| \); a contradiction. Thus, \( \ker(p_2) = \{\text{id}\} \), which implies that \( \Omega_1 \) and \( \Omega_2 \) are linked in \( G \).

Case 2: \( H = G(\Omega_2) \) (the case \( K = G(\Omega_1) \) is symmetric). Consider any pair \( (y, x) \in G(\Omega_1) \times G(\Omega_2) \). Since \( y \in G(\Omega_1) \) there is a \( z \in G(\Omega_2) \) such that \( (y, z) \in G \). Now, we can writes \( (y, z)(1, z^{-1}x) \) and note that \( (y, z) \in G \) and \( (1, z^{-1}x) \in \ker(p_1) \subseteq G \) by assumption on \( H \). Therefore, \( (y, x) \in G \) implying that \( G = G(\Omega_1) \times G(\Omega_2) \), and thus \( \text{Alt}(\Omega_1) \times \text{Alt}(\Omega_2) \leq G \).

Case 3. Finally, we are left with the possibility that \( G(\Omega_1) = \text{Sym}(\Omega_1) \), \( G(\Omega_2) = \text{Sym}(\Omega_2) \), \( H = \text{Alt}(\Omega_2) \) and \( K = \text{Alt}(\Omega_1) \). In this case \( G \) contains \( \text{Alt}(\Omega_1) \times \text{Alt}(\Omega_2) \).

The last ingredient for our algorithm is that when there are two linked orbits where the group action includes the alternating group, fixing a vertex in one orbit is equivalent to fixing some vertex of the other orbit.

Lemma 4.7 [5, Theorem 5.2A]. Let \( n = |V| > 9 \). Suppose \( G \) is a subgroup of \( \text{Alt}(V) \) of index strictly less than \( \binom{n}{2} \). Then, for some point \( u \in V \), the group \( G \) is the pointwise stabilizer subgroup \( \text{Alt}(V)_u \).

Corollary 4.8. Let \( \Omega_1 \) and \( \Omega_2 \) be two linked orbits of a permutation group \( G \leq \text{Sym}(V) \) with \( \text{Alt}(\Omega_1) \leq G(\Omega_1) \) and \( |\Omega_1| = |\Omega_2| > 9 \). Then for each \( u \in \Omega_1 \) there is a \( v \in \Omega_2 \) such that \( G_u = G_v \).
Proof. Let $\sigma : G(\Omega_1) \to G(\Omega_2)$ be the group isomorphism which witnesses that $\Omega_1$ and $\Omega_2$ are linked. As $G(\Omega_1) \in \{ \text{Alt}(\Omega_1), \text{Sym}(\Omega_1) \}$, the index of $G_\Omega(\Omega_1)$ in $G(\Omega_1)$ is $n!/(n-1)! = n$. As $\sigma$ is a group isomorphism, the index of $\sigma(G_\Omega(\Omega_1))$ in $\sigma(G(\Omega_1)) = G(\Omega_2)$ is also $n$. Thus Lemma 4.7 implies that there is $v \in \Omega_2$ such that $G_\Omega(\Omega_2) = \sigma(G_\Omega(\Omega_1))$. As this implies $\sigma^{-1}(G_\Omega(\Omega_2)) = G_\Omega(\Omega_1)$, it follows that $G_\Omega = G_v$. \qed

Algorithm 2: Exact-CNF-HGA$_d(X, k, F)$

\begin{verbatim}
1 Input: A hypergraph $X$ with hyperedge size bounded by $d$, a parameter $k$ and a formula $F$
2 Output: An automorphism $\sigma$ of $X$ with $|\text{supp}(\sigma)| = k$ that satisfies $F$, or $\perp$ if none exists
3 $G \leftarrow \langle \{ \sigma \in \text{Aut}(X) \mid \sigma \text{ has minimal complexity in } \text{Aut}(X) \text{ and } |\text{supp}(\sigma)| \leq k \} \rangle$
4 // see [2, Algorithm 3]
5 while $G$ contains an orbit of size more than $\frac{k}{2} \cdot \max\{(k-1)^2k, |T|+k, 9\}$ do
6     repeat
7         $\mathcal{O} \leftarrow$ the set of all $G$-orbits
8         for $\Omega \in \mathcal{O}$ do
9             $B(\Omega) \leftarrow$ a maximal block system of $\Omega$ in $G$
10            if $\exists \Delta \in B(\Omega) : |\Delta| > \frac{k}{2}$ or $|B(\Omega)| > (k-1)^2k$ and $\text{Alt}(B(\Omega)) \not\subseteq G(B(\Omega))$ then
11                $G \leftarrow G_{B(\Omega)} // the \text{ setwise stabilizer of all } \Delta \in B(\Omega)$
12            until $G$ remains unchanged
13        choose $\Omega_{\text{max}} \in \mathcal{O}$ such that $|B(\Omega_{\text{max}})| \geq |B(\Omega)|$ for all $\Omega \in \mathcal{O}$
14        if $|B(\Omega_{\text{max}})| > \max\{(k-1)^2k, |T|+k, 9\}$ then
15            $H \leftarrow G_{|T|} // the \text{ pointwise stabilizer of } T$
16            $\Omega_H \leftarrow$ the largest $H$-orbit that is contained in $\Omega_{\text{max}}$
17            $B_H \leftarrow \{ \Delta \in B(\Omega_{\text{max}}) \mid \Delta \subseteq \Omega_H \}$
18            choose $\Delta \in B_H$
19            $G \leftarrow G_{\{\Delta\}} // the \text{ setwise stabilizer of } \Delta$
20        $b \leftarrow \frac{k}{2} \cdot \max\{(k-1)^2k, |T|+k, 9\}$
21        $\mathcal{O} \leftarrow$ the set of all $G$-orbits
22 return ColorExactCNFGA$_d(X, \mathcal{O}, k, F)$ // see Algorithm 1
\end{verbatim}

Theorem 4.9. Algorithm 2 solves Exact-CNF-HGA in time $(d(k^k + |F|)!)^{O(k^2)} \text{poly}(N)$.

Proof. Suppose there is some $\pi \in \text{Aut}(X)$ of weight exactly $k$ that satisfies $F$.

By Lemma 2.1, the automorphism $\pi$ can be decomposed as a product of minimal-complexity automorphisms of weight at most $k$, which implies $\pi \in G$.

We will show that whenever the algorithm shrinks $G$, some weight $k$ automorphism of $X$ that satisfies $F$ survives. For the shrinking in line 11 we need to consider two cases. If $\Omega$ is an orbit with $|\Delta| > k/2$ for some (and thus all) $\Delta \in B(\Omega)$, then none of these blocks are moved by $\pi$. Indeed, if $\pi$ would move one block, it would have to move at least one further block, contradicting $|\text{supp}(\pi)| = k$. On the other hand, if $|B(\Omega)| > (k-1)^2k$ and $G(B(\Omega))$ does not contain the alternating group, then Theorems 4.1 and 4.2 imply that the primitive group $G(B(\Omega))$ contains no nontrivial element that moves at most $k$ elements of $B(\Omega)$. In particular, $\pi$ setwise stabilizes all $\Delta \in B(\Omega)$ and thus survives the shrinking.

We now turn to the other shrinking of $G$, which occurs in line 19. Note that this can only happen if $|B(\Omega_{\text{max}})| > (k-1)^2k$ because of the if-condition on line 14. This implies, as the
last execution of the repeat-loop resulted in no further shrinking of $G$, that $\text{Alt}(B(\Omega_{\max})) \leq G(B(\Omega_{\max}))$. Let $T = \bigcup_{\Omega \in \mathcal{O}} \{\Delta \in B(\Omega) \mid \Delta \cap T \neq \emptyset\}$ be the set of all blocks with vertices from $T$ and let $R = G_T$ be the setwise stabilizer of these blocks. Note that $H \leq R \leq G$. We next show that a sufficiently large part of $\text{Alt}(B(\Omega_{\max}))$ survives in $R$.

**Claim.** Let $\Omega_R$ be the largest orbit of $R$ that is contained in $\Omega_{\max}$. Then the set $B_R = \{\Delta \in B(\Omega_{\max}) \mid \Delta \subseteq \Omega_R\}$ is a maximal block system for the orbit $\Omega_R$ of $R$. Moreover, $|B_R| > k$ and $\text{Alt}(B_R) \leq R(B_R)$.

Let $\mathcal{O} = \{\Omega_1, \ldots, \Omega_k, \ldots, \Omega_l\}$ be an enumeration of the orbits of $G$ such that $\Omega_i$ is linked to $\Omega_{\max}$ if and only if $i > k$. Consider the sequence of subgroups $R = R^{(i)} \leq \cdots \leq R^{(0)} = G$, where $R^{(i)}$ is the subgroup of $R^{(i-1)}$ that setwise stabilizes all $\Delta \in T$ with $\Delta \subseteq \Omega_i$. As first step, we inductively show for $i \leq k$ that all orbits linked to $\Omega_{\max}$ in $G$ (including itself) remain orbits of $R^{(i)}$, and that $\text{Alt}(B(\Omega)) \leq R^{(i)}(B(\Omega))$ for all $\Omega \in \mathcal{O}$ with $|B(\Omega)| = |B(\Omega_{\max})|$ that are still an orbit of $R^{(i)}$.

**Case 1:** Suppose that $|B(\Omega_{\max})| > |B(\Omega_i)|$. Consider any $\Omega \in \mathcal{O}$ with $|B(\Omega)| = |B(\Omega_{\max})|$ that is an orbit of $R^{(i-1)}$; this includes all orbits linked to $\Omega_{\max}$ in $G$. By the induction hypothesis, we know $\text{Alt}(B(\Omega)) \leq R^{(i-1)}(B(\Omega))$, which implies $|R^{(i-1)}(B(\Omega))| > \left|R^{(i-1)}(B(\Omega_{\max}))\right|$. Thus we can apply Lemma 4.3 to $R^{(i-1)}(B(\Omega) \times B(\Omega_i))$. This gives us $\text{Alt}(B(\Omega)) \times |\Omega| \leq R^{(i-1)}(B(\Omega) \times B(\Omega_i))$. Thus fixing some blocks of $\Omega_i$ in $R^{(i-1)}$ preserves the alternating group $\text{Alt}(B(\Omega))$ in $R^{(i)}(B(\Omega))$. In particular, $\Omega$ is also an orbit of $R^{(i)}$.

**Case 2:** Suppose that $|B(\Omega_{\max})| = |B(\Omega_i)|$ and that $\Omega_i$ is no longer an orbit of $R^{(i-1)}$. This again implies $\left|R^{(i-1)}(B(\Omega))\right| > \left|R^{(i-1)}(B(\Omega_i))\right|$ for the orbits $\Omega \in \mathcal{O}$ we need to consider, and we can proceed as in case 1.

**Case 3:** Now suppose that $|B(\Omega_{\max})| = |B(\Omega_i)|$ and that $\Omega_i$ is still an orbit of $R^{(i)}$. Consider any $\Omega \in \mathcal{O}$ with $|B(\Omega)| = |B(\Omega_{\max})|$ that is still an orbit of $R^{(i-1)}$. If $\Omega$ is not linked to $\Omega_i$ (this includes $\Omega_{\max}$ and all orbits linked to the latter), then Lemma 4.6 implies $\text{Alt}(B(\Omega)) \times \text{Alt}(B(\Omega_i)) \leq G(B(\Omega) \times B(\Omega_i))$. Thus $\text{Alt}(\Omega) \leq R^{(i)}(B(\Omega))$. On the other hand, if $\Omega$ and $\Omega_i$ are linked in $G$ (and thus also in $R^{(i-1)}$), then Corollary 4.8 implies that setwise stabilizing $\Delta \subseteq \Omega$ is equivalent to stabilizing a block in the orbit $\Omega$, which thus is no longer an orbit of $R^{(i)}$.

Applying Corollary 4.8 repeatedly to $R^{(k)}(\times_{\ell=k+1} B(\Omega_i))$, we can obtain a set $T' \subseteq B(\Omega_{\max})$ with $R = R^{(k)}_{T'}$ and $|T'| \leq |T|$. Moreover, $\text{Alt}(B(\Omega_{\max})) \leq R^{(k)}(B(\Omega_{\max}))$ implies $\text{Alt}(\mathcal{S}) \leq R(\mathcal{S})$ for $\mathcal{S} = B(\Omega_{\max}) \setminus T'$. Note that $|\mathcal{S}| \geq |B(\Omega_{\max})| - |T| > k$. Thus $\Omega_R = \bigcup \mathcal{S}$ is the largest orbit of $R$ that is contained in $\Omega_{\max}$ and $\mathcal{B}_R = \mathcal{S}$, proving the claim.

**Claim.** $B_H$ is a maximal block system for the orbit $\Omega_H$ in $H$. Moreover, $|B_H| > k$ and $\text{Alt}(B_H) \leq H(B_H)$.

Let $\mathcal{T} = \{\Delta_1, \ldots, \Delta_m\}$ be an enumeration of the blocks with vertices from $T$. Consider the sequence of subgroups $H = H^{(m)} \leq \cdots \leq H^{(0)} = R$, where $H^{(i)} = H^{(i-1)}_{T \cap \Delta_i}$. As $|B_R| > k > |\Delta_i|$, Lemma 4.3 can be applied to $H^{(i)}(B_R \times \Delta_i)$. It follows that $\text{Alt}(B_R) \leq H^{(i)}(B_R)$. Thus we get $\Omega_H = \Omega_R$ and $B_H = B_R$, and the claim is shown.

The following claim concludes the correctness proof.
Claim. Let \( G \) and \( \Delta \) be as in the algorithm on line 18. Then for any \( \pi \in G_{\Delta} \) of weight \( k \) that satisfies \( F \), there is a \( \pi' \in G_{\Delta} \) of weight \( k \) that satisfies \( F \).

Choose \( \Delta' \in B_H \) with \( \Delta' \cap \text{supp}(\pi) = \emptyset \); this is possible because \( |B_H| > k \) and \( |\text{supp}(\pi)| \leq k \). As \( \text{Alt}(B_H) \leq H(B_H) \), there is a \( \rho \in H \) with \( \rho(\Delta) = \Delta' \). Thus \( \pi' = \rho \pi \rho^{-1} \) is in \( G_{\Delta} \). Clearly, conjugation preserves weight. Further, as \( \rho \in H \) implies for all \( v \in T \) that \( \rho(v) = v \) and thus \( \pi'(v) = \pi(v) \), we get that \( \pi' \) satisfies \( F \). This proves the claim.

Computing \( G \) on line 4 takes \( (dk)^{O(k^2)} \text{poly}(N) \) time by [2, Theorem 3.9]. On line 22, the call to ColorExactCNFGA takes \( (khl)^{O(k^2)k^{O(|F|)}} \text{poly}(N) \) time by Theorem 3.3. Each iteration of the while loop increases the number of orbits. The same is true for all except the last iteration of the repeat loop, and we can attribute the time of its last iteration to the containing while loop. Thus the number of iterations is bounded by \( n = |V| \). As all operations in the loops can be implemented in \( \text{poly}(n) \) time, this shows the claimed time bound of \( (d(k^k + |F|!) )^{O(k^2)} \text{poly}(N) \).

Now we are ready to turn to Exact-CNF-HGI. Our algorithm uses the following transformation on formulas. Given a formula \( F \) over \( \text{Var}(V) \) and \( \psi \in \text{Sym}(V) \), let \( \psi(F) \) denote the formula obtained from \( F \) by replacing each variable \( x_{uv} \) by \( x_{u\psi(v)} \).

**Lemma 4.10.** A product \( \sigma = \varphi \pi \in \text{Sym}(V) \) satisfies a formula \( F \) over \( \text{Var}(V) \) if and only if \( \varphi \) satisfies \( \pi^{-1}(F) \).

**Proof.** By definition, \( \sigma \) satisfies \( x_{uv} \) if and only if \( \sigma(u) = v \), which is equivalent to \( \varphi(u) = \pi^{-1}(v) \), i.e., to \( \varphi \) satisfying \( \pi^{-1}(x_{uv}) \).

Algorithm 3: Exact-CNF-HGI\(_d\)(\( X_1, X_2, k, F \))

1. **Input:** Two hypergraphs \( X_1 \) and \( X_2 \) on vertex set \( V \) with hyperedge size bounded by \( d \), a parameter \( k \in \mathbb{N} \) and a CNF formula \( F \) over \( \text{Var}(V) \)
2. **Output:** An isomorphism \( \sigma \) from \( X_1 \) to \( X_2 \) with \( |\text{supp}(\sigma)| = k \) that satisfies \( F \), or \( \bot \) if none exists
3. \( \pi \leftarrow \text{some isomorphism from } X_1 \text{ to } X_2 \text{ with } |\text{supp}(\pi)| \leq k \) // see [2, Theorem 3.8]
4. for \( U \subseteq \text{supp}(\pi) \) do // we will force \( u \notin \text{supp}(\varphi \pi) \) for \( u \in U \)
5. \( M \subseteq \text{supp}(\pi) \setminus U \) do // we will force \( u \notin \text{supp}(\varphi) \cap \text{supp}(\varphi \pi) \) for \( u \in M \)
6. \( I \leftarrow \text{supp}(\pi) \setminus (U \cup M) \) // we will force \( u \notin \text{supp}(\varphi) \) for \( u \in I \)
7. \( F' \leftarrow \pi^{-1}(F) \land \bigwedge_{u \in U} x_{u\pi^{-1}(u)} \land \bigwedge_{u \in M} (\neg x_{u\pi^{-1}(u)} \land \neg x_{u,u}) \land \bigwedge_{u \in I} x_{uu} \)
8. \( k' \leftarrow k - |I| + |U| \)
9. \( \varphi \leftarrow \text{Exact-CNF-HGA}(X_1, k', F') \) // see Algorithm 2
10. if \( \varphi \neq \bot \) then return \( \sigma = \varphi \pi \)
11. return \( \bot \)

**Theorem 4.11.** Algorithm 3 solves Exact-CNF-HGI in time \( (d(k^k + |F|!))^{O(k^2)} \text{poly}(N) \).

**Proof.** Suppose Algorithm 3 returns a permutation \( \sigma = \varphi \pi \). Then \( \pi \) is an isomorphism from \( X_1 \) to \( X_2 \) and \( \varphi \) is an automorphism of \( X_1 \) that satisfies \( F' \) and has weight \( k' \). As \( \varphi \) satisfies \( \pi^{-1}(F) \), Lemma 4.10 implies that \( \sigma \) satisfies \( F \). The additional literals in \( F' \) ensure \( |\text{supp}(\sigma)| = (|\text{supp}(\varphi)| \setminus U) \cup I \) and thus \( |\text{supp}(\sigma)| = k' - |U| + |I| = k \).

Now suppose there is an isomorphism \( \sigma \) from \( X_1 \) to \( X_2 \) that satisfies \( F \) and has weight \( k \); we need to show that the algorithm does not return \( \bot \) in this case. Let \( \pi \) be the isomorphism
computed on line 3. Then \( \varphi = \sigma \pi^{-1} \) is an automorphism of \( X_1 \); it satisfies \( \pi^{-1}(F) \) by Lemma 4.10. In the iteration of the loops where \( U = \{ u \in \text{supp}(\pi) \cap \text{supp}(\varphi) \mid u^{\sigma \pi} = u \} \) and \( M = (\text{supp}(\pi) \cap \text{supp}(\varphi)) \setminus U \), it holds that \( \varphi \) has weight \( k' \) and satisfies \( F' \). Thus \( \text{Exact-CNF-HGI}_d(X_1, k', F') \) does not return \( \bot \) in this iteration, and \( \text{Exact-CNF-HGI} \) does not return \( \bot \) either.

The isomorphism \( \pi \) can be found in \( (dk)^{O(k^2)} \text{poly}(N) \) time [2, Theorem 3.8]. The loops have at most \( 3k \) iterations, and \( \text{Exact-CNF-HGA}_d \) takes \( (d(k^k + |F|)!)^{O(k^2)} \text{poly}(N) \) time. The latter term thus bounds the overall runtime. □

5 Constrained isomorphisms with arbitrary weight

In this section, we show that finding graph isomorphisms with constraints and without weight restrictions is in \( \text{FPT}^\text{GI} \).

\( \text{CNF-HGI} \): Given two hypergraphs \( X_1 = (V, E_1) \) and \( X_2 = (V, E_2) \), and a CNF formula \( F \) over \( \text{Var}(V) \), decide whether there is an isomorphism from \( X_1 \) to \( X_2 \) that satisfies \( F \). The parameter is \( |F| \).

Let \( T \subseteq \text{Var}(V) \) be the variables that occur in the given formula \( F \). Our approach is to enumerate satisfying assignments to \( T \). We are only interested in assignments \( \alpha : T \rightarrow \{0, 1\} \) that are the restriction of the assignment given by some \( \sigma \in \text{Sym}(V) \), i.e., for all \( u \in V \) it holds that \( \sum_{x \in \{u,v\} \subseteq T} \alpha(x, u, v) \leq 1 \) and this sum is 1 if \( \{x_u, v \mid v \in T \} \subseteq T \). We call an assignment to \( T \) that satisfies these conditions a \text{partial permutation assignment}.

When a partial permutation assignment \( \alpha \) has \( \alpha(x_{u,v}) = 1 \), this can be easily encoded into the graph isomorphism instance using additional colors; we call the resulting graphs \( X' \) and \( Y' \). The challenge is to enforce that a permutation complies with \( \alpha(x_{u,v}) = 0 \). In the following algorithm, we use the inclusion-exclusion principle to count isomorphisms that avoid the set \( P = \{(u_1, v_1), \ldots, (u_k, v_k)\} \) of forbidden pairs given by \( \alpha \). For \( S \subseteq [k] \), we compute the size \( n_S \) of the set

\[
I_S = \{ \sigma \in \text{Iso}(X', Y') \mid \forall i \in S : u_i^\sigma = v_i \}.
\]

To do so, we encode the additional forced mappings using additional colors and use the GI oracle to decide whether the resulting graphs \( X_S \) and \( Y_S \) are isomorphic (otherwise we have \( n_S = 0 \), and if so, to compute a generating set for the automorphism group of \( X_S \), whose size then gives \( n_S \). Note that there is an isomorphism compatible with \( \alpha \) if and only if

\[
I_S \supseteq \bigcup_{i \in [k]} I_{\{i\}}.
\]

By the inclusion-exclusion principle, the size of this union can be computed as

\[
\left| \bigcup_{i \in [k]} I_{\{i\}} \right| = \sum_{\emptyset \neq S \subseteq [k]} (-1)^{|S|+1} \cdot \left| \bigcap_{i \in S} I_{\{i\}} \right| = \sum_{\emptyset \neq S \subseteq [k]} (-1)^{|S|+1} \cdot |I_S|.
\]

This solves the decision version of \( \text{CNF-HGI} \). To solve the search version, we check if a tentative mapping \( u \mapsto v \) leads to a solution by intersecting both sides of \( (*) \) with \( \{ \sigma \in \text{Sym}(V) \mid u^\sigma = v \} \); this restriction can again be encoded using additional colors in the oracle queries. The resulting condition can be decided using the inclusion-exclusion principle once again.
Theorem 5.1. Algorithm 4 solves CNF-HGI in $2^{2|F|}\text{poly}(N)$ time when given access to a GI oracle; the oracle queries have size poly($N$).

Proof. The correctness follows from the observations above. Regarding the time bound, the outer for-loop incurs a cost of $2^{|T|} \leq 2^{|F|}$. The for-loops over $S \subseteq [k]$ and the sums are not nested and contribute factor of $2^{|F|} \leq 2^{|F|}$. The remaining loops and operations are polynomial in the input size when given a GI oracle.

6 Exact complexity

The complexity of a permutation $\pi \in \text{Sym}(V)$ can be bounded by functions of its weight: $|\text{supp}(\pi)| - 1 \leq \text{compl}(\pi) \leq 2 \cdot |\text{supp}(\pi)|$. However, there is no direct functional dependence between these two parameters. And while the algorithms of Sections 3 and 4 can be modified to find isomorphisms of exactly prescribed complexity, we give an independent and more efficient algorithm in this section.

The main ingredient is an analysis of decompositions $\sigma = \sigma_1 \cdots \sigma_\ell$ of $\sigma \in \text{Sym}(V)$ into $\sigma_i \in \text{Sym}(V) \setminus \{\text{id}\}$ (for $1 \leq i \leq \ell$) with $\text{compl}(\sigma) = \sum_{i=1}^\ell \text{compl}(\sigma_i)$; we call such decompositions complexity-additive. For example, the decomposition into complexity-minimal permutations provided by Lemma 2.1 is complexity-additive.

For a sequence of permutations $\sigma_1, \ldots, \sigma_\ell \in \text{Sym}(V)$, its cycle graph $\text{CG}(\sigma_1, \ldots, \sigma_\ell)$ is the incidence graph between $\bigcup_{i=1}^\ell \text{supp}(\sigma_i)$ and the $\sigma_i$-orbits of size at least 2, i.e., the cycles of $\sigma_i$, Algorithm 4: CNF-HGI($X, Y, F$)

1. **Input:** Two hypergraphs $X$ and $Y$ on vertex set $V$ and a CNF formula $F$ over Var($V$)
   2. **Output:** An isomorphism $\sigma$ from $X$ to $Y$ that satisfies $F$, or $\bot$ if none exists
   3. $T \leftarrow$ the variables in $F$
   4. for each partial permutation assignment $\alpha : T \to \{0, 1\}$ do // at most $2^{|T|} \leq 2^{|F|}$ iterations
      5. $X' \leftarrow X$; $Y' \leftarrow X$; $P \leftarrow \emptyset$
      6. for $x_u, v \in T$ do
         7. if $\alpha(x_u, v) = 1$ then
            8. give color $u$ to $u$ in $X'$ and to $v$ in $Y'$
         9. else if $\exists v' \in V : x_u, v' \in T \land \alpha(x_u, v') = 1$ then
            10. $P \leftarrow P \cup \{(u, v)\}$ // forbidden pair
      11. let $\{(u_1, v_1), \ldots, (u_k, v_k)\} = P$
      12. for $S \subseteq [k]$ do // $2^{|F|} \leq 2^{|F|}$ iterations
         13. $n_S \leftarrow |\{\sigma \in \text{Iso}(X', Y') \mid \forall i \in S : u_i^\sigma = v_i\}|$ // use the GI oracle and additional colors
      14. if $n_S > \sum_{S \subseteq [k]} (-1)^{|S|+1} \cdot n_S$ then // there is a solution that is compatible with $\alpha$
         15. while there is a vertex $u$ in $X'$ whose color is not unique do
         16. for each vertex $v$ in $Y'$ that has the same color as $u$ in $X'$ do
            17. $n_{u,v,S} \leftarrow |\{\sigma \in \text{Iso}(X', Y') \mid u^\sigma = v \land \forall i \in S : u_i^\sigma = v_i\}|$
      18. if $n_{u,v,S} > \sum_{S \subseteq [k]} (-1)^{|S|+1} \cdot n_{u,v,S}$ then
         19. give color $u$ to $u$ in $X'$ and to $v$ in $Y'$
      20. continue with the next iteration of the while-loop
      21. return the unique color-preserving isomorphism from $X'$ to $Y'$
   22. return $\bot$
for 1 ≤ i ≤ ℓ. We call the former primal vertices and the latter cycle-vertices.

**Lemma 6.1.** Let \( σ ∈ \text{Sym}(V) \) and let \( σ = σ_1 ⋯ σ_ℓ \) be a complexity-additive decomposition. Then \( CG(σ_1, ⋯, σ_ℓ) \) is a forest.

**Proof.** If any of the permutations \( σ_i \) has more than one cycle, we further decompose it into its cycles. Note that this does not change the cycle graph. For the rest of this proof, we assume that each permutation \( σ_i \) has a single nontrivial orbit \( Ω_i \).

For the sake of contradiction, we assume that the graph \( CG(σ_1, ⋯, σ_ℓ) \) is no forest. Let \( ℓ' \) be the smallest index such that \( CG(σ_1, ⋯, σ_{ℓ'} ) \) is no forest. We first consider the case \( ℓ' = 2 \), i.e., that \( Ω_1 \) and \( Ω_2 \) have \( j ≥ 2 \) points in common. Note that \( \text{compl}(σ) = \sum_{i=1}^j \text{compl}(σ_i) \) implies \( \text{compl}(σ_1σ_2) = \text{compl}(σ_1) + \text{compl}(σ_2) \). By the definition of complexity, we get

\[
\text{compl}(σ_1) + \text{compl}(σ_2) = |Ω_1| - 1 + |Ω_2| - 1 \quad \text{and} \quad \text{compl}(σ_1σ_2) = |Ω_1| + |Ω_2| - j - c,
\]

where \( c \) is the number of cycles of \( σ_1σ_2 \). This yields a contradiction, as \( j ≥ 2 \) and \( c ≥ 1 \).

If \( ℓ' > 2 \), we know that \( X = CG(σ_1, ⋯, σ_{ℓ'-1}) \) contains a path between two points in \( Ω_{ℓ'} \); let us call them \( u \) and \( v \). As \( X \) is a forest, \( u \) and \( v \) are in the same cycle of \( ϕ = σ_1 ⋯ σ_{ℓ'-1} \). (This follows by induction on the size of the connected component, as the product of two cycles that share a single point is again a cycle.) Let \( ϕ = ϕ_1 ⋯ ϕ_j \) be a decomposition of \( ϕ \) into its cycles such that \( u, v ∈ \text{supp}(ϕ_i) \), and let \( ϕ_i' = σ_i^{-1}ϕ_iσ_i' \). Then we get the decomposition

\[
σ = ϕ_1σ_1ϕ_2 ⋯ ϕ_jσ_{j+1} ⋯ σ_ℓ. \tag{*}
\]

Note that each factor of this decomposition is a single cycle, because conjugation preserves cycle structure. As \( \sum_{i=1}^j \text{compl}(ϕ_i) = \text{compl}(σ_1 ⋯ σ_{ℓ'-1}) \) and \( \text{compl}(ϕ_i') = \text{compl}(ϕ_i) \), the decomposition \((*)\) is complexity-additive. Moreover, the graph \( CG(ϕ_1, σ_{ℓ'}) \) is no forest, as \( u, v ∈ \text{supp}(ϕ_1) \cap \text{supp}(σ_{ℓ'}) \). Thus we get the same contradiction as in the case \( ℓ' = 2 \). \( \square \)

Given a complexity-additive decomposition \( σ = σ_1 ⋯ σ_ℓ \) of a permutation \( σ ∈ \text{Sym}(V) \) and a coloring \( c : V → [k] \), the colored cycle graph \( CG_c(σ_1, ⋯, σ_ℓ) \) is obtained from the cycle graph \( CG(σ_1, ⋯, σ_ℓ) \) by coloring each primal vertex \( v ∈ V \) by \( c(v) \), and coloring each cycle-vertex that corresponds to a cycle of \( σ_i \) by \( i \). (Note that a vertex of this graph is a cycle-vertex if and only if it has odd distance to some leaf.) See Figure 1 for an example.

A cycle pattern \( P \) is a colored cycle graph \( CG_c(σ_1, ⋯, σ_ℓ) \) where all primal vertices have different colors. A complexity-additive decomposition \( σ' = σ'_1 ⋯ σ'_ℓ \) of a permutation \( σ' ∈ \text{Sym}(V) \) matches the cycle pattern \( P \) if there is a color-preserving isomorphism \( ϕ \) from \( CG_c(σ'_1, ⋯, σ'_ℓ) \) to \( P \) for some coloring \( c' : V → [k] \). Similarly, this decomposition weakly matches \( P \) if there is a coloring \( c' : V → [k] \) and a surjective color-preserving homomorphism \( ϕ \) from \( CG_c(σ'_1, ⋯, σ'_ℓ) \)

![Figure 1: The colored cycle graph CG_id((0, 1, 2)(4, 5, 6), (2, 3), (2, 4)(7, 8, 9)); the colors are depicted next to the vertices.](image)

Figure 1: The colored cycle graph \( CG_id((0, 1, 2)(4, 5, 6), (2, 3), (2, 4)(7, 8, 9)); the colors are depicted next to the vertices.
to \( P \) where \( \varphi(u) = \varphi(v) \) for \( u \neq v \) implies that \( u \) and \( v \) both belong to \( V \) and are in different \( \sigma' \)-orbits.

**Lemma 6.2.** Let \( P = (U, E) \) be a forest with vertex coloring \( c: U \to [k] \) such that

1. \( P \) contains no isolated vertices,
2. the set \( V \) of vertices that have even distance to some leaf and the set \( C \) of vertices that have odd distance to some leaf are disjoint,
3. the restriction \( c' \) of the coloring \( c \) to \( V \) is injective,
4. \( \{c(u) \mid u \in C\} = [\ell] \) for some \( \ell \in \mathbb{N} \), and
5. any two vertices \( u, v \in C \) with \( c(u) = c(v) \) have distance more than 2.

Then \( P \) is a cycle pattern. Moreover, there is \( \sigma_P \in \text{Sym}(V) \) and a decomposition \( \sigma_P = \sigma_1 \cdots \sigma_\ell \) that matches \( P \).

Note that any cycle pattern satisfies the properties (1) to (5).

**Proof.** For \( u \in C \), let \( \pi_u \) be a cycle on the neighbors of \( u \) in \( P \). For \( i \in [\ell] \), let \( \sigma_i \) be a product of the cycles \( \{\pi_u \mid u \in C, c(u) = i\} \). Note that the order of the multiplication does not matter, as these cycles are disjoint because of property (5). Then the colored cycle graph \( \text{CG}_{c'}(\sigma_1, \ldots, \sigma_\ell) \) is isomorphic to \( P \) via the isomorphism \( \varphi \) that maps each vertex from \( V \) to itself and the vertex of each cycle \( \pi_u \) of some \( \sigma_i \) to \( u \).

**Lemma 6.3.** Let \( P \) be a cycle pattern. Then for any \( \sigma \in \text{Sym}(V) \) that has a complexity-additive decomposition \( \sigma = \sigma_1 \cdots \sigma_\ell \) that weakly matches \( P \), it holds that \( \text{compl}(\sigma) = \text{compl}(\sigma_P) \), where \( \sigma_P \) is the permutation given by Lemma 6.2.

**Proof.** Let \( \sigma_P = \sigma'_1 \cdots \sigma'_\ell \) be the decomposition of \( \sigma_P \) given above; as it matches \( P \), there is a color-preserving isomorphism \( \psi \) from \( P \) to \( \text{CG}_{c'}(\sigma'_1, \ldots, \sigma'_\ell) \) for some coloring \( c': V \to [k'] \). As \( \sigma = \sigma_1 \cdots \sigma_\ell \) weakly matches \( P \), there is a coloring \( c: V \to [k] \) and a surjective color-preserving homomorphism \( \varphi \) from \( \text{CG}_c(\sigma_1, \ldots, \sigma_\ell) \) to \( P \) such that \( \varphi(u) = \varphi(v) \) for \( u \neq v \) implies that \( u \) and \( v \) both belong to \( V \) and are in different \( \sigma \)-orbits. Restricting \( \varphi \psi \) to the cycle-vertices yields a bijection from the cycles of \( \sigma_1, \ldots, \sigma_\ell \) to the cycles of \( \sigma'_1, \ldots, \sigma'_\ell \). Because of the coloring it follows that \( \ell' = \ell \) and that, for \( i \in [\ell] \), the restriction of \( \varphi \psi \) to the cycles of \( \sigma_i \) is a bijection to the cycles of \( \sigma'_i \). Let \( \Omega \) be a cycle of \( \sigma_i \). As Lemma 6.1 implies that all elements of \( \Omega \) are in the same \( \sigma \)-orbit, and as \( \varphi \) is surjective and only allowed to identify vertices from different \( \sigma \)-orbits, the degree of \( \Omega \) in \( \text{CG}_{c'}(\sigma_1, \ldots, \sigma_\ell) \) equals the degree of \( \varphi(\Omega) \) in \( P \) and thus also the degree of \( \varphi \psi(\Omega) \) in \( \text{CG}_{c'}(\sigma'_1, \ldots, \sigma'_\ell) \). This implies \( |\varphi \psi(\Omega)| = |\Omega| \) and thus \( \text{compl}(\sigma_i) = \text{compl}(\sigma'_i) \), which in turn implies \( \text{compl}(\sigma) = \text{compl}(\sigma_P) \).

**Lemma 6.4.** Let \( \sigma \in \text{Sym}(V) \). Then any complexity-additive decomposition \( \sigma = \sigma_1 \cdots \sigma_\ell \) (weakly) matches some cycle pattern that has at most \( 3 \cdot \text{compl}(\sigma) \) vertices.

**Proof.** Let \( t = \text{compl}(\sigma) \). We have \( k = |\text{supp}(\sigma)| \leq 2t \). Let \( c: V \to [k] \) be a coloring whose restriction to \( \text{supp}(\sigma) \) is injective. Then \( P = \text{CG}_c(\sigma_1, \ldots, \sigma_\ell) \) is a pattern, as the vertices in \( V \setminus \text{supp}(\sigma) \) are isolated in \( \text{CG}(\sigma_1, \ldots, \sigma_\ell) \) by Lemma 6.1 and thus are not contained in \( P \). The same lemma also implies that \( \sigma_1, \ldots, \sigma_\ell \) together have at most \( t \) cycles. Thus \( P \) has at most \( 3t \) vertices. It remains to observe that \( \varphi = \text{id} \) is an isomorphism from \( \text{CG}_c(\sigma_1, \ldots, \sigma_\ell) \) to \( P \).

As there are less than \( \mathcal{O}(3t)^{3t} \) forests on \( 3t \) vertices and \( 3t^2 \) ways to color them using \( 2t \) colors, Lemmas 6.2, 6.3 and 6.4 imply the following.
Corollary 6.5. For any $t \in \mathbb{N}$, there is a set $P_t$ of $t^{O(t)}$ cycle patterns such that a permutation $\sigma \in \text{Sym}(V)$ has complexity $t$ if and only if it has a complexity-additive decomposition $\sigma = \sigma_1 \cdots \sigma_\ell$ that weakly matches a pattern in $P_t$. Moreover, $P_t$ can be computed in $t^{O(t)}$ time.

For a pattern $P$, let $P_i$ denote the subgraph of $P$ induced by the cycle-vertices of color $i$ and their neighbors. A permutation $\sigma \in \text{Sym}(V)$ and a coloring $c : V \to [k]$ realize color $i$ of $P$ if there is an isomorphism $\varphi$ from $\text{CG}_c(\sigma)$ to $P_i$ that preserves colors of primal vertices.

Algorithm 5: ExactComplexityIso$_d(X,Y,t)$

```
1 Input: Two hypergraphs $X$ and $Y$ on vertex set $V$ with hyperedge size bounded by $d$, and $t \in \mathbb{N}$
2 Output: An isomorphism $\sigma$ from $X$ to $Y$ with $\text{compl}(\sigma) = t$, or $\perp$ if none exists
3 $A \leftarrow \{\sigma \in \text{Aut}(X) \mid \sigma \text{ has minimal complexity in } \text{Aut}(X) \text{ and } |\text{supp}(\sigma)| \leq 2t\}$
4 // see [2, Algorithm 3]
5 if $X = Y$ then $I \leftarrow A$ else
6 $I \leftarrow \{\sigma \in \text{Iso}(X,Y) \mid \sigma \text{ has minimal complexity in } \text{Iso}(X,Y) \text{ and } |\text{supp}(\sigma)| \leq 2t\}$
7 // see [2, Algorithm 2]
8 for $P \in P_t$ do // see Corollary 6.5
9 $k \leftarrow$ the number of primal vertices in $P$
10 $\ell \leftarrow$ the number of colors of cycle-vertices in $P$
11 for $h \in H_{V,k}$ do // $H_{V,k}$ is the perfect family of hash functions $h : V \to [k]$ from [8]
12 if there are $\sigma_1, \ldots, \sigma_{\ell-1} \in A$ and $\sigma_\ell \in I$ s.t. $(\sigma_i, h)$ realize color $i$ of $P$ then
13 return $\sigma = \sigma_1 \cdots \sigma_\ell$
14 return $\perp$
```

Theorem 6.6. Given two hypergraphs $X$ and $Y$ of hyperedge size at most $d$ and $t \in \mathbb{N}$, the algorithm ExactComplexityIso$_d(X,Y,t)$ finds $\sigma \in \text{Iso}(X,Y)$ with $\text{compl}(\sigma) = t$ (or determines that there is none) in $O((dt)^{O(t)} \text{poly}(N))$ time.

Proof. Suppose there is some $\sigma \in \text{Iso}(X,Y)$ with $\text{compl}(\sigma) = t$. Lemma 2.1 gives the complexity-additive decomposition $\sigma = \sigma_1 \cdots \sigma_\ell$ into minimal-complexity permutations $\sigma_1, \ldots, \sigma_{\ell-1} \in \text{Aut}(X)$ and $\sigma_\ell \in \text{Iso}(X,Y)$; all of them have complexity at most $t$. By the correctness of the algorithms from [2], we have $\sigma_1, \ldots, \sigma_{\ell-1} \in A$ and $\sigma_\ell \in I$. As $H_{V,k}$ is a perfect hash family, it contains some function $h$ whose restriction to $\text{supp}(\sigma)$ is injective. Then $\text{CG}_h(\sigma_1, \ldots, \sigma_\ell)$ is isomorphic to some $P \in P_t$ by Corollary 6.5. Thus $(\sigma_i, h)$ realize color $i$ of $P$, for $1 \leq i \leq \ell$, so the algorithm does not return $\perp$.

Now suppose that the algorithm returns $\sigma = \sigma_1 \cdots \sigma_\ell$ with $\sigma_1, \ldots, \sigma_{\ell-1} \in A \subseteq \text{Aut}(X)$ and $\sigma_\ell \in I \subseteq \text{Iso}(X,Y)$. This clearly implies $\sigma \in \text{Iso}(X,Y)$. To show $\text{compl}(\sigma) = t$, we observe that the algorithm only returns $\sigma$ if there is a pattern $P \in P_t$ whose cycle-vertices have $\ell$ colors and that contains $k$ primal vertices, and a hash function $h \in H_{V,k}$ such that $(\sigma_i, h)$ realize color $i$ of $P$, for $1 \leq i \leq \ell$. In particular, there is an isomorphism $\varphi_i$ from $\text{CG}_h(\sigma_i)$ to $P_i$ that preserves colors of primal vertices. As the primal vertices of $P$ all have different colors and as $P$ is a forest by Lemma 6.1, it follows that the decomposition $\sigma = \sigma_1 \cdots \sigma_\ell$ is complexity-additive. Now consider the function $\varphi = \bigcup_{i=1}^\ell \varphi_i$; it is well-defined, as $v \in \text{supp}(\varphi_i) \cap \text{supp}(\varphi_j)$ implies $\varphi_i(v) = \varphi_j(v)$ because $P$ contains only one primal vertex of color $h(v)$. It is surjective, as every vertex of $P$ occurs in at least one $P_i$. It is a homomorphism from $P_\sigma = \text{CG}_h(\sigma_1, \ldots, \sigma_\ell)$ to $P$, as every edge occurs in the support of one of the isomorphisms $\varphi_i$. Also, $\varphi(u) = \varphi(v)$
for \( u \neq v \) implies that \( u \) and \( v \) are in different connected components of \( P_\sigma \), as \( P \) is an forest; consequently \( u \) and \( v \) are in different orbits of \( \sigma \). Thus \( \sigma = \sigma_1 \cdots \sigma_\ell \) weakly matches \( P \). By Lemma 6.3 it follows that \( \text{compl}(\sigma) = t \).

It remains to analyze the runtime. The algorithms used to compute \( A \) and \( I \) each take \( \mathcal{O}(dt^{O(t^2)} \text{poly}(N)) \) time [2]. The pattern set \( \mathcal{P}_t \) can be computed in \( t^{O(t)} \) time by Corollary 6.5. As \( k \leq 2t \), the perfect hash family \( \mathcal{H}_{V,K} \) has size \( 2^{O(t)} \log^2 n \). As \( \ell \leq t \), this gives a total runtime of \( \mathcal{O}(dt^{O(t^2)} \text{poly}(N)) \).

7 Colored Graph Automorphism

In [2] we showed that the following parameterized version of Graph Automorphism is \( W[1] \)-hard. It was first defined in [6] and is a generalization of the problem studied by Schweitzer [15].

Col-GA: Given a graph \( X \) with its vertex set partitioned as \( \text{RED} \cup \text{BLUE} \), and a parameter \( k \), decide if there is a partition-preserving automorphism that moves exactly \( k \) \text{BLUE} vertices.

For an automorphism \( \pi \in \text{Aut}(X) \), we will refer to the number of \text{BLUE} vertices moved by \( \pi \) as the \text{BLUE} weight of \( \pi \). The graphs used in the \( W[1] \)-hardness reduction in [2] are designed to simulate the Circuit Value Problem for Boolean inputs of Hamming weight \( k \). \text{BLUE} vertices are used at the input level and are partitioned into color classes of size 2 (the pair of nodes in each color class can flip or not to simulate a Boolean value). Vertices in the graph gadgets used for simulating the circuit gates are the \text{RED} vertices. It turns out that in the \text{RED} part the color classes are of size at most 4. In this section, we show that Col-GA is in \( \text{FPT} \) when restricted to colored graphs where the \text{RED} color classes have size at most 3.

Given an input instance \( X = (V,E) \) with vertex partition \( V = \text{RED} \cup \text{BLUE} \) such that \text{RED} is refined into color classes of size at most 3 each, our algorithm proceeds as follows.

**Step 1: color-refinement.** \( X \) already comes with a color classification of vertices (\text{RED} and \text{BLUE}, and within \text{RED} color classes of size at most 3 each; within \text{BLUE} there may be color classes of arbitrary size). The color refinement procedure keeps refining the coloring in steps until no further refinement of the vertex color classes is possible. In a refinement step, if two vertices have identical colors but differently colored neighborhoods (with the multiplicities of colors counted), then these vertices get new different colors.

At the end of this refinement, each color class induces a regular graph, and each pair of color classes induce a semiregular bipartite graph.

**Step 2: local complementation.** We complement the graph induced by a color class if this reduces the number of its edges; this does not change the automorphism group of \( X \). Similarly, we complement the induced bipartite graph between two color classes if this reduces the number of its edges.

Now each \text{RED} color class induces the empty graph. Similarly, for \( b \in \{2,3\} \), the bipartite graph between any two color classes of size \( b \) is empty or a perfect matching. (Note that this does not necessarily hold for \( b \geq 4 \).) Color refinement for graphs of color class size at most 3 has been used in earlier work [10, 11].

Let \( C \subset \text{RED} \) and \( D \subset \text{BLUE} \) be color classes after Step 1. Because of the complementations we have applied, \( |C| = 1 \) implies that \( X[C,D] \) is empty, and if \( |C| \in \{2,3\} \) then \( X[C,D] \) is either empty or the degree of each \( D \)-vertex in \( X[C,D] \) is 1.
Step 3: fix vertices that cannot move. For any red color class $C \subset \text{RED}$ whose elements have more than $k$ blue neighbors, give different new colors to each vertex in $C$ (because of Step 2, each non-isolated red vertex is in a color class with more than one vertex). Afterwards, rerun Steps 1 and 2 so we again have a stable coloring.

Fixing the vertices in $C$ does not lose any automorphism of $X$ that has blue-weight at most $k$. Indeed, as every blue vertex has at most one neighbor in $C$, any automorphism that moves some $v \in C$ has to move all (more than $k$) blue neighbors of $v$.

Step 4: remove edges in the red part. We already observed that each red color class induces the empty graph. Let $X$ be the graph whose vertices are the red color classes, where two of them are adjacent if there is a perfect matching between them in $X$. For each $b \in \{1, 2, 3\}$, the red color classes of size $b$ get partitioned into components of $X$.

We consider each connected component $C$ of $X$ that consists of more than one color class. Let $X'$ be the subgraph of $X$ induced by vertices in $\bigcup C$ and their neighbors in blue. Because of Step 3, the graph $X'$ has color class size at most $3k$, so we can compute its automorphism group $H = \text{Aut}(X')$ in $2^{O(k^2)} \text{poly}(N)$ time [9]. We distinguish several cases based on the action of $H$ on an arbitrary color class $C \in C$:

Case 1: If $H(C)$ is not transitive, we split the color class $C$ into the orbits of $H(C)$ and start over with Step 1.

Case 2: If $H(C) = \text{Sym}(C)$, we drop all vertices in $(\bigcup C) \setminus C$ from $X$. And for each blue color class $D$ that has neighbors in at least one $C' \in C$, we replace the edges between a vertex $u \in D$ and $\bigcup C$ by the single edge $(u, v)$, where $v$ is the vertex in $C$ that is reachable via the matching edges from the neighbor of $u$ in $C'$.

Case 3: If $H(C)$ is generated by a 3-cycle $(v_1v_2v_3)$, we first proceed as in Case 2. Additionally, we add directed edges within each blue color class $D$ that now has neighbors in $C$. Let $D_i \subset D$ be the neighbors of $v_i$. We add directed edges from all vertices in $D_i$ to all vertices in $D_{(i+1) \mod 3}$ and color these directed edges by $C$.

After this step, there are no edges induced on the red part of $X$. Moreover, we have not changed the automorphisms on the induced subgraph, so the modified graph $X$ still has the same automorphism group as before.

Step 5: turn red vertices into hyperedges. We encode $X$ as a hypergraph $X' = (\text{BLUE} \cup \text{NEW}, E')$ in which each vertex in red is encoded as a hyperedge on the vertex set $\text{BLUE} \cup \text{NEW}$. Let $\text{NEW} = \{v_C \mid C \subset \text{RED} \text{ is a color class}\}$. Let $v \in C \subset \text{RED} \text{ be any red vertex}$. We encode $v$ as the hyperedge $e_v = \{v_C\} \cup \{u \in \text{BLUE} \mid (v, u) \in E(X)\}$.

In the hypergraph $X'$ we give distinct colors to each vertex in new in order to ensure that each color class $\{v_{C,1}, v_{C,2}, v_{C,3}\}$ in red is preserved by the automorphisms of $X'$.

Clearly, there is a 1-1 correspondence between the color-reserving automorphisms of $X$ and those of $X'$. Note that the hyperedges of $X'$ have size bounded by $k + 1$, as each red vertex in $X$ has at most $k$ blue neighbors after Step 3.

Step 6: bounded hyperedge size automorphism. We seek a weight $k$ automorphism of $X'$ using the algorithm of [2, Corollary 6.4]; this is possible in $d^{O(k^2)} 2^{O(k^3)} \text{poly}(N)$ time.

\footnote{There is a caveat that in addition to hyperedges in the graph $X'[[\text{BLUE}]]$ we also have colored directed edges. However, the algorithm of [2, Corollary 6.4] needs only minor changes to handle this.}
This algorithm gives us the following.

**Theorem 7.1.** The above algorithm solves Col-GA when the red part of the input graph is refined in color classes of size at most 3. It runs in \(d^{O(k)}2^{O(k^2)}\text{poly}(N)\) time.

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