Canonical structure of 3D gravity with torsion*

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Abstract

We study the canonical structure of the topological 3D gravity with torsion, assuming the anti-de Sitter asymptotic conditions. It is shown that the Poisson bracket algebra of the canonical generators has the form of two independent Virasoro algebras with classical central charges. In contrast to the case of general relativity with a cosmological constant, the values of the central charges are different from each other.

1 Introduction

Faced with enormous difficulties to properly understand fundamental dynamical properties of gravity, such as the nature of classical singularities and the problem of quantization, one is naturally led to consider technically simplified models with the same conceptual features. An important and useful model of this type is 3D gravity [1, 2]. In the last twenty years, 3D gravity has become an active research area, with a number of outstanding results. Here, we focus our attention on a particular line of development, characterized by the following achievements. In 1986, Brown and Henneaux introduced the so-called anti-de Sitter (AdS) asymptotic conditions in their study of 3D general relativity with a cosmological constant (GRΛ) [3]. They showed that the related behavior of the gravitational field allows for an extremely rich asymptotic structure—the conformal symmetry described by two independent canonical Virasoro algebras with classical central charges. Soon after that, Witten rediscovered and further explored the fact that GRΛ in 3D can be formulated as a Chern-Simons gauge theory [4]. The equivalence between gravity and an ordinary gauge theory was shown to be crucial for our understanding of quantum gravity. Then, in 1993, we had the discovery of the BTZ black hole [5], with a far-reaching impact on the development of 3D gravity. All these ideas have had a significant influence on our understanding of the quantum nature of 3D black holes [2,6-13].

Following a widely spread belief that general relativity is the most reliable approach for studying the gravitational phenomena, the analysis of these issues has been carried out mostly in the realm of Riemannian geometry. However, there is a more general conception of gravity based on Riemann-Cartan geometry, in which both the curvature and the torsion characterize the structure of gravity (see, for instance, Refs. [14, 15]). Riemann-Cartan

*Invited contribution to appear in Progress in General Relativity and Quantum Cosmology, vol. 2, ed. Frank Columbus (Nova Science Publishers, New York, 2005).
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geometry has been used in the context of 3D gravity since the early 1990s [16-18], with an idea to explore the influence of geometry on the dynamics of gravity. Recently, new advances in this direction have been achieved [19-24].

Asymptotic conditions are an intrinsic part of the canonical formalism, as they define the phase space in which the canonical dynamics takes place. Their influence on the dynamics is particularly clear in topological theories, where the propagating degrees of freedom are absent, and the only non-trivial dynamics is bound to exist at the asymptotic boundary. General action for topological 3D gravity with torsion, based on Riemann-Cartan geometry of spacetime, has been proposed by Mielke and Baekler [16, 17]. The objective of the present paper is to investigate the canonical structure of the general topological 3D gravity with torsion, including its asymptotic behavior, in the AdS sector of the theory. This will generalize the results of Refs. [3, 4] and [20], where the specific choice of parameters corresponds to Riemannian and telaparallel vacuum geometry, respectively. Combining this approach with another interesting result, the existence of the Riemann-Cartan black hole [19, 22], we shall be able to explore the full influence of torsion on the canonical and asymptotic structure of 3D gravity.

The paper is organized as follows. In Sect. 2 we review some basic features of Riemann–Cartan spacetime as the proper geometric arena for 3D gravity with torsion, and discuss the field equations derived from the Mielke-Baekler action. In Sect. 3 we describe the Riemann-Cartan black hole solution, a generalization of the BTZ black hole. Then, in Sect. 4, we introduce the concept of asymptotically AdS configuration, and derive the related asymptotic symmetry, which turns out to be the same as in general relativity—the conformal symmetry. In the next section, the asymptotic structure of the theory is incorporated into the Hamiltonian formalism by calculating the Poisson bracket (PB) algebra of the canonical generators. It has the form of two independent Virasoro algebras with classical central charges, the values of which differ from each other, in contrast to what we have in Riemannian GR and the teleparallel theory [3, 20]. Finally, Sect. 7 is devoted to concluding remarks, while Appendices contain some technical details.

Our conventions are given by the following rules: the Latin indices refer to the local Lorentz frame, the Greek indices refer to the coordinate frame; the first letters of both alphabets (a, b, c, ...; α, β, γ, ...) run over 1,2, the middle alphabet letters (i, j, k, ...; μ, ν, λ, ...) run over 0,1,2; the signature of spacetime is η = (+, −, −); totally antisymmetric tensor εijk and the related tensor density εμνρ are both normalized so that ε012 = 1.

2 Topological 3D gravity with torsion

Theory of gravity with torsion can be formulated as Poincaré gauge theory (PGT), with an underlying spacetime structure described by Riemann-Cartan geometry [14, 15].

PGT in brief. The basic gravitational variables in PGT are the triad field b^i and the Lorentz connection A^ij = −A^ji (1-forms). The field strengths corresponding to the gauge potentials b^i and A^ij are the torsion T^i and the curvature R^ij (2-forms): T^i = db^i + A^im ∧ b^m, R^ij = dA^ij + A^im ∧ A^mj. Gauge symmetries of the theory are local translations and local Lorentz rotations, parametrized by ξ^μ and ε^ij.
In 3D, we can simplify the notation by introducing the duals of $A_{ij}$, $R_{ij}$ and $\varepsilon^{ij}$:

$$\omega_i = -\frac{1}{2} \varepsilon_{ijk} A_{jk}, \quad R_i = -\frac{1}{2} \varepsilon_{ijk} R_{jk}, \quad \theta_i = -\frac{1}{2} \varepsilon_{ijk} \varepsilon^{jk}.$$

In local coordinates $x^\mu$, we can expand the triad and the connection 1-forms as

$$b_i = b_i^\mu dx^\mu, \quad \omega_i = \omega_i^\mu dx^\mu.$$

Gauge transformation laws have the form

$$\delta_0 b_i^\mu = -\varepsilon_{ijk} b_j^\rho \theta_k^\mu \equiv \delta_{\text{PGT}} b_i^\mu,$$

$$\delta_0 \omega_i^\mu = -\varepsilon_{ijk} \omega_j^\rho \theta_k^\mu - \xi^\rho \partial_\rho \omega_i^\mu \equiv \delta_{\text{PGT}} \omega_i^\mu,$$

and the field strengths are given as

$$T^i = \nabla b^i \equiv db^i + \varepsilon_{ijk} \omega^j \wedge b^k = \frac{1}{2} T^i_{\mu\nu} dx^\mu \wedge dx^\nu,$$

$$R^i = d\omega^i + \frac{1}{2} \varepsilon_{ijk} \omega^j \wedge \omega^k = \frac{1}{2} R^i_{\mu\nu} dx^\mu \wedge dx^\nu,$$

where $\nabla = dx^\mu \nabla_\mu$ is the covariant derivative.

To clarify the geometric meaning of the above structure, we introduce the metric tensor as a specific, bilinear combination of the triad fields:

$$g = \eta_{ij} b^i \otimes b^j, \quad g_{\mu\nu} = \eta_{ij} b^i_\mu b^j_\nu, \quad \eta_{ij} = (+, -, -).$$

Although metric and connection are in general independent dynamical/geometric variables, the antisymmetry of $A_{ij}$ in PGT is equivalent to the so-called \textit{metricity condition}, $\nabla g = 0$. The geometry whose connection is restricted by the metricity condition (metric-compatible connection) is called \textit{Riemann-Cartan geometry}. Thus, PGT has the geometric structure of Riemann-Cartan space.

The connection $A_{ij}$ determines the parallel transport in the local Lorentz basis. Being a true geometric operation, parallel transport is independent of the basis. This property is incorporated into PGT via the so-called \textit{vielbein postulate}, which implies the identity

$$A_{ijk} = \Delta_{ijk} + K_{ijk},$$

where $\Delta$ is Riemannian (Levi-Civita) connection, and $K_{ijk} = -\frac{1}{2} (T_{ijk} - T_{kij} + T_{jki})$ is the contortion.

\textbf{Topological action.} In general, gravitational dynamics is defined by Lagrangians which are at most quadratic in field strengths. Omitting the quadratic terms, Mielke and Baekler proposed a \textit{topological} model for 3D gravity [16, 17], with an action of the form

$$I = a I_1 + \Lambda I_2 + \alpha_3 I_3 + \alpha_4 I_4 + I_M,$$

where $I_M$ is a matter contribution, and

$$I_1 = 2 \int b^i \wedge R_i,$$

$$I_2 = -\frac{1}{3} \int \varepsilon_{ijk} b^i \wedge b^j \wedge b^k,$$

$$I_3 = \int \left( \omega^i \wedge d\omega_i + \frac{1}{3} \varepsilon_{ijk} \omega^i \wedge \omega^j \wedge \omega^k \right),$$

$$I_4 = \int b^i \wedge T_i.$$

(2.4b)
The first term, with $a = 1/16\pi G$, is the usual Einstein-Cartan action, the second term is a cosmological term, $I_3$ is the Chern-Simons action for the Lorentz connection, and $I_4$ is an action of the translational Chern-Simons type. The Mielke-Baekler model can be thought of as a natural generalization of Riemannian GR$_\Lambda$ (with $\alpha_3 = \alpha_4 = 0$) to a topological gravity theory in Riemann-Cartan spacetime.

**Field equations.** Variation of the action with respect to triad and connection yields the gravitational field equations:

\[
\varepsilon^{\mu\nu\rho}[aR_{i\nu\rho} + \alpha_4 T_{i\nu\rho} - \Lambda \varepsilon_{ijk}b^j_\nu b^k_\rho] = \tau^\mu_i,
\]

\[
\varepsilon^{\mu\nu\rho}[\alpha_3 R_{i\nu\rho} + aT_{i\nu\rho} + \alpha_4 \varepsilon_{ijk}b^j_\nu b^k_\rho] = \sigma^\mu_i,
\]

where $\tau^\mu_i = -\delta I_M/\delta b^i_\mu$ and $\sigma^\mu_i = -\delta I_M/\delta \omega^i_\mu$ are the matter energy-momentum and spin currents, respectively. For our purposes—to study the canonical structure of the theory in the asymptotic region—it is sufficient to consider the field equations in vacuum, where $\tau = \sigma = 0$. In the sector $\alpha_3\alpha_4 - a^2 \neq 0$, these equations take the simple form

\[
T_{ijk} = p\varepsilon_{ijk}, \quad R_{ijk} = q\varepsilon_{ijk},
\]

where

\[
p = \frac{\alpha_3 A + \alpha_4 a}{\alpha_3 \alpha_4 - a^2}, \quad q = -\frac{(\alpha_4)^2 + aA}{\alpha_3 \alpha_4 - a^2}.
\]

Thus, the vacuum configuration is characterized by constant torsion and constant curvature.

In Riemann-Cartan spacetime, one can use the identity (2.3) to express the curvature $R_{ij\mu\nu}(A)$ in terms of its Riemannian piece $\tilde{R}_{ij\mu\nu}(\Delta)$ and the contortion:

\[
R_{ij\mu\nu}(A) = \tilde{R}_{ij\mu\nu} + \left[\nabla_{\mu}K_{ij\nu} - K_{im\mu}K_{mj\nu} - (\mu \leftrightarrow \nu)\right].
\]

This relation, combined with the field equations (2.5), leads to

\[
\tilde{R}_{ij\mu\nu} = -\Lambda_{\text{eff}}(b^i_\mu b^j_\nu - b^j_\mu b^i_\nu), \quad \Lambda_{\text{eff}} \equiv q - \frac{1}{4}p^2,
\]

where $\Lambda_{\text{eff}}$ is the effective cosmological constant. Equation (2.6) can be considered as an equivalent of the second field equation (2.5b). Looking at (2.6) as an equation for the metric, one concludes that our spacetime has maximally symmetric metric [25]:

- for $\Lambda_{\text{eff}} < 0$ ($\Lambda_{\text{eff}} > 0$), the spacetime manifold is anti-de Sitter (de Sitter).

There are two interesting special cases of the general Mielke-Baekler model, which have been studied in the past.

- For $\alpha_3 = \alpha_4 = 0$, the vacuum geometry becomes Riemannian, $T_{ijk} = 0$. This choice corresponds to GR$_\Lambda$ [3, 4];
- for $(\alpha_4)^2 + a\Lambda = 0$, the vacuum geometry is teleparallel, $R_{ijk} = 0$. The vacuum field equations are “geometrically dual” to those of GR$_\Lambda$ [20].

In the present paper, we shall investigate the general model (2.4) with $\alpha_3\alpha_4 - a^2 \neq 0$, assuming that the effective cosmological constant is negative (anti-de Sitter sector):

\[
\Lambda_{\text{eff}} \equiv -\frac{1}{\ell^2} < 0.
\]

The de Sitter sector with $\Lambda_{\text{eff}} > 0$ is left for the future studies.
3 Exact vacuum solutions

Some aspects of the canonical analysis rely on the existence of suitable asymptotic conditions. A proper choice of these conditions is based, to some extent, on the knowledge of exact classical solutions in vacuum. For the Mielke-Baekler model (2.4), these solutions are well known [19, 22]. Their construction can be described by the following set of rules:

- For a given $\Lambda_{\text{eff}}$, use Eq. (2.6) to find a solution for the metric. This step is very simple, since the metric structure of maximally symmetric spaces is well known [25].
- Given the metric, find a solution for the triad field, such that $g = \eta_{ij} b^i \otimes b^j$.
- Finally, use Eq. (2.5a) to determine the connection $\omega^i$.

For exact solutions with non-vanishing sources, the reader can consult Ref. [24].

Riemann-Cartan black hole. For $\Lambda_{\text{eff}} < 0$, equation (2.6) has a well known solution for the metric — the BTZ black hole [5]. Using the static coordinates $x^\mu = (t, r, \varphi)$ (with $0 \leq \varphi < 2\pi$), and units $4G = 1$, it is given as

\[
ds^2 = N^2 dt^2 - N^{-2} dr^2 - r^2 (d\varphi + N_\varphi dt)^2,
\]

\[
N^2 = \left(-2m + \frac{r^2}{\ell^2} + \frac{J^2}{r^2}\right), \quad N_\varphi = \frac{J}{r^2}.
\]

The parameters $m$ and $J$ are related to the conserved charges—energy and angular momentum. Since the triad field corresponding to (3.1) is determined only up to a local Lorentz transformation, we can choose $b^i$ to have the simple, “diagonal” form:

\[
b^0 = N dt, \quad b^1 = N^{-1} dr, \quad b^2 = r (d\varphi + N_\varphi dt).
\]

Then, the connection is obtained by solving the first field equation (2.5a):

\[
\omega^0 = N \left( \frac{p}{2} dt - d\varphi \right), \quad \omega^1 = N^{-1} \left( \frac{p}{2} + \frac{J}{r^2} \right) dr,
\]

\[
\omega^2 = -\left( \frac{r}{\ell} - \frac{p \ell J}{2 r} \right) \frac{dt}{\ell} + \left( \frac{p}{2} - \frac{J}{r} \right) d\varphi.
\]

Equations (3.2) define the Riemann-Cartan black hole.

Riemann-Cartan AdS solution. In Riemannian geometry with negative $\Lambda$, the general solution with maximal number of Killing vectors is called the AdS solution [5, 25]. Although AdS solution and the black hole are locally isometric, they are globally distinct. The AdS solution can be obtained from (3.1) by the replacement $J = 0, 2m = -1$.

Similarly, there is a general solution with maximal symmetry in Riemann-Cartan geometry, the Riemann-Cartan AdS solution. It can be obtained from the black hole (3.2) by the same replacement ($J = 0, 2m = -1$). Using the notation $f^2 \equiv 1 + r^2/\ell^2$, we have:

\[
b^0 = f dt, \quad b^1 = f^{-1} dr, \quad b^2 = r d\varphi,
\]

\[
\omega^0 = f \left( \frac{p}{2} dt - d\varphi \right), \quad \omega^1 = \frac{p}{2f} dr, \quad \omega^2 = -\frac{r}{\ell} \left( \frac{dt}{\ell} - \frac{p \ell}{2} d\varphi \right).
\]
In order to understand symmetry properties of (3.3), we note that the form-invariance of a given field configuration in Riemann-Cartan geometry is defined by the requirements \( \delta_0 b^i_\mu = 0 \), \( \delta_0 \omega^i_\mu = 0 \), which differ from the Killing equation in Riemannian geometry, \( \delta_0 g_{\mu\nu} = 0 \) (\( \delta_0 \) is the PGT analogue of the geometric Lie derivative). When applied to the Riemann-Cartan AdS solution (3.3), these requirements restrict \((\xi^\mu, \theta^i)(k = 1, \ldots, 6)\) to the subspace defined by the basis of six pairs \((\xi^\mu_{(k)}, \theta^i_{(k)})\) given in Appendix A. The related symmetry group is the six-dimensional AdS group \(SO(2,2)\).

4 Asymptotic conditions

Spacetime outside localized matter sources is described by the vacuum solutions of the field equations (2.5). Thus, matter has no influence on the local properties of spacetime in the source-free regions, but it can change its global properties. On the other hand, global properties of spacetime affect symmetry properties of the asymptotic configurations, and consequently, they are closely related to the gravitational conservation laws.

In 3D gravity with \( \Lambda_{\text{eff}} < 0 \), maximally symmetric AdS solution (3.3) has the role analogous to the role of Minkowski space in the \( \Lambda_{\text{eff}} = 0 \) case. Following this analogy, we could choose (3.3) to be the field configuration to which all the dynamical variables approach in such a way, that the asymptotic symmetry is \(SO(2,2)\), the maximal symmetry of (3.3). However, such an assumption would exclude the important black hole geometries, which are not \(SO(2,2)\) invariant. Having an idea to maximally relax the asymptotic conditions and enlarge the set of asymptotic states (and the relevant group of symmetries), we introduce the concept of the AdS asymptotic behavior, based on the following requirements [3, 26]:

(a) asymptotic configurations should include the black hole geometries;
(b) they should be invariant under the action of the AdS group \(SO(2,2)\);
(c) asymptotic symmetries should have well defined canonical generators.

The conditions (a) and (b) together lead to an extended asymptotic structure, quite different from the standard, form-invariant vacuum configuration, while (c) is just a technical assumption.

AdS asymptotics. We begin our considerations with the point (a) above. The asymptotic behaviour of the black hole triad (3.2a) is given by

\[
b^i_\mu \sim \left( \begin{array}{ccc}
\frac{r}{\ell} - \frac{m\ell}{r} & 0 & 0 \\
0 & \frac{\ell}{r} + \frac{m\ell^3}{r^3} & 0 \\
J & 0 & r
\end{array} \right),
\]

where the type of higher order terms on the right hand side is not written explicitly. Similarly, the asymptotic behaviour of the connection (3.2b) has the form

\[
\omega^i_\mu \sim \left( \begin{array}{ccc}
\frac{p\ell}{2} \left( \frac{r}{\ell^2} - \frac{m}{r} \right) & 0 & -\frac{r}{\ell} + \frac{m\ell}{r} \\
0 & \frac{p\ell}{2r} + \frac{J\ell + pm\ell^3/2}{r^3} & 0 \\
-\frac{r}{\ell^2} + \frac{pJ}{2r} & 0 & \frac{pr}{2} - \frac{J}{r}
\end{array} \right).
\]
According to (a), asymptotic conditions should be chosen so as to include these black hole configurations.

In order to realize the requirement (b), we start with the above black hole configuration and act on it with all possible $SO(2, 2)$ transformations, defined by the basis of six pairs $(\xi_{(k)}, \theta_{(k)})$, displayed in Appendix A. The result has the form

$$\delta_{(k)}b^{i}_\mu \sim \begin{pmatrix} \mathcal{O}_1 & \mathcal{O}_4 & \mathcal{O}_1 \\ \mathcal{O}_2 & \mathcal{O}_3 & \mathcal{O}_2 \\ \mathcal{O}_1 & \mathcal{O}_4 & \mathcal{O}_1 \end{pmatrix}, \quad \delta_{(k)}\omega^{i}_\mu \sim \begin{pmatrix} \mathcal{O}_1 & \mathcal{O}_4 & \mathcal{O}_1 \\ \mathcal{O}_2 & \mathcal{O}_3 & \mathcal{O}_2 \\ \mathcal{O}_1 & \mathcal{O}_4 & \mathcal{O}_1 \end{pmatrix},$$

where $\mathcal{O}_n$ denotes a quantity that tends to zero as $1/r^n$ or faster, when $r \to \infty$.

The family of the black hole triads obtained in this way is parametrized by six real parameters, say $\sigma_i$; we denote it by $\mathcal{B}_6$. In order to have a set of asymptotic states which is sufficiently large to include the whole $\mathcal{B}_6$, we adopt the following asymptotic form for the triad field:

$$b^{i}_\mu = \begin{pmatrix} \frac{r}{\ell} + \mathcal{O}_1 & \mathcal{O}_4 & \mathcal{O}_1 \\ \mathcal{O}_2 & \frac{\ell}{r} + \mathcal{O}_3 & \mathcal{O}_2 \\ \mathcal{O}_1 & \mathcal{O}_4 & r + \mathcal{O}_1 \end{pmatrix} \equiv \begin{pmatrix} \frac{r}{\ell} & 0 & 0 \\ 0 & \frac{\ell}{r} & 0 \\ 0 & 0 & r \end{pmatrix} + B^{i}_\mu. \quad (4.1a)$$

The real meaning of this expression and its relation to $\mathcal{B}_6$ is clarified by noting that any $c/r^n$ term in $\mathcal{B}_6$ is transformed into the corresponding $c(t, \varphi)/r^n$ term in (4.1a), i.e. constants $c = c(\sigma_i)$ are promoted to functions $c(t, \varphi)$. Thus, (4.1a) is a natural generalization of $\mathcal{B}_6$.

The triad family (4.1a) generates the Brown–Henneaux asymptotic form of the metric,

$$g_{\mu\nu} = \begin{pmatrix} \frac{r^2}{\ell^2} + \mathcal{O}_0 & \mathcal{O}_3 & \mathcal{O}_0 \\ \mathcal{O}_3 & -\frac{\ell^2}{r^2} + \mathcal{O}_4 & \mathcal{O}_3 \\ \mathcal{O}_0 & \mathcal{O}_3 & -r^2 + \mathcal{O}_0 \end{pmatrix} \equiv \begin{pmatrix} \frac{r^2}{\ell^2} & 0 & 0 \\ 0 & -\frac{\ell^2}{r^2} & 0 \\ 0 & 0 & -r^2 \end{pmatrix} + G_{\mu\nu},$$

but clearly, it is not uniquely determined by it (any Lorentz transform of the triad produces the same metric).

Having found the triad asymptotics, we now use similar arguments to find the needed asymptotic behavior for the connection:

$$\omega^{i}_\mu = \begin{pmatrix} \frac{pr}{2\ell} + \mathcal{O}_1 & \mathcal{O}_2 & -\frac{r}{\ell} + \mathcal{O}_1 \\ \mathcal{O}_2 & \frac{p\ell}{2r} + \mathcal{O}_3 & \mathcal{O}_2 \\ -\frac{r}{\ell^2} + \mathcal{O}_1 & \mathcal{O}_2 & \frac{pr}{2} + \mathcal{O}_1 \end{pmatrix} \equiv \begin{pmatrix} \frac{pr}{2\ell} & 0 & -\frac{r}{\ell} \\ 0 & \frac{p\ell}{2r} & 0 \\ -\frac{r}{\ell^2} & 0 & \frac{pr}{2} \end{pmatrix} + \Omega^{i}_\mu. \quad (4.1b)$$

Note that the choice $\omega^{0}_1, \omega^{2}_1 = \mathcal{O}_2$, adopted in (4.1b), represents an acceptable generalization of the conditions $\omega^{0}_1, \omega^{2}_1 = \mathcal{O}_4$, suggested by the form of $\delta_{(k)}\omega^{i}_\mu$ (compare also with the conditions (C.1)).

As we have seen, the requirements (a) and (b) are not sufficient for a unique determination of the asymptotic behavior. Our choice of the asymptotics was guided by the idea to obtain the most general asymptotic behavior compatible with (a) and (b), with arbitrary
higher-order terms $B^i_\mu$ and $\Omega^i_\mu$. Although $B^i_\mu$ and $\Omega^i_\mu$ are arbitrary at this stage, certain relations among them will be established later (Appendix C), using some additional requirements. One can verify that the asymptotic conditions (4.1) are indeed invariant under the action of the AdS group $SO(2,2)$. In the next step, we shall examine whether there is any higher symmetry structure in (4.1), which will be the real test of our choice.

**Asymptotic symmetries.** Having chosen the asymptotic conditions in the form (4.1), we now wish to find the subset of gauge transformations that respect these conditions. Acting on a specific field satisfying (4.1), these transformations are allowed to change the form of the non-leading terms $B^i_\mu$, $\Omega^i_\mu$, as they are arbitrary by assumption. Thus, the parameters of the restricted gauge transformations are determined by the relations

\[
\varepsilon^{ijk} \theta_{b_k} b^j_\mu - (\partial_\mu \xi^\rho) b^j_\rho - \xi^\rho \partial_\rho b^j_\mu = \delta_0 B^i_\mu , \tag{4.2a}
\]

\[
- \partial_\mu \theta^i + \varepsilon^{ijk} \theta_{j_\omega} - (\partial_\mu \xi^\rho) \omega^j_\rho - \xi^\rho \partial_\rho \omega^j_\mu = \delta_0 \Omega^i_\mu . \tag{4.2b}
\]

The transformations defined in this way differ from those that are associated to the form-invariant vacuum configurations ($\delta_0 b^i_\mu = 0$, $\delta_0 \omega^i_\mu = 0$). The restricted gauge parameters are determined as follows [20].

The symmetric part of (4.2a) multiplied by $b^b_{\nu}$ (six relations) yields the transformation rule of the metric:

\[-(\partial_\mu \xi^\rho) g_{\nu\rho} - (\partial_\nu \xi^\rho) g_{\mu\rho} - \xi^\rho \partial_\rho g^\mu_{\nu} = \delta_0 G^\mu_{\nu}.\]

Expanding $\xi^\mu$ in powers of $r^{-1}$, we find the solution of these equations as

\[
\xi^0 = \ell \left[ T + \frac{1}{2} \left( \frac{\partial^2 T}{\partial t^2} \right) \frac{\ell^4}{r^2} \right] + O_4 , \tag{4.3a}
\]

\[
\xi^2 = S - \frac{1}{2} \left( \frac{\partial^2 S}{\partial \varphi^2} \right) \frac{\ell^2}{r^2} + O_4 , \tag{4.3b}
\]

\[
\xi^1 = -\ell \left( \frac{\partial T}{\partial t} \right) r + O_1 , \tag{4.3c}
\]

where the functions $T(t, \varphi)$ and $S(t, \varphi)$ satisfy the conditions

\[
\frac{\partial T}{\partial \varphi} = \ell \frac{\partial S}{\partial t} , \quad \frac{\partial S}{\partial \varphi} = \ell \frac{\partial T}{\partial t} . \tag{4.4}
\]

In GR$_\Lambda$, these equations define the two-dimensional conformal group at large distances [3].

The remaining three components of (4.2a) determine $\theta^i$:

\[
\theta^0 = -\frac{\ell^2}{r} \partial_0 \partial_2 T + O_3 ,
\]

\[
\theta^2 = \frac{\ell^3}{r} \partial_0^3 T + O_3 ,
\]

\[
\theta^1 = \partial_2 T + O_2 . \tag{4.3d}
\]

The conditions (4.2b) produce no new limitations on the parameters.
Introducing the light-cone coordinates $x^\pm = x^0/\ell \pm x^2$, the conditions (4.4) can be written in the form

$$\partial_\pm (T \mp S) = 0,$$

from which one easily finds the general solution for $T$ and $S$:

$$T + S = g(x^+), \quad T - S = h(x^-), \quad (4.5)$$

where $g$ and $h$ are two arbitrary, periodic functions.

The commutator algebra of Poincaré gauge transformations (2.1) is closed: we have $[\delta_0', \delta_0''] = \delta_0'''$, where $\delta_0' \equiv \delta_0(\xi', \theta')$ and so on, and the composition law reads:

$$\xi'''_{\mu} = \xi'^{\rho} \partial_{\rho} \xi''_{\mu} - \xi''^\rho \partial_{\rho} \xi'^{\mu},$$
$$\theta'''_{i} = \epsilon_{m n} \theta'^{m} \theta''^{n} + \xi'^{i} \cdot \partial \theta''^i - \xi''^i \cdot \partial \theta'^i.$$

Substituting here the restricted form of the parameters (4.3) and comparing the lowest order terms, we find the relations

$$T''' = T' \partial_{2} S'' + S' \partial_{2} T'' - T'' \partial_{2} S' - S'' \partial_{2} T',$$
$$S''' = S' \partial_{2} S'' + T' \partial_{2} T'' - S'' \partial_{2} S' - T'' \partial_{2} T', \quad (4.6)$$

that are expected to be the composition law for $(T, S)$. To clarify the situation, consider the restricted form of the gauge parameters (4.3), and separate it into two pieces: the leading terms containing $T$ and $S$, which define the $(T, S)$ transformations, and the higher order terms that remain after imposing $T = S = 0$, which define the residual (or pure) gauge transformations. If the relations (4.6) are to represent the composition law for the $(T, S)$ transformations, one has to check their consistency with higher order terms in the commutator algebra. As one can verify, the commutator of two $(T, S)$ transformations produces not only a $(T, S)$ transformation, with the composition law (4.6), but also an additional, pure gauge transformation. However, pure gauge transformations are irrelevant for our discussion of the conservation laws. Indeed, as we shall see in section 6, they do not contribute to the values of the conserved charges (their generators vanish weakly). Thus, we are naturally led to correct the non-closure of the $(T, S)$ commutator algebra by introducing an improved definition of the asymptotic symmetry [3, 26]:

- the asymptotic symmetry group is defined as the factor group of the gauge group determined by (4.3), with respect to the residual gauge group.

In other words, two asymptotic transformations are identified if they have the same $(T, S)$ pairs, and any difference stemming from the pure gauge terms is ignored. The asymptotic symmetry of our spacetime coincides with the conformal symmetry (see section 6).

In conclusion, the set of asymptotic conditions (4.1) is shown to be invariant under the conformal symmetry group, which is much larger then the original AdS group $SO(2, 2)$. The resulting configuration space respects the requirements (a) and (b) formulated at the beginning of this section. The asymptotic structure of the whole phase space, as well as the status of the last requirement (c), will be examined in the next two sections.
5 Gauge generator

In gauge theories, the presence of unphysical variables is closely related to the existence of gauge symmetries. The best way to understand the dynamical content of these symmetries is to explore the related canonical generator, which acts on the basic dynamical variables via the PB operation. To begin the analysis, we rewrite the action (2.4) as

\[ I = \int d^3 x \varepsilon^i_{\mu \nu \rho} \left[ ab^i_\mu R^\mu_{\nu \rho} - \frac{1}{3} \Lambda \varepsilon_{ijk} b^i_\mu b^j_\nu b^k_\rho \right. \\
\left. + \alpha_3 \left( \omega^i_\mu \partial_\nu \omega_i^\rho + \frac{1}{3} \varepsilon_{ijk} \omega^i_\mu \omega^j_\nu \omega^k_\rho \right) + \frac{1}{2} \alpha_4 b^i_\mu T^i_{\nu \rho} \right]. \] (5.1)

Hamiltonian and constraints. The basic Lagrangian variables \((b^i_\mu, \omega^i_\mu)\) and the corresponding canonical momenta \((\pi^i_\mu, \Pi^i_\mu)\) are related to each other through the set of primary constraints:

\[ \phi^0_i \equiv \pi^0_i \approx 0, \quad \Phi^0_i \equiv \Pi^0_i \approx 0, \quad \phi^\alpha_i \equiv \pi^\alpha_i - \alpha_4 \varepsilon^{\alpha \beta \gamma} b_i^\beta \approx 0, \quad \Phi^\alpha_i \equiv \Pi^\alpha_i - \varepsilon^{\alpha \beta \gamma} (2ab_i^\beta + \alpha_3 \omega_i^\beta) \approx 0. \] (5.2)

Explicit construction of the canonical Hamiltonian yields an expression which is linear in unphysical variables, as expected:

\[ H_c = b_i^0 H_i + \omega_i^0 K_i + \partial_\alpha D^\alpha, \]
\[ H_i = -\varepsilon^{0 \alpha \beta} \left( aR^\alpha_{\beta \gamma} + \alpha_4 T^\alpha_{\beta \gamma} - \Lambda \varepsilon_{ijk} b^i_\alpha b^j_\beta b^k_\gamma \right), \]
\[ K_i = -\varepsilon^{0 \alpha \beta} \left( aT^\alpha_{\beta \gamma} + \alpha_3 R^\alpha_{\beta \gamma} + \alpha_4 \varepsilon_{ijk} b^i_\alpha b^j_\beta b^k_\gamma \right), \]
\[ D^\alpha = \varepsilon^{0 \alpha \beta} \left[ \omega_i^0 \left( 2ab_i^\beta + \alpha_3 \omega_i^\beta \right) + \alpha_4 b_i^0 b_i^\beta \right]. \]

Going over to the total Hamiltonian,

\[ H_T = b_i^0 H_i + \omega_i^0 K_i + u_i^\mu \phi^\mu_i + v_i^\mu \Phi^\mu_i + \partial_\alpha D^\alpha, \] (5.3)

we find that the consistency conditions of the sure primary constraints \(\pi^0_i\) and \(\Pi^0_i\) yield the secondary constraints:

\[ H_i \approx 0, \quad K_i \approx 0. \] (5.4a)

These constraints can be equivalently written in the form:

\[ T_{i \alpha \beta} \approx p\varepsilon_{ijk} b^j_\alpha b^k_\beta, \quad R_{i \alpha \beta} \approx q\varepsilon_{ijk} b^j_\alpha b^k_\beta. \] (5.4b)

The consistency of the remaining primary constraints \(\phi^i_\alpha\) and \(\Phi^i_\alpha\) leads to the determination of the multipliers \(u^i_\beta\) and \(v^i_\beta\) (see Appendix B):

\[ u^i_\beta + \varepsilon^{ijk} \omega_j^\alpha b^k_\beta - \nabla_\beta b^i_j = p\varepsilon^{ijk} b_j^\alpha b^k_\beta, \]
\[ v^i_\beta - \nabla_\beta \omega^i_\alpha = q\varepsilon^{ijk} b_j^\alpha b^k_\beta. \] (5.5a)

Using the equations of motion \(b^i_\beta = u^i_\beta\) and \(\omega^i_\beta = v^i_\beta\), these relations reduce to the field equations

\[ T^i_{0 \beta} \approx p\varepsilon^{ijk} b_j^\alpha b^k_\beta, \quad R^i_{0 \beta} \approx q\varepsilon^{ijk} b_j^\alpha b^k_\beta. \] (5.5b)
The substitution of the determined multipliers (5.5a) into (5.3) yields the final form of the total Hamiltonian:

\[
H_T = \hat{H}_T + \partial_\alpha \hat{D}^\alpha,
\]

\[
\hat{H}_T = b^i_0 \hat{H}_i + \omega^i_0 \hat{K}_i + u^0_0 \pi^0_i + v^0_0 \Pi^0_i,
\]

(5.6a)

where

\[
\hat{H}_i = H_i - \nabla_\beta \phi_i^\beta - \varepsilon_{ijk} b^j_\beta \left( p\phi^k + q\Phi^k \right),
\]

\[
\hat{K}_i = \mathcal{K}_i - \nabla_\beta \Phi_i^\beta - \varepsilon_{ijk} b^j_\beta \phi^k,
\]

\[
\hat{D}^\alpha = D^\alpha + b^i_0 \phi_i^\alpha + \omega^i_0 \Phi_i^\alpha.
\]

(5.6b)

Further investigation of the consistency procedure is facilitated by observing that the secondary constraints \(\hat{H}_i, \hat{K}_i\) obey the PB relations (B.2). One concludes that the consistency conditions of the secondary constraints (5.4) are identically satisfied, which completes the Hamiltonian consistency procedure.

Complete classification of the constraints is given in the following table.

| First class | Second class |
|-------------|--------------|
| Primary     | Secondary    |
| \(\pi^0_i, \Pi^0_i\) | \(\phi^\alpha_i, \Phi^\alpha_i\) |

**Canonical gauge generator.** The results of the previous analysis are sufficient for the construction of the gauge generator [27]. Starting from the primary first class constraints \(\pi^0_i\) and \(\Pi^0_i\), one obtains:

\[
G[\epsilon] = \dot{\epsilon}^i \pi^0_i + \epsilon^i \left[ \hat{H}_i - \varepsilon_{ijk} \left( \omega^j_0 - p b^j_0 \pi^k 0 + q \varepsilon_{ijk} b^j_0 \Pi^k 0 \right) \right],
\]

\[
G[\tau] = \tau^i \Pi^0_i + \tau^i \left[ \hat{K}_i - \varepsilon_{ijk} \left( b^j_0 \pi^k 0 + \omega^j_0 \Pi^k 0 \right) \right].
\]

(5.7)

The complete gauge generator is given by the expression 

\[
G = G[\epsilon] + G[\tau],
\]

and its action on the fields, defined by \(\delta_0 \phi = \{\phi, G\}\), has the form:

\[
\delta_0 b^i_\mu = \nabla_\mu \epsilon^i - p\varepsilon_{ijk} b^j_\mu \tau^k + \varepsilon_{jk} b^j_\mu \tau^k,
\]

\[
\delta_0 \omega^i_\mu = \nabla_\mu \tau^i - q\varepsilon_{ijk} b^j_\mu \epsilon^k.
\]

This result looks more like a standard gauge transformation, with no trace of the expected local Poincaré transformations. However, after introducing the new parameters

\[
\epsilon^i = -\xi^\mu b^i_\mu, \quad \tau^i = - (\theta^i + \xi^\mu \omega^i_\mu),
\]

one easily obtains

\[
\delta_0 b^i_\mu = \delta_{\text{PGT}} b^i_\mu - \xi^\rho \left( T^i_{\mu \rho} - p \varepsilon_{ijk} b^j_\mu b^k_\rho \right),
\]

\[
\delta_0 \omega^i_\mu = \delta_{\text{PGT}} \omega^i_\mu - \xi^\rho \left( R^i_{\mu \rho} - q \varepsilon_{ijk} b^j_\mu b^k_\rho \right).
\]
Thus, on-shell, we have the transformation laws that are in complete agreement with (2.1). Expressed in terms of the new parameters, the gauge generator takes the form

\[ G = -G_1 - G_2, \]
\[ G_1 \equiv \dot{\xi}^\rho \left( b^{i}_\rho \pi_i^0 + \omega^{i}_\rho \Pi_i^0 \right) + \xi^\rho \left[ b_i^\rho \dot{\mathcal{H}}_i + \omega_i^\rho \dot{\mathcal{K}}_i + (\partial_\mu b_i^0) \pi_i^0 + (\partial_\mu \omega_i^0) \Pi_i^0 \right], \]
\[ G_2 \equiv \dot{\theta}^i \pi_i^0 + \dot{\theta}^i \left[ \dot{\mathcal{K}}_i - \varepsilon_{ijk} \left( b_j^0 \pi^k + \omega_j^0 \Pi_k^0 \right) \right], \tag{5.8} \]

where the time derivatives \( \dot{b}_\mu \) and \( \dot{\omega}_\mu \) are short for \( u_i^\mu \) and \( v_i^\mu \), respectively. Note, in particular, that the time translation generator is determined by the total Hamiltonian:

\[ G \left[ \xi^0 \right] = -\xi^0 \left( b_j^0 \pi^i + \omega_j^0 \Pi_j^0 \right) - \xi^0 \dot{\mathcal{H}}_T. \]

In the above expressions, the integration symbol \( \int d^3x \) is omitted for simplicity; later, when necessary, it will be restored.

**Asymptotics of the phase space.** In order to extend the asymptotic conditions (4.1) to the canonical level, one should determine an appropriate asymptotic behavior of the whole phase space, including the momentum variables. This step is based on the following general principle:

- the expressions than vanish on shell should have an arbitrary fast asymptotic decrease, as no solutions of the field equations are thereby lost.

By applying this principle to the primary constraints (5.2), one finds the following asymptotic behavior of the momentum variables:

\[ \pi_i^0 = \dot{\mathcal{O}}_i, \quad \pi_i^\alpha = \alpha_4 \varepsilon^{0\alpha\beta} b_i^\beta + \dot{\mathcal{O}}_i, \]
\[ \Pi_i^0 = \dot{\mathcal{O}}_i, \quad \Pi_i^\alpha = 2a \varepsilon^{0\alpha\beta} b_i^\beta + \alpha_3 \varepsilon^{0\alpha\beta} \omega_i^\beta + \dot{\mathcal{O}}_i. \tag{5.9} \]

We shall use this principle again in connection to the consistency requiremets (5.4b) and (5.5b), in order to refine the general asymptotic conditions (4.1) and (5.9) (Appendix C).

## 6 Canonical structure of the asymptotic symmetry

In this section, we study the influence of the adopted asymptotics on the canonical structure of the theory: we construct the improved gauge generators, examine their canonical algebra and prove the conservation laws.

**Improving the generators.** The canonical generator acts on dynamical variables via the PB operation, which is defined in terms of functional derivatives. A phase-space functional \( F = \int d^2x f(\phi, \partial \phi, \pi, \partial \pi) \) has well defined functional derivatives if its variation can be written in the form \( \delta F = \int d^2x \left[ A(x) \delta \phi(x) + B(x) \delta \pi(x) \right] \), where terms \( \delta \phi_{\mu} \) and \( \delta \pi_{\nu} \) are absent. In order to ensure this property for our generator (5.8), we have to improve its form by adding certain surface terms [28].

Let us start the procedure by examining the variations of \( G_2 \):

\[
\delta G_2 = \theta^i \delta \dot{\mathcal{K}}_i + R = \theta^i \delta \mathcal{K}_i + \partial \dot{\mathcal{O}} + R \\
= -2 \varepsilon^{0\alpha\beta} \theta^i \left( a \partial_\alpha \delta b_i^\beta + \alpha_3 \partial_\alpha \delta \omega_i^\beta \right) + \partial \dot{\mathcal{O}} + R \\
= -2 \varepsilon^{0\alpha\beta} \partial_\alpha \left( a \theta^i \delta b_i^\beta + \alpha_3 \theta^i \delta \omega_i^\beta \right) + \partial \dot{\mathcal{O}} + R = \partial \mathcal{O}_2 + R,
\]
where the last equality follows from the asymptotic relations \( \theta^i \delta b_{i\beta}, \theta^i \delta \omega_{i\beta} = O_2 \). The total divergence term \( \partial \mathcal{O}_2 \) gives a vanishing contribution after integration, as follows from the Stokes theorem:

\[
\int_{\mathcal{M}_2} d^2 x \partial_\alpha v^\alpha = \int_{\partial \mathcal{M}_2} v^\alpha df_\alpha = \int_0^{2\pi} v^1 d\varphi \quad (df_\alpha = \varepsilon_{\alpha\beta} dx^\beta),
\]

where the boundary of the spatial section \( \mathcal{M}_2 \) of spacetime is taken to be the circle at infinity, parametrized by the angular coordinate \( \varphi \). Thus, the boundary term for \( G_2 \) vanishes, and \( G_2 \) is regular as it stands, without any correction.

Going over to \( G_1 \), we have:

\[
\delta G_1 = \xi^\rho (b^i_{\rho} \delta \tilde{H}_i + \omega^i_{\rho} \delta \tilde{K}_i) + \partial \mathcal{O} + R
\]

\[
= -2\varepsilon^{0\alpha\beta}_0 \partial_\alpha \left[ \xi^\rho b^i_{\rho} \delta \left( a\omega_{i\beta} + \alpha_4 b_{i\beta} \right) + \xi^\rho \omega^i_{\rho} \delta \left( ab_{i\beta} + \alpha_3 \omega_{i\beta} \right) \right] + \partial \mathcal{O} + R.
\]

Using the adopted asymptotic conditions, the preceding result leads to

\[
\delta G_1 = -\partial_\alpha \left( \xi^0 \delta \mathcal{E}^\alpha + \xi^2 \delta \mathcal{M}^\alpha \right) + \partial \mathcal{O}_2 + R
\]

\[
= -\delta \partial_\alpha \left( \xi^0 \mathcal{E}^\alpha + \xi^2 \mathcal{M}^\alpha \right) + \partial \mathcal{O}_2 + R,
\]

where

\[
\mathcal{E}^\alpha \equiv 2\varepsilon^{0\alpha\beta}_0 \left[ \left( a + \frac{\alpha_3 \rho}{2} \right) \omega^0_{\beta} + \left( \alpha_4 + \frac{\alpha \rho}{2} \right) b^0_{\beta} + \frac{a}{\ell} b^2_{\beta} + \frac{\alpha_3}{\ell} \omega^2_{\beta} \right] b^0_{0},
\]

\[
\mathcal{M}^\alpha \equiv -2\varepsilon^{0\alpha\beta}_0 \left[ \left( a + \frac{\alpha_3 \rho}{2} \right) \omega^2_{\beta} + \left( \alpha_4 + \frac{\alpha \rho}{2} \right) b^2_{\beta} + \frac{a}{\ell} b^0_{\beta} + \frac{\alpha_3}{\ell} \omega^0_{\beta} \right] b^2_{2}. \tag{6.1}
\]

Thus, the improved form of the complete gauge generator (5.8) reads:

\[
\tilde{G} = G + \Gamma,
\]

\[
\Gamma = -\int df_\alpha \left( \xi^0 \mathcal{E}^\alpha + \xi^2 \mathcal{M}^\alpha \right) = -\int_0^{2\pi} d\varphi \left( \ell T \mathcal{E}^1 + S \mathcal{M}^1 \right). \tag{6.2}
\]

The adopted asymptotic conditions guarantee that \( \tilde{G} \) is finite and differentiable functional. The boundary term \( \Gamma \) depends only on \( T \) and \( S \), not on any pure gauge term in (4.3).

The improved time translation generator has the form

\[
\tilde{G}[\xi^0] = G[\xi^0] - E[\xi^0],
\]

\[
E[\xi^0] \equiv \int_0^{2\pi} d\varphi \xi^0 \mathcal{E}^1. \tag{6.3a}
\]

For \( \xi^0 = 1 \), the generator \( G \) reduces to \( -\hat{H}_T \), and the corresponding boundary term has the meaning of energy:

\[
\tilde{H}_T = \hat{H}_T + E, \quad E = \int_0^{2\pi} d\varphi \mathcal{E}^1. \tag{6.3b}
\]

The improved spatial rotation generator is given by

\[
\tilde{G}[\xi^2] = G[\xi^2] - M[\xi^2],
\]

\[
M[\xi^2] \equiv \int_0^{2\pi} d\varphi \xi^2 \mathcal{M}^1. \tag{6.4a}
\]
where $M$ is a finite integral. The boundary term for $\xi^2 = 1$,

$$M = \int_0^{2\pi} d\varphi \mathcal{M}^1,$$  \hspace{1cm} (6.4b)

is the angular momentum of the system.

**Canonical algebra.** The PB algebra of the improved generators could be found by a direct calculation, but we shall rather use another, more instructive method, based on the results of Refs. [29] and [20]. Let us first recall that our improved generator (6.2) is a differentiable phase space functional that preserves the asymptotic conditions (4.1) and (5.9), hence, it satisfies the conditions of the main theorem in Ref. [29]. Introducing a convenient notation, $\tilde{G}' \equiv \tilde{G}[T', S']$, $\tilde{G}'' \equiv \tilde{G}[T'', S'']$, the main theorem states that the PB $\{\tilde{G}'', \tilde{G}'\}$ of two differentiable generators is itself a differentiable generator. Taking into account that any differentiable generator $\tilde{G}$ is defined only up to an additive, constant phase-space functional $C$ (which does not change the action of $\tilde{G}$ on the phase space), the main theorem leads directly to

$$\{\tilde{G}'', \tilde{G}'\} = \tilde{G}''' + C''' ,$$ \hspace{1cm} (6.5)

where the parameters of $\tilde{G}'''$ are defined by the composition law (4.6), and $C'''$ is an unknown, field-independent functional, $C''' \equiv C''[T', S'; T'', S'']$. The term $C'''$ is known as the central charge of the PB algebra.

In order to calculate $C'''$, we note that $\{\tilde{G}'', \tilde{G}'\} = \delta_0\tilde{G}'' \approx \delta_0'\Gamma''$, where the weak equality is a consequence of the fact that $\delta_0'$ is a symmetry operation that maps constraints into constraints. Combining this result with $\tilde{G}''' \approx \Gamma'''$, Eq. (6.5) implies

$$\delta_0'\Gamma'' \approx \Gamma''' + C''' . \hspace{1cm} (6.6a)$$

This relation determines the value of $\Gamma'''$ only weakly, but since $\Gamma'''$ is a field-independent quantity, the weak equality is easily converted into the strong one. The calculation of $\delta_0'\Gamma''$ is based on the relations

$$\delta_0 (\ell\mathcal{E}^1) = -\mathcal{M}^1\partial_2 T - \ell\mathcal{E}^1\partial_2 S - \partial_2 \left(\mathcal{M}^1 T + \ell\mathcal{E}^1 S\right) + (2a + \alpha_3 p)\ell\partial_2^3 S - 2\alpha_3\partial_2^3 T + O_2 ,$$

$$\delta_0\mathcal{M}^1 = -\ell\mathcal{E}^1\partial_2 T - \mathcal{M}^1\partial_2 S - \partial_2 \left(\ell\mathcal{E}^1 T + \mathcal{M}^1 S\right) + (2a + \alpha_3 p)\ell\partial_2^3 T - 2\alpha_3\partial_2^3 S + O_2 ,$$

which follow from the refined asymptotic conditions derived in Appendix C, and the transformation rules defined by the parameters (4.3). Substituting the calculated expression for $\delta_0'\Gamma''$ into (6.6a) yields the following value for the central charge $C'''$:

$$C''' = (2a + \alpha_3 p)\ell \left(\int_0^{2\pi} d\varphi \left(\partial_2^7 T'' \partial_2^7 T' - \partial_2 S' \partial_2^7 T''\right) - 2\alpha_3 \int_0^{2\pi} d\varphi \left(\partial_2 T'' \partial_2^7 T' + \partial_2 S' \partial_2^7 S'\right) \right) . \hspace{1cm} (6.6b)$$
**Conservation laws.** As we noted in section 5, the improved total Hamiltonian is one of the generators, $\tilde{G}[1, 0] = -\ell \tilde{H}_T$. A direct calculation based on the PB algebra (6.5) shows that the asymptotic generator $\tilde{G}[T, S]$ is conserved [20]:

$$
\frac{d}{dt} \tilde{G} = \frac{\partial}{\partial t} \tilde{G} + \{ \tilde{G}, \tilde{H}_T \} \approx \frac{\partial}{\partial t} \Gamma[T, S] - \frac{1}{\ell} \Gamma[\partial_2 S, \partial_2 T] = 0.
$$

This also implies the conservation of the boundary term $\Gamma$.

Now, we wish to clarify the meaning of the conserved charges by calculating their values for the black hole solution (3.2). First, note that the black hole solution depends on the radial coordinate only, and consequently, the terms $E^1$ and $M^1$ in $\Gamma$ behave as constants.

Second, the parameters $(T, S)$ are periodic functions, equation (4.5), so that only zero modes in the Fourier expansion of $(T, S)$ survive the integration in $\Gamma$. Thus, there are only two independent non-vanishing charges for the black hole solution, given by two inequivalent choices of the constants $T$ and $S$. If we take, for instance, $(T = 1, S = 0)$ as the first choice, and $(T = 0, S = 1)$ as the second one, all the other non-zero charges will be given as linear combinations of these two.

For $(T = 1, S = 0)$ we have $\Gamma[1, 0] = -\ell E$, and the corresponding conserved charge is the energy $E$. Its value for the black hole solution is found to be

$$
E(\text{black hole}) = 4\pi \left[ m \left( a + \frac{\alpha_3 \rho}{2} \right) - \frac{\alpha_3 J}{\ell^2} \right].
$$

The second choice $(T = 0, S = 1)$ leads to $\Gamma[0, 1] = -M$. The corresponding conserved charge is the angular momentum $M$, and its black hole value reads

$$
M(\text{black hole}) = 4\pi \left[ J \left( a + \frac{\alpha_3 \rho}{2} \right) - \alpha_3 m \right].
$$

Our expressions for the conserved charges (6.8) coincide with the results obtained in Ref. [19]. In the sector $\alpha_3 = 0$ (GR$_\Lambda$ and the teleparallel theory), we have $E = m$ and $M = J$ (in units $4G = 1$), while for $\alpha_3 \neq 0$, the constants $m$ and $J$ do not have directly the meaning of energy and angular momentum, respectively. Geometrically, the two independent charges (6.8) parametrize the family of globally inequivalent, asymptotically AdS spaces.

**Central charge.** Using the Fourier expansion, one can rewrite the canonical algebra (6.5) in a more familiar form. The parameters $(T, S)$ can be Fourier decomposed as follows:

$$
T = \sum_{-\infty}^{+\infty} \left( a_n e^{i \pi x^+} + \bar{a}_n e^{i \pi x^-} \right), \quad S = \sum_{-\infty}^{+\infty} \left( a_n e^{i \pi x^+} - \bar{a}_n e^{i \pi x^-} \right).
$$

The asymptotic generator is a linear, homogeneous function of the parameters, so that:

$$
\tilde{G}[T, S] = -2 \sum_{-\infty}^{+\infty} \left( a_n L_n + \bar{a}_n \bar{L}_n \right).
$$

The previous relations imply:

$$
2L_n = -\tilde{G}[T = S = e^{i \pi x^+}], \quad 2\bar{L}_n = -\tilde{G}[T = -S = e^{i \pi x^-}].
$$

(6.9a)
Expressed in terms of the Fourier coefficients $L_n$ and $\bar{L}_n$, the canonical algebra takes the form of two independent Virasoro algebras with classical central charges:

\[
\begin{align*}
\{L_n, L_m\} &= -i(n - m)L_{n+m} - \frac{c}{12} in^3\delta_{n,-m}, \\
\{L_n, \bar{L}_m\} &= -i(n - m)\bar{L}_{m+n} - \frac{\bar{c}}{12} in^3\delta_{n,-m}, \\
\{L_n, \bar{L}_m\} &= 0.
\end{align*}
(6.9b)
\]

The central charges, given in the standard string theory normalization, have the form:

\[
\begin{align*}
c &= 12 \cdot 2\pi \left[ a \ell + \alpha_3 \left( \frac{p\ell}{2} - 1 \right) \right], \\
\bar{c} &= 12 \cdot 2\pi \left[ a \ell + \alpha_3 \left( \frac{p\ell}{2} + 1 \right) \right].
\end{align*}
(6.10)
\]

Thus, the gravitational sector with $\alpha_3 \neq 0$ has the conformal asymptotic symmetry with two different central charges, while $\alpha_3 = 0$ implies $c = \bar{c} = 3\ell/2G$.

- The general classical central charges $c$ and $\bar{c}$ differ from each other, in contrast to the results obtained in $\text{GR}_\Lambda$ and the teleparallel theory [3, 20].

By redefining the zero modes, $L_0 \rightarrow L_0 + c/24$, $\bar{L}_0 \rightarrow \bar{L}_0 + \bar{c}/24$, the Virasoro algebra takes its standard form. One should note that the central term for the $SO(2, 2)$ subgroup, generated by $(L_{-1}, L_0, L_1)$ and $(\bar{L}_{-1}, \bar{L}_0, \bar{L}_1)$, vanishes. This is a consequence of the fact that $SO(2, 2)$ is an exact symmetry of the AdS vacuum [3].

7 Concluding remarks

In this paper, we investigated the canonical structure of 3D gravity with torsion.

(1) The geometric arena for the topological 3D gravity with torsion, defined by the Mielke-Baekler action (2.4), has the form of Riemann-Cartan spacetime.

(2) There exists an exact vacuum solution of the theory, the Riemann-Cartan black hole (3.2), which generalizes the standard BTZ black hole in $\text{GR}_\Lambda$.

(3) Assuming the AdS asymptotic conditions, we constructed the canonical conserved charges. Energy and angular momentum of the Riemann-Cartan black hole are different from the corresponding BTZ values.

(4) The PB algebra of the canonical generators has the form of two independent Virasoro algebras with classical central charges. The values of the central charges are different from each other, in contrast to the situation in $\text{GR}_\Lambda$ and the teleparallel theory. The implications of this result for the quantum structure of black hole are to be explored.

Acknowledgements

This work was supported by the Serbian Science foundation, Serbia. One of us (MB) would like to thank Milovan Vasilić, Friedrich Hehl and Yuri Obukhov for a critical reading of the manuscript and many useful suggestions.
A Symmetries of the AdS vacuum

The invariance conditions $\delta_0 b_i^\mu = 0$ for the AdS triad (3.3a) yield the set of requirements on the parameters $(\xi^\mu, \theta^i)$, the general solution of which has the form [20]

$$\begin{align*}
\xi^0 &= \ell \sigma_1 - \frac{r}{f} \partial_2 Q, \\
\xi^1 &= \ell^2 f \partial_0 \partial_2 Q, \\
\xi^2 &= \sigma_2 - \frac{\ell^2 f}{r} \partial_0 Q,
\end{align*}$$

$$\begin{align*}
\theta^0 &= -\frac{\ell^2}{r} \partial_0 Q, \\
\theta^1 &= Q, \\
\theta^2 &= \frac{1}{f} \partial_2 Q,
\end{align*}$$

where

$$Q \equiv \sigma_3 \cos x^+ + \sigma_4 \sin x^+ + \sigma_5 \cos x^- + \sigma_6 \sin x^-,$$

and $\sigma_i$ are six arbitrary dimensionless parameters. The invariance conditions $\delta_0 \omega^i_{\mu} = 0$ for the AdS connection (3.3b) do not produce any new restrictions on $(\xi^\mu, \theta^i)$. For each $k = 1, 2, \ldots, 6$, we can choose $\sigma_k = 1$ as the only non-vanishing constant, and find the corresponding basis of six independent Killing vectors $\xi^\mu_{(k)}$:

$$\begin{align*}
\xi_{(1)} &= (\ell, 0, 0), \\
\xi_{(2)} &= (0, 0, 1), \\
\xi_{(3)} &= \left( \frac{r}{f} \sin x^+, -\ell f \cos x^+, \frac{\ell f}{r} \sin x^+ \right), \\
\xi_{(4)} &= \left( \frac{r}{f} \cos x^+, \ell f \sin x^+, \frac{\ell f}{r} \cos x^+ \right), \\
\xi_{(5)} &= \left( -\frac{r}{f} \sin x^-, \ell f \cos x^-, \frac{\ell f}{r} \sin x^- \right), \\
\xi_{(6)} &= \left( \frac{r}{f} \cos x^-, \ell f \sin x^-, -\frac{\ell f}{r} \cos x^- \right),
\end{align*}$$

and similarly for $\theta^i_{(k)}$. As one can explicitly verify, the six pairs $(\xi^\mu_{(k)}, \theta^i_{(k)})$ fall into the class of asymptotic parameters (4.3), and define the algebra of the AdS group $SO(2, 2)$.

B The algebra of constraints

The structure of the PB algebra of constraints is an important ingredient in the analysis of the Hamiltonian consistency conditions. For the nontrivial part of the PB algebra involving $(\phi_i^\alpha, \Phi_i^\alpha, \mathcal{H}_i, \mathcal{K}_i)$, we have the following result:

$$\begin{align*}
\{ \phi_i^\alpha, \phi_j^\beta \} &= -2\alpha_4 \varepsilon^{0\alpha\beta} \eta_{ij} \delta, \\
\{ \phi_i^\alpha, \Phi_j^\beta \} &= -2\alpha_3 \varepsilon^{0\alpha\beta} \eta_{ij} \delta, \\
\{ \phi_i^\alpha, \mathcal{H}_j \} &= 2\varepsilon^{0\alpha\beta} \left[ \alpha_4 \eta_{ij} \partial_\beta \delta - \varepsilon_{ijk} \left( \alpha_4 k^\beta_{\alpha} - \Lambda b^\beta_{\alpha} \right) \delta \right], \\
\{ \phi_i^\alpha, \mathcal{K}_j \} &= 2\varepsilon^{0\alpha\beta} \left[ \alpha_3 \eta_{ij} \partial_\beta \delta - \varepsilon_{ijk} \left( \alpha_3 k^\beta_{\alpha} + \alpha_3 b^\beta_{\alpha} \right) \delta \right], \\
\{ \Phi_i^\alpha, \mathcal{H}_j \} &= 2\varepsilon^{0\alpha\beta} \left[ \alpha_4 \eta_{ij} \partial_\beta \delta - \varepsilon_{ijk} \left( \alpha_4 k^\beta_{\alpha} + \alpha_4 b^\beta_{\alpha} \right) \delta \right], \\
\{ \Phi_i^\alpha, \mathcal{K}_j \} &= 2\varepsilon^{0\alpha\beta} \left[ \alpha_3 \eta_{ij} \partial_\beta \delta - \varepsilon_{ijk} \left( \alpha_3 k^\beta_{\alpha} + \alpha_3 b^\beta_{\alpha} \right) \delta \right].
\end{align*}$$
The essential part of the PB algebra involving the first class constraints (\(\mathcal{H}_i, \mathcal{K}_i\)) is given by the following relations:

\[
\{\phi_i^\alpha, \mathcal{H}_j\} = \varepsilon_{ijk} \left(p\phi^k\alpha + q\Phi^k\alpha\right) \delta, \\
\{\phi_i^\alpha, \mathcal{K}_j\} = -\varepsilon_{ijk}\phi^k\alpha \delta, \\
\{\Phi_i^\alpha, \mathcal{H}_j\} = -\varepsilon_{ijk}\phi^k\alpha \delta, \\
\{\Phi_i^\alpha, \mathcal{K}_j\} = -\varepsilon_{ijk}\Phi^k\alpha \delta, \\
\{\mathcal{H}_i, \mathcal{H}_j\} = \varepsilon_{ijk} \left(p\mathcal{H}^k + q\mathcal{K}^k\right) \delta, \\
\{\mathcal{H}_i, \mathcal{K}_j\} = -\varepsilon_{ijk}\mathcal{H}^k \delta, \\
\{\mathcal{K}_i, \mathcal{K}_j\} = -\varepsilon_{ijk}\mathcal{K}^k \delta.
\]

(B.2)

### C Asymptotic form of the constraints

Here, we analyze the influence of the secondary constraints (5.4) and relations (5.5) for the determined multipliers, on the basic asymptotic conditions (4.1) and (5.9), using the principle formulated at the end of section 5.

Let us start with the secondary constraints (5.4b). Using (4.1) and (5.9), these constraints imply the following asymptotic relations:

\[
\omega_0 = O_4, \quad \omega_2 = O_4, \\
\partial_1(re_2) = O_3, \quad \partial_1(rm_2) = O_3, \\
\partial_1 \left[r(B^2 - B^0)\right] = \ell \left(\Omega^2 - \Omega^0\right) + r^2\Omega^1 \\
\quad + \left(1 - \frac{p}{\ell}\right) \left(B^2 - B^0 + \frac{r^2}{\ell}B^1\right) + O_3, \\
\partial_1 (rB^2) = \frac{p}{\ell}B^0 + B^2 + \frac{r^2}{\ell}B^1 + \ell\Omega^0 + O_3, \\
\partial_1 e_0 - \partial_0 e_2 = O_3, \quad \partial_1 m_0 - \partial_0 m_2 = O_3, \\
\ell e_0 + m_2 = O_3, \quad \ell m_0 + e_2 = O_3.
\]

(C.1)

where:

\[
e_\mu = \left(a + \frac{\alpha_3^p}{2}\right) \omega^\mu_0 + \left(\alpha_4 + \frac{ap}{2}\right) b^\mu_0 + \frac{ab}{\ell} b^\mu_0 + \frac{\alpha_3}{\ell} \omega^2_\mu, \\
m_\mu = \left(a + \frac{\alpha_3^p}{2}\right) \omega^\mu_2 + \left(\alpha_4 + \frac{ap}{2}\right) b^\mu_2 + \frac{ab}{\ell} b^\mu_2 + \frac{\alpha_3}{\ell} \omega^0_\mu.
\]

From the expressions (C.1), one easily concludes that the terms \(E^\alpha\) and \(M^\alpha\), included in the surface integral (6.2) for \(\Gamma\), satisfy the following asymptotic conditions:

\[
\partial_1 E^1 = O_3, \quad E^2 = O_3, \quad \partial_1 M^1 = O_3, \quad M^2 = O_3.
\]

(C.2)

In a similar manner, equations (5.5b) lead to:

\[
\partial_1 (rB^0) = B^0 + \frac{p}{\ell}B^2_0 + \frac{r^2}{\ell^2}B^1_0 + \ell\Omega^0 + O_3, \\
\partial_1(re_0) = O_3, \quad \partial_1(rm_0) = O_3, \\
\partial_2 e_0 - \partial_0 e_2 = O_3, \quad \partial_2 m_0 - \partial_0 m_2 = O_3, \\
\ell e_0 + m_2 = O_3, \quad \ell m_0 + e_2 = O_3.
\]

(C.3)
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