A Procedure to Investigate the Collapse Behavior of Masonry Domes: Some Meaningful Cases

Milorad Pavlovic, Emanuele Reccia, and Antonella Cecchi
Department of Architecture Construction Conservation, University Iuav di Venezia, Terese, Dorsoduro 2206, 30123 Venice, Italy

ABSTRACT
Masonry domes represent an important part of the architectural heritage. However, the literature about domes analysis seems less consistent than that referred to other masonry structures. The collapses that have happened in recent years as a consequence of seismic actions or lack of maintenance show the need for detailed studies. Here a limit analysis to evaluate the masonry domes behavior is presented. An algorithm based on the kinematic approach has been developed to evaluate the geometric position of the hinges that determine the minimum collapse load multiplier. The proposed procedure is validated by a comparison with some meaningful cases—the collapse of Anime Sante Church in L’Aquila, the collapse of San Nicolò Cathedral in Noto, the crack pattern of San Carlo Alle Quattro Fontane Church in Rome, and the analysis developed on Hagia Sofia in Istanbul. The comparison with real cases shows a good agreement between the model results and the phenomenological crack patterns.

1. Why is it important to study masonry domes?

The conservation of the cultural heritage is a challenge for contemporary society. In recent decades, significant resources have been allocated for the conservation and restoration of architectural heritage. Historical buildings were restored, protected, and reinforced with the intent to limit the risks of degradation or loss, due to phenomena of structural damage and to external factors such as differential settlements and earthquake effects. The wide diffusion of historic masonry constructions in Italy, Europe, and Mediterranean area requires reliable tools for the evaluation of their structural safety.

Referring to the heritage, the problem consists not only in evaluating the safety of the structure under service loads and at collapse, but also in evaluating its response to a series of hazards:

- Variations of boundary conditions (soil settlements, induced, for example, by works carried out close to the construction or by groundwater level variations);
- Structural configuration (partial demolitions, integrations and modifications, strengthening and seismic retrofitting interventions);
- Loads (due to use variations);
- Materials (material decay due to long-time creep effects, material improvement associated to strengthening interventions); and
- Natural hazards (earthquake, wind).

Masonry domes represent a characteristic feature of the architectural heritage. However, despite of many efforts to protect the historical heritage, during past 30 years several collapses of masonry domes occurred in Mediterranean area as a consequence of seismic actions or lack of maintenance.

For example, the dome of the Noto Cathedral in Sicily collapsed in 1996 as a consequence of the earthquake of 1990 (Binda et al. 2003; Tringali 2003) and more recently the same happened to the dome of Anime Sante Church after the earthquake that hit in 2009 the city of L’Aquila. Earthquakes have always been a serious hazard for this typology of structure, as demonstrated for examples by the vicissitudes of Hagia Sophia in Istanbul (Mainstone 1988; Hidaka et al. 1993; Sato et al. 1996), which has been destroyed and rebuilt many times as a consequence of seismic events.

Regardless the damages caused by earthquakes, it is common to observe cracking in masonry domes. While vertical cracks, usually do not provide problems of
stability—for example, the dome of Santa Maria del Fiore Cathedral in Firenze was designed by Brunelleschi considering the possibility of vertical cracking (Bartoli et al. 1996)—instead, horizontal cracks may be related to the development of a kinematic mechanism. An example is the church of San Carlo alle Quattro Fontane in Rome, built by Borromini in the 17th century: the dome showed vertical cracking in the two main axes, the longitudinal and the transversal axes, and horizontal cracks at the base of the dome and at the base of the lantern (Degni 2008).

In this article, some of the domes previously listed have been studied. They have been chosen on the base of geometric parameters that typically may influence the structural behavior of domes: the presence or not of the lantern, the shape—hemispheric or lowered, the plan—circular or elliptical, the presence or not of the backfill. The analyses performed have been compared with the surveys taken after the collapse, in the case of Anime Sante church and Noto cathedral, or with the crack pattern, in the case of Hagia Sophia and San Carlo alle Quattro Fontane.

Masonry domes are very common structures in monumental buildings that characterize the historical centers of the European cities. Although their relevance and their vulnerability, domes have not been studied as much as other masonry structures. The collapses happened in recent years show the need of detailed studies.

2. Scientific background

An accurate modeling of the mechanical behavior of masonry domes is of fundamental importance for the correct evaluation of their structural safety to ensure their conservation. A dome is a vaulted structure having a circular plan and usually the form of a portion of sphere. Geometrically speaking, a dome is a surface that can be divided into a series of wedges, joined by parallel bands. Along the wedges act meridian forces, which are always compressive under vertical loads, while along the parallel bands act hoop forces, which are perpendicular to the meridian forces. Hoop forces, that restrain the out-of-plane movement of the wedges, are compressive in the upper zone of the dome and tensile in the lower zone, in most cases the passage from compressive to tensile forces occurs between 45 and 60 degrees respect to the vertical axis (Heyman 1982; Como 2010; Lucchesi 2007), as shown in Figure 1, that reports results obtained by membrane analysis of a semi-circular masonry dome (Pavlovic 2013).

Therefore, the load-bearing capacity of masonry domes is related to their shape: A dome is a double-curved shell that thanks to its shape exhibits an optimal structural behavior under axial-symmetric loads. However, in case of masonry domes this shape effect is affected by the low tensile resistance of masonry material. When the stresses due to self-weight and dead loads exceed the weak tensile strength of masonry, then the first damages occur: vertical cracks open at the base of dome due to the hoop forces, which may suggest an enlargement of the drum. Due to the development of vertical cracks, the dome is divided in wedges; membrane stresses become compressive stresses and the dome starts behave like a series of concentric masonry arches along the meridians. This phenomenon is quite common in masonry domes and it does not compromise their structural behavior (Heyman 1967; Como 2011). Instead horizontal cracks, which are due to meridian forces, are more dangerous and may denounce the development of a kinematic mechanism that may lead to collapse. In fact, like in a simple masonry arch, the position of the line of thrust changes, moving from the center of the section to its hedges. When it comes out from the core of the section, the opposite side of the section is not more compressed and should transmit tensile stress, which is not allowed for by masonry material. When the line of thrust touches the external or internal surface of the dome, a mechanism starts—in that point a hinge develops, while in the opposite side.

![Figure 1. Membrane analysis performed on Hagia Sophia dome: (a) stresses along parallels; (b) comparison between meridian and hoop stresses.](image-url)

---

"M. PAVLOVIC ET AL."
a crack opens. This kind of structure is basically designed to bear vertical dead and gravity loads only (Heyman 1966, 1967, 1977). For this reason masonry vaults are vulnerable to seismic actions, as highlighted in the previous section.

Although the construction of masonry domes dates back to the past, most of the domes were built before the 17th century, the most important studies on domes were made only starting from the 18th century. At that period, project was concerned the definition of geometric and material characteristics. In 1773, Coulomb has shifted the problem from planning the structure to its static verification and also introduced the concept of the friction in the mechanics of masonry structures. Thus, many following studies—Navier, Fontana, De La Hire—were developed in the same way, taking into account the friction in equilibrium questions. Many researchers aimed at the choice of the mechanisms in agreement with the experiments conducted on various geometries of arches. Regardless, the most important structural analysis of dome made in the past is the study made by Poleni (1748) on the dome of the Basilica di San Pietro in Rome. The method proposed by Poleni to assess the stability of the San Pietro’s dome—cracked along the meridians from the base to the lantern—was inspired by the Hooke’s law, and it can be considered as a first limited formulation of the static theorem of limit analysis of masonry structures (Como 2009b). A complete overview of the historical approaches to the study of arch and domes and the evolution of the structural theories may be found in Kurrer (2008).

In the first half of the 20th century, new methods of analysis for the evaluation of the behavior of masonry arch have been proposed. The most diffused approach to study the stability of masonry arch has been proposed by (Pippard and Ashby 1936; Pippard 1948). The method, that consider the arch as a two-hinge arch for which the minimum load is applied to a fixed position, allows determining the exact position of the two additional plastic hinges that take place when the arch start behaving as a four-hinge mechanism. This approach was further extended by (Heyman 1966, 1967, 1977) with the introduction of the line of thrust and the enunciation of the safety theorem.

Starting from the second half of the past century, the attention to the domes increased; in the literature numerous studies concerning the mechanical behavior of masonry domes may be found. The main approaches may be synthetized in simplified models and refined models. Models may be based on the limit analysis (Heyman 1967; Kooharian 1953), which are more rough but at the same time more immediate, or elastic analysis (Flügge 1973), but also limit analysis, taking into account the mechanical properties of masonry material or the three-dimensional behavior of domes (O’Dowyer 1999).

Simplified models (Heyman 1966, 1967, 1982; Oppenheim et al. 1989; Livesley 1992; Milani et al. 2009a; 2009b) focus on the collapse of masonry domes, and are devoted to the definition of possible kinematic mechanism shapes and the evaluation of collapse load at varying loading conditions or shape of masonry domes or other geometrical parameters, such as the curvature of the dome, the presence or not of lantern and its weight, the presence of oeil-de-boeuf or other types of opening. The greater part of these studies follows the approach of the limit analysis, in which masonry domes are modeled as kinematic chain of rigid blocks (Gilbert and Melbourne 1994). As well known, this modeling approach adopts for masonry material the hypotheses of infinite compressive strength, infinite sliding strength and tensile strength equal to zero, but does not require other specific mechanical parameters. Only few studies propose a non-linear analysis of masonry domes (Pesciullesi et al. 1997; Milani and Tralli 2012).

Other approaches focus on the elastic behavior of masonry domes, providing more refined models that usually represent the dome through its middle surface, fit to perform three-dimensional membrane analysis. The greater part of these models (Lucchesi et al. 2007) is devoted to the definition of the stresses distribution in parallels and meridians at varying the geometrical shape of masonry dome, the loading or the boundary conditions. Masonry material is modeled as isotropic or orthotropic material depending on the sophistication of modeling approach. In so doing, mechanical parameters of blocks and mortar joints are taken into account. The sensitivity to the arrangement of blocks of the masonry domes mechanical behavior has been widely acknowledged by historic treatises and literature (Alberti 1989; Choisy 1883; Nelli 1753; Doci 1992) and by more recent works. (Milani and Cecchi 2013).

The present study aims to propose a simple and fast procedure to provide reliable evaluation of the minimum collapse load multiplier and of the relative mechanism of collapse, finding the position of the hinges. The method is based on the kinematic approach to the limit analysis. The use of a simplified model is justified by the fact that this typology of structure may be damaged or may collapse because of problems due to instability, without involving the strength of the masonry material. The stability of
domes may easily be represented by the line of thrust and is basically related to the geometrical distribution of loads instead to the mechanical properties of masonry material, which may not be exactly established. The analysis here performed is in membrane regime, the reference is made to wedge of dome, hence a bi-dimensional analysis may be performed (Oppenheim et al. 1989). This assumption is motivated by the fact that the presence of vertical cracks due to hoop forces is common in masonry domes; segmental domes do not have problem of stability (Heyman 1977; Como 2011; Foraboschi 2004; Blasi and Foraboschi 1994). In the case of domes subjected to uniform distributed loads, with rigid boundary conditions, and in the absence of differential settlement these assumptions can be considered correct; therefore, it is possible to consider portion of them.

In the proposed method, domes have been divided in arches having a constant thickness. The difference in term of self-weight between arch and clove is negligible. In fact, comparing an arch having width $w_A$ equal to 1 meter with a clove having the same width $w_A$ of the arch at $37^\circ$ (Figure 2) the difference in term of volume is approximately 1% for domes with a thickness of 1/10 of the diameter. Moreover, the position of the center of gravity in the arch is at 0.62 R while in the clove is at 0.48 R (Figure 2), which implies an in-stabilizing effect that errs on the side of safety. The proposed method is compared with some meaningful cases of collapsed or cracked domes. The idea is to check if the position of the cracks in a masonry dome coincides with the position of the hinges related to the minimum collapse multiplier. To verify this condition, an algorithm has been developed: to define the position of the hinges that correspond to kinematic mechanism activated by the minimum collapse multiplier. The comparison between the results obtained by the limit analysis and both real crack patterns and surveys on some damaged domes allows validation of the procedure.

3. Procedure to define minimum load multiplier for limit analysis

Here a method for the limit analysis based on the kinematic approach of masonry domes is presented. The proposed method has been compared with some meaningful cases of collapsed or cracked domes in order to verify its reliability. The aim is to check if the position of the cracks in a masonry dome coincides with the position of the hinges related to the minimum collapse multiplier. Thus, an algorithm has been developed, which allows definition of the position of the hinges, along the intrados and the extrados, generated by a kinematic mechanism that may be activated by the minimum collapse multiplier.

Figure 2. Comparison between clove and slice.
3.1. General case

The proposed method was developed considering the section of a generic monocentric masonry dome loaded by a lantern, which weight is schematized by two symmetrical forces $F_1$ and $F_2$ equal to each other, as that shown in Figure 3. For the activation of a mechanism is necessary the formation of four hinges. Two hinges $(X_2Y_2)$ and $(X_3Y_3)$ are considered fixed, and the last one is supposed to be at the base of the dome, which is typical of masonry arches; the another one $(X_3Y_2)$ is supposed to be matching the lantern. These fixed hinges are supposed to be on extrados. The position of the other two hinges $(X_1Y_1)$ and $(X_3Y_3)$ is considered variable along the intrados. As shown in Figure 2a, for both of them the supposed rotation is approximately $70^o$, from the base of the dome to lantern $(\alpha_0)$ and from the lantern to the base $(\alpha_3)$. Each range has been subdivided in some significant portions and for every part have been calculated the geometric center, the weight and the polar coordinates of the hinges. The area of every portion has been calculated like the portion of an annulus, in which lower and upper arches are in function of variable angles $\alpha_0$ and $\alpha_3$. When the mechanism is activate the interested portions $(A_1, A_2$ and $A_3$) lose their connection and the structure collapse (Figure 4b).

The angle $\alpha_0$ defines the portion of the dome that is not interested in the mechanism, while the $A_1$ is defined by the angle encompassed between $\alpha_0$ and $\alpha_2$. The right part of the structure is subdivided in two portions $A_2$ and $A_3$ defined by the angle $\alpha_3$ and, respectively, $\alpha_2$ and $\alpha_4$. The weight of every portion is schematized by a single force applied on the center $G_i (X_{Gi}; Y_{Gi})$ where $i = 1, 3$.

The proposed algorithm allows definition of the minimum collapse multiplier in function of variable angles $\alpha_0$ and $\alpha_3$. In Table 1 is presented an application of it, regarding the geometry, presented in Figure 3. The minimum collapse multiplier $\lambda c_{(\alpha_0, \alpha_3)}$ which represents the minimal horizontal force that determines the structural collapse, has been calculated in function of the variable angles $\alpha_0$ and $\alpha_3$ that determine not only the position of the hinges $(X_3Y_1)$ and $(X_3Y_3)$, but also the areas of arch portion. Each variation covers an angle from the base of the dome to the altitude of the lantern. For the left side the variation is represent by the angle $\alpha_0$, while for the right side it’s schematized by the angle $\alpha_3$. The necessary condition that guarantees the minimum collapse multiplier is in function of the combination of the two angles which must respect these conditions:

$$0 < \alpha_0 < \alpha_1 \text{ and } \alpha_2 < \alpha_3 < \alpha_4$$

$$\lambda c_{(\alpha_0, \alpha_3)} = \frac{\sum_{i=1}^{n} P_i \cdot \eta_i}{\sum_{i=1}^{n} F_i \cdot \eta F_i}$$

(EQ1)

where $P_i$ is the self-loads ($P_i$) that represent every portion encompassed between two hinges and displacement of kinematic chain ($\eta_i$), as shown in Figure 3.

The weights of the portion of arch were calculated by:

$$P_i = A_i \cdot \gamma$$

(EQ2)

where $\gamma$ is the specific weight and $A_i$ represents the area of the considered annulus. Each portion is in function of angle $\alpha_i$, internal $(r)$ and external $(R)$ radius, while $(b)$ and $(H)$ are dimensions of the rectangular portion where present (i.e. an elliptic dome has a vertical part on the minor axe, as shown in Figure 3.

$$A_i = \frac{R \cdot \alpha_i + r \cdot \alpha_i}{2} \cdot (R - r) + b \cdot H$$

(EQ3)

Also the positions of different centers depend of the considered portion and of the angle encompassed between two hinges, given by:

$$X_{Gi} = \frac{4}{3} \cdot \frac{R^3 - r^3}{R^2 - r^2} \cdot \sin \frac{\alpha_i}{2} \cdot \cos \frac{\alpha_i}{2}$$

$$Y_{Gi} = \frac{4}{3} \cdot \frac{R^3 - r^3}{R^2 - r^2} \cdot \sin \frac{\alpha_i}{2} \cdot \sin \frac{\alpha_i}{2} + \frac{H}{2}$$

(EQ4)

The relative position of centers is necessary for the evaluation of the displacements ($\eta_i$), but also the

Figure 3. General case.
The position of the hinges is required. The rotation \((\varphi_1)\) is considered unitary, thus:

\[
\eta_1 = \varphi_1 \cdot (X_{G1} - X_i)
\]

\[(\text{EQ6})\]

\[
\eta_2 = \varphi_1 \cdot \left(\frac{X_1 - X_i}{X_{2ass} - X_2}\right) \cdot (X_{G2} - X_{2ass})
\]

\[(\text{EQ7})\]

\[
\eta_3 = -\varphi_1 \cdot \left(\frac{X_2 - X_i}{X_{2ass} - X_2}\right) \cdot \left(\frac{X_3 - X_{2ass}}{X_4 - X_3}\right) \cdot (X_{G3} - X_4)
\]

\[(\text{EQ8})\]

\[
\eta_F = -\varphi_1 \cdot (X_F - X_i)
\]

\[(\text{EQ9})\]

\[
Y_{2ass} = Y_1 \cdot \left(\frac{Y_2 - Y_i}{X_2 - X_i}\right) \cdot (X_{2ass} - X_i)
\]

\[(\text{EQ11})\]

The generic position of the hinges at intrados is obtained by:

\[
X_i = \pm r \cdot \cos \alpha_i
\]

\[(\text{EQ12})\]

\[
Y_i = r \cdot \sin \alpha_i + H
\]

\[(\text{EQ13})\]

For this generic example, represented in Figure 3, the minimum collapse multiplier \(\lambda_c(a_0, \alpha_3)\) is equal to 3.632; the combination of angles of the relative kinematic mechanism is reported in Table 2 (Focacci 2008).

\[
X_{2ass} = \frac{(X_4 - X_3) \cdot [X_1 \cdot (Y_2 - Y_i) + (X_3 - Y_i) \cdot (X_2 - X_i)] - X_3 \cdot (Y_4 - Y_3) \cdot (X_2 - X_1)}{(Y_2 - Y_i) \cdot (X_4 - X_3) - (Y_4 - X_3) \cdot (X_2 - X_1)}
\]

\[(\text{EQ10})\]
3.2. Specific cases

The general procedure proposed may be particularized for specific cases, specifically, in the case of hemispheric domes (Figure 5), the term \( H \) has zero value. The forces \( F_1 \) and \( F_2 \) represent the load of the lantern. Their variation produces different values of the collapse multiplier but does not vary the position of the hinges. This relationship means that the position of the hinges is determined by the geometry and not by the loads, while the horizontal force, which determines the collapse is in function of the loads itself. For this geometric configuration the minimum collapse multiplier \( \lambda_{c(a0, a3)} \) is equal to 1.113; the combination of angles of the relative kinematic mechanism are reported in Table 2.

In the same way it is also possible to analyze the kinematic configuration of a hemispheric dome without the lantern (Figure 6), or an arch, on which is applied only a single force \( F \) with the aim to simulate an eccentric load condition, instead of two as in the previous case. For this geometric configuration, the minimum collapse multiplier \( \lambda_{c(a0, a3)} \) is equal to 0.186; the combination of angles of the relative kinematic mechanism is reported in Table 2.

Table 2. Angles (rad) and minimum collapse multiplier of the different schematization of dome.

|                    | \( \alpha_0 \) | \( \alpha_1 \) | \( \alpha_2 \) | \( \alpha_3 \) | \( \alpha_4 \) | \( \lambda_c \) |
|--------------------|---------------|---------------|---------------|---------------|---------------|--------------|
| General case       | 0.620         | 1.238         | 1.903         | 2.523         | 3.147         | 3.623        |
| Hemispheric dome loaded by 2 forces | 0.615         | 1.230         | 1.911         | 2.526         | 3.147         | 1.113        |
| Hemispheric dome Without lantern | 0.569         | 1.222         | 2.428         | 3.147 /         | 0.186         |

In the same way it is also possible to analyze the kinematic configuration of a hemispheric dome without the lantern (Figure 6), or an arch, on which is applied only a single force \( F \) with the aim to simulate an eccentric load condition, instead of two as in the previous case. For this geometric configuration, the minimum collapse multiplier \( \lambda_{c(a0, a3)} \) is equal to 1.113; the combination of angles of the relative kinematic mechanism are reported in Table 2.

In the same way it is also possible to analyze the kinematic configuration of a hemispheric dome without the lantern (Figure 6), or an arch, on which is applied only a single force \( F \) with the aim to simulate an eccentric load condition, instead of two as in the previous case. For this geometric configuration, the minimum collapse multiplier \( \lambda_{c(a0, a3)} \) is equal to 1.113; the combination of angles of the relative kinematic mechanism are reported in Table 2.

3.3. Concluding remark of general procedure

After the calculation of the minimum collapse multipliers regarding the showed structures (Figure 3, Figure 5, Figure 6), the results are compared in order to study the variation of the collapse multiplier and its relative variation.

Figure 7a shows the trend of the coefficients \( \lambda_{c(a0, a3)} \) in function of variable angle \( \alpha_0 \). The values regarding the generic case, showed in Figure 3, are significantly higher than the other two because the major stability that provides vertical portions at the base and the lantern. This values, like those of Figure 5, have a symmetric distribution on the contrary of Figure 6, in which the hemispheric structure is loaded by an only single force and this determines more instability than the case in Figure 5.

Figure 7b represents the relative variation \( \lambda_{c_{min}}/\lambda_{c(F_{min}} \) (where \( \lambda_{c_{min}} \) is the minimum collapse multiplier of the figuration showed in Figure 3, while \( \lambda_{c(F_{min}} \) refers to the minimum collapse multiplier of Figure 6) of the three minimum collapse multipliers calculated by the proposed algorithm in function of variable angle \( \alpha_0 \). One can see that the results obtained from the geometry proposed in Figure 3 are almost 10 times bigger than those of the structure illustrated in Figure 6, while the configuration of Figure 5 has a value six times than Figure 6. This finding means that the major stability in a vaulted structure is ensured by the presence of a symmetric load (Figure 5) and also by the presence of vertical structures at the base, which allows to raise the center of gravity (Figure 3).

4. Some meaningful cases

In this section, the proposed method is validated by a comparison with some meaningful cases: the map of cracks of San Carlo Alle Quattro Fontane Church in Rome, the post-collapse configuration of San Nicolò Cathedral in Noto, the post-collapse configuration of Anime Sante Church in L’Aquila, and the survey
analysis made on Hagia Sofia in Istanbul. The comparison with real cases shows the reliability of the method.

4.1. San Carlo alle Quattro Fontane

The church of San Carlo alle Quattro Fontane has been planned by Borromini in Roma during the 17th century and is one of the most famous example of Roman Baroque. The church is based on a stretched central octagonal plant flanked by lateral chapels. Several restoration work have been carried out on the whole complex during its life, but the most significant were completed only in the early 2000 (Degni 2008). At the end of 1980s, the church and the monastery were in bad state of conservation, in particular the dome showed vertical cracking in the two main axis, as shown by the survey reported in Figure 8.

While those cracks are typical of the masonry domes, one can observe that horizontal cracks, which characterizes kinematic mechanism, are located at the base of the dome and at the base of the lantern. Hence in the analysis two of the four hinges have been positioned in correspondence of the existing horizontal cracks, while the position of the other two hinges that develop the four hinges mechanism activated by the minimum collapse multiplier have been calculated by the proposed algorithm: they show the possible position of horizontal cracks in case of earthquake. Two models of the dome have been realized along the two main axis because, due to the elliptic shape of the base of the dome, the radius and weights are different in the longitudinal and cross-sections. The weight of the lantern has been schematized by two symmetric equal forces $F_1$ and $F_2$.

The collapse multipliers calculated along the two main axes are quite similar, but not equal, and so the position of the hinges are. This finding is due to the different angles between the base of the dome to the lantern. The minimum collapse multiplier $\lambda_c$ obtained along the cross-
section is equal to 0.553, while on the longitudinal section $\lambda_c$ is equal to 3.623; the angles related to the two minimum collapse multipliers are reported in Table 3. Figure 9 shows the two models and the kinematic mechanisms, the results obtained are compared in Figure 10.

### 4.2. San Nicolò Cathedral

The construction of the Noto Cathedral dates back to the 1693. The original project developed by Gagliardi was modified in 1770 after the collapse of the first dome, which occurred due to the leak terrain strength. The reconstruction was entrust to the architect Stefano Ittar in 1789, but also this project has not been completed because of the earthquake of 1848. The project was modified again and the dome was rebuilt for the third time. During the 1950s the Cathedral was restored—maintenance operations have been carried out both on the main vertical and horizontal structures; the wooden floor has been substituted by a new one in pre-stressed concrete; and the cracks of the main pillars were filled by gypsum mortar injections (Tringali 2003). On December 13, 1990, an important earthquake hit the city of Noto: even though at the moment is seemed that the entity of the earthquake was insufficient to procure serious damages to the San Nicolò Cathedral, it can be stated that it was the beginning of a series of phenomena of instability that led to the collapse of the dome on March 13, 1996. The collapse is certainly to be attributed to a number of weaknesses and injuries suffered by the structures as a consequence of the earthquake of 1990, but also the low quality of the masonry material and presence of the pre-stressed concrete was damaged.

Table 3. Angles (rad) and minimum collapse multipliers for the different cases study.

| Case Study                        | $\alpha_0$ | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\alpha_4$ | $\lambda_c$ |
|----------------------------------|------------|------------|------------|------------|------------|-------------|
| San Carlo alle Quattro Fontane: cross section | 0.889      | 1.266      | 1.875      | 2.637      | 3.147      | 0.553       |
| San Carlo alle Quattro Fontane: longitudinal section | 0.620      | 1.238      | 1.903      | 2.523      | 3.147      | 3.623       |
| Cathedral of Noto                | 0.815      | 1.353      | 1.789      | 2.600      | 3.147      | 0.149       |
| Anime Sante: without backfill     | 0.788      | 1.313      | 1.827      | 2.615      | 3.147      | 0.077       |
| Anime Sante: with backfill        | 0.788      | 1.313      | 1.827      | 2.615      | 3.147      | 0.302       |
| Hagia Sophia: original dome      | 0.958      | 1.222      | 1.836      | 2.449      | /           | 0.294       |
| Hagia Sophia: original dome with backfill | 0.958      | 1.222      | 1.836      | 2.449      | /           | 1.095       |
| Hagia Sophia: rebuilt dome       | 0.855      | 1.222      | 2.565      | 3.147      | /           | 1.643       |
| Hagia Sophia: Rebuilt dome with backfill | 0.855      | 1.222      | 2.565      | 3.147      | /           | 2.317       |

**Figure 9.** San Carlo alle Quattro Fontane, mechanisms of collapse: (a) longitudinal section and (b) cross section.
slab was crucial in the collapse of the dome (Binda et al. 2003). In fact, the heavy loads of the concrete slab were transmitted to the pillars—which have been partially damaged during the earthquake—leading to their collapse that involved parts of the roof of the dome (Como 2010). The conservation state of the Cathedral after the collapse is reported in Figure 11.

The analysis performed shows a good agreement between the kinematic mechanism of collapse obtained and the damages that occurred during the collapse (Figure 12). It is possible to observe that the damages interested a big portion of the dome (bigger than the Anime Sante collapse), probably due to the incompatibility of the materials. When the earthquake occurred, the concrete slab had a different structural behavior than the other masonry structures, and it produced horizontal forces at the base of the dome. In the analysis, a hinge has been assumed in correspondence of that point. The minimum collapse multiplier and the angles that determine the position of the hinges of the relative mechanism—reported in Table 3—are compatible with the real situation.

4.3. Anime Sante Church

The construction of the Anime Sante Church begun in 1713 to commemorate the victims of the earthquake.
that destroyed the city of L’Aquila in 1703. The church consists of a rectangular hall with a barrel vault flanked by two chapels on each side, with eight windows placed to illuminate the presbytery from the hemispherical dome (Figure 13). On April the 6, 2009, an earthquake of Mw = 6.3 (ML = 5.8) hit L’Aquila and dozens of villages along the Aterno valley. The earthquake caused 308 victims and seriously damaged the historic center of L’Aquila. The Anime Sante Church had the same destiny—the earthquake provoked the collapse of the lantern and of the dome, diffused cracks of the key stone arches, detachment of the facade and apse and shear cracking of several walls (Figure 14).

Figure 13. Anime Sante Church before the earthquake.

Figure 14. Anime Sante Church after the earthquake.
The analysis has been performed considering the hemispheric portion of the dome loaded by two equal symmetric forces $F_1$ and $F_2$ to simulate the lantern. The lowest collapse multiplier $\lambda_c$ is equal to 0.077 and is provided by the kinematic chain illustrated in Figure 15. The mechanism of collapse is shown in Figure 16 together with a modal analysis performed by means of Finite Element in membrane regime. The first natural frequency corresponds to a local mode of vibration of the lantern, hence, during the seism, stresses were concentrated at the base of it leading to a local mechanism that caused the collapse of the lantern itself. Looking at the surveys carried out after the earthquake (Figure 14) it clearly appears that the lantern was the first element that collapsed dragging a portion of the dome: the results of the limit analysis are in good agreement with it, as shown in Figure 16.

A further limit analysis has been performed taking into account the backfill at the base of dome, which provides a stabilizing effect (Reccia et al. 2014). The analysis has been performed starting from the results obtained without backfill: the combination of angles that provides the minimum collapse multiplier has been used to calculate the variation of the minimum collapse multiplier considering the backfill. Once determined geometry, weight ($P_{rin}$) and barycenter ($X_{G_{rin}}$; $Y_{G_{rin}}$) of the backfill, the relative horizontal ($\delta_{rin}$) and vertical ($\eta_{rin}$) shifts have been calculated in order to find the minimum collapse multiplier (Figure 17). The collapse multiplier obtained considering the backfill is significantly bigger than the previous: $\lambda_{c_{rin}} = 0.302$, hence the backfill provide an increase of stability equal to approximately 23%, as shown in Figure 18, where results are compared; the values of multipliers and angles are reported in Table 3.

### 4.4. Hagia Sofia

Hagia Sophia was originally built in the 4th century under Constantine, but soon after was totally destroyed by an earthquake. In the 6th century it was rebuilt by Emperor Justinian but collapsed again during another earthquake in 558. The dome was then rebuilt in the 562 with a greater high (Mainstone 1988), giving Hagia Sofia the actual...
configuration (apart from the minarets made in the 16th century when it was transformed into a mosque). Afterward, in the course of its history, several collapses occurred, with different reconstructions: the western arch was damaged during the earthquake of 869, while the western part of the dome collapsed during that of the 989. The earthquake of 1346 provoked a further collapse of the dome, damaging about one-third of the area opposite to the one that collapsed of the 989. The reconstruction of this portion, which ended a few years later, still shows the discontinuity between the reconstructed area and the pre-existing. In the following centuries did not happen further collapses; however, the dome is weakened by the lack of homogeneity between the three portions it comprises and by the strong irregularities in the whole geometry (Hidaka et al. 1993; Sato et al. 1996).

The structure of the church is complex. The great dome, 34 m of diameter, is supported by a system of four pillars and six arches: it was the first example in which were used the so-called pendentives to join the rectangular base to the hemispheric dome. The stability of the dome is ensured by two different structural system: in the longitudinal section (east–west) there are two semi-domes that contrast the weight and the thrust of the dome, while in the cross section (north–south) the same task is given by a system of buttresses, which produce a passive way of stability.

The analysis has been carried out on the base of the surveys and studies of Mainstone (1988): both the original and the rebuilt domes have been analyzed. Figure 19 and Figure 20 show the two mechanisms of collapse: the rebuilt dome has a greater stability than the original one. It is also possible to notice that the first hinge coincide with the existing buttress at the base of the dome, that should provide an increase of stability. Figure 21 shows the effect of backfill in collapse mechanism. A further analysis of the two domes has been performed taking into account them. Similarly to what previously done in the analysis of Anime Sante, the geometry, weight \(P_{rin}\) and barycenter \((X_{G,rin}, Y_{G,rin})\) of the buttress, the relative horizontal \(\delta_{rin}\) and vertical \(\eta_{rin}\) shifts have been calculated in order to find the minimum collapse multiplier Figure 22). Also in this case, the collapse multipliers obtained considering the backfill are significantly bigger than the previous ones, as shown in Figure 23. This means that the buttresses guarantee a major stability of the whole structure, even if they’re made of different material because the stability is insured by the position of the center of gravity. The results are compared in Figure 23, the values of multipliers and angles are reported in Table 3.

5. Conclusions

Limit kinematic analysis may be a reliable procedure to investigate the behavior at failure of historical masonry domes. The algorithm here presented...
provides fast and reliable estimations of collapse multiplier and mechanism of collapse, as demonstrated by the comparison with some meaningful real cases. The analyses conducted in this work, if considered synoptically, allow to outline the following remarks on the cases of study analyzed:

- 2D analyses are suitable to evaluate the collapse behavior of masonry domes, the 3D effect may be neglected for the evaluation of the minimum collapse multiplier. However, when needed, 2D limit analysis may be coupled with more complex analysis able to take into account the
mechanical characteristics of materials and the 3D effect.

- The lantern is an essential element of masonry dome, which play an important role in the structural behavior of domes.

- Backfill and buttress increase the stability of masonry domes.

The conservation of masonry dome may be improved thanks to a fast and reliable tool of ana-
ysis such as the algorithm here presented, if combined with proper engineering reasoning.

References

Alberti, L.B. 1989. L’architettura. Traduzione di Giovanni Orlandi: Introduzione e note di Paolo Portoghesi. Milan, Italy: Il Polifilo.

Bartoli, G., A. Chiarugi, and V. Gusella. 1996. Monitoring systems on historic buildings: The Brunelleschi Dome. Journal of Structural Engineering 122(6):663–673.

Binda, L., Tiraboschi, C. and Baronio G. 2003. On-site investigation on the remains of the Cathedral of Noto. Construction and Building Materials 17(8):543–555.

Blasi, C., and P. Foraboschi. 1994. Analytical approach to collapse mechanisms of circular masonry arch. Journal of Structural Engineering 120(8):2288–2308.

Choisy, A. 1883. L’art de bâtiur chez les Byzantins. Parigi. Paris, France: Librarie de la Société Anonyme de Publications Periodiques.

Como, M. 2010. Statica delle costruzioni storiche in muratura. Rome, Italy: Aracne Editrice.

Degni, P. 2008. La “fabrica” di San Carlino alle Quattro Fontane: Gli anni del restauro, Rome, Italy: Istituto Poligrafico e Zecca dello Stato.

Doci, M. 1992. La geometria delle cupole sangallesche a spina-pesce in Saggi in onore di renato Bonelli, ed. G. Bozzi, G. Carbonara, and G. Villetti, in Quaderni dell’Istituto di Storia dell’Architettura, Università degli Studi di Roma la Sapienza. Rome, Italy: Multigrafica Editrice, 15–20.

Flügge, W. 1973. Stresses in shells. Berlin, Germany: Springer & Verlag.

Focacci, F. 2008. Riforzo delle murature con materiali compositi. Palermo, Italy: Dario Flaccovio Editore.

Foraboschi, P. 2004. Strengthening of masonry arches with fiber-reinforced polymer strips. Journal of Composites for Construction 8(3):191–202.

Gilbert, M., and Melbourne, C. 1994. Rigid block analysis of masonry structures. Structural Engineering 72(21):356–361.

Heyman, J. 1966. The stone skeleton. International Journal of Solids and Structures 2:249–279.

Heyman, J. 1967. On shell solutions of masonry domes. International Journal of Solids and Structures 2:227–240.

Heyman, J. 1977. Equilibrium of shell structures. Oxford, UK: Clarendon Press.

Heyman, J. 1982. The masonry arch. Chichester, UK: Helios Horvud

Hidaka, K., T. Sato, Y. Kawabe, and M. Yorulmaz. 1993. Photogrammetry of the eastern semi-dome of Hagia Sophia, Istanbul. In Proceedings of the IASS–MSU International Symposium on “Public Assembly Structures from Antiquity to the Present (May 24–28, 1993, Istanbul, Turkey). Istanbul, Turkey: Mimar Sinan Universitesi.

Kooharian, A. 1953. Limit analysis of voussoir (segmental) and concrete arches. In Proceedings of the American Concrete Institute (ACI) 49:317–328.

Kurren, K. E. 2008. The history of theories of structures: From arch analysis to computational mechanics. Berlin, Germany: Erst and Son.

Livesley, R. K. 1992. A computational model for the limit analysis of three-dimensional masonry structures. Meccanica 27(3):161–172.

Lucchesi, M., C. Padovani, G. Pasquinelli, and N. Zani. 2007. Static analysis of masonry vaults, constitutive model and numerical analysis. Journal of Mechanics of Materials and Structures 2(2):221–244.

Mainstone, R. J. 1988. Hagia Sophia, Architecture, structure and liturgy of Justinian’s Great Church. New York, NY: Thames and Hudson.

Milani, G., and A. Cecchi. 2013. Compatible model for herringbone bond masonry: Linear elastic homogenization, failure surfaces and structural implementation. International Journal of Solids and Structures 50(20–21):3274–3296.

Milani, G., E. Milani, and A. Tralli. 2009a. Upper Bound limit analysis model for FRP-reinforced masonry curved structures. Part I: Unreinforced masonry failure surfaces. Computers and Structures 87:1516–1533.

Milani, G., E. Milani, and A. Tralli. 2009b. Upper bound limit analysis model for FRP-reinforced masonry curved structures. Part II: Structural analyses. Computers and Structures 87:1534–1558.

Milani, G., and A. Tralli. 2012. A simple meso–macro model based on SQP for the non-linear analysis of masonry double curvature structures. International Journal of Solids and Structures 49(5):808–834.

Nelli, G. B. 1753. Ragionamento sopra la maniera di voltar le cupole senza adoperarvi le centine. Firenze, Italy: Discorsi di Architettura.

Oppenheim, I. J., D. J. Gunaratnam, and R. H. Allen. 1989. Limit state analysis of masonry domes. Journal of Structural Engineering 115(4):868–882.

O’Dowrey, D. 1999 Funicolar analysis of masonry vaults. Computer and Structures 73:187–197.

Pavlovic, M. 2013. Alcune considerazioni sulle cupole storiche. Tecnologos 24:32–45.

Pippard, A. J. S. 1948 The approximate estimation of safe loads on masonry bridges. The Civil Engineer in War: Institution of Civil Engineers 1:365.

Pippard, A. J. S., and E. R. J. Ashby. 1936. An experimental study of the voissour arch. Journal of the Institution of Civil Engineers 10:383–403.

Pesciullesi, C., M. Rapallini, A. Tralli, and A. Cianchi. 1997. Optimal spherical masonry domes of uniform
Poleni, G. B. 1748. Memorie Istoriche della Gran Cupola del Tempio Vaticano. Padua, Italy: Stamperia del Seminario, Padova.

Reccia, E., G. Milani, A. Cecchi, and A. Tralli. 2014. Full 3D homogenization approach to investigate the behaviour of masonry arch bridges: The Venice trans-lagoon railway bridge, submitted. Construction and Building Materials 66:567–586.

Sato, T., K. Hidaka, Y. Kawabe, T. Aoki, and K. Yamashita. 1996. Formal characteristics of the setting lines on the cornice of the main Dome of Hagia Sophia, Istanbul. Journal of Architecture, Planning and Environmental Engineering 485:219–226.

Tringali, S. 2003. The partial reconstruction design of the Cathedral of Noto—Part I: The social-economic impact on the town and on the territory and the cross-vaults, arches and dome system. Construction and Building Materials 17(8):595–602.