JACOBIANS AMONG ABELIAN THREEFOLDS: A FORMULA OF KLEIN AND A QUESTION OF SERRE

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Abstract. Let \( k \) be a field and \( f \) be a Siegel modular form of weight \( h \geq 0 \) and genus \( g > 1 \) over \( k \). Using \( f \), we define an invariant of the \( k \)-isomorphism class of a principally polarized abelian variety \( (A, a)/k \) of dimension \( g \). Moreover when \( (A, a) \) is the Jacobian of a smooth plane curve, we show how to associate to \( f \) a classical plane invariant. As straightforward consequences of these constructions when \( g = 3 \) and \( k \subset \mathbb{C} \) we obtain (i) a new proof of a formula of Klein linking the modular form \( \chi_{18} \) to the square of the discriminant of plane quartics; (ii) a proof that one can decide when \( (A, a) \) is a Jacobian over \( k \) by looking whether the value of \( \chi_{18} \) at \( (A, a) \) is a square in \( k \). This answers a question of J.-P. Serre. Finally, we study the possible generalizations of this approach for \( g > 3 \).

1. Introduction

1.1. Torelli theorem. Let \( k \) be an algebraically closed field. If \( X \) is a (nonsingular irreducible projective) curve of genus \( g \) over \( k \), Torelli’s theorem states that the map \( X \mapsto (\text{Jac} X, j) \), associating to \( X \) its Jacobian together with the canonical polarization \( j \), is injective. The determination of the image of this map is a long time studied question.

When \( g = 3 \), the moduli space \( A_g \) of principally polarized abelian varieties of dimension \( g \) and the moduli space \( M_g \) of nonsingular algebraic curves of genus \( g \) are both of dimension \( 3g - 3 = g(g + 1)/2 = 6 \). According to Hoyt [12] and Oort and Ueno [25], the image of \( M_3 \) is exactly the space of indecomposable principally polarized abelian threefolds. Moreover if \( k = \mathbb{C} \), Igusa [17] characterized the locus of decomposable abelian threefolds and that of hyperelliptic Jacobians making use of two particular modular forms \( \Sigma_{140} \) and \( \chi_{18} \) on the Siegel upper half space of degree 3.

Assume now that \( k \) is any field and \( g \geq 1 \). J.-P. Serre noticed in [22] that a precise form of Torelli’s theorem reveals a mysterious obstruction for a geometric Jacobian to be a Jacobian over \( k \). More precisely, he proved the following:

Theorem 1.1.1. Let \( (A, a) \) be a principally polarized abelian variety of dimension \( g > 0 \) over \( k \), and assume that \( (A, a) \) is isomorphic over \( \overline{k} \) to the Jacobian of a curve \( X_0 \) of genus \( g \) defined over \( \overline{k} \). The following alternative holds:

(i) If \( X_0 \) is hyperelliptic, there is a curve \( X/k \) isomorphic to \( X_0 \) over \( \overline{k} \) such that \( (A, a) \) is \( k \)-isomorphic to \( (\text{Jac} X, j) \).

(ii) If \( X_0 \) is not hyperelliptic, there is a curve \( X/k \) isomorphic to \( X_0 \) over \( \overline{k} \), and a quadratic character

\[ \varepsilon : \text{Gal}(\overline{k}_{\text{sep}}/k) \longrightarrow \{ \pm 1 \} \]

such that the twisted abelian variety \( (A, a)_{\varepsilon} \) is \( k \)-isomorphic to \( (\text{Jac} X, j) \).

The character \( \varepsilon \) is trivial if and only if \( (A, a) \) is \( k \)-isomorphic to a Jacobian.
Thus, only case (i) occurs if \( g = 1 \) or \( g = 2 \), with all curves being elliptic or hyperelliptic.

1.2. Curves of genus 3. Assume now \( k \subset \mathbb{C} \) and \( g = 3 \). Let there be given an indecomposable principally polarized abelian threefold \((A, a)\) defined over \( k \). In a letter to J. Top [28], J.-P. Serre asked a twofold question:

— How to decide, knowing only \((A, a)\), that \( X \) is hyperelliptic?
— If \( X \) is not hyperelliptic, how to find the quadratic character \( \varepsilon \)?

Moreover, he suggested a strategy in order to compute the twisting factor \( \varepsilon \). This strategy is based on a formula of Klein [20] relating the modular form \( \chi_{18} \) (in the notation of Igusa), to the square of the discriminant of plane quartics, see Th. 4.1.2 for a more precise formulation. In [21], two of the authors justified Serre’s strategy for a three dimensional family of abelian varieties and in particular determined the absolute constant involved in Klein’s formula.

In this article we prove that Serre’s strategy can be applied to any abelian threefolds. More precisely, we take a broader point of view.

(i) We look at the action of \( k \)-isomorphisms on Siegel modular forms defined over \( k \) and we define invariants of \( k \)-isomorphism classes of abelian varieties over \( k \).

(ii) We link Siegel modular forms, Teichmüller modular forms and invariants.

Then we derive a proof of Klein’s formula based on moduli spaces.

Once these two goals achieved, Serre’s strategy can be rephrased as finding a Siegel modular form whose locus has a good multiplicity on the Jacobian locus and then using point (i) to distinguish between Jacobians and their twists. For \( g = 3 \), the form \( \chi_{18} \) fulfills the criterion as can be seen thanks to Klein’s formula. On the other hand, we show that this is no longer the case for \( \chi_h \) when \( g > 3 \). We would like to point out that we do not actually need Klein’s formula to prove Sére’s strategy.

Indeed we do not need to go the full way from Siegel modular form to invariants and could instead use a formula due to Ichikawa relating \( \chi_{18} \) to the square of a Teichmüller modular form (see Rem. 4.2.2). However we think that the connection between Siegel modular forms and invariants is interesting enough in its own, besides the fact that it gives a new proof of Klein’s formula.

The paper is organized as follows. In [2] we review the necessary elements from the theory of Siegel and Teichmüller modular forms. Only [2, 3] is original: we introduce the action of isomorphisms and see how the action of twists is reflected on the values of modular forms. In [3] we link modular forms and certain invariants of ternary forms. Finally in [4] we deal with the case \( g = 3 \). We give first a proof of Klein’s formula and then we justify the validity of Serre’s strategy. Finally we explain the reasons behind the failure of the obvious generalization of the theory in higher dimensions and state some natural questions.

Acknowledgements. We would like to thank J.-P. Serre and S. Meagher for fruitful discussions and Y. F. Bilu and X. Xarles for their help in the final part of Sec. 4.3.
Hence, for any algebra $A$, assume now that $A$ is a commutative ring and $h$ a positive integer. A geometric Siegel modular form of genus $g$ and weight $h$ over $R$ is an element of the $R$-module $S_{g,h}(R) = \Gamma(A \otimes R, \omega^h)$.

Note that for any $n \geq 1$, we have an isomorphism $A_g \simeq A_{g,n} / \text{Sp}_{2g}(\mathbb{Z}/n\mathbb{Z})$.

If $g \geq 3$, as shown in [24], from the rigidity lemma of Serre [27], we can deduce that the moduli space $A_{g,n}$ can be represented by a smooth scheme over $\mathbb{Z}[\zeta_n, 1/n]$. Hence, for any algebra $R$ over $\mathbb{Z}[\zeta_n, 1/n]$, the module $S_{g,h}(R)$ is the submodule of $\Gamma(A_{g,n} \otimes \mathbb{Z}[\zeta_n, 1/n] R, \omega^h)$ consisting of the elements invariant under $\text{Sp}_{2g}(\mathbb{Z}/n\mathbb{Z})$.

Assume now that $R = k$ is a field. If $f \in S_{g,h}(k)$, $A$ is a p.p.a.v. of dimension $g$ defined over $k$ and $\alpha$ is a basis of $\omega_k[A]$, define

$$f(A, \alpha) = f(A)/\alpha^h.$$

In this way such a modular form defines a rule which assigns the element $f(A, \alpha) \in k$ to every such pair $(A, \alpha)$ and such that:

(i) $f(A, \lambda \alpha) = \lambda^{-h} f(A, \alpha)$ for any $\lambda \in k^\times$.
(ii) $f(A, \alpha)$ depends only on the $\mathbb{F}$-isomorphism class of the pair $(A, \alpha)$.

Conversely, such a rule defines a unique $f \in S_{g,h}(k)$. This definition is a straightforward generalization of that of Deligne-Serre [8] and Katz [19] if $g = 1$.

2.2. Complex uniformisation. Assume $R = \mathbb{C}$. Let $\mathbb{H}_g = \{ \tau \in \mathbb{M}_g(\mathbb{C}) \mid '\tau = \tau, \text{Im} \tau > 0 \}$ be the Siegel upper half space of genus $g$ and $\Gamma = \text{Sp}_{2g}(\mathbb{Z})$. As explained in [41 §2], the complex orbifold $A_g(\mathbb{C})$ can be expressed as the quotient $\Gamma \backslash \mathbb{H}_g$ where $\Gamma$ acts by

$$M \cdot \tau = (a \tau + b) \cdot (c \tau + d)^{-1} \quad \text{if} \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

The group $\mathbb{Z}^{2g}$ acts on $\mathbb{H}_g \times \mathbb{C}^g$ by

$$v.(\tau, z) = (\tau, z + \tau m + n) \quad \text{if} \quad v = \begin{pmatrix} m \\ n \end{pmatrix} \in \mathbb{Z}^{2g}.$$
If $U_g = \mathbb{Z}^{2g} \setminus (\mathbb{H}_g \times \mathbb{C}^g)$, the projection
\[ \pi : U_g \longrightarrow \mathbb{H}_g \]
defines a universal principally polarized abelian variety with fibres
\[ A_\tau = \pi^{-1}(\tau) = \mathbb{C}^g / (\mathbb{Z}^g + \tau \mathbb{Z}^g). \]

Let $j(M, \tau) = cq + d$ and define the action of $\Gamma$ on $\mathbb{H}_g \times \mathbb{C}^g$ by
\[ M. (\tau, (z_1, \ldots, z_g)) = (M. \tau, j(M, \tau)^{-1} \cdot (z_1, \ldots, z_g)) \quad \text{if } M \in \Gamma. \]
The map $j(M, \tau)^{-1} : \mathbb{C}^g \rightarrow \mathbb{C}^g$ induces an isomorphism:
\[ \varphi_M : A_\tau \longrightarrow A_{M. \tau}. \]
Hence, $V_g(\mathbb{C}) \simeq \Gamma \setminus U_g$ and the following diagram is commutative:
\[
\begin{array}{ccc}
\Gamma \setminus U_g & \sim & V_g(\mathbb{C}) \\
\downarrow & & \downarrow \\
\Gamma \setminus \mathbb{H}_g & \sim & A_g(\mathbb{C})
\end{array}
\]

As in [7, p. 141], let
\[ \zeta = \frac{dq_1}{q_1} \wedge \cdots \wedge \frac{dq_g}{q_g} = (2i\pi)^g dz_1 \wedge \cdots \wedge dz_g \in \Gamma(\mathbb{H}_g, \omega) \]
with $(z_1, \ldots, z_g) \in \mathbb{C}^g$ and $(q_1, \ldots, q_g) = (e^{2i\pi z_1}, \ldots, e^{2i\pi z_g})$. This section of the canonical bundle is a basis of $\omega[A_\tau]$ for all $\tau \in \mathbb{H}_g$ and the relative canonical bundle of $U_g/\mathbb{H}_g$ is trivialized by $\zeta$:
\[ \omega_{U_g/\mathbb{H}_g} = \wedge^g \Omega^1_{U_g/\mathbb{H}_g} \simeq \mathbb{H}_g \times \mathbb{C} : \zeta. \]

The group $\Gamma$ acts on $\omega_{U_g/\mathbb{H}_g}$ by
\[ M.(\tau, \zeta) = (M. \tau, \det j(M, \tau) \cdot \zeta) \quad \text{if } M \in \Gamma, \]
in such a way that
\[ \varphi_M^*(\zeta_{M. \tau}) = \det j(M, \tau)^{-1} \zeta_\tau. \]
Thus, a geometric Siegel modular form $f$ of weight $h$ becomes an expression
\[ f(A_\tau) = \tilde{f}(\tau) \cdot \zeta^{\otimes h}, \]
where $\tilde{f}$ belongs to the well-known vector space $\mathcal{R}_{g,h}(\mathbb{C})$ of analytic Siegel modular forms of weight $h$ on $\mathbb{H}_g$, consisting of complex holomorphic functions $\phi(\tau)$ on $\mathbb{H}_g$ satisfying
\[ \phi(M. \tau) = \det j(M, \tau)^h \phi(\tau) \]
for any $M \in \text{Sp}_{2g}(\mathbb{Z})$. Note that by Koeccher principle [10, p. 11], the condition of holomorphy at $\infty$ is automatically satisfied since $g > 1$. The converse is also true [7, p. 141]:

**Proposition 2.2.1.** If $f \in S_{g,h}(\mathbb{C})$ and $\tau \in \mathbb{H}_g$, let
\[ \tilde{f}(\tau) = f(A_\tau) / \zeta^{\otimes h} = (2i\pi)^g f(A_\tau) / (dz_1 \wedge \cdots \wedge dz_g)^{\otimes h}. \]
Then the map $f \mapsto \tilde{f}$ is an isomorphism $S_{g,h}(\mathbb{C}) \rightarrow \mathcal{R}_{g,h}(\mathbb{C})$. □
2.3. Teichmüller modular forms. Let $g > 1$ and $n > 0$ be positive integers and let $M_{g,n}$ denote the moduli stack of smooth and proper curves of genus $g$ with symplectic level $n$ structure [5]. Let $\pi : C_{g,n} \to M_{g,n}$ be the universal curve, and let $\lambda$ be the invertible sheaf associated to the Hodge bundle, namely

$$\lambda = \wedge^g \pi_* \Omega^1_{C_{g,n}/M_{g,n}}.$$ 

For an algebraically closed field $k$ the fibre over $C \in M_{g,n}(k)$ is the one dimensional vector space $\lambda[C] = \wedge^g \Omega^1_k[C]$.

Let $R$ be a commutative ring and $h$ a positive integer. A Teichmüller modular form of genus $g$ and weight $h$ over $R$ is an element of

$$T_{g,h}(R) = \Gamma(M_g \otimes R, \lambda^\otimes h).$$

These forms have been thoroughly studied by Ichikawa [13], [14], [15], [16]. As in the case of the moduli space of abelian varieties, for any $n \geq 1$ we have

$$M_g \simeq M_{g,n}/\text{Sp}_{2g}(\mathbb{Z}/n\mathbb{Z}),$$

and $M_{g,n}$ can be represented by a smooth scheme over $\mathbb{Z}[\zeta_n,1/n]$ if $n \geq 3$. Then, for any algebra $R$ over $\mathbb{Z}[\zeta_n,1/n]$, the module $T_{g,h}(R)$ is the submodule of

$$\Gamma(M_{g,n} \otimes \mathbb{Z}[\zeta_n,1/n] \otimes R, \lambda^\otimes h)$$
invariant under $\text{Sp}_{2g}(\mathbb{Z}/n\mathbb{Z})$.

Let $C/k$ be a genus $g$ curve. Let $\lambda_1, \ldots, \lambda_g$ be a basis of $\Omega^1_k[C]$ and $\lambda = \lambda_1 \wedge \cdots \wedge \lambda_g$ a basis of $\lambda[C]$. As for Siegel modular forms in (1), for a Teichmüller modular form $f \in T_{g,h}(k)$ we define

$$f(C,\lambda) = f(C)/\lambda^\otimes h \in k.$$ 

Ichikawa proves the following proposition:

**Proposition 2.3.1.** The Torelli map $\theta : M_g \to A_g$, associating to a curve $C$ its Jacobian $\text{Jac} C$ with the canonical polarization $j$, satisfies $\theta^* \omega = \lambda$, and induces for any commutative ring $R$ a linear map

$$\theta^* : S_{g,h}(R) = \Gamma(A_g \otimes R, \omega^\otimes h) \to T_{g,h}(R) = \Gamma(M_g \otimes R, \lambda^\otimes h),$$

such that $[\theta^* f](C) = [\theta^* f(\text{Jac} C)]$. Fixing a basis $\lambda$ of $\lambda[C]$, this is

$$f(\text{Jac} C,\alpha) = [\theta^* f](C,\lambda) \quad \text{if } \theta^* \alpha = \lambda.$$

\[\square\]

2.4. Action of isomorphisms. Suppose $\phi : (A',a') \to (A,a)$ is a $\overline{\mathbb{F}}$-isomorphism of principally polarized abelian varieties, then by definition

$$f(A,\alpha) = f(A',\beta)$$

where $\beta_i = \phi^*(\alpha_i)$ is a basis of $\Omega^1_{\overline{\mathbb{F}}}(A')$ and $\beta = \beta_1 \wedge \cdots \wedge \beta_g \in \omega[A']$. If $\alpha'_1, \ldots, \alpha'_g$ is another basis of $\Omega^1_{\overline{\mathbb{F}}}(A')$ and $\alpha' = \alpha'_1 \wedge \cdots \wedge \alpha'_g$, we denote by $M_{\phi} \in \text{GL}_g(\overline{\mathbb{F}})$ the matrix of the basis $(\beta_i)$ in the basis $(\alpha'_i)$. We can easily see that

**Proposition 2.4.1.** In the above notation,

$$f(A,\alpha) = \det(M_{\phi})^h \cdot f(A',\alpha').$$

\[\square\]

First of all, from this formula applied to the action of $-1$, we deduce that, if $k$ is a field of characteristic different from 2, then $S_{g,h}(k) = \{0\}$ if $gh$ is odd. From now on we assume that $gh$ is even and $\text{char} k \neq 2$. 

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Corollary 2.4.2. Let $(A, a)$ be a principally polarized abelian variety of dimension $g$ defined over $k$ and $f \in S_{g, h}(k)$. Let $\alpha_1, \ldots, \alpha_g$ be a basis of $\Omega^1_k[A]$, and put $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_g \in \omega[A]$. Then the quantity
\[
\bar{f}(A) = f(A, \alpha) \mod k^{\times h} \in k^{\times h}
\]
does not depend on the choice of the basis of $\Omega^1_k[A]$. In particular $\bar{f}(A)$ is an invariant of the $k$-isomorphism class of $A$.

Proof. Assume that $\phi$ is given by the quadratic character $\varepsilon$ of $\text{Gal}(\overline{k}/k)$. Then
\[
d^2 = \varepsilon(\sigma)^g \cdot d, \text{ where } d = \det(M_\sigma) \in \overline{k}, \quad \sigma \in \text{Gal}(\overline{k}/k).
\]
Assume that $g$ is odd. Then by our assumption $h$ is even, and $d^2 = \varepsilon(\sigma)^g d^\sigma \in k$. But $d \notin k$ since there exists $\sigma$ such that $\varepsilon(\sigma) = -1$. Using Prop. 2.4.1 we find that
\[
f(A, \alpha) = (d^2)^{h/2}f(A', \alpha').
\]
Since $d^2$ is not a square in $k$, if $\bar{f}(A) \neq 0$ then $\bar{f}(A)$ and $\bar{f}(A')$ belong to two different classes in $k^{h/2}/k^{\times h} \simeq k^{k/2}$. \hfill \square

Let now $(A, a)$ be a principally polarized abelian variety of dimension $g$ defined over $\mathbb{C}$. Let $\omega_1, \ldots, \omega_g$ be a basis of $\Omega^1_k[A]$ and $\omega = \omega_1 \wedge \cdots \wedge \omega_g \in \omega[A]$. Let $\gamma_1, \ldots, \gamma_{2g}$ be a symplectic basis (for the polarization $a$). The period matrix
\[
\Omega = [\Omega_1 \; \Omega_2] = \begin{pmatrix}
\int_{\gamma_1} \omega_1 & \cdots & \int_{\gamma_1} \omega_1 \\
\vdots & \ddots & \vdots \\
\int_{\gamma_g} \omega_g & \cdots & \int_{\gamma_g} \omega_g
\end{pmatrix}
\]
belongs to the set $R_g \subset M_{g, 2g}(\mathbb{C})$ of Riemann matrices, and $\tau = \Omega_2^{-1} \Omega_1 \in \mathbb{H}_g$.

Proposition 2.4.4. In the above notation,
\[
f(A, \omega) = (2i\pi)^{gh} \frac{\bar{f}(\tau)}{\det \Omega_2^h}.
\]

Proof. The abelian variety $A$ is isomorphic to $A_\Omega = \mathbb{C}^g/\mathbb{Z}^2g$ and $\omega \in \omega[A]$ maps to $\xi = dz_1 \wedge \cdots \wedge dz_g \in \omega[A_\Omega]$ under this isomorphism. The linear map $z \mapsto \Omega_2^{-1} z = z'$ induces the isomorphism
\[
\varphi : A_\Omega \longrightarrow A_r = \mathbb{C}^g/(\mathbb{Z}^g + \tau \mathbb{Z}^g).
\]

Let us denote $\xi' = dz'_1 \wedge \cdots \wedge dz'_g = (2i\pi)^{-g} \xi$ in $\omega[A_r]$. Thus, using Prop. 2.4.1 Equation (1) and Prop. 2.2.1 we obtain
\[
f(A, \omega) = f(A_\Omega, \xi) = \det \Omega_2^{-h} f(A_r, \xi')
\]
\[
= \det \Omega_2^{-h} f(A_r)/\xi'^{\otimes h} = (2i\pi)^{gh} \det \Omega_2^{-h} f(\tau)/\xi'^{\otimes h} = (2i\pi)^{gh} \frac{\bar{f}(\tau)}{\det \Omega_2^h},
\]
from which the proposition follows. \hfill \square

3. INTEGRALS AND MODULAR FORMS

In this section $k$ is an algebraically closed field of characteristic different from 2.
3.1. Invariants. We review some classical invariant theory. Let $E$ be a vector space of dimension $n$ over $k$. The left regular representation $\rho$ of $GL(E)$ on the vector space $X = \text{Sym}^d(E^*)$ of homogeneous polynomials of degree $d$ on $E$ is given by

$$\rho(u) : F(x) \mapsto (u \cdot F)(x) = F(u^{-1}x)$$

for $u \in GL(E)$, $F \in X$ and $x \in E$. If $U$ is an open subset of $X$ stable under $\rho$, we still denote by $\rho$ the left regular representation of $GL(E)$ on the $k$-algebra $O(U)$ of regular functions on $U$, in such a way that

$$\rho(u) : \Phi(F) \mapsto (u \cdot \Phi)(F) = \Phi(u^{-1} \cdot F),$$

if $u \in GL(E)$, $\Phi \in O(U)$ and $F \in U$. If $h \in \mathbb{Z}$, we denote by $O_h(U)$ the subspace of homogeneous elements of degree $h$, satisfying $\Phi(\lambda F) = \lambda^h \Phi(F)$ for $\lambda \in k^\times$ and $F \in U$. The subspaces $O_h(U)$ are stable under $\rho$. An element $\Phi \in O_h(U)$ is an invariant of degree $h$ on $U$ if

$$u \cdot \Phi = \Phi \quad \text{for every} \ u \in SL(E),$$

and we denote by $\text{Inv}_n(U)$ the subspace of $O_h(U)$ of invariants of degree $h$ on $U$. If $\text{Inv}_h(U) \neq \{0\}$, then $hd \equiv 0 \pmod{n}$, since the group $\mu_n$ of $n$-th roots of unity is in the kernel of $\rho$. Hence, if $\Phi \in O(U)$, and if $w$ and $n$ are two integers such that $hd = nw$, the following statements are equivalent:

(i) $\Phi \in \text{Inv}_h(U);$

(ii) $u \cdot \Phi = (\det u)^{-w} \Phi$ for every $u \in GL(E).$

If these conditions are satisfied, we call $w$ the weight of $\Phi$. The multivariate resultant $\text{Res}(f_1, \ldots, f_n)$ of $n$ forms $f_1, \ldots, f_n$ in $n$ variables with coefficients in $k$ is an irreducible polynomial in the coefficients of $f_1, \ldots, f_n$ which vanishes whenever $f_1, \ldots, f_n$ have a common non-zero root. One requires that the resultant is irreducible over $\mathbb{Z}$, i.e. it has coefficients in $\mathbb{Z}$ with greatest common divisor equal to 1, and moreover

$$\text{Res}(x_1^{d_1}, \ldots, x_n^{d_n}) = 1$$

for any $(d_1, \ldots, d_n) \in \mathbb{N}^n$. The resultant exists and is unique. Now, let $F \in X_d$, and denote $q_1, \ldots, q_n$ the partial derivatives of $F$. The discriminant of $F$ is

$$\text{Disc} F = c_{n,d}^{-1} \text{Res}(q_1, \ldots, q_n), \quad \text{with} \quad c_{n,d} = d^{(d-1)n - (-1)^n}/d,$$

the coefficient $c_{n,d}$ being chosen according to \cite{28}. Hence, the projective hypersurface which is the zero locus of $F \in X_d$ is nonsingular if and only if $\text{Disc} F \neq 0$. The discriminant is an irreducible polynomial in the coefficients of $F$, see for instance \cite{8} Chap. 9, Ex. 1.6(a)]. From now on we restrict ourselves to the case $n = 3$, i.e. we consider invariants of ternary forms of degree $d$, and summarize the results that we shall need.

**Proposition 3.1.1.** If $F \in X_d$ is a ternary form, the discriminant

$$\text{Disc} F = d^{-(d-1)(d-2)-1} \cdot \text{Res}(q_1, q_2, q_3)$$

where $q_1, q_2, q_3$ are the partial derivatives of $F$, is given by an irreducible polynomial over $\mathbb{Z}$ in the coefficients of $F$, and vanishes if and only if the plane curve $C_F$ defined by $F$ is singular. The discriminant is an invariant of $X_d$ of degree $3(d-1)^2$ and weight $d(d-1)^2$. \hfill \square

We refer to \cite{8} p. 118 and \cite{21} for a beautiful explicit formula for the discriminant, found by Sylvester.
Example 3.1.2 (Ciani quartics). We recall some results whose proofs are given in [21]. Let \( \text{Sym}_3(k) \) be the vector space of symmetric matrices of size 3 with coefficients in \( k \), and
\[
G_m(x, y, z) = \langle v \cdot m \cdot v \rangle, \quad v = (x, y, z),
\]
the quadratic form associated to \( m \in \text{Sym}_3(k) \). Then
\[
F_m(x, y, z) = G_m(x^2, y^2, z^2)
\]
is a ternary quartic, and the map \( m \mapsto F_m \) is an isomorphism of \( \text{Sym}_3(k) \) to the subspace of \( F \in X_4 \) which are invariant under the three involutions
\[
\sigma_1(x, y, z) = (-x, y, z), \quad \sigma_2(x, y, z) = (x, -y, z), \quad \sigma_3(x, y, z) = (x, y, -z).
\]
If
\[
m = \begin{bmatrix} a_1 & b_3 & b_2 \\ b_3 & a_2 & b_1 \\ b_2 & b_1 & a_3 \end{bmatrix} \in \text{Sym}_3(k),
\]
then
\[
F_m(x, y, z) = a_1x^4 + a_2y^4 + a_3z^4 + 2(b_1y^2z^2 + b_2x^2z^2 + b_3x^2y^2).
\]
For \( 1 \leq i \leq 3 \), let \( c_i = a_i a_k - b_i^2 \) the cofactor of \( a_i \). Then
\[
\text{Disc } F_m = 2^{40} a_1 a_2 a_3 (c_1 c_2 c_3)^2 \det(m)^4.
\]
Note that the discrepancy between the powers of 2 here and in [21, Prop.2.2.1] comes from the normalization by \( c_{n,d} \).

3.2. Geometric invariants for nonsingular plane curves. Let \( E \) be a vector space of dimension 3 over \( k \) and \( G = \text{GL}(E) \). The universal curve over the affine space \( X_d = \text{Sym}^d(E) \) is the variety
\[
\mathcal{Y}_d = \{ (F, x) \in X_d \times \mathbb{P}^2 \mid F(x) = 0 \}.
\]
The nonsingular locus of \( X_d \) is the principal open set
\[
X^\circ_d = (X_d)_{\text{Disc}} = \{ F \in X_d \mid \text{Disc}(F) \neq 0 \}.
\]
If \( \mathcal{Y}^0_d \) is the universal curve restricted to the nonsingular locus, the projection is a smooth surjective \( k \)-morphism
\[
\mathcal{Y}^0_d \longrightarrow X^\circ_d
\]
whose fibre over \( F \) is the nonsingular plane curve \( C_F \).

We recall the classical way to write down an explicit \( k \)-basis of \( \Omega^1[C_F] = H^0(C_F, \Omega^1) \) for \( F \in X^\circ_d(k) \) (see [3, p. 630]). Let
\[
\eta_1 = \frac{f(x_2dx_3 - x_3dx_2)}{q_1}, \quad \eta_2 = \frac{f(x_3dx_1 - x_1dx_3)}{q_2}, \quad \eta_3 = \frac{f(x_1dx_2 - x_2dx_1)}{q_3},
\]
where \( q_1, q_2, q_3 \) are the partial derivatives of \( F \), and where \( f \) belongs to the space \( X_{d-3} \) of ternary forms of degree \( d - 3 \). The forms \( \eta_i \) glue together and define a regular differential form \( \eta_f(F) \in \Omega^1[C_F] \). Since \( \dim X_{d-3} = (d-1)(d-2)/2 = g \), the linear map \( f \mapsto \eta_f(F) \) defines an isomorphism
\[
X_{d-3} \longrightarrow \Omega^1[C_F].
\]
This implies that the sections \( \eta_f \in \Gamma(X_d^0, \Omega^1_{X_d^0/X_d^0}) \) lead to a trivialization
\[
X^0_d \times X_{d-3} \longrightarrow \Omega^1_{X_d^0/X_d^0}.
\]
An element \( u \in G \) acts on \( \mathcal{Y}_d \) by
\[
u \cdot (F, x) = (u \cdot F, ux),
\]
and the projection \( \mathcal{Y}^0_d \longrightarrow X^\circ_d \) is \( G \)-equivariant.
We denote $\eta_1, \ldots, \eta_g$ the sequence of sections obtained by substituting for $f$ in $\eta_f$ the $g$ members of the canonical basis of $X_d$, enumerated according to the lexicographic order, the classical basis of $\Gamma(X^0_d, \Omega^1_{X^0_d}/\mathcal{O}_d)$. The section

$$\eta = \eta_1 \wedge \cdots \wedge \eta_g$$

is a basis of the one-dimensional space $\Gamma(X^0_d, \alpha)$, where

$$\alpha = \langle \pi g, \Omega^1_{X^0_d}/\mathcal{O}_d \rangle,$$

is the Hodge bundle of the universal curve over $X^0_d$. For every $F \in X^0_d$, an element $u \in G$ induces by restriction an isomorphism

$$\varphi_u : C_F \rightarrow C_{u^*},$$

which itself defines a linear automorphism $\varphi_u^*$ of $\alpha$.

For any $h \in \mathbb{Z}$, we denote by $\Gamma(X^0_d, \alpha^h)^G$ the subspace of sections $s \in \Gamma(X^0_d, \alpha^h)$ such that

$$\varphi_u^*(s) = s \quad \text{for every } u \in G.$$

If $\alpha \in \Gamma(X^0_d, \alpha)$ and $F \in X^0_d$, we define, in the same way as in Equation (1),

$$s(F, \alpha) = s(F)/\alpha^h.$$

Hence, $s \in \Gamma(X^0_d, \alpha^h)^G$ if and only if for all $u \in G$ and $F \in X^0_d$, one has

$$(\varphi_u^*(s))(F, \alpha) = s(F, \alpha).$$

**Proposition 3.2.1.** The section $\eta \in \Gamma(X^0_d, \alpha)$ satisfies the following properties.

(i) If $u \in G$, then

$$\varphi_u^* \eta = \det(u)^{w_0} \eta, \quad \text{with } w_0 = \left(\frac{d}{3}\right) = \frac{dg}{3} \in \mathbb{N}.$$

(ii) Let $h \geq 0$ be an integer. The linear map

$$\Phi \mapsto \tau(\Phi) = \Phi \cdot \eta^h$$

is an isomorphism

$$\tau : \text{Inv}_{gh}(X^0_d) \xrightarrow{\sim} \Gamma(X^0_d, \alpha^h)^G.$$  

**Proof.** Let $u \in G$. Since $\dim \alpha_{u, F} = 1$, there is $c(u, F) \in k^\times$ such that

$$(\varphi_u^*(\eta))(F, \eta) = c(u, F) \cdot \eta(F, \eta) = c(u, F).$$

and $c$ is a “crossed character”, satisfying

$$c(uu', F) = c(u, F) c(u', u \cdot F).$$

Now the regular function $F \mapsto c(u, F)$ does not vanishes on $X^0_d$. By Lemma 3.2.2 below and the irreducibility of the discriminant (Prop. 3.1.1), we have

$$c(u, F) = \chi(u)(\text{Disc } F)^{n(u)}$$

with $\chi(u) \in k^\times$ and $n(u) \in \mathbb{Z}$. The group $G$ being connected, the function $n(u) = n$ is constant. Since $c(1, F) = 1$, we have $(\text{Disc } F)^n = \chi(1)^{-1}$, and this implies $n = 0$. Hence, $c(u, F)$ is independent of $F$ and $\chi$ is a character of $G$. Since the group of commutators of $G$ is $\text{SL}_3(k)$, we have

$$\chi(u) = \det(u)^{w_0}$$

for some $w_0 \in \mathbb{Z}$. It therefore suffices to calculate $\chi(u)$ when $u = \lambda I_3$, with $\lambda \in k^\times$. In this case $u \cdot F = \lambda^{-d} F$. Moreover, the section $\eta_f$ is homogeneous of degree $-1$: if $\lambda \in k^\times$ and $F \in X^0_d$, then

$$\eta_f(\lambda^{-d} F)/\eta_f(F) = \lambda^d,$$
hence,
\[(\varphi_u^* \eta)(F, \eta) = \lambda^d \gamma = \det(u)^{w_0}.\]
This implies
\[\lambda^{3w_0} = \det(u)^{w_0} = \lambda^d,\]
and we have proven \(\square\).

By construction. This induces a morphism \(\Phi\). Then there is an \(m\) such that \(d > 2\) be an integer and \(g = \frac{d^2}{2}\). Since the fibres of \(Y_g^0 \to X_g^0\) are nonsingular non hyperelliptic plane curves of genus \(g\), by the universal property of \(M_g\) we get a morphism
\[p : X_g^0 \longrightarrow M_g^0,\]
where \(M_g^0\) is the moduli stack of nonhyperelliptic curves of genus \(g\) and \(p^* \lambda = \alpha\) by construction. This induces a morphism
\[p^* : \Gamma(M_g^0, \lambda_{\otimes h}) \longrightarrow \Gamma(X_g^0, \alpha_{\otimes h}).\]
Let \(s \in \Gamma(M_g^0, \lambda_{\otimes h})\). For every \(F \in X_g^0\), an element \(u \in G\) induces an isomorphism
\[\varphi_u : C_F \longrightarrow C_{u,F}.\]
By the universal property of \(M_g^0\), the diagram
\[
\begin{array}{ccc}
\lambda_{|p(X_g^0)} & \longrightarrow & \lambda_{|p(X_g^0)} \\
p^* & \downarrow & \downarrow p^* \\
\alpha & \varphi_u^* & \alpha
\end{array}
\]
is commutative. Hence
\[\varphi_u^* \circ p^*(s) = p^*(s),\]
and this means that \(p^* s \in \Gamma(X_g^0, \alpha_{\otimes h})\). Combining this result with Prop.3.2.1\(\square\), we obtain:

**Proposition 3.3.1.** For any integer \(h \geq 0\), the linear map \(\sigma = \tau^{-1} \circ p^*\) is a homomorphism:
\[\Gamma(M_g^0, \lambda_{\otimes h}) \longrightarrow \text{Inv}_{gh}(X_g^0)\]
such that
\[\sigma(f)(F) = f(C_F, (p^*)^{-1}\eta)\]
for any \(F \in X_g^0\) and any section \(f \in \Gamma(M_g^0, \lambda_{\otimes h})\). \(\square\)
4.1. Klein’s formula. We recall the definition of theta functions with (entire) characteristics
\[ [\varepsilon] = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \in \mathbb{Z}^g \oplus \mathbb{Z}^g, \]
following [2]. The (classical) theta function is given, for \( \tau \in \mathbb{H}_g \) and \( z \in \mathbb{C}^g \), by
\[ \theta[\varepsilon_1 \varepsilon_2](z, \tau) = \sum_{n \in \mathbb{Z}^g} q^{(n+\varepsilon_1/2)(n+\varepsilon_2/2)+2(n+\varepsilon_1/2)(z+\varepsilon_2/2)}. \]
The \textit{Thetanullwerte} are the values at \( z = 0 \) of these functions, and we write
\[ \theta[\varepsilon](\tau) = \theta[\varepsilon_1 \varepsilon_2](0, \tau) = \theta[\varepsilon_1 \varepsilon_2](0, \tau). \]
Recall that a characteristic is \textit{even} if \( \varepsilon_1 \varepsilon_2 \equiv 0 \pmod{2} \) and \textit{odd} otherwise. Let \( S_g \) (resp. \( U_g \)) be the set of even characteristics with coefficients in \( \{0,1\} \). For \( g \geq 2 \), we put \( h = |S_g|/2 = 2^{g-2}(2^g + 1) \) and
\[ \chi_h(\tau) = (2i\pi)^{gh} \prod_{\varepsilon \in S_g} \theta[\varepsilon](\tau). \]
In his beautiful paper [17], Igusa proves the following result [loc. cit., Lem. 10 and 11]. Denote by \( \Sigma_{140} \) the modular form defined by the thirty-fifth elementary symmetric function of the eighth power of the even Thetanullwerte. Recall that a principally polarized abelian variety \( (A, a) \) is decomposable if it is a product of principally polarized abelian varieties of lower dimension, and indecomposable otherwise.

Theorem 4.1.1. If \( g \geq 3 \), then \( \chi_h(\tau) \in \mathbb{R}_{g,h}(\mathbb{C}) \). Moreover, If \( g = 3 \) and \( \tau \in \mathbb{H}_3 \), then:

(i) \( A_\tau \) is decomposable if \( \chi_{18}(\tau) = \Sigma_{140}(\tau) = 0 \).
(ii) \( A_r \) is a hyperelliptic Jacobian if \( \bar{\chi}_{18}(\tau) = 0 \) and \( \bar{\Sigma}_{140}(\tau) \neq 0 \).

(iii) \( A_r \) is a non hyperelliptic Jacobian if \( \bar{\chi}_{18}(\tau) \neq 0 \).

Using Prop. 2.2.1, we define the geometric modular form of weight \( h \)
\[
\chi_h(A_r) = (2i\pi)^h \bar{\chi}_h(\tau)(dz_1 \wedge \cdots \wedge dz_g)^{\otimes h}.
\]

Then Ichikawa [15, 16] proved that \( \chi_h \in S_{g,h}(\mathbb{Q}) \). For \( g = 3 \), one finds
\[
\chi_{18}(A_r) = -(2\pi)^{54} \bar{\chi}_{18}(\tau)(dz_1 \wedge dz_2 \wedge dz_3)^{\otimes 18}.
\]

Now we are ready to give a proof of the following result [20, Eq. 118, p. 462]:

**Theorem 4.1.2** (Klein’s formula). Let \( F \) be a plane quartic defined over \( \mathbb{C} \) such that \( C_F \) is nonsingular. Let \( \eta_1, \eta_2, \eta_3 \) be the classical basis of \( \Omega^1[C_F] \) and \( \gamma_1, \ldots, \gamma_6 \) be a symplectic basis of \( H_1(C_F, \mathbb{Z}) \) for the intersection pairing. Let
\[
\Omega = [\Omega_1 \ \Omega_2] = \begin{pmatrix}
\int \gamma_1 \eta_1 & \cdots & \int \gamma_6 \eta_1 \\
\vdots & \ddots & \vdots \\
\int \gamma_1 \eta_3 & \cdots & \int \gamma_6 \eta_3
\end{pmatrix}
\]
be a period matrix of \( \text{Jac}(C) \) and \( \tau = \Omega_2^{-1} \Omega_1 \in \mathbb{H}_3 \). Then
\[
\text{Disc}(F)^2 = \frac{1}{2^{28}} (2\pi)^{54} \bar{\chi}_{18}(\tau) \det(\Omega_2)^{18}.
\]

**Proof.** The Cor. 3.2.2 shows that \( I = \sigma \circ \theta^* (\bar{\chi}_{18}) \) is an invariant of weight 54, and for any \( F \in X_4^0 \),
\[
I(F) = -(2\pi)^{54} \bar{\chi}_{18}(\tau) \det(\Omega_2)^{18}.
\]

Moreover Th. 4.1.11 shows that \( I(F) \neq 0 \) for all \( F \in X_4^0 \). Applying Lem. 3.2.2 for the discriminant, we find by comparison of the weights that \( I = c \text{Disc}^2 \) with \( c \in \mathbb{C} \) a constant. But if \( F_m \) is the Ciani quartic associated to a matrix \( m \in \text{Sym}_3(k) \) as in Example 3.1.2 and if \( \text{Disc} F_m \neq 0 \), then it is proven in [21, Cor. 4.2] that Klein’s formula is true for \( F_m \) and \( c = -2^{28} \).

**Remark 4.1.3.** The morphism \( \theta^* \) defines an injective morphism of graded \( k \)-algebras
\[
S_5(k) = \oplus_{h \geq 0} S_{5,h}(k) \longrightarrow T_3(k) = \oplus_{h \geq 0} T_{3,h}(k)
\]
In [14], Ichikawa proves that if \( k \) is a field of characteristic 0, then \( T_3(k) \) is generated by the image of \( S_5(k) \) and a primitive Teichmüller form \( \mu_{3,9} \in T_{3,9}(\mathbb{Z}) \) of weight 9, which is not of Siegel modular type. He also proves in [10] that
\[
\theta^* (\bar{\chi}_{18}) = -2^{28} (\mu_{3,9})^2.
\]
Th. 4.1.2 implies that \( \mu_{3,9} \) is actually equal to the discriminant up to a sign. This might probably be deduced from the definition of \( \mu_{3,9} \), although we did not sort it out (see also [18, Sec. 2.4]).

**Remark 4.1.4.** Besides [22] and [11] where an analogue of Klein’s formula is derived in the hyperelliptic case, there exists a beautiful algebraic Klein’s formula, linking the discriminant with irrational invariants [4, Th.11.1].
4.2. Jacobians among abelian threefolds. Let $k \subset \mathbb{C}$ be a field and let $g = 3$. We prove the following theorem which allows to determine whether a given abelian threefold defined over $k$ is $k$-isomorphic to a Jacobian of a curve defined over the same field. This settles the question of Serre recalled in the introduction.

**Theorem 4.2.1.** Let $(A, a)$ be a principally polarized abelian threefold defined over $k \subset \mathbb{C}$. Let $\omega_1, \omega_2, \omega_3$ be a basis of $\Omega_k^1[A]$ and $\gamma_1, \ldots, \gamma_6$ a symplectic basis of $H^1(A, \mathbb{Z})$, in such a way that

$$\Omega = [\Omega_1 \Omega_2] = \begin{pmatrix} \int_{\gamma_1} \omega_1 & \cdots & \int_{\gamma_6} \omega_1 \\ \vdots & \ddots & \vdots \\ \int_{\gamma_1} \omega_3 & \cdots & \int_{\gamma_6} \omega_3 \end{pmatrix}$$

is a period matrix of $(A, a)$. Put $\tau = \Omega_1^{-1} \Omega_1 \in \mathbb{H}_3$.

(i) If $\Sigma_{140}^2(\tau) = 0$ then $(A, \lambda)$ is decomposable. In particular it is not a Jacobian.

(ii) If $\Sigma_{140}^2(\tau) \neq 0$ and $\chi_{18}(\tau) = 0$ then there exists a hyperelliptic curve $X/k$ such that $(\text{Jac} X, j) \simeq (A, a)$.

(iii) If $\chi_{18}(\tau) \neq 0$ then $(A, a)$ is isomorphic to a Jacobian if and only if

$$-\chi_{18}(A, \omega_1 \wedge \omega_2 \wedge \omega_3) = (2\pi)^{54} \frac{\chi_{18}(\tau)}{\det(\Omega_2)^{18}}$$

is a square in $k$.

**Proof.** The first and second points follow from Th.4.1.1 and Th.4.1.1. Suppose now that $(A, a)$ is isomorphic over $k$ to the Jacobian of a non hyperelliptic genus 3 curve $C/k$. Let $F \subseteq X^0_4$ be a plane model of $C$. Using Th.4.1.2 we get that

$$-\chi_{18}(A, \omega_1 \wedge \omega_2 \wedge \omega_3) = (2\pi)^{54} \frac{\chi_{18}(\tau)}{\det(\Omega_2)^{18}} = 2^{28} \text{Disc}(F)^2$$

so it is a square in $k$. On the contrary, Cor.4.1.2 shows that if $(A', a')$ is a quadratic twist of a Jacobian $(A, a)$ then the expression

$$-f(A', \omega'_1 \wedge \omega'_2 \wedge \omega'_3) = (2\pi)^{54} \frac{\chi_{18}(\tau')}{\det(\Omega_2)^{18}}$$

is not a square. \hfill \Box

**Remark 4.2.2.** Note that one does not really need Klein’s formula. Alternatively, we could use (2) which also proves that $\theta^*(-\chi_{18})$ is a square.

**Corollary 4.2.3.** In the notation of Th.4.2.1 the quadratic character $\varepsilon$ of Gal($k_{\text{sep}}/k$) introduced in Theorem 1.1.1 is given by $\varepsilon(\sigma) = d/d^\sigma$, where

$$d = \sqrt{(2\pi)^{54} \frac{\chi_{18}(\tau)}{\det(\Omega_2)^{18}}}$$

with an arbitrary choice of the square root.

4.3. Beyond genus 3. It is natural to try to extend our results to the case $g > 3$. The first question to ask is

— Does there exist an analogue of Klein’s formula for $g > 3$?

Here we can give a partial answer. Using Sec.2.3 we can consider the Teichmüller modular form $\theta^*(\chi_h)$ with $h = 2^{g-2}(2^g + 1)$. In [10, Prop.4.5] (see also [29]), it is proven that for $g > 3$ the element

$$\theta^*(\chi_h)/2^{2^{g-1}(2^g-1)}$$
has as a square root a primitive element $\mu_{g,h/2} \in T_{g,h/2}(\mathbb{Z})$. If $g = 4$, in the footnote, p. 462 in [20] we find the following amazing formula

$$\frac{\tilde{\chi}_{68}(\tau)}{\det(\Omega_2)^{68}} = c \cdot \Delta(X)^2 \cdot T(X)^8.$$  

Here $\tau = \Omega_2^{-1}\Omega_1$, with $\Omega = [\Omega_1 \ \Omega_2]$ being a period matrix of a genus 4 non hyperelliptic curve $X$ given in $\mathbb{P}^3$ as an intersection of a quadric $Q$ and a cubic surface $E$. The elements $\Delta(X)$ and $T(X)$ are defined in the classical invariant theory as, respectively, the discriminant of $Q$ and the tact invariant of $Q$ and $E$ (see [26, p.122]). No such formula seems to be known in the non hyperelliptic case for $g > 4$.

Let us now look at what happens when we try to apply Serre’s apporoach for $g > 3$. To begin with, when $g$ is even, we cannot use Cor. 2.4.2 to distinguish between quadratic twists. In particular, using the previous result, we see that $\chi_h(A,\omega_k)$ is a square when $A$ is a principally polarized abelian variety defined over $k$ which is geometrically a Jacobian. A natural question is:

— What is the relation between this condition and the locus of geometric Jacobians over $k$?

Let us assume now that $g$ is odd. As we pointed out in Rem. 4.2.2 the existence of the square root is almost sufficient to answer Serre’s questions when $g = 3$. This is not the case when $g > 3$. The proof of the corollary 2.4.3 shows that

$$\chi_h(A') = (d^2)^{h/2} \chi_h(A)$$

for a Jacobian $A$ and a quadratic twist $A'$. What enables us to distinguish between $A$ and $A'$ when $g = 3$ is the following: if $A$ is the Jacobian of a curve then $\chi_h(A)$ is a square whereas $d^2$ is not and $h/2 = 9$ is odd. However as soon as $g > 3$, $2 \mid 2^{g-3}$, the power $g - 3$ being the maximal power of 2 dividing $h/2$, so it is not enough for $\chi(A)$ to be a square in $k$ to make a distinction between $A$ and $A'$. It must rather be an element of $k^{2^{g-2}}$.

It can be easily seen from the proof of [29, Th.1] that $\theta^* (\chi_h)$ does not admit a fourth root. According to [1] or [30] this implies $\chi_h(A) \notin k^{2^{g-2}}$ for infinitely many Jacobians $A$ defined over number fields $k$. So we can no longer use the modular form $\chi_h$ to easily characterize Jacobians over $k$. So the question is:

— Is it possible to find a modular form to replace $\chi_h$ in Serre’s strategy when $g > 3$?

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