PROFİNITE CONGRUENCES AND UNARY ALGEBRAS

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Abstract. Profinite congruences on profinite algebras determining profinite quotients are difficult to describe. In particular, no constructive description is known of the least profinite congruence containing a given binary relation on the algebra. On the other hand, closed congruences and fully invariant congruences can be described constructively. In a previous paper, we conjectured that fully invariant closed congruences on a relatively free profinite algebra are always profinite. Here, we show that our conjecture fails for unary algebras and that closed congruences on relatively free profinite semigroups are not necessarily profinite. As part of our study of unary algebras, we establish an adjunction between profinite unary algebras and profinite monoids. We also show that the Polish representation of the free profinite unary algebra is faithful.

1. Introduction

Our first concern in this paper is to try to understand congruences on profinite algebras such that the corresponding quotients are also profinite. Following [15], we call such congruences profinite. For instance, if one is interested in considering profinite presentations of algebras \( (A; R) \) (as in [12, 5]), one should take on the free profinite algebra on the generating set \( A \) the least profinite congruence \( \rho \) containing the relation \( R \). As in the theory of presentations of discrete algebras, it would be useful to have some sort of constructive description of \( \rho \). Other than specially well-behaved cases, such as in group theory, no such description is known in general. In all the exceptions found so far, the reason why such a description is known is that, unlike what happens in general, closed congruences are profinite. In the realm of semigroup theory, many favorable examples are presented in [6], where the aim is to find a general method to determine when a given pseudoidentity is a consequence, in all finite models, of a given set of pseudoidentities. Such examples are of special type in the sense that fully invariant closed congruences on relatively free profinite semigroups are concerned. They prompted the authors to ask whether such congruences are always profinite even in a more general algebraic context.

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Three types of examples are given in this paper. First, an easy example of a closed congruence on a relatively profinite semigroup that is not profinite. Second, a modification of the first example to make it a fully invariant closed congruence that turns out to be a profinite congruence on a finitely generated relatively free profinite semigroup whose quotient is countable and contains infinitely many idempotents. Third, an example, which is again derived from the first example, of a fully invariant closed congruence on a relatively profinite unary algebra which is not profinite, thereby giving a negative answer to the general case of our question raised in [6]. The latter example prompts a study of the relationship between profinite monoids and profinite unary algebras which leads us to exhibit a pair of adjoint functors between the two categories.

In the general framework of universal algebra, the standard Polish notation for terms induces a representation of the term algebra in a free monoid, which is faithful. In the profinite setting, this leads to a representation of the absolutely free profinite algebra in a suitable free profinite monoid. We show that, over a nonempty topological space of generators, such a representation is faithful if and only if the signature is at most unary, that is, it contains no operation symbols of arity greater than 1.

From a certain point of view, our main results and constructions involve representations of profinite unary algebras by appropriate profinite monoids. Each such profinite monoid appears as the set of implicit unary operations on a given unary algebra with a continuous multiplication given by composition. If one would generate a clone by these operations, each new n-ary operation may be identified with a unary operation of one of the arguments. Thus, in somewhat imprecise terms, we may say that this paper is also about profinite clones of profinite unary algebras. However, we did not introduce the notion of profinite clones formally, since profinite monoids are well-known and a more suitable notion here. Notice also that in the case of a clone on a non-unary (profinite) algebra, composition may not be simplified into a binary operation and the behavior is much less clear. In particular, we obtained only partial observations in this line of investigation.

2. Preliminaries

In this paper, we consider signatures to be sets $\Sigma = \bigsqcup_{n \geq 0} \Sigma_n$ graded by the natural numbers. The set $\Sigma_n$ consists of the n-ary operation symbols in the signature. The signature $\Sigma$ is a topological signature if each $\Sigma_n$ is a topological space; $\Sigma$ is a discrete signature if each $\Sigma_n$ is a discrete space.

An algebra of the signature $\Sigma$ is a nonempty set $S$ together with an evaluation mapping $E_n : \Sigma_n \times S^n \to S$ for each arity $n$; the image of the $(n + 1)$-tuple $(w, s_1, \ldots, s_n)$ is usually denoted $w_S(s_1, \ldots, s_n)$. In case the signature is a topological signature, by a topological algebra we mean an algebra $S$ endowed with a topology such that the evaluation mappings $E_n$ are continuous. In case the topology of the algebra is compact, we also say that the topological algebra is a compact algebra. All compact spaces in this paper are assumed to be Hausdorff.
A term is an element of an absolutely free algebra $T_{\Sigma}(X)$ on a set $X$ (of variables), which may be viewed as an expression in the variables using operation symbols as formal operations according to their arities. In case $X = \{x\}$, we write $T_{\Sigma}(x)$ instead of $T_{\Sigma}(X)$. One often writes $t = t(x_1, \ldots, x_n)$ to represent a term which involves, at most, the variables $x_1, \ldots, x_n$. For an algebra $S$, one may define recursively for each term $t(x_1, \ldots, x_n)$ an operation $t_S : S^n \to S$ as follows: in case $t = x_i$, then $t_S$ is the projection on the $i$th component; in case $t = w(t_1, \ldots, t_r)$ for an operation symbol $w$ of arity $r$, we put

$$t_S(s_1, \ldots, s_n) = w_S(t_1_S(s_1, \ldots, s_n), \ldots, t_r_S(s_1, \ldots, s_n)).$$

By a pseudovariety we mean a class of finite algebras of a fixed signature that is closed under taking homomorphic images, subalgebras and finite direct products. For a pseudovariety $\mathcal{V}$, a pro-$\mathcal{V}$ algebra is a compact algebra $S$ which is residually $\mathcal{V}$ in the sense that, given any two points $s, s' \in S$, there is a continuous homomorphism $\varphi : S \to T$ such that $T \in \mathcal{V}$ and $\varphi(s) \neq \varphi(s')$. Let $\text{Fin}_\mathcal{V}$ denote the pseudovariety of all finite $\Sigma$-algebras. A pro-$\text{Fin}_\mathcal{V}$ algebra is simply called a profinite algebra. Note that every profinite algebra is zero-dimensional.

In general, for two topological spaces $X$ and $T$, we denote $T^X$ the set of all functions from $X$ to $T$ and $\mathcal{C}(X,T)$ its subset consisting of the continuous functions. In particular, if $X$ is a discrete space, then $\mathcal{C}(X,T) = T^X$. Later we recall also the usual topologies on these function spaces.

Given a topological space $X$ and a pseudovariety $\mathcal{V}$, there is a free pro-$\mathcal{V}$ algebra over $X$, denoted $\overline{\Omega}_X \mathcal{V}$; it comes endowed with a continuous mapping $\iota : X \to \overline{\Omega}_X \mathcal{V}$ (if various pseudovarieties are present, we also denote it $\omega$) and it is characterized by the following universal property: for every continuous mapping $\varphi : X \to S$ into a pro-$\mathcal{V}$ algebra $S$, there is a unique continuous homomorphism $\hat{\varphi} : \overline{\Omega}_X \mathcal{V} \to S$ such that the following diagram commutes:

$$\begin{array}{ccc}
X & \xrightarrow{\iota} & \overline{\Omega}_X \mathcal{V} \\
\| \downarrow & & \downarrow \varphi \\
S. & \downarrow \hat{\varphi} & \end{array}$$

(2.1)

In the case of a pseudovariety of semigroups $\mathcal{V}$, since elements of free semigroups are usually called words, we may refer to elements of $\overline{\Omega}_X \mathcal{V}$ as $\mathcal{V}$-pseudowords on $X$ or, simply, pseudowords. The elements of $\overline{\Omega}_X \mathcal{V}$ that are not in the subalgebra generated by $X$ are said to be infinite pseudowords.

There is a natural interpretation of elements of $\overline{\Omega}_X \mathcal{V}$ as (partial) operations on a pro-$\mathcal{V}$ algebra $S$: for each $u \in \overline{\Omega}_X \mathcal{V}$, $u_S : \mathcal{C}(X,S) \to S$ maps each continuous function $\varphi : X \to S$ to $\hat{\varphi}(u)$, where $\hat{\varphi}$ is the unique continuous homomorphism of Diagram (2.1). If $X$ is a finite discrete space, which is the case that concerns us most in this paper, then $\mathcal{C}(X,S) = S^X$ and $u_S$ is a total operation of arity $|X|$. The following continuity result generalizes various statements appearing in the literature (see [2] Subsection 2.3 and [3] Proposition 4.7]) but does not seem to have been stated previously in its present generality. Although it may be considered folklore, the proof is presented here for the sake of completeness.
Theorem 2.1. For a finite set $X$ and a pro-$V$ algebra $S$, the mapping
\[\varepsilon^S_V : \Omega_X V \times S^X \to S\]
\[(u, \varphi) \mapsto \hat{\varphi}(u)\]
is continuous.

Proof. Let $W$ be a pseudovariety contained in $V$ and denote by $\pi_W = \hat{\iota}_W$ the unique continuous homomorphism $\Omega_X V \to \Omega_X W$ such that the diagram
\[
\begin{array}{ccc}
X & \xrightarrow{\iota_V} & \Omega_X V \\
\downarrow{\iota_W} & & \downarrow{\pi_W} \\
\Omega_X V & \xrightarrow{\pi_W} & \Omega_X W
\end{array}
\]
commutes. Let $h : S \to T$ be a continuous homomorphism where $S$ and $T$ are, respectively pro-$V$ and pro-$W$ algebras. The mapping $h$ induces the following diagram:
\begin{equation}
\begin{array}{cccc}
\Omega_X V \times S^X & \xrightarrow{\varepsilon^S_V} & S \\
\downarrow{\pi_W \times (h \circ \_)} & & \downarrow{h} \\
\Omega_X W \times T^X & \xrightarrow{\varepsilon^T_W} & T.
\end{array}
\end{equation}
We claim that it commutes. Indeed, given $\varphi \in S^X$, we may build the following diagram:
\[
\begin{array}{ccc}
\Omega_X V & \xrightarrow{\pi_W} & \Omega_X W \\
\downarrow{\iota_V} & & \downarrow{\iota_W} \\
S & \xrightarrow{\varphi} & T \\
\downarrow{h} & & \downarrow{h \circ \varphi}
\end{array}
\]
The left, upper, and lower triangles commute by the definition, respectively, of $\hat{\varphi}$, $\pi_W$, and $h \circ \varphi$. It follows that the whole diagram commutes. Hence, given $u \in \Omega_X V$, we obtain the following equalities
\[h(\varepsilon^S_V(u, \varphi)) = h(\hat{\varphi}(u)) = h \circ \varphi(\pi_W(u)) = \varepsilon^T_W(\pi_W(u), h \circ \varphi),\]
thereby showing that Diagram (2.2) commutes.

Now, since $S$ is zero-dimensional, to prove that the mapping $\varepsilon^S_V$ is continuous, it suffices to show that, for every clopen subset $K$ of $S$, the set $(\varepsilon^S_V)^{-1}(K)$ is open in the product space $\Omega_X V \times S^X$. By [1, Theorem 3.6.1], there is a continuous homomorphism $h : S \to T$ to an algebra $T$ from $V$ such that $K = h^{-1}(h(K))$. Let $W$ be the pseudovariety generated by $T$ and consider the corresponding commutative Diagram (2.2). Then the following equalities hold:
\[(\varepsilon^S_V)^{-1}(K) = (\varepsilon^S_V)^{-1}(h^{-1}(h(K))) = (\pi_W \times (h \circ \_))^{-1}((\varepsilon^T_W)^{-1}(h(K))).\]
Since $\Omega_X W \times T^X$ is a finite discrete space, the continuity of $\varepsilon^S_V$ reduces to that of the mapping $h \circ \_ : S^X \to T^X$, which is just $h$ on each component, whence continuous. □
The following are immediate consequences of Theorem 2.1 that we record here for later reference.

**Corollary 2.2.** Let $S$ be a pro-$V$ algebra and $X$ a finite set. Then, for every $w \in \Omega^X V$, the mapping $w_S : S^X \to S$ defined by $w_S(\varphi) = \hat{\varphi}(w)$ is continuous. □

**Corollary 2.3.** Let $h : S \to T$ be a continuous homomorphism between pro-$V$ algebras and $X$ a finite set. Then the following diagram commutes for each $w \in \Omega^X V$:

$$
\begin{array}{ccc}
S^X & \xrightarrow{w_S} & S \\
\downarrow{h_0} & & \downarrow{h} \\
T^X & \xrightarrow{w_T} & T.
\end{array}
$$

Proof. This follows from the commutativity of Diagram (2.2) where we consider $W = V$ and $\pi_W$ is the identity mapping. □

In particular, a pro-$V$ algebra may be viewed as a natural topological $\Sigma$-algebra for the signature $\Sigma = \bigcup_{n=1}^{\infty} \Omega^V X_n$, where $\{x_1, x_2, \ldots\}$ is a countable set of distinct variables and $X_n = \{x_1, \ldots, x_n\}$. Note that indeed the value of an $n$-ary operation depends continuously jointly on the $n$ arguments and the operation itself. In particular, taking the pro-$V$ algebra to be any of the free pro-$V$ algebras $\Omega^V X_n$, we see that $\Sigma$ is a clone.

An equivalence relation on a topological space $X$ is said to be closed (respectively clopen) if it is a subset of $X \times X$ with that property. In particular, it is well known that the equality relation $\Delta_X$ on $X$ is closed if and only if $X$ is Hausdorff. A congruence $\theta$ on a topological algebra $S$ is profinite if $S/\theta$ is a profinite algebra for the quotient algebraic and topological structures.

Sometimes, it is useful to consider presentations in the profinite context. Given a pseudovariety $V$, a set $X$, and a binary relation $R \subseteq (\Omega^X V)^2$, the pro-$V$ algebra presented by $(X; R)$ is the largest quotient $\Omega^X V/\theta$ where $\theta$ is a profinite congruence containing $R$. So, we are interested in determining the smallest profinite congruence on $\Omega^X V$ that contains $R$; note that it is the intersection of all clopen congruences containing $R$.

On the other hand, one may describe “constructively” the smallest closed congruence on $\overline{\Omega^X V}$ containing $R$ by applying the following natural “procedure” considered in [6]:

- start with the set $\Delta_{\overline{\Omega^X V}} \cup \{(t_{\overline{\Omega^X V}}(a, b_1, \ldots, b_n), t_{\overline{\Omega^X V}}(a', b_1, \ldots, b_n)) : t(x_1, \ldots, x_{n+1}) \text{ a term, } (a, a') \in R \cup R^{-1}, b_i \in \overline{\Omega^X V}\}$;
- alternate successively transitive closure and topological closure;
- repeat transfinitely until the relation obtained stabilizes.

If a fully invariant congruence is sought, it suffices to replace the occurrences of $a$ and $a'$ in the evaluation of the term operations in the above procedure by $\varphi(a)$ and $\varphi(a')$, respectively, where $\varphi$ runs over the monoid of continuous endomorphisms of $\overline{\Omega^X V}$. As argued in [6], the profiniteness of the smallest fully invariant closed congruence containing $R$ may be viewed as a sort of
completeness theorem in the proof theory for pseudoidentities. Unlike the expectation expressed in that paper, we show in this paper that such a completeness theorem does not hold in full generality.

It should be noted that whether a closed congruence on a profinite algebra is profinite is a purely topological property. Indeed, Gehrke [10, Theorem 4.3] proved that a quotient of a profinite algebra is profinite if and only if it is zero-dimensional.

For a profinite semigroup \( S \), we denote \( S^I \) the semigroup obtained from \( S \) by adding a new identity element, even if \( S \) already has one, which is an isolated point. Note that \( S^I \) is again a profinite semigroup. Moreover, if \( \varphi : S \rightarrow T \) is a continuous homomorphism between profinite semigroups, then so is its extension \( \varphi^I : S^I \rightarrow T^I \) sending the new identity element of \( S^I \) to the new identity element of \( T^I \).

Given an element \( s \) of a profinite semigroup \( S \), the sequence \( (s^n) \) converges to an idempotent, denoted \( s^\omega \); this follows from the simple combinatorial fact that, if the semigroup is finite, then the sequence in question is eventually constant with idempotent value.

For a positive integer \( n \), the pseudovariety \( K_n \) consists of all finite semigroups satisfying the identity \( x_1 \cdots x_n y = x_1 \cdots x_n \). The union of the ascending chain of pseudovarieties \( K_n \) is denoted \( K \). For a finite set \( A \), the structure of both \( \Omega A K_n \) and \( \Omega A K \) is well known and quite transparent: the former consists of all words in the letters of \( A \) of length at most \( n \) where the product of two such words is the longest prefix of their concatenation of length at most \( n \); the latter is realized as the set of all finite words \( a_1 \ldots a_m \) together with all right infinite words \( a_1 a_2 \ldots \), with each \( a_i \) in \( A \), where right infinite words are left zeros and otherwise multiplication is obtained by concatenation. By the content of a finite or infinite word we mean the set of letters that appear in it.

3. A non-profinite closed congruence

Our starting observation is the well-known fact that there is a continuous mapping \( C \rightarrow [0, 1] \) from the Cantor set \( C \) onto the interval \([0, 1]\) of real numbers. One simple description of such a mapping \( \varphi \) is obtained by first writing each element \( s \) of the Cantor set as a ternary expansion \( s = \sum_{n=1}^{\infty} a_n / 3^n \) with \( a_n \in \{0, 2\} \), which is identified with the infinite word \( a_1 a_2 \ldots \); replacing the digit 2 by 1 and viewing the result as a binary expansion \( \sum_{n=1}^{\infty} a_n / 2^{n+1} \) of a real number, we obtain the image \( \varphi(s) \). Because of the ambiguity in binary expansions resulting from the equality \( \sum_{n=1}^{\infty} 1/2^n = 1 \), we see that \( \varphi \) identifies two elements of \( C \), viewed as infinite words on the alphabet \( \{0, 2\} \), if and only if they constitute a pair of the form \( w02^\infty, w20^\infty \).

The above observations suggest considering the closed congruence \( \theta \) on the profinite semigroup \( \Omega A K \) generated by the pair \( (ab^\omega, ba^\omega) \), where \( A = \{a, b\} \).

**Lemma 3.1.** The closed congruence \( \theta \) is obtained at the first step of the procedure described in Section 2 that is, it consists of the diagonal \( \Delta_{\Omega A K} \) plus all pairs of the form \( (wab^\omega, wba^\omega) \) together with \( (wba^\omega, wab^\omega) \), with \( w \in A^* \).
Proof. It suffices to show that the relation described in the statement of the lemma is a closed congruence. It is clearly closed and a reflexive and symmetric binary relation that is stable under both left and right multiplication by the same element. Since each element is related with at most one element different from it, the relation is also transitive. □

The following lemma captures a key property that holds in finite quotients of $\Omega_AK/\theta$.

**Lemma 3.2.** If $S \in K_n$ and $s,t$ are elements of $S$ such that $st^\omega = ts^\omega$, then the subsemigroup $\langle s,t \rangle$ is nilpotent.

**Proof.** To prove the lemma, it suffices to establish the claim that $usv = utv$ whenever $u,v$ are products of factors $s$ and $t$ with a total number of factors for $uv$ equal to $n - 1$. To prove it, we proceed by induction on the number $k$ of factors in $v$. If $k = 0$, then we have $us = ust^\omega = uts^\omega = ut$. Assuming that $v = sv'$ and that the claim holds for right factors which are products of $k - 1$ factors $s$ and $t$, we obtain

$$usv = ussv' = usst^\omega = usts^\omega = ustt^\omega = uts^\omega = utsv' = utv$$

and similarly for $v = tv'$. □

**Corollary 3.3.** If $S \in K$, $s,t \in S$, and $st^\omega = ts^\omega$, then the equality $s^\omega = t^\omega$ holds.

□

We may now easily obtain the following result without referring to the topology of the real numbers.

**Corollary 3.4.** The quotient topological semigroup $\Omega_AK/\theta$ is not profinite.

**Proof.** If it were profinite, since $a^\omega$ and $b^\omega$ are not $\theta$-equivalent by Lemma 3.1, there would be a continuous homomorphism $\psi : \Omega_AK \to S$ onto a semigroup from $K$ such that $\psi(ab^\omega) = \psi(ba^\omega)$ and $\psi(a^\omega) \neq \psi(b^\omega)$. This is impossible by Corollary 3.3. □

Note that the minimum ideal $I$ of $\Omega_AK$ is homeomorphic with the Cantor set and carries a multiplication in which every element is a left-zero. By the considerations at the beginning of the section, the restriction $\theta'$ of the congruence $\theta$ to $I$ is such that the quotient topological semigroup $I/\theta'$ is homeomorphic with the unit real interval $[0,1]$, carrying also the left-zero multiplication. Of course, $\theta'$ is an easier example of a closed congruence on a profinite semigroup that is not profinite as no algebraic considerations are necessary to justify this statement. Yet, the example in this section has two advantages that justify considering it: the profinite semigroup $\Omega_AK$ is finitely generated and the arguments are essentially algebraic.

4. **A finitely generated countable profinite semigroup with infinitely many idempotents**

In [7], we have investigated in the realm of semigroup theory the property of a pseudovariety $V$ which consists in all finitely generated pro-$V$ algebras to be countable; such a pseudovariety $V$ is said to be locally countable. There, we have shown that countable finitely generated profinite semigroups with finitely many idempotents are rather well behaved. In particular, they are
algebraically generated by the given finite set of generators together with the idempotents. Thus, it is natural to ask whether the property of having only finitely many idempotents always holds for a countable finitely generated profinite semigroup. In this section, we present an example of a locally countable pseudovariety whose relatively free profinite semigroups on at least two generators have infinitely many idempotents.

The example in this section came about in an attempt to find a fully invariant closed congruence on a relatively free profinite semigroup that is not a profinite congruence; the existence of such an example remains an open problem. The starting point is the congruence of Section 3 but we need to suitably modify it to obtain a fully invariant closed congruence. While the example turns out not to be a profinite congruence, it still enjoys properties that justify considering it here. It also illustrates how difficult it seems to address the conjecture of [6] for semigroups.

Let \( A = \{a, b\} \) and let \( \theta \) be the fully invariant closed congruence on the profinite semigroup \( \Omega A K \) generated by the set of all pairs of the form

\[
(ab^n a^\omega, ab^{n+1} a^\omega) \quad (n \geq 1).
\]

For a pseudoword \( w \in \Omega A K \), denote by \( \tilde{0}(w) \) the shortest prefix of \( w \) with the same content as \( w \). Note that \( \tilde{0}(w) \) is a well-defined finite word. It turns out to define an invariant for \( \theta \)-classes.

**Lemma 4.1.** If \( u \theta v \) then \( \tilde{0}(u) = \tilde{0}(v) \).

**Proof.** The lemma is proved by transfinite induction on the construction of \( \theta \). In step 0, we consider the defining pairs (4.1), which do have the required property since both components start by \( ab \); this property is clearly preserved by substitution and multiplication on the left; multiplication on the right does not affect it since all nontrivial pairs have left-zero components. The property is also clearly preserved in the transitive closure steps. Finally, it is also preserved in the topological closure steps since the languages of the form \( w A^* \) are \( K \)-recognizable, which implies that the prefix of a fixed length defines a continuous function on \( \Omega A K \).

Next, we establish that \( \theta \) identifies various pairs of infinite pseudowords.

**Lemma 4.2.** The following \( \theta \) relations hold:

(i) \( ab^n a^\omega \theta ab^2 \) for \( n \geq 3 \);
(ii) \( ab^2 \theta (ab^n)^\omega \) for \( n \geq 2 \);
(iii) \( (ab)^2 a^\omega \theta (ab)^\omega \);
(iv) \( (ab)^\omega \theta (aba)^\omega \);
(v) \( (aba)^\omega \theta aba^\omega \);
(vi) \( (ab)^\omega \theta aba^2 \);
(vii) \( ab^2 \theta aba^\omega \);
(viii) \( ab^2 \theta abw \) for every \( w \in \Omega A K \setminus A^+ \).

**Proof.** To prove (i), respectively from the definition of \( \theta \) and its full invariance, we observe that

\[
ab^n a^\omega \theta ab^{n-1} ab^2 = ab^{n-1} a (b^{n-1})^\omega \theta ab^{2(n-1)} a^\omega.
\]

Hence, for every \( n \geq 3 \), there is \( m > n \) such that \( ab^n a^\omega \theta ab^m a^\omega \). Iterating this procedure, we obtain a strictly increasing sequence \( (n_k)_k \) of integers...
starting at \( n \) such that all \( ab^n a^\omega \) lie in the same \( \theta \)-class. Since \( \theta \) is closed, it follows that \( ab^\omega = \lim ab^n a^\omega \) also lies in the same \( \theta \)-class.

For (iii), note that, for \( n \geq 2 \),
\[
ab^\omega \theta ab^{2n} a^\omega = ab^n b^n a^\omega \theta ab^n a(b^n)^\omega.
\]

Hence, we have \( ab^\omega \theta ab^n ab^\omega \). Since \( \theta \) is stable under multiplication and transitive, it follows that \( ab^\omega \theta (ab^n)^m ab^\omega \) for every \( m \geq 1 \). Taking limits, we conclude that \( ab^\omega \theta (ab^n)^\omega \).

The relation (iii) follows from the following calculations:
\[
(ab)^2 a^\omega = a \cdot ba \cdot a^\omega \theta a(baba^2)^\omega = (a(ba^2)^\omega \theta a(ba)^\omega = (ab)^\omega.
\]

For (iv), note that
\[
aba \cdot (ab)^\omega = aba^2 (ba)^\omega = a \cdot ba \cdot a \cdot (ba)^\omega \theta a(ba)^2 a^\omega = (ab)^2 a^\omega \theta (ab)^\omega.
\]

Hence, we have \( (ab)^\omega \theta (aba)^n (ab)^\omega \) for every \( n \geq 1 \). Taking limits, yields (iv).

Next, for (v), we observe that
\[
aba^\omega \theta aba^2 b^\omega \theta aba^2 ba^\omega = aba \cdot aba^\omega.
\]

Iterating, we get \( aba^\omega \theta (aba)^n aba^\omega \) which, taking limits, yields (v).

Transitivity applied to (iv) and (v) gives (vi).

For (vii), we first note that
\[
ab^\omega \theta (ab^2)^\omega \theta ab^2 (ab)^\omega \theta ab^2 aba^\omega \theta ab^2 a^2 b^\omega = ab^2 a \cdot ab^\omega
\]

where, in the second step we use invariance of \( \theta \) under the substitution sending \( a \) to \( ab \) and fixing \( b \). Iterating and taking limits yields the first step in the following string of relations:
\[
ab^\omega \theta (ab^2 a)^\omega = a(b^2 a^2)^\omega \theta ab^2 a^\omega \theta ab \cdot ab^\omega.
\]

Again, iterating and taking limits gives \( ab^\omega \theta (ab)^\omega \) which, combined with (vi), yields (vii).

Finally, for (viii), we start by observing that, substituting \( b \) by \( b^n \) in (vii), we get \( ab^\omega \theta ab^m a^\omega \). Interchanging \( a \) and \( b \) and applying the same argument to the suffix \( ba^\omega \) of \( ab^m a^\omega \) yields \( ab^\omega \theta ab^m a^m b^\omega \). Thus, multiplying on the left any element of the \( \theta \)-class of \( ab^\omega \) by \( ab^m a^{m-1} \) with \( m, n \geq 1 \) we stay in the same class. We may, therefore obtain any infinite word in the letters \( a \) and \( b \) starting with \( ab \), which proves (viii). \( \square \)

Note that in the proof of Lemma 4.2(viii) we only used the relation \( ab^\omega \theta aba^\omega \). Combined with Lemma 4.2, this observation shows that \( \theta \) is also the fully invariant closed congruence generated by the pair \((aba^\omega, ab^\omega)\).

**Proposition 4.3.** The quotient topological semigroup \( \overline{\Pi_A K}/\theta \) is profinite.

**Proof.** We must show that distinct \( \theta \)-classes may be separated by clopen unions of \( \theta \)-classes. The \( \theta \)-classes of finite words are singleton sets. Thus, the \( \theta \)-class of a finite word is itself a clopen set separating it from any other \( \theta \)-class. On the other hand, by Lemmas 4.1 and 4.2(viii), two infinite pseudowords are \( \theta \)-equivalent if and only if they have the same value under \( \theta \).
Hence the $\theta$-class of an infinite pseudoword $w \in \overline{\Omega}_A K$ of full content is separated from any other class containing infinite pseudowords by the clopen set $0(w) \overline{\Omega}_A K$. The remaining classes are the singletons $\{a^w\}$ and $\{b^w\}$, which are separated by the clopen set $a \overline{\Omega}_A K$, which in turn is a union of $\theta$-classes by Lemma 4.1.

Note that $\overline{\Omega}_A K / \theta$ is a countable relatively free profinite semigroup with infinitely many idempotents.

5. A non-profinite fully invariant closed congruence

We originally proposed our conjecture that fully invariant closed congruences on relatively free profinite semigroups are profinite as being valid in all reasonable algebraic contexts [6]. We proceed to present a counterexample for unary algebras.

Throughout the remainder of the paper, we fix a finite set $A$ whose elements are viewed as symbols of unary operations. Note that an $A$-algebra is a nonempty set $U$ together with a function $a_U : U \to U$ for each $a \in A$. These functions generate a subsemigroup $S_U$ of the full transformation semigroup $T_U$ of the set $U$ and induce a homomorphism $\alpha_U : A^+ \to S_U$, where $\alpha_U(a) = a_U$. For $w \in A^+$ and $u \in U$, we write $w \cdot u$ for $\alpha_U(w)(u)$.

An onto homomorphism $\varphi : U \to V$ of $A$-algebras induces a homomorphism $\overline{\varphi} : S_U \to S_V$. Indeed, for $w, w' \in A^+$, $\alpha_U(w) = \alpha_U(w')$ means that, for every $u \in U$, $w \cdot u = w' \cdot u$, which implies that $w \cdot \varphi(u) = w' \cdot \varphi(u)$. Since $\varphi(u)$ represents an arbitrary element of $V$, we conclude that $\alpha_V(w) = \alpha_V(w')$. Hence, for the homomorphism $\overline{\varphi}$, the following diagram commutes:

\[
\begin{array}{ccc}
A^+ & \xrightarrow{\alpha_U} & S_U \\
\downarrow{\alpha_V} & \searrow{\overline{\varphi}} & \downarrow{\alpha_V} \\
S_V.
\end{array}
\]

Note that $\overline{\varphi}$ is onto and it is the unique homomorphism for which the diagram commutes. From these observations one may deduce that $\psi \circ \varphi = \psi \circ \overline{\varphi}$ for every pair of onto homomorphisms $\varphi : U \to V$ and $\psi : V \to W$. This property suggests that one may interpret the operator $S$ applied on $A$-algebras together with the operator $\overline{-}$ applied on homomorphisms between $A$-algebras as a functor from the category of $A$-algebras to the category of semigroups (or monoids).

Given an $A$-generated semigroup $S$, we may consider $S^I$ as an $A$-algebra where $a \cdot s = as$ for $a \in A$ and $s \in S^I$. Note that $S_{S^I} \simeq S$ is essentially the Cayley representation theorem for semigroups. To continue with the categorical point of view, one may see the operator $^I$ as a functor which is adjoint to the functor mentioned in the previous paragraph. For discrete algebras and monoids this was established in [14]. This idea is not needed for the purpose of this section, but we return to it in the next two sections, where we establish appropriate results in the realm of profinite $A$-algebras and monoids.

**Lemma 5.1.** Let $U$ be an $A$-algebra generated by an element $x$. Then the mapping $1 \mapsto x$ extends uniquely to an onto homomorphism of $A$-algebras $(S_U)^I \to U$. 
Proof. Given \( w, w' \in A^+ \) acting the same way on every element of \( U \), we have, in particular \( w \cdot x = w' \cdot x \). Hence, we may define a mapping \( \varphi : (S_U)i \rightarrow U \) by \( \varphi(\alpha_U(w)) = w \cdot x \) and \( \varphi(1) = x \). Clearly this mapping respects the action of \( A \) and it is onto. Moreover, \( \varphi \) is the unique homomorphism mapping 1 to \( x \).

A profinite \( A \)-algebra \( U \) is an inverse limit of finite \( A \)-algebras. Note that \( A \)-algebras in general do not have finitely determined syntactic congruences, a property that is known to entail that a compact zero-dimensional algebra is profinite (see [5]). Nevertheless, one may ask whether every compact zero-dimensional \( A \)-algebra is profinite. We proceed to describe a simple counterexample.

Example 5.2. Take \( A = \{a\} \) to be a singleton set of operation symbols. Let \( U \) be the one point compactification of the set of natural numbers, which is obtained by adding a point \( \infty \) to which all sequences without bounded subsequences converge. Define a structure of \( A \)-algebra on \( U \) by letting \( a(n) = \max\{0, n - 1\} \) for \( n \geq 0 \) and \( a(\infty) = \infty \). Then \( U \) is a compact zero-dimensional \( A \)-algebra. Given a continuous homomorphism \( \varphi : U \rightarrow F \) into a finite \( A \)-algebra \( F \), there must exist \( m > n \) such that \( \varphi(m) = \varphi(n) \). Then, we have \( \varphi(n) = \varphi(a(m-n)(m)) = \varphi(a(m-n)(\varphi(n))) \) and so \( \varphi(n) = \varphi(a(k(m-n))(m)) \) for all \( k \geq 1 \). Taking \( k \) sufficiently large, we conclude that \( \varphi(n) = \varphi(0) \) which yields \( \varphi(r) = \varphi(0) \) for all \( r \leq m \) as \( \varphi(0) \) is a fixed point under the action of \( a \). Since there must be arbitrarily large \( m \) in the above conditions, we deduce that \( \varphi \) is constant on natural numbers. Hence, \( U \) is not residually finite and, therefore, it is not profinite.

Let \( U = \lim_i U_i \) be an inverse limit of finite \( A \)-algebras with onto connecting homomorphisms \( \varphi_{ij} : U_i \rightarrow U_j \) \( (i \geq j) \). There is an associated inverse system of \( A \)-generated finite semigroups \( S_{U_i} \), with connecting (onto) homomorphisms \( \bar{\varphi}_{ij} : S_{U_i} \rightarrow S_{U_j} \) \( (i \geq j) \). Thus, we may consider the inverse limit \( \lim_i S_{U_i} \), which may be described as the closed subsemigroup of the product \( \prod_i S_{U_i} \) consisting of all \( (s_i) \) \( (\text{with} \ s_i \in S_{U_i}) \) such that \( \varphi_{ij}(s_i) = s_j \) whenever \( i \geq j \).

As defined above, \( S_{U_i} \) is the subsemigroup of \( S_U \) generated by the \( a_U \) \( (a \in A) \). For each \( i \), there is an onto homomorphism of \( A \)-algebras \( \varphi_i : U \rightarrow U_i \), which induces an onto semigroup homomorphism \( \bar{\varphi}_i : S_U \rightarrow S_{U_i} \).

Suppose that \( w, w' \in A^+ \) are such that there exists \( u \in U \) with \( w u \neq w'u \). Then, there exists \( i \) such that

\[
\varphi_i(w) = \varphi_i(w \cdot u) \neq \varphi_i(w' \cdot u) = \varphi_i(w') = \varphi_i(w). 
\]

Hence, the homomorphism \( S_U \rightarrow \prod_i S_{U_i} \) induced by the \( \bar{\varphi}_i \) is injective and it takes values in the profinite semigroup \( \lim_i S_{U_i} \), which we denote \( S_U \).

Lemma 5.3. Let \( V = \lim_i V_i \) be another inverse limit of finite \( A \)-algebras with onto connecting homomorphisms and let \( \rho : V \rightarrow U \) be an onto continuous homomorphism. Then there is a continuous semigroup homomorphism \( \hat{\rho} : \hat{S}_V \rightarrow \hat{S}_U \) respecting generators from \( A \).

Before establishing Lemma 5.3, we register the following immediate application which amounts to the independence of the profinite semigroup \( \hat{S}_U \)
on the particular expression $U = \varprojlim U_i$ of the profinite $A$-algebra $U$ as an inverse limit of finite $A$-algebras.

**Corollary 5.4.** Up to isomorphism, the profinite semigroup $\hat{S}_U$ defined above does not depend on the concrete inverse limit $U = \varprojlim U_i$ of finite $A$-algebras with onto connecting homomorphisms considered but only on $U$.

**Proof of Lemma 5.3.** Let $\psi_\gamma : V \to V_\gamma$ be the natural mapping. Let $\hat{S}_U = \varprojlim S_{U_i}$ and $\hat{S}_V = \varprojlim S_{V_\gamma}$. For every $i$, the composite continuous homomorphism $\varphi_i \circ \rho$ entails the existence of an index $\gamma_i$ and a homomorphism $\rho_{\gamma_i} : V_\gamma \to U_i$ such that the following diagram commutes (see, for instance, [13, Lemma 3.1.37]):

$$
\begin{array}{ccc}
V & \xrightarrow{\psi_\gamma} & V_\gamma \\
\downarrow\rho & & \downarrow\rho_{\gamma_i} \\
U & \xrightarrow{\varphi_i} & U_i,
\end{array}
$$

It follows that there is an onto homomorphism $S_{V_\gamma_i} \to S_{U_i}$ respecting the choice of generators from $A$, whence also an onto continuous homomorphism $\delta_i : \hat{S}_V \to \hat{S}_{U_i}$, which does not depend on the choice of $\gamma_i$ as it also respects the choice of generators from $A$. If $i \geq j$, it follows that the diagram

$$
\begin{array}{ccc}
\hat{S}_V & \xrightarrow{\delta_j} & \hat{S}_{U_j} \\
\downarrow\delta_i & & \downarrow\delta_{ij} \\
\hat{S}_U & \xrightarrow{\delta_i} & \hat{S}_{U_i}
\end{array}
$$

commutes. From the universal property of the inverse limit, we deduce that there exists a continuous homomorphism $\hat{S}_V \to \hat{S}_U$ respecting generators.

$\square$

We have shown how to associate an $A$-generated profinite semigroup with a profinite $A$-algebra. We now show how to, conversely, associate a profinite $A$-algebra with an $A$-generated profinite semigroup.

**Lemma 5.5.** Let $S$ be an $A$-generated profinite semigroup. Then $U = S^I$ is a profinite $A$-algebra and $\hat{S}_U$ is isomorphic with $S$ as a topological semigroup.

**Proof.** Let $S = \varprojlim S_i$ be a description of $S$ as an inverse limit of finite semigroups with onto connecting homomorphisms. Then, we have $U = S^I = \varprojlim S_i^I$ and so $\hat{S}_U = \varprojlim \hat{S}_{S_i^I} \simeq \varprojlim S_i = S$. $\square$

Recall that $S_U \subseteq T_U$. In other words, there is an action of $S_U$ on $U$, namely the mapping $S_U \times U \to U$ which maps $(\alpha_U(w), u)$ to $w \cdot u$. Elements of $\hat{S}_U$ may also be viewed as transformations of $U$ as the next result shows.

**Proposition 5.6.** For a profinite $A$-algebra $U$, there is a natural continuous action of $\hat{S}_U$ on $U$, that is a continuous mapping $\beta : \hat{S}_U \times U \to U$ such that $\beta(\alpha_U(w), u) = w \cdot u$ for every $w \in A^+$. The corresponding mapping $\hat{S}_U \to T_U$, which maps $s$ to $\beta(s, \_)$, is injective.

For $s \in \hat{S}_U$ and $u \in U$, we denote the image under $\beta$ of the pair $(s, u)$ also by $s \cdot u$. 

Proof. Let \( U = \lim U_i \) be an inverse limit of finite \( A \)-algebras with onto connecting homomorphisms \( \varphi_{ij} \). For the natural continuous homomorphism \( \varphi_1 : U \to U_i \), we have a corresponding continuous homomorphism \( \hat{\varphi}_1 : \hat{S}_U \to \hat{S}_{U_i} \). The two mappings provide a continuous mapping \( \hat{\varphi}_1 \times \varphi_i : \hat{S}_U \times U \to \hat{S}_{U_i} \times U_i \). Composing with the action \( \hat{S}_{U_i} \times U_i \to U_i \) gives a continuous mapping \( \psi_i : \hat{S}_U \times U \to U_i \). From the definition of these mappings, it is clear that, if \( i \geq j \), then \( \varphi_{ij} \circ \psi_i = \psi_j \). Hence, the mappings \( \psi_i \) induce a continuous mapping \( \beta : \hat{S}_U \times U \to U \) such that the following diagram commutes for every \( i \):

\[
\begin{array}{ccc}
\hat{S}_U \times U & \xrightarrow{\beta} & U \\
\downarrow{\varphi_i \times \varphi_i} & & \downarrow{\varphi_i} \\
\hat{S}_{U_i} \times U_i & \xrightarrow{\psi_i} & U_i.
\end{array}
\]

For \( s \in \hat{S}_U \), that is \( s = \alpha_U(w) \) with \( w \in A^+ \), we have \( \varphi_i(\beta(s, u)) = w \cdot \varphi_i(u) \), and consequently \( \beta(s, u) = w \cdot u \).

Given distinct \( s, t \in \hat{S}_U \), there is \( i \) such that \( \hat{\varphi}_i(s) \neq \hat{\varphi}_i(t) \). Since \( \hat{S}_{U_i} \) is a subsemigroup of \( \hat{T}_{U_i} \), there is \( u_i \in U_i \) such that \( \hat{\varphi}_i(s)(u_i) \neq \hat{\varphi}_i(t)(u_i) \). From the commutativity of the above diagram it follows that \( \beta(s, u) \neq \beta(t, u) \) for every \( u \in \varphi_i^{-1}(u_i) \), which proves the required injectivity. \( \square \)

Notice that, for every \( s \in \hat{S}_U \), the transformation \( \beta(s, \_ \,) \) is continuous as we show in the next section. This aspect is not important for the purpose of this section but plays a role later in the paper.

Let \( \overline{U} \) be the pseudovariety consisting of all finite \( A \)-algebras and let \( X = \{ x \} \). Abusing notation, we may write \( \overline{A}_X \) instead of \( \overline{A} \) for a pseudovariety \( V \) of \( A \)-algebras.

**Lemma 5.7.** The \( A \)-generated profinite semigroup \( S = \hat{\overline{S}}_{\overline{A} \overline{U}} \) is free.

**Proof.** Let \( T \) be an \( A \)-generated finite semigroup. Consider the unique continuous homomorphism of \( A \)-algebras \( \varphi : \overline{A}_X \overline{U} \to T' \) mapping \( x \) to \( 1 \). By Lemma 5.3 it induces an onto continuous homomorphism of semigroups \( S \to S_{T'} = T \) respecting the generators from \( A \). Hence, \( S \) is freely generated by \( A \) as a profinite semigroup. \( \square \)

Denote by \( \eta \) the unique continuous isomorphism \( \overline{A}_S \to \hat{\overline{S}}_{\overline{A} \overline{U}} \) respecting generators.

Consider now a pseudovariety \( V \) of semigroups. There is an associated pseudovariety \( V^u \) of \( A \)-algebras which is defined by the pseudoidentities of \( A \)-algebras \( \eta(u) \cdot x = \eta(v) \cdot x \), where \( u = v \) runs over all semigroup pseudoidentities on \( A \) satisfied by \( V \). Note that, in particular, \( \overline{U} = \overline{S}^u \). Let \( \lambda : \overline{A}_U \to \overline{A}_V^u \) and \( \psi : \overline{A}_S \to \overline{A}_V \) be the natural onto homomorphisms.

By definition, \( \eta \) induces a continuous homomorphism \( \eta_V : \overline{A}_V \to \hat{\overline{S}}_{\overline{A} \overline{V}^u} \). Indeed, let \( u, v \in \overline{A}_S \). If \( \psi(u) = \psi(v) \), then the pseudoidentity \( u = v \) holds in \( V \), so that by definition of \( V^u \), the pseudoidentity \( \eta(u) \cdot x = \eta(v) \cdot x \) holds in \( V^u \); substituting an arbitrary element of \( \overline{A}_U \) for \( x \), it follows that \( \lambda(\eta(u)) = \lambda(\eta(v)) \). Hence, there is indeed a continuous homomorphism \( \eta_V \).
such that the following diagram commutes:

\[
\begin{array}{ccc}
\Omega S & \xrightarrow{\eta} & \hat{S}_{\Pi, u} \\
\downarrow{\psi} & & \downarrow{\lambda} \\
\Omega V & \xrightarrow{\eta_V} & \hat{S}_{\Pi, V, u}
\end{array}
\]

The continuity of \(\eta_V\) follows from the compactness of \(\Omega S\).

**Proposition 5.8.** There is a continuous isomorphism of profinite \(A\)-algebras \(\varphi_V: \Omega_x V^u \to (\Omega A V)^I\) such that, for every \(w \in \Omega_x V^u\), the equality

\[
(5.1) \quad w = \eta_V^I(\varphi_V(w)) \cdot x
\]

holds.

In particular, \((\Omega A S)^I\) is the free 1-generated profinite \(A\)-algebra.

**Proof of Proposition 5.8.** Given \(u, v \in \Omega A S\) such that the pseudoidentity \(u = v\) is satisfied by \(V\), \(uw = vw\) is also valid in \(V\) for all \(w \in \Omega A S\). Hence the profinite \(A\)-algebra \((\Omega A V)^I\) satisfies the defining pseudoidentities for \(V^u\) and, whence, it is a pro-\(V^u\) \(A\)-algebra. Hence, there is a continuous homomorphism of profinite \(A\)-algebras \(\varphi_V: \Omega_x V^u \to (\Omega A V)^I\) mapping \(x\) to the identity element of \((\Omega A V)^I\).

Let \(w\) be an arbitrary point of the \(A\)-algebra \(\Omega_x V^u\). To establish that the equality \((5.1)\) holds and, therefore, that \(\varphi_V\) is an isomorphism, consider the following diagram:

\[
\begin{array}{ccc}
(\Omega A V)^I & \xrightarrow{\eta_V^I} & (\hat{S}_{\Pi, V^u})^I \\
\downarrow{\varphi_V} & & \downarrow{x} \\
\Omega_x V^u & \xrightarrow{\text{id}} & \Omega_x V^u
\end{array}
\]

where \(\nu\) is defined to make the upper triangle commutative. The desired equality means that the lower triangle also commutes. To prove it, we first claim that the mapping \(\nu\), sending each \(v \in (\Omega A V)^I\) to \(\eta_V^I(v) \cdot x\), is a homomorphism of \(A\)-algebras. Indeed, given \(v \in (\Omega A V)^I\) and \(a \in A\), we have

\[
\eta_V^I(av) \cdot x = (\eta_V^I(a)\eta_V^I(v)) \cdot x = \eta_V^I(a) \cdot (\eta_V^I(v) \cdot x) = a_{\Omega_x V^u}(\eta_V^I(v) \cdot x)
\]

Since \(\varphi_V\) is also a continuous homomorphism of \(A\)-algebras and \(\nu(\varphi_V(x)) = x\), we deduce that Diagram \((5.2)\) commutes.

The following is an easy consequence of the proof of Proposition 5.8.

**Corollary 5.9.** The mapping \(\eta_V\) is an isomorphism.

**Proof.** From the commutativity of Diagram \((5.2)\), we see that \(\nu\) is an isomorphism of \(A\)-algebras, namely \(\nu = \varphi_V^{-1}\). It follows that so is \(\eta_V^I\) and, therefore also \(\eta_V\).
**Lemma 5.10.** Let \( \theta \) be a closed congruence on \( \Omega_A V \) and \( \theta' \) its extension to the semigroup \( (\Omega_A V)^I \) that is obtained by making 1 a singleton class. In view of Proposition 5.8.5 that determines an equivalence relation \( \tilde{\theta} \) on \( \Omega_A V^u \). The relation \( \tilde{\theta} \) is a fully invariant closed congruence on \( \Omega_A V^u \).

**Proof.** We first show that \( \tilde{\theta} \) is a congruence. Suppose that \( u, v \in \Omega_A V^u \) are \( \tilde{\theta} \)-equivalent and let \( \varphi_V : \Omega_A V^u \rightarrow (\Omega_A V)^I \) be the isomorphism of \( A \)-algebras given by Proposition 5.8. Then, the pair \((\varphi_V(u), \varphi_V(v))\) belongs to \( \theta' \) and so do \((\varphi_V(a \cdot u), \varphi_V(a \cdot v)) = (a \varphi_V(u), a \varphi_V(v))\) for every \( a \in A \). Hence, \((a \cdot u, a \cdot v)\) belongs to \( \tilde{\theta} \).

Next, we verify that \( \tilde{\theta} \) is fully invariant. Given a continuous endomorphism \( \psi \) of \( \Omega_A V^u \) and a pair \((u, v)\) \( \in \tilde{\theta} \), if one of \( u \) and \( v \) is \( x \) then so is the other, and so obviously \((\psi(u), \psi(v)) \in \tilde{\theta} \). So, we may assume that \( u, v \neq x \). Let \( w = \psi(x) \); we may assume that \( w \neq x \). By Proposition 5.8 we have

\[
\psi(u) = \eta_V(\varphi_V(u) \cdot x) = \eta_V(\varphi_V(u)) \cdot w \quad \text{and} \quad \psi(v) = \eta_V(\varphi_V(v)) \cdot w \quad \text{so that} \quad \tilde{\theta} \ni \eta_V(\varphi_V(w)) \cdot x = \eta_V(\varphi_V(v)) \cdot w = \psi(\eta_V(\varphi_V(v)) \cdot x) = \psi(v).
\]

After all the above preparation, we are ready for our counterexample to Conjecture 3.3 for unary algebras.

**Theorem 5.11.** There is a pseudovariety \( U \) of unary algebras and a closed fully invariant congruence \( \theta \) on \( \Omega_X U \) that is not profinite.

**Proof.** Let \( A = \{a, b\} \) and let \( \theta \) be the equivalence relation on \( \Omega_A K \) of Section 6 namely the one that identifies two infinite words if and only if they are equal or constitute a pair of the form \((wab^2, wba^2)\), with \( w \) a finite word; finite words form singleton classes. Note that \( \theta \) is a closed congruence on \( \Omega_A K \). Hence, the associated relation \( \theta \) on \( \Omega_A K^u \) is a fully invariant closed congruence by Lemma 5.10. Note that the infinite words form a closed subspace of \( \Omega_A K \) homeomorphic with the Cantor set (it is the perfect kernel of \( \Omega_A K \), cf. [11] Section 6.2]). The relation \( \theta \) corresponds to the kernel of the Cantor function. Hence, \( \Omega_A K^u / \theta \) is not profinite (it contains a closed subspace homeomorphic with the interval \([0,1]\), which is not zero-dimensional) thereby showing that \( \tilde{\theta} \) is not a profinite congruence.

6. FROM PROFINITE UNARY ALGEBRAS TO PROFINITE MONOIDS

In the previous section we constructed, for a given profinite \( A \)-algebra \( U \), the profinite semigroup \( \hat{S}_U \). Now we exhibit an alternative construction, with the minor modification that the resulting structure is a profinite monoid.

Recall that \( A \) is a fixed unary signature. Terms over one single variable \( x \) may be written as \( w(x) \) with \( w \in A^* \). Every term \( w(x) \) induces, for a given \( A \)-algebra \( U \), the transformation \( \alpha_U(w) \in S_U \), with one exception, where \( w \) is the empty word, the considered term is \( x \), and the resulting transformation is the identity which is not a member of \( S_U \) as defined in Section 5. For that reason, we add the identity transformation of \( U \) to \( S_U \), if it is not already present, and denote the resulting monoid \( M_U \). Moreover, the considered mapping \( \pi : T_A(x) \rightarrow M_U \) is a monoid homomorphism if we consider the operation of substitution in \( T_A(x) \); more formally, we define the operation \( \cdot \) on \( T_A(x) \) by the rule \( w(x) \cdot u(x) = wu(x) \).
To imitate the previous construction in the case of profinite $A$-algebras and avoid working with inverse limits, we consider a pseudovariety $V$ of $A$-algebras, the free pro-$V$ algebra $\Omega_xV$ over the set $\{x\}$, and look at the transformations that are determined by members of $\Omega_xV$. We claim that $\Omega_xV$ is equipped with the binary operation of substitution generalizing that for $T_A(x)$. Indeed, by Theorem 2.1, the evaluation mapping $\Omega_xV \times \Omega_xV \to \Omega_xV \quad (u,v) \mapsto \sigma_v^V(u)$ is continuous where, in the notation of Section 2, $\sigma_v^V = \hat{\phi}_v$ for the mapping $\phi_v : \{x\} \to \Omega_xV$ sending $x$ to $v$. In case $V = Un$, we drop the superscript in the notation $\sigma_v^V$.

For each pair $u,v \in \Omega_xV$, we put $v \cdot u = \sigma_v^V(w)$. In this way we define a continuous multiplication on $\Omega_xV$. We claim that this binary operation is also associative. Indeed, for $u,v,w \in \Omega_xV$ we have
\[
(w \cdot v) \cdot u = \sigma_u^V(w \cdot v) = \sigma_u^V(\sigma_v^V(w)),
\]
and
\[
w \cdot (v \cdot u) = \sigma_v^V(w \cdot u) = \sigma_v^V(\sigma_u^V(w)).
\]
Hence, the associativity of the operation $\cdot$ is equivalent to stating that
\[
(6.1) \quad \sigma_{\sigma_v^V(u)} = \sigma_{\sigma_u^V(w)} = \sigma_u^V \circ \sigma_v^V,
\]
which follows from the commutativity of the following diagram:

Since $\Omega_xV$ is a topological monoid under the operation $\cdot$, with neutral element $x$, and a compact zero-dimensional space, by [13] it is a profinite monoid.

For a profinite $A$-algebra $U$ we consider the set $C(U)$ of all continuous transformations from $U$ to $U$. We endow $C(U)$ with the compact-open topology, with subbase of open sets formed by the subsets of the form
\[
[K, O] = \{ f \in C(U) : f(K) \subseteq O \}
\]
where $K$ is compact and $O$ is open. Since the profinite space $U$ is compact (and Hausdorff), we may choose as subbasic open sets only those $[K, O]$ where $O$ is clopen: indeed, $O$ is the union of a family $(O_i)_{i \in I}$ of clopen sets and, from the compactness of $K$ it follows that
\[
[K, O] = \bigcup_{F \text{ finite} \subseteq I} [K, \bigcup_{i \in F} O_i].
\]

Note that this smaller subbase of open sets actually consists of clopen sets:
\[
\mathcal{C}(U) \setminus [K, O] = \{ f \in U : f(K) \setminus O \neq \emptyset \} = \bigcup_{u \in K} \{ \{ u \}, U \setminus O \}.
\]

It is also a basic observation from general topology that \( \mathcal{C}(U) \) is Hausdorff, even under much weaker hypotheses \cite{16} Theorem 43.4. Moreover, since \( \mathcal{C}(U) \) is closed under composition, it is a monoid. Notice that composition in \( \mathcal{C}(U) \) is continuous with respect to the compact-open topology — more generally, this holds whenever \( U \) is locally compact, see \cite{17} Theorem 3.4.2.

Let \( V \) be a pseudovariety of \( A \)-algebras and \( U \) a pro-\( V \) algebra. We are now ready to state basic properties of the mapping \( \tau^V_U : \Omega_x V \to \mathcal{C}(U) \) defined by the rule \( \tau^V_U(w) = w_U \); this is well defined by Corollary \(2.2\).

**Lemma 6.1.** Let \( V \) be a pseudovariety of \( A \)-algebras, \( U \) be a pro-\( V \) algebra, and \( \tau^V_U : \Omega_x V \to \mathcal{C}(U) \) be the mapping defined above. Then

(i) \( \tau^V_U \) is a continuous homomorphism of topological monoids;
(ii) \( \tau^V_U \) is a closed mapping;
(iii) the image \( \text{Im}(\tau^V_U) \) is the closure of \( \mathcal{M}_U \) in \( \mathcal{C}(U) \).

**Proof.** We first show that \( \tau^V_U \) is a monoid homomorphism. So, we want to show that \( \tau^V_U(u \cdot v) \) and \( \tau^V_U(u) \circ \tau^V_U(v) \) are the same transformation of \( U \) for every \( u, v \in \Omega_x V \). We choose an arbitrary \( w \in U \) and check that \( (u \cdot v)_U(w) = u_U(v_U(w)) \). Let \( \psi \) be the unique continuous homomorphism \( \Omega_x V \to U \) that maps \( x \) to \( w \). Consider the following diagram:

\[
\begin{array}{ccc}
\Omega_x V & \xrightarrow{(u \cdot v)_V} & \Omega_x V \\
\downarrow \psi & & \downarrow \psi \\
U & \xrightarrow{u_U} & U,
\end{array}
\]

By Corollary \(2.3\) the two inner squares and the outer square commute while the upper triangle commutes by \(6.1\). Hence, we have
\[
(u \cdot v)_U(w) = (u \cdot v)_U(\psi(x)) = \psi((u \cdot v)_V(\psi(x))) = \psi(u_V(\psi(x))) = u_U(\psi(\psi(x))) = u_U(\psi(x)) = u_U(v_U(w)).
\]

Continuity of \( \tau^V_U \) follows from \cite{17} Theorem 3.4.7. This proves \(1\).

Since \( \tau^V_U \) is a continuous mapping from the compact space \( \Omega_x V \) to the Hausdorff space \( \mathcal{C}(U) \), it follows that \( \tau^V_U \) is a closed mapping, which gives \(1\).

From that we also get that \( \text{Im}(\tau^V_U) \) is a closed and compact subspace of \( \mathcal{C}(U) \). We may see \( T_A(x) \) as a dense subset of \( \Omega_x V \). Hence, \( \mathcal{M}_U \subseteq \text{Im}(\tau^V_U) \) and every element of \( \text{Im}(\tau^V_U) \) is a limit of some net consisting of elements of \( \mathcal{M}_U \). \( \square \)

We denote the topological monoid \( \text{Im}(\tau^V_U) \) by \( \mu_V(U) \). Notice that \( \mu_{\mu_V}(U) = \mathcal{M}_U \) whenever \( U \) is a finite \( A \)-algebra, because \( \mathcal{C}(U) \) is a discrete finite space in this case.
If \( \varphi : U \rightarrow V \) is a continuous homomorphism between pro-\( \mathcal{V} \) algebras and \( w \in \Omega_x \mathcal{V} \) is an arbitrary element, then the following diagram commutes by Corollary 2.3:

\[
\begin{array}{c}
U \xrightarrow{w_U} U \\
\downarrow \varphi \downarrow \varphi \\
V \xrightarrow{w_V} V.
\end{array}
\]

Now if, in addition, the mapping \( \varphi : U \rightarrow V \) is surjective, then we may define \( \tilde{\varphi} : \mu_{\mathcal{V}}(U) \rightarrow \mu_{\mathcal{V}}(V) \) by \( \tilde{\varphi}(w_U) = w_V \). To see that this definition is correct, let \( w_U = w_U' \) for a pair of elements \( w, w' \in \Omega_x \mathcal{V} \). Then for every \( v \in V \) we may choose \( u \in U \) such that \( \varphi(u) = v \). Now we have \( w_V(v) = w_V(\varphi(u)) = \varphi(w_U(u)) \) by the commutativity of the above diagram. Since \( w_U = w_U' \), we get \( \varphi(w_U(u)) = \varphi(w_U'(u)) = w_V(\varphi(u)) = w_V'(v) \). Altogether we deduce that \( w_V = w_V' \). Note that \( \tilde{\varphi} \) is the unique mapping such that the following diagram commutes:

\[
\begin{array}{ccc}
\hat{\Omega}_x \mathcal{V} & \xrightarrow{\tau_U} & \mu_{\mathcal{V}}(U) \\
\downarrow \tilde{\varphi} \downarrow \tau_V \\
\mu_{\mathcal{V}}(U) & \xrightarrow{\mu_{\mathcal{V}}(\cdot)} & \mu_{\mathcal{V}}(V).
\end{array}
\]

**Lemma 6.2.** If \( \varphi : U \rightarrow V \) is an onto continuous homomorphism between pro-\( \mathcal{V} \) algebras, then \( \tilde{\varphi} : \mu_{\mathcal{V}}(U) \rightarrow \mu_{\mathcal{V}}(V) \) is an onto continuous monoid homomorphism.

**Proof.** Clearly, the mapping \( \tilde{\varphi} \) is onto and it is a continuous monoid homomorphism because so are \( \tau_U \) and \( \tau_V \) by Lemma 6.1. \( \square \)

The expected statement follows.

**Proposition 6.3.** If \( U \) is a pro-\( \mathcal{V} \) \( A \)-algebra, then \( \mu_{\mathcal{V}}(U) \) is a profinite monoid.

**Proof.** In view of Lemma 6.1, \( \mu_{\mathcal{V}}(U) \) is a compact submonoid of the topological monoid \( C(U) \). Since \( C(U) \) is zero-dimensional, \( \mu_{\mathcal{V}}(U) \) is a topological monoid on a profinite space. By [13], it follows that \( \mu_{\mathcal{V}}(U) \) is a profinite monoid. \( \square \)

The following is a further application of Lemma 6.1.

**Proposition 6.4.** For a pseudovariety \( \mathcal{V} \) of \( A \)-algebras, the profinite monoids \( (\hat{\Omega}_x \mathcal{V}, \cdot) \) and \( (\mu_{\mathcal{V}}(\hat{\Omega}_x \mathcal{V}), \circ) \) are isomorphic.

**Proof.** We already know that \( (\hat{\Omega}_x \mathcal{V}, \cdot) \) is a profinite monoid and that \( \tau_U^\mathcal{V} : \hat{\Omega}_x \mathcal{V} \rightarrow \mu_{\mathcal{V}}(\hat{\Omega}_x \mathcal{V}) \) is an onto continuous monoid homomorphism by Lemma 6.1. We also know that \( \tau_U^\mathcal{V} \) is a closed mapping. If we show that \( \tau_U^\mathcal{V} \) is injective, then it is a bijective continuous closed mapping, and therefore it is a homeomorphism. And as a bijective monoid homomorphism it is also a monoid isomorphism. Thus, the whole statement is proved if we show that \( \tau_U^\mathcal{V} \) is injective.

For a pair of distinct elements \( w, w' \in \hat{\Omega}_x \mathcal{V} \), we want to show that the transformations \( w_U, w'_U \in \mu_{\mathcal{V}}(U) \) are distinct. Since \( U = \hat{\Omega}_x \mathcal{V} \) is freely
generated by $x$, we must show that $w_U(x) \neq w_U'(x)$. However, we have $w_U(x) = \sigma'_x(w) = w$. Hence, we get $w_U'(x) = w' \neq w = w_U(x)$. \hfill $\square$

To summarize, for any pro-$V$ algebra $U$, we construct the profinite monoid $\mu_V(U)$. Moreover, in view of Lemma 6.2, this construction respects onto homomorphisms. So, it is natural to ask whether the correspondence $\mu_V$ works as a functor from the point of view of category theory. We specify appropriate categories later, but now we just observe, that $\mu_V$ respects composition of surjective homomorphisms.

**Lemma 6.5.** Let $\varphi : U \to V$ and $\psi : V \to W$ be onto continuous homomorphisms between pro-$V$ algebras. Then the equality $\widetilde{\psi} \circ \varphi = \widetilde{\psi} \circ \varphi$ holds.

**Proof.** Let $w \in U$ be an arbitrary element. Then we may see that

$$(\widetilde{\psi} \circ \varphi)(w_U) = \widetilde{\psi}(\varphi(w_U)) = \widetilde{\psi}(w_V) = w_W.$$ 

For composite continuous homomorphism $\psi \circ \varphi : U \to W$, we also have $\widetilde{\psi} \circ \varphi(w_U) = w_W$. \hfill $\square$

Although, for a pro-$V$ algebra $U$, we have defined $\mu_V(U)$ using the pseudovariety $V$, it turns out that this profinite monoid depends only on $U$.

**Proposition 6.6.** Let $U$ be a pro-$V$ algebra. Then the equality $\mu_V(U) = \mu_U(U)$ holds.

**Proof.** Consider the following special case of Diagram (2.2):

$$
\begin{array}{c}
\overline{\Omega}_x U \times U & \xrightarrow{\epsilon_U^V} & U \\
\downarrow \pi_V \times \text{id}_U & & \downarrow \text{id}_U \\
\overline{\Omega}_x V \times U & \xrightarrow{\epsilon_U^V} & U 
\end{array}
$$

Since the diagram commutes, as was established in the proof of Theorem 2.1, we obtain the following chain of equalities for $w \in \overline{\Omega}_x U$ and $u \in U$:

$$w_U(u) = \epsilon_U^V(w, u) = \epsilon_U^V(\pi_V(w), u) = (\pi_V(w))_U(u).$$

This shows that $\mu_U(U) = \mu_V(U)$. \hfill $\square$

In view of Proposition 6.6, from hereon we drop the subscript $V$ in the notation $\mu_V$.

To conclude this section, we compare the construction of the profinite monoid $\mu(U)$ with the construction of the profinite monoid given via inverse limits in Section 5. Let $U = \varprojlim U_i$ be an inverse limit of finite $A$-algebras with onto connecting homomorphisms $\varphi_{ij} : U_i \to U_j$ ($i \geq j$). There is an associated inverse system of $A$-generated finite monoids $\mathcal{M}_{U_i} = \mu(U_i)$ with connecting homomorphisms $\tilde{\varphi}_{ij} : \mathcal{M}_{U_i} \to \mathcal{M}_{U_j}$ ($i \geq j$). Since, for every $i$, we have an onto continuous homomorphism $\varphi_i : U \to U_i$, we also have the corresponding continuous homomorphism of monoids $\tilde{\varphi}_i : \mu(U) \to \mu(U_i)$. By Lemma 6.5, all homomorphisms between $\mu(U)$ and the $\mu(U_i)$’s compose in the right way. To see that $\mu(U)$ is the inverse limit $\varprojlim \mu(U_i)$, we only need to show that, for every two distinct elements $w_U, w'_U \in \mu(U)$, there is an index $i$ such that $\tilde{\varphi}_i(w_U) \neq \tilde{\varphi}_i(w'_U)$. We show that in the next result.
Corollary 2.3, we know that \( \varphi \) \( \text{algebras} \) (Lemma 7.2. Let \( \gamma : \Sigma \to M \) be a homomorphism from the \( \Sigma \) \( \kappa \) a mapping. We denote by it remains to observe that the commutativity of the diagram means that \( \varphi \) \( \text{algebras} \) \( \mu(U) \) is an inverse limit of the corresponding system of finite monoids \( \mu(U) \). Proof. By the comment before the statement of the proposition, we have

\( \text{to show just one detail. Let} \ w, w' \in \overline{\Omega}_U \text{Un be such that} \ w_U \neq w'_U. \text{Then there is} \ u \in U \text{such that} \ w_U(u) \neq w'_U(u). \text{Since} \ U \text{is an inverse limit of A-}

\( \text{algebras} \) \( \{(U_i)_i\}, \text{there is} \ i \text{such that} \ \varphi_i(w_U(u)) \neq \varphi_i(w'_U(u)). \text{However, from Corollary 2.3 we know that} \ \varphi(w_U(u)) = w_U(\varphi(u)) = \varphi_i(w_U(\varphi(u))) \text{and, similarly,} \ \varphi_i(w'_U(u)) = \varphi_i(w'_U(\varphi(u))). \text{It follows that} \ \varphi(w_U) \neq \varphi(w'_U) \text{because they map} \ \varphi(u) \in U_i \text{to distinct points.} \)

7. From profinite monoids to profinite algebras

In Secion 4 we constructed, for a given semigroup, a certain \( A \)-algebra. We generalize that construction for an arbitrary signature. However, the unary case is the one we study in most detail. Once again, we work with monoids instead of semigroups in this section.

Let \( \Sigma \) be an arbitrary signature, let \( M \) be a monoid, and let \( \gamma : \Sigma \to M \) be a mapping. We denote by \( \kappa_{\gamma}(M) \) the following \( \Sigma \)-algebra: the domain is \( M \), and for every arity \( n \), we consider the evaluation mapping \( E_n : \Sigma_n \times M^n \to M \) given by

\[ E_n(w, m_1, \ldots, m_n) = \gamma(w) \cdot m_1 \cdot \cdots \cdot m_n. \]

This construction is robust in the sense of the following lemmas.

Lemma 7.1. Let \( \varphi : M \to N \) be a monoid homomorphism. Let \( \kappa_{\gamma}(M) \) be the \( \Sigma \)-algebra defined for a fixed mapping \( \gamma : \Sigma \to M \) and \( \kappa_{\varphi \circ \gamma}(N) \) be the \( \Sigma \)-algebra defined for the mapping \( \varphi \circ \gamma : \Sigma \to N \). Then \( \varphi : \kappa_{\gamma}(M) \to \kappa_{\varphi \circ \gamma}(N) \) is a homomorphism of \( \Sigma \)-algebras.

Proof. The following diagram commutes, for every arity \( n \):

\[ \begin{array}{ccc}
\Sigma_n \times M^n & \xrightarrow{E_n} & M \\
\downarrow{id_{\Sigma_n} \times \varphi^n} & & \downarrow{\varphi} \\
\Sigma_n \times N^n & \xrightarrow{E_n} & N.
\end{array} \]

It remains to observe that the commutativity of the diagram means that \( \varphi \) is a homomorphism from the \( \Sigma \)-algebra \( \kappa_{\gamma}(M) \) to the \( \Sigma \)-algebra \( \kappa_{\varphi \circ \gamma}(N) \).

Lemma 7.2. Let \( \kappa_{\gamma}(M) \) be the \( \Sigma \)-algebra defined for a fixed monoid \( M \) and a mapping \( \gamma : \Sigma \to M \). Then the following hold:

(i) If \( M \) is a topological monoid, then \( \kappa_{\gamma}(M) \) is a topological \( \Sigma \)-algebra.

(ii) If \( M \) is a profinite monoid, then \( \kappa_{\gamma}(M) \) is a profinite \( \Sigma \)-algebra.

Proof. The first statement is clear, since \( E_n \) is continuous whenever the multiplication on \( M \) is continuous.

Assume that \( M \) is profinite. For a pair of distinct elements \( m \neq m' \), there is a continuous homomorphism \( \varphi : M \to N \) onto a finite monoid \( N \) such that \( \varphi(m) \neq \varphi(m') \). By Lemma 7.1 we know that \( \varphi \) is a homomorphism of \( \Sigma \)-algebras, which is continuous, as topologies are kept.
Now we move our attention to the unary case, that is, the case when \( \Sigma_1 = A \), and \( \Sigma_n = \emptyset \) for every \( n > 1 \). If we look at Section 5, we see that the results deal with \( A \)-generated semigroups. In the setting of this section, this restriction may be expressed as \( \gamma(\Sigma) \) generating a dense submonoid of the profinite monoid \( M \). Moreover, the constructed profinite \( A \)-algebra \( \kappa_\gamma(M) \) is generated by the element \( 1 \in M \) (again in the algebraic-topological sense). To put it more formally, we let \( \Sigma_0 = \{1\} \) and we deal with profinite \( \Sigma \)-algebras which have no proper closed subalgebras. Then the natural homomorphisms between our structures are automatically surjective. This restriction fits to observations from Section 6. In this way we fix categories for which we want to relate constructions from Section 6 and this section.

Let \( \text{PUA}_A^1 \) be the category whose objects are profinite unary \( \Sigma \)-algebras without proper closed subalgebras and morphisms are continuous homomorphisms between such algebras. In particular, every morphism is an onto mapping, because every object \( U \) is a profinite \( A \)-algebra \( U \) generated by the single element \( 1_U \), where \( 1 \in \Sigma_0 \) is the unique nullary operational symbol. Notice that \( \text{PUA}_A^1 \) is a thin category, that is, for every two objects \( U \) and \( V \) there is at most one morphism from \( U \) to \( V \).

Furthermore, let \( \text{PM}_A \) be the category defined in the following way. The objects are \( A \)-generated profinite monoids, or more formally, mappings \( \gamma : A \to M \), where \( M \) is a profinite monoid such that the closed submonoid generated by \( \text{Im}(\gamma) \) is \( M \). And morphisms are surjective continuous monoid homomorphisms or, more formally, \( \varphi \) is a morphism from \( \gamma : A \to M \) to \( \beta : A \to N \) if \( \varphi : M \to N \) is a continuous monoid homomorphism and \( \varphi \circ \gamma = \beta \). Also the category \( \text{PM}_A \) is thin and all morphisms are surjective mappings.

Recall, that for an object \( U \) in \( \text{PUA}_A^1 \) we have constructed the profinite monoid \( \mu(U) \). This is not formally an object in the category \( \text{PM}_A \), since we need to fix a mapping \( \gamma : A \to \mu(U) \). However, this mapping is canonically determined in Lemma 6.1. Indeed, since \( M_U \) is generated (as a monoid) by transformations given by \( a \in A \), we have \( \gamma(a) = (ax)_U \in \mu(U) \). The same lemma ensures that this mapping \( \gamma : A \to \mu(U) \) is an object in the category \( \text{PM}_A \). We denote this \( \gamma \) by \( \Gamma(U) \). Moreover, if there is a morphism \( \varphi : U \to V \) in the category \( \text{PUA}_A^1 \), then \( \varphi \) is a morphism between the objects \( \Gamma(U) \) and \( \Gamma(V) \) by Lemma 6.2 and the considerations before that lemma. Thus, we denote \( \Gamma(\varphi) = \tilde{\varphi} \) and we see that \( \Gamma : \text{PUA}_A^1 \to \text{PM}_A \) is a functor of these thin categories by Lemma 6.5.

Now, for a given \( \gamma : A \to M \) in \( \text{PM}_A \), the \( A \)-algebra \( \kappa_\gamma(M) \) is profinite and generated by the element \( 1 \in M \). Thus, we may denote \( \Psi(\gamma) = \kappa_\gamma(M) \), which is an object in \( \text{PUA}_A^1 \). If \( \varphi \) is a morphism in \( \text{PM}_A \), in particular it is a surjective homomorphism \( \varphi : M \to N \), then by Lemma 7.1 the mapping \( \varphi : \Psi(\gamma) \to \Psi(\varphi \circ \gamma) \) is a continuous homomorphism of \( A \)-algebras. If we denote it as \( \Psi(\varphi) \), then we get a functor \( \Psi : \text{PM}_A \to \text{PUA}_A^1 \) between these thin categories. The only detail that might require some checking is that \( \Psi(\varphi) \circ \Psi(\varphi') = \Psi(\varphi \circ \varphi') \) for a pair of morphisms \( \varphi : \gamma \to (\varphi \circ \gamma) \) and \( \varphi' : (\varphi \circ \gamma) \to (\varphi' \circ \varphi \circ \gamma) \) in the category \( \text{PM}_A \). But this is trivial since \( \Psi(\psi) = \psi \) for every object \( \psi \) in \( \text{PM}_A \).
We may interpret the monoid analog of Lemma 5.5 with respect to Proposition 6.7 as the following statement.

**Lemma 7.3.** For every object $\gamma$ from $\mathbb{P}M_A$, we have $\Gamma(\Psi(\gamma)) \cong \gamma$. □

Composing the functors in the reverse order leads to the following result.

**Lemma 7.4.** For every object $U$ from $\mathbb{P}UA^1_A$, there exists a surjective continuous homomorphism from $\Psi(\Gamma(U))$ to $U$.

**Proof.** Recall that $\Psi(\Gamma(U)) = \kappa_U(\mu(U))$ is the $A$-algebra with domain $\mu(U) = \{w_U : w \in \overline{T}_U Un\}$, where $U$ is an $A$-algebra generated by the element $1_U$. We define the mapping $\beta : \mu(U) \to U$ by the rule $\beta(w_U) = w_U(1_U)$ for every $w \in \overline{T}_U Un$. This definition is correct as $w_U = w'_U$ implies $w_U(1_U) = w'_U(1_U)$. Moreover, $\beta$ is surjective because $U$ is generated by $1_U$ as a topological algebra.

By the definition of $w_U$, we have $\beta(w_U) = \varphi(w)$, where $\varphi : \overline{T}_U Un \to U$ is the unique continuous homomorphism of $\Sigma$-algebras. From the definition of $\beta$ we obtain that the following diagram commutes:

$$
\begin{array}{ccc}
\overline{T}_U Un & \xrightarrow{\tau_U} & \mu(U) \\
\downarrow{\varphi} & & \downarrow{\beta} \\
U & & U \end{array}
$$

Since $\tau_U$ is a continuous homomorphism of $A$-generated topological monoids, it is also a homomorphism of $A$-algebras. Since so is $\varphi$, it follows that $\beta$ is again a homomorphism of $A$-algebras. Since both mappings $\tau_U$ and $\varphi$ are continuous, we also conclude that $\beta$ is continuous: if we consider a closed subset $C$ of $U$, then $\beta^{-1}(C) = \tau_U(\varphi^{-1}(C))$ where $\varphi$ is continuous and $\tau_U$ is closed. □

We use the previous observations to establish the following result.

**Theorem 7.5.** The functor $\Psi : \mathbb{P}M_A \to \mathbb{P}UA^1_A$ is left-adjoint to the functor $\Gamma : \mathbb{P}UA^1_A \to \mathbb{P}M_A$.

**Proof.** To show that the functors are adjoint, we have to show that, for every $\gamma \in \mathbb{P}M_A$ and $U \in \mathbb{P}UA^1_A$, we have

$$\text{hom}_{\mathbb{P}UA^1_A}(\Psi(\gamma), U) \cong \text{hom}_{\mathbb{P}M_A}(\gamma, \Gamma(U)),$$

where the family of bijections is natural. Since the categories are thin, the naturality is trivial whenever we show that there exists a morphism from $\Psi(\gamma)$ to $U$ if and only if there exists a morphism from $\gamma$ to $\Gamma(U)$.

The implication from left to right is clear because it is enough to apply the functor $\Gamma$ and use Lemma 7.3. Similarly, if we have a morphism $\alpha : \gamma \to \Gamma(U)$, then we may use the functor $\Psi$ and we get $\Psi(\alpha) : \Psi(\gamma) \to \Psi(\Gamma(U))$. If we compose this morphism with the morphism given by Lemma 7.3, then we get an appropriate morphism from $\Psi(\gamma)$ to $U$. □

The main application of the previous result is that the functor $\Gamma$ is continuous in the sense that it preserves limits while $\Psi$ is cocontinuous, that is, it preserves inverse limits, which are a special case of limits with directed diagrams. Thus
Proposition 6.7 may be viewed as a consequence of Theorem 7.3. However, in fact, we used Proposition 6.7 in the proof of Theorem 7.3 via Lemma 7.3. It should also be pointed out that the discrete analog of Theorem 7.3 has been previously considered in [14].

8. The Polish representation

Recall the Polish notation for a term, which basically drops from the usual notation all parentheses and commas. For instance, for the term
\[ u(v(x_1, u(x_2, x_1), x_3), u(x_3, x_2)), \]
where \( u \in \Sigma_2 \) and \( v \in \Sigma_3 \), the Polish notation is the word \( uvx_1ux_2x_1x_3ux_3x_2 \) in the alphabet \( X \cup \Sigma \). The idea is that the arity of the operation symbols allows one to uniquely recover each term from its Polish notation. The Polish notations of terms thus live in the free monoid \( \Sigma^* \) obtained by restriction of the natural generating mapping \( \hat{\iota} : \Sigma \rightarrow \Omega_X \cup \Sigma \Omega_X \cup \Sigma \). Since \( \Sigma \) is self-free with basis \( \Omega_X \), the restriction \( \hat{\iota} \) is the identity on \( \Sigma \), \( \hat{\iota} \) is a homomorphism of \( \Sigma \)-algebras. Since \( \varphi(X) \subseteq \Sigma_V \), the restriction of \( \hat{\iota} \) to \( \Sigma_V \) takes its values in \( \Sigma_V \). Hence, \( \varphi \) does extend to a continuous endomorphism of \( \Sigma_V \).

**Proposition 8.1.** For every pseudovariety of monoids \( V \), the profinite \( \Sigma \)-algebra \( \Sigma_V \) is self-free with basis \( X \).

**Proof.** Consider the natural generating mapping \( \iota : X \rightarrow \Sigma_V \) and let \( \varphi : X \rightarrow \Sigma_V \) be an arbitrary continuous mapping. We may extend \( \varphi \) to a continuous function \( X \cup \Sigma \rightarrow \Omega_X \cup \Sigma \Omega_X \cup \Sigma \), which we still denote \( \varphi \), by letting the restriction of \( \varphi \) to \( \Sigma \) coincide with that of the natural generating mapping. By the universal property of \( \Omega_X \cup \Sigma \), there is a unique continuous (monoid) endomorphism \( \hat{\varphi} \) of \( \Omega_X \cup \Sigma \) such that \( \hat{\varphi} \circ \iota = \varphi \). Since \( \hat{\varphi} \) is the identity on \( \Sigma \), \( \hat{\varphi} \) is a homomorphism of \( \Sigma \)-algebras. Since \( \varphi(X) \subseteq \Sigma_V \), the restriction of \( \hat{\varphi} \) to \( \Sigma_V \) takes its values in \( \Sigma_V \). Hence, \( \varphi \) does extend to a continuous endomorphism of \( \Sigma_V \).

Combining Proposition 8.1 and [11] Theorem 2.16, we obtain the following result.

**Corollary 8.2.** For every pseudovariety \( V \) of monoids, discrete space \( X \), and discrete signature \( \Sigma \), there is a pseudovariety of \( \Sigma \)-algebras \( V \) such that the closed subalgebra \( \Sigma_V \) of \( \Omega_X \cup \Sigma \cup \Sigma \Omega_X \cup \Sigma \) defined above is isomorphic with \( \Omega_X \cup \Sigma \).

For the pseudovariety \( M \) of all finite monoids, there is a natural (onto) continuous homomorphism \( \Phi_X : \Omega_X \Fin_S \rightarrow S_M \) of \( \Sigma \)-algebras, namely the only one such that \( \Phi_X(x) = x \) for each \( x \in X \). We call it the Polish representation (of \( \Omega_X \Fin_S \)).

The next result shows that \( \Phi_X \) is in general not injective so that even the pseudovariety of finite algebras \( V \) of Corollary 8.2 corresponding to taking \( V = M \) is a proper subpseudovariety of \( \Fin_S \).
Proposition 8.3. Suppose that $\Sigma$ is a topological signature containing at least one symbol $u$ of arity at least $n \geq 2$. Let $X$ be a nonempty topological space. Then, the Polish representation $\Phi_X$ is not injective.

Proof. Let $x, y, z$ be three distinct variables, where we only require that $x \in X$. Then, writing $x^{\omega+1}$ for the product $xx^\omega$, the following equality holds in $\overline{\Omega}_X \cup \Sigma \mathbb{M}$ because $\omega$-powers are idempotents:

$$u^\omega((u^\omega x)^{\omega+1})^{\omega+1} = (u^\omega x^{\omega+1})^{\omega+1} = u^\omega u^\omega((u^\omega x)^{\omega+1})^{\omega+1}x^\omega.$$ (8.1)

We may consider the $\Sigma$-term

$$w_k(x, y, z) = u \left( \underbrace{u \left( \cdots u}^{k} (u(x, y, \ldots, y), z, \ldots, z), \ldots, z) \right).$$

We further let

$$t_k = w_k(w_k(x, x, x), w_k(x, x, x), w_k(x, x, x)).$$

Note that $\Phi_X(t_k) = u^k(u^k x^{k(n-1)+1})^{k(n-1)+1}$. Thus, if $t$ is any accumulation point in $\overline{\Omega}_X \mathbb{F}_\Sigma$ of the sequence $(t_r)_r$, then the continuity of $\Phi_X$ yields the equality

$$\Phi_X(t) = u^\omega((u^\omega x^{\omega+1})^{\omega+1}x^\omega).$$ (8.2)

Similarly, we may consider the $\Sigma$-term

$$s_k = w_k(t_k, x, x),$$

and let $s$ denote an accumulation point in $\overline{\Omega}_X \mathbb{F}_\Sigma$ of the sequence $(s_r)_r$. Again, continuity of $\Phi_X$ gives the equality

$$\Phi_X(s) = u^\omega u^\omega((u^\omega x^{\omega+1})^{\omega+1}x^\omega).$$ (8.3)

Combining the equations (8.1), (8.2), and (8.3), we conclude that, if $\Phi_X$ were injective, then the equality $t = s$ would hold in $\overline{\Omega}_X \mathbb{F}_\Sigma$. To show that this leads to a contradiction, we show that there is a continuous homomorphism $\xi : \overline{\Omega}_X \mathbb{F}_\Sigma \to A$ into a finite $\Sigma$-algebra such that $\xi(s) \neq \xi(t)$.

Let $A = \{a, b\}$ and define on $A$ a structure of $\Sigma$-algebra by interpreting each operation in $\Sigma_m$ with $m \neq n$ as the constant operation with value $a$ and each operation $v \in \Sigma_n$ by letting $v_A(a_1, \ldots, a_n) = \tilde{a}_n$, where $\tilde{a} = b$ and $\tilde{b} = a$. In this way, $A$ is a finite discrete topological algebra. Choose $\xi$ to be any continuous homomorphism $\xi : \overline{\Omega}_X \mathbb{F}_\Sigma \to A$ such that $\xi(x) = a$. In view of the definition of $u_A$, we have $\xi(w_k(x, x, x)) = \tilde{a} = b$ and so $\xi(t_k) = \tilde{b} = a$ while $\xi(s_k) = \tilde{a} = b$. Since $\xi$ is continuous, it follows that $\xi(t) = a$ and $\xi(s) = b$, which establishes the claim. $\square$

In contrast, free profinite unary algebras are faithfully represented through the Polish representation, which is yet another result showing that unary algebras are rather special.

Theorem 8.4. Let $\Sigma = \Sigma_0 \cup \Sigma_1$ be an at most unary topological signature and let $X$ be an arbitrary topological space. Then, the Polish representation $\Phi_X : \overline{\Omega}_X \mathbb{F}_\Sigma \to S_M$ is an isomorphism of topological algebras.

Proof. Since $\Phi_X$ is an onto continuous homomorphism, it suffices to show that it is injective. Suppose that $a$ and $b$ are distinct elements of $\Omega_X \text{Fin}_\Sigma$. Since $\Omega_X \text{Fin}_\Sigma$ is residually finite, there exists a continuous homomorphism $\varphi : \Omega_X \text{Fin}_\Sigma \to F$ into a finite $\Sigma$-algebra such that $\varphi(a) \neq \varphi(b)$. Let $M = M(F) \cup \{0\}$ be a disjoint union of finite discrete spaces, so that it is itself a finite discrete space. We define on $M$ a multiplication as follows:

- for $g, h \in M(F)$, we let $g \cdot h = g \circ h$;
- for $g \in M(F)$ and $s \in F$, we put $g \cdot s = g(s)$;
- all remaining products are set to be 0.

It is easy to verify that $M$ is a monoid for the above multiplication. Let $\iota : X \to \Omega_X \text{Fin}_\Sigma$ be the natural generating mapping and consider the evaluation mappings $E_0 : \Sigma_0 \to F$ and $E_1 : \Sigma_1 \times F \to F$. Note that $E_1$ induces a continuous mapping $\varepsilon : \Sigma_1 \to M(F)$ into the discrete monoid $M(F)$ defined by $\varepsilon(u)(s) = E_1(u, s)$ for $u \in \Sigma_1$ and $s \in F$. We may now define a continuous mapping $\psi : X \cup \Sigma \to M$ by letting $\psi(x) = \varphi(\iota(x))$ for $x \in X$, $\psi(c) = E_0(c)$ for $c \in \Sigma_0$, and $\psi(u) = \varepsilon(u)$ for $u \in \Sigma_1$. By the universal property of the free profinite monoid $\Omega_{X \cup \Sigma} M$, $\psi$ induces a continuous homomorphism of monoids $\hat{\psi} : \Omega_{X \cup \Sigma} M \to M$ such that $\hat{\psi} \circ \eta = \psi$, where $\eta : X \cup \Sigma \to \Omega_{X \cup \Sigma} M$ is the natural generating mapping. The following diagram may help the reader to keep track of all these continuous mappings.

Let $\gamma$ be the restriction of $\hat{\psi}$ to $S_M$. Note that $\gamma$ takes its values in $F$: all elements of $S_M$ are of the form $wa$, where $w$ belongs to the closed submonoid of $\Omega_{X \cup \Sigma} M$ generated by $\eta(\Sigma_1)$ and $a \in \eta(X \cup \Sigma_0)$; hence, we have $\hat{\psi}(wa) = \psi(w)\psi(a)$, where $\psi(w) \in M(F)$ and $\psi(a) \in F$, resulting in $\psi(wa) \in F$ according to the definition of the multiplication in $M$. Next, we claim that $\gamma$ is a homomorphism of $\Sigma$-algebras. Indeed, for $u \in \Sigma_1$ and $v \in S_M$, the following equalities hold:

$$\hat{\psi}(u_{S_M}(v)) = \hat{\psi}(\eta(u)v) = \hat{\psi}(\eta(u))\hat{\psi}(v) = \varepsilon(u)\hat{\psi}(v) = u_F(\hat{\psi}(v)).$$

Note that the diagram commutes as $\gamma \circ \Phi_X = \varphi$: since, by the above, both sides of the equation are continuous homomorphisms of $\Sigma$-algebras, it suffices to check that composing them with $\iota$ we obtain an equality and, indeed, for $x \in X$, we have

$$\varphi(\iota(x)) = \psi(x) = \hat{\psi}(\eta(x)) = \hat{\psi}(\Phi_X(\iota(x))) = \gamma(\Phi_X(\iota(x))).$$

Finally, since $\varphi(a) \neq \varphi(b)$, it follows that $\Phi_X(a) \neq \Phi_X(b)$, which establishes that $\Phi_X$ is injective. \qed
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