Hilbert function, generalized Poincaré series
and topology of plane valuations

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Abstract

To a multi-index filtration (say, on the ring of germs of functions
on a germ of a complex analytic variety) one associates several invari-
ants: the Hilbert function, the Poincaré series, the generalized Poincaré
series, and the generalized semigroup Poincaré series. The Hilbert func-
tion and the generalized Poincaré series are equivalent in the sense that
each of them determines the other one. We show that for a filtration
on the ring of germs of holomorphic functions in two variables defined
by a collection of plane valuations both of them are equivalent to the
generalized semigroup Poincaré series and determine the topology of the
collection of valuations, i.e. the topology of its minimal resolution.

Introduction

Let $\mathcal{O}_{X,0}$ be the ring of germs of functions on a germ $(X,0)$ of a complex
analytic variety. A (one-index) filtration by vector subspaces on the ring $\mathcal{O}_{X,0}$

$$\mathcal{O}_{X,0} = J(0) \supset J(1) \supset J(2) \supset \cdots$$

can be described by the function $\nu : \mathcal{O}_{X,0} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ defined by $\nu(g) = \sup\{i : g \in J(i)\}$. This function possesses the properties:

1) $\nu(\lambda g) = \nu(g)$ for $\lambda \in \mathbb{C}^*$, $\nu(0) = \infty$;

2) $\nu(g_1 + g_2) \geq \min\{\nu(g_1), \nu(g_2)\}$.

*Math. Subject Class. 14B05, 16W70, 13A18. Keywords: filtrations, Hilbert functions,
Poincaré series, plane valuations. Partially supported by the grant MTM2007-64704 (with
the help of FEDER Program) and MTM2012-36917-C03-01 / 02. Third author is also
partially supported by the Russian government grant 11.G34.31.0005, RFBR–13-01-00755,
NSh–4850.2012.1 and Simons-IUM fellowship.
Functions $\nu : \mathcal{O}_{X,0} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ with the properties 1) and 2) are called order functions. If, moreover,

3) $\nu(g_1g_2) = \nu(g_1) + \nu(g_2)$,

the function $\nu$ is a valuation on $\mathcal{O}_{X,0}$.

A multi-index filtration on the ring $\mathcal{O}_{X,0}$ will be defined by a collection $\nu_1, \ldots, \nu_r$ of order functions on $\mathcal{O}_{X,0}$: for $\underline{v} = (v_1, \ldots, v_r) \in \mathbb{Z}^r$

$$J(\underline{v}) = \{ g \in \mathcal{O}_{X,0} : \nu_i(g) \geq v_i \text{ for } i = 1, \ldots, r \} .$$

(It is sufficient to define $J(\underline{v})$ for $\underline{v} \in \mathbb{Z}_{\geq 0}^r$. However, below it will be convenient to assume it to be defined for all $\underline{v} \in \mathbb{Z}^r$).

**Remarks.** 1. One can consider a different notion of a multi-index filtration defined by a system of subspaces $J(\underline{v})$ numbered by $\underline{v} \in \mathbb{Z}_{\geq 0}^r$ such that for $\underline{v} \geq \underline{w}$, $J(\underline{v})$ is a valuation on $\mathcal{O}_{X,0}$.

Below we shall also consider more general order functions $\nu : \mathcal{O}_{X,0} \to S \cup \{\infty\}$ with values in an ordered semigroup $S$ and therefore filtrations indexed by $\underline{v} = (v_1, \ldots, v_r) \in S = S_1 \times \cdots \times S_r$. (The semigroup $S$ is partially ordered by the relation $\underline{v} \geq \underline{w}$ if and only if for any $\underline{v}$ and $\underline{w}$ in $\mathbb{Z}^r_{\geq 0}$ one has that $J(\underline{v}) \cap J(\underline{w}) = J(\max(\underline{v}, \underline{w}))$, where $\max(\underline{v}, \underline{w}) = (\max(v_1, w_1), \ldots, \max(v_r, w_r))$ ($\underline{v} = (v_1, \ldots, v_r)$, $\underline{w} = (w_1, \ldots, w_r)$).

2. Below we shall also consider more general order functions $\nu : \mathcal{O}_{X,0} \to S \cup \{\infty\}$ with values in an ordered semigroup $S$ and therefore filtrations indexed by $\underline{v} = (v_1, \ldots, v_r) \in S = S_1 \times \cdots \times S_r$. (The semigroup $S$ is partially ordered by the relation $\underline{v} \geq \underline{w}$ if and only if for any $\underline{v}$ and $\underline{w}$ in $\mathbb{Z}^r_{\geq 0}$ one has that $J(\underline{v}) \cap J(\underline{w}) = J(\max(\underline{v}, \underline{w}))$, where $\max(\underline{v}, \underline{w}) = (\max(v_1, w_1), \ldots, \max(v_r, w_r))$ ($\underline{v} = (v_1, \ldots, v_r)$, $\underline{w} = (w_1, \ldots, w_r)$).

To a multi-index filtration $\{J(\underline{v})\}$ ($\underline{v} \in \mathbb{Z}^r$) one associates several invariants:

1. The Hilbert function: $h(\underline{v}) = \dim \mathcal{O}_{X,0}/J(\underline{v})$. One can describe the Hilbert function by the power series $\tilde{H}(\underline{t}) = \sum_{\underline{v} \in \mathbb{Z}_{\geq 0}^r} h(\underline{v}) \underline{t}^\underline{v}$ (here $\underline{t} = (t_1, \ldots, t_r)$ and $\underline{t}^\underline{v} = t_1^{v_1} \cdots t_r^{v_r}$) or by a sort of a Laurent series $H(\underline{t}) = \sum_{\underline{v} \in \mathbb{Z}^r} h(\underline{v}) \underline{t}^\underline{v}$. They are defined if the subspaces $J(\underline{v})$ have finite codimensions.

2. The Poincaré series $P(\underline{t})$ defined in [5] (see also [1]):

$$P(\underline{t}) = \frac{L(\underline{t}) \prod_{i=1}^r (t_i - 1)}{t_1 \cdots t_r - 1}$$

(1)

where $L(\underline{t}) = \sum_{\underline{v} \in \mathbb{Z}^r} \dim(J(\underline{v})/J(\underline{v} + 1)) \underline{t}^\underline{v}$, $\underline{1} = (1, \ldots, 1)$. (One has $\underline{t} \cdot L(\underline{t}) = (1 - \underline{t})H(\underline{t})$.)

The Poincaré series can be also defined as an integral with respect to the Euler characteristic (see [1]):

$$P(\underline{t}) = \int_{\mathcal{O}_{X,0}} \underline{t}^{\nu(g)} d\chi ,$$

(2)
where \( \nu(g) = (\nu_1(g), \ldots, \nu_r(g)) \), \( t^\infty \) is assumed to be equal to zero.

If \((X, 0) = (\mathbb{C}^2, 0)\) and \(\nu_1, \ldots, \nu_r\) are curve valuations of rank one (type I.1 in the notations of [4]) corresponding to irreducible plane curve singularities \((C_i, 0) \subset (\mathbb{C}^2, 0), i = 1, \ldots, r\), then the Poincaré series \( P(t) \) coincides with the Alexander polynomial \( \Delta_C(t) \) of the link \( L = C \cap S^3_\varepsilon \) of the curve \( C = \bigcup_{i=1}^r C_i \) (see [11]).

3. The generalized Poincaré series \( P_g(t;q) \) defined in [3] as the same integral as in equation (2) with respect to the generalized Euler characteristic \( \chi_g \) with values in the Grothendieck ring \( K_0(\mathcal{V}_C) \) of quasi-projective varieties localized at \( L = [C] \):

\[
P_g(t;q) = \int_{\mathcal{O}_{X,0}} t^{\nu(g)} d\chi_g,
\]

where \( q = L^{-1} \). (One has \( P_g(t;q) \in \mathbb{Z}[q][[t_1, \ldots, t_r]] \); see [3].)

The generalized Poincaré series is related with zeta-functions of curves defined over finite fields (see [10]). It is conjectured that the generalized Poincaré series corresponding to a collection of plane curve valuations is closely related with the generating series of the Heegaard-Floer homologies of the link \( L = C \cap S^3_\varepsilon \) (see [9]).

In [3] it was explained that the Hilbert function of a filtration defines the Poincaré series and the generalized Poincaré series of it. On the other hand the Poincaré series does not define, in general, the Hilbert function and the generalized Poincaré series: see Example in [3] taken from [5].

Here we discuss relations between the Hilbert function and the generalized Poincaré series in the general setting, i.e., for order functions with values in ordered semigroups. We show that, if the Hilbert function is defined (i.e., if all the subspaces \( J(\nu) \) have finite codimensions), these two invariants are equivalent in the sense that each of them determines the other one. Thus they keep the same information about the filtration. However this information is encoded in different ways. In particular, as it was mentioned, for plane curve valuations the generalized Poincaré series is in the form related to classical knot invariants of the corresponding link.

Now let \((X, 0) \) be the plane \((\mathbb{C}^2, 0)\) and let the order functions \( \nu_i \) be valuations on the ring \( \mathcal{O}_{\mathbb{C}^2,0} \) of germs of functions in two variables. The classification of the valuations on \( \mathcal{O}_{\mathbb{C}^2,0} \) can be found in [14] (see also [4]).

It is well-known that the Alexander polynomial of a plane curve singularity, coinciding with the Poincaré series of the corresponding collection of curve valuations of rank one, determines the topology of the curve singularity: [12] (see also [2]). In [2] it was shown that the Poincaré series determines the topology of a collection of divisorial valuations in the plane, i.e., the topology
of its minimal resolution. (In [4] this was generalized to an arbitrary collection of valuations on the ring \( \mathcal{O}_{\mathbb{C}^2,0} \) does not containing curve valuations of rank one.) On the other hand, an example in [2] shows that the Poincaré series does not determine the topology of an arbitrary collection of valuations on \( \mathcal{O}_{\mathbb{C}^2,0} \).

Here we show that the generalized Poincaré series of a collection of plane valuations determines the topology of them. In particular this implies that the Hilbert function of a collection of plane valuations determines its topology. One can say that this is the main result of the paper formulated in “traditional” terms.

1 Hilbert function and generalized Poincaré series

Let \( S \) be a totally ordered abelian semigroup with the minimal element equal to zero. An order function on \( \mathcal{O}_{X,0} \) with values in \( S \) is a map \( \nu : \mathcal{O}_{X,0} \to S \cup \{\infty\} \) such that

1) \( \nu(\lambda g) = \nu(g) \) for \( \lambda \in \mathbb{C}^* \), \( \nu(0) = \infty \);

2) \( \nu(g_1 + g_2) \geq \min\{\nu(g_1), \nu(g_2)\} \).

Remarks. 1) If \( \nu(g_1 g_2) = \nu(g_1) + \nu(g_2) \), the order function \( \nu \) is a valuation. In this section the property to be a valuation will not be essential.

2) All the content of the section is valid for order functions (and thus for filtrations) on arbitrary complex vector spaces, e.g., on a module over \( \mathcal{O}_{X,0} \). We describe the situation in \( \mathcal{O}_{X,0} \), not trying to formulate in the most general context, for convenience.

A (finite) collection \( \nu = (\nu_1, \ldots, \nu_r) \) of order functions on \( \mathcal{O}_{X,0} \) with values in the semigroups \( S_1, \ldots, S_r \) respectively defines the \( r \)-index filtration

\[
J(\nu) = \{g \in \mathcal{O}_{X,0} : \nu_i(g) \geq v_i \text{ for all } i = 1, \ldots, r\},
\]

where \( \nu = (\nu_1, \ldots, \nu_r) \subseteq S = S_1 \times \cdots \times S_r \). (The semigroup \( S \) is partially ordered by \( \nu = (\nu_1, \ldots, \nu_r) \geq w = (w_1, \ldots, w_r) \) if and only if \( \nu_i \geq w_i \) for all \( i = 1, \ldots, r \).

Definition: The Hilbert function \( h \) of the filtration \( \{J(\nu)\} \) is the function on \( S \) defined by

\[
h(\nu) = \dim \mathcal{O}_{X,0}/J(\nu).
\]

We permit \( h(\nu) \) to be equal to infinity.
The set $\mathbb{Z}[[S]]$ of power series on the semigroup $S$ is the set of formal expressions of the form $\sum_{\underline{v} \in S} a_{\underline{v}} \underline{t}^{\underline{v}}$ ($\underline{v} = (v_1, \ldots, v_r) \in S$, $a_{\underline{v}} \in \mathbb{Z}$). One can write $\underline{t}^{\underline{v}}$ as $t_1^{v_1} \cdots t_r^{v_r}$. The set $\mathbb{Z}[[S]]$ is a free abelian group. It is a ring if in each semigroup $S_i$, each element $a \in S_i$ has only a finite number of different representations as the sum $a = a_1 + a_2$ of two elements of $S_i$. This takes place, in particular, if $\nu_i$ is a valuation on $\mathcal{O}_{X,0}$ and $S_i$ is the semigroup of its values.

**Definition:** The *Hilbert series* $\tilde{H}(t)$ of the filtration $\{J(\underline{v})\}$ is the generating series of the values of the Hilbert function:

$$\tilde{H}(t) = \sum_{\underline{v} \in S} h(\underline{v}) \underline{t}^{\underline{v}}.$$  

The Hilbert series is defined if all the values $h(\underline{v})$ are finite.

For $I \subset I_0 = \{1, \ldots, r\}$ and $\underline{v} \in S$, let

$$J^+(\underline{v}) := \{ g \in J(\underline{v}) : \nu_i(g) > v_i \text{ for all } i \in I \},$$

$$J^+(\underline{v}) := J^{+I_0}(\underline{v}).$$

**Definition:** The *Poincaré series* $P(t)$ of the filtration $J(\underline{v})$ is the element of $\mathbb{Z}[[S]]$ defined by

$$P(t) = \sum_{\underline{v} \in S} \left( \sum_{I \subset I_0} (-1)^{|I|} \dim J^+(\underline{v})/J^+(\underline{v}) \right) \underline{t}^{\underline{v}}.$$  

One can show that, for $S_i = \mathbb{Z}_{\geq 0}$, $i = 1, \ldots, r$, this definition coincides with $\bigoplus$: see, e.g., $\bigoplus$. The Poincaré series $P(t)$ is defined if all the factor spaces $J(\underline{v})/J^+(\underline{v})$ are finite dimensional.

Let $Y(\underline{v}) := \{ g \in \mathcal{O}_{X,0} : \nu(g) = \underline{v} \}$. One has

$$Y(\underline{v}) = J(\underline{v}) \setminus \bigcup_{i=1}^r J^{+\{i\}}(\underline{v}). \quad (4)$$

Let $F_{\underline{v}} = Y(\underline{v})/J^+(\underline{v})$. The union of the spaces $F_{\underline{v}}$ as a graded space, i.e. a space with the components numbered by the elements of $S$, is called the *extended semigroup of the filtration* $\{J(\underline{v})\}$: see, e.g., $\bigoplus$. The components $F_{\underline{v}}$ of the extended semigroup are called its *fibres*. The space $\bigsqcup_{\underline{v} \in S} F_{\underline{v}}$ is really a semigroup if all the order functions $\nu_i$ are valuations. However we keep the name for the general case as well.

Let $S$ be the subset of $S$ consisting of $\underline{v}$ with $Y(\underline{v}) \neq \emptyset$. If $\nu_i$ are valuations, the set $S$ is a semigroup: the *semigroup of values* of the collection $(\nu_1, \ldots, \nu_r)$.  

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One can see that the coefficient at $t^v$ in the Poincaré series $P(t)$ is equal to the Euler characteristic $\chi(\mathbb{P}F_v)$ of the projectivization $\mathbb{P}F_v = (F_v \setminus \{0\})/\mathbb{C}^*$ of the fibre of the extended semigroup.

Let $K_0(\mathcal{V}_C)$ be the Grothendieck ring of quasi-projective varieties. It is generated by classes $[X]$ of such varieties subject to the relations:

1) if $X_1 \cong X_2$, then $[X_1] = [X_2]$;
2) if $Y$ is Zariski closed in $X$, then $[X] = [Y] + [X \setminus Y]$ (the multiplication is defined by the Cartesian product). Let $\mathbb{L} = [\mathbb{C}]$ be the class of the complex affine line in $K_0(\mathcal{V}_C)$. The natural map $\mathbb{Z}[\mathbb{L}] \to K_0(\mathcal{V}_C)$ is an embedding.

Since the projectivization $\mathbb{P}F_v$ of the fibre $F_v$ of the extended semigroup is the complement to an arrangement of projective subspaces in a projective space, its class $[\mathbb{P}F_v]$ in the Grothendieck ring $K_0(\mathcal{V}_C)$ is a polynomial in $\mathbb{L}$:

$$[\mathbb{P}F_v] = p_v(\mathbb{L}).$$

Let $K_0(\mathcal{V}_C)(\mathbb{L})$ be the localization of the Grothendieck ring $K_0(\mathcal{V}_C)$ at $\mathbb{L}$ and let $q = \mathbb{L}^{-1} \in K_0(\mathcal{V}_C)(\mathbb{L})$. The natural map $\mathbb{Z}[q, q^{-1}] \to K_0(\mathcal{V}_C)(\mathbb{L})$ is an embedding.

**Definition:** The generalized Poincaré series of the filtration $\{J(\mathcal{V})\}$ is the element of $\mathbb{Z}[q][[S]]$ defined by

$$P_g(t; q) = \sum_{v \in S} q^{h^+(v)} p_v(q^{-1}) t^v,$$

where $h^+(v) = \dim \mathcal{O}_{X,0}/J^+(v)$.

In [3], there was described a relation of this definition with a motivic measure on the projectivization $\mathbb{P}\mathcal{O}_{X,0}$ of the ring $\mathcal{O}_{X,0}$ and an integral with respect to it (for, so called, finitely determined order functions). This identifies this definition with the one in [3].

Assume that, for each $v \in S$, $\dim \mathcal{O}_{X,0}/J(v) < \infty$ (and therefore the Hilbert series and the generalized Poincaré series are defined).

**Proposition 1** The Hilbert function, the Hilbert series and the generalized Poincaré series are equivalent in the sense that each of them determines the other two.

**Proof.** The Hilbert function determines the generalized Poincaré series by the equation (5). For $v \in S$, the coefficient at $t^v$ in the generalized Poincaré series $P_g(t; q)$ is not zero and the lower degree in $q$ of it is equal to $h(v)$. Therefore the generalized Poincaré series $P_g(t; q)$ determines $h(v)$ for all $v \in S$. For $v \in S \setminus S$, $h(v)$ is equal to the minimum of $h(w)$ for all $w \geq v, w \in S$. □
In [3], there was considered one more generalized Poincaré series corresponding to a filtration.

**Definition:** The generalized semigroup Poincaré series of the filtration \( \{J(v)\} \) is the element of \( \mathbb{Z}[L][[S]] \) defined by

\[
\hat{P}_g(t; L) = \sum_{v \in S} [F_v] t^v = \sum_{v \in S} p_v(L) t^v.
\] (6)

As in the generalized Poincaré series \( P_g(t; q) \), the coefficient at \( t^v \) in the generalized semigroup Poincaré series \( \hat{P}_g(t; L) \) is different from zero if and only if \( v \in S \).

### 2 Plane valuations and topology

The classification of plane valuations (i.e. valuations on the ring \( \mathcal{O}_{\mathbb{C}^2, 0} \)) can be found in [11] or [4]. We will follow the assumptions and the terminology from [4]. In particular we assume that each valuation \( \nu \) of rank one is normalized by the requirement that \( \min_{f \in \mathfrak{m}} \nu(f) = 1 \), where \( \mathfrak{m} \) is the maximal ideal of \( \mathcal{O}_{\mathbb{C}^2, 0} \). For a curve valuation of rank two (type II.1 in [4]) this will be the normalization of the second component. For the remaining valuations of rank two (exceptional curve valuations: case II.2 of [4]) we shall demand this for the first component of the value.

The notion of the minimal resolution of a valuation was described in [4]. The minimal resolution of a divisorial valuation is the minimal modification by a (finite) sequence of blowing-ups which produces the corresponding divisor. A rank 1 plane valuation can be defined as the limit of a sequence of divisorial valuations: see [4]. The minimal resolution of this valuation is the projective limit of the corresponding minimal resolutions.

**Remark.** Strictly speaking this is not a resolution in the usual sense (a modification of the plane) since the corresponding map is not proper. For example the minimal resolution of a curve valuation of rank 1 is the resolution of the curve followed by the infinite sequence of blowing-ups at the intersection points of the strict transform of the curve with the exceptional divisor. The minimal resolution of a curve valuation of rank 2 is the minimal resolution of the corresponding curve valuation of rank 1. The minimal resolution of an exceptional curve valuation (of rank 2) is the minimal resolution of the corresponding divisorial valuation (the first component of the considered one) followed by the additional blowing-up at the corresponding intersection point.
The minimal resolution of a finite collection of valuations is the projective limit of the corresponding (multi-index) system of modifications. It is simply the birational join of the minimal resolution of all the valuations.

The minimal resolution of a collection of plane valuations can be described by its dual graph $G$. Vertices of $G$ correspond to the irreducible components of the exceptional divisor. The set $\Gamma$ of vertices is a partially ordered set: a component $E_{\sigma}$ of the exceptional divisor is greater than a component $E_{\delta}$ if any modification containing $E_{\sigma}$ contains $E_{\delta}$ as well. Two vertices of $G$ are connected by an edge if the corresponding components of the exceptional divisor intersect. The graph $G$ is a tree.

**Remark.** There is another natural partial order on the set of vertices of the graph $G$ defined by the position of a vertex with respect to the root, i.e. to the vertex corresponding to the first born component of the exceptional divisor. A vertex $\sigma$ of the graph $G$ is greater than a vertex $\delta$ if $\delta$ lies on the geodesic between the root and the vertex $\sigma$. This partial order will be used in the proofs below, but not in the following definition.

**Definition:** Two collections of plane valuations are called topologically equivalent if the dual graphs of their minimal resolutions are isomorphic as graphs with partially ordered sets of vertices and with marked vertices corresponding to divisorial and exceptional curve valuations.

**Remark.** For plane curve valuations this definition coincides with the usual definition of the topological equivalence of plane curve singularities. According to this definition the curve valuations and the formal curve valuations are topologically indistinguishable. They have also equal Hilbert functions and generalized Poincaré series. In fact curve valuations and formal curve valuations constitute one single class of valuations on the ring of formal power series in two variables.

For convenience we shall exclude curve rank 2 valuations since, strictly speaking, the Hilbert function and the generalized Poincaré series are not defined for them (see the Remark after the Proof of Theorem 1).

**Theorem 1** Let $(\nu_1, \ldots, \nu_r)$ be a collection of plane valuations does not containing curve valuations of rank 2. Then the generalized Poincaré series $P_\partial(L, q)$ determines the topological type of the collection.

Together with Proposition 1 this implies the following statement.

**Corollary 1** The Hilbert function of a collection of plane valuations determines the topological type of the collection.
Proof of Theorem 1. As it was explained in Section 1, the generalized Poincaré series can be defined as the integral of the monomial function $t^\nu$ over the projectivization $\mathbb{P}O_{X,0}$ of the ring $O_{X,0}$ with respect to the generalized Euler characteristic (with values in the localization $K_0(\mathcal{V}_C)_L$ of the Grothendieck ring $K_0(\mathcal{V}_C)$ at $L = q^{-1}$).

Let $\hat{\mathcal{M}}$ be the completion of the localization $K_0(\mathcal{V}_C)_L$ of the Grothendieck ring $K_0(\mathcal{V}_C)$ with respect to the dimension filtration: see, e.g., [8]. One has the natural map $K_0(\mathcal{V}_C)_L \to \hat{\mathcal{M}}$. Its restriction to $\mathbb{Z}[[q]]$ is an embedding. We shall consider coefficients of the Poincaré series as elements of $\hat{\mathcal{M}}$. Under this assumption the formula for the generalized Poincaré series as an integral gives the following statement. Let $I$ be a subset of the set $I_0 = \{1, \ldots, r\}$ of indices and let $P_g,\{\nu_i\}_{i \in I}(t; q)$ be the generalized Poincaré series of the subcollection $\{\nu_i\}_{i \in I}$ of the collection $(\nu_1, \ldots, \nu_r)$. Then one has the following “projection formula”:

$$P_g,\{\nu_i\}_{i \in I}(t; q) = P_g(t; q)|_{t_j = 1} \quad \text{for } j \notin I.$$ 

This implies that the generalized Poincaré series of a collection of plane valuations determines the generalized Poincaré series of any subcollections, in particular, the generalized Poincaré series of each of them. On the other hand, the generalized Poincaré series of a valuation determines the usual Poincaré series of it:

$$P(t) = P_g(t; 1).$$

In [1] it was shown that the Poincaré series of a plane valuation determines the dual graph of the minimal resolution of it. Therefore, in order to describe the dual graph of the minimal resolution of a collection of plane valuations, one has to determine the splitting point $\alpha$ (i.e. the last common point counted from the root of the tree) of the resolution trees for each pair, say, $\nu_1$ and $\nu_2$ of the valuations. According of the description above the generalized Poincaré series $P_g(t; q)$ determines the generalized Poincaré series $P_g(t_1, t_2; q)$ of the pair $(\nu_1, \nu_2)$.

In turn the generalized Poincaré series $P_g(t_1, t_2; q)$ determines the semigroup of values of the pair $(\nu_1, \nu_2)$: this is simply the set of values $(v_1, v_2)$ with non-zero coefficients at the corresponding monomials. Now the splitting vertex of the resolution graph $G$ corresponds to the minimal element $(v_1^0, v_2^0)$ in the semigroup of values of the pair $(\nu_1, \nu_2)$ such that $v_1^0 = v_2^0$ (pay attention to the normalization of the valuations described at the beginning of the section) and there exists an element $(v_1, v_2)$ of the semigroup different from $(v_1^0, v_2^0)$ and such that either $v_1 = v_1^0$ or $v_2 = v_2^0$. The above fact for curve valuations of rank one can be found in [7], Remark (3.20). From the case of curve valuations it is easy to extend it to divisorial valuations (considering the valuations defined by the
corresponding curvettes, i.e., the pull-downs of non-singular curves transversal to the divisors) and, taking into account that any valuation could be regarded as a limit of divisorial ones, one can deduce the result for the general case.

This proves the statement. □

**Remark.** It is not difficult to see that one can permit curve rank 2 valuations in Theorem 1 and in the Corollary if one assumes that the Hilbert function $h(v)$ is defined with infinite values at the elements $v$ with non-zero first component of such a valuation and the coefficient at the corresponding monomial $t^v$ in the generalized Poincaré series is equal to zero. (The last assumption means that $q^{\infty}$ should be considered as 0.)

Though it is not clear whether the generalized semigroup Poincaré series of a filtration determines its Hilbert function (and thus the generalized Poincaré series), for plane valuations one can show that they are equivalent. This follows from the following statement (together with the fact that, for plane valuations, all three invariants: the Hilbert function and the both generalized Poincaré series are invariants of the topological type).

**Theorem 2** The generalized semigroup Poincaré series $\tilde{P}_g(t; L)$ determines the topological type of a collection $(\nu_1, \ldots, \nu_r)$ of plane valuations.

**Proof.** The generalized semigroup Poincaré series determines the semigroup of values of the collection $(\nu_1, \ldots, \nu_r)$ (as the set of values $v$ with non-zero coefficients at $t^v$) and therefore the semigroup of values of any subcollection of it. In particular it determines the semigroup of values of each valuation itself. In its turn this semigroup determines the topological type of the valuation in all cases except the following two:

1) The semigroup of values are the same for a plane curve valuation and for a divisorial valuation whose resolution graph is obtained from the resolution graph of the plane curve valuation by a truncation of the last infinite tail.  
2) Two divisorial valuations whose resolution graphs differs only by the length of the last tail have the same semigroup of values.

If the collection consists only of one valuation ($r = 1$), the generalized semigroup Poincaré series determines the usual Poincaré series which in turn determines the topological type.

If there are more than two valuations, then, say, the valuation $\nu_1$ is a curve valuation if and only if there exists $v_0^1, \ldots, v_0^r$ such that there are infinitely many elements in the semigroup of values of the form $(v_1, v_2^0, \ldots, v_r^0)$. The second problem arises in a particular situation which will be discussed later.

In order to restore the resolution graph one has to find the splitting point for each pair of valuations. The method to detect it from the semigroup of values of the pair of valuations is described in the Proof of Theorem 1.
The only remaining problem (formulated as 2) above) is the following one: assume that, say, \( \nu_1 \) is a divisorial valuation and the corresponding divisor \( E_\alpha \) is not on the resolution tree of the collection \( (\nu_2, \ldots, \nu_r) \). (If it is on that tree, it corresponds to the splitting point with the other valuations and was detected on the previous step.) In this case the vertex corresponding to this divisor is on an isolated branch of the resolution tree. Let \( g_\alpha \) be a function defining a curvette at the divisor \( E_\alpha \). In \cite{2} it was explained how one can determine the dual graph of the resolution of a divisorial valuation from its semigroup of values plus the value \( \nu_1(g_\alpha) \).

In this paragraph for each exceptional curve valuation (of rank two) as \( \nu_j(g) \) we use the first component of the value and for each curve valuation of rank two we use the second component. For each \( j = 2, \ldots, r \), the semigroup of values of the pair \( (\nu_1, \nu_j) \) lies in the half-plane \( v_j \geq \frac{\nu_j(g_\alpha)}{\nu_1(g_\alpha)} \nu_1 \) and the point \( (\nu_1(g_\alpha), \nu_2(g_\alpha), \ldots, \nu_r(g_\alpha)) \) is in the semigroup of values of the collection. Moreover this point is the first point of the semigroup of values on the ray

\[
v_j = \frac{\nu_j(g_\alpha)}{\nu_1(g_\alpha)} \nu_1 \quad \text{for } j = 2, \ldots, r
\]

(see e.g. \cite{2}). This proves the statement. \( \square \)

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