Abstract. Filter stability is a classical problem for partially observed Markov processes (POMP). For a POMP, an incorrectly initialized non-linear filter is said to be stable if the filter eventually corrects itself with the arrival of new measurement information. In the literature, studies on the stability of non-linear filters either focus on the ergodic properties on the hidden Markov process, or the observability of the measurement channel. While notions of observability exist in the literature, they are difficult to verify for many systems and specific examples of observable systems are mostly restricted to additive noise models. In this paper, we introduce a general definition of observability for stochastic non-linear dynamical systems and compare it with related findings in the literature. Our notion involves a functional characterization which is easily computed for a variety of systems as we demonstrate. Under this observability definition we establish filter stability results for a variety of criteria including weak merging and total variation merging, both in expectation and in an almost sure sense, as well as relative entropy. We consider the implications between these notions, which also unify various results in the literature in a concise manner. Our conditions, and the examples we study, complement and generalize the existing results on filter stability.

1. Introduction. In this paper we are interested in the merging of conditional distributions (known as the non-linear filter) for partially observed Markov processes (POMP) under different initial priors. We give conditions for when a sufficiently informative measurement channel can correct initialization errors in the filter recursion and result in merging with the correctly initialized filter under different convergence criteria.

Let $(\mathcal{X}, \mathcal{Y})$ be Polish spaces equipped with their Borel sigma fields $\mathcal{B}(\mathcal{X})$ and $\mathcal{B}(\mathcal{Y})$. $\mathcal{X}$ will be called the state space, and $\mathcal{Y}$ the measurement space. Let $\{Z_n\}_{n=0}^{\infty}$ be an independent identically distributed (i.i.d) $\mathcal{Z}$-valued noise process. We denote the probability measure associated with $Z_0$ by $Q$. We define $h : \mathcal{X} \times \mathcal{Z} \rightarrow \mathcal{Y}$ as the measurement function, which defines a regular conditional probability measure as follows: for a fixed $x \in \mathcal{X}$ we will denote $h(x, \cdot) = h_x(\cdot) : \mathcal{Z} \rightarrow \mathcal{Y}$ and denote the pushforward measure of $Q$ under $h_x$ as $h_xQ$, a measure on $\mathcal{B}(\mathcal{Y})$ for each $x \in \mathcal{X}$. That is, for a set $A \in \mathcal{B}(\mathcal{Y})$ we have $h_xQ(A) = Q(h_x^{-1}(A))$.

The system is initialized with a state $X_0 \in \mathcal{X}$ drawn from a prior measure $\mu$ on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. However, the state is not available at the observer, instead the observer sees the sequence $Y_n = h(X_n, Z_n)$. We then have for some set $A \in \mathcal{B}(\mathcal{X} \times \mathcal{Y})$,

$$P\left((X_0, Y_0) \in A\right) = \int_A dh_xQ(y)d\mu(x)$$

The system then updates via the transition kernel $T : \mathcal{X} \times \mathcal{B}(\mathcal{X}) \rightarrow [0,1]$

$$P((X_n, Y_n) \in A|(X,Y)_{[0,n-1]} = (x,y)_{[0,n-1]}) = \int_A dh_xQ(y)dT(x_n|x_{n-1})$$

It follows that $\{X_n, Y_n\}_{n=0}^{\infty}$ itself is a Markov chain, and we will denote $P^\mu$ as the probability measure on $\Omega = \mathcal{X}^{\mathbb{Z}_+} \times \mathcal{Y}^{\mathbb{Z}_+}$, endowed with the product topology, (this of course means $\omega \in \Omega$ is a sequence of states and measurements $\omega = \{(x_i, y_i)\}_{i=0}^{\infty}$) where $X_0 \sim \mu$.

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**Definition 1.1.** We define the one step predictor as $\pi^n_{n-1}(\cdot) = P^n(X_n \in \cdot | Y_0, ..., Y_{n-1})$

**Definition 1.2.** We define the non-linear filter as $\pi^n_0(\cdot) = P^n(X_n \in \cdot | Y_0, ..., Y_n)$

Which are both regular conditional probabilities on $\mathcal{X}$. Suppose that an observer runs a non-linear filter assuming that the initial prior is $\nu$, when in reality the prior distribution is $\mu$. The observer receives the measurements and computes the filter $\pi^n_0$ for each $n$, but the measurement process is generated according to the true measure $\mu$. The question we are interested in is that of filter stability, namely, if we have two different initial probability measures $\mu$ and $\nu$, when do we have that the filter processes $\pi^n_0$ and $\pi^n_0$ merge in some appropriate sense as $n \to \infty$. In the literature, there are a number of merging notions when one considers stability which we enumerate here.

**Definition 1.3.** Two sequences of probability measures $P_n, Q_n$ merge weakly if for every continuous and bounded function $f$ we have $\lim_{n \to \infty} \| \int f \, dP_n - \int f \, dQ_n \| = 0$.

**Definition 1.4.** For two probability measures $P$ and $Q$ we define the total variation norm as $\| P - Q \|_{TV} = \sup_{\| f \|_{\infty} \leq 1} \| \int f \, dP - \int f \, dQ \|$ where $f$ is assumed measurable.

We will also utilize the information theoretic notion of relative entropy (Kullback-Leibler divergence). Relative entropy is often utilized as a notion of distance between two probability measure as it is non-negative, although it is not a metric since it is not symmetric.

**Definition 1.5.**

(i) For two probability measures $P$ and $Q$ we define the relative entropy as $D(P \| Q) = \int \log \frac{dP}{dQ} \, dP = \int \frac{dP}{dQ} \log \frac{dP}{dQ} \, dQ$ where we assume $P \ll Q$ and $\frac{dP}{dQ}$ denotes the Radon-Nikodym derivative of $P$ with respect to $Q$.

(ii) Let $X$ and $Y$ be two random variables, let $P$ and $Q$ be two different joint measures for $(X,Y)$ with $P \ll Q$. Then we define the (conditional) relative entropy between $P(X|Y)$ and $Q(X|Y)$ as $D(P(X|Y) \| Q(X|Y)) = \int \log \left( \frac{dP_{X|Y}}{dQ_{X|Y}}(x,y) \right) dP(x,y) = \int \left( \int \log \left( \frac{dP_{X|Y}}{dQ_{X|Y}}(x,y) \right) dP(x|Y = y) \right) dP(y)$.

**Notions of Stability.**

**Definition 1.6.** A filter process is said to be stable in the sense of weak merging in expectation if for any $f \in C_b(\mathcal{X})$ and any prior $\nu$ with $\mu \ll \nu$ we have $\lim_{n \to \infty} E^\nu[\| f \, d\pi^n_0 - \int f \, d\pi^n_0 \|] = 0$.

**Definition 1.7.** A filter process is said to be stable in the sense of weak merging $P^n$ almost surely (a.s.) if there exists a set of measurement sequences $A \subset Y^{\mathbb{Z}_+}$ with $P^n$ probability 1 such that for any sequence in $A$, for any $f \in C_b(\mathcal{X})$ and any prior $\nu$ with $\mu \ll \nu$ we have $\lim_{n \to \infty} \| f \, d\pi^n_0 - \int f \, d\pi^n_0 \| = 0$.

**Definition 1.8.** A filter process is said to be stable in the sense of total variation in expectation if for any measure $\nu$ with $\mu \ll \nu$ we have $\lim_{n \to \infty} E^\nu[\| \pi^n_0 - \pi^n_0 \|_{TV}] = 0$.

**Definition 1.9.** A filter process is said to be stable in the sense of total variation $P^n$ a.s. if for any measure $\nu$ with $\mu \ll \nu$ we have $\lim_{n \to \infty} \| \pi^n_0 - \pi^n_0 \|_{TV} = 0$ $P^n$ a.s.

**Definition 1.10.** A filter process is said to be stable in relative entropy if for any measure $\nu$ with $\mu \ll \nu$ we have $\lim_{n \to \infty} E^\nu[D(\pi^n_0 \| \pi^n_0)] = 0$. Note that when we write $D(\pi^n_0 \| \pi^n_0)$, we are referring to the relative entropy conditioned on one specific measurement sequence. When we want to write the full relative entropy over all measurement sequences as in Definition 1.5 (ii) we write $E^\nu[D(\pi^n_0 \| \pi^n_0)]$. 


DEFINITION 1.11. Given $f : \mathcal{X} \to \mathbb{R}$ we define the Lipschitz norm $\|f\|_L = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : d(x, y) \neq 0 \right\}$. With $\text{Lip} := \{ f : \|f\|_L \leq 1, \|f\|_\infty \leq 1 \} \subset C_b(\mathcal{X})$ we define the bounded Lipschitz (BL) metric as $\|P - Q\|_{BL} = \sup_{f \in \text{Lip}} |P(f) - Q(f)|$. A system is then said to be stable in the sense of BL-merging $P^n$ a.s. if we have $\|\pi^n_0 - \pi^n_\nu\|_{BL} \to 0$ $P^n$ a.s.

Here we make a cautionary remark about the merging of probability measures compared to the convergence of a sequence of probability measures to a limit measure. In convergence, we have some sequence $P_n$ and a static limit measure $P$ and we wish to show $P_n \to P$ under some convergence notion. However, in merging we have two sequences $P_n$ and $Q_n$ which may not individually have limits, but come closer together for large $n$ in one of the merging notions defined previously. The distinction is important. Let us assume that $\mathcal{X}$ is a finite dimensional real space and let $C_0(\mathcal{X})$ denote the space of all continuous functions which decay to zero as $|x| \to \infty$ under the standard supremum norm. The topological dual space of such a space of functions is the set of all finite signed measures endowed with total variation [13] Chapter 1 and when the space is compact, merging under the weak* topology for two sequences of finite measures coincides with the merging notion given in Definition 1.3 that is considering all $C_b(\mathcal{X})$ functions. Likewise, in Definition 1.3 if $Q_n$ were replaced with a single probability measure (i.e. considering converging instead of merging), due to Prokhorov’s theorem [3] and resulting tightness, the convergence notions would still be equivalent. However, in general both $\pi^n_0$ and $\pi^n_\nu$ are time-varying and in this case, as elaborately noted in [12], weak* merging (that is, considering only $C_0(\mathcal{X})$ functions) is strictly weaker than the merging under all $C_b(\mathcal{X})$ functions (Definition 1.3) as the following example reveals:

Example 1.1. Consider two sequences of point masses $P_n = \delta_n$ and $Q_n = \delta_n + \frac{1}{n}$. These measures merge in the weak* sense since they both converge to the trivial (all zero) measure in the weak* sense. However, there exists a continuous and bounded function $f$ such that for large $n$ we have $P_n(f) = 1$ but $Q_n(f) = 0$, so $P_n$ and $Q_n$ do not merge in the sense of Definition 1.3.

From [12] we have that if $\mathcal{X}$ is compact (or if $Q_n = Q$ for a fixed probability measure $Q$), the merging notions are identical. We note that such subtleties involving merging notions were elaborately investigated by van Handel [26] who focused on merging in the bounded Lipschitz norm, Definition 1.11 which is strictly weaker than Definition 1.7 when the space considered is not compact.

We recall that total variation merging implies weak merging. Relative entropy merging is also a useful notion in studying stability for a number or reasons. The key relationship between relative entropy and total variation is Pinsker’s inequality (see e.g., [21] [11] [16]) which states that for two probability measures $P$ and $Q$ we have that $\|P - Q\|_{TV} \leq \sqrt{\frac{2}{\log(e)}} D(P||Q)$.

The rest of the paper is organized as follows. A review of the relevant existing results in the literature is presented in Section 2. Our main results are presented in Section 3. Results are of two types: results dealing with observability, and results comparing the different notions of merging and the implications between them. Results on observability and weak merging of the predictor are presented in Section 4. The weak merging of the predictor is extended to total variation merging almost surely in Section 5. In Section 5.1 we consider the general structure of Radon-Nikodym derivatives for conditioned and restricted sigma fields. In Section 5.2 we apply these structural results to show total variation merging in expectation is equivalent for the
filter and predictor. In Section 7 we study relative entropy merging. A number of examples of observable measurement channels are presented in Section 8. Conclusions are drawn in Section 9. Some technical results are presented in the Appendix.

2. Literature Review. We give here a review of the previous results in the field to frame the contributions of this paper. Filter stability is a classical problem and we refer the reader to [5] for a comprehensive review. In the literature, certain results draw on a quantitative approach (e.g., [4], [10] by using the signal to noise ratio to establish sufficient conditions for filter stability), however, qualitative results such as those in [24] show that stability can be achieved regardless of the signal to noise ratio, therefore this is not the underlying mechanism that gives rise to stability. In the mindset of [8], filter stability arises via two separate mechanisms:

1. The transition kernel is in some sense sufficiently ergodic, forgetting the initial measure and therefore passing this insensitivity (to incorrect initializations) on to the filter process.

2. The measurement channel provides sufficient information about the underlying state, allowing the filter to track the true state process.

In this work we focus on the latter of the two mechanisms. The question of interest is to find sufficient conditions for observability. Along these lines, the method taken in this paper first sees its origins in Chigansky and Liptser [7] and a series of papers by van Handel [23, 24, 26]. Chigansky and Liptser were not interested in proving full stability, arguing that such results usually rely on ergodicity conditions. Instead, they focused on informative observations for a specific continuous function \( f \), rather than over all continuous functions in the criterion of weak merging. Nonetheless, [7, Equation 1.7] captures the essence of our definition of one step observability (Definition 3.1). The idea is to express a continuous function \( f(x) \) by integrating a measurable function \( g(y) \) over the conditional distribution for \( Y \) given \( X = x \). That is, consider the functional \( S(g)(\cdot) \rightarrow \int_Z g(h(\cdot, z))Q(dz) \). We wish to take a continuous function \( f \) and solve for a measurable function \( g \) such that \( f \approx S(g) \).

A fundamental result which pairs with observability is that of Blackwell and Dubins [2], an implication of which Chigansky and Liptser independently arrived at. Blackwell and Dubins use martingale convergence theorem to show that if \( P \) and \( Q \) are two measures on a stochastic process with \( P \ll Q \), then the conditional distributions on the future based on the past merge in total variation \( P \) a.s.

The result of Blackwell and Dubins [2] allows for a more direct argument, which was established by van Handel in [23]. However, in [23] the author had a restrictive condition in the model in that the measurement channel must be additive noise, and the noise must have a characteristic function which disappears nowhere. In [24], van Handel introduces a definition of observability for non-linear systems. Namely, a system is observable if every initial prior results in a unique probability measure on the measurement sequence. In [26], van Handel extends these results to non-compact state spaces, where uniform observability is introduced: Given a uniformity class \( \mathcal{G} \subset C_b(\mathcal{X}) \), for two measures \( P, Q \) on \( B(\mathcal{X}) \) define \( \|P - Q\|_{\mathcal{G}} = \sup_{g \in \mathcal{G}} \left| \int gP - \int gQ \right| \). A filtering model is said to be \( \mathcal{G} \)-uniformly observable if the almost sure total variation merging of the conditional probability measures on the measurement sequence (arising from from Blackwell-Dubins when \( \mu \ll \nu \)) implies \( \|\pi^n_\mu - \pi^n_\nu\|_{\mathcal{G}} \rightarrow 0 \) for any priors \( \mu, \nu \). If \( \mathcal{G} \) is the uniformly bounded Lipschitz functions, the filtering model is simply called uniformly observable and \( \|\cdot\|_{\mathcal{G}} \) is the bounded Lipschitz distance. See previous discussion in introduction on BL-merging compared to our stability notion in Definition 1.7. For a compact state space, uniform observability and observability are equivalent.
notions [26]. We also note that for a finite state space and a non-degenerate measurement channel, stability can be fully characterised via observability and a detectability condition [24], [27] Theorem V.2 or [5] Theorems 2.7 and 3.1.

In view of the discussion surrounding Example 1.1, and the subtleties involving BL-merging and weak merging, [26] Remark 3.2 notes that when the state space is not compact, uniform observability by choosing the unit ball in $C_b(X)$ as the uniformity class may be too much to ask while in principle the definition of uniform observability could be stated as such. For our setup, we will be considering merging for each function in $C_b(X)$ individually. We also note that our approach to extend weak merging to total variation merging in Theorem 5.3 crucially uses tightness which arises due to Prokhorov’s theorem, and this strictly requires weak merging (and BL-merging is not sufficient). In any case there exist examples where it is possible (as in Section 8.4) to consider weak merging in the sense of Definition 1.3.

In this work we study a number of different stability notions introduced in Definition 1.6-1.10. Note that observability only implies weak merging almost surely, and for the discrete time case as studied here observability only implies weak merging of the predictor almost surely, not the filter directly. Methods are then needed to extend observability to imply more stringent notions of stability. A useful tool is the condition discovered by Kunita [17] and derived in full in [8] which states a necessary and sufficient condition for the merging of the filter in total variation in expectation based on comparing the sigma fields $\mathcal{F}_0^Y$ and $\bigcap_{n \geq 0} \mathcal{F}_n^X \vee \mathcal{F}_0^Y$.

Relative entropy as a measure of discrepancy between the true filter and the incorrectly initialized filter is studied by Clark, Ocone, and Coumarbatch in [9]. Here they consider the filtering problem in continuous time with the associated non-degeneracy assumptions. The authors establish that the relative entropy of the true filter and the incorrect filter as a supermartingale. The paper does not establish convergence to zero, however. A notable setup where actual convergence (of the relative entropy difference) to zero is established is the (rather specific) Beneš filter studied in [20].

3. Statements of the Main Results. We first introduce our notion of an “informative” or “observable” measurement channel.

**Definition 3.1.**

(i) [One Step Observability] A partially observed Markov process (POMP) is said to be one step observable if for every $f \in C_b(X)$, $\epsilon > 0$, $\exists$ a measurable and bounded function $g : Y \to \mathbb{R}$ such that

$$\|f(\cdot) - \int_Z g \circ h(\cdot, z) dQ(z)\|_{\infty} < \epsilon$$

(ii) [N Step Observability] A POMP is said to be $N$ step observable if for every $f \in C_b(X)$, $\epsilon > 0$, $\exists$ a measurable and bounded function $g : \mathcal{Y}^N \to \mathbb{R}$ such that

$$\|f(\cdot) - \int_{\mathcal{Y}^N} g(y_{[1,N]}) dP(y_{[1,N]|X_1 = \cdot})\|_{\infty} < \epsilon$$

(iii) [Observability] A POMP is said to be observable if it is $N$ step observable for some finite $N \in \mathbb{N}$.

One step observability is the specific case of $N$ step observability when $N = 1$. However, unlike the case with $N > 1$, this case leads to an observability condition that does not depend on the transition kernel $T$ of the underlying Markov process, and is determined only by $h$ and $Q$.

**Theorem 3.2.** Let $\mu \ll \nu$ and let the POMP be observable in the sense of Definition 3.1 (iii). Then $\pi_{n^-}^{\mu}$ and $\pi_{n^-}^{\nu}$ merge weakly as $n \to \infty$, $P^\mu$ a.s.
Assumption 3.1. The state space $\mathcal{X}$ is compact, and the measurement channel $h_x Q$ is continuous in total variation, that is $\|h_{x_n} Q - h_x Q\|_{TV} \to 0$ or
$$\|P(Y_0 \in \cdot | X_0 = x_n) - P(Y_0 \in \cdot | X_0 = x)\|_{TV} \to 0,$$
when $x_n \to x$.

This assumption allows us to use results in [14] and conclude the weak merging of the filter in expectation from the predictor a.s.

Theorem 3.3. Assume $\mu \ll \nu$, Assumption 3.1 holds, and let Definition 3.1 (iii) be satisfied. Then the filter is stable in the sense of Definition 1.6 (i.e., weak merging in expectation).

Proof. See Theorem 4.3.

Assumption 3.2. Assume $T(\cdot, x)$ is absolutely continuous with respect to the Lebesgue measure for every $x \in \mathcal{X}$ and denote the resulting pdf as $t(\cdot | x)$. Assume $t(\cdot | \cdot)$ is uniformly bounded. Further, assume that for some finite numbers $n, m$ $E^\mu [D(\pi_n || \pi_m)] < \infty$ and $E^\mu [D(P^\mu(Y_{0}, m)) || (P^\nu(Y_{0}, m))] < \infty$. Then the filter is stable in the sense of 1.10.

Proof. See Theorem 5.3, Corollary 6.8, and Theorem 6.9.

Theorem 3.5. Assume $\mu \ll \nu$, Assumption 3.2 holds, and let Definition 3.1 (iii) be satisfied. Further, assume that for some finite numbers $n, m$ $E^\mu [D(\pi_n || \pi_m)] < \infty$ and $E^\mu [D(P^\mu(Y_{0}, m)) || (P^\nu(Y_{0}, m))] < \infty$. Then the filter is stable in the sense of 1.10.

Proof. See Lemma 7.5 and the preceding discussion.

The proof of these theorems involves many of the supporting results proved in the paper. A diagram of the different merging notions and the results which extend them is seen in Figure 3.1. The dashed lines represent implications that are always true, and the solid lines are labelled with the theorems and assumptions that prove the implication.

3.1. Comparison with the literature. In view of the literature review and the stated results above, our contributions are as follows:

i) We have introduced in Definition 3.1 a very general definition of observability, building on and refining the approach laid out by Chigansky and Liptser [6], and in particular, van Handel [23, 24, 26]. This definition involves a functional characterization, and is explicit and testable/computable for many systems. Building on the the classical merging result of Blackwell and Dubins [2] which holds for predictive measures on finite dimensional events of the future, we can utilize more than just the one step look ahead measures and obtain a refined observability definition.

ii) Our results allow for stability under weak merging (with interpretations refined in the discussion in the introduction) and total variation both in expectation and a.s., as well as convergence in relative entropy. By considering these notions together, we have provided a unified analysis leading to complementary, and often more general, stability conditions. Theorem 3.2 generalizes prior results on the merging of predictor measures in view of the relaxed observability definition. Theorem 3.3 gives us the same conclusion as [23, Theorem 2.2], yet we allow for a more general measurement structure, that is any measurement channel that is observable (by Definition 3.1) and continuous in total variation, but require a compact $\mathcal{X}$. In Section 6.1 we generalize the structural results of [8] and [25, Lemma 5.6, Cor 5.7]
Fig. 3.1: Proof Program and Flow of Ideas and Conditions for Filter Stability

for general sigma fields. These are then used to establish the equivalence between merging of the predictor in total variation in expectation, and the filter in total variation in expectation and almost surely. Figure 3.1 provides a detailed visual summary of our results.

iii) That relative entropy of the filter is a non-increasing sequence was established in [9], but its convergence to zero was not established (the convergence to zero is established for the Benes filter in [20]) and the model considered was non-degenerate (as well as in continuous-time). By using the results of Barron [1, Theorem 2] combined with the structural results of Section 6.1, we show if the relative entropy is at any point not infinity, relative entropy merging is equivalent to total variation in expectation. This result has been hinted on in the literature, see [8, Remark 4.2] or [25, Remark 5.9], but has not been stated and established explicitly. Theorem 3.5 establishes this.

iv) We will provide examples in Section 8 where our definition of observability can be tested with little effort; these include invertible systems, measurement models with degeneracy (lacking a common reference measure for conditional probabilities), and systems characterized by polynomial measurements. Most previous examples of observable systems are restricted to additive measurement channels, e.g. [26, Proposition 3.11] considering models of the form \( y_n = h(x_n) + z \) where \( h \) has a uniformly continuous inverse, and the i.i.d noise process \( z \) has a characteristic function which disappears nowhere. We note also that for the compact state space setup (where BL-merging would be equivalent to weak merging), our observability definition would be a sufficiency test for the observability definition of van Handel
4. Weak Merging.

4.1. One Step Predictor and Observability. Lemma 4.1. Let \( g \) be a bounded and measurable function on \((Y^k, \mathcal{B}(Y))\). For any initial prior \( \mu \) we have

\[
\int_{Y^k} g(y_{[n,n+k]})dP^\mu(y_{[n,n+k]}|Y_{[0,n-1]}) = \int_{Y^k} \int_{X^k} g(y_{[n,n+k]})dP(y_{[n,n+k]}|x_n)d\pi^\mu_{n-}(x)
\]  

(4.1)

Proof. Since \( \{(X_n, Y_n)\}_{n=0}^\infty \) is a Markov chain, we can stop conditioning on the past measurements and the initial prior. 

Lemma 4.2. Let \( g \) be a bounded and measurable function on \((Y, \mathcal{B}(Y))\). For any initial prior \( \mu \) we have

\[
\int_Y g(y_n)dP^\mu(y_n|X_n = x) = \int_Z g(h_x(z))dQ(z)
\]  

(4.2)

Proof. \( Z \) is a random variable on the probability space \((Z, \mathcal{B}(Z), Q)\) and \( Y_n \) exists on the measurable space \((Y, \mathcal{B}(Y))\). Then, for every fixed choice of \( X_n = x \) we have that \( Y_n \) is a fixed function of \( Z \), that is \( Y_n = h_x(Z) \). For any set \( A \in \mathcal{B}(Y) \) we have \( P^\mu(Y_n \in A|X_n = x) = Q(h_x^{-1}(A)) \). Yet this means that \( P^\mu(Y_n \in \cdot|X_n = x) \) is exactly the pushforward measure of \( Q \) under the mapping \( h_x \), call this measure \( h_xQ(A) = Q(h_x^{-1}(A)) \). We then have:

\[
\int_Y g(y)dh_xQ(y) = \int_Z g(h_x(z))dQ(z)
\]

here we have applied Theorem 3.1 on the pushforward measure.

Notice that the inner integral in the RHS of Equation (4.1) is a function of \( x \). The LHS is then the term considered in the total variation merging of the predictive measures of the measurement sequences, while the RHS is the term considered in the weak merging of the one-step predictor. We can then leverage Blackwell and Dubin’s theorem to arrive at a sufficient condition for weak merging of the one-step predictor. Theorem 4.2 is closely related to [26] Prop. 3.11 and its proof is in essence a sufficient condition for uniform observability (of the predictor).

Proof of Theorem 4.2. Proof. Fix any \( f \in C_b(S) \) and \( \epsilon > 0 \). We wish to show that \( \exists N \) such that \( \forall n > N, \left| \int fd\pi^\mu_{n-} - \int fd\pi^\nu_{n-} \right| < \epsilon \).

By assumption our model is \( N' \) step observable, therefore we can find some \( g \) with \( \|g\|_\infty < \infty \) such that

\[
\hat{f}(x) = \int_{Y^{N'}} g(y_{[1,N']})dP(y_{[1,N']}|X_1 = x)
\]
and \(|f - \tilde{f}|_\infty < \frac{2}{3}\epsilon\). Then we have

\[
\left| \int f \pi^\mu_{n-} - \int f \pi^\nu_{n-} \right| \leq \left| \int \tilde{f} \pi^\mu_{n-} - \int \tilde{f} \pi^\nu_{n-} \right| + \left| \int (f - \tilde{f}) \pi^\mu_{n-} \right| + \left| \int (f - \tilde{f}) \pi^\nu_{n-} \right|
\]

Now, by assumption \(|f - \tilde{f}|_\infty < \frac{2}{3}\epsilon\) therefore the last two terms are less than \(2\epsilon/3\). We then apply Lemma 4.1 and we have

\[
\left| \int \tilde{f} \pi^\mu_{n-} - \int \tilde{f} \pi^\nu_{n-} \right| + \frac{2}{3}\epsilon
\]

\[
= \left| \int g(y|n,n+N^\pi) dP^\beta(\{n,n+N^\pi|Y_{[0,n-1]}\}) - \int g(y|n,n+N^\nu) dP^\nu(\{n,n+N^\nu|Y_{[0,n-1]}\}) \right| + \frac{2}{3}\epsilon
\]

Note that it always holds if \(\mu \ll \nu\) then we have \(P^\beta(\{0,\infty\} \in \cdot) \ll P^\nu(\{0,\infty\} \in \cdot)\). Then via a classic result by Blackwell and Dubins [2], we have that \(P^\beta(\{n\in N|Y_{[0,n-1]}\}) \ll P^\nu(\{n\in N|Y_{[0,n-1]}\})\) and \(P^\nu(\{n\in N|Y_{[0,n-1]}\})\) merge in total variation \(P^\mu\) a.s. Define \(\tilde{g} = \frac{g}{\|g\|_{\infty}}\). Then \(\exists N \in \mathbb{N}\) such that for all \(n > N\),

\[
\left| \int \tilde{g}(y|n,n+N^\pi) dP^\beta(\{n,n+N^\pi|Y_{[0,n-1]}\}) - \int \tilde{g}(y|n,n+N^\nu) dP^\nu(\{n,n+N^\nu|Y_{[0,n-1]}\}) \right| < \frac{\epsilon}{3\|g\|_{\infty}}
\]

we then have:

\[
\left| \int g(y|n,n+N^\pi) dP^\beta(\{n,n+N^\pi|Y_{[0,n-1]}\}) - \int g(y|n,n+N^\nu) dP^\nu(\{n,n+N^\nu|Y_{[0,n-1]}\}) \right| + \frac{2}{3}\epsilon
\]

\[
\leq |g|_{\infty} \epsilon \left( \frac{1}{3\epsilon} \right) + \frac{2}{3}\epsilon = \epsilon
\]

therefore, since \(f\) and \(\epsilon\) are arbitrary we have for any \(f \in C_b(S)\):

\[
\lim_{n \to \infty} \left| \int f \pi^\mu_{n-} - \int f \pi^\nu_{n-} \right| = 0,
\]

which means \(\pi^\mu_{n-}\) and \(\pi^\nu_{n-}\) merge weakly. \(\Box\)

**4.2. Weak Merging of Filter.** Here we will utilize results from [14]. This paper was concerned with a different topic than filter stability, namely the weak Feller property of the “filter update” kernel. However, some of the analysis is useful in providing concise arguments to connect the filter to the predictor. For the theorem in this section, we will provide a sketch of the proof and refer the reader to [14] for the full arguments.

**Theorem 4.3.** Let Assumption [17] hold, if the predictor merges weakly \(P^\mu\) a.s., then the filter merges weakly in expectation.

**Proof.** Begin by assuming that the predictor merges weakly almost surely. As is argued in [14], one can view the filter \(\pi^\mu_{n}\) as a function of \(\pi^\mu_{n-1}\) (the previous filter) and the current observation \(Y_n = y_n\), that is \(\pi^\mu_{n} = F(\pi^\mu_{n-1},y_n)\). Furthermore for a measure \(\mu\) and a function \(f\) we write \(\mu(f) = \int f d\mu\). Note that since we assume \(X\) is compact, BL merging is equivalent to weak merging. Then we have

\[
E^\mu[||\pi^\mu_{n} - \pi^\nu_{n}||_{BL}]
\]

\[
= E^\mu \left[ \sup_{f \in Lip} \left| \int Y F(\pi^\mu_{n-1},y_n)(f) dP(y_n|\pi^\mu_{n-1}) - \int Y F(\pi^\nu_{n-1},y_n)(f) dP(y_n|\pi^\nu_{n-1}) \right| \right]
\]

(4.3)
the notation of conditioning on the filter is inherited from [13], the notation is defined as
\[
P(y_n \in A|\pi_n^{\mu-1}) = \int_X Q(h_x^{-1}(A))d\pi_n^{\mu-1}(x) \tag{4.5}
\]
by the predictor merging weakly \(P^\mu\) a.s. and the channel \(h_xQ\) being continuous in total variation, we have that \(P(y_n|\pi_n^{\mu-1})\) is continuous in total variation [14]. We can then add and subtract \(\int_Y F(\pi_n^{\mu-1}, y_n)(f)dP(y_n|\pi_n^{\mu-1})\) inside the absolute value and by the triangle inequality we have that (4.4) is less that or equal to:
\[
E^\nu\left[\|P(y_n|\pi_n^{\mu-1}) - P(y_n|\pi_n^{\nu-1})\|_{TV}\right] +
\]
\[
E^\mu\left[\sup_{f \in L^p} \int_Y |F(\pi_n^{\mu-1}, y_n)(f) - F(\pi_n^{\nu-1}, y_n)(f)|dP(y_n|\pi_n^{\nu-1})\right] \tag{4.6}
\]
the first term above then goes to zero by the weak merging of the predictor and (4.5). For the second term, for a fixed choice of \(f\) define \(I^+_f\) to be the region \(\{y_n|F(\pi_n^{\mu-1}, y_n)(f) \geq F(\pi_n^{\nu-1}, y_n)(f)\}\) and its compliment as \(I^-_f\). By considering only \(I^+_f\) for the time being, we can drop the absolute value sign. We can then add and subtract \(\int_{I^+_f} F(\pi_n^{\mu-1}, y_n)(f)dP(y_n|\pi_n^{\mu-1})\) inside the supremum and (4.7) will be less than or equal to
\[
E^\nu\left[\|P(y_n|\pi_n^{\mu-1}) - P(y_n|\pi_n^{\nu-1})\|_{TV}\right] +
\]
\[
E^\mu\left[\sup_{f \in L^p} \int_X \int_{I^+_f} f(x)dh_x Q(y_n)d\pi_n^{\mu-1}(x) - \int_X \int_{I^-_f} f(x)dh_x Q(y_n)d\pi_n^{\nu-1}(x)\right] \tag{4.7}
\]
the first term above goes to zero as before, and the second term goes to zero by Lemma 2 in [14] and the weak merging of the predictor. The same can be done for \(I^-_f\) and the proof is complete. \(\square\)

Note that this result gives us the same conclusion as [23] Theorem 2.2. Yet we allow for a more general measurement structure, that is any measurement channel that is observable (by Definition 3.1 (iii)) and continuous in total variation, but require a compact \(X\).

5. Total Variation Merging of One Step Predictor. We now extend our results from weak merging to total variation.

**Lemma 5.1.** Let \(\exists\) some measure \(\mu\) such that \(T(\cdot, x) \ll \mu\) for every \(x \in X\). Then we have that \(\pi_n^{\mu-} \ll \mu\) for every \(n \in \mathbb{N}\)

**Proof.** For all \(n \geq 1\) we have
\[
\pi_n^{\mu-}(A) = \int_X T(A, x)d\pi_n^{\mu-1}(x) = \int_X \int_A \frac{dT}{\partial \mu}(a)d\mu(a)d\pi_n^{\mu-1}(x) = \int_A \left(\int_X \frac{dT}{\partial \mu}(a)d\pi_n^{\mu-1}(x)\right)d\mu(a)
\]
therefore \(\pi_n^{\mu-}\) is absolutely continuous with respect to \(\mu\) for every \(n \geq 1\). \(\square\)

**Lemma 5.2.** Let Assumption 3.2 hold and let \(f_n^{\mu-}\) denote the density function of \(\pi_n^{\mu-}\). Then we have \(\mathcal{F}^{\mu} = \{f_n^{\mu-}\}, \mathcal{F}^{\nu} = \{f_n^{\nu-}\}\) are uniformly bounded equicontinuous families.

**Proof.** As we see from the previous lemma,
\[
f_n^{\mu-}(x_n) = \frac{d\pi_n^{\mu-}}{d\lambda}(x_n) = \int_X t(x_n|x_n-1)d\pi_n^{\mu-1}(x_n-1)
\]
Where \( t \) is the pdf of the transition kernel since our dominating measure is now Lebesgue. We require \( \forall \epsilon > 0, \ x^* \in \mathcal{X} \exists \delta > 0 \) such that \( \forall \|x - x^*\| < \delta, \forall y \in [0, n-1] \in \mathcal{Y}_n \) we have \( |f^\mu_{n-}(x) - f^\nu_{n-}(x^*)| < \epsilon \). By Assumption 5.2, clearly \( f^\mu_{n-} \) is uniformly bounded since \( t \) is uniformly bounded. Then, for any \( \epsilon > 0, \forall x^* \in \mathcal{X} \) we can find a \( \delta > 0 \) such that \( |t(x_2|x_1) - t(x^*|x_1)| < \epsilon \) when \( |x_2 - x^*| < \delta \). Now, assume \( |x_2 - x^*| < \delta \), we have

\[
|f^\mu_{n-}(x_2) - f^\mu_{n-}(x^*)| = \left| \int_{\mathcal{X}} (t(x_2|x_1) - t(x^*|x_1))d\pi^\mu_{n-}(dx_1) \right| \leq \int_{\mathcal{X}} |t(x_2|x_1) - t(x^*|x_1)|d\pi^\mu_{n-} \leq \epsilon
\]

Which proves that \( \mathcal{F}^\mu \) and \( \mathcal{F}^\nu \) are uniformly bounded and equicontinuous families.

***

**Theorem 5.3.** Let Assumption 5.2 hold, if \( \pi^\mu_n \) and \( \pi^\nu_n \) merge weakly \( P^\mu \) a.s., then \( \|\pi^\mu_n - \pi^\nu_n\|_{\tau V} \to 0 \), \( P^\mu \) a.s.

Proof. By assumption we have a set of measurement sequences \( B \subset \mathcal{Y}_{\geq 1} \) with \( P^\mu(B) = 1 \) such that for every measurement sequence in \( B \) we have the predictor is stable in the weak sense along this measurement sequence. Fix any \( y \in [0, \infty) \in B \) and assume that measurement for the remainder of the proof. Via Lemma 5.1 and 5.2, \( \mathcal{F}^\mu \) and \( \mathcal{F}^\nu \) are uniformly bounded and equicontinuous families. Let \( \mathcal{F}^\mu \cap \mathcal{F}^\nu = \{f_n|f_n = f^\mu_n - f^\nu_n\} \), then the sequence \( \{f_n\}^\infty_{n=1} \) is a uniformly bounded and equicontinuous class of integrable functions. As in the proof of [18 Lemma 2], now pick a sequence of compact sets \( K_j \subset \mathcal{X} \) such that \( K_j \subset K_{j+1} \). By the Arzela-Ascoli theorem [22], for any subsequence we can find further subsequences \( f_{n_k} \) such that

\[
\lim_{k \to \infty} \sup_{x \in K_j} |f_{n_k}(X) - f^j(x)| = 0
\]

for some continuous function \( f^j : K_j \to [0, \infty) \). Via the \( K_j \) being nested, we can have \( \{f_{n_{k+1}}\} \) be a subsequence of \( \{f_{n_k}\} \), and therefore \( f^{j+1} = f^j \) over \( K_j \). Then define the function \( \tilde{f} \) on \( \mathcal{X} \) by \( \tilde{f}(x) = f^j(x), x \in K_j \). Using Cantor’s diagonal method, we can find an increasing sequence of integers \( \{m_i\} \) which is a subsequence of \( \{n_k\} \) for every \( j \). Therefore

\[
\lim_{i \to \infty} f_{m_i}(x) = \tilde{f}(x) \quad \forall x \in \mathcal{X}
\]

and the convergence is uniform over each \( K_j \) and \( \tilde{f} \) is continuous. Now, \( f_{m_i} \) converges weakly to the zero measure by assumption, and via uniform convergence for any Borel set \( \mathcal{B} \) we have

\[
\int_{\mathcal{B}} f_{m_i}(x)dx \to \int_{\mathcal{B}} \tilde{f}(x)dx,
\]

i.e. setwise convergence. Yet this implies weak convergence, so \( \tilde{f} = 0 \) almost everywhere, yet \( \tilde{f} \) is continuous so it is 0 everywhere.

Now, via Prokhorov theorem (Theorem 8.6.2 in [24]) we have that \( \mathcal{F}^\mu \) is a tight family. Therefore, for every \( \epsilon > 0 \) we can find a compact set \( K_\epsilon \) such that

\[
|\pi^\mu_{n-} - \pi^\nu_{n-}|(\mathcal{X} \setminus K_\epsilon) < \epsilon \quad \forall \ n \in \mathbb{N}.
\]

then we have

\[
\lim_{i \to \infty} \|\pi^\mu_{m_i} - \pi^\nu_{m_i}\|_{\tau V} \leq \lim_{i \to \infty} |\pi^\mu_{m_i} - \pi^\nu_{m_i}|(\mathcal{X} \setminus K_\epsilon) + |\pi^\mu_{m_i} - \pi^\nu_{m_i}|(K_\epsilon) \\
\leq \lim_{i \to \infty} \sup_{\|g\|_{\infty} \leq 1} \left| \int_{K_\epsilon} g(x)f_{m_i}(x)dx \right| + \epsilon
\]

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the case with the non-linear filter:
can also consider restricting and conditioning simultaneously, for example is the
P really discussing the conditional measure of A then have
P(A ∈ ·|F) = Eμ[1A|F0] = Eμ[1A|F0] = ∫ Eμ[1A|F0] dPμ(x0) = ∫ Eμ[1A|F0] dPμ(x0)

since we have already argued ˘f = 0. Now, over the compact set Kε, f_m converges to
˘f uniformly, therefore ∃N such that ∀k > N, ∥f − fn∥∞ < nKε. We then conclude that

limi→∞ ∥f − fn∥∞λ(Kε) + ε

Thus, for every subsequence of {f_n}_n≥1, we can find a subsequence that converges in
total variation, which implies that the original sequence converges in total variation.

6. From Predictor Stability to Filter Stability. Up to this point, we have
established the total variation merging of the predictor a.s. and the weak merging of
the filter in expectation. However, we would like to consider more stringent notions of
stability for the filter, as well as stability in relative entropy for both the predictor and
filter. Under different assumptions specific results can be developed, for example [23]
Lemma 4.2 which establishes the total variation merging of the filter in expectation
from that of the predictor using non-degeneracy. However, by examining the form of
the Radon Nikodym derivative of Pμ and Pν restricted and conditioned on different
sigma fields, we can gain significant insight into how these different notions of stability
relate to one another. These results are inspired as a generalization of Lemma 5.6 and
Corollary 5.7 in [23], or a similar derivation in the introduction of [8] which establish
the specific form of dPμ/dPν.

6.1. Some Structural Results. We first introduce some notation that is useful
when dealing with sigma fields rather than random variables directly. Strictly speaking,
we have two probability measures Pμ and Pν on (X^2+, Y^2+) ∪ B(X^2+) ∪ B(Y^2+).
We denote F_{a,b} as the sigma field generated by (X_a, · · · , X_b) and similarly for Y. We
then have B(X^2+) ∪ B(Y^2+) = F_{0,∞} ∪ F_{0,∞}, that is the sigma field generated by all
state and measurement sequences. When we write Pμ(X[0]|n) we are really discussing
the measure Pμ restricted to the sigma field F_{0,n} which we will denote Pμ|F_{0,n}. Similarly
for some set A ∈ F_{0,∞} ∩ F_{0,∞} when we write Pμ(X[0], Y[0] ∈ A|Y[0]|n) we are really
discussing the conditional measure of Pμ with respect to the sigma field F_{0,n}. We can
also consider restricting and conditioning simultaneously, this for example is the
the case with the non-linear filter:

πμ^n(·) = Pμ(X_n ∈ ·|Y[0]|n) = Pμ|F_{0,n} F_{0,n}

With the notation established, we proceed with the presentation of a number of
supporting technical results

LEMMA 6.1. For any sigma field G ⊆ F_{0,∞} ∩ F_{0,∞} we have:

dPμ|G

dPν|G = Eν dPμ|F_{0,∞} G

Proof. Begin with the largest sigma field, G = F_{0,∞} ∩ F_{0,∞}. Pick any A ∈ F_{0,∞} ∩ F_{0,∞}
we have

Pμ(X[0], Y[0]|n) = Eμ[1_A] = Eμ[Eμ[1_A|F_0]] = ∫ Eμ[1_A|F_0] dPμ(x_0) = ∫ Eμ[1_A|F_0] dPμ(x_0)
Now, conditioned on $\mathcal{F}_0^X$ the prior is irrelevant, therefore $E^\nu[1_A|\mathcal{F}_0^X] = E^\nu[1_A|\mathcal{F}_0^X]$ and we have:

$$
\int E^\nu[1_A|\mathcal{F}_0^X] \frac{d\mu}{d\nu}(x_0) d\nu(x_0) = \int E^\nu[1_A|\mathcal{F}_0^X] \frac{d\mu}{d\nu}(x_0) d\nu(x_0) = E^\nu[E^\nu[1_A|\mathcal{F}_0^X] \frac{d\mu}{d\nu}(X_0)]
$$

$$
= E^\nu[E^\nu(\frac{d\mu}{d\nu}(X_0)1_A|\mathcal{F}_0^X)] = E^\nu[1_A \frac{d\mu}{d\nu}(X_0)]
$$

Since $\frac{d\mu}{d\nu}(X_0)$ is $\mathcal{F}_0^X$ measurable so we can move it inside the conditional expectation. It is then clear that

$$
\frac{dP^\mu|\mathcal{F}_0^X \vee \mathcal{F}_0^Y}{dP^\nu|\mathcal{F}_0^X \vee \mathcal{F}_0^Y} = \frac{d\mu}{d\nu}(X_0) = E^\nu \left[ \frac{d\mu}{d\nu}(X_0) \right] \mathcal{F}_0^X \vee \mathcal{F}_0^Y P^\mu \text{ a.s.}
$$

Now pick some other field $\mathcal{G} \subset \mathcal{F}_0^X \vee \mathcal{F}_0^Y$, pick $A \in \mathcal{G}$ we have:

$$
P^\mu(X_0^\infty, Y_0^\infty \in A) = E^\nu[1_A] = E^\nu \left[ \frac{d\mu}{d\nu}(X_0)1_A \right] = E^\nu \left[ E^\mu \left[ \frac{d\mu}{d\nu}(X_0)1_A \mid \mathcal{G} \right] \right] = E^\nu \left[ 1_A E^\nu \left[ \frac{d\mu}{d\nu}(X_0) \mid \mathcal{G} \right] \right]
$$

Since $1_A$ is $\mathcal{G}$ measurable. It is then clear that

$$
\frac{dP^\mu|\mathcal{G}_1 \mid \mathcal{G}_2}{dP^\nu|\mathcal{G}_1 \mid \mathcal{G}_2} = E^\nu \left[ \frac{d\mu}{d\nu}(X_0) \right] \mathcal{G}_1 \vee \mathcal{G}_2 P^\mu \text{ a.s.}
$$

**Proof.** For any set $A \in \mathcal{G}_1$ we have:

$$
P^\mu(X_0^\infty, Y_0^\infty \in A) = E^\nu[1_A] = E^\nu \left[ E^\nu[1_A \mid \mathcal{G}_2] \right] = E^\nu \left[ E^\mu[1_A \mid \mathcal{G}_2] \frac{dP^\mu}{dP^\nu|\mathcal{G}_2} \right]
$$

$$
= E^\nu \left[ E^\mu[1_A \frac{dP^\mu}{dP^\nu|\mathcal{G}_2} \mid \mathcal{G}_2] \right] = E^\nu \left[ E^\nu[1_A E^\nu \left( \frac{d\mu}{d\nu}(X_0) \mid \mathcal{G}_2 \right)] \mathcal{G}_2 \right]
$$

where we can move $\frac{dP^\nu}{dP^\mu|\mathcal{G}_1 \mid \mathcal{G}_2}$ in between expectations since it is $\mathcal{G}_2$ measurable and we have applied the previous lemma in the final equality. We therefore have

$$
E^\nu[1_A] = E^\nu \left[ E^\nu[1_A E^\nu \left( \frac{d\mu}{d\nu}(X_0) \mid \mathcal{G}_2 \right)] \mid \mathcal{G}_2 \right] \quad \text{(6.1)}
$$

The Radon Nikodym derivative $\frac{dP^\nu}{dP^\mu|\mathcal{G}_1 \mid \mathcal{G}_2}$ is then the unique function $f$ such that

$$
E^\nu[1_A E^\nu \left( \frac{d\mu}{d\nu}(X_0) \mid \mathcal{G}_2 \right)] = E^\nu[1_A f E^\nu \left( \frac{d\mu}{d\nu}(X_0) \mid \mathcal{G}_2 \right)] \quad \text{(6.2)}
$$

We can also write

$$
E^\nu[1_A] = E^\nu \left[ E^\nu[1_A \mathcal{G}_1 \vee \mathcal{G}_2] \mathcal{G}_2 \right]
$$
= E^\nu[1_A E^{\nu\prime}(dP^\mu/dP^\nu)(G_1 \lor G_2)] = E^\nu[1_A E^{\nu\prime}(dP/d\nu)(X_0)|G_1 \lor G_2]

where we can move $1_A$ freely between expectations since it is $G_1$ and hence $G_1 \lor G_2$ measurable and we have again applied the previous lemma. We then have

$$E^\nu[1_A] = E^\nu[1_A E^{\nu\prime}(X_0)|G_1 \lor G_2]$$  \hspace{1cm} (6.3)

Equating (6.2) with (6.3) the Radon Nikodym derivative must satisfy

$$E^\nu[E^{\nu\prime}[1_A f E^{\nu\prime}(dP/d\nu)(X_0)|G_2]|G_2] = E^\nu[E^{\nu\prime}[1_A dP/d\nu](X_0)|G_2]$$  \hspace{1cm} (6.4)

which will be satisfied by $f = E^\nu[dP/d\nu](X_0)|G_2] \lor G_2]$ (note the denominator is positive $P^\mu$ a.s.).

$$E^\nu[E^{\nu\prime}[1_A f E^{\nu\prime}(dP/d\nu)(X_0)|G_2]|G_2] = E^\nu[E^{\nu\prime}[1_A \left( E^{\nu\prime}[dP/d\nu](X_0)|G_1 \lor G_2] \right) E^{\nu\prime}[dP/d\nu](X_0)|G_2]$$

$$= E^\nu[E^{\nu\prime}[1_A E^{\nu\prime}(dP/d\nu)(X_0)|G_1 \lor G_2]|G_2] = E^\nu[E^{\nu\prime}[1_A dP/d\nu](X_0)|G_2]$$

\textbf{Lemma 6.3.} For any two sigma fields $G_1, G_2 \subset \mathcal{F}_{0,\infty} \lor \mathcal{F}_{0,\infty}^Y$ we have:

$$\|P^\mu|_{G_1}|G_2 - P^\nu|_{G_1}|G_2\|_TV = \frac{E^\nu\left[ E^\nu\left[ dP/d\nu(X_0)|G_1 \lor G_2 \right] - E^\nu\left[ dP/d\nu(X_0)|G_2 \right] \right]}{E^\nu\left[ dP/d\nu(X_0)|G_2 \right]} \hspace{1cm} P^\mu \text{ a.s.}$$

\textbf{Proof.} An equivalent way to express total variation as opposed to that presented in Definition 4.1 is as

$$\|P^\mu|_{G_1}|G_2 - P^\nu|_{G_1}|G_2\|_TV = \int \left| \frac{dP^\mu|_{G_1}|G_2}{dP^\nu|_{G_1}|G_2} - 1 \right| dP^\nu|_{G_1}|G_2$$

$$= \int \frac{E^\nu\left[ dP/d\nu(X_0)|G_1 \lor G_2 \right]}{E^\nu\left[ dP/d\nu(X_0)|G_2 \right]} - 1 \left| dP^\nu|_{G_1}|G_2 \right|$$

via Lemma 6.2. We can then cross multiply and we have

$$\frac{E^\nu\left[ E^\nu\left[ dP/d\nu(X_0)|G_1 \lor G_2 \right] - E^\nu\left[ dP/d\nu(X_0)|G_2 \right] \right]}{E^\nu\left[ dP/d\nu(X_0)|G_2 \right]}$$

\textbf{6.2. Predictor and Filter Stability.} We will now use the previous lemma's to construct explicit expressions for the filter and predictor. Equations (6.3), (6.4), and Lemma 6.6 are known in the literature (see [25] Lemma 5.6 and Corollary 5.7] and [8] Equation 1.10]), however the general results we derive in Lemmas 6.1-6.3 have
not been not previously analysed outside of the context of the filter. Since our results apply to any general sigma field, not just the fields used in the analysis of the filter, we can perform a similar analysis for the predictor to determine Equation \((6.6), (6.8)\), and Lemma \([6.3]\). We then conclude in Corollary \([6.3]\) that the total variation merging of the predictor in expectation is equivalent to that of the filter.

**Lemma 6.4.** [24, Lemma 5.6] Assume \(\mu \ll \nu\). Then we have that \(\pi_n^\mu \ll \pi_n^\nu\), \(\pi_{n-}^\mu \ll \pi_{n-}^\nu\) \(P^\mu \) a.s. and we have

\[
\frac{d\pi_n^\mu}{d\pi_n^\nu}(x) = \frac{E^\nu\left[\frac{d\mu}{d\nu}(X_0)|Y_{[0,n]}\right]}{E^\nu\left[\frac{d\mu}{d\nu}(X_0)|Y_{[0,n]}\right]} \quad P^\mu \text{ a.s.} \quad (6.5)
\]
\[
\frac{d\pi_{n-}^\mu}{d\pi_{n-}^\nu}(x) = \frac{E^\nu\left[\frac{d\mu}{d\nu}(X_0)|Y_{[0,n-1]}\right]}{E^\nu\left[\frac{d\mu}{d\nu}(X_0)|Y_{[0,n-1]}\right]} \quad P^\mu \text{ a.s.} \quad (6.6)
\]

**Proof.** These results become clear from Lemma \(6.2\) when we state the filter as \(P^\mu\) restricted to \(F_n^\mu\) conditioned on \(F_0^\nu\), and the predictor is conditioned on \(F_{n-1}^\nu\).

**Lemma 6.5.** [25, Corollary 5.7] Assume \(\mu \ll \gamma\) for some measure \(\gamma\), then we can express

\[
\|\pi_n^\mu - \pi_n^\gamma\|_{TV} = \frac{E^\gamma\left[\left|E^\gamma\left[\frac{d\mu}{d\gamma}(X_0)|Y_{[0,n]}\right], X_{n} = \cdot\right| - E^\gamma\left[\frac{d\mu}{d\gamma}(X_0)|Y_{[0,n]}\right]\right]|Y_{[0,n]}\]}{E^\gamma\left[\frac{d\mu}{d\gamma}(X_0)|Y_{[0,n]}\right]} \quad (6.7)
\]
\[
\|\pi_{n-}^\mu - \pi_{n-}^\gamma\|_{TV} = \frac{E^\gamma\left[\left|E^\gamma\left[\frac{d\mu}{d\gamma}(X_0)|Y_{[0,n-1]}\right], X_{n-1} = \cdot\right| - E^\gamma\left[\frac{d\mu}{d\gamma}(X_0)|Y_{[0,n-1]}\right]\right]|Y_{[0,n-1]}\]}{E^\gamma\left[\frac{d\mu}{d\gamma}(X_0)|Y_{[0,n-1]}\right]} \quad (6.8)
\]

**Proof.** By Lemma \(6.3\) we can write

\[
\|\pi_n^\mu - \pi_n^\gamma\|_{TV} = \frac{E^\gamma\left[\left|E^\gamma\left[\frac{d\mu}{d\gamma}(X_0)|Y_{[0,n]}\right], X_{n} = \cdot\right| - E^\gamma\left[\frac{d\mu}{d\gamma}(X_0)|Y_{[0,n]}\right]\right]|Y_{[0,n]}\]}{E^\gamma\left[\frac{d\mu}{d\gamma}(X_0)|Y_{[0,n]}\right]}
\]
\[
\|\pi_{n-}^\mu - \pi_{n-}^\gamma\|_{TV} = \frac{E^\gamma\left[\left|E^\gamma\left[\frac{d\mu}{d\gamma}(X_0)|Y_{[0,n-1]}\right], X_{n-1} = \cdot\right| - E^\gamma\left[\frac{d\mu}{d\gamma}(X_0)|Y_{[0,n-1]}\right]\right]|Y_{[0,n-1]}\]}{E^\gamma\left[\frac{d\mu}{d\gamma}(X_0)|Y_{[0,n-1]}\right]}
\]

Now, via the Markov property of the system, we have that \((X_{[0,n-1]}, Y_{[0,n-1]})\) are independent of \((X_{[n+1,\infty)}, Y_{[n+1,\infty)})\) conditioned on \((X_n, Y_n)\) therefore we can state

\[
E^\gamma\left[\frac{d\mu}{d\gamma}(X_0)|Y_{[0,n]}, X_{n}\right] = E^\gamma\left[\frac{d\mu}{d\gamma}(X_0)|Y_{[0,\infty]}, X_{[n,\infty]}\right]
\]
\[
E^\gamma\left[\frac{d\mu}{d\gamma}(X_0)|Y_{[0,n-1]}, X_{n-1}\right] = E^\gamma\left[\frac{d\mu}{d\gamma}(X_0)|Y_{[0,\infty]}, X_{[n,\infty]}\right]
\]

**Lemma 6.6.** [25, Equation 1.10] The filter merges in total variation in expectation if and only if

\[
E^\nu\left[\frac{d\mu}{d\nu}(X_0)|\bigcap_{n \geq 0} F_{0,\infty}^\nu \vee F_{n,\infty}^X\right] = E^\nu\left[\frac{d\mu}{d\nu}(X_0)| F_{0,\infty}^\nu\right] \quad P^\nu \text{ a.s.} \quad (6.9)
\]
Proof.

\[ E^\nu \left[ \| \pi^\mu_n - \pi^\nu_n \|_{TV} \right] = E^\nu \left[ \frac{dP^\mu}{dP^\nu} (Y^\nu_0) \| \pi^\mu_n - \pi^\nu_n \|_{TV} \right] = E^\nu \left[ E^\nu \left[ \frac{d\mu}{d\nu} (X_0) \left| Y_{[0,n]} \right. \right] \| \pi^\mu_n - \pi^\nu_n \|_{TV} \right] \\
= E^\nu \left[ E^\nu \left[ |E^\nu \left[ \frac{d\mu}{d\nu} (X_0) \right] | Y_{[0,n]} \right|, X_{[n,\infty]} \right] - E^\nu \left[ \frac{d\mu}{d\nu} (X_0) \right| Y_{[0,n]} \right] \right] \\
= E^\nu \left[ |E^\nu \left[ \frac{d\mu}{d\nu} (X_0) \right| Y_{[0,n]}, X_{[n,\infty]} \right| - E^\nu \left[ \frac{d\mu}{d\nu} (X_0) \right| Y_{[0,n]} \right] \right] \right]

We then see that \( A_n = E^\nu \left[ \frac{d\mu}{d\nu} (X_0) \right| Y_{[0,n]} \) is a non-negative uniformly integrable martingale adapted to the increasing filtration \( \mathcal{F}^Y_{0,n} \). Hence the limit as \( n \to \infty \) in \( L^1(P^\nu) \) is \( E^\nu \left[ \frac{d\mu}{d\nu} (X_0) \right| \mathcal{F}^Y_{0,\infty} \). Similarly, we can view \( B_n = E^\nu \left[ \frac{d\mu}{d\nu} (X_0) \right| Y_{[0,\infty]}, X_{[n,\infty]} \right| \) as backwards non-negative uniformly integrable martingale with respect to the decreasing sequence of filtrations \( \mathcal{F}^Y_{0,\infty} \). Then by the backwards martingale convergence theorem, the limit as \( n \to \infty \) in \( L^1(P^\nu) \) is \( E^\nu \left[ \frac{d\mu}{d\nu} (X_0) \right| \mathcal{F}^Y_{0,\infty} \right| \mathcal{F}^X_{n,\infty} \). It is then clear the the total variation in expectation is zero if and only if equation (6.9) holds.

Lemma 6.7. The predictor merges in total variation in expectation if and only if

\[ E^\nu \left[ \frac{d\mu}{d\nu} (X_0) \right| \bigcap_{n \geq 1} \mathcal{F}^Y_{0,\infty} \cup \mathcal{F}^X_{n,\infty} \right] = E^\nu \left[ \frac{d\mu}{d\nu} (X_0) \right| \mathcal{F}^Y_{0,\infty} \right] P^\nu \text{ a.s.} \quad (6.10) \]

Proof. The proof is similar to the proof for Lemma 6.6.

Corollary 6.8. The filter merges in total variation in expectation if and only if the predictor merges in total variation in expectation.

Proof. From Lemma 6.6 and 6.7 it is clear that the two conditions for merging in total variation in expectation are equivalent since the sigma fields on the LHS of Equation (6.9) and (6.10) are equivalent.

Remark 6.1. Corollary 6.8 is a new result in view of the existing literature. We note first that much of the literature focuses on continuous time, where the predictor is not used in the analysis. In discrete time, [23, Lemma 4.2] proves that the merging of the predictor in total variation in expectation implies that of the filter. However this result relies on a non-degeneracy assumption in the observation channel and the specific structure of the filter recursion equation [3, Equation 1.4].

We have now established that the filter merges in total variation in expectation, but we would like to extend this result to almost surely. By a simple application of Fatou’s lemma, we can argue the liminf of the total variation of the filter is zero \( P^\mu \) a.s. Hence if the limit exists, it must be zero, yet it is not immediate that the limit will exist. In [23, p. 572], a technique is established to prove the existence of this limit. We now recall the following, where a proof is included for completeness.

Theorem 6.9. [23, p. 572] Assume the filter is stable in total variation in expectation. Then the filter is stable in total variation \( P^\mu \) a.s.

Proof. Let \( \gamma = \frac{d\mu}{d\nu} \), then we have that \( \mu \ll \gamma, \nu \ll \gamma \) and furthermore \( \| \frac{d\mu}{d\nu} \|_\infty < 2, \| \frac{d\nu}{d\gamma} \|_\infty < 2 \). The boundedness of the Radon-Nikodym derivatives is key, as this makes the expressions in the numerator of equation (6.7) uniformly integrable martingales, and by the martingale convergence theorem (see [2, Theorem 2]) converge as
dependent of past \( Y \). Therefore, if the limits exist a.s., they must be zero. Via Fatou’s lemma, we have that

\[
\lim_{n \to \infty} E[\|\pi_n^\mu - \pi_n^\nu\|_{TV}] = 0
\]

therefore, if the limits exist a.s., they must be zero. Via Fatou’s lemma, we have that

\[
\lim_{n \to \infty} E[\|\pi_n^\mu - \pi_n^\nu\|_{TV}] = 0
\]

7. Relative Entropy Merging. We will now show that the relative entropy merging of the filter is essentially equivalent to merging in total variation in expectation. Via Lemma 7.1 it is clear that the filter and predictor admit Radon-Nikodym derivatives. Therefore, working with \( D(\pi_n^\mu || \pi_n^\nu) \) and \( D(\pi_n^- || \pi_n^-) \) is well defined.

It has been established in [9] that the relative entropy of the filter is a decreasing sequence, but the analysis is in continuous time and the authors feel it is worth recreating the results here in discrete time. To this end, will extensively use the chain rule for relative entropy:

**Lemma 7.1.** For joint measures \( P, Q \) on random variables \( X, Y \) we have

\[
D(P(X,Y)||Q(X,Y)) = D(P(X)||Q(X)) + D(P(Y|X)||Q(Y|X))
\]

Note for two sigma fields \( F \) and \( G \) and two joint measures \( P \) and \( Q \) on \( F \vee G \) one could also express this relationship as

\[
D(P|_{F \vee G} || Q|_{F \vee G}) = D(P|_F || Q|_F) + D(P|_G || Q|_G)
\]

we will use either notations where it is most convenient. We now use the chain rule to establish the monotonicity and convergence of the respective relative entropy sequences.

**Lemma 7.2.**

\[
E^\mu[D(\pi_n^\mu || \pi_n^\nu)] \leq E^\mu[D(\pi_n^- || \pi_n^-)]
\]

**Proof.** Using chain rule (Lemma 7.1) we arrive at the following:

\[
D(P^\mu(\mathcal{X}_n, Y_n|\mathcal{Y}[0,n-1]) || P^\nu(\mathcal{X}_n, Y_n|\mathcal{Y}[0,n-1]))
\]

\[
= E^\mu[D(\pi_n^\mu || \pi_n^\nu)] + D(P^\mu(Y_n|\mathcal{Y}[0,n-1], \mathcal{X}_n) || P^\nu(Y_n|\mathcal{Y}[0,n-1], \mathcal{X}_n))
\]

\[
= E^\mu[D(\pi_n^\mu || \pi_n^\nu)]
\]

As was discussed in the proof of Theorem 3.2. \( Y_n \) conditioned on \( X_n \) is independent of past \( Y \) values and initial measure \( \nu \) or \( \mu \) since \( Y_n = h(X_n, Z) \), therefore

\[
D(P^\mu(\mathcal{Y}[0,n-1], \mathcal{X}_n)) || P^\nu(\mathcal{Y}[0,n-1], \mathcal{X}_n)) = 0.
\]

If we apply chain rule the other way we have

\[
D(P^\mu(\mathcal{X}_n, Y_n|\mathcal{Y}[0,n-1]) || P^\nu(\mathcal{X}_n, Y_n|\mathcal{Y}[0,n-1]))
\]

\[
= E^\mu[D(\pi_n^\mu || \pi_n^\nu)] + D(P^\mu(Y_n|\mathcal{Y}[0,n-1]) || P^\nu(Y_n|\mathcal{Y}[0,n-1]))
\]
Since relative entropy is always greater than zero, we can equate these two expressions and arrive at our conclusion, that the relative entropy of the one step predictor is is greater than the non-linear filter. 

**Lemma 7.3.**

\[ E^\mu[D(\pi^\mu_{n+1-}||\pi^\nu_{n+1-})] \leq E^\mu[D(\pi^\mu_n||\pi^\nu_n)] \]

**Proof.** Using chain rule in a similar fashion we have

\[
\begin{align*}
D(P^\mu(X_n, X_{n+1}|Y_{[0,n]})) & = E^\mu[D(\pi^\mu_n||\pi^\nu_n)] + D(P^\nu(X_{n+1}|Y_{[0,n]}, X_n)) \\
& = E^\mu[D(\pi^\mu_n||\pi^\nu_n)]
\end{align*}
\]

Now, \( Y_n \) is a noisy measurement of \( X_n \), and \( X \) is a Markov chain, therefore \( X_{n+1} \) conditioned on \( X_n \) is independent of \( Y_{[0,n]} \) and the initial measure, therefore the second term above is zero. Applying chain rule the other way we have

\[
\begin{align*}
D(P^\nu(X_{n+1}|X_n, Y_{[0,n]})) & = E^\mu[D(\pi^\nu_{n+1-}||\pi^\nu_{n-})] + D(P^\nu(X_n|X_{n+1}, Y_{[0,n]})) \\
& = E^\mu[D(\pi^\nu_{n+1-}||\pi^\nu_{n-})]
\end{align*}
\]

relative entropy is always non-negative, therefore we equate the two expressions and arrive at our conclusion. 

**Corollary 7.4.** The relative entropy of the one step predictor and the non-linear filter are monotonically decreasing sequences bounded below by zero, and therefore admit limits.

**Proof.** By a simply application of Lemma 7.2 and 7.3 we have

\[ E^\mu[D(\pi^\mu_{n+1-}||\pi^\nu_{n+1-})] \leq E^\mu[D(\pi^\mu_n||\pi^\nu_n)] \leq E^\mu[D(\pi^\mu_{n-}||\pi^\nu_{n-})] \]

therefore the one step predictor is a monotonically decreasing sequence bounded below by zero, and admits a limit. Similarly we have

\[ E^\nu[D(\pi^\nu_{n+1}||\pi^\nu_{n+1+})] \leq E^\mu[D(\pi^\nu_{n+1-}||\pi^\nu_{n+1-})] \leq E^\mu[D(\pi^\nu_n||\pi^\nu_n)] \]

so the no-linear filter also exhibits this property. 

In the literature it has been remarked that the relative entropy merging of the filter is in equivalent to the total variation merging in expectation. See for example \[8, Remark 4.2\] or \[25, Remark 5.9\]. In \[19\] it is shown that relative entropy is a non-increasing sequence, but not that the limit of this sequence is zero. The following result establishes this.

**Lemma 7.5.** Assume there exists some finite \( n \) such that \( E^\mu[D(\pi^\mu_n||\pi^\nu_n)] < \infty \) and some \( m \) such that \( E^\nu[D(P^\nu|_{\mathcal{F}_{d,m}^n}||P^\nu|_{\mathcal{F}_{d,m}^n})] < \infty \). Then the filter is stable in relative entropy if and only if it is stable in total variation in expectation.

**Proof.** First assume the filter is stable in relative entropy. Since the square root function is continuous, we have

\[
\lim_{n \to \infty} \sqrt{\frac{2}{\log(e)}} E^\mu[D(\pi^\mu_n||\pi^\nu_n)] = \lim_{n \to \infty} E^\mu\left[ \sqrt{\frac{2}{\log(e)}} D(\pi^\mu_n||\pi^\nu_n) \right] = 0
\]

where we have applied Jensen’s inequality. We then apply Pinsker’s inequality and we have \( \lim_{n \to \infty} E^\mu[||\pi^\mu_n - \pi^\nu_n||_{TV}] = 0 \).
For the converse direction, by chain rule, it is clear that
\[
E^\mu[D(\pi_n^\mu \| \pi_n^\nu)] = D(P^\mu |_{\mathcal{F}_n^X \vee \mathcal{F}_0^\nu}) \| P^\nu |_{\mathcal{F}_n^X \vee \mathcal{F}_0^\nu})
\]
\[
= D(P^\mu |_{\mathcal{F}_n^X \vee \mathcal{F}_0^\nu}) \| P^\nu |_{\mathcal{F}_n^X \vee \mathcal{F}_0^\nu}) - D(P^\mu |_{\mathcal{F}_n^X} \| P^\nu |_{\mathcal{F}_n^X})
\]
by the Markov Property we have
\[
X_n \independent \text{ given } Y_n
\]
Therefore, we have:
\[
D(P^\mu |_{\mathcal{F}_n^X \vee \mathcal{F}_0^\nu}) \| P^\nu |_{\mathcal{F}_n^X \vee \mathcal{F}_0^\nu}) = D(P^\mu |_{\mathcal{F}_n^X} \| P^\nu |_{\mathcal{F}_n^X})
\]
Then \(\mathcal{F}_n^X \vee \mathcal{F}_0^\nu\) is a decreasing sequence of sigma fields. By Theorem 2 we have that if the relative entropy is ever finite, the limit of the relative entropy restricted to these sigma fields is the relative entropy restricted to the intersection of the decreasing fields, that is
\[
\lim_{n \to \infty} D(P^\mu |_{\mathcal{F}_n^X \vee \mathcal{F}_0^\nu}) \| P^\nu |_{\mathcal{F}_n^X \vee \mathcal{F}_0^\nu}) = D(P^\mu |_{\mathcal{F}_n^X} \| P^\nu |_{\mathcal{F}_n^X})
\]
Likewise, \(\mathcal{F}_0^\nu\) is an increasing sequence of sigma fields, therefore by Theorem 3 we have that if the relative entropy is ever finite, the relative entropy restricted to these sigma fields is the relative entropy over the limit field, that is
\[
\lim_{n \to \infty} D(P^\mu |_{\mathcal{F}_n^X \vee \mathcal{F}_0^\nu}) \| P^\nu |_{\mathcal{F}_n^X \vee \mathcal{F}_0^\nu}) = D(P^\mu |_{\mathcal{F}_0^\nu}) \| P^\nu |_{\mathcal{F}_0^\nu})
\]
Therefore,
\[
\lim_{n \to \infty} E^\mu[D(\pi_n^\mu \| \pi_n^\nu)] = D(P^\mu |_{\mathcal{F}_n^X \vee \mathcal{F}_0^\nu}) \| P^\nu |_{\mathcal{F}_n^X \vee \mathcal{F}_0^\nu}) - D(P^\mu |_{\mathcal{F}_0^\nu}) \| P^\nu |_{\mathcal{F}_0^\nu})
\]
By Lemma 6.1 we have
\[
\frac{dP^\mu}{dP^\nu}|_{\mathcal{F}_n^X \vee \mathcal{F}_0^\nu} = E^\nu \left[ \frac{d\mu}{d\nu}(X_0) \right]_{\mathcal{F}_n^X \vee \mathcal{F}_0^\nu} = f_1
\]
\[
\frac{dP^\mu}{dP^\nu}|_{\mathcal{F}_0^\nu} = E^\nu \left[ \frac{d\mu}{d\nu}(X_0) \right]_{\mathcal{F}_0^\nu} = f_2
\]
Note that \(f_1 = \mathcal{F}_n^X \vee \mathcal{F}_0^\nu\), \(f_2 = \mathcal{F}_0^\nu\), \(\mathcal{F}_n^X \vee \mathcal{F}_0^\nu\), and \(\mathcal{F}_0^\nu\). By Lemma 6.1 we have that if the filter merges in total variation in expectation, then for a set of state and observation sequences \(\omega \in A \subset \mathcal{F}_n^X \vee \mathcal{F}_0^\nu\) with \(P^\nu(A) = 1\), we have \(f_1(\omega) = f_2(\omega)\). Yet this then means over the set \(A\) of \(P^\nu\) measure 1, \(f_1 = f_2\) is \(\mathcal{F}_0^\nu\) measurable. We then have
\[
D(P^\mu |_{\mathcal{F}_n^X \vee \mathcal{F}_0^\nu}) \| P^\nu |_{\mathcal{F}_n^X \vee \mathcal{F}_0^\nu}) = D(P^\mu |_{\mathcal{F}_0^\nu}) \| P^\nu |_{\mathcal{F}_0^\nu})
\]
\[
= \int_{\mathcal{F}_0^\nu} f_1(\omega) \log(f_1(\omega)) dP^\nu - \int_{\mathcal{F}_0^\nu} f_2(\omega) \log(f_2(\omega)) dP^\nu
\]
\[
= \int_{\mathcal{F}_0^\nu} f_1(\omega) \log(f_1(\omega)) dP^\nu - \int_{\mathcal{F}_0^\nu} f_2(\omega) \log(f_2(\omega)) dP^\nu
\]
\[
= \int_{\mathcal{F}_0^\nu} f_1(\omega) \log(f_1(\omega)) dP^\nu - \int_{\mathcal{F}_0^\nu} f_2(\omega) \log(f_2(\omega)) dP^\nu
\]
Therefore, if the relative entropy of the filter is ever finite, then total variation merging in expectation is equivalent to merging in relative entropy.

8. Observable Measurement Channel Examples. Let us now look at some measurement channels which satisfy Theorem 3.2.

8.1. Compact State and Noise Spaces with Affine Observation. Consider $\mathcal{X}, \mathcal{Z}$ as compact spaces and let $h(x, z) = a(z)x + b(z)$ for some functions $a, b$ where the image of $\mathcal{Z}$ under $a$ and $b$ is compact (this ensures that $\mathcal{Y}$ is compact). Note that for a fixed choice of $z$, this is an affine function of $x$. We will show sufficient conditions for one step observability. Since $\mathcal{X}$ is compact, the set of polynomials is dense in the set of continuous and bounded functions. Therefore, without loss of generality we assume $f$ is a polynomial. Consider then the mapping

$$S : \mathbb{R}^\mathcal{Z} \rightarrow \mathbb{R}^\mathcal{Z} \quad S(g)(\cdot) \mapsto \int_{\mathcal{Z}} g(h(\cdot, z))dQ(z)$$

Let $\mathbb{R}[x]_n$ represent the polynomials on the real line up to degree $n$. Then we have that $S(g)$ is invariant on $\mathbb{R}[x]_n$, that is if $g$ is polynomial of degree $n$ then $S(g)$ is a polynomial of degree $n$. Furthermore, the coefficients of $S(g)(x) = \sum_{i=0}^n \beta_i x^i$ can be related to the coefficients of $g(x) = \sum_{i=0}^n \alpha_i x^i$ by a linear transformation. Define $N(i, k) = \binom{i}{k}E(a(Z)^k b(Z)^{i-k})$ then by recursive application of binomial theorem we have

$$\begin{pmatrix}
\beta_0 \\
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_n
\end{pmatrix} =
\begin{pmatrix}
N(0, 0) & N(1, 0) & \cdots & N(n, 0) \\
0 & N(1, 1) & \cdots & N(n, 1) \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & N(n, n)
\end{pmatrix}
\begin{pmatrix}
\alpha_0 \\
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_n
\end{pmatrix}$$

if we want to generate any polynomial, we require this matrix to be invertible, and since it is upper triangular this amounts to none of the diagonal entries being zero, that is $E[a(z)^n] \neq 0 \forall n \in \mathbb{N}$. Furthermore, we want $g$ to be bounded so we must have $N(n, k) < \infty \forall n \in \mathbb{N}, k \in \{0, \ldots, i\}$.

Example. Consider $\mathcal{X} = [-10, 10], \mathcal{Z} = [-1, 1], Z \sim \text{Uni}([-1, 1])$ and $y = z^2x + z$. We then have $\mathcal{Y} = [-11, 11]$. For any $n \in \mathbb{N}$ we have

$$E[a(z)^n] = \frac{1}{2} \int_{-1}^1 z^{2n}dz = \frac{1}{2n+1} \neq 0$$

additionally, for any $n \in \mathbb{N}, k \in \{0, \ldots, n\}$ we have

$$N(n, k) = \binom{n}{k}E(a(z)^k b(z)^{n-k}) = \binom{n}{k}E(z^{n-k}) = \binom{n}{k} \frac{1}{n-k+1} < \infty$$

8.2. Compact State Space and Non-Compact Noise Space with Affine Observation. The result from the previous section can be extended for non-compact $\mathcal{Z}$. Assume $\mathcal{X}$ is compact, $\mathcal{Z} = \mathbb{R}$ and $h(x, z) = a(z)x + b(z)$. Assume that

i) For all $n \in \mathbb{N}, k \in \{0, \ldots, n\}$ we have

$$\int_{\mathcal{Z}} |a(z)^k b(z)^{n-k}|Q(dz) < \infty \quad (8.1)$$

ii) For every $n \in \mathbb{N}$, exists some finite value $0 < D$ such that over every compact set $[-M, M]$ we have

$$\inf\limits_{j \in \{0, \ldots, n-1\}} \int_{-M}^M |a(z)^j Q(dz)| > DQ(-M, M) \quad (8.2)$$
Then, the system is one step observable. See Appendix \[A\] for a proof of this result.

**Example.** Consider if \( Z \sim N(0, \sigma^2) \), and let \( a(z) = z + 3, b(z) = z \). The we have for any \( n \in \mathbb{N}, k \leq n \)

\[
\int_Z |a(z)b(z)^{n-k}| dQ(z) = \int_Z |(z+3)^k z^{n-k}| dQ(z) = \int_Z \sum_{i=0}^k \binom{n}{k} z^i 3^{i-k} z^{n-k} |dQ(z)| \\
\leq \sum_{i=0}^k \binom{n}{k} 3^{i-k} \int_Z |z^{n+i-k}| |dQ(z)| \\
\end{equation}

a Gaussian has finite moments, let \( V = \max_{k \leq n} E(|z^k|) < \infty \) and we have

\[
\sum_{i=0}^k \binom{n}{k} 3^{i-k} \int_Z |z^{n+i-k}| |dQ(z)| \leq \sum_{i=0}^k \binom{n}{k} 3^{i-k} V < \infty
\]

Which satisfies condition \[5.1\]. Then we have

\[
\int_{-M}^M (z+3)^n dQ(z) = \sum_{k=0}^n \binom{n}{k} 3^{n-k} \int_{-M}^M z^k Q(dz) = 3Q([-M,M]) + \sum_{k=1}^{2\lfloor \frac{n}{2k} \rfloor} \binom{n}{2k} 3^{n-2k} \int_{-M}^M z^{2k} Q(dz)
\]

Since all the odd moments are zero by symmetry. Now, clearly the minimum occurs at \( n = 1 \), since any higher power will add additional positive even moments to the sum. Therefore, we have

\[
\inf_{j \in \{0, \ldots, n-1\}} | \int_{-M}^M a(z)^j Q(dz)| > 3Q(-M, M)
\]

For any \( n \) and we satisfy condition \[5.2\].

**8.3. Indicator Function.** Consider \( \mathcal{X} \) as a compact state space, \( \mathcal{Z} = \mathbb{R} \). Let \( h(x, z) = 1_{x > z} x + 1_{x \leq z} z \) and assume that \( Q \) admits a density with respect to Lebesgue. We have

\[
\int_{\mathcal{Z}} g(h(x, z)) dQ(z) = \int_{-\infty}^x g(x) q(z) dz + \int_x^\infty g(z) q(z) dz \\
\]

again, we can approximate any continuous and bounded function \( f \) on \( \mathcal{X} \) as polynomial, so we assume \( f \) is differentiable. We have

\[
f(x) = \int_{-\infty}^x g(x) q(z) dz + \int_x^\infty g(z) q(z) dz \\
\]

\[
f'(x) = g(x) q(x) + \int_{-\infty}^x g'(x) q(z) dz - g(x) q(x) = g'(x) Q(Z \leq x)
\]

therefore, we only need to define \( g \) over \( \mathcal{X} \). Furthermore, we require \( g \) to be bounded, which is implied if \( g' \) is bounded since \( g \) is only defined over a compact space. Since \( \mathcal{X} \) is compact there exists some \( x_{\min} \in \mathbb{R} \) such that \( x_{\min} < \mathcal{X} \). We require for some \( \epsilon > 0 \) that \( Q(Z < x_{\min}) > \epsilon \). This condition says every \( x \in \mathcal{X} \) has some positive probability of being observed through \( h(x, z) \) and we will not always get pure noise.
8.4. Direct Observation. Consider now the case when \( y = h(x) \) for some invertible function \( h \). This can be written as \( y = h(x) + z \) where \( Q \sim \delta_0 \), that is a point mass at zero. We then have for any measurable bounded function \( g \)

\[
\int_Z g(h(x) + z) dQ(z) = g(h(x))
\]

Then for any continuous and bounded function \( f \), define \( g = f \circ h^{-1} \) and we have \( f(x) = \int_Z g(h(x) + z) dQ(z) \) and we satisfy Definition 8.1.

8.5. Finite State and Noise Space. Consider a finite setup \( \mathcal{X} = \{a_1, \cdots, a_n\} \) and \( \mathcal{Z} = \{b_1, \cdots, b_m\} \). Now, assume \( h(x, z) \) has \( K \) distinct outputs, where \( 1 \leq K \leq (n)(m) \) and \( \mathcal{Y} = \{c_1, \cdots, c_K\} \). We note that for such as setup, there is already a sufficient and necessary condition provided in [27] Theorem V.2]. However, we examine this case to show that our definition is equivalent to the sufficient direction of this theorem, which is van Handel’s notion of observability [24].

Then for each \( x, h \), can be viewed as a partition of \( \mathcal{Z} \), assigning each \( b_i \in \mathcal{Z} \) to an output level \( c_j \in \mathcal{Y} \). We can track this by the matrix \( H_x(i, j) = 1 \) if \( h_x(b_i) = y_j \) and zero else. Let \( Q \) be the \( 1 \times m \) vector representing the probability measure of the noise. Let us first consider the one step observability. Let \( g(c_i) = \alpha_i \) and \( \int_Z g(h(x, z)) dQ(z) = QH_x\alpha \). Therefore, any function \( f(x) \) can be expressed as a \( n \times 1 \) vector, and the system is one step observable if and only if the matrix \( M = \begin{pmatrix} QH_{a_1} \\ \vdots \\ QH_{a_n} \end{pmatrix} \) is rank \( n \).

Consider then \( N \) step observability. We wish to solve equations of the form

\[
f(x) = \int_{\mathcal{Y}^N} g(y_{[1,N]}) dP^\mu(y_{[1,N]}|x_1 = x)
\]

With knowledge of \( Q, h(\cdot, \cdot) \) and \( T \) we can directly compute the transition kernel for the joint measure \( y_{[1,N]}|X_1 \), however the size of this matrix is \( n \) by \( K^n \) where \( |\mathcal{X}| = n, |\mathcal{Y}| = K \) so complexity grows exponentially. We can deduce a sufficient, but not necessary, condition for \( n \) step observability using the marginal conditional measures. Consider that \( P^\mu(y_k \in \cdot | X_1 = a_j) = T(a_j) : T^{k-2}A, k \geq 2 \) where \( T(a_j) \) represents the \( j^\text{th} \) row of the transition matrix. Note that these are all \( 1 \times K \) vectors and represent the marginal measures of \( Y_k|X_1 \). Consider the class of functions \( \mathcal{G}^n = \{g : \mathcal{Y}^n \to \mathbb{R}\} \) and a subclass \( \mathcal{G}^n_{LC} = \{g(y_{[1,n]}) = \sum_{i=1}^{n} \alpha_i g_i(y_i) | \alpha_i \in \mathbb{R}, g_i \in \mathcal{G}^1\} \). That is, a linear combination of functions of the individual \( y_i \) values. We can use these functions to deduce a sufficient, but not necessary, condition for observability.

**Lemma 8.1.** Assume that \( |\mathcal{X}| = n \) and define the matrix

\[
M = \begin{pmatrix} A & TA & \cdots & T^{n-1}A \end{pmatrix}
\]

which is \( n \times nK \) where \( K = |\mathcal{Y}| \). If \( M \) is rank \( n \), then the system is \( n \) step observable. Furthermore, if \( M \) is not rank \( n \), appending more blocks of the form \( T^kA \) for \( k \geq n \) will not increase the rank of \( M \).

**Proof.** Begin with (8.3), consider a restriction to \( \mathcal{G}^n_{LC} \), that is we require \( g \) to be of the form \( g(y_{[1,n]}) = \sum_{i=1}^{n} g_i(y_i) \). Denote the \( n \times 1 \) vector \( \alpha = (g_1(c_1), \cdots, g_k(c_k), \cdots, g_k(c_1), \cdots, g_k(c_k)) \). Then we have

\[
f(x) = \sum_{i=1}^{n} P^\mu(y_i \in \cdot | X_1 = x) \begin{pmatrix} g_1(c_1) \\ \vdots \\ g_k(c_k) \end{pmatrix} = (QH_x T(x) : A \cdots T(x) : T^{n-2}A) \alpha
\]
We can then see that this matrix is the \( j \) row of \( M \) when \( x = a_j \), therefore we have

\[
\begin{pmatrix}
    f(a_1) \\
    \vdots \\
    f(a_n)
\end{pmatrix} = (A \ TA \ \cdots \ \ T^{n-1}A) \alpha .
\]

If \( M \) is rank \( n \), then any function \( f : \mathcal{X} \to \mathbb{R} \) can be expressed as a vector \( g \) put through matrix \( M \) and the system is observable.

Consider if \( M \) is not rank \( n \) and if we append another block \( T^nA \) to \( M \). By the Cayley-Hamilton theorem, \( T^n \) is a linear combination of lower powers of \( T \), e.g. \( T^n = \sum_{i=0}^{n} \alpha_i T^i \) for some coefficients \( \alpha_i \). Therefore this additional block is a linear combination of the previous blocks, and adds no dimension to the matrix \( M \). \( \square \)

If the conditions of this lemma fail, i.e. \( M \) is not rank \( n \), that means integrating \( g \) over the marginal measures cannot generate any \( f \) function. Yet the product of the marginal measures is not the joint measure since the \( Y_i|X_1 \) are not independent. Hence, working with the marginal measures only is not enough to determine observability as also noted by van Handel in [24, Remark 13] in a slightly different setup. Consider the following example

**Example.** Consider if \( \mathcal{X} = \{1, 2, 3, 4\} \) and \( Y = 1_{x \leq 2} \). This can be realized as

\[
A = \begin{pmatrix}
    QH_1 \\
    \vdots \\
    QH_4
\end{pmatrix} = \begin{pmatrix}
    0 & 1 \\
    0 & 1 \\
    1 & 0 \\
    1 & 0
\end{pmatrix}
\]

Consider the following transition kernel,

\[
T = \begin{pmatrix}
    0 & 1 & 1 & 1 \\
    \frac{1}{2} & 0 & 0 & \frac{1}{2} \\
    0 & 1 & \frac{1}{2} & \frac{1}{2} \\
    \frac{1}{4} & 0 & 0 & \frac{1}{4}
\end{pmatrix}
\]

Notice that the odd and even rows are identical. If we consider the marginal measures of \( Y_1|X_1, \ldots, Y_4|X_1 \) we have the matrix

\[
\begin{pmatrix}
    A & \cdots & T^nA \\
    0 & 1 & 0.75 & 0.25 & 0.5625 & 0.4375 & 0.69375 & 0.390625 \\
    0 & 1 & 0.50 & 0.50 & 0.6250 & 0.3750 & 0.59375 & 0.406250 \\
    1 & 0 & 0.75 & 0.25 & 0.5625 & 0.4375 & 0.69375 & 0.390625 \\
    1 & 0 & 0.50 & 0.50 & 0.6250 & 0.3750 & 0.59375 & 0.406250
\end{pmatrix}
\]

Which is only rank 3, not rank 4. Therefore, we cannot use the marginal measures to determine observability.

However, if we consider the joint measure of \( (Y_1, Y_2)|X_1 \) we have the matrix

\[
A' = \begin{pmatrix}
    0 & 0 & \frac{1}{2} & \frac{1}{2} \\
    0 & 0 & \frac{3}{4} & \frac{1}{4} \\
    \frac{1}{4} & \frac{1}{4} & 0 & 0 \\
    \frac{1}{4} & \frac{1}{4} & 0 & 0
\end{pmatrix}
\]

Where row \( i \) is conditioned on \( x = i \) and the columns are ordered in binary \( y_2y_1 \), e.g. \( P(y_1 = 1, y_2 = 0|x_1 = 2) \) is row 2 column 3. This matrix is full rank, hence the system is \( N \) step observable with \( N = 2 \), even though the marginal measures failed to be full rank.

Recall van Handel defines the term “observability” as every distinct prior resulting in a unique measure on \( \mathcal{Y}^{Z^+} \). Therefore, for a finite system our notion of \( N \) step observability is a sufficient condition for this notion.
8.6. Revisiting Results in [23] and [26]. In [23], van Handel has the following theorem. Assume \( Y_n = h(X_n) + Z_n \) where \( h \) has a uniformly continuous inverse and \( Z_n \) has a characteristic function which disappears nowhere. Then the one step predictor merges under the Bounded-Lipschitz norm a.s. Under these assumptions we have

\[
\int_Z g(h(x) + z)q(z)dz = \int_Z g(z - (-h(x)))q(z)dz = (g \ast Q)(-h(x))
\]

[26] Lemma C.1] proves that \( \{ f \ast Q | f \in U_b \} \) where \( U_b \) is the bounded uniformly continuous functions, is a dense set in \( U_b \) when \( Q \) has a characteristic function which vanishes nowhere. That is, for any bounded uniformly continuous \( f \), we can express \( f = k \circ (-h) \) where \( k = f \circ (-h)^{-1} \). Note that \( k \) is uniformly continuous since \( f \) and \( h^{-1} \) are uniformly continuous. Then for every \( \epsilon > 0 \) we can find a uniformly continuous (and hence measurable) function \( g \) such that

\[
\|k(-h(x)) - (g \ast Q)(-h(x))\|_\infty < \epsilon
\]

\[
\|f(x) - (g \ast Q)(-h(x))\|_\infty < \epsilon
\]

Now, the uniformly continuous functions are dense in the BL-functions, and thus the predictor will merge under Definition 1.11. For a compact space, this is equivalent to our Definition 1.7, yet in this setup \( \mathcal{X} \) is not necessarily compact. Therefore, our results do not strictly cover this example unless \( \mathcal{X} \) is compact.

In a related setup, in [26] van Handel considers the relationship between uniform observability and the rank condition usually used to define observability for a linear Gaussian system. Let \( \mathcal{X} = \mathbb{R}^d, \mathcal{Y} = \mathbb{R}^m \). Then let

\[
x_{n+1} = Ax_n + B\omega_n \quad y_n = Cx_n + D\nu_n
\]

Where \( A \in \mathbb{R}^{d \times d}, B \in \mathbb{R}^{d \times p}, C \in \mathbb{R}^{m \times d}, D \in \mathbb{R}^{m \times q} \) and \( \{ w_n \}_{n=0}^\infty, \{ v_n \}_{n=0}^\infty \) are iid \( p \) and \( q \) dimensional Gaussian noise processes. The system is observable in linear systems theory when we have the matrix

\[
\mathcal{O}(A,C) = \begin{pmatrix} C \\ CA \\ \ldots \\ CA^{d-1} \end{pmatrix}
\]

is rank \( d \). We now show that this definition is implied by Definition 3.1 but the converse direction only gives Definition 3.1 restricted to bounded Lipschitz functions rather than all continuous and bounded functions.

**Lemma 8.2.** [26] Proposition 3.7] If rank(\( \mathcal{O}(A,C) \)) = \( d \), then the system satisfies Definition 3.1 if we consider bounded Lipschitz functions instead of all continuous and bounded functions.

**Proof.** Conditioned on \( x_0 \), the randomness of \( (Y_{[0,n])} \) is determined by the \( \{ w_n \}_{n=0}^\infty \) and \( \{ v_n \}_{n=0}^\infty \). Furthermore \( Y_k = C(T^k x_0 + \sum_{i=0}^{n-1} T^{n-1-i}Bw_i) + Dv_k \), we can then write the vector of measurements as \( (Y_0, \ldots, Y_n|X_0 = x_0) = (C x_0, CT x_0, \ldots, C T^{n-1} x_0) + \xi \) where \( \xi \) is a \( nm \)-dimensional Gaussian random variable. Since rank(\( \mathcal{O}(A,C) \)) = \( d \), we can find some linear function \( \varphi : \mathbb{R}^{nm} \to \mathbb{R}^d \) such that \( \varphi ((C x_0, CT x_0, \ldots, C T^{n-1} x_0) + \xi) = x_0 + \varphi(\xi) \) where we will say \( \xi' = \varphi(\xi) \) is now an \( d \)-dimensional Gaussian random variable. Then for any \( g \in U_b \) we have

\[
\int_{Y_{n+1}} (g \circ \varphi)(y_{[0,n])} dP^\mu(dy_{[0,n])|x_0) = \int_{\mathbb{R}^d} g(x_0 + \xi')d\xi' = (g \ast \xi')(x_0)
\]
The previous discussion on density under convolution and bounded Lipschitz distance then carries over from the last example.

**Lemma 8.3.** If a system satisfies Definition 3.1 then we have $\text{rank}(O(A, C)) = d$.

*Proof. Assume that $\text{rank}(O(A, C)) \neq d$. Then there exists some vector $x_0^* \in \mathbb{R}^d$ such that $CT^kx_0^* = 0$ for any power of $i$. Then we have $(Y_0, \ldots, Y_n|X_0 = \alpha x_0^*) = \xi$ for any scalar $\alpha$. Define the projection mapping $\phi : \mathbb{R}^d \to \mathbb{R}$ by $\phi(x) = \{\alpha|x = \alpha x_0^* + \beta x_\perp\}$ where $x_0 \perp x_\perp$. Then define a continuous and bounded function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ and further define $f = \tilde{f} \circ \phi$, then $f$ is a continuous and bounded function on $\mathbb{R}^n$. Furthermore, for any $x = \alpha x_0^*$ and any finite $N$ we have that

$$\int_{\mathcal{Y}^N} g(y_{[0,N]})d\mu(y_{[0,N]}|X_0 = \alpha x_0^*) = \int g(\xi)d\xi$$

Which is not dependent on $x$. Therefore, along the subspace $x = \alpha x_0^*$, we have that integrating $g$ over the measurements conditioned on $x_0$ is a constant function, while $f(\alpha x_0^*) = f(\alpha)$ can be a non-constant function, hence the system cannot satisfy Definition 3.1. QED

**9. Conclusion.** We provide general sufficient conditions for the stability of nonlinear filters, under the informative measurements framework introduced by Chigan-sky and Liptser [6], and van Handel [23, 24, 26]. We consider stability in a weak sense, total variation, and relative entropy. The key condition for filter stability is a new notion of observability introduced in this work. This notion is explicit, is relatively easily computed, and is shown to apply to systems not analyzed under the prior studies in the literature. We have in addition presented a unified view of filter stability under various criteria.

**9.1. Connections with robust stochastic control.** A motivation for this work is to use filter stability as a sufficient condition to ensure robustness for partially observed stochastic control problems. We can slightly modify the construction of a POMP given in Section 1 to make it a partially observed Markov decision process (POMDP), by adding into the construction an action space $\mathcal{U}$ and a sequence of functions $\delta = \{\delta_n\}_{n=0}^\infty$ where $\delta_n : \mathcal{Y}^n \to \mathcal{U}$; this is a control policy. Further, we impose that the transition kernel $T$ is now a function of both $x_n$ and $u_n = \delta_n(y_{[0,n]})$. Finally, we introduce some cost function $c : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ and the controller’s objective is to choose a policy $\delta$ to minimize

$$J(\delta, \mu) = \limsup_{N \to \infty} \frac{1}{N} \sum_{n=0}^N E[\mu[c(x_n, \delta_n(y_{[0,n]}))]]$$

this is called an average cost stochastic control problem. A control policy $\delta$ can be realized as a sequence of functions of the filter $\pi_n^\mu$ (sometimes called the belief in the control literature). However, what if a control policy $\delta^\nu$ is designed assuming the true prior is $\nu$, when in actuality the prior is $\mu$. What be said about the cost $J(\delta^\nu, \mu)$?

Results in [15] show that, for infinite horizon discounted cost problems, if an optimal control policy is designed based on a false prior and then applied to the true system, the performance of this policy can approach that of a policy designed with knowledge of the true prior. For a certain class of discounted cost problems, if the false prior converges in total variation to the true prior, then the performance of the incorrectly designed policy will converge to the optimal cost. The authors explicitly show that weak convergence of the prior is not sufficient for robustness.
Based on our filter stability results, one can imagine running a POMDP for a number of iterations in a finite initial training phase \( n \in [0, N] \) where the incurred cost \( c(x_n, u_n) \) is not important. If the filter is stable in total variation \( P^n \) a.s. then an incorrectly initialized filter will merge with the true filter. The filter at the end of the training phase, \( \pi_n \), can then be used as a prior to design a control policy \( \delta^N \), which can then be utilized in the control phase of the problem \( n \in [N + 1, \infty) \) to minimize the expected cost. It is our intention to make the connection more explicit.

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Appendix A. Affine Observation With Non-compact Noise Space. We now show that conditions 8.1 and 8.2 ensure the affine observation channel \( y = a(z)x + b(z) \) is one step observable.

Pick any compact set \( K \subset \mathbb{R} \) with \( Q(K) > 0 \). We define

\[ N(i, k, K) = \left( \begin{array}{c} i \\ k \end{array} \right) \int_K a(z)^i b(z)^{i-k} Q(dz) \]

\[ A = \begin{pmatrix} N(0, 0, K) & N(1, 0, K) & N(2, 0, K) & \cdots & N(n-1, 0, K) \\ 0 & N(1, 1, K) & N(2, 1, K) & \cdots & N(n-1, 1, K) \\ 0 & 0 & N(2, 2, K) & \cdots & N(n-1, 2, K) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & N(n-1, n-1, K) \end{pmatrix} \]

Then if we define our \( S \) mapping as

\[ S_K : \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty \quad S_K(g) \mapsto \int_K g(h(x, z))Q(dz) \]

Then we note \( S \) is invariant on \( [x]_n \). Further, if \( g(y) = \sum_{i=0}^{n-1} \alpha_i y^i \) we will have \( S(g)(x) = \sum_{i=0}^{n-1} \beta_i x^i \) where \( \beta = A\alpha \) where \( \alpha \) and \( \beta \) are the column vectors of the coefficients of the polynomials.

Let the diagonal matrix be of \( A \) be \( \Lambda \) and let \( A_u = A - \Lambda \). Equation 8.2 implies that all the diagonal elements are then non-zero. Then since \( A \) is upper triangular, this implies it is invertible and we can express it’s inverse as \( A^{-1} = \sum_{j=0}^{n-1} (\Lambda^{-1} A_u)^j \Lambda^{-1} \).

We define the norm of a linear operator in the usual fashion, \( \|A\| = \sup_{v \in \mathbb{R}^n} \|Av\|/\|v\| \) where we work with the infinity norm on \( \mathbb{R}^n \). Then we define

\[ \lambda_1 = \sup_{i \in \{0, \ldots, n-1\}} \frac{1}{N(i, i, K)} \quad \lambda_2 = \sup_{i \in \{0, \ldots, n-1\}, k \in \{0, \ldots, i\}} N(i, k, K) \]

by (8.1), we can also define

\[ \max_{i \in \{0, \ldots, n-1\}, k \in \{0, \ldots, i\}} \int_K |a(z)^i b(z)^{i-k}|Q(dz) = V < \infty \]

We then have \( \lambda_1 \leq \frac{1}{DQ(K)} \) by assumption 8.2 and \( \lambda_2 \leq V \left( \frac{n}{\lambda_1} \right) \).

\[ \|A^{-1}\| \leq \sum_{j=0}^{n-1} \|\Lambda^{-1}\|^j \|A^{-1}\|^j \leq \sum_{j=0}^{n-1} (n\lambda_1)^j (n\lambda_2)^j \leq \sum_{j=0}^{n-1} (n)^{2j+1} \left( \frac{V \left( \frac{n}{\lambda_1} \right)}{DQ(K)} \right)^j < \infty \]
Let $\bar{x} = \max(|x| \in \mathcal{X})$. Viewing this we define

$$W = \sum_{j=0}^{n-1} (n)^{2j+1} \left( \frac{2V(n)}{D} \right)^j < \infty \quad C = \sum_{i=0}^{n} (\bar{x} + 1)^i \quad (A.1)$$

Note that $W$ is completely independent of the initial compact set $K \in \mathcal{Z}$ which we choose and is determined by $n$, the degree of the polynomial. Further $C$ is a constant determined only by $n$ and the not by the chosen compact set $K \subset \mathcal{Z}$. With this established, consider any function $f \in C_b(\mathcal{X})$. For any $\epsilon > 0$ we can find a finite degree polynomial $\tilde{f} = \sum_{i=0}^{n-1} \beta_i x^i$, $\beta = (\beta_0, \ldots, \beta_{n-1})$ such that $\|f - \tilde{f}\|_\infty \leq \epsilon$. Now define

$$\epsilon' = \frac{\epsilon}{\|\beta\|WC} \quad (A.2)$$

note that this value is determined only by $f$ and $\epsilon$. Now, by Equation (8.1), we can find some compact set $[-M, M]$ such that $\int_{[-M, M]^c} |a(z)|^2 |b(z)|^{i-k} Q(dz) < \epsilon'$ for all $i \in \{0, \ldots, n - 1\}, k \in \{0, \ldots, i\}$. We can then find some larger set $[-M', M]$ such that $M' > M$ and $Q([-M', M')) > \frac{1}{2}$. Now, define $g = \sum_{i=0}^{n-1} \alpha_i y^i$ such that $\tilde{f}(x) = \int_{-M'}^{M'} g(h(x, z)) dQ(z)$. Further, define $y_{\max} = \max_{x, z \in [-M', M']} h(x, z)$ and $y_{\min} = \min_{x, z \in [-M', M']} h(x, z)$ and force $g$ to be zero outside of $[y_{\min}, y_{\max}]$. This now makes $g$ a bounded function. Note that this will not violate the above equation since for all $x \in \mathcal{X}, z \in [-M', M']$ we have $h(x, z) \in [y_{\min}, y_{\max}]$. Now we have

$$\|\tilde{f} - \int_{\mathcal{Z}} g(h(\cdot, z)) dQ(z)\|_\infty = \|\int_{[-M', M']^c} g(h(\cdot, z)) dQ(z)\|_\infty \leq \sum_{k=0}^{n-1} x^k \|\alpha\| \sum_{i=k}^{n} \left( \frac{i}{k} \right) \int_{[-M', M']^c} |a(z)|^2 |b(z)|^{i-k} Q(dz) \leq \sum_{k=0}^{n-1} x^k \sum_{i=k}^{n} \|\alpha\| \left( \frac{i}{k} \right) \epsilon'$$

where we have applied equation (A.2). By our earlier derivations, we know that $\|\alpha\| \leq \|A^{-1}\|\|\beta\|$ and

$$\|A^{-1}\| \leq \sum_{j=0}^{n-1} (n)^{2j+1} \left( \frac{V(n)}{DQ([-M', M'])} \right)^j \leq \sum_{j=0}^{n-1} (n)^{2j+1} \left( \frac{2V(n)}{D} \right)^j = W < \infty$$

Since we have $Q([-M', M']) > \frac{1}{2}$. This is the key step, as we have an upper bound for $\|\alpha\|$ that depends only on the original polynomial $\tilde{f}$. Then we have

$$\|\tilde{f} - S(g)\| \leq \frac{\epsilon}{C} \sum_{k=0}^{n-1} x^k \sum_{i=k}^{n} \left( \frac{i}{k} \right) \leq \frac{\epsilon}{C} \sum_{i=0}^{n} \sum_{k=0}^{i} \left( \frac{i}{k} \right) x^k 1^{i-k} = \frac{\epsilon}{C} \sum_{i=0}^{\left(\bar{x} + 1\right)^i} = \epsilon$$

**Appendix B. Pushforward Measure.**

**Theorem B.1.** (Change of Variables Theorem for Measures). Let $\mathcal{S}_1, M_1, \mu)$ be a measure space and $S_2, M_2$ a measurable space. Let $F : S_1 \to S_2$ be a measurable function. We define the pushforward measure of $\mu$ under $F$ as $F \mu$ where for any $A \in M_2$, $F \mu(A) = \mu(F^{-1}(A))$. We then have for some function $g$ where $g \in L^1(F \mu)$ we have $\int_{S_2} g dF \mu = \int_{S_1} g \circ F d\mu$. 27
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