Some Classes of Co-Schwarzian Functions and Its Coefficient Inequality that Is Sharp

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Abstract — In our present work, we defined an inequality called Fekete – Szegö Inequality for functions \( f(z) \) in the classes of starlike functions and convex functions along with subclasses of these classes.

Keywords — Bounded analytic functions and concept of subordination, Convex functions, Fekete–Szegö Inequality, Starlike functions.

I. INTRODUCTION

We deal with geometric function theory and Koebe [6] proved that Riemann Mapping theorem is the main pillar of this theory. From this theorem, a conjecture called Bieberbach conjecture was produced. This conjecture was given by L. Bieberbach but finally proved by Louis De Branges[1]. While tackling with this conjecture, an equality arises called Fekete–Szegö Inequality, which was given by M. Fekete and G. Szegö [3]. Till now many researchers have solved this inequality for various classes and subclasses of starlike functions, convex functions, close-to-convex functions and for many other functions. Now, here we establish this inequality for different classes and subclasses of starlike functions and convex functions.

Firstly, we define some fundamental classes.

A consists of the family of analytic functions \( f \) with the normalization \( f(0) = 0, f'(0) = 1 \) and having functions of the type \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \);

\( S \) be the family of functions \( f \) normalized by \( f(0) = 0, f'(0) = 1 \) where \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \); univalent in the open disk: \( E = \{ z \in \mathbb{C} : |z| < 1 \} \) and \( S' \) the class of functions \( f \) for which \( \frac{zf'(z)}{f(z)} \) is subordinate to \( \phi(z) \); this was introduced by Ma and Minda [10].

In this equation, “\( \prec \)” denotes subordination [which states that let \( f(z) \) and \( g(z) \) are two analytic functions, if there exists a Schwarzian function \( w(z) \) (analytic in \( E \)) in such a way that \( |w(z)| < 1 \), \( w(0) = 0 \) and \( f(z) = g(w(z)) ; z \in E \), then the function \( f(z) \) is subordinate to \( g(z) \) and we write it as \( f(z) \prec g(z) \)].

The concept of subordination was given by Lindelöf [7]. Here, \( \phi(z) \) is an analytic function with positive real part on \( E \) which maps the unit disk \( E \) onto a region starlike with respect to 1 as well as symmetric with respect to real axis, satisfying conditions \( \phi(0) = 0 \) and \( \phi'(0) > 0 \) and Schwarzian function is an analytic function of the type \( w(z) = \sum_{n=1}^{\infty} c_n z^n \) with conditions \( w(0) = 0 \) and \( |w(z)| < 1 \).

Miller, S.S., Mocanu, P.T. and Reade, M.O. [9] proved the condition \( |c_1| \leq 1 \), \( |c_2| \leq 1 - |c_1|^2 \) for the above defined bounded analytic functions.

The class defined below is denoted by \( TK[a, b] \) and is a subclass of \( K \), satisfying the condition along with some subclasses:

\[
(1 - \alpha) \frac{f(z)}{z} + \alpha \left[ \frac{zf'(z) + \beta^2 f''(z)}{|f(z)|^2} \right] \prec \phi(z).
\]

II. MAIN RESULT

THEOREM-1:
Let \( f(z) \in TK[a, b] \) and \( \phi(z) = \frac{1 + w(z)}{1 - w(z)} \); \( w(z) \) is a Schwarzian function, then:
After comparing, we get:

$$a_2 = \frac{-2c_4}{1 + \alpha(1 + 4\beta)}$$

$$a_3 = \frac{2[1 + \alpha(1 + 4\beta)]c_2 + [16\alpha(1 + 2\beta) + 2(1 + \alpha(1 + 4\beta)]c_1^2}{(1 + 5\alpha + 18\alpha\beta)(1 + \alpha(1 + 4\beta))^2}$$

Using these values of $a_2$ and $a_3$, we get:

$$a_3 - \mu a_3^2 = \frac{2c_2}{1 + 5\alpha + 18\alpha\beta} + \left(\frac{16\alpha(1 + 2\beta) + 2(1 + \alpha(1 + 4\beta)]^2}{(1 + 5\alpha + 18\alpha\beta)(1 + \alpha(1 + 4\beta))^2} - \frac{4\mu}{(1 + \alpha(1 + 4\beta))^2}\right)c_1^2$$

After applying mode on both sides, we get:
\[ |a_3 - \mu a_2^2| \leq \left( \frac{2}{1 + 5\alpha + 18\beta} \right) |c_2| + \left| \frac{16\mu(1 + 2\beta + 2(1 + \alpha(1 + 4\beta)^2)}{(1 + 5\alpha + 18\beta)(1 + \alpha(1 + 4\beta))^2} - \frac{4\mu}{[1 + \alpha(1 + 4\beta)]^2} \right| |c_1|^2 \]

Using \(|c_2| \leq 1 - |c_1|^2\), we get:

\[ |a_3 - \mu a_2^2| \leq \frac{2}{1 + 5\alpha + 18\beta} + \left| \frac{16\mu(1 + 2\beta + 2(1 + \alpha(1 + 4\beta)^2)}{(1 + 5\alpha + 18\beta)(1 + \alpha(1 + 4\beta))^2} - \frac{4\mu}{[1 + \alpha(1 + 4\beta)]^2} \right| |c_1|^2 \]

Case 1:

\[ \mu \leq \frac{8\alpha(1 + 2\beta) + [1 + \alpha(1 + 4\beta)]^2}{2(1 + 5\alpha + 18\beta)} \]

Then

\[ |a_3 - \mu a_2^2| \leq \frac{2}{1 + 5\alpha + 18\beta} + \left| \frac{16\alpha(1 + 2\beta)}{(1 + 5\alpha + 18\beta)(1 + \alpha(1 + 4\beta))^2} - \frac{4\mu}{[1 + \alpha(1 + 4\beta)]^2} \right| |c_1|^2 \]

Subcase 1 (a):

\[ \mu \leq \frac{4 \alpha(1 + 2\beta)}{1 + 5\alpha + 18\beta} \]

Using \(|c_1| \leq 1\), we get:

\[ |a_3 - \mu a_2^2| \leq \frac{2}{1 + 5\alpha + 18\beta} + \left| \frac{16\alpha(1 + 2\beta) + 4(1 + \alpha(1 + 4\beta)^2)}{(1 + 5\alpha + 18\beta)(1 + \alpha(1 + 4\beta))^2} - \frac{4\mu}{[1 + \alpha(1 + 4\beta)]^2} \right| |c_1|^2 \] (2)

Subcase 1 (b):

\[ \mu \geq \frac{4 \alpha(1 + 2\beta)}{1 + 5\alpha + 18\beta} \]

Then

\[ |a_3 - \mu a_2^2| \leq \frac{2}{1 + 5\alpha + 18\beta} \] (3)

Case 2:

\[ \mu \geq \frac{8\alpha(1 + 2\beta) + [1 + \alpha(1 + 4\beta)]^2}{2(1 + 5\alpha + 18\beta)} \]

Then

\[ |a_3 - \mu a_2^2| \leq \frac{2}{1 + 5\alpha + 18\beta} + \left| \frac{4\mu}{[1 + \alpha(1 + 4\beta)]^2} - \frac{16\alpha(1 + 2\beta) + 4(1 + \alpha(1 + 4\beta)^2)}{(1 + 5\alpha + 18\beta)(1 + \alpha(1 + 4\beta))^2} \right| |c_1|^2 \]

Subcase 2 (a):

\[ \mu \geq \frac{4 \alpha(1 + 2\beta) + [1 + \alpha(1 + 4\beta)]^2}{1 + 5\alpha + 18\beta} \]

Then\[ |a_3 - \mu a_2^2| \leq \frac{2}{1 + 5\alpha + 18\beta} + \left| \frac{16\alpha(1 + 2\beta) + 4(1 + \alpha(1 + 4\beta)^2)}{(1 + 5\alpha + 18\beta)(1 + \alpha(1 + 4\beta))^2} + \frac{4\mu}{[1 + \alpha(1 + 4\beta)]^2} \right| |c_1|^2 \] (4)

Subcase 2 (b):

\[ \mu \leq \frac{4 \alpha(1 + 2\beta) + [1 + \alpha(1 + 4\beta)]^2}{1 + 5\alpha + 18\beta} \]
Then

\[ |a_3 - \mu a_z^2| \leq \frac{2}{1+5\alpha + 18\alpha \beta} \quad (5) \]

Combining (2), (3), (4) and (5), we get the required result. Extremal functions of this inequality are given by:

\[ f(z) = z \left( 1 - \frac{2\alpha(\alpha + 5) + 2\beta[3 + 4\alpha(1 + 2\beta)]}{(1 + \alpha + 4\alpha \beta)(1 + 5\alpha + 18\alpha \beta)} \right)^{-\frac{1+5\alpha + 18\alpha \beta}{\alpha(\alpha + 2\beta)(3 + 4\alpha(1 + 2\beta))}} \]

and

\[ f(z) = z \left( 1 + 2z^2 \right)^{\frac{1}{1+5\alpha + 18\alpha \beta}}. \]

**COROLLARY-2:**

TK [1,0] = K, as by substituting \( \alpha = 1 \) and \( \beta = 0 \) the result becomes:

\[ |a_3 - \mu a_z^2| \leq \begin{cases} 
1 - \mu, & \text{if } \mu \leq \frac{2}{3}; \\
\frac{1}{3}, & \text{if } \frac{2}{3} \leq \mu \leq \frac{4}{3}; \\
\mu - 1, & \text{if } \mu \geq \frac{4}{3}, 
\end{cases} \]

which is the required result for the class \( K \) given by Keogh and Merkes [5].

**THEOREM3:**

Let \( f(z) \in TK[\alpha, \beta, \delta] \) and \( \phi(z) = \frac{1+w(z)}{1-w(z)} ; w(z) \) is a Schwarzian function, then:

\[ |a_3 - \mu a_z^2| \leq \begin{cases} 
\frac{16\alpha(1+2\beta)\delta^2 + 2\beta^2[1 + \alpha(1+4\beta)]^2}{1 + 5\alpha + 18\alpha \beta} - \frac{4\mu \delta^2}{[1 + \alpha(1+4\beta)]^2}; \\
\mu \leq \frac{8\alpha(1+2\beta)\delta^2 + (\delta^2 - \delta)(1 + \alpha(1+4\beta))^2}{1 + 5\alpha + 18\alpha \beta}; \\
\frac{2\delta}{1 + 5\alpha + 18\alpha \beta}; \\
\mu \leq \frac{8\alpha(1+2\beta)\delta^2 + (\delta^2 - \delta)(1 + \alpha(1+4\beta))^2}{1 + 5\alpha + 18\alpha \beta}; \\
\mu \geq \frac{8\alpha(1+2\beta)\delta^2 + (\delta^2 - \delta)(1 + \alpha(1+4\beta))^2}{1 + 5\alpha + 18\alpha \beta}; \\
\mu \geq \frac{16\alpha(1+2\beta)\delta^2 + 2\beta^2[1 + \alpha(1+4\beta)]^2}{1 + 5\alpha + 18\alpha \beta}. 
\end{cases} \]

**PROOF:**

By definition of \( TK[\alpha, \beta, \delta] \),

\[ \frac{(1-\alpha)f(z)}{z} + \alpha \frac{z f'(z) + \beta z^2 f''(z)}{(f(z))^2} = \left( \frac{1+w(z)}{1-w(z)} \right)^\delta \quad (6) \]

where

\[ w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \]

\[ f(z) = z + a_2 z^2 + a_3 z^3 + \]

\[ f'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 \]

\[ f''(z) = 2a_2 + 6a_3 z + 12a_4 z^2 \]

Now by putting all these values in (6), we get:

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Case Subcase 
1 \[1+\alpha(1+4\beta)] a_2 z + [(1 + \alpha (5 + 18\beta)) a_3 - 4\alpha(1 + 2\beta)a_2^2] z^2 + = l + 2\delta c_1 z + 2(\delta c_2 + \delta^2 c_1^2)z^2 +

By comparing, we get:

\[
a_2 = \frac{2\delta c_1}{1+\alpha(1+4\beta)}
\]

and

\[
a_3 = \frac{2[1 + \alpha(1 + 4\beta)]^2 \delta c_2 + [16\alpha(1 + 2\beta) + 2[1 + \alpha(1 + 4\beta)]^2] \delta^2 c_1^2}{\{1 + \alpha(5 + 18\beta)\}\{1 + \alpha(1 + 4\beta)\}^2}
\]

Using these values of \(a_2\) and \(a_3\), we get:

\[
a_3 - \mu a_2^2 = \frac{2\delta c_2}{1+\alpha(5 + 18\beta)} + \left(\frac{16\alpha(1 + 2\beta) + 2[1 + \alpha(1 + 4\beta)]^2}{\{1 + \alpha(5 + 18\beta)\}\{1 + \alpha(1 + 4\beta)\}^2}\right) - \frac{4\mu}{1+\alpha(1+4\beta)^2} \delta^2 c_1^2
\]

After applying mode on both sides, we get:

\[
|a_3 - \mu a_2^2| \leq \left(\frac{2\delta}{1+\alpha(5 + 18\beta)}\right) |c_2| + \left(\frac{16\alpha(1 + 2\beta) + 2[1 + \alpha(1 + 4\beta)]^2}{\{1 + \alpha(5 + 18\beta)\}\{1 + \alpha(1 + 4\beta)\}^2}\right) - \frac{4\mu}{1+\alpha(1+4\beta)^2} \delta^2 |c_1| \]

Using \(|c_2| \leq 1 - |c_1|^2\), we get:

\[
|a_3 - \mu a_2^2| \leq \frac{2\delta}{1+\alpha(5 + 18\beta)} + \left(\frac{16\alpha(1 + 2\beta) + 2[1 + \alpha(1 + 4\beta)]^2}{\{1 + \alpha(5 + 18\beta)\}\{1 + \alpha(1 + 4\beta)\}^2}\right) - \frac{4\mu}{1+\alpha(1+4\beta)^2} \delta^2 |c_1| \]

**Case 1:**
If \(\mu \leq \frac{8\alpha(1+2\beta)+(1+\alpha(1+4\beta))^2}{2(1+\alpha(5+18\beta))}\), then,

\[
|a_3 - \mu a_2^2| \leq \frac{2\delta}{1+\alpha(5 + 18\beta)} + \left(\frac{16\alpha(1 + 2\beta)\delta^2 + 2(\delta^2 - \delta)(1 + \alpha(1 + 4\beta))}{\{1 + \alpha(5 + 18\beta)\}\{1 + \alpha(1 + 4\beta)\}^2}\right) - \frac{4\mu\delta^2}{1+\alpha(1+4\beta)^2} |c_1|^2
\]

**Subcase 1 (a):**
When

\[
\mu \leq \frac{8\alpha(1+2\beta)\delta^2 + (\delta^2 - \delta)(1 + \alpha(1 + 4\beta))}{2\delta^2(1+\alpha(5+18\beta))}
\]

Then, by using \(|c_1| \leq 1\), we get:

\[
|a_3 - \mu a_2^2| \leq \frac{2\delta^2(1+\alpha(1 + 4\beta))^2 + 16\alpha(1 + 2\beta)\delta^2}{\{1 + \alpha(5 + 18\beta)\}\{1 + \alpha(1 + 4\beta)\}^2} - \frac{4\mu\delta^2}{1+\alpha(1+4\beta)^2}
\]

(7)

**Subcase 1 (b):**
When \(\mu \geq \frac{8\alpha(1+2\beta)\delta^2 + (\delta^2 - \delta)(1 + \alpha(1 + 4\beta))}{2\delta^2(1+\alpha(5+18\beta))}\), then,

\[
|a_3 - \mu a_2^2| \leq \frac{2\delta}{1+\alpha(5 + 18\beta)} \]

(8)

**Case 2:**
If \(\mu \geq \frac{8\alpha(1+2\beta)+(1+\alpha(1+4\beta))^2}{2(1+\alpha(5+18\beta))}\), then,

\[
|a_3 - \mu a_2^2| \leq \frac{2\delta}{1+\alpha(5 + 18\beta)} + \left(\frac{4\mu\delta^2}{\{1+\alpha(1+4\beta)^2\}} - \frac{16\alpha(1 + 2\beta)\delta^2 + 2(\delta^2 - \delta)(1 + \alpha(1 + 4\beta))}{\{1 + \alpha(5 + 18\beta)\}\{1 + \alpha(1 + 4\beta)\}^2}\right) |c_1|^2
\]
Subcase-2 (a):
When \( \mu \geq \frac{8a(1+2\beta)\delta^2+(\delta^2+\delta)(1+a(1+4\beta))^2}{2\delta^2(1+a(5+18\beta))} \), then,
\[
|a_3 - \mu a_2^2 | \leq \frac{2\delta^2(1+a(1+4\beta))^2+16a(1+2\beta)\delta^2}{1+\alpha(5+18\beta)(1+a(1+4\beta))^2} + \frac{4\mu\delta^2}{(1+\alpha(1+4\beta))^2}
\]  
(9)

Subcase – 2 (b):
When \( \mu \leq \frac{8a(1+2\beta)\delta^2+(\delta^2+\delta)(1+a(1+4\beta))^2}{2\delta^2(1+a(5+18\beta))} \), then,
\[
|a_3 - \mu a_2^2 | \leq \frac{2\delta}{1+\alpha(5+18\beta)}
\]  
(10)

Combining (7), (8), (9) and (10), we get the required result.

EXTREMALS:
The result is sharp for extremal functions:
\[
f(z) = z \left( 1 - \frac{2\delta\alpha[(\alpha + 5) + 2\beta[3 + 4\alpha(1 + 2\beta)]]}{[1 + \alpha(1 + 4\beta)][1 + (5 + 18\beta)\alpha]} \right)^{-\frac{1+5\alpha+18\alpha\beta}{\alpha(5+\alpha)\beta^{3}+4\alpha(1+2\beta)]}}
\]
and \( f(z) = z(1 + 2\delta z^2)^{1+a(5+18\beta)} \).

COROLLARY-4:
\( TK[\alpha, \beta, 1] = TK[\alpha, \beta] \), as by substituting \( \delta = 1 \), the result becomes:
\[
|a_3 - \mu a_2^2 | \leq \left\{ \begin{array}{ll}
\frac{16a(1+2\beta)+2[1+a(1+4\beta)]^2}{[1+a(1+4\beta)]^2[1+(5+18\beta)a]} & \mu \leq \frac{4\alpha(1+2\beta)}{1+a(5+18\beta)}; \\
\frac{4\alpha(1+2\beta)}{1+a(5+18\beta)} & \frac{4\alpha(1+2\beta)}{1+a(5+18\beta)} \leq \mu \leq \frac{4\alpha(1+2\beta)+(1+a(1+4\beta))^2}{1+a(5+18\beta)}; \\
\frac{4\mu}{1+a(1+4\beta)^2} & \frac{4\alpha(1+2\beta)+(1+a(1+4\beta))^2}{1+a(5+18\beta)} \leq \mu \leq \frac{4\alpha(1+2\beta)+(1+a(1+4\beta))^2}{1+a(5+18\beta)}; \\
\mu & \geq \frac{4\alpha(1+2\beta)+(1+a(1+4\beta))^2}{1+a(5+18\beta)}.
\end{array} \right.
\]
which is the required result for the class \( TK[\alpha, \beta] \).

COROLLARY-5:
\( TK[1, 0, 1] = K \), as by substituting \( \alpha = 1, \beta = 0 \) and \( \delta = 1 \), the result becomes:
\[
|a_3 - \mu a_2^2 | \leq \left\{ \begin{array}{ll}
1 - \mu, & \text{if } \mu \leq \frac{2}{3}; \\
\frac{1}{3}, & \frac{2}{3} \leq \mu \leq \frac{4}{3}; \\
\mu - 1, & \text{if } \mu \geq \frac{4}{3}.
\end{array} \right.
\]
which is the required result for the class \( K \).

THEOREM6:
Let \( f(z) \in TK[\alpha, \beta, A, B] \) and \( \phi(z) = \frac{1+Aw(z)}{1+Bw(z)} \); \( w(z) \) is a Schwarzian function, then:
\[ |a_3 - \mu a_2^2| \leq \frac{4\alpha(1+2\beta)(A-B)^2 - B(A-B)[1 + \alpha(1+4\beta)]^2}{(1 + 5\alpha + 18\alpha^2)(1 + \alpha(1+4\beta))^2} - \frac{\mu(A-B)^2}{[1 + \alpha(1+4\beta)]^2}; \]

\[ \mu \leq \frac{4\alpha(1+2\beta)(A-B) - (B+1)[1 + \alpha(1+4\beta)]^2}{(A-B)(1 + 5\alpha + 18\alpha^2)}; \]

\[ \frac{A - B}{1 + 5\alpha + 18\alpha^2} \leq \frac{4\alpha(1+2\beta)(A-B) + (B+1)\{1 + \alpha(1+4\beta)]^2}{(A-B)(1 + 5\alpha + 18\alpha^2)}; \]

\[ \frac{\mu(A-B)^2}{[1 + \alpha(1+4\beta)]^2} \leq \frac{4\alpha(1+2\beta)(A-B)^2 - B(A-B)[1 + \alpha(1+4\beta)]^2}{(A-B)(1 + 5\alpha + 18\alpha^2)}; \]

\[ \mu \geq \frac{4\alpha(1+2\beta)(A-B) + (B+1)\{1 + \alpha(1+4\beta)]^2}{(A-B)(1 + 5\alpha + 18\alpha^2)}. \]

**PROOF:**

By definition of TK \([\alpha, \beta, A, B]\),

\[ (1-\alpha)\frac{f''(z)}{f'(z)} + \alpha\frac{\{f''(z) + \beta z^2 f''(z)\}'}{f'(z)} = \frac{1+\alpha w(z)}{1+\beta w(z)} \] (11)

where

\[ w(z) = c_1z + c_2z^2 + c_3z^3 \]

\[ f(z) = z + a_2z^2 + a_3z^3 \]

\[ f'(z) = 1 + 2a_2z + 3a_3z^2 + 4a_4z^3 \]

\[ f''(z) = 2a_2 + 6a_3z + 12a_4z^2 \]

Now, by putting all these values in (11), we get:

\[ (1-\alpha)(1+a_2z + a_3z^2) - \alpha\left(\frac{\{x+2a_2z^2+3a_3z^3+\beta 2a_2z^2+\beta 6a_3z^3+\beta 12a_4z^4+\ldots\}}{1+2a_2z+3a_3z^2+\ldots}\right) = \frac{1+\alpha c_1z + c_2z^2 + \ldots}{1+\beta (c_1z+c_2z^2+\ldots)} \]

By expanding the series, we get:

\[ 1 + [1+\alpha(1+4\beta)] a_2 z + [(1 + \alpha(5 + 18\beta)) a_3 - 4a(1+2\beta)a_2^2]z^2 = 1 + (A-B)c_1 z + [(A-B)c_2 - B(A-B)c_1^2]z^2 \]

By comparing, we get:

\[ a_2 = \frac{(A-B)c_1}{1+\alpha(1+4\beta)} \]

and

\[ a_3 = \frac{(A-B)[1 + \alpha(1+4\beta)]^2 c_2 + (A-B)[4\alpha(1+2\beta)(A-B) - B(1 + \alpha(1+4\beta)]}{\{1 + \alpha(5 + 18\beta)](1 + \alpha(1+4\beta)]} \]

Using these values of \(a_2\) and \(a_3\), we get:

\[ a_3 - \mu a_2^2 = \frac{(A-B)c_2}{1+\alpha(5+18\beta)} + \left(\frac{4\alpha(1+2\beta)(A-B) - B[1+\alpha(1+4\beta)]^2}{1+\alpha(5+18\beta)](1 + \alpha(1+4\beta)]^2} \frac{(A-B)^2 \mu}{[1+\alpha(1+4\beta)]^2} \right) c_1^2 \]

After applying mode on both sides, we get:

\[ |a_3 - \mu a_2^2| \leq \frac{(A-B)}{1+\alpha(5+18\beta)} |c_2| + \left|\frac{4\alpha(1+2\beta)(A-B) - B[1+\alpha(1+4\beta)]^2}{1+\alpha(5+18\beta)](1 + \alpha(1+4\beta)]^2} \frac{(A-B)^2 \mu}{[1+\alpha(1+4\beta)]^2} \right| |c_1|^2 \]
Using $|c_2| \leq 1 - |c_1|^2$, we get:

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)}{1+\alpha(5+18\beta)} + \left\{\frac{4\alpha(1+2\beta)(A-B)-(1+\alpha(14\beta))^2}{(1+\alpha(5+18\beta))(1+\alpha(14\beta))^2} - \frac{(A-B)^2\mu}{(1+\alpha(14\beta))^2}\right\}|c_1|^2$$

**Case 1:**
If $\mu \leq \frac{4\alpha(1+2\beta)(A-B)-(1+\alpha(14\beta))^2}{(A-B)(1+\alpha(5+18\beta))}$, then,

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)}{1+\alpha(5+18\beta)} + \left\{\frac{4\alpha(1+2\beta)(A-B)-(1+\alpha(14\beta))^2}{(1+\alpha(5+18\beta))(1+\alpha(14\beta))^2} - \frac{(A-B)^2\mu}{(1+\alpha(14\beta))^2}\right\}|c_1|^2$$

**Subcase – 1 (a):**
When $\mu \leq \frac{4\alpha(1+2\beta)(A-B)-(1+\alpha(14\beta))^2}{(A-B)(1+\alpha(5+18\beta))}$

By using $|c_1| \leq 1$, we get:

$$|a_3 - \mu a_2^2| \leq \frac{4\alpha(1+2\beta)(A-B)-(1+\alpha(14\beta))^2}{(1+\alpha(5+18\beta))(1+\alpha(14\beta))^2} - \frac{(A-B)^2\mu}{(1+\alpha(14\beta))^2} \tag{12}$$

**Subcase – 1 (b):**
When $\mu \geq \frac{4\alpha(1+2\beta)(A-B)-(1+\alpha(14\beta))^2}{(A-B)(1+\alpha(5+18\beta))}$

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)}{1+\alpha(5+18\beta)} \tag{13}$$

**Case – 2:**
If $\mu \geq \frac{4\alpha(1+2\beta)(A-B)-(1+\alpha(14\beta))^2}{(A-B)(1+\alpha(5+18\beta))}$, then,

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)}{1+\alpha(5+18\beta)} + \left\{\frac{(A-B)^2\mu}{1+\alpha(14\beta)^2} - \frac{4\alpha(1+2\beta)(A-B)+(1+\alpha(14\beta))^2}{(1+\alpha(5+18\beta))(1+\alpha(14\beta))^2}\right\}|c_1|^2$$

**Subcase-2 (a):**
When $\mu \geq \frac{4\alpha(1+2\beta)(A-B)+(1+\alpha(14\beta))^2}{(A-B)(1+\alpha(5+18\beta))}$

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)^2\mu}{1+\alpha(14\beta)^2} - \frac{4\alpha(1+2\beta)(A-B)-(1+\alpha(14\beta))^2}{(1+\alpha(5+18\beta))(1+\alpha(14\beta))^2} \tag{14}$$

**Subcase – 2 (b):**
When $\mu \leq \frac{4\alpha(1+2\beta)(A-B)+(1+\alpha(14\beta))^2}{(A-B)(1+\alpha(5+18\beta))}$, then,

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)}{1+\alpha(5+18\beta)} \tag{15}$$

Combining (12), (13), (14) and (15), we get the required result.

**EXTREMALS:**

Extremal functions of this inequality is given by:

$$f(z) = \left(1 + \frac{A(1-3\alpha) + B[1+2\alpha^2+7\alpha] + 2\beta B(1+7\alpha+8\alpha(1+2\beta))}{[1+\alpha(5+18\beta)][1+\alpha(1+4\beta)]}\right)z$$

and $f(z) = z [1 + (A - B)z^2]^{1\alpha(5+18\beta)}$.

**COROLLARY-7:**

$TK[\alpha, \beta, 1, -1] = TK[\alpha, \beta]$, as by substituting A =1 and B = -1, the result becomes:
\[ |a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{16\alpha(1+2\beta)+2\left[1+\alpha(1+4\beta)\right]^2}{1+5\alpha+18\alpha\beta} + \frac{4\mu}{1+\alpha(1+4\beta)}; \\
\frac{4\alpha(1+2\beta)}{1+5\alpha+18\alpha\beta}; \\
\frac{2}{1+5\alpha+18\alpha\beta}; \\
\frac{4\mu}{1+\alpha(1+4\beta)}; \\
\frac{16\alpha(1+2\beta) + 2\left[1+\alpha(1+4\beta)\right]^2(1+5\alpha+18\alpha\beta)}{1+\alpha(1+4\beta)}; \\
\mu \leq \frac{4\alpha(1+2\beta)}{1+5\alpha+18\alpha\beta}; \\
\mu \leq \frac{4\alpha(1+2\beta) + 1+\alpha(1+4\beta)}{2(1+5\alpha+18\alpha\beta)}; \\
\mu \leq \frac{4\alpha(1+2\beta) + 1+\alpha(1+4\beta)}{1+5\alpha+18\alpha\beta}. 
\end{cases} \]

which is the required result for the class \(TK[a, \beta]\).

**COROLLARY-8:**
\(TK [1,0,1, -1] = K\), as by substituting \(a = 1, \beta = 0, A = 1\) and \(B = -1\), the result becomes:
\[ |a_3 - \mu a_2^2| \leq \begin{cases} 
1 - \mu, \text{ if } \mu \leq \frac{2}{3}; \\
\frac{1}{3}, \text{ if } \mu^2 \leq \frac{4}{3}; \\
\mu - 1, \text{ if } \mu \geq \frac{4}{3}. 
\end{cases} \]

which is the required result for the class \(K\).

**THEOREM9:**
If \(f(z) \in TK[a, \beta, A, B, \delta]\) and \(\phi(z) = \left(\frac{1+Aw(z)}{1+Bw(z)}\right)^{\delta}\), then:
\[ |a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{\delta(A-B)}{1+5\alpha+18\alpha\beta}; \\
\frac{\delta(A-B)}{1+\alpha(1+4\beta)}; \\
\frac{\delta(A-B)}{1+5\alpha+18\alpha\beta}; \\
\frac{\delta(A-B)}{1+\alpha(1+4\beta)}; \\
\frac{\delta(A-B)}{1+5\alpha+18\alpha\beta}; \\
\frac{\delta(A-B)}{1+\alpha(1+4\beta)}; \\
\frac{\delta(A-B)}{1+5\alpha+18\alpha\beta}. 
\end{cases} \]

**PROOF:**
By definition of \(TK[a, \beta, A, B, \delta]\),
\[ (1-\alpha)\frac{f(z)}{z} + \frac{zf'(z) + \alpha zf''(z)}{f(z)^2} = \left(\frac{1+Aw(z)}{1+Bw(z)}\right)^{\delta} \]  \hspace{1cm} (16)

where
\[ w(z) = c_1 z + c_2 z^2 + c_3 z^3 \]
\[ f(z) = a_2 z^2 + a_3 z^3 \]
\[ f'(z) = 1 + 2 a_2 z + 3 a_3 z^2 + 4 a_4 z^3 \]
\[ f''(z) = 2 a_2 + 6 a_3 z + 12 a_4 z^2 \]

Now, by putting all these values in (16), we get:
\[(1 - \alpha) (1 + a_2 z + a_3 z^2 + \ldots) + \alpha \left[ \frac{z + 2a_2 z^2 + 3a_3 z^3 + \beta_2 a_2 z^2 + \beta_5 a_3 z^3 + \beta_1 12 a_4 z^4 + \ldots}{1 + 2a_2 z + 3a_3 z^2 + \ldots} \right]_t \]

By expanding the series, we get:

\[= \left( 1 + A(c_1 z + c_2 z^2 + \ldots) \right) \delta \]

\[= \left( 1 + B(c_1 z + c_2 z^2 + \ldots) \right) \delta \]

By expanding the series, we get:

\[= 1 + \delta (A - B) c_1 z + \delta (A - B) c_2 + \frac{\delta^2}{2} (A^2 + B^2) c_1^2 - \frac{\delta}{2} (A^2 - B^2) c_1^2 \]z^2.

By comparing, we get:

\[a_2 = \frac{\delta (A - B) c_1}{1 + \alpha (1 + 4\beta)}\]

and

\[\delta (A - B) [1 + \alpha (1 + 4\beta)]^2 c_2 + \frac{\delta}{2} (A^2 - B^2) + \frac{\delta^2}{2} (A^2 + B^2) c_1^2 = \frac{\delta (A - B) [1 + \alpha (1 + 4\beta)]^2 c_2 + \frac{\delta}{2} (A^2 - B^2) + \frac{\delta^2}{2} (A^2 + B^2) c_1^2}{1 + \alpha (1 + 4\beta)}\]

Using these values of \(a_2\) and \(a_3\), we get:

\[a_3 - \mu a_2^2 = \frac{\delta (A - B) c_2}{1 + \alpha (5 + 18\beta)} + \frac{\delta (A - B) [4\alpha \delta (1 + 2\beta) (A - B) + \frac{\delta}{2} (A - B) - (A + B)] (1 + \alpha (1 + 4\beta)^2)}{1 + \alpha (5 + 18\beta)(1 + \alpha (1 + 4\beta)^2)}\]

After applying mode on both sides, we get:

\[|a_3 - \mu a_2^2| \leq \left( \frac{\delta (A - B)}{1 + \alpha (5 + 18\beta)} \right) |c_2| + \frac{\delta (A - B) [4\alpha \delta (1 + 2\beta) (A - B) + \frac{\delta}{2} (A - B) - (A + B)] (1 + \alpha (1 + 4\beta)^2)}{1 + \alpha (5 + 18\beta)(1 + \alpha (1 + 4\beta)^2)} |c_1|^2\]

Using \(|c_2| \leq 1 - |c_1|^2\), we get:

\[|a_3 - \mu a_2^2| \leq \left( \frac{\delta (A - B)}{1 + \alpha (5 + 18\beta)} \right) + \frac{\delta (A - B) [4\alpha \delta (1 + 2\beta) (A - B) + \frac{\delta}{2} (A - B) - (A + B)] (1 + \alpha (1 + 4\beta)^2)}{1 + \alpha (5 + 18\beta)(1 + \alpha (1 + 4\beta)^2)} |c_1|^2\]

**Case 1:**

If \(\mu \leq \frac{4\alpha \delta (1 + 2\beta) (A - B) + \frac{\delta}{2} (A - B) - (A + B) - 1}{\delta (A - B) (1 + \alpha (5 + 18\beta))}\), then,

\[|a_3 - \mu a_2^2| \leq \left( \frac{\delta (A - B)}{1 + \alpha (5 + 18\beta)} \right) + \left( \frac{\delta (A - B) [4\alpha \delta (1 + 2\beta) (A - B) + \frac{\delta}{2} (A - B) - (A + B) - 1]}{1 + \alpha (5 + 18\beta)(1 + \alpha (1 + 4\beta)^2)} \right) |c_1|^2\]

**Subcase 1 (a):**

When \(\mu \leq \frac{4\alpha \delta (1 + 2\beta) (A - B) + \frac{\delta}{2} (A - B) - (A + B) - 1}{\delta (A - B) (1 + \alpha (5 + 18\beta))}\), then, by using \(|c_1| \leq 1\), we get:

\[|a_3 - \mu a_2^2| \leq \left( \frac{\delta (A - B) [4\alpha \delta (1 + 2\beta) (A - B) + \frac{\delta}{2} (A - B) - (A + B) - 1]}{1 + \alpha (5 + 18\beta)(1 + \alpha (1 + 4\beta)^2)} \right) - \frac{\delta^2 (A - B)^2 \mu}{(1 + \alpha (1 + 4\beta)^2)}\]

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Subcase – 1 (b):
When \( \mu \geq \frac{4a\delta(1+2\beta)(A-B)+\delta(A-B)(\frac{1}{2}\{A-B\} - \frac{1}{2}(A+B)+1\{1+a(1+4\beta)\})^2}{\delta(A-B)(1+a(5+18\beta))} \), then,

\[
|a_3 - \mu a_2^2| \leq \frac{\delta(A-B)}{1+a(5+18\beta)}
\]  

(18)

Case – 2:
If \( \mu \geq \frac{4a\delta(1+2\beta)(A-B)+\delta(A-B)(\frac{1}{2}\{A-B\} - \frac{1}{2}(A+B)+1\{1+a(1+4\beta)\})^2}{\delta(A-B)(1+a(5+18\beta))} \), then,

\[
|a_3 - \mu a_2^2| \leq \frac{\delta(A-B)}{1+a(5+18\beta)}\left(\frac{\delta^2(A-B)^2\mu}{(1+a(1+4\beta))^2} - \frac{\delta(A-B)[4a\delta(1+2\beta)(A-B)+\delta(A-B)(\frac{1}{2}\{A-B\} - \frac{1}{2}(A+B)+1\{1+a(1+4\beta)\})^2]}{(1+a(5+18\beta))(1+a(1+4\beta))^2}\right) |c_1|^2
\]  

(19)

Subcase 2 (a):
When \( \mu \geq \frac{4a\delta(1+2\beta)(A-B)+\delta(A-B)(\frac{1}{2}\{A-B\} - \frac{1}{2}(A+B)+1\{1+a(1+4\beta)\})^2}{\delta(A-B)(1+a(5+18\beta))} \), then,

\[
|a_3 - \mu a_2^2| \leq \frac{\delta^2(A-B)^2\mu}{(1+a(1+4\beta))^2} - \frac{\delta(A-B)[4a\delta(1+2\beta)(A-B)+\delta(A-B)(\frac{1}{2}\{A-B\} - \frac{1}{2}(A+B)+1\{1+a(1+4\beta)\})^2]}{(1+a(5+18\beta))(1+a(1+4\beta))^2}
\]  

(20)

By combining (17), (18), (19) and (20), we get the required result.

FOR EXTREMALS:
Extremal functions of this inequality is given by:

\[
f(z) = \left(1 + \frac{\delta(A-B)(1-3a+2a\beta)}{\delta(A-B)(1+a(1+4\beta))^2}\right)^{-1} \frac{\delta(A-B)(1-3a+2a\beta)}{\delta(A-B)(1+a(1+4\beta))^2}\[1 + \frac{\delta(A-B)(1-3a+2a\beta)}{\delta(A-B)(1+a(1+4\beta))^2}\]  

and \( f(z) = z [1 + \delta(A-B)z^2]^{-1} \).  

COROLLARY-10:
\( TK[\alpha, \beta, 1, -1, 1] = TK[\alpha, \beta] \), as by substituting \( A = 1, B = -1 \) and \( \delta = 1 \), the result becomes:

\[
|a_3 - \mu a_2^2| \leq \left\{ \begin{array}{c} \frac{4\alpha(1+2\beta)-\delta(A-B)(\frac{1}{2}\{A-B\} - \frac{1}{2}(A+B)+1\{1+a(1+4\beta)\})^2}{1+a(5+18\beta)}; & \mu \leq \frac{4\alpha(1+2\beta)}{1+5a+18a\beta} \; \text{and} \; \frac{4\alpha(1+2\beta)-\delta(A-B)(\frac{1}{2}\{A-B\} - \frac{1}{2}(A+B)+1\{1+a(1+4\beta)\})^2}{1+a(5+18\beta)}; \\
\frac{4\alpha(1+2\beta)}{1+5a+18a\beta}; & \mu \leq \frac{4\alpha(1+2\beta)}{1+5a+18a\beta} \; \text{and} \; \frac{4\alpha(1+2\beta)}{1+5a+18a\beta} \\
\end{array} \right.
\]

\( \mu \geq \frac{4\alpha(1+2\beta)}{1+5a+18a\beta} \),

which is the required result for the class \( TK[\alpha, \beta] \).

COROLLARY-11:
\( TK[\alpha, \beta, 1, -1, \delta] = TK[\alpha, \beta, \delta] \), as by substituting \( A = 1 \) and \( B = -1 \), the result becomes:
which is the required result for the class \( TK[\alpha, \beta, \delta] \).

**COROLLARY-12:**

\( TK[\alpha, \beta, A, B, 1] = TK[\alpha, \beta, A, B] \), as by substituting \( \delta = 1 \), the result becomes:

\[
|a_3 - \mu a_2^2| \leq \begin{cases}
16a(1+2\beta)(A-B)^2 - B(A-B)[1+a(1+4\beta)]^2 & \mu \leq \frac{4a(1+2\beta)(A-B)^2 - B(A-B)[1+a(1+4\beta)]^2}{(A-B)(1+5a+18a\beta)}; \\
\frac{1}{5a+18a\beta} & \mu \leq \frac{4a(1+2\beta)(A-B)^2 - B(A-B)[1+a(1+4\beta)]^2}{(A-B)(1+5a+18a\beta)}; \\
\frac{4a(1+2\beta)(A-B)^2 - B(A-B)[1+a(1+4\beta)]^2}{(A-B)(1+5a+18a\beta)} & \mu \geq \frac{4a(1+2\beta)(A-B)^2 - B(A-B)[1+a(1+4\beta)]^2}{(A-B)(1+5a+18a\beta)}.
\end{cases}
\]

which is the required result for the class \( TK[\alpha, \beta, A, B] \).

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