A parametrised family of Mordell curves

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Abstract

An elliptic curve defined by an equation of the type \( y^2 = x^3 + d \) is called a Mordell curve. We obtain a parametrised family of Mordell curves whose rank, in general, is at least three, and whose torsion group is \( \mathbb{Z}/3\mathbb{Z} \).

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Ever since Fermat’s assertion that the only solution in positive integers of the equation \( y^2 = x^3 - 2 \) is \((x, y) = (3, 5)\) [9], the diophantine equation, (1)

\[
y^2 = x^3 + d,\]

has been subjected to extensive investigations [1, 2, 3, 4, 6, 7, 8, 10, 11, Chapter 26, pp. 238–254, 12, 16]. Eq. (1) has now been solved for all integer values of \( d \) with \(|d| \leq 10^7\) [1].

The elliptic curve represented by Eq. (1) is known as a Mordell curve. Further, for various integer values of \( d \), we know the structure of the torsion subgroup of the group of rational points on the Mordell curve [5, Theorem 5.3, p. 134]. It is noteworthy that whenever the integer \( d \) is a nonzero perfect square different from 1, the torsion subgroup is necessarily \( \mathbb{Z}/3\mathbb{Z} \).

In this paper, we construct a parametrised family of Mordell curves defined by an equation of the type,

(2)

\[
y^2 = x^3 + k^2.\]
We shall show that the rank of the elliptic curves belonging to this family is, in general, at least three. The torsion subgroup of all curves defined by the equation (2) is \( \mathbb{Z}/3\mathbb{Z} \) with \((0, k)\) being a torsion point of order 3. Despite the considerable literature related to Eq. (1), it appears that such a parametrised family of curves has not been obtained earlier.

We will first solve the system of diophantine equations,

\[
\begin{align*}
(3) & \quad v_1^2 = u_1^3 + k^2, \\
(4) & \quad v_2^2 = u_2^3 + k^2, \\
(5) & \quad v_3^2 = u_3^3 + k^2.
\end{align*}
\]

On writing,

\[
\begin{align*}
(6) & \quad u_1 = am, \quad u_2 = bm, \quad u_3 = cm,
\end{align*}
\]

and eliminating \( m \) first between Eq. (3) and Eq. (4), and then between Eq. (3) and Eq. (5), we get the following two equations:

\[
\begin{align*}
(7) & \quad b^3(v_1^2 - k^2) = a^3(v_2^2 - k^2), \\
(8) & \quad c^3(v_1^2 - k^2) = a^3(v_3^2 - k^2).
\end{align*}
\]

To solve equations (7) and (8), we write,

\[
\begin{align*}
v_1 &= w_1t + k, \quad v_2 = w_2t + k, \quad v_3 = w_3t + k,
\end{align*}
\]

when each of the two equations (7) and (8) can be readily solved to get a nonzero solution for \( t \). Equating these two values of \( t \), we get the condition,

\[
\begin{align*}
(9) & \quad (b^3w_1 - a^3w_2)(c^3w_1^2 - a^3w_2^2) = (c^3w_1 - a^3w_3)(b^3w_1^2 - a^3w_3^2).
\end{align*}
\]

Now Eq. (9) is a homogeneous cubic equation in the variables \( w_1, w_2 \) and \( w_3 \) and it represents a cubic curve in the projective plane. Further, a rational point on this curve is easily seen to be \((w_1, w_2, w_3) = (a^3, b^3, c^3)\). The tangent to the cubic curve (9) at the aforementioned rational point
necessarily intersects the curve \((f)\) at another rational point which is thus easily found, and is given by,

\[
w_1 = a^3(b^3 + c^3 - a^3), \quad w_2 = b^3(c^3 + a^3 - b^3), \quad w_3 = c^3(a^3 + b^3 - c^3).
\]

With these values of \(w_1, w_2, w_3\), we obtain the following solution of the simultaneous equations \((7)\) and \((8)\):

\[
\begin{align*}
v_1 &= -(3a^6 - 2a^3b^3 - 2a^3c^3 - b^6 + 2b^3c^3 - c^6)r, \\
v_2 &= (a^6 + 2a^3b^3 - 2a^3c^3 - 3b^6 + 2b^3c^3 + c^6)r, \\
v_3 &= (a^6 - 2a^3b^3 + 2a^3c^3 + b^6 + 2b^3c^3 - 3b^6)r, \\
k &= (a^6 - 2a^3b^3 - 2a^3c^3 + b^6 - 2b^3c^3 + c^6)r,
\end{align*}
\]

(10)

where \(a, b, c\) and \(r\) are arbitrary parameters. Substituting the values of \(v_1, v_2, v_3\) and \(k\) given by \((10)\) and the value of \(u_1\) given by \((9)\) in \((3)\), we get,

\[
a^3m^3 = -8a^3r^2(a^3 + b^3 - c^3)(b^3 + c^3 - a^3)(c^3 + a^3 - b^3).
\]

(11)

Now Eq. \((11)\) is readily solved by taking \(m = r\), when we get,

\[
r = -8(a^3 + b^3 - c^3)(b^3 + c^3 - a^3)(c^3 + a^3 - b^3).
\]

We thus obtain a solution of the system of equations \((4), (11)\) and \((5)\) which is given by,

\[
\begin{align*}
k &= -8(a^3 + b^3 - c^3)(b^3 + c^3 - a^3)(c^3 + a^3 - b^3) \\
&\quad \times (a^6 - 2a^3b^3 - 2a^3c^3 + b^6 - 2b^3c^3 + c^6),
\end{align*}
\]

(12)
and

\begin{align*}
  u_1 &= -8a(a^3 + b^3 - c^3)(b^3 + c^3 - a^3)(c^3 + a^3 - b^3), \\
  u_2 &= -8b(a^3 + b^3 - c^3)(b^3 + c^3 - a^3)(c^3 + a^3 - b^3), \\
  u_3 &= -8c(a^3 + b^3 - c^3)(b^3 + c^3 - a^3)(c^3 + a^3 - b^3), \\
  v_1 &= 8(a^3 + b^3 - c^3)(b^3 + c^3 - a^3)(c^3 + a^3 - b^3) \\
   &\quad \times (3a^6 - 2a^3b^3 - 2a^3c^3 - b^6 + 2b^3c^3 - c^6), \\
  v_2 &= -8(a^3 + b^3 - c^3)(b^3 + c^3 - a^3)(c^3 + a^3 - b^3) \\
   &\quad \times (a^6 + 2a^3b^3 - 2a^3c^3 - 3b^6 + 2b^3c^3 + c^6), \\
  v_3 &= -8(a^3 + b^3 - c^3)(b^3 + c^3 - a^3)(c^3 + a^3 - b^3) \\
   &\quad \times (a^6 - 2a^3b^3 + 2a^3c^3 + b^6 + 2b^3c^3 - 3c^6),
\end{align*}

where \( a, b, c \) are arbitrary parameters.

It follows that when \( k \) is given by (12), there are three rational points \( P_1(a, b, c), P_2(a, b, c) \) and \( P_3(a, b, c) \) on the elliptic curve (2) with coordinates \((u_i, v_i)\), \( i = 1, 2, 3 \), where the values of \( u_i, v_i, i = 1, 2, 3 \), are given by (13).

We will now apply a theorem of Silverman [15, Theorem 11.4, p. 271] to show that these points are linearly independent. For this, we must find a specialisation \((a, b, c) = (a_0, b_0, c_0)\) such that the points \( P_1(a_0, b_0, c_0) \), \( P_2(a_0, b_0, c_0) \), and \( P_3(a_0, b_0, c_0) \) are linearly independent on the specialised curve over \( \mathbb{Q} \).

We take \((a, b, c) = (1, 2, 3)\), when we get the elliptic curve,

\begin{equation}
y^2 = x^3 + 28592640^2,
\end{equation}

on which we get the three points,

\begin{align*}
P_1(1, 2, 3) &= (97920, 41909760), \\
P_2(1, 2, 3) &= (195840, 91261440), \\
P_3(1, 2, 3) &= (293760, 161763840),
\end{align*}
each of which is of infinite order. The regulator of these three points, as determined by the software SAGE [13] is 33.9574760167017. As this is nonzero, it follows from a well-known theorem [14, Theorem 8.1, p. 242] that these three points are linearly independent. Hence the rank of the Mordell curve [14] is at least three.

It follows that, in general, the rank of the elliptic curves belonging to the parametrised family of Mordell curves [2], where $k$ is given by [12], is at least three.

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