Coherent Neutrino Propagation in a Dense Medium

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**Abstract**

Motivated by the effect of matter on neutrino oscillations (the MSW effect) we have studied the propagation of neutrinos in a dense medium. The dispersion relation for massive neutrinos in a medium is known to have a minimum at nonzero momentum $p \sim G_F \rho / \sqrt{2}$. We have studied in detail the origin and consequences of this dispersion relation for both Dirac and Majorana neutrinos both in a toy model with only neutral currents and a single neutrino flavour and in a realistic “Standard Model” with two neutrino flavours. For a range of neutrino momenta near the minimum of the dispersion relation, Dirac neutrinos are trapped by their coherent interactions with the medium. This effect does not lead to the trapping of Majorana neutrinos.

Motivated by the effect of matter on neutrino oscillations (the MSW effect), there have been several works in recent years aimed at understanding in a more complete way the propagation of one or more flavours of massive neutrinos in matter (such as in the Sun, in Neutron Stars or in Supernovae for which the density difference $\langle \rho_{\nu_e} - \rho_{\bar{\nu}_e} \rangle \neq 0$). In our recent paper we presented the results of our study of some of the unusual features of this problem. We were particularly interested in effects at low neutrino momentum and in effects due to the minimum which the dispersion relation has at nonzero momentum. In this presentation we begin by examining in some detail a simple model with only a single neutrino flavour in which the neutrino propagates in a background of electrons. This model will show many of the essential features of the more realistic models. We shall then simply state the results in these other cases including the case of the Standard Model both with Dirac and Majorana neutrinos. The reader is referred to our paper for more details.

We begin by considering a simplified model in which a Dirac neutrino propagates in an electron “gas” to which it couples only via the neutral current interaction. Our model may be described by the following Lagrangian

$$\mathcal{L} = \bar{\psi}_\nu (i \not{D} - m_\nu) \psi_\nu + \bar{\psi}_e (i \not{D} - m_e) \psi_e - \frac{1}{4} F^2 + \frac{m_e^2}{2} Z^2 - \mu_e \bar{\psi}_e \psi_e, \quad (1)$$

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where

\[ D^\pm_\mu = \partial_\mu \pm ig^2\frac{\sqrt{2}}{Z_\mu (1 - \gamma^5)}, \]

\[ F_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu. \]

The chemical potential term in the Lagrangian is included in order to give a non-zero value to the electron density; that is

\[ \rho_e = \langle \psi^+_e \psi_e \rangle \neq 0. \]

A detailed analysis of the above theory is given in our paper. An interesting way to gain insight into the Physics of our model is by examining the equations of motion following from the Lagrangian in Eq. (1). Varying the Lagrangian with respect to the \( Z^\mu \) field leads to

\[ \partial_\nu F^{\nu\mu} + m^2 Z^\mu = -g^2\sqrt{2}J^\mu, \]

where

\[ J^\mu = \bar{\psi}_e \gamma^\mu (1 - \gamma^5) \psi_e - \bar{\psi}_\nu \gamma^\mu (1 - \gamma^5) \psi_\nu. \]

If \( \rho_e = \langle \psi^+_e \psi_e \rangle \) is constant, then (5) leads to

\[ \langle Z^0 \rangle = -\frac{g\rho_e}{2\sqrt{2}m^2}, \]

that is, the \( Z^0 \) field has gained a vacuum expectation value. The equation of motion for the neutrino field is then given by

\[ [i\partial - m + \alpha\gamma^0 (1 - \gamma^5)] \psi_\nu = 0, \]

where \( \alpha = g^2\rho_e/8m^2_Z \). From this point of view the left-handed neutrino sees a mean (coherent) “scalar potential,” \( Z^0 \).

For constant electron density, the presence of the “chiral potential” in this expression leads to a shift in the frequency by \( \alpha \), but only for the left-handed (chiral) piece. This shift in the frequency is precisely the “index of refraction” familiar from the MSW effect. The shift in energy for the neutrino is opposite that for the anti-neutrino \[^b\]. If the neutrino is “attracted” by the medium, then the anti-neutrino is “repelled” by it.

The above Dirac Equation leads to four solutions for the energy:

\[ \omega = -\alpha \pm \sqrt{(p + \alpha s)^2 + m^2}. \]

\[^b\]This shift comes from a term in the effective Hamiltonian proportional to \( \psi^+_e \psi_e \) which equals the number density of neutrinos minus the number density of antineutrinos.
These dispersion relations are plotted in Figure 1 both for $m=0$ (dashed curves) and $m\neq0$ (solid curves.) Several key features of these plots should be noted. First of all, the “negative energy” states are, in this case, those which are unbounded from below as the momentum is increased. In the second quantized theory the correct energy of such a state is just the negative of its energy eigenvalue. We also note that when $m=0$ there are “level crossings.” These are avoided for $m\neq0$ by level repulsion due to the mixing of the levels.

![Figure 1: Dispersion relations for a neutrino in an electron “gas”](image)

Note that the minima of some of the dispersion relations occur at non-zero values of the momentum, $p=\pm\alpha$, instead of at the origin. In fact the minimum energy $\omega_{\text{min}}=-\alpha+m$ is less than the neutrino mass. Thus it is possible to produce a neutrino in the medium which has $\omega<m$. Such a neutrino will not have enough energy to survive in the vacuum and will thus be *trapped* by the medium. The condition for trapping is

$$p < p_{\text{trap}} \equiv \alpha + \sqrt{\alpha^2 + 2\alpha m}. \quad (10)$$

Thus, neutrinos produced in this medium with momentum $p<p_{\text{trap}}$ will not have enough energy to survive in the vacuum and will be trapped.

Before proceeding it is useful to get some idea of the overall magnitude of the effect of neutrino trapping. Setting $G\approx G_F$ and $m\approx10^{-3}$eV (which is a mass relevant for the MSW-resolution of the solar neutrino problem), we find that $p_{\text{trap}}\sim10^{-8}$eV in the sun (for which $\alpha\sim10^{-12}$eV) and $p_{\text{trap}}\sim100$eV in a supernova (for which $\alpha\sim100$eV.)

In the case of Majorana neutrinos, there is only a single (left-handed) field, $\chi_L$. The dispersion relations in this case can be obtained by solving the equations of motion, as was first done by Mannheim. The dispersion relation
for the negative helicity neutrino is given by

$$\omega = \pm \sqrt{(|\vec{p}| - 2\alpha)^2 + m^2}. \quad (11)$$

In this case the energy has a minimum value, $\omega = m$, which occurs at $|\vec{p}| = 2\alpha$. In fact, the dispersion relation in matter is identical to that in vacuum except for a lateral shift to the right. This implies in particular that, in contradistinction to the Dirac case, a neutrino cannot have an energy less than $m$ and there is thus no trapping of Majorana neutrinos in the medium.

We turn now to a more realistic case in which there are two neutrino flavours and in which there are both neutral current and charged current couplings to the medium. We have in mind, of course, the Standard Electroweak Model with massive neutrinos. The Dirac Equation in this case is given by:

$$\{\hat{p} - M + (\beta - \alpha Q) \gamma^0 (1 - \gamma_5)\} \psi = 0 \quad (12)$$

where $\psi$ has two flavour components, $M$ is the diagonal $2 \times 2$ mass matrix, $\beta \propto \rho G_F$ is the contribution of the neutral current which couples only to the left handed neutrinos, $\alpha \propto \rho e G_F$ represents the charged current contribution which couples only to $\nu_e$ and $Q$ is the mixing matrix

$$Q = \begin{pmatrix} \cos^2(\theta) & \sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) & \sin^2(\theta) \end{pmatrix}. \quad (13)$$

This Dirac equation leads to the following quartic equation for the dispersion relation:

$$[\omega^2 - p^2 - \mu^2 + (2\beta - \alpha)(\omega - sp)]^2 = \alpha^2(\omega - sp)^2 - \alpha\Delta^2 \cos(2\theta)(\omega - sp) + \frac{1}{4}\Delta^4,$$  

in which $s = \pm 1$ is the eigenvalue of $\sigma_3$, the spin projection in the $+z$ direction. The eight solutions to this equation (for $s = \pm 1$) lead to the eight dispersion relations (four positive energy and four negative energy) in the medium.

It is easy to see that both in the massless case and in the case when $\theta = 0$ the solutions to the quartic equation are precisely the dispersion relations expected in these cases. In order to analyze the dispersion relations when the coupling $\theta$ is nonzero but not too large it is to use a graphical approach. One begins by looking at the solutions when $\theta = 0$ in which case the two neutrino flavours decouple. Thus, for example, in Figure 2(a) the dotted curves represent the solutions for $\theta = 0$. Note that the dispersion relations for the two flavours of neutrinos cross at some points. When $\theta$ is “turned on” we expect that these levels will repel and will lead to a curve similar to the solid curve.
in that figure. The solid curve is, in fact, the solution to the quartic equation when \( \theta = 0.2 \). This graphical method is reasonably accurate when \( \theta \) is small but can be used as a guide even for larger values of \( \theta \). Notice that just as for the single neutrino case, the lightest Dirac neutrino is trapped by the medium.

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Figure 2: Dispersion relations for two neutrino flavours.

The case of Majorana neutrinos is interesting for two reasons. First of all, it is the favoured realistic scenario in models which have massive neutrinos, for example in models which employ the “see-saw” mechanism. Secondly the equations governing the dispersion relations are quadratic rather than quartic. See\(^2\) for details. Analysis of these dispersion relations shows that their minimum is always greater than or equal to the minimum mass and so again there appears to be no trapping in the Majorana case.

References

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