Approximating solutions of nonlinear periodic boundary value problems with maxima

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Abstract: In this paper we study a periodic boundary value problem of first order nonlinear differential equations with maxima and discuss the existence and approximation of the solutions. The main result relies on the Dhage iteration method embodied in a recent hybrid fixed point theorem of Dhage (2014) in a partially ordered normed linear space. At the end, we give an example to illustrate the applicability of the abstract results to some concrete periodic boundary value problems of nonlinear differential equations.

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1. Introduction

The study of fixed point theorems for the contraction mappings in partially ordered metric spaces is initiated by Ran and Reurings (2004) which is further continued by Nieto and Rodriguez-Lopez (2005) and applied to periodic boundary value problems of nonlinear first order ordinary differential equations for proving the existence results under certain monotonic conditions. Similarly, the study of hybrid fixed point theorems in a partially ordered metric space is initiated by Dhage (2013, 2014, 2015a) with applications to nonlinear differential and integral equation under weaker mixed conditions of nonlinearities. See Dhage (2015b, 2015c) and the references therein. In this paper we investigate the existence of approximate solutions of certain hybrid differential equations with maxima using the Dhage iteration method embodied in a hybrid fixed point theorem. We claim that the results of this paper are new to the theory of nonlinear differential equations with maxima.
Given a closed and bounded interval $J = [0, T]$ of the real line $\mathbb{R}$ for some $T > 0$, we consider the following hybrid differential equation (in short HDE) of first order periodic boundary value problems,

$$
\begin{align*}
&x'(t) = f(t, x(t)) + g\left(t, \max_{0 \leq \sigma \leq t} x(\sigma)\right), \\
&x(0) = x(T),
\end{align*}
$$

(1.1)

for all $t \in J = [0, T]$ and $f, g : J \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

By a solution of Equation (1.1) we mean a function $x \in C^1(J, \mathbb{R})$ that satisfies Equation (1.1), where $C^1(J, \mathbb{R})$ is the space of continuously differentiable real-valued functions defined on $J$.

Differential equations with maxima are often met in the applications, for instance in the theory of automatic control. Numerous results on existence and uniqueness, asymptotic stability as well as numerical solutions for such equations have been obtained. To name a few, we refer the reader to (Bainov & Hristova, 2011; Otrocol, 2014) and the references therein. The PBVP's of nonlinear first order ordinary differential equations have also been a topic of great interest since long time. The HDE (1.1) is a linear perturbation of first type of the PBVP of first order nonlinear differential equations. The details of different types of perturbation appears in Dhage (2010). The special cases of the HDE (1.1) are

$$
\begin{align*}
&x'(t) = f(t, x(t)), \quad t \in J, \\
&x(0) = x(T),
\end{align*}
$$

(1.2)

and

$$
\begin{align*}
&x'(t) = g\left(t, \max_{0 \leq \sigma \leq t} x(\sigma)\right), \quad t \in J, \\
&x(0) = x(T).
\end{align*}
$$

(1.3)

The HDE (1.2) has already been discussed in the literature for different aspects of the solutions using usual Picard as well as Dhage iteration method. See Zeidler (1986) and Dhage and Dhage (2015a). Similarly, the HDE (1.3) has been studied earlier using Picard method. See Bainov, and Hristova (2011) and the references therein. Very recently, Dhage and Octrocol (2016) have initiated the study of initial value problems of first order ordinary nonlinear differential equations via new Dhage iteration method, however to the best of author’s knowledge the HDE (1.3) is not discussed via Dhage iteration method. Therefore, the HDE (1.1) is new to the literature in the set up of Dhage iteration method. In this paper we discuss the HDE (1.1) for existence and approximation of the solutions via a new approach based upon Dhage iteration method which include the existence and approximation results for the HDEs (1.2) and (1.3) as special cases which are again new to the theory of differential equations.

In the following section we give some preliminaries and the key tool that will be used for proving the main result of this paper.

2. Preliminaries

Throughout this paper, unless otherwise mentioned, let $(E, \leq, \| \cdot \|)$ denote a partially ordered normed linear space. Two elements $x$ and $y$ in $E$ are said to be comparable if either the relation $x \leq y$ holds. A non-empty subset $C$ of $E$ is called a chain or totally ordered if all the elements of $C$ are comparable. It is known that $E$ is regular if $(x_n)$ is a nondecreasing (resp. nonincreasing) sequence in $E$ such that $x_n \to x^*$ as $n \to \infty$, then $x_n \leq x^*$ (resp. $x_n \geq x^*$) for all $n \in \mathbb{N}$. The conditions guaranteeing the regularity of $E$ may be found in Heikkilä and Lakshmikantham (1994) and the references therein.

We need the following definitions (see Dhage, 2013, 2014 and the references therein) in what follows.
Definition 2.1 A mapping $T : E \to E$ is called isotone or monotone nondecreasing if it preserves the order relation $\leq$, that is, if $x \leq y$ implies $Tx \leq Ty$ for all $x, y \in E$. Similarly, $T$ is called monotone nonincreasing if $x \leq y$ implies $Tx \geq Ty$ for all $x, y \in E$. Finally, $T$ is called monotonic or simply monotone if it is either monotone nondecreasing or monotone nonincreasing on $E$.

Definition 2.2 A mapping $T : E \to E$ is called partially continuous at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|Tx - Ta\| < \epsilon$ whenever $x$ is comparable to $a$ and $\|x - a\| < \delta$. $T$ called partially continuous on $E$ if it is partially continuous at every point of it. It is clear that if $T$ is partially continuous on $E$, then it is continuous on every chain $C$ contained in $E$.

Definition 2.3 A non-empty subset $S$ of the partially ordered Banach space $E$ is called partially bounded if every chain $C$ in $S$ is bounded. An operator $T$ on a partially normed linear space $E$ into itself is called partially bounded if $T(E)$ is a partially bounded subset of $E$. $T$ is called uniformly partially bounded if all chains $C$ in $T(E)$ are bounded by a unique constant.

Definition 2.4 A non-empty subset $S$ of the partially ordered Banach space $E$ is called partially compact if every chain $C$ in $S$ is a relatively compact subset of $E$. A mapping $T : E \to E$ is called partially compact if $T(E)$ is a partially relatively compact subset of $E$. $T$ is called uniformly partially compact if $T$ is a uniformly partially bounded and partially compact operator on $E$. $T$ is called partially totally bounded if for any bounded subset $S$ of $E$, $T(S)$ is a partially relatively compact subset of $E$. If $T$ is partially continuous and partially totally bounded, then it is called partially completely continuous on $E$.

Remark 2.1 Suppose that $T$ is a nondecreasing operator on $E$ into itself. Then $T$ is a partially bounded or partially compact if $T(C)$ is a bounded or relatively compact subset of $E$ for each chain $C$ in $E$.

Definition 2.5 The order relation $\leq$ and the metric $d$ on a non-empty set $E$ are said to be compatible if $\{x_n\}$ is a monotone sequence, that is, that monotone nondecreasing or monotone nonincreasing sequence in $E$ and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to $x'$ implies that the original sequence $\{x_n\}$ converges to $x'$. Similarly, given a partially ordered normed linear space $(E, \leq, \|\cdot\|)$, the order relation $\leq$ and the norm $\|\cdot\|$ are said to be compatible if $\leq$ and the metric $d$ defined through the norm $\|\cdot\|$ are compatible. A subset $S$ of $E$ is called Janhavi if the order relation $\leq$ and the metric $d$ or the norm $\|\cdot\|$ are compatible in it. In particular, if $S = E$, then $E$ is called a Janhavi metric or Janhavi Banach space.

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function $|\cdot|$ has this property. Similarly, the finite dimensional Euclidean space $\mathbb{R}^n$ with usual componentwise order relation and the standard norm possesses the compatibility property and so is a Janhavi Banach space.

Definition 2.6 An upper semi-continuous and monotone nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $D$-function provided $\psi(0) = 0$. An operator $T : E \to E$ is called partially nonlinear $D$-contraction if there exists a $D$-function $\psi$ such that

$$\|Tx - Ty\| \leq \psi(\|x - y\|)$$

for all comparable elements $x, y \in E$, where $0 < \psi(r) < r$ for $r > 0$. In particular, if $\psi(r) = kr, k > 0$, $T$ is called a partial Lipschitz operator with a Lipschitz constant $k$ and moreover, if $0 < k < 1$, $T$ is called a partial linear contraction on $E$ with a contraction constant $k$.

The Dhage iteration method embodied in the following applicable hybrid fixed point theorem of Dhage (2014) in a partially ordered normed linear space is used as a key tool for our work contained in this paper. The details of a Dhage iteration method is given in Dhage (2015b, 2015c), Dhage, Dhage, Graef (2016) and the references therein.
Theorem 2.1 (Dhage, 2014) Let \((E, \preceq, \| \cdot \|)\) be a regular partially ordered complete normed linear space such that every compact chain \(C\) of \(E\) is Janhavi. Let \(A, B : E \rightarrow E\) be two nondecreasing operators such that

(a) \(A\) is partially bounded and partially nonlinear \(I\)-contraction,
(b) \(B\) is partially continuous and partially compact, and
(c) there exists an element \(x_0 \in E\) such that \(x_0 \preceq Ax_0 + Bx_0\) or \(x_0 \succeq Ax_0 + Bx_0\). Then the operator equation \(Ax + Bx = x\) has a solution \(x^*\) in \(E\) and the sequence \(\{x_n\}\) of successive iterations defined by \(x_{n+1} = Ax_n + Bx_n, n = 0, 1, \ldots\), converges monotonically to \(x^*\).

Remark 2.2 The condition that every compact chain of \(E\) is Janhavi holds if every partially compact subset of \(E\) possesses the compatibility property with respect to the order relation \(\preceq\) and the norm \(\| \cdot \|\) in it.

Remark 2.3 We remark that hypothesis (a) of Theorem 2.1 implies that the operator \(A\) is partially continuous and consequently both the operators \(A\) and \(B\) in the theorem are partially continuous on \(E\). The regularity of \(E\) in above Theorem 2.1 may be replaced with a stronger continuity condition of the operators \(A\) and \(B\) on \(E\) which is a result proved in Dhage (2013, 2014).

3. Main results
In this section, we prove an existence and approximation result for the HDE (1.1) on a closed and bounded interval \(J = [0, T]\) under mixed partial Lipschitz and partial compactness type conditions on the nonlinearities involved in it. We place the HDE (1.1) in the function space \(C(J, \mathbb{R})\) of continuous real-valued functions defined on \(J\). We define a norm \(\| \cdot \|\) and the order relation \(\leq\) in \(C(J, \mathbb{R})\) by

\[
\|x\| = \sup_{t \in J} |x(t)|
\]

and

\[
x \leq y \iff x(t) \leq y(t) \quad \text{for all} \quad t \in J.
\]

Clearly, \(C(J, \mathbb{R})\) is a Banach space with respect to above supremum norm and also partially ordered w.r.t. the above partially order relation \(\preceq\). It is known that the partially ordered Banach space \(C(J, \mathbb{R})\) is regular and lattice so that every pair of elements of \(E\) has a lower and an upper bound in it. The following useful lemma concerning the Janhavi subsets of \(C(J, \mathbb{R})\) follows immediately form the Arzelà-Ascoli theorem for compactness.

Lemma 3.1 Let \((C(J, \mathbb{R}), \preceq, \| \cdot \|)\) be a partially ordered Banach space with the norm \(\| \cdot \|\) and the order relation \(\preceq\) defined by (3.1) and (3.2) respectively. Then every partially compact subset of \(C(J, \mathbb{R})\) is Janhavi.

Proof The proof of the lemma is well-known and appears in the works of Dhage (2015b, 2015c), Dhage and Dhage (2014), Dhage et al. (2016) and so we omit the details. \(\square\)

The following useful lemma is obvious and may be found in Dhage (2008) and Nieto (1997).

Lemma 3.2 For any function \(\sigma \in L^1(J, \mathbb{R})\), \(x\) is a solution to the differential equation

\[
\begin{align*}
x'(t) + Ax(t) &= \sigma(t), \quad t \in J, \\
x(0) &= x(T),
\end{align*}
\]

if and only if it is a solution of the integral equation

\[
x(t) = \int_0^T G_s(t, s) \sigma(s) \, ds
\]

(3.4)
where,

\[ G_j(t,s) = \begin{cases} 
\frac{e^{\lambda(s-t)}}{\lambda}, & \text{if } 0 \leq s \leq t \leq T, \\
\frac{e^{\lambda t}}{\lambda}, & \text{if } 0 \leq t < s \leq T.
\end{cases} \]  

(3.5)

Notice that the Green’s function \( G_j \) is continuous and nonnegative on \( J \times J \) and therefore, the number

\[ K_j := \max \{|G_j(t,s)| : t, s \in [0, T]\} \]

exists for all \( \lambda \in \mathbb{R}^+ \). For the sake of convenience, we write \( G_j(t,s) = G(t,s) \) and \( K_j = K \).

Another useful result for establishing the main result is as follows.

**Lemma 3.3** If there exists a differentiable function \( u \in C(J, \mathbb{R}) \) such that

\[
\begin{align*}
    u'(t) + \lambda u(t) &\leq \sigma(t), \quad t \in J, \\
    u(0) &\leq u(T),
\end{align*}
\]

(3.6)

for all \( t \in J \), where \( \lambda \in \mathbb{R}, \lambda > 0 \) and \( \sigma \in L^1(J, \mathbb{R}) \), then

\[ u(t) \leq \int_0^T G(t,s)\sigma(s) \, ds, \]

(3.7)

for all \( t \in J \), where \( G(t,s) \) is a Green’s function given by (3.5).

**Proof** Suppose that the function \( u \in C(J, \mathbb{R}) \) satisfies the inequalities given in (3.6). Multiplying the first inequality in (3.6) by \( e^{\lambda t} \),

\[ (e^{\lambda t}u(t))' \leq e^{\lambda t} \sigma(t). \]

A direct integration of above inequality from 0 to \( t \) yields

\[ e^{\lambda t}u(t) \leq u(0) + \int_0^t e^{\lambda s} \sigma(s) \, ds, \]

(3.8)

for all \( t \in J \). Therefore, in particular,

\[ e^{\lambda T}u(T) \leq u(0) + \int_0^T e^{\lambda s} \sigma(s) \, ds. \]

(3.9)

Now \( u(0) \leq u(T) \), so one has

\[ u(0)e^{\lambda T} \leq u(T)e^{\lambda T}. \]

(3.10)

From (3.9) and (3.10) it follows that

\[ e^{\lambda T}u(0) \leq u(0) + \int_0^T e^{\lambda s} \sigma(s) \, ds \]

(3.11)

which further yields

\[ u(0) \leq \int_0^T \frac{e^{\lambda s}}{(e^{\lambda T} - 1)} \sigma(s) \, ds. \]

(3.12)

Substituting (3.12) in (3.8) we obtain

\[ u(t) \leq \int_0^T G(t,s)\sigma(s) \, ds, \]

(3.13)
for all \( t \in J \). This completes the proof. \( \square \)

We need the following definition in what follows.

**Definition 3.1** A function \( u \in C^1(J, \mathbb{R}) \) is said to be a lower solution of the Equation (1.1) if it satisfies

\[
\begin{aligned}
  u'(t) &\leq f(t, u(t)) + g(t, \max_{0 \leq \xi \leq t} u(\xi)), \\
  u(0) &\leq u(T),
\end{aligned}
\]

for all \( t \in J \). Similarly, a differentiable function \( v \in C^1(J, \mathbb{R}) \) is called an upper solution of the HDE (1.1) if the above inequality is satisfied with reverse sign.

We consider the following set of assumptions in what follows:

\( (H_1) \) There exist constants \( \lambda > 0, \mu > 0 \) with \( \lambda \geq \mu \) such that

\[ 0 \leq \lambda \lambda x - \lambda \lambda y \leq \mu (x - y) \]

for all \( t \in J \) and \( x, y \in \mathbb{R} \), \( x \geq y \). Moreover, \( \lambda \lambda T < 1 \).

\( (H_2) \) There exists a constant \( M_g > 0 \) such that \( \|g(t, x)\| \leq M_g \) for all \( t \in J, x \in \mathbb{R} \);

\( (H_3) \) \( g(t, x) \) is nondecreasing in \( x \) for each \( t \in J \).

\( (H_4) \) HDE (1.1) has a lower solution \( u \in C^1(J, \mathbb{R}) \).

Now we consider the following HDE

\[
\begin{aligned}
x'(t) + \lambda x(t) &= \tilde{f}(t, x(t)) + g(t, \max_{0 \leq \xi \leq t} u(\xi)), \\
x(0) &= x(T),
\end{aligned}
\]

for all \( t \in J = [0, T] \), where \( \tilde{f}(t, x) = f(t, x) + \lambda x \), \( \lambda > 0 \).

**Remark 3.1** A function \( u \in C^1(J, \mathbb{R}) \) is a solution of the HDE (3.13) if and only if it is a solution of the HDE (1.1) defined on \( J \).

We also consider the following hypothesis in what follows.

\( (H_5) \) There exists a constant \( M_f > 0 \) such that \( \|f(t, x)\| \leq M_f \) for all \( t \in J \) and \( x \in \mathbb{R} \).

**Lemma 3.4** Suppose that the hypotheses \( (H_1), (H_2) \) and \( (H_5) \) hold. Then a function \( x \in C(J, \mathbb{R}) \) is a solution of the HDE (3.3) if and only if it is a solution of the nonlinear hybrid integral equation (in short HIE)

\[
x(t) = \int_0^T G(t, s) \tilde{f}(s, x(s)) \, ds + \int_0^T G(t, s) g(s, \max_{0 \leq \xi \leq s} x(\xi)) \, ds,
\]

for all \( t \in J \).

**Theorem 3.1** Suppose that hypotheses \( (H_4) - (H_5) \) hold. Then the HDE (1.1) has a solution \( x^* \) defined on \( J \) and the sequence \( \{x_n\} \) of successive approximations defined by
for all \( t \in J \), converges monotonically to \( x^* \).

**Proof** Set \( E = C(J, \mathbb{R}) \). Then, in view of Lemma 3.1, every compact chain \( C \) in \( E \) possesses the compatibility property with respect to the norm \( \| \cdot \| \) and the order relation \( \leq \) so that every compact chain \( C \) is Janhavi in \( E \).

Define two operators \( \mathcal{A} \) and \( \mathcal{B} \) on \( E \) by

\[
\mathcal{A}x(t) = \int_0^t G(t, s)\tilde{f}(s, x(s)) \, ds, \quad t \in J, \tag{3.16}
\]

and

\[
\mathcal{B}x(t) = \int_0^t G(t, s)g\left(s, \max_{a \leq s \leq \xi} x(s)\right) \, ds, \quad t \in J. \tag{3.17}
\]

From the continuity of the integral, it follows that \( \mathcal{A} \) and \( \mathcal{B} \) define the operators \( \mathcal{A}, \mathcal{B} : E \to E \). Applying Lemma (3.4), the HDE (1.1) is equivalent to the operator equation

\[ \mathcal{A}x(t) + \mathcal{B}x(t) = x(t), \quad t \in J. \]

Now, we show that the operators \( \mathcal{A} \) and \( \mathcal{B} \) satisfy all the conditions of Theorem 2.1 in a series of following steps.

**Step I:** \( \mathcal{A} \) and \( \mathcal{B} \) are nondecreasing on \( E \).

Let \( x, y \in E \) be such that \( x \geq y \). Then by hypothesis (\( H_1 \)), we get

\[
\mathcal{A}x(t) = \int_0^t G(t, s)\tilde{f}(s, x(s)) \, ds \\
\geq \int_0^t G(t, s)\tilde{f}(s, y(s)) \, ds \\
= \mathcal{A}y(t),
\]

for all \( t \in J \).

Next, we show that the operator \( \mathcal{B} \) is also nondecreasing on \( E \). Let \( x, y \in E \) be such that \( x \geq y \). Then \( x(t) \geq y(t) \) for all \( t \in J \). Since \( y \) is continuous on \([0, t]\), there exists a \( \xi^* \in [a, t] \) such that \( y(\xi^*) = \max_{a \leq s \leq \xi} y(s) \).

By definition of \( \leq \), one has \( x(\xi^*) \geq y(\xi^*) \). Consequently, we obtain

\[
\max_{a \leq s \leq \xi} x(s) = x(\xi^*) \geq y(\xi^*) = \max_{a \leq s \leq \xi} y(s).
\]

Now, using hypothesis (\( H_2 \)), it can be shown that the operator \( \mathcal{B} \) is also nondecreasing on \( E \).

**Step II:** \( \mathcal{A} \) is partially bounded and partially contraction on \( E \).

Let \( x \in E \) be arbitrary. Then by (\( H_2 \)) we have

\[
\int_0^s G(t, s)\tilde{f}(s, x(s)) \, ds \leq \|x\| \int_0^s |\tilde{f}(s, x(s))| \, ds \\
\leq \|x\| \int_0^s |\tilde{f}(s, x(s))| \, ds.
\]

Hence, \( \mathcal{A} \) is partially bounded on \( E \).
\[\|Ax(t)\| \leq \int_0^T G(t, s)\|\overline{f}(s, x(s))\| \, ds\]
\[\leq M_f \int_0^T G(t, s) \, ds\]
\[\leq M_f KT,\]
for all \(t \in J\). Taking the supremum over \(t\), we obtain \(\|Ax(t)\| \leq M_f KT\) and so, \(A\) is a bounded operator on \(E\). This implies that \(A\) is partially bounded on \(E\). Let \(x, y \in E\) be such that \(x \geq y\). Then by (H\(_1\)) we have
\[\|Ax(t) - Ay(t)\| \leq \int_0^T G(t, s)\|\overline{f}(s, x(s)) - \overline{f}(s, y(s))\| \, ds\]
\[\leq \int_0^T G(t, s)\|x(s) - y(s)\| \, ds\]
\[\leq \int_0^T G(t, s)\|x(s) - y(s)\| \, ds\]
\[\leq \lambda \int_0^T G(t, s)\|x - y\| \, ds\]
\[\leq \lambda KT\|x - y\|\]
for all \(t \in J\). Taking the supremum over \(t\), we obtain \(\|Ax - Ay\| \leq L\|x - y\|\) for all \(x, y \in E\) with \(x \geq y\), where \(L = \lambda KT < 1\). Hence \(A\) is a partially contraction on \(E\) and which also implies that \(A\) is partially continuous on \(E\).

**Step III:** \(H\) is partially continuous on \(E\).

Let \(\{x_n\}_{n \in \mathbb{N}}\) be a sequence in a chain \(C\) such that \(x_n \to x\), for all \(n \in \mathbb{N}\). Then
\[
\lim_{n \to \infty} Bx_n(t) = \lim_{n \to \infty} \int_0^T G(t, s)g\left(s, \max_{\xi \in [0, t]} x_n(\xi)\right) \, ds = \int_0^T G(t, s)\left[\lim_{n \to \infty} g\left(s, \max_{\xi \in [0, t]} x_n(\xi)\right)\right] \, ds = \int_0^T G(t, s)g\left(s, \max_{\xi \in [0, t]} x_n(\xi)\right) \, ds = Bx(t),
\]
for all \(t \in J\). This shows that \(Bx_n\) converges monotonically to \(Bx\) pointwise on \(J\).

Now we show that \(\{Bx_n\}_{n \in \mathbb{N}}\) is an equicontinuous sequence of functions in \(E\).

Let \(t_1, t_2 \in J\) with \(t_1 < t_2\). We have
\[
\|Bx_n(t_2) - Bx_n(t_1)\| = \left\| \int_0^T G(t_2, s)g\left(s, \max_{\xi \in [0, t_2]} x_n(\xi)\right) \, ds - \int_0^T G(t_1, s)g\left(s, \max_{\xi \in [0, t_1]} x_n(\xi)\right) \, ds \right\|
\leq \int_0^T \|G(t_2, s) - G(t_1, s)\| g\left(s, \max_{\xi \in [0, t_2]} x_n(\xi)\right) \, ds
\leq M_g \int_0^T |G(t_2, s) - G(t_1, s)| \, ds
\to 0 \quad \text{as} \quad t_2 \to t_1,
\]
uniformly for all $n \in \mathbb{N}$. This shows that the convergence $Bx_n \to Bx$ is uniform and hence $B$ is partially continuous on $E$.

**Step IV**: $B$ is partially compact operator on $E$.

Let $C$ be an arbitrary chain in $E$. We show that $B(C)$ is uniformly bounded and equicontinuous set in $E$. First we show that $B(C)$ is uniformly bounded. Let $y \in B(C)$ be any element. Then there is an element $x \in C$ such that $y = Bx$. By hypothesis $(H_1)$

$$\|y\| = \|Bx(t)\| = \left\| \int_0^T G(t, s)g\left(s, \max_{\xi \in \mathbb{D}} x(\xi)\right) \, ds \right\| \leq \int_0^T \|G(t, s)\| \left\| g\left(s, \max_{\xi \in \mathbb{D}} x(\xi)\right) \right\| \, ds \leq KT M_g = r,$$

for all $t \in J$. Taking the supremum over $t$ we obtain $\|y\| \leq \|Bx\| \leq r$, for all $y \in B(C)$. Hence $B(C)$ is uniformly bounded subset of $E$. Next we show that $B(C)$ is an equicontinuous set in $E$. Let $t_1, t_2 \in J$, with $t_1 < t_2$. Then, for any $y \in B(C)$, one has

$$\|y(t_2) - y(t_1)\| = \|Bx(t_2) - Bx(t_1)\| = \left\| \int_0^T G(t_2, s)g\left(s, \max_{\xi \in \mathbb{D}} x(\xi)\right) \, ds \right\| \leq \int_0^T |G(t_2, s) - G(t_1, s)| \left\| g\left(s, \max_{\xi \in \mathbb{D}} x(\xi)\right) \right\| \, ds \leq M_g \int_0^T |G(t_2, s) - G(t_1, s)| \, ds \to 0 \quad \text{as} \quad t_1 \to t_2,$$

uniformly for all $y \in B(C)$. This shows that $B(C)$ is an equicontinuous subset of $E$. So $B(C)$ is a uniformly bounded and equicontinuous set of functions in $E$. Hence it is compact in view of Arzelá-Ascoli theorem. Consequently $B: E \to E$ is a partially compact operator of $E$ into itself.

**Step V**: $u$ satisfies the inequality $u \leq Au + Bu$.

By hypothesis $(H_1)$ the Equation (1.1) has a lower solution $u$ defined on $J$. Then we have

$$\begin{cases} u'(t) \leq f(t, u(t)) + g\left(t, \max_{\xi \in \mathbb{D}} u(\xi)\right), & t \in J, \\ u(0) \leq u(T). \end{cases} \quad (3.18)$$

A direct application of lemma 3.3 yields that

$$u(t) \leq \int_0^T G(t, s)f(s, u(s)) \, ds + \int_0^T G(t, s)g\left(s, \max_{\xi \in \mathbb{D}} u(\xi)\right) \, ds,$$ \quad (3.19)

for $t \in J$. From definitions of the operators $A$ and $B$ it follows that $u(t) \leq Au(t) + Bu(t)$, for all $t \in J$. Hence $u \leq Au + Bu$. Thus $A$ and $B$ satisfy all the conditions of Theorem 2.1 and we apply it to conclude that the operator equation $Ax + Bx = x$ has a solution. Consequently the integral equation and the Equation (1.1) has a solution $x^*$ defined on $J$. Furthermore, the sequence $\{x_n\}_{n=0}^\infty$ of successive approximations defined by (3.5) converges monotonically to $x^*$. This completes the proof. \[\square\]
Remark 3.2  The conclusion of Theorem 3.1 also remains true if we replace the hypothesis \((H_2)\) with the following one.

\[(H_2')\] The HDE (1.1) has an upper solution \(v \in C^1(J, \mathbb{R})\).

Remark 3.3  We note that if the PBVP (1.1) has a lower solution \(u\) as well as an upper solution \(v\) such that \(u \leq v\), then under the given conditions of Theorem 3.1 it has corresponding solutions \(x_u\) and \(x_v\) and these solutions satisfy \(x_u \leq x_v\). Hence they are the minimal and maximal solutions of the PBVP (1.1) in the vector segment \([u, v]\) of the Banach space \(E = C^1(J, \mathbb{R})\), where the vector segment \([u, v]\) is a set in \(C^1(J, \mathbb{R})\) defined by

\([u, v] = \{x \in C^1(J, \mathbb{R}) \mid u \leq x \leq v\}\).

This is because the order relation \(\leq\) defined by (3.2) is equivalent to the order relation defined by the order cone \(\mathcal{K} = \{x \in C(J, \mathbb{R}) \mid x \geq \theta\}\) which is a closed set in \(C(J, \mathbb{R})\).

In the following we illustrate our hypotheses and the main abstract result for the validity of conclusion.

Example 3.1  We consider the following HDE

\[
\begin{align*}
x'(t) &= \arctan x(t) - x(t) + \tanh \left( \max_{\xi \in I} x(\xi) \right), \quad t \in J = [0, 1], \\
x(0) &= x(1).
\end{align*}
\]

(3.20)

Here \(f(t, x) = \arctan x(t) - x(t)\) and \(g(t, x) = \tanh x\). The functions \(f\) and \(g\) are continuous on \(J \times \mathbb{R}\). Next, we have

\[0 \leq \arctan x(t) - \arctan y(t) \leq \frac{1}{\epsilon^2 + 1}(x - y),\]

for all \(x, y \in \mathbb{R}, x > \xi > y\). Therefore \(\lambda = 1 > \frac{1}{\epsilon^2 + 1}\) = \(\mu\). Hence the function \(f\) satisfies the hypothesis \((H_1)\). Moreover, the function \(f(t, x) = \arctan x(t)\) is bounded on \(J \times \mathbb{R}\) with bound \(M_f = \pi/2\), so that the hypothesis \((H_2)\) is satisfied. The function \(g\) is bounded on \(J \times \mathbb{R}\) by \(M_g = 1\), so \((H_3)\) holds. The function \(g(t, x)\) is increasing in \(x\) for each \(t \in J\), so the hypothesis \((H_4)\) is satisfied. The HDE (3.20) has a lower solution \(u(t) = -2 \int_0^1 G(t, s) ds, t \in [0, 1]\), where \(G(t, s)\) is a Green’s function associated with the homogeneous PBVP

\[
\begin{align*}
x'(t) + x(t) &= 0, \quad t \in J, \\
x(0) &= x(1),
\end{align*}
\]

(3.21)

given by

\[
G(t, s) = \begin{cases} 
\frac{e^{\alpha t} - e^{\alpha s}}{\alpha}, & \text{if } 0 \leq s \leq t \leq 1, \\
\frac{e^{\alpha s} - e^{\alpha t}}{\alpha}, & \text{if } 0 \leq t < s \leq 1.
\end{cases}
\]

Finally, \(\lambda K T = \sup_{t \in [0, 1]} G(t, s) < 1\). Thus all the hypotheses of Theorem 3.1 are satisfied and hence the HDE (3.11) has a solution \(x^*\) defined on \(J\) and the sequence \(\{x_n\}_{n=0}^{\infty}\) defined by

\[
x_0 = -2 \int_0^1 G(t, s) ds, \\
x_{n+1}(t) = \int_0^1 G(t, s) \arctan x_n(s) ds \\
+ \int_0^1 G(t, s) \tanh \left( \max_{\xi \in I} x_n(\xi) \right) ds
\]
for each $t \in J$, converges monotonically to $x^*$.

Remark 3.4 Finally while concluding this paper, we mention that the study of this paper may be extended with appropriate modifications to the nonlinear hybrid differential equation with maxima,

$$
\begin{cases}
    x'(t) = f \left( t, x(t), \max_{s \leq t} x(s) \right) + g \left( t, x(t), \max_{s \leq t} x(s) \right), \\
    x(0) = x(T),
\end{cases}
$$

(3.22)

for all $t \in J = [a, b]$, where $f, g : J \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions. When $g \equiv 0$, the differential Equation (3.12) reduces to the nonlinear differential equations with maxima,

$$
\begin{cases}
    x'(t) = f \left( t, x(t), \max_{s \leq t} x(s) \right), \\
    x(0) = x(T),
\end{cases}
$$

(3.23)

which is new and could be studied for existence and uniqueness theorem via Picard iterations under strong Lipschitz condition. Therefore, the obtained results for differential Equation (3.12) with maxima via Dhage iteration method will include the existence and approximation results for the differential equation with maxima (3.13) under weak partial Lipschitz condition.

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