Long-time asymptotics for the generalized Sasa-Satsuma equation

Kedong Wang, Xianguo Geng, Mingming Chen and Ruomeng Li

School of Mathematics and Statistics, Zhengzhou University, Zhengzhou, 450001, China

* Corresondence: Email: chenmmzzu@163.com.

Abstract: In this paper, we study the long-time asymptotic behavior of the solution of the Cauchy problem for the generalized Sasa-Satsuma equation. Starting with the $3 \times 3$ Lax pair related to the generalized Sasa-Satsuma equation, we construct a Riemann-Hilbert problem, by which the solution of the generalized Sasa-Satsuma equation is converted into the solution of the corresponding Riemann-Hilbert problem. Using the nonlinear steepest decent method for the Riemann-Hilbert problem, we obtain the leading-order asymptotics of the solution of the Cauchy problem for the generalized Sasa-Satsuma equation through several transformations of the Riemann-Hilbert problem and with the aid of the parabolic cylinder function.

Keywords: nonlinear steepest descent method; generalized Sasa-Satsuma equation; long-time asymptotics

Mathematics Subject Classification: 35Q53, 35B40

1. Introduction

The Sasa-Satsuma equation

$$u_t + u_{xxx} + 6|u|^2 u_x + 3u(|u|^2)_x = 0,$$

so-called high-order nonlinear Schrödinger equation [1], is relevant to several physical phenomena, for example, in optical fibers [2, 3], in deep water waves [4] and generally in dispersive nonlinear media [5]. Because this equation describes these important nonlinear phenomena, it has received considerable attention and extensive research. The Sasa-Satsuma equation has been discussed by means of various approaches such as the inverse scattering transform [1], the Riemann-Hilbert method [6], the Hirota bilinear method [7], the Darboux transformation [8], and others [9, 10, 11]. The initial-boundary value problem for the Sasa-Satsuma equation on a finite interval was studied by the Fokas method [12], which is also effective for the initial-boundary value problems on the half-line [35, 36, 37]. In Ref. [13], finite genus solutions of the coupled Sasa-Satsuma hierarchy are obtained.
in the basis of the theory of trigonal curves, the Baker-Akhiezer function and the meromorphic functions [14, 15, 16]. In Ref. [17], the super Sasa-Satsuma hierarchy associated with the $3 \times 3$ matrix spectral problem was proposed, and its bi-Hamiltonian structures were derived with the aid of the super trace identity.

The nonlinear steepest descent method [18], also called Deift-Zhou method, for oscillatory Riemann-Hilbert problems is a powerful tool to study the long-time asymptotic behavior of the solution for the soliton equation, by which the long-time asymptotic behaviors for a number of integrable nonlinear evolution equations associated with $2 \times 2$ matrix spectral problems have been obtained, for example, the mKdV equation, the KdV equation, the sine-Gordon equation, the modified nonlinear Schrödinger equation, the Camassa-Holm equation, the derivative nonlinear Schrödinger equation and so on [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30]. However, there is little literature on the long-time asymptotic behavior of solutions for integrable nonlinear evolution equations associated with $3 \times 3$ matrix spectral problems [31, 32, 33]. Usually, it is difficult and complicated for the $3 \times 3$ case. Recently, the nonlinear steepest descent method was successfully generalized to derive the long-time asymptotics of the initial value problems for the coupled nonlinear Schrödinger equation and the Sasa-Satsuma equation with the complex potentials [33, 34]. The main differences between the $2 \times 2$ and $3 \times 3$ cases is that the former corresponds to a scalar Riemann-Hilbert problem, while the latter corresponds to a matrix Riemann-Hilbert problem. In general, the solution of the matrix Riemann-Hilbert problem can not be given in explicit form, but the scalar Riemann-Hilbert problem can be solved by the Plemelj formula.

The main aim of this paper is to study the long-time asymptotics of the Cauchy problem for the generalized Sasa-Satsuma equation [38] via nonlinear steepest decent method,

$$\begin{cases}
  u_t + u_{xxx} - 6a|u|^2 u_x - 6b|u|^2 u_x - 3a|u|^2 u_x - 3b^*|u|^2 u_x = 0, \\
  u(x, 0) = u_0(x),
\end{cases} \tag{1.2}$$

where $a$ is a real constant, $b$ is a complex constant that satisfies $a^2 \neq |b|^2$, the asterisk “*” denotes the complex conjugate. It is easy to see that the generalized Sasa-Satsuma equation (1.2) can be reduced to the Sasa-Satsuma equation (1.1) when $a = -1$ and $b = 0$. Suppose that the initial value $u_0(x)$ lies in Schwartz space $\mathcal{S}(\mathbb{R}) = \{ f(x) \in C^\infty(\mathbb{R}) : \sup_{x \in \mathbb{R}} |x^\alpha \partial^\beta f(x)| < \infty, \forall \alpha, \beta \in \mathbb{N} \}$. The vector function $\gamma(k)$ is determined by the initial data in (2.15) and (2.19), and $\gamma(k)$ satisfies the conditions (P1) and (P2), where

$$(P_1): \begin{cases}
  \gamma^*(k) B_1 \gamma(k) + \frac{|b|^2}{4} (\gamma^*(k) \sigma_3 \gamma(k))^2 < 1, \\
  2\gamma^*(k) B_1 \gamma(k) + |\gamma(k)|^2 + |B_1 \gamma(k)|^2 < 4,
\end{cases}$$

with

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} a & b^* \\ b & a \end{pmatrix}. \tag{1.3}$$

(P2): When $\det B_1 > 0$ and $a > 0$, $(2a - |B_1 \gamma(k)|^2)$ and $(2a - \det B_1 |\gamma(k)|^2)/(1 - \gamma^*(k) B_1 \gamma(k))$ are positive and bounded; otherwise, $(|B_1 \gamma(k)|^2 - 2a)$ and $(\det B_1 |\gamma(k)|^2 - 2a)/(1 - \gamma^*(k) B_1 \gamma(k))$ are positive and bounded.

The main result of this paper is as following:
Theorem 1.1. Let \( u(x,t) \) be the solution of the Cauchy problem for the generalized Sasa-Satsuma equation (1.2) with the initial value \( u_0 \in \mathcal{S}(\mathbb{R}) \). Suppose that the vector function \( \gamma(k) \) is defined in (2.19), the hypotheses \((P_1)\) and \((P_2)\) hold. Then, for \( x < 0 \) and \( \sqrt{\pi x} < C \),

\[
    u(x,t) = u_0(x,t) + O\left(c(k_0)t^{-1}\log t\right),
\]

where \( C \) is a fixed constant, and

\[
    u_0(x,t) = \sqrt{\frac{\nu}{12tk_0\gamma'(k_0)B_1\gamma(k_0)}} \left(-|\gamma_1(-k_0)|e^{i(\phi+\arg\gamma_1(-k_0))} + |\gamma_2(-k_0)|e^{-i(\phi+\arg\gamma_2(-k_0))}\right),
\]

\[
    k_0 = \sqrt{-\frac{x}{12t}}, \quad \nu = -\frac{1}{2\pi} \log(1 - \gamma^+(k_0)B_1\gamma(k_0)),
\]

\[
    \phi = \nu \log(196t_0^3) - 16tk_0^3 + \arg\Gamma(-i\nu) + \frac{1}{\pi} \int_{-k_0}^{k_0} \log|\xi| + k_0 d(1 - \gamma^+(\xi)B_1\gamma(\xi)) - \frac{\pi}{4},
\]

\( c(\cdot) \) is rapidly decreasing, \( \Gamma(\cdot) \) is the Gamma function, \( \gamma_1 \) and \( \gamma_2 \) are the first and the second row of \( \gamma(k) \), respectively.

Remark 1.1. The two conditions \((P_1)\) and \((P_2)\) satisfied by \( \gamma(k) \) are necessary. The condition \((P_1)\) guarantees the existence and the uniqueness of the solutions of the basic Riemann-Hilbert problem (2.16) and the Riemann-Hilbert problem (3.1). The boundedness of the function \( \delta(k) \) defined in subsection 3.1 relies on the condition \((P_2)\).

Remark 1.2. In the case of \( a = -1 \) and \( b = 0 \), the generalized Sasa-Satsuma equation (1.2) can be reduced to the Sasa-Satsuma equation. Then it is obvious that the condition \((P_1)\) is true, and the condition \((P_2)\) is reduced to the case that \( |\gamma(k)| \) is bounded. Therefore, the conditions \((P_1)\) and \((P_2)\) in this case are equivalent to the condition related to the reflection coefficient in [34], that is, \( |\gamma(k)| \) is bounded for the Sasa-Satsuma equation.

The outline of this paper is as follows. In section 2, we derive a Riemann-Hilbert problem from the scattering relation. The solution of the generalized Sasa-Satsuma equation is changed into the solution of the Riemann-Hilbert problem. In section 3, we deal with the Riemann-Hilbert problem via nonlinear steepest decent method, from which the long-time asymptotics in Theorem 1.1 is obtained at the end.

2. Basic Riemann-Hilbert problem

We begin with the \( 3 \times 3 \) Lax pair of the generalized Sasa-Satsuma equation

\[
    \psi_x = (ik\sigma + U)\psi, \quad (2.1a)
\]

\[
    \psi_t = (4ik^3\sigma + V)\psi, \quad (2.1b)
\]

where \( \psi \) is a matrix function and \( k \) is the spectral parameter, \( \sigma = \text{diag}(1, 1, -1) \),

\[
    U = \begin{pmatrix}
        0 & 0 & u \\
        0 & 0 & u^* \\
        au^* + bu & au + b^*u^* & 0
    \end{pmatrix}, \quad (2.2)
\]
We introduce a new eigenfunction $\mu$ through $\mu = \psi e^{-ikx - 4ik^3\sigma}$, where $e^\sigma = \text{diag}(e, e, e^{-1})$. Then (2.1a) and (2.1b) become

\[
\begin{align*}
\mu_x &= ik[\sigma, \mu] + U\mu, \\
\mu_t &= 4ik^3[\sigma, \mu] + V\mu,
\end{align*}
\]  

(2.4a) (2.4b)

where $[\cdot, \cdot]$ is the commutator, $[\sigma, \mu] = \sigma\mu - \mu\sigma$. From (2.4a), the matrix Jost solution $\mu_\pm$ satisfy the Volterra integral equations

\[
\mu_\pm(k; x, t) = I + \int_{-\infty}^{x} e^{ik(x - \xi)} U(\xi, t) \mu_\pm(k; \xi, t) e^{-ik(x - \xi)} d\xi, 
\]

(2.5)

Set $\mu_{\pm k}$ represent the first two columns of $\mu_\pm$, and $\mu_{\pm kR}$ denote the third column, i.e., $\mu_\pm = (\mu_{\pm kL}, \mu_{\pm kR})$. Furthermore, we can infer that $\mu_{\pm L}$ and $\mu_{\pm R}$ are analytic in the lower complex $k$-plane $C_-$, $\mu_{\pm L}$ and $\mu_{\pm R}$ are analytic in the the upper complex $k$-plane $C_+$. Then we can introduce sectionally analytic function $P_1(k)$ and $P_2(k)$ by

\[
\begin{align*}
P_1(k) &= (\mu_{-L}(k), \mu_{+R}(k)), \quad k \in C_-,
\end{align*}
\]

\[
\begin{align*}
P_2(k) &= (\mu_{+L}(k), \mu_{-R}(k)), \quad k \in C_+.
\end{align*}
\]

(2.6)

One can find that $U$ is traceless from (2.2), so $\det \mu_\pm$ are independent of $x$. Besides, $\det \mu_\pm = 1$ according to the evolution of $\det \mu_\pm$ at $x = \pm \infty$. Because all the $\mu_\pm e^{ikx + 4ik^3\sigma}$ satisfy the differential equations (2.1a) and (2.1b), they are linear related. So there exists a scattering matrix $s(k)$ that satisfies

\[
\mu_- = s(k) e^{-ikx - 4ik^3\sigma}, \quad \det s(k) = 1.
\]

In this paper, we denote a $3 \times 3$ matrix $A$ by the block form

\[
A = \begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix},
\]

where $A_{11}$ is a $2 \times 2$ matrix and $A_{22}$ is scalar. Let $q = (u, u')^T$ and we can rewrite $U$ of (2.2) as

\[
U = \begin{pmatrix}
0_{2 \times 2} & q \\
q^T B_1 & 0
\end{pmatrix},
\]

where $"^T"$ is the Hermitian conjugate. In addition, there are two symmetry properties for $U$,

\[
B^{-1}U^T(k)^tB = -U(k), \quad \tau U(-k)^t \tau = U(k),
\]

(2.7)

\[
B = \begin{pmatrix}
B_1 & 0 \\
0 & -1
\end{pmatrix}, \quad \tau = \begin{pmatrix}
\sigma_1 & 0 \\
0 & 1 \\
0 & 1
\end{pmatrix}, \quad \sigma_1 = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},
\]

(2.8)

where $B$ and $\tau$ are represented as block forms. Hence, the Jost solutions $\mu_\pm$ and the scattering matrix $s(k)$ also have the corresponding symmetry properties

\[
B^{-1} \mu_\pm^T(k)^tB = \mu_\pm^T(k), \quad \tau \mu_\pm(-k)^t \tau = \mu_\pm(k);
\]

(2.9)
where adj

We write s(k) as block form (s_{ij})_{2×2} and from the symmetry properties (2.10) we have

\begin{align*}
  s_{22}(k) &= \det[s_{11}^+(k^*)], \quad B_1^{-1} s_{21}^+(k^*) = \text{adj}[s_{11}(k)] s_{12}(k),
\end{align*}

where adjX denote the adjoint of matrix X. Then we can write s(k) as

\begin{equation}
  s(k) = \begin{pmatrix} s_{11}(k) & s_{12}(k) \\ s_{12}^+(k^*) \text{adj}[s_{11}^+(k^*)] B_1 & \det[s_{11}^+(k^*)] \end{pmatrix},
\end{equation}

where

\begin{equation}
  \sigma_1 s_{11}^+(−k^*) \sigma_1 = s_{11}(k), \quad \sigma_1 s_{12}^+(−k^*) = s_{12}(k).
\end{equation}

From the evaluation of (2.6) at \( t = 0 \), one infers

\begin{equation}
  s(k) = \lim_{x→+∞} e^{-ikxσ} \mu_−(k; x, 0) e^{ikxσ},
\end{equation}

which implies that

\begin{equation}
  \begin{cases}
    s_{11}(k) = I + \int_{−∞}^{+∞} q(ξ, 0) \mu_{−21}(k; ξ, 0) \, dξ, \\
    s_{12}(k) = \int_{−∞}^{+∞} e^{−2ikξ} q(ξ, 0) \mu_{−22}(k; ξ, 0) \, dξ.
  \end{cases}
\end{equation}

**Theorem 2.1.** Let \( M(k; x, t) \) be analytic for \( k ∈ \mathbb{C}\backslash \mathbb{R} \) and satisfy the Riemann-Hilbert problem

\begin{equation}
  \begin{cases}
    M_+(k; x, t) = M_−(k; x, t) J(k; x, t), \quad k ∈ \mathbb{R}, \\
    M(k; x, t) → I, \quad k → ∞,
  \end{cases}
\end{equation}

where

\begin{equation}
  M_±(k; x, t) = \lim_{ε→0^±} M(k ± iε; x, t),
\end{equation}

\begin{equation}
  J(k; x, t) = \begin{pmatrix} I − γ(k) γ^+(k^*) B_1 & −e^{2ikt} γ(k) \\ e^{2ikt} γ^+(k^*) B_1 & 1 \end{pmatrix},
\end{equation}

\begin{equation}
  \theta(k; x, t) = −\frac{x}{t} k − 4k^3, \quad γ(k) = s_{11}^+(k) s_{12}(k),
\end{equation}

\( γ(k) \) lies in Schwartz space and satisfies

\begin{equation}
  \sigma_1 γ^+(−k^*) = γ(k).
\end{equation}

Then the solution of this Riemann-Hilbert problem exists and is unique, the function

\begin{equation}
  q(x, t) = (u(x, t), u^∗(x, t))^T = −2i \lim_{k→∞} (k(M(k; x, t))_{12})
\end{equation}

and \( u(x, t) \) is the solution of the generalized Sasa-Satsuma equation.
Proof. The matrix \((J(k; x, t) + J^*(k; x, t))/2\) is positive definite because of the condition \((P_1)\) that \(\gamma(k)\) satisfies, then the solution of the Riemann-Hilbert problem \((2.16)\) is existent and unique according to the Vanishing Lemma [39]. We define \(M(k; x, t)\) by

\[
M(k; x, t) = \begin{cases} 
(\mu_{-L}(k), \mu_{+R}(k) \det[a^i(k^*)]), & k \in \mathbb{C}_- , \\
(\mu_{+L}(k) a(k), \mu_{-R}(k)), & k \in \mathbb{C}_+ .
\end{cases} \tag{2.22}
\]

Considering the scattering relation \((2.6)\) and the construction of \(M(k; x, t)\), we can obtain the jump condition and the corresponding Riemann-Hilbert problem \((2.16)\) after tedious but straightforward algebraic manipulations. Substituting the large \(k\) asymptotic expansion of \(M(k; x, t)\) into \((2.4a)\) and compare the coefficients of \(O(\frac{1}{k})\), we can get \((2.21)\).

\[\square\]

3. Long-time asymptotic behavior

In this section, we compute the Riemann-Hilbert problem \((2.16)\) by the nonlinear steepest decent method and study the long-time asymptotic behavior of the solution. We make the following basic notations. (i) For any matrix \(M\) define \(|M| = (\text{tr} M^* M)^{\frac{1}{2}}\) and for any matrix function \(A(\cdot)\) define \(|A(\cdot)|_\rho = \|A(\cdot)\|_\rho\). (ii) For two quantities \(A\) and \(B\) define \(A \leq B\) if there exists a constant \(C > 0\) such that \(|A| \leq CB\). If \(C\) depends on the parameter \(\alpha\) we shall say that \(A \preceq_\alpha B\). (iii) For any oriented contour \(\Sigma\), we say that the left side is \(+\) and the right side is \(−\).

3.1. The first transformation: reorientation

First of all, it is noteworthy that there are two stationary points \(±k_0\), where \(±k_0 = \pm \sqrt{\frac{\alpha}{\rho}\gamma(0)}\) satisfied \(\frac{d\gamma}{d\rho}|_{k=±k_0} = 0\). The jump matrix \(J(k; x, t)\) have a lower-upper triangular factorization and a upper-lower triangular factorization. We can introduce an appropriate Riemann-Hilbert problem to unify these two forms of factorizations. In this process, we have to reorient the contour of the Riemann-Hilbert problem.

The two factorizations of the jump matrix \(J\) are

\[
J = \begin{cases} 
\begin{pmatrix}
I & -e^{-2it\gamma(k)} \\
0 & 1
\end{pmatrix} \\
\begin{pmatrix}
I \gamma(k) & 0 \\
0 & 1
\end{pmatrix}
\end{cases} \begin{pmatrix}
I & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
I - e^{2it\gamma(k^*)B_1} & 0 \\
0 & (1 - \gamma^i(k^*)B_1\gamma(k))^{-1}
\end{pmatrix} \begin{pmatrix}
I & -e^{-2it\gamma(k)} \\
0 & 1
\end{pmatrix}.
\]

We introduce a \(2 \times 2\) matrix function \(\delta(k)\) to make the two factorization unified, and \(\delta(k)\) satisfies the following Riemann-Hilbert problem

\[
\begin{cases} 
\delta_+(k) = \delta_-(k)(I - \gamma(k)\gamma^i(k^*)B_1), & k \in (-k_0, k_0), \\
\delta_-(k), & k \in (-\infty, -k_0) \cup (k_0, +\infty), \\
\delta(k) \to I, & k \to \infty,
\end{cases} \tag{3.1}
\]

which implies a scalar Riemann-Hilbert problem

\[
\begin{cases} 
\det \delta_+(k) = \det \delta_-(k)(1 - \gamma^i(k^*)B_1\gamma(k)), & k \in (-k_0, k_0), \\
\det \delta_-(k), & k \in (-\infty, -k_0) \cup (k_0, +\infty), \\
\det \delta(k) \to 1, & k \to \infty.
\end{cases} \tag{3.2}
\]

\[\text{AIMS Mathematics} \quad \text{Volume 5, Issue 6, 7413–7437.}\]
The jump matrix \( I - \gamma(k) \gamma^\dagger(k^\ast)B_1 \) of Riemann-Hilbert problem (3.1) is positive definite, so the solution \( \delta(k) \) exists and is unique. The scalar Riemann-Hilbert problem (3.2) can be solved by the Plemelj formula,

\[
\det \delta(k) = \left( \frac{k - k_0}{k + k_0} \right)^{-i\nu} e^{i\epsilon(k)},
\]

(3.3)

where

\[
\nu = -\frac{1}{2\pi} \log(1 - \gamma^\dagger(k_0)B_1 \gamma(k_0)),
\]

\[
\chi(k) = -\frac{1}{2\pi i} \int_{k_0}^{k_0} \log \left( \frac{1 - \gamma^\dagger(\xi^\ast)B_1 \gamma(\xi)}{1 - \gamma^\dagger(k_0^\ast)B_1 \gamma(k_0)} \right) \frac{d\xi}{\xi - k}.
\]

Then we have by uniqueness that

\[
\delta(k) = B_1^{-1}(\delta^\dagger(k^\ast))^{-1} B_1, \quad \delta(k) = \sigma_1 \delta^\ast(-k^\ast) \sigma_1.
\]

(3.4)

Substituting (3.4) to (3.1), we have

\[
\delta^\dagger(k^\ast)B_1 \delta_+(k) = B_1 - B_1 \gamma(k) \gamma^\dagger(k^\ast)B_1,
\]

(3.5)

which means that

\[
\text{tr}[\delta^\dagger(k^\ast)B_1 \delta_+(k)] = 2a - |B_1 \gamma(k)|^2.
\]

(3.6)

Actually, the condition \((P_2)\) satisfied by \( \gamma(k) \) guarantee the boundedness of \( \delta_+(k) \) and we give a brief proof below. When \( \det B_1 > 0 \), we find that the Hermitian matrix \( B_1 \) can be decomposition. In other words, there exists a triangular matrix \( S \) that satisfies \( B_1 = aS^\dagger S \). So \( \text{tr}[\delta^\dagger S B_1 \delta_+] = aS \delta_+^2 \). When \( \det B_1 < 0 \) and \( |a| > 0 \), the matrix \( B_1 \) has a decomposition \( B_1 = S^\dagger D S \), where \( S \) is a triangular matrix and \( D \) is a diagonal matrix and the diagonal elements have opposite signs. In the case of \( a > 0 \), \( B_1 \) can be decomposed as below,

\[
B_1 = \begin{pmatrix}
-a & b^* \\
0 & 1
\end{pmatrix} \begin{pmatrix}
a \cdot \det B_1 & 0 \\
0 & a
\end{pmatrix} \begin{pmatrix}
a & 0 \\
-b & 1
\end{pmatrix}^{-1}.
\]

(3.7)

We denote \( S \delta_+(k) \) by \( (G_{ij})_{3 \times 3} \) and \( c_1 = 2a - |B_1 \gamma(k)|^2 \) is negative, then

\[
a \cdot \det B_1 (|G_{11}|^2 + |G_{21}|^2) + a(|G_{12}|^2 + |G_{22}|^2) = c_1.
\]

(3.8)

Noticing that \( \det B_1 < 0 \), we find a negative constant \( c_2 \) that satisfies \( c_2 \leq a \cdot \det B_1 (c_3 - 1)/(1 - \det B_1 c_3) \), where \( c_3 \) is a constant and \( 0 < c_3 < 1 \), which implies

\[
|S \delta_+(k)|^2 \leq \frac{c_1}{c_2} \leq 1.
\]

(3.9)

The case that \( a < 0 \) is similar. In particular, when \( a = 0 \), then \( |b| > 0 \), it is easy to see that \( B_1 \) is not definite. For \( |\text{Re} b| > 0 \), we have the decomposition

\[
B_1 = \begin{pmatrix}
\frac{b}{|b|^2 + |b|^2} & \frac{b^*}{|b|^2 + |b|^2} \\
\frac{b^*}{|b|^2 + |b|^2} & -\frac{1}{b+b^*}
\end{pmatrix} \begin{pmatrix}
\frac{b}{|b|^2 + |b|^2} & \frac{b^*}{|b|^2 + |b|^2} \\
\frac{b^*}{|b|^2 + |b|^2} & -\frac{1}{b+b^*}
\end{pmatrix}^{-1}.
\]

(3.10)
where the jump matrix $J$ for all $k$. Hence, by the maximum principle, we have

$$B_1 = \left( \begin{array}{cc} \frac{i}{2} & \frac{i}{2} \\ -\frac{i}{2} & \frac{i}{2} \end{array} \right) \left( \begin{array}{cc} -ib/2 & 0 \\ 0 & ib/2 \end{array} \right) \left( \begin{array}{cc} \frac{i}{2} & \frac{i}{2} \\ -\frac{i}{2} & \frac{i}{2} \end{array} \right) = \left( \begin{array}{cc} -ib/2 & 0 \\ 0 & ib/2 \end{array} \right) - \left( \begin{array}{cc} -ib/2 & 0 \\ 0 & ib/2 \end{array} \right) = 0. \quad (3.11)$$

So we get the boundedness of $|\delta_+(k)|$. The others have the same analysis,

$$\delta_+(k)B_1\delta_-(k) = (B_1 - \gamma(k)\gamma(k^*))^{-1}, \quad k \in (-k_0, k_0), \quad (3.12)$$

$$|\delta_+(k)|^2 = |\delta_-(k)|^2 = 2, \quad k \in (-\infty, -k_0) \cup (k_0, +\infty), \quad (3.13)$$

$$(\frac{1}{1 - \gamma(k)B_1\gamma(k)})^{1/2}, \quad k \in (-k_0, k_0), \quad (3.14)$$

$$|\det \delta_+(k)| = \begin{cases} 1 - \gamma(k)B_1\gamma(k), & k \in (-k_0, k_0), \\ 1, & k \in (-\infty, -k_0) \cup (k_0, +\infty), \end{cases} \quad (3.15)$$

$$|\det \delta_-(k)| = \begin{cases} \frac{1}{1 - \gamma(k)B_1\gamma(k)}, & k \in (-k_0, k_0), \\ 1, & k \in (-\infty, -k_0) \cup (k_0, +\infty). \end{cases} \quad (3.16)$$

Hence, by the maximum principle, we have

$$|\delta(k)| \leq \text{const} < \infty, \quad |\det \delta(k)| \leq \text{const} < \infty, \quad (3.16)$$

for all $k \in \mathbb{C}$. We define the functions

$$\rho(k) = \begin{cases} -\gamma(k), & k \in (-\infty, -k_0) \cup (k_0, +\infty), \\ \gamma(k), & k \in (-k_0, k_0), \end{cases} \quad (3.17)$$

$$\Delta(k) = \begin{pmatrix} \delta(k) & 0 \\ 0 & (\det \delta(k))^{-1} \end{pmatrix}. \quad (3.18)$$

Figure 1. The reoriented contour on $\mathbb{R}$.

We reverse the orientation for $k \in (-\infty, k_0) \cup (k_0, +\infty)$ as in Figure 1, and $M^\Delta(k; x, t) = M(k; x, t)\Delta^{-1}(k)$ satisfies the Riemann-Hilbert problem on the reoriented contour

$$M^\Delta(k; x, t) = M^\Delta(k; x, t)J^\Delta(k; x, t), \quad k \in \mathbb{R}, \quad (3.19)$$

where the jump matrix $J^\Delta(k; x, t)$ has a decomposition

$$J^\Delta(k; x, t) = (b_-)^{-1}b_+ = \begin{pmatrix} I & 0 \\ e^{2i\theta(k)}b_+ & e^{-2i\theta(k)} \end{pmatrix} \frac{1}{\det \delta_-(k)} \begin{pmatrix} I & 0 \\ 0 & e^{-2i\theta(k)} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & e^{-2i\theta(k)} \end{pmatrix} \begin{pmatrix} 1 \\ -e^{2i\theta(k)} \end{pmatrix} \delta_+(k) \rho(k) \det \delta_+(k) \quad (3.20)$$
3.2. Extend to the augmented contour

For the convenience of discussion, we define

\[
L = \{k_0 + \alpha k_0 e^{-\frac{2\pi i}{2}} : -\infty < \alpha \leq \sqrt{2}\} \cup \{-k_0 + \alpha k_0 e^{-\frac{2\pi i}{2}} : -\infty < \alpha \leq \sqrt{2}\},
\]

\[
L_\varepsilon = \{k_0 + \alpha k_0 e^{-\frac{2\pi i}{2}} : -\varepsilon < \alpha \leq \sqrt{2}\} \cup \{-k_0 + \alpha k_0 e^{-\frac{2\pi i}{2}} : -\varepsilon < \alpha \leq \sqrt{2}\}.
\]

**Theorem 3.1.** The vector function \( \rho(k) \) has a decomposition

\[
\rho(k) = h_1(k) + h_2(k) + R(k), \quad k \in \mathbb{R},
\]

where \( R(k) \) is a piecewise-rational function and \( h_2(k) \) has an analytic continuation to \( L \). Besides, they admit the following estimates

\[
|e^{-2i\theta(k)}h_1(k)| \leq \frac{1}{(1 + |k|^2)^{l}} , \quad k \in \mathbb{R},
\]

(3.21)

\[
|e^{-2i\theta(k)}h_2(k)| \leq \frac{1}{(1 + |k|^2)^{l}} , \quad k \in L,
\]

(3.22)

\[
|e^{-2i\theta(k)}R(k)| \leq e^{-16\varepsilon k_0^2}, \quad k \in L_\varepsilon,
\]

(3.23)

for an arbitrary positive integer \( l \). Considering the Schwartz conjugate

\[
\rho^*(k^*) = R^*(k^*) + h_1^*(k^*) + h_2^*(k^*),
\]

we can obtain the same estimate for \( e^{2i\theta(k)}h_1^*(k^*) \), \( e^{2i\theta(k)}h_2^*(k^*) \) and \( e^{2i\theta(k)}R^*(k^*) \) on \( \mathbb{R} \cup L^* \).

**Proof.** It follows from Proposition 4.2 in [18]. \( \square \)

A direct calculation shows that \( b_\delta \) of (3.20) can be decomposed further

\[
b_+ = b_\delta^* b_\delta^* = (I_{3x3} + \omega_\delta^*)(I_{3x3} + \omega_\delta^*)
\]

\[
= \begin{pmatrix} I_{2x2} & -e^{2i\theta_\delta}[\det\delta_+(k)]\delta_+(k)h_1(k) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I_{2x2} & -e^{2i\theta_\delta}[\det\delta_+(k)]\delta_+(k)h_2(k) + R(k) \\ 0 & 1 \end{pmatrix},
\]

\[
b_- = b_\delta^* b_\delta^* = (I_{3x3} - \omega_\delta^*)(I_{3x3} - \omega_\delta^*)
\]

\[
= \begin{pmatrix} I_{2x2} & 0 \\ -e^{2i\theta_\delta}h_1^*(k^*)B_1\delta_+^{-1}(k) & 1 \end{pmatrix} \begin{pmatrix} I_{2x2} & 0 \\ -e^{2i\theta_\delta}[h_2^*(k^*) + R^*(k^*)]B_1\delta_+^{-1}(k) & 1 \end{pmatrix}.
\]

![Figure 2. The contour Σ.](image)
Define the oriented contour $\Sigma$ by $\Sigma = L \cup L^*$ as in Figure 2. Let

$$M^b(k; x, t) = \begin{cases} 
M^\Lambda(k; x, t), & k \in \Omega_1 \cup \Omega_2, \\
M^\Lambda(k; x, t)(b^a)^{-1}, & k \in \Omega_3 \cup \Omega_4 \cup \Omega_5, \\
M^\Lambda(k; x, t)(b^c)^{-1}, & k \in \Omega_6 \cup \Omega_7 \cup \Omega_8.
\end{cases}$$

(3.24)

Lemma 3.1. $M^b(k; x, t)$ is the solution of the Riemann-Hilbert problem

$$\begin{cases}
M^b(k; x, t) = M^a(k; x, t)J^b(k; x, t), & k \in \Sigma, \\
M^b(k; x, t) \to I, & k \to \infty,
\end{cases}$$

(3.25)

where the jump matrix $J^b(k; x, t)$ satisfies

$$J^b(k; x, t) = (b^a)^{-1}b^b = \begin{cases} 
I^{-1}b^a_+, & k \in L, \\
(b^a)^{-1}I, & k \in L^*, \\
(b^c)^{-1}b^c_+, & k \in \mathbb{R}.
\end{cases}$$

(3.26)

Proof. We can construct the Riemann-Hilbert problem (3.25) based on the Riemann-Hilbert problem (3.19) and the decomposition of $b^a$. In the meantime, the asymptotics of $M^a(k; x, t)$ is derived from the convergence of $b^a$ as $k \to \infty$. For fixed $x$ and $t$, we pay attention to the domain $\Omega_3$. Noticing the boundedness of $\delta(k)$ and $\inf \delta(k)$ in (3.16), we arrive at

$$|e^{-2i\theta}[\det(\delta(k))[h_2(k) + R(k)]| \delta(k)| \leq |e^{-2i\theta}h_2(k)| + |e^{-2i\theta}R(k)|.$$  

Consider the definition of $R(k)$ in this domain,

$$|e^{-2i\theta}h_2(k)| \leq \frac{1}{|k + i|}, \quad |e^{-2i\theta}R(k)| \leq \frac{1}{|k + i|^{m+3}},$$

where $m$ is a positive integer and $\mu_i$ is the coefficient of the Taylor series around $k_0$. Combining with the boundedness of $h_2(k)$ in Theorem 3.1, we obtain that $M^b(k; x, t) \to I$ when $k \in \Omega_3$ and $k \to \infty$. The others are similar to this domain. \hfill $\Box$

The above Riemann-Hilbert problem (3.25) can be solved as follows. Set

$$\omega^x_\pm = \pm(b^a_\pm - I), \quad \omega^x = \omega^x_+ + \omega^x_-.$$

Let

$$(C \pm f)(k) = \int_{\Sigma} \frac{f(\xi)}{\xi - k_\pm} \frac{d\xi}{2\pi i}, \quad f \in \mathcal{L}^2(\Sigma)$$

(3.27)

denote the Cauchy operator, where $C_+ f$ ($C_- f$) denotes the left (right) boundary value for the oriented contour $\Sigma$ in Figure 2. Define the operator $C_{\omega^x} : \mathcal{L}^2(\Sigma) + \mathcal{L}^\infty(\Sigma) \to \mathcal{L}^2(\Sigma)$ by

$$C_{\omega^x} f = C_+(f \omega^x_+) + C_-(f \omega^x_-)$$

(3.28)

for the $3 \times 3$ matrix function $f$.  

AIMS Mathematics 
Volume 5, Issue 6, 7413–7437.
Lemma 3.2 (Beals-Coifman). If \( \mu^\#(k; x, t) \in L^2(\Sigma) + L^\infty(\Sigma) \) is the solution of the singular integral equation

\[
\mu^\# = I + C_{\omega^\#} \mu^\#.
\]

Then

\[
M^\#(k; x, t) = I + \int_{\Sigma} \frac{\mu^\#(\xi; x, t) \omega^\#(\xi; x, t)}{\xi - k} \frac{d\xi}{2\pi i}
\]

is the solution of the Riemann-Hilbert problem (3.25).

Proof. See [18], P. 322 and [40]. \( \square \)

Theorem 3.2. The expression of the solution \( q(x, t) \) can be written as

\[
q(x, t) = (u(x, t), u^*(x, t))^T = \frac{1}{\pi} \left( \left( (1 - C_{\omega^\#})^{-1} I \right)(\xi) \omega^\#(\xi) d\xi \right)_{12}.
\]

Proof. From (2.21), (3.24) and Lemma 3.2, the solution \( q(x, t) \) of the generalized Sasa-Satsuma equation is expressed by

\[
q(x, t) = \lim_{k \to \infty} -2i \left[ k(M^\#(k; x, t))_{12} \right]
\]

\[
= \frac{1}{\pi} \left( \int_{\Sigma} \mu^\#(\xi; x, t) \omega^\#(\xi) d\xi \right)_{12}
\]

\[
= \frac{1}{\pi} \left( \int_{\Sigma} \left( (1 - C_{\omega^\#})^{-1} I \right)(\xi) \omega^\#(\xi) d\xi \right)_{12}.
\]

3.3. Contour truncation

Set \( \Sigma' = \Sigma \setminus (\mathbb{R} \cup L_e \cup L_e^*) \) oriented as in Figure 3. We will convert the Riemann-Hilbert problem on the contour \( \Sigma \) to a Riemann-Hilbert problem on the contour \( \Sigma' \) and estimate the errors between the two Riemann-Hilbert problems. Let \( \omega^\# = \omega^c + \omega^r = \omega^b + \omega^c + \omega^r \), where \( \omega^r = \omega^\#|_{\mathbb{R}} \) is supported on \( \mathbb{R} \) and is composed of terms of type \( h_1(k) \) and \( h_1^*(k^*) \); \( \omega^b \) is supported on \( L \cup L^* \) and is composed of contribution to \( \omega^\# \) from terms of type \( h_2(k) \) and \( h_2^*(k^*) \); \( \omega^c \) is supported on \( L_e \cup L_e^* \) and is composed of contribution to \( \omega^\# \) from terms of type \( R(k) \) and \( R^*(k^*) \).

Lemma 3.3. For arbitrary positive integer \( l \), as \( t \to \infty \),

\[
\| \omega^r \|_{L^1(\mathbb{R}) \cup L^2(\mathbb{R}) \cup L^\infty(\mathbb{R})} \lesssim t^{-l},
\]

AIMS Mathematics
\[
\begin{align*}
\| \omega^h \|_{L^2(L^1,L^1)} & \lesssim t^{-1}, \\
\| \omega^f \|_{L^2(L^1,L^1)} & \lesssim e^{-16\delta k_0^3 t}, \\
\| \omega^o \|_{L^2(\Sigma)} & \lesssim (tk_0^3)^{-\frac{1}{2}}, \quad \| \omega^o \|_{L^2(\Sigma)} \lesssim (tk_0^3)^{-\frac{1}{2}}.
\end{align*}
\]

**Proof.** The proof of estimates (3.30), (3.31), (3.32) follows from Theorem 3.1. Afterwards, we consider the definition of \( R(k) \) on the contour \( k = k_0 + ak_0 e^{\frac{3\pi i}{2}} \), \(-\infty < a < \epsilon\),

\[ |R(k)| \lesssim (1 + |k|^5)^{-1}. \]

Resorting to \( \text{Re}(i\theta) \geq 8\alpha^2 k_0^3 \) and the boundedness of \( \delta(k) \) and \( \text{det} \delta(k) \) in (3.16), we can obtain

\[ |e^{-2i\theta} \text{det} \delta(k)| R(k) \delta(k) | \lesssim e^{-16\alpha^2} (1 + |k|^5)^{-1}. \]

Then we obtain (3.33) by simple computations. \( \square \)

**Lemma 3.4.** As \( t \to \infty \), \((1 - C_{\omega'})^{-1}: L^2(\Sigma) \to L^2(\Sigma)\) exists and is uniformly bounded:

\[ \|(1 - C_{\omega'})^{-1}\|_{L^2(\Sigma)} \leq 1. \]

Furthermore, \( \|(1 - C_{\omega'})^{-1}\|_{L^2(\Sigma)} \lesssim 1. \)

**Proof.** It follows from Proposition 2.23 and Corollary 2.25 in [18]. \( \square \)

**Lemma 3.5.** As \( t \to \infty \),

\[ \int_{\Sigma} ((1 - C_{\omega'})^{-1} I(\xi) \omega^h(\xi)) \, d\xi = \int_{\Sigma} ((1 - C_{\omega'})^{-1} I(\xi) \omega^h(\xi)) \, d\xi + O((tk_0^3)^{-\frac{1}{2}}). \]

**Proof.** A simple computation shows that

\[ ((1 - C_{\omega'})^{-1} I) \omega^h = ((1 - C_{\omega'})^{-1} I) \omega'+ \omega^h + ((1 - C_{\omega'})^{-1} (C_{\omega'} I)) \omega^h \\
+ ((1 - C_{\omega'})^{-1} (C_{\omega'} I)) \omega^f + ((1 - C_{\omega'})^{-1} C_{\omega'} (1 - C_{\omega'})^{-1} (C_{\omega'} I)) \omega^f. \]

After a series of tedious computations and utilizing the consequence of Lemma 3.4, we arrive at

\[ \| \omega^h \|_{L^2(\Sigma)} \lesssim \| \omega^h \|_{L^2(\Sigma)} + \| \omega^f \|_{L^2(L^1,L^1)} + \| \omega^f \|_{L^2(L^1,L^1)} \lesssim (tk_0^3)^{-\frac{1}{2}}, \]

\[ \| (1 - C_{\omega'})^{-1} (C_{\omega'} I) \omega^h \|_{L^2(\Sigma)} \lesssim \| (1 - C_{\omega'})^{-1} \|_{L^2(\Sigma)} \| C_{\omega'} I \|_{L^2(\Sigma)} \| \omega^h \|_{L^2(\Sigma)} \]

\[ \lesssim \| \omega^h \|_{L^2(\Sigma)} \| \omega^h \|_{L^2(\Sigma)} \lesssim (tk_0^3)^{-\frac{1}{2}}, \]

\[ \| (1 - C_{\omega'})^{-1} (C_{\omega'} I) \omega^f \|_{L^2(\Sigma)} \lesssim \| (1 - C_{\omega'})^{-1} \|_{L^2(\Sigma)} \| C_{\omega'} I \|_{L^2(\Sigma)} \| \omega^f \|_{L^2(\Sigma)} \]

\[ \lesssim \| \omega^f \|_{L^2(\Sigma)} \| \omega^f \|_{L^2(\Sigma)} \lesssim (tk_0^3)^{-\frac{1}{2}}. \]

Then the proof is accomplished as long as we substitute the estimates above into (3.35). \( \square \)
3.4. Noninteraction of disconnected contour components

Notice that $\omega'(k) = 0$ when $k \in \Sigma \setminus \Sigma'$, let $C_{\omega'}|_{L^2(\Sigma')}$ denote the restriction of $C_{\omega'}$ to $L^2(\Sigma')$. For simplicity, we write $C_{\omega'}|_{L^2(\Sigma')}$ as $C_{\omega'}$. Then

$$\int_{\Sigma}((1 - C_{\omega'})^{-1}I)(\xi)\omega'(\xi)\,d\xi = \int_{\Sigma}((1 - C_{\omega'})^{-1}I)(\xi)\omega'(\xi)\,d\xi.$$  

Lemma 3.6. As $t \to \infty$,

$$q(x, t) = (u(x, t), u'(x, t))^T = \frac{1}{\pi} \left( \int_{\Sigma}((1 - C_{\omega'})^{-1}I)(\xi)\omega'(\xi)\,d\xi, 1 \right) + O((tk_0^3)^{-1}). \quad (3.36)$$

Proof. From (3.29) and (3.34), we can obtain the result directly. □

Let $L' = L \setminus L_e$ and $\mu' = (1 - C_{\omega'})^{-1}I$. Then

$$M'(k; x, t) = I + \int_{\Sigma} \frac{\mu'(k; x, t)\omega'(k; x, t)}{\xi - k} \,\frac{d\xi}{2\pi i}$$

solves the Riemann-Hilbert problem

$$\begin{cases} M'_x(k; x, t) = M'_x(k; x, t)J'(k; x, t), & k \in \Sigma', \\ M'(k; x, t) \to I, & k \to \infty, \end{cases}$$

where

$$J' = (b')^{-1}b' = (I - \omega')^{-1}(I + \omega'),$$

$$\omega' = \omega'_+ + \omega'_-,$$

$$b'_+ = \begin{pmatrix} I & -e^{-2i\theta} [\det(\delta(k))]R(k) \\ 0 & 1 \end{pmatrix}, \quad b'_- = I, \quad \text{on } L',$$

$$b'_+ = I, \quad b'_- = \begin{pmatrix} I & 0 \\ 0 & \det(\delta(k))^{-1} \end{pmatrix}, \quad \text{on } (L')^*.$$
Lemma 3.8. As \( t \to \infty \),

\[
\int_{\Sigma'} ((1 - C_{\omega'})^{-1} I(\xi) \omega'(\xi) \, d\xi = \int_{\Sigma'_{A}} ((1 - C_{\omega'_{A}})^{-1} I(\xi) \omega'_{A}(\xi) \, d\xi \\
+ \int_{\Sigma'_{B}} ((1 - C_{\omega'_{B}})^{-1} I(\xi) \omega'_{B}(\xi) \, d\xi + O\left(\frac{c(k_0)}{t}\right). \tag{3.38}
\]

Proof. From identity

\[
(1 - C_{\omega'_{A}} - C_{\omega'_{B}})(1 + C_{\omega'_{A}}(1 - C_{\omega'_{A}})^{-1} + C_{\omega'_{B}}(1 - C_{\omega'_{B}})^{-1}) \\
= 1 - C_{\omega'_{A}} C_{\omega'_{A}}(1 - C_{\omega'_{A}})^{-1} - C_{\omega'_{A}} C_{\omega'_{B}}(1 - C_{\omega'_{B}})^{-1},
\]

we have

\[
(1 - C_{\omega'})^{-1} = 1 + C_{\omega'_{A}}(1 - C_{\omega'_{A}})^{-1} + C_{\omega'_{B}}(1 - C_{\omega'_{B}})^{-1} \\
+ [1 + C_{\omega'_{A}}(1 - C_{\omega'_{A}})^{-1} + C_{\omega'_{B}}(1 - C_{\omega'_{B}})^{-1}] \left[1 - C_{\omega'_{B}} C_{\omega'_{A}}(1 - C_{\omega'_{A}})^{-1} \\
- C_{\omega'_{A}} C_{\omega'_{B}}(1 - C_{\omega'_{B}})^{-1}\right]^{-1} \left[C_{\omega'_{B}} C_{\omega'_{A}}(1 - C_{\omega'_{A}})^{-1} + C_{\omega'_{A}} C_{\omega'_{B}}(1 - C_{\omega'_{B}})^{-1}\right].
\]

Based on Lemma (3.7) and Lemma (3.4), we arrive at (3.38). \( \square \)

For the sake of convenience, we write the restriction \( C_{\omega'_{A}}|_{\partial \Sigma'_{A}(\xi)} \) as \( C_{\omega'_{A}} \), similar for \( C_{\omega'_{B}} \). From the consequences of Lemma 3.6 and Lemma 3.8, as \( t \to \infty \), we have

\[
g(x, t) = -\left\{ \left( \int_{\Sigma'_{A}} ((1 - C_{\omega'_{A}})^{-1} I(\xi) \omega'_{A}(\xi) \, d\xi \right)_{12} \right. \\
- \left. \left( \int_{\Sigma'_{B}} ((1 - C_{\omega'_{B}})^{-1} I(\xi) \omega'_{B}(\xi) \, d\xi \right)_{12} + O\left(\frac{c(k_0)}{t}\right). \tag{3.39}
\]

3.5. Rescaling and further reduction of the Riemann-Hilbert problems

Extend the contours \( \Sigma'_{A} \) and \( \Sigma'_{B} \) to the contours

\[
\hat{\Sigma}'_{A} = \{ k = -k_0 + k_0 e^{i \alpha} \pi : \alpha \in \mathbb{R} \}, \tag{3.40}
\]

\[
\hat{\Sigma}'_{B} = \{ k = k_0 + k_0 e^{i \alpha} \pi : \alpha \in \mathbb{R} \}, \quad \tag{3.41}
\]

respectively. We introduce \( \hat{\omega}'_{A} \) and \( \hat{\omega}'_{B} \) on \( \hat{\Sigma}'_{A} \) and \( \hat{\Sigma}'_{B} \), respectively, by

\[
\hat{\omega}'_{A} = \left\{ \begin{array}{ll} \omega'_{A}(k), & k \in \Sigma'_{A}, \\
0, & k \in \hat{\Sigma}'_{A} \backslash \Sigma'_{A}, \end{array} \right. \quad \hat{\omega}'_{B} = \left\{ \begin{array}{ll} \omega'_{B}(k), & k \in \Sigma'_{B}, \\
0, & k \in \hat{\Sigma}'_{B} \backslash \Sigma'_{B}. \end{array} \right. \tag{3.42}
\]
Let $\Sigma_A$ and $\Sigma_B$ denote the contours \( \{ k = k_0 \alpha e^{\pm \frac{\pi i}{4}} : \alpha \in \mathbb{R} \} \) oriented inward as in $\Sigma'_A$, $\hat{\Sigma}'_A$, and outward as in $\Sigma'_B$, $\hat{\Sigma}'_B$, respectively. Define the scaling operators

\[
N_A : \mathcal{L}^2(\hat{\Sigma}'_A) \to \mathcal{L}^2(\Sigma_A), \\
f(k) \to (N_A f)(k) = f\left(\frac{k}{\sqrt{48tk_0}} - k_0\right),
\]

(3.43)

\[
N_B : \mathcal{L}^2(\hat{\Sigma}'_B) \to \mathcal{L}^2(\Sigma_B), \\
f(k) \to (N_B f)(k) = f\left(\frac{k}{\sqrt{48tk_0}} + k_0\right),
\]

(3.44)

and set

\[
\omega_A = N_A \hat{\omega}'_A, \quad \omega_B = N_B \hat{\omega}'_B.
\]

A simple change-of-variable arguments shows that

\[
C_{\omega_A} = N^{-1}_A C_{\omega_A} N_A, \quad C_{\omega_B} = N^{-1}_B C_{\omega_B} N_B,
\]

where the operator $C_{\omega_A}$ ($C_{\omega_B}$) is a bounded map from $\mathcal{L}^2(\Sigma_A)$ ($\mathcal{L}^2(\Sigma_B)$) into $\mathcal{L}^2(\Sigma_A)$ ($\mathcal{L}^2(\Sigma_B)$). On the part

\[
L_A = \left\{ k = \alpha k_0 \sqrt{48tk_0} e^{\pm \frac{3\pi}{4}} : -\epsilon < \alpha < +\infty \right\}
\]

de $\Sigma_A$, we have

\[
\omega_A = \omega_{A_+} = \begin{pmatrix} 0 & (N_A s_1)(k) \\ 0 & 0 \end{pmatrix},
\]

on $L^*_A$ we have

\[
\omega_A = \omega_{A_-} = \begin{pmatrix} 0 & 0 \\ (N_A s_2)(k) & 0 \end{pmatrix},
\]

where

\[
s_1(k) = -e^{-2i\theta(k)} [\det \delta(k)] \delta(k) R(k), \quad s_2(k) = \frac{e^{2i\theta} R^\dagger(k) \delta^{-1}(k)}{\det \delta(k)}.
\]
Lemma 3.9. As $t \to \infty$, and $k \in L_A$, then
\[ \left| (N_0 \tilde{\delta})(k) \right| \leq t^{-l}, \]  
(3.45)
where $\tilde{\delta}(k) = e^{-2i\theta(k)}[\delta(k)R(k) - (\det \delta(k))R(k)]$.

Proof. It follows from (3.1) and (3.2) that $\tilde{\delta}$ satisfies the following Riemann-Hilbert problem:
\[
\begin{aligned}
\tilde{\delta}_+(k) &= \tilde{\delta}_-(k)(1 - \gamma^+(k^*)B_1 \gamma(k)) + e^{-2i\theta}f(k), \quad k \in (-k_0, k_0), \\
\tilde{\delta}(k) &\to 0, \quad k \to \infty.
\end{aligned}
\]  
(3.46)
where $f(k) = \delta_-(k)[\gamma^+(k^*)B_1 \gamma(k)I - \gamma(k)\gamma^+(k^*)B_1]R(k)$. The solution for the above Riemann-Hilbert problem can be expressed by
\[
\tilde{\delta}(k) = X(k) \int_{-k_0}^{-k_0} e^{-2i\theta k} f(\xi) \frac{d\xi}{X_+(\xi)(\xi - k)} 2\pi i,
\]
\[
X(k) = \exp \left\{ \frac{1}{2\pi i} \int_{-k_0}^{-k_0} \log(1 - |\gamma(\xi)|^2) \frac{d\xi}{\xi - k} \right\}.
\]

Observing that
\[
(\gamma^+(k^*)B_1 \gamma(k)I - \gamma(k)\gamma^+(k^*)B_1)R(k) = (\gamma^+(k^*)B_1 \gamma(k)I - \gamma(k)\gamma^+(k^*)B_1)(R(k) - \rho(k))
\]
\[
= \text{adj}[B_1]\text{adj}[\gamma(k)\gamma^+(k^*)](h_1(k) + h_2(k)),
\]
we obtain $f(k) = O((k^2 - k_0^2)^l)$. Similar to the Lemma 3.1, $f(k)$ can be decomposed into two parts: $f(k) = f_1(k) + f_2(k)$, and
\[
|e^{-2i\theta(k)}f_1(k)| \leq \frac{1}{(1 + |k|^2)^l}, \quad k \in \mathbb{R},
\]  
(3.47)
\[
|e^{-2i\theta(k)}f_2(k)| \leq \frac{1}{(1 + |k|^2)^l}, \quad k \in L_l,
\]  
(3.48)
where $f_2(k)$ has an analytic continuation to $L_l$, $l$ is a positive integer and $l \geq 2$.

\[
L_l = \begin{cases} 
 k = k_0 + k_0 \alpha e^{\frac{3\pi i}{4}} : 0 \leq \alpha \leq \sqrt{2}(1 - \frac{1}{2l}) \\
\cup k = \frac{k_0}{t} - k_0 + k_0 \alpha e^{\frac{\pi i}{4}} : 0 \leq \alpha \leq \sqrt{2}(1 - \frac{1}{2l})
\end{cases}
\]
(see Figure 5).

![Figure 5. The contour $L_l$.](image-url)
As $k \in L_A$, we obtain
\[
(N_A \hat{\delta})(k) = X(\frac{k}{\sqrt{48t}k_0} - k_0) \int_{\frac{k}{\sqrt{48t}k_0} - k_0}^{\frac{k}{\sqrt{48t}k_0} - k_0} \frac{e^{-2i\theta(t)}f(\xi)}{X_+ (\xi + k_0 - \frac{k}{\sqrt{48t}k_0})} d\xi + \frac{k}{\sqrt{48t}k_0} - k_0 \int_{\frac{k}{\sqrt{48t}k_0} - k_0}^{\frac{k}{\sqrt{48t}k_0} - k_0} \frac{e^{-2i\theta(t)}f_1(\xi)}{X_+ (\xi + k_0 - \frac{k}{\sqrt{48t}k_0})} d\xi + \frac{k}{\sqrt{48t}k_0} - k_0 \int_{\frac{k}{\sqrt{48t}k_0} - k_0}^{\frac{k}{\sqrt{48t}k_0} - k_0} \frac{e^{-2i\theta(t)}f_2(\xi)}{X_+ (\xi + k_0 - \frac{k}{\sqrt{48t}k_0})} d\xi = I_1 + I_2 + I_3.
\]

As a consequence of Cauchy’s Theorem, we can evaluate $I_3$ along the contour $L_t$ instead of the interval $(\frac{k}{t} - k_0, k_0)$ and obtain $|I_3| \leq t^{-1/2}$. Therefore, (3.45) holds. 

\[\square\]

**Corollary 3.1.** As $t \to \infty$, and $k \in L^*_A$, then
\[
|(N_A \hat{\delta})(k)| \leq t^{-1}, \quad t \to \infty, \quad k \in L^*_A,
\]
where $\hat{\delta}(k) = e^{2i\theta(t)}R_1(k^*)B_1[\delta^{-1}(k) - (\det(\delta(k)))^{-1}]$.

Let $J^{a^0} = (I - \omega_A^{0,-1})^{-1}(I + \omega_A^{0,+})$, where
\[
\omega_A^{0} = \omega_A^{0,+} = \begin{pmatrix}
0 & -\delta_A^0 \gamma(-\xi) & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & -\delta_A^0 \gamma(-\xi) & \frac{1}{2} \\
0 & 0 & 0 & -\delta_A^0 \gamma(-\xi) \\
\end{pmatrix}, \quad k \in \Sigma_A,
\]
\[
\delta_A^0 = (196tk_0^3)^{\frac{-n}{2}}e^{8ik_0^3e^{\frac{1}{2}}},
\]
\[
\omega_A^{0} = \omega_A^{0,-} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
(\delta_A^0)^{-2}(-k)^{-2i\xi}e^\frac{\gamma^3}{1-\gamma^3}(-k_0) & 0 \\
-\delta_A^0 (-k)^{-2i\xi}e^\frac{\gamma^3}{1-\gamma^3}(-k_0) & 0 \\
\end{pmatrix}, \quad k \in \Sigma_A.
\]

It follows from (3.78) in [18] that
\[
||\omega_A - \omega_A^0||_{L^1(\Sigma_A) \cap L^2(\Sigma_A) \cap L^\infty(\Sigma_A)} \leq k_0 \frac{\log t}{\sqrt{tk_0^3}}.
\]

\[\square\]
There are similar consequences for \( k \in \Sigma_B \). Let \( J^{B_\epsilon} = (I - \omega_{B_\epsilon}^\gamma)^{-1}(I + \omega_{B_\epsilon}^\gamma) \), where

\[
\omega_{B_\epsilon}^\gamma = \omega_{B_\epsilon}^{\epsilon, +} = \begin{cases} 0 & k \in \Sigma_B^2, \\ 0 & k \in \Sigma_B^3, \end{cases} \]

(3.54)

\[
\delta_B^0 = (196t^{3/2}e^{-8t^{3/2}e^\gamma(k_0)} \right), \quad k \in \Sigma_B^2, \]

(3.55)

\[
\omega_{B_\epsilon}^\gamma = \omega_{B_\epsilon}^{\epsilon, -} = \begin{cases} 0 & k \in \Sigma_B^2, \\ 0 & k \in \Sigma_B^3, \end{cases} \]

(3.56)

**Theorem 3.3.** As \( t \to \infty \),

\[
q(x, t) = (u(x, t), u_\epsilon(x, t))^T
\]

\[
= \frac{1}{\pi \sqrt{48t^{k_0}}} \left( \int \Sigma_t \left( (1 - C_{\omega_{A}})^{-1} I \right)(\xi)\omega_{\epsilon, B}(\xi) \, d\xi \right)_{12} \]

\[
+ \frac{1}{\pi \sqrt{48t^{k_0}}} \left( \int \Sigma_t \left( (1 - C_{\omega_{A}})^{-1} I \right)(\xi)\omega_{\epsilon, \gamma}(\xi) \, d\xi \right)_{12} + O \left( \frac{c(k_0) \log t}{t} \right). \]

(3.57)

**Proof.** Notice that

\[
(1 - C_{\omega_{A}})^{-1} I \omega_A = (1 - C_{\omega_{A}})^{-1} I \omega_{A}^\gamma + \left( (1 - C_{\omega_{A}})^{-1} I \omega_A - (1 - C_{\omega_{A}})^{-1} I \omega_{A}^\gamma \right)
\]

Utilizing the triangle inequality and the boundedness in (3.53), we have

\[
\int \Sigma_t \left( (1 - C_{\omega_{A}})^{-1} I \right)(\xi)\omega_{A}(\xi) \, d\xi = \int \Sigma_t \left( (1 - C_{\omega_{A}})^{-1} I \right)(\xi)\omega_{A}^\gamma(\xi) \, d\xi + O \left( \frac{\log t}{\sqrt{t}} \right).
\]

According to (3.5) and a simple change-of-variable argument, we have

\[
\frac{1}{\pi} \int \Sigma_t \left( (1 - C_{\omega_{A}})^{-1} I \right)(\xi)\omega_{A}(\xi) \, d\xi \right)_{12}
\]

\[
= \frac{1}{\pi} \int \Sigma_t \left( (1 - C_{\omega_{A}})^{-1} I \right)(\xi)\omega_{A}^\gamma(\xi) \, d\xi \right)_{12}
\]

\[
\frac{1}{\pi} \int \Sigma_t \left( (1 - C_{\omega_{A}})^{-1} I \right)(\xi)\omega_{A}(\xi) \, d\xi \right)_{12}
\]

\[
= \frac{1}{\pi \sqrt{48t^{k_0}}} \left( \int \Sigma_t \left( (1 - C_{\omega_{B}})^{-1} I \right)(\xi)\omega_{A}(\xi) \, d\xi \right)_{12}
\]

\[
= \frac{1}{\pi \sqrt{48t^{k_0}}} \left( \int \Sigma_t \left( (1 - C_{\omega_{B}})^{-1} I \right)(\xi)\omega_{A}(\xi) \, d\xi \right)_{12} + O \left( \frac{c(k_0) \log t}{t} \right).
\]

There are similar computations for the other case. Together with (3.39), one can obtain (3.57). \( \square \)
For $k \in \mathbb{C} \setminus \Sigma_A$, set
\[
M^A_0(k; x, t) = I + \int_{\Sigma_A} \left( (1 - C_{\omega_A})^{-1} I \right) \frac{(\xi) \omega_A(\xi)}{\xi - k} \frac{d\xi}{2\pi i},
\]
(3.58)

Then $M^A_0(k; x, t)$ is the solution of the Riemann-Hilbert problem
\[
\begin{cases}
M^A_0(k; x, t) = M^A_0(k; x, t)J^A_0(k; x, t), & k \in \Sigma_A, \\
M^A_0(k; x, t) \to I, & k \to \infty,
\end{cases}
\]
(3.59)

In particular
\[
M^A_0(k) = I + \frac{M^A_0}{k} + O(k^{-2}), \quad k \to \infty,
\]
(3.60)

then
\[
M^A_0 = -\int_{\Sigma_A} \left( (1 - C_{\omega_A})^{-1} I \right) \frac{(\xi) \omega_A(\xi)}{\xi - k} \frac{d\xi}{2\pi i},
\]
(3.61)

There is a analogous Riemann-Hilbert problem on $\Sigma_B$,
\[
\begin{cases}
M^B_0(k; x, t) = M^B_0(k; x, t)J^B_0(k; x, t), & k \in \Sigma_B, \\
M^B_0(k; x, t) \to I, & k \to \infty,
\end{cases}
\]
(3.62)

where $J^B_0(k; x, t)$ is defined in (3.54) and (3.56). In the meantime, we have
\[
M^B_0(k) = I + \frac{M^B_0}{k} + O(k^{-2}), \quad k \to \infty.
\]
(3.63)

Next, we consider the relation between $M^A_1$ and $M^B_1$. From the expression (3.50), (3.52), (3.54) and (3.56), we have the symmetry relation
\[
J^A_0(k) = \tau(J^B_0(-k^*))^* \tau.
\]

By the uniqueness of the Riemann-Hilbert problem,
\[
M^A_0(k) = \tau(M^B_0(-k^*))^* \tau.
\]

Combining with the expansion (3.60) and (3.63), one can verify that
\[
M^A_1 = -\tau(M^B_1)^* \tau, \quad (M^A_1)_{12} = -\sigma_1(M^B_1)_{12}.
\]

Therefore, from (3.57) and (3.61), we have
\[
q(x, t) = (u(x, t), u^*(x, t))^T = \begin{align}
&- \frac{2i}{\sqrt{48\sqrt{k_0}}} \left( M^A_0 + M^B_0 \right)_{12} + O\left( \frac{c(k_0) \log t}{t} \right) \\
&= - \frac{i}{\sqrt{12\sqrt{k_0}}} \left( (M^A_0)_{12} - \sigma_1(M^A_0)_{12} \right) + O\left( \frac{c(k_0) \log t}{t} \right).
\]
(3.64)
3.6. Solving the model problem

In this subsection, we compute \((M_1^{A^0})_{12}\) explicitly. It is important to set

\[
\Psi(k) = H(k)(-k)^{i\nu}e^{-\frac{1}{2}ik^2}\sigma, \quad H(k) = (\delta^0_A)^{-\sigma} M^{A^0}(k)(\delta^0_A)^{\sigma}.
\] (3.65)

Then it follows from (3.59) that

\[
\Psi_+(k) = \Psi_-(k)v(-k_0), \quad v = e^{\frac{1}{2}ik^2(-k)^{i\nu}(\delta^0_A)^{-\sigma} J^{A^0}(k)(\delta^0_A)^{\sigma}(-k)^{i\nu}e^{-\frac{1}{2}ik^2}\sigma}.
\] (3.66)

The jump matrix is the constant one on the four rays \(\Sigma_{1}^A, \Sigma_{2}^A, \Sigma_{3}^A, \Sigma_{4}^A\), so

\[
\frac{d\Psi_+(k)}{dk} = \frac{d\Psi_-(k)}{dk} v(-k_0).
\] (3.67)

Then it follows that \((d\Psi/dk + i\kappa_\sigma \Psi)^{-1}\) has no jump discontinuity along any of the four rays. Besides, from the relation between \(\Psi(k)\) and \(H(k)\), we have

\[
\frac{d\Psi(k)}{dk} \Psi^{-1}(k) = \frac{dH(k)}{dk} H^{-1}(k) - \frac{ik}{2}H(k)\sigma H^{-1}(k) + i\frac{v}{k}H(k)\sigma H^{-1}(k) = O(k^{-1}) - \frac{i\kappa_\sigma}{2} + \frac{i}{2}(\delta^0_A)^{-\sigma}[\sigma, M_1^{A^0}] (\delta^0_A)^{-\sigma}.
\]

It follows by the Liouville’s Theorem that

\[
\frac{d\Psi(k)}{dk} + i\kappa_\sigma \Psi(k) = \beta \Psi(k),
\] (3.68)

where

\[
\beta = \frac{i}{2}(\delta^0_A)^{-\sigma}[\sigma, M_1^{A^0}] (\delta^0_A)^{-\sigma} = \begin{pmatrix} 0 & \beta_{12} \\ \beta_{21} & 0 \end{pmatrix}.
\]

Moreover,

\[
(M_1^{A^0})_{12} = -i(\delta^0_A)^{-2}\beta_{12}.
\] (3.69)

Set

\[
\Psi(k) = \begin{pmatrix} \Psi_{11}(k) & \Psi_{12}(k) \\ \Psi_{21}(k) & \Psi_{22}(k) \end{pmatrix}.
\]

From (3.68) and its differential, we obtain

\[
\frac{d^2\beta_{21}\Psi_{11}(k)}{dk^2} + \left(\frac{i}{2} + \frac{k^2}{4} - \beta_{21}\beta_{12}\right)\beta_{21}\Psi_{11}(k) = 0,
\]

\[
\Psi_{21}(k) = \frac{1}{\beta_{21}\beta_{12}} \left(\frac{d\beta_{21}\Psi_{11}(k)}{dk} + \frac{ik}{2}\beta_{21}\Psi_{11}(k)\right),
\]

\[
\frac{d^2\Psi_{22}(k)}{dk^2} + \left(\frac{i}{2} + \frac{k^2}{4} - \beta_{21}\beta_{12}\right)\Psi_{22}(k) = 0,
\]

\[
\beta_{21}\Psi_{12}(k) = \left(\frac{d\Psi_{22}(k)}{dk} - \frac{ik}{2}\Psi_{22}(k)\right).
\]
As is well known, the Weber’s equation
\[
\frac{d^2 g(\zeta)}{d\zeta^2} + \left( \varrho + \frac{1}{2} - \frac{\zeta^2}{4} \right) g(\zeta) = 0
\]
has the solution
\[
g(\zeta) = c_1 D_\varrho(\zeta) + c_2 D_\varrho(-\zeta),
\]
where \( D_\varrho(\cdot) \) denotes the standard parabolic-cylinder function, and \( c_1, c_2 \) are constants. The parabolic-cylinder function satisfies [41]
\[
\frac{dD_\varrho(\zeta)}{d\zeta} + \frac{\zeta}{2} D_\varrho(\zeta) - \varrho D_{\varrho-1}(\zeta) = 0,
\]
(3.70)
\[
D_\varrho(\pm \zeta) = \frac{\Gamma(\varrho + 1)e^{i\pi\varrho}}{\sqrt{2\pi}} D_{-\varrho-1}(\pm i\zeta) + \frac{\Gamma(\varrho + 1)e^{-i\pi\varrho}}{\sqrt{2\pi}} D_{-\varrho-1}(\mp i\zeta).
\]
(3.71)
As \( \zeta \to \infty \), from [42], we have
\[
D_\varrho(\zeta) = \begin{cases} 
\xi^\varrho e^{-\zeta^2/4}(1 + O(\zeta^{-2})), & \text{arg } \zeta < \frac{3\pi}{4}, \\
\xi^\varrho e^{-\zeta^2/4}(1 + O(\zeta^{-2})) - \frac{\sqrt{\pi}}{1!(\zeta^{-2})} e^{-\zeta^2/4} \zeta^{-1}(1 + O(\zeta^{-2})), & \frac{\pi}{4} < \text{arg } \zeta < \frac{5\pi}{4}, \\
\xi^\varrho e^{-\zeta^2/4}(1 + O(\zeta^{-2})) - \frac{\sqrt{\pi}}{1!(\zeta^{-2})} e^{-\zeta^2/4} \zeta^{-1}(1 + O(\zeta^{-2})), & -\frac{5\pi}{4} < \text{arg } \zeta < -\frac{3\pi}{4},
\end{cases}
\]
(3.72)
where \( \Gamma(\cdot) \) is the Gamma function. Set \( \varrho = i\beta_{21}\beta_{12} \).
\[
\beta_{21}\Psi_{11}(k) = c_1 D_\varrho \left( e^{\frac{\pi}{2}k} \right) + c_2 D_\varrho \left( e^{\frac{3\pi}{2}k} \right), \quad \beta_{21}\Psi_{22}(k) = c_3 D_{-\varrho} \left( e^{\frac{\pi}{2}k} \right) + c_4 D_{-\varrho} \left( e^{\frac{3\pi}{2}k} \right),
\]
(3.73)
(3.74)
where \( a_1, a_2, a_3, a_4 \) are constants. As \( \text{arg } k \in (-\pi, -\frac{3\pi}{4}) \cup (\frac{3\pi}{4}, \pi) \) and \( k \to \infty \), we arrive at
\[
\Psi_{11}(k)(-k)^{-i\varrho} e^{\frac{\pi}{2}k} \to I, \quad \Psi_{22}(k)(-k)^{i\varrho} e^{-\frac{\pi}{2}k} \to 1,
\]
then
\[
\beta_{21}\Psi_{11}(k) = \beta_{21} e^{\frac{\pi}{2}k} D_\varrho \left( e^{-\frac{3\pi}{2}k} \right), \quad \nu = \beta_{21}\beta_{12},
\]
\[
\Psi_{22}(k) = e^{\frac{\pi}{2}k} D_{-\varrho} \left( e^{\frac{3\pi}{2}k} \right).
\]
Consequently,
\[
\Psi_{21}(k) = \beta_{21} e^{\frac{\pi}{2}k} e^{-\frac{3\pi}{2}} D_{\varrho-1} \left( e^{-\frac{\pi}{2}k} \right),
\]
\[
\beta_{21}\Psi_{12}(k) = \varrho e^{\frac{\pi}{2}k} e^{-\frac{3\pi}{2}} D_{-\varrho-1} \left( e^{\frac{3\pi}{2}k} \right).
\]
For \( \text{arg } k \in (-\frac{3\pi}{4}, -\frac{\pi}{4}) \) and \( k \to \infty \), we have
\[
\Psi_{11}(k)(-k)^{-i\varrho} e^{\frac{\pi}{2}k} \to I, \quad \Psi_{22}(k)(-k)^{i\varrho} e^{-\frac{\pi}{2}k} \to 1,
\]
then
\[
\begin{align*}
\beta_{21} \Psi_{11}(k) &= \beta_{21} e^{-3\pi i 4} D_{e^{-\frac{3\pi}{4} k}}(e^{\frac{3\pi}{4} k}), \\
\Psi_{22}(k) &= e^{\frac{3\pi}{4} D_{e^{-\frac{3\pi}{4} k}}}.
\end{align*}
\]
Consequently,
\[
\begin{align*}
\Psi_{21}(k) &= \beta_{21} e^{-3\pi i 4} e^{\frac{3\pi}{4} D_{e^{-\frac{3\pi}{4} k}}}(e^{\frac{3\pi}{4} k}), \\
\beta_{21} \Psi_{12}(k) &= \phi e^{\frac{3\pi}{4} D_{e^{-\frac{3\pi}{4} k}}} (e^{\frac{3\pi}{4} k}).
\end{align*}
\]
Along the ray \( \arg k = -\frac{3\pi}{4} \),
\[
\begin{align*}
\Psi_+(k) &= \Psi_-(k) \begin{pmatrix} 1 & 0 \\ -\gamma^\dagger (-k_0) B_1 & 1 \end{pmatrix}.
\end{align*}
\] (3.75)

Notice the \((2, 1)\) entry of the Riemann-Hilbert problem,
\[
\begin{align*}
\beta_{21} e^{\frac{3\pi i}{4}} D_{e^{-\frac{3\pi}{4} k}}(e^{-\frac{3\pi}{4} k}) \\
= \beta_{21} e^{\frac{3\pi i}{4}} D_{e^{-\frac{3\pi}{4} k}}(e^{\frac{3\pi}{4} k}) - e^{\frac{3\pi}{4} D_{e^{-\frac{3\pi}{4} k}}} \gamma^\dagger (-k_0) B_1.
\end{align*}
\]
It follows from (3.71) that
\[
D_{e^{\frac{3\pi}{4} k}} = \frac{\Gamma(-\varrho + 1)e^{\frac{3\pi}{4}}}{\sqrt{2\pi}} D_{e^{-\frac{3\pi}{4} k}} + \frac{\Gamma(-\varrho + 1)e^{-\frac{3\pi}{4}}}{\sqrt{2\pi}} D_{e^{-\frac{3\pi}{4} k}}.
\]
Then we separate the coefficients of the two independent functions and obtain
\[
\beta_{21} = e^{-3\pi i 4} e^{\frac{3\pi}{4} \Gamma(-\varrho + 1)} \sqrt{2\pi} \gamma^\dagger (-k_0) B_1. \tag{3.76}
\]
Noting that \( B^{-1}(J^A(k^*)) B = (J^A(k))^{-1} \), we have \( \beta_{12} = -B^{-1}_1 \beta_{21} \), which means that
\[
\beta_{12} = -B^{-1}_1 B_1^\dagger \gamma(-k_0) e^{\frac{3\pi}{4}} e^{\frac{3\pi}{4} \Gamma(-\varrho + 1)} \sqrt{2\pi} \gamma^\dagger (-k_0) \tag{3.77}
\]
Finally, we can obtain (1.4) from (3.64), (3.69) and (3.77).

**Acknowledgments**

This work is supported by the National Natural Science Foundation of China (Grant Nos. 11871440 and 11931017).

**Conflict of interest**

The authors declare no conflict of interest.
References

1. N. Sasa, J. Satsuma, *New-type of soliton solutions for a higher-order nonlinear Schrödinger equation*, J. Phys. Soc. Jpn, 60 (1991), 409–417.

2. Y. Kivshar, G. Agrawal, *Optical Solitons: From Fibers to Photonic Crystals*, Waltham: Academic Press, 2003.

3. K. Porsezian, *Soliton models in resonant and nonresonant optical fibers*, Pramana, 57 (2001), 1003–1039.

4. A. V. Slunyaev, *A high-order nonlinear envelope equation for gravity waves in finite-depth water*, J. Exp. Theor. Phys., 101 (2005), 926–941.

5. M. Trippenbach, Y. B. Band, *Effects of self-steepening and self-frequency shifting on short-pulse splitting in dispersive nonlinear media*, Phys. Rev. A, 57 (1991), 4791–4803.

6. J. K. Yang, D. J. Kaup, *Squared eigenfunctions for the Sasa-Satsuma equation*, J. Math. Phys., 50 (2009), 023504.

7. C. Gilson, J. Hietarinta, J. Nimmo, et al. *Sasa-Satsuma higher-order nonlinear Schrödinger equation and its bilinearization and multisoliton solutions*, Phys. Rev. E, 68 (2003), 016614.

8. J. J. C. Nimmo, H. Yilmaz, *Binary Darboux transformation for the Sasa-Satsuma equation*, J. Phys. A, 48 (2015), 425202.

9. X. G. Geng, R. M. Li, B. Xue, *A vector general nonlinear Schrödinger equation with (m + n) components*, J. Nonlinear Sci., 30 (2020), 991–1013.

10. R. M. Li, X. G. Geng, *On a vector long wave-short wave-type model*, Stud. Appl. Math., 144 (2020), 164–184.

11. R. M. Li, X. G. Geng, *Rogue periodic waves of the sine-Gordon equation*, Appl. Math. Lett., 102 (2020), 106147.

12. J. Xu, Q. Z. Zhu, E. G. Fan, *The initial-boundary value problem for the Sasa-Satsuma equation on a finite interval via the Fokas method*, J. Math. Phys., 59 (2018), 073508.

13. Y. Y. Zhai, X. G. Geng, *The coupled Sasa-Satsuma hierarchy: trigonal curve and finite genus solutions*, Anal. Appl., 15 (2017), 667–697.

14. X. G. Geng, L. H. Wu, G. L. He, *Quasi-periodic solutions of the Kaup-Kupershmidt hierarchy*, J. Nonlinear Sci., 23 (2013), 527–555.

15. X. G. Geng, Y. Y. Zhai, H. H. Dai, *Algebra-geometric solutions of the coupled modified Korteweg-de Vries hierarchy*, Adv. Math., 263 (2014), 123–153.

16. J. Wei, X. G. Geng, X. Zeng, *The Riemann theta function solutions for the hierarchy of Bogoyavlensky lattices*, Trans. Amer. Math. Soc., 371 (2019), 1483–1507.

17. J. Wei, X. G. Geng, *A super Sasa-Satsuma hierarchy and bi-Hamiltonian structures*, Appl. Math. Lett., 83 (2018), 46–52.

18. C. Deift, X. Zhou, *A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation*, Ann. of Math., 137 (1993), 295–368.
19. K. Grunert, G. Teschl, *Long-time asymptotics for the Korteweg-de Vries equation via nonlinear steepest descent*, Math. Phys. Anal. Geom., 12 (2009), 287–324.

20. P. J. Cheng, S. Venakides, X. Zhou, *Long-time asymptotics for the pure radiation solution of the sine-Gordon equation*, Comm. Partial Differential Equations, 24 (1999), 1195–1262.

21. A. V. Kitaev, A. H. Vartanian, *Leading-order temporal asymptotics of the modified nonlinear Schrödinger equation: Solitonless sector*, Inverse Problem, 13 (1997), 1311–1339.

22. A. V. Kitaev, A. H. Vartanian, *Asymptotics of solutions to the modified nonlinear Schrödinger equation: Solution on a nonvanishing continuous background*, SIAM J. Math. Anal., 30 (1999), 787–832.

23. A. H. Vartanian, *Higher order asymptotics of the modified nonlinear Schrödinger equation*, Comm. Partial Differential Equations, 25 (2000), 1043–1098.

24. A. Boutet de Monvel, A. Kostenko, D. Shepelsky, G. Teschl, *Long-time asymptotics for the Camassa-Holm equation*, SIAM J. Math. Anal., 41 (2009), 1559–1588.

25. L. K. Arruda, J. Lenells, *Long-time asymptotics for the derivative nonlinear Schrödinger equation on the half-line*, Nonlinearity, 30 (2017), 4141–4172.

26. A. Boutet de Monvel, A. Its, V. Kotlyarov, *Long-time asymptotics for the focusing NLS equation with time-periodic boundary condition on the half-line*, Comm. Math. Phys., 290 (2009), 479–522.

27. P. Deift, J. Park, *Long-time asymptotics for solutions of the NLS equation with a delta potential and even initial data*, Int. Math. Res. Not. IMRN, 24 (2011), 5505–5624.

28. I. Egorova, J. Michor, G. Teschl, *Rarefaction waves for the Toda equation via nonlinear steepest descent*, Discrete Contin. Dyn. Syst., 38 (2018), 2007–2028.

29. H. Yamane, *Long-time asymptotics for the defocusing integrable discrete nonlinear Schrödinger equation*, J. Math. Soc. Japan, 66 (2014), 765–803.

30. J. Lenells, *The nonlinear steepest descent method: asymptotics for initial-boundary value problems*, SIAM J. Math. Anal., 48 (2016), 2076–2118.

31. A. Boutet de Monvel, D. Shepelsky, A Riemann-Hilbert approach for the Degasperis-Procesi equation, Nonlinearity, 26 (2013), 2081–2107.

32. A. Boutet de Monvel, J. Lenells, D. Shepelsky, *Long-time asymptotics for the Degasperis-Procesi equation on the half-line*, Ann. Inst. Fourier, 69 (2019), 171–230.

33. X. G. Geng, H. Liu, *The nonlinear steepest descent method to long-time asymptotics of the coupled nonlinear Schrödinger equation*, J. Nonlinear Sci., 28 (2018), 739–763.

34. H. Liu, X. G. Geng, B. Xue, *The Deift-Zhou steepest descent method to long-time asymptotics for the Sasa-Satsuma equation*, J. Differential Equations, 265 (2018), 5984–6008.

35. B. B. Hu, T. C. Xia, W. X. Ma, *Riemann-Hilbert approach for an initial-boundary value problem of the two-component modified Korteweg-de Vries equation on the half-line*, Appl. Math. Comput., 332 (2018), 148–159.

36. B. B. Hu, T. C. Xia, N. Zhang, et al. *Initial-boundary value problems for the coupled higher-order nonlinear Schrödinger equations on the half-line*, Int. J. Nonlinear Sci. Numer. Simul., 19 (2018), 83–92.
37. N. Zhang, T. C. Xia, E. G. Fan, A Riemann-Hilbert approach to the Chen-Lee-Liu equation on the half line, Acta Math. Appl. Sin. Engl. Ser., 34 (2018), 493–515.
38. X. G. Geng, J. P. Wu, Riemann-Hilbert approach and N-soliton solutions for a generalized Sasa-Satsuma equation, Wave Motion, 60 (2016), 62–72.
39. M. J. Ablowitz, A. S. Fokas, Complex variables: Introduction and applications, Cambridge: Cambridge University Press, 2003.
40. R. Beals, R. R. Coifman, Scattering and inverse scattering for first order systems, Comm. Pure Appl. Math., 37 (1984), 39–90.
41. R. Beals, R. Wong, Special functions and orthogonal polynomials, Cambridge: Cambridge University Press, 2016.
42. E. T. Whittaker, G. N. Watson, A Course of Modern Analysis 4th Ed, Cambridge: Cambridge University Press, 1927.