Three-Loop Effective Potential of $O(N) \phi^4$ Theory

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Abstract

The three-loop effective potential of the massless $O(N) \phi^4$ theory is calculated analytically using techniques of dimensional regularization. We see a complete agreement between our result and Jackiw’s result which was obtained only up to two-loop order using a different regularization (cutoff regularization) method, but the same renormalization conditions. For an easy check of the mutual cancellation of all the dangerous pole terms in each loop order, we give the $\epsilon$-expanded loop integrals in full detail.

I. INTRODUCTION

The effective potential plays a crucial role in determining the nature of the vacuum in a weakly coupled field theory, as was emphasized in the classic paper of Coleman and Weinberg [1]. Calculation of this object by summing infinite series of Feynman diagrams at zero momentum is an onerous task, especially when several interactions are present which complicate the combinatorial factors that multiply each diagram. Jackiw has succeeded in representing each loop order containing an infinite set of conventional Feynman diagrams

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by finite number of diagrams using his algebraic method which can be formally extended to the arbitrary higher loop order [2].

Self-interacting scalar field theory of $\phi^4$ model is one of the best analyzed field theories. Various renormalization group functions of $O(N) \phi^4$ theory are now available up to the five-loop order [3]. However, the three-loop effective potential of the same theory is less progressive. For renormalization of the two-point one-particle-irreducible (1PI) Green’s function, $\Gamma^{(2)}$, and the four-point 1PI Green’s function, $\Gamma^{(4)}$, of the $O(N) \phi^4$ theory only one type of propagator is involved in loop integrations, but for calculation of the effective potential two kinds of propagators due to two different induced mass-like terms are involved in the loop integrations. The two-loop effective potential for the massive $O(N) \phi^4$ theory in four dimensions has been calculated by Ford and Jones [4], and for the massless $O(N) \phi^4$ theory by Jackiw [2]. For the single-component massive $\phi^4$ theory in four dimensions, the two-loop effective potential has been calculated in Ref. [5], and the three-loop effective potential in Ref. [6].

Recently all the genuine three-loop integrals — genuine in the sense that they cannot be factorized into lower-loop integrals — appearing in the three-loop effective potential calculation of the massless $O(N) \phi^4$ theory has been calculated [8]. These integrals carry two kinds of propagator lines, A-type line and B-type line. The mass parameter of the B-type line is one-third as large as that of the A-type line.

The purpose of this paper is to calculate the three-loop effective potential of the massless $O(N) \phi^4$ theory in four dimensions of space-time in the dimensional regularization scheme [9]. In Sec. II the standard procedure for the renormalization is presented. The Feynman rules for the effective potential suitable for the Jackiw’s prescription are given and all the Feynman diagrams for the effective potential up to three-loop order are constructed. The values of all relevant integrals are listed in the Appendices A and B. Sec. III is devoted to concluding remarks.
II. RENORMALIZATION OF THE EFFECTIVE POTENTIAL

The Lagrangian for a theory of $N$ spinless fields $\phi_a$, with an $O(N)$-invariant interaction is given as

$$\mathcal{L}(\phi_a(x)) = \frac{1 + \delta Z}{2} \partial_\mu \phi_a \partial^\mu \phi_a - \frac{m^2 + \delta m^2}{2} \phi^2 - \frac{\lambda + \delta \lambda}{4!} \phi^4,$$  

where the quadratic and quartic expressions are $\phi^2 = \phi_a \phi_a$ and $\phi^4 = (\phi^2)^2$. The quantities $\phi_a$, $m$, and $\lambda$ are the renormalized field, the renormalized mass, and the renormalized coupling constant respectively, whereas $\delta Z$, $\delta m^2$, and $\delta \lambda$ are corresponding (infinite) counterterm constants. These are related to the usual bare quantities as follows:

$$m_0^2 = \frac{m^2 + \delta m^2}{1 + \delta Z}, \quad \lambda_0 = \frac{\lambda + \delta \lambda}{(1 + \delta Z)^2}, \quad \phi_0^2 = (1 + \delta Z) \phi^2.$$  

We will confine ourselves to the massless theory ($m = 0$). The effective potential is most suitably defined generically, when the effective action ($\Gamma[\phi_{cl}]$), being the generating functional of the one-particle-irreducible (1PI) Green’s functions ($\Gamma^{(n)}(x_1, ..., x_n)$), is expressed in the following local form (the so-called derivative expansion):

$$\Gamma[\phi_{cl}] = \int d^4 x \left[ - V(\phi_{cl}(x)) + \frac{1}{2} Z(\phi_{cl}(x)) \partial_\mu \phi_{cl}(x) \partial^\mu \phi_{cl}(x) + \cdots \right],$$  

where $\phi_{cl}(x)$ is the vacuum expectation value of the field operator $\phi(x)$ in the presence of an external source. By setting $\phi_{cl}(x)$ in $V(\phi_{cl}(x))$ to be a constant field $\hat{\phi}$, we obtain the effective potential $V_{eff}(\hat{\phi})$

$$V_{eff}(\hat{\phi}) \equiv V(\phi_{cl}(x)) \big|_{\phi_{cl}(x) = \hat{\phi}}.$$  

Following the field-shift method of Jackiw [2] for the calculation of the effective potential, we first obtain the shifted Lagrangian with the constant field configuration $\{ \hat{\phi}_a \}$

$$\mathcal{L}(\hat{\phi}_a; \phi_a(x)) = \frac{1 + \delta Z}{2} \partial_\mu \phi_a \partial^\mu \phi_a - \frac{1}{2} \phi_a \left[ \left( \delta m^2 + \frac{\lambda + \delta \lambda}{6} \hat{\phi}^2 \right) \delta_{ab} + \frac{\lambda + \delta \lambda}{3} \hat{\phi}_a \hat{\phi}_b \right] \phi_b - \frac{\lambda + \delta \lambda}{6} \hat{\phi}_a \phi_a \phi^2 - \frac{\lambda + \delta \lambda}{4!} \phi^4,$$  

(4)
where $\hat{\phi}^2 = \hat{\phi}_a \hat{\phi}_a$. Our perturbation theory differs from the bare or renormalized perturbation theory \[^{[10]}\], rather a mixed one between both of them. The vertices for the $\phi$-quadratic term with strength $\delta Z$ and $\delta m^2$ in the usual ‘renormalized’ perturbation theory are transferred to the propagator line in our treatment. Thus our perturbation theory follows the one used by Kastening \[^{[11]}\] in spirit. The Feynman rules of this shifted Lagrangian are

\begin{align*}
a \ b &= \lambda \frac{\hat{\phi}_a \hat{\phi}_b}{\hat{\phi}^2} + b \left[ \delta_{ab} - \frac{\hat{\phi}_a \hat{\phi}_b}{\hat{\phi}^2} \right] \\
&= \frac{ih}{(1 + \delta Z)k^2 - \delta m^2 - (\lambda + \delta \lambda)\hat{\phi}^2/2} \hat{\phi}_a \hat{\phi}_b \\
&\quad + \frac{ih}{(1 + \delta Z)k^2 - \delta m^2 - (\lambda + \delta \lambda)\hat{\phi}^2/6} \left[ \delta_{ab} - \frac{\hat{\phi}_a \hat{\phi}_b}{\hat{\phi}^2} \right],
\end{align*}

where the propagators without group indices $a$ and $b$, but with small capital letters $A$ and $B$ are given, as self-evident in the above expression, by

\begin{align*}
a &= \frac{ih}{(1 + \delta Z)k^2 - \delta m^2 - (\lambda + \delta \lambda)\hat{\phi}^2/2}, \\
b &= \frac{ih}{(1 + \delta Z)k^2 - \delta m^2 - (\lambda + \delta \lambda)\hat{\phi}^2/6}.
\end{align*}

The rules for vertices without group indices can also be defined:

\begin{align*}
\times &= -\frac{i(\lambda + \delta \lambda)}{\hbar}, \\
\bigcirc &= -\frac{i(\lambda + \delta \lambda)\sqrt{\hat{\phi}^2}}{\hbar}.
\end{align*}

The propagators and vertices without the group indices, Eqs. (5) and (6), will appear in the final formal expression when the group indices are contracted out. Without introducing any new loop-expansion parameter, which is eventually set to be unity, we will use $\hbar$ as a loop-counting parameter \[^{[12]}\]. This is the reason why we have kept all the traces of $\hbar$’s in the Feynman rules, Eqs. (5), (6), and (7), in spite of our employment of the usual “God-given” units, $\hbar = c = 1$. In addition to the above Feynman rules, Eq. (5), which are used in
constructing two- and higher-loop vacuum diagrams, we need another set of rules solely for
one-loop vacuum diagrams which are dealt with separately in Jackiw’s prescription and are
essentially the same as those of Coleman and Weinberg [1] from the outset:

\[
\begin{align*}
\Lambda &= -\ln \left(1 - \frac{(\lambda + \delta\lambda)\hat{\phi}^2/2}{(1 + \delta Z)k^2 - \delta m^2}\right), \\
\Omega &= -\ln \left(1 - \frac{(\lambda + \delta\lambda)\hat{\phi}^2/6}{(1 + \delta Z)k^2 - \delta m^2}\right).
\end{align*}
\] (8)

Using the rules, Eqs. (5) and (8), and including the terms of zero-loop order, we arrive
at the formal expression of the effective potential up to the three-loop order (see Fig. 1 to
Fig. 9):

\[
V_{\text{eff}}(\hat{\phi}_a) = \left[\frac{\delta m^2}{2} \hat{\phi}^2 + \frac{\lambda + \delta\lambda}{4!} \hat{\phi}^4\right] + [\text{Diag. 1}] + [\text{Diag. 2}] + [\text{Diag. 3}]
+ [\text{Diag. 4}] + [\text{Diag. 5}] + [\text{Diag. 6}] + [\text{Diag. 7}] + [\text{Diag. 8}] + [\text{Diag. 9}].
\] (9)

We have revealed both symmetry number factor and group factor (the second factor when
two factors appear simultaneously) explicitly in front of each diagram in Fig. 1 to Fig. 9. For
the purpose of renormalization we first expand the counterterm constants in power series,
beginning with order \( h \):

\[
\begin{align*}
\delta m^2 &= h\delta m_1^2 + h^2\delta m_2^2 + \cdots, \\
\delta\lambda &= h\delta\lambda_1 + h^2\delta\lambda_2 + \cdots, \\
\delta Z &= h\delta Z_1 + h^2\delta Z_2 + \cdots.
\end{align*}
\]

As is well known in our original \( \phi^4 \) theory of Eq. (1) \( \delta Z_1 \) vanishes [12]. Thus the field
renormalization counterterm appears in the three-loop calculation for the first time. In the
first square-bracket term on the right-hand side of Eq. (9), we need \( h^0 - \), \( h^1 - \), \( h^2 - \), and \( h^3 - \)
order terms in its expansion. In calculating [Diag. 1], [Diag. 2], and [Diag. 3] terms, the
careful \( h \) expansions are needed: \( h^1 - \), \( h^2 - \), and \( h^3 - \) order terms are needed from [Diag. 1] and
\( h^2 - \) and \( h^3 - \) order terms from [Diag. 2] and [Diag. 3]. In the Appendix B, the details of the
counterterm integrals are described. In the remaining terms, [Diag. 4] to [Diag. 9], the lowest
order in $h$ is $h^3$, which is already of the desired order. Thus with a simple replacement of every $\sigma^2$ in Eqs. (A3) and (A4) with $\lambda \hat{\phi}^2/2$ we readily evaluate the three-loop diagrams in Fig. 4 to Fig. 9.

In what follows we will use the following notation for the effective potential up to the $L$-loop order:

$$V^{[L]}_{\text{eff}}(\hat{\phi}_a) = \sum_{i=0}^{L} \bar{h}^i V^{(i)}_{\text{eff}}(\hat{\phi}_a),$$

and use $\tilde{N}$ for $N - 1$.

A. Up to Two-Loop Order

The zero-loop part of the effective potential is given as

$$V^{(0)}_{\text{eff}}(\hat{\phi}_a) = \frac{\lambda}{4!} \hat{\phi}^4.$$

We first remove all the $\epsilon$ poles in the subsequent contributions to the effective potential, $V^{(i)}_{\text{eff}}(\hat{\phi}_a) (i=1,2,3)$. The one-loop part of the effective potential is readily obtained as

$$V^{(1)}_{\text{eff}}(\hat{\phi}_a) = \frac{\delta m_1^2}{2} \hat{\phi}^2 + \frac{\delta \lambda_1}{4!} \hat{\phi}^4 - \frac{\lambda^2 \hat{\phi}^4}{(4\pi)^2\epsilon} \left[ \frac{1}{8} + \frac{\tilde{N}}{72} \right] + \frac{\lambda^2 \hat{\phi}^4}{(4\pi)^2} \left[ -\frac{3}{32} + \frac{\gamma}{16} + \frac{1}{16} \ln \left( \frac{\lambda \hat{\phi}^2/2}{4\pi M^2} \right) \right] + \tilde{N} \left[ -\frac{1}{96} + \frac{\gamma}{144} + \frac{1}{144} \ln \left( \frac{\lambda \hat{\phi}^2/6}{4\pi M^2} \right) \right] .$$

The $\epsilon$ poles in this equation are readily cancelled out by choosing the counterterm constants $\delta m_1^2$ and $\delta \lambda_1$ as follows:

$$\delta m_1^2 = \frac{\lambda}{(4\pi)^2} a_1 ,$$

$$\delta \lambda_1 = \frac{\lambda^2}{(4\pi)^2} \left[ \frac{1}{\epsilon} \left( 3 + \frac{\tilde{N}}{3} \right) + b_1 \right] ,$$

where $a_1$ and $b_1$ are unspecified but finite constants at this stage. One may put $a_1$ (and $a_2, a_3$ below) to be zero from the beginning because the theory is massless. In our dimensional regularization scheme the pole part of $\delta m_1^2$ vanishes, but this is not true in the cutoff regularization method.
The two-loop part of the effective potential is obtained as

\[
V^{(2)}_{\text{eff}}(\hat{\phi}_a) = \frac{\delta m^2}{2} \hat{\phi}^2 + \frac{\delta \lambda_2}{4!} \hat{\phi}^4 - \frac{a_1 \lambda^2 \hat{\phi}^2}{(4\pi)^4 \epsilon} \left[ \frac{1}{2} + \tilde{N} \right] - \frac{\lambda^3 \hat{\phi}^4}{(4\pi)^4 \epsilon^2} \left[ \frac{3}{8} + \tilde{N} + \frac{\tilde{N}^2}{216} \right] \\
+ \frac{\lambda^3 \hat{\phi}^4}{(4\pi)^4 \epsilon} \left[ \frac{1}{8} + \frac{5\tilde{N}}{216} \right] - \frac{b_1 \lambda^3 \hat{\phi}^4}{(4\pi)^4 \epsilon} \left[ \frac{1}{4} + \tilde{N} \right] \\
+ \frac{a_1 \lambda^2 \hat{\phi}^2}{(4\pi)^4} \left[ -\frac{1}{4} + \frac{\gamma}{4} + \frac{1}{4} \ln \left( \frac{\lambda \hat{\phi}^2}{4\pi M^2} \right) + \tilde{N} \left\{ -\frac{1}{12} + \frac{\gamma}{12} + \frac{1}{12} \ln \left( \frac{\lambda \hat{\phi}^2}{4\pi M^2} \right) \right\} \right] \\
+ \frac{b_1 \lambda^3 \hat{\phi}^4}{(4\pi)^4} \left[ -\frac{1}{8} + \frac{\gamma}{8} + \frac{1}{8} \ln \left( \frac{\lambda \hat{\phi}^2}{4\pi M^2} \right) + \tilde{N} \left\{ -\frac{1}{72} + \frac{\gamma}{72} + \frac{1}{72} \ln \left( \frac{\lambda \hat{\phi}^2}{4\pi M^2} \right) \right\} \right] \\
+ \frac{\lambda^3 \hat{\phi}^4}{(4\pi)^4} \left[ \frac{11}{32} + \frac{A}{8} - \frac{5}{16} \gamma + \frac{3}{32} \gamma^2 + \left( -\frac{5}{16} + \frac{3}{16} \gamma \right) \ln \left( \frac{\lambda \hat{\phi}^2}{4\pi M^2} \right) + \frac{3}{32} \ln^2 \left( \frac{\lambda \hat{\phi}^2}{4\pi M^2} \right) \right] \\
+ \tilde{N} \left\{ \frac{29}{432} + \frac{A}{108} - \frac{7}{108} \gamma + \frac{\gamma^2}{48} + \frac{13}{432} \ln 3 - \frac{\gamma}{48} \ln 3 \right\} \\
+ \left( -\frac{7}{108} + \frac{\gamma}{24} \right) \ln \left( \frac{\lambda \hat{\phi}^2}{4\pi M^2} \right) + \frac{1}{48} \ln \left( \frac{\lambda \hat{\phi}^2}{4\pi M^2} \right) \ln \left( \frac{\lambda \hat{\phi}^2}{4\pi M^2} \right) \right\} \\
+ \tilde{N}^2 \left\{ \frac{1}{864} - \frac{\gamma}{432} + \frac{\gamma^2}{864} + \left( -\frac{1}{432} + \frac{\gamma}{432} \right) \ln \left( \frac{\lambda \hat{\phi}^2}{4\pi M^2} \right) + \frac{1}{864} \ln^2 \left( \frac{\lambda \hat{\phi}^2}{4\pi M^2} \right) \right\} \right]. \quad (11)
\]

Notice that the so-called “dangerous” pole terms such as \((\hat{\phi}^4/\epsilon) \ln[\lambda \hat{\phi}^2/(4\pi M^2)]\), \((l = 0, 2, 4)\) in the above equation, which cannot be removed by terms of counterterm constants \([\delta m^2 \hat{\phi}^2/2\) and \(\delta \lambda \hat{\phi}^4/(4!)]\), have been completely cancelled out among each other. This serves as a strong check of the correctness of the calculation at this stage. The counterterm constants \(\delta m^2\) and \(\delta \lambda_2\) are determined as

\[
\delta m^2 = \frac{\lambda^2}{(4\pi)^4} \left[ \frac{a_1}{\epsilon} \left( 1 + \frac{\tilde{N}}{3} \right) + a_2 \right],
\]

\[
\delta \lambda_2 = \frac{\lambda^3}{(4\pi)^4} \left[ \frac{1}{\epsilon^2} \left( 9 + 2\tilde{N} + \frac{\tilde{N}^2}{9} \right) - \frac{1}{\epsilon} \left( 3 + \frac{5\tilde{N}}{9} \right) + b_1 \left( 6 + \frac{2\tilde{N}}{3} \right) + b_2 \right],
\]

where \(a_2\) and \(b_2\) are also unspecified but finite constants.

In the minimal subtraction scheme (MS scheme), the finite parts of the counterterm constants \((a_1, b_1, a_2, \text{ and } b_2)\) are taken to be zero. In our massless theory, however, we encounter the infrared singularity in the defining condition for a coupling constant. To avoid this difficulty we follow Coleman and Weinberg \[1\] and require

\[
\frac{d^2 V_{\text{eff}}}{d\hat{\phi}^2} \bigg|_{\hat{\phi}=0} = 0, \quad \frac{d^4 V_{\text{eff}}}{d\hat{\phi}^4} \bigg|_{\hat{\phi}=M} = \lambda, \quad (12)
\]
where $M$ is a renormalization scale. Then constants $a_1$, $b_1$, $a_2$, and $b_2$ are determined as

$$a_1 = a_2 = 0,$$
$$b_1 = -4 - \frac{3}{2} \gamma - \frac{3}{2} \ln \left(\frac{\lambda/2}{4\pi}\right) - \tilde{N} \left\{ \frac{4}{9} + \gamma + \frac{1}{6} \ln \left(\frac{\lambda/6}{4\pi}\right) \right\},$$
$$b_2 = \frac{139}{4} - 3\lambda + 15\gamma + \frac{9}{4} \gamma^2 + \left(15 + \frac{9}{2} \gamma\right) \ln \left(\frac{\lambda/2}{4\pi}\right) + \frac{9}{4} \ln^2 \left(\frac{\lambda/2}{4\pi}\right)$$
$$+ \tilde{N} \left\{ \frac{202}{27} - \frac{2}{9} \lambda + \frac{29}{9} \gamma + \frac{\gamma^2}{2} - \frac{14}{9} \ln 3 - \frac{\gamma}{2} \ln 3 \right.$$
$$+ \left(\frac{29}{9} + \gamma - \frac{1}{2} \ln 3\right) \ln \left(\frac{\lambda/2}{4\pi}\right) + \frac{1}{2} \ln^2 \left(\frac{\lambda/2}{4\pi}\right) \right\}$$
$$+ \tilde{N}^2 \left\{ \frac{113}{324} + \frac{4}{27} \gamma + \frac{\gamma^2}{36} + \left(\frac{4}{27} + \frac{\gamma}{18}\right) \ln \left(\frac{\lambda/6}{4\pi}\right) + \frac{1}{36} \ln^2 \left(\frac{\lambda/6}{4\pi}\right) \right\}. \quad (13)$$

Putting these constants into the finite parts of $V^{(1)}_{\text{eff}}$ and $V^{(2)}_{\text{eff}}$, we eventually obtain the finite effective potential, $V^{[2]}_{\text{eff}}$, which satisfies the renormalization conditions, Eq. (12):

$$V^{[2]}_{\text{eff}}(\hat{\phi}_a) = \left[ \frac{\lambda}{4!} \hat{\phi}^4 \right] + \frac{\hbar^2 \lambda \hat{\phi}^4}{(4\pi)^2} \left[ -\frac{25}{96} + \frac{1}{16} \ln \left(\frac{\hat{\phi}^2}{M^2}\right) + \tilde{N} \left\{ -\frac{25}{864} + \frac{1}{144} \ln \left(\frac{\hat{\phi}^2}{M^2}\right) \right\} \right]$$
$$+ \frac{\hbar^2 \lambda^2 \hat{\phi}^4}{(4\pi)^4} \left[ \frac{55}{24} - \frac{13}{16} \ln \left(\frac{\hat{\phi}^2}{M^2}\right) + \frac{3}{32} \ln^2 \left(\frac{\hat{\phi}^2}{M^2}\right) + \tilde{N} \left\{ \frac{635}{1296} - \frac{19}{108} \ln \left(\frac{\hat{\phi}^2}{M^2}\right) \right\} \right]$$
$$+ \frac{1}{48} \ln^2 \left(\frac{\hat{\phi}^2}{M^2}\right) \right\} + \tilde{N}^2 \left\{ \frac{85}{3888} - \frac{11}{1296} \ln \left(\frac{\hat{\phi}^2}{M^2}\right) + \frac{1}{864} \ln^2 \left(\frac{\hat{\phi}^2}{M^2}\right) \right\} \right\}. \quad (14)$$

**B. Three-Loop Order**

Having illustrated our strategy, we continue with three-loop structure of the model, which is appreciably more complicated than that of previous subsection II.A. In the $h^3$-order calculation we need another counterterm constant $\delta Z_2$ in addition to the counterterm constants, $\delta m_1^2$, $\delta \lambda_1$, $\delta m_2^2$, $\delta \lambda_2$, $\delta m_3^2$, and $\delta \lambda_3$ [9]. Instead of determining $\delta Z_2$ from the renormalization of $Z$ in Eq. (4) [9][13], we can obtain it from the renormalization of $\tilde{\Gamma}_{a,b}^{(2)}(p,-p)$:

$$\delta Z_2 = \frac{\lambda^2}{(4\pi)^4} \left[ 1 - \frac{1}{12} \left( -\frac{\tilde{N}}{36} \right) + c_2 \right], \quad (c_2 \ \text{a finite constant}).$$

Using the integrals in Appendices A and B, we have the rather long expression for the three-loop part of the effective potential. Thus we will separate it into two parts, the counterterm plus pole part and the finite part. The counterterm plus pole part is calculated as
\[
[V_{\text{eff}}^{(3)}]_{\text{ct+pl}} = \frac{\delta m_2^2}{2} \hat{\phi}^2 + \frac{\delta \lambda_3}{4!} \hat{\phi}^4 - \frac{a_1 \lambda_3 \hat{\phi}^2}{(4\pi)^6 \epsilon^2} \left[ 1 + \frac{7\tilde{N}}{18} + \frac{\tilde{N}^2}{18} \right] \\
+ \frac{\lambda^3 \hat{\phi}^2}{(4\pi)^6 \epsilon} \left[ a_1 \left\{ \frac{1}{4} + \frac{1}{2} + \frac{2}{3} \right\} - a_1 b_1 \left\{ \frac{1}{2} + \frac{5}{6} \right\} - a_2 \left\{ \frac{1}{2} + \frac{5}{6} \right\} \right] \\
- \frac{\lambda^4 \hat{\phi}^4}{(4\pi)^6 \epsilon^3} \left[ 9 + \frac{3\tilde{N}}{8} + \frac{\tilde{N}^2}{24} + \frac{\tilde{N}^3}{648} \right] \\
+ \frac{\lambda^4 \hat{\phi}^4}{(4\pi)^6 \epsilon} \left[ \frac{31}{36} + \frac{41\tilde{N}}{162} + \frac{17\tilde{N}^2}{972} \right] \left[ b_1 \lambda^4 \hat{\phi}^4 \left\{ \frac{9}{8} + \frac{\tilde{N}}{4} + \frac{\tilde{N}^2}{72} \right\} \\
- \frac{\lambda^4 \hat{\phi}^4}{(4\pi)^6 \epsilon} \left[ \frac{49}{192} + \frac{\zeta(3)}{6} + \frac{253}{3888} + \frac{5}{162} \zeta(3) + \frac{35\tilde{N}^2}{15552} \right] \\
+ \frac{\lambda^4 \hat{\phi}^4}{(4\pi)^6 \epsilon} \left[ b_1 \left\{ \frac{3}{8} + \frac{\tilde{N}}{72} \right\} - b_2 \left\{ \frac{1}{8} + \frac{\tilde{N}}{72} \right\} + c_2 \left\{ \frac{1}{4} + \frac{\tilde{N}}{36} \right\} - b_2 \left\{ \frac{1}{4} + \frac{\tilde{N}}{36} \right\} \right].
\]

Notice that the complete cancellation of all dangerous pole terms \([\hat{\phi}^l/e^m] \ln^n(\lambda \hat{\phi}^2/(4\pi M^2))\), 
\(l = 0, 2, 4; (m, n) = (1, 1), (1, 2), (2, 1)\) has taken place here too. Remaining harmless pole terms are eliminated by choosing the counterterm constants as follows:

\[
\delta m_2^2 = \frac{\lambda^3}{(4\pi)^6} \left[ a_1 \epsilon^2 \left\{ 2 + \frac{7\tilde{N}}{9} + \frac{\tilde{N}^2}{9} \right\} - \frac{a_1}{\epsilon} \left\{ \frac{1}{2} + \frac{\tilde{N}}{2} \right\} \left( \frac{5}{18} - \frac{2}{9} \gamma + \frac{2}{9} \ln 2 \right) \right] \\
+ \frac{a_1 b_1}{\epsilon} \left\{ 1 + \frac{\tilde{N}}{3} \right\} + \frac{a_2}{\epsilon} \left\{ 1 + \frac{\tilde{N}}{3} \right\} + a_3 \right]\]

\[
\delta \lambda_3 = \frac{\lambda^4 \hat{\phi}^4}{(4\pi)^6} \left[ \frac{1}{\epsilon^3} \left\{ \frac{27}{2} + 9\tilde{N} + \tilde{N}^2 + \frac{\tilde{N}^3}{27} \right\} \\
- \frac{1}{\epsilon^2} \left\{ \frac{62}{3} + \frac{164\tilde{N}}{27} + \frac{34\tilde{N}^2}{81} \right\} + \frac{b_1}{\epsilon^2} \left\{ 27 + 6\tilde{N} + \frac{\tilde{N}^2}{3} \right\} \\
+ \frac{1}{\epsilon} \left[ \frac{49}{8} + 4\zeta(3) + \tilde{N} \left( \frac{253}{162} + \frac{20}{27} \zeta(3) \right) + \frac{35\tilde{N}^2}{648} \right] \\
- \frac{b_1}{\epsilon} \left\{ \frac{9}{3} + \frac{5\tilde{N}}{3} \right\} - \frac{b_2}{\epsilon} \left\{ 3 + \frac{\tilde{N}}{3} \right\} + \frac{b_2}{\epsilon} \left\{ 6 + \frac{2\tilde{N}}{3} \right\} - \frac{c_2}{\epsilon} \left\{ 6 + \frac{2\tilde{N}}{3} \right\} + b_3 \right]^2, \tag{15}
\]

where \(a_3\) and \(b_3\) are arbitrary but finite constants as before. When taken in the MS scheme, \(\delta l_3\) in Eq. 15 completely coincides with that of Ref. 7 which was obtained from the renormalization of the \(\Gamma^{(2)}(p^2)\) and \(\Gamma^{(4)}(p^2)\). This fact shows another check for the correctness of our calculation.

The finite part is given as follows:

\[
[V_{\text{eff}}^{(3)}]_{\text{ln}} = \frac{a_1^2 \lambda^2}{(4\pi)^6} \left[ \frac{1}{4} + \frac{\tilde{N}}{4} \right] \ln \left( \frac{\lambda \hat{\phi}^2}{4\pi M^2} \right)
\]
From the renormalization conditions Eq. (12) we obtain

\[
\begin{align*}
\frac{\lambda^3}{(4\pi)^6} & \left[ a_1 \left\{ -\frac{3}{8} + \frac{\gamma}{2} - \frac{1}{2} \ln 2 + \tilde{N} \left( -\frac{2}{9} + \frac{13}{36} \gamma - \frac{13}{36} \ln 2 - \frac{1}{6} \ln 3 \right) \right\} 
+ \frac{\lambda^4}{(4\pi)^6} \left[ \frac{701}{384} + \frac{9}{16} A - \frac{143}{96} \gamma + \frac{27}{64} \gamma^2 + \frac{\zeta(3)}{4} + \frac{143}{96} \ln 2 - \frac{27}{32} \gamma \ln 2 + \frac{27}{64} \ln^2 2 \right. \\
& \quad \left. + \tilde{N} \left( \frac{2741}{5184} + \frac{9}{48} A - \frac{199}{432} \gamma + \frac{9}{64} \gamma^2 + \frac{5}{108} \zeta(3) + \frac{199}{432} \ln 2 - \frac{9}{32} \gamma \ln 2 + \frac{9}{64} \ln^2 2 \right) 
+ \frac{125}{864} \ln 2 - \frac{9}{32} \ln 2 + \frac{1}{64} \ln^2 2 + \frac{67}{2592} \ln 3 - \frac{\gamma}{48} \ln 3 + \frac{1}{48} \ln 2 \ln 3 + \frac{1}{192} \ln^2 3 \right] \\
& \quad + b_1 \left\{ -\frac{3}{4} + \frac{9}{16} \gamma - \frac{9}{16} \ln 2 + \tilde{N} \left( -\frac{11}{72} + \frac{\gamma}{8} - \frac{1}{8} \ln 2 - \frac{1}{16} \ln 3 \right) \right\} \\
& \quad + \tilde{N} \left( -\frac{1}{216} + \frac{\gamma}{144} - \frac{1}{144} \ln 6 \right) \right\} 
+ \frac{\lambda^4}{(4\pi)^6} \left[ \frac{199}{864} + \frac{9}{64} \gamma - \frac{9}{64} \ln 2 - \frac{3}{64} \ln 3 \right] \\
& \quad + \tilde{N} \left( -\frac{103}{5184} + \frac{\gamma}{64} - \frac{1}{64} \ln 2 - \frac{1}{96} \ln 3 \right) + \tilde{N} \left( -\frac{1}{1728} + \frac{\gamma}{1728} - \frac{1}{1728} \ln 6 \right) \\
& \quad + b_1 \left\{ \frac{9}{32} + \frac{\tilde{N}}{16} + \frac{\tilde{N}^2}{288} \right\} \right] \ln^2 \left( \frac{\lambda \phi^2}{4\pi M^2} \right) \\
& \quad + \frac{\lambda^4}{(4\pi)^6} \left[ \frac{9}{64} + \frac{3 \tilde{N}}{192} + \frac{\tilde{N}^2}{5184} \right] \ln^3 \left( \frac{\lambda \phi^2}{4\pi M^2} \right) + \cdots , \tag{16}
\end{align*}
\]

where three dots (\( \cdots \)) denote finite terms which can be absorbed into the counterterms.

If we choose the boundary condition \( d\tilde{\Gamma}^{(2)}(p^2)/dp^2|_{p^2=\lambda M^2/2} = 1 \), the constant \( c_2 \), which is used in obtaining \( b_3 \), is determined as

\[
c_2 = \frac{\lambda^2}{(4\pi)^4} \left( \frac{1}{12} + \frac{\tilde{N}}{36} \right) \left\{ -\frac{9}{4} + \gamma + \ln \left( \frac{\lambda/2}{4\pi} \right) \right\} . \tag{17}
\]

From the renormalization conditions Eq. (12) we obtain

\[
a_3 = 0 ,
\]

\[
10
\]


Putting the constants in Eqs. (13), (17), and (18) into Eq. (16) we finally arrive at the three-loop part of the effective potential:

\[
V_{\text{eff}}^{(3)}(\hat{\phi}_a) = \frac{h^3 4^4}{4!} \left[ -\frac{27035}{48} - \frac{75}{4} A - 25\zeta(3) + \left( -\frac{1957}{8} - \frac{9}{2} A - 6\zeta(3) \right) \ln\left(\frac{\lambda}{4\pi}\right) - \frac{359}{8} \ln^2\left(\frac{\lambda}{4\pi}\right) - \frac{27}{8} \ln^3\left(\frac{\lambda}{4\pi}\right) + \tilde{N}\left\{ -\frac{228725}{1296} - \frac{125}{36} A - \frac{125}{27} \zeta(3) + \frac{175}{216} \ln 3 \right. \\
+ \left( -\frac{16769}{216} - \frac{5}{6} A - \frac{10}{9} \zeta(3) + \frac{7}{36} \ln 3 \right) \ln\left(\frac{\lambda}{4\pi}\right) - \frac{523}{36} \ln^2\left(\frac{\lambda}{4\pi}\right) - \frac{9}{8} \ln^3\left(\frac{\lambda}{4\pi}\right) \right) \\
+ \tilde{N}^2\left\{ -\frac{62105}{3888} - \frac{25}{162} A + \frac{175}{648} \ln 3 + \left( -\frac{4775}{648} - \frac{A}{27} + \frac{7}{108} \ln 3 \right) \ln\left(\frac{\lambda}{4\pi}\right) \\
- \frac{319}{216} \ln^2\left(\frac{\lambda}{4\pi}\right) - \frac{1}{8} \ln^3\left(\frac{\lambda}{4\pi}\right) \right\} \\
+ \tilde{N}^3\left\{ -\frac{4655}{11664} - \frac{395}{1944} \ln\left(\frac{\lambda}{4\pi}\right) - \frac{5}{108} \ln^2\left(\frac{\lambda}{4\pi}\right) - \frac{1}{216} \ln^3\left(\frac{\lambda}{4\pi}\right) \right\} \right]. \tag{18}
\]

III. CONCLUDING REMARKS

In this paper, we have proceeded one loop further in the effective potential calculation of the massless \(O(N)\) \(\phi^4\) theory, that is, we have calculated the three-loop effective potential of the massless \(O(N)\) \(\phi^4\) theory in four dimensions for the first time. For an easy check of the cancellation of all the dangerous pole terms among themselves in each loop order, we have given the \(\epsilon\)-expanded loop integrals in full detail. The \(h^3\)-order part of Eq. (19) is our main result.
We see that the order $\hbar$- and $\hbar^2$- terms completely agree with previous calculations in which a different regularization scheme (cutoff regularization) was employed. In the cutoff regularization, the loop integrations in four dimensions are much more difficult in the three-loop diagrams with the overlap divergence.

We expect the present calculations will serve as a useful reference for a development of the three-loop effective potential of more realistic models containing gauge fields such as the scalar QED. Especially in relation to this, the Coleman-Weinberg mechanism is interesting, which appears in many contexts, from displacive phase transitions in solids to the thermodynamics of the early universe.

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APPENDIX A: LOOP INTEGRATION FORMULAS

In this Appendix A, the momenta appearing in the formulas are all (Wick-rotated) Euclidean ones and the abbreviated integration measure is defined as

$$\int_k = M^{4-n} \int \frac{d^n k}{(2\pi)^n},$$

where $n = 4 - \epsilon$ is the space-time dimension in the framework of dimensional regularization and $M$ is an arbitrary constant with mass dimension, usually taken as the renormalization scale. One-loop integrations are quite elementary. For the two-loop integrations one may refer to Ref. [14]. The genuine three-loop integrals which cannot be factorized into the lower-loop ones are quite involved. Here we simply quote the results for them. Details of the computation can be found in Ref. [8].

One-loop integrals:
\begin{align*}
S_0 & \equiv \int_k \ln \left(1 + \frac{\xi^2}{k^2 + \sigma^2}\right) = -\frac{(\xi^2 + \sigma^2)^2}{(4\pi)^2} \\
& \times \left(\frac{\xi^2 + \sigma^2}{4\pi M^2}\right)^{-\epsilon/2} \Gamma\left(\frac{\epsilon}{2} - 2\right) + \xi\text{-independent term,} \\
S_1 & \equiv \int_k \frac{1}{k^2 + \sigma^2} = \frac{\sigma^2}{(4\pi)^2}\left(\frac{\sigma^2}{4\pi M^2}\right)^{-\epsilon/2} \Gamma\left(\frac{\epsilon}{2} - 1\right), \\
S_2 & \equiv \int_k \frac{1}{k^2 + \sigma^2/3} = \frac{\sigma^2}{3(4\pi)^2}\left(\frac{\sigma^2/3}{4\pi M^2}\right)^{-\epsilon/2} \Gamma\left(\frac{\epsilon}{2} - 1\right), \\
S_3 & \equiv \int_k \frac{1}{(k^2 + \sigma^2)^2} = \frac{1}{(4\pi)^2}\left(\frac{\sigma^2}{4\pi M^2}\right)^{-\epsilon/2} \Gamma\left(\frac{\epsilon}{2}\right), \\
S_4 & \equiv \int_k \frac{1}{(k^2 + \sigma^2/3)^2} = \frac{1}{(4\pi)^2}\left(\frac{\sigma^2/3}{4\pi M^2}\right)^{-\epsilon/2} \Gamma\left(\frac{\epsilon}{2}\right). \quad \text{(A1)}
\end{align*}

Two-loop integrals:
\begin{align*}
W_1 & \equiv \int_k \frac{1}{(p^2 + \sigma^2)(p^2 + \sigma^2 + \sigma^2)} = \frac{\sigma^4}{(4\pi)^2}\left(\frac{\sigma^2}{4\pi M^2}\right)^{-\epsilon} \Gamma^2\left(\frac{\epsilon}{2} - 1\right), \\
W_2 & \equiv \int_k \frac{1}{(p^2 + \sigma^2/3)(p^2 + \sigma^2/3 + \sigma^2/3)} = \frac{\sigma^4}{9(4\pi)^4}\left(\frac{\sigma^2/3}{4\pi M^2}\right)^{-\epsilon} \Gamma^2\left(\frac{\epsilon}{2} - 1\right), \\
W_3 & \equiv \int_k \frac{1}{(p^2 + \sigma^2)(p^2 + \sigma^2 + \sigma^2/3)} = \frac{\sigma^4}{3(4\pi)^4}\left(\frac{\sigma^2}{4\pi M^2}\right)^{-\epsilon/2} \left(\frac{\sigma^2/3}{4\pi M^2}\right)^{-\epsilon/2} \Gamma^2\left(\frac{\epsilon}{2} - 1\right), \\
W_4 & \equiv \int_k \frac{1}{(k^2 + \sigma^2)(p^2 + \sigma^2)(p^2 + \sigma^2 + \sigma^2)} \\
& = \frac{\sigma^2}{(4\pi)^4}\left(\frac{\sigma^2}{4\pi M^2}\right)^{-\epsilon} \Gamma^2\left(1 + \epsilon/2\right) \frac{6}{\epsilon^2} - 3A + O(\epsilon) , \\
W_5 & \equiv \int_k \frac{1}{(k^2 + \sigma^2/3)(k^2 + \sigma^2/3 + \sigma^2/3)} \\
& = \frac{\sigma^2}{3(4\pi)^4}\left(\frac{\sigma^2/3}{4\pi M^2}\right)^{-\epsilon} \Gamma^2\left(1 + \epsilon/2\right) \frac{10}{\epsilon^2} + 6\ln 3 - 3\ln^2 3 - B + O(\epsilon) , \\
W_6 & \equiv \int_k \frac{1}{(k^2 + \sigma^2)(p^2 + \sigma^2)(p^2 + \sigma^2) + \sigma^2} \\
& = \frac{1}{(4\pi)^4}\left(\frac{\sigma^2}{4\pi M^2}\right)^{-\epsilon/2} \left[\frac{2}{\epsilon^2} + \frac{1}{\epsilon} \left(1 - 2\gamma\right) + \frac{1}{2} - \gamma + \gamma^2 + \frac{\pi^2}{12} + A + O(\epsilon) , \\
W_7 & \equiv \int_k \frac{1}{(k^2 + \sigma^2)(p^2 + \sigma^2)(p^2 + \sigma^2) + \sigma^2} \\
& = \frac{1}{(4\pi)^4}\left(\frac{\sigma^2/3}{4\pi M^2}\right)^{-\epsilon/2} \left[\frac{2}{\epsilon^2} + \frac{1}{\epsilon} \left(1 - 2\gamma\right) + \frac{1}{2} - \gamma + \gamma^2 + \frac{\pi^2}{12} + B + O(\epsilon) , \\
W_8 & \equiv \int_k \frac{1}{(k^2 + \sigma^2)(p^2 + \sigma^2))(p^2 + \sigma^2) + \sigma^2} \\
& = \frac{1}{(4\pi)^4}\left(\frac{\sigma^2}{4\pi M^2}\right)^{-\epsilon} \left[\frac{2}{\epsilon^2} + \frac{1}{\epsilon} \left(1 - 2\gamma\right) + \frac{1}{2} - \gamma + \gamma^2 + \frac{\pi^2}{12} + C + O(\epsilon) \right]. \quad \text{(A2)}
\end{align*}
In the above equation, \( \gamma \) is the usual Euler constant, \( \gamma = 0.5772156649 \cdots \), and numerical values of the constants, \( A, B, \) and \( C \) in Eq. (A2) are

\[
A = f(1, 1) = -1.1719536193 \cdots , \\
B = f(1, 3) = -2.3439072387 \cdots , \\
C = f \left( \frac{1}{3}, \frac{1}{3} \right) = 0.1778279325 \cdots ,
\]

where

\[
f(a, b) \equiv \int_0^1 dx \left[ \int_0^{1-z} dy \left( -\frac{\ln(1-y)}{y} \right) - \frac{z \ln z}{1-z} \right], \quad z = \frac{ax + b(1-x)}{x(1-x)}.
\]

These constants \( A, B, \) and \( C \) can be analytically integrated \[15\]. The results are expressed in terms of Clausen function:

\[
A = \frac{B}{2} = -\frac{3}{2} C - \frac{3}{4} \ln^2 3 = -\frac{2}{\sqrt{3}} \text{Cl}_2 \left( \frac{\pi}{3} \right),
\]

where

\[
\text{Cl}_2(\theta) = \int_0^\theta \ln[2 \sin(\theta'/2)]d\theta'.
\]

**Trivial three-loop integrals:**

\[
H_1 \equiv \int_k \left( \frac{1}{(k^2 + \sigma^2)^2} \right) \left( \int_p \frac{1}{p^2 + \sigma^2} \right)^2 = S_1^2 S_3, \\
H_2 \equiv \int_k \left( \frac{1}{(k^2 + \sigma^2)^2} \right) \int_p \frac{1}{p^2 + \sigma^2} \int_q \frac{1}{q^2 + \sigma^2/3} = S_1 S_2 S_3, \\
H_3 \equiv \int_k \left( \frac{1}{(k^2 + \sigma^2/3)^2} \right) \left( \int_p \frac{1}{p^2 + \sigma^2} \right)^2 = S_1^2 S_4, \\
H_4 \equiv \int_k \left( \frac{1}{(k^2 + \sigma^2)^2} \right) \left( \int_p \frac{1}{p^2 + \sigma^2/3} \right)^2 = S_2 S_3, \\
H_5 \equiv \int_k \left( \frac{1}{(k^2 + \sigma^2/3)^2} \right) \int_p \frac{1}{p^2 + \sigma^2} \int_q \frac{1}{q^2 + \sigma^2/3} = S_1 S_2 S_4, \\
H_6 \equiv \int_k \left( \frac{1}{(k^2 + \sigma^2/3)^2} \right) \left( \int_p \frac{1}{p^2 + \sigma^2/3} \right)^2 = S_2^2 S_4, \\
I_1 \equiv \int_k \frac{1}{k^2 + \sigma^2} \int_{p,q} \frac{1}{(p^2 + \sigma^2)^2(q^2 + \sigma^2)[(p + q)^2 + \sigma^2]} = S_1 W_6, \\
I_2 \equiv \int_k \frac{1}{k^2 + \sigma^2/3} \int_{p,q} \frac{1}{(p^2 + \sigma^2)^2(q^2 + \sigma^2)[(p + q)^2 + \sigma^2]} = S_2 W_6.
\]
\[ I_3 \equiv \int \frac{1}{k^2 + \sigma^2} \int_{p,q} \frac{1}{(p^2 + \sigma^2/3)^2(q^2 + \sigma^2/3)[(p+q)^2 + \sigma^2]} = S_1 W_7, \]
\[ I_4 \equiv \int \frac{1}{k^2 + \sigma^2/3} \int_{p,q} \frac{1}{(p^2 + \sigma^2/3)^2(q^2 + \sigma^2/3)[(p+q)^2 + \sigma^2]} = S_2 W_7, \]
\[ I_5 \equiv \int \frac{1}{k^2 + \sigma^2} \int_{p,q} \frac{1}{(p^2 + \sigma^2)^2(q^2 + \sigma^2/3)[(p+q)^2 + \sigma^2/3]} = S_1 W_8, \]
\[ I_6 \equiv \int \frac{1}{k^2 + \sigma^2/3} \int_{p,q} \frac{1}{(p^2 + \sigma^2)^2(q^2 + \sigma^2/3)[(p+q)^2 + \sigma^2/3]} = S_2 W_8. \] (A3)

Genuine three-loop integrals:

\[ J_1 \equiv \int_k \frac{1}{p} \left( \frac{1}{(p^2 + \sigma^2)[(p+k)^2 + \sigma^2]} \right)^2 \]
\[ = \Omega_2 \left[ \frac{16}{3} + \frac{1}{\epsilon^2} \left\{ \frac{92}{9} - 24\gamma \right\} + \frac{1}{\epsilon} \left\{ 35 - 46\gamma + 18\gamma^2 + \pi^2 \right\} \right], \]
\[ J_2 \equiv \int_{k^2} \frac{1}{p^2 + \sigma^2} \left( \frac{1}{(p+k)^2 + \sigma^2} \right)^2 \]
\[ = \Omega_2 \left[ \frac{176}{273} + \frac{1}{\epsilon^2} \left\{ \frac{332}{27} - \frac{88}{9}\gamma + \frac{28}{9}\ln 3 \right\} + \frac{1}{\epsilon} \left\{ \frac{365}{27} - \frac{166}{9}\gamma + \frac{22}{3}\gamma^2 + \frac{11}{27}\pi^2 + \frac{55}{9}\ln 3 - \frac{14}{3}\gamma \ln 3 + \ln^2 3 \right\} \right], \]
\[ J_3 \equiv \int_k \frac{1}{(p^2 + \sigma^2)[(p+k)^2 + \sigma^2]} \]
\[ = \Omega_2 \left[ \frac{16}{9\epsilon^3} + \frac{1}{\epsilon^2} \left\{ \frac{92}{27} - \frac{8}{3}\gamma + \frac{8}{3}\ln 3 \right\} + \frac{1}{\epsilon} \left\{ \frac{35}{9} - \frac{46}{9}\gamma + 2\gamma^2 + \frac{\pi^2}{9} + \frac{46}{9}\ln 3 - 4\gamma \ln 3 + 2\ln^2 3 \right\} \right], \]
\[ K_1 \equiv \int_k \frac{1}{k^2 + \sigma^2} \left( \int_p \frac{1}{(p^2 + \sigma^2)[(p+k)^2 + \sigma^2]} \right)^2 \]
\[ = \Omega_1 \left[ -\frac{8}{3} + \frac{1}{\epsilon^2} \left\{ \frac{68}{3} + 12\gamma \right\} + \frac{1}{\epsilon} \left\{ -\frac{134}{9} - 12A + 34\gamma - 9\gamma^2 - \frac{\pi^2}{2} \right\} \right], \]
\[ K_2 \equiv \int_{k^2} \frac{1}{(k^2 + \sigma^2)(p^2 + \sigma^2)[(p+k)^2 + \sigma^2]} \]
\[ = \Omega_1 \left[ -\frac{56}{9\epsilon^3} + \frac{1}{\epsilon^2} \left\{ \frac{52}{3} + \frac{28}{3}\gamma - \frac{4}{3}\ln 3 \right\} + \frac{1}{\epsilon} \left\{ \frac{302}{9} - 6A - \frac{2}{3}B + 26\gamma - 7\gamma^2 - \frac{7}{18}\pi^2 - 4\ln 3 + 2\gamma \ln 3 \right\} \right], \]
\[ K_3 \equiv \int_k \frac{1}{k^2 + \sigma^2/3} \left( \int_p \frac{1}{(p^2 + \sigma^2)[(p+k)^2 + \sigma^2/3]} \right)^2 \]
\[ = \Omega_1 \left[ -\frac{40}{9\epsilon^3} + \frac{1}{\epsilon^2} \left\{ \frac{116}{9} + \frac{20}{3}\gamma - \frac{8}{3}\ln 3 \right\} + \frac{1}{\epsilon} \left\{ -26 - \frac{4}{3}B + \frac{58}{3}\gamma - 5\gamma^2 - \frac{5}{18}\pi^2 + 4\gamma \ln 3 - \frac{22}{3}\ln^3 \right\} \right], \]
\[ K_4 \equiv \int_k \frac{1}{k^2 + \sigma^2} \left( \int_p \frac{1}{(p^2 + \sigma^2/3)[(p+k)^2 + \sigma^2/3]} \right)^2 \]
\[ = \Omega_1 \left[ -\frac{40}{9\epsilon^3} + \frac{1}{\epsilon^2} \left\{ -12 + \frac{20}{3}\gamma - \frac{8}{3}\ln 3 \right\} \right] \]
In the above equation the overall multiplying factors are

\[
\Omega_0 = \frac{1}{(4\pi)^6} \left( \frac{\sigma^2}{4\pi M^2} \right)^{-3\epsilon/2}, \quad \Omega_1 = \frac{\sigma^2}{(4\pi)^6} \left( \frac{\sigma^2}{4\pi M^2} \right)^{-3\epsilon/2}, \quad \Omega_2 = \frac{\sigma^4}{(4\pi)^6} \left( \frac{\sigma^2}{4\pi M^2} \right)^{-3\epsilon/2}.
\]

**APPENDIX B: DETAILS OF THE CONNTERTERM INTEGRALS**

Diagrams Fig. 1 to Fig. 3 contain various counterterm integrals in our perturbation theory. In this Appendix B they are shown explicitly:
Diag. 1a = \(- \frac{\hbar}{2(4\pi)^2} \left( \delta m^2 + (\lambda + \delta \lambda) \delta \phi^2/2 \right)^2 \left( \frac{\delta m^2 + (\lambda + \delta \lambda) \delta \phi^2/2}{4\pi M^2(1 + \delta Z)} \right)^{-\epsilon/2} \Gamma \left( -2 + \frac{\epsilon}{2} \right) \)

\[= \hbar \left[ - \frac{\lambda^2 \phi^4}{8(4\pi)^2} \left( \frac{\lambda \phi^2/2}{4\pi M^2} \right)^{-\epsilon/2} \Gamma \left( -2 + \frac{\epsilon}{2} \right) \right] \]

\[+ \hbar^2 \left[ - \frac{\lambda \phi^2}{2(4\pi)^2} \left( \frac{\delta m_1^2 + \delta \lambda_1 \phi^2}{2} \right) \left( 1 - \frac{\epsilon}{4} \right) \left( \frac{\lambda \phi^2/2}{4\pi M^2} \right)^{-\epsilon/2} \Gamma \left( -2 + \frac{\epsilon}{2} \right) \right] \]

\[+ \hbar^3 \left[ - \frac{1}{2(4\pi)^2} \left( \delta m_1^2 + \delta \lambda_1 \phi^2 \right) \left( 1 - 3\epsilon + \frac{\epsilon^2}{8} \right) \right] \]

\[+ \lambda \phi^2 \left( \delta m_2^2 + \delta \lambda_2 \phi^2 \right) (1 - \frac{\epsilon}{4}) + \lambda^2 \phi^4 \delta Z_2 \left( - \frac{1}{2} + \frac{\epsilon}{8} \right) \left( \frac{\lambda \phi^2/2}{4\pi M^2} \right)^{-\epsilon/2} \Gamma \left( -2 + \frac{\epsilon}{2} \right) \],

Diag. 1b = \(- \frac{\hbar(N - 1)}{2(4\pi)^2} \left( \frac{\delta m^2 + (\lambda + \delta \lambda) \delta \phi^2/6}{1 + \delta Z} \right)^2 \left( \frac{\delta m^2 + (\lambda + \delta \lambda) \delta \phi^2/6}{4\pi M^2(1 + \delta Z)} \right)^{-\epsilon/2} \Gamma \left( -2 + \frac{\epsilon}{2} \right) \)

\[= \hbar \left[ - \frac{(N - 1) \lambda^2 \phi^4}{72(4\pi)^2} \left( \frac{\lambda \phi^2/6}{4\pi M^2} \right)^{-\epsilon/2} \Gamma \left( -2 + \frac{\epsilon}{2} \right) \right] \]

\[+ \hbar^2 \left[ - \frac{(N - 1) \lambda \phi^2}{6(4\pi)^2} \left( \delta m_1^2 + \delta \lambda_1 \phi^2 \right) \left( 1 - \frac{\epsilon}{4} \right) \left( \frac{\lambda \phi^2/6}{4\pi M^2} \right)^{-\epsilon/2} \Gamma \left( -2 + \frac{\epsilon}{2} \right) \right] \]

\[+ \hbar^3 \left[ - \frac{N - 1}{2(4\pi)^2} \left( \delta m_1^2 + \delta \lambda_1 \phi^2 \right) \left( 1 - 3\epsilon + \frac{\epsilon^2}{8} \right) \right] \]

\[+ \lambda \phi^2 \left( \delta m_2^2 + \delta \lambda_2 \phi^2 \right) (1 - \frac{\epsilon}{4}) + \lambda^2 \phi^4 \delta Z_2 \left( - \frac{1}{2} + \frac{\epsilon}{8} \right) \left( \frac{\lambda \phi^2/6}{4\pi M^2} \right)^{-\epsilon/2} \Gamma \left( -2 + \frac{\epsilon}{2} \right) \],

Diag. 2a = \(- \frac{\hbar^2(\lambda + \delta \lambda)}{8(4\pi)^4(1 + \delta Z)^2} \left( \frac{\delta m^2 + (\lambda + \delta \lambda) \delta \phi^2/2}{1 + \delta Z} \right)^2 \left( \frac{\delta m^2 + (\lambda + \delta \lambda) \delta \phi^2/2}{4\pi M^2(1 + \delta Z)} \right)^{-\epsilon/2} \Gamma^2 \left( -1 + \frac{\epsilon}{2} \right) \)

\[= \hbar^2 \left[ \frac{\lambda^3 \phi^4}{32(4\pi)^4} \left( \frac{\lambda \phi^2/2}{4\pi M^2} \right)^{-\epsilon/2} \Gamma^2 \left( -1 + \frac{\epsilon}{2} \right) \right] \]

\[+ \hbar^3 \left[ \frac{1}{8(4\pi)^4} \left\{ \lambda^2 \phi^2 \delta m_1^2 \left( 1 - \frac{\epsilon}{2} \right) + \lambda^2 \phi^4 \delta \lambda_1 \left( \frac{3}{4} - \frac{\epsilon}{4} \right) \right\} \left( \frac{\lambda \phi^2/2}{4\pi M^2} \right)^{-\epsilon/2} \Gamma^2 \left( -1 + \frac{\epsilon}{2} \right) \right] \]

Diag. 2b = \(- \frac{\hbar^2(N - 1)(\lambda + \delta \lambda)}{12(4\pi)^4(1 + \delta Z)^2} \left( \frac{\delta m^2 + (\lambda + \delta \lambda) \delta \phi^2/2}{1 + \delta Z} \right)^2 \left( \frac{\delta m^2 + (\lambda + \delta \lambda) \delta \phi^2/6}{4\pi M^2(1 + \delta Z)} \right)^{-\epsilon/2} \Gamma^2 \left( -1 + \frac{\epsilon}{2} \right) \)

\[= \hbar^2 \left[ \frac{(N - 1) \lambda^3 \phi^4}{144(4\pi)^4} \left( \frac{\lambda \phi^2/2}{4\pi M^2} \right)^{-\epsilon/2} \left( \frac{\lambda \phi^2/6}{4\pi M^2} \right)^{-\epsilon/2} \Gamma^2 \left( -1 + \frac{\epsilon}{2} \right) \right] \]

\[+ \hbar^3 \left[ \frac{N - 1}{12(4\pi)^4} \left\{ \lambda^2 \phi^2 \delta m_1^2 \left( \frac{2}{3} - \frac{\epsilon}{3} \right) + \lambda^2 \phi^4 \delta \lambda_1 \left( \frac{1}{4} - \frac{\epsilon}{12} \right) \right\} \right] \]

\[\times \left( \frac{\lambda \phi^2/2}{4\pi M^2} \right)^{-\epsilon/2} \left( \frac{\lambda \phi^2/6}{4\pi M^2} \right)^{-\epsilon/2} \Gamma^2 \left( -1 + \frac{\epsilon}{2} \right) \],

Diag. 2c = \(- \frac{\hbar^2(N^2 - 1)(\lambda + \delta \lambda)}{24(4\pi)^4(1 + \delta Z)^2} \left( \frac{\delta m^2 + (\lambda + \delta \lambda) \delta \phi^2/6}{1 + \delta Z} \right)^2 \left( \frac{\delta m^2 + (\lambda + \delta \lambda) \delta \phi^2/6}{4\pi M^2(1 + \delta Z)} \right)^{-\epsilon/2} \Gamma^2 \left( -1 + \frac{\epsilon}{2} \right) \)
\[ \frac{N^2 - 1}{864(4\pi)^4} \left( \frac{\lambda \hat{\phi}^2 / 6}{4\pi M^2} \right)^{\epsilon} \Gamma^2 \left(-1 + \frac{\epsilon}{2} \right) \]

\[ + h^3 \left( \frac{N^2 - 1}{24(4\pi)^4} \left( \frac{\lambda^2 \hat{\phi}^2 \hat{m}_1^2}{3} \right)^{1 - \epsilon / 2} + \frac{\lambda^2 \hat{\phi}^4 \hat{\lambda}_1}{9} \left( \frac{3}{4} - \frac{\epsilon}{4} \right) \right) \left( \frac{\lambda \hat{\phi}^2 / 6}{4\pi M^2} \right)^{\epsilon} \Gamma^2 \left(-1 + \frac{\epsilon}{2} \right), \]

Diag. 3a =

\[ \frac{h^2}{12(4\pi)^4} \left( \frac{\delta m^2 + (\lambda + \delta \lambda) \hat{\phi}^2 / 2}{1 + \delta Z} \right) \left( \frac{\delta m^2 + (\lambda + \delta \lambda) \hat{\phi}^2 / 2}{4\pi M^2(1 + \delta Z)} \right)^{\epsilon} \]

\[ \times \frac{\Gamma^2(1 + \epsilon / 2)}{(1 - \epsilon)(1 - \epsilon / 2)} \left(-6 - 3A \right)^{\epsilon}, \]

Diag. 3b =

\[ - \frac{h^2}{36(4\pi)^4} \left( \frac{\lambda \hat{\phi}^2 / 2}{1 + \delta Z} \right)^{\epsilon} \left( \frac{\delta m^2 + (\lambda + \delta \lambda) \hat{\phi}^2 / 6}{4\pi M^2(1 + \delta Z)} \right) \]

\[ \times \frac{\Gamma^2(1 + \epsilon / 2)}{(1 - \epsilon)(1 - \epsilon / 2)} \left(-10 / \epsilon^2 + 6 / \epsilon \ln 3 - 3 / 2 \ln^2 3 - B \right) \]

\[ = h^3 \left( \frac{(N - 1) \lambda \hat{\phi}^4}{216(4\pi)^4} \left( \frac{\lambda \hat{\phi}^2 / 6}{4\pi M^2} \right)^{\epsilon} \Gamma^2(1 + \epsilon / 2) \right) \left(-10 / \epsilon^2 + 6 / \epsilon \ln 3 - 3 / 2 \ln^2 3 - B \right) \]

\[ + h^3 \left( \frac{N - 1}{36(4\pi)^4} \left( \frac{\lambda \hat{\phi}^2 \hat{m}_1^2}{1 - \epsilon} + \lambda^2 \hat{\phi}^4 \delta \lambda_1 \left( \frac{1}{2} - \frac{\epsilon}{6} \right) \right) \left( \frac{\lambda \hat{\phi}^2 / 6}{4\pi M^2} \right)^{\epsilon} \]

\[ \times \frac{\Gamma^2(1 + \epsilon / 2)}{(1 - \epsilon)(1 - \epsilon / 2)} \left(-10 / \epsilon^2 + 6 / \epsilon \ln 3 - 3 / 2 \ln^2 3 - B \right). \]
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FIGURES

\[ -\frac{\hbar}{i} \left[ \frac{1}{2} \begin{array}{c} \text{A} \\ \text{B} \end{array} + \frac{1}{2} (N - 1) \begin{array}{c} \text{B} \\ \text{A} \end{array} \right] \]

FIG. 1. [Diag. 1] = [Diag. 1a] + [Diag. 1b].

\[ -\frac{\hbar}{i} \left[ \frac{1}{2^3} \begin{array}{c} \text{A} \\ \text{A} \end{array} + \frac{1}{2^3} \frac{2(N - 1)}{3} \begin{array}{c} \text{A} \\ \text{B} \end{array} + \frac{1}{2^3} \frac{N^2 - 1}{3} \begin{array}{c} \text{B} \\ \text{B} \end{array} \right] \]

FIG. 2. [Diag. 2] = [Diag. 2a] + [Diag. 2b] + [Diag. 2c].

\[ -\frac{\hbar}{i} \left[ \frac{1}{3^2} \begin{array}{c} \text{A} \\ \text{A} \end{array} + \frac{1}{3^2} \frac{N - 1}{3} \begin{array}{c} \text{B} \\ \text{B} \end{array} \right] \]

FIG. 3. [Diag. 3] = [Diag. 3a] + [Diag. 3b].

\[ -\frac{\hbar}{i} \left[ \frac{1}{2^4} \begin{array}{c} \text{A} \\ \text{A} \\ \text{A} \\ \text{A} \end{array} + \frac{1}{2^4} \frac{2(N - 1)}{3} \begin{array}{c} \text{A} \\ \text{A} \\ \text{B} \\ \text{B} \end{array} + \frac{1}{2^4} \frac{N - 1}{3^2} \begin{array}{c} \text{B} \\ \text{B} \\ \text{A} \\ \text{A} \end{array} + \frac{1}{2^4} \frac{(N - 1)^2}{3^2} \begin{array}{c} \text{B} \\ \text{B} \\ \text{B} \\ \text{B} \end{array} \right. \]

\[ + \frac{1}{2^4} \frac{2(N^2 - 1)}{3^2} \begin{array}{c} \text{B} \\ \text{B} \\ \text{B} \\ \text{B} \end{array} + \frac{1}{2^4} \frac{(N - 1)(N + 1)^2}{3^2} \begin{array}{c} \text{B} \\ \text{B} \\ \text{B} \\ \text{B} \end{array} \]

FIG. 4. [Diag. 4] = [Diag. 4a] + [Diag. 4b] + [Diag. 4c] + [Diag. 4d] + [Diag. 4e] + [Diag. 4f].
\[-\frac{\hbar}{i} \left[ \frac{1}{2^3} \begin{array}{c} A \\ A \\ A \end{array} + \frac{1}{2^3} \frac{N-1}{3} \begin{array}{c} B \\ A \\ A \end{array} + \frac{1}{2^3} \frac{2(N-1)}{3^3} \begin{array}{c} A \\ B \\ A \end{array} + \frac{1}{2^3} \frac{2(N^2-1)}{3^3} \begin{array}{c} B \\ B \\ A \end{array} \\
+ \frac{1}{2^3} \frac{2(N^2-1)}{3^3} \begin{array}{c} B \\ B \\ A \end{array} + \frac{1}{2^3} \frac{N-1}{3^2} \begin{array}{c} A \\ A \\ A \end{array} + \frac{1}{2^3} \frac{2(N-1)}{3^3} \begin{array}{c} A \\ B \\ B \end{array} + \frac{1}{2^3} \frac{(N-1)^2}{3^3} \begin{array}{c} B \\ B \\ B \end{array} \right] \]

FIG. 5. [Diag. 5] = [Diag. 5a] + [Diag. 5b] + [Diag. 5c] + [Diag. 5d] + [Diag. 5e] + [Diag. 5f].

\[-\frac{\hbar}{i} \left[ \frac{1}{4!2} \begin{array}{c} A \\ A \\ A \end{array} + \frac{1}{4!2} \frac{2(N-1)}{3} \begin{array}{c} A \\ B \\ B \end{array} + \frac{1}{4!2} \frac{(N^2-1)}{3} \begin{array}{c} B \\ B \\ B \end{array} \right] \]

FIG. 6. [Diag. 6] = [Diag. 6a] + [Diag. 6b] + [Diag. 6c].

\[-\frac{\hbar}{i} \left[ \frac{1}{2^3} \begin{array}{c} A \\ A \\ A \end{array} + \frac{1}{2^3} \frac{2(N-1)}{3^2} \begin{array}{c} A \\ B \\ B \end{array} + \frac{1}{2^3} \frac{4(N-1)}{3^3} \begin{array}{c} B \\ B \\ B \end{array} + \frac{1}{2^3} \frac{N^2-1}{3^3} \begin{array}{c} B \\ B \\ B \end{array} \right] \]

FIG. 7. [Diag. 7] = [Diag. 7a] + [Diag. 7b] + [Diag. 7c] + [Diag. 7d].

\[-\frac{\hbar}{i} \left[ \frac{1}{2^4} \begin{array}{c} A \\ A \\ A \\ A \end{array} + \frac{1}{2^4} \frac{2(N-1)}{3^2} \begin{array}{c} A \\ A \\ B \\ A \end{array} + \frac{1}{2^4} \frac{4(N-1)}{3^4} \begin{array}{c} B \\ B \\ B \\ A \end{array} + \frac{1}{2^4} \frac{(N-1)^2}{3^4} \begin{array}{c} B \\ B \\ B \\ A \end{array} \right] \]

FIG. 8. [Diag. 8] = [Diag. 8a] + [Diag. 8b] + [Diag. 8c] + [Diag. 8d].
\[ \frac{-\hbar}{2} \left[ \frac{1}{4!} \frac{N - 1}{3^3} + \frac{1}{4!} \frac{4(N - 1)}{3^3} + \frac{1}{4!} \frac{N - 1}{3^3} \right] \]

FIG. 9. [Diag. 9] = [Diag. 9a] + [Diag. 9b] + [Diag. 9c].