An application of proof mining to the proximal point algorithm in CAT(0) spaces

Laurentiu Leuştean\textsuperscript{a,b} and Andrei Sipoş\textsuperscript{a,b}

\textsuperscript{a}Faculty of Mathematics and Computer Science, University of Bucharest, Academiei 14, 010014 Bucharest, Romania
\textsuperscript{b}Simion Stoilow Institute of Mathematics of the Romanian Academy, P. O. Box 1-764, 014700 Bucharest, Romania
E-mails: laurentiu.leustean@unibuc.ro, Andrei.Sipos@imar.ro

Dedicated to the memory of Professor Solomon Marcus (1925-2016)

Abstract

We compute, using techniques originally introduced by Kohlenbach, the first author and Nicolae, uniform rates of metastability for the proximal point algorithm in the context of CAT(0) spaces (as first considered by Bačák), specifically for the case where the ambient space is totally bounded. This result is part of the program of proof mining, which aims to apply methods of mathematical logic with the purpose of extracting quantitative information out of ordinary mathematical proofs, which may not be necessarily constructive.

Keywords: Proof mining; Convex optimization; Proximal point algorithm; CAT(0) spaces; Metastability; Fejér monotonicity; Total boundedness.

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1 Introduction

The proximal point algorithm is a fundamental tool of convex optimization, going back to Martinet [19], Rockafellar [20] and Brézis and Lions [3]. Since its inception, the schema turned out to be highly versatile, covering in its various developments, \textit{inter alia}, the problems of finding zeros of monotone operators, minima of convex functions and fixed points of nonexpansive mappings. For a general introduction to the field in the context of Hilbert spaces, see the book of Bauschke and Combettes [2].
A recent breakthrough was achieved by Bačák [1], who proved the weak convergence in complete CAT(0) spaces (that is, $\Delta$-convergence) of the variant of the algorithm used to find minima of convex, lower semicontinuous (lsc) proper functions. Let us detail the statement of his result. If $X$ is a complete CAT(0) space and $f : X \to (-\infty, +\infty]$ is a convex, lsc proper function that has minimizers, then, following Jost [8], we may define its resolvent by the relation

$$J_f(x) := \arg\min_{y \in X} \left[ f(y) + \frac{1}{2} d^2(x, y) \right].$$

For such an $f$, a starting point $x \in X$, and a sequence of weights $(\gamma_n)_{n \in \mathbb{N}}$, the proximal point algorithm $(x_n)_{n \in \mathbb{N}}$ is defined by setting:

$$x_0 := x, \quad x_{n+1} := J_{\gamma_n f}x_n \text{ for any } n \in \mathbb{N}.$$

Bačák’s result then states that, conditional on the fact that $\sum_{n=0}^{\infty} \gamma_n = \infty$, the sequence $(x_n)$ converges weakly to a minimizer of $f$. As a consequence, one gets (see [1, Remark 1.7])

**Theorem 1.1.** In the above hypotheses, assume, furthermore, that $X$ is a complete locally compact CAT(0) space. Then $(x_n)$ converges strongly to a minimizer of $f$.

The proof of Theorem 1.1 is what we are going to build upon, roughly, in our quantitative analysis from the viewpoint of proof mining.

Proof mining is a subfield of applied logic that seeks to use proof interpretations, like Gödel’s Dialectica or functional interpretation [6], originally developed with the purpose of giving consistency arguments for systems of arithmetic, in order to extract quantitative information out of proofs in ordinary mathematics. Under the name of “proof unwinding”, it was first proposed as a viable research program by G. Kreisel in the 1950s and after several decades of sporadic advances (one of the most significant being H. Luckhardt’s 1989 analysis [18] of the proof of Roth’s theorem on diophantine approximations) it was given maturity in the 1990s and the 2000s by U. Kohlenbach and his collaborators. The project has culminated into the general logical metatheorems developed by Kohlenbach [9] and by Gerhardy and Kohlenbach [5] for proofs in metric, (uniformly convex) normed and inner product spaces, as well as geodesic spaces like $W$-hyperbolic spaces and CAT(0) spaces. These logical metatheorems were extended to other classes of fundamental spaces in nonlinear and functional analysis, optimization, geometric group theory and geodesic geometry: Gromov hyperbolic spaces, $\mathbb{R}$-trees and a class of uniformly convex geodesic spaces [16], completions of metric and normed spaces [10], totally bounded metric spaces [10, 13], uniformly smooth normed spaces [12], Banach lattices and $C(K)$ spaces [7], $L^p$-spaces [7, 21] and CAT($\kappa$) spaces [15]. These logical metatheorems guarantee that from proofs of $\forall\exists$-sentences (satisfying some conditions) in formal systems associated to such abstract spaces $X$, one can extract effective uniform bounds on existentially quantified variables. Kohlenbach’s monograph
from 2008 [10] covers the major results in the field until then, while a survey of recent developments is [11].

The canonical example of an existentially quantified variable in ordinary mathematics comes from the definition of the limit of a sequence in a metric space \((X, d)\). If \((x_n)\) is a sequence in \(X\) and \(x \in X\), then \(\lim_{n \to \infty} x_n = x\) if and only if

\[
\forall k \in \mathbb{N} \exists N \in \mathbb{N} \forall n \geq N \left( d(x_n, x) \leq \frac{1}{k+1} \right).
\]

A witness for this existentially quantified \(N\), also called rate of convergence for the sequence, as it will be defined in more detail further below, would consist of a formula giving it in terms of the \(k\). Unfortunately, as the sentence above has three alternating quantifiers in a row (i.e. \(\forall \exists \forall\)), the techniques of proof mining preclude the extraction of such a computable rate if the proof is non-constructive in the sense of using at least once the law of excluded middle (one can show that the existence of a general procedure for these cases would contradict the impossibility of the halting problem). Four avenues have generally been tried so far in proof mining, if the convergence of a sequence was under discussion. The first one is the extraction of the full rate of convergence in the rare case that the proof is fully or at least partially constructive. The second one is to settle for a weaker property, like the limit inferior, which may have a tractable \(\forall \exists\) form (and if the sequence is nonincreasing, the extracted modulus of liminf would also be a rate of convergence). The third one is to use some uniqueness properties of the limit in order to extract the rate of convergence from a distantly related property like the rate of asymptotic regularity. Finally, the fourth way is what we are going to focus on here. It consists of considering instead of convergence the Cauchy property of the sequence

\[
\forall k \in \mathbb{N} \exists N \in \mathbb{N} \forall p \in \mathbb{N} \left( d(x_N, x_{N+p}) \leq \frac{1}{k+1} \right),
\]

and replacing it with an equivalent formulation (known in logic as its Herbrand normal form or its Kreisel no-counterexample interpretation), called metastability by Tao [22, 23]. The following sentence expresses the metastability of the sequence above:

\[
\forall k \in \mathbb{N} \forall g : \mathbb{N} \to \mathbb{N} \exists N \in \mathbb{N} \forall i, j \in [N, N + g(N)] \left( d(x_i, x_j) \leq \frac{1}{k+1} \right).
\]

It is immediately seen that this sentence is of a reduced \(\forall \exists\) logical complexity. It is, however, a simple exercise, to check that it is classically (but not intuitionistically) equivalent to the assertion that the sequence under discussion is Cauchy. Therefore, one can now say that the fourth way is focused on obtaining a rate of metastability for the sequence, i.e. a mapping \(\Phi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}\) satisfying, for all \(k \in \mathbb{N}\) and all \(g : \mathbb{N} \to \mathbb{N}\),

\[
\exists N \leq \Phi(k, g) \forall i, j \in [N, N + g(N)] \left( d(x_i, x_j) \leq \frac{1}{k+1} \right) . \quad (1)
\]
In a recent paper, Kohlenbach, the first author and Nicolae [13] have studied a general line of argument used in convergence proofs in nonlinear analysis and convex optimization. Specifically, it is often the case that an iterative sequence is proven to be convergent to a point in a certain set \( F \) (e.g. the set of fixed points of an operator using which the sequence was constructed) if it sits inside a compact space, it is Fejér monotone with respect to \( F \) (that is, for all \( q \in F \) and all \( n \in \mathbb{N} \), \( d(x_{n+1}, q) \leq d(x_n, q) \)) and it has “approximate \( F \)-points”, i.e. points which are, in a sense, near \( F \). The main result in [13] is that all this can be made effective. For that to work, however, the three hypotheses must also be transformed into a quantitative form. A “modulus of total boundedness” witnesses the space being compact. For the other two properties, one must formulate what exactly does it mean for a point to be “near” to \( F \). This is done in terms of an approximation \( F = \bigcap_{k \in \mathbb{N}} A F_k \), which helps formulate both the “modulus of uniform Fejér monotonicity” and the “approximate \( F \)-point bound”. The choice of an approximation to \( F \), as well as the computation of these moduli, has been done in [13, 17, 14] for some classical iterations associated to important classes of mappings and operators.

In this paper we apply the techniques developed in [13] to obtain a quantitative version of Theorem 1.1, providing an effective uniform rate of metastability for the proximal point algorithm in totally bounded CAT(0) spaces. The next section will give some preliminaries on the proximal point algorithm, while the last section of the paper is dedicated to the proof of our main quantitative result, Theorem 3.5.

We finish this Introduction with a recall of definitions from [13] and quantitative notions that will be used throughout the paper. We point out, first, that \( \mathbb{N} = \{0, 1, 2, \ldots \} \) and that we denote \( [m, n] = \{m, m + 1, \ldots , n\} \) for any \( m, n \in \mathbb{N} \) with \( m \leq n \).

Let \( (X, d) \) be a metric space. For any mapping \( T : X \rightarrow X \) we denote by \( \text{Fix}(T) \) the set of fixed points of \( T \).

A modulus of total boundedness for \( X \) is a function \( \alpha : \mathbb{N} \rightarrow \mathbb{N} \) such that for any \( k \in \mathbb{N} \) and any sequence \((x_n)\) in \( X \) there exist \( i < j \) in \( [0, \alpha(k)] \) such that

\[
d(x_i, x_j) \leq \frac{1}{k + 1}.\]

This notion was first used in [4] to analyze, using proof mining methods, the Furstenberg-Weiss proof of the Multiple Birkhoff Recurrence Theorem. One can easily see that \( X \) is totally bounded if and only if it has a modulus of total boundedness.

Let \( F \subseteq X \). We say that a family \((A F_k)_{k \in \mathbb{N}}\) of subsets of \( X \) is an approximation to \( F \) if

\[
F = \bigcap_{k \in \mathbb{N}} A F_k \quad \text{and} \quad A F_{k+1} \subseteq A F_k \quad \text{for all} \quad k \in \mathbb{N}.
\]

Elements of \( A F_k \) are also called \( k \)-approximate \( F \)-points.
Definition 1.2. [13] Let $F \subseteq X$ be a set with an approximation $(AF_k)$.

(i) $F$ is uniformly closed with respect to $(AF_k)$ with moduli $\delta_F, \omega_F : \mathbb{N} \to \mathbb{N}$ if for all $k \in \mathbb{N}$ and all $p, q \in X$ we have that
\[ q \in AF_{\delta_F(k)} \text{ and } d(p, q) \leq \frac{1}{\omega_F(k) + 1} \text{ imply } p \in AF_k. \]

(ii) $(x_n)$ is uniformly Fejér monotone with respect to $(AF_k)$ with modulus $\chi$ if for all $n, m, r \in \mathbb{N}$, all $p \in AF_{\chi(n,m,r)}$ and all $l \leq m$ we have that
\[ d(x_{n+l}, p) < d(x_n, p) + \frac{1}{r + 1}. \]

(iii) $(x_n)$ has approximate $F$-points with respect to $(AF_k)$ with modulus $\Phi$ (which is taken to be nondecreasing) if for all $k \in \mathbb{N}$ there is an $N \leq \Phi(k)$ such that $x_N \in AF_k$.

We refer to [13, Sections 3 and 4] for details and intuitions behind the above definitions. We remark that one can get nondecreasing moduli using the following transformation. For any $f : \mathbb{N} \to \mathbb{N}$, one defines $f^M : \mathbb{N} \to \mathbb{N}$ by
\[ f^M(n) := \max_{i \leq n} f(i). \]
Then $f^M$ is nondecreasing and for any $n$, we have that
\[ f(n) \leq f^M(n). \]

We now give some notions that are customary in quantitatively expressing some basic properties of real-valued sequences. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of nonnegative real numbers. If $(a_n)$ converges to 0, then a rate of convergence for $(a_n)$ is a mapping $\beta : \mathbb{N} \to \mathbb{N}$ such that for all $k \in \mathbb{N}$,
\[ \forall n \geq \beta(k) \left( a_n \leq \frac{1}{k + 1} \right). \]
If the series $\sum_{n=0}^{\infty} a_n$ diverges, then a function $\theta : \mathbb{N} \to \mathbb{N}$ is called a rate of divergence of the series if for all $P \in \mathbb{N}$ we have that
\[ \sum_{n=0}^{\theta(P)} a_n \geq P. \]

A modulus of liminf for $(a_n)$ is a mapping $\Delta : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, satisfying, for all $k, L \in \mathbb{N}$,
\[ \exists N \in [L, \Delta(k, L)] \left( a_N \leq \frac{1}{k + 1} \right). \]
Such a modulus exists if and only if $\liminf_{n \to \infty} a_n = 0$.

More generally, let $(b_n)_{n \in \mathbb{N}}$ be a sequence of real numbers and $b \in \mathbb{R}$. If $(b_n)$ converges to $b$, then a a rate of convergence of $(b_n)$ is a mapping $\beta : \mathbb{N} \to \mathbb{N}$ such that for all $k \in \mathbb{N}$,
\[ \forall n \geq \beta(k) \left( |b_n - b| \leq \frac{1}{k + 1} \right). \]
Thus, a rate of convergence of \((b_n)\) coincides with a rate of convergence of the sequence \(|b_n - b|\) of nonnegative reals.

## 2 Preliminaries on the proximal point algorithm

In the sequel, \(X\) is a CAT(0) space and \(f : X \to (-\infty, +\infty]\) is a convex, lower semicontinuous (lsc) proper function. Let us recall that a minimizer of \(f\) is a point \(x \in X\) such that \(f(x) = \inf_{y \in X} f(y)\). We denote the set of minimizers of \(f\) by \(\text{Argmin}(f)\) and we assume that \(\text{Argmin}(f)\) is nonempty.

The **proximal point mapping** or the **(Moreau-Yosida) resolvent**, as first introduced for CAT(0) spaces by Jost [8], is a tool for finding minimizers of such functions. For \(\gamma > 0\), the resolvent (or the proximal mapping) of \(f\) of order \(\gamma\) is the map \(J^\gamma_f : X \to X\), defined, for any \(x \in X\), by the following relation

\[
J^\gamma_f(x) := \argmin_{y \in X} \left[ \gamma f(y) + \frac{1}{2\gamma} d^2(x, y) \right].
\]

This is the definition from [1], as the factor of 2 does not appear in the original paper of Jost, but this is, obviously, insignificant. By [8, Lemma 2], the operator \(J^\gamma_f\) is well-defined. We shall denote \(J^1_f\) simply by \(J_f\). Then, for all \(\gamma > 0\) and for all \(x \in X\),

\[
J_\gamma f(x) = J^\gamma_f(x) = \argmin_{y \in X} \left[ \gamma f(y) + \frac{1}{2\gamma} d^2(x, y) \right] = \argmin_{y \in X} \left[ f(y) + \frac{1}{2\gamma} d^2(x, y) \right].
\]

The following property was also proved in [8].

**Proposition 2.1** ([8, Lemma 4]). For any \(\gamma > 0\), \(J_\gamma f\) is nonexpansive, that is, for all \(x, y \in X\),

\[
d(J_\gamma f x, J_\gamma f y) \leq d(x, y).
\]

We note that the definition of the proximal point mapping is motivated by the following proposition.

**Proposition 2.2.** Let \(x \in X\). Then \(x\) is a minimizer of \(f\) if and only if \(x\) is a fixed point of \(J_f\).

**Proof.** Suppose first that \(x\) is a minimizer of \(f\). It follows that for all \(y\),

\[
f(x) + \frac{1}{2} d^2(x, x) = f(x) \leq f(y) \leq f(y) + \frac{1}{2} d^2(x, y),
\]

therefore \(x\) is also the argmin of the right hand side w.r.t. \(y\) — that is, \(x = J_f(x)\).

Suppose now that \(J_f(x) = x\). Then for all \(y \in X\), as before,

\[
f(x) \leq f(y) + \frac{1}{2} d^2(x, y).
\]

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Let \( w \in X \). Using the fact that \( f \) is convex, we get, for any \( t \in (0, 1) \),
\[
    f(x) \leq f((1-t)x + tw) + \frac{1}{2}d^2(x, (1-t)x + tw) \\
    \leq (1-t)f(x) + tf(w) + \frac{1}{2}d^2(x, (1-t)x + tw) \\
    = (1-t)f(x) + tf(w) + \frac{1}{2}td^2(x, w).
\]
Subtracting \((1-t)f(x)\) and dividing by \( t \), we obtain that
\[
    f(x) \leq f(w) + \frac{1}{2}td^2(x, w),
\]
and by letting \( t \to 0 \), it follows that \( f(x) \leq f(w) \). Since \( w \) was chosen arbitrarily, we get that \( x \) is a minimizer of \( f \). \( \square \)

Since, trivially, \( \text{Argmin}(\gamma f) = \text{Argmin}(f) \), we get that

**Corollary 2.3.** For any \( \gamma > 0 \), \( \text{Fix}(J_{\gamma f}) = \text{Fix}(J_f) = \text{Argmin}(f) \).

We may now proceed to study the algorithm in itself. Let \((\gamma_n)_{n \in \mathbb{N}}\) be a sequence in \((0, \infty)\). The **proximal point algorithm** \((x_n)_{n \in \mathbb{N}}\) starting with \( x \in X \) is defined as follows:
\[
x_0 := x, \quad x_{n+1} := J_{\gamma_n f}x_n \quad \text{for all } n \in \mathbb{N}.
\]

Let us give some useful properties of the sequence \((x_n)\).

**Lemma 2.4.** For all \( n, m \in \mathbb{N} \) and all \( p \in X \),
\[
d(x_{n+1}, p) \leq d(x_n, p) + d(p, J_{\gamma_n}f p), \quad (3) \\
d(x_{n+m}, p) \leq d(x_n, p) + \sum_{i=n}^{n+m-1} d(p, J_{\gamma_i}f p). \quad (4)
\]

*Proof.* We have that
\[
d(x_{n+1}, p) = d(J_{\gamma_n}f x_n, p) \leq d(J_{\gamma_n}f x_n, J_{\gamma_n}f p) + d(J_{\gamma_n}f p, p) \\
\leq d(x_n, p) + d(p, J_{\gamma_n}f p).
\]
(4) follows immediately by induction on \( m \). \( \square \)

The following lemma contains results from [1].

**Lemma 2.5.** (i) The sequence \((f(x_n))\) is nonincreasing.

(ii) For all \( n \in \mathbb{N} \) and all \( p \in \text{Argmin}(f) \),
\[
2\gamma_n(f(x_{n+1}) - \min(f)) \leq d^2(x_n, p) - d^2(x_{n+1}, p) - d^2(x_n, x_{n+1}) \quad (5) \\
d^2(x_n, x_{n+1}) \leq d^2(x_n, p) - d^2(x_{n+1}, p) \quad (6) \\
f(x_{n+1}) - \min(f) \leq \frac{d^2(x, p)}{2 \sum_{i=0}^{n} \gamma_i} \quad (7)
\]
Proof. (i) This is used without proof in [1], and hence we shall justify it. Let $n \in \mathbb{N}$. By the definition of $J_{\gamma_n f}$ and considering that $x_{n+1} = J_{\gamma_n f}x_n$, we have that:

$$\gamma_n f(x_{n+1}) + \frac{1}{2}d^2(x_n, x_{n+1}) \leq \gamma_n f(x_n) + \frac{1}{2}d^2(x_n, x_n) = \gamma_n f(x_n),$$

and so, $\gamma_n f(x_{n+1}) \leq \gamma_n f(x_n)$, hence $f(x_{n+1}) \leq f(x_n)$.

(ii) With the assumption made in [1] that $\min(f) = 0$, (5) is the last inequality in the proof of (7) from [1], while (7) is obtained from the inequality before (8) in [1]. Note also that $\lambda_k$ in [1] corresponds to our $\gamma_{k-1}$. (6) follows immediately from (5).

We finish this section with two effective results on the behaviour of the proximal point algorithm, results that will be also used in the next section to get our main quantitative theorem.

Lemma 2.6. Let $b \in \mathbb{R}$ be such that $d(x, p) \leq b$ for some $p \in \text{Argmin}(f)$.

(i) $\liminf_{n \to \infty} d(x_n, x_{n+1}) = 0$ with modulus of liminf

$$\Delta_b(k, L) := [b^2(k+1)^2] + L - 1.$$  \hspace{1cm} (8)

(ii) Assume that $\sum_{n=0}^{\infty} \gamma_n = \infty$ with rate of divergence $\theta$. Then $\lim_{n \to \infty} f(x_n) = \min(f)$, with a (nondecreasing) rate of convergence

$$\beta_{b, \theta}(k) := \theta^M([b^2(k+1)/2]) + 1.$$  \hspace{1cm} (9)

Proof. (i) By (6), we get that for all $j \geq k$,

$$\sum_{n=j}^{k} d^2(x_n, x_{n+1}) \leq \sum_{n=0}^{k} d^2(x_n, x_{n+1}) \leq \sum_{n=0}^{k} (d^2(x_n, p) - d^2(x_{n+1}, p)) \leq d^2(x, p) \leq b^2.$$

Suppose that $d(x_n, x_{n+1}) > \frac{1}{k+1}$ for all $n \in [L, \Delta_b(k, L)]$. Then

$$(\Delta_b(k, L) - L + 1) \frac{1}{(k+1)^2} < \sum_{n=L}^{\Delta_b(k, L)} d^2(x_n, x_{n+1}) \leq b^2,$$

from which we get $\Delta_b(k, L) < b^2(k+1)^2 + L - 1$, a contradiction.

(ii) Since $f(x_n) \geq \min(f)$ for all $n \in \mathbb{N}$ and, by Lemma 2.5(i), $(f(x_n))$ is nonincreasing, all we have to show is that

$$f(x_{\beta_{b, \theta}(k)}) - \min(f) \leq \frac{1}{k+1}.$$  \hspace{1cm} 8
Assume that this is not true. Then, using (7) and the fact that $\theta$ is a rate of divergence, we get that
\[
\frac{1}{k+1} < f(x_{\beta_{b,\theta}(k)}) - \min(f) \leq \frac{b^2}{2 \sum_{i=0}^m (\lceil b^2(k+1)/2 \rceil) \lambda_i} \leq \frac{b^2}{2 \lceil b^2(k+1)/2 \rceil} \leq \frac{1}{k+1},
\]
a contradiction.

3 Quantitative results on the proximal point algorithm

We will now proceed to derive the moduli that are needed in order to apply the results of [13].

As in the previous section, $f : X \to (-\infty, +\infty]$ is a convex, lsc proper function, and we set $F := \text{Argmin}(f) \neq \emptyset$. For every $k \in \mathbb{N}$, let us define
\[
AF_k := \left\{ x \in X \mid \text{for all } i \leq k, d(x, J_{\gamma_i} f x) \leq \frac{1}{k+1} \right\}.
\] (10)

**Proposition 3.1.** $(AF_k)$ is an approximation to $F$.

*Proof.* Since, obviously, $(AF_k)$ is a nonincreasing sequence, it remains to prove that $F = \bigcap_{k \in \mathbb{N}} AF_k$.

“$\subseteq$” Let $x \in F$ and $k \in \mathbb{N}$ be arbitrary. Then, for all $i \leq k$, by Corollary 2.3, we have that $F = \text{Fix}(J_{\gamma_i} f)$, hence, in particular, $d(x, J_{\gamma_i} f x) \leq \frac{1}{k+1}$. Thus, $x \in AF_k$.

“$\supseteq$” Let $x \in \bigcap_{k \in \mathbb{N}} AF_k$. It follows, in particular, that for any $k \in \mathbb{N}$,
\[
d(x, J_{\gamma_k} f x) \leq \frac{1}{k+1}.
\]
As a consequence, we get that $x \in \text{Fix}(J_{\gamma_0} f) = F$, again by Corollary 2.3. □

This approximation will turn out to be convenient for the results we are aiming for.

**Proposition 3.2.** With respect to the above approximation, $F$ is uniformly closed with moduli
\[
\delta_F(k) := 2k + 1, \quad \omega_F(k) := 4k + 3.
\] (11)
Proof. Let \( k \in \mathbb{N} \) and \( p, q \in X \) be such that \( q \in AF_{2k+1} \) and \( d(p, q) \leq \frac{1}{k+1} \). We need to show that \( p \in AF_k \), i.e. that for all \( i \leq k \), \( d(p, J_{\gamma_i}(f)p) \leq \frac{1}{k+1} \). Let \( i \leq k \) be arbitrary. We get that
\[
d(p, J_{\gamma_i}(f)p) \leq d(p, q) + d(q, J_{\gamma_i}(f)p) \leq 2d(p, q) + d(q, J_{\gamma_i}(f)p) \leq 2 \cdot \frac{1}{k+1},
\]
where we have used at the second inequality the fact that \( J_{\gamma_i}(f) \) is nonexpansive.

\[\square\]

Lemma 3.3. The sequence \((x_n)\) is uniformly Fejér monotone w.r.t. \((AF_k)\) with modulus
\[
\chi(n, m, r) := \max\{n + m - 1, m(r + 1)\}. \tag{12}
\]

Proof. Let \( n, m, r \in \mathbb{N} \), \( p \in AF_{\chi(n, m, r)} \) and \( l \leq m \). We get that
\[
d(x_{n+l}, p) \leq d(x_n, p) + \sum_{i=n}^{i=n+l-1} d(p, J_{\gamma_i}(f)p) \leq d(x_n, p) + \sum_{i=n}^{i=n+m-1} d(p, J_{\gamma_i}(f)p) \leq d(x_n, p) + \frac{m}{\chi(n, m, r) + 1}.
\]

where at the second-to-last inequality we used that \( \chi(n, m, r) \geq n + m - 1 \), so \( d(p, J_{\gamma_i}(f)p) \leq \frac{1}{\chi(n, m, r) + 1} \) for all \( i = n, \ldots, n + m - 1 \), and at the last one, that \( \chi(n, m, r) \geq m(r + 1) \). \(\square\)

Proposition 3.4. Let \( b \in \mathbb{R} \) be such that \( d(x, p) \leq b \) for some \( p \in F \) and assume that \( \sum_{n=0}^{\infty} \gamma_n = \infty \) with rate of divergence \( \theta \). Suppose, moreover, that \( M : \mathbb{N} \to (0, \infty) \) is such that \( M(k) \geq \max_{0 \leq i \leq k} \gamma_i \) for all \( k \in \mathbb{N} \). Then \((x_n)\) has approximate \( F \)-points w.r.t. \((AF_k)\) with a (nondecreasing) modulus
\[
\Phi_{b, \theta, M}(k) := \left[b^2(k + 1)^2\right] + \beta_{b, \theta}([2(k + 1)^2 M(k)] - 1), \tag{13}
\]
where \( \beta_{b, \theta} \) is defined by \((9)\).

Proof. Let \( k \in \mathbb{N} \) be arbitrary. Denote, for simplicity,
\[
c := \beta_{b, \theta}([2(k + 1)^2 M(k)] - 1).
\]
Applying Lemma 2.6.(i), we obtain that there is an \( N \in [c, \Delta_b(k, c)] \) such that
\[
d(x_N, x_{N+1}) \leq \frac{1}{k+1},
\]

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where $\Delta_b$ is given by (8). We remark, first, that

$$N \leq \Delta_b(k, c) = [b^2(k + 1)^2] + c - 1 < [b^2(k + 1)^2] + c = \Phi_{b,\theta,M}(k).$$

Since $N \geq c$ and $\beta_{b,\theta}$ is a rate of convergence of $(f(x_n))$ towards $\min(f)$ (by Lemma 2.6.(ii)), we get that

$$f(x_N) \leq \min(f) + \frac{1}{[2(k + 1)^2M(k)]}.$$

On the other hand, for all $i \leq k$, we have, by the definition of $J_{\gamma_i}$, that

$$\gamma_i f(J_{\gamma_i}x_N) + \frac{1}{2}d^2(x_N, J_{\gamma_i}x_N) \leq \gamma_i f(x_N) + \frac{1}{2}d^2(x_N, x_N) = \gamma_i f(x_N).$$

As $f(J_{\gamma_i}x_N) \geq \min(f)$, it follows that, for all $i \leq k$,

$$d^2(x_N, J_{\gamma_i}x_N) \leq 2\gamma_i(f(x_N) - f(J_{\gamma_i}x_N)) \leq 2\gamma_i(f(x_N) - \min(f))$$

$$\leq 2M(k)\frac{1}{[2(k + 1)^2M(k)]} \leq \frac{1}{(k + 1)^2}.$$

Thus, we have proved that for all $k \in \mathbb{N}$ there is an $N \leq \Phi_{b,\theta,M}(k)$ such that for all $i \leq k$,

$$d(x_N, J_{\gamma_i}x_N) \leq \frac{1}{k + 1}.$$

That’s what was required. \qed

Now that all the necessary moduli have been computed, we may apply [13, Theorems 5.1 and 5.3] to get our main result, which finitarily expresses the strong convergence of the proximal point algorithm to a minimizer of $f$.

**Theorem 3.5.** Let $b > 0$, $\alpha, \theta : \mathbb{N} \to \mathbb{N}$ and $M : \mathbb{N} \to (0, \infty)$. Define $\Psi_{b,\theta,M,\alpha}, \Omega_{b,\theta,M,\alpha} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ as in Table 1. Then for all

(i) totally bounded CAT(0) spaces with modulus of total boundedness $\alpha$;

(ii) convex lsc proper mappings $f : X \to (-\infty, \infty]$ with $\text{Argmin}(f) \neq \emptyset$;

(iii) $x \in X$ such that $d(x, p) \leq b$ for some minimizer $p$ of $f$;

(iv) sequences $(\gamma_n)$ in $(0, \infty)$ such that $\sum_{n=0}^{\infty} \gamma_n = \infty$ with rate of divergence $\theta$ and $M(k) \geq \max_{0 \leq i \leq k} \gamma_i$ for all $k \in \mathbb{N}$;

we have that

(i) $\Psi_{b,\theta,M,\alpha}$ is a rate of metastability for the proximal point algorithm $(x_n)$ starting with $x$, i.e. for all $k \in \mathbb{N}$ and all $g : \mathbb{N} \to \mathbb{N}$ there is an $N \leq \Psi_{b,\theta,M,\alpha}(k, g)$ such that for all $i, j \in [N, N + g(N)]$,

$$d(x_i, x_j) \leq \frac{1}{k + 1}.$$
(ii) For all \( k \in \mathbb{N} \) and all \( g : \mathbb{N} \to \mathbb{N} \) there is an \( N \leq \Omega_{b,\theta,M,\alpha}(k, g) \) such that for all \( i, j \in [N, N + g(N)] \),

\[
d(x_i, x_j) \leq \frac{1}{k + 1}
\]

and for all \( i \in [N, N + g(N)] \) and all \( d \leq k \),

\[
d(x_i, J_{id\epsilon} x_i) \leq \frac{1}{k + 1}.
\]

**Proof.** Apply [13, Theorem 5.1] to get (i) and [13, Theorem 5.3] to obtain (ii). Remark that in our case, using the notations from [13], \( \alpha_G = \beta_H = \text{id}_{\mathbb{R}^+} \), hence \( P = \alpha(4k + 3) \) and, furthermore,

\[
\chi^M_g(n, r) = \max_{i \leq n} \chi_g(i, r) = \max_{i \leq n} \chi(i, g(i), r)
\]

\[
= \max_{i \leq n} \{ i + g(i) - 1, g(i)(r + 1) \}, \text{ by (12)}
\]

\[
k_0 = \max \left\{ k, \left\lfloor \frac{\omega_F(k) - 1}{2} \right\rfloor \right\} = 2k + 1, \text{ by (11)}
\]

\[
(\chi_{k,\delta_F}^M)(n, r) = \max \{ 2k + 1, \max_{i \leq n} \{ i + g(i) - 1, g(i)(r + 1) \} \}.
\]

We denote, for simplicity, \( \chi_{k,\delta_F}^M \) by \( \tilde{\chi} \).

---

| Table 1: Definitions of \( \Psi_{b,\theta,M,\alpha} \) and \( \Omega_{b,\theta,M,\alpha} \) |
|---|
| \( \Psi_{b,\theta,M,\alpha}(k, g) := (\Psi_0)_{b,\theta,M}(\alpha(4k + 3), k, g) \) |
| \( (\Psi_0)_{b,\theta,M}(0, k, g) := 0 \) |
| \( (\Psi_0)_{b,\theta,M}(n + 1, k, g) := \Phi_{b,\theta,M}(\chi_g^M((\Psi_0)_{b,\theta,M}(n, k, g), 4k + 3)) \) |
| \( \chi^M_g(n, r) = \max_{i \leq n} \max \{ i + g(i) - 1, g(i)(r + 1) \} \) |
| \( \Omega_{b,\theta,M,\alpha}(k, g) := (\Omega_0)_{b,\theta,M}(\alpha(8k + 7), k, g) \) |
| \( (\Omega_0)_{b,\theta,M}(0, k, g) := 0 \) |
| \( (\Omega_0)_{b,\theta,M}(n + 1, k, g) := \Phi_{b,\theta,M}(\chi_g^M((\Omega_0)_{b,\theta,M}(n, k, g), 8k + 8)) \) |
| \( \chi^M_g(n, r) = \max \{ 2k + 1, \max_{i \leq n} \max \{ i + g(i) - 1, g(i)(r + 1) \} \} \) |

The above theorem can be considered a “true” finitization (in the sense of Tao) of Theorem 1.1, since

(i) it involves only a finite segment of the proximal point algorithm \((x_n)\);
(ii) the existence of a rate of metastability $\Psi_{b,\theta,M,\alpha}$ is, as previously stated, classically equivalent to Cauchyness;

(iii) the existence of the second rate $\Omega_{b,\theta,M,\alpha}$ guarantees, for complete CAT(0) spaces, that the limit of the sequence is an element of $F = \text{Argmin}(f)$ (see [13, Remark 5.5]);

(iv) the modulus of total boundedness only needs to apply to the ball of radius $b$ considered in the proof, therefore we have derived strong convergence for locally compact CAT(0) spaces, as pointed out in [13, Remark 5.4].

Furthermore, both rates $\Psi_{b,\theta,M,\alpha}$ and $\Omega_{b,\theta,M,\alpha}$ are computable and are, moreover, expressed using primitive recursive functionals.

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