ON SUPERGROUPS AND THEIR SEMISIMPLIFIED
REPRESENTATION CATEGORIES

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Abstract. The representation category $\mathcal{A} = \text{Rep}(G, \epsilon)$ of a supergroup scheme $G$ has a largest proper tensor ideal, the ideal $\mathcal{N}$ of negligible morphisms. If we divide $\mathcal{A}$ by $\mathcal{N}$ we get the semisimple representation category of a pro-reductive supergroup scheme $G^{\text{red}}$. We list some of its properties and determine $G^{\text{red}}$ in the case $\text{Gl}(m|1)$.

Introduction

A fundamental fact about finite-dimensional algebraic representations of a reductive group over an algebraically closed field $k$ of characteristic 0 is complete reducibility: Every representation decomposes into a direct sum of irreducible representations. This is no longer true if we consider representations of supergroups. Indeed by a classical result of Djokovic-Hochschild [DH76] the representation category of a Lie superalgebra $g$ is semisimple if and only if $g$ is a semisimple Lie algebra or of the form $\text{osp}(1|2n)$ for $n \geq 1$. Correspondingly many standard techniques from Lie theory do not work for representations of supergroups. Although a lot of progress has been made on representations of special supergroups such as $\text{Gl}(m|n)$ and $\text{OSp}(m|2n)$, many classical questions are still open, most notably the tensor product decomposition of two irreducible representations. The category $\text{Rep}(G)$, $G$ a supergroup, is a tensor category. Every $k$-linear tensor category has a largest proper tensor ideal $\mathcal{N}$, the tensor ideal of negligible morphisms. By [AK02] the quotient category $\omega : \text{Rep}(G) \to \text{Rep}(G)/\mathcal{N}$ is an abelian semisimple $k$-linear tensor category.

0.1 Theorem. (Theorem 2.2) The quotient $\text{Rep}(G)/\mathcal{N}$ is a super-tannakian category, i.e. it is of the form $\text{Rep}(G^{\text{red}}, \epsilon)$ where $G^{\text{red}}$ is a supergroup scheme with semisimple representation category.

This result follows immediately from a characterization of representation categories due to Deligne [Del02]. A natural question is to understand and possibly determine $G^{\text{red}}$ for given $G$. This is very difficult and not even possible in the general case. We assemble a few general results about these quotients and then focus on the $\text{Gl}(m|n)$-case ($m \geq n$).

We show that the classification of the irreducible representations of $G^{\text{red}}$ is a wild problem for $n \geq 3$ in theorem 3.5. Hence the question should be modified as follows: We should study the subcategory in $\text{Rep}(G^{\text{red}}, \epsilon)$ generated by the images $\omega(L(\lambda))$ of the irreducible representations of $G$. To determine this subcategory

2010 Mathematics Subject Classification: 17B10, 18D10.
would amount to determine the tensor product decomposition of irreducible representations up to superdimension 0 and would give a parametrization of the indecomposable summands of non-vanishing superdimension. We study this problem in [HW15] in the case of $Gl(m|n)$. The cases $Gl(m|1)$ and $Sl(n|1)$ are rather special since the blocks are of tame representation type and the indecomposable representations have been classified [Ger98] and we can hope to determine the entire quotient category. From the classification it is easy to determine the irreducible elements in $G^{red}$ in lemma 4.2. We then compute their tensor product decomposition in theorem 4.12.

0.2 Theorem. (Theorem 4.13) We have $Gl(m|1)^{red} \simeq Gl(m-1) \times Gl(1) \times Gl(1)$.

In this statement we view $Gl(m|1)^{red}$ as a supergroup with trivial odd part and $Rep(G^{red})$ as the corresponding category of super representations. In order to determine the tensor product decomposition we use two tools: The theory of mixed tensors [Hei14] gives us the tensor product decomposition between the irreducible $Gl(m|1)$-representations. We then use cohomological tensor functors $DS : Rep(Gl(m|1)) \to Rep(Gl(m-1))$ akin to those of [DS05] [HW14] to reduce the tensor product decomposition between indecomposables to the irreducible case. The main point here is that $DS(V)$ is a $\mathbb{Z}$-graded object for any $V$, hence $DS$ could be interpreted as a functor to $\mathbb{Z} \times Rep(Gl(m-1))$.

To determine $G^{red}$ is probably in reach for the simple supergroups of maximal atypicality 1. To determine the subgroup of $G^{red}$ corresponding to the irreducible representations is already very difficult for $Gl(m|n)$ and even more so for the other simple supergroups $OSp(m|2n)$, $P(n)$ and $Q(n)$.

1. Preliminaries

Super Linear Algebra. Throughout the article $k$ is an algebraically closed field of characteristic 0. A super vector space is a finite-dimensional $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$ over $k$. Elements in $V_0$ respectively $V_1$ are called even respectively odd. An element is homogenous if it is either even or odd. For a homogenous element $v$ write $p(v)$ for the parity defined by

$$p(v) = \begin{cases} 
0 & v \in V_0 \\
1 & v \in V_1.
\end{cases}$$

We denote by $Hom(V, W)$ the set of $k$-linear parity-preserving morphism between two super vector spaces $V$ and $W$. The parity shift functor $\Pi : svec \to svec$ is defined by $(\Pi V)_0 = V_1$, $(\Pi V)_1 = V_0$ and on morphisms $f : V \to W$ via $\Pi f : v \mapsto f(v)$ where $v$ is viewed as an element of IIW and $f(v)$ as an element of IIV.

Supergroups. A superring is a $\mathbb{Z}_2$-graded ring $A = A_0 \oplus A_1$ such that the product map $A \times A \to A$ satisfies $A_iA_j \subset A_{i+j}$. A morphism of superrings is a $\mathbb{Z}_2$-grading preserving morphism of rings. A superring is commutative if $ab = (-1)^{p(a)p(b)}ba = 0$ for all $a, b \in A_0 \cup A_1$. A superring is a superalgebra if it is also a super vector space over $k$. We denote by $salg$ the category of commutative superalgebras with parity preserving morphisms. A group functor $G : salg \to sets$ is called an affine supergroup scheme $G$ if $G$ is a representable group functor. It is called a supergroup if the representing superalgebra is finitely generated. We define the functor
\(Gl(m|n)\) from the category of commutative superalgebras to the category of groups by sending \(A = A_0 \oplus A_1\) to the invertible \((m + n) \times (m + n)\) matrices of the form
\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]
where \(a\) is an \((m \times m)\)-matrix with entries in \(A_0\), \(b\) is an \((m \times n)\)-matrix with entries in \(A_1\), \(c\) is an \((n \times m)\)-matrix with entries in \(A_1\) and \(d\) is an \((n \times n)\)-matrix with entries in \(A_0\). A morphism \(f : A \to B\) is sent to the map that sends the matrix \((X_{ij}) \in Gl(m|n)(A)\) to the matrix \((f(X_{ij})) \in Gl(m|n)(B)\). This group functor is called the General Linear supergroup.

**Representations.** We define the group functor \(Gl_V : salg \to sets\) to be the functor that assigns to each commutative superalgebra \(A\) the even invertible elements of \(End_A(V \otimes A)\). For finite-dimensional \(V\) \(Gl_V\) is a supergroup. Let \(G\) be a group functor and \(V \in svec\) a finite-dimensional super vector space. A linear representation of \(G\) in \(V\) is a morphism of group functors \(G \to Gl_V\). If \(G\) has a linear representation on \(V\), we call \(V\) a \(G\)-module. We denote the category of finite-dimensional representations of \(G\) by \(Rep(G)\).

**Parity automorphisms.** Let \(G\) be a supergroup scheme and let \(\epsilon\) be an element of \(G(k)\) of order dividing 2 such that the automorphism \(int(\epsilon)\) of \(G\) is the parity automorphism defined by \(x \mapsto (-1)^{\rho(\epsilon)}x\) for homogenous \(x\). Then let \(Rep(G, \epsilon)\) be the category of (finite-dimensional) representations \(V = (V, \rho)\) such that \(\rho(\epsilon)\) is the parity automorphism of \(V\). If \(G\) is an affine group scheme, \(\epsilon\) is central. In this case the category \(Rep(G, \epsilon)\) identifies itself with \(Rep(G)\) with a new commutativity constraint: For every representation \((V, \rho)\) of \(G\) the involution \(\rho(\epsilon)\) defines a \(\mathbb{Z}_2\)-gradation on \(V\) and the commutativity isomorphism of the tensor product is given by the Koszul rule. If \(\epsilon\) is trivial, one recovers \(Rep(G)\). For the supergroup \(Gl(m|n)\) and \(\epsilon = diag(E_m, -E_n)\) we put \(Rep(Gl(m|n), \epsilon) = R_{m|n}\). For the whole category \(Rep(Gl(m|n))\) we also write \(T_{m|n}\). Then \(T_{m|n} = R_{m|n} \oplus IR_{m|n}\) [HW14].

The categories \(T_{m|n}\) and \(R_{m|n}\) are examples of super-Tannakian categories. For background on tensor categories we refer to [DM82]. We denote the unit object of a tensor category by \(\mathbb{1}\).

1.1 Definition. A \(k\)-linear, abelian, rigid tensor category \(A\) with \(k \simeq End(\mathbb{1})\) and with a \(k\)-linear exact faithful tensor functor \(\rho : A \to svec\) (a super fibre functor) is called a super-tannakian category.

1.2 Theorem. [Del02] Every super tannakian category \(A\) is tensor equivalent to the category \(A \simeq Rep(G, \epsilon)\) of representations of a supergroup scheme \(G\).

For a partition \(\lambda\) of \(n\) let \(S_\lambda(-)\) be the associated Schur functor [Del02]. We put \(Sym^n(X) = S_{(n)}(X)\) (the \(n\)-th symmetric power) and \(\Lambda^n(X) = S_{(1, \ldots, 1)}(X)\) (the \(n\)-th alternating power). An object \(X\) of \(A\) is called Schur-finite if there exists an integer \(n\) and a partition \(\lambda\) of \(n\) such that \(S_\lambda(X) = 0\).

1.3 Theorem. [Del02] If \(A\) is an abelian \(k\)-linear rigid tensor category with \(End(\mathbb{1}) \simeq k\) such that every object is Schur finite, then \(A\) is a super tannakian category, i.e. \(A \simeq Rep(G, \epsilon)\) for some supergroup scheme \(G\).
Following [Ger98] we call a small abelian $k$-linear category nice if morphism spaces are finite-dimensional, every object has a finite composition series and the category has enough projectives. An example is given by the category $\mathcal{R}_{m|n}$.

1.4 Lemma. [Ger98], lemma 1.1.1. If $\mathcal{A}$ is a nice category then

1. The endomorphism ring of any indecomposable object is a local ring.
2. Every object can be written as a direct sum of indecomposable objects.
3. Every module has a unique projective cover

2. The universal semisimple quotient

2.1. The universal semisimple quotient. An additive category $\mathcal{A}$ is a Krull-Schmidt category if every object has a decomposition in a finite direct sum of elements with local endomorphism rings. An ideal in a $k$-linear category is for any two objects $X, Y$ the specification of a $k$-submodule $T(X, Y)$ of $\text{Hom}_\mathcal{A}(X, Y)$, such that for all pairs of morphisms $f \in \text{Hom}_\mathcal{A}(X, X'), g \in \text{Hom}_\mathcal{A}(Y, Y')$ the inclusion $gT(X', Y) \subseteq T(X, Y')$ holds. Let $T$ be an ideal in $\mathcal{A}$, $\mathcal{A}/T$ is the category with the same objects as $\mathcal{A}$ and with $\text{Hom}_{\mathcal{A}/T}(X, Y) = \text{Hom}_\mathcal{A}(X, Y)/T(X, Y)$. It is again a Krull-Schmidt category [Liu09], [KZ08]. Suppose that $\mathcal{A}$ is abelian and that every object has finite length and let $X$ be an indecomposable element and $\phi$ an endomorphism of $X$. By Fitting’s lemma $\phi$ is either invertible or nilpotent. An element $X$ is indecomposable if and only if its endomorphism ring is a local ring.

We assume in the following that $\mathcal{A}$ is a super tannakian category or a pseudoabelian full tensor subcategory. Then all the above conditions hold.

An ideal in a tensor category is a tensor ideal if it is stable under $\text{id}_C \otimes -$ and $- \otimes \text{id}_C$ for all $C \in \mathcal{A}$. The ideal is then stable under tensor products from left or right with arbitrary morphisms. Let $Tr$ be the trace. For any two objects $A, B$ we define $\mathcal{N}(A, B) \subset \text{Hom}(A, B)$ by

$$\mathcal{N}(A, B) = \{ f \in \text{Hom}(A, B) \mid \text{ for all } g \in \text{Hom}(B, A), Tr(g \circ f) = 0 \}.$$  

The collection of all $\mathcal{N}(A, B)$ defines a tensor ideal $\mathcal{N}$ of $\mathcal{A}$ [AK02], the tensor ideal of negligible morphisms. By [AK02], 8.2.2a, we have the following theorem.

2.1 Theorem. (i) $\mathcal{N}$ is the largest proper tensor ideal of $\mathcal{A}$.
(ii) The only proper tensor ideal $\mathcal{I}$ of $\mathcal{A}$ such that the quotient $\mathcal{A}/\mathcal{I}$ is semisimple, is $\mathcal{I} = \mathcal{N}$.

The quotient $\mathcal{A}/\mathcal{N}$ will be called the universal semisimple quotient of $\mathcal{A}$.

2.2 Theorem. The quotient $\mathcal{A}/\mathcal{N}$ is again a super tannakian category. If $\mathcal{A}' \subset \mathcal{A}$ is a pseudoabelian full tensor subcategory, the quotient $\mathcal{A}'/(\mathcal{N} \cap \mathcal{A}')$ is a super tannakian category.

Proof. The quotient of a $k$-linear rigid tensor category by a tensor ideal is again a $k$-linear rigid tensor category. Since $\mathcal{N}$ is a tensor ideal the quotient functor $\omega : \mathcal{A} \to \mathcal{A}/\mathcal{N}$ is a tensor functor. The quotient category is semisimple by construction. Since $\text{Hom}$-spaces are finite-dimensional one has idempotent lifting, hence $\mathcal{A}/\mathcal{N}$ is pseudoabelian. A $k$-linear semisimple pseudoabelian category is abelian by [AK02]. By [Del02] an abelian tensor category is super tannakian if and only if for every
object \( A \) there exists a Schur functor \( S_\mu \) with \( S_\mu(A) = 0 \). Since \( \omega(S_\mu(A)) = S_\mu(\omega(A)) \) any object in \( \mathcal{A}/\mathcal{N} \) is also annihilated by a Schur functor. \( \Box \)

The category \( \mathcal{A}/\mathcal{N} \) has the following universal property.

**2.3 Proposition.** Let \( \omega : \mathcal{A} \to \mathcal{C} \) be a full tensor functor into a semisimple tensor category \( \mathcal{C} \). Then \( \omega \) factorises over the quotient \( \mathcal{A}/\mathcal{N} \).

**Proof.** Since \( \mathcal{C} \) is semisimple there are no negligible morphisms. However the image of a negligible morphism is negligible, since a tensor functor commutes with traces. Hence the image of a negligible morphism under \( \rho \) is zero, hence the functor factorizes. \( \Box \)

For completeness sake we assemble a few elementary lemmas about this quotient.

**2.4 Lemma.** An object \( X \) of \( \mathcal{A} \) maps to zero in \( \mathcal{A}/\mathcal{N} \) if and only if \( \text{id}_X \) belongs to \( \mathcal{N}(X,X) \).

The collection of these elements - called negligible objects - is denoted by \( N \). The dimension of an object \( X \) in a tensor category is defined \( Tr(\text{id}_X) \in \text{End}(1) \). If \( \mathcal{A} \subset \text{Rep}(G,\epsilon) \), then \( \dim_A(X) = s\dim(X) = \dim(X_0) - \dim(X_1) \).

**2.5 Lemma.** An indecomposable object is in \( N \) if and only if \( s\dim(X) = 0 \).

**Proof.** If \( X \in N \) we have \( Tr(g) = 0 \) for all \( g \in \text{End}(X) \), in particular for \( g = \text{id}_X \). Let \( s\dim(X) = 0 \). We have to show: \( \text{id}_X \in N(X,X) \), ie. \( Tr(g) = 0 \) for all \( g \in \text{End}(X) \). Since \( X \) is indecomposable \( g \) is either nilpotent or an isomorphism. If \( g \) is nilpotent \( Tr(g) = 0 \) \[\text{[Bru00], 1.4.3.}\] Let \( g \) be an isomorphism. Since \( X \) is indecomposable \( g \) has a unique eigenvalue \( \lambda \) and \( Tr(g) = \lambda s\dim(X) \), hence \( Tr(g) = 0 \). \( \Box \)

**2.6 Lemma.** \([\text{[Bru00], 1.4.}]\) The functor \( \mathcal{A} \to \mathcal{A}/\mathcal{N} \) induces a bijection between the isomorphism classes of indecomposable elements not in \( N \) and the isomorphism classes of irreducible elements in \( \mathcal{A}/\mathcal{N} \).

**Proof.** Let \( X \) be indecomposable, \( X \notin N \). Since \( \mathcal{A} \) and \( \mathcal{A}/\mathcal{N} \) are abelian and every object has finite length, an object \( X \) is indecomposable if and only if \( \text{End}(X) \) is a local ring. We have \( \text{End}_{\mathcal{A}/\mathcal{N}}(X) = \text{End}_{\mathcal{A}}(X)/\mathcal{N}(X) \). Since the quotient of a local ring by a (two-sided) ideal is again local, the image of \( X \) in \( \mathcal{A}/\mathcal{N} \) is indecomposable, hence irreducible. We show: If \( M \not\subset N \) in \( \mathcal{A} \) \( (M,N \text{ indecomposable}) \) we have \( \text{Hom}_{\mathcal{A}/\mathcal{N}}(M,N) = 0 \). Let \( f \in \text{Hom}_{\mathcal{A}}(M,N) \). Its image is zero in \( \text{Hom}_{\mathcal{A}/\mathcal{N}}(M,N) = \text{Hom}_{\mathcal{A}}(M,N)/\mathcal{N}(M,N) \) if and only if \( Tr(fg) = 0 \) for all \( g \in \text{Hom}_{\mathcal{A}}(N,M) \). Since \( M \) is indecomposable any endomorphism is invertible or nilpotent. The endomorphism \( fg \) is not bijective, hence nilpotent, hence \( Tr(fg) = 0 \) for all \( g \in \text{Hom}(N,M) \), hence \( \text{Hom}_{\mathcal{A}/\mathcal{N}}(M,N) = 0 \). \( \Box \)

Let \( I \) be an ideal in \( \mathcal{A} \). For \( X = \bigoplus X_i \) and \( Y = \bigoplus Y_j \) we have canonically \( I(X,Y) = \bigoplus_{i,j} I(X_i,Y_j) \) by \([\text{AK02}]\). Let \( X = \bigoplus X_i \) with \( X_i \in N \) for all \( i \), ie. \( \mathcal{N}(X_i,Y) = \text{Hom}(X_i,Y) \) and \( \mathcal{N}(Y,X_i) = \text{Hom}(Y,X_i) \) for all \( Y \in \mathcal{A} \). It follows
\( \mathcal{N}(X, X) = \text{Hom}(X, X) \), hence \( X \in \mathcal{N} \). If reciprocally \( X \in \mathcal{N} \) and \( X = \bigoplus X_i \), we have \( X_i \in \mathcal{N} \).

**2.7 Corollary.** (i) \( \mathcal{N} \) is closed under direct sums and direct summands. (ii) If \( X \in \mathcal{N} \) and \( Y \in \mathcal{A} \), we have \( X \otimes Y \in \mathcal{N} \) and each indecomposable summand of \( X \otimes Y \) has superdimension 0. (iii) Let \( X \notin \mathcal{N} \) and let \( X = \bigoplus X_i \) be its decomposition into indecomposable elements. Then \( \text{Hom}_{\mathcal{A}/\mathcal{N}}(X, X) = \bigoplus_i, \ sdim(X_i) \neq 0 \ k. \)

**2.2. The pro-reductive envelope.** Since the quotient \( \mathcal{A}/\mathcal{N} \) is again a super-tannakian category, this defines a reductive super group scheme \( G^{\text{red}} \) with \( \mathcal{A}/\mathcal{N} \cong \text{Rep}(G^{\text{red}}, \epsilon) \) with \( \epsilon : \mu_2 \to G \) such that the operation of \( \mu_2 \) gives the \( \mathbb{Z}_2 \)-gradation of the representations. We call \( G^{\text{red}} \) the pro-reductive envelope of \( G \) (following [AK02]). If \( G \) is an algebraic group, the pro-reductive envelope has been extensively studied by Andre and Kahn. Their proofs do not apply to the supergroup case. In the tannakian case \( \mathcal{N} = R \) is equal to the radical ideal. In particular no indecomposable elements maps to zero. Even in the tannakian case the pro-reductive cover will not be of finite type in general.

**2.8 Theorem.** [AK02], theorem C.5. The proreductive envelope of an affine \( k \)-group \( G \) is of finite type over \( k \) if and only if \( G \) is of finite type over \( k \) and the pronipotent radical of \( G \) is of dimension \( \leq 1 \).

Consider two examples. If \( G = \mathbb{G}_a \), then \( G^{\text{red}} = SL(2) \). If \( G = \mathbb{G}_a \times \mathbb{G}_a \), then \( G^{\text{red}} \) is no longer of finite type. In fact, the determination of \( G \cong G^{\text{red}} \) is unsolvable since it would include a classification of the indecomposable representations of \( G \) which is a wild problem [AK02], 19.7. More generally it seems plausible that \( \text{Rep}(G^{\text{red}}, \epsilon) \) is of finite or tame type if and only if \( \text{Rep}(G) \) is of finite or tame type. It is likely that if \( \text{Rep}(G) \) is of wild type, the problem of classifying indecomposable modules of non-vanishing superdimension is wild as well. Therefore we should not try to determine \( G^{\text{red}} \) in this case, but ask the following weaker questions: Given any object \( V \in \text{Rep}(G) \) or \( \text{Rep}(G, \epsilon) \), consider its image in \( \mathcal{A}/\mathcal{N} \). The tensor category generated by it is a semisimple algebraic tensor category (since \( \mathcal{A}/\mathcal{N} \) is semisimple). The semisimple algebraic tensor categories in characteristic zero were classified in [Wei09].

**2.9 Theorem.** Any supergroup \( G \) over \( k \) such that \( \text{Rep}(G) \) is semisimple is isomorphic to a semidirect product \( G' \rtimes H \) of a reductive algebraic \( k \)-group \( H \) with a product \( G' = \prod_{r \geq 1} \text{Sp}(1|2r)^{\text{red}} \) of simple supergroups of \( BC \)-type, where the semidirect product is defined by an abstract group homomorphism \( p : \pi_0(H) \to \prod_{r \geq 1} S_{a_r}. \)

Now consider an irreducible object \( V \in \text{Rep}(G, \epsilon) \) and consider the tensor category generated by \( \omega(V) \) in \( \text{Rep}(G, \epsilon)/\mathcal{N} \). This tensor subcategory corresponds to an algebraic group \( G_V \cong G^{\text{red}} \). Then this group is of finite type since it has a tensor generator. It is reductive since \( \text{Rep}(G^{\text{red}}, \epsilon) \) is semisimple.

**2.10 Lemma.** Let \( T \) be a maximal torus in \( G_V \) and \( X^*(T) \) its character group. Let \( R \) be the subgroup generated by the roots of \( G_V \). Then the center of \( G_V \) has cyclic character group \( X/R \) and \( (G_V)^{\text{red}} \) has cyclic center.
Proof. $\omega(V)$ is a tensor generator of $\text{Rep}(G_V)$ for the reductive group $G_V$ and likewise $\omega(V)$ is a tensor generator of $\text{Rep}(G_V)^0$. Now use that a reductive group has a faithful irreducible representation if and only if $X/R$ is cyclic and a semisimple group has a faithful irreducible representation if and only if its center is cyclic [McN].

2.3. The basic classical cases. Let $G$ be basic classical [Ser11b] with underlying basic classical Lie superalgebra $g$ [Kac78]. Dufo and Serganova [DS05] and [Ser11a] constructed for certain elements $x \in g_1$ with $[x,x] = 0$, where $g_1$ denotes the odd part of $g$, tensor functors $V \mapsto V_x : \text{Rep}(g) \to \text{Rep}(g_x)$ where $g_x$ is a classical Lie algebra or $\mathfrak{osp}(1|2n)$. These functors are not full, hence need not factorize over the quotient $\text{Rep}(G)/N$. However it should be expected that $G^{red}$ contains groups $G_x$ with Lie superalgebra $g_x$. For instance the superdimension of any irreducible representation in $\text{Rep}(G)$ equals the superdimension of some representation in $\text{Rep}(G^{red})$ and in $\text{Rep}(g_x)$. Note that this representation in $\text{Rep}(g_x)$ might not be irreducible.

For $gl(m|n)$ we have $g_x = gl(m-n)$ and for $\mathfrak{osp}(m|2n)$, $m = 2l$ or $2l+1$, we have $g_x = \mathfrak{osp}(m-2\min(l,n), n-2\min(l,n))$. For the exceptional Lie superalgebras the functor of Dufo-Serganov gives representations of the following Lie algebras:

- If $g = D(2,1,\alpha)$, then $g_x = gl(1)$.
- If $g = G_3$, then $g_x = sl(2)$.
- If $g = F_4$, then $g_x = sl(3)$.

Hence $G^{red}$ should contain $Gl(1)$ or $Sl(2)$ or $Sl(3)$ as a subgroup respectively. We determine $\text{Rep}(Gl(m|1))/N$ in this article. For the $Gl(n|n)$-case see [HW15]. The $OSp(2|2n)$-case can be treated similar to the $Gl(m|1)$-case. In this case we obtain $\text{Rep}(OSp(2|2n))/N \cong \text{Rep}(Sp(2n-2) \times Gl(1) \times Gl(1))$ (in the super sense).

3. ON THE GL(m|n)-CASE

3.1. Preliminaries. Let $g = gl(m|n)$ denote the Lie superalgebra of $Gl(m|n)$ with even part $g_0 = gl(m) \oplus gl(n)$. The Lie superalgebra has a $\mathbb{Z}$-grading $g(m|n) = g_{-\mathbb{N}} \oplus g_0 \oplus g_{\mathbb{N}}$ with $g_0 = g_0$ and $g_1 = g_{-1} \oplus g_1$ [Kac78]. Let $h$ be the Cartan algebra of diagonal matrices in $g$. We denote by $\epsilon_i$ the usual basis elements of $h^*$ [Ger98].

For $\lambda \in h^*$ let $L_0(\lambda)$ be the simple $g_0$-module of highest weight $\lambda \in h^*$ relative to the Borel subalgebra of upper triangular matrices $b_0$. The $g_0$-module $L_0(\lambda)$ can be extended trivially to $g_0 \oplus g_1$. The Kac-module and the AntiKac-module are by definition [Kac78]

$$K(\lambda) = \text{Ind}^{g_0 \oplus g_1}_{g_0}L_0(\lambda), \quad K'(\lambda) = \text{Ind}^{g_0 \oplus g_{-1}}_{g_0}L_0(\lambda).$$

3.1 Lemma. [Kac78] $K(\lambda)$ has irreducible top and socle. The top is given by the irreducible representation $L(\lambda)$. $K(\lambda)$ is finite-dimensional if and only if $L_0(\lambda)$ is finite-dimensional which is the case if and only if $L(\lambda)$ is finite-dimensional.

In particular the simple $g$-modules are up to a parity shift parametrised by the same set of highest weights as the simple $g_0$-modules. Hence the (integral dominant) highest weights $X^+$ of $gl(m|n)$ are of the form $\lambda = (\lambda_1, \ldots, \lambda_m | \lambda_{m+1}, \ldots, \lambda_{m+n})$. Here $\lambda_1 \geq \cdots \geq \lambda_m$ and $\lambda_{m+1} \geq \cdots \geq \lambda_{m+n}$ are integers and every $\lambda \in \mathbb{Z}^{m+n}$ with these properties parametrises a highest weight of an irreducible $g$-module. This set of highest weights is called $X^+$, so that the irreducible modules in $\text{Rep}(g)$ are given
by the \( \{ L(\lambda), \Pi L(\lambda) \mid \lambda \in X^+ \} \) where \( \Pi \) denotes the purity shift. In the \( \mathfrak{sl}(m|n) \)-case we have to identify two irreducible modules whose highest weights differ only from a weight of the form \((k, k, \ldots, k| -k, \ldots, -k)\) for \(k \in \mathbb{Z}\). We say that a module is a Kac-object if it has a filtration whose subquotients are Kac-modules. The full subcategory of these modules is denoted \( \mathcal{C}^+ \). Similarly we have the category \( \mathcal{C}^- \) of objects which have a filtration by AntiKac-modules. By \cite{Ger98} \( \mathcal{C}^+ \cap \mathcal{C}^- = \text{Proj} \). Both \( \mathcal{C}^+ \) and \( \mathcal{C}^- \) are tensor ideals in \( \mathcal{R} \). If \( K(\lambda) \) is irreducible the weight \( \lambda \) is called typical. If not, \( \lambda \) is called atypical. \( K(\lambda) \) is irreducible if and only if \( K(\lambda) \) is projective as a \( \mathfrak{g} \)-module \cite{Kac78}. It is well known that \( \text{Rep}(\mathfrak{gl}(m|n)) \) identifies with the category of integrable supermodules over \( \mathfrak{gl}(m|n) \) \cite{BS12}.

3.1.1. **Weight diagrams.** To each highest weight \( \lambda \in X^+ \) we associate, following \cite{BS12}, two subsets of cardinality \( n \) of the numberline \( \mathbb{Z} \)

\[
I_x(\lambda) = \{ \lambda_1, \lambda_2 - 1, \ldots, \lambda_n - n + 1 \}
\]

\[
I_o(\lambda) = \{ 1 - m - \lambda_{m+1}, 2 - m - \lambda_{m+2}, \ldots, n - m - \lambda_{m+n} \}.
\]

The integers in \( I_x(\lambda) \cap I_o(\lambda) \) are labeled by \( \vee \), the remaining ones in \( I_x(\lambda) \) resp. \( I_o(\lambda) \) are labeled by \( \times \) resp. \( \circ \). All other integers are labeled by a \( \wedge \). This labeling of the numberline \( \mathbb{Z} \) uniquely characterizes the weight \( \lambda \). If the label \( \vee \) occurs \( r \) times in the labeling, then \( r \) is called the degree of atypicality of \( \lambda \). Notice that \( 0 \leq r \leq n \), and \( \lambda \) is called maximal atypical if \( r = n \). Examples are the trivial module \( \mathbb{1} \) and the standard representation \( V \) of highest weight \( \lambda = (1, \ldots, 0)(0, \ldots, 0) \) for \( m \neq n \). Another example is the Berezin determinant \( \text{Ber} = L(1, \ldots, 1 \mid -1, \ldots, -1) \) of dimension 1. For the notion of a cup or cap diagram attached to a weight diagram we refer to \cite{BS11}[BS12].

3.2. **Wildness.** If \( \mathcal{A} \) is a nice category we can associate to it its Ext-quiver. The vertex set \( X^+ \) is given by the set of isomorphism classes of simple modules and the number of arrows from \( \lambda \) to \( \mu \) is given by \( \text{Ext}_\mathcal{A}(L(\lambda), L(\mu)) \).

3.2 Theorem. \cite{Ger98}, Thm 1.4.1. Let \( \mathcal{A} \) be a nice category and \( Q \) its Ext-quiver. Then there exists (an explicitly given) set of relations \( R \) on \( Q \) such that we have an equivalence of categories

\[
e : \mathcal{A} \rightarrow Q/R - \text{mod}
\]

such that \( e(M) = \bigoplus_{\lambda \in X^+} \text{Hom}_\mathcal{A}(P(\lambda), M) \) as graded vector spaces.

3.3 Lemma. \cite{ASS06} Let \( M \) be an indecomposable representation of a finite quiver \( Q \) which has no cyclic path. Then the number of composition factors of type \( L(\mu) \) in \( M \) are given by \( \text{dim}M_\mu \) where \( M_\mu \) is the vector space on the vertex \( \mu \).

A block \( \Gamma \) of \( X^+ \) is a connected component of the Ext-quiver. Let \( \mathcal{A}_\Gamma \) be the full subcategory of objects of \( \mathcal{A} \) such that all composition factors are in \( \Gamma \) (also called a block). This gives a decomposition \( \mathcal{A} = \bigoplus_\Gamma \mathcal{A}_\Gamma \) of full abelian subcategories. Every indecomposable module lies in a unique \( \mathcal{A}_\Gamma \) and all its simple submodules belong to \( \Gamma \). Two irreducible representations \( L(\lambda) \) and \( L(\mu) \) are in the same block if and only if the weights \( \lambda \) and \( \mu \) define labelings with the same position of the labels \( \times \) and \( \circ \). The degree of atypicality is a block invariant, and the blocks \( \Gamma \) of atypicality \( r \) are in 1-1 correspondence with pairs of disjoint subsets of \( \mathbb{Z} \) of cardinality \( m - r \) resp. \( n - r \).
3.4 Theorem. Assume $m, n \geq 2$. Then $\text{Gl}(m|n)^{\text{red}}$ is not of finite type.

Proof. This follows from the description of the Tannaka group generated by the irreducible elements in [HW15]. □

The statement also follows from the following lemma. This lemma should of course also hold for $m, n \geq 2$, but would require a more difficult argument. Let $Q$ denote the Ext-quiver of $R_{mn}$. Then there exists a system of relations $R$ on $Q$ such that $R_{mn} \cong kQ/R - \text{mod}$.

3.5 Lemma. Assume $m, n \geq 3$. Then the problem of classifying indecomposable representations of non-vanishing superdimension is wild.

Proof. We show that the classification is wild for every maximally atypical block for $n \geq 3$. Any such block is equivalent to the maximal atypical block $\Gamma$ of $\text{Gl}(n|n)$ [Ser06] [BS12]. Hence we show that the problem is wild in $\Gamma$. By [BS10], Cor. 5.15 for any two irreducible modules $L(\lambda), L(\mu) \in R_n$

$$\dim(\text{Ext}_{R_n}^1(L(\lambda), L(\mu))) = p^{(1)}_{\lambda, \mu} + p^{(1)}_{\mu, \lambda}$$

for the Kazhdan-Lusztig polynomials

$$p_{\lambda, \mu}(q) = \sum_{i \geq 0} p^{(i)}_{\lambda, \mu} q^i.$$ 

By [MS11], lemma 6.10 and [BS10], lemma 5.2 $p^{(1)}_{\lambda, \mu} \neq 0$ if and only if $\mu$ is obtained from $\lambda$ by interchanging the labels at the ends of one of the cups in the cup diagram of $\lambda$. For any $[\lambda] \in \Gamma$ with $\lambda_i > \lambda_{i+1} + 1$ for $i = 1, \ldots, n - 1$ the cup diagram looks like

The combinatorial rule from above shows that for every irreducible module $[\lambda]$ away from the diagonal $\dim \text{Ext}_1^1([\lambda], [\mu_i]) = \dim \text{Ext}_1^1([\mu_i], [\lambda]) = 1$ for exactly $2n$ different modules $\mu_i$ and $\dim \text{Ext}_1^1([\lambda], [\nu]) = 0$ for any $\nu \neq \mu_i$. In particular for any vertex away from the diagonal consider the subquiver with vertices $[\lambda], [\mu_1], \ldots, [\mu_{2n}]$ with arrows corresponding to $\dim \text{Ext}_1^1([\mu_i], [\lambda]) = 1$ and no arrows from $[\lambda]$ to any $[\mu_i]$ (so that $[\lambda]$ becomes a sink) (picture for $n = 3$):

Since this subquiver has no path of length $> 1$, it embeds fully into $k(Q)/R$. The classification of indecomposable representations of the $r$-subspace quiver is wild for $r \geq 5$. The superdimension formula of [Wei10] [HW14] shows that the superdimension is constant of alternating sign away from the diagonal: if $[\lambda]$ has superdimension $d$, the $[\mu_i]$ have superdimension $-d$. Hence an indecomposable representation of
this subquiver will give an indecomposable representation in $\Gamma$ of non-vanishing superdimension if and only if

$$\dim V_{[\lambda]} \neq \sum_{i=1}^{2n} \dim V_{[\mu_i]} \quad (\ast).$$

We are done when we have shown that the classification of indecomposable representations with $(\ast)$ is wild. Fix the vertex $[\mu_{2n}]$ and consider an indecomposable representation of the $(2n-1)$-subspace quiver by specifying a vector space for the vertices $[\lambda], [\mu_1], \ldots, [\mu_{2n-1}]$ with injections $V_{[\mu_i]} \to V_{[\lambda]}$. If $\dim V_{[\lambda]} \neq \sum_{i=1}^{2n-1} \dim V_{[\mu_i]}$ we put $V_{[\mu_{2n}]} = 0$. If $\dim V_{[\lambda]} = \sum_{i=1}^{2n-1} \dim V_{[\mu_i]}$ we put $V_{[\mu_{2n}]} = k$ and choose some injection of $k$ into $V_{[\lambda]}$. This defines a bijection between the isomorphism classes of indecomposable representations of the $(2n-1)$-subspace quiver with a subset of the indecomposable representations of the $2n$-subspace quiver satisfying $(\ast)$. \qed

As explained above this means that we should not try to determine $G^{red}$ for general $m, n$. Instead we should determine the tensor category generated by the irreducible elements in $A/N$ and the corresponding reductive supergroup. In the $n = 1$-case, where we have tame representation type, we show in this article:

3.6 Theorem. We have:

$$G^{red} = \begin{cases} 
\text{Gl}(m-1) \times \text{Gl}(1) \times \text{Gl}(1) & G = \text{Gl}(m|1) \\
\text{Sl}(m-1) \times \text{Gl}(1) \times \text{Gl}(1) & G = \text{Sl}(m|1), \ m \geq 3 \\
\text{Gl}(1) \times \text{Gl}(1) & G = \text{Sl}(2|1).
\end{cases}$$

This theorem should be understood in the super sense, i.e. $\text{Rep}(\text{Gl}(m|1))/N \simeq \text{Rep}(\text{Gl}(m-1) \times \text{Gl}(1) \times \text{Gl}(1)) \otimes \text{soc}_k$. We prove this by describing explicitly the image of an indecomposable representation $I$.

4. Determination of $G^{red}$ for $G = \text{Gl}(m|1)$

4.1. Irreducible elements. Assume from now on that we are in the $\text{Gl}(m|1)$-case and assume that weights are singly atypical. Recall that Kac-modules have a simple socle. The highest weight of the socle is denoted by $T^-\lambda$. The highest weight of the socle of the AntiKac-module $K'(\lambda)$ is denoted by $T^+\lambda$. If $\lambda$ is atypical $K(\lambda)$ is an extension of $L(\lambda)$ by $L(T^-\lambda)$:

$$0 \longrightarrow L(T^-\lambda) \longrightarrow K(\lambda) \longrightarrow L(\lambda) \longrightarrow 0.$$ 

Similarly we have the exact sequence

$$0 \longrightarrow L(T^+\lambda) \longrightarrow K'(\lambda) \longrightarrow L(\lambda) \longrightarrow 0.$$ 

By [Ger98] this sequences are up to equivalence all non-trivial extensions between simple modules: $\text{Ext}_{L}^{1}(L(\lambda), L(\mu)) = \mathbb{C}$ for $\mu \in \{T^+\lambda, T^-\lambda\}$ and zero else. An irreducible element is mapped to zero under $\omega$ if and only if it is typical or, equivalently, projective.
4.1.1. Indecomposable representations. We recall the results about the classification of indecomposable modules in the singly atypical case obtained by [Su00] and [Ger98]. We parametrise an atypical block as in [Ger98] by \( Z \) and denote the corresponding weight with \( a \in \mathbb{Z} \). By Germoni the indecomposable modules are either the Kac objects \( C^+ \) or the AntiKac objects \( C^- \) or \( V \) sits in an exact sequence

\[
0 \longrightarrow U \longrightarrow V \longrightarrow L \longrightarrow 0
\]

with \( U \in C^+ \) and \( L \) irreducible, or \( Q \) sits in an exact sequence

\[
0 \longrightarrow U' \longrightarrow L' \longrightarrow V \longrightarrow Q \longrightarrow 0
\]

with \( U \in C^+ \) and \( L' \) irreducible. In down to earth terms: Fix an arbitrary \( a \in \mathbb{Z} \) (that is, an arbitrary weight \( a \) in the block, or its corresponding simple module \( L(a) \)). From \( a \) we can either go a finite number of steps to the left to a point \( b \leq a \) using the extensions described by Kac-modules or a finite number of steps to the right to a point \( b \geq a \) using the extensions described by Anti-Kac-modules. To any such interval \([a, b]\) or \([b, a]\) corresponds a unique indecomposable module with composition factors \( L(a), L(a-1), \ldots, L(b) \) (ie. \( L(\lambda), L(T^-\lambda), \ldots, L(T^{-l}\lambda) \) where \( l = |b-a| \)) in the case of \( a \geq b \) resp with composition factors \( L(a), L(a+1), \ldots, L(b) \) (ie. \( L(\lambda), L(T^+\lambda), \ldots, L(T^{l}\lambda) \)). For \( b = a \) one obtains the simple modules \( L(a) \) and for \( b = a - 1 \) resp. \( b = a + 1 \) the Kac resp. Anti-Kac-modules. These two families of indecomposable modules are called ZigZag resp Anti-ZigZag-modules and denoted by \( Z^l(a) \) resp. \( \bar{Z}^l(a) \) where \( l \) is the number of composition factors. They form a complete system of representatives of the isomorphism classes of non-projective indecomposable modules in \( \mathcal{A} \).

4.1.2. Superdimension. The superdimension of any Kac resp. Anti-Kac-module is zero for any Type I Lie superalgebra. The superdimension is additive in short exact sequences, hence we obtain \( sdimT^-L(\lambda) = -sdimL(\lambda) \) for any atypical \( \lambda \) and likewise \( sdimT^+L(\lambda) = -sdimL(\lambda) \).

**4.1 Corollary.** The superdimension of the indecomposable modules \( Z^l(a) \) is given by

\[
sdimZ^l(a) = \sum_{i=1}^{l} (-1)^i sdimL(a) = \begin{cases} 
  sdimL(a) & l \text{ odd} \\
  0 & l \text{ even}.
\end{cases}
\]

The same for \( sdim\bar{Z}^l(a) \).

The only other remaining indecomposable modules are the projective covers \( P(\lambda) \) of the atypical simple modules which have superdimension 0.

**4.2 Corollary.** The irreducible objects in \( \mathcal{A}/N \) are up to isomorphism given by the

\[
\{ Z^l(\lambda), \bar{Z}^l(\lambda) \mid \lambda \text{ atypical, } l \text{ odd} \}.
\]

4.2. The toy example \( \mathfrak{sl}(2|1) \). The tensor products of the indecomposable modules have been computed by [GQS07] and can be used to compute the formulas in the quotient.
4.3 Lemma. The tensor product of the irreducible modules in $\mathcal{A}/\mathcal{N}$ is given by the following rules:

- $Z^{2p+1}(j_1) \otimes Z^{2p+1}(j_2) = Z^{2(p_1+p_2)+1}(j_1+j_2)$
- $Z^{2p+1}(j_1) \otimes Z^{2p+1}(j_2) = Z^{2(p_1+p_2)+1}(j_1+j_2)$
- $Z^{2p+1}(j_1) \otimes Z^{2p+1}(j_2) = Z^{2(p_2-p_1)+1}(j_1+j_2-p_1)$ for $p_1 \leq p_2$
- $Z^{2p+1}(j_1) \otimes Z^{2p+1}(j_2) = Z^{2(p_1-p_2)+1}(j_1+j_2-p_2)$ for $p_2 \leq p_1$

Proof. This is an inspection using [GQS07]. The tensor products between ZigZag-modules are given by Proposition 4 in loc.cit. as a direct sum of a $\Theta$-part and a $\Theta$-part. Since $\omega$ preserves direct sums we omit any projective module in the formulas. By definition $T(\_\_)$ consists of a direct sum of typical modules (formula (44)). For the contributions of the $\Theta$-part see p.836 in loc.cit. Note that $\Theta$ maps projective modules to projective modules. The $\mathfrak{gl}(1|1)$-formulas yield then the above result. □

We use the following reparametrization: We put

- $Z^{2p+1}(j) = (-p, -j)$
- $Z^{2p+1}(j) = (p, p - j)$

The irreducible elements in $\mathcal{A}/\mathcal{N}$ are then parametrized by $\mathbb{Z}_2 \times \mathbb{Z} \times \mathbb{Z}$. The $\mathbb{Z}_2$ comes from the fact that we can apply the parity shift $\Pi$ to an indecomposable representation. The rules for the tensor products read now:

\[
\begin{align*}
(p_1, p_1 - j_1) \otimes (p_2, p_2 - j_2) &= (p_1 + p_2, p_1 + p_2 - (j_1 + j_2)) \\
(-p_1, -j_1) \otimes (-p_2, -j_2) &= (-p_1 + p_2, -(j_1 + j_2)) \\
(p_1, p_1 - j_1) \otimes (-p_2, -j_2) &= (p_1 - p_2, p_1 - (j_1 - j_2)) \\
(p_1, p_1 - j_1) \otimes (p_2, -j_2) &= (p_1 - p_2, p_1 - (j_1 - j_2)).
\end{align*}
\]

Note that this is exactly the tensor product for the group $Gl(1) \times Gl(1)$. For the proof of the next statement recall that every representation can be replaced by its parity shift so that its superdimension is positive.

4.4 Corollary. $G^{red} = Gl(1) \times Gl(1)$

Proof. We assume without loss of generality that the superdimension of all objects $(p, q)$ is positive. We have to define a functor $\rho : \mathcal{A}/\mathcal{N} \to Rep(Gl(1) \times Gl(1))$ which is an equivalence of tensor categories. Use the parametrisation above of the irreducible elements in $\mathcal{A}/\mathcal{N}$ by $\mathbb{Z} \times \mathbb{Z}$. Define $\rho$ on objects by mapping the irreducible element corresponding to $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ to the irreducible representation $t^p \otimes t^q$ of $Gl(1) \times Gl(1)$. Note that $Hom$-spaces are either zero or one-dimensional by Schur’s lemma. The results on tensor products show that this is a tensor functor. □

Here we considered only indecomposable objects of positive superdimension. For the whole category we obtain $Rep(Sl(2|1))/\mathcal{N} \simeq Rep(Gl(1) \times Gl(1)) \otimes svec_k$.

4.3. Mixed tensors. We first determine the contribution from the irreducible modules. The subcategory $T \subset R_{mn}$ $(m \geq n)$ of mixed tensors is the pseudoabelian full subcategory of objects, which are direct summands in a tensor product
$V^\otimes r \otimes (V^\vee)^\otimes s$ for some $r, s$. We have the equivalence

$$T/N \simeq \text{Rep}(Gl(m-n))$$

of tensor categories by [Hei14]. Furthermore by loc.cit every singly atypical irreducible module is a Berezin twist of a mixed tensor. ZigZag or AntiZigZag-modules of length $> 1$ are never mixed tensors. The indecomposable objects in $T$ are parametrized by certain pairs of partitions $(\lambda^L, \lambda^R)$ [CW11] and we denote the corresponding indecomposable element by $R(\lambda^L, \lambda^R)$. By the above result every irreducible representation of non-vanishing superdimension can be written in the form $L(\lambda) \cong Ber^{s(\lambda)} \otimes R(\lambda^L, \lambda^R)$ for some integer $s(\lambda)$ and a pair of partitions $(\lambda^L, \lambda^R)$ satisfying $l(\lambda^L) + l(\lambda^R) \leq m-1$. It is clear from these results that the reductive supergroup corresponding to the tensor subcategory in $A/N$ generated by the irreducible modules is $Gl(1) \times Gl(m-1)$. Indeed the element $Ber^{s(\lambda)} \otimes R(\lambda^L, \lambda^R)$ $(\lambda^L = (\lambda^L_1, \ldots, \lambda^L_s, 0, \ldots), \lambda^L_1 > 0, \lambda^R = (\lambda^R_1, \ldots, \lambda^R_s, 0, \ldots), \lambda^R_1 > 0, t+s \leq m-1)$ corresponds to the representation $r(\lambda) \otimes L(wt(\lambda^L, \lambda^R))$ for the irreducible $Gl(m-1)$-representation

$$L(wt(\lambda^L, \lambda^R)) = L(\lambda^L_1, \ldots, \lambda^L_s, 0, \ldots, 0, -\lambda^R_1, \ldots, \lambda^R_s)$$

defined in [CW11][Hei14] and the determinantal representation $t$.

### 4.4. Indecomposable modules.

It remains to compute the tensor product of two ZigZag-modules up to superdimension 0. As the reparametrisation in the $sl(2|1)$-case shows it is preferable to deviate from the usual $Z^l(a)$-notation. We denote by $R(a, \ldots, b)$ the indecomposable module corresponding to the exact sequence

$$0 \to K \to R[a, \ldots, b] \to L(b) \to 0$$

where $K$ is a Kac-object with composition factors $L(a), \ldots, L(b-1)$. We call $R$ a roof module. Similarly we denote by $B[a, \ldots, b]$ the indecomposable module corresponding to

$$0 \to L(a) \to B[a, \ldots, b] \to K' \to 0$$

where $K'$ is a Kac-object with composition factors $L(a+1), \ldots, L(b)$. We call $B$ a bottom module.

### 4.5. Cohomological tensor functors.

We now define cohomological tensors $T_{m|1} \to T_{m-1|0}$. These were first defined in [DS05] and then later refined in [HW14]. We define a similar refined version in the $T_{m|1}$-case, but it can be easily extended to the general $T_{m|m}$-case [Heiar]. For any $x \in X = \{x \in g_1 \mid [x, x] = 0\}$ and any representation $(V, \rho)$, the operator $\rho(x)$ defines a complex since $\rho(x) \circ \rho(x)$ is zero, and we define $V_x = \text{ker}(\rho(x))/\text{im}(\rho(x))$. By [DS05] [Ser11a] this defines a tensor functor and $V_x \in T_{m-1|0}$. We fix the following element $x \in X$

$$x = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \in g_1(m|1) \text{ for } y = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 1 \end{pmatrix}.$$ 

and denote the corresponding tensor functor $V \to V_x$ by $DS : T_{m|1} \to T_{m-1|0}$.

**Parity considerations.** If $V$ is in $R_{m|1}$, $DS(V)$ may not be in $R_{m-1|0}$. We need to study this more closely. We embed $Gl(m-1|0)$ as an upper block matrix in
$Gl(m|1)$. More precisely we fix the embedding 
\[ Gl(m - 1|0) \times Gl(1|1) \to Gl(m|1) \]
\[ A \times \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} A & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}. \]

Recall that we defined the categories \( R_{m|n} \) in section 1. We now fix the morphism \( \epsilon : \mathbb{Z}/2\mathbb{Z} \to Gl(m) \times Gl(n) \) which maps $-1$ to \( \text{diag}(E_m, -E_n) = \epsilon_{m|n} \). Then \( R_{m|1} \) is the full subcategory of objects \( V \) such that \( p_V = \rho(\epsilon_{m|1}) \). If we use the embedding above, we obtain \( \epsilon_{m|1} = \epsilon_{m-1|0} \epsilon_{1|1} \). If we restrict a representation \( V \in R_{m|1} \) to \( Gl(m - 1) \), we get the decomposition
\[ V|_{Gl(m-1)} = V^+ \oplus V^- \]
where
\[ V^+ = \{ v \in V \mid \rho(\epsilon_{1|1})(v) = v \} \]
\[ V^- = \{ v \in V \mid \rho(\epsilon_{1|1})(v) = -v \}. \]

Since \( \rho(x) \) is an odd morphism
\[ \rho(x) : V^\pm \to V^\mp. \]
Hence \( \rho(x) \) induces the even morphism
\[ \rho(x) : V^\pm \to \Pi V^\mp. \]

We use the notation \( \partial \) for \( \rho(x) \).

\textbf{Z-grading.} We equip \( DS(V) \) with a \( \mathbb{Z} \)-grading as in [HW14]. Although \( DS(V) \) is in \( T_{m-1|0} \), it still has an action of the torus of diagonal matrices in \( Gl(m|1) \). Let \( V = \bigoplus \lambda V_\lambda \) be the weight decomposition and \( v = \sum \lambda v_\lambda \) in \( V \). An easy calculation shows the following lemma.

\textbf{4.5 Lemma.} \textit{The following holds for} \( \partial = \rho(x) \): a) \( \partial(V_\lambda) \subset V_{\lambda + \mu} \) \textit{for the odd simple root} \( \mu = \epsilon_n - \epsilon_{m+1} \), b) \( \partial v = 0 \) \textit{if and only if} \( \partial v_\lambda = 0 \) \textit{for all} \( \lambda \) \textit{and} c) \( v = \partial w \) \textit{if and only if} \( v_\lambda = \partial w_\lambda \) \textit{for all} \( \lambda \).

Hence \( DS(V) \) has a weight decompositon with respect to the weight lattice of \( gl(m|1) \). The weight decomposition with respect to \( gl(m - 1|0) \) is obtained by restriction. The kernel of this restriction map consists of the multiples \( \mathbb{Z}_\mu \). Hence \( DS(V) \) can be endowed with the weight structure coming from the \( gl(m|1) \)-module \( V \). This weight decomposition induces then on \( DS(V) \) a decomposition
\[ DS(V) = \bigoplus_{l \in \mathbb{Z}} DS(V)_l. \]

Now let \( H_s \) denote the torus in the diagonal matrices of elements of the form \( (1, \ldots, 1, t^{-1}) \) and denote by \( V_l \) the eigenspace of \( V \) where \( H_s \) acts by the eigenvalue \( t^l \). Another easy calculation shows the next lemma.

\textbf{4.6 Lemma.} \textit{If} \( v \in V \) \textit{satisfies} \( v \in V_\lambda \) \textit{and} \( v \in V_l \), \textit{then} \( \rho(x)v \in V_{l+1} \).
Since $\partial V_\lambda \subseteq V_{\lambda+\mu}$ we obtain a complex

$$
\cdots \xrightarrow{\partial} V_\lambda \xrightarrow{\partial} \Pi V_{\lambda+\mu} \xrightarrow{\partial} V_{\lambda+2\mu} \xrightarrow{\partial} \cdots
$$

which we can write by the last lemma as

$$
\cdots \xrightarrow{\partial} V_i \xrightarrow{\partial} \Pi V_{i+1} \xrightarrow{\partial} V_{i+2} \xrightarrow{\partial} \cdots
$$

We denote the cohomology of this complex by $H^i(V)$. Then $DS(V)_i = \Pi^i(H^i(V))$ and we obtain a direct sum decomposition of $DS(V)$ into $GL(m-1)$-modules

$$DS(V) = \bigoplus_i \Pi^i(H^i(V)).$$

This extra structure is very important since it carries a lot more information than just the $\mathbb{Z}_2$-graded version of $DS(L)$ in $T_{m-1|0}$.

4.7 Lemma. A short exact sequence in $R_{m|1}$ gives rise to a long exact sequence for $H^i$.

We denote by $\sigma$ the automorphism of $gl(m|1)$ defined by $\sigma(x) = -x^T$ where $(\cdot)^T$ denotes the supertranspose. Then $\sigma(x)$ is still a nilpotent element in $gl_1$. The corresponding tensor functor is denoted by $DS_\sigma : T_{m|1} \to T_{m-1|0}$. We can copy the arguments from above and endow $DS_\sigma(V)$ with a $\mathbb{Z}$-grading $DS_\sigma(V) = \bigoplus_i \Pi(H^i_\sigma(V))$. $DS$ and $DS_\sigma$ behave in the same way on irreducible objects, but differ on indecomposable elements, see lemma 4.9.

4.8 Corollary. For any representation $V$ $DS(V)$ and $DS_\sigma(V)$ are $\mathbb{Z}$-graded objects.

Another way of looking at this $\mathbb{Z}$-grading is the following. The collection of cohomology functors $H^i : R_{m|1} \to R_{m-1|0}$ for $i \in \mathbb{Z}$ defines a tensor functor

$$H^\bullet : R_{m|1} \to Gr^\bullet(R_{m-1|0})$$

to the category of $\mathbb{Z}$-graded objects in $R_{m-1|0}$. Using the parity shift functor $\Pi$, this functor can be extended to a tensor functor

$$H^\bullet : T_{m|1} \to Gr^\bullet(T_{m-1|0}).$$

As in [HW14] we conclude from the support variety calculations in [BKN09], the following lemma

4.9 Lemma. The kernel of $DS$ is $C^-$ and the kernel of $DS_\sigma$ is $C^+$. 

4.6 Cohomology computations. We can now calculate $DS(M)$ for any $M$ of nonvanishing superdimension. By [DS05] $DS(L(\lambda)) = mL^\text{core} \oplus mL^\text{core}$ for atypical $L(\lambda)$ where $L^\text{core}$ is the irreducible $GL(m-1)$-representation obtained from $L(\lambda)$ by replacing the single $\vee$ in the weight diagram by a $\wedge$. We recall from [Hei14] that
we have a commutative diagram

\[
\begin{array}{ccc}
\text{Rep}(\text{Gl}_{m-1}) & \downarrow F_{m|1} & \downarrow F_{m-1|0} \\
\mathcal{R}_{m|1} & \downarrow DS & R_{m-1|0} \\
\end{array}
\]

where \(\text{Rep}(\text{Gl}_{m-n})\) denotes Deligne’s category to the parameter \(\delta = m - n\). We obtain the same diagram if we replace \(DS\) by \(DS_\sigma\) since \(DS_\sigma\) sends the standard to the standard representation.

4.10 Lemma. Let \(L(\lambda)\) be atypical irreducible. Then

\[
DS(L(\lambda)) = DS_\sigma(L(\lambda)) = \begin{cases} 
L_{\text{core}} & \text{if } p(\lambda) \text{ even,} \\
\Pi L_{\text{core}} & \text{if } p(\lambda) \text{ odd.}
\end{cases}
\]

Proof. If \(L = R(\lambda)\) is the unique mixed tensor in a block \(\Gamma\), then \(DS(L) = L(\text{wt}(\lambda))\) by the commutativity of the diagram above. Hence the multiplicity of \(L_{\text{core}}\) is 1 (i.e. \(m = 1\) and \(m' = 0\)). Any other irreducible representation is a Berezin twist of \(L\) and \(DS(\text{Ber}^i) = \Pi^i L(i, \ldots, i)\). The Berezin twist does not change the multiplicity. The superdimension of \(L_{\text{core}}\) is positive and the superdimension of \(\Pi L_{\text{core}}\) is negative. Since the superdimension of \(L(\lambda)\) is positive if and only if \(p(\lambda)\) is even, the result follows.

\[\square\]

4.11 Lemma. We have the following equalities

\[
DS(B[a, \ldots, a]) = DS(L(a)) \quad \text{and} \quad DS_\sigma(B[a, \ldots, b]) = DS(L(a)),
\]

\[
DS(R[a, \ldots, b]) = DS(L(b)) \quad \text{and} \quad DS_\sigma(R[a, \ldots, b]) = DS_\sigma(L(a))
\]

of \(\mathbb{Z}\)-graded objects in \(T_{m-1|0}\).

Proof. Every roof or bottom module can be written in two ways, either as an extension of a Kac object or an AntiKac object. The roof module \(R(a, \ldots, b)\) is realized by the extension

\[
0 \to K \to R[a, \ldots, b] \to L(b) \to 0
\]

where \(K\) is a Kac-object and by the extension

\[
0 \to \tilde{K} \to R[a, \ldots, b] \to L(a) \to 0
\]

where \(K\) is an AntiKac-object. Similarly \(B[a, \ldots, b]\) is the indecomposable module corresponding to

\[
0 \to L(a) \to B[a, \ldots, b] \to K' \to 0
\]

where \(K'\) is a Kac-object and

\[
0 \to L(a) \to B[a, \ldots, a] \to \tilde{K}' \to 0
\]
where $K'$ is an AntiKac-object. We apply $DS$. By lemma 4.9 the kernel of $DS$ is $C^{-}$. Hence taking the long exact cohomology sequence arising from
\[ 0 \to K \to R[a, \ldots, b] \to L(b) \to 0 \]
gives, using $H^i(K) = 0$ for all $i$ and that the cohomology of every irreducible representation is concentrated in one degree, that
\[ H^i(R[a, \ldots, b]) \simeq H^i(L(b)) \]
for all $i$. Using $DS(V) = \bigoplus_i H^i(V)$ we obtain $DS(R[a, \ldots, b]) = DS(L(b))$ as a $\mathbb{Z}$-graded object in $T_{m-1|0}$ or $\mathbb{Z} \times T_{m-1|0}$. The exact sequence
\[ 0 \to \tilde{K} \to R[a, \ldots, b] \to L(a) \to 0 \]
gives, using $\text{Ker}(DS_n) = C^+$, the identification
\[ H^i_n(R[a, \ldots, b]) \simeq H^i_n(L(a)) \]
for all $i$. Hence $DS_n(R[a, \ldots, b]) = DS_n(L(a))$ as a $\mathbb{Z}$-graded object in $T_{m-1|0}$ or $\mathbb{Z} \times T_{m-1|0}$. In the same way we conclude
\[ H^i(B[a, \ldots, a]) \simeq H^i(L(a)) \] for all $i$
\[ H^i_n(B[a, \ldots, b]) \simeq H^i_n(L(a)) \] for all $i$.

4.7. Tensor products up to superdimension zero. We calculate the tensor product of two roof or bottom modules up to superdimension zero. We write $c_{a,b}^{\nu}$ for the coefficients in a $GL(m-1)$ tensor product $L(a) \otimes L(b) = \bigoplus c_{a,b}^{\nu}L(c)$.

4.12 Lemma. Denote the (ungraded) image of $L(a)$ under $DS$ by $L(\lambda_a)$.

\[ B(a_1, \ldots, a_s) \otimes B(c_1, \ldots, c_t) = \bigoplus_{\nu} c_{\lambda_a, \lambda_c}^{\nu} B(\nu, \ldots, \nu + \Delta), \ \Delta = (s - 1) + (t - 1) \]
\[ R(a_1, \ldots, a_s) \otimes R(c_1, \ldots, c_t) = \bigoplus_{\nu} c_{\lambda_a, \lambda_c}^{\nu} R(\nu, \ldots, \nu - \Delta), \ \Delta = (s - 1) + (t - 1) \]
\[ B(a_1, \ldots, a_s) \otimes R(c_1, \ldots, c_t) = \bigoplus_{\nu} c_{\lambda_a, \lambda_c}^{\nu} \begin{cases} B(\nu, \ldots, \nu + \Delta), & \Delta = (s - 1) - (t - 1), \ s > t \\ R(\nu - \Delta, \ldots, \nu), & \Delta = (s - 1) - (t - 1), \ s < t \\ L(\nu) & s = t \end{cases} \]

Proof. Consider the tensor product of two indecomposable modules of non-vanishing superdimension, say, for simplicity, $B(a, \ldots, a) \otimes B(c, \ldots, c)$. Under $DS$ they map to two irreducible elements of $\mathbb{Z} \times T_{m-1|0}$, say, $\tilde{L}(a) = a \times L(\lambda_a)$ and $\tilde{L}(c) = c \times L(\lambda_c)$. Their tensor product is given by the Littlewood-Richardson-rule
\[ \tilde{L}(a) \otimes \tilde{L}(c) = \bigoplus_{\nu} (a + c) \times c_{\lambda_a, \lambda_c}^{\nu} L(\nu). \]
Note that not only $DS : T_{m|1} \to T_{m-1|0}$ is a tensor functor, but that also the induced functor $DS : T_{m|1} \to \mathbb{Z} \times T_{m-1|0}$ is compatible with the tensor product by the Kunneth formula for the cohomology using that the cohomology of every indecomposable object is concentrated in one degree. The fiber of an element
a \times L(\lambda_a) under DS consists of a) the irreducible representation \(L(a)\), b) the roof module \(R(\alpha, \ldots, a)\) for any \(\alpha < a\) and c) the bottom modules \(B(a, \ldots, b)\) for any \(b > a\). Under the tensor functor \(DS_\sigma\), the two indecomposable elements map again to two indecomposable objects in \(\mathbb{Z} \times T_{m-1}\). Now \(DS_\sigma\) agrees with \(DS\) on irreducible representations, and on the indecomposable modules \(R\) and \(B\) the two functors differ by a \(\mathbb{Z}\)-shift: The representation \(B(a, \ldots, a)\) maps to \(\tilde{L}(\alpha) = a \times L(\lambda_a)\) and \(B(c, \ldots, \gamma)\) to \(\tilde{L}(\gamma) = \gamma \times L(\lambda_\gamma)\). Their tensor product is given by the Littlewood-Richardson-rule

\[
\tilde{L}(\alpha) \otimes \tilde{L}(\gamma) = \bigoplus_\nu (\alpha + \gamma) \times c_{\nu, \lambda_a, \lambda_\gamma}(\nu).
\]

Introduce \(\Delta = \alpha - a + \beta - b\). The tensor product in the two quotients differs then by a twist with \(t^\Delta\). We look at the fibre under \(DS_\sigma\) of an irreducible summand in this tensor product and compare it to the fibre under \(DS\). We search for common elements in the fibre of \(DS\) and \(DS_\sigma\): The possible bottoms which may appear in the tensor product are of the form \(B(a, \ldots, a_i)\) for varying \(a_i\) and the possible roofs of the form \(R(b_i, \ldots, a + \Delta)\) for varying \(b_i\). A possible equality happens only in the case \(b_i = a, a_i = a + \Delta\). Roofs can never appear in the tensor product: The possible roofs are of the form \(R(c_i, \ldots, a)\) (varying \(c_i\)) from \(DS\) and \(R(a + \Delta, \ldots, d_i)\) (varying \(d_i\)) from \(DS_\sigma\). Since \(\Delta > 0\) these can never be equal. The \(\Delta = 0\) case is easy, in that case both tensor products in the two quotients are equal, and hence neither roofs nor bottoms may appear in the tensor product. The same argument works in the case of the tensor product of two roof modules with a negative \(\Delta\). In the case of a tensor product of a roof with a bottom module \(R(a, \ldots, b) \otimes B(c, \ldots, d)\) one has roofs in the tensor product for \(\Delta < 0\), Bottoms for \(\Delta > 0\) and an irreducible element for \(\Delta = 0\). \(\square\)

4.8. The pro-reductive envelope. In this section we prove the following theorem.

4.13 Theorem. The quotient \(T_{m|1}/\mathcal{N}\) is equivalent as a tensor category to

\[
T_{m|1}/\mathcal{N} \simeq \text{Rep}(\text{Gl}(m-1) \times \text{Gl}(1) \times \text{Gl}(1)) \otimes \text{svec}_k.
\]

Proof. For convenience we replace every object by a parity shift so that its superdimension is positive. Then we have to define a functor \(\rho : T_{m|1}/\mathcal{N} \to \text{Rep}(\text{Gl}(m-1) \times \text{Gl}(1) \times \text{Gl}(1))\). Any atypical irreducible \(L(a)\) can be written as \(L(a) \cong \text{Ber}^{a_B} \otimes R(a^L, a^R)\) for some Berezin power and some mixed tensor attached to the bipartition \((a^L, a^R)\). The tensor product decomposition above forces the following Ansatz:

\[
\rho(L(a)) = 1 \times t^{a_B} \times L(\text{wt}(\lambda)).
\]

To each weight \(a\) in our block (with irreducible module \(L(a) = \text{Ber}^{a_B} \otimes R(a^L, a^R)\)) are attached the indecomposable modules

\[
R(a, \ldots, a-s), B(a, \ldots, a+r) \quad \text{for some } r, s \geq 0.
\]

For these we make the Ansatz

\[
\rho(R(a, \ldots, a-s)) = t^{-s} \otimes t^{a_B} \otimes L(\text{wt}(a^L, a^R))
\]

\[
\rho(B(a, \ldots, a+r)) = t^{-r} \otimes t^{a_B} \otimes L(\text{wt}(a^L, a^R)).
\]

Note that the Hom-spaces between the irreducible elements are either zero or one-dimensional since Schur’s lemma holds in any semisimple tensor category, hence the
functor is trivially defined for the morphisms. Our results on the tensor product decomposition above and on the image in $\mathcal{A}/\mathcal{N}$ show that $\rho$ is a tensor functor. It is clearly one-to-one on objects and fully faithful. This proves the theorem. □

4.9. The special linear case. In the $Sl(m|1)$-case ($m \geq 3$) $Ber$ is trivial and does not generate an extra $Gl(1)$-factor in the subgroup of $G^{red}$ generated by the irreducible representations. Otherwise the discussion is the same. Using $Gl(m) \cong Sl(m) \times Gl(1)$ for $n \geq 2$ we get

$$G^{red} = \begin{cases} Sl(m-1) \times Gl(1) \times Gl(1) & \text{for } m \geq 3 \\ Gl(1) \times Gl(1) & \text{for } m = 2. \end{cases}$$

Remark. By [VdJHKTM90] the character formula for an atypical representation of $Gl(m|1)$ has the form of a character formula of $Gl(m-1)$. Hence the superdimension of any irreducible representation equals up to a $(-1)^{p(\lambda)}$ factor the dimension of an irreducible $Gl(m-1)$-representation. Our results give a conceptual explanation for this.

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