On some problems of Harju concerning squarefree arithmetic progressions in infinite words

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December 6, 2018

Abstract
In a recent paper, Harju posed three open problems concerning square-free arithmetic progressions in infinite words. In this note we solve two of them.

1 Introduction
The study of infinite words avoiding squares is a classical problem in combinatorics on words. A square is a word of the form \( xx \), such as tartar. One of the most well-studied squarefree words \([10, 6]\) is the word \( \text{vtm} = 012021012102012 \cdots \) obtained by iterating the map \( 0 \rightarrow 012, 1 \rightarrow 02, 2 \rightarrow 1 \).

Harju \([7]\) studied the following question and showed that it has a positive solution for all \( k \geq 3 \):

Given \( k \), does there exist an infinite squarefree sequence \((w_n)_{n \geq 0}\) over a ternary alphabet such that the subsequence \((w_{kn})_{n \geq 0}\) is squarefree?

Carpi \([3]\), Currie and Simpson \([5]\), and Kao et al. \([8]\) also studied similar problems. Harju ended his paper with three open problems:

1. Does there exist a squarefree sequence \((w_n)_{n \geq 0}\) over a ternary alphabet such that for every \( k \geq 3 \), the subsequence \((w_{kn})_{n \geq 0}\) contains a square?

2. Do there exist pairs \((k, \ell)\) of relatively prime integers such that there exists a squarefree sequence \((w_n)_{n \geq 0}\) over a ternary alphabet for which both \((w_{kn})_{n \geq 0}\) and \((w_{\ell n})_{n \geq 0}\) are squarefree?

*The authors are supported by NSERC Discovery Grants 03901-2017 (Currie) and 418646-2012 (Rampersad).
3. It is true that for all squarefree words \((w_n)_{n\geq 0}\) over a ternary alphabet, there exists a word \((v_n)_{n\geq 0}\) and an integer \(k \geq 3\) such that \((v_{kn})_{n\geq 0} = (w_n)_{n\geq 0}\)?

In this note we show that the word \(vtm\) gives a positive answer to the first problem. We also show that there a positive answer to the third problem for every \(k \geq 23\).

2 The main results

We recall that \(vtm = (v_n)_{n\geq 0}\) is the fixed point (starting with 0) of the morphism that maps 0 \(\rightarrow\) 012, 1 \(\rightarrow\) 02, 2 \(\rightarrow\) 1.

**Theorem 1.** For each \(k \geq 2\) the sequence \((v_{kn})_{n\geq 0}\) contains either the square 00 or the square 22.

**Proof.** The first part of the proof relies on the fact that \(vtm\) is a 2-automatic sequence. Berstel \([1]\) studied several different ways to generate the sequence \(vtm\); in particular, he showed that \(vtm\) is generated by the 2-DFAO (deterministic finite automaton with output) in Figure 1. The automaton takes the binary representation of \(n\) as input, and if the computation ends in a state labeled \(a\), the automaton outputs \(a\), indicating that \(v_n = a\).

![Figure 1: 2-DFAO for vtm](image)

Since \(vtm\) is an automatic sequence, we can use Walnut \([9]\) to verify that it has certain combinatorial properties. We verify with Walnut that for every \(k \geq 2\), the sequence \(vtm\) contains a length \(k + 1\) factor of the form 0\(u\)0 or 2\(u\)2. The Walnut command to do this is:

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eval same_first_last "Ei (VTM[i]=0 & VTM[i+k]=0) | (VTM[i]=2 & VTM[i+k]=2)";
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The Walnut output for this command is the automaton in Figure 2 which shows that the given predicate holds for all \(k \geq 2\).

To complete the proof, it suffices to show that \(vtm\) contains an occurrence of the length \(k + 1\) factor 0\(u\)0 or 2\(u\)2 at a position congruent to 0 modulo \(k\). Blanchet-Sadri et al. \([2]\) Theorem 3 showed that for each odd \(k\) and each factor \(w\) of \(vtm\), the set of positions at which \(w\) occurs in \(vtm\) contains all congruence classes modulo \(k\), including, in particular, 0 modulo \(k\). This establishes the claim for all odd \(k \geq 2\).
If $k$ is even, write $k = 2^\ell k'$, where $k'$ is odd. Suppose that $k' \geq 3$. We have already seen that $\text{vtm}$ contains an occurrence of a length $h' + 1$ factor $0u0$ or $2u2$ at a position $i \equiv 0 \pmod{k'}$. From the automaton generating $\text{vtm}$, we see that if $v_i = 0$ (resp. $v_i = 2$), then $v_{2\ell+i} = 0$ (resp. $v_{2\ell+i} = 2$), which establishes the claim for $k' \geq 3$.

Finally, suppose that $k$ is a power of 2. Then the binary representations of $k$ and $2k$ have the form $10^\ell$ and $10^\ell + 1$ respectively, for some $\ell \geq 1$. From the automaton generating $\text{vtm}$, we see that $v_k = v_{2k} = 2$, as required. This completes the proof.

This resolves Harju’s first problem in the affirmative. For the third, we use the result of Currie [4] (also referenced in [7]), who showed that for $k \geq 23$ there exists a cyclic square-free $k$-uniform morphism on the ternary alphabet. Let $h_k : \{0, 1, 2\}^* \to \{0, 1, 2\}^*$ denote such a morphism. By cyclic we mean that if $\sigma$ denotes the morphism defined by the cyclic permutation of the alphabet $0 \to 1 \to 2 \to 0$, then for $i \in \{0, 1, 2\}$ we have $h_k(i) = \sigma^i(h_k(0))$. By $k$-uniform we mean that for $i \in \{0, 1, 2\}$ the images $h_k(i)$ all have length $k$. Finally, we say that the morphism $h_k$ is squarefree if it maps squarefree words to squarefree words. Without loss of generality, suppose that $h_k(i)$ begins with the letter $i$. Now if $w = (w_n)_{n \geq 0}$ is a squarefree word, then $v = h_k(w)$ is also squarefree and moreover $(v_{kn})_{n \geq 0} = (w_n)_{n \geq 0}$, as required. This resolves Harju’s third problem, and furthermore, shows that solutions exist for every $k \geq 23$.

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