KMS STATES, ENTROPY AND THE VARIATIONAL PRINCIPLE IN FULL C*-DYNAMICAL SYSTEMS

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Abstract. To any periodic and full C*-dynamical system \((A, \alpha, \mathbb{R})\), an invertible operator \(s\) acting on the Banach space of trace functionals of the fixed point algebra is canonically associated. KMS states correspond to positive eigenvectors of \(s\). A Perron–Frobenius type theorem asserts the existence of KMS states at inverse temperatures equal the logarithms of the inner and outer spectral radii of \(s\) (extremal KMS states). Examples arising from subshifts in symbolic dynamics, self-similar sets in fractal geometry and noncommutative metric spaces are discussed.

Certain subshifts are naturally associated to the system, and criteria for the equality of their topological entropy and inverse temperatures of extremal KMS states are given.

Unital completely positive maps \(\sigma_{\{x_j\}}\) implemented by partitions of unity \(\{x_j\}\) of grade 1 are considered, resembling the ‘canonical endomorphism’ of the Cuntz algebras. The relationship between the Voiculescu topological entropy of \(\sigma_{\{x_j\}}\) and the topological entropy of the associated subshift is studied. Examples where the equality holds are discussed among Matsumoto algebras associated to non finite type subshifts. In the general case \(h(\sigma_{\{x_j\}})\) is bounded by the sum of the entropy of the subshift and a suitable entropic quantity of the homogeneous subalgebra. Both summands are necessary.

The measure–theoretic entropy of \(\sigma_{\{x_j\}}\), in the sense of Connes–Narnhofer–Thirring, is compared to the classical measure–theoretic entropy of the subshift. A noncommutative analogue of the classical variational principle for the entropy is obtained for the ‘canonical endomorphism’ of certain Matsumoto algebras. More generally, a necessary condition is discussed. In the case of Cuntz–Krieger algebras an explicit construction of the state with maximal entropy from the unique KMS state is done.

¹On leave of absence from Dipartimento di Matematica, Università di Roma Tor Vergata, 00133 Roma. Supported by the EU and NATO–CNR.

²Supported by the Grant–in–aid for Scientific Research of JSPS.
Let $\mathcal{A}$ be a unital $C^*$–algebra endowed with a $2\pi$–periodic automorphic action $\alpha$ of $\mathbb{R}$. In algebraic statistical mechanics elements of $\mathcal{A}$ represent kinematic observables of an infinite quantum system, and $\alpha$ is the time evolution of the system. Equilibrium states of the system are states on $\mathcal{A}$ which satisfy the KMS condition with respect to $\alpha$.

We shall assume throughout the paper that $\alpha$ is $2\pi$–periodic, so it factors through an action $\gamma$ of the circle $\mathbb{T}$, and also that $\gamma$ is full, in the following sense. Let $\mathcal{A}^k$ be the spectral subspace of elements $a \in \mathcal{A}$ such that $\gamma_z (a) = z^k a$. We assume that for all $k \in \mathbb{Z}$, the closed linear span of $\{xy, x \in \mathcal{A}^k, y \in \mathcal{A}^{-k}\}$ is the fixed point algebra $\mathcal{A}^0$. An example is given by a crossed product $C^*$–algebra $\mathcal{A} = \mathcal{B} \rtimes \beta \mathbb{Z}$ by a single automorphism $\beta$, endowed with the dual action $\gamma = \hat{\beta}$. All KMS states on $\mathcal{A}$ are tracial, and are given by $\mathbb{T}$–invariant extensions to $\mathcal{A}$ of $\beta$–invariant tracial states on $\mathcal{B}$.

More generally, if $(\mathcal{A}, \gamma)$ is not a dual $C^*$–dynamical system, nontracial KMS states arise. Interesting examples are, in increasing generality, the Cuntz–Krieger algebras [CK], the Matsumoto algebras associated with a subshift [M], and the Pimsner algebras associated with a full finite projective Hilbert $C^*$–bimodule [P], all endowed with the canonical gauge action. While KMS states for the first two classes of $C^*$–algebras are now well understood (see [MFW], [E], [MYW]), the third class of $C^*$–algebras is the main motivation of the present paper. (We shall see that Pimsner $C^*$–algebras are in fact a typical example, in the sense that any unital, full and periodic $C^*$–dynamical system is isomorphic to a system constituted by a Pimsner $C^*$–algebra associated to a full, finite projective Hilbert bimodule, though not unique, and its canonical gauge action.)

We point out that, on the other hand, various authors, regarding the Cuntz–Krieger algebras as examples of noncommutative topological dynamical systems, have computed the Voiculescu topological entropy [V] of the so–called ‘canonical endomorphism’ $\sigma$ ([Ch], [BG]). However, among these, to the authors’ knowledge, it is only for the case of the Cuntz algebras $O_d$ that a close relationship is known between KMS states, Voiculescu’s topological entropy and measure–theoretic entropy in the sense of [CNT], see [Ch]. Choda’s result states that the CNT entropy of $\sigma$ computed with respect to the unique KMS state equals the topological entropy of $\sigma$, which is, in turn, $\log(d)$. This result can be regarded as a pivotal example of noncommutative dynamical system for which a variational principle for the entropy holds. Our ultimate goal is that of investigating the variational principle for the entropy and its relationship with KMS states, in full periodic $C^*$–dynamical systems.

KMS states for $2\pi$–periodic actions have already been considered in the literature by several authors. Olesen and Pedersen gave in [OPI] an existence and uniqueness theorem for KMS states of the Cuntz algebras. This was generalized to the case of Cuntz–Krieger algebras by Enomoto, Fuji and the second–named author in [EFW] and by Evans in [E].

In [BEH] Bratteli, Elliott and Herman construct, for any closed subset $F$ of the extended real line, a simple $C^*$–algebra $\mathcal{A}_F$ endowed with a $2\pi$–periodic one–parameter group, for which $F$ is precisely the set of inverse temperatures of KMS states, and such that for each $\beta \in F$, $\mathcal{A}_F$ has a unique KMS state at
inverse temperature $\beta$. We remark that if $F \subset \mathbb{R}$, then the $\mathbb{T}$–action is full, and if $F \subset (0, +\infty)$, $\mathcal{A}_F$ is purely infinite (cf. section 2). In a subsequent paper [BEK] Bratteli, Elliott and Kishimoto show that even the set of KMS states with a specified inverse temperature can be fairly arbitrary.

In a recent paper [MWY] Matsumoto, Yoshida and the second–named author study KMS states for the Matsumoto $C^*$-algebras associated with a subshift in symbolic dynamics. They develop a Perron–Frobenius type theorem for a suitable positive operator naturally acting on a certain subalgebra, and show that the logarithm of its spectral radius arises as the inverse temperature of some KMS state. Furthermore they show a connection with the topological entropy of the underlying subshift.

Our approach is close to that of [MWY], in that we emphasize the Perron–Frobenius theory. The starting point is that to any periodic and full $C^*$–dynamical system we associate certain completely positive maps on the underlying $C^*$–algebra, which we interpret as being Perron–Frobenius type operators. KMS states correspond then to the positively scaled tracial states on the fixed point algebra. We study the problem of existence of KMS states, thus proving a Perron–Frobenius type theorem, and the relationship with the variational principle in ergodic theory.

The paper is organized as follows. In the first section choose finite subsets $\{y_i\}$ and $\{x_j\}$ of $\mathcal{A}^1$ such that $\sum_j y_j^* y_j = I$ and $\sum_i x_i x_i^* = I$. Such multiplets exist because the group action is full. They can be regarded as playing the role of the canonical unitary in $\mathcal{B} \rtimes_\beta \mathbb{Z}$ implementing $\beta$. We then consider two completely positive (cp) maps: $T_{\{y_i\}} : T \rightarrow \sum_i y_i T y_i^*$ and $S_{\{x_j\}} : T \rightarrow \sum_j x_j^* T x_j$ on $\mathcal{A}^0$, and also, by transposition, operators $t'$ and $s'$ which are inverses of one another on the Banach space of trace functionals on $\mathcal{A}^0$. These operators are independent of the choice of the multiplets $\{y_i\}$ and $\{x_j\}$. KMS states for the system at finite inverse temperatures then correspond to tracial states on $\mathcal{A}^0$ which are positively scaled by those cp maps, or, equivalently, to tracial state eigenvectors of $s'$. In the next section we show that, under the necessary condition that the fixed point algebra has a tracial state, the inner and outer spectral radii of $s'$ correspond to inverse temperatures of ‘minimal’ and ‘maximal’ KMS states (see Theorem 2.5 and Corollary 2.6). This can be regarded as a Perron–Frobenius theorem. The key point in the proof is that one needs to consider the trace functionals of the enveloping von Neumann algebra of $\mathcal{A}^0$, endowed with its order structure.

In sections 3–5 we discuss some examples. In section 3 we apply our results to the Pimsner $C^*$–algebras generated by finite projective Hilbert bimodules, and we thus deduce a criterion for existence of KMS states which applies, in particular, to the case where the coefficient algebra is simple, unital and has a tracial state.

In section 4 we construct Hilbert bimodules, and hence full $C^*$–dynamical systems, via Pimsner’s construction, naturally arising from two different situations: subshifts of symbolic dynamics and self–similar sets in fractal geometry. In both cases the coefficient algebra is commutative, and the corresponding Hilbert bimodules are described by a finite set of endomorphisms. We show that in the former situation Pimsner’s construction yields the Matsumoto $C^*$–
algebras, while in the latter one gets a genuine Cuntz algebra. We also discuss
a generalization of the latter example to noncommutative metric spaces intro-
duced by Connes [Co]. It is interesting to compare our discussion with the
papers by Jørgensen-Pedersen [JP] and Bratteli-Jørgensen [BJ], where the au-
thors consider a relationship between Cuntz algebras and multiresolutions in
wavelet and fractal analysis.

In section 5 we look more closely at the subclass of the so called Cuntz–
Krieger bimodules (and the corresponding $C^*$–algebras). These are bimodules
for which the coefficient algebra is a finite direct sum of unital simple $C^*$–
algebras. The leading and simplest example is, of course, that of Cuntz–Krieger
algebras, where each summand algebra is a copy of the complex numbers. We
show in particular that if each of the summands has a unique trace and the
defining $\{0,1\}$–matrix $A$ is irreducible then the associated Pimsner $C^*$–algebra
has a unique KMS state at inverse temperature $\log(r(A))$, where $r(A)$ is the
spectral radius of $A$.

In the next section we associate to each pair $(\{y_i\}, \{x_j\})$ of finite subsets
of $A^1$ as above, a pair of one–sided subshifts, $(\ell_{\{x_j\}}, \ell'_{\{y_i\}})$, which roughly
correspond to the operator $s'$ and its inverse $t'$. We show that, under certain
conditions, the topological entropies of these subshifts are precisely the mini-
mal and maximal inverse temperatures of KMS states. Furthermore we give a
criterion for approximating such extremal temperatures with arbitrary (a priori
non KMS) tracial states satisfying suitable conditions.

In section 7 we introduce a ucp map $\sigma_{\{x_j\}} : T \to \sum_j x_j T x_j^*$ implemented
by a multiplet $\{x_j\}$ of grade 1 as above, which should be compared with the
map $S_{\{x_j\}}$. The main result of this section is the estimate

$$ht(\sigma_{\{x_j\}}) \leq h_{\text{top}}(\ell_{\{x_j\}}) + ht(\{\phi_{x_i,x_j}\})$$

where the l.h.s. is the Brown–Voiculescu topological entropy [B], [V] of $\sigma_{\{x_j\}}$
and the second summand at the r.h.s. is the topological entropy, suitable de-
defined, of the set of contractions $\phi_{x_i,x_j} : T \to x_i^* T x_j$ of the homogeneous $C^*$–
subalgebra $A^0$. Both summands at the r.h.s. of this inequality are necessary.
Indeed when $A = A^0 \rtimes_\alpha \mathbb{Z}$ then the associated subshift is trivial so its entropy is
zero, and the above inequality, when combined with monotonicity of topological
entropy ([B], [V]) leads to Brown’s result

$$ht_A(\text{Ad}(u)) = ht_{A^0}(\alpha),$$

where $u \in A$ is a unitary implementing $\alpha$ [B]. Another extreme case is that of
the Cuntz–Krieger algebras $O_A$. Now the second summand vanishes and the
previous estimate yields the result by Boca and Goldstein [BG] that $ht(\sigma_{\{x_j\}}) =
ht(\ell_{\{x_j\}}) = \log(r(A))$, see Corollary 7.8.

We then focus our attention on those algebras for which $ht(\{\phi_{x_i,x_j}\}) = 0$
and we show that if the $x_j$’s have pairwise orthogonal ranges

$$ht(\sigma_{\{x_j\}}) = h_{\text{top}}(\ell_{\{x_j\}}).$$

We next discuss new examples of this occurrence among Matsumoto algebras
[M] associated to a subshift.
Our assumption of orthogonality introduces a certain trivialization to the classical situation. In fact, in this case the algebra of continuous functions on the one–sided subshift \( \ell_{\{x_j\}} \), together with the endomorphism induced by the left shift epimorphism, sits naturally inside the noncommutative dynamical system \((\mathcal{A}, \sigma_{\{x_j\}})\). Therefore monotonicity of topological entropy implies that the topological entropy of \( \ell_{\{x_j\}} \) is \( \leq \) the topological entropy of the noncommutative subshift \( \sigma_{\{x_j\}} \), thus leading to the equality, see Theorem 7.7. We discuss new examples of this occurrence among the Matsumoto algebras associated to certain non finite type subshifts.

In section 8 we investigate the CNT dynamical entropy of \( \sigma_{\{x_j\}} \). We show that, under the orthogonality assumption, if \( \phi \) is a \( \sigma_{\{x_j\}} \)–invariant state of \( \mathcal{A} \) centralized by \( C(\ell_{\{x_j\}}) \) then \( h_\phi(\sigma_{\{x_j\}}) \geq h_\mu(\ell_{\{x_j\}}) \), where \( \mu \) is the shift–invariant probability measure on \( \ell_{\{x_j\}} \). We also find a condition on \( \sigma_{\{x_j\}} \) under which any such \( \mu \) arises as the restriction of some \( \phi \).

This enables us to obtain a variational principle for certain systems \((\mathcal{A}, \sigma_{\{x_j\}})\) for which \( h_{\{\phi_{x_i,x_j}\}} = 0 \). More precisely, our variational principle asserts, for those systems, the existence of \( \sigma_{\{x_j\}} \)–invariant states of \( \mathcal{A} \) with respect to which the CNT dynamical entropy equals the Voiculescu topological entropy of \( \sigma_{\{x_j\}} \).

In the last section we establish a closer relationship between KMS states and states with maximal entropy. The main point is that a KMS state \( \omega \) is to be understood as a quasi–invariant measure for the noncommutative shift \( \sigma_{\{x_j\}} \), as \( \omega \circ \sigma_{\{x_j\}} \) and \( \omega \) are equivalent. In classical ergodic theory, measures with this property are called conformal, and play an important role, as they lead to measures with maximal entropy. We thus show an explicit general way of constructing \( \sigma_{\{x_j\}} \)–invariant measures from KMS states. We then consider basic examples, which we may think of as being noncommutative Markov shifts: systems \((\mathcal{A}, \gamma)\) containing some Cuntz–Krieger algebra \( \mathcal{O}_A \) in a way that \( \gamma \) restricts to the canonical gauge action on \( \mathcal{O}_A \). We show that the \( \sigma_{\{x_j\}} \)–invariant state \( \phi \) previously derived from a KMS state with maximal entropy restricts, on the algebra of continuous functions on the classical Markov subshift \( \ell_A \), to the unique invariant measure \( \mu \) with maximal entropy. We thus conclude that if \( h_{\{\phi_{x_i,x_j}\}} = 0 \),

\[
\begin{align*}
    h_\mu(\ell_A) &= h_\phi(\sigma_{\{x_j\}}) = h_{\text{top}}(\sigma_{\{x_j\}}) = h_{\text{top}}(\ell_A) = \log(r(A)).
\end{align*}
\]

This yields a generalization of Choda’s result [Ch] to the Cuntz–Krieger algebras, and Matsumoto algebras associated to certain non finite type subshifts.

1. The scaling property

Recall that a state \( \omega \) over a \( C^* \)–algebra \( \mathcal{A} \) endowed with a one–parameter automorphism group \( \alpha \) is called a KMS state at inverse temperature \( \beta \in \mathbb{R} \) if

\[
    \omega(a_\alpha(b)) = \omega(ba),
\]

for all \( a, b \) in a dense \( * \)–subalgebra of \( \mathcal{A}_\alpha \), the set of entire elements for \( \alpha \) (which is in fact a dense \( * \)–subalgebra).

We will only consider \( 2\pi \)–periodic one–parameter groups, i.e. groups for which \( \alpha \) comes from an action \( \gamma \) of \( \mathbb{T} \) by \( \alpha_t := \gamma_{e^{it}} \). Furthermore, in view of
applications to the algebras generated by Hilbert bimodules, we will assume that $\mathcal{A}$ is unital and that the group action is full, in the sense explained in the introduction. Then we note that the spectral subspace $\mathcal{A}^k$, for positive $k$, is in fact the linear span of the product set of $k$ copies of $\mathcal{A}^1$. Moreover, by definition of full $C^*$-dynamical system, there exist, for all $n \in \mathbb{N}$, finite subsets $\{y_i\}$ and $\{x_j\}$ of $\mathcal{A}^n$ such that $\sum_i y_i^* y_i = I$ and $\sum_j x_j x_j^* = I$. We define correspondingly, for any tracial state $\tau$ on $\mathcal{A}^0$,

$$\delta_n(\tau) := \tau(\sum_i y_i y_i^*)$$

$$\epsilon_n(\tau) := \tau(\sum_j x_j^* x_j).$$

We shall usually write $\delta(\tau)$ and $\epsilon(\tau)$ for $\delta_1(\tau)$ and $\epsilon_1(\tau)$ respectively.

1.1. Lemma Let $(\mathcal{A}, \gamma, \mathbb{T})$ be a full $C^*$-dynamical system, with $\mathcal{A}$ unital, and let $\tau$ be a tracial state on $\mathcal{A}^0$. Then $\delta_n(\tau)$ and $\epsilon_n(\tau)$ do not depend on the finite subsets $\{y_i\}$ and $\{x_j\}$ of $\mathcal{A}^n$ satisfying the above relations. If in particular $\{y_i\}$, $\{x_j\} \subset \mathcal{A}^1$, one has, for all $n \in \mathbb{N}$,

$$\|\sum (x_{j_1} \ldots x_{j_n})^* x_{j_1} \ldots x_{j_n}\|^{-1/n} \leq \delta_n(\tau)^{1/n} \leq \|\sum y_{i_1} \ldots y_{i_n} (y_{i_1} \ldots y_{i_n})^*\|^{1/n}$$

$$\|\sum y_{i_1} \ldots y_{i_n} (y_{i_1} \ldots y_{i_n})^*\|^{-1/n} \leq \epsilon_n(\tau)^{1/n} \leq \|\sum (x_{j_1} \ldots x_{j_n})^* x_{j_1} \ldots x_{j_n}\|^{1/n}$$

Proof We shall only prove the statements relative to $\{y_i\}$, those relative to $\{x_j\}$ can be proved similarly. Let $\{z_1, \ldots, z_q\} \subset \mathcal{A}^n$ be another multiplet satisfying $\sum_k z_k^* z_k = I$, and write $\sum_k = \sum_i a_{k,i} y_i$, where $a_{k,i} := z_k y_i^* \in \mathcal{A}^0$. Then

$$\tau(\sum_k z_k z_k^*) = \tau(\sum_{k,i,j} a_{k,i} y_i y_j^* a_{k,j}) = \tau(\sum_{i,j,k} y_i y_j^* a_{k,j} a_{k,i}) = \tau(\sum_{i,j,k} y_i y_j^* z_k z_k^* y_i^*).$$

Note that $\delta_n(\tau) \leq \|\sum_i y_i y_i^*\|$. Furthermore

$$1 = \tau(\sum_{i,j} x_j x_j^* y_i^* x_j^*) = \tau(\sum_{i,j} x_j x_j^* y_i^*) \leq \|\sum_j x_j x_j^*\| \tau(\sum_i y_i y_i^*).$$

The conclusion follows choosing subsets in $\mathcal{A}^n$ of the form $\{y_1, \ldots, y_n\}$ and $\{x_{j_1}, \ldots, x_{j_n}\}$, where $\{y_i\}, \{x_j\} \subset \mathcal{A}^1$ and satisfy $\sum_i y_i^* y_i = I$ and $\sum_j x_j x_j^* = I$.

Let, for $\lambda > 0$, $\mathcal{T}\mathcal{S}_\lambda$ be the set of tracial states $\tau$ on $\mathcal{A}^0$ for which

$$\lambda \tau(x^* y) = \tau(y^* x), \quad x, y \in \mathcal{A}^1. \quad (1.2)$$

We shall show that a state of $\mathcal{A}$ satisfies the KMS condition w.r.t. $\alpha$ if and only if its restriction to $\mathcal{A}^0$ is an element of some $\mathcal{T}\mathcal{S}_\lambda$. We start with the following characterization of $\mathcal{T}\mathcal{S}_\lambda$. 
1.2. Lemma Let \( (\mathcal{A}, \gamma, \mathbb{T}) \) be a full \( C^* \)-dynamical system over a unital \( C^* \)-algebra. For a tracial state \( \tau \) on \( \mathcal{A}^0 \) and \( \lambda > 0 \), the following conditions are equivalent:

1. \( \tau \in T \mathcal{S}_\lambda \),
2. \( \tau(\sum_i y_i a y_i^*) = \lambda^{-1} \tau(a), \ a \in \mathcal{A}^0 \),
3. \( \tau(\sum_j x_j^* a x_j) = \lambda \tau(a), \ a \in \mathcal{A}^0 \).

Here \( \{y_i\} \) and \( \{x_j\} \) are finite subsets of \( \mathcal{A}^1 \) satisfying respectively

\[
\sum_i y_i^* y_i = I, \quad \sum_j x_j x_j^* = I.
\]

\( \lambda \) is uniquely determined by \( \tau \): \( \lambda = \epsilon(\tau) = \delta(\tau)^{-1} \).

Proof (1) \( \rightarrow \) (2) and (1) \( \rightarrow \) (3) are obvious. We show that (2) \( \rightarrow \) (1): for \( x, y \in \mathcal{A}^1 \), \( y^* x \in \mathcal{A}^0 \), so

\[
\tau(y^* x) = \lambda \tau(\sum_i y_i y_i^* x y_i^*) = \lambda \tau(\sum_i x y_i y_i^*) = \lambda \tau(x y^*).
\]

One similarly proves that (3) \( \rightarrow \) (1).

The following result characterizes the set of tracial states on \( \mathcal{A}^0 \) which gives rise to KMS states for \( (\mathcal{A}, \gamma) \). Let \( F_0 : \mathcal{A} \rightarrow \mathcal{A}^0 \) denote the projection onto the fixed point algebra obtained overaging over the circle group action.

1.3. Proposition The maps \( \omega \rightarrow \omega |_{\mathcal{A}^0}, \tau \rightarrow \tau \circ F_0 \) set up a bijective correspondence between the set of KMS states \( \omega \) for \( (\mathcal{A}, \gamma) \) at inverse temperature \( \beta \) and the set \( T \mathcal{S}_{\omega, \beta} \).

Proof If \( \omega \) is a KMS state at inverse temperature \( \beta \) then the KMS condition (1.1) can be formulated, equivalently, for any pair \( x, y \) in the dense linear span of the \( \mathcal{A}^k \)'s. Therefore the restriction \( \tau \) of \( \omega \) to \( \mathcal{A}^0 \) is a tracial state such that \( \omega = \tau \circ F_0 \) since \( \omega \) is \( \gamma \)-invariant. Furthermore, if \( x, y \in \mathcal{A}^1 \),

\[
\omega(y^* x) = \omega(x \alpha_{i, \beta}(y^*)) = e^{\beta} \omega(xy^*),
\]

therefore \( \tau \in T \mathcal{S}_{\omega, \beta} \). Conversely, if this condition is satisfied by some tracial state \( \tau \) on \( \mathcal{A}^0 \), then \( \omega := \tau \circ F_0 \) is an extension of \( \tau \) to a state on \( \mathcal{A} \), and it is not difficult to check that \( \omega(y^* x) = \omega(x \alpha_{i, \beta}(y^*)) \) for \( x, y \in \mathcal{A}^1 \), and hence inductively for \( x, y \in \mathcal{A}^1 \ldots \mathcal{A}^k = \mathcal{A}^k \). If \( x \in \mathcal{A}^k, y \in \mathcal{A}^k, h \neq k \) then \( \omega(y^* x) = 0 = \omega(x \alpha_{i, \beta}(y^*)) \), and the proof is complete.

Remark A simple argument shows that the sequences \( a_n := \inf \{ \epsilon_n(\tau), \tau \in T \mathcal{S}(\mathcal{A}^0) \} \) and \( b_n := \sup \{ \epsilon_n(\tau), \tau \in T \mathcal{S}(\mathcal{A}^0) \} \) are respectively supermultiplicative and submultiplicative, therefore the sequences \( a_n^{1/n}, b_n^{1/n} \) converge, and \( \lim_n a_n^{1/n} = \sup a_n^{1/n} \) and \( \lim_n b_n^{1/n} = \inf b_n^{1/n} \).

1.4. Corollary If

\[
(1) \lim_n \inf \{ \epsilon_n(\tau), \tau \in T \mathcal{S}(\mathcal{A}^0) \}^{1/n} > 1 \quad (\text{e.g.} \quad \inf \{ \epsilon(\tau), \tau \in T \mathcal{S}(\mathcal{A}^0) \} > 1)
\]

then every KMS state on \( (\mathcal{A}, \gamma) \) has positive inverse temperature,
(2) \( \lim_n \sup \{ \epsilon_n(\tau), \tau \in TS(A^0) \}^{1/n} < 1 \) (e.g. \( \sup \{ \epsilon(\tau), \tau \in TS(A^0) \} < 1 \))
then every KMS state on \((A, \gamma)\) has negative inverse temperature,
(3) \( \lim_n \inf \{ \epsilon_n(\tau), \tau \in TS(A^0) \}^{1/n} = \lim_n \sup \{ \epsilon_n(\tau), \tau \in TS(A^0) \}^{1/n} = 1 \)
(e.g. \( \epsilon(\tau) = 1 \) for all \( \tau \in TS(A^0) \)) then every KMS state on \((A, \gamma)\) is tracial.

We next give a criterion of faithfulness for KMS states.

1.5. Proposition Let \((A, \gamma)\) be a full periodic \(C^*\)-dynamical system with \(A\) unital, and consider the \(\ast\)-monomomorphism \(\alpha : A^0 \rightarrow M_p(A^0)\) associated to
a set \(\{y_i\}_{i=1}^p\) of \(A^1\) such that \(\sum_{i=1}^p y_i^* y_i = I\) and defined by \(\alpha(a) = (y_i a y_j^*)\).
If \(A^0\) has no proper closed ideal \(I\) such that \(\alpha(A^0) \cap M_p(I) = \alpha(I)\) (e.g. \(A\) is simple),
then any KMS state of \((A, \gamma)\) is faithful.

Proof Let \(\tau\) be the restriction of a KMS state \(\omega\) to \(A^0\). Then
\[
I := \{ a \in A^0 : \tau(a^* a) = 0 \}
\]
is a closed ideal of \(A^0\). Since for \(x, y \in A^1\), \(a \in A^0\),
\[
(xay^*)^*(xay^*) \leq \|x\|^2 y a^* y^*,
\]
we have, by (1.2), that \(xay^* \in I\) if \(a \in I\). This yields \(\alpha(I) \subset M_p(I) \cap \alpha(A^0)\).
We show the reverse inclusion. Let \(a \in A^0\) be such that \(y_i a y_j^* \in I\), \(i, j = 1, \ldots, p\).
Then \(\delta(\tau) \tau(a^* y_i^* y_i a y_j^* y_j) = \tau((y_i a y_j^*)^* y_i a y_j^*) = 0\).
Hence, summing up, we see that \(a \in I\), and this shows that \(\alpha(A^0) \cap M_p(I) \subset \alpha(I)\).
It follows from our assumption that \(I = \{0\}\), as, clearly, \(I \neq A^0\). Now the canonical conditional expectation \(F_0 : A \rightarrow A^0\) is faithful, so \(\omega\) is faithful.

We conclude this section recalling from [GP] a criterion for pure infinity of unital \(C^*\)-algebras, which we shall need in the sequel. We refrain from giving here the proof. We only point out that the arguments essentially go back to Cuntz’ proof of pure infinity of \(\mathcal{O}_d\) [C]. Also, the result is a generalization of Rørdam’s result (cf. [R]) about pure infinity of crossed products by proper corner endomorphisms.

Following [R], we say that a \(C^*\)-algebra \(\mathcal{B}\) has the comparability property if \(\mathcal{B}\) has at least one tracial state, and furthermore a projection \(e \in \mathcal{B}\) is equivalent to a subprojection of \(f\) if \(\tau(e) < \tau(f)\) for all tracial states of \(\mathcal{B}\). Assume that our \(C^*\)-algebra \(A\) has a nonunitary isometry \(S\) in some \(A^n\), \(n > 0\), and also that the fixed point algebra \(A^0\) has the comparability property. Then for all tracial states \(\tau\) on \(A^0\) one has, by Lemma 1.1,
\[
\delta_n(\tau) = \tau(SS^*) < 1. \tag{1.3}
\]
(Note that, by Corollary 1.4 and the following Proposition 2.2, all KMS states have positive inverse temperatures.) It is then natural to ask under which conditions (1.3) guarantees the existence of a nonunitary isometry in \(A^n\), or, better, pure infinity of \(A\).
1.6. Theorem [GP] Let \((A, \gamma, T)\) be a full \(C^{*}\)-dynamical system, and assume that \(A^0\) is unital, simple, separable, of real rank zero, and that every \(M_n(A^0)\) has the comparability property. If, for some \(n > 0\),

\[
\sup \{\delta_n(\tau), \tau \in \mathcal{T}(A^0) \} < 1,
\]

then \(A^n\) contains a nonunitary isometry. Furthermore, \(A\) is simple and purely infinite.

2. A Perron–Frobenius theorem

We associate to each pair of finite subsets \(\{y_i\}, \{x_j\} \subset A^1\) satisfying

\[
\sum_i y_i^* y_i = I, \\
\sum_j x_j x_j^* = I,
\]

a corresponding pair of completely positive maps \(T = T_{\{y_i\}}\) and \(S = S_{\{x_j\}}\) on the homogeneous subalgebra \(A^0\):

\[
T(a) := \sum_i y_i a y_i^*, \quad a \in A^0, \\
S(a) := \sum_j x_j^* a x_j, \quad a \in A^0.
\]

Let \(T', S' : A^0^* \rightarrow A^0^*\) denote the Banach space adjoints of \(T\) and \(S\) respectively, and let \(\mathcal{T}(A^0) \subset A^0^*\) the Banach subspace of trace functionals. Then one has the following result.

2.1. Proposition For any finite subset \(\{y_i\}\) (resp. \(\{x_j\}\)) of \(A^1\) satisfying \(\sum_i y_i^* y_i = I\) (resp. \(\sum_j x_j x_j^* = I\)) the associated operator \(T'\) (resp. \(S'\)) leaves \(\mathcal{T}(A^0)\) stable. Let \(t'\) (resp. \(s'\)) be the restriction of \(T'\) to \(\mathcal{T}(A^0)\). Then \(t'\) (resp. \(s'\)) does not depend on the set \(\{y_i\}\) (resp. \(\{x_j\}\)) satisfying \(\sum_i y_i^* y_i = I\) (resp. \(\sum_j x_j x_j^* = I\)). Furthermore \(t'\) and \(s'\) are inverses of one another.

Proof It is easy to check that \(T'\) transforms trace functionals into trace functionals. Let \(\{y'_k\}\) be another finite subset of \(A^1\) such that \(\sum_k y'_k^* y'_k = I\), and write \(y'_k = \sum a_{k,i} y_i\), with \(a_{k,i} = y'_k y_i^* \in A^0\). Then for any \(\tau \in \mathcal{T}(A^0), a \in A^0\),

\[
\tau(\sum_k y'_k^* y'_k) = \sum_{k,i,j} \tau(a_{k,i} y_i a y_j^* a_{k,j}) = \sum_{i,j,k} \tau(y_i a y_j^* a_{k,j}^* a_{k,i}) = \\
\sum_{i,j} \tau(y_i a y_j^* y_j y_i^*) = \tau(\sum_i y_i a y_i^*).
\]

Finally note that

\[
t'(s' (\tau)) = s' (\tau) \circ T = \tau \circ ST = \tau, \quad \tau \in \mathcal{T}(A^0),
\]
by that trace property of \( \tau \). Likewise, \( s'(\tau) = \tau, \ \tau \in T(A^0) \).

Note that KMS states of \((A, \gamma)\) correspond precisely to the tracial state eigenvectors for \( s' \) (or \( t' \)). The following result, which has its own interest, explains why \( \delta(\tau) = \epsilon(\tau)^{-1} \) when \( \tau \) corresponds to a KMS state.

2.2. Proposition The map \( h : TS(A^0) \to TS(A^0) \) taking a tracial state \( \tau \) to \( h(\tau) = \delta(\tau)^{-1}t'(\tau) \)

is a homeomorphism of \( TS(A^0) \) endowed with the weak*–topology. KMS states of \((A, \gamma)\) correspond, as in Prop. 1.3, to fixed points of \( h \). The inverse of \( h \) is the map \( k(\tau) = \epsilon(\tau)^{-1}s'(\tau) \).

We have:

\[
\epsilon_n(h^n(\tau)) = \delta_n(\tau)^{-1}, \\
\delta_n(k^n(\tau)) = \epsilon_n(\tau)^{-1},
\]

thus if \( \tau \in T(A^0) \) is the restriction of a KMS state, \( \epsilon(\tau) = 1/\delta(\tau) \).

Proof Clearly \( h(\tau) \) is a tracial state when \( \tau \) is. Furthermore \( \tau \to \delta(\tau) \) is a positive valued continuous function on a compact set, therefore \( h \) is continuous. For the same reason \( k \) is continuous on \( TS(A^0) \), and, by the trace property, \( h \) and \( k \) are inverses of one another.

Our next aim is to look more closely at the spectrum \( \sigma(s') \) of \( s' \). The previous proposition shows that

\[ 0 \notin \sigma(s') = \sigma(t')^{-1}. \]

So we can define the inner and outer spectral radius of \( s' \):

\[
r_{\text{max}}(s') := \max \{|\lambda|, \lambda \in \sigma(s')\}, \\
r_{\text{min}}(s') := r(t')_{\text{max}}^{-1} = \min \{|\lambda|, \lambda \in \sigma(s')\}.
\]

We give some estimates for \( r_{\text{min}}(s') \) and \( r_{\text{max}}(s') \).

2.3. Proposition Let \( \{y_i\}, \{x_j\} \subset A^1 \) satisfy \( \sum_i y_i^*y_i = I \) and \( \sum_j x_jx_j^* = I \).

Then one has

\[
r_{\text{min}}(s') \geq \lim_n \| \sum_{i_1, \ldots, i_n} y_{i_1} \cdots y_{i_n} (y_{i_1} \cdots y_{i_n})^* \|^{-1/n}, \\
r_{\text{max}}(s') \leq \lim_n \| \sum_{j_1, \ldots, j_n} (x_{j_1} \cdots x_{j_n})^*x_{j_1} \cdots x_{j_n} \|^{1/n}.
\]

Proof Note that

\[
T(a)^*T(a) \leq \| \sum_i y_i^*y_i \| T(a^*a), \quad a \in A^0,
\]
which, together with $T(I) = \sum_i y_i y_i^*$, implies $\|T\| = \|\sum_i y_i y_i^*\|$, hence inductively

$$\|T^n\| = \|\sum_{i_1, \ldots, i_n} y_{i_1} \cdots y_{i_n} (y_{i_1} \cdots y_{i_n})^*\|.$$  

Taking the $n$-th root and passing to the limit, one gets the spectral radius of $T$:

$$r(T) = \lim_n \|\sum_{i_1, \ldots, i_n} y_{i_1} \cdots y_{i_n} (y_{i_1} \cdots y_{i_n})^*\|^{1/n}.$$ 

Similarly, one has

$$r(S) = \lim_n \|\sum_{j_1, \ldots, j_n} (x_{j_1} \cdots x_{j_n})^* x_{j_1} \cdots x_{j_n}\|^{1/n}.$$  

The proof is completed recalling that $s'$ is the restriction of $S'$ to a closed subspace, so

$$r_{\text{max}}(s') \leq r(S') = r(S)$$

and similarly

$$r_{\text{min}}(s') = r_{\text{max}}(t^{-1}) \geq r(T)^{-1}.$$  

We next show that the inner and outer spectral radii of $s'$ correspond to inverse temperatures of KMS states, or, in other words, that they are in the point spectrum of $s'$, with corresponding positive eigenvalues. The fact that the outer spectral radius is in the point spectrum was first proved in [MWY] for the Matsumoto $C^*$–algebras associated with subshifts [M]. The key point in our situation is that one needs to consider the trace functionals of $A^0$ endowed with the order structure which arises when we extend such traces to normal traces on the enveloping von Neumann algebra.

We anticipate the following, possibly known, lemma.

**2.4. Lemma** Let $\sum_n \phi_n$ be a series of normal linear functionals on a von Neumann algebra $M$ weakly convergent to $\phi$. If each of the absolute values $|\phi_n|$ is tracial then

$$|\phi| \leq \sum_n |\phi_n|.$$  

**Proof** Let $\tau$ be a positive tracial linear functional, then

$$|\tau(xy)| \leq \|y\|\tau(|x|).$$

(see for example [T]). Consider the polar decompositions

$$\phi_n(x) = |\phi_n|(x u_n),$$

$$\phi(x) = |\phi|(x u).$$

Then, for a positive $x$, we have

$$|\phi|(x) = \phi(x u^*) = \sum_n \phi_n(x u^*) = \sum_n |\phi_n|(x u^* u_n) \leq$$

$$\lim_{n \to \infty} \sum_n |\phi_n|(x u^* u_n) = r_{\text{max}}(s') \leq r(S') = r(S).$$
We are now in the position of proving our main result of this section.

2.5. **Theorem** Let \((A, \gamma, \mathbb{T})\) be a full C*-dynamical system, and assume that \(A\) is unital, and that \(A^0\) has a tracial state. Then \(r_{\min}(s')\) and \(r_{\max}(s')\) are eigenvalues of \(s'\) with corresponding tracial state eigenvectors.

**Proof** We first show that \(r_{\max}(s')\) is a spectral value for \(s'\) and then that it is in fact an eigenvalue with a tracial state eigenvector. A similar argument will prove that \(r(t')\) is an eigenvalue for \(t' = s'^{-1}\) with a tracial state eigenvector.

By the uniform boundedness theorem, there exists a sequence \(\{z_n\}\) of complex numbers such that \(|z_n| \to r_{\max}(s') + \lambda\) and \(\|R(z_n)\tau_0\| \to \infty\) for some \(\tau_0 \in \mathcal{T}(A^0)\), where \(R(z)\) is the resolvent of \(s'\) in \(z\). Since \(\mathcal{T}(A^0)\) is linearly spanned by its tracial states, we may assume that \(\tau_0\) is a tracial state. Consider, for \(|z| > r_{\max}(s')\), the Neumann series:

\[
R(z) = \sum_{k=0}^{\infty} z^{-(k+1)} s'^k.
\]  

(2.1)

By the previous lemma, on the enveloping von Neumann algebra of \(A^0\),

\[
|R(z)\tau_0| \leq \sum_{k=0}^{\infty} |z|^{-(k+1)} s'^k(\tau_0) = R(|z|)\tau_0,
\]

so \(\|R(|z_n|)\tau_0\| \to \infty\), and this shows that \(r_{\max}(s') \in \sigma(s')\). (2.1) also shows that \(R(\lambda)\tau_0\) is a nonzero positive functional for \(\lambda > r_{\max}(s')\), hence, arguments similar to those of Lemma 3.1 in [MWY] prove that

\[
\tau_n := \frac{1}{\|R(|z_n|)\tau_0\|} R(|z_n|)\tau_0
\]

is a sequence of tracial states such that every weak*–limit point of it is a tracial state eigenvector with eigenvalue \(r_{\max}(s')\).

The previous theorem can be considered as an analogue of the Perron–Frobenius theorem for matrices with nonnegative entries.

2.6. **Corollary** Let \((\alpha, \mathbb{R})\) be a \(2\pi\)-periodic one–parameter automorphism group of a unital C*-algebra \(A\), such that the induced \(\mathbb{T}\)–action \(\gamma\) is full. If \(s'\) is defined as above, relatively to \(\gamma\), then the set of inverse temperatures of KMS states is a closed subset of the interval \([\log(r_{\min}(s')), \log(r_{\max}(s'))]\) containing the extreme points.

**Proof** The subset of \(\mathcal{T}S(A^0)\) corresponding to KMS states is weakly*–compact by Prop. 2.2, furthermore the map \(\epsilon : \mathcal{T}S(A^0) \to \mathbb{R}^+\) defined at the beginning of section 1 is weakly*–continuous. It follows that the set of elements of the form \(\log(\epsilon(\tau))\), when \(\tau\) ranges over all tracial states on \(A^0\) corresponding to KMS states, is compact. Now this set is precisely the set of possible inverse temperatures by Prop. 1.3. The rest follows from the previous Theorem.
A KMS state of \((A, \alpha)\) at inverse temperature
\[ \beta_{\text{min}} := \log(r_{\text{min}}(s')) \]
or
\[ \beta_{\text{max}} := \log(r_{\text{max}}(s')) \]
will be called extremal. Let \(\beta\) be the inverse temperature of a KMS state, and set, as in the previous section,
\[ a_n = \inf \{ \epsilon_n(\tau), \tau \in TS(A^0) \}, \]
\[ b_n = \sup \{ \epsilon_n(\tau), \tau \in TS(A^0) \}. \]
Then for all \(n\),
\[ a_n^{1/n} \leq e^\beta \leq b_n^{1/n}, \]
so
\[ \lim_{n} 1/n \log(a_n) \leq \beta \leq \lim_{n} 1/n \log(b_n). \]
It is then natural to ask for which tracial states \(\tau\), the sequence \(1/n \log(\epsilon_n(\tau))\) approximates the maximal or the minimal inverse temperature. In section 5 we shall give a sufficient condition.

We conclude this section with the discussion of two examples known in the literature. The first example, arising from ergodic theory, shows that in general, at a fixed inverse temperature, there may be more than one KMS state.

### 2.7. Example
Let \((X, T)\) be a topological dynamical system: \(X\) is a compact metric space endowed with a homeomorphism \(T\). We suppose that \(X\) is not a finite set. Then it is well known that the \(C^*\)-algebra \(A = C(X) \rtimes_{\alpha_T} Z\) is simple if and only if \(T\) is minimal, i.e. there is no nontrivial closed subset \(F \subset X\) such that \(T(F) = F\). Here \(\alpha_T\) is the automorphism of \(C(X)\) defined by \(\alpha_T(f) = f \circ T^{-1}\). Tracial states on \(C(X) \rtimes_{\alpha_T} Z\) are in one–to–one correspondence with \(T\)–invariant probability measures on \(X\), while there is no nontracial KMS state on \((A, \gamma)\). The operator \(s'\) therefore has spectrum contained in the unit circle. However, \(s'\) is the Banach space adjoint of \(\alpha_T\), so its spectrum is the same as that of \(\alpha_T\) which must be equal to \(T\) by simplicity of \(A\) [OP]. There is an important example, due to Furstenberg, of a minimal analytic diffeomorphism \(T\) of \(T^2\) with nonunique invariant measures (see, e.g, [Ma]), which thus leads to an example of nonuniqueness of tracial states on the simple crossed product \(C^*\)-algebra \(C(T^2) \rtimes_{\alpha_T} Z\).

The next example shows that the set of inverse temperatures can in general be an arbitrary closed subset of \(\mathbb{R}\).

### 2.8. Example
In [BEH] Bratteli, Elliott and Herman construct an example of a simple \(C^*\)-algebra \(B\) endowed with \(T\)–action for which the set of possible inverse temperatures can be any arbitrary closed subset \(F \subset \mathbb{R} \cup \{\pm \infty\}\). For each temperature the corresponding state is unique. More in detail, \(B\) is obtained by cutting down the crossed product \(A \rtimes_{\alpha} Z\) of an \(AF\)–algebra by some projection \(P\) in \(A\). If neither \(\pm \infty\) nor \(-\infty\) belongs to \(F\), \(A\) itself is simple, and this implies that the \(T\)–action is full since \(P\) is a full projection. If moreover \(F \subset (0, +\infty)\), one can choose \(\alpha\) so that \(\alpha(P) < P\), see [BEH], hence \(\alpha(B^0) \subset B^0\). Then \(\rho := \alpha |_{B^0}\) is a proper corner endomorphism of \(B^0\), and one has \(B = B^0 \rtimes_{\rho} \mathbb{N}\). Now by a result of Rørdam [R], \(B\) is purely infinite.
3. KMS states of the Pimsner algebras

In this section we discuss an application of the results of the previous section to the C*-algebra \( \mathcal{O}_X \) associated to a Hilbert C*-bimodule \( X \) over a C*-algebra \( \mathcal{B} \). We refer the reader to [P] for the construction of \( \mathcal{O}_X \). We just recall that both \( \mathcal{B} \) and \( \mathcal{B} \) embed isometrically respectively as a Hilbert bimodule in and a C*-subalgebra of \( \mathcal{O}_X \). We shall always assume that \( \mathcal{B} \) is finite projective and full, and that \( \mathcal{B} \) is unital. Therefore any finite basis \( \{x_j\} \) of \( X \) yields, in \( \mathcal{O}_X \), the relation

\[
\sum_j x_j x_j^* = I.
\]

Furthermore any finite subset \( \{y_i\} \) of \( X \) such that \( \sum_i < y_i, y_i >= I \) yields in \( \mathcal{O}_X \):

\[
\sum_i y_i^* y_i = I.
\]

\( \mathcal{O}_X \) is endowed with a canonical gauge action \( \gamma \) such that \( \gamma_z(x) = zx \), \( z \in \mathbb{T} \), \( x \in X \). Therefore we can conclude that \( (\mathcal{O}_X, \gamma) \) is a full periodic C*-dynamical system.

We start proving that systems of this form are typical examples, in the sense that we can always easily associate to any unital, full, periodic C*-dynamical system \((\mathcal{A}, \gamma)\), a finite projective Hilbert C*-bimodule \( X \) such that \( \mathcal{A} = \mathcal{O}_X \) and \( \gamma \) is the canonical \( \mathbb{T} \)-action. We should note, however, that the Hilbert bimodule \( X \) and its coefficient C*-algebra are, in general, not unique. In fact our construction leads to a maximal Hilbert bimodule. In applications, it may be more convenient to start with smaller Hilbert bimodules. It is well known the case of Cuntz–Krieger algebras discussed in [P], where the coefficient algebra is finite–dimensional. In section 4 we shall discuss the more general situation of Matsumoto algebras, and we will construct natural minimal generating Hilbert bimodules.

3.1. Theorem Let \((\mathcal{A}, \gamma)\) be a full C*-dynamical system over \( \mathbb{T} \) and assume that \( \mathcal{A} \) is unital. Then there exists a full finite projective Hilbert C*-bimodule \( X \) over a unital C*-algebra \( \mathcal{B} \) such that \((\mathcal{A}, \gamma)\) can be identified with \( \mathcal{O}_X \), endowed with its canonical gauge action.

Proof Choose finite subsets \( \{y_i\} \) and \( \{x_j\} \) of \( \mathcal{A}^1 \) such that \( \sum_i y_i^* y_i = I \) and \( \sum_j x_j x_j^* = I \). Let \( \mathcal{B} \) be the fixed point algebra \( \mathcal{A}^0 \). Set \( X = \sum_j x_j \mathcal{B} \). Then \( X \) is a right Hilbert \( \mathcal{B} \)-module in \( \mathcal{A} \) with \( \mathcal{B} \)-valued inner product: \( < x, y > = x^* y \).

The condition \( \sum_j x_j x_j^* = I \) shows that \( \{x_j\} \) is a finite basis of \( X \), and \( X \) is full by \( \sum_i y_i^* y_i = I \). Left \( \mathcal{B} \)-action is given by \( \phi(b)x = bx \) for \( b \in \mathcal{B} \) and \( x \in X \). For \( x = \sum_j x_j b_j \), we have \( bx = \sum_j x_i^* bx_j b_j \in X \). Thus \( \phi \) is well defined. Let \( \hat{x} \) the image of an element \( x \in X \) in \( \mathcal{O}_X \). By the universality of the Pimsner algebras, there exists a surjective \(*\)-homomorphism \( \varphi : \mathcal{O}_X \rightarrow \mathcal{A} \) such that \( \varphi(\hat{x}_j) = x_j \). Let \( F_0 : \mathcal{A} \rightarrow \mathcal{A}^0 \) and \( E : \mathcal{O}_X \rightarrow \mathcal{O}_X^0 \) be the natural conditional expectations. Since \( \varphi E = F_0 \varphi \) and \( E \) is faithful, \( \varphi \) is injective by a well known argument. Thus \( \varphi \) is the desired isomorphism.

A KMS state on \((\mathcal{O}_X, \gamma)\) at some inverse temperature \( \log(\delta) \in \mathbb{R} \) restricts to a tracial state \( \tau \) on the coefficient algebra \( \mathcal{B} \) satisfying, for \( a \in \mathcal{B} \), \( \tau(\sum_i <
3.2. Lemma Let \( \{x_1, \ldots, x_d\} \) be a basis of \( X \). Any tracial state \( \tau \) on the coefficient algebra \( B \) satisfying

\[
\tau \left( \sum_i < x_i, ax_i > \right) = \delta \tau(a), \quad a \in B,
\]

with \( \delta > 0 \), extends uniquely to a KMS state on \( \mathcal{O}_X \) at inverse temperature \( \log(\delta) \). This extension is faithful if \( \tau \) is faithful.

Proof We first prove uniqueness. A KMS state for \( (\mathcal{O}_X, \gamma) \) is determined by its restriction to the homogeneous \( C^* \)-subalgebra, and, by the trace–scaling property of the operator \( S_{\{x_i\}} \) on that subalgebra, it is in fact determined by its restriction to the coefficient algebra \( B \). Conversely, if one is given a tracial state \( \tau_0 \) on \( B \) as required then it is easy to check that

\[
\tau_n(a) := \frac{1}{\delta} \tau_{n-1} \left( \sum x_i^* ax_i \right), \quad a \in \mathcal{L}(X^n), n \geq 1,
\]

is a sequence of tracial states (faithful if \( \tau_0 \) is faithful) such that \( \tau_{n+1} |_{\mathcal{L}(X^n)} = \tau_n \), which thus gives rise to a tracial state \( \tau \) on the homogeneous subalgebra positively scaled by \( S_{\{x_i\}} \). Therefore \( \tau \) extends to a KMS state on \( \mathcal{O}_X \) at inverse temperature \( \log(\delta) \).

We now apply the results of the previous section to the Pimsner \( C^* \)-algebras.

3.3. Theorem Let \( B \) be a unital \( C^* \)-algebra with a tracial state, and let \( X \) be a full finite projective Hilbert \( C^* \)-bimodule over \( B \). Assume that for every tracial state \( \tau \) on \( B \), and any basis \( \{x_i\} \) of \( X \), \( \tau(\sum_i < x_i, x_i >) > 0 \). Then \( \mathcal{O}_X \) has a KMS state.

(1) Let \( s' : \mathcal{T}(\mathcal{O}_X^0) \to \mathcal{T}(\mathcal{O}_X^0) \) be the (invertible) operator obtained restricting to \( \mathcal{T}(\mathcal{O}_X^0) \) the Banach space adjoint of \( S(a) = \sum x_i^* ax_i \), where \( \{x_i\} \) is a basis of \( X \). Then the set of possible inverse temperatures is a closed subset of \( [\log(r_{\text{min}}(s')), \log(r_{\text{max}}(s'))] \) containing the extreme points.

(2) If \( \mathcal{O}_X \) is \( T \)-simple, every KMS state is faithful,

(3) if \( \tau(\sum_i < x_i, x_i >) > 1 \) for all \( \tau \in \mathcal{T}\mathcal{S}(B) \) then every KMS state has positive inverse temperature,

(4) if \( \tau(\sum_i < x_i, x_i >) = 1 \) for all \( \tau \in \mathcal{T}\mathcal{S}(B) \) then every KMS state is a tracial state,

(5) if \( \tau(\sum_i < x_i, x_i >) < 1 \) for all \( \tau \in \mathcal{T}\mathcal{S}(B) \) then every KMS state has a negative inverse temperature,

(6) if \( B \) has a unique trace then \( \mathcal{O}_X^0 \) has a unique trace, so \( \mathcal{O}_X \) has a unique KMS state.

Proof The function taking a tracial state \( \tau \) to the tracial state

\[
a \in B \to (\tau(\sum < x_i, x_i >))^{-1} \tau(\sum < x_i, ax_i >)
\]
is weakly$^*$-continuous, so, by the Schauder–Tychon fixed point theorem, there is a tracial state $\tau$ on $B$ such that
\[
\tau(\sum_i <x_i, ax_i>) = \tau(\sum_i <x_i, x_i>) \tau(a), \quad a \in B.
\]

We can now extend such a $\tau$ to a KMS state on $(\mathcal{O}_X, \gamma)$, by the previous lemma. (1) follows from Corollary 2.6. Since $\mathcal{O}_X$ is $\mathbb{T}$–simple, $\mathcal{O}_X^0$ has no ideal of the kind described in our faithfulness criterion, Prop. 1.5, hence (2) follows. (3)–(5) follow from Corollary 1.4. Finally, if $B$ has a unique tracial state then so does $\mathcal{O}_X^0$ since it is an inductive limit of $C^*$-algebras stably isomorphic to $B$ itself. Therefore $\mathcal{O}_X$ has a unique KMS state.

4. Examples of full dynamical systems arising from subshifts, self-similar sets and noncommutative metric spaces

In this section we continue our discussion of examples of full $C^*$-dynamical systems obtained via Pimsner’s construction. We start considering two different kinds of Hilbert bimodules both described by families of $^*$-endomorphisms on commutative $C^*$-algebras, arising respectively from symbolic dynamics and fractal geometry. We shall also discuss a generalization of the latter example to noncommutative metric spaces.

4.1. Subshifts in symbolic dynamics and Matsumoto algebras

We recall the construction of the Matsumoto algebra $\mathcal{O}_\Lambda$ associated with a two–sided subshift $\Lambda$. [M]. Fix a finite discrete set $\Sigma = \{1, 2, ..., d\}$, and let $\Sigma^\mathbb{Z}$ be the infinite product space endowed with the product topology. We denote by $\sigma$ the shift homeomorphism on $\Sigma^\mathbb{Z}$ defined by $(\sigma(x))_i = x_{i+1}$. For a shift–invariant closed subset $\Lambda$ of $\Sigma^\mathbb{N}$, the topological dynamical system $(\Lambda, \sigma | \Lambda)$ is called a subshift. We denote by $\Lambda_+$ the set of one–sided sequences $x \in \Sigma^\mathbb{N}$ such that $x$ appears in $\Lambda$. For example, $\Sigma^\mathbb{N} = \Sigma^\mathbb{Z}_+$. We shall still denote by $\sigma$ the left shift epimorphism of $\Sigma^\mathbb{N}$. The dynamical system $(\Lambda_+, \sigma | \Lambda_+)$ is called the one–sided subshift associated to $\Lambda$. A finite sequence $\mu = (\mu_1, \ldots, \mu_k)$ of elements $\mu_j \in \Sigma$ is called a word. We denote by $|\mu|$ the length $k$ of $\mu$. For $k \in \mathbb{N}$, let $\Lambda^k = \{\mu | \mu \text{ is a word with length } k \text{ appearing in some } x \in \Lambda\}$, $\Lambda_l = \bigcup_{k=0}^l \Lambda^k$ and $\Lambda^* = \bigcup_{k=0}^\infty \Lambda^k$, where $\Lambda^0$ denotes the set constituted by the empty word.

Let $\{e_1, \ldots, e_d\}$ be an orthonormal basis of a $d$-dimensional Hilbert space $H = \mathbb{C}^d$. Let $F^0$ be the one dimensional space $\mathbb{C}\Omega$ spanned by a normalized vector $\Omega$, called the vacuum vector, and let $F^k$ be the Hilbert space spanned by the vectors $e_{\mu} = e_{\mu_1} \otimes \cdots \otimes e_{\mu_k}$ for $\mu = (\mu_1, \ldots, \mu_k) \in \Lambda^k$. Consider the subspace $F_{\Lambda} = \oplus_{k=0}^\infty F^k$ of the full Fock space of $H$.

The creation operator $T_\nu$ by $e_\nu$ on $F_{\Lambda}$, for $\nu \in \Lambda^*$, is defined by
\[
T_\nu e_\mu = e_\nu \otimes e_\mu, \quad \text{if } \nu \mu \in \Lambda^*,
\]
\[
T_\nu e_\mu = 0 \quad \text{otherwise}
\]

The unital $C^*$-subalgebra $\mathcal{T}_{\Lambda}$ of the algebra of bounded linear operators on $F_{\Lambda}$ generated by $\{T_i | i = 1, \ldots, d\}$ is called the Toeplitz algebra associated with $\Lambda$, and contains the algebra $\mathcal{K}(F_{\Lambda})$ of compact operators on $F_{\Lambda}$. The Matsumoto algebra $\mathcal{O}_{\Lambda}$ associated with the subshift $\Lambda$ is the quotient algebra $\mathcal{T}_{\Lambda}/\mathcal{K}(F_{\Lambda})$. 
It is generated by the quotient image \( \{ S_i | i = 1, \ldots, d \} \) of \( \{ T_i | i = 1, \ldots, d \} \). The unitary representation of \( T \) on \( F_\Lambda \) defining the grading implements an automorphic action of \( T \) on \( F_\Lambda \) leaving \( \mathcal{K}(F_\Lambda) \) stable. We thus obtain an automorphic \( T \)-action \( \gamma \) on \( O_\Lambda \) such that

\[
\gamma_z(S_i) = zS_i, \quad z \in T, i = 1, \ldots, d.
\]

As \( \sum_i S_i S_i^* = I \) and \( \sum_i S_i^* S_i \geq I \), \( (O_\Lambda, \gamma) \) is a full periodic \( C^* \)-dynamical system. We set \( S_\mu = S_{\mu_1} \cdots S_{\mu_k} \), for \( \mu = (\mu_1, \ldots, \mu_k) \in \Lambda^* \).

For each \( i = 1, \ldots, d \), we define a (not necessarily unital) \(*\)-endomorphism \( \rho_i \) on \( \ell^\infty(\Lambda_+) \) by

\[
(\rho_i(f))(x) = \begin{cases} f(i, x_1, \ldots) & \text{if } (i, x_1, \ldots) \in \Lambda_+ \\ 0 & \text{otherwise} \end{cases}
\]

for \( f \in \ell^\infty(\Lambda_+) \) and \( x \in \Lambda_+ \). Then we have \( \cap_i \ker \rho_i = 0 \).

Consider the functions \( q_\mu \in \ell^\infty(\Lambda_+) \), for \( \mu = (\mu_1, \ldots, \mu_k) \in \Lambda^k \), defined by

\[
q_\mu(x) = \begin{cases} 1 & \text{if } \mu x \in \Lambda_+ \\ 0 & \text{otherwise} \end{cases}
\]

Thus \( \rho_i(I) = q_i \). (We should note that \( \rho_i(I) = q_i \) is not a continuous function, in general.)

Let \( \Lambda_A \) be the Markov subshift defined by a \( d \times d \) matrix \( A = (a_{i,j}) \) with entries in \( \{0, 1\} \) and with no zero rows or columns:

\[
\Lambda_A = \{ x \in \Sigma^\mathbb{Z} : a_{x_i, x_{i+1}} = 1, i \in \mathbb{Z} \}.
\]

Then each \( \rho_i \) preserves \( \mathcal{C}(\Lambda_A^+) \). For Markov subshifts, Matsumoto’s construction yields the Cuntz–Krieger algebras, see [M]:

\[
\mathcal{O}_\Lambda \simeq \mathcal{O}_A.
\]

4.2. Proposition Let \( \Lambda_A \) be the Markov subshift defined by a matrix \( A = (a_{i,j}) \in M_d(\{0, 1\}) \) and set \( \mathcal{B} = \mathcal{C}(\Lambda_A^+) \). Consider the right Hilbert \( \mathcal{B} \)-module \( X = \bigoplus_{i=1}^d q_i \mathcal{B} \) and the \(*\)-homomorphism \( \phi : \mathcal{B} \to \mathcal{L}(X) \) given by the diagonal matrix \( \phi(a) = \text{diag}(\rho_i(a)) \). Then the Pimsner algebra \( \mathcal{O}_X \) is isomorphic to the Cuntz-Krieger algebra \( \mathcal{O}_A \).

Proof The commutative \( C^* \)-algebra \( \mathcal{D}_A \) generated by \( \{ S_\mu S_\mu^* ; \mu \in \Lambda^* \} \) is isomorphic to \( \mathcal{C}(\Lambda_A^+) \) via an isomorphism which identifies \( S_\mu S_\mu^* \) with the characteristic function \( p_\mu = \chi_{[\mu]} \) of the cylinder set \( [\mu] = \{ x \in \Lambda_+ : (x_1, \ldots, x_k) = (\mu_1, \ldots, \mu_k) \} \) for \( \mu \in \Lambda^k \). Then the \(*\)-endomorphism \( \gamma_i \) on \( \mathcal{D}_A \) defined by \( \gamma_i(T) = S_i^* T S_i \) for \( T \in \mathcal{D}_A \) corresponds to the \(*\)-endomorphism \( \rho_i \) on \( \mathcal{C}(\Lambda_+) \).

We have

\[
\rho_i(p_r) = \delta_{i,r} \sum_{\{j : a_{ij} = 1\}} p_j \quad \text{and} \quad \rho_i(p_\mu) = \delta_{i,\mu_1} p_{(\mu_2, \ldots, \mu_k)}
\]

for \( \mu \in \Lambda^k \) with \( k \geq 2 \). The corresponding formulae for \( \gamma_i(S_\mu S_\mu^*) \) hold. Consider the right Hilbert \( \mathcal{D}_A \)-module \( Y = \bigoplus_{i=1}^d S_i \mathcal{D}_A \) and the \(*\)-homomorphism
A standard argument shows that \( \ell_t(S) \), the Hilbert bimodules \( X \) and \( Y \) are isomorphic. Now a standard argument shows that \( O_\Lambda \cong O_Y \cong O_X \).

If \( \Lambda \) is a general subshift, the endomorphisms \( \rho_i \), \( i = 1, \ldots, d \), do not leave \( C(\Lambda_+) \) stable. Thus we should replace \( C(\Lambda_+) \) by some unital \( C^* \)-subalgebra of \( \ell^\infty(\Lambda_+) \) which is invariant under \( \rho_i \), \( i = 1, \ldots, d \). We shall choose, to this aim, the smallest such \( C^* \)-algebra, which is related to the Krieger left cover of a sofic subshift and the past equivalence relation considered by Matsumoto.

Let \( A(\Lambda_+) \) be the unital \( C^* \)-subalgebra of \( \ell^\infty(\Lambda_+) \) generated by \( \{ q_\mu; \mu \in \Lambda^* \} \).

Since \( \rho_i(I) = q_i \) and \( \rho_i(q_\mu) = q_{\mu i} \), it is clear that \( A(\Lambda_+) \) is the smallest unital \( C^* \)-subalgebra of \( \ell^\infty(\Lambda_+) \) which is invariant under \( \rho_i \), \( i = 1, \ldots, d \).

### 4.3. Theorem
Let \( \mathcal{B} = A(\Lambda_+) \) be the commutative \( C^* \)-algebra associated to a subshift \( \Lambda \) as above. Consider the Hilbert right \( \mathcal{B} \)-module \( X = \oplus_{i=1}^d q_i \mathcal{B} \) and the \(*\)-homomorphism \( \phi: \mathcal{B} \to \mathcal{L}(X) \) given by the diagonal matrix \( \phi(a) = \text{diag}(\rho_i(a)) \). Then the Pimsner algebra \( O_X \) is isomorphic to the Matsumoto algebra \( O_\Lambda \).

**Proof** For \( l \in \mathbb{N} \), let \( A_l \) be the \( C^* \)-subalgebra of \( O_\Lambda \) generated by \( \{ S_\mu S_\mu, \mu \in \Lambda_l \} \) and \( A_\Lambda \) be the \( C^* \)-subalgebra of \( O_\Lambda \) generated by elements \( \{ S_\mu S_\mu, \mu \in \Lambda^* \} \).

Then \( (A_l)_l \) is an increasing sequence of commutative finite-dimensional algebras and \( A_\Lambda = \lim_{l \to \infty} A_l \). Similarly, let \( A_l(\Lambda_+) \) be the \( C^* \)-subalgebra of \( \ell^\infty(\Lambda_+) \) generated by \( \{ q_\mu, \mu \in \Lambda_l \} \). Then \( (A_l(\Lambda_+))_l \) is an increasing sequence of commutative finite-dimensional algebras and \( A(\Lambda_+) = \lim_{l \to \infty} A_l(\Lambda_+) \).

For \( x \in \Lambda_+ \), let \( A_l(x) = \{ \mu \in \Lambda_l; \mu x \in \Lambda_+ \} \). Matsumoto introduced in [M2] the following notion of past equivalence relation. Two points \( x \) and \( y \) in \( \Lambda_+ \) are called \( l \)-past equivalent, \( x \sim_l y \), if \( A_l(x) = A_l(y) \). The corresponding set of equivalent classes is denoted by \( \Omega_l := \Lambda_+/\sim_l \). For \( \mu \in \Lambda_l \), if \( x \sim_l y \), then \( q_\mu(x) = q_\mu(y) \). Thus \( q_\mu \) defines a function \( \hat{q}_\mu \in C(\Omega_l) \). The set \( \{ \hat{q}_\mu \in C(\Omega_l); \mu \in \Lambda_l \} \) separates the points in \( \Omega_l \), thus it generates \( C(\Omega_l) \). \( A_l(\Lambda_+) \) is precisely the set of functions in \( \ell^\infty(\Lambda_+) \) which have the same value on each \( l \)-past equivalent class and we have an isomorphism between \( A_l(\Lambda_+) \) and \( C(\Omega_l) \). We see directly that the commutative \( C^* \)-algebra \( A_\Lambda \) is isomorphic to \( B = A(\Lambda_+) \) via an isomorphism which identifies \( S_\mu S_\mu \) with \( q_\mu \), \( \mu \in \Lambda^* \). The \(*\)-endomorphism \( \gamma_\mu \) on \( A_\Lambda \) defined by \( \gamma_i(T) = S_\mu TS_\mu \) for \( T \in A_\Lambda \) corresponds to the \(*\)-endomorphism \( \rho_i \) on \( B = A(\Lambda_+) \). We have \( \rho_i(q_\mu) = q_{\mu i} \) and \( \gamma_i(S_\mu S_\mu) = S_{\mu i}S_{\mu i} \) for \( \mu \in \Lambda^* \).

Consider the right Hilbert \( A_\Lambda \)-module \( Y = \oplus_{i=1}^d S_iA_\Lambda \) and the \(*\)-homomorphism \( \phi: A_\Lambda \to \mathcal{L}(Y) \) given by the diagonal matrix \( \phi(a) = \text{diag}(\gamma_i(a)) \). Since \( (S_\mu S_\mu)S_i = S_i\gamma_i(S_\mu S_\mu) \), the Hilbert bimodules \( X \) and \( Y \) are isomorphic by the identification of \( B \) with \( A_\Lambda \). The universality of the Matsumoto algebra and the Pimsner algebra immediately shows that \( O_\Lambda \cong O_Y \cong O_X \).

### 4.4. Contractions of compact metric spaces

We next discuss an example associated with a self-similar set in fractal geometry. Let \( \Omega \) be a (separable) complete metric space and let \( \{ \gamma_1, \ldots, \gamma_d \} \) be a finite family of nonzero proper contractions of \( \Omega \) with Lipschitz constants \( c_i = \text{Lip}(\gamma_i) \). We shall assume that \( d \geq 2 \). Then there exists a unique nonempty compact set \( K \subset \Omega \) satisfying the (exact) invariance condition

\[ K = \bigcup_i \gamma_i(K). \]
The above invariance condition shows that the compact set $K$ is self-similar in a weak sense. For example the Cantor set, the Koch curve and the Sierpinski gasket are typical examples of self-similar sets. We refer the reader to the book of Hutchinson [H] for more information on fractal geometry. The topological dimension of $K$ is dominated by the Hausdorff dimension of $K$, and the Hausdorff dimension of $K$ is dominated by the similarity dimension of $K$, which is a finite number $D$ satisfying $\sum_i c_i^D = 1$. Thus $K$ has a finite topological dimension.

Consider the $C^*$-algebra $B = C(K)$ and the canonical Hilbert right $B$-module $X = B^d$. For each $i$, we define an endomorphism $\phi_i$ on $B$ by

$$ (\phi_i(a))(z) = a(\gamma_i(z)), \quad a \in B, \quad z \in K. $$

Left $B$-action $\phi : B \to \mathcal{L}_B(X)$ is defined by the diagonal matrix $\phi(a) = \text{diag}(\phi_i(a))$. We see that the (exact) invariance condition $K = \cup_i \gamma_i(K)$ is equivalent to the fact that $\phi$ is injective. The bimodule $X$ generates the Pimsner $C^*$-algebra $\mathcal{O}_X$. Let $\{x_1, \ldots, x_d\}$ be the canonical basis of $X$. Then the corresponding elements $\{S_1, \ldots, S_d\}$ of $\mathcal{O}_X$ generate a copy of the Cuntz algebra $\mathcal{O}_d \subset \mathcal{O}_X$. By construction, the Pimsner $C^*$-algebra $\mathcal{O}_X$ is isomorphic to the universal $C^*$-algebra generated by $B = C(K)$ and $\mathcal{O}_d$ satisfying the relations $aS_i = S_i\phi_i(a)$ for $a \in B$ and $i = 1, \ldots, d$.

In [H] Hutchinson shows that there exists a unique regular Borel probability measure $\mu$ on $K$ satisfying, for any measurable set $F$,

$$ \mu(F) = \sum_{i=1}^d \frac{1}{d^i} \mu(\gamma_i^{-1}(F)). $$

Consider the trace $\tau_0$ on $B$ corresponding to the probability measure $\mu$. Then $\tau_0$ satisfies

$$ \tau_0(\sum_i < x_i, ax_i >) = d\tau_0(a), \quad a \in B. $$

Since $< x_i, x_i > = 1$ for $i = 1, \ldots, d$, $\mathcal{O}_X$ has a KMS state at the inverse temperature $\beta$ if and only if $\beta = \log d$. Moreover the uniqueness of the probability measure implies that the corresponding KMS state is also unique. We shall show that the algebra $\mathcal{O}_X$ is in fact the Cuntz algebra $\mathcal{O}_d$. Before proving this, we study a more general situation to include standard $d$-times around embeddings.

Let $K$ be a compact metric space. Consider the $C^*$-algebra $B = C(K)$ and the state space $\mathcal{S}$ of $B$. Let $\text{Lip}(K)$ be the space of Lipschitz functions, and let $\text{Lip}(f)$ denote the Lipschitz constant of $f \in \text{Lip}(K)$. In [H] Hutchinson considers the following metric $L$ on $\mathcal{S}$:

$$ L(\varphi_1, \varphi_2) = \sup\{|\varphi_1(f) - \varphi_2(f)|; f \in \text{Lip}(K), \text{Lip}(f) \leq 1\}. $$

Then $(\mathcal{S}, L)$ is a complete metric space, and the topology defined by $L$ is precisely the weak$^*$-topology of $\mathcal{S}$.

Consider the canonical Hilbert right $B$-module $X = B^d$ and any injective unital $^*$-homomorphism $\phi : B \to \mathcal{L}_B(X)$. We identify $\phi(a)$ with the matrix $(\phi_{ij}(a))_{ij} \in B \otimes M_d(\mathbb{C})$ for $a \in B$. 
Then the Pimsner $C^*$-algebra $O_X$ is isomorphic to the universal $C^*$-algebra generated by $B = C(K)$ and the Cuntz algebra $O_d$ satisfying the relations $aS_j = \sum_i S_i \phi_{ij}(a)$ for $a \in B$ and $i, j = 1, \ldots, d$.

4.5. Proposition In the above situation, let $\Psi$ be the unital positive map on $B = C(K)$ defined by $\Psi(a) = \frac{1}{d} \sum_i \phi_{ii}(a)$. If the Banach space adjoint $\Psi^*$ induces a proper contraction on $S$ with respect to the metric $L$, then $O_X$ has a unique tracial state. Moreover $O_X$ has a KMS state at the inverse temperature $\beta$ if and only if $\beta = \log d$ and the corresponding KMS state is also unique.

Proof We identify $L(X^n)$ with $B \otimes M_d(\mathbb{C})$. Using the commutation relation, it is easy to see that the inclusion map $\Phi_n : L(X^n) \to L(X^{n+1})$ is described by matrices

$$\Phi_n((a_{\alpha,\beta})_{n,\beta}) = (\phi_{ij}(a_{\alpha,\beta}))_{(\alpha,i),(\beta,j)},$$

where $\alpha$ and $\beta$ run the set $\{1, \ldots, d\}^n$ of words with length $n$. Thus $\Phi_n = \phi \otimes id$ on $B \otimes M_d(\mathbb{C})$. A tracial state on $O_X^0 = \lim_n L(X^n)$ is described by a sequence of tracial states $(\tau_n)$, $n \geq 0$ on $L(X^n)$ such that $\tau_{n+1} |_{L(X^n)} = \tau_n$. Therefore one needs to assign a sequence of states $(\phi_n) \in S$ with $\tau_n = \phi_n \otimes tr$ on $L(X^n) \cong C(K) \otimes M_d(\mathbb{C})$. The coherence relations require that

$$\Psi^*(\phi_{n+1}) = \phi_n, \quad n \geq 0.$$

Since $K$ is compact, the diameter of $(S, L)$ is bounded. Hence $\bigcap_{n=1}^{\infty} \Psi^{*n}(S)$ consists of a single point $\omega_0$. The constant sequence of states $(\phi_n) \in S$ with $\phi_n = \omega_0$ gives a tracial state $\tau$ on $O_X^0$, because $\omega_0$ is the unique fixed point of $\Psi^*$ in $S$. Choose another tracial state. The coherence relations $\Psi^*(\phi_{n+1}) = \phi_n$ shows that any $\phi_n$ belongs to $\bigcap_{n=1}^{\infty} \Psi^{*n}(S)$. Therefore $\phi_n = \omega_0$. Thus $O_X$ has a unique KMS state at inverse temperature $\log d$.

We remark that the present situation is similar to that of the Cuntz-Krieger algebras associated to aperiodic matrices. In fact, for any state $\omega \in S$, $\Psi^*(\omega)$ converges to the unique $\omega_0$ in $S$ with respect to $L$. This resembles the Perron–Frobenius Theorem for aperiodic matrices.

4.6. Example In the fractal case, each proper contraction $\gamma_i$ induces an endomorphism $\phi_i$ on $B = C(K)$ satisfying $\text{Lip}(\phi_i(f)) \leq c_i \text{Lip}(f)$ for $f \in \text{Lip}(K)$. Hence $\Psi^* = \frac{1}{d} \sum_i \phi_i^*$ is a proper contraction on $S$ with respect to the metric $L$.

4.7. Example We next study the example of standard $d$-times around embeddings. Let $B = C(\mathbb{T})$ with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $X = \mathbb{R}^d$ be the natural right Hilbert $B$-module. Then a standard $d$-times around embedding is defined by a map $\phi : C(\mathbb{T}) \to C(\mathbb{T}, M_d(\mathbb{C}))$ of the form

$$(\phi(f))(t) = u_t \text{diag}(f\left(\frac{t}{d}\right), f\left(\frac{t+1}{d}\right), \ldots, f\left(\frac{t+d-1}{d}\right))u_t^*,$$

where $(u_t)_t$ is a continuous path of unitaries in $M_d(\mathbb{C})$ such that $u_0 = I$ and such that $u_1$ is the unitary matrix corresponding to the operator taking vectors $e_1, \ldots, e_d$ of the canonical basis of $\mathbb{C}^d$ to $e_d, e_1, \ldots, e_{d-1}$ respectively. We regard
\( \phi \) as a map \( \phi : \mathcal{B} \to \mathcal{L}(X) = \mathcal{B} \otimes M_d(\mathbb{C}) \). Then \( \Psi^* = \frac{1}{d} \sum_{i} \phi_i^* \) induces a proper contraction on \( S \) with respect to the metric \( L \). In fact

\[
(\Psi(f))(t) = \frac{1}{d} \left( f\left( \frac{t}{d} \right) + f\left( \frac{t+1}{d} \right) + \cdots + f\left( \frac{t+d-1}{d} \right) \right).
\]

We assume that \( d = 2 \) for the simplicity of notation. For \( f \in \text{Lip}(\mathbb{T}) \) and \( x, y \in \mathbb{T} \), choosing carefully the nearest pairs between \( \{\frac{x}{2}, \frac{x+1}{2}\} \) and \( \{\frac{y}{2}, \frac{y+1}{2}\} \), we have

\[
|(|\Psi(f)(x)| - |\Psi(f)(y)|| \leq \frac{1}{2} \text{Lip}(f)|d(x, y)|.
\]

Then \( \text{Lip}(\Psi(f)) \leq \frac{1}{2} \text{Lip}(f) \).

Therefore for \( \varphi_1, \varphi_2 \in S \), we have

\[
|(|\Psi^*(\varphi_1)) f\rangle - (|\Psi^*(\varphi_2)) f\rangle| = |\varphi_1(f)\Psi(f) - \varphi_2(f)\Psi(f)| \leq L(\varphi_1, \varphi_2) \frac{1}{2} \text{Lip}(f).
\]

Thus \( L(\Psi^*(\varphi_1), \Psi^*(\varphi_2)) \leq \frac{1}{2} L(\varphi_1, \varphi_2) \).

4.8. Remark One can easily show, using known results, that under suitable circumstances \( \mathcal{O}_X \) is simple and purely infinite. Indeed, assume that \( K \) is totally disconnected or connected and has a finite topological dimension. If \( \mathcal{O}_X = \lim_{\to} \mathcal{L}(X^n) \) is simple, then \( \mathcal{O}_X^0 \) is of real rank zero by [BDR], since \( \mathcal{O}_X \) has a unique trace. By a result of Martin and Pasnicu [MP], \( \mathcal{O}_X^0 \) has the comparability property on every matrix algebra. Thus we can apply a result by [GP] (Theorem 1.6) and conclude that the \( \mathcal{O}_X \) is simple and purely infinite. For example, in the case of a standard \( d \)-times around embeddings all the assumptions are satisfied. In fact \( \mathcal{O}_X^0 = \lim_{\to} \mathcal{L}(X^n) \) is a Bunce-Deddens algebra. Note that we have naturally embedded an AT algebra into a purely infinite simple \( C^* \)-algebra. Again, in the fractal case, it is easy to show that \( \mathcal{O}_X \) is simple. So we can apply the preceding argument. However, it is not difficult to show that in this case \( \mathcal{O}_X \) is canonically isomorphic to the Cuntz algebra \( \mathcal{O}_d \) (a fact which will be later generalized to noncommutative metric spaces): We identify \( \mathcal{L}(X^n) \) with \( \mathcal{L}(K, M_{d^n}(\mathbb{C})) \). Then the inclusion map \( \Phi_n : \mathcal{L}(X^n) \to \mathcal{L}(X^{n+1}) \) is described by block diagonal matrices

\[
(\Phi_n(f))(t) = \text{diag}(f(\gamma_1(t)), \ldots, f(\gamma_d(t))).
\]

Let \( \omega = (\omega_1, \ldots, \omega_k) \in \{1, \ldots, d\}^k \) be a finite word and \( \gamma_\omega = \gamma_{\omega_1} \cdots \gamma_{\omega_k} \). Then the inclusion map \( \Phi_{n+k,n} \) of \( \mathcal{L}(X^n) \) into \( \mathcal{L}(X^{n+k}) \) is given by

\[
(\Phi_{n+k,n}(f))(t) = \text{diag}(f(\gamma_{\omega}(t)))_{\omega}.
\]

By the uniform continuity of \( f, \Phi_{n+k,n}(f) \) is approximated by a constant matrix up to \( \varepsilon \) for a sufficient large \( k \). Thus \( \mathcal{O}_X^0 \) is a UHF algebra \( M_{d^n} \) and \( \mathcal{O}_X \) is exactly the Cuntz algebra \( \mathcal{O}_d \) generated by the original operators \( \{S_1, \ldots, S_d\} \).

4.9. Contraction of noncommutative metric spaces

The preceding argument suggests a generalization to noncommutative metric spaces introduced by Connes in [Co]. Our setting will be the following. Let \( \mathcal{A} \) and \( \mathcal{B} \) be unital \( C^* \)-algebras. Suppose that \( \mathcal{B} \) is a Banach bimodule over \( \mathcal{A} \). Let \( \delta : \mathcal{A} \supset \text{Dom}(\delta) \to \mathcal{B} \) be a densely defined \(*\)-derivation with \( \ker \delta = CI \). Let \( S \) be the state space of \( \mathcal{A} \). Consider the following metric \( L \) on \( S \):

\[
L(\varphi_1, \varphi_2) = \sup\{|\varphi_1(a) - \varphi_2(a)| ; a \in \text{Dom}(\delta), \|\delta(a)\| \leq 1\}.
\]
The metric is allowed to take the value $\infty$.

In [RiI], Rieffel considers the question of whether the metric topology agrees with the underlying weak$^*$ topology on the state space. His setting is, however, more general, as he works with normed vector spaces endowed with seminorms not necessarily arising from $^*$-derivations.

We assume that

$$\{a \in \text{Dom}(\delta) : \|\delta(a)\| \leq 1\}/\mathcal{C}I \text{ is bounded in } \mathcal{A}/\mathcal{C}I. \quad (4.1)$$

By Proposition 1.6 in [RiI] this condition is equivalent to the fact that the metric $L$ on $\mathcal{S}$ is bounded.

Let $\{\phi_1, \ldots, \phi_d\}$ be a finite family of unital $^*$-endomorphisms on $\mathcal{A}$, with $d \geq 2$. Recall that the crossed product $C^*$-algebra $C^*(A; \phi_1, \ldots, \phi_d)$ of $\mathcal{A}$ by $\{\phi_1, \ldots, \phi_d\}$ is the universal $C^*$-algebra generated by the image of a $C^*$-homomorphism $\pi : A \to C^*(A; \phi_1, \ldots, \phi_d)$ and the Cuntz algebra $\mathcal{O}_d$ with the generators $S_1, \ldots, S_d$ satisfying the relations $\pi(a)S_i = S_i\pi(\phi_i(a))$ for $a \in \mathcal{A}$ and $i = 1, \ldots, d$. We note that $\pi$ is isometric if and only if $\cap_i \ker \phi_i = 0$. In this case the crossed product $C^*$-algebra $C^*(A; \phi_1, \ldots, \phi_d)$ is isomorphic to $\mathcal{O}_X$, where $X$ is the trivial Hilbert right $\mathcal{A}$-module $X = \mathcal{A}^d$ endowed with the diagonal left $\mathcal{A}$-action: $\phi : A \to \mathcal{L}_d(X) \phi(a) = \text{diag}(\phi_1(a), \ldots, \phi_d(a))$.

4.10. Proposition In the above setting, assume that the restrictions $\gamma_i$ of the Banach space adjoint $\phi_i^* : A^* \to A^*$ to the state space $\mathcal{S}$ of $\mathcal{A}$ are proper contractions with respect to $L$. Then the endomorphism crossed product $C^*$-algebra $C^*(A; \phi_1, \ldots, \phi_d)$ is canonically isomorphic to the Cuntz algebra $\mathcal{O}_d$ and has a unique KMS state at inverse temperature $\log d$.

Proof Let $c$ be the maximum of the Lipschitz norms $c_i = \text{Lip}(\gamma_i)$, $i = 1, \ldots, d$. For any $a \in \text{Dom}(\delta)$ and $\varphi, \psi \in \mathcal{S}$, we have

$$|\varphi(a) - \psi(a)| \leq L(\varphi, \psi)\|\delta(a)\|.$$ 

For a finite word $\alpha = (\alpha_1, \ldots, \alpha_n) \in \{1, \ldots, d\}^n$, we use the multi-index notation $\alpha$ and $\gamma$. Then we have that

$$L(\gamma(\varphi), \gamma(\psi)) \leq c^n L(\varphi, \psi) \leq c^n \text{diam}(\mathcal{S}, L),$$

for any pair of states $\varphi, \psi \in \mathcal{S}$.

We shall show that $\pi(A)$ is included in the canonical UHF subalgebra $M_{d^n}$ of the Cuntz algebra $\mathcal{O}_d$. For any $a \in \text{Dom}(\delta)$ and $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$c^n \text{diam}(\mathcal{S}, L)\|\delta(a)\| \leq \varepsilon.$$ 

Fix a state $\omega_0 \in \mathcal{S}$. Consider the diagonal matrix $t = \text{diag}(\omega_0(\phi_\alpha(a)))_\alpha \in M_{d^n}(\mathbb{C})$. Then for any state $\omega \in \mathcal{S}$, we have

$$|\omega(t_{\alpha} - \phi_\alpha(a))| = |\omega_0(\phi_\alpha(a)) - \omega(\phi_\alpha(a))| \leq L(\gamma(\omega_0), \gamma(\omega))\|\delta(a)\| \leq \varepsilon.$$ 

Hence $\|t_{\alpha} - \phi_\alpha(a)\| \leq 2\varepsilon$. Since

$$\pi(a) = \pi(a) \sum_\alpha S_\alpha S_\alpha^* = \sum_\alpha S_\alpha \pi(\phi_\alpha(a))S_\alpha^*,$$

for any state $\omega \in \mathcal{S}$, we have

$$|\omega(a) - \omega(\pi(a))| \leq \varepsilon.$$ 

Hence $\pi(a)$ is a unitary element $U$ in $\pi(A)$ and $\omega (U) = \omega (\pi(a))$ for any state $\omega \in \mathcal{S}$.
we have that
\[ \| \sum_{\alpha} t_{\alpha \alpha} S_\alpha S^*_\alpha - \pi(a) \| = \| \sum_{\alpha} S_\alpha \pi(t_{\alpha \alpha} - \phi_\alpha(a)) S^*_\alpha \| \leq 2\varepsilon. \]

Thus \( C^*(A; \phi_1, \ldots, \phi_d) \) is precisely the Cuntz algebra \( \mathcal{O}_d \) generated by the original \( \{S_1, \ldots, S_d\} \).

Remark M. Rieffel has kindly pointed out to us that in the proof of the previous Proposition we never use the fact that \( L \) comes from a \( * \)-derivation, but rather only that it is a seminorm satisfying the boundedness condition (4.1). Seminorms of this kind were studied in [Rii].

4.11. Example Let \( \mathcal{D} \) be a noncommutative unital \( C^* \)-algebra. Let \( A = \{a \in C([0,1], \mathcal{D}); a(0) \in \mathbb{C} I \} \). Set \( Y = \{(x,y) \in [0,1] \times [0,1]; x \neq y\} \). Let \( B = C(Y, \mathcal{D}) \) be the set of \( \mathcal{D} \)-valued bounded continuous functions on \( Y \). Then \( B \) is a Banach bimodule over \( A \) by
\[
(a_1 f a_2)(x, y) = a_1(x) f(x, y) a_2(y).
\]

Let \( \delta : A \supset \text{Dom}(\delta) \rightarrow B \) be the densely defined \( * \)-derivation of De Leeuw, given by
\[
(\delta(a))(x, y) = \frac{a(x) - a(y)}{|x - y|},
\]
where \( \text{Dom}(\delta) \) is the set of Lipschitz functions in \( A \). Then ker\( \delta = \mathbb{C} I \). Let \( \alpha \) be the \( * \)-endomorphism on \( A \) defined by \( \alpha(f)(x) = f(\tfrac{x}{2}) \). Then the restriction \( \gamma \) of the Banach space adjoint \( \alpha^* : A^* \rightarrow A^* \) to the state space \( S \) is a proper contraction with respect to \( L \). Therefore the endomorphism crossed product \( C^* \)-algebra \( C^*(A; \alpha, \alpha) \) is isomorphic to the Cuntz algebra \( \mathcal{O}_2 \).

5. KMS states of Pimsner \( C^* \)-algebras associated to Cuntz–Krieger bimodules

In this section we illustrate Theorem 2.5 by examples. We shall discuss some situations where there is a unique KMS state, or more generally, where the set of KMS states can be easily characterized. The inspiring example is that of the Cuntz–Krieger algebras that we discuss here below. Some of the following facts are well known in terms of path algebras in subfactor theory.

5.1. KMS states of Cuntz–Krieger algebras

Let \( \mathcal{O}_A \) be the Cuntz–Krieger algebras associated to a matrix \( A = (a_{ij}) \in M_d(\{0,1\}) \). A KMS state \( \omega \) for the canonical circle action restricts to a tracial state \( \tau \) on the f.d. commutative subalgebra \( \mathcal{A} \) generated by the ranges \( P_1, \ldots, P_d \) of the generating partial isometries \( S_1, \ldots, S_d \). Let \( \lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{R}_+^d \) be defined by \( \lambda_i = \omega(P_i) \). Since \( \omega \) is normalized, we have \( \sum \lambda_i = 1 \). The scaling property \( s'(\tau) = \epsilon(\tau) \tau \) says, when checked on \( \mathcal{A} \), that \( \lambda \) is a nonnegative eigenvector of \( A \), and hence, when \( A \) is irreducible, it is the unique normalized Perron eigenvector for \( A \) by the Perron–Frobenius Theorem [G]. Let us analyse more in detail the structure of the Banach space \( T(\mathcal{O}_A^0) \) and the spectrum of the operator \( s' \). Let \( L_r \), \( r \geq 1 \), denote the unital finite-dimensional \( C^* \)-subalgebra of \( \mathcal{O}_A \) generated by elements of the form \( S_{i_1} \ldots S_{i_r} P_k(S_{j_1} \ldots S_{j_r})^* \).
Set $L_0 = \mathbb{C}P_1 + \cdots + \mathbb{C}P_d$. Then the set of minimal central projections for $L_r$ is \{\(\sigma^r(P_k), k = 1, \ldots, d\)\}, where \(\sigma\) is the canonical endomorphism of \(L_0' \cap \mathcal{O}_A\) defined by \(\sigma(T) = \sum_i S_iTS_i^*\). So \(L_r\sigma^r(P_k) \simeq M_{d_{r,k}}(\mathbb{C})\) where
\[
d_{r,k} = \text{Card}\{(i_1, \ldots, i_r) : S_{i_1} \cdots S_{i_r}P_k \neq 0\} = \text{Card}\{(i_1, \ldots, i_r) : a_{i_1,i_2}a_{i_2,i_3} \cdots a_{i_r,k} \neq 0\}.
\]
Therefore \(d_{r,k}\) is the sum of the entries in the \(k\)-th column of \(A^r\):
\[
d_{r,k} = \sum_{i_1, \ldots, i_r} a_{i_1,i_2}a_{i_2,i_3} \cdots a_{i_r,k}.
\]
The \((j,i)\)-entry of inclusion matrix of \(L_r \subset L_{r+1}\) can be computed by looking at the projection \(\sigma^r(P_i)\sigma^{r+1}(P_j) = \sigma^r(S_iP_jS_i^*)\) which is 0 when \(a_{i,j} = 0\), otherwise it is the sum \(d_{r,i}\) minimal projections of \(L_{r+1}\sigma^{r+1}(P_j)\). Thus the inclusion matrix of \(L_r \subset L_{r+1}\) is \(A^r\). A tracial state on \(\mathcal{O}_A^0\) is described by a sequence of positive traces \(\{\tau_r\}, r \geq 0\) on \(L_r\) such that \(\tau_0\) is normalized and \(\tau_{r+1} \mid_{L_r} = \tau_r\). Therefore one needs to assign a sequence of nonnegative column vectors \((t_r) \in \mathbb{R}^d_+\) which will be the values that \(\tau_r\) takes on the minimal projections of \(L_r\).

The coherence relations require that
\[
A(t_{r+1}) = t_r, \quad r \geq 0
\]
while the positivity and normalization properties translates into:
\[
t_r(k) \geq 0, \quad r \geq 0, k = 1, \ldots, d,
\]
\[
\sum_k t_0(k) = 1.
\]
The latter implies, as expected, normalization of each \(\tau_r\):
\[
\sum_k t_r(k)d_{r,k} = \sum_i (A^r t_r)(i) = \sum_i t_0(i) = 1, r \geq 0.
\]
Removing positivity and normalization, but requiring instead a norm bound for the sequence \((t_r)\), one finds that \(T(\mathcal{O}_A^0)\) is described by
\[
\{(t_r)_{r \geq 0} : t_r \in \mathbb{C}^d, t_r = A_{r+1}, r \geq 0, \sup_r \sum_i (A^r |t_r|)(i) < \infty\},
\]
with the Banach space norm
\[
\|(t_r)_{r \geq 0}\| = \sup_r \sum_i (A^r |t_r|)(i).
\]
The operator \(s'\) acts as:
\[
s'((t_r)_{r \geq 0}) = (A_{t_0}, t_0, t_1, \ldots)
\]
while its inverse is 
\[ t'(t_r)_{r \geq 0} = (t_1, t_2, \ldots). \]

5.2. Proposition If \( A = (a_{ij}) \in M_d(\{0, 1\}) \) is an irreducible symmetric matrix then \( T(\mathcal{O}_A^0) \) is linearly spanned by elements of the form \( (\lambda^{-r} t_0)_{r \geq 0} \) where \( t_0 \) is an eigenvector of \( A \) with eigenvalue \( \lambda \) and \( |\lambda| = r(A) \). Furthermore 
\[ \sigma(s') = \{ \lambda \in \sigma(A) : |\lambda| = r(A) \}. \]

Proof Let \( t_0 \) be an eigenvector for \( A \) with eigenvalue \( \lambda \) such that \( |\lambda| = r(A) \). Set \( t_r := \lambda^{-r} t_0 \), so that \( A t_{r+1} = t_r \). We have:
\[ \|A^r|t_r\|_2 = r(A)^{-r} \|A^r|t_0\|_2 \to \|E_0|t_0\|_2, \]
where \( E_0 \) is the rank one orthogonal projection onto the span of the Perron eigenvector. It follows that \( (t_r) \in T(\mathcal{O}_A^0) \). Furthermore \( (t_r) \) is also an eigenvector of \( s' \) with the same eigenvalue. The same argument shows that if \( t_0 \in T(\mathcal{O}_A^0) \) were an eigenvector with eigenvalue \( \lambda \) such that \( |\lambda| < r(A) \), and \( t_r \) is defined as above, then \( \|A^r|t_r\|_2 \) would be unbounded, so that \( (t_r) \) does not define an element of \( T(\mathcal{O}_A^0) \). With similar arguments one sees that \( T(\mathcal{O}_A^0) \) is linearly spanned by vectors of the form \( (\lambda^{-r} t_0) \) where \( |\lambda| = r(A) \). We show that if either \( \lambda \notin \sigma(A) \) or \( \lambda \in \sigma(A) \) but \( |\lambda| < r(A) \) then \( s' - \lambda \) is invertible. We start assuming that \( \lambda \notin \sigma(A) \). Then \( s' - \lambda \) is clearly injective. We show that it is also surjective. Given \( (v_r) \in T(\mathcal{O}_A^0) \) set \( t_r = (A - \lambda)^{-1}v_r \). Then
\[ A t_r = A(A - \lambda)^{-1}v_r = (A - \lambda)^{-1}Av_r = (A - \lambda)^{-1}v_{r-1} = t_{r-1}. \]
For any matrix \( B \) with complex entries let \( B^+ \) stand for the matrix with entries the absolute values of the corresponding elements of \( B \), and let \( M_B \) be the maximum of the absolute values of its entries. Then
\[ A^r|t_r| \leq A^r(A - \lambda)^{-1}t_r \leq M_{|A - \lambda|^{-1}} A^r|v_r| \]
which shows that \( (t_r) \in T(\mathcal{O}_A^0) \) and \( (s' - \lambda)(t_r) = v_r \). Assume now that \( \lambda \in \sigma(A) \) but \( |\lambda| < r(A) \). We have already noted that \( s' - \lambda \) is injective. Furthermore since for each \( t_r \in T(\mathcal{O}_A^0) \), any \( t_r \) belongs to the range of \( A - \lambda \), with similar arguments one shows that \( s' - \lambda \) is surjective.

Remark If \( A \) is aperiodic, then the homogeneous subalgebra of \( \mathcal{O}_A \) has a unique trace. More generally, if one drops the assumption that \( A \) is symmetric, then, with a more extensive use of Perron–Frobenius theory, one can still show that eigenvectors corresponding to nonmaximal eigenvalues do not appear in the point spectrum of \( s' \), hence our result shows that \( \sigma(s') \subset \{ \lambda : |\lambda| = r(A) \} \).

Note also that if we more generally start with a reducible matrix \( A \), then we are in a situation of nonuniqueness of KMS states for \( \mathcal{O}_A \) (corresponding to the minimal and maximal Perron eigenvalues of \( A \)).

5.3. KMS states as Markov traces arising from inclusions of finite algebras with finite Jones index
Let $N \subset M$ be an inclusion of $II_1$–factors with finite index or of finite-dimensional $C^*$–algebras such that $Z(N) \cap Z(M) = \mathbb{C}$. Let, in the latter case, $A$ be the inclusion matrix. Let $\tau$ be a faithful tracial state on $M$, and consider the unique $\tau$–preserving conditional expectation $E_\tau : M \to N$.

Endow $X = M$ with the $C^*$–bimodule structure over $N$ as follows. The structure of $N$–bimodule is defined by left and right multiplication, while the $N$–valued inner product is
\[ <x, y>_{N} := E_\tau(x^* y). \]

Then $X$ is full and finite projective as a right $N$–module ([GHJ]). It is not difficult to check that $L_N^{X_r} = M_{2r-1}$, where $M_{-1} = N \subset M_0 = M \subset M_1 \subset \ldots$ is the Jones tower. A KMS state at inverse temperature $\beta$ for $O_X$ corresponds precisely to a Markov trace for the tower, which is unique, and one has $\beta = \log([M : N])$.

See [K]. If $N \subset M$ are finite factors, each term of the tower is a finite factor, hence its trace space is one dimensional, and spanned by the Markov trace. So $\dim T(O_X^0) = 1$, and $s'$ acts multiplying by $[M : N]$. If $N \subset M$ are finite–dimensional $C^*$–algebras, the inclusion matrix of $L_N^{X_r} \subset L_N^{X_{r+1}}$ is $A^t A$, which is symmetric and irreducible ([GHJ]). It is not difficult to show, with arguments similar to those of the previous example, that $T(O_X^0)$ is again linearly spanned by traces corresponding to eigenvectors of $A^t A$ with maximal eigenvalue.

5.4. KMS states of Pimsner algebras associated with Cuntz–Krieger bimodules

After these motivating examples, we consider, more generally, systems of the form $(O_X, \gamma)$, where $X$ is what we call a Cuntz–Krieger Hilbert $C^*$–bimodule and $\gamma$ is the canonical gauge action. Such Hilbert bimodules, and simplicity of the corresponding $C^*$–algebras $O_X$, have been considered in [KPW]. Consider $d \geq 2$ unital simple $C^*$–algebras $A_1, \ldots, A_d$ and a matrix $A = (a_{i,j}) \in M_d(\{0, 1\})$ with no row and no column identically zero. Let, for any pair of indices $i, j$ such that $a_{i,j} = 1$, $X_{i,j}$ be a full, finite projective $A_i$–$A_j$ Hilbert bimodule, and let $X = \bigoplus_{i,j; a_{i,j} = 1} X_{i,j}$ be endowed with the natural structure of Hilbert bimodule over $\mathcal{A} := A_1 \oplus \cdots \oplus A_d$. Then since no row of $A$ is zero, left $\mathcal{A}$–action is faithful, and since no column of $A$ is zero, $X$ is full. Clearly $X$ is finite projective as a right module. We assume that there is a system of tracial states $\tau_1, \ldots, \tau_n$ on $A_1, \ldots, A_n$ respectively satisfying, for each pair of indices for which $a_{j,k} = 1$,
\[ \tau_k(\sum_r <x_r^k a x_r^n, k^r>) = \lambda_{j,k} \tau_j(a), \quad a \in A_j, \]

where $\lambda_{j,k}$ are such that $\sum_l \lambda_{j,k} = 1$. If $a_{i,j} = 1$, $X_{i,j}$ is full, finite projective and $\tau_i = 1$. Then, for any $a \in A_i$, one has $E_\tau(a) = \lambda_{i,j} E_\tau_j(a)$.
for some $\lambda_{j,k} > 0$. Here $\{x_{r}^{-k,j}\}_r$ if a basis of $X_{j,k}$. We set $\lambda_{j,k} = 0$ if $a_{j,k} = 0$. We will call $\{\tau_j\}$ a coherent set of traces. Note that $(\lambda_{j,k})$ is irreducible precisely when $A$ is. If each $A_j$ has a unique tracial state $\tau_j$, the set $\{\tau_j\}$ is coherent. This is indeed the case of Cuntz–Krieger algebras. Let $P_j$ be the identity of $A_j$.

Recall from [KPW] that $A = A' \cap O_X$ has a unique unital endomorphism $\sigma$ such that

$$\sigma(a)x = xa, \quad x \in X, a \in A' \cap O_X.$$ 

Recall also that if $A$ is irreducible, $O_X$ is simple [KPW], and if $A$ is aperiodic, $A$ is separable and has real rank zero and all $M_n(A_j)$ have the comparability property, then $O_X$ is simple and purely infinite (cf. Theorem 1.6). For all $r \geq 1 \mathcal{L}(X^r)$ is a finite direct sum of unital simple $C^*$-algebras, and its minimal central projections are $\sigma^r(P_1), \ldots, \sigma^r(P_d)$. One has $\mathcal{L}(X^r) = \mathcal{L}(A_j(X^r P_j))$, where $X^r P_j$ is regarded as an $A - A_j$ Hilbert bimodule.

Let $\tau_1, \ldots, \tau_d$ be a coherent choice of tracial states on $A_1, \ldots, A_d$ respectively. Let $\{u_i^{(r),j}\}$ be a basis of $X^r P_j$. Then the positive functional

$$a \in \mathcal{L}(A_j(X^r P_j)) \rightarrow \tau_j(\sum_i < u_i^{(r),j}, au_i^{(r),j}>)$$

is nonzero, tracial and independent of the basis. Let $\varepsilon_r(\tau_j)$ denote its norm. After normalization, we get a tracial state $T_j^{(r)}$ on $\mathcal{L}(X^r P_j)$. If $A_j$ has a unique tracial state, $T_j^{(r)}$ is the unique tracial state of $\mathcal{L}(X^r P_j)$. Consider a tracial state $\tau_r$ on $\mathcal{L}(X^r)$ which restricts to a multiple of $T_j^{(r)}$ on $\mathcal{L}(X^r P_j)$, and let $t_j^{(r)} = \tau(\sigma_r(P_j))$, so $t_j^{(r)} \geq 0$ and $\sum t_j^{(r)} = 1$. Then $\tau_r$ restricts to $\tau_{r-1}$ if and only if for all $a \in \mathcal{L}(X^{r-1} P_j)$, and all $j$,

$$t_j^{(r-1)}T_j^{(r-1)}(a) = \sum k t_k^{(r)} T_k^{(r)}(a \sigma^r(P_k)).$$

Now if $\{x_j\}$ is a basis of $X$, $\{x_{j_1} \ldots x_{j_r} P_k\}$ is a basis of $X^r P_k$, so

$$T_k^{(r)}(a \sigma^r(P_k)) = \varepsilon_r(\tau_k)^{-1} \varepsilon_r(\tau_k) \sum_{j_1 \ldots j_r} P_k x_{j_1}^* (x_{j_1} \ldots x_{j_r-1})^* ax_{j_1} \ldots x_{j_r-1} x_{j_r} P_k =$$

$$\lambda_{j,k} \varepsilon_r(\tau_k)^{-1} \varepsilon_r(\tau_k) ^{j} \tau_{j-1}(a).$$

So $\tau_r$ restricts to $\tau_{r-1}$ if and only if

$$\sum k \lambda_{j,k} t_k^{(r)} \varepsilon_r(\tau_k)^{-1} = t_j^{(r-1)} \varepsilon_r(\tau_j)^{-1}.$$

Set $v_k^{(r)} := \varepsilon_r(\tau_k)^{-1} t_k^{(r)}$. Then a solution is obtained choosing for $v^0$ the Perron–Frobenius eigenvector of the nonnegative matrix $(\lambda_{j,k})$ with the normalization $\sum v_k^{(0)} = 1$, and iteratively $v^r = \lambda^{-1} v^{r-1}$, where $\lambda$ is a positive eigenvalue of $(\lambda_{j,k})$. Note that if $A$ is aperiodic, then $(\lambda_{j,k})$ is aperiodic as well, so such a $v^0$ is the only possible solution. One can easily check that the tracial state $\tau$ thus obtained on $O_X^0$ satisfies

$$\tau(\sum_j x_j^* ax_j) = \lambda \tau(a), \quad a \in O_X^0,$$
and therefore gives rise to a KMS state of $O_X$.

Summarizing, we have proved the following result.

**5.5. Theorem** Let $A = (a_{i,j}) \in M_d(0,1)$ be an irreducible matrix, $A_1, \ldots, A_d$ unital simple $C^*$–algebras with a nonzero trace, and let, for any pair of indices for which $a_{i,j} = 1$, $X_{i,j}$ be a full, finite projective, Hilbert $A_i$–$A_j$ bimodule. Consider $X = \bigoplus_{i,j : a_{i,j} = 1} X_{i,j}$ as a Hilbert bimodule over $A_1 \oplus \cdots \oplus A_d$. Then any normalized Perron–Frobenius eigenvector of the irreducible matrix $(\lambda_{i,j})$ associated to a coherent system of tracial states on $A_1, \ldots, A_d$ defines, as described above, a KMS state of $O_X$ at inverse temperature $\beta = \log r((\lambda_{i,j}))$. In particular, if each $A_j$ has a unique tracial state, then there is a unique system of coherent tracial states, and the corresponding KMS state is the unique KMS state of $O_X$.

Note that if each $A_j$ has a unique trace and $A$ is aperiodic, then $O_X^0$ has a unique trace, which corresponds necessarily to the unique KMS state.

6. **Inverse Temperatures and Topological Entropy**

The aim of this section is to establish a relationship between inverse temperatures of extremal KMS states and the topological entropy of certain subshifts naturally associated to $(A, \gamma)$.

Let $\{y_i\}$ and $\{x_j\}$ be finite subsets of $A^1 \setminus \{0\}$ satisfying $\sum_i y_i^* y_i = I$ and $\sum_j x_j x_j^* = I$, and let $T = T_{\{y_i\}}$ and $S = S_{\{x_j\}}$ be the completely positive maps of $A^0$ defined in section 2. We define

$$h(T_{\{y_i\}}) := \lim_{n \to \infty} \frac{1}{n} \log \left( \sum_{i_1, \ldots, i_n} \|y_{i_1} \cdots y_{i_n} (y_{i_1} \cdots y_{i_n})^* \| \right)$$

$$h(S_{\{x_j\}}) := \lim_{n \to \infty} \frac{1}{n} \log \left( \sum_{j_1, \ldots, j_n} \|(x_{j_1} \cdots x_{j_n})^* x_{j_1} \cdots x_{j_n} \| \right).$$

In the case of the Matsumoto $C^*$–algebras $O_\Lambda$, $h(S)$ is the topological entropy of the shift homeomorphism $\sigma | \Lambda$, and it was shown to coincide with the maximal inverse temperture of certain KMS states in [MWY].

Note that since $(\sum \|y_{i_1} \cdots y_{i_n}\|^2)^{1/n} \geq 1$ for all $n$ and $\|y_i\| \leq 1$, for all $i$, then

$$0 \leq h(T_{\{y_i\}}) \leq \log(\text{Card}\{y_i\}),$$

and similarly,

$$0 \leq h(S_{\{x_j\}}) \leq \log(\text{Card}\{x_j\}).$$

Note however that if $A$ is a crossed product $C^*$–algebra by a single automorphism $\alpha$ then $h(\alpha) = h(\alpha^{-1}) = 0$. More generally, since the sequences defining $h(T)$ and $h(S)$ converge to their greatest lower bounds, we see that these are positive if and only if one has respectively

$$(\sum \|y_{i_1} \cdots y_{i_n}\|^2)^{1/n} \geq 1 + \varepsilon$$

$$(\sum \|x_{j_1} \cdots x_{j_n}\|^2)^{1/n} \geq 1 + \varepsilon$$
for all $n$ and some $\varepsilon > 0$.

We now associate to a fixed finite set of nonzero elements $\{x_j\} \subset A^1$ such that $\sum x_j x_j^* = I$, a one-sided subshift $\ell_{\{x_j\}}$, defined as follows. Set $\Sigma := \{1, \ldots, d\}$, where $d = \text{Card}\{x_j\}$. Then

$$\ell_{\{x_j\}} = \{\lambda \in \Sigma^\mathbb{N} : x_{\lambda_1} x_{\lambda_2} \cdots x_{\lambda_r} \neq 0, r \in \mathbb{N}\}.$$ 

Clearly $\ell_{\{x_j\}}$ is a closed subset of $\Sigma^\mathbb{N}$ mapped onto itself by the left shift homomorphism:

$$\sigma(\lambda)_i = \lambda_{i+1}.$$ 

Notice that if $(A, \gamma)$ does not result from a crossed product by an automorphism, or a proper corner endomorphism, $d \geq 2$.

Note also that, thank to the relation $\sum x_i x_i^* = I$, any $n$–tuple $(\lambda_1, \ldots, \lambda_r) \in \Sigma^r$ such that $x_{\lambda_1} x_{\lambda_2} \cdots x_{\lambda_r} \neq 0$ extends to an element of $\ell_{\{x_j\}}$. In particular, $\ell_{\{x_j\}}$ is nonempty.

Replacing the $T$–action $\gamma$ by the action $z \in \mathbb{T} \to \gamma'_z := \gamma_{z^{-1}}$, we see that we also have, for any finite subset $\{y_i\} \subset A^1 \setminus \{0\}$ such that $\sum y_i^* y_i = I$, a one–sided subshift:

$$\ell'_{\{y_j\}} = \{\lambda \in \Sigma'^\mathbb{N} : y_{\lambda_1} y_{\lambda_2} \cdots y_{\lambda_r} \neq 0, r \in \mathbb{N}\},$$

where $\Sigma'$ is the set of the first $d' := \text{Card}\{y_i\}$ positive integers.

We also introduce the following two–sided subshifts:

$$\Lambda_{\{x_j\}} = \{\lambda \in \Sigma^\mathbb{Z} : x_{\lambda_1} x_{\lambda_2} \cdots x_{\lambda_r+s} \neq 0, r \in \mathbb{Z}, s \in \mathbb{N}\},$$

and

$$\Lambda'_{\{y_j\}} = \{\lambda \in \Sigma'^\mathbb{Z} : y_{\lambda_1} y_{\lambda_2} \cdots y_{\lambda_r} \neq 0, r \in \mathbb{Z}, s \in \mathbb{N}\}.$$ 

Remark Even though it would seem more convenient to work with two–sided subshifts, we should point out that these may be rather small, in the sense that a finite word $(\lambda_1, \ldots, \lambda_r)$ occurring, e.g., in $\ell_{\{x_j\}}$ does not necessarily extend to a word in $\Lambda_{\{x_j\}}$. The following simple example well describes the situation. Consider the $C^*$–algebra $A = M_2(\mathbb{C}) \otimes \mathcal{C}(\mathbb{T})$, and define the following $2\pi$–periodic automorphic action $\alpha$ of $\mathbb{R}$:

$$\alpha_t = \text{adv}_t \otimes \beta_t$$

where

$$v_t = \text{diag}(e^{it}, 1)$$

and

$$\beta_t(f)(e^{ir}) = f(e^{i(r+t)}).$$

Let $e_{i,j}$, $i, j = 1, 2$, be a system of matrix units for $M_2(\mathbb{C})$, and define $x_1 = e_{1,2} \otimes I$, $x_2 = e_{2,1} \otimes u$, $y_1 = e_{1,2} \otimes I$, $y_2 = e_{1,1} \otimes u$, where $u(z) = z$, $z \in \mathbb{T}$. Then all the above elements are in $A^1$, and satisfy $x_1 x_1^* + x_2 x_2^* = I$, $y_1^* y_1 + y_2^* y_2 = I$, so $(A, \alpha)$ is a full $C^*$–dynamical system. Note that $x_1^2 = 0$, $x_1 x_2 \neq 0$, $x_2 x_1 = 0$, $x_2^2 \neq 0$, so

$$\Lambda_{\{x_j\}} = \{(\ldots, 2, 2, 2, \ldots)\}$$
while

$$\ell_{\{x_j\}} = \{(1, 2, 2, \ldots), (2, 2, 2, \ldots)\}.$$ 

Thus (1, 2) is a 2–word appearing in $\ell_{\{x_1, x_2\}}$ which can not be extended to any two–sided sequence of $\Lambda_{\{x_1, x_2\}}$.

We give a condition ensuring that $\ell_{\{x_j\}}$ and $\ell'_{\{y_i\}}$ are the positive parts of $\Lambda_{\{x_j\}}$ and $\Lambda'_{\{y_i\}}$ respectively.

**6.1. Proposition** Let $\{x_j\}$, be a finite subset of $A^1 \setminus \{0\}$ such that

$$\sum_j x_j x_j^* = I$$

and let $\ell_{\{x_j\}}$ and $\Lambda_{\{x_j\}}$ be the associated one–sided and two–sided subshifts respectively. If $\sum_j x_j^* x_j$ is invertible then $\Lambda_{\{x_j\}} \neq \emptyset$. Furthermore

$$\ell_{\{x_j\}} = \Lambda_{\{x_j\}}^+.$$ 

An analogous statement holds for $\ell'_{\{y_i\}}$ and $\Lambda'_{\{y_i\}}$.

**Proof** Since $\sum_j x_j x_j^* = I$, for any $(i_1, \ldots, i_r)$ such that $x_{i_1} \cdots x_{i_r} \neq 0$ there is an $r+1$ such that $x_{i_1} \cdots x_{i_r} x_{i_{r+1}} \neq 0$, and therefore there is a sequence $(i_n)_{n \geq 1}$ such that $x_{i_1} \cdots x_{i_n} \neq 0$ for all $n \in \mathbb{N}$. To complete the proof relative to the set $\{x_j\}$ it is now clear that it suffices to show, for any such $(i_n)$, the existence of $i_0 \in \Sigma$ such that $x_{i_0} x_{i_1} \cdots x_{i_n} \neq 0$ for all $n \geq 0$. If this were not the case, for all $k \in \Sigma$ there would exist $n_k$ such that $x_k x_{i_1} \cdots x_{i_{n_k}} = 0$. Letting $n = \max\{n_k, k \in \Sigma\}$, we must have $x_k x_{i_1} \cdots x_{i_n} = 0$ for all $k \in \Sigma$, and therefore, $\sum_k x_k^* x_k$ being invertible, $x_{i_1} \cdots x_{i_n} = 0$. This is now a contradiction. The statement relative to the set $\{y_i\}$ can be proved similarly.

In particular, if $A = (a_{ij})$ is a $\{0, 1\}$–matrix with no zero row or column, then the generating partial isometries $\{S_i\}$ of the Cuntz–Krieger algebra $O_A$ satisfy both

$$\sum_i S_i S_i^* = I,$$

and

$$S_i^* S_i = \sum_j a_{ij} S_j S_j^*,$$

thus $\sum_i S_i^* S_i$ is invertible. One has $\Lambda_{\{S_i\}} = \Lambda$ and $\ell_{\{S_i\}} = \Lambda_{\mathbb{N}}$. More generally, if $\Lambda$ is a nonempty subshift of $\Sigma^\mathbb{N}$ then the canonical set of generating partial isometries $\{S_i\}$ of the Matsumoto $C^*$–algebra $O_\Lambda$ still satisfy the above conditions (see 4.1) and we have also in this case $\Lambda_{\{S_i\}} = \Lambda$ and $\ell_{\{S_i\}} = \Lambda^+$. Another example is provided by the algebras generated by certain Cuntz–Krieger bimodules $X = \oplus_{(i,j) \in A_i \times A_j} X_{i,j}$ as described in section 5. More precisely, if $X_{i,j}$ is the Hilbert bimodule defined by a unital $^*$–isomorphism $\phi_{i,j} : A_i \rightarrow A_j$, then one can define $S_{i,j}$ to be the identity of $A_j$ regarded as an element of $X_{i,j}$. So $S_i = \sum_{j \in A_i} S_{i,j}$ are partial isometries of $O_X^1$ satisfying the Cuntz–Krieger relations with respect to $A = (a_{ij})$. Therefore $\Lambda_{\{S_i\}}$ is again the two–sided Markov subshift defined by the matrix $A = (a_{i,j})$. The example
arising from fractal geometry discussed in section 4 is in the same spirit, in that the natural basis of the generating module are generators of a Cuntz algebras, so the associated one or two–sided subshifts are full.

Note that all the examples above discussed have in common the fact that there is a multiplet \( \{ x_i \} \subset A^1 \) such that \( \sum_i x_i x_i^* = I \) consisting of elements with pairwise orthogonal ranges (and therefore they are necessarily partial isometries).

We start by establishing general estimates for the extremal inverse temperatures using the topological entropies of the associated subshifts. Recall [DGS] that for a one–sided (or two–sided) subshift \( (\ell, \sigma |_\ell) \) the topological entropy can be computed as

\[
h_{\text{top}}(\sigma |_\ell) = \lim_{n \to \infty} \frac{1}{n} \log(\theta_n(\ell)),
\]

where \( \theta_n(\ell) \) is the cardinality of the set \( \ell^n \) of distinct words of length \( n \) occurring in \( \ell \).

6.2. Proposition Let \( \{ y_i \}, \{ x_j \} \subset A^1 \setminus \{ 0 \} \) be finite subsets such that \( \sum_i y_i^* y_i = I \) and \( \sum_j x_j x_j^* = I \), and let \( \ell(\{ y_i \}) \) and \( \ell(\{ x_j \}) \) be the corresponding one–sided subshifts, defined as above. Then

\[
\beta_{\min} \geq -h(T(\{ y_i \})) \geq -h_{\text{top}}(\sigma |_{\ell(\{ y_i \})}),
\]

\[
\beta_{\max} \leq h(S(\{ x_j \})) \leq h_{\text{top}}(\sigma |_{\ell(\{ x_j \})}).
\]

Proof By Prop. 2.3 and the triangle inequality \( \beta_{\min} \geq -h(T_{\{ y_i \}}) \) and \( \beta_{\max} \leq h(S_{\{ x_j \}}) \). The rest follows from \( \sum_{i_1, \ldots, i_n} \| x_{i_1} \ldots x_{i_n} \|^2 \leq \theta_n(\ell(\{ x_j \})) \), and its analogue for \( \{ y_i \} \).

We shall see that in the general situation if it is possible to choose the multiplets \( \{ y_i \} \) and \( \{ x_j \} \) carefully, then the topological entropies of the corresponding subshifts lead to the extremal inverse temperatures of KMS states. We first present an intermediate result, which gives a sufficient condition for \( h(T) \) and \( h(S) \) to coincide with the topological entropy of the associated subshifts.

6.3. Proposition Set

\[
l_n := \min \{ \| y_{i_1} \ldots y_{i_n} \|^2 : y_{i_1} \ldots y_{i_n} \neq 0 \},
\]

\[
m_n := \min \{ \| x_{j_1} \ldots x_{j_n} \|^2 : x_{j_1} \ldots x_{j_n} \neq 0 \}.
\]

If

\[
\limsup_n l_n^{1/n} = 1
\]

then

\[
h(T_{\{ y_i \}}) = h_{\text{top}}(\sigma |_{\ell(\{ y_i \})}).
\]

Similarly, if

\[
\limsup_n m_n^{1/n} = 1
\]

then

\[
h(S_{\{ x_j \}}) = h_{\text{top}}(\sigma |_{\ell(\{ x_j \})}).
\]
Proof We shall prove only the first statement. Let $\theta_n$ denote the number of words of length $n$ occurring in $\ell' \{y_i\}$, i.e. the number on $n$-tuples $(i_1, \ldots, i_n) \in \Sigma^n$ such that $y_{i_1} \cdots y_{i_n} \neq 0$. Then given $\varepsilon > 0$, for infinitely many indices $n$,

$$(1 - \varepsilon)\theta_n^{1/n} \leq \theta_n^{1/n} \leq \left(\sum \|y_{i_1} \cdots y_{i_n}\|^2\right)^{1/n} \leq \theta_n^{1/n},$$

hence taking the logarithm of the limit over $n$,

$$h(T) = \lim \frac{1}{n} \log \theta_n = h_{\text{top}}(\sigma | \ell' \{y_i\}).$$

The previous result applies whenever one is working with a multiplet consisting of partial isometries with mutually orthogonal ranges.

We next show that, strengthening the hypotheses of the previous result, all the inequalities of Proposition 6.2 become equalities. More precisely, if the positive evaluations of a tracial state $\tau$ on the iterated basic monomials, e.g. $(x_{i_1} \cdots x_{i_n})^* x_{i_1} \cdots x_{i_n}$, do not get too small when $n$ increases, then the maximal inverse temperature $\beta_{\text{max}}$ can be approximated iterating the operator $s'$ on $\tau$.

The proof is inspired by an analogous result in [MWY] for the Matsumoto algebras associated to subshifts.

6.4. Theorem Let $\{y_i\}, \{x_j\}$ be finite subsets of $A^1$ such that $\sum y_i^* y_i = I$ and $\sum x_j x_j^* = I$. If $\tau$ is a tracial state on $A^0$, set

$$\mu_n(\tau) := \min\{\tau(y_{i_1} \cdots y_{i_n} (y_{i_1} \cdots y_{i_n})^*) : y_{i_1} \cdots y_{i_n} \neq 0\},$$

$$\nu_n(\tau) := \min\{\tau((x_{j_1} \cdots x_{j_n})^* x_{j_1} \cdots x_{j_n}) : x_{j_1} \cdots x_{j_n} \neq 0\}.$$

If

$$\lim \sup \frac{1}{n} \mu_n^{1/n}(\tau) = 1$$

then

$$\beta_{\text{min}} = -h(T_{\{y_i\}}) = -h_{\text{top}}(\sigma | \ell' \{y_i\}) = -\lim \sup \frac{1}{n} \log(\delta_n(\tau)).$$

In particular, if $\tau$ is the restriction of a KMS state $\omega$, then $\omega$ has minimal inverse temperature. If instead

$$\lim \sup \frac{1}{n} \nu_n^{1/n}(\tau) = 1$$

then

$$\beta_{\text{max}} = h(S_{\{x_j\}}) = h_{\text{top}}(\sigma | \ell' \{x_j\}) = \lim \sup \frac{1}{n} \log(\epsilon_n(\tau)).$$

If $\tau$ is the restriction of a KMS state $\omega$, then $\omega$ has maximal inverse temperature.

Proof We shall prove only the first statement. For infinitely many indices $n$,

$$(1 - \varepsilon)^n \theta_n \leq \theta_n \mu_n(\tau) \leq \delta_n(\tau) = \tau(\sum y_{i_1} \cdots y_{i_n} (y_{i_1} \cdots y_{i_n})^*) \leq \theta_n,$$
hence taking the \( n \)-th root and then the logarithm of the limit over a subsequence, by the arbitrariness of \( \varepsilon \), we get
\[
\limsup_n \frac{1}{n} \log(\delta_n(\tau)) = h_{\text{top}}(\sigma | \nu_{\{y_i\}}) = h(T_{\{y_i\}}).
\]
The last equality follows from Prop. 6.3. Now \( \delta_n(\tau)^{1/n} = \| t_n(\tau) \|^{1/n} \leq \| t^m(\tau) \|^{1/n} \to r(t') \), so
\[
h_{\text{top}}(\sigma | \nu_{\{y_i\}}) \leq -\beta_{\text{min}},
\]
which, together with Proposition 6.2, proves the first statement. If in particular \( \tau \) arises from a KMS state \( \omega \) at inverse temperature \( \beta \), then \( \delta_n(\tau) = e^{-n\beta} \), so \( \beta = \beta_{\text{min}} \).

The previous result can be regarded as an analogue of the well known fact from Perron–Frobenius theory that for an irreducible nonnegative matrix \( A \) the maximal eigenvalue can be approximated by \( r(A) = \limsup_n \| A^n(\tau) \|^{1/n} \), where \( \tau \) is any vector with positive entries [G].

6.5. Corollary If there is a finite subset \( \{y_i\} \) (resp. \( \{x_j\} \)) of \( \mathcal{A}^1 \) such that

1. \[
\sum_i y_i^* y_i = I \quad \text{(resp. } \sum_j x_j x_j^* = I),
\]
2. \[
y_i y_h^* = 0, \quad i \neq h \quad \text{(resp. } x_j x_k^* = 0, j \neq k),
\]
3. The algebra \( \mathcal{C} \) generated by all finite products of the form
\[
y_{i_1} \cdots y_{i_n} (y_{i_1} \cdots y_{i_n})^* \quad \text{(resp. } x_{j_1} \cdots x_{j_n} x_{j_1} \cdots x_{j_n})
\]
is finite–dimensional,
then the conclusions of the previous theorem hold for any faithful tracial state on \( \mathcal{A}^0 \).

Proof Just note that under our assumptions any of the nonzero basic monomials is a projection, and therefore it majorizes a minimal projection in \( \mathcal{C} \). It follows that \( \lim_n \mu_n(\tau)^{1/n} = 1 \) (resp. \( \lim_n \nu_n(\tau)^{1/n} = 1 \)) for any faithful trace \( \tau \) on \( \mathcal{A}^0 \), so the previous theorem applies.

In particular, this result applies to all the examples discussed at the beginning of this section.

7. Topological entropy of canonical ucp maps
Let \( \{x_j\} \) be a finite set of a \( C^* \)-algebra \( \mathcal{A} \) of grade 1 such that \( \sum_j x_j x_j^* = I \). In the previous section we have associated to this set a one–sided subshift \( (\ell_{\{x_j\}}, \sigma | \ell_{\{x_j\}}) \) of the Bernoulli shift \( (\Sigma^d, \sigma) \), where \( \Sigma \) is the state space of the first \( d \) positive integers, and \( d = \text{Card}\{j : x_j \neq 0\} \), in a way that, under suitable circumstances, its classical topological entropy equals an extremal inverse
temperature of KMS states. One can also associate to the subset \( \{x_j\} \) a unital completely positive map defined by

\[
\sigma_{\{x_j\}} : T \in \mathcal{A} \to \sum_j x_j T x_j^*.
\]

In view of the results of the previous section, we ask whether there is a relationship between the the Voiculescu topological entropy of this map and the classical topological entropy of the subshift \( \ell_{\{x_j\}} \). If \( \mathcal{A} = \mathcal{O}_n \) is the Cuntz algebra with generators \( S_1, \ldots, S_n \), Choda shows in [Ch] that the topological entropy of the canonical endomorphism \( \sigma_{\{S_j\}} \) is \( \log(n) \), i.e. the topological entropy of the associated full shift. In the more general case where \( \mathcal{A} = \mathcal{O_A} \) is a Cuntz–Krieger algebra, and \( \{S_j\} \) is the canonical set of generating partial isometries, Boca and Goldstein [BG] have recently computed the Voiculescu entropy [V] of this map, and they have shown that it equals the logarithm of the spectral radius of \( \mathcal{A} \), or, in other words, the classical topological entropy of the underlying finite type subshift \( \ell_A \) [DGS]. However, special cases, although extreme from a certain point of view, of full periodic \( C^* \)-dynamical systems are the crossed products by an automorphism \( \alpha \). Brown showed in [B] that \( h_{\mathcal{A} \otimes \mathbb{Z}}(Ad(u)) = h_{\mathcal{A} \otimes \mathbb{Z}}(\alpha) \), where \( u \) is a unitary implementing \( \alpha \). It is obvious that the associated subshift is in this case trivial, so its entropy is zero.

In Theorem 7.4 we give, for full periodic \( C^* \)-dynamical systems, an upper bound for \( h(\mathcal{A} \otimes \mathbb{Z}) \) which allows to recover the above discussed results as special cases. We will then apply this result to find new examples, among the Matsumoto algebras associated to non finite type subshifts, where

\[
h(\sigma_{\{x_j\}}) = h_{\text{top}}(\ell_{\{x_j\}})
\]

still holds.

In the beginning of this section the automorphic action of the circle plays no role, therefore we shall not assume that the \( x_j \)'s are of grade 1. We define the associated one–sided subshift \( \ell = \ell_{\{x_j\}} \) as in the previous section.

We now show that the ucp map \( \sigma_{\{x_j\}} \) can be understood as a noncommutative subshift. Let \( \sigma \upharpoonright \ell \) be the restriction of \( \sigma \) to \( \ell \) and \( T_{\ell} \) the *-monomorphism of \( \mathcal{C}(\ell) \) obtained by transposing \( \sigma \upharpoonright \ell \), i.e.

\[
T_{\ell} f(x) = f(\sigma(x)), \quad x \in \ell.
\]

Also, we will consider a natural basis of neighborhoods for \( \ell \). For each \( (i_1, \ldots, i_r) \in \ell^r \), consider the cylinder set

\[
[i_1 \ldots i_r] = \{(x_j)_{j} \in \ell : x_1 = i_1, \ldots, x_r = i_r\}.
\]

For a fixed \( r \in \mathbb{N} \), these constitute an open and closed cover of \( \ell \) with cardinality \( \theta_r = \text{Card} \ell^r \).

**7.1. Proposition** Let \( \{x_j\} \) be a finite subset of \( \mathcal{A} \) such that \( \sum_j x_j x_j^* = I \), and let \( (\ell, \sigma \upharpoonright \ell) \) be the associated one–sided subshift. Then there is a unique unital completely positive map \( \Phi : \mathcal{C}(\ell) \to \mathcal{A} \) taking the characteristic function
of \([i_1 \ldots i_r]\) to \(x_{i_1} \ldots x_{i_r} (x_{i_1} \ldots x_{i_r})^*\). One has \(\sigma_{\{x_j\}} \circ \Phi = \Phi \circ T_\ell\). Moreover, if the sequence \((m_n)_n\) defined in Proposition 6.3 does not converge to 0, then \(\Phi\) is faithful.

**Proof** We first notice that, for each \(r \in \mathbb{N}\), the map \(\Phi_r : \Sigma^N \to \Sigma^r\) projecting onto the first \(r\) coordinates takes \(\ell\) onto the subset \(\ell'\) of \(\subseteq \Sigma^r\) consisting of \(\theta_r\) elements. Therefore there is a natural *-monomorphism \(\phi_r : \mathbb{C}^{\theta_r} \to \mathbb{C}(\ell)\) taking a \(\theta_r\)-tuple assuming value 1 on \((i_1, \ldots, i_r)\) and zero elsewhere to the characteristic function of \([i_1 \ldots i_r]\). Similarly, there are, for \(r \leq s\), natural *-monomorphisms \(\phi_{r,s} : \mathbb{C}^{\theta_r} \to \mathbb{C}^{\theta_s}\) such that \(\phi_s \phi_{r,s} = \phi_r\). Since the cylinder sets \(\{i_r \ldots i_r\}, r \in \mathbb{N}\) form a basis of closed and open sets for \(\ell\), we see that the image of all the \(\phi_r\) is dense. It follows that \(\mathbb{C}(\ell)\) is the inductive limit of the \(\mathbb{C}^{\theta_r}\)'s under the maps \(\phi_{r,s}\).

We define the ucp map \(\Phi_r : \mathbb{C}^{\theta_r} \to \mathcal{A}\) which takes the characteristic function of \([i_1, \ldots, i_r]\) to the element \(x_{i_1} \ldots x_{i_r} (x_{i_1} \ldots x_{i_r})^*\). Since \(\Phi_s \phi_{r,s} = \Phi_r\), thanks to \(\sum_j x_j x_j^* = I\), we get a ucp map \(\Phi_r : \mathbb{C}^{\theta_r} \to \mathcal{A}\).

Since \(\|\Phi_r(f)\| \leq \|f\|\), \(\Phi_r\) extends to a ucp map on \(\mathbb{C}(\ell)\), which is the desired \(\Phi\). The relation \(\Phi \circ T_\ell = \sigma_{\{x_j\}} \circ \Phi\) can be easily checked on the total set of characteristic functions of cylinder sets.

We now construct a conditional expectation \(E_r : \mathbb{C}(\ell) \to \mathbb{C}^{\theta_r}\). Choose a faithful normalized Borel measure \(\mu\) on \(\ell\), and associate to a function \(f \in \mathbb{C}(\ell)\) the \(\theta_r\)-tuple with coordinates

\[
E_r(f)(i_1, \ldots, i_r) = \frac{1}{\mu([i_1 \ldots i_r])} \int_{[i_1 \ldots i_r]} f(x)d\mu(x),
\]

for each \((i_1, \ldots, i_r) \in \ell'\). One can easily check that \(E_r(I) = I\) and that \(E_r(fa) = E_r(f)a, a \in \mathbb{C}^{\theta_r}\). Clearly \((E_r)_r\) converges pointwise in norm to the identity. Assume now that \(f \in \mathbb{C}(\ell)\) is a positive element such that \(\Phi(f) = 0\). Then \(\Phi E_r(f)\) converges to 0. On the other hand

\[
\|\Phi E_r(f)\| = \| \sum_{i_1, \ldots, i_r} \left( \frac{1}{\mu([i_1 \ldots i_r])} \int_{[i_1 \ldots i_r]} f(x)d\mu(x) \right) x_{i_1} \ldots x_{i_r} (x_{i_1} \ldots x_{i_r})^* \| \geq \left( \frac{1}{\mu([i_1 \ldots i_r])} \int_{[i_1 \ldots i_r]} f(x)d\mu(x) \right) m_r,
\]

therefore if \(m_r\) does not converge to 0, a subsequence of \(E_r(f)\) converges to 0, so \(f = 0\).

Note that if \((m_n)_n\) does not converge to 0 then \(\limsup_n m_n^{1/n} = 1\) thus we are in the position of applying Proposition 6.3. A particularly important case is when the \(x_j\)'s have pairwise orthogonal ranges.

**7.2. Corollary** If there is a finite subset \(\{x_j\} \subset \mathcal{A}\) such that

\[
\sum_j x_j x_j^* = I, \quad x_i^* x_j = 0, \quad i \neq j,
\]

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then the ucp map $\Phi : C(\ell) \to A$ constructed in the previous proposition is in fact a $^*$–monomorphism. The restriction of $\sigma_{\{x_j\}}$ to $C(\ell)$ corresponds to the one–sided subshift $T_\ell$.

For any subshift $A$ the Matsumoto $C^*$–algebra $O_A$ satisfies the requirements of the previous result [M].

Let now $(A, \gamma)$ be a full $C^*$–dynamical system over $T$. Our next aim is to compare the topological entropy of the ucp map $\sigma_{\{x_j\}}$, when $\{x_1, \ldots, x_d\}$ is a finite subset of $A^1 \setminus \{0\}$ such that $\sum_j x_j x_j^* = I$, with entropic properties of the canonical homogeneous $C^*$–algebra. We refer the reader respectively to [V] for the notion of topological entropy for nuclear $C^*$–algebras and to [B] for its generalization to exact $C^*$–algebras, and to [BG] for the generalization of the topological entropy $ht(P)$ of a ucp map $P$ on a unital exact $C^*$–algebra.

We start defining an entropic quantity for the homogeneous subalgebra $A^0$ which, in the case where $A = A^0 \rtimes_{\alpha} Z$, reduces to the topological entropy of $\alpha$. We shall assume that $A^0$ is an exact $C^*$–algebra. We start fixing a choice of nonzero elements $\{x_i\}$ of $A^1$ such that $\sum_i x_i x_i^* = I$. Let us define, for $\mu = (i_1, \ldots, i_d) \in T^d$, $x_\mu := x_{i_1} \ldots x_{i_d}$. We set $x_0 = I$ and $|0| = 0$. We shall also consider the operators $q_{\alpha,\beta} = x_{i_\alpha} x_{i_\beta} \in A^0$ for $|\alpha| = |\beta| \geq 0$. Note that $q_{0,0} = I$.

Let be given $\pi : A^0 \to B(H)$ a faithful $^*$–representation, $\omega \subset A^0$ a finite subset and $\delta \geq 0$. Set, for $n \in \mathbb{N}$,

$$\omega^{(n)} = \{x_{i_\mu} q_{\delta,\delta} T q_{\epsilon,\epsilon} x_{i_\nu}, T \in \omega, |\mu| = |\nu| \leq n - 1, |\delta| = |\delta'| \leq n - 1, |\epsilon| = |\epsilon'| \leq n - 1\}.$$

Note that $\omega^{(n)}$ depends on the contractions $\phi_{x_i} : T \in A^0 \to x_i^* Tx_j$ rather then on the elements $\{x_i\}$. We define, for $a > 0$,

$$ht_a(\pi, \{\phi_{x_i} x_j\}, \omega, \delta) = \limsup_n \frac{1}{n} \log \text{rcp}(\pi, \omega^{(n)}, \delta, \theta_{n-1}^{-1}),$$

$$ht_a(\pi, \{\phi_{x_i} x_j\}, \omega, \delta) = \sup_{\delta > 0} ht_a(\pi, \{\phi_{x_i} x_j\}, \omega, \delta),$$

$$ht_a(\pi, \{\phi_{x_i} x_j\}) = \sup_{\omega \subset A^0 \text{finite}} \text{ht}_a(\pi, \{\phi_{x_i} x_j\}, \omega).$$

We will use the same notation as [B]. Thus in particular for a finite set $\Omega \subset A^0$, $\text{rcp}(\pi, \Omega, \delta)$ is computed with respect to factorizations of completely positive contractions, not necessarily unital, from $A^0$ to $B(H)$ via finite dimensional $C^*$–algebras.

Brown proves in [B] that $\text{rcp}(\pi, \Omega, \delta)$ is independent of the choice of $\pi$. We will regard $A$ faithfully represented on a Hilbert space $H$, and we take $\pi$ to be the inclusion $\iota_{A^0}$ of $A^0$ in $B(H)$. Moreover we will avoid indicating $\pi$ in the above definitions.

We anticipate, for later use, the following immediate consequence of the definition.

7.3. Lemma

(1) If $\alpha$ is an automorphism of a unital $C^*$–algebra $A^0$, $A = A^0 \rtimes_{\alpha} Z$, and $u$ is any unitary of $A^1$ implementing $\alpha$ on $A^0$,

$$ht_a(\phi_{u^* u}, \omega) = ht(a^{-1}).$$
(2) If $\omega$ is such that for some finite dimensional $C^*$-subalgebra $\mathcal{D} \subset \mathcal{A}^0$ which is the range of a conditional expectation, $\omega^{(n)} \subset \mathcal{D}$ except for finitely many $n$, then $ht_n(\{\phi_{x_i,x_j}\}, \omega) = 0$.

Proof (1) for $n \in \mathbb{N}$, $\theta_n = 1$ and $\omega^{(n)} = \omega \cup \ldots \cup \alpha^{-(n+1)}(\omega)$. (2) Let $E : \mathcal{A}^0 \to \mathcal{D}$ be a conditional expectation, and set $\phi := E$, $\psi := \iota_\mathcal{D}$, so that for $T \in \omega^{(n)}$ and infinitely many indices $n$, $\psi \circ \phi(T) = T$. This implies that $\text{rcp}(\omega^{(n)}, \frac{\delta}{\theta_{n-1}}) \leq \text{rank}(\mathcal{D})$.

Let now $\omega \subset \mathcal{A}^0$ be a finite subset and $n_0 \in \mathbb{N}$. A typical finite subset of $\mathcal{A}$ has the form

$$F(\omega, n_0) = \{x_i, T \in \omega, |\gamma| \leq n_0\}.$$ 

Our aim is to show the following result.

7.4. Theorem Let $(\mathcal{A}, \gamma, T)$ be a full $C^*$-dynamical system, with $\mathcal{A}^0$ exact. Let $\sigma_{\{x_i\}}$ and $\ell_{\{x_i\}}$ be the ucp map and the one-sided subshift associated to a set $\{x_i\} \subset \mathcal{A}^1$ satisfying $\sum_i x_i x_i^* = I$. Then for any finite subset $\omega \subset \mathcal{A}^0$ and $n_0 \in \mathbb{N}$,

$$ht(\sigma_{\{x_i\}}, F(\omega, n_0)) \leq h_{\text{tcp}}(\sigma |_{\ell_{\{x_i\}}}) + ht(\{\phi_{x_i,x_j}\}, \omega).$$

We will prove this theorem combining appropriate analogues of arguments of Brown [B] for crossed product $C^*$-algebras and Boca and Goldstein [BG] for Cuntz-Krieger algebras.

Motivated by [B], we define certain cp maps. Let $F \subset \mathcal{N}_0$ be a finite subset. Set

$$S_F : T \in \mathcal{A} \to (x_{\alpha}^* m_{|\alpha| - |\beta|}(T) x_{\beta})_{\alpha, \beta \in I_F} \in M_{\theta_F}(\mathcal{A}^0).$$

Here $I_F := \cup_{r \in F'} \iota_r$, $\theta_F = \sum_{r \in F'} \theta_r$, and, for $k \in \mathbb{Z}$, $m_k : \mathcal{A} \to \mathcal{A}^k$ is the natural projection. Note that $S_F$ is contractive and cp. For a contractive cp map $\phi : \mathcal{A}^0 \to \mathcal{B}$ set

$$\phi_F := \iota \otimes \phi \circ S_F : \mathcal{A} \to M_{\theta_F}(\mathcal{B})$$

which is contractive and cp.

Let $f \in l^2(\mathbb{N}_0)$ have support in $F$, and define the cp map

$$\tilde{S}_{F,f} : T = (T_{\alpha, \beta}) \in M_{\theta_F}(\mathcal{B}(\mathcal{H})) \to \sum_{\alpha, \beta \in I_F} f(|\alpha|) \overline{f(|\beta|)} x_{\alpha} T_{\alpha, \beta} x_{\beta}^* \in \mathcal{B}(\mathcal{H}).$$

Note that $\tilde{S}_{F,f}(I) = \|f\|^2 I$. Again, for a contractive cp map $\psi : \mathcal{B} \to \mathcal{B}(\mathcal{H})$, define

$$\psi_{F,f} := \tilde{S}_{F,f} \circ \iota \otimes \psi : M_{\theta_F}(\mathcal{B}) \to \mathcal{B}(\mathcal{H}).$$

So $\psi_{F,f}$ is cp contractive if $\|f\|_2 \leq 1$.

Finally, for a contractive cp map $\Lambda : \mathcal{A}^0 \to \mathcal{B}(\mathcal{H})$ set

$$\Phi_{\Lambda,F} := \tilde{S}_{F,f} \circ \iota \otimes \Lambda \circ S_F : \mathcal{A} \to \mathcal{B}(\mathcal{H}).$$
Note that in particular, if $\phi$ and $\psi$ are as above,

$$\Phi_{\psi \circ \phi, F} = \psi_{F, f} \circ \phi_F$$

which factors through the algebra $M_{\theta_F}(\mathcal{B})$. One can easily show that for an element of fixed degree $X \in \mathcal{A}^0$,

$$\Phi_{\ell, \mathcal{A}^0, F, f}(X) = \sum_{p \in F \cap (F + k)} f(p)\overline{f(p - k)}X.$$

The following lemma is our analogue of Lemma 3.4 in [B].

**7.5. Lemma** Let $\omega$ be a finite subset of the unit ball of $\mathcal{A}^0$, $n_0 \in \mathbb{N}$ and $\delta > 0$. Consider the set $F'(\omega, n_0) := \{Tx_\gamma, T \in \omega, |\gamma| \leq n_0\}$. Then there is a finite set $F \subset \mathbb{N}_0$ which depends only on $n_0$ and $\delta$ and not on $\omega$ such that

$$\theta_F rcp(F'(\omega, n_0), \delta) \leq \frac{\delta}{2 \max_{p \in F} \theta_p}.$$ 

**Proof** We proceed as in the proof of Lemma 3.4 in [B]. Let $f \in \ell^2(\mathbb{Z})$ be a function with finite support $E$, $\|f\|_2 \leq 1$ such that $|f \ast \overline{f} - 1| < \delta/2$, $n = 0, 1, \ldots, n_0$. Here $\overline{f}(p) = \overline{f(-p)}$. Replacing $f$ with a suitable translate if necessary, we may assume $E \subset \{n \in \mathbb{Z} : n \geq n_0\}$. Set $F = E \cup (E - 1) \cup \cdots \cup (E - n_0)$, which is a subset of $\mathbb{N}_0$. Note that $F$ depends on $\delta$ and $n_0$ but not on $\omega$. Consider contractive cp maps $\phi : \mathcal{A}^0 \to \mathcal{B}$ and $\psi : \mathcal{B} \to \mathcal{B}(\mathcal{H})$ with $\mathcal{B}$ finite dimensional such that

$$\|\psi \circ \phi(a) - a\| < \frac{\delta}{2 \max_{p \in F} \theta_p}, \quad a \in \bigcup_{|\gamma| \leq n_0} \bigcup_{|\alpha|, |\beta| \in F, |\alpha| = |\beta| + |\gamma|} x_\alpha^* x_\gamma x_\beta.$$

Let us choose $\mathcal{B}$ with minimal rank. Then the cp contractive map $\Phi_{\psi \circ \phi, F, f}$ factors through the finite dimensional algebra $M_{\theta_F}(\mathcal{B})$, which has rank $\theta_F \text{rank}(\mathcal{B})$. We are thus left to show that for $a \in F'(\omega, n_0)$,

$$\|\Phi_{\psi \circ \phi, F, f}(a) - a\| < \delta.$$ 

We write $a = Tx_\gamma$ with $T \in \omega, |\gamma| \leq n_0$. Then the l.h.s. is bounded by

$$\|\Phi_{\psi \circ \phi, F, f}(Tx_\gamma) - \Phi_{\ell, \mathcal{A}^0, F, f}(Tx_\gamma)\| + \|\Phi_{\ell, \mathcal{A}^0, F, f}(Tx_\gamma) - Tx_\gamma\|.$$ 

Now the computation of $\Phi_{\ell, \mathcal{A}^0, F, f}$ given before on elements with fixed degree shows that the second summand is bounded by

$$\sum_{p \in F \cap (F + |\gamma|)} |f(p)\overline{f(p - |\gamma|)} - 1| \|Tx_\gamma\| = \|f \ast \overline{f} - 1\| \|Tx_\gamma\| < \delta/2.$$ 

We now evaluate the first summand.

$$\|\Phi_{\psi \circ \phi, F, f}(Tx_\gamma) - \Phi_{\ell, \mathcal{A}^0, F, f}(Tx_\gamma)\| \leq \|\mathcal{L}_{\theta_F} \otimes (\psi \circ \phi - \ell_{\mathcal{A}^0}) \circ S_F(Tx_\gamma)\| =$$
\[ \| \sum_{p \in F \cap (F + |\gamma|)} (\sum_{|\alpha| = p, |\beta| = p - |\gamma|} e_{\alpha, \beta} \otimes (\psi \circ \phi - \iota_{A^0})(x_{\alpha}^* T x_{\beta})) \| = \max_{p \in F \cap (F + |\gamma|)} \| \sum_{|\alpha| = p, |\beta| = p - |\gamma|} e_{\alpha, \beta} \otimes (\psi \circ \phi - \iota_{A^0})(x_{\alpha}^* T x_{\beta}) \| \leq \delta/2. \]

The last inequality follows from our choice of \( \psi \) and \( \phi \) and from the fact that if a matrix \( A \in M_{n,k}(A^0) \) has entries of norm bounded by \( c \) then \( \| A \| \leq (hk)^{1/2}c. \)

Consider the contractive cp map

\[ \rho_r : A \to M_{\theta}(A) \]

taking \( T \in A \) to the matrix \( (x_{\mu}^* T x_{\nu})_{\mu, \nu} \). (One can easily check that \( \rho_r \) is a unital \( \ast \)-monomorphism with image the corner algebra \( P_r M_{\theta}(A) P_r \), where \( P_r = (x_{\mu}^* x_{\nu}) \) is an orthogonal projection.)

For \( n, n_0 \in \mathbb{N}, m \geq n + n_0 - 1, l = 0, \ldots, n - 1, |\alpha| \leq n_0, T \in A^0 \), we compute

\[ \rho_m(\sigma^l(x_{\alpha} T)) = \rho_m(\sum_{|\eta| = l} x_{\eta} T x_{\eta}^*) = (\sum_{|\eta| = l} x_{\eta}^* x_{\eta} T x_{\eta} x_{\eta}^*)_{\mu, \nu} \in \ell^m. \]

Writing \( \mu = \gamma \mu', |\gamma| = |\eta| + |\alpha| = l + |\alpha| \), and \( \nu = \delta \nu', |\delta| = |\eta| = l \) we have:

\[ \rho_m(\sigma^l(x_{\alpha} T)) = (\sum_{|\eta| = m} x_{\mu'}^* q_{\gamma, \eta} T q_{\eta, \delta} x_{\nu'})_{\mu, \nu} \in \ell^m. \]

Setting, again, \( \nu' = \epsilon \nu'' \), with \( |\epsilon| = |\mu'| = m - l - |\alpha| \), we obtain that

\[ \rho_m(\sigma^l(x_{\alpha} T)) = (\sum_{|\eta| = l} (x_{\mu'}^* q_{\gamma, \eta} T q_{\eta, \delta} x_{\nu''})_{\mu, \nu} \in \ell^m. \]

If now \( T \) ranges over a finite subset \( \omega \subset A^0 \), we see that the image of

\[ F(\omega, n_0) = \{ x_{\alpha} T, T \in \omega, |\alpha| \leq n_0 \} \subset A \]

under the maps \( \rho_{n+n_0-1} \circ \sigma^l, l = 0, \ldots, n - 1 \), is constituted by matrices of size \( \theta_{n+n_0-1} \) with entries sums of at most \( \theta_{n-1} \) elements in \( F(\omega(n+n_0), n_0) \).

**Proof of Theorem 7.4.** We apply the previous Lemma to the sets \( F^*(\omega(n+n_0), n_0) \) for fixed \( n_0 \) and \( \omega \) and all \( n \in \mathbb{N} \). Note that the corresponding set \( F \) can be chosen independent of \( n \). We can thus find for each \( n \in \mathbb{N} \) a contractive cp map \( \Lambda_n : A \to B(\mathcal{H}) \) factoring through a finite dimensional algebra \( B \) of rank

\[ \theta_{F}\text{rcp}(\cup_{|\gamma| \leq n_0} \cup_{|\alpha|, |\beta| \in F, |\alpha| = |\beta| + |\gamma|} x_{\alpha}^* \omega(n+n_0) x_{\beta}^* \leq \frac{\delta}{\theta_{n+n_0-1} \max_{p \in F} \theta_p} \]

\[ \theta_{F}\text{rcp}(\omega(n+n_0+\max F) \leq \frac{\delta}{2\theta_{n+n_0-1+\max F} \max_{p \in F} \theta_p}) \]

such that

\[ \| \Lambda_n(T x_{\gamma}) - \pi(T x_{\gamma}) \| < \frac{\delta}{\theta_{n+n_0-1} 2}, \quad T \in \omega(n+n_0), \quad |\gamma| \leq n_0. \]
Consider the ucp map $\Psi_m : M_\theta(B(H)) \to B(H)$ taking the matrix $(t_{\mu, \nu})$ to the operator
$$\sum_{|\mu|=|\nu|=m} \pi(x_{\mu}) t_{\mu, \nu} \pi(x_{\nu})^*.$$ Then the map $\Psi_{n+n_0-1} \circ \iota M_{n+n_0-1} \otimes \Lambda_n \circ \rho_{n+n_0-1} : \mathcal{A} \to B(H)$ factors through an algebra of rank bounded by
$$\theta_{n+n_0-1} \theta F \text{rcp}(\omega^{(n+n_0+\max F)}, \delta) \frac{2\theta^2}{2\theta n+n_0+\max F-1 \max F \theta_p}.$$ Thus if we show that
$$\|\Psi_{n+n_0-1} \circ \iota \otimes \Lambda_n \circ \rho_{n+n_0-1}(\sigma^l(x, T)) - \sigma^l(x, T)\| < \delta$$ for $T \in \omega, |\gamma| \leq n_0$ and $l = 0, \ldots, n-1$, we will deduce that
$$\text{rcp}(F(\omega, n_0) \cup \cdots \cup \sigma^{n-1}(F(\omega, n_0)), \delta) \leq \theta_{n+n_0-1} \theta F \text{rcp}(\omega^{(n+n_0+\max F)}, \delta) \frac{2\theta^2}{2\theta n+n_0+\max F-1 \max F \theta_p}$$ and the conclusion will follow. Now as $\Psi_m \circ \rho_m = \iota_{\mathcal{A}}$, it suffices to show that
$$||\iota M_{n+n_0-1} \otimes (\Lambda_n - \iota_{\mathcal{A}}) \circ \rho_{n+n_0-1}(\sigma^l(x, T))|| < \delta.$$ This follows from our choice of $\Lambda_n$ and from the fact that entries of $\rho_m(\sigma^l(x, T))$, $m = n + n_0 - 1$ are sums of at most $\theta_{n-1}$ elements of $F'(\omega^{(n+n_0)}, n_0)$.

7.6. Corollary Consider the same situation as in Theorem 7.4. Let $(\omega_\alpha)_{\alpha \in A}$ be a net of finite subsets of $A^0$ with total union. Then
$$\text{ht}(\sigma(x)) \leq h_{\top}(\sigma |_{\ell(x)}) + \lim_{\alpha} \text{ht}_2(\{\phi_{x, x} |_{\omega}, \omega\}).$$

Proof This is a straightforward consequence of the fact that $\cup_{\alpha, n_0} F(\omega, n_0) \cup F(\omega, n_0)^*$ is total in $\mathcal{A}$ and of the Kolmogorov–Sinai property of the entropy of a ucp map, [V], [B], [BG].

Remark If in particular $\mathcal{A} = A^0 \rtimes_\alpha \mathbb{Z}$ and $u \in A^1$ is a unitary implementing $\alpha$ on $A^0$ then $\ell_u$ is a single point space, so its entropy is zero. By Lemma 7.3 and the previous Corollary, we recover Brown’s result that $\text{ht}_A(\text{Ad}(u)) \leq \text{ht}_{A^0}(\alpha)$ (and therefore one deduces an equality by monotonicity of topological entropy [B].)

The case where we can choose the $x_j$’s with pairwise orthogonal ranges is of course of special interest, the Cuntz algebras, Cuntz–Krieger algebras and Matsumoto algebras belonging to this class. The next result shows that the estimate of the entropy can be made more precise in this case.
7.7. Theorem Let \((A,\gamma,\mathbb{T})\) be a full \(C^*\)-dynamical system, with \(A^0\) exact. Let \(\{x_j\} \subset A^1\) be a finite subset such that
\[
\sum_j x_j x_j^* = I, \\
x_i^* x_j = 0, \quad i \neq j, \\
\sum_j x_j x_j^* \text{ is invertible.}
\]
Then
\[
h_{\text{top}}(\sigma_{|\ell(x_j)}) \leq h(\sigma_{|\ell(x_j)}) \leq h_{\text{top}}(\sigma_{|\ell(x_j)}) + \lim_{\alpha} h\sigma_1(\{\phi_{x_i,x_j}\},\omega_\alpha),
\]
where \((\omega_\alpha)_{\alpha \in A}\) is any net of finite subsets of \(A^0\) with total union in \(A^0\). If in particular for some net \((\omega_\alpha)_{\alpha \in A}\) \(h_2(\{\phi_{x_i,x_j}\},\omega_\alpha) = 0, \alpha \in A\), then
\[
h(\sigma_{x_j}) = h_{\text{top}}(\sigma_{|\ell(x_j)}).
\]

Proof The proof of the second inequality \(\leq\) goes exactly as that of Theorem 7.4 with the only exception that entries of \(\rho_m(\sigma_{l(x_j)})\) are now already elements of \(F(\omega^{(n+n_0)}, n_0)\). We show that
\[
h(\sigma_{(x_j)}) \geq h_{\text{top}}(\sigma_{|\ell(x_j)}).
\]
By monotonicity of topological entropy \([B], [V]\) and Corollary 7.2,
\[
h(\sigma_{(x_j)}) \geq h(T_\ell),
\]
where, as before, \(T_\ell\) denotes the \(*\)-monomorphism of \(\mathcal{C}(\ell)\) implemented by the one-sided shift. We are thus left to show that \(h(T_\ell) \geq h_{\text{top}}(\sigma_{|\ell(x_j)})\). The proof is similar to that of \([BG]\), which in turn goes back to \([V, Proposition 4.6]\). Let \(\mu\) be a \(\sigma\)-invariant probability Borel measure on the two-sided subshift \(\Lambda = \Lambda_{(x_j)}\) defined before Prop. 6.1, and let us restrict it to a \(\sigma\)-invariant probability measure on \(\ell = \Lambda_+\). For any ucp map \(\gamma : M \to \mathcal{C}(\ell)\), with \(M\) finite dimensional, let \(h_{\mu,T_\ell}(\gamma)\) be defined as in \([CNT]\), by means of the function \(H_{\mu}(\gamma, T_\ell^\gamma, \ldots, T_\ell^{n-1}\gamma)\). Reasoning as in \([V, Proposition 4.6]\) we see that \(h_{\mu,T_\ell}(\gamma) \leq h(T_\ell)\). Choosing \(M = \mathbb{C}^{n_0}\) and \(\gamma : \mathbb{C}^{n_0} \to \mathcal{C}(\ell)\) the natural inclusion, then one finds, thanks to \([CNT, Remark III.5.2]\), that the classical measurable entropy \(H_\mu(\sigma_{|\Lambda})\) is \(\leq h(T_\ell)\). Taking the supremum over all invariant measures we obtain the claim, by the classical variational principle for topological entropy \([DGS, Theorem 18.8]\).

Remark It is natural to ask whether the upper bound for \(h(\sigma_{(x_j)})\) described in the previous result can be further improved to \(h_{\text{top}}(\sigma_{|\ell(x_j)}) + h_0(\{\phi_{x_i,x_j}\})\).

We next show that the estimates above obtained are good enough to compute \(h(\sigma_{(x_j)})\) in the case of Cuntz–Krieger algebras. This was first done by Boca and Goldstein \([BG]\).
7.8. Corollary [BG] Let $A = \mathcal{O}_\Lambda$ be a Cuntz–Krieger algebra defined by a $\{0,1\}$–matrix $A$, and let $\{S_i\}_{i=1}^d$ be the canonical set of generating partial isometries. We recall from [M] a few properties of $S_i$ which are projections. Furthermore the following commutation relations hold in $\mathcal{O}_\Lambda$:

\[ S_i^* S_j = 0, \quad i \neq j \]  
\[ \sum_i S_i S_i^* = I, \]  

which easily imply that for any pair of words with the same length, $q_{\alpha, \beta} := S_{\alpha}^* S_{\beta} = 0$ unless $\alpha = \beta$. We will write $q_\alpha$ for $q_{\alpha, \alpha}$, $\alpha \in \cup_\Lambda \Lambda^0$. Note that these are projections. Furthermore the following commutation relations hold in $\mathcal{O}_\Lambda$:

\[ q_\mu S_\nu = S_\nu q_\mu, \]  
\[ q_\mu q_\nu = q_\nu q_\mu. \]  

By (7.4) the algebra $Q_\Lambda$ generated by the projections $\{q_\alpha, \alpha \in \cup_\Lambda \Lambda^0\}$ is commutative and therefore finite dimensional.

These properties imply that the finite sets

\[ \omega_{k,l} := \{S_{\alpha} E S_{\beta}^*, |\alpha| = |\beta| \leq k, \ E \ minimal \ projection \ in \ Q^l\} \]

has total union in $\mathcal{O}_\Lambda^0$. It follows that $\mathcal{O}_\Lambda^0$ is AF [M], so $\mathcal{O}_\Lambda$ is a nuclear $C^*$–algebra.

Using properties (7.1)–(7.4) one can show with tedious computations that for all $n \in \mathbb{N}$,

\[ \omega_{k,l}^{(n)} \subset \{S_\alpha q S_{\beta}^*, |\alpha| = |\beta| \leq k, q \ projection \ in \ Q_{2n+\max(k,l)}\}. \]

This computation is aimed to give an estimate for $\text{ht}_n(\phi_{S_i, S_j}, \omega_{k,l})$.

7.9. Lemma If $\mathcal{O}_\Lambda$ is the Matsumoto $C^*$–algebra associated to a subshift $\Lambda$ we have, for $a > 0$, and $k, l \in \mathbb{N}_0$,

\[ \text{ht}_a(\phi_{S_i, S_j}, \omega_{k,l}) \leq 2\limsup_n \frac{1}{n} \log(\text{dim}(Q_n)). \]
Proof Let $\phi : O^0_\Lambda \to B$ and $\psi : B \to O^0_\Lambda$ be unital completely positive maps such that $\|\psi\phi(a) - a\| < \delta/\theta_k$, when $a$ ranges the projections of $Q_{2n+2\max(k,l)}$, and assume that $B$ has minimal rank. Consider as before the maps $\rho_k : O^0_\Lambda \to M_{\theta_k}(O^0_\Lambda)$, $\Psi_k : M_{\theta_k}(O^0_\Lambda) \to O^0_\Lambda$ which satisfy $\Psi_k \circ \rho_k = \iota_{O^0_\Lambda}$. Define $\phi' := \iota_{M_{\theta_k}} \circ \phi \circ \rho_k$ and $\psi' := \Psi_k \circ \iota_{M_{\theta_k}} \otimes \psi$. Then if $a$ is a projection of $Q_{2n+\max(k,l)}$ and $|\alpha| = |\beta| \leq k$,

$$\|\psi' \circ \phi'(S_\alpha a S_\beta^*) - S_\alpha a S_\beta^*\| \leq \| \sum_{|\gamma| = |\gamma'| = k} e_{\gamma,\gamma'} \otimes (\psi \circ \phi - \iota)(S_{\gamma'}^* S_\alpha a S_\beta^* S_{\gamma'}) \| < \delta$$

because $S_{\gamma'}^* S_\alpha a S_\beta^* S_{\gamma'}$ is a projection in $Q_{2n+2\max(k,l)}$. Any element of $\omega_{k,l}^{(n)}$ being of the form $S_\alpha a S_\beta$, we deduce that for all $n \in \mathbb{N}$, and $\delta > 0$,

$$\text{rcp}(\omega_{k,l}^{(n)}, \delta) \leq \theta_k \text{rcp}(\text{Proj}(Q_{2n+2\max(k,l)}), \delta/\theta_k) \leq \dim(Q_{2n+2\max(k,l)}).$$

The last inequality follows from the existence of a conditional expectation onto $Q_{2n+2\max(k,l)}$. The rest follows choosing $\delta$ of the form $\frac{\delta}{\sigma_n - 1}$, taking the logarithm, dividing by $n$ and passing to the lim sup.

We combine the previous result with Theorem 7.6.

7.10. **Theorem** If $\sigma_{\{S_1\}}$ is the ucp map associated to the canonical set of generators of a Matsumoto $C^*$-algebra $O_\Lambda$,

$$h_{\text{top}}(\Lambda) \leq h_{\text{top}}(\sigma_{\{S_1\}}) \leq h_{\text{top}}(\Lambda) + 2 \limsup_{n} \frac{1}{n} \log(\dim(Q_n)).$$

We conclude this section discussing two examples of subshifts for which $h(\sigma_{\{S_1\}}) = h_{\text{top}}(\Lambda)$. The first example beyond Markov shifts is that of sofic subshifts, see [DGS] and therein quoted references. By [M] a subshift is sofic if and only if $\cup_n Q_n$ is finite dimensional. Then Theorem 7.10 yields the desired equality. Another example is that of $\beta$-shifts associated to $\beta$-expansion of real numbers [Re], [Par], [Bl]. In this case it is proved in [KMW] that if the $\beta$-shift is not sofic, $\dim(Q_n) = n + 1$, and this leads again to the same conclusion.

8. CNT DYNAMICAL ENTROPY AND VARIATIONAL PRINCIPLE

In fact, if $A = O_n$ Choda shows in [Ch] not only that $h_{\text{top}}(\sigma) = \log(n)$ but also that, if $\phi$ is the unique KMS state of $O_n$, then $h_{\phi}(\sigma) = h_{\text{top}}(\sigma) = \log(n)$, where the l.h.s. denotes the Connes–Narnhofer–Thirring dynamical entropy of $\sigma$ [CNT]. This result has its own importance, as it exhibits a fundamental example where a noncommutative variational principle for the entropy holds true. (We refer the reader to [DGS] for a formulation of the variational principle for the entropy in ergodic theory for compact spaces.)

It is an open problem whether a noncommutative variational principle for the entropy of $C^*$-algebras holds. In this section we give a class of examples for which this is true, thus generalizing Choda’s result. Examples will be the canonical ucp map of the Cuntz–Krieger algebras, or certain Matsumoto algebras associated to non finite type subshifts.
In this section we show that the CNT dynamical entropy of the ucp map \(\sigma_{(x_i)}\), defined in the previous section is \(\geq\) the m.t. entropy of the associated subshift \(\Lambda_{(x_i)}\), as defined, e.g., in [DGS], see Theorems 8.5, 8.6. This inequality looks similar to that of Theorem 7.7 relative to the topological entropy, but it goes in the reverse order. We start establishing the setting of the CNT entropy.

Let \(A\) be a unital \(C^*\)-algebra, and let \(\gamma_i : A_i \to A, i = 1, \ldots, n\) be ucp maps from finite-dimensional \(C^*\)-algebras, and let \(\phi\) be a state on \(A\). Let us recall from [CNT] that an Abelian model for \((A,\phi,\gamma_1,\ldots,\gamma_n)\) is given by an Abelian finite-dimensional \(C^*\)-algebra \(B\), a state \(\mu\) on \(B\) and subalgebras \(B_1,\ldots,B_n\) of \(B\) for which there is a ucp map \(E : A \to B\) with \(\phi = \mu \circ E\). Consider first the entropy of the Abelian model \((B,\mu,B_1,\ldots,B_n)\) as defined in [CNT, III.3] and then the quantity \(H_\phi(\gamma_1,\ldots,\gamma_n)\), defined as the supremum of the entropies of all the Abelian models (see [CNT, Definition III.4]). The following result is an obvious consequence of the definition.

**8.1. Lemma** If, for \(i = 1, \ldots, n\), \(\gamma_i : A_i \to A\) is a \(*\)-monomorphism, \(\vee_{i=1}^n \gamma_i(A_i)\) is finite-dimensional and commutative and if there exists a conditional expectation \(E : A \to \vee_{i=1}^n \gamma_i(A_i)\) such that \(\phi \circ E = \phi\), then

\[
H_\phi(\gamma_1,\ldots,\gamma_n) \geq S(\phi |_{\vee_{i=1}^n \gamma_i(A_i)}),
\]

where the r.h.s. denotes the classical m.t. entropy of the restriction of \(\phi\) to \(\vee_{i=1}^n \gamma_i(A_i)\).

**Proof** Let us define \(B = \vee_{i=1}^n \gamma_i(A_i), B_i = \gamma_i(A_i), \mu = \phi |_B\). Then

\[(B,\mu,B_1,\ldots,B_n)\]

is an Abelian model for \((A,\phi,\gamma_1,\ldots,\gamma_n)\). Let \(E_i : B \to B_i\) denote the canonical conditional expectation associated to \(\mu\). Then \(E_i \circ E \circ \gamma_i : A_i \to \gamma_i(A_i)\) coincides with \(\gamma_i\), which is a \(*\)-isomorphism, thus its entropy defect is zero (see [CNT], section II). It follows from [CNT, Definition III.4] that

\[
H_\phi(\gamma_1,\ldots,\gamma_n) \geq S(\phi |_{\vee_{i=1}^n \gamma_i(A_i)}).
\]

Let now \(\sigma\) be a ucp map of our \(C^*\)-algebra \(A\) such that \(\phi \circ \sigma = \phi\), and let \(\gamma : M \to A\) be a ucp map from a finite-dimensional \(C^*\)-algebra \(M\). Define the m.t. dynamical entropy of \(\gamma\) with respect to \(\phi\) to be

\[
h_{\phi, \sigma}(\gamma) = \lim_{n \to \infty} \frac{1}{n} H_\phi(\gamma, \sigma \circ \gamma, \ldots, \sigma^{n-1} \circ \gamma),
\]

and, finally, define the m.t. dynamical entropy of \(\sigma\) as \(h_{\phi}(\sigma) = \sup_{\gamma} \{h_{\phi, \sigma}(\gamma)\}\), where the supremum is taken over all possible \(\gamma : M \to A\).

**8.2. Corollary** Let \(A\) be a unital \(C^*\)-algebra, \(\phi\) a state of \(A\), and let \(\sigma\) be a ucp map of \(A\) such that \(\phi \sigma = \phi\). Let \(\gamma : M \to A\) be a unital \(*\)-monomorphism from a commutative finite-dimensional \(C^*\)-algebra \(M\). Assume that the smallest \(\sigma\)-stable \(C^*\)-subalgebra \(C\) of \(A\) containing \(\gamma(M)\) is commutative and that \(\sigma \vert_C\)
is a \(\ast\)-monomorphism. If, for \(n \in \mathbb{N}\), there exists a conditional expectation \(E_n : \mathcal{A} \to \mathcal{C}_n := \bigvee_{i=0}^{n-1} \sigma^i \circ \gamma(M)\) such that \(\phi \circ E_n = \phi\), then
\[
h_{\phi,\sigma}(\gamma) \geq h_{\phi|_C}(\sigma|_C, \gamma(M)),
\]
where the r.h.s. denotes the classical m.t. dynamical entropy of the partition of the spectrum of \(\mathcal{C}\) defined by \(\gamma(M)\) with respect to \(\sigma|_C\) (see, e.g., [DGS, Def. 10.8]). It follows that
\[
h_\phi(\sigma) \geq h_{\phi|_C}(\sigma|_C),
\]
where the r.h.s. denotes the classical m.t. entropy of the epimorphism of the spectrum of \(\mathcal{C}\) defined by the restriction of \(\sigma\) ([DGS, Def. 10.10].

Proof. Just apply the previous lemma to \(\gamma_1 = \gamma, \ldots, \gamma_n = \sigma^{n-1}\gamma\) and then pass to the limit. The last assertion is a consequence of the classical Kolmogorov–Sinai property of the entropy.

In order to apply the above result, one needs to know under which conditions on the system \((\mathcal{A}, \sigma, \gamma)\) as in Cor. 8.2 every invariant measure \(\mu\) on \(\mathcal{C}\) extends to a \(\sigma\)-invariant state \(\phi\) on \(\mathcal{A}\) fulfilling all the requirements of the previous Corollary. We start giving a well known sufficient condition for the existence of invariant conditional expectations.

8.3. Lemma Let \(M \subset \mathcal{A}\) be a unital inclusion of \(\mathcal{C}\)–algebras, with \(M\) commutative and finite–dimensional, and let \(\phi\) be a state on \(\mathcal{A}\) faithful on \(M\) and such that
\[
\phi(am) = \phi(ma), \quad m \in M.
\]
Then there is a unique conditional expectation \(E : \mathcal{A} \to M\) such that \(\phi \circ E = \phi\).

Proof. Set \(E(\alpha) = \sum \phi(c)^{-1} \phi(\alpha c) c\), and check that \(E\) is the desired conditional expectation. Uniqueness follows easily from faithfulness of \(\phi\) on \(M\).

We next give a condition on \((\mathcal{A}, \sigma, \gamma)\) so that every invariant measure on the spectrum of \(\mathcal{C}\) extends to a \(\sigma\)-invariant state on \(\mathcal{A}\) containing \(\mathcal{C}\) in its centralizer. In view of the previous Lemma, this would imply the existence of conditional expectations \(E_n\) as in Corollary 8.2, satisfying all the necessary requirements.

8.4. Proposition Let \(\mathcal{A}\) be a unital \(\mathcal{C}\)–algebra endowed with a ucp map \(\sigma\), and let \(\mathcal{C} \subset \mathcal{A}\) be a unital \(\sigma\)-stable, AF, commutative, \(\mathcal{C}\)–subalgebra. If
\[
\sigma(ca) = \sigma(c) \sigma(a), \quad c \in \mathcal{C}, a \in \mathcal{A},
\]
then every state \(\mu\) on \(\mathcal{C}\) satisfying \(\mu \circ \sigma = \mu\) extends to a state \(\phi\) on \(\mathcal{A}\) such that
\[
\phi \circ \sigma = \phi,
\]
(1)
\[
\phi(ca) = \phi(ac), \quad c \in \mathcal{C}, a \in \mathcal{A}.
\]
(2)
In particular, if \( \mu \) is faithful, for every finite-dimensional \( C^* \)-subalgebra \( M \subset \mathcal{C} \) there exists a unique conditional expectations \( E_M : \mathcal{A} \to M \) such that
\[
\phi \circ E_M = \phi.
\]

**Proof** Let \( C_1 \subset C_2 \subset \ldots \) be an increasing sequence of unital finite-dimensional \( C^* \)-subalgebras of \( \mathcal{C} \) with dense union, and, for \( n \in \mathbb{N} \), let \( F_n \) be the set of minimal projections of \( C_n \). Set
\[
K_0 = \{ \phi \in \mathcal{S}(\mathcal{A}) : \phi \mid_{\mathcal{C}} = \mu \},
\]
which is a nonempty convex and compact subset of the state space \( \mathcal{S}(\mathcal{A}) \) in the weak*-topology. The function \( f_0 \) taking any state \( \phi \) on \( \mathcal{A} \) to the state \( a \to \sum_{e \in F_n} \phi(eae) \) is weakly*-continuous and leaves \( K_0 \) invariant, thus, by the Schauder–Tychonov fixed point theorem, the fixed point set
\[
K_1 = \{ \phi \in K_0 : f_0(\phi) = \phi \}
\]
is nonempty. Note that \( K_1 \) is still compact and convex. Define now the weakly*-continuous function \( f_1 : \mathcal{S}(\mathcal{A}) \to \mathcal{S}(\mathcal{A}) \) taking \( \phi \) to \( a \in \mathcal{A} \to \phi(\sum_{e \in F_n} eae) \) and check again that \( K_1 \) is invariant under \( f_1 \), so that
\[
K_2 = \{ \phi \in K_1 : f_1(\phi) = \phi \}
\]
is nonempty. We thus find iteratively a decreasing sequence \( K_0 \supset K_1 \supset K_2 \ldots \) of nonempty compact convex subsets of \( \mathcal{S}(\mathcal{A}) \). Consider the compact convex set \( K := \cap_{n \in \mathbb{N}} K_n \). A state \( \phi \) is in \( K \) if and only if
\[
\phi \mid_{\mathcal{C}} = \mu,
\]
\[
\phi(ae) = \phi(ea), \quad e \in \cup_n F_n, a \in \mathcal{A},
\]
and therefore, being \( \cup_n \mathcal{C}_n \) dense in \( \mathcal{C} \),
\[
\phi(ca) = \phi(ac), \quad c \in \mathcal{C}, a \in \mathcal{A}.
\]
We next define the weakly*-continuous function \( f_\sigma : \mathcal{S}(\mathcal{A}) \to \mathcal{S}(\mathcal{A}) \) taking \( \phi \) to \( \phi \circ \sigma \) and we check that \( f_\sigma \) leaves \( K \) invariant. First, any \( \phi \in K \) restricts to \( \mu \) on \( \mathcal{C} \), thus, being \( \mathcal{C} \) and \( \mu \sigma \)-invariant, \( f_\sigma(\phi) \) restricts to \( \mu \) on \( \mathcal{C} \), as well. We are left to show that for \( \phi \in K \), \( \mathcal{C} \) is in the centralizer of \( f_\sigma(\phi) \). We compute, for \( c \in \mathcal{C}, a \in \mathcal{A} \),
\[
f_\sigma(\phi)(ca) = \phi(\sigma(ca)) = \phi(\sigma(c)a) = \phi(\sigma(a)c) = f_\sigma(\phi)(ac).
\]
Thus, applying again the Schauder–Tychonov fixed point Theorem, we find a fixed point \( \phi \) of \( f_\sigma \), which is the desired extension of \( \mu \). The last assertion now follows from the previous lemma.

We now collect all the results we have obtained, in form of a Theorem.
8.5. Theorem Let $\mathcal{A}$ be a unital $C^*$-algebra, endowed with a faithful ucp map $\sigma$, and let $\gamma : M \to \mathcal{A}$ be a unital $*$-monomorphism from a commutative finite-dimensional $C^*$-algebra $M$. Assume that the smallest $\sigma$–stable $C^*$–subalgebra $C$ of $\mathcal{A}$ containing $\gamma(M)$ is commutative and that

$$\sigma(ca) = \sigma(c)\sigma(a), \quad c \in C, a \in \mathcal{A}.$$

Let $\mu$ be a faithful $\sigma$–invariant state on $C$ extended to a $\sigma$–invariant state $\phi$ on $\mathcal{A}$ centralized by $C$, this being possible by Prop. 8.4. Then

$$h_\phi(\sigma) \geq h_\mu(\sigma |_C),$$

where the r.h.s. denotes the classical m.t. entropy of the epimorph ism of the spectrum of $C$ defined by the restriction of $\sigma$ to $C$.

We now go back to the situation where $\sigma = \sigma_{\{x_j\}}$ is the ucp map implemented by a finite subset $\{x_j\}$ of $\mathcal{A}$ such that $\sum_j x_j x_j^* = I$.

8.6. Theorem Let $\mathcal{A}$ be a unital $C^*$–algebra, and let $\{x_j\}$ by a finite subset constituted by $d$ nonzero partial isometries satisfying

$$\sum_j x_j x_j^* = I,$$

$$\sum_j x_j^* x_j \text{ is invertible},$$

$$x_i^* x_j = 0, \quad i \neq j,$$

$$[x_{i_1} \ldots x_{i_r}(x_{i_1} \ldots x_{i_r})^*, x_j x_j^*] = 0, \quad j, i_1, \ldots, i_r = 1, \ldots, d, r \in \mathbb{N}.$$

Let $\sigma_{\{x_j\}}$ be the ucp map implemented by the multiplet $\{x_j\}$. Let

$$\Phi : C(\Lambda_{\{x_j\}}^+) \to \mathcal{A}$$

be the natural $*$–monomorphism defined in Corollary 7.2. Induce a shift–invariant measure $\mu_+$ on $\Lambda_{\{x_j\}}^+$ from a shift–invariant measure $\mu$ on $\Lambda_{\{x_j\}}$ and then extend $\mu_+$ to a $\sigma$–invariant state $\phi$ on $\mathcal{A}$ centralized by $\Phi(C(\Lambda_{\{x_j\}}^+))$, by Prop. 8.4. Then

$$h_\phi(\sigma_{\{x_j\}}) \geq h_\mu(\sigma |_{\Lambda_{\{x_j\}}}).$$

Proof Consider the finite dimensional commutative $C^*$–algebra $M$ of $C(\Lambda_{\{x_j\}}^+)$ generated by the characteristic functions of the cylinder sets $[i], i : x_i \neq 0$. $C(\Lambda_{\{x_j\}}^+)$ is naturally embedded in $\mathcal{A}$ via the $*$–monomorphism $\Phi$ defined in Cor. 7.2. Clearly $\Phi(C(\Lambda_{\{x_j\}}^+))$ is generated by the ranges of $\sigma_{\{x_j\}}^i \circ \Phi(M)$. Also, $\sigma$ is faithful as $\sum_j x_j^* x_j$ is invertible. The commutation relations between the domain projections and the range projections of the iterated products of the $x_j$’s easily show that $\sigma_{\{x_j\}}(ca) = \sigma_{\{x_j\}}(c)\sigma_{\{x_j\}}(a)$, $c \in \Phi(C(\Lambda_{\{x_j\}}^+))$, $a \in \mathcal{A}$. In particular, $\sigma_{\{x_j\}}$ is a $*$–monomorphism on $\Phi(C(\Lambda_{\{x_j\}}^+))$. Thus the previous theorem applies.
If for example $A = O_A$ or, more generally, $A = O_\Lambda$, the assumptions of the previous theorem hold true.

Note that, with the notation and assumptions of the previous result, we know that, using also Corollary 7.6,

$$h_\mu(\Lambda_{\{x_j\}}) \leq h_\phi(\sigma_{\{x_j\}}) \leq h_{\text{top}}(\Lambda_{\{x_j\}}) + \lim_{\alpha} h_\mu(\Phi(\Lambda_{\{x_j\}},\omega_\alpha)), \quad (8.1)$$

where $\omega_\alpha$ is any net of finite subsets of $\mathcal{A}^0$ with total union. The middle inequality is due to Voiculescu [V].

In classical ergodic theory, a probability measure $m$ on a dynamical system $(X,T)$ such that $m \circ T^* = m$ is called an equilibrium measure, or a measure with maximal entropy, if it maximizes the entropy, i.e. if $h_m(X) = h_{\text{top}}(X)$. It is well known that dynamical systems arising from subshifts admit equilibrium measures, see [DGS]. Applying this fact to our subshift $\Lambda_{\{x_j\}}$, we see that there exists a shift–invariant measure $\mu$ on $\Lambda_{\{x_j\}}$ with $h_\mu(\Lambda_{\{x_j\}}) = h_{\text{top}}(\Lambda_{\{x_j\}})$.

Combining with the previous inequality, we obtain, under the simplifying assumption that the second summand in (8.1) vanishes, an existence theorem of equilibrium states in the noncommutative situation above considered.

**8.7. Corollary** Consider the same situation as in Theorem 8.6. Let $\mu$ be a shift–invariant measure on $\Lambda_{\{x_j\}}$ with maximal entropy, and let us extend it to a $\sigma_{\{x_j\}}$–invariant state $\phi$ on $A$ centralized by $\Phi(C(\Lambda_{\{x_j\}}))$. Assume furthermore that for a net $\omega_\alpha$ of finite subsets of $\mathcal{A}^0$ with total union, $h_\mu(\Lambda_{\{x_j\}},\omega_\alpha) = 0$. Then

$$h_\mu(\Lambda_{\{x_j\}}) = h_\phi(\sigma_{\{x_j\}}) = h_{\text{top}}(\Lambda_{\{x_j\}}).$$

By virtue of the remark following Theorem 7.10, we obtain the following result.

**8.8. Corollary** Let $\Lambda$ be a subshift of one of the following types:

1. Markov shift,
2. sofic subshift
3. $\beta$–shift.

Let us extend an invariant measure $\mu$ on $\Lambda$ with maximal entropy to a state $\phi$ on $O_\Lambda$ centralized by the canonical commutative subalgebra $C(\Lambda_+) \subset O_\Lambda$. Then

$$h_\mu(\Lambda) = h_\phi(\sigma_{\{S_i\}}) = h_{\text{top}}(\Lambda),$$

where $\{S_i\}$ is the canonical set of generating partial isometries of the Matsumoto algebra $O_\Lambda$.

**9. From KMS states to equilibrium states**

In this section we make an attempt to show a closer connection between KMS states on full periodic $C^*$–dynamical systems studied in sections 1–6 and equilibrium states considered in sections 7–8. To motivate the result of this
section, we consider the classical situation of a topological dynamical system $(X, T)$ over a compact space $X$. A Borel probability measure $m$ on $X$ is called conformal if $m \circ T^*$ is equivalent to $m$. The study of conformal measures is of particular importance as it leads to equilibrium states of the system [DU]. Now in our noncommutative setting, where we replace $X$ by a unital $C^*$-algebra $A$ endowed with a full action of the circle, and $T$ by the ucp map $\sigma_{(x_j)}$, KMS states provide a natural class of states on $A$ which play the role of conformal measures. Indeed we have the following immediate result.

**9.1. Proposition** Let $(A, \gamma)$ be a full periodic $C^*$-dynamical system over a unital $C^*$-algebra $A$, and let $\{x_j\}$ be a finite subset of $A^1$ such that $\sum_j x_j x_j^* = I$ and $\sum_j x_j^* x_j$ is invertible. If $\omega$ is a KMS state at inverse temperature $\beta$ then

$$\omega \circ \sigma_{(x_j)} = \omega(a)$$

where $a = e^{-\beta} \sum_j x_j^* x_j$ is obviously a positive and invertible element of $A^0$.

We show how to produce $\sigma$–invariant states on $A$ from KMS states of the system $(A, \gamma)$. Consider the completely positive map $S_{(x_j)} : a \in A \to \sum_j x_j^* a x_j$ already considered in section 2. Let $\omega$ be a faithful KMS state for $(A, \gamma)$ at maximal inverse temperature $\beta_{\text{max}} = \log(\lambda_{\text{max}})$. Then, for $t > \lambda_{\text{max}}$, consider the series

$$\sum_{k=0}^{+\infty} \frac{S_{(x_j)}^k(I)}{t^{k+1}},$$

which we claim to be Cauchy for every seminorm $p_T$, $T \in A$, where $p_T(a) = |\omega(aT)|$, $a \in A$. We show the claim.

$$\omega\left(\sum_{n}^{m} \frac{S_{(x_j)}^k(I)}{t^{k+1}} T\right) = \lambda_{\text{max}}^{-1} \sum_{n}^{m} \left(\lambda_{\text{max}} \frac{t}{t^{k+1}}\right)^{k+1} \omega(\sigma_{(x_j)}^k(T)).$$

Now the r.h.s. is converging to 0 as $m, n \to \infty$, since $\sum_{k}^{+\infty} \frac{S_{(x_j)}^k}{\mu^{k+1}}$ is norm converging for $\mu > 1$. Being $\omega$ faithful, $b_t := \sum_{k=0}^{+\infty} \frac{S_{(x_j)}^k(I)}{t^{k+1}}$ lies in the enveloping von Neumann algebra of $A$. Set

$$a_t = (t - \lambda_{\text{max}})b_t,$$

so $\omega(a_t) = 1$, and define

$$\omega_t := \omega(a_t \cdot ).$$

Then for any $T \in A$,

$$\omega_t(T) - \omega_t(\sigma_{(x_j)}(T)) = \omega(a_t T) - \lambda_{\text{max}}^{-1} \omega(S_{(x_j)}(a_t) T) =$$

$$\lambda_{\text{max}}^{-1}(t - \lambda_{\text{max}})\omega((\lambda_{\text{max}} - S_{(x_j)})(b_t) T) \to 0$$

for $t \to \lambda_{\text{max}}$. So any weak$^*$–limit point $\phi$ of $\omega_t$ for $t \to \lambda_{\text{max}}$ is a $\sigma_{(x_j)}$–invariant state on $A$. Note that if every $x_{\mu}^* x_{\mu}$ commutes with any $x_{\nu} x_{\nu}^*$, for
all multiindices $\mu$ and $\nu$, then the Banach space generated by the $x_\mu x_\nu^*$ is in the centralizer of any such $\phi$.

The next result can be regarded as an example where the construction of equilibrium states out of KMS states is explicit. This is the noncommutative analogue of the well known relationship between the Perron–Frobenius Theorem and equilibrium states for Markov subshifts, see Proposition 17.14 in [DGS].

9.2. Theorem Let $(A, \gamma, T)$ be a unital, full, periodic $C^*$–dynamical system, and let $O_A \subset A$ be a unital $\mathbb{Z}$–graded inclusion of the Cuntz–Krieger algebra associated to an irreducible matrix $A$, in $A$. If $\omega$ is a faithful KMS state of $A$ at maximal inverse temperature $\log(\lambda_{\text{max}})$, then

1. $\lambda_{\text{max}} = r(A)$,
2. $\omega_t$ is norm convergent, for $t \to \lambda_{\text{max}}^+$, to a $\sigma_{\{S_i\}}$–invariant state $\phi$ centralized by $C(\Lambda A_+)$, where $\{S_i\}$ is the canonical set of generators of $O_A$.
3. $\phi$ restricts on $C(\Lambda A_+)$ to the unique probability measure $\mu$ for which

$$h_\mu(A) = h_{\text{top}}(A).$$

In particular, if for a net $\omega_\alpha$ of finite subsets of $A^0$ with total union

$$ht_2(\{\phi_{x_i,x_j}\}, \omega_\alpha) = 0$$

then

$$h_\mu(A) = h_\phi(\sigma_{\{x_j\}}) = h_{\text{top}}(\sigma_{\{x_j\}}) = h_{\text{top}}(A) = \log(r(A)).$$

Proof It is known that Markov subshifts defined by irreducible matrices have a unique maximal measure, see Theorem 19.14 in [DGS]. The elements $a_t$, $t \geq \lambda_{\text{max}}$, defined as in (9.1) belong to the finite–dimensional $C^*$–subalgebra of $C(\Lambda A_+)$ generated by $S_iS_i^*$, the characteristic functions of the cylinders $[i], i = 1, \ldots, d$. Since $\omega(a_1) = 1$ and $\omega$ is faithful, there exists a norm–limit point $a$ of $a_t$, for $t \to \lambda_{\text{max}}^+$. Inspection shows that $a$ is an eigenvector of $S(S_i)$ with eigenvalue $\lambda_{\text{max}}$, and therefore it corresponds to a left eigenvalue $(v_i)$ of $A$, normalized so that $\omega(a) = 1$. In particular, $a_t$ is convergent. Since $\omega$ is a KMS state of $A$, and hence of $O_A$ w.r.t. the gauge action, evaluating $\omega$ on $S_iS_i^*$ gives the unique, up to a scalar, positive right eigenvector $(u_i)$ of $A$. The normalization $\omega(a) = 1$ yields $\sum_i u_i v_i = 1$. Evaluating $\phi$ on $S_i \ldots S_i S_i \ldots S_i S_i^*$ gives

$$\phi(S_i \ldots S_i (S_i \ldots S_i)^*) = \omega(aS_i \ldots S_i (S_i \ldots S_i)^*) =$$

$$v_i \omega(S_i \ldots S_i (S_i \ldots S_i)^*) = \frac{v_i}{\lambda} \omega((S_i \ldots S_i)^* S_i \ldots S_i) =$$

$$\frac{v_i}{\lambda} a_{i_1,i_2} \ldots a_{i_{r-1},i_r} \sum_j \omega(a_{i_r,j} S_j S_j^*) = \frac{v_i u_{i_r}}{\lambda} a_{i_1,i_2} \ldots a_{i_{r-1},i_r}.$$

If we now compare with the formula given in Prop. 17.14 in [DGS], we see that $\mu = \phi |_{C(\Lambda A_+)}$ restricts precisely to the unique measure on $\Lambda A_+$ with maximal entropy.
Acknowledgments Part of this paper was written during a visit of C.P. at the Mathematics Department of the University of Orleans. She wishes to thank C. Anantharaman–Delaroche for invitation and for drawing attention to the Furstenberg’s example, and J. Renault for many fruitful discussions. We are also indebted to H. Matui for pointing out an error in section 7 of a previous version of this paper.

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