Naked Singularities and the Weyl Curvature Hypothesis

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Abstract
We examine the growth of the Weyl curvature in two examples of naked singularity formation in spherical gravitational collapse - dust and the Vaidya spacetime. We find that the Weyl scalar diverges along outgoing radial null geodesics as they meet the naked singularity in the past. The implications of this result for the Weyl curvature hypothesis are discussed. We mention the possibility that although classical general relativity admits naked singularity solutions arising from gravitational collapse, the second law of thermodynamics could forbid their occurrence in nature. The method can also be used to compare the relative importance of initial data and that of the energy-momentum tensor in deciding the metric solution in any general case.

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1 Introduction

As Penrose has emphasized [1], a fundamental explanation for the second law of thermodynamics can be had only after we first understand why the entropy of the Universe was extremely low in the beginning. There is good cosmological evidence that in the very early epochs, the matter distribution itself was in a thermal equilibrium state with maximal entropy. Hence the matter fields by themselves cannot provide an explanation for the low initial entropy. The low entropy constraint has to come from the spacetime geometry - a suitably constructed geometrical quantity plays the role of gravitational entropy. This gravitational entropy has to be initially so very small that the net sum of the matter and gravitational entropy takes a value far less than it could possibly have. As the Universe evolves, the gravitational entropy increases as a result of clumping of matter and formation of irregularities. The overall entropy of the Universe also increases.

A plausible geometrical quantity which could play the role of gravitational entropy is the Weyl curvature tensor [1]. This is because at the initial Big Bang singularity, the Ricci curvature necessarily diverges, whereas the Weyl curvature is zero, if the Universe is assumed to be Friedmannian. The development of inhomogeneities during evolution leads to an increase in the Weyl curvature. Ultimately, it is expected that inhomogeneities lead to the formation of gravitational singularities. If cosmic censorship holds, these final singularities will be black-holes, and as a result of the continuous increase of the Weyl tensor, we can expect that the Weyl tensor will diverge at the black-hole singularity. Thus the structure of the final singularity is very different from the initial singularity, in so far as the behaviour of the Weyl curvature is concerned.

In this way of looking at things, the validity of the second law requires that at the initial cosmological singularity, the Weyl curvature be zero, or at least ignorable in comparison with the Ricci curvature. This is the essence of Penrose’s Weyl curvature hypothesis. Various important investigations have been carried out in order to formulate a precise version of the hypothesis [2]. For natural reasons, these investigations are largely concerned with the initial cosmological singularity. Here we are interested in the Weyl hypothesis from a different viewpoint, namely the possible relevance of this hypothesis for naked singularities occurring in gravitational collapse of compact objects. A naked singularity should be regarded as a ‘T.I.F.’ as well as a ‘T.I.P.’,
because both ingoing and outgoing geodesics terminate at such a singularity. In one particular version of the hypothesis [3], Penrose enquires if given a naked singularity, the Weyl curvature would diverge along ingoing geodesics, and go to zero along outgoing geodesics terminating at the singularity. In other words, one is asking if the Weyl hypothesis that is usually applied to an initial cosmological singularity would also hold for a naked singularity forming in collapse. It is this version of the hypothesis that we examine in our brief note. Considering the similarity between a naked singularity and an initial cosmological singularity in some, though certainly not all, respects this version of the hypothesis deserves further examination.

This issue becomes interesting in the light of examples of naked singularities that have been found in recent years. Here we analyse the evolution of the Weyl tensor in the Tolman-Bondi spacetime, and in the Vaidya spacetime, both of which admit naked singularities for certain initial data. As is known, in both cases, it is only the central shell-focusing singularity which can be naked. For definiteness, we assume that it is the Weyl scalar $C \equiv C^{abcd}C_{abcd}$ which has to be zero at the initial singularity. We find that the Weyl scalar remains zero at the center for all epochs prior to the formation of the singularity. (The singularity is defined by the divergence of the Kretschmann scalar). Exactly at the (naked) singularity, however, the scalar blows up, both along ingoing as well as outgoing null geodesics. This appears to suggest that the version of the Weyl hypothesis under consideration does not hold if such naked singularities occur.

A knowledge of the evolution of Weyl curvature is of interest also for reasons other than the Weyl hypothesis. Various criteria for judging the seriousness of a naked singularity have been proposed - these include curvature strength and stability of the Cauchy horizon, among others. In addition to these, the divergence of the Weyl curvature at the naked singularity could imply that the singularity is to be taken seriously, because the Weyl tensor is not related to the local matter distribution, and is more directly associated with that part of the gravitational field which is determined by the matter non-locally. On the other hand, if the Weyl scalar were to be zero at the naked singularity, one could possibly conclude that these singularities are associated with the matter distribution.

The paper is organised as follows. In Section II we study the evolution of the Weyl tensor in the Tolman-Bondi dust collapse. We also use this opportunity to present a much simplified demonstration of the occurrence
of the naked singularity. From this demonstration one can see that while the early pioneering works on inhomogeneous dust collapse established the formation of a naked singularity through difficult calculations, we now have elementary ways of rederiving those early results. In Section III the Weyl calculation is repeated for the Vaidya spacetime. The possible implications of these results are discussed in Section IV.

2 Weyl tensor for dust collapse

The Tolman-Bondi metric for dust collapse in comoving coordinates \((t, r, \theta, \phi)\) is

\[
\begin{align*}
\text{ds}^2 &= -\frac{R(t,r)^2}{1 + f(r)} \text{d}t^2 - \frac{R^2(r,t)}{R'(r,t)} \text{d}r^2 - R^2(r,t) \text{d}\Omega^2
\end{align*}
\]

(1)

where \(R(t,r)\) is the area radius, and the free-function \(f(r)\) labels bound, marginally bound and unbound models, depending on whether \(f(r)\) is negative, zero or positive, respectively. The field equations are

\[
\begin{align*}
\epsilon(r,t) &= \frac{F'(r)}{R^2(r,t) R'(r,t)}, \quad \dot{R}^2 = f(r) + \frac{F(r)}{R},
\end{align*}
\]

(2)

where dot and prime indicate partial derivative with respect to \(t\) and \(r\), respectively. The quantity \(F(r)\) is equal to two times the mass inside the sphere labelled \(r\). The metric solution in terms of the parameters above yields a singularity in the spacetime when \(R(r,t) = 0\), where the Kretschmann scalar diverges.

As has been shown earlier \[4\], the central shell-focusing singularity is naked if the radial null-geodesic equation

\[
X = \lim_{r \to 0, R \to 0} \frac{R'(X, r)}{\alpha r^{\alpha - 1}} \left\{ 1 - \sqrt{\frac{\Lambda_0}{X}} \right\}
\]

(3)

admits one or more positive real roots \(X_0\). This equation is written by first eliminating the variable \(t\) in favour of \(R\), and then eliminating \(R\) in favour of \(X \equiv R/r^\alpha\), where \(X\) is the tangent to a possible outgoing geodesic. The constant \(\alpha > 1\) is chosen so that \(R'/r^{\alpha-1}\) has a unique, finite limit as
$R \to 0$, $r \to 0$. The constant $\Lambda_0$ is the limiting value of $\Lambda(r) = F(r)/r^\alpha$ as $r \to 0$.

At this stage the calculations in earlier papers \[4\], \[5\] proceed by giving a detailed expression for $R'$ - this is the most involved part of the calculation. The simplification we present here is in the calculation of $R'$. Consider first the marginally bound case, for which the solution is

$$R_3^3/2 = r_3^3/2 - \frac{3}{2}\sqrt{F(r)}t. \quad (4)$$

An initial scaling $R = r$ at the starting epoch $t = 0$ of the collapse has been assumed. From this equation, evaluate $R'$ and substitute for $t$ from (4). In the resulting expression, substitute $R = Xr^\alpha$, and perform a Taylor expansion of $F(r)$ around $r = 0$, so as to retain the leading non-vanishing term. We get that near $r = 0$,

$$\frac{R'}{r^\alpha - 1} = X + \frac{\theta}{\sqrt{X}}r^{q+3/2-3\alpha/2}. \quad (5)$$

Here, $\theta = -qF_q/3F_0$, and $q$ is defined such that in a series expansion of the initial density $\rho(r)$ near the center, the first non-vanishing derivative is the $q$th one ($=\rho_q$), and $F_q = \rho_q/(q+3)q!$. Now a unique definition of $\alpha$ is obtained by setting the power of $r$ as zero in the second term, giving $\alpha = 1 + 2q/3$. This then reproduces the result of the $R'$ calculation performed earlier \[3\], in a simpler manner. From this point on, the naked singularity analysis through evaluation of roots of (3) proceeds as before, but the overall calculation is now more economic.

For the non-marginally bound case, the solution of the Tolman-Bondi equations is

$$R^{3/2}G(-fR/F) = r^{3/2}G(-fr/F) - \sqrt{F(r)}t \quad (6)$$

where $G$ is a known elementary function \[3\]. $R'$ can be evaluated as before, and then we eliminate $t$ and substitute $R = Xr^\alpha$. The power-series expansion for $f(r)$ near $r = 0$ is of the form \[3\]

$$f(r) = f_2r^2 + f_3r^3 + f_4r^4 + ... \quad (7)$$

which implies that the argument $(-fR/F)$ of $G$ on the left-side of (3) goes to zero as $r \to 0$. The derivative of $G(-p)$ w.r.t. its argument $p \equiv rf/F$ is
obtained by differentiating (F), which gives
\[
\frac{dG}{d(-p)} = \frac{3G}{2p} - \frac{1}{p\sqrt{1 + p}}.
\]
Using this, one can now perform a Taylor-expansion of \(G, F\) and \(f\) about \(r = 0\) to get exactly the same expression for \(R'\) as given in (5) above, where \(\theta\) is now given by
\[
\theta = q \left( 1 - \frac{f_2}{2F_0} \right) \left( G(-f_2/F_0) \left[ \frac{F_q}{F_0} - \frac{3f_{q+2}}{2f_2} \right] \left[ 1 + \frac{f_2}{2F_0} \right] + \frac{f_{q+2}}{f_2} - \frac{F_q}{F_0} \right).
\]  
(8)
The constant \(q\) is now defined such that the first non-vanishing derivative of the initial density is the \(q\)th one, and/or the first non-vanishing term in the expansion for \(f(r)\), beyond the quadratic term, is of order \(r^{q+2}\). The constant \(\alpha\) is again equal to \((1 + 2q/3)\). Thus the \(R'\) calculation is simplified for the non-marginal case as well.

We return now to the calculation of the Weyl tensor, which is defined as
\[
C_{abcd} = R_{abcd} + g_{a[d}R_{c]b} + g_{b[c}R_{d]a} + \frac{1}{3} Rg_{a[c}g_{d]b},
\]
where square brackets denote antisymmetrization. The nonvanishing independent components for the Tolman-Bondi metric are
\[
\begin{align*}
C_{1010} &= \frac{R^2}{1+f} \left( \frac{F}{R^2} - \frac{F'}{3R^2R'} \right) \\
C_{2020} &= -\frac{F}{2R} + \frac{F'}{6R'} \\
C_{2121} &= -\frac{F'}{2R} + \frac{F(r)R^2}{2R(1+f)} \\
C_{3030} &= C_{2020} \sin^2 \theta \\
C_{3131} &= C_{2121} \sin^2 \theta \\
C_{3232} &= \left( -FR + \frac{F'R^2}{3R^3} \right) \sin^2 \theta 
\end{align*}
\]  
(9)
The Weyl scalar is given by
\[
C = \frac{12}{R^4} \left( \frac{F}{R} - \frac{F'}{3R'} \right)^2.
\]  
(10)
At the start of evolution at \(t = 0\), where the scaling \(R = r\) has been chosen, we see that the Weyl scalar at \(r = 0\) is zero (the mass-function \(F(r)\) grows as \(r^3\)
near \( r = 0 \). Using the above Tolman-Bondi solution we find that the scalar remains zero at \( r = 0 \) throughout the non-singular phase of the evolution. However, at the epoch of naked singularity formation, we know that in the neighborhood of the singularity, we have along an outgoing geodesic the relation \( R = X_0 r^\alpha \), \( X_0 \) being the finite tangent. Also, \( R' \) is given by (3), with \( X = X_0 \). With these substitutions, we see that in the limit of approach to the singularity, the scalar is equal to

\[
C(X_0, r) = \frac{12 F^2_0 \theta^2_0}{X_0^7 \left( X_0 + \theta_0 / \sqrt{X_0} \right)^2} r^{6(1-\alpha)}. \tag{11}
\]

For those initial data that lead to a naked singularity, the range of \( \alpha \) is \( 5/3 \leq \alpha \leq 3 \). Hence we see by letting \( r \) go to zero that the Weyl scalar diverges at the naked singularity along outgoing geodesics. In so far as ingoing geodesics are concerned, these exist for all initial data, i.e. \( \alpha \) takes all values greater than unity. Hence the Weyl scalar necessarily diverges along ingoing geodesics too.

The effect of the energy-momentum tensor on the geometry can be found by evaluating the scalar \( R^\alpha_\beta R^{\alpha\beta} \) and comparing it to the Weyl scalar. We get in the limit of approach to the naked singularity,

\[
R^\alpha_\beta R^{\alpha\beta} = \frac{F'^2}{R^2 R^4} = \frac{9F_0^2}{X_0^4 \left( X_0 + \theta_0 / \sqrt{X_0} \right)^2} r^{6(1-\alpha)}. \tag{12}
\]

We find that this scalar diverges at the same rate as the Weyl scalar, and the latter does not dominate over the former. Notice however that unlike the Weyl scalar, the invariant \( R^\alpha_\beta R^{\alpha\beta} \) is non-zero at the initial epoch, and evolves with time. Finally we note that for the points \( r \neq 0 \) the Weyl scalar is in general non-zero initially - it then evolves with time and diverges at the singularity \( R(t, r) = 0 \). We recall that the singularity at \( r \neq 0 \) is necessarily covered.
3 Weyl tensor for Vaidya spacetime

The collapse of null dust is described by the Vaidya metric

\[ ds^2 = \left(1 - \frac{2m(v)}{r}\right) dv^2 - 2dvdr - r^2 d\Omega^2 \]  

where \( v \) is the advanced time, and \( m(v) \) is the mass-function. The energy-momentum tensor is

\[ T_{ik} = \epsilon u_i u_k = \frac{1}{4\pi r^2} \frac{dm}{dv} u_i u_k \]  

where \( u_i = -\delta^v_i \) - this represents the radial inflow of radiation into an initially flat region of spacetime. The spacetime outside the radiation region is Schwarzschild. In the self-similar model, which is the case that we consider here, the mass-function in the cloud is given by \( 2m(v) = \lambda v \), i.e. it is linear. A curvature singularity forms when a shell hits the origin \( r = 0 \). It is known that the singularity at \( r = 0, v = 0 \) is naked for \( \lambda \leq 1/8 \) [6].

The non-zero independent components of the Weyl tensor are given by

\[ C_{1010} = \frac{2m(v)}{r} \left(1 - \frac{2m(v)}{r}\right) \]
\[ C_{2020} = -\frac{m(v)}{r} \left(1 - \frac{2m(v)}{r}\right) \]
\[ C_{2120} = \frac{m(v)}{r} \]
\[ C_{3232} = -2m(v)r \sin^2 \theta \]
\[ C_{3030} = C_{2020} \sin^2 \theta \]
\[ C_{3130} = C_{2120} \sin^2 \theta \]  

The Weyl scalar is given by

\[ C(v, r) = \frac{48m^2}{r^6} = \frac{12\lambda^2 v^2}{r^6} \]  

Prior to the formation of the singularity, the scalar is zero at the inner edge \( v = 0, r \neq 0 \) of the cloud. Naturally, it is also zero in the flat region to the interior of the collapsing cloud. When the naked singularity forms, the finite tangent along an outgoing geodesic from the singularity is defined as \( X = v/r \) (see [7]). Thus we get

\[ C(X_0, r) = \frac{12\lambda^2 X_0^2}{r^4} \]  

8
where $X_0$ is the limiting value of the tangent. The Weyl scalar diverges at
the naked singularity. The scalar $R_{\alpha\beta}R^{\alpha\beta}$ is however zero, and in this case,
the Weyl scalar dominates.

4 Discussion

Our aims in this paper were twofold. Firstly, to compute the Weyl tensor
in the approach to the naked singularity in the Tolman-Bondi and Vaidya
spacetimes, and secondly to consider what the results could mean for the
Weyl curvature hypothesis. We find that in these models, the Weyl scalar
behaves in a peculiar manner, remaining zero at the origin throughout the
non-singular phase of the evolution, and then suddenly jumping to infinity
along outgoing geodesics, at the epoch of singularity formation. The same is
true along ingoing geodesics. In so far as the Weyl hypothesis is concerned,
it appears reasonable that the hypothesis apply in the same manner to cos-
mological initial singularities and to naked singularities [3]. In the examples
considered here, we find that the curvature structure of the T.I.F.s is sim-
ilar to that of the T.I.P.s, which does not support the hypothesis. Also,
since naked singularities are expected to arise in strongly inhomogeneous
situations, for which the Weyl tensor is typically high, one could speculate
that the Weyl scalar should generically diverge in the past along outgoing
geodesics, rather than go to zero.

If we do assume that the Weyl hypothesis must be valid, then these results
offer an interesting possibility, namely that the second law of thermodynamics
could forbid the occurrence of naked singularities in nature. If the Weyl
hypothesis applies both to cosmological initial singularities and to naked
singularities arising from gravitational collapse, then validity of the second
law requires that only those naked singularities can actually occur in nature
for which the Weyl scalar goes to zero along outgoing geodesics. The second
law would hence disallow those naked singular solutions which disagree with
the Weyl hypothesis. If the Weyl scalar diverges generically along outgoing
geodesics, the second law would generically forbid naked singularities from
occurring in nature.

Thus while classical general relativity admits naked singularities in grav-
itational collapse, their occurrence in the real world could be thermodynam-
ically forbidden. The situation will then be analogous to the advanced wave
solutions in classical electrodynamics - solutions that are permitted by the theory but thermodynamically disallowed in the real world. The cosmic censorship hypothesis would then hold, though not in classical general relativity as such, but in the real universe, through the intervention of thermodynamics. However, this would mean attributing a fundamental status to the Weyl Curvature Hypothesis, which could possibly involve quantum gravity.

Although it was the notion of connection of the Weyl tensor with non-uniformity of energy momentum distribution which led Penrose to think of its relation to entropy, it must be kept in mind that the Weyl tensor has a certain part which is determined by the ‘initial’ conditions of the field equations. These are the metric and its derivatives specified on a chosen space-like surface. The Weyl scalar going to zero from TIF would mean that one would have to have unique initial conditions on a partial Cauchy surface chosen near the singularity formation event, as it approaches the ideal point. Thus, Penrose seems to indicate that there was only a unique choice of the metric solution \[ R \], presumably at a limiting space-like surface near the ideal point.

Also, given the fact that the divergence of the Weyl scalar and \( R_{\alpha\beta}R^{\alpha\beta} \) are comparable, if not the Weyl contribution dominating, one would be led to think that whenever a naked singularity occurs, the role played by the initial conditions in deciding the metric of the universe is as important as that of the energy momentum tensor, if not more.

It must be noted that the conjecture includes all singularities generically and not just the cosmological ones. We find that it is not true for Tolman Bondi dust and Vaidya radiation collapse scenarios.

This leaves the issue of the factors deciding the initial conditions open.

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