Affine Lie Algebras in Massive Field Theory
and
Form Factors from Vertex Operators

André LeClair
Newman Laboratory
Cornell University
Ithaca, NY 14853

We present a new application of affine Lie algebras to massive quantum field theory in 2 dimensions, by investigating the $q \to 1$ limit of the $q$-deformed affine $\hat{sl}(2)$ symmetry of the sine-Gordon theory, this limit occurring at the free fermion point. Working in radial quantization leads to a quasi-chiral factorization of the space of fields. The conserved charges which generate the affine Lie algebra split into two independent affine algebras on this factorized space, each with level 1 in the anti-periodic sector, and level 0 in the periodic sector. The space of fields in the anti-periodic sector can be organized using level-1 highest weight representations, if one supplements the $\hat{sl}(2)$ algebra with the usual local integrals of motion. Introducing a particle-field duality leads to a new way of computing form-factors in radial quantization. Using the integrals of motion, a momentum space bosonization involving vertex operators is formulated. Form-factors are computed as vacuum expectation values of vertex operators in momentum space. Based on talk given at the Berkeley Strings 93 conference, May 1993.
1. Introduction

The massive integrable quantum field theories in 2 dimensions are characterized as possessing an infinite number of local, commuting integrals of motion, which for instance imply the factorizability of the multi-particle S-matrix[1]. Over the past few years it has been shown that these theories also possess an infinite number of non-abelian symmetries, which generally correspond to $q$-deformations of affine Lie algebras[2][3][4]. It is now understood for example that the S-matrices are the minimal solutions to the quantum affine symmetry equations.

In order to make further progress toward understanding the full implications of these quantum affine symmetries, the author investigated the following problem[5]. Consider the sine-Gordon (SG) theory, with the action

$$S = \frac{1}{4\pi} \int d^2 z \left( \partial_z \phi \partial_{\bar{z}} \phi + 4\lambda \cos(\beta \phi) \right).$$

In [2][3] explicit conserved currents were constructed for the 6 generators corresponding to the simple roots of the $q - \hat{sl}(2)$ affine Lie algebra, where $q = \exp(-2\pi i/\beta^2)$. In this realization the central extension, or level, is zero, and the symmetry is actually a deformed loop algebra. When $\hat{\beta} = 1$, the SG theory is equivalent to a free massive Dirac fermion[6]. At this value of the SG coupling constant, $q$ becomes 1, and the results of [2] predict the existence of an ordinary undeformed affine $\hat{sl}(2)$ symmetry in the free Dirac theory. By studying this limit we were able to develop some new structures that we believe can be extended away from $q = 1$. Furthermore, this is not a completely trivial exercise, since there exist fields in the SG description, such as $\exp(i\alpha \phi)$ for $\alpha \notin Z$, which are not simply expressed in terms of the free fermion fields and thus do not have free-field form-factors or correlation functions.

In this talk we will mainly outline the results in [5]. We will first describe the full infinite set of conserved $\hat{sl}(2)$ charges directly in the free massive Dirac theory, and also the usual infinite number of abelian conserved charges $P_n$. Radial quantization will then be introduced as the natural way to obtain operators which diagonalize the Lorentz boost operator $L$. This introduces a fermionic fock space description of the space of fields $\mathcal{H}_F$. Furthermore, $\mathcal{H}_F$ factorizes into $\mathcal{H}_F^L \otimes \mathcal{H}_F^R$, where in the massless limit $\mathcal{H}_F^L$ ($\mathcal{H}_F^R$) is the left (right) ‘moving’ space of field-states. The conserved charges also factorize in their action on $\mathcal{H}_F^L \otimes \mathcal{H}_F^R$. This leads to two separate algebras $\hat{sl}(2)_L$ and $\hat{sl}(2)_R$, which each have level 1 in the anti-periodic sector, and level 0 in the periodic sector. In the same fashion,
the integrals of motion $P_n^{L,R}$ satisfy an infinite Heisenberg algebra in the anti-periodic sector, whereas they all continue to commute in the periodic sector. The spectrum of $\mathcal{H}_F$ in the anti-periodic sector can be obtained by supplementing the $\widehat{\mathfrak{sl}}(2)$ algebra with this Heisenberg algebra, the fields being organized into infinite highest weight modules. By introducing a particle-field duality, we describe a new way to compute form-factors in radial quantization. We then use these algebraic structures to formulate an exact momentum space bosonization. In this operator formulation, non-trivial form-factors of the SG fields $\exp(\pm i\phi/2)$ are computed as expectation values of vertex operators between level 1 highest weight states.

2. Affine $\widehat{\mathfrak{sl}}(2)$ Symmetry of the Massive Dirac Fermion

The Dirac theory is a massive free field theory of charged fermions. Introducing the Dirac spinors $\Psi_\pm = \begin{pmatrix} \psi_\pm \\ \bar{\psi}_\pm \end{pmatrix}$ of $U(1)$ charge $\pm 1$, the action reads

$$S = -\frac{1}{4\pi} \int dx dt \left( \bar{\psi}_- \partial_z \psi_+ + \psi_- \partial_z \bar{\psi}_+ + i\hat{m} (\bar{\psi}_- \psi_+ - \bar{\psi}_- \psi_+) \right). \quad (2.1)$$

We have continued to Euclidean space $t \to -it$, and $z, \bar{z}$ are the usual Euclidean light-cone coordinates: $z = (t + ix)/2, \ \bar{z} = (t - ix)/2$.

Generally, the conserved quantities follow from conserved currents $J_\mu$:

$$\partial_\bar{z} J_z + \partial_z J_{\bar{z}} = 0. \quad (2.2)$$

We will often display the two components of conserved currents by writing

$$Q = \int \frac{dz}{2\pi i} J_z - \int \frac{d\bar{z}}{2\pi i} J_{\bar{z}}, \quad (2.3)$$

i.e. without specifying the contour of integration.

Using the equations of motion:

$$\partial_z \bar{\psi}_\pm = i\hat{m} \psi_\pm, \quad \partial_{\bar{z}} \psi_\pm = -i\hat{m} \bar{\psi}_\pm, \quad (2.4)$$

it is easy to find an infinite number of conserved quantities. They are the following:

$$Q_{-n}^\pm = (-1)^{n+1} 2 \left( \int \frac{dz}{2\pi i} (\psi_\pm \partial_z^n \psi_\pm) - \int \frac{d\bar{z}}{2\pi i} (i\hat{m} \bar{\psi}_\pm \partial_{\bar{z}}^{n-1} \psi_\pm) \right)$$

$$Q_n^\pm = (-1)^{n+1} 2 \left( \int \frac{dz}{2\pi i} (-i\hat{m} \psi_\pm \partial_z^{n-1} \bar{\psi}_\pm) - \int \frac{d\bar{z}}{2\pi i} (\bar{\psi}_\pm \partial_{\bar{z}}^n \bar{\psi}_\pm) \right), \quad (2.5)$$

$$\alpha_{-n} = (-)^n \left( \int \frac{dz}{2\pi i} (\psi_+ \partial_z^n \psi_-) - \int \frac{d\bar{z}}{2\pi i} (i\hat{m} \bar{\psi}_+ \partial_{\bar{z}}^{n-1} \psi_-) \right)$$

$$\alpha_n = (-)^n \left( \int \frac{dz}{2\pi i} (-i\hat{m} \psi_+ \partial_z^{n-1} \bar{\psi}_-) - \int \frac{d\bar{z}}{2\pi i} (\bar{\psi}_+ \partial_{\bar{z}}^n \bar{\psi}_-) \right),$$
where \( n \geq 0 \), and \( n \) is odd for \( Q^\pm_{\pm n} \). (It turns out that \( Q^\pm_{\pm n} = 0 \) for \( n \) even.) The expressions for \( Q^\pm_{\pm 1} \) were originally derived by fermionizing the SG construction in \([2]\) at \( \hat{\beta} = 1 \) and noticing that the conservation of the resulting currents was a simple consequence of Dirac equations of motion\([7]\). Similar conserved charges were constructed for \( O(N) \) invariant fermions in \([8]\).

Define

\[
P_n \equiv \alpha_n \quad n \text{ odd}, \quad T_n \equiv \alpha_n \quad n \text{ even}.
\]

Then one can show using the fermionic commutation relations that the charges satisfy the following algebraic relations

\[
[P_n, P_m] = [P_n, T_m] = [P_n, Q^\pm_m] = [T_n, T_m] = [T_n, Q^\pm_m] = [Q^+_n, Q^-_m] = [Q^-_n, Q^+_m] = 0
\]

\[
[T_n, Q^\pm_m] = \pm 2 \frac{m^{|n|+|m|}}{m^{|n+m|}} Q^\pm_{n+m}
\]

\[
[L, \alpha_n] = -n \alpha_n, \quad [L, Q^\pm_n] = -n Q^\pm_n,
\]

where \( L \) is the Lorentz boost operator.

We now interpret this algebraic structure. The \( P_n \)'s are the usual infinity of commuting integrals of motion with Lorentz spin equal to an odd integer, where \( P_z = P_{-1}, \ P_0 = P_1 \), and the hamiltonian is \( P_1 + P_{-1} \). The additional charges \( T_n, Q^\pm_n \) all commute with the \( P_m \)'s; for \( m = \pm 1 \) this is just the statement that they are all conserved. The charge \( T_0 \) is simply the \( U(1) \) charge. The commutation relations of the \( T_n, Q^\pm_n \) are the defining relations of the level 0 \( \widehat{sl}(2) \) affine Lie algebra. More precisely this is a twisted \( \widehat{sl}(2) \) algebra. The twist has a simple explanation: the usual untwisted \( \widehat{sl}(2) \) algebra has an \( sl(2) \) subalgebra of Lorentz scalars, whereas the Dirac theory has only a \( U(1) \) symmetry; the twist breaks \( sl(2) \) to \( U(1) \). It was shown in \([3]\) how Ward identities for the \( \widehat{sl}(2) \) symmetry can fix the correlation functions of the fermion fields.

3. Radial Quantization

Let \( \mathcal{F} \) denote the complete space of fields, and let \( \mathcal{H}_F \) be spanned by the action of fields on the vacuum:

\[
\mathcal{H}_F = \{ \Phi_i(0)|0\} \equiv |\Phi_i\rangle, \quad \Phi_i \in \mathcal{F}.
\]

3
The space $\mathcal{H}_F$ diagonalizes the Lorentz boost operator $L$, but does not diagonalize the momentum operators $P_z, P_\bar{z}$. In order to construct the space $\mathcal{H}_F$ explicitly, one can work in radial quantization, since such a construction yields states which diagonalize $L$. Radial quantization was generally considered in [9], and specifically in 2d free fermion theories in [10]. Define the radial coordinates $(r, \varphi)$ as follows: $z = \frac{r}{2} \exp(i\varphi), \quad \bar{z} = \frac{r}{2} \exp(-i\varphi)$. In radial quantization one treats the $r$-coordinate as a ‘time’, and $\varphi$ as the ‘space’, and canonical commutation relations are specified at equal $\varphi$.

One begins by expanding the fermion fields in a basis of solutions to the Dirac equation of motion in radial coordinates. One finds that one can define both a periodic (p) and an anti-periodic (a) sector in this way. Namely,

$$\Psi^{(a,p)}_{\pm} = \begin{pmatrix} \bar{\psi}_{\pm} \\ \psi_{\pm} \end{pmatrix} = \sum_\omega b^\pm_\omega \Psi^{(a,p)}_{-\omega-1/2} + \bar{b}^\pm_\omega \overline{\Psi}^{(a,p)}_{-\omega-1/2}, \quad (3.2)$$

where for the periodic sector $\omega \in Z + 1/2$, and for the anti-periodic sector $\omega \in Z$. The basis spinors have the following explicit expressions:

$$\Psi^{(a)}_{-\omega-1/2} = \Gamma(\frac{1}{2} - \omega) \hat{m}^{\omega+1/2} \begin{pmatrix} i e^{i(\frac{1}{2} - \omega)\varphi} \ I_{\frac{1}{2} - \omega}(\hat{m}r) \\ e^{-i(\omega + \frac{1}{2})\varphi} \ I_{-\omega - \frac{1}{2}}(\hat{m}r) \end{pmatrix}$$

$$\overline{\Psi}^{(a)}_{-\omega-1/2} = \Gamma(\frac{1}{2} - \omega) \hat{m}^{\omega+1/2} \begin{pmatrix} e^{i(\frac{1}{2} + \omega)\varphi} \ I_{-\omega - \frac{1}{2}}(\hat{m}r) \\ -i e^{-i(\frac{1}{2} - \omega)\varphi} \ I_{\frac{1}{2} - \omega}(\hat{m}r) \end{pmatrix}, \quad (3.3)$$

$$\Psi^{(p)}_{-\omega-1/2} = \frac{2\hat{m}^{\omega+1/2}}{\Gamma(\frac{1}{2} + \omega)} \begin{pmatrix} -i e^{i(\frac{1}{2} - \omega)\varphi} \ K_{\omega - \frac{1}{2}}(\hat{m}r) \\ e^{-i(\omega + \frac{1}{2})\varphi} \ K_{\omega + \frac{1}{2}}(\hat{m}r) \end{pmatrix} \quad \omega \geq 1/2 \quad (3.4)$$

$$\overline{\Psi}^{(p)}_{-\omega-1/2} = \frac{2\hat{m}^{\omega+1/2}}{\Gamma(\frac{1}{2} + \omega)} \begin{pmatrix} e^{i(\frac{1}{2} + \omega)\varphi} \ K_{\omega + \frac{1}{2}}(\hat{m}r) \\ i e^{i(\omega - \frac{1}{2})\varphi} \ K_{\omega - \frac{1}{2}}(\hat{m}r) \end{pmatrix} \quad \omega \geq 1/2,$$

and $\Psi^{(p)}_{-\omega-1/2}, \overline{\Psi}^{(p)}_{-\omega-1/2}$ for $\omega \leq -1/2$ have the same expression as in the anti-periodic sector. In the quantum theory these modes satisfy:

$$\{b^+_\omega, \bar{b}^+_\omega\} = \{\bar{b}^-_\omega, \bar{b}^-_\omega\} = \delta_{\omega,-\omega}, \quad \{b_\omega, \bar{b}_\omega\} = 0. \quad (3.5)$$

In the massless limit $\hat{m} \to 0$, the operators $b^\pm_\omega, \bar{b}^\pm_\omega$ are the familiar fermionic oscillators of conformal field theory. Indeed,

$$\left( \begin{pmatrix} \bar{\psi}_{\pm} \\ \psi_{\pm} \end{pmatrix} \right) \xrightarrow{\hat{m} \to 0} \sum_\omega \left( \begin{pmatrix} b^+_\omega & \bar{b}^-_\omega \end{pmatrix} \right) \frac{e^{-\omega - 1/2}}{\omega - 1/2}. \quad (3.6)$$
What is remarkable is that since the oscillators are defined in the massive theory, the fermion fock spaces built from these oscillators continue to correspond precisely to the space of fields $\mathcal{H}_F$ in the massive theory. Furthermore, $\mathcal{H}_F$ factorizes into quasi-left/right pieces:

$$\mathcal{H}_F = \mathcal{H}_F^L \otimes \mathcal{H}_F^R.$$  

(3.7)

For more discussion on this point, see [5].

Consider first the periodic sector. We define the physical vacuum as follows:

$$b_0^\pm |0\rangle = \overline{b}_0^\pm |0\rangle = 0, \quad \omega \geq 1/2.$$  

(3.8)

Define the left and right periodic fock spaces:

$$\mathcal{H}_p^L = \left\{ b_{-\omega_1}^- b_{-\omega_2}^- \cdots b_{-\omega'_1}^- b_{-\omega'_2}^- \cdots |0\rangle \right\},$$

$$\mathcal{H}_p^R = \left\{ \overline{b}_{-\omega_1}^- \overline{b}_{-\omega_2}^- \cdots \overline{b}_{-\omega'_1}^- \overline{b}_{-\omega'_2}^- \cdots |0\rangle \right\},$$  

(3.9)

where $\omega, \omega' \geq 1/2$. One can see directly that in the periodic sector $\mathcal{H}_F = \mathcal{H}_p^L \otimes \mathcal{H}_p^R$. For example, using the explicit expansions (3.2), one finds

$$\partial_n \psi_\pm(0)|0\rangle = n! b_{-n-\frac{1}{2}}^\pm |0\rangle, \quad \partial_n \overline{\psi}_\pm(0)|0\rangle = n! \overline{b}_{-n-\frac{1}{2}}^\pm |0\rangle.$$  

(3.10)

Other states in $\mathcal{H}_p^{L,R}$ correspond to composite operators. For example, consider the $U(1)$ current $J_z = \psi_+ \psi_-$, $\overline{J}_z = \overline{\psi}_+ \overline{\psi}_-$. One has

$$J_z(0)|0\rangle = b_0^+ b_{-\frac{1}{2}}^- |0\rangle, \quad \overline{J}_z(0)|0\rangle = \overline{b}_0^+ \overline{b}_{-\frac{1}{2}}^- |0\rangle.$$  

(3.11)

We now turn to the anti-periodic sector. Due to the existence of the zero modes $b_0^\pm, \overline{b}_0^\pm$, the ‘vacuum’ in this sector is doubly degenerate for both left and right. Define these vacua as $|\pm \frac{1}{2}\rangle_L$ and $|\pm \frac{1}{2}\rangle_R$, characterized by

$$b_0^\pm |\mp \frac{1}{2}\rangle_L = |\pm \frac{1}{2}\rangle_L, \quad b_0^\pm |\pm \frac{1}{2}\rangle_L = 0, \quad b_n^\pm |\pm \frac{1}{2}\rangle_L = 0, \quad n \geq 1,$$

$$\overline{b}_0^\pm |\mp \frac{1}{2}\rangle_R = |\pm \frac{1}{2}\rangle_R, \quad \overline{b}_0^\pm |\pm \frac{1}{2}\rangle_R = 0, \quad \overline{b}_n^\pm |\pm \frac{1}{2}\rangle_R = 0, \quad n \geq 1.$$  

(3.12)

The vacua $\langle \pm \frac{1}{2}|$ are defined by the inner products

$$\langle \mp \frac{1}{2}| \pm \frac{1}{2}\rangle_L = \langle \frac{1}{2}| \pm \frac{1}{2}\rangle_R = 1.$$  

(3.13)
These vacuum states have $U(1)$ charge $\pm 1/2$. The anti-periodic fock spaces are defined as

$$H^L_{a\pm} = \left\{ b_{-n_1}^- b_{-n_2}^- \cdots b_{-n_1}'^- b_{-n_2}'^- \cdots | \pm \frac{1}{2} \rangle_L \right\}$$

$$H^R_{a\pm} = \left\{ b_{-n_1}^+ b_{-n_2}^- \cdots b_{-n_1}'^- b_{-n_2}'^+ \cdots | \pm \frac{1}{2} \rangle_R \right\},$$

(3.14)

for $n, n' \geq 1$. Based on a study of the massless limit the following identification was proposed in [5]:

$$e^{\pm i \phi(0)/2} |0\rangle = (|\pm \frac{1}{2}\rangle_L \otimes |\mp \frac{1}{2}\rangle_R) \equiv |\pm \frac{1}{2}\rangle,$$

(3.15)

where $\phi(z, \bar{z})$ is the local SG field.

Consider now the conserved charges of the last section in radial quantization. Given a conserved current $J_z, J_{\bar{z}}$, the conserved charge is

$$Q = \frac{1}{4\pi} \int_{-\pi}^{\pi} r \, d\varphi \left( e^{i\varphi} J_z + e^{-i\varphi} J_{\bar{z}} \right).$$

(3.16)

All of the conserved charges constructed in section 2 can thereby be expressed in terms of the radial modes $b^\pm_\omega, \bar{b}^\pm_\omega$. More specifically, define the inner product of two spinors $A = \begin{pmatrix} \pi \\ a \end{pmatrix}, B = \begin{pmatrix} \bar{b} \\ b \end{pmatrix}$ as

$$(A, B) = \frac{1}{4\pi} \int_{-\pi}^{\pi} r d\varphi \left( e^{i\varphi} a \bar{b} + e^{-i\varphi} \bar{a} b \right).$$

(3.17)

The conserved charges can all be expressed using the above inner product:

$$Q^-_n = \frac{(-)^{n+1}}{2} (\Psi_+, \partial^n \Psi^-), \quad Q^+_n = \frac{(-)^{n+1}}{2} (\Psi_-, \partial^n \Psi^+)$$

$$\alpha_- n = (-)^n (\Psi_+, \partial^n \Psi^-), \quad \alpha_+ n = (-)^n : (\Psi_+, \partial^n \Psi^-) :.$$

(3.18)

One finds that the charges split into left and right pieces:

$$Q^n_\pm = Q^n_\pm, L + Q^n_\pm, R, \quad \alpha_n = \alpha^L_n + \alpha^R_n.$$

(3.19)

The additional minus sign in the subscript $-n$ of the right piece of the charges in comparison to the left piece is dictated by Lorentz covariance. The explicit expressions are as follows:

$$\alpha^L_n = \hat{m} |n| + n \sum_{\omega \in S^n_{(a,p)}} \frac{\Gamma\left(\frac{1}{2} + \omega - n\right)}{\Gamma\left(\frac{1}{2} + \omega\right)} : b^n_{-\omega} \left. b^n_{\omega}^+ : \right.$$  

$$Q^n_\pm, L = \hat{m} |n| + n \sum_{\omega \in S^n_{(a,p)}} \frac{\Gamma\left(\frac{1}{2} + \omega - n\right)}{\Gamma\left(\frac{1}{2} + \omega\right)} b^n_{-\omega} \left. b^n_{\omega}^+ :,$$

(3.20)
where the sums over \( \omega \) run over \( S_n^{(p)} (S_n^{(a)}) \) for the periodic (anti-periodic) sector, and

\[
S_n^{(a)} = Z, \quad \forall \, n
\]

\[
S_n^{(p)} = \{ \omega \in Z + 1/2; \quad 0 > \omega > n \text{ if } n > 0; \quad n > \omega > 0 \text{ if } n < 0 \}\]  \hfill (3.21)

Identical expressions apply to \( Q_n^{\pm,R} \), \( \alpha_n^R \) with \( b_\omega^\pm \to b_\omega^\pm \).

Define as before \( P_n^{L,R} = \alpha_n^{L,R}, \quad n \text{ odd; } T_n^{L,R} = \alpha_n^{L,R}, \quad n \text{ even.} \) Then one can show that the above operators satisfy the relations

\[
\begin{align*}
[P_n^L, P_m^L] &= [T_n^L, T_m^L] = n \, k \, \hat{m}^{2|n|} \, \delta_{n,-m} \\
[P_n^L, T_m^L] &= [P_n^L, Q_m^\pm L] = 0 \\
[T_n^L, Q_m^\pm L] &= \pm 2 \, \frac{\hat{m}^{|n|+|m|}}{\hat{m}^{|n+m|}} \, Q_n^{\pm L} \\
[Q_n^{\pm L}, Q_m^{-L}] &= \frac{\hat{m}^{|n|+|m|}}{\hat{m}^{|n+m|}} \, T_{n+m}^L + \frac{n}{2} \, k \, \hat{m}^{2|n|} \, \delta_{n,-m},
\end{align*}
\]  \hfill (3.22)

where in the periodic sector the level \( k \) is zero, and in the anti-periodic sector \( k = 1 \). Identical results apply to the right pieces of the conserved charges, with the same level 1 in the antiperiodic sector. Note that the sum of the left and right operators (3.19) continues to satisfy a level \( k = 0 \) algebra in either sector, which is required for consistency with section 2.

Due to the non-zero level 1 in the anti-periodic sector, the space of fields in this sector can be organized using infinite highest weight modules. At level 1 there are only 2 highest weight states \( |\Lambda_j\rangle, \quad j = 0, 1/2, \) satisfying

\[
\begin{align*}
Q_n^{\pm L} \, |\Lambda_j\rangle_L &= T_n^L \, |\Lambda_j\rangle_L = 0 \quad n \geq 1 \\
T_0^L \, |\Lambda_0\rangle_L &= -\frac{1}{2} \, |\Lambda_0\rangle_L, \quad T_0^L \, |\Lambda_{1/2}\rangle_L = \frac{1}{2} \, |\Lambda_{1/2}\rangle_L.
\end{align*}
\]  \hfill (3.23)

These highest weight states are the ‘vacuum’ states defined above with the identification:

\[
| + \frac{1}{2} \rangle_L = |\Lambda_{1/2}\rangle, \quad | - \frac{1}{2} \rangle_L = |\Lambda_0\rangle.
\]  \hfill (3.24)

Let us denote the \( P_n \) extension of \( \hat{sl}(2) \) as \( \hat{\hat{sl}(2)} \), and define the \( \hat{\hat{sl}(2)}_L \) modules \( \hat{\hat{V}}_L \) as follows,

\[
\hat{\hat{V}}_{\Lambda_i} \equiv \{ Q_{-n_1}^{\pm L} Q_{-n_2}^{\pm L} \cdots T_{-n_1}^L T_{-n_2}^L \cdots P_{-n_1}^L P_{-n_2}^L \cdots |\Lambda_i\rangle \},
\]  \hfill (3.25)

for \( n, n', n'' \geq 1 \). Then one can show that

\[
\hat{\hat{H}}_a^L = \hat{\hat{V}}_{\Lambda_0}^L \oplus \hat{\hat{V}}_{\Lambda_{1/2}}^L,
\]  \hfill (3.26)

where \( \hat{\hat{H}}_a^L = \hat{\hat{H}}_{a,+}^L \oplus \hat{\hat{H}}_{a,-}^L \). This is proven by comparing the \( \text{Tr} \, (q^L) \) characters for the fermionic space \( \hat{\hat{H}}_a^L \) with the ones for the (twisted) \( \hat{\hat{sl}(2)} \) modules.
4. Particle-Field Duality and Form-Factors in Radial Quantization

In conventional canonical approaches to massive quantum field theory, one deals with the space of particles $\mathcal{H}_P$. In 2d, parameterizing the energy and momentum in terms of the usual rapidity, $E = m \cosh \theta$, $P = m \sinh \theta$, the space $\mathcal{H}_P$ is spanned by

$$\mathcal{H}_P = \{ |\theta_1, \ldots, \theta_n \rangle_{\epsilon_1 \cdots \epsilon_n} \}, \quad (4.1)$$

where $\epsilon_i$ are the quantum numbers of the particles. One can easily define a dual space $\mathcal{H}_P^*$, and a completeness relation

$$1 = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\epsilon_i} \int d\theta_1 \cdots d\theta_n |\theta_1, \ldots, \theta_n \rangle_{\epsilon_1 \cdots \epsilon_n} \langle \epsilon_1 \cdots \epsilon_n | \theta_1, \ldots, \theta_n \rangle. \quad (4.2)$$

Form-factors are matrix elements of fields in the space $\mathcal{H}_P$. Consider for example the form-factor

$$\langle \epsilon_1 \cdots \epsilon_n | \theta_1 \cdots \theta_n | \Phi(0) | 0 \rangle, \quad (4.3)$$

where $\Phi \in \mathcal{F}$. The completeness relation in $\mathcal{H}_P$ allows one to view $\Phi(0) | 0 \rangle$ as a element of $\mathcal{H}_P$. Conventionally, one thinks of expressing the fields $\Phi(0)$ in terms of the operators that create the states in $\mathcal{H}_P$, and one computes form-factors in the space $\mathcal{H}_P$.

We now propose an entirely different way to compute the same form-factors. Let us suppose that one can define a non-degenerate inner-product on the space of fields $\mathcal{H}_F$, and thereby construct the dual space $\mathcal{H}_F^*$ and the completeness relation:

$$1 = \sum_i |\Phi_i \rangle \langle \Phi_i |. \quad (4.4)$$

This implies that one can then consider states in $\mathcal{H}_F^*$ as elements of $\mathcal{H}_F^*$:

$$\langle \epsilon_1 \cdots \epsilon_n | \theta_1 \cdots \theta_n | \in \mathcal{H}_F^*. \quad (4.5)$$

Consequently, the form-factors can be computed directly in the space $\mathcal{H}_F$ rather than $\mathcal{H}_P$. Radial quantization provides us with an explicit construction of the spaces $\mathcal{H}_F, \mathcal{H}_F^*$ and the completeness relation, thus by the above reasoning, radial quantization leads to a new way to compute form-factors. The success of the method relies on being able to construct explicitly the map (4.5). We now describe how to determine this map in the model we are considering. Henceforth we will be working exclusively in the antiperiodic sector.

---

1 The periodic sector will be considered in [12].
In the standard temporal quantization, the fermion fields have the following plane wave expansions
\[
\Psi_{\pm} = \begin{pmatrix} \psi_{\pm} \\ \psi_{\mp} \end{pmatrix} = \pm \sqrt{m} \int_{-\infty}^{\infty} \frac{du}{2\pi i|u|} \hat{b}_{\pm}(u) \begin{pmatrix} 1/\sqrt{u} \\ -i/\sqrt{u} \end{pmatrix} e^{inz + i\pi / u},
\]
where \( u = \exp(\theta) \). We have combined the usual creation-annihilation operators into a single operator \( \hat{b}_{\pm}(u) \), where \( \hat{b}_{\pm}(u > 0) \) is a creation operator and \( \hat{b}_{\pm}(u < 0) \) is an annihilation operator. (See [5] for details.) The states in \( \mathcal{H}_P^* \) are constructed as follows:
\[
\epsilon_1 \cdots \epsilon_n \langle \theta_1 \ldots \theta_n | = \frac{1}{(2\pi i)^n} \langle 0 | \hat{b}_{\epsilon_1}(-u_1) \cdots \hat{b}_{\epsilon_n}(-u_n),
\]
where \( u_i = \exp(\theta_i) \).

The key to constructing explicitly the map from \( \mathcal{H}_P^* \) to \( \mathcal{H}_F^* \) is based on the following fact. By analytically continuing the \( u \) integral in (4.6) and appropriately redefining \( \hat{b}_{\pm}(u) \), one can obtain the expansions in radial quantization. More specifically, let
\[
\int_{-\infty}^{\infty} \frac{du}{2\pi i|u|} \hat{b}_{\pm}(u) \rightarrow \left( \int_{C_L^\varphi} \frac{du}{2\pi i u} b_{\pm}(u) - \int_{C_R^\varphi} \frac{du}{2\pi i u} b_{\pm}(u) \right),
\]
where \( C_L^\varphi, C_R^\varphi \) are contours depending on the angular direction \( \varphi \) of the cut displayed in figures 1,2, and

![Figure 1. The contour \( C_L^\varphi \). The cut (wavy line) is oriented at an angle \( \varphi \) from the negative \( y \)-axis. The circle is at \( |u| = 1 \).](image-url)
Figure 2. The contour $\mathcal{C}_\varphi^R$. The cut is oriented at an angle $\varphi$ from the positive $y$-axis. The circle is at $|u| = 1$.

\[ b^\pm(u) = \pm i \sum_{\omega \in \mathbb{Z}} \Gamma\left(\frac{1}{2} - \omega\right) \hat{m}^\omega b_\omega u^\omega \]
\[ (4.9) \]

\[ \overline{b}^\pm(u) = \pm \sum_{\omega \in \mathbb{Z}} \Gamma\left(\frac{1}{2} - \omega\right) \hat{m}^\omega \overline{b}_\omega u^{-\omega}. \]

Using integral representations for the Bessel functions, one obtains (3.3). This leads us to propose the following fundamental expression:

\[ \langle 0 \mid \hat{b}^{\epsilon_1}(u_1) \cdots \hat{b}^{\epsilon_n}(u_n) = \frac{1}{2} \left( \langle +\frac{1}{2} | + \langle -\frac{1}{2} | \right) \left( b^{\epsilon_1}(u_1) + \overline{b}^{\epsilon_1}(u_1) \right) \cdots \left( b^{\epsilon_n}(u_n) + \overline{b}^{\epsilon_n}(u_n) \right) \right). \]

\[ (4.10) \]

The RHS of the above equation is a state in $\mathcal{H}_F^*$, and this means that the computation of the form-factors can be carried out in $\mathcal{H}_F$.

As an example, consider the form-factors of the fields $\exp(\pm i\phi/2)$ in the SG theory. From the identification (3.13), one has

\[ \epsilon_1 \cdots \epsilon_n \langle \theta_1 \cdots \theta_n | \exp(\pm i\phi/2) | 0 \rangle = \frac{1}{(2\pi i)^n} L \langle \mp \frac{1}{2} | \hat{b}^{\epsilon_1}(-u_1) \cdots \hat{b}^{\epsilon_n}(-u_n) | \mp \frac{1}{2} \rangle_L \]
\[ = \frac{1}{(2\pi i)^n} R \langle \pm \frac{1}{2} | \overline{b}^{\epsilon_1}(-u_1) \cdots \overline{b}^{\epsilon_n}(-u_n) | \mp \frac{1}{2} \rangle_R \]

\[ (4.11) \]
Additional arguments for the above formula were given in [3]. One can compute the RHS explicitly and one finds agreement with the known result [13][14]. We will show this explicitly in the next section. Note that the RHS of the above expression has the structure of a conformal free-fermion correlator, but here we are in momentum space!

5. Vertex Operators and Momentum Space Bosonization

We will now describe a vertex operator construction for the form-factors of the last section. This involves constructing a bosonic representation of the operators $b^\pm(u), \bar{b}^\pm(u)$. Recall that in the anti-periodic section one has the infinite Heisenberg algebras of the chirally split integrals of motion:

$$\left[ \alpha^L_n, \alpha^L_m \right] = n \hat{\alpha}^{2|n|} \delta_{n,-m}, \quad \left[ \alpha^R_n, \alpha^R_m \right] = n \hat{\alpha}^{2|n|} \delta_{n,-m}. \quad (5.1)$$

Define some momentum space bosonic fields as follows:

$$-i\tilde{\phi}^L(u) = \sum_{n \neq 0} \hat{m}^{-|n|} \frac{\alpha^L_n}{\alpha^L_0} \frac{u^n}{n} + \alpha^L_0 \log(u) - \tilde{\alpha}^L_0$$

$$-i\tilde{\phi}^R(u) = \sum_{n \neq 0} \hat{m}^{-|n|} \frac{\alpha^R_n}{\alpha^R_0} \frac{u^{-n}}{n} + \alpha^R_0 \log(u) + \tilde{\alpha}^R_0, \quad (5.2)$$

where $[\alpha^L_0, \tilde{\alpha}^L_0] = [\alpha^R_0, \tilde{\alpha}^R_0] = 1$. The vacua $|\emptyset\rangle_{L,R}$ are defined to satisfy

$$\alpha^L_n |\emptyset\rangle_L = \alpha^R_n |\emptyset\rangle_R = 0, \quad n \geq 0; \quad \tilde{\alpha}^L_0 |\emptyset\rangle_L, \quad \tilde{\alpha}^R_0 |\emptyset\rangle_R \neq 0. \quad (5.3)$$

Note that $|\emptyset\rangle \equiv |\emptyset\rangle_L \otimes |\emptyset\rangle_R$ does not correspond to the physical vacuum since it is not annihilated by $P_1$ and $P_{-1}$. One has the following vacuum expectation values

$$L \langle \emptyset | \tilde{\phi}^L(u) \tilde{\phi}^L(u') | \emptyset \rangle_L = -\log(1/u - 1/u')$$

$$R \langle \emptyset | \tilde{\phi}^R(u) \tilde{\phi}^R(u') | \emptyset \rangle_R = -\log(u - u') \quad (5.4)$$

$$L \langle \emptyset | \prod_i e^{i\alpha_i \tilde{\phi}^L(u_i)} | \emptyset \rangle_L = \prod_{i<j} (1/u_i - 1/u_j)^{\alpha_i \alpha_j}$$

$$R \langle \emptyset | \prod_i e^{i\alpha_i \tilde{\phi}^R(u_i)} | \emptyset \rangle_R = \prod_{i<j} (u_i - u_j)^{\alpha_i \alpha_j}. \quad (5.5)$$
Using these free momentum space fields one can bosonize all of the ingredients of the last section:

\[ b^+(u) = \sqrt{\frac{\pi}{u}} : e^{i\phi^L(u)} : , \quad b^-(u) = \sqrt{\frac{\pi}{u}} : e^{-i\phi^L(u)} : \]

\[ b^+(u) = \sqrt{\frac{\pi}{u}} : e^{-i\phi^R(u)} : , \quad b^-(u) = -\sqrt{\frac{\pi}{u}} : e^{i\phi^R(u)} : , \]

and

\[ |\alpha\rangle_L = : \exp(i\alpha \phi^L(\infty)) : |\emptyset\rangle_L , \quad |\alpha\rangle_R = : \exp(-i\alpha \phi^R(0)) : |\emptyset\rangle_R \]

\[ L\langle \alpha \mid = \lim_{u\to 0} u^{-\alpha^2} \langle \emptyset \mid : \exp(i\alpha \phi^L(u)) : , \quad R\langle \alpha \mid = \lim_{u\to \infty} u^{\alpha^2} \langle \emptyset \mid : \exp(-i\alpha \phi^R(u)). \]

From this construction one finds

\[ L\langle \mp \frac{1}{2} | b^+(u_1) \cdots b^+(u_n)b^-(u_{n+1}) \cdots b^-(u_{2n}) | \pm \frac{1}{2} \rangle_L \]

\[ = R\langle \mp \frac{1}{2} | b^+(u_1) \cdots b^+(u_n)b^-(u_{n+1}) \cdots b^-(u_{2n}) | \mp \frac{1}{2} \rangle_R \]

\[ = \pi^n \sqrt{u_1 \cdots u_{2n}} \left( \prod_{i=1}^{n} \left( \frac{u_{i+n}}{u_i} \right)^{\pm 1/2} \right) \left( \prod_{i<j} (u_i - u_j) \right) \left( \prod_{n+1 \leq i < j} (u_i - u_j) \right) \left( \prod_{r=1}^{n} \prod_{s=n+1}^{2n} \frac{1}{u_r + u_s} \right). \]

These expressions agree with the known form-factors, though they were originally computed using rather different methods\[13 \text{ ] [14}].

6. Conclusions

We have described a new way to compute form-factors based on structures inherent in radial quantization, as vacuum expectation values of vertex operators. The integrals of motion played an essential role in the explicit construction of the vertex operators. We believe that these results can be extended away from the free fermion point, eventually leading to a vertex operator construction of the SG form-factors. These form-factors have been previously computed by Smirnov using bootstrap methods\[14]. Recently, Lukyanov has proposed a vertex operator construction for the SG form-factors at all values of the SG coupling\[15]. Comparing the latter construction with the results described here one finds some apparent differences. The free boson oscillators in \[15] have no apparent connection...
with the integrals of motion, in contrast with the above results. More importantly, it is proposed in [15] that the form-factors are computed as traces over free-field modules, whereas in our work they are vacuum expectation values. It would be interesting to understand more precisely if and how the construction in [15] is related to ours at the free-fermion point.

Acknowledgements

I would like to thank the organizers of the Berkeley Strings 93 conference for the opportunity to present this work. This work is supported by an Alfred P. Sloan Foundation fellowship, and the National Science Foundation in part through the National Young Investigator program.

References

[1] A. B. Zamolodchikov and Al. B. Zamolodchikov, Annals Phys. 120 (1979) 253.
[2] D. Bernard and A. LeClair, Commun. Math. Phys. 142 (1991) 99; Phys. Lett. B247 (1990) 309.
[3] G. Felder and A. LeClair, Int. Journ. Mod. Phys. A7 Suppl. 1A (1992) 239.
[4] F. A. Smirnov, Int. J. Mod. Phys. A7, Suppl. (1992).
[5] A. LeClair, Spectrum Generating Affine Lie Algebras in Massive Quantum Field Theory, CLNS 93/1220, hep-th/9305110.
[6] S. Coleman, Phys. Rev. D 11 (1975) 2088.
[7] R. K. Kaul and R. Rajaraman, Int. J. Mod. Phys. A8 (1993) 1815.
[8] E. Abdalla, M. C. B. Abdalla, G. Sotkov, and M. Stanishkov, Off Critical Current Algebras, Univ. Sao Paulo preprint, IFUSP-preprint-1027, Jan. 1993.
[9] S. Fubini, A. J. Hanson, and R. Jackiw, Phys. Rev. D7 (1973) 1732.
[10] H. Itoyama and H. B. Thacker, Nucl. Phys. B320 (1989) 541.
[11] P. Griffin, Nucl. Phys. B334, 637.
[12] C. Efthimion and A. LeClair, Particle-Field Duality and Form-Factors from Vertex Operators, in preparation.
[13] B. Schroer and T. T. Truong, Nucl. Phys. B144 (1978) 80; E. C. Marino, B. Schroer, and J. A. Swieca, Nucl. Phys. B200 (1982) 473.
[14] F. A. Smirnov, Form Factors in Completely Integrable Models of Quantum Field Theory, in Advanced Series in Mathematical Physics 14, World Scientific, 1992.
[15] S. Lukyanov, Free Field Representation for Massive Integrable Models, Rutgers preprint RU-93-30, hep-th/9307190.