A NOTE ON THE G-SARKISOV PROGRAM

ENRICA FLORIS

Abstract. The purpose of this note is to prove the $G$-equivariant Sarkisov program for a connected algebraic group $G$ following the proof of the Sarkisov program by Hacon and McKernan. As a consequence, we obtain a characterisation of connected subgroups of $\text{Bir}(\mathbb{Z})$ acting rationally on $\mathbb{Z}$.

1. Introduction

The result of the MMP on a pair with non pseudoeffective log-canonical divisor is a Mori fibre space by [BCHM10]. The outcome of the MMP is nevertheless not unique and the Sarkisov program describes the relation between two different Mori fibre spaces that are outcomes of two MMP on the same variety:

**Theorem 1.1** ([HM13], [Cor95]). Suppose that $\phi: X \to S$ and $\psi: Y \to T$ are two Mori fibre spaces. Then $X$ and $Y$ are birational if and only if they are related by a sequence of Sarkisov links.

A Sarkisov link between $\phi: X \to S$ and $\psi: Y \to T$ is a diagram of one of the four following types:

I

\[
\begin{align*}
X' & \rightarrow Y' \\
X & \downarrow \psi \\
S & \quad \phi \\
T & \\
\end{align*}
\]

II

\[
\begin{align*}
X' & \rightarrow Y' \\
X & \downarrow \psi \\
S & \quad \phi \\
T & \\
\end{align*}
\]

III

\[
\begin{align*}
X & \rightarrow Y' \\
S & \quad \psi \\
T & \downarrow \phi \\
\end{align*}
\]

IV

\[
\begin{align*}
X & \rightarrow Y' \\
S & \quad \psi \\
T & \downarrow \phi \\
R & \\
\end{align*}
\]

The vertical arrows in the diagrams are extremal contractions; the arrows to $X$ or $Y$ are divisorial contractions; the horizontal dotted arrows are composition of $K + \Xi$-flops where $\Xi$ is a divisor on the space on the top left such that $K + \Xi$ is klt and numerically trivial over the base.

Let $G$ be an connected algebraic group. If we have a klt $G$-pair $(W, \Lambda)$, that is, a klt pair $(W, \Lambda)$ together with a regular action of $G$ on $W$ preserving $\Lambda$, we can run a $G$-equivariant MMP on $(W, \Lambda)$, that is, an MMP where all the birational maps, divisorial contractions and flips, are compatible with the action of the group.

Date: October 12, 2018.

The author would like to thank Anne-Sophie Kaloghiros and Boris Pasquier for discussions, Andrea Fanelli for his comments on an earlier draft and Jérémie Blanc for his comments and for suggesting the problem and the application to subgroups of the Cremona group. The author is supported by the ANR Project FIBALGA ANR-18-CE40-0003-01.
If $K_W + \Lambda$ is not pseudoeffective, the outcome of such an MMP is a Mori fibre space $\phi: X \to S$ with a regular action of $G$ on $X$.

**Definition 1.2.** A $G$-Mori fibre space is a Mori fibre space with a regular action of a group $G$.

We say that two $G$-Mori fibre spaces $\phi: X \to S$ and $\psi: Y \to T$ are $G$-Sarkisov related if $X$ and $Y$ are results the $G$-equivariant MMP on $(Z, \Phi)$, for the same $\mathbb{Q}$-factorial klt $G$-pair $(Z, \Phi)$.

The purpose of this note is to prove that two Mori fibre spaces that are outcomes of two $G$-equivariant MMP on the same $G$-pair are related by a sequence of $G$-equivariant Sarkisov links.

**Theorem 1.3.** Let $G$ be a connected algebraic group. Let $(W, \Lambda)$ be a klt $G$-pair. Let $\phi: X \to S$ and $\psi: Y \to T$ be two $G$-Sarkisov related Mori fibre spaces. Then $X$ and $Y$ are related by a sequence of $G$-equivariant Sarkisov links and in every such link the horizontal dotted arrows are compositions of $G$-equivariant flops with respect to a suitable boundary.

As a direct consequence we obtain the following characterisation of subgroups of $\text{Bir}(W)$ that are maximal among the connected groups acting rationally on $W$.

**Corollary 1.4.** Let $W$ be an uniruled variety and let $G$ be a connected algebraic group acting rationally on $W$. Then $G$ is maximal among the connected groups acting rationally on $W$ if and only if $G = \text{Aut}^0(X)$ where $\phi: X \to S$ is a Mori fibre space and for every Mori fibre space $\psi: Y \to T$ which is related to $\phi$ by a finite sequence of $G$-Sarkisov links we have $G = \text{Aut}^0(Y)$.

We have the following application to subgroups of the Cremona group.

**Corollary 1.5.** Let $G$ be a connected algebraic group acting rationally on $\mathbb{P}^n$. Then $G$ is maximal among the connected groups acting rationally on $\mathbb{P}^n$ if and only if $G = \text{Aut}^0(X)$ where $\phi: X \to S$ is a rational Mori fibre space and for every Mori fibre space $\psi: Y \to T$ which is related to $\phi$ by a finite sequence of $G$-Sarkisov links we have $G = \text{Aut}^0(Y)$.

2. **Proof of Theorem 1.3**

In this section we present a proof of Theorem 1.3. We refer to [KM98] for the definitions of singularities of pairs of MMP and of Mori fibre space; to [DL16] for the definition of rational and regular action and to [HM13, Definition 3.1] for the definition of ample model.

First, we recall some well-known facts on $G$-equivariant MMP and some preliminary definitions and results from [HM13, Section 3].

**Definition 2.1.** We call a pair $(Z, \Phi)$ a $G$-pair if $G$ acts on $Z$ regularly and for all $g \in G$ we have $g \cdot \Phi = \Phi$.

**Remark 2.2.** The pair $(Z, 0)$ is a $G$-pair for every subgroup $G$ of $\text{Aut}(Z)$.

**Remark 2.3.** Let $G$ be a connected group and $(Z, \Phi)$ a $G$-pair. Then any MMP on $(Z, \Phi)$ is $G$-equivariant.

Indeed let $Z \dashrightarrow Z_1$ be the first step of the MMP. If it is an extremal contraction, then by the Blanchard’s lemma [BSU13, Prop 4.2.1] there is an induced action on $Z_1$ making the map $G$-equivariant. Equivalently, a connected group acts trivially
on the extremal rays contained in the \((K_Z + \Phi)\)-negative part of the Mori cone, that are discrete. Then the extremal ray corresponding to the contraction \(Z \rightarrow Z_1\) is \(G\)-invariant and so is the locus spanned by it.

Assume that the first step is a flip given by the composition of two small contrac-
tions \(\mu : Z \rightarrow Y\) and \(\mu^+ : Z_1 \rightarrow Y\). By the discussion above, there is an action on
\(Y\) making \(\mu \) \(G\)-equivariant. Moreover, \(Z_1 \cong \text{Proj}v \bigoplus_m \mu_* \mathcal{O}_X(m(K_Z + \Phi))\). Since
\(K_Z + \Phi\) is \(G\)-invariant, the group \(G\) acts on \(\mathcal{O}_X(m(K_Z + \Phi))\) for \(m\) sufficiently
divisible, and subsequently on \(Z_1\). See [KM98] and [CCdTdV+] for a discussion of
the case where \(G\) is finite.

**Remark 2.4.** If there is a commutative diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{f_1} & X_1 \\
\downarrow{f_2} & & \downarrow{g_1} \\
Y & \xrightarrow{g_2} & X_2
\end{array}
\]

where \(f_i\) is birational and \(G\)-equivariant for \(i = 1, 2\) and \(g_i\) is a morphism for
\(i = 1, 2\), then, by the Blanchard’s lemma [BSU13, Prop 4.2.1] there are two actions
of \(G\) on \(Y\). These actions coincide. Indeed, they coincide on an open set contained
in the image of the open set where \(f_2 \circ f_1^{-1}\) is an isomorphism.

Let \(Z\) be a smooth projective variety. From now on, we assume the setup of
[HM13, Section 3].

**Definition 2.5.** [BCHM10] Let \(V\) be a finite dimensional affine subspace of the real
vector space \(W\text{Div}_\mathbb{Q}(Z)\) of Weil divisors on \(Z\), which is defined over the rationals,
and \(A \geq 0\) an ample \(\mathbb{Q}\)-divisor on \(Z\).

\[
\mathcal{L}_A(V) = \{\Theta = A + B | B \in V, \ B \geq 0, \ K_Z + \Theta \text{ is log canonical and}\}, \\
\mathcal{E}_A(V) = \{\Theta \in \mathcal{L}_A(V) | K_Z + \Theta \text{ is pseudo-effective}\}.
\]

As in [HM13], we assume that there exists \(B_0 \in V\) such that \(K_Z + \Theta_0 =
K_Z + A + B_0\) is big and klt. Given a rational contraction \(f : Z \rightarrow X\), define

\[
\mathcal{A}_{A,f}(V) = \{\Theta \in \mathcal{E}_A(V) | f \text{ is the ample model of } (Z, \Theta)\}.
\]

In addition, let \(\mathcal{C}_{A,f}(V)\) denote the closure of \(\mathcal{A}_{A,f}(V)\). We recall the following
result from [HM13] for the benefit of the reader.

**Theorem 2.6.** [HM13, Theorem 3.3] There is a natural number \(m\) and there are
\(f_i : Z \rightarrow X_i\) rational contractions \(1 \leq i \leq m\) with the following properties:

1. \(\{\mathcal{A}_i = \mathcal{A}_{A,f_i} | 1 \leq i \leq m\}\) is a partition of \(\mathcal{E}_A(V)\). \(\mathcal{A}_i\) is a finite union of
   relative interiors of rational polytopes. If \(f_i\) is birational, then \(\mathcal{C}_i = \mathcal{C}_{A,f_i}\) is
   a rational polytope.

2. If \(1 \leq i \leq m\) and \(1 \leq j \leq m\) are two indices such that \(\mathcal{A}_j \cap \mathcal{C}_i \neq \emptyset\),
   then there is a contraction morphism \(f_{ij} : X_i \rightarrow X_j\) and a factorisation
   \(f_j = f_{ij} \circ f_i\).

Now suppose in addition that \(V\) spans the Néron-Severi group of \(Z\).

3. Pick \(1 \leq i \leq m\) such that a connected component \(\mathcal{C}_i\) of \(\mathcal{C}_i\) intersects the
   interior of \(\mathcal{L}_A(V)\). The following are equivalent:
\begin{itemize}
    \item $\mathcal{C}$ spans $V$.
    \item $f_i$ is birational and $X_i$ is $\mathbb{Q}$-factorial.
\end{itemize}

The following is a $G$-equivariant version of [HM13, Lemma 4.1].

**Lemma 2.7.** Let $G$ be a connected group. Let $\phi : X \to S$ and $\psi : Y \to T$ be two $G$-Sarkisov related $G$-Mori fibre spaces corresponding to two $\mathbb{Q}$-factorial klt projective $G$-pairs $(X, \Delta)$ and $(Y, \Gamma)$. Then we may find a smooth projective variety $Z$ with a regular action of $G$, two birational $G$-equivariant contractions $f : Z \dasharrow X$ and $g : Z \dasharrow Y$, a klt $G$-pair $(Z, \Phi)$, an ample $\mathbb{Q}$-divisor $A$ on $Z$ and a two-dimensional rational affine subspace $V$ of $W\text{Div}_{\mathbb{Q}}(Z)$ such that

1. if $\Theta \in \mathcal{L}_A(V)$ then $\Theta - \Phi$ is ample,
2. $A_{\phi \circ f}$ and $A_{\psi \circ g}$ are not contained in the boundary of $\mathcal{L}_A(V)$,
3. $V$ satisfies (1-4) of Theorem 2.6,
4. $C_{A,f}$ and $C_{A,g}$ are two dimensional, and
5. $C_{\phi \circ f}$ and $C_{\psi \circ g}$ are one dimensional.

**Proof.** By assumption we may find a $\mathbb{Q}$-factorial klt $G$-pair $(Z, \Phi)$ such that $f : Z \dasharrow X$ and $g : Z \dasharrow Y$ are both outcomes of the $G$-equivariant MMP on $(Z, \Phi)$. Let $p : W \to Z$ be any $G$-equivariant log resolution of $(Z, \Phi)$ which resolves the indeterminacy of $f$ and $g$. Such pair exists for instance by [Kol07, Proposition 3.9.1, Theorem 3.35, Theorem 3.36]. The rest of the proof goes as in [HM13, Lemma 4.1].

**Proof of Theorem 1.3.** The proof follows the same lines as [HM13, Theorem 1.3] but instead of choosing $(Z, \Phi)$ as in [HM13, Lemma 4.1] we choose the $G$-pair given by Lemma 2.7.

We prove now that any $X_i$ as in Theorem 2.6 carries a regular action of $G$ making $f_i$ $G$-equivariant. Since $\Theta \in \mathcal{L}_A(V)$ implies $\Theta - \Phi$ is ample, by Theorem 2.6(3) and [HM13, Lemma 3.6], for every $X_i$ corresponding to a full-dimensional polytope $\mathcal{A}_i$, the variety $X_i$ is the result of an MMP on $(Z, \Phi)$. By Remark 2.3 this MMP is $G$-equivariant and therefore there is a regular action of $G$ on $X_i$.

Let $X_j$ be a variety corresponding to a non full-dimensional polytope $\mathcal{A}_j$. Let $\mathcal{A}_i$ be a full-dimensional polytope such that $\mathcal{C}_i \cap \mathcal{A}_j \neq \emptyset$. By Theorem 2.6 there is a surjective morphism $f_{ij} : X_i \to X_j$. By the Blanchard’s lemma [BSU13, Prop 4.2.1] there is an action of $G$ on $X_j$ making the morphism $f_{ij}$ $G$-equivariant. By Remark 2.4 this action does not depend on the choice of $i$.

The links given by [HM13, Theorem 3.7] are $G$-equivariant. Indeed the maps appearing in the links are either

- morphisms of the form $f_{ij}$ as in Theorem 2.6, and those are $G$-equivariant by the discussion above; or
- flops (with respect to a suitable boundary) of the form $f_{ij} \circ f_{kj}^{-1}$, and those are again $G$-equivariant.

\[\square\]

3. Proof of Corollary 1.4

**Definition 3.1.** A connected subgroup $G < \text{Bir}(Z)$ is not maximal among the connected groups acting rationally on $Z$ if there is a connected subgroup of $\text{Bir}(Z)$ acting rationally on $Z$ such that $G \subset H$. It is maximal among the connected groups acting rationally on $Z$ otherwise. We will say maximal for short.
Proof of Corollary 1.4. By a theorem of Weil [Wei55] (see also [Kra]) there is a birational model $\widetilde{W}$ of $W$, such that $G$ acts regularly on $\widetilde{W}$. We then run a $G$-equivariant MMP on $\widetilde{W}$ (see Remark 2.3) and by [BDPP13] and [BCHM10] the result is a $G$-Mori fibre space $\phi: X \to S$.

We prove now that $G$ is maximal if and only if for every Mori fibre space $\psi: Y \to T$ which is related to $\phi$ by a finite sequence of $G$-Sarkisov links we have $G = \text{Aut}^0(Y)$.

Assume that $G$ is maximal. Let $X \to S \dashrightarrow Y \to T$ be a composition of $G$-Sarkisov links and let $\varphi: X \dashrightarrow Y$ be the corresponding birational map. Then $\varphi G \varphi^{-1} \subseteq \text{Aut}^0(Y)$ and if $G$ is maximal $\varphi G \varphi^{-1} = \text{Aut}^0(Y)$.

Assume that $G$ is not maximal and let $H$ be a connected subgroup of $\text{Bir}(Z)$ acting rationally on $Z$ and such that $G \subsetneq H$. By a theorem of Weil [Wei55] there is a birational model $\widetilde{W}$ of $W$, such that $H$ acts regularly on $\widetilde{W}$. We run an $H$-equivariant MMP on $\widetilde{W}$ and obtain a Mori fibre space $Y \to T$ such that $H < \text{Aut}^0(Y)$. This MMP is also $G$-equivariant. Therefore $X \to S$ and $Y \to T$ are $G$-Sarkisov related. By Theorem 1.3, there is a finite sequence of $G$-Sarkisov links $X \to S \dashrightarrow Y \to T$. □

References

[BCHM10] C. Birkar, P. Cascini, C. D. Hacon, and J. Mckernan. Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.*, 23(2):405–468, 2010. 1, 2, 5, 3

[BDPP13] S. Boucksom, J.-P. Demailly, M. Păun, and Th. Peterson. The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension. *J. Algebraic Geom.*, 22(2):201–248, 2013. 3

[BSU13] Michel Brion, Preema Samuel, and V. Uma. Lectures on the structure of algebraic groups and geometric applications, volume 1 of CMI Lecture Series in Mathematics. Hindustan Book Agency, New Delhi; Chennai Mathematical Institute (CMI), Chennai, 2013. 2, 3, 24, 2

[CCdTdV+] Daniel Chan, Kenneth Chan, Louis de Thanhoffer de Volcsey, Colin Ingalls, Kelly Jabbusch, Sándor J. Kovács, Rajesh Kulkarni, Boris Lerner, Basil Nanayakkara, Shinnosuke Okawa, and Michel Van den Bergh. The minimal model program for b-log canonical divisors and applications. 2.3

[Cor95] Alessio Corti. Factoring birational maps of threefolds after Sarkisov. *J. Algebraic Geom.*, 4(2):223–254, 1995. 1, 1

[DL16] Adrien Dubouloz and Alvaro Liendo. Rationally integrable vector fields and rational additive group actions. *Internat. J. Math.*, 27(8):1650060, 19, 2016. 2

[HM13] Christopher D. Hacon and James McKernan. The Sarkisov program. *J. Algebraic Geom.*, 22(2):389–405, 2013. 1, 2, 2, 2, 2.6, 2, 2

[KM98] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. 2, 2, 3

[Kol07] János Kollár. *Lectures on resolution of singularities*, volume 166 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2007. 2

[Kra] Hanspeter Kraft. Regularization of rational group actions. *arXiv:1808.08729*. 3

[Wei55] André Weil. On algebraic groups of transformations. *Amer. J. Math.*, 77:355–391, 1955. 3