Sharp Strichartz type estimates for the Schrödinger equation associated with harmonic oscillator

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Abstract
In this article we study the Schrödinger equation associated with Harmonic oscillator in the form of Strichartz type inequality. We give simple proofs for Strichartz type inequalities using purely the $L^2 \to L^p$ operator norm estimates of the spectral projections associated to the harmonic oscillator proved in [Jeong, Lee and Ryu, 2022 arxiv:2205.03036v1, 2022, and Koch and Tataru 2005 Duke Math. J. 128:369–392, 2005]. Our Strichartz type estimates are sharp in sense of regularity of initial data.

Keywords Harmonic Oscillator · Schrödinger equation · Strichartz inequality · Hermite expansion

Mathematics Subject Classification Primary 35Q41 · 42B37 · Secondary 42B35 · 26D99

1 Introduction and main results

Consider the free Schrödinger equation

$$\begin{aligned}
\begin{cases}
  iu_t(t, x) &= -\Delta u(t, x), \quad x \in \mathbb{R}^d, t \in \mathbb{R} \\
  u(0, x) &= f(x),
\end{cases}
\end{aligned}$$

(1.1)

where $\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$, Laplacian on $\mathbb{R}^d$. For $f \in L^2(\mathbb{R}^d)$, $e^{it\Delta}f$ is the unique solution to the initial value problem (1.1). The associated Strichartz inequality reads,

$$
\| e^{it\Delta} f \|_{L_t^q((0,\infty), L_x^p(\mathbb{R}^d))} \lesssim C \| f \|_{L_x^2(\mathbb{R}^d)},
$$

(1.2)

which holds, if and only if, $\frac{1}{q} = \frac{d}{2} \left( \frac{1}{2} - \frac{1}{p} \right)$, for $d = 1$ and $2 \leq p \leq \infty$.
\[ d = 2 \text{ and } 2 \leq p < \infty, \text{ and } 2 \leq p < \frac{2d}{d-2} \text{ if } d \geq 3. \]

This fundamental result proved by Keel and Tao [13] is the source of inspiration for several extensions of (1.1) to general compact and non-compact manifolds. The initial value problem (1.1) has been made for the Schrödinger equation of the form \( iu_t(t, x) + \Delta u(t, x) - V(x)u(t, x) = 0 \), for a suitable potential \( V \) by several authors, see [10, 12, 14]. In particular, when \( V(x) = |x|^2 \), the Strichartz inequalities have been studied in the literature see [13, 17]. In this case the initial value problem (1.1) turns out to be an initial value problem for the Schrödinger equation associated with harmonic oscillator \( H = -\Delta + |x|^2 \):

\[
\begin{align*}
    iu_t(t, x) + Hu(t, x) = 0, & \quad x \in \mathbb{R}^d, t \in \mathbb{R} \\
u(0, x) = f(x). & 
\end{align*}
\]

The author in [16, 17] proved the Strichartz inequalities for the above problem. If \( f \in L^2(\mathbb{R}^d) \), the solution of the initial value problem (1.3) is given by \( u(t, x) = e^{itH}f(x) \). The Strichartz inequality in this case reads as,

\[
\|e^{itH}f\|_{L^p_t(\mathbb{T}, L^q_x(\mathbb{R}^d))} \leq C\|f\|_{L^2(\mathbb{R}^d)},
\]

where \( f \in L^2(\mathbb{R}^d) \), \( \mathbb{T} = (-\pi, \pi) \) and \( p, q \geq 1 \) satisfying

\[
\left( \frac{d - 2}{d} \right) < \frac{1}{p} \leq 1 \text{ and } 1 \leq \frac{1}{q} \leq 2,
\]

or

\[
0 \leq \frac{1}{q} < 1 \text{ and } \frac{d}{q} + \frac{d}{p} \geq d.
\]

The key point for obtaining (1.2) and (1.4) are the following estimates

\[
\|e^{it\Delta}\|_{L^1(\mathbb{R}^d) \to L^{\infty}(\mathbb{R}^d)} \leq C|t|^{-\frac{d}{2}} \quad \text{and} \quad \|e^{itH}\|_{L^1(\mathbb{R}^d) \to L^{\infty}(\mathbb{R}^d)} \leq C|\sin t|^{-\frac{d}{2}},
\]

which express the dispersive property of the Schrödinger equation on \( \mathbb{R}^d \). Our main goal of this article is to study the modified version of inequality (1.4) by using \( L^2 \to L^p \) bounds of the spectral resolution of harmonic oscillator. For all unexplained notations we direct the reader to Section 2. Our main results are as follows:

**Theorem 1.1** Let \( d \geq 1, 2 \leq p \leq \infty, \text{ and } 2 \leq q < \infty. \) Then we have the following Strichartz type inequality

\[
\|e^{itH}f\|_{L^p_t(\mathbb{T}, L^q(\mathbb{R}^d))} \leq C_{p, q, s}\|f\|_{W^s(\mathbb{R}^d)},
\]

for all \( s \geq \kappa_{p, q} \), where \( d \geq 2, \)
\[ \mathcal{X}_{p,q} := \begin{cases} \frac{-1}{2} \left( \frac{1}{2} - \frac{1}{p} \right) + \left( \frac{1}{2} - \frac{1}{q} \right), & 2 \leq p < \frac{2(d + 3)}{d + 1} \\ \frac{-1}{6} + \frac{d}{6} \left( \frac{1}{2} - \frac{1}{p} \right) + \left( \frac{1}{2} - \frac{1}{q} \right), & \frac{2(d + 3)}{d + 1} < p \leq \frac{2d}{d - 2} \\ \frac{-1}{2} + \frac{d}{2} \left( \frac{1}{2} - \frac{1}{p} \right) + \left( \frac{1}{2} - \frac{1}{q} \right), & \frac{2d}{d - 2} \leq p \leq \infty \end{cases} \]

and while for \( d = 1 \),

\[ \mathcal{X}_{p,q} := \begin{cases} \frac{-1}{2} \left( \frac{1}{2} - \frac{1}{p} \right) + \left( \frac{1}{2} - \frac{1}{q} \right), & 2 \leq p < 4 \\ \frac{-1}{6} + \frac{1}{6} \left( \frac{1}{2} - \frac{1}{p} \right) + \left( \frac{1}{2} - \frac{1}{q} \right), & 4 < p \leq \infty \end{cases} \]

For \( q = 2 \), and \( s < \mathcal{X}_{p,2} \), the estimates (1.6) does not hold for any non trivial initial data \( f \in L^2(\mathbb{R}^d) \). So, in this case, the regularity order \( \mathcal{X}_{p,2} \) is sharp for \( d \geq 1 \) in the sense that (1.6) does not hold for all \( s < \mathcal{X}_{p,q} \).

For the case \( p = \frac{2(d + 3)}{d + 1} \) and \( d \geq 3 \) we have the following.

**Theorem 1.2** Let \( p = \frac{2(d + 3)}{d + 1} \), \( 2 \leq q < \infty \), and \( d \geq 3 \). Then we have the following Strichartz inequality

\[ \| e^{itH}f \|_{L^p_t(L^q_x(T))} \leq C_{p,q,s} \| f \|_{W^s(\mathbb{R}^d)}, \quad (1.7) \]

for \( s \geq -\frac{1}{2(d + 3)} + \left( \frac{1}{2} - \frac{1}{q} \right) \).

**Remark 1.3** The estimates (1.6) and (1.7) are different from those given in [16, 17] as the mixed norm on left hand sides of inequalities are different.

**Remark 1.4**

1. B. Bongioanni and K. M. Rogers proved the estimate (1.6) in Theorem 1.1 in the case \( q = 2 \) in [3].
2. D. Cardona in [6], proved the estimate similar to (1.6) of Theorem 1.1 with initial data of general \( L^p \)– Sobolev space associated with \( H \). When we compare those results for the case \( L^2 \)–Sobolev space, our results are sharper and give better range of the exponent \( p \).
3. The proofs of main results (Theorem 1.1 and 1.2) are consequence of boundedness of \( e^{itH} \) from suitable Triebel-Lizorkin spaces associated to the Hermite projection operators (see subsection 2.2) and sharp boundedness of Hermite projection operators (see Theorem 2.1). This idea is exploited by Cardona et al. in different settings, see [7, 8].

Organization of this article is as follows. In Section 2 we recall spectral decomposition of harmonic oscillator and Hermite expansion. We also define
Triebel-Lizorkin spaces associated to Hermite projections and their inclusion properties. We prove the main results along with supported lemmas in Section 3.

2 Preliminaries

In this section we will recall about Hermite functions and spectral decomposition of harmonic oscillator.

2.1 Harmonic oscillator and Hermite functions

The harmonic oscillator (Hermite operator) is denoted by $H$ and is defined by

$$H = -\Delta + |x|^2,$$

where $x \in \mathbb{R}^d$ and

$$\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}.$$

The Hermite operator $H$ is a positive operator densely defined on $L^2(\mathbb{R}^d)$. The Hermite functions are the eigenfunctions for the operator $H$. Hermite functions are defined in the following way. The one dimensional Hermite functions $h_k$ are defined by

$$h_k(x) = \frac{(-1)^k}{\sqrt{2^k k! \sqrt{\pi}}} \left( \frac{d^k}{dx^k} e^{-x^2} \right) e^{x^2}.$$

Then they form a complete orthonormal family in $L^2(\mathbb{R})$. Now for each multi index $\mu = (\mu_1, \mu_2, \ldots, \mu_d)$, ($\mu_i$ is non-negative integer), we define the $d$-dimensional Hermite functions by tensor product:

$$\Phi_\mu(x) = \prod_{i=1}^{d} h_{\mu_i}(x_i).$$

Then the function $\Phi_\mu$ are eigenfunctions of the Hermite operator $H$ with eigenvalue $(2|\mu| + d)$, where $|\mu| = \mu_1 + \cdots + \mu_d$, i.e.,

$$H \Phi_\mu = (2|\mu| + d) \Phi_\mu$$

and they form a complete orthonormal system in $L^2(\mathbb{R}^d)$. Let $f \in L^2(\mathbb{R}^d)$, then the Hermite expansion is given by

$$f = \sum_{\mu} \langle f, \Phi_\mu \rangle \Phi_\mu(x) = \sum_{k=0}^{\infty} P_k f,$$

where $\langle f, \Phi_\mu \rangle = \int_{\mathbb{R}^d} f(x) \Phi_\mu(x) dx$ and $\{P_k\}_{k \in \mathbb{N}_0}$ is the orthogonal projection operator to the eigenspace corresponding to the eigenvalue $(2k + d)$ which is given by
\[ P_k f(x) = \sum_{|\mu|=k} \langle f, \Phi_{\mu} \rangle \Phi_{\mu}(x). \]

Setting \( \Phi_k(x,y) = \sum_{|\mu|=k} \Phi_{\mu}(x) \Phi_{\mu}(y) \), the Hermite projection may be written as
\[ P_k f(x) = \int_{\mathbb{R}^d} \Phi_k(x,y)f(y)dy. \]

The functions \( \Phi_k(x,y) \) can be obtained by the following generating function identity called as Meheler’s formula:
\[ \sum_{k=0}^{\infty} \omega^k \Phi_k(x,y) = \pi^{-\frac{d}{4}} (1 - \omega^2)^{-\frac{d}{4}} e^{-\frac{1}{2} \frac{1}{\omega} (|x|^2 + |y|^2) + \frac{1}{\omega} xy}, \]
for \( |\omega| < 1 \), for more details (see [20]). Using the above formula it can be verify that \( e^{itH} \) is an integral operator see [19] with the kernel \( K_{it}(x,y) \) given by
\[ K_{it}(x,y) = \frac{(2\pi)^{-d/2}}{(-i \sin 2t)^{\frac{d}{2}}} e^{\frac{it}{2} \left( \cot 2t (|x|^2 + |y|^2) - \frac{2xy}{|x|^2 - |y|^2} \right)}. \]

The following theorem due to Koch and Tataru [15] and Sanghyuk Lee et al. [11], tells about the \( L^2 \rightarrow L^p \) operator norm of projections \( P_k \) associated with harmonic oscillator.

**Theorem 2.1** [11, 15] Let \( k \in \mathbb{N}_0 \), \( d \geq 1 \) and \( 2 \leq p \leq \infty \). Then we have
\[ \|P_k f\|_{L^p(\mathbb{R}^d)} \leq C \kappa_p\|f\|_{L^2(\mathbb{R}^d)}, \]
where \( d \geq 2 \)
\[ \kappa_p := \begin{cases} \frac{-1}{2} \left( \frac{1}{2} - \frac{1}{p} \right), & 2 \leq p < \frac{2(d + 3)}{d + 1} \\ \frac{-1}{6} + \frac{d}{6} \left( \frac{1}{2} - \frac{1}{p} \right), & \frac{2(d + 3)}{d + 1} < p \leq \frac{2d}{d - 2} \\ \frac{-1}{2} + \frac{d}{2} \left( \frac{1}{2} - \frac{1}{p} \right), & \frac{2d}{d - 2} \leq p \leq \infty \end{cases} \]
and while for \( d = 1 \),
\[ \kappa_p := \begin{cases} \frac{-1}{2} \left( \frac{1}{2} - \frac{1}{p} \right), & 2 \leq p < 4 \\ \frac{-1}{6} + \frac{1}{6} \left( \frac{1}{2} - \frac{1}{p} \right), & 4 < p \leq \infty \end{cases} \]
and for \( p = \frac{2(d + 3)}{d + 1} \) and \( d \geq 3 \) the \( \kappa_p = -\frac{1}{2(d + 3)} \). The exponent \( \kappa_p \) is sharp, in the sense that there is no \( f \in L^2(\mathbb{R}^d), \ f \neq 0 \), such that...
\[ \| P_k f \|_{L^p(\mathbb{R}^d)} \leq C k^s \| f \|_{L^2(\mathbb{R}^d)}, \quad (2.3) \]

for all \( s < \nu_p \).

Given a bounded function \( m \) on \( \mathbb{N}_0 \), we define \( m(H) \) using spectral theorem by

\[ m(H)f = \sum_{k=0}^{\infty} m(2k + d) P_k f, \quad (2.4) \]

where \( f \in L^2(\mathbb{R}^d) \). The operator \( m(H) \) is bounded on \( L^2(\mathbb{R}^d) \) if and only if \( m \) is bounded.

\[ \text{Dom}(m(H)) := \{ f \in L^2(\mathbb{R}^d) : \sum_{k=0}^{\infty} |m(2k + d)|^2 \| P_k f \|_{L^2(\mathbb{R}^d)}^2 < \infty \}. \]

There are some sufficient conditions on \( m \) so that \( m(H) \) can be extended to a bounded linear operator on \( L^p(\mathbb{R}^d) \), for more details see [20]. The \( r \)-nuclearity of multipliers associated with Hermite expansion has been studied by Barraza and Cardona in [1] and Pseudo multipliers for this Hermite expansion has been studied by Sayan and Thangavelu in [2]. In view of (2.4), for \( m(k) = e^{ikt} \) we have \( \text{Dom}(m(H)) = L^2(\mathbb{R}^d) \).

In this case

\[ m(H)f = e^{itH}f = \sum_{k=0}^{\infty} e^{i(2k+d)t} P_k f, \]

for \( f \in L^2(\mathbb{R}^d) \) and the kernel of \( e^{itH} \) given by (2.1).

### 2.2 Triebel-Lizorkin spaces associated to \( \{ P_k \}_{k \in \mathbb{N}_0} \)

**Definition 2.2** Let us consider \( 0 < p \leq \infty, \quad r \in \mathbb{R} \) and \( 0 < q \leq \infty \). The Triebel-Lizorkin spaces associated to the family of projections \( \{ P_k \}_{k \in \mathbb{N}_0} \) and to the parameter \( p, q \) and \( r \) is defined by those complex measurable functions \( f \) satisfying

\[ \| f \|_{F_p^q(\mathbb{R}^d)} = \| f \|_{F_p^q(\mathbb{R}^d, \{ P_k \}_{k \in \mathbb{N}_0})} := \left( \sum_{k=0}^{\infty} k^{rq} \| P_k f \|^q \right)^{\frac{1}{q}} < \infty. \quad (2.5) \]

For \( s \in \mathbb{R}, \quad 1 \leq p \leq \infty, \) \( W^{s,p}(\mathbb{R}^d) \) denotes \( L^p - \)Sobolev spaces associated with \( H \), and defined by the norm

\[ \| f \|_{W^{s,p}} = \| H^s f \|_{L^p} := \left( \int_{\mathbb{R}^d} |H^s f(x)|^p dx \right)^{\frac{1}{p}}. \]

We use the notation \( W^s(\mathbb{R}^d) = W^{s,2}(\mathbb{R}^d) \). It can be verified that
\[ \|f\|_{W^s(\mathbb{R}^d)} \lesssim \left( \sum_{k=0}^{\infty} k^{2s} \|P_k f\|_{L^2(\mathbb{R}^d)}^2 \right)^{\frac{1}{2}}. \]

We also have the following natural embedding properties for the Triebel-Lizorkin spaces associated to \( \{P_k\}_{k \in \mathbb{N}_0} \) which can be proved using similar analysis as in [9].

- \( F^r_{p,q_1} \hookrightarrow F^{r'}_{p,q_1}, \quad 0 < p \leq \infty, \quad 0 < q_1 \leq q_2 \leq \infty. \)
- \( F^r_{p,q_1} \hookrightarrow F^{r'}_{p,q_2}, \quad \epsilon > 0, \quad 0 < p \leq \infty, \quad 1 \leq q_2 < q_1 < \infty. \)
- \( F^0_{2,2} = L^2 \) and consequently, for every \( s \in \mathbb{R}, \ W^s = F^s_{2,2}. \)

More details of Sobolev spaces associated with Hermite expansion can be found in [4, 5, 18].

Throughout the article, \( \mathbb{T} \) denotes \((-\pi, \pi)\) and we shall write \( C \) or \( C' \) to denote positive constant of independent of significant quantities, the meaning of which can change one occurrence to another. For all \( 1 < p, q \leq \infty \), the mixed norms are defined as follows:

\[ \|h(t, x)\|_{L^q_x(\mathbb{R}^d, L^p_t(\mathbb{T}))} := \|h(\cdot, x)\|_{L^q_x(\mathbb{R}^d)} \|L^p_t(\mathbb{T}) < \infty \]

and

\[ \|h(t, x)\|_{L^q_x(T, L^p_x(\mathbb{R}^d))} := \|h(t, \cdot)\|_{L^q_x(\mathbb{R}^d)} \|L^p_t(\mathbb{T}) < \infty. \]

### 3 Proof of main result

In this section we will prove our main results. In the course of proof of main results, we will realize that the estimates of \( L^2 \rightarrow L^p \)-operator norm of the projections \( P_k f \) play an important role. Besides that we need the following lemmas. Let \( X \) denotes subspace of \( L^2(\mathbb{R}^d) \), spanned by Hermite functions.

**Lemma 3.1** For \( 1 \leq p \leq \infty \), we have

\[ \|u(t, x)\|_{L^q_x(\mathbb{R}^d, L^p_t(\mathbb{T}))} = \sqrt{2\pi} \|f\|_{F^0_{p,2}(\mathbb{R}^d)}, \tag{3.1} \]

where \( u \) is the solution of Schrödinger equation associated with Hermite operator (1.3) with initial data \( f \in F^0_{p,2}(\mathbb{R}^d). \)

**Proof** Since \( X \) is dense in \( F^0_{p,2}(\mathbb{R}^d) \), it is enough to consider \( f \in X \), we can write the solution of (1.3) as

\[ u(t, x) = e^{iHf}(x) = \sum_{k=0}^{\infty} e^{(2k+d)it} P_k f(x), \tag{3.2} \]

so we have,
\[ \|u(t,x)\|_{L^2_t(\mathbb{T})}^2 = \int_{-\pi}^{\pi} u(t,x)\overline{u(t,x)} dt = \int_{-\pi}^{\pi} \sum_{k,k'} e^{i(2k+d)t} e^{-i(2k'+d)t} P_k f(x)\overline{P_k f(x)} dt = \sum_{k,k'} \left( \int_{-\pi}^{\pi} e^{2i(k-k')t} dt \right) P_k f(x)\overline{P_k f(x)} \]

Observe that if \( k \neq k' \), then
\[ \int_{-\pi}^{\pi} e^{2i(k-k')t} dt = \begin{cases} 0, & \text{if } k \neq k' \\ 2\pi, & \text{if } k = k' \end{cases} \]
so we get
\[ \|u(t,x)\|_{L^2_t(\mathbb{T})}^2 = \sum_{k=0}^{\infty} 2\pi|P_k f|^2 = 2\pi \sum_{k=0}^{\infty} |P_k f|^2. \] (3.3)

Consider
\[ \|u(t,x)\|_{L^p_x(L^q_t(\mathbb{T}))} = \| \|u(t,x)\|_{L^2_t(\mathbb{T})}\|_{L^p_x(\mathbb{R}^d)} \]
\[ = \sqrt{2\pi} \left\| \left( \sum_{k=0}^{\infty} |P_k f|^2 \right)^{\frac{1}{2}} \right\|_{L^p_x(\mathbb{R}^d)} \]
\[ = \sqrt{2\pi} \|f\|_{F_{p,2}^0(\mathbb{R}^d)}, \] (3.4)
which completes the proof of the lemma.

**Lemma 3.2** Let \( 0 < p \leq \infty, 2 \leq q < \infty \) and \( s_q := \frac{1}{2} - \frac{1}{q} \). Then
\[ C_q' \|f\|_{F_{p,2}^0(\mathbb{R}^d)} \leq \|u(t,x)\|_{L^p_x(L^q_t(\mathbb{T}))} \leq C_{q,s} \|f\|_{F_{p,2}^0(\mathbb{R}^d)}, \] (3.5)
holds for every \( s \geq s_q \).

**Proof** Let us consider \( f \in X \). In order to estimate the norm \( \|u(t,x)\|_{L^p_x(L^q_t(\mathbb{T}))} \) we can use the Wainger Sobolev embedding Theorem:
\[ \left\| \sum_{n \in \mathbb{Z}, n \neq 0} |n|^{-2} \hat{F}(n)e^{int} \right\|_{L^q(T)} \leq C\|F\|_{L^r(T)}, \quad \alpha := \frac{1}{r} - \frac{1}{q} \] (3.6)
For \( s_q := \frac{1}{2} - \frac{1}{q} \) and \( f \in X \) we have
\[ \| u(t,x) \|_{L^q(T)} = \left\| \sum_{k=0}^{\infty} e^{i(2k+d)t} P_k f(x) \right\|_{L^q(T)} \]
\[ = \left\| \sum_{k=0}^{\infty} (2k+d)^{-s_q} e^{i(2k+d)t} (2k+d)^{s_q} P_k f(x) \right\|_{L^q(T)} \]
\[ \leq C_q \left\| \sum_{k=0}^{\infty} (2k+d)^{s_q} e^{i(2k+d)t} P_k f(x) \right\|_{L^2(T)} \]
\[ = C_q \left( \sum_{k=0}^{\infty} (2k+d)^{2s_q} |P_k f(x)|^2 \right)^{\frac{1}{2}} \]
\[ \leq C_q \left( \sum_{k=0}^{\infty} k^{2s_q} |P_k f(x)|^2 \right)^{\frac{1}{2}}. \]

Thus, we get
\[ \| u(t,x) \|_{L^q_x(\mathbb{R}^d)} \leq C_q \left( \sum_{k=0}^{\infty} k^{2s_q} |P_k f(x)|^2 \right)^{\frac{1}{2}} \]
\[ = C_q \| f \|_{F^s_{p,2}(\mathbb{R}^d)} \leq C_{q,p} \| f \|_{F^s_{p,2}(\mathbb{R}^d)} \]

Last inequality in the above is due to the embedding \( F^s_{p,2} \hookrightarrow F^s_{p,2} \) for every \( s \geq s_q \).

From Lemma 3.1 we know
\[ \| f \|_{F^s_{p,2}(\mathbb{R}^d)} = C \| u(t,x) \|_{L^q_x(\mathbb{R}^d)} . \] (3.7)

Since \( \| u(t,x) \|_{L^q_x(T)} \leq C_q \| u(t,x) \|_{L^q_x(T)} \) for \( 2 \leq q < \infty \), in view of (3.7), we will get
\[ \| f \|_{F^s_{p,2}(\mathbb{R}^d)} \leq C' \| u(t,x) \|_{L^q_x(\mathbb{R}^d)} . \]

This proves our Lemma.

Now, with the analysis developed above and by using Theorem 2.1 we will provide a short proof for the main results.

**Proof of Theorem 1.1 and 1.2** We have that from Lemma 3.2,
\[ C' \| f \|_{F^s_{p,2}(\mathbb{R}^d)} \leq \| u(t,x) \|_{L^q_x(\mathbb{R}^d)} \leq C_q \| f \|_{F^s_{p,2}(\mathbb{R}^d)} , \]
so, in view of the above it is enough to estimate the \( F^s_{p,2}(\mathbb{R}^d) \)-norm of the initial data \( f \) in terms of \( L^2 \)-Sobolev norm \( \| f \|_{W^s} \) of \( f \) for every \( s \geq s_q \). Moreover, by the embedding \( W^s \hookrightarrow W^{s,q} \) for every \( s \geq s_q \), it is enough to prove that
Now consider for \(2 \leq p < \infty\) together with the Minkowski integral inequality and Theorem 2.1, we get

\[
\|f\|_{F_{p,2}^{q_p}(\mathbb{R}^d)} \leq C\|f\|_{W^{p,q}}.
\]

where we have used that \(\kappa_{p,q} = s_q + \kappa_p\). Let us note that the previous estimates are valid for \(p = \infty\). We prove the sharpness results for the case \(q = 2\) by choosing \(f = P_k g\), where \(g\) is arbitrary \(g \neq 0 \in L^2(\mathbb{R}^d)\). Suppose the inequality (1.6) or (1.7) holds for some \(s < \kappa_p = \kappa_{p,2}\), then for the choice of \(f = P_k g\), we get that

\[
\|e^{itH}f\|_{L_t^2(\mathbb{R}^d, L^2(\mathbb{T}))} = \|P_k g\|_{L^p(\mathbb{R}^d)} \leq \|f\|_{W^s} = k^s \|P_k g\|_{L^2(\mathbb{R}^d)} \leq k^s \|g\|_{L^2(\mathbb{R}^d)},
\]

which leads to the improvement of the estimates for the spectral projection operators \(P_k\)’s which is not possible.

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Declarations

Conflict of Interest The authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.
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