Generalized Superconformal Symmetries and Supertwistor Dynamics

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Abstract

We show that in the supersymmetry framework described by a Poincaré superalgebra with tensorial central charges the role of generalized superconformal symmetry which contains all these central charges is played by $\text{OSp}(1|2^k)$, where $k = 3$ for $D = 4$. Following [1,2] we describe the free supertwistor model for $\text{OSp}(1|8)$. It appears that in such a scheme the tensorial central charges satisfy additional relations and the model describes the tower of supersymmetric massless states with an arbitrary (integer and half–integer) helicity spectrum.

1 Introduction

Let us recall that in the standard supersymmetry scheme [3,4] only scalar central charges are allowed. In particular $D = 4$ $N$-extended SUSY has the following most

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general form \( (A, B = 1, 2; i, j, k, l = 1 \ldots N; \, \tilde{Q}_{Ai} = (Q^i_A)^\dagger) \)

\[
\{Q^i_A, Q^j_B\} = \delta^j_i (\sigma^\mu)_{AB} P^\mu, \\
\{Q^i_A, \tilde{Q}^i_B\} = \epsilon_{AB} Z^{ij}, \\
\{\tilde{Q}_{Ai}, \tilde{Q}_{Bj}\} = \epsilon_{AB} \tilde{Z}_{ij},
\]

(1)

where \( Z^{ij} = -Z^{ji} \) describes complex scalar Abelian central charges

\[
[Q^i_A, Z^{kl}] = [\tilde{Q}^i_A, Z^{kl}] = 0, \quad [Q^i_A, P_\mu] = 0.
\]

(2)

In particular, if \( N = 1 \) the central charges are not present, and for \( N = 2 \) \((i, j = 1, 2)\) one can introduce one central charge \( Z = \epsilon_{ij} Z^{ij} \).

In a generalized supersymmetry scheme, in order to characterize all possible sources describing charged \( p \)-branes, domain walls etc. one uses nonvanishing tensorial central charges \( Z_{\mu_1 \ldots \mu_k} \), where \( k \leq \frac{|D|}{2} \). For example, in \( D = 4 \) the relations (1) are extended as follows [5]

\[
\begin{align*}
\{Q^i_A, \tilde{Q}^j_B\} & = \sigma^\mu_{AB} P^\mu_j + i (\sigma^\mu)_{AB} Y^i_{\mu j}, \\
\{Q^i_A, Q^j_B\} & = \sigma^\mu_{AB} \tilde{Z}^{ij} + \epsilon_{AB} Z^{ij}, \\
\{\tilde{Q}^i_A, Q^j_B\} & = \tilde{\sigma}^\mu_{AB} \tilde{Z}^{\mu ij} + \epsilon_{AB} \tilde{Z}^{ij},
\end{align*}
\]

(3)

where \( P_\mu \) and traceless \( Y^i_{\mu j} \) (\( Y^i_{\mu i} = 0 \)) are Hermitian, \( \tilde{Z}^{ij} = -\tilde{Z}^{ij} \) are complex and self–dual with respect to space–time indices, and their Hermitian conjugate \( \tilde{Z}^{\mu ij} = -\tilde{Z}^{\mu ij} \), \( \tilde{Z}^{\mu ij} = -\tilde{Z}^{\mu ij} \) are anti-selfdual. It should be mentioned that from the algebraic point of view it is also possible to introduce additional spinorial fermionic charges into \([P, Q]\) and \([Z, Q]\) commutators (see e.g. [7,8]). In this talk we shall consider only the case of bosonic tensorial central charges.

In particular, for \( N = 1, \, D = 4 \), using real Majorana supercharges we get

\[
\{Q_\alpha, Q_\beta\} = P_{\alpha \beta} = (\gamma^\mu C)_{\alpha \beta} P_\mu + (\sigma^{\mu \nu} C)_{\alpha \beta} Z^{\mu \nu},
\]

(4)

where the real tensor \( Z^{\mu \nu} \) describes six tensorial central charges. It has been firstly shown in [1] that the presence of nonvanishing tensorial central charges \( Z^{\mu \nu} \) allows one to describe the superparticle model with only one broken target space supersymmetry (i.e. 3/4 SUSY remains unbroken).

If we wish to realize the superalgebra \footnote{For simplicity, we consider here only the supercharge sector (“fermion-fermion” relations), which forms the subsuperalgebra of the full \( \mathcal{N} \)-extended Poincaré superalgebra.} \footnote{See [3] for recent discussion of such BPS states.} as an extension of the standard superspace framework one should introduce additional 6 bosonic central charge coordinates (see e.g. [1,2,9]). With increase in the number of space-time dimensions the number of these additional bosonic coordinates grow rapidly (e.g. for \( D = 11 \) one has...}
\[ \frac{32 \times 33}{2} - 11 = 517 \text{ central charge coordinates}. \] However, one can use the twistor and supertwistor framework [10, 11] and extend the Penrose formula for massless momenta

\[ P_{AB} = (\sigma^{\mu})_{AB} P_\mu = \lambda_A \lambda_B \] (5)

to the sector of central charges by assuming composite formulae

\[ Z_{AB} = \frac{1}{4} \sigma^\mu_{AB} Z_{\mu\nu} = \lambda_A \lambda_B, \quad Z_{\dot{A}\dot{B}} = \frac{1}{4} \sigma^\mu_{\dot{A}\dot{B}} Z_{\mu\nu} = \lambda_{\dot{A}} \lambda_{\dot{B}}. \] (6)

In such a way we obtain the tensorial central charges satisfying some constraints, and the number of independent degrees of freedom in dimension \( D \) is determined by the real dimension of the fundamental spinor representation. In particular, for \( D = 4 \) we have \( n = 4 \) degrees of freedom, for \( D = 5, 6 \) and \( 7 \) there are \( n = 8 \) degrees, for \( D = 8, 9 \) and 10 \( n = 16 \), and for \( D = 11 \) we have \( n = 32 \). If we consider the dimensions \( D = 3, 4, 6, 10 \) describing respectively the sequence of real, complex, quaternionic and octonionic pair of supercharges we see that subtracting \( D - 1 \) degrees of freedom describing massless momenta we obtain respectively \( m = 0, 1, 3, 7 \) internal variables parametrizing the spheres \( S^m \) [2].

2 Generalized Superconformal Symmetries and Tensorial Central Charges

Standard conformal algebras in \( D \) dimensions are isomorphic to the orthogonal algebras \( O(D, 2) \). In order to introduce the standard conformal superalgebra we should consider the fundamental spinorial realization of \( O(D, 2) \). The dimensions \( D = 3, 4 \) and 6 are specific in the sense that then the spin coverings of \( SO(D, 2) \) are described by the classical groups, namely, \( \text{Spin}(3, 2) = \text{Sp}(4; R) \), \( \text{Spin}(4, 2) = SU(2, 2) \) and \( \text{Spin}(6, 2) = U_\alpha(4, H) = U(4, 4) \cap \text{Sp}(8; C) = USp(4, 4; C) \) [3]. Supersymmetrization implies that

\[
\begin{align*}
D = 3 : & \quad \text{Sp}(4; R) \rightarrow \text{OSp}(N; 1|R) \\
D = 4 : & \quad \text{SU}(2, 2) \rightarrow \text{SU}(2, 2|N) \\
D = 6 : & \quad U_\alpha(4; H) \rightarrow UU_\alpha(N; 4|R)
\end{align*}
\] (7)

The standard conformal algebra \( O(D, 2) \) has the following three-fold grading

\[ O(D, 2) = P \oplus L \oplus K \] (8)

where \( P \) corresponds to the sector of \( D \) translation generators, \( L = M \oplus R \) contains Lorentz \( O(D - 1, 1) \) rotations \( M \) and the dilatation generator \( R \), and \( K \) describe \( D \)

5 In [12] tensorial central charges are, in contrast, the composites of the momenta for a multiparticle–multitime system.

6 We shall not include into this sequence \( D = 10 \) with octonionic nonassociative group-like matrices \( U_\alpha(4; O) \) (for more extensive discussion see [13]). By \( U_\alpha(N; F) \) we denote the group of symplectic–unitary matrices \((U^\dagger \Omega U = \Omega, \quad \Omega^T = -\Omega)\) over a field \( F \).
conformal boosts. The supersymmetrization of the graded structure \((8)\) for \(N = 1\) is performed by taking the "supersymmetric square roots" of the \(O(D, 2)\) generators in the following way (\(G = U(1)\) for \(D = 4\), and \(U(2)\) for \(D = 6\)):

\[
\{Q, Q\} \subset P, \quad \{S, S\} \subset K, \quad \{Q, S\} \subset L \oplus G.
\]  

(9)

where \(G\) describe internal symmetry generators. The superalgebra \((8, 9)\) is endowed with a five–fold graded structure \(P \oplus Q \oplus (L \oplus G) \oplus S \oplus K\).

In order to introduce generalized conformal superalgebras which include tensorial central charges we should generalize the set of relations \((9)\). In \(D = 4\) the generalized conformal algebra is obtained by replacing four-momenta generators \(P_\mu\) with ten generators \((P_\mu, Z_{\mu\nu})\), (see \((4)\)), and for \(D = 4\) in place of \((9)\) we obtain the following simple real superalgebra:

\[
\{Q_\alpha, Q_\beta\} = P_{\alpha\beta}, \quad \{S_\alpha, S_\beta\} = K_{\alpha\beta}, \quad \{Q_\alpha, S_\beta\} = L_{\alpha\beta}.
\]  

(10)

The relations \((10)\) contain two copies of the Poincaré superalgebra \((4)\) and 16 real bosonic charges \(L_{\alpha\beta}\) generate \(GL(4; R)\). We thus get a generalized \(D = 4\) conformal algebra given by \(Sp(8)\) (note that \(Sp(8)\) contains \(SU(2, 2)\))

\[
SU(2, 2) = Spin(4, 2) \subset Sp(8) = P \oplus L \oplus K,
\]  

(11)

and the generalized \(D = 4\) superconformal algebra \((10)\) is described by \(OSp(1; 8)\).

One can also introduce an N-extended \(D = 4\) generalized superconformal group \(OSp(N; 8)\) \((SU(2, 2)|N) \subset OSp(2N; 8)\) as well as the ones for \(D > 4\) containing \(D\)-dimensional super–Poincaré algebra with all possible tensorial charges [14, 15]. For example, in \(D = 10\) all 126 central charges are included in the simple generalized superconformal algebra \(OSp(1; 32)\) [15].

### 3 Supertwistorial Realizations of Generalized Superconformal Symmetries

Let us consider the following realization of the superalgebra \(OSp(1; 8)\) described by the relations \((10)\):

a) Bosonic sector

\[
P_{\alpha\beta} = \lambda_\alpha \lambda_\beta, \quad M_{\alpha\beta} = \lambda_\alpha \mu_\beta, \quad K_{\alpha\beta} = \mu_\alpha \mu_\beta.
\]  

(12a)

b) Fermionic sector \((\xi^2 = 1)\)

\[
Q_\alpha = \lambda_\alpha \xi, \quad S_\alpha = \mu_\alpha \xi
\]  

(12b)
where
\[ \left\{ \lambda_\alpha, \mu_\beta \right\} = \delta_\alpha^\beta, \quad \xi^2 = 1 \] (12c)

and raising of indices is performed with the help of an \( Sp(4) \) antisymmetric metric.

It is easy to see that the first relation (12a) is equivalent to the relations (5-6).

The relation with the superspace description is obtained via the following version of Penrose-Ferber formulae (see [11]) relating supertwistor coordinates with superspace coordinates
\[ \mu^\beta = \lambda_\alpha \left( X^{\alpha\beta} - i \theta^\alpha \theta^\beta \right), \quad \xi = \lambda_\alpha \theta^\alpha \] (13)

where \( (\lambda_\alpha, \mu_\alpha, \xi) \) describes an \( OSp(1|8) \) supertwistor and
\[ X^{\alpha\beta} = \frac{1}{4} (\gamma_\mu C)^{\alpha\beta} x^\mu - \frac{1}{8} (\sigma_{\mu\nu} C)^{\alpha\beta} y^{[\mu\nu]}. \] (14)

The four coordinates \( x^\mu \) describe \( D = 4 \) space-time, and \( y^{[\mu\nu]} = -y^{[\nu\mu]} \) are six central charge coordinates, dual to the tensorial central charges \( Z_{[\mu\nu]} \).

### 4 New Class of Massless Superparticle Models

Our aim here is

i) to construct the superparticle model with the momenta and tensorial charges given by the relations (5, 6) and

ii) to show that it is equivalent to the free particle model in supertwistor space.

We start with the following Brink–Schwarz–like action (see also [16]):
\[ S = \int d\tau (P^{\alpha\beta} \omega_{\alpha\beta} - e P_{\alpha\beta} P^{\alpha\beta}), \] (15)

where \( P^{\alpha\beta} \) is a symmetric \( 4 \times 4 \) matrix, \( e \) is an einbein and
\[ \omega_{\alpha\beta} = \frac{dX^{\alpha\beta}}{d\tau} - i \theta^\alpha \frac{d\theta^\beta}{d\tau}. \] (15a)

Substituting the ansatz (12a) \( P_{\alpha\beta} = \lambda_\alpha \lambda_\beta \) into (13) we get
\[ S = \int d\tau \chi^\alpha \chi^\beta \omega_{\alpha\beta} = \int d\tau \left( \lambda^A \bar{\lambda}^B \omega_{AB} + \lambda^A \lambda^B \omega_{AB} + \lambda^A \chi^B \omega_{AB} \right) \] (16)

where (we use the 2-component Weyl spinor notation)
\[ \omega_{AB} = \frac{dx_{AB}}{d\tau} + i \left( \frac{d\theta^A}{d\tau} \bar{\theta}^B - \theta^A \frac{d\bar{\theta}^B}{d\tau} \right), \]
\[ \omega_{AB} = \frac{dy_{AB}}{d\tau} - i \frac{d\theta^A}{d\tau} \theta^B, \quad \omega_{AB} = \frac{d\lambda_{AB}}{d\tau} - i \frac{d\bar{\lambda}^A}{d\tau} \bar{\theta}^B. \] (17)
The model (16) describes the generalized momenta $P_{\alpha\beta}$ satisfying the relations $P_{\alpha\beta}P^{\beta\gamma} = 0$ or more explicitly (using the notation of eqs. (14-15)):

$$
P_{AB}P^{AB} = 0, \quad Z_{AB}Z^{AB} = 0, \quad Z_{AB}P^{BC} = P_{AB}Z^{BC} = 0
$$

(18)

The relations (18) reduce 10 real degrees of freedom ($P_{\mu}, Z_{[\mu\nu]}$) to four real independent degrees of freedom. In particular, if we introduce three degrees describing $D = 4$ massless momenta ($\vec{p}, p_0 = |\vec{p}|$), one can describe the fourth degree of freedom $e^{i\alpha}$ as the phase of the spinor $\lambda_A$, which e.g. can be expressed as $e^{i\alpha} = Z_{12}/\bar{Z}_{12}$.

In such a way we obtain the $D = 4$ massless superparticle model with additional “internal” $U(1)$ degree of freedom. In particular if we perform quantization (see [2]) we obtain the superwave function $\Phi$ which depends only on independent variables $\lambda_A, \bar{\lambda}_A$ and one-dimensional Grassmann coordinate $\eta$ ($\eta^2 = 0$; see (13))

$$
\Phi(\lambda_A, \bar{\lambda}_A, \eta) = \Phi(\lambda_A, \bar{\lambda}_A) + i\eta \Psi(\lambda_A, \bar{\lambda}_A)
$$

(19)

Since the set of variables $(\lambda_A, \bar{\lambda}_A)$ is equivalent to $(p_\mu, (p^2 = 0); e^{i\alpha})$, where $p_\mu = \lambda \sigma_\mu \bar{\lambda}$, and $\lambda_1 \lambda_2 = |\lambda_1||\lambda_2|e^{2i\alpha}$, one gets

$$
\Phi(\lambda_A, \bar{\lambda}_A) = \sum_{k \in \mathbb{Z}} e^{2ki\alpha} \Phi_k(p_k)
$$

(20a)

$$
\Psi(\lambda_A, \bar{\lambda}_A) = \sum_{k \in \mathbb{Z}} e^{(2k+1)i\alpha} \Psi_{k+\frac{1}{2}}(p_k)
$$

(20b)

The massless fields collected in (20a) carry integer helicities ($s = k$), and the fields in (20b) are endowed with half-integer helicities ($s = k + \frac{1}{2}$) in accordance with the spin-statistics theorem for the $D = 4$ relativistic theories.

In $D = 4$ the model (16) contains superspace variables ($X^\mu, \theta^A, \bar{\theta}^\dot{A}$) extended by central charge coordinates $y^{\mu\nu}$ as well as by the spinors $\lambda_A, \lambda_\dot{B}$ describing half of the bosonic components of the $OSp(1|8)$ supertwistor. If we substitute the relations (13) adapted to $D = 4$ into the action (16) the latter can be expressed in terms of supertwistor components $Z_k = (\lambda_A, \mu^A, \xi)$ and thus becomes the free $OSp(1; 8)$ supertwistor action (see [4] for details)

$$
S = \int d\tau \left( \lambda_A \dot{\mu}^A + \bar{\lambda}_A \dot{\bar{\mu}}^\dot{A} + i\xi \dot{\xi} \right)
$$

(21)

where $\xi = \frac{1}{2} \left( \lambda_A \theta^A + \bar{\lambda}_A \bar{\theta}^\dot{A} \right)$.

5 Final Remarks

The classical and quantum version of $D = 4$ massless superparticle model with infinite spectra of helicities can be

- extended to dimensions $D > 4$, in particular to $D = 6$ and $D = 10$ [2]
- generalized to the super-Anti-de-Sitter background [9].
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References

[1] I. Bandos and J. Lukierski, Mod. Phys. Lett. 14, 1257 (1999).
[2] I. Bandos, J. Lukierski and D. Sorokin, hep-th/9904109, Phys.Rev.D, in press.
[3] R. Haag, J. Lopuszański and M. Sohnius, Nucl. Phys. B88, 257 (1995).
[4] T. Kugo and P. Townsend, Nucl. Phys. B221, 331 (1983).
[5] S. Ferrara and M. Porrati, Phys.Lett. B458, 43 (1999).
[6] J. Gauntlett and C.M. Hull, [hep-th/9909098].
[7] P. d’Auria and P. Fré, Nucl. Phys. B201, 101 (1982).
[8] E. Sezgin, Phys. Lett. B392, 323 (1997).
[9] C. Chryssomalakos, J.A. de Azcárraga J.M. Izquierdo and J.C. Pérez Bueno, hep-th/9904137, Nucl. Phys. B, in press.
[10] R. Penrose and M.A.H. Mac Callum, Phys. Rep. 6, 241 (1972).
[11] A. Ferber, Nucl. Phys. B131, 55 (1977).
[12] I. Bars and C. Deliduman, Phys.Lett. B417, 240 (1998), [hep-th/9710066].
[13] Z. Hasiewicz and J. Lukierski, Phys. Lett. 145B, 65 (1984).
[14] J. W. van Holten and A. van Proyen, J.Phys. A15, 3763 (1982).
[15] I. Bars, C. Deliduman and D. Minic, Phys. Lett. B466, 135 (1999); [hep-th/9904063].
[16] I. Rudychev and E. Sezgin, Phys.Lett. B415, 363 (1997), Addendum B424, 411 (1998).
[17] I. Bandos, J. Lukierski, C. Preitschopf and D. Sorokin, [hep-th/9907113], Phys.Rev. D, in press.