PRODUCTS IN EQUIVARIANT HOMOLOGY

SHIZUO KAJI AND HAGGAI TENE

Abstract. We refine the intersection product in homology to an equivariant setting, which unifies several known constructions. As an application, we give a common generalisation of the Chas-Sullivan string product and the Chataur-Menichi string product. We prove a vanishing result which enables us to define a secondary product, which is used to construct secondary versions of the Chataur-Menichi string product, and the equivariant intersection product in the Borel equivariant homology of a manifold with an action of a compact Lie group. The latter reduces to the product in homology of the classifying space defined by Kreck, which coincides with the cup product in negative Tate cohomology if the group is finite.

1. Introduction

Throughout this paper, we work in the category of compactly generated weak Hausdorff spaces. Fibrations and cofibrations are in the sense of Hurewicz. A multiplicative generalised (co)homology theory $h$ is in the sense of [13], and is always assumed to satisfy the weak equivalence axiom: a weak equivalence induces an isomorphism.

Let $M$ be a smooth manifold together with a smooth action of a compact (not necessarily connected) Lie group $G$. Denote by $M_G \rightarrow BG$ its Borel construction. Assume the action of $G$ on $M$ is $h$-oriented in an appropriate sense (see Definition 2.7). Given two spaces $X$ and $Y$ over $M_G$, that is, spaces together with maps $X \rightarrow M_G$ and $Y \rightarrow M_G$, consider the homotopy pullback diagram

$$
\begin{array}{ccc}
P & \rightarrow & X \\
\downarrow & & \downarrow \\
Y & \rightarrow & M_G.
\end{array}
$$

We define a product in this setting in §2 using umkehr maps:

$$
\mu_{M_G} : h_k(X) \otimes h_l(Y) \rightarrow h_{k+l+\dim(G)-\dim(M)}(P).
$$

Theorem A. The product $\mu_{M_G}$ is associative up to multiplication by a unit. It enjoys the following forms of naturality: with respect to maps of diagrams with the same base $M_G$, with respect to closed subgroup inclusions, and with respect to group extensions. Dually, there exists a homomorphism in cohomology of the form $h^m(P) \rightarrow h^{m-\dim(G)+\dim(M)}(X \times Y)$.

In general, this product is non-trivial. Some interesting examples related to string topology and equivariant homology are given later in the introduction. In [3] we focus on the case of the ordinary homology with coefficients in a commutative ring $R$, and show that $\mu_{M_G}$ vanishes under a certain degree condition. For a space $A$, denote by $\dim_R(A)$ the maximal $n$ such that $H_n(A; R) \neq 0$ and by $F_X$ (resp. $F_Y$) the homotopy fibre of the composition $X \rightarrow M_G \xrightarrow{m} BG$ (resp. $Y \rightarrow M_G \xrightarrow{m} BG$).

Theorem B. The product $\mu_{M_G} : H_k(X; R) \otimes H_l(Y; R) \rightarrow H_{k+l+\dim(G)-\dim(M)}(P; R)$ is trivial if $k > \dim_R(F_X) - \dim(G)$ or $l > \dim_R(F_Y) - \dim(G)$.

In particular, this implies that the pair-of-pants product defined by Chataur-Menichi vanishes whenever $k$ or $l$ is positive (see Example 3.4).

We use this vanishing phenomenon to introduce a secondary version $\overline{\mu}_{M_G}$ of $\mu_{M_G}$ in §4.

Date: May 23, 2016.

2010 Mathematics Subject Classification. Primary 55N91; Secondary 55R40, 55N45.

Key words and phrases. equivariant homology, free loop space, string topology, intersection product, Tate cohomology, stratifold, homotopy pullback, umkehr map.

The first named author was partially supported by KAKENHI, Grant-in-Aid for Young Scientists (B) 26800043 and JSPS Postdoctoral Fellowships for Research Abroad.
Theorem C. Let $k > \dim_R(F_X) - \dim(G)$ and $l > \dim_R(F_Y) - \dim(G)$. There exists a homomorphism
\[
\overline{\mu_{MG}} : H_k(X; R) \otimes H_l(Y; R) \to H_{k+l(\dim(G)-\dim(M)+1)}(P; R),
\]
which is natural with respect to maps of diagrams with the same base $M_G$, with respect to closed subgroup inclusions, and with respect to group extensions.

Now we list interesting examples of $\mu_{MG}$. In the following examples, we assume all actions are oriented without mentioning it.

(1) Let $M = pt$. For $G$-spaces $K$ and $L$ with classifying maps $K_G \to BG, L_G \to BG$, we have the following homotopy pullback diagram:
\[
\begin{array}{ccc}
(K \times L)_G & \longrightarrow & K_G \\
\downarrow & & \downarrow \\
L_G & \longrightarrow & BG.
\end{array}
\]
In this case, we obtain an exterior product in the homology of the Borel construction
\[
\mu_{BG} : h_k(K_G) \otimes h_l(L_G) \to h_{k+l(\dim(G))}(K \times L)_G,
\]
which reduces to the ordinary cross product when $G$ is trivial.

(2) When $X = Y = M_G$ with the identity maps, we obtain an “equivariant intersection product”
\[
m : h_k(M_G) \otimes h_l(M_G) \to h_{k+l(\dim(G) - \dim(M))}(M_G),
\]
which reduces to the intersection product when $G$ is trivial. If $h$ is the integral homology, equivariant homology classes can be represented by bordism classes of equivariant maps from closed, oriented stratifolds with free and orientation preserving actions of $G$ (see [27] and Appendix A). In this description, the product is given by intersection after making the cycles transversal.

(3) Let $X = Y = L(M_G)$ be the free loop space over $M_G$ with the evaluation map to $M_G$. In this case, $P \sim \text{Map}(S^1 \vee S^1, M_G)$ and by composing $\mu_{MG}$ with the induced map of the concatenation
\[
cat : \text{Map}(S^1 \vee S^1, M_G) \to L(M_G),
\]
we obtain a string product in the Borel construction:
\[
\psi : h_k(L(M_G)) \otimes h_l(L(M_G)) \to h_{k+l(\dim(G) - \dim(M))}(L(M_G)).
\]
This is similar to the product constructed by Behrend-Ginot-Noohi-Xu ([4]) in the language of stacks and to the one by Lupercio-Uribe-Xicotencatl [21] for orbifolds $[M/G]$ when $G$ is finite and $h$ is rational homology. It generalises two known constructions:

(a) When $G$ is trivial, this reduces to the Chas-Sullivan string product for $LM$ [9] [11].

(b) When $M = pt$, this reduces to the Chataur-Menichi pair-of-pants string product for $LBG$ [10]. Example (a).

(4) More generally, let $E \to M_G$ be a fibrewise monoid. Take $X = Y = E$ with the projection maps to $M_G$. The composition of $\mu_{MG}$ with the fibrewise multiplication gives a product
\[
h_k(E) \otimes h_l(E) \to h_{k+l(\dim(G) - \dim(M))}(E),
\]
which generalises Gruher-Salvatore’s construction [17].

The corresponding examples of $\overline{\mu_{MG}}$ are as follows.

(i) In the case corresponding to (1), we obtain a secondary exterior product in the Borel equivariant homology
\[
\overline{\mu_{BG}} : H^G_k(K; R) \otimes H^G_l(L; R) \to H^G_{k+l(\dim(G)+1)}(K \times L; R)
\]
for $k > \dim_R(K) - \dim(G)$ and $l > \dim_R(L) - \dim(G)$.

(ii) In the case corresponding to (2), we obtain a secondary equivariant intersection product
\[
\overline{\mu} : H^G_l(M; R) \otimes H^G_l(M; R) \to H^G_{k+l(\dim(G) - \dim(M)+1)}(M; R)
\]
for $k, l > \dim_R(M) - \dim(G)$. (We note $\dim(M) \geq \dim_R(M)$.) When $M$ is a point, this coincides (Proposition A.3) with the product in the integral homology of $BG$ defined in an unpublished work by Kreck (see Appendix A and [20]). The Kreck product was described in a geometric way using stratifolds and stratifold homology. When $G$ is finite, the product was shown in [26] to coincide with the cup product in negative Tate cohomology, under the identification $H_k(BG; \mathbb{Z}) \xrightarrow{\cong} \hat{H}^{k-1}(G; \mathbb{Z})$ for $k > 0$. 

Another product of the same grading can be defined by the tools developed in [16] using equivariant Poincaré duality and the cup product in negative Tate cohomology under an appropriate orientability condition. In the case when $G$ is finite and $M = pt$, it agrees with ours. In general we do not know if the two agree, though we expect that it is the case.

(iii) In the case corresponding to (3)-(b), composing $\mu_{BG}$ with the induced map of the concatenation, we obtain a product

$$\overline{\psi} : H_k(LBG; R) \otimes H_l(LBG; R) \to H_{k+l+\dim(G)+1}(LBG; R)$$

for $k, l > 0$. This is a secondary product to Chataur-Menichi’s string product.

In [5, 6] we prove some properties of the secondary product $\mu_{MG}$. In particular, we show in Theorem 5.6 a condition in which the secondary product vanishes. On the other hand, we describe several interesting examples in which $\mu_{MG}$ is non-trivial. We show in Example 6.2 that for every non-trivial compact Lie group $G$, there exists a manifold $M$ where the secondary intersection product $\mu_{MG}$ is non-trivial, and we see in Proposition 6.4 that $\overline{\psi}$ in the homology of $LBG$ is non-trivial in general. In §7 we focus on the concrete computation of $\mu_{BG}$ when all maps in diagram (1.1) are the identity maps. In Propositions 7.3 and 7.5 we give a complete description of the product in the case when $G$ is one of $S^1, SU(2)$ or $SO(3)$. We prove the following two vanishing results: Propositions 7.7 and 7.8. Briefly, they state respectively: The product in $H_*(BG; R)$ is trivial for

- any compact Lie group $G$ of rank greater than one with $R = \mathbb{Q}$,
- any connected compact classical (matrix) group of rank greater than one with $R = \mathbb{Z}$.

In the appendix, we show that the secondary intersection product generalises the Kreck product on $H_*(BG; \mathbb{Z})$.

We also define a third product $\mu_{MG}^*$ in [4] which is related to both $\mu_{MG}$ and $\overline{\mu_{MG}}$. Consider the homotopy pushout of the upper left corner of diagram (1.1)

$$\begin{array}{ccc}
P & \xrightarrow{P} & X \\
\downarrow & & \downarrow \\
Y & \xrightarrow{Y} & X \bowtie_{MG} Y.
\end{array}$$

The space $X \bowtie_{MG} Y$ is called the homotopy join [4].

**Theorem D.** There exists a homomorphism

$$\mu_{MG}^* : H_k(X; R) \otimes H_l(Y; R) \to H_{k+l+\dim(G)−\dim(M)+1}(X \bowtie_{MG} Y; R)$$

when $k, l > \dim(R(M) − \dim(G))$, which is natural with respect to maps of diagrams with the same base $MG$. It satisfies

$$\partial \circ \mu_{MG}^* = \mu_{MG}, \quad u_* \circ \overline{\mu_{MG}} = \mu_{MG}^*$$

where $\partial$ is the boundary operator in the Mayer–Vietoris sequence associated to the homotopy pushout, and $u$ is the composition $P \to X \to X \bowtie_{MG} Y$.

When $M = pt$ and $G$ is trivial, $\mu_{MG}^*$ reduces to the homomorphism $H_k(X; R) \otimes H_l(Y; R) \to H_{k+l+1}(X \ast Y; R)$ defined in [28 §2].

We finish the introduction with a remark on the homotopy pullback diagram (1.1), which looks somewhat artificial to have $MG$ as its base space. In fact, the diagram (1.1) can be viewed in a different but equivalent way. Given a map $X \to MG$, consider the pullback of the universal bundle $G \to EG \to BG$ via the composition $X \to MG \xrightarrow{m} BG$. Denote the total space by $K$, then we have a commutative diagram

$$\begin{array}{ccc}
K & \xrightarrow{f} & M \\
\downarrow & & \downarrow \\
X & \xrightarrow{f_G} & MG \xrightarrow{m} BG.
\end{array}$$
where \( f_G \) is the Borel construction of the \( G \)-map \( f \). Take the fibres of (1.1) mapping into \( BG \), and we obtain a homotopy pullback

\[
\begin{array}{ccc}
Z & \longrightarrow & K \\
\downarrow & & \downarrow f \\
L & \longrightarrow & M.
\end{array}
\]

Therefore, the diagram (1.1) can always be considered to be of the form

\[
\begin{array}{ccc}
Z_G & \longrightarrow & K_G \\
\downarrow & & \downarrow f_G \\
L_G & \longrightarrow & M_G,
\end{array}
\]

where \( g_G \) and \( f_G \) are the Borel constructions of some equivariant maps \( f : K \to M \) and \( g : L \to M \). Conversely, the homotopy pullback (1.3) consisting of equivariant maps gives rise to the homotopy pullback (1.4) by taking the Borel construction. (This can be shown by the five lemma.) We freely switch between the two equivalent diagrams in this paper.

**Acknowledgement:** We would like to thank Matthias Kreck for helpful discussions, Peter Teichner for suggesting Example 6.2, and the anonymous referee for the careful reading and the precious comments. The authors are grateful to the university of Bonn and the Mathematics Center Heidelberg (MATCH) for financial support during a part of this project.

2. THE CONSTRUCTION OF THE PRODUCT \( \mu_{M_G} \)

The construction of \( \mu_{M_G} \) is given by umkehr (or Gysin) maps in homology of two kinds: one with respect to cofibrations and one with respect to fibre bundles.

**The umkehr map with respect to a cofibration.** (compare [17]) Let \( i : A \to X \) be a cofibration, and \( \varphi \in h^d(X, X \setminus A) \) be a cohomology class. We define an umkehr map \( i^! : h_k(X) \to h_{k-d}(A) \) the following way (for simplicity, we assume that \( A \) is a subspace of \( X \)):

For every open neighbourhood \( U \) of \( A \) one has a homomorphism \( i_U^! : h_k(X) \to h_{k-d}(U) \):

\[
h_k(X) \to h_k(X, X \setminus A) \cong h_k(U, U \setminus A) \xrightarrow{i_* \cap \varphi} h_{k-d}(U).
\]

If \( U' \subseteq U \) is another open neighbourhood of \( A \) then by naturality of the cap product we have \( i_U^! = i_* \circ i_{U'}^! \), where \( i_* \) is the map induced by the inclusion. This implies that there is a homomorphism \( h_k(X) \to \lim h_{k-d}(U) \), where the inverse limit is taken over a fundamental system of neighbourhoods. According to [19] Corollary 2, the inclusion of \( A \) in its neighbourhoods induces an isomorphism \( h_*(A) \to \lim h_*(U) \), hence we obtain the desired umkehr map

\[
h_k(X) \to h_{k-d}(A).
\]

A typical example is when \( X \) is a smooth (Hilbert) manifold and \( A \) is a (finite codimensional) closed sub-manifold. Taking \( \varphi \) to be a Thom class, one recovers the usual Gysin map.

This construction is natural in the following sense:

**Lemma 2.1.** Suppose we are given the following topological pullback diagram

\[
\begin{array}{ccc}
\hat{A} & \longrightarrow & \hat{X} \\
\downarrow & & \downarrow f \\
A & \longrightarrow & X.
\end{array}
\]

If the horizontal maps are cofibrations, the umkehr maps with respect to a class \( \varphi \in h^d(X, X \setminus A) \) and its pullback \( f^*(\varphi) \) satisfy \( f_* \circ i_{\hat{U}}^! = i_U^! \circ f_* \).

**Proof.** Since \( f^{-1}(A) = \hat{A} \) we have a map of pairs \((\hat{X}, \hat{X} \setminus \hat{A}) \to (X, X \setminus A)\), hence \( f^*(\varphi) \in h^d(\hat{X}, \hat{X} \setminus \hat{A}) \). Let \( U \) be an open neighbourhood of \( A \) and \( \hat{U} = f^{-1}(U) \). By the naturality of the cap product we have:

\[
f_* \circ i_{\hat{U}}^! = i_U^! \circ f_*.
\]
Since $\hat{\partial}_U = i^*_U \circ \hat{\partial}$ and $\hat{\partial}_U = i^*_U \circ \hat{\partial}$, by the naturality of the induced maps in homology we obtain

$$i^*_U \circ f_* \circ \hat{\partial} = i^*_U \circ \hat{\partial} \circ f_*.$$ 

Now, the statement follows from the fact that this is true for every neighbourhood $U$ of $A$ and $h_{k-d}(A) \to \varinjlim h_{k-d}(U)$. 

**Remark 2.3.** In this paper we often encounter a diagram of the form (2.2) which is a homotopy pullback diagram with $\hat{\partial}$ being a closed embedding of a sub-manifold with an orientation of its normal bundle (a Thom class). In this case, replacing $f$ with a fibration, we can think of it as a strict pullback. Since a pullback of a cofibration along a fibration is a cofibration, we can use the lemma to define an umkehr map for $\hat{\partial}$ by pulling back the Thom class.

**The Grothendieck bundle transfer.** There are different ways to define umkehr maps with respect to “nice” fibrations. We use here Boardman’s approach, called the Grothendieck bundle transfer [6, Chap. V] which has the advantage allowing arbitrary homology theories and non-connected fibres. Note that the Grothendieck bundle transfer coincides with the integration along the fibre (which is considered in [10] for example) when both are defined.

We briefly recall the definition of the Grothendieck bundle transfer. Details can be found in [6, 3]. Let $G$ be a compact Lie group acting smoothly on a closed oriented manifold $F$. Let $G \to E' \overset{\mathcal{F}}{\to} B$ be a principal $G$-bundle and $F \to E \overset{\mathcal{F}}{\to} B$ be the associated $F$-bundle. The bundle of tangents along the fibre is defined to be the vector bundle $t(\mathcal{F}) : E' \times_G TF \to E' \times_G F = E$, where $TF$ is the tangent bundle of $F$. There is a fibrewise embedding $j : E \hookrightarrow U$ for some vector bundle $u : U \to B$. Then, by the Pontryagin-Thom construction one obtains a map between the Thom spaces $B^u \to E^o$, where $\nu$ is the normal bundle of $j$. This, together with the fact $p^*(u) = t \oplus \nu$, defines a map between the Thom spectra $B^0 \to E^{-l(p)}$. When $-l(p)$ is $h^*$-oriented, it induces a homomorphism

$$p^* : h_k(B) \to h_{k+\dim(F)}(E).$$

This construction can be relativised as in [6, Chap. V].

**The definition of $\mu_{M_G}$.** Observe that diagram (1.1) gives rise to another homotopy pullback

$$\begin{array}{ccc}
X \times Y & \xrightarrow{f \times g} & M_G \times M_G, \\
\downarrow & & \downarrow \Delta_{M_G}, \\
\Delta_{M_G} & \to & M_G,
\end{array}$$

where $\Delta_{M_G}$ is the diagonal. Recall that the Chas-Sullivan product is defined using the umkehr map with respect to the finite codimensional embedding $\Delta_M : M \to M \times M$ and the Chataur-Menichi product is defined using the Grothendieck bundle transfer with respect to $\Delta_{BG} : BG \to BG \times BG$. In contrast to these situations, in general neither $\Delta_{M_G} : M_G \to M_G \times M_G$ is finite codimensional nor its fibre $\Omega(M_G)$ is compact. Thus, we cannot just apply the umkehr maps reviewed at the beginning of this section. The key observation is that one can decompose $\Delta_{M_G}$ as

$$M_G \xrightarrow{\Delta_G} (M \times M)_{\Delta(G)} \overset{p}{\to} M_G \times M_G,$$

where $\Delta_G$ is the equivariant diagonal and $p$ is obtained by further quotienting by $G \times G$. Then, $\Delta_G$ is finite codimensional and $p$ has the fibre $(G \times G)/\Delta G \sim G$. Note that $p$ is the pullback of $\Delta_{BG} : BG \to BG \times BG$ along $m \times m : M_G \times M_G \to BG \times BG$, and $\Delta_{BG}$ is equivalent to the homogeneous bundle $(G \times G)/\Delta G \to (EG \times EG)_{\Delta(G)} \to BG \times BG$ with the structure group $G \times G$.

In what follows, we denote $\Delta_G$ simply by $G$ when it causes no confusion.

Given diagram (1.1), let $Q$ be the homotopy pullback:

$$\begin{array}{ccc}
Q & \to & X, \\
\downarrow & & \downarrow \\
Y & \to & BG,
\end{array}$$
where \( X \to M_G \to BG \) and \( Y \to M_G \to BG \) are compositions with the classifying map \( m : M_G \to BG \). Since \( p \) is the homotopy pullback of \( \Delta_{BG} \) via \( m \times m \), we have the diagram with all the squares being homotopy pullbacks:

\[
\begin{array}{cccc}
P & \xrightarrow{\Delta_G} & Q & \xrightarrow{\hat{p}} & X \times Y \\
\downarrow & & \downarrow & & \downarrow \\
M_G & \xrightarrow{\Delta_G} & (M \times M)_G & \xrightarrow{p} & M_G \times M_G \\
\downarrow & & \downarrow m \times m & & \downarrow \\
BG & \xrightarrow{\Delta_{BG}} & BG \times BG. & & \\
\end{array}
\]

We would like to define a homomorphism \( \mu_{M_G} \) by the following composition:

\[
h_k(X) \otimes h_l(Y) \xrightarrow{\varphi} h_{k+l}(X \times Y) \xrightarrow{\hat{p}} h_{k+l+\dim(G)}(Q) \xrightarrow{\Delta_G} h_{k+l+\dim(G)-\dim(M)}(P).
\]

In order to define the umkehr maps \( \hat{p} \) and \( \Delta_G^1 \), the bundle of tangents along the fibre for \( \hat{p} \) and the normal bundle of \( \Delta_G \) should be oriented (see Remark 2.3).

**Lemma 2.6.** (1) The normal bundle of \( \Delta_G \) is canonically isomorphic to the vector bundle \((TM)_G : EG \times_G TM \to EG \times_G M = M_G\).

(2) The bundle of tangents along the fibre \( t(\Delta_{BG}) \) for \( \Delta_{BG} : BG \to BG \times BG \) is canonically isomorphic to the universal adjoint bundle \( \text{ad}(EG) : \mathfrak{g} \to EG \times_G \mathfrak{g} \to BG \), where \( G \) acts on its Lie algebra \( \mathfrak{g} \) by the adjoint action.

**Proof.** (1) Observe that the bundle isomorphism given by the difference \((TM)\oplus TM)TM \to TM\) is \( G \)-equivariant, where \( G \) acts diagonally on the left hand side. Thus, we identify \((G\text{-equivariantly})\) the normal bundle of \( \Delta : M \to M \times M \) with \( TM \). The normal bundle of \( EG \times M \to EG \times M \times M \) is \( EG \times TM \to EG \times M \). Its quotient by \( G \) gives the normal bundle of \( \Delta_G \).

(2) By definition, the bundle of tangents along the fibre for \( \Delta_{BG} \) is

\[
E(G \times G) \times_{G \times G} T((G \times G)/G) \to E(G \times G) \times_{G \times G} (G \times G)/G = BG.
\]

Observe that \((G \times G)/G \to G\) given by \((g,h) \mapsto gh^{-1}\) is \( G \times G \)-equivariant homeomorphism, where \( G \) on the right hand side is acted by the sandwich action. Using the isomorphism \( TG \simeq G \times \mathfrak{g}\), this pullbacks to the \( G \times G \)-bundle isomorphism between \( T((G \times G)/G) \simeq ((G \times G)/G) \times \mathfrak{g} \), where \( \mathfrak{g} \) is acted by the sandwich action. In particular, the diagonal acts by adjunction. Thus, we have the bundle isomorphism \( E(G \times G) \times_{G \times G} T((G \times G)/G) \simeq EG \times_G \mathfrak{g} = ad(EG) \).

\[\square\]

**Definition 2.7.** Suppose that \( M \) is a smooth manifold together with a smooth action \( \rho \) of a compact Lie group \( G \). For \( h \), a multiplicative generalised (co)homology theory, we say that the **triple** \((M,G,\rho)\) is **\( h \)-orientable (oriented)** if the following two vector bundles (appearing in Lemma 2.6) are \( h \)-orientable (oriented)

- The Borel construction of the tangent bundle of \( M - (TM)_G : EG \times_G TM \to EG \times_G M = M_G \)
- The universal adjoint bundle of \( G - ad(EG) : EG \times_G \mathfrak{g} \to BG \).

We orient the bundle of tangents along the fibre \( t(p) \) and \( t(\hat{p}) \) in (2.5) by the pullback of the orientation of \( ad(EG) \) by Lemma 2.6. Note that in order to define \( \mu_{M_G} \) it is enough to orient \( t(p) \) rather than \( ad(EG) \), but for simplicity we assume an orientation of \( ad(EG) \).

This notion of orientability is generally stronger than the usual one:

**Lemma 2.8.** An \( h \)-orientation of a triple induces an \( h \)-orientation of \( M \) and \( \mathfrak{g} \) such that the action of \( G \) is orientation preserving, where \( G \) acts on \( \mathfrak{g} \) by adjunction. When \( h \) is the ordinary homology, the converse is also true.

**Proof.** Pulling back orientations of \((TM)_G \) and \( ad(EG) \), we obtain invariant orientations of \( TM \) and \( \mathfrak{g} \). For the converse, observe that the bundles \((TM)_G \) and \( ad(EG) \) are quotients of the equivariant bundles \( EG \times TM \) and \( EG \times \mathfrak{g} \) by the free \( G \)-actions. In the case of ordinary homology, an orientation of a vector bundle is equivalent to a continuous choice of orientations of the fibres, hence, an invariant orientation of an equivariant vector bundle with a free action gives rise to an orientation of its quotient. \[\square\]
To summarise,

**Theorem 2.9.** Let $M$ be a smooth manifold with a smooth action of a compact Lie group $G$, such that the triple $(M, G, \rho)$ is oriented. For the homotopy pullback, the umkehr maps $\hat{\Delta}_G$ and $\hat{p}$ are defined and induce a homomorphism $\mu_{MG} : h_k(X) \otimes h_l(Y) \to h_{k+l+\dim(G)-\dim(M)}(P)$. Dually, we have a homomorphism in cohomology of the form $h^m(P) \to h^{m-\dim(G)+\dim(M)}(X \times Y)$.

**Remark 2.10.** In [15], they define a homomorphism similar to our $\mu_{MG}$ when the base space in [14] is a Gorenstein space. The Borel construction $M_G$ is a typical example of Gorenstein spaces. They work algebraically on the level of the singular chain with coefficients in a field. In contrast, our construction is topological and allows us to work with any generalised homology theory. Also, our construction reveals the vanishing phenomenon Theorem 3.2 in a clear way. (Compare with [15, Theorem 14].)

Several examples related to equivariant homology and string topology are given in the introduction. The following example can be derived similarly.

**Example 2.11.** Let $(M, G, \rho)$ be an $h$-oriented triple. Consider the following homotopy pullback

\[
\begin{array}{ccc}
\text{Map}(S^1 \vee S^1, M_G) & \longrightarrow & L(M_G) \\
\downarrow & & \downarrow (\text{ev}_0, \text{ev}_{1/2}) \\
M_G & \longleftarrow & M_G \times M_G.
\end{array}
\]

From the factorisation (2.4) of $\Delta_{M_G}$, we obtain

\[h_* (L(M_G)) \xrightarrow{\hat{\Delta}_{G, \text{op}}} h_{*+\dim(G)-\dim(M)}(\text{Map}(S^1 \vee S^1, M_G)).\]

Composing it with the map induced by the inclusion $\text{Map}(S^1 \vee S^1, M_G) \hookrightarrow L(M_G) \times L(M_G)$, one obtains the “coproduct’’

\[h_* (L(M_G)) \to h_{*+\dim(G)-\dim(M)}(L(M_G) \times L(M_G)).\]

This reduces to [10, Example (b)] when $M = pt$ since $\text{Map}(I, M_G) \xrightarrow{(\text{ev}_0, \text{ev}_1)} M_G \times M_G$ is equivalent to $\Delta_{M_G}$.

**Remark 2.12.** Recall the homotopy equivalence $L B G \sim G_J^{ad} = E_G \times_G G_J^{ad}$, where $G_J^{ad}$ is $G$ acted by itself by adjunction. We have two different products on $h_* (L B G) \cong h_*(G_J^{ad})$ corresponding to (2) and (3) in §1:

\[m : h_k(G_J^{ad}) \otimes h_l(G_J^{ad}) \to h_{k+l}(G_J^{ad}),\]

\[\psi : h_k(L B G) \otimes h_l(L B G) \to h_{k+l+\dim(G)}(L B G).\]

The relation can be seen as follows. Notice that the following diagram commutes:

\[
\begin{array}{ccc}
G_J^{ad} & \xrightarrow{\text{mul}} & (G_J^{ad} \times G_J^{ad})_G \\
\downarrow \sim & & \downarrow \sim \\
L B G & \xrightarrow{\text{cal}} & \text{Map}(S^1 \vee S^1, M_G) \xrightarrow{\hat{p}} L B G \times L B G,
\end{array}
\]

where mul is induced by the multiplication of $G$ (which is equivariant with respect to the adjoint action). Composing $\hat{p}^2$ with $\text{mul}_*$, one obtains $\psi$, while with $\hat{\Delta}_G$, one obtains $m$.

**Properties of the product $\mu_{MG}$.** The following properties of $\mu_{MG}$ are easy to check:

**Proposition 2.13.** We have

1. $\mu_{MG}$ is associative up to multiplication by a unit.
2. $\mu_{MG}$ commutes with ring homomorphism between homology theories.
3. $\mu_{MG}$ is natural with respect to maps of diagrams with the same base $M_G$, that is, to commutative diagrams of the form

\[
\begin{array}{ccc}
X & \longrightarrow & M_G \\
\downarrow & & \downarrow \\
X' & \longrightarrow & Y.
\end{array}
\]
Next we show that the product \( \mu_{MG} \) commutes with restriction to closed subgroups. Consider a closed subgroup \( H \subseteq G \) and denote by \( \rho|_H \) the restriction of the action \( \rho \) of \( G \) on \( M \) to \( H \). Let \( G/H \to BH \overset{i}{\to} BG \) be the associated homogeneous bundle, and \( t(i) \) be its bundle of the tangents along the fibre. We first recall the following standard fact:

**Lemma 2.14.** (1) There is a \( G \)-equivariant isomorphism \( T(G/H) \cong G \times_H g/h \), where \( H \) acts on \( g \) by adjunction. Hence, we have \( t(i) = EG \times_G G \times_H g/h \cong EG \times_H g/h \) and the following short exact sequence of vector bundles over \( BH \)

\[
0 \to \text{ad}(EG) \to i^*(\text{ad}(EG)) \to t(i) \to 0
\]

since \( i^*(\text{ad}(EG)) = i^*(g_G) = g_H = EG \times_H g \).

(2) Furthermore, if \( K \subseteq H \) is a closed subgroup and \( i : BK \to BH \) is the classifying map of the inclusion, we have the following short exact sequence of vector bundles over \( BK \)

\[
0 \to EG \times_K h/k \to EG \times_K g/k \to EG \times_K g/h \to 0
\]

and hence, \( t(i \circ i') \simeq (i')^* (t(i)) \oplus t(i') \).

Orientations of two vector bundles in a short exact sequence determine that of the third one. (See, for example, [23 Chapter V, Proposition 1.10].) We always orient the bundle of tangents using the above short exact sequences so that we have \( (i \circ i')^2 = \pm (i')^2 \circ i^2 \) (\([33\) Chap. V (6.1)])

By abuse of notation, denote by \( i \) the map \( M_H \to M_G \) which is the homotopy pullback of \( i \) via the classifying map \( M_G \to BG \). We can pullback the whole square (1.4) at the end of \( §1 \) via \( i \) to have

![Diagram](attachment:diagram.png)

where we denote all the vertical maps by the same symbol \( i \). The orientations of their bundles of tangents along the fibre for \( i \) are given by that of \( BH \to BG \), which in turn is given by those of \( \text{ad}(EG) \) and \( \text{ad}(EH) \) by (2.15).

**Proposition 2.16** (Restriction). Suppose that \((M, G, \rho)\) and \((M, H, \rho|_H)\) are \( h \)-oriented in such a way that the orientation of \((TM)_H\) is the pullback orientation of \((TM)_G\). Then, the products \( \mu_{MG} \) and \( \mu_{MH} \) are compatible with the Grothendieck bundle transfer for \( i \), that is, the following diagram commutes up to sign:

\[
\begin{array}{ccc}
h_k(K_G) \otimes h_l(L_G) & \xrightarrow{\mu_{MG}} & h_{k+l+\dim(G)-\dim(M)}(Z_G) \\
\downarrow{\varphi \otimes i^2} & & \downarrow{i^2} \\
h_{k+N}(K_H) \otimes h_{l+N}(L_H) & \xrightarrow{\mu_{MH}} & h_{k+l+2N+\dim(H)-\dim(M)}(Z_H),
\end{array}
\]

where \( N = \dim(G) - \dim(H) \).

**Proof.** We show for the diagrams below

\[
\begin{array}{ccc}
M_H & \xrightarrow{\Delta_H} & (M \times M)_H \\
\downarrow{i} & & \downarrow{i} \\
M_G & \xrightarrow{\Delta_G} & (M \times M)_G
\end{array}
\begin{array}{ccc}
& & \\
\downarrow{i \times i} & & \\
& & \\
\end{array}
\begin{array}{ccc}
M_H & \xrightarrow{\Delta_H} & (M \times M)_H \xrightarrow{\rho_H} M_H \times M_H \\
\downarrow{i} & & \downarrow{i} \\
M_G & \xrightarrow{\Delta_G} & (M \times M)_G \xrightarrow{\rho_G} M_G \times M_G
\end{array}
\]

the umkehr maps commute up to sign, that is, \( \Delta_H^t \circ \varphi^2 = \pm \varphi^2 \circ \Delta_G^t \) and \( \rho_H^t \circ (i \times i)^2 = \pm i^2 \circ \rho_G^t \). Then, the similar commutativity of their pullbacks follows by the same argument.
For the left diagram, the commutativity follows from the naturality of the cap product \(i^\natural(x \cap c) = i^\natural(x) \cap i^\ast(c)\). For the right diagram, observe that it is a pullback of the following:

\[
\begin{array}{ccc}
BH & \xrightarrow{\Delta_{BH}} & BH \times BH \\
\downarrow i & & \downarrow i \times i \\
BG & \xrightarrow{\Delta_{BG}} & BG \times BG
\end{array}
\]

The bundle of tangents along the fibre for \(\Delta_{BG} \circ i = i \times i \circ \Delta_{BH} : BH \to BG \times BG\) is, by Lemma 2.14,

\[
i^\ast(t(\Delta_{BG})) \oplus t(i) \simeq \Delta_{BH}^\ast(t(i \times i)) \oplus t(\Delta_{BH}),
\]

where \(\Delta_{BH}^\ast(t(i \times i)) \simeq t(i) \oplus t(i)\). Recall from Lemma 2.6 that \(t(i \times i) \simeq \text{ad}(EG)\). Denote an orientation of a bundle \(\xi\) over \(B\) by an element in the cohomology of the Thom space \(u(\xi) \in \tilde{h}^*(B^\xi)\).

Now we compare the following two orientations for \(t(i \times i)\):

\[
\begin{align*}
u(t(\Delta_{BG} \circ i)) &= \pm v(i^\ast(-\text{ad}(EG))) \wedge u(-t(i)) \\
u(t(i \times i \circ \Delta_{BH})) &= \pm v(-t(i)) \wedge u(-t(i)) \wedge v(\text{ad}(EH)).
\end{align*}
\]

Since \(v(i^\ast(\text{ad}(EG))) = \pm v(-\text{ad}(EH)) \wedge u(-t(i))\), the two agree up to sign. \hfill \Box

We now see another kind of naturality of \(\mu_{M,G}\) with regard to the change of groups. Let \(1 \to N \to G \xrightarrow{\gamma} G/N \to 1\) be a short exact sequence of groups. Assume that \(N\) acts on \(M\) trivially so that the action \(\rho\) of \(G\) on \(M\) induces one of \(G/N\). Then, we have the homotopy pullback

\[
\begin{array}{ccc}
M_G & \xrightarrow{\gamma} & M_{G/N} \\
\downarrow m & & \downarrow m \\
BG & \xrightarrow{\gamma} & B(G/N).
\end{array}
\]

Assume further that \((M, G, \rho)\) and \((M, G/N, \rho)\) are oriented compatibly with respect to \(\gamma\). Given the diagram (1.1), we consider the homotopy pullback

\[
\begin{array}{ccc}
P' & \to & X \\
\downarrow & & \downarrow \\
Y & \to & M_{G/N},
\end{array}
\]

where \(X \to M_G \xrightarrow{\gamma} M_{G/N}\) and \(Y \to M_G \xrightarrow{\gamma} M_{G/N}\). We would like to compare \(\mu_{M_G}\) and \(\mu_{M_{G/N}}\). For this, we define an auxiliary group. Let \(K = \{(x, y) \in G \times G \mid \gamma(x) = \gamma(y)\}\) be the fibre product of two copies of \(G\) over \(G/N\). It is easy to see by the five lemma that the classifying space \(BK\) fits in the homotopy pullback diagram:

\[
\begin{array}{ccc}
BK & \to & B(G/N) \\
\downarrow j & & \downarrow \Delta_{G/N} \\
BG \times BG & \xrightarrow{\gamma \times \gamma} & B(G/N) \times B(G/N),
\end{array}
\]

where \(j\) is the classifying map of the inclusion. Observe that the diagonal \(\Delta_{BG}\) factorises into \(BG \xrightarrow{d} BK \xrightarrow{\gamma} BG \times BG\), where \(d\) is the classifying map of the diagonal inclusion. Thus, we obtain the
following homotopy commutative diagram

\[
\begin{array}{ccc}
P & \longrightarrow & \text{M}_G \\
\downarrow d & & \downarrow d \\
P' & \longrightarrow & \text{M}_K & \longrightarrow & \text{M}_{G/N} \\
\downarrow \Delta_K & & \downarrow \Delta_{G/N} \\
\text{(M \times M)}_K & \longrightarrow & \text{(M \times M)}_{G/N} \\
\downarrow j & & \downarrow p \\
X \times Y & \longrightarrow & \text{M}_G \times \text{M}_G & \longrightarrow & \text{M}_{G/N} \times \text{M}_{G/N}.
\end{array}
\]

The fibre of \( d \) is \( K/G \simeq N \) and that of \( j \) is \( (G \times G)/K \simeq G/N \).

**Proposition 2.17 (Extension).** Under the above setting, we have \( \mu_{MG} = \pm d^\ast \circ \mu_{M_{G/N}} \).

**Proof.** The orientation of \( j \) is given as the pullback of that of \( \Delta_{BG \times G} \). The orientation of \( d \) is given by those of \( \Delta_{BG} \) and \( j \) using Lemma 2.14. Then, we can argue just as in the proof of Proposition 2.16. \( \square \)

3. Vanishing of \( \mu_{MG} \)

From now on, we restrict to the case where \( h \) is the ordinary homology with coefficients in a commutative ring \( R \). We prove that in certain cases the product \( \mu_{MG} \) vanishes. In particular, it implies that the Chataur-Menichi pair-of-pants product vanishes in positive degrees. In the next section we use the vanishing result of \( \mu_{MG} \) to construct a secondary product \( \mu_{MG} \).

We will always assume that our triples are oriented. Recall from Lemma 2.8 that the triple \( (M, G, \rho) \) is \( R \)-orientable if and only if the action \( \rho \) on \( M \) is \( R \)-orientation preserving and the adjoint action of \( G \) on \( g \) is \( R \)-orientation preserving, so we will frequently state this condition.

**Lemma 3.1.** Let \( K \) and \( L \) be \( G \)-spaces. Given the following pullback diagram

\[
\begin{array}{ccc}
(K \times L)_G & \longrightarrow & K_G \\
\downarrow g_G & & \downarrow f_G \\
L_G & \longrightarrow & BG,
\end{array}
\]

where \( g_G \) and \( f_G \) are the classifying maps. The product \( \mu_{BG} : H^G_k(K; R) \otimes H^G_l(L; R) \rightarrow H^G_{k+l+\dim(G)}(K \times L; R) \) is trivial if \( k > \dim_R(K) - \dim(G) \) or \( l > \dim_R(L) - \dim(G) \).

**Proof.** Assume \( k > \dim_R(K) - \dim(G) \). Let \( T \) be a CW complex of dimension \( l \) and \( q : T \rightarrow L_G \) a map which induces a surjection \( q_\ast : H_l(T; R) \rightarrow H_l(L_G; R) \) (one can always find such \( T \) and \( q \) by taking a skeleton of a CW-approximation). Construct the following homotopy pullback diagram

\[
\begin{array}{ccc}
\tilde{T} & \longrightarrow & (K \times L)_G & \longrightarrow & K_G \\
\downarrow \tilde{q} & & \downarrow q & & \downarrow f_G \\
T & \longrightarrow & L_G & \longrightarrow & BG.
\end{array}
\]

By the naturality of the Grothendieck bundle transfer, it is enough to show that \( \mu_{MG} \) vanishes for the outer diagram which involves \( T \) and \( K_G \). By the Serre spectral sequence for the fibration \( K \rightarrow \tilde{T} \rightarrow T \) we have

\[
\dim_R(\tilde{T}) \leq \dim(T) + \dim_R(K) < l + k + \dim(G),
\]

since the homology of \( T \) with local coefficients vanishes above its dimension. Hence, by dimensional reasons, \( \mu_{BG} \) vanishes. \( \square \)

As a corollary, we have our main vanishing result in which \( M \) needs not be a point:

**Theorem 3.2 (Vanishing of the product).** For the diagram \( \{1,3\} \), the product \( \mu_{MG} : H^G_k(K; R) \otimes H^G_l(L; R) \rightarrow H^G_{k+l+\dim(G)-\dim(M)}(Z; R) \) is trivial if \( k > \dim_R(K) - \dim(G) \) or \( l > \dim_R(L) - \dim(G) \).
Proof. In this case $\mu_{MG}$ is defined to be the composition

$$
\mu_{MG} : H^G_k(K) \otimes H^G_l(L) \xrightarrow{\rho_{BG}} H^G_{k+l+\dim(G)}((K \times L)) \xrightarrow{\hat{\Delta}^G} H^G_{k+l+\dim(G) - \dim(M)}(Z)
$$

and by Lemma 3.1 the first map is trivial. \qed

Example 3.3. Assume the adjoint action of $G$ on $g$ is $R$-orientation preserving and $G$ acts on $M$ orientation preserving. When $K = L = M$, the equivariant intersection product

$$
m : H^G_k(M; R) \otimes H^G_l(M; R) \rightarrow H^G_{k+l+\dim(G)}(M; R)
$$

is trivial when $k > \dim_R(M) - \dim(G)$ or $l > \dim_R(M) - \dim(G)$.

Example 3.4. Assume the adjoint action of $G$ on $g$ is orientation preserving. Consider $\psi$ in §1 (3) when $M = pt$ and $h$ is the ordinary homology:

$$
\psi : H_k(LBG; R) \otimes H_l(LBG; R) \rightarrow H_{k+l+\dim(G)}(LBG; R).
$$

By Lemma 3.1 the product $\psi$ is trivial when $k > 0$ or $l > 0$. Moreover, by the gluing property [10, Proposition 13] and the pants decomposition of the surface $F_{g,p+q}$ (the connected compact oriented surface of genus $g$ with $p$ incoming boundary circles and $q$ outgoing boundary circles), the vanishing of the pair-of-pants product implies that the non-equivariant version of Chataur-Menichi’s stringy operation associated to $F_{g,p+q}$ with field coefficients $K$

$$
H_*(LBG; K)^{\otimes p} \rightarrow H_*(LBG; K)^{\otimes q}
$$

is trivial in positive degrees unless $g = 0$ and $p = 1$.

Remark 3.5. When $G$ is finite or connected, the vanishing of $\psi$ follows from Tamanoi’s vanishing theorem [23, §4] by invoking Hepworth and Lahtinen’s result [18] that asserts that the Chataur-Menichi’s TQFT admits a counit.

This fact does not always hold for generalised homology theories, as seen in the following example.

Example 3.6. The free loop fibration $LBG \rightarrow BG$ has a section $\iota$, hence $i_* : h_*(BG) \rightarrow h_*(LBG)$ is an injection. By Proposition 2.13(3), we have $i_* \circ m = \psi \circ i_*$, where $m$ is the equivariant intersection product for $h_*(BG)$ (§1 (2)). Now we show $m$ is non-trivial when $h = \Omega^{fr}$ is the framed bordism and $G = S^1$. Let $i = [pt] \in \Omega^{fr}_0$ and $\eta = [S^1] \in \Omega^{fr}_1$ be the generators, and denote by the same symbols their images in $h_*(BS^1)$ under the induced map of $pt \rightarrow BS^1$. By definition, $m(i \otimes \eta)$ can be computed by the pullback of the bundle

$$
\begin{array}{ccc}
S^1 & \rightarrow & S^1 \\
\downarrow & & \downarrow \\
S^1 & \rightarrow & BS^1 \\
\pi & \downarrow & \Delta_{BS^1} \\
pt & \rightarrow & BS^1 \times BS^1
\end{array}
$$

as $f_* \circ \pi^*(\eta)$. By [10] Lemma 6.17, Chap. V], we have $\pi^*(\eta) = [\pi_2 : S^1 \times S^1 \rightarrow S^1]$ (the projection on the second factor, where the framing on $S^1 \times S^1$ is the product of framings, which is not null bordant). Therefore, $m(i \otimes \eta) = [f \circ \pi_2 : S^1 \times S^1 \rightarrow BS^1]$ is non-trivial.

4. THE SECONDARY PRODUCT $\overline{\mu}_{MG}$

The vanishing of $\mu_{MG}$ in Theorem 3.2 suggests that in the case where both inequalities $k > \dim_R(K) - \dim(G)$ and $l > \dim_R(L) - \dim(G)$ hold, we can define a secondary product associated to

$$
\overline{\mu}_{MG} : H^G_k(K; R) \otimes H^G_l(L; R) \rightarrow H^G_{k+l+\dim(G) - \dim(M) + 1}(Z; R).
$$

In this section we describe the construction of $\overline{\mu}_{MG}$ and prove some basic properties. Note that there is no indeterminacy in the definition.

We first use some intermediate spaces $S$ and $T$, and later we show that the construction is independent of this choice. Given maps $f : S \rightarrow K_G$ and $g : T \rightarrow L_G$ where $S$ and $T$ have the homotopy type of $CW$ complexes of dimensions $k$ and $l$ respectively, we construct a homomorphism

$$
\varphi : H_k(S; R) \otimes H_l(T; R) \rightarrow H_{k+l+\dim(G) + 1}((K \times L)_G; R).
$$
Consider the following diagram, where all the squares are homotopy pullback:

\[ \begin{array}{ccc}
W & \xrightarrow{\phi} & S \\
\downarrow & & \downarrow \\
\hat{T} & \xrightarrow{(K \times L)_G} & K_G \\
\downarrow & & \downarrow_{m_{BG}} \\
\hat{T} & \xrightarrow{L_G} & BG.
\end{array} \]

We first consider the outer diagram and obtain a homomorphism

\[ \mu_{BG} : H_k(S; R) \otimes H_l(T; R) \to H_{k+l+\dim(G)}(W; R). \]

Next, we construct a homomorphism

\[ \phi_W : H_{k+l+\dim(G)}(W; R) \to H_{k+l+\dim(G)+1}((K \times L)_G; R). \]

For this, form the homotopy pushout of the the upper left corner of the diagram (4.1)

\[ \begin{array}{ccc}
W & \xrightarrow{\phi} & S \\
\downarrow & & \downarrow \\
\hat{T} & \xrightarrow{B} & (K \times L)_G,
\end{array} \]

where \( q \) is the whisker map. As in the proof of Theorem 3.1, the assumption on \( S \) and \( T \) implies that \( \dim_R(\hat{S}), \dim_R(\hat{T}) < k + l + \dim(G) \). Using this, we define \( \phi_W \) to be the composition \( q_* \circ \partial^{-1} \), where \( \partial \) is the boundary homomorphism of the Mayer-Vietoris sequence for the homotopy pushout

\[ 0 = H_{N+1}(\hat{S}; R) \oplus H_{N+1}(\hat{T}; R) \to H_{N+1}(B; R) \xrightarrow{\partial} H_N(W; R) \to H_N(\hat{S}; R) \oplus H_N(\hat{T}; R) = 0, \]

where \( N = k + l + \dim(G) \). By composition, we obtain a homomorphism

\[ (4.2) \quad \varphi : H_k(S; R) \otimes H_l(T; R) \xrightarrow{\phi_W \mu_{BG}} H_{k+l+\dim(G)+1}((K \times L)_G; R). \]

**Lemma 4.3.** Given \( \alpha_1, \alpha_2 \in H_k(S; R) \) such that \( f_*(\alpha_1) = f_*(\alpha_2) \) then for every \( \beta \in H_l(T; R) \) we have \( \varphi(\alpha_1 \otimes \beta) = \varphi(\alpha_2 \otimes \beta) \).

**Proof.** Since \( \varphi \) is a homomorphism it is enough to show that if \( \alpha \in \ker f_* \) then \( \varphi(\alpha \otimes \beta) = 0 \). Let \( f' : S' \to K_G \) be the \( (k+1) \)-skeleton of a CW-approximation of \( K_G \). By cellular approximation, \( f \) factors, up to homotopy, as \( S \xrightarrow{\mu} S' \xrightarrow{f'} K_G \) and \( \mu_*(\alpha) = 0 \). Consider the diagram (4.1) with \( S \) replaced by \( S' \). Similarly to what we did, take the homotopy pushout of the upper-left corner to obtain

\[ \begin{array}{ccc}
W' & \xrightarrow{\phi'} & S' \\
\downarrow & & \downarrow \\
\hat{T} & \xrightarrow{B'} & (K \times L)_G,
\end{array} \]

where \( \dim_R(\hat{S}') \leq N = k + l + \dim(G) \) and \( q' \) is the whisker map. Note that \( q \) factors up to homotopy through \( B \to B' \xrightarrow{q'} (K \times L)_G \). By the naturality of the Mayer-Vietoris sequence, we have the following commutative diagram of exact sequences:

\[ \begin{array}{cccc}
0 & \xrightarrow{} & H_{N+1}(B; R) & \xrightarrow{\partial} & H_N(W; R) & \xrightarrow{d_*} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \xrightarrow{} & H_{N+1}(B'; R) & \xrightarrow{} & H_N(W'; R) & \to & H_N(\hat{S}'; R) \oplus 0,
\end{array} \]
where $d : W \to W'$ is the whisker map. Since $\mu_{BG}$ is natural we have $d_*(\mu_{BG}(\alpha \otimes \beta)) = \mu_{BG}(\iota_* (\alpha \otimes \beta)) = 0$, which, together with the injectivity of the map $H_{N+1}(B'; R) \to H_N(W'; R)$ shows that $\varphi(\alpha \otimes \beta) = 0$. \hfill \Box

To define our product for classes $\alpha \in H_k(K_G; R)$ and $\beta \in H_l(L_G; R)$, choose two spaces $S, T$ as above, together with maps $f : S \to K_G$ and $g : T \to L_G$ and classes $\alpha \in H_k(S; R)$ and $\beta \in H_l(T; R)$ such that $\alpha = f_* (\overline{\alpha})$ and $\beta = g_* (\overline{\beta})$. (This is always possible as in the proof of Lemma 3.1.) Using the fact that $Q \sim (K \times L)_G$ in (4.4), we define
\[ \overline{\mu_{M_G}}(\alpha \otimes \beta) = \Delta^G_0 \circ \varphi(\overline{\alpha} \otimes \overline{\beta}). \]
By Lemma 4.3 we see that this is independent of the choices made.

To summarise,

**Theorem 4.5.** Let $M$ be a smooth $R$-oriented manifold with a smooth and orientation preserving action of $G$, and assume that the adjoint action of $G$ on $g$ is $R$-orientation preserving. For the homotopy pullback in (4.4), we have a homomorphism
\[ \overline{\mu_{M_G}} : H^G_k(K; R) \otimes H^G_l(L; R) \to H^G_{k+l+\dim(G)-\dim(M)+1}(Z; R), \]
when $k > \dim R(K) - \dim(G)$ and $l > \dim R(L) - \dim(G)$.

Under the same setting, we can define another homomorphism in a similar manner.

**Theorem 4.6.** Let $k, l > \dim R(M) - \dim(G)$. Given the diagram in (4.4), we have a homomorphism
\[ \mu_{M_G} : H_k(X \times Y; R) \otimes H_l(Y; R) \to H_{k+l+m(G)-m(M)+1}(X \times_{M_G} Y; R), \]
where $X \times_{M_G} Y$ is the homotopy join of $X \to M_G$ and $Y \to M_G$ as in (4.2). It satisfies
\[ \partial \circ \mu_{M_G} = \mu_{M_G}, \quad \iota_* \circ \mu_{M_G} = \mu_{M_G} \]
where $\partial$ is the boundary operator in the Mayer-Vietoris sequence associated to the homotopy pushout defining $X \times_{M_G} Y$, and $\iota$ is the composition $P \to X \times_{M_G} Y$.

**Example 4.7.** Let $G^m$ be the $n$-fold join of $G$ and $B_nG = G^{n+1}/G \to BG$ be the $n$-th stage of Milnor’s construction of the classifying space of $G$. Consider the following diagram, where the outer square is a homotopy pullback and the upper-left triangle is a homotopy pushout:
\[
\begin{array}{ccc}
(G^k \times G^l)/G & \rightarrow & B_f G \\
\downarrow \scriptstyle B_k G \quad \downarrow \scriptstyle B_{k+l+1} G \quad \downarrow \scriptstyle f_t \\
B_k G & \rightarrow & B. 
\end{array}
\]
Here, we identify $B_{k+l+1} G \sim B_k G \times_{BG} B_l G$. Denote a generator of $H_{(\dim(G)+1)n}(B_n G; \mathbb{Z}) \cong \mathbb{Z}$ by $\alpha_n$. Then, we have $\mu_{BG}^*(\alpha_k, \alpha_l) = \pm_\alpha_{k+l+1}$.

We give an alternative description of the map $\phi_W$ to clarify that $\overline{\mu_{M_G}}$ is indeed a “secondary” product. Consider a homotopy commutative square
\[ (4.8) \]
\[ \begin{array}{ccc} A & \overset{g}{\rightarrow} & B \\
\downarrow \scriptstyle f \quad \downarrow \scriptstyle f \quad \downarrow \scriptstyle f \\
C & \overset{g}{\rightarrow} & D. \end{array} \]
We define a map $\phi_A : H_N(A; R) \to H_{N+1}(D; R)$ when $N > \dim R(B), \dim R(C)$. Notice that the induced maps $(f \circ g)_* = (g \circ f)_* : H_*(A; R) \to H_*(D; R)$ are trivial for $* \geq N$. By replacing $D$ with the mapping cylinder of $f$ and $g$, we can assume that $f$ and $g$ are cofibrations with disjoint images. Let $H : (A \times I, A \times \{0\} \sqcup A \times \{1\}) \to (D, B \sqcup C)$ be the commuting homotopy, where $I = [0, 1]$ is the unit interval. Since $N > \dim R(B), \dim R(C)$, the map $H_{N+1}(D; R) \to H_{N+1}(D, B \sqcup C; R)$ is an isomorphism. We define $\phi_A$ by the composition
\[ H_N(A; R) \times [(I, \partial I)], \quad H_{N+1}(A \times I, A \times \partial I; R) \overset{H_*}{\rightarrow} H_{N+1}(D, B \sqcup C; R) \cong H_{N+1}(D; R). \]
By the naturality of the boundary operators in Mayer-Vietoris sequences, one can verify that $\phi_A$ coincides up to sign with the one defined using the Mayer-Vietoris sequence for the homotopy pushout in the diagram (4.1) when $A = W, B = \hat{S}, C = \hat{T}$ and $D = (K \times L)_G$.

The composition of $\mu_{BG} : H_k(S; R) \otimes H_l(T; R) \to H_{k+l}(W; R)$ with the induced map of $(f \circ \hat{g}) \sim (g \circ \hat{f}) : W \to (K \times L)_G$ is nothing but $\mu_{BG} : H_k(S; R) \otimes H_l(T; R) \to H_{k+l}(W; R)$, which is trivial by assumption. Hence, we use the homotopy between $(f \circ \hat{g})$ and $(g \circ \hat{f})$ to define our secondary product.

**Remark 4.9.** Under a stronger assumption on degrees than Theorem 4.5, we can give an alternative definition of $\mu_{MG}$. Consider the following diagram (instead of (4.1)) with all the squares being homotopy pullback

\[
\begin{array}{ccc}
\bigwedge & \rightarrow & \bar{S} \rightarrow S \\
\downarrow & & \downarrow \\
\hat{T} & \rightarrow & Z_G \rightarrow K_G \\
\downarrow & & \downarrow \\
T & \rightarrow & L_G \rightarrow M_G.
\end{array}
\]

Assume that $\dim_R(F_L) - \dim(G) + \dim(M) < l$ and $\dim_R(F_K) - \dim(G) + \dim(M) < k$, where $F_L$ (resp. $F_K$) is the homotopy fibre of $L_G \xrightarrow{\sigma_G} M_G$ (resp. of $K_G \xrightarrow{\sigma_G} M_G$). Since $\dim_R(S) \leq \dim(S) + \dim_R(F_L)$ and $\dim_R(T) \leq \dim(T) + \dim_R(F_K)$, the following is well-defined

\[
H_k(S; R) \otimes H_l(T; R) \xrightarrow{\mu_{MG}} H_{k+l}(W; R) \xrightarrow{\phi_{\mu}} H_{k+l}(W; R) \xrightarrow{\phi_{\mu}} H_{k+l}(W; R).
\]

Observe that $F_L$ is also the homotopy fibre of $L \xrightarrow{\phi} M$. Thus, $\dim_R(L) \leq \dim_R(F_L) + \dim(M)$. We see that the conditions $\dim_R(F_L) - \dim(G) + \dim(M) < l$ and $\dim_R(F_K) - \dim(G) + \dim(M) < k$ imply $\dim_R(L) - \dim(G) < l$ and $\dim_R(K) - \dim(G) < k$ as required in Theorem 4.5. In this case, we can see that both definitions agree up to sign by a similar argument as in the proof of Proposition 5.4

5. Properties of the secondary product $\mu_{MG}$

In this section, we discuss properties of the secondary product $\mu_{MG}$. We prove a vanishing result in some cases. On the other hand, we see interesting non-trivial examples.

**Proposition 5.1.** (Naturality) The product $\mu_{MG}$ is natural with respect to ring homomorphisms $R \to R'$ of the coefficients and with respect to equivariant maps $K \to K'$ and $L \to L'$ over $M$ in degrees where it is defined for both pairs.

*Proof.* The first assertion is obvious. The second one follows from Lemma 4.3. □

**Corollary 5.2.** For the homotopy pullback diagram

\[
\begin{array}{ccc}
K_G & \rightarrow & M_G \\
\downarrow & & \downarrow \\
K_G & \rightarrow & M_G,
\end{array}
\]

one obtains a natural pairing

\[
H_k^G(K; R) \otimes H_l^G(M; R) \to H_{k+l}^G(K; R)
\]

for $k > \dim_R(K) - \dim(G)$ and $l > \dim_R(M) - \dim(G)$.

**Corollary 5.3.** Assume that $K$ is a finite dimensional (rigid) $G$-CW complex. Let $n$ be either the maximal dimension of a free cell $D^n \times G$ in $K$ or $-1$ if there is no such cell. Let $K_\Sigma$ be the subcomplex consisting of those points with non-trivial stabiliser. Then, the inclusion $K_\Sigma \hookrightarrow K$ induces an isomorphism

\[
H_*^G(K_\Sigma; R) \cong H_*^G(K; R) \ (*>n)
\]

which is compatible with $\mu_{MG}$. 
Proof. Notice that $n \leq \dim(K) - G$ where $\dim(K)$ is the dimension of $K$ considered as a CW complex. The quotient map induces an isomorphism $H^G_\bullet(K, K_G) \cong H_\bullet(K/G, K_G/G)$ by \cite[Proposition 3.10.9]{2}. Since $H_\bullet(K/G, K_G/G; R) \cong \hat{H}_\bullet((K/G)/(K_G/G); R)$ vanishes for $* > n$, the long exact sequence asserts $H^G_\bullet(K; R) \cong H^G_\bullet(K; R)$ for $* > n$. Compatibility with $\mu_{MG}$ follows from the naturality. \qed

Similarly to the primary product, $\mu_{MG}$ commutes with restriction to closed subgroups.

**Proposition 5.4** (Restriction). Let $H \subseteq G$ be a closed subgroup, where the adjoint actions of $H$ and $G$ on their Lie algebras are $R$-orientation preserving. Denote by $i$ the classifying map of the inclusion. Assume that $M$ is an $R$-oriented manifold with an $R$-orientation preserving action of $G$. Then, the secondary products $\mu_{MG}$ and $\mu_{MH}$ are compatible with the Grothendieck bundle transfer for $i$, that is, the following diagram commutes up to sign:

\[
\begin{array}{ccc}
H_k(K_G; R) \otimes H_l(L_G; R) & \xrightarrow{\mu_{MG}} & H_{k+l+\dim(G)-\dim(M)+1}(Z_G; R) \\
\downarrow \hat{R} & & \downarrow \hat{R} \\
H_{k+N}(K_H; R) \otimes H_{l+N}(L_H; R) & \xrightarrow{\mu_{MH}} & H_{k+l+2N+\dim(H)-\dim(M)+1}(Z_H; R),
\end{array}
\]

where $N = \dim(G) - \dim(H)$.

**Proof.** The secondary product $\mu_{MG}$ is defined as the composition of the cross product, $\hat{R}$, $\phi_W$, and $\Delta_G^1$. We already showed in the proof of Proposition \cite[10]{2} that the cross product, $\hat{R}$, and $\Delta_G^1$ commute with the Grothendieck bundle transfer up to sign. Hence, it is enough to show the same for $\phi_W$.

Given a fibre bundle $i : D \to D$, pullback the whole diagram \cite[13]{2} via $i$. We obtain a relative bundle map $i : (D, B \sqcup C) \to (D, B \sqcup C)$ and the following homotopy pullback square

\[
\begin{array}{ccc}
(\hat{A} \times I, \hat{A} \times \partial I) & \xrightarrow{\hat{R}} & (\hat{D}, \hat{B} \sqcup \hat{C}) \\
\downarrow i & & \downarrow i \\
(A \times I, A \times \partial I) & \xrightarrow{R} & (D, B \sqcup C).
\end{array}
\]

Now, each square in

\[
\begin{array}{ccc}
H_N(A; R) \times [(I, \partial I)] & \xrightarrow{H_N+1(A \times I, A \times \partial I; R)} & H_{N+1}(D, B \sqcup C; R) \\
\downarrow \hat{R} & & \downarrow \hat{R} \\
H_{N+k}(\hat{A}; R) \times [(I, \partial I)] & \xrightarrow{H_{N+k+1}(\hat{A} \times I, \hat{A} \times \partial I; R)} & H_{N+k+1}(\hat{D}, \hat{B} \sqcup \hat{C}; R)
\end{array}
\]

commutes up to sign by the naturality of the relative version of the Grothendieck bundle transfer with the cross product and pullback. Therefore, we have $\phi_\hat{A} \circ \hat{i}^* = i^* \circ \phi_A$. \qed

Just as Proposition \cite[17]{2}, we have a naturality of $\mu_{MG}$ with respect to group extensions.

**Proposition 5.5** (Extension). Let $1 \to N \to G \overset{\gamma}{\to} G/N \to 1$ be a short exact sequence of groups. Assume that $N$ acts on $M$ trivially and that $(M, G, \rho)$ and $(M, G/N, \rho)$ are oriented compatibly with respect to $\gamma$. Under the same notation as in Proposition \cite[17]{2}, we have

\[
\mu_{MG} = \pm d^\hat{G} \circ \mu_{MG/N}.
\]

Next, we show a vanishing of $\mu_{MG}$ for direct products.

**Theorem 5.6** (Vanishing of the secondary product). Let $G = G_1 \times G_2$ act component-wise on $K = K_1 \times K_2, L = L_1 \times L_2$ and $M = M_1 \times M_2$. Set $k = k_1 + k_2$ and $l = l_1 + l_2$. When the secondary products for both

\[
\begin{array}{ccc}
(Z_1)_{G_1} & \xrightarrow{H^G_{k_1}} & (K_1)_{G_1} \\
\downarrow & & \downarrow \\
(L_1)_{G_1} & \xrightarrow{H^G_{l_1}} & (M_1)_{G_1}
\end{array}
\]

\[
\begin{array}{ccc}
(Z_2)_{G_2} & \xrightarrow{H^G_{k_2}} & (K_2)_{G_2} \\
\downarrow & & \downarrow \\
(L_2)_{G_2} & \xrightarrow{H^G_{l_2}} & (M_2)_{G_2}
\end{array}
\]

are defined, the secondary product

\[
\mu_{MG} : H^G_{k_1}(K; R) \otimes H^G_{l_1}(L; R) \to H^G_{k_1+l_1+\dim(G)-\dim(M)+1}(Z; R),
\]
vanishes on the image of the cross product\
\[(H_i^{G_1}(K_1; R) \otimes H_j^{G_2}(K_2; R)) \otimes (H_k^{G_1}(L_1; R) \otimes H_l^{G_2}(L_2; R)) \xrightarrow{\otimes \otimes} H_i^G(K; R) \otimes H_l^G(L; R).\]

**Proof.** It is not difficult to show that \(\mu_{(M_1)\otimes G_2}(\beta_1 \otimes \beta_2) = \pm \mu_{(M_1)\otimes G_2}(\beta_1 \otimes \beta_2)\) when the secondary product is defined. 

Taking \(M_i = K_i = L_i\) to be a point we get the following corollary, which will be used later.

**Corollary 5.7.** If \(k_i, l_i > -\dim R(G_i)\) for \(i = 1, 2\), then the secondary product
\[\mu_{BG} : H_k(BG; R) \otimes H_l(BG; R) \rightarrow H_{k+l+\dim(G)+1}(BG; R)\]
vanishes on the image of the cross product
\[(H_k(BG_1; R) \otimes H_k(BG_2; R)) \otimes (H_l(BG_1; R) \otimes H_l(BG_2; R)) \xrightarrow{\otimes \otimes} H_k(BG; R) \otimes H_l(BG; R).\]

We immediately obtain the following:

**Corollary 5.8.** The secondary product \(\mu_{BG}\) in the homology of \(BG\) vanishes when \(G\) is the torus \(T^n\) with \(n > 1\).

**Remark 5.9.** We do not know whether \(\mu_{MG}\) is associative up to multiplication by a unit, though we believe it is. The main difficulty is that in the definition we have to choose a CW-complex \(S\) and \(T\) of the right dimension to represent homology classes. However, in our construction, the image of \(\mu_{MG}\) is represented by \(B\) which is not necessarily a CW-complex of the right dimension. This prevents from directly iterating the construction.

## 6. Examples of the Secondary Product \(\mu_{MG}\)

Here we present several examples of \(\mu_{MG}\). Throughout this section, we assume all triples \((M, G, \rho)\) are oriented.

**Example 6.1.** Recall the diagram in Example 5.7. We can use this to compute partially
\[\overline{\mu} : H_k(BG; \mathbb{Z}) \otimes H_l(BG; \mathbb{Z}) \rightarrow H_{k+l+\dim(G)+1}(BG; \mathbb{Z}),\]
where \(\overline{\mu}\) is the secondary equivariant intersection product (§1 (ii)) for \(M = pt\).

For generators \(\alpha_i \in H_{(\dim(G)+1)k}(B_nG; \mathbb{Z})\), we have
\[\overline{\mu}((f_k)_*\alpha_k \otimes (f_l)_*\alpha_l) = \pm q_*(\alpha_{k+l+1}),\]
where \(q : B_{k+l+1}G \rightarrow BG\) is the whisker map. In particular,
\[\overline{\mu}(\alpha_0 \otimes \alpha_0) = \pm q_*(\Sigma([G])) \in H_{\dim(G)+1}(BG; \mathbb{Z}),\]
where \([G]\) is the fundamental class of \(G\) and \(q : \Sigma G \rightarrow BG\) is adjoint to the homotopy equivalence \(G \xrightarrow{\sim} \Omega BG\).

**Example 6.2.** Let \(H \subseteq G\) be a closed subgroup. Under an appropriate orientation of \((T(G/H))_G\), we show that the following diagram commutes up to sign
\[H^H(pt; \mathbb{Z}) \otimes H^*_H(pt; \mathbb{Z}) \xrightarrow{\overline{\mu}} H^*_H(pt; \mathbb{Z}) \cong H^G(G/H; \mathbb{Z}) \otimes H^*_G(G/H; \mathbb{Z}) \xrightarrow{\overline{\mu}} H^*_G(G/H; \mathbb{Z}),\]
where \(\overline{\mu}\) is the secondary equivariant intersection product (§1 (ii)) for \(pt\) and \(G/H\). To see this, we look at the orientations of the maps in the following diagram:
\[BH \xrightarrow{\Delta_{BH}} BH \times BH \cong (G/H)_G \xrightarrow{\Delta_G} (G/H \times G/H)_G \xrightarrow{pg} (G/H)_G \times (G/H)_G.\]
The normal bundle \(\nu(\Delta_G)\) of \(\Delta_G\) is \((T(G/H))_G\), which is isomorphic to \(t(i)\) in Lemma 2.14. Thus, we orient \(\nu(\Delta_G) = t(i)\) by the short exact sequence (2.15). Then, by Lemma 2.6, the orientations for \(-t(\Delta_{BH}) = -ad(EH)\) and \(\nu(\Delta_G) - \Delta_G^*(t(pg)) = \nu(\Delta_G) - i*(ad(EG))\) are equal.
When $G$ is non-trivial, there exists a finite non-trivial cyclic subgroup $H \subset G$. In this case, we will see in Proposition 6.3 that $\overline{m}$ in the first row is non-trivial. By the commutativity of the above diagram, we see that there always exists a $G$-manifold (we can take $G/H$) such that the secondary equivariant intersection product is non-trivial.

**Example 6.3 (Secondary string product in $H_*(LBG)$).** Assume the same setting as in Example 3.1. By composing $\overline{\rho}_{BG}$ with the induced map of the concatenation map, we obtain

$$\overline{\psi} : H_*(LBG;R) \otimes H_1(LBG;R) \to H_{k+l+\dim(G)+1}(LBG;R),$$

where $k, l > 0$. This can be thought of as a secondary product for Chataur-Menichi’s pair-of-pants product.

**Proposition 6.4.** The product $\overline{\psi}$ in $H_*(LBG;R)$ is non-trivial if $\overline{m}$ in $H_*(BG;R)$ is non-trivial. Moreover, if $G$ is abelian, the map $H_*(G;R) \otimes H_*(BG;R) \to H_*(LBG;R)$ induced by the equivalence $G \times BG \sim LBG$ is compatible up to sign with the products, where the product in $H_*(LBG;R)$ is given by $\overline{\psi}$, in $H_*(BG;R)$ by $\overline{m}$, and in $H_*(G;R)$ by the Pontryagin product.

**Proof.** The free loop fibration $LBG \to BG$ has a section $i$, hence $i_* : H_*(BG;R) \to H_*(LBG;R)$ is an injection. By naturality, we have the following commutative diagram:

$$
\begin{array}{ccc}
H_*(BG;R) \otimes H_*(BG;R) & \xrightarrow{\overline{m}} & H_*(BG;R) \\
\downarrow{i_*} & & \downarrow{i_*} \\
H_*(LBG;R) \otimes H_*(LBG;R) & \xrightarrow{\overline{\rho}_{BG}} & H_*(\text{Map}(S^1 \vee S^1, BG);R) \xrightarrow{c_*} H_*(LBG;R),
\end{array}
$$

where $d$ is the whisker map and $c$ is the concatenation. Since $c_* \circ d_* = i_*$ is injective, the first assertion holds. The second assertion follows from a straightforward diagram chasing. \qed

We give examples where the product $\overline{m}$ in $H_*(BG;R)$ is non-trivial in the next section.

### 7. Relation to the Kreck product

The Kreck product is a product in the homology of $BG$ with $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$ coefficients, where the adjoint action of $G$ on its Lie algebra is orientation preserving. It is defined in a geometric way using stratifolds. We will see in the Appendix:

**Proposition 7.1.** The secondary intersection product $\overline{m}$ coincides with the Kreck product in the case where the latter is defined, that is, when $M = X = Y = pt$ and $R = \mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$.

We denote $\overline{m}(a \otimes b)$ by $a \ast b$ when $M = X = Y = pt$. We give a complete computation of this product in the cases of finite cyclic groups, $S^1$, $SU(2)$, and $SO(3)$. We also prove that the product is torsion in the case where $G$ is of positive dimension and has rank greater than one.

It was shown in [26] that the when $G$ is finite and $R = \mathbb{Z}$ then the Kreck product coincides with the cup product in negative Tate cohomology. In particular, we have

**Corollary 7.2.** For a finite group $G$, the product $\ast$ in $H_*(BG;\mathbb{Z})$ coincides with the cup product in negative Tate cohomology.

The cup product in negative Tate cohomology is a non-trivial product. One family of groups with non-trivial products are those with periodic cohomology, which are exactly those groups such that every abelian subgroup is cyclic (see [1]). This includes, for example, all groups having a free and orientation preserving action on some sphere. Benson and Carlson showed that in many cases this product vanishes:

**Theorem** (Benson-Carlson [5]). Let $\mathbb{K}$ be a field of characteristic $p$. Suppose that the $p$-rank of $G$ is greater than one. If $H^*(G;\mathbb{K})$ is Cohen-Macaulay then the product vanishes.

This is the case, for example, if there is a $p$-Sylow subgroup with a non cyclic centre.
Computations of the Kreck product. We now use our construction to compute the Kreck product. Let $R$ be the coefficient ring. We need an $R$-orientation of the universal adjoint bundle $ad(EG)$ for the product to be defined. When $G$ is connected or finite, it admits a canonical orientation by that of the Lie algebra $g$. In what follows, we fix a choice of orientation of $ad(EG)$ without mentioning it specifically.

We start with a computation of the product when $G$ is a compact connected Lie group of rank one or a finite cyclic group, where it is non-trivial. We then show that if $G$ has positive dimension and its rank is greater than one, then the product is torsion, hence trivial in many cases (compare with [16, 9.11]). This is analogous to the theorem by Benson and Carlson stated before ([1, Theorem 3.1]) for finite groups of $p$-rank greater than one.

For $S^1$, $SU(2)$ and finite cyclic groups, we can compute the product directly from the definition using Example 6.1.

**Proposition 7.3.** Let $G$ be either $S^1$, $SU(2)$ or a finite cyclic group. Recall that $H_*(BG;\mathbb{Z})$ is generated by $a_k \in H_{(\dim(G)+1)k}(BG;\mathbb{Z})$ which are represented by the inclusions $S^{(\dim(G)+1)k+\dim(G)}/G \hookrightarrow S^{\infty}/G \sim BG$. For $k, k' > -\dim(G)$, we have

$$a_k \cdot a_{k'} = \pm a_{k+k'+1}.$$  

**Proof.** In these cases, $B_nG$ in Example 6.1 are nothing but the corresponding projective spaces (or lens spaces) $S^{(\dim(G)+1)k+\dim(G)}/G$. \hfill $\blacksquare$

In order to compute the product in $H_*(BSO(3);\mathbb{Z})$, we use the following:

**Proposition 7.4.** Let $\gamma: G' \to G$ be an $n$-covering of groups which preserves the orientation of the universal adjoint bundles. Then, for any $\alpha, \beta \in H_*(BG';R)$ we have

$$B\gamma_*(\alpha \cdot \beta) = n \cdot B\gamma_*(\alpha) \cdot B\gamma_*(\beta).$$  

**Proof.** Choose $S \xrightarrow{f} BG', T \xrightarrow{g} BG$ to represent classes $\alpha$ and $\beta$ as in $\blacksquare$. Consider the following diagram, where the top and the bottom squares are homotopy pullback and $d$ is the whisker map:

$$\begin{array}{ccc}
P' & \xrightarrow{\phi'} & S' \\
\downarrow g & & \downarrow f \\
BG' & \xrightarrow{\phi} & S \\
\downarrow d & & \downarrow B\gamma \\
BG & \xrightarrow{\phi} & B\gamma \\
\downarrow d & & \downarrow B\gamma \\
T & \xrightarrow{\phi} & T \\
\end{array}$$

Let $\phi_{P'}$ (resp. $\phi_P$) be as in $\blacksquare$ for the top square (resp. for the bottom square). Notice that we can use the same $S$ and $T$ to compute the products in the left and the right hand sides of (*). Since $\phi_P \circ d_* = B\gamma_* \circ \phi_P$, we have only to show $d_* \circ \mu_{BG} = n \mu_{BG'}$. By Proposition 2.17, we have $\mu_{BG'} = d^* \circ \mu_{BG}$. By [3, Chap. V (6.2) (d)],

$$d_* \circ \mu_{BG'} = d_* \circ d^* \circ \mu_{BG} = d_2(1) \cdot \mu_{BG},$$

where $d_2$ is the Grothendieck bundle transfer on the cohomology. Finally, by the commutative diagram of fibrations

$$\begin{array}{ccc}
G' & \xrightarrow{\gamma} & G \\
\downarrow P' & \xrightarrow{d} & P \\
\downarrow \bar{P} & & \downarrow \bar{P} \\
S \times T & \xrightarrow{\bar{P}} & S \times T. \\
\end{array}$$

we see $\iota^* \circ d_2(1) = \gamma_2(1)$ by naturality. Since $\gamma$ is $n$-covering and orientation preserving, we have $\gamma_2(1) = n$. \hfill $\blacksquare$
Now, we give a complete description of the product in $H_\ast(BSO(3); \mathbb{Z})$. We will see in the proof of Proposition 7.3 that the product vanishes on 2-torsion elements. For the free part, let $\gamma : SU(2) \to SO(3)$ be the double cover. By a Serre spectral sequence argument we have $\gamma_\ast(a_k) = 4^k b_k$, where $a_k \in H_{4k}(SU(2); \mathbb{Z})$ and $b_k \in H_{4k}(BSO(3); \mathbb{Z})$ are the generators for the free part. By Propositions 7.3 and 7.4 we conclude:

**Proposition 7.5.** The only non-trivial products in $H_\ast(BSO(3); \mathbb{Z})$ are

$$b_k * b_{k'} = \pm 2b_{k+k'+1} \quad (k, k' \geq 0).$$

In the case where $G$ has positive dimension and its rank is greater than one, we use our vanishing theorem (Corollary 5.7) to show that the product is torsion, and in many cases trivial.

First, we prove a small lemma:

**Lemma 7.6.** Let $\mathbb{K}$ be a field. Let $H \subseteq G$ be a closed subgroup with specified orientations for $ad(EH)$ and $ad(EG)$. Let $G/H \xrightarrow{j} BH \xrightarrow{i} BG$ be a homogeneous bundle whose bundle of tangents along the fibre $t(i)$ is oriented by $(2, 4)$. If $j_\ast : H_\ast(BH; \mathbb{K}) \to H_\ast(G/H; \mathbb{K})$ is surjective, then the Grothendieck bundle transfer $i_\ast : H_\ast(BG; \mathbb{K}) \to H_{\ast+\dim(G/H)}(BH; \mathbb{K})$ is injective.

**Proof.** Let $x \neq 0 \in H\ast(BG; \mathbb{K})$, then there exists $x^0 \in H\ast(BG; \mathbb{K})$ with the Kronecker pairing $\langle x^0, x \rangle \neq 0$. Since $j_\ast$ is surjective, we can choose an element $c \in H^{\dim(G/H)}(BH; \mathbb{K})$ such that $j_\ast(c)$ is Poincaré dual to the fundamental class of $G/H$. Then, by [8, Chap. V (6.2)] we have $i_\ast(c) = 1$, where $i_\ast$ is the Grothendieck bundle transfer on cohomology, and

$$\langle c \cup i_\ast(x^0), i_\ast(x) \rangle = \pm \langle i_\ast(c) \cup i_\ast(x^0), x \rangle = \pm \langle i_\ast(c) \cup x^0, x \rangle = \pm \langle x^0, x \rangle \neq 0,$$

which shows $i_\ast(x) \neq 0$. \qed

**Proposition 7.7.** Let $G$ be a compact Lie group of rank greater than one. Then, the product of two classes in $H_\ast(BG; \mathbb{Z})$ is a torsion element.

**Proof.** Since the product is natural with respect to coefficients (Proposition 5.1), it is sufficient to show that it vanishes with the rational coefficients. Let $T \subseteq G$ be the maximal torus of the identity component and $i : BT \to BG$ the classifying map of the inclusion. Since $H\ast(BT; \mathbb{Q}) \to H\ast(G/T; \mathbb{Q})$ is surjective, by Lemma 7.6 we see that $i_\ast$ is injective with the rational coefficients. By Corollary 5.8 the product vanishes in $H\ast(BT; \mathbb{Q})$, and so does in $H\ast(BG; \mathbb{Q})$ by the commutativity with $i_\ast$ (Proposition 5.3). \qed

**Proposition 7.8.** The product in $H_\ast(BG; \mathbb{Z})$ vanishes for $T^n, U(n)$ and $Sp(n)$ when $n > 1$, for $SU(n)$ when $n > 2$, and for $SO(n)$ when $n > 3$.

**Proof.** For $G = T^n, U(n), SU(n)$ and $Sp(n)$ the homology of $H_\ast(BG; \mathbb{Z})$ is torsion free. Therefore, in these cases the assertion follows from the previous proposition. For $G = SO(n)$, let $D_{n-1} = (\mathbb{Z}/2\mathbb{Z})^{n-1}$ be its maximal two-torus and $i : BD_{n-1} \to BSO(n)$ be the classifying map of the inclusion. By naturality with respect to coefficients (Proposition 5.1) and restriction (Proposition 5.3) the following diagram commutes up to sign

$$\begin{array}{ccc}
H_k(BSO(n); \mathbb{Z}) \otimes H_l(BSO(n); \mathbb{Z}) & \xrightarrow{m} & H_{k+l+\dim(SO(n))+1}(BSO(n); \mathbb{Z}) \\
\downarrow & & \downarrow \\
H_{k+\dim(SO(n))}(BD_{n-1}; \mathbb{Z}/2\mathbb{Z}) \otimes H_{l+\dim(SO(n))}(BD_{n-1}; \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{m} & H_{k+l+2\dim(SO(n))+1}(BD_{n-1}; \mathbb{Z}/2\mathbb{Z}),
\end{array}$$

where the vertical arrows are compositions of $i_\ast$ and mod-2 reduction. By the previous proposition, the image of the upper row lies in the torsion, which injects into $H_\ast(BD_{n-1}; \mathbb{Z}/2\mathbb{Z})$ by the universal coefficients theorem and Lemma 7.6. By Corollary 5.7 the product in the lower row vanishes. This implies that the product in the upper row vanishes. \qed

**Remark.** The same argument works when all the torsion elements in $H^n(BG; \mathbb{Z})$ are $p$-toral, that is, there is a $p$-torus $D$ of rank greater than one such that $H^n(BG; \mathbb{Z}/p\mathbb{Z})$ injects into $H^n(BD; \mathbb{Z}/p\mathbb{Z})$ for all the torsion primes $p$ of $G$. For example, the product in $H_\ast(BG_2; \mathbb{Z})$ is trivial for the compact simply-connected exceptional Lie group $G_2$.\[\]
APPENDIX A. THE KRECK PRODUCT

Throughout this Appendix, $G$ is assumed to be a compact Lie group whose adjoint action on its Lie algebra is $R$-orientation preserving. The Kreck product is a product in $H_*(BG; R)$ for $R = \mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$. Here, we describe a generalisation of the product in $H^G_*(M; R)$ for $R = \mathbb{Z}$. The case for $R = \mathbb{Z}/2\mathbb{Z}$ is obtained in an analogous way if one replaces oriented stratifolds by stratifolds.

First, we briefly recall stratifolds and stratifold homology, and refer to [20] for a complete treatment. Stratifolds are a generalisation of smooth manifolds. A stratifold is a pair $(S, F)$, where $S$ is a topological space and $F$ is a sub-sheaf of the sheaf of continuous real functions (considered as a sheaf of $\mathbb{R}$-algebras) which satisfies certain axioms. In particular, for every $r \in \mathbb{Z}_{\geq 0}$, the $r$-stratum is defined to be the set of points $x$ in $S$, where the tangent space at $x$ (which is defined as the space of derivations of germs at $x$) has dimension $r$. This stratum together with the restriction of the sheaf $F$ is required to be a smooth manifold. The dimension of the stratifold is defined to be the maximal integer $k$ such that the $k$-stratum is non-empty. A $k$-dimensional stratifold is called oriented if its $k$-th stratum is oriented and its $k-1$ stratum is empty. In a similar way stratifolds with boundary are defined. An example of a stratifold with boundary is the cone over a stratifold $S$, denoted by $CS$. Note that if $S$ is oriented and of positive dimension then $CS$ is oriented as well. This fact implies that the corresponding bordism theory, denoted by $SH$, is naturally isomorphic to integral homology when restricted to spaces having the homotopy type of a $CW$ complex.

We use here a certain class of stratifolds, called $p$-stratifolds (see [20, p. 24 example 10]). Those are constructed inductively. We start with a discrete set $S_0$ with the sheaf of all real functions. The $r^{th}$ skeleton $S_r$ is defined by attaching a smooth manifold $W_r$ of dimension $r$ to $S_{r-1}$ along a smooth map from its boundary $\partial W_r$ and extend the sheaf in an appropriate way. This process is similar to the construction of a $CW$ complex. The reason we use $p$-stratifolds is that they have the homotopy types of $CW$ complexes of the same dimension as the stratifolds themselves. This fact is needed in the comparison of our product with the Kreck product in Proposition A.3. From now on, when we say stratifold we mean $p$-stratifolds. Let $M$ be a smooth manifold. A smooth map $g : N \to M$ between manifolds and a map from a stratifold $f : S \to M$ are said to be transversal when the compositions $W_r \to S_r \to M$ and $\partial W_r \to S_{r-1} \to M$ are transversal to $g$ for all $r$. Two maps from stratifolds $f : S \to M, g : S' \to M$ are said to be transversal when $S \times S' \to M \times M$ is transversal to the diagonal $\Delta : M \to M \times M$. For an oriented $r$-dimensional stratifold $S$, the image of a fundamental class of $(W_r, \partial W_r)$ by the composition

$$H_r(W_r, \partial W_r; \mathbb{Z}) \to H_r(S_r, S_{r-2}; \mathbb{Z}) \cong H_r(S_r; \mathbb{Z})$$

is called a fundamental class of $S$. We need the following property of stratifolds:

**Lemma A.1.** A compact finite dimensional stratifold is an ANR. In particular, a closed inclusion of stratifolds is a cofibration.

**Proof.** A compact finite dimensional stratifold is a locally contractible, metrizable space, and hence, an ANR by a theorem of Borsuk [7]. By a theorem of Whitehead [29], a closed inclusion of compact ANR’s is a cofibration. □

Given a space $X$ together with an action of a compact Lie group $G$, the Borel $G$-equivariant homology of $X$ with the integral coefficients is identified with a bordism theory [27]. Denote by $SH^G_k(X)$ the set of bordism classes of equivariant maps $f : S \to X$ from compact oriented stratifolds $S$ of dimension $k$ with a smooth, free and orientation preserving $G$ action modulo $G$-bordism, i.e. a bordism with a smooth, free and orientation preserving $G$ action extending the given action on the boundary. Since we assumed that the adjoint representation of $G$ preserves the orientation of its Lie algebra, we can give a canonical orientation to $S/G$. The map $SH_k^G(X) \to SH_{k-\dim(G)}(X \times_G EG; \mathbb{Z}) \cong H_{k-\dim(G)}(X \times_G EG; \mathbb{Z})$ given by $[f : S \to X] \mapsto ([f]_G, [S/G]) \in H_{k-\dim(G)}(X \times_G EG; \mathbb{Z})$ is a natural isomorphism.

**Remark.** If the adjoint representation of $G$ does not preserve the orientation of the Lie algebra, then $S/G$ might be non orientable, for example, $O(2)$ acts on $SO(3)$ with orbit space $\mathbb{R}P^2$. In this case one cannot identify $SH_k^G(X)$ with $H^G_{k-\dim(G)}(X; \mathbb{Z})$. If one works with $\mathbb{Z}/2\mathbb{Z}$ coefficients then there is no such problem.

When $X$ is a point $pt$, we simply write a class $[S \to pt] \in SH_k^G(pt) \cong H_{k-\dim(G)}(BG; \mathbb{Z})$ by $[S]$. The Cartesian product $[S \times T]$ is trivial if at least one of $\dim(S), \dim(T)$ is positive (say $S$), since then
\( S \times T \) is the boundary of \( CS \times T \), which has a free and orientation preserving \( G \) action (if \( \dim(S) = 0 \) then \( CS \) has a non empty codimension 1 stratum). The Kreck product \([S] \ast [T]\) in \( H_{\ast}(BG; \mathbb{Z})\) is a secondary product defined when both \( \dim(S) \) and \( \dim(T) \) are positive. It is given by gluing together the two null bordisms \( CS \times T \) and \( S \times CT \) along their common boundary \( S \times T \), which is isomorphic to the join \( S \ast T \); that is,
\[
[S] \ast [T] = [S \ast T],
\]
The grading is given by
\[
H_k(BG; \mathbb{Z}) \otimes H_l(BG; \mathbb{Z}) \to H_{k+l+\dim(G)+1}(BG; \mathbb{Z}),
\]
where \( k, l > -\dim(G) \).

We generalise this construction as follows. Let \( X = M \) be a smooth oriented manifold with an orientation preserving action of a compact Lie group \( G \). Let \( S \to M \) and \( T \to M \) be maps from stratifolds \( S \) and \( T \). As is proved in [27], their transversal intersection product \( S \pitchfork T \to M \) is well-defined up to bordism. We have the following commutative diagram:
\[
\begin{array}{ccc}
(S \pitchfork T)/G & \longrightarrow & S/G \\
\downarrow & & \downarrow \\
T/G & \longrightarrow & M_G.
\end{array}
\]
Consider the following two diagrams:
\[
\begin{array}{ccc}
(S \times T)/G & \xrightarrow{\hat{p}} & S/G \times T/G \\
\downarrow & & \downarrow \\
BG & \xrightarrow{\Delta_{BG}} & BG \times BG,
\end{array}
\quad
\begin{array}{ccc}
(S \pitchfork T)/G & \xrightarrow{i} & (S \times T)/G \\
\downarrow & & \downarrow \\
M_G & \xrightarrow{\Delta_G} & (M \times M)_G.
\end{array}
\]
The diagram on the left is a homotopy pullback diagram where the vertical maps are the classifying maps, and the horizontal maps are fibre bundles with fibre \( G \). The diagram on the right is a topological pullback and \( i \) is a cofibration by Lemma 4.1.

**Lemma A.2.** Let \( \hat{p}^\natural \) be the Grothendieck bundle transfer with respect to \( p \). Then, \( \hat{p}^\natural([S/G] \times [T/G]) \) is a fundamental class of \( (S \times T)/G \). Let \( i^! \) be the umkehr map with respect to the pullback of the Thom class of \( \Delta_G \). Then, the image of a fundamental class of \( (S \times T)/G \) is a fundamental class of \( (S \pitchfork T)/G \).

**Proof.** The first assertion follows from [6, Lemma 6.17, Chap. V]. For the second, set \( Q = (S \times T)/G \) and \( Q' = (S \pitchfork T)/G \). We can relativise the construction of the umkehr map in [2] to have \( i^! : h_k(X, B) \to h_{k-d}(A, A \cap B) \) for a cofibered subspace \( B \subseteq X \) and a Thom class in \( h^d(X, X \setminus A) \). By naturality the following diagram commutes:
\[
\begin{array}{ccc}
h_k(X) & \longrightarrow & h_k(X, B) \\
\downarrow & & \downarrow \\
h_{k-d}(A) & \xrightarrow{i} & h_{k-d}(A, B \cap A),
\end{array}
\]
Set \( k = \dim(Q), X = Q, A = Q' \) and \( B = Q_{k-1} \) (then \( B \cap A = Q_{k-d-1} \)). Then, the horizontal maps are isomorphisms when \( h \) is integral homology. Therefore, it is enough to prove that the relative fundamental class in \( H_k(Q, Q_{k-1}) \) is mapped to a relative fundamental class in \( H_{k-d}(Q', Q_{k-d-1}) \). Let \( W \) be the manifold with boundary used to attach the top cell in \( Q \). Then, \( Q' \) is obtained by attaching \( W' = W \pitchfork M_G \) to \( Q_{k-d-1} \). By naturality, it is enough to show that the relative fundamental class for \( (W, \partial W) \) is mapped to one for \( (W', \partial W') \). This follows from the fact that the pullback of a Thom class along a transversal intersection is a Thom class. \( \square \)

We specify the fundamental classes as \([ (S \times T)/G ] = \hat{p}^\natural([S/G] \times [T/G]) \) and \([ (S \pitchfork T)/G ] = i^!([S \times T]'/G) \).

When \( k, l > \dim(M) - \dim(G) \), we have \( \dim(S/G), \dim(T/G) < \dim((S \pitchfork T)/G) = k + l + \dim(G) - \dim(M) \) so the map \( \phi_{(S \pitchfork T)/G} : H_{\dim((S \pitchfork T)/G)}((S \pitchfork T)/G; \mathbb{Z}) \to H_{\dim((S \pitchfork T)/G)+1}(M_G; \mathbb{Z}) \) in
is defined. Suppose we have two classes $\alpha \in H^G_k(M;\mathbb{Z}) \cong SH^G_k(M;\mathbb{Z})$ and $\beta \in H^G_l(M;\mathbb{Z}) \cong SH^G_{l+\dim(G)}(M)$, represented by the maps $S \to M$ and $T \to M$. We define a product

$$* : H^G_k(M;\mathbb{Z}) \otimes H^G_l(M;\mathbb{Z}) \to H^G_{k+l+\dim(G)−\dim(M)+1}(M;\mathbb{Z})$$

by $\alpha * \beta = \phi_{(S\times T)/G}([S \times T]/G)$. When $M = pt$, we have $S \times T = S \times T$ and the homotopy pushout of $S \leftrightarrow S \times T \to T$ is the join $S * T$. Therefore, this coincides with the Kreck product.

By the universal property, there is a map $d : (S \times T)/G \to P$, where $P$ is the homotopy pullback of the same diagram. We use it to give an more “geometric” interpretation of the products $\mu_{M_G}$ and $\overline{\mathbb{m}}$.

**Proposition A.3.**

1. The transversal intersection coincides with $\mu_{M_G}$. That is,

$$\mu_{M_G}([S/G],[T/G]) = d_*([S \times T]/G) \in H_{k+l+\dim(G)−\dim(M)}(P;\mathbb{Z}).$$

2. When $k, l > \dim(M) − \dim(G)$ we have

$$\overline{\mathbb{m}}(\alpha, \beta) = \pm \alpha * \beta,$$

where $\overline{\mathbb{m}}$ is the secondary intersection product (§3, (ii)).

**Proof.**

1. Recall from [2] and Lemma A.2 that

$$\mu_{M_G}([S/G],[T/G]) = \hat{\Delta}^G_G \circ \hat{p}([S/G] \times [T/G]) = \hat{\Delta}^G_G([S \times T]/G).$$

In order to compute $\hat{\Delta}^G_G$ we factor the map $(S \times T)/G \to (M \times M)_G$ as the composition

$$(S \times T)/G \xrightarrow{j} (S \times T)/G \xrightarrow{\pi} (M \times M)_G,$$

where $j$ is a weak equivalence and $\pi$ a fibration. This way the following is a pullback diagram:

$$
\begin{array}{ccc}
(S \times T)/G & \overset{d}{\longrightarrow} & P \\
\downarrow i & & \downarrow \Delta_G \\
(S \times T)/G & \overset{\sim}{\longrightarrow} & (S \times T)/G \xrightarrow{\pi} (M \times M)_G.
\end{array}
$$

By the naturality of the umkehr map with respect to cofibrations and Lemma A.2 we conclude

$$\hat{\Delta}^G_G([S \times T]/G) = d_* \circ i_*([S \times T]/G) = d_*([S \times T]/G).$$

2. Recall from [4] that

$$\overline{\mathbb{m}}(\alpha, \beta) = \Delta^G_G \circ \phi_{(S \times T)/G} \circ \hat{p}([S/G] \times [T/G]).$$

Consider the following diagram, where the top and the bottom squares are homotopy pullback:

and denote by $\phi_P$ the map defined as in [4] for the top square. In a similar way as in Proposition 5.4, we have

$$\Delta^G_G \circ \phi_{(S \times T)/G} = \pm \phi_P \circ \hat{\Delta}^G_G.$$
References

[1] A. Adem, R. J. Milgram, Cohomology of Finite Groups, Springer-Verlag Grundlehren 309 (2004).
[2] C. Allday and V. Puppe, Cohomological methods in transformation groups, Cambridge Studies in Advanced Mathematics, vol. 32, Cambridge University Press, Cambridge (1993).
[3] J. C. Becker and D. H. Gottlieb, The transfer map and fibre bundles, Topology 14 (1975), 1–12.
[4] K. Behrend, G. Ginot, B. Noohi, and P. Xu, String topology for stacks, Astérisque No. 343 (2012).
[5] D. Benson and J. Carlson, Products in negative cohomology, J. Pure Appl. Algebra 82 (1992), no. 2, 107–129.
[6] M. J. Boardman, Stable homotopy theory (mimeographed), University of Warwick, 1966.
[7] K. Borsuk, Über eine Klasse von lokal zusammenhängenden Räumen, Fundamenta Mathematicae, 19, (1932), 1, 220–242.
[8] M. C. Crabb and I. M. James, Fibrewise homotopy theory, Springer Monographs in Mathematics, Springer, (1998).
[9] M. Chas and D. Sullivan, String Topology, preprint, arxiv:math.GT/9911159.
[10] D. Chataur and L. Menichi, String topology of classifying spaces, J. Reine Angew. Math. 669 (2012), 1–45.
[11] R. Cohen and J. D. S. Jones, A homotopy theoretic realization of string topology, Math. Ann. 324 (2002), no. 4, 773–798.
[12] R. Cohen and J. Klein, Umkehr maps, Homology, Homotopy Appl. 11 (2009), no. 1, 17–33.
[13] T. tom Dieck, Algebraic topology. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich (2008).
[14] J. P. Doeraene, Homotopy pull backs, homotopy push outs and joins, Bull. of the Belg. Math. Soc. Simon Stevin 5-1 (1998), 15–37.
[15] Y. Félix and J. C. Thomas, String topology on Gorenstein spaces, Math. Ann., 345 (2009), no. 2, 417–452.
[16] J. P. C. Greenlees, and J. P. May, Generalized Tate cohomology, Mem. Amer. Math. Soc. 113 (1995), no. 543.
[17] K. Gruher and P. Salvatore, Generalized string topology operations, Proc. Lond. Math. Soc. (3) 96 (2008), no. 1, 78–106.
[18] R. Hepworth and A. Lahtinen, On string topology of classifying spaces, Advances in Mathematics, 281 (2015), pp. 394–507.
[19] P. Holm, Excision and cofibrations. Math. Scand. 19 (1966) 122–126.
[20] M. Kreck, Differential Algebraic Topology: From Stratified to Exotic Spheres, Graduate Studies in Mathematics, Vol. 110 (2010).
[21] E. Lupercio, B. Uribe, and M. A. Xicotencatl, Orbifold string topology, Geom. Topol. 12(4) (2008), 2203–2247.
[22] M. Mather, Pull-backs in homotopy theory, Canad. J. Math. 28 (1976), no. 2, 225–263.
[23] Y. B. Rudyak, On Thom spectra, orientability, and cobordism, Monographs in Mathematics, Springer (1998).
[24] R. E. Stong, Notes on cobordism theory, Princeton Univ. Press (1968).
[25] H. Tamaroi, Stable string operations are trivial. Int. Math. Res. Not. IMRN, 24 (2009), 4642–4685.
[26] H. Tene, On the product in negative Tate cohomology for finite groups, Algebr. Geom. Topol. 12 (2012), no. 1, 493–509.
[27] H. Tene, Stratifieds and Equivariant Cohomology Theories, PhD thesis, University of Bonn (2010).
[28] G. W. Whitehead, Homotopy groups of joins and unions, Transactions of AMS 83(1) (1956), 55–69.
[29] S. Whitehead, Note on a theorem due to Borsuk. Bull. Amer. Math. Soc. 54, (1948), 1125–1132.