A MODEL FOR FRAMED CONFIGURATION SPACES OF POINTS

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ABSTRACT. We study configuration spaces of framed points on compact manifolds. Such configuration spaces admit natural actions of the framed little discs operads, that play an important role in the study of embedding spaces of manifolds and in factorization homology. We construct real combinatorial models for these operadic modules, for compact smooth manifolds without boundary.

1. Introduction

Let $M$ be a smooth manifold of dimension $n$ and let $D^n$ denote the $n$-dimensional disc. Consider the space of embeddings of $n$-discs in $M$,

$$\text{Discs}^n_k(M) = \text{Emb}(D^n \sqcup \cdots \sqcup D^n, M).$$

The collection of spaces $\text{Discs}^n_k(M) \coloneqq \{\text{Discs}^n_k(M)\}_{k \geq 0}$ admits a natural action of the framed little $n$-discs operad, $E^n_{fr}$. The purpose of this paper is to compute the real homotopy type of the right $E^n_{fr}$-module $\text{Discs}^n_k(M)$, by providing combinatorial (graphical) models.

Our result is one step towards a real version of the Goodwillie–Weiss manifold calculus, as we shall briefly outline. The Goodwillie–Weiss manifold calculus interprets embedding spaces in terms of the operadic right modules $\text{Discs}^n_k(M)$. If $M$ and $N$ are smooth manifolds such that $\dim(N) - \dim(M) \geq 3$, then there is a weak equivalence [BW13; Tur13]:

$$\text{Emb}(M, N) \simeq \text{Map}^{h_{E^n_{fr}}}_{\text{mod}}(\text{Discs}^n_k(M), \text{Discs}^n_k(N)).$$

The right-hand side of this weak equivalence is still hard to compute, but one may hope to determine at least its real or rational homotopy type as follows. Denote by $\Omega(E^n_{fr})$ a cooperad in differential graded commutative algebras quasi-isomorphic to differential forms on $E^n_{fr}$, and similarly by $\Omega(\text{Discs}^n_k(M))$, and $\Omega(\text{Discs}^n_k(N))$ cooperadic comodules quasi-isomorphic to differential forms on $\text{Discs}^n_k(M)$ and $\text{Discs}^n_k(N)$. Then one has a natural map

$$\text{Map}^{h_{E^n_{fr}}}_{\text{mod}}(\text{Discs}^n_k(M), \text{Discs}^n_k(N)) \rightarrow \text{Map}^{h}_{\Omega(E^n_{fr})-\text{comod}}(\Omega(\text{Discs}^n_k(N)), \Omega(\text{Discs}^n_k(M))).$$

In good cases, one may hope that the right-hand side of (1) can be effectively computed, and that the map is a real or rational homotopy equivalence. For example, the case of higher dimensional long knots ($M = \mathbb{R}^n$, $N = \mathbb{R}^m$) has been successfully implemented in [FTW17]. The goal of the present paper is hence to contribute to the solution of the general case by providing combinatorial models for $\Omega(\text{Discs}^n_k(M))$, with $n = \dim M$.

Let us briefly describe our results. For technical reasons we will be working with a configuration space version of $\text{Discs}^n_k(M)$. The ordered configuration space of $k$ points on $M$ is given by

$$\text{Conf}_k(M) \coloneqq \{(x_1, \ldots, x_k) \in M^k \mid x_i \neq x_j, \text{ for } i \neq j\}.$$ 

Let us fix a Riemannian metric on $M$. The configuration space of $k$ framed points on $M$, $\text{Conf}_k^fr(M)$, consists of configurations in $\text{Conf}_k(M)$ with the additional prescription of positively oriented orthonormal bases of the tangent spaces at each of the points in the configuration,

$$\text{Conf}_k^fr(M) \coloneqq \{(x, B_1, \ldots, B_k) \mid x \in \text{Conf}_k(M), B_i \text{ oriented orthonormal basis of } T_{x_i}M\}.$$ 

Then one has a natural weak equivalence

$$\text{Discs}^n_k(M) \xrightarrow{\simeq} \text{Conf}_k^fr(M)$$

sending a configuration of discs to the configuration of their centers, equipped with the orthonormalization of the push-forward of the standard frame at the center of the discs. Furthermore we consider the Fulton–MacPherson–Axelrod–Singer compactification $\text{FM}_M$ of $\text{Conf}(M)$ and $\text{FM}_M^fr$ of $\text{Conf}^fr(M)$. The

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latter object carries a natural action of a Fulton-MacPherson-Axelrod-Singer-version of the framed $E_n$-operad $\text{FM}^\text{fr}_n$.

Some of the authors described in [CW16; Idr16] a graphical real model $\text{Graphs}_M(k)$ for $\text{Conf}_k(M)$. Elements of $\text{Graphs}_M(k)$ are linear combinations of undirected diagrams with $k$ numbered "external" vertices, some further "internal" vertices, and zero, one, or more decorations in $H(M)$ at each vertex:

\[ \omega_1 \quad \omega_2 \quad \omega_3 \]

There is a quasi-isomorphism
\[ \text{Graphs}_M(k) \to \Omega_{PA}(\text{FM}_M(k)) \]
into the (piecewise semi-algebraic) differential forms on $\text{FM}_M$ given by natural “Feynman rules”. In this paper we extend $\text{Graphs}_M$ to a model $\text{Graphs}_M^{\text{fr}}$ for the framed configuration space, which is merely obtained by (essentially) further decorating each external vertex by a model of $\text{SO}(n)$. Our graph complex comes with a zigzag:
\[ \text{Graphs}_M^{\text{fr}}(k) \leftrightarrow \cdots \to \Omega_{PA}(\text{FM}_M^{\text{fr}}(k)). \]

We furthermore describe an explicit cooperadic coaction of a graphical model $\text{Graphs}_n$ of $E_n$ on $\text{Graphs}_M^{\text{fr}}$. Our main result is then that this combinatorial coaction indeed models the desired topological action of $E_n$ on $\text{FM}_M$.

**Theorem 1** (See Theorem 22). The zigzag of maps $\text{Graphs}_n^{\text{fr}}(k) \leftrightarrow \cdots \to \Omega_{PA}(\text{FM}_M^{\text{fr}}(k))$ is a weak equivalence, and it is compatible with the action of $E_n$ on $\text{FM}_M^{\text{fr}}$.

We furthermore adopt an alternative viewpoint, which serves as an intermediary result in the proof of the theorem above. In general, the non-framed configuration spaces $\text{FM}_M$ do not carry an action of the little discs or Fulton–MacPherson operads. However, there exists a fiberwise version $\text{FM}_M^{\text{nfr}}$ of the Fulton–MacPherson operad. It is an operad in topological spaces over $M$, and the fiber over $x \in M$ is a compactification of the space of configurations of points in the tangent space $T_xM$. The collection $\text{FM}_M^{\text{nfr}}$ is equipped with an action of $\text{FM}_M$, almost tautologically.

Our second main result is to upgrade the dg commutative algebra model $\text{Graphs}_M$ of $\text{FM}_M$ so as to capture also the action of $\text{FM}_M^{\text{nfr}}$ on $\text{FM}_M^{\text{fr}}$.

**Theorem 2** (See Theorem 18). There is a graph complex $\text{Graphs}_M^n$, which is a model for $\text{FM}_M$, and which acts on the model $\text{Graphs}_M^{\text{fr}}$ of $\text{FM}_M$ from [CW16], so that the map (2) respects the (homotopy) comodule structures on both sides.

**Remark 3.** In this paper we will deviate notationally from [CW16] and denote the graphical commutative coalgebra model of the configuration space by $\text{Graphs}_M$ instead of $\star \text{Graphs}_M$. We hope that no confusion arises.

## 2. Background and Recollections

### 2.1. The Fulton-MacPherson-Axelrod-Singer compactifications

The configuration spaces of a manifold, even a compact one, are generally not compact. One way to fix this is through the Fulton–MacPherson–Axelrod–Singer compactification process. We will only give a quick account and refer to [FM94; AS94; Sin04; LV14] for more details.

First, let us consider $M = \mathbb{R}^n$. We can first mod out the translations and the positive rescaling in $\text{Conf}_k(\mathbb{R}^n)$ to obtain the space $\text{Conf}_k(\mathbb{R}^n)/\mathbb{R}^n \times \mathbb{R}_{>0}$, which is a manifold of dimension $nk - n - 1$ if $k \geq 2$ (otherwise it is reduced to a point). The Fulton–MacPherson compactification $\text{FM}_n(k)$ is a stratified manifold whose interior is $\text{Conf}_k(\mathbb{R}^n)/\mathbb{R}^n \times \mathbb{R}_{>0}$, and the inclusion is a homotopy equivalence. The elements of $\text{FM}_n(k)$ can be seen as configurations of $k$ points in $\mathbb{R}^n$, where the points are allowed to become “infinitesimally close” to each other. The collection $\text{FM}_n = \{ \text{FM}_n(k) \}_{k \geq 0}$ of all these spaces assembles to form a topological operad, the Fulton–MacPherson operad, obtained by considering “insertion” of infinitesimal configurations. The element obtained from the operadic composition in the picture below can be interpreted as follows: the points 1, 5 and 6 are infinitesimally close, and so do the points 2, 3 and 4. Moreover, the distance between the points 2 and 4 is infinitesimally small compared to the distance between 2 and 3.
Remark 4. This operad is weakly equivalent to the better-known little discs operad, i.e. it is an $E_n$-operad.

Let us now consider the case of $M$ being a closed $n$-manifold. The compactification $FM_M(k)$ is again a stratified manifold, with interior $Conf_k(M)$ (with no quotient), and the inclusion is a homotopy equivalence. Elements of $FM_M(k)$ can also be seen as configurations of $k$ points in $M$ where points can become infinitesimally close to each other. When they do become infinitesimally close, we see them as defining an infinitesimal configuration in the tangent space of $M$ at their location. If $M$ is framed, i.e. if we can coherently identify the tangent space at every point of $M$ with $\mathbb{R}^n$, then we can insert an infinitesimal configuration from $FM_n$ into a configuration of $FM_M$ and thus obtain the structure of a right operadic $FM_n$-module on $FM_M$.

We can restrict the canonical projections $p_i : M^k \to M$ (for $1 \leq i \leq k$) to $Conf_k(M)$, and then extend them to the compactification:

$$p_i : FM_M(k) \to M,$$

$$1 \leq i \leq k.$$

2.2. Homotopy (co)operads and (co)modules, rational homotopy theory of operads. A basic technical problem in the rational (or real) homotopy theory of operads is that for a topological operad $T$ the (PL or smooth, if defined) differential forms $\Omega(T)$ do not form a cooperad. This is due to the functor $\Omega$ being lax monoidal, but not oplax monoidal, so that the cocomposition maps are encoded by a zigzag

$$\Omega(T(k + l - 1)) \to \Omega(T(k) \times T(l)) \xrightarrow{\sim} \Omega(T(k)) \otimes \Omega(T(l)).$$

However, there is no natural direct map from the left to the right as would be required for a cooperad. One can use one of three workarounds for this problem: (i) use homotopy operads as in [LV14; KW17]; (ii) alter the functor $\Omega$ as in [Fre17a; Fre17b]; or (iii) work with topological vector spaces and the projectively completed tensor product so that the right-hand arrow above becomes an isomorphism. We will follow here the first approach, using an “ad hoc” notion of homotopy operad proposed in [LV14], see [KW17] for more details.

Concretely, let $Tree$ be category whose objects are forests of rooted trees from [KW17], and whose morphisms are generated by contracting edges of trees, and by cutting edges. The category $Tree$ is symmetric monoidal, the monoidal product being the disjoint union of forests.
We say that a (nonunital) homotopy operad in a symmetric monoidal category $C$ with weak equivalences is a symmetric monoidal functor

$$\mathcal{P} : \text{Tree} \to C,$$

such that all cutting morphisms are sent to weak equivalences. Concretely, a (nonunital) homotopy operad consists of the data

- For every tree $T$ an object $\mathcal{P}(T)$, on which the automorphisms of $T$ act.
- For every edge contraction $T \to T'$ (a map of trees) we have a corresponding map $\mathcal{P}(T) \to \mathcal{P}(T')$.
- If $T$ is obtained by grafting $T_1$ and $T_2$ then we have a weak equivalence $\mathcal{P}(T) \sim \mathcal{P}(T_1) \otimes \mathcal{P}(T_2)$.

These data must satisfy natural compatibility conditions. Any ordinary (nonunital) operad $\mathcal{P}$ is also a homotopy operad, by setting

$$\mathcal{P}(T) := \bigotimes_{v \in \text{vertices of } T} \mathcal{P}(\lfloor v \rfloor)$$

to be the treewise tensor product, the “contraction” morphism to agree with the operadic composition and the “cutting” maps are the isomorphisms

$$\mathcal{P}(T) \cong \mathcal{P}(T_1) \otimes \mathcal{P}(T_2).$$

There is also a variant for unital operads. One may define a category $\text{Tree}_*$ (see [KW17]) similar to $\text{Tree}$, but where in addition the trees may have a special type of univalent vertex representing the identity. A unital homotopy operad is then a symmetric monoidal functor $\text{Tree}_* \to C$.

Dually we define a homotopy cooperad in $C$ as a contravariant symmetric monoidal functor

$$C : \text{Tree} \to C^{\text{op}}.$$

The main example is as follows: Suppose $\mathcal{T}$ is a topological operad. Then the (PL) forms $\Omega(\mathcal{T})$ form a homotopy cooperad in the category $\text{Dgca}$ of dg commutative algebras. We will call such objects homotopy Hopf cooperads for short. The corresponding functor

$$\Omega(\mathcal{T}) : \text{Tree} \to \text{Dgca}$$

is defined such that

$$\Omega(\mathcal{T}) : T \mapsto \Omega(\times_T \mathcal{T}).$$

The contraction morphisms are the pullbacks of composition morphisms in $\mathcal{T}$ and the “cutting” morphisms are the natural maps

$$\Omega(T_1) \otimes \Omega(T_2) \rightarrow \Omega(T).$$

We will also work with the corresponding notion of homotopy operadic right modules. Let $\text{Tree}_*$ be a category whose objects are forests with one marked tree. The morphisms are generated by edge contractions and edge cuts. Cutting an edge in the marked tree will leave the upper (closer to the root) subtree marked, and the other subtree unmarked. The category $\text{Tree}_*$ is naturally a monoidal category module over $\text{Tree}$. Now suppose that

$$\mathcal{P} : \text{Tree} \to C$$

is a homotopy operad in the symmetric monoidal category $C$ (i.e., a symmetric monoidal functor), then a homotopy right operadic $\mathcal{P}$-module $\mathcal{M}$ is a functor

$$\mathcal{M} : \text{Tree}_* \to C$$

so that the pair $(\mathcal{P}, \mathcal{M})$ respects the given structure, and such that all cutting morphisms are sent to weak equivalences. More precisely, $\mathcal{M}$ is specified by the following data

1. A collection of objects $\mathcal{M}(T)$ for every (marked) tree $T$.
2. Contraction morphisms $\mathcal{M}(T) \to \mathcal{M}(T')$.
3. Cutting morphisms (weak equivalences) $\mathcal{M}(T) \to \mathcal{M}(T_1) \otimes \mathcal{P}(T_2)$.

Every operadic right module $\mathcal{M}$ over an operad $\mathcal{P}$ is in particular a homotopy operadic right module.

Dually, we define the notion of homotopy cooperadic right comodule. In particular we will consider homotopy cooperadic right comodules in the category $\text{Dgca}$, which we call Hopf right comodules. The main example will be as follows. Let $\mathcal{T}$ be again a topological operad, and $\mathcal{M}$ a topological operadic
right $\mathcal{T}$-module. Then the (PL) forms $\Omega(\mathcal{T})$ form a homotopy Hopf cooperad, as we saw. Furthermore the forms $\Omega(M)$, defined such that for a (marked tree)

$$T = T_1 \cdots T_r$$

we have

$$\Omega(M)(T) := \Omega(M(r) \times T_1 \cdots T_r),$$

naturally form a homotopy Hopf right comodule for $\Omega(\mathcal{T})$.

We will not fully develop the homotopy theory of homotopy (Hopf) (co)modules here. We just say that we equip the category of homotopy right comodules with a structure of a homotopical (or $\infty$-)category by declaring the weak equivalences to be the morphisms that are objectwise weak equivalences.

For us, understanding the “naive” homotopy type of the topological operad $\mathcal{T}$ acting on the topological operadic right module $M$ shall mean understanding the weak equivalence class (quasi-isomorphism type) of the pair consisting of the homotopy Hopf cooperad $\Omega(\mathcal{T})$ and its homotopy Hopf comodule $\Omega(M)$. For us a model of $(\mathcal{T}, M)$ shall be a pair consisting of a homotopy Hopf cooperad and a homotopy right comodule, such that the pair can be connected to $(\Omega(\mathcal{T}), \Omega(M))$ by a zigzag of quasi-isomorphisms.

We finally remark that a “proper” rational homotopy theory of topological operads has been developed by B. Fresse [Fre17a, Fre17b, Fre18]. Concretely, he constructs a model category structure on (ordinary) Hopf cooperads, together with a Quillen adjunction with the category of topological operads. Furthermore, he shows that morphisms of homotopy Hopf cooperads in our sense may be lifted to morphisms of ordinary dg Hopf cooperads in his framework, thus embedding our computations in a more satisfying homotopy theoretical framework.

**Remark 5.** We want to emphasize that the notion “homotopy operad” is a bit of a misnomer, since homotopy operads are not objects in the homotopy category of operads. “Lax operad” could be a better name.

### 2.3. Formality of $\text{FM}_n$.

The little discs operads are known to be formal over $\mathbb{Q}$, i.e. their rational cohomology completely determines their rational homotopy type as operads [Kon99, Tam03, LV14, Pet14, FW15]. There are several methods to prove this result. Here we recall the one pioneered by Kontsevich (which works over $\mathbb{R}$), based on graphical models, and that was recently applied to closed manifolds [CW16, Idr16] (see also Section 2.4) and compact manifolds with boundary [CILW18] by some of the authors and Lambrechts.

For a topological operad $P$ of finite cohomological type, its cohomology $H^*(P)$ (e.g. over $\mathbb{R})$ is naturally a Hopf cooperad, i.e. a cooperad in the category of commutative differential graded algebras (here, with a trivial differential). The forms on $P$ (for a suitable notion of “forms”) $\Omega^*(P)$ are a homotopy Hopf cooperad. The formality of $\text{FM}_n$ is then the statement that $H^*(\text{FM}_n)$ and $\Omega^*(\text{FM}_n)$ are quasi-isomorphic, i.e., they can be connected by a zigzag of quasi-isomorphisms of homotopy Hopf cooperads.

To set the notation, recall that the cohomology of $\text{FM}_n(k) \simeq \text{Conf}_k(\mathbb{R}^n)$ is given by the following algebra, with generators $\omega_{ij}$ of degree $n - 1$:

$$H^*(\text{FM}_n(k)) = S(\omega_{ij})_{1 \leq i, j \leq k} / (\omega_{ii} = \omega_{ij}^2 = \omega_{ijk}\omega_{ki} + \omega_{jki}\omega_{ij} = 0).$$

Kontsevich [Kon99] built a Hopf cooperad $\text{Graphs}_n$ to connect $H^*(\text{FM}_n)$ with the forms on $\text{FM}_n$ as follows. Elements of $\text{Graphs}_n(k)$ are linear combinations of graphs with two types of vertices: “external” vertices, numbered from 1 to $k$, and an arbitrary number of “internal” vertices, undistinguishable and usually drawn in black. The edges are formally directed, but an edge is identified with $(-1)^n$ times its opposite edge, so we will usually not draw the orientation. The total degree of a graph is $(n - 1)$ times the number of edges, minus $n$ times the number of internal vertices. We mod out by graphs containing connected components with only internal vertices. The differential of a graph is obtained as a sum over all possible ways of contracting an edge connected to an internal vertex. The product glues graphs along external vertices, while the cooperad structure maps collapse subgraphs.

**Remark 6.** To be consistent with what follows, note that we explicitly allow “tadpoles”, i.e. edges between a vertex and itself, as well as double edges, and we do not impose any condition on the valence of internal vertices. We then must clarify that in the differential, the so-called “dead ends”, i.e. edges connected to a univalent internal vertex, are not contracted.
Remark 7. The integrals are the reason that we are forced to work with PA forms. Indeed, the projections \( \text{FM}_n(k + l) \to \text{FM}_n(k) \) are not submersions in general, so we may not work with usual de Rham forms. However, they are semi-algebraic bundles.

Theorem 8 ([Kon99; LV14]). The morphisms defined above are quasi-isomorphisms of homotopy Hopf cooperads:

\[
H^*(\text{FM}_n) \xrightarrow{\sim} \text{Graphs}_n \xrightarrow{\sim} \Omega^*_\text{PA}(\text{FM}_n).
\]

Let us also recall the definition of the graph complex \( GC_n \). As a vector space, \( GC_n \) is spanned by connected graphs with only internal vertices. Edges are directed, but the elements of \( GC_n \) must be invariant under edge reversal, with a coefficient \(-1\) when an edge is reversed. Thus we draw undirected edges in pictures, which are to be understood as the sum of an edge with its symmetric (with a sign). Given a graph \( \gamma \in GC_n \) with \( e \) edges and \( v \) vertices, its (homological) degree is \( k(n - 1) - en + n \). The differential is dual of the differential in \( \text{Graphs}_n \) and splits vertices in two, summing over all possible ways of reconnecting incident edges to the two vertices. There is a (pre-)Lie algebra structure on \( \text{GC}_n \) given by insertion of graphs.

2.4. Graphical models for \( \text{FM}_M \). The methods described in Section 2.3 to build real models for \( \text{Conf}_k(\mathbb{R}^n) \) were enhanced by some of the authors to describe real models for \( \text{Conf}_k(M) \) when \( M \) is a closed manifold [CW16; Idr16]. We give here a quick account of the model found in the first reference.

The goal is to build a sequence of dgcas \( \text{Graphs}_{\text{M}}(k) \), equipped with an operadic right \( \text{Graphs}_{\text{M}} \)-comodule structure when \( M \) is framed. Just like \( \text{Graphs}_n \), the space \( \text{Graphs}_{\text{M}}(k) \) is spanned by graphs with two types of vertices: external vertices, numbered \( 1, \ldots, k \), and indistinguishable internal vertices of degree \(-n \). The edges are again undirected and of degree \( n - 1 \). Each vertex is decorated by zero, one, or more elements of the reduced cohomology \( \tilde{H}^*(M) \), in other words, by an element of the free unital symmetric algebra \( S(\tilde{H}^*(M)) \), and each decoration increases the total degree of the graph.

The differential \( \delta \) is a sum \( \delta_{\text{contr}} + \delta_{\text{cut}} \).
The contracting part $\delta_{\text{contr}}$ is the sum of all possible ways of contracting edges connected to an internal vertex, multiplying the decorations (in the free symmetric algebra). Note that dead ends are contractible here.

The cutting part $\delta_{\text{cut}}$ is the sum over all possible ways of cutting an edge and multiplying the endpoints of the edge by the diagonal class $\Delta_M \in H^*(M)^{\otimes 2}$. Recall that given a graded basis $\{e_i\}$ of $H^*(M)$ the diagonal class is expressed as follows: if $\{e^*_i\}$ is the dual basis with respect to the Poincaré duality pairing (i.e. $\int_M e_i e^*_j = \delta_{ij}$) then

$$(7) \quad \Delta_M := \sum_i (-1)^{|e_i|} e_i \otimes e^*_i.$$ 

Finally, there is a “partition function” $Z_M: GC^*_M \to \mathbb{R}$ which assigns a real number to graphs with only internal vertices, which is in general hard to compute. For example, if $\gamma$ is a graph with exactly one vertex and decorations $\alpha_1, \ldots, \alpha_k \in \hat{H}^*(M)$, then $Z_M(\gamma) = \int_M \alpha_1 \wedge \cdots \wedge \alpha_k$. Then in the definition of $\text{Graphs}_M$, a graph $\Gamma$ with a connected component $\gamma$ with only internal vertices is identified with $Z_M(\gamma) \cdot (\Gamma \setminus \gamma)$.

Moreover, if $M$ is framed, or more generally if the Euler class of $M$ vanishes, then there is an operadic right $\text{Graphs}_n$-comodule structure on $\text{Graphs}_M$, given by subgraph collapsing (multiplying all the decoration of the collapsed subgraph in the process).

To define the quasi-isomorphism $\text{Graphs}_M \to \Omega^*_\mathcal{PA}(FM_M)$, one first chooses representatives of the cohomology of $M$ via an injective quasi-isomorphism of chain complexes $\iota: H^*(M) \to \Omega^*_\mathcal{PA}(M)$. We will generally suppress it from the notation, viewing $H^*(M)$ as a subcomplex of $\Omega^*_\mathcal{PA}(M)$. Then there exists a “propagator”, a form $\varphi \in \Omega^{n-1}_\mathcal{PA}(FM_M(2))$, which satisfies the following properties:

- it is (anti-)symmetric, i.e. $\varphi^{2n} = (-1)^n \varphi$;
- its differential $d\varphi$ is the pullback of $\Delta_M$ under the canonical projection $FM_M(2) \to M^2$;
- its restriction to $\partial FM_M(2)$, which is a sphere bundle over $M$, is a global angular form (i.e. its integral on every fiber is 1);
- for all $\alpha \in H^*(M)$, one has $\int_y \varphi(x,y)\alpha(x) = 0$.

Then given a graph $\Gamma \in \text{Graphs}_M(k)$ with $l$ internal vertices, its image in $\Omega^*_\mathcal{PA}(FM_M(k))$ is the following integral along fibers:

$$(8) \quad \omega(\Gamma) := \int_{FM_M(k+l) \to FM_M(k)} \bigwedge_{(i,j) \in E_{\Gamma}} p_{ij}^*(\varphi).$$

**Theorem 9 ([CW16]).** The morphism described above define a quasi-isomorphism of dgcas:

$$\omega: \text{Graphs}_M(k) \xrightarrow{\sim} \Omega^*_\mathcal{PA}(FM_M).$$

If moreover $M$ is framed then this is compatible with the operadic comodule structure, respectively over $\text{Graphs}_n$ and $\Omega^*_\mathcal{PA}(FM_n)$.

For future use, note that

$$(9) \quad A := \text{Graphs}_M(1) \xrightarrow{\sim} \Omega^*_\mathcal{PA}(M)$$

is a real model for $M^1$ (if $M$ is not simply connected then this is a “naive” model, and we potentially need more information to recover the full real homotopy type of $M$). Moreover, we have maps:

$$(10) \quad A^{\otimes k} \to \text{Graphs}_M(k),$$

obtained by gluing the graphs at each external vertex, which represent the projection of Equation (3).

If $\dim M$ is even, then we have a canonical representative $E \in A$ of the Euler class of $M$. Recall the graded basis $\{e_i\}$ and dual basis $\{e^*_i\}$ of $H^*(M)$. Then our representative of the Euler class is given by a sum of graphs with two decorations:

$$(11) \quad E := \sum_i (-1)^{\deg e_i} e_i \odot e^*_i.$$ 

If $\dim M$ is odd then we merely set $E := 0$ for notational consistency later.

**Remark 10.** When $M$ is simply connected, it would be possible to replace $S(\hat{H}^*(M))$ by a Poincaré duality model $A$ of $M$, as is done in [Idr16]. However, this would add some technical complications due to the fact that there is no direct map $A \to \Omega^*_\mathcal{PA}(M)$ in general, so we will not go down this path. Most of the constructions below would work similarly.

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1One can take instead the smaller model $A = \text{Graphs}^\text{tree}_M(1)$, the subalgebra of $\text{Graphs}_M(1)$ consisting of graphs of genus zero since the inclusion $\text{Graphs}^\text{tree}_M(1) \to \text{Graphs}_M(1)$ is a quasi-isomorphism [CW16].
2.5. Equivariant graphical models. Throughout the paper we will abbreviate $G = SO(n)$ to shorten notation. There is an action of $G$ on $FM_n$ induced by the canonical action of $G$ on $\mathbb{R}^n$. The framed Fulton–MacPherson operad $FM_n^F$ is then obtained as the “framing product” or “semi-direct product” [SW03] of $FM_n$ with $G$:

\[(12) \quad FM_n^F := FM_n \circ G = \{FM_n(k) \times G^k\}_{k \geq 0}.
\]

The action of the group $G = SO(n)$ on $FM_n$ is not directly apparent on the model $Graphs_n$. Let us not describe it. To describe this action, consider the abelian Lie algebra

\[(13) \quad g = \bigoplus_{i \geq 0} \pi_i(SO(n)) \otimes \mathbb{R} = \begin{cases} \mathbb{R} p_3 \oplus \mathbb{R} p_5 \oplus \cdots \oplus \mathbb{R} p_{2n-7} \oplus \mathbb{R} E, & \text{if } n \text{ is even;} \\ \mathbb{R} p_3 \oplus \mathbb{R} p_5 \oplus \cdots \oplus \mathbb{R} p_{2n-7} \oplus \mathbb{R} p_{2n-3}, & \text{if } n \text{ is odd;}
\end{cases}
\]

where the elements $p_i$, living in degree $i$, are the Pontryagin classes and $E$, living in degree $n - 1$, is the Euler class. The action $G$ on $FM_n$ may be described by an $L_\infty$-action of $g$ on $Graphs_n$. This $L_\infty$ action has been identified in [KW17], and factors through the action of the graph complex $GC_n$ from Section 2.3. Concretely, the graph complex $GC_n$ acts on $Graphs_n$ by cooperadic bi-derivations, i.e., compatibly with the Hopf cooperad structure. Then an $L_\infty$-action of $g$ on $Graphs_n$ that factors through $GC_n$ may be described by a Maurer–Cartan element:

\[(14) \quad m \in C(g) \otimes GC_n = H(BG) \otimes GC_n.
\]

In [KW17] an explicit configuration space formula is given for $m$. Furthermore, the gauge equivalence type of $m$ is identified.

**Theorem 11** (Theorems 1.2, 1.3 of [KW17]). The Maurer-Cartan element $m \in H(BG) \otimes GC_n$ is gauge equivalent to

\[(15) \quad \begin{cases} E \mathcal{V}, & \text{for } n \geq 2 \text{ even;} \\ \sum_{j \geq 1} \frac{p_{2j-2}}{j} \frac{1}{(2j+1)!} \mathcal{V}^j, & \text{for } n \geq 3 \text{ odd;}
\end{cases}
\]

where (up to a normalization factor) $E \in H(BG)$ is the Euler class and $p_{2n-2} \in H(BG)$ is the top Pontryagin class.

We may lift the $L_\infty$-action of $g$ on $Graphs_n$ to an honest dg Lie action of a resolution

\[\hat{g} \xrightarrow{\sim} g.
\]

Concretely, $\hat{g}$ can be taken to be a quasi-free Lie algebra generated by the augmentation ideal of $H(BG)$, i.e. the cobar construction on $g$ on the Koszul dual (cocommutative coalgebra) of $g$. It is possible to take a smaller resolution given the particular expression of $m$ in Equation (15), cf. [KW17]). We furthermore have the identification $H(G) = U\hat{g}^*$, where $U$ denotes the universal coenveloping coalgebra, which is a cocommutative and cocommutative Hopf algebra. We similarly define the Hopf algebra $H(G) := U\hat{g}^*$.

Via the action of $\hat{g}$ we may equip $Graphs_n$ with a $\hat{H}(G)$-coaction. This is the model of [KW17] of $FM_n^F$ as an operad in $G$-spaces. To obtain a model for the framed little discs operads $FM_n^F$ one can take the semidirect product

\[Graphs_n \circ \hat{H}(G).
\]

Unfortunately, there is no direct map between the above models and the forms $\Omega^\bullet_{PA}(FM_n^F)$. The construction in [KW17] instead yields a zigzag of quasi-isomorphisms. In this paper we shall also need intermediate objects in this zigzag. In particular, one model for the equivariant forms on a $G$-space $X$ is given by the following dga $[KW17$, Section 4.1]:

\[(16) \quad \Omega^\bullet_G(X) := \text{Tot} \Omega(G^\bullet \times X) = \int_{n \in \Delta} \Omega(G^\bullet \times X) \otimes \Omega^\bullet_{\partial}(\Delta^\bullet),
\]

where $G^\bullet \times X = B_\bullet(\ast, G, X)$ is the simplicial bar construction and Tot is actually the fat totalization, i.e., the limit is only over $\Delta^+$, the cosimplicial category with objects $\underline{n} = \{1, \ldots, n\}$ ($n \geq 0$) and morphisms are strictly increasing maps. In particular,

\[(17) \quad B_G := \Omega^\bullet(\ast) = \text{Tot} \Omega(G^\bullet)\]
is a model for $*//G = BG$, and there is a quasi-isomorphism of dgcas $H(BG) \to BG$. The dgca $\Omega^* (X)$ is a $BG$ module. However, the category of $BG$-modules is only symmetric monoidal up to homotopy; to correct this, one can instead consider its free resolution given by the two-sided bar construction:

$$\Omega^*_G (X) := BG \otimes_{BG} \Omega_G^*(X) = \left( \bigoplus_{k \geq 0} BG \otimes (BG[1])^\otimes k \otimes \Omega_G^*(X) , d \right).$$

The first step in [KW17] is to find a model for the equivariant forms on $FM_n$. The model is denoted $\text{BGraphs}_n^m$. As a graded vector space,

$$(19) \quad \text{BGraphs}_n^m = \text{Graphs}_n \otimes H(BG).$$

The commutative algebra structure is defined term-wise. It is a cooperad in dgca over $H(BG)$, where the monoidal product is $\otimes_{H(BG)}$. The differential is the sum of the differential from $\text{Graphs}_n$ with a twist by a certain Maurer–Cartan element $m \in H(BG) \otimes GC_n$, which is gauge equivalent to an element with an explicit formula [KW17, Theorem 7.1]. Then there is a direct quasi-isomorphism [KW17, Theorem 6.7]

$$(20) \quad \omega_{\text{equivar}} : \text{BGraphs}_n^m \xrightarrow{\sim} \Omega_G^*(FM_n)$$

given by integral formulas similar to the ones of Section 2.3, using an “equivariant propagator” in $\Omega_G^{-1}(FM_n)(2)$ with an explicit formula [KW17, Appendix A] and Appendix A.

The second step in [KW17] is to recover a model for $\Omega(FM_n)$ together with its action of $G$. There is a (homotopy) pullback square:

$$\begin{array}{ccc}
FM_n & \xrightarrow{\iota} & EG \\
\downarrow & & \downarrow \\
FM_n // G & \xrightarrow{\iota} & BG.
\end{array}$$

Therefore, by the “pullback-to-pushout principle” [Hes07, Theorem 2.4] and the fact that $BG$ is simply connected, a model for $FM_n$ is given by a pushout of the models of the three other spaces in the diagram. The model of $FM_n // G$ is $\text{BGraphs}_n^m$ defined above. The model for $BG$ is merely $H(BG)$. Finally, the model for $EG$ is given by the “Koszul complex”,

$$K_G := (H(BG) \otimes H(G) , d_\varepsilon),$$

defined as follows. For any compact Lie group $G$, we have $H(BG) = \mathbb{R}[\alpha_1 , \ldots , \alpha_r]$ and $H(G) = \Lambda(\beta_1 , \ldots , \beta_r)$ for some classes with $\deg \alpha_i = \deg \beta_i + 1$. Then the complex $K_G$ is equipped with the differential such that $d_\varepsilon(\beta_i) = \alpha_i$. There is a homotopy $h_\kappa$ such that $h_\kappa(\alpha_i) = \beta_i$, and one checks easily that

$$d_\varepsilon h_\kappa + h_\kappa d_\varepsilon = \text{id}_{K_G} - \varepsilon(-) \cdot 1,$$

where $\varepsilon : K \to \mathbb{R}$ is the augmentation.

Using the pullback-to-pushout principle, we may then consider the tensor product:

$$B_n := K_G \otimes_{H(BG)} \text{BGraphs}_n^m.$$

We would like to connect it to $\Omega(FM_n)$, taking the action of $H(G)$ into account. However, there is no direct map, and the zigzag is built using the following method.

Like all Lie groups, $G$ is formal as a space, and there exists a direct quasi-isomorphism of dgcas $H(G) \xrightarrow{\sim} \Omega(G)$ (defined by choosing any closed representative of the classes $\beta_i$ above). However, $\Omega(G)$ is not a Hopf algebra, and the category of $\Omega(G)$-modules is not a symmetric monoidal category, which would cause problems later. One can strictify $\Omega(G)$ into a Hopf algebra using the $W$-construction [KW17, Section 3]. Let $I := \Omega(\Delta^1) = \mathbb{R}[t, dt]$ be a path object for $\mathbb{R}$ in the model category of dgca. Then the $W$-construction of $\Omega(G)$ is given by the end:

$$A_G := W\Omega(G) = \int_{n \in \Delta_+} \Omega(G^n) \otimes I^\otimes (n-1).$$

The category $\Delta_+$ of finite ordinals and strictly increasing maps acts on $\Omega(G^*) \otimes I^\otimes *$, using the product of $G$, the product of $I$, and the evaluations at 0 or 1, $\ev_{0,1} : I = \mathbb{R}[t, dt] \to \mathbb{R}$. One then checks that $A_G$ is a strict Hopf algebra, and there is a categorical quasi-isomorphism of cdgas $A_G \xrightarrow{\sim} \Omega(G)$ Moreover, $H(G) \xrightarrow{\sim} \Omega(G)$ can be lifted to a quasi-isomorphism of Hopf algebras $H(G) \xrightarrow{\sim} A_G$ [KW17, Proposition 4.5].
Similarly, one can consider the $W$ resolution of a homotopy $\Omega(G)$-comodule $X$, turning it into a strict $A_G$-comodule $WX$ (denoted by $\text{mod}_{A_G}(X)$ in the reference).

Let us now consider the simplicial resolution of $FM_n$ as a $G$-space obtained by considering the bar complex:

$$\overline{FM}_n^\bullet := G \times G^\bullet \times FM_n \to FM_n.$$  

(26)

Then there is a zigzag of quasi-isomorphisms compatible with the $G$-action [KW17, Theorem 5.5]:

$$B_n := K_G \otimes_{\hat{H}(BG)} B\text{Graphs}_n^m \xrightarrow{\sim} \text{Tot} \Omega(\overline{FM}_n^\bullet) \xleftarrow{\sim} \Omega(\overline{FM}_n) \xrightarrow{\sim} \Omega(FM_n),$$  

(27)

where the first map is defined by integral formulas, the second map is dual to the resolution $\overline{FM}_n^\bullet \to FM_n$, and the last map is the $W$-resolution.

The leftmost object in this diagram is quasi-isomorphic to $\text{Graphs}_n$, with the $\hat{H}(G)$-coaction considered in the beginning of this section. To see this, we may define the resolved Koszul complex $\hat{K}_G := (U\hat{g}^* \otimes H(BG), d_n)$ to get

$$\hat{B}_n := \hat{K}_G \otimes_{\hat{H}(BG)} B\text{Graphs}_n^m$$  

(28)

Then we have an explicit zigzag of quasi-isomorphisms of Hopf cooperads in $\hat{H}(G)$-comodules [KW17, Proposition 9.1]:

$$B_n \xrightarrow{\sim} \hat{B}_n \xleftarrow{\sim} \text{Graphs}_n.$$  

(29)

Combining everything, we get a zigzag of Hopf cooperads with a $G$-action:

$$\text{Graphs}_n \xrightarrow{\sim} \hat{B}_n \xleftarrow{\sim} \text{Tot} \Omega(\overline{FM}_n^\bullet) \xleftarrow{\sim} \Omega(\overline{FM}_n) \xrightarrow{\sim} \Omega(FM_n).$$  

(30)

The final step in [KW17] is to show that $(K_G \otimes_{\hat{H}(BG)} B\text{Graphs}_n^m) \circ H(G)$ can be connected to $\Omega(FM_n^\fr)$ by a zigzag of quasi-isomorphisms. This uses explicit $W$-resolutions of (co)modules over (co)operads. We use a similar construction in the proof of Theorem 22.

**Remark** 12. In [KW17], the group considered is the full orthogonal group $O(n)$. This adds difficulties, as $O(n)$ is disconnected. Compared to what we have written here, there is an additional step required, consisting of considering invariants under the action of $O(n)/SO(n) \cong \{\pm 1\}$. In what follows, we will only consider the $SO(n)$-action and oriented manifolds for simplicity. In order to obtain results for unoriented manifolds, one should consider the unoriented frame bundle $\text{Fr}_M^\text{unor}$ instead of the oriented frame bundle $\text{Fr}_M$, consider $SO(n)$-equivariant forms, and take the extra step of considering invariants under the action of $\{\pm 1\}$.

### 3. THE FIBER-WISE LITTLE DISCS OPERAD

**3.1. Motivation.** Let $M$ be an oriented manifold. In general, if $M$ is not framed, then the spaces $FM_n^M$ do not form a right $FM_n$-module. Indeed, in order to insert an infinitesimal configuration in a point $x \in M$, one needs to identify the tangent space $T_x M$ with $\mathbb{R}^n$. If $M$ is not framed, there is no way to do this coherently for all $x \in M$.

To correct this, we build a new operad $FM_n^M$ in topological spaces over $M$, which we call the fiber-wise Fulton–MacPherson operad over $M$. The operad is defined such that the fiber over the map $FM_n^M \to M$ at a point $x$ is (essentially) the Fulton–MacPherson-compactified configuration space of points in the tangent space $T_x M$. Given such an element, one can insert the infinitesimal configuration into the tangent space at $x$ using the given frame, so that we have composition maps:

$$c_i : FM_n^M(r) \times^M_{x} FM_n^M(s) \to FM_n^M(r + s - 1),$$  

(31)

where the pullback on the LHS is obtained by considering the projection $p_i : FM_n^M(r) \to M$ which forgets all but the $i$-th point.

**3.2. Definition.** Let us now describe this operad more precisely and in more generality. Let $G = SO(n)$ and let $Y \to B$ be a principal $G$-bundle – the example that we will care the most about being the oriented frame bundle $Y = \text{Fr}_M$ over $B = M$. We define an operad $FM_n^{Y \to B}$ by:

$$FM_n^{Y \to B}(r) := Y \times_G FM_n,$$  

(32)
keeping in mind that in this notation, the index in $\times_G$ denotes a quotient by the action of $G$, not a pullback. This is an operad in the category $\text{Top}/B$ of spaces over $B$. The unit is given by the identity $B \to \text{FM}^n_M \to B(1) = Y/G \to B$, and the composition is defined by:

$$o_i : \text{FM}^n_M \to B(r) \times_B \text{FM}^n_M \to B(s) \to \text{FM}^n_M \to B(r+s-1)$$

$$([y,c],[y',c']) \mapsto [y,c \circ_i (y/y',c')]$$

where $Y \times_B Y \to G$, $(y,y') \mapsto y/y'$ is defined using the principal bundle structure, and the action of $G$ on $\text{FM}_n$ is by rotations.

Fix some Riemannian metric on $M$, which allows us to define the notion of “orthonormal basis” in tangent spaces of $M$. In the special case that $Y = \text{Fr}_M$ is the oriented orthonormal frame bundle over $M$, we abbreviate the operad defined above to:

$$\text{FM}^n_M := \text{FM}^n_{\text{Fr}_M}.$$  

The object $\text{FM}_M$ carries a structure which we call right operadic multimodule for $\text{FM}^n_M$. Concretely, we have the projections of Equation (3), and natural “insertion” operations

$$\text{FM}_M \circ_M \text{FM}^n_M \to \text{FM}_M$$

that are compatible with the operad structure on $\text{FM}^n_M$. Here, $\circ_M$ denotes the corresponding notion of plethysm given by the fact that $\text{FM}_M(r)$ comes with $r$ maps to $M$, i.e. $\text{FM}_M \circ_M \text{FM}^n_M \subset \text{FM}_M \circ \text{FM}^n_M$ corresponds to the tuples $(\underbrace{c_{\text{FM}^n_M(k)}}_{\text{FM}^n_M}, \ldots, a_k)$ such that for all $i = 1, \ldots, k$, the projection $p_i(c)$ agrees with the corresponding location of $a_i$. In particular, we note that $\text{FM}_M$ is not an operadic right $\text{FM}^n_M$ module in $\text{Top}/M$.

Furthermore we note that the notion of right operadic multimodule has a natural “homotopy” equivalent, similarly to the notion of homotopy operads and modules recalled in section 2.2.

Remark 13. As explained in Remark 12, if we were dealing with an unoriented manifold, we would need to look at the principal $O(n)$-bundle given by the unoriented frame bundle $\text{Fr}^\text{unor}_M$ in the definition of $\text{FM}^n_M$.

Remark 14. It is possible to give the following interpretation of these algebraic structures. On any topological space $X$, there exists a unique coalgebra structure, with comut the unique map $\varepsilon : X \to s$, and coproduct $\Delta : X \to X \times X$ given by the diagonal $\Delta(x) = (x,x)$, which is automatically cocommutative. A right (or left) $X$-comodule $M$ is nothing but a space $M$ equipped with a map $f : M \to X$, as the coaction $M \to M \times X$ is forced to be of the form $m \mapsto (m,f(m))$ by the counit axiom.

Such a coalgebra $X$ naturally defines a cooperad $\underline{X}$ concentrated in arity 1. An operadic left $\underline{X}$-comodule is the same thing as a $\Sigma$-collection $F = \{F(k)\}_{k \geq 0}$ equipped with arbitrary maps $f^k : F(k) \to X$ for all $k \geq 0$. Similarly, an operadic right $\underline{X}$-comodule is the same thing as a $\Sigma$-collection $G = \{G(k)\}_{k \geq 0}$ equipped with arbitrary $g^k_i : G(k) \to X$ for all $k \geq 1$ and $1 \leq i \leq k$ - in other words, it is precisely what we call an $\underline{X}$-multimodule. Given a left $\underline{X}$-comodule $G$ and a right $\underline{X}$-comodule $F$, one can define their usual composition product over the cooperad $\underline{X}$ using pullbacks:

$$(G \o\underline{X} F)(k) := \bigoplus_{l_1 + \ldots + l_r = k} \left( G(r) \times X^{\times r} (F(l_1) \times \ldots \times F(l_r)) \right).$$

Note that a left $\underline{X}$-comodule structure induces a right $\underline{X}$-comodule structure by setting $g^k_i = f^k$ for all $i$. An operad in the category of spaces over $X$ (such as $\text{FM}^n_M$) is then the same thing as a left $\underline{X}$-comodule $P$ equipped with a monoid structure in the category of $\underline{X}$-bicomodule. A right operadic multimodule $F$ (such as $\text{FM}_M$) is a right $\underline{X}$-comodule equipped with a composition map compatible with the operadic structure maps of $P$:

$$F \circ\underline{X} P \to F.$$  

3.3. A model for the frame bundle. The oriented frame bundle $\text{Fr}_M$, like all principal $G$-bundles, fits in a pullback diagram

$$\begin{array}{ccc}
\text{Fr}_M & \longrightarrow & EG \\
\downarrow & & \downarrow \\
M & \longrightarrow & BG
\end{array}$$

where $M \to BG$ classifies the (oriented) tangent bundle of $M$. 

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Recall that we take $A = \text{Graphs}_M(1)$ as a model for $M$ (see Equation (9)). The algebraic version of the above square is the following pushout diagram:

\[
\begin{array}{ccc}
\text{Fr}^\text{alg}_M & \leftrightarrow & K_G \\
\downarrow & & \downarrow \\
A & \leftrightarrow & H(BG)
\end{array}
\]

where $K_G = (H(BG) \otimes H(G), d_k)$ is the almost acyclic “Koszul complex”, see Section 2.5.

The map $H(BG) \to A$ sends the Pontryagin classes of $H(BG) = \mathbb{R}[p_4, p_8, \ldots]$ to graphs with a single external vertex, decorated by some representative of the respective Pontryagin class of $M$. For the Euler class (in case $n$ is even), we send it to the sum of graphs with two decorations ($e_i$ and $e_i^*$) of Equation (11).

This is done so that real dgca model for $\text{FM}_M$ as right $\text{FM}_n^M$-module is compatible with the differentials.

It follows that the algebraic model for the frame bundle is

\[
\text{Fr}^\text{alg}_M := A \otimes H(BG) K = (A \otimes H(G), d),
\]

where the differential takes a generator from $H(G)$ and attaches to the external vertex the corresponding Pontryagin/Euler class.

The map from $\text{Fr}^\text{alg}_M$ is however not directly compatible with the $G$-action. For this reason we will also consider a “resolution” of the frame bundle, namely its $W$ construction as right $G$-space $\text{Fr}_W^M$. This $W$-construction comes with a natural $W$-action. Furthermore, we may extend the map of complete dg Hopf algebras $\hat{H}(G) \to \Omega'(WG) := W(\Omega \text{PA}(G))$ from (essentially) [KW17, Proposition 4.5] by a map of Hopf comodules $\text{Fr}_W^\text{alg}_M := A \otimes \hat{H}(BG) \hat{K} = (A \otimes \hat{H}(G), d) \to \Omega'(\text{Fr}_M^W)$.

3.4. Graphical model for the fiber-wise little discs operad. We now define the dgca model for $\text{FM}_n^M$. It works as follows. If $Y$ is a $G$-space and $X$ a right $G$-space with a free $G$-action, then

\[
\begin{array}{ccc}
X \times_G Y & \rightarrow & Y/G \\
\downarrow & & \downarrow \\
X//G & \rightarrow & BG
\end{array}
\]

is a homotopy pullback square. We can apply this to $X = \text{Fr}_M$ and $Y = \text{FM}_n$ to obtain $\text{FM}_n^M = \text{FM}_n \times_G \text{Fr}_M$ fits in a homotopy cartesian square:

\[
\begin{array}{ccc}
\text{FM}_n^M & \rightarrow & \text{FM}_n//G \\
\downarrow & & \downarrow \\
M = \text{Fr}_M/G & \rightarrow & BG
\end{array}
\]

Using the pullback-to-pushout lemma [Hes07, Theorem 2.4], [FHT15, Proposition 15.8], we then obtain that:

\[
\text{Graphs}_n^M(r) := A \otimes_{H(BG)} \text{BGraphs}_n
\]

is a dgca model of $\text{FM}_n^M$.

Let us now give a more concrete description of $\text{Graphs}_n^M$. This dgca is isomorphic to:

\[
\text{Graphs}_n^M(r) \cong (A \otimes \text{Graphs}_n(r), d_A + \delta_{\text{contr}} + ET \cdot),
\]

where $d_A$ is the differential on $A$, $\delta_{\text{contr}}$ is the differential on $\text{Graphs}_n(r)$, $E \in A$ is the Euler class as in the previous section, and $T \cdot$ is the action of the tadpole graph on $\text{Graphs}_n$:

\[
T = \quad .
\]

Concretely, this last part of the differential is the sum over all possible ways of removing an edge from the graph and multiplying the element of $A$ by the Euler class. The product is the product of $A$ and the product of $\text{Graphs}_n$. The cooperad structure is given by maps:

\[
\text{Graphs}_n^M(r + s - 1) \to \text{Graphs}_n^M(r) \otimes_A \text{Graphs}_n^M(s)
\]
that are defined using the cooperad structure of $\text{Graphs}_n$. Finally, a connected component $\gamma$ with only internal vertices is identified with a form $z(\gamma)$ on $M$, given by an integral defined using the Feynman rules below.

We have a direct morphism of Hopf cooperads

$$\omega: \text{Graphs}_n^M \to \Omega^*_\text{PA}(\text{FM}_n^M).$$

defined as follows. Let us take a fiberwise volume form $\varphi \in \Omega^{-1}_\text{PA}(\text{FM}_n^M(2))$ on the sphere bundle on $M$, which is (anti-)symmetric and such that $d\varphi$ is the pullback image of the Euler class $E \in A$ (if $n$ is odd we may simply require $d\varphi = 0$). Moreover, given $a \in A$, we can consider its image in $\Omega^*_\text{PA}(M)$ under the quasi-isomorphism of Equation (9), then pull it back to $\text{FM}_n^M$ using the projection. By abuse of notation, we still denote by $a$ this element in $\Omega^*_\text{PA}(\text{FM}_n^M)$. Then for an element $a \otimes \Gamma \in \text{Graphs}_n^M(r)$, such that $\Gamma$ has $s$ internal vertices, we define $\omega(\Gamma) \in \Omega^*_\text{PA}(\text{FM}_n^M(r))$ by:

$$\omega(\Gamma) := a \wedge \int_{\text{FM}_n^M(r)\to \text{FM}_n^M(r)} \bigwedge_{(i,j) \in E_\Gamma} p_{ij}^*\varphi.$$

(44)

Let $\gamma \in \text{GC}_n$ (see Section 2.3) be a connected graph with only internal vertices. Recall that in $\text{Graphs}_n$, such graphs are set to zero. In $\text{Graphs}_n^M$, however, we identify these graphs with a certain form, given by a “partition function” $z$ defined as follows:

$$z(\gamma) := \begin{cases} E, & \text{if } \gamma = T \text{ is a tadpole;} \\ \int_{\text{FM}_n^M(k)} \bigwedge_{(i,j) \in E_\Gamma} p_{ij}^*\varphi, & \text{otherwise.} \end{cases}$$

(45)

More precisely, given an element $a \otimes \Gamma \in \text{Graphs}_n^M(r)$, if $\Gamma$ can be written as a disjoint union $\Gamma = \Gamma' \sqcup \gamma$ where $\gamma$ has only internal vertices, then we have the identification $a \otimes \Gamma \equiv (a \wedge z(\gamma)) \otimes \Gamma'$. There could a priori be a problem in the definition of $\text{Graphs}_n^M$, as the form $z(\gamma)$ might not be in the image of $A \to \Omega^*_\text{PA}(M)$. However, thanks to the following lemma, this cannot happen:

**Lemma 15.** The partition function $z$ vanishes on all graphs other than tadpoles.

**Proof.** The argument is similar to the one of [LV14, Lemma 9.4.3]. Let $\gamma$ be a graph with only internal vertices, different from a tadpole, and consider $z(\gamma)$. If the graph $\gamma$ has univalent or isolated vertices, then $z(\gamma) = 0$ by a simple dimension argument. If $\gamma$ has a bivalent vertex, then Kontsevich’s trick [Kon94, Lemma 2.1] shows that $z(\gamma)$ vanishes by a symmetry argument. Finally, if $n \geq 3$ and $\gamma$ only has vertices that are at least trivalent, then $z(\gamma)$ vanishes by a degree counting argument. For $n = 2$, then one must use a more sophisticated proof technique found in [Kon03, Lemma 6.4].

**Remark 16.** This lemma also follows from [KW17, Theorem 7.1], which is a much more general statement where the space $M$ is roughly speaking replaced by the classifying space $BO(n)$. Our argument is much simpler here due to the fact that $\dim M = n$.

**Theorem 17.** The morphism $\omega$ above defines a quasi-isomorphism of Hopf cooperads:

$$\omega: \text{Graphs}_n^M \xrightarrow{\sim} \Omega^*_\text{PA}(\text{FM}_n^M).$$

**Proof.** Checking that $\omega$ commutes with all the structures involved is done by arguments very similar to the ones found in [LV14; CW16; Idr16], using theorems of [HLTV11]. It is compatible with:

- the differential: by the Stokes formula [HLTV11, Prop. 8.3] and the additivity of integration along fibers [HLTV11, Prop. 8.11];
- the products: by the multiplicative property of integration along fibers [HLTV11, Prop. 8.15];
- the cooperad structure: by an immediate check on generators;
- the identification of internal components $\gamma$ with $z(\gamma)$: by the multiplicative property of integral along fibers and the double pushforward formula [HLTV11, Prop. 8.13].

Finally, it is a quasi-isomorphism by the fact that the model of the homotopy pullback is the homotopy pushout of the models (and since the maps we consider are (co)fibrations then we can recover the adjective “homotopy”).

**Theorem 18.** The symmetric sequence $\text{Graphs}_M$ is a Hopf right $\text{Graphs}_n^M$-comodule, and we have a quasi-isomorphism of (homotopy) Hopf right comultimodules:

$$\langle \omega, \omega \rangle : (\text{Graphs}_M, \text{Graphs}_n^M) \xrightarrow{\sim} (\Omega^*_\text{PA}(\text{FM}_M), \Omega^*_\text{PA}(\text{FM}_n^M)).$$
Proof. The proof is an direct extension of the proof of the main theorem of [CW16]. We can define maps
\[ \phi_i^*: \text{Graphs}_\mathcal{M}(r + s - 1) \to \text{Graphs}_\mathcal{M}(r + s - 1) \otimes \text{Graphs}_\mathcal{M}(s) \]
in a straightforward way, using subgraph contraction. Checking that these maps commute with the differential and the maps \( \omega \) is immediate from the definitions. \( \square \)

4. The framed configuration module

4.1. Constructions for right multimodules. Let \( X \) be a topological space and \( \mathcal{P} \) be an operad in spaces over \( X \). Let \( \mathcal{M} \) be a right \( \mathcal{P} \)-multimodule. Suppose that \( f: Y \to X \) is some map of topological spaces. Then we may define the pullback \( Y \times_f \mathcal{P} \) such that \( (Y \times_f \mathcal{P})(r) = Y \times_f \mathcal{P}(r) \). It is an operad in spaces over \( Y \). Similarly, the pullback \( Y \times_f \mathcal{M} \), defined such that \( (Y \times_f \mathcal{M})(r) := Y^r \times_f \mathcal{M}(r) \) is a right \( f^*\mathcal{P} \)-multimodule.

Let \( \phi: Q \to \mathcal{P} \) be a map of operads in spaces over \( X \). Then \( \mathcal{M} \) can be naturally made into a right \( Q \)-multimodule, which we shall denote by \( \phi^*\mathcal{M} \) (accepting a slight clash in notation).

Next suppose that \( \mathcal{P} = X \times \mathcal{R} \), where \( \mathcal{R} \) is an ordinary topological operad. Then, \( \mathcal{M} \) can be made into a right \( \mathcal{R} \)-module \( \mathcal{M}|_{\mathcal{R}} \), via \( \mathcal{M}|_{\mathcal{R}} \circ \mathcal{R} = \mathcal{M} \circ_X \mathcal{P} \to \mathcal{M} \).

4.1.1. Framing construction. Consider now the following input data:

1. An operad \( \mathcal{P} \) in spaces over \( X \) as above;
2. A right \( \mathcal{P} \)-multimodule \( \mathcal{M} \);
3. A bundle over \( X, f: F \to X \);
4. A trivializing morphism

\[ \phi: F \times \mathcal{R} \to F \times_f \mathcal{P} \]
of operads in spaces over \( X \), where \( \mathcal{R} \) is an ordinary operad.

To these input data we associate the right operadic \( \mathcal{R} \)-module

\[ \text{Fra}^\prime_{\mathcal{P}, \mathcal{M}, f, \phi} := (\phi^*(F \times_f \mathcal{M}))(\mathcal{R}). \]

It is clear that the construction is functorial in the input data. If in addition \( \mathcal{R} \) is an operad in \( G \)-spaces for a topological group \( G \), and the bundle \( F \) carries a \( G \)-action such that \( \phi \) is \( G \)-equivariant, then \( \text{Fra}^\prime_{\mathcal{P}, \mathcal{M}, f, \phi} \) is a right \( \mathcal{R} \)-module in \( G \)-spaces. By this we mean (abusively) that \( \text{Fra}^\prime_{\mathcal{P}, \mathcal{M}, f, \phi}(r) \) carries an action of \( G^r \) such that the composition morphisms are \( G \) equivariant in a natural sense. This implies in particular that \( \text{Fra}^\prime_{\mathcal{P}, \mathcal{M}, f, \phi} \) carries a right action of the framed operad \( \mathcal{R} \circ G \).

Next suppose that the above input data is of the following special form.

- The operad \( \mathcal{R} \) is an operad in \( G \)-spaces, for \( G \) some topological group.
- The bundle \( F \to X \) is a principal \( G \) bundle and \( \mathcal{P} = F \times_G \mathcal{R} \).

Then there is a natural trivializing morphism \( \phi: F \times \mathcal{R} \to F \times_f \mathcal{P} \), defined such that

\[ \phi(a, b) = (a, (a \times_G b)). \]

It is furthermore \( G \)-equivariant. In this special case we denote the right \( \mathcal{R} \)-multimodule \( \text{Fra}^\prime_{\mathcal{P}, \mathcal{M}, f, \phi} \) alternatively by

\[ \text{Fra}^\prime_{\mathcal{R}, F, \mathcal{M}} := \text{Fra}^\prime_{F \times_G \mathcal{R}, \mathcal{M}, f, \phi}. \]

Example 19. The example to keep in mind is \( \mathcal{R} = \mathcal{R}_M, \mathcal{P} = \mathcal{P} = \mathcal{M} \times \mathcal{M}_n, \mathcal{M} = \mathcal{M}_M, \mathcal{M} = \mathcal{F}_M \).

Then we have that \( F \times_G \mathcal{R} = \mathcal{F}_M^\mathcal{M}, \) and \( \text{Fra}_{\mathcal{R}, F, \mathcal{M}} = \text{Fra}^\prime_{\mathcal{R}, F, \mathcal{M}} \).

4.1.2. Functoriality. In particular, we want to stress that the constructions \( \text{Fra}^\prime_{\mathcal{P}, \mathcal{M}, f, \phi} \) and \( \text{Fra}_{\mathcal{R}, F, \mathcal{M}} \) depend functorially on the data. For example, suppose that we have two tuples \( (\mathcal{P}, \mathcal{M}, f, \phi), (\mathcal{P}', \mathcal{M}', f', \phi') \) as above. Suppose that we have morphisms

\[ \alpha: \mathcal{P} \to \mathcal{P}', \ \beta: \mathcal{R} \to \mathcal{R}', \ \gamma: \mathcal{M} \to \mathcal{M}', \]

and a morphism of bundles \( \delta: F \to F' \) from the bundle \( f: F \to X \) to \( f': F' \to X \). Suppose that our morphisms respect the naturally given structure on objects, and make in particular the following

\[ \phi(a, b) = (a, (a \times_G b)). \]
diagrams commute:

\[(48)\]
\[
\begin{array}{ccc}
\mathcal{M} & \overset{\gamma}{\longrightarrow} & \mathcal{P} \\
\downarrow & & \downarrow \\
\mathcal{M}' & \overset{\alpha}{\longrightarrow} & \mathcal{P}'
\end{array}
\]
\[
\begin{array}{ccc}
\mathcal{P} \times_f F & \overset{\phi}{\longleftarrow} & \mathcal{R} \times F \\
\downarrow^{\alpha \times_f \delta} & & \downarrow^{\beta \times \delta} \\
\mathcal{P}' \times_{f'} F' & \overset{\phi'}{\longleftarrow} & \mathcal{R}' \times F'.
\end{array}
\]

Here the dashed arrows denote the operadic right action. In this situation it is clear that the maps given provide a morphism of operads and their right modules

\[
\begin{array}{c}
\text{Fra}_{\mathcal{P}, \mathcal{M}, f, \phi} \\
\downarrow_{\gamma \times \delta} \\
\text{Fra}_{\mathcal{P}', \mathcal{M}', f', \phi'}
\end{array}
\]

Furthermore, if the maps \(\alpha, \beta, \delta\) above are compatible with the \(G\)-action, then the operads \(\mathcal{R}\) and \(\mathcal{R}'\) on the right-hand side of the above diagram may be replaced by their \(G\)-framed versions. Furthermore, if \(\alpha, \beta, \gamma, \delta\) are weak equivalences, then so is the induced map of operadic right modules above. (Here we use that \(f, f'\) are supposed to be fiber bundles, hence in particular fibrations.)

Let us also state a slightly laxer version of the above functoriality result. Suppose next that we have maps \(\beta, \gamma, \delta\) as above, but in addition two homotopic maps of operads over \(X\)

\[(49)\]
\[
\begin{array}{ccc}
\mathcal{M} & \overset{\gamma}{\longrightarrow} & \mathcal{P} \\
\downarrow & & \downarrow \\
\mathcal{M}' & \overset{\alpha_0}{\longrightarrow} & \mathcal{P}'
\end{array}
\]
\[
\begin{array}{ccc}
\mathcal{P} \times_f F & \overset{\phi}{\longleftarrow} & \mathcal{R} \times F \\
\downarrow^{\alpha_1 \times \delta} & & \downarrow^{\beta \times \delta} \\
\mathcal{P}' \times_{f'} F' & \overset{\phi'}{\longleftarrow} & \mathcal{R}' \times F'.
\end{array}
\]

Then we claim that still we have a (homotopy) morphism \(\text{Fra}_{\mathcal{P}, \mathcal{M}, f, \phi} \rightarrow \text{Fra}_{\mathcal{P}', \mathcal{M}', f', \phi'}\). Concretely, suppose that the homotopy is realized by a path in the mapping space

\[
\alpha : I \times \mathcal{P} \rightarrow \mathcal{P}',
\]

with \(I = [0,1]\), whose endpoints agree with \(\alpha_0, \alpha_1\) respectively. Let us define the bundle \(f_I : F_I := I \times F \rightarrow X\) by trivial extension of \(F\). We will also define the trivializing morphism (of operads over \(F_I\))

\[
\phi_I : \mathcal{R} \times F_I \rightarrow \mathcal{P}' \times_{f_I} F_I
\]

as the composition

\[
\mathcal{R} \times F_I \cong I \times \mathcal{R} \times F \xrightarrow{id \times \phi} I \times \mathcal{P} \times_f F \rightarrow (I \times P) \times_{f_I} (I \times F) \xrightarrow{\alpha \times_f id} \mathcal{P}' \times_{f_I} F_I.
\]

Here we used the diagonal \(I \rightarrow I \times I\) for the middle arrow. Furthermore let us define the map

\[
\tilde{\phi} : \mathcal{R} \times F \rightarrow \mathcal{P}' \times_{f'} F
\]

as the composition

\[
\mathcal{R} \times F \xrightarrow{\phi} \mathcal{P} \times_f F \xrightarrow{\alpha \times_f id} \mathcal{P}' \times_{f} F.
\]

Then we build the following zigzag

\[(50)\]
\[
\begin{array}{c}
\text{Fra}_{\mathcal{P}, \mathcal{M}, f, \phi} \\
\downarrow_{p_0} \\
\text{Fra}_{\mathcal{P}', \mathcal{M}', f_I, \phi_I} \\
\uparrow_{\sim} \\
\text{Fra}_{\mathcal{P}', \mathcal{M}', f, \tilde{\phi}} \\
\downarrow_{p_2} \\
\text{Fra}_{\mathcal{P}', \mathcal{M}', f', \phi'}
\end{array}
\]

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Let us explain the construction of the vertical maps, which are all obtained from maps on the input data of Fra′ and functoriality. The top vertical map on the left ρ₀ is obtained by the maps α₀ : P → P′, γ : M → M′ and the map i₀ : F → F₁ = I × F sending an element u ∈ F to (0, u). We have to check that the following diagrams commute, which is evident by construction.

\[
\begin{align*}
\mathcal{M} & \longleftarrow \mathcal{P} & \mathcal{P} \times_f F & \longleftarrow \mathcal{R} \times F \\
\downarrow \gamma & \downarrow \alpha_0 & \downarrow_{\alpha_0 \times f_1} & \downarrow id \times i_0 \\
\mathcal{M}' & \longleftarrow \mathcal{P}' & \mathcal{P}' \times_f F & \longleftarrow \mathcal{R} \times F_1.
\end{align*}
\]

Similarly the map ρ₁ is induced by the maps i₁ : F → F₁ = I × F sending an element u ∈ F to (0, u). One readily checks that the relevant diagrams commute:

\[
\begin{align*}
\mathcal{M}' & \longleftarrow \mathcal{P}' & \mathcal{P}' \times_f F & \longleftarrow \mathcal{R} \times F \\
\downarrow = & \downarrow = & \downarrow_{id \times f_1} & \downarrow id \times i_1 \\
\mathcal{M}' & \longleftarrow \mathcal{P}' & \mathcal{P}' \times_f F & \longleftarrow \mathcal{R} \times F_1.
\end{align*}
\]

Finally the map ρ₂ is defined by functoriality of Fra′ and the maps on input data β : R → R′ and δ : F → F′. One again checks that the relevant diagrams commute. Overall we have constructed a zigzag (the LHS of Equation (50)), in which the only arrow pointing in the upward direction is a weak equivalence. In other words, we have constructed a homotopy equivalence. Furthermore if all data respect the G-actions, then we may pass to modules for the G-framed operads. The constructions above readily extend to homotopy operads and modules, and dualize to (homotopy) Hopf cooperads and comodules.

4.2. The framed configuration module \( FM^f_M \). Using the above general construction we can now define the framed configuration module of \( M \) as

\[
FM^f_M := \text{Fra}_{FM_n \text{Fr}_M, FM_M}.
\]

More concretely, we have

\[
FM^f_M(r) = (FM_M \circ_M Fr_M)(r) = FM_M \times M \cdot Fr^r_M,
\]

where the fiber product is defined using the maps \( p_1, \ldots, p_r \) of Equation (3).

By construction \( FM^f_M \) carries a natural operadic right action of the framed Fulton–MacPherson operad \( FM^n_\text{fr} \). In particular, \( FM^f_M(r) \) carries a natural \( G^r \) action

\[
FM^f_M \circ G \to FM^f_M,
\]

and an operadic right \( FM_n \)-module structure. The composition morphisms implicit in the construction \( \text{Fra}_{FM_n \text{Fr}_M, FM_M} \) are given explicitly by the following composition:

\[
FM^f_M \circ FM = FM_M \circ_M Fr_M \circ FM_n \xrightarrow{\Delta} FM_M \circ_M Fr_M \circ FM_n \circ_M Fr_M \xrightarrow{\pi} FM_M \circ_M Fr_M \circ FM_n \circ M Fr_M = FM^f_M,
\]

where \( \Delta \) is the diagonal map of \( Fr_M \) and \( \pi : Fr_M \times FM_n \to FM_M \) is the projection.
4.3. The graphical model $\text{Graphs}^fr_M$. Let us first define our (at this point of the paper tentative) graphical model for $\text{FM}_M^r$. In the next subsection we will then relate this graphical model to the forms on (a version of) $\text{FM}_M^r$ by an explicit zigzag of quasi-isomorphisms of homotopy Hopf comodules.

By definition, the space $\text{FM}_M^r(r)$ fits into the following pullback diagram:

$$
\begin{array}{c}
\text{FM}_M^r(r) \\ \downarrow \quad \downarrow \\
(\text{Fr}_M)^{\times r} \\ \text{M}^{\times r}
\end{array}
$$

Recall the model $\text{FM}_M^W^{\text{alg}} = (A \otimes \widehat{H}(G), d)$ of $\text{Fr}_M$ obtained in Section 3.3. We can take as an algebraic model for $\text{FM}_M^r(r)$:

$$
\text{Graphs}^fr_M(r) := (A \otimes \widehat{H}(G), d)^{\otimes r} \otimes_{A^{\otimes r}} \text{Graphs}_M(r)
$$

Here some care has to be taken since one cannot immediately apply the pullback-to-pushout lemma, since the base $M'$ is not necessarily simply connected. We may still conclude that $\text{Graphs}^fr_M(r)$ is a dgca model for $\text{FM}_M^r(r)$ since $\text{FM}_M^r(r) \rightarrow \text{FM}_M^r(r)$ is a $G'$-principal bundle, and hence one can explicitly write down a model given a model of the base as in [GHV76, Theorem 1, section 9.3], which in this case agrees with $\text{Graphs}^fr_M(r)$.

We can now define the $\text{Graphs}_n$-comodule structure on $\text{Graphs}^fr_M$ by writing a diagram which is dual to Equation (53):

$$
\text{Graphs}^fr_M = (\text{Graphs}_M \circ \widehat{H}(G), d) \rightarrow \text{Graphs}_M \circ_A (A \otimes \text{Graphs}_n, d) \circ_A (A \otimes \widehat{H}(G), d) \xrightarrow{\pi^*} \\
\text{Graphs}_M \circ_A ((A \otimes \widehat{H}(G), d) \otimes \text{Graphs}_n) \circ_A (A \otimes \widehat{H}(G), d) \xrightarrow{\text{mult}_{\text{Fr}_M^W^{\text{alg}}}} \\
(\text{Graphs}_M \circ \widehat{H}(G), d) \circ \text{Graphs}_n = \text{Graphs}_M^r \circ \text{Graphs}_n.
$$

Here $\text{mult}_{\text{Fr}_M^W^{\text{alg}}}$ is the product of $\text{Fr}_M^{\text{alg}}$ (dual of the diagonal), and $\pi^* : \text{Graphs}_n^M \rightarrow (A \otimes \widehat{H}(G), d) \otimes \text{Graphs}$ is the map

$$
\pi^* : (A \otimes \text{Graphs}_n, d) \rightarrow (A \otimes \widehat{H}(G), d) \otimes \text{Graphs}_n
$$

$$
a \otimes \Gamma \mapsto \sum b \otimes \Gamma'' ,
$$

where we use Sweedler notation $\Gamma \mapsto \sum b \otimes \Gamma''$ to describe the $\widehat{H}(G)$-coaction on $\text{Graphs}_n$ of section 2.5. It can be seen from the discussion following below that $\pi^*$ in fact is the model of the projection $\pi : \text{Fr}_M \times \text{FM}_n \rightarrow \text{FM}_n^M$.

In addition to the coaction of the cocommutative Hopf algebra $\widehat{H}(G)$ on $\text{Graphs}_n$ we have, compatibly, coactions of $\widehat{H}(G)'$ on $\text{Graphs}^fr_M$. We can hence pass to the framed Hopf cooperad $\text{Graphs}^fr_n = \text{Graphs}_n \circ \widehat{H}(G)$, which then inherits an action on $\text{Graphs}^fr_M$. Finally, our graphical model for the pair $(\text{FM}_M^r, \text{FM}_n^r)$ of topological operad and operadic right module is the pair $(\text{Graphs}^fr_M, \text{Graphs}^fr_n)$ of a Hopf cooperad and a Hopf cooperadic right module. Note that our graphical model is indeed an honest Hopf cooperad/Hopf comodule, not just a homotopy cooperad/comodule.

4.4. The zigzag. Our next goal is to build a dgca model of $\text{FM}_M^r = \text{FraFM}_n, \text{Fr}_M, \text{FM}_M$ as an operadic right $\text{FM}_n^r$-module. We will do this by going through the steps of the construction $\text{FraFM}_n, \text{Fr}_M, \text{FM}_M$ from Section 4.1. The input of that construction is

1. The operad $\text{FM}_n$ in $G = \text{SO}(n)$-spaces;
2. The right $\text{FM}_n^r = \text{FM}_n \times_G \text{Fr}_M$-multimodule $\text{FM}_M$;
3. The principal $G$-bundle $\text{Fr}_M$;
4. The trivializing morphism $\pi : \text{Fr}_M \times \text{FM}_n \rightarrow \text{FM}_n^r$.

We will use the following combinatorial models for the above objects:
(1) We will use two different models for the operad \( FM_n \) in \( G \)-spaces. First \( \hat{H}(G) \otimes B\text{Graphs}_n \) is a homotopy Hopf cooperad with a \( \hat{H}(G) \)-coaction modelling \( FM_n \) (or rather the homotopy equivalent realization of \( WG \times G^* \times FM_n \)) via the map

\[
\hat{H}(G) \otimes B\text{Graphs}_n \to \text{Tot}(\Omega_{PA}(WG \times G^* \times FM_n)),
\]

see [KW17] or section 2.5. The map is a quasi-isomorphism of homotopy cooperads in homotopy \( \Omega_{PA}(WG) \)-comodules. Secondly we may take as a model \( \text{Graphs}_n \) with the \( \hat{H}(G) \)-coaction of section 2.5. In contrast to the former model, this is an honest dg Hopf cooperad in \( \hat{H}(G) \)-comodules. There is a comparison quasi-isomorphism

\[
\text{Graphs}_n \xrightarrow{\sim} \hat{H}(G) \otimes B\text{Graphs}_n
\]

by taking the coaction.

(2) For the principal bundle \( \text{Fr}_M \), we will use the model \( \text{Fr}_M^W = (A \otimes \hat{H}(G), d) \) of section 3.3.

(3) For \( FM_n \) as right \( FM_n^M \)-module, we use the model \( \text{Graphs}_M \), coacted upon by \( A \otimes \text{Graphs}_n \).

The fourth step in the construction \( \text{Fr}_{FM_n,\text{Fr}_M,FM_n} \) is to build a model of the trivializing morphism (47), i.e. the map \( \pi : \text{Fr}_M \times FM_n \to FM_n^M \). This is provided by the following commutative diagram of homotopy Hopf cooperads under \( A \otimes \hat{H}(G) \).

\[
\begin{array}{ccc}
(A \otimes \hat{H}(G) \otimes A \text{Graphs}_n^M, d) & \xrightarrow{\sim} & (A \otimes \hat{H}(G) \otimes \text{Graphs}_n, d) \\
\downarrow & & \downarrow \\
(A \otimes \hat{H}(G) \otimes A \otimes \hat{H}(G) \otimes B\text{Graphs}_n, d) & \xrightarrow{\sim} & (A \otimes \hat{H}(G) \otimes \hat{H}(G) \otimes B\text{Graphs}_n, d) \\
\downarrow & & \downarrow \\
\text{Tot} \Omega_{PA}(\text{Fr}_M^W \times_M \text{Fr}_M^W \times G^* \times FM_n) & \xrightarrow{\sim} & \text{Tot} \Omega_{PA}(\text{Fr}_M^W \times WG \times G^* \times FM_n)
\end{array}
\]

The diagram also clearly respects the homotopy \( \Omega_{PA}(WG) \)-coaction.

The next complication is that in the above diagram we used \( \text{Fr}_M^W \times G^* \times FM_n \) as our replacement of \( FM_n^M \), while in the model of the action on \( FM_n \) used \( FM_n^M \). Thus we have to check commutativity of the diagram of homotopy Hopf cooperads under \( A \).

\[
\begin{array}{ccc}
\text{Graphs}_n^M = (A \otimes \text{Graphs}_n, d) & \xrightarrow{\sim} & (A \otimes \hat{H}(G) \otimes B\text{Graphs}_n, d) \\
\downarrow & & \downarrow \\
\Omega_{PA}(FM_n^M) & \xrightarrow{\sim} & \text{Tot} \Omega_{PA}(\text{Fr}_M^W \times G^* \times FM_n)
\end{array}
\]

Unfortunatly, this diagram does not commute. However, we check in Proposition 24 and Corollary 25 that it homotopy commutes, which will be enough for our purposes, though it adds additional complications. More concretely, we will dualize the argument of section 4.1.2 to create our final zigzag of quasi-morphisms of homotopy Hopf cooperads and their homotopy right Hopf comodules (actions being depicted as dashed arrows) as follows.

\[
\begin{array}{ccc}
\Omega_{PA}(FM_n \circ_M \text{Fr}_M^W) & \xrightarrow{\nu_1} & \text{Tot} \Omega_{PA}(WG \times G^* \times FM_n) \\
\text{Graphs}_M \circ_A \Omega_{PA}(\text{Fr}_M^W)[t, dt] & \xrightarrow{\nu_2} & \text{Tot} \Omega_{PA}(WG \times G^* \times FM_n)
\end{array}
\]

A few explanations are in order, since the notation is somewhat compressed. The reader is advised to follow the diagram (50) in parallel, of which (58) is the reformulation in the dual and homotopy setting. We now describe each line and each vertical arrow in the diagram.
In the first line, \( \text{Tot} \Omega_{PA}(WG \times G^* \times FM_n) \) is a homotopy Hopf cooperad. The underlying functor sends a tree \( T \) to the dgca

\[ \text{Tot} \Omega_{PA}(WG \times G^* \times FM_n(T)). \]

In particular, mind that for each tree there are multiple factors of \( FM_n(r) \), one for each node, but only one factor \( WG \) for the whole tree – otherwise we would not know how to define the contraction and gluing morphisms properly.

Secondly, \( \Omega_{PA}(FM_M \circ_M Fr_M^W) \) is a homotopy right comodule over the aforementioned cooperad. Concretely, let \( T \) be a marked tree as in (4), i.e. the marked vertex has children \( T_1, \ldots, T_r \). The functor \( \Omega_{PA}(FM_M \circ_M Fr_M^W) \) maps \( T \) to the dgca

\[ \text{Tot} \Omega_{PA}(FM_M(r) \times_M (Fr_M^W)^r \times \prod_{j=1}^r (WG \times G^* \times FM_n(T_j))). \]

The contraction morphisms (comodule structure) are defined as pullbacks of the topological composition, in the natural way.

- In the second line, the Hopf cooperad \( \text{Tot} \Omega_{PA}(WG \times G^* \times FM_n) \) is the same. The homotopy Hopf comodule \( \text{Graphs}_M \circ_A \Omega_{PA}(Fr_M^W)[t, dt] \) assigns to a tree \( T \) as before the dgca

\[ \text{Tot} \text{Graphs}_M(r) \otimes_A \Omega_{PA}((Fr_M^W \times I)^r \times \prod_{j=1}^r (WG \times G^* \times FM_n(T_j))), \]

where \( I = [0, 1] \) is again the interval.

The contraction morphisms are defined (dually and) analogously to the action on the module in the second line of (50). More concretely, for the contraction (or rather expansion) of a top edge we first take the corresponding \( \text{Graphs}_M \)-coaction on the factor \( \text{Graphs}_M \). The resulting element in \( \text{Graphs}_n \) is then sent to \( \text{Tot} \Omega_{PA}(Fr_M^W \times G^* \times FM_n)[t, dt] \) using the explicit morphism from Corollary 25. Finally the result can naturally be mapped to \( \text{Tot} \Omega_{PA}(Fr_M^W \times WG \times G^* \times FM_n)[t, dt] \) using the pullback of the \( WG \) action on \( Fr_M^W \).

- The left vertical upwards arrow \( \nu_0 \) is induced by the map \( \text{Graphs}_M \to \Omega_{PA}(FM_M) \) of section 3.4, and by the restriction to the endpoint \( t = 0 \) of the intervals \( I \).

- In the third line, the Hopf cooperad is still the same, but we restrict the comodule to \( t = 1 \). This comodule is defined analogously to the one in the second line, except that in the (co)contraction morphisms one does not see the full homotopy from Proposition 24; we just see the map at the \( t = 1 \) end of the interval, i.e., the upper composition in (57).

- Correspondingly, the map \( \nu_1 \) is merely the restriction to the endpoints \( t = 1 \) of the intervals \( I \).

- In the last line, we see the \( \text{Graphs}_n \)-comodule \( \text{Graphs}_M \circ H(G) \) from Section 2.5.

- Finally, the map \( \nu_2 \) is defined using the morphism \( A \otimes H(G), d) \to \Omega_{PA}(Fr_M^W) \) from Section 3.3.

The diagram (58) realizes or desired zigzag of morphisms of homotopy Hopf cooperads and comodules. Additionally, it is not hard to check that all arrows in the diagram are quasi-isomorphisms. (This is due to the fact that \( \text{Graphs}_M(r) \) is free as an \( A^* \)-module.) We thus get as an intermediary result:

**Proposition 20.** The Hopf right comodule \( (\text{Graphs}_M^fr, \text{Graphs}_n^fr) \) is a model for \( (FM_M^fr, FM_n) \).

Finally, note that all objects in the diagram have a homotopy \( \Omega_{PA}(WG) \simeq H(G) \)-coaction, compatible with all structures and maps. We then desire to apply the framing construction to pass from \( FM_n \) to the framed counterpart \( FM_n^fr = FM_n \circ G \).

**Proposition 21.** Let \( T \) be an operad in \( G \)-spaces and \( M \) be a right \((T \circ G)\)-module. Suppose that \( H \)

is a Hopf coalgebra modeling \( G \), that \( C \) is a Hopf cooperad in \( H \)-comodules modeling \( T \), and that \( N \) is a homotopy \( C \)-comodule with a compatible \( H \)-coaction, modeling \( M \). Then the natural coaction of \( C \circ H \) on \( M \) models the action of \((T \circ G)\) on its right module \( M \).

**Proof.** It is shown in [KW17, Theorem 5.5] that the algebraic semidirect product \( C \circ H \) models the framed operad \( T \circ G \). Note that this is a priori not completely obvious: for example, we do not know how to apply the framing construction directly to homotopy \((co)\)operad. The proof of that theorem extends immediately to \((co)\)modules. 

Using this proposition, we can conclude:
Theorem 22. The Hopf right comodule \((\text{Graphs}^r_M, \text{Graphs}_n)\) is a model for \((\text{FM}^r_M, \text{FM}_n)\). Furthermore, the \(\text{SO}(n)\)-actions on \((\text{FM}^r_M, \text{FM}_n)\) are modelled by the natural \(\hat{H}(\text{SO}(n))\)-actions on the pair \((\text{Graphs}^r_M, \text{Graphs}_n)\), therefore the pair \((\text{Graphs}^r_M, \text{Graphs}^s_n)\) is a model for \((\text{FM}^r_M, \text{FM}^s_n)\).

5. Parallelized manifolds and the relation to \([\text{CW16}; \text{Idr16}]\)

5.1. Choosing a framing. Suppose that \(M\) is a framed manifold, i.e., it comes equipped with a section of the frame bundle

\[s : M \to \text{Fr}_M.\]

In this case we may define an \(\text{FM}_n\) action on the (non-framed) configuration space \(\text{FM}_M\) directly. It is shown in \([\text{CW16}; \text{Idr16}]\) that, provided proper choices are made in the construction, there is a natural direct \(\text{Graphs}_n\)-coaction on our model \(\text{Graphs}^r_M\) for \(\text{FM}_M\), that models the \(\text{FM}_n\)-action on \(\text{FM}_M\). In this section we shall elucidate this action and describe the relation to the present work.

To this end we note that the action of \(\text{FM}_n\) on \(\text{FM}_M\) in the parallelized setting may be obtained from the action of \(\text{FM}_n\) on \(\text{FM}^r_M\) and the section of the frame bundle \(s\) as follows. First, \(\text{FM}^r_M\) may be considered as the pullback

\[
\text{FM}_M(r) \to \text{FM}^r_M(r) \quad \downarrow \quad \downarrow \\
M^r \quad s^r \quad \text{Fr}_M^s
\]

Then the action of \(\text{FM}_n\) may be recovered by functoriality of the pullback, as the dashed edge in the following map of pullback squares:

\[
\begin{array}{ccc}
\text{FM}_M(r + s - 1) & \to & \text{FM}^r_M(r + s - 1) \\
\text{FM}_M(r) \times \text{FM}_n(s) & \to & \text{FM}^r_M(r) \times \text{FM}_n(s) \\
\downarrow & & \downarrow \\
M^r + s - 1 & \Delta & \text{Fr}_M^{r+s-1} \\
\downarrow & & \downarrow \\
M^r & \Delta & \text{Fr}_M^s
\end{array}
\]

In the diagram the map \(\circ j\) denotes the action on \(\text{FM}^r_M\). The arrows labeled \(\Delta\) are \(s - 1\)-fold diagonals, applied to the \(j\)-th factor in the product.

Let us dualize the above diagram and constructions, using our model \((\text{Graphs}^r_M, \text{Graphs}_n)\) for the framed configuration space comodule. We suppose that our section \(s\) is modeled by the dgca morphism

\[
\sigma : (A \otimes \hat{H}(G), d) \to A
\]

Then we obtain a model for \(\text{FM}_n(r)\) as the pushout

\[
\text{Graphs}^r_M(r) \otimes (A \otimes \hat{H}(G))^r \xrightarrow{\sigma^r} \text{Graphs}^r_M(r) \leftarrow \text{Graphs}^r_M(r)
\]

Here we again cannot readily apply the pullback-to-pushout Lemma since the base is not simply connected. However, one nevertheless verifies explicitly that the Lemma still holds, i.e., that the induced map \(\text{Graphs}^r_M(r) \otimes (A \otimes \hat{H}(G))^r \to \Omega_{PA}(\text{FM}_M)\) is a quasi-isomorphism, analogously to the proof of Theorem 18, using the result of \([\text{GHV76}, \text{Theorem 1, section 9.3}]\).

The (collection of the) objects in the upper left corner of the diagram (isomorphic to \(\text{Graphs}^r_M\)) inherit a natural \((\text{Graphs}^r_M)\)-coaction from the \(\text{Graphs}_n\)-coaction on \(\text{Graphs}^r_M(r)\). Let us describe that coaction explicitly. Start with a graph \(\Gamma \in \text{Graphs}_M\). Then we take the \(\text{Graphs}^r_n\)-coaction, producing a product

\[
\sum \Gamma' \otimes \gamma \in \text{Graphs}_M \otimes \text{Graphs}_n,
\]

by contracting subgraphs \(\gamma\) of \(\Gamma\). Next we coact by \(\hat{H}(G)\) on \(\gamma\), producing

\[
\sum \Gamma' \otimes h \otimes \gamma' \in \text{Graphs}_M \otimes \hat{H}(G) \otimes \text{Graphs}_n,
\]
with $h$ elements of $\hat{H}(G)$. Finally we use $\sigma$ to map $h$ to $A$ and multiply that factor inside $\Gamma'$, producing

$$\sum \Gamma' p_\ast^\prime(\sigma(h)) \otimes \gamma' \in \text{Graphs}_M \otimes \text{Graphs}_n,$$

which is the desired result of the $\text{Graphs}_n$-coaction. Let us denote the right $\text{Graphs}_n$-comodule obtained from $\text{Graphs}_M$ via the map $\sigma$ above by $\sigma^\ast \text{Graphs}_M^\ast$.

If we make careful choices, then the trivialization of the frame bundle may be used to define the model $(A \otimes \hat{H}(G), d)$ for the frame bundle so that the differential is $d = 0$. Furthermore, in this case we also have that $\text{Graphs}_M^\ast(r) = \text{Graphs}_M(r) \otimes \hat{H}(G)^\ast$, with no piece of the differential going from $\hat{H}(G)$ to $\text{Graphs}_M$. In that case we may take our section $\sigma$ to be the trivial map, which is just the projection to the first factor $A$. The $\text{Graphs}_n$-coaction on $\text{Graphs}_n$ we obtain then agrees with the action of $[CW16]$.

There is also an alternative viewpoint yielding the same formula. Our section $s$ of the frame bundle gives rise to a trivialization of the fiberwise little discs operad

$$M \times \text{FM}_n \xrightarrow{\times \text{id}} \text{Fr}_M \times \text{FM}_n \rightarrow \text{Fr}_M \times_G \text{FM}_n \cong \text{FM}_n^M.$$

Here the right-hand arrow is the quotient under the $G = \text{SO}(n)$-action, and the whole composition is an isomorphism of operads over $M$. Using this trivialization we may then pull back the right $\text{FM}_n^M$-module structure on $\text{FM}_M$ to a right $M \times \text{FM}_n$-module structure, which then restricts trivially to an $\text{FM}_n$-module structure.

Translating this construction into algebraic models, the trivialization is represented by the following composition

$$\text{Graphs}_n^M \rightarrow ((A \otimes \hat{H}(G), d) \otimes \text{Graphs}_n)^{\hat{H}(G)} \rightarrow A \otimes \hat{H}(G) \otimes \text{Graphs}_n \xrightarrow{\sigma \otimes \text{id}} A \otimes \text{Graphs}_n.$$

Here we used $\text{Graphs}_n$ with the $\hat{H}(G)$-coaction as a model for $\text{FM}_n$. The first arrow is induced by the $\hat{H}(G)$-coaction on $\text{Graphs}_n$. Finally, the (co)trivialization morphism above allows us to (co)restrict the $A\text{Graphs}_n$-coaction on $\text{Graphs}_M$ to a coaction of $\text{Graphs}_n$. The formula for this coaction is evidently precisely the same as obtained from the previous, alternative derivation.

5.2. Computation: Changing the frame. Suppose still that $M$ is parallelizable, i.e., that the tangent bundle is a trivial bundle. In this section we want to study the effect of changing the trivialization of the tangent bundle on our graphical models for $\text{FM}_n$ as $\text{FM}_n$-module. We shall see that the dependence on the chosen framing is relatively minor.

For the purposes of the following computation, assume that we have chosen some reference trivialization, and that we have made choices so that our model $A\text{Graphs}_n = (A \otimes \text{Graphs}_n, d)$ in fact has the form $(A, d) \otimes (\text{Graphs}_n, d)$, with no piece of the differential between $\text{Graphs}_n$ and $A$. Similarly, in this case our dgca model for the frame bundle can be taken to be the tensor product of dgcas $A \otimes \hat{H}(G)$. Suppose also that the MC element $m \in H(BG) \otimes GC_n$ controlling the $\text{SO}(n)$-action on $\text{FM}_n$ has already been brought to the form of Theorem 11 by a gauge transformation for simplicity.

Then suppose that we have some other trivialization of the tangent bundle, which is modeled by a dgca map $\sigma : (A \otimes \hat{H}(G), d) \rightarrow A$ as in (59) above. Since $\hat{H}(G) \cong H(G)$ as dgcas we may suppose that $\sigma$ is given as a composition

$$A \otimes \hat{H}(G) \rightarrow A \otimes H(G) \rightarrow A.$$

Such $\sigma$ is completely determined by providing the images of the generators of $H(G)$, i.e., by the images of the Pontryagin classes and, for even $n$, the Euler class.

Next let us write down explicitly the composition (60) describing the trivialization map

$$f_\sigma : \text{Graphs}_n^M \rightarrow A \otimes \text{Graphs}_n$$

associated to $\sigma$. Using the explicit form of the action as encoded by $m$ of Theorem 11, we see that for even dimension $n$ the maps sends a graph $\Gamma \in \text{Graphs}_n$ to

$$f_\sigma(\Gamma) = (1 + \sigma(E) \mathcal{V} \cdot) \Gamma,$$

while for $n$ odd the corresponding map is

$$f_\sigma(\Gamma) = (1 + \sigma(P_{\text{top}}}) \theta) \Gamma,$$

where $\theta$ stand for the $\theta$-graph

$$\theta = \begin{array}{c}
\circ \\
21
\end{array}.$$
Remark. A basis for $E$ where $z$.

Then one may in fact check that one has an isomorphism of right $\mathbb{R}$-modules as a right $\mathbb{R}$-module.

Given the maps $f_\sigma$, let us denote the corresponding $\text{Graphs}_n$-right comodule $f_\sigma^* \text{Graphs}_M$. (It is the same as $\text{Graphs}_n$ as a Hopf collection, but the operadic coaction is twisted by $f_\sigma$.) Furthermore, let us note that the construction of $\text{Graphs}_M$ depends on a choice of MC element $z \in \text{GC}_M$, which generally depends on choices made.

Remark 23. Let us make the dependence explicit in the notation and write $\text{Graphs}_M^z$ for the moment. Then one may in fact check that one has an isomorphism of right $\text{Graphs}_n$ comodules

$$f_\sigma^* \text{Graphs}_M^z \cong \text{Graphs}_M^{z'},$$

where $z' \in \text{GC}_M$ is another Maurer-Cartan element which can be obtained through a natural $A \otimes \text{GC}_n$-action on $\text{GC}_M$ as

$$z' = \begin{cases} \exp(\sigma(E) \partial \cdot)z & \text{for } n \text{ even} \\ \exp(\sigma(P_{\text{top}})\theta)z & \text{for } n \text{ odd} \end{cases}.$$ 

For details we refer to forthcoming work. One may say in summary that the choice of framing essentially only affects the coefficient of the tadpole graph in the MC element $z$ if $n$ is even, only affects the coefficient of the $\theta$-graph if $n = 3$, and has no effect for $n \geq 5$ odd.

Appendix A. Explicit form of propagator

For completeness, let us give an explicit form of the equivariant propagator, i.e., an equivariant form on the $n - 1$-sphere extending the (round) volume form whose equivariant differential is 0 for $n$ odd or proportional to the Euler class for $n$ even. Such a formula has been given within the toric Cartan model in [KW17, Appendix A]. Here we will alternatively use the (non-toric) Cartan model instead:

$$\left(S(\text{so}_n[-2]) \otimes \Omega(S^{n-1})\right)^G.$$ 

A basis for $\text{so}_n[-2]$ (dual of antisymmetric matrices) is denoted by symbols $u_{ij} = -u_{ji}$, with $1 \leq i \neq j \leq n$. The map from this model to forms on $\text{FM}_n(2) = S^{n-1}$ are defined similarly to the map $\Phi$ from [KW17, Section 4.7].

Define the operator

$$I := \sum_{i<j} u_{ij} x_i x_j.$$ 

Then define the equivariant volume form to be

$$\Omega^G_{sm} := C_n L E \sum_{0 \leq k < n/2} (n - 2k - 2)! I^k(dx_1 \cdots dx_n),$$

where $E = \sum_{j=1}^n x_j \frac{\partial}{\partial x_j}$ is the Euler vector field, $C_n$ is some normalization constant, and it is understood that (only) after contraction one restricts the form to the sphere. We set $(-1)!! := 1$.

Let us now verify that $\Omega^G_{sm}$ satisfies the defining equations, see [KW17, Section 6.4]. The well-definedness, the fact that it is a volume form of area 1 on the sphere, and that it is equivariant under the antipodal map and the action of $G$, are proved the same way as in [KW17, Lemma A.1]. Finally we must check that its image under the differential $d_n := d + \sum_{i,j} u_{ij} x_i x_j$ is proportional to the Euler class (for even $n$) or zero (for odd $n$). First, we have that:

$$d\Omega^G_{sm} = C_n L E \sum_{0 \leq k < n/2} (n - 2k - 2)! I^k(dx_1 \cdots dx_n)$$

$$= C_n \sum_{0 \leq k < n/2} (n - 2k)! I^k(dx_1 \cdots dx_n)$$

$$= C_n \sum_{1 \leq k < n/2} (n - 2k)! I^k(dx_1 \cdots dx_n).$$
Next, we see that:
\[
\sum_{i,j} u_{ij} x_{ij} \omega_{sm}^C = -C_n t_E \sum_{0 \leq k < n/2} (n - 2k - 2)! I^k \sum_{i,j} u_{ij} x_{ij} (dx_1 \cdots dx_n)
\]
\[
= -C_n t_E \sum_{0 \leq k < n/2} (n - 2k - 2)! I^k \sum_{i,j} u_{ij} x_{ij} (dx_1 \cdots dx_n)
\]
\[
= -C_n t_E \sum_{0 \leq k < n/2} (n - 2k - 2)! I^k \sum_{l=1}^{n} x_l dx_1 \sum_{i,j} u_{ij} t_{ij} (dx_1 \cdots dx_n)
\]
\[
= -C_n t_E \sum_{0 \leq k < n/2} (n - 2k - 2)! I^k+1 \sum_{l=1}^{n} x_l dx_1 (dx_1 \cdots dx_n)
\]
\[
= -C_n \sum_{0 \leq k < n/2} (n - 2k - 2)! I^k+1 (dx_1 \cdots dx_n)
\]
\[
= -C_n \sum_{1 \leq k < n/2 + 1} (n - 2k)! I^k (dx_1 \cdots dx_n).
\]

Hence we find that:
\[(d + \sum_{i,j} u_{ij} x_{ij}) \omega_{sm}^C = \begin{cases} 0, & \text{for } n \text{ odd;} \\ -C_n I^{n/2} (dx_1 \cdots dx_n) \propto E, & \text{for } n \text{ even.} \end{cases}\]

**Appendix B. Homotopy commutativity of a diagram**

The purpose of this section is to show the following Proposition.

**Proposition 24.** The following diagram of homotopy Hopf cooperads under \( H(BG) \) is homotopy commutative:
\[
\begin{array}{cccc}
\text{BGraphs}_n & \longrightarrow & \text{Tot}(\Omega_{PA}(G^* \times FM_n)) \\
\downarrow & & \downarrow \\
(H(BG) \otimes \hat{H}(G) \otimes H(BG) \otimes \text{Graphs}_n, d) & \longrightarrow & \text{Tot}(\Omega_{PA}(G^* \times WG \times G^* \times FM_n))
\end{array}
\]

where the left vertical arrow is induced through the coaction (according to the MC element \( m \))
\[\text{Graphs}_n \to \hat{H}(G) \otimes \text{Graphs}_n.\]

Pulling back via the map \( A \to \Omega_{PA}(M) \) we obtain the following corollary, which is a key part of Section 4.4 (see diagram (57)):

**Corollary 25.** The following diagram of homotopy operads under \( A \simeq \Omega_{PA}(M) \) is homotopy commutative:
\[
\begin{array}{cccc}
\text{Graphs}_n^M & \longrightarrow & \text{Tot}(\Omega_{PA}(FM_n^M)) \\
\downarrow & & \downarrow \\
(A \otimes \hat{H}(G) \otimes H(BG) \otimes \text{Graphs}_n, d) & \longrightarrow & \text{Tot}(\Omega_{PA}(Fr_M^W \times G^* \times FM_n)).
\end{array}
\]

We will in fact show that one has a homotopy in the naive sense that one has a map
\[\text{BGraphs}_n \to \text{Tot}(\Omega_{PA}(G^* \times WG \times G^* \times FM_n))[t, dt]\]
compatible with the homotopy cooperadic structure, whose restriction to \( t = 0 \) agrees with the upper rim of the above diagram, and whose restriction to \( t = 1 \) agrees with the lower rim.

To show the statement we need several auxiliary constructions and Lemmas.

**B.1. Two models for \( H(BG) \).** First, we study two models for \( BG \). In the diagram above (Equation (61)), the dga \( (H(BG) \otimes \hat{H}(G) \otimes H(BG), d) \) appears. Here, \( \hat{H}(G) \) is the canonical (Koszul) resolution of the coalgebra \( H(G) \), i.e., it is a cofree coassociative coalgebra cogenerated by \( H(BG)[-1] \) (see also Section 2.5). Put differently, elements of \( \hat{H}(G) \) are words in a basis of monomials in the Pontryagin and Euler classes. A typical element of \( \hat{H}(G) \) is, for example,
\[\{P_3x_2^2\} \{P_{16}(P_8)\} \in \hat{H}(G).\]
The product on such words is the shuffle product, and the differential on $\hat{H}(G)$ merges two adjacent monomials. Within the cdga $H(BG) \otimes \hat{H}(G) \otimes H(BG)$, there is an additional piece of the differential, which takes the first (respectively last) monomial in the word and identifies it with an element of the left (respectively right) factor $H(BG)$. In other words, it is the two-sided cobar construction $\Omega(H(BG), H(BG), H(BG))$, and as such it is a resolution of $H(BG)$.

The second model of $H(BG)$ we consider is as follows. We consider a version of the Cartan model

$$\text{Car} := (\mathbb{R}[u_{ij}, v_{ij}, \ldots])^G,$$

where the indices run from 1 to $n$ each. The generators $u_{ij} = -u_{ji}$ (of $\mathfrak{so}_n[\mathbb{Z}]$) and $v_{ij} = -v_{ji}$ (of a different copy of $\mathfrak{so}_n[\mathbb{Z}]$) have degree +2, while the generators $v_{ij} = -v_{ji}$ have degree +1. The differential is defined such that $dv_{ij} = u_{ij} - \bar{u}_{ij}$.

There is a direct map

$$\left( H(BG) \otimes \hat{H}(G) \otimes H(BG), d \right) \rightarrow \text{Car}$$

defined as follows:

- The left (resp. right) $H(BG) \cong \mathbb{R}[u_{ij}, v_{ij}, \ldots]^G$ is mapped by replacing $u_{ij}$ by $u_{ij}$ (resp. $\bar{u}_{ij}$).

Generally, for $f \in H(BG)$ a polynomial in Euler and Pontryagin classes, we denote (slightly abusively) the image polynomial in variables $u_{ij}$ by

$$f(u_{ij}, \ldots).$$

- Let us formally denote, for $t \in \mathbb{R}$,

$$x_{ij,t} = (1 - t)u_{ij} + t\bar{u}_{ij} - dtv_{ij}.$$

The map $\hat{H}(G) \rightarrow \text{Car}$ sends a word

$$f_1 \cdots f_k$$

to the $k$-fold iterated integral

$$\iiint_{0 \leq t_1 \leq t_2 \leq \ldots \leq t_k \leq 1} f_1(\ldots, x_{ij,t_1}, \ldots) \cdots f_k(\ldots, x_{ij,t_k}, \ldots).$$

**Lemma 26.** The map (63) above is a quasi-isomorphism of dgcas $(H(BG) \otimes \hat{H}(G) \otimes H(BG), d) \rightarrow \text{Car}$.

**Proof.** First, we check that the map intertwines the commutative products. This is clear on the factors $H(BG)$. On the factor $\hat{H}(G)$ it follows from the usual shuffle formula for iterated integrals.

Second, we check that the map commutes with the differentials. This follows easily from Stokes’ Theorem, and the fact that the $x_{ij,t}$ are closed under the combined differential (the differential on the complex plus the de Rham differential in $t$).

Finally, we check the quasi-isomorphism property. Indeed, both inclusions of $H(BG)$ as the first factor in both the domain and the target are a quasi-isomorphism. The result follows by the 2-out-of-3 property of quasi-isomorphisms. \qed

### B.2. Variants of graph complexes and graph cooperads

We will consider here graph complexes $\text{GC}^\text{bi}_n$ and $\text{Graphs}^\text{bi}_n$, which are defined similarly to $\text{GC}_n$ and $\text{Graphs}_n$, except that we distinguish three types of edges. We call these type $u$-edges, $\bar{u}$-edges and $v$-edges, marked by an appropriate letter in drawings. We impose the differential on $\text{Graphs}^\text{bi}_n$

$$d\left(\begin{array}{c} v \\ u \end{array}\right) = \begin{array}{c} u \\ \bar{u} \end{array}$$

In particular the $v$-edges have degree $n - 2$, while the $u$-edges and $\bar{u}$-edges have degree $n - 1$ in $\text{Graphs}^\text{bi}_n$. The other summands of the differential contract the $u$- and $\bar{u}$-type edges, just like in $\text{Graphs}_n$. For $\text{GC}^\text{bi}_n$ we correspondingly have the dual differential and grading conventions. We have natural maps

$$\begin{array}{ccc}
\text{Graphs}_n & \xrightarrow{\phi_0} & \text{Graphs}^\text{bi}_n \\
\phi_1 & \xrightarrow{\phi_1} & \text{Graphs}_n
\end{array}$$

where the map $\phi_0$ on the left send a graph to the same graph with all edges marked by $u$, the map $\phi_1$ marks all edges by $\bar{u}$, and the right-hand map identifies (forgets) colors $u$ and $\bar{u}$ and sends $v$ to zero. All maps here are quasi-isomorphisms.

**Remark 27.** Note that, in fact, $\text{Graphs}^\text{bi}_n$ is a cylinder object for $\text{Graphs}_n$. 24
Dually, we have likewise defined maps on the dual complexes of connected graphs with only internal vertices:

\[(66) \quad GC_n \to GC_n^{bi} \cong GC_n.\]

Recall the Cartan model \(Car = (\mathbb{R}[u_{ij}, v_{ij}, \tilde{u}_{ij}])^G\) of \(H(BG)\). We now define a differential on \(Car \otimes \text{Graphs}^{bi}_n\) such that there is a natural map of homotopy cooperads over \(Car\),

\[\text{BGBGraphs}^{bi}_n := (\text{Car} \otimes \text{Graphs}^{bi}_n, d) \to \text{Tot} \Omega PA(G^* \times G \times G^* \times F\text{M}_n)^{Wr},\]
given as follows:

- The \(u\)-edges are sent to the corresponding equivariant propagators in the \(u_j\).
- The \(\tilde{u}\)-edges are sent to the propagators in the \(\tilde{u}_j\).
- The \(v\)-edges are sent to interpolating forms between the two equivariant propagators, defined similarly to \(x_{ij,t}\) from Equation (64).
- On \(Car\), the map is defined similarly to the map from the non-toric Cartan model from Appendix A.

B.3. Construction. We now describe a general construction that we will apply in the next section. Suppose that \(g\) and \(\mathfrak{h}\) are dg Lie algebras, acting on modules \(U\) and \(V\), respectively. Suppose that we have maps of dg Lie algebras \(f : \mathfrak{h} \to g\) and of modules \(F : U \to V\). Concretely, for \(u \in U\) and \(h \in \mathfrak{h}\):

\[h \cdot F(u) = F(f(h) \cdot u)\]

Suppose that \(\mu \in \mathfrak{g}\) and \(m \in \mathfrak{h}\) are Maurer-Cartan elements. Finally suppose that we have a gauge transformation

\[f(m) \simeq \mu.\]

Such a gauge transformation may be integrated (provided suitable (pro-)nilpotence properties) to a group-like element \(A \in \mathcal{U}g\) satisfying

\[A^{-1} f(m) A = \mu.\]

Under these conditions we may build a map of the twisted dg vector spaces (provided again suitable nilpotence conditions guaranteeing convergence)

\[F^m : U^m \to V^m\]

\[F^m(u) = F(A \cdot u).\]

It is an elementary exercise to verify that this map indeed intertwines the differentials.

Let us remark on a special case of this construction that will be used later. Suppose that in fact our Lie algebras and modules are defined over the ground ring \(\mathbb{R}[t, dt]\), and more specifically, assume that

\[\mathfrak{g} = \mathfrak{g}'[t, dt]\]
\[U = U'[t, dt]\]
\[\mathfrak{h} = \mathfrak{h}'[t, dt]\]
\[V = V'[t, dt],\]

where the actions are extended from actions of \(\mathfrak{g}'\) on \(U'\) and \(\mathfrak{h}'\) on \(V'\). We assume that our MC element \(m\) above has no \(t\)-dependence, i.e., that \(m \in \mathfrak{h}'\). Then \(\tilde{m} = \tilde{m}_t + dt h_t := f(m) \in \mathfrak{g}\) encodes a family of gauge equivalent MC elements \(\tilde{m}_t \in \mathfrak{g}\). Let us choose for the MC element entering the above construction \(\mu := \tilde{m}_0 \in \mathfrak{g}' \subset \mathfrak{g}\). Then indeed \(\mu\) and \(f(m)\) are gauge equivalent MC elements. The gauge equivalence is encoded by the MC element

\[f(m(t \mapsto st)) \in \mathfrak{g}[s, ds]\]

obtained by formally replacing \(t\) by \(st\) in \(\tilde{m} = f(m)\). In this case, one can check that the element \(A \in \mathcal{U}g\) above is (as function of \(t\)) the gauge flow encoded by \(\tilde{m}\) up to time \(t\). In other words, making explicit the time dependence, \(A_t\) is obtained by solving the ODE

\[\dot{A}_t = h_t A_t.\]

Eventually, our construction then produces an \(\mathbb{R}[t, dt]\)-linear map

\[F = U^m \to V^m = (V')^m[t, dt],\]

This map can be seen as an explicit homotopy for the family of maps

\[F^m_t : (U^m)^t \to (V')^m\]

\[F^m_t(u) = F_t(A_t u).\]
In this construction we use only natural operations. Hence, if there is additional structure (dgcas, cooperads) on our spaces $U$ and $V$ and this structure is preserved by the Lie actions, then our homotopy also is compatible with the structure given.

**B.4. Two maps, and the proof of Proposition 24.** We describe next two maps of Hopf cooperads under $H(BG)$

$$F_0, F_1 : B\text{Graphs}_n \to B\text{GBGraphs}_{bi}^n = (\text{Car} \otimes \text{Graphs}_{bi}^n, d).$$

The first sends a graph $\Gamma \in \text{Graphs}_n$ to the same graph with all edges colored $u$. In other words $F_0$ agrees with the $H(BG)$-linear extension of (65). The second map $F_1$ is the composition

$$\Gamma \mapsto F_1(\Gamma) = \phi_1(A\Gamma),$$

where $\phi_1$ is, up to $H(BG)$-linear extension, the map from (65) coloring all edges by $\tilde{u}$. The element $A \in BTB \otimes UGC_n$ is a group-like element obtained as the parallel transport from $t = 0$ to $t = 1$ of the MC element (and in particular flat connection on the interval)

$$m(\ldots, x_{ij,t}, \ldots) \in \text{Car} \otimes \text{GC}_n[t, dt].$$

More concretely, the characteristic property of the element $A$ is that

$$A^{-1}m_1A = m_0,$$

where $m_0 = m(\ldots, u_{ij}, \ldots)$ and $m_1 = m(\ldots, \tilde{u}_{ij}, \ldots)$ are the MC elements at the endpoints of the interval. The construction of the maps $F_0, F_1$ fits exactly the construction of the previous subsection. In particular they intertwine the differentials properly. Furthermore, as we saw in the previous subsection $F_0$ and $F_1$ are homotopic. We shall mark this result in the following Lemma:

**Lemma 28.** The two maps

$$F_0, F_1 : B\text{Graphs}_n \to B\text{GBGraphs}_{bi}^n$$

are homotopic.

Finally we note the following result, which follows by explicit computation.

**Lemma 29.** The upper and lower compositions

$$B\text{Graphs}_n \xrightarrow{F_0} B\text{GBGraphs}_{bi}^n \xrightarrow{F_1} \text{Tot}(\Omega PA(G^* \times WG \times G^* \times FM_n))$$

agree with the upper and lower composition in the diagram (61).

Hence Proposition 24 follows immediately from the preceding two Lemmas and the construction of Section B.3.

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