Weak-Hamiltonian dynamical systems

by

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ABSTRACT. A big-isotropic structure $E$ is an isotropic subbundle of $TM \oplus T^*M$, endowed with the metric defined by pairing. The structure $E$ is said to be integrable if the Courant bracket $[\mathcal{X}, \mathcal{Y}] \in \Gamma E, \forall \mathcal{X}, \mathcal{Y} \in \Gamma E$. Then, necessarily, one also has $[\mathcal{X}, \mathcal{Z}] \in \Gamma E^\perp, \forall \mathcal{Z} \in \Gamma E^\perp$. A weak-Hamiltonian dynamical system is a vector field $X_H$ such that $(X_H, dH) \in E^\perp (H \in C^\infty(M))$. We obtain the explicit expression of $X_H$ and of the integrability conditions of $E$ under the regularity condition $\text{dim}(\text{pr}_{T^*M}E) = \text{const}$. We show that the port-controlled, Hamiltonian systems (in particular, constrained mechanics) [1, 4] may be interpreted as weak-Hamiltonian systems. Finally, we give reduction theorems for weak-Hamiltonian systems and a corresponding corollary for constrained mechanical systems.

1 Big-isotropic structures

In this section we recall some basic facts concerning the big-isotropic structures that were studied in our paper [9]. All the manifolds and mappings are of class $C^\infty$ and we use the standard notation of Differential Geometry, e.g., [5]. In particular, $M$ is an $m$-dimensional manifold, $\chi^k(M)$ is the space of $k$-vector fields, $\Omega^k(M)$ is the space of differential $k$-forms, $\Gamma$ indicates the space of global cross sections of a vector bundle, $X, Y, ..$ are either contravariant vectors or vector fields, $\alpha, \beta, ...$ are either covariant vectors or 1-forms, $d$ is the exterior differential and $L$ is the Lie derivative.

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The vector bundle $T^{big} M = TM \oplus T^* M$ is called the \textit{big tangent bundle}. It has the natural, non degenerate metric of zero signature (neutral metric)

\begin{equation}
  g((X, \alpha), (Y, \beta)) = \frac{1}{2}(\alpha(Y) + \beta(X)),
\end{equation}

the non degenerate, skew-symmetric 2-form

\begin{equation}
  \omega((X, \alpha), (Y, \beta)) = \frac{1}{2}(\alpha(Y) - \beta(X))
\end{equation}

and the Courant bracket of cross sections \cite{3} \cite{3}

\begin{equation}
  [(X, \alpha), (Y, \beta)] = ([X,Y], L_X \beta - L_Y \alpha + \frac{1}{2}d(\alpha(Y) - \beta(X))).
\end{equation}

**Definition 1.1.** A $g$-isotropic subbundle $E \subseteq T^{big} M$ of rank $k$ ($0 \leq k \leq m$) is called a \textit{big-isotropic structure} on $M$. A big-isotropic structure $E$ is \textit{integrable} if $\Gamma E$ is closed by the Courant bracket operation.

From the properties of the Courant bracket (axiom (v) of the definition of a Courant algebroid \cite{6}, see \cite{9}) it follows that if

$$(X, \alpha) \in \Gamma(E), \ (Y, \beta) \in \Gamma(E), \ (Z, \gamma) \in \Gamma(E'),$$

where $E' = E^\perp$ is the $g$-orthogonal bundle of $E$, then

$$g([(X, \alpha), (Z, \gamma)], (Y, \beta)) + g((Z, \gamma), [(X, \alpha), (Y, \beta)]) = 0,$$

whence we see that the integrability of $E$ is equivalent with the property that $[E, E'] \subseteq E'$ (Courant bracket).

The big-isotropic structures are a generalization of the (almost) Dirac structures which are obtained if $k = m$. The reader can find many examples in \cite{9}, in particular the following one which we will use later.

**Example 1.1.** Let $\Sigma$ be a subbundle of rank $k$ of $T^* M$ and $P \in \chi^2(M)$ a bivector field. Then

\begin{equation}
  E_P = graph(\sharp_P|_{\Sigma}) = \{(\sharp_P \sigma = i(\sigma)P, \sigma) / \sigma \in \Sigma\}
\end{equation}

is a big-isotropic structure on $M$ with the $g$-orthogonal bundle

\begin{equation}
  E'_P = \{(\sharp_P \beta + Y, \beta) / \beta \in T^* M, Y \in S = \text{ann} \Sigma\}.
\end{equation}
The structure (1.4) is integrable iff \([9]\): 1) \(\Sigma\) is closed with respect to the bracket of 1-forms defined by
\[
\{\alpha, \beta\}_P = L_{\sharp P} \alpha \beta - L_{\sharp P} \beta \alpha - d(P(\alpha, \beta)),
\]
2) the Schouten-Nijenhuis bracket \([P, P]\) (e.g., \([8]\)) satisfies the condition
\[
[P, P](\sigma_1, \sigma_2, \beta) = 0, \quad \forall \sigma_1, \sigma_2 \in \Sigma, \forall \beta \in T^* M.
\]

We may also define a big-isotropic structure on a vector space (or a vector bundle) \(V\) as an isotropic subspace \(E \subseteq V \oplus V^*\). Then, we get the subspaces \(U_E = \text{pr}_V E\), \(U_{E'} = \text{pr}_V E'\) and a bilinear mapping \(\varpi : U_E \times U_{E'} \to \mathbb{R}\) given by
\[
\varpi(v_1, v_2) = \omega((v_1, a_1), (v_2, a_2)) = a_1(v_2) = -a_2(v_1),
\]
where \((v_1, a_1) \in E, (v_2, a_2) \in E'\) (the equalities hold and the result is independent of the choice of \(a_1, a_2\) because \((v_1, a_1) \perp_g (v_2, a_2)\)). The following result is Proposition 2.1 plus formula (2.17) of \([9]\):

**Proposition 1.1.** For any pair of subspaces \(U_E \subseteq U_{E'} \subseteq V\) and any bilinear mapping \(\varpi : U_E \times U_{E'} \to \mathbb{R}\) with a skew-symmetric restriction to \(U_E \times U_E\), there exists a unique, big-isotropic subspace \(E \subseteq V \oplus V^*\) such that \(U_E = \text{pr}_V E\), \(U_{E'} = \text{pr}_V E'\) and \(\varpi\) is the mapping (1.8). The space \(E\) and the orthogonal space \(E'\) are given by
\[
E = \{ (v, a) / v \in U_E, \forall w \in U_{E'}, a(w) = \varpi(v, w) \},
\]
\[
E' = \{ (w, b) / w \in U_{E'}, \forall v \in U_E, b(v) = -\varpi(v, w) \}.
\]
The dimensions of the spaces above satisfy the following equalities
\[
\text{dim } E = \text{dim } U_E + \text{dim } \text{ann } U_{E'}, \quad \text{dim } E' = \text{dim } U_{E'} + \text{dim } \text{ann } U_E.
\]

**2 Weak-Hamiltonian vector fields**

The aim of this paper is to show that the big-isotropic structures are interesting for physics and control theory because they define a Hamiltonian formalism that may be used in applications.
**Definition 2.1.** A function \( f \in C^\infty(M) \) is a *Hamiltonian*, respectively *weak-Hamiltonian*, function if there exists a vector field \( X_f \in \chi^1(M) \) such that \((X_f, df) \in \Gamma E\), respectively \((X_f, df) \in \Gamma E'\). The vector field \( X_f \) is a *Hamiltonian*, respectively *weak-Hamiltonian*, vector field of \( f \).

The vector field \( X_f \) is not unique; in the Hamiltonian case \( X_f \) is defined up to the addition of any \( Z \in \text{ann } \text{pr}_{T^*M} E' \) and in the weak-Hamiltonian case up to \( Z \in \text{ann } \text{pr}_{T^*M} E \). We denote by \( C^\infty_{\text{ham}}(M, E) \) the set of Hamiltonian functions, by \( C^\infty_{\text{wham}}(M, E) \) the set of weak-Hamiltonian functions and by \( \chi_{\text{ham}}(M, E), \chi_{\text{wham}}(M, E) \), respectively, the sets of Hamiltonian and weak-Hamiltonian vector fields. It follows that \( Z \in \chi_{\text{ham}}(M, E) \) is Hamiltonian, for two functions \( f_1, f_2 \) iff \( df_2 - df_1 \in \text{ann } U_{E'} \) and \( Z \in \chi_{\text{wham}}(M, E) \) is weak-Hamiltonian for \( f_1, f_2 \) iff \( df_2 - df_1 \in \text{ann } U_E \).

Furthermore, if \( f \in C^\infty_{\text{ham}}(M, E) \) and \( h \in C^\infty_{\text{wham}}(M, E) \) the following bracket is well defined

\[
\{f, h\} = \omega(X_f, X_h) = X_fh = -X_hf
\]

and does not depend on the choice of the Hamiltonian vector fields of the function \( f, h \). The bracket \((2.1)\) is called the *Poisson bracket* of the two functions.

Even though it is defined in the general case, the Poisson bracket has interesting properties if \( E \) is an integrable, big-isotropic structure, which we assume for the moment. Then, formula \((1.3)\) shows that \( \{f, h\} \in C^\infty_{\text{wham}}(M, E) \) and one of its weak-Hamiltonian vector fields is \([X_f, X_h]\). If both \( f, h \in C^\infty_{\text{ham}}(M, E) \), their Poisson bracket is skew symmetric and belongs to \( C^\infty_{\text{ham}}(M, E) \). Furthermore, the Poisson bracket satisfies the *Leibniz rule*

\[
\{l, \{f, h\}\} = \{\{l, f\}, h\} + \{f, \{l, h\}\},
\]

\( \forall f \in C^\infty_{\text{ham}}(M, E), h \in C^\infty_{\text{wham}}(M, E) \). Property \((2.2)\) restricts to the Jacobi identity on \( C^\infty_{\text{ham}}(M, E) \). Thus, \( C^\infty_{\text{ham}}(M, E) \) with the Poisson bracket is a Lie algebra and \( C^\infty_{\text{wham}}(M, E) \) is a module over this Lie algebra. Also, \( \chi_{\text{ham}}(M, E) \) is a Lie subalgebra of \( \chi^1(M) \) and \( \chi_{\text{wham}}(M, E) \) is a module over the former for the usual Lie bracket of vector fields.

In what follows integrability will hold only if explicitly postulated. In the remaining part of this section we discuss some big-isotropic structures where one has an explicit expression of a weak-Hamiltonian vector field, a fact that
is important in applications. For instance, for a big-isotropic structure of the form (1.4) formula (1.5) provides such an expression:

\[ (2.3) \quad X_H = \sharp p dH + Y, \quad \forall Y \in S, \quad H \in C^\infty(M). \]

The following proposition extends a result given in [4] for almost Dirac structures.

**Proposition 2.1.** Let \( E \) be a big-isotropic structure on \( M \) such that

\[ (2.4) \quad \dim(\text{pr}_{T^*_xM}E_x) = \text{const.} \quad (x \in M). \]

Then, there exist bivector fields \( \Pi \in \chi^2(M) \) such that if \( H \) is a Hamiltonian, respectively a weak-Hamiltonian, function the formulas

\[ (2.5) \quad X_H = \sharp \Pi dH + Z, \quad Z \in \text{ann } \text{pr}_{T^*_xM}E'_x, \]

respectively

\[ (2.6) \quad X_H = \sharp \Pi dH + W, \quad W \in \text{ann } \text{pr}_{T^*_xM}E, \]

define the Hamiltonian, respectively weak-Hamiltonian, vector fields of \( H \).

**Proof.** For a simpler notation put

\[ (2.7) \quad \Sigma = \text{pr}_{T^*_xM}E, \quad S = \text{ann } \Sigma = (TM \oplus 0) \cap E', \]

\[ \Sigma' = \text{pr}_{T^*_xM}E', \quad S' = \text{ann } \Sigma' = (TM \oplus 0) \cap E; \]

notice that

\[ (2.8) \quad \Sigma \subseteq \Sigma', \quad S' \subseteq S. \]

We shall use Proposition [1.1] for the fibers \( E_x \) of \( E \) (\( x \in M \)) taking \( V = T^*_xM \) and denoting the corresponding bilinear mapping \( \varpi \) by \( P_x : \Sigma_x \times \Sigma'_x \rightarrow \mathbb{R} \). Then, after changing the order of the terms of a pair, formulas (1.9) become

\[ (2.9) \quad E_x = \{(X, \alpha) / \alpha \in \Sigma_x, \beta(X) = P_x(\alpha, \beta), \forall \beta \in \Sigma'_x\}, \]

\[ E'_x = \{(Y, \beta) / \beta \in \Sigma'_x, \alpha(Y) = -P_x(\alpha, \beta), \forall \alpha \in \Sigma_x\}, \]

where

\[ (2.10) \quad P_x(\alpha, \beta) = \frac{1}{2}(\alpha(Y) - \beta(X)), \]
for any choice of $X,Y$ such that $(X,\alpha) \in E_x, (Y,\beta) \in E'_x$ and the result is independent of this choice. Hypothesis (2.4), which will be called the $*$-regularity condition, together with formulas (1.10), show that $\Sigma, \Sigma'$ are subbundles of $T^*M$. Therefore, we may choose bundle decompositions

$$\Sigma' = \Sigma \oplus Q_1, \quad T^*M = \Sigma \oplus Q_1 \oplus Q_2.$$  

Then, we can extend $P$ to a bivector field $\Pi$ by means of the formula

$$\Pi(\lambda, \mu) = P(\lambda', \mu') + P(\lambda', \mu'') - P(\mu', \lambda'') \quad (\lambda, \mu \in T^*M),$$

where $'$ and $''$ denote the first and second projection in the decomposition (2.11) of $T^*M$, and the expressions (2.9) become

$$E_x = \{(X, \alpha) / \alpha \in \Sigma_x, \; X|_{\Sigma_x} = (\sharp_H \alpha)|_{\Sigma_x}\},$$
$$E'_x = \{(Y, \beta) / \sharp_H \beta - Y \in S_x, \; \beta \in \Sigma'_x\}.$$  

The required formulas (2.5), (2.6) are a straightforward consequence of (2.13). \hfill \square

It is obvious that, in fact, only the values of the mapping $P$ actually appear in the expressions of the vector fields (2.5), (2.6) and two bivector fields $\Pi_1, \Pi_2$ produce the same values $X_H$ if they have the same restriction $P$ to $\Sigma \times \Sigma'$. Notice also that the formulas (2.6) and (2.3) differ only by the fact that the former includes the restriction $dH \in \Sigma'$. In view of (2.7), if $(TM \oplus 0) \cap E = 0$ this restriction is void, therefore, any function $H \in C^\infty(M)$ is a weak-Hamiltonian function and formulas (1.4), (1.5) with $P$ replaced by $\Pi$ hold. Still, $\Pi$ is not uniquely defined.

Remark 2.1. It is always possible to consider an arbitrary “Hamiltonian function” $H \in C^\infty(M)$, then restrict to the subset of the points of $M$ where $dH \in \Sigma'$ \[1\].

The following proposition yields the integrability conditions of a $*$-regular, big-isotropic structure.

Proposition 2.2. Let $E$ be a $*$-regular, big-isotropic structure with the associated subbundles $\Sigma, S, \Sigma', S'$ and let $\Pi \in \chi^2(M)$ be such that formulas (2.5), (2.6) hold. Then

$$E = \{(\sharp_H \alpha + Z, \alpha) / \alpha \in \Sigma, Z \in S'\},$$
$$E' = \{(\sharp_H \beta + W, \beta) / \beta \in \Sigma', W \in S\}.$$
The structure $E$ is integrable iff the following conditions are satisfied:

1) the distribution $S'$ is integrable and $S$ is projectable to the space of leaves of $S'$ (see Section 5 of [9] for this notion of projectability);

2) the subbundle $\Sigma$ is closed by the $\Pi$-brackets (1.6) and $\forall \alpha \in \Gamma \Sigma$, $\forall \beta \in \Gamma \Sigma'$ one has $\{\alpha, \beta\} \in \Gamma \Sigma'$;

3) $\forall \alpha_1, \alpha_2 \in \Gamma \Sigma, \beta \in \Gamma \Sigma'$ one has

$$[\Pi, \Pi](\alpha_1, \alpha_2, \beta) = 0.$$ 

Proof. The formulas (2.14) have the same proof like (2.5), (2.6).

If we use the Gelfand-Dorfman formula

$$\Pi(\{\alpha_1, \alpha_2\}\Pi, \beta) = \gamma([\sharp_\Pi \alpha_1, \sharp_\Pi \alpha_2]) + \frac{1}{2}[\Pi, \Pi](\alpha_1, \alpha_2, \beta),$$

we get

$$[(\sharp_\Pi \alpha_1 + Z_1, \alpha_1), (\sharp_\Pi \alpha_2 + Z_2, \alpha_2)]$$

$$= (\sharp_\Pi(\{\alpha_1, \alpha_2\}\Pi - L_{Z_2} \alpha_1 + L_{Z_1} \alpha_2) - \sharp L_{Z_2} \Pi \alpha_1 + \sharp L_{Z_1} \Pi \alpha_2$$

$$+ [Z_1, Z_2] - \frac{1}{2}i(\alpha_1 \wedge \alpha_2)[\Pi, \Pi], \{\alpha_1, \alpha_2\}\Pi - L_{Z_2} \alpha_1 + L_{Z_1} \alpha_2),$$

where $Z_1, Z_2 \in S', \alpha_1, \alpha_2 \in \Sigma$.

The structure $E$ is integrable iff the right hand side of (2.16) belongs to $E$ and we may brake this condition into the cases: a) $\alpha_1 = \alpha_2 = 0$, b) $Z_1 = Z_2 = 0$, c) $Z_1 = 0, \alpha_2 = 0$ (equivalently $Z_2 = 0, \alpha_1 = 0$).

In case a) the condition becomes $([Z_1, Z_2], 0) \in E$, which is equivalent to the first assertion of condition 1) of the proposition.

Furthermore in case b) the bracket (2.16) belongs to $E$ iff the first assertion of condition 2) and condition 3) hold.

Finally, a technical computation shows that if $\alpha \in \Sigma, \beta \in \Sigma', Z \in S' \subseteq S$ then

$$L_{Z}\Pi(\alpha, \beta) = \{\alpha, \beta\}\Pi(Z).$$

Now, in case c) the right hand side of (2.16) is

$$-(\sharp_\Pi(L_{Z_2} \alpha_1) + \sharp L_{Z_2} \Pi \alpha_1, L_{Z_2} \alpha_1),$$
which belongs to $E$ iff

$$L_Z \alpha_1 \in \Sigma, \; \sharp L_Z \Pi \alpha_1 \in S'.$$

From (2.7) and (2.17) it follows that the two conditions mentioned above are equivalent with the second assertions of 1) and 2), respectively. \qed

**Remark 2.2.** Let $E$ be an integrable, $*$-regular, big-isotropic structure. Then, Corollary 5.1 of [9] shows that $E$ is projectable with respect to the foliation $S'$, and the projection of $E$ onto the local spaces of the slices of $S'$ is an integrable, big-isotropic structure of the type discussed in Example 1.1.

### 3 Port-controlled dynamical systems

In this section we present some applications where weak-Hamiltonian vector fields can be used. Following [4], a physical network is a sum of port-controlled, generalized, Hamiltonian systems with interconnections. Many concrete examples, in particular constrained mechanics, are discussed in [1, 4]. We shall give weak-Hamiltonian interpretations of such port-controlled systems.

With the notation of [4], a port-controlled, generalized, Hamiltonian system is a system of equations of the following form

$$\begin{align*}
\dot{x} &= J(x) \frac{\partial H}{\partial x}(x) + g(x)f, \\
\dot{e} &= g^T(x) \frac{\partial H}{\partial x}(x),
\end{align*}$$

where a dot denotes time-derivative and one uses the matrix notation. In (3.1) $x = (x_i) \ (i = 1, \ldots, n)$ is the column of energy variables, which are local coordinates on a manifold $N$ seen as the phase space, $H$ is the total stored energy, $J$ is a skew-symmetric $(n, n)$-matrix, $f = (f_j) \in \mathbb{R}^p \ (j = 1, \ldots, p)$ is the column of flows, $g$ is an $(n, p)$-matrix, $e = (e_j)$ is the column of efforts and $T$ denotes matrix transposition.

The evolution of the system is defined by the differential equations on the first line of (3.1) where a choice of functions $f_j = f_j(x)$ is made. If we see $J$ as a bivector field on $N$ and $g$ as a vector bundle morphism $g : N \times \mathbb{R}^p \to TN$, these differential equations are equivalent with the weak-Hamiltonian vector field

$$X_H = \sharp J dH + gf$$
of the function $H$ with respect to the big-isotropic structure

$$E_J = graph(\sharp J|\Sigma), \quad \Sigma = ann S,$$

where $S$ is any distribution on $N$ such that $img \subseteq S$. If $\text{rank } g = \text{const.}$ and $S = img$, formula (3.2) is that of all the weak-Hamiltonian vector fields of $H$. Since (3.3) is of the type (1.4) we see that a port-controlled system has a weak-Hamiltonian interpretation with respect to an integrable big-isotropic structure iff there exists a subbundle $S \subseteq TM$ that contains $img$, $ann S$ is closed by the bracket (1.6) for $P = J$ and $J$ satisfies the condition (1.7).

Moreover, we can show that the whole system (3.1) may be seen as a weak-Hamiltonian vector field on $M = N \times \mathbb{R}^p$. For this purpose, notice that $g$ defines a bivector field $G \in \chi^2(M)$ given by

$$G_{(x,f)}(\alpha' + \alpha'', \beta' + \beta'') = \beta''(g^T(x)\alpha') - \alpha''(g^T(x)\beta'),$$

where $x \in N$, $f \in \mathbb{R}^p$, $\alpha', \beta' \in T_x^* N$, $\alpha'', \beta'' \in T_f^* \mathbb{R}^p \approx \mathbb{R}^p$ and $g^T(x) : T_x^* N \rightarrow \mathbb{R}^p$. Then, we have the bivector field $P = J + G \in \chi^2(M)$ and the weak-Hamiltonian vector field

$$X_H = \sharp J + G dH + gf$$

of $H$ with respect to any big-isotropic structure $graph(\sharp J+G|\Sigma)$, where $S = ann \Sigma$ is a regular distribution on $N$ that contains $img$. The integral lines of the vector field (3.5) are given by (3.1) where $e_j$ are the time derivatives of the coordinates of the factor $\mathbb{R}^p$ of $M$ and one uses the natural identification of $T \mathbb{R}^p$ with $\mathbb{R}^p$. The integrability conditions of $graph(\sharp J+G|\Sigma)$ are provided by (1.6) and (1.7) again.

In [4] one also defines port-controlled Hamiltonian systems with constraints, which have the form

$$\dot{x} = J(x) \frac{\partial H}{\partial x}(x) + g(x)f + b(x)\lambda,$$
$$e = g^T(x) \frac{\partial H}{\partial x}(x), \quad 0 = b^T(x) \frac{\partial H}{\partial x}(x)$$

where the notation is like in (3.1), $b$ is an $(n,k)$-matrix and $\lambda \in \mathbb{R}^k$. As in the non-constrained case, the system (3.3) is a weak-Hamiltonian system on $M = N \times \mathbb{R}^{p+k}$, where the Hamiltonian function $H$ is required to satisfy the constraint $b^T(dH) = 0$.

Consider the port-controlled system (3.1) again. It is called energy-preserving [4] if the vectors $f \in \mathbb{R}^p, e \in \mathbb{R}^p \approx (\mathbb{R}^p)^*$ are assumed to satisfy
the condition \((f, e) \in \Delta(x)\) where \(\Delta(x)\) is a maximal (i.e., \(p\)-dimensional), isotropic subspace of \(\mathbb{R}^p \times \mathbb{R}^*\) parameterized by \(x \in N\). The reason for this name is that, then, the energy \(H\) is preserved along the integral lines of the vector field (3.2) of the system. Indeed, in view of the second equation (3.1) and since \((f, e) \in \Delta\) implies \(e(f) = 0\), we have

\[ \dot{H} = X_H H = 0. \]

Then, it turns out that the differential equations of the first line of (3.1) are equivalent with a Hamiltonian vector field with respect to an almost Dirac structure. We give a more conceptual proof of this result proven differently in Proposition 2.2 of [4].

With the notation of (3.2), put

\[ D = \left\{ (\sharp_J \alpha + g f, \alpha) / (f, g^T \alpha) \in \Delta \right\} \subseteq TN \oplus T^* N. \]

The isotropy of \(\Delta\) implies that \(D\) is a big-isotropic structure on \(N\) and we shall compute \(\dim D\) for any fixed point \(x \in N\). Denote \(\Delta' = \Delta \cap (\mathbb{R}^p \times \text{im } g^T)\). Then the correspondence

\[ (\sharp_J \alpha + g f, \alpha) \mapsto (f, g^T \alpha) \]

produces a surjective homomorphism

\[ \phi : D \to \Delta'/\Delta' \cap (\ker g \oplus 0) \]

with

\[ \ker \phi = \left( (\sharp_J \alpha, \alpha) / g^T \alpha = 0 \right), \]

whence,

\[ \dim \ker \phi = n - \text{rank } g \ (n = \dim N). \]

On the other hand, if we notice that

\[ \mathbb{R}^p \times \text{im } g^T = (\ker g)^\perp \]

(perpendicularity is with respect to the neutral metric of \(\mathbb{R}^p \oplus \mathbb{R}^{p*}\) and the result holds because the two spaces are orthogonal and the sum of their dimensions is \(2p\)), we get

\[ \Delta' = \Delta \cap (\ker g)^\perp = \Delta^\perp \cap (\ker g)^\perp = (\Delta + \ker g)^\perp \]
($\Delta^\perp = \Delta$ because of the maximal isotropy of $\Delta$). Now, if $\dim(\Delta \cap \ker g) = i$ the known formula

$$\dim(\Delta + \ker g) = \dim \Delta + \dim \ker g - i,$$

implies

$$\dim \Delta' = 2p - \dim(\Delta + \ker g) = \rang g + i.$$

Together with (3.8) and (3.9), the previous result gives $\dim D = (n - \rang g + [(\rang g + i) - i] = n$, hence, $D$ is an almost Dirac structure. Furthermore, for $X_H$ given by (3.2) and since we asked that $(f, e) \in \Delta$, we have $(X_H, dH) \in \Gamma D$ and $X_H$ is a Hamiltonian vector field of $H$.

**Remark 3.1.** The systems discussed in [1, 4] are direct sums of port-controlled systems on a product manifold where the components may not be energy preserving but the sum is such. These are energy-preserving physical networks and the corresponding $\Delta$ is a power-preserving interconnection between the port-controlled components [4].

**Remark 3.2.** The structure (3.7) may present a technical difficulty: even if $\Delta(x)$ is differentiable with respect to $x \in N$, $D$ may not be differentiable. For instance, if $\Delta(x) = \mathbb{R}^p \oplus 0$ one has

$$D = \{(\sharp_J \alpha + Z, \alpha) / \alpha \in \ann \im g, Z \in \im g\}$$

(3.10)

and $D$ is not differentiable if $\rang g$ is not constant. If $\rang g = \text{const.}$, (3.10) has the same form as $E$ of (2.14), with $E' = E = D$, and the integrability conditions will be like in Proposition 2.2 i.e., : 1) $\im g$ is integrable, 2) $\ann \im g$ is closed by the $J$-bracket of 1-forms, 3) $[J, J]|_{\ann \im g} = 0$.

It was shown in [1, 4] that the dynamical systems (3.1) include the constrained mechanical systems. Here, we give a straightforward, weak-Hamiltonian interpretation of a constrained mechanical system.

A mechanical system has a configuration space, which is a manifold $Q$, the space of the velocities, which is the tangent bundle $TQ$, and the space of the phases, which is the cotangent bundle $T^*Q$. Constraints consist of a $k$-dimensional distribution $L$ on $Q$. In Hamiltonian mechanics, the differential equations of the motion are those of the integral lines of a vector field of the form

$$X = \sharp p dH + \sharp_p (\pi^* \alpha) \in \chi^1(T^*Q),$$

(3.11)
where $P$ is defined by $\sharp_P \circ \flat_{\omega} = -Id$, $\omega$ being the canonical symplectic form of $T^*Q$ ([7], Section 6.2), $H$ is the Hamiltonian of the system, $\pi : T^*Q \to Q$ is the natural projection and $\alpha \in \text{ann } L$ (e.g., [1]).

The constraint distribution $L$ produces a natural, $\omega$-isotropic subbundle

$$S_L = \{\sharp_P(\pi^*\alpha) / \alpha \in \text{ann } L\} \subseteq T(T^*Q).$$

The corresponding $\omega$-orthogonal subbundle is

$$S_L^\perp = \text{ann}(\pi^*(\text{ann } L)) = \{X \in T(T^*Q) / \pi^*X \in L\}.$$

A comparison with formula (2.3) shows that the vector field (3.11) is weak-Hamiltonian with respect to the big isotropic structure $E_L = \text{graph}(\sharp_P|_{\text{ann } S_L})$.

The structure $E_L$ is integrable iff $\text{ann } S_L$ is closed by the bracket (1.6); the fact that $\omega$ is a symplectic form implies the Poisson condition $[P, P] = 0$, hence, (1.7) holds too. Notice that $\sigma \in \text{ann } S_L$ is equivalent with $\sharp_P\sigma \in S_L^\perp$ and, since [8]

$$\sharp_P\{\sigma_1, \sigma_2\}_P = [\sharp_P\sigma_1, \sharp_P\sigma_2],$$

it follows that $E_L$ is integrable iff the distribution $S_L^\perp$ is integrable. Now, let us recall that $L$ itself is integrable iff, $\forall \alpha \in \text{ann } L$, $d\alpha$ belongs to the ideal spanned by $\text{ann } L$. Since $\text{ann } S_L^\perp = \pi^*(\text{ann } L)$ and $\pi^*$ is injective, the same condition characterizes the integrability of $S_L^\perp$. Therefore, like in the Dirac interpretation of [1], the structure $E_L$ is integrable iff $L$ is integrable, i.e., iff the system has holonomic constraints.

### 4 Symmetries and reduction

In this section we extend some results on symmetries and reduction from Hamiltonian to weak-Hamiltonian systems. The case of Hamiltonian systems on a Dirac manifold was treated in [12].

**Definition 4.1.** A vector field $Z \in \chi^1(M)$ is an infinitesimal symmetry of a big-isotropic structure $E$ if

$$\langle L_ZX, L_Z\alpha \rangle \in \Gamma E, \quad \forall (X, \alpha) \in \Gamma E.$$

A diffeomorphism $\varphi : M \to M$ is a symmetry of $E$ if

$$\langle \varphi_*X, \varphi^*^{-1}\alpha \rangle \in \Gamma E, \quad \forall (X, \alpha) \in \Gamma E.$$
Obviously, the flow of an infinitesimal symmetry consists of symmetries of $E$. Furthermore, for (infinitesimal) symmetries the conditions required for $E$ also hold for the $g$-orthogonal space $E'$ of $E$ because the neutral metric $g$ is invariant by any (infinitesimal transformation) diffeomorphism of $M$.

**Proposition 4.1.** Let $E$ be a $\ast$-regular, big-isotropic structure defined by formulas (2.14). 1). The diffeomorphism $\varphi : M \to M$ is a symmetry of $E$ iff the subbundles $S, S'$ are invariant by $\varphi_\ast$ and for all $\beta \in \Sigma'$ one has $\sharp_{\varphi_\ast(\Pi_\varphi^{-1})\beta} = \sharp_{\Pi \beta}$. 2). The vector field $Y \in \chi^1(M)$ is an infinitesimal symmetry of $E$ iff $\forall Z \in \Gamma S', \forall \alpha \in \Gamma \Sigma, \forall \beta \in \Gamma \Sigma'$ one has

$$[Y, Z] \in \Gamma S', L_Y \alpha \in \Gamma \Sigma, L_Y \Pi(\alpha, \beta) = 0.$$  

The conditions stated in 1), 2) depend only on the mapping $P$ defined by (2.10).

**Proof.** The notation used here is that of formula (2.14).

1). If $\varphi$ is a symmetry then, for all $\alpha \in \Sigma$, $Z \in S'$, we have

$$\varphi_\ast (\sharp_{\Pi \alpha} + Z), \varphi^\ast \alpha) = (\sharp_{\varphi_\ast \Pi}(\varphi^\ast \alpha) + \varphi_\ast Z, \varphi^\ast \alpha) \quad \text{(2.14)}$$

where $U \in S'$. The same must hold for all $\alpha \in \Sigma'$, $Z \in S$ with $U \in S$ because $\varphi$ also preserves the orthogonal subbundle $E'$. It is easy to derive 1) from (4.4) and to see that 1) also is the sufficient condition for (4.4) to hold.

2). From the first formula (2.14), we see that $Y$ is an infinitesimal symmetry iff for all $\alpha \in \Sigma$ one has $L_Y \alpha \in \Sigma$ and $\sharp_{L_Y \Pi \alpha} + [Y, Z] \in \Gamma S'$. By looking at the cases $\alpha = 0$ and $Z = 0$ separately we get the required conclusion.

The last assertion of the proposition is obvious.

An infinitesimal symmetry $Z$ acts on Poisson brackets as a derivation. Indeed, take $f \in C^\infty_{ham}(M, E)$, $h \in C^\infty_{wham}(M, E)$ and corresponding pairs $(X_f, df) \in \Gamma E, (X_h, dh) \in \Gamma E'$. From (4.1), it follows that

$$([Z, X_f], d(Z f)) \in \Gamma E, ([Z, X_h], d(Z h)) \in \Gamma E',$$

whence

$$Z \{f, h\} = Z(X_f h) = [Z, X_f ] h + X_f (Z h) = \{ Z f, h \} + \{ f, Z h \}.$$
If the structure $E$ is integrable, any Hamiltonian vector field $Z \in \chi_{\text{ham}}(M, E)$ is an infinitesimal symmetry. Indeed, assume that $Z = X_f, f \in C^\infty(M)$, and $(X, \alpha) \in \Gamma E$. The integrability of $E$ implies
\[
[(X_f, df), (X, \alpha)] = ([X_f, X], L_{X_f} \alpha) \in \Gamma E,
\]
which is the required symmetry property.

Let $H$ be a weak-Hamiltonian function on $(M, E)$. Then, we are interested in $H$-preserving, infinitesimal and global symmetries, i.e., vector fields $Z$ that satisfy (1.1) and $ZH = 0$, and diffeomorphisms $\varphi : M \to M$ that satisfy (1.2) and $H \circ \varphi = H$. The following proposition is in the spirit of Noether’s theorem [1, 7].

**Proposition 4.2.** Let $E$ be an integrable, big-Hamiltonian structure on $M$. A Hamiltonian vector field $Z \in \chi_{\text{ham}}(M, E)$ is an $H$-preserving infinitesimal symmetry for $H \in C^\infty_{\text{wham}}(M, E)$ iff $Z$ is the Hamiltonian vector field of a function $f$ such that $\{f, H\} = 0$.

**Proof.** We already know that $Z$ is an infinitesimal symmetry. Then, the orthogonality of the pairs $(Z, df), (X_H, dH)$ gives
\[
ZH - \{f, H\} = 0
\]
and this shows the equivalence between $ZH = 0$ and the condition required by the proposition. \hfill \Box

We may define a first integral of a weak-Hamiltonian dynamical system $X_H$ to be a function $f \in C^\infty_{\text{wham}}(M, E)$ such that $\{f, H\} = 0$ But, the usual properties of first integrals hold only in the integrable case; then, the Hamiltonian vector fields $X_f$ of the first integral $f$ are $H$-preserving infinitesimal symmetries and the Poisson bracket of two first integrals of $X_H$ is a first integral again because of the Leibniz property (2.2).

Now, let us refer to reduction. In [9] we discussed the reduction of a big-isotropic structure $E$ on $M$ and we recall the main results. Let $\iota : N \to M$ be an embedded submanifold of $M$. Then, the formula
\[
(4.5) \quad \iota^*(E_x) = \{(X, \iota^* \alpha) / X \in T_xN, \alpha \in T^*_xM, (X, \alpha) \in E_x\},
\]
where $x \in N$, defines the pullback $\iota^*E$ of $E$ to $N$. $\iota^*E$ is a field of big-isotropic subspaces of $T^{\text{big}}(M)$ and, if this field is a differentiable subbundle of $T^{\text{big}}(M)$, we say that the submanifold $N$ is $E$-proper with the induced big-isotropic structure $\iota^*E$. Moreover, if $E$ is integrable the same holds for $\iota^*E$. 

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Remark 4.1. A $E$-proper submanifold $\iota : N \to M$ of $(M, E)$ may be seen as a \textit{general constraint} and a \textit{constrained weak-Hamiltonian system} may be defined as a weak-Hamiltonian vector field $X_H \ (H \in C^\infty(M))$ on the manifold $(N, \iota^*E)$.

Furthermore, assume that the $E$-proper submanifold $N$ of $M$ has a foliation $\mathcal{F}$ with a paracompact, Hausdorff quotient manifold $Q = N/\mathcal{F}$ and the natural projection $\pi : N \to Q$. Then, the formula

$$\pi_*(\iota^*E_x) = \left\{ (\pi_*X, \alpha) \mid X \in T_xN, \alpha \in T_{\pi(x)}^*Q, (X, \pi^*\alpha) \in \iota^*E_x \right\}$$

defines a big-isotropic subspace of $T^\text{big}_{\pi(x)}Q, \forall x \in N$.

Assume that the following two \textit{reducibility conditions} are satisfied:

\begin{enumerate}
  \item R1) $T_{\mathcal{F}} \oplus 0 \subseteq \iota^*E$,
  \item R2) every vector field $Y \in \chi^1(N)$ that is tangent to $\mathcal{F}$ is an infinitesimal symmetry of $\iota^*E$.
\end{enumerate}

Then, $E^\text{red} = \pi_*(\iota^*E)$ given by (4.6) is a well defined, big-isotropic structure on $Q$ called the \textit{reduced structure} of $E$ via $(N, \mathcal{F})$. Moreover, if $E$ is integrable condition R1) implies R2) and the reduced structure $E^\text{red}$ is integrable too [9].

Theorem 4.1. Let $E$ be a big-isotropic structure on the manifold $M$. Assume that the connected, Lie group $G$ acts on $M$ by symmetries of $E$ that keep fixed an embedded submanifold $\iota : N \to M$. Assume that the restriction of the action of $G$ to $N$ is proper and free and denote by $\mathcal{F}$ the foliation of $N$ by the orbits of $G$. Finally, assume that the following reducibility condition holds

$$R) \text{ for any infinitesimal transformation } Z \text{ of } G, \exists \alpha \in \text{ann} \, T_N N \text{ such that } (Z|_N, \alpha) \in E|_N.$$  

Then, there exists a Hausdorff manifold $Q = N/\mathcal{F}$ endowed with a reduced, big-isotropic structure $E^\text{red}$ and if $E$ is integrable $E^\text{red}$ is integrable too.

Proof. If $E$ is integrable, this is Corollary 5.2 of [9]. But, the fact that condition R) is equivalent with R1) holds in the non-integrable case too. Condition R2) holds for the infinitesimal transformations $Z$ of $G$ on $N$ because of the invariance of $E$ and $N$. This implies the fact that any vector field spanned by such infinitesimal transformations is also an infinitesimal symmetry of $\iota^*E$.

Indeed, for any $f \in C^\infty(N)$ and $(X, \iota^*\alpha) \in \Gamma \iota^*E$ one has

$$(L_fZX, L_fZ(\iota^*\alpha)) = f(L_ZX, L_Z(\iota^*\alpha)) - (Xf)(Z, 0) + (\iota^*\alpha)(Z)(0, df),$$

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where \((L_Z X, L_Z (\iota^* \alpha)) \in \iota^* E\) because \(Z\) is an infinitesimal symmetry, \((Z, 0) \in \iota^* E\) by R1), and \((\iota^* \alpha)(Z) = 0\) because the isotropy of \(\iota^* E\) implies \((X, \iota^* \alpha) \perp_g (Z, 0)\). Hence, R2) holds as stated and we are done.

Theorem 4.1 is straightforwardly enhanced by the following result, which we call a theorem because of its in-principle importance.

**Theorem 4.2.** Assume that the notation and hypotheses of Theorem 4.1 hold and that we have a \(G\)-invariant, weak-Hamiltonian function \(H \in C_{\text{wham}}^\infty(M, E)\) with a weak-Hamiltonian vector field \(X_H\) such that \(X_H(x) \in T_x N, \forall x \in N\) and \(X_H|_N\) is \(F\)-projectable. Then, the function \(H|_N\) is the lift by \(\pi\) of a function \(H^{\text{red}} \in C_{\text{wham}}^\infty(Q, E^{\text{red}})\) and \(\pi_* (X_H|_N)\) is a weak-Hamiltonian vector field \(X^{\text{red}}_H \in \chi^1(Q)\) of \(H^{\text{red}}\).

**Proof.** Notice that \((X_H|_N, d(H \circ \iota)) \in \iota^* E'\), where the latter is defined like \(\iota^* E\) and is equal to the orthogonal space \((\iota^* E)'\) [9]. The existence of \(H^{\text{red}}\) and \(X^{\text{red}}_H\) is obvious and (4.6) shows that \((X^{\text{red}}_H, dH^{\text{red}}) \in (E^{\text{red}})'\).

**Remark 4.2.** Each of the following two conditions: i) \(X_H\) is \(G\)-invariant, ii) \(\iota^* E' \cap (TN \oplus 0) = TF\) implies the \(F\)-projectability of \(X_H|_N\). Under condition i), it is obvious that \(X_H|_N\) is \(F\)-projectable. Furthermore, if \(Z \in \chi^1(M)\) is an infinitesimal action of \(G\), \(Z\) is an infinitesimal symmetry of \(E\) and \((X_H, dH) \in E'\) implies \((L_Z X_H, L_Z dH) = ([Z, X_H], 0) \in E'\). Since both \(Z\) and \(X_H\) are tangent to \(N\), we get \(([Z, X_H]|_N, 0) \in \iota^* E'\) and, if hypothesis ii) holds, \([Z, X_H]|_N \in T F\). Therefore, again, \(X_H|_N\) is projectable to \(Q\).

Thus, we can simplify the integration of a weak-Hamiltonian, dynamical system by reduction if we have a convenient group of symmetries and a nice invariant submanifold.

Like for the usual Hamiltonian systems, the required submanifold may come from a momentum map. We will say that an \(E\)-preserving action of a connected Lie group \(G\) on \((M, E)\) is a Hamiltonian action if the infinitesimal transformations \(Z\) of \(G\) are Hamiltonian vector fields, i.e., \(\exists f \in C^\infty(M)\) such that \((Z, df) \in \Gamma E\). Like in the Poisson case (e.g., [8], Proposition 7.25), it follows that the action is Hamiltonian iff it preserves \(E\) and \(\exists J \in C^\infty(M, \mathcal{G}^*)\) such that

\[
(4.7) \quad (\xi_M, d(\xi \circ J)) \in \Gamma E, \quad \forall \xi \in \mathcal{G},
\]

where \(\mathcal{G}\) is the Lie algebra of \(G\) and \(\xi_M\) is the infinitesimal action of \(\xi\) on \(M\). Such a function \(J\) is a momentum map. Notice that if \(E\) is integrable and the
action has a momentum map then the action necessarily preserves $E$ because the Hamiltonian vector fields are infinitesimal symmetries of $E$. Finally, a momentum map $J$ is equivariant if $J(g(x)) = coad_g(J(x)), \forall g \in G, x \in M$.

From Theorems 4.1 and 4.2 and with the notation there, we get

**Corollary 4.1.** Consider an action of $G$ on $M$ that preserves $E$ and has an equivariant momentum map $J : M \to G^*$ such that $0$ is a regular value of $J$. Assume that $G$ acts properly and freely on the $G$-invariant submanifold $N = J^{-1}(0) \subseteq M$ giving rise to the quotient manifold $Q = N/F$ where the leaves of $F$ are the orbits of $G|_N$. Then $Q$ has the reduced, big-isotropic structure $E_{\text{red}}$ of $E$, which is integrable if $E$ is integrable. Furthermore, consider a pair $(X_H, dH) \in \Gamma E'$ where $H \in C^\infty_{\text{wham}}(M, E)$ is $G$-invariant and $X_H|_N$ is $F$-projectable. Then $(X_H, dH)|_N$ projects to a pair $(X_{H_{\text{red}}}, dH_{\text{red}}) \in \Gamma(E_{\text{red}})'$, and one has a reduced, weak-Hamiltonian system on $Q$.

**Proof.** For the first assertion we just have to check condition R). If $Z = \xi_M$ for $\xi \in G$ then $(Z, d(\xi \circ J)) \in \Gamma E$ and, since $J$ is constant on $N$, $d(\xi \circ J) \in \text{ann} TN$, which is the required condition. For the second assertion we have to check that $X_H$ is tangent to $N$. This holds because the invariance of $H$ implies $X_H(\xi \circ J) = -X_{\xi_0J}H = -\xi_M H = 0$.

**Remark 4.3.** The $G$-invariance of $H$ is equivalent with $\{\xi \circ J, H\} = 0$, $\forall \xi \in G$. Hence, like in symplectic mechanics, given a system $X_{H^\gamma}$ on $(M, E)$, we should look for symmetry groups $G$ that lead to reduction by looking for first integrals $f_i$ of $H$ such that $X_{f_i}$ are infinitesimal symmetries of $E$ and $\text{span}\{X_{f_i}\}$ is a Lie algebra.

**Remark 4.4.** We can reformulate Corollary 4.1 for an arbitrary non-critical value $\gamma$ of $J$ and the level set $N = J^{-1}(\gamma)$. Indeed, if the group $G$ satisfies the hypotheses of Corollary 4.1 and $G'$ is a connected subgroup of $G$ with the Lie algebra $i : G' \subseteq G$, it follows easily that $J' = i^T \circ J$ is an equivariant momentum map of the action of $G'$ on $M$. In particular, if $G' = G_{\gamma}$ is the isotropy subgroup of $\gamma \in G^*$ with respect to the coadjoint action we have $J^{-1}(\gamma) = J^{-1}(0)$, and we may use Corollary 4.1 for the connected component of the unit of $G_{\gamma}$ instead of $G$. The result will be a version of the Marsden-Weinstein reduction theorem in the present context.

We finish by discussing the application of Corollary 4.1 to the constrained mechanical system described at the end of Section 3, with the notation used there, i.e., the configuration space is $Q$, the constraint distribution is $L \subseteq TQ$.
and the associated big-isotropic structure is $E_L$. Assume that $G$ is a connected Lie group acting on $Q$ such that the distribution $L$ is strongly invariant, by which we mean the following two conditions: a) $\forall g \in G, g_*(L) = L$, b) $\forall x \in M, T_x(G(x)) \subseteq L_x$ (G-orbit of the point $x$). Then, the derivative mappings yield a group $G^g\ast$ that acts on the phase space $T^*Q$ by symplectomorphisms of the canonical symplectic form $\omega$ and preserves the big-isotropic structure $E_L$. Furthermore, there exists a well known, equivariant, momentum map $J : T^*Q \to G^\ast$ for the symplectic structure of $T^*Q$ defined by

$$< J(\alpha), \xi > = < \alpha, \xi_Q > \quad (\alpha \in T^*Q, \xi \in G)$$

(e.g., [7], Theorem 12.1.4).

The fact that $J$ is a momentum map for $\omega$ means that we have

$$\xi_{T^*Q} = \sharp_P d(\xi \circ J) \quad (\sharp_P \omega = -Id). \quad (4.8)$$

But, condition b) of the strong invariance of $L$ also implies

$$< d(\xi \circ J), \sharp_P(\pi^*\alpha) > = -\pi^*\alpha(\xi_{T^*Q}) = -\alpha(\xi_Q),$$

which vanishes because of b). Thus, $J$ also is a momentum map with respect to the structure $E_L$ and we get

**Corollary 4.2.** Let $(Q, L)$ be a constrained mechanical system with the Hamiltonian function $H$ and the Hamiltonian vector field $X_H$. Assume that the connected Lie group $G$ acts on $Q$ such that: 1) $G$ strongly preserves $L$, 2) $G^{g\ast}$ preserves the pair $(H, X_H)$. Let $J : T^*Q \to G^\ast$ be the naturally associated momentum map and assume that 0 is a regular value of $J$ and that the orbits of $G^{g\ast}\big|_N$ are the leaves of a foliation $\mathcal{F}$ of $N$ by the leaves of a submersion $\pi : N \to Q$, where $Q$ is a Hausdorff, differentiable manifold. Then the system admits a reduction to $Q$ via $(N, \mathcal{F})$.

Notice that the constraints may be non-holonomic.

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References

[1] G. Blankenstein and T. S. Ratiu, Lectures on Dirac Structures and Port-Hamiltonian Systems, Summer School and Conference on Poisson Geometry, SMR1665, ICTP, Trieste, Italy, 2005, http://poisson.zetamu.com/Notes%20and%20Slides/BlankensteinRatiu_notes.pdf.

[2] G. Blankenstein, A. J. van der Schaft, Symmetries and conservation laws for implicit port-controlled Hamiltonian systems, Proceedings IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control, Editors N.E. Leonard, R. Ortega, Princeton, U.S.A., 98-103, 2000.

[3] T. J. Courant, Dirac Manifolds, Transactions Amer. math. Soc., 319 (1990), 631-661.

[4] M. Dalsmo and A. J. van der Schaft, On representations and integrability of mathematical structures in energy conserving physical systems, SIAM J. Control Optim., 37 (1998), 54-91.

[5] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry I,II. Intersc. Publ., New York, 1963, 1969.

[6] Z.-J. Liu, A. Weinstein and P. Xu, Manin triples for Lie bialgebroids, J. Diff. Geom., 45 (1997), 547-574.

[7] J. E. Marsden and T. S. Ratiu, Introduction to Mechanics and Symmetry, Texts in Appl. Math., vol. 17, Second ed., Springer-Verlag, New York, 1999.

[8] I. Vaisman, Lectures on the geometry of Poisson manifolds, Progress in Math., Vol. 118, Birkhäuser Verlag, Basel, 1994.

[9] I. Vaisman, Isotropic subbundles of $TM \oplus T^*M$, arXiv:math.DG/0610522

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