ONE VERSION OF THE CLARK REPRESENTATION
THEOREM FOR ARRATIA FLOW

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Abstract. The article contains description of the functionals from the family of coalescing Brownian particles. New type of the stochastic integral is introduced and used.

Introduction

The aim of this article is to establish the Clark representation for the functionals from the Arratia flow of coalescing Brownian particles [1-4]. The following description of this flow will be used. We consider the random process \( \{x(u); u \in \mathbb{R}\} \) with the values in \( C([0; 1]) \) such, that for every \( u_1 < ... < u_n \)

1) \( x(u_k, \cdot) \) is the standard Wiener process starting at the point \( u_k \),

2) \( \forall t \in [0; 1] \)

\[
x(u_1, t) \leq ... \leq x(u_n, t),
\]

3) The distribution of \( (x(u_1, \cdot), ..., x(u_n, \cdot)) \) coincides with the distribution of the standard \( n \)-dimensional Wiener process starting at \( (u_1, ..., u_n) \) on the set

\[
\{ f \in C([0; 1], \mathbb{R}^n) : f_k(0) = u_k, k = 1, ..., n, f_1(t) < ... < f_n(t), t \in [0; 1] \}. \]

Roughly speaking the process \( x \) can be described as a family of Wiener particles which start from every point of \( \mathbb{R} \), move independently up to the moment of the meeting then coalesce and move together.

The following fact is well-known. If \( \{w(t); t \in [0; 1]\} \) is a standard Wiener process and square-integrable random variable \( \alpha \) is measurable with respect to \( w \), then \( \alpha \) can be represented as a sum

\[
\alpha = E\alpha + \int_0^1 f(t)dw(t),
\]

with the usual Ito stochastic integral in the right side. Our aim is to establish the variant of this theorem for the case, when \( \alpha \) is measurable with respect to Arratia flow \( \{x(u, t); u \in [0; U], t \in [0; 1]\} \). It follows from the description above, that the Brownian motions \( \{x(u, \cdot); u \in [0; U]\} \) are not jointly Gaussian. So, the original Clark theorem can not be used in this situation. The article is divided onto three parts. In the first part the construction of the stochastic integral with respect to Arratia flow is presented. The next part is devoted to the variants of the Clark theorem for finite number of the Brownian motions stopped in the random times. In the last part the modification of the construction from the first part is applied to the representation of the functionals from the flow.

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1. Spatial stochastic integral with respect to Arratia flow

Let \( \{x(u) : u \in \mathbb{R}\} \) be the Arratia flow, i.e., the flow of Brownian particles with coalescence described above. For \( U > 0 \) consider a partition \( \pi \) of the interval \([0; U]\), \( \pi = \{u_0 = 0, \ldots, u_n = U\} \). For \( k = 1, \ldots, n \) define

\[
\tau(u_k) = \inf\{1, t \in [0; 1] : x(u_k, t) = x(u_{k-1}, t)\}.
\]

Note, that \( \tau(u_k), k = 1, \ldots, n \) are stopping moments with respect to the flow \( F^\tau_t = \sigma(x(u_k, s), k = 1, \ldots, n, s \leq t) \).

Let us consider for a bounded measurable function \( a : \mathbb{R} \to \mathbb{R} \) the sum

\[
S_\pi = \sum_{k=1}^{n} \int_0^{\tau(u_k)} a(x(u_k, s)) dx(u_k, s).
\]

Our aim is to investigate the limit of \( S_\pi \) under

\[
|\pi| = \max_{k=0, \ldots, n-1} (u_{k+1} - u_k) \to 0
\]

and its properties depending on the function \( a \) and the spatial variable \( U \).

Let us begin with the moments of \( S_\pi \). It follows from the standard properties of Ito stochastic integral, that

\[
ES_\pi = 0,
\]

and

\[
ES^2_\pi = E \sum_{k=1}^{n} \int_0^{\tau(u_k)} a^2(x(u_k, s)) ds.
\]

Let us denote

\[
\overline{S}_\pi = \sum_{k=1}^{n} \int_0^{\tau(u_k)} a^2(x(u_k, s)) ds.
\]

Consider the sequence of increasing partitions \( \{\pi_n : n \geq 1\} \) of the interval \([0; U]\) with \(|\pi_n| \to 0, n \to \infty\).

**Lemma 1.1.** There exists a limit

\[
\lim_{n \to \infty} \overline{S}_{\pi_n} \ a.s.
\]

**Proof.** To prove the lemma we will check two properties of the sequence \( \{\overline{S}_{\pi_n} : n \geq 1\} :\)

\[
\forall n \geq 1 : \overline{S}_{\pi_n} \leq \overline{S}_{\pi_{n+1}},
\]

and

\[
\sup_{n \geq 1} E\overline{S}_{\pi_n} < +\infty.
\]
Note that it is enough to prove (1.4) in the case, when \( \pi_{n+1} \) contains only one additional point \( v_0 \) comparing with \( \pi_n \). Suppose that \( \pi_n = \{u_0 = 0, ..., u_n = U\} \) and \( \pi_{n+1} = \{u_0 = 0, ..., u_k, v_0, u_{k+1}, ..., u_n = U\} \). Denote
\[
\hat{\tau}(u_{k+1}) = \inf\{1, t \in [0; 1] : x(u_{k+1}, t) = x(v_0, t)\}.
\]
Now
\[
\mathfrak{S}_{\pi_{n+1}} - \mathfrak{S}_{\pi_n} = \int_0^{\hat{\tau}(u_{k+1})} a^2(x(u_{k+1}, s)) ds + \int_{\hat{\tau}(u_{k+1})}^{\tau(u_{k+1})} a^2(x(v_0, s)) ds - \int_0^{\tau(u_{k+1})} a^2(x(u_{k+1}, s)) ds.
\]
There are two possibilities. In the first one \( \tau(v_0) < \tau(u_{k+1}) \). Now \( \hat{\tau}(u_{k+1}) = \tau(u_{k+1}) \). So in this case
\[
\mathfrak{S}_{\pi_{n+1}} - \mathfrak{S}_{\pi_n} = \int_0^{\tau(u_{k+1})} a^2(x(v_0, s)) ds \geq 0.
\]
The next case is \( \tau(v_0) \geq \tau(u_{k+1}) \). This possibility can be realized only if \( \tau(v_0) = \tau(u_{k+1}) \). Now \( \hat{\tau}(u_{k+1}) \leq \tau(v_0) \) and
\[
\int_0^{\tau(u_{k+1})} a^2(x(u_{k+1}, s)) ds = \int_0^{\hat{\tau}(u_{k+1})} a^2(x(u_{k+1}, s)) ds + \int_{\hat{\tau}(u_{k+1})}^{\tau(u_{k+1})} a^2(x(v_0, s)) ds.
\]
So, in this case
\[
\mathfrak{S}_{\pi_{n+1}} - \mathfrak{S}_{\pi_n} = \int_0^{\tau(v_0)} a^2(x(v_0, s)) ds - \int_{\hat{\tau}(u_{k+1})}^{\tau(u_{k+1})} a^2(x(v_0, s)) ds = \int_0^{\tau(v_0)} a^2(x(v_0, s)) ds - \int_0^{\tau(v_0)} a^2(x(v_0, s)) ds = \int_0^{\tau(u_{k+1})} a^2(x(v_0, s)) ds \geq 0.
\]
Hence (1.4) is true. Let us estimate the expectation of \( S_{\pi_n} \). Consider two independent standard Wiener processes \( w_1, w_2 \) which start from 0 and \( u > 0 \) correspondingly. Denote
\[
\tau = \inf\{1, t : w_1(t) = w_2(t)\}.
\]
Then
\[
E\tau = \int_0^1 \int_u^u p_2(t) dt + \int_{-u}^u p_2(t) dt,
\]
where \( p_t \) is the density of the normal distribution with zero mean and covariance \( t \). It follows from (1.6) that
\[
E\tau \sim \frac{3u}{2\sqrt{\pi}}, \ u \to 0 + .
\]
Consequently,
\[
\lim_{n \to \infty} E\mathfrak{S}_{\pi_n} \leq U \frac{3}{2\sqrt{\pi}} \sup_R a^2.
\]
Now the statement of the lemma follows from (1.4) and (1.7).

Remark 1. It follows from the proof of the lemma that there exists a limit
\[
\lim_{n \to \infty} E\mathfrak{S}_{\pi_n}.
\]
Lemma 1.2. There exists a limit

\[ m(U) = L_2 - \lim_{n \to \infty} S_{\pi_n}. \]

Proof. Let the partitions \( \pi_n, \pi_{n+1} \) be the same as in the proof of the previous lemma. Then

\[
ES_{\pi_n} S_{\pi_{n+1}} = E \sum_{j=1}^{n} \int_{0}^{\tau(u_{j+1})} a(x(u_{j+1}, s)) dx(u_{j+1}, s). \\
\cdot \left( \sum_{j_2 \neq k+1} \int_{0}^{\tau(u_{j_2})} a(x(u_{j_2}, s)) dx(u_{j_2}, s) + \int_{0}^{\tau(v_0)} a(x(v_0, s)) dx(v_0, s) + \right. \\
\left. + \int_{0}^{\tau(u_{k+1})} a^2(x(u_{k+1}, s)) dx(u_{k+1}, s) \right) = \\
= E \sum_{j \neq k+1} \int_{0}^{\tau(u_j)} a^2(x(u_j, s)) ds + E \int_{0}^{\tau(u_{k+1})} a(x(u_{k+1}, s)) dx(u_{k+1}, s). \\
\cdot \left( \int_{0}^{\tau(v_0)} a(x(v_0, s)) dx(v_0, s) + \int_{0}^{\tau(u_{k+1})} a(x(u_{k+1}, s)) dx(u_{k+1}, s) \right) = \\
= E \sum_{j \neq k+1} \int_{0}^{\tau(u_j)} a^2(x(u_j, s)) ds + E \int_{0}^{\tau(u_{k+1}) \wedge \tau(v_0)} a^2(x(u_{k+1}, s)) ds + \\
\cdot E \int_{\tau(u_{k+1}) \wedge \tau(v_0)} a^2(x(u_{k+1}, s)) ds = \\
= E \sum_{j \neq k+1} \int_{0}^{\tau(u_j)} a^2(x(u_j, s)) ds + E \int_{0}^{\tau(u_{k+1})} a^2(x(u_{k+1}, s)) ds = \\
= ES_{\pi_n}^2.
\]

Consequently for all \( n \leq m \)

\[ ES_{\pi_n} S_{\pi_m} = ES_{\pi_n}. \]

Now the statement of the lemma follows from the remark 1.

Remark 2. Note, that the limit \( m(U) \) does not depend on the choice of the sequence of partitions \( \{\pi_n; n \geq 1\} \).

To prove this we need in an estimation of the rate of convergence \( ES_{\pi_n}^2 \) to its limit.

Lemma 1.3. There exists a constant \( C \), such that for every partition \( \pi \) of the interval \([0; U]\)

\[ |ES_{\pi}^2 - Em^2(U)| \leq C|\pi| \sup_{\mathbb{R}} a^2. \]

Proof. First consider the partitions \( \pi', \pi'' \) where \( \pi'' \) is obtained from \( \pi' \) by adding one point on the interval \([u_k, u_{k+1}]\). As it was mentioned in the proof of the lemma 1

\[ \overline{S}_{\pi''} - \overline{S}_{\pi'} = \int_{0}^{\tau(v_0)} a^2(x(v_0, s)) ds. \]
Here \( \zeta(v_0) = \inf \{1; t : (x(v_0, t) - x(u_k, t))(x(v_0, t) - x(u_{k+1}, t)) = 0\} \).

Let us estimate \( E\zeta(v_0) \). Consider the standard Wiener process \( \vec{w} \) on the plane, which is starting from the point \( \vec{r} \). Suppose that this point lies inside the angle with the vertex in the origin. Let the value of the angle be less then \( \pi/2 \) and the angle lies in the part of the plane where the both coordinate are nonnegative. Define \( \zeta \) the first exit time of \( \vec{w} \) from the angle. Then enlarging the angle up to \( \pi/2 \) and using one-dimension expressions like (1.6) we can check that there exists \( C > 0 \) such, that

\[
E\zeta \wedge 1 \leq Cr_1r_2,
\]

where \( \vec{r} = (r_1, r_2) \).

From this remarks we can conclude that there exists \( C_1 > 0 \) such, that

\[
E\zeta(v_0) \leq C_1(u_{k+1} - v_0)(v_0 - u_k).
\]

This conclusion can be obtained if we note, that \( \zeta(v_0) \) is the minimum of 1 and the first exit time of the 3-dimensional Wiener process from the space angle with the value \( \pi/3 \). It follows from (1.9) and (1.11) that

\[
E(S_{\pi'} - S_{\pi''}) \leq C_1 \sup_{ \mathbb{R} } a^2(u_{k+1} - v_0)(v_0 - u_k).
\]

Now let us consider the general case when \( \pi'' \) is obtained from \( \pi' \) by the adding of a few new points. Denote by \( v_1 < \ldots < v_m \) the new points on the interval \([u_k, u_{k+1}]\). Then the new amount which is obtained in \( E(S_{\pi''} - S_{\pi'}) \) from these points can be estimated due to (1.12) by the sum

\[
C_1 \sup_{ \mathbb{R} } a^2 \sum_{j=1}^{m} (v_j - v_{j-1})(u_{k+1} - v_j),
\]

where we suppose, that \( v_0 = u_k \). Consequently

\[
E(S_{\pi''} - S_{\pi'}) \leq C_1 \sup_{ \mathbb{R} } a^2 \sum_{k=0}^{n-1} (u_{k+1} - u_k)^2 \leq C_1 \sup_{ \mathbb{R} } a^2 U|\pi'|.
\]

This inequality leads to the existence of the limit

\[
\lim_{|\pi| \to 0} ES_{\pi}.
\]

It follows from the proof of the lemma 2, that

\[
Em^2(U) = \lim_{|\pi| \to 0} ES_{\pi}.
\]

It is clear now that (1.8) holds.

The independence \( m(U) \) from a choice of the sequence \( \{\pi_n; n \geq 1\} \) now follows in standard way.

Define for \( U \geq 0 \) the \( \sigma \)-field

\[
\tilde{F}_U = \sigma(x(u, \cdot); 0 \leq u \leq U).
\]
Lemma 1.4. The process \( \{m(U); U \geq 0\} \) is \((\tilde{F}_U)\)-martingale.

Proof. The measurability of \( m(U) \) with respect to \( \tilde{F}_U \) is evident. Let \( 0 \leq U_1 < U_2 \). Consider the partition \([\pi]\) of \([0; U_2]\) which contains the point \( U_1 \). Then

\[
E(S_n/\tilde{F}_{U_1}) = \sum_{u_k \leq U_1} \int_0^{\tau(u_k)} a(x(u_k, s))dx(u_k, s) + \\
E(\sum_{u_k > U_1} \int_0^{\tau(u_k)} a(x(u_k, s))dx(u_k, s)/\tilde{F}_{U_1}).
\]

To prove that the last summand is equal to zero it is enough to consider the expression

\[
E(\int_0^{\tau(u)} a(x(u, s))dx(u, s)/\tilde{F}_{U_1})
\]

for \( u > U_1 \). Take \( 0 \leq u_1 < \ldots < u_n = U_1 \). For a bounded Borel function \( f : C([0; 1]^n) \to \mathbb{R} \) the expectation

\[
E \int_0^{\tau(u)} a(x(u, s))dx(u, s)f(x(u_1, \cdot), \ldots, x(u_n, \cdot))
\]

can be rewritten as

\[
E \int_0^{\tilde{\tau}} a(w(s))dw(s)f(w_1, \ldots, w_n),
\]

where \( w \) and \( w_1, \ldots, w_n \) are independent standard Wiener processes starting from the points \( u \) and \( u_1, \ldots, u_n \) correspondingly, and \( \tilde{\tau} \) is a stopping time for \((w, w_1, \ldots, w_n)\). \( \tilde{f} \) is a bounded Borel function on \( C([0; 1]^n) \). Denote by \( \Gamma \) the \( \sigma \)-field corresponding to \( \tilde{\tau} \). Then

\[
E \int_0^{\tilde{\tau}} a(w(s))dw(s)f(w_1, \ldots, w_n) = \\
= E \int_0^{\tilde{\tau}} a(w(s))dw(s)F(w_1, \ldots, w_n)/\Gamma = \\
= E \int_0^{\tilde{\tau}} a(w(s))dw(s)f(\tilde{w}_1, \ldots, \tilde{w}_n),
\]

where \( \tilde{w}_k(s) = w_k(s \wedge \tilde{\tau}), k = 1, \ldots, n \), and \( \tilde{f} \) is new bounded Borel function. Due to the Clark representation theorem

\[
\tilde{f}(\tilde{w}_1, \ldots, \tilde{w}_n) = c + \sum_{k=1}^n \int_0^{\tilde{\tau}} \eta_k(s)dw_k(s),
\]

where for \( k = 1, \ldots, n \eta_k \) is the square-integrable random function adapted to the flow

\[
\Gamma_t = \sigma(w(s), w_1(s), \ldots, w_n(s), s \leq t).
\]

Consequently,

\[
E \int_0^{\tilde{\tau}} a(w(s))dw(s)f(\tilde{w}_1, \ldots, \tilde{w}_n) = E \int_0^{\tilde{\tau}} a(w(s))dw(s) \\
\cdot \left( c + \sum_{k=1}^n \int_0^{\tilde{\tau}} \eta_k(s)dw_k(s) \right) = 0.
\]
Hence (1.13) also equal to zero. Finally

\[ E( \sum_{k>U_1} \int_0^{\tau(k)} a(x(u_k, s))dx(u_k, s)/F_{U_1} ) = 0. \]

Taking the limit under the diameter of partition tends to 0 we get the statement of the lemma.

2. Clark representation for the finite family of coalescing Brownian motions

This section is devoted to the integral representation of the functionals from \( x(u_1, \cdot), \ldots, x(u_n, \cdot) \) \( u_1 < u_2 < \ldots < u_n \). Let us start with the following simple lemma, which was already used in the previous section.

**Lemma 2.1.** Let \( w \) be the standard Wiener process on \([0; 1]\) and \( 0 \leq \tau \leq 1 \) be the stopping time for \( w \). Suppose, that the square-integrable random variable \( \alpha \) is measurable with respect to \( \{ w(\tau \wedge t); t \in [0; 1] \} \). Then \( \alpha \) can be represented as

\[ \alpha = E\alpha + \int_0^\tau f(t)dw(t) \]

with the certain adapted square-integrable random function \( f \).

**Proof.** Note, that

\[ \sigma(\tau(\tau \wedge t); t \in [0; 1]) = F_\tau, \]

where \( F_\tau \) is the \( \sigma \)-field corresponding to the stopping moment \( \tau \). Now, due to the original Clark theorem

\[ \alpha = E\alpha + \int_0^1 f(t)dw(t). \]

It remains now to apply the conditional expectation with respect to \( F_\tau \) to the both sides of this equality. Lemma is proved.

Consider the following situation. Let \( w_1, w_2 \) be an independent standard Wiener processes on \([0; 1]\) and \( \tau \) be a stopping time with respect to its join flow of \( \sigma \)-fields. The processes \( w_1, w_2 \) and the random variable \( \tau \) can be considered on the product of probability spaces \( \Omega_1 \times \Omega_2 \). Here \( \Omega_1 \) is related to \( w_1 \) and \( \Omega_2 \) is related to \( w_2 \).

**Lemma 2.2.** For every fixed \( \omega_1 \in \Omega_1 \) the random variable \( \tau(\omega_1, \cdot) \) on \( \Omega_2 \) is the stopping moment for \( w_2 \) on \( \Omega_2 \).

**Proof.** The set \( \{ \omega_2 : \tau(\omega_1, \omega_2) < t \} \) is the cross section of \( \{ \tau < t \} \) in \( \Omega_1 \times \Omega_2 \). Hence its measurability with respect to \( \sigma(w_2(s); s \leq t) \) follows from the usual arguments of measure theory.

The previous two lemmas lead to the following result.

**Theorem 2.1.** Let \( w_0, w_1, \ldots, w_n \) be an independent standard Wiener processes on \([0; 1]\) and for every \( k = 1, \ldots, n \) \( \tau_k \) is the stopping time for the process \((w_0, w_1, \ldots, w_k)\). Suppose, that the square-integrable random variable \( \alpha \) is measurable with respect to the set \((w_0(\cdot), w_1(\tau_1 \wedge \cdot), \ldots, w_n(\tau_n \wedge \cdot))\). Then \( \alpha \) can be represented as

\[ \alpha = E\alpha + \sum_{k=0}^n \int_0^{\tau_k} f_k(t)dw_k(t), \]
where \( \tau_0 = 1 \) and \( f_k \) is adapted to the flow generated by \( w_k \) under fixed \( w_j, j \neq k \).

**Proof.** Denote \( \tilde{w}_k(t) = w_k(\tau_k \land t), \ k = 1, \ldots, n \). Consider the random variable \( \alpha - E(\alpha/\tilde{w}_0, \ldots, \tilde{w}_{n-1}) \). It is measurable with respect to \( \tilde{w}_n \) under fixed \( \tilde{w}_0, \ldots, \tilde{w}_{n-1} \) and has zero mean. Due to the previous lemma it can be written as

\[
\alpha - E(\alpha/\tilde{w}_0, \ldots, \tilde{w}_{n-1}) = \int_0^{\tau_n} f_n(t)dw_n(t),
\]

where the random function \( f_n \) under fixed \( \tilde{w}_0, \ldots, \tilde{w}_{n-1} \) is adapted to the flow generated by \( w_n \). Repeat the same procedure to the random variable \( E(\alpha/\tilde{w}_0, \ldots, \tilde{w}_{n-1}) \). Then

\[
E(\alpha/\tilde{w}_0, \ldots, \tilde{w}_{n-1}) - E(\alpha/\tilde{w}_0, \ldots, \tilde{w}_{n-2}) = \int_0^{\tau_n-1} f_{n-1}(t)dw_{n-1}(t).
\]

After \( n \) steps we will get the statement of the theorem.

**Remark.** Note, that the representation from the theorem has the following property

\[
E\alpha^2 = (E\alpha)^2 + \sum_{k=0}^{n} E \int_0^{\tau_k} f_k(t)^2 dt.
\]

Consider an example of application of the theorem 2.1.

**Example 2.1.** Let \( \alpha \) be the square-integrable random variable measurable with respect to \( x(u_0, \cdot), \ldots, x(u_n, \cdot) \), where \( u_0, \ldots, u_n \) are the different points. Define the random moments

\[
\tau_0 = 1, \ \tau_k = \inf\{1, t : x(u_k, t) \in \{x(u_0, t), \ldots, x(u_{k-1}, t)\}\}, k = 1, \ldots, n.
\]

Then \( \alpha \) can be represented as

\[
\alpha = E\alpha + \sum_{k=0}^{n} \int_0^{\tau_k} f_k(t)dx(u_k, t),
\]

where for every \( k \) the random function \( f_k \) is measurable with respect \( x(u_0, \cdot), \ldots, x(u_k, \cdot) \) and (under fixed \( x(u_0, \cdot), \ldots, x(u_{k-1}, \cdot) \)) is adapted to the flow \( x(u_k, \tau_k \land \cdot) \). In this representation

\[
E\alpha^2 = (E\alpha)^2 + \sum_{k=0}^{n} E \int_0^{\tau_k} f_k(t)^2 dt.
\]

3. **Clark representation**

Let \( \alpha \) be the square-integrable random variable measurable with respect to \( \{x(u, \cdot); u \in [0; U]\} \). Suppose, that \( \{u_n; n \geq 0\} \) is a dense set in \([0; U]\) containing 0 and \( U \). Define the random moments \( \{\tau_k; k \geq 0\} \) as in the theorem 2.1. The following analog of the Clark representation holds.
Theorem 3.1. The random variable $\alpha$ can be represented as an infinite sum

\begin{equation}
\alpha = E\alpha + \sum_{n=0}^{\infty} \int_{0}^{\tau_k} f_k(t)dx(u_k,t),
\end{equation}

where $\{f_k\}$ satisfy the same conditions as in the theorem 2.1 and the series converges in the square mean. Moreover

$E\alpha^2 = (E\alpha)^2 + \sum_{n=0}^{\infty} E \int_{0}^{\tau_k} f_k(t)dx(u_k,t)$.

Proof. As it was mentioned in the first section $x$ has a cadl`ag trajectories as a random process in $C([0; 1])$. Consequently

$\sigma(x(u,\cdot); u \in [0; U]) = \sigma(x(u_n,\cdot); n \geq 0) = \bigvee_{n=0}^{\infty} \sigma(x(u_0,\cdot), \ldots, x(u_n,\cdot))$.

Hence due to the Levy theorem

$\alpha = L_{2^{-}} \lim_{n \to \infty} E(\alpha/x(u_0,\cdot), \ldots, x(u_n,\cdot))$.

Due to the theorem 2.1

$E(\alpha/x(u_0,\cdot), \ldots, x(u_n,\cdot)) = E\alpha + \sum_{k=0}^{n} \int_{0}^{\tau_k} f_k(t)dx(u_k,t)$,

where $\tau_k$ and $f_k$ do not change with $n$. So, taking the limit under $n \to \infty$ we get the statement of the theorem.

Note, that the sum in (3.1) is closely related to the spatial stochastic integral which was built in the first section. Really, suppose, that $a$ is bounded measurable function on $\mathbb{R}$ and the set $\{u_n; n \geq 0\}$ is dense in $[0; U]$ with $u_0 = 0, u_1 = U$.

Lemma 3.1.

$\sum_{n=0}^{\infty} \int_{0}^{\tau_n} a(x(u_n,t))dx(u_n,t) = m(U) + \int_{0}^{1} a(x(0,t))dx(0,t)$,

where $m(U)$ was defined in the first section.

Proof. Note, that for every $n \geq 1$ the points $u_0, \ldots, u_n$ if ordered in the growing order form a partition of $[0; U]$. Under $n \to \infty$ these partitions increase and their diameters tend to zero. To prove the lemma it remains to note that for every $n \geq 1$ the sum $\sum_{k=0}^{n} \int_{0}^{\tau_k} a(x(u_k,t))dx(u_k,t)$ consider with the sum $S_\pi$ for the corresponding partition. Lemma is proved.

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