ISOPERIMETRIC FUNCTIONAL INEQUALITIES VIA THE MAXIMUM PRINCIPLE: THE EXTERIOR DIFFERENTIAL SYSTEMS APPROACH

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Abstract. The goal of this note is to give the unified approach to the solutions of a class of isoperimetric problems by relating them to the exterior differential systems studied by R. Bryant and P. Griffiths.

1. Introduction: a function and its gradient

In this note we list several classical by now isoperimetric inequalities which can be proved in a unified way. This unified approach reduces them to the so-called exterior differential systems studied by Robert Bryant and Phillip Griffiths. To the best of our knowledge, this is the first article where this connection is made.

Let \( d\gamma(x) \) be the standard \( n \)-dimensional Gaussian measure \( d\gamma(x) = \frac{1}{\sqrt{(2\pi)^n}} e^{-\frac{|x|^2}{2}} dx \). Set \( \Omega \subset \mathbb{R} \) to be a bounded closed interval and let \( \mathbb{R}^+ := \{ x \in \mathbb{R} : x \geq 0 \} \). By symbol \( C^\infty(\mathbb{R}^n; \Omega) \) we denote the smooth functions on \( \mathbb{R}^n \) with values in \( \Omega \).

We prove the following theorem:

Theorem 1. If a real valued function \( M(x, y) \) is such that \( M(x, \sqrt{y}) \in C^2(\Omega \times \mathbb{R}^+ \) and it satisfies the differential inequalities

\[
\begin{pmatrix}
M_{xx} + \frac{M_y}{y} & M_{xy} \\
M_{xy} & M_{yy}
\end{pmatrix} \leq 0
\]

then

\[
\int_{\mathbb{R}^n} M(f, \|\nabla f\|) d\gamma \leq M \left( \int_{\mathbb{R}^n} f d\gamma, 0 \right) \quad \text{for all} \quad f \in C^\infty(\mathbb{R}^n; \Omega).
\]

One can obtain the similar result for uniformly log-concave probability measures, and the short way to see this is based on the mass transportation argument. In fact, let \( d\mu = e^{-U(x)} dx \) be a probability measure such that \( U(x) \) is smooth and \( \text{Hess} U \geq R \cdot \text{Id} \) for some \( R > 0 \). By the result of Caffarelli (see [16]) there exists a Brenier map \( T = \nabla \phi \) for some convex function \( \phi \) such that \( T \) pushes forward \( d\gamma \) onto \( d\mu \), moreover \( 0 \leq \text{Hess} \phi \leq \frac{1}{\sqrt{R}} \cdot \text{Id} \). We apply (1.1) to \( f(x) = g(\nabla \phi(x)) \) and use the fact \( M_y \leq 0 \) which follows from (1.1). Since

\[
\|\nabla f(x)\| = \|\nabla \phi(x) \nabla g(\nabla \phi)\| \leq \frac{1}{\sqrt{R}} \|\nabla g(\nabla \phi)\|
\]

we obtain:

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Corollary 1. If \( M(x,y) \) satisfies \( M(x,\sqrt{y}) \in C^2(\Omega \times \mathbb{R}_+) \) and (1.1) then for any \( g \in C^\infty(\mathbb{R}^n; \Omega) \) we have
\[
\int_{\mathbb{R}^n} M \left( g, \frac{\|\nabla g\|}{\sqrt{R}} \right) d\mu \leq M \left( \int_{\mathbb{R}^n} g d\mu, 0 \right),
\]
where \( d\mu = e^{-U(x)} dx \) is a probability measure such that \( \text{Hess} U(x) \geq R \cdot I_d \).

In Section 1.1 we present applications of the functional inequality (1.3). In Section 2 we prove a theorem about equivalence of some functional inequalities and partial differential inequalities. Corollary 1 is just a consequence of this result. We will notice that our proof of Corollary 1 for general log-concave measure will not differ from the case of Gaussian measures and it will be completely self-contained (it will not need the mass transportation argument).

In Section 3 we describe solutions of (1.1) (in the important case for us when the determinant of the matrix in (1.1) is zero) by reducing it to the exterior differential system (EDS) studied by R. Bryant and P. Griffiths.

This allows us to linearize the underlying non-linear PDE that appeared by the requirement of determinant of the matrix in (1.1) to vanish. In Section 4 we investigate one dimensional case of the results obtained in Section 2, and in Section 5 we present further applications. In particular, we sharpen Beckner–Sobolev inequality (already sharp of course), and we show other examples of new isoperimetric inequalities, which one obtains through EDS method.

Acknowledgement. We are very grateful to Robert Bryant from whom we learned how to solve an important for our goals non-linear PDE (see [15]). In Section 3.1 this allows us to explain how one could find the right functions \( M(x,y) \) for all the applications mentioned in Section 1.1, and how to find new functions \( M \) each responsible for a particular isoperimetric inequality.

1.1. A unified approach to classical inequalities via one and the same PDE. In this section we list classical isoperimetric inequalities that can be obtained by choosing different solution of the one and the same PDE
\[
y(M_{xx}M_{yy} - M_{xy}^2) + M_yM_{yy} = 0
\]
corresponding to different initial values at \( y = 0 \). In the next sections we will show how exterior differential systems (EDS) method allows us to reduce it to a linear PDE, and thus match this classical isoperimetric inequalities with interesting solution of a linear PDE that happened to be just a reverse heat equation.

Then later, starting with subsection 1.1.4, we show that one can choose other interesting solutions of (1.4), and, in its turn, this translates to new isoperimetric inequalities. In particular, we will show an instance when Beckner–Sobolev inequality can be further sharpened in an ultimate way.

1.1.1. Log-Sobolev inequalities: entropy estimates. Log-Sobolev inequality of Gross (see [20]) states that
\[
\int_{\mathbb{R}^n} |f|^2 \ln |f|^2 d\gamma - \left( \int_{\mathbb{R}^n} |f|^2 d\gamma \right) \ln \left( \int_{\mathbb{R}^n} |f|^2 d\gamma \right) \leq 2 \int_{\mathbb{R}^n} \|\nabla f\|^2 d\gamma
\]
whenever the right hand side of (1.5) is well-defined and finite for complex-valued \( f \). This implies that if \( f \) and \( \|\nabla f\| \) are in \( L^2(d\gamma) \) then \( f \) is in the Orlicz space \( L^2 \ln L \). A proof of Gross uses two-point inequality which by central limit theorem establishes hypercontractivity of the Ornstein–Uhlenbeck semigroup \( \|e^{t(\Delta - x \cdot \nabla)}\|_{L^p(d\gamma) \rightarrow L^q(d\gamma)} \leq 1 \) for all \( t \geq 0 \) such that
\[ e^{-2t} \leq \frac{e^{-1}}{q-1}. \] Then as a corollary differentiating this estimate at point \( t = 0 \) for \( q = 2 \) one obtains (1.5). Earlier than Gross similar two-point inequality was proved by Aline Bonami (see [9, 10]). For more on two-point inequalities we refer the reader to [33]. For the simple proof of hypercontractivity of Ornstein–Uhlenbeck semigroup we refer the reader to [25, 29], and also to earlier works [22, 31]. Bakry and Emery [2] extended the inequality for log-concave measures. Namely the inequality

\[
\int_{\mathbb{R}^n} f^2 \ln f^2 d\mu - \left( \int_{\mathbb{R}^n} f^2 d\mu \right) \ln \left( \int_{\mathbb{R}^n} f^2 d\mu \right) \leq \frac{2}{R} \int_{\mathbb{R}^n} \|\nabla f\|^2 d\mu \quad (1.6)
\]

holds for a bounded real-valued \( f \in C^1 \) and a log-concave probability measure \( d\mu = e^{-U(x)} dx \) such that \( \text{Hess} U(x) \geq R \cdot \text{Id} \). For further remarks we refer the reader to [3].

**Proof of (1.6):** Take

\[
M(x, y) = x \ln x - \frac{y^2}{2x}, \quad x > 0 \quad \text{and} \quad y \geq 0. \quad (1.7)
\]

We have

\[
\begin{pmatrix}
M_{xx} + \frac{M_y}{y} & M_{xy} \\
M_{xy} & M_{yy}
\end{pmatrix} = \begin{pmatrix}
-\frac{y^2}{x^2} & \frac{y}{x} \\
\frac{y}{x} & -\frac{1}{x}
\end{pmatrix} \leq 0. \quad (1.8)
\]

By Corollary 1.3 we obtain

\[
\int_{\mathbb{R}^n} \left( g \ln g - \frac{1}{2R} \frac{\|\nabla g\|^2}{g} \right) d\mu \leq \left( \int_{\mathbb{R}^n} g d\mu \right) \ln \left( \int_{\mathbb{R}^n} g d\mu \right). \quad (1.9)
\]

Taking \( g = f^2 \) for positive \( f \) and rearranging terms in (1.9) we arrive at (1.6).

**Remark 1.** The proof we just presented has an obstacle: \( M(x, \sqrt{y}) \notin C^2(\mathbb{R}_+ \times \mathbb{R}_+) \). In order to avoid this obstacle one has to consider \( M^\varepsilon(x, y) := M(x + \varepsilon, y) \) for some \( \varepsilon > 0 \). Then surely \( M^\varepsilon(x, y) \) will satisfy (1.1), what is more \( M^\varepsilon(x, \sqrt{y}) \in C^2(\mathbb{R}_+ \times \mathbb{R}_+) \) and we can repeat the same proof as above for \( M^\varepsilon(x, y) \). Finally, we just send \( \varepsilon \to 0 \) assuming that \( \int f^2 d\mu \neq 0 \) and we obtain the desired estimate. We should use the same idea in the applications presented below.

1.1.2. Bobkov’s inequality: Gaussian isoperimetry. In [7] Bobkov obtained the following functional version of Gaussian isoperimetry. Let \( \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2} dx \), and let \( \Phi'(x) \) be a derivative of \( \Phi \). Set \( I(x) := \Phi'(\Phi^{-1}(x)) \). Then for any locally Lipschitz function \( f : \mathbb{R}^n \to [0, 1] \), we have

\[
I \left( \int_{\mathbb{R}^n} f d\mu \right) \leq \int_{\mathbb{R}^n} \sqrt{I^2(f) + \frac{\|\nabla f\|^2}{R}} d\mu \quad (1.10)
\]

where \( d\mu = e^{-U(x)} dx \) is a log-concave probability measure such that \( \text{Hess} U \geq R \cdot \text{Id} \). Bobkov’s proof uses a two-point inequality: for all \( 0 \leq a, b \leq 1 \) we have

\[
I \left( \frac{a + b}{2} \right) \leq \frac{1}{2} \sqrt{I^2(a) + \frac{|a - b|^2}{2}} + \frac{1}{2} \sqrt{I^2(b) + \frac{|a - b|^2}{2}}. \quad (1.11)
\]

Iterating (1.11) appropriately and using central limit theorem Bobkov obtained (1.10) for the Gaussian measures. By the mass transportation argument one immediately obtains (1.10) for
uniformly log-concave measures. Notice that \( I(0) = I(1) = 0 \). Testing (1.10) for \( d\mu = d\gamma \) and \( f(x) = 1_A \) where \( A \) is a Borel subset of \( \mathbb{R}^n \) one obtains Gaussian isoperimetry: for any Borel measurable set \( A \subset \mathbb{R}^n \)

\[
\gamma^+(A) \geq \Phi'(\Phi^{-1}(\gamma(A))) \quad \text{where} \quad \gamma^+(A) := \liminf_{\varepsilon \to 0} \frac{\gamma(A_\varepsilon) - \gamma(A)}{\varepsilon}
\]

(1.12) denotes Gaussian perimeter of \( A \), here \( A_\varepsilon = \{ x \in \mathbb{R}^n : \text{dist}(\mathbb{R}^n, A, x) < \varepsilon \} \). For further remarks on (1.10) see [4]. Gaussian isoperimetry (1.12) can be derived also from Ehrhard’s inequality (see for example [25]).

**Proof of (1.10).** Take

\[
M(x, y) = -\sqrt{I^2(x) + y^2} \quad \text{where} \quad x \in [0, 1], \quad y \geq 0.
\]

(1.13)

We have

\[
\begin{pmatrix}
M_{xx} + \frac{M_x}{y} & M_{xy} \\
M_{xy} & M_{yy}
\end{pmatrix} = \begin{pmatrix}
-\frac{(I'(x))^2}{(I^2(x) + y^2)^{3/2}} & \frac{I(x)I''(x) + 1}{\sqrt{I^2(x) + y^2}} - \frac{y}{(I^2(x) + y^2)^{3/2}} \\
\frac{y}{(I^2(x) + y^2)^{3/2}} & -\frac{I'(x)}{(I^2(x) + y^2)^{3/2}}
\end{pmatrix}.
\]

(1.14)

Notice that \( I''(x)I(x) + 1 = 0 \) therefore (1.14) is negative semidefinite. So by Corollary 1 we obtain

\[
\int_{\mathbb{R}^n} -\sqrt{I^2(f) + \|\nabla f\|^2} \, d\mu \leq -I \left( \int_{\mathbb{R}^n} f \, d\mu \right)
\]

(1.15)

rearranging terms in (1.15) we obtain (1.10) for differentiable \( f : \mathbb{R}^n \to [0, 1] \). Notice that (1.15) still holds if \( I''(x)I(x) + 1 \geq 0 \) for arbitrary smooth \( I(x) \).

\( \square \)

1.1.3. **Poincaré inequality and spectral gap.** Classical Poincaré inequality for the Gaussian measure obtained by J. Nash [32] (see p. 941) states that

\[
\int_{\mathbb{R}^n} f^2 \, d\gamma - \left( \int_{\mathbb{R}^n} f \, d\gamma \right)^2 \leq \int_{\mathbb{R}^n} \|\nabla f\|^2 \, d\gamma.
\]

(1.16)

The inequality also says that the spectral gap i.e. the first nontrivial eigenvalue of the self-adjoint positive operator \( L = -\Delta + x \cdot \nabla \) in \( L^2(\mathbb{R}^n, d\gamma) \) is bounded from below by 1. If \( d\mu = e^{-U(x)} \, dx \) is a probability measure such that \( \text{Hess} \, U \geq R \cdot \text{Id} \) then we have

\[
\int_{\mathbb{R}^n} g^2 \, d\mu - \left( \int_{\mathbb{R}^n} g \, d\mu \right)^2 \leq \frac{1}{R} \int_{\mathbb{R}^n} \|\nabla g\|^2 \, d\mu.
\]

(1.17)

It is a folklore that inequality (1.17), besides of mass transportation argument, follows from the log-Sobolev inequality (1.6): apply (1.6) to the function \( f(x) = 1 + \varepsilon g(x) \) where \( \int g \, d\mu = 0 \), and send \( \varepsilon \to 0 \). Then the left hand side of (1.6) is

\[
2\varepsilon^2 \int g^2 \, d\mu + o(\varepsilon^2)
\]

whereas the right hand side of (1.6) is

\[
\frac{2\varepsilon^2}{R} \int \|\nabla g\|^2 \, d\mu.
\]

This gives (1.17). In [11] Brascamp and Lieb obtained the improvement of (1.17): instead of \( \frac{\|\nabla g\|^2}{R} \) one can put \( (\text{Hess} \, U)^{-1} \nabla g, \nabla g \) in the right hand side of (1.17), where we assume that \( \text{Hess} \, U \) is just positive. For a simple proof of this improvement we refer the reader to [17] (see also [8] by using Prekopa–Leindler inequality).

More subtle result of Bobkov [6] in this direction says that for any log-concave probability measure \( d\mu = e^{-U(x)} \, dx \) one can put \( K\| x - \int x d\mu \|_{L^2(d\mu)} \|\nabla g\|^2 \) instead of \( \frac{\|\nabla g\|^2}{R} \) for some
universal constant $K > 0$. This implies that nonnegative operator $L = -\Delta + \nabla U \cdot x$ has a spectral gap.

In [5] Beckner found an inequality which interpolates in a sharp way between Poincaré inequality and log-Sobolev inequality. The inequality was obtained for Gaussian measures but, again, by mass transportation argument it can be easily translated to a log-concave probability measure. Beckner–Sobolev inequality states that for $f \in L^2(d\mu)$ and $1 \leq p \leq 2$ we have

$$\int |f|^2 d\mu - \left( \int |f|^p \right)^{2/p} \leq \frac{(2 - p)}{R} \int \|\nabla f\|^2 d\mu$$  \hspace{1cm} (1.18)

where $d\mu = e^{-U(x)} dx$ is a probability measure such that $\text{Hess} U \geq R \cdot \text{Id}$. Case $p = 1$ gives Poincaré inequality (1.17) and case $p \to 2$ after dividing (1.18) by $2 - p$ gives (1.6). Beckner–Sobolev inequality was studied for different measures in [28].

Proof of (1.18): Take

$$M(x, y) = x^2 - \frac{2 - p}{p^2} x^{2-p} y^2 \quad \text{where} \quad x, y \geq 0 \quad 1 \leq p \leq 2.$$  \hspace{1cm} (1.19)

Notice that

$$\begin{pmatrix} M_{xx} + \frac{M_x}{y} & M_{xy} \\ M_{xy} & M_{yy} \end{pmatrix} = \begin{pmatrix} -\frac{2(2-p)(1-p)(2-3p)x^{2-p}y^2}{p^4} & -\frac{4(2-p)(1-p)x^{2-3}y}{p^3} \\ -\frac{4(2-p)(1-p)x^{2-3}y}{p^3} & -\frac{4(2-p)x^{2-2}}{p^2} \end{pmatrix} \leq 0.$$  \hspace{1cm} (1.20)

By Corollary 1 we have

$$\int_{\mathbb{R}^n} \frac{g^{2-p}}{p^2} - \frac{2 - p}{p^2} g^{2-p} \|\nabla g\|^2 \frac{1}{R} \, d\mu \leq \left( \int_{\mathbb{R}^n} g \, d\mu \right)^{2/p}$$  \hspace{1cm} (1.21)

for positive (in fact nonnegative) functions $g$. Now set $g = |f|^p$, and notice that $\|\nabla |f|\| \leq \|\nabla f\|$. After rearranging terms in (1.21) we obtain (1.18).

\hfill $\square$

1.1.4. $3/2$ function. Beckner’s inequality (1.18) can be rewritten in an equivalent form

$$\int_{\mathbb{R}^n} f^p d\gamma - \left( \int_{\mathbb{R}^n} f \, d\gamma \right)^p \leq \frac{p(p-1)}{2} \int_{\mathbb{R}^n} f^{p-2} \|\nabla f\|^2 d\gamma, \quad p \in [1, 2].$$  \hspace{1cm} (1.22)

In fact inequality (1.22) can be essentially improved for $p \in (1, 2)$. We will illustrate the improvement in the case $p = 3/2$ and for the general case we should refer the reader to our recent paper [24] which is based on the application of Theorem 1.

The following inequality valid for all smooth bounded nonnegative $f$ was proved in our recent paper [24]:

$$\int_{\mathbb{R}^n} f^{3/2} d\gamma - \left( \int_{\mathbb{R}^n} f \, d\gamma \right)^{3/2} \leq$$

$$\int_{\mathbb{R}^n} \left( f^{3/2} - \frac{1}{\sqrt{2}} (2f - \sqrt{f^2 + \|\nabla f\|^2}) \sqrt{f + \sqrt{f^2 + \|\nabla f\|^2}} \right) d\gamma.$$  \hspace{1cm} (1.23)

Inequality (1.23) improves Beckner’s bound (1.22) for $p = 3/2$. Indeed, notice that we have the following pointwise inequality

$$x^{3/2} - \frac{1}{\sqrt{2}} \left( 2x - \sqrt{x^2 + y^2} \right) \sqrt{x + \sqrt{x^2 + y^2}} \leq \frac{3}{8} x^{-1/2} y^2, \quad x, y \geq 0.$$  \hspace{1cm} (1.24)
which follows from the homogeneity, i.e., take \( x = 1 \). By plugging \( f \) for \( x \), \( |\nabla f| \) for \( y \) and integrating we see that (1.23) improves on (1.22).

Inequality (1.24) is always strict except when \( y = 0 \). Also notice that when \( y \to +\infty \) the right hand side of (1.24) increases as \( y^2 \) whereas the left hand side of (1.24) increases as \( y^{3/2} \). It should be mentioned as well that when \( x \to 0 \) the difference in (1.24) goes to infinity. The only place where the quantities in (1.24) are comparable is when \( y/x \to 0 \). We should notice that since the left hand side of (1.24) is decreasing function in \( x \) (see [24]), and when \( x = 0 \) it becomes \( \frac{y^{1/2}}{\sqrt{x}} \) then it follows from (1.23)

\[
\int_{\mathbb{R}^n} f^{3/2} d\gamma - \left( \int_{\mathbb{R}^n} f d\gamma \right)^{3/2} \leq \frac{1}{\sqrt{2}} \int_{\mathbb{R}^n} |\nabla f|^{3/2} d\gamma. \tag{1.25}
\]

Inequality (1.25) gives some information about the measure concentration of \( \gamma \).

**Proof of** (1.23). Take

\[
M(x, y) = \frac{1}{\sqrt{2}} \left( 2x - \sqrt{x^2 + y^2} \right) \sqrt{x + \sqrt{x^2 + y^2}} \text{ where } x, y \geq 0. \tag{1.26}
\]

We have

\[
\begin{pmatrix}
M_{xx} + \frac{M_x}{y} M_{xy} \\
M_{xy} & M_{yy}
\end{pmatrix} = \frac{3\sqrt{2}}{8\sqrt{x^2 + y^2}} \begin{pmatrix}
\frac{y^2}{(x + \sqrt{x^2 + y^2})^{3/2}} & \frac{y}{\sqrt{x + \sqrt{x^2 + y^2}}} \\
\frac{y}{\sqrt{x + \sqrt{x^2 + y^2}}} & -\frac{1}{\sqrt{x + \sqrt{x^2 + y^2}}}
\end{pmatrix}. \tag{1.27}
\]

Clearly (1.27) is negative semidefinite. So by Corollary 1 we obtain

\[
\int_{\mathbb{R}^n} \frac{1}{\sqrt{2}} \left( 2f - \sqrt{f^2 + \frac{||\nabla f||^2}{R}} \right) \sqrt{f + \sqrt{f^2 + \frac{||\nabla f||^2}{R}}} d\mu \leq \left( \int_{\mathbb{R}^n} fd\mu \right)^{3/2}. \tag{1.28}
\]

This is of course (1.23) for the Gaussian measure \( \gamma \): by taking \( R = 1 \) and rearranging terms in (1.28) we obtain (1.23).

1.1.5. **Banaszczyk’s problem:** (B) **Theorem.** The problem was proposed by W. Banaszczyk (see for example [27]) which says that given symmetric convex body \( K \subset \mathbb{R}^n \) the function \( \phi(t) = \gamma(e^tK) \) is log-concave on \( \mathbb{R} \). The problem was solved in [19]: clearly one only needs to check log-concavity at one point: \( (\ln \phi(t))''|_{t=0} \leq 0 \). This is the same as

\[
\int_{\mathbb{R}^n} ||x||^4 d\gamma_k - \left( \int_{\mathbb{R}^n} ||x||^2 d\gamma_k \right)^2 \leq 2 \int_{\mathbb{R}^n} ||x||^2 d\gamma_k \tag{1.29}
\]

where

\[
d\gamma_k = \frac{1}{K(x)} e^{-||x||^2/2} dx = e^{-||x||^2/2 - \psi(x)} dx
\]

where a convex function \( \psi \) is a constant on \( K \) and it is \( +\infty \) outside of the set \( K \). In other words one can assume that \( d\gamma_k = e^{-U(x)} dx \) is a probability measure where \( U(x) \) is even and such that \( \text{Hess } U \geq Id \). Setting \( f(x) = ||x||^2 \) then inequality (1.29) can be rewritten as follows

\[
\int_{\mathbb{R}^n} f^2 d\mu - \left( \int_{\mathbb{R}^n} f d\mu \right)^2 \leq \frac{1}{2} \int_{\mathbb{R}^n} ||\nabla f||^2 d\mu. \tag{1.30}
\]
which is better than Poincaré inequality (1.17). This is a key ingredient in (B) Theorem and it was proved by Cordero-Erausquin–Fradelizi–Maurey in [19] that (1.30) holds provided that \( \int_{\mathbb{R}^n} \nabla f d\mu = 0 \), and \( d\mu = e^{-U(x)} dx \) is a probability measure such that \( \text{Hess} U \geq \text{Id} \) (which is true for \( f(x) = \|x\|^2 \)).

If one tries to apply Corollary 1 then the right choice of the function \( M \) must be

\[
M(x, y) = x^2 - \frac{y^2}{2} \tag{1.31}
\]

but unfortunately this function does not satisfy (1.1). However, we want (1.30) to hold only for the functions such that \( \int \nabla f d\mu = 0 \) therefore one can slightly modify the proof of Theorem 1 in order to obtain (1.30). In Section 4 we will show how it works and we will present a different proof of (1.30) with the extra conditions that \( f \) is even and \( d\mu \) is even (which definitely is enough for the (B) Theorem).

1.1.6. \( \Phi \)-entropy. Let \( \Phi : \Omega \to \mathbb{R} \) be a convex function. Given a probability measure \( d\mu \) on \( \mathbb{R}^n \) define \( \Phi \)-entropy (see [18]) as follows

\[
\text{Ent}_{\mu}^{\Phi}(f) \overset{\text{def}}{=} \int_{\mathbb{R}^n} \Phi(f) d\mu - \Phi \left( \int_{\mathbb{R}^n} f d\mu \right).
\]

Corollary 1 provides us with systematic approach to finding the bounds of \( \Phi \)-entropy for uniformly log-concave measures \( d\mu \). Indeed, let us illustrate it on the example of the Gaussian measure. Given a convex function \( \Phi \) on \( \Omega \subset \mathbb{R} \) let \( M(x, y) \) be such that \( M(x, 0) = \Phi(x) \), \( M(x, \sqrt{y}) \in C^2(\Omega \times \mathbb{R}_+) \) and \( M \) satisfies (1.1). Then by Theorem 1 we obtain

\[
\int_{\mathbb{R}^n} \Phi(f(x)) d\gamma - \Phi \left( \int_{\mathbb{R}^n} f d\gamma \right) \leq \int_{\mathbb{R}^n} [M(f, 0) - M(f, \|\nabla f\|)] d\gamma.
\]

In our recent paper [24] we do find the bounds of \( \Phi \) entropy as an application of Theorem 1 for the following fundamental examples

\[
\begin{align*}
\Phi(x) &= x^p \quad \text{for} \quad p \in \mathbb{R} \setminus [0, 1]; \\
\Phi(x) &= -x^p \quad \text{for} \quad p \in (0, 1); \\
\Phi(x) &= e^x; \\
\Phi(x) &= -\ln x.
\end{align*}
\]

Finding the best possible \( M \) is based on solving a PDE problem (1.37) with boundary conditions (1.32, 1.33, 1.34, 1.35) (see Section 1.1.8, Section 3 and [24]).

1.1.7. Yet another isoperimetric inequality obtained by EDS method. In Section 3 we consider a peculiar example (see Section 3.1.5) of the elliptic solution of PDE (1.4) with initial data

\[
M(x, 0) = x \arccos(-x) + \sqrt{1 - x^2} \quad \text{for} \quad x \in [-1, 1]
\]

which is not related to the applications that we have discussed before, but which gives yet another example of a new isoperimetric inequality. It looks like a useful one in particular because Poincaré inequality is its corollary.
1.1.8. **Concluding remarks.** As we shall notice in order to use Theorem 1 for the applications to functional (and thereby isoperimetric) inequalities, there is a difficulty: one has to find the right function \( M(x, y) \), for example such as (1.7), (1.19), (1.13), (1.26), (1.31) and functions \( M \) mentioned in Section 1.1.6 (see [24]).

If one knows what inequality should be proved then one can try to guess what function \( M(x, y) \) one has to choose: in the integrand one needs to set \( g = x \) and \( \|\nabla g\| = y \) and then integrand in terms of \( x \) and \( y \) will be \( M(x, y) \).

In general finding \( M(x, y) \) will be based purely on solving PDEs. Let us recall the discussions of Section 1.1.6. First notice that given, for example, a convex function \( \Phi : \Omega \to \mathbb{R} \) and suppose one wants to find an optimal *error term* in the Jensen’s inequality (\( \Phi\)-entropy (see [18]))

\[
0 \leq \int_{\mathbb{R}^n} \Phi(f(x)) d\gamma - \Phi \left( \int_{\mathbb{R}^n} f d\gamma \right) \leq \int_{\mathbb{R}^n} \text{Error}(f, \|\nabla f\|) d\gamma \quad \text{for all } f \in C^\infty(\mathbb{R}^n; \Omega).
\]

If we find \( M(x, y) \in C^2(\Omega \times \mathbb{R}_+) \) such that \( M(x, 0) = \Phi(x) \) and \( M(x, y) \) satisfies (1.1) then by Theorem 1 we can find a possible error term as follows

\[
\int_{\mathbb{R}^n} \Phi(f(x)) d\gamma - \Phi \left( \int_{\mathbb{R}^n} f d\gamma \right) \leq \int_{\mathbb{R}^n} [M(f, 0) - M(f, \|\nabla f\|)] d\gamma. \tag{1.36}
\]

In fact we would like to minimize the error term which corresponds to maximize \( M(x, y) \) under the constraints (1.1) and \( M(x, 0) = \Phi(x) \). This shows that partial differential inequality (1.1) should degenerate. Indeed, if \( \lambda_1(x, y) \) and \( \lambda_2(x, y) \) denote eigenvalues of the matrix in (1.1) then condition (1.1) becomes \( \lambda_1 + \lambda_2, 0 \leq 0 \) and \( \lambda_1 \cdot \lambda_2 \geq 0 \). If we have strict inequality \( \lambda_1 \cdot \lambda_2 > 0 \) then \( \lambda_1 \cdot \lambda_2 < 0 \). In this case we can slightly perturb \( M \) at a point \( (x, y) \) so that to make \( M(x, y) \) larger but still keep the inequality \( \lambda_1 \cdot \lambda_2 > 0 \). Clearly the condition \( \lambda_1 + \lambda_2 < 0 \) still holds. We can continuo perturbing \( M \) until (1.1) degenerates. Therefore we will seek \( M(x, y) \) among those functions which in addition with (1.1) also satisfy a *degenerate elliptic Monge–Ampère equation of general type*:

\[
\det \begin{pmatrix} M_{xx} + \frac{M_y}{M_{yy}} M_{xy} & M_{xy} \\ M_{xy} & M_{yy} \end{pmatrix} = M_{xx} M_{yy} - M_{xy}^2 + \frac{M_y M_{yy}}{y} = 0 \quad \tag{1.37}
\]

for \( (x, y) \in \Omega \times \mathbb{R}_+ \).

For example in log-Sobolev (1.6) and in Bobkov’s inequality (1.10) determinant of the matrices (1.8) and (1.14) are zero. In Beckner–Sobolev inequality (1.18) determinant of (1.20) is zero if and only if \( p = 1, 2 \). Notice that these are exactly cases when Beckner–Sobolev inequality interpolates Poincaré and log-Sovbolev inequality. Moreover, since the determinant in Beckner–Sobolev inequality is not zero for \( p \in (1, 2) \) this indicates that one should improve the inequality, and this is exactly what was done in (1.23). We refer the reader to our recent paper [24] where we do improve Beckner–Sobolev inequality by solving elliptic Monge–Ampère equation (1.37) with a boundary condition \( M(x, 0) = x^p \) with \( p \in \mathbb{R} \).

In Section 3 we will show that thanks to the exterior differential systems studied by R. Bryant and P. Griffiths (see [12, 13, 14]) nonlinear equation (1.37) can be reduced (after suitable change of variables) to linear backwards heat equation. In Section 3.1 we will illustrate this on the examples

\[
M(x, 0) = x \ln x, \ M(x, 0) = x^2, \ M(x, 0) = -I(x) \quad \text{and} \quad M(x, 0) = x^{3/2}
\]

which correspond to log-Sobolev, Poincaré, Bobkov and 3/2 inequalities.
To justify the claim that Section 3 makes approach to bounds of \( \Phi \)-entropy systematic we do consider a peculiar example (see Section 3.1.5)

\[
M(x,0) = x \arccos(-x) + \sqrt{1 - x^2} \quad \text{for} \quad x \in [-1, 1]
\]

which is not related to the applications that we have discussed before.

2. Function of variables \( D^\alpha f \). The proof of Theorem 1

We prove here Theorem 1 and even more general Theorem 2. The reader who a priori believes in Theorem 1 can skip to the next Section 3 devoted to the exterior differential systems (EDS) method of finding the elliptic solutions of PDE (1.4) (by elliptic solutions we mean the solutions \( M \) satisfying the condition (1.1) on \( M \)).

Let \( dm = e^{-U(x)}dx \) be a log-concave measure such that \( U \) is smooth and \( \text{Hess} U \geq R \cdot \text{Id} \). Set \( L = \Delta - \nabla U \cdot \nabla \). Then \(-L\) is a self-adjoint positive operator in \( L^2(\mathbb{R}^n, dm) \), moreover by (1.17) it has a spectral gap. Let \( P_t := e^{tL} \) be the corresponding semigroup generated by \( L \). Let \( \alpha = (\alpha_0, \ldots, \alpha_m) \) where \( \alpha_j = (\alpha_j^1, \ldots, \alpha_j^n) \) is a multi index of size \( n \) and \( \alpha_j^i \in \mathbb{N} \cup \{0\} \) for each \( j = 0, \ldots, m \) and \( i = 1, \ldots, n \). Let \( |\alpha_j| \) be the length of the multi index i.e. \( |\alpha_j| = \alpha_j^1 + \ldots + \alpha_j^n \). By \( D^\alpha_j \) we denote the differential operator

\[
D^\alpha_j = \frac{\partial^{|\alpha_j|}}{\partial x_1^{\alpha_j^1} \cdots \partial x_n^{\alpha_j^n}}.
\]

Further we fix some multi-multi index \( \alpha = (\alpha_0, \ldots, \alpha_m) \) where each \( \alpha_j \) is a multi index of size \( n \) as above.

**Test functions** \( C^\infty(\mathbb{R}^n; \Lambda) \). Let \( \Lambda \) be a closed convex subset of \( \mathbb{R}^m \). By \( C^\infty(\mathbb{R}^n; \Lambda) \) we denote the set of test functions \( f = (f_0, \ldots, f_m) : \mathbb{R}^n \to \Lambda \) i.e., smooth bounded vector functions with values in \( \Lambda \). Let

\[
D^\alpha f = (D^{\alpha_0} f_0, \ldots, D^{\alpha_m} f_m) \quad \text{and} \quad P_t f := (P_t f_0, \ldots, P_t f_m).
\]

We require that \( C^\infty(\mathbb{R}^n; \Lambda) \) is closed under taking \( D^\alpha \), i.e., \( D^\alpha C^\infty(\mathbb{R}^n; \Lambda) \subset C^\infty(\mathbb{R}^n; \Lambda) \). Linearity and positivity of \( P_t \) implies that \( P_t f, P_t D^\alpha f \in \Lambda \) for any \( f \in C^\infty(\mathbb{R}^n; \Lambda) \).

Let \( B(u_1, \ldots, u_m) : \Lambda \to \mathbb{R} \) be a smooth (at least \( C^2 \)) function, such that \( P_t B(D^\alpha f) \) is well defined for all \( t \geq 0 \). Set

\[
[L, D^\alpha] \overset{\text{def}}{=} ([L, D^{\alpha_0}] f_0, \ldots, [L, D^{\alpha_m}] f_m) \quad \text{and} \quad \Gamma(D^\alpha f) \overset{\text{def}}{=} \{ \langle \nabla D^{\alpha_i} f_i, \nabla D^{\alpha_j} f_j \rangle \}_{i,j=0}^m
\]

where \( \Gamma(D^\alpha f) \) denotes \( (m+1) \times (m+1) \), and \([A, B] = AB - BA\) denotes commutator of \( A \) and \( B \).

**Theorem 2.** The following conditions are equivalent:

(i) \( \nabla B(D^\alpha f) \cdot [L, D^\alpha] f + \text{Tr} \{ \text{Hess} B(D^\alpha f) \} \Gamma(D^\alpha f) \leq 0 \) for all \( f \in C^\infty(\mathbb{R}^n; \Lambda) \).

(ii) \( P_t[B(D^\alpha f)](x) \leq B(D^\alpha[P_t f](x)) \) for all \( t \geq 0, x \in \mathbb{R}^n \) and \( f \in C^\infty(\mathbb{R}^n; \Lambda) \).

**Proof.** (i) implies (ii): let \( V(x, t) = P_t[B(D^\alpha f)](x) - B(D^\alpha[P_t f](x)) \). Notice that

\[
(\partial_t - L)V(x, t) = (L - \partial_t)B(D^\alpha[P_t f](x)) = \sum_j \frac{\partial B}{\partial u_j} LD^{\alpha_j} P_t f_j + \sum_{i,j} \frac{\partial^2 B}{\partial u_i \partial u_j} \nabla D^{\alpha_i} P_t f_i \cdot \nabla D^{\alpha_j} P_t f_j - \sum_j \frac{\partial B}{\partial u_j} D^{\alpha_j} L P_t f_j = \nabla B(D^\alpha P_t f) \cdot [L, D^\alpha] P_t f + \text{Tr} \{ \text{Hess} B(D^\alpha P_t f) \} \Gamma(D^\alpha P_t f) \leq 0
\]
The last inequality follows from (i) and the fact that $P_t f(x) \in \Lambda$. Indeed, we can find a function $g \in C^\infty(\mathbb{R}^n; \Lambda)$ such that $g = P_t f$ in a neighborhood of $x$ and we can apply (i) to $g$.

By maximum principle we obtain $V(x,t) \leq \sup_x V(x,0) = 0$. Another way (without maximum principle) is that

$$V(x,t) = \int_0^t \frac{\partial}{\partial s} P_s B(D^\alpha P_{t-s} f) ds = \int_0^t P_s \left[ \left( L - \frac{\partial}{\partial t} \right) B(D^\alpha P_{t-s} f) \right] ds,$$

and the integrand in (2.3) is non positive by (2.2).

(ii) implies (i): for all $f \in C^\infty(\mathbb{R}^n; \Lambda)$ we have

$$0 \geq \lim_{t \to 0} \frac{V(x,t)}{t} = \lim_{t \to 0} \frac{V(x,t) - V(x,0)}{t} = \frac{\partial}{\partial t} V(x,t)|_{t=0} = \nabla B(D^\alpha f) \cdot \left[ L, D^\alpha \right] f + \text{Tr}(\text{Hess } B(D^\alpha f)) \Gamma(D^\alpha f)).$$

□

Remark 2. We notice that if one considers diffusion semigroups generated by

$$L = \sum_{i,j} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_j b_j(x) \frac{\partial}{\partial x_j}$$

where $A = \{a_{ij}\}_{i,j=1}^{n}$ is positive then absolutely nothing changes in Theorem 2 except the matrix $\Gamma(D^\alpha f)$ takes the form

$$\Gamma(D^\alpha f) = \{\nabla D^{\alpha_i} f_i A (\nabla D^{\alpha_j} f_j)^T\}_{i,j=0}^m.$$

2.1. Proof of Theorem 1. Consider a special case when $n = m$, $f = (f, \ldots, f)$, $\alpha_0 = (0, \ldots, 0)$, $\alpha_1 = (1, 0, \ldots, 0), \ldots, \alpha_n = (0, \ldots, 0, 1)$. Then $D^\alpha f = (f, \nabla f)$, and given that $L = \Delta - \nabla U \cdot \nabla$ we obtain

$$\nabla B(D^\alpha f) \cdot \left[ L, D^\alpha \right] f = \nabla_{1,\ldots,n} B (\text{Hess } U)(\nabla f)^T.$$

Here $\nabla_{1,\ldots,n} B$ is a gradient of $B(u_0, \ldots, u_n)$ taken with respect to $u_1, \ldots, u_n$ variables. Assume that $f$ takes values in the closed convex set $\Omega \subset \mathbb{R}$. Take

$$B(u_0, \ldots, u_n) = M \left( u_0, \sqrt{\frac{u_1^2 + \cdots + u_n^2}{R}} \right),$$

where $M(x, \sqrt{y}) \in C^2(\Omega \times \mathbb{R}_+)$ satisfies (1.1). Notice that $M_y \leq 0$. Indeed, if we multiply the first diagonal entry of (1.1) by $y$ and send $y \to 0$ we obtain $M_y(x,0) \leq 0$. On the other hand since the second diagonal entry of (1.1) is nonpositive we obtain $M_y(x,y) \leq 0$ for all $y$.

Next we notice

$$\nabla_{1,\ldots,n} B(D^\alpha f) = \frac{M_y}{\|\nabla f\| \sqrt{R}} \nabla f.$$

Since $M_y \leq 0$ and $\text{Hess } U \geq R \cdot Id$, we have

$$\nabla B(D^\alpha f) \cdot \left[ L, D^\alpha \right] f = \frac{M_y}{\|\nabla f\| \sqrt{R}} \nabla f (\text{Hess } U)(\nabla f)^T \leq \sqrt{R \|\nabla f\| M_y}.$$
Therefore
\[ \nabla B(D^\alpha f) \cdot [L, D^\alpha] f + \text{Tr}(\text{Hess } B(D^\alpha f)) \Gamma(D^\alpha f) \leq \text{Tr}(W \Gamma(D^\alpha f)) \]
where
\[
W = \begin{bmatrix}
\partial_{00}^2 B + \frac{\sqrt{R} M_{00}}{\|\nabla f\|} & \partial_{01}^2 B & \cdots & \partial_{0n}^2 B \\
\partial_{10}^2 B & \partial_{11}^2 B & \cdots & \partial_{1n}^2 B \\
\vdots & \vdots & \ddots & \vdots \\
\partial_{n0}^2 B & \partial_{n1}^2 B & \cdots & \partial_{nn}^2 B \\
\end{bmatrix}
\]
where \( \partial_{ij}^2 B = \frac{\partial^2 B}{\partial x_i \partial x_j} \). We will show that \( W \leq 0 \), and then we will obtain \( \text{Tr}(W \Gamma(D^\alpha f)) \leq 0 \) because \( \Gamma(D^\alpha f) \geq 0 \).

We have \( \partial_{00}^2 B = M_{xx}, \partial_{0j}^2 B = \frac{M_{xy}}{\|\nabla f\|\sqrt{R}} f_j \) for all \( j \geq 1 \) and
\[
\partial_{ij}^2 B = \frac{M_{xy} \delta_{ij}}{\|\nabla f\|^3 \sqrt{R}} f_i f_j - \frac{M_y}{\|\nabla f\| \sqrt{R}} f_i f_j + \frac{M_y \delta_{ij}}{\|\nabla f\| \sqrt{R}} \quad \text{for } i, j \geq 1
\]
where \( \delta_{ij} \) is Kronecker symbol.

Notice that since \( M(x, \sqrt{y}) \in C^2(\Omega \times \mathbb{R}_+) \) we have that \( B \in C^2(\Omega \times \mathbb{R}^n) \). If \( \nabla f = 0 \) then there is nothing to prove because \( W \) becomes diagonal matrix with negative entries on the diagonal. Further assume \( \|\nabla f\| \neq 0 \).

Now notice that
\[
W = S \left( W_1 + \frac{M_y \sqrt{R}}{\|\nabla f\|} W_2 \right) S
\]
where \( S \) is a diagonal matrix with diagonal \( (1, \frac{\nabla f}{\|\nabla f\| \sqrt{R}}) \), and
\[
W_1 = \begin{bmatrix}
M_{xx} + \frac{\sqrt{R} M_{xy}}{\|\nabla f\|} & M_{xy} & \cdots & M_{xy} \\
M_{xy} & M_{yy} & \cdots & M_{yy} \\
\vdots & \vdots & \ddots & \vdots \\
M_{xy} & M_{yy} & \cdots & M_{yy} \\
\end{bmatrix}
\quad \text{and} \quad
W_2 = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & \frac{\|\nabla f\|^2}{(f_j)_x^2} - 1 & -1 & \cdots & -1 \\
0 & -1 & \frac{\|\nabla f\|^2}{(f_j)_y^2} - 1 & \cdots & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & -1 & \cdots & -1 & \frac{\|\nabla f\|^2}{(f_{nx})^2} - 1 \\
\end{bmatrix}
\]

It is clear that \( W_1 \leq 0 \) because \( M \) satisfies (1.1) at point \( x \) and \( \frac{\nabla f}{\|\nabla f\| \sqrt{R}} \).

For the \( W_2 \), first notice that if \( f_j \neq 0 \) for all \( j \geq 1 \) then \( W_2 \) is well defined and \( W_2 \geq 0 \). Otherwise if \( f_j = 0 \) for some \( j \), then consider initial expression \( SW_2 S \) and notice that \( SW_2 S = SW_2 S + D \), where \( W_2 \) is the same as \( W_2 \) except \( j \)th column and row are replaced by zeros, and \( D \) is zero matrix except the element \( (j, j) \) is equal to \( \frac{1}{R} \). We again see that \( SW_2 S \geq 0 \). Hence \( M_y SW_2 S \leq 0 \) as soon as (1.1) holds.

Thus we have proved that if \( M(x, \sqrt{y}) \in C^2(\Omega \times \mathbb{R}_+) \), \( M \) satisfies (1.1) then by Theorem 2 we have
\[
P_t M(f, \|\nabla f\|) \leq M(P_t f, \|\nabla P_t f\|) \quad \text{for all } f \in C^\infty(\mathbb{R}^n; \Omega). \quad (2.5)
\]

We send \( t \to \infty \) and because of the fact \( \|\nabla P_t f\| \leq e^{-tR} P_t \|\nabla f\| \) (see [3]) we obtain
\[
\int_{\mathbb{R}^n} M(f, \|\nabla f\|) d\mu \leq M\left( \int_{\mathbb{R}^n} f d\mu, 0 \right),
\]
where \( d\mu = e^{-U(x)} dx \) is a probability measure.
Remark 3. It is worth mentioning but not necessary for our purposes that (2.5) also implies (1.1) in case of Gaussian measure. This follows from the fact that the matrix $\lambda \Gamma(D^2 f)$ can be an arbitrary positive definite matrix where $\lambda > 0$ and $f \in C^\infty(\mathbb{R}^n; \Lambda)$. Then condition $\text{Tr}(W \Gamma(D^2 f)) \leq 0$ implies that $W \leq 0$ and this gives us condition (1.1).

2.2. Relation to stochastic calculus and $\Gamma$-calculus approach. Inequality (2.5) implies that the map

$$t \to \int_{\mathbb{R}^n} M(P_t f, \|\nabla P_t f\|) d\gamma$$

is monotone provided that $M$ satisfies (1.1). Indeed, by sending $t \to 0$ in (2.5) we obtain its infinitesimal form $LM(f, \|\nabla f\|) \leq \frac{d}{ds} M(P_s f, \|\nabla P_s f\|) \bigg|_{s=t}$. Finally, if the last inequality is true for any $f$ then it is true for any $f$ of the form $P_t f$. This implies that

$$LM(P_t f, \|\nabla P_t f\|) \leq \frac{d}{ds} M(P_s f, \|\nabla P_s f\|) \bigg|_{s=t},$$

and it gives monotonicity of (2.6).

Interpolation (2.5) (or even monotonicity (2.6)) plays a fundamental role in functional inequalities and it was known before for some particular functions $M(x, y)$ as a consequence of their special properties and some linear algebraic manipulations (see [3], [1]). The purpose of Theorem 1 was to exclude the linear algebra involved in the interpolation (2.5) and to show that in fact (2.5) boils down (actually it is equivalent) to the fact that $M$ satisfies an elliptic Monge–Ampère equation of a general form (1.1). Monge–Ampère equation (1.1), apparently, was not noticed before or it was hidden in the literature from the wide audience. Equations of Monge–Ampère type are of course among the most important fully nonlinear partial differential equations (see [35], [36]).

Next we will show that in fact (1.1) gives monotonicity of the type (2.6) in different settings as well.

2.2.1. Stochastic calculus approach.

Proposition 1. For $t \geq 0$, let $W_t, N_t$ be $\mathcal{F}_t$ real-valued martingales with $W_t = W_0 + \int_0^t w_s dB_s, N_t = N_0 + \int_0^t a_s dB_s$, and let $A_t = A_0 + \int_0^t a_s dW_s$ where $A_0, a_s \geq 0$. Assume that $A_t$ is bounded, $a_t|N_t|^2 \geq |w_t|^2$ and $W_t \in \Omega$ for $t \geq 0$. Assume that $M(x, \sqrt{\gamma}) \in C^2(\Omega \times \mathbb{R})$ satisfies (1.1). Then

$$z_t = M(W_t, |N_t| \sqrt{A_t})$$

is a supermartingale for $t \geq 0$.

Proof. The proof of the proposition proceeds absolutely in the same way as in [1], which treat the case of a particular function $M$ involved in Bobkov’s inequality. In fact, it is (1.1) which makes the drift $\Delta(t)$ nonpositive where $dz_t = u_t dB_t + \Delta(t) dt$. \qed

One may obtain another proof of Theorem 1 for the case $n = 1$ using Proposition 1 for the special case $W_t = \mathbb{E}[f(B_t)|\mathcal{F}_t], A_t = t$ and $N_t = \mathbb{E}[\nabla f(B_t)|\mathcal{F}_t]$ where $0 \leq t \leq 1$, $B_0 = 0$, and $f$ is a real valued smooth bounded function. Indeed, in this case by optional stopping theorem one obtains

$$M(\mathbb{E}[f(B_t)|\mathcal{F}_0], 0) = z_0 \geq \mathbb{E}z_1 = \mathbb{E}M(f(B_1), |\nabla f(B_1)|).$$
2.2.2. Γ-calculus approach. One may obtain Theorem 1 using the remarkable Γ-calculus (see [3]). In fact, setting \( \Gamma(f, g) = \nabla f \cdot \nabla g \) one can easily show that if \( M(x, \sqrt{y}) \in C^2(\Omega \times \mathbb{R}_+) \) satisfies (1.1) then the following map

\[
s \to P_s B(P_{t-s} f, \Gamma(P_{t-s} f, P_{t-s} f))
\]

is monotone for \( 0 \leq s \leq t \) for any given \( t > 0 \), where \( B(x, y^2) = M(x, y) \). Initially this was the way we obtained Theorem 1 (see for example [23]). Later it became clear to us that one does not have to be limited with notations \( \Gamma, \Gamma_2, \Gamma_3 \) etc in order to enjoy interpolations of the form (2.5). In fact, one can directly work with an arbitrary differential operator \( D^\alpha \) (2.1), and Theorem 1 is just a consequence of Theorem 2 for an appropriate choice of Bellman function (2.4). In support of classical notations we should say that it is not clear for us how Γ-calculus can be used in proving Theorem 2 which is a simple statement if working with the classical notations of differential operators \( D^\alpha \).

3. Reduction to the exterior differential systems and backwards heat equation

As we have already mentioned in Section 1.1.8 (and it also follows from the proof of Theorem 1) in order inequality (1.2) to be sharp we need to assume that (1.1) degenerates i.e.,

\[
\det \begin{pmatrix} M_{xx} + \frac{M_y}{M_{xy}} & M_{xy} \\ M_{xy} & M_{yy} \end{pmatrix} = M_{xx}M_{yy} - M_{xy}^2 + \frac{M_yM_{yy}}{y} = 0. \tag{3.1}
\]

Let us make the following observation: consider 1-graph of \( M(x, y) \) i.e.,

\[
(x, y, p, q) = (x, y, M_x(x, y), M_y(x, y))
\]

in \( xypq \)-space. This is a simply-connected surface \( \Sigma \) in 4-space on which \( \Upsilon = dx \wedge dy \) is non-vanishing but to which the two 2-forms

\[
\Upsilon_1 = dp \wedge dx + dq \wedge dy \quad \text{and} \quad \Upsilon_2 = (ydp + qdx) \wedge dq
\]

pull back to be zero.

Conversely, suppose given simply connected surface \( \Sigma \) in \( xypq \)-space (with \( y > 0 \)) on which \( \Upsilon \) is non-vanishing but to which \( \Upsilon_1 \) and \( \Upsilon_2 \) pull back to be zero. The 1-form \( pdx + qdy \) pull back to \( \Sigma \) to be closed (since \( \Upsilon_1 \) vanishes on \( \Sigma \)) and hence exact, and therefore there exists a function \( m : \Sigma \to \mathbb{R} \) such that \( d \tau = pdx + qdy \) on \( \Sigma \). We then have (at least locally), \( m = M(x, y) \) on \( \Sigma \) and, by its definition, we have \( p = M_x(x, y) \) and \( q = M_y(x, y) \) on the surface. Then fact that \( \Upsilon_2 \) vanishes when pulled back to \( \Sigma \) implies that \( M(x, y) \) satisfies the desired equation.

Thus, we have encoded the given PDE as an exterior differential system on \( \mathbb{R}^4 \). Note, that we can make a change of variables on the open set where \( q < 0 \): Set \( y = qr \) and let \( t = \frac{1}{2}q^2 \). then, using these new coordinates on this domain, we have

\[
\Upsilon_1 = dp \wedge dx + dt \wedge dr \quad \text{and} \quad \Upsilon_2 = (rdp + dxr) \wedge dt.
\]

Now, when we take an integral surface \( \Sigma \) on these 2-forms on which \( dp \wedge dt \) is vanishing, it can be written locally as a graph of the form

\[
(p, t, x, r) = (p, t, u_p(p, t), u_t(p, t))
\]

(since \( \Sigma \) is an integral of \( \Upsilon_1 \)), where \( u(p, t) \) satisfies \( u_t + u_{pp} = 0 \) (since \( \Sigma \) is an integral of \( \Upsilon_2 \)). Thus, “generically” our PDE is equivalent to the backwards heat equation, up to a change of
variables. Thus the function $M(x, y)$ can be parametrized as follows

$$x = u_p \left( p, \frac{1}{2} q^2 \right); \quad y = q u_t \left( p, \frac{1}{2} q^2 \right);$$

$$M(x, y) = p u_p \left( p, \frac{1}{2} q^2 \right) + q^2 u_t \left( p, \frac{1}{2} q^2 \right) - u \left( p, \frac{1}{2} q^2 \right). \quad (3.2)$$

Note that $y \geq 0$, $q = M_y \leq 0$ then $u_t \left( p, \frac{1}{2} q^2 \right) \leq 0$. Let us rewrite the conditions $M_{yy} \leq 0$ and $M_{xx} + \frac{M_y}{y} \leq 0$ in terms of $u(p, t)$. In other words we want $q_y$ and $p_x + \frac{q}{y} \leq 0$. We have

$$0 = u_{pp} p_y + u_{pt} q q_y \quad \text{and} \quad 1 = q_y u_t + q_p y u_{tp} + q^2 q_y u_{tt}.$$

Then

$$1 = q_y u_t + q^2 q_y \left( \frac{u_{pt}}{u_{pp}} \right)^2 + q^2 q_y u_{tt} \quad \text{and} \quad M_{yy} = q_y = \frac{u_t}{u_t^2 - 2t(u_{tt} u_{pp} - u_{pt}^2)}.$$

Thus the negative definiteness of the matrix (1.1) (if its determinant is known to be zero) is equivalent to

$$u_t^2 - 2t \det(\text{Hess } u) \geq 0. \quad (3.3)$$

Let us show that the function $u(p, t)$ must satisfy a boundary condition:

$$u(f'(x), 0) = x f'(x) - f(x) \quad \text{for} \quad x \in \Omega \quad \text{where} \quad f(x) = M(x, 0). \quad (3.4)$$

Indeed, we know that $M(x, \sqrt{y}) \in C^2(\Omega \times \mathbb{R}_+)$ therefore $M_y(x, 0) = 0$. By choosing $y = 0$ in (3.2), we have $q = 0$, and we obtain the desired boundary condition:

$$M(x, 0) = x M_x(x, 0) - u(M_x(x, 0), 0).$$

Now it is clear how to find the function $M(x, y)$ provided that $M(x, 0)$ is given: First we try to find a function $u(p, t)$ such that

$$u_{pp} + u_t = 0, \quad u_t \leq 0, \quad (3.5)$$

$$u(M_x(x, 0), 0) = x M_x(x, 0) - M(x, 0) \quad x \in \Omega, \quad (3.6)$$

$$u_t^2 - 2t \det(\text{Hess } u) \geq 0. \quad (3.7)$$

Then a candidate for $M(x, y)$ will be given by (3.2). We should mention that if $M(x, 0)$ is convex then (3.6) simply means that $u_p(p, 0)$ is Legendre transform of $M_x(x, 0)$. Indeed, if we take derivative in (3.6) with respect to $x$ we obtain $u_p(M_x(x, 0), 0) = x$.

### 3.1. Back to the applications, old and new

**Revisiting Section 1.1 with our new tool.** Further we assume that we know the expression $M(x, 0)$ and we would like to restore the function $M(x, y)$ which satisfies conditions of Theorem 1, PDE (3.1) and hence it gives us inequality (1.3), or the error term in Jensen’s inequality (see Section 1.1.8 for the explanations).
3.1.1. Gross function. In this case we have \( M(x,0) = x \ln x \). Condition (3.6) can be rewritten as follows \( u(p,0) = e^{p-1} \) for all \( p \in \mathbb{R} \). If we set \( D = \frac{\partial^2}{\partial p^2} \) then

\[
\begin{align*}
  u(p,t) &= e^{-tD}e^{p-1} = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} e^{p-1} = e^{p-t-1} \quad \text{for all } \ t \geq 0.
\end{align*}
\]

Clearly \( u(p,t) \) satisfies (3.7) because \( \det(\text{Hess } u) = 0 \). Notice that we have \( u_t < 0 \),

\[
\begin{align*}
  \begin{cases}
    x = e^{p-\frac{q^2}{2}} - 1; \\
    y = -q e^{p-\frac{q^2}{2}} - 1;
  \end{cases}
  \quad \text{then} \quad
  \begin{cases}
    q = -\frac{y}{x}; \\
    p = \ln x + \frac{y^2}{2x^2} + 1.
  \end{cases}
\end{align*}
\]

Therefore we obtain

\[
M(x,y) = xp + qy - u \left( p, \frac{1}{2} q^2 \right) = x \ln x + \frac{y^2}{2x^2} + x - \frac{y^2}{x} - x = x \ln x - \frac{y^2}{2x}.
\]

3.1.2. Nash’s function. In this case we have \( M(x,0) = x^2 \). Condition (3.6) takes the form

\[
u(p,0) = \frac{p^2}{4} \quad \text{for all } \ p \in \mathbb{R}.
\]

Then

\[
u(p,t) = e^{-tD}D^2 - \frac{1}{4} = (1-tD)D^2 - \frac{1}{4} = \frac{p^2}{4} - \frac{t}{2} \quad t \geq 0.
\]

\( u(p,t) \) satisfies (3.7) because \( \det(\text{Hess } u) = 0 \). We have \( u_t < 0 \)

\[
\begin{align*}
  \begin{cases}
    x = \frac{p}{2}; \\
    y = -\frac{q}{2};
  \end{cases}
  \quad \text{then} \quad
  \begin{cases}
    p = 2x; \\
    q = -2y.
  \end{cases}
\end{align*}
\]

We obtain

\[
M(x,y) = 2x^2 - 2y^2 - (x^2 - y^2) = x^2 - y^2.
\]

3.1.3. Bobkov’s function. It is not clear at all where the function \( M(x,y) = -\sqrt{I(x^2 + y^2)} \) comes from. Apparently it was a pretty good guess.

Let us show how easily it can be restored by solving Monge–Ampère equation (1.37). In this case we have \( M(x,0) = -I(x) \). Condition (3.6) takes the form

\[
u(p,0) = p \Phi(p) + \Phi'(p) \quad \text{for all } \ p \in \mathbb{R}.
\]

Now we will try to find the usual heat extension of \( u(p,0) \) (call it \( \tilde{u}(p,t) \)) which satisfies \( \tilde{u}_{pp} = \tilde{u}_t \), and then we try to consider the formal candidate \( u(p,t) := \tilde{u}(p,-t) \).

It is easier to find the heat extension of \( \tilde{u}_p(p,0) \) and then take the antiderivative in \( p \). Indeed, notice that (3.8) implies \( u_p(p,0) = \Phi(p) \). the heat extension of \( \Phi(p) \) is \( \Phi(p) \left( \frac{p}{\sqrt{1+2t}} \right) \).

Indeed, the heat extension of the function \( 1_{(-\infty,0]}(p) \) at time \( t = 1/2 \) is \( \Phi(p) \). Then by the semigroup property the heat extension of \( \Phi(p) \) at time \( t \) will be the heat extension of \( 1_{(-\infty,0]}(p) \) at time \( 1/2 + t \) which equals to \( \Phi(p) \left( \frac{p}{\sqrt{1+2t}} \right) \). Thus \( \tilde{u}_p(p,t) = \Phi(p) \left( \frac{p}{\sqrt{1+2t}} \right) \). Taking antiderivative in \( p \) and using (3.8) if necessary we obtain

\[
\tilde{u}(p,t) = \sqrt{1 + 2t} \Phi' \left( \frac{p}{\sqrt{1+2t}} \right) + p \Phi \left( \frac{p}{\sqrt{1+2t}} \right).
\]

This expression is well defined even for \( t \in (-1/2,0) \). Therefore if we set

\[
u(p,t) = \tilde{u}(p,-t) = \sqrt{1 - 2t} \Phi' \left( \frac{p}{\sqrt{1-2t}} \right) + p \Phi \left( \frac{p}{\sqrt{1-2t}} \right) \quad \text{for} \ p \in \mathbb{R}, \ t \in \left[ 0, \frac{1}{2} \right],
\]
direct computations show that \( u(p, t) \) satisfies (3.5), (3.8) and (3.7) because \( \det(\text{Hess} \, u) = -\left( \frac{\Phi'(\sqrt{1 - 2t})}{\sqrt{1 - 2t}} \right)^2 < 0 \). We have \( u_t = -\Phi'(\sqrt{1 - 2t}) < 0 \) and \( u_p = \Phi \left( \frac{p}{\sqrt{1 - 2t}} \right) \). Therefore,

\[
\begin{cases}
  x = \frac{p}{\sqrt{1 - q^2}}; \\
  y = \frac{-q}{\sqrt{1 - q^2}} \Phi' \left( \frac{p}{\sqrt{1 - q^2}} \right);
\end{cases}
\]

From the last equalities we obtain \( M_y = q = -\frac{y}{\sqrt{I^2(x) + y^2}} \) and \( M_x = p = \frac{I(x) \Phi^{-1}(x)}{\sqrt{I^2(x) + y^2}} \) where we remind that \( I(x) = \Phi'(\Phi^{-1}(x)) \). Then it follows that

\[
M(x, y) = -\sqrt{I^2(x) + y^2}.
\]

3.1.4. Function \( 3/2 \). In this case we have \( M(x, 0) = x^{3/2} \) for \( x \geq 0 \). It follows from (3.6) that \( u(p, 0) = \frac{1}{4\pi} p^2 \) for \( p \geq 0 \). The solution of the backwards heat is the Hermite polynomial, i.e., we have \( u(p, t) = \frac{1}{\sqrt{\pi t}} (p^2 - 6tp) \). \( u(p, t) \) satisfies (3.7) because \( \text{Hess} \, u < 0 \). Since \( p \geq 0 \) we have \( u_t \leq 0 \). Next we obtain

\[
\begin{cases}
  x = \frac{4}{9} (p^2 - q^2); \\
  y = -\frac{8}{9} pq.
\end{cases}
\]

Finally

\[
M(x, y) = xp + yq - u \left( p, \frac{1}{2} q^2 \right) = \frac{1}{2} (2x - \sqrt{x^2 + y^2}) \sqrt{x + \sqrt{x^2 + y^2}}.
\]

3.1.5. Function \( \arccos(x) \). Consider an increasing convex function

\[
M(x, 0) = x \arccos(-x) + \sqrt{1 - x^2}.
\]

It follows from (3.6) that \( u(p, 0) = -\sin(p) \) for \( p \in [0, \pi] \). The solution of the backwards heat (3.5) becomes \( u(p, t) = -e^t \sin(p) \). Notice that \( u_t \leq 0 \) for \( p \in [0, \pi] \), and

\[
u_t^2 - 2t \det(\text{Hess} \, u) = e^{2t} (2t + \sin^2(x)) \geq 0.
\]

Conditions (3.2) can be rewritten as follows

\[
x = -e^{q^2} \cos(p); \\
y = -e^{q^2/2} \sin(p); \\
M(x, y) = px + qy + e^{q^2/2} \sin(p) = px + qy - \frac{y}{q}, \quad x \in [-1, 1], \quad y \geq 0.
\]

It follows that the negative number \( q \) satisfies the equation

\[
-q \sqrt{e^{q^2} - x^2} = y, \quad (3.9)
\]

and \( p = \arccos(-xe^{-q^2/2}) \). Thus we obtain

\[
M(x, y) = x \arccos(-xe^{-q^2/2}) + (1 - q^2) \sqrt{e^{q^2} - x^2}
\]
where a negative number $q$ is the unique solution of (3.9). Thus we obtain
\[ \int_{\mathbb{R}^n} f \arccos(-fe^{-r/2}) + (1-r)\sqrt{e^r - f^2}d\gamma \leq \] (3.10)
\[ \left( \int_{\mathbb{R}^n} f d\gamma \right) \arccos \left( -\int_{\mathbb{R}^n} f + \sqrt{1 - \left( \int_{\mathbb{R}^n} f d\gamma \right)^2} \right) \]
for any smooth bounded $f : \mathbb{R}^n \to (-1,1)$ where $r > 0$ solves the equation
\[ \|\nabla f\|^2 = r(e^r - f^2) . \]

One can obtain Poincaré inequality from (3.10). Indeed, take $f_\varepsilon = \varepsilon f$ and send $\varepsilon \to 0$.

Notice that
\[ r = \varepsilon^2\|\nabla f\|^2 + O(\varepsilon^2) ; \]
\[ M \left( \int_{\mathbb{R}^n} f_{\varepsilon} d\gamma, 0 \right) = 1 + \frac{\pi}{2} \varepsilon \int_{\mathbb{R}^n} f d\gamma + \frac{1}{2} \left( \int_{\mathbb{R}^n} f d\gamma \right)^2 \varepsilon^2 + O(\varepsilon^2) ; \]
\[ M(f_{\varepsilon}, \|\nabla f\|) = 1 + \frac{\pi}{2} \varepsilon f + \frac{1}{2} (f^2 - \|\nabla f\|^2) \varepsilon^2 + O(\varepsilon^2) . \]

Substituting these expressions into (3.10) and sending $\varepsilon \to 0$ we obtain Poincaré inequality.

### 4. One dimensional case

Let $n = 1$, and set $\alpha = (\alpha_0, \ldots, \alpha_m)$ where $\alpha_0 = 0, \alpha_1 = 1, \ldots, \alpha_n = n$. Take $f = (f_1, \ldots, f_{m+1})$,

where $f \in C_0^\infty(\mathbb{R} ; \Omega)$. Then $D^\alpha f = (f_1, f_1', f_1'', \ldots, f_1^{(m)})$. Given a log-concave probability measure $e^{-U(x)}dx$ such that $U''(x) \geq 0$, the associated semigroup $P_t$ has the generator
\[ L = d^2x - U'(x)dx . \]

Let $u = (u_0, \tilde{u})$ where $\tilde{u} = (u_1, \ldots, u_m) \in \mathbb{R}^m$ is arbitrary and $u_0 \in \Omega$. Let the function $B(u_0, \ldots, u_m) \in C^2(\Omega \times \mathbb{R}^m)$. Let $B_j := \frac{\partial B}{\partial u_j}$ and $B_{ij} := \frac{\partial^2 B}{\partial u_i \partial u_j}$. Set
\[ L_j(u, y) = \sum_{k=j+1}^m \binom{k}{j} B_k(u) U^{(k-j+1)}(y) \quad \text{for} \quad j = 0, \ldots, m-1 . \]

**Remark 4.** Notice that if $e^{-U(x)}dx = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx$ then $L_j(u, y) = (j + 1)B_{j+1}(u)$.

Further we assume that $B_{mm} \neq 0$. Theorem 2 implies the following corollary:

**Corollary 2.** The following conditions are equivalent:

(i) For all $u \in \Omega \times \mathbb{R}^m$ we have
\[ B_{mm} \leq 0, \quad \tilde{u} \{ B_{m} - B_{mm} B_{i} \} \leq 0 . \]

(ii) For all $f \in C_0^\infty(\mathbb{R}^m ; \Omega)$ and $t \geq 0$ we have
\[ P_t B(f, f', \ldots, f^{(m)}) \leq B(P_t f, P_t f', \ldots, P_t f^{(m)}) . \]

**Remark 5.** If we send $t \to \infty$ then (ii) in the corollary implies an inequality
\[ \int_{\mathbb{R}} B(f, f', \ldots, f^{(m)}) d\mu(x) \leq B \left( \int_{\mathbb{R}} f d\mu, 0, \ldots, 0 \right) \quad \text{for all} \quad f \in C_0^\infty(\mathbb{R}^m ; \Omega) . \]
Proof. It is enough to show that (i) in Corollary 2 is the same as (i) in Theorem 2. Notice that

\[ [L, D^{α_0}] = 0 \quad \text{and} \quad [L, D^{α_k}] = \sum_{ℓ=1}^{k} \left( \binom{k}{ℓ} U^{(ℓ+1)}(x) d^{k+1-ℓ} x \right) \quad \text{for} \quad 1 \leq k \leq m. \]

Thus

\[ \nabla B[L, D^{α}] f = \sum_{k=1}^{n} \frac{∂B}{∂u_k} \left( \sum_{ℓ=1}^{k} \left( \binom{k}{ℓ} U^{(ℓ+1)}(x) (P_t f)^{k+1-ℓ} \right) \right) \]

and

\[ \Gamma(D^{α} f) = \begin{bmatrix} g' \cdot g' & g' \cdot g'' & \cdots & g' \cdot g^{(m+1)} \\ g'' \cdot g' & g'' \cdot g'' & \cdots & g'' \cdot g^{(m+1)} \\ \vdots & \vdots & \ddots & \vdots \\ g^{(m+1)} \cdot g' & g^{(m+1)} \cdot g'' & \cdots & g^{(m+1)} \cdot g^{(m+1)} \end{bmatrix} \]

Therefore quantity (i) in Theorem 2 takes the following form

\[ \sum_{k=1}^{m} B_k(u) \left[ \sum_{ℓ=1}^{k} \left( \binom{k}{ℓ} U^{(ℓ+1)}(y) u_{k+1-ℓ} \right) \right] + \sum_{i,j=0}^{m} B_{ij}(u) u_{i+1} u_{j+1} \]

where \( u_1, \ldots, u_{n+1}, y \) are arbitrary real numbers and \( u_0 \) takes values in \( Ω \). Notice that the above expression can be rewritten as follows

\[ B_{mn} u_{m+1}^2 + 2u_{m+1} \left( \sum_{j=0}^{m-1} B_{mj} u_{j+1} \right) + \sum_{i,j=0}^{m-1} B_{ij} u_{i+1} u_{j+1} + \sum_{k=1}^{m} B_k(u) \left[ \sum_{ℓ=1}^{k} \left( \binom{k}{ℓ} U^{(ℓ+1)}(y) u_{k+1-ℓ} \right) \right] \]

This expression is nonpositive if and only if condition (i) of Corollary 2 holds. \( \square \)

5. Further applications

Houdré–Kagan [21] obtained an extension of the classical Poincaré inequality:

\[ \sum_{k=1}^{2d} \frac{(-1)^{k+1}}{k!} \int_{ℝ^n} ||∇^k f||^2 dγ \leq \int_{ℝ^n} f^2 dγ - \left( \int_{ℝ^n} f \right)^2 \leq \sum_{k=1}^{2d-1} \frac{(-1)^{k+1}}{k!} \int_{ℝ^n} ||∇^k f||^2 dγ \quad (5.1) \]

for all compactly supported functions \( f \) on \( ℝ^n \), and any \( d \geq 1 \). Here by symbol \( ||∇^k f|| \) we denote

\[ ||∇^k f||^2 = \sum_{|α|=k} (D^α f)^2. \]

We refer the reader to [34] for further remarks on (5.1) in one dimensional case \( n = 1 \). A remarkable paper [30] explains (5.1) via integration by parts.

We will illustrate the use of Corollary 2 on (5.1) in case \( n = 1 \).

Proof of (5.1) in case \( n = 1 \).

Consider

\[ B(u_0, u_1, \ldots, u_m) = \sum_{k=0}^{m} \frac{(-1)^k}{k!} u_k^2, \]
and $d\mu = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$. If $m$ is odd then $B_{mm} \leq 0$ and condition (i) of Corollary 2 holds. Indeed, in this case $L_j(u, y) = B_{j+1}(u)(j + 1) = u_{j+1}(-1)^{j+1} \frac{2}{j!}$, and
\[
\tilde{u}\{B_{mj}(u)B_{mi}(u) - B_{mm}(u)B_{ij}(u) - \delta_{ij} \frac{B_{mm}}{u_{j+1}} L_j(u, y)\}_{i,j=0}^{m-1} \tilde{u}^T =
\]
\[
- B_{mm} \tilde{u}\left\{B_{jj} + B_{j+1} \frac{j + 1}{u_{j+1}} \right\}_{i,j=0}^{m-1} \tilde{u}^T = 0.
\]
Thus by (ii) of Corollary 2 we obtain that for all $f \in C_0^\infty(\mathbb{R})$
\[
\left(\int_{\mathbb{R}^n} f^2 d\mu\right)^2 - \left(\int_{\mathbb{R}^n} f d\mu\right)^2 \leq \frac{1}{2} \int_{\mathbb{R}^n} \|\nabla f\|^2 d\mu.
\]
for odd $m$, and similarly we obtain the opposite inequality for even $m$.

**Proof of (1.30) (Banaszczyk conjecture).** We will show that if $f$ is even and $d\mu = e^{-U(x)} dx$ is an even log-concave measure such that $\text{Hess}U \geq \text{Id}$ then
\[
\left(\int_{\mathbb{R}^n} f^2 d\mu\right)^2 - \left(\int_{\mathbb{R}^n} f d\mu\right)^2 \leq \frac{1}{2} \int_{\mathbb{R}^n} \|\nabla f\|^2 d\mu.
\]
(5.2)
Indeed, take $M(x, y)$ as in (1.31) i.e.,
\[
M(x, y) = x^2 - \frac{y^2}{2}
\]
for $x \in \mathbb{R}$, $y \geq 0$.
Unfortunately $M(x, y)$ does not satisfy (1.1) (because $M_{xx} + M_{yy}/y = 1 > 0$) therefore we cannot directly apply Theorem 1.

Let $P_t$ be the associated semigroup to $d\mu$ and let $L$ be its generator. Consider the function $V(x, t) = P_t M(f, \|\nabla f\|) - M(P_t f, \|\nabla P_t f\|)$ as in the proof of Theorem 2. Then
\[
(\partial_t - L)V(x, t) = -\nabla P_t f(\text{Hess}U)(\nabla P_t f)^T + 2\|\nabla P_t f\|^2 - \|\nabla^2 P_t f\|^2 \leq \|\nabla P_t f(x)\|^2 - \|\nabla^2 P_t f\|^2.
\]
Clearly it is not true that the above expression is pointwise i.e., for all $x \in \mathbb{R}^n$, non positive (consider $t = 0$). Therefore we cannot directly apply maximum principle as in the proof of Theorem 2 in order to get pointwise bound $V(x, t) \leq 0$. Actually we do not need pointwise estimate $V(x, t) \leq 0$ in order to get (5.2), for example $\int_{\mathbb{R}^n} V(x, t) d\mu \leq 0$ will be enough. Notice that
\[
\int_{\mathbb{R}^n} V(x, T) d\mu = \int_0^T \int_{\mathbb{R}^n} (\partial_t - L)V(x, t) d\mu dt \leq \int_0^T \int_{\mathbb{R}^n} \|\nabla P_t f(x)\|^2 - \|\nabla^2 P_t f\|^2 d\mu ds \leq 0
\]
for all $T \geq 0$. The last inequality follows from the application of Poincaré inequality (1.17) to the functions $\partial_{x_j} P_t f(x)$ for all $j = 1, \ldots, n$, and the fact that $\int_{\mathbb{R}^n} \partial_{x_j} P_t f(x) d\mu = 0$ because $P_t f(x)$ is even function. Thus we obtain that
\[
\int_{\mathbb{R}^n} M(f, \|\nabla f\|) d\mu \leq \int_{\mathbb{R}^n} M(P_T f, \|\nabla P_T f\|) d\mu
\]
for all $T \geq 0$.
By sending $T \to \infty$ we arrive at (5.2) because $\lim_{T \to \infty} \|\nabla P_T f\| = 0$.

In the end we should mention that even though the current paper is self-contained it should be considered as a continuation of the ideas developed in our recent papers [25, 26] where similar to (1.1) PDEs happen to rule some functional inequalities.
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