BOMBIERI-TYPE THEOREM FOR CONVOLUTION OF ARITHMETIC FUNCTIONS ON NUMBER FIELD

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Abstract. Let $K$ be an imaginary quadratic number field of class number one and $O_K$ be its ring of integers. We show that, if the arithmetic functions $f, g : O_K \rightarrow \mathbb{C}$ both have level of distribution $\vartheta$ for some $0 < \vartheta \leq 1/2$ then the Dirichlet convolution $f \ast g$ also have level of distribution $\vartheta$.

1. Introduction and statements of results

Let $\Lambda(n)$ be the usual Van-Mangoldt function. For $x > 1$ Siegel-Walfisz theorem states that for any $D > 0$

$$\sum_{n \leq x} \chi(n) \Lambda(n) = O\left( \frac{x}{(\log x)^D} \right)$$

for any non-principal character $\chi \pmod{q}$ if $q \ll (\log x)^{3D}$.

An arithmetic function $f$ is said to have level of distribution $\vartheta$ for $0 < \vartheta \leq 1$ if for any $A > 0$ there exists a constant $B = B(A)$ such that

$$\sum_{q \leq N^{\vartheta}} \max_{M \leq N} \max_{a \equiv \chi(q)} \left| \sum_{n \leq M} f(n) - \frac{1}{\varphi(q)} \sum_{n \leq M} f(n) \right| \leq A \frac{N}{(\log N)^A}.$$ (1.1)

The Bombieri-Vinogradov theorem states indicator function of primes have level of distribution $\vartheta$ for any $\vartheta \leq 1/2$ and the Elliott-Halberstam conjecture predicts the level of distribution to be 1.

A complex valued arithmetic function $f$ is said to satisfy Siegel-Walfisz condition if there exist positive constants $C, D$ such that

$$f(n) = O\left( \tau(n)^C \right) \quad \text{and} \quad \sum_{n \leq x} f(n) \chi(n) = O\left( \frac{x}{(\log x)^D} \right),$$ (1.2)

for any non-principal Dirichlet character $\chi \pmod{q}$ where $q$ is an ideal of $O_K$ of norm $q \ll (\log x)^{3D}$.

If arithmetic function $f$ and $g$ both satisfies (1.2) condition and have level of distribution 1/2 then Motohashi [8] obtained that the Dirichlet convolution $f \ast g$ does so.

In this article, we extend Motohashi’s [8] result to arithmetic functions on imaginary quadratic number fields of class number one.

Let $K$ be a number field of degree $d$, class number one with $r_1$ real and $r_2$ non-conjugate complex embeddings and $O_K$ be its ring of integers. An element $w \in O_K$ is said to be a prime number in $K$, if the principal ideal $wO_K$ is a prime ideal. Let $\mathcal{P}$ be the set of prime numbers in $K$. 

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Now we first introduce the notion of Siegel-Walfisz condition and level of distribution in number field.

For $Y' \geq 1, Y \geq 0$ and $N > 1$, let $A_0^0(Y', Y, N)$ be the set of $\xi \in \mathcal{O}_K$ which satisfies $Y' \leq \sigma(\xi) \leq Y + N^b$ for all real embeddings and $Y' \leq |\sigma(\xi)| \leq Y + N^b$ for all complex embeddings of $K$. We also define $A^0(Y', N + Y') = A_0^0(Y', Y, N)$ and $A^0(N) = A_0^0(1, 0, N)$.

A complex valued arithmetic function $f : \mathcal{O}_K \to \mathbb{C}$ is said to satisfy Siegel-Walfisz condition if there exist positive constants $C, D$ such that

\[(S-W) \quad f(a) = O \left( \tau(a)^C \right) \quad \text{and} \quad \sum_{a \in A^0(N)} f(a)\chi(a) = O \left( \frac{|A^0(N)|}{(\log N)^{3D}} \right), \]

for any non-principal Dirichlet character $\chi \mod q$ with $|q| \ll (\log N)^D$.

An arithmetic function $f : \mathcal{O}_K \to \mathbb{C}$ is said to have level of distribution $\vartheta$ for $0 < \vartheta \leq 1$ if for any $A > 0$ there exists a constant $B = B(A)$ such that if $Q = \frac{|A^0(N)|^\vartheta}{(\log N)^A}$ then

\[(1.3) \quad \sum_{|q| \leq Q} \max_{M \leq N} \max_{\gamma, a=1} |\varepsilon(M; q, a; f)| \ll_{A,K} |A^0(N)|^{1-\vartheta}, \]

where

\[\varepsilon(M; q, a; f) = \sum_{\substack{\alpha \in A^0(M) \\ \alpha \equiv \gamma \mod q}} f(a) - \frac{1}{\varphi(q)} \sum_{\substack{\alpha \in A^0(M) \\ \alpha \equiv \gamma \mod q}} f(a).\]

An analogue of Elliott-Halberstam conjecture for number fields predicts that the prime element in $\mathcal{O}_K$ have level of distribution $\vartheta$ with any $\vartheta$ in $0 < \vartheta \leq 1$. Hinz [6] showed that primes have level of distribution $1/2$ in totally real algebraic number fields and have level of distribution $2/5$ in imaginary quadratic fields. Huxley [7] obtained level of distribution $1/2$ for an weighted version of (1.3).

Remark. Method applied in this paper relies on the equality $|\sigma(w)| = |w|^{1/2}$ for each $w \in \mathcal{O}_K$ and embeddings $\sigma : K \to \mathbb{C}$ where $|w|$ denoted the norm of $w$. In general for a number field of degree $d > 1$, a lemma of Siegel [10] gives the existence of two positive constants $c_1$ and $c_2$ depending only on $K$ with $c_1c_2 = 1$ and a unit $\epsilon$ of $K$ such that the inequalities

\[c_1|\alpha|^{1/d} \leq |\sigma(\alpha)\sigma(\epsilon)| \leq c_2|\alpha|^{1/d}\]

holds for all $\alpha \in \mathcal{O}_K$ and all embeddings $\sigma$ of $K$. Now $c_1 = c_2 = 1$ implies that all embeddings give equivalent norms. This is possible only in imaginary quadratic number fields.

The following theorem is a number field version of a general result by Motohashi [8].

The main theorems of this paper are as follows.

**Theorem 1.1.** Let $K$ be an imaginary quadratic field of class number one and $\zeta_0$ be a generator of the group of roots of unity. Let $f$ and $g$ be complex valued arithmetic functions on $\mathcal{O}_K$ satisfying $f(\zeta_0^r a) = f(a), g(\zeta_0^r a) = g(a)$ for all $a \in \mathcal{O}_K$ and positive integer $r$. If $f$ and $g$ both satisfies $(S-W)$ and have common level of distribution $1/2$ then their Dirichlet convolution $f * g$ also satisfies $(S-W)$ and have common level of distribution $1/2$.

The following corollary is an iterative version of the above Theorem 1.1.

**Corollary 1.2.** Let $K$ be an imaginary quadratic field of class number one. Let $f_i (i = 1, \ldots, n)$ be complex valued arithmetic functions on ring of integers $\mathcal{O}_K$ having common level
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of distribution 1/2 such that \( f_1(\zeta^r_0 a) = f_1(a) \) for all \( r \) and satisfies (S-W). Then the Dirichlet convolution \( f_1 \ast \ldots \ast f_n \) also satisfies (S-W) and have common level of distribution 1/2.

Another application of Theorem 1.1 is with \( f = \mathbb{1}_w \) the indicator function which takes value 1 if \( w \) is a prime element in \( A^0(N) \) otherwise it is 0.

**Corollary 1.3.** Let \( K \) be an imaginary quadratic field of class number one. If primes in \( A^0(N) \) have level of distribution 1/2 then product of two primes in \( A^0(N) \) also have level of distribution 1/2.

**Remark.** If \( f \) and \( g \) have level of distribution \( \vartheta \) for \( 0 < \vartheta < 1/2 \) then it is clear from the proof of the Theorem 1.1 that \( f \ast g \) also have level of distribution \( \vartheta \).

Hinz [6] showed that primes have level of distribution 2/5 in imaginary quadratic number field. Using this result, an application of Corollary 1.3 we get the following.

**Corollary 1.4.** Let \( K \) be an imaginary quadratic field of class number one. Then product of two primes in \( A^0(N) \) have level of distribution 2/5.

2. Preliminary Lemmas

The following lemma is Perron summation formula in imaginary quadratic number field.

**Lemma 2.1.** Let \( K \) be an imaginary quadratic number field. Let \( \tilde{f} \) be complex valued arithmetic functions on ring of integers \( \mathcal{O}_K \) such that \( \tilde{f}(\zeta^r_0 a) = \tilde{f}(a) \) and satisfy \( \tilde{f}(a) = O(\tau(a)^C) \). Then for \( k \geq 1 \) we have,

\[
\sum_{w \in A^0(N)} \tilde{f}(w) \log^k \left( \frac{N^2}{|w|} \right) = \frac{w_K k!}{2\pi i} \int_{\sigma - iT}^{\sigma + iT} \tilde{F}(s) \frac{(N^2)^s}{s^{k+1}} ds + O \left( \frac{N^2}{T^k} \right)
\]

where \( w_K \) is the number of roots of unity of \( K \), \( \sigma = \Re(s) > 1 \) and

\[
\tilde{F}(s) = \sum_{w \in \mathcal{O}_K} \frac{\tilde{f}(w)}{|w|^s}.
\]

**Proof.** We know that the number of roots of unity in imaginary quadratic field is 2, 4 or 6. Using this and a Theorem from Tenenbaum [page 134, [11]] we have

\[
\sum_{w \in A^0(N)} \tilde{f}(w) \log^k \left( \frac{N^2}{|w|} \right) = w_K \sum_{|w| \leq N^2} \tilde{f}(w) \log^k \left( \frac{N^2}{|w|} \right)
\]

\[
= \frac{w_K k!}{2\pi i} \int_{\sigma - iT}^{\sigma + iT} \tilde{F}(s) \frac{(N^2)^s}{s^{k+1}} ds + O \left( \frac{N^2}{T^k} \right).
\]

We next state the large sieve inequality for number field \( K \) of degree \( d > 1 \). Let, \( \theta_1, \ldots, \theta_d \) be an integral basis of \( K \) so that every integer \( \xi \) of \( K \) is representable uniquely as

\[
\xi = n_1 \theta_1 + \cdots + n_d \theta_d
\]

where \( n_1, \ldots, n_d \) are rational integers.

If we take an element say \( \xi \in A^0(Y', N + Y') \) then as we take a fixed integral basis \( \theta_1, \ldots, \theta_d \) of \( K \) so the element \( \xi \) can be written as

\[
\xi = n_1 \theta_1 + \cdots + n_d \theta_d
\]
where \( C_2 Y' < |n_i| < C_1 (Y' + N) \), \( i = 1, 2, \ldots, d \) and \( C_1, C_2 \) are depending on \( K \).

**Lemma 2.2** ([5]). Let, \( f(x) \) be a positive decreasing continuous function on \( Q_1 < x \leq Q_2 \). Then we have,

\[
\sum_{Q_1 < |q| \leq Q_2} f(|q|) \frac{|q|}{\varphi(q)} \sum_{\chi(q)} \sum_{\xi \in A^0(Y', N+y')} c(\xi) \chi(\xi) \leq \left( f(Q_1^2 + |A^0(N)|) + \int_{Q_1}^{Q_2} xf(x)dx \right) \sum_{\xi \in A^0(Y', N+y')} |c(\xi)|^2
\]

where \( \sum^* \) denotes summation over primitive multiplicative characters \( \chi \) (mod \( q \)).

As an application of the Lemma 2.2 with \( f(x) = 1/x \) we get the following lemma.

**Lemma 2.3.** For any positive numbers \( Q_1 \) and \( Q_2 \) with \( Q_1 < Q_2 \) we have,

\[
\sum_{Q_1 < |q| \leq Q_2} \frac{1}{\varphi(q)} \sum_{\chi(q)} \sum_{\xi \in A^0(Y', N+y')} c(\xi) \chi(\xi) \leq \left( \frac{|A^0(N)|}{Q_1} + Q_2 \right) \sum_{\xi \in A^0(Y', N+y')} |c(\xi)|^2
\]

where \( \sum^* \) denotes summation over primitive multiplicative characters \( \chi \) (mod \( q \)).

The following lemma is a consequence of Minkowski’s lattice point theorem (see [2, page 12]).

**Lemma 2.4.** Let \( A^0(N) \) be defined as above. We have,

\[
|A^0(N)| = (1 + o(1))(2\pi)^{r_2} N^d \frac{1}{|D_K|}
\]

where \( D_K \) is the discriminant of number field \( K \) of degree \( d \).

**Lemma 2.5.** Let \( K \) be an algebraic number field. For any natural number \( R \), we have

\[
\sum_{u \subset \mathcal{O}_K \atop |u| < R} \frac{1}{|u|} \ll_K \log R,
\]

and

\[
\sum_{p \in \mathcal{P} \atop |p| \leq R} \frac{1}{|p|} \ll_K \log \log R
\]

where first sum is over all non-zero integral ideals of \( \mathcal{O}_K \) whose norm is less than or equal to \( R \).

### 3. Proof of Theorem 1.1

**Proof.** We assume that \( M > N^{1/2} \). For \( M \leq N \) and \((\gamma, a) = 1\) we have

\[
\varepsilon(M; q, a; f \ast g) = \sum_{\xi, \eta \in A^0(M) \atop \xi \equiv a(q)} f(\xi)g(\eta) - \frac{1}{\varphi(q)} \sum_{\xi \equiv a(q)} f(\xi)g(\eta).
\]

Now, since \( \xi, \eta \in A^0(M) \), we can divide the range of summation over \( \xi \) and \( \eta \) as follows: \(|\xi| \leq (\log N)^{A'}, (\log N)^{A'} < |\xi| \leq |A^0(M)|(\log N)^{-B'}, |\eta| \leq (\log N)^{B'}\).
Therefore using \( |A^0\left(\frac{M}{|\eta|^{1/2}}\right) | = (1 + o(1))\frac{1}{|\eta|} |A^0(M)| \) the term \( \varepsilon(M; q, a; f \ast g) \) can be written as

\[
\varepsilon(M; q, a; f \ast g) = \sum_{|\xi| < |\log N| A'} f(\xi) \varepsilon\left( \frac{M}{|\xi|^{1/2}}; q, \xi^{-1} a; g \right)
\]

\[
+ \sum_{(\log N)^{A'} < |\xi| \leq |A^0(M)| (\log N)^{A'}} f(\xi) \varepsilon\left( \frac{M}{|\xi|^{1/2}}; q, \xi^{-1} a; g \right)
\]

\[
+ \sum_{|\eta| \leq (\log N)^{B'}} g(\eta) \left\{ \varepsilon\left( \frac{M}{|\eta|^{1/2}}; q, \eta^{-1} a; f \right) - \varepsilon\left( \min\left( \frac{M}{|\eta|^{1/2}}; \frac{N}{(\log N)^{A'/2}} \right); q, \eta^{-1} a; f \right) \right\}
\]

=: \Sigma_1 + \Sigma_2 + \Sigma_3.

Since \( |\xi| \leq (\log N)^{A'} \), using (1.3) and \( |A^0\left(\frac{N}{|\xi|^{1/2}}\right) | = (1 + o(1))\frac{1}{|\xi|} |A^0(N)| \), by taking summation over norm of \( q \) of the sum \( \Sigma_1 \) we have,

\[
\sum_{|q| \leq Q} \max_{M \leq N} \sum_{(\gamma, q) = 1} \Sigma_1 \ll \sum_{|\xi| < (\log N)^{A'}} |f(\xi)| \sum_{|q| \leq |A^0(N)| (\log N)^{A'}} \max_{M \leq N} \varepsilon\left( \frac{M}{|\xi|^{1/2}}; q, \xi^{-1} a; g \right)
\]

\[
\ll \sum_{|\xi| < (\log N)^{A'}} \tau(\xi)^C \frac{|A^0(N)|}{|\xi| \log^A (N/|\xi|^{1/2})} \ll \frac{|A^0(N)|}{(\log N)^D'}
\]

where \( D' \) is a constant depending on \( A' \) and \( C \).

Similarly as above we have,

\[
\sum_{|q| \leq Q} \max_{M \leq N} \sum_{(\gamma, q) = 1} \Sigma_3 \ll \frac{|A^0(N)|}{(\log N)^D'}.
\]

Therefore finally we have to estimate the following sum:

\[
\Sigma_4 = \sum_{|q| \leq Q} \max_{M \leq N} \sum_{(\gamma, q) = 1} \sum_{(\log N)^{A'} < |\xi| \leq |A^0(M)| (\log N)^{A'}} f(\xi) \varepsilon\left( \frac{M}{|\xi|^{1/2}}; q, \xi^{-1} a; g \right).
\]

Now using the orthogonality of characters in algebraic number field the innermost sum of \( \Sigma_4 \) can be written as

\[
\sum_{A_2 < |\xi| \leq A_2} f(\xi) \varepsilon\left( \frac{M}{|\xi|^{1/2}}; q, \xi^{-1} a; g \right) = \sum_{\chi \neq \chi_0} \frac{1}{\varphi(q)} \sum_{\chi \neq \chi_0} \chi(a) \sum_{A_2 < |\xi| \leq A_2} f(\xi) \chi(\xi) \sum_{w \in A^0\left(\frac{M}{|\xi|^{1/2}}\right)} \chi(w) g(w)
\]
where \( \chi_o \) be the principal character \((\text{mod } q)\) and \( A_1 := (\log N)^{A'}, A_2 := \frac{|A^0(M)|}{(\log N)^{B'}}. \) Therefore, using this estimation, the sum \( \Sigma_4 \) can be written as

\[
\Sigma_4 = \sum_{|q| \leq D_1} \max_{M \leq N} \frac{1}{\varphi(q)} \sum_{\chi \neq \chi_o} \sum_{\xi(q)} \bar{\chi}(a) A_1 < |\xi| \leq A_2 f(\xi)\chi(\xi) \sum_{w \in A^0} \chi(w)g(w)
\]

\[
+ \sum_{D_1 < \xi \leq Q} \max_{M \leq N} \frac{1}{\varphi(q)} \sum_{\chi \neq \chi_o} \sum_{\xi(q)} \bar{\chi}(a) A_1 < |\xi| \leq A_2 f(\xi)\chi(\xi) \sum_{w \in A^0} \chi(w)g(w) =: \Sigma_5 + \Sigma_6.
\]

where \( D_1 := (\log N)^B. \)

To calculate the sum \( \Sigma_5 \) we will use (S-W) condition and (1.3) directly for each arithmetic functions \( f \) and \( g \) and for calculating sum \( \Sigma_6 \) we will use Lemma 2.1 together with large sieve inequality for algebraic number field by extracting primitive characters from the sum over all non-principal characters \((\text{mod } q)\).

**Estimation of \( \Sigma_5 \).** Using (S-W) condition and (1.3) we have

\[
\Sigma_5 \leq \sum_{|q| \leq D_1} \frac{1}{\varphi(q)} \sum_{\chi \neq \chi_o} \sum_{|\xi| \leq A_2} f(\xi)\chi(\xi) \sum_{w \in A^0} \sum_{|\xi| \leq A_2} g(w)\chi(\xi) |\xi| \log B^r (N/|\xi|^{1/2})
\]

\[
+ \sum_{|q| \leq D_1} \frac{1}{\varphi(q)} \sum_{\chi \neq \chi_o} \sum_{|\xi| \leq A_2} f(\xi)\chi(\xi) \sum_{w \in A^0} \sum_{|\xi| \leq A_2} g(w)\chi(\xi) |\xi| \log B^r (N/|\xi|^{1/2})
\]

\[
\ll D_1 \sum_{A_1 < |\xi| \leq A_2} \tau(\xi)^C \frac{|A^0(N)|}{|\xi| \log B^r (N/|\xi|^{1/2})} + (\log N)^{d'} \ll \frac{|A^0(N)|}{\log B^r N}
\]

for some sufficiently large constant \( B' \) depending on \( B \) and \( C. \)

**Estimation of \( \Sigma_6 \).** To calculate sum \( \Sigma_6 \) we have to calculate the following sum.

\[
\Sigma_6' := \sum_{D_1 < |q| \leq Q} \max_{M \leq N} \frac{1}{\varphi(q)} \sum_{\chi \neq \chi_o} \sum_{\xi(q)} \bar{\chi}(a) A_1 < |\xi| \leq A_2 f(\xi)\chi(\xi) \sum_{w \in A^0} \chi(w) \log^2 \left( \frac{|A^0(M)|}{|\xi||w|} \right).
\]

First we will show that \( \Sigma_6' = O \left( \frac{|A^0(N)|}{(\log N)^{B''}} \right) \) for some large \( D' > 2 \) and then by using partial summation formula we have, \( \Sigma_6 = O \left( \frac{|A^0(N)|}{(\log N)^{B''}} \right). \)
Each character \( \chi \neq \chi_0 \) occurring here is induced by a primitive character \( \chi^*(q_1) \) with \( q_1 | q \). So \( \Sigma_6 \) can be written as \( \Sigma'_6 = \sum_{D_1 < |q| \leq Q} \max_{\varphi(q) \leq N} \left\{ \frac{1}{d(q)} \sum_{\chi \in \chi(q_1)} \sum_{A_1 < |\xi| \leq A_2} f(\xi) \chi(\xi) \sum_{w \in A^0 \left( \frac{M}{|\xi|^1/2} \right)} g(\xi) \chi(w) \log^2 \left( \frac{|A^0(M)|}{|\xi||w|} \right) \right\} \)

Writing \( q_1 q_2 = q \) and using Lemma 2.5 we have

\[ \Sigma'_6 \ll \log N \max_{M \leq N} \max_{|q_2| \leq Q} I_{M,q_2} \]

where

\[ I_{M,q_2} := \sum_{D_1 < |q_1| \leq Q} \left\{ \frac{1}{\varphi(q_1)} \sum_{\chi \in \chi(q_1)} \int_{|\sigma-iT|}^{\sigma+iT} \tilde{f}(\chi, s) \tilde{g}(\chi, s) \frac{|A^0(M)|^s}{s^3} ds + O \left( \frac{|A^0(M)|^2}{T^2} \right) \right\} \]

and \( \tilde{f}(\xi) = f(\xi), \tilde{g}(\xi) = g(\xi), \) if \( (\xi, q_2) = 1 \), \( \tilde{f}(\xi) = \tilde{g}(\xi) = 0 \) otherwise.

By using \( |A^0 \left( \frac{M}{|\xi|^1/2} \right)| = (1 + o(1)) \frac{1}{|\xi|} |A^0(M)| \), Lemma 2.4 and Lemma 2.1 to the innermost sum of \( I_{M,q_2} \) we have,

\[ I_{M,q_2} = \frac{w_K}{\pi} \sum_{D_1 < |q_1| \leq Q} \left\{ \frac{1}{\varphi(q_1)} \sum_{\chi \in \chi(q_1)} \left| I_1 + I_2 + I_3 \right| + O \left( \frac{|A^0(M)|^2}{T^2} \right) \right\} = I_{M,q_2}^1 + I_{M,q_2}^2 + I_{M,q_2}^3 + E, \]

where

\[ I_1 := \int_{|\sigma-iT|}^{\sigma+iT} \tilde{f}(\chi, s) \tilde{g}(\chi, s) \frac{|A^0(M)|^s}{s^3} ds, \quad \tilde{f}(\chi, s) = \sum_{A_1 < |\xi| \leq A_2} \frac{\tilde{f}(\xi) \chi(\xi)}{|\xi|^s} \]

and

\[ \tilde{g}(\chi, s) = \sum_{w \in \mathcal{O}_K} \frac{\tilde{g}(w) \chi(w)}{|w|^s}, \quad \sigma = 1 + \frac{1}{2 \log N}. \]

For the above choice of \( \sigma \) it is easy to see that for some \( Y < T \),

\[ \tilde{g}(\chi, s) \ll (1 + |s|)|Y|^{1-\sigma}(\log Y)^d \] and \( \tilde{g}(\chi, s) \ll A_2^{1-\sigma}(\log A_2)^d + A_1^{1-\sigma}(\log A_1)^d. \)

Therefore integrals \( I_2 \) and \( I_3 \) are bounded above by

\[ \ll Y^{1-\sigma}(\log Y)^d (\log N)^d \frac{|A^0(N)|}{T}. \]

Write, \( I_1 = \int_{|\sigma-iT|}^{\sigma+iT} \tilde{f}(\chi, s) (\tilde{g}_1(\chi, s) + \tilde{g}_2(\chi, s) + \tilde{g}_3(\chi, s)) \frac{|A^0(M)|^s}{s^3} ds =: I_4 + I_5 + I_6, \)

where

\[ \tilde{g}_1(\chi, s) := \sum_{|w| \leq Y} \frac{\tilde{g}(w) \chi(w)}{|w|^s}, \quad \tilde{g}_2(\chi, s) := \sum_{w; Y < |w| \leq T} \frac{\tilde{g}(w) \chi(w)}{|w|^s} \]

and

\[ \tilde{g}_3(\chi, s) := \sum_{w; |w| > T} \frac{\tilde{g}(w) \chi(w)}{|w|^s}. \]
By using calculations of integrals $I_2$ and $I_3$ we can say that the integral $I_6$ also bonded above by

$$\ll T^{1-σ}(\log T)^{d'}(\log N)^d|A^0(N)|.$$ 

By splitting intervals $[D_1, Q]$ and $[A_1, A_2]$ into Dyadic intervals we have,

$$I_{M,q_2}^k \ll \sum_{j=0}^J \sum_{i=0}^I I_{M,q_2}^k(j, i),$$

where $2^JD_1 < Q \leq 2^{J+1}D_1, 2^IA_1 < A_2 \leq 2^{I+1}A_1,$

$$I_{M,q_2}^k(j, i) = \int_{σ-iT}^{σ+iT} S_{j,i}(s)|A^0(M)|^s \frac{ds}{|s|^3}$$

and,

$$S_{j,i}(s) = \sum_{2^jD_1<|q|\leq 2^{j+1}D_1} \frac{1}{\varphi(q)} \sum_{\chi \neq \chi_0} \left| \tilde{f}_i(\chi, s) \tilde{g}_k(\chi, s) \right| (k = 1, 2).$$

Observe that,

$$\int_{σ-iT}^{σ+iT} \tilde{f}_i(\chi, s) \tilde{g}_1(\chi, s) \frac{|A^0(M)|^s}{s^3} ds - \int_{1/2-iT}^{1/2+iT} \tilde{f}_i(\chi, s) \tilde{g}_1(\chi, s) \frac{|A^0(M)|^s}{s^3} ds = O\left( \frac{|A^0(M)|}{T^3} \left( \sum_{|w|t|w| \leq Y} \tau(w)C \right) \left( \sum_{\xi|\xi| \leq A_2} \frac{\tau(\xi)C}{|\xi|^{1/2}} \right) \right) = O\left( \frac{|A^0(M)|^{3/2} \sqrt{Y} (\log Y)^{d'} (\log N)^{d'}}{T^3} \right).$$

Therefore using above observations we have

$$I_{M,q_2}(j, i) \ll \int_{1/2-iT}^{1/2+iT} S_{j,i}(s) \frac{|A^0(M)|^s}{|s|^3} ds + \int_{1/2-iT}^{1/2+iT} S_{j,i}(s) \frac{|A^0(M)|^s}{|s|^3} ds$$

$$+ O\left( \frac{|A^0(M)|^{2} \sqrt{Y} (\log Y)^{d'} (\log N)^{d'}}{T^2 (\log N)^{B'}} \right).$$

Now using Cauchy-Schwarz inequality on $\chi$ sum and then again on $q$ sum we have

$$S_{j,i}(s) \leq \left( \sum_{2^jD_1<|q|\leq 2^{j+1}D_1} \frac{1}{\varphi(q)} \sum_{\chi \neq \chi_0} \left| \tilde{g}_k(\chi, s) \right|^2 \right)^{1/2} \left( \sum_{2^jD_1<|q|\leq 2^{j+1}D_1} \frac{1}{\varphi(q)} \sum_{\chi \neq \chi_0} \left| \tilde{f}_i(\chi, s) \right|^2 \right)^{1/2}.$$ 

Therefore, using Lemma 2.3 we have for $s = 1/2 + it (-T \leq t \leq T)$,

$$S_{j,i}(s) \ll \left( \frac{2^{j+1}D_1 + 2^{j}A_1}{2^jD_1} \right) \sum_{\xi|\xi| \leq A_2} \frac{\tau(\xi)C}{|\xi|} \left( \frac{2^{j+1}D_1 + Y}{2^jD_1} \right) \sum_{|w|\leq A_2} \frac{\tau(w)C}{|w|}^{1/2} \ll (Y + 2^jA_1)^{1/2} (\log N)^{d'}.$$ 

Let us choose

$$Y := (2^jD_1)^2 \quad \text{and} \quad T := e^{2(\log |A^0(N)|)^4}. $$
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Using the above choice of $Y$ and $T$ we have

$$S_{j,i}^1(s) \ll |A^0(N)|^{1/2} (\log N)^{-B'}.$$

Let $2^R Y < T \leq 2^{R+1} Y$ and for $0 \leq r \leq R$

$$\tilde{g}_2^{(r)}(\chi, s) = \sum_{2^r Y < |w| \leq 2^{r+1} Y} \tilde{g}(w) \chi(w) |w|^s.$$

Therefore we have,

$$R \ll (\log N)^2 \quad \text{and} \quad \tilde{g}_2(\chi, s) = \sum_{r=0}^R \tilde{g}_2^{(r)}(\chi, s).$$

Now using Lemma 2.3 we have for $s = \sigma + it (-T \leq t \leq T)$,

$$S_{j,i}^2(s) \ll \max_{0 \leq r \leq R} \left( \left( 2^{j+1} D_1 + \frac{2^j A_1}{2^j D_1} \right) \sum_{\xi : |\xi| \geq 2^r A_1} \frac{\tau(\xi)^C}{|\xi|^2} \right)^{1/2} \times \left( \left( 2^{j+1} D_1 + \frac{2^r Y}{2^j D_1} \right) \sum_{w : |w| \geq 2^r Y} \frac{\tau(w)^C}{|w|^2} \right)^{1/2} \left( \log N \right)^2 \ll \left( \frac{2^j D_1}{A_1} + \frac{1}{2^j D_1} \right)^{1/2} \left( \frac{2^j D_1}{Y} + \frac{1}{2^j D_1} \right)^{1/2} \left( \log N \right)^{d+2} \ll \log^{-B'} N.$$

Using the above choice of $Y, T$ and Substituting above estimations into (3.2), (3.1) we have,

$$\Sigma_6' \ll |A^0(N)| \log^{-B'} N.$$

\[\square\]

4. PROOF OF COROLLARY 1.3

\textbf{Proof.} We need the following lemma.

\textbf{Lemma 4.1} (Lemma 2, [5]). If $|q| \ll \log^D N$ with a positive constant $D$, then we have for a non-principal character $\chi \pmod{q}$

$$\sum_{w \in A^0(N)} \chi(w) \ll |A^0(N)| \exp \left( -c (\log N)^{1/2} \right),$$

for some $c = c(D, K) > 0$.

Using Lemma 4.1 we can say that the function $f(w) = 1_w$ satisfies (S-W) condition. Therefore under hypothesis that prime have level of distribution $1/2$, Corollary follows from Theorem 1.1 and Corollary 1.2. \[\square\]
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