Geometric constraints on the dynamics of networks

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The functional and computational power of a network emerges as a property of the system operating as a coherent whole at a global scale, not at the scale of individual nodes. A natural question to ask is what are the physical principles that allow a collection of nodes to interact in such a way that they produce emergent global network dynamics? Here, we address this question by developing a generalized theoretical framework that formally describes how in any physically constructible geometric network, the geometry of the network i.e. its physical structure, constrains and bounds signaling and the flow of information through the network. This then in turn predicts the emergent dynamics of the network. Our results illustrate how the interplay between strictly local geometric and temporal process at the scale of individual node pairs directly affects the behavior of the whole network. We also provide empirical evidence that at least some important examples of widely different naturally occurring and engineered networks follow these theoretical principles.
1 Introduction

In the most general sense, networks consist of nodes, subsets of which are interconnected via edges that exchange information. But the functional and computational power of a network emerges as a property of the system operating as a coherent whole at a global scale, not at the scale of individual nodes. Each node individually cannot accomplish what the network itself can do, even though each contributes to the collective that is the network. This dichotomy is even more pronounced given that individual nodes have no information, i.e. knowledge, about the states, actions, internal processes, or anything else about other nodes, other than the input signals they receive from their own incoming edges. A natural question to ask is what are the physical principles that allow a collection of nodes to interact in such a way that they produce emergent global network dynamics? In other words, what drives the collective behavior of the system (i.e. what a network can represent and compute), even though the individual components (i.e. the nodes) are independently participating?

Here, we address this question by developing a generalized theoretical framework that formally describes how in any physically constructible geometric network, the geometry of the network i.e. its physical structure, constrains and bounds signaling and the flow of information through the network. This then in turn predicts the emergent dynamics of the network. Our results illustrate how the interplay between strictly local geometric and temporal process at the scale of individual node pairs directly affects the behavior of the whole network. One of the main results is the derivation of conditions necessary for optimal signaling in a network from a consideration of an intuitive fundamental principle: in order for a network to sustain recurrent signaling there needs to be a temporal consistency between the amount of time it takes individual nodes to internally process an incoming signal and the signaling latency between nodes, i.e. the temporal delay on the edges as a function of the signaling speed and path length of the edges. From an applied perspective, we provide empirical evidence that at least some important examples of widely different naturally occurring and engineered networks, specifically, the prevalence of the small world network topology, axonal branching of pyramidal neurons in the visual cortex, the internet router network, and dynamic signaling in biological neural network are capable of approaching such a state of optimal dynamical efficiency. The technical construction of the framework was developed from an abstraction of key principles underlying biological neural cellular signaling, which we were able to generalize to the abstract concept of a network modeled as a graph endowed with dynamical properties.

2 Theoretical development

We define a network as a physically realizable and constructible object consisting of nodes with unique internal dynamics and a physical connectivity in space between specific pairs of nodes. Because it is a physical network there must exist a finite signaling speed for the
propagation of information between nodes and a finite temporal delay associated with the flow of information between nodes. Formally, we will consider graph models of such physical networks. In general we will explicitly refer to the graph model of a network, and therefore refer to the vertices and edges of the graphs, but it should be understood that they model the nodes and connections of the physical network of interest. At times we will interchange the terminology between vertices and nodes, and connections and edges, when there is no chance for confusion.

We consider the flow of information, or signaling, along the path lengths of directed edges that have a physical representation in space (or the plane) that connect two vertices of the graph in space (or the plane; Fig. 1). In this sense edges are represented by smooth and continuous Jordan arcs. Such a graph represents a model of a physically constructible network, in that any real world network can be modeled as such a graph. The term 'geometric graph' is often used in a somewhat ambiguous way depending on the author or context. Topological graph theory typically refers to the embedding and study of graphs in some topological space, including three dimensional Euclidean spaces and surfaces [6, 7]. However, other authors for example have distinguished geometric graphs from topological graphs as graphs in the plane with distinct points connected by straight line edges in the former case versus graphs in the plane with distinct points connected by Jordan arcs in the latter case, i.e. both in \( \mathbb{R}^2 \) [16]. In this paper we only need to consider graphs with a physical representation in the Euclidean plane \( \mathbb{R}^2 \) or three dimensional space \( \mathbb{R}^3 \), and not in any other topological space. As such, we will use the term 'geometric graph' to refer to any graph in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \). We formalize these notions below.

**Definition 2.1.** A graph is an ordered pair of finite disjoint sets \( G = (V, E) \) such that \( v \) is the set of vertices in the graph and \( E \) is the set of edges joining the vertices.

**Definition 2.2.** A geometric graph \( G = (\bar{V}, \bar{E}) \) is a graph who’s vertices have a physical position in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \) in some appropriate coordinate system. Physical coordinates \( \bar{v}_i := \bar{x} \) for an ordered triplet \( \bar{x} = (x_1, x_2, x_3) \) are assigned to each vertex \( v_i \) for all vertices \( i = 1 \ldots N \) in the graph, with the set of all vertices given by \( \bar{V} = \{v_i\} \).

**Definition 2.3.** A complete geometric graph \( G = (\bar{V}, \bar{E}) \) is a geometric graph with directed edges that also have a physical definition in space: An edge \( e_{ij} \) is defined if there is a directed connection from vertex \( i \) to vertex \( j \), i.e. from \( v_i \) to \( v_j \). Let an edge between \( v_i \) and \( v_j \) be defined geometrically as \( e_{ij} = f(\bar{x}) \) for some function \( f(\cdot) \) in \( \mathbb{R}^3 \), restricted to simple non-intersecting Jordan arcs. The physical distance of the edge between any two vertices is determined by the path integral taken by the edge in space where for some scalar field \( f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R} \). The path integral distance \( d_{ij} \) from \( \bar{v}_i \) to \( \bar{v}_j \) formed by a piecewise smooth curve \( C \subset U \) is defined as

\[
|e_{ij}| = d_{ij} := \int_C f \, ds = \int_{\bar{v}_i}^{\bar{v}_j} f(\bar{r}(t)) \, |\bar{r}'(t)| \, dt
\]
where \( r : [\bar{v}_i, \bar{v}_j] \to C \) is an arbitrary bijective parametrization of the curve \( C \) such that \( r(\bar{v}_i) \) and \( r(\bar{v}_j) \) give the endpoints of \( C \) for \( \bar{v}_i < \bar{v}_j \).

As a technical comment, it should be noted that our use of the term ‘complete geometric graph’ here differs from the use of the term ‘complete graph’ referring to a simple undirected graph in which every pair of distinct vertices is connected by a unique edge, or that of a complete digraph which is a directed graph in which every pair of distinct vertices is connected by a pair of unique edges. In addition, we do not notionally distinguish between ‘geometric graph’ and ‘complete geometric graph’, i.e. between definitions 2.2 and 2.3 in the rest of this paper in order to avoid introducing additional (unnecessary) notation. This distinction is not necessary because we strictly deal with complete geometric graphs in the work that follows, and the notation \( G = (\bar{V}, \bar{E}) \) should be assumed to refer accordingly.

**Definition 2.4.** The set of all geometric edges for the geometric graph \( G = (\bar{V}, \bar{E}) \) is given by \( \bar{E} = \{e_{ij}\} \) for all existing edges.

**Definition 2.5.** We distinguish a non-minimal complete geometric graph, denoted by \( G \), from the ‘minimal’ complete geometric graph representation \( G_{\text{min}} \), in the sense that the edges of \( G_{\text{min}} \) are all (geodesic) straight line edges, i.e. \( E_{\text{min}} \) with \( \langle d_{ij} \rangle_{\text{min}} < d_{ij} \), for any edges \( d_{ij} \) in \( G \).

The graphs \( G_{\text{min}} \) and \( G \) share the same vertex set \( \bar{V} \) and structural (physical) connectivity, but the edge set of \( G_{\text{min}} \) given by \( E_{\text{min}} \) are straight lines while the edge set of \( G \) given by \( \bar{E} \) follow longer convoluted geometric paths in space determined by the actual physical nature of the network under consideration, c.f. definition 2.3 above. As an example of this distinction, consider the network shown in fig. 1. From a classical graph theoretic perspective both networks in panels a. and b. are equivalent because they have the same connectivity and adjacency matrix. However, dynamically, the delay associated with the arrival of a signal propagating along an edge at a constant finite speed will be a function of the propagation speed and path length of the edge. A notion that is not necessarily fully captured by a single ‘weight’ value in the adjacency matrix itself. This consideration is critical for computing the dynamics of the network as a whole for nodes that require a finite amount of processing time associated with how they use and respond to any incoming signals. Real world networks typically look much more like the network in panel b. than the one in a. Mathematically though, there will exist a mapping from \( G \to G_{\text{min}} \) by increasing or decreasing the temporal delays in \( G_{\text{min}} \) to match those that would occur in \( G \). However, this is not always possible and it is not always obvious how to produce such a mapping when considering real world networks. This can also lead to an over simplification and loss of insights into the physical mechanisms that underlie the dynamics and behavior of the network.

**Definition 2.6.** The subgraph \( H_j \) is the reverse geometric tree that consists of all vertices \( v_i \) with directed geometric edges into \( v_j \). We write \( H_j(v_i) \) to represent the set of all vertices \( v_i \) in \( H_j \) and \( H_j[v_i] \) to refer to a specific \( v_i \in H_j(v_i) \).
Definition 2.7. The edge set of $H_j$ is denoted by $H_j(E)$

Definition 2.8. There exists a signaling speed $s_{ij}$ for the flow of a signal (information) on the edge $e_{ij}$.

Corollary 2.9. $0 < s_{ij} < \infty$ for any physically constructible network, i.e. it must be finite.

Definition 2.10. The absolute refractory period of a vertex $v_j$ is given by $R_j$, determined by a given internal dynamic model of the node being represented by the vertex $v_j$.

What constitutes $R_j$ and the processes that contribute to it will depend on the physical details of individual networks, but it represents the summation of any and all dynamical processes in $v_j$ that contribute to a processing delay from the instant a signal arrives at $v_j$ to the moment when the node produces some actionable output as a causal response to the input.

Corollary 2.11. $R_j > 0$ for any physically constructible network, i.e. there cannot exist an infinitely fast or instantaneous response.

Definition 2.12. By $H_j[v_i] \rightsquigarrow v_j$ we mean a vertex $v_i \in H_j$ that causally leads to the activation of $v_j$.

2.1 Competitive refractory dynamics

Consider a vertex $v_i$ with a directed geometric edge to a vertex $v_j$. The length of the edge can be the minimum geodesic, a straight line $\langle d_{ij}\rangle_{min}$ in $\mathbb{R}^3$, or it can be a spatially convoluted path such that $d_{ij} > \langle d_{ij}\rangle_{min}$, c.f. definition 2.3. For $v_i$ to signal or communicate with $v_j$, there must be some physical signal $s_{ij}$ representing a flow of information from $v_i$ to $v_j$ over the edge that connects them which travels at some finite speed (c.f. corollary 2.9). Note that in the following development we assume that $s_{ij}$ is a constant value $s$ for all edges in the network, but this not need be in the general case. See the Discussion section below.

If $v_i$ and $v_j$ share the same internal dynamics, i.e. are both the same type of node and their internal dynamics can be modeled in the same way, then $R_i = R_j \equiv R$. This would be the case for example, if one were considering a homogeneous population of neurons in a specific neural circuit. In the construction of this framework, once $v_j$ receives a signal from $v_i$ it becomes refractory for the period $R_j$ as determined by the internal dynamic model of the node and will not be able to respond to another incoming signal during this period of time. Note that there is no restriction on the internal dynamic model that a node can take. Again using biological neurons as an example, any model that produces an action potential could be applied (see [10] for details).

Let $\tau_{ij}$ represent the time delay it takes a signal to reach $v_j$ from $v_i$. Then
Figure 1: Classical versus complete geometric graph models of networks. Complete geometric graphs as we define them here are defined as graphs who’s vertices have a physical position in space (or the plane), with directed edges that also have a physical (path integral) representation in space (or the plane). a. A classical representation of a graph consisting of vertices connected by edges, where the only important consideration is the connectivity of the vertices (i.e. adjacency matrix). There is no geometric significance to how the graph is drawn. b. A “non-minimal” complete geometric representation of the same graph. While both networks in panels a. and b. are equivalent in classical graph theory, they are very different as we define them here, with their structures directly determining their dynamics.

\[ \tau_{ij} = \frac{d_{ij}}{s} \]  \hspace{1cm} (1a)

\[ \langle \tau_{ij} \rangle_{\text{min}} = \frac{\langle d_{ij} \rangle_{\text{min}}}{s} \]  \hspace{1cm} (1b)

which ensures that \( \langle \tau_{ij} \rangle_{\text{min}} < \tau_{ij} \) always.

Let \( y_j(\Omega, t) \) represent the instantaneous state of vertex \( j \) as a function of time and some parameter set \( \Omega \) determined by the internal model that represents the dynamics of the node. In the context of biological neural networks we have previously defined a geometric (although not complete geometric) state space framework of network dynamics with this form and showed how such a framework can be used to represent, compute, and simulate the temporal signaling evolution of a dynamic network [10]. That previous work provided an analytical description and algorithm for computing the instantaneous internal states of a node in a geometric network and its subsequent advancement in discrete time steps using a transfer function definition of the internal model. The framework was designed to accommodate essentially any specific model of single node dynamics, and naturally extends to any physical network (not just neurobiological). For our purposes here though, how \( y_j(\Omega, t) \) is computed is not of immediate concern, only that it can be. The important consideration is that the internal state can be interpreted as a binary function at any time \( t \) determined by the internal machinery of \( v_j \) that results in \( y_j(\Omega, t) \). We can then define
this function at some observation time $T_o$ as

$$y_j(\Omega, T_o) = \begin{cases} 1, & \text{iff } v_j \text{ can respond to an input} \\ 0, & \text{iff it is refractory to any input} \end{cases}$$

Given this construction, in theory one can compute $y_j$ for any $T_o$ as a function of $R_j$ and $\tau_{ij}$ for all $v_i v_j$ vertex pairs. This then determines for all nodes $v_i$ into a given node $v_j$ which node $v_i$ is able to 'win' and 'activate' node $j$ if $y_j = 1$. Once the winning node 'activates' $v_j$ it will become refractory for a period of time $R_j$ during which $y_j = 0$, determined by its internal dynamic model. Intuitively though, note that if the state of $v_j$ at $T_o$ is $y_j = 0$ it could be refractory for some time $< R_j$ if had become refractory prior to $T_o$. This situation is interesting because we have to take into account phase shifts in the temporal properties of individual $\tau_{ij}$ and $R_j$ at the sampling time $T_o$ in order to understand the dynamics of the network. In fact, it is the core of much of the dynamical richness underlying distributed signaling in a geometric network. In the next section we formalize these notions and arrive at formal description that determines the winning vertex $v_i$. We then use this construction to prove a result about the bounds that define optimal efficient signaling within a network. And also show a number of practical applications of this analysis to real world networks.

Before we continue though, we emphasize again (since we discuss this in the main text also) that it is critical to note that the response of each vertex is causally independent from whatever all the other vertices in the network are doing. All that matters for computing $y_j$ at any instantaneous time are its own internal dynamics, which determine $R_j$, and when signals are produced by (competing) input vertices into $v_j$. This is because each vertex does not have information about what all the other vertices that make up the network are doing or what their states are. It can only know its own internal state and react to the inputs it is receiving in order to decide if and when it will send out an output itself. Thus, there is a natural independence in the dynamics of the vertices that make up a network. This allows us to independently compute the interacting states of any $v_i v_j$ vertex pair at any time $T_o$. This is not dependent on any 'average' metric of the state or behavior of the network as a whole or on any statistical probability densities associated with the frequency of occurrence of events such as in Markovian processes. It is a deterministic process at a local scale that affects the dynamics of the network at the global scale. In theory, by computing in parallel all $v_i v_j$ state pairs one could arrive at the state of the overall network. From a practical engineering perspective, this has the advantage that the computation of the $v_i v_j$ states that make up the network can be computed using high performance computing such as graphical processing unit (GPU) architectures [10].

### 2.2 Signal flow dynamics between node pairs and the refraction ratio

We begin by defining a simple relationship between $R_j$ and $\tau_{ij}$ for $v_j$ that predicts the state $y_j$ of $v_j$. It is a detailed consideration of this ratio that then provides the insights into the
dynamics between \( v_j \) and the nodes that connect into it.

**Definition 2.13.** The refraction ratio between the refractory period \( R_j \) for vertex \( v_j \) and the temporal signaling delay \( \tau_{ij} \) into \( v_j \) for a directed connected vertex \( v_i \) on the edge \( e_{ij} \) is given by

\[
\Delta_{ij} = \frac{R_j}{\tau_{ij}} = \frac{R_j \cdot s}{d_{ij}}
\]

(3)

where \( R_j > 0 \) due to corollary 2.11 and \( \tau_{ij} > 0 \) due to corollaries 2.9 and 2.11.

**Definition 2.14.** Let \( v_i(\Delta_{ij}) \) be to the vertex \( H_j[v_i] \) with refraction ratio \( \Delta_{ij} \) for \( i = 1, 2, \ldots, N \).

Our analysis and enumeration of the combinatorial signaling space between \( v_i \) and \( v_j \) will proceed by a consideration of this ratio.

### 2.2.1 Unallowable conditions

There are a number of unallowable conditions that are necessitated by the physical construction of a real world network and the definitions given in section ??.

- \( R = 0 \) implies a non-refractory vertex capable of instantaneous recovery to an incoming signal from an upstream vertex, a condition which is not allowed (c.f. corollary 2.11).
- \( \tau_{ij} \rightarrow 0 \) \( \Delta_{ij} \) becomes undefined, which is equivalent to stating \( d_{ij} \rightarrow 0 \) since \( \tau_{ij} \propto d_{ij} \) for a fixed signaling speed \( s \), with \( s \) of course as the constant of proportionality. The theoretical limit occurs when \( R_j \) and \( \tau_{ij} \rightarrow 0 \). But this implies \( s = \infty \) or, functionally equivalently, \( d_{ij} = 0 \), i.e. no geometric distances to an edge, which we discuss further below, and simultaneously infinitely fast recovery times of the internal node dynamics. But these conditions are unattainable by corollaries 2.9 and 2.11. \( \Delta_{ij} \) therefore necessarily implies finite dynamic signaling and information flow in a network, as required.

### 2.2.2 Trivial bounds

The trivial lower bound occurs as \( R_j \rightarrow 0, y_i = 1 \forall \tau_{ij} \). Intuitively, for any \( v_i \) into \( v_j \) when \( y_i = 1 \), the vertex with the shortest edge path integral will win and activate \( v_j \). In other words, assuming a constant signaling speed \( s \) for \( H_j(E) \) if we let \( D_{ij} := \{d_{ij} : i = 1, 2, \ldots, N \} \) be the set of all edge path integrals for \( H_j(E) \), then \( H_j[v_i] \sim v_j = v_i(\min_i d_{ij}) \) for \( d_{ij} \in D_{ij} \).

The trivial upper bound occurs as \( R_j \rightarrow \infty, y_j = 0 \forall \tau_{ij} \), in which case there would be no information flow or signaling in the network ever at all.

### 2.2.3 Refraction ratio analysis with no temporal offset

The dynamics at the network scale for any physically constructible network will depend on \( R_j \), which in turn of course is determined by the node’s internal dynamics. For any \( R_j > 0 \) the temporal delay \( \tau_{ij} \) of a signal traveling between \( v_i \) and \( v_j \) must be such that \( \tau_{ij} > 0 \).
The ideal will occur when $\tau_{ij}$ is as small as possible but matched to $R_j$; in other words, when it is just a bit larger than $R_j$ so that $v_j$ responds as soon as it stops being refractory. In the informal description introduced above, if there are a number of nodes $v_i$ with directed edges into a node $v_j$, the $v_i$ that manages to achieve this condition, i.e. the signal from the upstream node $v_i$ that reaches $v_j$ first, will ‘win’ and activate $v_j$ while immediately making it refractory to later arriving signals for a period $R_j$. We formalize these ideas in the following way. Begin by considering any two vertices $v_i$ and $v_j$ in a complete directed geometric graph $G(\bar{V}, \bar{E})$. Consider what happens when $v_i$ signals $v_j$ at time $t = t_i$. The shortest physically possible reaction time for $v_j$ in all cases will be a signal reaching it from $v_i$ just as its refractory period is ending. This occurs when $\tau_{ij} \to R_j^+\text{, i.e. approaches } R_j$ from the right, that is, is slightly longer than $R_j$. Let $\Delta_{ij}$ represent the set of all $\Delta_{ij}$ ratios for all $v_i$ vertices with directed edges $e_{ij}$ into $v_j$ for which the condition $\tau_{ij} \to R_j^+$ is met:

$$\Delta_{ij}^o := \{ \Delta_{ij} : i = 1, 2, \ldots, N, R_j/\tau_{ij} \text{ for } \tau_{ij} \to R_j^+ \}$$ (4)

This then implies that

$$\forall \Delta_{ij} \in \Delta_{ij}^o \implies \Delta_{ij} < 1$$

We can then prove the following relation

**Lemma 2.15.** Let $t_i \forall i \in \Delta_{ij}^o = t_o$, i.e. all vertices into $v_j$ initiate a signaling event at the same time $t_o$. Assume $v_j$ becomes refractory exactly at $t_o$. The refraction ratio $\Delta_{ij}$ for the ‘winning’ vertex $H_j[v_i^*] \rightsquigarrow v_j$ with $v_i^*(\Delta_{ij}^*)$ is given by

$$\Delta_{ij}^* = \max_i \Delta_{ij} \text{ for } \Delta_{ij} > 0 \in \Delta_{ij}^o$$ (5)

*Proof.* Given the set $\Delta_{ij}^o$, assume it is a well ordered set, i.e. there exists a smallest element in $\Delta_{ij}^o$. Then order the elements $\Delta_{ij} \in \Delta_{ij}^o$ such that $< \Delta_{ij} \forall i = 1, 2, \ldots, N$. By construction of the dynamics of the model, in the limit as $\tau_{ij} \to R_j^+$ the winning vertex $v_i^*(\Delta_{ij}^*)$ will be the vertex associated with the refraction ratio $\Delta_{Nj} = \max_i \Delta_{ij}$ as required.  

In addition, we make the following observation about the limit of the value $\Delta_{ij}$ can take.

**Lemma 2.16.** Given the set $\Delta_{ij}^o$, $\forall \Delta_{ij} \in \Delta_{ij}^o$, if we let $\Delta_{ij,\text{max}}$ represent the largest attainable value by an element of $\Delta_{ij}$,

$$\Delta_{ij,\text{max}} = \lim_{\tau_{ij} \to R_j^+} \frac{R_j}{\tau_{ij}} \to 1$$ (6)

*Proof.* This follows directly from the definition of $\Delta_{ij}$ in equation 3.

Alternatively, we can re-write equation 5 as an inequality condition of a difference
Lemma 2.17. Let $t_i \forall i \in \Delta_{ij}^0 = t_o$. Assume $v_j$ becomes refractory exactly at $t_o$. The 'winning' vertex $H_j[v_i^*] \rightsquigarrow v_j$ is given by

$$H_j[v_i^*] = v_i \in H_j \text{ such that } \min_i[(\tau_{ij} - R_j) > 0]$$

(7)

Proof. The necessary condition for $H_j[v_i^*] \rightsquigarrow v_j$ is $\tau_{ij} \rightarrow R_j^+$ (c.f. lemma 2.15). By lemma 2.16 in the limit $\Delta_{ij} \rightarrow \Delta_{ij,max}$ when $\tau_{ij} \rightarrow R_j$, which represents the largest value attainable by $v_i^*$. This implies that $(\tau_{ij} - R_j) \rightarrow 0$. Therefore, in every case $v_i^*$ will be the smallest positive value of $(\tau_{ij} - R_j) \rightarrow 0$, or $\min_i[(\tau_{ij} - R_j) > 0]$ as required. Note that the condition for positive values of the difference is necessary because negative values imply that the signal from $v_i$ arrives at $v_j$ while it is still refractory.

Algorithmically, equation 7 is much more efficient to implement because one only needs to compute a difference compared to equation 5 which necessitates computing a division. This becomes significant when computing in parallel all $H_j \in G(\bar{V}, \bar{E})$.

2.2.4 Refraction ratio analysis with temporal offset

Under most conditions, there is likely to be a temporal offset between when $v_i$ signals at $t_i$ and how far along $v_j$ is in its recovery from its refractory period due to a previous signaling event relative to an observation time $T_o$. This would be the case for all situations other than when $v_j$ becomes refractory exactly at $t_i$ as in section 2.2.3. We first consider the case where $v_i$ signals $v_j$ at the same $t_i$ for every $v_i \in H_j$, i.e. $t_i \forall i \in \Delta_{ij}^0 = t_o$.

Definition 2.18. Let $\phi_j$ represent a temporal offset from $R_j$ resulting in an 'effective' refractory period, such that at $t_i$

$$R_j = R_j - \phi_j \text{ where } 0 \leq \phi_j \leq R_j$$

(8)

We re-write equation 3 as

$$\bar{\Delta}_{ij} = \frac{R_j}{\tau_{ij}} = \frac{\bar{R}_j}{d_{ij}}$$

(9)

When $\phi_j = 0$ it implies $v_j$ became refractory exactly when $v_i$ signaled at $t_i$. This is effectively the special case described by equations 5 and 7. When $\phi_j = R_j$ it implies that $v_j$ is not refractory and can respond to an input from any $v_i$ at any time. Note how when $\phi_j = R_j$ $v_j$ may have been refractory at some time $t \leq t_o - R_j$, but assures the condition that $\bar{R}_j = 0$ at $t_o$.

Furthermore, the following lemma applies.

Lemma 2.19. Let $t_i \forall i \in \Delta_{ij}^0 = t_o$. If $\phi_j = R_j$ then the edge path integral for the 'winning' vertex $H_j[v_i^*] \rightsquigarrow v_j$ with $v_i^*(\bar{\Delta}_{ij})$ will be $\min_i d_{ij} \forall d_{ij} \in \Delta_{ij}^0$.
Proof. Given the definition of $\Delta_{ij}$ in equation 3, for a constant $s$, $\Delta_{ij}^s = \lim_{r_{ij} \to R_j^+} \max_i \Delta_{ij}$ when $\tau_{ij} \in \Delta_{ij}^0 = \min_i \tau_{ij}$ for $v_{ij}^*$, since if $\phi_j = 0 \forall i \in H_j[|v_{ij}|] y_j = 1 \forall i \in H_j[v_{ij}]$. This condition will be met when $d_{ij} \in \Delta_{ij}^0 = \min_i d_{ij}$. □

And by direct extension of lemmas 2.15, 2.16 and 2.17 we can write the equivalent expressions for $0 \leq \phi_j \leq R_j$.

**Lemma 2.20.** Let $t \forall i \in \Delta_{ij}^o = t_o$. Assume $v_j$ has an effective refractory period given by $R_j$ for some value of $\phi_j$ at $t_o$. The refraction ratio $\hat{\Delta}_{ij}$ for the 'winning' vertex $H_j[v_j^*] \leadsto v_j$ with $v_j^*(\hat{\Delta}_{ij})$ is given by

$$\hat{\Delta}_{ij} = \max_i \Delta_{ij} \text{ for } \hat{\Delta}_{ij} > 0 \in \Delta_{ij}^o$$  \hspace{1cm} (10)

**Lemma 2.21.** Given the set $\Delta_{ij}^o, \forall \Delta_{ij} \in \Delta_{ij}^o$, and assuming $v_j$ has an effective refractory period given by $R_j$ for some value of $\phi_j$ at $t_o$, if we let $\Delta_{ij,max}$ represent the largest attainable value by an element of $\Delta_{ij}^o$,

$$\Delta_{ij,\text{max}} = \lim_{\tau_{ij} \to R_j^+} \frac{R_j}{\tau_{ij}} \to 1$$ \hspace{1cm} (11)

**Lemma 2.22.** Let $t \forall i \in \Delta_{ij}^o = t_o$. Assume $v_j$ has an effective refractory period given by $R_j$ for some value of $\phi_j$ at $t_o$. The 'winning' vertex $H_j[v_j^*] \leadsto v_j$ is given by

$$H_j[v_j^*] = v_i \in H_j \text{ such that } \min_i [(\tau_{ij} - R_j) > 0]$$ \hspace{1cm} (12)

In the most general case $t_i$ for $v_i \in H_j$, i.e. the times at which each vertex initiates a signal, would not be expected to be all the same. One would expect that $t_i \neq t_o \forall i$. At any given instantaneous moment $T_o$ a signal from any $v_i$ may be traveling part way along $e_{ij}$ at a speed $s$, effectively shortening $\tau_{ij}$. Or it may be delayed in signaling if $v_i$ signals some time after $T_o$, effectively lengthening $\tau_{ij}$. At time $T_o$ we need to take into account the degree of signaling progression for each edge and define $\tau_{ij}$ analogous to $R$. (Note that we write $T_o$ to distinguish the case when $t_i \forall i \in \Delta_{ij}^o \neq t_o$ since $t_i$ here represents the time that vertex $v_i$ signals, which could be different than the time $T_o$ at which the network is observed. In effect both $t_o$ and $T_o$ represent an instantaneous sampling or observation time of the state of the network, but we reserve writing $t_o$ to indicate an observation moment that coincides with all $v_i \in H_j$ signaling at the same time.)

**Definition 2.23.** Let $\tau_{ij}$ represents the temporal delay, i.e. latency period, for a signal that travels on the edge $e_{ij}$ for vertex $v_i \in H_j$ when $v_i$ initiates a signaling event at $t_i$ which could come before, right at, or after the instantaneous observation time $T_o$, i.e. $t_i \geq T_o$ or $t_i < T_o$, and $y_{ij} = 1$, i.e. $v_j$ is not refractory. We then define a temporal offset for $\tau_{ij}$, an effective shortening or lengthening of $\tau_{ij}$ as follows

$$\bar{\tau}_{ij} = \tau_{ij} + \delta_{ij} \text{ where, } \delta_{ij} \in \mathbb{R}$$ \hspace{1cm} (13)
We extend equation 9 as
\[ \Lambda_{ij} = \frac{\bar{R}_j}{\bar{\tau}_{ij}} \] (14)

\[ \delta_{ij} > 0 \] represents an effective delay or elongation beyond \( \tau_{ij} \). In other words, it represents the vertex \( v_i \) initiating a signal at some time after \( T_o \). Values \( -\tau_{ij} < \delta_{ij} < 0 \) represent an effective shortening of \( \tau_{ij} \). This would be the case when \( v_i \) had initiated a signal that was traveling part way along the edge \( e_{ij} \) towards \( v_j \) prior to the sampling time \( T_o \). When \( \delta_{ij} = 0 \) it implies that \( v_i \) signals exactly at the moment the network is observed. And when \( \delta_{ij} = -\tau_{ij} \) it implies that the signal arrives at \( v_j \) at the moment the network is observed.

Values of \( \delta_{ij} < \tau_{ij} \), which result in \( \bar{\tau}_{ij} < 0 \), represent a signal arriving at \( v_j \) prior to the observation time \( T_o \).

For completeness, we re-write equation 2 to include \( \bar{R}_j \) and \( \bar{\tau}_{ij} \) as
\[ \bar{y}_j(\Omega, T_o) = \begin{cases} 1, & \text{iff } v_j \text{ can respond to an input from any } v_i \\ 0, & \text{iff it is refractory to any input for a period } \bar{R}_j \text{ that begins at } T_o \end{cases} \] (15)

We also similarly define
\[ \Lambda^0_{ij} := \{ \Lambda_{ij} : i = 1, 2, \ldots N, \bar{R}_j/\bar{\tau}_{ij} \text{ for } \bar{\tau}_{ij} \to \bar{R}_j^+ \} \]

analogous to equation 4.

We can now formally state the conditional relationship between \( \bar{\tau}_{ij} \) and \( \bar{R}_j \) necessary for signaling for the general case.

\textbf{Lemma 2.24.} Let \( G = (\bar{V}, \bar{E}) \) represent a complete geometric graph model of a network consisting of subgraphs \( H_j \) such that all \( v_i \in H_j \), i.e. \( H_j(v_i) \), contain directed edges \( H_j(\bar{E}) \) into vertex \( v_j \). Assume a signaling speed \( s_{ij} \) between \( v_i \) and \( v_j \). \( v_i \) may activate \( v_j \) iff \( \bar{\tau}_{ij} > \bar{R}_j \).

\textit{Proof.} \( v_j \) will be in a state where it is capable of being activated in response to receiving a signal from \( v_i \in H_j \) only when \( \bar{y}_j(\Omega, T_o) = 1 \). This is the case only following the completion of the effective refractory period \( \bar{R}_j \). At an observation time \( T_o \) \( \bar{R}_j \) could be at any point in its temporal evolution over its range of \( 0 \leq \bar{R}_j \leq R_j \). In order for the signal from \( v_i \) to arrive at \( v_j \) when \( \bar{y}_j(\Omega, T_o) = 1 \) then, in every case \( \bar{\tau}_{ij} > \bar{R}_j \).

And by direct extension of lemma 2.19 we can write

\textbf{Lemma 2.25.} If \( \phi_j = R_j \) then \( \bar{R}_j = 0 \Rightarrow \Lambda_{ij} = 0 \) and the ‘winning’ vertex \( H_j[v^*_j] \) will be given by the vertex with the delay min \( \bar{\tau}_{ij} \forall d_{ij} \in \Lambda^0_{ij} \).

By further extending lemmas 2.20, 2.21 and 2.22 we arrive at the general theorems that completely describe the competitive refractory dynamics framework introduced in section 2.1.
Theorem 2.26. Assume \( t_i \forall i \in \Lambda_{ij}^0 \) at \( T_0 \) for some value of \( \delta_{ij} \), such that all \( t_i \) need not necessarily be equivalent. Assume \( v_j \) has an effective refractory period given by \( R_j \) for some value of \( \phi_j \) at \( T_0 \). The refraction ratio \( \Lambda_{ij} \) for the ‘winning’ vertex \( H_j[v_j^\ast] \rightsquigarrow v_j \) with \( v_j^\ast(\Lambda_{ij}^0) \) is given by
\[
\Lambda_{ij}^0 = \max_i \Lambda_{ij} \text{ for } \Lambda_{ij} > 0 \in \Lambda_{ij}^0
\] (16)

Proof. The proof parallels the proof of lemma [2.15](#). Given the set \( \Lambda_{ij}^0 \), assume it is a well ordered set. Then order the elements \( \Lambda_{ij} \in \Lambda_{ij}^0 \) such that \( < \Lambda_{ij} \forall i = 1, 2, \ldots N \). Due to the dynamics of the model it is still the case that in the limit as \( \bar{\tau}_{ij} \to \bar{R}_j^+ \) the winning vertex \( v_j^\ast(\Lambda_{ij}^0) \) will be the vertex associated with the refraction ratio \( \Lambda_{Nj} = \max_i \Delta_{ij} \) as required. In other words, in every case \( H_j[v_j^\ast] \rightsquigarrow v_j \) will always be the vertex that arrives at \( v_j \) first subject to it recovering from its effective refractory period \( \bar{R}_j \) given the temporal evolution of \( \bar{\tau}_{ij} \) for all \( i \) and \( \bar{R}_j \) at the instantaneous time \( T_0 \) at which the network is observed. \( \square \)

Theorem 2.27. Given the set \( \Lambda_{ij}^0, \forall \Lambda_{ij} \in \Lambda_{ij}^0 \), and assuming \( v_j \) has an effective refractory period given by \( \bar{R}_j \) for some value of \( \phi_j \) at \( T_0 \), i.e. \( \bar{R}_j \neq 0 \), if we let \( \Lambda_{ij,max} \) represent the largest attainable value by an element of \( \Lambda_{ij}^0 \),
\[
\Lambda_{ij,max} = \lim_{\bar{\tau}_{ij} \to \bar{R}_j^+} \frac{\bar{R}_j}{\bar{\tau}_{ij}} \to 1
\] (17)

Proof. The proof follows directly from lemma [2.16](#) and the definition of equations [9](#) and [13](#).

If \( \bar{\Lambda}_{ij} = 0 \) then we immediately know that \( v_i \) with \( \min \bar{\tau}_{ij} \forall v_i \in H_j(v_i) \) will win, i.e. \( \min d_{ij} \forall v_i \in H_j(v_i) \), since \( \bar{\Lambda}_{ij} = 0 \) only when \( \bar{R}_j = 0 \) which implies that \( y_j(\Omega, t) = 1 \). \( \square \)

Theorem 2.28. Let \( t_i \forall i \in \Lambda_{ij}^0 = \bar{\tau}_{ij} \) at \( T_0 \) for some value of \( \delta_{ij} \) and \( C_{ij} \). Assume \( v_j \) has a remaining refractory period given by \( \bar{R}_j \) for some value of \( \phi_j \) at \( T_0 \). The ‘winning’ vertex \( H_j[v_j^\ast] \rightsquigarrow v_j \) is given by
\[
H_j[v_j^\ast] = v_i \in H_j \text{ such that } \min_i[\bar{\tau}_{ij} - \bar{R}_j] > 0
\] (18)

Proof. This proof parallels the proof of lemma [2.17](#) due again to the dependency of \( v_j^\ast(\Lambda_{ij}^0) \) given the temporal evolution of \( \bar{\tau}_{ij} \) for all \( i \) and \( \bar{R}_j \) at the \( T_0 \). The necessary condition for \( H_j[v_j^\ast] \rightsquigarrow v_j \) in this case is \( \bar{\tau}_{ij} \to \bar{R}_j^+ \). By theorems [2.26](#) and [2.27](#) in the limit \( \Lambda_{ij} \to \Lambda_{ij,max} \) when \( \bar{\tau}_{ij} \to \bar{R}_j \), which represents the largest value attainable by \( v_j^\ast \). This implies that \( (\bar{\tau}_{ij} - \bar{R}_j) \to 0 \). Therefore, in every case \( v_j^\ast \) will be the smallest positive value of \( (\bar{\tau}_{ij} - \bar{R}_j) \to 0 \), or \( \min_i[(\bar{\tau}_{ij} - \bar{R}_j)] > 0 \) as required. In this case also, note that the condition for positive values of the difference is necessary because negative values imply that the signal from \( v_{ij} \) arrive at \( v_j \) while it is still refractory. \( \square \)
3 Optimized information flow

3.1 Optimized bounded refraction ratio

Given the effective refractory period $\bar{R}_j$ and the effective delay time $\bar{\tau}_{ij}$ along the edge $e_{ij}$, the condition for the winning vertex $v_i$ that achieves activation of $v_j$, i.e. $H_j[v_i^+] \sim v_j$, is dependent on the $\lim_{\bar{\tau}_{ij} \to \bar{R}_j^+}$. This implies a balance between how fast information is capable of propagating through the network relative to how quickly its nodes can process incoming signals. When a mismatch between network geometry and dynamics exists, it can render the nodes unable to process any information at all or can result in highly inefficient network dynamics. If the signaling speeds $s_{ij}$ are too fast, or equivalently, if the delay times $\bar{\tau}_{ij}$ are too short compared to the amount of time a node requires to process an input and generate an output, determined by its effective refractory period $\bar{R}_j$, the network will not be able to sustain any internal recurrent activity. (See section 4.4 below for an example.) If $s_{ij}$ is too slow or the set of $\bar{\tau}_{ij}$ too long then that can render the network highly inefficient.

In this section we formalize these concepts.

We begin by deriving upper and lower bounds on the signaling dynamics which in turn define what we mean by an optimized refraction ratio and, from that, an optimally efficient network.

**Theorem 3.1** (Optimized information flow theorem). Let $G = (\bar{V}, \bar{E})$ represent a complete geometric graph model of a network consisting of subgraphs $H_j$ such that all $v_i \in H_j$, i.e. $H_j(v_i)$, contain directed edges $H_j(E)$ into vertex $v_j$. For each $v_i v_j$ vertex pair, given a signaling speed $s_{ij}$ between $v_i$ and $v_j$, assume that the state of $v_j$ and when each $v_i$ into $v_j$ signaled are observed or measured at time $T_o$. An optimally efficient network is one where the internal dynamics are matched to the flow of signaling (information) on its edges that results in an optimized refraction ratio $\langle \Lambda_{ij} \rangle_{opt} \forall v_i \in G = (\bar{V}, \bar{E})$ defined by the bounds

$$\langle \Lambda_{ij} \rangle_{opt} = \lim_{\tau_{ij} \to R_j^+} \Lambda_{ij} \text{ when } \phi_j = R_j \text{ and } \delta_{ij} = 0 \quad \text{[Upper bound]} \quad (19a)$$

$$\langle \Lambda_{ij} \rangle_{opt} \Rightarrow \lim_{\delta_{ij} \to -\delta_j^+} \Lambda_{ij} \text{ when } \phi_j = R_j \quad \text{[Lower bound]} \quad (19b)$$

where $\tau_{ij}$ is the absolute signaling delay time on the edge $e_{ij}$ and $R_j$ is the absolute refractory period for $v_j$.

**Proof.** By lemma 2.24 the necessary condition for the activation of $v_j$ by $v_i \in H_j$ is $\bar{\tau}_{ij} > \bar{R}_j$. By equation 3 $\bar{R}_j = R_j - \phi_j$ where $0 \leq \phi_j \leq R_j$, which implies that $0 \leq \bar{R}_j \leq R_j$. $\bar{R}_j$ is bounded by its very construction. The absolute lower bound on $\bar{R}_j$ implies that activation of $v_j$ by a $v_i \in H_j$ will be achieved when $\bar{\tau}_{ij} > 0$, and the absolute upper bound implies that $\bar{\tau}_{ij} > R_j$. But note how $\bar{\tau}_{ij}$ can always achieve these bounds independent of $\tau_{ij}$ for a given $v_i v_j$ pair at some observation time $T_o$ because by equation 13 $\delta_{ij} \in \mathbb{R}$, i.e. any vertex $v_i$ can activate $v_j$ independent of the absolute delay time $\tau_{ij}$ by delaying the initiation of...
an output signal at \( v_i \) long enough if \( \tau_{ij} \) is too short or initiating a signal at \( v_i \) prior to \( T_o \) if \( \tau_{ij} \) is too long. However, by theorems 2.27 and 2.28, \( \tau_{ij} \) need only be slightly larger than \( R_j \) in order to successfully signal \( v_j \): \( \tau_{ij} \rightarrow \bar{R}_j^+ \). Because \( \bar{R}_j \) is naturally bounded by \( 0 \leq \bar{R}_j \leq R_j \), it follows intuitively that the optimal signaling condition will be given by \( \tau_{ij} \rightarrow \bar{R}_j^+ \) for values of \( \delta_{ij} \) not too smaller than zero or not too greater than zero in order to meet the condition that \( \tau_{ij} \rightarrow \bar{R}_j^+ \) while avoiding compensation by \( \delta_{ij} \). In other words, the response dynamic range for any \( v_j \) will always be bounded by the limits of \( R_j \) in the sense that these limits determine the temporal properties of when \( v_j \) can actively participate in network signaling and when it cannot. Ultimately, of course, this is a function of \( v_j \)'s internal dynamics, which in turn determines \( R_j \) and \( \bar{R}_j \). No value of \( \tau_{ij} \) need be much greater than \( R_j \) for any \( v_i \) with a directed edge \( e_{ij} \) into \( v_j \). When the condition \( \tau_{ij} \rightarrow \bar{R}_j^+ \) is met, it ensures that such a \( \tau_{ij} \) is guaranteed to be able to operate over the entire response dynamic range of \( v_j \), i.e. all values of \( R_j \) (see corollaries 3.2 and 3.3 below).

For the upper bound this optimized boundary condition will occur when \( \tau_{ij} \rightarrow \bar{R}_j^+ \) when \( \phi_j \) and \( \delta_{ij} = 0 \) because it represents the upper achievable limit for \( R_j \) (when \( \phi_j = 0 \)) and forces the optimal condition that \( \tau_{ij} \rightarrow \bar{R}_j^+ \) without compensating with \( \delta_{ij} \). For the lower bound the optimal condition is given by \( \bar{\tau}_{ij} \rightarrow 0^+ \Rightarrow \delta_{ij} \rightarrow -\phi_j^+ \) when \( \phi_j = R_j \), since when \( \phi_j = R_j \Rightarrow \bar{R}_{ij} = 0 \). Forcing the condition that \( \phi_j = R_j \) implies that \( \tau_{ij} \) on its own is capable of meeting the lower bound without compensation by \( \delta_{ij} \). Formally, we can define an optimized bound as \( |\tau_{ij} - R_j| < \epsilon \) for some bounded error \( \epsilon \). If \( \tau_{ij} \) is too short, either because the path length of \( e_{ij} \) is too short or \( s_{ij} \) is too fast, this implies that given \( R_j \), \( \bar{\tau}_{ij} \rightarrow \bar{R}_j^+ \) if \( \delta_{ij} = 0 \). To achieve the lower bound it would require \( \delta_{ij} < 0 \) so that \( \bar{\tau}_{ij} < \tau_{ij} \). To achieve the upper bound it would require \( \delta_{ij} > 0 \) so that \( \bar{\tau}_{ij} > \tau_{ij} \). If \( \tau_{ij} \) is too long, either because the path length of \( e_{ij} \) is too long or \( s_{ij} \) is too slow, this implies that given \( R_j \), \( \bar{\tau}_{ij} \rightarrow 0^+ \) if \( \delta_{ij} \rightarrow -\phi_j^+ \).

For the lower bound the important condition is that \( \bar{\tau}_{ij} \rightarrow 0^+ \) when \( \delta_{ij} \rightarrow -\phi_j^+ \). It is trivial what \( \delta_{ij} \) is, since this condition will always be met when \( \delta_{ij} = -\tau_{ij} \). But for the upper bound the important condition is that \( \tau_{ij} \rightarrow \bar{R}_j^+ \) when \( \delta_{ij} = 0 \), which implies that in every such case \( \bar{\tau}_{ij} = \tau_{ij} \).

What these bounds imply is that if \( v_i \) satisfies the bounding conditions, it will always be within a range where it could ‘win’ and activate \( v_j \) for any value of \( R_j \) at any observation time \( T_o \) and time of signaling initiation of \( v_i \) \( t_i \). The given \( v_i \) may not of course always ‘win’ in activating \( v_j \) but it is insured to be as efficient as possible, i.e. as efficient as any other node in its signaling of \( v_j \) over all values of \( \bar{R}_j \). If \( R_j \) is the same for all nodes in the network, i.e. \( R_j = R \forall j \in G(\bar{V}, \bar{E}) \), this in effect bounds the dynamic window over which all network dynamics, that is, all temporal information (signaling) processes, will occur on that network. Explicitly, this dynamic window is given by \( \bar{R}_j = R_j - \phi_j \) for \( 0 \leq \phi_j \leq R_j \). So there is no need or reason for any \( \tau_{ij} \) to go far beyond this dynamic window in order to satisfy the optimality condition \( \bar{\tau}_{ij} \rightarrow \bar{R}_j^+ \). This is in essence the intuitive basis of using the
bounds derived in theorem 3.1 to define optimized information flow between node pairs.

Note that we must keep the explicit condition that $\phi_j = R_j$ because that forces $\Lambda_{ij} = 0$ only when $\delta_{ij} \to -\phi^+$. Otherwise, in the general case any value of $\tilde{\tau}_{ij} > 0$ will result in $\Lambda_{ij} = 0$ for any value of $R_j$.

We also have the following corollaries.

**Corollary 3.2.** Given the conditions for lower and upper bounds in theorem 3.1, let $\langle \delta_{ij} \rangle_{\text{upper}}$ denote the magnitude that $\delta_{ij}$ must take in order to achieve the upper bound condition for some value of $\tau_{ij}$ that does not necessarily satisfy $\langle \Lambda_{ij} \rangle_{\text{opt}}$. Similarly, let $\langle \delta_{ij} \rangle_{\text{lower}}$ denote the value that $\delta_{ij}$ must take in order to achieve the lower bound condition for some value of $\tau_{ij}$ that does not necessarily satisfy $\langle \Lambda_{ij} \rangle_{\text{opt}}$. In every case, the relationship between $\langle \delta_{ij} \rangle_{\text{upper}}$ and $\langle \delta_{ij} \rangle_{\text{lower}}$ is given by

$$\langle \delta_{ij} \rangle_{\text{lower}} = \langle \delta_{ij} \rangle_{\text{upper}} - R_j$$

**Proof.** The condition for the upper bound is $\tau_{ij} \to R_j^+$ for $\delta_{ij} = 0$ (and $\phi_j = 0$). With a bit of abuse of the relational terms, since $\bar{\tau}_{ij} = \tau_{ij} + \delta_{ij}$, under these conditions if $\tau_{ij} \neq R_j$ then $\bar{\tau}_{ij} \to R_j^+$. For a given $\tau_{ij}$ and for a known or measurable $R_j$, $\langle \delta_{ij} \rangle_{\text{upper}} = - (\tau_{ij} - R_j)$, since this is what the value of $\delta_{ij}$ would have to be in order to achieve the optimal condition. For the lower bound $\bar{\tau}_{ij} \to 0^+$ when $\delta_{ij} = -\phi_j$ given that $\phi_j = R_j$. Thus, if $\tau_{ij} - \phi_j \equiv \tau_{ij} - R_j > 0$, or more correctly if $\bar{\tau}_{ij} \to 0^+$, it implies that $\langle \delta_{ij} \rangle_{\text{lower}} = -\tau_{ij}$ would be needed to meet the lower bound optimality condition. The difference between $\langle \delta_{ij} \rangle_{\text{lower}}$ and $\langle \delta_{ij} \rangle_{\text{upper}}$ is therefore $-\tau_{ij} - [-\tau_{ij} - R_j] = -R_j$. Thus, $\langle \delta_{ij} \rangle_{\text{lower}} = \langle \delta_{ij} \rangle_{\text{upper}} - R_j$. \qed

**Corollary 3.3.** If a signal $v_i \in H_j(v_i)$ characterized by a delay time $\tau_{ij}$ on the edge $e_{ij}$ is able to achieve either the optimal upper bound or optimal lower bound as defined in theorem 3.1 then it is guaranteed to be able to achieve the other optimal bound.

**Proof.** Asking if a signal capable of achieving the upper bound can also achieve the lower bound is equivalent to asking if $\bar{\tau}_{ij} \to 0^+$ when $\tau_{ij} = R_j$. But the condition for the lower bound is $\bar{\tau}_{ij} \to 0^+$ when $\delta_{ij} = -\phi_j$. Substituting for these explicit variables we arrive at

$$\bar{\tau}_{ij} = \tau_{ij} + \delta_{ij} = R_j - \phi_j$$

but since $\phi_j = R_j$ for the lower bound, it implies that $\bar{\tau}_{ij} = 0$, or more appropriately, $\bar{\tau}_{ij} \to 0^+$. Asking if a signal that satisfies the optimality condition for the lower bound can also
achieve the upper bound is equivalent to asking if $\bar{\tau}_{ij} \rightarrow R_j^+$ when $\delta_{ij} \rightarrow -\phi_j$. Similarly,

$$\bar{\tau}_{ij} = \tau_{ij} + \delta_{ij}$$

$$0 = \tau_{ij} - R_j$$

$$\tau_{ij} = R_j$$

which implies that the optimality condition for upper bound is satisfied.

3.2 Optimally efficient networks

The definition of an optimally efficient network then follows:

**Definition 3.4.** (Optimally efficient network) In every case, as a function of the effective refractory period $\bar{R}_j$ and effective delay time $\bar{\tau}_{ij}$ along the edge $e_{ij}$, the condition for the winning vertex $v_i$ that achieves activation of $v_j$, i.e. $H_j[v^*_i] \sim v_j$, is dependent on the $\lim_{\bar{\tau}_{ij} \rightarrow \bar{R}_j^+} \forall v_i \in G(\bar{V}, \bar{E})$. When this condition is satisfied for all edges $e_{ij} \in \bar{E}$, i.e. $\bar{E} = \{e_{ij}\}$, for all $v_i,v_j$ node pairs by the upper and lower bound definitions for $\langle \Lambda_{ij} \rangle_{opt}$ (equation 19) such that $\Delta_{ij} = R_j/\tau_{ij} \rightarrow 1$, the network is optimally efficient, i.e. $\langle \Lambda_{ij} \rangle_{opt} \forall v_i \in G(V,E)$. This is equivalent to requiring the condition $|\tau_{ij} - R_j| < \epsilon \forall v_i \in G(\bar{V}, \bar{E})$ for some arbitrarily small value of $\epsilon$.

4 Optimal efficiency in real world natural and engineered networks

In this section we show empirical results that suggest that at least some naturally occurring and engineered networks have evolved or have been designed so that they approach $\langle \Lambda_{ij} \rangle_{opt}$. We empirically explored this in four distinct types of networks, the prevalence of the small world network topology, axonal branching of pyramidal neurons in the visual cortex, the internet router network, and dynamic signaling in geometric simulations of biological neuronal networks.

4.1 The pervalence of the small world network topology

In 1998 Watts and Strogatz published their seminal paper introducing small world networks, and suggested that this topology could be pervasive across both natural and engineered networks [5]. The small world network connectivity structure lives between a completely random network and a regular lattice. It provides an opportunity for nodes that would normally not be connected to be connected, resulting in a 'short circuiting' of dynamical behaviors and communication between different parts of the network that would normally not be in such immediate and direct contact. A key observation was that the transition to
a small world topology from a regular lattice is essentially undetectable at the local scale, but can have significant effects on the dynamics and the spread of information.

What is less obvious though, is why these networks are so prevalent. Our theory suggests an intriguing explanation. We considered the small world topology by first defining a cost function associated with deviation of $\Lambda_{ij}$ for a $v_iv_j$ node pair. We can define a simple cost function as $C_{ij} = |\tau_{ij} - R_j|$ for every connected vertex pair and then average over the entire network: $C_N = \frac{\sum_{k=1}^K C_k}{K}$ for $k = 1, 2, 3, \ldots, K$ edges for each connected $v_iv_j$ pair. A regular lattice network with probability $p$ of long range random re-wirings equal to zero (see [5] for details), can be made to exhibit arbitrarily near optimal signaling dynamics approaching $\langle \Lambda_{ij} \rangle_{\text{opt}}$ since the speed of signal propagation can balance the length of the edges such that they match any given internal node dynamics. Because it is a regular lattice network, with all edges being equal, this will apply to the entire network and $C_N \to 0$. In the context of a small world networks however, this leads to an interesting interpretation of why such few long range connections have such a significant effect on network dynamics: assuming consistent internal node dynamics and a constant signaling speed on all edges, sparse long range random re-wiring events sufficient to produce a meaningful effect on network dynamics can be constructed from an optimally efficient regular lattice network with essentially no effect on a loss of optimization. To see this, consider first the clustering coefficient in a small world network, $C(p)$, which is a measure of the degree of local connectivity. It reflects the deviation from a regular lattice network at the local scale. (See Fig. 2 in [5] for a formal definition.) $C(p)$ is nearly constant and unchanging from a lattice network until about $p \approx 0.01$ ($C(p)/C(0) \approx 1$, where $C(0)$ represents a value of $p = 0$, i.e. no random rewiring and a lattice network structure.) In contrast, the characteristic path length $L(p)$ for $p = 0.01$ as a ratio of the characteristic path length of a lattice network $L(0)$ is $L(p)/L(0) \approx 0.2$, indicating a significant degree of long range re-wiring events at the scale of the whole network. $L(p)$ is a measure of the shortest path between two vertices, averaged over the entire network. If we then consider the network optimization cost $C_N$ as a function of $p$ we observe that at the critical transition probability $p = 0.01$ there is essentially a negligible change in $C_N$ relative to an optimized regular lattice network (Fig. 1). We considered $C_N$ for families of small world networks with increasing random re-wirings starting from a regular lattice network that had a ratio $\Lambda_{ij} = 1.2$ for all vertex pairs. We then investigated how $C_N$ changed as a function of increasing $p$ for small world networks where the random re-wirings produced signaling delays that resulted in 2x, 10x, and 20x $> \langle \Lambda_{ij} \rangle_{\text{opt}}$ for re-wired vertex pairs. In all cases, $\Delta C_N < 0.019$ at $p = 0.01$ even though $\Lambda_{ij} = 1.667$ (for the curve labeled 2x; see inset in Fig. 1), 8.333 (curve labeled 10x), and 16.667 (curve labeled 20x) for the fraction of re-wired edges. In other words, even though the signaling efficiency for each of the three networks (by design) progressively deviated from $\langle \Lambda_{ij} \rangle_{\text{opt}}$, the additional deviation from $\langle \Lambda_{ij} \rangle_{\text{opt}}$ introduced by random re-wirings associated with a small world network, measured by the cost function $C_N$ was essentially negligible at re-wiring probabilities (i.e. $p = 0.01$) that produce significant effects on network dynamics.
attributed to the small world topology. We then explored how $C_N$ changes at fixed $p$ values as a function of increasing re-wiring signaling delays, expressed again as $x$ times greater than the signaling speed associated with $\langle \Lambda_{ij} \rangle_{opt}$ (Fig. 2). The change in $C_N$ was linear but with different slopes, reflecting the value of $p$. While there is an increase in $C_N$ associated with re-wired edges that have signaling speeds that progressively move $\Lambda_{ij}$ away from $\langle \Lambda_{ij} \rangle_{opt}$, as would be expected, the change is linear and rather flat. Lastly, we explored how $C_N$ changed as a function of $p$ for different deviations from $\langle \Lambda_{ij} \rangle_{opt}$ for the starting regular lattice network (Fig. 3). Increasing the initial deviation from $\langle \Lambda_{ij} \rangle_{opt}$ for all vertex pairs in the starting lattice network did not change the dynamics of how $C_N$ varied, but did affect the starting value of $C_N$ across all values of $p$, as would be expected.

Small world networks represent a connectivity class which can be designed to display arbitrarily optimal signaling dynamics with essentially negligible deviation from regular lattice networks while simultaneously displaying sufficient long range random edge re-wirings (e.g. at $p = 0.01$) to produce a significant impact on the dynamics of the network. It makes sense for both natural and engineered networks to take advantage of this topological structure, since it provides a simple set of construction rules to produce tailored effects on dynamics and the propagation of information through the network (via the re-wiring probability $p$) with essentially no resource costs in doing so. In other words, the analysis we suggest here allows us to mechanistically understand why and how small world networks are able to achieve such dramatic effects on network dynamics.

### 4.2 Axonal branching of pyramidal neurons in the visual cortex

Spiking biological neurons that produce action potentials have an absolute refractory period extending out to about 0.8-1 ms. The absolute refractory period is mediated by the inactivation of populations of sodium channels responsible for the depolarizing phase of the action potential. During this period other incoming stimuli cannot elicit another response from that neuron. The integrative combinatorial nature of the interactions from presynaptic neurons synapsing on a post-synaptic neuron is exceedingly complex due to synaptic dendritic integration prior to reaching the threshold potential at the initial segment that triggers an action potential. Neurons also have a relative refractory period that typically lasts to about 5 ms due to the potassium mediated after hyperpolarization and recovery times of the sodium channels that result in necessitating larger than typical stimuli in order to have a chance to produce subsequent action potentials. Furthermore, typical chemical synapses have synaptic delays on the order of 1-5 ms, and longer in some cases. We do not elaborate further on the details of the neurophysiology here but direct the interested reader to the standard texts in the field. In the context of the theory we develop here, collectively the biological absolute and relative refractory periods represent the refractory period necessitated by the theory, $R_j$. Axonal conduction delays, the period of time it takes for action potentials to propagate down the length of the axon, represents the delay time $\tau_{ij}$ between the presynaptic neuron through which the action
Figure 2: Network cost function $C_N$ as a function of small world re-wiring probability $p$ for three networks with differing deviations of dynamic signaling optimality for re-wired edges. The starting lattice network was considered optimally efficient with $\Lambda_{ij} = 1.2 := \langle \Lambda_{ij} \rangle_{\text{opt}}$. Randomly re-wired long range connections had signaling delays that resulted in $2x$, $10x$, and $20x > \langle \Lambda_{ij} \rangle_{\text{opt}}$. Inset: Magnified $C_N$ scale up to $p = 0.01$.

Figure 3: Network cost function $C_N$ as a function of increasing re-wiring deviation from $\langle \Lambda_{ij} \rangle_{\text{opt}}$ for small world networks. $C_N$ was computed for value of $p = 0.001$, 0.01, and 0.1.
potential is propagating (node $v_i$) and the postsynaptic neuron it is passing along the signal to (node $v_j$). Conduction delays are a function of the action potential (signaling) propagation speed down the axon, which is itself a function of several variables including axon diameter and axonal myelination, and the length of the axon itself. Axon lengths can vary significantly depending on whether they are long range or local connections for example. As such, there is a tremendous degree of variability in axonal conduction delays across different neuronal populations, brain regions, and across species. (See Table 1 in http://www.scholarpedia.org/article/Axonal_conduction_delays for example). Yet, in every case for all neurons there seems to be a regime that can be found in the conduction delays such that the effective temporal delays (equation 13 $\tilde{\tau}_{ij} = \tau_{ij} + \delta_{ij}$ where, $\delta_{ij} \in \mathbb{R}$) matches the effective refractory period (equation 9 $\tilde{R}_{ij} = R_j - \phi_j$ where $0 \leq \phi_j \leq R_j$), resulting in an optimized refraction ratio $\langle \Lambda_{ij} \rangle_{opt}$.

Budd and colleagues recently investigated in detail the relationship between the amount of material a neuron commits to its axonal arbor, the branching points where the axon synapses with various postsynaptic neurons, and conduction delay times [9]. Originally proposed by Santiago Ramon y Cajal and subsequently adopted by many groups, the prevailing dogma was that a neuron optimizes resources in order to minimize the amount of cellular material it uses in its axonal arborization. However, as Budd and colleagues showed this comes at the price of increased conduction delays. These authors showed that

Figure 4: Network cost function $C_N$ as a function of small world re-wiring probability $p$ for three networks with differing deviations from $\langle \Lambda_{ij} \rangle_{opt}$ for the starting lattice network. Signaling delays for the starting lattice network were 1.2, 3x and 5x $> \langle \Lambda_{ij} \rangle_{opt}$.
a neuron sacrifices material cost in order to achieve shorter conduction delays, representing
a trade off between anatomical optimization and dynamics. They first carried out detailed
tract tracing of excitatory spiny and pyramidal cells and inhibitory large basket cell in the
in vivo cat neocortex, and then compared axonal path lengths and conduction delays (given
knowledge of the range of conduction speeds in these neurons) with graph theoretic models
optimized for shortest physical paths (i.e. conservation of material) or paths that provided
the most efficient conduction times. Their results suggest that neurons are neither optimized
for conservation of cellular material nor for the shortest conduction delays possible, but
instead fall in a regime in between these two extremes. They also showed that there is
a scaling effect where this balance is maintained independent of the number of synaptic
boutons (terminals) a neuron branchings into. The authors go on to suggest that the design
of intracortical arbors are tailored to support a high degree of temporal precision in the
timing of when action potentials occur, which underlies how they contribute to neuronal
signaling, so called temporal dispersion- which is critical to rate code models of neuronal
signaling, see for example [12,18,19]. They further hypothesize that neurons closely preserve
and regulate the relationship between distance and latency.

What this paper does not strictly address is why such a trade off exists in the first
place, it only provides empirical evidence for the relationship. Why is that the neuron
compromises between material costs and temporal latencies in the way that it does? Similar
to our consideration of small world networks, interpreting their results in the context of our
theory suggests a possible answer to this question. The results of Budd et. al. imply that
neurons put significant resources in the form of cellular material into maintaining a constant
conduction delay for all synapses, independent of the specifics of the axonal arborization.
In other words, all other things being equal two neurons synapsing onto the same post
synaptic neuron would have equal opportunity for contributing to the generation of an action
potential as a result of dendritic integration at any instantaneous moment in time, omitting
of course other important considerations such as synaptic coupling strength and ‘by design’
differences in how different classes of neurons synapse on a postsynaptic dendritic arbor,
e.g. clustering of inhibitory synapses closer to the soma. This makes sense etiologically
because the establishment of synaptic connections is not deterministic but stochastic, in
the sense that while there is a high level pre-programmed plan for how a developing brain
should be wired up, individual synaptic connections between specific neurons are not hard
coded. If such a trade off between distance and signaling speed was not the case then
stochastic connectivity effects at the level of cellular neural circuits and networks could have
a disproportionate degree of influence on neural signaling and information processing at the
network scale, which would ultimately affect the ability of the brain to function properly.
Given the definition of \( \langle \Lambda_{ij} \rangle_{opt} \) and theorem 3.1 at a constant signaling speed, a lack of
distance (path length) preservation in axonal arbors would stochastically make specific
pre- postsynaptic neuronal node pairs more or less efficient. Information in the brain at
the level of individual neurons and neural circuits is tightly tied to the timing effects of
action potentials. So in order for neurons to be on equal footing with regards to temporal

\[ \Lambda_{ij} \]
dispersion, conduction delays must be tightly regulated and preserved. This is what our theory predicts and what Budd et. al. have independently demonstrated experimentally. Our theoretical interpretation suggests a putative reason why neurons and neuronal networks are designed the way they are.

We further explored this in simulations. It is surprisingly difficult to get specific and accurate data on axonal and axonal arbor cellular anatomical path lengths along with simultaneous conduction velocities. To a first approximation we used data provided in [9], who carried out detailed anatomical tract tracing reconstructions of sample spiny cortical neurons and basket cells in adult cat primary visual cortex (see for example their Figs. 4, 5, and 12). Spiny cell axonal arbors, which we considered in our simulations, range from just over zero to just under 4 mm in length (median = 1.66 mm with a standard deviation of 0.77 mm). In our simulations we used a path length range of 1 to 3.8 mm. Axonal conduction velocity is a function of axonal diameter, branching, ion channel sub-types and density, and myelination [9][11], although numerical simulations suggest that path length is the most significant determinant of conduction velocity [13]. Reported values of neuronal conduction velocities in adult cat visual cortex differ rather significantly depending on the study. We used a range of 0.1 to 0.6 m/sec, which is a range that covers most of the reported values [8, 13, 14]. Using this as a starting point, we first computed conduction signaling delays $\tau_{ij}$ as a function of the path length and conduction velocity ranges (Fig. 5a.). This yielded a maximum conduction delay of just under 40 ms. We then computed the refraction ratio $\Lambda_{ij}$ as a function of $\tau_{ij}$ and the refractory period $R_j$. There is very little direct data on the refractory periods of spiny neurons that we are aware of, and as such we assumed a refractory period range of 0.8 to 5 ms, as described above. Given these assumptions and ranges, the surface of $\Lambda_{ij}$ extends from a maximum value of 3 to a minimum of 0.021. There exists a subset of $\Lambda_{ij}$ that displays near optimal efficiency, indicating that within the experimental limitations of the available measured data to date, spiny neurons are capable of achieving dynamic optimally efficient signaling that approaches $\langle \Lambda_{ij} \rangle_{opt}$ for a range of appropriate combinations of the key contributing parameters, namely, path length, conduction velocity, and refractory period. We considered parameter ranges that resulted in refraction ratios bounded by $0.8 \leq \Lambda_{ij} \leq 1.2$ as near optimal, i.e. approaching $\langle \Lambda_{ij} \rangle_{opt}$. The smallest value of $R_j$ that resulted in a ratio of $\Lambda_{ij} = 0.8$ occurred at 1.34 ms, with $\tau_{ij} = 1.6667$ ms. The smallest value of $\tau_{ij}$ that resulted in a ratio of $0.8 = 5$ ms, with $R_j = 4$. The smallest value of $R_j$ that produced a ratio of 1.2 occurred at 2 ms, with $\tau_{ij} = 1.6667$ ms; while the smallest value of $\tau_{ij}$ that resulted in a ratio of 1.2 was at 3.333 ms, with $R_j = 4$ ms.

4.3 Internet packet processing and routing

As a very different example, we also considered the internet routing network. This case is particularly interesting because the theory suggests a scenario that is contrary to a 'standard' interpretation of optimized packet routing. Ramaswamy et. al. provide a good
Figure 5: Computed conduction (signaling) delays $\tau_{ij}$ and refraction ratios $\Lambda_{ij}$ (panel b.) for spiny cortical neurons in the adult cat primary visual cortex. The two dimensional surface of $\tau_{ij}$ was computed from experimentally measured ranges of path length and axonal conduction velocities derived from the published literature (panel a.), while $\Lambda_{ij}$ (panel b.) was computed from the data in panel a. along with refractory period estimates of mammalian central neurons.

A concise introduction to the current demands placed on packet routing on the internet [17]. Borrowing from these authors’ own description: The internet has evolved from its original functionality as a store and forward network to a much more complex and sophisticated communication infrastructure. The needs required by network security and performance necessitates network traffic that is processed within the network in addition to packet forwarding and routing. This processing occurs primarily on routers as opposed to end systems, and include critical processes and applications such as network address translation (NAT), firewalling, and virtual private network (VPN) tunneling. These authors then go on to make the point that other complex services place even greater demands on packet processing at the router level, including virus scanning, content adaptation for wireless clients, and ad insertion in web page requests. In their paper they investigated the (increasing) temporal delays associated with increasing router level processing of packets. Sending a packet from one node in the network to another can experience delays associated with transmission delays, the time it takes to send the packet to the system, propagation delays, the time it takes to actually transmit a packet, the processing delay, which is the time it takes to handle the packet on the network, and queuing delays, the amount of time the packet is buffered before it can be sent. Within our theory the combination of transmission and propagation delays collectively contribute to the representation of $\tau_{ij}$. The processing and queuing delays map to $R_j$. Table 1 provides measured delay times for the different classes of delays associated with packet routing (c.f. table 1 in [17]), along with the computed refraction ratio $\Lambda_{ij}$ for each set of data. The computed values of $\Lambda_{ij}$ in the table intentionally ignore queuing delays, since these can range from zero to
infinite, depending on the amount of buffering taking place. But we will come back to this consideration. While $\tau_{ij}$, the time it takes to actually transport a packet through the network, does not change as the packet goes from a simple to a more complex payload, $R_j$ increases by two orders of magnitude. The increased demands on the processing time associated with complex packet manipulations necessary to meet the needs of ever increasing advanced and sophisticated applications that depend on such increased functionality is understandably a consideration and even concern for the efficient design of the internet infrastructure. However, in contrast to this intuitive concern, comparing the refraction ratios for simple versus complex packets suggests that simple packet routing is inefficient and far from optimal because of a mismatch between the transmission and propagation times versus the processing time ($\Lambda_{ij} = 0.0099$, assuming all routers and the links between them display the same dynamics), the former being much slower than the amount of time a simple packet is handled by a router itself. While the processing time of a complex packet increases significantly by two orders of magnitude, at current estimates the increase in the amount of time a router needs to hold onto a packet to carry out these additional operations is such that it almost perfectly matches the temporal delay associated with the propagation of packets through the network, which brings the refraction ratio to optimal ($\Lambda_{ij} = 0.99$).

From a network design and resource allocation and management perspective, this simple analysis suggests that no changes to the network need occur for existing complex packets. However, it does point to a saturation of the temporal resources available to the router as the demands on it continue to increase given existing protocols and algorithms. Any future increases in processing times will cause a deviation from dynamic optimality, but this time with $\Lambda_{ij} > 1$ due to increasing $R_j$. These are considerations that could play an important role in the design and resource allocation considerations for future iterations of the internet. Finally, we point out how the theory we develop here can provide a guide for network design and dynamic analysis, but ultimately domain specific considerations must determine how the refraction ratio is computed. In the example here we chose to ignore queuing delays due to their significant variability. Clearly though, the inclusion of such a delay will impact the value of $\Lambda_{ij}$, but deciding how such a variable is included, when, and the weight it carries is dependent on the specifics of the network and system under consideration. However, an analysis similar to the simple example we discuss here could provide a powerful tool for testing the relative contributions of different variables to the dynamic efficiency of a network, in support of informed and systematic network design decisions.

4.4 Dynamic signaing in a biological neuronal network

Lastly, we have previously shown numerically in a simulated geometric neuronal network consisting of Izhekivitch neurons (a model of the internal neuronal machinery responsible for the generation of action potentials in each neuron) that the ability of the network to sustain recurrent signaling is a function of the balance between the internal dynamics of a
Table 1: Internet router network delay times and computed refraction ratios. Transmission and propagation delays contribute to $\tau_{ij}$, while processing and queuing delays contribute to $R_j$. Note that the computed values of $\Lambda_{ij}$ in the table intentionally ignore queuing delays. See text for details. Data adapted from Table 1 in [17].

| Type of delay          | Simple packet forwarding (µs) | Complex payload modifications (µs) | Contribution to the refraction ratio $\Lambda_{ij}$ |
|-----------------------|-------------------------------|-----------------------------------|-----------------------------------------------|
| Transmission delay    | 10                            | 10                                | $\tau_{ij}$                                   |
| Propagation delay     | 1000                          | 1000                              | $\tau_{ij}$                                   |
| Processing delay      | 10                            | 1000                              | $R_j$                                        |
| Queuing delay         | 10 to infinity                | 10 to infinity                    | $R_j$                                        |
| Refraction ratio $\Lambda_{ij}$ | 0.0099                      | 0.99                              |                                               |

neuron and the signaling propagation speed between neurons [10] (Fig. 6). This particular example is significant because it allows for a direct comparison between different signaling regimes and illustrates the consequences of deviating from $\langle \Lambda_{ij} \rangle_{opt}$. In fact, this example was the original motivation for the work we present in this paper. Note that here we do not provide a full analysis of this network in the context of the theory, which is the subject of other but related on-going work, only as a numerical example of the consequences of a mismatch in the refraction ratio. Computationally such a geometric Izhelivitch network can be reduced to a relationship between the positions of the neurons in space (or in the plane in this particular example) and the temporal delays with which signals propagate between them. The longer, e.g. more convoluted, an axon is the longer it will take a signal to reach from one neuron to another. In recurrent distributed networks like biological neural networks the relationship between network geometry and signaling delays serves as a form of signal storage, essentially giving the neurons time to recover from the refractory period between activations, which in turn results in sustainable recurrent signaling, but only if the internal dynamics match the network signaling dynamics. In Fig. 6, the network shown in panels a. and b. was driven by an external input for the first 500 ms. If the signaling speeds were too fast (200 pixels/ms in the example here) incoming signals from upstream neurons never had an opportunity to activate downstream cells because they were still refractory and not able to respond. This led to unsustainable signaling that quickly died away (Fig. 6c. bottom panel). As the signaling delays increased at an appropriate rate, recurrent sustained activity emerged (at a signaling speed of 20 pixels/ms: panel c., middle), and became stable as the signaling speed matched the neuronal dynamics (2 pixels/ms, panel c, top). Note that no parameter other than the signaling speed (or inversely, signaling delays)
on the edges connecting the nodes changed between the three numerical experiments in panel c. The geometric structure of the network, including its physical connectivity, the internal model of the dynamics of individual neurons, and the coupling (synaptic) weights between connected neurons did not change, only the signaling speed. The impact on the global network dynamics, however, is strikingly significant, producing behaviors that range from a complete inability to sustain any inherent recurrent activity, to a persistent and stable oscillatory-like state.

5 Discussion

In this paper we present an intuitively and conceptually simple framework that describes the dynamics of signaling and information flows in complete geometric networks derived from foundational principles of biological cellular neural signaling. Beginning from these foundations, the resultant theory provides a conceptual and quantitative basis for analytically and deterministically understanding the dynamic function and behavior of networks from some starting state. Our results provide a mechanistic explanation for how the interplay between strictly local geometric and temporal process at the scale of individual node pairs directly affects the behavior of the whole network. It is insightful to note how the response of each vertex is causally independent from whatever all the other vertices in the network are doing. Computing \( y_j \) at any instantaneous time is only dependent on the internal dynamics of \( v_j \), which in turn determines \( R_j \), and when signals arrive from (competing) input vertices into \( v_j \). This allows the independent computation of the interacting states of any \( v_i v_j \) vertex pair at the observation time \( T_o \). The computation of \( y_j \) is not dependent on any ‘average’ metric of the state or behavior of the network as a whole, or on any statistical probability densities associated with the frequency of occurrence of events such as in Markovian processes. It is a deterministic process at a local scale that affects the dynamics of the network at the global scale. Computing in parallel all \( v_i v_j \) state pairs reports back the state of the overall network.

An important result of this work is the derivation of the bounds that define the optimized refraction ratio, \( \langle \Lambda_{ij} \rangle_{opt} \) via the optimized information flow theorem (theorem 3.1). This theorem provides a quantitative definition for what it means for a network to be dynamically optimally efficient in how signals (information) propagates through its edges. In this paper we explored this in the context of biological neural networks, the internet routing network, the small world topology, and simulated biological neuronal networks. Some theoretical aspects of the framework were intentionally left open in order to provide sufficient application specific flexibility in how the theory can be used. For example, the situation where there is a tie with regards to two signals reaching a node \( v_j \) at exactly the same moment. We did not attempt to explicitly account for this because the details of how such a tie is broken is a property of the specifics of the system itself. Consider as an example Watt’s and Strogatz’s simulation of the spread of infections on small world networks (see their Fig. 3 in 5 and
Figure 6: Relationship between network geometry, signaling delays and their effect on network dynamics. The three dimensional geometric network shown in panel A was assigned random weights uniformly distributed between -1 and 1 on each physical edge (panel B). An Izhikevitch model of bursting neurons was used to model the individual vertex dynamics. C. By varying the speed of signal propagations between cells (in arbitrary units of pixels/ms), signaling delays have a critical impact on the resultant spike dynamics. See text for details. Adapted from [10].
its accompanying text). The tie breaking rule in that case is that if two vertices are at the same time step attempting to infect the same \( v_j \) there is no competition at all and no tie breaking rule is needed. Either \( v_j \) will be infected given the respective probabilities of each \( v_i \) (or the same probability for a given \( r_i \) equal for the whole network) or it will not. The competitive refractory dynamics in this model only exists at the level of nodes attempting to reach uninfected nodes at a given time step first, and do not directly compete with themselves when they infect at the same time.

As a technical comment, we note that we have purposely ignored any coupling weight between \( v_i \) and \( v_j \), such as a synaptic weight, and focused only on the dynamic considerations that constrain the flow of signaling and information in the network. Nothing is lost by doing this; the description does not change and can accommodate a modulating weight term in \( y_j \) that represents some form of signal coupling. We show this explicitly in a related model in [10] (see equations equations 2.9 and 2.10 in that paper).

From an engineering perspective, we propose that the framework and its analysis can be applied to any physically constructible real world network to provide insights into their behaviors and dynamics. We anticipate that any real world network will have spatial and temporal considerations that map to the geometric spatial and temporal properties of the framework, thereby making the practical application of the theory to any such system meaningful. This potentially results in some important applications. The theory could be used to guide the design of a network, as opposed to analyzing an existing network, since dynamics can be predicted by the theory at the individual node pair scale by testing different interactions of the internal processing times of the nodes and expected signaling delays, which themselves could be subject to possible experimental design considerations. This would allow the design of dynamically optimally efficient networks from the ground up. This consideration applies not just to the construction of physical systems, but also to information systems. For example, we are using the framework to derive an algorithm that computes, in parallel, the shortest paths between all sets of connected nodes in a network as a function of the states and dynamics of the nodes at the moment that the algorithm is run. In other words, the algorithm computes all allowable paths in the network as a function of the available resources at any given moment in time. It effectively reduces all possible physical paths through a network, which are ultimately determined by its structural connectivity, to only the allowable or realizable paths given the dynamic and resource constrains imposed on the network by its state at the moment it is observed, i.e. the moment the algorithm is run. This work has a number of practical applications we are exploring, including analyzing the functional (i.e. statistically related) and effective (i.e. causally related) dynamics of biological neural networks derived from structural connectomics work, i.e. functional connectomics.

Finally, from a theoretical perspective, in many ways the work we introduce here is just a beginning. Future theoretical work will address a more general version of the framework capable of accommodating heterogeneity in the network, such as variable \( s_{ij} \) across different \( v_i,v_j \) node pairs or even on different parts of the same edge, and varying
classes of nodes within the same network that result in differing $R_j$ values. We anticipate that this will lead to very interesting signaling and information flow dynamics on the network. Such a generalized theory will be applicable to broader classes of networks, such as for example signaling across different networks of distinct populations of neurons in the brain. Other theoretical work will explore possible connections that map the concepts and formal descriptions in our theory to other theoretical models, such the growing literature on adaptive networks. More abstractly, there may be some interesting opportunities to explore new theoretical terrains. For example, the development of the competitive dynamics framework and optimized information flow bounds in non-Euclidean geometric networks.

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