Building bulk from Wilson loops

Koji Hashimoto

Department of Physics, Osaka University
Toyonaka, Osaka 560-0043, JAPAN
*E-mail: koji@phys.sci.osaka-u.ac.jp

We provide formulas for holographically building a bulk metric from given expectation values of rectangular Wilson loops. As a corollary, we prove that any confining quark potential necessarily leads to the existence of a bulk IR bottom.

Contents

1 Introduction: building a bulk from QFT data ........................................... 1
2 Formulas ................................................................................................. 3
3 Rebuilding near horizon geometries ....................................................... 5
4 Theorem: confinement leads to a bulk IR bottom ................................. 7
5 Discussions ............................................................................................ 8

1. Introduction: building a bulk from QFT data

Building a bulk gravity spacetime from a given quantum field theory (QFT) data is one of the central issues in the holographic principle and the AdS/CFT correspondence [1–3]. The difficulty resides in its inverse problem nature. Normally in AdS/CFT, once a bulk spacetime or a bulk gravity system is given, then boundary QFT data such as correlators are calculated by solving classical physics in the bulk. On the other hand, building a bulk from a given QFT data is going backward, and going from lower dimensions to higher dimensions\(^1\), by decoding the holographic principle. The issue of building a bulk is concerned with the unrevealed mechanism of the emergent holographic dimension, which also signals the importance and the difficulty of the issue.

There have been a great advance in research on building a bulk spacetime metric from given boundary QFT data. The holographic renormalization [4] provides a bulk metric for a given expectation value of a QFT energy momentum tensor. A bulk reconstruction using bulk geodesics and light cones [5–11] uses divergent behavior of QFT correlators. A bulk-building method using holographic entanglement entropy [12, 13] has been developed extensively [14–29]. Relatedly, the identification of the bulk with tensor networks [30, 31] through the entanglement properties has been utilized to build a bulk space [32–35].

\(^{1}\)The easiest analogue of this difficulty is the problem of finding a quantum mechanical potential when only energy eigenvalues are provided. QFTs are infinite-dimensional and many-body generalization of the quantum mechanics, which suggests the difficulty.
In spite of these developments, there are obstacles to use the existing technologies to build a bulk spacetime metric from genetic QFT’s. This can be manifested in a question: what is a gravity dual of the Yang-Mills theory? Generically in AdS/CFT, the boundary QFT is strongly coupled, so the preparation of any QFT data needs non-perturbative methods such as lattice simulations. Unfortunately, the real-time correlators and the entanglement entropies used above are currently difficult to evaluate on the lattice. While energy momentum tensors were evaluated on the lattice recently, the holographic renormalization assumes bulk Einstein equations.

In this paper, we use Wilson loops as the QFT data to build a bulk spacetime metric. The Wilson loops in lattice simulations have been well studied and are well defined. The gravity dual of the Wilson loops [36–39] is known and does not assume the bulk Einstein equations. And fortunately, it is a minimal surface in the bulk, thus the bulk-building technologies developed for the entanglement entropy [15, 16] with the Ryu-Takayanagi surface can be employed.2

Our target bulk spacetime is assumed to be given by a bulk metric in string frame in any spacetime dimensions (≥ 3),

\[ ds^2 = -f(\eta)dt^2 + g(\eta)d\vec{x}^2 + d\eta^2, \]

where \( \eta \) is the emergent holographic radial coordinate. Generically the bulk metric can always be cast into this form, when the QFT is invariant under spacetime translation and space rotation, which is often favored in measuring Wilson loops on the lattice in the continuum limit. We take the gauge \( g_{\eta\eta} = 1 \) so that the proper distance along the radial direction is measured simply by the \( \eta \) coordinate. We also assume that \( f(\eta) \) and \( g(\eta) \) are monotonic functions of \( \eta \), and \( f(\infty) = g(\infty) = \infty \) which defines the boundary of the bulk geometry.

According to [36–39], once the bulk metric (1) is given, the quark potential \( E \) and the interquark distance \( R \) are calculated by a Nambu-Goto string hanging down in the bulk from the boundary, as

\[ E(\eta_0) = \frac{1}{\pi\alpha'} \int_{\eta_0}^{\infty} d\eta \sqrt{f(\eta)} \sqrt{\frac{f(\eta)g(\eta)}{f(\eta)g(\eta) - f(\eta_0)g(\eta_0)}}, \]

\[ R(\eta_0) = 2 \int_{\eta_0}^{\infty} d\eta \frac{1}{\sqrt{g(\eta)}} \sqrt{\frac{f(\eta_0)g(\eta_0)}{f(\eta)g(\eta) - f(\eta_0)g(\eta_0)}}, \]

with a parameter \( \eta_0 \). This \( \eta = \eta_0 \) is the deepest bulk location of the smooth tip of the string. Eliminating \( \eta_0 \) from (2) and (3) provides the quark potential \( E(R) \) in the boundary QFT.

The quark potential \( E(R) \) is related to the expectation value \( \exp[-E(R)T] \) of a temporal Wilson loop of a rectangular path of the size \( T \times R \), in the infinite limit of the time domain \( T \to \infty \). In the same manner, another observable available is a spatial Wilson loop of the size \( R_s \times R' \), which defines \( E_s(R_s) \) in the similar limit \( R' \to \infty \). The formula for \( E_s(R_s) \) is the same as (2) and (3) except for replacing \( f \) by \( g \).

Our goal in this paper is to provide an inverse formula: given \( E(R) \) and \( E_s(R_s) \), build \( f(\eta) \) and \( g(\eta) \). In particular, at zero temperature, since the Lorentz symmetry shows \( f = g \), we

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2 For the interplay between the Ryu-Takayanagi surface and the holographic Wilson loop in thermalization and confinement, see [40–43].
need only $E(R)$ to build a bulk metric. Furthermore, our formula derives a theorem stating that any confining quark potential results in the existence of an IR bottom of the built bulk geometry.

The organization of this paper is as follows. In Section 2, we provide our analytic formulas for building a bulk spacetime from a given Wilson loop data. In Section 3, we use the formulas with a power-law quark potential, to find that the built bulk is near-horizon geometries of $D_p$-branes. In Section 4, we provide a theorem that the confinement leads to a bulk IR bottom. Section 5 is for discussions on various aspects of our formulas.

2. Formulas

The formulas we obtain to build a bulk metric (1) at zero temperature and at non-zero temperature are as follows.

Zero temperature case

Given a quark potential $E(R)$, solve

$$ f_0 = 2\pi\alpha' \frac{dE(R)}{dR} $$

(4)

to get $R$ as a function of $f_0$. Then substitute it to the following differential equation

$$ \frac{d\eta}{df} = \frac{1}{\pi} \sqrt{f} \frac{d}{df} \int_{f_0}^{f} \frac{R(f_0)}{\sqrt{f_0^2 - f^2}}. $$

(5)

Integrate this to find $\eta = \eta(f)$. Finally, invert it to find a bulk metric $f(\eta)$.

Non-zero temperature case

Given a potential $E_s(R_s)$ for a spatial Wilson loop and $E(R)$ for a temporal Wilson loop, solve

$$ g_0 = 2\pi\alpha' \frac{dE_s(R_s)}{dR_s}, \quad h_0 = 2\pi\alpha' \frac{dE(R)}{dR}, $$

(6)

to get $R_s(g_0)$ and $R(h_0)$. First, substitute $R_s(g_0)$ to the differential equation

$$ \frac{d\eta(g)}{dg} = \frac{1}{\pi} \sqrt{g} \frac{d}{dg} \int_{g_0}^{g} \frac{R_s(g_0)}{\sqrt{g_0^2 - g^2}}. $$

(7)

Integrate it to find $\eta = \eta(g)$. Invert it to find a bulk metric component $g(\eta)$. Then substitute the explicit $g(\eta)$ and also $R(h_0)$ to the differential equation

$$ \frac{d\eta(h)}{dh} = \frac{1}{\pi} \sqrt{g(\eta(h))} \frac{d}{dh} \int_{h_0}^{h} \frac{R(h_0)}{\sqrt{h_0^2 - h^2}}. $$

(8)

Solve this to find $\eta(h)$, which is inverted to $h(\eta)$. Then obtain another component of the bulk metric as $f(\eta) = h(\eta)^2/g(\eta)$.

Proof. First, let us provide a proof of our formulas (4) and (5) for the case of zero temperature, $f(\eta) = g(\eta)$. We follow the strategy developed in [15, 16] for the entanglement entropy.
To simplify equations, we rewrite (2) and (3) with a new notation,
\[ E \equiv \pi \alpha E, \quad R \equiv R/2, \quad V(\eta) \equiv f(\eta)^2, \quad f_0 \equiv f(\eta_0). \]  

Then (2) and (3) are written as
\[ \tilde{E} = \int_{\eta_0}^{\infty} d\eta \frac{V^{3/4}}{\sqrt{V - f_0^2}}, \quad \tilde{R} = \int_{\eta_0}^{\infty} d\eta \frac{V^{-1/4} f_0}{\sqrt{V - f_0^2}}. \]  

We shall use the following equalities:
\[ d \left( \frac{\sqrt{V - f_0^2}}{V'} \right) = \frac{1}{2} \frac{V'}{\sqrt{V - f_0^2}} \] \[ d \left( \arctan \left( \frac{V - f_0^2}{V' f_0} \right) \right) = \frac{1}{2} \frac{V' f_0}{\sqrt{V - f_0^2}}. \]  

Making a partial integration, we find
\[ \tilde{E} = \int_{\eta_0}^{\infty} d\eta \frac{1}{2} \frac{V'}{\sqrt{V - f_0^2}} \frac{2V^{3/4}}{V'} \] \[ = \int_{\eta_0}^{\infty} d\eta \frac{d}{d\eta} \left( \frac{\sqrt{V - f_0^2}}{V'} \right) \frac{2V^{3/4}}{V'} \] \[ = - \int_{\eta_0}^{\infty} d\eta \sqrt{V - f_0^2} \left( \frac{2V^{3/4}}{V'} \right)' \left. + \left( \frac{\sqrt{V - f_0^2} 2V^{3/4}}{V'} \right) \right|_{\eta=\infty} - \left( \frac{\sqrt{V - f_0^2} 2V^{3/4}}{V'} \right) \right|_{\eta=\eta_0} \] \[ = - \int_{\eta_0}^{\infty} d\eta \sqrt{V - f_0^2} \left( \frac{2V^{3/4}}{V'} \right)' \left. + \left( \frac{\sqrt{V - f_0^2} 2V^{3/4}}{V'} \right) \right|_{\eta=\infty}. \]  

In the last equality we used \( V(\eta = \eta_0) = f(\eta_0)^2 = f_0^2 \). In the same manner, we find
\[ \tilde{R} = - \int_{\eta_0}^{\infty} d\eta \left( \arctan \left( \frac{\sqrt{V - f_0^2}}{f_0} \right) \right) \left( \frac{2V^{3/4}}{V'} \right)' \left. + \left( \frac{\sqrt{V - f_0^2} 2V^{3/4}}{V'} \right) \right|_{\eta=\infty}. \]  

Comparing (12) with (13), we find
\[ \frac{d\tilde{R}}{df_0} = \frac{1}{f_0} \frac{d\tilde{E}}{d\tilde{R}}. \]  

This is equivalent to
\[ f_0 = \frac{d\tilde{E}}{d\tilde{R}}, \]  

which proves (4).

Next, we shall prove (5). We rewrite \( \tilde{R} \) in (10) as follows:
\[ \tilde{R}(f_0) = \int_{f_0}^{f(\eta=\infty)} df \left[ \frac{df}{f} \left( \frac{f^{-1/2}}{\sqrt{f^2 - f_0^2}} \right) \right] \] \[ = \int_{f_0}^{f(\eta=\infty)} df \left[ \frac{df}{f} \right] \frac{f_0}{\sqrt{f^2 - f_0^2}}. \]  

Then, using the following formula
\[ F(x) = \int_{x}^{0} dt y(t) \frac{x}{\sqrt{t^2 - x^2}} \Rightarrow y(t) = \frac{-2}{\pi} \frac{d}{dt} \int_{t}^{a} dx \frac{F(x)}{\sqrt{x^2 - t^2}}, \]  

which can be proven by just substituting the first expression \( F(x) \) to the last integral, the equation (16) can be inverted and we find
\[ \frac{d\eta(f_0)}{df} f^{-1/2} = -\frac{2}{\pi} \frac{d}{df} \int_{f}^{f(\eta=\infty)} df_0 \frac{\tilde{R}(f_0)}{\sqrt{f_0^2 - f^2}}. \]  

Using \( f(\infty) = \infty \), this proves (5).
Next, let us provide a proof of the formulas (6), (7) and (8) of the non-zero temperature case. The proof for (7) is exactly the same as the zero temperature case above. We simply replace $f$ by $g$, then (7) with the first equation of (6) is proven. Next, to prove (8), we define

$$V \equiv f(\eta)g(\eta)$$

and rewrite $\tilde{E}$ and $\tilde{R}$ as

$$\tilde{E} = \int_{\eta_0}^{\infty} d\eta \frac{1}{\sqrt{g(\eta)}} \frac{V(\eta)}{\sqrt{V(\eta) - V(\eta_0)}}, \quad \tilde{R} = \int_{\eta_0}^{\infty} d\eta \frac{1}{\sqrt{g(\eta)}} \frac{\sqrt{V(\eta_0)}}{\sqrt{V(\eta) - V(\eta_0)}}. \quad (20)$$

Then, putting $V(\eta_0) \equiv h_0^2$, and using exactly the same trick of partial integration, we find

$$h_0 = \frac{d\tilde{E}}{d\tilde{R}}$$

which is the second equation of (6). Writing $h(\eta) \equiv \sqrt{V(\eta)}$, (20) can be written as

$$\tilde{R} = \int_{\eta_0}^{\infty} d\eta \frac{1}{\sqrt{g(\eta)}} \frac{h_0}{\sqrt{h(\eta)^2 - h_0^2}}. \quad (22)$$

Changing the integration variable to $h$ and using the inversion formula (17) gives

$$\frac{dh(h)}{dh} \frac{1}{\sqrt{g(\eta(h))}} = \frac{-2}{\pi} \frac{d}{dh} \int_h^{h(\eta = \infty)} dh_0 \frac{\tilde{R}(h_0)}{\sqrt{h_0^2 - h^2}}, \quad (23)$$

which is nothing but (8).

3. Rebuilding near horizon geometries

In this section, to demonstrate the applicability of our formulas (5) and (4), we adopt, as the input Wilson loop, a simple power-law potential

$$E(R) = -\frac{c}{2\pi\alpha'} \frac{1}{R^{n-1}} \quad (24)$$

with $n \geq 2$ and with a positive constant $c$. In the following we find that our formulas build BPS near-horizon geometries of Dp-branes (except for the internal space part).

Let us start with the case of $n = 2$,

$$E(R) = -\frac{c}{2\pi\alpha'} \frac{1}{R}. \quad (25)$$

This corresponds to Wilson loops of conformal field theories (CFTs) at zero temperature, because of the scaling symmetry. We will see that the formula (5) with (4) builds an AdS
We substitute this to the right hand side of (5), then changing the variable of integration to \( x = f_0 / f \) gives
\[
\frac{1}{\pi} \sqrt{f} \frac{d}{df} \int_0^f df_0 \frac{R(f_0)}{\sqrt{f_0^2 - f^2}} = \frac{1}{\pi} \sqrt{f} \frac{d}{df} \frac{\sqrt{c}}{\sqrt{f}} \int_0^1 dx \frac{1}{\sqrt{x(x^2 - 1)}} = \frac{\Gamma(5/4)}{\Gamma(3/4)\sqrt{\pi}} \sqrt{c} \frac{1}{f} .
\] (27)
Thus integrating the differential equation (5) gives
\[
f(\eta) = \exp \left[ \frac{2}{L} (\eta - \text{const.}) \right]
\] (28)
with
\[
L = \frac{2\Gamma(5/4)\sqrt{c}}{\Gamma(3/4)\sqrt{\pi}} .
\] (29)
This (28) is nothing but an AdS spacetime, with \( L \) being its AdS radius. Note that this rebuilt AdS is not necessarily five-dimensional. Starting with the conformal potential (25) in any CFT in \( d \) dimensions, an AdS\(_{d+1} \) spacetime is obtained in the bulk.

As a trivial check of our formulas, we look at our AdS radius (29) at \( d = 4 \). Using the explicitly known result
\[
E = -\frac{2\sqrt{2}\Gamma(3/4)^2}{\Gamma(1/4)^2} \sqrt{\lambda} \frac{1}{R}
\] (30)
for the Wilson loop [36] evaluated by an AdS\(_5 \) through the AdS/CFT correspondence, we calculate the right hand side of (29), to find
\[
L = (2\lambda)^{1/4} \sqrt{\alpha'} .
\] (31)
This is the standard AdS/CFT dictionary, relating the AdS radius and \( \alpha' \) with the ’t Hooft coupling \( \lambda \) of the \( \mathcal{N} = 4 \) supersymmetric Yang-Mills theory in 4 dimensions.

Next, let us consider the general case (24) with \( n > 2 \). Substituting it to (4), we find \( R \propto f_0^{-1/n} \). Then, (5) gives
\[
\frac{d\eta(f)}{df} \propto f^{-1/n - 1/2} .
\] (32)
When \( n > 2 \), integrating this finds \( (\eta(f) - \text{const.}) \propto f^{-1/n + 1/2} \), which gives the bulk metric
\[
f(\eta) \propto (\eta - \text{const.})^{2n/(n - 2)} .
\] (33)
So, we conclude that the power-law quark potential (24) for \( n > 2 \) leads to a power-law \( f(\eta) \) of the bulk geometry, and for \( n = 2 \) it leads to an AdS spacetime.

The power-law behavior of \( f(\eta) \) is popularly known for the near horizon geometry of Dp-branes [44],
\[
ds^2 = \eta^{2(7-p)/p - 2} \left( -dt^2 + \sum_{i=1}^p (dx^i)^2 \right) + d\eta^2 + \text{(internal space)} ,
\] (34)
where we ignored numerical coefficients. If we employ this expression, the relation between the power \( n \) of the quark potential and the spatial dimensionality \( p \) of the QFT is found as \( n = (7 - p)/(5 - p) \). This relation is the one obtained in the holographic Wilson loop calculations in [45], so it serves as another check of our formulas.
4. Theorem: confinement leads to a bulk IR bottom

In [46], a beautiful theorem about the IR behavior of the metric \( f(\eta) \) and confinement was provided. It states that when there exists an “IR bottom” \( \eta = 0 \) such that \( f(\eta) \) can be expanded around it as \( f(\eta) = f(0) + a\eta^k \) with \( f(0) > 0 \) and \( k > 0 \), the quark potential shows confinement, \( E(R) \propto R \) at large \( R \).

Here, using the formula (5) with (4) at the zero-temperature case, we provide the inverse theorem of it: when the quark potential shows confinement \( E(R) \propto R \) at large \( R \), the bulk needs to have its IR bottom.

**Theorem 1.** Assume the linear confinement: at large \( R \), the quark potential is given by

\[
\frac{dE(R)}{dR} = \sigma + \frac{c}{R^n} + \text{(higher in } 1/R \text{)}.
\]  

Here \( \sigma > 0 \) is the confining string tension. The second term (with \( c > 0 \) and \( n > 0 \)) is the leading correction.

Then the bulk metric function \( f(\eta) = g(\eta) \) in (1) has an IR bottom: \( f(\eta) \) approaches a minimum \( f = 2\pi\alpha'\sigma \) at which the gradient \( df/d\eta \) vanishes. The location of the IR bottom in the \( \eta \) coordinate is

- at a finite value of \( \eta \), when \( n > 2 \) (or when the correction vanishes faster than the power-law).
- at \( \eta = -\infty \), when \( 2 \geq n > 0 \).

**Proof.** The confinement condition (35), in which we safely ignore the higher terms for large \( R \), is substituted to (4) to give

\[
R(f_0) = \left( \frac{f_0 - \tilde{\sigma}}{\tilde{c}} \right)^{-\frac{1}{n}},
\]  

where \( \tilde{\sigma} \equiv 2\pi\alpha'\sigma \) and \( \tilde{c} \equiv 2\pi\alpha'c \). This is substituted to the right hand side of (5), and is evaluated, with the change of the integration variable \( f_0 = fx \), as

\[
\frac{d\eta(f)}{df} = \frac{1}{\pi} \sqrt{\tilde{f}} \frac{d}{df} \int_\infty^1 dx \frac{\tilde{c}^{\frac{1}{n}}}{(fx - \tilde{\sigma})^{\frac{1}{2} - \frac{1}{n}}} \sqrt{x^2 - 1}
\]

\[
= \tilde{c}^{\frac{1}{n}} \int_1^\infty dx \frac{x}{\left(x - \frac{\tilde{\sigma}}{f}\right)^{\frac{1}{2} + \frac{1}{n}}} \sqrt{x^2 - 1}
\]

\[
\cong \tilde{c}^{\frac{1}{n}} \frac{1}{n\pi} f^{-\frac{1}{2} - \frac{1}{n}} \int_1^\infty \frac{1}{\left(1 - \frac{\tilde{\sigma}}{f}\right)^{\frac{1}{2} + \frac{1}{n}}} \frac{\sqrt{\pi/2} \Gamma\left(\frac{1}{2} + \frac{1}{n}\right)}{\Gamma(1 + \frac{1}{n})}
\]

\[
= \frac{\tilde{c}^{\frac{1}{n}}}{n\sqrt{2\pi}} \frac{\Gamma\left(\frac{1}{2} + \frac{1}{n}\right)}{\Gamma(1 + \frac{1}{n})} (f - \tilde{\sigma})^{-\frac{1}{2} - \frac{1}{n}}.
\]  

\( \tilde{c} \) is positive, since normally the large-\( R \) quark potential \( E(R) \) approaches its linear behavior from below, as it is consistent with the short-distance behavior.
In “≃” above, we have used \( f \simeq \tilde{\sigma} \) since \( R \) is sufficiently large in (36). The last expression is integrated to give

\[
\eta = \begin{cases} 
\sqrt{\frac{2\tilde{c}}{\pi}} \cdot \frac{\Gamma\left(\frac{1}{2} + \frac{1}{n}\right)}{(n-2)\sqrt{\pi} \Gamma(1 + \frac{1}{n})} (f - \tilde{\sigma})^{\frac{1}{2} - \frac{1}{n}} + \text{const.} & (n \neq 2) \\
\frac{\sqrt{\tilde{c}}}{\sqrt{2\pi}} \cdot \log(f - \tilde{\sigma}) + \text{const.} & (n = 2)
\end{cases}
\]

(38)

This is easily inverted to give \( f(\eta) \), as

\[
\eta = \begin{cases} 
\frac{\sqrt{2\tilde{c}}}{\pi} \cdot \frac{\Gamma\left(\frac{1}{2} + \frac{1}{n}\right)}{(n-2)\sqrt{\pi} \Gamma(1 + \frac{1}{n})} (f - \tilde{\sigma})^{\frac{1}{2} - \frac{1}{n}} - \text{const.} & (n \neq 2) \\
\frac{\sqrt{\tilde{c}}}{\sqrt{2\pi}} \cdot \log(f - \tilde{\sigma}) + \text{const.} & (n = 2)
\end{cases}
\]

(39)

where \( d_{n > 2} \) (\( d_{n < 2} \)) is a positive (negative) constant. The expression proves the theorem, and at the IR bottom of the bulk geometry, we find \( f = \tilde{\sigma} \).

The theorem is also true for the correction \( c/R^n \) replaced by \( \exp(-cR^m) \) with \( c > 0 \) and \( m > 0 \), that is, an exponentially fast approach to the linear confinement potential. The proof goes in the same manner as above, and we find

\[
\eta = e_m \sqrt{f - \tilde{\sigma}} \left( \log \frac{\tilde{\sigma}}{f - \tilde{\sigma}} \right)^{\frac{1}{m} - 1} + \text{const.}
\]

(40)

with a positive constant \( e_m \) which depends on \( c \) and \( m \). Inverting it results in the existence of the IR bottom \( f = \tilde{\sigma} \) at a finite bulk location in \( \eta \).

The famous holographic geometry showing the confinement is a doubly Wick-rotated black 4-brane solution [47]. The IR bottom of the geometry is located at a finite value of \( \eta \). In fact, the leading correction of the holographic Wilson loop using the geometry, evaluated in [48], is exponentially suppressed as \( \exp[-cR] \), thus it is consistent with our theorem.

5. Discussions

Various comments and discussions about our formulas and the theorem are in order.

Lüscher term. It would be very interesting to plug in lattice results of Wilson loops to our formulas, to obtain a bulk geometry. When substituting lattice results, there is a subtlety. The quark potential at large \( R \) in confining gauge theories has a \( 1/R \) correction called Lüscher term [49]. This correction stems from a quantum fluctuation of the Nambu-Goto string, and the effect in the gravity dual was evaluated in [50]. Note that our formulas are for a classical string in the bulk, so the Lüscher term is dropped. Because lattice results include the Lüscher term in general (and the Lüscher term is found to be rather consistently seen in lattice simulations [51]), in applying the simulation results to our formulas, the Lüscher term contribution needs to be subtracted in the lattice results. To precisely argue the quantum corrections, the relevance to the large \( N \) limit and the strong coupling limit may need to be worked out.\(^5\)

\(^5\)In fact, in the holographic setup of pure Yang-Mills theory which is dimensionally reduced from 5-dimensional supersymmetric Yang-Mills theory [47], the QCD scale and the QCD string tension are parametrically separate by the factor of \('t\) Hooft coupling, so the subtraction of the Lüscher term would be a tractable problem. However, in realistic Yang-Mills theory these two scales may not be distinguishable.
Unstable branch of string. In the formula (8) for the non-zero temperature case, note that the connected fundamental string corresponding to the temporal Wilson loop with the minimum energy does not reach the horizon [38, 39], therefore the formula (8) is not sufficient to build the temporal component of the metric near the horizon. The phenomena is the Debye screening of color charges, and in the bulk a pair of straight separate strings is favored. Our formulas work for a single curved string connecting the two boundary points, and in general, for a given black hole geometry, there exist two such configurations: one is the minimum energy configuration that is favored when $R$ is small. The other is a local maximum energy configuration that is always unfavored. The latter configuration is classical and probes the bulk region very close to the horizon (see [52] for its horizon behavior and the relevance to chaos), so it could be more appropriate for our purpose. When one wants to probe $f(\eta)$ of the region very close to the horizon, one needs this local maximum $E(R)$ which may be hidden in lattice data. This $E(R)$ is related to quantum string breaking and quantum chaos of Wilson loops, but the details on how to extract the local maximum $E(R)$ from lattice data requires some study.

Machine learning holography. For solving the inverse problem of building a bulk spacetime from QFT data, recently an effort has been put to utilize machine learning, where the holographic bulk spacetime is identified with neural networks [53–61]6. Among the work, the method which uses only QFT spectra or one-point function [54, 55, 58–61] would be suitable for lattice simulation data. A combination of the machine learning method and the results of the present paper would further constrain possible bulk geometries.

Distinguishing holographic QFTs from non-holographic QFTs. For a given QFT, it would be ideal if one can judge the existence of a gravity dual by just examining QFT observables obtained in lattice simulations. A protocol for getting a bulk spacetime geometry from Wilson loops is given in this paper, thus it would be desirable to calculate holographically some other physical observables with this geometry and compare them with lattice data of the same QFT. For example, non-rectangular Wilson loops (such as circular Wilson loops) would be the first choice, as the holographic computation of it needs only the string frame metric in the bulk. If the holographic result is different from the lattice result, then it means that there is no consistent bulk gravity dual. This kind of falsification test of the existence of the gravity dual will help developing study of possible borders between holographic QFTs and non-holographic QFTs.

References

[1] J. M. Maldacena, “The Large N limit of superconformal field theories and supergravity.” Int. J. Theor. Phys. 38 (1999), 1113-1133 doi:10.1023/A:1026531531582 [arXiv:hep-th/9711200 [hep-th]].
[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from noncritical string theory,” Phys. Lett. B 428 (1998), 105-114 doi:10.1016/S0370-2693(98)01373-9 [arXiv:hep-th/9802109 [hep-th]].
[3] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2 (1998), 253-291 doi:10.4310/ATMP.1998.v2.n2.a2 [arXiv:hep-th/9802150 [hep-th]].

6See [62] for a review of data science approach to string theory, and also see [63] for applications of machine learning to material sciences.
[4] S. de Haro, S. N. Solodukhin and K. Skenderis, “Holographic reconstruction of space-time
and renormalization in the AdS / CFT correspondence,” Commun. Math. Phys. 217 (2001), 595-622
doi:10.1007/s002200100381 [arXiv:hep-th/0002230 [hep-th]].

[5] J. Hammersley, “Extracting the bulk metric from boundary information in asymptotically AdS
spacetimes,” JHEP 12 (2006), 047 doi:10.1088/1126-6708/2006/12/047 [arXiv:hep-th/0609202 [hep-th]].

[6] V. E. Hubeny, H. Liu and M. Rangamani, “Bulk-cone singularities & signatures of horizon formation
in AdS/CFT,” JHEP 01 (2007), 009 doi:10.1088/1126-6708/2007/01/009 [arXiv:hep-th/0610041 [hep-th]].

[7] N. Engelhardt and G. T. Horowitz, “Towards a Reconstruction of General Bulk Metrics,” Class. Quant.
Grav. 34 (2017) no.1, 015004 doi:10.1088/1361-6382/34/1/015004 [arXiv:1605.01070 [hep-th]].

[8] N. Engelhardt and G. T. Horowitz, “Recovering the spacetime metric from a holographic dual,” Adv.
Theor. Math. Phys. 21 (2017), 1635-1653 doi:10.4310/ATMP.2017.v21.n7.a2 [arXiv:1612.00391 [hep-th]].

[9] R. S. Roy and D. Sarkar, “Bulk metric reconstruction from boundary entanglement,” Phys. Rev. D 98
(2018) no.6, 066017 doi:10.1103/PhysRevD.98.066017 [arXiv:1801.07280 [hep-th]].

[10] P. Burda, R. Gregory and A. Jain. “Holographic reconstruction of bubble spacetimes,” Phys. Rev. D
99 (2019) no.2, 026003 doi:10.1103/PhysRevD.99.026003 [arXiv:1804.05202 [hep-th]].

[11] S. Hernández-Cuenca and G. T. Horowitz, “Bulk reconstruction of metrics with a compact space
asymptotically,” [arXiv:2003.08409 [hep-th]].

[12] S. Ryu and T. Takayanagi, “Holographic derivation of entanglement entropy from AdS/CFT,”
Phys. Rev. Lett. 96 (2006), 181602 doi:10.1103/PhysRevLett.96.181602 [arXiv:hep-th/0603001 [hep-th]].

[13] S. Ryu and T. Takayanagi, “Aspects of Holographic Entanglement Entropy,” JHEP 08 (2006), 045
doi:10.1088/1126-6708/2006/08/045 [arXiv:hep-th/0605073 [hep-th]].

[14] J. Hammersley, “Numerical metric extraction in AdS/CFT,” Gen. Rel. Grav. 40 (2008), 1619-1652
doi:10.1007/s10714-007-0564-6 [arXiv:0705.0159 [hep-th]].

[15] S. Bilson, “Extracting spacetimes using the AdS/CFT conjecture,” JHEP 08 (2008), 073
doi:10.1088/1126-6708/2008/07/073 [arXiv:0807.3695 [hep-th]].

[16] S. Bilson, “Extracting Spacetimes using the AdS/CFT Conjecture: Part II,” JHEP 02 (2011), 050
doi:10.1007/JHEP02(2011)050 [arXiv:1012.1812 [hep-th]].

[17] V. E. Hubeny, “Extremal surfaces as bulk probes in AdS/CFT,” JHEP 07 (2012), 093
doi:10.1007/JHEP07(2012)093 [arXiv:1203.1044 [hep-th]].

[18] B. Balasubramanian, B. Czech, B. D. Chowdhury and J. de Boer, “The entropy of a hole in spacetime,”
JHEP 10 (2013), 220 doi:10.1007/JHEP10(2013)220 [arXiv:1305.0856 [hep-th]].

[19] V. Balasubramanian, B. D. Chowdhury, B. Czech, J. de Boer and M. P. Heller, “Bulk curves from
boundary data in holography,” Phys. Rev. D 89 (2014) no.8, 086004 doi:10.1103/PhysRevD.89.086004
[arXiv:1310.4204 [hep-th]].

[20] R. C. Myers, J. Rao and S. Sugishita, “Holographic Holes in Higher Dimensions,” JHEP 06 (2014), 044
doi:10.1007/JHEP06(2014)044 [arXiv:1403.3416 [hep-th]].

[21] B. Czech, X. Dong and J. Sully, “Holographic Reconstruction of General Bulk Surfaces,” JHEP 11
(2014), 015 doi:10.1007/JHEP11(2014)015 [arXiv:1406.4889 [hep-th]].

[22] B. Czech and L. Lamprou, “Holographic definition of points and distances,” Phys. Rev. D 90 (2014),
106005 doi:10.1103/PhysRevD.90.106005 [arXiv:1409.4473 [hep-th]].

[23] S. A. Gentle and C. Keeler, “On the reconstruction of Lifshitz spacetimes,” JHEP 03 (2016), 195
doi:10.1007/JHEP03(2016)195 [arXiv:1512.04538 [hep-th]].

[24] M. Freedman and M. Headrick, “Bit threads and holographic entanglement,” Commun. Math. Phys.
352 (2017) no.1, 407-438 doi:10.1007/s00220-016-2796-3 [arXiv:1604.00354 [hep-th]].

[25] A. Saha, S. Karar and S. Gangopadhyay, “Bulk geometry from entanglement entropy of CFT,” Eur.
Phys. J. Plus 135 (2020) no.2, 132 doi:10.1140/epjp/s13360-020-00110-7 [arXiv:1807.04646 [hep-th]].

[26] N. Bao, C. Cao, S. Fischetti and C. Keeler, “Towards Bulk Metric Reconstruction from Extremal
Area Variations,” Class. Quant. Grav. 36 (2019) no.18, 185002 doi:10.1088/1361-6382/ab377f
[arXiv:1904.04834 [hep-th]].

[27] C. Cao, X. L. Qi, B. Swingle and E. Tang, “Building Bulk Geometry from the Tensor Radon Transform,”
[arXiv:2007.00004 [hep-th]].

[28] N. Jokela and A. Pönni, “Towards precision holography,” [arXiv:2007.00010 [hep-th]].

[29] C. A. Agón, E. Cáceres and J. F. Pedraza, “Bit threads, Einstein’s equations and bulk locality,”
[arXiv:2007.07907 [hep-th]].

[30] B. Swingle, “Entanglement Renormalization and Holography,” Phys. Rev. D 86 (2012), 065007
doi:10.1103/PhysRevD.86.065007 [arXiv:0905.1317 [cond-mat.str-el]].

[31] F. Pastawski, B. Yoshida, D. Harlow and J. Preskill, “Holographic quantum error-correcting codes: Toy
models for the bulk/boundary correspondence,” JHEP 06 (2015), 149 doi:10.1007/JHEP06(2015)149
[arXiv:1503.06257 [hep-th]].

[32] B. Swingle, “Constructing holographic spacetimes using entanglement renormalization,”
[arXiv:1209.3304 [hep-th]].
[33] N. Bao, G. Penington, J. Sorce and A. C. Wall, “Beyond Toy Models: Distilling Tensor Networks in Full AdS/CFT,” JHEP 19 (2020), 069 doi:10.1007/JHEP11(2019)069 [arXiv:1812.01171 [hep-th]].

[34] A. Milsted and G. Vidal, “Geometric interpretation of the multi-scale entanglement renormalization ansatz,” [arXiv:1812.00529 [hep-th]].

[35] N. Bao, G. Penington, J. Sorce and A. C. Wall, “Holographic Tensor Networks in Full AdS/CFT,” [arXiv:1902.10157 [hep-th]].

[36] J. M. Maldacena, “Wilson loops in large N field theories,” Phys. Rev. Lett. 80 (1998), 4859-4862 doi:10.1103/PhysRevLett.80.4859 [arXiv:hep-th/9803002 [hep-th]].

[37] S. J. Rey and J. T. Yee, “Macroscopic strings as heavy quarks in large N gauge theory and anti-de Sitter supergravity,” Eur. Phys. J. C 22 (2001), 379-394 doi:10.1007/s100520100799 [arXiv:hep-th/9803001 [hep-th]].

[38] A. Brandhuber, N. Itzhaki, J. Sonnenschein and S. Yankielowicz, “Wilson loops in the large N limit at finite temperature,” Phys. Lett. B 434 (1998), 36-40 doi:10.1016/S0370-2693(98)00730-8 [arXiv:hep-th/9803137 [hep-th]].

[39] S. J. Rey, S. Theisen and J. T. Yee, “Wilson-Polyakov loop at finite temperature in large N gauge theory and anti-de Sitter supergravity,” Nucl. Phys. B 527 (1998), 171-186 doi:10.1016/S0550-3213(98)00471-4 [arXiv:hep-th/9803135 [hep-th]].

[40] V. Balasubramanian, A. Bernamonti, J. de Boer, N. Copland, B. Craps, E. Keski-Vakkuri, B. Muller, A. Schafer, M. Shigemori and W. Staessens, “Thermalization of Strongly Coupled Field Theories,” Phys. Rev. Lett. 106 (2011), 191601 doi:10.1103/PhysRevLett.106.191601 [arXiv:1012.4753 [hep-th]].

[41] V. Balasubramanian, A. Bernamonti, J. de Boer, N. Copland, B. Craps, E. Keski-Vakkuri, B. Muller, A. Schafer, M. Shigemori and W. Staessens, “Holographic Thermalization,” Phys. Rev. D 84 (2011), 026010 doi:10.1103/PhysRevD.84.026010 [arXiv:1103.2683 [hep-th]].

[42] U. Kol, C. Nunez, D. Schofield, J. Sonnenschein and M. Warschawski, “Confinement, Phase Transitions and non-Locality in the Entanglement Entropy,” JHEP 06 (2014), 005 doi:10.1007/JHEP06(2014)005 [arXiv:1403.2721 [hep-th]].

[43] N. Bao, A. Chatwin-Davies, B. E. Niehoff and M. Usatyuk, “Bulk Reconstruction Beyond the Entanglement Wedge,” Phys. Rev. D 101 (2020) no.6, 066011 doi:10.1103/PhysRevD.101.066011 [arXiv:1911.00519 [hep-th]].

[44] N. Itzhaki, J. M. Maldacena, J. Sonnenschein and S. Yankielowicz, “Supergravity and the large N limit of theories with sixteen supercharges,” Phys. Rev. D 58 (1998), 046004 doi:10.1103/PhysRevD.58.046004 [arXiv:hep-th/9802042 [hep-th]].

[45] A. Brandhuber, N. Itzhaki, J. Sonnenschein and S. Yankielowicz, “Wilson loops, confinement, and phase transitions in large N gauge theories from supergravity,” JHEP 06 (1998), 001 doi:10.1088/1126-6708/1998/06/001 [arXiv:hep-th/9803263 [hep-th]].

[46] Y. Kinar, E. Schreiber and J. Sonnenschein, “Quantum anti-Q potential from strings in curved space-time: Classical results,” Nucl. Phys. B 566 (2000), 103-125 doi:10.1016/S0550-3213(99)00652-5 [arXiv:hep-th/9811192 [hep-th]].

[47] E. Witten, “Anti-de Sitter space, thermal phase transition, and confinement in gauge theories,” Adv. Theor. Math. Phys. 2 (1998), 505-532 doi:10.4310/ATMP.1998.v2.n3.a3 [arXiv:hep-th/9803131 [hep-th]].

[48] J. Greensite and P. Olesen, “Remarks on the heavy quark potential in the supergravity approach,” JHEP 08 (1998), 009 doi:10.1088/1126-6708/1998/08/009 [arXiv:hep-th/9806235 [hep-th]].

[49] M. Luscher, “Symmetry Breaking Aspects of the Roughening Transition in Gauge Theories,” Nucl. Phys. B 180 (1981), 317-329 doi:10.1016/0550-3213(81)90423-5

[50] Y. Kinar, E. Schreiber, J. Sonnenschein and N. Weiss, “Quantum fluctuations of Wilson loops from string models,” Nucl. Phys. B 583 (2000), 76-104 doi:10.1016/S0550-3213(00)00238-8 [arXiv:hep-th/9911123 [hep-th]].

[51] M. Luscher and P. Weisz, “Quark confinement and the bosonic string,” JHEP 07 (2002), 049 doi:10.1088/1126-6708/2002/07/049 [arXiv:hep-lat/0207003 [hep-lat]].

[52] K. Hashimoto, K. Murata and N. Tanahashi, Phys. Rev. D 98 (2018) no.8, 086007 doi:10.1103/PhysRevD.98.086007 [arXiv:1803.06756 [hep-th]].

[53] Y. Z. You, Z. Yang and X. L. Qi, “Machine Learning Spatial Geometry from Entanglement Features,” Phys. Rev. B 97 (2018) no.4, 045153 doi:10.1103/PhysRevB.97.045153 [arXiv:1709.01223 [cond-mat.dis-nn]].

[54] K. Hashimoto, S. Sugishita, A. Tanaka and A. Tomiya, “Deep learning and the AdS/CFT correspondence,” Phys. Rev. D 98 (2018) no.4, 046019 doi:10.1103/PhysRevD.98.046019 [arXiv:1802.08313 [hep-th]].

[55] K. Hashimoto, S. Sugishita, A. Tanaka and A. Tomiya, “Deep Learning and Holographic QCD,” Phys. Rev. D 98 (2018) no.10, 106014 doi:10.1103/PhysRevD.98.106014 [arXiv:1809.10536 [hep-th]].

[56] H. Y. Hu, S. H. Li, L. Wang and Y. Z. You, “Machine Learning Holographic Mapping by Neural Network Renormalization Group,” Phys. Rev. Res. 2 (2020) no.2, 023369 doi:10.1103/PhysRevResearch.2.023369

11/12
