Three pion nucleon coupling constants

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There exist four pion nucleon coupling constants, \( f_{\pi^0_{pp}} \), \( f_{\pi^0_{nn}} \), \( f_{\pi^+_{pn}}/\sqrt{2} \) and \( f_{\pi^-_{np}}/\sqrt{2} \) which coincide when up and down quark masses are identical and the electron charge is zero. While there is no reason why the pion-nucleon-nucleon coupling constants should be identical in the real world, one expects that the small differences might be pinned down from a sufficiently large number of independent and mutually consistent data. Our discussion provides a rationale for our recent determination
\[ f_{\pi^0_{pp}} = 0.0759(4), \quad f_{\pi^0_{nn}} = 0.079(1), \quad f_{\pi^+_{pn}}/\sqrt{2} = 0.0763(6), \]

based on a partial wave analysis of the 3\( \sigma \) self-consistent nucleon-nucleon Granada-2013 database comprising 6713 published data in the period 1950-2013.

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1. Introduction

Four score and a year ago Yukawa brought forth a new theory of nuclear forces and dedicated to the proposition that protons and neutrons exchange pions\(^\text{[1]}\). But

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created and annihilated pions are all not equal. The quest for isospin violations in particle and nuclear physics has been a permanent goal ever since Kemmer \cite{Kemmer} invented the concept and generalized the Pauli principle. A readable account of the early developments can be found in Ref. \cite{Ref3}. Actually, the neutral pion was sought and found because isospin symmetry required it. While the mass of the pion may be deduced directly from their decays $\pi^0 \rightarrow \gamma \gamma$ and $\pi^\pm \rightarrow \mu^\pm + \nu$, the determination of the coupling constant to nucleons is more intricate and needs further theoretical elaboration. Although this is not a fundamental constant of QCD, the pion-nucleon-nucleon coupling constant is the strong hadronic charge of neutrons and protons, which appear as the effective constituents of atomic nuclei.

In 1940 Bethe obtained the value $f^2 = 0.077 - 0.080$ from the study of deuteron properties \cite{Bethe}. Subsequent determinations based on a variety of processes can be traced from recent compilations \cite{Ref5,Ref6}. Attempts to make a microscopic distinction via radiative vertex corrections aiming at predictive power have been made in the past (see \cite{Ref7} for pre-QCD account). At the hadronic level and within the meson exchange picture there have been many attempts to determine the many possible causes of isospin breaking, $\rho^0 - \omega$ and $\pi - \eta$ mixing, pion mass differences in the two-pion exchange interaction, $\pi \gamma$ exchange, etc. (see e.g. \cite{Ref8,Ref9} for post QCD comprehensive accounts). All these many complications contribute to the belief that isospin violations in strong interactions remains one of the least understood issues in the nuclear force.

In this contribution we provide our point of view on this subject and the elements underlying ongoing work on the determination of the three coupling constants. Although many of the issues we address here are well known for the experts after 60 years of NN Partial Wave Analysis fits, we try in this short account to be pedagogical for the non-experts. For comprehensive and concise reviews we suggest Ref. \cite{Ref5,Ref6} where the latest NN based determination $f^2 = 0.0750(9)$ is recommended.

2. Charge Dependent One Pion Exchange (CD-OPE)

The coupling constant is rigorously defined as the $\pi NN$ three point vertex function when all three particles are on the mass shell, a condition that cannot generally be satisfied for real momenta. At the hadronic level the pion-nucleon-nucleon vertex is described by the Lagrangian (we use the standard convention \cite{Ref10})

$$\mathcal{L} = \frac{f_{eab}}{m_{\pi^+}} \bar{B}_a \gamma^\mu \gamma_5 B_0 \partial_\mu \phi_e,$$

where $\bar{B}_a, B_0$ and $\phi_e$ are antibaryon, baryon and pseudoscalar meson field respectively and $\gamma^\mu$ and $\gamma_5$ are Dirac matrices. This yields four possible vertices

$$p \rightarrow \pi^+ n, \quad n \rightarrow \pi^- p, \quad p \rightarrow \pi^0 p, \quad n \rightarrow \pi^0 n,$$

so their amplitudes are

$$\mathcal{A}(p \rightarrow \pi^+ n) = f_{\pi^+ np} \bar{u}_n(p_n, s_n) \gamma^\mu \gamma_5 u_p(p_p, s_p) q_\mu,$$

$$\mathcal{A}(n \rightarrow \pi^- p) = f_{\pi^- np} \bar{u}_p(p_p, s_p) \gamma^\mu \gamma_5 u_n(p_n, s_n) q_\mu,$$

$$\mathcal{A}(p \rightarrow \pi^0 p) = f_{\pi^0 pp} \bar{u}_p(p_p, s_p) \gamma^\mu \gamma_5 u_p(p_p, s_p) q_\mu,$$

$$\mathcal{A}(n \rightarrow \pi^0 n) = f_{\pi^0 nn} \bar{u}_n(p_n, s_n) \gamma^\mu \gamma_5 u_n(p_n, s_n) q_\mu.$$
Fig. 1. Feynman diagrams contributing to the charge dependent one pion exchange interaction. The couplings are assumed to be in the isospin limit $g = g_{\pi^0 pp} = g_{\pi^+ np}/\sqrt{2} = -g_{\pi^0 nn}$.

and so on. The relevant relationships between the pseudoscalar pion coupling constants, $g_{\pi NN}$, and the pseudovector ones, $f_{\pi NN}$, are given by

$$
\frac{g_{\pi^0 pp}}{\sqrt{4\pi}} = 2M_p f_{\pi^0 pp}, \quad \frac{g_{\pi^0 nn}}{\sqrt{4\pi}} = 2M_n f_{\pi^0 nn}, \quad \frac{g_{\pi^+ pn}}{\sqrt{4\pi}} = \frac{M_p + M_n}{m_{\pi^+}} f_{\pi^+ pn}.
$$

(4)

with $M_p = 938.272$ MeV the proton mass, $M_n = 939.566$ MeV the neutron mass, and $m_{\pi^\pm} = 139.570$ MeV the mass of the charged pion. From these vertices one may obtain the NN scattering amplitude to lowest order in perturbation theory, see Fig. 1. For instance, the unpolarized differential $nn$-cross section reads

$$
d\sigma_{nn} = \frac{g_{\pi^0 nn}^4}{32\pi s(s - 4M^2_n)} \left[ \frac{t^2}{(t - m_{\pi^0}^2)^2} - \frac{2\left(\frac{t}{2} - M_n^2\right)^2 - \left(\frac{t}{2} - M_n^2\right)\left(\frac{u}{2} - M_n^2\right)}{(t - m_{\pi^0}^2)(u - m_{\pi^0}^2)} \right] + (u \leftrightarrow t),
$$

(5)

where $s = 4(p^2 + M_n^2)$, $t = -q^2 = -4p^2\sin(\theta/2)^2 < 0$ and $s + t + u = 4M_n^2$ with $(p, \theta)$ the CM momentum and scattering angle respectively. This is perhaps the most straightforward way of checking OPE from data, i.e. extrapolating $(t - m_{\pi^0}^2)d\sigma/dt$ from the physical $t < 0$ kinematics to the unphysical limit $t \rightarrow m_{\pi^0}^2$. Unfortunately, the Born approximation violates elastic unitarity and there are many ways to restore it. For reasons to be explained below we use the phenomenological potential approach and a partial wave expansion analysis \[\text{[1]}\]. The concept of a potential is essentially non-relativistic and the procedure to obtain it is to match perturbatively the non-relativistic quantum mechanical potential to the same scattering amplitude obtained in quantum field theory for the direct term, namely

$$
f_{\text{QFT}}^{\text{Born}}(\theta, E) = -\frac{2\mu}{4\pi} \int d^3 x e^{-ik' \cdot \vec{x}} V(\vec{x}, \vec{p}) e^{ik \cdot \vec{x}},
$$

(6)

where the on-shell condition is understood $k' = k = p$. This already incorporates an ambiguity, since one may add terms which vanish on the mass-shell. In the static limit of heavy nucleons the ambiguity dissapears and the potential deduced from
field theory becomes local, which is of great advantage since we obtain a Schrödinger equation. The CD-OPE potential in the pp and np channels so obtained reads

\[ V_{pp \rightarrow pp}(r) = f_{\pi}^2 V_{m,OPE}(r), \]
\[ V_{np \rightarrow np}(r) = V_{pn \rightarrow pn}(r) = f_{\pi}^2 V_{m,OPE}(r), \]
\[ V_{nn \rightarrow nn}(r) = f_{\pi}^2 V_{m,OPE}(r), \]

where \( V_{m,OPE} \) is given by

\[ V_{m,OPE}(r) = \left( \frac{m}{m_{\pi}} \right)^2 \frac{1}{3} \frac{e^{-mr}}{r} \left\{ \sigma_1 \cdot \sigma_2 + \left( 1 + \frac{3}{mr} + \frac{3}{(mr)^2} \right) S_{12} \right\}, \]

with \( \sigma_1 \) and \( \sigma_2 \) the single nucleon Pauli matrices and \( S_{12} = 3\sigma_1 \cdot \hat{r}\sigma_2 \cdot \hat{r} - \sigma_1 \cdot \sigma_2 \) the tensor operator. Using the standard notation we make the identifications

\[ f_{\pi}^2 = f_{\pi}^0 f_{\pi}^0, \quad f_{\pi}^2 = f_{\pi}^0 f_{\pi}^0, \quad 2f_{\pi}^2 = f_{\pi}^0 f_{\pi}^0. \]

In many derivations of the OPE potential some emphasis is placed on the contact and singular piece which is proportional to a Dirac delta function located at the origin. We omit them here, since they will play no role in our subsequent discussion.

3. The static nuclear potential

The saturation property of nuclei suggests that there is an equilibrium distance between two nucleons. While one talks about nuclear forces, the truth is that they have never been measured directly in experiment. From a purely classical viewpoint, this would require to pull two nucleons apart at distances larger than their size and measure the necessary force, similarly as Coulomb and Cavendish did for the electric and gravitational forces 250 years ago. For such an ideal experiment the behaviour of the system at shorter distances would be largely irrelevant. This situation would naturally occur if nucleons were truly infinitely heavy sources of baryon charge. From a fundamental point of view the nucleon force is defined from the static energy between two nucleons which in QCD are made of three quarks and any number of quark-antiquark pairs and gluons. Nucleons and pions are composite and extended particles which can be characterized by any gauge invariant combination of interpolating fields with the proper quantum numbers. This generates some ambiguity except for heavy quarks. In any case the static energy reads

\[ E_{NN}(R) = 2M_N + \sum_{q \in N,q' \in N'} V_{q,q'}(\vec{x}_q - \vec{x}_{q'}). \]

Here, the quarks in each nucleon are located at the same point \( \vec{x}_q = \vec{R}/2 \) and \( \vec{x}_{q'} = -\vec{R}/2 \). On the lattice static baryon sources have been placed at a fixed distance and in fact there exist lattice calculations addressing the pion-nucleon coupling constant. Momentum dependence of the strong form factor and coupling constant determination in lattice QCD gives \( g_{\pi NN} \sim 10 - 12 \) and more recently \( g_{\pi NN} \sim \)
13(1) for \( m_\pi \sim 560\text{MeV} \) using QCD sum rules. On the lattice in the quenched approximation it has been found that for a pion mass of \( m_\pi = 380 \text{MeV} \) the value \( g_{\pi NN}^2/(4\pi) = 12.1 \pm 2.7 \) which is encouraging. These are courageous efforts which still are far from the 1% accuracy needed for witnessing isospin breaking in the couplings (see below).

Some intuition may be gathered from a chiral quark model picture. At scales above the confinement radius we expect exchange of purely hadronic states and at long distances the OPE mechanism will dominate, but because the quarks in each nucleon are on top of each other, they will contribute coherently to the couplings. Of course, \( u \) and \( d \) quarks are not heavy, but spontaneous chiral symmetry breaking will provide them with a constituent mass \( M_0 \) which will give a total mass \( M_q = M_0 + m_q \). Thus, neglecting em contributions, the proton and neutron masses are \( M_p = 3M_0 + 2m_u + m_d \) and \( M_n = 3M_0 + m_u + 2m_d \), so that \( M_p - M_n = m_d - m_u \), a reasonable value. Within this picture, relative corrections \( \delta g/g \) at the nucleon or quark level are the same. The \( \pi qq \) coupling is the residue of the Bethe-Salpeter equation at the pion pole. In a model where the pion is composite such as the NJL model (see e.g. for a review) one has for \( m_u, m_d \ll M_0 \) (\( \Lambda \) is the NJL cut-off)

\[
\delta g|_{\pi qq} = A(M_0, \Lambda) \frac{m_q + m_\bar{q}}{M_0} + \ldots
\]

with \( A(M_0, \Lambda) > 0 \) so the coupling to the pion grows with the quark mass. Thus

\[
\frac{\delta g}{g}|_{\pi^0 nn} > \frac{\delta g}{g}|_{\pi^+ np} > \frac{\delta g}{g}|_{\pi^0 pp}.
\]

4. Scattering and analytical properties

Unfortunately we cannot carry out the classical Cavendish experiment for nucleons in the laboratory. We may instead analyze the about 8000 pp and np scattering data collected in accelerators below LAB energy 350MeV in the period 1950-2013. Quantum mechanically wave-particle duality implies a finite wavelength resolution for a given relative momentum, \( \Delta r \sim 1/p \). This will allow to sample the potential at coarse grained distance scales, \( r_n \sim n\Delta r \). Fluctuations of the interaction below that scale will be unobservable and manifest as correlations among the interaction at the sampled points \( V(r_n) \). The “force” is defined as the average change of \( V(r_n) \) over the resolution scale \( \Delta r \), for instance \( F(r_n) = -(V(r_n + \Delta r/2) - V(r_n - \Delta r/2))/\Delta r \).

The NN scattering amplitude has five independent complex components which are a function of energy and scattering angle,

\[
M = a + m(\sigma_1 \cdot n)(\sigma_2 \cdot m) + (g - h)(\sigma_1 \cdot m)(\sigma_2 \cdot m)
+ (g + h)(\sigma_1 \cdot l)(\sigma_2 \cdot l) + c(\sigma_1 + \sigma_2) \cdot n.
\]

We use the three unit vectors \( k_f \) and \( k_i \) are relative final and initial momenta,

\[
l = \frac{k_f + k_i}{|k_f + k_i|}, \quad m = \frac{k_f - k_i}{|k_f - k_i|}, \quad n = \frac{k_f \wedge k_i}{|k_f \wedge k_i|}.
\]
Fig. 2. Two complementary views of the anatomy of the NN interaction. Left panel: The LAB energy complex energy plane, showing the partial waves left cut structure due to multipion and $\sigma, \rho, \omega$ exchange and the right cut structure due to pion production. Right panel: NN Potential as a function of distance, compared with a free wave, $\sin(pr)$, with $p = 2k_F$, the relative momentum corresponding to back-to-back scattering in nuclear matter; the most energetic process inside a heavy nucleus.

For this amplitude the partial wave expansion in this case reads

$$M^{J}_{m_J, m'_J}(\theta) = \frac{1}{2ik} \sum_{J,l} \sqrt{4\pi(2l+1)} Y_{m'_J - m_J}(\theta, 0) \times C^{J, S}_{m_J, m'_J, m_s, m_s} I^{J - l'}(S_{l,l'} - \delta_{l,l'}) C^{J, S}_{0, m_J, m_J}$$

\[(18)\]

where $S$ is the unitary coupled channel S-matrix, and the $C'$s are Clebsch-Gordan coefficients. One has that $S^{JS} = (M^{JS} - i1)(M^{JS} + i1)^{-1}$ with $(M^{JS})^\dagger = M^{JS}$ a hermitian coupled channel matrix (also known as the K-matrix). At the level of partial waves the multipion exchange diagrams generate left hand cuts in the complex s-plane, which come in addition to the NN elastic right cut and the $\pi NN, 2\pi NN$ etc., pion production cuts. At low energies for $|p| \leq m_\pi/2$ we have

$$p^{J+l'+1} M_{l,l'}^{JS}(p) = -\alpha^{-1} l_{l'}^{JS} + \frac{1}{2} (r) l_{l'}^{JS} p^2 + (v) l_{l'}^{JS} p^4 + \ldots$$

\[(19)\]

which is the coupled channels effective range expansion. We sketch the situation in Fig. 2 (left panel) where a possible contour for a dispersion relations study is also depicted. When the cuts are explicitly taken into account there still remains the question on the number of subtraction constants (see e.g. Ref. and references therein). While one can pursue such an analysis, comparison with experimental data goes beyond just partial waves and generates complications coming from long range effects, which are most efficiently treated in coordinate space within the potential approach preferred by nuclear physicists. The magnetic moment interaction which...
decreases as $1/r^3$ is crucial to describe the data but remains a challenge in momentum space. Even in the friendly coordinate space these terms usually need summing about 1000-2000 partial waves. This is a serious bottleneck for any analysis aiming at a direct comparison to experimental data. Fortunately, any pion cut generates a contribution to the potential which at long distances falls off as $e^{-nm_{\pi}r}$. The potential in coordinate space, sketched in Fig. 2 (right panel), has the same analytical structure (left panel) but permits incorporating these otherwise difficult long distance effects by simply solving the Schrödinger equation and using the sampled, coarse grained potential $V(r_n)$, as fitting parameters themselves.

5. An elementary determination of the coupling constant

One motivation to study the NN force is to apply it to nuclear structure and nuclear reactions. Many studies are conducted with this application in mind, when not specifically designed to produce a potential friendly to some preferred computational many-body method. This introduces a bias and hence a systematic error in the analysis of nuclear forces which is often forgotten. Thus, fits rarely go much beyond the pion production region, since the inelasticity becomes important above 350 MeV LAB energy and the addition of $\pi NN$ channel does not improve the description.

On the other hand, being composite particles made out of quarks and gluons, nucleons have a finite size which can be determined by a variety of methods, mostly by electron and neutrino scattering. To what extent this finite size is relevant for nuclear structure calculations is not completely obvious, but in practice nucleons in nuclear physics are treated as elementary and pointlike. Excitations such as the $\Delta$ resonance are explicitly included themselves as elementary point like particles as well. The most dense known nuclear system is nuclear matter where an average equilibrium separation distance between nucleons is $d_{NN} \sim 1.8$ fm. Thus, nuclear binding is obviously related to this mid-range distance. On the other hand the pion production threshold happens at LAB energy $\sim 2m_{\pi} = 280$ MeV.

An extreme situation corresponds to assume that the elementarity radius $r_e$ is arbitrarily small. The lightest nucleus, the deuteron, is a bound np system with $J^C = 1^+$, and corresponds to a $^3S_1 - ^3D_1$ mixed state. The tensor part of the OPE potential diverges as $1/r^3$ as already pointed out by Bethe. Thus, the deuteron equations are singular at short distances, but they can be renormalized by imposing physical renormalization conditions. In Fig. 3 we show results for a number of renormalized observables and requiring the state to be normalizable at short distances, showing that $f^2 = 0.072 - 0.074$, not far from Bethe’s venerable value.

6. NN interaction and Effective elementarity of the Nucleon

The elementarity size $r_e$ of the nucleon can operationally be characterized by looking at departures from point-like behaviour. If we look for instance at the electromagnetic interaction, Fig. 4 (left panel), it clearly suggests that protons interact as point-like charges, $1/r$, above $r > r_e = 2$ fm. The electric charge screening feature
Fig. 3. Renormalized OPE deuteron properties using the physical binding energy, average pion mass and asymptotic D/S ratio $\eta = 0.0256(4)$ as a function $f^2_{\pi NN}$ compared with experiment.

we see in the electromagnetic case also holds for the strong interaction but here, it corresponds to axial charge screening. This can be illustrated within a Quark Cluster picture using a Chiral Quark model. At long distances one can determine the NN potential in the Born-Oppenheimer approximation, allowing for $\Delta$ isobar intermediate states, but include the $\pi N \Delta$ transition form factor. In the Chiral Quark model, where PCAC holds the vertices $\pi NN$, $\pi N \Delta$ and $\pi \Delta \Delta$ are proportional to the Axial Form Factors, for which we may use axial-vector meson dominance. This provides a dominating contribution to the Two Pion Exchange...
Fig. 5. NN Potentials in the Born-Oppenheimer, including OPE and TPE via Δ excitation, as a function of NN distance. We show the effect of axial-vector (transition) form factors, generating an axial screening of the interaction below the elementarity radius \( r_e \sim 2 \text{fm} \).

(TPE) for \( r > r_e \). The result is shown in Fig. 5 and, as we see, the elementarity radius turns out, again, to be about \( r_e = 2 \text{fm} \sim d_{NN} \) the average distance of nucleons in nuclear matter, suggesting that nucleon compositeness should be play a marginal role in nuclei.

The Quark model estimate suggests that we may pin down the interaction as if nucleons were elementary particles down to the elementarity radius \( r_e = 2 \text{fm} \). The practical utility of this information is that we can compare different contributions stemming from meson exchange above the elementarity radius \( r_e \), showing that (unregularized) OPE is the main contribution for distances larger than a cut-off scale \( r > r_c = 3 \text{fm} \) below which TPE starts contributing significantly. This pattern occurs also with the exchange of heavier mesons including \( \sigma, \rho, \omega, \eta, \delta, a_1 \) etc. which allows us to discard them in our analysis as well as TPE. In fact, one common feature of all the high quality interactions is that they contain unregularized OPE above 3 fm. Thus, we will take

\[
V_{NN}(r)|_{\text{strong}} = V_{NN}^{OPE}(r), \quad r > r_c = 3 \text{fm}.
\]  

(20)

Until 1993 most “high quality fits” carried out by many groups in Bonn, Paris Washington and Nijmegen never improved over a \( \chi^2/\nu \sim 2 \). From the point of view of statistical analysis this is undesirable; one finds the closest theory to experiment but the difference between theory and experiment cannot be identified as a random fluctuation, precluding any sensible error analysis. The great achievement of the
Nijmegen group in the 1990’s was to pursue all effects which could explain the very accurate pp and np measurements. Besides Coulomb this includes three main effects: vacuum polarization (VP), relativistic corrections and magnetic moments (MM) interactions. For instance VP dominates over MM for \( r < 1000 \text{fm} \) whereas beyond the effect is the opposite, see Fig. 4. The em piece becomes,

\[
V_{NN}(r)_{\text{em}} = V_{NN}^C(r) + V_{NN}^{MM}(r) + V_{NN}^{VP}(r) + V_{NN}^{C_{\text{Rel}}}(r), \quad r > r_c = 3 \text{fm}.
\] (21)

As mentioned before we cannot pin down details below \( \Delta r = 1/p_{\text{max}} \), so we explicitly coarse grain the interaction in the innermost unknown region, which means in practice sampling the “unknown” original potential at equidistant values separated by a distance \( \Delta r = 0.6 \text{fm} \), \( r_n = n \Delta r \), using Dirac delta-shells. This was an idea suggested by Aviles long ago and rediscovered recently, which means

\[
V(r)|_{\text{strong}} = V(r)_{\text{DS}} \equiv \Delta r \sum_{n=1}^{5} V_{\Delta r}(r_n) \delta(r - r_n) \quad r \leq r_c = \text{fm}
\] (22)

where \( V_{\Delta r}(r_n) \) become the fitting parameters which depend on the resolution scale \( \Delta r \) and whose total number can be estimated a priori. This can be seen by inspecting Fig. 3 (right panel) by counting how many partial waves and how many points \( r_n \) per partial wave sample the interaction with resolution \( \Delta r \). This viewpoint provides a rationale for the number of parameters, \( N_{\text{Par}} \), (typically about 40-50) that were traditionally needed in the past for high quality fits and allows at the same time the most succesful pp and np fit to date.

7. Fitting and selecting data

Fitting and selecting data are intertwined, particularly when there are incompatible data, as it is the case in NN scattering. We have collected \( N_{\text{Dat}} = 8000 \) np+pp scattering data, \( O_i^{\text{exp}} \), measured between 1950 and 2013 below \( E_{\text{LAB}} = 350 \text{MeV} \), with given experimental uncertainty \( \Delta O_i \). On the other hand the NN potential

\[
V_{NN}(r) = V_{DS}(r)\theta(r_c - r) + \{V_{NN}(r)_{\text{OPE}} + V_{NN}(r)_{\text{em}}\} \theta(r - r_c)
\] (23)

will generate \( O_i^{\text{th}} \), and we can minimize the distance between theory and experiment by tuning the fitting parameters, the potential at the coarse grained distances, \( [V(r_n)]_{ij}^{\text{DS}} \), and the pion-nucleon coupling constants \( f_0^2, f_p^2, f_c^2 \).

Even in the case of mutually consistent data, we can never be sure that the phenomenological theory is correct, so one poses the classical (and admittedly twisted) statistical question as follows: Assuming that the theory is correct, what is the probability \( q \) that the data are not described by the theory ?. This corresponds to find the probability that for the measured observables, we cannot say that the true values fulfill the relation

\[
O_i = O_i^{\text{th}} + \xi_i \Delta O_i,
\] (24)

with \( i = 1, \ldots, N_{\text{Dat}} \) and \( \xi_i \) are independent random normal variables with vanishing mean value \( \langle \xi_i \rangle = 0 \) and unit variance \( \langle \xi_i \xi_j \rangle = \delta_{ij} \), implying that \( \langle O_i \rangle = O_i^{\text{th}} \). If this
probability is large then we can discard the phenomenological theory and look for a better one. But if it turns out to be small, there is no good reason to discard it, and moreover we can vary the parameters in such a way that they cover the fluctuations in the data. The p-value is \( p = 1 - q \) and determines the confidence level we have on the theory. This is the standard set up for the \( \chi^2 \) least square fitting, since the sum of \( \nu \)-gaussians, \( \sum_{i=1}^{\nu} \xi_i^2 \) has a \( \chi^2 \)-distribution with \( \nu \)-degrees of freedom. Of course, one can only check this assumption after the optimal fit has been carried out, and determining whether the incoming residuals,

\[
R_i = \frac{O_i^{\text{exp}} - O_i^{\text{th}}}{\Delta O_i}
\]

belong to a gaussian distribution as we initially assumed. If this is the case the fit is self-consistent. The most popular \( \chi^2 \)-test provides a p-value of 68% when

\[
1 - \sqrt{2/\nu} \leq \chi^2_{\text{min}}/\nu \leq 1 + \sqrt{2/\nu}
\]

where \( \nu = N_{\text{Dat}} - N_{\text{Par}} \). In previous works we extensively discuss more stringent tests as well as the conditions allowing a legitimate global scaling of errors.\(^{31,32,33}\)

One important issue here is the role played by the number of fitting parameters, which we claim to be optimally fixed by the maximum scattering LAB energy. Obviously, if we have too few parameters a successful fit will not be accomplished.\(^a\) On the other hand, although there is no limit in principle to include more parameters, we expect strong correlations among them which display explicitly a undesirable parameter redundancy and no real fit improvement.\(^b\) In our case we found about 50 parameters to be realistic and with moderate correlations.\(^{31,32}\)

All this is fine provided we have a collection of mutually compatible data. When this is not the case, we may ask which experiment or datum including its error estimate is correct \( O_i^{\text{exp}} \pm \Delta O_i \). This may not necessarily mean genuinely wrong experiments, but rather unrealistic error estimates. Note that the main purpose of a fit is to determine the true values of certain parameters with a given and admissible confidence level, so we search for a maximization of experimental consensus by excluding data sets inconsistent with the rest of the database within the fitting model. We extend the Nijmegen 3\( \sigma \) criterion\(^{41}\) by the following selection process:

1. Fit to all data. If \( \chi^2/\nu < 1 \) you can stop. If not proceed further.
2. Remove data sets with improbably high or low \( \chi^2 \) (3\( \sigma \) criterion)
3. Refit parameters for the remaining data.
4. Re-apply 3\( \sigma \) criterion to all data.
5. Repeat until no more data are excluded or recovered.

\(^a\)Of course with decreasing energy the number of essential parameters decreases; for instance at threshold only two scattering lengths are needed for the only non-vanishing S-wave contributions.

\(^b\)In addition, the covariance matrix size increases and, furthermore, may become numerically singular, preventing both efficiently finding an optimum and making an assessment of uncertainties.
The effect of the selection criterion is to go from $\chi^2/\nu|_{\text{all}} = 1.41$ to $\chi^2/\nu|_{\text{selected}} = 1.05$ with a reduction in the number of data from $N_{\text{Data}} = 8173$ to $N_{\text{Data}} = 6713$. While this seems a drastic rejection of data it allows to perform the largest self-consistent fit to date below 350 MeV. For such a large number of data this is not a minor improvement; it makes the difference between having $p \ll 1$ or $p \sim 0.68$.

In the process of selecting and fitting we have learned some features of the phenomenological interaction within the coarse grained approach where no a priori condition on the fitting parameters was imposed. We can fit the pp database independently. However, the isovector phases in the pn system are largely uncontrolled by the np data. Therefore it is preferable to fit the pp first, and to refit the pp+np system simultaneously by making some statistically testable isospin assumptions.

We have also tried to use normality of the residuals as a rejection tool, without much success. The reason is that a normality test checks the excess or deficit of residuals as compared to the guess distribution, but does not indicate specifically which data are responsible for normality deviations.

Likewise, we have analyzed the robustness of the database with respect to restricting or enlarging the rejection level from $3\sigma$ to $2\sigma$ or $4\sigma$, respectively. In the first case, it was not possible to find a self-consistent database with the number of accepted data fluctuating from subsequent fits. This is probably related to the grouping of data in a common experiment, preventing a stable decision of accepting/rejecting data groups. In the $4\sigma$-rejection level case, the self-consistent database exists but does not comply to the normality test at the imposed significance level.

Why are these apparently small details important?. One reason is that nuclear structure calculations are insensitive to long range potential, but quite dependent on the least known mid range part, so that errors are propagated to binding energies or matrix elements $^{22,23,24,34,35}$ The role played by $\chi$TPE and determination of chiral constants $c_1$, $c_3$ $c_4$ $^{37,38}$ (where $N_{\text{Par}} = 30$ and $r_c = 1.8$) and inclusion $\Delta$-resonance $^{39}$ have been analyzed with the Granada-2013 database. In all, the present situation regarding both the selection of data and the normality of residuals is highly satisfactory. In our view, this combined consistency of the statistical assumptions and the model analyzing them provides a good starting point to proceed further to determine the pion-nucleon coupling constants.

8. Determination of $\pi\text{NN}$ coupling constants

The charge symmetry breaking is restricted to mass differences by setting $f_p = -f_n = f_c = f$ and the value $f^2|_{\text{Nij}} = 0.0750(9)$ recommended by the Nijmegen group $^{40}$ has been used in most of the potentials since the seminal 1993 partial wave analysis $^{41}$ In their 1997 status report $^{42}$ the Nijmegen group wrote: “The present accuracies in the determination of the various coupling constants are such, that with a little improvement in the data and in the analyses these charge-independence breaking effects could be checked.” The Granada-2013 database has 6713 data compared to the 4313 of Nijmegen-1993. Can this be the invoked little improvement ?.
We try to answer this by recalling that electroweak corrections scale with the fine structure constant $\alpha = 1/137$ and the light quark mass differences. Thus
\[
\frac{\delta g}{g} = O\left(\alpha, \frac{m_u - m_d}{\Lambda_{\text{QCD}}}\right) = O\left(\alpha, \frac{M_p - M_n}{\Lambda_{\text{QCD}}}\right)
\]
for the relative change around a mean value. These are naturally at the $1-2\%$ level, a small effect. The question is on how many independent measurements are needed to achieve this desired accuracy. According to the central limit theorem, for $N$ direct independent measurements the relative standard deviation scales as
\[
\frac{\Delta g}{g} = O\left(\frac{1}{\sqrt{N}}\right)
\]
and $\delta g \sim \Delta g$ for $N = 7000-10000$. We cannot carry out these direct measurements of $g$ but we can proceed indirectly by considering a set of mutually consistent NN scattering measurements $O_{\text{exp}}^i$ with $i = 1, \ldots, N_{\text{Dat}}$ and use a model with $g$ and $\vec{\lambda} = (\lambda_1, \ldots, \lambda_{N_{\text{par}}})$ parameters, which produces $O_{\text{th}}^i(g, \vec{\lambda})$. We can then eliminate the parameters $\vec{\lambda}$, in favor of $N_{\text{par}}$ experiments and we are left with $N = N_{\text{Dat}} - N_{\text{par}}$ independent observables which depend just on $g$ providing $N$ independent determinations. Of course, these measurements will have some statistical error, so that $O_i = \bar{O}_i \pm O_{\delta i}$, which means that $O_i$ is a random variable, and a $\chi^2$-fit is nothing but a democratic way of eliminating the parameters. Since $N_{\text{Dat}} \gg N_{\text{par}}$, we need about $N_{\text{Dat}} \sim 7000-10000$ to witness isospin breaking with the coarse grained interaction, and in our recent work we do. From our full covariance matrix analysis we get for the $g'$s
\[
g_2^c/(4\pi) = 14.91(39), \quad g_2^p/(4\pi) = 13.81(11), \quad g_2^p/(4\pi) = 13.72(7).
\]
We thus confirm the premonition of the Nijmegen group, although further “little improvements” are still needed to confirm an ordering pattern, such as e.g. Eq. (15).

In Fig. 6 we show a chronological recreation of $f_2^p$ and $f^2$ determinations using at any rate the NN data of the complete database measured up to a given year, which expectedly resembles the historic plot. We consider in any case and, when needed, the corresponding 3\sigma-consistent database. We also plot several Nijmegen determinations including $f^2|_{\text{Nij}}$. Assuming a unique pion-nucleon coupling constant we obtain $f^2 = 0.07611(33)$ which is 1\sigma compatible but almost three times more accurate. The latest most accurate $\pi N$ scattering determinations has are based on the GMO rule with $g_2^c/(4\pi) = 14.11(20)$, use fixed-t dispersion relations with $g_2^c/(4\pi) = 13.76(8)$ and are based on $\pi N$ scattering lengths and $\pi^{-}d$ scattering and the GMO sum rule yielding $g_2^c/(4\pi) = 13.69(19)$. Our value, Eq. (28) is compatible with this last determination, but twice more accurate.

9. Conclusions
According to our analysis neutrons interact more strongly than protons above $r_c = 3\text{fm}$, but we cannot check what is the influence on neutron-neutron scattering as
we have not determined the nn-interaction below $r_c$. The traditional and compelling explanation that the nn scattering length is larger than the strong contribution to the pp-scattering length (a model dependent quantity) would rest on the assumption that there is no relevant isospin breaking below 3 fm, a fact that is not supported by our analysis and requires further understanding.

Isospin breaking at short distances has always been a difficult subject. We suggest to cut the gordian knot by separating the NN interaction in two distinct regions marked by a short distance radius $r_c$ and assuming a charge dependent one pion exchange above $r_c$, as the unique strong contribution. Below this radius we purposely ignore the specific form of the interaction by coarse graining it down to $\Delta r = 0.6$ fm, the shortest deBroglie wavelength before pions are produced. This distance turns out to be $r_c = 3$ fm and we can describe 6727 NN scattering data with a total $\chi^2 = 6907$ and 55 short distance parameters plus $f_p^2, f_0^2$ and $f_c^2$.

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