SUPERSINGULAR REPRESENTATIONS OF RANK 1 GROUPS

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Abstract. We prove that any connected reductive group of semisimple \( F \)-rank 1 over a \( p \)-adic field admits an irreducible admissible supersingular mod-\( p \) representation. This establishes one of the missing cases in Vignéras’ existence proof for general reductive groups.

1. Introduction

The mod-\( p \) representation theory of connected reductive groups over local fields has seen a tremendous amount of activity recently. Most notably, the seminal work of Abe–Henniart–Herzig–Vignéras ([AHHV17]) gives a classification of the irreducible admissible mod-\( p \) representations of such groups. Their construction requires as input a supply of supersingular (or, equivalently, supercuspidal) representations. We may therefore ask the question: does every reductive group over a local field admit an irreducible admissible supersingular representation?

Let \( F \) be a finite extension of \( \mathbb{Q}_p \), and \( G \) a connected reductive group over \( F \). Aside from a few choices of \( G \), Vignéras in [Vig17a] answers the above question affirmatively for the group \( G(F) \) by reducing mod \( p \) certain integral structures on complex representations. The purpose of this short note is to establish one of the missing cases, namely when \( G \) has semisimple \( F \)-rank 1. The remaining case is the subject of forthcoming work of Herzig, and will settle the existence question for reductive groups over (finite extensions of) \( \mathbb{Q}_p \). In contrast, the case where \( F \) has positive characteristic remains open in general.

Our method, which we now sketch, uses the theory of coefficient systems and diagrams, as utilized by Paškūnas in [Pas04] (building on earlier work of Schneider–Stuhler and Ronan–Smith). We may assume that our reductive group \( G \) is absolutely simple and simply connected, of \( F \)-rank 1. Given a supersingular module \( \Xi \) for the pro-\( p \)-Iwahori–Hecke algebra of \( G := G(F) \), we can naturally construct a \( G \)-equivariant coefficient system \( D_\Xi \) on the Bruhat–Tits tree of \( G \). The homology of \( D_\Xi \) admits a smooth \( G \)-action, and our task is to construct an appropriate irreducible quotient of this homology space. To this end, we define an auxiliary coefficient system \( D'_\Xi \), which is built out of injective envelopes of representations of certain parahoric subgroups, along with a morphism \( D_\Xi \to D'_\Xi \). The image of the induced map on homology is admissible, and we construct a quotient \( \pi' \) of this image which is irreducible and admissible. Further, the construction of \( \pi' \) implies that its pro-\( p \)-Iwahori invariants contain the module \( \Xi \); using a criterion of Ollivier–Vignéras, we conclude that \( \pi' \) is supersingular.

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2. Construction

2.1. Notation and preliminaries. Let \( F \) be a finite extension of \( \mathbb{Q}_p \), and let \( k_F \) denote the residue field of \( F \). For an algebraic group \( H \) over \( F \), we denote by the italicized letter its group of \( F \)-points: \( H := H(F) \). We denote by \( C \) an algebraically closed field of characteristic \( p \), and fix an embedding of \( k_F \) into \( C \). The field \( C \) will serve as the field of coefficients for all modules and representations appearing.

We let \( G \) denote a connected reductive group of semisimple \( F \)-rank equal to 1. We will show that \( G \) admits irreducible, admissible, supersingular representations. By [Vig17a, Prop. 9.1], it suffices to assume \( G \) is an absolutely simple and adjoint group of \( F \)-rank 1. Let \( G^{sc} \) denote the simply connected cover of \( G \):

\[
1 \to A \to G^{sc} \to G \to 1
\]

Proceeding as in the proof of [Vig17a, Prop. 9.5], we see that \( G^{sc} \) admits an irreducible, admissible, supersingular representation on which \( A \) acts trivially if and only if \( G \) does. Therefore, we may assume that our group
G is absolutely simple, simply connected, and has F-rank equal to 1. We will then construct irreducible, admissible, supersingular representations of G on which its (finite) center acts trivially.

2.2. Parahoric subgroups. We let $\mathcal{B}$ denote the Bruhat–Tits building of G. By our assumptions on G, $\mathcal{B}$ is a one-dimensional contractible simplicial complex, i.e., a tree. Fix a maximal F-split torus $S$ (which has rank 1), and let $\mathcal{A} := X_*(S) \otimes_\mathbb{Z} \mathbb{R}$ denote the standard apartment in $\mathcal{B}$ corresponding to $S$. We let $v_0$ and $v_1$ denote two adjacent vertices of $\mathcal{A}$, and let $\varepsilon$ denote the edge joining them. We let $K_0$ and $K_1$ denote the stabilizers of $v_0$ and $v_1$, respectively. The group $I := K_0 \cap K_1$ is then the stabilizer of $\varepsilon$.

The vertices $v_0$ and $v_1$ are representatives of the two orbits of G on the set of vertices of $\mathcal{B}$, and the edge $\varepsilon$ is a representative of the unique orbit of G on the edges of $\mathcal{B}$. By [Ser03, §4, Thm. 6], we may therefore write the group G as an amalgamated product:

$$G \cong K_0 *_I K_1.$$  

Since the group G is semisimple and simply connected, the stabilizers of vertices and edges in $\mathcal{B}$ are parahoric subgroups. For $i \in \{0, 1\}$, we let $K^+_i$ denote the pro-$p$ radical of $K_i$, that is, the largest open, normal, pro-$p$ subgroup of $K_i$. The quotient $G_i := K_i/K^+_i$ is isomorphic to the group of $k_F$-points of a connected reductive group over $k_F$ (see [HV15, §3.7]). Likewise, if we let $I^+$ denote the maximal, normal pro-$p$ subgroup of $I$, then $T := I/I^+$ is isomorphic to the group of $k_F$-points of a torus over $k_F$. The image of $I$ in $G_i$ is equal to a minimal parabolic subgroup $B_i$, with Levi decomposition $B_i = T_i U_i$. Thus, we identify the quotient $T$ with $T_1$.

2.3. Hecke algebras. We let $\mathcal{H}$ denote the pro-$p$–Iwahori–Hecke algebra of G with respect to $I^+$, that is, the convolution algebra of $C$-valued, compactly supported, $I^+$-bi-invariant functions on G (see [Vig16, §4] for more details). For $g \in G$, we let $T_g$ denote the characteristic function of $I^+ g I^+$. The algebra $\mathcal{H}$ is generated by two operators $T_{s_0}, T_{s_1}$, along with operators $T_z$, where $s_0$ and $s_1$ represent affine reflections fixing $v_0$ and $v_1$, respectively, and $z \in T$. For $i \in \{0, 1\}$, we let $H_i$ denote the subalgebra of $\mathcal{H}$ generated by $T_{s_i}$ and $T_z$ for $z \in T_i$; this is exactly the subalgebra of functions in $\mathcal{H}$ with support in $K_i$. The algebra $H_i$ is canonically isomorphic to the finite Hecke algebra $\mathcal{H}(G_i, U_i)$.

Since $K^+_i$ is an open normal pro-$p$ subgroup of $K_i$, the smooth irreducible representations of $K_i$ and $G_i$ are in bijection. Further, the finite group $G_i$ possesses a strongly split BN pair of characteristic $p$ ([Vig16, Prop. 3.25]). Therefore, by [CE04, Thm. 6.12], the functor $\rho \mapsto p^\rho$ induces a bijection between (isomorphism classes of) smooth irreducible representations of $K_i$ and (isomorphism classes of) simple right $H_i$-modules, all of which are one-dimensional.

We briefly recall some facts about supersingular $\mathcal{H}$-modules. We refer to [Vig17b, Def. 6.10] for the precise definition, and give instead the classification of simple supersingular $\mathcal{H}$-modules. Since $G$ is simply connected, every supersingular $\mathcal{H}$-module is a character. The characters $\Xi$ of $\mathcal{H}$ are parametrized by pairs $(\chi, J)$, where $\chi : T \rightarrow C^\times$ is a character and $J$ is a subset of:\n
$$S_\chi := \{ s \in \{ s_0, s_1 \} : \chi(c_s) \neq 0 \}$$

(here $c_s$ is a certain element of the group algebra of $T$ which appears in the quadratic relation for $T_z$; note also that the definition of $S_\chi$ is independent of the choice of lift $\tilde{s}$). The correspondence is given as follows (cf. [Vig17b, Thm. 1.6]): for $z \in T$, we have $\Xi(T_z) = \chi(z)$, and for $s \in \{ s_0, s_1 \}$, we have

$$\Xi(T_z) = \begin{cases} 0 & \text{if } s \in J, \\ \chi(c_s) & \text{if } s \notin J. \end{cases}$$

Since $G$ is simple, [Vig17b, Thm. 1.6] implies that $\Xi$ is supersingular if and only if

$$(S_\chi, J) \neq (\{ s_0, s_1 \}, \emptyset), (\{ s_0, s_1 \}, \{ s_0, s_1 \}).$$

2.4. Diagrams. Since the group G is an amalgamated product of two parahoric subgroups, the formalism of diagrams used in [KX15] applies to the group G. We recall that a diagram $D$ is a quintuple $(\rho_0, \rho_1, \sigma, \iota_0, \iota_1)$ which consists of a smooth representation $\rho_i$ of $K_i$, a smooth representation $\sigma$ of $I$, and $I$-equivariant morphisms $\iota_i : \sigma \rightarrow \rho_i|_I$. We depict diagrams as
Morphisms of diagrams are defined in the obvious way.

Let $\Xi$ denote a supersingular character of $H$, associated to a pair $(\chi, J)$. We define a diagram $D_\Xi$ as follows:

- set $\sigma := \chi^{-1}$, which we view as a character of $I$ by inflation;
- we let $\rho_{\Xi,i}$ denote a smooth irreducible $K_i$-representation such that $\rho_{\Xi,i}^I \cong \Xi|_H$ as $H_i$-modules (by the discussion above, $\rho_{\Xi,i}$ is unique up to isomorphism);
- let $\iota_i$ denote the $I$-equivariant map given by $\sigma = \chi^{-1} \mapsto \rho_{\Xi,i}^I \hookrightarrow \rho_{\Xi,i}|_I$.

Thus

$$D_\Xi = \begin{array}{ccc}
\sigma & \rho_0 \\
\sigma & \rho_1
\end{array}$$

We now wish to construct an auxiliary diagram $D'$ into which $D_\Xi$ injects. This will be done with the use of injective envelopes.

**Lemma 2.1.** Let $i \in \{0, 1\}$. We then have

$$\text{inj}_{K_i}(C[G_i])_I \cong \bigoplus_\xi \text{inj}_{I}(\xi)^{\oplus |B_i \backslash G_i|},$$

where $\xi$ runs over $C$-characters of $I$ (or, equivalently, of $T_1$), and $\text{inj}_{H}(\tau)$ denotes a fixed choice of injective envelope of $\tau$ in the category of smooth $H$-representations.

**Proof.** By [Pas04, Lem. 6.19], we have

$$\text{inj}_{K_i}(C[G_i])_I \cong \bigoplus_\xi \text{inj}_{I}(\xi)^{\oplus m_\xi}$$

where

$$m_\xi = \dim_C \left( \text{Hom}_{T_i}(\xi, \text{inj}_{G_i}(C[G_i]^{U_i})) \right)$$

(note that the proof of [Pas04, Lem. 6.19] does not require that $\rho$ be irreducible, only that $K_i^+$ acts trivially). Since $C[G_i]$ is injective as a representation of $G_i$, we have isomorphisms of $T_i$-representations

$$\text{inj}_{G_i}(C[G_i]^{U_i}) \cong C[U_i \backslash G_i] \cong \bigoplus_\xi \xi^{\oplus |B_i \backslash G_i|},$$

which gives the claim. \hfill \Box

**Lemma 2.2.** Set $\alpha := \text{lcm}\{|B_0 \backslash G_0|, |B_1 \backslash G_1|\}$. There exists a diagram $D'$ of the form

$$D' = \begin{array}{ccc}
\text{inj}_{K_0}(C[G_0])^{\oplus \alpha \cdot |B_0 \backslash G_0|^{-1}} & \oplus \xi \text{inj}_{I}(\xi)^{\oplus \alpha} \\
\kappa_0 & \kappa_1
\end{array}$$

where $\kappa_0$ and $\kappa_1$ are isomorphisms, and a morphism of diagrams
\[
\begin{array}{ccc}
\psi : & \chi^{-1} & \rightarrow \bigoplus \mathfrak{inj}_f(\xi)^{\oplus a} \\
& \psi \downarrow & \downarrow \kappa_0 \\
\chi^{-1} & \rightarrow & \mathfrak{inj}_{K_0}(C[G_0])^{\oplus a \cdot |B_0 \backslash G_0|^{-1}} \\
\rho_{\Xi, 0} & \rightarrow & \mathfrak{inj}_{K_0}(C[G_0])^{\oplus a \cdot |B_0 \backslash G_0|^{-1}} \\
\end{array}
\]

in which all arrows are injective.

**Proof.** We fix the following injections, which are equivariant for the relevant groups:

- injective envelopes \( j_{\xi} : \xi \inj \mathfrak{inj}_f(\xi) \) for each \( C \)-character \( \xi \) of \( I \);
- injective envelopes \( j_i : C[G_i]^{\oplus a \cdot |B_i \backslash G_i|^{-1}} \inj \mathfrak{inj}_{K_i}(C[G_i])^{\oplus a \cdot |B_i \backslash G_i|^{-1}} \) for \( i \in \{0, 1\} \);
- an inclusion \( \mathfrak{inc} : \chi^{-1} \inj \bigoplus \xi^{\oplus a} \);
- an inclusion \( c_i : \rho_{\Xi, i} \inj \mathfrak{inj}_{K_i}(C[G_i])^{\oplus a \cdot |B_i \backslash G_i|^{-1}} \) for \( i \in \{0, 1\} \).

Let \( i \in \{0, 1\} \). We first construct the \( \kappa_i \). We have an \( I \)-equivariant sequence of maps

\[
\chi^{-1} \xrightarrow{\ell_i} \rho_{\Xi, i} \xrightarrow{c_i} C[G_i]^{\oplus a \cdot |B_i \backslash G_i|^{-1}} \xrightarrow{j_i} \mathfrak{inj}_{K_i}(C[G_i])^{\oplus a \cdot |B_i \backslash G_i|^{-1}}
\]

and thus we obtain

\[
\chi^{-1} \xrightarrow{\ell_i \circ c_i \circ j_i} \mathfrak{inj}_{K_i}(C[G_i])^{\oplus a \cdot |B_i \backslash G_i|^{-1}} I^+.
\]

By Lemma 2.1 and [Pas04, Lem. 6.14], we have \( \bigoplus \xi^{\oplus a} \cong (\mathfrak{inj}_{K_i}(C[G_i])^{\oplus a \cdot |B_i \backslash G_i|^{-1}})^I \). We fix an isomorphism \( \alpha_i : \bigoplus \xi^{\oplus a} \xrightarrow{\alpha_i} (\mathfrak{inj}_{K_i}(C[G_i])^{\oplus a \cdot |B_i \backslash G_i|^{-1}} I^+ \xrightarrow{\kappa_i} \mathfrak{inj}_{K_i}(C[G_i])^{\oplus a \cdot |B_i \backslash G_i|^{-1}} | I.

By [Pas04, Lem. 6.13], the above map extends to an \( I \)-equivariant injection

\[
\kappa_i : \bigoplus \mathfrak{inj}_f(\xi)^{\oplus a} \xrightarrow{\kappa_i} \mathfrak{inj}_{K_i}(C[G_i])^{\oplus a \cdot |B_i \backslash G_i|^{-1}} | I
\]

such that

\[
\kappa_i \circ \bigoplus \mathfrak{inj}_f(\xi)^{\oplus a} = \alpha_i.
\]

Since both \( \bigoplus \mathfrak{inj}_f(\xi)^{\oplus a} \) and \( \mathfrak{inj}_{K_i}(C[G_i])^{\oplus a \cdot |B_i \backslash G_i|^{-1}} | I \) are injective \( I \)-representations and \( \kappa_i \) induces an isomorphism between their \( I^+ \)-invariants (cf. Lemma 2.1), we see that \( \kappa_i \) must in fact be an isomorphism.

We now construct the morphism of diagrams. Set \( \psi_{K_i} := j_i \circ c_i \) and \( \psi_I := (\bigoplus j_i^{\oplus a}) \circ \mathfrak{inc} \). We have

\[
\psi_{K_i} \circ \ell_i \xrightarrow{(1)} \alpha_i \circ \mathfrak{inc} \xrightarrow{(2)} \kappa_i \circ \psi_I,
\]

and therefore we obtain the desired morphism of diagrams.

\[ \square \]

2.5. **Supersingular representations.** We let \( D_\Xi \) and \( D' \) denote the coefficient systems on \( \mathcal{B} \) associated to \( D_\Xi \) and \( D' \), respectively (cf. [KX15, §6.3]). The homology of coefficient systems gives smooth \( G \)-representations, and we define

\[
\pi := \text{im} \left( H_0(\mathcal{B}, D_\Xi) \xrightarrow{\psi_*} H_0(\mathcal{B}, D') \right),
\]

the image of the induced map \( \psi_* \) on homology.

**Theorem 2.3.** The \( G \)-representation \( \pi \) admits an irreducible, admissible, supersingular quotient.

**Proof.** We use language and notation from [Pas04] and [KX15].
Our construction shares some similarities with that of Vignéras in [Vig17a]. In Remark 2.6.

(1) \( \pi \) is nonzero. Fix a basis \( v \) for \( \chi^{-1} \), and let \( \omega_{0,v}(v) \) denote the 0-chain with support \( v_0 \) satisfying \( \omega_{0,v}(v)(v_0) = t_0(v) \) (here we identify the \( \mathbb{K}_0 \)-representation \( D_{\mathbb{K},v_0} \) with \( \rho_{v,0} \)). Since the maps \( \iota_0, \iota_1 \) are injective, [Pas04, Lem. 5.7] implies that the image \( \omega_{0,v}(v) \) is nonzero in \( H_0(\mathcal{B}, D_{\mathbb{K}}) \). Now set

\[
\tilde{\omega} := \psi_* (\omega_{0,v}(v)) \in H_0(\mathcal{B}, D'_{\mathbb{K}}).
\]

This is the image in \( H_0(\mathcal{B}, D'_{\mathbb{K}}) \) of a \( D_{\mathbb{K}} \)-valued 0-chain supported on \( v_0 \), and since the maps \( \iota_0, \iota_1 \) are isomorphisms and \( \psi \) is injective, we have \( \tilde{\omega} \not\equiv 0 \) ([Pas04, Lem. 5.7] again).

(2) \( \pi \) is admissible. Since \( \iota_0, \iota_1 \) are isomorphisms, [Pas04, Prop. 5.10] gives

\[
\pi|I \subset H_0(\mathcal{B}, D'_{\mathbb{K}})|_I \cong D'_{\mathbb{K}} \cong \bigoplus_{\xi} \text{ind}_I (\xi)^{\otimes a},
\]

which implies \( \pi^{I^+} \subset \bigoplus_{\xi} \xi^\otimes a \), so that \( \pi \) is admissible.

(3) \( \pi \) contains \( \Xi \). The element \( \tilde{\omega}_{0,v}(v) \in H_0(\mathcal{B}, D_{\mathbb{K}}) \) is \( I^+ \)-invariant and stable by the action of \( \mathcal{H} \), and the vector space it spans is isomorphic to \( \Xi \) as an \( \mathcal{H} \)-module (for all of this, see the proof of [KX15, Prop. 7.3]). Since \( \psi_* \) is \( G \)-equivariant, the same holds for \( \tilde{\omega} \in \pi \).

(4) \( \tilde{\omega} \) generates \( \pi \). Since \( \tilde{\omega}_{0,v}(v) \) generates \( H_0(\mathcal{B}, D_{\mathbb{K}}) \) as a \( G \)-representation and \( \psi_* \) is \( G \)-equivariant, \( \tilde{\omega} \) generates \( \pi \) as a \( G \)-representation.

(5) Construction and properties of the quotient. The representation \( \pi \) is finitely generated. By Zorn’s lemma, there exists a maximal \( G \)-subrepresentation \( \pi_1 \) such that \( \tilde{\omega} \not\equiv \pi_1 \). We then define \( \pi' := \pi/\pi_1 \). By construction, \( \pi' \) is nonzero and irreducible, by maximality of \( \pi_1 \). Since \( F \) has characteristic 0, admissibility is preserved under taking quotients ([Hen09, §4, Thm. 1]), and therefore \( \pi' \) is admissible.

(6) \( \pi' \) contains \( \Xi \). Since \( \tilde{\omega} \not\equiv \pi_1 \), we obtain an injection of \( \mathcal{H} \)-modules \( \Xi \cong C\tilde{\omega} \hookrightarrow (\pi')^{I^+} \).

(7) \( \pi' \) is supersingular. This follows from the inclusion \( \Xi \subset (\pi')^{I^+} \) and [OV17, Thm. 5.3].

\[ \square \]

Remark 2.4. The proofs above do not actually require that \( C \) be algebraically closed; it suffices to assume that \( C \) is a sufficiently large finite extension of \( \mathbb{F}_p \).

Remark 2.5. Since the center of \( G \) is finite, it is contained in \( I \cap Z_G(S) \). Hence, taking \( \Xi \) to be associated to \((I, J)\), where \( I \) is the trivial character of \( I \) and \( J \not= 0, \{s_0, s_1\} \), we obtain an irreducible, admissible, supersingular representation \( \pi' \) with trivial action of the center. This achieves our initial goal.

Remark 2.6. Our construction shares some similarities with that of Vignéras in [Vig17a]. In op. cit., supersingular representations are constructed as subquotients of \( C[G\backslash G] \cong \text{c-ind} G^G (1_\Gamma) \), where \( \Gamma \) denotes a discrete, cocompact subgroup of \( G \) and \( 1_\Gamma \) denotes the trivial character of \( \Gamma \). Taking \( \Gamma \) to be torsion-free, we use the Mackey formula to obtain

\[
\text{c-ind}^G (1_\Gamma) |_{K_i} \cong \bigoplus_{G/K_i} \text{c-ind}^G (1) \cong \bigoplus_{G/K_i} \text{ind}^G (1) \cong \text{ind}_K (C[G_i])^\otimes a_i
\]

where \( a_i = a \cdot |G_i|^{-1} \) (cf. [Pas04, Prop. 5.10]).

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