Finite population games of optimal execution

D. Evangelista\textsuperscript{a}, Y. Thamsten\textsuperscript{b}

\textsuperscript{a}Escola de Matemática Aplicada (EMAp), Fundação Getúlio Vargas (FGV), 22250-900, Rio de Janeiro, RJ, Brasil
\textsuperscript{b}Instituto de Matemática e Estatística (IME), Universidade Federal Fluminense (UFF), 24020-140, Niterói, RJ, Brasil

Abstract

We investigate finite population games of optimal execution, taking place at a market with friction. The models over which we develop our results are akin to the standard Almgren-Chriss model with linear price impacts. On the one hand, at a temporary level, our perspective is rather similar to that of the aforementioned model. On the other hand, all players in the model will impact the asset’s public price, yielding an aggregate permanent price impact. We propose to analyze two different settings. The first one comprises the case where there is no hierarchy among players, and there is a symmetry of information. In this setting, we obtain closed-form formulas to the Nash equilibrium in the most general setting, i.e., when players’ preferences are completely heterogeneous. Particularizing to the case of homogeneous parameters, we show that the average optimal inventory of the finite population converges to its mean-field counterpart, uniformly over a fixed trading horizon, as the population size grows to infinity. In the second framework, we consider a major player, also called a leader, with the first move advantage, and a population of minor players, also known as followers, thought of as high-frequency traders, which trade on informational advantage against the leader. This leads to a model of McKean-Vlasov type for the dynamics of the asset’s midprice. We prove the existence and uniqueness of the Stackelberg-Nash equilibrium for a reasonable set of model parameters. We also characterize it as the solution of an abstract vector forward-backward stochastic differential equation system.

Keywords: Finite Population Games, Optimal Execution, Price Impacts, Hierarchic Games, Asymmetric Information.

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1. Introduction

It is often the case that large institutional investors have to execute large trades. For example, when there is a market shock, it is common that institutional investors have to diminish their exposure in certain assets in order to comply with regulatory requirements (see, e.g. \cite{5, 30}). Another situation that leads to the study of this problem is when a market maker finds herself holding too much inventory (either long or short). In these circumstances, agents incur transaction costs. Such costs can be intelligible in simple terms, such as fees from brokerage firms, or can have a more complex nature, such as certain indirect costs.

There are two main research directions on the optimal trade execution problem in the context of markets with frictions. One was initiated by Bertsimas and Lo, and Almgren and Chriss, in the seminal works \cite{4, 1, 2}, and the other by Obizhaeva and Zhang, see \cite{27}. We will follow the perspective of the former. Many works have been developed based on it, and we mention the advances exposed in the books \cite{12, 19}. In these, costs take the form of two types of price impact, which are modeled directly as static functions: the temporary and the permanent price impacts. The second line of work aforementioned introduces supply-demand functions for the Limit Order Book (LOB). It derives a price impact process for a LOB having an

Email addresses: david.evangelista@fgv.br (D. Evangelista), ythamsten@id.uff.br (Y. Thamsten)

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aspect called resilience. The first approach can be seen as a particular case of the second one (see [27]), namely, as its high resilience limit.

Thus, the types of costs considered in this work will be the ones known as permanent and temporary price impacts, such as described in [1, 2, 12]. These are types of liquidity costs. On the one hand, the effect of the trading rate of a given agent in the dynamics of the asset price is what is understood as a permanent price impact. On the other hand, the temporary impact refers to the additional cost per share the investor incurs by virtue of having to consume additional layers of liquidity (a process referred to as “walking the book”) in order to have her order completely fulfilled. In particular, we work under the order book resilience hypothesis, as opposed to considering a finite resilience level. The reasonableness of this model has been assessed in [12, 29].

In this paper, we are concerned with finite stochastic differential games, played by rational competing agents, all of which are found to be carrying out optimal execution programs. We will propose an endogenous way in which the aggregate of players’ actions affects the asset’s price. For a more detailed discussion in this direction, see [6, 15]. This is key because the information exchange between agents is assumed to occur through the dynamics of the price of the asset being traded.

In our setting, each trader submits only marketable orders. These can be to either buy or sell some amount of shares. It is understood that the goal of each agent is to finish the day with zero inventory; hence, a positive initial inventory corresponds to a liquidation program, whereas a negative one leads to an acquisition routine. In principle, all of their initial inventories are a considerable proportion of the daily volume.

At first, we will assume there is no hierarchy among players and symmetry of information in such a way that there are no informed traders. We refer to this first framework as the standard market model from now on. In this circumstance, closed-form solutions are available for the most general setting, that is, even if we assume that the parameters among the population are completely heterogeneous. Namely, in this setup, we find closed-form expressions for the inventories and corresponding speeds of trading of all the agents. The market equilibrium resulting from these interactions between players of a finite population leads us to inquire about what happens in the limiting case, as the population size grows to infinity. In this way, we are led to consider an instance of Mean-Field Games (MFG). We provide a discussion on these in the sequel.

MFG is a branch of game theory developed to study the behavior of large populations of competing rational players. These models combine differential games with the evolution of a population of players that seek to optimize a value function by choosing appropriate controls. In turn, the evolution of the population is determined by aggregating individual contributions of players’ strategies. Since the seminal works by J-M. Lasry and P-L. Lions [23, 24, 25], and by M. Huang, P. Caines and R. Malhamé [20, 21], MFG models were widely studied in the mathematical and engineering communities. Recently, MFG models have been considered in some instances of questions within the scope of optimal execution. We mention the works [9, 13, 16, 14, 22]. We emphasize that the standard market model we propose serves as a basis for many studies of MFG models, such as in the aforementioned [10, 9, 22].

In contradistinction to MFG, it is commonly understood that finite population games are seldom analytically treatable (see e.g. [22]). In particular, the availability of closed-form formulas is a rather rare event. Fortunately, we are apt to explicit solutions in all settings analyzed here. Next, when we particularize for the setting of homogeneous parameters, whose MFG model has been studied in [9], we can find a closed-form formula for the finite average field of inventories, akin to the formula found in that work. In fact, we provide an alternative representation for the formula presented in [9], and this allows us to show that the finite average field of inventories converges uniformly to the mean-field, over the fixed time horizon, as the size of the population grows to infinity. We also establish a rate for this convergence.

Subsequently, we will consider a setting where there is a single major player (the leader) and a population of minor traders (the followers). The leader is assumed to have a first move advantage, whereas the followers, interpreted as high-frequency traders, are considered to trade on informational advantage against the major player. We refer to the fact that the followers trade on informational advantage against the leader as information asymmetry, which is considered in this part of the paper, apart from a hierarchy structure. A similar viewpoint has been mentioned in [18], and a MFG model on the hierarchical game with symmetric information has been treated in [22].
Therefore, our next results are developed under the viewpoint of the latter framework, that is, the one comprising major-minor player games. In this context, the strategy we look for is of Stackelberg-Nash type, related to the works \cite{7,8}. The procedure is that, for each strategy of the major player, minor players will always accommodate to a Nash Equilibrium (NE). This feeds back into the optimization criterion of the leader. Our contribution at this point is to prove the well-posedness of the Stackelberg-Nash equilibrium. We mean well-posedness in the sense of establishing the aspects of existence, uniqueness, and Hilbertian continuity with respect to initial data of the equilibrium. Furthermore, we provide a characterization of it via abstract forward-backward stochastic differential equations of the McKean-Vlasov type.

The mathematical methodology we use to obtain the aforementioned results consists mainly in variational analysis. This approach has been successfully employed in several works within the scope of finance, see \cite{3,13}, to name a few. Our optimization problems will be phrased as maximization of functionals on certain Hilbert spaces, all of which will be proved strictly concave and coercive for a reasonable set of model parameters. Then, we take the Gâteaux derivative, characterizing the unique maximum as a solution of a forward-backward stochastic differential equation (FBSDE). In the standard market model, we will find closed-form solutions for these. In the hierarchic case with asymmetric information, we will resort to establishing well-posedness of the equilibrium.

The remainder of the paper is organized as follows. In Section 2 we present the basic state processes and pose our optimal control problem comprising a finite player of optimal trade execution, without hierarchy, and with symmetry of information. In Section 3 we recall the MFG developed in \cite{9}, which will serve as our benchmark, and propose an alternative representation formula for the mean inventory field. In Section 4 we analyze the finite player game in depth. We begin with the case of heterogeneous parameters and then particularize to the homogeneous setting. In Section 5 is where we modify the model introduced in Section 2 to account for a hierarchy structure and asymmetric information. In this section, we identify the Stackelberg-Nash equilibrium as a solution to an abstract vector FBSDE, proving a well-posedness result of the equilibrium. Finally, it is in Section 6 where we conclude and leave our further remarks.

2. The standard market model

Let us consider a stochastic differential game model comprising \( N \) traders seeking to carry out an optimal execution program on a single financial asset. The model we use is also described in \cite{10} Chapter 1. Let \( \mathcal{N} = \{1, \ldots, N\} \) be the set indexing the players, where \( i \in \mathcal{N} \) denotes the player \( i \). We may refer to an element of \( \mathcal{N} \) as either a player or a trader. The control variable of a representative trader, \( \nu^i_t \), accounts for the instantaneous trading speed, at time \( t \), that she desires to sell or buy. Accordingly, \( \nu^i > 0 \) (resp. \( \nu^i < 0 \)) represents the trading speed of an optimal trader that seeks to buy (resp. sell).

**Notations and assumptions.** Let us consider a fixed time horizon \( T > 0 \), filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_t)\), with \( \mathcal{F}_T = \mathcal{F} \). We denote by \( W^0_t \) and \( \{W^i_t\}_t, i \in \mathcal{N}, N + 1 \) independent Brownian motions adapted to \( \{\mathcal{F}_t\}_t \). We assume that \( \mathcal{A}_i \) is the set of scalar \( \{\mathcal{F}_t\}_t \)-progressively measurable processes \( \{\nu^i_t\}_{0 \leq t \leq T} \) satisfying

\[
\mathbb{E} \int_0^T |\nu^i|^2 dt < \infty.
\]

Furthermore, we write \( \mathcal{A}_i(N) := \mathcal{A}_1 \times \ldots \times \mathcal{A}_N \) where, for each \( i \in \mathcal{N} \), \( \mathcal{A}_i \) is the space of control strategies of player \( i \) irrespective of what the other players do.

**State variables and optimization criteria.** Let \( \{s_t\}_{0 \leq t \leq T} \) denote the asset’s mid-price process assumed to evolve according to

\[
\begin{align*}
\begin{cases}
&ds_t = \frac{1}{N} \sum_{j \in \mathcal{N}} h^j(\nu^j_t) dt + \sigma^0 dW^0_t, &\text{on } [0,T] \\
&s_0 = s.
\end{cases}
\end{align*}
\] (2.1)

The functions \( h^i \) above, \( i \in \mathcal{N} \), are all deterministic. For each \( i \), \( h^i(\nu) \) measures the impact that the trader exerts on the price dynamics when negotiating with speed \( \nu \). Our case of interest comprises the linear price impact model of \cite{11}, in which case \( h^i(\nu) = \alpha^i \nu^i \), for \( i \in \mathcal{N} \) and \( \alpha^i > 0 \).
Next, we describe the state variables of a representative trader \( i \in \mathcal{N} \). Her inventory process \( \{q^i_t\}_{0 \leq t \leq T} \) has the following dynamics

\[
\begin{aligned}
dq^i_t &= \nu^i_t dt + \sigma^i dW^i_t, & & \text{on } [0, T] \\
q^i_0 &= q^i,
\end{aligned}
\]  
(2.2)

where \( \sigma^i \geq 0 \) for each \( i \in \mathcal{N} \). We propose to consider uncertainty in (2.2), namely, the noise term \( \sigma^i dW^i_t \), because the investor may be a broker, causing her to receive a random stream of payments (see [3]).

We represent by \( \{x^i_t\}_{0 \leq t \leq T} \) the cash process of the trader; thus, it satisfies

\[
\begin{aligned}
dx^i_t &= -(\nu^i_t)^2 dt + \sigma^i dW^i_t, & & \text{on } [0, T] \\
x^i_0 &= x^i.
\end{aligned}
\]  
(2.3)

The function \( \nu \mapsto c'(\nu) \) is a nonnegative cost function satisfying \( c'(0) = 0 \). It represents the temporary impact for trading at a rate \( \nu \) from the perspective of agent \( i \in \mathcal{N} \).

The last state process for our representative player is the book value of her inventory on the asset under study, denoted by \( \{v^i_t\}_{0 \leq t \leq T} \). We define it as the sum of the cash held by the trader with the value of the inventory as marked to the mid-price, that is

\[
v^i_t = x^i + q^i_t s_t, & & \text{on } [0, T].
\]  
(2.4)

Using Itô’s product rule, we infer that instantaneous wealth changes are given by

\[
dx^i_t = dx^i_t + q^i_t ds_t + s_t dq^i_t + d(q^i, s)_t
\]

\[
= -c'(\nu^i_t) + q^i_t \frac{1}{N} \sum_{j \in \mathcal{N}} h^i(\nu^j_t) dt + \sigma^i q^i_t dW^0_t + \sigma^i s_t dW^i_t.
\]

Note that the terminal wealth is

\[
v_T^i = v^i + \int_0^T \left[ -c'(\nu^i_t) + q^i_t \frac{1}{N} \sum_{j \in \mathcal{N}} h^i(\nu^j_t) \right] dt
\]

\[
+ \int_0^T \sigma^0 q^i_t dW^0_t + \int_0^T \sigma^i s_t dW^i_t,
\]  
(2.5)

where \( v^i := x^i + q^i s \). We remark that \( v^i \) represents the book value of the initial inventory of player \( i \).

Each trader \( i \in \mathcal{N} \) has an execution constraint that is modeled by a function \( c^i_s \) of the inventory they hold. This represents a sense of urgency of the corresponding player. Moreover, we assume there is a terminal liquidation constraint relative to the trading horizon \([0, T]\), represented by a function \( g^i \). The latter function can be interpreted as the penalization of a sizeable marketable order submitted at the final time \( T \). Thus, given \( i \in \mathcal{N} \) and \( \nu^i \in \mathcal{A}_j \), for \( j \in \mathcal{N} \setminus \{i\} \), trader \( i \) wants to choose \( \nu^i \in \mathcal{A}_i \) in such a way as to maximize the performance criterion given by

\[
J_i(\nu^i, \nu^{-i}) = \mathbb{E}_{0, s, q^i} \left[ v^i_T - g^i(q^i_T) - \int_0^T c^i_s(q^i_t) dt \right] - v^i,
\]  
(2.6)

where

\[
\mathbb{E}_{0, s, q^i}[] := \mathbb{E}[][s_0 = s, q^i_0 = q^i],
\]

and \( (\nu^i, \nu^{-i}) \) stands for the strategy profile

\[
(\nu^i, \nu^{-i}) := (\nu^1, \ldots, \nu^{i-1}, \nu^i, \nu^{i+1}, \ldots, \nu^N).
\]

In other words, \( (\nu^i, \nu^{-i}) \) denotes the strategy in which player \( i \) chooses the control \( \nu^i \) while the others, indexed by \( j \in \{1, \ldots, N\} \setminus \{i\} \), follow \( \nu^j \). This notation will be employed for the remainder of the paper.
Here, we focus on the case in which
\[ c_i(\nu) = \kappa i \nu^2, \quad c_i(q) = \phi_i q^2, \quad g_i(q) = A_i q^2, \quad h_i(\nu) = \alpha_i \nu, \] (2.7)
for each player \( i \in \mathcal{N} \). Since the controls are square integrable and the instantaneous market impact function \( h \) is linear, we deduce that the processes \( q_t \) and \( s_t \) are square integrable and the stochastic integrals in (2.5) are martingales. Therefore, the expectation of the stochastic integrals appearing in (2.5) vanish; hence, together with (2.7), this leads the payoff functional of the trader to take the form
\[ J_i(\nu_i, \nu_i^t) = E_0, q_i \left[ \int_0^T \left[ -\phi_i(q_t) - \kappa_i(\nu_t^i)^2 + q_t \sum_{j \in \mathcal{N}} \alpha_j \nu_j^i \right] dt - A_i(q_T^i)^2 \right], \] (2.8)

The above payoff functional (2.8), referred to as the revenue functional in optimal control literature, has been widely used in market microstructure and algorithmic trading models. We especially mention the works authored by Cartea, Jaimungal, and their collaborators, see e.g. [12]. It is worthwhile to mention that the above functional describes optimization from the perspective of a type of order called Implementation Shortfall. It can be modified to emulate other types of orders provided by brokers and dealing desks of asset managers, such as Target Close or Volume-weighted Average Price, see [19].

Our objective is to find a Nash equilibrium for the set of functionals \{J_1, ..., J_N\} in terms of the following definition.

**Definition 2.1** (Nash equilibrium). A set of admissible strategy profiles \( \nu^* = (\nu_1^*, ..., \nu_N^*) \in A^N \) is said to be a Nash equilibrium for the game if, for each \( i \in \{1, ..., N\} \), the inequalities
\[ J_i(\nu^i, \nu^{\ast -i}) \geq J_i(\nu^i, \nu^*) \] (2.9)
are verified for all \( \nu^i \in A_i \).

Our first objective in this work is to prove the existence and uniqueness of a Nash equilibrium, according to Definition 2.1. The next step is to compare the NE obtained for the finite population game with its counterpart in the MFG of controls. We discuss this benchmark in the next section.

3. The benchmark model

Here, we recall the application MFG of controls to trade crowd, developed in [9]. We emphasize that it will serve as a benchmark for our finite game model. We do not identify a representative player as in the previous section. Rather, we write the dynamics for a generic player without the superscript \( i \). Moreover, we assume that the heterogeneity comes only from initial conditions, \( q_0 \). Therefore, in this section, all players share the same market parameters: \( A, \alpha, \kappa, \phi, \sigma \), meaning that either they are homogeneous or they have identical preferences.

Let \( \nu_t(q) \) be the control of a representative trader at time \( t \) with an inventory level \( q \). From (2.8), it is clear that \( \nu_t \) depends only on the state variable, \( q \), and on market parameters. Thus, we assume that the distribution density \( m \in C([0,T]; \mathcal{P} (\mathbb{R})) \) models the evolution of traders’ inventory level, \( q \). Instead of the empirical measure in (2.1), we regard the distribution on the space of controls as:
\[ \widetilde{\mu}_t = \int \mu_\nu_t(q) m_t(dq). \]

Here, since we choose \( h \) as in (2.7), \( \widetilde{\mu}_t \) can be written as
\[ \widetilde{\mu}_t = \alpha \mu_t, \]
where
\[ \mu_t = \int \nu_t^*(q) m_t(dq). \]

As a consequence, the reference price evolves according to
\[
\begin{cases}
    ds_t = \alpha \mu_t dt + \sigma dW_t^0, \\
    s_0 = s,
\end{cases}
\]

where, as before, \( \alpha > 0 \) models the slope of the linear permanent price impact. For a generic trader, her inventory, cash, and wealth processes follow the corresponding dynamics as described in (2.2), (2.3), and (2.4), respectively. The solution of the MFG studied in [9] is phrased as a two-step problem:

(i) For each flow \( \{\mu_t\}_{0 \leq t \leq T} \) of probability measures on \( \mathbb{R} \), solve the standard stochastic control problem:
\[
\sup_{\nu \in \mathcal{A}} J_{\text{MFG}}^\nu,
\]
where the performance is assessed via the functional
\[
J_{\text{MFG}}^\nu := \mathbb{E}_0,q \left[ \int_0^T \left[ -\phi(q_t)^2 - \kappa \nu_t^2 + \alpha q_t \mu_t \right] dt - A(q_T)^2 \right],
\]
subject to
\[
\begin{cases}
    dq_t = \nu_t dt + \sigma dW_t, \\
    q_0 = q.
\end{cases}
\]

(ii) Find a flow \( \{\nu_t^*\}_{0 \leq t \leq T} \) such that, for all \( t \in [0,T] \),
\[
\text{Law}[q_t] = m_t, \quad \int \nu_t^*(q) m_t(dq) = \mu_t^*,
\]
where \( \{\nu_t^*\}_{0 \leq t \leq T} \in \mathcal{A} \) is an optimal control corresponding to \( \{\mu_t^*\}_{0 \leq t \leq T} \) according to step (i).

Here, the assumption that each trader’s particular contribution to the overall distribution of controls is negligible is essential. This allows us to find an optimal criterion for a trader, while the distribution \( \mu_t \) is fixed.

First, we note that the value function associated with \( J_{\text{MFG}} \) in (3.1) solves the HJB equation:
\[
\begin{cases}
    \partial_t V(t,q) + \frac{\sigma^2}{2} \partial^2_{qq} V(t,q) + \alpha q_t \mu_t - \phi q^2 \\
    + \sup_{\nu \in \mathbb{R}} \{-\kappa \nu^2 + \nu \partial_q V(t,q)\} = 0, \quad \text{in } [0,T[ \times \mathbb{R},
    \\
    V(T,q) = -Aq^2, \quad \text{in } \mathbb{R}.
\end{cases}
\]

The optimal control in feedback form is
\[
\nu_t^* = \frac{\partial_q V(t,q)}{2\kappa}.
\]

Thus, the HJB equation becomes
\[
\begin{cases}
    \partial_t V(t,q) + \frac{\sigma^2}{2} \partial^2_{qq} V(t,q) + \alpha q_t \mu_t - \phi q^2 + \frac{\partial_q V(t,q)^2}{4\kappa} = 0, \quad \text{in } [0,T[ \times \mathbb{R},
    \\
    V(T,q) = -Aq^2, \quad \text{in } \mathbb{R}.
\end{cases}
\]
Next, the Kolmogorov-Fokker-Planck (KFP) equation associated with the stochastic flow of (3.2) is given by

\[
\begin{cases}
\partial_t m_t - \frac{\sigma^2}{2} \partial^2_{qq} m_t + \partial_q \left( m_t \frac{\partial_q V(t, q)}{2\kappa} \right) = 0, & \text{in } ]0, T[ \times \mathbb{R}, \\
m_0 \text{ given,} & \text{in } \mathbb{R}.
\end{cases}
\]

Therefore, the MFG of interacting strategies is given by

\[
\begin{cases}
\partial_t V(t, q) + \frac{\sigma^2}{2} \partial^2_{qq} V(t, q) + \alpha q \mu_t - \phi q^2 \\
\quad + \frac{\partial_q V(t, q)}{4\kappa} = 0, & \text{in } ]0, T[ \times \mathbb{R}, \\
\partial_t m_t - \frac{\sigma^2}{2} \partial^2_{qq} m_t + \partial_q \left( m_t \frac{\partial_q V(t, \cdot)}{2\kappa} \right) = 0, & \text{in } ]0, T[ \times \mathbb{R}, \\
\mu_t = \int q \partial_q V(t, q) dq, & \text{on } [0, T], \\
V(T, q) = -Aq^2, & m_0 \text{ given,} & \text{in } \mathbb{R}, \\
\int q m_t (dq) = 1.
\end{cases}
\] (3.4)

Next, we introduce the average inventory level of all players, which is given by

\[E(t) = \int q m_t (dq).\] (3.5)

Assuming \(\sigma = 0\), it is proven in [9] that it is a solution of the second-order system of ODEs:

\[
\begin{cases}
2\kappa E''(t) + \alpha E'(t) - 2\phi E(t) = 0, & \text{for } t \in ]0, T[, \\
E(0) = E_0, & \kappa E'(T) + A E(T) = 0.
\end{cases}
\] (3.6)

The solution of the system (3.6) is found in closed-form in this work, and we recall their result below.

**Proposition 3.1** (Closed-form solution for the net inventory dynamics). For any \(\alpha \in \mathbb{R}\), problem (3.6) has a unique solution, \(E\), given by

\[E(t) = E_0 a (e^{r+ t} - e^{r- t}) + E_0 e^{r- t},\]

where \(a\) is given by

\[a = \frac{(\alpha/4 + \kappa \theta - A) e^{-\theta T}}{\frac{\alpha^2}{4} \sinh(\theta T) + 2\kappa \theta \cosh(\theta T) + 2A \sinh(\theta T)}.\]

The denominator of the above expression is positive, and

\[r_- = -\frac{\alpha}{4\kappa} - \theta, \quad r_+ = -\frac{\alpha}{4\kappa} + \theta, \quad \theta = \frac{1}{\kappa} \sqrt{\kappa \phi + \frac{\alpha^2}{16}}.\]

**Proof.** See e.g. [9] Proposition 3.1.

We will utilize an alternative representation for this solution. This is pertinent because it viabilizes the proof of convergence of the average finite inventory field in a more straightforward manner. The method of proof here is also akin to the one we will use in the next section when treating the corresponding finite game with homogeneous preferences.
Proposition 3.2. The solution $E$ of the second-order ODE system (3.6) can be represented as

$$E(t) = E_0 \frac{y(t)}{y(0)},$$

where

$$y(t) := - \left( r^+ + \frac{A}{\kappa} \right) \frac{e^{-r^+(T-t)}}{2\theta} + \left( r^- + \frac{A}{\kappa} \right) \frac{e^{-r^-(T-t)}}{2\theta}.$$

Proof. We begin by introducing the matrix

$$\Lambda := \begin{bmatrix} 0 & 1 \\ \frac{\kappa}{\pi} & - \frac{A}{2\theta} \end{bmatrix}.$$

Notice that the eigenvalues of $\Lambda$ are precisely $r^\pm$. The corresponding eigenvectors are $(1, r^\pm)^T$. Let us take $Y(t) := (y_1(t), y_2(t))^T$ as the solution of the backward first-order ODE system

$$\begin{cases}
\dot{Y}(t) = \Lambda Y(t), \\
Y(T) = \begin{bmatrix} 1 \\
-\frac{A}{\kappa} \end{bmatrix}.
\end{cases}$$

(3.7)

Thus,

$$Y(t) = e^{-\Lambda(T-t)} \begin{bmatrix} 1 \\
-\frac{A}{\kappa} \end{bmatrix}.$$

We can compute the above matrix exponential as follows:

$$e^{-\Lambda t} = \frac{1}{2\theta} \begin{bmatrix}
\frac{r^+ e^{-r^+(T-t)} - r^- e^{-r^-(T-t)}}{r^+ r^-} & e^{-r^+ t} - e^{-r^- t} \\
\frac{r^+ e^{-r^+ t} - r^- e^{-r^- t}}{r^+ r^-} & e^{-r^+ t} - e^{-r^- t}
\end{bmatrix},$$

where we have written $\tau := T - t$. In this way, we infer

$$\begin{cases}
y_1(t) = - \left( r^+ + \frac{A}{\kappa} \right) \frac{e^{-r^+(T-t)}}{2\theta} + \left( r^- + \frac{A}{\kappa} \right) \frac{e^{-r^-(T-t)}}{2\theta}, \\
y_2(t) = - \left( r^+ r^- + \frac{\kappa^2 A}{2\theta} \right) \frac{e^{-r^+(T-t)}}{2\theta} + \left( r^+ r^- + \frac{\kappa^2 A}{2\theta} \right) \frac{e^{-r^-(T-t)}}{2\theta}.
\end{cases}$$

The first of these identities allows us to conclude that $y_1 \equiv y$. We claim $y(t) > 0$ for all $t$. Indeed, the relations

$$e^{r^+(T-t)} e^{-r^-(T-t)} = e^{2\theta(T-t)} > 1 > \frac{r^- + \frac{A}{\kappa}}{r^+ + \frac{A}{\kappa}}$$

imply

$$\left( r^+ + \frac{A}{\kappa} \right) e^{-r^-(T-t)} > \left( r^- + \frac{A}{\kappa} \right) e^{-r^+(T-t)},$$

and this is equivalent to $y(t) > 0$. Having established this fact, we proceed to define

$$\overline{y}(t) := \frac{y_2(t)}{y(t)} = \frac{\dot{y}(t)}{y(t)},$$

and we verify that

$$E(t) := E(0) e^{\int_0^t \overline{y}(u) du} = E_0 \frac{y(t)}{y(0)},$$

(3.8)

solves (3.6). Observe that $E$ satisfies

$$E'(t) = \overline{h}(t) E(t) \text{ and } E''(t) = \left( \overline{h}'(t) + \overline{h}^2(t) \right) E(t).$$
From this, it follows that

\[ E'(T) = \bar{h}(T)E(T) = -\frac{A}{\kappa}E(T). \]  (3.9)

On the other hand, with the aid of (3.7), we obtain

\[ \bar{h}'(t) = \frac{y_2(t)}{y(t)} - \frac{\varphi(t)}{y(t)^2}y_2(t), \]

\[ = \left( \frac{\varphi(t)}{y(t)} - \frac{\alpha}{2\kappa}y_2(t) \right) - \left( \frac{y_2(t)}{y(t)} \right)^2 \]

\[ = \frac{\varphi}{\kappa} - \frac{\alpha}{2\kappa}\bar{h}(t) - \bar{h}^2(t). \]  (3.10)

Consequently,

\[ E''(t) + \frac{\alpha}{2\kappa}E'(t) - \frac{\varphi}{\kappa}E(t) = \left( \bar{h}'(t) + \bar{h}^2(t) + \frac{\alpha}{2\kappa}\bar{h}(t) - \frac{\varphi}{\kappa} \right) E(t) = 0, \]  (3.11)

with the last equality above coming from (3.10). Putting together (3.9), (3.11), and the identity \( E(0) = E_0 \), valid by construction (see (3.8)), we infer that \( E \) does indeed solve (3.6).

**4. Analysis of the finite player game**

**4.1. The case of completely heterogeneous preferences**

Our approach here is based on the variational formulation. It allows us to characterize the speeds of trading and corresponding inventories of the Nash equilibrium as the solution of an FBSDE system. We develop this below.

We recall the definition of the functional of player \( i \) described in Section 2

\[ J_i(w^i, \nu^{-i}) := \mathbb{E} \left[ \int_0^T \left\{ -\phi^i(q^i_t - \bar{q}^i_t)^2 - \kappa^i(w^i_t - \bar{w}^i_t)^2 + \frac{1}{N} \sum_{j \in \mathcal{N}} \alpha^j \nu^j_t \right\} dt - A^i(q^i_T)^2 \right], \]

Our starting point is a lemma.

**Lemma 4.1.** Assume the model parameters satisfy \( A^i > \alpha^i/(2N) \). Then, for each \( i \in \mathcal{N} \) and each \( \nu^{-i} \), the functional \( w^i \mapsto J_i(w^i, \nu^{-i}) \) is strictly concave. Furthermore, it is coercive.

**Proof.** Let us fix \( w^i, \bar{w}^i \in \mathcal{A}_i, \nu^{-i} \in \mathcal{A}_{-i}, 0 \leq \lambda \leq 1 \). Denote by \( q^i \) and \( \bar{q}^i \) the corresponding inventory processes associated with \( w^i \) and \( \bar{w}^i \), respectively. We have

\[ \lambda J_i(w^i, \nu^{-i}) + (1 - \lambda) J_i(\bar{w}^i, \nu^{-i}) = J_i(\lambda w^i + (1 - \lambda)\bar{w}^i, \nu^{-i}) \]

\[ = \lambda(1 - \lambda) \mathbb{E} \left[ \int_0^T \left\{ -\phi^i(q^i_t - \bar{q}^i_t)^2 - \kappa^i(w^i_t - \bar{w}^i_t)^2 \right\} dt \right. \]

\[ + \left. \int_0^T \frac{\alpha^i}{N} (q^i_t - \bar{q}^i_t)(w^i_t - \bar{w}^i_t)dt - A^i(q^i_T)^2 \right] \]

\[ = \lambda(1 - \lambda) \mathbb{E} \left[ \int_0^T \left\{ -\phi^i(q^i_t - \bar{q}^i_t)^2 - \kappa^i(w^i_t - \bar{w}^i_t)^2 \right\} dt \right. \]

\[ - \left. \left( \frac{A^i}{2N} \right)(q^i_T - \bar{q}^i_T)^2 \right] \]

\[ \leq 0, \]

9
with equality holding above if, and only if, \( w^i = \bar{w}^i \). This shows the strict concavity of \( w^i \mapsto J_i(w^i, \nu^{-i}) \).

To see that this functional is coercive, we notice the following

\[
J_i(w^i, \nu^{-i}) = \mathbb{E} \left[ \int_0^T \left\{ -\phi'(q_i^j)^2 - \kappa^i(w_i^j)^2 + \frac{q_j^i}{N} \sum_{j \in N \setminus \{i\}} \alpha^j \nu_i^j \right\} dt \right.
\]

\[
- \left( A^i - \frac{\alpha^i}{2N} \right) (q_i^j)^2 \right] \right) dt \right] \rightarrow -\infty,
\]

as \( \mathbb{E} \left[ \int_0^T (w_i^j)^2 dt \right] \rightarrow \infty. \)

This lemma allows us to deduce a characterization of the Nash equilibrium of the \( N \)-player game in terms of partial derivatives of the functionals \( J_1, \cdots, J_N \). In order to state this result, we employ some notations: given \( \nu \in \mathbb{A}^N \), and \( w^i \in \mathbb{A}_i \), we set

\[
D_i J_i(\nu^i, \nu^{-i}) \cdot w^i := \lim_{\epsilon \to 0} \frac{J_i(\nu^i + \epsilon w^i, \nu^{-i}) - J_i(\nu^i, \nu^{-i})}{\epsilon} \quad \text{(i \in \{1, \ldots, N\}).}
\]

From a standard concave analysis, we conclude:

**Corollary 4.2.** A strategy \( \nu^* \in \mathbb{A}^N \) is a Nash equilibrium of the finite \( N \)-player game if, and only if,

\[
D_i J_i(\nu^*, \nu^{-i}) \cdot w^i = 0,
\]

for all \( w^i \in \mathbb{A}_i \).

Next, we further describe the Nash equilibrium thus obtained, exhibiting a closed-form formula for it.

**Theorem 4.3.** Assume \( A^i > \alpha^i/(2N) \), for all \( i \in N \). Then, the Nash equilibrium \( \nu^* \in \mathbb{A}^N \) of the \( N \)-player game, together with corresponding inventory processes \( \{q^i_1\}, \ldots, \{q^i_N\} \), solve the FBSDE system

\[
\begin{cases}
 dq_i^j = \nu_i^j \, dt + \sigma_i^j \, dW_i^j, \\
 -2\kappa^i \, dq_i^j = -2\phi(q_i^j) \, dt + \frac{1}{N} \sum_{j \in N \setminus \{i\}} \alpha^j \nu_i^j \, dt - dM_i^i, \\
 q_0^i = q_i^i, \quad \nu_i^j = \left( \frac{\alpha^i}{2\kappa^i N} - \frac{A^i}{\alpha^i} \right) q_i^j, \quad i \in \{1, \ldots, N\},
\end{cases}
\]

(4.1)

for suitable square integrable martingales \( \{M_i^i, \mathcal{F}_i\}, \ldots, \{M_i^N, F_i^N\} \).

Moreover, if we write \( \nu^* := (\nu^1, \ldots, \nu^N)^T, q^* = (q^1, \ldots, q^N)^T, \sigma = (\sigma^1, \ldots, \sigma^N)^T, \text{ and } W = (W^1, \ldots, W^N)^T, \) then the solution of the vector FBSDE system (4.1) is given in closed-form by

\[
\begin{cases}
 \nu_i^* = h(t) q_i^*, \\
 q_i^* = e^{\int_0^t h(\tau) \, d\tau} q + \int_0^t e^{\int_0^\tau h(\sigma) \, d\sigma} \sigma \, dW_i,
\end{cases}
\]

where \( h : [0, T] \rightarrow \mathbb{R}^{N \times N} \) is a deterministic continuous function, and solves the matrix Ricatti backward ODE

\[
\begin{cases}
 h'(t) + h(t)^2 + PH(t) - Q = 0, \quad \text{for } t \in [0,T], \\
 h(T) = R,
\end{cases}
\]

(4.2)

the matrices \( P, Q \) and \( R \) being given by

\[
P := (P_{ij}), \quad P_{ij} := (1 - \delta_{ij}) \frac{\alpha^j}{2\kappa^i N},
\]
\[ Q := \text{diag} \left( \frac{\phi^i}{\kappa^i} \right), \]
and
\[ R := \text{diag} \left( \frac{\alpha^i}{2\kappa^i N} - \frac{A^i}{\kappa^i} \right). \]

Furthermore, \( h \) is given by the explicit formula
\[ h(t) = Y_2(t)Y_1(t)^{-1}, \quad (4.3) \]
where \( Y(t) = (Y_1(t), Y_2(t))^T \in \mathbb{R}^{2N \times N} \) is given by
\[ Y(t) = e^{-B(T-t)} \begin{bmatrix} I^{N \times N} \\ R \end{bmatrix}, \quad (4.4) \]
\[ B := \begin{bmatrix} 0^{N \times N} \\ I^{N \times N} \\ Q & -P \end{bmatrix}. \quad (4.5) \]

Proof. Let \( \nu \in A^{(N)} \). It is straightforward to compute the partial derivative
\[ D_iJ_1(\nu^i, \nu^{-i}) \cdot w^i = \mathbb{E} \left[ \int_0^T \left\{ -2\kappa^i \nu^i_t - 2\phi \int_t^T q^i_u du - \frac{1}{N} \sum_{j \in N \setminus \{i\}} \alpha^j q^j_t + \frac{1}{N} \sum_{j \in N} \alpha^j q^j_t \right\} \right] \]
\[ = \mathbb{E} \left[ \int_0^T \left\{ -2\kappa^i \nu^i_t + 2\phi \int_t^T q^i_u du - \frac{1}{N} \sum_{j \in N \setminus \{i\}} \alpha^j q^j_t + \frac{1}{N} \sum_{j \in N} \alpha^j q^j_t \right\} \right] \]
\[ + \mathbb{E} \left[ -2\phi \int_0^T q^i_t dt + \frac{1}{N} \sum_{j \in N} \alpha^j q^j_t \right] \]
whence the first-order optimality condition, together with the concavity of the functionals \( J_1, \ldots, J_N \), imply that \( \nu^* \) is a Nash equilibrium to the \( N \)-player game if, and only if, \( \{ q_1^*, \ldots, q_N^* \} \) and \( \{ \nu_1^*, \ldots, \nu_N^* \} \) solve system (4.1). In order to achieve the announced characterization, we propose the ansatz
\[ q^*_t = h(t)\nu^*_t, \]
where \( h : [0, T] \to \mathbb{R}^{N \times N} \) is deterministic. On the one hand, we have
\[ d\nu^*_t = (h'(t) + h(t)^2) q^*_t dt + (\sigma^T h(t)) \cdot dW_t. \]
On the other hand, the FBSDE system reads as
\[ d\nu^*_t = (Q - Ph(t)) q^*_t dt + dM^*_t. \]
Thus, it must be that
\[ h'(t) + h(t)^2 + Ph(t) - Q = 0, \quad h(T) = R, \]
and then the \( N \)-dimensional martingale \( M^* = (M^{*1}, \ldots, M^{*N})^T \) is identified via
\[ dM^*_t = (\sigma^T h(t)) \cdot dW_t, \quad M^*_0 = 0. \]
To assert that the Ricatti ODE does indeed have a continuous solution defined in \([0, T]\), we refer to Theorem 2.3 of \cite{17}. We expose the statement of this result in the sequel, for the convenience of the reader. It assures that, if we are able to two matrices \( C \) and \( D \), with \( C \) symmetric, for which
\[ L := \begin{bmatrix} CB_{11} + DB_{21} & CB_{12} + B_{11}^T D + DB_{22} \\ 0^{N \times N} & B_{12}^T D \end{bmatrix} \]
satisfies $L + L^T \leq 0$, and such that

$$C + DR + R^T D^T > 0,$$

then $t \mapsto h(t)$ given by (4.3) has the following properties: (i) It is well-defined, in the sense that $Y_2(t)$ stemming from (4.4) is indeed invertible for each $t \leq T$; (ii) It is continuous for $t \leq T$ (continuously differentiable for $t < T$); (iii) Based on [17, 28], it guarantees that $h$ solves the ODE (4.2).

In the present context, upon selecting $C := 0^N \times N$ and $D := -I^N \times N$, equation (4.5) implies

$$L = \begin{bmatrix} -Q & P \\ 0^N \times N & -I^N \times N \end{bmatrix},$$

whence it follows promptly that

$$L + L^T \leq 0. \quad (4.6)$$

Moreover, we observe that

$$C + DR + R^T D^T = -2R > 0. \quad (4.7)$$

From (4.6) and (4.7), we infer that it is licit to invoke Theorem 2.3 of [17].

In the sequel, we present some numerical illustrations of Theorem 4.3. We consider two settings of finite games. The first one consists of a population of two players, whereas the other has five players. The parameters employed in both can be seen in Tables 1 and 2. We take $T = 1$ and $\sigma = 0$ for the remainder of this subsection.

\begin{table}[h]
\centering
\begin{tabular}{cccccc}
\hline
$i$ & $\alpha_i$ & $\kappa_i$ & $\phi_i$ & $A_i$ & $q^i$ \\
\hline
1 & $1 \times 5 \times 1$ & $6$ & $1 \times 4$ & $-2$ & $5$ \\
2 & $1 \times 4$ & $5 \times 1 \times 5$ & $1 \times 3$ & $1 \times 10$ & \\
\hline
\end{tabular}
\caption{Parameters for the first setting: the game with two players.}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{cccccc}
\hline
$i$ & $\alpha_i$ & $\kappa_i$ & $\phi_i$ & $A_i$ & $q^i$ \\
\hline
1 & $1 \times 5 \times 1$ & $6$ & $1 \times 4$ & $1 \times 2$ & $5$ \\
2 & $1 \times 4$ & $5 \times 1 \times 5$ & $1 \times 3$ & $2 \times 5 \times 2$ & $1 \times 2$ & $6.25$ \\
3 & $1 \times 5 \times 5$ & $5 \times 1 \times 5$ & $5 \times 1 \times 4$ & $5 \times 1 \times 2$ & $7.5$ \\
4 & $1 \times 4$ & $5 \times 1 \times 5$ & $7.5 \times 1 \times 4$ & $7.5 \times 1 \times 2$ & $8.75$ \\
5 & $1 \times 5 \times 1$ & $5 \times 1 \times 5$ & $1 \times 3$ & $1 \times 10$ & \\
\hline
\end{tabular}
\caption{Parameters for the second setting: the game with five players.}
\end{table}

In our numerical experiments, we have chosen completely distinct parameters for all players. We consider a situation in which there are players with more inventory to liquidate and, at the same time, have larger impact slopes, a larger sense of urgency, and larger terminal penalization (for ending up holding some inventory). Moreover, the players added in the second setup all have parameters in between those of agents 1 and 2 of the first game. Notice that the first and last players in both settings have the same preferences.
Figure 1: Inventories of all agents in games comprising populations of size two (left panel) and five (right panel).

In Figure 1 we see that Agent 1 is incentivized, in both settings, to liquidate rather fast. Then, she proceeds to short the stock to seize the impact on the price stemming from the other players. An aspect which is worthwhile mentioning is that, although player one’s preferences do not change from one game to the other, the aforementioned incentive is seen to be mitigated. Similarly, player two in the first game and player five in the last one have the same parameters, but the strategies in the two different settings are distinct, even though this effect is less intense for this agent than it is for individual 1; see Figure 2 below.

Figure 2: Inventories of the first (left panel) and last (right panel) agents in both games.

We can compute the average inventories held by the population in the two frameworks illustrated above. The effect of adding intermediary players was to tranquilize the overall sense of urgency of the market, as shown in Figure 3.

Figure 3: Comparison of average fields for two finite games with heterogeneous preferences.
4.2. The case of homogeneous preferences

One particularly important case, under which we put ourselves into for the remainder of the present section, is that where the agents are assumed to have homogeneous preferences: $\alpha^i \equiv \alpha$, $\kappa^i \equiv \kappa$, $\phi^i \equiv \phi$, $A^i \equiv A$. From here on, we also suppose $\sigma \equiv 0$. For each $t \in [0, T]$, we set

$$E_N(t) := \frac{1}{N} \sum_{j=1}^{N} q_{t,j}^i.$$  

**Theorem 4.4.** The average inventory $E_N$ solves the second-order ODE

$$\begin{cases}
E''_N + \frac{\alpha^2}{N} (1 - \frac{1}{N}) E'_N - \frac{\phi}{\kappa} E_N = 0, \\
E_N(0) = \frac{1}{N} \sum_{j \in N} q^j, \kappa E'_N(T) + AE_N(T) = \frac{\alpha^2}{2N} E_N(T).
\end{cases} \quad (4.8)$$

It is explicitly given by

$$E_N(t) = \left( \frac{1}{N} \sum_{j=1}^{N} q^j \right) \frac{y_N(t)}{y_N(0)}, \quad (4.9)$$

where

$$y_N(t) := -\left[ r_N^- + \frac{1}{\kappa} \left( A - \frac{\alpha}{2N} \right) \right] e^{-r_N^-(T-t)} \frac{2\theta_N}{2\theta_N},$$

$$+ \left[ r_N^+ + \frac{1}{\kappa} \left( A - \frac{\alpha}{2N} \right) \right] e^{-r_N^+(T-t)} \frac{2\theta_N}{2\theta_N}, \quad (4.10)$$

for the parameters

$$\theta_N := \frac{1}{4\kappa} \left( \alpha^2 \left( 1 - \frac{1}{N} \right)^2 + 16\phi \kappa, \right.$$\n
$$r_N^\pm := -\frac{\alpha}{4\kappa} \left( 1 - \frac{1}{N} \right) \pm \theta_N.$$\n
**Proof.** We introduce the mean speed $\nu_N$,

$$\nu_{N,t} := \frac{1}{N} \sum_{j \in N} \nu^j_{t,j}, \text{ for each } t \in [0, T].$$

It follows from (4.1) that $(E_N, \nu_N)$ solve the FBSDE

$$\begin{cases}
dE_{N,t} = \nu_{N,t} dt, \\
-d\nu_{N,t} = -\frac{\nu}{N} E_{N,t} dt + \frac{\kappa}{N} (1 - \frac{1}{N}) \nu_{N,t} dt - d\mathcal{M}_t, \\
E_{N,0} = \frac{1}{N} \sum_{j \in N} q^j, \nu_{N,T} = \left( \frac{\alpha}{2\kappa N} - \frac{A}{\kappa} \right) E_{N,T}. \quad (4.11)
\end{cases}$$

for some square integrable martingale $(\mathcal{M}_t, \mathcal{F}_t)$. Proceeding as in the proof of Theorem 4.3, this FBSDE is seen to have a unique solution, for which $\mathcal{M} \equiv 0$. In particular, we check that the relation

$$\nu_N(t) = h_N(t) E_N(t), \quad (4.12)$$

holds for the deterministic function $h_N$ solving the Ricatti ODE

$$h'_N + h_N^2 + \frac{\alpha}{2\kappa} \left( 1 - \frac{1}{N} \right) h_N - \frac{\phi}{\kappa} = 0, \quad h_N(T) = \frac{1}{\kappa} \left( A - \frac{\alpha}{2N} \right).$$
Proceeding as in the aforementioned theorem, we are apt to prove that
\[ h_N(t) = \frac{y_N'(t)}{y_N(t)}, \]
where \( y_N(t) \) is given by (4.10). From (4.12) and \( E_N' = \nu_N \), this promptly yields the representation formula (4.9).

Replacing \( \nu_N = E_N' \) in the BSDE of (4.11), we infer that \( E_N \) solves the second-order ODE (4.8). Then, carrying out the analogous diagonalization procedure, as done in the proof of Proposition 3.2, we deduce the representation (4.9) for the finite average field.

Consequently, we can prove that the average inventory of the players in the NE converges to the MFG benchmark. The formulas obtained allow us to go even further, obtaining a rate of convergence. This is the content of the next result we present.

**Theorem 4.5.** Let \( E \) be given by (3.5), for some known \( E_0 \). For simplicity, let us suppose the initial data are given real numbers, i.e., \( q^j \in \mathbb{R} \), for \( j \in \{1, \ldots, N\} \), and also \( E_0 \in \mathbb{R} \). Then, assuming the convergence of the numerical sequences
\[ \frac{1}{N} \sum_{j=1}^N q^j \xrightarrow{N \to \infty} E_0, \]
the following limit,
\[ \lim_{N \to \infty} E_N = E, \]
is valid uniformly on \([0, T]\). Furthermore, it holds the following rate of convergence:
\[ \sup_{t \in [0, T]} |E_N(t) - E(t)| = O \left( \left| \frac{1}{N} \sum_{j \in \mathcal{N}} q^j - E_0 \right| + \frac{1}{N} \right). \]  

**Proof.** In the present proof, \( C > 0 \) will denote a generic constant, which possibly depends on model parameters, but does not depend upon \( N \). Throughout the estimates made here, \( C \) may change from line to line.

Let us fix \( t \in [0, T] \) and write
\[ E_N(t) - E(t) = \frac{y(t)}{y_N(0)y(0)} \left[ y(0)(E_N(0) - E(0)) + E(0)(y(0) - y_N(0)) \right] \]
\[ + \frac{E_N(0)}{y_N(0)}(y_N(t) - y(t)). \]  

Since the numerical sequences \( \{E_N(0)\} \) and \( \{y_N(0)\} \) are convergent, the latter converging to \( y(0) > 0 \), we deduce that
\[ |E_N(0)| \leq C \text{ and } \frac{1}{C} \leq y_N(0) \leq C, \]  
for large enough \( N \). Moreover, \( y : [0, T] \to [0, \infty) \) is continuous, whence bounded above and away from zero, i.e.,
\[ \frac{1}{C} \leq \sup_{u \in [0, T]} y(u) \leq C. \]  

From the identity (4.14), together with (4.15) and (4.16), we are apt to estimate
\[ |E_N(t) - E(t)| \leq C \left( |E_N(0) - E(0)| + |y(0) - y_N(0)| + |y_N(t) - y(t)| \right). \]
Next, let us employ the notations
\[
\begin{align*}
    c_{\pm}^N &:= \frac{1}{2N} [r_{\pm}^N + \frac{1}{\kappa} (A - \frac{\alpha}{2N})], \\
    c_{\pm} &:= \frac{1}{2N} [r_{\pm} + \frac{A}{\kappa}],
\end{align*}
\]
in such a way that
\[
y_N(t) - y(t) = (c_{\pm} - c_{\pm}^N) e^{-r_{\pm}^N (T-t)} + c_{\pm}^N e^{-r_{\pm}^N (T-t)} - e^{-r_{\pm} (T-t)} \tag{4.18}
\]
\[
+ (c_{\pm} + c_{\pm}^N - c_{\pm}^N) e^{-r_{\pm} (T-t)} + c_{\pm}^- e^{-r_{\pm} (T-t)} - e^{-r_{\pm}^N (T-t)}.
\]

We compute
\[
c_{\pm} - c_{\pm}^N = \frac{1}{2\theta N} \left[ r_{\pm}^N + \frac{1}{\kappa} (A - \frac{\alpha}{N}) \right] (\theta - \theta_N) + \frac{1}{2\theta} \left( r_{\pm}^N - r_{\pm} - \frac{\alpha}{2\kappa N} \right).
\]

As in (4.15), we are able to infer
\[
|r_{\pm}^N| \leq C \text{ and } \frac{1}{C} \leq \theta_N \leq C, \tag{4.19}
\]
as long as \(N\) is sufficiently large. It is also straightforward to derive the relations
\[
|\theta - \theta_N| \leq \frac{C}{N} \text{ and } |r_{\pm}^N - r_{\pm}| \leq \frac{C}{N}. \tag{4.20}
\]

Moreover, applying the mean value theorem to the function \(r \mapsto e^{-r^2(T-t)}\) allows us to conclude
\[
|e^{-r_{\pm}^N (T-t)} - e^{-r_{\pm} (T-t)}| \leq (T-t)^2 e^{|r_{\pm}||r_{\pm}^N - r_{\pm}|} \leq \frac{C}{N}. \tag{4.21}
\]

Utilizing (4.19), (4.20) and (4.21) in (4.18), we obtain
\[
|y_N(t) - y(t)| \leq \frac{C}{N}. \tag{4.22}
\]

We emphasize that the constant \(C\) figuring in (4.22) is independent of \(t\) and \(N\). Thus, from (4.17) and (4.22), it follows that
\[
|E_N(t) - E(t)| \leq C \left( |E_N(0) - E_0| + \frac{1}{N} \right),
\]
and this proves the desired result.

We finish this subsection with some numerical experiments intended to illustrate the convergence deduced in Theorem 4.5. The parameters used are exposed in Table 3.

| \(T\) | \(\alpha\) | \(\kappa\) | \(\phi\) | \(A\) | \(E_0\) |
|---|---|---|---|---|---|
| 1 | 1e - 4 | 1e - 4 | 1e - 4 | 1e - 4 | 10 |

Table 3: Parameters for the finite games with homogeneous market parameters.

We begin plotting in Figure 4 the average inventories \(E_1, \ldots, E_5\) initiated at the same level \(E_0 = 10\). The speed of convergence is noticeably fast in the present example.

We also consider in Figure 5 the corresponding deviations of the average finite optimal inventories relative to the benchmark. There, we evidence that \(|E_N - E|/|E| = O(1/N)|\).

Next, we investigate in Figure 6 the behavior \(E_N\), \(N \in \{2, \ldots, 5\}\), when they are initiated at a value equal to \(10(1 - 1/N)\). We also present, analogously as in the case of equal initial datum among \(E, E_1, \ldots, E_5\), the deviations of them relative to the benchmark.
Figure 4: In the left panel, we show the mean-fields for the five first finite games and for the benchmark. In the right panel, we zoomed the plot to show these trajectories for times $0.25 \leq t \leq 0.5$.

Figure 5: Plots of the differences among the average inventory fields in the finite games and the benchmark.

Figure 6: Inventories starting with initial value $10(1 - 1/N)$ in the corresponding $N$-player game (left) and difference between them and the benchmark (right).
5. Accounting for a hierarchy structure and asymmetric information

In this section, we describe a model including one major player, having the first move advantage, and a finite population of minor players, all individuals constituting it being high-frequency informed traders. The information in our model is represented via an adequate inclusion between the sigma-algebras to which each of the corresponding state processes for each of the populations is adapted.

The hierarchy in the present model is introduced via the kind of equilibrium we seek for, namely, a Stackelberg-Nash type of equilibrium.

In order to identify the Stackelberg-Nash equilibrium, we proceed as follows. First, given \( \nu^0_t \), we find a Nash equilibrium for the finite player game of the minor players in terms of it. Next, we feed this Nash equilibrium back into the functional the major player intends to maximize and optimize it. Since we characterize the controls chosen by all players at each step, the equilibrium we find is unique under the Stackelberg-Nash paradigm.

Below, we expand on this approach, adopting the previously developed variational methodology.

5.1. Basic state processes

We assume \( \mathbb{F}^0 := \{\mathcal{F}^{0}_t\} \) is the sigma-algebra to which the rate of trading of the major player, \( \{\nu^0_t\} \), is adapted to, and \( \mathbb{F} := \{\mathcal{F}_t\} \) is the one to which those of the minor players, \( \{\nu^i_t : i = 1, \ldots, N\} \), are adapted to. As was previously announced, we assume players labeled by \( i > 0 \) are informed traders, on the sense that, for each \( t \in [0, T] \), the inclusion \( \mathcal{F}_t \supseteq \mathcal{F}^{0}_t \) holds; in this connection, see [18].

The stock price process has dynamics
\[
dS_t = \left( \alpha^0 \nu^0_t + \alpha \mathbb{E} \left[ \frac{1}{N} \sum_{j=1}^{N} \nu^j_t \big| \mathcal{F}^0_t \right] \right) dt + \sigma dW_t.
\]

Notice that \( \{S_t\} \) must be \( \mathbb{F} \)-progressively measurable. The inventory process is assumed to evolve according to
\[
dq^i_t = \nu^i_t dt, \quad i \geq 0.
\]

The wealth process of the major player \( \{X^{0}_t\} \) satisfies
\[
dX^{0}_t = -\nu^0_t (S_t + \kappa \nu^0_t) dt.
\]

On the other hand, we suppose here and henceforth that preferences are homogeneous among the population of minor players so that their wealth processes solve
\[
dX^i_t = -\nu^i_t (S_t + \kappa \nu^i_t) dt, \quad i > 0.
\]

5.2. For each major strategy, minors accommodate to an NE

Assume \( i > 0 \). For each \( (\nu^i, \nu^{-i}, \nu^0) \), we set
\[
J_i(\nu^i, \nu^{-i}, \nu^0) := \mathbb{E} \left[ X^{i}_T + Q^{i}_T S_T - A(Q^{i}_T)^2 - \phi \int_{0}^{T} (Q^{i}_u)^2 du \right] - (x^i + q^i S)
\]
\[
= \mathbb{E} \left[ \int_{0}^{T} \left\{ -\kappa (\nu^i)^2 + q^i \left( \alpha^0 \nu^0_t + \alpha \mathbb{E} \left[ \frac{1}{N} \sum_{j=1}^{N} \nu^j_t \big| \mathcal{F}^0_t \right] \right) \right\} dt
\]
\[
- A(q^i)^2 \right].
\]
Proceeding as before, for an adequate set of parameters, we can prove the strict concavity and coercivity properties of the functional \( w \mapsto J_i(w, \nu^{-1}, \nu^0) \). Thus, in order to characterize the NE of the minor players, we just have to investigate the first-order optimality conditions.

Let us set
\[
L^2 := \left\{ x : x \text{ is } \mathbb{F} - \text{progressively measurable and } \mathbb{E} \left[ \int_0^T x_t^2 dt \right] < \infty \right\},
\]
and we then write
\[
(x, y) := \mathbb{E} \left[ \int_0^T x_t y_t dt \right],
\]
as well as
\[
\|x\| := (x, x)^{1/2}.
\]
We also consider the spaces
\[
H^1(0, T; \mathbb{L}^2) := \left\{ x \in \mathbb{L}^2 : x, \dot{x} \in \mathbb{L}^2 \right\},
\]
(5.1)
\[
H := \left\{ x \in H^1(0, T; \mathbb{L}^2) : x(0) = 0 \right\}.
\]
(5.2)
The space, \( H \), is endowed with the norm
\[
\|x\|_H := (\|x\|^2 + \|\dot{x}\|^2)^{1/2}.
\]
If \( x \in H \), then
\[
x_t = \int_0^t \dot{x}_u du.
\]
In particular, from the Cauchy-Schwarz inequality, we infer
\[
\sup_{t \in [0, T]} |x_t| \leq T^{1/2} \|\dot{x}\|
\]
and also
\[
\|x\| \leq \frac{T}{\sqrt{2}} \|\dot{x}\|.
\]
Let us designate by \( P : \mathbb{L}^2 \to L^2 \) the projection operator, which is such that
\[
P(x)_t := \mathbb{E} \left[ x_t | \mathcal{F}_0^t \right].
\]
The operator \( P \) projects onto the space of \( \mathbb{F}^0 \)-progressively measurable processes. Taking the Gâteaux derivative of \( J_i \) in the corresponding direction, we obtain
\[
\langle D_i J_i(\nu^i, \nu^{-1}; \nu^0), w^i \rangle = \mathbb{E} \left[ \int_0^T w_t^i \left\{ -2 \kappa \nu_t^i - 2\phi \int_t^T q_u^j du - 2Aq_T^i 
\right.
\right.
\]
\[
+ \int_t^T \left( \alpha \nu_t^0 + \alpha \mathbb{E} \left[ \frac{1}{N} \sum_{j=1}^{N} \nu_u^j | \mathcal{F}_u^0 \right] \right) du \right) dt 
\]
\[
+ \frac{\alpha}{N} \int_0^T q_t^i \mathbb{E} \left[ w_t^i | \mathcal{F}_t^0 \right] dt 
\]
\[
= \mathbb{E} \left[ \int_0^T w_t^i \left\{ -2 \kappa \nu_t^i - 2\phi \int_t^T q_u^j du - 2Aq_T^i 
\right.
\right.
\]
\[
+ \int_t^T \left( \alpha \nu_t^0 + \alpha \mathbb{E} \left[ \frac{1}{N} \sum_{j=1}^{N} \nu_u^j | \mathcal{F}_u^0 \right] \right) du 
\]
\[
+ \frac{\alpha}{N} \left( P^* q^i_t \right) dt \right].
\]
We infer that the processes \( \{q^i\} \) and \( \{\nu^i\}\) characterizing an NE for the minor players should be solutions of the abstract FBSDE system

\[
\begin{align*}
dq^i_t &= \nu^i_t dt, \\
-2\kappa d\nu^i_t &= -2\phi q^i_t + \left( \alpha^0 \nu^0_t + \alpha \mathbb{E} \left[ \frac{1}{N} \sum_{j=1}^{N} \nu^j_t \right] \mathcal{F}^0_t \right) dt \\
q^0_0 &= q^i, -2\kappa \nu^0_t = 2A \nu^0_t - \frac{\alpha}{N} (P^* q^0_t) T.
\end{align*}
\]

for suitable square integrable martingales \( \{q^i\} \text{ and } \{\nu^i\} \) for some constant \( C > 0 \).

For now on, we denote the NE of the followers corresponding to each \( \nu^0 \) by \( \{\nu^i(\nu^0)\}_i \), and the corresponding inventories by \( \{q^i(\nu^0)\}_i \). Let us also write

\[
E_N(\nu^0)_t := \frac{1}{N} \sum_{i=1}^{N} q^i(\nu^0)_t,
\]

and

\[
\mu_N(\nu^0)_t := \frac{1}{N} \sum_{i=1}^{N} \nu^i(\nu^0)_t.
\]

The pair \( (E_N, \mu_N) \) solves the FBSDE

\[
\begin{align*}
dE_N(\nu^0)_t &= \mu_N(\nu^0)_t dt, \\
-2\kappa d\mu_N(\nu^0)_t &= -2\phi E_N(\nu^0)_t dt + \left( \alpha^0 \nu^0_t + \alpha \mathbb{E} \left[ \mu_N(\nu^0)_t \right] \mathcal{F}^0_t \right) dt \\
E_N(\nu^0)_0 &= \frac{1}{N} \sum_{j=1}^{N} q^j, \\
-2\kappa \mu_N(\nu^0)_T &= 2A E_N(\nu^0)_T - \frac{\alpha}{N} (P^* E_N(\nu^0)_T),
\end{align*}
\]

for an adequate square integrable martingale \( \{M^N_t, \mathcal{F}_t\}_t \). We now proceed to prove a well-posedness theorem for the above system. In order to do so, we define a natural notion of a solution for it below.

**Definition 5.1.** We say that (5.3) is well-posed if, for any \( (E_{N,0}, \nu^0) \in (L^2)^2 \), there exists a unique pair \( (E_N - E_{N,0}, \mu_N) \in H \times L^2 \) such that \( E_N = \mu_N \),

\[
\|E_N\| + \|\mu_N\| \leq C \left( \|E_{N,0}\| + \|\nu^0\| \right),
\]

for some constant \( C > 0 \) (possibly dependent of the model parameters), and for all \( w \in L^2 \), we have

\[
0 = \mathbb{E} \left[ \int_0^T w_t \left\{ -2\kappa \mu_{N,t} - 2\phi \int_t^T E_{N,u} du - 2AE_{N,T} \right. \\
\left. + \int_t^T \left( \alpha^0 \nu^0_u + \alpha \mathbb{E} \left[ \mu_{N,u} | \mathcal{F}^0_u \right] \right) du + \frac{\alpha}{N} (P^* E_{N,t}) \right\} dt \right].
\]

This definition leads to the following result.

**Theorem 5.2.** Suppose that the functional

\[
w \in L^2 \mapsto \mathbb{E} \left[ \int_0^T w_t \left\{ -2\kappa \mu_{N,t} - 2\phi \int_t^T E_{N,u} du - 2AE_{N,T} \right. \\
\left. + \int_t^T \left( \alpha^0 \nu^0_u + \alpha \mathbb{E} \left[ \mu_{N,u} | \mathcal{F}^0_u \right] \right) du + \frac{\alpha}{N} (P^* E_{N,t}) \right\} dt \right] \in \mathbb{R}
\]

20
is strictly concave and coercive, and also that the model parameters satisfy

\[
\begin{align*}
\phi > 0 \text{ and } 2\kappa > \frac{\phi^2}{4\phi} \left(1 + \frac{1}{N}\right)^2, \\
or else \ 2\kappa > \alpha \left(1 + \frac{1}{N}\right) \frac{T}{\sqrt{2}}.
\end{align*}
\]  

(5.5)

Then, the system (5.3) is well-posed. Furthermore, if \(E_{N,0} \equiv 0\), then

\[
\|\mu_N(\nu^0)\| \leq \frac{\alpha^0 T}{\theta \sqrt{2}} \|\nu^0\|,
\]  

(5.6)

where \(\theta := 2\kappa - \max \left(\frac{\alpha^2}{4\phi} \left(1 + \frac{1}{N}\right)^2, \alpha \left(1 + \frac{1}{N}\right) \frac{T}{\sqrt{2}}\right)\).

Proof. Writing \(x = E_N - E_{N,0}, \dot{y} = w\), it follows that (5.4) admits the alternative expression

\[
\mathcal{B}[x, y] = \langle \ell, y \rangle, \text{ for all } y \in L^2,
\]

where

\[
\langle \ell, y \rangle := \mathbb{E} \left[ \int_0^T \dot{y}_t \left\{\frac{\alpha}{N} (P^* x)_t \right\} dt \right].
\]

Claim 1: \(\mathcal{B}\) is a continuous bilinear form.

We write \(\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3 + \mathcal{B}_4 + \mathcal{B}_5\), where

\[
\mathcal{B}_1 := 2\kappa \mathbb{E} \left[ \int_0^T \dot{x}_t \dot{y}_t dt \right],
\]

\[
\mathcal{B}_2 := 2\phi \mathbb{E} \left[ \int_0^T \dot{y}_t \int_t^T x_u du dt \right] = 2\phi \mathbb{E} \left[ \int_0^T x_t y_t dt \right],
\]

\[
\mathcal{B}_3 := 2\alpha \mathbb{E} \left[ \int_0^T \dot{y}_t x_T dt \right] = 2\alpha \mathbb{E} [x_T y_T],
\]

\[
\mathcal{B}_4 := -\alpha \mathbb{E} \left[ \int_0^T \dot{y}_t \int_t^T (P^* x) u du dt \right] = -\alpha \mathbb{E} \left[ \int_0^T y_t (P^* x) dt \right],
\]

\[
\mathcal{B}_5 := -\frac{\alpha}{N} \mathbb{E} \left[ \int_0^T \dot{y}_t (P^* x) dt \right].
\]

Since

\[|\mathcal{B}[x, y]| \leq 2\kappa \|\dot{x}\| \|\dot{y}\|,\]
we conclude that
\[ |B(x, y)| \leq C \|x\| \|y\|, \]
for some constant \( C = C(\kappa, \phi, A, \alpha, T) > 0 \). This proves Claim 1.

**CLAIM 2:** \( \ell \) is a continuous linear form.

This time we set
\[
\langle \ell_1, y \rangle := \mathbb{E} \left[ \int_0^T y_t \left\{ -2\phi(T-t)E_{N,0} - 2AE_{N,0} + \frac{\alpha}{N} (P^* E_{N,0})_t \right\} dt \right],
\]
\[
\langle \ell_2, y \rangle := \alpha^0 \mathbb{E} \left[ \int_0^T \int_t^T \nu_0^d du dt \right] = \alpha^0 \mathbb{E} \left[ \int_0^T \nu_0^d y_t dt \right],
\]
so that \( \ell = \ell_1 + \ell_2 \). From this decomposition, we infer, as in the above claim, the estimate
\[ |\langle \ell, y \rangle| \leq \left( 2AT^{1/2} + 2\phi + \frac{\alpha}{N} \right) \|E_{N,0}\| \|y\|_H + \alpha^0 \|\nu^0\| \|y\|. \]

This finishes the demonstration of Claim 2.

**CLAIM 3:** \( \overline{B} \) is coercive.

Indeed, we have
\[
\overline{B}_1[x, x] = 2\kappa \|\dot{x}\|^2,
\]
\[
\overline{B}_2[x, x] = 2\phi \|x\|^2,
\]
\[
\overline{B}_3[x, x] = 2AE \left[ (x_T)^2 \right] \geq 0,
\]
\[
\overline{B}_4[x, x] + \overline{B}_5[x, x] = -\alpha \left( x, P\dot{x} \right) + \frac{1}{N} (\dot{x}, P^* x) = -\alpha \left( 1 + \frac{1}{N} \right) (x, P\dot{x}).
\]

In case \( \phi > 0 \), we can proceed to estimate
\[ \overline{B}_4[x, x] + \overline{B}_5[x, x] \geq -\phi \|x\|^2 - \frac{\alpha^2}{4\phi} \left( 1 + \frac{1}{N} \right)^2 \|\dot{x}\|^2. \]

Alternatively, we can instead consider
\[ \overline{B}_4[x, x] + \overline{B}_5[x, x] \geq -\alpha \left( 1 + \frac{1}{N} \right) \frac{T}{\sqrt{2}} \|\dot{x}\|^2 \]
This implies that there exists \( \tilde{\theta} > 0 \) for which
\[ \overline{B}[x, x] \geq \tilde{\theta} \|x\|_H^2. \]

This establishes Claim 3.
Accordingly, we define the spaces \( E_N, M \) as follows:

\[
E_N = \{ x \in H : \int_0^T \| \dot{x} \|^2 \, dt < \infty \},
\]

\[
M = \{ x \in H : \int_0^T \| \dot{x} \|^2 + \frac{\alpha}{N} \| x \| \, dt < \infty \}.
\]

We conclude from Theorem 5.2 that

\[
\theta \| \dot{x} \|^2 \leq \langle \mathcal{F}, x \rangle = \langle \mathcal{F}, x \rangle \leq (2AT^{1/2} + 2\phi + \frac{\alpha}{N}) \| E_N \| \| x \| + \alpha \| \nu^0 \| \| x \|.
\]

In particular, \( E_N \equiv 0 \) allows us to conclude (5.6). □

Note that the mapping \( L : \nu^0 \in L^2 \rightarrow (E_N(\nu^0) - E_N(0), \mu_N(\nu^0) - \mu_N(0)) \in H \times L^2 \) is linear. Explicitly, writing \( L\nu^0 = (L_1\nu^0, L_2\nu^0) = (\tilde{q}, \tilde{\nu}) \), we have

\[
\begin{align*}
\frac{d\tilde{q}_t}{dt} &= \tilde{\nu}_t dt,
\frac{1}{2}d\tilde{\nu}_t &= -2\phi\tilde{\nu}_t dt + (\alpha^0\nu_t^0 + \alpha E [\tilde{\nu}_t | F^0_t]) dt - \frac{\alpha}{N} d(P^*\tilde{q}_t) - d\tilde{M}_t, \\
\tilde{q}_0 &= 0, -2\kappa\tilde{\nu}_T = 2\tilde{\nu}_T - \frac{\alpha}{2} (P^*\tilde{q}_0)^T,
\end{align*}
\]

where \( \{\tilde{M}_t, F_t\} \) is a square integrable martingale. Since the above system has the same form as (5.3), we conclude from Theorem 5.2 that \( L : H \times L^2 \rightarrow L^2 \) is continuous. In particular, from (5.6), we also deduce the next corollary:

**Corollary 5.3.** The operator \( L_2 : L^2 \rightarrow L^2 \) is continuous, and moreover

\[
\| L_2\nu^0 \| \leq \frac{\alpha^0 T}{\theta \sqrt{2}} \| \nu^0 \|.
\]

### 5.3. Major optimizes conditional on minors following NE

We begin the present subsection by defining the Hilbert spaces relative to the major player. Firstly, we set

\[
L^{2,0} := \left\{ x : x \text{ is } F^0 \text{- progressively measurable and } \mathbb{E} \left[ \int_0^T x_t^2 \, dt \right] < \infty \right\}.
\]

Accordingly, we define the spaces \( H^1(0, T; L^{2,0}) \) and \( H^0 \) in a way analogous to (5.1) and (5.2). The functional for the major player, denoted by \( J : \| \cdot \|_0 \rightarrow \mathbb{R} \), is given by

\[
J_0(\nu^0) := \mathbb{E} \left[ X^0_T + q^0_T S^0_T - A^0(\nu_T^0)^2 - \phi^0_0 \int_0^T (q_u^0)^2 \, du \right]
- \left[ q^0 + \frac{\alpha^0}{2} (q^0)^2 \right]
= \mathbb{E} \left[ \int_0^T \left\{ -\kappa^0(\nu_t^0)^2 - \phi^0(\nu_t^0)^2 + \alpha q_t^0 \mathbb{E} [\mu_N(\nu^0) | F_t^0] \right\} dt \right.
- \left( A^0 - \frac{\alpha^0}{2} \right) (q_T^0)^2
= \mathbb{E} \left[ \int_0^T \left\{ -\kappa^0(\nu_t^0)^2 - \phi^0(\nu_t^0)^2 + \alpha q_t^0 (P \mu_N(\nu_0))_t \right\} dt \right.
- \left( A^0 - \frac{\alpha^0}{2} \right) (q_T^0)^2].
\]
It is pertinent to observe that the functional of the major player depends only on the average turnover rates on the minor players, and not on each of the speeds of the minors separately. This is a consequence of our assumption of homogeneous preferences among the population of minor players.

Once again, we assume here and henceforth that $J_0$ is strictly concave and coercive, which is valid under mild assumptions on the major player’s preferences. Upon differentiating $J_0$, we deduce that the maximizer must solve

\[
\begin{align*}
J_0'(v^0) \cdot w^0 &= E \left[ \int_0^T w_0^0 \left\{ -2\kappa^0 \nu_0^0 - 2\phi^0 \int_t^T q_0^0 du + \alpha \int_t^T P_{\mu_N}(v^0)_u du \\
&\quad + \alpha (PL_2)^* q^0_0 + (2A^0 - \alpha^0) q_0^0 \right\} dt \right].
\end{align*}
\]

we deduce that the maximizer must solve

\[
\begin{align*}
\begin{cases}
dq_0^0 = \nu_0^0 dt, \\
-2\kappa^0 dq_0^0 = \left\{ -2\phi^0 q_0^0 + \alpha \left[ P_{\mu_N}(v^0) \right]_t \right\} dt - \alpha \left[ P(PL_2)^* q^0 \right]_t, \\
-\mu_0^0, \\
q_0^0 = q^0, -2\kappa^0 \nu_0^0 = (2A^0 - \alpha^0) q_0^0 - \alpha \left[ P(PL_2)^* q^0 \right]_T,
\end{cases}
\end{align*}
\]

(5.8)

where \( \{M_0^0, F_0^0\} \) is a suitable square integrable \( \mathbb{F}^0 \)-martingale.

We can prove a well-posedness result for (5.8) in a similar manner as done for (5.3). This is the following result.

**Theorem 5.4.** Assume $J_0$ is strictly concave and coercive, that the model parameters satisfy $2A^0 - \alpha^0 > 0$, and further

\[
\begin{align*}
\phi^0 > 0 \text{ and } 2\kappa^0 \geq \frac{\alpha^2 (\alpha^0)^2 T^2}{2P(\nu^0)^2}, \\
or else \ 2\kappa^0 > \frac{\alpha^0 \alpha T^2}{\mathbb{q}}.
\end{align*}
\]

Then, given \((E_{N,0}, q_0^0) \in \mathbb{L} \times \mathbb{L}^{2,0}\), the variational problem

\[
J_0'(v^0) \cdot w^0 = 0, \text{ for all } w^0 \in \mathbb{L}^{2,0},
\]

has a unique solution \((q^0, \nu^0) \in (\mathbb{L}^{2,0})^2 \) such that $q^0 - q_0^0 \in H^0$. Furthermore, there exists a constant $C = C(\alpha, \alpha^0, \kappa, \kappa^0, A, A^0, T) > 0$ for which

\[
\|q^0\| + \|
u^0\| \leq C \left( \|E_{N,0}\| + \|q_0^0\| \right).
\]

**Proof.** Now we write $x = q - q_0^0$ and $\hat{y} = w^0$. The variational problem (5.9) can be cast in the following form:

\[
E \left[ \int_0^T \hat{y}_t \left\{ 2\kappa^0 \dot{x}_t + 2\phi^0 \int_t^T x_u du - \alpha \int_t^T (PL_2)^* x_t du - \alpha [(PL_2)^* x]_t \\
+ (2A^0 - \alpha^0) x_T \right\} dt \right]
\]

\[
= E \left[ \int_0^T \hat{y}_t \left\{ -2\kappa^0 q_0^0 (T - t) + \alpha \int_0^T P_{\mu_N}(0) du + \alpha \left[ (PL_2)^* q_0^0 \right]_t \\
- (2A^0 - \alpha^0) q_0^0 \right\} dt \right],
\]

or equivalently,

\[
B[x, y] = \langle \ell, y \rangle, \text{ for all } y \in H.
\]
where the bilinear form $B : H \times H \to \mathbb{R}$ and the linear form $\ell : H \to \mathbb{R}$ are defined as

$$B[x, y] := E \left[ \int_0^T \dot{y}_t \left( 2\kappa^0 \dot{x}_t + 2\phi^0 \int_t^T x_u du - \alpha \int_t^T (PL_2 \dot{x})_u \, du \right) \right]$$

$$- \alpha \left[ (PL_2)^* x \right]_t + (2A^0 - \alpha^0)x_T \right] dt,$$

$$\langle \ell, y \rangle := E \left[ \int_0^T \dot{y}_t \left( -2\kappa^0 \dot{y}_0(T-t) + \alpha \int_t^T P\mu N(0)_u du + \alpha \left[ (PL_2)^* y_0 \right]_t \right) dt \right].$$

**CLAIM 1:** The bilinear form $B$ is continuous.

We decompose $B$ in the following form

$$B[x, y] = B_1[x, y] + B_2[x, y] + B_3[x, y] + B_4[x, y] + B_5[x, y],$$

where

$$B_1[x, y] := 2\kappa^0 E \left[ \int_0^T \dot{x}_t \dot{y}_t \, dt \right],$$

$$B_2[x, y] := 2\phi^0 E \left[ \int_0^T \dot{y}_t \int_t^T x_u du \, dt \right] = 2\phi^0 E \left[ \int_0^T x_t \dot{y}_t \, dt \right],$$

$$B_3[x, y] := -\alpha E \left[ \int_0^T \dot{y}_t \int_t^T (PL_2 \dot{x})_u du \, dt \right] = -\alpha E \left[ \int_0^T \dot{y}_t (PL_2 \dot{x})_t \, dt \right],$$

$$B_4[x, y] := -\alpha E \left[ \int_0^T \dot{y}_t [(PL_2)^* x]_t \, dt \right],$$

$$B_5[x, y] := (2A^0 - \alpha^0) E \left[ \int_0^T \dot{y}_t x_T \, dt \right] = E \left[ x_T y_T \right].$$

Thus,

$$|B_1[x, y]| \leq 2\kappa^0 \| \dot{x} \| \| \dot{y} \|,$$

$$|B_2[x, y]| \leq 2\phi^0 \| x \| \| y \||\dot{x}|,$$

$$|B_3[x, y]| \leq \alpha \| y \| L_2^2 \| \dot{x} \|,$$

$$|B_4[x, y]| \leq \| \dot{y} \| L_2^2 \| \dot{x} \|,$$

$$|B_5[x, y]| \leq (2A^0 - \alpha^0) T \| \dot{x} \| \| \dot{y} \|.$$

whence

$$|B[x, y]| \leq C_0 \| x \| H \| y \| H$$

and therefore $B$ is continuous, proving Claim 1.

**CLAIM 2:** The bilinear form $B$ is coercive.

Maintaining the notations above, we have

$$B_1[x, x] = 2\kappa^0 \| \dot{x} \|^2,$$

$$B_2[x, x] = 2\phi^0 \| x \|^2,$$

$$B_3[x, x] + B_4[x, x] = -\alpha [(x, PL_2 \dot{x}) + (\dot{x}, (PL_2)^* x)] = -2\alpha (x, PL_2 \dot{x});$$

$$B_5[x, x] = (2A^0 - \alpha^0) \| x \| T \| \dot{x} \| \| \dot{y} \|.$$
In the case when $\phi^0 > 0$, we can estimate

$$B_3[x, x] + B_4[x, x] \geq -\phi^0 \|x\|^2 - \frac{\alpha^2 \|L_2\|^2}{\phi^0} \|\dot{x}\|^2.$$ 

In general, we can always consider the lower bound

$$B_3[x, x] + B_4[x, x] \geq -\alpha T \sqrt{2} \|L_2\| \|\dot{x}\|^2.$$ 

Finally, it is clear that

$$B_5[x, x] \geq 0.$$ 

Altogether, we conclude

$$B[x, x] \geq \theta_0 \|x\|_H^2,$$ 

for an adequate constant $\theta_0 > 0$; hence, Claim 2 is established.

It can be proven in a similar way that $\ell : H \to \mathbb{R}$ is continuous, with $\|\ell\| \leq C \left( \|E_{N,0}\| + \|q_0^0\| \right)$. Here, we omit the details.

Invoking the Lax-Milgram Lemma one more time (see [26]), we deduce that the variational problem (5.9) admits a unique solution. The continuity with respect to the initial data follows from

$$\theta_0 \|x\|_H^2 \leq B[x, x] = \langle \ell, x \rangle \leq \|\ell\| \|x\|_H,$$

whence

$$\|x\|_H \leq \frac{\|\ell\|}{\theta_0} \leq C \left( \|E_{N,0}\| + \|q_0^0\| \right).$$

6. Conclusions

We studied game-theoretical models of optimal trade execution from two different perspectives. On the one hand, we began with the finite player game, assuming neither informational imbalance nor hierarchy among the agents. On the other hand, we presented a way to consider the situation in which there is a large institutional investor optimally executing against a crowd of high-frequency informed traders. Thus, we stipulated a hierarchical game, with the major player having first move advantage, whereas the population of competing rational minor players had an informational advantage, utilizing it to trade against the major one.

In the first model we investigated, we were able to apply variational techniques to obtain a closed-form solution for the inventories and corresponding rates of trading in the most general framework, i.e., the one in which preferences are assumed completely heterogeneous. We proceeded to particularize to the case where preferences are homogeneous. The MFG counterpart of the latter was analyzed in [9]. We obtained a convenient representation for the MFG, which allowed us to prove that the average finite inventory field converges uniformly to the mean-field limit, over the fixed time horizon, as the population size grows to infinity. Moreover, we were able to obtain a rate of convergence for this uniform limit.

The variational approach was likewise successful in the second model we analyzed. It led us to a characterization of the Stackelberg-Nash equilibrium as solutions to abstract FBSDE systems. We defined a notion of weak solutions to these and proceeded to show well-posedness results in adequate Hilbert spaces. These proofs depended upon a relation between model parameters, pointing in the direction of an interplay among the urgency of the players and the size of the time horizon.

A pertinent question is how the Stackelberg-Nash equilibrium we identified in the last model, when particularized to the case of symmetric information, relates to the MFG studied in [22]. In this case, closed-form formulas for the strategies we obtained for the finite game may be available. It is worthwhile to investigate whether they converge to the corresponding solutions found in [22], as the population size grows to infinity.
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References

[1] Almgren, R. and Chriss, N. (1999). Value under liquidation. Risk, 12(12):61–63.
[2] Almgren, R. and Chriss, N. (2001). Optimal execution of portfolio transactions. Journal of Risk, 3:5–40.
[3] Bank, P., Soner, H. M., and Voß, M. (2017). Hedging with temporary price impact. Mathematics and Financial Economics, 11(2):215–239.
[4] Bertsimas, D. and Lo, A. W. (1998). Optimal control of execution costs. Journal of Financial Markets, 1(1):1–50.
[5] Braouezec, Y. and Wagalath, L. (2018). Risk-based capital requirements and optimal liquidation in a stress scenario. Review of Finance, 22(2):747–782.
[6] Bucci, F., Mastromatteo, I., Eisler, Z., Lillo, F., Bouchaud, J.-P., and Lehalle, C.-A. (2020). Co-impact: Crowding effects in institutional trading activity. Quantitative Finance, 20(2):193–205.
[7] Cardaliaguet, P., Cirant, M., and Porretta, A. (2018). Remarks on nash equilibria in mean field game models with a major player. arXiv preprint arXiv:1811.02811.
[8] Cardaliaguet, P., Cirant, M., and Porretta, A. (2020). Splitting methods and short time existence for the master equations in mean field games. arXiv preprint arXiv:2001.10406.
[9] Cardaliaguet, P. and Lehalle, C.-A. (2017). Mean field game of controls and an application to trade crowding. Mathematics and Financial Economics, pages 1–29.
[10] Carmona, R. and Delarue, F. (2018). Probabilistic Theory of Mean Field Games with Applications I. Springer.
[11] Carmona, R., Delarue, F., and Nadirashvili, N. (2015). A probabilistic weak formulation of mean field games and applications. The Annals of Applied Probability, 25(3):1189–1231.
[12] Cartea, Á., Jaimungal, S., and Penalva, J. (2019). Trading algorithms with learning in latent alpha models. Mathematical Finance, 29(3):735–772.
[13] Freiling, G., Jank, G., and Sarychev, A. (2000). Non-blow-up conditions for riccati-type matrix differential and difference equations. Results in Mathematics, 37(1-2):84–103.
[14] Fudenberg, D. and Levine, D. (1998). The Theory of Games and Decision. MIT Press.
[15] GUÉANT, O. (2016). The Financial Mathematics of Market Liquidity: From optimal execution to market making, volume 33. CRC Press.
[16] Huang, M., Caines, P. E., and Malhamé, R. P. (2007). Large-population cost-coupled LQG problems with nonuniform agents: individual-mass behavior and decentralized e-Nash equilibria. IEEE Trans. Automat. Control, 52(9):1560–1571.
[17] Huang, M., Malhamé, R. P., and Caines, P. E. (2006). Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle. Commun. Inf. Syst., 6(3):221–251.
[18] Huang, X., Jaimungal, S., and Nourian, M. (2019). Mean-field game strategies for optimal execution. Applied Mathematical Finance, 26(2):153–185.
[19] Lasry, J.-M. and Lions, P.-L. (2006a). Jeux à champ moyen. I. Le cas stationnaire. C. R. Math. Acad. Sci. Paris, 343(9):619–625.
[20] Lasry, J.-M. and Lions, P.-L. (2006b). Jeux à champ moyen. II. Horizon fini et contrôle optimal. C. R. Math. Acad. Sci. Paris, 343(10):679–684.
[21] Lasry, J.-M. and Lions, P.-L. (2007). Mean field games. Jpn. J. Math., 2(1):229–260.
[22] Lax, P. and Milgram, A. (1964). Parabolic equations. Contributions to the Theory of Partial Differential Equations, Annals of Mathematics Studies 33, 33:167–190.
[23] Obizhaeva, A. A. and Wang, J. (2013). Optimal trading strategy and supply/demand dynamics. Journal of Financial Markets, 16(1):1–32.
[24] Reid, W. T. (1972). Riccati differential equations. Elsevier.
[25] Sophie, L. and Charles-Albert, L. (2018). Market microstructure in practice. World Scientific.
[26] Wagalath, L. and Zubelli, J. P. (2018). A liquidation risk adjustment for value at risk and expected shortfall. International Journal of Theoretical and Applied Finance, 21(03):1850010.