AN ALGORITHM TO GENERATE RANDOM FACTORED SMOOTH INTEGERS

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Abstract. Let $x \geq y > 0$ be integers. We present an algorithm that will generate an integer $n \leq x$ at random, with known prime factorization, such that every prime divisor of $n$ is $\leq y$. Further, asymptotically, $n$ is chosen uniformly from among all integers $\leq x$ that have no prime divisors $> y$.

In particular, if we assume the Extended Riemann Hypothesis, then with probability $1 - o(1)$, the average running time of our algorithm is

$$O\left(\frac{(\log x)^3}{\log \log x}\right)$$

arithmetic operations.

We also present other running times based on differing sets of assumptions and heuristics.

1. Introduction

Given integers $x$ and $y$, with $x \geq y > 0$, we say a positive integer $n \leq x$ is $y$-smooth if every prime divisor of $n$ is $\leq y$. (In the literature, friable is sometimes used in place of smooth.)

In this paper, we describe and analyze a new algorithm to generate a positive integer $n \leq x$ that is $y$-smooth, together with a complete list of $n$’s prime divisors. Further, we want $n$ to be chosen uniformly at random from among all $y$-smooth integers $\leq x$. Let $\Psi(x, y)$ count the number of $y$-smooth integers $\leq x$. Then we want the probability a particular $n$ is chosen to be $1/\Psi(x, y)$.

To achieve this, we start by describing an algorithm that lists all $y$-smooth integers $\leq x$. This method is recursive, and uses Buchstab’s identity as its central control mechanism. We then make use of a random input $r$, with $0 \leq r < 1$, and return the smooth number at position $k = \lfloor r\Psi(x, y) \rfloor$ from that listing. If we view the algorithm’s execution as a tree, then each leaf is a smooth number, and the listing is composed of all the leaves. We then prune the algorithm so that its execution generates the leaf for the smooth number at position $k$, while doing as little extra work as possible.

We then analyze the running time of our algorithm, and look at tradeoffs we can make to improve performance. In particular, we look at relaxing the uniformity requirement, we consider assuming unproven conjectures like the
ERH and a conjecture on the maximum size of gaps between primes, and we look at using randomized prime tests. We also discuss average-case versus worst-case running times.

We examine two special cases likely to arise in practice: What if a list of primes up to $y$ is already available to the algorithm? This might happen if $y$ is small. On the other extreme, if we set $y = x$, our algorithm generates random factored integers, and we look at the circumstances under which our new method is competitive with the algorithm from [4]. Also, can we generalize our work to generate random semismooth integers?

1.1. Model of Computation. We measure algorithm complexity by counting the number of arithmetic operations on integers of $O(\log x)$ bits. This includes addition/subtraction, multiplication/division with remainder, and other basic operations such as array indexing, comparisons, and branching. As a consequence, for example, performing a base-2 strong pseudoprime test on an integer $\leq x$ would take $O(\log x)$ arithmetic operations.

We also work with floating-point real numbers with up to $O(\log x)$ bits of precision, and again each operation on such a number takes unit time.

In practice, we would expect integers $\leq y$ to fit into 64-bit machine words, and floating-point numbers would be represented using a `long double` data type, which is typically 64 or 80 bits. Integers between $y$ and $x$ are not used often, and so multi-precision software (such as GMP) should not be needed except in rare cases. In fact, it might be possible to represent $x$ in a double precision floating point representation.

1.2. Paper Outline. The rest of our paper is organized as follows: We begin in §2 with an algorithm to compute $\Psi(x, y)$ exactly, and then modify it to enumerate all $y$-smooth integers $\leq x$. Then in §3 we show how to modify the algorithm to selectively list just one $y$-smooth integer $\leq x$ by pruning our first algorithm. In §4 we give a detailed running time analysis of our algorithm, and prove two running times with different sets of assumptions. We then discuss some special cases and applications in §5 and conclude with an example run in §6.

2. Enumerating All Smooth Integers

2.1. Buchstab’s Identity. As mentioned in the Introduction, our algorithms are based on a version of Buchstab’s identity,

$$\Psi(x, y) = 1 + \sum_{p \leq y} \Psi(x/p, p),$$

which decomposes $\Psi(x, y)$ by its largest prime divisor [29 §5.3]. Combine this with some base cases,

- $\Psi(x, y) = 0$ if $x < 1$ or $y < 1$, and
- $\Psi(x, 1) = 1$ if $x \geq 1$, and
- $\Psi(x, 2) = \lfloor \log_2 x \rfloor + 1$ if $x \geq 1$, and
- $\Psi(x, y) = x$ if $y \geq x$, and of course
\( \Psi(x, y) = \Psi(\lfloor x \rfloor, \lfloor y \rfloor) \),
and you have a simple recursive algorithm to compute the exact value of
\( \Psi(x, y) \), that runs in time \( O(\Psi(x, y)) \).

2.2. Example. Let’s illustrate this with an example to compute \( \Psi(15, 3) \).
We have
\[
\Psi(15, 3) = 1 + \Psi(15/2, 2) + \Psi(15/3, 3).
\]
Here \( \Psi(15/2, 2) = \Psi(7, 2) = \lfloor \log_2 7 \rfloor + 1 = 3 \), leaving \( \Psi(15/3, 3) = \Psi(5, 3) \)
as a recursive case:
\[
\Psi(5, 3) = 1 + \Psi(5/2, 2) + \Psi(5/3, 3) = 1 + (\lfloor \log_2 2.5 \rfloor + 1) + 1 = 4.
\]
Plugging back in, we get \( \Psi(15, 3) = 8 \), which is correct.

To have this method generate a list of numbers, we merely need to re-
member the prime \( p \) from the sum in Buchstab’s identity used to construct
the recursive call. We will annotate these primes as subscri pts to \( \Psi \) and
redo our example. We have
\[
\Psi(15, 3) = 1 + \Psi_2(15/2, 2) + \Psi_3(15/3, 3).
\]
This generates the 1. The call to \( \Psi(7, 2) \) generates powers of 2, namely
\( 2^0, 2^1, 2^2 \). The 2 coming in via the subscript tells us multiply everything
generated by 2, giving us the list \( 2 \cdot 2^0, 2 \cdot 2^1, 2 \cdot 2^2 \). \( \Psi(5, 3) \) would recursively
generate the list \( 1, 2, 3, 4 = 2^2 \), and applying the 3 subscript gives the list
\( 3, 3 \cdot 2, 3 \cdot 3, 3 \cdot 3 \cdot 2^2 \). The complete list, then, is
\[
(1, 2 \cdot 2^0, 2 \cdot 2^1, 2 \cdot 2^2, 3, 3 \cdot 2, 3 \cdot 3, 3 \cdot 3 \cdot 2^2),
\]
or \( 1, 2, 4, 8, 3, 6, 9, 12 \).

2.3. Intuition. If we take Buchstab’s identity and divide through by \( \Psi(x, y) \),
we see that when constructing a number \( n \leq x \), each prime \( p \leq y \) has a
chance to be \( n \)’s largest prime divisor, and that probability is \( \Psi(x/p, p)/\Psi(x, y) \).
No prime is chosen, or \( n = 1 \), with probability \( 1/\Psi(x, y) \).

Also note that if we omit the 4th base case (\( \Psi(x, y) = x \) if \( y \geq x \)), the
enumeration produced is in lexicographic order based on prime factorization.
So from our example, the enumeration could be written like this:

\[
\begin{align*}
1 \\
2 \\
2 \cdot 2 \\
2 \cdot 2 \cdot 2 \\
3 \\
2 \cdot 3 \\
2 \cdot 2 \cdot 3 \\
3 \cdot 3
\end{align*}
\]

Here’s another example starting at position 100 in the enumeration of 7-
smooth integers \( \leq 1000 \):
2.4. The Algorithm. It’s time to describe the algorithm in detail. The stack $S$ is used to store the primes we wrote as subscripts to $\Psi$, and the vector/array $V$ holds the enumeration, so $V = [1, 2, 4, 8, 3, 6, 9, 12]$ from our example, or, with complete factorizations,

$$V = [1, [2], [2, 2], [2, 2, 2], [3], [3, 2], [3, 3], [3, 2, 2]].$$

Procedure \textbf{Enumerate}($x, y, S, V$):

- If $x < 1$ Then do nothing and return;
- If $S$ is empty Then Append 1 onto $V$;
- Else Append a copy of $S$ onto $V$;
- For each prime $p \leq y$ Do:
  - Push $p$ onto $S$;
  - \textbf{Enumerate}($x/p, p, S, V$);
  - Pop $p$ from $S$;

Our example, then, would be produced using \textbf{Enumerate}(15, 3, $S = []$, $V = []$). The $S$ and $V$ arguments must be pass-by-reference, but $x$ and $y$ should be ordinary pass-by-value arguments.

To generate one randomly chosen smooth number, we simply construct the list $V$ and choose one entry uniformly at random.

\textbf{Algorithm 1.}

1. Set $S := []$ and $V := []$.
2. Choose $r$ uniformly at random, with $0 \leq r < 1$.
3. Set $k := \lfloor r \Psi(x, y) \rfloor$.
4. Call \textbf{Enumerate}($x, y, S, V$).
5. Output $V[k]$.

The running time, up to a constant factor, is the total number of calls to the \textbf{Enumerate}() function, which is also the total number of primes in the vector $V$ at the end. But this is just the total number of prime divisors of all $y$-smooth numbers $\leq x$. Due to the work of Alladi [2], Hensley [16], and
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Hildebrand [18] we know the average number of prime divisors is

\[ O \left( \log \log x + \frac{\log x}{\log y} \right). \]

So our running time is \( O(\Psi(x, y)(\log \log x + (\log x)/\log y)) \) arithmetic operations.

Wait a second. We had to find all the primes \( \leq y \). But this can be done in time \( O(y/\log \log y) \) using the Atkin-Bernstein sieve [3], and since \( \Psi(x, y) \geq \lfloor y \rfloor \) when \( x \geq y \), this cost is negligible. Even a simpler sieve of Eratosthenes is only \( O(y \log \log y) \) time, and would be fine here.

3. PRUNING ALGORITHM 1

Algorithm 1 above works, but is quite slow because it constructs all smooth numbers before selecting one at random. We only need to construct one. So, to improve our algorithm, instead of working through all the primes \( p \leq y \) in the main loop of the Enumerate function, we use the value of \( k = \lfloor r\Psi(x, y) \rfloor \) to figure out the prime \( p \leq y \) on which to recurse. From Buchstab’s identity, this means finding the consecutive primes \( p_1 \) and \( p_2 \) such that

\[ 1 + \sum_{p \leq p_1} \Psi(x/p, p) < k + 1 \leq 1 + \sum_{p \leq p_2} \Psi(x/p, p), \]

or, rewriting this applying Buchstab’s identity,

\[ \Psi(x, p_1) < k + 1 \leq \Psi(x, p_2). \]

Then, we recurse on the branch that computes \( \Psi(x/p_2, p_2) \). The value of \( k \) for the recursive call, \( k' \), is simply \( k - \Psi(x, p_1) \). The understanding, here, is that if \( k = 0 \) then we are returning 1, but if \( k + 1 \leq \Psi(x, 2) \), then we recurse on \( p_2 = 2 \), which means in this special case, \( p_1 \) is set to 1, even though 1 isn’t a prime.

This gives the following structure for our algorithm.

Algorithm 2.

1. Set \( S := [] \).
2. Choose \( r \) uniformly at random, with \( 0 \leq r < 1 \).
3. Set \( k := \lfloor r\Psi(x, y) \rfloor \).
4. Output \( \text{Branch}(x, y, k, S) \).

Function \( \text{Branch}(x, y, k, S) \):

If \( x < 1 \) Then return nothing/null;
If \( k = 0 \) Then
   If \( S \) is empty Then return 1;
   Else return \( S \);
Find consecutive primes \( p_1 < p_2 \) such that \( \Psi(x, p_1) < k + 1 \leq \Psi(x, p_2) \).
Push \( p_2 \) onto \( S \);
Set \( k' = k - \Psi(x, p_1) \);
Return \textbf{Branch}(x/p_2,p_2,k',S);

The running time of Algorithm 2 is at worst proportional to the recursion depth of the \textbf{Branch} function, times the time for one execution of the \textbf{Branch} function (excluding the recursive call at the end). The recursion depth is exactly the number of prime divisors, with multiplicity, in the number \( n \) generated by the algorithm. From [2] above, this is \( O(\log \log x + (\log x)/\log y) \) in the average case, and \( O(\log x) \) in the worst case. It should be easy to see that the bottleneck in one \textbf{Branch} execution is finding the primes \( p_1, p_2 \), the \textit{Find} step, which we will discuss in detail below. Note that the computation of \( k' = k - \Psi(x,p_1) \) can re-use the value of \( \Psi(x,p_1) \) from the \textit{Find} step. Also note that the single computation of \( \Psi(x,y) \) in step (3) of Algorithm 2 is negligible in the overall running time, as the \( \Psi \) function will be evaluated many times.

At this point we have several questions that require our attention.

1. How do we compute/estimate the \( \Psi \) function?
2. How do we find the primes \( p_1, p_2 \)?
3. Should we use a prime testing algorithm, or does it make more sense to find all primes \( \leq y \) with a sieve? Perhaps a combination of the two?

3.1. \textbf{Algorithms to Estimate} \( \Psi(x,y) \). In addition to our recursive method above for exactly computing \( \Psi(x,y) \), there are a number of algorithms to estimate \( \Psi \) in the literature:

- A faster method to compute \( \Psi(x,y) \) exactly [7],
- Methods to give upper and lower bounds on \( \Psi(x,y) \) using formal power series [8, 23],
- Methods based on the saddle-point approach of Hildebrand and Tenenbaum [19, 20, 26, 27, 28], (see also [6] for an LMO-style algorithm), and
- Methods based on the Dickman-deBruijn function [12, 30].

Given the right circumstances, any of these approaches might be optimal. It turns out that currently, the fastest method is the oldest,

\[
\Psi(x,y) \approx x \cdot \rho(u),
\]

where \( \rho(u) \) is the Dickman-deBruijn function, and \( u = u(x,y) = (\log x)/\log y \). To be precise, Hildebrand proved that

\[
\Psi(x,y) = x\rho(u) \left( 1 + O\left(\frac{\log(u+1)}{\log y}\right)\right)
\]

under the condition that \( \log y > (\log \log x)^{5/3+\epsilon} \) [29, Cor. 9.3]. Under the assumption of the ERH, the range on \( y \) can be extended down to \( y > (\log x)^{2+\epsilon} \).

We will use this estimate so long as \( y \) is in the correct range, because it asymptotically estimates \( \Psi \) correctly, and the \( \rho \) function is fast and easy to compute.
3.1.1. Computing $\rho$ for Large $y$. The idea is to pre-compute a table of values $\rho(u)$ for all $u$ up to a bound $B$ we will determine later, using the trapezoid rule (see [30]).

We know that the absolute error on a subinterval of size $h$ is proportional to $\rho''(u_h) \cdot h^3$, where $u_h$ is a point in the interval of size $h$ (See [11 §7.2]). From [29, Cor. 8.3] we have that $\rho''(u) \sim (\log u)^2 \rho(u)$, giving us a relative error of $(1 + O(h^2/(\log u)^2))$. If we choose $h = 1/\log x$, then the relative error on our computation of $\rho$ will be smaller than the relative error in Hildebrand’s result above.

This will take $O(B(\log x)^2)$ arithmetic operations to build the table, which will contain $B \log x$ floating point numbers. Computing estimates for $\Psi(x, y)$ will then involve only a table lookup and a multiplication, or constant time.

3.1.2. The Cutoff $L(x)$. Define $L(x)$ to be the cutoff point where, if $y < L(x)$, then the $x\rho(u)$ estimate in (4) is no longer valid. So then

$$L(x) = \begin{cases} (\log x)^{2+\epsilon} & \text{if we assume the ERH, and} \\ \exp \left[ (\log \log x)^{5/3+\epsilon} \right] & \text{otherwise.} \end{cases}$$

If $u \leq (\log x)/\log L(x)$, then (4) is valid, and we use (3) to estimate $\Psi(x, y)$. This also means we can take $B = B(x) = (\log x)/\log L(x)$ when computing our table of $\rho$ values. Note that as the algorithm progresses, $x$ gets smaller, and $B(x)$ decreases as $x$ does. Also, observe that when a value of $u$ arises at any point in our computation, if $u > B$, then (4) is not valid, and we don’t need a value of $\rho$ for this $u$.

3.1.3. Saddle-Point Methods for Small $y$. We are all set for when $y \geq L(x)$. When $y < L(x)$, we will use a theorem of Hildebrand and Tenenbaum based on the saddle-point method to estimate $\Psi$ [29, Thm 10]. This allows $y$ to be as small as 2.

If we are assuming the ERH, then we will use the HT-fast algorithm of [20], which gives a running time of $O((\sqrt{y}/\log y) \log \log x)$ assuming we have a list of primes up to $\sqrt{y}$. Without the ERH, we use algorithm HT from [19] with LMO summation as described in [6] for a running time of $y^{2/3+o(1)} \log \log x$.

3.1.4. When $x, y$ are Both Small. If we discover that $\Psi(x, y) \log x$ is smaller than the time already spent constructing $n$, we can revert to using Algorithm 1 to finish off the computation. When $x, y$ are both small, the asymptotics break down and $n$ is much less likely to be chosen suitably uniformly, and so the exact Algorithm 1 is preferable in this case.

3.2. Searching for $p_1, p_2$. Next, we look at the Find step. Our first task is to address the special case when $p_1 = 1, p_2 = 2$. This only takes constant time, since $\Psi(x, 2) = \lfloor \log_2 x \rfloor + 1$. So going forward, we can assume $p_1 > 1$.

Since we are using an approximation algorithm to compute $\Psi$, we break this down into two steps:
Find a real number \( t \) such that \( \Psi(x, t) - (k + 1) \) is near zero.

Find consecutive primes \( p_1 < p_2 \) with \( p_1 < t \leq p_2 \).

### 3.2.1. Finding \( t \) when \( t \geq L(x) \)

We can compute \( \Psi(x, L(x)) \) using the \( x^\rho(u) \) method in constant time to quickly determine whether our solution \( t \) is larger or smaller than \( L(x) \). Our first case is when \( t \geq L(x) \). Writing \( v = v(x, t) = (\log x)/\log t \), this means we can use the \( x^\rho(v) \) estimate for the rest of this step.

The estimate \( x^\rho(v) \) is continuous, strictly increasing in \( t \), and differentiable so long as \( v \geq 1 \). Thus we can use Newton’s method to find \( t \) in at most \( O(\log \log y) \) evaluations of \( \Psi \). We only need to know \( t \) to the nearest integer.

To be precise, we are finding a root \( t \) to the function \( f(t) = \Psi(x, t) - (k+1) \). Using the defining equation \( \rho'(v) = -\rho(v - 1)/v \), we obtain the iteration function

\[
g(t) = t - \frac{f(t)}{f'(t)} = t - \frac{\rho(v) - (k + 1)/x}{\rho(v - 1)} t \log t.
\]

That said, bisection or binary search is fine here, since evaluations of \( \Psi \) take constant time, for an overall cost of \( O(\log \log y) \) arithmetic operations, which we will see is negligible compared to other steps in the algorithm.

### 3.2.2. Finding \( t \) when \( t < L(x) \)

The estimates based on the saddle-point method involve sums over primes, and so will not be continuous in general. However, \( f(t) = \Psi(x, t) - (k + 1) \) is still a non-decreasing function of \( t \) with a simple root, so we can use the Illinois algorithm \([13]\) or Brent’s algorithm \([10]\), both of which converge super-linearly, to find a root in \( O(\log \log y) \) evaluations of \( \Psi \). Once the size of the interval to search has shrunk to length \( 2(\log y)^2 \), we switch to bisection to avoid any issues with discontinuity, and with no effect on the asymptotic running time. See also \([11, \S3.3, \S3.5]\).

### 3.2.3. Computing \( p_1, p_2 \) from \( t \)

With \( t \) in hand, we want to find the largest prime \( p_1 < t \) and the smallest prime \( p_2 \geq t \). If we happen to have a list of primes up to and a bit past \( t \), we can do a quick interpolation search on the list to find \( p_1, p_2 \) in \( O(\log \log t) \) time, which is negligible compared to finding \( t \) itself. So for now we will assume we don’t have access to such a list.

Prime testing can be expensive, so to minimize the number of prime tests, we do the following. Set an interval length \( w = 2[\log t] \), and sieve an interval of length \( w \) with midpoint \( t \) by the primes \( \leq \log t \). We then perform a base-2 strong pseudoprime test on the \( O(w/\log \log t) \) integers that pass the sieve. This takes \( O(w(\log t)/\log \log t) \) time. Identify candidate values for \( p_1, p_2 \), if they exist on the interval, and prime test them. If two such candidates are found that pass, then we are done. If not, set \( w := 2 \cdot w \) and repeat the process until two candidates are found. Of course, if one candidate is found but not the other, just continue doubling the interval above or below \( t \), but not both, as appropriate.
In the average case, non-prime integers that pass the base-2 strong pseudoprime test are very rare; see [24] for example. So on average, only two prime tests are needed. Also, by the prime number theorem, we expect to find these primes without having to double \( w \) at all, or maybe just a constant number of times. So on average, the cost is \( O((\log t)^2 / \log \log t) \) plus a constant number of prime tests.

In the worst case, every base-2 strong pseudoprime test requires a follow-up full prime test. Worse than that, even if we assume the ERH, we cannot guarantee a prime will show up until \( w \gg \sqrt{\log t} \). If we are sieving an interval of length \( \sqrt{t} \) or larger, then we may as well sieve by primes up to \( \sqrt{t} \) so that no prime tests are required. So with the ERH, the cost is \( O(\sqrt{t}(\log t) / \log \log t) \) to run the Atkin-Bernstein sieve on an interval of size \( O(\sqrt{t}\log t) \).

Without any conjectures, in the worst case we may as well find all primes up to \( y \) to avoid the problem, or perhaps simply use Algorithm 1. Later, we discuss how having a list of primes \( \leq y \) makes things easier.

For a more reasonable worst-case running time, we can choose to use the following conjecture.

**Conjecture 1.** There exist absolute constants \( n_0, c > 0 \) such that if \( n > n_0 \), then \( p_{n+1} - p_n \leq c(\log p_n)^2 \).

See [15] for a discussion of the reasonableness of this conjecture. With this conjecture, we have \( w = O((\log t)^2) \) for a bound of \( O((\log t)^2 / \log \log t) \) prime tests.

### 3.3. Prime Testing

In practice, it makes sense to use a fast, probabilistic prime test such as Miller-Rabin [22, 25], to find \( p_1, p_2 \) and then optionally verify their primality using a more rigorous test. Since the Miller-Rabin test fails with probability at most \( 1/4 \), we can use each test at most \( O(\log \log x) \) times, on each of \( p_1, p_2 \), and still get the error probability down to \( o(1) \) over the entire algorithm. A single test is \( O(\log y) \) operations to perform, the same as a base-2 strong pseudoprime test, giving a cost of \( O((\log y) \log \log x) \) operations for each candidate prime.

Possible options for the more rigorous prime test include the following:

- If we are assuming the ERH, Miller’s test [22] takes \( O((\log y)^3) \) arithmetic operations.
- If we allow random numbers, Bernstein’s variant of the AKS test [9] takes expected \( (\log y)^{3+o(1)} \) arithmetic operations. Note that this is much slower in practice than Miller’s algorithm. Also note that the randomness is only in the running time, not correctness.
- If we don’t allow the ERH or random number use, but have access to a substantially large table of pseudosquares, the pseudosquares prime test [21] is fast at \( O((\log y)^2) \) arithmetic operations.
And then there’s the AKS test [1] which has a variant due to Lenstra and Pomerance that takes \((\log y)^{5+o(1)}\) arithmetic operations. No ERH or random numbers are required.

If we are in a theoretical situation where random bits are a scarce resource, building a table of pseudosquares is probably the way to go.

4. Analysis and Tradeoffs

The running time of our algorithm is proportional to the following structure:

\[ T + D \cdot (S \log \log y + P) \]

where

- \(T\) is the cost to build the table of \(\rho(u)\) values,
- \(D\) is the number of prime divisors or the recursion depth,
- \(S\) is the cost of evaluating \(\Psi(x, y)\), and
- \(P\) is the cost of finding \(p_1, p_2\) from \(t\).

Let’s look at each one in detail.

We know \(T = O((\log x)^3 / \log (\log x))\) (with ERH), and \(L(x)\) depends only on whether we assume the ERH or not:

\[
T = O\left(\frac{(\log x)^3}{\log \log x}\right) \quad \text{(with ERH)}
\]

\[
T = O\left(\frac{(\log x)^3}{(\log \log x)^{5/3+\epsilon}}\right) \quad \text{(without ERH)}
\]

Note that if we are generating large numbers of random smooth factored integers, computing the \(\rho\) table can be viewed as one-time preprocessing.

From (2) we have

\[
D = O(\log x) \quad \text{(worst case)},
\]

\[
D = O\left(\log \log x + \frac{\log x}{\log y}\right) \quad \text{(average case)}.
\]

As we look at \(S\), we’ll find it helpful to define \(z\) as the smaller of \(y\) and \(L(x)\). When \(y \geq L(x)\), we know \(S\) is constant time. When \(y < L(x)\), if we are assuming the ERH, then we use algorithm HT-fast, and we know that \(L(x) = (\log x)^{2+\epsilon}\), so that

\[ S = O\left(\sqrt{z} \cdot \frac{\log z}{\log x}\right) \leq (\log x)^{1+\epsilon+o(1)} \quad \text{(with ERH)}.
\]

Note that we could drop the \(o(1)\) in the exponent here by choosing a different \(\epsilon\), but we leave it there to remind the reader that we are masking lower-order multiplicative terms. Without the ERH, we use the LMO version of algorithm HT, and \(L(x) = \exp[(\log \log x)^{5/3+\epsilon}]\), giving

\[
S = O(z^{2/3+o(1)} \log \log x) \leq \exp\left[(\log \log x)^{5/3+\epsilon+o(1)}\right] \quad \text{(without ERH)}.
\]

Again, we leave the \(o(1)\) to indicate the masking of lower order factors.
Our last one, $P$, is the most interesting.

If we use an average-case analysis, then $P = O((\log y)^2/\log \log y + M)$, where the first term is the cost of sieving and using base-2 strong pseudo-prime tests, and $M$ is the cost of one prime test. If we are assuming the ERH, we can set $M = O((\log y)^3)$, or better yet, if we are using probabilistic prime tests, $M = O((\log y) \log \log x)$. With neither the ERH nor randomization, we use the AKS test for $M = (\log y)^{5+o(1)}$:

\[
P = O\left(\frac{(\log y)^2}{\log \log y}\right) \quad \text{(avg. case, random)},
\]

\[
P = O\left(\frac{(\log y)^3}{\log \log y}\right) \quad \text{(avg. case, ERH, not random)},
\]

\[
P = (\log y)^{5+o(1)} \quad \text{(avg. case, no ERH, not random)}.
\]

If we use a worst-case analysis, we have more options to consider. If we assume Conjecture[1] then $P = O(M(\log y)^2/\log \log y)$. If we further allow the use of random numbers, we can use a probabilistic prime test. In this case, the expected (not average!) running time is $O(\log y)$ to reject a number as not prime. This gives $P = O((\log y)^3/\log \log y)$, with the final two probabilistic prime tests taking only $O((\log y) \log \log x)$ time. Without randomization, $M = O((\log y)^3)$ with the ERH, giving $P = O((\log y)^6/\log \log y)$, and $M = (\log y)^{5+o(1)}$ without the ERH, for $P = (\log y)^{7+o(1)}$:

\[
P = O\left(\frac{(\log y)^3}{\log \log y}\right) \quad \text{(worst case, Conj[1] random)},
\]

\[
P = O\left(\frac{(\log y)^6}{\log \log y}\right) \quad \text{(worst case, Conj[1] ERH, not random)},
\]

\[
P = (\log y)^{8+o(1)} \quad \text{(worst case, Conj[1] no ERH, not random)}.
\]

If we don’t assume Conjecture[1] but do assume the ERH, then we will sieve an interval of size $O(\sqrt{\log y})$ for primes, searching for $t$ on that interval quickly finds $p_1, p_2$, giving us

\[
P = O\left(\frac{\sqrt{\log y}}{\log \log y}\right) \quad \text{(worst case, no Conj[1] ERH)}.
\]

With neither conjecture, the approach is the same but the length of the interval is much larger, giving

\[
P = y/\exp\left[O\left(\frac{(\log y)^{3/5}}{(\log \log y)^{1/5}}\right)\right] \quad \text{(worst case, no Conj[1] no ERH),}
\]

essentially the best currently known error term for the prime number theorem.

**Complexity Results.** Given all our options for conjectures, the use of randomness, and worst-case versus average-case, we can obtain a wide range of running times for our algorithm. Rather than go through all of these, we present two versions that seem reasonably useful.
Theorem 1. Let \( x \geq y > 0 \) be integers, and let \( r \in [0,1) \) be chosen uniformly at random. Under the assumption of the Extended Riemann Hypothesis (ERH), With probability \( 1 - o(1) \), Algorithm 2 will construct an integer \( n \) with known factorization such that \( n \leq x \) and \( n \) possesses no prime divisor larger than \( y \). The average running time of this algorithm is

\[
O \left( \frac{(\log x)^3}{\log \log x} \right)
\]

arithmetic operations. If a table of values of \( \rho(u) \) is precomputed for \( u \) up to \( O((\log x)/\log \log x) \), then the running time drops to

\[
O \left( \frac{(\log x)(\log y)}{\log \log y} + \frac{(\log x)^{2+\epsilon}}{\log y} \right)
\]

arithmetic operations, where \( \epsilon > 0 \).

Here \( n \) is chosen asymptotically uniformly, in the sense that as \( x, y \to \infty \), \( n \)'s relative position in the full enumeration of \( y \)-smooth integers \( \leq x \) is \( \lfloor r \Psi(x,y)(1 + o(1)) \rfloor \).

This theorem follows from choosing our options to obtain the smallest possible running time, namely ERH, randomized prime tests, and an average-case running time analysis.

Theorem 2. Let \( x \geq y > 0 \) be integers, and let \( r \in [0,1) \) be chosen uniformly at random. Algorithm 2 will construct an integer \( n \) with known factorization such that \( n \leq x \) and \( n \) possesses no prime divisor larger than \( y \). As in the sense of Theorem 1, \( n \) is chosen asymptotically uniformly. Let \( \epsilon > 0 \) and let \( L(x) = \exp[(\log \log x)^{5/3+\epsilon}] \) as required by (4). Let \( z \) be the smaller of \( y \) and \( L(x) \). Under the assumption of Conjecture 1, Algorithm 2 has a worst-case running time of

\[
z^{2/3+o(1)}(\log x) \log \log x + O \left( \frac{(\log x)^3}{(\log \log x)^{5/3+\epsilon}} \right).
\]

Here the proof follows from our discussion above, with the no-ERH option, a worst-case running time, with Conjecture 1, and no randomized prime tests (they don’t help the overall running time). The second term is from precomputing \( \rho(u) \), and could dominate if \( y \) is very small, say \( y < \log x \).

5. Special Cases

5.1. We Have Primes! If we happen to have a list of primes \( \leq y \) available, then that helps in several ways:

- We don’t need Conjecture 1 and our running time can be worst-case with no downside.
- Searching for \( p_1, p_2 \) becomes a simple binary or interpolation search in our list of primes. No prime testing required.
• With a list of primes, we can assume that using $O(\pi(y))$ time is reasonable, and so we can use the accurate Algorithm HT to estimate $\Psi$ all the time, and skip the use of $\rho(u)$. We could use the LMO version, but in practice it has not been implemented to our knowledge.
• If running times seem slow in practice, switch to Algorithm HT-fast, which has a running time that is roughly proportional to $\sqrt{y}$. This means assuming the ERH.

If $y$ is very small, say $y \leq (\log x)^2$, finding all primes up to $y$ and then using this approach makes a good deal of sense.

5.2. **Bach’s Result.** We can use Algorithm 2 to produce random factored numbers in the sense of Bach’s algorithm [4] by simply setting $y = x$. The running time would be $O((\log x)^3/\log \log x)$ to precompute the table of $\rho$ values, and then $O((\log x)^2/\log \log x)$ operations on average to generate each integer $n$ (using Theorem 1).

5.3. **Semismooth Numbers.** The central control structure of our algorithm is Buchstab’s identity. With some straightforward modifications, we could adapt our approach to make use of the generalized Buchstab identity given in [6] for generating random factored semismooth integers.

6. **Example Run**

We implemented our algorithm in C++ on a linux desktop workstation and ran it with $x = 10^{100}$, $y = 10000$, and $r = 0.5$. It generated the following list of prime divisors for $n$:

2 3 5 7 29 31 97 113 113 157 223 241 503 509 569 691
727 1033 1367 1571 2141 2339 2617 2741 3041 3221 3547 3989
4021 4513 4999 5573 6577 7573 9463

The resulting $n$ is roughly $4.29 \cdot 10^{97}$, which occupies a position near $2.05 \cdot 10^{61}$ in the enumeration. The run took less than 0.35 seconds of wall time.

For this small of a value of $y$, we found all primes $\leq y$ and used Algorithm HT to estimate $\Psi(x, y)$ throughout.

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