The exit times for the diffusion risk model with constant interest

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Abstract. This paper investigates the diffusion risk model with constant interest. The Laplace-Stieltjes transforms (LST) of some exit times of the risk process are obtained.

1 Introduction

The diffusion risk model with constant interest is described by

\[ U(t) = u + \int_0^t C(U(s))ds + \sigma B(t), \tag{1} \]

where \( u \) denotes the initial capital, \( C(x) = c + rx, c > 0 \) represents the premiums income per unit time and \( r > 0 \) is the constant force of interest. \( \{B(t), t \geq 0\} \) is a standard Brownian motion and \( \sigma > 0 \) is the diffusion coefficient. The model (1) is a diffusion risk model with interest which represents that the company can earn investment income at a constant force of interest \( r \) when the surplus is positive. When the surplus turns negative, the company is allowed to borrow money at the same force of interest \( r \).

For any interval \([b, a]\), where \( b < u < a \), define the first hitting time of the upper barrier \( a \) for the risk process \( \{U(t), t \geq 0\} \) to be

\[ T_a = \left\{ \begin{array}{ll}
\inf\{t \geq 0, U(t) = a\}, & \\
\infty, & \text{if } U(t) \neq a \text{ for all } t \geq 0.
\end{array} \right. \]

Correspondingly, define the first hitting time of the lower barrier \( b \) for the risk process \( \{U(t), t \geq 0\} \) to be

\[ T_b = \left\{ \begin{array}{ll}
\inf\{t \geq 0, U(t) = b\}, & \\
\infty, & \text{if } U(t) \neq b \text{ for all } t \geq 0.
\end{array} \right. \]

Then \( T_{a,b} = T_a \wedge T_b \) is the first exit time of the process \( \{U(t), t \geq 0\} \) from the interval \((b, a)\). This paper investigates the Laplace-Stieltjes transforms (LST) of the exit times. The similar subject of this paper is considered by some authors. Alili, Patie and Pedersen\cite{1} mainly considered the first hitting time of an Ornstein-Uhlenbeck process. Chiu and Yin\cite{2-4} investigated some passage times of the reserve-dependent risk process and the spectrally negative Lévy process. dos Reis\cite{5} and Gerber\cite{6} mainly studied some stopping times of the classical risk process. Jacobsen and Jensen\cite{7} considered the exit times for a class of piecewise exponential Markov processes with two-sided jumps. Kella and Stadje\cite{8} and Perry and Stadje\cite{9} mainly some exit times of the processes with compound Poisson process.

The remainder of the paper is organized as follows. In section 2 we give some preliminaries of the diffusion process with constant interest. In section 3 we obtain the LST of some exit times.

2 Preliminaries

The model (1) is a time-homogeneous Markov process (see Klebaner\cite{10}) taking values in \( \mathbb{R} \) with generator \( A \) that satisfies
\[Af(x) = \frac{\sigma^2}{2} f''(x) + (c + rx) f'(x)\]

where \( f \) belongs to the domain \( D(A) \) of the generator \( A \) of \( \{U(t), t \geq 0\} \). Furthermore \( (U(t), t) \) is also Markovian with generator \( A' \) that satisfies

\[A' h(x, t) = Ah(x, t) + \frac{\partial}{\partial t} h(x, t).\]

If \( h(x, \cdot) \) has a continuous first derivative for each \( x \) and for each \( t \), \( h(\cdot, t) \) is in the domain of \( A \), then \( h(x, t) \in D(A') \). Denote by \( F_t = \sigma \{U(s), 0 < s \leq t\} \) the natural filtration. For later use, we give the following Lemma.

**Lemma 2.1** If \( h(x, t) \) is a twice continuously differentiable in \( x \) and once in \( t \) function with bounded first derivative in \( x \), then \( h(x, t) \in D(A') \) and furthermore

\[M_h(t) = h(U(t), t) - \int_0^t A' h(U(s), s) ds\]

is a martingale.

In order to obtain the LST of the first exit time \( T_{a,b} \), for any \( \alpha > 0 \), we will try to find a solution to the equation

\[Af(x) = \alpha f(x),\]

that is

\[\frac{\sigma^2}{2} f''(x) + (c + rx) f'(x) = \alpha f(x).\]  

(3) is a second order linear differential equation, which has two positive independent solutions \( f_1, f_2 \) such that \( f_1 \) is strictly decreasing and \( f_2 \) is strictly increasing. Then every solution is a linear combination of the form

\[C_1 f_1(x) + C_2 f_2(x),\]

where \( C_1, C_2 \) are arbitrary constants. From Cai et al.[11], we know that

\[f_1(x) = \exp\{-\frac{(c + rx)^2}{r \sigma^2}\} U\left(1 + \frac{\alpha}{2r}, \frac{1}{2}, \frac{(c + rx)^2}{r \sigma^2}\right),\]

and

\[f_2(x) = (c + rx) \exp\{-\frac{(c + rx)^2}{r \sigma^2}\} M\left(1 + \frac{\alpha}{2r}, \frac{3}{2}, \frac{(c + rx)^2}{r \sigma^2}\right),\]

where \( M \) and \( U \) are called the confluent hypergeometric functions of the first and second kind respectively. It is easy to verify that \( f_1(x) \rightarrow 0 \) as \( x \rightarrow +\infty \). More details on confluent hypergeometric functions can be found in Abramowitz and Stegun[12].

### 3 The LST of some exit times

**Theorem 3.1** Given that the initial state \(-\frac{c}{r} < a < u\), the LST of the time to hit \( a \) is given by

\[E_{\alpha}[e^{-\alpha T_{a,b}}] = \frac{f_1(u)}{f_1(a)}.\]  

**Proof.** Assume that \( h(x, t) \) takes the form \( h(x, t) = e^{-\alpha t} f_1(x) \), it follows from Lemma 2.1 that \( h(x, t) = e^{-\alpha t} f_1(x) \) is in the domain of \( A' \) and

\[A' h(x, t) = Ah(x, t) + \frac{\partial}{\partial t} h(x, t) = 0.\]

By Dynkin's formula, we conclude that
\[ e^{-at} f_1(U(t)) - f_1(U(0)) = h(U(t), t) - h(U(0), 0) - \int_0^t A' h(U(s), s) \, ds \]
is a zero-mean martingale. Thus, for stopping time \( T_a \) and initial condition \( u \), we have that
\[
E_u \left[ e^{-a(T_a)} f_1(U(T_a)) \right] = f_1(u). \tag{5}
\]
Because \( f_1(x) \) is bounded on the range of possible values of \( \{U(t \wedge T_a), t \geq 0\} \), letting \( t \to +\infty \) in (5), dominated convergence theorem yields
\[
E_u \left[ e^{-aT_a} f_1(U(T_a)) \right] = f_1(u),
\]
so that
\[
E_u \left[ e^{-aT_a} \right] = \frac{f_1(u)}{f_1(a)}.
\]
This completes the proof.

**Theorem 3.2** For \( b < u < a \), the LST of the first exit time from the upper barrier \( a \) is given by
\[
E_u \left[ e^{-aT_a} 1_{T_a < T_b} \right] = \frac{f_3(u)}{f_3(a)}, \tag{6}
\]
Where \( f_3(x) = C_1 f_1(x) + C_2 f_2(x) \) and \( C_1, C_2 \) satisfy \( f_3(b) = C_1 f_1(b) + C_2 f_2(b) = 0 \).

**Proof.** It follows from Lemma 2.1 that \( h(x, t) = f_3(x) e^{-at} \) is in the domain of \( A' \) and \( A' h(x, t) = A h(x, t) + \frac{\partial}{\partial t} h(x, t) = 0 \).

By Dynkin's formula, we conclude that
\[
f_3(U(t)) e^{-at} - f_3(U(0)) = h(U(t), t) - h(U(0), 0) - \int_0^t A' h(U(s), s) \, ds
\]
is a zero-mean martingale. Thus, for stopping time \( T_{a,b} \) and initial condition \( u \), we have that
\[
E_u \left[ f_3(U(T_{a,b})) e^{-a(t-T_{a,b})} \right] = f_3(u). \tag{7}
\]
Because \( f_3(x) \) is bounded on the range of possible values of \( \{U(t \wedge T_{a,b}), t \geq 0\} \), letting \( t \to +\infty \) in (7), dominated convergence theorem yields
\[
E_u \left[ f_3(U(T_{a,b})) e^{-aT_{a,b}} \right] = f_3(u),
\]
hence
\[
E_u \left[ f_3(U(T_a)) e^{-aT_a} 1_{T_a < T_b} \right] + E_u \left[ f_3(U(T_b)) e^{-aT_b} 1_{T_b < T_a} \right] = f_3(u),
\]
thus
\[
E_u \left[ f_3(U(T_a)) e^{-aT_a} 1_{T_a < T_b} \right] = f_3(u),
\]
so that
\[
E_u \left[ e^{-aT_a} 1_{T_a < T_b} \right] = \frac{f_3(u)}{f_3(a)}.
\]
This completes the proof.

Using the same argument, we can obtain the following Theorem.

**Theorem 3.3** For \( b < u < a \), the LST of the first exit time from the lower barrier \( b \) is given by
\[
E_u \left[ e^{-aT_b} 1_{T_a < T_b} \right] = \frac{f_4(u)}{f_4(b)}, \tag{8}
\]
Where \( f_4(x) = C_1 f_1(x) + C_2 f_2(x) \) and \( C_1, C_2 \) satisfy \( f_4(a) = C_1 f_1(a) + C_2 f_2(a) = 0 \).
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