SELF-SIMILAR SOLUTIONS OF THE LOCALIZED INDUCTION APPROXIMATION: SINGULARITY FORMATION

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1. Introduction

In this paper we continue the investigation started in [8] about the creation of singularities in a self-similar form for the binormal flow

\[ X_t = X_s \times X_{ss}, \]

with \( X(\cdot, t) \) a curve in \( \mathbb{R}^3 \) parametrized by \( s \) the arclength parameter for all time \( t \). This flow was proposed by Da Rios [6], and rediscovered later on by Arms and Hama [11], and Betchov [3], as an approximation of the evolution of a vortex tube of infinitesimal cross section \( X(\cdot, t) \) under Euler equations. This approximation\(^1\) only takes into consideration the local effects of the corresponding Biot-Savart integral. For this reason (1) is also known as the Localized Induction Approximation (LIA). Notice however that, if use is made of the Serret-Frenet formulae

\[
\begin{aligned}
T_s &= cn \\
n_s &= -cT + \tau b \\
b_s &= -\tau n,
\end{aligned}
\]

then (1) can be also written as

\[ X_t = cb, \]

where \( c \) stands for the curvature, and \( b \) for the binormal vector.

Intimately related to the above flow is the one described by the tangent vector \( T(s, t) \),

\[ T_t = T \times T_{ss}. \]

Calling \( \Gamma \) the circulation of the vortex tube, the singular vectorial measure \( \vec{\omega} = \Gamma T \, ds \) gives the corresponding vorticity. Therefore, if we want to keep in the model the fundamental property that the circulation along the tube be constant, we have to take \( T \) a unit vector-a property which is preserved under the flow. This has as a serious drawback -see [20]-, that if the filament is closed, then the total length has to be preserved (at least if a minimal regularity of the filament is assumed). We wish to emphasise that \( T \) represents the direction of the vorticity vector \( \frac{\vec{\omega}}{|\vec{\omega}|} \). It was proved in [5] that, among other conditions, the divergence of the integral

\[ \int_I \sup_x |\nabla_x \left( \frac{\vec{\omega}}{|\vec{\omega}|} \right)|^2 \, dt = \infty \]

\(^1\)We refer the reader to [2], [19] and [20] for a detailed analysis of the model.
is necessary for the formation of a singularity in Euler equations for some time in the interval $I$. In our setting the above integral reduces to

\[ \int_I \sup_s |T_s(s,t)|^2 \, dt = \int_I \sup_s |c(s,t)|^2 \, dt = \infty. \]

In [8] we looked at self-similar solutions of (1) with respect to the unique scaling that preserves arclength. In this case, it can be easily proved -see [4], [16], and [17]- that

\[ X(s,t) = \sqrt{t} G(c_0(s/\sqrt{t})), \]

with $G_{c_0}$ the curve determined by $c(s) = c_0$, $c_0$ denoting a constant, and $\tau(s) = s/2$. Here $\tau$ stands for the torsion. As a consequence, $c(s,t) = c_0/\sqrt{t}$ and (4) holds true in this case.

The main purpose of this paper is to characterize all the possible solutions of (1) such that (4) is verified in a self-similar way, that is to say such that the curvature is self-similar with respect to the scaling

\[ \tilde{c}(s,t) = \frac{1}{\sqrt{t}} c(s/\sqrt{t}). \]

Accordingly we take $\tilde{\tau}(s,t) = \tau(s/\sqrt{t})/\sqrt{t}$. Also recall that from the Hasimoto transformation -see [10]-, if

\[ \tilde{\psi}(s,t) = \tilde{c}(s,t)e^{i \int_0^s \tilde{\tau}(s',t) \, ds'}, \]

then

\[ i\tilde{\psi}_t + \tilde{\psi}_{ss} + \frac{\tilde{\psi}}{2} (|\tilde{\psi}|^2 + \alpha(t)) = 0. \]

Writing $\tilde{\psi}(s,t) = \psi(s/\sqrt{t})/\sqrt{t}$ and assuming $\alpha(t) = \alpha/t$ for some constant $\alpha$, we get that

\[ \psi'' - \frac{i}{2} (\psi + s\psi') + \frac{\psi}{2} (|\psi|^2 + \alpha) = 0. \]

Either looking at (7) or at (8) we conclude that for any $\alpha$ there is in principle a two parameter family of possible solutions. As we have already seen -see [8]-, the solutions (5), except by rotations, are characterized just by one parameter ($c_0$). However, if we consider (7), the initial conditions that can develop into self-similar solutions have to be homogenous of degree $-1$, which gives two free parameters. One is the $\delta$ function which on integrating the Frenet system of equations leads to the kink solution of (1) obtained in [8]. The other candidate is to take $\tilde{\psi}(s,0) = p.v.(1/s)$, as initial condition of (7). In this case, when we perform the first integration in $s$, a logarithmic term appears which breaks the scaling
symmetry. Therefore, in order to see these “self-similar solutions” in the framework of \(1\) we have to make a modification of the usual ansantz given in \(5\).

Take \(\mathcal{A}\) a real antisymmetric \(3 \times 3\) matrix and define for some \(G\)
\[
X(s,t) = e^{\mathcal{A} \log t} \sqrt{t} G(s/\sqrt{t}).
\]
If we ask that \(X(s,t)\) be a solution of \(1\), we get that \(G\) has to solve the system of ordinary differential equations (O.D.E. for short)
\[
(I + \mathcal{A}) G - sG' = 2G' \times G'', \quad |G'(s)|^2 = 1.
\]
Multiplication by \(G'\) and from the vectorial identity
\[
F \times (G \times H) = (F \cdot H) G - (F \cdot G) H,
\]
we get
\[
G'' = \frac{1}{2}((I + \mathcal{A}) G \times G').
\]
Given any initial condition \((G(0), G'(0))\) with \(|G'(0)|^2 = 1\), it is very easy to prove global existence of a solution \(G\) which solves the above equation (see section 2).

In section 2, associated with the solution \(G\), we shall prove the existence of regular \(X(s,t)\) of the form \(9\) solving the following problem:
\[
\begin{cases}
X_t = X_s \times X_{ss}, & t > 0, \quad s \in \mathbb{R}, \\
X(s,0) = se^{A \log |s|} \chi_{[0,\infty)}(s) + se^{A \log |s|} \chi_{(-\infty,0]}(s),
\end{cases}
\]
for some \((A^+, A^-) \in \mathbb{R}^3 \times \mathbb{R}^3\). Here, \(\chi_E\) denotes the characteristic function of the set \(E\) and, due to the invariance of LIA under rotations, we can assume without loss of generality that
\[
\mathcal{A} = \begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a \in \mathbb{R}.
\]

The initial data \(X(s,0)\) in \(12\) includes a wide variety of 3d-spirals, whose rotation axis is the OZ-axis under the assumption \(13\) on the matrix \(\mathcal{A}\). The initial data \(X(s,0)\) is singular at \(s = 0\) whenever, either \(a = 0\) and \(A^+ - A^- \neq 0\), or \(a \neq 0\) and the first two components of either \(A^+\) or \(A^-\) are different from zero. The singularity of \(X(s,0)\) becomes clear on studying the behaviour of the tangent vector \(T(s,0)\) at \(s = 0\): in the case when \(a \neq 0\), the
singularity at $s = 0$ comes from the non-existence the limit $\lim_{s \to 0} T(s, 0)$, whereas if $a = 0$ we find that $T(s, 0)$ has a jump singularity at $s = 0$ (see [3]).

It is important to mention that, because $X(s, t)$ is of the form [H], the shape of the curve $X(s, 0)$ is directly related to the asymptotic behaviour of the solutions of [11]. In this setting we have the following theorem:

**Theorem 1.** Given $(G(0), G'(0))$ with $|G'(0)|^2 = 1$, let $G(s)$ be the solution of [11] associated to this initial data. Then, there exist unique vectors $B^\pm$ with $|B^\pm| = 1$ and $A^\pm = (I + A)^{-1}B^\pm$ such that the following asymptotics hold as $s \to \pm \infty$:

1. $G(s) = s e^{A \log |s|} A^\pm + e^{A \log |s|} \left\{ 2e^{2A} B^\pm - AB^\pm \times B^\pm \right\} - 4 \frac{c_{\pm \infty} n}{s^2} + O \left( \frac{1}{s^3} \right)$,

2. $T(s) = e^{A \log |s|} B^\pm - 2 \frac{c_{\pm \infty} b}{s} + O \left( \frac{1}{s^2} \right)$.

3. Moreover, if $a \neq 0, B_3^\pm \neq \pm 1$, and $c_{\pm \infty} \neq 0$, then

$$c_{\pm \infty} (n - ib)(s) = \frac{b_\pm e^{ia}}{|AB^\pm|^2} e^{i\phi(s)} e^{A \log |s|} \left\{ AB^\pm \times B^\pm - iAB^\pm \right\} + O \left( \frac{1}{s} \right).$$

4. If $a \neq 0$ and either $B_3^+ \neq 3$ or $B_3^- \neq \{-1, -1\}$, then $G(s) = (0, 0, \pm s)$.

Here\(^2\), $\phi(s) = (s^2/4) - \gamma \log |s|$, $a_\pm \in [0, 2\pi)$,

$$\gamma_\pm = 3a B_3^\pm + \alpha, \quad c_\pm^2 = -a B_3^\pm - \alpha,$$

$$b_\pm^2 = a^2 (-a B_3^\pm - \alpha) (1 - (B_3^\pm)^2) \quad \text{with} \quad b_\pm \geq 0 \quad \text{and}$$

$$\alpha = -a T_3(0) - \frac{1}{4}(I + A) G(0)^2.$$

**Remark 1.** In (i) and (ii) we understand that $c_{\pm \infty} n = c_{\pm \infty} b = 0$, whenever $c_{\pm \infty} = 0$.

Conversely, we can fix the data at infinity and find $(G(0), G'(0))$ such that the asymptotics at infinity of the corresponding solution $G$ is prescribed by the given data. More precisely:

**Theorem 2.** Given $a \neq 0$, $B^+ = (B_1^+, B_2^+, B_3^+) \in S^2$ with $B_3^+ \neq \pm 1$, $a_+ \in [0, 2\pi)$ and $b_+ \geq 0$, there exists a solution of [11] satisfying (i), (ii) and (iii) of Theorem 1 if $s \to +\infty$.

\(^2\{T, n, b\} is the Frenet frame associated to $G$, $c(s)$ the curvature function, and $c_{\pm \infty} = \lim_{s \to \pm \infty} c(s)$.

The existence of the limits $c_{\pm \infty}$ will be proved in Corollary [1].
and \( B^+ = (I + A)A^+ \). Moreover, if \( b_+ > 0 \), the solution is unique. A similar result can be obtained at \( s = -\infty \).

The proofs of Theorems 1 and 2 are related to the asymptotic behaviour of the solutions of the complex O.D.E. \( \mathbf{8} \). In order to integrate \( \mathbf{8} \), it turns out that, it is better to introduce the variable \( f(s) \) defined by

\[
\psi(s) = f(s)e^{is^2/4},
\]

and consider the equation

\[
f'' + i\frac{s}{2}f' + \frac{f}{2}(|f|^2 + \alpha) = 0.
\]

The theorem below gathers the main properties that we prove for the solutions of the latter equation:

**Theorem 3.** Let \( f \) be a solution of the equation

\[
f'' + i\frac{s}{2}f' + \frac{f}{2}(|f|^2 + \alpha) = 0, \quad \alpha \in \mathbb{R}.
\]

Then

i) There exists \( E(0) \geq 0 \) such that the identity

\[
|f'|^2 + \frac{1}{4}(|f|^2 + \alpha)^2 = E(0)
\]

holds true for all \( s \in \mathbb{R} \).

ii) The limits \( \lim_{s \to +\infty} |f|^2(s) = |f|_{+\infty}^2 \) and \( \lim_{s \to -\infty} |f'|^2(s) = |f'|_{-\infty}^2 \) do exist.

iii) Moreover, if \( |f|_{+\infty} \neq 0 \) or \( |f|_{-\infty} \neq 0 \), then

\[
f(s) = |f|\pm\infty e^{i\phi_2(s)} + 2i |f'|\pm\infty \frac{e^{id}}{s} e^{i\phi_3(s)} + O\left( \frac{1}{|s|^2} \right),
\]

\[
f'(s) = |f'|\pm\infty e^{id} e^{i\phi_3(s)} + i |f|\pm\infty (|f|^2\pm\infty + \alpha) \frac{e^{ic}}{s} e^{i\phi_2(s)} + O\left( \frac{1}{|s|^2} \right).
\]

Here, \( |f|\pm\infty, |f'|\pm\infty \geq 0 \), \( c_\pm \) and \( d_\pm \) are arbitrary constants in \([0, 2\pi]\),

\[
\phi_2(s) = (|f|^2\pm\infty + \alpha) \log |s|, \quad \text{and} \quad \phi_3(s) = -(s^2/4) - (2|f|^2\pm\infty + \alpha) \log |s|.
\]

Moreover, we also prove the following converse of the above theorem:
Theorem 4. With the same notation as in Theorem 3, given complex numbers \(|f| + e^{i\theta_1}\) and \(|f'| + e^{i\theta_2}\), with \(|f| + \infty, |f'| + \infty \geq 0\) and \(\theta_1, \theta_2 \in [0, 2\pi)\), there exists a unique solution \(f \in C^2(\mathbb{R})\) of (15) satisfying (iii) of Theorem 3, if \(s \rightarrow +\infty\). A similar result can be obtained at \(s \rightarrow -\infty\).

The asymptotic behaviour for the solutions of (15) given in Theorem 3 is not only used in the proof of Theorems 1 and 2, but also it is used to prove an ill-posedness result for the following initial value problem (IVP for short) related to non-linear cubic Schrödinger equations:

\[
\begin{aligned}
\psi_t + \psi_{ss} + \frac{\psi}{2}(|\psi|^2 - \alpha t) &= 0, \quad t > 0, \quad s \in \mathbb{R}, \quad \alpha \geq 0 \\
\psi(s, 0) &= c_1 p.v.(1/s), \quad \text{with} \quad c_1 \in \mathbb{C} \setminus \{0\}.
\end{aligned}
\]

This paper is organized as follows. Section 2 contains the results related to solutions of the binormal flow. For the reader’s convenience, we split Section 2 into different subsections. In Subsection 2.1 we prove the existence of solutions of LIA that converge uniformly to an initial data in the shape of a 3d-spiral. The exhaustive asymptotics for \(G\) and \(T\) in Theorem 1 will be proved in Subsection 2.2. Finally, the proof of Theorem 2 is included in Subsection 2.3.

The key point for the asymptotics of \(T\) and \(c(n - i b)\) is to look at the quantities

\[
y = \frac{d|T'|^2}{ds} \quad \text{and} \quad h = -\frac{1}{2} AT \cdot T'.
\]

In fact, \((y, h)\) solves the system -see (105) and Remark 6-

\[
\begin{aligned}
y' &= (|T'|^2 + \alpha)/2, \\
g(|T'|^2) &= 2E(0) - (3|T'|^2 + \alpha)(|T'|^2 + \alpha)/2,
\end{aligned}
\]

where \(E(0) = a^2/4\).

In the case \(c(s) \neq 0\), then

\[
\begin{aligned}
h &= c^2(\tau - s/2), \\
y &= \frac{dc^2}{ds}, \quad \text{and} \quad \frac{y}{2} + ih = \bar{f}f',
\end{aligned}
\]

where \(f\) is given in (12) and is a solution of (15). In fact, the equation (15) will be reduced to solve the system (18). The study of the properties of the solutions \((y, h)\) of the latter system of equations, together with the proofs of Theorem 3 and its partial converse Theorem 4 are saved for Section 4.
We conclude this paper with Section 5. This section is devoted to proving more specific facts and consequences on the results obtained in the previous sections. Here, we consider two special symmetric cases of solutions of LIA. Also, we discuss the question of ill-posedness for the IVP and the binormal flow. In particular, in Proposition 4, we prove the lack of uniqueness of weak solutions for the IVP associated to LIA when the initial data considered is a curve in the shape of a corner.

We finish this section giving some of the notation that will be used throughout this paper. In the sequel, \( f', f_s \), or \( df/ds \) will denote the derivative with respect to the variable \( s \) of \( f(s) \). Unless it is explicitly stated otherwise, we will use bold and gothic letters to denote vectors and matrices, respectively, and the overbar will indicate the complex conjugate. Finally, \( S^{n-1} \) will be the unit sphere in the euclidean space \( \mathbb{R}^n \).

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2. Proofs of the Theorems

Given \( a \in \mathbb{R} \), define for some \( G \)
\[
X_a(s, t) = e^{\frac{2}{t} \log t \sqrt{t} G(s/\sqrt{t})}, \quad t > 0,
\]
where, as we have already said in the introduction, \( A \) is assumed to be, with out loss of generality, the matrix
\[
A = \begin{pmatrix}
0 & -a & 0 \\
a & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad a \in \mathbb{R}.
\]

It is easy to see that \( X_a(s, t) \) is a solution of LIA if and only if \( G(s) \) satisfies
\[
(\mathcal{I} + A)G - sG' = 2G' \times G'', \quad |G'| = 1.
\]
Observe that the equation can be written equivalently as
\[
G'' = \frac{1}{2}(\mathcal{I} + A)G \times G',
\]
whenever
\[
|G'(0)| = 1 \quad \text{and} \quad (\mathcal{I} + A)G(0) \cdot G'(0) = 0.
\]
Indeed, the equation (23) easily follows from (22) by taking the outer product of \( \mathbf{G}' \) and \( \mathbf{G} \), and using the vectorial identity

\[
\mathbf{F} \times (\mathbf{G} \times \mathbf{H}) = (\mathbf{F} \cdot \mathbf{H}) \mathbf{G} - (\mathbf{F} \cdot \mathbf{G}) \mathbf{H}.
\]

Now, assume that \( \mathbf{G} \) is a solution of (23) satisfying the initial conditions (24). Then, we first notice that

\[
\frac{d}{ds} |\mathbf{G}'(s)|^2 = 2 \mathbf{G}'' \cdot \mathbf{G}' = ((\mathbf{I} + \mathbf{A}) \mathbf{G} \times \mathbf{G}') \cdot \mathbf{G}' = 0,
\]

so that

\[
|\mathbf{G}'(s)|^2 = |\mathbf{G}'(0)|^2 = 1, \quad \forall s \in \mathbb{R}.
\]

Secondly, the outer product of \( \mathbf{G}' \) and \( \mathbf{G} \) together with the above identity yields

\[
\mathbf{G}' \times \mathbf{G}'' = \frac{1}{2}((\mathbf{I} + \mathbf{A}) \mathbf{G} - (\mathbf{G}' \cdot (\mathbf{I} + \mathbf{A}) \mathbf{G}) \mathbf{G}').
\]

As a consequence, it is enough to prove that \( \mathbf{G}' \cdot (\mathbf{I} + \mathbf{A}) \mathbf{G} = s \). To this end, notice that

\[
\frac{d}{ds} (\mathbf{G}' \cdot (\mathbf{I} + \mathbf{A}) \mathbf{G}) = \mathbf{G}'' \cdot (\mathbf{I} + \mathbf{A}) \mathbf{G} + \mathbf{G}' \cdot (\mathbf{I} + \mathbf{A}) \mathbf{G}' = 1,
\]

and by taking into account that the outer product of \( \mathbf{G}' \) and \( \mathbf{G}'' \) gives that \( \mathbf{G}'' \cdot (\mathbf{I} + \mathbf{A}) \mathbf{G} = 0 \), and that \( \mathbf{A} \mathbf{v} \cdot \mathbf{v} = 0 \). Here we have used the antisymmetry of \( \mathbf{A} \). Integrating (28) and using the initial conditions in (24) allow us to conclude that \( \mathbf{G}' \cdot (\mathbf{I} + \mathbf{A}) \mathbf{G} = s \).

The previous observation reduces the problem of finding solutions of LIA of the form (20) to proving the existence of solutions of (23) with initial conditions satisfying (24). In this setting, notice that the local existence of solution of the initial value problem (23)-(24) follows from the classical theory for first order O.D.E. system. The global existence of \( C^\infty(\mathbb{R}; \mathbb{R}^3) \)-solution follows from (27).

The proposition below summarizes the obtained results.

**Proposition 1.** Given \( \mathbf{a} \in \mathbb{R} \), define

\[
\mathbf{X}_\mathbf{a}(s,t) = e^{\frac{2}{t} \log t \sqrt{t} \mathbf{G}(s/\sqrt{t})}, \quad t > 0,
\]

with

\[
\mathbf{A} = \begin{pmatrix}
0 & -\mathbf{a} & 0 \\
\mathbf{a} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

where \( \mathbf{G}(s) \) is the solution of

\[
\mathbf{G}'' = \frac{1}{2}((\mathbf{I} + \mathbf{A}) \mathbf{G} \times \mathbf{G}').
\]
associated to a given initial data \((G(0), G'(0))\) such that

\[ |G'(0)| = 1, \quad \text{and} \quad (I + A)G(0) \cdot G'(0) = 0. \]

Then, \(X_a(s, t)\) is an analytic solution of LIA for all \(t > 0\), and \(c^2(s)\) is also analytic.

**Remark 2.** The condition \((I + A)G(0) \cdot G'(0) = 0\) can be removed: Let \(G\) be a solution of \((23)\) associated to a given initial data \((G(0), G'(0)) \in \mathbb{R}^3 \times S\), and consider \(\tilde{G}(s) = G(s - s_0)\) with \(s_0 = (I + A)G(0) \cdot G'(0)\). Then, \(\tilde{G}\) satisfies \((22)\) and the result in Proposition 1 holds true for \(X_a(s, t) = e^{\frac{4}{4} \log t} \tilde{G}(s/\sqrt{t})\).

Hereafter, we will assume without loss of generality that \(s_0 = 0\), i.e. \((I + A)G(0) \cdot G'(0) = 0\). In the case when \(s_0 \neq 0\), the results that will be proved in the sequel for \(G\) will be valid for \(\tilde{G}\).

**Figure 1.** \(G(0) = (0, 0, 2c_0), \ G'(0) = (1, 0, 0), \) with \((a, c_0) = (10, 1), \ (15, 5)\) and \((20, 3)\), respectively.
Figure 1 depicts different examples of the curve $X(s,t)$ at time $t = 1$, that is of $G(s)$. The curves $G(s)$ have been obtained solving the equation (23) with the initial conditions $G(0) = (0,0,2c_0)$ and $G'(0) = (1,0,0)$. The subsequent plots correspond with different choices of the parameters $c_0$ and $a$. As we will continue to prove, the evolution of each of these curves $G(s)$ under the relation in (20) leads to solutions of LIA which converge as $t \to 0^+$ to different curves in the shape of a 3d-spiral.

In what follows, we will often drop the subscript $a$ in the definition of $X_a(s,t)$ in Proposition 1 and just write $X(s,t)$. Hereafter, $c(s)$ and $\tau(s)$ will be the curvature and torsion functions related to $X(s,1) = G(s)$, respectively. Also, we will write $c_0 = c(0)$ and $c_{\pm \infty} = \lim_{s \to \pm \infty} c(s)$.

2.1. Convergence to the initial data.

As in [8], the study of $\lim_{t \to 0^+} X(s,t)$ is based on finding an explicit expression of $G(s)$. Here, we also obtain a formula of the tangent vector $T(s)$ which will be used in the following subsection.

For simplicity of the exposition, through this subsection we will write $G' \times G'' = cb$ even when $c(s) = 0$, in which case the both sides of this identity will be understood to be zero.

We begin obtaining a closed formula for $G(s)$. Consider the quantity

$$e^{-A \log |s|} \frac{G}{s}, \quad s \neq 0.$$  

Then, deriving with respect to $s$ and taking into account that $G$ is a solution of (22), we obtain

$$\left(e^{-A \log |s|} \frac{G}{s}\right)_s = -e^{-A \log |s|} \frac{AG}{s^2} + e^{-A \log |s|} \left(\frac{sG_s - G}{s^2}\right)$$

$$= -2e^{-A \log |s|} \frac{cb}{s^2},$$

(29)

and therefore, integrating the above identity in the interval $[s_1, s]$, we get

$$e^{-A \log |s|} \frac{G(s)}{s} - e^{-A \log |s_1|} \frac{G(s_1)}{s_1} = -2 \int_{s_1}^{s} e^{-A \log |s'|} \frac{cb}{(s')^2} ds'.$$  

(30)

We need the following lemma.
Lemma 1. If \( G \) is a solution of (22), then

\[
|\mathbf{T}'|_2(s) = -aT_3(s) - \alpha, \quad \text{with} \quad \alpha = -aT_3(0) - \frac{1}{4} |(I + A)G(0)|^2
\]

or, equivalently,

\[
c^2(s) = -aT_3(s) - \alpha, \quad \text{with} \quad \alpha = -aT_3(0) - c_0^2 \quad \text{and} \quad c_0^2 = |(I + A)G(0)|^2/4.
\]

Remark 3. If \( \alpha > 0 \) and \( a > 0 \) (resp. \( a < 0 \)), then from (31) it follows that \( G_3(s) \) is monotone decreasing (increasing), and therefore \( G(s) \) has no self-intersections (see Figure 3).

Proof of Lemma 1. Firstly, notice that (22) rewrites as

\[
(I + A)G = s\mathbf{T} + 2\mathbf{T} \times \mathbf{T}', \quad \text{and} \quad |\mathbf{T}| = 1.
\]

Deriving (32), we obtain

\[
\mathcal{A}\mathbf{T} = s\mathbf{T}' + 2\mathbf{T} \times \mathbf{T}'', \quad \mathbf{T} \cdot \mathbf{T}' = 0.
\]

Hence,

\[
\mathcal{A}\mathbf{T} \times \mathbf{T} = -s\mathbf{T} \times \mathbf{T}' - 2(\mathbf{T} \cdot \mathbf{T}'')\mathbf{T} + 2|\mathbf{T}|^2\mathbf{T}'
\]

\[
= -\frac{s}{2}(I + A)G + \frac{s^2}{2} \mathbf{T} - 2(\mathbf{T} \cdot \mathbf{T}'')\mathbf{T} + 2\mathbf{T}'',
\]

by bearing in mind the vectorial identity (25) and (32).

Next, from (32), \( (I + A)G \cdot \mathbf{T}' = 0 \), so that if we take the scalar product of the above identity and \( \mathbf{T}' \), we obtain

\[
(\mathcal{A}\mathbf{T} \times \mathbf{T}) \cdot \mathbf{T}' = \frac{d|\mathbf{T}'|^2}{ds}.
\]

Finally, a simple calculation proves that, if \( \mathcal{A} \) is defined by (24), then the latter identity rewrites

\[
\frac{d|\mathbf{T}'|^2}{ds} = -aT_3'(s).
\]

The result follows by integrating the previous identity and noticing that, from (22), \( |\mathbf{T}'(0)|^2 = |(\mathbf{T}' \times \mathbf{T})(0)|^2 = |(I + A)G(0)|^2/4. \)

Notice that, from the previous lemma, in particular we get that \( |c(s)| \leq C, \ \forall s \in \mathbb{R}, \) and \( |a| \leq M. \)
Proposition 2. 

Precisely, we will prove the following result: 

\[ A^+ = \lim_{s \to +\infty} e^{-A \log |s|} \frac{G(s)}{s} = G(1) - 2 \int_{-\infty}^{+\infty} e^{-A \log |s'|} \frac{cb}{(s')^2} ds', \tag{33} \]

\[ A^- = \lim_{s \to -\infty} e^{-A \log |s|} \frac{G(s)}{s} = -G(-1) + 2 \int_{-\infty}^{-1} e^{-A \log |s'|} \frac{cb}{(s')^2} ds'. \]

Now, choosing in (30) \( s = +\infty, \ s_1 = s > 0 \) and \( s = -\infty, \ s_1 = s < 0 \), we respectively obtain

\[ G(s) = se^{A \log |s|} A^+ + 2se^{A \log |s|} \int_{s}^{+\infty} e^{-A \log |s'|} \frac{cb}{(s')^2} ds', \quad s > 0, \tag{34} \]

\[ G(s) = se^{A \log |s|} A^- - 2se^{A \log |s|} \int_{-\infty}^{s} e^{-A \log |s'|} \frac{cb}{(s')^2} ds', \quad s < 0. \]

Let us define

\[ B^\pm = (I + A) A^\pm = \lim_{s \to \pm\infty} e^{-A \log |s|} T(s), \]

(notice that, because \(|T(s)| = 1\), from the above identity easily follows that \(|B^\pm| = 1\)).

Then, deriving (34) and using the above notation, we obtain the following expressions for the tangent vector

\[ T(s) = e^{A \log |s|} B^+ - 2 \frac{cb}{s} + 2e^{A \log |s|} \int_{s}^{+\infty} e^{-A \log |s'|} \frac{(I + A)(cb)}{(s')^2} ds', \quad s > 0 \tag{35} \]

\[ T(s) = e^{A \log |s|} B^- - 2 \frac{cb}{s} - 2e^{A \log |s|} \int_{-\infty}^{s} e^{-A \log |s'|} \frac{(I + A)(cb)}{(s')^2} ds', \quad s < 0. \]

Corollary 1. The limits \( \lim_{s \to \pm\infty} T_3(s) \) do exist, and therefore (see 11) it follows the existence of \( c_{t=\infty} = \lim_{s \to \pm\infty} c(s) \).

We will now continue with the proof of the convergence of \( \mathbf{X}(s,t) \) to the initial data. Precisely, we will prove the following result:

Proposition 2. Let \( \mathbf{X}(s,t) \) be the solution of LIA defined in Proposition 1. Then, there exist \( A^+ \) and \( A^- \in \mathbb{R}^3 \) (not necessarily unitary vectors) such that

\[ \lim_{t \to 0^+} \mathbf{X}_a(s,t) = \lim_{t \to 0^-} e^{\frac{A}{2} \log t} \sqrt{t} G_a(s/\sqrt{t}) = \begin{cases} 
se^{A \log |s|} A^+, \quad s \geq 0 \\
se^{A \log |s|} A^-, \quad s \leq 0
\end{cases} \]
Hence,

\[ |X_n(s, t) - se^{-A \log |s|} A^\pm| \leq 2\sqrt{t} \left( \sup_{s \in \mathbb{R}} |c(s)| \right). \]

Moreover, the maps \( T^\pm : (G(0), G'(0), a) \in \mathbb{R}^3 \times S^2 \times \mathbb{R} \rightarrow A^\pm \) are continuous.

**Proof.** From the definition of the matrix \( A \) in (21), it can be easily shown that \( e^{A \log(|s|/\sqrt{t})} \) is an orthogonal matrix \( \forall (s, t) \) such that \( |s|/\sqrt{t} \neq 0 \). Also, the norm \( \|(I + A)^{-1}\| \leq 1 \).

Now, let \( s > 0 \). By taking into account the just mentioned properties related to \( A \), from the expression of \( G(s) \) in (33), we get that

\[
|X(s, t) - se^{-A \log |s|} A^\pm| = |e^{\frac{1}{2} \log t} G(s/\sqrt{t}) - se^{-A \log |s|} A^\pm| = 2se^{-A \log |s|} \int_{s/\sqrt{t}}^{+\infty} e^{-A \log |s'|} \left( \frac{cb(s')}{(s')^2} \right) ds' \leq 2s\sqrt{t} \int_{s/\sqrt{t}}^{+\infty} \frac{c(s'\sqrt{t})}{(s')^2} ds'' \leq 2\sqrt{t} \left( \sup_{s > 0} |c(s)| \right).
\]

The same argument as above, where we now use (34) for \( s < 0 \), gives that

\[
|X(s, t) - se^{-A \log |s|} A^-| \leq 2\sqrt{t} \left( \sup_{s > 0} |c(s)| \right).
\]

Next, consider \( s = 0 \). Notice that, because \( X(s, t) = e^{\frac{1}{2} \log t} \sqrt{t} G(s/\sqrt{t}) \), we know that \( G(s) \) is solution of (22). In particular, the evaluation of (22) at \( s = 0 \) yields

\[ G(0) = 2(I + A)^{-1}(G' \times G'')(0), \quad \text{i.e.,} \quad G(0) = 2(I + A)^{-1}(cb(0)). \]

Hence,

\[
|X(0, t) - 0| = |e^{\frac{1}{2} \log t} \sqrt{t} G(0) - 0| = \sqrt{t}|G(0)| = \sqrt{t}2(I + A)^{-1}(cb(0))| \leq 2c_0 \sqrt{t}(I + A)^{-1} = 2c_0 \sqrt{t},
\]

with \( c_0 = c(0) \). The convergence result of \( X(s, t) \) stated in the proposition is an immediate consequence of the above inequalities.

Finally, the continuity property of the maps \( T^\pm : (G(0), G'(0), a) \in \mathbb{R}^3 \times S^2 \times \mathbb{R} \rightarrow A^\pm \) easily follows from the expressions of \( A^\pm \) in (33), the fact that \( c(s) \) is bounded (see Lemma 4),
and the continuous dependence of the solutions of the ODE \( (G(0), G'(0), a) \).

\[ \square \]

2.2. Asymptotics.

In the sequel, we will be devoted to study the asymptotic behaviour of \( G(s) \) and \( T(s) \), quantifying their wavelike behaviour through the vector \( c(n - ib) \). To this end, we will need a more exhaustive study of the properties of the curvature and torsion functions related to the curve \( G(s) \), in the case when \( c_{\pm \infty} \neq 0 \). In order to clarify the exposition we have included this analysis in a separated section (see Section 3).

We will continue to prove the following theorem:

**Theorem 5.** Let \( G \) be a solution of \( (22) \). Then, the following asymptotics hold as \( s \to \pm \infty \):

i) \[ G(s) = s e^{A \log |s|} A^\pm + \frac{e^{A \log |s|}}{s} \{ 2c^{\pm \infty}_B B^\pm - AB^\pm \times B^\pm \} - 4 \frac{c^{\pm \infty}_B n}{s^2} + O \left( \frac{1}{s^3} \right), \]

ii) \[ T(s) = e^{A \log |s|} B^\pm - 2 \frac{c^{\pm \infty}_B b}{s} + O \left( \frac{1}{s^2} \right). \]

iii) Moreover, if \( a \neq 0 \), \( B^\pm_3 \neq \pm 1 \) and \( c_{\pm \infty} \neq 0 \), then

\[ c_{\pm \infty} (n - ib)(s) = \frac{b \pm e^{ia} \mp A \log |s|}{} \{ AB^\pm \times B^\pm - iAB^\pm \} + O \left( \frac{1}{s^7} \right), \]

where \( \phi(s) = (s^2/4) - \gamma \pm \log |s| \).

iv) If \( a \neq 0 \) and either \( B^\pm_3 \) or \( B^\pm_3 \in \{+1, -1\} \), then \( G(s) = (0, 0, \pm s) \).

Here, the vectors \( B^\pm, A^\pm \), and the constants \( a_\pm, b_\pm, c_{\pm \infty} \) and \( \gamma \pm \) are defined as in the main theorem (see Theorem 1).

**Remark 4.** The maps \( (G(0), G'(0), a) \to (b_\pm, a_\pm) \) are continuous. (See the expression of \( z_+ = b_+ e^{ia} \) in \([123], [124]\), and notice the continuous dependence of the solutions of the O.D.E. which satisfies \( w_1 \) with respect to the initial data. A similar expression can be found for \( z_- = b_- e^{ia} \).

**Proof of Theorem 2** In what follows, we will reduce ourselves to consider the case \( s \to +\infty \). The case \( s \to -\infty \) will follow using the same arguments.
Recall that in the previous subsection we have obtained the following expressions for the vectors $\mathbf{G}(s)$ and $\mathbf{T}(s)$ (see (34) and (35))

$$(36) \quad \mathbf{G}(s) = se^{A \log |s|} \mathbf{A} + 2se^{A \log |s|} \int_s^{+\infty} e^{-A \log |s'|} \frac{\mathbf{b}}{(s')^2} ds', \quad s > 0,$$

and

$$(37) \quad \mathbf{T}(s) = e^{A \log |s|} \mathbf{B} - 2\frac{\mathbf{b}}{s} + 2e^{A \log |s|} \int_s^{+\infty} e^{-A \log |s'|} \frac{(I + A)(c\mathbf{b})}{(s')^2} ds', \quad s > 0.$$

Observe that, from the Serret-Frenet system (see (2))

\[ n_s = -c_T + (\tau - s/2) b + (s/2) b, \]

with \(|T| = |n| = |b| = 1\). Then,

\[ \int_s^{+\infty} e^{-A \log |s'|} \frac{\mathbf{b}}{(s')^2} ds' = 2 \int_s^{+\infty} e^{-A \log |s'|} \frac{c}{(s')^2} (s'/2) \mathbf{b} ds' \]

$$(38) \quad = 2 \int_s^{+\infty} e^{-A \log |s'|} \frac{c}{(s')^2} (\mathbf{n}_s + c\mathbf{T} - (\tau - s'/2) \mathbf{b}) ds'.$$

Now, notice that

\[ e^{-A \log |s|} \frac{c\mathbf{n}_s}{s^2} = \left( e^{-A \log |s|} \frac{c\mathbf{n}}{s^3} \right)_s - e^{-A \log |s|} \left\{ -c\frac{\mathbf{A} \mathbf{n}}{s^4} + \left( \frac{c_s}{s^3} - \frac{3c}{s^2} \right) \mathbf{n} \right\}. \]

Therefore, after an integration by parts and having into account that $c(s)$ is bounded (see Lemma 1), the identity (35) rewrites as

\[ \int_s^{+\infty} e^{-A \log |s'|} \frac{c\mathbf{b}}{(s')^2} ds' = -2e^{-A \log |s|} \frac{c\mathbf{n}}{s^3} + 2 \int_s^{+\infty} e^{-A \log |s'|} \frac{(A + 3I)(c\mathbf{n})}{(s')^4} ds' \]

$$+ 2 \int_s^{+\infty} e^{-A \log |s'|} \frac{(c^2 \mathbf{T} - c_s \mathbf{n} - c(\tau - s'/2) \mathbf{b})}{(s')^3} ds'.$$

Recall that $\mathbf{G}(s)$ satisfies that

\[ (I + A) \mathbf{G} - s \mathbf{G}' = 2(\mathbf{G'} \times \mathbf{G}''), \quad \text{i.e.,} \quad (I + A) \mathbf{G} - s \mathbf{T} = 2c\mathbf{b}. \]

Then, by using the Serret-Frenet formulae, the derivation of the above equation with respect to $s$ concludes that the unit vector $\mathbf{T}(s)$ is a solution of

$$\mathcal{A} \mathbf{T} = c(s - 2\tau) \mathbf{n} + 2c_s \mathbf{b},$$

from which it follows that

$$\mathcal{A} \mathbf{T} \times \mathbf{T} = 2(c_s \mathbf{n} + c(\tau - s/2) \mathbf{b}).$$
The substitution of (41) into (39) yields
\[
\int_{s}^{+\infty} e^{-A \log |s'|} \frac{cb}{(s')^2} ds' = -2 e^{-A \log |s|} \frac{cm}{s^2} + 2 \int_{s}^{+\infty} e^{-A \log |s'|} \frac{(A + 3I)T_s}{(s')^4} ds' \\
\int_{s}^{+\infty} e^{-A \log |s'|} \frac{AT \times T}{(s')^3} ds' + 2 \int_{s}^{+\infty} e^{-A \log |s'|} \frac{cT}{(s')^3} ds',
\]
(42)

(notice that, since \(cn = T_s\), the above identity holds true even if \(c = 0\), what can be checked directly).

As a consequence, we conclude the following expression for \(G(s)\) (36)
\[
G(s) = \frac{se^{-A \log |s|}}{A} + \frac{4 cn}{s^2} \\
+ 2s e^{-A \log |s|} \left\{ \int_{s}^{+\infty} e^{-A \log |s'|} \frac{(A + 3I)T_s}{(s')^4} ds' - \int_{s}^{+\infty} e^{-A \log |s'|} \frac{AT \times T}{(s')^3} ds' \\
+ 2 \int_{s}^{+\infty} e^{-A \log |s'|} \frac{c^2 T}{(s')^3} ds' \right\}.
\]
(43)

Also, from Lemma 11, it is not difficult to see that the integral in (42) is a n error term of order \(O(1/s^2)\), as \(s \to +\infty\). Then, from (37), we get that
\[
T(s) = e^{-A \log |s|} B + \frac{2cb}{s} + O \left(\frac{1}{s^3}\right), \quad s \to +\infty.
\]
(44)

Now, we come back to the proof of the asymptotics of \(G(s)\) in \(i)\). To this end, we will analyze each of the integrals in (43).

Firstly, since \(|T| = 1\), notice that
\[
e^{-A \log |s|} \frac{(A + 3I)T_s}{s^4} = \left( e^{-A \log |s|} \frac{(A + 3I)T}{s^4} \right) + O \left(\frac{1}{s^6}\right), \quad s \to +\infty.
\]
Then,
\[
\int_{s}^{+\infty} e^{-A \log |s'|} \frac{(A + 3I)T_s}{(s')^4} ds' = O \left(\frac{1}{s^4}\right).
\]
(45)

Secondly, by taking into account (35) and the asymptotic development of \(c^2(s)\) in Theorem 7 part \(iv)\), precisely
\[
c^2(s) = c^2_{+\infty} + 2 \frac{b_+ s}{s^2} \sin \tilde{\phi}(s) + O \left(\frac{1}{s^2}\right), \quad \text{with} \quad \tilde{\phi}(s) = a_+ + (s^2/4) - \gamma_+ \log(s),
\]
we obtain
\[
c^2 T(s) = e^{A \log |s|} B + \left( c^2_{+\infty} + 2 \frac{b_+ s}{s^2} \sin \tilde{\phi}(s) \right) - 2c^2_{+\infty} \frac{cb}{s} + O \left(\frac{1}{s^2}\right).
\]
so that
\[ \int_{s}^{+\infty} e^{-A\log|s'|} \frac{c^2 B}{(s')^3} ds' = c^2_+ \frac{B^+}{2s^2} + 2b_+ B^+ \int_{s}^{+\infty} \frac{\sin \tilde\phi(s)}{(s')^4} ds' \]
(46)
\[ - 2c^2_+ \int_{s}^{+\infty} e^{-A\log|s'|} \frac{cb}{(s')^4} ds' + O\left(\frac{1}{s^4}\right).\]

On the one hand, notice that \( \tilde\phi'(s) = s/2 - \gamma_+/s \neq 0 \) for \( s \) sufficiently large. Therefore, an integration by parts argument gives that
\[ \int_{s}^{+\infty} e^{-A\log|s'|} \frac{\sin \tilde\phi(s)}{(s')^4} ds' = O\left(\frac{1}{s^4}\right). \]
(47)

On the other hand, a similar argument to that given in obtaining (42), and using the fact that \( c(s) \) is bounded (see Lemma 1) conclude that
\[ \int_{s}^{+\infty} e^{-A\log|s'|} \frac{cb}{(s')^4} ds' = O\left(\frac{1}{s^4}\right), \quad \text{as} \quad s \to +\infty, \]
by using once again that \( c(s) \) is bounded (see Lemma 1). Substituting (47) and (48) into (46), one gets
\[ \int_{s}^{+\infty} e^{-A\log|s'|} \frac{c^2 B}{(s')^3} ds' = c^2_+ \frac{B^+}{2s^2} + O\left(\frac{1}{s^4}\right). \]
(49)

Finally, from (45), we have
\[ \mathcal{A}T \times T = e^{A\log|x|} (A B^+ \times B^+) - \frac{2}{s} \left( e^{A\log|x|} A B^+ \right) \times (c b) \]
\[ - \frac{2}{s} A (c b) \times \left( e^{A\log|x|} B^+ \right) + O\left(\frac{1}{s^2}\right). \]
Therefore, from (48)
\[ \int_{s}^{+\infty} e^{-A\log|s'|} \frac{\mathcal{A}T \times T}{(s')^3} ds' = \frac{1}{2s^2} A B^+ \times B^+ + O\left(\frac{1}{s^4}\right). \]
(50)

The substitution of (45), (49) and (50) into (43) concludes the asymptotic behaviour of \( G(s) \) stated in \( i) \), that is
\[ G(s) = s e^{A\log|x|} A^+ + e^{A\log|x|} s \left\{ 2c^2_+ B^+ - A B^+ \times B^+ \right\} - 4 \frac{c n}{s^2} + O\left(\frac{1}{s^3}\right). \]
(51)

From (44) and (51), we see that giving more accurate asymptotics of \( G \) and \( T \) implies the study of the associated vectors \( c b \) and \( c n \). We will continue to obtain a closed formula for \( c(n - i b) \) which is valid for \( s \) sufficiently large, \( a \neq 0, B_3^+ \neq \pm 1 \) and \( c_{+\infty} \neq 0 \).

We firstly observe that, because \( A \) is an antisymmetric matrix, \( \mathcal{A}T \cdot T = 0. \) Then, under the above assumptions on \( a \) and \( B_3^+ \), the expression for \( T(s) \) in (44) asserts us that
\( |T_3(s)| < 1 \) when \( s \) is sufficiently large. Then \( \mathcal{A}T \neq 0 \) and we can consider \( \{ \mathcal{A}T, T, \mathcal{A}T \times T \} \) as a basis of \( \mathbb{R}^3 \). Thus, the vector \( c(n - ib) \) can be written as a linear combination of the elements of this basis.

To this end, recall that (see (40) and (41))

\[
\mathcal{A}T = -2c(\tau - s/2)n + 2c_s b, \quad \text{and} \quad \mathcal{A}T \times T = 2c_s n + 2c(\tau - s/2)b.
\]

Therefore, because \( n \perp b \), \( b \perp T \), \( n \perp T \), \( |n| = |b| = |T| = 1 \), we obtain

\[
c(n - ib) \cdot \mathcal{A}T = -2c^2(\tau - s/2) - i\frac{dc^2}{ds} = -i\left( \frac{dc^2}{ds} - 2ic^2(\tau - s/2) \right),
\]

\[
c(n - ib) \cdot T = 0, \quad \text{and} \quad c(n - ib) \cdot (\mathcal{A}T \times T) = \frac{dc^2}{ds} - 2ic^2(\tau - s/2).
\]

As a consequence, we conclude that

\[
c(n - ib) = \left( \frac{dc^2}{ds} - 2ic^2(\tau - s/2) \right) \left( \frac{\mathcal{A}T \times T}{|\mathcal{A}T \times T|^2} - i\frac{\mathcal{A}T}{|\mathcal{A}T|^2} \right)
\]

(notice that (52) is valid whenever \( T_3(s) \neq \pm 1 \) and \( a \neq 0 \). In particular, it is valid if \( c = 0 \). In that case, both sides are understood to be zero).

Now, by using the asymptotic behaviour of \( T(s) \) given in (44), we get that

\[
\mathcal{A}T \times T - i\mathcal{A}T = e^{4\log|s|} \{ AB^+ \times B^+ - iAB^+ \} + O \left( \frac{1}{|s|} \right).
\]

Also, it is satisfied that

\[
|\mathcal{A}T \times T|^2(s) = |\mathcal{A}T|^2(s) = |AB^+|^2 + O \left( \frac{1}{|s|} \right).
\]

On the other hand, from the asymptotics related to \( c(s) \) and \( \tau(s) \) in Theorem 7, it follows that

\[
\frac{dc^2}{ds} - 2ic^2(\tau - s/2) = b_+ e^{ia} e^{i\phi(s)} + O \left( \frac{1}{|s|} \right), \quad s \to +\infty,
\]

with \( \phi(s) = (s^2/4) - \gamma_+ \log |s| \).

From (52) - (55) we obtain

\[
c(n - ib) = \frac{b_+ e^{ia}}{|AB^+|^2} e^{i\phi(s)} e^{4\log|s|} \{ AB^+ \times B^+ - iAB^+ \} + O \left( \frac{1}{|s|} \right),
\]

for some constant \( a_+ \in [0, 2\pi] \).

The proof of (i)-(iii) is now an immediate consequence of the above identity, (44), (51), and (110). Notice that (110) follows from (107), which is also true if \( c_{\pm \infty} = 0 \) because it
just involves $|f|^2 = c^2$ and $h = -(AT \cdot T')/2$. In fact, \cite{101} easily follows from Lemma \cite{102} and using that, from the equation \cite{22}, $h$ rewrites as $h = -a(G_3 - sT_3)/4$ (see Remark \cite{4}). Besides, observe that the constants in Theorem \cite{5} (or, equivalently in Theorem \cite{1} in the introduction) are directly deduced from the ones in Theorem \cite{7} and Lemma \cite{1}. Finally, we prove the part (iv). Let $a \neq 0$ and $B_3^+ = +1$. Then,

$$a^2(1 - T_3^2)|_{s=\infty} = |AT \times T'|^2|_{s=\infty} = 0,$$

and therefore $h(+\infty) = y(+\infty) = 0$ (recall $h = -(AT \cdot T')/2$ and $y = (AT \times T) \cdot T'$).

Assume now that $c_{+\infty} = 0$, so that $a^2 = a^2$, by using Lemma \cite{11}. Then, $y = h = 0$ solves \cite{103} (recall that $E(0) = a^2/4$ -see \cite{100}) so that $(dc^2/ds) = 0$, and since $c_{+\infty} = 0$, we obtain $c(s) = 0$. Hence, $G(0, 0, s)$.

Secondly, we assume that $c_{+\infty} \neq 0$. Then $c(s) \neq 0$ for $s$ large enough and \cite{32} is valid whenever $AT \neq 0$ (i.e., $a \neq 0$ and $T_3 \neq \pm 1$).

Let us prove that $T_3(s) \neq \pm 1$ for $s$ large enough. Notice that, since $B_3^+ = +1$, it is enough to see that $T_3(s) \neq 1$. To this end, assume on the contrary that there exists $s_0$ large enough such that $T_3(s_0) = 1$. Then, there exists $\{s_n\}_n \rightarrow s_0$, as $n \rightarrow +\infty$, such that $T_3(s_n) \neq \pm 1$ and $T_3(s_n) \rightarrow 1$. Hence, from \cite{11}, we get that $c(s_0) = 0$, which contradicts the assumption $c_{+\infty} \neq 0$. Therefore, \cite{32} holds true for $s$ large enough and letting $s \rightarrow +\infty$ we get that $c_{+\infty} = 0$, which also leads to contradiction.

The same arguments as above prove that, in the cases when $B_3^- = 1$ or $B_3^\pm = -1$, $G(s) = (0, 0, +s)$ or $G(s) = (0, 0, -s)$, respectively. This concludes the proof.

\[ \square \]

2.3. Scattering problem for the curve $G$.

We will finish this section proving the Theorem \cite{2} concerning with the scattering problem for $G(s)$. More precisely,

**Theorem 6.** Given $a \neq 0$, $B^+ = (B_1^+, B_2^+, B_3^+) \in S^2$ with $B_3^+ \neq \pm 1$, $a_+ \in [0, 2\pi)$ and $b_+ \geq 0$, there exists a solution $G(s)$ of

\[
(I + A)G - sG' = 2G' \times G'', \quad |G'| = 1
\]
satisfying the following identities:

\[(57) \quad \lim_{s \to +\infty} e^{-A \log |s|} \frac{G(s)}{s} = A^+,
\]

\[(58) \quad \lim_{s \to +\infty} e^{-A \log |s|} T(s) = B^+,
\]

\[(59) \quad \lim_{s \to +\infty} e^{-i\phi(s)} e^{-A \log |s|} (T' - iT \times T')(s) = \frac{b_+ e^{i\alpha+}}{|AB|^2} (AB \times B^+ - iAB^+),
\]

where \(A^+ = (I + A)^{-1} B^+\), \(\phi(s) = (s^2/4) - \gamma_+ \log |s|\),

\[
\gamma_+ = 3aB_3^+ + \alpha \quad \text{and} \quad \alpha = -aB_3^+ - \frac{b_+^2}{a^2[1 - (B_3^+)^2]}.
\]

Moreover, if \(b_+ > 0\), the solution is unique. A similar result can be obtained at \(s \to -\infty\).

**Proof of Theorem 6.** Existence. Firstly, observe that, whenever \(T\) is a solution of

\[(60) \quad \mathcal{A} T = s T' + 2 T \times T''
\]

with initial data

\[(61) \quad |T(0)| = 1 \quad \text{and} \quad T(0) \cdot T'(0) = 0,
\]

then \(G(s)\) such that

\[(62) \quad G'(s) = T(s) \quad \text{and} \quad (I + A)G(0) = 2 T(0) \times T'(0)
\]
satisfies \(\mathcal{M}\), and \((I + A)G(0) \cdot G'(0) = 0\).

Indeed, from \(\mathcal{M}\),

\[
\frac{d}{ds}|G'|^2 = \frac{d|T|^2}{ds} = 2 T \cdot T' = 0, \quad s \neq 0 \quad \text{and}
\]

\[
\frac{d}{ds}(I + A)G = \mathcal{A} T + T = s T' + 2 T \times T'' + T = \frac{d}{ds}(s T + 2 T \times T').
\]

Then, \(\mathcal{M}\) follows from the above identities and the initial conditions in \(\mathcal{L}\) and \(\mathcal{M}\), respectively.

Moreover, from the results in Section 4, the condition \(\mathcal{M}\) is satisfied with

\[
A^+ = (I + A)^{-1} \left( \lim_{s \to +\infty} e^{-A \log |s|} T(s) \right).
\]

Previous remarks allow us to reduce ourselves to prove the existence of a solution \(T\) of \(\mathcal{M}\) and \(\mathcal{L}\) satisfying the limiting conditions \(\mathcal{N}\) and \(\mathcal{O}\).
It is easy to see the existence of a global solution of (60) and (61): Given a fixed initial data \((T(0), T'(0))\) as in (61), consider \(G(s)\) such that
\begin{equation}
G'' = \frac{1}{2} (\mathcal{I} + A) G \times G',
\end{equation}
\begin{equation}
(\mathcal{I} + A) G(0) = 2 T(0) \times T'(0) \quad \text{and} \quad G'(0) = T(0).
\end{equation}
Notice that (61) and (64) imply that \(|G'(0)| = 1\) and \((\mathcal{I} + A) G(0) \cdot G'(0) = 0\) and thus, there exists \(G(s)\) global solution of (63)-(64) (see Section 2). Moreover, \(G \in C^\infty(\mathbb{R}; \mathbb{R}^3)\) and \(|G'(s)| = 1, \ \forall s \in \mathbb{R}\).

Now take \(T(s) = G'(s)\). Then, deriving (56) we obtain that \(T(s)\) is a solution of (60).

On the other hand, from (63) at \(s = 0\), \(2 G''(0) = (\mathcal{I} + A) G(0) \times G'(0)\). Therefore,
\[T(0) = G'(0), \quad |G'(0)| = 1 \quad \text{and} \quad T(0) \cdot T'(0) = G'(0) \cdot G''(0) = 0,\]
so that the initial conditions in (61) are satisfied.

Finally, recall in the sequel that, if \(T\) is a solution of (60) and (61), then \(G\) defined in (62) verifies (60). Hence, we have already proved (see (26), (27) and Lemma 1) that \(T\) satisfies the following properties:
\begin{equation}
|T(s)| = 1, \quad T(s) \cdot T'(s) = 0, \quad \text{and} \quad |T'(s)|^2 = -aT_3(s) - \beta, \quad \forall s \in \mathbb{R},
\end{equation}
with \(\beta = -|T'(0)|^2 - aT_3(0)\).

We now come back to the proof of Theorem 6. For a fixed \(a \neq 0\), \(B^+ \in \mathbb{S}^2\) with \(B^+_3 \neq \pm 1, a_+ \in [0, 2\pi)\) and \(b_+ \geq 0\), there exists a unique \(\alpha \in \mathbb{R}\) such that
\begin{equation}
b_+ = \sqrt{a^2(-aB^+_3 - \alpha)(1 - (B^+_3)^2)}, \quad \text{i.e.,} \quad \alpha = -aB^+_3 - \frac{b_+^2}{a^2(1 - (B^+_3)^2)}.
\end{equation}
In particular, as we have already observed at the beginning of the proof, the result in Theorem 6 will follow from the existence, for this value of \(\alpha\), of initial data \((T(0), T'(0))\) such that
\begin{equation}
|T(0)| = 1, \quad T(0) \cdot T'(0) = 0 \quad \text{and} \quad |T'(0)|^2 = -aT_3(0) - \alpha,
\end{equation}
and \(T(s)\) solution of
\begin{equation}
\mathcal{A} T = s T' + 2 T \times T'',
\end{equation}
associated to this initial data, satisfying the limiting conditions \((68)\) and \((69)\). The existence of such this solution is based on a compactness argument.

To this end, consider the compact set

$$K^\alpha = \{(u, v) \in \mathbb{R}^3 \times \mathbb{R}^3 / |u| = 1, \ u \cdot v = 0, \ |v|^2 = -au_3 - \alpha\},$$

with \(u = (u_1, u_2, u_3)\), and for \(s \geq 1\) define the operator \(W(s)\) given by

$$W(s) \begin{pmatrix} T(0) \\ T'(0) \end{pmatrix} = \begin{pmatrix} e^{-A\log |s|} T(s) \\ e^{-i\tilde{\phi}(s)}e^{-A\log |s|}(T' - iT \times T')(s) \end{pmatrix}, \quad \forall (T(0), T'(0)) \in K^\alpha,$$

where \(T\) is a solution of \((67)\) and \(\tilde{\phi}(s) = (s^2/4) - (3T_3(s) + \alpha)\log |s|\).

Recall that we have already proved the global existence of a solution of the problem \((67)-\(68)\), so that \(W(s)\) is well-defined. Also, it is easy to see that \(W(s)\) is a continuous operator, \(\forall s \geq 1\).

Assume momentarily that the following claim holds:

**Claim.** Let be \(B^+, a_+, b_+\) and \(\alpha\) as above, and \(s_0 \geq 1\). Then, there exists initial data \((T(0), T'(0)) \in K^\alpha\), depending on \(s_0\), and \(T(s)\) solution of \((68)\) associated to this initial data such that

$$e^{-A\log |s_0|} T(s_0) = B^+, \quad \text{and}$$

$$e^{-i\tilde{\phi}(s_0)}e^{-A\log |s_0|}(T' - iT \times T')(s_0) = \frac{b_+ e^{ia_+}}{|AB^+|^2} (AB^+ \times B^+ - iAB^+),$$

with \(T_3(s_0) = B_3^+\).

Now, choose a sequence \(\{s_n : s_n \geq 1\}_{n \in \mathbb{N}}\) such that \(s_n \to +\infty\), as \(n \to +\infty\). Then, for any fixed \(n \in \mathbb{N}\) the above claim ensures the existence of initial data \((T_n(0), T'_n(0)) \in K^\alpha\), and \(T_n(s)\) solution of \((68)\) with this initial data such that the following identity holds

$$W(s_n) \begin{pmatrix} T_n(0) \\ T'_n(0) \end{pmatrix} = \begin{pmatrix} e^{-A\log |s_n|} T_n(s_n) \\ e^{-i\tilde{\phi}(s_n)}e^{-A\log |s_n|}(T'_n - iT_n \times T'_n)(s_n) \end{pmatrix}$$

$$= \begin{pmatrix} B^+ \\ \frac{b_+ e^{ia_+}}{|AB^+|^2} (AB^+ \times B^+ - iAB^+) \end{pmatrix}, \quad \forall n \in \mathbb{N}.$$  \hspace{1cm} (71)

The compactness of the set \(K^\alpha\) yields the existence of \((T(0), T'(0)) \in K^\alpha\) such that

$$T(0) = \lim_{k \to +\infty} T_{n_k}(0) \quad \text{and} \quad T'(0) = \lim_{k \to +\infty} T'_{n_k}(0),$$

(72)
for some subsequence of \( \{ (T_n(0), T'_n(0)) \}_{n \in \mathbb{N}} \).

Next, consider the operator \( W(\infty) = \lim_{s \to +\infty} W(s) \) defined as follows

\[
W(\infty) \begin{pmatrix} T(0) \\ T'(0) \end{pmatrix} = \lim_{s \to +\infty} W(s) \begin{pmatrix} T(0) \\ T'(0) \end{pmatrix}
\]

(73)

\[
= \lim_{s \to +\infty} \left( e^{-A \log |s|} T(s) e^{-i \phi(s)} e^{-A \log |s|} (T' - iT \times T')(s) \right),
\]

where \( T(s) \) satisfies (48) with initial data \((T(0), T'(0)) \in K^\alpha\).

Notice that, from (63) and the fact that \((T(0), T'(0)) \in K^\alpha\), it follows that

\[
|T(s)| = 1 \quad \text{and} \quad |T'(s)|^2 = -\alpha T_3(s) - \alpha \quad \text{with} \quad \alpha = -\alpha T_3(0) - |T'(0)|^2.
\]

Also, from the Serret-Frenet formulae (see (22)), \( c(n - i b) = T' - iT \times T' \), where we understand that both sides are zero if \( T' = 0 \).

The above remarks and the asymptotic behaviour of \( T \) and \( c(n - i b) \) (see parts i)-iii) of Theorem 1 and Remark 1 assert the existence of the limits in the definition of \( W(\infty) \), so that \( W(\infty) \) is well-defined. Moreover, from Remark 1 it follows that \( W(\infty) \) is continuous.

Now take the initial data \((T(0), T'(0)) \) in (72). Then, the continuity property of \( W(\infty) \) and the identity (71) yield

\[
W(\infty) \begin{pmatrix} T(0) \\ T'(0) \end{pmatrix} = \lim_{k \to \infty} \lim_{s \to +\infty} \left( e^{-A \log |s_n|} T_{n_k}(s_{n_k}) e^{-i \phi(s_{n_k})} e^{-A \log |s_n|} (T'_{n_k} - iT_{n_k} \times T'_{n_k})(s_{n_k}) \right)
\]

(74)

\[
= \begin{pmatrix} B^+ \\ \frac{b_k e^{i n_s \phi}}{|AB^+|^2} (AB^+ \times B^+ - iAB^+) \end{pmatrix},
\]

because \( s_{n_k} \to +\infty \), as \( k \to +\infty \).

From (74) and the definition of the action of \( W(\infty) \) on the initial data in (72), we conclude that the solution of (67) associated to the initial data in (72) satisfies the limiting conditions in (68) and (69), that is

\[
\lim_{s \to +\infty} \left( e^{-A \log |s|} T(s) e^{-i \phi(s)} e^{-A \log |s|} (T' - iT \times T')(s) \right) = \begin{pmatrix} B^+ \\ \frac{b_k e^{i n_s \phi}}{|AB^+|^2} (AB^+ \times B^+ - iAB^+) \end{pmatrix}.
\]
The result in Proposition 6 is an immediate consequence of this identity, by noticing that the first limiting condition gives that \( \lim_{s \to +\infty} T_3(s) = B_3^+ \).

We now come back to the proof of the claim: Let \( s_0 \geq 1 \). Firstly, we notice that the global existence of solution of (67)-(68) and the identities in (65) reduce the problem to finding \((T(s_0), T'(s_0)) \in K^\alpha\) satisfying the conditions in the statement of the claim, that is (69) and (70).

Then, from (69), it follows that \( T(s_0) \) is given by

\[
T(s_0) = e^{A\log|s_0|}B^+,
\]

and in particular \( T_3(s_0) = B_3^+ \).

On the other hand, by taking into account (69) in (70), we obtain that \( T'(s_0) \) should satisfy

\[
e^{-i\phi(s_0)}e^{-A\log|s_0|}(T' - iT \times T')(s_0) = \frac{b_+e^{ia+}}{|AB^+|} e^{-A\log|s_0|}(AT \times T - iTAT)(s_0),
\]

or, equivalently,

\[
(T' - iT \times T')(s_0) = \omega(AT \times T - iTAT)(s_0),
\]

for some known \( \omega \in \mathbb{C} \) which depends on \( B_3^+, a_+, b_+ \) and \( \alpha \), and such that \( |\omega|^2 = b_+^2 / |AB^+|^4 \).

In order to prove the existence of \( T'(s_0) \) solution of the above equation, firstly observe that, because \( B_3^+ \neq \pm1 \) and \( T(s_0) = e^{A\log|s_0|}B^+ \) (see (75)), it follows that \( T_3(s_0) \neq \pm1, s_0 \geq 1 \), and therefore \( AT(s_0) \neq 0 \). As a consequence, the set of vectors

\[
\mathcal{B} = \{T(s_0), (AT \times T)(s_0), AT(s_0)\}
\]

is a basis in \( \mathbb{R}^3 \) (recall that, because \( A \) is an antisymmetric matrix, \( (AT \cdot T)(s_0) = 0 \)).

In the coordinate system associated to \( \mathcal{B} \), \( T = (1, 0, 0) \) and, by taking into account that \( (T \cdot T')(s) = 0 \) (see (69)), \( T'(s_0) = \beta_1 (AT \times T)(s_0) + \beta_2 AT(s_0) \), where the (real) scalars \( \beta_1 \) and \( \beta_2 \) are uniquely determined by the equation \( \beta_1 + i\beta_2 = \omega \), just by rewriting the equation (76) in the coordinate system associated to \( \mathcal{B} \).

Now it is easy to check that \( (T(s_0), T'(s_0)) \in K^\alpha \). Indeed, on the one hand, from (75) and (76),

\[
|T(s_0)| = 1 \quad \text{and} \quad T(s_0) \cdot T'(s_0) = 0.
\]
On the other hand, from (76), \( T_3(s_0) = B_3^+ \) and \( |\omega|^2 = b_+^2/|AB|^4 \), we get that
\[
2|T'(s_0)|^2 = 2|\omega|^2|AT(s_0)|^2 = 2 \frac{b_+^2}{|AB|^4} |AB^+|^2.
\]
Then, using the definition of \( b_+ \) in (66) and the fact that \( |AB^+|^2 = a^2(1 - (B_3^+)^2) \), from previous identity we conclude that
\[
|T'(s_0)|^2 = -aB^+ - \alpha = -aT_3(s_0) - \alpha
\]
This finishes the proof of the claim.

The proof of the uniqueness is based on a uniqueness result for the solutions of the self-similar Schrödinger equation that will be proved in Section 4 (see Lemma 2 in Section 4).

**Uniqueness.** Assume \( G_j, j = 1, 2 \), are two solutions of (56) satisfying (57), (58) and (59) for some given \( a \neq 0 \), \( B^+ \in S^2 \) with \( B_3^+ \neq \pm 1 \), \( a_+ \in [0, 2\pi) \) and \( b_+ > 0 \). Let \( c_j \) and \( \tau_j \) denote respectively the curvature and torsion related to
Consider also the functions \( f_j(s), j = 1, 2 \) defined through the relations (96), i.e,
\[
f_j(s)e^{it^2} = c_j(s)e^{i\int_0^s \tau_j(s') \, ds'}, \quad j = 1, 2,
\]
and the associated functions \( h_j = \Im(f_j f_j') \) and \( y_j = d|f_j|^2/\, ds \).

We begin to prove that \( f_j, j = 1, 2 \) satisfy the hypothesis in Lemma 2.

To this end, recall that it has been already shown that
\[
(77) \quad AT \times T = 2c(\tau - s/2)b + 2c_3n \quad \text{and} \quad c^2(s) = -aT_3(s) + \alpha,
\]
for any \( G(s) \) solution of equation (56) (from (11) and by using the Serret-Frenet formulae in (66)).

Firstly, notice that from (69), it is easy to check that
\[
2 \frac{|b_+|^2}{|AB^+|^2} = \lim_{s \to +\infty} |T_j' - iT_j \times T_j'|^2(s) = \lim_{s \to +\infty} 2 |T_j'|^2(s) = 2 c^2_{+\infty, j}, \quad j = 1, 2.
\]
Then,
\[
(78) \quad c^2_{+\infty, j} = c^2_{+\infty} \quad \text{with} \quad c^2_{+\infty} = \frac{|b_+|^2}{|AB^+|^2} \neq 0.
\]
because \( B_3^+ \neq \pm 1, a \neq 0 \) and \( b_+ > 0 \) or, equivalently,
\[
(79) \quad |f_j|^2_{+\infty} = |f_j|^2_{+\infty} \quad \text{with} \quad |f_j|^2_{+\infty} = \frac{|b_+|^2}{|AB^+|^2} \neq 0.
\]
Also, from \(65\),

\[
\lim_{s \to +\infty} |AT_j \times T_j|(s) = |AB^+ \times B^+|, \quad j = 1, 2.
\]

Besides, from \(77\),

\[
|AT_j \times T_j|^2(s) = 4 ((c_j')^2 + c_j^2(\tau_j - s/2)^2), \quad \forall s \in \mathbb{R}, \quad j = 1, 2,
\]

and, by taking into account the above identities together with \(97\), we obtain

\[
|f'_j|_{+\infty} = |f'|_{+\infty}, \quad \text{with} \quad |f'|_{+\infty} = |AB^+ \times B^+|/4.
\]

Finally, we will prove that

\[
\lim_{s \to +\infty} e^{i(s^2/4 - \gamma_{+j} \log |s|)}(\bar{f}_j f'_j)(s) = \frac{b_{+j}}{2} e^{-ia_{+j}}, \quad j = 1, 2.
\]

To this end, notice that the part \(iv)\) in Theorem \(8\) asserts that

\[
\left(\frac{1}{2} \frac{d|f_j|^2}{ds} - i \Re m(\bar{f}_j f'_j)\right)(s) = \frac{b_{+j}}{2} e^{ia_{+j} + i\phi_{1,j}(s)} + O\left(\frac{1}{|s|}\right), \quad s \to +\infty,
\]

where \(\phi_{1,j}(s) = (s^2/4) - \gamma_{+j} \log |s|\), and the constants \(a_{+j}, \gamma_{+j}\) and \(b_{+j}\) are defined in Theorem \(8\).

Let us continue by proving that \(a_{+j}\) and \(b_{+j}\) do not depend on \(j = 1, 2\) under the limiting conditions \(65\) and \(77\).

Indeed, from the Serret-Frenet equations, \(97\), \(98\), \(99\) and \(100\), it is easy to see that

\[
\lim_{s \to +\infty} e^{-i\phi(s)} e^{-A \log |s|} (T' - iT \times T')(s) = \lim_{s \to +\infty} e^{-i\phi(s)} e^{-A \log |s|} c_j(s) (n_j - ib_j)(s) =
\]

\[
\lim_{s \to +\infty} e^{-i(s^2/2 + (3|c_j(s)|^2 + 2a) \log |s|)} e^{-A \log |s|} \left(\frac{dc_j}{ds} - 2ic_j^2(\tau_j - s/2)\right) (AT_j \times T_j - iAT_j)(s) = b_{+j} e^{ia_{+j}} (AB^+ \times B^+ - iAB^+).
\]

Here, it has been used that \((-3|c_j(s)|^2 + 2a) - \gamma_{+j} = -3(|c_j(s)|^2 - c_{+\infty,j}^2) = o(1)\), as \(s \to +\infty\), in obtaining the last identity.

Then, the above identity and \(101\) yield

\[
b_{+j} = b_+ \quad \text{and} \quad a_{+j} = a_+, \quad j = 1, 2.
\]

The identity \(102\) easily follows by substituting the above identities into \(103\).

Now, notice that under the conditions \(77\) and \(80\), Lemma \(2\) concludes that

\[
\frac{d|f_1|^2}{ds}(s) = \frac{d|f_2|^2}{ds}(s) \quad \text{and} \quad \Re m(\bar{f}_1 f'_1) = \Re m(\bar{f}_2 f'_2), \quad s \geq s_0 \gg 1.
\]
Hence,
\[ \frac{dc_1^2}{ds}(s) = \frac{dc_2^2}{ds}(s) \quad \text{and} \quad c_1^2(\tau_1 - s/2) = c_2^2(\tau_2 - s/2), \]
by taking into account (97).

Next recall that \( c_\pm^2, \pm = 1 \neq 0 \) (see (78)). Then, from the above identities we obtain
\[ c_1(s) = c_2(s) \neq 0 \quad \text{and} \quad \tau_1(s) = \tau_2(s), \quad \forall \ s \geq s_0 \gg 1. \]

Therefore, there exists \( \rho \) rotation in \( \mathbb{R}^3 \) such that
\[ (84) \quad T_1^1(s) = \rho(T_2^2(s)), \quad \forall \ s \geq s_0. \]

Now observe that, from (84) and (58), we get that
\[ \lim_{s \to +\infty} e^{-A \log |s|} \rho(T_2^2(s)) = \lim_{s \to +\infty} e^{-A \log |s|} T_1^1(s) = B^+. \]

Then, writing \( \rho = (a_{i,j})_{i,j=1}^3 \) from the above identity follows that
\[ \lim_{s \to +\infty} (a_{31}T_1^1(s) + a_{32}T_2^1(s) + a_{33}T_3^1(s)) = B_3^+, \]
where \( T_1^1 \) satisfies (58), so that \( a_{31} = a_{32} = 0 \) and \( a_{33} = 1 \).

As a consequence, \( \rho = \left( \begin{array}{ccc} e^{i\delta} & 0 \\ 0 & 1 \end{array} \right) \), for some \( \delta \in [0, 2\pi] \), and
\[ e^{-A \log |s|} \rho = \rho e^{-A \log |s|}, \quad s \neq 0. \]

Next, using the above identity and the fact that \( T_1^1 = \rho(T_2^2) \), from (59) we get that
\[ (85) \quad AB^+ \times B^+ - iAB^+ = \rho(AB^+ \times B^+ - iAB^+). \]

Denoting as before \( (B_1^+, B_2^+, B_3^+) \) the components of \( B^+ \) and taking into account that \( \rho = \left( \begin{array}{ccc} e^{i\delta} & 0 \\ 0 & 1 \end{array} \right) \), the first two components of the vectorial identity (59) can be rewritten as
\[ a (B_1^+ + i B_2^+) (B_3^+ + 1) (e^{i\delta} - 1) = 0, \]
and, because \( B_3^+ \neq \pm 1 \) and \( a \neq 0 \), we conclude that \( e^{i\delta} = 1, \delta \in [0, 2\pi] \), that is \( \delta = 0 \). Then, \( \rho \equiv 1_{3 \times 3} \), so that
\[ T_1^1(s) = T_2^2(s), \quad \forall \ s \geq s_0 \gg 1, \]
and therefore there exist \( \mathbf{a} \in \mathbb{R}^3 \) such that
\[ G^1(s) = G^2(s) + \mathbf{a}, \quad \forall \ s \geq s_0 \gg 1, \]
Finally, observe that from (57)

$$G_j(s) = se^{4\log|s|A^+} + o(|s|), \quad \text{as } s \to +\infty \quad \text{and} \quad j = 1, 2.$$  

Then, $a = (0, 0, 0)$ and we conclude that

$$G^1(s) = G^2(s), \quad s \geq s_0 \gg 1.$$ 

\[\square\]

3. Properties of the curvature and torsion associated to the curve $G$

In this section, we analyze the properties of the curvature and torsion functions related to the curve $G$. We will assume through this section that $c_{\pm\infty} \neq 0$.

We will now continue to see how the Hasimoto transform allows to reduce the study of these properties to analyzing those of the (self-similar) solutions of the cubic non-linear Schrödinger equation. Though the argument can be found in some classical references as [10] and [15], we have included it here for the sake of completeness.

Firstly, notice that if

$$X(s, t) = e^{\frac{4}{\sqrt{t}} \log t G(s/\sqrt{t})},$$

then the associated curvature and torsion have the self-similar form

$$c(s, t) = \frac{1}{\sqrt{t}} c(s/\sqrt{t}) \quad \text{and} \quad \tau(s, t) = \frac{1}{\sqrt{t}} \tau(s/\sqrt{t}),$$

with $c(s) = c(s, 1)$ and $\tau(s) = \tau(s, 1)$.

Indeed, from [39] we get that

$$T(s, t) = e^{\frac{4}{\sqrt{t}} \log t} T(s/\sqrt{t}), \quad n(s, t) = e^{\frac{4}{\sqrt{t}} \log t} n(s/\sqrt{t}) \quad \text{and}$$

$$b(s, t) = e^{\frac{4}{\sqrt{t}} \log t} b(s/\sqrt{t}),$$

where $\{T(s), n(s), b(s)\}$ is the Serret-Frenet frame of the curve $G(s) = X(s, 1)$ (i.e. $T(s) = T(s, 1)$, $n(s) = n(s, 1)$ and $b(s) = b(s, 1)$). Then,

$$T_s(s, t) = \frac{1}{\sqrt{t}} e^{\frac{4}{\sqrt{t}} \log t} T_s(s/\sqrt{t}), \quad b_s(s, t) = \frac{1}{\sqrt{t}} e^{\frac{4}{\sqrt{t}} \log t} b_s(s/\sqrt{t})$$
and, from the Serret-Frenet system (see (2)) we conclude that
\[ c(s, t) = n(s, t) \cdot T_s(s, t) = \frac{1}{\sqrt{t}} (n \cdot T_s)(s/\sqrt{t}) = \frac{1}{\sqrt{t}} c(s/\sqrt{t}) \quad \text{and} \]
\[ \tau(s, t) = -n(s, t) \cdot b_s(s, t) = -\frac{1}{\sqrt{t}} (n \cdot b_s)(s/\sqrt{t}) = \frac{1}{\sqrt{t}} \tau(s/\sqrt{t}). \]

Secondly, defining as in [10] (see also [15, p. p 195])
\[ \psi(s, t) = e^{i \int_0^s \tau(s', t) \, ds'} \]
\[ N(s, t) = (n + ib)(s, t) e^{i \int_0^s \tau(s', t) \, ds'}, \]
then \( \psi \) solves the nonlinear cubic Schrödinger equation
\[ i \psi_t + \psi_{ss} + \frac{\psi}{2} (|\psi|^2 + \alpha(t)) = 0, \]
with
\[ \alpha(t) = -i (N_t \cdot N)(0, t) - \frac{c_0^2}{t}, \]
or, equivalently,
\[ \alpha(t) = ((A b \cdot n)(0) - c_0^2)/t. \]

Indeed, from the identities in (87) and (88), and the definition of the complex vector \( N \) in (89), it follows that
\[ N(s, t) = (n + ib)(s, t) e^{i \int_0^s \tau(s', t) \, ds'} = e^{\frac{A}{2} \log t (n + ib)(s/\sqrt{t}) e^{i \int_0^{s/\sqrt{t}} \tau(s') \, ds'}}. \]

Now, by deriving this identity with respect to the time variable, we get that
\[ (N_t)(s, t) = \frac{1}{2t} e^{\frac{A}{2} \log t (A(n + ib)(\eta) - \eta(n + ib) \cdot \eta - i \eta(n + ib)(\eta) \tau(\eta)) e^{i \int_0^\eta \tau(s') \, ds'}} \]
\[ \bigg|_{\eta = s/\sqrt{t}}. \]

Then, evaluating these identities at \( s = 0 \),
\[ N(0, t) = e^{\frac{A}{2} \log t (n + ib)(0)}, \quad N_t(0, t) = \frac{1}{2t} e^{\frac{A}{2} \log t A(n + ib)(0)}, \]
and therefore,
\[ (N_t \cdot N)(0, t) = \frac{1}{2t} e^{\frac{A}{2} \log t A(n + ib)(0) \cdot e^{\frac{A}{2} \log t (n - ib)(0)}} \]
\[ = \frac{1}{2t} A(n + ib)(0) \cdot (n - ib)(0) = \frac{i}{t} A b \cdot n(0). \]

Here, we have used that \( A \) is an antisymmetric matrix, so that \( A v \cdot v = 0 \), and \( e^{\frac{A}{2} \log t} \) is a rotation in \( \mathbb{R}^3 \).
Next, notice that, because $c(s, t)$ and $\tau(s, t)$ satisfy (87), the function $\psi$ can be written as

$$
\psi(s, t) = \frac{1}{\sqrt{t}} \psi(s/\sqrt{t}),
$$

where

$$(92) \quad \psi(s) = c(s)e^{i \int_0^s \tau(s') \, ds'},$$

and, from the equation (90) and (91), it follows that $\psi(s)$ solves

$$(93) \quad \psi'' - \frac{i}{2} (\psi + s \psi') + \frac{\psi}{2} (|\psi|^2 + \alpha) = 0,$$

with $\alpha = \alpha(1)$, that is

$$(94) \quad \alpha = A b \cdot n(0) - c_0^2.$$

Finally, if we introduce the function $f$ through the definition

$$(95) \quad \psi(s) = f(s)e^{i s^2/4},$$

then $f$ satisfies

$$
f'' + \frac{i}{2} f' + \frac{f}{2} (|f|^2 + \alpha) = 0.
$$

On the one hand, recall that from the definitions of $\psi$ and $f$ in (92) and (95), it follows that

$$(96) \quad f(s)e^{i s^2/4} = c(s)e^{i \int_0^s \tau(s') \, ds'},$$

so that

$$(97) \quad |f|^2 = c^2, \quad |f'|^2 = c_0^2 + c^2 (\tau - s/2)^2 \quad \text{and} \quad \Im (\bar{f} f') = c^2 (\tau - s/2).$$

On the other hand, we have already seen that if $X(s, t)$ of the form (86) is a solution of LIA, then $T = G$ satisfies—see Lemma 11 and (92)—

$$
\mathcal{A} T = c(s - 2\tau) n + 2c_0 b \quad \text{and} \quad c^2 = -a T_3(s) - \alpha, \quad \text{3}
$$

from which the above identities rewrite as

$$(98) \quad |f|^2 = -a T_3(s) - \alpha, \quad |f'|^2 = \frac{1}{4} |\mathcal{A} T \times T'|^2 \quad \text{and} \quad \Im (\bar{f} f') = -\frac{1}{2} \mathcal{A} T \cdot T'.
$$

3Define $v = 2c_0 b(0)$, so that $|v|^2 = 4c_0^2$. From the definition of $\alpha$ in (91) and the evaluation of (92) at $s = 0$, it is easy to see that $\alpha = T(0) \cdot \left[ A \left( \frac{\lambda}{|v|} \right) \times \frac{\lambda}{|v|} \right] = -\frac{|\lambda|^2}{4} = -a T_3(0) - c_0^2 = -a T_3(0) - \frac{1}{4} (\mathcal{I} + A) G(0)|^2.$
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Finally, recall the definitions of the pair \((y, h)\) in (17) and (19):

\[
y = \frac{dT'}{ds}, \quad h = -\frac{1}{2} \mathbf{A} \cdot \mathbf{T}' \quad \text{and} \quad \frac{y}{2} + ih = \bar{f}f'.
\]

The identities in (97) reduce the problem of studying the properties of \(c(s)\) and \(\tau(s)\) to analyze the properties related to the solutions \(f\) of the latter equation. In fact, the following theorem is an immediate consequence of (97) and the results that will be proved in the next section for the solutions of the latter O.D.E. (see Theorem 8).

**Theorem 7.** Let \(X(s, t) = e^{\frac{4}{a} \log t} \sqrt{t} \mathbf{G}(s/\sqrt{t})\) and let \(c(s)\) and \(\tau(s)\) denote the curvature and torsion related to the curve \(\mathbf{G}(s) = X(s, 1)\). Then,

i) There exists \(4E(0) \geq 0\) such that the identity

\[
c^2_s + c^2(\tau - s/2)^2 + \frac{1}{4}(c^2 + \alpha)^2 = E(0)
\]

holds true for all \(s \in \mathbb{R}\).

ii) \(c(s), c_s(s)\) and \(\tau - s/2\) are bounded globally defined functions.

iii) The limits \(\lim_{s \to \pm \infty} c(s) = c_{\pm \infty}\) do exist and

\[
c^2(s) - c^2_{\pm \infty} = O(1/|s|), \quad s \to \pm \infty.
\]

iv) Moreover, the following asymptotics hold as \(s \to \pm \infty:\)

\[
\left(\frac{1}{2} \frac{dc^2}{ds} + ic^2(\tau - s/2)\right)(s) = \frac{b_{\pm}}{2} e^{-ia_{\pm}} e^{-i\phi(s)} - i \frac{\tilde{\gamma}_s}{s} + i \frac{b_{\pm}}{2} \left(3\tilde{\gamma}_s - \frac{\tilde{\gamma}_s^2}{2}\right) \frac{e^{-ia_{\pm}}}{s^2} e^{-i\phi(s)} - \frac{b_{\pm} \gamma_{\pm}}{2} \frac{e^{ia_{\pm}}}{s^2} e^{i\phi(s)} + \frac{b_{\pm} \gamma_{\pm}}{2} \frac{e^{-ia_{\pm}}}{s^2} e^{-i\phi(s)} + O\left(\frac{1}{|s|^3}\right).
\]

Here, \(\phi(s) = (s^2/4) - \gamma_{\pm} \log |s|\), \(a_{\pm}\) is a constant in the interval \([0, 2\pi)\),

\[
\gamma_{\pm} = -3c^2_{\pm \infty} - 2\alpha, \quad \tilde{\gamma}_s = 2E(0) - (3c^2_{\pm \infty} + \alpha) (c^2_{\pm \infty} + \alpha)/2,
\]

\[
b_{\pm}^2 = 4c^2_{\pm \infty}(E(0) - (c^2_{\pm \infty} + \alpha)^2/4) \quad \text{with} \quad b_{\pm} \geq 0,
\]

\[
\alpha = -aT_3(0) - \frac{1}{4} |(I + \mathbf{A}) \mathbf{G}(0)|^2 \quad \text{and} \quad E(0) = \frac{a^2}{4}.
\]

\(^4\)Notice that, from (97) and (98), it follows that \(E(0) = a^2/4\).
4. Self-similar solutions of the Cubic Non-linear Schrödinger equation

**Theorem 8.** Let \( f \) be a solution of the equation

\[
  f'' + i \frac{s}{2} f' + \frac{f}{2}(|f|^2 + \alpha) = 0, \quad \alpha \in \mathbb{R}.
\]

Then

i) There exists \( E(0) \geq 0 \) such that the identity

\[
  |f'|^2 + \frac{1}{4}(|f|^2 + \alpha)^2 = E(0)
\]

holds true for all \( s \in \mathbb{R} \).

ii) \( f(s), f'(s) \) and \( \mathcal{I}(\hat{f}'')(s) \) are bounded globally defined functions.

iii) The limits \( \lim_{s \to \pm \infty} |f|^2(s) = |f|^2_{\pm \infty} \) and \( \lim_{s \to \pm \infty} |f'|^2(s) = |f'|^2_{\pm \infty} \) do exist and

\[
  |f|^2 - |f|_{\pm \infty}^2 = O \left( \frac{1}{|s|} \right), \quad s \to \pm \infty.
\]

iv) The following asymptotics hold as \( s \to \pm \infty \):

\[
  |f|^2(s) = |f|_{\pm \infty}^2 - \frac{4}{s} \Re m(\hat{f}'')(s) - 2\gamma_{\pm}^2 + O \left( \frac{1}{|s|^3} \right),
\]

\[
  \hat{f}''(s) = \left( \frac{1}{2} \frac{d|f|^2}{ds} + i \Re m(\hat{f}'') \right)(s)
\]

\[
  = \frac{b_{\pm}}{2} e^{-ia_{\pm}} e^{-i\phi_1(s)} - i \frac{\gamma_{\pm}}{s} + \frac{b_{\pm}}{2} \left( 3\gamma_{\pm}^2 - \frac{\gamma_{\pm}^4}{2} \right) \frac{e^{-ia_{\pm}}}{s^2} e^{-i\phi_1(s)}
\]

\[
  - \frac{b_{\pm} \gamma_{\pm}}{2} \frac{e^{ia_{\pm}}}{s^2} e^{i\phi_1(s)} + \frac{b_{\pm} \gamma_{\pm}}{2} \frac{e^{-ia_{\pm}}}{s^2} e^{-i\phi_1(s)} + O \left( \frac{1}{|s|^3} \right).
\]

v) Moreover, if \( |f|_{\pm \infty} \neq 0 \) or \( |f|_{-\infty} \neq 0 \), then

\[
  f(s) = |f|_{\pm \infty} e^{ic_{\pm}} e^{i\phi_{2}(s)} + 2i |f'|_{\pm \infty} \frac{e^{id_{\pm}}}{s} e^{i\phi_{3}(s)} - |f|_{\pm \infty} (|f|^2_{\pm \infty} + \alpha) \frac{e^{ic_{\pm}}}{s^2} e^{i\phi_{2}(s)} + O \left( \frac{1}{|s|^3} \right),
\]

\[
  f'(s) = |f'|_{\pm \infty} e^{id_{\pm}} e^{i\phi_{3}(s)} + i |f|_{\pm \infty} (|f|^2_{\pm \infty} + \alpha) \frac{e^{id_{\pm}}}{s} e^{i\phi_{2}(s)}
\]

\[
  + |f'|_{\pm \infty} \left( |f|^2_{\pm \infty} + \alpha \right) \frac{e^{id_{\pm}}}{s^2} e^{i\phi_{3}(s)}
\]

\[
  - 2|f'|_{\pm \infty} (|f|^2_{\pm \infty} + \alpha) \frac{e^{id_{\pm}}}{s^2} e^{i(\phi_{2} + \phi_{1})(s)} + O \left( \frac{1}{|s|^3} \right).
\]
Here,

\[
\begin{align*}
\phi_1(s) &= \left(\frac{s^2}{4}\right) - \gamma_\pm \log |s|, \\
\phi_2(s) &= (|f|_{\pm \infty}^2 + \alpha) \log |s|, \\
\phi_3(s) &= -(s^2/4) - (2|f|_{\pm \infty}^2 + \alpha) \log |s|,
\end{align*}
\]

(i.e. \(\phi_3 = \phi_2 - \phi_1\))

\(|f|_{\pm \infty}, |f'|_{\pm \infty} \geq 0, a_\pm \) and \(c_\pm\) are arbitrary constants in \([0, 2\pi)\), \(d_\pm = c_\pm - a_\pm\),

\[
\begin{align*}
\gamma_\pm &= -3|f|_{\pm \infty}^2 - 2\alpha, \\
\tilde{\gamma}_\pm &= 2E(0) - (3|f|_{\pm \infty}^2 + \alpha) (|f|_{\pm \infty}^2 + \alpha)/2, \quad \text{and} \\
b_\pm &= \frac{4|f|_{\pm \infty}^2 (E(0) - (|f|_{\pm \infty}^2 + \alpha)^2/4)}{4|f|_{\pm \infty}^2 + 2|\Im m(\bar{f}f')|^2} \quad \text{with} \quad b_\pm \geq 0.
\end{align*}
\]

Remark 5. The maps \((f(0), f'(0)) \mapsto (|f|_{\pm \infty}, c_\pm, |f'|_{\pm \infty}, a_\pm)\) are continuous. This follows from the construction of the solution. In particular, from \([107], [109], [124]\), and from the convergence of the integrals \([128], [129]\).

Proof of Theorem 8. Firstly, by multiplying the equation \((101)\) by \(\bar{f}f'\) and taking the real part, it is easy to see that

\[
\frac{dE}{ds} = \frac{d}{ds} \left[|f'|^2 + \frac{1}{4}(|f|^2 + \alpha)^2\right] = 0,
\]

so that the following quantity is preserved for all \(s\)

\[
|f|^2 + \frac{1}{4}(|f|^2 + \alpha)^2 = E(0).
\]

As a consequence, we get that \(f\) and \(f'\) are globally well-defined,

\[
|f(s)|, |f'(s)| \leq C \quad \text{and} \quad |\Im m(\bar{f}f')(s)| \leq C, \quad \forall s \in \mathbb{R}.
\]

This concludes \(i)\) and \(ii)\). We will now continue with the proof of \(iii)\)-\(v)\) in the case \(s \to +\infty\).

The case \(s \to -\infty\) follows using the same arguments. To this end, we consider the functions \(h(s)\) and \(y(s)\) defined through the following identities:

\[
(104) \quad h = \Im m(\bar{f}f') \quad \text{and} \quad y = \frac{d|f|^2}{ds}.
\]

On the other hand, since \(\bar{f}\) solves \((101)\), we obtain

\[
h'(s) = \Im m(\bar{f}'f' + \bar{f}f'') = -s \frac{d|f|^2}{ds} = -\frac{s}{4}y(s),
\]

and

\[
y' = 2|f'|^2 + 2\Re e(f'' \bar{f}) = sh + 2|f'|^2 - |f|^2 (|f|^2 + \alpha) = sh + g(|f|^2),
\]

where \(g(|f|^2) = 2E(0) - (3|f|^3 + \alpha)(|f|^2 + \alpha)/2\), by using the conservation law \((103)\).
Therefore, we conclude that the pair \((y, h)\) satisfies the coupled system of equations:

\[
\begin{cases}
\frac{d|f|^2}{ds} = y \\
y' = sh + g(|f|^2); \\
g(|f|^2) = 2E(0) - (3|f|^2 + \alpha)(|f|^2 + \alpha)/2 \\
h' = -\frac{s}{4}y.
\end{cases}
\]

Also, since \(f\) and \(f'\) are bounded, \(h(s)\) and \(y(s)\) are bounded functions. Notice also that multiplication by \(\bar{f}\) in (105) yields

\[
f'' \bar{f} + \frac{i}{2} f' \bar{f} + \frac{|f|^2}{2} (|f|^2 + \alpha) = 0,
\]

from which we obtain

\[
\Im m(f'' \bar{f})(s) + \frac{s}{4} \frac{d|f|^2}{ds} = 0.
\]

Then,

\[
\int_0^s \Im m(f'' \bar{f})(s') ds' + \frac{1}{4} \int_0^s s' \frac{d|f|^2}{ds'} (s') ds' = 0,
\]

and integration by parts gives

\[
|f|^2 - \frac{1}{s} \int_0^s |f|^2 = -\frac{4}{s} (h(s) - h(0)),
\]

so that

\[
\left(\frac{1}{s} \int_0^s |f|^2\right)_s = +\frac{1}{s} \left(|f|^2 - \frac{1}{s} \int_0^s |f|^2\right) = -\frac{4}{s^2} (h(s) - h(0)), \quad s \neq 0.
\]

Since \(h\) is bounded, from the above identity we get that the limit \(\lim_{s \to +\infty} (1/s) \int_0^s |f|^2\) exists and, from (106), it follows that

\[
\lim_{s \to +\infty} \frac{1}{s} \int_0^s |f|^2 = |f|^2_{+\infty}.
\]

Notice that, from the conservation law (108) and the above identity, we also obtain the existence of the limit \(\lim_{s \to +\infty} |f'(s)| = |f'|_{+\infty}.

Now the integration of (107) from \(s > 0\) to \(+\infty\) yields

\[
|f|^2_{+\infty} - \frac{1}{s} \int_0^s |f|^2(s') ds' = 4 \frac{h(0)}{s} - 4 \int_s^{+\infty} \frac{h(s')}{(s')^2} ds', \quad s > 0.
\]

From (106) and (108), we get

\[
|f|^2(s) - |f|^2_{+\infty} = -\frac{4}{s} h + 4 \int_s^{+\infty} \frac{h(s')}{(s')^2} ds',
\]
and, by taking into account that $h$ is bounded, we conclude (iii), that is

$$|f|^2(s) - |f|_{+\infty}^2 = O\left(\frac{1}{|s|}\right), \quad s \to +\infty. \quad (110)$$

We now continue to prove the asymptotics in (iv).

Defining $\tilde{\gamma}_+^-$ to be the limiting value of $g(|f|^2(s))$ as $s \to +\infty$, that is

$$\tilde{\gamma}_+^- = 2E(0) - \frac{1}{2} (3|f|_{+\infty}^2 + \alpha) (|f|_{+\infty}^2 + \alpha) = 2|f|_{+\infty}^2 - |f|_{+\infty}^2 (|f|_{+\infty}^2 + \alpha), \quad (111)$$

(recall the conservation law (103)) and taking into account that $h = y'/s - g(|f|^2(s))$ (see (103)), it can be shown that

$$|f|^2(s) - |f|_{+\infty}^2 = -4\frac{h}{s} + 4 \int_{-\infty}^{+\infty} \frac{h(s')}{(s')^2} ds' = -4\frac{h}{s} + 4 \frac{y(s')}{(s')^3}\bigg|_{s=\infty}^{+\infty}$$

$$+ 12 \int_{-\infty}^{+\infty} \frac{y(s')}{(s')^5} ds' - 2\tilde{\gamma}_+^- s^2 - 4 \int_{-\infty}^{+\infty} g(|f|^2(s')) - \tilde{\gamma}_+^- ds', \quad s > 0. \quad (112)$$

Also, defining $\gamma_+$ to be the limiting value of $g'(|f|^2(s))$ as $s \to +\infty$, that is

$$\gamma_+ = g'(|f|_{+\infty}^2) = -3|f|_{+\infty}^2 - 2\alpha, \quad (113)$$

from the definition of $g(|f|^2)$ in (105) and (110), it follows that

$$g(|f|^2) - \tilde{\gamma}_+^- = \gamma_+ (|f|^2(s) - |f|_{+\infty}^2) - \frac{3}{2} (|f|^2(s) - |f|_{+\infty}^2)^2 = O\left(\frac{1}{|s|}\right), \quad \text{as } s \to +\infty. \quad (114)$$

Therefore, taking into account the above observations in (112), we conclude that

$$|f|^2(s) = |f|_{+\infty}^2 - 4\frac{h}{s} - 2\tilde{\gamma}_+^- s^2 + O\left(\frac{1}{|s|^3}\right), \quad \text{as } s \to +\infty. \quad (115)$$

This finishes the proof of the asymptotic development related to $|f|^2(s)$ in iv). In order to derive the behaviour of $y(s)$ and $h(s)$ for $s$ sufficiently large, we will consider in (105) two new variables $u$ and $v$ defined through $y(s)$ and $h(s)$ as follows

$$y(s) = u(s^2/4) \quad \text{and} \quad h(s) = v(s^2/4). \quad (115)$$

Then, defining $t = s^2/4$, i.e., $|s| = 2\sqrt{t}$, from (105) and (115), we obtain that $u$ and $v$ satisfy:

$$u'(t) = 2v(t) + \frac{g(|f|^2(2\sqrt{t}))}{\sqrt{t}}, \quad \text{and} \quad v' = \frac{1}{2} u(t). \quad (116)$$

Therefore,

$$u'' = -u + \frac{\gamma_+}{t} - \frac{\tilde{\gamma}_+^-}{2t^{3/2}} + \frac{g'(|f|^2(2\sqrt{t}))}{t} - \gamma_+ u - \frac{g(|f|^2(2\sqrt{t}))}{2t^{3/2}} - \tilde{\gamma}_+^- \quad \text{as } t \to +\infty. \quad (117)$$
Observe that, because \( f \) and \( f' \) are bounded, \(|f|^2\), \( y \) and \( h \) are bounded. Then, from (115) and (116), it follows that

(118) \[ |u(t)| \leq C \quad \text{and} \quad |u'(t)| \leq C, \quad \forall t > 1, \quad \text{and} \]

(119) \[ h(2\sqrt{t}) = v(t) = \frac{u'}{2} - \frac{\gamma_+}{2\sqrt{t}} - \frac{g(|f|^2(2\sqrt{t})) - \tilde{\gamma}_+}{2\sqrt{t}} = \frac{u'}{2} + O\left(\frac{1}{\sqrt{t}}\right), \quad t \to +\infty. \]

Next, notice that (117) rewrites as

(120) \[ \begin{pmatrix} u \\ u' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 + \frac{\gamma_+}{t} & 0 \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix} + \begin{pmatrix} 0 \\ F_2 \end{pmatrix}, \quad \text{with} \]

\[ F_2(t) = -\frac{\tilde{\gamma}_+}{2t^{3/2}} + \frac{(g')(|f|^2(2\sqrt{t})) - \gamma_+}{t} u - \frac{g(|f|^2(2\sqrt{t})) - \tilde{\gamma}_+}{2t^{3/2}}. \]

From the definition of \( g, \gamma_+ \) and \( \tilde{\gamma}_+ \) (see (105), (111) and (113), respectively), we get that

(121) \[ g(|f|^2(2\sqrt{t})) - \tilde{\gamma}_+ = \gamma_+ (|f|^2(2\sqrt{t}) - |f|^2_{+\infty}) - \frac{3}{2} (|f|^2(2\sqrt{t}) - |f|^2_{+\infty})^2, \quad \text{and} \]

where, using (114) and (115),

\[ |f|^2(2\sqrt{t}) - |f|^2_{+\infty} = -\frac{u'}{\sqrt{t}} + \frac{\gamma_+}{2t} + \frac{g(|f|^2(2\sqrt{t})) - \tilde{\gamma}_+}{t} + O\left(\frac{1}{t^{3/2}}\right). \]

Substituting the above identity into expression (121), after some straightforward calculations we find that

(122) \[ g(|f|^2(2\sqrt{t})) - \tilde{\gamma}_+ = -\gamma_+ \frac{u(t)}{\sqrt{t}} + \frac{u' + \gamma_+}{2t} - \frac{3}{2} \frac{(u')^2(t)}{t} + O\left(\frac{1}{t^{3/2}}\right), \quad t \to +\infty, \]

(123) \[ (g')(|f|^2(2\sqrt{t})) - \gamma_+ = \frac{3 u(t)}{\sqrt{t}} - \frac{3 \gamma_+}{2t} + O\left(\frac{1}{t^{3/2}}\right), \]

so that

(124) \[ \begin{pmatrix} u \\ u' \end{pmatrix} = P \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \quad \text{with} \quad P = \begin{pmatrix} 1 & 1 \\ i\lambda_+ & -i\lambda_+ \end{pmatrix}; \quad \lambda_+ = \sqrt{1 - \gamma_+/t}. \]
give that the new variables $w_1$ and $w_2$ satisfy
\[
\begin{pmatrix}
e^{-i f^+_1 \lambda^+} & 0 \\
0 & e^{-i f^+_1 \lambda^+}
\end{pmatrix}
\begin{pmatrix}
w_1 \\
w_2
\end{pmatrix}
(t) =
\begin{pmatrix}
e^{-i f^+_1 \lambda^+} & 0 \\
0 & e^{i f^+_1 \lambda^+}
\end{pmatrix}
\left\{-\frac{i}{2 \lambda^+} F_2(t) + \frac{\gamma^+}{4 t^2 (1 - \gamma^+/t)} (w_2 - w_1) \right\}.
\]

Notice that (124) and (125) yield
\[
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\]
\[
\text{that the new variables } w_1(t) \text{ and } w_2 \text{ in (124), it is easy to see that}
\]
\[
(125) \quad w_2 = \bar{w}_1.
\]

As a byproduct, in order to analyze the solution of the latter system of equations, we can reduce ourselves to study its first component, that is
\[
\left( e^{-i f^+_1 \lambda^+} w_1 \right)'(t) = e^{-i f^+_1 \lambda^+} \left\{-\frac{i}{2 \lambda^+} F_2(t) + \frac{\gamma^+}{4 t^2 (1 - \gamma^+/t)} (w_2 - w_1) \right\} = e^{-i f^+_1 \lambda^+} I(t).
\]

To this end, we integrate the previous identity from 1 to $t \gg 1$ to obtain
\[
(126) \quad e^{-i f^+_1 \lambda^+} w_1(t) = w_1(1) + \int_1^t e^{-i f^+_1 \lambda^+} I(t') dt'.
\]

Notice that (124) and (125) yield
\[
\left( \frac{w_1}{w_1} \right) = P^{-1} \left( \begin{array}{c} u \\ u' \end{array} \right) = \frac{1}{2} \left( \begin{array}{c} u - \frac{i}{\lambda^+} u' \\ u + \frac{i}{\lambda^+} u' \end{array} \right), \quad \text{so that } w_1 - w_1 = \frac{i}{\lambda^+} u'.
\]

Then, from (125) and the above observation, we get that
\[
(127) \quad I(t) = \frac{i}{2 \lambda^+} \left( \frac{\gamma^+}{2 t^{3/2}} - 3 \frac{u u'}{t^{3/2}} + \frac{3 \gamma^+}{2} \frac{u}{t^2} \right) + O \left( \frac{1}{t^{5/2}} \right).
\]

Here we have used the asymptotic behaviour of $(1 - \gamma^+/t)^{-1}$ when $t \gg 1$, and the fact that both $u$ and $u'$ are bounded for $t$ sufficiently large (see Remark in (118)).

Therefore, from (126) and (127), we obtain
\[
(128) \quad e^{-i f^+_1 \lambda^+} w_1(t) = z_+ - \frac{i}{2} \int_t^{+\infty} e^{-i f^+_{1'} \lambda^+} \left\{ \frac{\gamma^+}{2 (t')^{3/2}} - 3 \frac{u u'}{(t')^{3/2}} + \frac{3 \gamma^+}{2} \frac{u}{(t')^2} \right\} dt' + O \left( \frac{1}{t^{5/2}} \right),
\]

with $z_+ = \lim_{t \to +\infty} e^{-i f^+_1 \lambda^+} w_1(t)$, that is,
\[
(129) \quad z_+ = w_1(1) + \int_1^{+\infty} e^{-i f^+_1 \lambda^+} \left( -\frac{i}{2 \lambda^+} F_2(t') + \frac{\gamma^+}{4 (t')^2 (1 - \gamma^+/t') (w_1 - w_1)} dt' \right).
\]
In particular,

\[(130) \quad w_1(t) = z_+ e^{i f_1^t \lambda_+} (1 + O(1/\sqrt{t})) \quad \text{as} \quad t \to +\infty.\]

Now, recall that \(\lambda_+ = (1 - \gamma_+ / t)^{-1/2} = 1 + O(1/t)\), as \(t \to +\infty\). Also, from \((124)\) and the fact that \(w_2 = \bar{w}_1\), we obtain

\[\text{(131)} \quad \begin{pmatrix} u \\ u' \end{pmatrix} = P \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_1 + \bar{w}_1 \\ i \lambda_+(w_1 - \bar{w}_1) \end{pmatrix}.\]

Then, from \((130)\), it follows that

\[u = w_1 + \bar{w}_1 = z_+ e^{i f_1^t \lambda_+} + z_+ e^{-i f_1^t \lambda_+} + O(1/\sqrt{t}), \quad \text{and} \quad uu' = i \lambda_+(z_+^2 e^{2i f_1^t \lambda_+} - (\bar{w}_1)^2 e^{-2i f_1^t \lambda_+} + O(1/\sqrt{t})).\]

On the one hand, if we write \(e^{-i f_1^t \lambda_+} = (e^{-i f_1^t \lambda_+})_a / (-i \lambda_+\), an integration by parts argument shows that

\[\int_t^{+\infty} \frac{e^{-i f_1^t \lambda_+}}{\lambda_+} \frac{dt'}{(t')^{3/2}} = \int_t^{+\infty} \frac{e^{-i f_1^t \lambda_+}}{(t')^{3/2}} dt' + O\left(\frac{1}{t^{3/2}}\right) = O\left(\frac{1}{t^{3/2}}\right).\]

On the other hand, a similar integration by parts argument in the expression of \(uu'(t)\) yields

\[\int_t^{+\infty} \frac{e^{-i f_1^t \lambda_+}}{\lambda_+} \frac{uu'}{(t')^{3/2}} dt' = O\left(\frac{1}{t}\right), \quad \text{as} \quad t \to +\infty.\]

Hence, from \((128)\), we get that

\[w_1(t) = z_+ e^{i f_1^t \lambda_+} (1 + O(1/t)), \quad \text{as} \quad t \to +\infty.\]

Next, arguing with this new expression for \(w_1(t)\) instead of with \((130)\), the same ideas as before show that

\[\int_t^{+\infty} \frac{e^{-i f_1^t \lambda_+}}{\lambda_+} \frac{uu'}{(t')^{3/2}} dt' = O(t^{-3/2}) \quad \text{and} \quad \int_t^{+\infty} \frac{e^{-i f_1^t \lambda_+}}{\lambda_+} \frac{u}{(t')^{2}} dt' = \frac{z_+}{t} + O(t^{-2}).\]

By replacing these identities into \((128)\) and taking into account \((132)\), it follows that

\[e^{-i f_1^t \lambda_+}w_1(t) = z_+ \left(1 - \frac{3}{4} \frac{\gamma_+}{t} i\right) + O\left(\frac{1}{t^{3/2}}\right), \quad \text{as} \quad t \to +\infty,\]

so that, writing \(z_+ = b_+ e^{ia_+}\), for certain \(a_+ \in [0, 2\pi)\) and \(b_+ \geq 0\), we conclude

\[w_1(t) = b_+ e^{i f_1^t \lambda_+ + a_+} \left(1 - \frac{3}{4} \frac{\gamma_+}{t} i\right) + O\left(\frac{1}{t^{3/2}}\right), \quad \text{as} \quad t \to +\infty.\]

In the sequel, the constant \(a_+ \in [0, 2\pi)\) may change its value at each occurrence.
Coming back to (131), from (133) we obtain the following asymptotics for \( u(t) \) and \( u'(t) \), as \( t \to +\infty \):

\[
\begin{align*}
(134) \quad u(t) &= b_+ \cos \phi(t) + \frac{b_+}{4} \left( 3\gamma_+ - \frac{\gamma_+^2}{2} \right) \frac{\sin \phi(t)}{t} + O \left( \frac{1}{t^{3/2}} \right), \\
&
\end{align*}
\]

\[
\begin{align*}
(134) \quad u'(t) &= -b_+ \sin \phi(t) + \frac{b_+}{4} \left( 3\gamma_+ - \frac{\gamma_+^2}{2} \right) \frac{\cos \phi(t)}{t} + \frac{b_+}{2} \gamma_+ \frac{\sin \phi(t)}{t} + O \left( \frac{1}{t^{3/2}} \right), \\
&
\end{align*}
\]

with \( a_+ \in [0, 2\pi) \) and \( \phi(t) = a_+ + t - \frac{\gamma_+}{2} \log t, \ t > 0 \).

Finally, recall that \( y(s) = u(s^2/4) \) and \( h(s) = v(s^2/4) \) (see (113)). Also, from (119) and (132), it follows that

\[
(135) \quad v = \frac{u'}{2} - \frac{\gamma_+}{2\sqrt{t}} + \frac{\gamma_+ u^2}{2t^2} + O \left( \frac{1}{t^{3/2}} \right).
\]

Therefore, from the asymptotics related to \( u \) and \( u' \) and these two observations, we get that

\[
\begin{align*}
y(s) &= b_+ \cos \phi(s^2/4) + b_+ \left( 3\gamma_+ - \frac{\gamma_+^2}{2} \right) \frac{\sin \phi(s^2/4)}{s^2} + O \left( \frac{1}{s^3} \right), \\
h(s) &= -b_+ \sin \phi(s^2/4) - \frac{\gamma_+}{s} + \frac{b_+}{2} \left( 3\gamma_+ - \frac{\gamma_+^2}{2} \right) \frac{\cos \phi(s^2/4)}{s^2} \\
&\quad - \frac{\gamma_+ b_+}{2} \frac{\sin \phi(s^2/4)}{s^2} + O \left( \frac{1}{s^3} \right) \quad \text{as} \quad s \to +\infty.
\end{align*}
\]

The asymptotics of \( \tilde{f} \tilde{f}' \) in (iv) is now an immediate consequence of (130), and the fact that \( \tilde{f} \tilde{f}' = y/2 + ih \) (see (101)). Besides, in particular from (130), \( b_+ = \lim_{s \to +\infty} |y + 2ih|(s) = 2|f|_{+\infty}\tilde{f}'|_{+\infty} \), that is

\[
(137) \quad b_+ = 2 |f|_{+\infty}|\tilde{f}'|_{+\infty} \geq 0 \quad \text{and} \quad b_+^2 = 4 |f|_{+\infty}^2 (E(0) - (|f|_{+\infty}^2 + \alpha)^2/4),
\]

using the conservation law (103).

Finally, we give the proof of \( v \). We start the study of \( f(s) \) in the case when \( |f|_{+\infty} \neq 0 \). To this end, by using polar coordinates in the plane, we write \( f(s) \) as

\[
f(s) = \rho(s) e^{i\varphi(s)}.
\]

The asymptotics for \( f(s) \) will follow from the ones concerning with \( \rho(s) \) and \( \varphi(s) \). Notice that \( \rho^2 = |f|^2 \). Then, from (112) and (130), we obtain

\[
\begin{align*}
(138) \quad \rho(s) = |f|_{+\infty} \left( 1 + \frac{b_+}{|f|_{+\infty}^2} \frac{\sin \phi(s^2/4)}{s} + \frac{1}{|f|_{+\infty}^2} \left( \tilde{\gamma}_+ - \frac{b_+^2}{4|f|_{+\infty}^2} \right) \frac{1}{s^2} \\
&\quad + \frac{b_+^2}{4|f|_{+\infty}^2} \frac{\cos 2\phi(s^2/4)}{s^2} + O \left( \frac{1}{|s|^3} \right) \right).
\end{align*}
\]
Secondly, since $f(s) = \rho(s) e^{i\varphi(s)}$, $3m(\tilde{f}f') = \varphi'\rho^2$ and, from the asymptotics of $h(s) = 3m(\tilde{f}f')$ in $(130)$, we obtain

$$
\varphi'\rho^2 = -\frac{b_+}{2} \sin(\phi(s^2/4)) + \frac{b_+}{2} \left( 3\gamma_+ - \frac{\gamma_+^2}{2} \right) \frac{\cos(\phi(s^2/4))}{s^2} - \gamma_+ \frac{\sin(\phi(s^2/4))}{s^2} + O\left( \frac{1}{|s|^3} \right).
$$

If we now use $(138)$ in the previous identity and integrate the result from $1$ to $s \gg 1$, we obtain

$$
\varphi(s) = c_+ - \frac{1}{|f|_{+\infty}^2} \left( \frac{b_+^2}{2|f|_{+\infty}^2} - \tilde{\gamma}_+ \right) \log |s| + \frac{b_+}{|f|_{+\infty}^2} \frac{\cos(\phi(s^2/4))}{s} - \frac{b_+^2}{2|f|_{+\infty}^2} \frac{\sin(2\phi(s^2/4))}{s^2} + O\left( \frac{1}{|s|^3} \right),
$$

with (see $(139)$ and $(111)$)

$$
\frac{1}{|f|_{+\infty}^2} \left( \frac{b_+^2}{2|f|_{+\infty}^2} - \tilde{\gamma}_+ \right) = |f|_{+\infty}^2 + \alpha. \tag{140}
$$

Finally, recall that $f(s) = \rho(s) e^{i\varphi(s)}$. Then, after a few simplifications where we use the identities $\phi(s^2/4) = a_+ + \phi_1(s)$, $b_+ = 2|f|_{+\infty} |f'_{+\infty}|$ and $(140)$, from $(138)$ and $(139)$ we easily get that

$$
f(s) = |f|_{+\infty} e^{ic_+} e^{i\phi_2(s)} + 2i |f|_{+\infty} e^{id_+} e^{i\phi_3(s)} - |f|_{+\infty} (|f|_{+\infty}^2 + \alpha) \frac{e^{ic_+}}{s^2} e^{i\phi_2(s)} + O\left( \frac{1}{|s|^4} \right), \tag{141}
$$

with $\phi_2(s) = (|f|_{+\infty}^2 + \alpha) \log |s|$, $\phi_3(s) = (\phi_2 - \phi_1)(s) = -(s^2/4) - (2|f|_{+\infty}^2 + \alpha) \log |s|$ and $d_+ = c_+ - a_+$. This concludes the proof of the asymptotic behaviour of $f(s)$, when $|f|_{+\infty} \neq 0$.

We now consider the case $|f|_{+\infty} = 0$. Then, $b_+ = 0$ (see $(140)$), and from the results in the part iv) it follows that

$$
|f(s)|^2 = 2 \frac{\tilde{\gamma}_+}{s^2} + O\left( \frac{1}{s^3} \right) \quad s \to +\infty, \tag{142}
$$

with $\tilde{\gamma}_+ = 2|f'_{+\infty}|$.

Next, define $g = e^{-i\beta \log s} f$, where $\beta = |f|_{+\infty}^2 + \alpha = \alpha$. Since $f$ solves $(101)$, we get that $g$ solves

$$
\left[ e^{i(\frac{\tilde{\gamma}_+}{s^2} + 2\beta \log s)} \right]' = e^{i(\frac{\tilde{\gamma}_+}{s^2} + 2\beta \log s)} G(s), \tag{143}
$$
with
\[ G(s) = -\frac{g}{2}(|g|^2 - |g|^2_{+\infty}) + \frac{\beta(\beta + i)}{s^2} g = \frac{g}{s^2}(\beta(\beta + i) - \tilde{\gamma}_+) + O\left(\frac{1}{|s|^4}\right), \]
bearing in mind (142).

Simple computations give the asymptotic behaviour of \( f(s) \) in the case \( |f|_{+\infty} = 0 \).

In order to obtain the asymptotics of \( f'(s) \), we observe that
\[ |f|^2 f' = f(\bar{f}f') = f \left( \frac{1}{2} \frac{df}{ds} + i\Im(ff') \right), \]
so that
\[ f' = \frac{f}{|f|^2} \left( \frac{1}{2} \frac{df}{ds} + i\Im(ff') \right), \tag{144} \]
whenever \( |f|^2(s) \neq 0 \).

Recall that \( \rho(s) = |f(s)|, \phi(s^2/4) = a_+ + \phi_1(s), b_+ = 2|f| + |f'|_{+\infty} \) and \( \tilde{\gamma}_+ = 2|f|^2_{+\infty} - |f|^2_{+\infty}(|f|^2_{+\infty} + \alpha) \). Then, the behaviour related to \( f'(s) \) in the case \( |f|_{+\infty} \neq 0 \) easily follows from (144), the asymptotic of \( \bar{f}f' \) given in the part (iv), (138) and (141). The same argument is valid in the case \( |f|_{+\infty} = 0 \), by using (142) instead of (138). To this end, it is enough to observe that \( |f'|_{+\infty} \neq 0 \), if \( |f|_{+\infty} = 0 \), unless \( f = 0 \) (see Lemma 2 in the following pages), so that \( \tilde{\gamma}_+ = 2|f'|_{+\infty} \neq 0 \). This finishes the proof of the Theorem 8.

\[ \square \]

Remark 6. The parts (iii) and (iv) also hold true for the quantities
\[ y = \frac{d|T'|^2}{ds} \quad \text{and} \quad h = -\frac{1}{2} \mathcal{A} \mathbf{T} \cdot \mathbf{T}', \]
with \( \mathbf{T} = \mathbf{G}' \) and \( \mathbf{G} \) solving the equation (22), that is
\[ (\mathcal{I} + \mathcal{A}) \mathbf{G} - s \mathbf{G}' = 2 \mathbf{G}' \times \mathbf{G}'' \quad \text{and} \quad |\mathbf{G}'|^2 = 1. \tag{145} \]

In order to prove this remark, it is enough to observe that the identity (107) is also satified because it just involves \( |f|^2 = c^2 \) and \( h \). In fact, (107) is a consequence of Lemma 1 and the identity \( h = -a(G_3 - sT_3)/4 \), which easily follows from the equation (145).

Also, notice that the pair \((y, h)\) defined as above is a solution of the system (105) with \( E(0) = a^2/4 \). To this end, we firstly observe that the derivation of (145) gives that
\[ \mathcal{A} \mathbf{T} = s\mathbf{T}' - 2\mathbf{T} \times \mathbf{T}'' \quad \text{and} \quad |\mathbf{T}| = 1. \tag{146} \]
from which it follows that

$$y = \frac{d|T'|^2}{ds} = (\mathcal{A}T \times T) \cdot T'.$$

Then,

$$h' = -\frac{1}{2} \mathcal{A}T \cdot T'' = -\frac{1}{2} (sT' + 2T \times T') \cdot T'' = -\frac{s}{4} y.$$

Also,

$$y' - sh = (\mathcal{A}T \times T)' \cdot T' + (\mathcal{A}T \times T) \cdot T'' + \frac{s}{2} \mathcal{A}T \cdot T',$$

where, from Lemma 1, we get that

$$(\mathcal{A}T \times T)' = ((0, 0, -a) + aT_3 T)' \cdot T' = -(|T'|^2 + \alpha)|T'|^2$$

and

$$(\mathcal{A}T \times T) \cdot T'' + \frac{s}{2} \mathcal{A}T \cdot T' = -\mathcal{A}T \cdot (T'' \times T) + \frac{s}{2} \mathcal{A}T \cdot T'$$

$$= -\frac{1}{2} |\mathcal{A}T|^2 = \frac{a^2}{2} (1 - T_3^2) = \frac{1}{2} (a^2 - (|T'|^2 + \alpha)^2),$$

so that

$$y' - sh = \frac{a^2}{2} - \frac{1}{2} (3|T'|^2 + \alpha)(|T'|^2 + \alpha).$$

Finally, from Lemma 1 observe that $y$ and $h$ are bounded functions.

**Remark 7.** Using a fixed point argument in (128), we obtain: Given $z_+ \in \mathbb{C}$, $\gamma_+$ and $\tilde{\gamma}_+$, there exists $t_0$ sufficiently large and a unique $w_1 \in \mathcal{C}(t \geq t_0)$, with $\sup_{t \geq t_0} |w_1(t)| \leq 2|z_+|$, solving the integral equation (128). Moreover, $w_1(t)$ satisfies the following limiting condition

$$\lim_{t \to +\infty} e^{-i(t - \frac{\gamma_+}{2} \log |t|)} w_1(t) = z_+.$$

To this end, consider $X$ to be the following set

$$X = \{ w_1 \in \mathcal{C}(t \geq t_0) / ||w_1||_X = \sup_{t \geq t_0} |w_1(t)| \leq 2|z_+| \},$$

and define the operator $T w_1$ as the righthand side of (128)—see also (120)–(122). Then it is easy to prove that $T : X \to X$ is a contraction on $X$. 
To this end, firstly define

\[ f(s_0) = |f|_{+\infty} e^{i\theta_1}, \quad \text{and} \quad f'(s_0) = |f'|_{+\infty} e^{i\theta_2}. \]

(148)

A similar result can be obtained as \( s \to -\infty \).

Proof of Theorem 9: Existence. Given \( \theta_1, \theta_2 \in [0, 2\pi) \), \( |f'|_{+\infty} \) and \( |f|_{+\infty} \geq 0 \), define the compact set

\[ D^\alpha = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} / |z_2|^2 + \frac{1}{4}(|z_1|^2 + \alpha)^2 = |f'|_{+\infty}^2 + \frac{1}{4}(|f|^2_{+\infty} + \alpha)^2 \}. \]

We continue to prove that for fixed \( \alpha \), \( f(s_0) \in D^\alpha \) and \( f \) solution of (101) associated to this initial data such that

\[ e^{-i(|f(s_0)|^2 + \alpha) \log |s_0|} f(s_0) = |f|_{+\infty} e^{i\theta_1} \quad \text{and} \]

\[ e^{i(\frac{\alpha}{2} + (2|f(s_0)|^2 + \alpha) \log |s_0|)} f'(s_0) = |f'|_{+\infty} e^{i\theta_2}. \]

(149)

To this end, firstly define

\[ f(s_0) = |f|_{+\infty} e^{i\theta_1} e^{-i(|f'|_{+\infty}^2 + \alpha) \log |s_0|} \quad \text{and} \quad f'(s_0) = |f'|_{+\infty} e^{i\theta_2} e^{-i(\frac{\alpha}{2} + (2|f|^2_{+\infty}) \log |s_0|)}. \]

Notice that, from the above identities, it follows that \( |f(s_0)| = |f|_{+\infty} \) and \( |f'(s_0)| = |f'|_{+\infty} \).

Hence \( f(s_0), f'(s_0) \in D^\alpha \).

Secondly, the ODE's theory asserts us the existence of a local solution of (101), in particular for any \( D^\alpha \). Moreover, by taking into account the conservation law (103), we observe that such this solution is globally well-defined. As a by-product, we can consider \( (f(s_0), f'(s_0)) \), and from (103) we conclude that \( (f(s_0), f'(s_0)) \in D^\alpha \) (because \( (f(s_0), f'(s_0)) \in D^\alpha \)).

Next, for \( s \geq 1 \), define the map \( F(f) \) by

\[ F(s) \begin{pmatrix} f(0) \\ f'(0) \end{pmatrix} = \begin{pmatrix} e^{-i(|f(s)|^2 + \alpha) \log |s|} f(s) \\ e^{i(\frac{\alpha}{2} + (2|f(s)|^2 + \alpha) \log |s|)} f'(s) \end{pmatrix}, \quad \forall (f(0), f'(0)) \in \mathbb{C} \times \mathbb{C}, \]

where \( f \) is a solution of (101) associated to the initial data \( (f(0), f'(0)) \).
Let \( \{s_n : s_n \geq 1\} \) be a sequence such that \( s_n \to +\infty \), as \( n \to +\infty \). For any fixed \( n \in \mathbb{N} \), we have proved the existence of \((f_n(0), f'_n(0)) \in D^\alpha \) and \( f_n \) solution of (101) associated to this initial data such that (see (149))

\[
F(s_n) \begin{pmatrix} f_n(0) \\ f'_n(0) \end{pmatrix} = \begin{pmatrix} e^{-i(|f_n(s_n)|^2 + \alpha) \log |s_n| f_n(s_n)} \\ e^{i(\frac{2}{\pi} + (2|f_n(s_n)|^2 + \alpha) \log |s_n|) f'_n(s_n)} \end{pmatrix} = \begin{pmatrix} |f|_{+\infty} e^{i\theta_1} \\ |f'|_{+\infty} e^{i\theta_2} \end{pmatrix}.
\]

Notice that, because \( D^\alpha \) is a compact set, there exists a subsequence of \( \{(f_n(0), f'_n(0))\}_{n \in \mathbb{N}} \) such that

\[
\lim_{k \to +\infty} f_{n_k}(0) = f(0) \quad \text{and} \quad \lim_{k \to +\infty} f'_{n_k}(0) = f'(0), \quad \text{for some} \quad (f(0), f'(0)) \in D^\alpha.
\]

Now, define \( F(\infty) = \lim_{s \to +\infty} F(s) \), i.e.,

\[
F(\infty) \begin{pmatrix} f(0) \\ f'(0) \end{pmatrix} = \lim_{s \to +\infty} F(s) \begin{pmatrix} f(0) \\ f'(0) \end{pmatrix} = \lim_{s \to +\infty} \begin{pmatrix} e^{-i(|f(s)|^2 + \alpha) \log |s| f(s)} \\ e^{i(\frac{2}{\pi} + (2|f(s)|^2 + \alpha) \log |s|) f'(s)} \end{pmatrix}.
\]

Theorem 8, part iv), and v), asserts us the existence of the above limits, and therefore \( F(\infty) \) is a well-defined operator. Besides, \( F(\infty) \) is continuous (see remark [3]).

We now consider the initial data (151). Then, by taking into account the continuity property of \( F(\infty) \) and the identities in (150), we obtain

\[
F(\infty) \begin{pmatrix} f(0) \\ f'(0) \end{pmatrix} = \lim_{k \to +\infty} F(\infty) \begin{pmatrix} f_{n_k}(0) \\ f'_{n_k}(0) \end{pmatrix} = \lim_{k \to +\infty} \lim_{s \to +\infty} \begin{pmatrix} e^{-i(|f_{n_k}(s)|^2 + \alpha) \log |s| f_{n_k}(s)} \\ e^{i(\frac{2}{\pi} + (2|f_{n_k}(s)|^2 + \alpha) \log |s|) f'_{n_k}(s)} \end{pmatrix} = \begin{pmatrix} |f|_{+\infty} e^{i\theta_1} \\ |f'|_{+\infty} e^{i\theta_2} \end{pmatrix}.
\]

The limiting conditions in (148) are a consequence of the definition of \( F(\infty) \) and the above identity, by taking into account that \( |f(s)|^2 - |f|_{+\infty}^2 = o(1) \), as \( s \to +\infty \). This concludes the proof of the existence.

We will now continue to prove the uniqueness of such this solution. We will need the following lemma.

**Lemma 2.** Let \( f_j, j = 1, 2 \), two solutions of (101) such that

\[
|f_j|_{+\infty} = |f|_{+\infty}, \quad |f'_j|_{+\infty} = |f'|_{+\infty} \quad \text{and}
\]

(152)
\[
(153) \quad \lim_{s \to +\infty} e^{i \phi_{1,j}(s)}(\vec{f}_j f_j')(s) = |f'|_{+\infty} |f|_{+\infty} e^{ia_+},
\]
with \( \phi_{1,j}(s) = \frac{s^2}{4} - \gamma_{+j} \log |s| \) and \( \gamma_{+j} = -3|f_j|^2_{+\infty} - \alpha \), for some \( |f|_{+\infty}, |f'|_{+\infty} \geq 0 \), and \( a_+ \in [0,2\pi) \). Then,
\[
\frac{df_1^2}{ds}(s) = \frac{df_2^2}{ds}(s) \quad \text{and} \quad \Imm(\vec{f}_1 f_1')(s) = \Imm(\vec{f}_2 f_2')(s) \quad \forall s \geq s_0 \gg 1,
\]
that is \( \vec{f}_1 f_1' = \vec{f}_2 f_2' \), \( \forall s \geq s_0 \gg 1 \). A similar result can be obtained at \( s = -\infty \).

Proof. Assume \( f_j, j = 1,2 \), are two solutions of \( 101 \) satisfying \( 152 \) and \( 153 \), and consider \( h_j = \Imm(\vec{f}_j f_j'), y_j = d|f_j|^2/\text{ds}, \) and the associated \( w_{1,j}(t) \) defined through the change of variables \( 115 \) and \( 124 \).

By using \( 116, 117 \) together with \( 124 \) and \( 125 \), after some straightforward calculations one gets that
\[
\lim_{s \to +\infty} e^{i \phi_{1,j}(s)}(\vec{f}_j f_j')(s) = \lim_{s \to +\infty} e^{i \phi_{1,j}(s)} \left( \frac{y_j}{2} + ih_j \right)(s) =
\]
\[
\lim_{t \to +\infty} e^{i (\phi_{1,j}(2\sqrt{t}) + ih_j)}(t) = \lim_{t \to +\infty} e^{i(t - \frac{2\pi j}{\gamma_{+j}} \log |t|) e^{-i\gamma_{+j} \log 2 a_+}} w_{1,j}(t).
\]
Here, we have made use that \( \lambda_{+j} = \sqrt{1 - (\gamma_{+j}/t)} = 1 + o(1) \), as \( t \to +\infty \) in obtaining the last identity.

Therefore, from the previous identity and the limiting condition \( 153 \), we get that
\[
(154) \quad \lim_{t \to +\infty} e^{-i(t - \frac{2\pi j}{\gamma_{+j}} \log |t|) w_{1,j}(t) = b_+ e^{-i a_+ e^{-i\gamma_{+j} \log 2}}.}
\]

On the other hand, from \( 152 \) it follows that
\[
(155) \quad \tilde{\gamma}_{+} = \gamma_{+j}, \quad \gamma_{+} = \gamma_{+j} \quad \text{and} \quad \lambda_{+} = \lambda_{+j} = \sqrt{1 - (\gamma_{+}/t)}, \quad j = 1,2,
\]
for some \( \tilde{\gamma}_{+}, \gamma_{+} \) and \( \lambda_{+} \) depending only on \( |f|_{+\infty} \) and \( |f'|_{+\infty} \) (independent of \( j \)).

Next, notice that because the r.h.s. in \( 128 \) is \( O(1/\sqrt{t}) \), as \( t \to +\infty \) and \( 155 \), from \( 128 \) and \( 129 \), it is easy to check that
\[
\tilde{z}_{+j} = z_{+j}, \quad j = 1,2,
\]
for some \( z_{+j} \) depending on the fixed values \( |f|_{+\infty}, |f'|_{+\infty}, \theta_1 \) and \( \theta_2 \).

Previous analysis shows that \( w_{1,j}, j = 1,2 \), are two solutions of the integral equation in \( 128 \), for the above values of \( z_{+j}, \gamma_{+} \) and \( \tilde{\gamma}_{+} \), and in particular, \( |w_{1,j}(t)| < 2 |z_{+j}| \), when \( t \) is
large enough. Then, from Remark 7, it follows that \( w_{1,1}(t) = w_{1,2}(t), \forall t \geq t_0 \) and \( t_0 \) large enough. Therefore, after undoing the change of variables, we get that
\[
y_1(s) = y_2(s) \quad \text{and} \quad h_1(s) = h_2(s), \quad \forall s \text{ large enough.}
\]
\( \square \)

**Uniqueness.** Given \( |f|_{+\infty}, |f'|_{+\infty} \geq 0 \) and \( \theta_1, \theta_2 \in [0, 2\pi) \), assume \( f_j, j = 1, 2 \), are two solutions of (101) satisfying (148), that is
\[
\lim_{s \to +\infty} e^{-i\phi_{2,j}(s)} f_j(s) = |f|_{+\infty} e^{i\theta_1} \quad \text{and} \quad \lim_{s \to +\infty} e^{-i\phi_{3,j}(s)} f_j'(s) = |f'|_{+\infty} e^{i\theta_2}.
\]

**Case \( |f|_{+\infty} \neq 0 \):** Define \( h_j = \mathcal{I}(f_j f'_j) \) and \( y_j = d|f_j|^2/ds \). Then \( h_j \) and \( y_j \) (see (104)) are solutions of the system of ODE's in (105), and therefore the associated \( w_{1,j}, j = 1, 2 \), defined through the change of variables (115) and (124), is a solution of the integral equation (128), with \( z_{+,j} \in \mathbb{C}, \lambda_{+,j} = \sqrt{1 - \gamma_{+,j}/t} \) and \( \tilde{\gamma}_{+,j} \) and \( \gamma_{+,j} \) given by (111) and (113).

Now, from (156) and (157), we firstly observe that
\[
|f|_{+\infty} = |f_j|_{+\infty} \quad \text{and} \quad |f'|_{+\infty} = |f_j'|_{+\infty}, \quad j = 1, 2.
\]

Besides, from (156) and (157), we obtain
\[
\lim_{s \to +\infty} e^{i\phi_{1,j}(s)} (\tilde{f}_j f'_j)(s) = |f|_{+\infty} |f'|_{+\infty} e^{i(\theta_2 - \theta_1)},
\]
where recall that \( \phi_{1,j}(s) = (\phi_{2,j} - \phi_{3,j})(s) = (s^2/4) - \gamma_{+,j} \log |s| \).

Then, Lemma 2 yields that
\[
\frac{d|f_1|^2}{ds}(s) = \frac{d|f_2|^2}{ds}(s) \quad \text{and} \quad \mathcal{I}(f_1 f'_1)(s) = \mathcal{I}(f_2 f'_2)(s), \quad s \text{ large enough}.
\]

We will continue to prove that \( f_1(s) = f_2(s) \). To this end, we describe \( f_j(s), j = 1, 2 \) as
\[
f_j(s) = \rho_j(s) e^{i\varphi_j(s)}, \quad \rho_j(s) \geq 0 \quad \text{and} \quad \varphi_j(s) \in \mathbb{R},
\]
so that,
\[
|f_j| = \rho_j \quad \text{and} \quad h_j = \mathcal{I}(f_j f'_j) = \varphi'_j \rho_j, \quad j = 1, 2.
\]

From the first identity in (159), one gets that \( |f_1(s)| = |f_2(s)| \neq 0, \forall s \gg s_1 \) and \( s_1 \) large enough, because \( |f_1|_{+\infty} = |f_2|_{+\infty} = |f|_{+\infty} \neq 0 \) (see (158)). Then,
\[
\rho_1(s) = \rho_2(s) \neq 0, \quad \forall s \geq s_1.
\]
Now, recall that \( h_1 = h_2 \) and \( h_j = \varphi_j' \rho_j \), so that
\[
\varphi_1(s) = \varphi_2(s) + \theta_0, \quad \forall s \geq s_1 > 0,
\]
for some \( \theta_0 \in \mathbb{R} \), because \( \rho_1(s) = \rho_2(s) \neq 0, \forall s \geq s_1 \). As a consequence,
\[
f_1(s) = f_2(s) e^{i\theta_0}, \quad \forall s \geq s_1.
\]
Finally, by taking into account the latter identity in the limiting condition (156), it is easy to check that \( \theta_0 = 2k\pi \), for some \( k \in \mathbb{Z} \). Therefore, we conclude that
\[
f_1(s) = f_2(s), \quad \forall s \geq s_1 \quad \text{c.q.d.}
\]

**Case \(|f|_{+\infty} = 0\):** Define \( g_j = e^{-i\alpha \log s} f_j \), for \( j = 1 \) and \( 2 \), and \( \phi(s) = (s^2/4) + \alpha \log |s| \).

On the one hand, by using (156) and (157) we obtain
\[
|f_j|_{+\infty} = 0, \quad \lim_{s \to +\infty} g_j(s) = 0 \quad \text{and} \quad \lim_{s \to +\infty} e^{i\phi(s)} g_j(s) = |f_j|_{+\infty} e^{i\theta_2},
\]
so that, from (156), it is easy to see that \( g_j \) are solutions of the following integral equation:
\[
g(s) = z_2^+ - z_1^+ \int_{+\infty}^{+\infty} e^{-i\phi(\eta)} d\eta + \int_{+\infty}^{+\infty} e^{-i\phi(\eta)} \int_{\eta}^{+\infty} e^{i\phi(s')} ds' d\eta,
\]
where \( z_2^+ = 0, z_1^+ = |f_j|_{+\infty} e^{i\theta_2} \), and
\[
G(s) = -\frac{g}{2} (|g|^2(s) - |g|^2_{+\infty}) + \frac{\alpha(\alpha + i)}{s^2} g.
\]
Also, since \( f_j \) solves (101), and \( |f_j|_{+\infty} = 0, |g|^2 - |g|^2_{+\infty} = O(1/|s|^2) \), as \( s \to +\infty \) (see iv) in Theorem 3.

On the other hand, a fixed point argument proves that: there exists \( s_0 \) sufficiently large such that (161) has an unique solution in the space
\[
X = \{ g \in C([s_0, +\infty)), \lim_{s \to +\infty} g(s) = 0 \quad \text{and} \quad |g|^2 - |g|^2_{+\infty} = O(1/|s|^2) \}.
\]
As a by-product, we obtain \( g_1(s) = g_2(s), \forall s \geq s_0 \). Thus \( f_1(s) = f_2(s), s \geq s_0 \). □

**Remark 8.** If \( f \) solves (101) and \( |f|_{+\infty} = |f|_{-\infty} = 0 \), then \( f(s) = 0 \).

Indeed, let \( f \) be a solution of (101) such that \( |f|_{\pm\infty} = 0 \). Then, the asymptotics in (iv) of Theorem 3 imply that
\[
|f|^2(s) = \frac{2}{s^2} \tilde{\gamma} + O\left( \frac{1}{|s|^2} \right), \quad \text{with} \quad \tilde{\gamma} = 2|f|^2_{\pm\infty}, \quad s \to \pm\infty.
\]
Now define
\[
\psi(s, t) = \frac{1}{\sqrt{t}} f(s/\sqrt{t}) e^{i t \left( \frac{s^2}{4t} - \frac{\alpha^2}{2} \log t \right)} \quad t > 0.
\]
It is easy to prove that \(\psi(s, \cdot)\) is a solution of the cubic non-linear Schrödinger equation, that is,
\[
i\psi_t + \psi_{ss} + \frac{\psi}{2} |\psi|^2 = 0, \quad (NLS)
\]
and, from (162), \(\psi(s, \cdot) \in L^2(\mathbb{R})\).

On the other hand, the conservation of the \(L^2\)-norm of the solutions of (NLS) yields
\[
+\infty > \lim_{t \downarrow 0} \|\psi(\cdot, t)\|_{L^2(\mathbb{R})}^2 = \lim_{t \downarrow 0} \frac{1}{t} \int_{\mathbb{R}} \left| f(s/\sqrt{t}) e^{i t \left( \frac{s^2}{4t} - \frac{\alpha^2}{2} \log t \right)} \right|^2 ds
\]
with \(\int_{|\eta| \geq 1} |f(\eta)|^2 d\eta < +\infty\), because of (162). As a consequence, we get that \(|f(\eta)| = 0\) a.e. \(|\eta| \geq 1\), and using (162) we conclude that \(\tilde{\gamma}_\pm = 2|f'|_{\pm\infty} = 0\).

The result in the remark now follows from Lemma 2 and the fact that \(|f|_{\pm\infty} = |f'|_{\pm\infty} = 0\) (notice also that \(f = 0\) is a solution of (101)).

5. Further results

5.1. Symmetric solutions.

Here, we consider two particular cases of solutions of the equation
\[
G'' = \frac{1}{2} (A + I) G \times G'.
\]
These solutions come from the symmetry properties of the above equation. For these special cases, we will able to obtain more specific properties that the ones obtained in Section 2.

We study what we will refer as to “Odd Case” and “Mixed Case”.

I. Odd Case: Let \(G_\delta\) be a solution of (163) with the initial conditions
\[
G_\delta(0) = (0, 0, 0) \quad \text{and} \quad G'_\delta(0) = (0, \sqrt{1 - \delta^2}, \delta), \quad \delta \in [-1, 1].
\]
Then,
\[
G_\delta(s) = -G_\delta(-s).
\]
Indeed, notice that if \(G_\delta\) satisfies (163) and (164), then \(\tilde{G}(s) = -G_\delta(-s)\) is also a solution of (163),
\[
\tilde{G}(0) = -G_\delta(0) = (0, 0, 0) \quad \text{and} \quad \tilde{G}'(0) = G'_\delta(0).
\]
Therefore, from the uniqueness assumption, we conclude that $G_\delta(s) = -G_\delta(-s)$.

For a fixed $a \in \mathbb{R}$, consider the map $T^+$ defined by

$$(G_\delta(0) = 0, G_\delta'(0) = (0, \sqrt{1 - \delta^2}, \delta)) \rightarrow A^+_3(\delta),$$

with $A^+(\delta) = (A^+_j(\delta))_{j=1}^3$ the vector which prescribes the asymptotic behaviour of the solution $G_\delta$ at $s = \pm \infty$.

We have already proved the continuity of $T^+$ (see Proposition 2). On the other hand, observe that $G_1(s) = (0, 0, s)$ and $G_{-1}(s) = (0, 0, -s)$, so that $A^+_3(1) = 1$ and $A^+_3(-1) = -1$, respectively. As a consequence, for any $z^+_3 \in (-1, 1)$, there exists $\delta \in (-1, 1)$ such that $A^+_3(\delta) = z^+_3$.

In particular, if $z^+_3 = 0$, there exists $\delta_0 \in (-1, 1)$ such that $A^+_3(\delta_0) = 0$. Thus, we conclude the existence of $G_{\delta_0}$ solution of (163) and (164) such that is asymptotically a plane spiral, that is

$$G_{\delta_0}(s) \approx e^{A^+ \log |s|} A^+(\delta_0), \text{ with } A^+_3(\delta_0) = 0, \text{ as } s \to \pm \infty$$

(recall that $G_{\delta_0}(s) = -G_{\delta_0}(-s)$, so that $A^+(\delta_0) = A^-(\delta_0)$, and the asymptotics in Proposition 3).

Notice that, due to the invariance of (163) under rotations with respect to the OZ-axe, from previous remark it follows that we can “generate” all the solutions of (163) which asymptotically are plane spirals.

In Figure 2, Figure 3 and Figure 4, we display the graphics of different solutions of (163) associated to an initial data of the form (164). The right-handside pictures represent the solution near the point $s = 0$.

**Figure 2.** $G(0) = 0, G'(0) = (0, \sqrt{1 - \delta^2}, \delta), a = 10$ and $\delta = 0.956$. 

In Figure 2, Figure 3 and Figure 4, we display the graphics of different solutions of (163) associated to an initial data of the form (164). The right-handside pictures represent the solution near the point $s = 0$. 
II. Mixed Case: Now assume that $G(s)$ is a solution of (163) and define

$$\tilde{G}(s) = (G_1(-s), G_2(-s), -G_3(-s))$$
Then, a straightforward calculation gives that \( \tilde{G}(s) \) is also a solution of (163). Moreover, in the particular case when \( G(s) \) also satisfies the initial conditions
\[
G(0) = \left( \frac{-2c_0}{\sqrt{1 + a^2}}, 0, 0 \right) \quad \text{and} \quad G'(0) = (0, 0, 1), \quad (\text{resp. } G'(0) = (0, 0, -1)),
\]
with \( c_0 > 0 \), from the uniqueness assumption and the invariance of the equation under the transformation, it follows that \( G(s) \) satisfies
\[
\begin{align*}
G_1(s) &= G_1(-s) \\
G_2(s) &= G_2(-s) \\
G_3(s) &= -G_3(-s).
\end{align*}
\]
In particular, from (167) and the definitions of \( A^\pm \) in (33), we get that
\[
A^+_1 = -A^-_1, \quad A^+_2 = -A^-_2, \quad \text{and} \quad A^+_3 = A^-_3.
\]
Also, from Lemma 4 and the initial conditions in (166), it follows that
\[
c^2(s) = -aT_3(s) - \alpha \quad \text{with} \quad \alpha = -a - c_0^2 \quad (\text{resp. } \alpha = a - c_0^2).
\]
In [8], we studied the case when \( a = 0 \). In the general frame when \( a \) is not necessarily zero, only some of the results that were established there can be recovered. We continue to see some remarks on this respect.

**Remark 9.** If \( a = 0 \) (i.e., \( A = 0 \)), we proved in [8] that given any \( \theta \in (0, \pi] \), there exists \( G \) solution of (163) such that the angle between \( B^+ = A^+ \) and \( -A^- = B^- \) is precisely \( \theta \). It would be interesting to prove that the same result holds for any \( A \in \mathcal{M}_{3 \times 3} \).

**Remark 10.** (Self-intersections)

(i) Given \( a \in \mathbb{R} \), if \( |c_0| > c_2 \), with \( c_2 \) large enough, then there exists \( 0 < s \ll 1 \) such that
\[
G(s) = G(-s) \quad (G \text{ has self-intersections}).
\]

In fact, arguing as in the proof of (ii) in [8] Proposition 4], we prove that \( G(s) = G(-s) \) if and only if \( G_3(s) = 0 \), with \( G_3(s) \) the solution of the following initial value problem:
\[
\begin{align*}
G''''_3 + \left( c^2 + \frac{s^2}{4} \right) G'_3 - \frac{s}{4} G_3 &= -\frac{a}{2}(1 - (G'_3)^2) \\
G_3(0) &= 0 \quad G'_3(0) = \pm 1 \quad G''_3(0) = 0.
\end{align*}
\]
By bearing in mind that \( c^2 = -aT_3(s) - \alpha \) (see (168)), similar arguments to those given in \( \text{[8]} \) proves that, for values of the parameter \( c_0 \) large enough, the solution \( G_3(s) \) of (169) vanishes for some \( 0 < s \ll 1 \).

(ii) If \( a \neq 0 \), then Remark \( \text{[4]} \) asserts that the solution \( G(s) \) does not self-intersect whenever \( \alpha > 0 \), that is whenever \( c_0^2 \leq -a \) (resp. \( c_0^2 \leq a \))-see (168). Also recall that in \( \text{[8, Proposition 4]} \) we proved that, if \( a = 0 \) and \( c_0 \) is sufficiently small, then \( G(s) \) has no self-intersections.

Two different examples of solutions of (163) associated to a data of the type (166) are plotted in Figures 5, 6 and 7. Figure 5 represents a plane spiral at infinity. Notice that in any of the latter figures, we can clearly see the wavelike behaviour of the solutions. As
before, in both figures the r.h.s. pictures are a zoom at the origin of the solution plotted in the l.h.s. picture. All figures in this paper have been generated using Mathematica 4.2.

5.2. On an ill-posedness result for the initial value problem associated to cubic Schrödinger equations.

Here, we address the question of existence and uniqueness for the initial value problem (IVP) related to cubic Schrödinger equations with the principal value distribution as initial data:

\[
\begin{align*}
    i\psi_t + \psi_{ss} + \frac{\psi}{2}(|\psi|^2 + \frac{\alpha}{t}) &= 0, \quad t > 0, \quad s \in \mathbb{R}, \quad \alpha \in \mathbb{R} \\
    \psi(s, 0) &= c_1 \text{p.v.}(1/s), \quad \text{with} \quad c_1 \in \mathbb{C} \setminus \{0\}.
\end{align*}
\]

More precisely, the following ill-posedness result is true:

**Proposition 3.** Let \( \alpha \geq 0 \) and consider the IVP defined in (170). Then, either there is no weak solution \( \psi \) for the IVP (170) in the class

\[
t^{\frac{1}{2}}\psi, \quad t^{\frac{3}{2}}|\psi|^2\psi \in L^\infty([0, +\infty) : \mathcal{S}'(\mathbb{R})) \quad \text{with} \quad \lim_{t \downarrow 0} \psi(s, 0) = c_1 \text{p.v.}(1/s), \quad \text{in} \quad \mathcal{S}'(\mathbb{R}), \quad c_1 \in \mathbb{C} \setminus \{0\},
\]

or there is more than one.
Proof of Proposition 3. The proof follows the argument in the proof of Theorem 1.5 in [14]. Briefly, assuming the existence of a unique solution of the problem (170), then it has to be a self-similar solution of the form

\[ \psi(s, t) = \frac{1}{\sqrt{t}} e^{\frac{i s^2}{2t}} f(s/\sqrt{t}), \]

with \( f \) an odd solution of

\[ f'' + i \frac{s}{2} f' + \frac{f}{2} (|f|^2 + \alpha) = 0, \quad \alpha \in \mathbb{R}, \]

On the other hand, since \( \alpha \geq 0 \), we get that \( 2|f|_{+\infty} + \alpha \neq 0 \), and therefore, from the asymptotics in (ii) in this section, it follows that either there is no limit for the solution \( \psi(s, t) \), as \( t \to 0 \), or the limit is identically zero, which contradicts the fact that \( \psi(s, t) \) solves (170).

Of special interest in Proposition 3 is the case \( \alpha = 0 \). If this is the case, the equation in (170) reduces to the commonly referred in the literature as to be the cubic non-linear Schrödinger equation:

\[ i\psi_t + \psi_{ss} + \frac{\psi}{2} |\psi|^2 = 0. \quad (NLS) \]

In this setting, it is important to be mentioned that in [14] Kenig, Ponce and Vega proved an analogous ill-posedness result to the one in Proposition 3 \( \alpha = 0 \) for the delta distribution as initial datum. They left opened the same question for the principal value.

5.3. On the ill-posedness of the Localized Induction Approximation.

We consider the following IVP:

\[
\begin{cases}
X_t = X_s \times X_{ss}, & s \in \mathbb{R}, \quad t > 0, \\
X(s, 0) = X_0(s).
\end{cases}
\]

(171)

Take \( \tilde{X}(s, t) = \sqrt{t} \tilde{G}(s/\sqrt{t}) \) with \( \tilde{G} \) solving

\[ \frac{1}{2} \tilde{G} - \frac{s}{2} \tilde{G}' = \tilde{e} \tilde{b} \]

and the initial conditions

\[ \tilde{T}(0) = \mathbf{e}_1, \quad \tilde{n}(0) = \mathbf{e}_2, \quad \tilde{b}(0) = \mathbf{e}_3, \]

(173)

where \( \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \) is the canonical basis in \( \mathbb{R}^3 \).
Observe that evaluating (172) at \( s = 0 \), we obtain

\[
\tilde{G}(0) = 2\tilde{c}_0 \tilde{b}(0) = (0, 0, 2\tilde{c}_0),
\]

with \( \tilde{c}_0 = \tilde{c}(0) \) and \( \tilde{c}(s) \) the curvature associated to the curve \( \tilde{G}(s) \).

In [8], it was proved that \( \tilde{X} \) solves (171) with initial data

\[
\tilde{X}_0(s) = s \left( A^+ \chi_{[0, +\infty)}(s) + A^- \chi_{(-\infty, 0]}(s) \right),
\]

for some \( A^+ \) and \( A^- \) unitary vectors in \( \mathbb{R}^3 \), depending on the parameter \( \tilde{c}_0 \).

Now define \( G(s) \) as

\[
G(s) = \tilde{G}(s) \chi_{[0, +\infty)}(s) + \rho \tilde{G}(s) \chi_{(-\infty, 0]}(s)
\]

with \( \rho = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \).

We claim that:

\[
X(s, t) = \sqrt{t} G(s/\sqrt{t})
\]

is also a solution of (171) with initial data

\[
X_0(s) = \tilde{X}_0(s) \chi_{[0, +\infty)}(s) + \rho \tilde{X}_0(s) \chi_{(-\infty, 0]}(s).
\]

We prove the claim by direct computation.

Firstly, notice that \( X(s, t) \) is a solution of LIA if and only if \( G \) satisfies

\[
\frac{1}{2} G(s/\sqrt{t}) - \frac{s}{2\sqrt{t}} G'(s/\sqrt{t}) = (G' \times G'')(s/\sqrt{t}), \quad t > 0.
\]

Now, from (174), we obtain

\[
G'(s) = \tilde{G}'(s) \chi_{(0, +\infty)}(s) + \rho \tilde{G}'(s) \chi_{(-\infty, 0]}(s) + [\tilde{G}(0) - \rho \tilde{G}(0)] \delta_0,
\]

with \( \tilde{G}(0) = \rho \tilde{G}(0) \), because of (174). Therefore,

\[
G'(s) = \tilde{G}'(s) \chi_{(0, +\infty)}(s) + \rho \tilde{G}'(s) \chi_{(-\infty, 0]}(s).
\]

Next, differentiating the above identity we obtain

\[
G''(s) = \tilde{G}''(s) \chi_{(0, +\infty)}(s) + \rho \tilde{G}''(s) \chi_{(-\infty, 0]}(s) + [\tilde{G}'(0) - \rho \tilde{G}'(0)] \delta_0,
\]

and from (174)

\[
\tilde{G}'(0) = e_1 = -\rho \tilde{G}'(0),
\]
so that
\begin{equation}
G''(s) = G''(s) \chi_{(0, +\infty)}(s) + \rho G''(s) \chi_{(-\infty, 0)}(s) + 2\mathbf{e}_1 \delta_0.
\end{equation}

We firstly compute \((G' \times G'')(s)\). From (180) and (181), we obtain
\begin{equation}
(G' \times G'')(s) = (\tilde{G}' \times \tilde{G}'')(s) \chi_{(0, +\infty)}(s) + \rho (\tilde{G}' \times \tilde{G}'')(s) \chi_{(-\infty, 0)}(s) + 2G'(s) \times \mathbf{e}_1 \delta_0.
\end{equation}

Notice that \(G'(s) \times \mathbf{e}_1\) can be extended to a continuous function up to \(s = 0\) because, from (180) and (173),
\[
\lim_{s \to 0^\pm} G'(s) \times \mathbf{e}_1 = 0.
\]

As a by-product
\[
(G' \times G'')(s) = (\tilde{G}' \times \tilde{G}'')(s) \chi_{(0, +\infty)}(s) + \rho (\tilde{G}' \times \tilde{G}'')(s) \chi_{(-\infty, 0)}(s).
\]

Moreover, from the initial conditions in (173), it is easy to check that \(G' \times G''\) can be extended continuously at \(s = 0\) and
\[
\lim_{s \to 0^\pm} (G' \times G'')(s) = \tilde{c}_0 \tilde{b}(0).
\]

In order to compute \((G' - sG)/2\) we use (176), (180) and (172). Then, we obtain the following
\[
\frac{1}{2} G(s) - \frac{s}{2} G'(s) = \left( \frac{1}{2} \tilde{G}(s) - \frac{s}{2} \tilde{G}'(s) \right) \chi_{(0, +\infty)}(s) + \rho \left( \frac{1}{2} \tilde{G}(s) - \frac{s}{2} \tilde{G}'(s) \right) \chi_{(-\infty, 0)}(s)
\]
\[
= (\tilde{c} \tilde{b})(s) \chi_{(0, +\infty)}(s) + \rho (\tilde{c} \tilde{b})(s) \chi_{(-\infty, 0)}(s),
\]
and
\[
\lim_{s \to 0^\pm} \left( \frac{1}{2} G(s) - \frac{s}{2} G'(s) \right) = \tilde{c}_0 \tilde{b}(0).
\]

From previous identities we conclude (179), and therefore \(X(s, t) = \sqrt{t} G(s/\sqrt{t})\) satisfies
\begin{equation}
X_t = X_s \times X_{ss}, \quad s \in \mathbb{R} \quad t > 0.
\end{equation}

Notice that \(X(s, t) = \sqrt{t} G(s/\sqrt{t})\) is continuous for all \(s\) and \(t > 0\) (see (176) and (174)), and \(X_s(s, t) = G'(s/\sqrt{t})\) is a real analytic function except at \(s = 0\), where it has a jump singularity. In fact, using (180) and (173) it is easy to check that
\[
\lim_{s \to 0^+} G'(s/\sqrt{t}) = \mathbf{e}_1, \quad \text{and} \quad \lim_{s \to 0^-} G'(s/\sqrt{t}) = -\mathbf{e}_1.
\]
Then, here (182) is understood for 
\[ X(s,t) \in C \left( (0, +\infty) : \text{Lip}^1(\mathbb{R}) \right) \]
and 
\[ (X_t)(s,\cdot) \in C(\mathbb{R}), \quad \text{for all} \quad t > 0. \]
Finally, using the same arguments as in the proof of in [8, Proposition 1], we observe that 
\[ |X(s,t) - X_0(s)| \leq 2\tilde{c}_0 \sqrt{t}, \quad \forall s \in \mathbb{R}. \]
This concludes the proof of the claim.

The claim asserts the existence of a (singular) solution of LIA associated to a given initial data \( X_0(s) \) as in (178). Such this solution is different from the (regular) solution obtained in [8] for the same data. The following proposition gathers the above obtained ill-posedness result:

**Proposition 4.** Given \((A^+, A^-) \in \mathbb{S}^1 \times \mathbb{S}^1\) with \(A^+ - A^- \neq 0\), consider the IVP:

\[
\begin{align*}
X_t &= X_s \times X_{ss}, \quad s \in \mathbb{R}, \quad t > 0, \\
X(s,0) &= s \left(A^+\chi_{[0, +\infty)}(s) + A^-\chi_{(-\infty, 0]}(s) \right).
\end{align*}
\]

Then, there exist more than one solution for the IVP (183) in the class 
\[ X(s,t) \in C \left( (0, +\infty) : \text{Lip}^1(\mathbb{R}) \right), \]
\[ (X_t)(s,\cdot) \in C(\mathbb{R}) \quad \text{for all} \quad t > 0. \]

Figure 8 illustrates the vortex line evolution for same value of \(c_0\) (i.e, \(c_0 = 0.8\)). Precisely, the subsequent plots correspond to vortex position at time \(t = 10^{-4}, 0.1, 1, 1.5, 2\) and 2.5 (c.f. [8, Figure 1]).

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