Temperature dependent effective mass renormalization in a Coulomb Fermi liquid

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We calculate numerically the quasiparticle effective mass ($m^*$) renormalization as a function of temperature and electron density in two- and three-dimensional electron systems with long-range Coulomb interaction. In two dimensions, the leading temperature correction is linear and positive with the slope being a universal density independent number in the high density limit. We predict an enhancement of the effective mass at low temperatures and a non-monotonic temperature dependence at higher temperatures ($T/T_F \sim 0.1$) with the peak shifting toward higher temperatures as density decreases. In three dimensions, we find that the effective mass temperature dependence is non-linear and non-universal, and depends on the electron density in a complicated way. At very high densities, the leading correction is positive, while at lower densities it changes sign and the effective mass decreases monotonically from its zero temperature value with increasing temperature.

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I. INTRODUCTION

In the Fermi liquid theory the interacting electron system is composed of weakly interacting quasiparticles at low energies with long quasiparticle lifetimes. The effective mass of a quasiparticle, which can be viewed as the bare mass of a free electron being renormalized by electron-electron interactions, is an important and fundamental Fermi liquid parameter. For decades theorists have been exploring the effective mass renormalization in two- and three-dimensional interacting electron systems (2DES and 3DES). In spite of this great deal of theoretical activity concentrating almost entirely on the density dependence of the effective mass renormalization, the temperature dependence of the effective mass has not been studied until very recently. Besides the considerable difficulties involved in the finite temperature numerical and analytical many-body calculations in 2DES and 3DES, the reason that this issue has not been addressed before can also be explained by the fact that the Fermi energy in 3D metals is typically $10^4$K, and therefore any finite temperature effects are negligible. In the past decade, however, low density 2DES have been attracting attention, and several experiments have been performed to measure the 2D effective mass $m^*$. The temperature dependence of the quasiparticle effective mass in 2DES is of considerable experimental interest since the Fermi energy in realistic 2DES may be $1K$ or lower, which makes the issue of the temperature dependence of 2D Fermi liquid parameters extremely important. In addition, the temperature dependence of the Fermi liquid parameters such as the effective mass is obviously of considerable fundamental theoretical significance.

The $T = 0$ quasiparticle effective mass renormalization in an electron system interacting through the long-range Coulomb interaction is one of the oldest many-body problems in theoretical condensed matter physics, and a number of theoretical calculations of 3D and 2D electron effective mass have been carried out in the literature. In fact, the Coulomb interaction induced electron effective mass renormalization at $T = 0$ is standard text-book material in electronic many-body theory. Essentially all of these calculations, both analytical and numerical (and both 2D and 3D), are based on the leading-order dynamically screened interaction one-loop self-energy evaluation (the so-called RPA or ‘GW’ self-energy approximation) because this approximation is really the only meaningful nontrivial calculation that can actually be carried out, and (perhaps more importantly) because this RPA self-energy is asymptotically exact in the weakly interacting high density regime. There have been a few finite temperature RPA self-energy calculations over the years mostly in the context of low dimensional systems, but none for the temperature dependence of the effective mass renormalization in interacting electron systems. Very recently, Chubukov and Maslov considered the problem of temperature corrections to the 2D Fermi liquid theory for the case of a short-ranged interaction. In particular, they showed that the leading many-body temperature correction is linear in 2D similar to the results which we reported recently for the long-range Coulomb interaction in 2DES.

In the current paper we present a calculation of the density and temperature dependent effective mass renormalization by the Coulomb interaction in 2DES and 3DES at arbitrary densities and temperatures (i.e. not necessarily restricted to high densities and low temperatures). We work within the random-phase approximation (RPA), or equivalently in the ring-diagram approximation for the self-energy, which gives exact results in the high-density limit ($r_s \ll 1$) but is known to be qualitatively reliable at relatively low densities as well. RPA is perhaps the only manageable way to perform any non-trivial quantitative calculations in electronic many-body systems, and the finite temperature RPA effective mass renormalization is certainly a problem of intrinsic interest. In two dimensions, our numerical results predict a non-monotonic effective mass temperature dependence.
The leading temperature dependence is linear and positive, with the low-temperature slope being independent of the electron density in the high density limit. The temperature at which the effective mass is maximum at a particular density moves toward higher temperatures as density decreases. In three dimensions, we find that the effective mass temperature dependence is non-universal and depends on the electron density in a complicated way. At very high densities, the leading correction is positive, while at lower densities it changes sign and decreases monotonically from its zero temperature value. This is in contrast to the 2D results where the effective mass always increases (linearly) with temperature at low temperatures, and then decreases with temperature beyond a density-dependent characteristic temperature. We find the 3D temperature correction to the effective mass to be nonlinear in contrast to our 2D results.

We express the quasiparticle effective mass \( m^*(n, T) = m^*(r_s, T/T_F) \) in units of the bare band mass \( m \) (which is, by definition, a constant) and present our results as a function of the usual dimensional interaction parameter \( r_s \) (the average inter-electron separation measured in the units of Bohr radius) and the dimensionless temperature \( T/T_F \) where \( T_F = E_F/k_B \) is the Fermi temperature. Note that \( r_s \propto n^{-1/2} (n^{-1/3}) \) in 2D (3D) systems, and \( T_F \propto n (n^{2/3}) \) in 2D (3D) systems, where \( n \) is the appropriate 2D (per unit area) or 3D (per unit volume) electron density. Note that the dimensionless interaction and temperature parameters \( r_s \) and \( T/T_F \) are not independent parameters since they both depend on the electron density. We also note that \( r_s \ll 1 \) (high-density) and \( r_s \gg 1 \) (low density) limits are respectively the weak- and the strong-interaction limits of the electron system (at \( T = 0 \)), and \( T/T_F \ll 1 \) and \( T/T_F \gg 1 \) are respectively the low-temperature (quantum) and the high-temperature (classical) limits. We consider the electron system to be a uniform jellium system with the noninteracting kinetic energy dispersion being the usual parabolic dispersion. We use \( \hbar = k_B = 1 \) throughout.

The structure of our paper is as follows: In section II we provide the formalism which we will use in this paper. In section III we explain in detail the numerical method we are using in the effective mass calculations. In section IV we present all our numerical results for 2D and 3D effective mass, comparing to analytical results in the high density limit. In section V we discuss a special approximation method, the plasmon-pole approximation, and present our effective mass results using this method. In section VI we calculate the imaginary self-energy of quasiparticles and discuss the validity of the quasiparticle approximation at finite temperatures. We provide a conclusion and discussion of our results in section VII.

II. FORMALISM

In this section we give the theoretical formalism, the basic equations, and the notations which will be used throughout the paper.

### A. Effective mass

In a system of interacting fermions the retarded Green’s function can be written as

\[
G_R(k, \omega) = \frac{1}{\omega - e_0(k) + \mu + \Sigma(k, \omega)},
\]  

(1)

where \( e_0(k) = k^2/2m \) is the spectrum of non-interacting fermions, \( \mu \) is the chemical potential, and \( \Sigma(k, \omega) \) is the quasiparticle self-energy, the imaginary (real) part of which determines the lifetime (effective mass) of the quasiparticle. The quasiparticle energy can be obtained by solving the Dyson’s equation

\[
\epsilon(k) = e_0(k) + \text{Re} \Sigma(k, \epsilon(k)).
\]

(2)

The quasiparticle effective mass can be written by definition as

\[
\frac{m^*}{m} = \frac{m}{k} \frac{d}{dk} \epsilon(k)\bigg|_{k=k_F} = \frac{1 - \frac{\partial}{\partial \omega} \text{Re} \Sigma(k, \omega)}{1 + \frac{\partial}{\partial \omega} \text{Re} \Sigma(k, \omega)} \bigg|_{k=k_F, \omega=0}.
\]

(3)

Note that in the above equation \( \omega = 0 \) is measured from the renormalized chemical potential \( \mu^* \), which is given by

\[
\mu^* = \mu + \text{Re} \Sigma(k_F, 0).
\]

(4)

All the above equations are exact, while the RPA approximation for \( \Sigma(k, \omega) \) that we are going to use is the first order perturbation theory in the dynamically screened interaction. There has been extensive discussion on whether one should use exact Eq.(3) for calculating the effective mass or it is more consistent to use the so-called on-shell approximation, keeping only the first order interaction terms in the expression for the effective mass (since \( \Sigma \) is calculated only to first order in the dynamically screened interaction):

\[
\frac{m^*}{m} = \frac{1}{1 + \frac{m}{k} \frac{d}{dk} \Sigma(k, \xi_k) \bigg|_{k=k_F}},
\]

(5)

where \( \xi_k = k^2/(2m) - \mu \). Note that all the quantities on the right side of Eq. (5) are in the leading order in effective interaction. There are compelling arguments in favor of the latter choice: the on-shell approximation is believed to be more accurate as it effectively accounts for some higher order diagrams and satisfies the Ward identity. We have extensively discussed this issue elsewhere.

Obviously, the two equations for calculating \( m^* \) are identical in the high-density limit \( r_s \ll 1 \). However, in the region of \( r_s > 1 \), they give very different results. In
what follows, we use Eq. 5 for all the numerical results shown in this paper because we believe the on-shell approximation to be the superior one in our case. Both formulae give similar temperature dependence for $m^*(T)$. The main qualitative results of the paper are insensitive to the choice of the on-shell or off-shell formula for the effective mass.

B. Self-energy in the RPA approximation.

\[
\begin{align*}
\Sigma(k, \omega) &= -T \sum_{\omega_m} G(k, \omega_m) D(q, \omega_m),
\end{align*}
\]

FIG. 1: Feynman diagram for RPA self-energy calculation. Solid lines denote the free electron Green’s function and the dashed lines the bare Coulomb potential.

Within RPA, the finite temperature electron self-energy can be expressed in terms of the Feynman diagrams shown in Fig. 1 and can be written in the Matsubara formalism as [7]:

\[
\Sigma(k, \nu_n) = -T \sum_{\omega_m} G(k - q, \nu_n - \omega_m) D(q, \omega_m),
\]

where $\gamma_n = \pi(2n + 1)T$ is the fermion Matsubara frequency, $\omega_m = 2\pi T n$ is the boson Matsubara frequency with $n$ and $m$ integers, and $T$ the temperature. The function $D(q, \omega_m)$ denotes the coupling to a collective mode (phonon, plasmon, electron-hole excitation, etc.), i.e., $D$ is the bosonic propagator for the effective interaction. In our case, the function is the dynamically screened Coulomb interaction given by the sum of the ring or bubble diagrams:

\[
D(q, \omega_m) = \frac{v_0(q)}{1 + v_0(q)\Pi(q, \omega_m)},
\]

where $v_0(q)$ is the bare Coulomb interaction and $\Pi(q, \omega_m)$ is the (bare) polarization operator, which is defined as

\[
\Pi(q, \omega_m) = 2 \sum_{\nu_n} \int \frac{d^d p}{(2\pi)^d} G(0)(p, \nu_n) \cdot G(0)(q + p, \nu_n + \omega_m),
\]

where $d$ is the dimension of the system and “(0)” denotes the non-interacting system. We mention that Eqs. 7 and 8 together form what is called the RPA for an electron gas, where the bare Coulomb interaction is dynamically screened by the electron dielectric function, which is formed from the infinite series of the polarization bubbles. The corresponding electron self-energy, obtained in the leading-order expansion in the dynamically screened interaction $D$, is conventionally called the RPA self-energy approximation, although the “dynamical Hatree-Fock” approximation or the “Ring Diagram Approximation” may be a more appropriate terminology.

For calculations, it is more convenient to use the self-energy defined as a function of the real frequency $\omega$ rather than the Matsubara one. Using the standard procedure of analytic continuation, one obtains the following expression for the analytically continued self-energy:

\[
\Sigma^R(k, \omega) = -\int \frac{d^d q}{(2\pi)^d} \int \frac{d \nu}{2\pi} \left[ \Im G_R^{(0)}(k - q, \nu + \omega) D_R(q, -\nu) \tanh(\frac{\nu + \omega}{2T}) \\
+ G_R^{(0)}(k - q, \nu + \omega) \Im D_R(q, \nu) \coth(\frac{\nu}{2T}) \right],
\]

where functions labeled with index “R” are retarded functions, i.e., functions analytical in the upper half-planes of the complex frequency. The corresponding effective interaction can be written as:

\[
D_R(q, \omega) = \frac{v(q)}{1 + v(q)\Pi_R(q, \omega)},
\]

where the retarded polarizability can be obtained from Eq. 5 using the following identities:

\[
\Pi_R(q, \omega) = \Pi(q, i\omega_n \to \omega + i\eta),
\]

where $\eta$ is a real infinitesimal positive number.

Note that we will almost always use retarded quantities unless otherwise stated. Thus without causing any confusion, we can drop the superscript “R”.

C. Effective interaction

The next step toward deriving the renormalization of mass is to obtain expressions for the effective coupling $D(q, \omega)$. We use the long-range bare Coulomb interaction to get

\[
\begin{align*}
&v_0^{2D}(q) = \frac{2\pi e^2}{q}, \\
v_0^{2D}(q) = \frac{4\pi e^2}{q^2},
\end{align*}
\]

and the effective interaction

\[
D(q, \omega) = \frac{v_0(q)}{1 + v_0(q)\Pi(q, \omega)} = \frac{v_0(q)}{\epsilon(q, \omega)},
\]

where $\epsilon(q, \omega) \equiv 1 + v_0\Pi$ is the RPA dynamical dielectric function. In the RPA the full polarizability is approximated by the bare polarizability as in Eq. 8, which is just the bare bubble diagram.
Analytical properties of the propagator \( \Pi_0 \) (where \( \Pi_0 \) denotes the \( T = 0 \) form for the bare polarizability with \( \Pi \) denotes the finite \( T \) bare polarizability) are non-trivial even at zero temperature. The zero temperature polarization for 2DES and 3DES are well-known and shown below. For 2D \( T = 0 \) case we have

\[
\Pi_{0}^{2D}(q, \omega, \mu) = -\frac{m}{\pi} + \frac{m^2}{\pi q^2} \left[ \sqrt{(\omega + \frac{q^2}{2m})^2 - \frac{2\mu q^2}{2m}} - \sqrt{(\omega - \frac{q^2}{2m})^2 - \frac{2\mu q^2}{2m}} \right],
\]

(14)

where \( \mu \) is the chemical potential, the frequency \( \omega \) can be any complex number, and the branch cut of the square roots are taken so that the imaginary part is positive. For 3D \( (T = 0) \) case we have

\[
\Pi_{0}^{3D}(q, \omega, \mu) = \frac{k_F m}{2\pi^2 q^2} \left\{ 1 + \frac{m^2}{2k_F^2 q^3} \left[ 4\mu_\epsilon q - (\epsilon_q + \omega)^2 \right] \ln \left( \frac{\epsilon_q + q\mu_{\mu} + \omega}{\epsilon_q + q\mu_{\mu} + \omega} \right) \\
+ \frac{m^2}{2k_F^2 q^3} \left[ 4\mu_\epsilon q - (\epsilon_q - \omega)^2 \right] \ln \left( \frac{\epsilon_q + q\mu_{\mu} - \omega}{\epsilon_q + q\mu_{\mu} - \omega} \right) \right\},
\]

(15)

where \( \epsilon_q = q^2/2m \), \( \mu \) is the chemical potential and \( \mu = \xi q^2/2m = m\nu q^2/2m \), and the frequency \( \omega \) can be any complex number.

Finite temperature polarizability can be easily obtained from those at zero temperature using the following identity:

\[
\Pi(q, \omega, \mu; T) = \int_{0}^{\infty} d\mu' \frac{\Pi_0(q, \omega, \mu')}{4T \cosh^2 \left( \frac{\mu - \mu'}{2T} \right)}.
\]

(16)

We find Eq. (16) to be the most convenient numerical method for obtaining the finite-\( T \) polarizability.

D. Dimensionless parameters

Our 2D and 3D electron system can be characterized by two parameters, namely density \( (n) \) and temperature \( (T) \). This immediately leads to two dimensionless parameters \( r_s \) and \( T/T_F \) characterizing the system with \( r_s \) being the effective zero-temperature interaction strength and \( T/T_F \) being the effective temperature (note that they are not independent). The definition of \( r_s \) is the following. In 2DES \( r_s \) is defined such that

\[
4\pi n a_B^3 r_s^2 = 1,
\]

(17)

\[
k_F r_s a_B = \sqrt{2},
\]

(18)

where \( n \) is the 2D electron density, \( k_F \) is the Fermi momentum, and \( a_B = (m e^2)^{-1} \) is the Bohr radius. In 3DES \( r_s \) is defined such that

\[
4\pi n a_B^3 r_s^3/3 = 1,
\]

(19)

\[
k_F r_s a_B = (9\pi/4)^{1/3}.
\]

(20)

The Fermi temperature \( T_F \) \( \equiv E_F \equiv k_F^2/(2m) \), which goes as \( T_F \propto r_s^{-2} \) in both 2D and 3D.

III. Numerical methods in \( m^* \) calculations in RPA

In this section we explain in detail our numerical approach for the effective mass calculation within RPA. In carrying out the integrations of self-energy in Eq. (9) in order to obtain the effective mass, we use three different techniques, namely frequency sum, frequency integration, and plasmon-pole approximation (PPA). The first two techniques are equivalent, and we explain them in detail in this section. PPA is a further approximation of RPA, which has been extensively used in the literature [6, 12, 17]. We discuss the PPA in section V. Since there is no existing literature on the finite temperature effective mass or self-energy calculation to check our numerical results, it is crucial for us to use these different techniques to ensure the correctness of our numerical calculations. We mention here that our frequency sum results and frequency integration results agree well with each other. The frequency integration result is numerically relatively more noisy and therefore in this paper we will only show the frequency sum results. We also check our numerical results against the already known results at \( T = 0 \) and against analytical calculations in the \( T/T_F, r_s \to 0 \) limit.

A. Frequency integration technique

Eq. (9) gives the general formula for the RPA self-energy at real frequencies. It can also be written in a more succinct way as

\[
\Sigma(k, \omega) = -\int \frac{d^d q}{(2\pi)^d} v_0(q) n_F(\xi_{q-k}) \\
- \int \frac{d^d q}{(2\pi)^d} \int \frac{d\varepsilon}{2\pi} 2v_0(q) \text{Im}^{-1}(q, \varepsilon) \left| n_F(\xi_{q-k}) + n_B(\varepsilon) \right|,
\]

(21)

where \( n_F(x) = 1/(\exp(x/T) + 1) \) is the Fermi function and \( n_B(x) = 1/(\exp(x/T) - 1) \) the Bose function. This method of calculating the self-energy involves integration over real frequencies, and therefore we call it the frequency integration method. It is also known as the spectral or the Lehmann representation of the self-energy. The derivation of Eq. (21) from Eq. (9) is given in the appendix.
The self-energy of Eq. (21) is composed of two parts: the exchange part and the correlation part. The (frequency independent) exchange part is also known as the Hartree-Fock self-energy, and its contribution to the effective mass at \( T = 0 \) is singular in both 2D and 3D. Not surprisingly, this singularity is cancelled out by contributions from the correlation part of the self-energy. Effective mass is derived from the self-energy through Eq. (5), and we therefore need to obtain the real part of Eq. (21) by putting \( i\eta \) to be 0 and regarding the frequency integration as a principal value integration. It is easy to derive from Eq. (21) that the imaginary part of the self-energy can be written as

\[
\text{Im}\Sigma(k, \omega) = -\int \frac{d^d q}{(2\pi)^d} v_0(q) \text{Im}^{-1}(q, \xi_{q-k} - \omega)) \cdot [n_B(\xi_{q-k} - \omega) + n_F(\xi_{q-k})].
\] (22)

The \( \text{Im}\Sigma \) is not needed in the effective mass calculation since \( m^* \) is a Fermi surface property. But it is important to have some idea of the magnitude of \( \text{Im}\Sigma \) in order to ensure that quasi-particles are well defined at finite \( T \).

Numerically carrying out the integration in Eq. (21) is non-trivial: for each momentum \( q \) and frequency \( \omega \), a three dimensional integration is required to obtain \( \Sigma(q, \omega) \), and what makes the problem even more difficult is that the \( \text{Im}^{-1} \) term in the integrand is highly non-monotonic. A careful examination of the dynamical dielectric function tells us that at \( T = 0 \), \( \text{Im}^{-1}(q, \omega) \) contains delta-functions at plasmon excitation frequencies, and at finite temperatures these delta-functions broaden into sharp peaks. Integration over these sharp frequencies requires special care. For each \( q \), the position (i.e. frequency) of the sharp peaks can be determined by solving \( \text{Re}\epsilon(q, \omega) = 0 \), and their weight can be determined from \( \text{Re}\epsilon(q, 0) \) using the Kramers-Kronig relations.

One advantage of the frequency integration method is that in Eq. (21) we can directly put \( T = 0 \) to obtain the zero temperature result, in contrast to the frequency sum method which we will describe in detail below.

### B. Frequency sum technique

Due to the great numerical difficulty in carrying out the frequency integration method introduced above (because it involves integration over highly non-monotonic or singular functions), it is advisable to seek alternatives. At zero temperature, previous works in calculating self-energy and related quantities often transform the real frequency integration into integrations over imaginary frequencies using the analytic properties of the dielectric function. The purpose of this contour distortion is to avoid singularities along the real axis. At finite temperature, a similar approach can be adopted. At finite temperature, what is different from the zero temperature case is that we transform the integration into an imaginary frequency summation (or Matsubara frequency summation). Hu at al. [11] showed in detail how to perform such a transformation from the real-frequency integration to an imaginary frequency summation. Following the technique of contour distortion introduced in Ref. [11] we can write the RPA self-energy as a sum of the Matsubara frequency along the imaginary axis:

\[
\Sigma(k, \omega) = -\int \frac{d^d q}{(2\pi)^d} v_0(q) n_F(\xi_{q-k})
\]

\[
-\int \frac{d^d q}{(2\pi)^d} v_0(q) \left[ \frac{1}{\epsilon(q, \xi_{q-k} - \omega) - 1} \cdot [n_B(\xi_{q-k} - \omega) + n_F(\xi_{q-k})] \right.
\]

\[
-\int \frac{d^d q}{(2\pi)^d} T \sum_{\omega_n} v_0(q) \left[ \frac{1}{\epsilon(q, i\omega_n) - 1} \cdot \frac{1}{i\omega_n - (\xi_{q-k} - \omega)} \right],
\] (23)

where the frequency sum is over even Matsubara frequencies \( \omega_n = i2n\pi T \) with \( n \) integers. The above expression contains three parts, namely the exchange part, the residue part and the line part from top to bottom in Eq. (23). The proof of the equivalence between Eq. (23) and Eq. (21) is provided below.

Since the exchange part exists in both Eq. (23) and Eq. (21), we only need to consider the correlation part of the self-energy

\[
\Sigma^{\text{COF}}(k, \omega) = -\int \frac{d^d q}{(2\pi)^d} \int \frac{d\nu}{2\pi} \frac{2v_0(q)\text{Im}^{-1}(q, \nu)}{2\pi i \nu + \omega + i\eta - \xi_{q-k}}
\]

\[
\cdot [n_F(\xi_{q-k}) + n_B(\nu)].
\] (24)

We choose the contour as in Fig. 4. It is easy to see that the integration over real axis can be transformed into integration over contour \( \mathcal{C} \), so that we have

\[
\Sigma^{\text{COF}}(k, \omega) = -\int \frac{d^d q}{(2\pi)^d} \int \frac{d\nu}{2\pi} \frac{v_0(q)(\epsilon^{-1}(q, \nu) - 1)}{2\pi i \nu + \omega + i\eta - \xi_{q-k}}
\]

\[
\cdot [n_F(\xi_{q-k}) + n_B(\nu)].
\] (25)

This is because \( \epsilon(q, \omega - i\eta) = \epsilon^*(q, \omega + i\eta) \), and therefore the integration of the real part of the integrand right above the real axis in the positive direction and right below the real axis in the negative direction cancel each other, and the corresponding integration of the imaginary part on these two lines are equal to each other. The \( -1 \) after \( \epsilon^{-1}(q, \omega) \) is inserted to make the integration on the arc part of contour \( \mathcal{C} \) vanish as the radius of the contour approaches infinity. Now we are left to evaluate the residues within contour \( \mathcal{C} \), the positions of which are denoted by crosses in Fig. 2. Note that the analytic property of the dielectric function \( \epsilon^{-1}(q, \omega) \) is very important in this approach. The transformation requires that \( \epsilon^{-1}(q, \omega) \) is analytic in the upper and the lower half of the complex plane, which is true for electron gas systems. The single residue at \( \xi_{q-k} - \omega - i\eta \) right below the real axis produced by the denominator of the integrand produces the residue part of the self-energy. This part
can be easily derived as
\[ \Sigma^\text{res} = -\int \frac{d^d q}{(2\pi)^d} v_0(q) \left[ \frac{1}{\epsilon(q, \xi_{q-k} - \omega)} - 1 \right] \cdot [n_B(\xi_{q-k} - \omega) + n_F(\xi_{q-k})]. \] (26)

The residues at \( \omega_n = 2n\pi T \) on the imaginary axis (the third term in Eq. (24)), which are produced by the Bose function \( n_B(\nu) \), lead to the line part of the self-energy. This part can be written as
\[ \Sigma^\text{line} = -\int \frac{d^d q}{(2\pi)^d} T \sum_{\omega_n} v_0(q) \left[ \frac{1}{\epsilon(q, i\omega_n)} - 1 \right] \cdot \frac{1}{i\omega_n - (\xi_{q-k} - \omega)}. \] (27)

From Eq. (26) and Eq. (27), we have \( \Sigma^\text{cor} = \Sigma^\text{res} + \Sigma^\text{line} \), and we thus obtain Eq. (24).

![Contour of integration for the derivation of self-energy formula for the frequency sum method.](image)

**Fig. 2:** Contour of integration for the derivation of self-energy formula for the frequency sum method. The thick lines on real axis denote the branch cut for \( \epsilon^{-1}(q, \omega) \). The crosses mark the poles due to the integrand; the ones on the imaginary axis are due to \( n_B(\omega) \), and the isolated pole is due to the denominator.

The frequency sum method proves to be a far more efficient numerical technique for calculating the self-energy than the frequency integration method due to the absence of the strong non-monotonicity and singularity in the real frequency dependence of the integrand. One thing to notice is that at high temperatures, higher Matsubara frequency terms can be neglected because \( (\epsilon^{-1}(q, i\omega_n) - 1) \to 0 \) when \( \omega_n \to \infty \), while at low temperatures a large number of Matsubara terms have to be kept in the sum in order to ensure accuracy. At zero temperature, the frequency sum turns into an integration over imaginary frequencies, and we have
\[
\Sigma(k, \omega) = -\int_{R_1} \frac{d^d q}{(2\pi)^d} v_0(q) \\
+ \int_{R_2} \frac{d^d q}{(2\pi)^d} \frac{v_0(q)}{\epsilon(q, \xi_{q-k} - \omega)} \\
- \int \frac{d^d q}{(2\pi)^d} \int dv \left[ \frac{1}{\epsilon(q, iv) - 1} \right] \frac{\omega - \xi_{q-k}}{\nu^2 + (\omega - \xi_{q-k})^2},
\] (28)

where the integration region \( R_1 \) denotes the region where \( |k - q| < k_c \), and \( R_2 \) denotes the integration region where\( |k - q| \) is in between \( k_c \) and \( k_F \). This explicit formula for self-energy is exactly what previous works (see, e.g. Ref. [5]) used to calculate the zero temperature RPA self-energy.

It is obvious that the frequency independent exchange part of the self-energy is real. By noticing that \( \epsilon(q, i\omega_n) = \epsilon^s(q, \omega_n) \), we can see that the line part of the self-energy is real as well. Thus the only contribution to the imaginary part of the self-energy comes from the residue part, which gives the same result as Eq. (22) in the frequency integration method.

**IV. RPA RESULTS FOR \( m^*(r_s, T/T_F) \)**

In this section we present our numerical results for effective mass in 2D and 3D electron systems within RPA. We first present in section IV A results for the zero temperature effective mass to compare with earlier works. Our finite temperature results for 2DES are presented in section IV B and those for 3DES in section IV C. In section IV D we present results for a model bare potential where the Coulomb interaction is cut-off by a finite length so that the bare interaction is short-ranged. We do this in order to investigate the model dependence of our results.

**A. Zero temperature effective mass**

We first present our extreme low temperature results \( (T/T_F \approx 10^{-4}) \) in Fig. 3 and Fig. 4 to be compared with the existing \( T = 0 \) results [3, 4, 5, 6, 7, 8, 9]. We calculate \( m^*(r_s) \) in the \( r_s = 0 - 10 \) range, showing that the effective mass renormalization could be as large as 4.5 for dilute \( (r_s \sim 10) \) 2DES and 3 for \( (r_s \sim 10) \) 3DES. We emphasize that the results presented in Fig. 3 and Fig. 4 are entirely based on the \( T \to 0 \) limit of our finite temperature theory. They are in quantitative agreement with the existing \( T = 0 \) 2D RPA effective mass calculations [3] (which are restricted to the \( r_s < 5 \) regime) and are consistent with the existing \( T = 0 \) 3D effective mass calculations at low \( r_s \) [3]. This serves as a stringent check on our numerical approaches.
It is clear from Figs. 3 and 4 that both our 2D and 3D results show the non-monotonic dependence of $m^*(r_s)$ on $r_s$ in the high-density regime (i.e. in the $r_s \ll 1$ regime). This nonmonotonic low-$r_s$ behavior for $m^*(r_s)$ at $T = 0$ has been reported in the earlier literature [3, 5].

We emphasize that the numerical results given in Figs. 5 and 6 are obtained by putting $T/T_F \approx 10^{-4}$ in our finite-temperature formalism.

**B. Finite temperature effective mass in 2DES**

In Fig. 5 and Fig. 6 we show our calculated 2D $m^*(T)$ as a function of $T/T_F$ for different values of the 2D interaction parameter $r_s$ (= 0.1−10). In the low temperature region the effective mass first rises to some maximum, and then decreases as temperature increases. This non-monotonic trend is systematic, and the value of $T/T_F$ where the effective mass reaches the maximum increases with increasing $r_s$. The initial increase of $m^*(T)$ is almost linear in $T/T_F$ as $T \to 0$, and the slope $\frac{d(m^*/m)}{dT/T_F}$ is almost independent of $r_s$ for very small $r_s$ (< 1) (which is shown in Fig. 5), but increases with $r_s$ for larger $r_s$ values. It is important to notice that this non-monotonic temperature dependence of $m^*(T)$ with a maximum around $T/T_F \lesssim 1$ persists all the way to $r_s \to 0$, which suggests that it is not an artifact of our approximation scheme since RPA become exact as $r_s \to 0$.

In Fig. 7 and Fig. 8 we show the dependence of the effective mass renormalization as a function of the interaction parameter $r_s$ for a few values of fixed temperature (rather than fixed $T/T_F$, remembering that $T_F \propto r_s^{-2}$ since $T_F \propto n$ and $r_s \propto n^{-1/2}$). Fig. 7 shows the effective mass for high $T$ and large $r_s$ values while Fig. 8 concentrates on the low $T$ region. The calculated $m^*(r_s)$ for fixed $T$ values are quite striking: For low fixed values of $T$, $m^*/m$ initially increases with $r_s$ even faster than the corresponding $T = 0$ result, eventually decreasing with $r_s$ at large enough values (where the corresponding $T/T_F$ values become large enough). This nonmonotonic behavior of $m^*(r_s)$ as a function of $r_s$ for fixed temperatures showing a temperature-dependent maximum (with the value of $r_s$ at which the $m^*$ peak occurs decreasing with increasing $T$ as in Fig. 7) is complementary to the nonmonotonicity of $m^*(T)$ in Fig. 5 as a function of
$T/T_F$ (at fixed $r_s$) and arises from the relationship between the dimensionless variables $T/T_F$ ($\propto r_s^{-2}$) and $r_s$ ($\propto T_F^{-1/2}$) due to their dependence on the carrier density (i.e. $T_F \propto n$ and $r_s \propto n^{-1/2}$). At large $r_s$ and high temperature, Fig. 7 shows that the effective mass increases from below unity with increasing $r_s$. This is the region where the exchange part of the self-energy dominates, and it can be easily shown that the exchange self-energy produces this peculiar effect on the $r_s$ dependence of $m^*(r_s)$ at fixed high $T$ values. Since this region is hardly accessible by experiments, and moreover the quasiparticles may not even be well-defined at such high $T/T_F$ values, we do not further discuss the physics related to this region.

One immediate consequence of our results shown in Figs. 5 and 7 is that $m^*(T/T_F, r_s) \equiv m^*(T, n)$ in 2DES could show a strong enhancement at low (but finite) temperatures and low electron densities (large $r_s$). Comparing with the actual system parameters for 2D electrons in Si inversion layers and GaAs heterostructures (and taking into account the quasi-2D form factor effects [18 neglected in our strictly 2D calculation) we find that, consistent with recent experimental findings [11], our theoretical calculations predict (according to Figs. 5 and 7 as modified by subband form factors) $m^*/m$ to be enhanced by a factor of 2 $-$ 4 for the experimental densities and temperatures used in recent measurements [11]. Due to the approximate (i.e. RPA) nature of our theory we do not further pursue the comparison with experimental data in this paper since the main goal of this paper is to discuss the temperature dependence of $m^*(r_s, T/T_F)$ which has not yet been reported in the literature. A direct experimental observation of an increasing $m^*(T)$ at low temperatures in 2DES will be a striking confirmation of our theory.

C. Finite temperature effective mass in 3DES

In Fig. 9 and Fig. 10 we show our calculated 3D $m^*(T)$ as a function of $T/T_F$ for different $r_s$ values. In Fig. 9, $r_s$ varies from 1 to 10 while in Fig. 10, $r_s$ is from 0.1 to 1. The 3D temperature dependence of the effective mass shows very different characteristics from that of 2D. Fig. 9 shows that for $r_s > 1$ the effective mass decreases monotonically with increasing $T$ at low temperatures. However for $r_s < 1$, as shown in Fig. 10, the effective mass increases with increasing $T$ in the temperature region we are considering. We therefore conclude that in 3DES the sign of the slope $\frac{d(m^*/m)}{d(T/T_F)}|_{T=0}$ is non-universal, which differs from that of 2DES where the above mentioned slope is always positive for all $r_s$. Another interesting feature is that the sign of $\frac{d(m^*/m)}{d(r_s)}|_{T=0}$ matches the sign of $\frac{d(m^*/m)}{d(r_s)}|_{T=0}$ very well. In particular, $m^*(T)$ decreases with increasing $T$ (at low $T$) in the “larger” $r_s$-regime where the corresponding $T = 0$ $m^*(r_s)$ shows an increasing mass with increasing $r_s$. Similarly, $m^*(T)$ increases (at low $T$) with increasing $T$ in the $r_s$-regime where the corresponding $m^*(r_s; T = 0)$ shows decreasing $m^*$ with increasing $r_s$.

FIG. 7: Calculated 2D effective mass as a function of $r_s$ at fixed value of temperatures. $T$ is in the unit of $T_F$ at $r_s = 1$.

FIG. 8: Calculated 2D effective mass as a function of $r_s$ at fixed value of temperatures. $T$ is in the unit of $T_F$ at $r_s = 1$. This plot is similar to Fig. 7 but concentrating on the low temperature region.

FIG. 9: Calculated 3D effective mass as a function of $T/T_F$ for different $r_s$: $r_s = 10 \rightarrow 1$ from top to bottom.
D. Model short-range bare interaction

So far in all of our calculations we have been using the realistic long-ranged Coulomb interaction for the bare potential as in Eq. (12). A question naturally arises: how is the temperature dependence that we find in our calculations related to the long range nature of the interaction between electrons? Therefore we also calculate the effective mass in 2DES and 3DES using simple (parameterized) finite-range interaction model

$$v^{2D}(q) = \frac{2\pi e^2}{q + a},$$

$$v^{3D}(q) = \frac{4\pi e^2}{q^2 + a^2},$$  \hspace{1cm} (29)

where $a$ is the cut-off wavevector which eliminates the long wavelength Coulomb divergence.

Our numerical calculation shows that as $a/k_F \to 0$, we recover the $m^*(T)$ behavior of the bare Coulomb interaction results in both 2D and 3D. As $a/k_F$ increases, the mass renormalization in both 2D and 3D is suppressed, but all the qualitative features of the temperature dependence persist. In 2DES, as $a/k_F$ increases, the temperature where the effective mass reaches the maximum decreases, and the effective mass enhancement (from the $T = 0$ value to the maximum) decreases, but the linear-$T$ dependence at low $T$ and the non-monotonic trend remain unchanged. In 3DES, as $a/k_F$ increases, the $r_s$ region where $\frac{d(m^*/m)}{d(r_s)}|_{T=0} < 0$ shrinks, but the consistency between the sign of $\frac{d(m^*/m)}{d(T/T_F)}|_{T=0}$ and the sign of $\frac{d(m^*/m)}{d(T/T_F)}|_{T=0}$ remains.

From these result we conclude that the qualitative features of the temperature dependence are model independent and not peculiar to the bare interaction being Coulombic. This conclusion is further reinforced by the recent report of a linearly $T$ dependent electronic specific heat in a short-range interaction model \cite{14}. It may be worth while, however, to note that RPA is specific to the long-range Coulomb interaction in giving an exact result in the high-density $r_s \to 0$ limit, and there is nothing special about RPA in the case of short-range interaction.

V. Plasmon-pole approximation

We now apply a simple-to-use dynamical approximation to calculate $m^*(T)$. The plasmon-pole approximation (PPA) has often been used \cite{6, 12, 17} to obtain the electron self-energy in the literature. It is a simple technique for carrying out the frequency sum or integration in the RPA self-energy calculation by using a spectral pole (i.e. a delta function) ansatz for the dynamical dielectric function $\epsilon(k, \omega)$. In other words, it is an approximation to the RPA. The PPA ansatz assumes that

$$-2\text{Im} \frac{1}{\epsilon(k, \omega)} = C_k (\delta(\omega - \tilde{\omega}_k) - \delta(\omega + \tilde{\omega}_k)), \hspace{1cm} (30)$$

where the pole $\tilde{\omega}_k$ and the spectral weight $C_k$ of the PPA propagator in Eq. (30) are determined by using the the Kramers-Krönig relation (i.e. causality)

$$\text{Re} \left( \frac{1}{\epsilon(k, 0)} \right) = 1 + \frac{2}{\pi} \int_0^\infty \frac{d\omega \text{Im} \frac{1}{\epsilon(k, \omega)}}{\omega} \hspace{1cm} (31)$$

and the $f$-sum rule (i.e. current conservation)

$$\int_0^\infty \omega d\omega \text{Im} \frac{1}{\epsilon(k, \omega)} = -\frac{\pi}{2} C_k \omega^2 \hspace{1cm} (32)$$

Putting Eq. (30) in Eq. (31) and Eq. (32) we have

$$C_k = \pi \omega_p(k) \sqrt{1 - \text{Re}^{-1}(k, 0)}, \hspace{1cm} (33)$$

$$\tilde{\omega}_k = \frac{\omega_p(k)}{\sqrt{1 - \text{Re}^{-1}(k, 0)}}, \hspace{1cm} (34)$$

where $\omega_p(k)$ in Eqs. (32), (36) and (37) is the long wavelength plasmon frequency which is defined as

$$\lim_{\omega \to \infty} \text{Re} [\epsilon(k, \omega)] = 1 - \frac{\omega_p^2}{\omega^2}. \hspace{1cm} (35)$$

It is well-known that in 2DES

$$\omega_p^2(k) = \frac{2\pi n e^2}{\hbar^2} k, \hspace{1cm} (36)$$

and in 3DES

$$\omega_p^2(k) = \frac{4\pi n e^2}{\hbar^2}. \hspace{1cm} (37)$$

We mention that $\tilde{\omega}_k$ in Eq. (30) does not correspond to the real plasmon dispersion in the electron liquid, but simulates the whole excitation spectra of the system behaving as an effective plasmon at low momentum and as the single-particle electron-hole excitation at large momentum, as constrained by the Kramers-Krönig relation and the $f$-sum rule. Details on the PPA are available in

![Fig. 10: Calculated 3D effective mass as a function of $T/T_F$ for low $r_s$ values: $r_s = 1.0 \to 0.1$.](image)
literature \cite{6,12,17}, including its finite-temperature generalization \cite{12}. The PPA, which is known to give results close to the full RPA calculation of self-energy, allows a trivial carrying out of the frequency sum in the retarded self-energy function leading to:

\[
\text{Re} \Sigma(k, \omega) = -\int \frac{d^2q}{(2\pi)^2} \frac{\omega q}{n_F(\epsilon_{q-k})} \frac{d^2q}{(2\pi)^2} \frac{\omega q}{n_F(\epsilon_{q-k})} \frac{d^2q}{(2\pi)^2} \frac{\omega q}{n_F(\epsilon_{q-k})} + \frac{n_B(-\omega_q) + n_F(\epsilon_{q-k})}{\omega_q + (\epsilon_{q-k} - \omega)} \frac{n_B(-\omega_q) + n_F(\epsilon_{q-k})}{\omega_q + (\epsilon_{q-k} - \omega)}. \hspace{1cm} (38)
\]

where \(C_q\) and \(\tilde{\omega}_k\) only depend on \(\epsilon(k,0)\) at finite temperatures, and are determined by Eq. (33) and Eq. (34). Obviously the PPA provides a great simplification of the problem since the most numerically demanding part of the calculation (the frequency sum or integration) is trivially done. It should be noted, however, that although the PPA is known to produce a reliable approximation \(\text{Re} \Sigma\), it, by definition, fails completely for \(\text{Im} \Sigma\).

We present our PPA results for the 2D effective mass as a function of \(T/T_F\) at fixed \(r_s\) values in Fig. 11. One immediate observation by comparing Fig. 14 and Fig. 11 is that even though PPA provides a very good approximation for the self-energy (indeed, our numerical results for PPA self-energy and RPA self-energy match very well), it fails to provide accurate result for the effective mass. The zero temperature effective mass generated by PPA is almost half of that from RPA, and the temperatures where \(m^*\) maximizes shift to higher \(T\) values in the PPA compared with RPA. But the qualitative behavior of \(m^*(r_s, T/T_F)\) is similar in the PPA and RPA for the 2DES as is clear by comparing Figs. 11 and 14.

![Figure 11](image1.png)

**FIG. 11:** Calculated 2D PPA effective mass as a function of \(T/T_F\) at fixed value of \(r_s\).

From our results of 3D PPA effective mass calculation presented in Fig. 12 we can see that they are different from RPA results even qualitatively. In fact, our RPA results for \(m^*(r_s, T/T_F)\) are similar in both 2D and 3D.

![Figure 12](image2.png)

**FIG. 12:** Calculated 3D PPA effective mass as a function of \(T/T_F\) at fixed value of \(r_s\).

**VI. QUASIPARTICLE DECAY**

The quasiparticle decay rate (or the inverse lifetime) is given \(\text{Im} \Sigma\) by the imaginary part of the self-energy. As we have discussed in section \(\text{III B}\), the imaginary part of the quasiparticle self-energy can be calculated from Eq. (23). It is also obvious that only the second term in Eq. (23) contributes to the imaginary self-energy: the first term is obviously real, and the last term is also real because \(\epsilon(q, -\omega_n) = \epsilon^*(q, \omega_n)\). Thus we have

\[
\text{Im} \Sigma(k, \omega) = -\int \frac{d^d q}{(2\pi)^d} \frac{\omega_q}{n_F(\epsilon_{q-k})} \frac{1}{\epsilon(q, \epsilon_{q-k})} \frac{d^d q}{(2\pi)^d} \frac{\omega_q}{n_F(\epsilon_{q-k})} + \frac{n_B(\epsilon_{q-k} - \omega) + n_F(\epsilon_{q-k})}{\omega_q + (\epsilon_{q-k} - \omega)}. \hspace{1cm} (39)
\]

Fig. 13 and Fig. 14 show the calculated imaginary self-energy on the Fermi surface in 2D and 3D respectively. The quasiparticle decay (i.e., finite \(\text{Im} \Sigma\)) here arises entirely from having a finite temperature. The results show that the magnitude of the imaginary self-energy increases with increasing \(r_s\) and \(T/T_F\). It is obvious from Eq. (39) that the imaginary self-energy vanishes on the Fermi surface at \(T = 0\) as it must since the quasiparticles are perfectly well-defined at \(T = 0\) for \(k = k_F\). As \(T\) increases, the magnitude of imaginary self-energy remains small compared to the Fermi energy up to a certain temperature, and the quasiparticles on the Fermi surface remain well-defined up to that temperature. The important question is whether the finite temperature quasiparticles are sufficiently well-defined for the interesting behavior of \(m^*(T)\) we discussed in section \(\text{IV}\) to be experimentally observable. If the quasiparticles are ill-defined (i.e., \(\text{Im} \Sigma(k_F) > E_F\) in the temperature regime of interest) then obviously all the interesting temperature dependence of \(m^*(T)\) predicted by us is only of academic interest since the large broadening will make it impossible to define quasiparticles, let alone their effective mass. Examining the results of Figs. 13 and 14 compared with those presented in section \(\text{IV}\), it is clear that there is a well-defined regime of \((r_s, T/T_F)\) values.
where \( m^*(T/T_F) \) shows non-trivial temperature dependence with the condition \( E_F \gg |\text{Im} \Sigma(k_F)| \) well-satisfied so that quasiparticles are well-defined. Although this is not unexpected since \( |\text{Im} \Sigma(T)| \sim T^2 \) for \( T/T_F \ll 1 \) whereas \( m^*(T)/m \approx 1 + O(T) \) in 2D, it is nevertheless important to see that \( \text{Im} \Sigma \) remains small in magnitude in the \( (r_s, T/T_F) \) regime of interest.

![FIG. 13: Calculated magnitude of the 2D RPA imaginary self-energy of quasiparticles on Fermi surface as a function of \( r_s \) at different values of \( T/T_F \).](image)

![FIG. 14: Calculated 3D RPA imaginary self-energy of quasiparticles on Fermi surface as a function of \( r_s \) at different values of \( T/T_F \).](image)

Earlier theoretical work on the quasiparticle damping of 2D interacting electron systems can be found in Refs. 8-10, 11a, 11c.

**VII. DISCUSSION AND CONCLUSION**

In this work we have obtained detailed results for the temperature dependence of the quasiparticle effective mass, \( m^*(r_s, T/T_F) \), at arbitrary values of temperature and density in 2D and 3D electron systems interacting via the long range Coulomb interaction. Our central approximation is the RPA (i.e. the dynamically screened Hartree-Fock self-energy approximation), which is the leading-order one-loop self-energy calculation in a dynamically screened effective interaction expansion. RPA is exact in the high-density (\( r_s \to 0 \)) limit at \( T = 0 \), and is therefore a controlled nontrivial approximation which is empirically known to work well for \( r_s > 1 \) (e.g. metals with \( r_s \sim 3 - 6 \) and 2D semiconductors with \( r_s \sim 1 - 10 \)). We also calculate the finite-temperature imaginary self-energy (i.e. the quasiparticle decay rate or broadening) to ensure that the broadening remains small in the \( (r_s, T/T_F) \) parameter regime of our interest where \( m^*(T) \) shows interesting temperature dependence.

As mentioned earlier in the paper, it is well-known that at \( T = 0 \), \( m^*(r_s) \) can be exactly calculated (in both 2D and 3D) in the asymptotic \( r_s \to 0 \) limit by systematically expanding the RPA self-energy since ring diagrams (included in the RPA) are the most divergent diagrams in the \( r_s \to 0 \) limit. Such a zero temperature \( r_s \)-expansion of RPA gives the following formula for \( m^*(r_s) \) in both 2D and 3D:

\[
\left. \frac{m^*(r_s)}{m} \right|_{r_s \to 0} = 1 + ar_s(b + \ln r_s) + O(r_s^2), \quad (40)
\]

where \( a, b \) are constants of order unity. What we find numerically is that the leading temperature correction to this effective mass formula is linear in \( T/T_F \) in 2D and nonlinear in \( T/T_F \) in 3D. In this paper we have calculated \( m^*(r_s, T/T_F) \) numerically for the one-loop dynamically screened Hartree-Fock RPA self-energy theory for arbitrary \( r_s \) and \( T/T_F \) finding nontrivial temperature dependence of the effective mass at all densities.

Our most important result is the unexpected discovery of a strong temperature-dependent quasiparticle effective mass \( m^*(T) \) at low temperatures in 2DES. Since the temperature scale for the temperature dependence of \( m^*(T) \) is the Fermi temperature which tends to be high (\( \sim 10^4K \)) in the 3D electron liquids (i.e. metals), our temperature-dependent effective mass results for 3D systems are mostly of theoretical interest since any actual \( T \)-dependence of \( m^*(T) \) in the \( T/T_F \lesssim 10^{-4} \) regime will be miniscule. Our numerical results for the calculated \( m^*(T) \) in 2D systems are consistent with a linear leading-order temperature correction for the 2D quasiparticle effective mass: Results in Figs. 4 and 6 can be well fitted to the formula \( m^*(T) \approx 1 + A^{2D}(r_s) + B^{2D}(r_s)(T/T_F) + \cdots \) for small \( T/T_F \) where the slope \( B^{2D}(r_s) \) seems to be a constant independent of \( r_s \) (i.e. density) at least in the high-density \( (r_s \ll 1) \) limit; for \( r_s > 1 \) the slope \( B^{2D}(r_s) \) has a weak density dependence increasing somewhat with increasing \( r_s \) (but our approximation scheme, RPA, becomes less quantitatively reliable at large \( r_s \), therefore it is possible that the slope \( d(m^*/m)/d(T/T_F) \) is indeed independent of \( r_s \) in the \( T \to 0 \) limit). In addition to this interesting (and unexpected) linear leading-order temperature correction to the quasiparticle effective mass, we also find \( B^{2D}(>0) \) to be positive for all \( r_s \), indicating that in 2DES, the leading-order temperature correction to the effective mass is positive. Thus, \( m^*(T) \) increases with increasing \( T \) at first before eventually decreasing as \( T/T_F \) increases substantially, leading to a maximum in
We also emphasize a point in this context that seems have recently argued [16] that RPA remains extremely systems) RPA is by no means an exact theory, but we necessarily small. In this non-asymptotic regime (where non-asymptotic region where increase in $3D$. Our numerical results are consistent with this extension Im$\Sigma$, remains small) all the way to $T^*$ and perhaps even above $T^*$. This is reasonable since the many-body correction to $m^*$ is linear in $T/T_F$ whereas the broadening Im$\Sigma \sim (T/T_F)^2$, ensuring that for $T/T_F < 1$, the quasiparticle effective mass is a well-defined quantity. In contrast to the linear (with positive slope) leading-order $T$-dependence we find for all $r_s$ in our calculated 2D $m^*(T)$, our 3D results show non-universal $m^*(r_s,T/T_F)$ behavior. In 3D, $m^*(T/T_F)$ increases with increasing $T/T_F$ at low temperatures only for very high densities (small $r_s$) – for larger $r_s$ values $m^*(T)$ decreases monotonically with increasing temperature (in sharp contrast to the striking non-monotonicity in $m^*(T)$ in 2D) and this decrease is more consistent with a nonlinear leading-order temperature dependence (rather than a linear one as in 2D). Our best guess for our numerical results shown in Fig. 9 and 10 is the following equation: 

$$m^*(r_s,T/T_F) = 1 + A(r_s) + B(r_s) \left( \frac{T}{T_F} \right) + C(r_s) \left( \frac{T}{T_F} \right)^2 \ln \left( \frac{T}{T_F} \right) + \cdots$$

(41)

with $B(r_s) \equiv B^{2D}$, a constant, in 2D, and $B(r_s) \equiv 0$ in 3D. Our numerical results are consistent with this exact result, but our numerical results apply also in the non-asymptotic region where $T/T_F$ and $r_s$ are not necessarily small. In this non-asymptotic regime (where $r_s$ is not small, actually $r_s$ may be large in 2D semiconductor systems) RPA is by no means an exact theory, but we have recently argued [16] that RPA remains extremely well-valid (if somewhat uncontrolled) even for $r_s \gg 1$. We also emphasize a point in this context that seems not to have been widely appreciated in the literature. The point is that RPA becomes a progressively better approximation as $T/T_F$ increases at a fixed $r_s$ (for any $r_s$), because the system is becoming more classical in the $T/T_F \gg 1$ regime where RPA is again exact. Thus in the $(r_s,T/T_F)$ parameter space (see Fig. 15) RPA is exact as $r_s \to 0$ (the high density limit) and as $T_F \to 0$ (the high-temperature or equivalently the low density limit) or as $T \to \infty$. Thus the regime of validity of RPA is greatly enhanced at finite temperature, and in fact even at very large $r_s$ (i.e. very low density) RPA becomes exact as $T$ is raised (because $T/T_F \gg 1$ limit is more easily achieved at low densities).

![FIG. 15: Schematic validity of RPA. The shaded area denotes the region where RPA is considered to be valid. Line A denotes a certain density value above which RPA is valid at $T = 0$ (e.g. the vertical line A may correspond to the $r_s = 1$ condition so that for higher density, i.e. to the right to line A, RPA is valid even at $T = 0$). Line B denotes the line of $T_F \propto n$. In the region above line B, RPA is again valid. Therefore for any fixed value of density $n$ (or equivalently fixed $r_s$), RPA is a better approximation with increasing $T$, whereas for fixed value of temperature, low density values (or large $r_s$) counter-intuitively makes RPA to valid again since RPA is valid for $T > T_F$.]

Finally, we comment on the anomalous (often referred to as “nonanalytic” [14]) nature of the temperature corrections to the quasiparticle effective mass in 2D systems (but not in 3D) as manifested in the linear leading-order temperature correction we find in interacting 2D electron systems. This particular feature is apparently generic in 2D and not due merely to our using the long-range bare Coulomb interaction, because in Ref. [14] the same linear-$T$ correction is found in calculations using a zero-range bare interaction although the sign of the slope is negative in the zero-range interaction case. This kind of (leading-order) linear temperature correction is quite common in 2D electron systems due to the peculiar form of the 2D polarizability with a $T = 0$ cusp at $2k_F$. This leading-order linear-$T$ correction is interesting because the naive expectation in a Fermi system (based on the
usual Sommerfeld expansion of the Fermi functions) is that the leading-order correction in a “normal” situation should always be $\mathcal{O}(T/T_F)^2$ for all electronic properties. In 2D electron systems it seems that the generic situation is “anomalous”, i.e. the leading-order temperature correction is $\mathcal{O}(T/T_F)$ rather than the “normal” quadratic Fermi behavior expected on the basis of analytic Sommerfeld expansion of Fermi functions. In 2D interacting electron systems, therefore, all leading-order thermal corrections to electronic properties are much stronger (by a factor of $T_F/T$, which is a large number as $T \to 0$) than the quadratic Fermi gas behavior. This anomalous nonanalyticity, which may have important consequences for fermionic quantum critical phenomena, obviously has important experimental implications since it is much easier to observe a linear temperature correction than a quadratic one at low temperatures.

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Appendix

Here we provide a proof of for equivalence between Eq. (28) and Eq. (21).

$$
\Sigma(k, \omega) = -\int \frac{d^d q}{(2\pi)^d} \int \frac{dv}{2\pi} \left\{ \Im G_R(k - q, \nu + \omega) D_R(q, \nu) \tanh\left(\frac{\nu + \omega}{2T}\right) + G_R(k - q, \nu + \omega) \Im D_R(q, \nu) \coth\left(\frac{\nu}{2T}\right) \right\}
$$

$$
= -\int \frac{d^d q}{(2\pi)^d} \int \frac{dv}{2\pi} v_0(q) \left\{ -\pi \delta(\nu + \omega - \xi_{q-k}) \epsilon^{-1}(q, \nu) \tanh\left(\frac{\nu + \omega}{2T}\right) + \frac{1}{\nu + \omega + i\eta - \xi_{q-k}} \Im \epsilon^{-1}(q, \nu) \coth\left(\frac{\nu}{2T}\right) \right\}. \quad (42)
$$

Using Kramers-Krönig relations for $\epsilon^{-1}(q, \nu)$ in the above equation, we have

$$
\Sigma(k, \omega) = -\int \frac{d^d q}{(2\pi)^d} \int \frac{dv}{2\pi} v_0(q) \left\{ \right. 
-\pi \delta(\nu + \omega - \xi_{q-k}) \epsilon^{-1}(q, \nu) \tanh\left(\frac{\nu + \omega}{2T}\right) 
+ \frac{1}{\nu + \omega + i\eta - \xi_{q-k}} \Im \epsilon^{-1}(q, \nu) \coth\left(\frac{\nu}{2T}\right) \right\}
$$

$$
= \int \frac{d^d q}{(2\pi)^d} v_0(q) \frac{1}{2} \tanh\left(\frac{\xi_{q-k}}{2T}\right) 
- \int \frac{d^d q}{(2\pi)^d} \int \frac{dv}{2\pi} v_0(q) \Im \epsilon^{-1}(q, \nu) \frac{1}{2\pi + \nu + \omega + i\eta - \xi_{q-k}} \tanh\left(\frac{\xi_{q-k}}{2T}\right) 
- \int \frac{d^d q}{(2\pi)^d} \int \frac{dv}{2\pi} \Im \epsilon^{-1}(q, \nu) \frac{1}{2\pi + \nu + \omega + i\eta - \xi_{q-k}} \tanh\left(\frac{\xi_{q-k}}{2T}\right) 
$$

$$
- \int \frac{d^d q}{(2\pi)^d} \int \frac{dv}{2\pi} \Im \epsilon^{-1}(q, \nu) \frac{1}{2\pi + \nu + \omega + i\eta - \xi_{q-k}} \tanh\left(\frac{\xi_{q-k}}{2T}\right) \left[ n_F(\xi_{q-k}) + n_B(\nu) \right]. \quad (43)
$$

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