Renormalon Singularities of the QCD Vacuum Polarization Function to Leading Order in $1/N_f$

C.N. Lovett-Turner and C.J. Maxwell

Centre for Particle Theory, University of Durham
South Road, Durham, DH1 3LE, England

We explicitly determine the residues and orders of all the ultra-violet (UV) and infra-red (IR) renormalon poles in the Borel plane for the QCD vacuum polarization function (Adler D-function), to leading order in an expansion in the number of quark flavours, $N_f$. The singularity structure is precisely as anticipated on general grounds. In particular, the leading IR renormalon is absent, in agreement with operator product expansion ideas. There is a curious and unexplained symmetry between the third and higher UV and IR renormalon residues. We are able to sum up separately UV and IR contributions to obtain closed form results involving $\zeta$-functions. We argue that the leading UV renormalon should have a more complicated structure than conventionally assumed. The disappearance of IR renormalons in flavour-saturated SU($N$) QCD is shown to occur for $N = 3, 6$ or $9$. 

1
1 Introduction

The topic of the large-order behaviour of the expansion coefficients in perturbative field theory calculations continues to be vigorously pursued in the literature [1–9]. Notwithstanding the importance of establishing to what extent physical observables can be formally reconstructed from their divergent perturbative expansions, there is the more pragmatic motivation of the need to assess the reliability of the growing number of higher-order perturbative calculations in QCD and QED.

In this paper we wish to make use of recent progress [6, 8] in exact all-orders QED calculations to leading order in the $1/N_f$ expansion, with $N_f$ the number of fermions, to explicitly determine the singularity structure of the perturbative QCD vacuum polarization function in the Borel plane. On very general grounds one anticipates branch point singularities evenly spaced along the positive and negative real axis in the Borel variable $[\bar{\beta}]$. Those on the positive axis, referred to as infra-red (IR) renormalons, are supposedly correlated with the absence from the formal perturbation series of infra-red non-perturbative effects, vacuum condensates, present in the operator product expansion (OPE). They are responsible for fixed-sign factorial growth of the series coefficients and represent a genuine ambiguity in reconstructing the physical observable from the formal perturbation series. Those on the negative axis, so-called ultra-violet (UV) renormalons, correspond to alternating-sign factorial growth of the series coefficients and do not prevent the reconstruction of the observable by Borel summation.

Whilst the above singularity structure is well-motivated theoretically, there have been various problematic issues. In particular the connection with the OPE suggests that the leading IR renormalon should be absent for the case of the QCD vacuum polarization function since there is no relevant operator of dimension two; the first contribution being the gluon condensate of dimension four. This conclusion has been questioned on various grounds by several authors [10, 11].

The leading asymptotic growth of the perturbative coefficients will be determined by the Borel plane singularity nearest the origin; for the case of the QCD vacuum polarization function this is the first UV renormalon. We point out that the conventionally expected structure of this singularity, with a single branch point exponent, would enable one to obtain the asymptotic growth of the coefficients to all orders in the $1/N_f$ expansion given an exact
large-$N_f$ result. We suggest that this is unlikely and indicate a more complicated structure for the first UV renormalon in accordance with recent results of Vainshtein and Zakharov obtained using their “UV renormalon calculus” [7].

We shall show that the actual singularity structure of the QCD vacuum polarization function is precisely as expected; in particular the leading IR renormalon singularity is indeed absent. This has also been noted for the singularities in the QED vacuum polarization function in reference [8]. We further demonstrate that there is an unexpected symmetry between the third and higher UV and IR singularities. We are able to sum up the UV and IR contributions separately to obtain a closed form result involving $\zeta$-functions.

We finally show that in SU(3) QCD, with $N_f = 15$ or 16, the IR renormalon singularities are absent [12, 13]; and that they first vanish when the instanton/anti-instanton singularity becomes leading. The requirement that this happens for an SU($N$) theory uniquely selects $N = 3$.

The organisation of the paper is as follows. In section 2 we shall introduce the Adler D-function and define its $1/N_f$ expansion. We shall also discuss the exact large-$N_f$ QCD result for its coefficients [9]. In section 3 we define the Borel plane and review the anticipated singularity structure. We discuss reasons for which one may expect a more complicated singularity structure for the first UV renormalon and make contact with the results of reference [7]. In section 4 we shall explicitly determine the Borel plane singularity structure for the exact large-$N_f$ result for D; and show that the anticipated UV and IR renormalon poles are present. In section 5 we discuss the vanishing of the IR renormalons for certain values of $N_f$ in SU($N$) QCD. Section 6 contains our conclusions.

2 The Adler D-function and the $1/N_f$ Expansion

We shall be interested in the SU($N$) QCD vacuum polarization function with $N_f$ flavours of massless quarks,

$$\Pi(-q^2)(q_\mu q_\nu - g_{\mu\nu}q^2) = 16\pi^2 i \int d^4x e^{iq\cdot x}\langle 0| T\{ J_\mu(x) J_\nu(0) \}|0 \rangle . \quad (1)$$
To avoid an unspecified constant we shall actually focus on the related Adler D-function,

\[ D(Q^2) = -\frac{3}{4} Q^2 \frac{d}{dQ^2} \Pi(Q^2) \quad (2) \]

where \( Q^2 = -q^2 \) is the spacelike Euclidean squared momentum transfer. This quantity is related to the experimentally-relevant R-ratio in \( e^+e^- \) annihilation,

\[ R = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} \quad (3) \]

where, taking \( s \) to be the physical timelike Minkowski squared momentum transfer, \( R \) and \( D \) are related by the dispersion relation

\[ R(s) = \frac{1}{2\pi i} \int_{-s-i\epsilon}^{-s+i\epsilon} dQ^2 \frac{D(Q^2)}{Q^2}. \quad (4) \]

In QCD perturbation theory we have

\[ D(Q^2) = d(R) \sum_f Q_f^2 \left(1 + \frac{3}{4} C_F \tilde{D}\right) + \left(\sum_f Q_f\right)^2 \tilde{D} \quad (5) \]

where \( Q_f \) denotes the electric charge of the quarks and the summation is over the flavours accessible at a given energy. \( d(R) \) is the dimension of the quark representation of the colour group (here \( d(R) = N \)). We define the SU\((N)\) Casimirs \( C_A = N, C_F = (N^2 - 1)/2N \).

The correction to the parton model result has the perturbative expansion

\[ \tilde{D} = a + d_1 a^2 + d_2 a^3 + \ldots + d_k a^{k+1} + \ldots, \quad (6) \]

with ‘\( a \)’ the renormalization group (RG) improved coupling \( \alpha_s(\mu^2)/\pi \). The \( \tilde{D} \) contribution first enters at \( O(a^3) \) due to the existence of diagrams of the “light-by-light” type. Our interest here is in the asymptotic growth of the \( d_k \) coefficients in large orders.

The RG-improved coupling \( a(\mu^2) \) will evolve with renormalization scale \( \mu^2 \) according to the beta-function equation

\[ \frac{da}{d\ln \mu} = -b a^2 (1 + ca + c_2 a^2 + \ldots) \quad (7) \]
where $b$ and $c$ are universal with $[14]$

$$
\begin{align*}
  b &= \frac{(11C_A - 2N_f)}{6}, \\
  c &= \left[ -\frac{7}{8} C_A^2 b - \frac{11}{8} C_A C_F + \frac{5}{4} C_A + \frac{3}{4} C_F \right],
\end{align*}
$$

(8)

$c_2$ and higher coefficients are renormalization scheme (RS) dependent. We shall usually consider the $\overline{\text{MS}}$ scheme with $\mu^2 = Q^2$. For the R-ratio there is an analogous expansion for the quantity $\tilde{R}$ with perturbative coefficients $r_k$, defined as in equations (5) and (6). The dispersion relation (4) means that the $r_k$ are directly related to the $d_k$. For instance $r_1 = d_1$ and $r_2 = d_2 - \pi^2 b^2/12$. The $\pi^2$ terms arise due to analytic continuation. Given knowledge of the asymptotic growth of the $d_k$ one can obtain that of the $r_k$ using equation (4).

We shall continue to focus on the $d_k$ for the moment.

The coefficients $d_k$ will be polynomials of degree $k$ in $N_f$:

$$
d_k = d_k^{[k]} N_f^k + d_k^{[k-1]} N_f^{k-1} + \ldots + d_k^{[0]},
$$

(9)

where each term is a sum of multinomials in $C_A$, $C_F$ and $N_f$ of degree $k$ so that $d_k^{[r]}$ has the structure $C_A^{k-r-s} C_F^s$ (note the prefactor of $C_F$ in equation (5)). The first two coefficients $d_1$ and $d_2$ have been computed $[15]$ and the result using the $\overline{\text{MS}}$ scheme with $\mu^2 = Q^2$, expanded in $N_f$ as in equation (9), is

$$
\begin{align*}
  d_1 &= \left( -\frac{11}{12} + \frac{2}{3} \zeta_3 \right) N_f + C_A \left( \frac{41}{8} - \frac{11}{3} \zeta_3 \right) - \frac{1}{8} C_F, \\
  d_2 &= \left( \frac{151}{162} - \frac{19}{27} \zeta_3 \right) N_f^2 + C_A \left( -\frac{970}{81} + \frac{224}{27} \zeta_3 + \frac{5}{9} \zeta_5 \right) N_f \\
    &+ C_F \left( -\frac{29}{96} + \frac{19}{6} \zeta_3 - \frac{10}{3} \zeta_5 \right) N_f + C_A^2 \left( \frac{90445}{2592} - \frac{2737}{108} \zeta_3 - \frac{55}{18} \zeta_5 \right) \\
    &+ C_A C_F \left( -\frac{127}{48} + \frac{143}{12} \zeta_3 + \frac{55}{3} \zeta_5 \right) + C_F^2 \left( -\frac{23}{32} \right) .
\end{align*}
$$

(10)

Here $\zeta_p$ denotes the Riemann zeta function,

$$
\zeta_p \equiv \sum_{l=1}^{\infty} l^{-p} .
$$

(11)
1/$N_f$ expansions as in equation (9) have been widely used in the past in the investigation of large-order behaviour and renormalons [6]. As we shall emphasise, it is actually more useful for these purposes to consider an expansion in powers of $b$. We can write
\[ d_k = d_k^{(k)} b^k + d_k^{(k-1)} b^{k-1} + \ldots + d_k^{(0)} . \] (12)
The leading coefficient in the $1/N_f$ expansion is exactly related to that in the “1/b expansion” with
\[ d_k^{[k]} = (-1/3)^k d_k^{(k)} . \] (13)
As before, we can write out the known $d_1$ and $d_2$ coefficients now expanded according to equation (12):
\[
\begin{align*}
d_1 &= \left( \frac{11}{4} - 2\zeta_3 \right) b + \frac{C_A}{12} - \frac{C_F}{8} \\
d_2 &= \left( \frac{151}{18} - \frac{19}{3} \zeta_3 \right) b^2 + C_A \left( \frac{31}{6} - \frac{5}{3} \zeta_3 - \frac{5}{3} \zeta_5 \right) b \\
&\quad + C_F \left( \frac{29}{32} - \frac{19}{2} \zeta_3 + 10\zeta_5 \right) b + C_A^2 \left( -\frac{799}{288} - \zeta_3 \right) \\
&\quad + C_A C_F \left( -\frac{827}{192} + \frac{11}{2} \zeta_3 \right) + C_F^2 \left( -\frac{23}{32} \right).
\end{align*}
\] (14)
We note in passing that the “1/b expansion” has a somewhat more compact structure than the $1/N_f$. In particular the $\zeta_3$ terms in $d_1$, which are present in all orders of the $1/N_f$ expansion, are now present only in the leading term in the “1/b expansion”; and the $\zeta_5$ terms present in all but the leading coefficient in the $1/N_f$ expansion are now present only in the $d_2^{(1)}$ coefficient in the “1/b expansion”.

Continuing progress in applying the $1/N_f$ expansion in QED [10] has led Broadhurst to an elegant generating function for the leading order (large-$N_f$) coefficients of the QED Gell-Mann–Low function (MOM scheme beta-function) [8].
\[
\Psi_n^{[n]} = \frac{3^{2-n}}{2} \left( \frac{d}{dx} \right)^{n-2} P(x) \bigg|_{x=1} \] (15)
where
\[
P(x) = \frac{32}{3(1+x)} \sum_{k=2}^{\infty} \frac{(-1)^k k}{(k^2 - x^2)^2} . \] (16)
\(\Psi^{[n]}\) can be explicitly evaluated in closed form:

\[
\frac{\Psi^{[n]}}{(n-2)!} = \frac{(n-1)}{(-3)^{n-1}} \left[ -2n + 4 - \frac{n+4}{2^n} \right] + \frac{16}{n-1} \sum_{s > 0} s(1 - 2^{-2s})(1 - 2^{2s-n})\zeta_{2s+1}.
\]

Using this result one can then obtain the leading-order large-\(N_f\) result for the QCD Adler D-function. In the \(\overline{\text{MS}}\) scheme with \(\mu^2 = Q^2\) one has:

\[
d^{[k]}_k = 2T_f k! \int_{0}^{\infty} \frac{(-5/9)^m}{m!} \frac{\Psi^{[k+2-m]}_{k+2-m}}{(k-m)!},
\]

where \(T_f\) is a group theory factor; \(T_f = 1/2\) for the standard fermion representation. The \((-5/9)^m\) factors enter since one is converting from the MOM scheme Adler function to that in the \(\overline{\text{MS}}\) scheme. The results of (18) and (17) are in agreement with the exactly known coefficients \(d_1^{[1]}\) and \(d_2^{[2]}\) in equation (10).

Our aim is to make use of the exact large-\(N_f\) result of equation (18) to obtain as much information as possible about the singularity structure of the QCD D-function in the Borel plane; and hence about the large-order behaviour of its perturbative coefficients. To this end we shall begin by reviewing what can be inferred on very general grounds about this structure; and then we shall compare the exact result with these expectations.

3 The Borel Plane and Renormalons

Consider a general quantity \(D\), calculated in perturbative field theory with coupling \(a\),

\[
D = (a + d_1 a^2 + d_2 a^3 + \ldots + d_k a^{k+1} + \ldots)
\]

Using the integral representation of \(k!\):

\[
k! = \int_{0}^{\infty} dt \, e^{-t} t^k
\]

we can formally write

\[
D = \sum_{m=0}^{\infty} a^{m+1} \frac{d_m}{m!} \int_{0}^{\infty} dt \, e^{-t} t^m
\]
with $d_0 = 1$. Then exchanging the order of summation and integration

\[
D = \int_0^\infty \! dt \, a e^{-t} \sum_{m=0}^\infty (ta)^m \frac{d_m}{m!}
\]

\[
= \int_0^\infty \! dz \, e^{-z/a} \sum_{m=0}^\infty \frac{z^m d_m}{m!}
\]

\[
= \int_0^\infty \! dz \, e^{-z/a} B[D](z)
\]

(22)

where we have set $z = ta$. $B[D](z)$ is called the Borel transform of $D(a)$ and is defined by

\[
B[D](z) \equiv \sum_{m=0}^\infty \frac{z^m d_m}{m!}.
\]

(23)

The idea will be that even if the coefficients $d_k$ grow like $k!$, as is expected in field theory, the Borel transform defined in equation (23) may still be defined and hence the integral representation of equation (22), the Borel sum, may exist.

As an example suppose that $d_k = (-1)^k k!$; then $B[D](z) = 1 - z + z^2 - z^3 + \ldots = 1/(1 + z)$, where we have assumed analytic continuation along the whole real line. More generally, if $d_k = (1/\gamma z_i)^k \gamma k!$ ($\gamma > 0$), then $B[D](z)$ has a singularity proportional to $(z - z_i)^{-\gamma - 1}$; so if $\gamma$ is a positive integer we have a pole; and for non-integer $\gamma$ a branch point in the $z$-plane (Borel plane) at $z = z_i$. If all the singularities are located off the positive $z$-axis it may be possible, if certain analyticity properties of $D$ are satisfied, to reconstruct $D(a)$ from its formal divergent series using the Borel sum of equation (22). This, however, is not our interest here. We want to use the $z$-plane singularities to encode the large-order behaviour of the perturbative coefficients.

Returning to the specific example of the Adler D-function in QCD, let us now ask on rather general grounds where the singularities $z_i$ could be located. In the large-$N_f$ limit we know that, if the $d_k$ have factorial growth, asymptotically we must have $d_k \approx N_f^k k!$. Similarly we can consider a large-$N$ limit where necessarily $d_k \approx N^k k!$. This follows since the $k^{th}$ order Feynman diagrams for $d_k$ necessarily have factors which are multinomials in $N_f$, $C_A$ and $C_F$ of degree $k$. We therefore have singularities at positions $z_i \sim 1/N_f$ in the large-$N_f$ limit and $z_i \sim 1/N$ in the large-$N$ limit. If singularities are present and visible in both limits then the simplest possibility
is \( z_i \sim 1/(AN + BN_f) \) involving some unspecified linear combination of \( N \) and \( N_f \). These are the renormalon singularities and in fact they lie at \( z_k = 2k/b \) where \( k = \pm 1, \pm 2, \pm 3, \ldots \). There are also singularities due to instanton/anti-instanton solutions of the classical equations of motion \([2, 17, 18]\). These lie at \( z_k = 4k, k = 1, 2, 3, \ldots \). Since their positions are independent of \( N \) and \( N_f \), the above arguments suggest that they are invisible in the large-\( N \) and large-\( N_f \) limits. In fact they are invisible at all orders of the \( 1/N \) and \( 1/N_f \) expansions, so we shall not learn about them from the exact large-\( N_f \) result.

We can motivate the \( 2/b \) spacing of the renormalons by considering the connection between the IR renormalons and the absence from the formal perturbation theory of non-perturbative vacuum condensates present in the OPE. Performing a short distance OPE on the time-ordered product of currents in equation (1) one has

\[
\tilde{D}(Q^2) = C_{PT}(Q^2/\mu^2, a(\mu^2)) + C_{GG}(Q^2/\mu^2, a(\mu^2)) \frac{\langle 0|GG|0\rangle(\mu^2)}{Q^4} + O(Q^{-6}),
\]

(24)

where \( C_{PT} \) corresponds to the perturbation series of equation (6). There are then higher terms consisting of perturbatively calculable coefficient functions multiplied by non-perturbative vacuum condensates of increasing mass dimension. The lowest such is the gluon condensate with dimension 4 and a \( Q^{-4} \) scaling behaviour. There will also be dimension 6, 8, \ldots condensates corresponding to \( Q^{-6}, Q^{-8}, \ldots \) behaviour. Crucially, there is no relevant dimension 2 operator and hence no condensate with \( Q^{-2} \) behaviour. These condensates are supposedly associated with the IR renormalon singularities at \( z = 4/b, 6/b, \ldots \). The connection follows from a consideration of the \( Q^2 \) behaviour of the large-order perturbative coefficients \( d_k \). Suppose that, for \( \mu^2 = Q^2 \), \( d_k \) behaves asymptotically as

\[
d_k \approx A \left( \frac{1}{z_i} \right)^k k!(
\),

(25)

where \( A \) is a constant and the bracket denotes possible additional powers or logarithms of \( k \). Then, assuming a trivial beta-function with one term (\( c=0 \)), the renormalization group fixes the asymptotic behaviour for general \( \mu^2 \),
\[ d_k \approx A \left( \frac{\mu^2}{Q^2} \right)^{\frac{b\gamma_i}{2}} \left( \frac{1}{z_i} \right)^k \Gamma(k) . \] (26)

So such a singularity at \( z = z_i \) corresponds to \( Q^{-b\gamma_i} \) scaling behaviour. To reproduce the \( Q^{-4}, Q^{-6}, \ldots \) behaviour of the OPE condensates one then requires Borel plane singularities at \( z = 4/b, 6/b, \ldots \), the IR renormalons.

In QED, one-loop vacuum polarization diagrams with a chain of vacuum polarization bubbles inserted lead to fixed-sign factorial growth and UV renormalon singularities \[^7\]. These are associated with the Landau pole in QED. For the QCD vacuum polarization function one can also consider one-loop diagrams with a single gluon line inserted. Applying a cut-off on the momentum of this line, inserting the running QCD coupling and integrating over the high-momentum region, Mueller \[^9\] has shown that one can explicitly derive the form of the leading UV renormalon at \( z = -2/b \); and, by integrating over momenta less than the cut-off, of the leading IR renormalon at \( z = 4/b \).

The final conclusion is that one expects UV renormalon singularities, UV\(_\ell\), at \( z = z_\ell = -2\ell/b, \ell = 1, 2, 3, \ldots; \) and IR renormalon singularities, IR\(_\ell\), at \( z = z_\ell = 2\ell/b, \ell = 1, 2, 3, \ldots \). For the specific case of QCD vacuum polarization one expects IR\(_1\) to be absent since, as discussed, there is no dimension two condensate.

These singularities at \( z = z_\ell \) in the Borel plane should be branch points of the form \[^9\]

\[ B[\tilde{D}](z) = \frac{A_0 + A_1 (1 - z/z_\ell) + O((1 - z/z_\ell)^2)}{(1 - z/z_\ell)^{p + cz_\ell}}, \] (27)

where \( p \) is a positive integer. The exponent \( cz_\ell \) is fixed by the renormalization group in order to give the required \( Q^2 \) scaling, taking into account a realistic two-term beta-function with \( c \neq 0 \). UV\(_1\) should be the singularity closest to the origin (assuming that IR\(_1\) is absent) and so will give the overall asymptotic large-order behaviour of the \( d_k \) coefficients. Corresponding to a branch point in the Borel plane of the form of equation (27), one has the large-order behaviour

\[ d_k = \frac{A_0}{\Gamma(p + cz_\ell)} \left( \frac{1}{z_\ell} \right)^k \Gamma(k + p + cz_\ell) . \] (28)
So for UV$_1$ we have the leading behaviour

\[
    d_k \approx \frac{A_0}{\Gamma(p - 2c/b)} \left( -\frac{b}{2} \right)^k k^{p-1} k^{-2c/b} k! (1 + O(1/k)),
\]

where we have expanded the second $\Gamma$-function of equation (28) for large $k$. The $A_1$ and higher terms in the numerator Taylor series about $z = z_\ell$ in equation (27) are $O(1/k)$ sub-asymptotic effects. The explicit diagrammatic evaluations \[3, 4, 19\] give $A_0 = \frac{4}{9} \Gamma(2 - 2c/b)$, $p = 2$ for UV$_1$, in the MOM scheme with $\mu^2 = Q^2$.

To make contact with the $1/N_f$ and $1/b$ expansions of equations (9) and (12) we note that from (8)

\[
    \frac{c}{b} = \left[ -\frac{7}{8} \frac{C_A^2}{b^2} + \frac{5}{4} \frac{C_A}{b} - \frac{11}{8} \frac{C_A C_F}{b^2} + \frac{3}{4} \frac{C_F}{b} \right].
\]

Then writing $k^{-2c/b}$ as $e^{-2c \ln k/b}$ and expanding we have

\[
    d_k \approx \frac{A_0}{\Gamma(p - 2c/b)} \left( -\frac{b}{2} \right)^k k^{p-1} k! \left[ 1 - \frac{5}{2} \frac{C_A}{b} \ln k - \frac{3}{2} \frac{C_F}{b} \ln k \right]
    + \left( \frac{25}{8} \ln^2 k + \frac{7}{4} \ln k \right) \frac{C_A^2}{b^2} + \ldots \].
\]

We shall define

\[
    \frac{A_0}{\Gamma(p - 2c/b)} = A_{01} + \frac{A_{02}}{b} + \frac{A_{03}}{b^2} + \ldots,
\]

where $A_{0r}$ is a sum of multinomials in $C_A$ and $C_F$ of degree $r - 1$ and $A_{01}$ is a pure number. One can then obtain the asymptotic behaviour of the coefficients to all orders in the $1/N_f$ and $1/b$ expansions. The $1/b$ expansion corresponds to the successive terms in equation (31). So we have

\[
    d_k^{(k)} \approx A_{01} \left( -\frac{1}{2} \right)^k k^{p-1} k!
    d_k^{(k-1)} \approx A_{01} \left( -\frac{1}{2} \right)^k k^{p-1} k! \left( -\frac{5}{2} C_A \ln k - \frac{3}{2} C_F \ln k \right),
\]

\[
    \vdots
\]

(33)
For the $1/N_f$ expansion asymptotics the leading terms will come from the binomial expansion of $b^k = \left( \frac{11CA-2N_f}{6} \right)^k$,

$$d_k^{[k-r]} \approx A_{01} \frac{6^{-k}}{r!} \left( -\frac{11}{2} C_A \right)^r k^{p+r-1} k!$$

(34)

where $r = 0, 1, 2, \ldots$. Note that an additional factor of $k$ is gained for each additional order in the $1/N_f$ expansion. The leading term in the large-$N$ expansion, $d_k^{[0]}$, is given by

$$d_k^{[0]} \approx \tilde{A}_{01} 12^{-k} (-11C_A)^{k} k^{p-1} k^{-102/121} k!$$

(35)

where $\tilde{A}_{01}$ is the large-$N$ limit of equation (32). Of course $\tilde{A}_{01}$ cannot be determined from the large-$N_f$ result.

Assuming a UV$_1$ singularity as in equation (27) we therefore apparently have the remarkable conclusion that, given an exact large-$N_f$ (large-$b$) result for $d_k^{[k]}$ ($d_k^{[k]}$), we can obtain $A_{01}$ and $p$; and hence the asymptotics (up to an $O(1/k)$ correction) of the coefficients to all orders in the $1/N_f$ (1/b) expansion are determined by equations (33) and (34). We shall check in due course that the exact large-$N_f$ result of equation (18) gives $A_{01} = 4/9$ and $p = 2$ (MOM scheme, $\mu^2 = Q^2$) in agreement with other evaluations [6, 7, 19].

The conclusion that the large-$N_f$ result can determine the full asymptotics beyond leading order in $1/N_f$ seems too good to be true; and indeed recent work by Vainshtein and Zakharov [4] casts doubt on it. These authors have systematically developed a “UV renormalon calculus” in which the leading ultra-violet behaviour of loop diagrams with different numbers of chains of vacuum polarization graphs inserted is extracted using an operator product expansion. Including one chain they reproduce a UV$_1$ result of the form of equation (27) in agreement with the result quoted above. Explicitly evaluating the two-chain (three-loop) result in a simplified U(1) model they find a contribution to the UV renormalon asymptotics which is $1/N_f$ down on the one-chain result; but has additional powers of $k$ and so dominates the one-chain result. Conventional wisdom would have expected additional chains to have a suppression by powers of $k$ and hence not to contribute to the $A_0$ coefficient in equation (27). Combinatorial factors conspire to modify this, however.
The UV renormalon calculus results \cite{7} imply that the Borel plane singularity at \( z = -2/b \) in equation (27) should be modified to

\[
B[\tilde{D}](z) = \sum_{m=1}^{\infty} \frac{A_0^{(m)} + A_1^{(m)}(1 + bz/2) + O((1 + bz/2)^2)}{(1 + bz/2)^{\gamma_m - 2c/b}} \tag{36}
\]

where \( m \) labels the number of vacuum polarization chains, \( m = 1 \) corresponding to the previous one-chain results. Counting powers of \( N_f \), one expects each additional chain to give a \( 1/N_f \) suppression and hence equation (32) should generalise to

\[
\frac{A_0^{(m)}}{\Gamma(\gamma_m - 2c/b)} \approx \frac{A_0^{(m)}}{b^{m-1}} + \frac{A_0^{(m)}}{b^{m}} + \ldots \tag{37}
\]

The exponent \( \gamma_m \) will be related to the anomalous dimensions of the operators appearing in the operator product expansion of reference \cite{7}. There will be a number of operators and hence a number of contributions to equation (36) for any \( m \). Each such \( \gamma_m \) will have a \( 1/N_f, 1/b \) expansion and we can define the leading term \( p_m \), typically a positive integer. We select for each \( m \) the \( \gamma_m \) contribution with largest \( p_m \); and this is the single term displayed in equation (36). Providing that \( p_{m+1} > p_m + 1 \) for any \( m \), then (36) and (37) lead to the \( 1/N_f, 1/b \) expansion asymptotics

\[
d_{k}^{(k-r)} \approx A_0^{(r+1)} \left( \frac{-3}{6^k} \right) k^{p_{r+1}-1} k! \\
d_{k}^{(k-r)} \approx A_0^{(r+1)} \left( \frac{-1}{2^{k}} \right) k^{p_{r+1}-1} k! \tag{38}
\]

with \( r = 0, 1, 2, \ldots \). The \( A_0^{(r+1)} \) will consist of multinomials in \( C_A, C_F \) of degree \( r \). The inequalities on \( p_m \) are required since, from equation (34), we see that, at fixed \( m \), one gains an extra factor of \( k \) for each additional \( 1/N_f \) order. The results of (38) imply that one does not get something for nothing after all; but requires \( O((1/N_f)^r) \) results (diagrams with \( r + 1 \) chains) to obtain the leading asymptotics to this order in the \( 1/N_f \) expansion.

Having discussed the general expectations for the Borel plane singularity structure we return to the exact large-\( N_f \) result of equation (18) and exhibit what its singularity structure actually is.
4 Borel Plane Singularities of the Exact Large-$N_f$ Result

Returning to equation (15) we can use Taylor’s theorem to write

$$\Psi_n = \frac{3^{2-n}}{2} \cdot (n-2)! \cdot \Phi_{n-2} \cdot \hat{P}(u)$$

where $\Phi_n F(x)$ denotes the coefficient of $x^n$ in the expansion of $F(x)$ as a power series.

$$\hat{P}(u) = \frac{32}{3(2+u)} \sum_{k=2}^{\infty} \frac{(-1)^k k}{(k^2 - (1+u)^2)^2}$$

which is obtained from equation (16) by writing $u = x - 1$. Using partial fractions on equation (40) and isolating the required coefficient one finds the rather simple result

$$\frac{\Psi_n}{(n-2)!} = \frac{6n-1}{3^{n+1}} - \frac{6n+1}{6^{n+1}} + 12 \sum_{\ell=3}^{\infty} (-1)^{\ell+1} \left\{ n \left( \frac{1}{\ell + 1} - \frac{1}{\ell + 2} \right) \right. \\
+ \left. \left( - \frac{1}{(\ell + 1)^2} + \frac{2}{(\ell + 2)^2} \right) \right\} \left( \frac{1}{3\ell} \right)^n \\
- \frac{3}{(-6)^{n-1}} + 12 \sum_{\ell=3}^{\infty} (-1)^{\ell+1} \left\{ n \left( \frac{1}{\ell - 1} - \frac{1}{\ell - 2} \right) \right. \\
+ \left. \left( - \frac{2}{(\ell - 1)^2} - \frac{2}{(\ell - 2)^2} \right) \right\} \left( \frac{-1}{3\ell} \right)^n. \tag{41}$$

Putting in factors of $T^n_f N^n_f = (1/2)^n N^n_f$ and replacing $N_f$ by $(-3b)$, we see that successive terms in equation (41) are proportional to $(-b/2)^n, (-b/4)^n, (-b/2\ell)^n, (b/4)^n$ and $(b/2\ell)^n$ corresponding exactly to the poles UV$_1$, UV$_2$, UV$_{\ell}$, IR$_2$ and IR$_{\ell}$ respectively in the Borel plane. There is no term involving $(1/(-3))$ and so IR$_1$ is indeed absent as anticipated. The linear factor proportional to $n$ in all but IR$_2$ means that all the poles are double poles ($p = 2$) except for IR$_2$ which is a simple pole ($p = 1$). Notice that the branch point exponent $cz_{\ell}$ is sub-leading in the $1/N_f, 1/b$ expansion and so one sees poles and not branch points in the large-$N_f$ limit.
Using equation (18) we can then obtain the leading-$b$ result for $d_k$ (MS scheme, $\mu^2 = Q^2$):

\[
d_k^{(k)} b^k = k! \left\{ \sum_{\ell=1}^{\infty} [A_0(\ell) k + A_1(\ell)] \left( -\frac{b}{2\ell} \right)^k \right. \\
+ \left. \sum_{\ell=1}^{\infty} [B_0(\ell) k + B_1(\ell)] \left( \frac{b}{2\ell} \right)^k \right\}
\]

(42)

where

\[
A_0(1) = \frac{4}{9} e^{-5/3}; \quad A_1(1) = \frac{14}{9} e^{-5/3}; \\
A_0(2) = -\frac{1}{3} e^{-10/3}; \quad A_1(2) = -\frac{11}{6} e^{-10/3}; \\
A_0(\ell)(\ell \geq 3) = \frac{8}{3} \frac{1}{\ell^2} \left( \frac{1}{\ell+1} + \frac{1}{\ell+2} \right) e^{-5\ell/3}; \\
A_1(\ell)(\ell \geq 3) = \frac{8}{3} \frac{1}{\ell^2} \left[ \left( 2 + \frac{5\ell}{3} \right) \left( \frac{1}{\ell+1} + \frac{1}{\ell+2} \right) \\
+ \left( -\frac{1}{(\ell+1)^2} + \frac{2}{(\ell+2)^2} \right) \right] e^{-5\ell/3}; \\
B_0(1) = B_1(1) = 0; \\
B_0(2) = 0; \quad B_1(2) = e^{10/3} \\
B_0(\ell) = -A_0(-\ell); \quad B_1(\ell) = -A_1(-\ell) \quad (\ell \geq 3).
\]

(43)

To obtain the result in the MOM scheme with $\mu^2 = Q^2$ one simply omits the exponential factors in equations (43). The first sum in equation (42) generates UV$_\ell$ singularities of the form of equation (27) and the second sum the IR$_\ell$ singularities. IR$_1$ is absent ($B_0(1) = B_1(1) = 0$) as required from the OPE; and all poles are double ($p = 2$) except IR$_2$ for which $B_0(2) = 0$ giving a simple pole ($p = 1$). For $\ell \geq 3$ there is a curious and unexplained symmetry between the residues of UV$_\ell$ and IR$_\ell$ with $A(\ell) = -B(-\ell)$. As promised the coefficient $A_0(1) = \frac{4}{9} e^{-5/3}$ is in agreement with other calculations of UV$_1$ [6, 7, 19]. The coefficient $B_1(2) = e^{10/3}$ is consistent with the result of [19] for the first IR renormalon, IR$_3$.

Since equation (41) is split into contributions from UV and IR renormalons we can sum up these contributions separately. Summing the first
three UV terms in (41) one obtains

\[
\Psi[n]_{\text{UV}} = \frac{2(\zeta_2 - 2)}{(-3)^{n-1}} - \frac{2\zeta_2 - 3}{(-6)^{n-1}} + \frac{4}{(-3)^{n-1}} \sum_{m=1}^{n-1} (-1)^m m(1 - 2^{-m})(1 - 2^{m-n})\zeta_{m+1},
\]

(44)

and summing the final two IR terms gives

\[
\Psi[n]_{\text{IR}} = \frac{2(n^2 - 3n + 4 - \zeta_2)}{(-3)^{n-1}} - \frac{\frac{1}{2}(n^2 + 3n + 2 - 4\zeta_2)}{(-6)^{n-1}} + \frac{4}{(-3)^{n-1}} \sum_{m=1}^{n-1} m(1 - 2^{-m})(1 - 2^{m-n})\zeta_{m+1}.
\]

(45)

The UV and IR pieces separately contain even and odd \(\zeta\)-functions but the even \(\zeta\)-functions cancel in the sum of (44) and (45) to reproduce equation (17) which contains only odd \(\zeta\)-functions.

We finally discuss the connection between the Borel plane singularity structure of \(\tilde{D}\), which we have discussed extensively, and that of the more experimentally-relevant \(\tilde{R}\), related to it by the dispersion relation of equation (4). It is easy to show that in the large-\(b\) limit \((c = 0)\) \[11\]

\[
B[\tilde{R}](z) = \frac{\sin(\pi bz/2)}{\pi bz/2} B[\tilde{D}](z)
\]

(46)

Since \(\sin(\pi bz/2)\) has single zeros \(\sim (z - z_\ell)\) at the same positions as the renormalon singularities one finds that renormalon poles of order \(p\) in \(B[\tilde{D}]\) are converted to poles of order \(p - 1\) in \(B[\tilde{R}]\). This implies that the poles in \(B[\tilde{R}]\) are simple poles except for IR\(_2\) which was a simple pole in \(B[\tilde{D}]\) and hence apparently vanishes \[11\]. The absence of the IR\(_1\) singularity at \(z = 2/b\) in \(B[\tilde{D}](z)\) implies from equation (46) that \(B[\tilde{R}](z)\) must have a compensating zero at this position. Brown and Yaffe \[11\] considered this unlikely and hence cast doubt on the absence of IR\(_1\). The exact large-\(N_f\) result shows that there is indeed no IR\(_1\) singularity in \(B[\tilde{D}]\) and hence such a zero is present in \(B[\tilde{R}]\).

The leading asymptotics of the coefficients \(r_k\) will be given by UV\(_1\) for \(B[\tilde{R}](z)\). Expanding around \(z = -2/b\) we have

\[
\frac{\sin(\pi bz/2)}{\pi bz/2} = \left(1 + \frac{bz}{2}\right) + O\left(\left(1 + \frac{bz}{2}\right)^2\right)
\]

(47)
and hence from equation (46) we find that the asymptotic behaviour of $r_k$ is given by changing $p \to p - 1$ in the results for $d_k$ (equations (33) and (34)). Even assuming the more complicated UV$_1$ structure of equation (36) one simply changes $p_m \to p_m - 1$. This implies that on very general grounds one expects

$$\frac{r_k}{d_k} \approx \frac{1}{k}(1 + O(1/k))$$

so that the $r_k$ coefficients grow more slowly asymptotically.

We conclude by noting that exact large-$N_f$ results do exist also for other QCD observables. In particular, reference [9] contains a leading-$N_f$ result for the perturbative coefficients of the radiative corrections to the Gross–Llewellyn-Smith (GLS) sum rule. This reveals that in the Borel plane the GLS corrections have simple poles at $z = \pm 2/b, \pm 4/b$. So the first two UV and IR renormalons are present but the remaining renormalons are absent to leading order in $1/N_f$. It would be interesting to compare this with the OPE structure for these corrections.

5 Vanishing IR Renormalons

The singularity structure in the Borel plane for SU($N$) QCD will change as $N$ and $N_f$ are varied. In particular, if $N_f$ approaches $11N/2$ from below then $b = (11N - 2N_f)/6$ will approach zero from above; and, as more flavours of quark are added, the IR$_\ell$ and UV$_\ell$ singularities, which are spaced at intervals of $2/b$, will move outwards away from the origin in the $z$-plane. The instanton/anti-instanton (I\(\bar{I}\)) singularities remain fixed at $z = 4, 8, 12, \ldots$ independent of $N$ and $N_f$. When $b = 1/2$ ($N_f = 15$ for SU(3)), IR$_1$ will be at the same position as the leading I\(\bar{I}\) singularity at $z = 4$; and for flavour saturation (maximum $N_f$ for which $b > 0$, $N_f = 16$ for SU(3)) $b = 1/6$ and the leading singularity on the positive axis will be the I\(\bar{I}\) at $z = 4$.

For SU(3) QCD a remarkable phenomenon first occurs at $b = 1/2$ ($N_f = 15$). One finds that the branch point exponent of IR$_\ell$, $2\ell/b$ in equation (27), becomes a negative integer and the structure of IR$_\ell$ in the Borel plane is then

$$B[\tilde{D}](z) = A_0 \left(1 - \frac{z}{4\ell}\right)^{88\ell - p}.$$

The IR$_\ell$ singularity disappears provided that $p < 88\ell$. For the particular case of the D-function we have apparently $p = 1$ or 2 and so all of the IR
renormalons disappear at $b = 1/2$ and the only singularities on the positive $z$-axis are those due to instantons. Notice that the UV renormalons are still present but become poles.

As first noted by White [12, 13] the IR renormalon singularities also disappear for $N_f = 16$ flavour-saturated SU(3). The exponent $2c \ell/b$ again becomes a negative integer and the IR singularities disappear for $p < 906 \ell$. The implication is that $N_f = 15$ and 16 SU(3) QCD are very special instanton-dominated theories. At precisely the point when instantons become the leading singularities all the other IR renormalon singularities on the positive $z$-axis vanish. It is interesting to ask if this scenario is unique to SU(3) or can be realised for other values of $N$. $b = 1/2$ will occur for integer $N_f$ only for $N$ odd. For $N$ even the IR singularity becoming leading and flavour saturation are telescoped into the single value $b = 1/3$.

Considering $N$ odd first, we require that $c/b$ is integer for $b = 1/2$;

$$\left( \frac{c}{b} \right)_{b = 1/2} = \frac{(-25N^3 + 13N^2 + 11N - 3)}{4N}.$$  (51)

A necessary condition for this to be an integer is that $3 \equiv 0 \pmod{N}$, which uniquely fixes $N = 3$. Then $(c/b)_{b = 1/2} = -44$.

For $N$ odd and flavour saturation $b = 1/6$;

$$\left( \frac{c}{b} \right)_{b = 1/6} = \frac{(-225N^3 + 39N^2 + 99N - 9)}{4N}.$$  (52)

A necessary condition for this to be an integer is that $9 \equiv 0 \pmod{N}$ so $N = 3$ or 9. The $N = 9$ case gives $(c/b)_{b = 1/6} = -4444$; and, for $N = 3$, $(c/b)_{b = 1/6} = -453$.

For $N$ even and $b = 1/3$

$$\left( \frac{c}{b} \right)_{b = 1/3} = \frac{(-225N^3 + 78N^2 + 99N - 18)}{4N}.$$  (53)
A necessary condition for this to be an integer is that $18 \equiv 0 \pmod{N}$ and so $N = 2, 6$ or $18$. $N = 6$ is the only case for which $c/b$ is an integer and then $(c/b)|_{b=\frac{1}{3}} = -1884$.

So for flavour-saturated SU($N$) IR renormalons are absent only for $N = 3, 6$ or $9$. For SU(9) the $b = 1/2$ case where the $\mathbf{1}$ singularity becomes leading still has IR renormalons.

We conclude that SU(3) QCD is a very special theory from yet another point of view.

6 Conclusions

In this paper we have discussed and sought to extend our knowledge of the Borel plane singularity structure of the Adler D-function (QCD vacuum polarization). This singularity structure succinctly encodes the large-order asymptotic behaviour of the perturbation theory coefficients.

We pointed out that an expansion of the perturbative coefficients in powers of $b$, the first QCD beta-function coefficient, rather than in $N_f$, was natural when comparing with QCD renormalon expectations. We further noted that, if the leading UV renormalon indeed has the expected structure of a simple branch point, then knowledge of the perturbative coefficients to leading order in $1/N_f$, $1/b$ allows the large-order behaviour to all-orders in $1/N_f$, $1/b$ to be inferred, equations (33) and (34). This seems unlikely and a more complicated structure was proposed, equation (36), which is consistent with the UV renormalon calculus results of Vainshtein and Zakharov [7].

Using the exact large-$N_f$ result for the D-function of reference [8] we exhibited the explicit singularity structure in the z-plane and found the expected UV and IR renormalon singularities. They appear as poles in the large-$N_f$ limit. In particular, the first IR renormalon, which would correspond to $Q^{-2}$ behaviour not present in the OPE, is absent. We gave explicit expressions for the residues at all of these poles (equations (43)) and unexpectedly found those for the third and higher UV and IR renormalons to be symmetrically related. We were also able to sum up separately the UV and IR renormalon contributions in closed form (equations (44) and (45)) and obtained expressions containing even and odd $\zeta$-functions. When UV and IR contributions are combined the even $\zeta$-functions cancel.

We finally noted that in flavour-saturated SU($N$) QCD the IR renor-
malons are absent for $N = 3, 6$ and $9$. These theories then have ambiguities dominated by instantons. For SU(3) the IR renormalons first disappear when $N_f = 15$ ($b = 1/2$), at which point the IT singularity is in the same position as IR$_1$ and becomes leading.

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