MINIMALITY OF INTERVAL EXCHANGE TRANSFORMATIONS WITH RESTRICTIONS

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Abstract. It is known since 40 years old paper by M. Keane that minimality is a generic (i.e. holding with probability one) property of an irreducible interval exchange transformation. If one puts some integral linear restrictions on the parameters of the interval exchange transformation, then minimality may become an “exotic” property. We conjecture in this paper that this occurs if and only if the linear restrictions contain a Lagrangian subspace of the first homology of the suspension surface. We partially prove it in the ‘only if’ direction and provide a series of examples to support the converse one. We show that the unique ergodicity remains a generic property if the restrictions on the parameters do not contain a Lagrangian subspace (this result is due to Barak Weiss).

1. Introduction

Interval exchange transformations are maps from an interval to itself that look like suggested by the name: the interval is cut into several subintervals that are “glued back” in a different way. On each subinterval the map has the form of a translation $x \mapsto x + \text{const.}$

The first examples of interval exchange transformations were studied by V. Arnold (see, e.g. [1]) and A. Katok and A. Stepin [19] in early 1960s. The general notion was introduced by V. Oseledets in [33].

Definition 1. An interval exchange transformation is specified by a natural number $n$, a permutation $\pi \in S_n$, and a probability vector $a = (a_1, a_2, \ldots, a_n)$, $\sum_i a_i = 1$, of lengths of the subintervals. The transformation $T_\pi, a$ determined by this data is the map $[0,1) \to [0,1)$ defined by the formula:
$$T_\pi, a(x) = x - x_i + \bar{x}_{\pi^{-1}(i)} \quad \text{if } x \in [x_{i-1}, x_i),$$

where
$$x_0 = \bar{x}_0 = 0, \quad x_i = \sum_{j=1}^i a_j, \quad \bar{x}_i = \sum_{j=1}^i a_{\pi(j)}, \quad i = 1, \ldots, n.$$

We will only be interested in interval exchange transformations $T_\pi, a$ with an irreducible permutation $\pi \in S_n$, i.e. such that no subset of the form $\{1, 2, \ldots, k\}$ with $1 \leq k < n$ is invariant under $\pi$.

Any interval exchange transformation is invertible with the inverse given by $T_{\pi,a}^{-1} = T_{\pi^{-1}, a} \cdot \pi$, where $a \cdot \pi = (a_{\pi(1)}, \ldots, a_{\pi(n)})$.

Definition 2. Let $T$ be an interval exchange transformation. The $T$-orbit of a point $x \in [0,1)$ is the subset $\{T^k(x) \mid k \in \mathbb{Z}\}$.

Interval exchange transformations appear naturally as the first return map on a transversal for singular foliations defined by a closed 1-form on a surface. More precisely, for each interval exchange transformation one can construct a translation surface such that the transformation will be the first return map of the vertical foliation on some horizontal interval (see [27, 39, 45] for details); and vice versa, for each foliation on an oriented surface that is defined by a closed one-form the first return map on any transversal is an interval exchange transformation.

Definition 3. An interval exchange transformation $T$ is called minimal if all $T$-orbits are everywhere dense in $[0,1)$.
The question of finding conditions under which an interval exchange transformation is minimal was posed by M. Keane in [20]. He shows that all interval exchange transformations with irreducible permutation and rationally independent lengths $a_i$ satisfy the following condition, which, in turn, implies that the transformation is minimal.

**Definition 4.** We say that an interval exchange transformation $T_{\pi, a}$ satisfies Keane’s condition if the $T_{\pi, a}$-orbits of the points $x_1 = a_1, x_2 = a_1 + a_2, \ldots, x_{n-1} = \sum_{i=1}^{n-1} a_i$ are pairwise disjoint and infinite.

Moreover, Keane posed a conjecture in [20], which was later proved by H. Masur [26] and W. Veech [37], that almost all irreducible interval exchange transformations are uniquely ergodic.

There are, however, several ways how families of irreducible non-generic interval exchange transformations may arise, in which case the results mentioned above do not apply. In such a family, almost all interval exchange transformations may still be minimal (and even uniquely ergodic), but it may also happen that all or almost all are not. We illustrate this with two simple examples.

**Example 1.** Let $n = 2k$ and

$$(1) \quad \pi = \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 2k-3 & 2k-2 & 2k-1 & 2k \\ 4 & 3 & 6 & 5 & \ldots & 2k & 2k-1 & 2 & 1 \end{pmatrix} \in S_n,$$

$a = (a_1, a_2, a_1, a_2, \ldots, a_1, a_2) \in \mathbb{R}^n$. If $a_1$ and $a_2$ are incommensurable, then $T_{\pi, a}$ is minimal and uniquely ergodic. In the particular case $k = 2$ such interval exchange transformations appear in [31].

**Example 2.** Let $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$ and $a_1 = a_4$. Then for any $x \in [x_2, x_3)$ we have $T_{\pi, a}(x) = x$, so, the transformation is not minimal unless $a_3 = 0$, i.e. $x_2 = x_3$.

In general, the situation can be more complicated. Let $\mathcal{U}$ be a subspace of $\mathbb{R}^n$ defined by a homogeneous system of linear equations with integral coefficients, and $\pi \in S_n$ a fixed permutation. We denote by $\Delta^{n-1}$ the simplex $\{a \in \mathbb{R}^n : a_i \geq 0, \sum_i a_i = 1\}$. The subset $M(\pi, \mathcal{U}) = \{a \in \Delta^{n-1} \cap \mathcal{U} ; T_{\pi, a} \text{ is minimal}\}$

can be a nontrivial part of $\Delta^{n-1} \cap \mathcal{U}$ as the following two examples show.

**Example 3.** Let $\pi$ be as in the previous example and $a$ satisfy the following restriction: $3a_1 = a_2 + a_4$. Then:

1. if $a_2 > a_1$ then for any $x \in [\max(0, a_1 - a_4), \min(a_4, a_2 - a_1))$ we have $T_{\pi, a}^4(x) = x$ (this is a straightforward check), and $T_{\pi, a}$ is not minimal;
2. if $a_2 < a_1$ and $a_1, a_2, a_3$ are linearly independent over $\mathbb{Q}$, then $T_{\pi, a}$ is minimal (this will be shown in Subsection 5.6, Example 5).

see Fig. on the left.

**Example 4.** Let $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 6 & 5 & 2 & 7 & 4 & 1 \end{pmatrix}$ and $a = (a_1, a_2, a_3, a_3, a_1, a_1, a_2), 3a_1 + 2a_2 + 2a_3 = 1$.

Then the set of points $(a_1, a_2)$ for which the interval exchange transformation $T_{\pi, a}$ is minimal forms a fractal set shown in Fig. on the right. Up to a projective transformation this set is the so called Rauzy gasket [24][12][6]. As shown in [5] it has Hausdorff dimension between 1 and 2.

If, in this example, we take the parameter vector $a$ of the form $a = (a_1, a_2, a_3, a_3 - e, a_1, a_1 - e, a_2 + e)$ with $e \neq 0, -a_2 < e < \min(a_1, a_3)$, then the transformation $T_{\pi, a}$ will never be minimal. This will be discussed in more detail in Subsection 5.6.

These two examples demonstrate two possible behaviors of the set minimal interval exchange transformation with integral linear restrictions on parameters. We refer to those behaviors as stable (as in Example 3) and unstable (as in Example 4).

We introduce below an easily checkable condition for a set of integral linear restrictions, and conjecture that the condition is exactly what is responsible for instability of minimal interval exchange transformations with integral linear restrictions on the parameters. The set of restrictions satisfying the condition
is termed rich and otherwise poor. We show that minimal and uniquely ergodic interval exchange transformations that are subject to a poor set of restrictions are always stable.

We also demonstrate several natural sources of families of interval exchange transformations with integral linear restrictions on parameters. These sources include:

1. rank two interval exchange transformations;
2. singular measured foliations on Riemann surfaces defined by a quadratic differential;
3. interval exchange transformations with flips;
4. measured foliations on non-orientable surfaces;
5. Novikov’s problem on plane sections of triply periodic surfaces;
6. systems of partial isometries;
7. interval translation mappings.

In several cases, the set of restrictions appears to be rich. In most such cases, our conjecture about instability of minimal interval exchange transformations translates into a statement that is either known or has been conjectured to be true.

M. Boshernitzan in [8] considered interval exchange transformations $T_{\pi,a}$ with $\dim_{\mathbb{Q}}(a_1,\ldots,a_n) = 2$ (rank two interval exchange transformations) and proposed a way to test such transformations for minimality. A family of rank two interval exchange transformations having the same discrete pattern, in the terminology of [8], is an instance of a family of transformations satisfying a fixed set of integral linear restrictions, in our terminology. Theorem 9.1 of [8] states that minimality is a stable property in such a family. We show in Subsection 5.1 that a rank two interval exchange transformation can be minimal only if its set of restrictions is poor.

The paper is organized as follows. In Section 2 we introduce all basic notions and define poor and rich restriction spaces as well as the notion of stability for minimal interval exchange transformations with restrictions.

In Section 3 we consider interval exchange transformations with poor restrictions. We show that, in this case, minimal uniquely ergodic ones are always stably minimal. We also show that in a small neighborhood of such a transformation almost all transformations satisfying the same family of restrictions are uniquely ergodic.

In Section 4 we conjecture that minimal interval exchange transformations with rich restrictions are never stably minimal and, moreover, that minimality occurs with probability 0 in this case.
In Section 5 we discuss how interval exchange transformations with rich or poor restrictions arise in different subjects. We show that some known problems are particular instances of our conjectures.

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2. Preliminaries

Throughout Sections 2–4 we assume that \( \pi \) is a fixed irreducible permutation \( \pi \in S_n \).

2.1. Suspension surface. Interval exchange transformations (see the Introduction for the definition) are intimately related with singular foliations that can be defined by a closed 1-form on an oriented surface. The leaves of such a foliation can also be considered as trajectories of a Hamiltonian system on the surface. The corresponding interval exchange transformation appears as the first return map for some transversal of the foliation (or flow). The surface and the 1-form are constructed from an interval exchange transformation as follows.

Definition 5. Let \( a \in \Delta^{n-1} \). Take the square \([0,1] \times [0,1]\) and make the following identifications:

\[
\begin{align*}
(x, 1) & \sim (T_\pi, a(x), 0), \quad x \in [0, 1); \\
(x, 1) & \sim \left( \lim_{y \to x-0} T_\pi, a(y), 0 \right), \quad x \in (0, 1]; \\
(0, y) & \sim (0, 0), \quad y \in [0, 1); \\
(1, y) & \sim (1, 0), \quad y \in [0, 1].
\end{align*}
\]

The obtained surface will be denoted \( \Sigma_{\pi, a} \) and called the suspension surface of \( T_{\pi, a} \). A smooth structure can be chosen on \( \Sigma_{\pi, a} \) so that \( dx \) becomes a smooth differential 1-form on \( \Sigma_{\pi, a} \), which we denote by \( \omega_{\pi, a} \) or simply \( \omega \) if \( a \) and \( \pi \) are fixed.

Remark 1. From topological point of view, this construction is equivalent to the one in [26, page 174] or [39, Section 12], where the surface comes with more structure, namely, a specific Riemann surface structure and a holomorphic 1-form of which \( \omega \) is the real part.

The fibration of \([0,1] \times [0,1]\) by vertical arcs \( \{x\} \times [0,1] \) becomes, after the identifications, a singular foliation on \( \Sigma_{\pi, a} \), which we denote by \( \mathcal{F}_{\pi, a} \). Its leaves are locally defined by the equation \( \omega = 0 \). The foliation \( \mathcal{F}_{\pi, a} \) has finitely many singularities, which may occur only at the points of the subset

\[ S_{\pi, a} = \{(x_1, 1), \ldots, (x_{n-1}, 1)\}/\sim \subset \Sigma_{\pi, a}. \]

All singularities have the form of a saddle or a multiple saddle, see Fig. 2.

![Figure 2. Singularities of \( \mathcal{F}_{\pi, a} \)](image)

The points of \( S_{\pi, a} \) that are regular for \( \mathcal{F}_{\pi, a} \) are called marked points.

One can see that, under an appropriate orientation of leaves, the first return map of the foliation \( \mathcal{F}_{\pi, a} \) on the parametrized transversal \( \gamma(t) = (t, 1/2)/\sim \in \Sigma_{\pi, a} \) coincides with \( T_{\pi, a} \) except at finitely many points of discontinuity of \( T_{\pi, a} \), where the first return map is not well defined. So, minimality of \( T_{\pi, a} \)
is equivalent to that of $F_{\pi, a}$, where minimality is understood in a weak sense: $F_{\pi, a}$ is minimal if every regular leaf is everywhere dense.

Keane’s condition for $T_{\pi, a}$ means exactly that every leaf of $F_{\pi, a}$ is simply connected and contains at most one marked point or singularity of $F_{\pi, a}$. In particular, there are no saddle connections.

Note that we consider a singular point of $F_{\pi, a}$ and separatrices coming from it as parts of a single singular leaf of $F_{\pi, a}$.

Let $\Sigma$ be a closed oriented surface of genus $g \geq 1$, $S$ a finite subset of $\Sigma$, and $m : S \to \mathbb{N} \cup \{0\}$ a function such that $\sum_{P \in S} m(P) = 2g - 2$. The construction above gives a local parametrization of the moduli space $\mathcal{M}(\Sigma, S, m)$ of closed 1-forms $\omega$ on a surface $\Sigma$ with the following properties:

1. $\omega$ is the real part of a holomorphic 1-form on $\Sigma$ for some complex structure;
2. $\omega$ has no zeros outside $S$;
3. every point $P \in S$ has a neighborhood in which $\omega$ can be written as $\text{Re}(d(x + iy)^m(P)+1)$ for some local coordinates $x, y$.

Two forms $\omega$ and $\omega'$ are considered equivalent if there is a diffeomorphism $\varphi : (\Sigma, S) \to (\Sigma, S)$ isotopic to the identity modulo $S$ such that $\omega = \text{const} \cdot \varphi^* \omega'$ with $\text{const} > 0$.

In order to get a local parametrization of $\mathcal{M}(\Sigma, S, m)$ around a point represented by a 1-form $\omega$ one needs to find a simple transversal arc (or loop) $\gamma$ of the induced foliation with endpoints in $S$ such that $\gamma$ intersects all leaves and all saddle connections if any. One parametrizes $\gamma$ so that $dt = \text{const} \cdot \omega|_{\gamma}$ with $\text{const} > 0$, where $t$ is the parameter. Then the first return map of the induced foliation on $\gamma$ (extended to the ambiguity points to be continuous on the right) is an interval exchange map $T_{\rho, a}$ such that $(\Sigma_{\rho, a}, S_{\rho, a}, \omega_{\rho, a})$ can be identified with $(\Sigma, S, \omega)$. A perturbation of $\omega$ such that $\gamma$ remains a transversal leaves the permutation $\rho$ unchanged, so, the components of $a$ become local coordinates in $\mathcal{M}(\Sigma, S, m)$ around the point represented by $\omega$. These parameters are also interpreted as coordinates of the relative cohomology class of $\omega$ in $H^1(\Sigma, S; \mathbb{R})$.

For more details about suspension surfaces of interval exchange transformations we refer the reader to [39].

2.2. Restrictions.

**Definition 6.** An integral linear restriction (or simply a restriction) is a linear function $r \in (\mathbb{R}^n)^*$ with integral coefficients. An interval exchange transformation $T_{\pi, a}$ is said to satisfy a restriction $r$ if $r(a) = 0$. A real subspace $\mathcal{R} \subset (\mathbb{R}^n)^*$ spanned by a family of integral restrictions satisfied by $T_{\pi, a}$ is called a restriction space of $T_{\pi, a}$. The subspace generated by all such integral restrictions is called the full restriction space of $T_{\pi, a}$ and denoted by $\mathcal{R}(T_{\pi, a})$.

The reader is alerted that the normalization condition $\sum_i a_i = 1$ in the definition of an interval exchange transformation is introduced only to eliminate the rescaling degree of freedom, which is redundant, and has nothing to do with integral linear restrictions.

We will use the interpretation of integral linear restrictions in terms of the suspension surface $\Sigma_{\pi, a}$ and the 1-form $\omega_{\pi, a}$. Denote by $\gamma_1, \ldots, \gamma_n$ the images of the oriented intervals $[0, x_1], [x_1, x_2], \ldots, [x_{n-1}, 1]$, respectively, in $\Sigma_{\pi, a}$ under identifications [3]. They are regarded as 1-chains, and their homology classes $h_i = [\gamma_i]$ form a basis in the relative homology group $H_1(\Sigma_{\pi, a}, S_{\pi, a}; \mathbb{Z})$, since $\gamma_1, \ldots, \gamma_n$ form the 1-skeleton of a cell decomposition of $\Sigma_{\pi, a}$ with 0-skeleton $S_{\pi, a}$. We obviously have

$$a_i = \int_{h_i} \omega_{\pi, a}.$$

Thus all possible restrictions for $T_{\pi, a}$ can be regarded as elements of the relative integral homology group $H_1(\Sigma_{\pi, a}, S_{\pi, a}; \mathbb{Z})$. Namely, the restriction corresponding to a homology class $r \in H_1(\Sigma_{\pi, a}, S_{\pi, a}; \mathbb{Z})$ has the form

$$\int_r \omega_{\pi, a} = 0.$$

The set of all restrictions satisfied by $T_{\pi, a}$ is then a subgroup of $H_1(\Sigma_{\pi, a}, S_{\pi, a}; \mathbb{Z})$, and the full restriction space $\mathcal{R}(T_{\pi, a})$ is a subspace of $H_1(\Sigma_{\pi, a}, S_{\pi, a}; \mathbb{R}) = H_1(\Sigma_{\pi, a}, S_{\pi, a}; \mathbb{Z}) \otimes \mathbb{R}$. 

Definition 7. A restriction \( r \in \mathcal{R}(T_{\pi,a}) \) will be called an arc restriction if it can be presented by a single arc with distinct endpoints (which must be from \( S_{\pi,a} \)). A restriction \( r \in \mathcal{R}(T_{\pi,a}) \) will be called a loop restriction if it can be presented by a simple closed curve. If \( r \in \mathcal{R}(T_{\pi,a}) \) is a loop restriction or an arc restriction we say that \( r \) is a simple restriction.

We say that a simple restriction \( r \) is realized by \( T_{\pi,a} \) if it can be presented by a simple piecewise smooth arc or a simple closed piecewise smooth curve contained in a leaf (typically, singular leaf) of \( F_{\pi,a} \). In this case, the corresponding arc or closed curve will be called a realization of \( r \). The number of intersections of this curve or arc with the transversal \( (0,1) \times \{1/2\} / \sim \) will be called the length of the realization. A realization \( \rho \) of \( r \) is elementary if there is no decomposition \( r = r_1 + r_2, \rho = \rho_1 \cup \rho_2 \) with \( r_1 \) and \( r_2 \) simple restrictions realized by \( \rho_1 \) and \( \rho_2 \), respectively.

Realizing a simple restriction is a stable property in the following sense.

Proposition 1. Suppose \( T_{\pi,a} \) realizes a simple restriction \( r \). Then there exists an open neighborhood \( U \subset \Delta^{n-1} \) of \( a \) such that \( T_{\pi,a'} \) also realizes \( r \) whenever \( a' \in U \) and \( \mathcal{R}(T_{\pi,a'}) \supset \mathcal{R}(T_{\pi,a}) \).

This statement generalizes one of the arguments of [42].

Proof. When \( a' \) varies within a small neighborhood of \( a \) the corresponding surface \( \Sigma_{\pi,a'} \) can be identified with \( S_{\pi,a'} \) so that \( S_{\pi,a'} \) coincides with \( S_{\pi,a} \) and the 1-form \( \omega_{\pi,a'} \) remains close to \( \omega_{\pi,a} \).

Let \( r = r_1 + \ldots + r_k \) be a decomposition of \( r \) into restrictions that admit an elementary realization, which clearly always exists. Let \( \rho_1, \ldots, \rho_k \) be elementary realizations of \( r_1, \ldots, r_k \), respectively. Each \( \rho_i \) is either a regular closed leaf (this may occur only if \( k = 1, r = r_1 \)) or a saddle connection. Saddle connections and closed leaves persist under small perturbations if the integral of the 1-form over them remains zero. The latter does occur for \( \omega_{\pi,a'} \) if \( \mathcal{R}(T_{\pi,a'}) \supset \mathcal{R}(T_{\pi,a}) \) since \( r_i \in \mathcal{R}(T_{\pi,a}), i = 1, \ldots, k \).

2.3. Rich and poor restriction spaces. Let \( \mathcal{V} \) be a finite dimensional vector space over \( \mathbb{R} \) or \( \mathbb{Q} \) equipped with a skew-symmetric bilinear form \( \langle , \rangle \). Recall that a subspace \( \mathcal{W} \subset \mathcal{V} \) is called isotropic if the restriction of \( \langle , \rangle \) vanishes on \( \mathcal{W} \). A subspace \( \mathcal{W} \subset \mathcal{V} \) is called coisotropic if \( \langle u, v \rangle = 0 \) \( \forall v \in \mathcal{W} \) implies \( u \in \mathcal{W} \).

If \( \mathcal{V} \) is symplectic, i.e. the form \( \langle , \rangle \) is non-degenerate, then any maximal isotropic subspace \( \mathcal{W} \subset \mathcal{V} \) is called a Lagrangian subspace of \( \mathcal{V} \). The dimension of a Lagrangian subspace is always one half of \( \text{dim} \mathcal{V} \). A subspace \( \mathcal{W} \subset \mathcal{V} \) of a symplectic space is coisotropic if and only if \( \mathcal{W} \) contains a Lagrangian subspace.

Denote by \( \varphi \) the map from \( \mathcal{V} \) to the dual space \( \mathcal{V}^* \) given by \( v \mapsto \langle v, \cdot \rangle \). The space \( \mathcal{V} / \ker \varphi \) is symplectic with respect to \( \langle , \rangle \) (which is well-defined on the quotient space by construction), and \( \varphi \) transfers the symplectic structure to \( \text{Im} \varphi \subset \mathcal{V}^* \) via the natural isomorphism \( \mathcal{V} / \ker \varphi \to \text{Im} \varphi \).

Let \( \mathcal{A} \subset \mathcal{V}^* \) be a vector subspace. The following three conditions are equivalent:

1. the intersection \( \mathcal{A} \cap \text{Im} \varphi \) is coisotropic;
2. the subspace \( \text{Ann}(\mathcal{A}) = \{ v \in \mathcal{V} : \langle v, r \rangle = 0 \ \forall r \in \mathcal{A} \} \) is isotropic;
3. the image of \( \langle , \rangle \) in \( \mathcal{V}^* \wedge \mathcal{V}^* \) in \( (\mathcal{V}^*/\mathcal{A}) \wedge (\mathcal{V}^*/\mathcal{A}) \) under the natural projection \( \mathcal{V}^* \wedge \mathcal{V}^* \to (\mathcal{V}^*/\mathcal{A}) \wedge (\mathcal{V}^*/\mathcal{A}) \) is zero.

Definition 8. A subspace \( \mathcal{A} \subset \mathcal{V}^* \) will be called rich with respect to \( \langle , \rangle \) if the three equivalent conditions above are satisfied.

Definition 9. A restriction space for an interval exchange transformation \( T_{\pi,a} \) is said to be rich if it is rich with respect to the bilinear form \( \langle , \rangle_\pi \) on \( \mathbb{R}^n \) defined by

\[
\langle e_i, e_j \rangle_\pi = \begin{cases} 
1 & \text{if } i < j \text{ and } \pi^{-1}(i) > \pi^{-1}(j), \\
-1 & \text{if } i > j \text{ and } \pi^{-1}(i) < \pi^{-1}(j), \\
0 & \text{otherwise},
\end{cases}
\]

where \( e_1, \ldots, e_n \) is the standard basis of \( \mathbb{R}^n \).

Now we describe the topological meaning of the above definitions. Let \( (\Sigma_{\pi,a}, \omega_{\pi,a}) \) be the surface and the closed 1-form associated with an interval exchange transformation \( T_{\pi,a} \) as described in Section 2.1.
Let $\mathcal{Y}$ be the first relative cohomology group $H^1(\Sigma_{\pi,a}, S_{\pi,a}; \mathbb{R})$. The dual vector space $\mathcal{Y}^*$ is naturally identified with $H_1(\Sigma_{\pi,a}, S_{\pi,a}; \mathbb{R})$. Let $e_1, \ldots, e_n$ be the basis in $\mathcal{Y}$ dual to the basis $h_1, \ldots, h_n$ introduced in Section 2.2.

Then the bilinear form $(\cdot, \cdot)_\pi$ defined by $\mathfrak{3}$ is nothing else but “the intersection form” on $H^1(\Sigma_{\pi,a}, S_{\pi,a}; \mathbb{R})$:

$$(\eta_1, \eta_2)_\pi = \int_{\Sigma_{\pi,a}} \eta_1 \wedge \eta_2$$

(provided that the orientation of $\Sigma_{\pi,a}$ is chosen appropriately).

In the notation introduced in the beginning of the section the map $\varphi$ is the composition $p \circ \text{PD}$, where $\text{PD} : H^1(\Sigma_{\pi,a}, S_{\pi,a}; \mathbb{R}) \to H_1(\Sigma_{\pi,a} \setminus S_{\pi,a}; \mathbb{R})$ is the Poincaré duality operator and $p : H_1(\Sigma_{\pi,a} \setminus S_{\pi,a}; \mathbb{R}) \to H_1(\Sigma_{\pi,a}; \mathbb{R}) \subset H_1(\Sigma_{\pi,a}, S_{\pi,a}; \mathbb{R})$ the natural projection.

Thus, the symplectic spaces $\mathcal{Y} / \ker \varphi$ and $\text{Im} \varphi$ are $H^1(\Sigma_{\pi,a}; \mathbb{R})$ and $H_1(\Sigma_{\pi,a}; \mathbb{R})$, respectively, and the respective symplectic forms on them are the $\sim$ and $\sim$ products.

**Definition 10.** The Sah–Arnoux–Fathi (SAF) invariant of $T_{\pi,a}$ is the following element of the rational vector space $\mathbb{R} \wedge_{\mathbb{Q}} \mathbb{R}$:

$$\text{SAF}(T_{\pi,a}) = \sum_{i=1}^{n} a_i \wedge_{\mathbb{Q}} (\bar{x}_{\pi^{-1}(i)} - x_i) = \sum_{i<j} (a_i \wedge_{\mathbb{Q}} a_j - a_{\pi(i)} \wedge_{\mathbb{Q}} a_{\pi(j)}).$$

The SAF invariant was introduced by P. Arnoux in [2] who also showed that $\text{SAF}(T_{\pi,a})$ is an invariant of the measured foliation induced by $\omega_{\pi,a}$ on $\Sigma_{\pi,a}$ (in particular, it is invariant under the Rauzy induction).

The SAF invariant has many applications in the study of interval exchanges. In particular, it has been used to characterize group properties of interval exchange transformations [38] and to study the Veech group of translation surfaces [33, 22].

It is known that the SAF invariant vanishes for periodic (i.e. such that every orbit is finite) interval exchange transformations. Arnoux and Yoccoz constructed in [4] the first example of minimal and uniquely ergodic interval exchange transformation for which SAF is equal to zero.

The Galois flux introduced in [28] can be viewed as an instance of the SAF invariant in the case when the parameters of an interval exchange transformation belong to a quadratic field.

The following statement appears in [2] with an attribution to G. Levitt.

**Proposition 2.** The full restriction space $\mathcal{R}(T_{\pi,a})$ is rich if and only if $\text{SAF}(T_{\pi,a}) = 0$.

**Proof.** Denote by $\mathcal{R}_0$ the intersection of $\mathcal{R} = \mathcal{R}(T_{\pi,a}) \subset H_1(\Sigma_{\pi,a}, S_{\pi,a}; \mathbb{R})$ with $H_1(\Sigma_{\pi,a}, S_{\pi,a}; \mathbb{Q})$. Since the form $(\cdot, \cdot)_\pi$ has integer coefficients and $\mathcal{R}(T_{\pi,a})$ is generated by integral relative cycles, $\mathcal{R}$ is a rich subspace of $H_1(\Sigma_{\pi,a}, S_{\pi,a}; \mathbb{R})$ if and only if $\mathcal{R}_0$ is a rich subspace of $H_1(\Sigma_{\pi,a}, S_{\pi,a}; \mathbb{Q})$.

By construction $\mathcal{R}_0$ is the kernel of the $\mathbb{Q}$-linear map $A : H_1(\Sigma_{\pi,a}, S_{\pi,a}; \mathbb{Q}) \to \mathbb{R}$ defined by $A(h_i) = a_i$, $i = 1, \ldots, n$. The claim now follows from the fact that $A$ induces a map $H_1(\Sigma_{\pi,a}, S_{\pi,a}; \mathbb{Q}) \wedge_{\mathbb{Q}} H_1(\Sigma_{\pi,a}, S_{\pi,a}; \mathbb{Q}) \to \mathbb{R} \wedge_{\mathbb{Q}} \mathbb{R}$ that takes the form $(\cdot, \cdot)_\pi$ to $\text{SAF}(T_{\pi,a})$.  

**Remark 2.** It immediately follows from Proposition 2 that, for a fixed $\pi$, the set of $a$ such that $\text{SAF}(T_{\pi,a}) = 0$ has measure zero.

2.4. Asymptotic cycle and separating cycle. Let $\pi \in S_n$ and $a \in \Delta^{n-1}$ be fixed. Following A. Zorich [33, 44], for any $x \in [0, 1)$, we denote by $c_{\pi,a,k}(x)$ the homology class from $H_1(\Sigma_{\pi,a}; \mathbb{R})$ presented by the closed curve

$$\left( \bigcup_{i=1}^{k} \{T_{\pi,a}^{i-1}(x) \times [0, 1) \} \cup \alpha \right),$$

where $\alpha = [x, T_{\pi,a}^{k}(x)] \times \{0\}$ or $[T_{\pi,a}(x), x] \times \{0\}$, under identifications [2]. (The orientation is chosen so that “vertical” segments are directed upwards.)

The following is a particular case of the notion of asymptotic cycle introduced by S. Schwartzman [30] in different terms.
Definition 11. The homology class $c_{\pi,a}$ that is Poincaré dual to $[\omega_{\pi,a}] \in H^1(\Sigma_{\pi,a};\mathbb{R})$ is called the asymptotic cycle of $T_{\pi,a}$.

By definition the asymptotic cycle belongs to the absolute homology $H_1(\Sigma_{\pi,a};\mathbb{R})$, which is a subspace of the relative homology $H_1(\Sigma_{\pi,a}, S_{\pi,a};\mathbb{R})$.

Proposition 3. If $T_{\pi,a}$ is uniquely ergodic, then for any $x \in [0,1)$ there exists a limit

$$
\lim_{k \to \infty} \frac{c_{\pi,a,k}(x)}{k}.
$$

This limit does not depend on $x$ and is equal up to a non-zero multiple to the asymptotic cycle $c_{\pi,a}$. Moreover, the convergence is uniform with respect to $x$.

The existence of the limit (4) for almost all $x$ is a simple consequence of Birkhoff’s ergodic theorem. The fact that for a generic irreducible interval exchange transformation the convergence is uniform and holds for all $x \in [0,1)$ was observed by Zorich [33, 41]. A proof of Proposition 3 in exactly this form can be found in [40, Section 3]. Due to the normalization condition $\sum a_i = 1$ ‘a non-zero multiple’ in the formulation of Proposition 3 is actually $\pm 1$.

Proposition 4. The full restriction space of $T_{\pi,a}$ is rich if and only if it contains the asymptotic cycle $c_{\pi,a}$.

Proof. The space $\mathcal{W} = \mathcal{A}(T_{\pi,a}) \cap H_1(\Sigma_{\pi,a};\mathbb{R})$ is generated by all $c \in H_1(\Sigma_{\pi,a};\mathbb{Z})$ such that $c \sim c_{\pi,a} = 0$. If $\mathcal{W}$ is coisotropic this implies $c_{\pi,a} \in \mathcal{W}$.

If $\mathcal{W}$ is not coisotropic, then there exists $s \in H_1(\Sigma_{\pi,a};\mathbb{Z}) \setminus \mathcal{W}$ such that $s \sim r = 0$ for all $r \in \mathcal{W}$. Since $s \notin \mathcal{W}$ we have $s \sim c_{\pi,a} \neq 0$. Therefore, $c_{\pi,a} \notin \mathcal{W}$, which implies $c_{\pi,a} \notin \mathcal{A}(T_{\pi,a})$. \hfill \Box

Definition 12. An element $s \in H_1(\Sigma_{\pi,a};\mathbb{Z})$ such that $s \sim r = 0$ for all $r \in \mathcal{A}(T_{\pi,a}) \cap H_1(\Sigma_{\pi,a};\mathbb{R})$ and $s \sim c_{\pi,a} \neq 0$ will be called a separating cycle for $T_{\pi,a}$. As we have just seen, it exists if and only if the full restriction space $\mathcal{A}(T_{\pi,a})$ is poor.

Proposition 5. If a separating cycle for $T_{\pi,a}$ can be presented by a closed transversal of the foliation $\mathcal{F}_{\pi,a}$ such that it intersects all separatrices (and hence, all leaves) of $\mathcal{F}_{\pi,a}$, then $T_{\pi,a}$ is minimal.

Proof. Let $\xi$ be a transversal representing a separating cycle $s$ such that $\xi$ intersects all saddle connections of $\mathcal{F}_{\pi,a}$. If $\mathcal{F}_{\pi,a}$ is not minimal, then there must be a realization $\rho$ of a loop restriction $r$. It must intersect $\xi$, and the contributions of all intersections to $s \sim r$ will have the same sign. So, we will have $s \sim r \neq 0$, a contradiction. \hfill \Box

Remark 3. If a homology class $s \in H_1(\Sigma_{\pi,a};\mathbb{Z})$ can be presented by a transversal intersecting all the leaves of $\mathcal{F}_{\pi,a}$, then we always have $s \sim c_{\pi,a} \neq 0$.

Remark 4. A separating cycle can alternatively be defined as an element $s$ of $H_1(\Sigma_{\pi,a} \setminus S_{\pi,a};\mathbb{Z})$ such that $s \sim r = 0$ for any restriction $r$ and $s \sim c_{\pi,a} \neq 0$.

In this setting, the assertion of Proposition 5 will be that $T_{\pi,a}$ satisfies Keane’s condition.

Example 5. In Example 3 the set $S_{\pi,a}$ consists of a single point, in which $\mathcal{F}_{\pi,a}$ has a double saddle singularity. There is a single restriction, up to a multiple, which is $r = 3h_1 - h_2 - h_4$. It can also be presented by the closed curve $\rho$ consisting of the following three oriented straight line segments (under identifications 2): $\rho_1 = [(a_2,0), (a_1,1)], \rho_2 = [(a_2 + a_4 - a_1,0), (a_1 + a_2,1)], \rho_3 = [(a_2 + a_3 + a_4,0), (a_2 + a_3 + a_4,1)]$.

To see that $\rho$ represents $3h_1 - h_2 - h_4$ we compute

$$
\int_{\rho} dx = \int_{\rho_1} dx + \int_{\rho_2} dx + \int_{\rho_3} dx = (a_1 - a_2) + (2a_1 - a_4) + 0 = 3a_1 - a_2 - a_4,
$$

which holds for any $a \in \Delta^3$. 

The cycle $s = 2h_1 - h_2 + h_3$ is separating, and if $a_2 < a_1$ it can be presented by a transversal $\xi$ obtained by a small deformation from the closed curve consisting of the following oriented straight line segments:

$\xi_1 = [(a_2, 0), (a_1, 1)]$, $\xi_2 = [(a_2, 0), (a_1 + a_2, 1)]$, $\xi_3 = [(a_1 + a_2, 1), (a_1 + a_2 + a_3, 1)]$,

see Fig. 3. Indeed, for any $a \in \Delta^3$ we have

$$\int_{\xi} dx = \int_{\xi_1} dx + \int_{\xi_2} dx + \int_{\xi_3} dx = (a_1 - a_2) + a_1 + a_3 = 2a_1 - a_2 + a_3,$$

which implies that $\xi$ represents $s$. The transversal $\xi$ is disjoint from $\rho$, so, we have $s \sim r = 0$.

It is quite obvious (consult Fig. 3) that separatrices emanating from $(x_1, 1), (x_3, 1)$ in the downward direction and from $(\tilde{x}_1, 0), (\tilde{x}_2, 0)$ in the upward direction hit the transversal $\xi$. One can see that the two remaining separatrices can avoid meeting $\xi$ only if they form a saddle connection. This means that for some $k \geq 1$ we have $T_{\pi, a}^{-k}(x_2) = x_2 + k(a_1 + a_3) = \tilde{x}_3$, and $a_1 + k(a_1 + a_3) = a_4 + a_3$. This is an integral restriction that $T_{\pi, a}$ is supposed not to satisfy.

Thus $\xi$ intersects all the separatrices and hence all the leaves of $F_{\pi, a}$. Therefore, according to Proposition 5 the foliation $F_{\pi, a}$ is minimal.

![Figure 3. A transversal representing a separating cycle (bold line) and the restriction cycle (dashed line) in Example 5.](image)

It also follows from Theorem 2 below that the transformation $T_{\pi, a}$ is uniquely ergodic for almost all $a$ such that $a_1 > a_2$ and $\rho(a) = 0$.

If $a_1 < a_2$, then this construction fails because $\int_{\xi_1} < 0$ and $\int_{\xi_2} > 0$, hence $\xi_1 \cup \xi_2 \cup \xi_3$ cannot be made transverse to the foliation by a small deformation.

3. Stability of minimal interval exchange transformations with poor restrictions

**Definition 13.** Let $\mathcal{R}$ be a restriction space of a minimal interval exchange transformation $T_{\pi, a}$. We say that $T_{\pi, a}$ is $\mathcal{R}$-stably minimal if there exists an open neighborhood $U$ of $a$ in $\Delta^m$ such that $T_{\pi, a'}$ is minimal whenever $a' \in U$ and $\mathcal{R}(T_{\pi, a'}) = \mathcal{R}$.

**Theorem 1.** Let $T_{\pi, a}$ be a minimal and uniquely ergodic interval exchange transformation such that the full restriction space $\mathcal{R}(T_{\pi, a})$ is poor. Let $\mathcal{R}_0$ be a subspace of $\mathcal{R}(T_{\pi, a})$ generated by all restrictions realized by $T_{\pi, a}$ (see Definition 7). Then the transformation $T_{\pi, a}$ is $\mathcal{R}$-stably minimal for any restriction space $\mathcal{R}$ such that $\mathcal{R}_0 \subset \mathcal{R} \subset \mathcal{R}(T_{\pi, a})$.

**Proof.** If $\mathcal{R} = 0$, then the assertion follows from Keane’s result [20]. In what follows we assume $0 \neq \mathcal{R} \subset \mathcal{R}(T_{\pi, a})$, and $\mathcal{R} = \mathcal{R}(T_{\pi, a})$ if $T_{\pi, a}$ does not satisfy Keane’s condition.

By Imanishi’s theorem [15] the foliation $F_{\pi, a}$, and hence the transformation $T_{\pi, a}$ is not minimal if and only if some union of realizations of restrictions that is contained in finitely many leaves of $F_{\pi, a}$ cut the surface $M_{\pi, a}$ into two non-trivial pieces. Keane’s condition is equivalent to the absence of any realized restrictions.

Thus, in order to prove that $T_{\pi, a'}$ is minimal it suffices to show that $T_{\pi, a'}$ does not realize any simple restriction that is not already realized by $T_{\pi, a}$.
Let \( s \in H_1(\Sigma_{\pi,a}; Z) \) be a separating cycle. Without loss of generality we can assume \( s \sim c_{\pi,a} > 0 \). Take an arbitrary preimage \( \tilde{s} \in H_1(\Sigma_{\pi,a} \setminus S_{\pi,a}; Z) \) under the natural projection \( H_1(\Sigma_{\pi,a} \setminus S_{\pi,a}; Z) \to H_1(\Sigma_{\pi,a}; Z) \). Note that the \( \sim \) product is a well defined pairing \( H_1(\Sigma_{\pi,a} \setminus S_{\pi,a}; Z) \times H_1(\Sigma_{\pi,a}; Z) \to Z \).

**Lemma 1.** There exists a constant \( C \) such that \( |\tilde{s} \sim r| < C \) for any simple restriction \( r \in \mathcal{R}(T_{\pi,a}) \).

**Proof.** If \( r \) is an absolute cycle, then \( s \sim r = 0 \) by Definition 12.

If \( r_1 \) and \( r_2 \) are two arc restrictions such that \( \partial r_1 = \partial r_2 \), then \( r_1 - r_2 \in H_1(\Sigma_{\pi,a}; Z) \) is an absolute cycle, so, we have \( (r_1 - r_2) \sim \tilde{s} = (r_1 - r_2) \sim s = 0 \). Therefore, \( r_1 \sim \tilde{s} = r_2 \sim \tilde{s} \). The assertion of the lemma now follows from the fact that there are only finitely many different possibilities for \( \partial r \) if \( r \) is an arc restriction.

For \( x, y \in S_{\pi,a} \), denote by \( r_{xy} \) the relative 1-cycle presented by the arc \([x, y] \times \{1\}\) if \( x < y \) and \([y, x] \times \{1\}\) otherwise directed from \( x \) to \( y \). If \( r \) is an arc restriction with \( \partial r = y - x \) we denote by \( \hat{r} \) the absolute cycle \( r - r_{xy} \in H_1(\Sigma_{\pi,a}; Z) \). If \( r \) is a loop restriction we set \( \hat{r} = r \).

It follows from Proposition \#\# and Lemma \#\# that, for large enough \( k \), we have

\[
s \sim c_{\pi,a,j}(x) > s \sim \hat{s}
\]

whenever \( j \in [k, 2k] \cap \mathbb{Z} \), \( x \in [0, 1] \), and \( r \) is a simple restriction. We fix such a large \( k \) from now on.

Denote \( C_{\pi,a,j} = \{ c_{\pi,a,j}(x) : x \in [0, 1] \} \).

**Lemma 2.** There exists a neighborhood \( U \) of \( a \) in \( \Delta^{n-1} \) such that whenever \( a' \in U \) and \( \mathcal{R}(T_{\pi,a'}) = \mathcal{R} \) we have:

1. \( c_{\pi,a',j} = c_{\pi,a,j} \) for all \( j \in [1, 2k] \cap \mathbb{Z} \);
2. \( T_{\pi,a'} \) does not realize any restriction of length \( < 2k \) that is not realized by \( T_{\pi,a} \).

**Proof.** One can see that, for any \( j \), the finite set \( C_{\pi,a,j} \) is defined by the permutation \( \pi \) and the relative positions of points from \( \bigcup_{i=1}^{k-1} (T_{\pi,a})^i(S_{\pi,a}) \) (we abuse notation here slightly by thinking of \( S_{\pi,a} \) as a subset of \([0, 1]\)). If \( a \) is perturbed slightly, then the order of points in this union does not change provided that all coincidences, if any, between these points are preserved.

Coincidences may occur only if Keane’s condition is not satisfied. Indeed, an equation \( (T_{\pi,a})^i(x) = y \) with \( x, y \in S_{\pi,a} \) means that an arc restriction is realized in the singular leaf passing through \( x \).

If \( T_{\pi,a'} \) satisfies all restrictions realized by \( T_{\pi,a} \) and \( a' \) is close enough to \( a \), then \( (T_{\pi,a'})^i(x) = y \) holds true, too.

If \( a' \) is sufficiently close to \( a \) then no new coincidences occur in \( \bigcup_{i=0}^{2k} (T_{\pi,a'})^i(S_{\pi,a}) \) compared to \( \bigcup_{i=0}^{2k} (T_{\pi,a})^i(S_{\pi,a}) \), which implies Condition (2) from the Lemma assertion.

We fix \( U \) as in Lemma \#\# Let \( a' \in U \), \( \mathcal{R}(T_{\pi,a'}) = \mathcal{R} \).

**Lemma 3.** For any simple restriction \( r \) and any \( j \geq k \) we have

\[
s \sim c_{\pi,a',j+k}(x) > s \sim \hat{s}
\]

**Proof.** We apply induction in \([j/k]\), where \([x]\) stands for the integral part of \( x \). For \( k \leq j \leq 2k \) the assertion follows from Lemma \#\# and inequality (5).

The induction step is obtained by applying the following relation

\[
c_{\pi,a',j+k}(x) = c_{\pi,a',j}(x) + c_{\pi,a',k}((T_{\pi,a'})^j(x)) \in c_{\pi,a',j} + C_{\pi,a',k}.
\]

The inequality

\[
s \sim c > \max_{r \text{ is a simple restriction}} s \sim \hat{r}
\]

holds for any \( c \in C_{\pi,a',j} \) by the induction hypothesis and for any \( c \in C_{\pi,a',k} \) by the induction base. Therefore, it also holds for any \( c \in C_{\pi,a',j+k} \subset C_{\pi,a',j} + C_{\pi,a',k} \).

We are now ready to conclude the proof of Theorem \#\#.

Suppose that \( T_{\pi,a'} \) realizes some simple restriction that is not realized by \( T_{\pi,a} \). Let \( r \) be such a restriction with minimal possible length, and \( j \) the length of \( r \). By the choice of \( U \) we must have \( j > 2k \).
There must exist \( x, y \in S_{\pi, a'} \) (possibly \( x = y \)) such that \( (T_{\pi, a'})^i(x) = y \) holds, and \( r \) is presented by an arc (or loop if \( x = y \)) contained in the singular fiber through \( x \).

We have \( c_{\pi, a', j}(x) = \tilde{r} \), which contradicts Lemma 6.

Therefore, \( T_{\pi, a'} \) does not realize any simple restriction that is not realized by \( T_{\pi, a} \) and, hence, is minimal.

Barak Weiss drew our attention to the fact that the results of paper 29 allow us to prove an analogue of the Masur–Veech theorem 26–27 (Keane’s conjecture 21) in our settings. The next theorem is essentially due to him though we modified the formulation and the proof from his original suggestion in order to keep the style of the paper uniform.

**Theorem 2.** Let \( T_{\pi, a} \) be a minimal and uniquely ergodic interval exchange transformation such that the full restriction space \( \mathcal{B}(T_{\pi, a}) \) is poor. Let \( \mathcal{R}_1 \) be a subspace of \( \mathcal{B}(T_{\pi, a}) \) generated by all restrictions realized by \( T_{\pi, a} \) (see Definition 7), and \( \mathcal{B} \) is a restriction space for \( T_{\pi, a} \) such that \( \mathcal{R}_1 \subset \mathcal{B} \subset \mathcal{B}(T_{\pi, a}) \). Then there exists an open neighborhood \( V \) of \( a \) in \( \{ b \in \Delta^{n-1} : \mathcal{B}(T_{\pi, b}) \subset \mathcal{B} \} \) such that \( T_{\pi, a'} \) is uniquely ergodic for almost all \( a' \in V \) with respect to the Lebesgue measure.

**Proof.** First we consider the case when \( T_{\pi, a} \) satisfies Keane’s condition, i.e. when \( F_{\pi, a} \) has no saddle connections. Denote \( P = \{ b \in \Delta^{n-1} : \mathcal{B}(T_{\pi, b}) \subset \mathcal{B} \} \). \( P \) is a polyhedron in an affine subspace of \( \mathbb{R}^n \) with the latter being identified with \( H^1(\Sigma, \pi, a; \mathbb{R}) \) (see Section 2.3), and \( a \) is an interior point of \( P \).

For a neighborhood \( U \subset P \) of \( a \) we can identify all surfaces \( \Sigma_{\pi, a'}, a' \in U \), with a fixed surface \( \Sigma \) so that \( S_{\pi, a'} \) is identified with a fixed subset \( S \subset \Sigma \), and the form \( \omega_{\pi, a'} \) viewed as a 1-form on \( \Sigma \) depends on \( a' \) continuously (in \( C^1 \) topology, say). We may also assume that \( \omega_{\pi, a'} \) does not depend on \( a' \) in a small neighborhood of the subset \( S \), and the image of the oriented interval \([0, 1] \times \{1/2\} \) under identifications (2) in \( S_{\pi, a'} \) is identified with a fixed oriented curve \( \gamma \) in \( \Sigma \) for all \( a' \in U \). It defines a relative homology class \([\gamma] \in H_1(\Sigma, \pi, a; \mathbb{Z}) \).

Let \( s \in H_1(\Sigma, \pi, a; \mathbb{Z}) \) be a separating cycle for \( T_{\pi, a} \). It has a preimage \( s' \in H_1(\Sigma \setminus S; \mathbb{Z}) \) such that \( s' \sim r = 0 \) for all restrictions \( r \in \mathcal{R} \). Denote by \( c_{\pi, a'} \) the homology class from \( H_1(\Sigma \setminus S; \mathbb{R}) \) that is Poincaré dual to the relative cohomology class \([\omega_{\pi, a'}] \in H^1(\Sigma, \pi, a; \mathbb{R}) \). (The image of \( c_{\pi, a'} \) in \( H_1(\Sigma; \mathbb{R}) \) under the projection induced by the natural inclusion \( \Sigma \setminus S \to \Sigma \) is the asymptotic cycle \( c_{\pi, a} \).)

Let \( s'' = s - (\gamma - s')c_{\pi, a'} \). We will have \( [\gamma] - s'' = 0 \), and still \( s'' \sim r = 0 \) for any restriction \( r \in \mathcal{R} \), and \( s'' \sim c_{\pi, a} \neq 0 \). Denote by \( b \in H^1(\Sigma, \pi, a; \mathbb{R}) \) the cohomology class that is Poincaré dual to \( s'' \). It can be viewed as a tangent vector to \( P \) (at \( a \) or any other point as \( P \) is a polyhedron in an affine space) since \( b([\gamma]) = 0 \) and \( b(r) = 0 \) for any \( r \in \mathcal{R} \). We also have \( b(c_{\pi, a}) \neq 0 \). By changing the sign of \( b \) we may assume \( b(c_{\pi, a}) > 0 \).

It follows from 29, Theorem 1.1, that the cohomology class \( b \) can be represented by a closed 1-form \( \sigma \) on \( \Sigma \) such that \( \omega_{\pi, a} + i\sigma \) is a holomorphic 1-form with respect to some Riemann surface structure on \( \Sigma \) (this complex structure need not agree with the smooth structure on \( \Sigma \) at \( S \)). Since \( \gamma \) is transverse to the foliation defined by \( \omega_{\pi, a} \), the 1-form \( \sigma \) can be chosen so that \( \sigma|_{\gamma} \) vanishes at the endpoints of \( \gamma \).

The 1-form \( \omega_{\pi, a} + i\sigma \) defines a locally Euclidean metric \( ds^2 = |\omega_{\pi, a} + i\sigma|^2 \) on \( \Sigma \setminus S \). With respect to this metric, the transversal \( \gamma \) intersects the leaves of \( F_{\pi, a} \) at an angle bounded from below by a positive constant \( \delta \). Since \( \omega_{\pi, a'} \) depends on \( a' \) continuously and coincides with \( \omega_{\pi, a} \) in a small neighborhood of \( S \) when \( a' \) is close enough to \( a \), there is an open neighborhood \( V \subset U \) of \( a' \) such that for all \( a' \in V \) the 1-form \( \omega_{\pi, a'} + i\sigma \) is a holomorphic 1-form for some complex structure on \( S \) (depending on \( a' \)), and, with respect to the locally Euclidean metric \( |\omega_{\pi, a'} + i\sigma|^2 \), the curve \( \gamma \) remains transverse to the leaves of \( F_{\pi, a'} \) and intersects them at an angle bounded from below by some \( \delta > 0 \). This \( \delta \) can be chosen to be independent of \( a' \), though this is actually not important.

We have that \( \gamma \) is transverse to the leaves of the foliation defined by the 1-form

\[ \zeta_{a', \phi} = \text{Re} (e^{i\phi}(\omega_{\pi, a'} + i\sigma)) \]

if \( a' \in V \) and \( |\phi| < \delta \). If, additionally, \( a' + \tan\phi \cdot b \) lies in the interior of \( P \), then the first return map of the foliation defined by \( \zeta_{a', \phi} \) on \( \gamma \) is, after an appropriate choice of the coordinate on \( \gamma \), an interval exchange transformation \( T_{\pi, a'} + \tan\phi \cdot b \).
By Kerekhov–Masur–Smillie [23], for a fixed \( a' \in V \), the foliation defined by \( \zeta_{a',\phi} \) is uniquely ergodic for almost all \( \phi \). It means that if \( \ell \) is a straight line in \( \mathbb{R}^n \) having direction \( b \) and passing through a point in \( V \), then for almost any \( a' \in \ell \cap V \) the interval exchange transformation \( T_{\pi,a'} \) is ergodic. Fubini’s theorem implies that \( T_{\pi,a} \) is uniquely ergodic for almost all \( a' \in V \) provided that the set of all such \( a' \) is measurable.

By using the Rauzy induction [31] one can show that the set of all \( a \in \Delta^{n-1} \) such that \( T_{\pi,a} \) is uniquely ergodic is a Borel set. Therefore, so is the intersection of this set with \( P \).

We have completed the proof under the Keane’s condition assumption.

Now we reduce the general case to the one when Keane’s condition is satisfied. We keep using notation \( P = \{ b : b \in \Delta^{n-1} ; \mathcal{R}(T_{\pi,b}) \supseteq \mathcal{R} \} \). There is a small neighborhood \( U \subseteq P \) of \( a \) such that whenever \( b \in U \) the interval exchange transformation \( T_{\pi,b} \) realizes any restriction realized by \( T_{\pi,a} \).

For \( b \in U \), denote by \( \Sigma_{\pi,b} \) the singular surface obtained from \( \Sigma_{\pi,a} \) by collapsing to a point each saddle connection realizing a restriction from \( \mathcal{R}_0 \). In general, \( \Sigma_{\pi,b} \) may have singularities. Namely, if \( T_{\pi,a} \) realizes some loop restrictions, then \( \Sigma_{\pi,b} \) for \( b \in U \), will have finitely many points with a neighborhood that has the form of a wedge sum of several open discs. Thus, \( \Sigma_{\pi,b} \) can be obtained from a surface, which we denote by \( \tilde{\Sigma}_{\pi,b} \), by making finitely many identifications of points.

Thus, we have two projections:

\[
\Sigma_{\pi,b} \overset{p}{\rightarrow} \tilde{\Sigma}_{\pi,b} \leftarrow \tilde{\Sigma}_{\pi,b},
\]

of two smooth surfaces to a singular one. The first projection, \( p \), collapses all the saddle connections, and the second, \( \tilde{\rho} \), is one-to-one outside of a finite subset.

We denote by \( \tilde{S}_{\pi,b} \subseteq \tilde{\Sigma}_{\pi,b} \) the image of \( S_{\pi,b} \) under the projection \( p \), and by \( \tilde{\Sigma}_{\pi,b} \subseteq \tilde{\Sigma}_{\pi,b} \) the preimage of \( \tilde{S}_{\pi,b} \) under \( \tilde{\rho} \). The kernel of the map \( p_* : H_1(\Sigma_{\pi,b}, S_{\pi,b}; \mathbb{R}) \rightarrow H_1(\tilde{\Sigma}_{\pi,b}, \tilde{S}_{\pi,b}; \mathbb{R}) \) induced by the projection \( p \) is clearly \( \mathcal{R}_0 \), and the projection \( \tilde{\rho} \) induces an isomorphism \( \tilde{\rho}_* : H_1(\tilde{\Sigma}_{\pi,b}, \tilde{S}_{\pi,b}; \mathbb{R}) \rightarrow H_1(\tilde{\Sigma}_{\pi,b}, \tilde{S}_{\pi,b}; \mathbb{R}) \). Thus we have a natural epimorphism \( \tilde{\rho}_*^{-1} \circ p_* : H_1(\Sigma_{\pi,b}, S_{\pi,b}; \mathbb{R}) \rightarrow H_1(\tilde{\Sigma}_{\pi,b}, \tilde{S}_{\pi,b}; \mathbb{R}) \), which we denote by \( \iota \), whose kernel is \( \mathcal{R}_0 \).

There is an obvious way to transfer the 1-form \( \omega_{\pi,b} \) from \( \Sigma_{\pi,b} \) to \( \tilde{\Sigma}_{\pi,b} \). The obtained 1-form on \( \tilde{\Sigma}_{\pi,b} \) will be denoted by \( \tilde{\omega}_{\pi,b} \), and the foliation it defines by \( \tilde{\mathcal{F}}_{\pi,b} \). Clearly, \( \tilde{\mathcal{F}}_{\pi,b} \) is minimal if and only if so is \( \mathcal{F}_{\pi,b} \).

By construction the foliation \( \tilde{\mathcal{F}}_{\pi,b} \) has no saddle connections, so it induces an interval exchange map that satisfies Keane’s condition at every transversal connecting two points from \( \tilde{\Sigma}_{\pi,a} \). We choose such a transversal so that, additionally, it intersects all the leaves of \( \tilde{\mathcal{F}}_{\pi,b} \). We can use this transversal for all \( b \) close enough to \( a \). Let \( T_{\pi,b} \) be interval exchange map induced at this transversal by \( \tilde{\mathcal{F}}_{\pi,b} \). We can think of \( \tilde{\Sigma}_{\pi,b} \) as being identified with \( \Sigma_{\pi,b} \).

For a cycle \( c \in H_1(\Sigma_{\pi,b}, S_{\pi,b}; \mathbb{Z}) \), we obviously have \( \int_c \omega_{\pi,b} = \int_{\iota(c)} \tilde{\omega}_{\pi,b} \). Therefore, the restriction space \( \mathcal{R}(T_{\pi,b}) \) is naturally identified with \( \mathcal{R}(\tilde{T}_{\pi,b}) / \mathcal{R}_0 \).

The image \( \iota(c) \) of an absolute cycle \( c \in H_1(\Sigma_{\pi,b}; \mathbb{R}) \) is also an absolute cycle, i.e. lies in \( H_1(\tilde{\Sigma}_{\pi,b}; \mathbb{R}) \), if and only if

\[
\iota(c) - r = 0 \quad \text{for any } r \in \mathcal{R}_0 \cap H_1(\Sigma_{\pi,b}; \mathbb{R}).
\]

This holds, in particular, for the asymptotic cycle \( c_{\pi,b} \) of \( T_{\pi,b} \), and one can see that \( \iota(c_{\pi,b}) \) is the asymptotic cycle of \( \tilde{T}_{\pi,b} \). Condition (7) also holds for any separating cycle \( s \) of \( T_{\pi,b} \), and the image \( \iota(s) \) is then a separating cycle for \( \tilde{T}_{\pi,b} \).

Therefore, if \( b = a \), then the full restriction space \( \mathcal{R}(\tilde{T}_{\pi,b}) = \iota(\mathcal{R}) \) is poor. Varying \( b \) in a small neighborhood of \( a \) so that \( \mathcal{R}(\tilde{T}_{\pi,b}) \supseteq \mathcal{R}_0 \) holds is equivalent to varying \( \tilde{b} \) in a small neighborhood of \( \tilde{a} \). Thus, the passage from \( T_{\pi,a} \) to \( \tilde{T}_{\pi,b} \) described above reduces the general case to the case when Keane’s condition is satisfied.

The assumption on the unique ergodicity of \( T_{\pi,a} \) in Theorems 1 and 2 can be weakened by requiring only that there exists a cycle \( s \in H_1(\Sigma_{\pi,a} \setminus S_{\pi,a}; \mathbb{Z}) \) such that
(1) \( s \prec r = 0 \) for all \( r \in \mathcal{R}(T_{\pi,a}) \) and

(2) \( s \prec c > 0 \) for any cycle \( c \) that can be obtained as the limit in (1) for some \( x \).

This is a ‘weak’ version of a separating cycle. Due to Minsky–Weiss [29], condition (2) holding either for \( s \) or \(-s\) can be shown to be equivalent to the existence of a closed transversal representing \( s \) and intersecting all regular leaves of \( \mathcal{F}_{\pi,a} \).

The proof of Theorem 2 generalizes quite directly with the weaker assumption. Our proof of Theorem 1 does not generalize directly, but there is another proof that does. Namely, instead of applying Proposition 3, one can show that the existence of a separating cycle (in the weak sense) is a stable property and that it implies minimality (cf. Proposition 4).

The unique ergodicity requirement as well as the just mentioned weaker assumption look artificial to us in the present context. We find it plausible that such assumptions can be dropped completely in both theorems. So, we pose

**Conjecture 1.** Theorems 1 and 2 remain true without the assumption on the unique ergodicity of \( T_{\pi,a} \).

We conclude the section by a remark about dimensions. As follows from the definition, a restriction space that is a subspace of a poor restriction space is also poor, however, richness or poorness of a restriction space is not strongly related to its dimension. The following statement is easy.

**Proposition 6.** (i) Any minimal rich restriction space has dimension equal to the genus of the suspension surface.

(ii) Any maximal poor restriction space has codimension two.

Taking into account that the genus of the suspension surface is bounded from above by \( n/2 \) we see that maximal poor restriction spaces are larger in dimension than minimal rich ones if \( n > 4 \).

**Example 6.** Consider in more detail Example 1 from the Introduction. The restriction space \( \mathcal{R} \) is \((n-2)\)-dimensional and generated by \( h_1 - h_3, h_2 - h_4, \ldots, h_{n-2} - h_n \) in the notation of Section 2.2.

The restriction space
\[
\langle h_1 - h_3, h_2 - h_4, \ldots, h_{n-2} - h_n \rangle
\]
for interval exchange transformations with permutation (1) is poor.

Indeed, the space \( \text{Ann}(\mathcal{R}) \) is 2-dimensional and generated by \( e_1 + e_3 + \ldots + e_{n-1} \) and \( e_2 + e_4 + \ldots + e_n \).

For the bilinear form (2) we have
\[
\langle e_1 + e_3 + \ldots + e_{n-1}, e_2 + e_4 + \ldots + e_n \rangle = 3k - 1,
\]
where \( n = 2k \). Thus, \( \text{Ann}(\mathcal{R}) \) is not isotropic.

Since codimension of \( \mathcal{R} \) is two it is a maximal poor restriction space.

4. **Instability of Minimal Interval Exchange Transformations with Rich Restrictions**

Suppose we have a minimal interval exchange transformation \( T_{\pi,a} \) such that the full restriction space \( \mathcal{R}(T_{\pi,a}) \) is rich. Assume for simplicity that \( T_{\pi,a} \) is uniquely ergodic. What happens under small perturbation of the parameters?

In this case, as we have seen above, the asymptotic cycle of \( T_{\pi,a} \) lies in \( \mathcal{R}(T_{\pi,a}) \). Therefore, the cycles \( c_{\pi,a,k}(x) \) form a smaller and smaller angle with \( T_{\pi,a} \) when \( k \) grows. If we take \( a' \) close to \( a \) and satisfying the same restrictions, for large enough \( k \), some new cycles appear among \( c_{\pi,a',k}(x) \), which do not belong to \( C_{\pi,a,k} \) but are linear combinations with natural coefficients of cycles from \( C_{\pi,a,j} \) with \( j < k \).

These new cycles still form a small angle with \( \mathcal{R}(T_{\pi,a}) \) and there is a good chance that some of them will have the form \( \tilde{r} \) with \( r \in \mathcal{R}(T_{\pi,a}) \). This means that some new restriction that was not realized by \( T_{\pi,a} \) will be realized by \( T_{\pi,a'} \). The new realized restrictions may cause \( T_{\pi,a'} \) to be non-minimal. And if this happens, it remains true for \( T_{\pi,a''} \) if \( a'' \) is sufficiently close to \( a' \) and satisfies the same restrictions.

So, it is natural to expect that minimal interval exchange transformations \( T_{\pi,a} \) will never be \( \mathcal{R}(T_{\pi,a}) \)-stably minimal.

**Conjecture 2.** If \( \mathcal{R} \) is a rich restriction space with respect to \( \langle , \rangle_\pi \) and \( T_{\pi,a} \) satisfies all restrictions from \( \mathcal{R} \), then \( T_{\pi,a} \) is not \( \mathcal{R} \)-stably minimal.
Due to stability of non-minimality (Proposition 1) this conjecture immediately implies the following.

**Corollary 1** (to Conjecture 2). Let $\mathcal{F}$ be a rich restriction space with respect to $\langle \cdot \rangle_{\pi}$. Then the subset $M_{\pi, \mathcal{F}}$ of minimal exchange transformations from $X_{\pi, \mathcal{F}} = \{ T_{\pi, a} : \mathcal{F}(T_{\pi, a}) \supset \mathcal{F} \}$ is nowhere dense in $X_{\pi, \mathcal{F}}$.

In some cases, $M_{\pi, \mathcal{F}}$ is contained in a codimension one subset of $X_{\pi, \mathcal{F}}$, and in some other cases, the codimension of $M_{\pi, \mathcal{F}}$ in $X_{\pi, \mathcal{F}}$ is known to be between 0 and 1, see Example 4 and Subsection 5.6. This motivates us to conjecture that the following general statement is true.

**Conjecture 3.** Let $\mathcal{F}$ be a rich restriction space with respect to $\langle \cdot \rangle_{\pi}$. Then the subset $M_{\pi, \mathcal{F}}$ has zero Lebesgue measure in $X_{\pi, \mathcal{F}}$.

### 5. Examples

#### 5.1. Rank two interval exchange transformations.

The rank of an interval exchange transformation $T_{\pi, a}$ is defined as rank$_0(\{a_1, \ldots, a_n\})$. M. Boshernitzan [8] shows—in different terms—that a rank two minimal interval exchange transformations is always stably minimal and uniquely ergodic. This can be viewed as a simple illustration to the phenomenon discussed in this paper as we have the following.

**Proposition 7.** If $T_{\pi, a}$ is a rank two minimal interval exchange transformation, then $\mathcal{F}(T_{\pi, a})$ is poor.

**Proof.** If $T_{\pi, a}$ has rank two, then $\omega_{\pi, a}$ has the form $f^*(\lambda_1 d\varphi_1 + \lambda_2 d\varphi_2)$, where $f : \Sigma_{\pi, a} \to \mathbb{T}^2$ is a smooth map, $\varphi_1, \varphi_2$ are angular coordinates on the torus $\mathbb{T}^2$, and $\lambda_1, \lambda_2$ are incommensurable reals. The large freedom in choosing the map $f$ can be used to make the number of preimages of any point of $\mathbb{T}^2$ uniformly bounded from above by some $k \in \mathbb{N}$.

Let $\gamma$ be a regular leaf of $F_{\pi, a}$. Its image $f(\gamma)$ in $\mathbb{T}^2$ is contained in a straight line $\ell$, which is a leaf of an irrational winding of $\mathbb{T}^2$. If $T_{\pi, a}$ is minimal, then $\gamma$ is not closed. Since $f(\gamma)$ visits every point of $\ell$ at most $k$ times it follows that for a certain parametrization $\gamma(t)$ there must be a non-zero limit

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t df(\gamma(t)) = \infty.$$

This means that the image of the asymptotic cycle $c_{\pi, a}$ under the induced map $f_* : H_1(\Sigma_{\pi, a}) \to H_1(\mathbb{T}^2)$ is not zero. The kernel of this map is $\mathcal{F}(T_{\pi, a}) \cap H_1(\Sigma_{\pi, a})$, so we have $c_{\pi, a} \notin \mathcal{F}(T_{\pi, a})$. By Proposition 4 $\mathcal{F}(T_{\pi, a})$ is poor.

#### 5.2. Quadratic differentials.

It is well known that the measured singular foliation defined on a Riemann surface by the real part of a generic quadratic differential with prescribed multiplicities of zeros is minimal. H. Masur even shows in [26] that almost all such foliations are uniquely ergodic.

Let $M$ be an oriented surface. The family of all measured foliations on $M$ that can be defined by a quadratic differential $q$ on $M$ (for some complex structure) with zeros at fixed points with prescribed multiplicities gives rise to a family of foliations on the double cover $\hat{M}$ branched at zeros of $q$ of odd multiplicity. A foliation from this family can be defined by a closed 1-form on $\hat{M}$ but this 1-form is already not generic even if so is $q$.

More precisely, the family of 1-forms $\omega$ that we obtain in this way is characterized locally by the set $\mathcal{S}$ of zeros with fixed multiplicities and the condition $\iota^* \omega = -\omega$, where $\iota$ is the involution of $\hat{M}$ that exchanges the sheets of the covering map $\hat{M} \to M$.

Thus, we have a family of closed 1-forms with restrictions

$$\int_{\iota + \iota^*c} \omega = 0,$$

where $c \in H_1(\hat{M}, S; \mathbb{Z})$. The restriction space here is

$$\mathcal{R} = \{ c \in H_1(\hat{M}, S; \mathbb{R}) : c = \iota^*c \}$$

**Proposition 8.** Let $\iota : \hat{M} \to \hat{M}$ be an orientation preserving involution. Then the restriction space $\mathcal{R}$ is poor.
Let \( \hat{\tau} \) transformations with flips have periodic orbits. The suspension surface \( \Sigma \) there exists a non-zero cycle \( \omega \in H_1(M;\mathbb{R}) \) such that \( \iota_\ast \omega = -\omega \). Since \( \iota \) preserves the orientation of \( M \) it also preserves the intersection index. So, for any \( c' \in \mathcal{R} \cap H_1(M;\mathbb{R}) \) we have \( c \sim c' = \iota_\ast (c) \sim \iota_\ast (c') = -c \sim c' \), hence \( c \sim c' = 0 \). On the other hand, \( c \notin \mathcal{R} \). Thus \( \mathcal{R} \cap H_1(M;\mathbb{R}) \) is not coisotropic. \( \square \)

Masur’s result on the unique ergodicity for quadratic differentials \([20]\) supports our Conjecture \([1]\) in the part related to Theorem \([2]\) in this case.

5.3. Interval exchange maps with flips. A. Nogueira proved in \([32]\) that almost all interval exchange transformations with flips have periodic orbits. The suspension surface \( M \) of an interval exchange transformation with flips is defined similarly to the case of ordinary interval exchange transformations, but now it is non-orientable. The foliation \( \mathcal{F} \) induced on \( M \) has orientable leaves but it is not coorientable. Let \( \hat{M} \) be the orientation double cover of \( M \). The preimage of \( \mathcal{F} \) on \( \hat{M} \) can be defined by a closed 1-form \( \omega \).

Let \( \iota \) be the involution of \( \hat{M} \) that exchanges the sheets of the covering map \( \hat{M} \to M \). Similarly to the previous case we will have \( \iota^\ast \omega = -\omega \), which gives rise to the same restriction space \([8]\). However, now \( \iota \) flips the orientation of \( \hat{M} \), which inverses the conclusion.

**Proposition 9.** Let \( \iota : \hat{M} \to \hat{M} \) be an orientation reversing involution. Then the restriction space \([8]\) is rich.

**Proof.** The space \( H_1(\hat{M},\mathbb{R}) \) is symplectic, and the involution \( \iota_\ast \) restricted to it changes the sign of the symplectic form. This implies that \( H_1(\hat{M},\mathbb{R}) \) splits into a direct sum of \( \pm 1 \)-eigenspaces, which are both Lagrangian. The \(-1\)-eigenspace is \( \mathcal{R} \). \( \square \)

5.4. Measured foliations on non-orientable surfaces. C. Danthony and A. Nogueira proved in \([16]\) that almost all measured foliations on non-orientable surfaces have a compact leaf.

Let \( M \) be a non-orientable surface and \( \mathcal{F} \) a measured singular foliation on \( M \). Let \( \hat{M} \) be the double branched cover of \( M \) on which the preimage of \( \mathcal{F} \) becomes orientable, and \( \iota \) the corresponding involution of \( \hat{M} \).

If \( \hat{M} \) is orientable then the preimage of \( \mathcal{F} \) on \( \hat{M} \) is coorientable, and hence, can be defined by a closed 1-form \( \omega \). We come to exactly the same situation as in the previous example: \( \iota^\ast \omega = -\omega \), \( \iota_\ast [M] = -[M] \). Thus, foliations close to \( \mathcal{F} \) (with the same singularities) give rise to a family of 1-forms satisfying rich restrictions.

If \( \hat{M} \) is non-orientable, then the first return map of \( \mathcal{F} \) on a suitable transversal is an interval exchange map with flips and restrictions. So, after proceeding to the orientation double cover of \( \hat{M} \) measured foliations close to \( \mathcal{F} \) will translate to a family of 1-forms satisfying all restrictions from \([8]\) and some additional restrictions. But the restriction space \([8]\) is already rich, so the larger restriction space will also be rich.

5.5. Novikov’s problem. Let \( M \) be a closed and homologous to zero surface smoothly embedded in the 3-torus \( \mathbb{T}^3 \) and \( H \in \mathbb{R}^3 \) a non-zero vector. Denote by \( \eta \) the following 1-form on \( \mathbb{T}^3 \): \( \eta = H_1 \, dx_1 + H_2 \, dx_2 + H_3 \, dx_3 \), where \( x_i \) are the angular coordinates (defined up to \( 2\pi k \) with \( k \in \mathbb{Z} \)) on \( \mathbb{T}^3 \), and by \( \omega \) the restriction \( \eta|_M \).

The closed 1-form \( \omega \) defines a foliation on \( M \) whose leaves lifted to \( \mathbb{R}^3 \) may or may not be open and have an asymptotic direction. In 1982 S. Novikov suggested to study their behavior in connection with the theory of conductivity in normal metals \([20]\).

The forms \( \omega \) that arise in this situation are not generic since \( \int_M \omega = 0 \) whenever \( c \in H_1(M,\mathbb{Z}) \) has zero image in \( H_1(\mathbb{T}^3,\mathbb{Z}) \) under the embedding \( M \to \mathbb{T}^3 \). Locally these are the only restrictions, so, the problem can be reduced to studying interval exchange transformations with the restriction space \( \mathcal{R}(i) \) generated by all cycles \( c \in H_1(\Sigma,\mathbb{Z}) \) such that \( \iota_\ast (c) = 0 \), where \( \Sigma \) is the suspension surface and \( \iota : \Sigma \to \mathbb{T}^3 \) is some embedding.

Note, however, that it is not clear whether all interval exchange transformations obtained in this way can come from the restriction of a 1-form on \( \mathbb{T}^3 \) with constant coefficients to a surface. So, the family of transformations that can arise in Novikov’s problem is probably a proper part of the family of all interval
exchange transformations with restrictions of the form \( \mathcal{R}(i) \), but it has the same number of degrees of freedom.

**Proposition 10.** If \( i_*([\Sigma]) = 0 \in H_2(\mathbb{T}^3, \mathbb{Z}) \), then the restriction space \( \mathcal{R}(i) \) is rich.

**Proof.** Let \( \omega_1 \) and \( \omega_2 \) be two closed 1-forms on \( \Sigma \) whose homology classes satisfy all restrictions from \( \mathcal{R}(i) \). Then there are closed 1-forms \( \eta_1, \eta_2 \) on \( \mathbb{T}^3 \) such that \( \omega_j = i^*\eta_j, j = 1, 2 \). We have

\[
\langle \omega_1, \omega_2 \rangle = \int_\Sigma \omega_1 \wedge \omega_2 = \int_\Sigma i^*(\eta_1 \wedge \eta_2) = \int_\Sigma \eta_1 \wedge \eta_2 = 0
\]

since \( i_*[\Sigma] = 0 \).

We see that \( \text{Ann}(\mathcal{R}(i)) \) is isotropic, so \( \mathcal{R}(i) \) is rich. \( \square \)

A. Zorich’s result \[42\] can be interpreted in terms of interval exchange transformations as follows: minimal foliations arising in Novikov’s problem are never \( \mathcal{R}(i) \)-stable. It was shown later in \[13\] that almost all foliations in Novikov’s problem are not minimal. Moreover, all minimal foliations reside in a codimension one subset \( X \) of the set of all relevant foliations.

The subset \( X \) is defined (locally) by another restriction, and if we add this restriction to the corresponding \( \mathcal{R}(i) \) the question of minimality becomes highly nontrivial. It is conjectured (in different terms) that almost all foliations from \( X \) are not minimal \[25\].

In the genus 3 case, a two parameter subfamily of \( X \) is studied in \[12\] and it is shown that minimal foliations in \( X \) form a subset known as the Rauzy gasket.

### 5.6. Systems of partial isometries.

**Definition 14.** By a system of partial isometries we mean a collection \( \Psi = \{\psi_1, \ldots, \psi_k\} \) of interval isometries \( \psi_i : [a_i, b_i] \to [c_i, d_i] \), where \( [a_i, b_i] \) and \( [c_i, d_i] \) are subintervals of \([0, 1]\) of equal length for each \( i = 1, \ldots, k \). It is required that \( \{a_1, c_1, \ldots, a_k, c_k\} \) contains 0 and \( \{b_1, d_1, \ldots, b_k, d_k\} \) contains 1. The isometries \( \psi_i \) can preserve or reverse orientation. The system is called orientation preserving if all \( \psi_i \)'s preserve orientation.

Two systems of partial isometries obtained from each other by permuting \( \psi_i \)'s and replacing some \( \psi_i \) by \( \psi_i^{-1} \) are considered equal.

The \( \Psi \)-orbit of a point \( x \in [0, 1] \) is the set of images of \( x \) under all compositions \( \psi_i^{\pm 1} \circ \ldots \circ \psi_1^{\pm 1}, m = 0, 1, 2, \ldots, \) that are defined at \( x \).

With every system of partial isometries \( \Psi \) one associates a foliated 2-dimensional complex called the suspension complex of \( \Psi \) and denoted here \( \Sigma_\Psi \), see \[17\]. Such foliated complexes are particular cases of band complexes that were studied in connection with geometric group theory and the theory of \( \mathbb{R} \)-trees \[7\], \[17\].

Minimal band complexes (the notion of minimality is slightly more subtle here but for \( \Sigma_\Psi \), outside of a codimension one subset of such complexes, it is equivalent to saying that orbits of \( \Psi \) are everywhere dense) are classified by E. Rips into toral type, surface type, and thin type \[17\].

Oriented systems of partial isometries can be viewed as a generalization of interval exchange transformations so that interval exchange transformations will appear as systems of partial isometries satisfying certain integral linear restrictions on the parameters. There is, however, another relation between these objects that associates an interval exchange transformation satisfying a number of restrictions to any orientation preserving system of partial isometries.

**Definition 15.** Let \( \Psi = \{\psi_i : [a_i, b_i] \to [c_i, d_i] : i = 1, \ldots, k\} \) be an orientation preserving system of partial isometries and \( y = (y_1, y_2, \ldots, y_{2k}) \) a 2k-tuple of pairwise distinct points of \((0, 1)\). Such a 2k-tuple will be referred to as admissible for \( \Psi \). The surface \( \Sigma_{\Psi, y} \) that is constructed below will be called a double suspension surface of \( \Psi \).

Choose \( \varepsilon > 0 \) so small that the intervals \([y_i, y_i + \varepsilon]\) are pairwise disjoint and contained in \([0, 1]\). The surface \( \Sigma_{\Psi, y} \) is obtained from

\[
D = [0, 1] \times [0, 1] \setminus \bigcup_{i=1}^k ([a_i, b_i] \times (y_i, y_i + \varepsilon) \cup (c_i, d_i) \times (y_{i+k}, y_{i+k} + \varepsilon))
\]
by making the following, orientation and measure preserving, identifications:

\[ [0,1] \times \{0\} \text{ with } [0,1] \times \{1\}, \]

\[ [a_i, b_i] \times \{y_i\} \text{ with } [c_i, d_i] \times \{y_{i+k} + \varepsilon\}, \]

\[ [a_i, b_i] \times \{y_i + \varepsilon\} \text{ with } [c_i, d_i] \times \{y_{i+k}\} \]

and collapsing to a point every straight line segment in the boundary of the domain \( D \), namely, \( \{0\} \times [0,1], \{1\} \times [0,1], a_i \times \{y_i, y_i + \varepsilon\}, b_i \times \{y_i, y_i + \varepsilon\}, c_i \times \{y_{i+k}, y_{i+k} + \varepsilon\}, \) and \( d_i \times \{y_{i+k} + \varepsilon\}, i = 1, \ldots, k. \)

The surface \( \hat{\Sigma}_{\Psi, Y} \) comes with a closed differential 1-form \( \omega \) whose pullback in \( D \) is \( d\xi \) (a smooth structure on \( \hat{\Sigma}_{\Psi, Y} \) can be chosen so that \( \omega \) is smooth). The 1-form \( \omega \) defines a measured orientable foliation \( \hat{\mathcal{F}}_{\Psi, Y} \), whose first return map on the transversal \( \gamma = 0,1] \times \{0\} \sim [0,1] \times \{1\} \) is an interval exchange. We denote this transformation by \( T_{\Psi, Y} \).

It is not, however, necessarily true that all leaves of the foliation \( \hat{\mathcal{F}}_{\Psi, Y} \) meet \( \gamma \). If it is we say that the transformation \( T_{\Psi, Y} \) fills the surface \( \hat{\Sigma}_{\Psi, Y} \). If it is not, then the foliation \( \hat{\mathcal{F}}_{\Psi, Y} \) is not minimal.

If \( T_{\Psi, Y} \) fills \( \hat{\Sigma}_{\Psi, Y} \), then the minimality of \( \hat{\mathcal{F}}_{\Psi, Y} \) is equivalent to that of \( T_{\Psi, Y} \). Indeed, in this case, the foliated surface \( \hat{\Sigma}_{\Psi, Y} \) can be identified with the suspension surface of \( T_{\Psi, Y} \), possibly after collapsing to a point some saddle connections. Such collapsing will not be essential in the sequel, so we will think of \( \hat{\Sigma}_{\Psi, Y} \) as the suspension surface for \( T_{\Psi, Y} \) provided that the latter fills \( \hat{\Sigma}_{\Psi, Y} \).

**Proposition 11.** If \( T_{\Psi, Y} \) fills \( \hat{\Sigma}_{\Psi, Y} \), then the full restriction space \( \mathcal{R}(T_{\Psi, Y}) \) is rich.

**Proof.** The point is that \( \hat{\Sigma}_{\Psi, Y} \) can be realized as the boundary of a handlebody \( H_{\Psi, Y} \) such that \( \omega \) is continued to \( H_{\Psi, Y} \) as a closed 1-form. The handlebody is obtained from \([0,1]^3\) by the following identifications:

1. each of the two squares \( \{0\} \times [0,1] \times [0,1] \) and \( \{1\} \times [0,1] \times [0,1] \) is collapsed to a point;
2. for any \( x \in (0,1) \) the straight line segment \([x,0,0], (x,1,0)\) is collapsed to a point;
3. the square \([0,1] \times \{0\} \times [0,1] \) is identified with \([0,1] \times \{1\} \times [0,1] \) by the projection along the second axis;
4. for each \( i = 1, \ldots, k \) the rectangle \([a_i, b_i] \times \{y_i, y_i + \varepsilon\} \times \{1\} \) is identified with \([c_i, d_i] \times \{y_{i+k}, y_{i+k} + \varepsilon\} \times \{1\} \) by the map \((x, y, 1) \mapsto (x + c_i - a_i, y_{i+k} + \varepsilon - y_i - 1, 1)\);
5. collapse each straight line segment \([a_i] \times \{y_i, y_i + \varepsilon\} \times \{1\} \) and \([b_i] \times \{y_i, y_i + \varepsilon\} \times \{1\} \) to a point, \( i = 1, \ldots, k. \)

One can see that this produces a handlebody whose boundary is obtained from \( D \times \{1\} \) by the same identifications as in Definition [15].

Now, it is a general fact that any closed 1-form on a handlebody \( H \) restricted to \( \partial H \) satisfies a rich set of restrictions, which always contains the kernel of the map \( H_1(\partial H, \mathbb{Z}) \rightarrow H_1(H, \mathbb{Z}) \) induced by the inclusion \( \partial H \rightarrow H \). Indeed, for any two such forms \( \omega_1, \omega_2 \) the product \( \omega_1 \wedge \omega_2 \) is exact as \( H^2(H) = 0 \). Therefore, \( \int_{\partial H} \omega_1 \wedge \omega_2 = 0 \).

So, according to Conjecture [3] we expect that interval exchange transformations coming from systems of partial isometries and the double suspension surface construction will typically be non-minimal if they happen to fill the double suspension surface.

With every band complex \( \Sigma \) Bestvina and Feighn associate a number \( c(\Sigma) \) called the excess of \( \Sigma \), see [7] for the definition. For any \( \Sigma \), the Rips machine generates a sequence \( \Sigma_0 = \Sigma, \Sigma_1, \Sigma_2, \ldots \) of band complexes such that \( c(\Sigma_{i+1}) \leq c(\Sigma_i) \) for all \( i \). Moreover, for some \( n \), we have \( c(\Sigma_n) = c(\Sigma_{n+i}) \) for all \( i \geq 0 \). We call \( c(\Sigma_n) \) the asymptotic excess of \( \Sigma \).

In the case when \( \Psi = \{\psi_i : [a_i, b_i] \rightarrow [c_i, d_i] ; i = 1, \ldots, k\} \) is an orientation preserving system of partial isometries such that the differences \( c_i - a_i, i = 1, \ldots, k \), are independent over \( \mathbb{Q} \) the excess of the suspension complex is simply

\[ c(\Sigma_\Psi) = \sum_{i=1}^{k} (b_i - a_i) - 1, \]

and it coincides with the asymptotic excess of \( \Sigma_\Psi \).
In any case, the asymptotic excess of $\Sigma_\Psi$ is equal to a non-trivial integral linear combination of the parameters $a_i, b_i, c_i, d_i, i = 1, \ldots, k$ (by definition one of $a_i$’s is zero, so it is understood that this $a_i$ is excluded from consideration when we speak about dependencies and linear combinations of the parameters). So, if the parameters are rationally independent the asymptotic excess is not zero. We think that this implies non-minimality of $\hat{F}_\Psi, y$ for any $y$.

**Conjecture 4.** Let $\Psi$ be an orientation preserving system of partial isometries. If $\Sigma_\Psi$ has non-zero asymptotic excess, then, for any admissible $y$, the corresponding interval exchange transformation $T_{\Psi, y}$ either is non-minimal or does not fill $\hat{\Sigma}_{\Psi, y}$.

One half of this conjecture is easy: if the asymptotic excess of $\Sigma_\Psi$ is negative, then $\Sigma_\Psi$ has compact leaves (equivalently, $\Psi$ has finite orbits), which immediately implies that $\hat{F}_{\Psi, y}$ has compact leaves for any admissible $y$, and thus, is not minimal.

If the asymptotic excess is positive, then $\Sigma_\Psi$ is either not minimal (and then the conjecture also follows easily) or of toral type. So, it remains to establish the conjecture in the toral case. So far we can do this in the case $k = 3$ for a special choice of $y$.

**Proposition 12.** Let $\Psi = \{\psi_i : [a_i, b_i] \to [c_i, d_i] : i = 1, 2, 3\}$ be an orientation preserving system of partial isometries and let $y_1 < y_2 < \ldots < y_k$. Then the inequality $\sum_{i=1}^{k}(b_i - a_i) > 1$ (which holds whenever the asymptotic excess of $\Sigma_\Psi$ is positive) implies that $\hat{F}_{\Psi, y}$ is not minimal.

A proof of this fact can be extracted from [14] where $\Psi$ and the double suspension surface are realized in the 3-torus so that the foliations on both are induced by plane sections of a fixed direction.

**Definition 16.** A system of partial isometries $\Psi = \{\psi_i : [a_i, b_i] \to [c_i, d_i] : i = 1, \ldots, k\}$ will be called balanced if $\sum_{i=1}^{k}(b_i - a_i) = 1$.

**Proposition 13.** Let $\Psi$ be a balanced and orientation preserving system of partial isometries, and $y$ an admissible vector for $\Psi$. If, in addition, $\Psi$ is of thin type, then the transformation $T_{\Psi, y}$ is minimal and fills $\hat{\Sigma}_{\Psi, y}$.

**Proof.** We give only a sketch leaving details to the reader, who is assumed being familiar with the theory of band complexes and the Rips machine.

As noted above, $T_{\Psi, y}$ is minimal and fills $\hat{\Sigma}_{\Psi, y}$ if and only if $\hat{F}_{\Psi, y}$ is minimal.

There is a leafwise embedding of the suspension complex $\Sigma_\Psi$ into the foliated handlebody $H_{\Psi, y}$ introduced in the proof of Proposition 11 such that, for any leaf $L$ of $\Sigma_\Psi$, the embedding of $L$ into the corresponding leaf of $H_{\Psi, y}$ is a quasi-isometry and a homotopy equivalence. The image of $\Sigma_\Psi$ in $H_{\Psi, y}$ is the union of the straight line segment $[0, 1] \times \{0\} \times \{0\}$ and the rectangles

$$\bigcup_{i=1}^{k}([a_i, b_i] \times \{y_i + \varepsilon/2\} \times [0, 1] \cup [c_i, d_i] \times \{y_i + k + \varepsilon/2\} \times [0, 1])$$

under the identifications from the proof of Proposition 11.

Since $\Psi$ is balanced, the presence of compact leaves in $\Sigma_\Psi$ is equivalent to the presence of regular leaves that are not simply connected. So, if $\Psi$ is minimal, then all regular leaves of $\Sigma_\Psi$ are infinite trees.

If $\Psi$ is of thin type, then there is an everywhere dense leaf $L$ of $\Sigma_\Psi$ having form of a 1-ended infinite tree. The corresponding leaf of $H_{\Psi, y}$ has connected boundary, which is everywhere dense in $\partial H_{\Psi, y} = \hat{\Sigma}_{\Psi, y}$. Therefore, the foliation in $\hat{\Sigma}_{\Psi, y}$, and hence, the foliation $\hat{F}_{\Psi, y}$, is minimal. \square

Propositions 11 and 13 together with Conjecture 2 (if proven) imply that being of thin type is an unstable property of a balanced system of partial isometries. We will prove that instability does occur in the case when the balancedness is the only linear restriction on the parameters of $\Psi$.

**Proposition 14.** Let $\Psi$ be a balanced and orientation preserving system of partial isometries with parameters that do not satisfy any linear integral restriction except balancedness. Let $y$ be an admissible vector such that $T_{\Psi, y}$ fills the double suspension surface $\hat{\Sigma}_{\Psi, y}$. Then $T_{\Psi, y}$ is not $\mathcal{A}(T_{\Psi, y})$-stably minimal.
Figure 4. Double suspension surface. Each vertical straight line segment in $\partial D$ is collapsed to a point, the bottom side of the square is identified with the top one.

**Proof.** For any open neighborhood $U$ of $\Psi$ one can find a system of partial isometries $\Psi' = \{\psi'_i : [a'_i, b'_i] \to [c'_i, d'_i] ; i = 1, \ldots, k\}$ in $U$ such that

1. the shifts $c'_i - a'_i$ are pairwise commensurable;
2. no integral linear relation for the parameters of $\Psi'$ holds that is not a consequence of this commensurability and the balancedness of $\Psi'$.

Then the 1-form $\omega = dx$ on the handlebody $H_{\Psi', y}$ has pairwise commensurable periods over all integral cycles. Therefore, it is proportional to the pullback of the angular form $d\phi$ on the circle $S^1$ under a map $\theta : H_{\Psi', y} \to S^1$. The connected components of the sets $\theta^{-1}(x), x \in S^1$, are the leaves of the foliation $F_{\Psi', y}$ induced by $\omega$ on $H_{\Psi', y}$.

The absence of additional relations on the parameters implies that any subset $\theta^{-1}(x)$ contains only one singularity of $F_{\Psi', y}$. Therefore, the Euler characteristics $\chi(\theta^{-1}(x))$, which is a piecewise constant function of $x$, can jump only by one when $x$ varies continuously. On the other hand, the balancedness of $\Psi'$ implies that $\chi(\theta^{-1}(x))$ averages to zero on $S^1$. Therefore, for some $x \in S^1$, we must have $\chi(\theta^{-1}(x)) > 0$, which means that at least one of the connected components of $\theta^{-1}(x)$ is a disc. The boundary $\rho$ of this disc is a realization of some restriction $r$ that holds for $T_{\Psi, y}$ since $\rho$ is homologous to zero in $H_{\Psi', y}$. For any $\Psi''$ close enough to $\Psi'$ the restriction $r$ will be realized also by $T_{\Psi'', y}$. \qed

Now we consider a particular family of systems of partial isometries with $k = 3$ in which the subset of systems giving rise to minimal interval exchange transformations is precisely known, and explain the origin of Example [4]

Let $\Psi$ be an orientation preserving system of partial isometries of the form

$$\{\psi_i : [0, b_i] \to [c_i, 1], \ i = 1, 2, 3\}, \quad b_1 < b_2 < b_3 < c_2, \ c_3 < b_3, \ 2b_3 < b_2 + c_1, \ b_i + c_i = 1,$$

and $y \in (0, 1)^6$ be an arbitrary vector with $y_1 < y_2 < \ldots < y_6$. The domain $D$ and identifications in it to obtain $\tilde{\Sigma}_{\Psi, y}$ are shown in Fig. [3]

One can see from the left picture that the interval exchange transformation $T_{\Psi, y}$ fills the double suspension surface and is associated with the following permutation:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 5 & 7 & 2 & 6 & 1 & 4 \end{pmatrix}$$
and the following vector of parameters:

\[ \mathbf{a} = (b_1, b_2 + c_1 - 2b_3, b_3 - c_3, b_3 - b_2, c_2 - b_3, b_2 - b_1, b_1). \]

The dashed lines in the left picture of Fig. 4 show the segments of separatrices between the singularity and the first intersection with \([0,1] \times \{0\}\). The corresponding foliation on the double suspension surface has two singularities each of which is a double saddle.

If the excess \( e(\Psi) = b_1 + b_2 + b_3 - 1 \) is not equal to zero, then \( T_{\Psi,y} \) is not minimal. In the case \( e(\Psi) = 0 \) the double suspension surface can be identified with the quotient of the regular skew polyhedron \( \{4,6\mid 4\} \) by \( \mathbb{Z}^3 \), embedded in the 3-torus with the foliation induced by the 1-form \( (b_2 + b_3) \ dx_1 + (b_3 + b_1) \ dx_2 + (b_1 + b_2) \ dx_3 \), see \( [12,15] \). The minimality of \( T_{\Psi,y} \) is equivalent to \( \Psi \) being of thin type. This occurs if and only if the point \( (b_1 : b_2 : b_3) \in \mathbb{RP}^2 \) belongs to the Rauzy gasket, which is defined as follows.

Denote by \( \Delta \) the following subset of the projective plane \( \mathbb{RP}^2 \): \( \Delta = \{(x_1 : x_2 : x_3) : x_i \geq 0, i = 1, 2, 3\} \), and by \( P_1, P_2, P_3 \), the projective transformations with matrices

\[
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix},
\]

respectively.

Points of the Rauzy gasket are in one-to-one correspondence with infinite sequences \((i_1, i_2, \ldots) \in \{1, 2, 3\}^\mathbb{N}\) in which every element of \( \{1, 2, 3\} \) appears infinitely often. For any such sequence the intersection

\[
\bigcap_{m=1}^{\infty} (P_{i_1} \circ \ldots \circ P_{i_m})(\Delta)
\]

consists of a single element, which is the point of the Rauzy gasket corresponding to the sequence.

The interval exchange transformations from Example 4 was produced by using the construction above with the additional assumption \( b_1 + b_2 < b_3 \). One can see from the right picture in Fig 4 that the first return map for the shorter transversal \([0, b_2 + c_3]\) will be the interval exchange transformation associated with the permutation

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
3 & 6 & 5 & 2 & 7 & 4 & 1
\end{pmatrix}
\]

and the following vector of parameters:

\[
\frac{1}{b_2 + c_3}(b_1, b_2 + c_1 - 2b_3, b_3 - c_3, b_3 - b_1 - b_2, b_1, c_2 - b_3, 2b_2 - b_3). \]

The parameters \( a_i, i = 1, 2, 3 \), and \( e \) from Example 4 are related to these as follows:

\[
a_1 = \frac{b_1}{b_2 + c_3}, \quad a_2 = \frac{b_2 + c_1 - 2b_3}{b_2 + c_3}, \quad a_3 = \frac{b_3 - c_3}{b_2 + c_3}, \quad e = \frac{e(\Psi)}{b_2 + c_3} = \frac{b_1 + b_2 + b_3 - 1}{b_2 + c_3}. \]

The triangle defined by

\[
a_i \geq 0, \quad 3a_1 + 2a_2 + 2a_3 = 1
\]

coincides with \( P_3(P_2(\Delta)) \), so it intersects the mentioned above Rauzy gasket in a subset that is also projective equivalent to the Rauzy gasket.

5.7. Interval translation mappings. Interval translation mappings were introduced by M. Boshernitzan and I. Kornfeld in \( [10] \).

**Definition 17.** An interval translation mapping is a map \( T : [0,1) \to [0,1) \) of the form

\[
T(x) = x + t_i \quad \text{if} \quad \sum_{j=1}^{i-1} \lambda_j \leq x < \sum_{j=1}^{i} \lambda_j,
\]
where $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a probability vector and $t = (t_1, \ldots, t_k)$ is a real vector whose coordinates satisfy the inequalities

$$-\sum_{j=1}^{i-1} \lambda_j \leq t_i \leq 1 - \sum_{j=1}^{i} \lambda_j.$$ 

The map defined by these parameters will be denoted $T_{\lambda, t}$.

An interval translation mapping $T$ is said to be of finite type if for some $m \in \mathbb{N}$ the image of $T^{m+1}$ coincides with that of $T^m$, and otherwise of infinite type.

M. Boshernitzan and I. Kornfeld ask in [10] whether almost all interval translation mappings are of finite type. This question is answered in the positive in the particular cases of so called double rotations, see [35], [11], and [9], and for interval translation mappings on three intervals (i.e. in the case $k = 3$), see [11].

In general, this problem is an instance of our Conjecture 3. Indeed, an interval translation mapping can be viewed as an orientation preserving system of partial isometries $\Psi = \{\psi_i : [a_i, b_i] \rightarrow [c_i, d_i] : i = 1, \ldots, k\}$ such that $a_1 = 0$, $a_i = b_{i-1}$ for $i = 2, \ldots, k$, $b_k = 1$. The only difference is that $\psi_i$’s are defined on the whole interval $[a_i, b_i]$, including the right endpoint, which is inessential. ‘Infinite type’ for an interval translation mappings means exactly ‘thin type’ for the corresponding system of partial isometries.

One can show that for this kind of systems of partial isometries the converse statement to Proposition 13 is also true: the foliation $\mathcal{F}_{\Psi, y}$ is minimal if and only if $\Psi$ is of thin type. So, the problem of studying finite type interval translation mappings translates exactly into a problem of studying interval exchange maps satisfying certain rich set of integral linear restrictions.

Note, however, that the minimality of $\mathcal{F}_{\Psi, y}$ is broken here in a ‘non-standard’ way. Typically, for a generic system of isometries $\Psi$ the reason for $\mathcal{F}_{\Psi, y}$ to be non-minimal is the presence of a closed regular leaf, which does not occur for $\Psi$ corresponding to a generic interval translation mapping. If $\Psi$ comes from a finite type interval translation mapping $T$, then $\mathcal{F}_{\Psi, y}$ decomposes into two halves having the same qualitative behavior. In particular, if $T$ is a minimal interval exchange transformation, then $\mathcal{F}_{\Psi, y}$ will have two minimal components each having $T$ as the first return map for some transversal.

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