Polydimensional Relativity, a Classical Generalization of the Automorphism Invariance Principle *

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Abstract
The automorphism invariant theory of Crawford has shown great promise, however its application is limited by the paradigm to the domain of spin space. Our conjecture is that there is a broader principle at work which applies even to classical physics. Specifically, the laws of physics should be invariant under polydimensional transformations which reshuffle the geometry (e.g. exchanges vectors for trivectors) but preserves the algebra. To complete the symmetry, it follows that the laws of physics must be themselves polydimensional, having scalar, vector, bivector etc. parts in one multivector equation. Clifford algebra is the natural language in which to formulate this principle, as vectors/tensors were for relativity. This allows for a new treatment of the relativistic spinning particle (the Papapetrou equations) which is problematic in standard theory. In curved space the rank of the geometry will change under parallel transport, yielding a new basis for Weyl’s connection and a natural coupling between linear and spinning motion.

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*FTP://www.clifford.org/clf-alg/preprints/1996/pezza9601.latex
I. Introduction

There has been relatively few new physical principles proposed which are based upon the unique structure of geometric algebra. A notable exception is the form of spin gauge theory put forth by Crawford\cite{8}. His proposition is that quantum mechanics should be form invariant under local changes in spinor space basis (equivalently the matrix representation of the Dirac algebra can be different at each point in space). The motivation is to have a unified theory in which the gauge fields of curvature describe gravity as well as all the other fundamental forces.

The action of this local metric-preserving automorphism transformation is to “mix up” the basis elements of the full Clifford algebra, such that the basis vector generators $\gamma_\mu$ at one point could be a mixture of the bivector, trivector, etc. at another point. However, a reshuffling of this “spin” geometry $\gamma_\mu$ (i.e. the “soldering forms” which connect the spinor basis to the tangent basis of spacetime) will not change the physical basis vectors $e_\mu$ of real geometric space-time into something other than a vector. The two algebras are independent; any element of the “spin” Clifford algebra $\gamma_\mu$ will necessarily commute\cite{11} with the basis vectors $e_\alpha$. In order to get the curvature of the spin space to “create” curvature in coordinate spacetime, a constraint must be imposed by fiat. For example, in general relativity, the condition that the covariant derivative of the metric tensor will vanish is equivalent to stating that the universe has the geometric structure of a Riemann space. In spin gauge theory, the different constraint choices imposed by various authors (usually obscured in some reasonable sounding assumption) is making some sort of classification of the type of spin space plus geometry space in which unified phenomena exists.

The most unambiguous way to choose the connection between spinor space and coordinate space is to simply have one unified geometric language for classical fields and quantum mechanics. Column spinors are replaced by geometric multispinors (aggregates of scalar, vector, bivector, etc.) which are left ideals of the algebra\cite{10}. Now the Dirac matrices $\gamma_\mu$ can be varying linear combinations of only the basis vectors $e_\alpha$ at each point in space, with necessarily vanishing covariant derivative (whereas Crawford has it to be non-vanishing). The general automorphism transformation must be disallowed because it would reshuffle the full spin algebra. Except for electromagnetic and gravitational fields, all of Crawford’s interesting features are necessarily suppressed. In order to describe other interactions, Chisholm and Farwell\cite{9} resort to introducing higher dimensions to the Clifford algebra (at last count 7 extra dimensions on top of the 4 of spacetime).

Our own approach has been to stay within the 4D algebra, but make use of all 16 geometric degrees of freedom in the multivector wavefunction to describe multiple generations of particles\cite{10}. In order to accommodate all the known couplings, we were heuristically led to consider a new form of bilateral (left and right sided) multiplication on the wavefunction that can not be derived from a gauge transformation. The action of this operation is equivalent to a linear transformation on the full Clifford algebra, and hence can be cast into a form
which resembles automorphism gauge theory. The problem is that Crawford’s principle [8] is limited by the paradigm to spin space. We take a big leap and propose that classical physics obeys the automorphism principle. This has broad consequences to both special and general relativity, some examples of which are explored in the following sections.

II. Extension of Special Relativity

Einstein required the laws of physics to be invariant under Lorentz transformations, which “rotate” between scalar time and vector space. We propose a generalization: the laws should be invariant under Automorphism transformations which reshuffle vector space with bivectors, trivectors, etc.

A. Review of Standard Theory

According to Minkowski, the world is a four dimensional continuum, which we often call spacetime. Events \( \Sigma \) are points in the manifold with coordinates \( (t, x, y, z) \), where the fourth dimension is “time”. One of the postulates that Einstein put forth is that the speed of light is the same for all observers; equivalently the speed of light “\( c \)” is a physical limit which cannot be exceeded. Geometrically this forces the metric measure of time to be the opposite sign as the other dimensions, such that distance \( d \Sigma \) between two points is measured as the root of,

\[
    c^2 d\tau^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2).
\]

(1)

The affine parameter \( \tau \) is commonly called the proper time. The other postulate upon which relativity is based is that the laws of physics are invariant in all inertial (nonaccelerated) frames. Specifically this means that physical formulations must be the same in reference frames which differ only by constant velocity; equivalently formulas [such as eq. (1)] must be invariant under the Lorentz group \( SL(2, \mathbb{C}) \).

The principle of least action states that a particle will “choose” to take the path of least distance (in spacetime). Using the calculus of variations, one minimizes the action integral, based upon eq. (1),

\[
    A = \int L d\tau = \int m_0 c d\tau = \int m_0 c \sqrt{u^\alpha u_\alpha} \, d\tau,
\]

(2)

which is clearly invariant under the Lorentz group. The integrand \( L \) is called the Lagrangian, which is generally a function of the coordinates \( x^\alpha \) and the velocities \( u^\alpha = \dot{x}^\alpha = dx^\alpha / d\tau \). The four-momentum \( p^\mu \),

\[
    p^\mu = \frac{\delta L}{\delta u^\mu} = m_0 u^\mu,
\]

(3a)

is conserved in time. In addition to having one more component, it differs from
the non-relativistic three-momentum $\vec{P} = m\vec{v}$ by the Lorentz factor $\gamma$, 

$$\gamma = \frac{dt}{d\tau} = \frac{v^0}{c} = \left(1 - \frac{v^2}{c^2}\right)^{-1},$$  

which appears in kinematic formulas as the “relativistic correction” (e.g. mass increases by: $m = \gamma m_0$).

Non-relativistically, rotational motion is “uncoupled” from the linear motion. This is not the case in relativistic theory where the Pauli-Lubanski spin polarization four-vector $s^\mu$ must everywhere be perpendicular to the four-momentum: $p_\mu s^\mu = 0$, (known as the Dixon transversality condition, other authors use the slightly inequivalent Frenkel condition: $\dot{x}_\mu s^\mu = 0$). Hence if the linear motion changes with time, so must the spin. One can argue for reciprocal effects. When a particle is boosted in a direction perpendicular to its spin, the mass on one side is moving faster than on the other, causing an asymmetric relativistic mass distribution resulting in a sideways shift of the center-of-mass. Under either linear or angular acceleration this causes a sideways contribution to the momentum. Hence the conserved momentum is no longer parallel to the velocity, 

$$p^\mu = m\dot{x}^\mu + \dot{S}^{\mu\alpha} \dot{x}_\alpha,$$

$$p^\mu \dot{x}_\mu = -m_0 c,$$

$$\dot{S}^{\mu\alpha} = \frac{1}{2} \epsilon^{\mu\alpha\beta\delta} p_\beta s_\delta,$$

$$\dot{S}_{\mu\nu} = \dot{x}_\mu p_\nu - \dot{x}_\nu p_\mu.$$  

There is some disagreement over the proper form of these equations (we have followed Barut). Interestingly, the equations of motion appear to admit self-sustaining circular solutions with no net momentum, for which there are various possible physical interpretations. This feature may be an artifact of the coordinates no longer being a true description of the center-of-mass of a spinning particle. Regardless, the problem at hand is that it is difficult to find a generalization of eq. (2) which will simultaneously give both the equation of motion for the translation and the spin. A recent review of the various methods is given by Frydryszak.

B. The Clifford Manifold

We propose that space is a fully polydimensional continuum. Each event $\Sigma$ is a generalized “point” in a Clifford manifold which has a coordinate $q^A$ associated with each basis multivector element $E_A$ of the geometry. As an example, consider a disk (hockey puck) constrained to move on a 2D (flat) Euclidean surface. The set of basis elements \{\(E_A\}\} generated by two anticommuting basis vectors is: \(\{E_0, E_1, E_2, E_3\} = \{1, e_1, e_2, e_1 \wedge e_2\}\). The event’s coordinates are \(\Sigma = \Sigma(c, x, y, R\theta)\), where the position is given by \((x, y)\) and the scalar time needs the universal constant of the speed of light “c” applied to convert the
scale to distance units. The bivector coordinate $\theta$ tells the angular position of the hockey puck. In order to have units of distance, we need another fundamental physical constant $R$ which we loosely interpret as the radius of gyration (for a fundamental particle it will be within a geometric factor of the Compton wavelength).

The Clifford algebra associated with the $(++)$ metric signature is: $R(2) = M(2, R) = \text{End} \ R^{2,0}$, isomorphic with two-by-two real matrices. The unit bivector $E_3 = e_1 \wedge e_2$ must then square to negative unity. The differential element $d\Sigma$ and its main involution $d\Sigma$ are,

from which we can construct a scalar quadratic form analogous to eq. (1),

which is invariant under the six parameter correlated automorphism group $O(2, 2; R)$.

In special relativity, the affine “proper time” is not the same as the ordinary time of non-relativistic space. In polydimensional relativity, the new affine parameter $d\lambda$ of eq. (5c) for the spinning particle is not the same as the proper time of special relativity. The latter corresponds instead to an equivalent colinear non-spinning particle. In analogy to the introduction of the Lorentz factor eq. (3c) to make equations relativistic, a new spin correction factor $\Gamma$ is introduced,

where the “dot” refers to differentiation with respect to the new affine parameter $\lambda$, and $\omega = d\theta/d\tau = \dot{\theta}/\Gamma$ is the angular velocity relative to the “old” proper time. In special relativity the speed of light cannot be exceeded, here the angular velocity $\dot{\theta}$ (with respect to parameter $\lambda$) cannot exceed $c/R$, although $\omega$ can go to infinity.

C. Polydimensional Mechanics

Let’s continue with our 2D example of a hockey puck. We propose a generalization of eq. (2), where the Lagrangian is based upon the polydimensional form of eq. (5c), which is invariant under the Automorphism group $O(2, 2)$,

When re-parameterized in terms of the more familiar proper time using eq. (6), and compared with eq. (2) it appears as if the spin has increased the rest
mass by a factor of the inverse spin correction factor eq. (6). Indeed the four-momentum derived from the Lagrangian gives the momentum: $\mathbf{P} = m\mathbf{v}$, and energy: $E = mc^2$, where the spin-corrected linear mass is,

$$m = \frac{\gamma m_0}{\Gamma} = \gamma m_0 \sqrt{1 + \frac{R^2 \omega^2}{c^2}}.$$  

(8)

These are physically reasonable results, however they do not agree with the standard formulas eq. (4abcd). In particular, eq. (8) differs significantly from what one might derive from standard special relativity for the total energy of a macroscopic rotating object with center-of-mass speed $\mathbf{v}$,

$$m' = \frac{\gamma m_0}{\sqrt{1 - \frac{R^2 \omega^2}{c^2}}}.$$  

(9)

Note however that this last statement is also not derivable from eq. (4a).

One desirable feature of “standard” eq. (9) over eq. (8) is that there is a limit on the angular velocity: $\omega = R/c$, such that the tangent speed of the rim of the object will not exceed the speed of light. However, the spin angular momentum (of say a ring of mass): $L = m' R^2 \omega$, will go to infinity as the angular velocity approaches this limit. In our interpretation however, the angular velocity may well go to infinity, but the angular momentum,

$$L = \frac{\delta L}{\delta \omega} = \Gamma m_0 R^2 \omega,$$  

(10)

approaches a finite limit: $\lim_{\omega \to \infty} L = m_0 R c$. The appearance is that the “rim” speed for the bare mass approaches $c$ as a limit as desired. This is a very pleasing result for if we quantize the spin angular momentum to be $\hbar/4\pi$ (where $\hbar = \text{Planck's constant}$), the radius of gyration $R$ will be the Compton wavelength (over $4\pi$). Another interesting feature is that the spin correction to the mass is $(\Gamma m_0)$ in the rotational motion of eq. (10); differing from $(m_0/\Gamma)$ in eq. (8) for the linear motion.

### III. Extension of General Relativity

Einstein’s general theory of relativity requires the laws of physics to be form invariant (covariant) under general coordinate transformations. Physical quantities are represented by tensors, which necessarily preserve their rank under coordinate transformations, e.g. a vector is a vector to all observers. Even in a curved space, under parallel transport a vector cannot change into a bivector (nor change length, although it may twist). In our generalization, this will no longer be the case.

#### A. Review of Standard Theory

The weak equivalence principle states that the trajectory of a freely falling body in a gravitational field is independent of its internal structure and composition.
(e.g. heavy balls fall just as fast as light ones). The strong equivalence principle states that an accelerated reference frame is equivalent to gravitation, or that mass curves space, and accelerated motion is due to the curvature.

In general coordinates, the tangent basis vectors: \( \mathbf{e}_\mu = \partial_\mu \Sigma \) at event \( \Sigma \) are a function of the coordinates. Under differential displacement the basis vectors change,

\[
\partial_\alpha \mathbf{e}_\mu = \Gamma^\beta_{\alpha\mu} \mathbf{e}_\beta,
\]

where in a space without torsion the affine connections are symmetric in the lower indices: \( \Gamma^\alpha_{\beta\mu} = \Gamma^\beta_{\mu\alpha} \). The generalization of eq. (1) requires the introduction of the metric tensor \( g_{\alpha\beta} \), which contains all the information necessary to describe gravitation,

\[
c^2 d\tau^2 = dx^\alpha dx^\beta g_{\alpha\beta},
\]

\[
g_{\alpha\beta} = \mathbf{e}_\alpha \cdot \mathbf{e}_\beta = \frac{1}{2} \{ \mathbf{e}_\alpha, \mathbf{e}_\beta \}.
\]

The latter equation is the definition of a Clifford algebra in general coordinates. The differential of the metric tensor can hence be computed directly from eq. (12a) and eq. (11) only if the Leibniz rule for differentiation holds. While not generally true (e.g. in a Weyl space), it is the condition for a Riemann space,

\[
\partial_\mu g_{\alpha\beta} = (\partial_\mu \mathbf{e}_\alpha) \cdot \mathbf{e}_\beta + \mathbf{e}_\alpha \cdot (\partial_\mu \mathbf{e}_\beta) = \Gamma^\delta_{\mu\alpha} g_{\delta\beta} + \Gamma^\delta_{\mu\beta} g_{\delta\alpha},
\]

By permutation, one can solve for the affine connections in terms of the metric tensor.

Trajectories in curved space can be derived from the action integral of eq. (2) by substituting eq. (12a) for the proper time. The result is known as the geodesic equation, which describes the shortest path between two points in curved space,

\[
\ddot{x}^\mu = -\dot{x}^\alpha \dot{x}^\beta \Gamma^\mu_{\alpha\beta}.
\]

This is consistent with the weak equivalence principle, in that all particles follow the same path independent of mass (e.g. big balls fall at the same rate as small balls). In a Riemann space, the parallel displacement of a vector over a small closed loop will not change its length, but may rotate the vector in proportion to the amount of curvature (due to gravity),

\[
\Delta V^\nu = R^\nu_{\alpha\beta\mu} V^\mu \Delta A^{\alpha\beta},
\]

\[
R^\alpha_{\beta\mu\nu} = \mathbf{e}_\beta \cdot [\partial_\mu, \partial_\nu] \mathbf{e}_\alpha,
\]

where \( \Delta A^{\alpha\beta} \) is the oriented area of the loop, and \( R^\alpha_{\beta\mu\nu} \) is the Riemann curvature tensor.

In a Weyl space however, the length of a vector can change under parallel displacement. The Leibniz rule is no longer valid, such that eq. (13) no longer holds. It is replaced by flat with,

\[
\partial_\mu g_{\alpha\beta} = \Gamma^\delta_{\mu\alpha} g_{\delta\beta} + \Gamma^\delta_{\mu\beta} g_{\delta\alpha} + \phi_\mu g_{\alpha\beta},
\]
where $\phi_\mu$ was originally intended by Weyl\cite{6} to be the electromagnetic vector potential, however the approach did not yield the correct electrodynamical equations. The parallel transport of a scalar (such as length of vector $V^2$) around a closed loop would yield: $\Delta V^2 = V^2 F_{\mu\nu} \Delta A^{\mu\nu}$, where: $F_{\mu\nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu$.

It has been argued by Papapetrou\cite{7} that a fully covariant equation of motion for a spinning particle would differ from eq. (14),

$$p^\mu = -\rho^\alpha \dot{x}^\beta \Gamma_\alpha^\mu - \frac{1}{2} R^\mu_{\rho\sigma} \dot{x}^\rho S^{\sigma\alpha},$$

where the spin tensor is still given by eq. (4c) and the momentum by eq. (4a). While a non-spinning particle will follow a geodesic, a spinning one will travel a different path, which clearly violates the weak equivalence principle.

### B. Polydimensionally Affine Space

The tangent basis multivectors $E_A = \partial_A \Sigma$ of the polydimensional Clifford manifold are functions of the full set of generalized coordinates,

$$E_A(q^B) = \Delta^B_A(q^C) \tilde{E}_B.$$  

We call $\Delta^B_A$ the geobeins ("geometry legs"), which are completely analogous to Crawford’s dreibeins, except here we are reshuffling physical geometry at every point. The fiducial basis $\tilde{E}_A$ is assumed to be the Clifford group generated by an orthonormal basis which satisfies eq. (12b). However, eq. (12b) will no hold for the generalized tangent basis vectors: $e_\alpha$ unless the geobeins are severely restricted. For example, eq. (12b) can not accommodate an idempotent/nilpotent basis which does not have an identity element.

The general form which would allow for that possibility could be expressed as a Jordan algebra: $\{ E_A, E_B \} = 2 G^C_{AB} E_C$, where $G_{AB} = G_{AB} G^{00}$ would be the Cartan metric. However, for the purposes of this paper we propose the mild generalization as an ansatz,

$$\frac{1}{2} \{ e_\alpha, e_\beta \} = g_{\alpha\beta} \ E_0,$$  

(19a)

which among other enhancements, generalizes eq. (12b) to include Weyl space. We further propose the simplifying restriction that the basis scalar $E_0$ commutes locally with all elements, and $\{ E_0, E_0 \} = 2 g_{00} E_0$, where $g_{00}$ is the "scale" or metric of the scalar coordinate. We assume that the wedge product of basis vectors is still given by the Lie product: $[e_\alpha, e_\beta] = 2 e_\alpha \wedge e_\beta$, so that the Clifford product of two basis vectors may now be written,

$$e_\alpha e_\beta = g_{\alpha\beta} \ E_0 + e_\alpha \wedge e_\beta.$$  

(19b)

The generalized polydimensional connection $\Lambda^C_{AB}$ is defined,

$$\frac{\partial E_A}{\partial q_B} = \Lambda^C_{AB} E_C.$$  

(20)
In a Clifford manifold, the basis elements are interdependent; a bivector is the outer product of two basis vectors. Hence the connection of a multivector may be derived from the connections of the basis vectors. Note however that the Leibniz rule is no longer valid for the inner (dot) or outer (wedge) products because the definitions of these products involved an alternating sign depending upon the rank of the geometry, which is no longer fixed. The Leibniz rule is however valid for the Clifford direct geometric product.

At this point we take an epagogic approach by using simple examples to illustrate the new features. Let us return to our 2D “hockey puck” problem. The explicit form of eq. (20) for the two basis vectors is,

$$\partial_A e_\mu = \sigma_{A\mu} e_0 + \Gamma_{A\mu} e_\nu + \lambda_{A\mu} e_1 \wedge e_2.$$  \hspace{1cm} (21a)

Then the connection for the bivector $E_3 = e_1 \wedge e_2$ is hence completely determined from eq. (21a),

$$\partial_A (e_1 \wedge e_2) = \frac{1}{2}[\partial_A e_1, e_2] + \frac{1}{2}[e_1, \partial_A e_2] = \Gamma_{A\alpha}^\alpha E_3 + Q_{A}^\nu e_\nu.$$  \hspace{1cm} (21b)

$$Q_1^1 = g_{00} (\lambda_{A1} g_{22} - \lambda_{A2} g_{12}),$$  \hspace{1cm} (21c)

$$Q_2^1 = g_{00} (\lambda_{A2} g_{11} - \lambda_{A1} g_{12}).$$  \hspace{1cm} (21d)

The connection for the basis scalar,

$$\partial_A E_0 = -\phi_A E_0 + M_{A}^\mu e_\mu + N_A e_1 \wedge e_2,$$  \hspace{1cm} (21e)

can be simplified by differentiating eq. (19a). Equating terms of similar geometry, the bivector terms give us that $N_A = 0$ under our restrictions. Further the scalar portion recovers eq. (16) showing that $\phi_A$ is Weyl’s connection coefficient. The vector part shows,

$$M_{A}^\mu g_{\mu\delta} = \sigma_{A\delta} g_{00}.$$  \hspace{1cm} (21f)

The parallel displacement of a vector around a closed loop could now return as a completely different object (e.g. a bivector). One would generalize the curvature formula eq. (15b) to something like,

$$[\partial_A, \partial_B]E_C = \mathcal{F}_{ABC}^D E_D.$$  \hspace{1cm} (22a)

In our 2D case, a loop in the x-y plane would yield something like,

$$[\partial_\mu, \partial_\nu] e_\alpha = R_{\mu\nu\alpha}^\beta e_\beta + W_{\mu\nu\alpha} E_0 + V_{\mu\nu\alpha} e_1 \wedge e_2,$$  \hspace{1cm} (22b)

Now this becomes more acceptable if you start out with objects that are multivectorial in the first place; in fact we propose that particles have scalar+vector+bivector parts to represent their mass, linear motion, spin, etc. The curvature which bends one type of geometry into another is simply a coupling of these various portions (e.g. a spin contribution to linear momentum). Even more strange however is that we can have closed paths which are not in the ordinary vector coordinates, but involve the coordinates associated with the other basis multivectors. Hence a particle which is “spun” then translated will be in a different state than one which is translated then spun. We can even have multivector paths, which are not just one-dimensional lines, but part scalar, part linear and part area.
C. Polygeometrodynamics

We generalize by proposing a new equivalence principle, that the laws of physics should be fully covariant under local automorphism transformations. Generalized forces will be associated with curvature which bends one type of geometry into another (e.g. vector twisted into scalar).

It remains to be shown that the connection coefficients can be derived from some sort of generalized metric (e.g. the Cartan metric). Further it would be nice to have some generalized form of the action integral eq. (7) from which the equations of motion can be derived. By induction we believe that with the proper development one obtains generalized polygeodesics, which resemble eq. (14): \( \ddot{q}^A + \dot{q}^B \dot{q}^C \Lambda_{BC}^A = 0 \), where the differentiation is in terms of the affine parameter \( \lambda \) which is defined by the generalization of eq. (5c) to polydimensionally curved space.

We present a simplified case, where there is no change in the scale, such that eq. (21e) is zero, hence \( \sigma_{A\mu} \) in eq. (21a) also vanishes. The geodesics are of the form,

\[
\ddot{x}^\nu = -\dot{x}^\alpha \dot{x}^\beta \Gamma_{\alpha\beta}^\nu - \left( R \dot{\theta} \right)^2 Q_{3\nu} - R \dot{\theta} \dot{x}^\alpha (Q_{3\nu} + \Gamma_{3\nu}^\nu), \tag{23a}
\]

\[
R \ddot{\theta} = -R \dot{\theta} \dot{x}^\beta (\Gamma_{\beta\alpha} + \lambda_{3\beta}) - \dot{x}^\alpha \dot{x}^\beta \lambda_{\alpha\beta} - \left( R \dot{\theta} \right)^2 \Gamma_{3\alpha}, \tag{23b}
\]

where the subscript 3 is associated with the spin coordinate: \( q^3 = R \theta \). The second equation shows that the spin geodesic has a new torque proportional to the linear motion coupled by \( \lambda_{\alpha\beta} \). Comparing the first equation to the Papapetrou equation (17) suggests that we might want to make the identification:

\[
2 \Gamma_{3\mu} = R R_{1\mu}, \tag{24}
\]

In other words, the commutator derivative of general relativity might be equivalent to differentiation with respect to a bivector.

IV. Summary of Principles

We summarize our explorations epagogically, by proposing several broad organizing principles. Just as tensors were the natural language in which to formulate general relativity, Clifford algebra is the natural language in which to express the polydimensional theory.

**Principle of Relative Dimension.** In standard relativity, a scalar (point) is the same to all observers, in all coordinate systems. While a line may be bent due to curvature, its length is unchanged. Now Dimension is in the eye of the beholder. The geometric rank that an observer assigns to an object (e.g. bivector) is a function of the observer’s frame. It might be possible to logically extend this statement to say that there is no absolute dimension to the universe.
Polydimensional Isotropy. ‘No preferred direction’ is extended to mean that there is no absolute direction to which you can assign the geometry of a vector. For example, if we turn out the lights and exchange the basis vectors for their dual trivectors in all formulas in 4D, you can’t tell that a change was made.

The Greider Maxima. To be complete, the laws of physics must be multivectorial in form (having scalar, vector, bivector etc. parts). Every geometric piece of a multivector equation must be physically interpretable. A separate ‘Spin space’ is an unneeded construct.

Polydimensional Covariance. The laws of physics should be form invariant under local automorphism transformations, which reshuffle the physical geometry. Spin gauge theory (in spinor space) is not therefore an artifact of spin space, it is a manifestation of this broader classical principle.

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