Box model for channels of human migration

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Abstract

We discuss a mathematical model of migration channel based on the truncated Waring distribution. The truncated Waring distribution is obtained for a more general model of motion of substance through a channel containing finite number of boxes. The model is applied then for case of migrants moving through a channel consisting of finite number of countries or cities. The number of migrants in the channel strongly depends on the number of migrants that enter the channel through the country of entrance. It is shown that if the final destination country is very popular then large percentage of migrants may concentrate there.

1 Introduction

Population migration involves the relocation of individuals, households or moving groups between geographical locations. Much efforts are directed to the study of internal migration in order to understand this migration and to make projection of the internal migration flows that may be very important for taking decisions about economic development of regions of a country (Armitage (1986), Braken and Bates (1983), Champion et al. (2002)). In addition the study of migration becomes very actual after the large migration flows directed to Europe in September 2015. From the point of view of migrating units migration models may be classified as macromodels or
micromodels (Stillwell and Congdon (1991), Cadwallader (1989, 1992)). Micromodels are based on the individual migrating unit (person, group or household) and on the processes underlying the decision of the migrant to remain in the current location or to move somewhere else (Maier and Weiss (1991)). Macromodels are for the aggregate migration flows. Many macromodels are explanatory and they use non-demographic information. An important class of these models are the gravity models having their roots in the ideas of Ravenstein about the importance of distance for migration and their first applications in the study of intercity migration by Zipf (Ravenstein (1885), Zipf (1946)). Further development of these models was made by the concept of spatial interactions (Wilson (1970), Stillwell (1978), Fotheringham et al. (2001)) or by introduction of statistical spatial interaction models (Congdon (1991), Flowerdew (1991), Nakaya (2001), Fotheringham et al. (2002)).

Other kinds of macromodels use demographic information from the field of multi-state demography in order to generate projections of migration flows. In the first models of this kind one used a cohort component model which involved the estimation of the population in the studied region at the beginning of a projection period. Then a projection of the number of births during the future time period and the survival of those in existence or being born during the period was made (Bowley (1924), Whelpton(1936)). In the course of the time these models become multi-regional stochastic models and the requirement to the used information was changed: the first models required little information about migration and the more sophisticated models require maximum information about migration, e.g. a migration flow information disaggregated by single year of age and sex (Rees and Wilson (1977), Rogers (1990, 1995)).

Migration models may be classified as probability models or deterministic models with respect to their mathematical features. Many probability models of migration (Willekens (1999, 2008)) are focused on the change of address in the process of migration involving the crossing of an administrative boundary. The migration data can be connected to the events of migration (event data or movement data) or to the place of residence (migrant status) at a given point of time (status data). Interesting kind of data are the duration data where the time is measured as a duration of since a reference event (event-origin that may be associated with the start of the process of migration). The observation of place of residence of an individual at two points in time leads to collection of transition data (Ledent (1990), Willekens (1999)). Data obtained by recording the migrant status at several points of time are called
Different kinds of data are connected to different probability models. The duration data may be used in the exponential model of migration (Willekens (1999)). This model considers two states - origin state and a destination state. It is assumed that at the onset of migration all individuals are in the origin state. Individuals may leave the origin state for different reasons but only the leaving connected to migration are considered. And the event of migration is assumed to be experienced by an individual only once (non-repeatable event). Let the size of population (of identical individuals) be $m$ and let $T$ be the time at which an individual migrates ($T=0$ denotes the birth of the individual). One can define probability distribution $F(t) = P(T < t)$ ($T < t$ means that the migration event happens between $T = 0$ and $T = t$). $f(t)$ is the probability density function connected to $F(t)$ and $S(t) = 1 - F(t)$ is called survival function. It is connected to the size $mS(t)$ of the risk set of individuals (those who have not migrated but are exposed to the risk of migration in the future). One can introduce the conditional density function $\mu(t) = f(t)/S(t)$ which represents probability that the migration event occurs in a small interval following $t$ provided that it was not occurred before $t$. The survival function and the probability density functions then are

$$S(t) = \exp \left[ - \int_0^t dt\mu(t) \right]; \quad f(t) = \mu(t) \exp \left[ - \int_0^t dt\mu(t) \right].$$  \hspace{1cm} (1)

If $\mu(t)$ is constant (equal to $\mu$) it is referred to as a migration rate. Migration rate $\mu$ can be determined on the basis of an appropriate likelihood function (Willekens (1999)). The exponential model of migration can be extended in different directions, e.g. to the case of non-identical individuals; the life span can be split into age intervals (of one or several years) (Blossfeld and Rohwer (2002), Blossfeld et al. (2007)); the number of possible destination states may become larger than 1 (Hachen (1988)).

The exponential model of migration is obtained on the basis of assumption that migration is a non-repeatable event. Let us now assume that the migration is a repeatable event and there is no upper limit on the number of migrations in a time interval. If the event data are available the Poisson model of migration can be used. The Poisson model describes the number of migrations during an interval of unit length (e.g. year, month, etc.). In this model the probability of observing $n$ migrations during the unit interval is
given by the Poisson distribution

\[ P(N = n) = \frac{\lambda^n}{n!} \exp(-\lambda) \]  

(2)

where \( \lambda \) is the expected number of migrations during the unit interval.

As we have mentioned above the Poisson model is applicable when event data are available. When status data for migration are available, i.e., migration is measured by comparing the places of residence at two consecutive points of time, then the probability of observing \( n \) migrants among a sample population of \( m \) individuals is given by the binomial distribution (\emph{binomial model of migration})

\[ P(N = n) = \frac{m!}{n!(m-n)!} p^n (1-p)^{m-n} \]  

(3)

of index \( m \) and parameter \( p \) representing the probability of being migrant. If the number of possible destinations for migration is larger than 1 (let us assume that the number of possible destinations be \( K \)) then the probability distribution connected to migration is given by the multinomial distribution (\emph{multinomial model of migration})

\[ P(N_1 = n_1, \ldots, N_K = n_K) = \frac{m!}{n_1! \ldots n_K!} \prod_{i=1}^{K} \frac{p_i^{n_i}}{n_i!} \prod_{i=1}^{K} p_i = 1; \sum_{i=1}^{K} n_i = m \]  

(4)

The transition between the states (e.g. addresses occupied in different ages) is given by transitional probabilities \( p_{ij}(x) \) which correspond to the probability that an individual who resides in state \( i \) at \( x \) resides in state \( j \) at \( t + 1 \). \( p_{ij} \) may be represented as (Willekens (2008))

\[ p_{ij}(x) = \frac{\exp[\beta_{j0}(x) + \beta_{j1}(x)Y_i(x)]}{\sum_{r=1}^{K} \exp[\beta_{r0}(x) + \beta_{r1}(x)Y_i(x)]} \]  

(5)

for the case when only the most recent state occupancy is relevant and the person has single relevant attribute (covariate) \( Y_i \) that is equal to 1 if the state is occupied and is equal to 0 otherwise. Then the state probability \( \pi_i \) that an individual occupies state \( i \) is given by

\[ \logit(\pi_j) = \ln \left( \frac{\pi_j}{1 - \pi_j} \right) = \beta_{j0}(x) + \beta_{j1}(x)Y_i(x) \]  

(6)
From the transition probabilities for discrete time processes one may turn to transition rates and this way leads to the Markov chain models (Singer and Spilerman (1979)). The parameters of these models may be estimated even if some data are missing (McLachlan and Krishnan (1997)). Markov chain models are useful to demographers concerned with problems of movement of people (Collins (1972, 1975)) as these models are appropriate for describing and analysing the nature of changes generated by the movement from one state of a system to another possible state and in some cases Markov models may be useful for forecasting future changes. Bayesian modelling framework may be used for generating estimates of place-to-place migration flows too (Raymer (2007), Bierley et al. (2008)).

From the deterministic models we shall give brief additional information about the gravity models and about models connected to the replicator dynamics. Finally we shall mention some urn models of interest for our study.

The gravity model of migration (already mentioned above in the text) is a place-to-place migration model that assumes that interregional migration is directly related to the population of the origin and of the destination regions and inversely related to the distance between them (Greenwood (2005)). The classic gravity model can be written as

\[ \ln(M_{ij}) = \ln(G) + \beta_1 \ln(P_i) + \beta_2 \ln(P_j) - \alpha \ln(D_{ij}) \]  

(7)

where \( M_{ij} \) is the migration from region \( i \) to region \( j \); \( G \) is a constant; \( P_i \) and \( P_j \) are populations of regions \( i \) and \( j \) and \( D_{ij} \) is the distance between the regions. The gravity model may be extended in different ways, e.g., to include the income and unemployment in the two regions as well as in the ways discussed above in the text.

A simple version of the replicator equation describes the time change of the part \( p_i \) of the population of \( n \) individuals \( (p_i = n_i/n) \) that use strategy \( s_i \) at time \( t \) \( (n = \sum_{i=1}^{m} n_i) \) for the case of symmetric game (Cressman and Tao (2014), Karev and Kareva (2014)). The form of the replicator equation is

\[ \frac{dp_i}{dt} = p_i[\pi(s_i, p) - \pi(p, p)], \quad i = 1, \ldots, m \]  

(8)

where \( \pi(s_i, p) = \sum_{i=1}^{m} p_i \pi(s_i, p) \) are the payoff of the individuals that follows the strategy \( s_i \) and the the population mean payoff.
The urn models that may be studied analytically usually consider a single urn containing balls of different colours and a fixed set of rules that specify the way the urn composition evolves: at each discrete instant, a ball is picked up at random, its color is inspected, and, in compliance with the rules, a collection of balls of various colours is added to the urn (Flajolet et al. (2006)). The balls may be exchanged between two or more urns - an example of such model is the Bernoulli-Laplace model for exchange of balls between two urns or the model of transfer of balls from one urn to another one (Blom et al. (1994)). As we shall see below our model possesses some features of replicator model as well as of an urn model.

Our interest in the problems of human migration arose in the course of the research on ideological struggles (Vitanov et al. (2010, 2012)) and waves and statistical distributions in population and other systems (Vitanov et al. (2009a,b,2011, 2013a, 2013b, 2015), Vitanov and Vitanov (2013)), Vitanov and Dimitrova (2010, 2014). Migration channels are one possibility for movement of migrants. Below we shall discuss a mathematical model of migration channels based on the truncated Waring distribution. The short review of the models above allows us to mention the place of our model within the set of models of migration. Our model is a multi-regional macromodel. In addition we shall observe that the discussed model doesn’t belong to the class of the probability models of migration despite the fact that it is closely connected to the Waring distribution. We shall see that: (i) From the point of view of the model equations our model is a deterministic model that is close to the class of replication models and to the sub-class of the replication-mutation models; (ii) From the point of view of distribution of migrants in the countries of the channel our model may be considered as an urn model.

The text below is organized as follows. In Sect.2 the truncated Waring distribution is obtained as a result of mathematical model of a substance that moves through a finite sequence of cells. In Sect. 3 the discussed model is applied to the channels of migration where the moving substance are the migrants and the sequence of cells are countries or cities that are on the route of migration. A discussion on: (i) the place of our models in the set of models of migration; (ii) the meaning of the parameters of the model, and (iii) application of the model for purposes other than explanatory ones, is presented in Sect.4.
2 The model and the truncated Waring distribution

Let us consider the following simple model (Schubert and Glänzel (1984)). We have an array of $N+1$ cells (boxes) indexed in succession by non-negative integers, i.e., the first cell has index 0 and the last cell has index $N$. In the model discussed by Schubert and Glänzel $N$ is infinite. We shall treat $N$ as finite number and this will lead us to an important effect (concentration of migrants in the final destination country). We assume that there exists an amount $x$ of some substance that is distributed among the cells. Let $x_i$ be the amount of the substance in the $i$-th cell. Then

$$x = \sum_{i=0}^{N} x_i$$  \hspace{1cm} (9)

The fractions $y_i = x_i/x$ can be considered as probability values of distribution of a discrete random variable $\zeta$

$$y_i = p(\zeta = i), \; i = 0, 1, \ldots, N$$  \hspace{1cm} (10)

The expected value of the random variable $\zeta$ is

$$E(\zeta) = \sum_{i=0}^{N} iy_i$$  \hspace{1cm} (11)

The amount $x_i$ can change due to the following 3 processes:

1. Some amount $s$ of the substance $x$ may enter the system of cells from the external environment through the 0-th cell;

2. Amount $f_i$ may be transferred from the $i$-th cell into the $i + 1$-th cell;

3. Amount $g_i$ may leak out the $i$-th cell into the external environment.

The above processes can be modeled mathematically by the system of ordinary differential equations:

$$\frac{dx_0}{dt} = s - f_0 - g_0;$$

$$\frac{dx_i}{dt} = f_{i-1} - f_i - g_i, \; i = 1, 2, \ldots, N - 1$$

$$\frac{dx_N}{dt} = f_{N-1} - g_N.$$  \hspace{1cm} (12)
The following forms of the amounts of the moving substance are assumed in (Schubert and Glänzel (1984)) \((\alpha, \beta, \gamma, \sigma)\) are parameters

\[ s = \sigma x; \quad \sigma > 0 \rightarrow \text{self-reproducing property} \]

\[ f_i = (\alpha + \beta i)x_i; \quad \alpha > 0, \beta \geq 0 \rightarrow \text{cumulative advantage of higher cells} \]

\[ g_i = \gamma x_i; \quad \gamma \geq 0 \rightarrow \text{uniform leakage over the cells} \quad (13) \]

Substitution of Eqs.(13) in Eqs.(12) leads to the relationships

\[
\begin{align*}
\frac{dx_0}{dt} &= \sigma x - \alpha x_0 - \gamma x_0; \\
\frac{dx_i}{dt} &= [\alpha + \beta(i-1)]x_{i-1} - (\alpha + \beta i + \gamma)x_i, \quad i = 1, 2, \ldots, N-1 \\
\frac{dx_N}{dt} &= [\alpha + \beta(N-1)]x_{N-1} - \gamma x_N
\end{align*}
\quad (14)
\]

Let us sum the equations from (14). The result of the summation is

\[
\frac{dx}{dt} = (\sigma - \gamma)x
\quad (15)
\]

and the solution for \(x\) is

\[ x = x(0) \exp[(\sigma - \gamma)t] \quad (16) \]

where \(x(0)\) is the amount of \(x\) at \(t = 0\).

The distribution of \(y_i\) will lead us to the truncated Waring distribution. From Eqs.(14) and with the help of Eq.(16) and the relationship

\[
\frac{dy_i}{dt} = \frac{1}{x^2} \left[ x \frac{dx_0}{dt} - x_i \frac{dx_i}{dt} \right]
\]

one obtains

\[
\begin{align*}
\frac{dy_0}{dt} &= \sigma - (\alpha + \sigma)y_0; \\
\frac{dy_i}{dt} &= [\alpha + \beta(i-1)]y_{i-1} - (\alpha + \beta i + \sigma)y_i, \quad i = 1, 2, \ldots, N-1 \\
\frac{dy_N}{dt} &= [\alpha + \beta(N-1)]y_{N-1} - \sigma y_N
\end{align*}
\quad (17)
\]

We search for solution of Eq.(17) in the form

\[ y_i = y_i^* + F_i(t) \quad (18) \]
where \( y_i^* \) is the stationary solution of Eqs.(18) given by the relationships

\[
\begin{align*}
y_0^* &= \frac{\sigma}{\sigma + \alpha} \\
y_i^* &= \frac{\alpha + \beta(i - 1)}{\alpha + \beta i} y_{i-1}^*, \quad i = 1, 2, \ldots, N - 1; \\
y_N^* &= \frac{\alpha + \beta(N - 1)}{\sigma} y_{N-1}^*
\end{align*}
\]

For the functions \( F_i \) we obtain the system of equations

\[
\begin{align*}
\frac{dF_0}{dt} &= -(\alpha + \sigma) F_0; \\
\frac{dF_i}{dt} &= [\alpha + \beta(i - 1)] F_{i-1} - (\alpha + \beta i + \sigma) F_i, \quad i = 1, 2, \ldots, N - 1 \\
\frac{dF_N}{dt} &= [\alpha + \beta(N - 1)] F_{N-1} - \sigma F_N
\end{align*}
\]

The solutions of these equations are

\[
\begin{align*}
F_0(t) &= b_{00} \exp[-(\alpha + \sigma)t] \\
F_1(t) &= b_{10} \exp[-(\alpha + \sigma)t] + b_{11} \exp[-(\alpha + \beta + \sigma)t] \\
\ldots
\end{align*}
\]

\[
F_i(t) = \sum_{j=0}^{i} b_{ij} \exp[-(\alpha + \beta j + \sigma)t]; \quad i = 1, 2, \ldots, N - 1
\]

\[
F_N(t) = \sum_{j=0}^{N} b_{Nj} \exp[-(\alpha + \beta j + \sigma)t].
\]

where

\[
\begin{align*}
b_{ij} &= \frac{\alpha + \beta(i - 1)}{\beta(i - j)} b_{i-1,j}; \quad i = 1, \ldots, N - 1; \quad j = 0, \ldots, i - 1; \\
b_{Nj} &= \frac{\alpha + \beta(N - 1)}{\alpha + j\beta} b_{N-1,j}; \quad j = 0, \ldots, N - 1 \\
b_{NN} &= 0.
\end{align*}
\]

\( b_{ij} \) that are not determined by Eqs.(25) may be determined by the initial conditions. In the exponential function in \( F_i(t) \) there are no negative coefficients and because of this when \( t \to \infty \), \( F_i(t) \to 0 \) and the systems comes
to the stationary distribution from Eqs. (19). The form of this stationary distribution is:

\[
P(\zeta = i) = \frac{a}{a + k} \frac{(k - 1)^{[i]}}{(a + k)^{[i]}}, \quad k^{[i]} = \frac{(k + i)!}{k!}; \quad i = 0, \ldots, N - 1
\]

\[
P(\zeta = N) = \frac{1}{a + k} \frac{(k - 1)^{[N]}}{(a + k)^{[N-1]}},
\]

with parameters \( k = \alpha/\beta \) and \( a = \sigma/\beta \).

The obtained distribution is close to the Waring distribution that can be obtained for the case of channel with infinite number of cells (see Appendix 1). The distribution (26) has a concentration of substance in the last cell (i.e. in the \( N \)-th cell). For the case of Waring distribution the same substance is distributed in the cells \( N, N + 1, \ldots \).

Let us calculate one example connected to the obtained distribution. Let us have 6 cells (\( N = 5 \)). Then

\[
P(0) = \frac{\sigma}{\sigma + \alpha};
\]

\[
P(1) = \frac{\alpha}{\alpha + \beta + \sigma} \frac{\sigma}{\sigma + \alpha};
\]

\[
P(2) = \frac{\alpha + 2\alpha}{\alpha + 2\beta + \sigma} \frac{\sigma}{\sigma + \alpha};
\]

\[
P(3) = \frac{\alpha + 3\beta}{\alpha + 2\beta} \frac{\sigma}{\sigma + \alpha};
\]

\[
P(4) = \frac{\alpha + 4\beta}{\alpha + 2\beta + \sigma} \frac{\sigma}{\sigma + \alpha};
\]

\[
P(5) = \frac{\alpha + 5\beta}{\sigma + \alpha} \frac{\sigma}{\sigma + \alpha}
\]

Let now we assume that \( \alpha = \sigma/2 \) and \( \beta = \sigma/4 \). Then

\[
P(0) = \frac{2}{3}; \quad P(1) = \frac{4}{21}; \quad P(2) = \frac{1}{14}; \quad P(3) = \frac{2}{63}; \quad P(4) = \frac{1}{63}; \quad P(5) = \frac{1}{42}.
\]

We note that \( P(5) > P(4) \), i.e there can be a concentration of substance in the last cell. This is difference with respect to the Waring distribution discussed in Appendix A. We note also that the sum of all probabilities is equal to 1 (as it can be expected).
3 Application of the model to migration channels

Let us consider a sequence of \( N + 1 \) countries (or cities). A flow of migrants flows through this migration channel from the country of entrance to the final destination country. We can consider this sequence of countries as sequence of boxes (cells). The entry country will be the box with label 0 and the final destination country will be the box with label \( N \). Let us have a number \( x \) of migrants that are distributed among the countries. Let \( x_i \) be the number of migrants in the \( i \)-th country. This number can change on the basis of the following three processes: (i) A number \( s \) of migrants enter the channel from the external environment through the 0-th cell (country of entrance); (ii) A number \( f_i \) of migrants can be transferred from the \( i \)-th country (city) to the \( i+1 \)-th country (city); (iii) A number \( g_i \) from migrants change their status (e.g. they are not anymore migrants and become citizens of the corresponding country).

Let us assume that the number of migrants is large and continuum approximation may be used. Then the values of \( x_i \) can be determined by Eqs. (12). The relationships (13) mean that: (i) The number of migrants \( s \) that enter the channel is proportional of the current number of migrants in all countries (cities) that form the channel; (ii) There may be preference for some countries (cities), e.g. migrants may prefer the countries that are around the end of the migration channel (and the final destination country may be the most preferred one); (ii) It is assumed that the conditions along the channel are the same with respect to 'leakage' of migrants, e.g. the same proportion \( \gamma \) of migrants leave the flow of migrants (e.g. they may become citizens of corresponding country, etc.)

As it can be seen from Eq. (16) the change of the number of migrants depends on the values of \( \sigma \) (characteristic parameter for the migrants that enter the channel) and \( \gamma \) (characteristic parameter for migrants that change their status. If \( \sigma > \gamma \) the number of migrants in the channel increases exponentially. If \( \sigma < \gamma \) the number of the migrants in the channel decreases exponentially. Thus there are three main regimes of functioning of the channel.

The dynamics of the distribution of the migrants in the channel is modelled by Eqs. (17). When the time since the beginning of the operation of the channel become large enough then the distribution of the migrants in the
cells of the channel (i.e., in the countries or cities that form the migration channel) becomes close to the stationary distribution described by Eqs. (19). Let us stress that the stationary distribution described by (19) is very similar to the Waring distribution but there is a substantial difference between the two distributions due to the finite length of the migration channel: there may be large concentration of migrants in the last cell of the channel (in the final destination country). In order to illustrate this let us consider the case of large $\beta$ (final destination country is very popular among the migrants). Let $\beta \gg \alpha + \sigma$ and $\alpha \gg \sigma$. Then $y_0^* \approx \sigma/\alpha \to 0$; $y_i^* \approx \sigma/(\alpha + \beta i) \to 0$ for $i = 1, 2, \ldots, N - 1$ and $y_N^* \approx 1$, i.e., almost all migrants may reach the final destination country. Let us refine this example by substituting numbers in Eqs. (27). Let $\alpha = 1/10$, $\beta = 1$, $\sigma = 1/100$. Then approximately in the entry country (0-th cell of the channel) there will be 9.09% of the migrants; in the following country (the first cell) there will be 0.82% of the migrants. In the third country (second cell) there will be 0.43% of the migrants. In the following country (third cell of the channel) there will be 0.29% of the migrants. In the fifth country (fourth cell of the channel) there will be 0.22% of the migrants. In the final destination country there will be 89.15% of the migrants. Hence if the final destination country becomes very popular this may lead to a large concentration of migrants there.

4 Discussion

The model of migration channels presented in sections 2 and 3 is connected to the Waring distribution. Nevertheless the model is not a probability model. It is a deterministic model that leads to several results and one of these results is a probability distribution. The deterministic results of the model are connected to the system of deterministic ordinary differential equations. The probability result of the model is connected to the fact that a migration channel is discussed. Thus more than 1 country is considered and because of this a probability distribution connected to the numbers of migrants in the countries of the channel can be derived. The essence of the discussed model are the deterministic differential equations and not the calculation of probabilities as in the case of exponential model, Poisson model, multinomial model or Markov chain models. The model is not of the kind of gravity models as the distance among the origins and destinations of migration is not presented explicitly in the model. The model possesses some features of
replicator models. In more detail the migrants in each country of the channel (except for the final destination country) may follow one of the strategies: (a) to remain in the corresponding country without change of their status; (b) to remain in the country and to change their status from migrants to non-migrants; (c) to move to the next country of the channel. Replication is connected to the migrants that follow strategy (a). The change of strategy of the members of the other two classes of migrants may be treated as mutation. The changes of the number $x_i$ of migrants that have strategy to stay as migrants in the $i$-th country of the channel is given by an ordinary differential equation as in the case of replicator model. But in our model we do not consider the payoffs connected to the different strategies explicitly. What is taken into account is that the countries that are closer to the final destination country are more preferred by the migrants. This determines the direction of the movement of migrants (from entry country of the channel to the final destination country of the channel).

From the point of view of the distribution of the migrants in the countries of the channel our model is a case of an urn model. Indeed the countries of the channel may be considered as urns and the migrants may be considered as balls of the same color. Urns are numbered and at each time step there is addition of a number of balls in the urn with number 0. This addition is proportional to the number of the balls in all urns. Then some balls are removed from the 0-th urn (corresponding to migrants that change their status) and some balls are moved to the urn with number 1. The same actions of removing and transferring balls are performed to all urns (urn by urn following the increasing urn numbers). Continuous approximation is assumed next. The goal is to obtain the asymptotic stationary distribution of the balls in the urns. This asymptotic stationary distribution is the truncated Waring distribution.

The difference between the Waring distribution and the truncated Waring distribution from the point of view of our model is that there is a concentration of substance (migrants) in the last cell (final destination country) of the sequence of cells (countries). If the final destination country is attractive then this concentration may be very significant: almost all migrants will go (and may remain) there.

The parameters that govern the distribution of migrants in the countries that form the channels are $\sigma$, $\alpha$, $\beta$ and $\gamma$. $\sigma$ is the "gate" parameter as it regulates the number of migrants that enter the channel. $\sigma$ is responsible for the (exponential) growth of the number of migrants in the channel. If $\sigma$ is
large then the number of migrants may increase very fast and this can lead to
problems in the corresponding countries. Because of this the states may try
to keep $\sigma$ at satisfactory small value (and to decrease the illegal migration) by
official boundary entrances and even by fences. We note the $\sigma$ participates
in each term of the truncated Waring distribution. This means that the
situation at the entrance of the migration channel influences significantly the
distribution of migrants in the countries of the channel. If the gates are open
then a flood of migrants may be observed.

The parameter $\gamma$ regulates the "absorption" of the channel as it regulates
the change of the status of some migrants. They can settle in the corre-
sponding country (may obtain citizenship); may die on the route through
the channel, etc. The large value of $\gamma$ may compensate the value of $\sigma$ and
even may lead to decrease of the number of migrants in the channel. The
large value of $\gamma$ may lead to integration problems connected to migrants if
the integration capacity of the corresponding country is limited. We shall
discuss this question elsewhere.

The parameter $\alpha$ regulates the motion of the migrants from one country
to the next country of the channel. Small value of $\alpha$ means that the way of
the migrants through the channel is more difficult and because of this the
migrants tend to concentrate in the entry country (and eventually in the
second country of the channel). The countries that are at the second half
of the migration channel and especially the final destination country may
try to decrease $\alpha$ by agreements that commit the entry country to keep the
migrants on its territory. Any increase of $\alpha$ leads to increase of the proportion
of migrants that reach the second half of the migration channel and especially
the final destination country.

The parameter $\beta$ regulates the attractiveness of the countries along the
migration channel. Large values of $\beta$ mean that the final destination country
is very attractive for some reason. This increases the attractiveness of the
countries from the second half of the channel (migrants want more to reach
these countries as in such a way the distance to the final destination country
decreases). If for some reason $\beta$ is kept at high value a flood of migrants
may reach the final destination country which may lead to large logistic and
other problems there.

The discussed model of migration channels helps us to understand the
functioning of such channels and especially the effect of concentration of
migrants in a popular final destination country. In addition the discussed
model may be used for evaluation of different scenarios of distribution of
migrants in the countries that belong to a migration channel. Let us discuss one possible scenario about the West-Balkan migration route from Greece to Germany. The countries from this route are Greece, Macedonia, Serbia, Croatia, Slovenia, Austria and Germany. The question we want to answer is about the number of migrants without change of the status that are expected to be at the end of the period in each of the countries of the channel. The UN agency of refugees estimated that from 1.10.2015 till 14.11.2015 (3/2 months) 200 000 migrants are distributed in the countries of this route. Let us assume that in 2016 (from March till November, i.e., for 9 months) the intensity of motion through this migration channel remains the same. This means that one may expect that 1 200 000 migrants will be distributed among the countries of the channel at the end of the period. Let us assume that $\sigma = 0.005$ (which means that at the end of period around 6000 migrants enter the channel through entry country per unit time, i.e., per day). Let us assume that $\alpha = 0.025$ and $\beta = 0.006$ (which means that 2.5% of the migrants move from country to country because of reasons not connected to popularity of the final destination country and the popularity of the final destination country is such that about 5% of the migrants from the country next to the final destination country move to the final destination country per day). Then according to the discussed model the contribution of the West-Balkan route to the numbers of migrants without status in the corresponding countries will be approximately as follows: Greece: around 200 000; Macedonia: around 139 000; Serbia: around 102 000; Croatia: around 81 000; Slovenia: around 63 000; Austria: around 51 000 and Germany: around 564 000. Note that for some countries like Germany other migration channels may contribute too and the total number of migrants without status in these countries may be larger. The quantitative information obtained by the discussed model may be helpful in addition to information from other sources and models, e.g., it may be used for estimation of different kinds of expenses connected with the migrants. In addition the model allows quick estimation of the distribution of migrants for the case of other scenarios (other values of the parameters of the channel). Thus the model may contribute to the process of management and control of migration flows.
A  Waring distribution: properties and derivation

The Waring distribution (named after Edward Waring - a Lucasian professor of Mathematics in Cambridge in the 18th century) is a probability distribution on non-negative integers (Irwin (1963, 1968), Diodato (1994))

\[ p_i = \rho \frac{\alpha(i)}{(\rho + \alpha)(i+1)}; \quad \alpha(i) = \alpha(\alpha + 1) \ldots (\alpha + i - 1) \quad (29) \]

Waring distribution may be written also as follows

\[ p_0 = \rho \frac{\alpha(0)}{(\rho + \alpha)(1)} = \frac{\rho}{\alpha + \rho} \]

\[ p_i = \frac{\alpha + (i - 1)}{\alpha + \rho + i} p_{i-1}. \quad (30) \]

The mean \( \mu \) (the expected value) of the Waring distribution is

\[ \mu = \frac{\alpha}{\rho - 1} \text{ if } \rho > 1 \quad (31) \]

The variance of the Waring distribution is

\[ V = \frac{\alpha \rho(\alpha + \rho - 1)}{(\rho - 1)^2(\rho - 2)} \text{ if } \rho > 2 \quad (32) \]

\( \rho \) is called the tail parameter as it controls the tail of the Waring distribution. This can be see from the relationship for Waring distribution when \( i \to \infty \). Then

\[ p_i \approx \frac{1}{i^{(1+\rho)}}. \quad (33) \]

which is the frequency form of the Zipf distribution (Chen (1980)).

The Waring distribution contains as particular cases other interesting distributions. One example is the famous Yule distribution (called also Yule-Simon distribution). In this case \( \alpha \to 0 \) and the Waring distribution is reduced to the Yule-Simon distribution (Simon (1955))

\[ p(\zeta = i \mid \zeta > 0) = \rho B(\rho + 1, i) \quad (34) \]

where \( B \) is the beta-function. Another example is the geometric distribution. Let \( \alpha = \sigma/\beta \) and \( \rho = a/\beta \). Then we obtain the following variant of the
Waring distribution: \( p_0 = a/\left(\sigma + a\right) \) and \( p_i = \left[(\sigma + \beta(i - 1))/(a + \sigma + \beta i)\right]p_{i-1} \). Let us now set \( \beta = 0 \) in this variant of the distribution. We obtain \( p_i = \left[\sigma/(\sigma + a)\right]p_{i-1} \) and the corresponding distribution

\[
p(\zeta = i) = q(1 - q)^i; \quad q = \frac{a}{\sigma + a}
\] (35)

is called geometric distribution (Frank (1962), Coleman (1964)).

The Waring distribution is a distribution with a very long tail. Because of this property the Waring distribution is very suitable to describe characteristics of many systems from the areas connected to research on biology and society. The Waring distributions is obtained when the model discussed in the main text is applied for the case of an infinite sequence of cells \( N \to \infty \).

In this case we have infinite array of cells (boxes) indexed in succession by non-negative integers, i.e., the first cell has index 0. We assume again that there exists an amount \( x \) of some substance that is distributed among the cells. If \( x_i \) is the amount of the substance in the \( i \)-th cell then \( x = \sum_{i=0}^{\infty} x_i \). We introduce \( y_i = p(\zeta = i) \), \( i = 0, 1, \ldots \) and assume that the expected value of the random variable \( \zeta \) is finite \( E(\zeta) = \sum_{i=0}^{\infty} iy_i \).

The content \( x_i \) of any cell can change due to the same 3 processes as in the main text: (i) Some amount \( s \) of the substance \( x \) may enter the system of cells from the external environment through the 0-th cell; (ii) Amount \( f_i \) can be transferred from the \( i \)-th cell into the \( i + 1 \)-th cell; (iii) Amount \( g_i \) may leak out the \( i \)-th cell into the external environment. These processes can be modeled mathematically by the system of ordinary differential equations:

\[
\begin{align*}
\frac{dx_0}{dt} & = s - f_0 - g_0; \\
\frac{dx_i}{dt} & = f_{i-1} - f_i - g_i, \quad i = 1, 2, \ldots
\end{align*}
\] (36)

The following relationships for the amount of the moving substances may be assumed (Schubert and Glänzel (1984)) \( (\alpha, \beta, \gamma, \sigma \) are constants): \( s = \sigma x \); \( \sigma > 0 \) (self-reproducing property); \( f_i = (\alpha + \beta i)x_i \); \( \alpha > 0, \beta \geq 0 \) (cumulative advantage of higher cells); \( g_i = \gamma x_i \); \( \gamma \geq 0 \) (uniform leakage over the cells). The substitution of these conditions in Eqs.\((36)\) leads to the relationships

\[
\begin{align*}
\frac{dx_0}{dt} & = \sigma x - \alpha x_0 - \gamma x_0; \\
\frac{dx_i}{dt} & = \left[\alpha + \beta(i - 1)\right]x_{i-1} - (\alpha + \beta i + \gamma)x_i
\end{align*}
\] (37)
Let us sum the equations from (37). The result of the summation is \[ \frac{dx}{dt} = (\sigma - \gamma)x \] and the solution for \( x \) is \[ x = x(0) \exp[(\sigma - \gamma)t] \] where \( x(0) \) is the amount of \( x \) at \( t = 0 \).

The distribution of \( y_i \) will lead us to the Waring distribution. From Eqs. (37) and with the help of the relationship for \( \frac{dx}{dt} \) and the relationship \[ \frac{dy_i}{dt} = \frac{1}{x_i} \left[ x_i \frac{dx}{dt} - x_i \frac{d^2 x}{dt^2} \right] \] one obtains

\[ \begin{align*}
\frac{dy_0}{dt} &= \sigma - (\alpha + \sigma)y_0; \\
\frac{dy_i}{dt} &= [\alpha + \beta(i - 1)]y_{i-1} - (\alpha + \beta i + \sigma)y_i
\end{align*} \] (38)

The solution of Eq. (38) is

\[ y_i = y_i^* + \sum_{j=0}^{i} b_{ij} \exp[-(\alpha + \beta j + \sigma)t] \] (39)

where \( y_i^* \) is the stationary solution of Eqs. (39) given by the relationships

\[ \begin{align*}
y_0^* &= \frac{\sigma}{\sigma + \alpha} \\
y_i^* &= \frac{\alpha + \beta(i - 1)}{\alpha + \beta i + \sigma} y_{i-1}^*, \quad i = 1, 2, \ldots
\end{align*} \] (40)

Above \( b_{ij} \) are determined by the initial conditions. In the exponential function there are no negative coefficients and because of this when \( t \to \infty \) the sum in Eq. (39) vanishes and the systems comes to the stationary distribution from Eqs. (40). This distribution is called Waring distribution. The explicit form of the Waring distribution is:

\[ P(\zeta = i) = \frac{a}{a + k(a + k)^{[i]}} \left( \frac{(k - 1)^{[i]}}{k!} \right); \quad k^{[i]} = \frac{(k + i)!}{k!}; \quad i = 0, 1, \ldots \] (41)

with parameters \( k = \alpha/\beta \) and \( a = \sigma/\beta \).

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