On the Complexity of Distance-

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Abstract

For a fixed positive integer \(d \geq 2\), a distance-

Distance-\(d\) Independent Set Reconfiguration (D\(d\)ISR) problem under TS/TJ asks if there is a corresponding sequence of adjacent D\(d\)ISs that transforms one given D\(d\)IS into another. The problem for \(d = 2\), also known as the Independent Set Reconfiguration problem, has been well-studied in the literature and its computational complexity on several graph classes has been known. In this paper, we study the computational complexity of D\(d\)ISR on different graphs under TS and TJ for any fixed \(d \geq 3\). On chordal graphs, we show that D\(d\)ISR under TJ is in \(P\) when \(d\) is even and PSPACE-complete when \(d\) is odd. On split graphs, there is an interesting complexity dichotomy: D\(d\)ISR is PSPACE-complete for \(d = 2\) but in \(P\) for \(d = 3\) under TS, while under TJ it is in \(P\) for \(d = 2\) but PSPACE-complete for \(d = 3\). Additionally, certain well-known hardness results for \(d = 2\) on perfect graphs and planar graphs of maximum degree three and bounded bandwidth can be extended for \(d \geq 3\).

Keywords and phrases: reconfiguration problem, distance-\(d\) independent set, computational complexity, token sliding, token jumping

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1. Introduction

Recently, reconfiguration problems have attracted the attention from both theoretical and practical viewpoints. The input of a reconfiguration problem consists of two feasible solutions of some source problem (e.g., Satisfiability, Independent Set, Vertex Cover, Dominating Set, etc.) and a reconfiguration rule that describes an adjacency relation between solutions. One of the primary goals is to decide whether one feasible solution can be transformed into the other via a sequence of adjacent feasible solutions where each intermediate member is obtained from its predecessor by applying the given reconfiguration rule exactly once. Such a sequence, if exists, is called a reconfiguration sequence.

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In general, the (huge) set of all feasible solutions is not given, and one can verify in polynomial time (with respect to the input’s size) the feasibility and adjacency of solutions. Another way of looking at reconfiguration problems involves the so-called reconfiguration graphs—a graph whose vertices/nodes are feasible solutions and their adjacency relation is defined via the given reconfiguration rule. Naturally, a reconfiguration sequence is nothing but a walk in the corresponding reconfiguration graph. Readers may recall the classic Rubik’s cube puzzle as an example of a reconfiguration problem, where each configuration of the Rubik’s cube corresponds to a feasible solution, and two configurations (solutions) are adjacent if one can be obtained from the other by rotating a face of the cube by either 90, 180, or 270 degrees. The question is whether one can transform an arbitrary configuration to the one where each face of the cube has only one color. For an overview of this research area, we refer readers to the surveys [6, 13, 16, 30].

Reconfiguration problems involving vertex-subsets (e.g., clique, independent set, vertex cover, dominating set, etc.) of a graph have been extensively considered in the literature. In such problems, to make it more convenient for describing reconfiguration rules, one usually views a vertex-subset of a graph as a set of tokens placed on its vertices. Some well-known reconfiguration rules in this setting are:

- **Token Sliding (TS)**: a token can only move to one of the unoccupied adjacent vertices;
- **Token Jumping (TJ)**: a token can move to any unoccupied vertex; and
- **Token Addition/Removal (TAR(k))**: a token can either be added to an unoccupied vertex or removed from an occupied one, such that the number of tokens is always (upper/lower) bounded by some given positive integer $k$.

Let $G$ be a simple, undirected graph. An independent set of a graph $G$ is a set of pairwise non-adjacent vertices. The INDEPENDENT SET (IS) problem, which asks if $G$ has an independent set of size at least some given positive integer $k$, is one of the fundamental $NP$-complete problems in the computational complexity theory [44]. Given an integer $d \geq 2$, a distance-$d$ independent set (also known as $d$-scattered set or sometimes $d$-independent set) of $G$ is a set of vertices whose pairwise distance is at least $d$. This “distance-$d$ independent set” concept generalizes “independent set”: an independent set is also a distance-$d$ independent set but may not be a distance-$d$ independent set for $d \geq 3$. Given a fixed integer $d \geq 2$, the distance-$d$ independent set (DiIS) problem asks if there is a distance-$d$ independent set of $G$ whose size is at least some given positive integer $k$. Clearly, DiIS is $NP$-complete but $NP$-complete for every fixed $d \geq 3$ on general graphs [42] and on regular bipartite graphs when $d \in \{3, 4, 5\}$ [39]. Eto et al. [28] proved that DiIS is $NP$-complete for every fixed $d \geq 3$ even for planar bipartite graphs of maximum degree three. They also proved that on chordal graphs, DiIS is $P$ for any even $d \geq 2$ and remains $NP$-complete for any odd $d \geq 3$. The complexity of DiIS on several other graphs has also been studied [15, 22, 36]. Additionally, DiIS and its variants have been studied extensively from different viewpoints, including exact exponential algorithms [14], approximability [11, 15, 28], and parameterized complexity [10].

In this paper, we take DiIS as the source problem and initiate the study of distance-$d$ INDEPENDENT SET RECONFIGURATION (DiISR) from the classic complexity viewpoint. The problem for $d = 2$, which is usually known as the INDEPENDENT SET RECONFIGURATION (ISR) problem, has been well-studied from both classic and parameterized complexity viewpoints. ISR under TS was first introduced by Hearn and Demaine [35]. Ito et al. [32] introduced and studied the problem under TAR along with several other reconfiguration problems. Under TJ, the problem was first studied by Kamiński et al. [31]. Very recently, the study of ISR on directed graphs has been initiated by Ito et al. [7]. Additionally, research on the structure and realizability of reconfiguration graphs for ISR has been initiated both under TS [2, 3] and TJ [4]. Readers are referred to [6, 16] for a complete overview of recent developments regarding ISR. We now briefly mention some known results regarding the computational complexity of ISR on different graph classes. ISR remains $PSPACE$-complete under any of TS, TJ, TAR on general graphs [32], planar graphs of maximum degree three and bounded bandwidth [18, 26, 35] and perfect graphs [31]. Under TS, the problem is $PSPACE$-complete even on split graphs [8]. Interestingly, on bipartite graphs, ISR under TS is $PSPACE$-complete while under any of TJ and TAR it is $NP$-complete [12]. On the positive side, ISR under any of TJ and TAR is in $P$ on even-hole-free graphs [31] (which also contains chordal graphs, split graphs, interval graphs, trees, etc.), cographs [21], and claw-free graphs [27]. ISR under TS is in $P$ on cographs [31], claw-free graphs [27], trees [23], bipartite permutation graphs and bipartite distance-hereditary graphs [24], and interval graphs [9, 19]. Very recently, Bartier et al. [1] proved that...
ISR under any of TS and TJ is PSPACE-complete on $H$-free graphs when $H$ is neither a path nor a subdivision of the claw. Additionally, they designed a polynomial-time algorithm to solve ISR under TS when the input graph is fork-free.

DdISR for $d \geq 3$ was first studied by Siebertz [17] from the parameterized complexity viewpoint. More precisely, in [17], Siebertz proved that DdISR under TAR is in FPT for every $d \geq 2$ on “nowhere dense graphs” (which generalized the previously known result for $d = 2$ of Lokshtanov et al. [25]) and it is $\text{w}[1]$-hard for some value of $d \geq 2$ on “somewhere dense graphs” that are closed under taking subgraphs.

Since the TJ and TAR rules are somewhat equivalent [31], i.e., any TJ-sequence between two size-$k$ token-sets can be converted into a TAR-sequence between them whose members are token-sets of size at least $k - 1$ and vice versa; in this paper, we consider DdISR ($d \geq 3$) under only TS and TJ. Most of our results are summarized in Table 1. In short, we show the following results.

- It is worth noting that
  - Even though it is well-known that DdIS on $G$ and IS on its $(d - 1)$th power (this concept will be defined later) are equivalent, this does not necessarily hold for their reconfiguration variants. (Section 3.1.)
  - The definition of DdIS implies the triviality of DdISR for large enough $d$ on graphs whose (connected) components’ diameters are bounded by some constant, including cographs and split graphs. (Section 3.2.)

- On chordal graphs and split graphs, there are some interesting complexity dichotomies. (Section 4.)
  - Under TJ on chordal graphs, DdISR is in $\mathbb{P}$ for even $d$ and PSPACE-complete for odd $d$.
  - On split graphs, DdISR under TS is PSPACE-complete for $d = 2$ [8] but in $\mathbb{P}$ for $d = 3$. Under TJ, it is in $\mathbb{P}$ for $d = 2$ [31] but PSPACE-complete for $d = 3$.

- The known results for $d = 2$ on perfect graphs [31] and on planar graphs of maximum degree three and bounded bandwidth [18, 26, 35] can be extended for $d \geq 3$. (Section 5.)

| Graph                | TS                        | TJ                        |                      |
|----------------------|---------------------------|---------------------------|----------------------|
|                      | $d = 2$                   | $d \geq 3$                | $d = 2$              |
|                      |                           |                           | $d \geq 3$           |
|                      |                           |                           |                      |
| chordal              | PSPACE-C [8]              | unknown                   | P [31]               |
|                      | (Thm. 3.4 and 4.4)        |                           |                      |
| split ($\subseteq$ chordal) | PSPACE-C [8]         | $P$ (Thm. 4.1)           | odd $d$: PSPACE-C    |
|                      |                           |                           | (Thm. 4.2)           |
| tree ($\subseteq$ chordal) | P [23]                 | unknown                   | P [31]               |
|                      |                           |                           |                      |
|                      |                           | $d = 3$: PSPACE-C         |                      |
|                      |                           | (Thm. 4.3)                |                      |
|                      |                           | $d \geq 4$: P             |                      |
|                      |                           | (Thm. 3.4)                |                      |
| perfect              | PSPACE-C [31]             | PSPACE-C                  | P [31]               |
|                      | (Thm. 5.1)                | (Thm. 5.2)                | (Thm. 3.2)           |
| planar $\cap$ subcubic $\cap$ bounded bandwidth | PSPACE-C [18, 26, 35] | PSPACE-C                  | PSPACE-C [31]       |
|                      | (Thm. 5.1)                | (Thm. 5.2)                | (Thm. 5.2)           |
| cograph ($\subseteq$ perfect) | P [31]                 | P [21]                    |                      |
|                      | (Thm. 3.5)                | (Thm. 3.5)                |                      |
| interval ($\subseteq$ chordal) | P [9, 19]               | unknown                   | P [31]               |
|                      |                           |                           | (Thm. 3.2)           |

Table 1: The computational complexity of DdISR ($d \geq 3$) on some graphs considered in this paper. For comparison, we also mention the corresponding known results for $d = 2$. PSPACE-C stands for PSPACE-complete.

2. Preliminaries

For terminology and notation not defined here, see [20]. Let $G$ be a simple, undirected graph with vertex-set $V(G)$ and edge-set $E(G)$. For two sets $I, J$, we sometimes use $I - J$ and $I + J$ to indicate $I \setminus J$ and $I \cup J$, respectively. Additionally, we simply write $I - u$ and $I + u$ instead of $I - \{u\}$ and $I + \{u\}$, respectively. The neighborhood of a vertex $v$ in $G$, denoted by $N_G(v)$, is the set $\{w \in V(G) : vw \in E(G)\}$.

$^2$They proved the result for ISR, but it is not hard to extend it for DdISR.
The closed neighborhood of \( v \) in \( G \), denoted by \( N_G[v] \), is simply the set \( N_G(v) + v \). Similarly, for a vertex-subset \( I \subseteq V(G) \), its neighborhood \( N_G(I) \) and closed neighborhood \( N_G[I] \) are respectively \( \bigcup_{v \in I} N_G(v) \) and \( N_G(I) + I \). The degree of a vertex \( v \) in \( G \), denoted by \( \text{deg}_G(v) \), is \( |N_G(v)| \). The distance between two vertices \( u, v \) in \( G \), denoted by \( \text{dist}_G(u, v) \), is the number of edges in a shortest path between them. For convenience, if there is no path between \( u \) and \( v \) then \( \text{dist}_G(u, v) = \infty \). The diameter of \( G \), denoted by \( \text{diam}(G) \), is the largest distance between any two vertices. A (connected) component of \( G \) is a maximal subgraph in which there is a path connecting any pair of vertices. An independent set (IS) of \( G \) is a vertex-subset \( I \subseteq V(G) \) such that for any \( u, v \in I \), we have \( uv \notin E(G) \). A distance-\( d \) independent set (DiIS) of \( G \) for an integer \( d \geq 2 \) is a vertex-subset \( I \subseteq V(G) \) such that for any \( u, v \in I \), \( \text{dist}_G(u, v) \geq d \).

Unless otherwise noted, we denote by \( (G, I, J, R) \) an instance of DiISR under \( R \in \{\text{TS}, \text{TJ}\} \) where \( I \) and \( J \) are two distinct DiISs of a given graph \( G \), for some fixed \( d \geq 2 \). Since \( (G, I, J, R) \) is obviously a no-instance if \( |I| \neq |J| \), from now on, we always assume that \( |I| = |J| \). Imagine that a token is placed on each vertex in a DiIS of a graph \( G \). A TS-sequence in \( G \) between two DiISs \( I \) and \( J \) is the sequence \( S = \langle I = I_0, I_1, \ldots, I_q = J \rangle \) such that for \( i \in \{0, \ldots, q-1\} \), the set \( I_i \) is a DiIS of \( G \) and there exists a pair \( x_i, y_i \in V(G) \) such that \( I_i - I_{i+1} = \{x_i\} \), \( I_{i+1} - I_i = \{y_i\} \), and \( x_iy_i \in E(G) \). By simply removing the restriction \( x_iy_i \in E(G) \), we immediately obtain the definition of a TJ-sequence in \( G \). Depending on the considered rule \( R \in \{\text{TS}, \text{TJ}\} \), we can also say that \( I_{i+1} \) is obtained from \( I_i \) by immediately sliding/jumping a token from \( x_i \) to \( y_i \) and write \( x_i \xrightarrow{G} \text{R} \ y_i \). As a result, we can also write \( S = \langle x_0 \xrightarrow{G} \text{R} \ y_0, \ldots, x_{q-1} \xrightarrow{G} \text{R} \ y_{q-1} \rangle \). In short, \( S \) can be viewed as a (ordered) sequence of either DiISs or token-moves. (Recall that we defined \( S \) as a sequence between \( I \) and \( J \). As a result, when regarding \( S \) as a sequence of token-moves, we implicitly assume that the initial DiIS is \( I \).) With respect to the latter viewpoint, we say that \( S \) slides/jumps a token \( t \) from \( u \) to \( v \) in \( G \) if \( t \) is originally placed on \( u \in I_0 \) and finally on \( v \in I_q \) after performing \( S \). The length of a R-sequence is simply the number of times the rule \( R \) is applied. In particular, the R-sequence \( S = \langle x_0 \xrightarrow{G} \text{R} \ y_0, \ldots, x_{q-1} \xrightarrow{G} \text{R} \ y_{q-1} \rangle \) has length \( q \).

We conclude this section with the following remark: since DiIS is in \( \text{NP} \), DiISR is always in \( \text{PSPACE} \) [32]. As a result, to show the \( \text{PSPACE} \)-completeness of DiISR, it is sufficient to construct a polynomial-time reduction from a known \( \text{PSPACE} \)-hard problem and prove its correctness.

## 3. Observations

### 3.1. Graphs and Their Powers

An extremely useful concept for studying distance-\( d \) independent sets is the so-called graph power. For a graph \( G \) and an integer \( s \geq 1 \), the \( s \)-th power of \( G \) is the graph \( G^s \) whose vertices are \( V(G) \) and two vertices \( u, v \) are adjacent in \( G^s \) if \( \text{dist}_G(u, v) \leq s \). Observe that \( I \) is a distance-\( d \) independent set of \( G \) if and only if \( I \) is an independent set of \( G^{d-1} \). Therefore, DiISR in \( G \) is equivalent to IS in \( G^{d-1} \).

However, this may not apply for their reconfiguration variants. More precisely, the statement “the DiISR’s instance \((G, I, J, R)\) is a yes-instance if and only if the ISR’s instance \((G^{d-1}, I, J, R)\) is a yes-instance” holds for \( R = \text{TJ} \) but not for \( R = \text{TS} \). The reason is that under \( \text{TJ} \) we do not care about edges when performing token-jumps (as long as they result new DiISs), therefore whatever token-jump we perform in \( G \) can also be done in \( G^{d-1} \) and vice versa. Therefore, we have the following proposition.

**Proposition 3.1.** Let \( G \) and \( H \) be two graph classes and suppose that for every \( G \in G \) we have \( G^{d-1} \in H \) for some fixed integer \( d \geq 2 \). If ISR under \( \text{TJ} \) on \( H \) can be solved in polynomial time, so does DiISR under \( \text{TJ} \) on \( G \).

Recall that the power of any interval graph is also an interval graph [37, 38] and ISR under \( \text{TJ} \) on even-hole-free graphs (which contains interval graphs) is in \( \text{P} \) [31]. Along with Proposition 3.1, we immediately obtain the following corollary.

**Corollary 3.2.** DiISR under \( \text{TJ} \) is in \( \text{P} \) on interval graphs for any \( d \geq 2 \).

On the other hand, when using token-slides, we need to consider which edges can be used for moving tokens, and clearly \( G^{d-1} \) has much more edges than \( G \), which means certain token-slides we perform in \( G^{d-1} \) cannot be done in \( G \). More precisely, we have the following proposition.

**Proposition 3.3.** Let \( d \geq 3 \) be a fixed integer. There exists a DiISR’s no-instance \((G, I, J, \text{TS})\) such that the ISR’s instance \((G^{d-1}, I, J, \text{TS})\) is a yes-instance, where \( I, J \) are DiISs of \( G \) (and therefore ISs of \( G^{d-1} \)) of size \( k \geq 2 \).
Thus, when $d \geq 3$, any path $P_{ij}$ between $v_i$ and $v_j$ has length at least $d$, and the equality happens when either $\text{dist}_G(v_i, v_j) = \ell$ or $\text{dist}_G(v_i, v_j) = \ell$. It follows that $\text{dist}_G(v_i, v_j) \geq d$. Since this holds for any pair $v_i, v_j \in I$ (1 \leq i < j \leq k), the set $I$ is a DdISR of $G$.

Next, we prove that $(G, I, J, TS)$ is a no-instance of DdISR. To show this, we prove that no tokens in $I$ can be moved even to one of its adjacent vertices and (by applying similar arguments) so do those in $J$. For each $i \in \{1, 2, \ldots, k\}$, let $x_{ij}$ be the unique neighbor of $v_i$ in the path $Q_{v_i, v_j}$ where $j \in \{1, 2, \ldots, k\} - i$. From the construction of $G$, $\text{dist}_G(v_i) = \bigcup_{j \in \{1, 2, \ldots, k\} - i} \{x_{ij}\} \cup \{v_i^*\}$. We now prove that for each $i \in \{1, 2, \ldots, k\}$, a token $t_i$ on $v_i$ can be slid to one of its neighbors if all other tokens $t_j$ on
\[v_j, \text{ for } j \in \{1, 2, \ldots, k\} - i, \text{ must be moved before moving } t_i. \text{ If this claim holds, no tokens in } I \text{ (and similarly in } J) \text{ can be moved, and therefore } (G, I, J, TS) \text{ is a no-instance.} \]

If \(t_i\) is moved to \(v_i\), since all \(Q_j^i, v_i\) are of length \(d - 1\), the tokens \(t_j\) must be moved before \(t_i\), for \(j \in \{1, 2, \ldots, k\} - i\). If \(t_i\) is moved to some \(x_{ij}\), since \(Q_j^i, v_i\) is of length \(d - 1\), the token \(t_j\) on \(v_j\) must be moved before \(t_i\), for \(j \in \{1, 2, \ldots, k\} - 1\). By applying the above arguments for \(t_j\) instead of \(t_i\), we obtain that there must be some token \(t_{ij}\) where \(\ell \in \{1, 2, \ldots, k\} - \{i, j\}\) that must be moved before \(t_j\), which clearly also before \(t_i\). By repeatedly applying these arguments, we finally obtain that all other tokens must be moved before \(t_i\).

It remains to show that \((G^{d-1}, I, J, TS)\) is a yes-instance of ISR. Observe that for each \(i \in \{1, 2, \ldots, k\}\), we have \(\text{dist}_G(v_i, w_i) = d - 1\) and \(\text{dist}_G(v_i, w_j) \geq 2d - 3 \geq d\) for every \(d \geq 3\) and \(j \in \{1, 2, \ldots, k\} - i\). Therefore, in \(G^{d-1}\), there is an edge between \(v_i\) and \(w_i\), while there is no edge joining \(v_i\) and \(w_j\), for \(i, j \in \{1, 2, \ldots, k\}\) and \(i \neq j\). Thus, \(S = \langle v_1 \xrightarrow{G^{d-1}} \tau_S w_1, v_2 \xrightarrow{G^{d-1}} \tau_S w_2, \ldots, v_k \xrightarrow{G^{d-1}} \tau_S w_k \rangle\) is a TS-sequence that transforms \(I\) into \(J\).

\[\square\]

\subsection*{3.2. Graphs With Bounded Diameter Components}

The following observation is straightforward.

**Proposition 3.4.** Let \(G\) be a graph class such that there is some constant \(c > 0\) satisfying \(\text{diam}(C_G) \leq c\) for any \(G \in G\) and any component \(C_G\) of \(G\). Then, \(\text{DDISR on } G\) under \(R \in \{\text{TS}, TJ\}\) is in \(P\) for every \(d \geq c + 1\).

**Proof.** When \(d \geq c + 1\), any DDISR contains at most one vertex in each component of \(G\), and the problem becomes trivial: under TJ, the answer is always “yes”; under TS, compare the number of tokens in each component.

As a result, on cographs (a.k.a \(P_4\)-free graphs), one can immediately derive the following corollary.

**Corollary 3.5.** \(\text{DDISR on cographs under } R \in \{\text{TS}, TJ\}\) is in \(P\) for any \(d \geq 2\).

**Proof.** It is well-known that the problems for \(d = 2\) is in \(P\) [21, 31]. Since a connected cograph has diameter at most two, Proposition 3.4 settles the case \(d \geq 3\).

\[\square\]

\section*{4. Chordal Graphs and Split Graphs}

In this section, we will focus on chordal graphs and split graphs. Recall that the odd power of a chordal graph is also chordal [38, 43] and ISR under TJ on even-hole-free graphs (which contains chordal graphs) is in \(P\) [31]. Therefore, it follows from Proposition 3.1 that the following corollary holds.

**Corollary 4.1.** \(\text{DDISR is in } P\) on chordal graphs under TJ for any even \(d \geq 2\).

In the following, we have the contrast.

**Theorem 4.2.** \(\text{DDISR is PSPACE-complete on chordal graphs under TJ for any odd } d \geq 3\).

**Proof.** We reduce from the ISR problem, which is known to be PSPACE-complete under TJ [32]. Let \((G, I, J, TJ)\) be an ISR’s instance. We construct a DDISR’s instance \((G', I', J', TJ)\) (\(d \geq 3\) is odd) as follows. We note that a similar reduction was used by Eto et al. [28] for showing the NP-completeness of DDISR on chordal graphs for odd \(d \geq 3\). We first describe how to construct \(G'\) from \(G\). For each vertex \(v \in V(G)\), we add \(v\) to \(V(G')\). For each edge \(uv \in V(G)\), we add a vertex \(x_{uv}\) to \(V(G')\) and create an edge in \(G'\) between \(x_{uv}\) and both \(u\) and \(v\). Next, we add an edge in \(G'\) between \(x_{uv}\) and \(x_{wv}\) for any pair of distinct edges \(uv, wv' \in E(G)\). Finally, for each \(v \in V(G)\), we add to \(G'\) a new path \(P_v\) on \((d - 3)/2\) vertices and then add an edge in \(G'\) between \(v\) and one of \(P_v\)'s endpoints. One can verify that \(G'\) is indeed a chordal graph: it is obtained from a split graph by attaching new paths to certain vertices. Clearly this construction can be done in polynomial time. (For example, see Figure 2.)

For each \(u \in V(G)\), we define \(f(u) \in V(G')\) to be the vertex whose distance in \(G'\) from \(u\) is largest among all vertices in \(V(P_u) + u\). Let \(f(X) = \bigcup_{x \in X} \{f(x)\}\) for a vertex-subset \(X \subseteq V(G)\). From the construction of \(G'\), note that if \(u\) and \(v\) are two vertices of distance 2 in \(G\), one can always find a shortest path \(Q\) between \(f(u)\) and \(f(v)\) whose length is exactly \(d\). Indeed, \(Q\) can be obtained by joining the paths from \(f(u)\) to \(u\), from \(u\) to \(x_{uw}\), from \(x_{uw}\) to \(x_{wv}\), from \(x_{wv}\) to \(v\), and from \(v\) to \(f(v)\), where
Therefore, we have the following corollary.

It remains to consider the case \( d = 2 \) has diameter at most 3. It remains to show the if direction. Let \( x \in N_G(u) \cap N_G(v) \). It follows that if \( I \) is an independent set of \( G \) then \( f(I) \) is a distance-\( d \) independent set of \( G' \). Therefore, we can set \( I' = f(I) \) and \( J' = f(J) \).

From the construction of \( G' \), note that for each \( uv \in E(G) \), \( x_{uv} \) is of distance exactly one from each \( wz \in E(G) - uv \), at most two from each \( v \in V(G) \), and at most \( 2 + (d - 3)/2 \leq d - 1 \) from each vertex in \( P_u \) for \( u \in V(G) \). It follows that any distance-\( d \) independent set of \( G' \) of size at least two must not contain any vertex in \( \bigcup_{uv \in E(G)} \{ x_{uv} \} \).

We now show that there is a TJ-sequence between \( I \) and \( J \) in \( G \) if and only if there is a TJ-sequence between \( I' \) and \( J' \) in \( G' \). If \( |I| = |J| = 1 \), the claim is trivial. As a result, we consider the case \( |I| = |J| \geq 2 \). Since \( f(I) \) is a distance-\( d \) independent set in \( G' \) if \( I \) is an independent set in \( G \), the only-if direction is clear. It remains to show the if direction. Let \( S' \) be a TJ-sequence in \( G' \) between \( I' \) and \( J' \). We modify \( S' \) by repeating the following steps:

- Let \( x \xrightarrow{G'}_{TJ} y \) be the first token-jump that move a token from \( x \in f(V(G)) \) to some \( y \in V(P_u) + u - f(u) \) for some \( u \in V(G) \). If no such token-jump exists, we stop. Let \( I_x \) and \( I_y \) be respectively the distance-\( d \) independent sets obtained before and after this token-jump. In particular, \( I_y = I_x - x + y \).

- Replace \( x \xrightarrow{G'}_{TJ} y \) by \( x \xrightarrow{G'}_{TJ} f(u) \) and replace the first step after \( x \xrightarrow{G'}_{TJ} f(u) \) of the form \( y \xrightarrow{G'}_{TJ} z \) by \( f(u) \xrightarrow{G'}_{TJ} y \). From the construction of \( G' \), note that any path containing \( f(u) \) must also contains all vertices in \( V(P_u) + u \). Additionally, \( I_y = V(P_u) + u = \emptyset \), otherwise no token in \( I_z \) can jump to \( y \). Therefore, the set \( I_x - x + f(u) \) is also a distance-\( d \) independent set. Moreover, \( \text{dist}_{G'}(f(u), z) \geq \text{dist}_{G'}(y, z) \) for any \( z \in V(G') - V(P_u) \). Roughly speaking, this implies that no token-jump between \( x \xrightarrow{G'}_{TJ} y \) and \( y \xrightarrow{G'}_{TJ} z \) breaks the “distance-\( d \) restriction”. Thus, after the above replacements, \( S' \) is still a TJ-sequence in \( G' \).

- Repeat the first step.

After modification, the final resulting TJ-sequence \( S' \) in \( G' \) contains only token-jumps between vertices in \( f(V(G)) \). By definition of \( f \), we can construct a TJ-sequence between \( I \) and \( J \) in \( G \) simply by replacing each step \( x \xrightarrow{G'}_{TJ} y \) in \( S' \) by \( f^{-1}(x) \xrightarrow{G}_{TJ} f^{-1}(y) \). Our proof is complete.

Now, we consider the split graphs. Proposition 3.4 implies that on split graphs (where each component has diameter at most 3), D3ISR is in \( P \) under \( R \in \{TS, TJ\} \) for any \( d \geq 4 \). Interestingly, recall that when \( d = 2 \), the problem under TS is \( PSPACE \)-complete even on split graphs [8] while under TJ it is in \( P \) [31]. It remains to consider the case \( d = 3 \).

Observe that the constructed graph \( G' \) in the proof of Theorem 4.2 is indeed a split graph when \( d = 3 \). Therefore, we have the following corollary.

**Corollary 4.3.** D3ISR is \( PSPACE \)-complete on split graphs under TJ.

In contrast, under TS, we have the following proposition.
Proposition 4.4. D3ISR is in P on split graphs under TS.

Proof. Let \((G, I, J, TS)\) be an instance of D3ISR and suppose that \(V(G)\) can be partitioned into a clique \(K\) and an independent set \(S\). One can assume without loss of generality that \(G\) is connected, otherwise each component can be solved independently. If \(|I| = |J| = 1\), the problem becomes trivial: \((G, I, J, R)\) is always a yes-instance. Thus, we now consider \(|I| = |J| \geq 2\). Observe that for every \(u \in V(G)\) and \(v \in K\), we have \(\text{dist}_G(u, v) \leq 2\). Therefore, in this case, both \(I\) and \(J\) are subsets of \(S\). Note that \(I \neq J\). Now, no token in \(I \cup J\) can be slid, otherwise such a token must be slid to some vertex in \(K\), and each vertex in \(K\) has distance at most two from any other token, which contradicts the restriction that tokens must form a D3IS. Hence, \((G, I, J, TS)\) is always a no-instance if \(|I| = |J| \geq 2\).

5. Extending Some Known Results for \(d = 2\)

In this section, we prove that several known results on the complexity of DdISR for the case \(d = 2\) can be extended for \(d \geq 3\).

5.1. Perfect Graphs

In this section, we prove the \(\text{PSPACE}\)-completeness of DdISR \((d \geq 3)\) on perfect graphs by extending the corresponding known result for ISR of Kamiński et al. [31].

Theorem 5.1. DdISR is \(\text{PSPACE}\)-complete on perfect graphs under \(R \in \{TS, TJ\}\) for any \(d \geq 3\).

Proof. Recall that Kamiński et al. [31] proved the \(\text{PSPACE}\)-hardness of the problem for \(d = 2\) by reducing from a known \(\text{PSPACE}\)-complete problem called \textit{Shortest Path Reconfiguration} (SPR) [29]. In a SPR’s instance \((G, u, v, P, Q)\), two shortest \(uv\)-paths are adjacent if one can be obtained from the other by exchanging exactly one vertex. The SPR problem asks whether there is a sequence of adjacent shortest \(uv\)-paths between \(P\) and \(Q\). We will slightly modify their reduction to prove the \(\text{PSPACE}\)-hardness for \(d \geq 3\).

We first describe the reduction by Kamiński et al. [31]. Let \(G\) be a given graph and let \(u, v \in V(G)\). We delete all vertices and edges that does not appear in any shortest \(uv\)-path and let the resulting graph be \(\tilde{G}\). Let \(k = \text{dist}_{\tilde{G}}(u, v)\) and for each \(i \in \{0, \ldots, k\}\) let \(D_i\) be the set of vertices in \(\tilde{G}\) of distance \(i\) from \(u\) and \(k - i\) from \(v\). Let \(G'\) be the graph obtained from \(\tilde{G}\) by turning each \(D_i\) into a clique and complementing (i.e., if there is an edge between two vertices, remove it; otherwise, create one) the edges of \(\tilde{G}\) between two consecutive layers \(D_i\) and \(D_{i+1}\). Kamiński et al. [31] proved that \(G'\) is a perfect graph. They also claimed that there is a one-to-one correspondence between \(uv\)-shortest paths of \(G\) and independent sets of size \(k + 1\) of \(G'\): the \(k + 1\) vertices of a shortest \(uv\)-path in \(G\) become independent in \(G'\). This is used for showing that there is a sequence of adjacent \(uv\)-shortest paths in \(G'\) if and only if there is a R-sequence between size-\((k + 1)\) independent sets of \(G'\), for \(R \in \{TS, TJ, TAR\}\).

Our modification is simple. We construct a graph \(G''\) from \(G'\) as follows. First, we replace each edge between two consecutive layers \(D_i\) and \(D_{i+1}\) by a new path of length \(d - 1\), and let the resulting graph be \(\tilde{G}'\). For each \(j \in \{1, \ldots, d - 2\}\), let \(D_{ij}\) be the set of vertices in \(\tilde{G}'\) of distance \(j\) from some vertex in \(D_i\) and distance \(d - 1 - j\) from some vertex in \(D_{i+1}\). Let \(G''\) be the graph obtained from \(\tilde{G}'\) by turning each \(D_{ij}\) into a clique. Since \(G''\) is perfect, the above construction implies that \(G''\) is perfect too. Clearly the construction of both \(G'\) and \(G''\) can be done in polynomial time. In fact, if \(d = 2\), we have \(G'' = G'\). (For example, see Figure 3.)

It is sufficient to show that there is a R-sequence of size-\((k + 1)\) independent sets in \(G''\) if and only if there is a R-sequence of size-\((k + 1)\) distance-d independent sets in \(G''\), for \(R \in \{TS, TJ\}\). From the construction of \(G''\), it follows that any independent set of \(G'\) of size \(k + 1\) is also a distance-d independent set of \(G''\), since a shortest path of length \(2\) in \(G'\) between two vertices \(x \in D_i\) and \(y \in D_{i+1}\) becomes a shortest path of length \(d - 1 + 1 = d\) in \(G''\). Thus, any TJ-sequence in \(G'\) between two size-\((k + 1)\) independent sets is also a TJ-sequence in \(G''\). On the other hand, note that a size-\((k + 1)\) distance-d independent set of \(G''\) must not contain any new vertex in \(V(G'') - V(G')\), otherwise one can verify that the set must be of size at most \(k\), which is a contradiction. (Note that a DdIS must contain at most one vertex in \(\bigcup_{i=1}^{d-2} D_{ij}\).) As a result, a size-\((k + 1)\) distance-d independent set of \(G''\) is also an independent set of \(G'\). Thus, any TJ-sequence in \(G''\) between two size-\((k + 1)\) distance-d independent sets is also a TJ-sequence in \(G''\). We note that in both \(G'\) and \(G''\), a token can never move out of the layer where it belongs, and therefore any token-jump is also a token-slide. Our proof is complete.
5.2. Planar Graphs

In this section, we claim that the PSPACE-hard reduction of Hearn and Demaine [35] for ISR under TS can be extended to DoISR (\(d \geq 3\)) under \(R \in \{\text{TS}, \text{TJ}\}\). We now briefly introduce the powerful tool Hearn and Demaine [34, 35] used as the source problem for proving several hardness results, including the PSPACE-hardness of ISR: the Nondeterministic Constraint Logic (NCL) machine. An NCL “machine” is an undirected graph whose edges are assigned with weight 1 (thin) or 2 (thick). An (NCL) configuration of this machine is an orientation of the edges such that the sum of weights of in-coming arcs at each vertex is at least two. Figure 4(a) illustrates a configuration of an NCL machine. Two NCL configurations are adjacent if they differ in a single edge direction. Given an NCL machine along with two configurations, Hearn and Demaine [35] proved that it is PSPACE-complete to determine whether there exists a sequence of adjacent NCL configurations which transforms one into the other.

Indeed, even when restricted to AND/OR constraint graphs, the above problem remains PSPACE-complete. An AND/OR constraint graph is a NCL machine that contains exactly two type of vertices: “NCL AND vertices” and “NCL OR vertices”.

More precisely, a vertex of degree three is called an NCL AND vertex if its three incident edges have weights 1, 1 and 2. (See Figure 4(b).) The “behavior” of an NCL AND vertex \(u\) is similar to a logical AND: the weight-2 edge can be directed outward for \(u\) if and only if both two weight-1 edges are directed inward for \(u\). Note that, however, the weight-2 edge is not necessarily directed outward even when both weight-1 edges are directed inward. A vertex of degree three is called an NCL OR vertex if its three incident edges have weights 2, 2 and 2. (See Figure 4(c).) The “behavior” of an NCL Or vertex \(v\) is similar to a logical OR: one of the three edges can be directed outward for \(v\) if and only if at least one of the other two edges is directed inward for \(v\).
For example, the NCL machine in Figure 4(a) is an And/Or constraint graph.

**Theorem 5.2.** $D_d\text{ISR}$ is PSPACE-complete under $R \in \{TS, TJ\}$ on planar graphs of maximum degree three and bounded bandwidth for any $d \geq 2$.

**Proof.** We note that Hearn and Demaine [35]'s proof can be applied with the vertex gadgets shown in Figure 5. Indeed, when $d = 2$, the gadgets in Figure 5 are exactly those used in [35]. We rephrase their proof here for the sake of completeness. From now on we consider only TS rule. We will see later that in the constructed graph, any token-jump is also a token-slide, and therefore our reduction also holds for TJ. The maximum degree will also be clear from the construction of gadgets and how they are joined. The final complexity result follows from the result of van der Zanden [26] which combines [35] and [18]: NCL remains PSPACE-complete even if an input NCL machine is planar and bounded bandwidth.

The NCL And and Or vertex gadgets are constructed as in Figure 5(a) and (b). The edges that cross the dashed-line gadget borders are “port” edges. A token on an outer port-edge vertex represents an inward-directed NCL edge, and vice versa. Given an And/Or graph and configuration, we construct a corresponding graph for $D_d\text{ISR}$ under TS, by joining together And and Or vertex gadgets at their shared port edges, placing the port tokens appropriately.

![Figure 5](image)

Figure 5: Vertex gadgets for $D_d\text{ISR}$ under TS: (a) And, (b) Or. The dotted edges represent paths of length $d - 2$.

First, observe that no port token may ever leave its port edge. Choosing a particular port edge $E$, if we inductively assume that this condition holds for all other port edges, then there is never a legal move outside $E$ for its token—another port token would have to leave its own edge first.

The And gadget clearly satisfies the same constraints as an NCL And vertex; the upper token can slide in just when both lower tokens are slid out. Likewise, the upper token in the Or gadget can slide in when either lower token is slid out—the internal token can then slide to one side or the other to make room. It thus satisfies the same constraints as an NCL Or vertex. As a result, it follows that a sequence of adjacent NCL configurations in a given And/Or graph can indeed be transformed into a TS-sequence in the constructed graph and vice versa. Since no port token ever leaves its port edge, it follows from the construction of gadgets that in the constructed graph any token-jump is also a token-slide, which means the theorem also holds for TJ. Our proof is complete.

6. Open Problem: Trees

In this section, we discuss the problems on trees. Since the power of a tree is a (strongly) chordal graph [40, 41] and ISR on chordal graphs under TJ is in P [31], Proposition 3.1 implies that the following
corollary holds.

**Corollary 6.1.** DdISR under TJ on trees is in P for any \( d \geq 3 \).

On the other hand, the complexity of DdISR under TS for \( d \geq 3 \) remains unknown.

**Conjecture 6.2.** DdISR under TS on trees is in P for \( d \geq 3 \).

Demaine et al. [23] showed that the problem for \( d = 2 \) can be solved in linear time. Their algorithm is based on the so-called rigid tokens. Given a tree \( T \) and a DdIS \( I \) of \( T \) (\( d \geq 2 \)), a token \( t \) on \( u \in I \) is \((T, I, d)\)-rigid if there is no TS-sequence in \( T \) that slides \( t \) from \( u \) to some vertex \( v \in N_T(u) \). We denote by \( R(T, I, d) \) the set of all vertices where \((T, I, d)\)-rigid tokens are placed. Demaine et al. proved that \( R(T, I, 2) \) can be found in linear time. (In general, deciding whether a token is \((G, I, J, 2)\)-rigid on some graph \( G \) is PSPACE-complete [24].) Clearly, it holds for any \( d \geq 2 \) that every DdISR’s instance \((T, I, J, 2)\) where \( R(T, I, d) \neq R(T, I, d) \) is a no-instance. When \( R(T, I, 2) = R(T, I, 2) = \emptyset \), they proved that \((T, I, J, TS)\) is always a yes-instance of ISR. Based on these observations, one can derive a polynomial-time algorithm for solving ISR on trees under TS.

Indeed, one can extend the recursive characterization of \((T, I, 2)\)-rigid tokens given by Demaine et al. as follows. For a given vertex \( u \) of a graph \( G \) and an integer \( s \geq 0 \), we denote by \( N_G^s(u) \) the \( s \)-neighborhood of \( u \) in \( G \) which contains all vertices of distance \( s \) from \( u \) in \( G \). Additionally, we denote by \( N_G[u] = \bigcup_{s \geq 0} N_G^s(u) \) the set of all vertices of distance at most \( s \) from \( u \) in \( G \) and called it the \( s \)-closed neighborhood of \( u \) in \( G \). For example, \( N_G^0(u) = \{u\} \), \( N_G^1(u) = N_G(u) \), and \( N_G[0] = N_G[u] \). For two vertices \( u, v \) of a tree \( T \), we denote by \( T_w \) the tree induced by \( v \) and its descendants when regarding \( u \) as the root of \( T \).

**Lemma 6.3.** Let \( I \) be a DdIS of a tree \( T \) for \( d \geq 2 \) and let \( u \in I \). The token \( t \) on \( u \) is \((T, I, d)\)-rigid if and only if

\[
(a) \quad |V(T)| = |\{u\}| = 1; \quad \text{or} \\
(b) \quad |V(T)| \geq 2 \quad \text{and for each} \quad v \in N_T(u) \quad \text{there exists a vertex} \quad w \in (N_T^{d−1}(v) − u) \cap I \quad \text{such that} \quad t_w \quad \text{on} \quad w \quad \text{is} \quad (T_w, I \cap V(T_w^w), d)\)-rigid.
\]

**Proof.** Our proof is analogous to the one given by Demaine et al. [23, Lemma 1]. For the sake of completeness, we present it here. Part (a) is trivial. We prove part (b). Before going into more details, we note that for any \( v \in (N_T^{d−1}(v) − u) \), if \( v \notin V(T_w^w) \) then \( \text{dist}_T(w, u) \leq d − 2 \) and therefore \( w \notin I \). In other words, \( (N_T^{d−1}(v) − u) \cap I \subseteq V(T_w^w) \).

\((\Rightarrow)\) Suppose to the contrary that there exists a TS-sequence that slides \( t \) to some vertex \( v \in N_T(u) \). On the other hand, from the assumption, there exists a vertex \( w \in (N_T^{d−1}(v) − u) \cap I \) such that the token \( t_w \) on \( w \) is \((T_w^w, I \cap V(T_w^w), d)\)-rigid. Since \( \text{dist}_T(u, w) = d \), \( t_w \) must be moved before \( t \) and therefore can only be moved to some vertex in \( T_w^w \). However, since the token \( t_w \) on \( w \) is \((T_w^w, I \cap V(T_w^w), d)\)-rigid, this cannot be done. The reason is that if there is a TS-sequence \( S \) in \( T \) that slides \( t_w \), we can “restrict” it to a TS-sequence in \( T_w^w \) by taking the intersection of each independent set in \( S \) with \( V(T_w^w) \), which implies \( t_w \) is \((T_w^w, I \cap V(T_w^w), d)\)-movable.

\((\Rightarrow)\) Suppose that \( |V(T)| \geq 2 \). We claim that if there exists a vertex \( v \in N_T(u) \) such that \( (N_T^{d−1}(v) − u) \cap I = \emptyset \) or for every \( v \in (N_T^{d−1}(v) − u) \cap I \), the token \( t_w \) on \( w \) is \((T_w^w, I \cap V(T_w^w), d)\)-movable then there exists a TS-sequence \( S \) in \( T \) that moves the token \( t \) on \( v \). If \( (N_T^{d−1}(v) − u) \cap I = \emptyset \), there are no tokens of distance \( d − 1 \) from \( v \) except \( t \), and therefore we can immediately slide \( t \) to \( v \). Thus, it remains to consider the case for every \( v \in (N_T^{d−1}(v) − u) \cap I \), the token \( t_w \) on \( w \) is \((T_w^w, I \cap V(T_w^w), d)\)-movable. Since \( t_w \) is \((T_w^w, I \cap V(T_w^w), d)\)-movable, there exists a TS-sequence \( S_w \) in \( T_w^w \) that slides \( t_w \) to some vertex in \( N_{T_w^w}(w) \). Moreover, note that for any DdIS \( I \) of \( T_w^w \), we have \( I \cup (I^* \cap V(T_w^w)) \) is also a DdIS of \( T_w^w \) for any DdIS \( I^* \) of \( G \) satisfying \( I^* \cap V(T_w^w) = I \cap V(T_w^w) \). (Intuitively, we move tokens “away from \( w \)” inside \( T_w^w \). These moves do not break any “distance-\( d \)” restriction” between a token inside \( T_w^w \) and a token outside \( T_w^w \).) In other words, one can “extend” \( S_w \) to form a TS-sequence in \( T \) by simply taking the union of each DdIS in \( S_w \) with \( I^* \cap V(T_w^w) \). Thus, for each \( v \in (N_T^{d−1}(v) − u) \cap I \), we can perform \( S_w \) to slides \( t_w \) to some vertex in \( N_{T_w^w}(w) \) and finally slide \( t \) from \( u \) to \( v \). It remains to verify that for any pair of distinct vertices \( w, w' \in (N_T^{d−1}(v) − u) \cap I \) we can indeed perform \( S_w \) and \( S_w' \) independently. To see this, note that by definition the subtrees \( T_w^w \) and \( T_w^w' \) are distinct, i.e., \( V(T_w^w) \cap V(T_w^w') = \emptyset \). Moreover, since both \( w, w' \) are in \( I \), there distance must be at least \( d \).
Using this characterization, one can naturally derive a recursive algorithm that decides if the token on \( u \in I \) is \((T, I, d)-\text{rigid}\) in \(O(n)\) time, where \( n = |V(T)| \). Therefore, one can find \( R(T, I, d) \) in \(O(n^2)\) time. However, for \( d \geq 3 \), even when \( R(T, I, d) = R(T, J, d) = \emptyset \), \((T, I, J, TS)\) may be a no-instance of \( DdISR \). An example of such instances is described in Figure 6. We remark that the instance described in Figure 6 does not apply for \( d = 2 \), because in that case no tokens can move. As a result, characterizing these no-instances may be crucial in proving Conjecture 6.2. Additionally, a first step toward proving Conjecture 6.2 may be to consider \( DdISR \) under \( TS \) on caterpillars— a subclass of trees. Here a caterpillar is a tree having a backbone path containing all vertices of degree at least 3 (called backbone vertices). Each hair of the caterpillar is a path starting from a backbone vertex and not containing any other backbone vertex. For example, the graph in Figure 6 is a caterpillar with the backbone path having exactly one vertex \( u \) (whose degree is at least 3) and four hairs of length exactly \( d - 1 \) starting from \( u \).

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Appendix

A. A Reduction under TJ on General Graphs

Recall that Ito et al. [32] proved the PSPACE-completeness of ISR under TJ/TAR by reducing from 3-SATISFIABILITY RECONFIGURATION (3SAT-R). In this section, we present a simple proof for the PSPACE-hardness of DdISR (d ≥ 3) on general graphs under TJ by reducing from ISR instead of 3SAT-R.

**Theorem A.1.** DdISR is PSPACE-complete under TJ for any d ≥ 3.

**Proof.** We reduce from the ISR problem, which is known to be PSPACE-complete under TJ [32]. Let \((G, I, J, TJ)\) be an ISR’s instance. We construct a DdISR’s instance \((G', I, J, TJ)\) \((d ≥ 3)\) as follows. Unless otherwise noted, we always assume \(p = (d − 1)/2\) if \(d\) is odd and \(p = (d − 2)/2\) if \(d\) is even. To construct \(G'\) from \(G\), we first replace each edge \(uv \in E(G)\) with a new path \(P_{uv} = x_{0}^{uv} \ldots x_{d−1}^{uv}\) of length \(d − 1\), where \(x_{0}^{uv} = u\) and \(x_{d−1}^{uv} = v\). We emphasize that the ordering of endpoints is important, i.e., \(P_{uv} = x_{0}^{uv} \ldots x_{d−1}^{uv}\) is the path where \(x_{i}^{uv} = x_{i−1}^{uv}\) for \(0 ≤ i ≤ d−1\). Basically, \(P_{uv}\) and \(P_{vu}\) describe the same path with different vertex-labels. If \(d\) is odd, we create a new clique \(K\) whose vertices are \(\bigcup_{uv \in E(G)} \{x_{p}^{uv}\}\). If \(d\) is even, we add a new vertex \(x^*\) and create an edge between it and every vertex in \(\bigcup_{uv \in E(G)} \{x_{p}^{uv}, x_{p+1}^{uv}\}\). The resulting graph is \(G'\). (For example, see Figure 7.) Clearly this construction can be done in polynomial time.

![Figure 7: An example of constructing \(G'\) from a given graph \(G\) for some values of \(d\) ∈ \{2, 3, 4\} (in case \(d = 2\), we have \(G' = G\)). Vertices in \(V(G') − V(G)\) are marked with the gray color.](image)

Now, we show that an independent set of \(G\) of size \(k ≥ 2\) is also a distance-\(d\) independent set of \(G'\) and vice versa. From the construction, for every \(uv \in E(G)\), it follows that \(\text{dist}_{G'}(u, v) ≤ d − 1\) and therefore both \(u\) and \(v\) cannot be in the same distance-\(d\) independent set of \(G'\). Additionally, observe that for every \(uv \notin E(G)\), any path between \(u\) and \(v\) in \(G'\) must contain \(x_{p}^{uw}\) and \(x_{p}^{vz}\), for some \(w \in N_{G}(u)\) and \(z \in N_{G}(v)\). Thus, a shortest path between \(u\) and \(v\) in \(G'\) must be of the form \(x_{0}^{uw} \ldots x_{p}^{uw} x_{p}^{vz} \ldots x_{d−1}^{vz}\) if \(d\) is odd and \(x_{0}^{uw} \ldots x_{p}^{uw} x_{p}^{vz} \ldots x_{d−1}^{uw}\) if \(d\) is even. One can verify that such a shortest path has length exactly \(d\). Therefore, \(\text{dist}_{G'}(u, v) ≥ d\). As a result, if \(I\) is an independent set of \(G\) (of size \(k ≥ 2\)) then it is also a distance-\(d\) independent set of \(G'\). On the other hand, suppose that \(I'\) is a distance-\(d\) independent set of \(G'\) of size \(k ≥ 2\). We claim that \(I'\) does not contain any new vertex. Let \(X = \bigcup_{uv \in E(G)} \{x_{1}^{uv}, \ldots, x_{d−2}^{uv}\} \cup \{x^*\}\) be the set of all new vertices. (Note that \(x^*\) only appears when \(d\) is even.) To show that \(I' \cap X = \emptyset\), we will show that the distance between a vertex in \(X\) and any other
Theorem B.1. \(TJ\) is a yes-instance of ISR if and only if \((G', I, J, TJ)\) is a yes-instance of \(DdISR\). Suppose that \(|I| = |J| = k\). If \(k = 1\), the claim is trivial. Therefore, we consider \(k \geq 2\). In this case, since any independent set of \(G\) is also a distance-\(d\) independent set of \(G'\) and vice versa, it follows that any \(TJ\)-sequence in \(G\) is also a \(TJ\)-sequence in \(G'\) and vice versa. Our proof is complete.

B. Proofs for Section 5 (Extending Some Known Results for \(d = 2\))

B.1. General Graphs

Ito et al. [32] proved that ISR is \(PSPACE\)-complete on general graphs under \(TJ/TAR\). Indeed, their proof uses only maximum independent sets, which implies that any token-jump is also a token-slide [27], and therefore the \(PSPACE\)-completeness also holds under \(TS\). We will show that the reduction of Ito et al. [32] can be extended for showing the \(PSPACE\)-completeness of \(DdISR\) for \(d \geq 3\).

Theorem B.1. \(DdISR\) is \(PSPACE\)-complete under \(R \in \{TS, TJ\}\) for any \(d \geq 3\).

Proof. Recall that Ito et al. [32] proved the \(PSPACE\)-hardness of the problem for \(d = 2\) by reducing from a known \(PSPACE\)-complete problem called 3-Satisfiability Reconfiguration (3SAT-R) [33]. A 3SAT formula \(\varphi\) consists of \(n\) variables \(x_1, \ldots, x_n\) and \(m\) clauses \(c_1, \ldots, c_m\), each clause contains at most three literals (a literal is either \(x_i\) or \(\overline{x_i}\)). A truth assignment \(\phi\) is a way of assigning either 1 (true) or 0 (false) to each \(x_i\). A truth assignment \(\phi\) satisfies a clause if at least one literal is 1 and it satisfies the formula \(\varphi\) if all clauses are satisfied. In a 3SAT-R instance \((\varphi, \phi_s, \phi_t)\), two satisfying assignments \(\phi\) and \(\phi'\) are adjacent if they differ in assigning exactly one variable. The 3SAT-R problem asks whether there is an adjacent satisfying assignments for \(\varphi\) between two given satisfying assignments \(\phi_s\) and \(\phi_t\). We will slightly modify their reduction to prove the \(PSPACE\)-hardness for \(d \geq 3\).

We first describe the reduction by Ito et al. [32]. Let \(\varphi\) be a given 3SAT formula. We construct a graph \(G\) as follows. For each variable \(x_i\) (\(1 \leq i \leq n\)) in \(\varphi\), we add an edge \(e_{x_i}\) to the graph; its two endpoints are labeled \(x_i\) and \(\overline{x_i}\). Then, for each clause \(c_j\) (\(1 \leq j \leq m\)) in \(\varphi\), we add a clique whose nodes correspond to literals in \(c_j\). Finally, we add an edge between two nodes in different components if and only if the nodes correspond to opposite literals. Ito et al. [32] proved that there is a one-to-one correspondence between satisfying assignments for \(\varphi\) and maximum independent sets of size \(n + m\) in \(G\): \(n\) vertices are chosen from the endpoints of edges corresponding to the variables; a literal is 1 (true) if the corresponding endpoint is chosen. They use this observation to show that there is a sequence of adjacent satisfying assignments between \(\phi_s\) and \(\phi_t\) if and only if there is a \(TJ\)-sequence between two corresponding size-\((n + m)\) independent sets of \(G\). (Indeed, they proved under TAR rule. However, since only independent sets of size at least \(n + m - 1\) are used, any TAR-sequence in \(G\) can be converted into a \(TJ\)-sequence [31], the result under \(TJ\) also holds.)

Our modification is simple. Instead of joining two nodes in different components by an edge, we join them by a path of length \(d - 1\). Let \(G'\) be the resulting graph. Figure 8 describes an example of constructing \(G'\) from the same 3SAT formula used by Ito et al. [32]. Indeed, when \(d = 2\), our constructed graph \(G'\) and the graph \(G\) constructed by Ito et al. are identical. One can verify that any independent set of \(G\) of size \(n + m\) is also a \(DdISR\) of \(G'\). Additionally, note that no token can ever leave its corresponding component. To see this, choose a token \(t\) and assume inductively that the statement holds for all other tokens, then \(t\) can never jump/slide to any vertex outside its corresponding component because some other token must be moved first in order to make room for \(t\). Thus, any token-jump is also a token-slide. (Recall that each component is either a single vertex, an edge, or a triangle.) Hence, \((G', I, J, R)\) is a yes-instance of ISR if and only if \((G', I, J, R)\) is a yes-instance of \(DdISR\), where \(I\) and \(J\) are size-\((n + m)\) independent sets of \(G\) (which are also \(DdISs\) of \(G'\)) and \(R \in \{TS, TJ\}\). Our proof is complete.
Figure 8: An example of constructing the graphs $G$ and $G'$ from a 3SAT formula $\varphi$ used in [32] having three variables $x_1$, $x_2$, and $x_3$ and three clauses $c_1 = \{x_1, \overline{x}_2\}$, $c_2 = \{\overline{x}_1, x_2, x_3\}$, and $c_3 = \{x_2, \overline{x}_3\}$. Paths of length $d - 1$ are represented by dotted edges.