Large $N$ limit of Yang–Mills partition function and Wilson loops on compact surfaces

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Abstract

We compute the large $N$ limit of several objects related to the two-dimensional Euclidean Yang–Mills measure on closed, connected, orientable surfaces $\Sigma$ with genus $g \geq 1$, when a structure group is taken among the classical groups of order $N$. We first generalise to all classical groups the convergence of partitions functions obtained by the second author for unitary groups. We then apply this result to prove convergence of Wilson loop observables for loops included within a topological disc of $\Sigma$. This convergence solves a conjecture of B. Hall and shows moreover that the limit is independent of the topology of $\Sigma$ and is equal to an evaluation of the planar master field. Besides, using similar arguments, we show that Wilson loops vanish asymptotically for all non-contractible simple loops.

1 Introduction

The purpose of this article is to study a model of random matrices originating from 2D Euclidean quantum field theory, known as the two-dimensional Yang–Mills measure. This model is defined for a classical compact Lie group and a compact surface endowed with an area measure and can be understood as a gauge theory. Each loop on the surface defines an observable called Wilson loop. We are interested here in Wilson loops when the size $N$ of the matrices goes to infinity, while the loop and the surface are fixed.

Let us attempt to give an account of the origin and state of this problem. The question seems to have first appeared in a mathematical paper in [54]. Therein, the candidate limit of Wilson loops on an arbitrary surface $\Sigma$ is called master field.

This paper was inspired by the large $N$ limits considered in quantum gauge theories [13, 29, 44, 41, 24, 23], which started after the landmark work of t’Hooft [56]. Since then, the Yang–Mills measure in two dimensions has been rigorously defined1 [25, 18, 26, 47, 49, 51, 32], and the latter question has received the attention of several mathematicians2 [58, 52, 2, 36, 20, 19, 12, 28]. It is now known that the

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See [39] for a historical review on the motivations of these approaches.

See [37] for a recent reviews paper on the Yang–Mills measure and the master fields in two dimensions. See also [7] for a study of large $N$ limits of Wilson loops, for discrete Yang–Mills measure in dimension higher than 2.
limit indeed exists when the surface is the plane [2, 35] or the sphere [12], giving rise to two different objects. Simultaneously to and independently from [12], conditional convergence results were obtained in [28] for loops within a topological disc embedded in a compact surface, with different set of assumptions for simple loops. Both works [12, 28] relied on the so-called Makeenko–Migdal equations, which allow to tackle the problem recursively in the complexity of loops. These equations first appeared in physics in [41, 29] and were proved rigorously on the plane in [35, 9, 20] and on surfaces in [20, 19].

Our main result, Theorem 2.16, shows that for all loops included within a topological disc of $\Sigma$, the limit Wilson loop converges and its limit is given by the value of the master field on the plane evaluated at the loop obtained by embedding the same disc in the plane instead of $\Sigma$, in a way that preserves the area measure. In that sense, the behaviour of the Yang–Mills measure within a topological disc is not affected asymptotically by the topological constraint imposed by the surface. The idea of a "trivial" behaviour of the large $N$ limit of the Yang–Mills measure on closed surfaces of genus $g \geq 1$ appeared previously in the physics literature. Nonetheless, we could not find mentioned the specific manifestation of this triviality appearing in our main result.

Our second result Theorem 2.17 shows that for any simple non-contractible loop the Wilson loop of any iteration vanishes asymptotically. This leaves the case of loops which are neither embedded in a topological disc nor an iteration of a simple loop. We investigate this question in the next paper in the series [11].

Our argument does not rely here on the Makeenko–Migdal equations but mainly on the convergence of another quantity: the partition function. We prove the convergence of the partition function using harmonic analysis on the classical compact groups, generalising a result of [27] and of the second author [31].

The rest of the paper is divided in three sections. The first one recalls the setup of the problem, presents the main results and an idea of proof. The second section recalls the necessary notions from representation theory of compact groups and the result of [27, 31] on partition functions which we generalise to all group series of compact matrix Lie groups and all area parameter. The last one gives the proof of our main results on Wilson loops.

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2 Setup and statement of results

2.1 Heat kernel on compact Lie groups

2.1.1 Heat kernel
In this text, $G$ will denote a compact Lie group endowed with a bi-invariant inner
product. The heat kernel on $G$ is the family of smooth functions $(p_t, t \in (0, \infty))$ on
$G$ satisfying
\[ \frac{d}{dt} p_t(g) = \frac{1}{2} \Delta(p_t)(g) \text{ for } t > 0 \text{ and } g \in G \]
and for any continuous function $f$,
\[ \lim_{t \to 0} \int_G f(g)p_t(g)dg = f(1). \]
We denote here by $dg$ and $\Delta$ the Haar probability measure on $G$ and the Laplace–Beltrami operator associated to its inner product, while we write $1$ for the identity element of $G$. The functions $p_t$ are central and invariant by inversion for all $t > 0$, and form a semigroup for the convolution product on $G$: that is, for $t > 0$,
\[ p_t(xgx^{-1}) = p_t(g) \text{ and } p_t(g^{-1}) = p_t(g) \quad \forall x, g \in G, \]
and
\[ p_t * p_s = p_{t+s}, \forall s > 0. \]

2.1.2 Classical compact Lie groups
In the sequel, we shall say that $G_N$ is a classical group of size $N^4$ if it is equal to
one of the following matrix Lie groups:\footnote{Although Sp($N$) is a group of complex matrices of size $2N$, the denomination “size $N$” is not misleading, as it can be also considered as a subgroup of GL$_N$(H). We do not exploit this property, and choose such a terminology only for the sake of simplicity.}
1. The unitary group \( \text{U}(N) = \{ U \in \text{GL}_N(\mathbb{C}) : UU^* = I_N \} \),
2. The special unitary group \( \text{SU}(N) = \{ U \in \text{U}(N) : \det(U) = 1 \} \),
3. The special orthogonal group \( \text{SO}(N) = \{ O \in \text{GL}_N(\mathbb{R}) : OO^t = I_N \} \),
4. The compact symplectic group \( \text{Sp}(N) = \text{U}(2N) \cap \{ S \in \text{GL}_{2N}(\mathbb{C}) : S^t JS = J \} \) where \( J \in \text{GL}_{2N}(\mathbb{C}) \) is defined by
   \[ J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}. \]

We fix the bi-invariant metric as follows. Assume that \( G_N \) is a subgroup of \( \text{GL}_n(\mathbb{C}) \) (so that \( n = N \) for \( \text{U}(N) \), \( \text{SU}(N) \), \( \text{SO}(N) \) and \( n = 2N \) for \( \text{Sp}(N) \)). We set an integer parameter \( \beta \) which is equal to 1 if \( G_N = \text{SO}(N) \), 2 if \( G_N = \text{U}(N) \), \( \text{SU}(N) \), and 4 if \( G_N = \text{Sp}(N) \). We endow the Lie algebra \( g_N \) of \( G_N \) with the scalar product
   \[ \langle X,Y \rangle = \frac{\beta n}{2} \text{Tr}(X^*Y), \forall X,Y \in g_N. \]  \hspace{1cm} (1)

Here \( \text{Tr} \) denotes the trace of the above matrix, that is the non-normalised sum of diagonal coefficients. It is conjugation-invariant in the sense that
   \[ \langle gXg^{-1},gYg^{-1} \rangle = \langle X,Y \rangle, \text{ for all } X,Y \in g_N \text{ and } g \in G_N, \]
and defines therefore a bi-invariant metric on \( G_N \).

Remark. Except in case 1., there is up to constant a unique invariant inner product. The above choice of scaling is standard in random matrices. For instance, the gaussian vector on Hermitian matrices obtained by composition of the above scalar product with the multiplication by \( i \) is the classical Gaussian Unitary Ensemble.

2.2 2D-maps and multiplicative functions

Assume that \( \Sigma \) is a two dimensional compact Riemannian manifold, with Riemannian distance \( d_\Sigma \) and genus \( g \geq 0 \). It is homeomorphic to a \( 4g \)-gon whose sides are, counterclockwise, \( a_1, b_1, a_1^{-1}, b_1^{-1}, \ldots, a_g, b_g, a_g^{-1}, b_g^{-1} \), such that each two edges with same letter are glued together, while respecting the orientation. The polygon is called the fundamental domain of the surface. If \( \Sigma \) has a nonempty boundary \( \partial \Sigma \), the latter can be described as the union of \( n \) connected components \( (C_i)_{1 \leq i \leq n} \), each homeomorphic to the unit circle \( S^1 \). We introduce here some notations used throughout this text.

Denote by \( P(\Sigma) \) the set of continuous maps \([0,1] \rightarrow \Sigma\) with positive and finite length, up to Lipschitz reparametrisation. When \( \gamma \in P(\Sigma) \), \( \mathcal{L}(\gamma) \) denotes its length, \( \gamma^{-1} \in P(\Sigma) \) its reverse, \( \gamma = \tilde{\gamma}(0) \) its starting point and \( \tilde{\gamma} = \tilde{\gamma}(1) \) its endpoint, where \( \tilde{\gamma} \) is some parametrisation. When \( \tilde{\gamma} = \gamma \), we say that \( \gamma \) is a loop of \( \Sigma \) and write
   \[ L(\Sigma) = \{ \gamma \in P(\Sigma) : \tilde{\gamma} = \gamma \}. \]

\( ^5 \)This parameter corresponds to the Dyson index in random matrix theory. For details over its significance, see for instance Section 4.1 of [1]. We introduce this parameter so that standard Brownian motions on these Lie algebras all converge to the same process when \( N \rightarrow \infty \), see for instance [10, Section 2] for an explanation.
When \( \gamma_1, \gamma_2 \in P(\Sigma) \) with \( \gamma_1 = \tau_1, \gamma_1 \gamma_2 \in P(\Sigma) \) stands for their concatenation. A distance on \( P(\Sigma) \) is defined\cite{35} setting for all \( \gamma_1, \gamma_2 \in P(\Sigma) \),

\[
d(\gamma_1, \gamma_2) = |\mathcal{L}(\gamma_1) - \mathcal{L}(\gamma_2)| + \inf_{t \in [0, 1]} \{d_{\Sigma}(\gamma_1(t), \gamma_2(t))\}
\]  

(2)

where the first infimum is taken over all Lipschitz parametrisations \( \tilde{\gamma}_1, \tilde{\gamma}_2 \) of respectively \( \gamma_1 \) and \( \gamma_2 \). In the following paragraphs, we will consider subsets of \( P(\Sigma) \) corresponding to loops traced in embedded graphs.

If \( \Psi \) is a diffeomorphism of \( \Sigma \) and \( \ell \in L(\Sigma) \), \( \psi(\ell) \in L(\Sigma) \) is the loop of \( \Sigma \) obtained by composition of a parametrisation of \( \ell \) with \( \Psi \).

### 2.2.1 Topological maps on compact surfaces

We follow here conventions of \cite[Section 1.3.2]{30}. A graph \( \mathcal{G} \) is a triple \((V, E, I)\) consisting of a set \( V \) of vertices, a set \( E \) of edges and an incidence relation \( I \) such that an edge is incident to either one vertex or two distinct vertices, called endpoints. It can be given an orientation by setting, for any \( e \in E \), a source \( s \in V \) and a target \( t \in V \). An oriented graph can be then represented by a quadruple \((V, E, s, t)\) where \( V \) is the set of vertices, \( E \) the set of oriented edges, and \( s, t : E \to V \) are the functions that map respectively an edge \( e \) to its source and target. An isomorphism between two graphs \( \mathcal{G}_1 = (V_1, E_1, s_1, t_1) \) and \( \mathcal{G}_2 = (V_2, E_2, s_2, t_2) \) is a bijection \( \phi : V_1 \cup E_1 \to V_2 \cup E_2 \) that sends \( V_1 \) (resp. \( E_1 \)) onto \( V_2 \) (resp. \( E_2 \)), and such that

\[
s_2(\phi(e)) = \phi(s_1(e)), \quad t_2(\phi(e)) = \phi(t_1(e)), \quad \forall e \in E.
\]

A topological map \( M \) on \( \Sigma \) is a finite oriented graph \( \mathcal{G} = (V, E, s, t) \) endowed with an embedding \( \theta : \mathcal{G} \to \Sigma \), called drawing of the graph, such that:

- \( \Sigma \) is a compact, connected and orientable surface,
- the vertices are drawn as distinct points of \( \Sigma \),
- oriented edges are drawn as oriented continuous curves that only intersect at their endpoints,
- for any \( e \in E \), there is an edge \( e^{-1} \in E \) such that \( \theta(e^{-1}) = \theta(e)^{-1} \),
- the set \( F = \Sigma \setminus \bigcup_{e \in E} \theta(e) \) is given by a union of open discs called faces of the map.

In order to avoid too heavy notations, we will always identify vertices and edges with their drawing, so that a map on \( \Sigma \) can be represented by \((V, E, F)\) (the applications \( s \) and \( t \) will remain implicit from now on). \( \Sigma \) is called the underlying surface of \( M \). If \( v \in V \) is a vertex in \( M \), we denote by \( P(M) \) (resp. \( L(M), L_1(M) \)) the set of paths (resp. loops, loops with base \( v \)) in \( M \) obtained by concatenation of oriented edges. A loop is simple when each vertex of \( M \) is the source of at most one of its edges.

Two maps \( M_1 = (V_1, E_1, F_1) \) and \( M_2 = (V_2, E_2, F_2) \), respectively on \( \Sigma_1 \) and \( \Sigma_2 \) and with associated graphs \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \), are equivalent if there is an orientation-preserving homeomorphism \( \phi : \Sigma_1 \to \Sigma_2 \) such that the restriction of \( \phi \) to \( M_1 \) induces an isomorphism between \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \). The genus of a topological map in \( \Sigma \) is the genus of \( \Sigma \), and it does not depend on its equivalence class.

Mind that in the above definition, the boundary of \( \Sigma \) might be non-empty. The following result, which is a particular case of Proposition 1.3.10 in \cite{35}, allows to identify boundary components of \( \Sigma \) with loops of a map, up to reorienting.
**Proposition 2.1.** Assume that $M$ is a topological map on a compact, connected, orientable surface $\Sigma$ endowed with a Riemannian metric, such that $\partial \Sigma$ has positive and finite length. Then, any connected component $C$ of $\partial \Sigma$ is the drawing of an element of $L(M)$.

When there is a metric on $\Sigma$ such that each edge of a topological map $M$ is mapped to a curve of finite length, we call $M$ rectifiable. When the boundary of $\Sigma$ is empty, we shall say that the map $M$ is closed. This latter property does not depend on the equivalence class of $M$.

### 2.2.2 Topological maps on $\mathbb{R}^2$ or an open disc

A topological map $M$ on the plane $\mathbb{R}^2$ or an open disc is a topological map on the sphere $S^2$ together with a marked face $f_{\infty}$. The latter is called the infinite face of $M$, and it contains the point at infinity when one makes the identification $S^2 \cong \mathbb{R}^2 \cup \{\infty\}$. Unless mentioned otherwise, when $M$ is a topological map on the plane or an open disc, $F$ will denote the set of bounded faces, and $\hat{F}$ will denote $F \cup \{f_{\infty}\}$ whenever it makes sense.

### 2.2.3 Multiplicative functions

For any $P \subset P(\Sigma)$ and any compact group $G$, a multiplicative function $h : P \to G$ is a function that satisfies $h_{\gamma^{-1}} = h_{\gamma}^{-1}$ for any $\gamma \in P$ such that $\gamma^{-1} \in P$, and $h_{\gamma_1 \gamma_2} = h_{\gamma_2} h_{\gamma_1}$ for any $\gamma_1, \gamma_2 \in P$ such that $\gamma_1 \gamma_2 \in P$. We denote by $\mathcal{M}(P,G)$ the space of multiplicative functions from $P$ to $G$. The gauge group $\Gamma = \mathcal{C}^\infty(\Sigma,G)$ acts on it by gauge transformations: for any $j \in \Gamma$ and $h \in \mathcal{M}(P,G)$,

$$(j \cdot h)_\gamma = j(\overline{\gamma})^{-1} h_{\gamma} j(\gamma), \forall \gamma \in P.$$  

Considering topological maps, gauge-invariance of multiplicative functions is equivalent to a $G$-invariance, as stated by the following lemma which is a particular instance of [35, Lemma 2.1.5].

**Lemma 2.2.** Let $M$ be a topological map in $\Sigma$, $v \in \mathbb{V}$ and $\gamma_1, \ldots, \gamma_n \in P(M)$. There exist $\ell_1, \ldots, \ell_m \in \mathcal{L}_v(M)$ such that for any $f : G^n \to \mathbb{C}$ with $h \mapsto f(h_{\gamma_1}, \ldots, h_{\gamma_n})$ gauge-invariant on $\mathcal{M}(P(M),G)$,

$$f(h_{\gamma_1}, \ldots, h_{\gamma_n}) = \tilde{f}(h_{\ell_1}, \ldots, h_{\ell_m}), \forall h \in \mathcal{M}(P(M),G),$$
for some function \( \tilde{f} : G^m \to \mathbb{C} \) which is invariant by the diagonal action of \( G \).

Two sigma-fields may be put on \( \mathcal{M}(P(M), G) \):
- The smallest sigma-field \( \mathcal{C} \) such that for any \( \gamma \in P(M) \) the evaluation function
  \[
  H_\gamma : \begin{cases} \mathcal{M}(P(M), G) & \to & G \\ h & \mapsto & h_\gamma \end{cases}
  \]
  is measurable: it is the \textit{cylindrical sigma-field}.
- The smallest sigma-field \( \mathcal{J} \) that makes
  \[
  h \mapsto f(h_{\ell_1}, \ldots, h_{\ell_n})
  \]
  measurable, for all \( v \in V, n \in \mathbb{N}, \ell_1, \ldots, \ell_n \in L_v(M) \) and \( f : G^n \to \mathbb{C} \) \( G \)-invariant: it is the \textit{invariant sigma-field}.

We will mainly work with \( \mathcal{C} \) but some results will only hold on \( \mathcal{J} \), therefore we will specify which sigma-field we consider.

### 2.2.4 Wilson loops

If \( \chi : G \to \mathbb{C} \) is a central function, for any \( \ell \in L(M) \), \( \chi(h_\ell) \) does not depend on the choice of based point. The function

\[
\mathcal{M}(L(M), G) \longrightarrow \mathbb{C} \\
\begin{array}{ccc}
h & \mapsto & \chi(h_\ell) 
\end{array}
\]

is then called a Wilson loop. When \( G \) belongs to one of the four series given above, we will be interested in the Wilson loops

\[
W_\ell : \begin{cases} \mathcal{M}(L(M), G) & \to & \mathbb{C} \\ h & \mapsto & \text{tr}(h_\ell), \end{cases}
\]

where \( \ell \) is a loop of \( M \), while for any \( M \in M_d(\mathbb{C}) \), \( \text{tr}(M) = \frac{1}{d} \sum_{i=1}^{d} M_{i,i} \).

Remark (Gauge equivalence). For most compact Lie groups \( G \), it can be shown that the family of Wilson loops separate points of \( \mathcal{M}(P(M), G)/\Gamma \) endowed with the quotient topology. When \( G \) belongs to one of the four series of the previous section, it can further be shown \([48, 33]\) that it is enough to consider the family \( \{W_\ell, \ell \in L(M)\} \).

### 2.2.5 Area-weighted maps

An \textit{area-weighted map} is a topological map \( M \) together with\(^7\) a function \( a : F \to \mathbb{R}_+^* \).

Two maps \((M, a)\) and \((M', a')\) are equivalent if \( M \) and \( M' \) are equivalent as maps and the associated homomorphism \( \phi \) of \( \Sigma \) defines a bijection \( \tilde{\mathcal{F}} : \mathbb{F} \to \mathbb{F}' \) with

\[
a' \circ \tilde{\mathcal{F}} = a.
\]

\(^6\)It is also called holonomy in the literature.

\(^7\)In the case of \( \mathbb{R}^2 \), the marked faces are excluded because their area is considered to be infinite.
For any $T > 0$ we set

$$\Delta_M(T) = \{a : F \to \mathbb{R}_+^* : \sum_{f \in F} a_f = T\}$$

when $M$ is a closed topological map, and when $(M, a)$ is an area-weighted map we write $|a| = \sum_{f \in F} a_f$. If the surface is endowed with a Riemannian volume $\text{vol}$, then it induces in particular an area function $\text{vol} : F \to \mathbb{R}_+^*$.

### 2.3 Two dimensional Yang–Mills measure

In this section we recall a definition of the Yang–Mills measure in three steps:

1. Given a topological map $M$, we define a uniform measure $U_{M,C,G}$ on $\mathcal{M}(\mathcal{P}(M), G)$ with and without constraints.

2. We define the discrete Yang–Mills measure $\text{YM}_{M,C,a,G}$ on an area-weighted topological map $(M, a)$ as an absolutely continuous measure with respect to $U_{M,C,G}$.

3. We define the Yang–Mills holonomy field $(H_\ell)_\ell \in \mathcal{P}(\Sigma)$ on any compact, connected, orientable Riemann surface $\Sigma$ with volume form $\text{vol}$, whose distribution $\text{YM}_{\Sigma,C,G}$ is the continuous version of $\text{YM}_{M,C,a,G}$.

#### 2.3.1 Uniform measure on $\mathcal{M}(\mathcal{P}(M), G)$

For any topological map $M$, the space $\mathcal{M}(\mathcal{P}(M), G)$ is a compact Lie group when endowed with the pointwise multiplication and has a unique Haar measure $\mu_M$ on both sigma-fields $C$ and $J$. Fixing a set $E^+$ of positively oriented edges (so that for any $e \in E$, only $e$ or $e^{-1}$ belongs to $E^+$), $\mathcal{M}(\mathcal{P}(M), G)$ can be identified with $G^{E^+}$ and $\mu_M$ is the push-forward of the direct product of Haar measures on $G$. Let us call it the unconstrained uniform measure and denote it by $U_{M,G}$. In contrast, we need to define a constrained version to put boundary conditions when necessary. If $\Sigma$ is a compact surface with boundary, denote by $B(\Sigma)$ the set of connected components of $\partial \Sigma$, each taken twice, one for each orientation. There is a natural action of $\mathbb{Z}/2\mathbb{Z}$ on $B(\Sigma)$ by orientation-reversal.

A set of boundary conditions on a compact surface $\Sigma$ is a $\mathbb{Z}/2\mathbb{Z}$-equivariant map $C : B(\Sigma) \to G/\text{Ad}$, where $G/\text{Ad}$ is the set of conjugation classes of $G$ where $\mathbb{Z}/2\mathbb{Z}$ acts by inversion. The fact that boundary conditions take values in $G/\text{Ad}$ rather than simply $G$ will be explained later. If $(M, a)$ is an area-weighted map on $\Sigma$, then the boundary conditions also apply to $M$, as according to Proposition 2.1, elements of $B(\Sigma)$ can be identified with loops $L(M)$ up to re-rooting. Denoting by $L_1, \ldots, L_p$ the elements of $B(\Sigma)$ oriented positively, there is a bijection between $(G/\text{Ad})^{B(\Sigma)}/(\mathbb{Z}/2\mathbb{Z})$ and $(G/\text{Ad})^p$, so that any set $C$ of boundary conditions can be identified with a tuple $C = (t_1, \ldots, t_p)$ with $t_1, \ldots, t_p \in G/\text{Ad}$.

For any $t \in G/\text{Ad}$ and $n \geq 1$, the set $t(n) = \{(x_1, \ldots, x_n) \in G^n : x_1 \cdots x_n \in t\}$ is a homogeneous space for the $G^n$-action

$$(g_1, \ldots, g_n) \cdot (x_1, \ldots, x_n) = (g_1 x_1 g_2^{-1}, \ldots, g_n x_n g_1^{-1}).$$
We denote by $\delta_{\ell(n)}$ the extension to $G^n$ of the unique $G^n$-invariant probability measure on $t(n) \subset G^n$. It can be thought of as the conditional Haar measure on $G^n$, under the condition $x_1 \cdots x_n \in t$.

Let us now define a measure on $(M(P(M), G), C)$. Therefore, let us choose a specific labelling of $E$. If we denote again by $L_1, \ldots, L_p$ the connected components of $\partial \Sigma$, we choose the labels of the oriented edges $e \in E^+$ such that $L_i = e_{i,1} \cdots e_{i,n_i}$ for every $i$. Denote finally by $e_1, \ldots, e_m$ the remaining edges of $E^+$. Then $M(P(M), G)$ is isomorphic to $G^n \times G^{n_1} \times \cdots \times G^{n_p}$ where $h \in M(P(M), G)$ is mapped to the tuple $g_i = h_{e_i,1} 1 \leq l \leq m, g_{l,n_k} = h_{e_i,n_k}, 1 \leq i \leq n_i, 1 \leq k \leq p$. The uniform measure on $M(P(M), G)$ with boundary conditions $C = (t_1, \ldots, t_p)$ is the measure $U_{M,C,G}$ on $(M(P(M), G), C)$ defined by

$$U_{M,(t_1, \ldots, t_p),G}(dh) = dg_1 \otimes \cdots \otimes dg_m \otimes \prod_{i=1}^p \delta_{t_i(n)}(dg_{i,1} \cdots dg_{i,n_i}).$$

By convention, when $C = \emptyset$, we write $U_{M,C,G} = \mu_M$. The translation invariance of the Haar measure and the choice $G/Ad$-valued boundary conditions yield the following invariance proved given in [35, Prop. 2.3.6.].

**Proposition 2.3.** The measure $U_{M,(t_1, \ldots, t_p),G}$ is left invariant by the action of $\Gamma$ on $M(P(M), G)$.

We shall also consider constrained measures. Assume that $M$ is closed, $\ell \in L(M)$ is a simple loop and $t \in G/Ad$. Denoting by $\ell = e_{0,1} \cdots e_{0,n}$ the edge decomposition of $\ell$ in $M$, and labeling $e_1, \ldots, e_m$ the other edges of $M$, the uniform measure on $M(P(M), G)$ with constraint $t$ on $\ell$ is the measure $U_{M,C,\ell=t,G}$ on $(M(P(M), G), C)$ defined by

$$U_{M,C,\ell=t,G}(dh) = dg_1 \otimes \cdots \otimes dg_m \otimes \delta_{t(n)}(dg_{0,1} \cdots dg_{0,n}).$$

### 2.3.2 Discrete Yang–Mills measure

We follow here §1.5.2 of [32]. Consider a compact, connected, orientable surface $\Sigma$ with genus $g \geq 0$, $p$ boundary components, and an area-weighted map $(M, a)$ with graph $G$ embedded in $\Sigma$. For each face $f \in F$, its boundary $\partial f$ is a loop of $G$ given by the concatenation of edges bordering $f$ considered up to the choice of base point and of orientation. Set $C = (t_1, \ldots, t_p) \in (G/Ad)^p$, when $p \geq 1$, and $C = \emptyset$ when $p = 0$. The Yang–Mills measure on $(M, a)$ with structure group $G$ and boundary conditions $C$ is the probability measure $YM_{M,C,a,G}$ on $(M(P(M), G), C)$ with density

$$\frac{1}{Z_{M,a,G}(C)} \prod_{f \in F} p_a(f \partial f)$$

with respect to $U_{M,C,G}$, where $Z_{M,a,G}(C) > 0$ is a constant, equal to $Z_{M,a,G}$ if $p = 0$, and defining a function of $t_1, \ldots, t_p$ when $p \geq 1$. Since the heat kernel defines

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This measure seems to have been introduced first in the physics literature in [44], see also [43] and [57] where it was used to compute symplectic volumes of flat connexions. Statistical physics models with heat kernel weight in lattice gauge theories, or continuous spin systems, also bear the name of Villain model, see for instance [45]. The term "discrete Yang-Mills measure" might have been introduced and formally defined in [50, Sect. 2].
a functions on $G$ which are central and invariant by inversion, each term of the above product depends neither on the base point, nor on the orientation for the boundary of the faces, and defines therefore a well defined Wilson loop. Using the semigroup property leads to the following elementary but remarkable lemma.

**Lemma 2.4** ([49, 35]). The constant

$$Z_{M,a,G}(t_1, \ldots, t_p) = \int_{\mathcal{M}(\mathcal{P}(M), G)} \prod_{f \in F} p_a(f h_{af}) U_{M,(t_1, \ldots, t_p), G}(dh)$$

and

$$Z_{M,a,G} = \int_{\mathcal{M}(\mathcal{P}(M), G)} \prod_{f \in F} p_a(f h_{af}) U_{M,G}(dh)$$

depend only on the genus $g$ of $\Sigma$, the total area $T = \sum_{a \in F} a_f$ and in the first case on the boundary conditions $C = (t_1, \ldots, t_p)$ up to reordering. When $M$ is not closed, $Z_{M,a,G} = 1$.

This constant is called the *partition function*. We denote it by $Z_{(g,p),T,G}(t_1, \ldots, t_p)$, or $Z_{g,T,G}$ when $p = 0$, and we drop the subscript $G$ if there is no ambiguity on the group. When $h \in G$, we also write $Z_{(g,p),T}(h_1, \ldots, h_p)$ and $Z_{(g,p),T}(h_1, \ldots, h_p)$ where $[g] \in G/Ad$ denotes the conjugacy class of an element $g \in G$. We give explicit expressions using representation theory in section 3.1, and an example of discrete Yang–Mills measure in section 2.5.1.

When $p \geq 1$, the Yang–Mills measure with or without boundary conditions are related by the disintegration formula

$$Z_{M,a,G} \mathbb{YM}_{M,a,G} = \int_{(G/Ad)^p} Z_{M,a,G}(t_1, \ldots, t_p) \mathbb{YM}_{M,(t_1, \ldots, t_p),G} dt_1 \ldots dt_p. \quad (5)$$

When $(M', a')$ is another map with genus $g$ and volume $T$, we say that $(M', a')$ is finer than $(M, a)$ if any face $f$ of $M$ is a disjoint union of faces $f_1, \ldots, f_k$ of $M'$ with $a_f = a'_{f_1} + \cdots + a'_{f_k}$. In this case, the restriction of $h \in \mathcal{M}(\mathcal{P}(M'), G)$ to the edges of $M$ defines an element $\mathcal{R}_M^{M'}(h)$ of $\mathcal{M}(\mathcal{P}(M), G)$. We have the following lemma.

**Lemma 2.5** ([49, 35]). When $C$ is a set of boundary conditions on $\Sigma$ and $(M', a')$ is finer than $(M, a)$, then

$$(\mathcal{R}_M^{M'})_* (\mathbb{YM}_{M', C,a',G}) = \mathbb{YM}_{M,C,a,G}.$$ 

In particular if $\ell \in L(M)$, the random variable $W_\ell$ has same law under $\mathbb{YM}_{M', C,a',G}$ and $\mathbb{YM}_{M,C,a,G}$.

We further need to define the Yang–Mills measures with a constraint along a single loop, together with a disintegration formula. Assume that $M$ is closed, $\ell \in L(M)$ is a simple loop and $t \in G/Ad$. Denoting by $\ell = \ell_{0,1} \ldots \ell_{0,n}$ the edge decomposition of $\ell$ in $M$, and labeling $e_1, \ldots, e_n$ the other edges of $M$, the Yang–Mills on $M(\mathcal{P}(M), G)$ with structure group $G$ and constraint $C_{\ell \sim t}$ is the probability measure $\mathbb{YM}_{M,C_{\ell \sim t},a,G}$ on $(\mathcal{M}(\mathcal{P}(M), G), C)$ with density

$$\frac{1}{Z_{M,a,G}(\ell; t)} \prod_{f \in F} p_a(f h_{af})$$

with respect to $U_{M,C_{\ell \sim t},G}$, where $Z_{M,a,G}(\ell; t) > 0$ is a positive constant.
Proposition 2.6 ([35, 32]). Let $(M, a)$ be a closed, area-weighted map, and let $\ell \in L(M)$ be a simple loop. Then for any $t \in G/\text{Ad}, a \in \Delta(T)$,

$$Z_{M,a,G}Y_{M,a,G} = \int_{G/\text{Ad}} Z_{M,a,G}(\ell; t) Y_{M,C,t^{-1},a,G} dt.$$

(7)

Remark. Remarkably, the Yang–Mills measure with constraints is related to the Yang–Mills measure with boundary conditions, see for instance Theorem 4.2 below. This relation plays the role of a Markov property. The holonomy fields under the Yang–Mills measure can be furthermore understood as an example of a two-dimensional Markovian holonomy fields as defined in [35].

2.4 Yang–Mills holonomy field

The compatibility relation of Lemma 2.5 suggests that the measures considered can be obtained as the image of a single measure on a larger probability space. This has indeed been achieved in [32] and was then generalised in [35], leading to the notion of (continuous) Markovian holonomy fields, allowing to consider a very large family of loops at once. Though it is not crucial to our argument, we recall here their definition, as it allows to reformulate our main results in a unified continuous model.

A different rigorous continuous approach has been given earlier by [18, 47, 49]. It relies on stochastic analysis of the white noise on the plane and the formula (6) is obtained as a consequence of the construction and it is called the Driver–Sengupta formula. The random holonomy field can be understood as the parallel transport of a random connection with curvature given by a white noise. Recently, yet another continuous construction, defining a random connection one form has been given in [8] recovering the latter formula.

Let $\Sigma$ be a closed, connected, orientable surface, an open disc of $\mathbb{R}^2$ or $\mathbb{R}^2$ itself, endowed with an area measure $\text{vol}$. For any $\ell \in L(\Sigma)$, we use the same notation as in (3), for the random variable

$$W_\ell = \text{tr}(H_\ell).$$

Furthermore, whenever $h \in \mathcal{M}(\mathcal{P}(\Sigma), G)$, the restriction of $h$ to a topological map $M$ on $\Sigma$ defines an element $\mathcal{R}_M(h)$ of $\mathcal{M}(\mathcal{P}(M), G)$. Let us finally mention that the space $\mathcal{M}(\mathcal{P}(\Sigma), G)$ can be endowed with two sigma-fields $\mathcal{C}$ and $\mathcal{J}$ defined as in the discrete case. The continuous extension to the discrete Yang–Mills measure is provided by the following result.

Theorem 2.7 ([35]). Assume that $\Sigma$ is a closed, connected, orientable two dimensional Riemannian manifold, an open disc of $\mathbb{R}^2$ or $\mathbb{R}^2$ itself, with area measure $\text{vol}$. For any set $C = (t_1, \ldots, t_p)$ of boundary conditions, there exist probability measures $Y_{M,\Sigma,G}$ and $Y_{M,\Sigma,C,G}$ on $(\mathcal{M}(\mathcal{P}(\Sigma), G), \mathcal{C})$, such that

1. for any rectifiable map $M$ on $\Sigma$,

$$\mathcal{R}_M)_*(Y_{M,\Sigma,G}) = Y_{M,\text{vol},G} \text{ and } (\mathcal{R}_M)_*(Y_{M,\Sigma,C,G}) = Y_{M,\text{vol},G},$$

2. $Z_g,T Y_{M,\Sigma,G} = \int_{(G/\text{Ad})^p} Z_{(g,p),T}(t_1, \ldots, t_p) Y_{M,(t_1, \ldots, t_p),\text{vol},G} dt_1 \ldots dt_p,$
3. If \((\gamma_n)_{n \geq 0}\) is a sequence of \(P(\Sigma)\) and \(\gamma \in P(\Sigma)\) with \(\gamma_n = \gamma, \forall n \geq 0\) and \(d(\gamma_n, \gamma) \to 0\) as \(n \to \infty\), then, under \(YM_{\Sigma,G,C}\), \((H_{\gamma})_{n \geq 0}\) converges in distribution towards \(H_{\gamma}\).

Moreover, if \(\Psi\) is a diffeomorphism of \(\Sigma\) preserving \(\text{vol}\), then under \(YM_{\Sigma,G}\) and \(YM_{G}\), \(W_t\) has same law as \(W_{\Psi(t)}\).

**Proposition 2.8** ([35]). When \(\ell \in L(\Sigma)\) is a simple loop of a closed, connected, orientable two dimensional Riemannian manifold \(\Sigma\) with area measure \(\text{vol}\), and \(t \in G/\text{Ad}\), there is a measure \(YM_{\Sigma,C,G}\) on \((\mathcal{M}(P(\Sigma), G), C)\) such that

1. For any rectifiable map \(M\) on \(\Sigma\) such that \(\ell\) is a drawing of a loop of \(M\),

\[
(\mathcal{F}_M)_*(YM_{\Sigma,C,G}) = YM_{M,C,G}.
\]

Moreover the constant \(Z_{M,a,G}(\ell; t)\) only depends on \(\ell \in L^2(\Sigma)\) and \(t\). We denote it by \(Z_{\Sigma,G}(\ell; t)\).

2. If \(\Sigma\) has genus \(g\) and total volume \(T\),

\[
Z_{g,T}YM_{\Sigma} = \int_{G/\text{Ad}} Z_{\Sigma,G}(\ell; t)YM_{\Sigma,C,G} d\ell.
\]  

(8)

The random process \((H_{\gamma}, \gamma \in P(\Sigma))\) with distribution given by \(YM_{\Sigma,G}\) is called the **Yang–Mills holonomy field** on \(\Sigma\). We are primarily interested in the random variables \(W_\ell\) for \(\ell \in L(\Sigma)\), with structure group a classical group \(G_N\) of size \(N\).

### 2.5 Master fields, conjectures and main results

Following the physics literature, I. M. Singer raised in [54] the following question, that we will reformulate here in a slightly modified form as a conjecture.

**Conjecture 2.9.** Let \(\Sigma\) be a closed, connected, orientable, two dimensional, Riemannian manifold \(\Sigma\), or the Euclidean plane \(\mathbb{R}^2\), or a disc of \(\mathbb{R}^2\). Assume that \(G_N\) is a classical group of size \(N\) with metric given by (1). Then for any loop \(\ell \in L(\Sigma)\), there is a constant \(\Phi_{\Sigma}(\ell)\) such that, under \(YM_{\Sigma,G}\),

\[
W_\ell \to \Phi_{\Sigma}(\ell) \text{ in probability as } N \to \infty.
\]  

(9)

If \(\Psi\) is a diffeomorphism of \(\Sigma\) preserving its volume form,

\[
\Phi_{\Sigma}(\Psi(\ell)) = \Phi_{\Sigma}(\ell).
\]

The limit function

\[
\Phi_{\Sigma} : L(\Sigma) \to \mathbb{C}
\]

is called a **master field**. This conjecture has been partly proved for the plane, working with a smaller class of loops in [58, 2], for unitary groups. In [36] , it was simultaneously proved for the plane and for all above group series. Recently another argument using Makeenko–Migdal equations was also given in [28].

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*the specific family of loops was not specified in [54].
Remark. For any loop, $\ell \in L(\Sigma)$, since $H_\ell$ is a unitary matrix, $|\text{tr}(H_\ell)| \leq 1$ and the convergence in probability (9) is equivalent to

$$E[|\text{tr}(H_\ell) - \Phi_\Sigma(\ell)|] \to 0$$

as $N \to \infty$. When we say that (9) holds uniformly in the set of area vectors $\alpha$, it is equivalent to the uniformity of the convergence (*) in $\alpha$. To show (*), it is sufficient to show

$$E[|\text{tr}(H_\ell)\text{tr}(H_\ell^\dagger)|^2] \to 0$$

Remark. The linear extension of a master field $\Phi_\Sigma$ to $\mathbb{C}[L(\Sigma)]$ comes automatically with a structure of non-commutative probability space [36, 12]. In the case of the plane, this non-commutative distribution can be characterised using free probability ([36] and [6]). In that case, one can recover the distribution of a free Brownian motion from the master field ([3, 36, 6]).

**Theorem 2.10** ([58, 2, 36] and [12]). The conjecture 2.9 holds true when $\Sigma$ is an open disc of the plane or a sphere of total area $T > 0$. Consider $\ell \in L(M)$ where $(M, a)$ is a closed, area-weighted map with fixed total area $T$. Then under $Y_{M,a}$,

$$W_\ell \to \Phi_{M,a}(\ell)$$

in probability, where the right-hand side is deterministic and depends continuously on $a$, over $a \in \mathbb{R}^*_F$ in the case of the plane and $a \in \Delta_M(T)$ in the case of the sphere.

The work [36] was the first to show rigorously that the master field $\Phi_{\mathbb{R}^2}$ satisfies a set of differential equations named after Makeenko–Migdal equations, that appeared earlier in the physics papers [41, 29]. In [12], the conjecture was proved for unitary groups, in the case of the sphere.

**Example 2.11** ([58, 2, 36, 12]). Assume that $\ell \in L(\mathbb{R}^2)$ is simple and encloses an area $t$. For $n \geq 1$,

$$\mu_t(n) := \Phi_{\mathbb{R}^2}(\ell^n) = \frac{e^{-\frac{n^2}{2}}}{n} \sum_{m=0}^{n-1} \frac{(-1)^m}{m!} \left( \frac{n}{m+1} \right).$$

Denote by $S^2_T$ the two-dimensional Euclidean sphere with total volume $T$. When $T \leq \pi^2$ and $\ell \in L(S^2_T)$ is simple, cutting the sphere into two domains of area $t$ and $T - t$,

$$\mu_{t,T}(n) := \Phi_{S^2_T}(\ell^n) = J_1(2n\sigma) = \int_{-2}^{2} \exp(in\sigma x) \frac{\sqrt{4-x^2}}{\pi} dx,$$

where $\sigma = \sqrt{\frac{T-t}{T}}$ and

$$J_1(x) = \sum_{m \geq 0} \frac{(-1)^m}{m!(m+1)!} \left( \frac{x}{2} \right)^{2m}$$

is a Bessel function of the first kind.

\footnote{An the expression for any $T > 0$ was proved in [12, Thm 2.4].}
Remark. 1. Note that when \( n \geq 1 \) and \( T \leq \pi^2 \) are fixed, \( \mu_t(n) \) and \( \mu_{t,T}(n) \) are different functions of \( t \). This is also true \([12]\) when \( T > \pi^2 \), though the expression of \( \mu_{t,T} \) is different. Therefore, when \( \ell \) is a simple loop enclosing a disc of area \( t \), the expression of the master field is not the same when the surface in which \( \ell \) is drawn is the plane or the sphere.

2. Let us highlight nonetheless two relations between the master field on the sphere and on the plane. On the one hand, it can be shown \([12]\) that
\[
\lim_{T \to \infty} \mu_{t,T}(n) = \mu_t(n), \forall n \geq 1. \tag{10}
\]
On the other hand, it follows from dominated convergence that for all \( t \geq 0 \)
\[
\lim_{k \to \infty} \mu_{t,k^2}(nk) = J_1(2n\sqrt{t}).
\]
Therefore, for all \( 0 \leq t \leq T \leq \pi^2 \),
\[
\lim_{k \to \infty} \mu_{k^2}(nk) = \mu_{t,T}(n), \forall n \geq 1. \tag{11}
\]
3. The sequences \( \mu_t(n) \) and \( \mu_{t,T}(n), n \geq 1 \) are moment sequences of measures \( \mu_t \) and \( \mu_{t,T} \) on the unit circle, associated to a time marginal of the free Brownian motion \([3]\) and of the free brownian bridge \([12]\). Since both \( \mu_t \) and \( \mu_{t,T} \) are invariant by complex conjugation, \((10)\) and \((11)\) imply the weak convergences
\[
\mu_{t,T} \to \mu_t \text{ as } T \to \infty
\]
and for any \( t \leq T \leq \pi^2 \),
\[
\mu_{k^2} \to \mu_{t,T} \text{ as } k \to \infty,
\]
where for any measure on \( \nu \) on the unit circle, \( \nu^k \) denotes the push forward of \( \nu \) by \( z \mapsto z^k \).

The conjecture 2.9 remains open for general surfaces. Though, using the Makeenko–Migdal equations on surfaces proved in \([19]\), it was realised in \([12, 28]\) that it is sometimes enough to show the convergence for a restricted family of loops. This idea was exploited for general surfaces in \([28]\) yielding the following theorem.\(^{12}\) Let us say that a loop \( \ell \) of a map \( M \) is included in a disc if its drawing is included in an open, contractible set \( U \) of \( \Sigma \).

**Theorem 2.12** \([28]\). Consider \( G_N = U(N) \). Assume that whenever \( \ell = s^n \) with \( n \geq 0 \) and \( s \) is a simple loop included in a disc, of a closed, area-weighted topological map\(^{13}\) \((M,a)\), under \( \text{YM}_{M,a,G_N} \), the random variable \( W_\ell \) converges in probability towards a constant as \( N \to \infty \). Then this also holds true for \( W_\ell \) for any combinatorial loop \( \ell \) included in a disc.

\(^{11}\)In the physics literature, the regimes \( T \leq \pi^2 \) and \( T > \pi^2 \) are respectively called the weak and the strong regimes \([16]\).

\(^{12}\)Another similar result is obtained in \([28]\), where only an assumption on simple loops is made. For \( g \geq 1 \), the conclusion is then weaker and holds for loops with constrained area vector.

\(^{13}\)Recall that we assumed here that, by convention, a closed topological map can be embedded in a closed, connected, orientable surface.
B. Hall conjectured further that the above assumption can be removed.

Conjecture 2.13 ([28]). Consider $G_N = U(N)$. Whenever $\ell = s^n$ with $n \geq 0$ and $s$ is a simple loop included in a disc of a closed, area-weighted topological map $(M, a)$, $W_\ell$ converges in probability towards a constant as $N \to \infty$, under $\text{YM}_{M, a, G_N}$.

Our main result implies that this conjecture holds true for any closed surface $\Sigma$ and any group series of classical groups. We simultaneously prove Theorem 2.12 without using the Makeenko–Migdal equations. Instead we shall use the convergence of the partition of function; a result proved by the first author in [12] for unitary groups that we recall and generalise to other group series in section 3. Remarkably the limit in the above conjecture is given in terms of the master field on the plane as follows.

Assume that $\ell$ is a loop of an area weighted map $(M, a)$ included in a closed disc $U$ of $\Sigma$. Considering only vertices, edges and area-weighted faces of $M$ which are mapped by the embedding of $M$ in $U$, and replacing all faces intersecting $\Sigma \setminus U$ by a single marked face yields an area-weighted map $(\tilde{M}_U, a_U)$ of $\mathbb{R}^2$. We then denote by the same symbol the loop of $\tilde{M}_U$ obtained by concatenating the edges of $\ell$.

Theorem 2.14. Consider an area-weighted topological map $(M, a)$ on a closed surface $\Sigma$ of genus $g \geq 1$ and total volume $T$. Assume that $\ell$ is a loop of $(M, a)$ included in a disc $U$ of $M$. Then for any classical group $G_N$ of size $N$, under $\text{YM}_{M, a, G_N}$,

$$W_\ell \to \Phi_{\tilde{M}_U, a_U}(\ell)$$

in probability as $N \to \infty$, uniformly in $a \in \Delta_M(T)$.

Example 2.15. In particular, if $\ell = s^n$ where $s$ is a simple contractible loop enclosing an area $t \in (0, T]$, then

$$W_\ell \to \frac{e^{-nt}}{n} \sum_{m=0}^{n-1} \frac{(-nt)^m}{m!} \left(\frac{n}{m+1}\right)$$

in probability as $N \to \infty$.

Using a continuous construction and uniformity estimates obtained in [35], this result can be generalised as follows to a continuous setting allowing to consider a much wider family of loops.

Theorem 2.16. Assume that $\Sigma$ is a closed, connected, orientable Riemann surface with Riemannian volume $\text{vol}$, $\Psi : U \to D_R = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 < R\}$ is a diffeomorphism, where $\pi R^2 < T$, $U$ is an open set of $\Sigma$ and $\Psi_*(\text{vol}_U)$ is the Lebesgue measure. Then, for any $\ell \in L(U)$, and any classical group $G_N$ of size $N$, under $\text{YM}_{\Sigma, G_N}$,

$$W_\ell \to \Phi_{\mathbb{R}^2}(\Psi(\ell))$$

in probability as $N \to \infty$.

Assuming furthermore that $\Psi^{-1} : D_R \to U$ can be extended continuously to $\overline{D_R}$, with piecewise continuous derivatives, for any loop $\ell \in L(\overline{D_R})$,

$$W_{\Psi^{-1}(\ell)} \to \Phi_{\mathbb{R}^2}(\ell)$$

in probability as $N \to \infty$.

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14For instance, in Fig. 2, we consider a map $M$ with 3 faces embedded in the torus, and in Fig. 3, an additional embedded disc $U$. The resulting map $\tilde{M}_U$ with 2 vertices, two edges and 2 bounded faces $F_1, F_2$, is drawn in Fig. 4.
Note that a simple loop is included in a disc if and only if it is contractible. Our second result below is concerned with simple loops \( \ell \) which are not contractible.

**Theorem 2.17.** Assume that \( \Sigma \) is a closed, connected Riemann surface of genus \( g \geq 1 \), with Riemannian volume \( \text{vol} \). Then, for all simple, non-contractible loop \( \ell \) on \( \Sigma \), for any \( k \in \mathbb{Z}^* \) and any classical group \( G_N \) of size \( N \), under \( \text{YM}_{\Sigma,G_N} \),

\[
W_{\ell,k} \to 0 \text{ in probability as } N \to \infty.
\]

Our result covers all simple loops on closed, connected, orientable surfaces. We further believe it holds true for non-orientable surfaces. In a sequel to the current work [11], we investigate the master field question for all loops with self-intersections.

2.5.1 Example and idea of proof for Theorem 2.14

In this section we illustrate the key objects, namely the discrete Yang–Mills measure, the Wilson loops and the master field, with an explicit example on a torus of area \( T \). We then give an idea of the proof of Theorem 2.14 for this example.

![Figure 2: An example of area-weighted map on a torus.](image)

Consider the area-weighted map \((M, a)\) in Fig. 2. It has 3 vertices, 6 edges labelled \( a, b, c, d, e, f \), and 3 faces whose boundaries are described clockwise by \( \partial F_1 = b^{-1}a^{-1}bcdef^{-1} \), \( \partial F_2 = f^{-1}ed^{-1} \) and \( \partial F_3 = e^{-1} \). Denote by \( a_i \) the weight of the face \( F_i \), \( i \in \{1, 2, 3\} \). Denote respectively by \( x_1, \ldots, x_6 \) the values of the edge variables.

\(^{15}\)Let us recall an argument for the converse statement. Consider the surface \( \Sigma' \) obtained by cutting \( \Sigma \) along \( \ell \). It has either one or two connected components. If \( \Sigma' \) is connected, then, according to [55, Sect. 6.3.1] or [22, Sect. 1.3.1], we can assume that \( \ell \) is one of the generators of the fundamental group of \( \Sigma \). Hence \( \ell \) is not contractible. If \( \Sigma' \) has two connected components \( \Sigma_1 \) and \( \Sigma_2 \), we want to show that both have zero genus. Would both of them have positive genus, then \( \ell \) would not be contractible in either of them. By Seifert–van Kampen theorem, it would follow that \( \ell \) is not contractible in \( \Sigma \).
Consider the area weighted map identical to \( a \) and the partition function \( Z \). The unconstrained partition function \( Z_{M,a,G} \) is given by

\[ Z_{M,a,G} = Z_{1,T,G} = \int_{G^2} p_T([x,y]) dx dy, \tag{13} \]

and the partition function \( Z_{M,a,G}(b;[y]) \) for the Yang-Mills measure with constraint \( C_{b;[y]} \), where \( y \in G \), is

\[ Z_{M,a,G}(b;[y]) = \int_{G} p_T([x,zyz^{-1}]) dx = \int_{G} p_T([x,y]) dx. \tag{14} \]

Consider the area weighted map \( (M^*, a^*) \) with one boundary component, which is identical to \( M \) but with \( F_2 \) and \( F_3 \) removed. Then for all \( x \in G \),

\[ Z_{M^*,a^*,G}(x) = \int_{G^2} p_a(x_3^{-1}x_4x_6x_3[x_1,x_2]) dx_1 dx_2 dx_3 \delta_{[x]}(dx_4 dx_6) \]

\[ = \int_{G^2} p_a(x[x_1,x_2]) dx_1 dx_2 = Z_{(2,1),a_1}(x). \]

Lastly, instead of the first map above, consider an area weighted map \( (M,a) \) with an additional edge, labeled \( t \), bounding a closed disc \( U \) and splitting the face \( F_1 \) into two faces \( F_0 \) and \( F_1 \) as in Fig. 3. The discrete Yang–Mills measure is then

\[ dYM_{M,a,G}(x_1,\ldots,x_7) = \frac{1}{Z_{1,T,G}} p_{a_0}(x_3^{-1}x_7x_3[x_1,x_2]) p_a(x_7^{-1}x_6x_4) \]

\[ \times p_{a_2}(x_4^{-1}x_5x_6^{-1}) p_{a_3}(x_5^{-1}) dx_1 \cdots dx_7, \tag{15} \]

with the same partition function as the initial area-weighted map.

Consider the loop \( \ell = efg \), included in the disc \( U \) whose boundary is \( t \). Under \( YM_{M,a,G} \), the Wilson loop \( W_\ell \) is a complex random variable whose expectation is

\[ E[W_\ell] = \frac{1}{Z_{1,T,G}} \int_{G^7} \text{tr}(x_6x_5x_4)p_{a_0}(x_3^{-1}x_7x_3[x_1,x_2]) p_a(x_7^{-1}x_6x_4) \]

\[ \times p_{a_2}(x_4^{-1}x_5x_6^{-1}) p_{a_3}(x_5^{-1}) dx_1 \cdots dx_7. \]

The weighted map \( (\tilde{M}_U,a_0) \) has then only two bounded faces, \( F_2 \) and \( F_3 \), as shown on figure 4. We have, under the corresponding planar Yang–Mills measure \( YM_{\tilde{M}_U,a_U} \),

\[ E[W_\ell] = \int_{G^3} \text{tr}(x_3x_2x_1)p_{a_2}(x_1^{-1}x_2x_3^{-1}) p_{a_3}(x_2^{-1}) dx_1 dx_2 dx_3. \]

According to Theorem 2.16, the limit of \( E[W_\ell] \) with respect to \( YM_{M,a} \) is the same as the one with respect to \( YM_{\tilde{M}_U,a_U} \). We can then use the table of the master field on the plane [36] to compute \( \Phi_{\tilde{M}_U,a_U}(\ell) \), and we obtain that

\[ \Phi_{\tilde{M}_U,a_U}(\ell) = e^{-\frac{a_2}{2} - a_3}(1 - a_3). \]
Figure 3: An area-weighted map on the torus. The loop \( def \) is included in the disc \( U \) with boundary \( t \).

Figure 4: The lifting of the loop \( \ell \) of Fig. 2 in the plane.

Let us consider another loop, using that the convergence of Theorem 2.16 holds uniformly on \( \Delta_M(T) \). Consider the loop \( \gamma = f^{-1}d^{-1} \) in \( M \). Then under \( YM_{M,a} \),

\[
E(W_\gamma) = \frac{1}{Z_{1,T,G}} \int_{G^6} \text{tr}(x_6x_4^{-1})p_{a_0}(x_3^{-1}x_7x_3[x_1, x_2])p_{a_1}(x_7^{-1}x_6x_4)
\]
\[
\times p_{a_2}(x_4^{-1}x_5x_6^{-1})p_{a_3}(x_5^{-1})dx_1 \cdots dx_7
\]
\[
= \frac{1}{Z_{1,T,G}} \int_{G^4} \text{tr}(x_6x_4^{-1})p_{a_0+a_1}(x_6x_4[x_1, x_2])p_{a_2+a_3}(x_4^{-1}x_6^{-1})dx_1dx_2dx_3dx_6,
\]

and

\[
\lim_{a_0,a_1 \rightarrow 0} E(W_\gamma) = \frac{1}{Z_{1,T,G}} \int_{G^2} \text{tr}([x_1, x_2])p_T([x_1, x_2])dx_1dx_2 = E(W_{a,b}),
\]

where the expectation on the right-hand side is with respect to the map with respect to the Yang-Mills measure on the torus of area \( T \).

According to Theorem 2.16 and the above identity, the limit of \( E[W_\gamma] \) with respect to \( YM_{(\Delta_M,a)} \) is the same as the one of \( E[W_\gamma] \) with respect to \( YM_{(\tilde{M}_U,a_U)} \) as
$N \to \infty$ and it holds uniformly in $a \in \Delta_M(T)$. In particular, using the value [36] of the planar master field on simple loops, we find
\begin{equation}
\lim_{N \to \infty} \mathbb{E}[W_{[a,b]}] = \lim_{a_0, a_1 \downarrow 0} \lim_{a \in \Delta_M(T)} \lim_{N \to \infty} \mathbb{E}[W_a] = \lim_{a_0, a_1 \downarrow 0} \frac{e^{-23.53}}{a_1^{4.5}} = e^{-\frac{23.53}{4.5}}.
\end{equation}

_Idea of proof for Theorem 2.14 and $\ell$ in figure 2._ Let us give now an idea of the proof of Theorem 2.14 for $\ell = \text{def}$ as in figure 2. Keeping the same notation for the value of the edge variables, the Yang-Mills measure in the area weighted map $(\tilde{M}_U, a_U)$ is
\begin{equation}
dYM_{\tilde{M}_U, a_U}(x_4, x_5, x_6) = p_{a_2}(x_4^{-1} x_5 x_6^{-1}) p_{a_3}(x_5^{-1}) dx_4 dx_5 dx_6.
\end{equation}
From the expression (15) for the discrete Yang-Mills measure in the map $(M, a)$ of figure 3, it follows that under $YM_{M(a)}, a_U$ the random variable $(h_e, h_e, h_f)$ is absolutely continuous with respect to $YM_{\tilde{M}_U, a_U}$ with density
\begin{equation}
F(x_4, x_5, x_6) = \frac{1}{Z_{1, T, G}} \int_{G^3} p_{a_2}(x_3^2 x_2 x_4) p_{a_1}(x_2 x_4) dx_2 dx_4 dx_5 dx_6.
\end{equation}
We conclude that for any $\varepsilon > 0$, $n_{YM_{M(a)}, a_U}(|W_\ell - \Phi_{\tilde{M}, a_U}(\ell)| > \varepsilon)$ is bounded by
\begin{equation}
\frac{\|Z_{1, a_0 + a_1, 1} \|_{L^\infty(G)}}{Z_{1, T, G}} n_{YM_{\tilde{M}, a_U}}(|W_\ell - \Phi_{\tilde{M}, a_U}(\ell)| > \varepsilon).
\end{equation}
By [36], the last probability vanishes when $G = G_N$ is as in section 2.1.2 and $N \to \infty$. To conclude, it is enough to bound the ratio uniformly in $N$. This will be a consequence of the observation that
\begin{equation}
\|Z_{1, a_0 + a_1, 1} \|_{L^\infty(G)} = Z_{1, a_0 + a_1, 1} (1) = Z_{1, a_0 + a_1, 1} (1) = Z_{1, a_0 + a_1, G_N}
\end{equation}
combined with Theorem 3.1. The full proof of Theorem 2.14 for a fixed area follows this argument and is given in section 4.1.

## 3 Asymptotics of partition functions

Our argument relies on the convergence of partition functions of closed orientable surfaces, for classical groups, generalising results of [27] and of the second author [31]. This result was mentioned without proof in [46] for $g > 1$ and in [17] for $g = 1$ and $G = U(N)$ or $SU(N)$. For $g \geq 2$, for all group series except the unitary group series, it is a simple consequence of [27]. For $g = 1$ it is proved in [31] for $U(N)$ and $SU(N)$. For all classical groups series, the limit involves the Jacobi theta function and the Euler phi function, both defined for $q \in \mathbb{C}$ such that $|q| < 1$, with
\begin{equation}
\theta(q) = \sum_{n \in \mathbb{Z}} q^{n^2} \quad \text{and} \quad \phi(q) = \prod_{m=1}^{\infty} (1 - q^{m^2}).
\end{equation}

\footnote{It can be shown that the ratio $\frac{\sup_{W \in \mathcal{E}} Z_{1, a_0 + a_1, G_N}}{Z_{1, T, G_N}}$ diverges as $N \to \infty$, and an additional argument is required to prove the uniform convergence in $\Delta_M(T)$.}

\footnote{The Jacobi theta function only appears for the unitary group series.}
Let \( r \geq 1 \) be an integer, we denote respectively by \( \tilde{A}_r, A_r, B_r, C_r, D_r \) the type of \( U(r), SU(r+1), SO(2r+1), Sp(r) \) or \( SO(2r) \), in reference to the type of their root system (see [5] for instance). Note that we will alternatively use \( r \) as the rank of the root system or \( N \) as the size of the classical group, and their relations will be implicit: for instance, if we consider \( SU(N) \), it will be of type \( A_r \) with \( r = N - 1 \). Conversely, if we consider the classical group of type \( D_r \), it will be \( SO(2r) \). This change of index may be confusing at first, but it will be helpful in order to deal simultaneously with all considered root systems. Considering limits when \( r \to \infty \) and when \( N \to \infty \) is equivalent.

**Theorem 3.1.** For any \( T > 0 \), \((r, g) \in (\mathbb{N}^*)^2 \) and any type \( X_r \), with \( X \in \{B, C, D\} \), let us denote by \( Z_{g,T,X_r} \), the Yang–Mills partition function on an orientable compact surface of genus \( g \) and area \( T \) with structure group of type \( X_r \) (we will also write \( Z_{g,T,r} \) when the type of the structure group is unambiguous). Set \( q_T = e^{-\frac{T}{2}} \).

1. For all \( g \geq 1 \) and \( T > 0 \), \( \lim_{r \to \infty} Z_{g,T,X_r} \) exists and is given by the table (21) below.

2. Moreover if \( g = 2 \), for all \( X \in \{A, B, C, D\} \), \( Z_{g,T,X_r} = \lim_{T \to 0} Z_{g,T,X_r} \) is well defined and

\[
\lim_{r \to \infty} Z_{g,0,X_r} = 1.
\]

As we can see, the limits for odd and even orthogonal groups are different when \( g = 1 \), which justify our choice to make the distinction between the root systems rather than the groups themselves.

Point 2 was proved for all types except \( \tilde{A} \) in [27], whereas 1 was proved in [31] for group series \( A, \tilde{A} \), proving independently point 2 for types \( \tilde{A} \). Theorem 3.1 will be proved in Section 3. Let us also mention that, whereas the partition function on the plane or a disc in the plane is equal to 1 for any group, the case of the sphere behaves very differently, as the partition function goes to zero exponentially fast, at a speed of order \( r^2 \) ([14]). It further displays a phase transition that was discovered by and named after Douglas and Kazakov [16] and proved in the case of unitary groups by Lévy and Maïda [38]. See also Boutet de Monvel and Shcherbina [4], where the convergence of the right-hand-side below was first proven for a more general class of models.

**Theorem 3.2** (Douglas–Kazakov phase transition). For any \( T \geq 0 \), set \( Z_{T,N} \) as the Yang–Mills partition function on the sphere of area \( T \) with structure group \( U(N) \). Then the quantity

\[
F(T) = \lim_{N \to \infty} \frac{1}{N^2} \log Z_{T,N}
\]

exists. It defines a function \( F \in \mathcal{C}^2(\mathbb{R}_+^*) \cap \mathcal{C}^\infty(\mathbb{R}_+^* \setminus \{\pi^2\}) \) which admits a third-order jump at the area \( \pi^2 \).

We will prove Theorem 3.1 in section 3.2. Beforehand, we recall in the next section a well known expression of the partition function using representation theory of compact groups.
3.1 Character decomposition of the partition function

We shall express the partition functions $Z_{g,T,X}$, for any $X \in \{\tilde{A}, A, B, C, D\}$ as a sum over non-increasing sequences of integers. The result of this section are well known [49, 32]. This will follow from standard representation theory of compact groups, and we will prove it succinctly for the sake of completeness. The main result we want to prove is Prop. 3.7, and the remaining of the section may be skipped by anyone familiar with representation theory. Most of the results we will present can be found in [5] and [21].

Definition 3.1. Let $G$ be a compact Lie group.

1. A complex representation of $G$ is a couple $(\varphi, V)$, where $V$ is a complex vector space and $\varphi : G \to \text{GL}(V)$ is a smooth group morphism.
2. A representation $(\varphi, V)$ is irreducible if $V$ is the only nontrivial subspace left invariant by $\varphi$.
3. The dimension $d_\varphi$ of a representation $(\varphi, V)$ is defined as the dimension of the vector space $V$.
4. The character of a representation $(\varphi, V)$ is the function $\chi_\varphi : G \to \mathbb{C}$ defined by
   \[ \chi_\varphi(g) = \text{Tr}(\varphi(g)). \]

In the following, we will only consider finite-dimensional representations, unless stated otherwise. The two main results of the representation theory that we will be using are the celebrated Schur’s lemma and Plancherel’s theorem.

Lemma 3.3 (Schur’s lemma). 1. Let $(\varphi_1, V_1)$ and $(\varphi_2, V_2)$ be two irreducible representations of $G$. If $A : V_1 \to V_2$ is a nonzero linear map such that
   \[ A\varphi_1(g) = \varphi_2(g)A, \forall g \in G, \] (22)

then $A$ is an isomorphism.

2. Let $(\varphi, V)$ be an irreducible representation of $G$. If $A \in \text{End}(V)$ satisfies
   \[ A\varphi(g) = \varphi(g)A, \forall g \in G, \] (23)

then there exists $\alpha \in \mathbb{C}$ such that $A = \alpha I$, where $I$ denotes the identity of $V$.

Among the numerous consequences of this lemma, one can state that the relation (22) defines an equivalence relation between irreducible representations of $G$, and it enables to define the set $\hat{G}$ of equivalence classes of irreducible representations. This set, sometimes called dual space of $G$, is countable whenever $G$ is compact, and it is also a group when $G$ is abelian. Another consequence of Schur’s lemma is that two irreducible representations within the same class $\lambda \in \hat{G}$ have same dimension $d_\lambda$ and same character $\chi_\lambda$.

Theorem 3.4 (Plancherel’s theorem). For any continuous function $f \in C^\infty(G)$, the following sum absolutely converges

\[ f(g) = \sum_{\lambda \in \hat{G}} d_\lambda(f * \chi_\lambda)(g), \forall g \in G, \] (24)

where for any $\chi \in C(G)$, $f * \chi(h) = \int_G f(h)\chi(h^{-1}g)dh, \forall g \in G.$
From these results one can prove the following, which is a particular instance of Thm. 4.2 in [40].

**Theorem 3.5.** Let \((p_t)_{t>0}\) be the heat kernel on \(G\). Then for all \(t > 0\), \(p_t \in C^\infty(G)\) and the following sum

\[
p_t(g) = \sum_{\lambda \in \hat{G}} e^{-\frac{t}{c_\lambda}} d_\lambda \chi_\lambda(g), \quad \forall g \in G,
\]

absolutely converges, where \(c_\lambda \geq 0\) is the non-negative real number such that

\[
\Delta_G \chi_\lambda = -c_\lambda \chi_\lambda.
\]

Before we state the Fourier decomposition of Yang–Mills partition function, let us also mention an easy but useful result about the irreducible characters of compact groups.

**Proposition 3.6.** Let \(\lambda \in \hat{G}\) be an equivalence class of irreducible representations of a compact group \(G\). We have

\[
\int_G \chi_\lambda(x[y,z]) dy = d_\lambda^{-1} \chi_\lambda(xz^{-1}) \chi_\lambda(z), \quad \forall x, z \in G
\]

and

\[
\int_G \chi_\lambda(x[y,z]) dydz = \frac{\chi_\lambda(x)}{d_\lambda}, \quad \forall x \in G.
\]

**Proof.** Consider an element \((\varphi, V)\) of the equivalence class \(\lambda\) and set

\[
A_z = \int_G \varphi(yzy^{-1}) dy, \quad \forall z \in G.
\]

Since the Haar measure is invariant by multiplication, \(A_z\) satisfies (23) of Schur’s lemma. Therefore, \(A_z = \alpha_z I\) for some scalars \(\alpha_z \in \mathbb{C}\) with

\[
\alpha_z d_\lambda = \text{Tr}(A_z) = \int_G \chi_\lambda(yzy^{-1}) dy = \chi_\lambda(z).
\]

We can now write the left-hand side of (26) as

\[
\text{Tr}(\varphi(x)A_z \varphi(z^{-1})) = \frac{\chi_\lambda(z)}{d_\lambda} \text{Tr}(\varphi(z^{-1}x)) = d_\lambda^{-1} \chi_\lambda(z) \chi_\lambda(z^{-1}x), \forall x, z \in G,
\]

while using Plancherel theorem, the left-hand side of (27) reads

\[
d_\lambda^{-1} \int_G \chi_\lambda(z) \chi_\lambda(z^{-1}x) dz = \frac{1}{d_\lambda} \chi_\lambda(x) = \chi_\lambda(x), \forall x \in G.
\]

**Proposition 3.7.** Let \(\Sigma\) be a connected orientable surface of genus \(g \geq 1\). If \(\partial \Sigma\) is connected, then the Yang–Mills partition function on \(\Sigma\) with structure group of type \(X_r\) and boundary condition \(t \in G/\text{Ad}\) is given by

\[
Z_{(g,1),T,X_r}(t) = \sum_{\lambda \in \hat{G}} e^{-\frac{2g}{c_\lambda}} d_\lambda^{1-2g} \chi_\lambda(t).
\]
If $\Sigma$ has no boundary, then the Yang–Mills partition function on $\Sigma$ with structure group $G$ is given by

$$Z_{g,T,X_r} = \sum_{\lambda \in \hat{G}} e^{-\frac{T}{d_\lambda} c_\lambda} d_{2g} - 2g.$$  

(29)

Proof. Let us start with the simplest case, which is (29). Consider the area-weighted map $(M,a)$ of genus $g$ with 1 vertex, $2g$ edges $(a_1,b_1,\ldots,a_g,b_g)$ and 1 face $f$ with area $T > 0$ and boundary

$$\partial f = b_g^{-1}a_g^{-1}b_g a_g \cdots b_1^{-1}a_1^{-1}b_1 a_1.$$

Then

$$Z_{g,T,X_r} = Z_{M,a,G} = \int_{G^{2g}} p_T([x_1,y_1] \cdots [x_g,y_g]) dx_1 dy_1 \cdots dx_g dy_g.$$

Using the Fourier decomposition of the heat kernel (25), we get

$$Z_{g,T,X_r} = \sum_{\lambda \in \hat{G}(N)} e^{-\frac{T}{d_\lambda} c_\lambda} \int_{G^{2g}} \chi_\lambda([x_1,y_1] \cdots [x_g,y_g]) dx_1 dy_1 \cdots dx_g dy_g.$$

We now integrate out all commutators using (27), which yields

$$\int_{G^{2g}} \chi_\lambda([x_1,y_1] \cdots [x_g,y_g]) dx_1 dy_1 \cdots dx_g dy_g = \frac{1}{d_\lambda^{2g-1}}.$$

The results follows. The proof of (28) is similar, using an area-weighted map with one face whose boundary is $\partial f = b_g^{-1}a_g^{-1}b_g a_g \cdots b_1^{-1}a_1^{-1}b_1 a_1 \ell$, where $\ell$ is the simple loop corresponding to $\partial \Sigma$, oriented positively, for any $h \in G$ to

$$Z_{(g,1),T,X_r}(h) = Z_{M,a,G}(h) = \int_{G^{2g}} p_T(h[x_1,y_1] \cdots [x_g,y_g]) dx_1 dy_1 \cdots dx_g dy_g. \quad (30)$$

Now we will specify (29) to all compact classical groups: to do this, we only need to know the dual space $\hat{G}$, and the numbers $c_\lambda$ and $d_\lambda$ for every $\lambda \in \hat{G}$. This can be done thanks to their root systems. Most definitions and results are borrowed from [5], and can also be recovered with much clarity from Sections 2.2 and 2.3 in [42].

Definition 3.2. Let $G$ be a compact connected Lie group.

1. A weight of a representation $(\pi,V)$ of $G$ is a group morphism $\omega : T \to U(1)$ where $T$ is a maximal torus of $G$, such that the space $V^\omega = \{ v \in V : \omega(t)v = \omega(t)v, \forall t \in T \}$ is not trivial. The weights form a lattice $L = Z \Omega = \bigoplus_{i=1}^r Z \omega_i$, where $\{\omega_i\}$ is a distinguished basis of the lattice and $r$ is the rank of the weight lattice. The normaliser of $T$ in $G$ acts by conjugation on $T$ yielding an action on $L$. The vector space $\mathfrak{t}^\ast = \mathbb{R}^\Omega = L^\ast \otimes_{\mathbb{Z}} \mathbb{R}$ can be endowed with an invariant inner product $\langle \cdot, \cdot \rangle$. In fact, if $t$ is the Lie algebra of the maximal torus $T \subset G$, we have $\mathbb{R}^\Omega \simeq t^\ast$ and $\langle \cdot, \cdot \rangle$ can be taken as the dual of the inner product (1).

18 The following formula goes back to Frobenius for finite groups, see e.g. [53, Sect. 7.9].
2. A root of $G$ is a non-zero weight of the adjoint representation of $G$. The root system $\Phi$ can be split into $\Phi_+ \cup \Phi_-$ with $\Phi_- = -\Phi_+$. The elements of $\Phi_+$ are called positive roots.

3. The Weyl chamber $C_r$ is the set

$$C_r = \{ x \in V_r : \langle x, \alpha \rangle > 0 \ \forall \alpha \in \Phi_+ \}. $$

It is an open cone in $V_r$.

4. A dominant weight is a weight that belongs to the closure $C_r$ of the Weyl chamber. We denote by $\Lambda_r$ the set of dominant weights.

5. The element

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha$$

has the property that for any $\omega \in I_r$

$$\omega \in C_r \text{ if and only if } \rho + \omega \in C_r.$$ (31)

Below are listed the root systems corresponding to the classical groups depending on their root systems. Note that we do not treat the root systems of exceptional Lie algebras $E_6$, $E_7$, $E_8$, $F_4$ and $G_2$, as we focus in this paper on unitary, special unitary, orthogonal and symplectic groups.

| Group     | Type | $\Phi$                                            |
|-----------|------|--------------------------------------------------|
| $U(r)$    | $A_r$ | $\{e_i - e_j, 1 \leq i, j \leq r, i \neq j \}$    |
| $SU(r+1)$ | $A_r$ | $\{e_i - e_j, 1 \leq i, j \leq r+1, i \neq j \}$  |
| $SO(2r+1)$ | $B_r$ | $\{\pm e_i \pm e_j, 1 \leq i < j \leq r \} \cup \{\pm e_i, 1 \leq i \leq N \}$ |
| $Sp(r)$   | $C_r$ | $\{\pm e_i \pm e_j, 1 \leq i < j \leq r \} \cup \{\pm 2e_i, 1 \leq i \leq N \}$ |
| $SO(2r)$  | $D_r$ | $\{\pm e_i \pm e_j, 1 \leq i < j \leq r \}$     |

For $SU(r+1)$, we identify $Y_r$ with $Y_r = \mathbb{R}^{r+1}/\mathbb{R}(1,\ldots,1)$ endowed with the norm

$$\|x\|^2 = \frac{1}{N+1} \sum_{i=1}^{N+1} (x_i - \frac{1}{N+1} \sum_{j=1}^{N+1} x_j)^2, \ \forall [x] \in \mathbb{R}^{N+1}/\mathbb{R}(1,\ldots,1).$$

For any classical group $G$ associated with the type $X_r$, with $X \in \{ \tilde{A}, A, B, C, D \}$, there is a bijection $\hat{G} \simeq \Lambda_r$: this is a consequence of the highest-weight theory (see [5] for instance). From now on, $\lambda$ will indistinctly represent an equivalence class of irreducible representations or an element of $\Lambda_r$.

**Proposition 3.8.** Let $G_N$ be a classical group associated with one of the types $\tilde{A}_r, A_r, B_r, C_r, D_r$ and $\lambda \in G_N \simeq \Lambda_r$ be an equivalence class of irreducible representations. We have

$$c_\lambda = \langle \lambda + 2\rho, \lambda \rangle$$

and

$$d_\lambda = \prod_{\alpha \in \Phi_+} (\lambda + \rho, \alpha) / \prod_{\alpha \in \Phi_+} (\rho, \alpha).$$ (33)

We list below some of the objects defined previously, specified for the corresponding classical groups. Most of the explicit computations can be found in [5].

24
\[ \Phi_+ = \{ e_i - e_j, \ 1 \leq i < j \leq r \} \]
\[ \mathcal{C}_r = \{ x \in \mathbb{R}^r : x_1 \geq \cdots \geq x_r \} \]
\[ \Lambda_r = \{ \lambda \in \mathbb{Z}^r : \lambda_1 \geq \cdots \geq \lambda_r \} \]
\[ \rho = \left( \frac{r-1}{2}, \frac{r-3}{2}, \ldots, \frac{3-r}{2}, \frac{1-r}{2} \right) \]
\[ c_\lambda = \frac{1}{r} \sum_{i=1}^{r} \lambda_i (\lambda_i + r + 1 - 2i), \ \forall \lambda \in \Lambda_r \]
\[ d_\lambda = \prod_{1 \leq i < j \leq r} \frac{\lambda_i - \lambda_j + j - i}{j - i}, \ \forall \lambda \in \Lambda_r \]

\[ \text{SO}(2r + 1): \text{ type } B_r \]
\[ \Phi_+ = \{ e_i \pm e_j, \ 1 \leq i < j \leq r \} \]
\[ \mathcal{C}_r = \{ x \in \mathbb{R}^r : x_1 \geq \cdots \geq x_r \geq 0 \} \]
\[ \Lambda_r = \{ \lambda \in \mathbb{N}^r : \lambda_1 \geq \cdots \geq \lambda_r \} \]
\[ \rho = \left( \frac{r}{2}, \frac{r-3}{2}, \ldots, \frac{1}{2} \right) \]
\[ c_\lambda = \frac{1}{2r+1} \sum_{i=1}^{r} \lambda_i (\lambda_i + 2r + 1 - 2i), \ \forall \lambda \in \Lambda_r \]
\[ d_\lambda = \prod_{1 \leq i < j \leq r} \frac{\lambda_i - \lambda_j + j - i}{j - i} \prod_{1 \leq i < j \leq r} \frac{\lambda_i + \lambda_j + 2r + 1 - i - j}{2r + 1 - i - j}, \ \forall \lambda \in \Lambda_r \]
• \( \text{Sp}(r) \): type \( C_r \)

\[
\Phi_+ = \{ e_i \pm e_j, 1 \leq i < j \leq r \} \cup \{ 2e_i, 1 \leq i \leq r \}
\]
\[
\mathcal{C}_r = \{ x \in \mathbb{R}^r : x_1 \geq \cdots \geq x_r \geq 0 \}
\]
\[
\Lambda_r = \{ \lambda \in \mathbb{N}^r : \lambda_1 \geq \cdots \geq \lambda_r \}
\]
\[
\rho = (r, r - 1, \ldots, 1)
\]
\[
c_\lambda = \frac{1}{2r} \sum_{i=1}^{r} \lambda_i (\lambda_i + 2r - 2i), \quad \forall \lambda \in \Lambda_r
\]
\[
d_\lambda = \prod_{1 \leq i < j \leq r} \frac{\lambda_i - \lambda_j + j - i}{j - i} \prod_{1 \leq i \leq j \leq r} \frac{\lambda_i + \lambda_j + 2r - 2 - i - j}{2r + 2 - i - j}, \quad \forall \lambda \in \Lambda_r
\]

• \( \text{SO}(2r) \): type \( D_r \)

\[
\Phi_+ = \{ e_i \pm e_j, 1 \leq i < j \leq r \} \cup \{ e_i, 1 \leq i \leq r \}
\]
\[
\mathcal{C}_r = \{ x \in \mathbb{R}^r : x_1 \geq \cdots \geq x_r \geq 0 \}
\]
\[
\Lambda_r = \{ \lambda \in \mathbb{N}^r : \lambda_1 \geq \cdots \geq \lambda_r \}
\]
\[
\rho = (r - 1, r - 2, \ldots, 0)
\]
\[
c_\lambda = \frac{1}{2r} \sum_{i=1}^{r} \lambda_i (\lambda_i + 2r - 2i), \quad \forall \lambda \in \Lambda_r
\]
\[
d_\lambda = \prod_{1 \leq i < j \leq r} \frac{(\lambda_i - \lambda_j + j - i)(\lambda_i + \lambda_j + 2r - 2 - i - j)}{(j - i)(2r - i - j)}, \quad \forall \lambda \in \Lambda_r
\]

We will prove Theorem 3.1 using an asymptotic estimation of \((29)\).

### 3.2 Proof of convergence of partition functions

We first recall the result of [27] and give an alternative proof based on the result of [31] for \( A_r \) series.

#### 3.2.1 Witten zeta function

When \( g \geq 2 \), for the root systems \( A_r, B_r, C_r \) and \( D_r \), a proof relies on an asymptotic estimation of the Witten zeta function\(^{19}\)

\[
\zeta_{X_r}(s) = \sum_{\lambda \in \Lambda_r} \frac{1}{d_\lambda}, \quad \forall s > 1, \quad \forall X \in \{ A, B, C, D \}. \tag{34}
\]

The first claim we can do is that, for any root system \( X_r \) with \( X \in \{ A, B, C, D \} \), any \( T \in \mathbb{R}_+ \) and any \( g \geq 2 \), we have

\[
1 \leq \sum_{\lambda \in \Lambda_r} e^{-\frac{T}{d_\lambda}} \leq \zeta_{X_r}(2). \tag{35}
\]

Indeed, the sum in the middle contains only nonnegative terms, therefore is bounded from below by the term corresponding to \( \lambda = (0, \ldots, 0) \in \Lambda_r \), which is equal to 1.

\(^{19}\)The following function was introduced in [57] and seems to have been named after Witten in [59].
The upper bound comes from the fact that for any \( \lambda \in \Lambda_r \) and \( g \geq 2 \), the number \( c_\lambda \) is nonnegative, and \( d_\lambda^{2g-2} \geq d_\lambda^2 \). Thanks to (35) we only need to prove
\[
\lim_{r \to \infty} \zeta_X, (2) = 1
\] (36)
for \( X \in \{A, B, C, D\} \) in order to prove Theorem 3.1 for \( g \geq 2 \).

**Proposition 3.9** ([27]). For any real \( s > 1 \) and \( X \in \{A, B, C, D\} \), one has
\[
\lim_{r \to \infty} \zeta_X, (s) = 1.
\]

**Proof.** The case of \( A_r \) is detailed in [31], we argue here that it implies the other cases. We will compare the dimensions and the sets of dominant weights for different root systems, thus we will denote by \( \Lambda_{Xr} \) the set of dominant weights of type \( X_r \) and \( d_{X_r, \lambda} \) the dimension of an irreducible representation of the compact connected group of type \( X_r \) with highest weight \( \lambda \). First, let us remark that the sets of dominant weights are the same for \( A_r, B_r \) and \( C_r \). Let \( X \in \{B, C\} \) and \( \lambda \in \Lambda_{Xr} \) a dominant weight different from \((0, \ldots, 0)\). As we have \( \lambda_i \geq 0 \) for any \( i \), it is clear that
\[
d_{X_r, \lambda} \geq \prod_{1 \leq i < j \leq r} \frac{\lambda_i - \lambda_j + j - i}{j - i} = d_{A_r, \lambda}.
\]
It follows that, for any \( s > 1 \):
\[
0 \leq \sum_{\lambda \in \Lambda_{Xr}, \lambda \neq (0, \ldots, 0)} \frac{1}{(d_{X_r, \lambda})^s} \leq \sum_{\lambda \in \Lambda_{A_r}, \lambda \neq (0, \ldots, 0)} \frac{1}{(d_{A_r, \lambda})^s}.
\]
We get the right limit by letting \( r \) go to infinity.

In the case of \( D_r \), some dominant weights have a negative coefficient, therefore an extra care must be taken. Let \( \lambda \in \Lambda_{Dr} \) and \( 1 \leq i < j \leq r - 1 \), we have \( \lambda_i + \lambda_j \geq 0 \) as the sum of nonnegative integers. Now, if \( 1 \leq i \leq r - 1 \), we also know that \( \lambda_i \leq \lambda_{r-1} = |\lambda_r| \) therefore \( \lambda_i - \lambda_r \geq 0 \) as well. In any case, we deduce that
\[
d_{Dr, \lambda} \geq \prod_{1 \leq i < j \leq r} \frac{\lambda_i - \lambda_j + j - i}{j - i} = d_{A_r, \tilde{\lambda}},
\]
where \( \tilde{\lambda} = (\lambda_1, \ldots, \lambda_{r-1}, |\lambda_r|) \). Any dominant weight \( \lambda \) of type \( A_r \) corresponds to at most two different dominant weights of type \( rN \): the same weight, and the one obtained by changing the sign of \( \lambda_r \). Hence, we have
\[
0 \leq \sum_{\lambda \in \Lambda_{Dr}, \lambda \neq (0, \ldots, 0)} \frac{1}{(d_{Dr, \lambda})^s} \leq 2 \sum_{\lambda \in \Lambda_{A_r}, \lambda \neq (0, \ldots, 0)} \frac{1}{(d_{A_r, \lambda})^s}.
\]
The right-hand side converges to 0 when \( r \to \infty \) therefore we obtain the expected limit for \( D_r \).

**Proof of Theorem 3.1 with \( g \geq 2 \).** According to Prop. 3.9, for any \( g \geq 2 \) and \( X \in \{A, B, C, D\} \) we have
\[
\lim_{r \to \infty} \zeta_X, (2g - 2) = 1.
\]
Together with (35), we get
\[
\lim_{r \to \infty} Z_{g, T, X_r} = 1
\]
\[27\]
3.2.2 Discrete Gaussian random variables in a cone

In this section we prove that the convergence of $Z_{T,X_r}$ as $r \to \infty$ and $T > 0$ is fixed for types $\{B,C,D\}$.

**Definition 3.3.** Let $r \geq 1$ be an integer and $\Lambda_r$ be the set of dominant elements of type $X_r$, with $X \in \{\tilde{A}, A, B, C, D\}$. A random variable $\mu$ on $\Lambda_r$ is Gaussian with parameter $t$ if for any $\Lambda \subset \Lambda_r$

$$P(\mu \subset \Lambda) = \frac{1}{Z_{r,T}} \sum_{\lambda \in \Lambda} e^{-\frac{t}{2}c_\lambda}$$  \hspace{1cm} (37)

and the associated partition function is

$$Z_{r,T} = \sum_{\lambda \in \Lambda_r} e^{-\frac{t}{2}c_\lambda}.$$  \hspace{1cm} (38)

The denomination ‘Gaussian’ is justified by the following remark. Recall that for any classical group $G_N \subset GL_n(\mathbb{C})$ of type $X_r$ and any $\lambda \in \hat{G}_N$,

$$c_\lambda = \frac{1}{n} \langle \lambda + \rho, \lambda \rangle = \frac{1}{n} (\|\lambda + \rho\|^2 - \|\rho\|^2).$$

It follows that for any $\lambda \in \Lambda_r$,

$$P(\mu = \lambda) \propto e^{-\frac{t}{2\pi} \|\lambda + \rho\|^2},$$

which thanks to (31) defines, after a shift by $\rho$, a discrete Gaussian distribution conditioned to belong to $\rho + (I_r \cap \mathcal{C}_r) = (\rho + I_r) \cap \mathcal{C}_r$. Note that if $G_N$ is a classical group of type $X_r$, the partition function $Z_{r,T}$ of the Gaussian distribution on $\Lambda_r$ with parameter $T$ is equal to the Yang–Mills partition function $Z_{1,T,X_r}$ on a torus of area $T$ with structure group $G_N$. Before we prove Theorem 3.1 for $g = 1$, let us give a few notations that we will use. When $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a sequence of non-negative real numbers, we denote by $|\lambda|$ its total sum. We also denote by

$$\mathcal{P} = \{ (\alpha_1, \alpha_2, \ldots) \in \mathbb{N}^{\mathbb{N}} : \alpha_1 \geq \alpha_2 \geq \cdots \}$$

the set of integer partitions. For any $\alpha \in \mathcal{P}$ we write $\ell(\alpha)$ for its number of non-zero parts. It is well known that the generating function of partitions is given by the inverse of Euler function:

$$\sum_{\alpha \in \mathcal{P}} q^{|\alpha|} = \frac{1}{\phi(q)}, \ \forall q \in \mathbb{C} \text{ s.t. } |q| < 1.$$  \hspace{1cm} (39)

**Proof of Thm 3.1 with $g = 1$.** Types $B$ and $C$. Let us define an increasing sequence setting $\tilde{\rho} = (r - \rho_1, \ldots, r - \rho_r)$ and write

$$\langle \lambda + 2\rho, \lambda \rangle = \|\lambda\|^2 + 2 \sum_{i=1}^{r} \lambda_i \rho_i = 2r \|\lambda\| + \|\lambda\|^2 - 2 \sum_{i=1}^{\ell(\lambda)} \lambda_i \tilde{\rho}_i.$$

For the types $B_r$ and $C_r$ we have $\Lambda_r \simeq \{ \lambda \in \mathcal{P} : \ell(\lambda) \leq r \}$ therefore we have

$$Z_{r,T} = \sum_{\lambda \in \mathcal{P}, \ell(\lambda) \leq r} e^{-\frac{2r}{2\pi} \|\lambda\|^2} e^{-\frac{T}{2\pi} \|\lambda\|^2 - 2 \sum_{i=1}^{\ell(\lambda)} \lambda_i \tilde{\rho}_i}.$$
Since $\hat{\rho}$ is increasing and $\lambda$ is non-increasing,
$$\sum_{1 \leq i, j \leq \ell(\lambda)} (\lambda_i - \lambda_j)(\hat{\rho}_i - \hat{\rho}_j) \leq 0.$$  
It follows that
$$0 \leq 2 \sum_{i=1}^{\ell(\lambda)} \lambda_i \hat{\rho}_i \leq \frac{2}{\ell(\lambda)} \sum_{i=1}^{\ell(\lambda)} \lambda_i \sum_{i=1}^{\ell(\lambda)} \hat{\rho}_i \leq |\lambda|\ell(\lambda),$$
where we used for the last inequality that in $B_r$ and $C_r$ cases, $\sum_{i=1}^{\ell(\lambda)} \hat{\rho}_i$ is either $\ell(\lambda)^2$ or $\ell(\lambda)\ell(\ell(\lambda)-1)/2$. We obtain
$$\sum_{\lambda \in \mathcal{P}, \ell(\lambda) \leq r} e^{-2\frac{r}{\ell(\lambda)}|\lambda|} \leq Z_{r,T} \leq \sum_{\lambda \in \mathcal{P}, \ell(\lambda) \leq r} e^{-2\frac{r}{\ell(\lambda)}|\lambda|}.$$  
Recalling that $n$ is $2r+1$ or $2r$, both sides converge to $\Phi(q_T)^{-1}$ when $r \to \infty$ by dominated convergence, and we therefore obtain the expected limit.

**Type $D$.** In this case, let us introduce two notations. For $\lambda = (\lambda_1, \ldots, \lambda_r)$ such that $\lambda_1 \geq \cdots \geq \lambda_r$ and $m \in \mathbb{Z}$, we set $\lambda = (\lambda_1, \ldots, \lambda_{r-1})$ and $\lambda + m = (\lambda_1 + m, \ldots, \lambda_r + m)$. We have
$$\langle \lambda + 2\mu, \rho \rangle = ||\lambda||^2 + 2 \sum_{i=1}^{r-1} \lambda_i (r-i) = 2r|\lambda| + \lambda_2^2 + ||\lambda||^2 - 2 \sum_{i=1}^{r-1} i \lambda_i.$$  
We find then
$$Z_{r,T} = \sum_{\lambda \in \mathcal{P}, k \in \mathbb{Z}, \ell(\lambda) \leq r-1, |k| \leq \lambda_{r-1}} e^{2\frac{r}{\ell(\lambda)}|\lambda| - \frac{k^2}{2}(2||\lambda||^2 - 2 \sum_{i=1}^{r-1} i \lambda_i)}$$
$$= \sum_{m \geq 0} \sum_{\lambda \in \mathcal{P}, k \in \mathbb{Z}, \ell(\lambda) \leq r-1, \lambda_{r-1} = m, |k| \leq m} e^{2\frac{r}{\ell(\lambda)}|\lambda| - \frac{k^2}{2}(2||\lambda||^2 - 2 \sum_{i=1}^{r-1} i \lambda_i)}$$
$$= \sum_{m \geq 0} \left( \sum_{|k| \leq m} e^{-\frac{k^2}{2m^2}} \right)^m q_T^{2m} \sum_{\mu \in \mathcal{P}, \ell(\mu) \leq r-2} q_T^{2|m|} e^{-\frac{2r}{\ell(\mu)}||\mu + m||^2 - 2 \sum_{i=1}^{r-1} i (\mu_i + m)}.$$  
Using the same argument as we did for the types $B$ and $C$, we have
$$2 \sum_{i=1}^{r-1} i (\mu_i + m) \leq \ell(\mu)|\mu + m|,$$
it follows that
$$Z_{r,T} \leq \sum_{m \geq 0} \left( \sum_{|k| \leq m} e^{-\frac{k^2}{2m^2}} \right)^m q_T^{2m} \sum_{\mu \in \mathcal{P}, \ell(\mu) \leq r-2} q_T^{2|m|} e^{-\frac{2r}{\ell(\mu)}||\mu + m||^2}$$
and
$$Z_{r,T} \geq \sum_{m \geq 0} \left( \sum_{|k| \leq m} e^{-\frac{k^2}{2m^2}} \right)^m q_T^{2m} \sum_{\mu \in \mathcal{P}, \ell(\mu) \leq r-2} q_T^{2|m|} e^{-\frac{2r}{\ell(\mu)}||\mu + m||^2 - \ell(\mu)|\mu + m|}.$$
By dominated convergence both bounds converge to
\[
\sum_{m \geq 0} (2m + 1) q^m \sum_{\lambda \in \mathcal{P}} q^{\lambda} = \frac{1 + q^r}{(1 - q^r)^2 \phi(q^r)},
\]

hence we deduce the expected limit for \( Z_{r,T} \).

**Remark.** For any \( T > 0 \), it is also possible to deduce the case \( g \geq 2 \) from the case \( g = 1 \) using that when \( G \) is of type \( X_r \), for any non-trivial representation \( \lambda \in \hat{G} \),
\( d_{\lambda} \geq r \).

**Remark.** A proof for the groups \( U(N) \) and \( SU(N + 1) \) using highest weights can be produced similarly to the other cases yielding a different proof of the result of \([31]\), the latter being formulated in terms of Young diagrams.

## 4 Proof of Wilson loops convergence

In this section, we will prove theorems 2.14 and 2.17. The proof relies on an absolute continuity relation and the boundedness of the Yang–Mills partition function,\(^{20}\) except for a class of loops considered in 4.2.2, which requires an independent argument. Theorem 2.14 and its proof can be extended to the continuous setting of \([35, \text{Sect. 3}]\) and \([6, \text{Sect. 4}]\) will only be sketched at the end of the first following section. From here onwards, \( G_N \) will denote a classical group of size \( N \).

### 4.1 Loops within in a disc

Assume that \((M, a)\) is an area-weighted map on \( \Sigma \) of genus \( g \geq 1 \) and area \(|a| = T\), that \( U \) is a topological closed disc of \( \Sigma \) obtained by union of faces of \( M \), with area \(|a_U|\), such that \( u = T - |a_U| > 0 \). The boundary of \( U \) is given by a loop \( \partial U \in \mathcal{L}(\tilde{M}_U) \).

**Lemma 4.1.** The measure \((\mathcal{R}_{M_U}^M)_*(YM_{M,a,G_N})\) has density
\[
\frac{Z_{g,1,u,G_N}(h_{\partial U}^{-1})}{Z_{g,T,G_N}}
\]

with respect to \( YM_{\tilde{M}_U,a \cup \partial U,G_N} \).

**Proof.** Thanks to Lemma 2.5, we can assume that \( M \) has same number of faces as \( \tilde{M}_U \), that is, that exactly one face of \( M \) is not included in \( U \). Denote by \( k \) be the number of edges of \( \partial U \). Let \( M' \) be the map with \( k + 1 \) vertices, \( k + 2g + 1 \) edges \( u_1, \ldots, u_k, e, a_1, b_1, \ldots, a_g, b_g \) and two faces \( U' \) and \( V_g \), such that \( \partial U' = u_k \cdots u_1 \) and \( \partial V_g = e^{-1} [b_g^{-1}, a_g^{-1}] \cdots [b_1^{-1}, a_1^{-1}] \). Let us assume that \( M \) is obtained by gluing \( M' \) with \( \tilde{M}_U \), identifying \( \partial U \) with \( \partial U' \) so that the base of \( \partial U \) is sent to an endpoint of \( e \). For any \( h \in \mathcal{M}(\partial U) \), let us write \( x_i = h_{a_i}, y_i = h_{b_i} \) for all \( 1 \leq l \leq g \) and \( z = h_e \). Denote the set of non-marked faces of \( \tilde{M}_U \) by \( \mathcal{F}_U \). Then \( YM_{M,a,G_N}(dh) \) can be written as
\[
\frac{1}{Z_{g,T,G_N}} p_u(h_{\partial U}^{-1}z[x_1,y_1] \cdots [x_g,y_g]z^{-1}) \prod_{i=1}^g dx_idy_i \prod_{f \in \mathcal{F}_U} p_u(h_{\partial f}) U_{\tilde{M}_U,G_N}(dh).
\]

\(^{20}\)which follows from the convergence discussed in the Section 3.
Integrating over \((x_i, y_i)\) and \(z\) the result follows by invariance by conjugation of the Haar measure and (30).

**Proof of Thm. 2.14.** Without loss of generality, up to refinement of \(M\), we can assume that \(U\) is a topological closed disc, that is the union of faces of \(M\). Let us first assume that \(|a_U| < T = |a|\). For any \(h \in G_N\) and \(\lambda \in G_N\), since the representation is unitary, \(\|\chi_\lambda\|_\infty = \chi_\lambda(1) = d_\lambda\). For any \(s > 0\), we deduce from (28) that

\[
\|Z(\gamma, 1, a, G_N)\|_\infty = \sum_{\lambda \in G_N} d_\lambda^{2-2s} e^{-\frac{2}{\epsilon} \epsilon \lambda} = Z_{g, a, G_N}.
\]

Together with the previous lemma, \(\mathcal{D}_{M, a}^g (YM_{M, a, G_N})\) has a density with respect to \(YM_{M, a, G_N}\) which is bounded from above by

\[
\frac{Z_{g, |a| - |a_U|, G_N}}{Z_{g, |a|, G_N}}.
\]

According to Theorem 3.1, the right-hand side is uniformly bounded in \(N\). In particular for any \(\epsilon > 0\) and any loop \(\ell \in L(M_U)\),

\[
P_{YM, a}(|W_\ell - \Phi_{M_U, a_U}(\ell)| > \epsilon) \leq \frac{Z_{g, |a| - |a_U|, G_N}}{Z_{g, |a|, G_N}} P_{YM_{M_U, a_U}}(|W_\ell - \Phi_{M_U, a_U}(\ell)| > \epsilon).
\]

Together with Theorem 2.10 for the plane, we conclude that the right-hand side converges towards zero as \(N \to \infty\).

To conclude it remains to prove the uniformity of the convergence in the set \(A\) of \(\alpha \in \Delta_M(T)\) with \(a(f) > 0\) for any face \(f\) and \(|a_U| < T\). It is enough to show that for any faces \(f_1, f_2\) of \(M\) adjacent in \(M_U\), there is a constant \(K > 0\) such that for all \(a, a' \in A\), with \(a(f) = a'(f)\) for all faces distinct from \(f_1, f_2\) and \(|a - a'| < T\),

\[
|E_{YM, a} [W_\ell - \Phi_{\tilde{M}_U, a_U}(\ell)] - E_{YM, a'} [W_\ell - \Phi_{\tilde{M}_U, a_U}(\ell)]| \leq K|a - a'|^{1/2}.
\]

When \(f_1, f_2\) and \(a, a'\) are as above with \(a(f_1) < a'(f_1)\), while \(e\) is an edge of \(\tilde{M}_U\) between \(f_1\) and \(f_2\) with \(f_2\) on its right, consider an area weighted map \((M', a')\) finer than \((M, a)\), with an additional edge \(e'\) with same endpoints as \(e\), an additional face \(f\) included in \(f_2\), bounded by the simple loop \(s = e(e')^{-1}\), with \(a'(f) = |a - a'| = 2(a'(f_1) - a(f_1))\), while all other faces can be identified with a face of \(M\), with \(a'(f_1) = a(f_1), a'(f_2) = a(f_2)\) and \(a'(f) = a(f)\) for all \(f \notin \{f_1, f_2\}\). Denote by \(\ell'\) the loop of \(M'\) obtained from \(\ell\) by replacing all occurrences of \(e\) in \(\ell\) by \(e'\). Then under \(YM_{M_U, a'}\), \(W_{\ell'}\) has same law as \(W_\ell\) under \(YM_{M, a'}\). two paths \(\alpha, \beta\) of \(M'\) with \(\ell = \alpha \beta\) and \(W_{\alpha \beta}\) has same law as \(W_\ell\) under \(YM_{M, a'}\). Using this identity in law, the left-hand-side of (41) equals

\[
|E_{YM, a'} [W_\ell - \Phi_{\tilde{M}_U, a_U}(\ell)] - E_{YM, a'} [W_{\ell'} - \Phi_{\tilde{M}_U, a_U}(\ell')]| \leq E_{YM, a'} [W_\ell - W_{\ell'}] + |\Phi_{\tilde{M}_U, a_U}(\ell) - \Phi_{\tilde{M}_U, a_U}(\ell')|.
\]

To conclude it is enough to prove that

\[
E_{YM, a'} [W_\ell - W_{\ell'}] \leq K|a - a'|^{1/2},
\]

for some \(K > 0\) independent of \(N\). Now by induction on the number of occurrences of \(s\) in \(\ell'\), we can assume that \(\ell' = \alpha \beta\) and \(\ell = \alpha \beta\) for some paths \(\alpha, \beta\) of \(M'\).

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Denoting by $V$ an open disc of $\Sigma$ associated to $\tilde{f}$, we can then rewrite and bound the latter left-hand side as
\[
E_{YM_{\Sigma,G_N}}[\|\text{tr}(H_{\alpha\beta} - H_{\alpha\beta}H_s)\|] = E_{YM_{\Sigma,G_N}}[\|\text{tr}(H_{\beta\alpha} - H_{\beta\alpha}H_s)(H_{\beta\alpha}H_s - H_{\beta\alpha}H_s^*)\|]^{1/2} \\
\leq \sqrt{2}E_{YM_{\Sigma,G_N}}[1 - \Re(\text{tr}(H_s))]^{1/2} \\
\leq \frac{\sqrt{2}Z_{g,T-a'^r(\tilde{f})}}{Z_{g,T}}E_{YM_{\Sigma,G_N}}[1 - \Re(\text{tr}(H_s))]^{1/2} \\
\leq \frac{\sqrt{2}Z_{g,T/2}}{Z_{g,T}}E_{YM_{\Sigma,G_N}}[1 - \Re(\text{tr}(H_s))]^{1/2},
\]
where we used the same argument as in the first part of the proof for the penultimate inequality, and $2a'^r(\tilde{f}) = |a - a'| < T$. Now under $YM_{\tilde{f},a'V}$, $H_s$ has same law as a Brownian motion at time $|a - a'|$. It follows from [34] or [10] that for all group series considered, there is a constant $c > 0$ independent of $N$ with
\[
E_{YM_{\Sigma,G_N}}[1 - \Re(\text{tr}(H_s))] \leq 1 - e^{-c|a - a'|} \leq c|a - a'|,
\]
leading to the required bound (42).

Proof of Theorem 2.16. We only sketch here the additional arguments needed for the two convergence claims and refer respectively to [35] and [6] for more details. For any multiplicative function $h \in M(P(\Sigma),G_N)$, the composition of its restriction to $P(U)$ with $\Psi$ defines an element $\Psi(h)$ of $M(P(D_R),G_N)$. According to the splitting property of Markovian holonomy fields (property (A3) of [35, Def. 3.1.2]) and Lemma 4.1, $\Psi_s(YM_{\Sigma,G_N})$ has density
\[
\frac{Z_{(g,1),T-\text{vol}(U)}(H_{\partial D_R})}{Z_{g,T,G_N}}
\]
with respect to $YM_{D_R,G_N}$. The first claim then follows with the same argument as for the proof of Theorem 2.14. To prove the second claim, for any $\psi(\ell)$ with $\ell \in L(D_R)$, consider a sequence of loops piecewise geodesic loops, included in an open disc, converging towards $\psi(\ell)$ for (2). The claim then follows adapting the uniform continuity estimate\(^{21}\) (11) of [6, Sect. 4].

4.2 Simple non-contractible loops
Consider a simple non-contractible loop $\ell$ on a compact surface $\Sigma$ of genus $g \geq 1$. Two possibilities occur when cutting $\Sigma$ along $\ell$. If the surface is cut into two surfaces with exactly one connected boundary component, each with genus at least $1$, then the loop is separating. Otherwise, the new surface has one connected component and two boundary components, and the loop is called non-separating. Both cases are illustrated in Fig. 5 below. We refer to [55] for details on these loops. It is now important to understand how the Yang–Mills measure interacts with surgery: we therefore begin with a few results that will help us in the next section.

\(^{21}\)The latter estimate is presented in [6] for the plane; we claim it applies to any compact surface. See [12, Sect. 5] where this argument is detailed for the sphere.

\(^{22}\)Indeed, if one of the boundary components has genus 0, it implies that $\ell$ is contractible.
4.2.1 Separating loops

Consider first two compact connected orientable surfaces $\Sigma_1$ and $\Sigma_2$ such that $\partial \Sigma_1$ and $\partial \Sigma_2$ are connected. Denote by $L_1$ and $L_2$ the corresponding loops, oriented positively. Let $\psi : L_1 \to L_2$ be an orientation-reversing diffeomorphism, and $\Sigma$ be the gluing of $\Sigma_1$ and $\Sigma_2$ along $\psi$. Let $\ell$ be the corresponding simple loop in the surface $\Sigma$. For $i \in \{1, 2\}$, denote by $\tilde{J}_i$ the sigma-field generated by $(H_{\ell_1}, \ldots, H_{\ell_n}, n \geq 1, \ell_i \in L(\Sigma_i))$. For any $i$, a function $f : \mathcal{M}(P(\Sigma_i), G_N) \to \mathbb{C}$ is $\tilde{J}_i$-measurable if and only if $f \circ R_{\Sigma_i} : \mathcal{M}(P(\Sigma_i), G_N) \to \mathbb{C}$ is $J_i$-measurable. The next theorem is a particular case of [32, Thm. 5.1.1]; a different version is given in [50, Thm. 1] (and informally in (1)).

**Theorem 4.2.** The sigma-fields $\tilde{J}_1$ and $\tilde{J}_2$ are independent on $\mathcal{M}(P(\Sigma), G_N)$ under $\text{YM}_{\Sigma, G_N}$ conditionally on the random variable $[H_\ell]$. Moreover, for any $\tilde{J}_i$-measurable function $f_i : \mathcal{M}(P(\Sigma_i), G_N) \to \mathbb{C}$, for $i \in \{1, 2\}$, the product $f_1 f_2$ is measurable with respect to $J$ and the following equality holds true for any $t \in G_N/\text{Ad}$:

$$
\int f_1(h) f_2(h) \text{YM}_{\Sigma, G_N}(dh) = \int f_1 \circ R_{\Sigma_1}(h) \text{YM}_{\Sigma_1, t, G_N}(dh) \int f_2 \circ R_{\Sigma_2}(h) \text{YM}_{\Sigma_2, t^{-1}, G_N}(dh).
$$

(43)

This theorem provides a sort of spatial Markov property, which is a consequence of the semigroup property of the heat kernel. It has the following application.

**Corollary 4.3.** Let $\ell$ be a separating loop in a closed, area-weighted map $(M, a)$ of genus $g \geq 2$, splitting $M$ into two respective maps $M_1$ and $M_2$ of genus $g_1, g_2$ and total area $T_1$ and $T_2$. Let $f : \mathcal{M}(P(M), G_N) \to \mathbb{C}$ be a bounded $\tilde{J}_1$-measurable function. We have

$$
\int_{\mathcal{M}(P(M), G_N)} f(h) \text{YM}_{M, a, G_N}(dh) = \int_{\mathcal{M}(P(M_1), G_N)} f \circ R^M_{M_1}(h) I(h) \text{YM}_{M_1, a, G_N}(dh).
$$

(44)

\[^{23}\] applied to non-trivial bundles
Finally, using the disintegration formula (5) leads to the expected equality.

Proof. To lighten the notation, we will drop in this proof the subscripts $G_N$, as the structure group remains fixed. First of all, using (7), the left-hand-side of (44) equals

$$
\frac{1}{Z_{g,T}} \int_{G/ad} \int_{\mathcal{M}(P(M), G_N)} f(h) Z_{g,T}(\ell; t) YM_{M, t^{-1}, a}(dh) dt.
$$

Let us compute $Z_{g,T}(\ell; t)$. Using the invariance property of Yang–Mills measure by subdivision, one can assume without loss of generality that that for $i \in \{1, 2\}$, $M_i$ has 1 vertex $v^{(i)}$, $2g_i + 1$ edges $a_1^{(i)}, b_1^{(i)}, \ldots, a_g^{(i)}, b_g^{(i)}, e^{(i)}$ all with $v^{(i)}$ as source and target, and 1 face whose boundary is given by $\epsilon_{1}^{(i)}[a_1^{(i)}, b_1^{(i)}], \ldots, [a_g^{(i)}, b_g^{(i)}]$ where $\epsilon_1 = 1$ and $\epsilon_2 = -1$. The map $M$ can then be defined as the quotient of $M_1 \cup M_2$ by the relations $v^{(1)} \sim v^{(2)}$ and $(e^{(1)})^{-1} \sim e^{(2)}$. We obtain

$$
Z_{g,T}(\ell; t) = \int_{\epsilon_{g+1}^{(1,2)}} p_{T_1}(x[y_1, z_1] \cdots [y_{g_1}, z_{g_1}])
\times p_{T_2}(x^{-1}[y_1', z_1'] \cdots [y_{g_2}', z_{g_2}']) dx \prod_{1 \leq i \leq g_1, 1 \leq j \leq g_2} dy_i dz_i dy_j' dz_j'.
$$

From this result and from Theorem 4.2 we deduce

$$
\int_{\mathcal{M}(P(M), G_N)} f(h) YM_{M, a}(dh)
= \frac{1}{Z_{g,T}} \int_{G/ad} \int_{\mathcal{M}(P(M), G_N)} f \circ \mathcal{R}_M^{M_1}(h) Z_{(g_1, 1), T_1}(h) YM_{M_1, t^{-1}, a}(dh)
\times \int_{\mathcal{M}(P(M_2), G_N)} Z_{(g_2, 1), T_2}(t^{-1}) YM_{M_2, t^{-1}, a}(dh) dt.
$$

The integral over $\mathcal{M}(P(M_2), G_N)$ is simply equal to $Z_{(g_2, 1), T_2}(t^{-1})$; furthermore, under $YM_{M_1, t^{-1}, a}$, $[H_{\ell}] = t$, so that we have

$$
Z_{g,T} \int_{\mathcal{M}(P(M), G_N)} f(h) YM_{M, a}(dh)
= \int_{G/ad} \int_{\mathcal{M}(P(M), G_N)} f \circ \mathcal{R}_M^{M_1}(h) Z_{(g_1, 1), T_1}(h) Z_{(g_2, 1), T_2}(h^{-1}) YM_{M_1, t^{-1}, a}(dh) dt.
$$

Finally, using the disintegration formula (5) leads to the expected equality. \qed

To prove Theorem 2.17, we shall need the following similar but simpler lemma.

**Lemma 4.4.** Let $\ell$ be a separating loop in a closed, area-weighted map $(M, a)$ of genus $g \geq 2$, splitting $M$ into two respective maps $M_1$ and $M_2$ of genus $g_1, g_2$ and total area $T_1$ and $T_2$. Under $YM_{M, a, G_N}$, the random variable $H_\ell$ has density

$$
\frac{Z_{(g_1, 1), T_1, G_N}(h) Z_{(g_2, 1), T_2, G_N}(h^{-1})}{Z_{g, T, G_N}}.
$$

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with respect to the Haar measure on $G_N$.

Proof. Let us use the same $M, M_1$ and $M_2$ as in the previous proof. For any central function $f : G_N \to \mathbb{C}$ measurable and bounded, we have

$$E_{YM,a,G} [f(H)] = \frac{1}{Z_{g,T,G_N}} \int_{G_N^{g+1}} f(x)p_{T_1}(x[y_1, z_1] \cdots [y_{g_1}, z_{g_1}])$$

$$\times p_{T_2}(x^{-1}[y_1', z_1'] \cdots [y_{g_2}', z_{g_2}'])dx \prod_{1 \leq i \leq g_1 \leq j \leq g_2} dy_idz_idy_jdz_j,$$

thus by integrating over all variables but $x$ we obtain the expected result. 

Proof of Theorem 2.17 for separating, non-contractible, simple loops. Using Lemma 4.4 and the bound (40), we find that under YM$_M$, the density of $h_\ell$ with respect to the Haar measure is uniformly bounded by

$$Z_{g_1, T_1, G_N} Z_{g_2, T_2, G_N} Z_{g, T, G_N}.$$

Since $\ell$ is not contractible, $g_1, g_2 \geq 1$, and according to Theorem 3.1, the above density is uniformly bounded in $N$. Now for a Haar distributed random variable $H$ on $G_N$ belonging to $X_r$, according to [15], whenever $n \neq 0$, $tr(H^n)$ converges in probability towards 0, as $N \to \infty$. The claim now follows by absolute continuity similarly to the proof of Theorem 2.14.

4.2.2 Non-separating loops

Let $\Sigma'$ be a compact connected orientable surface such that $\partial \Sigma'$ has two connected components; denote the corresponding loops, oriented positively, by $L_1$ and $L_2$. Let $\psi : L_1 \to L_2$ be an orientation-reversing diffeomorphism, and $\Sigma$ be the gluing of $\Sigma'$ along $\psi$. Let $\ell$ be the corresponding loop in $\Sigma$. We have the following, which is a particular case of [32, Thm. 5.4.1]. We use the same notation $\tilde{\mathcal{J}}'$ for the sigma-field on $M(P(\Sigma), G)$ generated by the loops in $\Sigma'$.

Theorem 4.5. Let $f : M(P(\Sigma), G) \to \mathbb{C}$ be a $\tilde{\mathcal{J}}'$-measurable function. For any $t \in G/\text{Ad}$, the following equality holds true:

$$\int f(h)YM_{\Sigma, G_{t^{-1}G}}(dh) = \int f \circ R_{\Sigma'}(h)YM_{\Sigma', (t^{-1})G}(dh).$$

(45)

The consequence in terms of absolute continuity is the following.

Corollary 4.6. Let $\ell$ be a non-separating simple loop in a closed, area-weighted map $(M, a)$ of genus $g \geq 1$. Under YM$_{M, a, G}$, the random variable $H_\ell$ has density given by

$$\frac{\varphi_{g,T}(h)}{Z_{g, [a], G}}, \forall h \in G$$

with respect to the Haar measure, and for all $g \geq 1, T > 0$,

$$\varphi_{g,T}(h) = \sum_{\lambda \in \hat{G}} d_{\lambda}^{2g-2g} |\chi_\lambda(h)|^2 e^{-T \xi_{\lambda}}, \forall h \in G.$$

(46)
Remark. The above density can also be written as $Z_{(g,2),|a|}(h, h^{-1})$.

Proof. According to [55, Sect. 6.3.1], or [22, Sect. 1.3.1], since $\ell$ is not separating, we can choose $M$ with 1 vertex, 1 face and $2g$ edges, one of which is $\ell$, and writing $u = h\ell$,

$$YM_{M,a}(dh) = p_T([u, v][x_2, y_2] \cdots [x_g, y_g])dudv \prod_{i=2}^{g} dx_idy_i.$$  \hfill (47)

It follows that the law of $h\ell$ under $YM_{M,a}$ has density

$$\varphi_{g,T}(u) = Z_{g,T,N}^{-1} \int_{G^{g-1}} p_T([u, v][x_2, y_2] \cdots [x_g, y_g])dv \prod_{i=2}^{g} dx_idy_i, \forall u \in G,$$

with respect to the Haar measure. Expanding in characters and using (27) yields

$$\varphi_{g,T}(h) = \sum_{\lambda \in \hat{G}} e^{-\frac{T}{2}c_\lambda} I_\lambda(u)$$

with

$$I_\lambda(u) = d_\lambda^{1-2(g-1)} \int_G \chi_\lambda([u,v])dv.$$  

Now (26) yields $I_\lambda(u) = d_\lambda^{2-2g} \chi_\lambda(u)\chi_\lambda(u^{-1}), \forall u \in G$ and the claim. \hfill $\Box$

Proof of Theorem 2.17 for $g \geq 2$ and non-separating, simple loops. Then, using that $\|\chi_\lambda\|_{\infty} = \chi_\lambda(1)$ for all $\lambda \in \hat{G},$

$$\|\varphi_{g,T}\|_{\infty} = \varphi_{g,T}(1) = Z_{g-1,T,G_N}, \forall g \geq 1, T > 0.$$

It follows that under $YM_{M,a}$, the density of $h\ell$ with respect to the Haar measure is bounded by

$$\frac{Z_{g-1,|a|,G_N}}{Z_{g,|a|,G_N}}.$$

When $g \geq 2$, it follows from 3.1 that this sequence is uniformly bounded when $N \to \infty$. As in the argument of the former section for the separating case, the claim follows from [15] and absolute continuity. \hfill $\Box$

When $g = 1$, the above argument of absolute continuity argument fails. Indeed for a total area $T$, the maximum of the density is then given by $Z_{0,T,N} = Z_{1,T,N}$. For $\tilde{A}_N$, thanks to Theorem 3.2, and more precisely formula (28) of [38],

$$\lim_{N \to \infty} \frac{1}{N^2} \log(Z_{0,T,N}) = F(T) = \frac{T}{24} + \frac{3}{4} - \frac{1}{2} \log(T), \forall 0 < T \leq \pi^2.$$  

Since $F(T) \geq F(\pi^2) > 0$ for all $T \in (0, \pi^2]$, $\lim_{N \to \infty} Z_{0,T,N} = +\infty$. Similarly to the strategy of [36, 12, 28], we shall consider the expectation and the variance of the random variable $W_\ell$.

Remark. For any $z$ in the center of $G$, under $YM_{M,a}$, $zh\ell$ has same law as $h\ell$. This can be checked changing variable in formula (47). Alternatively, Schur’s lemma implies that $z$ acts by a unitary scalar in any irreducible representation and (47) implies that for any $\varphi_{g,T}(zh) = \varphi_{g,T}(h)$ for any $h \in G$. Consequently whenever
$D_z \in G$ is a diagonal matrix of multiplication by a scalar $z \in \mathbb{C}$, under $\text{YM}_{M,a}$, \(\text{tr}(h_{\ast n}) = \text{tr}(h_n^0)\) has same law as $z^n \text{tr}(h_n)$. For $X_N = \{A_N, A_N\}$, the center is given by $\{\text{Id}_{N+1} : \zeta^{N+1} = 1\}$ and $\{z \text{Id}_N : z \in \mathbb{C}, |z| = 1\}$. It follows that for $X = \tilde{A}, \mathbb{E}[W_{r s}] = 0$ for any $n \neq 0$, whereas for $X = A, \mathbb{E}[W_{r s}] = 0$ for any $n \neq 0$ modulo $N + 1$. In particular, when $X \in \{A, \tilde{A}\}, \lim_{N \to \infty} \mathbb{E}[W_{r s}] = 0$.

Unfortunately, the argument given in the above remark does not apply when $X = \{B, C, D\}$. Also, it does not give any information about $\text{Var}(W_I) = \mathbb{E}[|W_I|^2] - |\mathbb{E}[W_I]|^2$. Instead, we shall use the expansion in characters of the heat kernel and the following lemma, which is a consequence of the expression of characters as a ratio of alternating functions.

**Lemma 4.7.** Let $G_N \subset \text{GL}_n(\mathbb{C})$ be a classical group of size $N$ and type $X_r$, for $X \in \{B, C, D\}$. For any $k \neq 0$ and $\lambda \in \tilde{G}_N$, \[\text{Tr}(g^k)\chi_\lambda(g) = \sum_{\mu \in G_N} c_{\lambda,k}^g \chi_\mu(g),\] with \[c_{\lambda,k}^g \in \{-1, 0, 1\}, \forall \mu \in \tilde{G}_N\] and \[\sum_{\mu \in G_N} |c_{\lambda,k}^g| \leq n\] \[\text{Proof.}\] Let us introduce a few common notations for the different cases. Set $Z_{\text{sym}} = Z$ if $r$ is odd or $Z + \frac{1}{2}$ if $r$ is even. The mapping \[\lambda \in \tilde{G}_N \mapsto \lambda + \rho\] establishes a bijection between the set of highest weights and $\{\mu \in Z_{\text{sym}} : \mu_1 > \cdots > \mu_r\}$. The symmetric group $\Sigma_r$ acts in an obvious way on \[\Delta_r = \{\mu \in Z_{\text{sym}} : \mu_1 > \cdots > \mu_r\} \cong \{\mu \in Z_{\text{sym}} : \mu_i \neq \mu_j, \forall i \neq j\}/\Sigma_r,\] where the symmetric group $\Sigma_r$ acts on $\mu$ by permutation of its components. For $\mu$ with $\mu_i \neq \mu_j, \forall i \neq j$, we denote by $[\mu]$ its decreasing rearrangement and $\sigma_{\mu} \in \Sigma_r$ the unique permutation such that $[\mu]_{\sigma} = [\mu]_{\sigma(i)}$ for all $i$. We will prove each case separately using the previous notations. The explicit formulae for the characters $\chi_{\lambda}$ that we will use can be found in [5] or in [42, Sect. 2.3]. When $\mu \in Z_{\text{sym}}$ with $\mu_i = \mu_j$ for some $i \neq j$, by convention $\chi_{[\mu]}(\rho) = 0$ and $\varphi(\mu) = 0$. The proof for all cases will rely on the following computation: for any $(z_1, \ldots, z_r) \in (\mathbb{C}^*)^r$, any $(m_1, \ldots, m_r) \in Z_r$ and any $k \in \mathbb{N}^r$, we have \[\sum_{\ell=1}^{r} (z_{\ell}^{m_{\ell}} - z_{\ell}^{m_{\ell}}) \det(z_{\ell}^{m_{\ell}} - z_{\ell}^{m_{\ell}}) = \sum_{\ell=1}^{r} (\det(z_{\ell}^{m_{\ell}} + k \delta_{\ell}) + \det(z_{\ell}^{m_{\ell}} - k \delta_{\ell})).\] \[\text{Let us prove (51). Using the Leibniz formula for determinant and the invariance of } \sum_{\ell} (z_{\ell}^{k} + z_{\ell}^{-k}) \text{ by the action } \ell \mapsto \sigma(\ell) \text{ of the symmetric group, we have} \]
\[\sum_{\ell=1}^{r} (z_{\ell}^{k} + z_{\ell}^{-k}) \det(z_{\ell}^{m_{\ell}} - z_{\ell}^{m_{\ell}}) = \sum_{\sigma \in \Sigma_r} \varphi(\sigma) \sum_{\ell=1}^{r} (z_{\ell}^{k} + z_{\ell}^{-k}) \prod_{i=1}^{r} (z_{\sigma(i)}^{m_{\ell}} - z_{\sigma(i)}^{m_{\ell}}) \]
\[= \sum_{\sigma \in \Sigma_r} \varphi(\sigma) \sum_{\ell=1}^{r} (z_{\sigma(\ell)}^{k} + z_{\sigma(\ell)}^{-k}) \prod_{i=1}^{r} (z_{\sigma(i)}^{m_{\ell}} - z_{\sigma(i)}^{m_{\ell}}).\]
We can then put the sum over $\mathcal{G}_r$ back in by linearity, and for any $\sigma$ and any $\ell$ we have
\[
(z_{\sigma(\ell)}^k + z_{\sigma(\ell)}^{-k}) \prod_{i=1}^{r} (z_{\sigma(i)}^{m_i} - z_{\sigma(i)}^{-m_i}) = (z_{\sigma(\ell)}^{m_\ell + k} - z_{\sigma(\ell)}^{-m_\ell - k}) \prod_{i \neq \ell} (z_{\sigma(i)}^{m_i} - z_{\sigma(i)}^{-m_i}).
\]
Summing over $\ell$ leads to (51). We can now prove (48) for each type of group.

Case $B_r$: let $g \in \text{SO}(2r+1)$ be an element with eigenvalues $(z_{1}^{\pm 1}, \ldots, z_{r}^{\pm 1}, 1)$. We have
\[
\chi_\lambda(g) = \frac{\det(z_{i}^{\lambda_j + \rho_j} - z_{i}^{-\lambda_j - \rho_j})}{\det(z_{i}^{\rho_j} - z_{i}^{-\rho_j})}.
\]
If we substitute $\mu = \lambda + \rho \in \Delta_r$, we can write
\[
\chi_\lambda(g) = \chi_{\mu - \rho}(g) = \frac{\det(z_{i}^{\mu_j} - z_{i}^{-\mu_j})}{\det(z_{i}^{\rho_j} - z_{i}^{-\rho_j})},
\]
so that
\[
\text{Tr}(g^k) \chi_\lambda(g) = \left(1 + \sum_{\ell=1}^{r} (z_{i}^{\ell} + z_{i}^{-\ell}) \right) \frac{\det(z_{i}^{\mu_j} - z_{i}^{-\mu_j})}{\det(z_{i}^{\rho_j} - z_{i}^{-\rho_j})},
\]
and by (51) we obtain that
\[
\text{Tr}(g^k) \chi_\lambda(g) = \chi_\lambda(g) + \sum_{\ell=1}^{r} \left( \epsilon(\sigma_{\mu_{k,\ell}}^+) \chi_{[\mu_{k,\ell}] - \rho}(g) + \epsilon(\sigma_{\mu_{k,\ell}}^-) \chi_{[\mu_{k,\ell}] - \rho}(g) \right),
\]
which yields (48) with $c_{\lambda,k}^\mu = 1$, $c_{[\mu_{k,\ell}] - \rho} = \epsilon(\sigma_{[\mu_{k,\ell}]})$ for all $1 \leq \ell \leq r$ and $c_{\lambda,k}^\mu = 0$ for any other $\mu \in \hat{G}_N$, which also proves (49). Indeed, each $[\mu_{k,\ell}] - \rho$ corresponds to an element of $\hat{G}_N$ if and only if $[\mu_{k,\ell}] \in \Delta_r$, and if it is not the case, then $\chi_{[\mu_{k,\ell}]}$ and summand in the right-hand-side of (52) vanish. Finally, there are $2r + 1$ nonzero coefficients, whose absolute value is always equal to 1, hence (50) is also satisfied.

Case $C_r$: let $g \in \text{Sp}(r)$ be an element with eigenvalues $(z_{1}^{\pm 1}, \ldots, z_{r}^{\pm 1})$. We have again
\[
\chi_\lambda(g) = \chi_{\mu - \rho}(g) = \frac{\det(z_{i}^{\mu_j} - z_{i}^{-\mu_j})}{\det(z_{i}^{\rho_j} - z_{i}^{-\rho_j})},
\]
It follows that
\[
\text{Tr}(g^k) \chi_\lambda(g) = \sum_{\ell=1}^{r} \left( \epsilon(\sigma_{[\mu_{k,\ell}]^+}) \chi_{[\mu_{k,\ell}]} - \rho(g) + \epsilon(\sigma_{[\mu_{k,\ell}]}^-) \chi_{[\mu_{k,\ell}]} - \rho(g) \right),
\]
as expected.
Case $D_r$: let $g \in \text{SO}(2r)$ be an element with eigenvalues $(z_i^{\pm 1}, \ldots, z_r^{\pm 1})$. We have

$$\chi_\lambda(g) = x_{\mu-\rho}(g) = \frac{\det(z_i^{\mu_j} - z_i^{-\mu_j}) + \det(z_i^{\mu_j} + z_i^{-\mu_j})}{\det(z_i^{\rho_j} + z_i^{-\rho_j})},$$

therefore

$$\text{Tr}(g^k) \chi_\lambda(g) = \left( \sum_{\ell=1}^r (z_i^{k} + z_i^{-k}) \right) \frac{\det(z_i^{\mu_j} - z_i^{-\mu_j}) + \det(z_i^{\mu_j} + z_i^{-\mu_j})}{\det(z_i^{\rho_j} + z_i^{-\rho_j})} = \sum_{\ell=1}^r \left( \varepsilon(\sigma_{\mu_{\ell,j}}) \chi_{\mu_{\ell,j}}(g) + \varepsilon(\sigma_{\mu_{\ell,j}}) \chi_{\mu_{\ell,j}}(g) \right).$$

Proof of Theorem 2.17 for $g = 1$ and non-separating, simple loops. Let us set $T = |a|$. We shall compute the expectation and the variance of $W_T$ under $YM_{M,a,G_N}$, and show that both converge to zero as $N \to \infty$. Assume that $G_N$ is of type $X_r$ and a subgroup of $\text{GL}_N(\mathbb{C})$ as in section 2.1.2. Using Corollary 4.4 and expanding in characters yield

$$n \mathbb{E}[W_{\rho}] = Z_{1,T,X_r}^{-1} \int_{G_N} \varphi_{T,1}(h) \text{Tr}(h^k) dh = Z_{1,T,X_r}^{-1} \sum_{\lambda \in \widehat{G}_N} e^{-\frac{k}{n} c_{\lambda}} I_{\lambda},$$

where for all $\lambda \in \widehat{G}_N$, using the orthogonality of characters,

$$I_{\lambda} = \int_{G_N} \text{Tr}(h^k) \chi_\lambda(h) \chi_\lambda(h^{-1}) dh = \sum_{\mu \in \widehat{G}_N} c_{\lambda,k} \int_{G_N} \chi_\mu(h) \chi_\lambda(h^{-1}) dh = c_{\lambda,k}^\mu.$$

Thanks to (49), it follows that

$$|\mathbb{E}[W_{\rho}]| \leq (nZ_{1,T,X_r})^{-1} \sum_{\lambda \in \widehat{G}_N} e^{-\frac{k}{n} c_{\lambda}} = \frac{1}{n}.$$

Similarly,

$$n^2 \mathbb{E}(|W_{\rho}|^2) = Z_{1,T,X_r}^{-1} \int_{G_N} |\text{Tr}(h^k)|^2 \varphi_{T,1}(h) dh = Z_{1,T,X_r}^{-1} \sum_{\lambda \in \widehat{G}_N} e^{-\frac{k}{n} c_{\lambda}} J_{\lambda},$$

where

$$J_{\lambda} = \int_{G_N} \text{Tr}(h^k) \chi_\lambda(h) \text{Tr}(h^{-k}) \chi_\lambda(h^{-1}) dh = \sum_{\mu,\nu} c_{\lambda,k}^\mu c_{\lambda,k}^\nu \int_{G_N} \chi_\mu(h) \chi_\nu(h^{-1}) dh = \sum_{\mu \in \widehat{G}_N} (c_{\lambda,k}^\mu)^2.$$

\(^{24}\)so that \(n\) is respectively \(r, r + 1, 2r + 1\) for types $\tilde{A}_r$, $A_r$, $B_r$, and $2r$ for types $\{C_r, D_r\}$. 39
Using Lemma 4.7, we conclude that
\[ 0 \leq J_\lambda = \sum_{\mu \in \hat{G}_N} |c_{\lambda,k}^\mu| \leq n \]
and
\[ \mathbb{E}[|W_{\ell k}|^2] \leq n^{-2} Z^{-1}_{1,T,N} \sum_{\lambda \in \hat{G}_N} e^{-\frac{n}{2} c_{\lambda,k} J_\lambda} = \frac{1}{n}. \quad (54) \]

From (53) and (54), we conclude that the expectation and variance of \( W_{\ell k} \) under \( \text{YM}_{M,a,G_N} \) both vanish as \( N \to \infty \). Therefore, \( W_{\ell k} \) converges to 0 in probability. \( \square \)

Remark. For the \( A_r \) type, for all \( k \neq 0 \) and \( \lambda \in \Lambda_r \), \( c_{\lambda,k}^\lambda = 0 \), and the first part of the above proof yields another argument for \( \mathbb{E}[W_{\ell k}] = 0 \) for all \( r \geq 1 \).

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References

[1] Greg W. Anderson, Alice Guionnet, and Ofer Zeitouni. *An introduction to random matrices*, volume 118 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010.

[2] Michael Anshelevich and Ambar N. Sengupta. Quantum free Yang–Mills on the plane. *J. Geom. Phys.*, 62(2):330–343, 2012.

[3] Philippe Biane. Free Brownian motion, free stochastic calculus and random matrices. In *Free probability theory (Waterloo, ON, 1995)*, volume 12 of *Fields Inst. Commun.*, pages 1–19. Amer. Math. Soc., Providence, RI, 1997.

[4] A. Boutet de Monvel and M. V. Shcherbina. On free energy in two-dimensional \( U(n) \)-gauge field theory on the sphere. *Teoret. Mat. Fiz.*, 115(3):389–401, 1998.

[5] Theodor Bröcker and Tammo tom Dieck. *Representations of compact Lie groups*, volume 98 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[6] Guillaume Cébron, Antoine Dahlqvist, and Franck Gabriel. The generalized master fields. *J. Geom. Phys.*, 119:34–53, 2017.

[7] Sourav Chatterjee. Rigorous solution of strongly coupled so(n) lattice gauge theory in the large n limit. *Communications in Mathematical Physics*, 366(1):203–268, 2019.
[8] Ilya Chevyrev. Yang-Mills measure on the two-dimensional torus as a random distribution. *Comm. Math. Phys.*, 372(3):1027–1058, 2019.

[9] Antoine Dahlqvist. Free energies and fluctuations for the unitary Brownian motion. *Comm. Math. Phys.*, 348(2):395–444, 2016.

[10] Antoine Dahlqvist. Integration formulas for Brownian motion on classical compact Lie groups. *Ann. Inst. Henri Poincaré Probab. Stat.*, 53(4):1971–1990, 2017.

[11] Antoine Dahlqvist and Thibaut Lemoine. Large N limit of the yang-mills measure on compact surfaces II: Makeenko-migdal equations and planar master field. 2021.

[12] Antoine Dahlqvist and James R. Norris. Yang–mills measure and the master field on the sphere. *Communications in Mathematical Physics*, 377(2):1163–1226, 2020.

[13] Jean-Marc Daul and Vladimir A. Kazakov. Wilson loop for large N Yang-Mills theory on a two-dimensional sphere. *Phys. Lett. B*, 335(3-4):371–376, 1994.

[14] A. B. de Monvel and M. V. Shcherbina. Free energy of the two-dimensional $u(n)$-gauge field theory on the sphere. *Theoretical and Mathematical Physics*, 115(3):670–679, 1998.

[15] Persi Diaconis and Steven N. Evans. Linear functionals of eigenvalues of random matrices. *Trans. Amer. Math. Soc.*, 353(7):2615–2633, 2001.

[16] M. R. Douglas and V. A. Kazakov. Large N Phase Transition in continuum QCD$_2$. *Physics Letters B*, 319, 1993.

[17] Michael R. Douglas. Conformal field theory techniques in large N Yang-Mills theory. In *Quantum field theory and string theory (Cargèse, 1993)*, volume 328 of *NATO Adv. Sci. Inst. Ser. B Phys.*, pages 119–135. Plenum, New York, 1995.

[18] Bruce K. Driver, YM$_2$: continuum expectations, lattice convergence, and lassos. *Comm. Math. Phys.*, 123(4):575–616, 1989.

[19] Bruce K. Driver, Franck Gabriel, Brian C. Hall, and Todd Kemp. The Makeenko-Migdal equation for Yang-Mills theory on compact surfaces. *Comm. Math. Phys.*, 352(3):967–978, 2017.

[20] Bruce K. Driver, Brian C. Hall, and Todd Kemp. Three proofs of the Makeenko-Migdal equation for Yang-Mills theory on the plane. *Comm. Math. Phys.*, 351(2):741–774, 2017.

[21] Jacques Faraut. *Analysis on Lie groups*, volume 110 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2008.

[22] Benson Farb and Dan Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.

[23] Rajesh Gopakumar. The master field in generalised QCD$_2$. *Nuclear Phys. B*, 471(1-2):246–260, 1996.

[24] David J. Gross and Andrei Matytsin. Some properties of large-n two-dimensional yang-mills theory. *Nuclear Physics B*, 437(3):541–584, 1995.

[25] Leonard Gross. The Maxwell equations for Yang-Mills theory. In *Mathematical quantum field theory and related topics (Montreal, PQ, 1987)*, volume 9 of *CMS Conf. Proc.*, pages 193–203. Amer. Math. Soc., Providence, RI, 1988.
[26] Leonard Gross, Christopher King, and Ambar Sengupta. Two-dimensional Yang-Mills theory via stochastic differential equations. *Ann. Physics*, 194(1):65–112, 1989.

[27] Robert Gurahnick, Michael Larsen, and Corey Manack. Low degree representations of simple lie groups. *Proceedings of the American Mathematical Society*, 140(5):1823–1834, 2012.

[28] Brian C. Hall. The large-$N$ limit for two-dimensional Yang-Mills theory. *Comm. Math. Phys.*, 363(3):789–828, 2018.

[29] V. A. Kazakov and I. K. Kostov. Nonlinear strings in two-dimensional $U(\infty)$ gauge theory. *Nuclear Phys. B*, 176(1):199–215, 1980.

[30] Sergei K. Lando and Alexander K. Zvonkin. *Graphs on surfaces and their applications*, volume 141 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2004.

[31] Thibaut Lemoine. Large $N$ behaviour of the two-dimensional Yang–Mills partition function. *Combinatorics, Probability and Computing*, pages 1–22, 2021.

[32] Thierry Lévy. Yang-Mills measure on compact surfaces. *Mem. Amer. Math. Soc.*, 166(790):xiv+122, 2003.

[33] Thierry Lévy. Wilson loops in the light of spin networks. *J. Geom. Phys.*, 52(4):382–397, 2004.

[34] Thierry Lévy. Schur-Weyl duality and the heat kernel measure on the unitary group. *Adv. Math.*, 218(2):537–575, 2008.

[35] Thierry Lévy. Two-dimensional Markovian holonomy fields. *Astérisque*, (329):172, 2010.

[36] Thierry Lévy. The master field on the plane. *Astérisque*, (388):ix+201, 2017.

[37] Thierry Lévy. *Two-Dimensional Quantum Yang–Mills Theory and the Makeenko–Migdal Equations*, pages 275–325. Springer International Publishing, Cham, 2020.

[38] Thierry Lévy and Mylène Maïda. On the Douglas-Kazakov phase transition. Weighted potential theory under constraint for probabilists. In *Modélisation Aléatoire et Statistique—Journées MAS 2014*, volume 51 of *ESAIM Proc. Surveys*, pages 89–121. EDP Sci., Les Ulis, 2015.

[39] Thierry Lévy and Ambar Sengupta. Four chapters on low-dimensional gauge theories. In *Stochastic geometric mechanics*, volume 202 of *Springer Proc. Math. Stat.*, pages 115–167. Springer, Cham, 2017.

[40] Ming Liao. *Lévy processes in Lie groups*, volume 162 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2004.

[41] Yuri Makeenko and Alexander A. Migdal. Exact equation for the loop average in multicolor QCD. *Physics Letters B*, 88:135–137, 1979.

[42] Pierre-Loïc Méliot. The cut-off phenomenon for Brownian motions on compact symmetric spaces. *Potential Anal.*, 40(4):427–509, 2014.

[43] P. Menotti and E. Onofri. The action of $su(n)$ lattice gauge theory in terms of the heat kernel on the group manifold. *Nuclear Physics B*, 190(2):288–300, 1981.
[44] Alexander A. Migdal. Recursion equations in gauge field theories. *Sov. Phys. JETP*, pages 413–418, 1975.

[45] K. Osterwalder and E. Seiler. *Lattice gauge theories*, pages 26–36. Springer Berlin Heidelberg, Berlin, Heidelberg, 1978.

[46] Boris Rusakov. Large-N quantum gauge theories in two dimensions. *Physics Letters B*, 303(1):95–98, 1993.

[47] Ambar Sengupta. The Yang-Mills measure for $S^2$. *J. Funct. Anal.*, 108(2):231–273, 1992.

[48] Ambar Sengupta. Gauge invariant functions of connections. *Proc. Amer. Math. Soc.*, 121(3):897–905, 1994.

[49] Ambar Sengupta. Gauge theory on compact surfaces. *Mem. Amer. Math. Soc.*, 126(600):viii+85, 1997.

[50] Ambar N. Sengupta. Sewing Yang-Mills measures for non-trivial bundles. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 6(suppl.):39–52, 2003.

[51] Ambar N. Sengupta. Gauge theory in two dimensions: topological, geometric and probabilistic aspects. In *Stochastic analysis in mathematical physics*, pages 109–129. World Sci. Publ., Hackensack, NJ, 2008.

[52] Ambar N. Sengupta. Traces in two-dimensional QCD: the large-N limit. In *Traces in number theory, geometry and quantum fields*, Aspects Math., E38, pages 193–212. Friedr. Vieweg, Wiesbaden, 2008.

[53] Ambar N. Sengupta. *Representing finite groups*. Springer, New York, 2012. A semisimple introduction.

[54] Isadore M. Singer. On the master field in two dimensions. In *Functional analysis on the eve of the 21st century, Vol. 1 (New Brunswick, NJ, 1993)*, volume 131 of *Progr. Math.*, pages 263–281. Birkhäuser Boston, Boston, MA, 1995.

[55] John Stillwell. *Classical topology and combinatorial group theory*, volume 72 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1993.

[56] Gerard ’t Hooft. A planar diagram theory for strong interactions. *Nuclear Physics B*, 72(3):461 – 473, 1974.

[57] Edward Witten. On quantum gauge theories in two dimensions. *Comm. Math. Phys.*, 141(1):153–209, 1991.

[58] Feng Xu. A random matrix model from two-dimensional Yang-Mills theory. *Comm. Math. Phys.*, 190(2):287–307, 1997.

[59] Don Zagier. Values of zeta functions and their applications. In *First European Congress of Mathematics, Vol. II (Paris, 1992)*, volume 120 of *Progr. Math.*, pages 497–512. Birkhäuser, Basel, 1994.