Local Hydraulic Resistance in Heterogeneous Porous Media

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Abstract We examine the validity of the commonly used Hagen-Poiseuille model of local resistance of porous media using direct numerical simulations. We provide theoretical arguments that highlight possible limitations of this model and formulate a new constitutive model that is based on the circularity of iso-pressure surfaces. We compare the performance of both models on three different three-dimensional artificial porous media. We show that the new model improves the root-mean-squared-relative error from 59%, 48% and 32% for the HP model to 12%, 14% and 18% for the three porous media respectively. We anticipate that our approach may find broad application in network models of porous media that are typically build from 3D images with intricate pore geometries.

Plain Language Summary The flow of liquids through porous materials is an everyday problem with many applications ranging from groundwater flow, packed bed reactors, filter devices to blood flow through kidneys. The classical flow modeling approach has been to represent the porous medium as a lattice of circular tubes that represent the pore network and the hydraulic resistance in individual tubes is modeled assuming uniform flow with a parabolic velocity profile, called "Hagen-Poiseuille" law. In this paper we show that this approach has limitations and provide a refined model for local hydraulic resistance based on the pore geometry of complex three-dimensional porous media. In particular, we show that the circularity, a parameter that quantifies the "roundness" of a surface of equal pressure traversing a pore cross section, is key to predict the local hydraulic resistance.

1. Introduction

Porous media flow is important for a wide range of applications in nature and technology, spanning from groundwater remediation and oil recovery to packed bed reactors and particle filters. In these flows, the highly complex and three-dimensional pore geometries give rise to complicated pore velocity fields that form the backbone for transport, mixing and chemical reaction processes. Detailed knowledge of these velocity fields is important for the modeling of effective parameters, most notably the permeability and the prediction of transport in porous media (Bear, 1972; Scheidegger, 1974). Despite its importance and extensive research, the relation between geometrical features of porous media and the resulting flow is still not fully understood.

Given that detailed knowledge of geometrical features of porous media is often unavailable, the classical flow modeling approach has been to represent the porous medium as a lattice of circular tubes that represent the pore network (Scheidegger, 1974). The flow in the tubes is assumed to be uniform along the direction of the flow and the velocity profile parabolic. While this is a rather crude approximation of the real geometry and flow behaviors, it has provided useful predictions for flow and transport. Early studies have modeled velocity distributions (Haring & Greenkorn, 1970), permeability (Fatt, 1956; Katz & Thompson, 1986) and particle dispersion (Saffman, 1959) based on bulk statistics of the medium geometry such as pore size distributions. These early studies have spurred many subsequent works on statistical pore scale models (e.g., de Anna et al., 2017; Dentz et al., 2018; Dullien, 1975; Kutovsky et al., 1996 & Maier et al., 1999). The second class of models that hinges on the simplified lattice representation of porous media are the so-called pore network models (Thompson & Fogler, 1997). For both classes of models, the simplified modeling of local hydraulic resistance of individual pores based on Hagen-Poiseuille is a central element.

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Many authors have tried to relate the statistics of pore velocity to statistics of pore geometry represented by e.g., the local pore radius and the connectivity between pores. For example, one of the simplest models is the so-called capillary bundle model, in which the porous medium is conceptualized as a parallel arrangement of capillaries with given pore sizes (Scheidegger, 1974). Extensions of this model include parallel arrangements of wavy tubes (Le Borgne et al., 2011). These simple models are not appropriate for complex porous media, for which the network aspect is important. That is, in general, the connectivity between pores cannot be neglected and the concept of the linear pore breaks down (Dentz et al., 2018). An ad-hoc model that conceptualizes flow in porous media as a system of serial and parallel pore arrangements can be found in Holzner et al. (2015), and the resulting dispersion of tracers was predicted by Fouixon and Holzner (2016). Hyman et al. (2012) and Siena et al. (2014) statistically related velocity distributions to pore size distributions of statistically generated 3D porous media. Based on direct numerical simulations in 2D porous media composed of disks, de Anna et al. (2017) showed that the low velocity tail of the pore velocity distribution is governed by local pore size. This is a notable result as it suggests that the slow flow velocities are not strongly dependent on the connectivity between pores. Alim et al. (2017) showed that pore velocity distributions are governed by local correlations of pore sizes that organize flux ratios at pore junctions, while pore size itself was a poor predictor of flux ratios. They simplified two-dimensional porous medium flow by a network of tubes with varying diameter and the flow within each tube was calculated by solving for Kirchhoff’s circuit law for two-dimensional Poiseuille flow within the tubes of rectangular cross section. Even though using tubes with varying diameter is a refinement compared to simpler models with tubes of constant diameter, the simplification with respect to real geometries is still strong. Despite this, a comparison of simulation results with the experimentally obtained velocity distribution in a two-dimensional micromodel composed of pillars showed reasonable agreement (Alim et al., 2017). As mentioned, the statistical models in these works are based on the concept that local velocity profiles are parabolic. Some recent papers provided qualitative examples comparing pore velocity profiles to a parabola that suggest the assumption may be reasonable (de Anna et al., 2017; Dentz et al., 2018) and some evidence is provided by the comparison between simulations and experiments by Alim et al. (2017). However, a rigorous assessment of local hydraulic resistance in three-dimensional pore geometries is still missing in the literature.

With the advent of experimental techniques, like micro-computed tomography, there is now access to impressive details of three dimensional porous media architectures. These can be used in conjunction to numerical methods such as direct numerical simulations (DNS) (Weller et al., 1998) and Lattice-Boltzmann methods (LBM) (Boek & Ventuoli, 2010), which solve the Navier-Stokes equations. The obtained flow field enables to compare for example, local velocities to shear rates (Berg & van Wunnik, 2017). Under certain conditions, multiphase flow LBM methods can be significantly faster than DNS, but for complex geometries at low Reynolds number, DNS can be faster or comparable due to local grid refinement compared to high computer memory requirements for the LBM method (Hoef et al., 2005). Also modern approaches such as deep-learning methods (Kamrava et al., 2021) show promising results. Even though today’s computing facilities make it possible to solve flow and transport with unprecedented accuracy in these complex geometries using numerical simulations, this approach is only feasible in small domains. Simple models that reduce the full complexity of real porous media are still needed. A lattice representation is the standard approach in so-called pore network models in which a Kirchhoff-type system of equations is solved to model single or multiphase porous medium flows (Thompson & Fogler, 1997) especially in absence of a detailed microstructure. The lattice is usually constructed based on data from experimental pore scale characterization measurements, for example, imaging or mercury intrusion porosimetry. These pore network models are a valuable tool for understanding meso-scale phenomena, linking single pore processes and continuum porous media used in engineering (Xiong et al., 2016). Besides a sound network construction approach that mimics the real media, another critical aspect for the accuracy of modeling is the representation of local hydraulic resistance in the pores.

In this paper we start with a theoretical background where we define local pores based on consecutive iso-pressure surfaces, followed by a new model for the local hydraulic conductivity. In the methods section we describe our numerical experiment consisting of direct numerical simulations (DNS) from which we obtain local velocity and pressure data in heterogeneous porous media. Based on a post-processing of the DNS data we can extract the local hydraulic conductivity of a local pore. In the results we compare the measured hydraulic resistances to the Hagen-Poiseuille model and to the newly formulated model based a local shape
parameter circularity. In the discussion we treat the limitations and extrapolations of the newly proposed model. In the final chapter we provide our conclusions.

2. Theoretical Background

For low Reynolds numbers and incompressible flow, the local flow in a porous media is described by the Stokes equations,

\[ \nabla p = \mu \nabla^2 \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0 \]  (1)

with pressure \( p \) and velocity \( \mathbf{u} \) and dynamic viscosity \( \mu \). For an arbitrary volume \( \mathcal{V} \) in a porous media, enclosed by surface \( \partial \mathcal{V} \) given by two iso-pressure surfaces \( S_{p_1} \) and \( S_{p_2} \) and solid-liquid surface boundary \( \Gamma \), we can write down the integral form of the Stokes equations using the divergence theorem.

\[ \int_{\mathcal{V}} (\mathbf{u} \cdot \nabla p - \mu \nabla^2 \mathbf{u}) \, dV = \int_{\partial \mathcal{V}} \mathbf{u} \cdot \mathbf{n} \, dS - \mu \int_{\partial \mathcal{V}} (\nabla \otimes \mathbf{u}) \mathbf{n} \, dS + \mu \int_{\mathcal{V}} (\nabla \otimes \mathbf{u})^2 \, dV = 0, \]  (2)

with \( \mathbf{n} \) the normal vector pointing outwards of surface \( \partial \mathcal{V} \), and \( \otimes \) the dyadic product. Given that at the porous media boundary domain \( \Gamma \) we have a no-slip condition \( \mathbf{u} = 0 \) we can write

\[ Q \Delta p = -\mu \int_{S_{p_1} + S_{p_2}} \mathbf{u} \cdot (\nabla \otimes \mathbf{u}) \mathbf{n} \, dS + \mu \int_{\mathcal{V}} (\nabla \otimes \mathbf{u})^2 \, dV, \]  (3)

with the volumetric flow rate through any cross section defined by

\[ Q = \int \mathbf{u} \cdot \mathbf{n} \, dS. \]  (4)

Here we introduce the notion of disconnected iso-pressure surface \( S_i(p) \) for a given pressure value \( p \). Iso-pressure surfaces are usually disconnected because they exist in the fluid domain only and are thus interrupted by the solid phase of the media. The first term of Equation 2, the boundary term, will be less significant when the total volume \( \mathcal{V} \) is enlarged by increasing \( \delta p \). Furthermore, when we have saturated conditions, the complete pore space can be compartmentalized in a network of enclosed volumes \( V(p, p + \delta p) \), which we will call pores. We have assessed the relevance of the boundary term to \( Q \Delta p \) for ten pores in the Supporting Information S1. We found that they contribute generally below 5\% for the shortest available pores to below 1\% for average size pores. Therefore it is reasonable to estimate Equation 3 by

\[ Q \Delta p \approx \mu \int (\nabla \otimes \mathbf{u})^2 \, dV. \]  (5)

In the following we apply a decomposition of the velocity vector \( \mathbf{u} = u_p \mathbf{\hat{p}} + u_r \mathbf{\hat{r}} \) with \( \mathbf{\hat{p}} = \nabla p / |\nabla p| \) and \( \mathbf{\hat{r}} \) perpendicular to \( \mathbf{\hat{p}} \). We assume that the velocity vector is mostly aligned with the pressure gradient and that the most important contributions to the velocity gradient tensor are given by \( \nabla u_p \), that is, \( \nabla u_r \ll \nabla u_p \). This is to be expected because in Stokesian flow the high dissipation regions are located near the porous media boundary, and there the boundary condition enforces the velocity gradient to be perpendicular to the flow, and the flow aligned with \( \mathbf{\hat{p}} \). Hence we write

\[ |\nabla u_r| \approx |\nabla u_r| + |\nabla u_p|, \]  (6)

Equations 1–6 are valid for arbitrary volumes \( \mathcal{V} \). When we consider viscous dissipation \( \langle |\nabla u_r|^2 \rangle \) in an infinitesimal volume \( \mathcal{d}V \) enclosed by \( S(p), S(p + \delta p) \) (with respective areas \( A(p), A(p + \delta p) \)), separated by average distance \( \mathcal{d}x \) defined by \( dV = A(p) \mathcal{d}x \), we can estimate (analogous to Mortensen et al., 2005) the average value of the first term of Equation 6 by

\[ |\nabla u_r|^2 \approx 8\pi (\alpha_0 + \alpha_1) \frac{Q^2}{A^3}, \]  (7)
with circularity parameter $C = \mathcal{L}^2 / 4\pi A(p)$ with perimeter $\mathcal{L} = \int_{\partial\Omega} dl$. The circularity parameter is related to the compactness factor $C = C / 4\pi$ in (Mortensen et al., 2005), and for HP flow it is equal to one. The coefficients $a_0$ and $a_1$ can be calculated in first order of circularity analytically or numerically for simple shapes of the iso-pressure surfaces, such as squares, triangles, or a perturbation of a sphere by spherical harmonics (Mortensen et al., 2005). For heterogeneous media, the class of shapes are generally unknown and not symmetric, and therefore $a_0$ and $a_1$ are expected to be intrinsically dependent on the pore geometry and therefore to change from pore to pore. Note that in (Mortensen et al., 2005) the flow is exactly perpendicular to the pressure gradient.

For the second, longitudinal term, we can assume that $Q$ remains constant for $p \rightarrow p + dp$, and the change of the velocity in longitudinal direction is caused by a change in cross-sectional area $A(p) \rightarrow A(p + \delta p)$. We estimate $u_\parallel$ by the total flux $Q / A$, that is,

$$[\nabla_x u_\parallel]^2 = 8\pi a_2 \frac{Q^2}{A^2} \left| \frac{dA}{dx} \right|^2,$$

with proportionality factor $a_2$. Again, $a_2$ is intrinsically dependent on pore geometry and changes from pore to pore. We rewrite Equation 5 in infinitesimal form, and replace the velocity tensor by Equation 6. Using expressions Equation 7 and Equation 8 leads to

$$\frac{dp}{dx} = \frac{8\pi \mu}{Q} (\nabla \otimes u)^2 A = \frac{8\pi \mu}{A(p)} f(a_i, S(p)),$$

with

$$f(a_i, S(p)) = a_0 + a_1 C + a_2 \frac{1}{A} \left| \frac{dA}{dx} \right|^2.$$

This parametrization is consistent with the Hagen-Poiseuille equation for pipe geometries $f(a_i) \rightarrow 1$, for which the local infinitesimal pressure gradient is given by

$$\frac{dp}{dx} = \frac{8\pi \mu}{A^2} Q.$$

As long as the volumetric flow rate $Q$ remains constant and $dV = Adx$ remains valid, (no changes in the topology of $S(p)$ will occur because of splitting/merging of the flow), Equation 9 can be integrated over $\Delta p$. We define a pore by the integrated volume, bound by iso-pressure surfaces $S_i(p), S_j(p + \Delta p)$ and the porous media boundary. The hydraulic resistance of a pore is then given by $R = \frac{p}{Q}$. The right-hand side of Equation 5 gives us therefore a statistical model for the hydraulic resistance $R_m$ is given by

$$R_m = 8\pi \mu \int_0^{L_{eff}} \frac{1}{A^2} f(a_i, S(p)) \, dx,$$

with $L_{eff} = \int dx$ the total effective length of the pore. When the geometry of a medium is given by a long pipe, the hydraulic resistance is given by the Hagen-Poiseuille (HP) model $R_{HP}$, given by $R_m(f \rightarrow 1)$. The HP model therefore only depends on the cross-sectional area $A(p)$.

3. Methods

To generate heterogeneous porous media we make use of the Gaussian Random Fields (GRF), which are increasingly used to represent realistic porous media (Liu et al., 2019). We used a fast Fourier transform and a spectral density function to generate GRF scalar functions (Hyman et al., 2012; Siena et al., 2014; Teubner, 1991). A threshold on the GRF function is used to define the porous media-fluid interface $\Gamma$ with porosities 0.68, 0.34 and 0.17 respectively. For details on the GRF functions and geometrical parameters such as average pore size and surface roughness are given in the Supporting Information S1. These porous media are used as input for direct numerical simulations (DNS, OpenFOAM v. 4.1, Weller et al., 1998), that solve
the Stokes equations (Equation 1) in the pore space. The boundary conditions are defined at the inlet \( p_1 \) and outlet \( p_2 \) and a no-slip condition for the porous media-fluid interface. A visualization of the three porous media is shown in Figure 2. Next, a chain of visualization toolkit (VTK) based image analysis techniques (Hernderson, 2007; Schroeder et al., 2006) is employed to extract iso-pressure surfaces \( S(p) \) and enumerate the disconnected areas identified as an iso-pressure patch \( S_i(p) \). This patch is part of a pore and has a surface area \( A_i(p) \), circularity \( C_i(p) \), center of “mass” of iso-pressure surface \( X_i(p) \) and the volumetric flow rate \( Q_i(p) \). For each \( S_i(p) \), we identify its closest neighbor \( S_j(p + \delta p) \). This neighboring iso-pressure patch (building up a pore) is found by calculating the distance function \( f_d(x, S) \), between any given point \( x \in S(p) \) and all iso-pressure patches \( S_j(p + \delta p) \). This distance function is defined by

\[
f_d(x, S) = \min \{ ||x - y|| \} \quad | \quad y \in S.
\]

for each \( i, k \) we define the averaged distance matrix

\[
d_{i,k} = \frac{1}{A_i(p)} \int_{S_j(p)} f_d(x, S_j(p + \delta p)) dS_i.
\]

The closest neighbor \( S_j(p + \delta p) \) is found by the minimum value of \( d_{i,j} = \min \{ d_{i,k} \} \). When \( \delta p \) is chosen sufficiently small the enclosed volume can be estimated by \( V_i(p, \delta p) = A_i \delta x \approx A_i d_{i,j} \). We use forward integration of consecutive patches until merging or splitting takes place. This is translated into constraints on the volumetric flow rate conservation and an upper bound for \( d_{i,j} \). We noticed however that the distances between \( X_i(p) \) and \( X_j(p + \delta p) \) of two consecutive pores are more sensitive to topology changes, and are therefore used instead. The precise values for these constraints can be found in the Supporting Information S1. A demonstration of the correct identification of pores by forward integration is shown in Figure 1 (Right), showing a merging of two pores. Although the proposed definition of individual pores deals naturally with junctions, a straight forward pore-network implementation is still missing. This is partly due to the exclusion of iso-pressure patches that are singular, and have no neighboring patches due to rapid changing topologies. The percentages of excluded surface area patches are 18%, 26% and 2%, which, at least for the first two porous media, prevents a continuous reconstruction of the network-topology. This is not an issue for the present work which aims at validating the novel constitutive relation on the level of individual pores and a continuous pore network is not required. However, for a pore-network implementation of the approach to be applicable, this should be resolved in future work.

The main challenge with the data format of the OpenFoam simulations is that it is unstructured, and the meshing is refined toward the boundary of the porous media. Although this ensures that the geometry is accurately described and that the simulation converges, it also causes challenges in the extraction of \( S(p) \) by using a VTK contour filter. Since it is based on a threshold on \( p \) it breaks up the mesh close to \( \Gamma \) into many
disconnected noisy area patches. These are removed by applying a filter on the area size of the patches, resulting in a reduction of total surface area of maximally 1%. Extracting circularity $\mathcal{A} \approx \iota(p)$ is achieved by applying a contour filter on $\mathcal{A} \approx \iota(p)$ with a threshold on the velocity of $|u| = 10^{-9}$ ms$^{-1}$, which is numerically zero.

For each of the three porous media, we evaluate Equation 9 for all consecutive iso-pressure pairs $\mathcal{A} \approx \iota(p)$. To obtain measured values for the resistance of a pore, we divide the total pressure difference $\mathcal{A} \Delta p$ by the total flux $\mathcal{A} \mathcal{A}$. We fit Equation 16 to all pores belonging to one porous media, yielding three sets of $\mathcal{A} \mathcal{A} i$. 

4. Results

The result of the DNS for the three porous media is shown in Figure 2. The Reynolds numbers are calculated by $\text{Re} \sim \epsilon_q q/\nu$, with $q$ the average flux through the porous media and $\epsilon_q$, the average pore size defined by the total porous media volume to total porous media interface ratio $\epsilon_q = 4\phi V / |\Gamma|$. For all porous media, Re is smaller than $10^{-2}$. In Figure 2 (bottom), the results of the infinitesimal pressure gradients vs. the HP model (Equation 11) are shown. We observe that the HP model underestimates the pressure gradient by up to two orders of magnitude for the first porous media and a relative good estimate for the third. We notice that $C$ is the lowest for smallest pores, indicating that smaller iso-pressure surfaces are more circular than larger, more complex shaped iso-pressure surfaces. Besides the fact that the data covers different ranges we see no visual distinction between the three porous media, that is, their data overlap and behave uniformly with respect to $C$ and size $\mathcal{A} \mathcal{A}$, see Figure 3 (top, left).

For each porous media, Equation 9 has been fitted by minimizing the least-squared error independently to obtain estimates for $a$. The contribution of the term $a_2$ is insignificant for all three porous media and is reported in the Supporting Information S1. A simple fit, excluding $a_2$, resulted in three values for $a_0 = 0.48, 0.52, 0.19$ and $a_1 = 0.90, 0.87, 1.16$ for the corresponding porous media respectively. The result of the fitting is shown in Figure 3 (top, right). Using Equation 9 with the fitted values for $a_0$ and $a_1 (a_2 = 0)$
we obtain a new model for local pore resistances $R_m$, Equation 16, which performs much better than the HP model. The result is shown in Figure 3 (bottom, right). We notice that obtained values for $a_1$ are underestimated by $1/e$ given that the circularity is overestimated by $e$.

The Pearson correlation coefficients $R^2$ for all models of the resistances are higher than 0.88. For the HP model the values for $R^2$ are given by 0.91, 0.88 and 0.99. The coefficients $R^2$ for $R_m$ are given by 0.97, 0.95 and 0.99. The high values are caused by the large domain size spanning several orders of magnitude. The deviations of $R_{HP}$ with the measured values $\Delta p/Q$ are not uniform across the scales and therefore $R^2$ is not a reliable parameter when it comes to expressing the improvement over the HP model (Wilcox, 2009). We therefore calculated the reduction of the root-mean-squared-relative-error (RMSRE) from $R_{HP}$ to $R_m$. For the first porous media we found a reduction in the RMSRE from 59% to 12%, for the second from 46% to 13% and for the last from 31% to 15%.

We visually observe that the circumferences of the iso-pressure surfaces are not smooth and lead to an overestimation of $C(p)$. In the Supporting Information S1 we have evaluated this error to be a factor $e = 1.15$, $1.11$, $1.08$ for the three porous media respectively. The origin of this error is the grid refinement near the boundary. Given that this is uniform throughout the porous media, we expect the error to be similar for all $C(p)$. Considering this observation and the fact that the last term in Equation 10 is insignificant, it is interesting to test the robustness of the linearity in $C$ and the consistency of the function $f \rightarrow 1$ when $C \rightarrow 1$. We introduce an alternative function $g$ to $f$ with two fit parameters $\alpha$ and $\beta$,

$$g(C) = 1 - \alpha \left[1 - (C/e)^\beta\right],$$

(15)

with a reduction factor $1/e$ for $C$ to compensate for the known overestimation of $C$. This function is consistent with the HP model as long as $\beta = 1$. The fitting resulted in three values for $\alpha = 0.45$, $0.58$, $0.34$, and $\beta = 1.09$, $1.05$, $1.08$. The latter suggesting that non-linear contributions of $C$ can be present but are expected to be relatively small. A model including a quadratic term, reported in the Supporting Information S1, gives the same performance, but with the danger of over-fitting. By using $g(C)$ the RMSREs are higher than for the model resulting from fitting Equation 16. This is reflected in a central spread of $R_m$ around the $1:1$ line, whilst with Equation 15, the model predicts generally values above the $1:1$ line. The Pearson correlation coefficients of the uniform model are similar, given by $R^2 = 0.97$, $0.95$ and $0.99$ indicating a similar performance. All models and their parameters including performances are listed in the Supporting Information S1.

The range of local resistances in Figure 3 (bottom) show that low resistances are correlated with high values for $C$, and are poorly estimated by the HP model. These pores are crucial for predicting preferential flow paths since they depend on the paths of least resistances throughout a network. For this purpose we have computed the RMSRE weighted by the volumetric flow rate. We found a reduction from 86%, 60%, 32% for the HP model to 10%, 12%, 13% for the new model for the three heterogeneous media respectively, which shows a remarkable improvement for high flux pores.

5. Discussion

The most important finding is the prediction of the resistance of a pore by $R_{HP}$ is highly underestimated with an average RMSRE of 0.45. This is most pronounced when the pores have a complex geometry, which are usually correlated with large pore areas, see Figure 3. An average underestimation of the resistances
leads to an average overestimation of the mean fluxes in a network model. This will affect transport predictions for example, breakthrough times will be underestimated (Dentz et al., 2018). Network models such as (Alim et al., 2017), often base their local resistances on the smallest distances to the porous media boundary. In general this will underestimate the cross-sections and therefore obtain higher resistances for the HP-based model, potentially reducing the error with respect to our HP model. Since anomalous diffusion has been correlated with the degree of heterogeneity of the porous media, it is important that low flux regions are included. The inaccurate representation of the low velocity regions of larger cross-sectional areas will therefore contribute to poor estimates of anomalously long residence times. Estimating these residence times properly is important because they underlie non-Fickian scaling behavior of the dispersion of flow tracers (Dentz et al., 2018; Dentz & Tartakovsky, 2006).

The variation in the obtained values for $\alpha$ are smaller than the variations obtained by (Mortensen et al., 2005), which is consistent with the observation that the iso-pressure surfaces are much more alike than the difference between triangles and perturbed circles.

Given the errors of the model, it is insightful to have a closer look at the validity of the assumptions leading to Equation 6. We have tested them for high and low dissipation regions (see Supporting Information S1). The main conclusion is that for low dissipation regions (comprising $\approx 40\%$ of the total dissipation) the fraction of the leading term $|\nabla \cdot \mathbf{u}|^2$ is given by $83\%, 87\%$ and $100\%$ for the porous media respectively. The contribution to the error of the model for off-parallel velocity gradients are not expected to be larger than $7\%$ for highly heterogeneous materials such as porous media 1 and 2.

We expect that our main results are transferable to other media such as packed-beads, sandstone and disordered media, since iso-pressure surfaces are quite heterogeneous even if grains are regular. In ordered and/or high porosity media we expect iso-pressure surfaces that are highly connected, similar to porous media 1 in this paper, and sometimes even consisting of a singular patch. In these cases extracting statistics can be challenging. One possible strategy for separation of highly connected iso-pressure surfaces into smaller patches could be a watershed or Morse-Smale-Complex segmentation (Tierny et al., 2018).

One of the key observations of our work is the possibility of introducing a local geometric factor that provides for the ratio of pressure difference and mass flux in a given pore. The factor depends on the considered pore and not the rest of the complicated shape of the medium boundary. The non-triviality of this observation is made transparent by employing the boundary integral representation which is equivalent to the Stokes equations obeyed by the flow. The representation gives the flow as an integral over the medium surface where the points of the surface appear as sources that produce the flow as superposition. The “charge” of such sources is proportional to the stress tensor at the boundary and the flow that each charge induces in space is given by an appropriate Green’s function, (see e.g., Pozrikidis, 1992). Our result demonstrates contributions other than those from the boundary of the considered pore can be neglected in the superposition. The mechanism by which this occurs, consists of both screening effect and destructive interference between different pores. This deserves further studies which are beyond our scope here. We have used circularity, as a single measure for the shape of $S(p)$, but to improve on this result it might be necessary to include other shape parameters such as curvature measures of $S(p)$, and/or inclusion of a model for the boundary term (first term of Equation 3), which however may lead to non-linear behavior in a circuit model. The errors in the model exceed the errors that can be attributed to finite volume method and mesh-refinements, and part of the uncertainty can be caused by the assumption that the flow is directed along the pressure gradient.

Although a pore-network implementation is still missing due to the incomplete evaluation of all surface area patches, an alternative option is to use a statistical network representation based on our results. Given that the distributions of the resistances show similarity with a log-normal distribution (see Supporting Information S1), a pathway for a statistical network based on these distributions seems feasible.

### 6. Conclusion

We have proposed a new iso-pressure surface based definition for individual pores in heterogeneous porous media with the aim of measuring and modeling the local hydraulic resistance which can potentially be used in a pore-network model. This new definition uses constant volumetric flow rates as a constraint on the...
length of the pore. The definition of the pores allows us to estimate the local hydraulic resistance in terms of the velocity gradient tensor. This can be modeled by Equation 16, with $\alpha_i = 0$ resulting in

$$R = 8\pi \mu \int_0^{\frac{1}{\lambda_{\text{eff}}}} \frac{1}{A^2} \left( \alpha_0 + \alpha_1 \frac{C}{A} \right) \, dx.$$  \hspace{1cm} (16)

This model significantly improves the Hagen-Poiseuille model for heterogeneous media.

Data Availability Statement
The results of the DNS simulations and the results of the postprocessing on which the figures are based can be accessed here (Krol, 2021, September).

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