Large deviations for small noise hypoelliptic diffusion bridges on sub-Riemannian manifolds

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Abstract

In this paper we study a large deviation principle of Freidlin-Wentzell type for pinned hypoelliptic diffusion measures associated with a natural sub-Laplacian on a compact sub-Riemannian manifold. To prove this large deviation principle, we use rough path theory and manifold-valued Malliavin calculus.

Keywords. Large deviation principle; Sub-Riemannian geometry; Pinned diffusion process; Malliavin calculus; Rough path theory;

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1 Introduction

In the theory of stochastic differential equations (SDEs), small noise problems SDEs are considered very important and have been studied intensively and extensively. A large deviation principle (LDP) associated with them is called Freidlin-Wentzell’s LDP. One of its typical formulations is as follows. Let $\mathcal{M}$ be a Euclidean space or a manifold and let $V_i, 0 \leq i \leq d$, be sufficiently nice vector fields on $\mathcal{M}$. For a standard $d$-dimensional Brownian motion $(w_t)_{0 \leq t \leq 1}$, consider the following Stratonovich-type SDE:

$$dX^\varepsilon_t = \varepsilon \sum_{i=1}^{d} V_i(X^\varepsilon_t) \circ dw^i_t + \varepsilon^2 V_0(X^\varepsilon_t)dt, \quad X^\varepsilon_0 = x.$$ 

Here, $x \in \mathcal{M}$ is a given initial point and $0 < \varepsilon \ll 1$ is a small parameter. Note that $X^\varepsilon = (X^\varepsilon_t)_{0 \leq t \leq 1}$ is the diffusion process associated with the generator $\varepsilon^2(\frac{1}{2} \sum_{i=1}^{d} V_i^2 + V_0)$ and the starting point $x$. Then, as is well-known, the law of $X^\varepsilon$ satisfies an LDP as $\varepsilon \searrow 0$.

Let us consider the case that $X^\varepsilon_t$ has a (sufficiently nice) strictly positive density with respect to a reference measure on $\mathcal{M}$ (e.g. the Lebesgue measure when $\mathcal{M}$ is a Euclidean space). Then, the pinned diffusion process from $x$ to $a$ associated with the above generator exists, where $a$ is a given end point. It seems quite natural to ask whether an LDP of Freidlin-Wentzell type holds for these scaled pinned diffusion measures as $\varepsilon \searrow 0$. In this work we take up this problem.
Although there are a large number of papers on the standard version of Freidlin-Wentzell’s LDP, only a few papers have been published on this type of LDP. To the author’s knowledge, the first one is Hsu [11]. He proved a pinned version of Freidlin-Wentzell’s LDP for pinned Brownian motion on a compact Riemannian manifold (i.e., the pinned diffusion process associated with one half of Laplace-Beltrami operator). Then, by studying SDEs under a suitable bracket-generating condition on the coefficient vector fields, the author [15, 16] proved this type of LDP in the Euclidean setting for rather general pinned diffusion processes. His method is a combination of rough path theory and quasi-sure analysis, which is a potential theoretic part of Malliavin calculus. Also, Bailleul [3] proved this type of LDP on a compact manifold for pinned diffusions associated with the sum-of-square type generator as above with a suitable bracket-generating condition on these vector fields. He combined a probabilistic method (rough path theory) and an analytic method (Sanchez-Calle’s estimate for the semi-group generated by a sum-of-square type operator).

The purpose of this paper is to prove an analogous LDP for the pinned diffusion process on a sub-Riemannian manifold $\mathcal{M}$ associated with the generator $\varepsilon^2 (\Delta_{\text{sub}}/2 + V)$, where $\Delta_{\text{sub}}$ is a natural “div-grad type” sub-Laplacian on $\mathcal{M}$ and $V$ is an arbitrary smooth vector field on $\mathcal{M}$. Our proof is basically similar to those in the preceding works [15, 16]. However, there are two new ingredients in this work. First, in order to realize the $\varepsilon^2 (\Delta_{\text{sub}}/2 + V)$-diffusion process via an SDE, we use Eells-Elworthy’s construction on a frame bundle over $\mathcal{M}$. Second, since we work on the manifold $\mathcal{M}$ and its frame bundle, we need manifold-valued Malliavin calculus developed in Taniguchi [28].

The organization of this paper is as follows. In Section 2 we formulate our LDP precisely and then state our main result (Theorem 2.1). Section 3 is devoted to recalling the stochastic parallel transport over a sub-Riemannian manifold. Stochastic tools such as Malliavin calculus, quasi-sure analysis, and rough path theory are collected in Section 4. In Section 5 we provide an LDP for the rough path lifts of certain positive Watanabe distributions on the geometric rough path space (Theorem 5.1). Our main result is almost immediate from this, thanks to Lyons’ continuity theorem. The lower estimate of the LDP in Theorem 5.1 is proved in Section 6 while the upper estimate is proved in Section 7. In Appendix A we show the strict positivity of the heat kernel associated with $\varepsilon^2 (\Delta_{\text{sub}}/2 + V)$, which ensures the well-definedness of the pinned diffusion measure.

Throughout this paper we will use the following notation.

- $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_+ = \{0\} \cup \mathbb{N}_+$. The set of all real numbers is denoted by $\mathbb{R}$.

- The time interval of (rough) paths is basically $[0, 1]$ unless otherwise specified.

- Let $U$ be an open set of a manifold $\mathcal{N}$ and $\mathcal{V}$ be a vector (or a fiber) bundle over $U$. We denote by $\Gamma(U; \mathcal{V})$ the set of all smooth sections of $\mathcal{V}$ on $U$. When $U = \mathcal{N}$, we will often simply write $\Gamma(\mathcal{V})$. For instance, we will write $\Gamma(T\mathcal{N})$ or $\Gamma(\mathcal{N}; T\mathcal{N})$ for the set of all smooth vector fields on $\mathcal{N}$. 

2
• For a manifold \( \mathcal{N} \), \( y \in \mathcal{N} \), and a subbundle \( \mathcal{V} \) of \( T\mathcal{N} \) with a metric, we set

\[
\mathcal{H}_y(\mathcal{N}, \mathcal{V}) = \{ \gamma: [0, 1] \to \mathcal{N} \mid \text{absolutely continuous, } \gamma_0 = y, \\
\gamma'_t \in \mathcal{V}_\gamma_t \text{ for almost all } t, \quad \mathcal{E}(\gamma) := \int_0^1 |\gamma'_t|^2 dt < \infty \}.
\]

We call \( \mathcal{E}(\gamma) \) the energy of \( \gamma \).

• Cameron-Martin space \( \mathcal{H}^d \) over \( \mathbb{R}^d \) is a real Hilbert space defined by

\[
\mathcal{H}^d = \{ h: [0, 1] \to \mathbb{R}^d \mid \text{absolutely continuous, } h_0 = 0, \\
\text{with } \|h\|_{\mathcal{H}^d}^2 := \int_0^1 |h'_t|^2 dt < \infty \}.
\]

In other words, \( \mathcal{H}^d = \mathcal{H}_0(\mathbb{R}^d, T\mathbb{R}^d) \) and \( \mathcal{E}(h) = \|h\|_{\mathcal{H}^d}^2 \). When \( d \) is obvious from the context, we simply write \( \mathcal{H} \).

• The classical \( d \)-dimensional Wiener space \( (\mathcal{W}, \mathcal{H}, \mu) = (\mathcal{W}^d, \mathcal{H}^d, \mu^d) \) is defined as follows. (1) \( \mathcal{H} \) is Cameron-Martin Hilbert space as above. (2) \( \mathcal{W} \) is the Banach space of all continuous paths from \([0,1]\) to \( \mathbb{R}^d \) which start at the origin. The topology of \( \mathcal{W} \) is that of uniform convergence as usual. (3) \( \mu \) is the \( d \)-dimensional Wiener measure. A generic element of \( \mathcal{W} \) is denoted by \( w \). The coordinate process \((w_t)_{0 \leq t \leq 1}\) defined on \( (\mathcal{W}, \mu) \) is called the canonical realization of \( d \)-dimensional Brownian motion.

• Let \( Z_1, \ldots, Z_m \) be smooth vector fields on an open subset \( U \) of a manifold \( \mathcal{N} \). We set \( \Sigma^1 = \{ Z_1, \ldots, Z_m \} \) and \( \Sigma^k = \{ [Z_i, Y] \mid Y \in \Sigma^{k-1} \} \) for \( k \geq 2 \), recursively. We next set for \( k \geq 1 \)

\[
\text{Lie}^{(k)}(Z_1, \ldots, Z_m) = \text{span} \left[ \cup_{j=1}^k \Sigma^j \right] \text{ and } \text{Lie}(Z_1, \ldots, Z_m) = \text{span} \left[ \cup_{j=1}^\infty \Sigma^j \right].
\]

Here, \( \text{span} \ A \) means the linear span of \( A \). For \( x \in U \), we set

\[
\text{Lie}^{(k)}(Z_1, \ldots, Z_m)(x) = \{ Y(x) \mid Y \in \text{Lie}^{(k)}(Z_1, \ldots, Z_m) \} \subset T_x \mathcal{N}
\]

and also set \( \text{Lie}(Z_1, \ldots, Z_m)(x) \subset T_x \mathcal{N} \) in the same way.

As for the last item, one should note the following simple fact. Suppose that there is a smooth function \( F: U \to \text{GL}(m, \mathbb{R}) \) such that

\[
[Z_1, \ldots, Z_m] = [\hat{Z}_1, \ldots, \hat{Z}_m] \quad \text{on } U.
\]

Then, for every \( x \in U \), we have \( \text{Lie}^{(k)}(Z_1, \ldots, Z_m)(x) = \text{Lie}^{(k)}(\hat{Z}_1, \ldots, \hat{Z}_m)(x) \) for every \( k \geq 1 \) and therefore \( \text{Lie}(Z_1, \ldots, Z_m)(x) = \text{Lie}(\hat{Z}_1, \ldots, \hat{Z}_m)(x) \).
2 Setting and main result

First we recall the essentials of sub-Riemannian geometry following [24]. We say that \((M, D, g)\) is a sub-Riemannian manifold if (i) \(M\) is a connected, smooth manifold of dimension \(n\), (ii) \(D \subset T M\), \(T M\) being the tangent bundle of \(M\), is a smooth distribution of constant rank \(d (1 \leq d \leq n)\) which satisfies the Hörmander condition at every \(x \in M\) and (iii) \(g = (g_x)_{x \in M}\), where each \(g_x\) is an inner product on the fiber \(D_x\), and \(x \mapsto g_x\) is smooth. (When there is no risk of confusion, we simply say that \(M\) is a sub-Riemannian manifold.) When \(n = d\), this definition coincides with that of a (connected) Riemannian manifold. Throughout this paper \(M\) is assumed to be compact.

The precise statement of the Hörmander condition on \(D\) at \(x \in M\) is as follows: If \(\{Z_1, \ldots, Z_d\}\) is a local frame of \(D\) over a coordinate neighborhood \(U\) around \(x\), then \(\text{Lie}(Z_1, \ldots, Z_d)(x) = T_x M\). As is well-known, this condition does not depend on the choice of \(U\) and \(\{Z_1, \ldots, Z_d\}\).

Now we recall a “div-grad type” sub-Laplacian on a sub-Riemannian manifold \(M\). Let \(\text{vol}\) be a smooth volume on \(M\), that is, \(\text{vol}\) is a measure on \(M\) whose restriction to every local coordinate chart is written as a strictly positive smooth density function times the Lebesgue measure on the chart. We consider the second-order differential operator of the form \(\Delta_{\text{sub}} = \text{div} \circ \text{grad}_D\), where \(\text{grad}_D\) is the horizontal gradient in the direction of \(D\) and \(\text{div}\) is the divergence with respect to \(\text{vol}\) (i.e. \(\text{div} = -(\text{grad}_D)^*\) at least formally, where the adjoint is taken with respect to \(\text{vol}\)).

A continuous path \(\gamma: [0, 1] \to M\) is said to be an admissible path \(1\) if \(\gamma\) is absolutely continuous, \(\gamma_t' \in D_{\gamma_t}\) for almost all \(t \in [0, 1]\), and

\[
E(\gamma) := \int_0^1 |\gamma'_t|_{g_{\gamma_t}}^2 \, dt < \infty. \quad (2.1)
\]

We call \(E(\gamma)\) the energy of \(\gamma\). (If \(\gamma\) is not admissible, we set \(E(\gamma) = +\infty\) by convention.) By Chow-Rashevsky’s theorem [24 Theorem 1.14], for every \(x, y \in M\) there exists an admissible path \(\gamma\) such that \(\gamma_0 = x\) and \(\gamma_1 = y\).

We define \(d_{SR}: M \times M \to [0, \infty)\) by

\[
d_{SR}(x, y) = \inf \left\{ \int_0^1 |\gamma'_t|_{g_{\gamma_t}} \, dt \mid \gamma: [0, 1] \to M, \text{ admissible with } \gamma_0 = x, \gamma_1 = y \right\}.
\]

Then, \(d_{SR}(x, y) < \infty\) for every \(x, y \in M\). It is well-known that \(d_{SR}\) becomes a distance on \(M\), which generates the same topology as the original manifold topology of \(M\). This is called the sub-Riemannian distance of \(M\). According to [24 Proposition 2.1], it holds that

\[
d_{SR}(x, y)^2 = \inf \{ E(\gamma) \mid \gamma: [0, 1] \to M, \text{ admissible with } \gamma_0 = x, \gamma_1 = y \}. \quad (2.2)
\]

\(^1\)It is called a “horizontal” path in [24]. We avoid this term, however, because the term “horizontal” is also used in the theory of connections on a principal bundle.
Let $V$ be any smooth vector field on $\mathcal{M}$ (i.e. $V \in \Gamma(T\mathcal{M})$) and $\varepsilon \in (0, 1]$. Then, the diffusion process on $\mathcal{M}$ associated with $\varepsilon^2(\Delta_{sub}/2 + V)$ starting at $x \in \mathcal{M}$ has a density $p^\varepsilon_t(x,a)$ with respect to $vol(da)$ at every $t > 0$. In fact, $p^\varepsilon_t(x,a) > 0$ for all $x, a \in \mathcal{M}$ and $t, \varepsilon \in (0, 1]$ and $a \mapsto p^\varepsilon_t(x,a)$ is smooth for all $x \in \mathcal{M}$ and $t, \varepsilon \in (0, 1]$. The pinned diffusion measure $Q^\varepsilon_{x,a}$ associated with $\varepsilon^2(\Delta_{sub}/2 + V)$ from $x$ to $a$ is a unique probability measure on $\mathcal{M}$ that the following holds: For every $k \geq 1$, $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = 1$ and $G \in C^\infty(\mathcal{M}^k)$,

\[
\int_{C([-1,1],\mathcal{M})} G(\xi_{t_1}, \ldots, \xi_{t_k}) Q^\varepsilon_{x,a}(d\xi) = p^\varepsilon_t(x,a)^{-1} \int_{\mathcal{M}^k} G(x_1, \ldots, x_k) \prod_{i=0}^{k} p^\varepsilon_{t_{i+1} - t_i}(x_i, x_{i+1}) \prod_{i=1}^{k} vol(dx_i)
\]

with the convention that $x_0 = x$ and $x_{k+1} = a$. (We will see that $Q^\varepsilon_{x,a}$ does exist.) Here, $C([-1,1],\mathcal{M})$ is the set of all continuous paths on $\mathcal{M}$ equipped with the compact-open topology. The closed subset of all continuous paths which start at $x$ and end at $a$ is denoted by $C_{x,a}([-1,1],\mathcal{M})$, in which $Q^\varepsilon_{x,a}$ is supported.

Now we state our main theorem. This can be viewed as a version of Freidlin-Wentzell type LDP for pinned hypoelliptic pinned diffusion processes on a sub-Riemannian manifold. As one can easily expect, the rate function equals one half of the energy functional type LDP for pinned hypoelliptic pinned diffusion processes on a sub-Riemannian manifold.

Theorem 2.1. Let $(\mathcal{M},\mathcal{D},g)$ be a compact sub-Riemannian manifold with a smooth volume $vol$. For $x, a \in \mathcal{M}$, $V \in \Gamma(T\mathcal{M})$ and $\varepsilon \in (0, 1]$, let $Q^\varepsilon_{x,a}$ be the pinned diffusion measure as above. Then, $(Q^\varepsilon_{x,a})_{0 < \varepsilon \leq 1}$ satisfies an LDP on $C_{x,a}([-1,1],\mathcal{M})$ as $\varepsilon \searrow 0$ with the speed $\varepsilon^2$ and the good rate function $J: C_{x,a}([-1,1],\mathcal{M}) \rightarrow [0, \infty]$, where

\[
J(\gamma) = \frac{1}{2} \{ \mathcal{E}(\gamma) - d_{\mathcal{M}}(x,a)^2 \}.
\]

Here, $\mathcal{E}(\gamma)$ is the energy of $\gamma$ defined by (2.3).

Remark 2.2. Take any distance $d$ on $\mathcal{M}$ which generates the topology of $\mathcal{M}$. Then,

\[
dist(\gamma, \hat{\gamma}) := \sup_{0 \leq t \leq 1} d(\gamma_t, \hat{\gamma}_t)
\]

defines a distance on $C([-1,1],\mathcal{M})$. It is known that dist generates the compact-open topology of $C([-1,1],\mathcal{M})$ regardless of the choice of $d$. Typical examples of $d$ include (1) the sub-Riemannian distance $d_{\mathcal{M}}$ on $\mathcal{M}$, (2) the Riemannian distance on $\mathcal{M}$ with respect to any Riemannian metric on $\mathcal{M}$, (3) the Euclidean distance of $\mathbb{R}^m$ restricted to $\mathcal{M}$ for any embedding $\mathcal{M} \hookrightarrow \mathbb{R}^m$. 

\[5\]
Remark 2.3. In Theorem 2.1 above and Proposition 2.4 below, the topology of the path spaces can be slightly strengthened as we now explain. In this remark, \( \alpha \in (1/3, 1/2) \).

Let \( \mathcal{N} \) be a compact smooth manifold and let \( \iota: \mathcal{N} \hookrightarrow \mathbb{R}^m \) be an embedding for some \( m \in \mathbb{N}_+ \). We denote by \( C^{\alpha-H}([0,1], \mathbb{R}^m) \) be the set of all \( \alpha \)-Hölder continuous paths taking values in \( \mathbb{R}^m \) and define

\[
C^{\alpha-H}([0,1], \mathcal{N}) := C^{\alpha-H}([0,1], \mathbb{R}^m) \cap C([0,1], \mathcal{N}),
\]

whose distance is the restriction of the natural one of \( C^{\alpha-H}([0,1], \mathbb{R}^m) \).

Let \( \mathcal{N}' \) be another compact smooth manifold and let \( \iota': \mathcal{N}' \hookrightarrow \mathbb{R}^{m'} \) be an embedding for some \( m' \in \mathbb{N}_+ \) and suppose that \( \phi: \mathcal{N} \to \mathcal{N}' \) is a smooth map. Noting that \( \phi \) extends to a smooth map from \( \mathbb{R}^m \) to \( \mathbb{R}^{m'} \) with compact support, \( \phi \) naturally induces a continuous map from \( C^{\alpha-H}([0,1], \mathcal{N}) \) to \( C^{\alpha-H}([0,1], \mathcal{N}') \) in an obvious way. In particular, this topology of \( C^{\alpha-H}([0,1], \mathcal{N}) \) is independent of the choice of \( \iota \).

The reason why the LDP in Theorem 2.1 (and in Proposition 2.4 below) also holds on \( C^{\alpha-H}([0,1], \mathcal{M}) \) is as follows. When we derive Theorem 2.1 from Theorem 5.1, we consider a rough differential equation (RDE) on \( \mathcal{P} \), a principle bundle over \( \mathcal{M} \), and embed \( \mathcal{P} \) into a Euclidean space and then use Lyons’ continuity theorem with respect to the \( \alpha \)-Hölder rough path topology, together with the contraction principle for LDPs. So, one can easily see that our LDP holds on \( C^{\alpha-H}([0,1], \mathcal{M}) \), too. (However, we do not write this fact in our main theorem since this set does not look very beautiful from the geometric viewpoint.)

Before closing this section, we claim that our method can reprove a very similar LDP in [3] on a compact manifold when the generator of the diffusion process is of sum-of-squares type.

Let \( \mathcal{M}' \) be a compact smooth manifold and \( V_0, V_1, \ldots, V_k \ (k \in \mathbb{N}_+) \) be smooth vector fields on \( \mathcal{M}' \). Consider the second order differential operator \( \varepsilon^2 (V_0 + \frac{1}{2} \sum_{i=1}^k V_i^2) \). Assume the bracket generating condition that \( \text{Lie}(V_1, \ldots, V_k)(x) = T_x \mathcal{M}' \) for every \( x \in \mathcal{M}' \). Then, the pinned diffusion measure from \( x \) to \( a \) associated with this operator exists uniquely, which is denoted by \( \tilde{Q}^\varepsilon_{x,a} \).

The following proposition, together with Remark 2.3, is the main result of [3, Theorem 1]. See [3, Eq. (4)] for an explicit expression of \( J' \), which may not always have a deep geometric meaning.

**Proposition 2.4.** Let the notation and assumptions be as above. Then, for every \( x, a \in \mathcal{M}' \), \( \{\tilde{Q}^\varepsilon_{x,a}\}_{0 < \varepsilon \leq 1} \) satisfies an LDP on \( C_{x,a}([0,1], \mathcal{M}') \) as \( \varepsilon \searrow 0 \) with the speed \( \varepsilon^2 \) and a good rate function \( J' \).

**Proof.** We can prove this proposition using the same method for Theorem 2.1. The actual proof of this proposition is much simpler than that of Theorem 2.1 since we need not use a principal bundle over \( \mathcal{M}' \). So, we omit the proof. \( \square \)
Remark 2.5. Suppose that $\Delta_{\text{sub}}$ admits a sum-of-squares form in the following sense: there exist $k \geq d$ and $V_i \in \Gamma(TM)$ ($1 \leq i \leq k$) such that the following condition holds:

$$\Delta_{\text{sub}} - \sum_{i=1}^{k} V_i^2 \in \Gamma(TM), \quad \text{Lie}(V_1, \ldots, V_k)(x) = T_xM' \text{ for every } x \in M'. \quad (2.4)$$

When $\Delta_{\text{sub}}$ admits the above expression, it is in a sense true that the LDP in Proposition 2.4 immediately implies the LDP in Theorem 2.1. However, it does not seem easy to obtain in this way the explicit expression (2.3) of the rate function $J$ from $J'$. The reason is as follows. In this case, $k$ equals the dimension of Brownian motion that plays a key role in the proof and can be very large. Hence, a one-to-one correspondence of Cameron-Martin paths and admissible paths on $M$ as in Proposition 3.2 breaks down in general. Therefore, even when (2.4) holds, we believe Theorem 2.1 is worth proving. (Unfortunately, the author does not know for which sub-Riemannian manifolds Condition (2.4) holds. For instance, for the Laplace-Beltrami operator on a Riemannian manifold, (2.4) is known to be satisfied.)

3 Stochastic parallel transport over sub-Riemannian manifold

In this section we recall the stochastic parallel transport over a compact sub-Riemannian manifold $(M, D, g)$. The history of Eells-Elworthy’s construction of Itô’s stochastic parallel transport is quite long. The Riemannian case is classical (see [12, Chapter 2] or [14, Section V-4] for example), but non-Riemannian cases have also been studied. For diffusion processes associated with semielliptic second-order differential operators, see [7, 8]. For those associated with sub-Laplacians on various kinds of sub-Riemannian manifolds, see [4, 5, 10, 18, 27] among others. In this section we will mainly follow [10].

Now we construct a principle bundle over $M$ and a connection which are associated with the sub-Riemannian structure of $(M, D, g)$. We write $n = \dim M$ and the rank of $D$ with $1 \leq d \leq n$. Since the Riemannian case (i.e. the case $d = n$) is classical and simpler, we only elaborate the case $1 \leq d < n$ in this section. But, the results in this section still hold for the case $d = n$.

First, we take a subbundle $D^\perp = (D^\perp_x)_{x \in M}$ of the tangent bundle $TM$ such that

$$T_xM = D_x \oplus D^\perp_x, \quad x \in M \quad (3.1)$$

holds. To make sure the existence of such $D^\perp$, one just need to take any Riemannian metric of $M$ and set $D^\perp_x$ to be the orthogonal complement of $D_x$ in $T_xM$ with respect to the metric. The projections associated with (3.1) are denoted by $\text{pr}_x : T_xM \to D_x$ and $\text{pr}_x^\perp : T_xM \to D^\perp_x$, respectively. Then, $\text{pr} = (\text{pr}_x)_{x \in M}$ and $\text{pr}^\perp = (\text{pr}_x^\perp)_{x \in M}$ belong to $\Gamma(\text{End}(TM, D))$ and $\Gamma(\text{End}(TM, D^\perp))$, respectively. For any metric $h$ on $D^\perp$, we set

$$\hat{g}_x(v, \hat{v}) = g_x(\text{pr}_x v, \text{pr}_x \hat{v}) + h_x(\text{pr}_x^\perp v, \text{pr}_x^\perp \hat{v}), \quad v, \hat{v} \in T_xM.$$


Then, \( \hat{g} \) is a Riemannian metric on \( \mathcal{M} \) which tames \( g \), that is, the restriction of \( \hat{g} \) to \( \mathcal{D}_x \times \mathcal{D}_x \) equals \( g_x \). Moreover, the decomposition (3.1) is an orthogonal decomposition with respect to \( \hat{g}_x \).

Next, choose metric Koszul connections \( \hat{\nabla} \) and \( \tilde{\nabla} \) on \( \Gamma(\mathcal{D}) \) and \( \Gamma(\mathcal{D}^\perp) \) with respect to the metrics \( g \) and \( h \), respectively. Note that they always exist. Define

\[
\nabla_X V = \hat{\nabla}_X (\text{pr} V) + \tilde{\nabla}_X (\text{pr}^\perp V), \quad X, V \in \Gamma(TM). \tag{3.2}
\]

Since the projections are pointwise operations, \( \nabla \) is a \( \hat{g} \)-metric Koszul connection on \( \Gamma(TM) \). It is immediate from the definition that, for all \( X \in \Gamma(TM) \), \( Y \in \Gamma(\mathcal{D}) \) and \( Z \in \Gamma(\mathcal{D}^\perp) \), we have \( \nabla_X Y \in \Gamma(\mathcal{D}) \) and \( \nabla_X Z \in \Gamma(\mathcal{D}^\perp) \).

Now we introduce a principal bundle over \( \mathcal{M} \). We choose any \( (\mathcal{D}, h) \) and \( \nabla \) (or \( \hat{\nabla} \) and \( \tilde{\nabla} \perp \)) as above and will fix them in what follows. The product \( O(d) \times O(n - d) \) of two orthogonal groups naturally acts on it from the right.

\[
O(\mathcal{M}; \mathcal{D} \oplus \mathcal{D}^\perp)_x = \{ u = \xi \oplus \eta: \mathbb{R}^d \oplus \mathbb{R}^{n-d} = \mathbb{R}^n \to T_x \mathcal{M} = \mathcal{D}_x \oplus \mathcal{D}_x^\perp \mid \xi: \mathbb{R}^d \to \mathcal{D}_x \text{ and } \eta: \mathbb{R}^{n-d} \to \mathcal{D}_x^\perp \text{ are linear isometries}\},
\]

\[
O(\mathcal{M}; \mathcal{D} \oplus \mathcal{D}^\perp) = \bigsqcup_{x \in \mathcal{M}} O(\mathcal{M}; \mathcal{D} \oplus \mathcal{D}^\perp)_x.
\]

This is a subbundle of the orthonormal frame bundle over the Riemannian manifold \( (\mathcal{M}, \hat{g}) \). For notational simplicity we will write \( \mathcal{P} := O(\mathcal{M}; \mathcal{D} \oplus \mathcal{D}^\perp) \) and \( G := O(d) \times O(n - d) \). The Lie algebra of \( G \) is \( \mathfrak{o}(d) \times \mathfrak{o}(n - d) \), which will be denoted by \( \mathfrak{g} \). Here, \( \mathfrak{o}(d) \) stands for the set of real \( d \times d \) anti-symmetric matrices. In the usual way, we view \( G \) as a subgroup of \( O(n) \) via the following injection:

\[
G \ni (U, V) \mapsto \left( \begin{array}{c} U \\ O \\ V \end{array} \right) \in O(n).
\]

We denote the natural projection by \( \pi: \mathcal{P} \to \mathcal{M} \). The right action on \( \mathcal{P} \) by \( a \in G \) is denoted by \( R_a \). The vertical vector (field) associated with \( X \in \mathfrak{g} \) is denoted by \( X^* \), which is defined by \( X^*(u) = (d/dt)_{t=0} R_{e^{tX}u}(u) \) at \( u \in \mathcal{P} \).

Let \( U \subset \mathcal{M} \) be a local coordinate neighborhood with a local chart \( (x^1, \ldots, x^n) \) and let \( \{Z_1, \ldots, Z_d\} \) and \( \{Z_{d+1}, \ldots, Z_n\} \) be a local orthonormal frame over \( U \) of \( \mathcal{D} \) and \( \mathcal{D}^\perp \), respectively. The canonical orthonormal basis of \( \mathbb{R}^n \) is denoted by \( \{e_1, \ldots, e_n\} \). For \( x \in U \), set \( \sigma_U(x) \oplus \tilde{\sigma}_U(x) \in \mathcal{P}_x \) by \( \sigma_U(x)(e_i) = Z_i(x) \) for all \( 1 \leq i \leq d \) and \( \tilde{\sigma}_U(x)(e_i) = Z_i(x) \) for all \( d+1 \leq i \leq n \). Then, for every \( \xi \oplus \eta \in \mathcal{P}_x \), there is a unique \( e = \{e_{\alpha\beta}\}_{1 \leq \alpha, \beta \leq n} \in G \) such that

\[
\xi \oplus \eta = (\sigma_U(x) \oplus \tilde{\sigma}_U(x)) \circ e = (\sigma_U(x) \circ \{e_{\alpha\beta}\}_{1 \leq \alpha, \beta \leq d}) \oplus (\tilde{\sigma}_U(x) \circ \{e_{\alpha\beta}\}_{d+1 \leq \alpha, \beta \leq n}).
\]

Of course, \( e_{\alpha\beta} = 0 \) if \( (\alpha, \beta) \) belongs to

\[
(\{1, \ldots, d\} \times \{d+1, \ldots, n\}) \cup (\{d+1, \ldots, n\} \times \{1, \ldots, d\}). \tag{3.3}
\]
In this way, we can identify $\pi^{-1}(U) \cong U \times G$ with a local chart $(x^1, \ldots, x^n; \{e_{\alpha\beta}\})_{1 \leq \alpha, \beta \leq n}$.

Next, set $\hat{\omega} = \{\hat{\omega}_{\beta}^\alpha\}_{1 \leq \beta \leq n}$ by

$$\nabla Z_\beta = \sum_{\alpha=1}^n \hat{\omega}_{\beta}^\alpha Z_\alpha \in \Gamma(U; T^*M \otimes TM), \quad 1 \leq \beta \leq n. \quad (3.4)$$

Since $\hat{\nabla}$ and $\hat{\nabla}^\perp$ are metric and $\nabla$ is defined by \((3.2)\), we can easily see that $\hat{\omega}_{\alpha}^\beta = -\hat{\omega}_{\beta}^\alpha$ for all $(\alpha, \beta)$ and $\hat{\omega}_{\beta}^\beta = 0$ if $(\alpha, \beta)$ belongs to \((3.3)\). Hence, $\hat{\omega}$ is a $g$-valued one-form on $U$, i.e. $\hat{\omega} \in \Gamma(U; T^*M \otimes g)$. Unfortunately, $\hat{\omega}$ does not define a global one-form on $M$.

We need to lift it to obtain a globally defined one-form on the principal bundle $\mathcal{P}$.

Define $\omega \in \Gamma(U \times G; T^*\mathcal{P} \otimes g)$ by

$$\omega = e^{-1}\hat{\omega}e + e^{-1}de. \quad (3.5)$$

This is a $g$-valued one-form on $\pi^{-1}(U)$. Note that $e^{-1}de$ stands for Maurer-Cartan form (i.e. the left translation of tangent vectors to the unit element) on $G$. By a standard argument, we can see that $\omega$ actually defines a global one-form on $\mathcal{P}$, namely, we have $\omega \in \Gamma(\mathcal{P}; T^*\mathcal{P} \otimes g)$. It is easy to see that $R^a_\ast \omega = a^{-1}\omega a$ for all $a \in G$ and that $\omega(X^\ast) = X$ for all $X \in g$. Thus, $\omega$ is an Ehresmann connection on $\mathcal{P}$.

On the above coordinate chart $\pi^{-1}(U)$, we set $\Gamma_{\beta\gamma}^\alpha$ by $\Gamma_{\beta\gamma}^\alpha = \hat{\omega}_{\beta}^\alpha(Z_\gamma)$, or equivalently,

$$\nabla Z_\gamma Z_\beta = \sum_{\alpha=1}^n \Gamma_{\beta\gamma}^\alpha Z_\alpha, \quad 1 \leq \alpha, \beta, \gamma \leq n. \quad (3.6)$$

We can easily see that $\Gamma_{\beta\gamma}^\alpha = 0$ if $(\alpha, \beta)$ belongs to \((3.3)\) and that $\Gamma_{\beta\gamma}^\alpha = -\Gamma_{\alpha\gamma}^\beta$ for all $\alpha, \beta, \gamma$. For $u = \xi \oplus \eta \in \pi^{-1}(x)$, we define a linear injection $\ell_u : T_xM \rightarrow T_u\mathcal{P}$ by

$$\ell_u \left( \sum_{\gamma=1}^n c_{\gamma} Z_\gamma(x) \right) = \sum_{\gamma=1}^n c_{\gamma} \left( Z_\gamma(x) - \sum_{\alpha, \beta, \delta = 1}^n \Gamma_{\gamma\delta}^\beta(x) e_{\beta\delta} \frac{\partial}{\partial e_{\alpha\delta}} \right), \quad (c_1, \ldots, c_n) \in \mathbb{R}^n \quad (3.7)$$

and set $\mathcal{K}_u = \ell_u(D_x)$ and $\mathcal{K}_u^\perp = \ell_u(D_x^\perp)$. It is known that

$$\ker(\omega_u : T_u\mathcal{P} \rightarrow g) = \mathcal{K}_u \oplus \mathcal{K}_u^\perp.$$ 

This is called the horizontal subspace and $\ell_u$ is called the horizontal lift. The subspace $\mathcal{V}_u = \{X^\ast(u) \mid X \in g\}$ is called the vertical subspace and it holds that $T_u\mathcal{P} = \ker \omega_u \oplus \mathcal{V}_u$. (From these explicit expression, too, we can see that the horizontal lift and the horizontal subspaces are compatible with the right action of $G$.)

We define the canonical horizontal vector fields $A_i = \ell_u(u\langle e_i \rangle)$ \((1 \leq i \leq n)\) on $\mathcal{P}$.

Since $u\langle e_i \rangle = \sum_{\gamma=1}^n e_{\gamma i} Z_\gamma(x)$, $A_i$ reads

$$A_i(u) = \sum_{\gamma=1}^n e_{\gamma i} \left( Z_\gamma(x) - \sum_{\alpha, \beta, \delta = 1}^n \Gamma_{\gamma\delta}^\beta(x) e_{\beta\delta} \frac{\partial}{\partial e_{\alpha\delta}} \right). \quad (3.7)$$
in the local chart. At every \( u \), \( \{A_i(u)\}_{i=1}^d \) and \( \{A_i(u)\}_{i=d+1}^n \) are linear base of \( \mathcal{K}_u \) and \( \mathcal{K}_u^\perp \), respectively. We equip \( \mathcal{K}_u \oplus \mathcal{K}_u^\perp \) a unique inner product so that \( \{A_i(u)\}_{i=1}^n \) becomes an orthonormal basis. Set a linear isometry \( \theta_u: \mathcal{K}_u \oplus \mathcal{K}_u^\perp \to \mathbb{R}^n \) by \( \theta_u(A_i(u)) = e_i \) for all \( 1 \leq i \leq n \). We can naturally view \( \theta \) as an \( \mathbb{R}^n \)-valued one-form on \( \mathcal{P} \). To summarize, the following four bijective linear maps are all isometric. (In either pair of maps, those in the opposite directions are inverses to each other.)

\[
\mathbb{R}^n = \mathbb{R}^d \oplus \mathbb{R}^{n-d} = \mathcal{K}_u \oplus \mathcal{K}_u^\perp \quad \text{ker} \omega_u = \mathcal{K}_u \oplus \mathcal{K}_u^\perp \quad T_x \mathcal{M} = D_x \oplus D_x^\perp.
\]

By restricting this to the first components, we obtain the following isometric correspondance:

\[
\mathbb{R}^d \xrightarrow{\theta_u^{-1}} \mathcal{K}_u \quad \text{ker} \omega_u \xrightarrow{\ell_u} \mathbb{R} \quad \mathcal{K}_u^\perp \xrightarrow{\ell_u} D_x.
\]

**Lemma 3.1.** Canonical horizontal vector fields \( \{A_1, \ldots, A_d\} \) on \( \mathcal{P} \) satisfy the partial Hörmander condition at every \( u \in \mathcal{P} \), that is,

\[
\pi_* \text{Lie}(A_1, \ldots, A_d)(u) = T_{\pi(u)} \mathcal{M}, \quad u \in \mathcal{P}.
\]

**Proof.** Let \( u \in \pi^{-1}(x) \) and use the local chart as above. Note that

\[
[A_1, \ldots, A_d] = [\ell(Z_1), \ldots, \ell(Z_d)] E, \quad \text{where } E = (\epsilon_{\alpha\beta})_{1 \leq \alpha, \beta \leq d}.
\]

Since \( E = E(u) \) is a smooth \( O(d) \)-valued function in \( u \), we have

\[
\text{Lie}(A_1, \ldots, A_d)(u) = \text{Lie}(\ell(Z_1), \ldots, \ell(Z_d))(u).
\]

Since we have \( \ell(Z_i)(u) = Z_i(x) + \) (a vertical vector field) from (3.6), we see that

\[
[\ell(Z_i), \ell(Z_j)](u) = [Z_i, Z_j](x) + \) (a vertical vector field).
\]

Repeating this, we have

\[
\pi_* \text{Lie}(\ell(Z_1), \ldots, \ell(Z_d))(u) = \text{Lie}(Z_1, \ldots, Z_d)(x).
\]

Since \( \mathcal{M} \) is sub-Riemannian, the right hand side equals \( T_x \mathcal{M} \). \( \square \)

Now we turn to the (anti-)development of finite energy paths. Let \( \mathcal{H}^n \) be Cameron-Martin space over \( \mathbb{R}^n \). For \( h \in \mathcal{H}^n \) and \( u \in \mathcal{P} \), we consider the following controlled ODE (skeleton ODE):

\[
d\phi_t = \sum_{i=1}^n A_i(\phi_t)dh_t^i, \quad \phi_0 = u,
\]

and set \( \psi_t = \pi(\phi_t) \). To emphasize the dependency on \( h \), we sometimes write \( \phi(h) \) and \( \psi(h) \). It is clear that \( \phi(h) \in \mathcal{H}_u(\mathcal{P}, \mathcal{K} \oplus \mathcal{K}^\perp) \) and \( \psi(h) \in \mathcal{H}_{\pi(u)}(\mathcal{M}, T\mathcal{M}) \). Moreover, the
energy is preserved, that is, \( \|h\|^2_{H^n} = \mathcal{E}(\phi(h)) = \mathcal{E}(\psi(h)) \). It should also be noted that \( \psi(h)'_t = \phi(h)_t(h'_t) \).

The map \( h \mapsto \psi(h) \) is bijective. The inverse of

\[ h \in H^n \mapsto \phi(h) \in H_u(P, K \oplus K^\perp) \]

is given by the line integral of \( \theta \):

\[ (u_t)_{t \in [0, 1]} \in H_u(P, K \oplus K^\perp) \mapsto \int_0^1 \theta_{u_t}(u'_t) \in H^n. \]

The inverse of the projection \( \pi: H_u(P, K \oplus K^\perp) \rightarrow H_{\pi(u)}(\mathcal{M}, TM) \) is the horizontal lift of paths. Recall that the horizontal lift of \( (x_t)_{t \in [0, 1]} \in H_{\pi(u)}(\mathcal{M}, TM) \) is a unique \( (u_t)_{t \in [0, 1]} \in H_u(P, K \oplus K^\perp) \) such that \( u'_t = \ell_{u_t}(x'_t) \) for almost all \( t \). Locally, \( (u_t) \) satisfies a simple controlled ODE as follows. Suppose that \( 0 < \tau \leq 1 \) and that \( (x_t)_{t \in [0, \tau]} \) stays in a local chart. Then, there exists a unique Cameron-Martin path \( (k_t)_{t \in [0, \tau]} \) over \( \mathbb{R}^n \) such that

\[ dx_t = \sum_{i=1}^n Z_i(x_t)k'_idt = \sum_{i=1}^n Z_i(x_t)dk_t \quad \text{on } [0, \tau] \quad \text{with } x_0 = x = \pi(u). \]

Therefore, the horizontal lift \( (u_t)_{t \in [0, \tau]} \) solves the following controlled ODE:

\[ du_t = \sum_{i=1}^n \ell(Z_i)(u_t)dk_t \quad \text{on } [0, \tau] \quad \text{with } u_0 = u. \]

The local expression of \( \ell(Z_i) \) was given in (3.6). Hence, we can write down this ODE for \( u_t = (x^1_t, \ldots, x^n_t; \{ e_{\alpha,\beta,t} \}_{1 \leq \alpha,\beta \leq n}) \) concretely using these coordinates. (However, we do not elaborate it because it is well-known and cumbersome. The point here is to explain that the lift map is explicitly computable and dependency on the data such as \( x, k \) can be tracked.) Thus, we have seen that the development map \( h \mapsto \phi(h) \) is bijective and preserves energy.

Restricting this correspondence to the first component, we have the following proposition. Note that we can naturally view \( H^n = H^d \oplus H^{n-d} \) as a direct sum of Hilbert spaces.

**Proposition 3.2.** For \( h \in H^d \) and \( u \in P \), consider the following ODE:

\[ d\phi(h)_t = \sum_{i=1}^d A_i(\phi(h)_t)dh'_t, \quad \phi(h)_0 = u, \quad (3.9) \]

and set \( \psi(h)_t = \pi(\phi(h)_t) \). Then, the development map \( h \mapsto \psi(h) \) is an energy-preserving bijection from \( H^d \) to \( H_{\pi(u)}(\mathcal{M}; D) \).

We now provide a generalization of Chow-Rashevsky’s theorem for future purpose. We will use this in Section A.
Proposition 3.3. Let $V \in \Gamma(TM)$. Then, for every $x, y \in M$ and $\tau \in (0, 1]$, there exists $k \in \mathcal{H}_x(M, TM)$ such that $k_\tau = y$ and $k'_t - V(k_t) \in \mathcal{D}_{k_t}$ for almost all $t \in [0, \tau]$.

Proof. We may simply work on $[0, \tau]$ since we can just set $k$ to be constant on $[\tau, 1]$. Note that if we take $l \in \mathbb{N}_+$ large enough, then we can find $B_1, \ldots, B_l \in \Gamma(D)$ such that \[
\text{Lie}(B_1, \ldots, B_l)(x) = T_xM \quad \text{for every } x \in M.
\]

Let us consider the following ODE on $M$ controlled by an $l$-dimensional Cameron-Martin path $h: [0, \tau] \to \mathbb{R}^l$:
\[
dk_t = \sum_{i=1}^l B_i(k_t)dh_i + V(k_t)dt, \quad k_0 = x.
\]

Thanks to the above condition on Lie brackets of $B_i$’s, this controlled ODE is strongly completely controllable. It implies that for every $x, y, \tau$, we can find $h$ such that the solution $k = k(h)$ satisfies $k_\tau = y$ (see [19, Section 5] for example). Hence, this solution $k$ is a desired path.

Define a second-order differential operator $\tilde{\Delta}$ on $\mathcal{M}$ by
\[
\tilde{\Delta}f = \text{Trace}_\mathcal{D}(\nabla_{\text{grad}\_\mathcal{D}}f), \quad f \in C^2(\mathcal{M}).
\]

The precise meaning is as follows. First, $\nabla_{\text{grad}\_\mathcal{D}}f \in \Gamma(T^*\mathcal{M} \otimes \mathcal{D})$ and therefore $v \mapsto \nabla_v \text{grad}\_\mathcal{D}f$ can be viewed a linear map from $\mathcal{D}_x$ to itself at every $x \in \mathcal{D}$. The right hand side at $x$ is defined to be the trace of this linear map.

Lemma 3.4. Let the notation be as above. Then, $\Delta_{\text{sub}} - \tilde{\Delta} \in \Gamma(\mathcal{D})$.

Proof. Take a local orthonormal frame $\{Z_1, \ldots, Z_d\}$ of $\mathcal{D}$ on a coordinate neighborhood $U \subset \mathcal{M}$. Then, it is well-known that
\[
\Delta_{\text{sub}} f = \sum_{i=1}^d \{Z_i^2 f + (\text{div} Z_i)Z_i f\},
\]
where div stands for the divergence with respect to the measure $\text{vol}$.

On the other hand, since $\text{grad}\_\mathcal{D} f = \sum_{i=1}^d (Z_i f)Z_i$, we see from [34] that
\[
\tilde{\Delta} f = \sum_{i=1}^d Z_i^2 f + \sum_{i,j=1}^d \hat{\omega}^j_i (Z_j)Z_i f
\]
and hence
\[
\Delta_{\text{sub}} - \tilde{\Delta} = \sum_{i=1}^d (\text{div} Z_i)Z_i - \sum_{i,j=1}^d \hat{\omega}^j_i (Z_j)Z_i \in \Gamma(U; \mathcal{D}).
\]
The left hand side is a globally defined at most second-order operator. However, as the right hand side shows, its second-order part vanishes. Hence, this is a globally defined first-order operator, i.e. a vector field.
Lemma 3.5. Let the notation be as above. Then, we have
\[ \sum_{i=1}^{d} A_i^2(\pi^* f) = \pi^*(T f), \quad f \in C^2(M). \]
Here, \( \pi^* f := f \circ \pi \in C^2(P) \) is the pullback of \( f \) by the projection \( \pi: P \to M \).

Proof. We work with the local chart \( (x^1, \ldots, x^n; \{e_{\alpha \beta}\}_{1 \leq \alpha, \beta \leq n}) \) on \( \pi^{-1}(U) = U \times G \) as above. We see from the local expression (3.7) that, for \( 1 \leq i, j \leq d \),
\[
A_i(\pi^* f) = \sum_{\gamma=1}^{d} e_{\gamma i} \pi^*(Z_{\gamma} f),
\]
\[
A_j A_i(\pi^* f) = \sum_{\gamma, \delta=1}^{d} e_{\delta j} e_{\gamma i} \pi^*(Z_{\delta} Z_{\gamma} f) - \sum_{\gamma, \delta, \epsilon=1}^{d} \Gamma_{\delta \epsilon}^{\gamma} e_{\epsilon j} e_{\delta i} \pi^*(Z_{\gamma} f).
\]

On the other hand, noting that \( \{u(e_1), \ldots, u(e_d)\} \) is an orthonormal basis at \( \pi(u) \), we compute as follows:
\[
\langle \nabla_{u(e_i)} \text{grad}_D f, u(e_i) \rangle_D = \sum_{\gamma, \delta, \epsilon=1}^{d} \epsilon_{\delta j} \epsilon_{\gamma i} \langle \nabla Z_{\epsilon}(Z_{\gamma} f), Z_{\delta} Z_{\gamma} f \rangle_D
\]
\[
= \sum_{\delta, \epsilon=1}^{d} \epsilon_{\delta j} \epsilon_{\gamma i} \Gamma_{\delta \epsilon}^{\gamma} Z_{\delta} Z_{\gamma} f - \sum_{\gamma, \delta, \epsilon=1}^{d} \epsilon_{\delta j} \epsilon_{\gamma i} \Gamma_{\delta \epsilon}^{\gamma} Z_{\gamma} f
\]
\[
= A_j A_i(\pi^* f)(u),
\]
where we used \( \Gamma_{\delta \epsilon}^{\gamma} = -\Gamma_{\epsilon \delta}^{\gamma} \). Setting \( i = j \) and summing them up, we prove the lemma. \( \square \)

Now we introduce an SDE on \( P \). For \( V \in \Gamma(TM) \), we set \( A_0 \in \Gamma(TP) \) by
\[
A_0(u) = \ell_u \langle V_0(\pi(u)) \rangle, \quad u \in P, \quad \text{where} \quad V_0 := V + (\Delta_{\text{sub}} - \bar{T})/2 \in \Gamma(TM).
\]

Let \( (w_t)_{t \in [0,1]} \) be a standard \( d \)-dimensional Brownian motion and consider the following Stratonovich-type SDE for \( u \in P \) and \( 0 < \varepsilon \leq 1 \):
\[
U^\varepsilon_t = \varepsilon \sum_{i=1}^{d} A_i(U^\varepsilon_t) \circ dw^i_t + \varepsilon^2 A_0(U^\varepsilon_t)dt, \quad U^\varepsilon_0 = u.
\]

From the scaling property of Brownian motion, the two processes \( (U^\varepsilon_t) \) and \( (U^1_{\varepsilon t}) \) have the same law. We will write \( X^\varepsilon_t := \pi(U^\varepsilon_t) \).

Lemma 3.6. Let \( x \in M \). Choose \( u \in \pi^{-1}(x) \) and consider SDE (3.10). Then, the law of the process \( (X^\varepsilon_t)_{t \in [0,1]} \) is independent of the choice of \( u \) and this process is a diffusion process on \( M \) associated with the generator \( \varepsilon^2(\Delta_{\text{sub}}/2 + D) \).
Proof. By the same reason (the rotational invariance of \((w_i)\)) as in the Riemannian case, the law of \((X^i_t)_{t \in [0,1]}\) is independent of the choice of \(u\). By Itô formula, one can show that the generator of \((U^i_t)\) is \(\varepsilon^2(\frac{1}{2} \sum_{i=1}^d A^2_i + A_0)\). From this and Lemma 3.5 we can easily see that \((X^i_t)\) is still a diffusion process and its generator is \(\varepsilon^2(\Delta_{\text{sub}}/2 + V)\).

Remark 3.7. For manifold-valued SDEs, the manifold need not be a submanifold of a Euclidean space. However, for manifold-valued RDEs and Malliavin calculus, the manifold is usually embedded into a Euclidean space. Therefore, in what follows we choose an embedding \(\iota: \mathcal{P} \rightarrow \mathbb{R}^M\) and extend the vector fields \(A_i, 0 \leq i \leq d\), to \(C^\infty\) vector fields on \(\mathbb{R}^M\) with compact support. In this way, we may view SDE (3.10) as an SDE on \(\mathbb{R}^M\).

For our purpose, any \(M\) and \(\iota\) will do. (Recall that, thanks to the tubular neighborhood theorem, every smooth function or vector field on \(\mathcal{P}\) extends to smooth function or vector field on \(\mathbb{R}^M\) with compact support, respectively.)

4 Preliminaries from stochastic analysis

In this section we recall several important probabilistic results which we will use in the proof of our main results. Basically, all results in this section (except Lemma 4.3) are either known or easily derived from known facts.

4.1 Elements of (manifold-valued) Malliavin calculus

We first recall Watanabe’s theory of generalized Wiener functionals (i.e. Watanabe distributions) in Malliavin calculus. Most of the contents and the notations in this section are contained in [14, Sections V.8–V.10] with trivial modifications. Also, [25, 23, 13, 22] are good textbooks of Malliavin calculus. For manifold-valued Malliavin calculus, see [28]. For basic results of quasi-sure analysis, we refer to [21, Chapter II].

Let \((\mathcal{W}, \mathcal{H}, \mu)\) be the classical \(d\)-dimensional Wiener space. (The results in this subsection also hold on any abstract Wiener space, however.) Let us recall the following:

(a) Basics of Sobolev spaces \(D_{p,r}(\mathcal{K})\) of \(\mathcal{K}\)-valued (generalized) Wiener functionals, where \(p \in (1, \infty), r \in \mathbb{R},\) and \(\mathcal{K}\) is a real separable Hilbert space. As usual, we will use the spaces \(D_{\infty}(\mathcal{K}) = \cap_{k=1}^\infty \cap_{1<p<\infty} D_{p,k}(\mathcal{K}), \bar{D}_{\infty}(\mathcal{K}) = \cap_{k=1}^\infty \cup_{1<p<\infty} D_{p,k}(\mathcal{K})\) of test functionals and the spaces \(D_{-\infty}(\mathcal{K}) = \cup_{k=1}^\infty \cap_{1<p<\infty} D_{p,-k}(\mathcal{K}), \bar{D}_{-\infty}(\mathcal{K}) = \cup_{k=1}^\infty \cap_{1<p<\infty} D_{p,-k}(\mathcal{K})\) of Watanabe distributions as in [14]. When \(\mathcal{K} = \mathbb{R}\), we simply write \(D_{p,r}\), etc.

(b) Meyer’s equivalence of Sobolev norms. (See [14, Theorem 8.4]. A stronger version can be found in [25, Theorem 4.6].)

(c) Pullback \(T \circ F = T(F) \in \bar{D}_{-\infty}\) of tempered Schwartz distribution \(T \in \mathcal{S}'(\mathbb{R}^n)\) on \(\mathbb{R}^n\) by a non-degenerate Wiener functional \(F \in D_{\infty}(\mathbb{R}^n)\). (See [14, Sections 5.9]. In fact, this is very strongly related to Item (d) below.)
(d) A generalized version of integration by parts formula (IbP formula) in the sense of Malliavin calculus for Watanabe distribution, which is given as follows (See [14, p. 377]):

For \( F = (F^1, \ldots, F^n) \in D_\infty(\mathbb{R}^n) \), we denote by \( \sigma^{ij}_F(w) = \langle DF^i(w), DF^j(w) \rangle_H \) the \((i,j)\)-component of Malliavin covariance matrix (\( 1 \leq i, j \leq n \)). We denote by \( \gamma^{ij}_F(w) \) the \((i,j)\)-component of the inverse matrix \( \sigma^{-1}_F \) (if the inverse exists in a certain sense). Recall that \( F \) is called non-degenerate in the sense of Malliavin if \( (\det \sigma_F)^{-1} \in \cap_{1 < p < \infty} L^p \). Note that \( \sigma^{ij}_F \in D_\infty \) and \( D\gamma^{ij}_F = -\sum_{k,l} \gamma^{ik}_F (D\sigma^{kl}_F) \gamma^{lj}_F \).

Hence, derivatives of \( \gamma^{ij}_F \) can be written in terms of \( \gamma^{ij}_F \)'s and the derivatives of \( \sigma^{ij}_F \)'s, which implies that \( \gamma^{ij}_F \in D_\infty \), too.

Suppose \( G \in D_\infty \) and \( T \in S'(\mathbb{R}^n) \). Then, the following integration by parts holds:

\[
\mathbb{E}\left[ \partial_i T(F) \cdot G \right] = \mathbb{E}\left[ T(F) \cdot \Phi_i(\cdot; G) \right],
\]

(4.1)

where \( \Phi_i(w; G) \in D_\infty \) is given by

\[
\Phi_i(w; G) = \sum_{j=1}^d D^* \left( \gamma^{ij}_F \cdot G \cdot DF^j \right)(w).
\]

Note that \( \mathbb{E} \) on the both sides of (4.1) is in fact the generalized expectation, that is, the pairing of \( D_{-\infty} \) and \( D_\infty \). Here, \( D \) and \( D^* \) are the \( H \)-derivative (i.e. the gradient operator in the sense of Malliavin calculus) and its adjoint (i.e. the divergence operator).

(e) If \( \eta \in D_{-\infty} \) satisfies that \( \langle \eta, F \rangle \geq 0 \) for every non-negative \( F \in D_\infty \), it is called a positive Watanabe distribution. According to Sugita’s theorem [26], for every positive Watanabe distribution \( \eta \), there uniquely exists a finite Borel measure \( \mu_\eta \) on \( \mathcal{W} \) such that

\[
\langle \eta, F \rangle = \int_{\mathcal{W}} \hat{F}(w) \mu_\eta(dw), \quad F \in D_\infty
\]

holds, where \( \hat{F} \) stands for \( \infty \)-quasi-continuous modification of \( F \). If \( \eta \in D_{p, k} \) is positive, then it holds that

\[
\mu_\eta(A) \leq \| \eta \|_{p, -k} \text{Cap}_{q, k}(A) \quad \text{for every Borel subset } A \subset \mathcal{W},
\]

where \( p, q \in (1, \infty) \) with \( 1/p + 1/q = 1 \), \( k \in \mathbb{N}_+ \), and \( \text{Cap}_{q, k} \) stands for the \((q, k)\)-capacity associated with \( D_{q,k} \). (For more details, see [21, Chapter II].)

We will also use a localized version of the Watanabe distribution theory, which can be found in [29, pp. 216–217]. (For proofs, see [17, Propositions 3.1 and 3.2].)

Let \( \rho > 0 \), \( \xi \in D_\infty \) and \( F \in D_\infty(\mathbb{R}^n) \) and suppose that

\[
\inf_{w \in \mathbb{S}^{n-1}} v^* \sigma_F v \geq \rho \quad \text{on } \{ w \in \mathcal{W} \mid |\xi(w)| \leq 2 \},
\]

(4.3)

where \( \mathbb{S}^{n-1} \) is the unit ball of \( \mathbb{R}^n \). Let \( \chi : \mathbb{R} \rightarrow \mathbb{R} \) be a smooth function whose support is contained in \([-1, 1]\). Then, the following proposition holds (see [28, Proposition 6.1]).
Proposition 4.1. Assume (4.3). For every \( T \in \mathcal{S}'(\mathbb{R}^n) \), \( \chi(\xi) \cdot T(F) \in \mathcal{D}_{-\infty} \) can be defined in a unique way so that the following properties hold:

(i) If \( T_k \rightarrow T \in \mathcal{S}'(\mathbb{R}^n) \) as \( k \rightarrow \infty \), then \( \chi(\xi) \cdot T_k(F) \rightarrow \chi(\xi) \cdot T(F) \in \mathcal{D}_{-\infty} \).

(ii) If \( T \) is given by \( g \in \mathcal{S}(\mathbb{R}^n) \), then \( \chi(\xi) \cdot T(F) = \chi(\xi)g(F) \in \mathcal{D}_{\infty} \).

We also provide an asymptotic theorem. It is a very special case of [29, Proposition 6.2]. Let \( \{F_\varepsilon\}_{0 \leq \varepsilon \leq 1} \subset \mathcal{D}_\infty(\mathbb{R}^n) \) and \( \{\xi_\varepsilon\}_{0 \leq \varepsilon \leq 1} \subset \mathcal{D}_\infty \) be families of Wiener functionals such that the following asymptotics hold:

\[
F_\varepsilon = F_0 + O(\varepsilon) \quad \text{ in } \mathcal{D}_\infty(\mathbb{R}^n) \text{ as } \varepsilon \searrow 0,
\]

\[
\xi_\varepsilon = \xi_0 + O(\varepsilon) \quad \text{ in } \mathcal{D}_\infty \text{ as } \varepsilon \searrow 0.
\]

Here, \( O(\varepsilon) \) is the large Landau symbol. Recall that \( \mathcal{D}_\infty \) and \( \mathcal{D}_\infty(\mathbb{R}^n) \) are endowed with a natural topology as Fréchet spaces.

Proposition 4.2. Assume (4.4), (4.5) and \( |\xi_\varepsilon| \leq 1/8 \). Moreover, assume that there exists \( \rho > 0 \) independent of \( \varepsilon \) such that (4.3) with \( F = F_\varepsilon \) and \( \xi = \xi_\varepsilon \) holds for every \( \varepsilon \in (0, 1] \). Let \( \chi : \mathbb{R} \rightarrow \mathbb{R} \) be a smooth function whose support is contained in \([-1, 1]\) such that \( \chi(x) = 1 \) if \( |x| \leq 1/2 \). Then, we have

\[
\lim_{\varepsilon \searrow 0} \chi(\xi_\varepsilon) \cdot T(F_\varepsilon) = T(F_0) \quad \text{ in } \mathcal{D}_{-\infty}.
\]

More precisely, there exists \( k \in \mathbb{N}_+ \) such that the above convergence takes place in \( \mathcal{D}_{p,-k} \) for every \( p \in (1, \infty) \).

Let us quickly review manifold-valued Malliavin calculus. Malliavin calculus for SDEs on manifolds was developed by Taniguchi [28]. Roughly speaking, under suitable assumptions, almost all of important results in the Euclidean case still hold true in the manifold case with natural modifications.

Let \( \mathcal{N} \) be a compact manifold of dimension \( m \), which is equipped with a smooth volume \( \text{vol}_\mathcal{N} \). (A measure on \( \mathcal{N} \) is said to be a smooth volume if it is expressed on each coordinate chart as a strictly positive smooth density function times the Lebesgue measure.) Choose a Riemannian metric on \( \mathcal{N} \) so that the determinant of the (deterministic) Malliavin covariance of \( \mathcal{N} \)-valued functionals are well-defined. Any choice of the Riemannian metric and the smooth volume will do.

An \( \mathcal{N} \)-valued Wiener functional \( F : \mathcal{W} \rightarrow \mathcal{N} \) is said to belong to \( \mathcal{D}_{p,k}(\mathcal{N}) \), \( p \in (1, \infty) \) and \( k \in \mathbb{N}_+ \), if \( f(F) \in \mathcal{D}_{p,k} \) for every \( f \in C^\infty(\mathcal{N}) \). If \( \iota : \mathcal{N} \rightarrow \mathbb{R}^M \) is an embedding, then \( F \in \mathcal{D}_{p,k}(\mathcal{N}) \) holds if and only if \( \iota(F) \in \mathcal{D}_{p,k}(\mathbb{R}^M) \) since every \( f \in C^\infty(\mathcal{N}) \) extends to a smooth function on \( \mathbb{R}^M \) with compact support. The same holds true for \( F \in \mathcal{D}_\infty(\mathcal{N}) := \cap_{k=1}^\infty \cap_{l \leq p < \infty} \mathcal{D}_{p,k}(\mathcal{N}) \). For \( F \in \cup_{1 \leq p < \infty} \mathcal{D}_{p,1}(\mathcal{N}) \), \( D_hF(w) \in T_{F(w)}\mathcal{N} \). Hence, the Malliavin covariance \( \sigma_F(w) \) in this case is a symmetric bilinear form on \( T_{F(w)}\mathcal{N} \times T_{F(w)}\mathcal{N} \). Thanks to the Riemannian metric, \( \det \sigma_F(w) \) can still be defined.

One of two main results in [28] is as follows. As in the Euclidean case, if \( F \in \mathcal{D}_\infty(\mathcal{N}) \) is non-degenerate in the sense of Malliavin, i.e. \( (\det \sigma_F)^{-1} \in \cap_{1 \leq p < \infty} L^p \), then
the composition \( T(F) = T \circ F \in \hat{D}_{-\infty} \) is well-defined as a Watanabe distribution for every distribution \( T \circ F \) and \( U \). Moreover, the law of \( F \) on \( \mathcal{N} \) has a smooth density \( p_F \) function with respect to \( \mathbf{vol}_\mathcal{N} \). In particular, \( \delta_a(Y_t) \) is a positive Watanabe distribution and \( p_F(a) = \mathbb{E}[\delta_a(F)] \) for every \( a \in \mathcal{N} \). One should note here that \( \delta_a \) and \( \delta_a(Y_t) \) depend on the choice of \( \mathbf{vol}_\mathcal{N} \). (Since any other smooth volume can be expressed as \( \mathbf{vol}_\mathcal{N}(dy) = \rho(y)\mathbf{vol}_\mathcal{N}(dy) \) for some strictly positive smooth function \( \rho \) on \( \mathcal{N} \), the delta function with respect to \( \mathbf{vol}_\mathcal{N} \) is given by \( \hat{\delta}_a = \rho(a)^{-1}\delta_a \).) The other main result in [28] is proving non-degeneracy for the projected solution of an SDE whose coefficient vector fields satisfies the partial Hörmander condition.

From here we consider SDE \( \{3.10\} \) (with \( x \in \mathcal{M} \) and \( u \in \pi^{-1}(x) \)) and set \( \mathcal{N} \) to be either \( \mathcal{M} \) or \( \mathcal{P} \). By Remark \textbf{3.7} we can easily see from the corresponding result in the Euclidean case that \( U^\varepsilon_t \in D^\infty(\mathcal{P}) \) for every \((\varepsilon, t) \in [0, 1]^2\) and \( X^\varepsilon_t = \pi(U^\varepsilon_t) \in D^\infty(\mathcal{M}) \) and that, for every \( f \in C^\infty(\mathcal{P}), p \in (1, \infty), k \in \mathbb{N}, D_{p,k}\)-norm of \( f(U^\varepsilon_t) \) is bounded in \((\varepsilon, t) \in [0, 1]^2\). In Lemma \textbf{3.1} we checked the partial Hörmander condition. Therefore, \( X^\varepsilon_t \) is non-degenerate for every \( t, \varepsilon \in (0, 1) \). More precisely, the following Kusuoka-Stroock’s estimate is known: There exist a constant \( \nu > 0 \) independent of \( p \) and a constant \( C_p > 0 \) such that, for every \( 1 < p < \infty \),

\[
\|(\det \sigma_{X^\varepsilon_t})^{-1}\|_{L^p} \leq C_p \varepsilon^{-\nu}, \quad \varepsilon \in (0, 1]. \tag{4.6}
\]

Combining this with Lemma \textbf{3.6}, the transition probability of \( \varepsilon^2(\Delta_{\text{sub}}/2 + V)\)-diffusion has a density \( p^\varepsilon_t(x, a) \) with respect to \( \mathbf{vol}(da) \), which is smooth in \( a \in \mathcal{M} \). It holds that

\[
p^\varepsilon_t(x, a) = p^{1/2}_t(x, a) = \mathbb{E}[\delta_a(X^\varepsilon_t)]. \tag{4.7}
\]

We will show in Section \textbf{A} that \( p^\varepsilon_t(x, a) > 0 \) for all \( \varepsilon, t \in (0, 1] \) and \( x, a \in \mathcal{M} \), which enables us to define the pinned diffusion measure associated with \( \varepsilon^2(\Delta_{\text{sub}}/2 + V) \) from every \( x \) to every \( a \). (We will later make sure that the measure actually exists.)

4.2 Elements of rough path theory

In this subsection we recall the geometric rough path space with Hölder or Besov norm and quasi-sure properties of the rough path lift. For basic properties of geometric rough path space, we refer to [20, 9]. For the geometric rough path space with Besov norm, we refer to [4, Appendix A.2]. The quasi-sure properties of the rough path lift is summarized in [15]. In this paper we assume \( \alpha \in (1/3, 1/2) \) for the Hölder parameter. We also assume that the Besov parameter \((\alpha, 4m)\) satisfy the following conditions:

\[
\frac{1}{3} < \alpha < \frac{1}{2}, \quad m \in \mathbb{N}_+, \quad \alpha - \frac{1}{4m} > \frac{1}{3}, \quad 4m\left(\frac{1}{2} - \alpha\right) > 1. \tag{4.8}
\]

We work in Lyons’ original formulation of RDEs (see [20], but we basically study the first level paths of solutions only. For brevity we will write \( \lambda^\varepsilon_t := \varepsilon^2 t \) for \( \varepsilon \in (0, 1] \).

We denote by \( G\Omega^H_\alpha(\mathbb{R}^d) \) the \( \alpha \)-Hölder geometric rough path space over \( \mathbb{R}^d \). A generic element of \( G\Omega^H_\alpha(\mathbb{R}^d) \) is denoted by \( \boldsymbol{w} = (\mathbf{w}^1, \mathbf{w}^2) \). For \( \beta \in (0, 1] \), let \( C^\beta_0([0, 1], \mathbb{R}^d) \)
be the Banach space of all \( \mathbb{R}^k \)-valued \( \beta \)-Hölder continuous paths that start at 0. If \( \alpha + \beta > 1 \), then the Young pairing

\[
G^{\alpha}_{\omega}(\mathbb{R}^d) \times C_0^{\beta-H}([0, 1], \mathbb{R}^k) \ni (\mathbf{w}, \lambda) \mapsto (\mathbf{w}, \lambda) \in G^{\alpha}_{\omega}(\mathbb{R}^{d+k})
\]

is a well-defined, locally Lipschitz continuous map. (See [9] Section 9.4 for example.)

Now we consider a system of RDEs driven by the Young pairing \((\mathbf{w}, \lambda) \in G^{\alpha}_{\omega}(\mathbb{R}^{d+1})\) of \( \mathbf{w} \in G^{\alpha}_{\omega}(\mathbb{R}^d) \) and \( \lambda \in C_0^{1-H}([0, 1], \mathbb{R}^1) \). (The main example we have in mind is \( \lambda_t = \text{const} \times t \).) For vector fields \( V_i : \mathbb{R}^n \rightarrow \mathbb{R}^n \) (0 \( \leq \) \( i \) \( \leq \) \( d \)), consider

\[
dx_t = \sum_{i=1}^{d} V_i(x_t)dw^i_t + V_0(x_t)d\lambda_t, \quad x_0 = x \in \mathbb{R}^n. \tag{4.9}
\]

The RDEs for the Jacobian process and its inverse are given as follows;

\[
dJ_t = \sum_{i=1}^{d} \nabla V_i(x_t)J_tdw^i_t + \nabla V_0(x_t)J_t d\lambda_t, \quad J_0 = \text{Id}_n, \quad \tag{4.10}
\]

\[
dK_t = -\sum_{i=1}^{d} K_t\nabla V_i(x_t)dw^i_t - K_t\nabla V_0(x_t)d\lambda_t, \quad K_0 = \text{Id}_n. \tag{4.11}
\]

Note that \( J, K, \) and \( \nabla V_i \) are \( \text{Mat}(n, n) \)-valued. Here, \( \text{Mat}(n, m) \) stands for the set of all real \( n \times m \) matrices and \( \text{Id}_n \) stands for the identity matrix of size \( n \).

For simplicity we assume that \( V_i, 0 \leq i \leq d, \) is of \( C_0^\infty \), that is, when viewed as an \( \mathbb{R}^n \)-valued function, \( V_i \) is a bounded smooth function with bounded derivatives of all order. It is then known that a unique global solution of (4.9)–(4.11) exists for any \( \mathbf{w} \) and \( \lambda \). Moreover, Lyons’ continuity theorem holds. In that case, the following map is continuous:

\[
G^{\alpha}_{\omega}(\mathbb{R}^d) \times C_0^{1-H}([0, 1], \mathbb{R}^1) \ni (\mathbf{w}, \lambda) \mapsto (\mathbf{x}, \mathbf{J}, \mathbf{K}) \in G^{\alpha}_{\omega}(\mathbb{R}^n \oplus \text{Mat}(n, n)^{\oplus 2}).
\]

The map \((\mathbf{w}, \lambda) \mapsto \mathbf{x}\) is denoted by \( \Phi : G^{\alpha}_{\omega}(\mathbb{R}^d) \times C_0^{1-H}([0, 1], \mathbb{R}^1) \rightarrow G^{\alpha}_{\omega}(\mathbb{R}^n) \). (We adopt Lyons’ formulation of RDEs as in [20]. So, the initial values of the first level paths must be adjusted.) If \( w \in C_0^{1-H}([0, 1], \mathbb{R}^d) \) or \( w \in \mathcal{H} = \mathcal{H}^d \) and \( \mathbf{w} \) is its natural lift, then the path

\[
t \mapsto (x + x^1_{0,t}, \text{Id} + J^1_{0,t}, \text{Id} + K^1_{0,t}) \tag{4.12}
\]

coincides with the solution of a system (4.9)–(4.11) of ODEs understood in the usual Riemann-Stieltjes sense. Recall that \( x^1_{0,t} \) is the first level path of \( x \) evaluated at \((0, t)\). Keep in mind that \((\text{Id} + J^1_{0,t})^{-1} = \text{Id} + K^1_{0,t}\) always holds.

When \( \mathbf{w} \) is Brownian rough path \( \mathbf{W} \) i.e. the natural lift of \((w_t)\), and \( \lambda = \lambda^1 \), the process in (4.12) coincides \( \mu \)-a.s. with the corresponding system of usual Stratonovich SDEs with drift. In this case \( x_t := x + x^1_{0,t} \in \mathbf{D}_\infty(\mathbb{R}^n) \) for every \( t \). If \( G : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is a
smooth map with bounded derivatives of all order \((\geq 1)\), then \(G(x_t) \in D_\infty(\mathbb{R}^m)\) and for every \(h \in \mathcal{H}\) and \(t\), it holds that
\[
D_h G(x_t) = (\nabla G)(x + x^1_{0,t})(\text{Id} + J^1_{0,t}) \int_0^t (\text{Id} + K^1_{0,s})V(x + x^1_{0,s})dh_s, \quad \text{a.s.} \quad (4.13)
\]
(with \(w = W\)). Here, we view \(V := [V_1, \ldots, V_d] \in \text{Mat}(n, d)\) and \(\nabla G \in \text{Mat}(m, n)\).

We define a continuous function
\[
\Gamma : G\Omega^H_\alpha(\mathbb{R}^d) \times C_0^{-H}([0, 1], \mathbb{R}^1) \times [0, 1] \to \text{Mat}(m, m) \quad (4.14)
\]
as follows: Set
\[
\Gamma(w, \lambda)_t = (\nabla G)(x + x^1_{0,t})(\text{Id} + J^1_{0,t})C(w, \lambda)_t(\text{Id} + J^1_{0,t})^*(\nabla G)(x + x^1_{0,t})^*,
\]
where
\[
C(w, \lambda)_t := \int_0^t (\text{Id} + K^1_{0,s})V(x + x^1_{0,s})V(x + x^1_{0,s})^*(\text{Id} + K^1_{0,s})^*ds.
\]
Here, the superscript \(*\) stands for the transpose of a matrix. Then, (4.13) implies that when \(w = W\) and \(\lambda = \lambda^1\), Malliavin covariance matrix of \(G(x + x^1_{0,t})\) equals \(\Gamma(W, \lambda^1)_t\), \(\mu\)-a.s. Similarly for \(h \in \mathcal{H}\), the deterministic Malliavin covariance matrix of \(G(x + x^1_{0,t})\) equals \(\Gamma(h, \lambda^1)_t\), where \(w = h\) is the natural rough path lift of \(h\). (When \(G\) is the identity map, these formulas are well-known. The general case is just a straightforward modification.)

For \((\alpha, 4m)\) which satisfies (4.14), \(G\Omega^B_{\alpha,4m}(\mathbb{R}^d)\) denotes the geometric rough path space over \(\mathbb{R}^d\) with \((\alpha, 4m)\)-Besov norm. Recall that the distance on this space is given by
\[
d(w, \hat{w}) = \|w^1 - \hat{w}^1\|_{\alpha,4m-B} + \|w^2 - \hat{w}^2\|_{2\alpha,2m-B} \\
= \left(\int_0^1 \int_{|t-s|^{1+4ma}} \frac{\|w^1_{t,s} - \hat{w}^1_{t,s}\|_{4m}^{2m}}{|t-s|^{1+4ma}}dsdt\right)^{1/2m} + \left(\int_0^1 \int_{|t-s|^{1+4ma}} \frac{\|w^2_{t,s} - \hat{w}^2_{t,s}\|^{2m}}{|t-s|^{1+4ma}}dsdt\right)^{1/2m}.
\]
By the Besov-Hölder embedding theorem for rough path spaces, there is a continuous embedding \(G\Omega^B_{\alpha,4m}(\mathbb{R}^d) \hookrightarrow G\Omega^H_{\alpha - (1/4m)}(\mathbb{R}^d)\). If \(\alpha < \alpha' < 1/2\), there is a continuous embedding \(G\Omega^H_{\alpha'}(\mathbb{R}^d) \hookrightarrow G\Omega^B_{\alpha,4m}(\mathbb{R}^d)\). Basically, we will not write the first embedding explicitly. (For example, if we write \(\Phi(w, \lambda)\) for \((w, \lambda) \in G\Omega^B_{\alpha,4m}(\mathbb{R}^d) \times C_0^{-H}([0, 1], \mathbb{R}^1)\), then it is actually the composition of the first embedding map above and \(\Phi\) with respect to \(\{\alpha - 1/(4m)\}\)-Hölder topology.) It is known that the Young translation by \(h \in \mathcal{H}\) works well on \(G\Omega^B_{\alpha,4m}(\mathbb{R}^d)\) under (4.15). The map \((w, h) \mapsto \tau_h(w)\) is continuous from \(G\Omega^B_{\alpha,4m}(\mathbb{R}^d) \times \mathcal{H}\) to \(G\Omega^B_{\alpha,4m}(\mathbb{R}^d)\), where \(\tau_h(w)\) is the Young translation of \(w\) by \(h\) (see [15] Lemma 5.1).

Now we review quasi-sure properties of rough path lift map \(\mathcal{L}\) from \(\mathcal{W}\) to \(G\Omega^B_{\alpha,4m}(\mathbb{R}^d)\). For \(k = \mathbb{N}_+\) and \(w \in \mathcal{W}\), we denote by \(w(k)\) the \(k\)th dyadic piecewise linear approximation of \(w\) associated with the partition \(\{j2^{-k} | 0 \leq j \leq 2^k\}\) of \([0, 1]\). We denote
the natural lift of \( w(k) \) by \( \mathcal{L}(w(k)) \), which is defined by Riemann-Stieltjes (or Young) integral. We set

\[
\mathcal{Z}_{α,m} := \{ w ∈ \mathcal{W} \mid \{ \mathcal{L}(w(k)) \}_{k=1}^∞ \text{ is Cauchy in } GΩ_{α,m}^{B}(\mathbb{R}^d) \}.
\]

We define \( \mathcal{L} : \mathcal{W} → GΩ_{α,m}^{B}(\mathbb{R}^d) \) by \( \mathcal{L}(w) = \lim_{m→∞} \mathcal{L}(w(k)) \) if \( w ∈ \mathcal{Z}_{α,m} \) and we define \( \mathcal{L}(w) \) to be the zero rough path if \( w \notin \mathcal{Z}_{α,m} \). We will use this version of \( \mathcal{L} \), which is Borel measurable, and write \( \mathbf{W} := \mathcal{L}(w) \) (as before) when it is regarded as a rough path space-valued random variable defined on \( \mathcal{W} \).

Note that \( \mathcal{H} \) and \( C_0^{α-H}([0, 1], \mathbb{R}^d) \) with \( β ∈ (1/2, 1] \) are subsets of \( \mathcal{Z}_{α,m} \) and the two definition of rough path lift coincide. (We will often write \( h = \mathcal{L}(h) \) for \( h ∈ \mathcal{H} \).) Under the scalar multiplication (i.e. the dilation) and the Cameron-Martin translation, \( \mathcal{Z}_{α,m} \) is left invariant. Moreover, \( c\mathcal{L}(w) = \mathcal{L}(cw) \) and \( τ_h(\mathcal{L}(w)) = \mathcal{L}(w+h) \) for any \( w ∈ \mathcal{Z}_{α,m}, c ∈ \mathbb{R} \), and \( h ∈ \mathcal{H} \). It is known that \( \mathcal{Z}^c_{α,m} \) is slim, that is, the \( (p, r) \)-capacity of this set is zero for any \( p ∈ (1, ∞) \) and \( r ∈ \mathbb{N}_+ \). Therefore, from a viewpoint of quasi-sure analysis, the lift map \( \mathcal{L} \) is well-defined. Moreover, the map \( \mathcal{W} ⊇ w → \mathcal{L}(w) ∈ GΩ_{α,m}^{B}(\mathbb{R}^d) \) is \( ∞ \)-quasi-continuous. (This kind of \( ∞ \)-quasi-continuity was first shown in [I].)

Let \( ε ∈ (0, 1] \) be a small parameter. We now recall that the unique solution of an RDE driven by \( ε\mathbf{W} = \mathcal{L}(εw) \) coincides with that of the corresponding scaled Stratonovich SDE given as follows:

\[
dX_\varepsilon^x = \varepsilon \sum_{i=1}^d V_i(X_\varepsilon^x) \circ dw_\varepsilon^i + \varepsilon^2 V_0(X_\varepsilon^x) dt, \quad X_0^x = x ∈ \mathbb{R}^n. \tag{4.15}
\]

When necessary, we will write \( X_\varepsilon^x = X_\varepsilon^x(t, x, w) \) or \( X_\varepsilon^x(t, x) \). Then, \( X_\varepsilon^x(⋅, x, w) = x + Φ(ε\mathbf{W}, X_\varepsilon^x) \) holds \( µ \)-a.s., which means that the right hand side is an \( ∞ \)-quasi-continuous modification of the left hand side as a \( C^{α-H}([0, 1], \mathbb{R}^n) \)-valued Wiener functional. Similarly, for every \( t, ε ∈ (0, 1], Γ(ε\mathbf{W}, X_\varepsilon^x)_t = ε^{-2} σ_{X_\varepsilon^x} \) holds, \( µ \)-a.s. Therefore, not just \( X_\varepsilon^x \) itself, but also its Malliavin covariance matrix \( σ_{X_\varepsilon^x} \) is a continuous function of Brownian rough path.

The skeleton ODE (without drift) associated with small noise problems for the above SDE [4.15] is given as follows: For \( h ∈ \mathcal{H} \),

\[
dΦ_t = \sum_{i=1}^d V_i(Φ_t) dh_t^i, \quad Φ_0 = x ∈ \mathbb{R}^n. \tag{4.16}
\]

We write the unique solution \( φ = φ(h) \) when necessary (which equals \( x + Φ(h, 0)^1 \)). The deterministic Malliavin covariance matrix of \( φ(h)_t \) at \( h \) is denoted by \( σ_{φ_t}(h) \). As is well-known, \( Γ(h, 0)_t = σ_{φ_t}(h) \). Recall that \( det σ_{φ_t}(h) > 0 \) if and only if the tangent map of \( φ_t : \mathcal{H} → \mathbb{R}^n \) at \( h \) is surjective.

Set \( X^{ε,h} := X^{ε}(x, w + (h/ε)) = x + Φ(τ_h(ε\mathbf{W}), ε^x)^1 \) for \( h ∈ \mathcal{H} \). In other words, \( X^{ε,h} \) uniquely solves the following Stratonovich SDE:

\[
dX^{ε,h}_\varepsilon = \sum_{i=1}^d V_i(X^{ε,h}_\varepsilon) \circ d(εw_\varepsilon^i + h_i) + ε^2 V_0(X^{ε,h}_\varepsilon) dt, \quad X^{ε,h}_0 = x.
\]
Small noise asymptotics of $X^\varepsilon,h$ has been extensively studied. One of basic results is the following asymptotics.

$$X^\varepsilon,h_1 = \phi(h_1) + \varepsilon \Xi^h_1 + O(\varepsilon^2) \quad \text{in } D_\infty(\mathbb{R}^n) \text{ as } \varepsilon \searrow 0,$$

where $\Xi^h_1$ is the element of the first order Wiener chaos given by the following Wiener integral:

$$\Xi^h_1(w) = (\text{Id} + J(h,0)_1) \int^t_0 (\text{Id} + K(h,0)_1) \nabla (\phi(h)_s) dw_s.$$

Hence, for $G: \mathbb{R}^n \to \mathbb{R}^m$ as above,

$$G(X^\varepsilon,h_1) = G(\phi(h)_1) + \varepsilon (\nabla G)(\phi(h)_1) \Xi^h_1 + O(\varepsilon^2) \quad \text{in } D_\infty(\mathbb{R}^m) \text{ as } \varepsilon \searrow 0.$$

Note that $(\nabla G)(\phi(h)_1) \Xi^h_1$ belongs to the first order Wiener chaos and therefore induces a mean-zero Gaussian measure on $\mathbb{R}^m$. Its covariance matrix equals $\Gamma(h,0)_1$, which in turn equals the deterministic Malliavin covariance matrix of $H \ni k \mapsto G(\phi(k)_1)$ at $h$.

The skeleton ODE (with drift) associated with the above SDE (4.15) with $\varepsilon = 1$ is given as follows: For $h \in H$,

$$d\zeta_t = \sum_{i=1}^d V_i(\zeta_t) dh^i_t + V_0(\zeta_t) dt, \quad \zeta_0 = x \in \mathbb{R}^n. \quad (4.17)$$

We write the unique solution $\zeta = \zeta(h)$ when necessary (which equals $x + \Phi(h,\lambda^1)$).

The deterministic Malliavin covariance matrix of $\zeta(h)_t$ at $h$ is denoted by $\sigma_\zeta(h)$. As is well-known, $\Gamma(h,\lambda_1)_t = \sigma_\zeta(h)$.

**Lemma 4.3.** Consider SDE (4.15) with $\varepsilon = 1$ and ODE (4.17). We assume that at $t \in (0,1]$, $\det \sigma_{X^1_t} > 0$ holds, $\mu$-a.s. Then, $\{h = \mathcal{L}(h) \mid h \in H, \det \sigma_\zeta(h) > 0\}$ is dense in $G \Omega^H_\alpha(\mathbb{R}^d)$ for any $1/3 < \alpha < 1/2$.

**Proof.** Take $m \in \mathbb{N}_+$ so large that $(\alpha + (1/4m), 4m)$ still satisfies (4.18) and set

$$A := \{w \in Z_{\alpha + (1/4m), 4m} \mid \det \sigma_{X^1_t}(w) > 0\} \subset \mathcal{W}.$$

This subset is of full $\mu$-measure and hence $\mathcal{L}(A)$ is of full measure with respect to the law of Brownian rough path. Note that $\mathcal{L}(A) \subset G \Omega^H_\alpha(\mathbb{R}^d)$ due to the Besov-Hölder embedding theorem. Thanks to the support theorem for Brownian rough path (see [23, Theorem 13.54]), $\mathcal{L}(A)$ must be dense in $G \Omega^H_\alpha(\mathbb{R}^d)$. For every $w \in A$, we have $\lim_{k \to \infty} \mathcal{L}(w(k)) = \mathcal{L}(w)$ in $\alpha$-Hölder topology and therefore

$$\lim_{k \to \infty} \sigma_\zeta(w(k)) = \lim_{k \to \infty} \Gamma(\mathcal{L}(w(k)), \lambda)_t = \Gamma(\mathcal{L}(w), \lambda)_t = \sigma_{X^1_t}(w).$$

This implies that $\det \sigma_\zeta(w(k)) > 0$ for large enough $k$. This proves the lemma. \qed
5 Large deviations for rough path lift of positive Watanabe distributions

In this section we formulate an LDP for the rough path lifts of Watanabe’s pull-back of the delta functions, from which our main theorem (Theorem 2.1) easily follows.

For $x \in \mathcal{M}$, we take any $u \in \pi^{-1}(x)$ and consider SDE (3.10) driven by the canonical realization of $d$-dimensional Brownian motion $(w_t)_{t \in [0,1]}$. By (4.16), $\delta_\nu(X_t^\nu) \in \hat{D}_{-\infty}$ is a well-defined positive Watanabe distribution for every $a \in \mathcal{M}$. By the positivity of the heat kernel (which will be proved in Section A), we see that $\nu_\nu(X_t^\nu) > 0$ for all $t, \varepsilon \in (0,1]$ and $x, a \in \mathcal{M}$. By Sugita’s theorem [26], the positive Watanabe distribution $\delta_\nu(X_t^\nu)$ at time $t = 1$ is in fact a non-trivial finite Borel measure on $\mathcal{W}$, which will be denoted by $\theta_{\nu,a}$.

Since $\mathcal{L}$ is defined outside a slim set, we can lift the measure $\theta_{\nu,a}$ to a measure on $G\Omega_{3,4m}^B(\mathbb{R}^d)$. We write $\nu_{\nu,a}^\varepsilon = (\varepsilon \mathcal{L})[\theta_{\nu,a}^\varepsilon]$. Here, $\varepsilon \mathcal{L}$ is the composition of $\mathcal{L}$ and the dilation by $\varepsilon$. Since the complement of $Z_{\alpha,4m}$ is slim, $\nu_{\nu,a}^\varepsilon$ does not depend on how $\mathcal{L}$ is defined on this complement. We denote by $\hat{\theta}_{\nu,a}$ and $\hat{\nu}_{\nu,a}$ the normalized measure of $\theta_{\nu,a}$ and $\nu_{\nu,a}$, respectively. (Since the total mass of $\theta_{\nu,a}^\varepsilon$ or of $\nu_{\nu,a}^\varepsilon$ equals $\mathbb{E}[\delta_\nu(X_1^\nu)] > 0$, this normalization is well-defined.)

Let $\phi(h)$ be the solution of ODE (3.9) and write $\psi(h) = \pi(\phi(h))$. In what follows, we write $\mathcal{H} = D^d$ for simplicity. We set

$$Q_{\nu,a}^u = \{h \in \mathcal{H} \mid \psi(h)_1 = a\}. \quad (5.1)$$

By Chow-Rashevsky’s theorem [24, Theorem 1.14], there exists an admissible path $(x_t)_{t \in [0,1]} \in \mathcal{H}_2(\mathcal{M}, \mathcal{D})$ such that $x_1 = a$. By Proposition 3.2, its anti-development belongs to $Q_{\nu,a}^u$. This implies that $Q_{\nu,a}^u \neq \emptyset$ for any $u$ and $a$.

Define a rate function $I_1 : G\Omega_{3,4m}^B(\mathbb{R}^d) \to [0, \infty]$ as follows;

$$I_1(w) = \begin{cases} \frac{1}{2}\|h\|_H^2 & \text{(if } w = \mathcal{L}(h) \text{ for some } h \in Q_{\nu,a}^u), \\ \infty & \text{(otherwise)}. \end{cases}$$

From the Schilder-type LDP for Brownian rough path [9, Theorem 13.42], we can easily see that $I_1$ is good. Also define $I_1(w) = I_1(w) - \min\{\|h\|_H^2/2 \mid h \in Q_{\nu,a}^u\}$, which is also good. From the goodness of $I_1$ and Proposition 3.2, we can easily see that the minimum above exists and equals $d_{SR}(x,a)^2/2$.

Our main purpose of this section is to prove that $\{\nu_{\nu,a}^\varepsilon\}_{0 < \varepsilon \leq 1}$ satisfies an LDP of Schilder type on $G\Omega_{3,4m}^B(\mathbb{R}^d)$ as $\varepsilon \searrow 0$. As we will see, our main result easily follows from the following theorem.

**Theorem 5.1.** Let the notation as above. Let $u \in \mathcal{P}$ and $a \in \mathcal{M}$ and assume (4.3) for the Besov parameter $(\alpha, 4m)$. Then, the following (i) and (ii) hold:

(i) The family $\{\nu_{\nu,a}^\varepsilon\}_{0 < \varepsilon \leq 1}$ of finite measures is exponentially tight and satisfies an LDP
on $G\Omega_{\alpha,4m}^B(\mathbb{R}^d)$ as $\varepsilon \searrow 0$ with the speed $\varepsilon^2$ and a good rate function $I_1$, that is, for every Borel set $A \subset G\Omega_{\alpha,4m}^B(\mathbb{R}^d)$, the following inequalities hold;

$$\inf_{w \in A^c} I_1(w) \leq \liminf_{\varepsilon \searrow 0} \varepsilon^2 \log \nu_{\varepsilon,u,a}(A^c) \leq \limsup_{\varepsilon \searrow 0} \varepsilon^2 \log \nu_{\varepsilon,u,a}(A) \leq \inf_{w \in A} I_1(w).$$

(ii) The family $\{\hat{\nu}_{\varepsilon,u,a}\}_{0<\varepsilon \leq 1}$ of probability measures is exponentially tight and satisfies an LDP on $G\Omega_{\alpha,4m}^B(\mathbb{R}^d)$ as $\varepsilon \searrow 0$ with the speed $\varepsilon^2$ and a good rate function $\hat{I}_1$.

Since the whole set is both open and closed, Theorem 5.1 (i) implies that

$$\lim_{t \searrow 0} t \log p_t^1(x, a) = \lim_{\varepsilon \searrow 0} \varepsilon^2 \log \mathbb{P}[\delta_u(X^\varepsilon_t)] = \lim_{\varepsilon \searrow 0} \varepsilon^2 \log \theta_{\varepsilon,u,a}(\mathcal{W}) = \lim_{\varepsilon \searrow 0} \varepsilon^2 \log \nu_{\varepsilon,u,a}(G\Omega_{\alpha,4m}^B(\mathbb{R}^d)) = -\min\{\|h\|_H^2/2 | h \in \mathcal{Q}^{\varepsilon,a}\} = -d_{SR}(x, a)^2/2.$$

We have also used (4.7) above. Due to this Varadhan-type asymptotic formula, Theorem 5.1 (ii) is immediate from (i). We will prove Theorem 5.1 (i) in Sections 6 and 7.

**Proof of Theorem 2.1.** Now we consider $G\Omega_{\alpha}^H(\mathbb{R}^d)$ with $\alpha \in (1/3, 1/2)$. Due to the Besov-Hölder embedding, $\hat{\nu}_{\varepsilon,u,a}$ actually sits on this space and Theorem 5.1 (ii) still holds even if $G\Omega_{\alpha,4m}^B(\mathbb{R}^d)$ is replaced by this space. Obviously, the family $\{\delta_{\lambda^c}\}_{0<\lambda \leq 1}$ is exponentially tight and satisfies an LDP on $C_0^{1-H}([0, 1], \mathbb{R})$ with the good rate function $+\infty \cdot 1_{\{0\}}^c$ with the convention that $\infty \cdot 0 = 0$. By a general result for LDPs for product measures (see [6, Exercise 4.2.7]), in which the exponential tightness plays a key role, $\{\hat{\nu}_{\varepsilon,u,a} \otimes \delta_{\lambda^c}\}_{0<\lambda \leq 1}$ satisfies an LDP on $G\Omega_{\alpha}^H(\mathbb{R}^d) \times C_0^{1-H}([0, 1], \mathbb{R})$ with the good rate function which is defined for $(w, \lambda)$ as follows:

$$\hat{I}_1(w) + \infty \cdot 1_{\{0\}}^c(\lambda) = \begin{cases} \frac{1}{\varepsilon^2} \|h\|_H^2 & \text{(if $w = \mathcal{L}(h)$ for some $h \in \mathcal{Q}^{\varepsilon,a}$ and $\lambda = 0$)}, \\ \infty & \text{(otherwise)}. \end{cases}$$

Choose an embedding as in Remark 3.7 and consider RDE (4.9) with $V_i = A_i$, $x = u$ and $n = M$. Define a continuous map $\Psi: G\Omega_{\alpha}^H(\mathbb{R}^d) \times C_0^{1-H}([0, 1], \mathbb{R}) \to C_{x,a}([0, 1], \mathcal{M})$ by $\Psi(w, \lambda)_t = \pi_x + \Phi(w, \lambda)_0^t$. Then, $\Psi(\varepsilon W, \lambda^c)$ is an $\infty$-quasi-continuous modification of $X^c = \pi(U^c)$, where $U^c$ solves SDE (3.10). In what follows we use this version of $X^c$. Note that $\psi(h) = \Psi(\mathcal{L}(h), 0)$ for $h \in \mathcal{H}$.

Now we claim that the law of $\Psi$ under $\hat{P}_{\varepsilon,u,a} \otimes \delta_{\lambda^c}$ is the pinned $\varepsilon^2(\Delta_{sub}/2 + V)$-diffusion measure $\mathcal{Q}_{x,a}^c$ from $x = \pi(u)$ to $a$. Let $k \geq 1$, $G \in C^\infty(\mathcal{M}^k)$ and $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = 1$ be arbitrary. Then, we have

$$\int G(\Psi(w, \lambda)_1, \ldots, \Psi(w, \lambda)_{t_k}) \hat{P}_{\varepsilon,u,a} \otimes \delta_{\lambda^c}(dw d\lambda) = \int G(\Psi(\varepsilon W, \lambda^c)_1, \ldots, \Psi(\varepsilon W, \lambda^c)_{t_k}) \hat{P}_{\varepsilon,u,a}(dw)$$

23
\[ = p_t^1(x,a)^{-1}\mathbb{E}[G(X^t_{i_1}, \ldots, X^t_{i_k})\delta_a(X^t_1)] \]
\[ = p_t^1(x,a)^{-1}\int_{M^k} G(x_1, \ldots, x_k) \prod_{i=0}^{k} p_{t_{i+1}-t_i}^1(x_i, x_{i+1}) \prod_{i=1}^{k} \text{vol}(dx_i) \]
as desired. Here, we set \( x_0 = x \) and \( x_{k+1} = a \) for simplicity. This proves our claim. Note that this argument also proves the existence of the pinned diffusion measure \( Q^\varepsilon_{x,a} \).

By the above fact and Lyons’ continuity theorem, we can use the contraction principle ([6, Theorem 4.2.1]) to prove that \( \{Q^\varepsilon_{x,a}\}_{0<\varepsilon\leq 1} \) satisfies an LDP with a good rate function. Since \( \psi \) is a bijection that preserves the energy (Proposition 3.2), the rate function \( J \) is given by [23]. This completes the proof of Theorem 2.1.

6 Lower estimate

In this section we prove the lower estimate of LDP in Theorem 5.1 (i). Take any \( u \in \mathcal{P} \) and \( a \in \mathcal{M} \). For this \( u, \phi(h) \) denotes the unique solution of ODE (3.9) and \( \psi(h) \) is its projection on \( \mathcal{M} \). The subset \( Q^{u,a} \subset H \) is defined in (5.1).

**Lemma 6.1.** For every \( h \in Q^{u,a} \) there exists a sequence \( \{h_j\}_{j=1}^\infty \) in \( Q^{u,a} \) which satisfies the following conditions:

(i) \( \lim_{j \to \infty} \|h_j - h\|_H = 0. \)

(ii) For all \( j \), \( D\psi(h_j)_1 : H \to T_{h_j} \mathcal{M} \) is surjective, where \( D\psi(h_j)_1 \) denotes the tangent map of \( \mathcal{H} \ni k \mapsto \psi(k)_1 \in \mathcal{M} \) at \( h_j \).

(iii) For all \( j \), \( \langle h_j, \cdot \rangle \in \mathcal{W}^\ast \), that is, this linear functional on \( \mathcal{H} \) extends to a bounded linear functional on \( \mathcal{W} \).

**Proof.** First we will find \( \{h_j\}_{j=1}^\infty \) which satisfies (i) and (ii) only. Let \( \{Z_1, \ldots, Z_d\} \) be an orthonormal frame of \( \mathcal{D} \) on a coordinate neighborhood \( U \) of \( x = \pi(u) \). We view \( U \) as an open subset of \( \mathbb{R}^n \) and extend \( Z_i, 1 \leq i \leq d \), as a smooth vector field on \( \mathbb{R}^n \) with compact support. We denote by \( \mathcal{H}_\tau \), \( \tau \in (0,1) \), the Hilbert space of \( \mathbb{R}^d \)-valued Cameron-Martin paths defined on the time interval \([0,\tau]\).

Consider the following ODE on \( \mathbb{R}^n \) driven by a Cameron-Martin path \( k \):

\[ dx(k)_t = \sum_{i=1}^{n} Z_i(x(k)_t)dk_t, \quad x(k)_0 = x. \]

The following fact was proved in [10, Section 3]. For every \( j \) large enough, there exists a Cameron-Martin path \( k_j \in \mathcal{H}_{1/j} \) such that \( |k_j| \leq 1 \) for almost all \( t \in [0,1/j] \) and \( Dx(k_j)_{1/j} \) is surjective. (Precisely, \( \{Z_i\} \) is assumed in to satisfy the bracket generating condition at every point of \( \mathbb{R}^n \) in [10], while the condition is assumed only on \( U \) here. However, this difference does not matter at all since when \( j \) is large enough, the Cameron-Martin norm of \( x(k_j) \) is small enough and therefore \( x(k_j) \) stays inside \( U \), anyway.)
Then, since \( \{Z_i\} \) are orthonormal, \( |x(k_j)| \leq 1 \) for almost all \( t \in [0,1/j] \). Denote by \( l_j := \psi^{-1}(x(k_j)) \in \mathcal{H}_{1/j} \) the anti-development of \( x(k_j) \), then \( |l_j'| \leq 1 \) for almost all \( t \in [0,1/j] \), too. Define \( h_j \in \mathcal{H} \) by

\[
h_{j,t} := \begin{cases} 
    l_j, & \text{on } t \in [0,1/j], \\
    l_j, (2j/t - t) & \text{on } t \in [1/j, 2/j], \\
    h_{\tau} & \text{on } t \in [2/j, 1].
\end{cases}
\]

Since ODE (3.9) has no drift term, we can easily see that \( \phi \) can find a partition of \( 0 = s \leq Z \psi \) of \( \xi \). When \( \psi(T)(\cdot) \) is fixed.) Hence, \( \text{End} := \text{End} \) there exist sufficiently small \( s \) that \( \psi(h_j) \) satisfies the following ODE with the initial condition \( \psi(h_j)|_{\mathbb{R}^d} \) of \( \mathcal{D} \) can be chosen. We may assume \( s = 1/j, U_1 = U \) and \( Z_i |_{\mathbb{R}^d} = Z_i \) for all \( 1 \leq i \leq d \).

Next we show (ii). Fix \( j \) and consider the admissible path \( \psi(h_j): [0,1] \rightarrow \mathcal{M} \). We can find a partition of \( 0 = s_0 < s_1 < \cdots < s_N = 1 \) of \( [0,1] \) such that, for all \( 1 \leq r \leq N \), \( \psi(h_j)|_{[s_{r-1},s_r]} \) is contained in a coordinate neighborhood \( U_r \) on which an orthonormal frame \( \{Z_1^{(r)},\ldots,Z_d^{(r)} \} \) of \( \mathcal{D} \) can be chosen. We may assume \( s_1 = 1/j, U_1 = U \) and \( Z_i |_{\mathbb{R}^d} = Z_i \) for all \( 1 \leq i \leq d \).

Obviously, there exists a unique Cameron-Martin path \( k^{(r)}: [s_{r-1}, s_r] \rightarrow \mathbb{R}^d \) such that \( \psi(h_j)|_{[s_{r-1},s_r]} \) satisfies the following ODE with the initial condition \( x_{s_{r-1}}^{(r)} = \psi(h_j)|_{s_{r-1}} \):

\[
dx_i^{(r)} = \sum_{i=1}^{n} Z_i^{(r)}(x_i^{(r)})dk_i^{(r)} \quad \text{on } [s_{r-1}, s_r].
\]

When the initial value of ODE (6.1) is replaced by \( \xi^{(r-1)} \), we write \( \text{End}^{(r)}(\xi^{(r-1)}) := x_{s_{r-1}}^{(r)} \). When \( \xi^{(r-1)} \) is close enough to \( \psi(h_j)|_{s_{r-1}} \), \( \text{End}^{(r)}(\xi^{(r-1)}) \) is well-defined. By the theory of flow of diffeomorphisms for ODEs, \( \text{End}^{(r)} \) is a local diffeomorphism from a neighborhood of \( \psi(h_j)|_{s_{r-1}} \) to a neighborhood of \( \psi(h_j)|_{s_r} \). (Note that here and in what follows, \( k^{(r)} \) is fixed.) Hence, \( \text{End} := \text{End}^{(N)} \circ \cdots \circ \text{End}^{(2)} \) is a local diffeomorphism from a neighborhood of \( \psi(h_j)|_{s_1} \) to a neighborhood of \( \psi(h_j)|_{s_N} \). In particular, the tangent map of \( \text{End} \) is bijection from \( T_{h_j}\mathcal{M} \) to \( T_{q_1}\mathcal{M} \), where we write \( b := x(k_j)|_{s_1} = \psi(h_j)|_{s_1} \).

Now consider \( \psi(h_j)|_{[0,s_1]} = x(k_j) \). Since \( Dk_j|_{s_1} \) is surjective, for every \( v \in T_{h_j}\mathcal{M} \) there exist sufficiently small \( \varepsilon > 0 \) and a \( C^1 \)-curve \( (-\varepsilon,0) \ni \varepsilon \mapsto k^{(r)}_j \in \mathcal{H}_{1/j} \) such that \( k^{(r)}_j = k_j \) and \( (d/d\varepsilon)|_{\varepsilon=0} x(k^{(r)}_j)|_{s_1} = v \). (Below \( \varepsilon_0 \) may change from line to line.)

Take \( u \in T_{a}\mathcal{M} \) arbitrarily and let \( v \) be the unique element of \( T_{b}\mathcal{M} \) which corresponds to \( u \) through the bijection. For this \( v \), we take \( k^{(r)}_j \) as above. Define \( h_j^{(r)} \) to be the unique element in \( \mathcal{H} \) such that (1) \( \psi(h_j^{(r)}) \) coincides with \( x(k^{(r)}_j) \) on \( [0,s_1] \) and (2) \( \psi(h_j^{(r)})|_{[s_{r-1},s_r]} \) solves ODE (6.1) for all \( 2 \leq r \leq N \). By way of construction, \( h_j^{(r)} = h_j^{(r)} \) and \( (d/d\varepsilon)|_{\varepsilon=0} \psi(h_j^{(r)})|_{s_1} = v \). Thus, we have shown (ii).

Finally, we cope with (iii). Since \( h \mapsto \psi(h) \) is Fréchet-\( C^1 \) and the inclusion \( \mathcal{W}^* \hookrightarrow \mathcal{H}^* \) is continuous and dense, we may use a topological lemma [15, Lemma 7.3]. It implies that we can find \( \hat{h}_j \in Q^{\mathbb{R}_+} \) which satisfies the following condition for sufficiently large \( j \): (1) \( \|h_j - \hat{h}_j\|_{\mathcal{H}} \leq 1/j \), (2) \( D\psi(\hat{h}_j)|_{\mathcal{H}} \rightarrow T_{a}\mathcal{M} \) is surjective, and (3) \( \langle \hat{h}_j, \cdot \rangle_{\mathcal{H}} \in \mathcal{W}^* \).
\( W^* \). Hence, \( \{ \hat{h}_j \}_{j=1}^\infty \) is the desired sequence. (Precisely, [15] Lemma 7.3 is for \( \mathbb{R}^n \)-valued Fréchet-\( C^1 \) maps. However, it still holds for our manifold-valued case without modification since information outside a sufficiently small neighborhood of \( a \) is not used in its proof.)

For \( R > 0 \), we set

\[
\hat{B}_R = \{ w \in G \Omega^B_{\alpha,4m}(\mathbb{R}^d) \mid \| w^1 \|_{4m-B}^{4m} + \| w^2 \|_{2n,2m-B}^{2m} < R^{4m} \} \quad (6.2)
\]

and set \( \hat{B}_R(h) = \tau_h(\hat{B}_R) \), where \( \tau_h \) is the Young translation by \( h \in \mathcal{H} \) on \( G \Omega^B_{\alpha,4m}(\mathbb{R}^d) \). Since \( \tau_h \) is a homeomorphism from \( G \Omega^B_{\alpha,4m}(\mathbb{R}^d) \) to itself, \( \{ \hat{B}_R(h) \mid R > 0 \} \) forms a fundamental system of open neighborhoods around \( h = \mathcal{L}(h) \).

**Proposition 6.2.** Assume that \( h \in Q^{n,a} \) satisfies that \( D\psi(h)_1 : \mathcal{H} \to T_a \mathcal{M} \) is surjective and that \( (h, \bullet)_{\mathcal{H}} \in W^* \). Then, there exists a constant \( c = c(h) > 0 \) independent of \( R \) such that

\[
\liminf_{\varepsilon \to 0} \varepsilon^2 \log \nu^\delta_{a,h}(\hat{B}_R(h)) \geq -\frac{1}{2} \| h \|_\mathcal{H}^2 - cR \quad (6.3)
\]

holds for every sufficiently small \( R > 0 \).

**Proof.** In this proof \( R \in (0, R_0) \) and \( \varepsilon \in (0, \varepsilon_0) \), where \( R_0 > 0 \) and \( \varepsilon_0 > 0 \) are sufficiently small constants and may vary from line to line.

Take a coordinate neighborhood \( \tilde{U} \) of \( a \). We also view \( \tilde{U} \) as a bounded open subset of \( \mathbb{R}^n \). By applying a dilation on \( \mathbb{R}^n \) to \( \tilde{U} \) if necessary we may also assume that \( \delta_a \) on \( \mathcal{M} \) (with respect to \( \text{vol} \)) corresponds to the usual \( \delta_a \) on \( \mathbb{R}^n \) (with respect to the standard Lebesgue measure). We also take another open subset \( U \) so that \( a \in U \subset \tilde{U} \subset \tilde{U} \).

For \( F \in D_{2,1}(\mathcal{M}) \) that take values in \( \tilde{U} \) (or \( F \in D_{2,1}(\mathcal{M}) \) restricted to a subset of \( \{ F \in \tilde{U} \} \)), there are two Malliavin covariance matrices. One is the original one defined with respect to the Riemannian metric \( \mathcal{M} \), while the other is the standard one for \( \mathbb{R}^n \)-valued Wiener functionals via the inclusion \( \tilde{U} \subset \mathbb{R}^n \). Since there is a constant \( C = C(\tilde{U}) > 0 \) such that

\[
C^{-1} \det \sigma_F(w) \leq \det \tilde{\sigma}_F(w) \leq C \det \sigma_F(w) \quad \text{on } \{ w \in W \mid F(w) \in \tilde{U} \},
\quad (6.4)
\]

either one of the two works. In what follows we will use the standard one for \( \mathbb{R}^n \)-valued Wiener functionals and denote it by \( \sigma_F \) again by slightly abusing the notation.

For \( b \in \pi^{-1}(a) \), there exist a smooth map \( \tilde{\pi} : \mathcal{P} \to \mathbb{R}^n \) and an open neighborhood \( V \) of \( b \) such that \( \tilde{U} \subset \pi^{-1}(\tilde{U}) \) and \( \pi |_V \equiv \tilde{\pi} |_V \). We can extend \( \tilde{\pi} \) again so that it is a smooth function \( \tilde{\pi} : \mathbb{R}^M \to \mathbb{R}^n \) with compact support. (Recall the embedding \( \iota : \mathcal{P} \hookrightarrow \mathbb{R}^M \) in Remark 3.7.) We write \( \tilde{X}_i^\epsilon = \tilde{\pi}(U_i^\epsilon) \). When we need to specify the dependency on \( u \) and \( w \), we write \( \tilde{X}_i^\epsilon = \tilde{X}_i^\epsilon(t,u,w) \). (We will use the same notation for \( X_i^\epsilon \) and \( U_i^\epsilon \), too.)

Let \( \chi : \mathbb{R} \to \mathbb{R} \) be as in Proposition 4.1. Moreover, we assume that \( \chi \) is even and non-increasing on \([0, \infty)\) so that \( \chi \) takes values in \([0, 1]\). Take any \( f \in C^\infty(\mathbb{R}^n, [0, \infty]) \) whose support is contained in the unit ball and set \( f_j = f^\epsilon f(j \cdot) \). Then, \( \lim_{j \to \infty} f_j = \delta_0 \)
in $S'(\mathbb{R}^n)$. If we set $f^\varepsilon_j = \varepsilon^{-n} f_j((\cdot - a)/\varepsilon)$, then $\lim_{j \to \infty} f^\varepsilon_j = \delta_a$ in $S'(\mathbb{R}^n)$. There exists $j_0 > 0$ such that the support of $f^\varepsilon_j$ is contained in $U$ for every $j \geq j_0$ and $\varepsilon \in (0, 1]$. In that case, $f^\varepsilon_j$ can also be viewed as a function on $\mathcal{M}$. We will assume $j \geq j_0$ and set $b := \phi(h)_1 \in \mathcal{P}$.

Then, it holds that

$$\nu^\varepsilon_{u,a}(\bar{B}_R(h)) = \int I_{B_R(h)}(w)\nu^\varepsilon_{u,a}(dw) = \int I_{B_R(\varepsilon W)}(\tau_h(w))\nu^\varepsilon_{u,a}(dw)$$

$$\geq \int \chi \left( \frac{\|\tau_h(\varepsilon W)\|_{1,4m-B} + \|\tau_h(\varepsilon W)\|_{2,2m-B}}{R^{4m}} \right) \theta^\varepsilon_{u,a}(dw)$$

$$= \mathbb{E} \left[ \chi \left( \frac{\|\tau_h(\varepsilon W)\|_{1,4m-B} + \|\tau_h(\varepsilon W)\|_{2,2m-B}}{R^{4m}} \right) \delta_a(X^\varepsilon_1) \right]$$

$$= \lim_{j \to \infty} \mathbb{E} \left[ \chi \left( \frac{\|\tau_h(\varepsilon W)\|_{1,4m-B} + \|\tau_h(\varepsilon W)\|_{2,2m-B}}{R^{4m}} \right) f^\varepsilon_{j,a}(X^\varepsilon_1) \right]$$

$$= \lim_{j \to \infty} \mathbb{E} \left[ \chi \left( \frac{\|\tau_h(\varepsilon W)\|_{1,4m-B} + \|\tau_h(\varepsilon W)\|_{2,2m-B}}{R^{4m}} \right) f^\varepsilon_{j,a}(X^\varepsilon_1) \right]$$

$$= e^{-\|h\|_h^2/2\varepsilon^2} \lim_{j \to \infty} \mathbb{E} \left[ e^{-\langle h, w \rangle / \varepsilon} \chi \left( \frac{\|\varepsilon W\|_{1,4m-B} + \|\varepsilon W\|_{2,2m-B}}{R^{4m}} \right) \right]$$

$$\times \varepsilon^{-n} f_j \left( \varepsilon^{-1} [X^\varepsilon(1, u, w + \frac{h}{\varepsilon}) - a] \right)$$

$$\geq e^{-\|h\|_h^2/2\varepsilon^2} e^{-c R / \varepsilon} \lim_{j \to \infty} \mathbb{E} \left[ \chi \left( \frac{\|\varepsilon W\|_{1,4m-B} + \|\varepsilon W\|_{2,2m-B}}{R^{4m}} \right) \right]$$

$$\times f_j \left( \varepsilon^{-1} [X^\varepsilon(1, u, w + \frac{h}{\varepsilon}) - a] \right) \right].$$

(6.5)

Note that when $R > 0$ and $\varepsilon > 0$ are sufficiently small, $U^\varepsilon_i$ is close enough to $b$ and therefore we have $X^\varepsilon_i = X^\varepsilon_i$. We used Cameron-Martin formula, too. We now check the last inequality. Form the assumption that $\langle h, \cdot \rangle \in \mathcal{W}^n$ and the fact that $(\alpha, 4m)$-Besov norm (of the first level path) is stronger than the usual sup-norm, we have $|\langle h, w \rangle| \leq c R / \varepsilon$ for a certain constant $c > 0$ if $\|\varepsilon W\|_{1,4m-B} \leq R$.

Set

$$\xi^\varepsilon = R^{-4m} (\|\varepsilon W\|_{1,4m-B} + \|\varepsilon W\|_{2,2m-B})$$

$$F^\varepsilon = \varepsilon^{-1} \left[ X^\varepsilon \left( 1, x, w + \frac{h}{\varepsilon} \right) - a \right]$$

Let $\Gamma$ as in (4.14) (with $(x_t) = (U_t)$, $G = \hat{\pi}$ and $m = M$). Then, Malliavin covariance of $F^\varepsilon$ equals $\Gamma(\pi_{\varepsilon}(\varepsilon W), X^\varepsilon)_1$, which tends to $\Gamma(h, 0)_1$ as $(\varepsilon W, \lambda) \to (0, 0)$. Note that $\Gamma(h, 0)_1$ is the deterministic Malliavin covariance of $\psi_1$ at $h$. As is well-known, $D\psi(h)_1$
is surjective if and only if \( \Gamma(h, 0)_1 \) is a strictly positive symmetric matrix. Hence, when \( R_0 > 0 \) and \( \varepsilon_0 > 0 \) are sufficiently small, there exists a constant \( \rho > 0 \) such that Condition (4.3) holds for every \( \varepsilon \in (0, \varepsilon_0) \) and \( R \in (0, R_0] \). (\( \rho \) does not depend on \( \varepsilon \) or \( R \).) Now we can apply Proposition 4.1 to the right hand side of (6.5) to obtain

\[
\nu_{u,a}^\varepsilon(\hat{B}_R(h)) \geq e^{-\|h\|_2^2/2\varepsilon^2} e^{-cR/\varepsilon^2} \varepsilon^{-\eta} \mathbb{E}[\chi(\varepsilon)\delta_0(F_{\varepsilon})].
\]

It suffices to prove that \( \lim_{\varepsilon \searrow 0} \mathbb{E}[\chi(\varepsilon)\delta_0(F_{\varepsilon})] \) exists and strictly positive. Since \( \chi \) is constant near the origin, (4.5) clearly holds with \( \xi_0 = 0 \). We will check (4.4).

Since \( (U_{\varepsilon}^\varepsilon) \) is viewed as the solution of the vector space-valued SDE with small noise, its asymptotic behavior as \( \varepsilon \searrow 0 \) is well-known:

\[
U_{\varepsilon}^\varepsilon(1, u, w + \frac{h}{\varepsilon}) = b + \varepsilon \eta_1 + O(\varepsilon^2) \text{ in } D_\infty(\mathbb{R}^M),
\]

where \( \eta_1 = \eta_1(w) \) is a certain element of the first order Wiener chaos (which can actually be written down explicitly as a Wiener integral). It is easy to see from this that

\[
F_{\varepsilon} = \nabla \hat{\pi}(b)\langle \eta_1 \rangle + O(\varepsilon) \text{ in } D_\infty(\mathbb{R}^n).
\]

\( \nabla \hat{\pi}(b)\langle \eta_1 \rangle \) is also an element of the first order Wiener chaos and therefore induces a Gaussian measure of mean zero on \( \mathbb{R}^n \). Since its covariance matrix equals \( \Gamma(h, 0)_1 \), the Gaussian measure is non-degenerate and hence its probability density function is strictly positive at the origin. From this and Proposition 4.2 we see that

\[
\lim_{\varepsilon \searrow 0} \mathbb{E}[\chi(\varepsilon)\delta_0(F_{\varepsilon})] = \mathbb{E}[\delta_0(\nabla \hat{\pi}(b)\langle \eta_1 \rangle)] \in (0, \infty).
\]

This completes the proof of Proposition 6.2. \( \square \)

**Proof of the lower estimate in Theorem 5.1 (i).** Let \( O \subset G_{\Omega^B_{\alpha, m}}(\mathbb{R}^d) \) be an open set with \( \inf_{w \in O} I_1(w) < \infty \). Then, we see from Lemma 6.1 that for every \( \kappa > 0 \) we can find \( h \in \mathcal{Q}^{u,a} \) such that (1) \( h \in O \) and (2) \( 0 \leq (\|h\|_2^2/2) - \inf_{w \in O} I_1(w) < \kappa \) and (3) \( h \) satisfies the assumption of Propositions 6.2. Using Proposition (6.2), we have

\[
\liminf_{\varepsilon \searrow 0} \varepsilon^2 \log \nu_{u,a}(O) \geq \liminf_{\varepsilon \searrow 0} \varepsilon^2 \log \nu_{u,a}(\hat{B}_R(h)) \geq - \inf_{w \in O} I_1(w) - cR - \kappa
\]

for every sufficiently small \( R > 0 \). Letting \( R \searrow 0 \) and then \( \kappa \searrow 0 \), we have the desired lower estimate. \( \square \)

### 7 Upper estimate

In this section we prove the upper estimate of LDP in Theorem 5.1 (i). By a standard argument in the large deviation theory, we can easily prove it by using Proposition 7.3 and Proposition 7.5, which will be given below. (That \( \nu_{u,a}^\varepsilon \) may not be a probability
measure is irrelevant in this part.) In this section the positive constant $\nu$ varies from line to line.

The key of proving the upper estimate is a localized version of Itô PDE. This type of Itô PDE is not new. For example, it was used in the proofs of Propositions 4.1 and 4.2. In the manifold-valued Malliavin calculus in [28], a quite similar argument already appeared. Concerning this, see also [30].

We consider the projected process $X^\varepsilon = \pi(U^\varepsilon)$ defined by SDE (3.10). Let $a \in U \subset \bar{U} \subset \bar{U}$ as in (6.4). By taking $\bar{U} \subset \mathbb{R}^n$ slightly smaller if necessary, we can extend the coordinate functions on $\bar{U} \ni x \mapsto x^i \in \mathbb{R}, 1 \leq i \leq n$, to a smooth function on $\mathcal{M}$, which is denoted by $\beta^i$. If we set $\hat{X}^\varepsilon_i = \{\beta^i(X^\varepsilon_i)\}^n_{i=1}$, then $\hat{X}^\varepsilon_i \in D_{\infty}(\mathbb{R}^n)$ and $\hat{X}^\varepsilon_i$ on $\{w \mid X_i^\varepsilon(w) \in \bar{U}\}$. It should be recalled that the Sobolev norm $||\hat{X}^\varepsilon_i||_{p,k}$ is bounded in $\varepsilon \in (0, 1]$ for every $1 < p < \infty$ and $k \in \mathbb{N}$.

**Lemma 7.1.** Let the notation be as above and write $F_\varepsilon := \hat{X}^\varepsilon_i$ for notational simplicity. Suppose that $\eta_1, \eta_2 : \mathbb{R}^n \to [0, 1]$ be smooth functions with compact support in $\bar{U}$ such that $\eta_2 \equiv 1$ on the support of $\eta_1$. Then, for every $f \in \mathcal{S}(\mathbb{R}^n)$, $G \in D_{p,k} (1 < p < \infty, k \in \mathbb{N}_+)$ and $1 \leq i \leq n$, the following assertions hold true:

(i) $\gamma_{F_\varepsilon, \eta_1}(F_\varepsilon) \in D_{\infty}(\text{Mat}(n, n))$. Here, $\gamma_{F_\varepsilon}$ is the inverse of the Malliavin covariance matrix $\sigma_{F_\varepsilon}$.

(ii) If we set

$$\Phi_i^{\eta_1}(\cdot; G) = \sum_{j=1}^d D^{ij} \left( \gamma_{F_\varepsilon, \eta_1}(F_\varepsilon) \cdot G \cdot DF_i^j \right),$$

then $\Phi_i^{\eta_1}(\cdot; G) \in D_{p', k-1}$ for every $p' \in (1, p)$. Moreover, there exist positive constant $c = c_{p,p'}$ and $\nu$ such that

$$\|\Phi_i^{\eta_1}(\cdot; G)\|_{p', k-1} = c\varepsilon^{-\nu} \|G\|_{p,k}, \quad 0 < \varepsilon \leq 1. \quad (7.1)$$

Here, $c = c_{p,p'}$ and $\nu$ are independent of $\varepsilon$ and $G$. Moreover, $\nu$ does not depend on $(p, p')$, either.

(iii) We have

$$\mathbb{E}[\partial_i f(F_\varepsilon) \eta_1(F_\varepsilon) G] = \mathbb{E}[f(F_\varepsilon) \eta_2(F_\varepsilon) \Phi_i^{\eta_1}(\cdot; G)]. \quad (7.2)$$

**Proof.** By (6.6) and (6.4), there exist positive constants $C_p, \nu$ such that

$$\mathbb{E}[\partial_i f(F_\varepsilon) G_1^{\eta_1}(F_\varepsilon)]^{1/p} \leq C_p \varepsilon^{-\nu}, \quad 0 < \varepsilon \leq 1. \quad (7.3)$$

Here, $C_p$ and $\nu$ are independent of $\varepsilon$. Moreover, $\nu$ does not depend on $p$, either.

Now we prove (i). It may be heuristically obvious, but we must take care of the possibility that $\gamma_{F_\varepsilon}$ is not defined outside $\{F_\varepsilon \subset \bar{U}\}$. For $m \in \mathbb{N}_+$ we set $\sigma_{F_\varepsilon}^m := \sigma_{F_\varepsilon} + m^{-1}\text{Id}_n$, where $\text{Id}_n$ stands for the identity matrix of size $n$. Then, $(\det \sigma_{F_\varepsilon}^m)^{-1} \leq m^n$, a.s. and its inverse $\gamma_{F_\varepsilon}^m := (\sigma_{F_\varepsilon}^m)^{-1}$ exists. Moreover, $(\det \sigma_{F_\varepsilon}^m)^{-1} \not\Rightarrow (\det \sigma_{F_\varepsilon})^{-1}$ in $[0, \infty]$, a.s. Note that $\gamma_{F_\varepsilon}^m$ and $(\det \sigma_{F_\varepsilon}^m)$ are both $D_{\infty}$-functionals defined on $\mathcal{W}$. Recall that

$$\gamma_{F_\varepsilon}^m = (\det \sigma_{F_\varepsilon}^m)^{-1} \times [\text{the adjugate matrix of } \sigma_{F_\varepsilon}^m].$$
From this and (7.3), we can easily see that \((\gamma_{F_\varepsilon}^m)^{ij} \eta_1(F_\varepsilon) \to \gamma_{F_\varepsilon}^{ij} \eta_1(F_\varepsilon)\) in \(L^p\) as \(m \to \infty\) (1 < \(p < \infty\).

Now we calculate the first order derivative. Note that

\[
D\{(\gamma_{F_\varepsilon}^m)^{ij} \eta_1(F_\varepsilon)\} = -\sum_{k,l} (\gamma_{F_\varepsilon}^m)^{ik} \cdot D\sigma_{F_\varepsilon}^{kl} \cdot (\gamma_{F_\varepsilon}^m)^{ij} \cdot \eta_1(F_\varepsilon) + \sum_{l} (\gamma_{F_\varepsilon}^m)^{ij} \cdot \partial_l \eta_1(F_\varepsilon) \cdot DF_{F_\varepsilon}^l.
\]

By the same reason as above, this belongs to \(D_\infty(\mathcal{H})\) and converges in \(L^p(\mathcal{H})\), 1 < \(p < \infty\), as \(m \to \infty\). The closability of \(D\) implies that

\[
D\{\gamma_{F_\varepsilon}^{ij} \eta_1(F_\varepsilon)\} = -\sum_{k,l} \gamma_{F_\varepsilon}^{ik} \cdot D\sigma_{F_\varepsilon}^{kl} \cdot \gamma_{F_\varepsilon}^{lj} \cdot \eta_1(F_\varepsilon) + \sum_{l} \gamma_{F_\varepsilon}^{ij} \cdot \partial_l \eta_1(F_\varepsilon) \cdot DF_{F_\varepsilon}^l
\]

and \(\|\gamma_{F_\varepsilon}^{ij} \eta_1(F_\varepsilon)\|_{p,1} = O(\varepsilon^{-\nu})\) for every 1 < \(p < \infty\). Repeating essentially the same argument for higher order derivatives, we can prove that \(\|\gamma_{F_\varepsilon}^{ij} \eta_1(F_\varepsilon)\|_{p,k} \leq O(\varepsilon^{-\nu})\) for every 1 < \(p < \infty\) and \(k \in \mathbb{N}_+\) (if we adjust the value of \(\nu > 0\)). Thus, we have shown (i).

To prove (ii), just recall that \(D^*\) is a bounded linear map from \(D_{p,k}(\mathcal{H})\) to \(D_{p,k-1}\) for every \(p\) and \(k\) (see [13] p. 365) for instance.

Finally, we prove (iii). From a well-known formula for \(D^*\), we see that

\[
\Phi_i^{\tilde{\eta}}(\cdot; G) = -\sum_{j=1}^d \{\langle D(\gamma_{F_\varepsilon}^{ij} \cdot \eta_1(F_\varepsilon) \cdot G), DF_{\varepsilon}^{j}\rangle_{\mathcal{H}} + \gamma_{F_\varepsilon}^{ij} \cdot \eta_1(F_\varepsilon) \cdot G \cdot LF_{\varepsilon}^{j}\}.
\]

where \(L = -D^* D\) is the Ornstein-Uhlenbeck operator. Since this vanishes outside \(\{F_\varepsilon \in \text{supp}(\eta_1)\}\), we have \(\Phi_i^{\tilde{\eta}}(\cdot; G) = \eta_2(F_\varepsilon) \Phi_i^{\eta} (\cdot; G)\). As in the proof for the standard It\'o formula in (7.2), we see from the definition of \(D^*\) that

\[
\mathbb{E}[\partial_i f(F_\varepsilon) \eta_1(F_\varepsilon) G] = \mathbb{E}[\langle D f(F_\varepsilon); \sum_{j=1}^d \gamma_{F_\varepsilon}^{ij} DF_{\varepsilon}^{j}\rangle_{\mathcal{H}} \eta_1(F_\varepsilon) G] = \mathbb{E}[f(F_\varepsilon) \Phi_i^{\eta} (\cdot; G)].
\]

This completes the proof of Lemma 7.1. \(\square\)

Take a smooth function \(\eta_j : \mathbb{R}^n \to [0, 1], 1 \leq j \leq 2n + 1\), with compact support in \(\tilde{U}\) with the following properties: (1) \(\eta_1 \equiv 1\) on \(U\), (2) \(\eta_{j+1} \equiv 1\) on the support of \(\eta_j\) for all \(1 \leq j \leq 2n\). We write \(\eta := \{\eta_i\}_{i=1}^{2n+1}\).

Let \(\beta := (i_1, i_2, \ldots, i_s)\) be a multi-index of length at most \(2n\), that is, \(s \leq 2n\) and \(1 \leq i_1, \ldots, i_s \leq n\). For \(\beta = (i_1)\), just set \(\Phi_{\beta}^{\eta}(\cdot; G) = \Phi_i^{\eta} (\cdot; G)\). For \(\beta = (i_1, i_2)\), set \(\Phi_{\beta}^{\eta}(\cdot; G) = \Phi_i^{\eta} (\cdot, \Phi_i^{\eta} (\cdot; G))\). When \(s \geq 3\), write \(\beta' = (i_1, \ldots, i_{s-1})\) and set recursively \(\Phi_{\beta'}^{\eta}(\cdot; G) = \Phi_{\beta_s}^{\eta} (\cdot, \Phi_{\beta'}^{\eta} (\cdot; G))\).

**Proposition 7.2.** There exists a constant \(\nu > 0\) such that

\[
\|\delta_a(X_1^\varepsilon)\|_{2,-2n} = O(\varepsilon^{-\nu}) \quad \text{as } \varepsilon \downarrow 0.
\]
Proposition 7.3. The family \( \{g_k\}_{k \in \mathbb{N}} \) is a sequence of smooth functions supported in \( U \) such that \( g_k \to \delta_a \) in \( \mathcal{S}'(\mathbb{R}^n) \) as \( k \to \infty \), then we see from the results in [28] that
\[
\mathbb{E}[\delta_a(X^0_1) G] = \lim_{k \to \infty} \mathbb{E}[g_k(X^0_1) G] = \lim_{k \to \infty} \mathbb{E}[g_k(F_\varepsilon) \eta_1(F_\varepsilon) G] = \lim_{k \to \infty} \mathbb{E}[g_k(F_\varepsilon) \eta_1(F_\varepsilon) G]
\]
for every \( G \in \mathcal{D}_\infty \), where we wrote \( F_\varepsilon = \hat{X}^0_1 \) again.

Next, recall that if we set \( f(x) := \prod_{j=1}^n (x_j - a_j)^+ \) and \( \beta = (1, 1, 2, \ldots, n, n) \), then \( \partial^\beta \mathcal{F} = \delta_a \) in the distribution sense. Take \( \kappa > 0 \) so small that \( \prod_{j=1}^n [a_j - \kappa, a_j + \kappa] \subset U \) holds. For this \( \kappa \), we can find a sequence \( \{\lambda_k\}_{k=1}^\infty \) of smooth and non-decreasing function on \( \mathbb{R} \) such that (1) \( \lambda_k(z) \) coincides with \( z^+ := z \vee 0 \) outside \( [\kappa, \kappa] \) for all \( k \), (2) \( \lambda_k(z) \) converges to \( z^+ \) uniformly on \( [\kappa, \kappa] \) as \( k \to \infty \). Set \( f_k(x) = \prod_{j=1}^n \lambda_k(x_j - a_j) \). Then \( f_k \) is smooth on \( \mathbb{R}^n \) and \( \lim_{k \to \infty} f_k = f \) uniformly on \( \mathbb{R}^n \). Note that \( \lim_{k \to \infty} \partial^\beta f_k = \delta_a \) in the distribution sense and \( \partial^\beta f_k \equiv 0 \) outside \( U \). So, we may take \( g_k = \partial^\beta f_k \) in (7.5) above.

By (7.2) in Lemma 7.1 and the way \( \Phi_\beta^\nu(\cdot; G) \) is defined, we have
\[
\mathbb{E}[\partial^\beta f_k(F_\varepsilon) \eta_1(F_\varepsilon) G] = \mathbb{E}[f_k(F_\varepsilon) \eta_{2n+1}(F_\varepsilon) \Phi_\beta^\nu(\cdot; G)].
\]
From this and (7.5), we have
\[
\mathbb{E}[\delta_a(X^0_1) G] = \mathbb{E}[f(F_\varepsilon) \eta_{2n+1}(F_\varepsilon) \Phi_\beta^\nu(\cdot; G)].
\]
Note that \( |f(F_\varepsilon) \eta_{2n+1}(F_\varepsilon)| \) is dominated by a polynomial in \( |F_\varepsilon| \). So, \( \|f(F_\varepsilon) \eta_{2n+1}(F_\varepsilon)\|_{L^p} \) is bounded in \( \varepsilon \) for every \( 1 < p < \infty \). By using (7.2) and (7.1) in Lemma 7.1 repeatedly, we can easily show that
\[
\|\Phi_\beta^\nu(\cdot; G)\|_{L^{3/2}} = c\varepsilon^{-\nu}\|G\|_{2, 2n}, \quad 0 < \varepsilon \leq 1,
\]
if the value of \( \nu > 0 \) is adjusted. Therefore, the right hand side of (7.6) is also dominated by \( c\varepsilon^{-\nu}\|G\|_{2, 2n} \), where \( c \) is independent of \( \varepsilon \) and \( k \). This proves the proposition.

**Proposition 7.3.** The family \( \{\nu_{u,a}^\varepsilon\}_{0 < \varepsilon \leq 1} \) is exponentially tight on \( G\Omega^B_{\alpha,4m}(\mathbb{R}^d) \), that is, for every \( M \in (0, \infty) \), there exists a compact set \( K = K_M \) such that
\[
\limsup_{\varepsilon \downarrow 0} \varepsilon^2 \log \nu_{u,a}^\varepsilon(K^c) \leq -M.
\]

**Proof.** Let \( \hat{B}_R = \hat{B}_{\alpha,4m}^R \) be “ball of radius \( R > 0 \)” as in (6.2). It is shown in [15] Lemma 7.6] that there exists a constant \( c = c_{\alpha,4m} > 0 \) independent of \( R \) such that
\[
\text{Cap}_{2,2n}(\{w \in W \mid \mathcal{L}(w) \in (\hat{B}_{\alpha,4m}^R)^c\}) \leq e^{-cR^2} \quad \text{for sufficiently large } R > 0.
\]

Take \( \kappa > 0 \) so small that Besov parameter \( (\alpha + \kappa, 4m) \) still satisfies (4.8). It is well-known that \( G\Omega^B_{\alpha+k,4m}(\mathbb{R}^d) \) is embedded in \( G\Omega^B_{\alpha,4m}(\mathbb{R}^d) \) and every bounded subset of the former is precompact in the latter.
By the way $\nu^\epsilon_{a,a}$ is defined, it holds that

$$
\nu^\epsilon_{u,a}(\hat{B}^\alpha_{R/\epsilon}) = \theta^\epsilon_{u,a}(\{w \in W \ | \ \epsilon L(w) \in (\hat{B}^\alpha_{R/\epsilon})\})
$$

$$
\leq \|\delta_a(X^\epsilon)\|_{L^2} \inf_{\nu \in \nu^\epsilon_{a,a}} \{\{w \in W \ | \ L(w) \in (\hat{B}^\alpha_{R/\epsilon})\} \}
$$

$$
\leq C\epsilon^{-\nu e^{-cR^2/\epsilon^2}} \quad \text{for sufficiently large } R > 0.
$$

Here, we used the inequality in Item (e) in Subsection 4.1 and Proposition 7.2. Hence, we have

$$
\limsup_{\epsilon \searrow 0} \epsilon^2 \log \nu^\epsilon_{u,a}(\hat{B}^\alpha_{R/\epsilon}) \leq -cR^2.
$$

For given $M > 0$, we choose $R$ so large that $cR^2 \geq M$ holds. Since $\hat{B}^\alpha_{R/\epsilon}$ is precompact in $(\alpha, 4m)$-Besov topology, the proof is completed.

For $R > 0$ and $w \in GOL_{\alpha,4m}(\mathbb{R}^d)$, we set

$$
\tilde{B}_R(w) = \{v \in GOL_{\alpha,4m}(\mathbb{R}^d) | \|v^1 - w^1\|_{\alpha,4m_B} + \|v^2 - w^2\|_{2m_{2m_B}} \leq R^4m\}.
$$

Clearly, $\{\tilde{B}_R(w) | R > 0\}$ forms a fundamental system of neighborhoods around $w$. Set

$$
\Xi_\epsilon = \{(\epsilon w)^1 - w^1\|_{\alpha,4m_B} + \{(\epsilon w)^2 - w^2\|_{2m_{2m_B}}.
$$

Then, for every $\epsilon \in (0,1]$ and $w, \Xi_\epsilon = \Xi_\epsilon(\cdot, w)$ is $\infty$-quasi continuous and belongs to $D_\infty$. Moreover, $\{\Xi_\epsilon\}_\epsilon$ is bounded in $\epsilon$ in $D_{p,k}$ for every $1 < p < \infty$ and $k \in \mathbb{N}$. For a smooth, non-increasing function $\chi: [0, \infty) \rightarrow [0, 1]$ such that $\chi \equiv 1$ on $[0,1]$ and $\chi \equiv 0$ on $[2, \infty)$, $\chi(\Xi_\epsilon/R^4m) \in D_\infty$ satisfies $1_{\tilde{B}_R(w)} \circ (\epsilon L) \leq \chi(\Xi_\epsilon/R^4m) \leq 1_{\tilde{B}_2R(w)} \circ (\epsilon L)$ quasi-surely.

Let us recall that the law of $\epsilon W = \epsilon L(w)$ satisfies the standard version of Schild-type LDP on $GOL_{\alpha,4m}(\mathbb{R}^d)$ with a good rate function $I$, where

$$
I(w) = \begin{cases} 
\frac{1}{2} \|h\|_{L^2}^2 & \text{(if } w = L(h) \text{ for some } h \in H) \\
\infty & \text{(otherwise)}.
\end{cases}
$$

By a general result on LDPs (see [6, Lemma 4.1.6] for example), we then have

$$
\lim_{R \searrow 0} \limsup_{\epsilon \searrow 0} \epsilon^2 \log \mu(\{w \in W \ | \ \epsilon W \in \tilde{B}_R(w)\})
$$

$$
\leq -\lim_{R \searrow 0} \inf \{I(v) \ | \ v \in \tilde{B}_R(w)\} = -I(w), \quad w \in GOL_{\alpha,4m}(\mathbb{R}^d).
$$

\textbf{Lemma 7.4}. Let the notation be as above. For every $p \in (1, \infty)$ and $k \in \mathbb{N}$, it holds that

$$
\lim_{R \searrow 0} \limsup_{\epsilon \searrow 0} \epsilon^2 \log \|\chi(\Xi_\epsilon(\cdot, w)/R^4m)\|_{p,k}^p \leq -I(w), \quad w \in GOL_{\alpha,4m}(\mathbb{R}^d).
$$
Proof. We will write \( \Xi_\epsilon = \Xi_\epsilon(\cdot, w) \). By Meyer’s equivalence, it is enough to estimate \( \sum_{j=0}^k \| D^j \chi(\Xi_\epsilon / R^{4m}) \|_{L^p}^p \), where \( D \) is the \( \mathcal{H} \)-derivative. Hence, it amounts to compute
\[
\max_{0 \leq j \leq k} \limsup_{\epsilon \searrow 0} \epsilon^2 \log \| D^j \chi(\Xi_\epsilon / R^{4m}) \|_{L^p}^p.
\]

For \( j = 0 \), we can easily see that \( \| \chi(\Xi_\epsilon / R^{4m}) \|_{L^p} \leq \mu(\{ \epsilon w \in \tilde{B}_2R(w) \}) \). For \( j = 1 \), we have \( D \chi(\Xi_\epsilon / R^{4m}) = \chi'(\Xi_\epsilon / R^{4m})(D \Xi_\epsilon)R^{-4m} \). Hence, for every \( q \in (1, \infty) \) and \( R > 0 \), there exists a positive constant \( C = C_{q,R} \) (independent of \( \epsilon \)) such that
\[
\| D \chi(\Xi_\epsilon / R^{4m}) \|_{L^p} \leq \mathbb{E}[|\chi'(\Xi_\epsilon / R^{4m})|^p]^{1/q} \mathbb{E}[\| D \Xi_\epsilon \|_{L^q}R^{-4mr}]^{1/r} \leq C \mu(\{ \epsilon w \in \tilde{B}_2R(w) \})^{1/q},
\]
where \( 1/q + 1/r = 1 \). By repeating similar computations, we have for all \( 0 \leq j \leq k \) that
\[
\| D^j \chi(\Xi_\epsilon / R^{4m}) \|_{L^p} \leq C \mu(\{ \epsilon w \in \tilde{B}_2R(w) \})^{1/q}.
\]

Using (7.8), we have \( \lim_{R \searrow 0} \limsup_{\epsilon \searrow 0} \epsilon^2 \log \| \chi(\Xi_\epsilon / R^{4m}) \|_{L^p}^p \leq -(1/q)I_0(w) \). Letting \( q \searrow 1 \), we finish the proof.

**Proposition 7.5.** Let the notation be as above. Then, we have
\[
\lim_{R \searrow 0} \limsup_{\epsilon \searrow 0} \epsilon^2 \log \nu_{u,a}(\tilde{B}_R(w)) \leq -I_1(w), \quad w \in G\Omega_{\alpha,4m}(\mathbb{R}^d).
\]

**Proof.** Write \( \mathring{a} := \Psi(w, 0) \) for simplicity. First, we consider the case \( a \neq \mathring{a} \). Let \( U \) and \( \mathring{U} \) (with \( U \cap \mathring{U} = \emptyset \)) be a neighborhood of \( a \) and \( \mathring{a} \), respectively. By Lyons’ continuity theorem, there exists \( R_0 > 0 \) and \( \epsilon_0 > 0 \) such that \( \Psi(v, \lambda')_1 \in \mathring{U} \) if \( v \in \mathring{B}_{2R_0}(w) \) and \( \epsilon \in (0, \epsilon_0) \). Suppose that \( \{ g_k \}_{k=1}^\infty \) be a sequence of smooth functions with supports in \( U \) that approximates \( \delta_a \). Then, we have
\[
\nu_{u,a}^\epsilon(\tilde{B}_R(w)) \leq \mathbb{E}[|\chi(\Xi_\epsilon / R^{4m})\delta_a(X_1^\epsilon)|] = \lim_{k \to \infty} \mathbb{E}[\chi(\Xi_\epsilon / R^{4m})g_k(X_1^\epsilon)]
\]
\[
\leq \lim_{k \to \infty} \mathbb{E}[\mathbf{1}_{\{ \epsilon w \in \tilde{B}_{2R}(w) \}}g_k(\Psi(\epsilon w, \lambda')_1)] = 0
\]
if \( \epsilon \in (0, \epsilon_0) \) and \( R \in (0, R_0) \). Thus, we have shown (7.9) for this case.

Next, we consider the case \( a = \mathring{a} \). As in the proof of Proposition 7.2, we use the localized IbP formula. Let \( \eta := \{ \eta_i \}_{i=1}^{2n+1} \) as in Proposition 7.2. Then, we can see that, for every \( p \in (1, \infty) \), there exist positive constants \( c \) and \( \nu \) (independent of \( \epsilon \)) such that
\[
\nu_{u,a}^\epsilon(\tilde{B}_R(w)) \leq \mathbb{E}[|\chi(\Xi_\epsilon / R^{4m})\delta_a(X_1^\epsilon)|]
\]
\[
= \mathbb{E}[f(F_v)\eta_{2n+1}(F_v)\Phi^\nu_\delta(\cdot; \chi(\Xi_\epsilon / R^{4m}))]
\]
\[
\leq c\epsilon^{-\nu} \| \chi(\Xi_\epsilon / R^{4m}) \|_{L^p}.
\]

Here, we used (7.11) and (7.2) in Lemma 7.4 repeatedly. Using Lemma 7.4 and then letting \( p \searrow 1 \), we have \( \lim_{R \searrow 0} \limsup_{\epsilon \searrow 0} \epsilon^2 \log \nu_{u,a}^\epsilon(\tilde{B}_R(w)) \leq -I_1(w) \) in this case. Thus, we have proved (7.9).
A  Positivity of heat kernel

Let \( p_\varepsilon(x,a) = \mathbb{E}[\delta_a(X^\varepsilon_t)] \) be the heat kernel (or the density function) on \( \mathcal{M} \) associated with the \( \varepsilon^2(\Delta_{sub}/2+V) \)-diffusion process. Here, \( (X^\varepsilon_t) \) is the projection of the solution of SDE \( (3.10) \) with the starting point \( u \). The skeleton ODE with drift which corresponds to SDE \( (3.10) \) with \( \varepsilon = 1 \) is given as follows:

\[
d\bar{\phi}(h)_t = \sum_{i=1}^{d} A_i(\bar{\phi}(h)_t)dh^i_t + A_0(\bar{\phi}(h)_t)dt, \quad \bar{\phi}(h)_0 = u. \quad (A.1)
\]

We write \( \bar{\psi}(h)_t = \pi(\bar{\phi}(h)_t) \).

The purpose of this appendix is to verify that \( \bar{\psi}(h)_t \) solves ODE \( (A.1) \). By extending the coefficient vector fields with compact support, we also view \( \bar{\psi}(h)_t \) as a coordinate neighborhood of \( \bar{\psi}(h)_t \).

Second, consider \( \zeta \) is a path on \( \mathcal{M} \) of finite energy such that \( \zeta'_t - V_0(\zeta_t) \in D_{\mathcal{G}} \) for almost all \( t \). Conversely, if \( \zeta \) is a path on \( \mathcal{M} \) of finite energy such that \( \zeta'_t - V_0(\zeta_t) \in D_{\mathcal{G}} \) for almost all \( t \), then there uniquely exists a Cameron-Martin path \( h \) such that \( \zeta = \zeta(h) \).

Lemma A.1. Let \( x,a \in \mathcal{M} \) and \( u \in \pi^{-1}(x) \). Then, for every \( \tau \in (0,1] \), there exists \( h \in \mathcal{H} \) such that \( \bar{\psi}(h)_\tau = a \) and \( D\bar{\psi}(h)_\tau : \mathcal{H} \to T_a\mathcal{M} \) is surjective.

Proof. It is sufficient to prove the case \( \tau = 1 \). The general case can be done with trivial modifications.

We now make two simple remarks. First, let \( \{Z_1, \ldots, Z_d\} \) be an orthonormal frame of \( \mathcal{D} \) on a coordinate neighborhood \( U \subset \mathcal{M} \). Consider the following ODE on \( U \) driven by a Cameron-Martin path \( h \):

\[
d\zeta(h)_t = \sum_{i=1}^{d} Z_i(\zeta(h)_t)dh^i_t + V_0(\zeta(h)_t)dt. \quad (A.3)
\]

Then, \( \zeta = \zeta(h) \) is of finite energy and satisfies that \( \zeta'_t - V_0(\zeta_t) \in D_{\mathcal{G}} \) for almost all \( t \). Conversely, if \( \zeta \) is a path on \( \mathcal{M} \) of finite energy such that \( \zeta'_t - V_0(\zeta_t) \in D_{\mathcal{G}} \) for almost all \( t \), there uniquely exists a Cameron-Martin path \( h \) such that \( \zeta = \zeta(h) \). In this sense, we have a one-to-one correspondence \( h \leftrightarrow \zeta(h) \).

By using Proposition 3.3 and Lemma 4.3, we can now prove Lemma A.1 in a similar way to the proof of Lemma 6.1.
as an ODE on $\mathbb{R}^n$. Since these vector fields satisfy Hörmander’s condition at $x$, we can use Lemma 4.3. It implies that there exists a Cameron-Martin path $k: [0, 1/2] \to \mathbb{R}^d$ such that (1) the tangent map of $\zeta(\cdot)_{1/2}$ is surjective at $k$ and (2) $[0, 1/2] \ni t \mapsto \zeta(k)_t$ stays inside $U$. By Proposition 3.3 there exist a finite energy path $\eta: [1/2, 1] \to M$ such that $\eta_{1/2} = \zeta(k)_{1/2}$, $\eta_1 = a$ and $\eta_t' - V_0(\eta_t) \in D_{\eta_t}$ for almost all $t$.

Define $\zeta \in H_x(M, TM)$ to be the concatenation of $\zeta(k)$ and $\eta$. Then, $\zeta_1 = a$ and $\zeta_t' - V_0(\zeta_t) \in D_{\zeta_t}$ for almost all $t$. The corresponding $h \in H$ is the desired element. (The proof of the surjectivity of $D\hat{\psi}(h)_1$ is essentially the same as in the proof of Lemma 6.1 and is therefore omitted.)

Remark A.2. Precisely, the positivity theorem for the density in [2] is for SDEs on a Euclidean space. But, after slightly modifying it, one can verify that it still holds for SDEs on a compact manifold.

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