A Mixed Mimetic Spectral Element Model of the 3D Compressible Euler Equations on the Cubed Sphere

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Abstract

A model of the three-dimensional rotating compressible Euler equations on the cubed sphere is presented. The model uses a mixed mimetic spectral element discretization which allows for the exact exchanges of kinetic, internal and potential energy via the compatibility properties of the chosen function spaces. A Strang carryover dimensional splitting procedure is used, with the horizontal dynamics solved explicitly and the vertical dynamics solved implicitly so as to avoid the CFL restriction of the vertical sound waves. The function spaces used to represent the horizontal dynamics are discontinuous across vertical element boundaries, such that each horizontal layer is solved independently so as to avoid the need to invert a global 3D mass matrix, while the function spaces used to represent the vertical dynamics are similarly discontinuous across horizontal element boundaries, allowing for the serial solution of the vertical dynamics independently for each horizontal element. The model is validated against a standard test case for a baroclinic instability within an otherwise hydrostatically and geostrophically balanced atmosphere.

Keywords: Mimetic, Spectral element, Euler equations, Cubed sphere, Horizontally explicit/vertically implicit

1. Introduction

Mimetic finite element families are an appealing choice for the discretization of geophysical flow problems. This is on account of their capacity to preserve both conservation laws and leading order balance relations in the discrete form [1–4], and their ability to represent complex geometries such as the surface of the sphere [5–7].

The use of mimetic discretizations to represent the solution variables, and the adjoint properties of the differential operators implied by those spaces, allows for the conservation of energy via the exact balance of kinetic, internal and potential energy exchanges [2–4]. Several mimetic finite volume models of the compressible Euler equations on the sphere have previously been presented [8–10], and the Raviart-Thomas family of compatible finite elements has been chosen to form the basis of the LFric atmospheric model [11, 12]. Compatibility properties have also been used to preserve energy conservation in moist hydrostatic atmospheric models using collocated spectral elements [13]. In the present formulation we make use of the mixed mimetic spectral element method [14, 15], a compatible family of function spaces with spectral error convergence, and extend on previous work on the rotating shallow water equations [3, 5] in order to develop a solver for the 3D rotating compressible Euler equations on the cubed sphere.

The remainder of this article proceeds as follows: In Section 2 the rotating compressible Euler equations, and their energetic properties will be introduced in the continuous form. Section 3 will provide a brief introduction to the mixed mimetic spectral element method. Readers are referred to references therein for more detailed discussions. Section 4 will discuss the construction of discrete function spaces built off the mixed mimetic spectral element method required to solve the compressible Euler equations, as well as the use of those spaces to ensure consistent energetic exchanges in the discrete form and the associated metric transformations for these spaces. The details of the time stepping scheme, including the implicit vertical solver will be discussed in Section 5, and the results for a standard
baroclinic instability test case will be presented in Section 6. Section 7 will discuss the conclusions of this work and the future directions we intend to pursue.

2. The rotating compressible Euler equations

The compressible Euler equations for a shallow atmosphere may be expressed as \[9, 16\]

\[
\begin{align*}
\frac{\partial u}{\partial t} + (\omega + f) \times u + \nabla \left( \frac{1}{2} \|u\|^2 + gz \right) + \theta \nabla \Pi &= 0, \\
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) &= 0, \\
\frac{\partial \rho \theta}{\partial t} + \nabla \cdot (\rho \theta u) &= 0,
\end{align*}
\]

where \( u = u e_\lambda + v e_\phi + w e_z \) are the zonal, meridional and vertical velocity components respectively, \( \rho \) is the density, \( f = fe_z \) is the Coriolis term, \( g \) is the acceleration due to gravity, \( \theta \) is the potential temperature, and \( \Pi \) is the Exner pressure (including the specific heat at constant pressure). The last two are defined with respect to the standard thermodynamic variables of temperature, \( T \), and pressure, \( p \), as

\[
\Pi := c_p \left( \frac{p}{p_0} \right)^{\frac{R}{c_p}},
\]

\[
\theta := \frac{c_p T}{\Pi},
\]

\[
p = \rho RT.
\]

For these identities we used \( c_p \) for the specific heat at constant pressure, \( p_0 \) for the reference pressure, and \( R \) for the ideal gas constant. We may remove the direct dependence on pressure from the system by simply substituting expression (2c) for pressure into (2a) and (2b), obtaining

\[
\Pi = c_p \left( \frac{\rho R \theta \Pi}{p_0 c_p} \right)^{\frac{R}{c_p}} = c_p \left( \frac{\rho R \theta}{p_0} \right)^{\frac{R}{c_p}},
\]

\[
\theta = \frac{\rho RT - cv}{p_0},
\]

where \( cv = c_p - R \) is the specific heat at constant volume.

The potential temperature/Exner pressure form of the pressure gradient term, \( \theta \nabla \Pi \), in (1a) is equivalent to the standard density/pressure form since

\[
\frac{1}{\rho} \nabla p = \frac{RT}{p} \nabla p + \frac{R \theta}{p} \left( \frac{p}{p_0} \right)^{\frac{R}{c_p}} \nabla p = \frac{R \theta}{p_0} \frac{c_p}{R} p^{R/c_p} \nabla p = \frac{R \theta}{p_0} \frac{c_p}{R} \nabla p = c_p \theta \nabla \left( \frac{p}{p_0} \right)^{\frac{R}{c_p}} \theta \nabla \Pi.
\]

One advantage of the Exner pressure/potential temperature representation of the thermodynamics is the formulation of the temperature equation in flux form (1c), which allows us to exploit the adjoint relationship between gradient and divergence in the mimetic framework in order to preserve energetic exchanges.

To obtain a closed system for the solution of the compressible Euler equations, (1) and (3) must be supplemented by Dirichlet and Neumann boundary conditions. The following identities are imposed as Dirichlet boundary conditions on the \( z \)-component of velocity, \( w \), and on the potential temperature, \( \theta \),

\[
w|_{z=0} = w|_{z=z_{top}} = 0,
\]

\[
\theta|_{z=0} = \theta^b,
\]

\[
\theta|_{z=z_{top}} = \theta^t,
\]
where \(z^{top}\) corresponds to the \(z\)-coordinates of the top boundary of the domain, and \(\theta^b\) and \(\theta^t\) correspond to the potential temperature at the bottom and top boundaries of the computational domain, respectively. For Neumann boundary conditions the following identities are imposed

\[
\frac{\partial \Pi}{\partial z} \bigg|_{z=0} = \frac{\partial \Pi}{\partial z} \bigg|_{z=\xi_{top}} = 0.
\]

Note that in this formulation we have invoked the shallow atmosphere approximation, for which gravity is constant throughout the fluid column, the height of the fluid column is negligible with respect to the earth’s radius, and the horizontal components of the Coriolis term are omitted [17].

2.1. Energetics

Before introducing the discrete form of the Euler equations, we analyse the energetics of the continuous system. This will help to guide our choice of function spaces for the various solution variables for the discrete form.

2.1.1. Kinetic, potential, and internal energy

The kinetic energy, \(K\), is defined as

\[
K := \frac{1}{2} \langle u, \rho u \rangle = \frac{1}{2} \int_\Omega \rho \|u\|^2.
\]

(7)

where \(\|u\| := \langle u, u \rangle\), and \(\langle \cdot , \cdot \rangle\) is the \(L^2\) inner product given as usual as

\[
\langle f, g \rangle := \int_\Omega fg \, d\Omega,
\]

(8)

for scalar fields, and as

\[
\langle u, v \rangle := \int_\Omega u \cdot v \, d\Omega,
\]

(9)

for vector fields.

The time variation of kinetic energy is obtained by summing the \(L^2\) inner product, between the momentum equation, (1a), and \(\rho u\), and between the continuity equation, (1b), and \(\frac{1}{2} \|u\|^2\)

\[
\frac{\partial K}{\partial t} = -\langle g, \rho w \rangle - \langle \rho u, \theta \nabla \Pi \rangle,
\]

(10)

where again \(w\) is the \(z\)-component of the velocity field, \(u\).

The potential energy, \(P\), is given by

\[
P := \langle \rho , gz \rangle = \int_\Omega \rho gz \, d\Omega,
\]

(11)

and its time derivative follows directly

\[
\frac{\partial P}{\partial t} = \left\langle gz, \frac{\partial \rho}{\partial t} \right\rangle + \langle gz, \nabla \cdot (\rho u) \rangle = \langle g, \rho \omega \rangle,
\]

(12)

where we have used integration by parts on the last identity and homogeneous boundary conditions for the vertical component of the velocity field, \(\xi\).

The internal energy, \(I\), is defined as

\[
I := \int_\Omega c_v \rho T \, d\Omega \quad \text{and} \quad \int_\Omega c_v \rho \theta \Pi \, d\Omega \quad \text{and} \quad \int c_v \rho \theta \left( \frac{R \rho \theta}{p_0} \right) \frac{\xi}{\xi} \, d\Omega = \int_\Omega c_v \left( \frac{R}{p_0} \right) \left( \rho \theta \right) \frac{\xi}{\xi} \, d\Omega.
\]

(13)

After some manipulation, the time variation of internal energy is given by

\[
\frac{\partial I}{\partial t} = -\langle \nabla \cdot (\rho \theta u) , \Pi \rangle = \langle \rho u , \theta \nabla \Pi \rangle.
\]

(14)

where integration by parts was used on the last identity, together with homogeneous boundary conditions for \(u\) and periodic boundary conditions on the horizontal directions.
2.1.2. Conservation of total energy

Following [9], the total energy of the system, $\mathcal{H}$, is given as the sum of kinetic, $K$, potential, $P$, and internal, $I$, energy

$$\mathcal{H} := K + P + I = \int_{\Omega} \frac{1}{2} \rho u^2 \, d\Omega + \int_{\Omega} \rho g z \, d\Omega + \int_{\Omega} \frac{c_v}{\Theta} \Pi \, d\Omega. \quad (15)$$

where we used $\Theta := \rho \theta$.

For the proof of conservation of total energy $\mathcal{H}$, (15), first consider the column vectors

$$\mathbf{a} := \begin{bmatrix} u & \rho & \Theta \end{bmatrix}^T, \quad (16)$$

and

$$\mathbf{h} := \begin{bmatrix} U & \Phi & \Pi \end{bmatrix}^T, \quad (17)$$

where $U := \rho u$, and $\Phi := \frac{1}{2} u^2 + gz$. Introducing the skew-symmetric operator

$$\mathbf{B} := \begin{bmatrix} -q \times (\cdot) & -\nabla (\cdot) & -\Theta \nabla (\cdot) \\ -\nabla \cdot (\cdot) & 0 & 0 \\ -\nabla \cdot (\Theta \cdot (\cdot)) & 0 & 0 \end{bmatrix}, \quad (18)$$

where $q = (\omega + f)/\rho$ is the potential vorticity, the original prognostic equations, (1a)-(1c), may be rewritten as

$$\frac{\partial \mathbf{a}}{\partial t} = \mathbf{B} \mathbf{h}. \quad (19)$$

Note now that the variational derivatives of $\mathcal{H}$ with respect to the prognostic variables $u$, $\rho$, and $\Theta$, are

$$\frac{\delta \mathcal{H}}{\delta u} = \rho u = U, \quad \frac{\delta \mathcal{H}}{\delta \rho} = \frac{1}{2} u^2 + gz = \Phi, \quad \frac{\delta \mathcal{H}}{\delta \Theta} = \Pi, \quad (20)$$

and therefore

$$\mathbf{h} = \frac{\delta \mathcal{H}}{\delta \mathbf{a}}. \quad (21)$$

Substituting (21) into (19) yields

$$\frac{\partial \mathbf{a}}{\partial t} = \mathbf{B} \frac{\delta \mathcal{H}}{\delta \mathbf{a}}. \quad (22)$$

Conservation of total energy follows directly since (18)

$$\frac{\partial \mathcal{H}}{\partial t} = \frac{\delta \mathcal{H}}{\delta \mathbf{a}} \cdot \frac{\partial \mathbf{a}}{\partial t} = \mathbf{B} \frac{\delta \mathcal{H}}{\delta \mathbf{a}} \left( \mathbf{B} \frac{\delta \mathcal{H}}{\delta \mathbf{a}} \right) = 0, \quad (23)$$

where the last identity follows from the skew-symmetry of $\mathbf{B}$. Note that here the dot product involves not only a summation over the elements of the vectors but also an integration over $\Omega$, e.g.

$$\frac{\delta \mathcal{H}}{\delta \mathbf{a}} \cdot \frac{\partial \mathbf{a}}{\partial t} = \int_{\Omega} \sum_{i=1}^{3} \delta \mathcal{H} \frac{\partial a_i}{\partial t} \, d\Omega.$$

3. Mimetic polynomial basis functions

3.1. 1D mimetic polynomial function spaces

The mixed mimetic spectral element method is built off two types of one-dimensional polynomials: one associated with nodal interpolation, and the other with integral interpolation (histopolation) [14][19]. Subsequently, these two types of polynomials will be combined to generate the family of three-dimensional basis functions used to discretize the system.
3.1.1. Nodal polynomial basis functions

Consider the canonical interval $I = [-1, 1] \subset \mathbb{R}$ and the Legendre polynomials, $L_N(\xi)$ of degree $N$ with $\xi \in I$. The $N + 1$ roots, $\xi_i$, of the polynomial $\left(1 - \xi^2\right) \frac{d}{d\xi}$ are called Gauss-Lobatto-Legendre (GLL) nodes and satisfy $-1 = \xi_0 < \xi_1 < \cdots < \xi_{N-1} < \xi_N = 1$. Let $l_i^N(\xi)$ be the Lagrange polynomial of degree $N$ through the GLL $\xi_i$, such that

$$l_i^N(\xi) = \delta_{i,j}, \quad i, j = 0, \ldots, N, \quad (24)$$

where $\delta_{i,j}$ is the Kronecker delta. The explicit form of these Lagrange polynomials is given by

$$l_i^N(\xi) = \prod_{k \neq i}^{N} \frac{\xi - \xi_k}{\xi_i - \xi_k}. \quad (25)$$

Let $q_h(\xi)$ be a polynomial of degree $N$ defined on $I = [-1, 1]$ and $q_i = q_h(\xi_i)$, then the expansion of $q_h(\xi)$ in terms of Lagrange polynomials is given by

$$q_h(\xi) := \sum_{i=0}^{N} q_i l_i^N(\xi). \quad (26)$$

Because the expansion coefficients in (26) are given by the value of $q_h$ in the nodes $\xi_i$, we refer to this interpolation as a nodal interpolation and we will denote the Lagrange polynomials in (25) by nodal polynomials.

3.1.2. Histopolant polynomial basis functions

Using the nodal polynomials we can define another set of basis polynomials, $e_i^N(\xi)$, as

$$e_i^N(\xi) := -\sum_{k=0}^{i-1} \frac{d l_k^N(\xi)}{d\xi}, \quad i = 1, \ldots, N. \quad (27)$$

These polynomials $e_i^N(\xi)$ have polynomial degree $N - 1$ and satisfy,

$$\int_{\xi_{i-1}}^{\xi_i} e_j^N(\xi) d\xi = \delta_{i,j}, \quad i, j = 1, \ldots, p. \quad (28)$$

Using (27), the integral of $e_j^N(\xi)$ becomes [14][19]

$$\int_{\xi_{i-1}}^{\xi_i} e_j^N(\xi) d\xi = \int_{\xi_{i-1}}^{\xi_i} \sum_{k=0}^{i-1} \frac{d l_k^N(\xi)}{d\xi} = -\sum_{k=0}^{i-1} \int_{\xi_{i-1}}^{\xi_i} \frac{d l_k^N(\xi)}{d\xi} = -\sum_{k=0}^{i-1} (l_k^N(\xi_i)-l_k^N(\xi_{i-1})) = -\sum_{k=0}^{i-1} (\delta_{k,j} - \delta_{k,j-1}) = \delta_{i,j}. \quad (29)$$

Let $g_h(\xi)$ be a polynomial of degree $(N - 1)$ defined on $I = [-1, 1]$ and $g_i = \int_{\xi_{i-1}}^{\xi_i} g_h(\xi) d\xi$, then its expansion in terms of the polynomials $e_i^N(\xi)$ is given by

$$g_h(\xi) := \sum_{i=1}^{N} g_i e_i^N(\xi). \quad (29)$$

Because the expansion coefficients in (29) are the integral values of $g_h(\xi)$, we denote the polynomials in (27) by histopolant polynomials and refer to (29) as histopolation. It can be shown, [14][19], that if $q_h(\xi)$ is expanded in terms of nodal polynomials, as in (26), then the expansion of its derivative $\frac{d q_h(\xi)}{d\xi}$ in terms of histopolant, or edge polynomials is

$$\left(\frac{d q_h(\xi)}{d\xi}\right)_h = \sum_{i=1}^{N} \left(\int_{\xi_{i-1}}^{\xi_i} \frac{d q_h(\xi)}{d\xi} d\xi\right) e_i^N(\xi) = \sum_{i=1}^{N} (q_h(\xi_i) - q_h(\xi_{i-1})) e_i^N(\xi)$$

$$= \sum_{i=1}^{N} (q_i - q_{i-1}) e_i^N(\xi) = \sum_{i=1,j=0}^{p} E_{i,j}^0 q_j e_i^N(\xi), \quad (30)$$

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Figure 1: Basis polynomials associated to $p = 4$. Left: nodal polynomials, $l^p_i(\xi)$. Right: edge polynomials, $e^p_i(\xi)$.

where $E^{1,0}_{i,j}$ are the coefficients of the $N \times (N + 1)$ matrix $E^{1,0}$ for the one dimensional case, hereafter referred to as an *incidence* matrix. The following identity holds (Commuting property)

$$\left( \frac{dq(\xi)}{d\xi} \right)_h = \frac{dq_h(\xi)}{d\xi}. \quad (31)$$

For an example of the one-dimensional basis polynomials corresponding to $N = 4$, see Figure 1.

### 3.2. 3D mimetic polynomial function spaces

A fundamental element in the proposed discretization for the compressible Euler equations, (1), is the de Rham sequence of function spaces in the domain $\Omega \subset \mathbb{R}^3$

$$\mathbb{R} \rightarrow H^1(\Omega) \xrightarrow{\nabla} H(\text{curl}, \Omega) \xrightarrow{\nabla} H(\text{div}, \Omega) \xrightarrow{\nabla} L^2(\Omega) \rightarrow 0, \quad (32)$$

where, as usual, the space $H^1(\Omega)$ represents square integrable functions over $\Omega$ whose gradient is also square integrable, the function spaces $H(\text{curl}, \Omega)$ and $H(\text{div}, \Omega)$ contain square integrable vector fields over $\Omega$ with square integrable curl and divergence, respectively, and the function space $L^2(\Omega)$ contains square integrable functions.

More specifically, this work relies on approximating polynomial spaces $\mathcal{P}_h(\Omega) \subset H^1(\Omega)$, $\mathcal{W}_h(\Omega) \subset H(\text{curl}, \Omega)$, $\mathcal{U}_h(\Omega) \subset H(\text{div}, \Omega)$, and $\mathcal{Q}_h(\Omega) \subset L^2(\Omega)$ such that

$$\mathbb{R} \rightarrow \mathcal{P}_h(\Omega) \xrightarrow{\nabla} \mathcal{W}_h(\Omega) \xrightarrow{\nabla} \mathcal{U}_h(\Omega) \xrightarrow{\nabla} \mathcal{Q}_h(\Omega) \rightarrow 0. \quad (33)$$

These 3-dimensional polynomial function spaces may be constructed from tensor products of the 1-dimensional function spaces presented in Section 3.1. Moreover, each of these polynomial function spaces has an associated finite set of basis functions $\epsilon^p_i$, $\epsilon^W_i$, $\epsilon^U_i$, and $\epsilon^Q_i$, such that

$$\mathcal{P}_h = \text{span}\{\epsilon^p_1, \ldots, \epsilon^p_{d_p}\}, \quad \mathcal{W}_h = \text{span}\{\epsilon^W_1, \ldots, \epsilon^W_{d_w}\}, \quad \mathcal{U}_h = \text{span}\{\epsilon^U_1, \ldots, \epsilon^U_{d_u}\}, \quad \text{and} \quad \mathcal{Q}_h = \text{span}\{\epsilon^Q_1, \ldots, \epsilon^Q_{d_q}\}. \quad (34)$$
As previously discussed, e.g. [20], these basis functions are given by
\[ e_m^\xi (\xi, \eta, \zeta; N) := l_i^\xi (\xi) l_j^\eta (\eta) l_k^\zeta (\zeta), \quad m = i + jN + kN^2, \quad i, j, k = 0, \ldots, N, \]
\[ e_m^\omega (\xi, \eta, \zeta; N) := \begin{cases} 
  l_i^\xi (\xi) l_j^\eta (\eta) l_k^\zeta (\zeta) e_1, & m = i + jN + kN^2 + 1, 
  i = 1, \ldots, N, j = 0, \ldots, N, k = 0, \ldots, N, 
  \end{cases} \]
\[ e_m^W (\xi, \eta, \zeta; N) := \begin{cases} 
  l_i^\xi (\xi) l_j^\eta (\eta) l_k^\zeta (\zeta) e_2, & m = i + jN + kN^2, 
  i = 1, \ldots, N, j = 0, \ldots, N, k = 0, \ldots, N, 
  \end{cases} \]
\[ e_m^H (\xi, \eta, \zeta; N) := \begin{cases} 
  l_i^\xi (\xi) l_j^\eta (\eta) l_k^\zeta (\zeta) e_3, & m = i + jN + kN^2 + 1, 
  i = 1, \ldots, N, j = 0, \ldots, N, k = 0, \ldots, N, 
  \end{cases} \]
\[ e_m^\omega (\xi, \eta, \zeta; N) := e_m^\omega (\eta) e_m^\omega (\zeta), \quad m = i + jN + kN^2, \quad i, j, k = 0, \ldots, N, \]
where for compactness, the subscripts + and − mean addition and subtraction of 1, e.g. \( N_+ := N + 1 \) and \( i_- := i - 1 \).

Moreover, the basis functions satisfy the following identities, see for example [20]:
\[ \nabla e^\omega_1 = \sum_{k=0}^{d_0} E^1_{\xi,j} e^W_k, \quad \nabla \times e^W_1 = \sum_{k=0}^{d_0} E^2_{\xi,j} e^H_k, \quad \text{and} \quad \nabla \cdot e^H_1 = \sum_{k=0}^{d_0} E^3_{\xi,j} e^\omega_k, \]
(35)

where \( E^1_{\xi,j} \), \( E^2_{\xi,j} \), and \( E^3_{\xi,j} \) are the incidence matrices corresponding to the discrete versions of the differential operators grad, curl, and div, respectively.

4. Numerical discretization

4.1. Splitting into horizontal and vertical contributions

Consider the following splitting into the horizontal, \( u_\parallel \), and vertical, \( u_\perp \), components of the velocity field \( u = u e_\perp + v e_\parallel + w e_z \):
\[ u_\parallel := u e_\perp + v e_\parallel, \quad u_\perp := w e_z. \]
(36)

Moreover, let \( \nabla_\parallel \) and \( \nabla_\perp \) represent the horizontal and vertical components of the gradient operator of a scalar field \( \phi \):
\[ \nabla_\parallel \phi = \frac{\partial \phi}{\partial x} e_1 + \frac{\partial \phi}{\partial y} e_2 + \frac{\partial \phi}{\partial z} e_3, \quad \nabla_\perp \phi = \frac{\partial \phi}{\partial x} e_1 + \frac{\partial \phi}{\partial y} e_2. \]
(37)

In a similar way, \( \nabla_\parallel \times \) and \( \nabla_\perp \times \) represent, respectively, the horizontal and vertical components of the curl of a vector field \( u = u e_\perp + v e_\parallel + w e_z \):
\[ \nabla_\parallel \times u := \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) e_1 + \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) e_2 + \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) e_3, \quad \nabla_\perp \times u := \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) e_1. \]
(38)

From (37) and (38) follows directly that
\[ \nabla \phi = \nabla_\parallel \phi + \nabla_\perp \phi, \quad \text{and} \quad \nabla \times u = \nabla_\parallel \times u + \nabla_\perp \times u. \]

With (36) and (38) it is possible to rewrite the definition of vorticity, \( \omega := \nabla \times u \), as
\[ \omega = \frac{\omega_{\parallel}}{\omega_\parallel} + \frac{\omega_{\perp}}{\omega_\perp} + \frac{\omega_{\parallel}}{\omega_\perp} u_1. \]
(39)
Using (36), (37), (38), and (39), it is possible to split the compressible Euler equations, (1), into horizontal and vertical components

\[
\frac{\partial u_{i}}{\partial t} + (\omega_{i} + f_{i}) \times u_{i} + \omega_{i} \times u_{i} + \frac{1}{2} \nabla ||u_{i}||^{2} + \theta \nabla \Pi = 0, \quad (40a)
\]

\[
\frac{\partial u_{\perp}}{\partial t} + \omega_{\perp} \times u_{\perp} + \nabla_{\perp} \left( \frac{1}{2} ||u_{\perp}||^{2} + g_{z} \right) + \theta \nabla_{\perp} \Pi = 0, \quad (40b)
\]

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u_{i}) + \nabla \cdot (\rho u_{\perp}) = 0, \quad (40c)
\]

\[
\frac{\partial (\rho \theta)}{\partial t} + \nabla \cdot (\rho \theta u_{i}) = 0, \quad (40d)
\]

where, as in (39), \(\omega_{i} := \nabla_{i} \times u_{i}\) and \(\omega_{\perp} := \nabla \times u_{\perp}\).

The same splitting into horizontal and vertical components may be done for the basis functions, Section 3.2

\[
\epsilon^{W}_{j} = (\epsilon^{W}_{j})_{i} + (\epsilon^{W}_{j})_{\perp} := \epsilon^{W}_{ji} + \epsilon^{W}_{j\perp}, \quad (41)
\]

and

\[
\epsilon^{U}_{j} = (\epsilon^{U}_{j})_{i} + (\epsilon^{U}_{j})_{\perp} := \epsilon^{U}_{ji} + \epsilon^{U}_{j\perp}. \quad (42)
\]

Recalling the definition of \(\epsilon^{W}_{j}\) and \(\epsilon^{U}_{j}\), Section 3.2 we can explicitly write their horizontal and vertical components as

\[
\epsilon^{W}_{m}(\xi, \eta, \zeta; N) := \begin{cases} 
\epsilon^{W}_{m}(\xi)\epsilon^{W}_{m}(\eta)\epsilon^{W}_{m}(\zeta)e_{i}, & m = i + jN + kNN_{\perp} - 1, \quad i, j, k = 0, \ldots, N, \\
\epsilon^{W}_{m}(\xi)e^{N}_{m}(\eta)\epsilon^{W}_{m}(\zeta)e_{i}, & m = i + jN + kNN_{\perp} + NN_{\perp}^{2}, \quad i, k = 0, \ldots, N, \quad j = 1, \ldots, N,
\end{cases} \quad (40b)
\]

\[
\epsilon^{U}_{m}(\xi, \eta, \zeta; N) := \begin{cases} 
\epsilon^{W}_{m}(\xi)e^{N}_{m}(\eta)\epsilon^{W}_{m}(\zeta)e_{i}, & m = i + jN + kNN_{\perp}, \quad i, j = 0, \ldots, N, \quad k = 1, \ldots, N, \\
\epsilon^{W}_{m}(\xi)\epsilon^{U}_{m}(\eta)\epsilon^{W}_{m}(\zeta)e_{i}, & m = i + jN + kNN_{\perp} + NN_{\perp}^{2}, \quad i, k = 1, \ldots, N, \quad j = 0, \ldots, N.
\end{cases} \quad (40c)
\]

Using this splitting of the basis functions we can also split the discrete function spaces such that (34) becomes

\[
\mathcal{W}_{h} = \mathcal{W}_{h, i} \oplus \mathcal{W}_{h, \perp} = \text{span}[\epsilon^{W}_{i}, \ldots, \epsilon^{W}_{i}] \oplus \text{span}[\epsilon^{W}_{i}, \ldots, \epsilon^{W}_{i}], \quad (43)
\]

\[
\mathcal{U}_{h} = \mathcal{U}_{h, i} \oplus \mathcal{U}_{h, \perp} = \text{span}[\epsilon^{U}_{i}, \ldots, \epsilon^{U}_{i}] \oplus \text{span}[\epsilon^{U}_{i}, \ldots, \epsilon^{U}_{i}], \quad (44)
\]

where

\[
d_{W_{i}} := 2NN_{\perp}^{2}, \quad d_{W_{\perp}} := NN_{\perp}^{2}, \quad d_{U_{i}} := 2N_{i}NN_{\perp}^{2}, \quad d_{U_{\perp}} := N_{\perp}NN_{\perp}^{2}. \quad (45)
\]

**Remark 1.** An important point relevant in the derivations that follow is that

\[
\epsilon^{W}_{j} = \epsilon^{W}_{j}, \quad j = 0, \ldots, d_{W_{i}} - 1,
\]

\[
\epsilon^{W}_{j} = \epsilon^{W}_{j}, \quad j = 0, \ldots, d_{W_{\perp}} - 1,
\]

\[
\epsilon^{U}_{j} = \epsilon^{U}_{j}, \quad j = 0, \ldots, d_{U_{i}} - 1,
\]

\[
\epsilon^{U}_{j} = \epsilon^{U}_{j}, \quad j = 0, \ldots, d_{U_{\perp}} - 1.
\]
4.2. Unsplit discretization

To discretize the unsplit form of the compressible Euler equations, \([1]\), we first introduce the weak form: Given a domain \(\Omega \subset \mathbb{R}^3\) and a Coriolis term \(f = fe_z \in H(\text{curl}, \Omega)\), find \(u, U, F, P \in H(\text{div}, \Omega), \omega \in H(\text{curl}, \Omega)\), and \(\Pi, \rho, \Theta \in L^2(\Omega)\) such that

\[
\begin{align*}
\left( \sigma, \frac{\partial u}{\partial t} \right)_{\Omega} + \langle \sigma, (\omega + f) \times u \rangle_{\Omega} + \left( \nabla \cdot \sigma, \frac{1}{2} \|u\|^2 + g \zeta \right)_{\Omega} &+ \langle \sigma, \theta P \rangle_{\Omega} = 0, \quad \forall \theta \in H(\text{div}, \Omega), \\
\left( \alpha, \frac{\partial \rho}{\partial t} \right)_{\Omega} + \langle \alpha, \nabla \cdot U \rangle_{\Omega} = 0, \quad \forall \alpha \in L^2(\Omega), \\
\left( \alpha, \frac{\partial \Theta}{\partial t} \right)_{\Omega} + \langle \alpha, \nabla \cdot F \rangle_{\Omega} = 0, \quad \forall \alpha \in L^2(\Omega), \\
\langle \nabla \cdot \sigma, \Pi \rangle_{\Omega} - \langle \sigma, P \rangle_{\Omega} = 0, \quad \forall \sigma \in H(\text{div}, \Omega), \\
\langle \sigma, \rho u \rangle_{\Omega} - \langle \sigma, U \rangle_{\Omega} = 0, \quad \forall \sigma \in H(\text{div}, \Omega), \\
\langle \sigma, \theta U \rangle_{\Omega} - \langle \sigma, F \rangle_{\Omega} = 0, \quad \forall \sigma \in H(\text{div}, \Omega), \\
\langle \nabla \times \beta, u \rangle_{\Omega} - \langle \beta, \omega \rangle_{\Omega} = 0, \quad \forall \beta \in H(\text{curl}, \Omega), \\
\langle \alpha, \Theta \rangle_{\Omega} + \left( \alpha, \rho(\theta_0 + \theta_{1}) \right)_{\Omega} - \langle \alpha, \rho \theta \rangle_{\Omega} = 0, \quad \forall \alpha \in L^2(\Omega), \\
\langle \alpha, \Pi \rangle_{\Omega} - c_p \left( R \frac{\Theta^0}{\rho_0} \right) \left( \alpha, \Theta^{R/c} \right)_{\Omega} = 0, \quad \forall \alpha \in L^2(\Omega).
\end{align*}
\]

Consider now the domain \(\Omega \subset \mathbb{R}^3\) and its tessellation \(T(\Omega)\) consisting of \(M\) arbitrary quadrilaterals (curved), \(\Omega_m\), with \(m = 1, \ldots, M\). We assume that all quadrilateral elements \(\Omega_m\) can be obtained from a map \(\Phi_m : (\xi, \eta, \zeta) \in I^3 \mapsto (x, y, z) \in \Omega_m\). Then the pushforward \(\Phi_m^*\) maps functions in the reference element \(I^3\) to functions in the physical element \(\Omega_m\), see for example \([21, 22]\). For this reason it suffices to explore the analysis on the reference domain \(I^3\). Additionally, the multi-element case follows the standard approach in finite elements.

**Remark 2.** If a differential geometry formulation was used, the physical quantities would be represented by differential \(k\)-forms and the map \(\Phi_m : (\xi, \eta, \zeta) \in I^3 \mapsto (x, y, z) \in \Omega_m\) would generate a pullback, \(\Phi_m^*\), mapping \(k\)-forms in physical space, \(\Omega_m\), to \(k\)-forms in the reference element, \(I^3\) [20].

The discrete weak formulation can be stated as: Given \(\Omega = I^3\), the polynomial degree \(N\) and a Coriolis term \(f_h \in W_{h,1}(\Omega)\), for any time \(t \in (0, t_f]\) find \(u_h, U_h, F_h, P_h \in U_h(\Omega), \theta_h \in U_{h,1}(\Omega), \omega_h \in W_{h}(\Omega)\), and \(\Pi_h, \rho_h, \Theta_h \in Q_h(\Omega)\) such that
\[
\left\langle \sigma_h, \frac{\partial u_h}{\partial t} \right\rangle_{\Omega} + \left\langle \sigma_h, (\omega_h + f_h) \times u_h \right\rangle_{\Omega} + \left\langle \nabla \cdot \sigma_h, \frac{1}{2}||u_h||^2 + gz \right\rangle_{\Omega} + \\
\left\langle \sigma_h, \theta_h p_h \right\rangle_{\Omega} = 0, \quad \forall \sigma_h \in U_h(\Omega),
\]
\[
\left\langle \alpha_h, \frac{\partial \rho_h}{\partial t} \right\rangle_{\Omega} + \left\langle \alpha_h, \nabla \cdot U_h \right\rangle_{\Omega} = 0, \quad \forall \alpha_h \in Q_h(\Omega)
\]
\[
\left\langle \alpha_h, \frac{\partial \theta_h}{\partial t} \right\rangle_{\Omega} + \left\langle \alpha_h, \nabla \cdot F_h \right\rangle_{\Omega} = 0, \quad \forall \alpha_h \in Q_h(\Omega).
\]
\[
\langle \nabla \cdot \sigma_h, \Pi_h \rangle_{\Omega} - \langle \sigma_h, P_h \rangle_{\Omega} = 0, \quad \forall \sigma_h \in U_h(\Omega)
\]
\[
\langle \sigma_h, \rho_h u_h \rangle_{\Omega} - \langle \sigma_h, U_h \rangle_{\Omega} = 0, \quad \forall \sigma_h \in U_h(\Omega)
\]
\[
\langle \sigma_h, \theta_h U_h \rangle_{\Omega} - \langle \sigma_h, F_h \rangle_{\Omega} = 0, \quad \forall \sigma_h \in U_h(\Omega)
\]
\[
\langle \nabla \times \beta_h, u_h \rangle_{\Omega} - \langle \beta_h, \omega_h \rangle_{\Omega} = 0, \quad \forall \beta_h \in W_h(\Omega)
\]
\[
\langle \sigma_{h,\perp}, \Theta_h \rangle_{\Omega} + \left\langle \sigma_{h,\perp}, \rho_h (\theta_h^{0} + \theta_h^{1}) \right\rangle_{\Omega} - \langle \sigma_{h,\perp}, \rho_h \theta_h \rangle_{\Omega} = 0, \quad \forall \alpha_h \in U_{h,\perp}(\Omega)
\]
\[
\langle \alpha_h, \Pi_h \rangle_{\Omega} - c_p \left( \frac{R}{P_0} \right) \left\langle \alpha_h, \Theta_h^{R/c_v} \right\rangle_{\Omega} = 0, \quad \forall \alpha_h \in Q_h(\Omega).
\]

Using the expansions for all unknowns, (47) may be written as: Find \(u, U, F, P \in \mathbb{R}^{d_u}, \theta \in \mathbb{R}^{d_u}, \omega \in \mathbb{R}^{d_w}\), and
4.3. Split discretization

It is important to note that functions in $W_{1,h}$ and $U_{1,h}$ are discontinuous across vertical element boundaries, while those in $U_{L,h}$ are discontinuous across horizontal boundaries. Similarly functions in $Q_h$ are discontinuous across both vertical and horizontal boundaries. These properties allow us to split the three-dimensional problem presented in (48) into separate horizontal and vertical problems, and in doing so avoid solving any global three-dimensional implicit systems.

The split discretization is obtained by using (36), (37), (38), and (39), together with (41) and (42) in (48) in order
to obtain a discrete version of the split Euler equations (40). The horizontal discrete equations are given by

\[
\sum_{i=0}^{d_{\parallel t}-1} \left( \epsilon_j^{\parallel t}, \epsilon_i^{\parallel t} \right)_\Omega \frac{d u_{\parallel i}}{d t} + \sum_{i=0}^{d_{\parallel t}-1} \left( \epsilon_j^{\parallel t}, (\omega_{h,\perp} + f_{h,\perp}) \times \epsilon_i^{\parallel t} \right)_\Omega u_{\parallel i} +
\]

\[
\sum_{i=0}^{d_{\parallel t}-1} \left( \epsilon_j^{\perp t}, \omega_{h,\parallel} \times \epsilon_i^{\perp t} \right)_\Omega u_{\perp i} +
\]

\[
\sum_{i,j=0}^{d_{\parallel t}-1,d_{\perp t}-1} \left( E_{\parallel}^{1,2} \right)_{k,j} \left( \epsilon_k^{\parallel t}, \epsilon_j^{\parallel t} \right)_\Omega \Pi_i - \sum_{i=0}^{d_{\parallel t}-1} \left( \epsilon_j^{\parallel t}, \epsilon_i^{\parallel t} \right)_\Omega P_{\parallel i} = 0, \quad j = 0, \ldots, d_{\parallel t} - 1, (49a)
\]

\[
\sum_{i,j=0}^{d_{\parallel t}-1,d_{\perp t}-1} \left( E_{\parallel}^{1,2} \right)_{k,j} \left( \epsilon_k^{\parallel t}, \epsilon_j^{\parallel t} \right)_\Omega \Pi_i - \sum_{i=0}^{d_{\parallel t}-1} \left( \epsilon_j^{\parallel t}, \epsilon_i^{\parallel t} \right)_\Omega P_{\parallel i} = 0, \quad j = 0, \ldots, d_{\parallel t} - 1, (49b)
\]

\[
\sum_{i=0}^{d_{\parallel t}-1} \left( \epsilon_j^{\parallel t}, \rho_h \epsilon_i^{\parallel t} \right)_\Omega u_{\parallel i} - \sum_{i=0}^{d_{\parallel t}-1} \left( \epsilon_j^{\parallel t}, \epsilon_i^{\parallel t} \right)_\Omega U_{\parallel i} = 0, \quad j = 0, \ldots, d_{\parallel t} - 1. (49c)
\]

\[
\sum_{i=0}^{d_{\parallel t}-1} \left( \epsilon_j^{\parallel t}, \theta_h \epsilon_i^{\parallel t} \right)_\Omega u_{\parallel i} - \sum_{i=0}^{d_{\parallel t}-1} \left( \epsilon_j^{\parallel t}, \epsilon_i^{\parallel t} \right)_\Omega U_{\parallel i} = 0, \quad j = 0, \ldots, d_{\parallel t} - 1. (49d)
\]

\[
\sum_{i,j=0}^{d_{\parallel t}-1,d_{\perp t}-1} \left( E_{\parallel}^{2,1} \right)_{k,j} \left( \epsilon_k^{\parallel t}, \epsilon_j^{\parallel t} \right)_\Omega W_i - \sum_{i=0}^{d_{\parallel t}-1} \left( \epsilon_j^{\parallel t}, \epsilon_i^{\parallel t} \right)_\Omega \omega_{h,\parallel} = 0, \quad j = 0, \ldots, d_{\parallel t} - 1. (49e)
\]

\[
\sum_{i,j=0}^{d_{\parallel t}-1,d_{\perp t}-1} \left( E_{\parallel}^{2,1} \right)_{k,j} \left( \epsilon_k^{\parallel t}, \epsilon_j^{\parallel t} \right)_\Omega W_i - \sum_{i=0}^{d_{\parallel t}-1} \left( \epsilon_j^{\parallel t}, \epsilon_i^{\parallel t} \right)_\Omega \omega_{h,\perp} = 0, \quad j = 0, \ldots, d_{\parallel t} - 1. (49f)
\]

where we have introduced \( \omega_{h,\parallel} := \sum_{i=0}^{d_{\perp t}-1} \omega_{h,\parallel} \epsilon_i^{\perp t} \), \( \omega_{h,\perp} := \sum_{i=0}^{d_{\perp t}-1} \omega_{h,\perp} \epsilon_i^{\perp t} \). In the same way, the vertical discrete
equations are 

\[
\begin{align*}
\sum_{i=0}^{d_u-1} \left( \epsilon_j^u, \epsilon_i^u \right)_\Omega \frac{du_i^\|}{dt} + \sum_{i=0}^{d_u-1} \left( \epsilon_j^u, \epsilon_i^u \right)_\Omega u_{i0}^\| + \\
\sum_{i,k=0}^{d_u-1,d_u-1} \left( E_{i}^{u^2} \right)_{k,j} \left( \epsilon_j^u, \frac{1}{2} d_{h,\|} \cdot \epsilon_i^u \right)_\Omega u_{i,\|} + \\
\sum_{i=k=0}^{d_u-1,d_u-1} g \left( E_{i}^{u^2} \right)_{k,j} \left( \epsilon_j^u, \epsilon_i^u \right)_\Omega z_i + \sum_{i=0}^{d_u-1} \left( \epsilon_j^u, \theta^u_h \epsilon_i^u \right)_\Omega P_{i,\|} = 0, \quad j = 0, \ldots, d_u-1,
\end{align*}
\]

(50a)

\[
\begin{align*}
\sum_{i=0}^{d_u-1} \left( \epsilon_j^u, \epsilon_i^u \right)_\Omega u_{i,\|} - \sum_{i=0}^{d_u-1} \left( \epsilon_j^u, \epsilon_i^u \right)_\Omega U_{i,\|} = 0, \quad j = 0, \ldots, d_u-1,
\end{align*}
\]

(50b)

\[
\begin{align*}
\sum_{i=0}^{d_u-1} \left( \epsilon_j^u, \theta^u_h \epsilon_i^u \right)_\Omega U_{i,\|} - \sum_{i=0}^{d_u-1} \left( \epsilon_j^u, \epsilon_i^u \right)_\Omega F_{i,\|} = 0, \quad j = 0, \ldots, d_u-1,
\end{align*}
\]

(50c)

\[
\begin{align*}
\sum_{i,k=0}^{d_u-1,d_u-1} \left( E_{i}^{u^2} \right)_{k,j} \left( \epsilon_j^u, \epsilon_i^u \right)_\Omega u_{i,\|} - \sum_{i=0}^{d_u-1} \left( \epsilon_j^u, \epsilon_i^u \right)_\Omega W_{i,\|} = 0, \quad j = 0, \ldots, d_u-1,
\end{align*}
\]

(50d)

where we have introduced \( \omega_{h,\|} := \sum_{i=0}^{d_{w,\|}} \omega_{i,\|} W_{i,\|} \). Note that in practice we do not assemble the second term in (50a) (and as such we do not apply the diagnostic equation (50c) for \( \omega_{h,\|} \)). This is because the vertical scales are small with respect to the horizontal scales, and so the horizontal gradients of vertical terms have minimal impact on the dynamics, and also because the rotational terms have no projection onto the energy of the system within the mimetic discretization [13] (though they do re-arrange kinetic energy). At non-hydrostatic resolutions it may become necessary to include this term however.

The only equation in the above system that cannot be effectively split between the horizontal and vertical systems due to the \( C^0 \) continuity of function spaces is the vorticity diagnostic equation (49c). However if we limit ourselves to elements of degree \( p = 1 \) in the vertical, then by the orthogonality of the nodal polynomials, the \( \omega_{h,\|} \) term may be effectively diagnosed from the velocity field in the two layers above and below this interface only, thus avoiding the need for a global solve.

Additionally, we also have the flux form equations for density and density weighted potential temperature transport that contain both vertical and horizontal components. While we have not included these in the split systems described in (49) and (50), since doing so incurs a temporal splitting error, in practice these equations are also split between their horizontal and vertical components. For the Strang carryover scheme detailed in Section 5.3 this results in a second order temporal error for the full system.

\[
\begin{align*}
\sum_{i=0}^{d_{\theta}} \left( \epsilon_j^\theta, \epsilon_i^\theta \right)_\Omega \frac{d\theta_i}{dt} + \sum_{i=0}^{d_{\theta}-1,d_{\theta}-1} \left( E_{i}^{\theta^2} \right)_{k,j} \left( \epsilon_j^\theta, \frac{1}{2} d_{h,\|} \cdot \epsilon_i^\theta \right)_\Omega \theta_{i,\|} + \\
\sum_{i=0}^{d_{\theta}-1} \left( \epsilon_j^\theta, \epsilon_i^\theta \right)_\Omega \theta_{i,\|} - \sum_{i=0}^{d_{\theta}-1} \left( \epsilon_j^\theta, \epsilon_i^\theta \right)_\Omega \theta_{i,\|} = 0, \quad j = 0, \ldots, d_{\theta}-1 \quad (51a)
\end{align*}
\]

\[
\begin{align*}
\sum_{i=0}^{d_{\theta}} \left( \epsilon_j^\theta, \epsilon_i^\theta \right)_\Omega \frac{d\theta_i}{dt} + \sum_{i=0}^{d_{\theta}-1,d_{\theta}-1} \left( E_{i}^{\theta^2} \right)_{k,j} \left( \epsilon_j^\theta, \frac{1}{2} d_{h,\|} \cdot \epsilon_i^\theta \right)_\Omega F_{i,\|} + \\
\sum_{i=0}^{d_{\theta}-1} \left( \epsilon_j^\theta, \epsilon_i^\theta \right)_\Omega F_{i,\|} - \sum_{i=0}^{d_{\theta}-1} \left( \epsilon_j^\theta, \epsilon_i^\theta \right)_\Omega F_{i,\|} = 0, \quad j = 0, \ldots, d_{\theta}-1 \quad (51b)
\end{align*}
\]
Note that the diagnostic equations for potential temperature (48h) and the equation of state (48i) are also included in both the horizontal and vertical systems.

**Remark 3.** In (49), (50), and (51), we have used the fact that the incidence matrices can be written as

\[
E_{\parallel}^{1,0} = \begin{bmatrix} E_{\parallel}^{1,0} & E_{\parallel}^{2,1} \\ E_{\perp}^{1,0} & 0 \end{bmatrix}, \quad E_{\parallel}^{3,1} = \begin{bmatrix} E_{\parallel}^{3,1} & E_{\parallel}^{2,1} \\ E_{\perp}^{3,1} & 0 \end{bmatrix}, \quad \text{and} \quad E_{\perp}^{3,2} = \begin{bmatrix} E_{\parallel}^{3,2} & E_{\perp}^{3,2} \end{bmatrix},
\]

where \( E_{\parallel}^{1,0} \) is a \( d_{W_\perp} \times d_P \) matrix, \( E_{\parallel}^{1,0} \) is a \( d_{W_\perp} \times d_P \) matrix, \( E_{\parallel}^{2,1} \) is a \( d_{\eta} \times d_{W_\perp} \) matrix, \( E_{\perp}^{2,1} \) is a \( d_{\eta} \times d_{W_\perp} \) matrix, \( E_{\parallel}^{3,1} \) is a \( d_{\eta} \times d_{W} \) matrix, \( E_{\perp}^{3,1} \) is a \( d_{\eta} \times d_{W} \) matrix, and \( E_{\perp}^{3,2} \) is a \( d_Q \times d_{\eta} \) matrix.

**Remark 4.** In (49), (50), and (51), two important points to note are

\[
\begin{align*}
\mathbf{u}_i &= \mathbf{u}_{i\parallel}, \quad i = 0, \ldots, d_{\eta} - 1 \quad \text{and} \quad \mathbf{u}_{i+d_{\eta}} = \mathbf{u}_{i\perp}, \quad i = 0, \ldots, d_{\eta} - 1
\end{align*}
\]

and

\[
\mathbf{\omega}_h = \mathbf{\omega}_{h\parallel} + \mathbf{\omega}_{h\perp} = \sum_{i=0}^{d_{\eta}-1} \mathbf{\omega}_{i\parallel} + \sum_{i=0}^{d_{\eta}-1} \mathbf{\omega}_{i\perp}.
\]

In compact matrix notation we can write (49) as

\[
M_{\parallel}^{\eta} \frac{d\mathbf{u}_i}{dt} + R^{\parallel} \mathbf{u}_i + R^{\perp} \mathbf{u}_i + \left( E_{\perp}^{3,1} \right)^T T^{\eta} \mathbf{u}_i + S^{\eta} \mathbf{H}^{\parallel} = 0,
\]

with

\[
M_{\parallel}^{\eta} := \left( \mathbf{e}_i \mathbf{e}_j \right)_\Omega, \quad R_{\parallel}^{\parallel} := \left( \mathbf{e}_i \mathbf{e}_j + \mathbf{f}_{i\perp} \times \mathbf{e}_j \right)_\Omega, \quad R_{\parallel}^{\perp} := \left( \mathbf{e}_i \mathbf{e}_j \times \mathbf{e}_j \right)_\Omega,
\]

\[
M_{\perp}^{\eta} := \left( \mathbf{e}_i \mathbf{e}_j \right)_\Omega, \quad S_{\perp}^{\parallel} := \left( \mathbf{e}_i \times \mathbf{e}_j \right)_\Omega, \quad S_{\perp}^{\perp} := \left( \mathbf{e}_i \mathbf{e}_j \right)_\Omega, \quad T_{\perp}^{\eta} := \left( \mathbf{e}_i \mathbf{e}_j \right)_\Omega, \quad T_{\parallel}^{\eta} := \left( \mathbf{e}_i \mathbf{e}_j \right)_\Omega.
\]

In a similar manner, the vertical equations, (50), can be written in compact matrix notation as

\[
M_{\parallel}^{\eta} \frac{d\mathbf{u}_i}{dt} + R^{\parallel} \mathbf{u}_i + \left( E_{\perp}^{1,2} \right)^T T^{\eta} \mathbf{u}_i + gM^2 \mathbf{z} + S^{\parallel} \mathbf{P}^{\perp} = 0,
\]

with

\[
R_{\parallel}^{\perp} := \left( \mathbf{e}_i \mathbf{e}_j + \mathbf{f}_{i\perp} \times \mathbf{e}_j \right)_\Omega, \quad S_{\parallel}^{\parallel} := \left( \mathbf{e}_i \mathbf{e}_j \times \mathbf{e}_j \right)_\Omega, \quad S_{\parallel}^{\perp} := \left( \mathbf{e}_i \mathbf{e}_j \right)_\Omega, \quad N_{\parallel}^{\eta} := \left( \mathbf{e}_i \mathbf{e}_j \right)_\Omega.
\]
Following the same procedure we may obtain the weak equations for the variational derivative with respect to \( \Theta \):

\[
(\epsilon_j^W, \Phi) = \left( \frac{1}{2} [u_{h,j} \cdot \epsilon_j^W], \Phi \right) \Omega, \quad j = 0, \ldots, d_u.
\]

Finally, (51) may be written in compact matrix notation as

\[
M^\Omega \frac{d\rho}{dt} + M^\Omega E_{1,2} U^1 + M^\Omega E_{1,2} U^2 = 0, \quad (55a)
\]

\[
M^\Omega \frac{d\Theta}{dt} + M^\Omega E_{1,2} F^1 + M^\Omega E_{1,2} F^2 = 0, \quad (55b)
\]

\[
L^{\Delta t, \Omega} \Theta + N^{\Delta t, \Omega} (\theta^b + \theta^l) - N^{\Delta t, \Theta} = 0, \quad (55c)
\]

\[
e_p \left( R \left( \frac{d}{p_0} \right)^{R/c_v} \sum_{i=0}^{d_u-1} \left< \epsilon_i^Q, (\epsilon_i^Q \Theta)^{R/c_v} \right> \right) - M^\Omega \Pi = 0, \quad (55d)
\]

with

\[
L^{\Delta t, \Omega} := \left< \epsilon_i^\Omega, \rho \epsilon_j^\Omega \right> \Omega.
\]

### 4.4. Discrete energetics

The conservation of energy for the rotating shallow water equations via balanced kinetic-potential exchanges has previously been analysed [3] and experimentally verified [5] for the mixed mimetic spectral element method. In terms of energetics, the qualitative difference between the rotating shallow water equations and the compressible Euler equations is the presence of kinetic and internal energy exchanges. As such we here extend the previous analysis to demonstrate that these exchanges are balanced in the discrete form.

As seen before, we have that the discrete velocity field, \( u_h \), density, \( \rho_h \), and density weighted potential temperature, \( \Theta_h \), are

\[
u = \sum_{i=0}^{d_u} u_i \epsilon_i^\Omega, \quad \rho = \sum_{i=0}^{d_u} \rho_i \epsilon_i^\Omega, \quad \text{and} \quad \Theta = \sum_{i=0}^{d_u} \Theta_i \epsilon_i^\Omega.
\]

The discrete Hamiltonian \( H_h := \mathcal{H}[u_h, \rho_h, \Theta_h] \) is then given by

\[
\mathcal{H}[u_h, \rho_h, \Theta_h] = \int_\Omega \frac{1}{2} \rho_h |u_h|^2 \, d\Omega + \int_\Omega \rho_h g z_h \, d\Omega + \int_\Omega c_v \left( R \left( \frac{d}{p_0} \right)^{R/c_v} \right) \Theta_h \, d\Omega.
\]

Using the definition of the variational derivative, see for example [23], we can compute the variational derivative of the Hamiltonian with respect to the velocity, \( \frac{\partial H}{\partial u} \),

\[
\frac{d}{dt} \mathcal{H}[u_h + \epsilon \nu_h, \rho_h, \Theta_h] = \left< \nu_h, \frac{\delta \mathcal{H}}{\delta u_h} \right>, \quad \forall \nu_h \in \mathcal{U}_h.
\]

The left hand side of this expression may be evaluated to yield

\[
\left< \nu_h, \frac{\delta \mathcal{H}}{\delta u_h} \right> = \left< \nu_h, \rho_h u_h \right>, \quad \forall \nu_h \in \mathcal{U}_h.
\]

Since this expression is valid for all \( \nu_h \in \mathcal{U}_h \), then

\[
\left< \epsilon_j^\Omega, \frac{\delta \mathcal{H}}{\delta u_h} \right> = \left< \epsilon_j^\Omega, \rho_h u_h \right>, \quad j = 0, \ldots, d_u.
\]

Following the same procedure we may obtain the weak equations for the variational derivative with respect to \( \rho_h \)

\[
\left< \epsilon_j^\Omega, \frac{\delta \mathcal{H}}{\delta \rho_h} \right> = \left< \epsilon_j^\Omega, \frac{1}{2} |u_h|^2 + g z_h \right>, \quad j = 0, \ldots, d_Q.
\]
such that

\[ R_i = q_i, \quad \Theta_{h,i} = q_i^2, \quad j = 0, \ldots, d_Q. \]  (62)

The discrete compressible Euler equations (47) may then be formulated as a skew-symmetric system for the discrete analogue of (16)-(19) as

\[
\begin{pmatrix}
M^u u_j \\
M^q p_j, t \\
M^q \Theta_j
\end{pmatrix}
= \begin{pmatrix}
-R_i \\
-M^q E^{3,2} \\
-M^q E^{3,2} (M^u)^{-1} S^u
\end{pmatrix}
\begin{pmatrix}
\partial u_j \\
\partial q_j \\
\partial \Theta_j
\end{pmatrix}
+ \begin{pmatrix}
(E^{1,2})^T M^q S^u (M^u)^{-1} (E^{1,2})^T M^q \\
0 \\
0
\end{pmatrix}
\begin{pmatrix}
\Phi \\
\Pi
\end{pmatrix}.
\]  (63)

Multiplying both sides by \([U^T \quad \Phi^T \quad \Pi^T]\), gives

\[
U^T M^u \frac{\partial u}{\partial t} + u^T T u \frac{\partial \rho}{\partial t} + g z^M E^{1,2} \frac{\partial \rho}{\partial t} + \Pi^T M^q \frac{\partial \Theta}{\partial t} = \frac{\partial \Phi}{\partial t} + \frac{\partial \Pi}{\partial t} + \frac{\partial I}{\partial t} = 0.
\]  (64)

where

\[
K_h = \frac{1}{2} U^T M^u u = U^T \tau^d u, \quad P_h = g z^M E^{1,2} \rho, \quad I_h = \frac{c_p}{c_v} \Pi^T M^q \Theta = \frac{c_p}{c_v} \left[ \frac{R}{{p}_0} \right] (64',)
\]  (65)

Note that for \(R_{q,i} := \left[ \epsilon^u_j, \ q_h \times \epsilon^u_j \right]_j \), where \(q_h\) is the potential vorticity [3], this is itself a skew-symmetric operator such that \(U^T R_q U = U^T RU = 0\). As such neither \(R_q\) nor \(R\) projects onto the energy in the discrete form.

Note also that the pressure gradient diagnostic equation (48) and the temperature flux diagnostic equation (49) appear within the skew-symmetric operator in (63) within the top right and bottom left blocks respectively. The discrete energy exchanges are therefore given as

\[
\frac{\partial \Phi}{\partial t} = g U^T (E^{1,2})^T M^q z + U^T S^u (M^u)^{-1} (E^{1,2})^T M^q \Pi,
\]  (66)

\[
\frac{\partial \Pi}{\partial t} = -g z^M E^{1,2} U,
\]  (67)

\[
\frac{\partial \Pi}{\partial t} = -\Pi^T M^q E^{1,2} (M^u)^{-1} S^u U.
\]  (68)

The right hand side terms of (66) exactly balance those of (67) and (68), thus allowing for the exact balances of kinetic to potential and kinetic to internal energy respectively. This holds for both the horizontal and vertical discretisations presented above, assuming periodic boundary conditions in the horizontal and Dirichlet boundary conditions for the vertical velocity and potential temperature [5] and Neumann conditions for the Exner pressure [6] in the vertical. As an aside we note that energetic consistency is satisfied independent of the choice of function space for \(\vartheta_h\), since this only appears within \(S^u\), thus justifying our choice to represent \(\vartheta_h \in \mathcal{U}_L\).

4.5. Metric terms

The Jacobian matrix between local element coordinates \(\xi := (\xi, \eta, \zeta)\) and global coordinates \(x := (x, y, z)\) is given as

\[
J = \begin{bmatrix}
x_\xi & x_\eta & 0 \\
y_\xi & y_\eta & 0 \\
0 & 0 & z_\zeta
\end{bmatrix},
\]  (69)

where the subscripts represent derivatives with respect to local element coordinates and we have assumed that all horizontal layers are perfectly flat, such that the projection of vertical local coordinates onto horizontal vertical coordinates and horizontal local coordinates onto vertical global coordinates are zero. The \(H(\text{curl}, \Omega)\), \(H(\text{div}, \Omega)\) and \(L^2(\Omega)\) forms of the Piola transformation are given respectively as [11, 12]

\[
J^{-\tau}, \quad \frac{1}{J} J, \quad \frac{1}{J}.
\]  (70)
where $J$ is the determinant of the Jacobian matrix. The metric transformations for the respective mass matrices are therefore

$$ (J^{-T})^T J^{-1} = \frac{1}{J^2} \begin{bmatrix}
(z_{\xi})^2((x_{\eta})^2 + (y_{\eta})^2) & -(z_{\xi})^2(x_{\xi} y_{\eta} + x_{\eta} y_{\eta}) & 0 \\
-(z_{\xi})^2(x_{\xi} y_{\eta} + x_{\eta} y_{\eta}) & (z_{\xi})^2((x_{\xi})^2 + (y_{\xi})^2) & 0 \\
0 & 0 & (x_{\xi} y_{\eta} - x_{\eta} y_{\xi})^2
\end{bmatrix} $$

(71)

$$ \frac{1}{J^2} J^T J = \frac{1}{J^2} \begin{bmatrix}
(x_{\xi})^2 + (y_{\xi})^2 & x_{\xi} x_{\eta} + y_{\xi} y_{\eta} & 0 \\
x_{\xi} x_{\eta} + y_{\xi} y_{\eta} & (x_{\eta})^2 + (y_{\eta})^2 & 0 \\
0 & 0 & (z_{\xi})^2
\end{bmatrix} $$

(72)

$$ \frac{1}{J^2} = \frac{1}{(z_{\xi})^2(x_{\xi} y_{\eta} - x_{\eta} y_{\xi})^2}. $$

(73)

Since the horizontal and vertical components of both the $H(\text{curl}, \Omega)$ and $H(\text{div}, \Omega)$ transformations are orthogonal, these metric transformations are further simplified for these spaces as

$$ W_{h,\parallel} = \frac{1}{(x_{\eta})^2 - (x_{\xi})^2} \begin{bmatrix}
(x_{\xi})^2 + (y_{\xi})^2 & -x_{\xi} x_{\eta} - y_{\xi} y_{\eta} \\
x_{\xi} x_{\eta} - y_{\xi} y_{\eta} & (x_{\xi})^2 + (y_{\xi})^2
\end{bmatrix} $$

$$ W_{h,\perp} = \frac{1}{(z_{\xi})^2} \begin{bmatrix}
1 \\
1
\end{bmatrix} $$

$$ U_{h,\parallel} = \frac{1}{(x_{\eta})^2 - (x_{\xi})^2} \begin{bmatrix}
(x_{\xi})^2 + (y_{\xi})^2 & x_{\xi} x_{\eta} + y_{\xi} y_{\eta} \\
x_{\xi} x_{\eta} + y_{\xi} y_{\eta} & (x_{\eta})^2 + (y_{\eta})^2
\end{bmatrix} $$

(74)

$$ U_{h,\perp} = \frac{1}{(z_{\xi})^2} \begin{bmatrix}
1 \\
1
\end{bmatrix} $$

$$ Q_0 = \frac{1}{(z_{\xi})^2(z_{\eta} y_{\xi} - x_{\xi} y_{\eta})^2} \begin{bmatrix}
1 \\
1
\end{bmatrix} $$

As a special case, we also note that an inner product between basis functions in $U_{h,\parallel}$ and functions in $Q_0$ has the metric term

$$ \frac{z_{\xi}}{J} \frac{1}{J} = \frac{1}{(z_{\xi})(x_{\xi} y_{\eta} - x_{\eta} y_{\xi})^2}. $$

(75)

As stated these metric terms do not account for any projection of horizontal vector components onto vertical components and vice versa. As such we limit ourselves here to the case where the horizontal and vertical degrees of freedom are strictly orthogonal, and we are unable to account for the tilting of layers or bottom topography in the current implementation. We note however that the tilting of horizontal layers is naturally represented in the finite element formulation via these cross terms, and these may be implemented at a later date with minimal disruption to the current formulation.

Note that the vorticity term $\langle \sigma_h, \omega_{h,\parallel} \times u_{h,\perp} \rangle$ is alternatively formulated as $\langle \sigma_h, u_{h,\perp} \nabla \cdot u_{h,\parallel} \rangle$ (with the vertical derivative derived in the weak form). In this form the vertical velocity derivatives may be interpreted as being oriented normal to the edges, rather than tangent to them, and as such the $H(\text{curl}, \Omega)$ form of the Piola transformation is used to construct the metric term for this operator as well as for (49c).

5. Time stepping

5.1. Implicit vertical solve

The fast time scales of the sound waves and small spatial scales of the vertical motions of the atmosphere with respect to the horizontal make it impractical to solve for the vertical dynamics explicitly. Furthermore since the vertical dynamics are dominated by hydrostatic pressure gradients in the momentum equation, it is necessary to solve for the nonlinear vertical momentum equation using an implicit iterative procedure. Our method for doing so is described as follows.

Following [2] we begin by taking the logarithm of the equation of state (3) as

$$ \ln(\Pi) = \frac{R}{c} \left( \ln(p) + \ln \left( \frac{R}{P_0} \right) \right) + \ln(c_p). $$

(76)
We define the additional matrix operators discretized in time (with the vorticity components omitted, as discussed in Section 4.3) as

$$\frac{\Pi^{n+1} - \Pi^n}{\Delta t} = R \frac{\rho \theta^{n+1} - (\rho \theta)^n}{c_v}.$$  \hspace{1cm} (77)

Substituting in the potential temperature equation (1c), we then express the evolution of the Exner pressure as

$$(\rho \theta)^n \Pi^{n+1} = (\rho \theta)^n \Pi^n - \frac{\Delta R}{c_v} \Pi^n \nabla \cdot (\rho \theta u).$$  \hspace{1cm} (78)

Note the similarity between (78) and the internal energy evolution equation (14). The vertical dynamics may then be discretized in time (with the vorticity components omitted, as discussed in Section 4.3) as

$$u^{n+1}_l + \frac{\Delta t}{2} \nabla_\perp (u^{n+1}_l)^2 + \Delta t \nabla_\perp \Pi^{n+1}_l = u^n_l - \Delta t g \nabla_\perp z$$  \hspace{1cm} (79)

$$\rho^{n+1} + \Delta t \nabla_\perp \cdot (\rho^{n+1}_l u^{n+1}_l) = \rho^n$$  \hspace{1cm} (80)

$$\Theta^{n+1} + \Delta t \nabla_\perp \cdot (u^{n+1}_l \Theta^{n+1}) = \Theta^n$$  \hspace{1cm} (81)

$$\Theta^n \Pi^{n+1} = \Theta^n \Pi^n - \Delta \frac{R}{c_v} \Pi^n \nabla_\perp \cdot (u^{n+1}_l \Theta^{n+1})$$  \hspace{1cm} (82)

We define the additional matrix operators $M^n_{0,\theta,ij}$, $M^n_{0,u,ij}$ and $M^{u,ij}$, for which

$$M^n_{0,\theta,ij} := \langle \epsilon_i^2, \Theta_i \epsilon_j^2 \rangle_\Omega,$$

$$M^n_{0,u,ij} := \langle \epsilon_i \Omega_i \epsilon_j^2 \rangle_\Omega,$$

$$M^{u,ij} := \langle \epsilon_i \epsilon_j, \Theta_i \epsilon_j \rangle_\Omega.$$  

Dropping the $n+1$ superscripts for variables at the new time level (but keeping those for variables at time level n), the discrete form of (78) is then given as:

$$M_{0,\theta}^n \Pi = M_{0,\theta}^n - \frac{\Delta R}{c_v} M_{0,\theta}^n \left( M_{u}^{2,ij} \right)^{-1} M_{0,u}^{u,ij} u_\perp,$$  \hspace{1cm} (83)

and the discrete form of (79) as:

$$M^{u,ij} u_\perp + \frac{1}{2} \Delta t (E^{2,ij}_L) T U^u u_\perp + \Delta t S^{u,ij} (M^{u,ij} E^{2,ij}_L)^{-1} M^n_0 \Pi = M^{u,ij} u_\perp + \Delta t g (E^{2,ij}_L) T M^2 z.$$  \hspace{1cm} (84)

Substituting (83) into (84) gives

$$M^{u,ij} u_\perp + \frac{\Delta t}{2} (E^{2,ij}_L) T U^u u_\perp + \Delta t S^{u,ij} (M^{u,ij} E^{2,ij}_L)^{-1} M^n_0 \Pi = M^{u,ij} u_\perp + \Delta t g (E^{2,ij}_L) T M^2 z.$$  \hspace{1cm} (85)

Finally, rearranging gives an expression for the vertical velocity at each fixed point Picard iteration as:

$$\left[ M^{u,ij} + \frac{\Delta t}{2} (E^{2,ij}_L) T U^u - \frac{\Delta t R}{c_v} S^{u,ij} (M^{u,ij} E^{2,ij}_L)^{-1} M^n_0 \Pi \right] u_\perp = M^{u,ij} u_\perp + \Delta t g (E^{2,ij}_L) T M^2 z + S^{u,ij} (M^{u,ij} E^{2,ij}_L)^{-1} M^n_0 \Pi.$$  \hspace{1cm} (86)

In order to incorporate the pressure gradient term into the left hand side in (86) so as to ensure the stable implicit solution of vertical motions, we have sacrificed the energetically consistent formulation of the vertical pressure gradient term as presented in (50a), (50b) and (63). As such we do not expect the vertical kinetic and internal exchanges to exactly balance, as we do for the horizontal explicit discretization.
The Exner pressure at the current Picard iteration is then derived from (83), and the corresponding Picard iteration solves for the other variables are then given (omitting the horizontal terms) as

\[ N^{tt} \theta = L^{tt} \psi \theta + N^{tt} (\theta^h + \theta^v) \]

\[ U_\perp = (M^{tt})^{-1} N^{tt} u_\perp \]

\[ \rho = \rho^h - \Delta t E_{1}^{tt} U_\perp \]

\[ \Theta = \Theta^h - \Delta t E_{1,2}^{tt} (M^{tt})^{-1} S^{tt} U_\perp. \]

5.2. Explicit horizontal solve

The horizontal dynamics are solved using an explicit stably stable Runge-Kutta scheme of the form [24, 25]

\[ A_h^1 = A_h^0 + \Delta t H(A_h^0), \]

\[ A_h^2 = \frac{3}{4} A_h^1 + \frac{1}{4} A_h^0 + \frac{\Delta t}{4} H(A_h^1), \]

\[ A_h^" = \frac{1}{3} A_h^2 + \frac{2}{3} A_h^0 + \frac{2\Delta t}{3} H(A_h^2), \]

where \( A_h = [v_h, \rho_h, \Theta_h, \Pi_h]^T \), \( H(A_h) \) is the right hand side forcing terms for the horizontal dynamics as presented in (53). \( A_h^0 \) is the state vector at the end of the first vertical half step and \( A_h^" \) is the state vector at the end of the horizontal solve.

5.3. Directional splitting

We use a Strang carryover splitting scheme to partition the horizontal and vertical dynamics [25, 26]. On the first time step we solve for a half step, \( \Delta t/2 \) in the vertical, then a full step, \( \Delta t \) in the horizontal, then a second half step in the vertical. For each successive time step the vertical solve from the previous time step is carried over for a half step as

\[ B_h^x = B_h^x + \frac{\Delta t}{2} V(B_h^x), \]

where \( B_h = [w_h, \rho_h, \Theta_h, \Pi_h]^T \), and \( V(B_h) \) is the forcing terms for the vertical dynamics as given in (86), (83), (87)-(90). Following this we solve for the horizontal dynamics for a full step (91)-(93), and then for the vertical dynamics for another half step via the iterative solution of (86), (83), (87)-(90). A nonlinear Picard iteration, \( m \) is applied as an outer loop over this system, which is assumed to converge once \( |B_h^{m-1} - B_h^m|_2 / |B_h^{m-1}|_2 < 10^{-6} \) for \( B_h \in v_h, \Pi_h, \rho_h, \Theta_h \) and where \( | \cdot |_2 \) is the \( L^2 \) norm. Since all the solution variables in \( B_h \) require to solve for the vertical dynamics are discontinuous across horizontal element boundaries, the inner linear system above is solved in serial for each horizontal element, via a direct LU solve using the PETSc library [27, 29].

Similarly, since all the solution variables involved in the vertical dynamics, \( A_h \) are discretized on function spaces that are discontinuous across vertical element boundaries, this system may be solved independently, in parallel for each horizontal layer. The mass matrices are solved using the PETSc GMRES solver, with a block Jacobi preconditioner for each element, as was done for the shallow water equations in our previous work [5].

5.4. Dissipative terms

In order to stabilise the model we also include various dissipative terms. These include a biharmonic viscosity on both the horizontal momentum and temperature equations [5, 30] with a value of 0.072\( \Delta x^{1.2} \), where \( \Delta x \) is the average spacing between GLL nodes. A Rayleigh friction term with a coefficient of 0.2 is also applied to the top layer of the vertical momentum equation, as is often used in atmospheric models to suppress orographically forced gravity waves [31]. While this Rayleigh friction term is not strictly necessary for the stability of the simulation presented here, it greatly reduces the noise in the energetic profiles as the model adjusts to a state of hydrostatic balance from its initial conditions. No viscosity is required to stabilize the vertical solution, perhaps because this is only second order accurate, so the internal dissipation of this low order discretization is sufficient to prevent nonlinear instabilities.
6. Results

We validate the model using a dry baroclinic instability test case [32] with the shallow atmosphere approximation. The appeal of this test case is that the initial condition is specified for a $z$–level vertical coordinate, whereas other such test cases that are defined on pressure level vertical coordinates require the solution of a nonlinear problem in order to compute the corresponding $z$–level configuration. The initial state is one of geostrophic horizontal and hydrostatic vertical balance, overlaid with a small, $O(1\text{m/s})$, perturbation to the zonal and meridional velocity components.

The model was run with $24 \times 24$ elements of degree $p = 3$ on each face of the cubed sphere (and linear elements in the vertical), for an averaged resolution of $\Delta x \approx 128\text{km}$ and 30 vertical levels on 96 processors ($4^2 = 16$ per face of the cubed sphere) with a time step of $\Delta t = 120\text{s}$. While the vertical dynamics are all solved implicitly and so do not limit the time step size, the explicit horizontal dynamics present both diffusive and advective CFL restrictions due to the biharmonic viscosity and sound waves respectively.

Figures 2 and 3 show the zonal averages of density $\rho$, Exner pressure $\Pi$, potential temperature $\theta$ and zonal velocity $u$ at day 10. These profiles are almost indistinguishable from their initial states, with the exception of the zonal velocity, which exhibits a small kink near the bottom boundary where the baroclinic instability occurs, demonstrating that the leading order geostrophic and hydrostatic balances in the horizontal and vertical are well satisfied.

![Figure 2: Zonal averages of density, $\rho$, in kg·m$^{-3}$ (left) and Exner pressure, $\Pi$, in m$^2$·s$^{-2}$·K$^{-1}$ (right) at day 10.](image-url)

Figures 4 and 5 show the evolution of the kinetic (horizontal and vertical), potential and internal energy with time, and the associated exchanges. These are shown on both logarithmic scales for their normalised absolute difference from initial value (Fig. 4), and as a straight difference between their current and initial value (Fig. 5).

The growth in the baroclinic instability is evident in the increase in kinetic energy, and the reduction in both potential and internal energy as isopycnals flatten in the region of the instability. Note that the total amounts of potential and internal energy are approximately $3.6 \times 10^{23}$ and $9.2 \times 10^{23}\text{kg} \cdot \text{m}^2\text{s}^{-2}$ respectively, and so are several orders of magnitude greater than the amounts of horizontal and vertical kinetic energy (approximately $4.0 \times 10^{20}$ and $2.5 \times 10^{13}\text{kg} \cdot \text{m}^2\text{s}^{-2}$ respectively). As such the flattening of the density contours from which the baroclinic instability draws energy are barely evident in Fig. 2.

Figure 5 also shows the power associated with the sum of vertical energy exchanges and horizontal exchanges as a function of time. That the vertical exchanges sum to such a small value, $O(10^9\text{kg} \cdot \text{m}^2\text{s}^{-3})$, compared to the total amounts of potential and internal energy, $O(10^{23}\text{kg} \cdot \text{m}^2\text{s}^{-2})$, suggests that the transfer of potential to internal energy and vice versa through vertical motions are well balanced. This is in spite of the fact that the kinetic-internal energy exchanges are not exactly balanced in the vertical as they are in the horizontal.

The observation that potential energy is greater than its initial value for most of the simulation, while internal energy is smaller, as shown in Figs. 4 and 5, is explained by the fact that the initial condition is not precisely in hydrostatic balance. As such the initial adjustment process leads to a slight rise in the fluid, resulting in an increase in
Figure 3: Zonal averages of potential temperature, $\theta_h$ in K (left) and zonal velocity, $u_h$ in m·s$^{-1}$ (right) at day 10.

Figure 4: Left: difference in energy with respect to initial values. Right: vertical kinetic energy.

Figure 5: Left: difference in energy with respect to initial values. Right: vertical kinetic energy.
potential energy, and a corresponding reduction in pressure, leading to a reduction of internal energy via the equation of state. These changes are of $O(10^{-4})$ compared to the total amounts of potential and internal energy in the system. The high frequency oscillation in the internal and potential energies observed over the first 24 hours of the simulation during this adjustment process is reduced by the application of the Rayleigh friction to the top layer of the vertical momentum equation.

We present the bottom level pressure, $p_b$, and temperature, $T$, and the vertical component of the relative vorticity, $\omega$, at $z \approx 1.5km$ at days 8 and 10 in Figs. 6, 7, and 8 as well as a meridional cross section of the pressure perturbation at 50°N in Fig. 9. In the cases of the pressure and temperature, these are reconstructed from the model variables as $p = p_0 \left( \Pi / \rho_p \right)^{y/R}$ and $T = \theta \Pi / \rho_p$. The pressure perturbation in Fig. 9 is derived by removing the average pressure at the corresponding vertical level at 50°S. These results compare well with the previously published test case results [32] in both shape and magnitude, and clearly show the signal of the baroclinic instability.

For completeness we also show the results at days 8 and 10 for the original model variables of bottom level Exner pressure and potential temperature and relative vertical vorticity at $z \approx 1.5km$ in Figs. 10, 11, and 12 respectively. These are presented for the northern hemisphere only, looking down from the north pole. These results are perhaps
Figure 8: Vertical component of the relative vorticity, $\omega_h$ (in s$^{-1}$) at $z = 1.5$km, day 8 (left) and 10 (right).

Figure 9: Vertical cross section of the pressure perturbation, $p_h - \bar{p}_h$ (in hPa) at 50°N, day 8 (left) and day 10 (right).
slightly sharper than the reconstructed temperature and pressure fields since they are interpolated directly from the degrees of freedom.

Figure 10: Bottom level Exner pressure, $\Pi_h$ (in m$^2$s$^{-2}$K$^{-1}$) day 8 (left) and 10 (right).

Figure 11: Potential temperature, $\theta_h$ (in °K) at $z = 1.5$km, day 8 (left) and 10 (right).

7. Conclusions

A model of the rotating compressible Euler equations on the cubed sphere using the mixed mimetic spectral element method is presented. The model uses a Strang carryover directional splitting scheme with an implicit Picard solver for the vertical dynamics in order to negate the CFL condition of the vertical sound waves. The discontinuities in the discrete function spaces are exploited so as to solve for each horizontal layer and each vertical element independently in order to avoid the need to invert a global 3D mass matrix. The forcing and flux terms are constructed so as to take advantage of the adjoint relations between the discrete gradient and divergence operators and balance the exchanges of kinetic, potential and internal energy. The exception to this is the construction of the vertical pressure gradient operator, which violates this principal in order to allow for stable implicit solve of the vertical dynamics.
Despite this lack of exact balance in the vertical kinetic-internal energy exchanges, the sum of the kinetic to potential and kinetic to internal vertical exchanges are observed to be small with respect to the horizontal kinetic-internal exchanges and do not appear to corrupt the dynamics, at least over the time scale of the simulation presented here. The signal of the baroclinic instability is well developed and compares well against previous published results. In future work we will explore fully implicit formulations so as to avoid the horizontal vertical splitting and restore the balance of vertical kinetic internal energy exchanges to the discrete system.

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References

[1] C. J. Cotter, J. Shipton, Mixed finite elements for numerical weather prediction, J. Comp. Phys. 231 (2012) 7076–7091.
[2] A. T. T. McRae, C. J. Cotter, Energy- and enstrophy-conserving schemes for the shallow-water equations, based on mimetic finite elements, Q. J. R. Meteorol. Soc. 140 (2014) 2223–2234.
[3] D. Lee, A. Palha, M. Gerritsma, Discrete conservation properties for shallow water flows using mixed mimetic spectral elements, J. Comp. Phys. 357 (2018) 282–304.
[4] C. Eldred, T. Dubos, E. Kritsikis, A quasi-Hamiltonian discretization of the thermal shallow water equations, J. Comp. Phys. 379 (2019) 1–31.
[5] D. Lee, A. Palha, A mixed mimetic spectral element model of the rotating shallow water equations on the cubed sphere, J. Comp. Phys. 375 (2018) 240–262.
[6] W. Bauer, C. J. Cotter, Energy-entrophy conserving compatible finite element schemes for the rotating shallow water equations with slip boundary conditions, J. Comp. Phys. 373 (2018) 171–187.
[7] J. Shipton, T. H. Gibson, C. J. Cotter, Higher-order compatible finite element schemes for the nonlinear rotating shallow water equations on the sphere, J. Comp. Phys. 375 (2018) 1121–1137.
[8] W. C. Skamarock, J. B. Klemp, M. G. Duda, L. F. Fowler, S.-H. Park, T. D. Ringler, A multiscale nonhydrostatic atmospheric model using centroidal Voronoi tessellations and C-grid staggering, Mon. Wea. Rev. 140 (2012) 3090–3105.
[9] A. Gassmann, A global hexagonal C-grid nonhydrostatic dynamical core (ICONAP) designed for energetic consistency, Q. J. R. Meteorol. Soc. 139 (2013) 152–175.
[10] T. Dubos, S. Dubey, M. Tort, R. Mittal, Y. Meurdesoif, F. Hourdin, DYNAMICO-1.0, an icosahedral hydrostatic dynamical core designed for consistency and versatility, Geosci. Model Dev. 8 (2015) 3131–3150.
[11] A. Natale, J. Shipton, C. J. Cotter, Compatible finite element spaces for geophysical fluid dynamics, Dyn. Stat. Climate Sys. 1 (2016) 1–31.
[12] T. Melvin, T. Benacchio, B. Shipway, N. Wood, J. Thuburn, C. Cotter, A mixed finite-element, finite-volume, semi-implicit discretisation for atmospheric dynamics: Cartesian geometry, Q. J. R. Meteorol. Soc.
