$\mathcal{P}\mathcal{T}$-symmetric rational Calogero model with balanced loss and gain

Debdeep Sinha\textsuperscript{a} and Pijush K. Ghosh\textsuperscript{b}

Department of Physics, Siksha-Bhavana, Visva-Bharati University, Santiniketan, PIN 731 235, India

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Abstract. A two-body rational Calogero model with balanced loss and gain is investigated. The system yields a Hamiltonian which is symmetric under the combined operation of parity ($\mathcal{P}$) and time reversal ($\mathcal{T}$) symmetry. It is shown that the system is integrable and exact, stable classical solutions are obtained for particular ranges of the parameters. The corresponding quantum system admits bound state solutions for the exactly same ranges of the parameters for which the classical solutions are stable. The eigenspectrum of the system is presented with a discussion on the normalization of the wave functions in proper Stokes wedges. Finally, the Calogero model with balanced loss and gain is studied classically, when the pair-wise harmonic interaction term is replaced by a common confining harmonic potential. The system admits stable solutions for particular ranges of the parameters. However, the integrability and/or exact solvability of the system is obscure due to the presence of the loss and gain terms. The perturbative solutions are obtained and are compared with the numerical results.

1 Introduction

The damped harmonic oscillator with a friction term linear in velocity is not a Hamiltonian system. In order to make the system Hamiltonian, one needs to introduce a time reversed version of the original oscillator which may be considered as a thermal bath [1]. A system consisting of these two oscillators considered together yields a $\mathcal{P}\mathcal{T}$-symmetric Hamiltonian and the total energy is conserved. The Hamiltonian formulation necessarily implies that loss and gain are equally balanced. The quantization of this kind of coupled oscillators having balanced loss and gain is also discussed in the literature [2–5]. Neither classically stable solutions nor quantum bound states can be obtained for this system. However, the situation changes significantly if these two oscillators are coupled through interactions which are $\mathcal{P}\mathcal{T}$-symmetric. An investigation in this regard has been carried out recently [6,7], where a system of coupled oscillators having balanced loss and gain is considered. Both classically stable solutions as well as quantum bound states are obtained within the unbroken $\mathcal{P}\mathcal{T}$-symmetric region. Further, this system exhibits $\mathcal{P}\mathcal{T}$-symmetric phase transitions which occur at the same ranges of coupling parameter for the classical as well as quantum cases. This mathematical model is motivated by an experiment performed on two coupled $\mathcal{P}\mathcal{T}$-symmetric whispering-gallery–mode optical resonators [8]. The results of this experiment are well explained by the model considered in ref. [6].

The oscillator systems having balanced loss and gain with different types of couplings are studied extensively in the literature. For example, a pair of mutually coupled active LCR circuits, one with amplification and the other with equivalent attenuation, is used to realize the $\mathcal{P}\mathcal{T}$-symmetric phase transition [9]. A chain of linearly coupled oscillators with its continuum limit is considered in [10]. Further, $\mathcal{P}\mathcal{T}$-symmetric dimer of coupled nonlinear oscillators with cubic nonlinearities is considered in [11]. The system is not amenable to Hamiltonian formulation. Subsequently, a Hamiltonian system of nonlinear oscillators with balanced loss and gain is considered in [12]. The small amplitude oscillations in this system are shown to be governed by a $\mathcal{P}\mathcal{T}$-symmetric nonlinear Schrödinger dimer. The common feature of all of these systems is that the unbroken $\mathcal{P}\mathcal{T}$-symmetric regime content classically stable solutions.

The Calogero model [13–18] is an exactly solvable model in one dimension where each particle interacts with all other via a long-range inverse square potential. There are reviews [19–22] on the topic discussing various aspects of this model. The Calogero type of systems have its influences in diverse branches of physics such as in exclusion statistics [23],

\textsuperscript{a} e-mail: debdeep.sinha.rs@visva-bharati.ac.in
\textsuperscript{b} e-mail: pijushkanti.ghosh@visva-bharati.ac.in
quantum chaos [24,25], spin chains [26,27] algebraic and integrable structure [28,29], self-adjoint extensions [30–33], collective field formulation of many-particle systems [34,35], etc. Therefore, the effect of Calogero type of potential in the case of a coupled oscillator system having balanced loss and gain is an obvious curiosity. One of our main objectives in this work is to examine how the integrable properties of the Calogero model get modified due to the presence of balanced loss and gain terms. Another motivation of our study is to investigate \( \mathcal{P}\mathcal{T}\)-symmetric phase transition in the presence of specified type of nonlinear interaction governed by inverse square potential. Finally, it is expected that the additional inverse square interaction term may be realized in the context of whispering-gallery–mode optical resonators.

In this article, we investigate a two-body rational Calogero model with balanced loss and gain. The system yields a Hamiltonian which is \( \mathcal{P}\mathcal{T}\)-symmetric. We obtain exact stable classical solutions for the particular ranges of the parameters for which the \( \mathcal{P}\mathcal{T} \) symmetry remains unbroken. A quantization of this classical model is carried out. This quantized version admits bound state solutions for exactly the same ranges of the parameters for which the classical solutions are stable. The eigenstates and eigenstates of the system are presented. The eigenfunctions are not normalizable along the real line. We define the proper Stoke wedges and discuss the normalization of the ground state wave function in this Stokes wedges. Finally, the Calogero model with balanced loss and gain is studied classically, when the pair-wise harmonic interaction term is replaced by a common confining harmonic potential. In this case the system admits stable solutions in the unbroken \( \mathcal{P}\mathcal{T}\)-symmetric regime. However, the exact solvability of the system is obscure due to the presence of the loss and gain terms. For this case we obtained perturbative solutions in the unbroken \( \mathcal{P}\mathcal{T}\)-symmetric regime. This perturbative results are compared with the exact numerical calculations.

The plan of this paper is as follows. In the next section we introduce the main model. In sect. 2.1, we discuss the classical results for a two-body rational Calogero model with balanced loss and gain. The exact stable solution for this system is obtained. In sect. 2.2, the classical perturbative solutions for the Calogero model with balanced loss and gain are discussed when the pair-wise harmonic interaction term is replaced by a common confining harmonic potential. These results are compared with the numerical calculations. In sect. 3, the quantum case for the two-body rational Calogero model with balanced loss and gain is considered. In the last section we make a summary and discuss the results.

## 2 Classical model

The model we consider has the following Lagrangian:

\[
L = \dot{x} \dot{y} + \gamma (x \dot{y} - y \dot{x}) - \omega^2 x y - \frac{\epsilon}{2} (x^2 + y^2) - \frac{g}{2(x - y)^2},
\]

where the dot over the variables denotes derivative with respect to time. The first four terms describe a pair of two oscillators, having common frequency \( \omega \), with balanced loss and gain and are coupled linearly via coupling parameter \( \epsilon \). The last term is the reminiscent of the two-body Calogero potential describing a system of two particles interacting with each other via long-range inverse square potential. Thus the Lagrangian of eq. (1) describes a dissipative harmonic oscillator system along with its time reversed version interacting with each other via a two-body inverse square potential plus a linear interacting term. The whole system yields a Hamiltonian

\[
H = P_x P_y + \gamma (y P_y - x P_x) + (\omega^2 - \gamma^2) x y + \frac{g}{2(x - y)^2} + \frac{\epsilon}{2} (x^2 + y^2),
\]

where

\[
P_x = \dot{y} - \gamma y, \quad P_y = \dot{x} + \gamma x
\]

are, respectively, the momenta conjugate to the \( x \) and \( y \) variables. The total energy of the system is conserved.

The following equations of motion may be obtained either from the Lagrangian (1) or from the Hamiltonian (2):

\[
\ddot{x} + 2 \gamma \dot{x} + (\omega^2 x + \epsilon y) + \frac{g}{(x - y)^2} = 0,
\]

\[
\ddot{y} - 2 \gamma \dot{y} + (\omega^2 y + \epsilon x) - \frac{g}{(x - y)^2} = 0.
\]

The rational Calogero model with the balanced loss and gain is obtained in the limit \( \epsilon = -\omega^2 \) for which two particles interact with each other via pair-wise harmonic plus inverse square interaction. The limit \( \epsilon = 0 \) corresponds to a system with balanced loss and gain where two particles are confined in a common harmonic potential and interacting with each other through inverse square potential. The \( \epsilon = 0 \) case may be considered as Sutherland model [16] in the presence of balanced loss and gain terms. One of the purposes of this article is to investigate whether or not the
quantum and classical integrability of the Calogero-Sutherland model is preserved after the inclusion of balanced loss and gain terms. The classical equations of motion are highly nonlinear for \( g \neq 0 \). It is also under the purview of the present article to investigate \( \mathcal{PT} \)-symmetric phase transition in the presence of nonlinear interaction. Finally, the system with \( g = 0 \) is experimentally realized [8,6]. It is expected that the additional inverse square interaction term may be realized in the context of whispering-gallery-mode optical resonators.

It may be noted that the Hamiltonian (2) is \( \mathcal{PT} \)-symmetric, where the action of parity \( \mathcal{P} \) is to interchange the gain and loss oscillators,

\[
\mathcal{P} : x \rightarrow -y, \quad \mathcal{P} : y \rightarrow -x, \quad \mathcal{P} : P_x \rightarrow -P_y, \quad \mathcal{P} : P_y \rightarrow -P_x,
\]

and the action of time reversal \( \mathcal{T} \) is to change the sign of the momenta

\[
\mathcal{T} : x \rightarrow x, \quad \mathcal{T} : y \rightarrow y, \quad \mathcal{T} : P_x \rightarrow -P_x, \quad \mathcal{T} : P_y \rightarrow -P_y.
\]

It may be noted that if a real, time-independent potential \( V(x,y) \) is added to the Lagrangian (1), then it remains \( \mathcal{PT} \)-symmetric provided \( V(x,y) = V(-x,-y) \).

It will be convenient to cast the equations of motion in the following new coordinates:

\[
z_1 = x + y, \quad z_2 = x - y.
\]

In this coordinates eqs. (4) take the following form:

\[
\ddot{z}_1 + (\omega^2 + \varepsilon)z_1 + 2\gamma \dot{z}_2 = 0,
\]

\[
\ddot{z}_2 + (\omega^2 - \varepsilon)z_2 + 2\gamma \dot{z}_1 + \frac{2g}{z_2^3} = 0.
\]

It is interesting to see that under parity \( (\mathcal{P}) \), \( z_1 \) and \( z_2 \) transform as

\[
\mathcal{P} : z_1 \rightarrow -z_1, \quad \mathcal{P} : z_2 \rightarrow z_2.
\]

Thus the parity operator has its usual meaning of spatial reflection in the new coordinate system.

### 2.1 Two-body Calogero model with balanced loss and gain

In the coordinates \((z_1, z_2)\), the Lagrangian of eq. (1) takes the following form, for \( \varepsilon = -\omega^2 \):

\[
L = \frac{1}{4} \left( \ddot{z}_1^2 - \ddot{z}_2^2 + \frac{\gamma}{2} (z_2 \dot{z}_1 - z_1 \dot{z}_2) \right) + \frac{\omega^2}{2} z_2^2 - \frac{g}{2z_2^3}.
\]

This Lagrangian describes a coupled oscillators having balanced loss and gain and are interacting with each other via long-range inverse square potential and a two-body harmonic term. The equations of motion can be obtained as

\[
\ddot{z}_1 + 2\gamma \dot{z}_2 = 0,
\]

\[
\ddot{z}_2 + 2\omega^2 z_2 + 2\gamma \dot{z}_1 + \frac{2g}{z_2^3} = 0.
\]

The above equations can also be derived from the following Hamiltonian:

\[
H = (P_{z_1}^2 - P_{z_2}^2) - \gamma (z_1 P_{z_2} + z_2 P_{z_1}) - \frac{\omega^2}{2} z_2^2 - \frac{\gamma^2}{4} (z_1^2 - z_2^2) + \frac{g}{2z_2^3},
\]

where \( P_{z_1} \) and \( P_{z_2} \) are, respectively, the momenta conjugate to the normal coordinates \( z_1 \) and \( z_2 \). Equations (12) will be decoupled in terms of \( z_1 \) and \( z_2 \) as \( \gamma \) tends to zero with \( z_1 \) describing a free particle and \( z_2 \) describing a harmonic oscillator in an inverse square potential. Integrating eq. (11), we get

\[
\Pi = \dot{z}_1 + 2\gamma \dot{z}_2 = 2P_{z_1} + \gamma z_2,
\]

with \( \Pi \) being a constant of integration which can be determined by fixing the initial conditions. It may be noted that \( \Pi \) is an integral of motions related to the translational symmetry of the action. The Poisson bracket of \( \Pi \) with the Hamiltonian \( H \) is zero. Thus the existence of two integral of motions \( H \) and \( \Pi \) in involution imply that the system is integrable.
Substituting eq. (14) in (12), we get
\[
\ddot{z}_2 + \Omega^2 z_2 + \frac{2g}{\dot{z}_2} = -2\gamma \Pi, \quad \Omega^2 = 2(\omega^2 - 2\gamma^2).
\]
(15)
The frequency \(\Omega\) is real for the range \(-\frac{\omega}{\sqrt{2}} \leq \gamma \leq \frac{\omega}{\sqrt{2}}\). This indicates \(\mathcal{PT}\)-symmetric phase transitions one at \(-\frac{\omega}{\sqrt{2}}\) and the other at \(\frac{\omega}{\sqrt{2}}\). It is interesting to note that in [6] the phase transition point depends on the linear coupling strength \(\epsilon\) but in the present case the phase transition point does not depend directly on the coupling parameter \(g\) rather, as we shall see, the solution of \(z_1\) and \(z_2\) put a restriction on the possible range of \(g\). If we chose the initial conditions as to set \(H = 0\), then eq. (15) reduces to the Ermakov-Pinney equation of the following form:
\[
\ddot{z}_2 + \Omega^2 z_2 + \frac{2g}{\dot{z}_2} = 0.
\]
(16)
This equation describes the motion of a particle in a harmonic plus inverse square interaction. The system admits stable solutions for attractive inverse square potential implying that \(g < 0\). One of the possible ways to make \(H = 0\), is to take the ratio \(\frac{\dot{z}_2(0)}{z_2(0)} = -2\gamma\). One such choice is \(z_2(0) = -2\gamma b\), \(z_2(0) = b\) and \(z_2(0) = a\). In this case the solution of (16) is given as
\[
z_2(t) = \left[ \frac{1}{b^2\Omega^2} (a^2b^2 - 2g) \sin^2 \Omega t + \frac{2ab}{\Omega} \sin \Omega t \cos \Omega t + b^2 \cos^2 \Omega t \right]^{\frac{1}{2}}.
\]
(17)
The expression for \(z_1\) can be obtained by integrating eq. (14),
\[
z_1 = -2\gamma \int \left[ \frac{1}{b^2\Omega^2} (a^2b^2 - 2g) \sin^2 \Omega t + \frac{2ab}{\Omega} \sin \Omega t \cos \Omega t + b^2 \cos^2 \Omega t \right]^{\frac{1}{2}} dt + I,
\]
(18)
with \(I\) being the constant of integration which can be fixed by the value of \(z_1(t)\) at time \(t = 0\). We make the following change of variable:
\[
\sin \phi = \sqrt{\frac{D + (A - C) \cos 2\Omega t - B \sin 2\Omega t}{2D}}
\]
(19)
where
\[
A = \frac{1}{b^2\Omega^2} (a^2b^2 - 2g), \quad B = \frac{ab}{\Omega}, \quad C = b^2, \quad D = \sqrt{(C - A)^2 + B^2}.
\]
(20)
In the variable \(\phi\) the integration of eq. (18) takes the following form:
\[
z_1(t) = 2\gamma \sqrt{\frac{C + A + D}{2\Omega}} \int_0^\phi \left( 1 - \frac{2D}{(C + A + D) \sin^2 \phi} \right)^{\frac{1}{2}} d\phi + I
\]
(21)
\[
= 2\gamma \sqrt{\frac{C + A + D}{2\Omega}} E[\phi, k^2] + I,
\]
(22)
with \(E(\phi, k^2)\) being the elliptical integral of second kind having the argument \(\phi\) and \(k^2 = \frac{2D}{C + A + D}\). The argument \(k^2\) of the elliptical integral satisfies the condition \(0 < \frac{2D}{C + A + D} < 1\). In the unbroken \(\mathcal{PT}\)-symmetric region, \(A\) and \(B\) are always real implying that \(D\) is also real in this region. In order that the argument of the elliptical function be real the following condition must be satisfied:
\[
D + (A - C) \cos 2\Omega t \geq B \sin 2\Omega t.
\]
(23)
The expression of \(D\) in (20) implies that \(D\) is real and positive and \(D > B\) as well as \(D > (C - A)\). The maximum value of the right-hand side of the inequality in (23) is \(B\) but at that time the left-hand side becomes \(D\). Again, the minimum value of the left-hand side of the inequality in (23) is \(D \pm (C - A)\) depending on the relative value of \(A\) and \(C\) but in this case the right-hand side of (23) is zero and we have \(D \pm (C - A) \geq 0\), which is always true. Thus, we can infer that the relation in (23) is true for all \(t\).
Some observations regarding the nature of the solutions are as follows:
- As \(\gamma\) approaches the transition value \(\pm \frac{\omega}{\sqrt{2}}, \Omega \to 0\), and \(z_2\) becomes singular indicating the occurrence of the phase transition.
The stability of the system depends on the nature of the solutions for \( z_1 \) and \( z_2 \). The solution (17) for \( z_2 \) is well behaved and does not introduce any instability in the system. However, the nature of the solution for \( z_1 \) depends on the initial conditions imposed and may introduce instability in the system. For example, for \( a = 1, b = 1 \), the expression for \( z_1 \) can be obtained from eq. (22):

\[
z_1 = 2\gamma \frac{\sqrt{1 + A + D}}{\sqrt{2}\Omega} E[\phi, k^2],
\]

with \( A, B, C, D \) given by eq. (20) and the constant of integration is taken to be zero. This solution has a periodic nature and does not introduce instability in the system. However, for \( a = 0, b = 1 \), the expression for \( z_1 \),

\[
z_1 = 2\gamma \frac{E[\Omega t, 1 + \frac{2g}{z^2}]}{\Omega},
\]

with \( A, B, C, D \) given by eq. (20) and \( I = 0 \), increases linearly with time and introduces instability in the system. Thus, the nature of the solutions for \( z_1 \) and therefore the stability of the system depends on the initial conditions imposed.

### 2.2 Two-body Sutherland model with balanced loss and gain

The Lagrangian in eq. (1) for \( \epsilon = 0 \) describes a two-body Sutherland [16] model with balanced loss and gain. In this case eqs. (8) take the following form:

\[
\begin{align*}
\ddot{z}_1 + \omega^2 z_1 + 2\gamma \dot{z}_2 & = 0, \\
\ddot{z}_2 + \omega^2 z_2 + 2\gamma \dot{z}_1 + \frac{2g}{z^2} & = 0.
\end{align*}
\]

In the limit \( \gamma \) tends to zero the above two equations decoupled in terms of \( z_1 \) and \( z_2 \), one gives a harmonic oscillator and other a harmonic oscillator in an inverse square potential. It is the sole effect of the dissipative term that makes the system a coupled and nontrivial one. Interestingly the coupling arising due to the inverse square potential in \((x, y)\) coordinates disappears in \((z_1, z_2)\) coordinates.

We define \( \dot{z}_1 = p \) and \( \dot{z}_2 = q \) in order to investigate the possible equilibrium points and the stability of the system. The stationary points are those for which the following equations are satisfied:

\[
\begin{align*}
\dot{p} + \omega^2 z_1 + 2\gamma q & = 0, \\
\dot{q} + \omega^2 z_2 + 2\gamma p + \frac{2g}{z^2} & = 0, \\
\dot{z}_1 & = 0, \\
\dot{z}_2 & = 0.
\end{align*}
\]

If we solve the above equations, we get \((\dot{p}, \dot{q}, \dot{z}_1, \dot{z}_2) = (0, 0, 0, (\frac{-2g}{z^2})^\frac{1}{2})\) as the equilibrium point of the system which indicates that the coupling strength \( q \) must be negative. In order to do the linear stability analysis, we consider a small variation about the equilibrium point \((p, q, z_1, z_2)\) as \((\dot{p} + v_1, \dot{q} + v_2, \dot{z}_1 + v_3, \dot{z}_2 + v_4)\). A Taylor series expansion of eqs. (27) about the equilibrium point then yields up to first order, the following set of equations:

\[
\begin{pmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4
\end{pmatrix} =
\begin{pmatrix}
0 & -2\gamma & -\omega^2 & 0 \\
-2\gamma & 0 & 0 & -4\omega^2 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4
\end{pmatrix}.
\]

The eigenvalues of the \( 4 \times 4 \) matrix in eq. (28), given by

\[
\lambda = \pm \left[ P \pm \sqrt{P^2 - 16\omega^2} \right]^{\frac{1}{2}}, \quad P = (5\omega^2 - 4\gamma^2),
\]

will determine the nature of the equilibrium point. Since positive eigenvalues indicate a growing solution, for stable solution we must have all the eigenvalues either be negative or imaginary. The only possibility of having stable solution
Fig. 1. Numerical result (dashed line) vs. perturbative result (continuous line). In this figure, $\gamma = 0.1$, $g = -0.5$, $\omega = 1.0$. The upper graph shows variation of $x$ as a function of time and the lower one shows variation of $y$ as a function of time.

is to take $\gamma = \pm \sqrt{\frac{5}{4}}\omega$, which makes all the eigenvalues imaginary. Therefore, for the stable solutions the value of $\gamma$ should be in the range $-\sqrt{\frac{5}{4}}\omega < \gamma < \sqrt{\frac{5}{4}}\omega$.

Equations (26) define a coupled set of second-order nonlinear differential equations. Unlike the case of rational Calogero model, we are unable to find any exact solutions. This is primarily due the fact that a constant of motion similar to $\Pi$ cannot be found. We investigate the solutions by employing Lindstedt-Poincaré perturbation method. It may be noted in this regard that the system becomes decoupled in the limit $\gamma \to 0$. We treat the $\gamma$-dependent term as perturbation with $\gamma$ being a small parameter. The unperturbed equations have the following form:

$$
\ddot{z}_1 + \omega^2 z_1 = 0,
\ddot{z}_2 + \omega^2 z_2 + \frac{2g}{z_2^3} = 0.
$$

(30)

We employ the following initial conditions:

$$
z_1(0) = 0.5, \quad z_2(0) = 1,
\dot{z}_1(0) = 0, \quad \dot{z}_2(0) = 0.
$$

(31)

(32)

Up to first order in $\gamma$, the solutions may be written as

$$
z_1(t) = 0.5 \cos \omega t + O(\gamma^2),
$$

(33)

$$
z_2(t) = -\frac{2g}{\omega^2} + \gamma \left[ -\frac{5}{3\omega^2} \cos 2\omega t + \frac{5}{3\omega^2} \left( 2 \sin^3 \omega t \sin 2\omega t + (1 + \sin 2\omega t) \cos \omega t \cos 2\omega t \right) \right] + O(\gamma^2).
$$

(34)

The integrability of this system is obscure and we obtain numerical solutions. This numerical results are in well agreement with the perturbative results at the initial time (fig. 1).

3 Quantum case

In order to quantize the classical Hamiltonian (13), we replace the classical variables $P_{z_1}$, $P_{z_2}$, $z_1$, $z_2$ by operators satisfying the commutation relations $[z_1, P_{z_1}] = i$, $[z_2, P_{z_2}] = i$ and the rest of the commutators are zero. We replace $P_{z_1}$ by $-i\partial_{z_1}$ and $P_{z_2}$ by $-i\partial_{z_2}$ and obtain the following expressions for the Hamiltonian $\hat{H}$ and the conserved quantity $\hat{\Pi}$ in the quantum theory:

$$
\hat{H} = -\left( \partial_{z_1}^2 - \partial_{z_2}^2 \right) + i\gamma \left( z_1 \partial_{z_2} + z_2 \partial_{z_1} \right) - \frac{\omega^2}{2} z_2^2 + \frac{g}{2z_2^3} - \frac{\gamma^2}{4} \left( z_1^2 - z_2^2 \right),
$$

(35)

$$
\hat{\Pi} = -2i \partial_{z_1} + \gamma z_2.
$$

(36)
The operators $\hat{H}$ and $\hat{\Pi}$ commute and constitute two integrals of motion for the system which in turn implies that the system is integrable. Since $\hat{H}$ and $\hat{\Pi}$ commute, it is always possible to chose a basis in which simultaneous eigenstates of $\hat{H}$ and $\hat{\Pi}$ may be constructed. An eigenfunction of the operator $\hat{H}$ with continuous eigenvalues $k$ has the following form:

$$\psi = \exp \left[ i \frac{\gamma}{2} (k - \gamma z_2) \right] \tilde{\phi}(z_2),$$

where $\tilde{\phi}(z_2)$ is an arbitrary function of $z_2$, whose functional form is to be determined by demanding that $\psi$ is also an eigenfunction of the Hamiltonian $H$. In the limit of vanishing $\gamma$, the $z_1$-dependent part of $\psi$ is the wave function of a free particle with wave vector $k/2$. This is consistent with the fact that in the same limit $H$ reduces to conjugate momentum operator $P_{\gamma}$ up to an overall multiplication factor of two. Substituting (37) in the time-independent Schrödinger equation,

$$\left[ -\left( \partial_{z_2} - \partial_2^2 \right) + i \gamma (z_1 \partial_{z_2} + z_2 \partial_{z_1}) - \frac{\omega^2}{2} z^2_2 + \frac{g}{2z_2^2} - \frac{\gamma^2}{4} (z_1^2 - z_2^2) \right] \psi = E \psi,$$

we get the following equation:

$$\partial^2_{z_2} \tilde{\phi}(z_2) - \frac{1}{4} \Omega^2 z^2_2 \tilde{\phi}(z_2) + \frac{g}{2z_2^2} \tilde{\phi}(z_2) + \frac{k}{4} (k - 2\gamma z_2) \tilde{\phi}(z_2) = E \tilde{\phi}(z_2).$$

This is a differential equation of only one variable $z_2$. For vanishing $\gamma$, the centre of mass and the relative coordinates separate out in the Hamiltonian (13) and eq. (39) describes the motion of an isotonic oscillator. However, for $\gamma \neq 0$, the centre-of-mass modes are coupled to the equation governed by $z_2$. Similar situation arises in ref. [6], for which the Hamiltonian separates out into a free particle in the centre-of-mass frame and a simple harmonic oscillator in the relative coordinate for $\gamma = 0$. The term linear in $k$ in eq. (39) for $\gamma \neq 0$ can always be absorbed by a shift of the relative coordinate for harmonic oscillator, but, not for isotonic oscillator. This poses difficulty in solving eq. (39).

The series method solution allows normalizable solution of (39) only for $k = 0$. Other nontrivial exact solutions are also not apparent. This is consistent with the fact that we obtain exact classical solutions when the value of the constant of motion $H$ is zero. Therefore, an obvious choice is to consider the exact normalizable solution corresponding to $k = 0$. In this case, eq. (39) reduces to the following form:

$$\partial^2_{z_2} \tilde{\phi}(z_2) - \frac{1}{4} \Omega^2 z^2_2 \tilde{\phi}(z_2) + \frac{g}{2z_2^2} \tilde{\phi}(z_2) = E \tilde{\phi}(z_2).$$

This equation is invariant under the operation $z_2 \rightarrow -z_2$. Therefore the solutions may always be chosen to be either even or odd under parity transformation. The potential has a singularity at $z_2 = 0$ which breaks the space into two disjoint regions ($z_2 > 0$ or $z_2 < 0$) and the wave function vanishes at $z_2 = 0$. The ground state wave function and the corresponding energy of (40) are, respectively, given as

$$\tilde{\phi}(z_2) = z_2^\lambda \exp[-Cz^2_2], \quad E_0 = -(2 + 4\lambda)C,$$

with $C = \pm \frac{1}{2} \Omega$ and $\lambda$ satisfying the relation $\lambda(\lambda - 1) = -\frac{g}{2z_2^2}$. It is interesting to note that $C$ is real for the range $-\frac{\sqrt{2}}{\sqrt{\gamma}} \leq \gamma \leq \frac{\sqrt{2}}{\sqrt{\gamma}}$. This indicates $PT$-symmetric phase transitions one at $-\frac{\sqrt{2}}{\sqrt{\gamma}}$ and other at $\frac{\sqrt{2}}{\sqrt{\gamma}}$. Outside this range the energy is complex and comes in complex conjugate pairs indicating a broken $PT$-symmetric region. Interestingly the classical and quantum $PT$-symmetric phase transitions occur at the same value of the parameter $\gamma$. In order to have the complete spectra we make the following substitution:

$$\tilde{\phi}(z_2) = z_2^\lambda \exp \left[ -\frac{1}{4} \Omega z^2_2 \right] \phi(z_2),$$

(42)

with which eq. (40) reduces to the following form:

$$\partial^2_{z_2} \phi + \left( \frac{2\lambda}{z_2} - 4Cz_2 \right) \partial_{z_2} \phi = (E + 2C + 4\lambda C) \phi.$$ 

(43)

We demand a series solution of eq. (43) having the form

$$\phi(z_2) = \sum_{n=0}^{\infty} a_n z_2^n,$$

(44)
Fig. 2. Stoke wedges. For negative coefficient of $z_2$, the Stoke wedge is centered about the positive imaginary axis with a opening angle $\frac{\pi}{2}$. For positive coefficient of $z_2$, the Stoke wedge is centered about the negative imaginary axis having the same opening angle.

The recursion relation for $a_n$ is given by

$$a_{n+2} = \frac{(E + 2C + 4\lambda C) + 4Cn}{(n + 2)[(n + 1) + 2\lambda]}a_n,$$

with $a_1 = 0$, and $a_0$ is obtained from the normalization condition. Thus the series for $\phi(z_2)$ contains only even powers of $z_2$. For normalizable solutions, the series must be terminated and we get the following expression for the energy states:

$$E = -2C(2n + 1 + 2\lambda),$$

with $E = -2C(1 + 2\lambda)$ being the ground state energy. If we now chose the positive sign of $C$, then the ground state wave function is normalizable on the real line but the system becomes unbounded from below, i.e. the system does not have a stable ground state. So we chose the negative sign for $C$ which makes the system bounded from below but in this case the normalization of the wave function becomes crucial and we discuss it below. If we write $\phi(z_2) = \phi_{2m}(z_2)$ with $m = 0, 1, 2, \ldots$, then first few polynomials may be written as

$$\phi_2 = a_0 \left(1 - \frac{4C}{(1 + 2\lambda)}z_2^2\right),$$

$$\phi_4 = a_0 \left(1 - \frac{8C}{(1 + 2\lambda)}z_2^2 + \frac{16C^2}{(3 + 2\lambda)(1 + 2\lambda)}z_2^4\right).$$

All the eigenstates of $\hat{H}$ of the form $\phi(z_2)\psi_0$ with $\psi_0 = z_2^{\lambda}\exp[-\{Cz_2 + \frac{i\gamma}{2}z_1z_2\}]$, are the eigenstates of $\hat{H}$ belonging to the zero eigenvalue.

We now consider the normalization of the ground state wave function

$$\psi_0 = z_2^{\lambda}\exp\left[\frac{\Omega}{4}z_2^2 - \frac{i\gamma}{2}z_1z_2\right], \quad C = -\frac{1}{4}\Omega.$$

Clearly this wave function is not normalizable along the real $z_2$ line. We have to fix the Stoke wedges in the complex $z_2$-plane where the wave function $\psi_0$ is normalizable. The first part ($z_2^{\lambda}$-part) of the exponential vanishes in a pair of Stoke wedges with opening angle $\frac{\pi}{2}$ and centered about the positive and negative imaginary axes in the complex $z_2$-plane. The second part ($\frac{i\gamma}{2}z_1z_2$) of the exponential vanishes in the upper half of the complex $z_2$-plane if the coefficient ($\frac{i\gamma}{2}z_1$) of $z_2$ is negative and vanishes in the lower half of the complex $z_2$ plane if the coefficient ($\frac{i\gamma}{2}z_1$) of $z_2$ is positive. Therefore $\psi_0$ vanishes in a single Stoke wedge either with opening angle $\frac{\pi}{2}$ and centered about the positive imaginary axes in the complex $z_2$-plane or with opening angle $\frac{\pi}{2}$ and centered about the negative imaginary axes in the complex $z_2$-plane (fig. 2).
4 Conclusion

We have considered a two-body rational Calogero model having balanced loss and gain. The Hamiltonian for the system is obtained which is found to be $PT$-symmetric. This system admits two integral of motions in involution. This system is integrable both classically and quantum mechanically. In particular, the classical equations of motion for the system are solved exactly for the particular ranges of the parameters. We obtained exact, stable classical solutions. We also quantized this classical model. The quantized system yields bound state solutions for exactly the same range of the parameters for which the classical solutions are stable. The normalization of the wave functions in the proper Stoke wedges are discussed. Further, the Calogero model with balanced loss and gain is studied classically, when the pair-wise harmonic interaction term is replaced by a common confining harmonic potential. This system may be considered as the Sutherland model in the presence of balanced loss and gain. The integrability of this system is obscure. In the classical level, the stability analysis is carried out and perturbative solutions are obtained. Finally, this perturbative results are compared with the numerical results. In our study we only focus on two-body problem. The question of many-body generalization of coupled oscillators system having balanced loss and gain and are interacting via Calogero-Sutherland type of potential will be very much interesting.

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