Effective field theory of pairing rotations

T. Papenbrock\textsuperscript{1,2}

\textsuperscript{1}Department of Physics and Astronomy, University of Tennessee, Knoxville, Tennessee 37996, USA
\textsuperscript{2}Physics Division, Oak Ridge National Laboratory, Oak Ridge, Tennessee 37831, USA

Pairing rotations are the low-energy excitations of finite superfluid systems, connecting systems that differ in their number of Cooper pairs. This paper presents a model-independent derivation of pairing rotations within an effective theory that exploits the emergent breaking of \(U(1)\) phase symmetries. The symmetries are realized nonlinearly and the Nambu-Goldstone modes depend only on time because the system is finite. Semi-magic nuclei exhibit pairing rotational bands while the pairing spectrum becomes an elliptical paraboloid for open-shell nuclei. Model-independent relations between double charge-exchange reactions and \(\alpha\) particle capture or knockout in open-shell nuclei are in analogy to the pair transfer reactions in a single superfluid. Odd semi-magic nuclei are described by coupling a fermion to the superfluid. The leading-order theories reproduce data for pairing rotational bands within uncertainty estimates.

I. INTRODUCTION

Atomic nuclei are finite superconductors. Hallmarks of nuclear pairing are excitation gaps in even-even nuclei \([1]\), reduced moments of inertia due to superfluidity \([2]\), odd-even staggering in many observables, and pairing vibrations \([3]\) and rotations \([4]–[7]\) (see Ref. \([8]\) for an overview). Pairing rotational spectra are the analogue to rotational bands in deformed nuclei; they are quadratic in the difference of Cooper pairs and are associated with the Nambu-Goldstone mode of a broken \(U(1)\) phase symmetry \([9]–[11]\). They explain why two-nucleon transfer is enhanced between nuclei within a pairing-rotational band \([7]–[16]\). Pairing rotational modes have been studied via pairing models, see e.g., Refs. \([6]–[17], [18]\), and in Hartree-Fock-Bogoliubov \([19]–[20]\) and relativistic mean-field computations \([21]\).

In this paper we revisit pairing rotations in the framework of effective field theory \([22]–[30]\). This brings simplicity and model independence to an old subject. The approach requires us to be conscious about the breakdown scale, and the power counting allows us to estimate or quantify \([31]–[32]\) uncertainties. Open-shell nuclei are described as two interacting superfluids starting from the most general Lagrangian compatible with the symmetry breaking. As we will see, the model-independent approach yields relations between double charge-exchange reactions and two-nucleon transfer, and these can be tested experimentally.

The approach presented in this work differs from the one by Furnstahl et al. \([24]\). That work proposed an effective field theory for dilute Fermi systems. Here we merely exploit the dynamics of Nambu-Goldstone modes corresponding to the emergent breaking of phase symmetries in finite systems. Then quantum field theory reduces to quantum mechanics \([33]\) and a fermion only appears in odd systems.

This paper is organized as follows. Section \([1]\) presents the effective field theory for even and odd semi-magic nuclei, respectively. The theory for open-shell even-even nuclei is derived in Sect. \([11]\). The theory is confronted with data in Sect. \([V]\). The summary in Sect. \([V]\) discusses the main results.

II. EFFECTIVE THEORY FOR A SINGLE SUPERFLUID

A. Even semi-magic nuclei

1. Leading-order Hamiltonian

Let us consider a finite superfluid of a single fermion species with spin \(1/2\) and assume that all fermions are in Cooper pairs. Examples are even isotopes of tin or lead, or even isotones with neutron number \(N = 82\). The corresponding nuclear ground states must be invariant under \(U(1)\) phase transformations which are generated by

\[
g(\alpha) = e^{i\alpha n}. \tag{1}\]

Here \(\alpha\) is the phase angle and \(\hat{n}\) is the operator that yields the number of pairs. A finite system displays emergent rather than spontaneous symmetry breaking \([34]\). Nevertheless, we can follow the standard approach to spontaneous symmetry breaking via non-linear realizations \([35]–[38]\), and the Nambu-Goldstone mode parameterizes the coset \(U(1)/1 \sim U(1)\) of the broken symmetry. For finite systems, however, a tremendous simplification occurs because the Nambu-Goldstone “field” \(\alpha = \alpha(t)\) depends only on time \([33], [39]\), and quantum field theory thus reduces to quantum mechanics. In our case, the phase velocity

\[
\dot{\alpha} \equiv \partial_t \alpha \tag{2}
\]

is the only quantity that can enter the Lagrangian. The leading-order Lagrangian then becomes

\[
L_{LO} = \frac{a}{2} \dot{\alpha}^2 + n_0 \dot{\alpha}. \tag{3}
\]

Here, \(a\) and \(n_0\) are low-energy constants. The constant \(a\) is akin to a mass term while \(n_0\) is a constant gauge
potential. A Legendre transformation yields the Hamiltonian
\[ H_{\text{LO}} = \frac{1}{2a} (p_\alpha - n_0)^2 . \] (4)

Here,
\[ p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} \] (5)
is the canonical momentum. We see that \( p_\alpha \) is a constant of motion. For an interpretation of \( p_\alpha \) as the number of pairs \( N \) we consider phase transformations \( g(\beta)g(\alpha) = g(\alpha + \beta) \). Thus, the phase \( \alpha \) changes to \( \alpha + \beta \) and this is a nonlinear realization of the phase symmetry. Applying Noether’s theorem to infinitesimal phase transformations then yields that \( p_\alpha \) is conserved and therefore must be identified with the number of pairs.

We quantize the Hamiltonian (4) as usual via
\[ p_\alpha \to \hat{p}_\alpha = -i\hat{\alpha}_\alpha , \] (6)
(and of course also identify the pair number operator as \( \hat{n} = -i\hat{\alpha}_\alpha . \) Thus, the Hamiltonian is
\[ H = \frac{1}{2a} (-i\hat{\alpha}_\alpha - n_0)^2 \\
= \frac{1}{2a} (\hat{n} - n_0)^2 \] (7)

Requiring that the wave function \( \psi(\alpha) \) is single-valued under gauge transformations \( \psi(\alpha) \to \psi'(\alpha) = e^{i\lambda \alpha} \psi(\alpha) \) with constant \( \lambda \) then shows that \( n_0 \) must be an integer. Eigenfunctions are
\[ \psi_n(\alpha) = \langle \alpha|n \rangle = \frac{1}{\sqrt{2\pi}} e^{in\alpha} , \] (8)
and these describe a system of \( N \) pairs. The corresponding energies
\[ \varepsilon_n = \frac{1}{2a} (n - n_0)^2 \] (9)
are in a pairing rotational band.

Let us discuss time reversal invariance. The pair-number operator \( \hat{n} = \hat{p}_\alpha \) is even under time reversal. This implies that the phase \( \alpha \) is odd. As \( \dot{\alpha} \) is even under time reversal, higher-order contributions to the effective theory can also contain odd powers of the phase velocity. Under time reversal, the eigenfunction \( \psi_n(\alpha) \to \psi_n(-\alpha) = \psi_n^\ast(\alpha) = \psi_{-n}(\alpha) . \) Formally, we could admit negative pair numbers \( n \) (and negative \( n_0 \)), and the spectrum is invariant under this change. In this case, we would interpret \( n \) as the number of hole pairs.

Besides the number operator \( \hat{n} \), the other operator of interest is the pair-removal operator \( \hat{P} \) with matrix elements
\[ \langle \alpha'|\hat{P}|\alpha \rangle = P_0 e^{-i\alpha} \delta(\alpha' - \alpha) . \] (10)
Here, \( P_0 \) is a constant that denotes the overall strength. Clearly \( P|n\rangle = P_0|n - 1\rangle \), and \( P^\dagger|n\rangle = P_0^*|n + 1\rangle \). The effective theory then predicts that \( \langle n + 1|\hat{P}|n \rangle = P_0 \), i.e. pair transfer within the nuclei of a pairing rotational band is independent of the number of pairs in a given nucleus. This hallmark of pairing rotations has been confirmed experimentally in two-nucleon transfer reactions, see \([7, 12, 14]\).
\(n_0\) we replace \((n_b - n_0)^2\) by its average \(n_b^2/3\), taken over the shell. This then yields

\[
|g| \approx \frac{9}{an_b^2}, \tag{15}
\]

and the uncertainty estimate for leading-order results is

\[
\Delta \varepsilon_n \approx \frac{3|n - n_0|^3}{an_b^6}. \tag{16}
\]

Below the breakdown energy, the term proportional to \(g\) is then suppressed by a factor \(1/n_b \ll 1\) compared to the leading term.

It is clear how to generalize this approach to even higher orders: The Lagrangian is expanded in powers of \(\hat{\chi}\), and the Hamiltonian becomes an expansion in powers of \(\hat{n} - n_0\); subsequent orders are suppressed by increasing factors of \(\sqrt{\varepsilon}/A_b \sim n_b^{-1}\).

The assumptions underlying the power counting can be tested by extracting the low-energy coefficients (2\(\alpha\))\(^{-1}\) and \(g\) from data. In analogy to rotations of deformed nuclei, one can also think about subleading corrections in the framework of a variable moment of inertia [30]. This introduces the \(n\)-dependent pairing rotational constant as

\[
\frac{1}{2}\frac{\partial^2 \varepsilon_n}{\partial n^2} = \frac{1}{2a} + g(n - n_0). \tag{17}
\]

This expression will be used below to extract \(g\) from data.

### B. Odd semi-magic nuclei

Pairing rotations in odd systems were previously considered in Ref. [11] using a BCS state within a pairing model. Within the effective theory they can be described as a spin-1/2 fermion coupled to the superfluid.

The Lagrangian is

\[
L = \frac{a}{2} \hat{\chi}^2 + n_0 \hat{\chi} + L_X + L_{\text{int}}. \tag{18}
\]

The fermion Lagrangian is

\[
L_X = \int d^3r \hat{\chi}^\dagger \left( i \partial_t + \frac{\hbar^2 \Delta_r}{2m} - V \right) \hat{\chi}(r), \tag{19}
\]

and the interaction \(L_{\text{int}}\) will be specified shortly. Here, we have introduced the two-component fermion field

\[
\hat{\chi}(r) = \begin{pmatrix} \hat{\chi}_+^\dagger(r) \\ \hat{\chi}_-^\dagger(r) \end{pmatrix}, \tag{20}
\]

and its adjoint. The operators \(\hat{\chi}_+^\dagger(r)\) and \(\hat{\chi}_-^\dagger(r)\) create and annihilate a fermion with spin projection \(s = \pm 1/2\) at the position \(r\), respectively. They fulfill the usual anti-commutation relations. In Eq. (19) the potential is denoted as \(V\) and the fermion’s mass as \(m\). We neglected fermion-fermion interactions because we are only interested in a single fermion coupled to a superfluid.

The fermion-pair number operator is

\[
\hat{n}_\chi = \frac{1}{2} \int d^3r \hat{\chi}^\dagger(r) \hat{\chi}(r), \tag{21}
\]

and this operator couples the fermion to the superfluid, i.e. we have

\[
L_{\text{int}} = -\hat{n}_\chi \dot{\chi}. \tag{22}
\]

The superfluid-fermion interaction [22] is so simple because (i) the coupling of the fermion to the superfluid must be via the phase velocity (as we deal with a Nambu-Goldstone mode) and (ii) it can only happen in gauge space, i.e. via the fermion-pair number operator [21]. The sign is chosen for convenience. The canonical momentum of the superfluid is \(p_\alpha = \partial L/\partial \dot{\chi}_\alpha\), and the behavior of the superfluid under phase transformations is as before.

Under phase transformations with an infinitesimal angle \(\delta \beta\), the fermion field changes by

\[
e^{i\delta \beta \hat{n}_\chi} \hat{\chi}_\alpha(r) e^{-i\delta \beta \hat{n}_\chi} = \hat{\chi}_\alpha(r) - i\delta \beta \hat{\chi}_\alpha(r). \tag{23}
\]

Introducing the fermion canonical momenta

\[
\hat{\Pi}_\alpha(r) = \frac{\delta L}{\delta \dot{\chi}_\alpha(r)} = i\dot{\chi}_\alpha^\dagger(r), \tag{24}
\]

and applying Noether’s theorem to the coupled system then shows that the total number of pairs

\[
\hat{n}_{\text{tot}} = p_\alpha + \hat{n}_\chi \tag{25}
\]

is conserved under phase rotations. This is as expected. After the quantization in the eigenstates of \(\hat{n}_{\text{tot}}\) are products of a superfluid state \(|n\rangle\) with \(n\) pairs and a fermion state. We denote the latter as \(|qjj_z\rangle\) where \(j\) denotes the fermion’s total angular momentum, \(j_z\) its projection onto an arbitrary axis, and \(q\) accounts for any other quantum numbers. Thus

\[
\hat{n}_{\text{tot}}|n\rangle|qjj_z\rangle = \left(n + \frac{1}{2}\right)|n\rangle|qjj_z\rangle, \tag{26}
\]

and we have half integer numbers of pairs \(n_{\text{tot}} = n + 1/2\).

A Legendre transform yields the Hamiltonian

\[
\hat{H} = \frac{1}{2a} (-\hat{n}_\alpha + \hat{n}_\chi - n_0)^2 + \hat{H}_X
\]

\[
= \frac{1}{2a} (\hat{n}_{\text{tot}} - n_0)^2 + \hat{H}_X, \tag{27}
\]

with

\[
\hat{H}_X = \int d^3r \hat{\chi}^\dagger(r) \left( -\frac{\hbar^2 \Delta_r}{2m} + V \right) \hat{\chi}(r). \tag{28}
\]

The eigenstates (26) of the total pair-number operator are also eigenstates of the Hamiltonian (27). Using

\[
\hat{H}_X|qjj_z\rangle = e_{q,j}|qjj_z\rangle, \tag{29}
\]
we find the spectrum
\[ \varepsilon_{n_0 \pm q_j} = \frac{1}{2a} (n_{\text{tot}} - n_0)^2 + \varepsilon_{q_j} . \] (30)

This shows that we also have pairing rotational bands in odd-mass nuclei. These connect states that differ by the number of pairs but have equal spin and parity. In contrast to pairing-rotational bands in even-even nuclei, these are not necessarily ground states. The Hamiltonian (27) must reduce to Eq. (7) when acting onto the fermion vacuum. Thus, \( n_0 \) is an integer. Except for the uninteresting constant \( \varepsilon_{q_j} \), the pairing rotational bands in odd and even nuclei have the same parabolic form. As in the case of even isotopes, the theory for odd nuclei also predicts that pair transfer and removal is equal in strength for states of a pairing rotational band. Subleading corrections are similar as for even nuclei, i.e. we have an expansion of the Hamiltonian in powers of \((n_{\text{tot}} - n_0)\).

### C. Adjustment of low-energy constants

The spectra (1) and (22) relate superfluid systems in the vicinity of integer \( n_0 \) pairs to each other. Before we can apply the effective theory of pairing rotations to nuclei, however, we need to include the dominant energy contributions to nuclear states. These consist of an overall constant and a term linear in the number of pairs.

Let us discuss even nuclei first. Adding the contributions \( E_{n_0} + S(\tilde{n} - n_0) \) to the Hamiltonian (7) yields the energy spectrum
\[ E_n = E_{n_0} - S_{n_0}(n - n_0) + \varepsilon_n . \] (31)

Here, \( E_{n_0} \) is the ground-state energy of the nucleus with \( n_0 \) pairs, and \( S_{n_0} \) denotes the pair removal energy, and \( \varepsilon_n \) is from Eq. (9). As \( E_{n_0} \approx -8.4 \text{ MeV} \) for a nucleus with mass number \( A \) and \( S_{n_0} \approx 16 \text{ MeV} \) for heavy nuclei, we see that the pairing rotation energies \( \varepsilon_n \) yield a small correction [except when \((n - n_0)^2 \gg 1\)] because the low-energy scale \( \xi \) is much smaller than \( S_{n_0} \).

The expansion (31) presents us with an ambiguity [10]. One could generally argue that the ground-state energy \( E_n \) can be expanded around \( n_0 \) in powers of \((n - n_0)\). Then, our leading-order theory for pairing would be just one contribution to the quadratic term, and other contributions are hard to pin down without a microscopic theory. However, the leading-order effective theory predicts that the pairing rotational constant \( a \) in Eq. (9) does not depend on which nucleus (identified by the number of pairs \( n_0 \)) the band is centered. Within the effective theory, any variation of \( a \) must be attributed to subleading corrections, see Eq. (17). Thus, when exploring pairing rotational bands, we can vary \( n_0 \) and find out if any observed variation of \( a \) is consistent with the size of subleading contributions.

This leaves us with the following approach. We will assume that pairing yields the dominant quadratic term in the energy expansion and adjust the low-energy constants \( E_{n_0}, S_{n_0}, a \) to the binding energies of the nuclei with \( n_0 \) and \( n_0 \pm 1 \) pairs. We use
\[ S_{n_0} = \frac{1}{2} (E_{n_0-1} - E_{n_0+1}) , \]
\[ a^{-1} = E_{n_0+1} - 2E_{n_0} + E_{n_0-1} . \] (32)

We see that \( S_{n_0} \) is the average of two two-nucleon separation energies, while the rotational constant is a three-point difference of even nuclei. Clearly, when adjusting \( a \) this way it becomes an \( n_0 \)-dependent quantity, and the variations of \( a \) with \( n_0 \) inform us about the size of subleading corrections.

Figure 1 shows the pairing rotational constant \((2a)^{-1}\), computed via Eq. (32), for even isotopes of tin (as a function of pairs above neutron number \( N = 50 \)), of \( N = 82 \) isotopes (as function of pairs above proton number \( N = 50 \)) and lead (as a function of pair holes below the neutron number \( N = 126 \)). We see that the pairing rotational constant is approximately \( n_0 \) independent for isotopes of lead while the \( N = 82 \) isotones and the isotopes of tin exhibit more variations. This suggests that higher-order corrections are significant in those nuclei. We also see that the variations are not smooth as the number of pairs (or pair holes in lead) changes. This suggests that the (smooth) subleading contributions discussed in Sect. II A 2 are only part of the corrections beyond quadratic order. The non-smooth fluctuations are outside the scope of the effective theory. They also prevent us from adjusting subleading low-energy constants locally, i.e. in a vicinity of a given \( n_0 \).

![Figure 1: Pairing rotational constants for even isotopes of tin (as a function of pairs above neutron number \( N = 50 \)), of \( N = 82 \) isotones (as function of pairs above proton number \( N = 50 \)) and lead (as a function of pair holes below the neutron number \( N = 126 \)).](image)
TABLE I. Average values of pairing rotational constants \((2a)^{-1}\) and the absolute average scale \(\langle |g| \rangle\) for subleading correction (both in MeV) for isotopes of tin and lead, and \(N = 82\) isotones. Also shown are the maximum number of pairs \(n_0\) in the relevant major shell, and – in the last column – the estimate \(g^n\) for the size of the low-energy constant \(g\) (in MeV).

| Element | \(\langle |g| \rangle\) | \(n_0\) | \(9/(\langle |g| \rangle^2)\) |
|---------|----------------|------|----------------|
| Sn      | 0.38           | 16   | 0.027          |
| Pb      | 0.26           | 22   | 0.0037         |
| \(N = 82\) | 0.97       | 16   | 0.068          |

Therefore, let us consider global adjustments of \(g\) in Eq. (14) and check the power counting. The average slopes of the lines in Fig. 1 are small compared to the rotational constants, and this suggests that the smooth subleading correction could be systematic. We can use Eq. (16) and identify the average slope as \(g\). Table I presents the average values of the rotational constant \((2a)^{-1}\) and \(g\) for the tin and lead isotopes and the \(N = 82\) isotones. Also shown is the maximum number of pairs for the major shell corresponding to the nuclei of interest, and the estimate \(3/\langle |g| \rangle^2\) from Eq. (15) for the size of the coupling \(g\). The theoretical estimates correctly identify the scale of \(\langle |g| \rangle\) (they are about twice of what was extracted from data), and this gives us confidence in the power counting proposed in Sect. II A 2.

Thus, the uncertainty estimate (16) is expected to capture the smooth corrections to the leading-order pairing rotational bands. In what follows we will assume that pairing does yield the dominant quadratic contribution to the expansion (31), limit the discussion to the leading-order theory, and use the uncertainty estimate (16).

For odd nuclei, we expand the pairing rotational contribution as \(\langle n_0 - n_0 \rangle^2 = \langle n_0 - n_0 - 1/2 \rangle^2 + \langle n_0 - n_0 \rangle - 1/4\). The constant and linear terms \(\langle n_0 - n_0 \rangle - 1/4\) can then be absorbed in an expansion of the energy (31). Thus, we will employ Eqs. (31) and (32) for even nuclei (by using integer \(n_0\)) and for odd nuclei (by using half integer \(n_0\)). Inspection shows that the pairing rotational constants for the odd nuclei are close to their even neighbors. This allows us to use the data in Table I also for uncertainty estimates in odd nuclei.

### III. EFFECTIVE THEORY FOR TWO SUPERFLUIDS

#### A. Even-even nuclei

In heavy open-shell nuclei, neutrons form isovector pairs and so do protons, and both superfluids interact. Thus, we do not consider proton-neutron pairing but will include interactions between proton and neutron pairs. The effective theory is based on the emergent symmetry breaking from \(U(1) \times U(1) \to 1\), and the coset is isomorph to \(U(1) \times U(1)\). The phases \(\alpha(t)\) and \(\beta(t)\) denote the Nambu-Goldstone modes corresponding to neutron and proton pairs, respectively. The most general Lagrangian up to quadratic terms in phase velocities is

\[
L = \frac{1}{2} (\dot{\alpha}, \dot{\beta}) \hat{M} \left( \begin{array}{c} \dot{\alpha} \\ \dot{\beta} \end{array} \right) + (n_0, z_0) \left( \begin{array}{c} \dot{\alpha} \\ \dot{\beta} \end{array} \right).
\]  

Here, \(n_0\) and \(z_0\) are low-energy constants and \(\hat{M}\) is a symmetric \(2 \times 2\) “mass” matrix with three parameters, and we employed a matrix-vector notation. The off-diagonal elements of \(\hat{M}\) introduce an interaction between the two superfluids. Introducing the canonical momenta \(p_\alpha \equiv \partial L / \partial \dot{\alpha}\) and \(p_\beta \equiv \partial L / \partial \dot{\beta}\), and performing a Legendre transform yields the Hamiltonian

\[
H = \frac{1}{2} (p_\alpha - n_0, p_\beta - z_0) \hat{M}^{-1} \left( \begin{array}{c} p_\alpha - n_0 \\ p_\beta - z_0 \end{array} \right).
\]

Quantization proceeds as in the previous Section, and single valuedness of the wavefunction under simple gauge transformations requires that \(n_0\) and \(z_0\) are integers. The resulting Hamiltonian is

\[
H = \frac{1}{2} (\dot{n} - n_0, \dot{z} - z_0) \hat{M}^{-1} \left( \begin{array}{c} \dot{n} - n_0 \\ \dot{z} - z_0 \end{array} \right),
\]

where \(\dot{n} \equiv -i\partial_\alpha\) and \(\dot{z} \equiv -i\partial_\beta\) count the conserved number of pairs in each superfluid. Energies are obtained by replacing these number operators by their eigenvalues, i.e.

\[
\varepsilon_{n,z} = \frac{1}{2a} \left( \frac{n-n_0}{2} \right)^2 + \frac{1}{2b} \left( \frac{z-z_0}{2} \right)^2 + \frac{1}{c} (n-n_0)(z-z_0).
\]

Here, we have chosen the constants \(1/a\), \(1/b\), and \(1/c\) as the diagonal and off-diagonal entries of \(\hat{M}^{-1}\), respectively. We see that all even-even nuclei in an entire region are connected via pairing, and the spectrum is an elliptical paraboloid; any section of this paraboloid is a Mambu-Goldstone modes corresponding to neutron and proton pairs, respectively. The most general Lagrangian up to quadratic terms in phase velocities is

\[
L = \frac{1}{2} (\dot{\alpha}, \dot{\beta}) \hat{M} \left( \begin{array}{c} \dot{\alpha} \\ \dot{\beta} \end{array} \right) + (n_0, z_0) \left( \begin{array}{c} \dot{\alpha} \\ \dot{\beta} \end{array} \right).
\]

The spectrum (36) recovers the results of Refs. 17, 19, 40, 42. The effective theory thus supports the recent proposal by Hinohara and Nazarewicz 10 to employ the pairing rotational tensor \(\hat{M}^{-1}\) as a model-independent
indicator for pairing. Its eigenvectors are expected to point into the directions of the valley of $\beta$ stability and perpendicular to it; the corresponding eigenvalues are expected to be small and large in magnitude, respectively.

The eigenstates of the Hamiltonian (35) are product states $|n, z\rangle$ that specify the number of pairs in each fluid, i.e.

$$\hat{n}|n, z\rangle = n |n, z\rangle,$$
$$\hat{\varepsilon}|n, z\rangle = z |n, z\rangle.$$  \hspace{1cm} (38)

Analogous to the case of one superfluid [see Eq. (10)] we can introduce pair removal (or pair addition) operators for each superfluid via

$$\hat{P} = P_0 e^{-i\alpha},$$
$$\hat{Q} = Q_0 e^{-i\beta}.$$  \hspace{1cm} (39)

We thus see that the leading-order theory of pairing predicts that four different reactions involve the same absolute squared nuclear matrix element, which is independent of $n$ and $z$. As pair transfer in single superfluid systems, these are testable predictions for two coupled superfluids.

Let us briefly discuss subleading corrections of the Hamiltonian (35). These are in powers of $(\hat{n} - n_0)^k (\hat{\varepsilon} - z_0)^l$ with $k + l = 3$. Alternatively, and with view on Eq. (37), one could also include powers $(T - T_0)^k (A - A_0)^l$. Following the steps in Sect. II A 2 that led to Eq. (11) we can also here estimate the uncertainties and find

$$\Delta \varepsilon_{n, z_0} = \frac{3|n - n_0|^3}{an^2_0},$$
$$\Delta \varepsilon_{n_0, z} = \frac{3|z - z_0|^3}{bn^2_0},$$
$$\Delta \varepsilon(T, A_0) = \frac{3}{4} \left( \frac{1}{2a} + \frac{1}{2b} - \frac{1}{c} \right) \frac{|T - T_0|^3}{\min(z_0^2, n_0^2)},$$
$$\Delta \varepsilon(T_0, A) = \frac{3}{4} \left( \frac{1}{2a} + \frac{1}{2b} + \frac{1}{c} \right) \frac{|A - A_0|^3}{\min(z_0^2, n_0^2)},$$

for pairing rotational bands in isotopes, isotones, isobars, and nuclei with the isospin projection, respectively, of the nucleus with $n_0$ neutron and $z_0$ proton pairs.

The effective field theory can also be extended to odd and to odd-odd nuclei, and one can easily write down the leading-order result. However, in practical applications, it is difficult to trace how states with non-zero spins evolve as neutron and proton numbers are changed, and this is particularly so for odd-odd nuclei. For this reason, such extensions of the theory are not pursued in this paper.

**B. Adjustment of low-energy constants**

As was the case for a single superfluid, we have to add the dominant contributions $E_{n_0, z_0} - S_{n_0} (\hat{n} - n_0) - S_{z_0} (\hat{\varepsilon} - z_0)$ to the Hamiltonian (35) and find the energy spectrum

$$E_{n, z} = E_{n_0, z_0} - S_{n_0} (n - n_0) - S_{z_0} (z - z_0) + \varepsilon_{n, z}.$$  \hspace{1cm} (40)

Here, $\varepsilon_{n, z}$ is from Eq. (36) and contains the low-energy constants $a$, $b$, while $c$, and $S_{n_0}$ and $S_{z_0}$ are (approximately) pair separation energies. We adjust the parameters $S_{n_0}$ and $a$ (and $S_{z_0}$ and $b$) similarly as in the case of a single superfluid [see Eq. (32)] and have

$$S_{n_0} = \frac{1}{2} (E_{n_0-1, z_0} - E_{n_0+1, z_0}),$$
$$a^{-1} = E_{n_0+1, z_0} - 2E_{n_0, z_0} + E_{n_0-1, z_0},$$
$$S_{z_0} = \frac{1}{2} (E_{n_0, z_0-1} - E_{n_0, z_0+1}),$$
$$b^{-1} = E_{n_0, z_0+1} - 2E_{n_0, z_0} + E_{n_0, z_0-1}.$$  \hspace{1cm} (45)

We need one more datum to determine $c$ and choose the symmetric expression

$$c^{-1} = \frac{1}{4} (E_{n_0-1, z_0+1} + E_{n_0+1, z_0-1} + E_{n_0-1, z_0-1} - E_{n_0-1, z_0+1}).$$  \hspace{1cm} (46)

**IV. APPLICATIONS**

**A. Single superfluid: semi-magic nuclei**

Figure 2 shows the pairing rotational band in tin isotopes centered on neutron number $N_0$ as indicated. Different bands are shifted by 25 MeV as $N_0$ is increased from 54 to 78. The number of pairs is $n = N/2$ and $n_0 = N_0/2$. Experimental data $E_n - E_{n_0} + S_{n_0}(n - n_0)$ is compared with the theory prediction $\varepsilon_n$, see Eq. (11). Here and in what follows, the $y$-axis is simply labelled as $\varepsilon$. Errorbars show the uncertainty estimates (16) using the average value of $a$ from Table I. We see that theory describes data accurately within errorbars. For each band, the three lowest-energy points with $-2 \leq N - N_0 \leq 2$ have been adjusted to data.

Figure 3 shows the pairing rotational bands in lead isotopes centered on neutron number $N_0$ as indicated. Different bands are shifted by 25 MeV as $N_0$ is increased...
FIG. 2. Pairing rotational bands in tin isotopes, centered on nuclei with $N_0$ neutrons as indicated: Experimental data $E_n - E_{n_0} + S_{n_0}(n - n_0)$ is compared with the theory prediction $\varepsilon_n = (n - n_0)^2/(2a)$ for nuclei with $n$ pairs around $n_0$. Errorbars are uncertainty estimates from omitted subleading terms. Bands are shifted by multiples of 25 MeV as $N_0 = 2n_0$ is increased from 54 to 78. In each band, the energies with $|N - N_0| \leq 2$ have been adjusted to data.

from 98 to 122. Errorbars again show the uncertainty estimates using the average value of $a$ from Table 1. Theory describes data accurately within the uncertainty estimates.

Figure 4 shows the pairing rotational bands in $N = 82$ isotones centered on nuclei with proton number $Z_0$ as indicated. Different bands are shifted by 25 MeV as $Z_0$ is increased from 52 to 68. The number of pairs is $n = Z/2$ and $n_0 = Z_0/2$. Uncertainty estimates are based on Eq. (16) and the value of $a$ from Table 1. Again, theory describes data accurately within uncertainties.

In summary, the leading-order Hamiltonian (7) yields an accurate description of pairing rotational bands within uncertainty estimates. This gives confidence in the power counting and the underlying separation of scales in even semi-magic nuclei.

B. Odd semi-magic nuclei

Let us also test the effective field theory prediction for odd nuclei. The ground-state spin typically evolves across an isotopic or isotonic chain, and we focus therefore on low-lying states with constant spin and parity. The excitation energy of such states must be added to the ground-state energies $E_n$ in Eq. (16) when computing the low-energy constants.

In the odd tin isotopes we focus on the $J^\pi = 7/2^+$ state.

FIG. 3. Pairing rotational bands in lead isotopes centered on nuclei with neutron number $N_0$ as indicated: Experimental data $E_n - E_{n_0} + S_{n_0}(n - n_0)$ is compared with the theory prediction $\frac{1}{2a} (n - n_0)^2$ for nuclei with $n$ pairs around $n_0$. Errorbars are uncertainty estimates for omitted subleading terms. Bands are shifted by multiples of 25 MeV as $N_0 = 2n_0$ is increased from 98 to 122. In each band, the energies with $|N - N_0| \leq 2$ have been adjusted to data.

FIG. 4. Pairing rotational bands in $N = 82$ isotones, centered on nuclei with proton number $Z_0$ as indicated. Experimental data $E_n - E_{n_0} + S_{n_0}(n - n_0)$ is compared with the theory prediction $\frac{1}{2a} (n - n_0)^2$ for nuclei with $n$ pairs around $n_0$. Uncertainties estimate the omitted contributions from subleading terms. Bands are shifted by 25 MeV as $Z_0 = 2n_0$ is increased from 52 to 68. In each band, the energies with $|Z - Z_0| \leq 2$ have been adjusted to data.
that is low in energy and compute the pairing rotational band for the nucleus with neutron number $N_0 = 65$. The results are shown in Fig. 5 and compared to a pairing rotational band in the neighboring even isotopes (centered at $N = 64$ and shifted by 10 MeV). The uncertainty estimates (16) with $a$ from Table I reflect the scale of deviations from data but do not capture them quantitatively for the larger values of $N - N_0$.

FIG. 5. Pairing rotational bands in odd (blue squares) and even (red circles) tin isotopes. The odd nuclei have spin/parity $J^\pi = 7/2^+$ with $^{115}$Sn ($N_0 = 65$) at the center, while the even nuclei are centered at $^{114}$Sn. Data is shown as black crosses. In each band, the central three points are adjusted to data.

The agreement between theory and experiment is better in $N = 82$ isotones. In the odd isotones we focus on the $J^\pi = 5/2^+$ and $7/2^+$ states that are low in energy and can easily be traced across the chain, taking $Z = 50$ (element Pr) as the central nucleus of the pairing rotational band. The results are shown in Fig. 6 and compared to the pairing rotational band in even isotones, centered at the Nd nucleus ($Z = 60$). The uncertainty estimate (16) uses the value of $a$ from Table I and captures the differences between theory and data.

Finally we turn to lead. Here, an isomeric $J^\pi = 13/2^+$ state is known in odd isotopes lighter than $^{208}$Pb, although its exact spacing with respect to the ground state is only known for $^{195}$Pb and heavier isotopes; we use tentative spin assignments for more neutron-deficient isotopes and take $^{197}$Pb as the center for the computation of the pairing rotational band. The results are shown in Fig. 7 and compared to an even isotope. The uncertainty estimate (16) with $a$ from Table I captures the discrepancies between data and theory.

Overall, the results of this Subsection show that the ef-
effective field theory also delivers accurate results for pairing rotational bands in odd semi-magic nuclei. We also see that the pairing rotational bands for even and odd semi-magic nuclei have the same pairing rotational constant to a very good approximation.

C. Two superfluids: open-shell nuclei

Let us take $^{166}\text{Yb}$ as the $(Z_0 = 70, N_0 = 96)$ nucleus in the center of the rare-earth region and adjust the low-energy constants from Eqs. (45) and (46) to its immediate even-even neighbors. This yields the pairing rotational constants $1/(2a) \approx 0.30$ MeV, $1/(2b) \approx 0.94$ MeV, $1/c \approx -0.95$ MeV. The proton and neutron pairing rotational constants are consistent with those presented in Table I for Pb isotopes and $N = 82$ isotones, respectively. The size of the off-diagonal coupling $1/c$ shows that the interaction of the two superfluids is strong. The curvature is small for pairing at constant isospin projection and large for isobars [see Eq. (17)].

Figure 8 shows the proton-pairing rotational bands (shifted by multiples of 12 MeV) for fixed neutron number $N$. Overall, theory and data agree reasonably well, and only for large values of $|Z - Z_0|$, and significant away from $N = 96$ do we see disagreement. The error estimates are based on $\Delta \varepsilon_{n,z}$ from Eqs. (43). They are too small to capture the deviations for large $N$ and small $Z$. The coupling between the two superfluids makes it interesting to also study other “directions” of pairing rotational bands [19], e.g., the isobar section and the section of constant isospin projection $T_z$ of the pairing elliptical paraboloid (36). The former section consists of nuclei that are connected via double charge exchange reactions, while the latter section describes nuclei that are linked by $\alpha$ particle capture or removal. The nucleus $^{166}\text{Yb}$ is kept at the center. Figure 10 shows the isobar section. Uncertainty estimates, taken as $\Delta \varepsilon(T_z, A_0)$ from Eqs. (43), capture the scale of the difference to data but are not quantitatively correct.

Figure 11 shows the section with constant isospin projection. Here, the uncertainties are taken as $\Delta \varepsilon(T_z, A)$ from Eqs. (43). They capture the scale of differences between theory and data well.

The comparison of the isobar and constant $T_z$ pairing rotational bands with the $N = 96$ proton pairing band of Fig. 8 and the $Z = 70$ neutron pairing band of Fig. 9 shows that the rotational constants differ considerably for each section of the elliptical paraboloid. Diagonalization of the mass matrix yields eigenvalues 0.09 and 2.4 MeV, and the corresponding eigenvectors have an angle of $28^\circ$ and $118^\circ$ with the neutron axis on the

![Figure 8](image_url)

**FIG. 8.** Proton-pairing rotational bands as sections of a pairing elliptical paraboloid. Bands for neutron numbers as indicated in the rare earth region around $^{166}\text{Yb} (Z_0 = 70, N_0 = 96)$: Experimental data is compared with the theory prediction. A total of six low-energy constant has been adjusted for all shown bands. Different bands are shifted by multiples of 12 MeV.

![Figure 9](image_url)

**FIG. 9.** Neutron-pairing rotational bands as sections of a pairing elliptical paraboloid for proton numbers as indicated in the rare earth region around $^{166}\text{Yb} (Z_0 = 70, N_0 = 96)$: Experimental data is compared with the theory prediction. A total of six low-energy constant has been adjusted for all shown bands. Different bands are shifted by multiples of 12 MeV.
D. Estimating energy gains from particle number projection

Let us also consider another application of Eq. (44). Calculations based on nuclear energy density functionals \[44, 45\] or Hamiltonians \[46, 47\] often do not employ particle number projections. Then, one really computes a symmetry-breaking state (with a fixed orientation in gauge space), that consists of a superposition of states with different numbers of pairs. Such a localized state clearly has too much kinetic energy in gauge space, and the formula (46) allows one to estimate this. Using \(\langle \hat{N} \rangle = N_0\) and \(\Delta N^2 \equiv \langle (\hat{N} - N_0)^2 \rangle\), and similar for \(\hat{Z}\), one finds

\[
\Delta E = \frac{1}{8a} \langle \Delta N^2 \rangle + \frac{1}{8b} \langle \Delta Z^2 \rangle + \frac{1}{4c} \langle \Delta N \Delta Z \rangle .
\] (47)

Here, the coefficients \(a\), \(b\), and \(c\) may be determined from computations or data via Eqs. (45) and (46).

As an example, let us consider the computation of semi-magic \(^{64}\)Ni within Bogoliubov many-body perturbation theory in Ref. [48]. The number variance is about \(\Delta N^2 \approx 16\), (see Fig. 9 of that work) and \((2a)^{-1} \approx 0.72\text{ MeV} \) (from data). This yields \(\Delta E \approx 2.9\text{ MeV}\).

V. SUMMARY

This paper revisited pairing rotations in a model-independent way within an effective field theory. It followed the standard approach to emergent symmetry breaking via a nonlinear realization of the broken phase symmetry. This led to pairing rotational bands in semi-magic nuclei and to a pairing elliptical paraboloid in systems where paired protons and neutrons interact. Coupling a fermion to the superfluid extends the theory to odd semi-magic nuclei. The expansion of the effective Hamiltonians is in powers of differences of Cooper-pair numbers, and subleading corrections are suppressed by inverse powers of the maximum number of pairs in a shell. The key input for the effective field theory consist of the matrix containing the pairing rotational constants. The eigenvalues of this model-independent quantity are given by the curvatures of the nuclear ground-state energies as a function of proton and neutron numbers. A comparison with data shows that the leading-order theory is accurate (within uncertainty estimates) for heavy semi-magic nuclei and for nuclei sufficiently far away from shell closures.

The theory predicts that pair transfer is constant for nuclei in a pairing rotational band. For nuclei on a pairing elliptical paraboloid, the nuclear matrix element for pair transfer, double charge exchange reactions, and \(\alpha\) particle knockout or capture are nucleus independent and related to each other.

Segrè chart, respectively. (This is essentially along the valley of \(\beta\) stability and perpendicular to it.) Consistent with this, the neutron pairing bands and the constant-\(T_z\) pairing band have the smallest curvature because they are oriented mainly along the valley of \(\beta\) stability.
It is interesting to compare the effective theory of this work with the those for deformed nuclei [27, 39, 49–51]. For axially symmetric deformations, one exploits the emergent symmetry breaking of rotational $SO(3)$ down to axial $SO(2)$. Then the coset spaces is the two-sphere and Nambu-Goldstone modes parameterize that manifold. Finite ground-state spins and fermions introduce couplings to gauge potentials (which usually are referred to as Coriolis forces). The treatment of pairing is technically somewhat simpler than deformation because the broken symmetry groups are Abelian. Otherwise, however, one follows the same path.

One could combine both approaches, simultaneously capturing deformation and superfluidity. Then, the low-energy physics of nuclei away from shell closures becomes extremely simple: The pattern of the emergent symmetry breaking – from a product of rotational $SO(3)$ times pairing $U(1) \times U(1)$ down to axial $SO(2)$ – is all that matters. The symmetries are realized nonlinearly, and low-lying excitations are the quantized excitations of the corresponding Nambu-Goldstone modes in finite systems. Each nucleus exhibits a ground-state rotational band and pairing rotations connect ground-state energies of different nuclei. While we have, of course, many nuclear models that break symmetries or incorporate the effects of symmetry breaking, the effective field theory approach makes it front and center, is aware about its breakdown scale, and allows one to make systematic improvements and uncertainty estimates.

ACKNOWLEDGMENTS

This work has been supported by the U.S. Department of Energy under grant No. DE-FG02-96ER40963 and under contract DE-AC05-00OR22725 with UT-Battelle, LLC (Oak Ridge National Laboratory).

[1] A. Bohr, B. R. Mottelson, and D. Pines, “Possible analogy between the excitation spectra of nuclei and those of the superconducting metallic state,” Phys. Rev. 110, 936–938 (1958).
[2] A. B. Migdal, “Superfluidity and the moments of inertia of nuclei,” Nuclear Physics 13, 655-674 (1959).
[3] D. R. Bès and R. A. Broglia, “Pairing vibrations,” Nuclear Physics 80, 289-313 (1966).
[4] Yukihisa Nogami, “Improved superconductivity approximation for the pairing interaction in nuclei,” Phys. Rev. 134, B313–B321 (1964).
[5] A. Bohr, “Pair correlations and double transfer reactions,” in Nuclear Structure: Dubna Symposium 1968 (Dubna, 4-11 July 1968) Proceedings Series (International Atomic Energy Agency, Vienna, 1969) p. 179.
[6] D. R. Bès, R. A. Broglia, R. P. J. Perazzo, and K. Kumar, “Collective treatment of the pairing hamiltonian: (i). formulation of the model,” Nuclear Physics A 143, 1–33 (1970).
[7] R. A. Broglia, O. Hansen, and C. Riedel, “Two-neutron transfer reactions and the pairing model,” in Advances in Nuclear Physics Vol. 6, edited by M. Baranger and E. Vogt (Springer, Boston, MA, 1973) Chap. 3, p. 287.
[8] David M. Brink and Ricardo A. Broglia, Nuclear Superfluidity: Pairing in Finite Systems Cambridge Monographs on Particle Physics, Nuclear Physics and Cosmology (Cambridge University Press, 2005).
[9] R. A. Broglia, J. Terasaki, and N. Giovanardi, “The anderson–goldstone–nambu mode in finite and in infinite systems,” Physics Reports 335, 1–18 (2000).
[10] Nobuo Hinohara and Witold Nazarewicz, “Pairing nambu-goldstone modes within nuclear density functional theory,” Phys. Rev. Lett. 116, 152502 (2016).
[11] G. Potel, A. Idini, F. Barranco, E. Vigezzi, and R. A. Broglia, “From bare to renormalized order parameter in gauge space: Structure and reactions,” Phys. Rev. C 96, 034606 (2017).
[12] W. von Oertzen and A. Vitturi, “Pairing correlations of nucleons and multi-nucleon transfer between heavy nuclei,” Rep. Prog. Phys. 64, 1247–1337 (2001).
[13] G. Potel, F. Barranco, F. Marini, A. Idini, E. Vigezzi, and R. A. Broglia, “Calculation of the transition from pairing vibrational to pairing rotational regimes between magic nuclei $^{100}$Sn and $^{132}$Sn via two-neutron transfer reactions,” Phys. Rev. Lett. 107, 092501 (2011).
[14] G. Potel, A. Idini, F. Barranco, E. Vigezzi, and R. A. Broglia, “Cooper pair transfer in nuclei,” Rep. Prog. Phys. 76, 106301 (2013).
[15] G. Potel, A. Idini, F. Barranco, E. Vigezzi, and R. A. Broglia, “Quantitative study of coherent pairing modes with two-neutron transfer: Sn isotopes,” Phys. Rev. C 87, 054321 (2013).
[16] Hirotaka Shimoyama and Masayuki Matsuo, “Anomalous pairing vibration in neutron-rich sn isotopes beyond the $n = 82$ magic number,” Phys. Rev. C 84, 044317 (2011).
[17] Rainer Beck, Manfred Kleber, and Hartwig Schmidt, “Pairing rotations and separation energies,” Zeitschrift fur Physik 250, 155–165 (1972).
[18] Masayuki Matsuo, “Treatment of Nucleon-Number Conservation in the Selfconsistent Collective-Coordinate Method: -Coupling between Large-Amplitude Collective Motion and Pairing Rotation—,” Progress of Theoretical Physics 76, 372–386 (1986).
[19] Nobuo Hinohara, “Collective inertia of the nambu-goldstone mode from linear response theory,” Phys. Rev. C 92, 034321 (2015).
[20] Nobuo Hinohara, “Extending pairing energy density functional using pairing rotational moments of inertia,” J. Phys. G: Nucl. Part. Phys. 45, 024004 (2018).
[21] Taiki Kouno, Chikako Ishizuka, Tsunenori Inakura, and Satoshi Chiba, “Pairing strength in the relativistic mean-field theory determined from the fission barrier heights of actinide nuclei and verified by pairing rotation and binding energies,” Progress of Theoretical and Experimental Physics 2022, 10.1093/ptep/ptab167 023D02.
[22] U. Van Kolck, “Effective field theory of nuclear forces,” Prog. Part. Nucl. Phys. 43, 337–418 (1999).
[23] H.-W. Hammer and R.J. Furnstahl, “Effective field the-
theory for dilute fermi systems,” Nuclear Physics A 678, 277 – 294 (2000).

[24] P. F. Bedaque and U. van Kolck, “Effective field theory for few-nucleon systems,” Annual Review of Nuclear and Particle Science 52, 339–396 (2002) [nucl-th/0203055].

[25] R. J. Furnstahl, H.-W. Hammer, and S. J. Puglia, “Effective field theory for dilute fermions with pairing,” Annals of Physics 322, 2703 – 2732 (2007).

[26] E. Epelbaum, H.-W. Hammer, and Ulf-G. Meißner, “Modern theory of nuclear forces,” Rev. Mod. Phys. 81, 1773–1825 (2009).

[27] T. Papenbrock, “Effective theory for deformed nuclei,” Nucl. Phys. A 852, 36 – 60 (2011).

[28] H. W. Grießhammer, J. A. McGovern, D. R. Phillips, and G. Feldman, “Using effective field theory to analyze low-energy compton scattering data from protons and light nuclei,” Prog. Part. Nucl. Phys. 67, 841 – 897 (2012).

[29] H.-W. Hammer, C. Ji, and D. R. Phillips, “Effective field theory description of halo nuclei,” Journal of Physics G: Nuclear and Particle Physics 44, 103002 (2017).

[30] H.-W. Hammer, Sebastian König, and U. van Kolck, “Nuclear effective field theory: Status and perspectives,” Rev. Mod. Phys. 92, 025004 (2020).

[31] M. R. Schindler and D. R. Phillips, “Bayesian methods for parameter estimation in effective field theories,” Ann. Phys. 324, 682 – 708 (2009).

[32] R. J. Furnstahl, D. R. Phillips, and S. Wesolowski, “A recipe for eft uncertainty quantification in nuclear physics,” Journal of Physics G: Nuclear and Particle Physics 42, 034028 (2015).

[33] J. Gasser and H. Leutwyler, “Spontaneously broken symmetries: Effective lagrangians at finite volume,” Nuclear Physics B 307, 763 – 778 (1988).

[34] C. Yannouleas and U. Landman, “Symmetry breaking and quantum correlations in finite systems: studies of quantum dots and ultracold Bose gases and related nuclear and chemical methods,” Rep. Prog. Phys. 70, 2067 (2007).

[35] Steven Weinberg, “Nonlinear realizations of chiral symmetry,” Phys. Rev. 166, 1568–1577 (1968).

[36] Curtis G. Callan, Sidney Coleman, J. Wess, and Bruno Zumino, “Structure of phenomenological lagrangians. ii,” Phys. Rev. 177, 2247–2250 (1969).

[37] S. Coleman, J. Wess, and Bruno Zumino, “Structure of phenomenological lagrangians. i,” Phys. Rev. 177, 2239–2247 (1969).

[38] T. Brauner, “Spontaneous symmetry breaking and Nambu-Goldstone bosons in quantum many-body systems,” Symmetry 2, 609–657 (2010) [arXiv:1001.5212].

[39] T. Papenbrock and H. A. Weidenmüller, “Effective field theory for finite systems with spontaneously broken symmetry,” Phys. Rev. C 89, 014334 (2014).

[40] H. J. Krappe, “On the use of a variable moment of pairing,” Zeitschrift für Physik A Hadrons and Nuclei 275, 297–304 (1975).

[41] Teruo Kishimoto and Tetsuo Kammuri, “Pair Rotation in the Dynamical Nuclear Field Theory,” Progress of Theoretical Physics 74, 1245–1263 (1985).

[42] E.R. Marshalek, “The rpa at high spin and conservation laws,” Nuclear Physics A 275, 416–444 (1977).

[43] X. B. Wang, J. Dobaczewski, M. Kortelainen, L. F. Yu, and M. V. Stoitsov, “Lipkin method of particle-number restoration to higher orders,” Phys. Rev. C 90, 014312 (2014).

[44] Michael Bender, Paul-Henri Heenen, and Paul-Gerhard Reinhard, “Self-consistent mean-field models for nuclear structure,” Rev. Mod. Phys. 75, 121–180 (2003).

[45] T. Nikšić, D. Vretenar, and P. Ring, “Relativistic nuclear energy density functionals: Mean-field and beyond,” Prog. Part. Nucl. Phys. 66, 519 – 548 (2011).

[46] W.H. Dickhoff and C. Barbieri, “Self-consistent green’s function method for nuclei and nuclear matter,” Prog. Part. Nucl. Phys. 52, 377 – 496 (2004).

[47] V. Somà, C. Barbieri, and T. Duguet, “Ab initio gorkov-green’s function calculations of open-shell nuclei,” Phys. Rev. C 87, 011303 (2013).

[48] Alexander Tichai, Robert Roth, and Thomas Duguet, “Many-body perturbation theories for finite nuclei,” Frontiers in Physics 8 (2020), 10.3389/fphy.2020.00164.

[49] Q. B. Chen, N. Kaiser, Ulf-G. Meiβner, and J. Meng, “Effective field theory for triaxially deformed nuclei,” The European Physical Journal A 53, 204 (2017).

[50] T. Papenbrock and H. A. Weidenmüller, “Effective field theory for deformed odd-mass nuclei,” Phys. Rev. C 102, 044324 (2020).

[51] I. K. Alnamlah, E. A. Coello Pérez, and D. R. Phillips, “Effective field theory approach to rotational bands in odd-mass nuclei,” Phys. Rev. C 104, 064311 (2021).