THE BOUNDARY TERM IN HUISKEN’S MONOTONICITY FORMULA AND THE ENTROPY OF TRANSLATORS

BRIAN WHITE

Abstract. For a manifold-with-boundary moving by mean curvature flow, the entropy at a later time is bounded by the entropy at an earlier time plus a boundary term. This paper controls the boundary term in a geometrically natural way. In particular, it shows (under mild hypotheses) that the entropy of a compact translator is less than or equal to the entropy of the boundary plus the maximal cone density of the boundary.

1. Introduction

For a closed surface moving by mean curvature flow in Euclidean space, Huisken’s monotonicity formula [Hui90, §3] implies that a certain weighted area decreases in time. That in turn implies that the entropy of the surface is a decreasing function of time.

For mean curvature flow of surfaces with boundary (where the motion of the boundary is prescribed), the entropy need not decrease, because the analog of Huisken’s monotonicity formula includes a spacetime boundary integral. In order to bound entropy at one time in terms of the entropy at an early time, it is necessary to control the boundary integral.

In this paper, we control the boundary integral in a geometrically natural way. (For the easier case of non-moving boundaries, see [Whi21, Theorem 7.1].) In particular, we show that it is bounded by the Gaussian area of the surface swept out by a certain time-dependent rescaling of the the boundary.

As an application, we prove a simple, explicit bound for the entropy of a compact translator with boundary, provided the boundary lies in a hyperplane or finite union of hyperplanes perpendicular to the direction of translation. The surface can be of any dimension and codimension. For example, as a special case, we have

Theorem 1. Consider an m-dimensional compact surface $M$ in $\mathbb{R}^{m+1}$ that translates with velocity $ve_{m+1}$ under mean curvature flow. Suppose that $\partial M$ consists of $k$ components, each of which is the boundary of a convex region in a horizontal $m$-plane. Then the entropy of $M$ is at most

$$k \left( 1 + \frac{m \omega_m}{\omega_{m-1}} \right).$$

Here $\omega_n$ denotes the volume of the unit ball in $\mathbb{R}^n$. 

Date: 4 April, 2022. Revised 23 July, 2023.
2020 Mathematics Subject Classification. Primary 53E10.
Key words and phrases. Mean curvature flow, monotonicity, entropy, boundary.
Corollary 2. For such a surface $M$,

$$\frac{\text{area}(M \cap B(x, r))}{\omega_m r^m} \leq C_m k$$

for all balls $B(x, r)$, where $C_m$ depends only on $m$.

The corollary follows from the theorem because density ratios (for any surface) are bounded by a constant times the entropy; see (2) below. The bound (1) plays a key role in constructing families of complete, non-rotationally invariant translating annuli in $\mathbb{R}^3$ [HMW22].

See Section # for the analogous results for boundaries $\partial M$ that may also contain some vertical components.

We remark that Ilmanen’s elliptic regularization [Ilm94] obtains very general mean curvature flows by taking suitable limits of sequences of translators. The bounds in this paper imply entropy bounds for such limits.

We now describe the general result for translators.

If $\Sigma$ is a $(m-1)$-dimensional submanifold of Euclidean space, we let $C(\Sigma) := \{rx : x \in \Sigma, r > 0\}$ be the cone over $\Sigma$ with vertex at the origin. The density of the cone is

$$\Theta(C(\Sigma)) := \frac{\text{area}(C(\Sigma) \cap B(0, r))}{\omega_m r^m},$$

where $\omega_m$ is the volume of the unit ball in $\mathbb{R}^n$. (Note that the right hand side does not depend on $r$.) Here, area should be counted with multiplicity.

We define the maximal cone density of $\Sigma$ to be

$$\text{mcd}(\Sigma) := \sup_{v \in \mathbb{R}^n} \Theta(C(\Sigma + v)).$$

Theorem 3. Suppose that $M$ is a compact $m$-dimensional manifold in $\mathbb{R}^n$ that translates with velocity $ve_n$ under mean curvature flow. Suppose that $\partial M$ lies in a horizontal hyperplane. Then

$$\text{entropy}(M) \leq \text{entropy}(\partial M) + \text{mcd}(\partial M).$$

More generally, if $\partial M = \bigcup_i \Sigma_i$, where each $\Sigma_i$ lies in a horizontal hyperplane $P_i$, then

$$\text{entropy}(M) \leq \sum_i (\text{entropy}(\Sigma_i) + \text{mcd}(\Sigma_i)).$$

Recall that if $S$ is a $d$-dimensional embedded submanifold of $\mathbb{R}^n$, then the maximal density ratio of $S$ is

$$\text{mdr}(S) := \sup_{x \in \mathbb{R}^n, r > 0} \frac{H^d(S \cap B(x, r))}{\omega_d r^d}.$$

(If $S$ is an immersed variety, then the $d$-dimensional area should be counted with multiplicity.) Entropy and maximal density ratio are closely related:

$$\text{entropy}(S) \leq \text{mdr}(S) \leq c_d \text{entropy}(S),$$

where $d = \dim(S)$. (See [Whi21 Theorem 9.1].) Thus as a consequence of Theorem 3 we have
Corollary 4. If $M$ is compact, $m$-dimensional translator in $\mathbb{R}^n$ with velocity $ve_n$ and if $\partial M$ lies in a horizontal hyperplane, then

$$\text{entropy}(M) \leq \text{mdr}(\partial M) + \text{mcd}(\partial M),$$

and

$$\text{mdr}(M) \leq c_m(\text{mdr}(\partial M) + \text{mcd}(\partial M)).$$

More generally, if $\partial M = \cup_i \Sigma_i$, where each $\Sigma_i$ lies in a horizontal hyperplane, then

$$\text{entropy}(M) \leq \sum_i (\text{mdr}(\Sigma_i) + \text{mcd}(\Sigma_i)),$$

and

$$\text{mdr}(M) \leq c_m \sum_i (\text{mdr}(\Sigma_i) + \text{mcd}(\Sigma_i)).$$

Theorem 1 follows from Corollary 4 because if $\Sigma$ is the boundary of a convex region in a $m$-plane, then

$$\text{mdr}(\Sigma) \leq \frac{m \omega_m}{\omega_{m-1}}$$

(see Proposition 15), and

$$\text{mcd}(\Sigma) = 1.$$
we let \( \nu_M(x,t) \) be the unit vector that it tangent to \( M(t) \), normal to \( \Gamma(t) \), and that points out from \( M(t) \). We let \( \check{M}(t) \) be the rescaled surface

\[
\check{M}(t) = \frac{M(t)}{|t|^{1/2}}.
\]

(The reader may wonder why we use the flow \( \check{M} \) rather than the standard renormalized flow \( \tau \mapsto \check{M}(-e^{-\tau}) \).

The latter flow has a nicer equation of motion, but in this paper, there is no advantage in changing the time variable.)

More generally, \( M(\cdot) \) can be a Brakke flow with boundary \( \Gamma(\cdot) \). (See [Whi21, Definition 8.1].) In this case, the vector \( \nu(x,t) \) is a vector of length \( \leq 1 \) that is perpendicular to \( \Gamma(t) \) at \( x \).

If we think of \( M(t) \) as a non-equilibrium soap film, then \( -\nu(x,t) \) is the force per unit \( (m-1) \)-dimensional measure that the soap film exerts on the boundary \( \Gamma(t) \) at \( x \).

Define

\[
\rho_m(x,t) = \frac{1}{(4\pi t)^{m/2}} \exp \left(-\frac{|x|^2}{4t}\right),
\]

\[
\Phi_m(x) = \frac{|x|^{m-2}}{(4\pi)^{m/2}} \exp \left(-\frac{1}{4}|x|^2\right).
\]

Thus

\[
\rho_m(x,t) = |t|^{-m/2} \Phi_m(x/|t|^{1/2}).
\]

We will sometimes write \( \rho \) and \( \Phi \) for \( \rho_m \) and \( \Phi_m \) when the \( m \) is clear from the context.

If \( S \) is an \( m \)-dimensional submanifold of \( \mathbb{R}^n \), we define its \( \Phi \)-area to be

\[
\Phi_m[S] := \int_S \Phi_m \, d\mathcal{H}^m.
\]

The entropy of \( S \) is the supremum of \( \Phi_m[S'] \) among all surfaces \( S' \) obtained from \( S \) by translation and dilation.

Note that

\[
\int_{x \in M(t)} \rho_m(x,t) \, d\mathcal{H}^m x = \int_{x \in \check{M}(t)} \Phi_m(x/|t|^{1/2}) |t|^{-m/2} \, d\mathcal{H}^m x
\]

\[
= \int_{y \in \check{M}(t)} \Phi_m(y) \, d\mathcal{H}^m y
\]

\[
= \Phi_m[\check{M}(t)].
\]

We now use Huisken’s Monotonicity Inequality, modified for surfaces with boundary: see [Whi21 Theorem 18.3]. (Note: the terms \( K \) and \( A \) in that theorem are 0 here because the ambient space is Euclidean.) The monotonicity theorem states that for \( a < b < 0 \),

\[
\Phi_m[\check{M}(b)] - \Phi_m[\check{M}(a)] \leq \int_{t=a}^{b} \int_{\Gamma(t)} \nu_M \cdot \left( \dot{\Gamma} - \frac{\nabla \rho}{\rho} \right) \rho \, d\mathcal{H}^{m-1} \, dt
\]

\[
= \int_{t=a}^{b} \int_{q \in \Gamma(t)} \nu_M \cdot \left( \dot{\Gamma} + \frac{q}{2t} \right) \rho \, d\mathcal{H}^{m-1} \, dt.
\]
(Here $\rho = \rho_m$.) We can express this last quantity, $Q$, in terms of $F$:

$$Q = \int_{t=a}^{b} \int_{\Sigma} \nu_M \cdot \left( \frac{\partial F}{\partial t} \right) \parallel + \frac{F}{2t} \right) \rho(F, t) \, d\mu \, dt$$

$$= \int_{t=a}^{b} \int_{\Sigma} \nu_M \cdot \left( \frac{\partial F}{\partial t} + \frac{F}{2t} \right) \rho(F, t) \, d\mu \, dt$$

$$= \int_{t=a}^{b} \int_{\Sigma} \nu_M \cdot (-t)^{1/2} \frac{\partial}{\partial t} \left( \frac{F}{(-t)^{1/2}} \right) |t|^{-m/2} \Phi \left( \frac{F}{|t|^{1/2}} \right) \, d\mu \, dt$$

$$= \int_{t=a}^{b} \int_{\Sigma} \nu_M \cdot \frac{\partial}{\partial t} \left( \frac{F}{|t|^{1/2}} \right) \Phi \left( \frac{F}{|t|^{1/2}} \right) |t|^{-(m-1)/2} \, d\mu \, dt$$

$$= \int_{t=a}^{b} \int_{\Sigma} \nu_M \cdot \frac{\partial \tilde{F}}{\partial t} \Phi(\tilde{F}) \, d\tilde{\mu} \, dt$$

Here we have used

$$\nu_M \cdot \left( \frac{\partial F}{\partial t} \right) \parallel = \nu_M \cdot \frac{\partial F}{\partial t},$$

which is true since $\nu_M$ is perpendicular to $\Gamma(t)$.

Recall that $\nu_M$ is a unit vector perpendicular to $\Gamma(t)$ and therefore also to $\tilde{F}(M, t)$. (In the case of Brakke Flow, $\nu_M$ is a vector of length at most one that is perpendicular to $\Gamma(t)$.) Therefore,

$$\nu_M \cdot \frac{\partial \tilde{F}}{\partial t} \leq \left| \left| \frac{\partial \tilde{F}}{\partial t} \right| \right|.$$

Thus

$$Q \leq A(\tilde{F}, a, b)$$

where

$$A(\tilde{F}, a, b) = \int_{t=a}^{b} \int_{\Sigma} \left| \left| \frac{\partial \tilde{F}}{\partial t} \right| \right| \Phi(x, t) \, d\tilde{\mu} \, dt$$

The expression for $A(\tilde{F}, a, b)$ has a simple geometric meaning: it is the $m$-dimensional $\Phi$-area swept out by the $\tilde{F}(\Sigma, t)$ from $t = a$ to $t = b$. In other words, $A(\tilde{F}, a, b)$ is the $\Phi$-area of the immersion

$$\tilde{F} : \Sigma \times [a, b] \to \mathbb{R}^n.$$

We have shown:

**Theorem 6.** Suppose $t \in (-\infty, 0) \to M(t)$ is an $m$-dimensional Brakke flow with boundary $\Gamma(t)$, where $\Gamma(t) = F(\Sigma, t)$. For each $T < 0$, let $N(T)$ be the surface swept out by

$$\hat{\Gamma}(t) := \frac{\Gamma(t)}{|t|^{1/2}}$$

from $t = T$ to 0:

$$N(T) = \bigcup_{T \leq t < 0} \hat{\Gamma}(t).$$

In terms of the parametrization $F$,

$$N(T) = \tilde{F}(\Sigma \times [T, 0]).$$

Then

$$\Phi_m[\tilde{M}(t)] + \Phi_m[N(t)]$$
is a decreasing function of $t$.

Proof. We showed above that

$$\Phi[\tilde{M}(b)] - \Phi[\tilde{M}(a)] \leq \Phi[F|\Sigma \times [a, b]]$$

$$= \Phi[F|\Sigma \times [a, 0]] - \Phi[F|\Sigma \times [b, 0]]$$

$$= \Phi[N(a)] - \Phi[N(b)].$$

\[ \square \]

**Corollary 7.** If the areas of the $M(t)$ are bounded above, or, more generally, if

$$\lim_{t \to -\infty} \Phi_m[\tilde{M}(t)] = 0,$$

then

$$\Phi_m[\tilde{M}(t)] \leq \Phi_m[\tilde{F}|\Sigma \times (-\infty, t)]$$

$$\leq \Phi_m[\tilde{F}|\Sigma \times (-\infty, 0)]$$

for all $t \in (-\infty, 0)$.

Because of the corollary, it is useful to have upper bounds for $\Phi_m[\tilde{F}|\Sigma \times (-\infty, 0)]$. The next section gives an upper bound in the case of boundaries that move by translation.

3. Translators

**Theorem 8.** Suppose that $\Sigma$ is a compact, embedded $(m-1)$-dimensional manifold in $\mathbb{R}^{n-1}$, and suppose that

$$F : \Sigma \times \mathbb{R} \to \mathbb{R}^n,$$

$$F(x, t) = (x, 0) + (a + t)v e_n.$$

Let $S = \tilde{F}(\Sigma \times (-\infty, 0))$. Then

$$\Phi_m[S] \leq \text{entropy}(\Sigma) + \Theta(\{rx : x \in \Sigma, r \geq 0\})$$

$$\leq \text{entropy}(\Sigma) + \text{mc}(\Sigma).$$

Proof. According to Theorem [4] below, for any $m$-dimensional submanifold $S$ of $\mathbb{R}^n$,

(6) $$\Phi_m[S] \leq \Phi_m[\Pi(S)] + \int_{y \in \mathbb{R}} \Phi_{m-1}[S^y]|\Phi_1(y) dy,$$

where $\Phi_m[\Pi(S)]$ is the $\Phi$-area (counting multiplicity) of the projection of $S$ to the horizontal $(n-1)$-plane, and where $S^y$ is the horizontal slice

$$S^y := \{x \in \mathbb{R}^{n-1} : (x, y) \in S\}.$$

In our case,

$$F(x, t) = (x, (a + t)v)$$

so

$$\tilde{F}(x, t) = ([t]^{-1/2}x, [t]^{-1/2}(a + t)v).$$

Write $r = |t|^{-1/2}$, so $t = -r^{-2}$. Thus

(7) $$S = \{(rx, (ra - r^{-1})v) : r > 0, x \in \Sigma\}.$$
From (7), we see that the projection \( \Pi(S) \) of \( S \) to the horizontal plane is precisely
\[
C(\Sigma) = \{ rx : r > 0, x \in \Sigma \},
\]
the cone over \( \Sigma \):
\[
(8) \quad \Pi(S) = C(\Sigma).
\]
For each \( y \in \mathbb{R} \), let \( \mathcal{R}(y) \) be the set of \( r > 0 \) such that
\[
(ra - r^{-1})v = y,
\]
and let \( n(y) \) be the number of elements of \( \mathcal{R}(y) \). From (7), we see that
\[
S^y = \cup_{r \in \mathcal{R}(y)} r \Sigma.
\]
Thus
\[
(9) \quad \Phi_m[S^y] = \sum_{r \in \mathcal{R}(y)} \Phi_m[r \Sigma] \leq \sum_{r \in \mathcal{R}(y)} \text{entropy } \Sigma = n(y) \text{ (entropy } \Sigma). \]

By (6), (8), and (9),
\[
\Phi_m[S] \leq \Phi_m[C(\Sigma)] + \text{(entropy } \Sigma) \int n(y) \Phi_1(y) \, dy.
\]
The asserted inequality follows, because simple calculations (see Lemmas 9 and 10 below) show that
\[
\int n(y) \Phi_1(y) \, dy \leq 1,
\]
and that, for any \( m \)-dimensional cone \( C \) with vertex 0,
\[
\Phi_m[C] = \Theta(C).
\]

**Lemma 9.** Let \( n(y) \) be the number of \( r > 0 \) such that \((ra - r^{-1})v = y\), i.e., such that \(avr^2 - yr + v = 0\). Then
\[
\int n(y) \Phi_1(y) \, dy \leq 1.
\]

**Proof.** The roots \( r \) are given by
\[
r = \frac{y \pm \sqrt{y^2 - 4av^2}}{2av}.
\]
If \( a < 0 \), then exactly one of the roots is positive, so \( n(y) = 1 \) for all \( y \) and therefore
\[
\int n(y) \Phi_1(y) \, dy = \int \Phi_1(y) \, dy = 1.
\]
Now suppose that \( a > 0 \). Let us also suppose that \( v > 0 \). (The case \( v < 0 \) is essentially the same.) If \( y > 2\sqrt{av} \), then there are two real roots, both positive. If
If \(|y| < 2\sqrt{av}\), there are no real roots. If \(y < -2\sqrt{av}\), there are two real roots, both negative. Thus
\[
\int n(y)\Phi_1(y)\,dy = 2\int_{-2\sqrt{av}}^{\infty} \Phi_1(y)\,dy \\
\leq 2\int_{0}^{\infty} \Phi_1(y)\,dy \\
= 1.
\]

\[\square\]

**Lemma 10.** If \(C\) is an \(m\)-dimensional cone with vertex at the origin, then \(\Phi_m[C] = \Theta(C)\).

**Proof.** The volume of the cone in \(B(0, r + dr) \setminus B(0, r)\) is \(d(\omega_m \Theta(C)r^m)\). Thus
\[
\Phi_m[C] = \int_{r=0}^{\infty} \frac{1}{(4\pi)^{m/2}} e^{-r^2/4} d(\omega_m \Theta(C)r^m) \\
= \Theta(C) \int_{r=0}^{\infty} \frac{1}{(4\pi)^{m/2}} e^{-r^2/4} d(\omega_m r^m) \\
= \Theta(C) \int_{\mathbb{R}^m} \Phi_m(x)\,dx \\
= \Theta(C).
\]

\[\square\]

**Theorem 11.** Suppose that \(M\) is an \(m\)-dimensional compact surface in \(\mathbb{R}^n\) that translates with velocity \(ve_n\) under mean curvature flow, and suppose that \(\partial M\) lies in a horizontal hyperplane. Then
\[
\text{entropy}(M) \leq \text{mcd}(\partial M) + (\text{entropy}(\partial M)).
\]

More generally, if \(\partial M = \bigcup_{i=1}^{k} \Gamma_i\), where each \(\Gamma_i\) lies in a horizontal plane \(P_i\), then
\[
\text{entropy}(M) \leq \sum_{i=1}^{k} (\text{mcd}(\Gamma_i) + (\text{entropy}(\Gamma_i))).
\]

**Proof.** Consider the MCF \(M(t) := M + (t + 1)ve_n\).

By Corollary \(\square\) and Theorem \(\square\)
\[
\Phi_m[M(t)] \leq \text{mcd}(\partial M) + (\text{entropy}(\partial M))
\]
for all \(t < 0\). Now \(\dot{M}(-1) = M(-1) = M\), so, in particular,
\[
\Phi_m[M] \leq \text{mcd}(\partial M) + (\text{entropy}(\partial M)).
\]

Now let \(M'\) be any surface obtained from \(M\) by translating and dilating. Then, by the same argument,
\[
\Phi_m[M'] \leq \text{mcd}(\partial M') + (\text{entropy}(\partial M')).
\]
But \(\text{mcd}(\partial M') = \text{mcd}(\partial M)\) and \(\text{entropy}(\partial M') = \text{entropy}(\partial M)\). Thus
\[
\Phi_m[M'] \leq \text{mcd}(\partial M) + (\text{entropy}(\partial M)).
\]
Taking the supremum over all such $M'$ gives
\[ \text{entropy}(M) \leq \text{mcd}(|\partial M|) + (\text{entropy}(\partial M)). \]
The assertion for the case $\partial M = \bigcup \Gamma_i$ is proved in the same way. \hfill \Box

4. Slicing

**Lemma 12.** Let $S$ be a smooth, $m$-dimensional manifold (possibly with boundary) in $\mathbb{R}^n$. For $x \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}$, let
\[
S_x := \{ y \in \mathbb{R} : (x, y) \in M \},
\]
\[
S^y = \{ x \in \mathbb{R}^{n-1} : (x, y) \in M \}.\]
Let
\[
\Pi : \mathbb{R}^n \to \mathbb{R}^{n-1},
\]
\[
\Pi(x, y) = x \quad (x \in \mathbb{R}^{n-1}, y \in \mathbb{R}).
\]
Suppose $f : \mathbb{R}^{n-1} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are smooth, nonnegative functions. Then
\[
\int_{(x,y) \in S} f(x)g(y) \, d\mathcal{H}^m(x,y) \leq \int_{x \in \Pi(S)} \left( \sum_{y \in S_x} g(y) \right) f(x) \, d\mathcal{H}^{m-1}x
\]
\[
+ \int_{y \in \mathbb{R}} \left( \int_{x \in S^y} f(x) \, d\mathcal{H}^{m-1}x \right) g(y) \, dy.
\]

**Proof.** Let
\[
h : (x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} \mapsto y
\]
be the height function. Then $\nabla_S h$ is the projection of $\nabla h = e^n$ to $\text{Tan}(S, \cdot)$. Note that the Jacobian of the map $\Pi : S$ is $J = \sqrt{1 - |\nabla_M h|^2}$, and thus that
\[
J + |\nabla_S h| \geq 1.
\]
By the area formula for $\Pi|S$,
\[
\int_{x \in \Pi(S)} \left( \sum_{y \in S_x} g(y) \right) f(x) \, d\mathcal{H}^{m-1}x = \int_{x \in \Pi(S)} \left( \sum_{y \in S_x} f(x)g(y) \right) \, d\mathcal{H}^{m-1}x
\]
\[
= \int_{(x,y) \in S} f(x)g(y)J(x,y) \, d\mathcal{H}^m(x,y).
\]
Likewise, by the coarea formula for $h|S$,
\[
\int_{y \in \mathbb{R}} \left( \int_{x \in S^y} f(x) \, d\mathcal{H}^{m-1}x \right) g(y) \, dy = \int_{y \in \mathbb{R}} \int_{x \in S^y} f(x)g(y) \, d\mathcal{H}^{m-1}x \, dy
\]
\[
= \int_{(x,y) \in S} f(x)g(y) |\nabla_S h| \, d\mathcal{H}^m(x,y).
\]
Now add (12) and (13) and use the inequality (11). \hfill \Box

**Remark 13.** Note that if $g \leq 1$, then $\sum_{y \in S_x} g(y)$ is less than or equal to the number of points in $S_x$, which is the multiplicity of the projection $\Pi|S$. Thus (in this case) the first integral on the right hand side in (10) is bounded above by the integral of $f$ over $\Pi(S)$, counting multiplicity.
**Theorem 14.** Let $S$ be an $m$-dimensional submanifold in $\mathbb{R}^n$. Then

$$\Phi_m[S] \leq \Phi_m[\Pi|S] + \int_{y \in \mathbb{R}} \Phi_{m-1}[S^y] \Phi_1(y) \, dy$$

where $\Phi_m[\Pi|S]$ is the $\Phi_m$-area of $\Pi(S)$, counting multiplicity.

**Proof.** Note that for $(x, y) \in \mathbb{R}^{m-1} \times \mathbb{R}$,

$$\Phi_m(x, y) = \frac{1}{(4\pi)^{m/2}} \exp \left(-\frac{|x|^2 - |y|^2}{4} \right)$$

$$= \frac{1}{(4\pi)^{(m-1)/2}} \exp \left(-\frac{|x|}{4} \right) \frac{1}{(4\pi)^{1/2}} \exp \left(-\frac{|y|^2}{4} \right)$$

$$= \Phi_{m-1}(x) \Phi_1(y).$$

The assertion of the theorem follows immediately from Lemma 12 (letting $f(x) = \Phi_{m-1}(x)$ and $g(y) = \Phi_1(y)$) and Remark 13. \(\square\)

5. The Maximal Density Ratio of a Convex Surface

**Proposition 15.** Let $U$ be a bounded, convex, open region in $\mathbb{R}^{m+1}$. Then

$$\text{mdr}(\partial U) \leq \frac{(m+1)\omega_{m+1}}{\omega_m}.$$  

**Proof.** For $x \in \mathbb{R}^m$, let $\pi(x)$ be the point in $\overline{U}$ closest to $x$. Then

$$|\pi(x) - \pi(y)| \leq |x - y|$$

for all $x, y \in \mathbb{R}^m$. Let $Q = \partial B(x, r) \setminus U$. Then $\pi$ is a distance-decreasing map from $Q$ onto $B(x, r) \cap \partial U$. Thus

$$\mathcal{H}^m(\partial B(x, r)) \geq \mathcal{H}^m(Q) \geq \mathcal{H}^m(B(x, r) \cap \partial U),$$

so

$$\frac{\mathcal{H}^m(B^m(x, r) \cap \partial U)}{\omega_m r^m} \leq \frac{\mathcal{H}^m(\partial B(x, r))}{\omega_m r^m} = \frac{(m+1)\omega_{m+1}}{\omega_m}.$$ \(\square\)

6. Boundaries with Vertical Pieces

Now suppose that $t \rightarrow M(t)$ is a mean curvature flow of $m$-dimensional manifolds in $\mathbb{R}^n$ and that the boundary of $M(t)$ is a fixed $(m-1)$-plane $\Gamma$. According to Theorem 6

$$\Phi_m[\dot{M}(t)] + \Phi_m[\bigcup_{t \leq \tau < 0} |\tau|^{-1/2} \Gamma]$$

is a decreasing function of $t$ for $t < 0$. In particular, for $T < t < 0,$

$$\Phi_m[\dot{M}(t)] \leq \Phi_m[\dot{M}(T)] + \Phi_m[\bigcup_{T \leq \tau < 0} |\tau|^{-1/2} \Gamma]$$

$$\leq \Phi_m[\dot{M}(T)] + \Phi_m[\bigcup_{-\infty \leq \tau < 0} |\tau|^{-1/2} \Gamma]$$

$$\leq \Phi_m[\dot{M}(T)] + \frac{1}{2}$$

since $Q := \bigcup_{-\infty < \tau < 0}$ is a halfspace if $0 \notin \Gamma$ (in which case $\Phi_m[Q] = 1/2$) or is the $(m-1)$-plane $\Gamma$ if $0 \in \Gamma$ (in which case $\Phi_m[Q] = 0$).
Likewise, if $\Gamma$ is the union of $\ell$ fixed $(m-1)$-planes, then
\[ \Phi_m[\tilde{M}(t)] \leq \Phi_m[\tilde{M}(T)] + \frac{\ell}{2}. \]

Combining this reasoning with the analysis in the previous sections (see, in particular, Theorem 11 and Proposition 15) gives

**Theorem 16.** Suppose that $M$ is a compact $m$-dimensional translator in $\mathbb{R}^n$ and that
\[ \partial M = (\bigcup_{i=1}^k \Gamma_i) \cup (\bigcup_{i=1}^\ell \Gamma'_i), \]
where each $\Gamma_i$ is contained in a horizontal plane $P_i = \mathbb{R}^{n-1} \times \{p_i\}$, and where each $\Gamma'_i$ is contained in $S_i \times \mathbb{R}$, where $S_i$ is an $(m-2)$-plane in $\mathbb{R}^{n-1}$. Then
\[
\text{entropy}(M) \leq \sum_{i=1}^k (\text{mcd}(\Gamma_i) + \text{entropy}(\Gamma_i)) + \frac{\ell}{2} \\
\leq \sum_{i=1}^k (\text{mcd}(\Gamma_i) + \text{mdr}(\Gamma_i)) + \frac{\ell}{2}.
\]
In particular, if $n = m + 1$ and if each $\Gamma_i$ is contained in the boundary of a convex region in $P_i$, then
\[
\text{entropy}(M) \leq k \left(1 + \frac{m\omega_m}{\omega_{m-1}}\right) + \frac{\ell}{2}.
\]

**References**

[EWW02] Tobias Ekholm, Brian White, and Daniel Wienholtz, *Embeddedness of minimal surfaces with total boundary curvature at most $4\pi$*, Ann. of Math. (2) **155** (2002), no. 1, 209–234, DOI 10.2307/3062155. MR1888799

[HMW22] David Hoffman, Francisco Martín, and Brian White, *Translating Annuli for Mean Curvature Flow* (2022). In preparation.

[Hui90] Gerhard Huisken, *Asymptotic behavior for singularities of the mean curvature flow*, J. Differential Geom. **31** (1990), no. 1, 285–299. MR1068760

[Ilm94] Tom Ilmanen, *Elliptic regularization and partial regularity for motion by mean curvature*, Mem. Amer. Math. Soc. **108** (1994), no. 520, x+90, DOI 10.1090/memo/0520. MR1196160

[Whi21] Brian White, *Mean Curvature Flow with Boundary*, ArxiVendi Analytica (2021). arXiv:1901.03008 [math.DG]. ↑1, 2, 4