Scale-invariant tensor spectrum from conformal gravity

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Abstract

We study cosmological tensor perturbations generated during de Sitter inflation in the conformal gravity with mass parameter $m^2 = 2H^2$. It turns out that tensor power spectrum is scale-invariant.

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1 Introduction

It is well-known that the Einstein-Weyl gravity has seven propagating gravitational modes, even the massive spin-2 state is a ghost and it violates unitarity [1]. The conformal gravity of $\sqrt{-g}C_{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}(=C^2)$ is invariant under the Weyl transformation of $g_{\mu\nu} \rightarrow \Omega^2(x)g_{\mu\nu}$ and thus, it has its own interests in gravity and cosmology [2]. The Weyl invariance removes one of seven gravitational modes and it propagates only six modes [3, 4, 5]. In the flat background, these are a healthy massless spin-2, a ghost massless spin-2 and a massless spin-1.

On the other hand, the Einstein-Weyl gravity provides surely massive scalar and vector propagations generated during de Sitter inflation in addition to massive tensor ghost and massless tensor [6, 7, 8, 9]. The authors have shown that in the limit of $m^2 \rightarrow 0$ (keeping the conformal gravity only), the vector and tensor power spectra became constant [10]. Recently, the Harrison-Zel’dovich scale-invariant spectrum was obtained from the Lee-Wick scalar model in de Sitter spacetime [11]. This model is a fourth-order scalar theory whose mass parameter is given by $M^2 = 2H^2$.

It is very important to look for a scale-invariant tensor spectrum which is the most robust and model-independent prediction of inflation. We note that the tensor amplitude is a direct measure of the expansion rate $H$ during inflation. This is in contrast to the scalar amplitude which depends on both $H$ and $\varepsilon$ in the slow-roll inflation. It is worth noting that the Einstein-gravity provides a scale-invariant spectrum in the superhorizon region of $k \ll aH(z \ll 1)$ only. This is known to be a scale-invariant superhorizon spectrum of tensor perturbation.

In this Letter, we will find a scale-invariant tensor power spectrum generated during de Sitter inflation from conformal gravity with mass parameter $m^2 = 2H^2$.

2 Einstein-Weyl gravity

We start with the Einstein-Weyl gravity whose action is given by

$$S_{\text{EW}} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left[ R - 2\kappa\Lambda - \frac{1}{2m^2}C_{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} \right],$$

(1)

where the Weyl-squared term takes the form

$$C_{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} = 2\left( R^{\mu\nu}R_{\mu\nu} - \frac{1}{3}R^2 \right) + (R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2).$$

(2)
In the limit of $m^2 \to 0$, one recovers the conformal gravity action \cite{12, 10}

$$S_{\text{CG}} = -\frac{1}{4km^2} \int d^4x \sqrt{-g} \left[C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma}\right].$$  \hspace{1cm} (3)

Here we have $\kappa = 8\pi G = 1/M_p^2$, $M_p$ being the reduced Planck mass and a mass parameter $m^2$ is introduced to make the action dimensionless. Greek indices run from 0 to 3 with conventions $(-+++)$, while Latin indices run from 1 to 3, throughout the paper. Further, we note that the Weyl-squared term ($\sqrt{-g}C^2$) is invariant under the Weyl transformation of $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu}$. The Einstein equation is given by

$$G_{\mu\nu} + \kappa \Lambda g_{\mu\nu} - \frac{1}{m^2} B_{\mu\nu} = 0,$$  \hspace{1cm} (4)

where the Einstein tensor $G_{\mu\nu}$ and the Bach tensor $B_{\mu\nu}$ take the forms

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu},$$  \hspace{1cm} (5)

$$B_{\mu\nu} = 2\nabla^\rho \nabla^\sigma C_{\mu\rho\nu\sigma} + G^{\rho\sigma} C_{\mu\rho\sigma\sigma}.$$  \hspace{1cm} (6)

The solution is given by the de Sitter spacetime whose curvature quantities are

$$\bar{R}_{\mu\nu\rho\sigma} = H^2(\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho}), \quad \bar{R}_{\mu\nu} = 3H^2\bar{g}_{\mu\nu}, \quad \bar{R} = 12H^2$$  \hspace{1cm} (7)

with $H^2 = \frac{\kappa \Lambda}{3}$. We describe de Sitter spacetime explicitly by choosing either a conformal time $\eta$ or a cosmic time $t$

$$ds^2_{\text{dS}} = \bar{g}_{\mu\nu}dx^\mu dx^\nu = a(\eta)^2 \left[ -d\eta^2 + \delta_{ij}dx^i dx^j \right]$$  \hspace{1cm} (8)

$$= -dt^2 + a^2(t)\delta_{ij}dx^i dx^j,$$  \hspace{1cm} (9)

where the conformal and cosmic scale factors are given by

$$a(\eta) = -\frac{1}{H\eta}, \quad a(t) = e^{Ht}.$$  \hspace{1cm} (10)

The choice of Newtonian gauge $B = E = 0$ and $\vec{E}_i = 0$ leads to $10 - 4 = 6$ degrees of freedom (DOF)[Considering a fourth-order equation for $h_{ij}$, it amounts 8 DOF]. In this case, the cosmologically perturbed metric can be simplified to be

$$ds^2 = a(\eta)^2 \left[ - (1 + 2\Psi)d\eta^2 + 2\Psi_i d\eta dx^i + \left( (1 + 2\Phi)\delta_{ij} + h_{ij} \right) dx^i dx^j \right]$$  \hspace{1cm} (11)

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with the transverse vector $\partial_i \Psi^i = 0$ and transverse-traceless tensor $\partial_i h^{ij} = h = 0$. In order to get the cosmological perturbed equations, one is first to obtain the bilinear action and then, varying it to yield the perturbed equations. We expand the Einstein-Weyl gravity action (1) up to quadratic order in the perturbations of $\Psi, \Phi, \Psi_i$, and $h_{ij}$ around the de Sitter background [7]. Since the scalar, vector, and tensor are decoupled from each other, we will consider the tensor perturbed equation only

$$\Box^2 h_{ij} - m^2 a^2 \Box h_{ij} + 2m^2 a^3 H h_{ij}' = 0,$$

(12)

where $\Box = -d^2/d\eta^2 + \partial_i \partial^i$ and the prime (‘) denotes the differentiation with respect to $\eta$. We introduce Fourier modes for $h_{ij}$

$$h_{ij}(\eta, x) = \frac{1}{(2\pi)^\frac{3}{2}} \int d^3k \sum_{s=+,-} p^s_{ij}(k) h^s_k(\eta) e^{ik \cdot x},$$

(13)

where $p^s_{ij}$ is a linear polarization tensor with $p^{+/-}_{ij, +/-} = 1$ and $h^e_{ij}$ is a linearly polarized tensor mode. Plugging (13) into (12) leads to the fourth-order differential equation

$$(h^s_k)^{'''} + 2k^2(h^s_k)^{''} + k^4 h^s_k + m^2 a^2 (h^s_k)^{''} + 2m^2 a^3 H (h^s_k) + m^2 a^2 k^2 h^s_k = 0,$$

(14)

which can be factorized to be

$$\left[ \frac{d^2}{d\eta^2} - \frac{2}{\eta} \frac{d}{d\eta} + k^2 \right] \left[ \eta^2 \frac{d^2}{d\eta^2} - 2\eta \frac{d}{d\eta} + 2 + k^2 \eta^2 + \frac{m^2}{H^2} \right] h^s_k = 0.$$

(15)

In the limit of $m^2 \to \infty$, one recovers the second-order (tensor) equation for the Einstein gravity. In the other limit of $m^2 \to 0$, one finds the fourth-order equation for the conformal gravity

$$\left[ \frac{d^2}{d\eta^2} - \frac{2}{\eta} \frac{d}{d\eta} + k^2 \right] \left[ \eta^2 \frac{d^2}{d\eta^2} - 2\eta \frac{d}{d\eta} + 2 + k^2 \eta^2 \right] h^s_k = 0,$$

(16)

which reduces further to a degenerate fourth-order equation

$$\left[ \frac{d^2}{d\eta^2} + k^2 \right]^2 h^s_k = 0 \quad \eta \to z = -k\eta \left[ \frac{d^2}{dz^2} + 1 \right]^2 h^s_k = 0$$

(17)

which is equivalent to the equation

$$\Box^2 h_{ij} = 0$$

(18)
for $h_{ij}$. Then, Eq. (16) is considered as the Fourier transform of the Weyl transformation for $\Box$:

$$\Box^2 \eta_{\mu\nu} \rightarrow \Delta_4 = \hat{\nabla}^2 (\hat{\nabla}^2 - 2H^2), \quad \hat{\nabla}^2 = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^\mu\nu \partial_\nu),$$

(19)

where $\Delta_4$ is the fourth-order Lee-Wick operator in the dS background. Also, we note the similar fourth-order equation (59) in Ref. [12] for the covariant approach to the conformal gravity with the transverse gauge. Hence, the fourth-order equation (16) can be interpreted to be the tensor version of the Lee-Wick scalar equation. Accordingly, Eq. (16) implies two second-order equations with $z = -k\eta = k/(aH)$

$$\left[ \frac{d^2}{dz^2} - 2 \frac{d}{z \frac{dz}{dz}} + 1 \right] h_{k1}^s = 0,$$

(20)

$$\left[ \frac{d^2}{dz^2} - 2 \frac{d}{z \frac{dz}{dz}} + 1 + 2 \frac{2}{z^2} \right] h_{k2}^s = 0$$

(21)

whose solutions to (20) and (21) are easily found to be

$$h_{k1}^s = c_1 (i + z) e^{iz},$$

(22)

$$h_{k2}^s = c_2 z e^{iz},$$

(23)

where $c_1$ and $c_2$ are constants to be determined. (22) is the tensor solution to the Einstein gravity [solution to a massless minimally coupled scalar], while (23) is the other solution to the conformal gravity [solution to a massless conformally coupled scalar] in de Sitter spacetime. In this sense, we insist that the conformal gravity includes the Einstein gravity as a part [13].

It is noted that (22) and (23) are also the solutions to the Lee-Wick scalar model [11]. In the Lee-Wick scalar model, its power spectrum is Harrison-Zel’dovich scale-invariant as

$$P_{\text{LW}} = \left( \frac{H}{2\pi} \right)^2,$$

(24)

whereas the spectrum for a massless minimally coupled scalar takes the form

$$P_{\phi} = \left( \frac{H}{2\pi} \right)^2 \left[ 1 + \frac{k^2}{(aH)^2} \right] = \left( \frac{H}{2\pi} \right)^2 \left[ 1 + z^2 \right],$$

(25)

which is obviously a scale-variant spectrum and a scale-invariant superhorizon spectrum.

The tensor power spectrum for the Einstein-Weyl gravity was given by [8]

$$P_{\text{EW}} = \frac{2H^2}{\pi^2 M_p^2 m^2 + 2H^2} \left[ 1 + z^2 - \frac{\pi}{2} z^3 e^{i\left( \frac{\pi}{2} + \frac{\pi}{4} \right) H_{\nu}^{(1)}(z)} \right]^{2}, \quad \nu = \sqrt{1 - \frac{m^2}{H^2}.}$$

(26)
However, for the case of \( m^2 = 0 (\nu = 1/2) \), one finds the zero power spectrum

\[
P_{\text{EW}}^{m^2=0} = \frac{2H^2}{\pi^2M_P^2} \frac{0}{0 + 2H^2} \cdot 1 = 0
\]  

(27)

which implies that the \( m^2 \to 0 \)-limit power spectrum of conformal gravity is not properly produced from that of Einstein-Weyl gravity. One may attempt to interpret it so that the power spectrum is not gravitationally produced because the Weyl-squared term \( (\mathcal{C}_2) \) is a conformally invariant quantity and thus, is decoupled from the expanding gravitational (de Sitter) background [14]. However, this is not true. Considering (24) and the tensor theory, one expects to have the scale-invariant tensor spectrum like as

\[
P_{\text{CG}} = 2 \times \left( \frac{2}{M_P} \right)^2 \times \mathcal{P}_{\text{LW}} = \frac{2H^2}{\pi^2M_P^2},
\]  

(28)

which will be shown explicitly in the next section.

3 Conformal gravity

In order to compute tensor power spectrum in the conformal gravity [10], we begin with the action (3). In this case, there is no restriction on \( m^2 \). Then, the fourth-order differential equation is given by

\[
\Box^2 h_{ij} = 0 \to \left( h_k^{ss'} \right)^{''''} + 2k^2 \left( h_k^{ss} \right)^{''} + k^4 h_k^{ss} = 0,
\]  

(29)

which is further rewritten to be (17). This is the same equation for a degenerate Pais-Uhlenbeck (PU) oscillator [15] and its solution is given by

\[
h_k^{ss}(z) = \frac{N}{2k^2} \left[ i(a_2^s + a_1^s z) e^{iz} + c.c. \right]
\]  

(30)

with \( N \) the normalization constant. After quantization, \( a_2^s \) and \( a_1^s \) are promoted to operators \( \hat{a}_2^s(k) \) and \( \hat{a}_1^s(k) \), which leads to the expression \( \hat{h}_k^{ss}(z) \). The presence of \( z \) in \((\cdots)\) reflects clearly that \( \hat{h}_k^{ss}(z) \) is a solution to the degenerate equation (17). Together with \( N = \sqrt{2\kappa m^2} \), the canonical quantization could be accomplished by introducing commutation relations between \( \hat{a}_1^s(k) \) and \( \hat{a}_j^{s'}(k') \) as

\[
[\hat{a}_1^s(k), \hat{a}_j^{s'}(k')] = 2k \delta^{s s'} \begin{pmatrix} 0 & -i \\ i & 1 \end{pmatrix} \delta^3(k - k').
\]  

(31)
The tensor power spectrum is defined by

$$\langle 0 | \hat{h}_{ij}(\eta, \mathbf{x}) \hat{h}^{ij}(\eta, \mathbf{x}^\prime) | 0 \rangle = \int d^3 k \frac{P_{\text{h}}^m}{4\pi k^3} e^{i k \cdot (\mathbf{x} - \mathbf{x}^\prime)}. \quad (32)$$

Here we choose the Bunch-Davies vacuum $| 0 \rangle$ by imposing $\hat{a}_s^s(k) | 0 \rangle = 0$. Now denoting $P_{\text{h}}^m e^{i k \cdot (\mathbf{x} - \mathbf{x}^\prime)} \equiv \sum_{s, s'} = +, - \ P_{ss'}^m \ h_{ij}(k) \ P_{ij}(k') | 0 \rangle e^{i k \cdot (\mathbf{x} - \mathbf{x}^\prime)}$ and substituting (13) together with $\hat{h}^s(k)$ into (32), then one finds that $P_{ss'}^m$ takes the form

$$P_{ss'}^m = \frac{m^2}{2\pi^2 M_P^2} \int \frac{d^3 k'}{k'^3} \left( \frac{1}{k^2} P_{ij}^s(k) P_{ij'}^{s'}(k') \times \langle 0 | \left( [\hat{a}_2^s(k), \hat{a}_2^{s'}(k')] + z[\hat{a}_1^s(k), \hat{a}_1^{s'}(k')] \right) \right) e^{i k \cdot (\mathbf{x} - \mathbf{x}^\prime)} \langle 0 | e^{i k' \cdot (\mathbf{x} - \mathbf{x}^\prime)} \right). \quad (33)$$

In obtaining (34), we used the commutation relations of (31) which reflect the quantum nature of a degenerate PU oscillator. This is in contrast to a non-degenerate PU oscillator for the Lee-Wick scalar theory [11].

Finally, we obtain the tensor power spectrum

$$P_{\text{h}}^m = \frac{m^2}{\pi^2 M_P^2}, \quad (35)$$

which leads to the scale-invariant tensor spectrum when choosing $m^2 = 2H^2$

$$P_{\text{h}}^{m^2=2H^2} = \frac{2H^2}{\pi^2 M_P^2}. \quad (36)$$

This is surely compared to the scale-variant tensor spectrum for the Einstein gravity [17]

$$P_h = \frac{2H^2}{\pi^2 M_P^2} \left( 1 + \frac{k^2}{(aH)^2} \right). \quad (37)$$

4 Discussions

We have found the scale-invariant tensor power spectrum generated during de Sitter inflation from conformal gravity with mass parameter $m^2 = 2H^2$. This scale-invariant tensor spectrum could be understood because the conformal gravity is invariant under the Weyl transformation. This contrasts to the scale-variant tensor spectrum (37) of Einstein gravity which is not Weyl-invariant. This is considered as a tensor version of Harrison-Zel’dovich
scale-invariant spectrum obtained from the Lee-Wick scalar model with mass parameter $M^2 = 2H^2$. We summarize below the difference between second-order theory and fourth-order theory and similarity between scalar and tensor in the same order. Their DOF are shown explicitly.

| power spectrum       | scalar theory (DOF)                  | tensor theory (DOF)            |
|----------------------|--------------------------------------|-------------------------------|
| scale-variant        | massless minimally coupled scalar theory (1) | Einstein gravity (2)         |
| (second-order)       |                                       |                               |
| scale-invariant      | Lee-Wick model with $M^2 = 2H^2$ (2)  | Conformal gravity with $m^2 = 2H^2$ (4) |
| (fourth-order)       |                                       |                               |

Consequently, the scale-invariant scalar and tensor spectra could be obtained from fourth-order Lee-Wick scalar theory and conformal gravity.

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References

[1] K. S. Stelle, Phys. Rev. D 16, 953 (1977).

[2] R. J. Riegert, Phys. Rev. Lett. 53, 315 (1984).

[3] S. Deser, E. Joung and A. Waldron, J. Phys. A 46, 214019 (2013) arXiv:1208.1307 [hep-th].

[4] S. Deser, E. Joung and A. Waldron, Phys. Rev. D 86, 104004 (2012) arXiv:1301.4181 [hep-th].

[5] S. F. Hassan, A. Schmidt-May and M. von Strauss, arXiv:1303.6940 [hep-th].

[6] T. Clunan and M. Sasaki, Class. Quant. Grav. 27, 165014 (2010) arXiv:0907.3868 [hep-th].

[7] N. Deruelle, M. Sasaki, Y. Sendouda and A. Youssef, JCAP 1103, 040 (2011) arXiv:1012.5202 [gr-qc].

[8] N. Deruelle, M. Sasaki, Y. Sendouda and A. Youssef, JHEP 1209, 009 (2012) arXiv:1202.3131 [gr-qc].

[9] Y. S. Myung and T. Moon, JCAP 1408, 061 (2014) arXiv:1406.4367 [gr-qc].

[10] Y. S. Myung and T. Moon, arXiv:1407.0441 [gr-qc].

[11] Y. S. Myung and T. Moon, arXiv:1412.7263 [gr-qc].

[12] P. D. Mannheim, Phys. Rev. D 85, 124008 (2012) arXiv:1109.4119 [gr-qc].

[13] J. Maldacena, arXiv:1105.5632 [hep-th].

[14] K. Dimopoulos, Phys. Rev. D 74, 083502 (2006) hep-ph/0607229.

[15] P. D. Mannheim and A. Davidson, Phys. Rev. A 71, 042110 (2005) hep-th/0408104.

[16] Y. W. Kim, Y. S. Myung and Y. J. Park, Phys. Rev. D 88, 085032 (2013) arXiv:1307.6932.

[17] D. Baumann, arXiv:0907.5424 [hep-th].

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