Simple polytopes without small separators, II: Thurston’s bound

Lauri Loiskekoski
Institut für Mathematik, FU Berlin
Arnimallee 2
14195 Berlin, Germany
lauri.loiskekoski@fu-berlin.de

Günter M. Ziegler
Institut für Mathematik, FU Berlin
Arnimallee 2
14195 Berlin, Germany
ziegler@math.fu-berlin.de

August 22, 2017

Abstract

We show that there are simple 4-dimensional polytopes with \( n \) vertices such that all separators of the graph have size at least \( \Omega(n/\log n) \). This establishes a strong form of a claim by Thurston, for which the construction and proof had been lost.

We construct the polytopes by cutting off the vertices and then the edges of a particular type of neighborly cubical polytopes. The graphs of simple polytopes thus obtained are 4-regular; they contain 3-regular “cube-connected cycle graphs” as minors of spanning subgraphs.

1 Introduction

For an arbitrary fixed constant \( c \) with \( 0 < c < \frac{1}{2} \) (where traditionally \( c = \frac{1}{3} \) is used) and a simple graph with vertex set \( V \), a separator with separation constant \( c \) is a partition of \( V \) into sets \( A, B, C \) with \( cn \leq |A| \leq |B| \leq (1-c)n \) such that \( C \) separates \( A \) from \( B \), that is, there is no edge between a vertex in \( A \) and a vertex in \( B \). The size of the separator is defined as the size of the set \( C \).

The Lipton–Tarjan planar separator theorem from 1979 [5] states that each planar graph on \( n \) vertices—and in particular the graph of any 3-dimensional polytope—has a separator of size \( |C| = O(\sqrt{n}) \). No such general result can exist for \( d \)-dimensional polytopes, as their graphs may be complete. However, the situation for simple \( d \)-polytopes, whose graphs are \( d \)-regular, is interesting: Kalai had conjectured in [4, Conj. 20.2.12] that for any fixed \( d \geq 2 \), the graphs of all simple \( d \)-polytopes on \( n \) vertices have separators of size

\[
O\left(n^{1-\frac{1}{d-1}}\right).
\]

\*The first author was funded by DFG through the Berlin Mathematical School. Research by the second author was supported by the DFG Collaborative Research Center TRR 109 “Discretization in Geometry and Dynamics.”
In our previous paper [6] we have shown that cutting off the vertices, and then the edges, from any neighborly cubical 4-polytope (a cubical 4-dimensional polytope with the graph of an \( m \)-cube) yields a simple 4-dimensional polytope with \( n = \Theta(m2^m) \) vertices such that any vertex separator has size at least \( \Omega(n/\log^{3/2} n) \).

This disproved Kalai’s conjecture, but it fell short of establishing a claim by Thurston, documented by Kalai in [4, p. 460], who refers to [7]. Indeed this does not appear there, but Gary Miller told us (personal communication) that in the context of the work towards [7], Thurston had claimed that there are simplicial 3-spheres on \( n \) tetrahedra for which all separators of the dual graph contain at least \( \Omega(n/\log n) \) vertices. Miller reports that for this “Thurston gave an embedding of the cube-connected cycle graph in \( \mathbb{R}^3 \) as linear tets [tetrahedra]”. Thurston’s construction for this seems to be lost.

In this paper, we prove a stronger version of this: There is not only an embedding of tetrahedra in \( \mathbb{R}^3 \), or a triangulated simplicial 3-sphere with \( n \) facets for which every separator of the dual graph has size at least \( \Omega(n/\log n) \), but indeed there is a simplicial convex 4-polytope with this property. Our main theorem describes the dual polytope, which is simple:

**Theorem 1.** There is a family of simple convex 4-dimensional polytopes \( \text{NC}_c^4(m)^n \) with \( n = \Theta(m2^m) \) vertices for which the smallest separators of the graph have size \( \Theta(2^m) = \Theta(n/\log n) \).

We will obtain such polytopes by cutting off first all the vertices, and then the edges, from a specific sequence of neighborly cubical 4-polytopes. In Section 2 we describe the construction of the polytopes \( \text{NC}_c^4(m), \text{NC}_c^4(m)^\prime, \) and finally \( \text{NC}_c^4(m)^\prime\prime \), and derive the combinatorial properties, and in particular collect all information that will be needed on their graphs, which we denote by \( \text{G}_m, \text{G}_m^\prime, \) and \( \text{G}_m^\prime\prime \), respectively. The graphs \( \text{G}_m^\prime\prime \) contain as spanning subgraphs the cube-connected cycle graphs of Preparata & Vuillemin [8] that Thurston presumably referred to.

Then in Section 3 we review Sinclair’s canonical paths method, which allows one to show that a graph has large edge expansion, which in turn implies that all separators are large.

Finally, in Section 4 we specify canonical paths in \( \text{G}_m^\prime\prime \) and show that not too many paths are routed through any edge of any of the four different types of edges in \( \text{G}^\prime\prime \), which we call “long,” “medium,” “extra,” and “short”. This completes the proof.

## 2 Neighborly cubical 4-polytopes and their vertex and edge truncations

### 2.1 Neighborly cubical polytopes

A *neighborly cubical \( d \)-polytope* is a \( d \)-dimensional polytope \( \text{NC}_d(m) \) that is cubical (that is, its facets are combinatorially equivalent to the \( (d-1) \)-cube \( [0,1]^{d-1} \)) and whose graph is isomorphic to the graph of an \( m \)-cube, for some \( m \geq d \geq 4 \). (Except for the last section, we will in this paper mainly refer to the case \( d = 4 \).)
The boundary complex of any such polytope is combinatorially equivalent to a subcomplex of the $m$-cube $[0,1]^m$, so its non-empty faces can be identified with vectors in $\{-1,0,1\}^m$, or equivalently in $\{-,+,\ast\}^m$.

The definition of neighborly cubical 4-polytopes immediately yields $f_0$ and $f_1$. Using the Euler equation and double-counting, which yields $3f_3 = f_2$, we conclude that every neighborly cubical 4-polytope has the $f$-vector

$$f(\text{NC}_4(m)) = (f_0, f_1, f_2, f_3) = 2^{m-2}(4, 2m, 3m - 6, m - 2).$$

### 2.2 The cyclic neighborly cubical polytopes

Joswig & Ziegler [2] established the existence of neighborly cubical $d$-polytopes on $2^m$ vertices, for all $m \geq d \geq 4$, by constructing one specific example of a neighborly cubical $d$-polytope $\text{NC}_d'(m)$, which we here call a cyclic neighborly cubical polytope $\text{NC}_d(m)$. This name is chosen to reflect that – according to the analysis of Sanyal & Ziegler [10] – all vertex figures of $\text{NC}_d'(m)$ are combinatorially isomorphic to a pyramid over a triangulation of the cyclic polytope $C_{d-2}(m-1)$.

The polytopes $\text{NC}_d'(m)$ were originally obtained by a somewhat subtle linear algebra/matrix theory construction. However, the “Cubical Gale Evenness Criterion” [2, Thm. 18] provides a complete description of the combinatorics of the polytope $\text{NC}_4'(m)$.

The graph $G_m$ of $\text{NC}_d'(m)$ is isomorphic to the graph of the $m$-cube. Thus we label each vertex with an entry in $\{-,+,\ast\}^m$. The edges are labeled by vectors in $\{-,+,\ast\}^m$ with exactly one $\ast$-entry. The coordinate $i \in [m] = \{1, \ldots, m\}$ with the $\ast$-entry will be called the direction of the edge. There are $2^{m-1}$ edges in each direction.

We now specialize to the case $d = 4$. The Cubical Gale Evenness Criterion is quite complicated even in this case, but we will not need the full description.

**Theorem 2** (Part of the Cubical Gale Evenness Criterion, for $d = 4$ [2, Thm. 18]). The facets of the cyclic neighborly polytope $\text{NC}_4'(m)$ are given by vectors in $\{-,+,\ast\}^m$ with exactly three $\ast$s.

If the first component of a vector in $\{-,+,\ast\}^m$ is $\ast$, then it corresponds to a facet of $\text{NC}_4'(m)$ if and only if the rest of the vector satisfies the Gale Evenness criterion, that is, if between any two non-$\ast$-entries there is an even number of $\ast$-entries. Equivalently, this happens if in the rest of that vector the two $\ast$-entries are cyclically adjacent.

This theorem shows that at any vertex $v \in \{-,+,\ast\}^m$, the two incident edges in directions $i$ and $i + 1 \pmod{m}$ span a 2-face, as $v_2v_3 \ldots v_{i-1}\ast v_{i+2} \ldots v_n \in \{-,+,\ast\}^m$ corresponds to a facet of $\text{NC}_4'(m)$ for $2 \leq i < m$. As the 2-faces correspond to edges of the vertex figure, this implies that the vertices of the vertex figure are cyclically connected in a consistent way independent of which vertex figure we are looking at: There is a Hamilton cycle $123 \ldots m$ in the graph of each vertex figure.

Thus the boundary complex of $\text{NC}_4'(m)$ contains a copy of the polyhedral surface described by Ringel [9] in 1955; this was pointed out explicitly in Ziegler [12, Sect. 3]. See also Joswig & R"orig [1].
2.3 Truncating the vertices

The polytope \( NC_4^c(m) \) is obtained by truncating all the vertices of the cyclic neighborly cubical polytope \( NC_4^c(m) \).

The resulting polytope has

- \((m-2)2^{m-2}\) facets that arise from the “old” facets (of \( NC_4^c(m) \)); which are vertex-truncated cubes (and hence simple 3-polytopes), and
- \(2^m\) “new” facets, which are simplicial 3-polytopes with \( m \) vertices each.

Moreover, the vertices of each “new” facet are naturally labeled by 1, 2, \ldots, \( m \), where the label is given by the direction of the edge of \( G_m \) that the new vertex lies on.

Thus the graph \( G'_m \) of \( NC_4^c(m) \) – illustrated in the middle of Figure 1 – has \( m2^m \) vertices, which may be labeled by \((v, i) \in \{0, 1\}^m \times [m]\), where \( v \in \{0, 1\}^m \) denotes the vertex of \( NC_4^c(m) \) which has been truncated, and \( i \in [m] \) is the direction of the edge of \( NC_4^c(m) \) which has been cut.

Indeed, there are three types of edges in \( G'_m \): First, there are the \( m2^m-1 \) “long” edges between vertices \((v, i)\) and \((v', i)\), where \( v \) and \( v' \) differ only in coordinate \( i \). Then there are the \( m2^m \) “medium” edges between vertices \((v, i)\) and \((v, i+1 \pmod m)\) on the Hamilton cycle on the graph of the vertex figure at \( v \), which is given by \((1, 2, \ldots, m)\). The “long” and “medium” edges together form the cube-connected cycle graphs of Preparata & Vuillemin [8]. Finally there are \((2m-6)2^m\) additional “extra” edges between vertices \((v, i)\) and \((v, j)\) with \( j - i \neq \pm 1 \pmod m \), which we will not use in the following, but which together with the “medium” edges yield the graphs of the \( 2^m \) (simplicial) vertex figures.

In describing \( G'_m \), we refer to the \( 2^m \) subgraphs formed by the “medium” and “extra” edges as clusters. These cluster subgraphs are maximal planar graphs on \( m \) vertices and \( 3m - 6 \) edges. The clusters are connected by “long” edges in a cube-like fashion.

2.4 Then cutting off the edges

The polytope \( NC_4^c(m)'' \) is then obtained from \( NC_4^c(m)' \) by cutting off what remains of the \( m2^{m-1} \) original edges of \( NC_4^c(m) \). The new facets formed by this have the combinatorial type of prisms over \( k \)-gons, with \( 3 \leq k \leq m - 1 \). Here \( k \) is the number of facets that contained the edge that was cut off, or equivalently the degree of its end vertices in the cluster triangulations (that is, in the vertex figures from the previous step). The graph of each prism consists of two cycles of short edges, which both are \( k \)-cycles, and of set of \( k \) long edges that we refer to as a parallel class of long edges.

The graph \( G''_m \) of the resulting polytope \( NC_4^c(m)'' \) – illustrated in the bottom part of Figure 1 – has \( 2^m \) planar subgraphs that we again refer to as clusters, which are connected in an \( m \)-cube like pattern by parallel classes of at least 3 and at most \( m - 1 \) “long” edges. Each cluster in \( G''_m \) is a 3-connected 3-regular planar graph on \( 6m - 12 \) vertices, which consists of \( m \) cycles formed by “short” edges and \( m \) “medium” and \( 2m - 6 \) “extra” edges.

The cycles of short edges correspond bijectively to the vertices of \( G'_m \); thus they get labels of the form \((v, i)\). “Medium” edges connect the subsequent cycles \((v, i)\) and \((v, j)\) with \( j = i + 1 \pmod m \), while the “extra” edges connect non-subsequent cycles.
Figure 1: The graphs $G_m$, $G'_m$, and $G''_m$, for $m = 5$: Local situation at one cluster.
2.5 Minor relations

There are minor relations

\[ G''_m \rightarrow G'_m \rightarrow G_m. \]

Here \( G'_m \) is obtained from \( G''_m \) by contraction of the “short” edges in \( G''_m \) and identification of “long” edges that become parallel after the contractions. Similarly, \( G_m \) is obtained from \( G'_m \) by contraction of the “medium” and “extra” edges in \( G'_m \).

2.6 Greater generality

The construction and analysis in this section can be extended to the large number of combinatorial types of neighborly cubical 4-polytopes described and analyzed by Sanyal & Ziegler [10]: We can find a cycle going through the directions in a cluster for any labeling of the vertices of the polygon; [10, Thm. 3.7] tells how the triangulation is done. In particular, the combinatorial type of the vertex figure for a vertex \( v \in NC_4(m) \) is created by pushing the vertices of the polygon in the specified order until we hit a plus sign in the vertex label, which corresponds to pulling and completes the triangulation. Changing the order of the vertices means our cycle would not be \((1, 2, \ldots, m)\), but some other (fixed!) permutation of \([m]\).

3 Edge expansion and vertex separators for simple graphs

3.1 Definitions

Let \( G = (V, E) \) be a simple graph. For any set \( S \subset V \), the edge boundary \( \delta(S) \) is the set of edges with one end in \( S \) and the other in \( V \setminus S \). The edge expansion \( \chi(G) \) of \( G \) is then defined by

\[
\chi(G) := \min \left\{ \frac{|\delta(S)|}{|S|} : S \subset V, S \neq \emptyset, S \leq \frac{|V|}{2} \right\}.
\]

This quantity is crucial for the mixing properties of Markov chains on \( G \). We consider it because of its close connection to separators: For regular graphs with large expansion, separators have to be large as well (see below).

For some examples (including ours), the expansion can very efficiently be estimated by the method of “canonical paths” pioneered by Sinclair [11]. A nice introduction and overview was given by Kaibel [3]. (A number of powerful generalizations of Sinclair’s method exist, for example using random paths or, equivalently, flows. We will not use these.)

3.2 Sinclair’s canonical paths method

To estimate the edge expansion, Sinclair tells us to specify, for each pair of vertices \((s, t) \in V \times V\), a path from \( s \) to \( t \). Let \( \phi : E(G) \rightarrow \mathbb{N} \) count for each edge the number of
paths that use it. We can now bound the edge expansion in terms of
\[ \phi_{\text{max}} := \max \{ \phi(e) : e \in E(G) \}. \]

**Lemma 3** (Sinclair’s lemma). Let \( \phi_{\text{max}} \) be defined as in the previous paragraph. Then the edge expansion of the graph \( G \) satisfies
\[ X(G) \geq \frac{n}{2 \phi_{\text{max}}}. \]

**Proof.** Let \( \phi(\delta(S)) \) be the sum of \( \phi(e) \) over all edges in \( \delta(S) \), which is at least the number of paths that use edges in \( \delta(S) \). Thus we get \( \phi(\delta(S)) \geq |S| |V \setminus S| \). On the other hand we clearly have \( \phi(\delta(S)) \leq \phi_{\text{max}} |\delta(S)| \). This means that for any \( |S| \leq \frac{|V|}{2} \),
\[ X(G) \geq \frac{|\delta(S)|}{|S|} \geq \frac{\phi(\delta(S))}{\phi_{\text{max}} |S|} \geq \frac{|S| |V \setminus S|}{\phi_{\text{max}} |S|} \geq \frac{n}{2 \phi_{\text{max}}}. \]

### 3.3 Relating edge expansion and vertex separators

**Lemma 4.** In a \( d \)-regular graphs \( G \) on \( n \) vertices with edge expansion \( X(G) \), all separators have size at least
\[ \frac{c}{d} X(G)n = \Omega(X(G)n), \]
where \( d \) is the constant from the definition of a separator.

**Proof.** Let \( G \) be a \( d \)-regular graph with expansion \( X(G) \) and let \((A, B, C)\) be a separator of \( G \), with \(|B| \geq |A| \geq cn\). By definition there are at least \( X(G)|A| \) edges in the boundary \( \delta(A) \). The other ends of these edges lie in \( C \), as there are no edges between \( A \) and \( B \). Since \( G \) is \( d \)-regular, \(|C| \) has size at least \( X(G)|A|/d \geq X(G)cn/d = (c/d)X(G)n \).

### 4 Canonical paths

#### 4.1 Defining the paths

In order to lower-bound the expansion of \( G''_m \), we now define paths between every pair of vertices of \( NC_3(m) \).

Let \( v_0 \) be a vertex in the cycle of short edges labeled \((v, i)\) and \( w_0 \) be a vertex in the cycle labeled \((w, j)\). Now route the path from \( v_0 \) to \( w_0 \) as follows. For each of the coordinates, taken in cyclic order \( i, i + 1, \ldots, m, 1, \ldots, i - 1 \), perform the following procedure:

**Procedure P:** If the path as constructed up to now ends at a vertex \( u_0 \in (u, k) \), then
- If \( u \) and \( w \) differ in coordinate \( k \), then take the long edge incident to \( u_0 \), which leads to the cycle \((u', k)\), where \( u \) and \( u' \) differ only in the coordinate \( k \). Then use only short edges and then a medium edge to get to the cycle labeled \((u', k + 1)\).
• If \( u \) and \( w \) do not differ in coordinate \( k \), then use only short edges and then one medium edge to get to the cycle labeled \((u, k + 1)\).

This procedure is performed at most \( m \) times until the path constructed reaches a vertex in the cluster labeled \( w \), and then at most \( m - 1 \) times within this cluster until the path constructed reaches a vertex in the cycle labeled \((w, j)\), and then in one final iteration we take less than \( m \) steps within the cycle to reach the vertex \( w_0 \).

We will now proceed to show that the maximum number of paths through any edge of any of the four types is \( O(m^22^m) \).

4.2 Long edges
There are altogether \( n^2 \) paths, and for each coordinate direction \( i \), half of the paths take a long edge in direction \( i \).

By symmetry, as the two possible values of the other coordinates are not distinguished by our construction of the paths, all \( 2^{m-1} \) parallel classes of long edges are used the same number of times. Thus for each of the parallel classes of long edges, exactly \((n^2/2)/2^{m-1}\) paths use (exactly) one edge from this class. Thus at most \( n^2/2^m \) paths use any individual long edge. With \( n = O(m2^m) \), we get \( n^2/2^m = O(m^22^m) \).

4.3 Medium and short edges
Each path is constructed, by the algorithm described above, in not more than \( 2m \) runs of the Procedure P, where in each iteration step we may take a long edge, then we may take a few steps in a cycle of short edges, and then take one single step on a medium edge if needed.

Extra edges are not used at all by our canonical paths.

Thus, if all \( n^2 \) paths are taken together, the Procedure P is performed not more than \( 2mn^2 \) iterations times.

Moreover, due to the cyclic symmetry of the construction, which treats all directions equally, and due to the symmetry between the two values in each coordinate direction, all \( m2^m \) different (and disjoint!) edge sets of “a cycle of short edges plus the medium edge leaving it” is used by \emph{the same} number of paths. (This reflects the fact that the symmetry group of the subgraph \( G'_m \) formed by the medium and the long edges, which is a cyclically connected cube graph, is transitive on the medium edges.) Thus no short edge on a cycle, or medium edge leaving it, is used by more than \((2mn^2)/(m2^m) = 2n^2/2^m \) paths. With \( n = O(m2^m) \), we get \( 2n^2/2^m = O(m^22^m) \).

4.4 Wrapping things up
For the graphs \( G''_m \) of the polytopes NC\(_2\)(m)”, we have obtained in Section 2.4 that they are 4-regular on \( n = (6m - 12)2^m \) vertices. In Section 4 we have constructed canonical paths such that no edge is used more than \( 2n^2/2^m \) times. With Sinclair’s Lemma from
Section 3.2, this implies the expansion bound

\[ \mathcal{X}(G''_m) \geq \frac{n}{2} \phi_{\text{max}} \geq \frac{n}{2n^2/2^m} = \frac{2^m}{2n} = \frac{2^m}{2(6m - 12)2^m} = \frac{1}{12(m - 2)}. \]

With the lemma from Section 3.3 this yields that all separators of $G''_m$ have size at least

\[ \frac{c}{4} \mathcal{X}(G''_m)n \geq \frac{cn}{48(m - 2)} = \Omega \left( \frac{n}{\log n} \right), \]

as $n = (6m - 12)2^m$ and thus $m = \Theta(\log n)$.

On the other hand, it is easy to see that separators of this order of magnitude exist: For this, for example, look at all the $\frac{n}{m}$ long edges of one particular direction (each vertex is incident to one long edge, and there is the same number of long edges in all directions), and take as vertex separator one end of each of these. This yields a separator of size $\frac{m}{4} = O(\frac{n}{\log n})$.

\[ \square \]

5 Final comments

Our construction can be extended to higher dimensions by taking prisms over $N_{c^4}(m)^''$. For a fixed dimension $d$ the resulting polytope has $2^{d-4}$ times as many vertices as the original. There are $d - 4$ new directions and every vertex has an edge in each of these directions. We can amend our algorithm for example by taking edges in these directions first and then proceeding in the usual way.

It is still unknown if there can be graphs of simple polytopes with even larger smallest separators. In particular, it would be interesting to know if these graphs can be true expanders, that is, have smallest separators with $\Omega(n)$ vertices.

Acknowledgement

Thanks to Johanna Steinmeyer for the TikZ pictures.

References

[1] Michael Joswig and Thilo Rörgig, Neighborly cubical polytopes and spheres, Israel J. Math. 159 (2007), 221—242.

[2] Michael Joswig and Günter M. Ziegler, Neighborly cubical polytopes, Discrete Comput. Geometry (Grünbaum Festschrift: G. Kalai, V. Klee, eds.) 24 (2000), 325–344.

[3] Volker Kaibel, On the expansion of graphs of $0/1$-polytopes, in: “The Sharpest Cut: The Impact of Manfred Padberg and his Work”, MOS-SIAM Series on Optimization, SIAM, Philadelphia PA, 2004, pp. 199–216.

[4] Gil Kalai, Polytope skeletons and paths, Handbook of Discrete and Computational Geometry (J. E. Goodman and J. O’Rourke, eds.), CRC Press, Boca Raton, second ed., 2004, First edition 1997, pp. 455–476.
[5] Richard J. Lipton and Robert E. Tarjan, *A separator theorem for planar graphs*, SIAM J. Applied Math. **36** (1979), 177–189.

[6] Lauri Loiskekoski and Günter M. Ziegler, *Simple polytopes without small separators*, Preprint, October 2015, 7 pages, arXiv:1510.00511; Israel J. Math., to appear.

[7] Gary L. Miller, Shang-Hua Teng, William Thurston, and Stephen A. Vavasis, *Separators for sphere-packings and nearest neighbor graphs*, J. ACM **44** (1997), 1–29.

[8] Franco P. Preparata and Jean Vuillemin, *The cube-connected cycles: a versatile network for parallel computation*, Communications of the ACM **24** (1981), no. 5, 300–309.

[9] Gerhard Ringel, *Über drei kombinatorische Probleme am n-dimensionalen Würfel und Würfelgitter*, Abh. Math. Sem. Univ. Hamburg **20** (1955), 10–19.

[10] Raman Sanyal and Günter M. Ziegler, *Construction and analysis of projected deformed products*, Discrete Comput. Geometry **43** (2010), 412–435.

[11] Alistair Sinclair, *Algorithms for Random Generation and Counting: A Markov Chain Approach*, Progress in Theoretical Computer Science, Birkhäuser, Boston, 1993.

[12] Günter M. Ziegler, *Polyhedral surfaces of high genus*, in: “Discrete Differential Geometry”, Oberwolfach Seminars, vol. 38, Birkhäuser, Basel 2008, pp. 191–213.