Bits, Matrices and $1/N$

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**Abstract**

We propose a simple string bit formalism for interacting strings in a plane wave background, in terms of supersymmetric quantum mechanics with a symmetric product target-space. We construct the light-cone supersymmetry generators and Hamiltonian at finite string coupling. We find a precise match between string amplitudes and the non-planar corrections to the correlation functions of BMN operators computed from gauge theory, and conjecture that this correspondence extends to all orders in perturbation theory. We also give a simple RG explanation for why the effective string coupling is $g_s = J^2/N$ instead of $g_s = g_{ym}^2$. 
Introduction

Recently a new framework to describe type IIB string dynamics from $\mathcal{N} = 4$ supersymmetric Yang-Mills theory has been proposed in [1]. It was argued that for states with large R-charge $J$, the gauge theory in effect amounts to a (discrete) light cone quantization of strings moving in the background of a pp-wave geometry [2]

$$ds^2 = -4dx^+dx^- - \mu^2(\vec{r}^2 + \vec{y}^2)(dx^+)^2 + d\vec{y}^2 + d\vec{r}^2, \quad F_{+1234} = F_{+5678} = \frac{\mu}{4\pi^3 g_s \alpha'^2}. \quad (1)$$

The specific dictionary proposed in [1] involves a one-to-one identification of a class of single trace operators in the gauge theory with single string states in this background. At the non-interacting string level, at leading order in the large $N$ expansion, supporting evidence was found via the computation of the anomalous scale dimensions of the gauge theory operators, confirming the identification

$$P_ - = \Delta - J. \quad (2)$$

Subsequent work has extended this correspondence to low orders in string perturbation theory.

In this note, we will propose a simple effective description of the string dynamics in the pp-wave background, in terms of a string bit language [4] inspired by the formalism of [1] as well as by matrix string theory [5][6]. This reduced description is based on a simple supersymmetric quantum mechanical model with a symmetric product target space. We will present evidence that this effective description exactly reproduces the complete perturbation expansion of the $\mathcal{N} = 4$ SYM theory in the BMN limit. In particular, we will find a precise match between the amplitudes of the string bit model and the three-point proposed in [10] and one-loop amplitudes computed in [9] and [10]. The philosophy of our approach, however, is to compare the gauge and string theory at a more microscopic level, by matching both systems directly via their light-cone Hamiltonian evolution.

The light-cone gauge worldsheet Lagrangian of a string propagating in the background (1) takes the quadratic form

$$\mathcal{L} = \frac{1}{2}(\partial_+ x_i \partial_- x_i - m^2 x_i^2) + i(\theta_a \partial_+ \theta_a + \bar{\theta}_a \partial_- \bar{\theta}_a - 2m\bar{\theta}_a \Pi_{ab} \theta_b), \quad (3)$$

with $m = \mu p_+$ and $\Pi = \gamma_{1234}$. The quantization and symmetries of this Lagrangian have been studied in detail in [3]. In accordance with the symmetries of the pp-wave background, it exhibits a maximally extended supersymmetry (for all details, see [3]).

In the following, our aim is to construct an interacting version of this string theory, with a discretized world-sheet and with the same space-time symmetries. The only generators that are
expected to receive corrections at finite string coupling are the light-cone generators $Q^-$ and $P^-$. Hence, the part of the supersymmetry algebra that will be most important to us, is

$$\{Q^a, Q^b\} + \{\tilde{Q}^a, \tilde{Q}^b\} = \delta^{ab} H + m J^{ab}, \quad [H, Q_a] = 0$$ (4)

where $H \equiv P^-$ and $J^{ab}$ is a suitable contraction of gamma matrices with the $SO(4) \times SO(4)$ Lorentz generators $J^{ij}$, see [3].

A convenient formalism for describing the full multi-string Hilbert space, is to use a single orbifold field theory on the symmetric product target space

$$\text{Sym}^J \mathcal{M} = \mathcal{M}^J / S_J$$ (5)

with $S_J$ the permutation group and $J$ the total DLCQ momentum. In the case of IIA string theory in flat space, this formalism naturally arises as the IR limit of matrix string theory [5] [6]. Moreover, it allows for a remarkably compact description of the string splitting and joining, via the interaction vertex [6]

$$g_s V_{int} = \frac{g_s}{2} \sum_{n<m} \int d\sigma \left( \tau^i \tau^j \otimes \Sigma^i \Sigma^j \right)_{nm},$$ (6)

where each term on the right-hand side represents a suitable twist-field that implements a simple permutation $(nm)$, interchanging the $n$-th and $m$-th copy of the target space. This description of matrix string interactions will be useful to keep in mind in what follows.

**Bits**

Motivated by the BMN formalism, we will now set up the string bit language, by introducing $J$ copies of supersymmetric phase space coordinates $\{p^i_n, x^i_n, \theta^a_n, \bar{\theta}^a_n\}$, with $n = 1, \ldots, J$, satisfying canonical commutation relations

$$[p^i_n, x^j_m] = \delta^{ij} \delta_{mn}, \quad \{\theta^a_n, \theta^b_m\} = \delta^{ab} \delta_{mn}, \quad \{\bar{\theta}^a_n, \bar{\theta}^b_m\} = \delta^{ab} \delta_{mn}.$$ (7)

Following the example of matrix string theory, we consider these $J$ copies as obtained via the quantization of the $J$-th symmetric product $\text{Sym}_J \mathcal{M}$ of the plane wave target space $\mathcal{M}$. In other words, we divide out the symmetric group $S_J$, acting via permutation of on the labels $n$, thus defining a quantum mechanical orbifold model.

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1Here, relative to the notation in [3], $Q = Q^{-1}$ and $\tilde{Q} = Q^{-2}$.

2For some recent work on matrix (string) theory in pp-wave backgrounds, see [13][14][12].
In the context of 2-d field theory, it is well known that the Hilbert space of such an orbifold field theory decomposes as a direct sum over twisted sectors. Although the occurrence of such twisted sectors is often thought of as a purely “stringy” phenomenon, they do in fact also arise in point-particle quantum mechanics. For a clear discussion of this in the context of cohomology of symmetric orbifold spaces, see [7], pp 56-57. Following the prescription outlined in [7], we construct our Hilbert space as a direct sum of “twisted sectors”

$$\mathcal{H} = \bigoplus_{\gamma} \mathcal{H}_{\gamma}$$

labeled by conjugacy classes $\gamma$ of the symmetric group $S_J$. Each twisted sector $\mathcal{H}_{\gamma}$ can be thought of as made up from states localized at the fixed point set of $\gamma$. This fixed point set is mapped onto itself by the stabilizer subgroup $N_\gamma$ of permutations $\sigma$ that commute with $\gamma$. Correspondingly, in $\mathcal{H}_{\gamma}$, we can act with arbitrary operators $O(p, x, \theta)$ that are left invariant under the action of $N_\gamma$:

$$\{p^i_n, x^i_n, \theta^a_n\} \rightarrow \{p^i_{\sigma(n)}, x^i_{\sigma(n)}, \theta^a_{\sigma(n)}\}, \quad \sigma \in N_\gamma.$$  

By copying (with only slight modification) the discussion in [6] (on pp 4-5), it is readily verified that the resulting Hilbert space takes the form of a sum over multi-string Hilbert spaces, each string with a discretized worldsheet consisting of $J_\ell$ bits with $\sum_\ell J_\ell = J$. As in matrix string theory, we interpret $J_\ell$ as the discrete light-cone momentum of the string. The above invariance under the stabilizer subgroup imposes the constraint

$$(U_\ell)^{J_\ell} = 1 \quad U_\ell = e^{2\pi i (L_0^{(\ell)} - \tilde{L}_0^{(\ell)}) / J_\ell}$$

on each string, with $U_\ell$ the operator that translates the string bit $x_n$ by one unit. Here the adjacency of two sites is specified by the twisted boundary condition $\gamma$. So in particular the “overall” translation operator $U = \otimes_\ell U_\ell$ is defined to act via

$$UX_nU = X_{\gamma(n)}$$

with $X_n = \{p^i_n, x^i_n, \theta^a_n\}$. It translates the bits only within each individual string, i.e. it does not translate any bit from one string to the next. Finally, we notice that each multi-string state is required to be (anti-)symmetric (according to statistics) under interchange of strings with equal discrete light-cone momentum $J_\ell$.

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3Here and in the following, we use the notation $\gamma$ both for the conjugacy class as for a specific representative in this class.
The light-cone supersymmetry generators and Hamiltonian of the free string theory read

\[ Q_0 = Q_0^{(0)} + \lambda Q_0^{(1)}, \quad H_0 = H^{(0)} + \lambda H^{(1)} + \lambda^2 H^{(2)} \]  

with

\[ Q_0^{(0)} = \sum_n (p_n^i \gamma_i \theta_n - x_n^i (\gamma_i \Pi) \tilde{\theta}_n), \quad Q_0^{(1)} = \sum_n (x_n^i (\gamma_i(n)) - x_n^i \gamma_i \theta_n) \]  

\[ H^{(0)} = \sum_n \left( \frac{1}{2} (p_{i,n}^2 + x_{i,n}^2) + 2i \tilde{\theta}_n \Pi \theta_n \right), \]  

\[ H^{(1)} = -\sum_n i (\theta_n \theta_{\gamma(n)} - \tilde{\theta}_n \tilde{\theta}_{\gamma(n)}), \quad H^{(2)} = \sum_n \frac{1}{2} (x_{\gamma(n)}^i - x_n^i)^2 \]  

These expressions are a straightforward discretization of the expressions in [3]. Here \( \lambda \) is a parameter that controls the size of the string bits relative to the mass scale \( m \) in the light-cone worldsheet Lagrangian (3).

Via [1] and [11], there now exists convincing evidence that the above free light-cone Hamiltonian exactly summarizes the propagation of the BMN operators in the leading order large \( N \) limit, via the identification (2) and of \( \lambda \) with the ‘t Hooft coupling

\[ \lambda^2 = g^2_{ym} N. \]  

Our next goal is to extend this correspondence to include string splitting and joining.

**Interactions**

We now add interaction terms to the light-cone generators, as follows

\[ Q^{\hat{a}} = Q_0^{\hat{a}} + g_s S_1^{\hat{a}}, \quad H = H_0 + g_2 V_1 + g_2^2 V_2, \]  

with \( g_2 \) the (effective) string coupling. Imposing the light-cone supersymmetry algebra (4) produces the relations

\[ \{Q_0^{\hat{a}}, S_1^{\hat{b}}\} + \{\tilde{Q}_0^{\hat{a}}, \tilde{S}_1^{\hat{b}}\} + (\hat{a} \leftrightarrow \hat{b}) = \delta^{\hat{a}\hat{b}} V_1 \]  

\[ [H_0, S_1^{\hat{a}}] + [V_1, Q_0^{\hat{a}}] = 0 \]  

\[ \{S_1^{\hat{a}}, S_1^{\hat{b}}\} + \{\tilde{S}_1^{\hat{a}}, \tilde{S}_1^{\hat{b}}\} = \delta^{\hat{a}\hat{b}} V_2 \]  

\[ [V_2, Q_0^{\hat{a}}] + [V_1, S_1^{\hat{a}}] = 0. \]
We wish to solve these relations via suitable interaction terms $S_1$ and $V_1$ that induce the splitting and joining of strings. It will be a non-trivial result that the above algebra can be satisfied, without the need of introducing any higher order interaction terms than $V_2$.

Following the matrix string theory example, it is natural to look for operators analogous to the twist-field interaction (6). Consider operators $\Sigma_{mn}$ that implement a simple transposition of two string bits via

$$\Sigma_{nm} X_m = X_n \Sigma_{nm}, \quad \Sigma_{nm} X_k = X_k \Sigma_{nm}, \quad k \neq m, n$$

(22)

with $X_n = \{p_n^i, x_n^i, \theta_n^a\}$. Clearly, by acting with $\Sigma_{mn}$ on a given multi-string sector, we get a different multi-string sector via

$$\Sigma_{mn} : \mathcal{H}_\gamma \to \mathcal{H}_{\tilde{\gamma}}, \quad \text{with } \tilde{\gamma} = \gamma \circ (mn).$$

(23)

Depending on whether the two sites $n$ and $m$ in the sector $\gamma$ correspond to one single string (say of length $J_0$) or two separate ones (say of length $J_1$ and $J_2$), the new sector $\tilde{\gamma}$ is obtained by either splitting the single string in two pieces of length $(m - n)$ and $(J_0 - m + n)$, or by joining the two strings to one of length $J_1 + J_2$.

Let us now introduce the $S_J$ invariant operator

$$\Sigma = \frac{1}{J^2} \sum_{n < m} \Sigma_{nm}$$

(24)

and define the interaction terms via

$$S_{\dot{1}} = [Q_{\dot{0}}^{\dot{1}}, \Sigma], \quad V_1 = [H_0, \Sigma].$$

(25)

This form of the interaction term has several motivations. First, it manifestly solves the equations (18) and (19) imposed by the space-time supersymmetry algebra. Secondly, by the preceding discussion, it represents a simple splitting and joining interaction. Finally, the matrix string theory vertex (6), light-cone string field theory [8] and the $\mathcal{N} = 4$ gauge theory calculations [10], all suggest that the interaction vertex should contain, besides the simple splitting and joining operator, a prefactor quadratic in the string oscillators. Moreover, we expect that the interaction term should vanish when $\lambda = 0$, i.e. in the free SYM theory. (We will further motivate this requirement in the next subsection.) Notice that, since the leading order generators $Q^{(0)}$ and $H^{(0)}$ are permutation invariant, both interaction terms are indeed (at least) of order $\lambda$.

It is straightforward, via (22) and the definition of the light-cone generators, to explicitly compute $S_1$ and $V_1$. For $S_1$ we find

$$S_1 = \frac{1}{J^2} \sum_{m,n} \Sigma_{mn} \left( \theta_m \gamma^i (x^i_m - x^i_n) + \theta^{-1}_m \gamma^i (x^i_n - x^i_m) - \delta_{m\gamma} (\theta_m \gamma^i x^i_n - \theta_n \gamma^i x^i_m) \right)$$

(26)
From this explicit form, it is directly verified that the $S_1$ terms in fact satisfy a commutation relation of the form (20), which can thus be taken as the definition of the second order interaction term $V_2$. The relation (21) is then automatically satisfied.

Thus the interacting string-bit theory has all the required symmetries, as well as all required dynamic properties (at least all obvious ones). We will now make a comparison with the gauge theory amplitudes.

Matrices

The interacting string bit theory is constructed to correspond with the non-planar gauge theory amplitudes in the BMN limit, via the identification of the effective string coupling $g_2$ with

$$g_2 = J^2/N,$$

and of the light-cone energy with scale dimension $\Delta - J$. In view of this last correspondence, it seems appropriate to view the gauge theory amplitudes as obtained via radial evolution starting from an initial state defined at some given point, say at $r = 0$, which then evolves via the dilation operator to a final state defined at $r = \infty$. Preferably, from a given initial operator at $r = 0$, one would then like to be able to keep track of how the index structure of the state evolves with $r$, so that, via an appropriate dictionary, we can identify the number of strings propagating at this instant and make a precise comparison with the corresponding light-cone string diagram.

A technical fact, that seems to be at odds with this philosophy, is that the two-point functions of single trace operators in the gauge theory in fact have a non-trivial $1/N$ expansion even at $g_{ym} = 0$, [9][16] and [10]. This would seem to indicate that strings can split and join even without any non-trivial SYM interaction taking place, thus preventing a precise determination of the number of strings at given light-cone time $r$. Our point of view, however, is that the $1/N$ corrections that are present in the free SYM theory, do not correspond to proper string interactions, but need to be absorbed into a redefinition of the precise operator dictionary (using techniques similar to [15]).4 This point of view was also taken in [10].

Modulo this subtlety, single trace operators in first order correspond to single strings. The cyclicity of the trace then naturally implements the $L_0 - \tilde{L}_0$ constraint (10), while via permutations of the operators in a single or multi-trace expression, we can (similarly as discussed above) induce string splitting or joining. Combined with the original analysis of [1], it seems

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4This redefinition is possible, because string S-matrix elements between in- and out-states, via the above prescription, in fact correspond to two-point functions in SYM theory of corresponding multi-trace operators placed at $r = 0$ and $r = \infty$, rather than multi-point functions. In particular, the three-point function is an S-matrix element between an one-string and a two-string state (see below), and not equal to a three-point function of the SYM theory.
natural to look for a combinatoric language by which we can characterize the SYM interactions as a composition of a small number of elementary operations, such as nearest neighbor hopping or simple permutations. The general dictionary relies on a one-to-one correspondence between interaction terms in the SYM and string bit Hamiltonian, via

$$\text{Tr}(\theta[Z,\theta]) \leftrightarrow H^{(1)} + g_2 [H^{(1)}, \Sigma]$$

$$\text{Tr}[[Z,X]]^2 \leftrightarrow H^{(2)} + g_2 [H^{(2)}, \Sigma] + g_2^2 V_2^{(2)}$$

with $V_2 = \lambda^2 V_2^{(2)}$. We leave a more detailed check of this dictionary to a future publication.

Instead, to illustrate the correspondence, let us compare the predictions for some low order amplitudes with existing gauge theory calculations. This task turns out to be almost trivial, since many of the computations done in [9] [10] can be readily transferred to the string bit language.

First, consider the elementary three point interaction between an initial single string state $|i\rangle$, splitting into a two-string final state $|f_1, f_2\rangle$. Quite generally, since $V_1 = [H, \Sigma]$, we have that

$$\langle i | V_1 | f_1, f_2 \rangle = (\Delta_i - \Delta_{f_1} - \Delta_{f_2}) \langle i | \Sigma | f_1, f_2 \rangle$$

The reduced matrix element on the right-hand side corresponds to the three-point function computed in the free SYM gauge theory. Equation (30) is therefore in direct accordance with the three-point string vertex proposed in [10].

Let us focus on the same class of operators considered in [9] [10]

$$O_p^J = \frac{1}{\sqrt{JN^2}} \sum_{l=1}^{J} e^{2\pi i l/J} \text{Tr}(\phi Z^l \psi Z^{J-l}) \leftrightarrow O_p^J = \frac{1}{J^{3/2}} \sum_{k,l=1}^{J} a_k^\dagger b_l^\dagger e^{2\pi i p(l-k)/J}$$

and compute the three point function

$$\langle O_p^J | V_1 | O_q^{J_1} \rangle_{J_1 J_2}$$

by taking the overlap between the initial and final state in the string bit theory. The calculation is a (suspiciously) exact copy of that in the gauge theory: the overlap between the in- and out-state involves the sum

$$\sum_{k,l=1}^{J_1} e^{2\pi i (l-k)(p/J-q/J_1)} = \frac{\sin^2(\pi p J_1 / J)}{\pi^2 (p/J - q/J_1)^2}$$

(assuming $J_1$ is large), while

$$\Delta_p - \Delta_q = \lambda (p^2 - q^2 J^2 / J_1^2).$$
After accounting for normalization factors of $\sqrt{J}$ and such, one obtains the exact same result as in [10], eqn (5.10).

As a perhaps somewhat more instructive example, let us compute (from the string bit Hamiltonian) the anomalous dimension $\Delta_p$ of $O_\mu^I$ to second order in the effective string coupling $g_2$. To zeroth order we have $H_0|p\rangle = (\Delta^{(0)} - J)|p\rangle$. Next we note that when acting on the space of one-string and two-string states, the total Hamiltonian takes the form

$$\left( \begin{array}{cc} H_0 + g_2^2 V_2 & g_2 V_1 \\ g_2 V_1^* & H_0 + g_2^2 V_2 \end{array} \right) \left( \begin{array}{c} |p\rangle_s \\ |q\rangle_{1,2} \end{array} \right).$$

(35)

So we find

$$\Delta_p = \Delta_p^{(0)} + g_2^2 \Delta_p^{(2)}, \quad \Delta_p^{(2)} = \langle p | V_2 | p \rangle + \sum_q \frac{||p | V_1 | q \rangle||^2}{\Delta_p^{(0)} - \Delta_q^{(0)}}$$

(36)

Inserting (25), we can rewrite the last term as

$$\sum_q (\Delta_p^{(0)} - \Delta_q^{(0)})|\langle p | \Sigma | q \rangle|^2.$$ 

(37)

This contribution was evaluated in [10], with the end-result (5.17). The first term in (36), upon inserting (20), can be written as

$$\langle p | V_2 | p \rangle = \langle p | (Q \hat{\alpha} \Sigma S_{\hat{1}}^\dagger + \text{h.c.}) | p \rangle + \langle p | V_1 \Sigma | p \rangle$$

(38)

(no sum over $\hat{a}$).

We can compare this expression with the analysis in section 3.3 of [10]. Since $Q \hat{a}$ in the first term directly acts on the single string state, it represents an nearest neighbor hopping term. We thus recognize the first term as representing the semi-nearest neighbor interactions, while the second term on the right-hand side is the non-nearest neighbor interaction (see figs 8 and 9 in [10]). The semi-nearest neighbor interactions turn out to cancel due to anti-symmetry. Upon inserting a complete set of intermediate states into the second term in (38), one quickly verifies that it takes the exact same form as the contribution (37), again in accordance with the results in [9] [10]. We thus conclude that the total shift in the anomalous dimension is twice (37), and thus twice (5.17) in [10].

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5Here h.c. is in fact short hand for three additional terms: a term $\widetilde{Q}_0 \Sigma \widetilde{S}_1$ plus the two hermitian conjugate terms. Further, to obtain (36), we used that $Q$ and $\Sigma^2$ commute when placed between two single string states, since $\langle 1 | \Sigma^2 | 1 \rangle = \sum_{mn} \langle 1 | \Sigma_m n | 1 \rangle$.

6Our answer thus differs by a factor of two from [10]. Our interpretation, however, is that to compute the total shift in dimensions, one should diagonalize the full light-cone Hamiltonian, leading to a mixing between one and two string states. This may look at odds with an S-matrix philosophy, where one and two string states would not mix. However, both from the gauge theory and string theory perspective, there’s no reason such mixing would not occur, since strings with non-zero $p_\perp$ never truly separate from each other in the asymptotic region.
The physical meaning of the dimensionless parameter $\lambda$ is that it represents the (inverse) size of the string bit relative to the worldsheet mass-scale $m$ created by the plane wave background. Calling the string bit size $\ell$, we have

$$m \ell = 1/\lambda.$$  

The SYM perturbative regime therefore corresponds to the limit where the string bit size is much larger than the length scale set by $m$. There is no reason, however, certainly not within the string bit model, to assume that $\lambda$ is small.

In the limit of large $\lambda$ and large $J$, there will be an intermediate regime of worldsheet scales $L$ at which the string bits look infinitesimal, while the mass perturbation $m$ is still small. In this regime, the worldsheet dynamics should look close to that of strings in flat space. In particular, the string interactions should have an accurate effective description by means of the matrix string interaction vertex (6). In addition, the effective string coupling should be equal to $g_s$ rather than $g_2$.

Let us compare the two effective interaction vertices (6) and (25). Both are a product of a pure twist-field (splitting/joining) interaction and a quadratic expression in the string oscillators. It indeed looks like it should be possible to continuously connect the two operators by means of a suitable renormalization group trajectory, connecting the small $\lambda$ regime (25) with the large $\lambda$ regime (6). This problem is presently under study in [17].

On general grounds, however, one can already deduce how the effective couplings should be related in the two regimes. Namely, a pure twist field interaction $V_{\text{twist}} = \bar{\sigma} \bar{\Sigma} \Sigma \bar{\Sigma}$ in the massless continuum worldsheet theory has scale dimension 2. In the massive theory, however, this conformal dimension will depend non-trivially on the ratio (40): it interpolates from 2 for $\lambda$ large, to 0 for small $\lambda$, [17]. Since the typical length scale of the worldsheet dynamics is of

$J^2/N$ versus $g_s$

A remaining puzzle is to explain why the effective string coupling needs to be identified with $g_2 = J^2/N$, rather than with the fundamental string coupling

$$g_s = g_2 \lambda^2 / J^2$$  

predicted by the AdS/CFT dictionary. An intuitive explanation is that strings in the plane wave background are in effect confined to a finite transverse volume, which amounts to an effective dimensional reduction to two dimensions and a corresponding rescaling of the Newton constant [10]. It is important, however, to understand this renormalization also directly from the worldsheet perspective.\(^7\)

Ideas similar to the following discussion were put forward in [12].
order $J\ell$, one expect that under the RG flow the twist field gets renormalized with a factor of $(mJ\ell)^2$. That is

$$V_{\text{twist}} \leftrightarrow \Sigma J^2/\lambda^2 \quad (41)$$

with $\Sigma$ the string bit version (24) of the twist field. This explains the renormalization (39) of the effective string coupling.

**Conclusion and outlook**

We have presented a simple string bit system that describes interacting strings in a plane wave background. The model is complete in the sense that the space-time symmetry algebra closes at finite string coupling, without the need for higher order terms other than that we have described here. The BMN conjecture is that the free string bit Hamiltonian correctly summarizes all planar interactions of $\mathcal{N} = 4$ gauge theory in the BMN limit; via our model, one is now free to extend this conjecture to arbitrary orders in the string loop expansion. While in principle it is possible that extra terms may (need to) be added to $H$, these are highly constrained by the requirement that they preserve the space-time supersymmetry algebra, as well as the existence of a continuum limit. In this sense, such possible extra terms would seem to amount to a marginal or irrelevant deformation of the model, rather than a necessary correction term.

What is the regime of validity of the model? From the interaction Hamiltonian, we find that string interactions remain weak as long as

$$\frac{g_2 \lambda}{J} = \frac{g_{ym} J}{\sqrt{N}}, \quad \text{and} \quad \frac{g_2 \lambda^2}{J^2} = g_s, \quad (42)$$

are both sufficiently small. Outside this regime, the continuum limit ceases to exist and the string worldsheet description breaks down. The region $g_2 \lambda > J$ has indeed been identified in [1] as the regime where the strings start to expand into giant gravitons. It seems likely that a matrix (string) theory type description takes over in this regime. Indeed it would be interesting to derive or construct a dual $U(J)$ matrix model that reduces to our bit model in its strong coupling limit.

It should be straightforward to extend the string bit model to the quiver set-up of [18], which should in particular clarify the relation with matrix string theory. The central role of the permutation group further suggests that similar ideas as presented here may be used to construct an interacting string bit theory in the Penrose limit of $AdS^3$. 
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Note added:

Since the time of writing this paper, several further developments led to a more precise understanding of both the gauge theory and string theory side of the story. These new insights made clear that the original proposal as formulated here needs some modification. To clarify the present status of our results, a brief summary of this subsequent history will be helpful.

Initially, the interaction vertex as proposed in eqn (25) coincided not only with the reported values of the gauge theory amplitudes [9][10] but also with the three-point function in string field theory, as reported in the original version of [19]. Soon after, however, more careful study of the gauge theory amplitudes [20][21], (in part prompted by our suggestion that operator mixing must in fact be taken into account) resulted in a new and different answer for the three-point function, as well as for the shift in the conformal dimension of the two-impurity states [20][21]. Simultaneously, it was pointed out [22] that the original string theory calculation of [19] contained a subtle sign error, and needed correction as well.\footnote{An alternative construction of the string interaction vertex, which in fact bears close resemblance to the vertex (25), was proposed in [23][24].}

In [25], the following refinement of the bit string vertex (25) was proposed

$$S_1^a = [\hat{Q}_0^\hat{a}, \Sigma], \quad Q_0 = Q_0^- - Q_0^+, \quad (43)$$

where $Q_0 = Q_0^- + Q_0^+$ is the free supercharge of the bit string theory and the superscripts indicate the projection onto the term with fermionic creation ($\langle$) or annihilation ($\rangle$) operators only. This vertex was shown to reproduce all known gauge theory amplitudes [20][21]. In addition, a basis transformation was proposed that relates the gauge theory basis of single and multi-trace operators with the string theory basis of single and multi-string states. This proposed dictionary was verified to linear order in $g_2$ in [26], where a first precise match between the gauge theory amplitudes and string theory interactions was found. This dictionary was independently proposed in [27] and [28], and verified to second order in [29]. Finally, in [30] an effective quantum mechanical model for capturing the gauge theory interactions was constructed, which was shown in [31] to be in accord with the string bit vertex (43). Also our general approach of identifying string scattering amplitudes with two-point functions of multi-trace operators, rather than multi-point functions, has been supported by several later results.
A remaining challenge is to construct the continuum limit of the bit theory. Some aspects of this problem, including (ways of avoiding) fermion doubling, have been analyzed in [32][33][34].

References

[1] D. Berenstein, J. Maldacena and H. Nastase, “Strings in flat space and pp waves from $N = 4$ super Yang Mills,” arXiv:hep-th/0202021.

[2] M. Blau, J. Figueroa-O’Farrill, C. Hull and G. Papadopoulos, “A new maximally supersymmetric background of IIB superstring theory,” hep-th/0110242; “Penrose limits and maximal supersymmetry,” hep-th/0201081.

[3] R. R. Metsaev, “Type IIB Green Schwarz superstring in plane wave Ramond Ramond background,” hep-th/0112044; R. R. Metsaev and A. A. Tseytlin, Phys. Rev. D 65, 126004 (2002) [arXiv:hep-th/0202109].

[4] C. B. Thorn, “Supersymmetric quantum mechanics for string-bits,” Phys. Rev. D 56, 6619 (1997) [arXiv:hep-th/9707048].

[5] L. Motl, “Proposals on nonperturbative superstring interactions,” arXiv:hep-th/9701025; T. Banks and N. Seiberg, “Strings from matrices,” Nucl. Phys. B 497, 41 (1997) [arXiv:hep-th/9702187].

[6] R. Dijkgraaf, E. Verlinde and H. Verlinde, “Matrix string theory,” Nucl. Phys. B 500, 43 (1997) [arXiv:hep-th/9703030].

[7] C. Vafa and E. Witten, Nucl. Phys. B 431, 3 (1994) [arXiv:hep-th/9408074].

[8] A. Volovich, M. Spradlin, “Superstring interactions in a pp-wave background,” arXiv:hep-th/0204146; M. x. Huang, arXiv:hep-th/0205311; C. S. Chu, V. V. Khoze and G. Travaglini, arXiv:hep-th/0206005.

[9] C. Kristjansen, J. Plefka, G. W. Semenoff and M. Staudacher, arXiv:hep-th/0205033; N. R. Constable, D. Z. Freedman, M. Headrick, S. Minwalla, L. Motl, A. Postnikov and W. Skiba, “PP-wave string interactions from perturbative Yang-Mills theory,” arXiv:hep-th/0205089.

[10] D. J. Gross, A. Mikhailov and R. Roiban, “Operators with large R charge in $N = 4$ Yang-Mills theory,” arXiv:hep-th/0205066.

[11] R. Gopakumar, “String interactions in PP-waves,” arXiv:hep-th/0205174.

[12] K. Dasgupta, M. M. Sheikh-Jabbari and M. Van Raamsdonk, JHEP 0205, 056 (2002) [arXiv:hep-th/0205185].

[13] Y. Hikida and Y. Sugawara, “Superstrings on PP-wave backgrounds and symmetric orbifolds,” arXiv:hep-th/0205220; G. Bonelli, “Matrix strings in pp-wave backgrounds from deformed super Yang-Mills theory,” arXiv:hep-th/0205213.
[15] S. Corley, A. Jevicki and S. Ramgoolam, arXiv:hep-th/0111222;
[16] D. Berenstein and H. Nastase, arXiv:hep-th/0205048.
[17] D. Vaman, J. Pearson, and H. Verlinde, in preparation.
[18] S. Mukhi, M. Rangamani and E. Verlinde, “Strings from quivers, membranes from moose,” JHEP 0205, 023 (2002) [arXiv:hep-th/0204147]; M. Alishahiha and M. Sheikh-Jabbari, “Strings in pp-waves and worldsheet deconstruction,” arXiv:hep-th/0204174.
[19] M. Spradlin and A. Volovich, “Superstring interactions in a pp-wave background. II,” arXiv:hep-th/0206073.
[20] N. Beisert, C. Kristjansen, J. Plefka, G. W. Semenoff and M. Staudacher, “BMN correlators and operator mixing in N = 4 super Yang-Mills theory,” arXiv:hep-th/0208178.
[21] N. R. Constable, D. Z. Freedman, M. Headrick and S. Minwalla, “Operator mixing and the BMN correspondence,” arXiv:hep-th/0209002.
[22] A. Pankiewicz, “More comments on superstring interactions in the pp-wave background,” JHEP 0209, 056 (2002) [arXiv:hep-th/0208209].
[23] C. S. Chu, V. V. Khoze, M. Petrini, R. Russo and A. Tanzini, “A note on string interaction on the pp-wave background,” arXiv:hep-th/0208148.
[24] P. Di Vecchia, J. L. Petersen, M. Petrini, R. Russo and A. Tanzini, “The 3-string vertex and the AdS/CFT duality in the pp-wave limit,” arXiv:hep-th/0304025.
[25] D. Vaman and H. Verlinde, “Bit strings from N = 4 gauge theory,” arXiv:hep-th/0209215.
[26] J. Pearson, M. Spradlin, D. Vaman, H. Verlinde and A. Volovich, “Tracing the string: BMN correspondence at finite J**2/N,” JHEP 0305, 022 (2003) [arXiv:hep-th/0210102].
[27] D. J. Gross, A. Mikhailov and R. Roiban, “A calculation of the plane wave string Hamiltonian from N = 4 super-Yang-Mills theory,” arXiv:hep-th/0208231.
[28] J. Gomis, S. Moriyama and J. w. Park, “SYM description of SFT Hamiltonian in a pp-wave background,” arXiv:hep-th/020153.
[29] R. Roiban, M. Spradlin and A. Volovich, arXiv:hep-th/0211220.
[30] N. Beisert, C. Kristjansen, J. Plefka and M. Staudacher, Phys. Lett. B 558, 229 (2003) [arXiv:hep-th/0212269].
[31] M. Spradlin and A. Volovich, Phys. Lett. B 565, 253 (2003) [arXiv:hep-th/0303220].
[32] J. G. Zhou, “PP-wave string interactions from string bit model,” arXiv:hep-th/0208232.
[33] S. Bellucci and C. Sochichiu, “Fermion doubling and BMN correspondence,” Phys. Lett. B 564, 115 (2003) arXiv:hep-th/0302104; arXiv:hep-th/0307253.
[34] U. Danielsson, F. Kristiansson, M. Lubcke and K. Zarembo, arXiv:hep-th/0306147.