Box dimension of mixed Katugampola fractional integral of two-dimensional continuous functions

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Abstract
The goal of this article is to study the box dimension of the mixed Katugampola fractional integral of two-dimensional continuous functions on \([0, 1] \times [0, 1]\). We prove that the box dimension of the mixed Katugampola fractional integral having fractional order \((\alpha = (\alpha_1, \alpha_2); \alpha_1 > 0, \alpha_2 > 0)\) of two-dimensional continuous functions on \([0, 1] \times [0, 1]\) is still two. Moreover, the results are also established for the mixed Hadamard fractional integral. Our new results are to improve the existing studies. We pose also some open problems for further research.

Keywords  Box dimension · Mixed Katugampola fractional integral · Two-dimensional continuous functions · Mixed Hadamard fractional integral · Bounded variation

Mathematics Subject Classification  26A33 · 28A80 · 31A10

1 Introduction
Fractional Calculus (FC) and Fractal Geometry (FG) have become rapidly growing fields, both in theoretical and applicable aspects. Their interrelation has not been clearly recognized almost by end of the 20th century, being still a challenging conjecture discussed at some of the first conferences specialized on FC, like these in 1989 (FC, Japan), in 1996 (TMSF, Bulgaria), and next ones, see for example the notes [11]. Since the random fractals are better examples of highly irregular functions, the FC operators are the best mathematical tools for analyzing such functions. Tatom [18]...
proposed some general relation between fractional calculus and fractal functions. The FG is more prominent than the classical geometry to study irregular sets. A theory on FG is given, for example, in reference [1]. For various notions and definitions of fractional integrals and derivatives the readers may be referred to FC handbooks, as [9, 16].

The box dimension plays an important role to study the smoothness of any irregular function. A connection between FC and fractal dimensions can be found in [8, 13–15, 21]. Liang [12] proved that the box dimension of a function which is of bounded variation and continuous on $[0, 1]$ is 1 and also the box dimension of its Riemann-Liouville fractional integral $I^v f$ on $[0, 1]$ is 1, where

$$(I^v f)(x) = \frac{1}{\Gamma(v)} \int_0^x (x - s)^{v-1} f(s) \, ds, \quad v > 0. \quad (1.1)$$

Liang [14] investigated the fractal dimension of the fractional integral of Riemann-Liouville type of a continuous function having box dimension 1.

We try to establish more general results for the fractional integral of mixed Katugampola type. Box dimension of a bivariate function which is of bounded variation in Arzelá sense and continuous on $[a, b] \times [c, d]$ has been investigated in [19]. It has been shown that box dimension of a bivariate function which is of bounded variation in Arzelá sense and continuous on $[a, b] \times [c, d]$ is 2. Also, some examples of two-dimensional functions are given which are not of bounded variation. Additionally, they have proved that the box dimension of the fractional integral of mixed Riemann-Liouville type of a continuous function which is of bounded variation in Arzelá sense is 2. Analogous results can be seen for the mixed Katugampola fractional integral $(G^\alpha f)$ in Verma and Viswanathan [20], where

$$(G^\alpha f)(x, y) = \frac{(\rho_1 + 1)^{1-\alpha_1}(\rho_2 + 1)^{1-\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_a^x \int_c^y \left(x^{\rho_1+1} - s^{\rho_1+1}\right)^{\alpha_1-1} \times \left(y^{\rho_2+1} - t^{\rho_2+1}\right)^{\alpha_2-1} s^{\rho_1} t^{\rho_2} f(s, t) \, ds \, dt.$$
similar result, in particular for the mixed Hadamard fractional integral and discuss some open problems.

2 Preliminaries

In this section, we give definitions of some fractional integrals and other required terminologies related to the present study.

2.1 Katugampola fractional integral

Among the most common operators of FC that are studied for their theory and applications are the Riemann-Liouville, the Erdélyi-Kober and the Hadamard fractional integrals.

We recall first a version of fractional integral introduced by Katugampola [6], which includes two of these well-known fractional integrals.

Definition 1 ([6]) Let \( f \) be a function defined on \([a, b]\), \( a \geq 0 \). Assuming that the following integral exists, the Katugampola fractional integral of \( f \) is defined by

\[
(\mathcal{I}_\nu f)(x) = \frac{(\rho + 1)^{1-\nu}}{\Gamma(\nu)} \int_a^x (x^\rho s^{\rho+1} - s^{\rho+1})^{\nu-1} s^\rho f(s) \, ds, \quad (2.1)
\]

where \( \nu > 0 \) and \( \rho \neq -1 \).

Next, the mentioned operators of classical FC are defined as follows ([9, 10, 16]).

Definition 2 ([16,§18.3],[9,§2.7]) The Hadamard fractional integral is defined by

\[
(\mathcal{H}_\nu f)(x) = \frac{1}{\Gamma(\nu)} \int_a^x (\log u)^{\nu-1} \frac{f(u)}{u} \, du, \quad \nu > 0. \quad (2.2)
\]

Definition 3 ([17],[10,Ch.2], see also [16,§18.1] and [9,§2.6]) Let \( f \) be a function defined on \([a, b]\), \( a \geq 0, \nu > 0 \), and \( \rho > -1, \eta \) be additional real parameters that compared to the Riemann-Liouville integral (1.1) allow much more wide range of applications. Assuming that the following integral exists, the Erdélyi-Kober fractional integral of \( f \) is defined by

\[
(\mathcal{J}_{\rho,\eta} f)(x) = \frac{(\rho + 1)x^{-(\rho+1)(\nu+\eta)}}{\Gamma(\nu)} \int_a^x (x^\rho s^{\rho+1} - s^{\rho+1})^{\nu-1} s^{(\rho+1)\eta+\rho} f(s) \, ds. \quad (2.3)
\]

For \( \rho = 1, \) arbitrary \( \eta \) and \( a = 0, \) this is the “classical” Erdélyi-Kober operator, initially introduced in [4].

Remark 1 The Katugampola fractional operator (2.1) incorporates both the Riemann-Liouville fractional operator (1.1) (for \( \rho = 0 \)) and the Hadamard fractional operator (2.2) (in limiting case \( \rho \to -1^+ \)). Since the Erdélyi-Kober operator (2.3) is a
generalization of the Riemann-Liouville fractional operator, we note that the Riemann-
Liouville operator can be derived from both Katugampola operator and Erdélyi-Kober
operator. While, the Hadamard operator (2.2) can be derived from the Katugampola
operator but not from the Erdélyi-Kober operator.

Note that a relation between the Katugampola integral (2.1) and the Erdélyi-Kober
integral (2.3) can be found only under certain parameters’ conditions. For example,
when \( \eta = 0 \), the relation is
\[
(J^\nu_{\rho,0} f)(x) = \rho^\nu x^{-\rho \nu} (J^\nu f)(x).
\]
If we choose additionally \( \rho = 1 \) in above, we have
\[
(J^\nu_{1,0} f)(x) = x^{-\nu} (J^\nu f)(x).
\]
But because of the presence of the multiplier \( \rho^\nu x^{-\rho \nu} \) in above relations, the results
related to Erdélyi-Kober operator cannot be directly derived from the results of the
Katugampola operator (2.1). It can be possible only for some particular cases.

Remark 2 However, it is interesting to attract attention to a modified version of the
Katugampola fractional integral, as appeared later in [7]. Namely, in this preprint, an
extension of operator (2.1) is proposed as:
\[
(J^\nu_{\rho,\beta,\kappa,\eta} f)(x) = \frac{(\rho + 1)^{1-\beta} x^\kappa}{\Gamma(\nu)} \int_a^x (x^\rho + 1 - s^\rho + 1)^{\nu - 1} s^\rho \eta + \rho f(s) ds,
\]
with \( \nu > 0 \), \( \rho \neq -1 \) and additional real parameters \( \beta, \kappa, \eta \).

Note that for \( \beta = \nu, \kappa = 0 \) and \( \eta = 0 \), this reduces to the initial Katugampola
integral (2.1), namely: \( (J^\nu f)(x) = (J^\nu_{\rho,\nu,0,0} f)(x) \).

However, as pointed out in [7], the new multiplier \( x^\kappa \) (with arbitrary exponent \( \kappa \))
helps the possibility to reduce (2.4) to a particular case of Erdélyi-Kober integral (2.3),
if taking \( \beta = 0 \) and \( \kappa = -(\rho + 1)(\nu + \eta) \). Therefore, by definition (2.4) one can
indeed incorporate more cases of known fractional integrals.

2.2 Mixed Katugampola Fractional Integral

Definition 4 (Verma and Viswanathan [20]) Let \( f \) be a function which is defined on a
closed rectangle \([a, b] \times [c, d]\) and \( a \geq 0, c \geq 0 \). Assuming that the following integral
exists, the mixed Katugampola fractional integral of \( f \) is defined by
\[
(\mathcal{I}_\alpha f)(x, y) = \frac{(\rho_1 + 1)^{1-\alpha_1} (\rho_2 + 1)^{1-\alpha_2}}{\Gamma(\alpha_1) \Gamma(\alpha_2)} \int_a^x \int_c^y (x^\rho_1 + 1 - s^\rho_1 + 1)^{\alpha_1 - 1} \times
\]
\[
(y^\rho_2 + 1 - t^\rho_2 + 1)^{\alpha_2 - 1} s^{\rho_1} t^{\rho_2} f(s, t) ds dt,
\]
where \( \alpha = (\alpha_1, \alpha_2) \) with \( \alpha_1 > 0, \alpha_2 > 0 \) and \( (\rho_1, \rho_2) \neq (-1, -1) \).
2.3 Mixed Hadamard Fractional Integral

By using L’Hospital rule and $\rho_1, \rho_2 \rightarrow -1^+$, the mixed Katugampola reduces to the mixed Hadamard fractional integral.

**Definition 5** Let $f$ be a function which is defined on a closed rectangle $[a, b] \times [c, d]$ and $a \geq 0, c \geq 0$. Assuming that the following integral exists, the *mixed Hadamard fractional integral* of $f$ is defined by

\[
(\mathcal{H}_\gamma f)(x, y) = \frac{1}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \int_a^x \int_c^y \left(\log \frac{x}{u}\right)^{\gamma_1-1} \left(\log \frac{y}{v}\right)^{\gamma_2-1} \frac{f(u, v)}{uv} \, du \, dv, \tag{2.6}
\]

where $\gamma = (\gamma_1, \gamma_2)$ with $\gamma_1 > 0, \gamma_2 > 0$.

For the above definitions of the two-dimensional fractional integrals and some properties, the readers can be referred to Verma and Viswanathan [20].

2.4 Range of $f$

**Definition 6** Let $A = [a, b] \times [c, d]$ be a rectangle. For a function $f : A \rightarrow \mathbb{R}$, the maximum range of $f$ over $A$ is given by

\[
R_f[A] := \sup_{(t,u),(x,y) \in A} |f(t,u) - (x,y)|.
\]

2.5 Box Dimension

**Definition 7** Let $S \neq \emptyset$ be a bounded subset of $\mathbb{R}^n$. Let the smallest number of sets having diameter at most $\delta$ is denoted by $N_\delta(S)$ which can cover $S$. The lower box dimension and upper box dimension of $S$, respectively, are defined as follows:

\[
\dim_B(S) = \lim_{\delta \to 0} \frac{\log N_\delta(S)}{-\log \delta} \tag{2.7}
\]

and

\[
\overline{\dim}_B(S) = \lim_{\delta \to 0} \frac{\log N_\delta(S)}{-\log \delta}. \tag{2.8}
\]

If $\dim_B(S) = \overline{\dim}_B(S)$, the common value is called the *box dimension* of $S$. That is,

\[
\dim_B(S) = \lim_{\delta \to 0} \frac{\log N_\delta(S)}{-\log \delta}.
\]

Let $C(I \times J)$ be the set of continuous functions, where $I \times J = [0, 1] \times [0, 1]$. Sometimes we will use the term fractional integral of mixed Katugampola type in the place of mixed Katugampola fractional integral.
3 Main results

In this section, first we give the following lemmas which act as preludes to our main theorem.

**Lemma 1** Let $f : [0, 1] \times [0, 1] \to \mathbb{R}$ be continuous and $0 < \delta < 1, \frac{1}{3} < m, n < 1 + \frac{1}{\delta}$, for some $m, n \in \mathbb{N}$. If the number of $\delta$-cubes is denoted by $N_{\delta}(Gr(f))$ which intersect the graph $Gr(f)$ of the function $f$, then

$$\sum_{j=1}^{n} \sum_{i=1}^{m} \max \left\{ \frac{R_f[A_{ij}]}{\delta}, 1 \right\} \leq N_{\delta}(Gr(f)) \leq 2mn + \frac{1}{\delta} \sum_{j=1}^{n} \sum_{i=1}^{m} R_f[A_{ij}], \quad (3.1)$$

where $A_{ij}$ is the $(i, j)$-th cell corresponding to the net under consideration.

**Proof** If $f(x, y)$ is continuous on $I \times J$, the number of cubes having side $\delta$ in the part above $A_{ij}$ which intersect $Gr(f, I \times J)$ is at least

$$\max \left\{ \frac{R_f[A_{ij}]}{\delta}, 1 \right\}$$

and at most

$$2 + \frac{R_f[A_{ij}]}{\delta}.$$

By summing over all such parts, we get the required result. \( \square \)

**Lemma 2** Let $f(x, y) \in C(I \times J)$ and $0 < \alpha_1 < 1, 0 < \alpha_2 < 1$.

If $h_1 > 0, h_2 > 0$ and $x + h_1 \leq 1, y + h_2 \leq 1$, then

$$(\mathcal{I}^\alpha f)(x+h_1, y+h_2) - (\mathcal{I}^\alpha f)(x, y)$$

$$= \frac{(\rho_1 + 1)^{-\alpha_1}(\rho_2 + 1)^{-\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{0}^{1} \int_{0}^{1} (1-s)^{\alpha_1-1}(1-t)^{\alpha_2-1}$$

$$\times \left[ \left( x + h_1 \right)^{\alpha_1} \left( y + h_2 \right)^{\alpha_2} f \left( x + h_1 s \frac{1}{\rho_1+1}, y + h_2 t \frac{1}{\rho_2+1} \right) 
- \left( x^{\rho_1+1} \right)^{\alpha_1} \left( y^{\rho_2+1} \right)^{\alpha_2} f \left( x s \frac{1}{\rho_1+1}, y t \frac{1}{\rho_2+1} \right) \right] dsdt$$

**Proof** By considering the conditions of the lemma, we have

$$(\mathcal{I}^\alpha f)(x+h_1, y+h_2) - (\mathcal{I}^\alpha f)(x, y)$$

$$= \frac{(\rho_1 + 1)^{-\alpha_1}(\rho_2 + 1)^{-\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{0}^{x+h_1} \int_{0}^{y+h_2} \left( x + h_1 \right)^{\alpha_1} \left( y + h_2 \right)^{\alpha_2}$$

$$\times f \left( x s \frac{1}{\rho_1+1}, y t \frac{1}{\rho_2+1} \right) dsdt.$$

\( \square \)
Thus, we have
\[-\frac{1}{\Gamma(\alpha_1) \Gamma(\alpha_2)} \int_0^x \int_0^y (x^{\alpha_1+1} - s^{\rho_1+1})^{\alpha_1-1} \times (y^{\rho_2+1} - t^{\rho_2+1})^{\alpha_2-1} s^{\rho_1 t^{\rho_2}} f(s, t) \, ds \, dt.\] Applying integral transformation, let
\[
\left(\frac{s}{x+h_1}\right)^{\rho_1+1} = u, \quad \left(\frac{t}{y+h_2}\right)^{\rho_2+1} = v.
\]
Then,
\[
ds \, dt = |J| \, du \, dv,
\]
where
\[
J = \begin{bmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{bmatrix} = \frac{(x+h_1)^{\rho_1+1}(y+h_2)^{\rho_2+1}}{(\rho_1+1)(\rho_2+1)s^{\rho_1 t^{\rho_2}}}.
\]
Thus, we have
\[
\frac{1}{\Gamma(\alpha_1) \Gamma(\alpha_2)} \int_0^x \int_0^y ((x+h_1)^{\rho_1+1} - s^{\rho_1+1})^{\alpha_1-1} \times ((y+h_2)^{\rho_2+1} - t^{\rho_2+1})^{\alpha_2-1} s^{\rho_1 t^{\rho_2}} f(s, t) \, ds \, dt
\]
\[
=\frac{1}{\Gamma(\alpha_1) \Gamma(\alpha_2)} \int_0^1 \int_0^1 (1-s)^{\alpha_1-1} (1-t)^{\alpha_2-1} \left((x+h_1)^{\rho_1+1}\right)^{\alpha_1}
\times \left((y+h_2)^{\rho_2+1}\right)^{\alpha_2} f\left((x+h_1)s^{\frac{1}{\rho_1^{\alpha_1}}} , (y+h_2)t^{\frac{1}{\rho_2^{\alpha_2}}}\right) \, ds \, dt.
\]
Similarly,
\[
\frac{1}{\Gamma(\alpha_1) \Gamma(\alpha_2)} \int_0^x \int_0^y (x^{\rho_1+1} - s^{\rho_1+1})^{\alpha_1-1} \times (y^{\rho_2+1} - t^{\rho_2+1})^{\alpha_2-1} s^{\rho_1 t^{\rho_2}} f(s, t) \, ds \, dt
\]
\[
=\frac{1}{\Gamma(\alpha_1) \Gamma(\alpha_2)} \int_0^1 \int_0^1 (1-s)^{\alpha_1-1} (1-t)^{\alpha_2-1}
\times \left(x^{\rho_1+1}\right)^{\alpha_1} \left(y^{\rho_2+1}\right)^{\alpha_2} f\left(xs^{\frac{1}{\rho_1^{\alpha_1}}} , yt^{\frac{1}{\rho_2^{\alpha_2}}}\right) \, ds \, dt.
\]
Consequently, we get the desired result by using these two values in (3.2). \(\Box\)

Now, we establish our main result.

**Theorem 1** Let a non-negative function \(f(x, y) \in C(I \times J)\) and \(0 < \alpha_1 < 1, \ 0 < \alpha_2 < 1, \ \rho_1 < -1, \ \rho_2 < -1.\)
If
\[ \dim_B \text{Gr}(f, I \times J) = 2, \] (3.3)
then the box dimension of the mixed Katugampola fractional integral of \( f(x, y) \) of order \( \alpha = (\alpha_1, \alpha_2) \) exists and is equal to 2 on \( I \times J \), as
\[ \dim_B \text{Gr}(\mathfrak{I}^\alpha f, I \times J) = 2. \] (3.4)

Proof Since \( f(x, y) \in C(I \times J) \), \( (\mathfrak{I}^\alpha f)(x, y) \) is also continuous on \( I \times J \) (from Theorem 3.4 in [20]). From the definition of the box dimension, we can get
\[ \dim_B \text{Gr}(\mathfrak{I}^\alpha f, I \times J) \geq 2. \] (3.5)

Now, to establish equation (3.4), we have to show that
\[ \dim_B \text{Gr}(\mathfrak{I}^\alpha f, I \times J) \leq 2. \] (3.6)

Suppose that \( 0 < \delta < \frac{1}{2}, \frac{1}{3} < m, n < 1 + \frac{1}{3} \) and \( N_\delta(\text{Gr}(f)) \) is the number of \( \delta \)-cubes that intersect \( \text{Gr}(f) \). From equation (3.3), it holds
\[ \lim_{\delta \to 0} \frac{\log N_\delta(\text{Gr}(f))}{-\log \delta} = 2. \]

Let \( N_\delta(\text{Gr}(\mathfrak{I}^\alpha f)) \) be the number of \( \delta \)-cubes that intersect \( \text{Gr}(\mathfrak{I}^\alpha f) \). Thus inequality (3.6) can be written as
\[ \lim_{\delta \to 0} \frac{\log N_\delta(\text{Gr}(\mathfrak{I}^\alpha f))}{-\log \delta} \leq 2. \] (3.7)

Now, we have to prove inequality (3.7).
For \( 0 < \delta < \frac{1}{2}, \frac{1}{3} < m, n < 1 + \frac{1}{3} \), let non-negative integers \( i \) and \( j \) be such that \( 0 \leq i \leq m, 0 \leq j \leq n \). Then
\[
\left| (\rho_1 + 1)^{-\alpha_1}(\rho_2 + 1)^{-\alpha_2} \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{A_{ij}} \mathfrak{I}^\alpha f \right|
\]
\[ = \sup_{(x+h_1,y+h_2),(x,y) \in A_{ij}} \left| (\mathfrak{I}^\alpha f)(x+h_1, y+h_2) - (\mathfrak{I}^\alpha f)(x, y) \right|,
\]
where \( A_{ij} = [i\delta, (i+1)\delta] \times [j\delta, (j+1)\delta] \). Here,
\[
\left| (\mathfrak{I}^\alpha f)(x+h_1, y+h_2) - (\mathfrak{I}^\alpha f)(x, y) \right|
\]
\[ = \left| \int_0^1 \int_0^1 (1-s)\alpha_1^{-1}(1-t)\alpha_2^{-1} \left( (x+h_1)^{\rho_1+1} \right)^{\alpha_1} \left( (y+h_2)^{\rho_2+1} \right)^{\alpha_2}
\]
\[ \times f \left( (x+h_1)\delta^{\alpha_1}, (y+h_2)t^{\alpha_2} \right) ds dt
\]
\[ - \int_0^1 \int_0^1 (1-s)\alpha_1^{-1}(1-t)\alpha_2^{-1} \left( (x+h_1)^{\rho_1+1} \right)^{\alpha_1} \left( (y+h_2)^{\rho_2+1} \right)^{\alpha_2}
\]
\[ \times f \left( x s^{\frac{1}{\alpha_1 + 1}}, y t^{\frac{1}{\alpha_2 + 1}} \right) \, dsdt + \int_0^1 \int_0^1 (1 - s)^{\alpha_1 - 1}(1 - t)^{\alpha_2 - 1} \left( (x + h_1)^{\alpha_1 + 1} (y + h_2)^{\alpha_2 + 1} \right) \, dsdt \]

Let \( i \geq 1, j \geq 1 \). On the one hand, we have

\[ \left| \int_0^1 \int_0^1 (1 - s)^{\alpha_1 - 1}(1 - t)^{\alpha_2 - 1} \times f \left( (x + h_1)s^{\frac{1}{\alpha_1 + 1}}, (y + h_2)t^{\frac{1}{\alpha_2 + 1}} \right) \, dsdt \right| = \left| \int_0^j \int_0^{\frac{j}{\alpha_2 + 1}} (1 - s)^{\alpha_1 - 1}(1 - t)^{\alpha_2 - 1} \times f \left( (x + h_1)s^{\frac{1}{\alpha_1 + 1}}, (y + h_2)t^{\frac{1}{\alpha_2 + 1}} \right) \, dsdt \right| + \sum_{l=1}^j \int_0^l \int_0^{\frac{l}{\alpha_2 + 1}} (1 - s)^{\alpha_1 - 1}(1 - t)^{\alpha_2 - 1} \times f \left( (x + h_1)s^{\frac{1}{\alpha_1 + 1}}, (y + h_2)t^{\frac{1}{\alpha_2 + 1}} \right) \, dtds. \]
\[
\times \left[ f \left( (x + h_1)s^{\frac{1}{\alpha_1+1}}, (y + h_2)t^{\frac{1}{\alpha_2+1}} \right) - f \left( xs^{\frac{1}{\alpha_1+1}}, yt^{\frac{1}{\alpha_2+1}} \right) \right] dsdt \\
+ \sum_{r=1}^{i} \int_{\frac{r-1}{\alpha_1+1}}^{\frac{r}{\alpha_1+1}} \int_{\frac{r-1}{\alpha_2+1}}^{\frac{r}{\alpha_2+1}} (1-s)^{\alpha_1-1}(1-t)^{\alpha_2-1} \\
\times \left[ f \left( (x + h_1)s^{\frac{1}{\alpha_1+1}}, (y + h_2)t^{\frac{1}{\alpha_2+1}} \right) - f \left( xs^{\frac{1}{\alpha_1+1}}, yt^{\frac{1}{\alpha_2+1}} \right) \right] dsdt \\
+ \sum_{r=1}^{i} \sum_{l=1}^{j} \int_{\frac{r-1}{\alpha_1+1}}^{\frac{r}{\alpha_1+1}} \int_{\frac{l-1}{\alpha_2+1}}^{\frac{l}{\alpha_2+1}} (1-s)^{\alpha_1-1}(1-t)^{\alpha_2-1} \\
\times \left[ f \left( (x + h_1)s^{\frac{1}{\alpha_1+1}}, (y + h_2)t^{\frac{1}{\alpha_2+1}} \right) - f \left( xs^{\frac{1}{\alpha_1+1}}, yt^{\frac{1}{\alpha_2+1}} \right) \right] dsdt \\
\leq \frac{1}{(i+1)(j+1)} R_f [[0, \delta] \times [0, \delta]] \\
+ \sum_{l=1}^{j} \frac{1}{(i+1)(j+1)} (R_f [[0, \delta] \times [(l-1)\delta, l\delta]] + R_f [[0, \delta] \times [l\delta, (l+1)\delta]]) \\
+ \sum_{r=1}^{i} \frac{1}{(i+1)(j+1)} (R_f [[(r-1)\delta, r\delta] \times [0, \delta]] + R_f [[r\delta, (r+1)\delta] \times [0, \delta]]) \\
+ \sum_{r=1}^{i} \sum_{l=1}^{j} \frac{1}{(i+1)(j+1)} (R_f [[(r-1)\delta, r\delta] \\
\times [(l-1)\delta, l\delta]] + R_f [[(r-1)\delta, r\delta] \times [l\delta, (l+1)\delta]]) \\
+ R_f [[r\delta, (r+1)\delta] \times [(l-1)\delta, l\delta]] + R_f [[r\delta, (r+1)\delta] \times [l\delta, (l+1)\delta]]) .
\]

By using Bernoulli’s inequality \((1+u)^r \leq 1 + ru'\) for \(0 \leq r' \leq 1\) and \(u \geq -1\), we can see that
\[
\int_{0}^{\frac{1}{\alpha_1+1}} \int_{0}^{\frac{1}{\alpha_2+1}} (1-s)^{\alpha_1-1}(1-t)^{\alpha_2-1} dsdt \leq \frac{1}{(i+1)(j+1)} .
\]

On the other hand, for a suitable absolutely positive constant \(C\), we have
\[
\int_{0}^{1} \int_{0}^{1} (1-s)^{\alpha_1-1}(1-t)^{\alpha_2-1} f \left( xs^{\frac{1}{\alpha_1+1}}, yt^{\frac{1}{\alpha_2+1}} \right) \\
\times \left[ \left( (x + h_1)^{\alpha_1+1} \right)^{\alpha_1} \left( (y + h_2)^{\alpha_2+1} \right)^{\alpha_2} - \left( x^{\alpha_1+1} \right)^{\alpha_1} \left( y^{\alpha_2+1} \right)^{\alpha_2} \right] dsdt \\
\leq \frac{C \max_{0 \leq (x, y) \leq 1} f(x, y)}{\alpha_1 \alpha_2} .
\]
From Lemma 1, we have

\[ N_\delta(Gr(\mathcal{T}^a f)) \leq 2mn + \frac{1}{\delta} \sum_{j=1}^{n} \sum_{i=1}^{m} R_{\mathcal{T}^a f}[A_{ij}] \leq 2mn \]

\[ + \frac{1}{\delta} \sum_{j=1}^{n} \sum_{i=1}^{m} \left( \frac{1}{(i+1)(j+1)} R_f[[0, \delta] \times [0, \delta]] \right) \]

\[ + \sum_{i=1}^{j} \frac{1}{(i+1)(j+1)} \left( R_f[[0, \delta] \times [(l-1)\delta, l\delta]] + R_f[[0, \delta] \times [l\delta, (l+1)\delta]] \right) \]

\[ + \frac{1}{\delta} \sum_{j=1}^{n} \sum_{i=1}^{m} \frac{C \max_{0 \leq (x, y) \leq 1} f(x, y)}{\alpha_1 \alpha_2} \]

\[ \leq \frac{1}{\delta} \left( C + \sum_{j=1}^{n} \sum_{i=1}^{m} \left( \frac{1}{(i+1)(j+1)} R_f[[0, \delta] \times [0, \delta]] \right) \right) \]

\[ + \sum_{i=1}^{j} \frac{1}{(i+1)(j+1)} \left( R_f[[0, \delta] \times [(l-1)\delta, l\delta]] + R_f[[0, \delta] \times [l\delta, (l+1)\delta]] \right) \]

\[ + \frac{1}{\delta} \sum_{j=1}^{n} \sum_{i=1}^{m} \frac{C \max_{0 \leq (x, y) \leq 1} f(x, y)}{\alpha_1 \alpha_2} \]

\[ \leq \frac{C}{\delta} \left( \sum_{j=0}^{n} \sum_{i=0}^{m} \frac{1}{(i+1)(j+1)} \right) \left( \sum_{j=0}^{n} \sum_{i=0}^{m} R_f[A_{ij}] \right) \]

\[ \leq \frac{C}{\delta} (\log m)(\log n) \sum_{j=0}^{n} \sum_{i=0}^{m} R_f[A_{ij}] \leq C (\log m)(\log n) N_\delta(Gr(f)). \]
Therefore,
\[
\log N_\delta(Gr(\mathcal{I}^\alpha f)) - \log \delta \leq \log \{C \log m \log n \log N_\delta(Gr(f)) \} - \log \delta
\leq \frac{\log C}{-\log \delta} + \frac{\log(\log m)}{-\log \delta} + \frac{\log(\log n)}{-\log \delta} + \frac{\log N_\delta(Gr(f))}{-\log \delta}.
\]

So, we obtain
\[
\dim_B Gr(\mathcal{I}^\alpha f, I \times J) = \lim_{\delta \to 0} \log N_\delta(Gr(\mathcal{I}^\alpha f)) - \log \delta
\leq \lim_{\delta \to 0} \left( \frac{\log C}{-\log \delta} + \frac{\log(\log m)}{-\log \delta} + \frac{\log(\log n)}{-\log \delta} + \frac{\log N_\delta(Gr(f))}{-\log \delta} \right)
\leq \lim_{\delta \to 0} \frac{\log N_\delta(Gr(f))}{-\log \delta} = \lim_{\delta \to 0} \frac{\log N_\delta(Gr(f))}{-\log \delta} = 2.
\]

So, inequality (3.7) holds. From inequalities (3.5) and (3.7), we get (3.4), completing the proof.

Corollary 1 Let \( 0 < \alpha_1 < 1, 0 < \alpha_2 < 1, \rho_1 < -1, \rho_2 < -1 \) and \( f \) be a continuous function of bounded variation on \([0, 1] \times [0, 1]\). Then
\[
\dim_B Gr(\mathcal{I}^\alpha f) = 2.
\]

Proof From Lemma 3.7 in [20], for a function \( f \) which is continuous and of bounded variation in Arzelá sense on \([0, 1] \times [0, 1]\), we have
\[
\dim_B Gr(f) = 2.
\]

So, from Theorem 1, directly we obtain
\[
\dim_B Gr(\mathcal{I}^\alpha f) = 2.
\]

This completes the proof.

Remark 3 In [20], Verma and Viswanathan proved that the fractional integral of mixed Katugampola type of a bounded variation function is again of bounded variation in Arzelá sense. By using this result they deduced that the box dimension of the fractional integral of mixed Katugampola type is 2. Their results are more on analytical aspects. But, we have proved that if \( f \) is a continuous function having box dimension 2, then the box dimension of the fractional integral of mixed Katugampola type of \( f \) is also 2. So, our results are more on dimensional aspects because we are using the dimension of function to compute the dimension of its fractional integral of mixed Katugampola type.
Now, we establish similar result for the mixed Hadamard fractional integral (2.6).

**Theorem 2** Let a non-negative function $f(x, y) \in C(I \times J)$ and $0 < \gamma_1 < 1$, $0 < \gamma_2 < 1$.

If

$$\dim B \text{Gr}(f, I \times J) = 2,$$

then, the box dimension of the mixed Hadamard fractional integral of $f(x, y)$ of order $\gamma = (\gamma_1, \gamma_2)$ exists and is equal to 2 on $I \times J$, as

$$\dim B \text{Gr}(\mathcal{H}_\gamma f, I \times J) = 2.$$ (3.9)

The proof of Theorem 2 follows as in Theorem 1.

**Lemma 3** ([19]) Suppose a continuous function $h : [c, d] \rightarrow \mathbb{R}$. Define a set $H = \{(x, y, h(y)) : x \in [a, b], y \in [c, d]\}$ and $a < b$. Then, $\overline{\dim}_B(H) \leq \overline{\dim}_B(\text{Gr}(h)) + 1$.

**Remark 4** Let $g : [a, b] \rightarrow \mathbb{R}$ and $h : [c, d] \rightarrow \mathbb{R}$ be two continuous maps. Now, define $g_1, g_2 : [a, b] \times [c, d] \rightarrow \mathbb{R}$ such that

$$g_1(x, y) = g(x) + h(y), \quad \text{and} \quad g_2(x, y) = g(x)h(y).$$

By using Lemma 3, we have $\overline{\dim}_B(\text{Gr}(g_1)) \leq \overline{\dim}_B(\text{Gr}(h)) + 1$ and $\overline{\dim}_B(\text{Gr}(g_2)) \leq \overline{\dim}_B(\text{Gr}(h)) + 1$.

In the following remark, we corroborate Theorem 2 by using existing results.

**Remark 5** Let $h : [a, b] \rightarrow \mathbb{R}$ be a continuous function having box dimension 1. We define a bivariate continuous function $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ such that $f(x, y) = h(x)$. Remind Definition 5 of the mixed Hadamard fractional integral:

$$(\mathcal{H}_\gamma f)(x, y) = \frac{1}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \int_a^x \int_c^y (\log \frac{x}{u})^{\gamma_1-1}(\log \frac{y}{v})^{\gamma_2-1} \frac{f(u, v)}{uv} dudv.$$ (2.6)

For $\gamma_2 = 1$, we get

$$(\mathcal{H}_\gamma f)(x, y) = \frac{1}{\Gamma(\gamma_1)} \int_a^x \int_c^y (\log \frac{x}{u})^{\gamma_1-1} \frac{f(u, y)}{uv} dudv.$$ (2.2)

From the definition of $f(x, y) = h(x)$, we obtain

$$(\mathcal{H}_\gamma f)(x, y) = \frac{\log(\frac{y}{c})}{\Gamma(\gamma_1)} \int_a^x (\log \frac{x}{u})^{\gamma_1-1} \frac{h(u)}{u} du.$$ (2.5)

Now, we have the following relation between the fractional integral of Hadamard type (2.2) and of mixed Hadamard type (2.6):

$$(\mathcal{H}_\gamma f)(x, y) = \log(\frac{y}{c}) (\mathcal{H}_\gamma h)(x).$$ (2.6)
where the Hadamard fractional integral defined by (2.2) is

$$(\mathcal{H}^\gamma h)(x) = \frac{1}{\Gamma(\gamma)} \int_a^x (\log \frac{x}{u})^{\gamma-1} \frac{h(u)}{u} \, du.$$  

From Remark 4, we have $\dim B \text{Gr}(\mathcal{H}^\gamma f) \leq \dim B \text{Gr}(\mathcal{H}^\gamma h) + 1$. Since, $\dim B \text{Gr}(h) = 1$, from [22] it follows that $\dim B \text{Gr}(\mathcal{H}^\gamma h) = 1$, and hence $\dim B \text{Gr}(\mathcal{H}^\gamma f) = 2$. This corroborates Theorem 2. In similar way, we can deduce for the mixed Katugampola fractional integral.

**Remark 6** The fractional integral of mixed Katugampola type is a kind of unification of the fractional integral of mixed Riemann-Liouville type and the fractional integral of mixed Hadamard type. Thus, our results are more general and hold for both fractional integral of mixed Riemann-Liouville type and the fractional integral of mixed Hadamard type.

## 4 Open Problems

It will be interesting to explore next the following problems:

(i) The box dimension of fractional integral of mixed Katugampola type of a Hölder continuous function.

(ii) Is the fractional integral of mixed Katugampola type of a bivariate fractal interpolation function (FIF) again bivariate FIF? This problem we have explored for the mixed Riemann-Liouville case in [3]. One can see that if $(\rho_1, \rho_2) = (0, 0)$, then the fractional integral of mixed Katugampola type converts into the fractional integral of mixed Riemann-Liouville type. Now, it will be interesting to see if it also holds for some $(\rho_1, \rho_2) \neq (0, 0)$.

(iii) Consider a modification of the mixed Katugampola type operator (2.5) in the style of “modified Katugampola integral” (2.4) from [7]. Then one could encompass also studies on two-dimensional R-L operators ([16,§24.2-§24.3]), and two-dimensional Erdélyi-Kober integrals (for a geometrical interpretation, see in Herrmann [5]).

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**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest.

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