A Hydrodynamic Approach
to Superconductivity

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Abstract
Recently Tsai et.al. (cond-mat/0406174) have used the renormalization group approach to study strong coupling superconductors without assuming a broken symmetry phase. We use the hydrodynamic formulation to study the same problem with the same intention. We recast the electron-phonon + electron-electron problem in the hydrodynamic language and compute the one-particle electron Green function at finite temperature. From this we extract the dynamical density of states at finite temperature and look for sign of a gap.

1 Introduction

Motivated by a series of recent experiments[1], [2],[3], [4], [5] Tsai et.al[6] have studied the question of reproducing the results of Eliashberg’s strong coupling theory of superconductivity without making an assumption of a broken symmetry phase. The need for this stems from fact that in real systems, there may be many sources of instability and there is no general guide (other than through experiments) to determine which of these instabilities dominate. Thus from a theoretical standpoint, as Tsai et.al. [6] argue, it is desirable to have a theory that is unbiased in the sense that it makes no assumption about the system being in a given phase. Rather all sources instability are treated on an equal footing and a sufficiently powerful theory should be able to pick out which of these phases dominate in what regions of the coupling constant/temperature plane. The RG approach of Shankar[7] as applied to the problem of strongly coupled superconductors by Tsai et.al. [6] is one such. Here we present an alternative to this interesting and important work, namely the hydrodynamic approach. In the hydrodynamic approach, we simply recast the electron-phonon
+ electron-electron problem in terms of the electron’s hydrodynamic variables namely the density fluctuations and the conjugate namely the velocity potential. The crucial fermionic nature of the electron is captured by a nontrivial phase functional that enters into the framework in the lagrangian formulation. The hamiltonian is identical to the Dashen-Sharp formula in terms of currents and densities.

Unlike the RG approach\(^1\), which requires dividing the Fermi surface up into patches, we do not have to pay conscious attention to the Fermi surface. The properties of the free Fermi theory are automatically encoded in the phase functional. However our approach is by no means exact. The phase functional that encodes Fermi statistics has to be determined by expanding in powers of density fluctuations. We retain only the leading linear term, thereby implying that three-body density correlations are implicitly ignored. This means we have to work in the high density limit, where the Migdal’s theorem applies. This is exactly the regime in which Tsai et. al. operate. However, our approach can be generalised in principle to regimes of lower density as well by including three-body terms and so on (but the calculations are obviously harder to carry out in practice). Thus the purpose of the present article is to highlight the usefulness of the hydrodynamic approach by comparing favorably with the results of well-established methods authored by famous physicists\(^2\).

2 The Hydrodynamic Action

In an earlier preprint\(^[8]\), we recall that we had written down the action for electrons coupled to phonons in the hydrodynamic language. We rederive that here for the sake of completeness. We assume following Tsai et.al.\(^[6]\) that the electron-electron interaction is short-ranged as is the electron-phonon interaction. In their work, the bare electron-electron coupling constant is denoted by \(u_0\) and the bare electron-phonon coupling \(g\). We try to follow the notation of Tsai et.al. as closely possible. The full action may be written down as follows.

\[
S = \int_0^{-i\beta} dt \sum_{\sigma} \int d^3 x \ \psi^\sigma(x\sigma, t) \left( i\partial_t + \frac{\nabla^2}{2m_e} \right) \psi(x\sigma, t) + \int_0^{-i\beta} dt \sum_q \phi_q^\sigma(t) (i\partial_t - w_q) \phi_q(t)
\]

\[
+ \frac{q}{\sqrt{V}} \int_0^{-i\beta} dt \sum_q \rho_q(t) (\phi_q(t) + \phi_{-q}^*(t)) - \frac{u_0}{2V} \int_0^{-i\beta} dt \sum_q \rho_q(t) \rho_{-q}(t) \quad (1)
\]

The prime over the summation \(q\) indicates that we restrict the values of \(|q| < \Lambda_D\), the Debye cutoff. The free part may be recast in the hydrodynamic language

\(^1\)which the author claims no expertise of
\(^2\)sardonic tone unintentional
as follows.

\[ S_{\text{free}} = \int \left( \rho \partial_t \Pi - \rho \partial_t \Phi_e - \frac{(\nabla \rho)^2}{4m_e} + \rho (\nabla \Pi)^2 \right) \]  

(2)

Here the phase function \( \Phi_e(\rho; x\sigma) \) encodes Fermi statistics. Without it, we
would be describing bosons rather than fermions. Also the Fermi current is given
by the hydrodynamic expression which implicitly defines \( \Pi \) namely,
\( J = -\rho \nabla \Pi \).

The phase functional \( \Phi_e \) has to be fixed by making contact with the correlation
functions of the free Fermi theory. This has been done in an earlier preprint and
we may write down the leading term in the expansion in powers of the density
fluctuations.

\[ \Phi_e(\rho; x\sigma) = \sum_{q n} e^{i q \cdot x z n t} C(q, n\sigma) \rho_{-q\sigma, -n} \]  

(3)

\[ \beta z_n C(q, n\sigma) = \frac{1}{2 \langle \rho_{q\sigma, n} \rho_{-q\sigma, -n} \rangle_0} - \frac{\beta z_n^2}{2 N^0 \epsilon_q} - \frac{\beta \epsilon_q}{2 N^0} \]  

(4)

Here \( N^0 \) is the total number of electrons and,
\[ \langle \rho_{q\sigma}(t) \rho_{-q\sigma}(t') \rangle_0 = e^{iz_n(t - t')} \langle \rho_{q\sigma, n} \rho_{-q\sigma, -n} \rangle_0 \]  

(5)

is the density correlation function of the free Fermi theory. The slow part of the
field variable is given by,
\[ \psi_{\text{slow}}(x\sigma, t) = e^{2i \sum_q e^{i q \cdot x z n t} C(q, n\sigma) \rho_{-q\sigma, -n} e^{-i \sum_q e^{i q \cdot x z n t} X_{q\sigma, n}} \]  

(6)

Also \( z_n = \frac{2\pi n}{\beta} \) is the bosonic Matsubara frequency. Furthermore, the mysterious factor of two in the exponential is to ensure that the exponents in the one-dimensional case come out right. By expanding in powers of the density fluctuations and retaining the leading terms we may write the full action as follows.

\[ S = \sum_{q \sigma, n} (-i \beta z_n) \rho_{q, n, \sigma} X_{q, n, \sigma} + \sum_{q \sigma, n} (i \beta z_n) C(q, n, \sigma) \rho_{q, n, \sigma} \rho_{-q, -n, \sigma} \]  

\[ + \frac{i \beta N^0}{2} \sum_{q \sigma, n} \epsilon_q X_{q, n, \sigma} X_{-q, -n, \sigma} + (-i \beta) \sum_{q n} \phi_{q, n}^* (iz_n - w_q) \phi_{q, n} \]  

\[ - \frac{i \beta g}{\sqrt{V}} \sum_{q n} \rho_{q, n} (\phi_{q, n} + \phi_{-q, -n}^*) + \frac{i \beta u_0}{2V} \sum_{q n} \rho_{q, n} \rho_{-q, -n} \]  

(7)

Here \( \rho_{q, n} = \rho_{q^\uparrow, n} + \rho_{q^\downarrow, n} \). Following Tsai et.al. we choose the phonons to be Einstein-like namely constant dispersion \( w_q = w_E \). Since the action is purely
quadratic in the phonons, we may integrate it out and write down an effective action for
the electrons.

\[ S_{\text{eff}} = \sum_{q,n,\sigma} (-i\beta z_n) \rho_{q,n,\sigma} X_{q,n,\sigma} + \sum_{q,n,\sigma} (i\beta z_n) C(q,n,\sigma) \rho_{q,n,\sigma} \rho_{q,-n,\sigma} \]

\[ + \frac{i\beta N^0}{2} \sum_{q,n,\sigma} \epsilon_q X_{q,n,\sigma} X_{-q,-n,\sigma} + \frac{i\beta}{2V} \sum_{q,n} \left( u_0 - \frac{2g^2 w_E}{w_E^2 + z_n^2} \right) \rho_{q,n} \rho_{-q,-n} \]

Now we wish to compute the propagator. More specifically, we wish to simply compute the momentum distribution and from that extract the quasiparticle residue at finite temperature. Tsai et al. use this procedure to ascertain the transition temperature for the metal-superconductor transition. However the crucial point is that the vanishing of the quasiparticle residue is a necessary but not sufficient condition for the system to be a superconductor. It could also be an insulator. Thus either one has to demonstrate the divergence of the d.c. conductivity or more illuminatingly demonstrate phase coherence. In other words the nonvanishing nature of Yang’s offdiagonal long-range order correlation function. However since we know from prior experience, that this is a superconducting transition, we shall content ourselves in just computing the momentum distribution. After integrating out the phonons we may write,

\[ S_{\text{eff}} = \sum_{q,n} (-i\beta z_n) \rho_{q,n,\uparrow} X_{q,n,\uparrow} + \sum_{q,n} (-i\beta z_n) \rho_{q,n,\downarrow} X_{q,n,\downarrow} \]

\[ + \sum_{q,n,\sigma} (i\beta z_n) C(q,n,\sigma) \rho_{q,n,\uparrow} \rho_{-q,-n,\downarrow} + \sum_{q,n,\sigma} (i\beta z_n) C(q,n,\sigma) \rho_{q,n,\downarrow} \rho_{-q,-n,\uparrow} \]

\[ + \frac{i\beta N^0}{2} \sum_{q,n} \epsilon_q X_{q,n,\uparrow} X_{-q,-n,\uparrow} + \frac{i\beta N^0}{2} \sum_{q,n} \epsilon_q X_{q,n,\downarrow} X_{-q,-n,\downarrow} \]

\[ + \frac{i\beta}{2V} \sum_{q,n} \left( u_0 - \frac{2g^2 w_E}{w_E^2 + z_n^2} \right) \left( \rho_{q,n} \rho_{-q,-n} + 2\rho_{q\uparrow,n} \rho_{-q\uparrow,-n} + \rho_{q\downarrow,n} \rho_{-q\downarrow,-n} \right) \]

The full propagator may be written as follows.

\[ \langle T \psi_{\text{slow}}(x, t) \psi_{\text{slow}}^\dagger(x', t') \rangle = \left\{ e^{i \sum_{q,n} \left( e^{i q \cdot x} e^{i q \cdot x'} + e^{i q \cdot x'} e^{i q \cdot x} \right) (2C(q,n,\uparrow) \rho_{-q\uparrow,-n} - X_{q\uparrow,n})} \right\} \]

\[ = e^{-\frac{1}{2} \sum_{q,n} \left( 2 - e^{i q \cdot (x-x')} - e^{i q \cdot (x'-x)} \right) e^{i q \cdot (t-t')} E(q,n) \}

\[ E(q,n) = (4C(q,n,\uparrow) C(-q,-n,\uparrow) \rho_{q\uparrow,n} \rho_{-q\uparrow,-n} - 4C(-q,-n,\uparrow) \langle X_{q\uparrow,n} \rangle) \]

\[ + \langle X_{q\uparrow,n} X_{-q\uparrow,-n} \rangle \]

(11)
Now we make use of the trick outlined earlier namely multiply and divide by the free propagator, in the denominator use the the bosonized version and in the numerator use the one obtained from elementary considerations. Thus we may write (see appendices for definitions notation e.t.c.),

\[
\frac{\langle T \psi(x↑,t)\psi(x↑′,t′) \rangle}{\langle T \psi(x↑,t)\psi(x↑′,t′) \rangle_0} = e^{-\frac{1}{2} \sum_{q,n} \left( -e^{i\epsilon_q(x-x')} e^{z_n(t-t')} - e^{i\epsilon_q(x-x')} e^{z_n(t-t')} \right) E_{di,ff}(q,n)}
\]

Thus we have an asymptotically exact formula for the propagator provided we evaluate \( E \). To this end we first integrate out the down spin variables and write,

\[
S_{eff,↑} = \sum_{q,n} (-i\beta z_n) \rho_{q,n,↑} X_{q,n,↑} + \frac{i\beta N^0}{2} \sum_{q,n} \epsilon_q X_{q,n,↑}X_{-q,-n,↑} + \sum_{q,n} G(q,n) \rho_{q↑,n} \rho_{-q↑,-n}
\]

where,

\[
G(q,n) = \frac{i\beta z_n^2}{2N^0 \epsilon_q} + (i\beta z_n) C(q,n,↓) + \frac{i\beta}{2V} \left( u_0 - \frac{2g^2 w_E}{w^2_E + z_n^2} \right)
\]

\[
G↑(q,n) = (i\beta z_n) C(q,n,↑) + \frac{i\beta}{2V} \left( u_0 - \frac{2g^2 w_E}{w^2_E + z_n^2} \right) + \frac{\beta^2}{V^2} \left( u_0 - \frac{2g^2 w_E}{w^2_E + z_n^2} \right)^2 \]

We may now read off the correlation functions.

\[
\langle \rho_{q,n,↑} \rho_{-q,-n} \rangle = \frac{1}{\beta z_n^2 \frac{1}{N^0 \epsilon_q} - 2iG↑(q,n)}
\]

\[
\langle X_{q,n,↑} X_{-q,-n} \rangle = \frac{1}{\beta z_n^2 \frac{1}{N^0 \epsilon_q} + i\frac{(\beta z_n)^2}{2G↑(-q,-n)}}
\]

\[
\langle \rho_{q↑,n} X_{q↑,n} \rangle = \frac{(\beta z_n)^2}{2\epsilon_q \beta N^0 G↑(q,n) - (\beta z_n)^2}
\]

\[
\langle \rho_{q,n,↑} \rho_{-q,-n} \rangle_0 = \frac{1}{\beta z_n^2 \frac{1}{N^0 \epsilon_q} + (2\beta z_n) C(q,n,↑)}
\]

\[
\langle X_{q,n,↑} X_{-q,-n} \rangle_0 = \frac{1}{\beta z_n^2 \frac{1}{N^0 \epsilon_q} + \frac{(\beta z_n)}{2 C(q,n,↑)}}
\]

\[
\langle \rho_{q↑,n} X_{q↑,n} \rangle_0 = \frac{-1}{2\epsilon_q \beta N^0 \frac{1}{C(q,n,↑)} + \beta z_n}
\]
We use the current algebra constraint to determine the approximate long-wavelength nature of the correlation functions of the free theory. This means the unknown $C$ has to be fixed so that the current-current correlation are related in the usual manner to density-density correlation functions.

\[
j_{q^\uparrow,n} = X_{-q^\uparrow,-n}(\frac{N^0}{2}q) \quad (22)
\]

\[
\langle j_{q^\uparrow,n} \cdot j_{-q^\uparrow,-n} \rangle = \frac{(N^0)^2 q^2}{4} \langle X_{q^\uparrow,n}X_{-q^\uparrow,-n} \rangle = k_F^2 \langle \rho_{q^\uparrow,n}\rho_{-q^\uparrow,-n} \rangle \quad (23)
\]

This means,

\[
\frac{k_F^2}{\frac{\beta z^2}{N\epsilon_q} + (2\beta z_n)C(q,n,\uparrow)} = \frac{(N^0)^2 q^2}{4\beta N^0 \epsilon_q + \frac{(2\beta z_n)}{C(q,n,\uparrow)}} \quad (24)
\]

Thus we may deduce,

\[
C(q,n,\uparrow) \approx \frac{2k_F^2}{z_n N^0 (2m)} \quad (25)
\]

This in turn means,

\[
\langle \rho_{q,n,\uparrow}\rho_{-q,-n} \rangle_0 = \frac{1}{\beta \frac{z^2}{n} + v_F q^2} \quad (26)
\]

Just to verify that this is sensible, we compute the static density-density correlation at zero temperature which we know is $(N^0/2)q/(2k_F)$.

\[
\sum_n \langle \rho_{q,n,\uparrow}\rho_{-q,-n} \rangle_0 = \int_{-\infty}^{\infty} dn \frac{N^0 q^2}{2m \beta \frac{z^2}{n} + v_F^2 q^2} = \frac{N^0 q}{2 \frac{2}{k_F}} \quad (27)
\]

as required. It is important to point out that there is no such simple connection between density-density and current-current correlation functions for interacting systems, since the four-point functions are not obligated to resemble the non-interacting values that enable the correspondence. Therefore, in particular one should not look to similarly relate Eq.(16) and Eq.(17). Finally, we wish to ascertain that the last correlation function is consistent with current algebra. We know that,

\[
\langle \rho_{q^\uparrow,n}X_{q^\uparrow,n} \rangle_0 = \frac{-z_n}{\beta v_F^2 q^2 + \beta z_n^2} \quad (28)
\]

In other words,

\[
-2i\frac{q}{N^0} \langle \rho_{q^\uparrow,n}j_{-q^\uparrow,-n} \rangle_0 = \frac{-z_n}{\beta v_F^2 q^2 + \beta z_n^2} \quad (29)
\]
\[
\langle \rho_{\uparrow}\mathbf{\cdot}\mathbf{j}_{\uparrow}\rangle_0 = \sum_{\mathbf{k}} \mathbf{k} \cdot \mathbf{q} n_F(\mathbf{k} + \mathbf{q}/2)(1 - n_F(\mathbf{k} - \mathbf{q}/2)) = \sum_{\mathbf{k}} \mathbf{k} \cdot \mathbf{q} n_F(\mathbf{k} + \mathbf{q}/2) = -\frac{N^0 q^2}{4}
\]

Thus we must have,
\[
(- \frac{2}{i q^2 N^0})(- \frac{N^0 q^2}{4}) = \frac{v_F q}{2 i v_F q}
\]
an identity. Now we may proceed to evaluate the full propagator.

3 Dynamical Density of States

Here we compute the full propagator (see appendices for more details and definitions of the various terms). From there we extract the dynamical density of states. An examination of this should tell us whether or not a gap is present. The one-particle spectral function is given by,

\[
2\pi A(\mathbf{k}, \omega) = \int_{-\infty}^{\infty} dt \ e^{i\omega t} \left( \langle \mathbf{c}_{\mathbf{k}\uparrow}(t) \mathbf{c}_{\mathbf{k}\uparrow}^\dagger(0) \rangle + \langle \mathbf{c}_{\mathbf{k}\uparrow}^\dagger(0) \mathbf{c}_{\mathbf{k}\uparrow}(t) \rangle \right) \quad (32)
\]

We define the dynamical density of states as,

\[
D(\omega) = \frac{1}{V} \sum_{\mathbf{k}} A(\mathbf{k}, \omega)
\]

Thus the relevant quantities are the unequal-time, equal-space Green functions : \( \left\langle \mathbf{T} \psi(\mathbf{x} \uparrow, t) \psi^\dagger(\mathbf{x} \uparrow, t') \right\rangle \). From the appendices we find that we may simplify these in two spatial dimensions,

\[
\left\langle \psi(\mathbf{x} \uparrow, t) \psi^\dagger(\mathbf{x} \uparrow, t') \right\rangle = e^{-F_>(t - t')}
\]

\[
\left\langle \psi^\dagger(\mathbf{x} \uparrow, t) \psi(\mathbf{x} \uparrow, t') \right\rangle = e^{-F_<(t - t')}
\]

\[
F_>(t - t') \approx \left( \tilde{u}_r - k_F \right) \frac{\pi \rho^0}{\beta} i \Lambda_D(t - t')
\]

\[
F_<(t - t') \approx \left( \tilde{u}_r - k_F \right) \frac{\pi \rho^0}{\beta} i \Lambda_D(t' - t)
\]

\[
\tilde{u}_r - k_F \approx \frac{m \rho^0 (u_0 - 2 g^2/w_F)}{2 k_F}
\]

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To first order, we take the zero temperature free Green functions.

\[
\langle \psi(x \uparrow, t) \psi(x \uparrow, 0) \rangle_0 = \frac{2m\pi}{(2\pi)^2} \frac{e^{-i\epsilon_F t}}{it}
\]

\[
\langle \psi^\dagger(x \uparrow, 0) \psi(x \uparrow, t) \rangle_0 = \frac{2m\pi}{(2\pi)^2} \frac{1 - e^{-i\epsilon_F t}}{it}
\]

In this case,

\[
2\pi D_0(\omega) = \int_{-\infty}^{\infty} \frac{dt}{2\pi i} e^{i\omega t} \frac{m}{t} = m \theta(\omega)
\]

Thus,

\[
D_0(\omega) = \frac{m}{2\pi} \theta(\omega)
\]

as required. In general,

\[
\langle \psi(x \uparrow, t) \psi^\dagger(x \uparrow, 0) \rangle = \frac{2m\pi}{(2\pi)^2} \frac{e^{-i\epsilon_F t}}{it} e^{-\frac{(k_x - k_F)^2}{2\rho \beta}} i\Lambda_D t
\]

\[
\langle \psi^\dagger(x \uparrow, 0) \psi(x \uparrow, t) \rangle = \frac{2m\pi}{(2\pi)^2} \frac{1 - e^{-i\epsilon_F t}}{it} e^{-\frac{(k_x - k_F)^2}{2\rho \beta}} i\Lambda_D t
\]

Therefore,

\[
2\pi D_0(\omega) = \int_{-\infty}^{\infty} \frac{dt}{2\pi i} e^{i(\omega - \Delta) t} \frac{m}{t} = \frac{m}{2\pi} \theta(\omega - \Delta)
\]

where,

\[
\Delta = k_B T \left\{ \frac{m(2g^2/w_E - u_0)\Lambda_D}{2\pi k_F} \right\}
\]

We may provisionally identify \(\Delta\) with a gap. Thus we find that there is no superconductivity unless the (screened) phonon strength exceeds the (screened) electron-electron repulsion. Thus we have the (necessary) condition for superconductivity \(g^2 > u_0 w_E/2\). In particular we may also conclude that for \(g^2 = 0\) and \(u_0 < 0\) also we have a gap and hence possibly also superconductivity. Strictly speaking we have to compute the d.c. conductivity also and show that it diverges in order to be convinced that the transition is superconducting (or demonstrate phase coherence). But since we already know this to be a superconducting transition we shall content ourselves with doing the bare minimum as do Tsai et al.

**This preprint is Incomplete**

I seem to be making some errors in the computation of integrals. Perhaps some more knowledgeable people can
help me out. In any case this is meant to show members of hiring committees that I am desperately trying to do something important.

4 Appendix A

Here we provide some details of the computations of the propagator.

\[ G(\mathbf{q}, n) = \frac{2mi\beta z_n^2}{2N^0q^2} + \frac{2i\beta k^2_{\rho}}{N^0(2m)} + \frac{i\beta}{2V} \left( u_0 - \frac{2g^2wE}{w_E^2 + z_n^2} \right) \]  \hspace{1cm} (44)

\[ G_\uparrow(\mathbf{q}, n) = \frac{(i\beta z_n)}{z_nN^0(2m)} \left( \frac{2k^2_{\rho}}{2} \right) + \frac{i\beta}{2V} \left( u_0 - \frac{2g^2wE}{w_E^2 + z_n^2} \right) + \frac{\beta^2}{V^2} \left( \frac{u_0 - \frac{2g^2wE}{w_E^2 + z_n^2}}{4G(-\mathbf{q}, -n)} \right)^2 \]  \hspace{1cm} (45)

\[ G_\uparrow(\mathbf{q}, n) = \frac{(i\beta(-2g^2\rho^0wE(2k^2_{\rho}q^2 + m^2z_n^2)) + (w_E^2 + z_n^2)(2k^2_{\rho}q^2(m\rho^0u_0 + k^2_{\rho}) + m^2(m\rho^0u_0 + 2k^2_{\rho}z_n^2)))}{(mN^0(-2g^2\rho^0q^2wE + (w_E^2 + z_n^2)(q^2(m\rho^0u_0 + 2k^2_{\rho}) + m^2z_n^2)))} \]  \hspace{1cm} (46)

\[ G_\uparrow(\mathbf{q}, n) = \frac{i\beta(-2g^2\rho^0wE(2k^2_{\rho}q^2 + m^2z_n^2)) + (w_E^2 + z_n^2)(2v^2_{\rho}q^2(m\rho^0u_0 + k^2_{\rho}) + (m\rho^0u_0 + 2k^2_{\rho}z_n^2))}{-2g^2\rho^0v^2_{\rho}wE + (w_E^2 + z_n^2)(q^2(m\rho^0u_0 + 2k^2_{\rho}) + 2z_n^2))} \]  \hspace{1cm} (47)

\[ \langle \rho_{\mathbf{q}, n\uparrow\rho_{-\mathbf{q}, -n\uparrow}} \rangle = \frac{1}{\frac{(\beta z_n)}{N^0\epsilon_{\mathbf{q}}} - 2iG_\uparrow(\mathbf{q}, n)} \]  \hspace{1cm} (48)

\[ \langle X_{\mathbf{q}, n\uparrow X_{-\mathbf{q}, -n\uparrow}} \rangle = \frac{1}{\beta N^0\epsilon_{\mathbf{q}} + i \frac{2G_\uparrow(-\mathbf{q}, -n)}{(\beta z_n)^2}} \]  \hspace{1cm} (49)

\[ \langle \rho_{\mathbf{q}\uparrow n\downarrow X_{\mathbf{q}\uparrow n\downarrow}} \rangle = \frac{(\beta z_n)}{2i\epsilon_{\mathbf{q}}\beta N^0 G_\uparrow(\mathbf{q}, n) - (\beta z_n)^2} \]  \hspace{1cm} (50)

\[ \langle \rho_{\mathbf{q}, n\uparrow\rho_{-\mathbf{q}, -n\uparrow}} \rangle = \frac{N^0\epsilon_{\mathbf{q}}}{\beta} \left( \frac{z_n^2 + 2v^2_{\rho}wE(2k^2_{\rho}q^2 + m^2z_n^2) - 2w_E^2 + z_n^2)(2k^2_{\rho}q^2(mN^0u_0 + k^2_{\rho}V) + m^2(mN^0u_0 + 2k^2_{\rho}V)z_n^2)}{m(2g^2\rho^0q^2wE - (w_E^2 + z_n^2)(q^2(mN^0u_0 + 2k^2_{\rho}V) + m^2Vz_n^2))} \right)^{-1} \]  \hspace{1cm} (51)

\[ \langle \rho_{\mathbf{q}, n\uparrow\rho_{-\mathbf{q}, -n\uparrow}} \rangle_0 = \frac{N^0\epsilon_{\mathbf{q}}}{\beta} \left( z_n^2 + v^2_{\rho}q^2 \right)^{-1} \]  \hspace{1cm} (52)
Consider the following identity from complex analysis.

\[ E_\text{diff}(q,n) = e^{-\frac{\beta(-16m^4n_1^4\pi^4 + 8\beta^2k_F^2m^2n_1^2\pi^2q^2 + \beta^4k_F^4q^4)(4n_1^2\pi^2u_0 + \beta^2w_E(-2g^2 + u_0w_E))}{4n_1^2\pi^2(4m^2n_1^2\pi^2 + \beta^2k_F^2q^2)(16m^2n_1^4\pi^4V + \beta^4k_F^4q^4(-2g^2mN_0 + mN_0u_0w_E + k_F^2Vw_E) + 4\beta^2n_1^2\pi^2(q^2(mN_0u_0 + k_F^2V) + m^2Vw_E))}} \]

(53)

\[
\frac{\langle T \psi(x \uparrow, t)\psi^\dagger(x' \uparrow, t') \rangle}{\langle T \psi(x \uparrow, t)\psi^\dagger(x' \uparrow, t') \rangle_0} = e^{-\frac{1}{2}\sum_{n_0} \left(2 - e^{iq_n(x-x')}e^{iz_n(t-t')} - e^{iq_n'(x-x')e^{iz_n(t-t')}}\right)}(E(q,n)-E_0(q,n))
\]

(54)

Consider the following identity from complex analysis.

\[
\oint_{c.p.} \frac{dz}{e^{2\pi i z} - 1} f(z) = \sum_{n=\infty} f(n) + \sum_{m=\text{alt.poles.of.f}} \frac{f_r(z_m)}{e^{2\pi i z_m} - 1}
\]

(55)

\[
f_r(z_m) = (2\pi i) L_{z\rightarrow z_m} (z-z_m)f(z)
\]

(56)

If \( f \) falls off fast enough,

\[
\sum_{n=\infty} f(n) = -\sum_{m=\text{alt.poles.of.f}} \frac{f_r(z_m)}{e^{2\pi i z_m} - 1}
\]

(57)

We note that the pathological \( n = 0 \) should be excluded from consideration. The reasons for this are not entirely clear but doing so enables the right exponent of the Hubbard model to be recovered[9]. When \( m = 1, 2 \) we have,

\[
f_r(z_{1,2}) = \frac{(2\pi i)\beta(-16m^4n_1^4\pi^4 + 8\beta^2k_F^2m^2n_1^2\pi^2q^2 + \beta^4k_F^4q^4)(4n_1^2\pi^2u_0 + \beta^2w_E(-2g^2 + u_0w_E))}{4n_1^2\pi^2(8m^2n_1^2\pi^2)(16m^2n_1^4\pi^4V + \beta^4k_F^4q^4(-2g^2mN_0 + mN_0u_0w_E + k_F^2Vw_E) + 4\beta^2n_1^2\pi^2(q^2(mN_0u_0 + k_F^2V) + m^2Vw_E))}
\]

(58)

\[
z_1 = \frac{i\beta k_F q}{2m\pi}
\]

(59)

\[
z_2 = -\frac{i\beta k_F q}{2m\pi}
\]

(60)

\[
f_r(z_{3,4}) = \frac{(2\pi i)\beta(-16m^4n_3^4\pi^4 + 8\beta^2k_F^2m^2n_3^2\pi^2q^2 + \beta^4k_F^4q^4)(4n_3^2\pi^2u_0 + \beta^2w_E(-2g^2 + u_0w_E))}{4n_3^2\pi^2(4m^2n_3^2\pi^2 + \beta^2k_F^4q^2)(64m^2n_3^4\pi^4V + \beta^4k_F^4q^4(-2g^2mN_0 + mN_0u_0w_E + k_F^2Vw_E) + 4\beta^2n_3^2\pi^2(q^2(mN_0u_0 + k_F^2V) + m^2Vw_E))}
\]

(61)

\[
z_3 = \frac{i\beta w_E}{2\pi}
\]

(62)

\[
z_4 = -\frac{i\beta w_E}{2\pi}
\]

(63)
\[ f_r(z_{5.6}) = \frac{(2\pi i)\beta(-16m^4n_5^2\pi^4 + 8\beta^2k_F^2m^2n_5^2\pi^2q^2 + \beta^4k_F^4q^4)(4n_5^2\pi^2u_0 + \beta^2w_E(-2g^2 + u_0w_E))}{4n_5^2\pi^2(4m^2n_5^2\pi^2 + \beta^2k_F^2q^2)(64m^2n_5^2\pi^4V + 8\beta^2n_5^2\pi^2(q^2(mN^0u_0 + k_F^2V) + m^2Vw_E^2))} \]

\[ z_5 = i\frac{\beta q}{2\pi m\sqrt{w_E}}(-2g^2m\rho^0 + m\rho^0u_0w_E + k_F^2w_E)^* = \frac{i\beta}{2\pi}v_+q \]

\[ z_6 = -i\frac{\beta q}{2\pi m\sqrt{w_E}}(-2g^2m\rho^0 + m\rho^0u_0w_E + k_F^2w_E)^* = -\frac{\beta}{2\pi}v_+q \]

\[ f_r(z_1) = -\frac{k_F}{N^0q} \]

\[ f_r(z_2) = \frac{k_F}{N^0q} \]

\[ f_r(z_3) = \frac{g^2}{Vw_E^2} \]

\[ f_r(z_4) = -\frac{g^2}{Vw_E^2} \]

\[ f_r(z_5) = -\frac{q}{(2\rho^0V\sqrt{w_E}(-2g^2m\rho^0 + (k_F^2 + m\rho^0u_0)w_E)^*)} = \frac{\hat{u}_r}{N^0q} \]

\[ f_r(z_6) = -\frac{q}{(2\rho^0V\sqrt{w_E}(-2g^2m\rho^0 + (k_F^2 + m\rho^0u_0)w_E)^*)} = -\frac{\hat{u}_r}{N^0q} \]

\[ \left\langle \psi(x^\uparrow, t^\uparrow)\psi(x^\uparrow, t) \right\rangle \bigg/ \left\langle \psi(x^\uparrow, t^\uparrow)\psi(x^\uparrow, t) \right\rangle \bigg|_0 = e^{-\sum_n (1-\cos[q_n(x-x')])} \sum_n f(q_n) \]

\[ \sum_{n=-\infty}^{\infty} E_{diff}(q, n) = -\frac{k_F}{e^{-\beta v_Fq} - 1} - \frac{k_F}{e^{\beta v_Fq} - 1} - \frac{g^2}{Vw_E^2} - 1 - \frac{g^2}{e^{\beta w_E} - 1} - \frac{\hat{u}_r}{N^0q} - \frac{\hat{u}_r}{e^{\beta v_q} - 1} \]

In the noninteracting limit, this sum is zero since \( v_+ \to v_F \) and \( \hat{u}_r \to k_F \) and \( g^2 \to 0 \).
Appendix B

Here we calculate the equal-space unequal-time Green functions.

\[
\langle T \psi(x \uparrow, t) \overline{\psi}(x \uparrow, t') \rangle = e^{-\frac{1}{2} \sum_{q_n} \left( 2 - e^{z_n(t-t')} - e^{z_n(t'-t)} \right)(E(q,n) - E_0(q,n))} \tag{75}
\]

Let us first assume \(\text{Im}[t-t'] < 0\). This means,

\[
\langle \psi(x \uparrow, t) \overline{\psi}(x \uparrow, t') \rangle = e^{-\frac{1}{2} \sum_{q < \Lambda_D} F_\geq(q,t-t')} \tag{76}
\]

where,

\[
F_\geq(q,t-t') = \frac{1}{2} \sum_n \left( 2 - e^{z_n(t-t')} - e^{z_n(t'-t)} \right) E_{\text{diff}}(q,n) \tag{77}
\]

\[
F_\geq(q,t-t') = \left( -\frac{g^2}{\Delta_{q,E}} - \frac{g^2}{\Delta_{q,E}} \right) \left( 1 - e^{-i\nu F(q)(t-t')} \right) + \left( -\frac{\gamma}{N q} \right) \left( 1 - e^{-i\tau q(t-t')} \right) \tag{78}
\]

Next we assume \(\text{Im}[t-t'] > 0\) then,

\[
\langle \overline{\psi}(x \uparrow, t) \psi(x \uparrow, t') \rangle = e^{-\frac{1}{2} \sum_{q < \Lambda_D} F_\leq(q,t-t')} \tag{79}
\]

where,

\[
F_\leq(q,t-t') = \frac{1}{2} \sum_n \left( 2 - e^{z_n(t-t')} - e^{z_n(t'-t)} \right) E_{\text{diff}}(q,n) \tag{80}
\]

\[
F_\leq(q,t-t') = \left( -\frac{g^2}{\Delta_{q,E}} - \frac{g^2}{\Delta_{q,E}} \right) \left( 1 - e^{i\nu F(q)(t-t')} \right) + \left( -\frac{\gamma}{N q} \right) \left( 1 - e^{i\tau q(t-t')} \right) \tag{81}
\]
\begin{align}
F_\geq(q, t' - t) &\approx -\frac{2k_F/v_F}{N_0\beta q^2} \left(1 - e^{-iv_F q(t' - t)}\right) + \frac{2\tilde{u}_r/v_\ast}{N_0\beta q^2} \left(1 - e^{-iv_\ast q(t' - t)}\right) \\
&\quad + \left(\frac{g^2}{V w_E}\right) \left(1 - e^{-iw_E (t' - t)}\right)
\end{align}

(82)

First we focus on two dimensions. We have to make use of the temperature constraint repeatedly namely $v_F \Lambda_D \ll k_B T \ll w_E$. Then,

\begin{align}
F_\geq(t' - t) &\approx \frac{1}{2\pi} \left(-\frac{2k_F}{v_F} - \frac{2\tilde{u}_r/v_\ast}{N_0\beta}\right) \int_{0}^{\Lambda_D} dq \frac{1 - e^{-iv_F q(t' - t)}}{q} + \frac{1}{2\pi} \int_{0}^{\Lambda_D} dq \frac{1 - e^{-iv_\ast q(t' - t)}}{q} \\
&\quad + \left(\frac{g^2}{V w_E}\right) \left(\frac{\pi \Lambda_D^2}{2(2\pi)^2}\right)
\end{align}

(84)

Since $t' - t \sim 1/(k_B T)$ the temperature constraint tells us that we may expand in powers of small $q$.

\begin{align}
F_\geq(t' - t) &\approx \frac{(\tilde{u}_r - k_F)}{\pi \rho^2 \beta} i\Lambda_D (t' - t) + \frac{g^2}{V w_E} \frac{\pi \Lambda_D^2}{(2\pi)^2}
\end{align}

(85)

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