Multivector Contractions Revisited, Part I

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Abstract

We reorganize, simplify and expand the theory of contractions or interior products of multivectors, and related topics like Hodge star duality. Many results are generalized and new ones are given, like: geometric characterizations of blade contractions and regressive products, higher-order graded Leibniz rules, determinant formulas, improved complex star operators, etc. Different contractions and conventions found in the literature are discussed and compared, in special those of Clifford Geometric Algebra. Applications of the theory are developed in a follow-up paper.

Keywords: Grassmann exterior algebra, Clifford geometric algebra, contraction, interior product, inner derivative, insertion operator, Hodge star

MSC: 15A66, 15A75

1 Introduction

Contractions, interior products or inner derivatives of multivectors or forms date back, in essence, to Grassmann [15], and have since been used in Differential Geometry [1, 14], Physics [17, 33], Computer Science [2, 9], etc. Still, they are often seen as somewhat obscure operations.

A difficulty is the various kinds of contraction (left, right, for vectors, multivectors, forms, tensors), some very abstract [3]. Contraction by vectors is prevalent, even when multivectors could be of great use, as in [42]. Contraction between multivectors is simpler than with forms, but needs an inner product. Hestenes inner product, of Clifford Geometric Algebra (GA) [9, 18], is a symmetrized contraction with worse properties.

Also, contractions are often presented in ways that obfuscate their simple nature as adjoints of exterior products. For example, geometers view the contraction or insertion of vector fields on differential forms as an antiderivation linked to exterior and Lie derivatives ([1, p.429], [6, p.207], [21, p.35]), making it seem more complicated than necessary.

Different conventions are another source of confusion: notations vary, and nonequivalent definitions give contractions with distinct properties.

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For example, \[3, 4, 9, 14, 18, 32, 34, 35, 36, 38\] have each a different contraction. Most authors present their favorite one without warning about the others, and their differences are barely discussed in the literature. Appendix A fills this void, and those who are familiar with some contraction (including Hestenes product) may want to take a look at it first.

This work organizes, simplifies and extends the theory of contractions and related subjects. Most results can be found throughout the literature, but usually only for simple or homogeneous elements, and in forms that seem at odds due to the various conventions. Here they are presented in a more general, uniform and streamlined way, with simpler proofs. This is possible thanks to our notation, an improved multi-index formalism, a mirror principle, and use of general multivectors right from the start.

New results include: geometric characterizations of contractions and regressive products; higher order graded Leibniz rules for contractions with exterior and Clifford products; determinantal formulas; etc. We also study star operators akin to the Hodge star or the dual of GA \[9, 36\], and a new involution simplifies their use. A natural concept of complex orientation gives simpler stars, better suited for complex geometry \[20, 40\].

For simplicity, we use contractions of multivectors in Euclidean or Hermitian vector spaces, but most results adapt for contractions with forms, in pseudo-Euclidean spaces, or on manifolds. The complex case, often neglected but important for geometry and quantum theory, differs from the real one in that contractions are sesqui-linear.

Use of general multivectors whenever possible simplifies the theory. It requires left and right contractions, but a mirror principle facilitates their use. Authors who focus on homogeneous elements find it redundant to have both, as in such case they differ only by grade dependent signs. But these signs clutter the algebra, force one to keep track of grades, and make it hard to work with non-homogeneous elements. These are important since they result from Clifford products \[18\]; represent quantum superpositions of states with variable numbers of fermions \[33\]; appear in Graph Theory \[5\], via Berezin calculus (whose derivatives and integrals are indeed contractions \[23\]); and can store data about sets of subspaces of mixed dimensions, which have many applications \[11, 30\].

In a follow-up paper \[31\], we use contractions to study subspaces associated to a general multivector, special factorizations and decompositions, new simplicity criteria and Plücker-like relations, supercommutators of multi-fermion creation and annihilation operators, etc.

Section 2 sets up notation and concepts we use. Section 3 defines contractions and studies their properties. Section 4 describes star operators and the regressive product. Appendix A discusses different contractions and conventions found in the literature, in special those of GA.

### 2 Preliminaries

In this article, \(X\) is an \(n\)-dimensional Euclidean or Hermitian space, with inner product \((\cdot, \cdot)\) (Hermitian product in the complex case, conjugate-linear in the left entry). When we mention linearity, it is to be understood in the complex case as sesquilinearity, if appropriate.
2.1 Multi-index formalism

For $1 \leq p \leq q$, let $I_p^q = \{(i_1, \ldots , i_p) \in \mathbb{N}^p : 1 \leq i_1 < \cdots < i_p \leq q\}$ and $M^q = \{(i_1, \ldots , i_q) \in \mathbb{N}^q : 1 \leq i_1 \leq q, i_j \neq i_k \text{ if } j \neq k\}$. Also, let $I_0 = M_0 = \{\emptyset\}, \mathcal{I} = \bigcup_{p=0}^{\infty} I_p, \mathcal{M} = \bigcup_{p=0}^{\infty} M_p, T = \bigcup_{q=0}^{\infty} T^q$ and $\mathcal{M} = \bigcup_{q=0}^{\infty} M^q$. Let $|i| = p$ and $||i|| = i_1 + \cdots + i_p$ for $i = (i_1, \ldots , i_p)$, and $|\emptyset| = ||\emptyset|| = 0$. We also write $(i_1, \ldots , i_p)$ as $i_1 \cdots i_p$, and, in general, use $i, j, k$ for elements of $I$, and $r, s, t$ for those of $M$.

For $r, s \in M$, form $r \circ s \in M$ by removing from $r$ any indices of $s$. If they are disjoint (no common indices), $r \circ s \in M$ equals $r$ followed by $s$. We write $r \subset s$ if all indices of $r$ are in $s$. Ordering $r$ we form $r \in I$, and $\epsilon_r$ is the sign of the permutation that orders it ($\epsilon_\emptyset = 1$). The number of pairs $(r, s) \in r \times s$ with $r > s$ is $|r| > |s|$. For $i, j \in I$, form $i \cup j$, $i \cap j$ and $i \triangle j \in I$ by ordering their union, intersection and symmetric difference. For $i \in T^q$, let $i' = (1, \ldots , q)\backslash i$ (its dependence on $q$ is left implicit).

Proposition 2.1. Let $r, s \in M$ and $i, j, k \in I$ be pairwise disjoint.

- $i)$ $\epsilon_{r \circ s} = (-1)^{|r||s|}\epsilon_r \epsilon_s$.
- $ii)$ $\epsilon_{r \circ s} = \epsilon_r \epsilon_s$.
- $iii)$ $\epsilon_{ij} = (-1)^{|i|>|j|}$.
- $iv)$ $\epsilon_{i'j'} = (-1)^{|i|\left(|i'|+1\right)}+|i|$.
- $v)$ $\epsilon_{ijk} = \epsilon_{ij} \epsilon_{ik} \epsilon_{jk}$.
2.2 Grassmann algebra

Grassmann’s exterior algebra [3, 9, 36] of a subspace \( V \subseteq X \) is a graded algebra \( \bigwedge V = \bigoplus_{p \in \mathbb{Z}} \bigwedge^p V \), with \( \bigwedge^0 V = \{ \text{scalars} \} = \mathbb{R} \) or \( \mathbb{C} \), \( \bigwedge^1 V = V \), and \( \bigwedge^p V = \{ 0 \} \) if \( p \notin [0, \dim V] \). Its bilinear associative exterior product \( \wedge \) is alternating, with \( A \wedge B = (-1)^{pq} B \wedge A \in \bigwedge^{p+q} V \) for \( A \in \bigwedge^p V \) and \( B \in \bigwedge^q V \). For \( u, v \in V \), \( u \wedge v = -v \wedge u \), so \( u \wedge v = 0 \).

**Definition 2.2.** Given \( v_1, \ldots, v_q \in X \) and \( r = (r_1, \ldots, r_p) \in \mathcal{M}_p \), let \( v_r = v_{r_1} \wedge \cdots \wedge v_{r_p} \), and \( v_0 = 1 \).

**Definition 2.3.** For a proposition \( P \), let \( \delta_P = \begin{cases} 1 & \text{if } P \text{ is true,} \\ 0 & \text{otherwise.} \end{cases} \)

We have \( v_r \wedge v_s = \delta_{r \cap s = \emptyset} v_{rs} = \delta_{r \cap s = \emptyset} \epsilon_{rs} \epsilon_{rs} \) for \( r, s \in \mathcal{M}^q \), and so \( v_i \wedge v_j = \delta_{i \cap j = \emptyset} \epsilon_{ij} v_{ij} \) for \( i, j \in \mathcal{I}^q \). A basis \( \beta_V^p = \{ v_1, \ldots, v_p \} \) of \( V \) gives bases \( \beta_{A^p} = \{ v_1 \}_{1 \leq i \leq p} \) of \( \bigwedge^p V \), and \( \beta_A = \{ v_1 \}_{1 \leq i \leq \dim A} \) of \( V \). For \( V = \{ 0 \} \), \( \beta_A = \beta_{A^0} = \{ 1 \} \). If \( U \subseteq V \) then \( U \subset \bigwedge \).

**Example 2.4.** If \( \beta_V = (v_1, v_2, v_3) \) then \( \beta_{A^2} = (v_1, v_2, v_3) \) and \( \beta_A = (1, v_1, v_2, v_3, v_12, v_13, v_23, v_123) \), for \( v_12 = v_1 \wedge v_2, v_13 = v_1 \wedge v_3, v_23 = v_2 \wedge v_3 \) and \( v_123 = v_1 \wedge v_2 \wedge v_3 \). Also, \( v_2 \wedge v_13 = v_2 v_13 = -v_123 \) and \( v_2 \wedge v_13 = 0 \).

Any \( M \in \bigwedge X \) is a multivector, and \( (M)_p \) is its component in \( \bigwedge^p X \). Any \( H \in \bigwedge^p X \) is homogeneous of grade \( |H| = p \), or a p-vector. For \( v_1, \ldots, v_p \in X \), \( B = v_1 \wedge \cdots \wedge v_p \) is a simple p-vector, or p-blade. We have \( B \neq 0 \Leftrightarrow v_1, \ldots, v_p \) are linearly independent, in which case its space is \( [B] = \text{span} \{ v_1, \ldots, v_p \} = \{ v \in X : v \wedge B = 0 \} \). A scalar \( \lambda \) is a 0-blade, with \( \langle \lambda \rangle = \{ 0 \} \), and 0 is a p-blade for all \( p \). For a p-dimensional subspace \( V \) and a p-blade \( B \neq 0 \), \( V = [B] \Leftrightarrow \bigwedge^p V = \text{span} \{ B \} \). A blade \( A \) is a subblade of \( B \) if \( |A| \subseteq |B| \). They have same orientation if \( A = AB \). Any \( M \in \bigwedge V \) has a (non-unique) blade decomposition \( M = \sum_i B_i \) for blades \( B_i \in \bigwedge V \).

To help distinguish results that only hold for certain kinds of multivectors, we usually (but not always) use \( L, M, N \) for general multivectors, \( F, G, H \) for homogeneous ones, and \( A, B, C \) for blades.

The inner product of \( A = v_1 \wedge \cdots \wedge v_p \) and \( B = w_1 \wedge \cdots \wedge w_p \) is \( \langle A, B \rangle = \det \langle (v_i, w_j) \rangle \). It is extended linearly, with distinct \( \bigwedge^p X \)’s being orthogonal and \( \langle \kappa, \lambda \rangle = \bar{\kappa} \lambda \) for \( \kappa, \lambda \in \bigwedge^p X \), where \( \bar{\kappa} \) is the complex conjugate. If a basis \( \beta_V \) is orthonormal, so are \( \beta_{A^p} \) and \( \beta_{A^p} \). The norm of \( M \) is \( ||M|| = \sqrt{\langle M, M \rangle} \). In the real case, \( ||A|| \) is the p-dimensional volume of the parallelepiped spanned by \( v_1, \ldots, v_p \). In the complex case, \( ||A||^2 \) gives the 2p-dimensional volume of that spanned by \( v_1, i v_1, \ldots, v_p, i v_p \).
Any linear map $T : X \to Y$ extends to an outermorphism, a linear $T : \bigwedge X \to \bigwedge Y$ with $\text{dim}(M \wedge N) = \text{dim}(TM \wedge TN)$ for $M, N \in \bigwedge X$, and $T(1) = 1$. If $B$ is a $p$-blade, so is $TB$, and $|TB| = |T||B|$ if $TB \neq 0$. Also, $T(\bigwedge^p V) = \bigwedge^p(T(V))$ for $V \subset X$. Note that a scalar $\lambda$ times the outermorphism of $T$ is not an outermorphism (so, it is not that of $\lambda T$).

We use $P_V : X \to V$ and $P_M : \bigwedge X \to V$ for orthogonal projections onto subspaces $V \subset X$ and $V \subset \bigwedge X$, and $P_B = P_{\{B\}}$ for a blade $B$. As an outermorphism, $P_V = P_{\Lambda V}$. For $p$-blades $A$ and $B \neq 0$, $P_B A = (\frac{|B|}{\|B\|}) B$.

If $A \neq 0$ we have $P_V A \neq 0 \Leftrightarrow [P_V] = [A]$.

Graded involution $\hat{\cdot}$ and reversion $\hat{\cdot}$ are linear maps $\bigwedge X \to \bigwedge X$ given by $\hat{M} = \sum_p (-1)^p \hat{M}_p$ and $\hat{M} = \sum_p (-1)^{\frac{p(p-1)}{2}} \hat{M}_p$. For $M, N \in \bigwedge X$, $(\hat{M})^\dagger = \hat{M}$, $(\hat{M}, \hat{N}) = (\hat{M}, \hat{N}) = (M, N)$ and $(M \wedge N)^\dagger = \hat{M} \wedge \hat{N}$, but $\hat{\cdot}$ reverses the order, $(M \wedge N)^\dagger = \hat{N} \wedge \hat{M}$.

**Definition 2.5.** Let $M^\pm$ be a composition of $k$ grade involutions of $M$, and $M = M^{-(n+1)}$ for $n = \text{dim} X$.

So, $\hat{M} = \sum_p (-1)^{\rho_{n+1}} \hat{M}_p$, and $H \wedge M = M^{\rho} \wedge H$ if $H \in \bigwedge^p X$. Note that $M = M$ if $n$ is odd, $M \in \bigwedge^n X$ or $M \in \bigwedge^+ X$, where $\bigwedge^+ X = \bigoplus_{k \in \mathbb{Z}} \bigwedge^{2k} X$ is the even subalgebra.

### 3 Constructions

We will consider contractions between multivectors, for simplicity, but most of the theory adapts for the other kinds discussed in Appendix A. It also adapts for pseudo-Euclidean spaces [34, 36, 37], but some results hold only for non-null blades or have signature-dependent signs: e.g., (2) becomes $v_1 \wedge v_j = \delta_{i,j} \epsilon_{ij} v_j v_{j+1}$, with $\sigma_i = (v_i, v_i) = \pm 1$. Constructions can also be defined in spaces with degenerate metrics, via their basic operational properties ([9, p.73], [25, p.223]).

**Definition 3.1.** The left contraction $M \ll N$ of a contractor $M \in \bigwedge X$ on a contractee $N \in \bigwedge X$, and the right contraction $N \ll M$ of $M \in \bigwedge N$ by $M$, are the unique elements of $\bigwedge X$ satisfying, for all $L \in \bigwedge X$,

$$\langle L, M \ll N \rangle = \langle L \wedge M, N \rangle \quad \text{and} \quad \langle L, N \ll M \rangle = \langle L \wedge M, N \rangle. \quad (1)$$

So, $M \ll$ and $\ll M$ are the adjoint operators of $M \wedge$ and $\wedge M$, respectively. Constructions are bilinear in the real case, but in the complex one they are conjugate-linear in the contractor and linear in the contractee. We often prove results only for the left contraction, as the right one is similar.

**Proposition 3.2.** For an orthonormal basis $(v_1, \ldots, v_n)$ and $i, j \in T^n$,

$$v_i \wedge v_j = \delta_{i,j} \epsilon_{i\{j\}} v_{j+1}, \quad \text{and} \quad v_j \ll v_i = \delta_{i,j} \epsilon_{i\{j\}} v_{j+1}. \quad (2)$$

**Proof.** For $k \in T^n$, $(v_k, v_i \wedge v_j) = (v_k \wedge v_k, v_j) = 0$ vanishes unless $i \cap k = \emptyset$ and $j = k$ (so $i \subseteq j$ and $k = j \setminus i$), in which case it gives $\epsilon_{i\{j\}}$. \quad \Box

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1By convention, maps take precedence over products: $TM \wedge TN$ means $(TM) \wedge (TN)$.
Proof. (i–iii) Follow from (2) and linearity, with \( G = \sum_{i \in T_q} \lambda_i V_i \) and \( H = \sum_{j \in T_q} \kappa_j v_j \) for scalars \( \lambda_i, \kappa_j \). (iv) Likewise, extending an orthonormal basis \( (v_1, \ldots, v_p) \) of \( V \) to \( X \), so \( M = \sum_{i \in T_r} \lambda_i V_i \) and \( N = \sum_{j \in T_r} \kappa_j v_j \). (vi) \( \langle L, (M \lhd N) \rangle = \langle \tilde{L}, M \lhd N \rangle = \langle M \land L, N \rangle = (\langle M \lhd \tilde{L} \rangle, \tilde{N}) = \langle L \land M, \tilde{N} \rangle = \langle L, \tilde{N} \lhd M \rangle \) Likewise for ‘\( \lhd \) ’ and ‘\( \land \) ’, but without swapping \( L \) and \( M \).

So, contractions generalize inner products, giving multivectors instead of scalars if grades differ. They vanish if the contractor has larger grade, and this will make the asymmetry \( M \lhd N \neq N \lhd M \) useful. While iv gives \( G \lhd H = (-1)^{(p+1)} H \lhd G \), in general \( M \lhd N \neq \pm N \lhd M \). In vi, contractor and contractee keep their roles as ‘\( \lhd \) ’ swaps them and switches \( \lhd \) and \( \land \).

A mirror principle follows by applying ‘\( \lhd \) ’ to a formula (with \&, \lhd, \& or sideless operators that commute with ‘\( \lhd \) ’, like \( \land \) ), distributing over all terms, and renaming them: if the formula is valid for generic elements, so is its mirror version, with terms in reversed order, and \( \lhd \) and \( \land \) switched. For example, ‘\( \lhd \) ’ applied to i below gives \( \tilde{N} \lhd (M \land \tilde{L}) = (\tilde{N} \lhd \tilde{L}) \lhd \tilde{M} \), and relabeling \( \tilde{L}, M, N \) as \( L, M, N \) we find the mirror formula \( \tilde{N} \lhd (M \land L) = (\tilde{N} \lhd L) \lhd \tilde{M} \). The principle would be more general if notations were designed for this: e.g., \( \langle \cdot, \cdot \rangle \) would have to show which entry is conjugate-linear (not the worst idea). It is best to allow a certain flexibility, with some elements keeping their order (learning which ones takes just a little practice).

The following are the main tools for operating with contractions.

**Proposition 3.5.** Let \( v, w_1, \ldots, w_q \in X, H \in \Lambda^p X \) with \( p \leq q \), and \( L, M, N \in \Lambda X \). Then:

i) \( (L \land M) \lhd N = M \lhd (L \lhd N) \).

ii) \( H \lhd w_1 \ldots w_q = \sum_{i \in T_p} \epsilon_{ij} (H, w_i) w_i \).

iii) \( v \lhd w_1 \ldots w_q = \sum_{i=1}^q (-1)^{i-1} (v, w_i) w_i \ldots \hat{w}_i \ldots w_q \), where \( \hat{w} \) means \( i \) is absent.

iv) \( v \lhd (M \land N) = (v \lhd M) \land N + M \land (v \lhd N) \).
Proof. (i) Follows from (1) and associativity of ∧. (ii) Linearity lets us assume \( H \) is a blade. For a \((q-p)\)-blade \( B \), Laplace determinant expansion gives

\[
\langle B, H \downarrow w_1 \cdots w_q \rangle = \langle H \wedge B, w_1 \cdots w_q \rangle = \sum_{i \in \mathbb{Z}_2} \epsilon_i \langle H, w_i \rangle \langle B, w_i \rangle.
\]

Follows from ii. (iv) Follows from iii, as we can assume \( M = w_{1 \cdots p} \) and \( N = w_{(p+1) \cdots (p+q)} \) for \( w_1, \ldots, w_{p+q} \in X \).

The reordering of \( L \) and \( M \) in i reflects the adjoint nature of contractions. Some conventions avoid it, but have other difficulties (see Appendix A.2). The mirror of ii is \( w_{1 \cdots q} \downarrow L = \sum_{i \in \mathbb{Z}_2} \epsilon_i \langle H, w_i \rangle w_{1 \cdots q} \) (after some relabeling). In iii, the sign is + at the first term of the sum, while \( w_{1 \cdots q} \downarrow v = \sum_{i=1}^{q} (-1)^{q-i} \langle v, w_i \rangle w_{1 \cdots i-1}v_{i+1} \cdots w_{q} \) has + at the last one. As iv is a graded Leibniz rule, \( v \downarrow \) is a graded derivation. The notation makes it, and \((M \wedge N) \downarrow v = M \wedge (N \downarrow v) + (M \downarrow v) \wedge N\), look natural: as \( v \) ‘approaches’ \( M \wedge N \) from either side, it applies \( \downarrow \) on the term over which it ‘jumps’. Some authors switch left and right contractions, losing this.

Example 3.6. Let \( v, w_1, \ldots, w_4 \in X \) and \( H \in \bigwedge^2 X \). Then \( v \downarrow (v \wedge w_2) = \langle v \rangle w_1w_3 - \langle v, w_2 \rangle w_1 \), while \( (v \wedge w_2) \downarrow v = -(v, w_2)w_1 + \langle v, w_2 \rangle w_1 \). Also, \( H \downarrow w_{1234} = \langle H, w_{12} \rangle w_{34} - \langle H, w_{13} \rangle w_{24} + \langle H, w_{14} \rangle w_{23} + \langle H, w_{23} \rangle w_{14} - \langle H, w_{24} \rangle w_{13} - \langle H, w_{34} \rangle w_{12} = w_{1234} \downarrow H \).

Corollary 3.7. \( v \downarrow \downarrow M = v_p \downarrow (\cdots (v_1 \downarrow M) \cdots) \), for \( v, \ldots, v_p \in X \) and \( M, N \in \bigwedge X \).

Corollary 3.8. \( v \wedge (M \downarrow N) = (M \downarrow v) \downarrow N + \hat{M} \downarrow (v \wedge N) \), for \( v \in X \) and \( M, N \in \bigwedge X \).

Proof. For \( L \in \bigwedge X \), we have \( \langle L, \hat{M} \downarrow (v \wedge N) \rangle = \langle v \downarrow (\hat{M} \wedge L), N \rangle = \langle (v \downarrow M) \wedge L + M \wedge (v \downarrow L), N \rangle = \langle L, (M \downarrow v) \downarrow N + v \wedge (M \downarrow N) \rangle \).

In terms of operators, this is the adjoint of iv, arranged for convenience.

Corollary 3.9. Let \( M \in \bigwedge V \) and \( N \in \bigwedge W \).

i) If \( L \in \bigwedge (W^\perp) \) then \( L \downarrow (M \wedge N) = (L \downarrow M) \wedge N \).

ii) If \( H \in \bigwedge (V^\perp) \) then \( H \downarrow (M \wedge N) = M^{\perp p} \wedge (H \downarrow M) \).

Proof. (i) Linearity and Corollary 3.7 let us assume \( L = v \in W^\perp \), in which case it follows as in the proof of Proposition 3.5iv. (ii) Likewise.

In [31], we show what the solutions of \( v \wedge M = 0 \) and \( v \downarrow M = 0 \), with \( v \in X \), reveal about the structure of a multivector \( M \in \bigwedge X \). For now, note that iv gives \( v \downarrow (v \wedge M) + v \wedge (v \downarrow M) = \|v\|^2 M \), so:

Corollary 3.10. \( v \wedge M = v \downarrow M = 0 \iff v = 0 \) or \( M = 0 \).

Corollary 3.11. For \( 0 \neq v \in X \) and \( M \in \bigwedge X \):

i) \( v \wedge M = 0 \iff M = v \downarrow N \) for \( N \in \bigwedge X \). In particular, \( N = v v^\downarrow M \|v\|^2 \).

ii) \( v \downarrow M = 0 \iff M = v \wedge N \) for \( N \in \bigwedge X \). In particular, \( N = v^\downarrow M \|v\|^2 \).

Corollary 3.12. If \( M \neq 0 \), \( \{v \in X : v \wedge M = 0\} \perp \{w \in X : w \downarrow M = 0\} \).

Proof. \( v \wedge M = w \downarrow M = 0 \iff 0 = w \downarrow (v \wedge M) = \langle w, v \rangle M \Rightarrow w \perp v \).

Proposition 3.13. For \( v \in X \) and nonzero \( M \in \bigwedge V \) and \( N \in \bigwedge W \) in disjoint subspaces \( V \) and \( W \), \( v \downarrow (M \wedge N) = 0 \iff v \downarrow M = v \wedge N = 0 \).
Proof. \((\Leftarrow)\) Follows from Proposition 3.5iv. \((\Rightarrow)\) If the largest grade in \(M\) is \(r \neq 0\), it is at most \(r - 1\) in \(v \Join M \in \bigwedge V\). Given bases \((v_1, \ldots, v_p)\) of \(V\) and \((w_1, \ldots, w_q)\) of \(W\), \((v \Join M) \wedge N\) has no component with \(i \in \mathbb{I}^p\) in the basis \(\{v_i \wedge w_j\}_{i \in \mathbb{I}^p, j \in \mathbb{I}^q}\) of \(\bigwedge(V \oplus W)\). Unless \(v \Join N = 0\), \(M \wedge (v \Join N)\) has, contradicting \((v \Join M) \wedge N + M \wedge (v \Join N) = 0\). Likewise, \(v \Join M = 0\).

In \([31]\), we show how \(M \Join v = 0\) is linked, in a sense, to orthogonality (after all, contractions generalize inner products). For now, we have:

**Proposition 3.14.** \(v \Join M = 0 \iff M \in \bigwedge ([v]^+)\), for \(v \in X\) and \(M \in \bigwedge X\).

**Proof.** \((\Rightarrow)\) Assume \((v, w_1, \ldots, w_{n-1})\) is an orthonormal basis of \(X\), so \(\bigwedge X = \text{span}\{v_1, v \wedge w_1\}_{i \in \mathbb{I}^{n-1}}\). By Proposition 3.5iii, \(v \Join w_1 = 0\) and \(v \Join (v \wedge w_1) = w_1\). By Corollary 3.11ii, \(M \in \text{span}\{v, v \wedge w_1\}_{i \in \mathbb{I}^{n-1}} = \text{span}\{w_1\}_{i \in \mathbb{I}^{n-1}} = \langle([v]^+)\rangle\). \((\Leftarrow)\) \(M \in \text{span}\{w_1\}_{i \in \mathbb{I}^{n-1}}\), so \(v \Join M = 0\).

**Corollary 3.15.** \([B] = \{v \in X : v \Join B = 0\}^\perp\), for a blade \(B\).

**Definition 3.16.** \(U\) is partially orthogonal (\(\perp\)) to \(V\) if \(V^\perp \cap U = \{0\}\).

For a blade \(B \neq 0\), \([B] \perp V \iff PVB = 0\).

**Proposition 3.17.** Given a blade \(B \neq 0\) and a subspace \(V \subseteq X\), we have \([B] \perp V \iff B \Join M = 0\) for all \(M \in \bigwedge V\).

**Proof.** \((\Rightarrow)\) Follows from Propositions 3.7 and 3.14. \((\Leftarrow)\) \(|PVB|^2 = \langle B, PVB \rangle = B \Join PVB = 0\).

**Corollary 3.18.** \(M \in \bigwedge V\), \(N \in \bigwedge (V^\perp) \Rightarrow M \Join N = \lambda N\) for \(\lambda = (M)_0\).

In such case, if \(M\) has no scalar component then \(M \Join N = 0\).

Next, we combine left and right contractions.

**Proposition 3.19.** \(L \Join (M \wedge N) = (L \Join M) \wedge N\), for \(L, M, N \in \bigwedge X\).

**Proof.** Follows from (1) and associativity of \(\wedge\).

This lets us write just \(L \Join M \wedge N\). Note the order of \(\Join\) and \(\wedge\), as, in general, \((L \Join (M \wedge N)) \neq (L \Join M) \wedge N\).

**Proposition 3.20.** Let \(A\) and \(B\) be blades, and \(M, N \in \bigwedge X\).

i) If \([A] \subseteq [B]\) then \((A \wedge M) \Join B = (PA)M \wedge (A \Join B)\).

ii) If \(N \subseteq [A]\) then \((M \wedge N) \Join B = N \wedge (M \Join B)\).

**Proof.** Extend an orthonormal basis of \([A]\) to \([B]\) and then to \((v_1, \ldots, v_n)\) of \(X\), and assume \(N = A = v_1, B = v_j\) and \(M = v_k\) for \(i, j, k \in \mathbb{I}^n\) with \(i \subset j\). (i) If \(k \not\subseteq i\) both sides vanish, otherwise \(i = \mathbb{I}^k\) and \(j = \mathbb{I}^m\) for \(l, m \in \mathbb{I}^p\), and so \((v_i \wedge v_k) \wedge v_j = \epsilon_{i k} v_i \wedge v_j = \epsilon_{i k} v_i (v_k \wedge v_m) = v_k \wedge (v_i \wedge v_j) = (P_{v_i} v_k) \wedge (v_j \wedge v_k)\). (ii) Both sides vanish unless \(i \subset k \subset j\), in which case \(i\) gives \((v_i \wedge v_j) \wedge v_k = (P_{v_i} v_k) \wedge (v_j \wedge v_k)\).

In \([31]\), we show how to replace \(B\) by a general multivector. By \(i\) and its mirror, \((B \wedge M) \Join B = B \wedge (M \Join B)\), so we can write \(B \wedge M \Join B\).

**Corollary 3.21.** \(B \wedge M \Join B = PB\), for a unit blade \(B\) and \(M \in \bigwedge X\).

**Corollary 3.22.** If \(M = N \Join B\) for \(N \in \bigwedge X\) and a blade \(B \neq 0\) then \(M = \frac{B}{\|B\|^2} \Join B\).
The method used in Proposition 3.20 also gives other triple products. For any blade \( B \), we find \( B \lhd (M \lhd B) = \lambda \|B\|^2 \) for \( \lambda = (M)_0 \), and \((B \lhd M) \lhd B = (M \lhd B) \lhd B = \langle M, B \rangle B = (P_B M)_0 \) if \( B \) is a real unit \( p \)-blade). If it is nonscalar, \( B \lhd M \lhd B = 0 \).

3.1 Geometric interpretation

Now we obtain complete geometric characterizations for contractions of nonzero blades \( A \in \Lambda^p X \) and \( B \in \Lambda^q X \).

**Definition 3.23.** \( B = B_P \land B_\perp \) is a projective-orthogonal (PO) factorization w.r.t. \( A \) if \( B_P \) and \( B_\perp \) are subblades of \( B \) of grades \( m = \min(p, q) \) and \( q - m \), respectively, with \([B_\perp]\) orthogonal to \([A]\) and \([B_P]\).

Note that \( P_B A = P_{B_P} A = \langle (P_B A)^2 \rangle \) if \( B = 0 \).

**Proposition 3.24.** \( A \lhd B = \langle A, B_P \rangle B_\perp \).

Proof. As \([A] \perp [B_\perp]\), \( A \lhd (B_P \land B_\perp) = (A \lhd B_P) \land B_\perp \). And \( A \lhd B_P = \langle A, B_P \rangle \) (immediate if \( p = m \), and if \( p > m \) both vanish).

So, \( A \lhd B \) takes an inner product of \( A \) with a subblade of \( B \) where it projects, contracting this subblade and leaving another orthogonal to it. Likewise, \( B_\perp \lhd A = \langle A, B_P \rangle B'_\perp \) for \( B'_\perp = (B_\perp)^* \) (so \( B = B'_\perp \land B_P \)).

**Definition 3.25.** \( \Theta_{V,W} = \cos^{-1} \frac{|\langle P_B A \rangle|}{\|A\|} \) is the asymmetric angle of \( V = [A] \) with \( W = [B] \).

Formerly called Grassmann angle [27], it is linked to the various products of GA [29], and gives an asymmetric Fubini-Study metric in the Total Grassmannian [30]. As blade norms (squared, in the complex case) give products of GA, and gives an asymmetric Fubini-Study metric in the Total Grassmannian [30]. As blade norms (squared, in the complex case) give products of GA [29], and gives an asymmetric Fubini-Study metric in the Total Grassmannian [30]. As blade norms (squared, in the complex case) give products of GA [29], and gives an asymmetric Fubini-Study metric in the Total Grassmannian [30]. As blade norms (squared, in the complex case) give products of GA [29], and gives an asymmetric Fubini-Study metric in the Total Grassmannian [30]. As blade norms (squared, in the complex case) give products of GA [29], and gives an asymmetric Fubini-Study metric in the Total Grassmannian [30].

**Theorem 3.26.** For nonzero blades \( A \in \Lambda^p X \) and \( B \in \Lambda^q X \), we have \( A \lhd B = 0 \iff [A] \perp [B] \). If \( A \lhd B \neq 0 \), it is the only \((q-p)\)-blade such that:

i) \( [A \lhd B] = [A]^{\perp} \cap [B] \).

ii) \( \|A \lhd B\| = \|P_B A\| \|B\| = \|A\| \|B\| \cos \Theta_{[A],[B]} \).

iii) \( (P_B A) \land (A \lhd B) \) has the orientation of \( B \).

Proof. \( A \lhd B = 0 \iff \langle A, B_P \rangle = 0 \iff P_B A = 0 \iff [A] \perp [B] \). Otherwise, \([A \lhd B] = [B_\perp] = [A]^{\perp} \cap [B] \), as \( P_B ([A]) = [P_B A] = [B_P] \), \( \|A \lhd B\| = \|\langle A, B_P \rangle\| \|B_\perp\| \|B\| = \|P_B A\| \|B\| \), and \((P_B A) \land (A \lhd B) = \frac{|\langle A, B_P \rangle|^2}{\|B_P\|^2} B \).

The right contraction is similar, except that \((B \perp \lhd A) \land (P_B A) \) has the orientation of \( B \) (Fig. 1). Note that ii holds for all blades, as \([A] \perp [B] \iff \Theta_{[A],[B]} = \frac{\pi}{2} \).
Example 3.27. If $A = v_2 - 2v_3 + iv_4$ and $B = v_{134}$ for an orthonormal basis $(v_1, \ldots, v_4)$ of $C^4$ then $A \wedge B = 2v_{14} - iv_{13}$, so $[A] = [v_1] \oplus [2v_4 - iv_3]$. As $B_\perp = \frac{A - B}{\sqrt{v_4}}$ and $B_P = B \wedge B_\perp = \frac{2v_3 + iv_4}{\sqrt{v_4}}$ give a PO factorization, $P_B A = -2v_3 + iv_4$ and $(P_B A) \wedge (A \wedge B) = 5B$. As $\cos \Theta = \sqrt{\frac{v_4}{v_4}}$, areas in the real plane $[A]$ contract by $\frac{v_4}{v_4}$ if orthogonally projected on $[B]$ (each real dimension contracts by $\frac{\sqrt{v_4}}{v_4}$). As 6-dimensional volumes of $[B]$ vanish if projected on $[A]$, $\Theta_{[B],[A]} = \frac{v_4}{v_4}$ and $B \wedge A = 0$.

3.2 Exterior and interior product operators

Some properties are better expressed in terms of the following operators.

**Definition 3.28.** (Left) exterior and interior products by $M \in \bigwedge X$ are given, respectively, by $e_M(N) = M \wedge N$ and $i_M(N) = M \wedge N$, for $N \in \bigwedge X$.

Both are linear in $N$. In $M$, $e_M$ is linear and $i_M$ is conjugate-linear.

**Proposition 3.29.** $e_M e_N = e_{M \wedge N}$ and $i_M i_N = i_{N \wedge M}$, for $M, N \in \bigwedge X$.

**Proof.** Follows from associativity of $\wedge$, and Proposition 3.5i. □

Note the order of $M$ and $N$ in $i_{N \wedge M}$. If $M$ is odd, or a non-scalar blade, then $e_M^2 = i_M^2 = 0$, so $\text{Im } e_M \subset \text{ker } e_M$ and $\text{Im } i_M \subset \text{ker } i_M$.

As $e_M$ and $i_M$ are adjoints, $\text{ker } e_M = (\text{Im } i_M) = 0$ and $\text{ker } i_M = (\text{Im } e_M) = 0$. Also, $i_M e_M$ and $e_M i_M$ are self-adjoint.

**Proposition 3.30.** Let $L, M, N \in \bigwedge X$.

i) $M \wedge (M \wedge N) = 0 \Leftrightarrow M \wedge N = 0$.

ii) $M \wedge (M \wedge N) = 0 \Leftrightarrow M \wedge N = 0$.

iii) $L = M \wedge N \Leftrightarrow L = M \wedge (M \wedge K)$ for some $K \in \bigwedge X$.

iv) $L = M \wedge N \Leftrightarrow L = M \wedge (M \wedge K)$ for some $K \in \bigwedge X$.

**Proof.** Follows from usual properties of adjoint operators. □

Let $M \wedge \bigwedge V = \{M \wedge N : N \in \bigwedge V\}$, for $M \in \bigwedge X$ and $V \subset X$.

**Proposition 3.31.** $\text{Im } e_B = B \wedge \bigwedge (B) = \bigwedge (B)$ for any blade $B$, and if $B \neq 0$ then $\text{Im } i_B = \bigwedge (B)$. 

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Figure 2: \( B \wedge (B \perp A) = P_{B \wedge A \perp} A \), if \( \| B \| = 1 \). Among all planes containing \( B \), that of \( B \wedge A \perp \) is the closest one to \( A \) (it forms the smallest angle \( \Theta \)). The PO-factorization was chosen with \( \| A_P \| = 1/\| P_A B \| \), so \( \| A \perp \| = \| B \perp A \| \).

**Proof.** Given orthonormal bases \( (v_1, \ldots, v_p) \) of \([B]\) and \( (w_1, \ldots, w_{n-p})\) of \([B]^\perp\) \((p = 0 \text{ or } n \text{ is trivial})\), \( \{ v_i \wedge w_j \} \in \mathcal{T}_p \), \( j \in \mathcal{T}_n-p \) is one for \( \wedge X \). For \( B = v_1 \cdots w \) we have \( e_B (v_1 \wedge w_j) = \delta_{i=j} B \wedge w_j \) and \( \iota_B (v_1 \wedge w_j) = \delta_{i=1-p} w_j \). □

**Proposition 3.32.** Let \( B \) be a unit blade, and \( M \in \wedge X \).
\[ \begin{align*}
&\text{i) } B \perp (B \wedge M) = \mathcal{P}_{\text{Im } B} M = P_{[B]} M. \\
&\text{ii) } B \wedge (B \perp M) = \mathcal{P}_{\text{Im } B} M.
\end{align*} \]

**Proof.** (i) If \( M \in \text{Im } e_B = \wedge([B]^\perp) \), Corollary 3.9i gives \( B \perp (B \wedge M) = M \). If \( M \in \{ \text{Im } e_B \}^\perp \) then \( B \perp (B \wedge M) = 0 \). (ii) If \( M \in \text{Im } e_B \) then \( M = B \wedge N \) for \( N \in \text{Im } \iota_B \), so \( B \perp (B \wedge M) = B \perp (B \wedge (B \wedge N)) = B \wedge N = M \). If \( M \in \{ \text{Im } e_B \}^\perp \) then \( B \perp (B \wedge M) = 0 \). □

In [31], \( e_B \) and \( \iota_B \) are linked to multi-fermion creation and annihilation operators, and this result lets us interpret \( \iota_B e_B \) and \( e_B \iota_B \) as vacancy and occupancy operators, related to the quantum number operator.

Some cases of ii are worth mentioning. For a unit \( v \in X \) and \( M \in \wedge X \), \( v \wedge (v \perp M) = M - P_{[v]} M \), and for \( w \in X \), \( v \wedge (v \parallel w) = P_{w} w \). With a PO factorization \( A_P \wedge A_{\perp} \) of a q-blade \( A \) w.r.t. a unit p-blade \( B \) w.r.t, one obtains \( B \wedge (B \perp A) = P_{B \wedge A \perp} A \) (Fig. 2). The geometric relevance of \([B] \wedge A \perp\) is that, among all \( q \)-dimensional subspaces \( \wedge X \supset [B] \), it is the closest one to \([A]\), in the sense that it minimizes \( \Theta_{V,[A]} \) [27].

**Corollary 3.33.** Let \( B \neq 0 \) be a blade, and \( M \in \wedge X \).
\[ \begin{align*}
&\text{i) } \text{If } M = B \perp N \text{ for } N \in \wedge X \text{ then } M = B \perp \frac{B \wedge M}{\| B \|^2}.
\end{align*} \]

**Corollary 3.34.** The restricted maps \( \wedge([B]^\perp) \Rightarrow \frac{e_B}{\iota_B} B \wedge ([B]^\perp) \) are mutually inverse isometries, for a unit blade \( B \).

Fig. 3 shows how \( e_B \) and \( \iota_B \) act in \( \wedge X \).

For \( 0 \neq v \in X \), \( \ker e_v = \text{Im } e_v \) and \( \ker e_v = \text{Im } e_v \), by Corollary 3.11, so we have exact sequences \( 0 \xrightarrow{e_v} \wedge^0 X \xrightarrow{e_v} \wedge^1 X \xrightarrow{e_v} \cdots \xrightarrow{e_v} \wedge^n X \xrightarrow{e_v} 0 \) and \( 0 \xrightarrow{\iota_v} \wedge^0 X \xrightarrow{\iota_v} \wedge^1 X \xrightarrow{\iota_v} \cdots \xrightarrow{\iota_v} \wedge^n X \xrightarrow{\iota_v} 0 \). If \( \| e \| = 1 \), \( e_v e_v + e_v e_v = 1 \).
3.3 Higher order Leibniz rule

Now we obtain higher order versions of the graded Leibniz rule and of its adjoint. In [31] we interpret them in terms of supercommutators.

**Theorem 3.35.** For \(v_1, \ldots, v_p \in X\) and \(M, N \in \bigwedge X\),

\[
v_{1 \ldots p} \cdot (M \wedge N) = \sum_{i \in I^p} \epsilon_W (v_{1 \ldots (i-1)} M^{-|i|}) \wedge (v_i \cdot N), \quad \text{and} \quad (3)
v_{1 \ldots p} \wedge (M \cdot N) = \sum_{i \in I^p} \epsilon_W (M^{-|i|} \cdot v_i) \cdot (v_{1 \ldots (i-1)} \wedge N). \quad (4)
\]

**Proof.** Proposition 3.5iv gives (3) for \(p = 1\) and, assuming it for \(v_{1 \ldots (p-1)}\), also \(v_{1 \ldots p} \cdot (M \wedge N) = v_p \cdot (v_{1 \ldots (p-1)} \cdot (M \wedge N)) = v_p \cdot \sum_{i \in I^{p-1}} \epsilon_W (v_{1 \ldots (i-1)} M^{-|i|}) \wedge (v_i \cdot N) = I + II\), where

\[
I = \sum_{i \in I^{p-1}} \epsilon_W (v_p \cdot (v_{1 \ldots (i-1)} M^{-|i|})) \wedge (v_i \cdot N)
\]

\[
= \sum_{i \in I^{p-1}} \epsilon_W (v_{1 \ldots (i-1)} M^{-|i|}) \wedge (v_i \cdot N)
\]

\[
= \sum_{j \in I^{p-1}} \epsilon_W (v_{1 \ldots (j-1)} M^{-|j|}) \wedge (v_j \cdot N),
\]

for \(j = i\) and \(j' = i'\), since \(\epsilon_W = \epsilon_W, p = \epsilon_W\), and

\[
II = \sum_{i \in I^{p-1}} \epsilon_W (v_{i'} M^{-|i'|} \cdot (v_{1 \ldots (i-1)} \wedge (v_i \cdot N))
\]

\[
= \sum_{i \in I^{p-1}} \epsilon_W (v_{1 \ldots (i-1)} M^{-|i|}) \wedge (v_{1 \ldots (i-1)} \wedge (v_i \cdot N))
\]

\[
= \sum_{j \in I^{p-1}, p \notin j} \epsilon_W (v_{1 \ldots (j-1)} M^{-|j|}) \wedge (v_j \cdot N),
\]

for \(j = i\) and \(j' = i'\), since \(\epsilon_W = \epsilon_W, p = (-1)^{|i'|} \epsilon_W\).

Corollary 3.8 gives (4) for \(p = 1\) and, assuming it for \(v_{1 \ldots (p-1)}\), also

\[
v_{1 \ldots p} \wedge (M \cdot N) = v_p \wedge (v_{1 \ldots (p-1)} \wedge (M \cdot N)) = v_p \wedge \sum_{i \in I^{p-1}} \epsilon_W (M^{-|i|} \cdot v_i \wedge N),
\]
\[ \hat{v}_i \wedge (\hat{v}_y \wedge N) = I + II, \]
where
\[
I = \sum_{i \in I^p-1} \epsilon_i \langle (M^{-i}[y] \wedge \hat{v}_i) \wedge \hat{v}_p \rangle \wedge (\hat{v}_y \wedge N),
\]

\[
II = \sum_{i \in I^p-1} \epsilon_i \langle (M^{-i}[y] \wedge \hat{v}_i) \wedge (v_p \wedge \hat{v}_y \wedge N),
\]

which are then developed as above.

The higher order graded Leibniz rule (3) shows that contraction by a \( p \)-blade is a graded derivation of order \( p \). Writing in (3) and (4) the terms for \( I = \emptyset \) and \( i = 1 \cdot \cdots \cdot p \), we find

\[
v_{1 \cdot \cdots \cdot p} \wedge (M \wedge N) = (v_{1 \cdot \cdots \cdot p} \wedge M) \wedge N + \cdots + M \wedge (v_{1 \cdot \cdots \cdot p} \wedge N),
\]

and

\[
v_{1 \cdot \cdots \cdot p} \wedge (M \wedge N) = M \wedge (v_{1 \cdot \cdots \cdot p} \wedge N) + \cdots + (M \wedge v_{1 \cdot \cdots \cdot p} \wedge N).
\]

Though unobvious, (4) is equivalent to the adjoint of (3), \( M \wedge (v_{1 \cdot \cdots \cdot p} \wedge N) = \sum_{i \in I^p} \epsilon_i \langle v_1 \wedge \langle (v_i \wedge M^{-i}[y]) \wedge N \rangle \rangle \rangle = (v_{1 \cdot \cdots \cdot p} \wedge M) \wedge N + \cdots + v_{1 \cdot \cdots \cdot p} \wedge (M \wedge v_{1 \cdot \cdots \cdot p} \wedge N),\]

and vice-versa.

**Example 3.36.** Let \( v_1, v_2 \in X \) and \( M, N \in \bigwedge X \). By (3), \( v_{12} \wedge (M \wedge N) = (v_{12} \wedge M) \wedge N + (v_2 \wedge M) \wedge (v_1 \wedge N) - (v_{1} \wedge M) \wedge (v_{2} \wedge N) + M \wedge (v_{12} \wedge N). \)

By (4), \( v_{12} \wedge (M \wedge N) = M \wedge (v_{12} \wedge N) + (M \wedge v_1) \wedge (v_2 \wedge N) - (M \wedge v_2) \wedge (v_1 \wedge N) + (M \wedge v_{12}) \wedge (v_1 \wedge N) + (M \wedge v_{12}) \wedge (v_2 \wedge N) + (M \wedge v_{12}) \wedge (v_2 \wedge N).

For more complex calculations one can use Proposition 2.1iv, noting that \( (-1)^{[i ([\theta + 1])]} \) follows the pattern + -- + for \( [i] \mod 4 = 0, 1, 2, 3, \) and \( (-1)^{[i]} = (-1)^{[\theta]} \), with \( \theta \) = number of odd indices in \( i \).

**Example 3.37.** For \( v_1, \ldots, v_4 \in X \) and \( M, N \in \bigwedge X \), (3) gives

\[
v_{1234} \wedge (M \wedge N) = + (v_{1234} \wedge M) \wedge N
- (v_{234} \wedge M) \wedge (v_1 \wedge N) + (v_{134} \wedge M) \wedge (v_2 \wedge N)
- (v_{24} \wedge M) \wedge (v_3 \wedge N) + (v_{12} \wedge M) \wedge (v_4 \wedge N))
- (v_{13} \wedge M) \wedge (v_{24} \wedge N) - (v_{12} \wedge M) \wedge (v_{34} \wedge N))
+ (v_{2} \wedge M) \wedge (v_{134} \wedge N) - (v_{1} \wedge M) \wedge (v_{234} \wedge N))
+ M \wedge (v_{1234} \wedge N).
\]

### 3.4 Outermorphisms and contractions

Let \( T : X \to Y \) be a linear map into a Euclidean or Hermitian space \( Y \) (same as \( X \)). The outermorphism of its adjoint \( T^\dagger : Y \to X \) is the adjoint of its outermorphism, i.e. \( \langle M, T^\dagger N \rangle = \langle TM, N \rangle \) for \( M \in \bigwedge X \) and \( N \in \bigwedge Y \). Likewise for its inverse, if it exists. We say \( T \) is an isometry if \( \langle Tx, Ty \rangle = \langle x, y \rangle \) for all \( x, y \in X \), in which case \( T^\dagger T = 1 \), and \( (TM, TN) = \langle M, N \rangle \) for all \( M, N \in \bigwedge X \). By convention, maps take precedence over contractions: \( TM \wedge TN \) means \( (TM) \wedge (TN) \).
Proposition 3.38. \(T(T^\dagger M \bowtie N) = M \bowtie TN\), for \(M \in \bigwedge Y\) and \(N \in \bigwedge X\).

Proof. For \(L \in \bigwedge Y\), \(\langle L, M \bowtie TN \rangle = \langle M \land L, TN \rangle = \langle T^\dagger (M \land L), N \rangle = \langle T^\dagger M \land T^\dagger L, N \rangle = \langle L, (T^\dagger M \bowtie N) \rangle\).

Corollary 3.39. If \(T\) is invertible then \(T(M \bowtie N) = (T^\dagger)^{-1} M \bowtie TN\), for \(M, N \in \bigwedge X\).

Corollary 3.40. \(T\) is an isometry \(\iff\) \(T(M \bowtie N) = TM \bowtie TN\) for all \(M, N \in \bigwedge X\).

Proof. \((\Rightarrow) T^\dagger = T^{-1}\), so \(M = T^\dagger L\) for \(L = TM\). \((\Leftarrow)\) For \(x, y \in X\), \(\langle Tx, Ty \rangle = Tx \bowtie Ty = T(x \bowtie y) = T\langle x, y \rangle = \langle x, y \rangle\).

3.5 Determinant formulas

For blades \(A = v_1 \land \cdots \land v_p\) and \(B = w_1 \land \cdots \land w_q\), let \(A_{p \times q} = \{(v_i, w_j)\}\), \(B_{q \times q} = \{(w_i, w_j)\}\) and \(M_{(p+q) \times (p+q)} = (A^\dagger B)\) where \(^\dagger\) is the conjugate transpose. Also, let \(W_{n \times q}\) have as columns the \(w_j\)'s decomposed in an arbitrary basis \((u_1, \ldots, u_n)\). The following expansions can be useful when the determinants can be computed efficiently (e.g. if they are sparse).

Proposition 3.41. If \(p \leq q\) then

\[
A \bowtie B = \sum_{j \in I_p^q} \epsilon_{jy} \det(A_j) w_y = \sum_{i \in I_p^{n-p}} \det(W_i) u_i, \tag{5}
\]

where \(A_j\) is the submatrix of \(A\) formed by the columns with indices in \(j\), and \(W_i\) is the submatrix of \(W\) formed by the lines with indices in \(i\).

Proof. The first equality is Proposition 3.5ii, as \(\det A_j = \langle A, w_j \rangle\). The other follows by Laplace expansion of \(\det(W_i)\) w.r.t. the first \(p\) rows, as if \(w_j = \sum_{i=1}^n \lambda_{ij} u_i\) then \(w_y = \sum_{i \in I_p^{n-p}} \lambda_{iy} u_i\) for \(\lambda_{iy} = \det(\lambda_{ij})_{i \in I, j \in y}\).

Proposition 3.42. \(\|A \bowtie B\| = \sqrt{\det M}\).

Proof. If \(p > q\), Laplace expansion w.r.t. the first \(p\) lines gives \(M = 0\). If \(p \leq q\), and \(N_y = (A^\dagger B_y)\) is the submatrix of \(M\) formed by \(A^\dagger\) and the columns of \(B\) with indices not in \(j\), the same expansion and (5) give \(\|A \bowtie B\|^2 = \langle B, A \land (A \bowtie B) \rangle = \sum_{j \in I_p^q} \epsilon_{jy} \det(A_j) \langle B, A \land w_y \rangle = \sum_{j \in I_p^q} \epsilon_{jy} \det(A_j) \cdot \det(N_y) = (-1)^p \det M\).

If \(B \neq 0\), \(\|A \bowtie B\| = \sqrt{\det B \cdot \det(AB^{-1}A^\dagger)}\), by Schur’s determinant identity. In [30], we use it to compute an asymmetric Fubini-Study metric.

Example 3.43. Let \(v_1, v_2, w_1, w_2, w_3 \in \mathbb{R}^3\) have \(A = (-1 0 1)\), and \(W = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\) in basis \(\beta = (u_1, u_2, u_3)\). By (5), \(v_1 \bowtie w_{123} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} u_2 + \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 1 & 0 \end{pmatrix} u_3 = 4u_1 + 6u_2 + 8u_3\), same as expanding the \(w_i\'s\) in \(v_1 \bowtie w_{123} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} w_1 + \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} w_2 + \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 1 & 0 \end{pmatrix} w_3\). With \(A = (-1 0 1)\), \(v_1 \bowtie w_{123} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} u_1 + \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} u_2 + \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 1 & 0 \end{pmatrix} u_3 = u_1 - 4u_2 + 8u_3\), same as \(v_1 \bowtie w_{123} = -u_1 + 4u_2 + 8u_3\). If \(\beta\) is orthonormal, so \(B = W^\dagger W\), Proposition 3.42 gives \(\|v_1 \bowtie w_{123}\| = \sqrt{115}\) and \(\|v_1 \bowtie w_{123}\| = 9\).
3.6 Clifford product and contractions

In this section, \( X \) is real\(^2\). Its Clifford Geometric Algebra \([9, 18, 36]\) is \( \bigwedge X \) with a bilinear associative Clifford geometric product \( MN \) for \( M, N \in \bigwedge X \), which has \( uv = \|v\|^2 \) for \( v \in X \), and \( v_1v_2\cdots v_p = v_1 \wedge v_2 \wedge \cdots \wedge v_p \) for orthogonal \( v_1, \ldots, v_p \in X \). So, \( uv = \langle v, w \rangle + v \wedge w \) for \( v, w \in X \). For an orthonormal basis \((v_1, \ldots, v_n)\), as \( v_i^2 = 1 \) and \( v_iv_j = -v_jv_i \) if \( i \neq j \), we have \( v_iv_j = (-1)^{|i\wedge j|}v_i \wedge v_j \) for \( i, j \in \mathbb{T}^n \). For \( G \in \bigwedge X \) and \( H \in \bigwedge X \), \( GH \) can have homogeneous components of grades \( \|q-p\|, |q-p| + 2, \ldots, p + q \), with \( \langle GH \rangle_{p+q} = G \wedge H \). In a versor \( M = v_1v_2\cdots v_p \), for \( v_1, \ldots, v_p \in X \), all components have the parity of \( p \), so \( M = (-1)^p M \) and \( M = (-1)^{p(n+1)} M \), but \( M \neq (-1)^{\frac{n(n-1)}{2}} M \). Blades are versors, as they can be factored into orthogonal vectors. Using an orthonormal basis, for \( L, M, N \in \bigwedge X \) one finds \( \langle MN \rangle^* = MN, \langle MN \rangle = MN, \langle MN \rangle = \hat{N}M, \langle M, N \rangle = \langle MN \rangle_0 \), and \( \langle L, MN \rangle = \langle ML, N \rangle = \langle LN, M \rangle \). The mirror principle holds with Clifford products.

**Proposition 3.44.** \( G \wedge H = (GH)_{q-p} \) and \( G \vee H = (GH)_{p-q} \), for \( G \in \bigwedge X \) and \( H \in \bigwedge X \).

**Proof.** For \( F \in \bigwedge X \), \( \langle F, G \wedge H \rangle = \langle G \wedge F, H \rangle = \langle (GF), H \rangle = \langle GF, H \rangle = \langle F, (GH)_{q-p} \rangle \). Likewise for \( G \vee H \).

**Proposition 3.45.** Let \( v, w_1, \ldots, w_q \in X \) and \( M, N \in \bigwedge X \).

i) \( vM = v \wedge M \).

ii) \( v \wedge M = \frac{vM - Mv}{2} \) and \( v \wedge M = \frac{vM + Mv}{2} \).

iii) \( v \wedge (MN) = (v \wedge M)N + \hat{M}(v \wedge N) \).

iv) \( v \wedge (MN) = (v \wedge MN) - \hat{M}(v \wedge N) \).

v) \( v \wedge (MN) = (v \wedge M)N + \hat{M}(v \wedge N) \).

vi) \( v \wedge (MN) = (v \wedge M)N - \hat{M}(v \wedge N) \).

vii) \( \langle w_1w_2\cdots w_q \rangle = \sum_{i=1}^q (-1)^{q-2} \langle v, w_i \rangle w_1w_2\cdots w'_i w_q, \) where \( w'_i \)

means \( w_i \) is absent.

viii) \( \langle v \rangle \wedge M \wedge N = (v \wedge M) \wedge N + M \wedge (v \wedge N) \).

**Proof.** (i) Assume \( v = v_1 \) and \( M = v_1 \) for an orthonormal basis \((v_1, \ldots, v_n)\) and \( i \in \mathbb{T}^n \). If 1 \( \in i \) then \( v_1 \wedge v_1 = 0 \) and \( v_1v_1 = v_1 \wedge v_1 \). If 1 \( \notin i \), \( v_1 \wedge v_1 = 0 \) and \( v_1v_1 = v_1 \wedge v_1 \). (ii) \( \hat{M}v = M \wedge v + M \wedge v = -v \wedge M + v \wedge M \).

(iii) \( \langle v \rangle \wedge (MN) = \frac{vMN - MNv}{2} = \frac{vM - Mv}{2} N + \hat{M} \frac{vN - Nv}{2} \). (iv–vi) Similar. (vii) Follows from (iii) by induction. (viii) Follows from (i).

By iii, \( v \wedge \) is a graded derivation w.r.t. the Clifford product as well. With some rearrangements, iii and v are adjoint formulas, while iv and vii are self-adjoint. Since viii is not as simple as Proposition 3.5i, there is no easy formula like Corollary 3.7 for when the contractor is a versor. We find \( (uvw) \hat{M} = \langle u, v \rangle w \hat{M} - \langle u, w \rangle v \hat{M} + \langle v, w \rangle u \hat{M} + w \hat{M} \). (v–vi) Similar. (vii) Follows from (iii). (viii) Follows from (i).

\(^2\)Complex Clifford algebras fail to reflect complex geometry: \( vw \neq \langle v, w \rangle + v \wedge w \), since \( vw \) is bilinear but \( \langle v, w \rangle \) is sesquilinear. As complex numbers can be represented in real Clifford algebras, one can use these, but this is not always convenient.
Corollary 3.46. Let $M \in \bigwedge V$ and $N \in \bigwedge W$.

i) If $L \in \bigwedge (W^\perp)$ then $L \wedge (MN) = (L \wedge M)N$.

ii) If $H \in \bigwedge^p (V^\perp)$ then $H \wedge (MN) = M^\ast p (H \wedge N)$.

Proof. (i) Linearity and Corollary 3.7 let us assume $L = v \in W^\perp$, in which case it follows from Proposition 3.45iii. (ii) Similar.

Likewise, $L \wedge (MN) = (L \wedge M)N$ and $H \wedge (MN) = M^\ast p (H \wedge N)$.

Corollary 3.47. For $v \in X$ and nonzero $M \in \bigwedge V$ and $N \in \bigwedge W$ in disjoint subspaces $V$ and $W$, $v \wedge (MN) = 0 \Leftrightarrow v \wedge M = v \wedge N = 0$.

Proof. As in Proposition 3.13, but using Proposition 3.45iii.

Theorem 3.48. For $v_1, \ldots, v_p \in X$ and $M, N \in \bigwedge X$,

$$v_1 \ldots p \wedge (MN) = \sum_{\epsilon \in iF^p} \epsilon (v_\epsilon \wedge M^\ast |\epsilon|)(v_\epsilon \wedge N),$$

and

$$v_1 \ldots p \wedge (MN) = \sum_{\epsilon \in iF^p} \epsilon (v_\epsilon \wedge M^\ast |\epsilon|)(v_\epsilon \wedge N).$$

Proof. The proof of (6) is like that of (3), but with Proposition 3.45iii and Clifford products instead of $\wedge$. Proposition 3.45v gives (7) for $p = 1$ and, assuming it for $v_1 \ldots (p-1)$, also $v_1 \ldots p \wedge (MN) = v_p \wedge (v_1 \ldots (p-1) \wedge (MN) = v_p \wedge \sum_{\epsilon \in iF^{p-1}} \epsilon (\tilde{v}_\epsilon \wedge M^\ast |\epsilon|)(\tilde{v}_\epsilon \wedge N) = I + II$, where

$$I = \sum_{\epsilon \in iF^{p-1}} \epsilon (v_p \wedge (\tilde{v}_\epsilon \wedge M^\ast |\epsilon|))(\tilde{v}_\epsilon \wedge N),$$

and

$$II = \sum_{\epsilon \in iF^{p-1}} \epsilon (\tilde{v}_\epsilon \wedge M^\ast |\epsilon|)(v_p \wedge \tilde{v}_\epsilon \wedge N),$$

which are developed as before, using $\tilde{v}_1 \wedge v_p = v_p \wedge \tilde{v}_1 = \tilde{v}_p$.

4 Star duality

Hodge-like star operators can be defined by contraction with an orientation element (also called a unit pseudoscalar or volume element).

Definition 4.1. An orientation of $X$ is a unit $\Omega \in \bigwedge^n X$, for $n = \dim X$. It gives left and right star operators $\star : \bigwedge X \rightarrow \bigwedge X$, and left and right duals of $M \in \bigwedge X$, respectively, via $\star M = \Omega \wedge M$ and $\star^* = M \wedge \Omega$.

When we use $\star$, it is implicit an $\Omega$ was chosen. If an orthonormal basis $(v_1, \ldots, v_n)$ is fixed, assume $\Omega = v_1 \ldots n$. A real space has 2 orientations $\pm \Omega$, but a complex one has a continuum of them, the unit circle in $\bigwedge^n X$. Our approach is unorthodox: complex spaces are usually seen as canonically oriented [20, p. 25], with the complex structure inducing a real orientation $\Omega_R \in \bigwedge^{2n} X_R$ in the underlying real space $X_R$.

We also write $\star_L$ and $\star_R$ for left and right stars. Having both simplifies the algebra, and the mirror principle switches them. Note that $\star_R$ uses a left contraction, and vice-versa ($\star$ is at the same side as $\Omega$). By convention, $\star$ takes precedence over products: $\star M \wedge N^\ast$ means $(\star M) \wedge (N^\ast)$.
In the real case, our $M^*$ and $^*M$ correspond, respectively, to the Hodge dual $^*M$ and $^{-1}M$ [1, 20], to $M^*$ and $^*M$ in [36], and, up to signs (see Appendix A.3), to the undual $M^{−*}$ and dual $M^*$ of GA [9]. In the complex case, our stars are conjugate-linear, simpler than the Hodge star of complex analysis\(^3\) [40], and better suited for complex geometry, relating blades of orthogonal complex subspaces.

**Proposition 4.2.** For a $p$-blade $B \neq 0$, $B^*$ is the unique $(n-p)$-blade such that $[B^*] = [B]^+$, $\|B^*\| = \|B\|$ and $B \wedge B^*$ has the orientation of $\Omega$.

**Proof.** Follows from Theorem 3.26.

Likewise for $^*B$, except that $^*B \wedge B$ has the orientation of $\Omega$. Note that $[0^*] = [0]^+$.

**Proposition 4.3.** $v_i^* = \epsilon v_i v_i$ and $^*v_i = \epsilon v_i v_i$ for $i \in I^n$, an orthonormal basis $(v_1, \ldots, v_n)$, and orientation $v_1 \ldots n$.

**Proof.** Follows from (2).

**Example 4.4.** Let $(v_1, \ldots, v_4)$ be an orthonormal basis of $C^4$, and $B = 3v_1 + iv_3 + v_4$. Then $B^* = 3v_1^* - iv_3^* + v_4^* = 3v_{234} - iv_{123} = v_2 \wedge (v_3 - 3v_4) \wedge (v_1 + iv_3)$, and so $[B]^+ = [v_2] \oplus [v_3 - 3v_4] \oplus [v_1 + iv_3]$. Also, $B \wedge B^* = 11\Omega$.

**Proposition 4.5.** Let $M \in \bigwedge X$.

1. $(M^*)^* = (\hat{M})^*$.
2. $^*M = M$.
3. $^*M = \check{M}^*$.

**Proof.** (i) $(M^*)^* = (M \otimes \Omega)^* = \hat{M} \otimes \hat{\Omega} = (\check{M})^*$. (ii) Follows from Corollary 3.21. (iii) Follows from Proposition 3.4iv.

This shows $^*L$ and $^*R$ are inverses, and lets us (re)position $^*$ as needed. As $M^* = \check{M}$, if $n$ is odd then $^*L = ^*R$ is an involution of $\bigwedge X$. If $n$ is even this holds in $\bigwedge^+ X$. While $^*$ and $^\dagger$ commute, $(M^*)^\dagger = (\check{M})^\dagger = (-1)^n (\check{M})^\dagger$ and $(M^*)^\dagger = ^\dagger(\check{M}) = (-1)^{\frac{n(n-1)}{2}} (\check{M})^\dagger$, with stars $^\dagger$ and $^\ddagger$ for $\hat{\Omega}$ and $\check{\Omega}$.

**Proposition 4.6.** Let $M, N \in \bigwedge X$.

1. $(M \wedge N)^* = N \wedge M^*$ and $(M \vee N)^* = N \wedge M^*$.
2. $M^* \wedge N = M \wedge N^*$ and $^*M \wedge N = M \wedge N^*$.
3. $\langle M^*, N^* \rangle = \langle N, M \rangle$.

**Proof.** (i) $(M \wedge N)^* \wedge \Omega = N \wedge (M \otimes \Omega)$, and, with its mirror, $(M \wedge N)^* = (^*M \wedge N)^* = (N \wedge M^*)^* = N \wedge M^*$. (ii) $(M \otimes \Omega) \wedge N = M \wedge (\Omega \wedge N)$, and $^*M \wedge N = ^*M \wedge N^* = M \wedge N^*$.* (iii) As elements of distinct grades are orthogonal, we can assume $M, N \in \bigwedge^p X$, so that $\langle N, M \rangle = N \wedge M = N \wedge M^* = N^* \wedge M = \langle M^*, N^* \rangle$.

\(^3\)A C-linear (sometimes conjugate-linear, and denoted by $^\ddagger$) extension of the real Hodge star of the dual space $X_R^*$ of $X_R$ to its complexification $X_R^* \otimes_R \mathbb{C}$, relating $\mathbb{C}$-valued $R$-linear $p$- and $(2n-p)$-forms [40, pp. 156–159].
So, $\star$ can turn $\wedge$ into $\downarrow$ or $\downarrow$ and vice-versa, or switch $\downarrow$ and $\perp$, but one must heed the sides. Other formulas are obtained via mirror principle and Proposition 4.5, like $(M \downarrow N)^\star = N^\star \wedge M$ and $M^\star \perp N^\star = M \downarrow N$. The complex case can help identify errors: e.g. $(M \downarrow N)^\star \neq N \wedge M^\star$, as the first one is linear in $M$, and the other is conjugate-linear. The reordering in $i$ is again due to the adjoint nature of contractions. The corresponding formulas in GA [9, p. 82] avoid it, and use only the left contraction, but this simplicity comes at the cost of extraneous signs (see Appendix A.3).

By $i$, a subspace can be given by a blade $B$ or its dual $B^\star$, as the solution set of $v \wedge B = 0$ or $v \downarrow B^\star = 0$ (like a plane given by a normal vector in $\mathbb{R}^3$). In [31], we study such equations for $M \in \bwedge X$. By iii, stars are orthogonal operators (anti-unitary, in the complex case), giving isometries (anti-isometries, in the complex case) of $\bwedge^n X$ with $\bwedge^{n-2} X$. In the real case, $\star_L$ is the adjoint of $\star_R$ (if $n$ is odd, $\star$ is self-adjoint).

**Corollary 4.7.** $||M^\star|| = ||M||$, for $M \in \bwedge X$.

**Corollary 4.8.** $G \wedge H^\star = \langle H, G \rangle \Omega$, for $G, H \in \bwedge X$.

Some authors use this to define stars, or to, given a star, determine an inner product by $(G, H) = (G \wedge H^\star)^\star$.

**Example 4.9.** In $\mathbb{R}^3$, the cross product is $u \times v = (u \wedge v)^\star = v \downarrow u^\star$. The usual triple product is $\langle u, v \times w \rangle = u \downarrow (v \wedge w)^\star = (u \wedge v \wedge w)^\star$, whose modulus is the volume of $u \wedge v \wedge w$. We also easily obtain $(u \times v) \times w = w \downarrow (u \times v)^\star = w \downarrow (u \wedge v) = (\langle u, w \rangle v - \langle v, w \rangle u)$, and $(u \times v, w \times y) = \langle (u \wedge v)^\star, (w \wedge y)^\star \rangle = \langle u \wedge v, w \wedge y \rangle = \langle u, w \rangle \langle v, y \rangle - \langle u, y \rangle \langle v, w \rangle$.

In the real case, stars and Clifford products are related as follows.

**Proposition 4.10.** Let $v, v_1, \ldots, v_p \in X$ and $M, N \in \bwedge X$.

i) $M^\star = \hat{M} \Omega$.

ii) $(MN)^\star = \tilde{N} M^\star$.

iii) $(v_1 v_2 \cdots v_p)^\star = v_p \cdots v_2 v_1^\star$.

iv) $(M^\star) N = \hat{M} (\tilde{N})^\star$.

v) $(v M)^\star = (v^\star M)^\star + M \downarrow v^\star$.

**Proof.** (i) If $M = v_1$ and $\Omega = v_1 \cdots$, for an orthonormal basis $(v_1, \ldots, v_n)$ and $i \in \mathbb{Z}_n^+$ then $v_i^\star = v_{i+1} \cdots = (\hat{v}_i = \tilde{v}_i)_{n-q} = \tilde{v}_i$, as $\hat{v}_i \Omega = \pm v_{i+1} \cdots = \pm v_n$. (ii) $(MN)^\star = (MN) \Omega = \hat{N} \tilde{M} \Omega$. (iii) Follows from ii. (iv) Follows from i and its mirror. (v) Follows from Propositions 3.45i and 4.6i. □

Stars with respect to oriented subspaces are also useful.

**Definition 4.11.** A unit $q$-blade $B \in \bwedge X$ gives left and right stars $\star_B : \bwedge X \rightarrow \bwedge [B]$, and left and right duals of $M \in \bwedge X$ w.r.t. $B$, respectively, by $^\star_B M = B \perp M$ and $M^{\star_B} = M \downarrow B$.

For a $p$-blade $A \neq 0$, $A^{\star_B}$ is a $(q-p)$-blade, and $A^{\star_B} \neq 0 \Leftrightarrow [A] \not\subseteq [B]$, in which case $[A^{\star_B}] = [A] \perp [B]$. For $M \in \bwedge X$, $^\star_B M^{\star_B} = P_B M$ and $M^{\star_B} = \star_B (M^{\star_B})$. In $\bwedge [B]$, $\star_B$ has properties as $\star$, which with $M^{\star_B} = (P_B M)^{\star_B}$ can be extended to $\bwedge X$: e.g. $(M^{\star_B}, N^{\star_B}) = (P_B N, P_B M)$.

**Proposition 4.12.** Let $B$ be a unit blade, and $M, N \in \bwedge X$. 18
4.1 Regressive product

Stars induce a regressive product dual to the exterior one. These two are the basic products of Grassmann-Cayley algebra [41], in which, however, their symbols are usually swapped.

**Definition 4.13.** The regressive product \( M \lor N \) of \( M, N \in \wedge X \) is given by \((M \lor N)^* = M^* \lor N^*\).

It is bilinear, associative, and satisfies \( G \lor H = (-1)^{(n-p)(n-q)} H \lor G \in \wedge^{p+q-n} X \) for \( G, H \in \wedge^p X \) and \( n = \dim X \). Also, \( G \lor H = 0 \) if \( p + q < n \).

**Proposition 4.14.** Let \( M, N \in \wedge X \).

i) \((M \lor N)^* = \tilde{M} \lor \tilde{N}\).

ii) \((M \lor N)^* = M^* \lor N^*\).

iii) \(M \lor N = N \lor M^*\).

*Proof.* (i) Follows via Proposition 4.5i. (ii) \((M \lor N)^* = (M^* \lor N^*)^* = (M \lor N)^* = M^* \lor N^*\). (iii) \(M \lor N = (M \lor N)^* = N \lor M^*\).

Also, \((M \lor N) = \tilde{M} \lor \tilde{N} = (-1)^n \tilde{M} \lor \tilde{N} \) and \((M \lor N) = \tilde{N} \lor \tilde{M} = (-1)^{n-q} \tilde{N} \lor \tilde{M}\), where \(\lor\) and \(\lor\) are regressive products w.r.t. \(\tilde{\Omega}\) and \(\tilde{\Omega}\). Relabeling \(\lor\) as \(\lor\), the mirror principle holds.

**Proposition 4.15.** \(v_i \lor v_j = \delta_{i,j=1\ldots n} \epsilon_{i'j'} v_{i'j'}\) for \(i, j \in I^n\), orthonormal basis \(v_1, \ldots, v_n\), and orientation \(v_1 \ldots v_n\).

*Proof.* \(v_i \lor v_j = v_{i\cup j} v_{i\cup j'} = \epsilon_{i'j'} v_{i\cup j} v_{i\cup j'} \) is 0 unless \(i \cup j = 1 \ldots n\) (so \(i' \subset j\)), in which case \(j = (i \cup j') j\) and \(v_i \lor v_j = \epsilon_{i'j'} v_{i\cup j}\), as Proposition 2.1 gives \(\epsilon_{i'j'} \epsilon_{i'j'} = \epsilon_{(i'j')j'} \epsilon_{i'j'} = \epsilon_{i'j'} \epsilon_{i'j'} = \epsilon_{i'j'}\).

The following geometric characterization dualizes the fact that, for nonzero blades, \(A \lor B \neq 0 \iff [A] \cap [B] = \{0\}\), in which case \([A \lor B] = [A] \oplus [B]\), and \(\|A \lor B\| = \|A\|\|B\| \cos \Theta_{[A][B]^\perp}\) (see [29]).

**Theorem 4.16.** For nonzero blades \(A \in \wedge X\) and \(B \in \wedge X\), \(A \lor B = 0 \iff [A] + [B] = \{0\}\). If \(A \lor B \neq 0\), it is the only \((p + q - n)\)-blade with:

i) \([A \lor B] = [A] \cap [B]\).

ii) \(\|A \lor B\| = \|A\|\|B\| \cos \Theta_{[A][B]^\perp}\).\)

iii) \(A \lor ((A \lor B) \lor B)\) has the orientation of \(\Omega\).
Figure 4: For unit blades $A, B \in \Omega^2 \mathbb{R}^3$ with $\Theta_{[A], [B]} = 60^\circ$, $A \lor B$ is a vector $v$ of norm $\frac{1}{2}$ in $[A] \cap [B]$, oriented so if $B = v \land B'$ then $A \land B'$ has the orientation of $\mathbb{R}^3$.

**Proof.** $(A \lor B)^* = A^* \land B^* = 0 \iff \{0\} \neq [A^*] \cap [B^*] = ([A] + [B])^\perp$. If $A \lor B = B \land A^* \neq 0$ then $[A \lor B] = [A^*]^\perp \cap [B]$, $\langle \Omega, A \land ((A \lor B) \land B) \rangle = \langle A^*, B, A^* \land B \rangle = \langle A^*, \|B\|^2 P_B(A^*) \rangle = \|B\|^2 \|P_B(A^*)\|^2$, and $P_B(A^*) \neq 0$.

Also, $\|A \lor B\| = \|B \land A^*\| = \|A^*\| \|B\| \cos \Theta_{[A], [B]}$.

So, $A \lor B$ describes ‘necessary’ intersections, that occur once the subspaces fill up $X$. As $[A] + [B] \neq X \iff [A]^\perp \lor [B] \iff \Theta_{[A]^\perp, [B]} = \frac{\pi}{2}$, and areas of $V^\perp$ orthogonally projected on $W$ contract by $\cos \Theta_{V \lor W} = \frac{\|A\|^2}{\|A\|^2} = \frac{\sqrt{2}}{4}$. Note that $\cos \Theta_{V \lor W} = \frac{\|A\|^2}{\|A\|^2} = 1$,

$\Theta_{V \lor W} = \Theta_{\scriptscriptstyle V,W} = \Theta_{\scriptscriptstyle W,V}$ as dim $V^\perp = \text{dim} W$, but $\cos \Theta_{V \lor W} = \frac{\|A\|^2}{\|A\|^2} = 1$

**Example 4.17.** If $A = v_{13}$ and $B = (v_1 + v_2) \land (v_3 + \sqrt{3} v_4)$ for an orthonormal basis $(v_1, \ldots, v_4)$ of $\mathbb{R}^4$ then $A \lor B = \sqrt{3} v_{13} \lor v_{24} = \sqrt{3} \varepsilon_{1234} v_9 = -\sqrt{3}$, so $V \lor W = \{0\}$ for $V = [A]$ and $W = [B]$. Also, $A \land ((A \lor B) \land B) = 3 \varepsilon_{1234}$, and areas of $V^\perp$ orthogonally projected on $W$ contract by $\cos \Theta_{V \lor W} = \frac{\|A\|^2}{\|A\|^2} = \frac{\sqrt{2}}{4}$, and so $\Theta_{V \lor W} + \Theta_{V \lor W} \neq \frac{\pi}{2}$.

**Example 4.18.** If $A = v_1 \land (v_2 + v_4) \land (iv_3 + v_4) = iv_{123} + v_{124} - iv_{134}$ and $B = v_{123}$ for an orthonormal basis $(v_1, \ldots, v_4)$ of $\mathbb{C}^4$ then $A \lor B = \varepsilon_{12} v_{12} - i \varepsilon_{23} v_{23} = -v_{12} + iv_{13}$, so $V \lor W = [v_1] \lor [iv_3 - v_2]$ for $V = [A]$ and $W = [B]$. Also, $A = A' \land C$ for $C = A \lor B$ and $A' = \frac{A}{\|A\|^2} = -\frac{-12 + 2 i 4 + 1 i 3}{2}$, $B = C \land B'$ for $B' = \frac{C}{\|C\|^2} = \frac{-v_{13} - 4 v_2}{2}$, and $A \land B' = A' \land B = v_{1234}$.

As $\cos \Theta_{V \lor W} = \frac{\sqrt{2}}{4}$, areas of $V^\perp$ orthogonally projected on $W$ contract by $\frac{\sqrt{2}}{4}$. As $\cos \Theta_{V \lor W} = \frac{1}{2}$ and $\Theta_{V \lor W} = 90^\circ$, 6-dimensional volumes of $V$ contract by $\frac{1}{2}$ if orthogonally projected on $W$, and vanish on $W^\perp$.

### 4.1.1 join and meet

The usefulness of $\lor$ for finding intersections is limited by the condition $[A] + [B] = X$, which is too restrictive for low grade blades in large spaces. A workaround is to reduce the space to $[A] + [B]$, in which case another
notation is used [9] (but note that some authors use the terms join and meet for and ).

**Definition 4.19.** The join \( A \cup B \) and meet \( A \cap B \) of blades \( A \) and \( B \) are defined, up to scalar multiples, as nonzero blades \( J \) and \( M \), respectively, such that \( [J] = [A] + [B] \) and \( [M] = [A] \cap [B] \).

These operations are nonlinear and restricted to blades. If a join \( J \) is known, a meet can be obtained as in the regressive product, but with \( * \).

And given a meet we can obtain a join.

**Proposition 4.20.** Let \( A, B \in \bigwedge X \) be nonzero blades.

i) Given a unit join \( J \), a meet is \( A \cap B = B \cdot A^* \).

ii) Given a meet \( M \), a join is \( A \cup B = A \cap (M \cdot B) \).

**Proof.** (i) \([A] \cup [B] = [J]\) implies \([A^*] = [A] \cap [J] \neq [B] \), so \([B \cdot A^*] = [A^*] \cap [B] = ([A] + [J^*]) \cap [B] = [B] \cap [B] \). (ii) \( B = M \cap B' \) with \([B'] \perp [M] \), so \( M \cdot B = [M][B'] \) and \( [A \cap (M \cdot B)] = [A \cap B'] = [A] + [B] \).

The meet \( A \cap B \) in \( i \) has properties like \( A \vee B \), but considering the space as \([J]: \) e.g., \([A \cap B] = \|[A]\|\|B\| \cos \Theta_{[A] \perp \cap [J], [B]} \). Its bilinear formula remains valid if \( A \) or \( B \) changes but \([A] + [B] \) does not.

**Example 4.21.** If \( A = (v_1 + v_4) \wedge (v_2 + 2v_3) \) and \( B = v_1 \) for an orthonormal basis \((v_1, v_2, v_3, v_4)\) of \( \mathbb{R}^4 \) then \( A \wedge B = 0 \) does not give \([A] \cap [B] \).

As \([A] \) and \([B] \) are distinct planes in \([v_{124}] \), a unit join is \( J = v_{124} \), a meet is \( M = B \cup (A \cdot J) = v_{124} \cup (v_4 - 2v_2 - v_1) = -2v_1 + v_2 \), and \([A] \cap [B] = [v_2 - 2v_1] \). Lengths in \([A] \perp [J] = [A \cdot J] = [v_4 - 2v_2 - v_1] \) contract by \( \cos \Theta_{[A] \perp \cap [J], [B]} = \frac{v_1^2}{5} \) if orthogonally projected on \([B] \).

### 4.2 Outermorphisms and duality

Let \( T : X \to Y \) be a linear map into a Euclidean or Hermitian space \( Y \) (same as \( X \)) with orientation \( \Omega_Y \) and star \( * \) (beware: \( * \) is in \( X \), \( \ast \) in \( Y \)). We have \( T \Omega \neq 0 \iff T \) is injective. If \( T \) is invertible then \( T \Omega = \Delta_T \cdot \Omega_Y \) for a scalar \( \Delta_T \neq 0 \). If \( Y = X \) then \( T \Omega = (\det T) \cdot \Omega \).

**Proposition 4.22.** \( T((T^\dagger N)^\dagger) = N \cdot T \Omega \), for \( N \in \bigwedge Y \). If \( T \) is injective then \( T((T^\dagger N)^\dagger) = \|T\Omega\| N^* \) with \( B = \frac{T\Omega}{\|T\Omega\|} \), otherwise \( T((T^\dagger N)^\dagger) = 0 \).

**Proof.** Follows from Proposition 3.38. \( \square \)

**Corollary 4.23.** If \( T \) is an isometry then \( T(M^*) = (TM)^* \), for \( M \in \bigwedge X \). If \( T \) is also invertible then \( T(M^*) = \Delta_T \cdot (TM)^* \) and \( |\Delta_T| = 1 \).

**Proof.** \( M = T^\dagger N \) for \( N = TM \), and \( \|T\Omega\| = 1 \).

**Corollary 4.24.** If \( T \) is invertible, the following diagram is commutative.

Stars can be left or right, but must have equal (resp. opposite) sides for equal (resp. opposite) arrows.

\[
\begin{align*}
\bigwedge X & \xrightarrow{T^\dagger} \bigwedge X \\
\bigwedge Y & \xleftarrow{T^{-1}} \bigwedge Y
\end{align*}
\]
Proof. \( T((T^\dagger N)') = \Delta_T \cdot N^* \) for \( N \in \bigwedge Y \), as above. Other relations follow from it, like \( T(M^*) = \Delta_T \cdot ((T^\dagger)^{-1}M)^* \) for \( M = T^\dagger N \in \bigwedge X \).

**Corollary 4.25.** \( T^{-1}N = \frac{1}{\Delta_T}((T^\dagger(N^*)) \) for \( N \in \bigwedge Y \), if \( T \) is invertible.

**Example 4.26.** If \( T = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix} \), \( N = w_1 + w_{13} \), \( \Omega = v_{123} \) and \( \Omega_Y = w_{123} \) in orthonormal bases \((v_1, v_2, v_3)\) of \( X \) and \((w_1, w_2, w_3)\) of \( Y \) then \( \Delta_T = 3i \), \( N^* = w_{23} - w_2 \) and \( T^\dagger(N^*) = (2v_1 - iv_2) \wedge (v_1 + iv_3) - (2v_1 - iv_2) \), so \( T^{-1}N = \frac{1}{3}(2iv_{13} + iv_{12} + v_{23} - 2v_1 + iv_2) = \frac{2v_{23} - iv_1 + 2iv_{23} + v_{13}}{3} \).

**Corollary 4.27.** Let \( T : X \to X \) be linear, and \( M \in \bigwedge X \).

i) \( T((T^\dagger M)^*) = (\det T) \cdot M^* \).

ii) \( T(M^*) = (\det T) \cdot ((T^\dagger)^{-1}M)^* \) if \( T \in GL(X) \).

iii) \( T(M^*) = (\det T) \cdot (TM)^* \) if \( T \in U(X) \).

iv) \( T(M^*) = (TM)^* \) if \( T \in SU(X) \).

In the real case, the general and special unitary groups \( U(X) \) and \( SU(X) \) become the orthogonal ones, \( O(X) \) and \( SO(X) \).

**Proposition 4.28.** If \( T : X \to Y \) is invertible and \( M, N \in \bigwedge X \) then \( T(M \land N) = \frac{1}{\det T}TM \land TN \), where the second \( \land \) is w.r.t. \( \ast \) (in \( Y \)).

**Proof.** \( T(M \land N) = T((M^* \land N^*)) = \Delta_T \cdot ((T^\dagger)^{-1}(M^* \land N^*)) = \Delta_T \cdot ((TM)^* \land (TN)^*) = \frac{T(M \land TN)}{\Delta_T} \).

**Corollary 4.29.** \( T(M \land N) = \frac{1}{\det T}TM \land TN \), for \( M, N \in \bigwedge X \) and \( T \in GL(X) \).

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### A The contractions zoo

The literature has many contractions (also called interior products, inner multiplications, inner derivatives, insertion operators, etc.), all playing similar roles, but with subtle differences which can be confusing. Here we explain their differences and the reason for such diversity.

#### A.1 Different contractions

In this section, \( X \) does not need an inner product at first, \( X' \) is its dual space, and we use \( A, B, C \) for any multivectors.

The simplest contraction is the pairing of \( \varphi \in X' \) and \( v \in X \), giving a scalar \( \langle \varphi, v \rangle = \varphi(v) \). For general tensors [3], contractions trace out selected pairs \((i, j_k)\) of covariant and contravariant indices, giving a product of pairings \( \varphi_i(v_{j_1}) \) and a lower order tensor. For multivectors and forms
(multi-covectors), alternativity requires adding the results of all correspondences of indices of one element with those of the other, choosing an initial one as positive, and changing sign for each index transposition.

Let \( A = v_1 \wedge \cdots \wedge v_p \in \Lambda^p X \) and \( \alpha = \varphi_1 \wedge \cdots \wedge \varphi_q \in \Lambda^q (X') \). If \( p = q \), their contraction is a pairing \( \langle \alpha, A \rangle = \alpha(v_1, \ldots, v_p) = \det (\varphi_i(v_j)) \), with \( \varphi_1(v_1) \cdots \varphi_p(v_p) \) chosen as positive. Setting \( \langle \alpha, A \rangle = 0 \) if \( p \neq q \), and extending linearly, we have a nondegenerate pairing of \( \Lambda^p X \) and \( \Lambda^q (X') \), and an isomorphism \( (\Lambda^p X) \cong \Lambda^q (X') \). Contractions differ from this pairing if \( p \neq q \), giving, instead of a scalar, a lower grade multivector or form built with the \( \varphi_i \)'s or \( v_j \)'s left out of each index correspondence. Two choices for an initial positive correspondence give left or right contractions:\(^4\) first covectors with first vectors, or last covectors with last vectors.

For \( p = 1 \), contractions of \( v \in X \) on \( \alpha \) are \((q - 1)\)-forms. The left one, \( v \lrcorner \alpha = \sum_{i=1}^q (-1) \varphi_1 \wedge \cdots \wedge \varphi_i (v) \wedge \cdots \wedge \varphi_q \), matches \( v \) positively with \( \varphi_1 \), and the right one, \( \alpha \lrcorner v = \sum_{i=1}^q (-1)^{q-i} \varphi_1 \wedge \cdots \wedge \varphi_i (v) \wedge \cdots \wedge \varphi_q \), with \( \varphi_q \). They are partial evaluations: for \( u_1, \ldots, u_{q-1} \in X \), \( v \) inserted in the first entry of \( \alpha \) in \( (v \lrcorner \alpha)(u_1, \ldots, u_{q-1}) = \alpha(v, u_1, \ldots, u_{q-1}) \), or in the last one in \( (\alpha \lrcorner v)(u_1, \ldots, u_{q-1}) = \alpha(u_1, \ldots, u_{q-1}, v) \). Equivalently, \( \langle v \lrcorner \alpha, B \rangle = \langle \alpha, v \wedge B \rangle \) and \( \langle \alpha \lrcorner v, B \rangle = \langle \alpha, B \wedge v \rangle \) for \( B \in \Lambda^{q-1} X \).

Likewise, for \( q = 1 \), contractions of \( \varphi \in X' \) on \( A \) are \((p - 1)\)-vectors given by \( \langle \beta, \varphi \lrcorner A \rangle = \langle \varphi \lrcorner \beta, A \rangle \) and \( \langle \beta, A \lrcorner \varphi \rangle = \langle \beta \wedge \varphi, A \rangle \) for \( \beta \in \Lambda^{p-1} X' \), so \( \varphi \) is applied positively on \( v_1 \) for \( \varphi \lrcorner A \), or on \( v_p \) for \( A \lrcorner \varphi \).

Generalizing, we have four contractions, \( A \lrcorner \alpha, \alpha \lrcorner A, A \lrcorner \alpha \), \( \alpha \lrcorner A \in \Lambda^{q-p} X' \) and \( \alpha \lrcorner A, A \lrcorner \alpha \in \Lambda^{p-q} X \), given, for \( B \in \Lambda^{q-p} X \) and \( \beta \in \Lambda^{p-q} X' \), by

\[
\begin{align*}
\langle A \lrcorner \alpha, B \rangle &= \langle \alpha, A \wedge B \rangle, \\
\langle \alpha \lrcorner A, B \rangle &= \langle \alpha, B \wedge A \rangle, \\
\langle \beta, A \lrcorner \alpha \rangle &= \langle \beta \lrcorner \alpha, A \rangle, \\
\langle \beta, \alpha \lrcorner A \rangle &= \langle \beta \lrcorner A, \alpha \rangle.
\end{align*}
\]

They extend linearly for all \( A \in \Lambda X \) and \( \alpha \in \Lambda (X') \). Left (resp. right) contractions match the contractor (the element switching sides) positively with the first (resp. last) components of the contractee (the other element). The result is of the same kind (multivector or form) as the contractee, vanishing if the contractor has larger grade.

**Example A.1.** For \( A \in \Lambda^3 X \) and \( \alpha = \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4 \in \Lambda^4 X' \), we have \( A \lrcorner \alpha = A \lrcorner \alpha = 0 \), and

\[
\begin{align*}
A \lrcorner \alpha &= + \langle \varphi_1 \wedge \varphi_2 \wedge \varphi_3, A \rangle \varphi_4 - \langle \varphi_1 \wedge \varphi_2 \wedge \varphi_4, A \rangle \varphi_3 \\
&\quad + \langle \varphi_1 \wedge \varphi_3 \wedge \varphi_4, A \rangle \varphi_2 - \langle \varphi_2 \wedge \varphi_3 \wedge \varphi_4, A \rangle \varphi_1, \\
\alpha \lrcorner A &= + \langle \varphi_2 \wedge \varphi_3 \wedge \varphi_4, A \rangle - \langle \varphi_2 \wedge \varphi_3 \wedge \varphi_4, A \rangle \\
&\quad + \langle \varphi_3 \wedge \varphi_2 \wedge \varphi_4, A \rangle - \langle \varphi_3 \wedge \varphi_2 \wedge \varphi_3, A \rangle.
\end{align*}
\]

An inner/Hermitian product \( \langle \cdot, \cdot \rangle \) in \( X \) gives the musical isomorphism \( b : X \to X', \) \( \psi^i(v) = \langle v, w \rangle \) for \( v, w \in X \), whose outermorphism enables contractions \( A \lrcorner B = A^t \lrcorner B \) and \( B \lrcorner A = B^t \lrcorner A \) of multivectors \( A, B \in \Lambda X \). Though not so common outside of GA, they are simpler and have more direct geometric interpretations, just as it is easier to work with inner product spaces than dual ones. In the complex case, \( b \) is conjugate-linear.

\(^4\)This is convention I of Appendix A.2; II switches left and right contractions; and III matches last covectors with first vectors, or vice-versa.
so these contractions are sesquilinear, while those of multivectors with forms were bilinear. This construction is equivalent to Definition 3.1: we have $\langle A^\flat, B \rangle = \langle A, B \rangle$, where the first $\langle \cdot, \cdot \rangle$ is the pairing and the other is the inner product, and so $\langle C, A \cdot B \rangle = \langle C^\flat, A^\flat \cdot B \rangle = \langle (A \wedge C)^\flat, B \rangle = \langle A \wedge C, B \rangle$ for $C \in \Lambda X$.

### A.2 Different conventions

If having left and right contractions between different kinds of elements is not confusing enough, one must be aware of the various conventions. To make matters worse, usually these are not clearly identified. For simplicity, here $A, B, C$ can be multivectors or forms.

Most authors use $\cdot$ for the left contraction, with the lower side of the ‘hook’ towards the contractor. But in [35] it is towards the contractee; in [32] the contractor is on the left of either $\cdot$ or $\cdot$; [12, 38] use $\cdot$ to contract a multivector on a form, and $\cdot$ for the opposite (but their $\cdot$‘s differ). In Differential Geometry [1, 21], contraction by a vector $v$ often appears as an operator $i_v$ or $\iota_v$. Other symbols used are $\cdot\cdot$ [37], $\lceil$ [9], $\lfloor$ [2], $\ominus$ [4] and $:$ [34]. We use $\cdot, \cdot, \cdot$ to distinguish conventions I, II, III below, but this is not common practice. In GA, some authors [9, 19, 39] use $\lceil$, but there is an effort to standardize $\cdot$ (in our opinion, this is unfortunate, as $\cdot$ helps identify their convention).

Definitions and properties differ as well. Table 2 shows how some properties of the left contraction vary in common conventions. We use convention I [16, 36, 37]. In II, used by many authors [12, 13, 14, 22, 24], sides are switched: their left contraction $A \cdot B$ is our right one $B \cdot A$, and vice-versa. In III, used in GA [9, 19, 25] and by Bourbaki [3], there is a reversion $\cdot$ in the contractor: its $A \cdot B$ is our $\bar{A} \cdot B$.

In I (resp. II), property (a) means the left contraction by $A$ is the adjoint of the left (resp. right) exterior product by $A$. In III, it is the adjoint of the left exterior product by the reversed $\bar{A}$.

In II and III, (b) shows $\Lambda X$ is a left $(\Lambda X)$-module w.r.t. the left contraction. In I, it is a right module, as the order of $A$ and $B$ is reversed, but the notation does not make this evident.

|   | (a) $\langle A \cdot B, C \rangle = \langle B, A \wedge C \rangle$ | (b) $(A \wedge B) \cdot C = B \cdot (A \wedge C)$ | (c) $v \cdot (B \wedge C) = (v \cdot B) \wedge C + (-1)^{|B|} B \wedge (v \cdot C)$ |
|---|---|---|---|
| I | | | |
| II | | | |
| III | | | |

Table 2: Properties of the left contraction in conventions I, II and III.
In I and III, (c) is a graded Leibniz rule, but in II the sign is at the ‘wrong’ term. In I and III it may seem misplaced for the right contraction, \((B \wedge C)_\wedge v = (-1)^{|C|}(B_\wedge v) \wedge C + B \wedge (C_\wedge v)\), but becomes natural if we think of \(v\) as ‘coming from the right’, as the notation suggests.

Contractions I, II and III differ only by grade dependent signs, so which one to use is a matter of choice. But fixing a standard one, preferably that with more intuitive formulas, would reduce the confusion.

We advocate for I. Some authors see the reordering of \(A\) and \(B\) in \((b)\) as a drawback, hiding the module structure. But thinking in terms of modules seems to bring little advantage here, while the reordering fits well with the nature of the contraction as an adjoint operator.

The weird ‘Leibniz rule’ of II is for us a deal breaker. Many authors seem content with it, but its popularity may be an accident of history: according to [14, p. 112], Bourbaki used II in the 1958 edition of [3], which might explain its early dissemination.

Bourbaki’s switch to III in the 1970 edition seems to have been ill-assimilated, and III only became popular with its use in GA. The reversion in \((a)\) enforces the left module structure while preserving the Leibniz rule, but it makes orientations harder to interpret, as we discuss below.

A.3 Geometric algebra contractions

The constructions of III were introduced in GA by Lounesto [25], with the reversion ” used “to absorb some inconvenient signs” [7, p. 134]. But these force their way back, requiring more adjustments: e.g., our formula \(A \cdot B = \langle A, B \rangle\), for equal grades, becomes \(A|B = A*p\), with ” hidden in a scalar product \(A*p = \langle A, B \rangle\); then \((a)\) becomes \(C*(A|B) = (C \wedge A)*B\) [8, p. 38], with \(A\) at the right side of \(\wedge\); and so on.

Convention III serves another purpose in GA: for \(A \in \Lambda^p X\) and \(B \in \Lambda^q X\), it lets \(A|B \equiv (AB)_{p-q}\) and \(A|B = (AB)_{p+q}\) be components of the Clifford product \(AB\), as are other GA products: \(A*B = (AB)_0\), and \(A\wedge B = (AB)_{p+q}\). But \(AB\) reflects the orientations of \(A\) and \(B\) [29], so to have results with orientations directly related to those of \(A\) and \(B\) one must often use \(AB\): e.g., \(\|A\|^2 = A*A = (AA)_0\). The exterior product is not affected by this [29, p. 25], but contractions are.

Interpreting the orientation of \(A|B = (-1)^{\frac{p(p-1)}{2}} A \wedge B\) is less immediate than in Theorem 3.26, taking some thought and knowledge of \(p\). For example, \((i \wedge j)(i \wedge j \wedge k) = -k\) for the canonical basis of \(\mathbb{R}^3\), is algebraically easy result, but with a sign whose meaning is not obvious. The only interpretation for the orientation of \(A|B\) we could find ([9, p. 76], [10, p. 29], [26, p. 45]) is for \(p = 1\), when \(A|B = A \wedge B\).

Another case of signs gone awry in GA is the dual \(A^* = A|\Omega\) (for a unit pseudoscalar \(\Omega\) in Euclidean \(\mathbb{R}^n\)), which differs by \((-1)^{\frac{p(p-1)}{2} + n(n-1)}\) from the usual Hodge dual: e.g., \(i^* = -j \wedge k\) for \(\Omega = i \wedge j \wedge k\) (strangely, a figure in [9, p. 82] presents this as the usual right-hand rule).

For beginners, these signs with no obvious meaning are an off-putting aspect of GA. One soon learns to put some to good use: e.g. identifying \(\mathbb{C}\) with \(\Lambda^+\mathbb{R}^2\), with imaginary unit \(i = i \wedge j\), as \(i^2 = -1\). But most signs

\[\text{Bourbaki used instead a homomorphism into the opposite algebra.}\]
remain a nuisance: \( I \star I = -1 \) gives no new information, is less useful since \( \star \) is not as flexible as the Clifford product, and to interpret its sign one must stop and think that
\[
I \star I = (-1)^{\frac{p(p-1)}{2}} \langle I, I \rangle.
\]
Such details, and the use, with altered meanings, of misleadingly familiar symbols and terms, might explain why GA is still not as widely used as it should be.

It seems the theory was thrown a little off-track by the idea of all products being components of \( AB \), perhaps due to a sense of algebraic elegance trumping geometric interpretation. Its intuitiveness and familiarity might improve if instead of \( A \star B \) and \( A \sqcap B \), we use \( \langle A, B \rangle \) and \( A \sqsubseteq B \), which are components of \( \tilde{A}B \). This requires adapting some formulas (e.g., the projection \( P_B A = (A|B)1_B^{-1} \) becomes \( P_B A = \frac{B \cdot A}{|B|^2} \)), but does not seem to cause a loss of computational power. Lounesto [26] uses \( \langle A, B \rangle \), and Dorst [9, p. 71] has suggested absorbing \( \tilde{\cdot} \) into \( A \star B \), but as they still use \( A|B = A \cdot B \), this half-way solution becomes less convenient. Rosén [36] uses \( \langle A, B \rangle \) and \( A \sqsubseteq B \), but with multi-covectors.

Another product that should be avoided is Hestenes inner product [18], a symmetrized contraction:
\[
A \cdot B = (AB)|_{q-p} = A|B \text{ if } p \leq q, A|B \text{ otherwise.}
\]
The symmetry \( A \cdot B = \pm B \cdot A \), for homogeneous elements, seems handy, but blurs the distinction between contractor and contractee. One must compare grades to know the role of each term, which affects how contractions operate with exterior products. So, formulas with \( \cdot \) often carry grade conditionals: e.g., equations (1.25b) and (1.25c) in [18, p. 7] give different results for \( A \cdot (B \cdot C) \), depending on the grades (compare with our Propositions 3.5i and 3.19). Also, the adjoint duality of formulas with \( \sqsubseteq \) and \( \wedge \) is partially lost with \( \cdot \) (e.g., compare (1.42) and (1.43) in [18, p. 12] with Propositions 3.5iv and 3.8, which have no grade restrictions). Grade conditionals hamper the use of non-homogeneous elements (which are an intrinsic part of GA, arising from Clifford products), and force us to track grades and analyze various grade dependent cases in proofs. The asymmetry of contractions, which appropriately vanish when grade conditions are not satisfied, often lets us treat all cases at once, allowing simpler proofs for more general results (as can be seen throughout this work). For more discussions of the advantages of contractions over Hestenes product, see [7, pp. 134–136], [8, pp. 39,45], [25, pp. 224–225] or [29, pp. 22–24].

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\[6\]In [18], Hestenes defined \( A \cdot B = 0 \) if \( A \) or \( B \) were scalars, but later changed it [17].
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