On the maximum number of distinct intersections in an intersecting family

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Abstract

For \( n > 2k \geq 4 \) we consider intersecting families \( \mathcal{F} \) consisting of \( k \)-subsets of \( \{1, 2, \ldots, n\} \). Let \( \mathcal{I}(\mathcal{F}) \) denote the family of all distinct intersections \( F \cap F', \ F \neq F' \) and \( F, F' \in \mathcal{F} \). Let \( \mathcal{A} \) consist of the \( k \)-sets \( A \) satisfying \( |A \cap \{1, 2, 3\}| \geq 2 \). We prove that for \( n \geq 50k^2 \) \( |\mathcal{I}(\mathcal{F})| \) is maximized by \( \mathcal{A} \).

1 Introduction

Let \( n, k \) be positive integers, \( n > 2k \). Let \( X = \{1, 2, \ldots, n\} \) be the standard \( n \)-element set and let \( \binom{X}{k} \) be the collection of all its \( k \)-subsets. For a family \( \mathcal{F} \subset \binom{X}{k} \) let \( \mathcal{I}(\mathcal{F}) := \{ F \cap F' : F, F' \in \mathcal{F}, F \neq F' \} \) be the family of all distinct pairwise intersections. Recall that a family \( \mathcal{F} \) is called intersecting if \( F \cap F' \neq \emptyset \) for all \( F, F' \in \mathcal{F} \).

One of the cornerstones of extremal set theory is the Erdős–Ko–Rado Theorem:

**Theorem 1.1 ([EKR]).** Suppose that \( \mathcal{F} \subset \binom{X}{k} \) is intersecting. Then

\[
|\mathcal{F}| \leq \binom{n-1}{k-1}.
\]
For a fixed element $x \in X$ define the full star $S_x$ by $S_x := \{ S \in \binom{X}{k} : x \in S \}$. Clearly $S_x$ is intersecting and it provides equality in (1.1). Subfamilies of $S_x$ are called stars. If we permit $n = 2k$ then there are many other intersecting families attaining equality in (1.1). However, Hilton and Milner [HM] proved that for $n > 2k$ the full stars are the only intersecting families with this property.

For a family $G \subset 2^X$ define the family of transversals:

$$\mathcal{T}(G) := \{ T \subset X : |T| \leq k, T \cap G \neq \emptyset \text{ for all } G \in G \}.$$  

With this definition $G \subset \binom{X}{k}$ is intersecting iff $G \subset \mathcal{T}(G)$. For $G \subset 2^X$ and $0 \leq \ell \leq n$ define the $\ell$-th level of $G$ by $G^{(\ell)} := \{ G \in G, |G| = \ell \}$.

An intersecting family $\mathcal{F} \subset \binom{X}{k}$ is called saturated if $\mathcal{F} \cup \{ G \}$ ceases to be intersecting for all $G \in \binom{X}{k} \setminus \mathcal{F}$.

**Observation 1.2.** An intersecting family $\mathcal{F} \subset \binom{X}{k}$ is saturated iff $\mathcal{F} = \mathcal{T}(\mathcal{F})^{(k)}$.

The aim of the present paper is to investigate the maximum size of $\mathcal{I}(\mathcal{F})$ over intersecting families $\mathcal{F} \subset \binom{X}{k}$. Since $\mathcal{F} \subset \tilde{\mathcal{F}}$ implies $\mathcal{I}(\mathcal{F}) \subset \mathcal{I}(\tilde{\mathcal{F}})$, in the process we may assume that $\mathcal{F}$ is saturated.

Unless otherwise stated, all considered intersecting families are supposed to be saturated. We need the following lemma that was essentially proved in [F78]. In order to state it, recall that a family $\mathcal{B}$ is called an antichain if $B \not\subset B'$ holds for all distinct members $B, B' \in \mathcal{B}$. Recall also that an antichain $\{ A_1, \ldots, A_p \}$ is called a sunflower of size $p$ with center $C$ if

$$A_i \cap A_j = C \text{ for all } 1 \leq i < j \leq p.$$  

**Lemma 1.3.** Suppose that $\mathcal{F} \subset \binom{X}{k}$ is a saturated intersecting family. Let $\mathcal{B} = \mathcal{B}(\mathcal{F})$ be the family of minimal (w.r.t. containment) sets in $\mathcal{T}(\mathcal{F})$. Then

(i) $\mathcal{B}$ is an intersecting antichain,

(ii) $\mathcal{F} = \{ H \in \binom{X}{k} : \exists B \in \mathcal{B}, B \subset H \}$,

(iii) $\mathcal{B}$ contains no sunflower of size $k + 1$.

The proof is given in the next section.

Define the intersecting family $\mathcal{A} = \mathcal{A}(n,k)$ on the ground set $X = \{1, \ldots, n\}$ by

$$\mathcal{A} := \left\{ A \in \binom{X}{k} : |A \cap \{1, 2, 3\}| \geq 2 \right\}.$$
The main result of the present paper is

**Theorem 1.4.** Suppose that \( n \geq 50k^2 \), \( k \geq 2 \) and \( \mathcal{F} \subset \binom{X}{k} \) is intersecting. Then

\[
|I(\mathcal{F})| \leq |I(A)|. 
\]

Let us note that it is somewhat surprising that the maximum is attained for \( A \) and not the full star which is much larger. Let us present the formula for \( |I(A)| \).

**Proposition 1.5.**

\[
|I(A)| = 3 \sum_{0 \leq i \leq k-2} \binom{n-3}{i} + 3 \sum_{0 \leq i \leq k-3} \binom{n-3}{i} + \sum_{0 \leq i \leq k-4} \binom{n-3}{i}. 
\]

**Proof.** Let \( A, A' \in \mathcal{A} \). Then there are seven possibilities for \( A \cap A' \cap \{1, 2, 3\} \), namely, all non-empty subsets of \( \{1, 2, 3\} \). If \( A \cap A' \cap \{1, 2, 3\} = \{1\} \) then \( A \cap \{1, 2, 3\} \) and \( A' \cap \{1, 2, 3\} \) are \( \{1, 2\} \) and \( \{1, 3\} \) in some order. Since \( n > 2k \) it is easy to see \( A \cap A' = \{1\} \cup D \) is possible for all \( D \subset \{4, \ldots, n\} \), \( |D| \leq k-2 \).

The remaining six cases can be dealt similarly.

Note that the RHS of (1.3) can be simplified to

\[
3 \sum_{0 \leq i \leq k-2} \binom{n-3}{i} + \sum_{0 \leq i \leq k-4} \binom{n-3}{i}. 
\]

In comparison

\[
|I(S_x)| = \sum_{0 \leq i \leq k-2} \binom{n-1}{i} = 2 \sum_{0 \leq i \leq k-2} \binom{n-2}{i} - \binom{n-2}{k-2}. 
\]

That is,

\[
|I(S_x)| < \frac{2}{3} |I(A)|. 
\]

Doing more careful calculations, one can replace \( \frac{2}{3} \) with \( \frac{n}{3(n-k)} \).

The paper is organized as follows. In Section 2 we prove Lemma 1.3 and the main lemma (Lemma 2.3) which provides some upper bounds concerning \( \mathcal{B}(\mathcal{F}) \). In Section 3 we prove Theorem 1.4. In Section 4 we mention some related problems.
2 Preliminaries and the main lemma

Proof of Lemma 1.3. The fact that $B$ is an antichain is obvious. Suppose for contradiction that $B, B' \in B$ but $B \cap B' = \emptyset$. If $|B| = |B'| = k$ then $B, B' \in \mathcal{F}$ and $B \cap B' \neq \emptyset$ follows. By symmetry suppose $|B| = |B'| < k$. Now $B, B' \in \mathcal{T}(\mathcal{F})$ implies that $B' \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. Choose a $k$-element superset $F'$ of $B'$ with $B \cap F' = \emptyset$. Since $B' \in \mathcal{T}(\mathcal{F})$, we have $F' \in \mathcal{T}(\mathcal{F})$. By Observation 1.2, $F' \in \mathcal{F}$. However, $B \cap F' = \emptyset$ contradicts $B \in \mathcal{T}(\mathcal{F})$. This proves (i). Statement (ii) is immediate from the definition of $B$.

To prove (iii) suppose for contradiction that $B_0, B_1, \ldots, B_k \in B$ form a sunflower with center $C$. Since $B$ is an antichain, $C$ is a proper subset of $B_0$. Consequently $C \notin \mathcal{T}(\mathcal{F})$. Thus we may choose $F \in \mathcal{F}$ satisfying $C \cap F = \emptyset$. However this implies $F \cap B_i = \emptyset$, a contradiction. □

In what follows, $B := B(\mathcal{F})$ is as in Lemma 1.3: the family of minimal transversals of $\mathcal{F}$. Let us recall the Erdős–Rado Sunflower Lemma.

Lemma 2.1 ([ER]). Let $\ell \geq 1$ be an integer and $\mathcal{D} \subset \binom{X}{\ell}$ a family which contains no sunflower of size $k + 1$. Then

\[(2.1) \quad |\mathcal{D}| \leq \ell! k^\ell.\]

The following statement is both well-known and easy.

Lemma 2.2. Suppose that $\mathcal{E} \subset \binom{X}{2}$ is intersecting, then either $\mathcal{E}$ is a star or a triangle.

We are going to use the standard notation: for integers $a \leq b$ we set $[a, b] = \{i: a \leq i \leq b\}$ and $[n] = [1, n]$. We also write $(x, y)$ instead of $\{x, y\}$ if $x \neq y$.

Based on Lemma 2.1 we could prove (1.2) for $n > k + 50k^3$. To get a quadratic bound we need to improve it under our circumstances. To state our main lemma we need some more definitions.

Define $t = t(\mathcal{B}) := \min\{|B|: B \in \mathcal{B}\}$. The covering number $\tau(\mathcal{B})$ is defined as follows: $\tau(\mathcal{B}) := \min\{|T|: T \cap B \neq \emptyset \text{ for all } B \in \mathcal{B}\}$. Since $\mathcal{F}$ is a saturated intersecting family, using (ii) of Lemma 1.3 we have $\tau(\mathcal{B}) = t$.

Now we can present our main lemma that is a sharpening of a similar result in [F17]. Put $\mathcal{B}^{(\leq \ell)} := \bigcup_{i=1}^{\ell} \mathcal{B}^{(i)}$. 

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Lemma 2.3. Let $\ell$ be an integer, $k \geq \ell \geq 2$. Suppose that $F \subset \binom{X}{k}$ is a saturated intersecting family, $B = B(F)$, $t \geq 2$. Assume that $\tau(B^{(\leq \ell)}) \geq 2$. Then

\begin{equation}
|B(\ell)| \leq t \cdot \ell \cdot k^{\ell-2}.
\end{equation}

Proof. For the proof we use a branching process. We need some notation. During the proof a sequence is an ordered sequence of distinct elements of $X$: $(x_1, x_2, \ldots, x_s)$. Sequences are denoted by $S, S_1$ etc. and we let $\hat{S}$ denote the underlying unordered set: $\hat{S} = \{x_1, x_2, \ldots, x_s\}$.

To start the branching process, we fix a set $B_1 \in B$ with $|B_1| = t(B)$ and for each element $y_1 \in B_1$ we assign weight $t(B) - 1$ to the sequence $(y_1)$.

At the first stage, we replace each 1-sequence $(y_1)$ with at most $\ell^2$-sequences: using $\tau(B^{(\leq \ell)}) \geq 2$ we choose an arbitrary $B(y_1) \in B$ satisfying $y_1 \notin B(y_1), |B(y_1)| \leq \ell$, and assign weight $(t(B) \cdot |B(y_1)|)^{-1} \geq (t(B) \cdot \ell)^{-1}$ to each sequence $(y_1, y_2), y_2 \in B(y_1)$. Note that the total weight assigned is exactly 1.

At each subsequent stage we pick a sequence $S = (x_1, \ldots, x_p)$ with weight $w(S)$ such that there exists $B \in B$ satisfying $\hat{S} \cap B = \emptyset$. Then we replace $S$ by the $|B|$ sequences $(x_1, \ldots, x_p, y), y \in B$, and assign weight $\frac{w(S)}{|B|}$ to each of them.

We continue until $\hat{S} \cap B \neq \emptyset$ holds for all sequences $S$ and all $B \in B$. Since $X$ is finite, this eventually happens. Importantly, the total weight assigned is still 1.

Claim 2.4. For each $B \in B^{(\ell)}$ there is some sequence $S$ with $\hat{S} = B$.

Proof. Let us suppose the contrary. Since $B$ is intersecting, a sequence with $\hat{S} = B$ is not getting replaced by a longer sequence during the process. Let $S = (x_1, \ldots, x_p)$ be a sequence of maximal length that occurred at some stage of the branching process satisfying $\hat{S} \subsetneq B$. Since $\hat{S}$ is a proper subset of $B$, $\hat{S} \cap B' = \emptyset$ for some $B' \in B$. Thus at some point we picked $S$ and chose some $\tilde{B} \in B$ disjoint to it. Since $B$ is intersecting, $B \cap \tilde{B} \neq \emptyset$. Consequently, for each $y \in B \cap \tilde{B}$ the sequence $(x_1, x_2, \ldots, x_p, y)$ occurred in the branching process. This contradicts the maximality of $p$. \hfill \Box

Let us check the weight assigned to $S$ with $|\hat{S}| = \ell$. It is at least $1 / (t(B) \ell k^{\ell-2})$. Since the total weight is 1, (2.2) follows. \hfill \Box
We should remark that the same $B \in \mathcal{B}(\ell)$ might occur as $\widehat{S}$ for several sequences $S$ and for many sequences $|\widehat{S}| \neq \ell$ might hold. This shows that there might be room for considerable improvement.

Let us mention that if $\mathcal{F} \subset \binom{X}{k}$ is a saturated intersecting family with $\tau(\mathcal{F}) = k$ then $\mathcal{B}(\mathcal{F}) = \mathcal{F}$ and (2.2) reduces to $|\mathcal{F}| \leq k^k$, an important classical result of Erdős and Lovász [EL].

3 The proof of Theorem 1.4

Since the case $k = 2$ trivially follows from Lemma 2.2, we assume $k \geq 3$. Take any saturated intersecting $\mathcal{F}$ and let $\mathcal{B} = \mathcal{B}(\mathcal{F})$. First recall that for the full star $S_x$,

$$|\mathcal{I}(S_x)| = \sum_{0 \leq \ell \leq k-2} \binom{n-1}{\ell},$$

which is less than $|\mathcal{I}(\mathcal{A})|$, and so we may assume that $\mathcal{F}$ is not the full star, i.e., $\mathcal{B}^{(1)} = \emptyset$.

Recall that $t = \min\{|B| : B \in \mathcal{B}\}$. Let us first present two simple inequalities for sums of binomial coefficients that we need in the sequel.

$$\frac{n-a}{i} / \frac{n-a}{i-1} = \frac{n-a-i+1}{i} \geq \frac{n-k}{k-1}$$

for $k > i > 0$, $a \geq 0$ and $i + a \leq k + 1$. Thus for every $1 \leq s \leq k-1$

$$\sum_{0 \leq i \leq s} \binom{n-2}{i} \geq \frac{n-k}{k-1} \sum_{0 \leq i \leq s-1} \binom{n-2}{i}.$$  

Thus

$$\frac{n-2}{i} / \frac{n}{i} = \frac{(n-i)(n-i-1)}{n(n-1)} > \left(1 - \frac{k}{n}\right)^2 \text{ for } 1 < i < k.$$

Let us partition $\mathcal{F}$ into $\mathcal{F}^{(1)} \cup \ldots \cup \mathcal{F}^{(k)}$ where $F \in \mathcal{F}^{(\ell)}$ if $\ell = \max\{|B| : B \in \mathcal{B}, B \subset F\}$. 

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Set $\mathcal{I}_\ell = \{F \cap F' : F \in \mathcal{F}(\ell), F' \in \mathcal{F}(\ell') \cup \ldots \cup \mathcal{F}(\ell)\}$. In human language, if $F \in \mathcal{F}(\ell)$, $F' \in \mathcal{F}(\ell')$ then we put $F \cap F'$ into $\mathcal{I}_\ell$ iff $\ell' \leq \ell$. It should be clear that

$$|\mathcal{I}(\mathcal{F})| \leq \sum_{\ell \leq k} |\mathcal{I}_\ell|.$$

The point is that for $F \in \mathcal{F}(\ell)$ and $B \subset F$, $B \in \mathcal{B}(\ell)$, for an arbitrary $F' \in \mathcal{F}$, $F \cap F'$ is partitioned as

$$F \cap F' = (B \cap F') \cup ((F \setminus B) \cap F').$$

Here there are at most $2^\ell - 1$ possibilities for $B \cap F'$ and $(F \setminus B) \cap F'$ is a subset of $X$ of size at most $k - \ell$. This proves

**Lemma 3.1.** For any $t \leq \ell \leq k$ such that $\tau(\mathcal{B}(\leq \ell)) \geq 2$ we have

$$(3.4) \quad |\mathcal{I}_\ell| \leq (2^\ell - 1)|\mathcal{B}(\ell)| \sum_{0 \leq i \leq k-\ell} \binom{n}{i} < 2^\ell \cdot \ell^2 k^{\ell-2} \sum_{0 \leq i \leq k-\ell} \binom{n}{i} =: f(n, k, \ell).$$

Note that if $\tau(\mathcal{B}(2)) = 2$ then $\mathcal{F}$ coincides with $\mathcal{A}$ and we have nothing to prove. Let $\alpha$ be the smallest integer such that $\tau(\mathcal{B}(\leq \alpha)) \geq 2$. We have $\alpha \geq 3$. The family $\mathcal{F}' := \bigcup_{i=1}^{\alpha-1} \mathcal{F}(i)$ is a trivial intersecting family, and thus

$$(3.5) \quad \left| \bigcup_{i=1}^{\alpha-1} \mathcal{I}_i \right| \leq |\mathcal{I}(\mathcal{S}_x)|.$$

On the other hand, using $(3.2)$ it is clear that for $\ell \geq 2$

$$f(n, k, \ell) / f(n, k, \ell + 1) > \frac{(n - k)\ell^2}{2k^2(\ell + 1)^2} \geq 6 \quad \text{for } n \geq 50k^2.$$

Hence

$$(3.6) \quad \sum_{\ell=\alpha}^{k} |\mathcal{I}_\ell| < \sum_{3 \leq \ell \leq k} f(n, k, \ell) < \frac{6}{5} f(n, k, 3).$$

Summing the right hand sides of $(3.5)$ and $(3.6)$, we get that

$$|\mathcal{I}(\mathcal{F})| \leq \sum_{0 \leq i \leq k-2} \binom{n-1}{i} + \frac{432}{5} \sum_{0 \leq i \leq k-3} \binom{n}{i}.$$
\[
\begin{align*}
&< \binom{n-2}{k-2} + \left( \frac{432k}{5} + 2 \right) \sum_{0 \leq i \leq k-3} \binom{n}{i} \\
&\quad \leq \binom{n-2}{k-2} + 90 \frac{k(k-1)}{n-k} \sum_{0 \leq i \leq k-2} \binom{n}{i} \leq \binom{n-2}{k-2} + 1.8 \sum_{0 \leq i \leq k-2} \binom{n}{i} \\
&\quad < \binom{n-2}{k-2} + 2 \sum_{0 \leq i \leq k-2} \binom{n-2}{i} < |\mathcal{I}(\mathcal{A})|.
\end{align*}
\]

4 Concluding remarks

Let \( n_0(k) \) be the smallest integer such that Theorem 1.4 is true for \( n \geq n_0(k) \). We proved \( n_0(k) \leq 50k^2 \). One can improve on the constant 50 by being more careful in the analysis. The following example shows that \( n_0(k) \geq (3 - \varepsilon)k \).

For \( 1 \leq p \leq k, n > 2k \), define the family \( \mathcal{B}_p(n,k) \) by

\[
\mathcal{B}_p(n,k) := \left\{ A \in \binom{[n]}{k} : |A \cap [2p-1]| \geq p \right\}.
\]

Note that \( \mathcal{S}_1 = \mathcal{B}_1(n,k) \) and \( \mathcal{A} = \mathcal{B}_2(n,k) \). It is easy to verify that

\[
|\mathcal{I}(\mathcal{B}_p(n,k))| = \sum_{i=1}^{p-1} \binom{2p-1}{i} \sum_{j=0}^{k-p} \binom{n-2p+1}{j} + \sum_{i=p}^{2p-1} \binom{2p-1}{i} \sum_{j=0}^{k-i-1} \binom{n-2p+1}{j}.
\]

By doing some calculations, one can see that \( |\mathcal{I}(\mathcal{B}_3(n,k))| > |\mathcal{I}(\mathcal{B}_2(n,k))| \) for \( n < (3 - \varepsilon)k \). It would be interesting do decide, whether for \( n > (1 + \varepsilon)k \) the maximum is always attained on one of the families \( \mathcal{B}_p(n,k) \).

Note that for \( n = 2k, k \geq 14 \), it is possible to construct an intersecting family \( \mathcal{F} \) with \( |\mathcal{I}(\mathcal{F})| = \sum_{i=0}^{k-1} \binom{n}{i} \) using an argument from [FKKP]. We say that a family \( \mathcal{F} \) almost shatters a set \( X \subset [n] \) if for any \( A \subset X, A \not\in \{\emptyset, X\} \), there is \( F \in \mathcal{F} \) such that \( F \cap X = A \). Take a random intersecting family \( \mathcal{F} \) by picking a \( k \)-set from each pair \( (A, [n] \setminus A) \) independently at random. In [FKKP, Theorem 7] it is proved, that with positive probability \( \mathcal{F} \) almost shatters every \( X \in \binom{[2k]}{k} \). Fix such a family \( \mathcal{F} \); then, by applying the almost shattering property two times, it is easy to show that, for each \( I \subset [n], 1 \leq |I| < k \), there are two sets \( F_1, F_2 \in \mathcal{F} \), such that \( I \subset F_1 \) and \( F_1 \cap F_2 = I \).
Another natural problem is to consider \( \tilde{I}(\mathcal{F}) = \{ F \cap F': F, F' \in \mathcal{F} \} = \mathcal{I}(\mathcal{F}) \cup \mathcal{F} \). Essentially the same proof shows that \( \tilde{I}(\mathcal{F}) \) is maximised by \( \mathcal{F} = \mathcal{S}_x \) for \( n \geq 50k^2 \) and one can verify that \( |\mathcal{I}(\mathcal{B}_2(n,k))| > |\mathcal{I}(\mathcal{S}_x)| \) for \( n < (5 - \varepsilon)k \).

In [F20] the analogous problem for the number of distinct differences \( F \setminus F' \) was considered. Improving those results in [FKK], we proved that for \( n > 50k \cdot \log k \) the maximum is attained for the full star, \( \mathcal{S}_x \). We showed also that it is no longer true for \( n = ck, 2 \leq c < 4, k > k_0(c) \).

The methods used in [FKK] are completely different.

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