THE KODAIRA DIMENSION OF SOME MODULI SPACES OF ELLIPTIC K3 SURFACES

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Abstract. We study the moduli spaces of elliptic K3 surfaces of Picard number at least 3, i.e. \( U \oplus \langle -2k \rangle \)-polarized K3 surfaces. Such moduli spaces are proved to be of general type for \( k \geq 220 \). The proof relies on the low-weight cusp form trick developed by Gritsenko, Hulek and Sankaran. Furthermore, explicit geometric constructions of some elliptic K3 surfaces lead to the unirationality of these moduli spaces for \( k \leq 11 \) and for other 21 isolated values up to \( k = 100 \).

Introduction

Moduli spaces of complex K3 surfaces are a fundamental topic of interest in algebraic geometry. One of the first geometric properties one wants to understand is their Kodaira dimension. Towards this direction, the seminal work \([GHS07b]\) of Gritsenko, Hulek and Sankaran proved that the moduli space \( \mathcal{F}_{2d} \) of polarized K3 surfaces of degree \( 2d \) is of general type for \( d > 61 \) and for other smaller values of \( d \). It is then natural to address the general question about the Kodaira dimension of moduli spaces of lattice polarized K3 surfaces. We are interested in studying a particular class of such surfaces, namely elliptic K3 surfaces of Picard number at least 3.

A K3 surface \( X \) is called elliptic if it admits a fibration \( X \to \mathbb{P}^1 \) in curves of genus one together with a section. The classes of the fiber and the zero section in the Néron-Severi group generate a lattice isomorphic to the hyperbolic plane \( U \), and they span the whole Néron-Severi group if the elliptic K3 surface is very general. The geometry of elliptic surfaces can be studied via their realization as Weierstrass fibrations. By using this description, Miranda \([Mir81]\) constructed the moduli space of elliptic K3 surfaces and showed its unirationality as a by-product. Later, Lejarraga \([Lej93]\) proved that this space is actually rational. We want to study the divisors of the moduli space of elliptic K3 surfaces which parametrize the surfaces whose Néron-Severi groups contain primitively \( U \oplus \langle -2k \rangle \). These are the moduli spaces \( \mathcal{M}_{2k} \) of \( U \oplus \langle -2k \rangle \)-polarized K3 surfaces. Geometrically we are considering elliptic K3 surfaces admitting an extra class in the Néron-Severi group: if \( k = 1 \), it comes from a reducible fiber of the elliptic fibration, while if \( k \geq 2 \) it is represented by an extra section, intersecting the zero section in \( k - 2 \) points with multiplicity (cf. Remark 5.6).

In the present article, we aim at computing the Kodaira dimension of the moduli spaces \( \mathcal{M}_{2k} \).

Theorem 0.1. The moduli space \( \mathcal{M}_{2k} \) is of general type for \( k \geq 220 \), or

\[
\begin{align*}
k &\geq 208, \quad k \neq 211, 219, \quad \text{or} \quad k \in \{170, 185, 186, 188, 190, 194, 200, 202, 204, 206\}. 
\end{align*}
\]

Moreover, the Kodaira dimension of \( \mathcal{M}_{2k} \) is non-negative for \( k \geq 176 \), or

\[
\begin{align*}
k &\geq 164, \quad k \neq 169, 171, 175 \quad \text{or} \quad k \in \{140, 146, 150, 152, 154, 155, 158, 160, 162\}.
\end{align*}
\]
The Torelli theorem for K3 surfaces (see [PS72]) allows the moduli spaces \( \mathcal{M}_{2k} \) to be realized as quotients of bounded hermitian symmetric domains \( \Omega_{L_{2k}} \) of type IV and dimension 17 by the stable orthogonal groups \( \tilde{O}^+(L_{2k}) \), where the lattice \( L_{2k} \) is the orthogonal complement of \( U \oplus \langle -2k \rangle \) in the K3 lattice \( \Lambda_{K3} := 3U \oplus 2E_8(-1) \). Via this description, one can apply the low-weight cusp form trick (Theorem 2.1) developed in [GHS07b]. This tool provides a sufficient condition for an orthogonal modular variety to be of general type. Namely, one has to find a non-zero cusp form on \( \Omega_{L_{2k}} \) of weight strictly less than the dimension of \( \Omega_{L_{2k}} \) vanishing along the ramification divisor of the projection \( \Omega_{L_{2k}} \to \tilde{O}^+(L_{2k})/\Omega_{L_{2k}} \). In our case, to construct a suitable cusp form, we use the quasi-pullback method (Theorem 2.3) to pull back the Borcherds form \( \Phi_{12} \) along the inclusion \( \Omega_{L_{2k}} \hookrightarrow \Omega_{L_{2,26}} \) induced by a lattice embedding \( L_{2k} \hookrightarrow L_{2,26} \). Here, the lattice \( L_{2,26} \) denotes the unique (up to isometry) even unimodular lattice of signature \( (2,26) \). The lattice embedding \( L_{2k} \hookrightarrow L_{2,26} \) determines the number \( N(L_{2k}) \) of effective roots in \( L_{2k} \). If \( N(L_{2k}) \) is positive, the embedding governs the weight \( 12 + N(L_{2k}) \) of the cusp form. Therefore the whole proof of Theorem 0.1 boils down to finding the values of \( k \) for which there exists a suitable primitive embedding \( L_{2k} \hookrightarrow L_{2,26} \), whose orthogonal complement contains at least 2 and at most 8 roots (cf. Problem 4.1).

In the second part of the article we give a geometric construction of all \( U \oplus \langle -2k \rangle \)-polarized K3 surfaces as double covers of the Hirzebruch surface \( \mathbb{F}_4 \) branched over a suitable smooth curve admitting a rational curve intersecting the branch locus with even multiplicities. We then review that, for \( k \geq 4 \) even, all \( U \oplus \langle -2k \rangle \)-polarized K3 surfaces admit a structure as hyperelliptic quartic K3 surfaces, i.e. double covers of \( \mathbb{P}^1 \times \mathbb{P}^1 \) branched over a curve of bidegree \( (4,4) \). Finally, we recall the realization of elliptic K3 surfaces as Weierstrass fibrations. These geometric constructions lead to the following:

**Theorem 0.2.** The moduli space \( \mathcal{M}_{2k} \) is unirational for \( k \leq 11 \) and for \( k \in \{13, 16, 17, 19, 21, 25, 26, 29, 31, 34, 36, 37, 39, 41, 43, 49, 59, 61, 64, 73, 100\} \).

In Section §1 we review the general construction for the moduli spaces of lattice polarized K3 surfaces as orthogonal modular varieties. We give a description of the moduli spaces \( \mathcal{M}_{2k} \), which are the main object of study in this article. In Section §2 we describe the method used in proving Theorem 0.1 namely the low-weight cusp form trick (Theorem 2.1). The desired form is cooked up as a quasi-pullback of the Borcherds form \( \Phi_{12} \) (Theorem 2.3). Section §3 is devoted to the proof of Proposition 3.1. Indeed, we study some special reflections in the stable orthogonal group \( \tilde{O}^+(L_{2k}) \). This is then used to impose the vanishing of the quasi-pullback \( \Phi|_{L_{2k}} \) of the Borcherds form along the ramification divisor of the quotient map \( \Omega_{L_{2k}} \to \mathcal{M}_{2k} \). In Section §4 we tackle Problem 4.1 of finding primitive embeddings \( L_{2k} \hookrightarrow L_{2,26} \) with at least 2 and at most 8 orthogonal roots. First, we prove that for any \( k \geq 4900 \) such an embedding exists. Then, we perform a computer analysis (see Algorithm 4.6) to find explicit embeddings for the remaining values of \( k \). In Section §5 we review the classical constructions of elliptic K3 surfaces as double covers of \( \mathbb{P}^1 \times \mathbb{P}^1 \) and \( \mathbb{F}_4 \), and Weierstrass fibrations. Finally, in Section §6 explicit geometric constructions, as the ones presented in Section §5 are used to prove Theorem 0.2.

**Conventions.** Throughout the article we will always work over \( \mathbb{C} \). We have used the software **Magma** to implement Algorithm 4.6.
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1. Moduli spaces of lattice polarized K3 surfaces

In this section we review the construction of the moduli spaces of lattice polarized K3 surfaces. An excellent reference to this subject is [Dol96].

First we recall some basic notions of lattice theory. Let $L$ be an integral lattice of signature $(2, n)$. Let $\Omega_L$ be one of the two connected components of

$$\{[w] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (w, w) = 0, (w, \bar{w}) > 0\}.$$

It is a hermitian symmetric domain of type IV and dimension $n$. We denote by $O^+(L)$ the index two subgroup of the orthogonal group $O(L)$ preserving $\Omega_L$. If $\Gamma < O^+(L)$ is of finite index we denote by $F_L(\Gamma)$ the quotient $\Gamma \backslash \Omega_L$. By a result of Baily and Borel [BB66], $F_L(\Gamma)$ is a quasi-projective variety of dimension $n$.

For every non-degenerate integral lattice $L$ we denote by $L^\vee := \text{Hom}(L, \mathbb{Z})$ its dual lattice. If $L$ is even, the finite group $A_L := L^\vee / L$ is endowed with a quadratic form $q_L$ with values in $\mathbb{Q}/2\mathbb{Z}$, induced by the quadratic form on $L$. We define:

$$\tilde{O}(L) := \ker(O(L) \to O(A_L))$$
and

$$\tilde{O}^+(L) := \tilde{O}(L) \cap O^+(L).$$

A compact smooth complex surface $X$ is a K3 surface if $X$ is simply connected and $H^0(X, \Omega^2_X)$ is spanned by a non-degenerate holomorphic 2-form $\omega_X$. The cohomology group $H^2(X, \mathbb{Z})$ is naturally endowed with a unimodular intersection pairing, making it isomorphic to the K3 lattice

$$\Lambda_{K3} := 3U \oplus 2E_8(-1),$$
where $U$ is the hyperbolic plane and $E_8(-1)$ is the unique (up to isometry) even unimodular negative definite lattice of rank 8. In particular the signature of $H^2(X, \mathbb{Z})$ is $(3, 19)$.

Fix an integral even lattice $M$ of signature $(1, t)$ with $t \geq 0$. An $M$-polarized K3 surface is a pair $(X, j)$ where $X$ is a K3 surface and $j : M \hookrightarrow \text{NS}(X)$ is a primitive embedding. Let

$$N := j(M)_{\Lambda_{K3}}^+$$
be the orthogonal complement of $M$ in $\Lambda_{K3}$. It is an integral even lattice of signature $(2, 19 - t)$.

By the Torelli theorem [PS72] (see also [Dol96 Corollary 3.2]), the moduli spaces of $M$-polarized K3 surfaces can be identified with the quotient of a classical hermitian symmetric domain of type IV and dimension $19 - t$ by an arithmetic group. More precisely, the 2-form $\omega_X$ of a $M$-polarized K3 surface $X$ determines a point in the period domain

$$\Omega_N := \{[w] \in \mathbb{P}(N \otimes \mathbb{C}) \mid (w, w) = 0, (w, \bar{w}) > 0\}^+,$$
modulo the action of the group [Dol96 Proposition 3.3]

$$\tilde{O}^+(N) = \{g \in O^+(\Lambda_{K3}) \mid g|_M = \text{id}\}.$$

**Theorem 1.1.** [Dol96 §3] The variety $F_N(\tilde{O}^+(N))$ is isomorphic to the coarse moduli space of $M$-polarized K3 surfaces.
In the following, we will study the moduli spaces of $M$-polarized K3 surfaces with $M = U \oplus \langle -2k \rangle$, i.e. elliptic K3 surfaces of Picard rank at least 3. Since the embedding $U \oplus \langle -2k \rangle \hookrightarrow \Lambda_{K3}$ is unique up to isometry by [Nik79, Theorem 1.14.4], we get the isomorphism

$$L_{2k} := U \oplus 2E_8(-1) \oplus \langle 2k \rangle \cong (U \oplus \langle -2k \rangle)^\perp_{\Lambda_{K3}}.$$  

As we discussed above, the quotient variety

$$\mathcal{M}_{2k} := \mathcal{F}_{L_{2k}}(\tilde{O}^+(L_{2k}))$$

is the moduli space of $U \oplus \langle -2k \rangle$-polarized K3 surfaces. Notice that all these surfaces are elliptic, since they contain a copy of the hyperbolic plane $U$.

2. Low-weight cusp form trick

The computation of the Kodaira dimension of modular orthogonal varieties relies on the low-weight cusp form trick developed by Gritsenko, Hulek and Sankaran [GHS07b]. In order to describe it, we need a little theory of modular forms on orthogonal groups.

Let $L$ be an integral even lattice of signature $(2,n)$. A modular form of weight $k$ and character $\chi : \Gamma \to \mathbb{C}^*$ for a finite index subgroup $\Gamma < O^+(L)$ is a holomorphic function $F : \Omega_L^* \to \mathbb{C}$ on the affine cone $\Omega_L^*$ over $\Omega_L$ such that

$$F(tZ) = t^{-k}F(Z) \forall t \in \mathbb{C}^*, \text{ and } F(gZ) = \chi(g)F(Z) \forall g \in \Gamma.$$  

A modular form is a cusp form if it vanishes at every cusp. We denote the vector spaces of modular forms and cusp forms of weight $k$ and character $\chi$ for $\Gamma$ by $M_k(\Gamma,\chi)$ and $S_k(\Gamma,\chi)$ respectively.

**Theorem 2.1.** [GHS07b, Theorem 1.1] Let $L$ be an integral lattice of signature $(2,n)$ with $n \geq 9$, and let $\Gamma < O^+(L)$ be a subgroup of finite index. The modular variety $\mathcal{F}_L(\Gamma)$ is of general type if there exists a nonzero cusp form $F \in S_k(\Gamma,\chi)$ of weight $k < n$ and character $\chi$ that vanishes along the ramification divisor of the projection $\pi : \Omega_L \to \mathcal{F}_L(\Gamma)$ and vanishes with order at least 1 at infinity.

If $S_n(\Gamma, \det) \neq 0$ then the Kodaira dimension of $\mathcal{F}_L(\Gamma)$ is non-negative.

2.1. Ramification divisor. First, we need to describe the ramification divisor of the orthogonal projection, which turns out to be the union of rational quadratic divisors associated to reflective vectors.

For any $v \in L \otimes \mathbb{Q}$ such that $v^2 < 0$ we define the rational quadratic divisor

$$\Omega_v(L) := \{ [Z] \in \Omega_L \mid (Z,v) = 0 \} \cong \Omega_{v_L}^\perp$$

where $v_L^\perp$ is an even integral lattice of signature $(2, n - 1)$.

The reflection with respect to the hyperplane defined by a non-isotropic vector $r \in L$ is given by

$$\sigma_r : l \mapsto l - 2 \frac{\langle l, r \rangle}{r^2} r.$$  

If $r$ is primitive and $\sigma_r \in O(L)$, then we say that $r$ is a reflective vector. We notice that $r$ is always reflective if $r^2 = \pm 2$, and we call it root in this case.

If $v \in L^\vee$ and $v^2 < 0$, the divisor $\Omega_v(L)$ is called a reflective divisor if $\sigma_v \in O(L)$.  


\textbf{Theorem 2.2.} [GHS07b, Corollary 2.13] For \( n \geq 6 \), the ramification divisor of the projection \( \pi_\Gamma : \Omega_L \to \mathcal{F}_L(\Gamma) \) is the union of the reflective divisors with respect to \( \Gamma < \text{O}^+(L) \):

\[
\text{Rdiv}(\pi_\Gamma) = \bigcup_{Z \in \mathcal{L}} \Omega_r(L)
\]

2.2. Quasi pullback. To apply Theorem 2.1 we need a supply of modular forms for \( \Gamma \). These are provided by quasi-pullbacks of modular forms with respect to some higher rank orthogonal group. In our case, let \( L_{2,26} \) denote the unique (up to isometry) even unimodular lattice of signature \((2, 26)\):

\[
L_{2,26} := 2U \oplus 3E_8(-1).
\]

Borcherds proved [Bor95] that \( M_{12}(\text{O}^+(L_{2,26}), \det) \) is a one dimensional complex vector space spanned by a modular form \( \Phi_{12} \), called the \textit{Borcherds form}. The zeroes of \( \Phi_{12} \) lie on rational quadratic divisors defined by \((-2)\)-vectors in \( L_{2,26} \), i.e. \( \Phi_{12}(Z) = 0 \) if and only if there exists \( r \in L_{2,26} \) with \( r^2 = -2 \) such that \((Z, r) = 0\). Moreover the multiplicity of the rational quadratic divisor of zeroes of \( \Phi_{12} \) is one.

Given a primitive embedding of lattices \( L \hookrightarrow L_{2,26} \), with \( L \) of signature \((2, n)\), we define

\[
R_{L_{2,26}}(L) := \{ r \in L_{2,26} \mid r^2 = -2, \ (r, L) = 0 \}.
\]

To construct a modular form for some subgroup of \( \text{O}^+(L) \), one might try to pull back \( \Phi_{12} \) along the closed immersion \( \Omega^*_L \hookrightarrow \Omega^*_{L_{2,26}} \). However, for any \( r \in R(L) \) one has \( \Omega^*_L \subset \Omega^*_{r^\perp} \) and hence \( \Phi_{12} \) vanishes identically on \( \Omega^*_L \). The method of the quasi-pullback, first developed by Gritsenko, Hulek, and Sankaran [GHS07b], deals with this issue by dividing out by appropriate linear factors:

\textbf{Theorem 2.3.} [GHS13, Theorem 8.2] Let \( L \hookrightarrow L_{2,26} \) be a primitive non-degenerate sublattice of signature \((2, n)\), \( n \geq 3 \), and let \( \Omega_L \hookrightarrow \Omega_{L_{2,26}} \) be the corresponding embedding of the homogeneous domains. The set of \((-2)\)-roots \( R_{L_{2,26}}(L) \) in the orthogonal complement of \( L \) is finite. We put \( N(L) := |R_{L_{2,26}}(L)|/2 \). Then the function

\[
\Phi|_L(Z) := \frac{\Phi_{12}(Z)}{\Pi_{r \in R_{L_{2,26}}(L) \pm 1}(Z, r)} \mid_{\Omega_L} \in M_{12+N(L)}(\tilde{\Omega}^+(L), \det)
\]

is non-zero, where in the product over \( r \) we fix a finite system of representatives in \( R_{L_{2,26}}(L) \). The modular form \( \Phi|_L \) vanishes only on rational quadratic divisors of type \( \Omega_v(L) \) where \( v \in L^\vee \) is the orthogonal projection with respect to \( L^\perp \) of a \((-2)\)-root \( r \in L_{2,26} \) on \( L^\vee \).

Moreover, if \( N(L) > 0 \), then \( \Phi|_L \) is a cusp form.

We want to apply the low-weight cusp form trick and Theorem 2.3 to the orthogonal variety isomorphic to the moduli space of \( U \oplus \langle -2k \rangle \)-polarized K3 surfaces.

In our situation, we need to find a suitable primitive embedding of \( L_{2k} \hookrightarrow L_{2,26} \), such that the quasi-pullback \( \Phi|_{L_{2k}} \) is a cusp form of weight (strictly) less than 17 which vanishes along the ramification divisor of the projection

\[
\pi : \Omega_{L_{2k}} \to M_{2k} = \tilde{\Omega}^+(L_{2k})/\Omega_{L_{2k}}.
\]

\textbf{Remark 2.4.} By [GHS99] Theorem 1.7] the abelianization of \( \tilde{\Omega}^+(L_{2k}) \) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \). This is because \( L_{2k} \) is isomorphic to \( 2U \oplus E_8(-1) \oplus \langle -2k \rangle_{E_8(-1)} \), since the embedding \( U \oplus \langle -2k \rangle \hookrightarrow \Lambda_{K3} \) is unique up to isometry (cf. [Nik79, Theorem 1.14.4]). As a consequence,
the Albanese varieties of the moduli spaces \( \mathcal{M}_{2k} \) are all trivial (cf. [Kon88 Theorem 2.5]). Moreover, [GHS09 Corollary 1.8] implies that the unique non-trivial character of \( \tilde{O}^+(L_{2k}) \) is det.

3. Special reflections

Let \( L_{2k} \hookrightarrow L_{2,26} \) be a primitive embedding. Since the embedding \( U \oplus 2E_8(-1) \hookrightarrow L_{2,26} \) is unique up to isometry by [Nik79 Theorem 1.14.4], we can assume that every summand of \( U \oplus 2E_8(-1) \) is mapped identically onto the corresponding summand of \( L_{2,26} \). Therefore, any choice of a primitive vector \( l \in U \oplus E_8(-1) \) of norm \( l^2 = 2k \) gives a primitive embedding

\[
L_{2k} = U \oplus 2E_8(-1) \oplus \langle 2k \rangle \hookrightarrow L_{2,26}.
\]

In this section we prove the following:

**Proposition 3.1.** The quasi-pullback \( \Phi|_{L_{2k}} \) defined in Theorem 2.3 vanishes along the ramification divisor of

\[
\pi: \Omega_{L_{2k}} \to \mathcal{M}_{2k} = \tilde{O}^+(L_{2k}) \setminus \Omega_{L_{2k}}
\]

for any primitive embedding \( L_{2k} \hookrightarrow L_{2,26} \) such that \( (L_{2k})^{±}_{L_{2,26}} \) does not contain a copy of \( E_8(-1) \).

For any \( l \in L \) we define its divisibility \( \text{div}(l) \) to be the unique \( m > 0 \) such that \( (l, L) = m\mathbb{Z} \) or, equivalently, the unique \( m > 0 \) such that \( l/m \in L' \) is primitive. Since \( \text{div}(r) > 0 \) is the smallest intersection number of \( r \) with any other vector, \( \text{div}(r) \) divides \( r^2 \). Moreover, if \( r \) is reflective, the number \( 2\frac{(lr)}{r^2} \) must be an integer, so \( r^2 \) divides \( 2(l, r) \) for all \( l \in L \), i.e. \( r^2 \mid 2\text{div}(r) \). Summing up

\[
\text{div}(r) \mid r^2 \mid 2\text{div}(r).
\]

**Proposition 3.2.** Let \( r \in L_{2k} \) be a reflective vector. Then \( \sigma_r \) induces \( \pm \text{id} \) in \( A_{L_{2k}} \), i.e. \( \pm \sigma_r \in \tilde{O}(L) \), if and only if \( r^2 = \pm 2 \) or \( r^2 = \pm 2k \) and \( \text{div}(r) \in \{k, 2k\} \).

**Proof.** Similar to [GHS07b Proposition 3.2, Corollary 3.4]. \( \square \)

Now \( \sigma_r \in O^+(L \otimes \mathbb{R}) \) if and only if \( r^2 < 0 \) (see [GHS07a]). Recall that an integral lattice \( T \) is called 2-elementary if \( A_T \) is an abelian 2-elementary group.

**Proposition 3.3.** Let \( r \in L_{2k} \) be primitive with \( r^2 = -2k \) and \( \text{div}(r) \in \{k, 2k\} \). Then \( L_r := r^+_L \) is a 2-elementary lattice of signature \( (2, 16) \) and determinant 4.

**Proof.** We have the following well-known formula for \( \det(L_r) \) (see for instance [GHS07b Equation 20]):

\[
\det(L_r) = \frac{\det(L_{2k}) \cdot r^2}{\text{div}(r)^2} \in \{1, 4\}.
\]

Since \( L_{2k} \) has signature \( (2, 17) \) and \( r^2 < 0 \), we have that \( L_r \) has signature \( (2, 16) \). Therefore \( \det(L_r) \) cannot be 1, because there are no unimodular lattices with signature \( (2, 16) \) (see [Nik79 Theorem 0.2.1]). This shows that \( \text{div}(r) = k \). Therefore the reflection \( \sigma_r \) acts as \( -\text{id} \) on the discriminant group \( A_{L_{2k}} \) (see [GHS07b Corollary 3.4]). Now we can extend \( -\sigma_r \in \tilde{O}(L_{2k}) \) to an element \( \tilde{\sigma}_r \in O(A_{K3}) \) by defining \( \tilde{\sigma}_r|_{U \oplus (-2k)} = \text{id} \) on the orthogonal complement of \( L_{2k} \hookrightarrow A_{K3} \). Put \( S_r := (L_r)^{±}_{A_{K3}} \). It is easy to realize that

\[
\tilde{\sigma}_r|_{L_r} = -\text{id} \quad \text{and} \quad \tilde{\sigma}_r|_{S_r} = \text{id}.
\]
Then $L_r$ is 2-elementary by \cite[Corollary I.5.2]{Nik79}.

**Proposition 3.4.** Given any embedding $L_{2k} \hookrightarrow L_{2,26}$, let $r \in L_{2k}$ be a primitive reflective vector with $r^2 = -2k$, and consider $L_r = r_{L_{2k}}^\perp$ as above. Under the chosen embedding, the orthogonal complement $(L_r)_{L_{2,26}}^\perp$ is isomorphic to either $D_{10}(-1)$ or $E_8(-1) \oplus 2A_1(-1)$.

**Proof.** Since $L_{2,26}$ is unimodular, the discriminant groups of $L_r$ and $(L_r)_{L_{2,26}}^\perp$ are isometric up to a sign. The previous proposition thus implies that $(L_r)_{L_{2,26}}^\perp$ is a 2-elementary, negative definite lattice of rank 10 and determinant 4. By \cite[Proposition 1.8.1]{Nik79}, any 2-elementary discriminant form is isometric to a direct sum of 4 finite quadratic forms, represented by the 2-elementary lattices $A_1$, $A_1(-1)$, $U(2)$, $D_4$. Since $(L_r)_{L_{2,26}}^\perp$ has signature $-2 \pmod 8$ and determinant 4, it is immediate to realize that its discriminant form must be isometric to the discriminant form of $2A_1(-1)$. Now we notice that the lattice $E_8(-1) \oplus 2A_1(-1)$ is a 2-elementary, negative definite lattice of rank 10 with the desired discriminant form. Finally it is enough to compute the genus of $E_8(-1) \oplus 2A_1(-1)$. A quick check with Magma yields that the whole genus consists of $E_8(-1) \oplus 2A_1(-1)$ and $D_{10}(-1)$. Alternatively, one can use the Siegel mass formula \cite{CSS} and check that the mass of the quadratic form $f$ associated to the lattice $E_8(-1) \oplus 2A_1(-1)$ is

$$m(f) = 2^8 \cdot 4! \cdot 1814400 = \frac{1}{2229534720}.$$  

Since a straightforward check shows that $D_{10}(-1)$ is in the genus of $E_8(-1) \oplus 2A_1(-1)$, and the equality

$$\frac{1}{|O(D_{10}(-1))|} + \frac{1}{|O(E_8(-1) \oplus 2A_1(-1))|} = \frac{1}{3715891200} + \frac{1}{5573836800} = \frac{1}{2229534720} = m(f)$$

holds, we deduce that $\{D_{10}(-1), E_8(-1) \oplus 2A_1(-1)\}$ is the whole genus of $E_8(-1) \oplus 2A_1(-1)$. \hfill \Box

Now we are ready to prove Proposition 3.1.

**Proof of Proposition 3.1.** In order to prove that $\Phi|_{L_{2k}}$ vanishes along the ramification divisor of the projection $\pi$, we have to show that $\Phi|_{L_{2k}}$ vanishes on the $(-2k)$-divisors $\Omega_r(L_{2k})$ given by reflective vectors $r \in L_{2k}$ of norm $-2k$ (see Theorem 2.2). Hence let $r$ be a $(-2k)$-reflective vector. By Proposition 3.4, $(L_r)_{L_{2,26}}^\perp$ is a root lattice with at least 180 roots ($E_8(-1) \oplus 2A_1(-1)$ has 244 and $D_{10}(-1)$ has 180). Since by assumption the orthogonal complement of $L_{2k}$ in $L_{2,26}$ does not contain a copy of $E_8(-1)$, the root lattice generated by $R_{L_{2,26}}(L_{2k})$ has rank at most 9 and does not contain a copy of $E_8(-1)$. By checking all such root lattices, we obtain $|R_{L_{2,26}}(L_{2k})| \leq |\{\text{roots of } D_9\}| = 144$ (just recall that $A_n$ has $n(n+1)$ roots, $D_n$ has $2n(n-1)$ roots, $E_6$, $E_7$ have 72 and 126 roots respectively). Consequently $\Phi|_{L_{2k}}$ vanishes along the $(-2k)$-divisor $\Omega_r(L_{2k})$ given by $r$ with order $\geq 180 - 144 > 0$, as claimed. \hfill \Box

4. Lattice Engineering

By the previous discussion, we have transformed our original question of determining the Kodaira dimension of $M_{2k}$ to the following
**Problem 4.1.** For which $2k > 0$ does there exist a primitive vector $l \in U \oplus E_8(-1)$ with norm $l^2 = 2k$ such that $l$ is orthogonal to at least 2 and at most 8 roots?

We want to find a lower bound for the values $2k$ answering Problem 4.1 positively (see Proposition 4.5). Since $U \oplus E_8(-1)$ contains infinitely many roots, we want to start by reducing to the more manageable case of $E_8(-1)$, whose number of roots is finite.

For simplicity we define

$$R(l) := \{ r \in U \oplus E_8(-1) : r^2 = -2, (r, l) = 0 \} = R_{L_{2,2k}}(L_{2k}).$$

The following is a slight generalization of [TV19] Lemma 4.1, 4.3.

**Lemma 4.2.** Let $l = \alpha e + \beta f + v$, where $U = \langle e, f \rangle$ such that $e^2 = f^2 = 0$ and $ef = 1$, $v \in E_8(-1)$ and $\alpha, \beta \in \mathbb{Z}$, with norm $l^2 = 2k > 0$. Let $r = \alpha' e + \beta' f + v'$ be a vector of $R(l)$, where $v' \in E_8(-1)$ and $\alpha', \beta' \in \mathbb{Z}$. If $\alpha \neq \beta$, $\alpha, \beta > \sqrt{k}$ and $\alpha \beta < \frac{5}{4}k$, then $\alpha' = \beta' = 0$.

**Proof.** See [Pet19] Lemma 3.3 and 3.4. \qed

In other words, if $l = \alpha e + \beta f + v \in U \oplus E_8(-1)$ is a vector of norm $2k$ satisfying the assumptions of the previous lemma, then the roots of $U \oplus E_8(-1)$ orthogonal to $l$ are roots of $E_8(-1)$. Therefore the set $R(l)$ coincides with the set of roots in $v_{E_8(-1)}^\perp$. The following lemma, inspired by [GHS07b] Theorem 7.1, controls the number of roots of $E_8(-1)$ orthogonal to $v$.

**Lemma 4.3.** There exists $v \in E_8$ with $v^2 = 2n$ and such that $v_{E_8}^\perp$ contains at least 2 and at most 8 roots if the inequality

$$2N_{E_7}(2n) > 28N_{E_6}(2n) + 63N_{D_6}(2n),$$

holds, where $N_L(2n)$ denotes the number of representations of $2n$ by the positive definite lattice $L$.

**Proof.** We follow closely [GHS07b] Theorem 7.1. Let $a \in E_8$ be a root. Its orthogonal complement $E_7^{(a)} := a_{E_8}^\perp$ is isometric to $E_7$. The set of 240 roots in $E_8$ consists of the 126 roots in $E_7^{(a)}$ and other 114 roots, forming the subset $X_{114}$. Assume that every $v \in E_7^{(a)}$ with $v^2 = 2n$ is orthogonal to at least 10 roots in $E_8$, including $\pm a$. By [GHS07b] Lemma 7.2 we know that every such $v$ is contained in the union

$$\bigcup_{i=1}^{28} (A_2^{(i)})_{E_8}^\perp \cup \bigcup_{j=1}^{63} (A_1^{(j)})_{E_7^{(a)}}^\perp,$$

where $A_2^{(i)}$ (resp. $A_1^{(j)}$) are root systems of type $A_2$ (resp. $A_1$) contained in $X_{114}$ (resp. $E_7^{(a)}$).

Denote by $n(v)$ the number of components in the union containing $v$. Since $(A_2^{(i)})_{E_8}^\perp \cong E_6$ and $(A_1^{(j)})_{E_7^{(a)}}^\perp \cong D_6$, we have counted the vector $v$ exactly $n(v)$ times in the sum

$$28N_{E_6}(2n) + 63N_{D_6}(2n).$$

We distinguish three cases.

(i) If $v \cdot c \neq 0$ for every $c \in X_{114} \setminus \{ \pm a \}$, then $v$ is orthogonal to at least 4 copies of $A_1$ in $E_7^{(a)}$, so $n(v) \geq 4$. 
(ii) If \( v \) is orthogonal to only one \( A_2^{(i)} \) (6 roots), then \( v \) is orthogonal to at least 2 copies of \( A_1 \) in \( E_7^{(a)} \), so \( n(v) \geq 3 \).

(iii) If \( v \) is orthogonal to at least two \( A_2^{(i)} \), then \( n(v) \geq 2 \).

In conclusion \( n(v) \geq 2 \) for every \( v \in E_7^{(a)} \). Therefore, under our assumption that every \( v \in E_7^{(a)} \) with \( v^2 = 2n \) is orthogonal to at least 10 roots, we have shown that any such \( v \) is contained in at least 2 sets of the union \( \bigcup_i \), i.e.

\[
2N_{E_7}(2n) \leq 28N_{E_6}(2n) + 63N_{D_6}(2n).
\]

\[\square\]

**Proposition 4.4.** Let \( n \geq 952 \). Then there exists \( v \in E_8(-1) \) with \( v^2 = -2n \) such that \( v_{E_8(-1)} \) contains at least 2 and at most 8 roots.

**Proof.** [GHS07b] Equations (31), (33) and (34)] give the following estimates:

\[
N_{E_7}(2n) > 123.8 n^{5/2}, \quad N_{E_6}(2n) < 103.69 n^2, \quad N_{D_6}(2n) < 75.13 n^2.
\]

By Lemma 4.3 we immediately obtain the claim.\[\square\]

We are now ready to answer Problem 4.1.

**Proposition 4.5.** Let \( k \geq 4900 \). Then there exists a primitive \( l \in U \oplus E_8(-1) \) with \( l^2 = 2k \) and \( 2 \leq |R(l)| \leq 8 \).

**Proof.** Pick \( k > 0 \) and consider \( l = \alpha e + \beta f + v \), where \( l^2 = 2k \), \( v^2 = -2n \), so that \( \alpha \beta = n + k \). Suppose that there exist \( \alpha \) and \( \beta \) satisfying the hypotheses of Lemma 4.2 such that \( n = \alpha \beta - k \geq 952 \). Then Proposition 4.3 implies that we can find a \( v \in E_8(-1) \) with \( v^2 = -2n \) such that \( v_{E_8(-1)} \) contains at least 2 and at most 8 roots. Moreover Lemma 4.2 assures that the roots of \( U \oplus E_8(-1) \) orthogonal to \( l = \alpha e + \beta f + v \) are contained in \( E_8(-1) \), so that \( l_{U \oplus E_8(-1)} \) also contains at least 2 and at most 8 roots. Therefore the existence of such \( \alpha, \beta \) is sufficient for the existence of \( l \in U \oplus E_8(-1) \) with \( 2 \leq |R(l)| \leq 8 \).

Now let \( k \geq 4900 = 70^2 \), and consider

\[
\alpha = \lfloor \sqrt{k} + 6 \rfloor, \quad \beta = \alpha + 1.
\]

Clearly \( \alpha \neq \beta \), \( \gcd(\alpha, \beta) = 1 \) and \( \alpha, \beta > \sqrt{k} \). Moreover

\[
\frac{5}{4}k - \alpha \beta \geq \frac{5}{4}k - (\sqrt{k} + 7)(\sqrt{k} + 8) = \frac{1}{4}k - 15\sqrt{k} - 56 > 0,
\]

and

\[
n = \alpha \beta - k \geq (\sqrt{k} + 6)(\sqrt{k} + 7) - k = 13\sqrt{k} + 42 \geq 952,
\]

completing the proof.\[\square\]

In order to deal with the remaining cases, we perform an exhaustive computer analysis. More precisely, for each \( k < 4900 \) we search for a primitive vector \( l \in U \oplus E_8(-1) \) with \( l^2 = 2k \) and \( 2 \leq |R(l)| \leq 8 \) (or \( 2 \leq |R(l)| \leq 10 \), if we want to prove that \( M_{2k} \) has non-negative Kodaira dimension). We have implemented the following algorithm in Magma.

**Algorithm 4.6.**

- Construct the list \( \text{Lst} \) of all vectors \( v \in E_8(-1) \) with norm \( |v^2| \leq 2 \cdot 4900 \) orthogonal to at most 4 of the 8 effective roots of a given basis of \( E_8(-1) \).
• Fix $k < 4900$. For every $v \in \text{List}$, and for every positive divisor $2\alpha$ of $2k - v^2$, let $\beta = \frac{2k-v^2}{2\alpha}$ and, if $\gcd(\alpha, \beta) = 1$, consider the vector $l = \alpha e + \beta f + v \in U \oplus E_8(-1)$. By construction $l$ is primitive of norm $2k$.

• We compute the negative definite lattice $l^\perp \subset U \oplus E_8(-1)$.

• If the minimum norm of the lattice $l^\perp$ is $-2$ and $l^\perp$ contains at most 8 (or 10) roots, we return the vector $l$.

This search, exhaustive in the range specified by the algorithm, shows that a vector $l \in U \oplus E_8(-1)$ with $l^2 = 2k < 2 \cdot 4900$ and $2 \leq |R(l)| \leq 8$ exists if

$$k \geq 208, \ k \neq 211, 219 \text{ or } k \in \{170, 185, 186, 188, 190, 194, 200, 202, 204, 206\}. \quad (5)$$

Moreover a similar vector $l$ with $2 \leq |R(l)| \leq 10$ exists if

$$k \geq 164, \ k \neq 169, 171, 175 \text{ or } k \in \{140, 146, 150, 152, 154, 155, 158, 160, 162\}. \quad (6)$$

The interested reader can find the list of such vectors in the ArXiv distribution of this article. We also attach a code in Magma to verify that such vectors actually have the desired properties.

We are now ready to prove Theorem 0.1.

**Proof of Theorem 0.1.** Proposition 4.5 combined with the previous search assures that there exists a primitive $l \in U \oplus E_8(-1)$ with $l^2 = 2k < 2 \cdot 4900$ and $2 \leq |R(l)| \leq 8$ if $k \geq 4900$ or $k$ belongs to the list $(5)$, in particular for any $k \geq 220$. Such an $l \in U \oplus E_8(-1)$ determines an embedding $L_{2k} \hookrightarrow L_{2,26}$ with the property

$$1 \leq N(L_{2k}) \leq 4,$$

where $N(L_{2k})$ is the number of effective roots in the orthogonal complement $(L_{2k})_{L_{2,26}}^\perp$. Hence Theorem 2.3 provides a non-zero cusp form $\Phi|_{L_{2k}}$ of weight $12 + N(L_{2k}) \leq 12 + 4 < 17 = \dim(\mathcal{M}_{2k})$, which vanishes along the ramification divisor of $\pi : \Omega_{L_{2k}} \to \mathcal{M}_{2k}$ in view of Proposition 3.1, since $l^2$ does not contain $E_8(-1)$, otherwise $l$ would be orthogonal to at least 240 roots. Then the low-weight cusp form trick (Theorem 2.1) ensures that $\mathcal{M}_{2k}$ is of general type.

An analogous argument shows that $\mathcal{M}_{2k}$ has non-negative Kodaira dimension if $k$ belongs to the list $(5)$, in particular for any $k \geq 176$. \hfill $\square$

5. Geometric constructions

In this section we recall three well-known geometric constructions of K3 surfaces. Namely, double covers of the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$ (see §5.1) and of the Hirzebruch surface $\mathbb{F}_4$ (see §5.2) branched over suitable curves define lattice polarized K3 surfaces with respect to the lattices $U(2)$ and $U$ respectively. Furthermore, every elliptic K3 surface can be reconstructed from its Weierstrass fibration (see §5.3).

5.1. Double covers of $\mathbb{P}^1 \times \mathbb{P}^1$. Let $\mathbb{F}_0 := \mathbb{P}^1 \times \mathbb{P}^1$ be the smooth quadric surface in $\mathbb{P}^3$. Its Picard group is generated by the classes of the two pencils $\ell_1, \ell_2$ of lines, hence $\text{Pic}(\mathbb{F}_0)$ endowed with the intersection form on $\mathbb{F}_0$ is isomorphic to the hyperbolic plane $U$. The canonical bundle is $K_{\mathbb{F}_0} = \mathcal{O}_{\mathbb{F}_0}(-2, -2)$. 


Now let $\pi : X \to \mathbb{F}_0$ be the double cover branched over a smooth curve $B \in |-2K_{\mathbb{F}_0}| = |\mathcal{O}_{\mathbb{F}_0}(4,4)|$. Then $X$ is a smooth K3 surface. The pullbacks $E_i = \pi^*\ell_i$ for $i = 1, 2$ are smooth elliptic curves, and $E_1E_2 = 2\ell_1\ell_2 = 2$, so that

$$\langle E_1, E_2 \rangle = U(2) \hookrightarrow \text{NS}(X).$$

This embedding is primitive, and $\text{NS}(X) = U(2)$ for a very general branch divisor $B$.

Assume now that there exists a smooth rational curve $C \in |\mathcal{O}_{\mathbb{F}_0}(1, d)|$ for $d \geq 0$ intersecting $B$ with even multiplicities. For instance, $C$ can be simply tangent to $B$ in exactly $2d + 2$ points. Then we have the following (cf. [Fes18, Proposition 5.1]):

**Lemma 5.1.** Let $\nu : X \to Y$ be a double cover of smooth projective surfaces branched over a smooth curve $B$, and assume that there exists a smooth rational curve $C \subseteq Y$ intersecting $B$ with even multiplicities. Then the pullback $\nu^*C$ splits into two disjoint irreducible components, both isomorphic to $C$.

**Proof.** Let $D := \nu^{-1}(C) \subseteq X$. The double cover $\nu$ induces a double cover $\overline{\nu} : D \to C$, which is isomorphic to an unbranched double cover. This is because the branch locus of $\overline{\nu}$ coincides with the set $b(C) := \{ x \in C \mid \operatorname{mult}_x(C, B) \equiv 1 \pmod{2} \} = \emptyset$. The unique unbranched cover of $C \cong \mathbb{P}^1$ is given by a disjoint union of two smooth rational curves isomorphic to $C$. $\square$

In the case $Y = \mathbb{F}_0$ as above, the pullback $D = \pi^*C = D_1 + D_2$ splits into the union of two irreducible components $D_1, D_2 \cong \mathbb{P}^1$. Since $D_1$ is smooth rational, we have $D_1^2 = -2$, and moreover $D_1E_1 = 1$, $D_1E_2 = d$. This implies that there exists an embedding (not necessarily primitive)

$$\langle E_1, E_2, D_1 \rangle = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & d \\ 1 & d & -2 \end{pmatrix} \cong U \oplus \langle -2(2d + 4) \rangle \hookrightarrow \text{NS}(X).$$

If instead the branch divisor $B$ is not smooth, but has simple singularities, the double cover $\pi : X \to \mathbb{F}_0$ is a K3 surface with isolated simple singularities. Therefore the desingularization $\widetilde{X} \to X$ is a smooth K3 surface, since simple singularities do not change adjunction.

The following result is well known, but we include its proof for the sake of completeness.

**Proposition 5.2.** Let $X$ be an elliptic K3 surface with $\text{NS}(X) \cong U \oplus \langle -2k \rangle$ for some $k \geq 1$. Then $X$ can be realized as a double cover of $\mathbb{F}_0$ if and only if $k$ is even and $k \geq 4$.

**Proof.** If $X$ is a double cover of $\mathbb{F}_0$, the pullback map induces a primitive embedding

$$U(2) \hookrightarrow \text{NS}(X) = U \oplus \langle -2k \rangle.$$  

It is then easy to notice that any even lattice of rank 3 containing primitively $U(2)$ must have the discriminant divisible by 4, so we conclude that $k = \frac{1}{2} \det(\text{NS}(X))$ is even.

Conversely assume that $\text{NS}(X) = U \oplus \langle -2k \rangle$ for a certain $k \geq 4$ even. Then as above we have an isomorphism

$$\begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & d \\ 1 & d & -2 \end{pmatrix} \cong U \oplus \langle -2k \rangle.$$
for $d = \frac{1}{2}(k - 4) \geq 0$, so there are two genus one fibrations $|E_1|, |E_2| : X \to \mathbb{P}^1$ induced by the two elements $E_1, E_2$ of the previous basis of square zero. We can now consider the surjective map

$$
\pi = (|E_1|, |E_2|) : X \to \mathbb{P}_0.
$$

It is a morphism of degree 2, since the preimage of any point of $\mathbb{P}_0$ consists of the two points of intersection of two elliptic curves in $|E_1|$ and $|E_2|$, as $E_1E_2 = 2$. Consider the branch divisor $B$; if $B$ is smooth, then $\pi$ is a double cover, as claimed. Assume by contradiction that $B$ is singular. $B$ must have simple singularities, since otherwise the canonical divisor of $X$ would be strictly negative. Thus $X$ is the desingularization of the double cover $\tilde{\pi} : \tilde{X} \to \mathbb{P}_0$ branched over $B$, and therefore $\text{NS}(X)$ contains the class of a smooth rational curve orthogonal to $U$. This is however absurd, since $\text{rkNS}(X) = 3$ and $\text{NS}(X) \not\cong U \oplus A_1(-1)$.

It only remains to deal with the case $k = 2$, so consider a K3 surface $X$ with $\text{NS}(X) = U \oplus \langle -4 \rangle$. If by contradiction $X$ is a double cover of $\mathbb{P}_0$, then $\text{NS}(X)$ contains primitively $U(2)$, so that

$$
U \oplus \langle -4 \rangle \cong \begin{pmatrix}
0 & 2 & a \\
2 & 0 & b \\
a & b & -2c
\end{pmatrix}
$$

for $a, b, c \in \mathbb{Z}$, $c \geq 1$. Say that the previous isomorphism is given by the choice of a basis $\{E_1, E_2, D\}$. The determinant of $\text{NS}(X)$ is 4, and this forces $ab + 2c = 1$. Thus $a, b$ are odd, and without loss of generality $a < 0$, $b > 0$. Now choose $n \geq 0$ such that $a + 2n = 1$ and consider the divisor $D + nE_2$. It is effective by Riemann-Roch, since

$$(D + nE_2)^2 = -2c + 2nb = -2c + b(1 - a) = -2c - ab + b = b - 1 \geq 0$$

and $D + nE_2$ has intersection $1 \geq 0$ with the nef divisor $E_1$. Moreover $(D + nE_2)E_1 = 1$ means that $D + nE_2$ coincides with $kE_1 + S$ for a certain $k \geq 0$ and a section $S$ of the elliptic pencil $|E_1|$. In other words, $\text{NS}(X)$ is generated by the three elements $E_1, E_2, S$. However the intersection form of $X$ with respect to this basis is

$$
\begin{pmatrix}
0 & 2 & 1 \\
2 & 0 & \alpha \\
1 & \alpha & -2
\end{pmatrix}
$$

and this matrix has determinant 4 only if $\alpha = -1$, which is a contradiction, as $E_2$ is nef and $S$ is effective. \hfill \Box

**Remark 5.3.** Let $X$ be a K3 surface with $\text{NS}(X) = U \oplus \langle -2k \rangle$ for a certain $k \geq 4$ even. Then an argument as above shows that a basis of $\text{NS}(X)$ is given by $\{E_1, E_2, D\}$ with intersection matrix

$$
\begin{pmatrix}
0 & 2 & 1 \\
2 & 0 & d \\
1 & d & -2
\end{pmatrix}
$$

where $d = \frac{1}{2}(k - 4)$, $\pi = (|E_1|, |E_2|) : X \to \mathbb{P}_0$ is the double cover branched over a $(4, 4)$-curve $B$, and $C = \pi(D)$ is a smooth $(1, d)$-curve meeting $B$ with even multiplicities.
5.2. Double covers of $F_4$. Consider the Hirzebruch surface $F_4 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(4))$. We denote by $p : F_4 \to \mathbb{P}^1$ the $\mathbb{P}^1$-bundle structure. We have that $\text{Pic}(F_4) = \mathbb{Z}(f,s)$, where $f$ is the class of a fiber $F$ of the projection $p$, while $s$ is the class of the unique curve $S \subseteq F_4$ with negative self-intersection. The intersection form on $\text{Pic}(F_4)$ with respect to this basis is

$$
\begin{pmatrix}
0 & 1 \\
1 & -4
\end{pmatrix} \cong U.
$$

The canonical bundle of $F_4$ is given by $K_{F_4} = -2s - 6f$. Notice that $\varphi = \varphi_{|s+4f|} : F_4 \to C_4$ is the desingularization of the quartic cone $C_4 \subseteq \mathbb{P}^5$ over the normal rational curve $C = \text{Im}(|\mathcal{O}_{\mathbb{P}^1}(4)|) \subseteq \mathbb{P}^4$.

Now consider the double cover $\pi : X \to F_4$ branched over a curve $B \in | -2K_{F_4}| = |4s + 12f|$. The linear system $|4s + 12f|$ has a fixed part, given by the curve $S$, and a moving part $|3s + 12f|$. Assume that $B$ splits as the sum $S + B_0$, where $B_0 \in |3s + 12f|$ is a smooth irreducible curve disjoint from $S$, as $s(3s + 12f) = 0$. Then the surface $X$ is a smooth K3 surface. The pullback $E = \pi^*F$ is a smooth elliptic curve, since the restricted double cover $E \to F$ is branched over $(4s + 12f)f = 4$ points. Moreover $\pi$ is totally ramified over $S \subseteq B$, so $\pi^*S = 2C$, where $C = \pi^{-1}(S) \cong \mathbb{P}^1$ is a smooth rational curve. Since $EC = \frac{1}{2}(\pi^*F)(\pi^*S) = FS = 1$, we have a primitive embedding

$$
\begin{pmatrix}
0 & 1 \\
1 & -2
\end{pmatrix} \cong U \hookrightarrow \text{NS}(X).
$$

For a general branch divisor $B$, we simply have $\text{NS}(X) \cong U$.

Consider the linear system $|s + 2kf|$ for $k \geq 2$. Its general member $D$ is a smooth rational curve meeting $F$ in $1$ point, $S$ in $2k - 4$ points and $B$ in $(s + 2kf)(4s + 12f) = 8k - 4$ points. Assume further that the curve $D$ intersects the branch divisor $B$ with even multiplicities. Then Lemma 5.1 assures that the pullback $\pi^*D = D_1 + D_2$ splits into two disjoint components $D_1, D_2 \cong \mathbb{P}^1$. This implies that there exists an embedding (not necessarily primitive)

$$
\langle E, C, D_1 \rangle = \begin{pmatrix}
0 & 1 & 1 \\
1 & -2 & k - 2 \\
1 & k - 2 & -2
\end{pmatrix} \cong U \oplus \langle -2k \rangle \hookrightarrow \text{NS}(X),
$$

since $D_1E = \frac{1}{4}(\pi^*D)(\pi^*F) = DF = 1$ and $D_1C = \frac{1}{4}(\pi^*D)(\pi^*S) = \frac{1}{2}DS = k - 2$.

Proposition 5.4. Every elliptic K3 surface $X$ is the desingularization of a double cover of the Hirzebruch surface $F_4$.

Proof. Assume that $U \hookrightarrow \text{NS}(X)$, and denote by $E, C$ the smooth curves in $X$ generating $U$ such that $E^2 = 0$, $C^2 = -2$. Consider the linear system $|4E + 2C|$. By [Huy16, Corollary 8.1.6] the divisor $4E + 2C$ is nef, as it has non-negative intersection with every smooth rational curve. Moreover $4E + 2C$ has intersection $0$ with the curve $C$. Since $(4E + 2C)^2 = 8$ and $\dim |4E + 2C| = 5$, $\psi = \varphi_{|4E+2C|} : X \to \mathbb{P}^5$ is a morphism onto a surface $Y \subseteq \mathbb{P}^5$ contracting $C$. $C$ is a smooth ($-2$)-curve, so $Y$ is singular. Now the elliptic curve $E$ has intersection $(4E + 2C)E = 2$ with $4E + 2C$, so $\psi$ has degree $2$ by [Sai74, Theorem 5.2]. This implies that $\deg(Y) = 4$, so $Y \subseteq \mathbb{P}^5$ is a singular surface of minimal degree, thus $Y$ is the quartic cone $C_4$ (see [del87]). Therefore $\psi$ factors through the minimal resolution of $C_4$, which is $F_4$, giving a morphism $\pi : X \to F_4$ of degree 2. Now we can repeat the argument
in the proof of Proposition 5.2 obtaining that $X$ is the desingularization of a double cover of $\mathbb{F}_4$. □

Remark 5.5. Every K3 surface $X$ with $\text{NS}(X) = U \oplus \langle -2k \rangle$ for a certain $k \geq 2$ can be obtained as a double cover $\pi : X \to \mathbb{F}_4$ branched over a smooth curve $B \in |4s + 12f|$ admitting a rational curve $D \in |s + 2kf|$ intersecting $B$ with even multiplicities.

If instead $X$ is a K3 surface with $\text{NS}(X) = U \oplus \langle -2 \rangle$, then it is the desingularization of the double cover of $\mathbb{F}_4$ branched over a curve $B$ with a unique singularity of type $A_1$.

5.3. Weierstrass fibrations. Let $X$ be a smooth K3 surface. Recall that $X$ is said elliptic if it admits an elliptic fibration, i.e. a morphism $\pi : X \to \mathbb{P}^1$ whose general fiber is a curve of genus one, together with a distinguished section. The Néron-Severi group of an elliptic K3 surface contains primitively a copy of the hyperbolic plane $U$, spanned by the classes of the fiber and the zero section of the elliptic fibration.

Let $X$ be a smooth elliptic K3 surface. By [Mir89, Section §II.3] $X$ is the desingularization of a Weierstrass fibration $\pi' : Y \to \mathbb{P}^1$, where $Y$ is defined by an equation

$$Y^2Z = X^3 + AXZ^2 + BZ^3$$

in $\mathbb{P} (\mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(6) \oplus \mathcal{O}_{\mathbb{P}^1})$ with $A \in H^0(\mathcal{O}_{\mathbb{P}^1}(8))$ and $B \in H^0(\mathcal{O}_{\mathbb{P}^1}(12))$ minimal and with $\Delta = 4A^3 + 27B^2$ not identically zero. Conversely, every such Weierstrass fibration desingularizes to a smooth elliptic K3 surface. We will usually restrict to the chart $\{Z \neq 0\}$ over the affine base $\mathbb{A}^1_t \subseteq \mathbb{P}^1$, where the equation (7) becomes

$$y^2 = x^3 + A(t)x + B(t),$$

with $A$ and $B$ polynomials in $t$ of degree at most 8 and 12 respectively. Notice that this is the equation of the generic fiber of the Weierstrass fibration, which is an elliptic curve over $\mathbb{C}(t)$. Under this identification, sections of the fibration $\pi$ (or $\pi'$) correspond to $\mathbb{C}(t)$-rational points of equation (8). In particular the distinguished zero section is located at the point of infinity $S_0 = (0 : 1 : 0)$. Moreover we will write $S = (u(t), v(t))$ to denote the section $S$ of $\pi$ corresponding to the $\mathbb{C}(t)$-rational point $(u(t), v(t))$ of equation (8). By the above description, $u, v \in \mathbb{C}(t)$ are rational functions of degree at most 4, 6 respectively.

Remark 5.6. Let $X$ be a $U \oplus \langle -2k \rangle$-polarized K3 surface. If $k \geq 2$, the given elliptic fibration on $X$ admits an extra section $S$ such that $SS_0 = k - 2$. This follows from the isomorphism of lattices

$$U \oplus \langle -2k \rangle \cong \begin{pmatrix} 0 & 1 & 1 \\ 1 & -2 & k-2 \\ 1 & k-2 & -2 \end{pmatrix}.$$}

Conversely, if $X$ is an elliptic K3 surface and $S$ is an extra section with $SS_0 = k - 2$, then there exists an embedding

$$U \oplus \langle -2k \rangle \hookrightarrow \text{NS}(X).$$

This embedding is not necessarily primitive. However, it is primitive if the lattice $U \oplus \langle -2k \rangle$ has no non-trivial overlattices (for instance if $2k$ is square-free, cf. [Nik79 Proposition 1.4.1]).
6. Unirationality of $\mathcal{M}_{2k}$ for small $k$

The aim of this section is to prove Theorem 0.2, i.e. the unirationality of $\mathcal{M}_{2k}$ for $k \leq 11$ and $k \in \{13, 16, 17, 19, 21, 25, 26, 29, 31, 34, 36, 37, 39, 41, 43, 49, 59, 61, 64, 73, 100\}$. For some of the cases we will use the geometric constructions of Section 5. For the others, we will find projective models of $U \oplus (-2k)$-polarized K3 surfaces given by (quasi-)polarizations of degree $\leq 8$. More precisely, the strategy will consist in finding $\mathbb{Z}$-bases of $U \oplus (-2k)$ given by the (quasi-)polarization and $(-2)$-curves of small degree.

6.1. $k = 1$. The variety $\mathcal{M}_2$ is the moduli space of $U \oplus (-2)$-polarized K3 surfaces. If $X$ is a general K3 surface in $\mathcal{M}_2$, then $X$ is the desingularization of an elliptic K3 surface $Y$ with an $A_1$ singularity. Hence $X$ admits an elliptic fibration with a unique reducible fiber, consisting of two irreducible smooth rational curves. A quick inspection of the Kodaira fibers [Mir89, Table I.4.1] yields that this reducible fiber can be either of type $I_2$ (two smooth rational curves meeting transversely at two distinct points) or $III$ (two smooth rational curves simply tangent at one point). This depends on whether the $A_1$ singularity on $Y$ belongs to a nodal or cuspidal rational curve respectively. After moving the singular fiber at $t = 0$, $Y$ can be written as a Weierstrass equation

$$y^2 = x^3 + a(t)x^2 + b(t)x + c(t)$$

satisfying $t \mid b(t)$ and $t^2 \mid c(t)$. Up to a change of coordinates in $x$, this equation is equivalent to the one in $[\mathfrak{S}]$. Conversely, a general such Weierstrass equation desingularizes to an elliptic K3 surface with an $I_2$ or a $III$ fiber at $t = 0$. From this description we can define a dominant rational map

$$\mathcal{P}_2 := \{(a, b, c) \in H^0(\mathcal{O}_{\mathbb{P}^1}(4)) \times H^0(\mathcal{O}_{\mathbb{P}^1}(8)) \times H^0(\mathcal{O}_{\mathbb{P}^1}(12)) : t \mid b(t), \ t^2 \mid c(t)\} \longrightarrow \mathcal{M}_2$$

sending the polynomials $(a, b, c)$ into the isomorphism class of the desingularization of the corresponding Weierstrass equation. Since $\mathcal{P}_2$ is an affine space, $\mathcal{M}_2$ is unirational.

6.2. $k = 2$. An $U \oplus (-4)$-polarized K3 surface $X$ is an elliptic K3 surface admitting a section $S$ disjoint from the zero section $S_0$ of the given elliptic fibration by Remark 5.6. Let

$$y^2 = x^3 + a(t)x^2 + b(t)x + c(t)$$

be a Weierstrass equation for $X$, where the point at infinity $S_0 = (0 : 1 : 0)$ is the zero section. Let $S = (u(t), v(t))$ be the extra section. Notice that the points of intersection of $S$ and $S_0$ coincide with the poles of $v$ (or equivalently of $u$), as $(u(t_0) : v(t_0) : 1) = (0 : 1 : 0)$ if and only if $t_0$ is a pole for $v$. But $S$ and $S_0$ are disjoint by assumption, thus $u, v$ are simply polynomials of degree at most 4,6 respectively. After the change of variables $x \mapsto x - u$, $y \mapsto y - v$, the Weierstrass equation becomes

$$y^2 + 2v(t)y = x^3 + d(t)x^2 + e(t)x,$$

for polynomials $d, e, v$ of degree at most 4,8,6 respectively. Conversely, a general Weierstrass equation as in $[\mathfrak{S}]$ defines an elliptic K3 surface containing the disjoint sections $S_0 = (0 : 1 : 0)$, $S = (0, 0)$, and therefore an $U \oplus (-4)$-polarized K3 surface. This implies that there exists a dominant rational map

$$\mathcal{P}_4 := \{(d, e, v) \in H^0(\mathcal{O}_{\mathbb{P}^1}(4)) \times H^0(\mathcal{O}_{\mathbb{P}^1}(8)) \times H^0(\mathcal{O}_{\mathbb{P}^1}(6))\} \longrightarrow \mathcal{M}_4.$$

$\mathcal{P}_4$ is an affine space, thus $\mathcal{M}_4$ is unirational.
6.3. $k = 3$. Let $X$ be the desingularization of a double cover of $\mathbb{P}^2$ branched over a sextic $B$ with an $A_2$ singularity. Then $X$ is a K3 surface with

$$\langle 2 \rangle \oplus A_2(-1) \cong U \oplus \langle -6 \rangle \hookrightarrow \text{NS}(X).$$

Since 6 is square-free, the previous embedding is primitive. Conversely, if $X$ is a K3 surface with $\text{NS}(X) = \langle 2 \rangle \oplus A_2(-1)$, the linear system associated to the first element of the basis induces a morphism $X \to \mathbb{P}^2$ of degree 2 contracting the two $(-2)$-curves, thus $X$ is the desingularization of a double cover of $\mathbb{P}^2$ branched over a sextic with an $A_2$ singularity. Up to a projective transformation, we can assume that the sextic $B \subseteq \mathbb{P}^2$ has an $A_2$ singularity at $P = (0 : 0 : 1) \in \mathbb{P}^2$, and that the unique line of $\mathbb{P}^2$ meeting $B$ in $P$ with multiplicity 3 is $V(x_0)$. This forces $B$ to be given by an equation $f(x_0, x_1, x_2) \in H^0(\mathcal{O}_{\mathbb{P}^2}(6))$ with coefficients of the terms $x_0^6, x_0x_1^5, x_1^6, x_0x_1x_2^3, x_1^3x_2$ zero. We denote by $\mathcal{P}_6$ the linear subspace of $H^0(\mathcal{O}_{\mathbb{P}^2}(6))$ parametrizing the polynomials with this vanishing of the coefficients. Therefore there exists a dominant rational map

$$\mathcal{P}_6 \dashrightarrow \mathcal{M}_6.$$

$\mathcal{P}_6$ is an affine space, hence $\mathcal{M}_6$ is unirational.

6.4. $k = 4$. By Proposition 5.2 and Remark 5.3 a general $U \oplus \langle -8 \rangle$-polarized K3 surface $X$ is the double cover of $\mathbb{P}_0$ branched over a smooth $(4, 4)$-curve $B$ admitting a line $C$ simply tangent to $B$ at 2 points. Assume that $\mathbb{P}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ has coordinates $((x_0 : x_1), (y_0 : y_1))$, and without loss of generality that $C$ is given by $x_0 = 0$. If $B$ is given by a bihomogeneous polynomial $f(x_0, x_1, y_0, y_1)$ of bidegree $(4, 4)$, then $B$ is tangent to $C$ at 2 points if and only if

$$f(x_0, x_1, y_0, y_1) = x_0g(x_0, x_1, y_0, y_1) + x_1^4h_1(y_0, y_1)^2h_2(y_0, y_1)^2$$

for $g \in H^0(\mathcal{O}_{\mathbb{P}_0}(3, 4)), h_1, h_2 \in H^0(\mathcal{O}_{\mathbb{P}^1}(1))$. Therefore we get a dominant rational map

$$\mathcal{P}_8 := \{(g, h_1, h_2) \in H^0(\mathcal{O}_{\mathbb{P}_0}(3, 4)) \times H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \times H^0(\mathcal{O}_{\mathbb{P}^1}(1))\} \dashrightarrow \mathcal{M}_8$$

sending $(g, h_1, h_2)$ to the isomorphism class of the double cover of $\mathbb{P}_0$ branched along the divisor $f = 0$ defined above. It follows that $\mathcal{M}_8$ is unirational.

6.5. $k = 5$. Let $X$ be the desingularization of a double cover of $\mathbb{P}^2$ branched over a sextic $B$ with a simple node and admitting a tritangent line. Then $X$ is a K3 surface with

$$\left(\begin{array}{ccc} 2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{array}\right) \cong U \oplus \langle -10 \rangle \hookrightarrow \text{NS}(X)$$

(see Lemma 5.1). Since 10 is square-free, the previous embedding is primitive. Conversely, let $X$ be a K3 surface with $\text{NS}(X)$ isometric to the previous lattice, with basis $\{H, L, C\}$. The linear system $|H|$ induces a morphism $X \to \mathbb{P}^2$ of degree 2 contracting $C$. Let $Y \to \mathbb{P}^2$ denote the double cover obtained contracting $C$. Then $Y$ has a singular point of type $A_1$, so the branch locus $B \subseteq \mathbb{P}^2$ has a node. Moreover $L$ is mapped onto a line of $\mathbb{P}^2$ meeting $B$ with even multiplicities, so generically it will be a tritangent of $B$. Now, up to a projective transformation, we can assume that the tritangent line is given by $V(x_0)$, so that $B$ is given by an equation of the form

$$f = x_0g(x_0, x_1, x_2) + h_1(x_1, x_2)^2h_2(x_1, x_2)^2h_3(x_1, x_2)^2.$$ (10)
We can also assume that the node of $B$ is located at $P = (1 : 0 : 0)$. This forces the coefficients of $g$ of the terms $x_0^2, x_0^1x_1, x_0^4x_2$ to be zero. We denote by $Q_{10}$ the linear subspace of $H^0(O_{\mathbb{P}^2}(5))$ parametrizing the polynomials with this vanishing of the coefficients. Then there exists a dominant rational map

$$\mathcal{P}_{10} = Q_{10} \times H^0(O_{\mathbb{P}^2}(1))^3 \dashrightarrow \mathcal{M}_{10}.$$ 

sending $(g, h_1, h_2, h_3)$ to the isomorphism class of the double cover of $\mathbb{P}^2$ branched over $f$ defined as in equation 10. As $\mathcal{P}_{10}$ is an affine space, $\mathcal{M}_{10}$ is unirational.

6.6. $k = 6$. By Proposition 5.2 and Remark 5.3, a general such K3 surface is the double cover of $\mathbb{P}_0$ branched over a $(4, 4)$-curve $B$ admitting a smooth $(1, 1)$-curve $C$ intersecting $B$ in 4 points with multiplicity 2. Up to an automorphism of $\mathbb{P}_0$ we can assume that $C = V(x_0y_1 - x_1y_0)$; moreover we can assume that $B$ doesn’t pass through the point $((0 : 1), (0 : 1)) \in C$, so that the intersection $B \cap C$ is contained in the chart $U = \{x_0 \neq 0, y_0 \neq 0\}$, with coordinates $(1 : u), (1 : v)$. Say that $B$ is given by the equation

$$f(x_0, x_1, y_0, y_1) = \sum_{i+j=k+l=4} \alpha_{ijkl}x_0^ix_1^jy_0^ky_1^l$$

with $\alpha_{0404} = 1$. Since $C|_U = V(u - v)$, the intersection $B \cap C \subseteq U$ is given by the vanishing of

$$g(u) = f(1, u, 1, u) = \sum_{i+j=k+l=4} \alpha_{ijkl}u^j+l = \sum_{\eta=0}^8 \beta_{\eta}u^\eta,$$

where $\beta_{\eta} = \sum_{j+l=\eta} \alpha_{ijkl}$ and $\beta_8 = \alpha_{0404} = 1$. Now $g(u)$ has 4 double roots at $u = \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ if and only if

$$g(u) = (u - \varepsilon_1)^2(u - \varepsilon_2)^2(u - \varepsilon_3)^2(u - \varepsilon_4)^2.$$

The choice of $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ uniquely determines the coefficients $\beta_{\eta}$ for $\eta \leq 7$, which in turn uniquely determine 8 of the $\alpha_{ijkl}$. The other 17 coefficients $\alpha_{ijkl}$ are free parameters so, if we denote them by $\alpha'_1, \ldots, \alpha'_7$, we have that there exists a rational dominant map

$$\mathcal{P}_{12} := \{(\varepsilon_i, \alpha'_j) \in (\mathbb{A}^1)^4 \times (\mathbb{A}^1)^{17}\} \dashrightarrow \mathcal{M}_{12}.$$ 

$\mathcal{P}_{12}$ is an affine space, so $\mathcal{M}_{12}$ is unirational.

6.7. $k = 7$. Let $X' \subseteq \mathbb{P}^3$ be a quartic surface containing a line $L$ and with an $A_1$ singularity $P$ located on the line $L$. Then the desingularization $X$ of $X'$ is a smooth K3 surface with

$$\begin{pmatrix} 4 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix} \cong U \oplus \langle -14 \rangle \hookrightarrow \text{NS}(X).$$

The embedding is primitive, since 14 is square-free. Conversely, let $X$ be a K3 surface with $\text{NS}(X)$ isometric to the previous lattice, with basis $\{H, L, N\}$. The unique $(-2)$-divisor in $H^\perp$ is $\pm N$. The linear system $|H|$ induces a map $\varphi : X \rightarrow \mathbb{P}^3$ contracting $N$. $\varphi$ is an embedding outside of $N$ by [Sai74, Theorem 5.2], since there is no isotropic vector $E$ with $EH = 2$. The image $\varphi(X) \subseteq \mathbb{P}^3$ is a quartic containing a line $\varphi(L)$ (since $LH = 1$) and a singular point $\varphi(N)$ of type $A_1$ located on $L$. 

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Now, if \( X' \subseteq \mathbb{P}^3 \) is a quartic as above, containing the line \( L = V(x_0, x_1) \) and with an \( A_1 \) singularity at \( P = (0 : 0 : 0 : 1) \), it is given by an equation

\[
f(x_0, x_1, x_2, x_3) = x_0g(x_0, x_1, x_2, x_3) + x_1h(x_0, x_1, x_2, x_3)
\]

with \( g(P) = h(P) = 0 \). For a general choice of \( g \) and \( h \), the singularity at \( P \) is of type \( A_1 \).

This shows that there exists a dominant rational map

\[
\mathcal{P}_{14} := \{(g, h) \in H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \times H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \mid g(P) = h(P) = 0\} \dashrightarrow \mathcal{M}_{14}.
\]

\( \mathcal{P}_{14} \) is an affine space, thus \( \mathcal{M}_{14} \) is unirational.

6.8. \( k = 8 \). By Proposition 5.2 and Remark 5.3, a general such K3 surface is the double cover of \( \mathbb{P}^3 \) branched over a \((4, 4)\)-curve \( B \) admitting a smooth \((1, 2)\)-curve \( C \) intersecting \( B \) in 6 points with multiplicity 2. Fix \( \mathbb{P}^3 = V(x_0x_3 - x_1x_2) \subseteq \mathbb{P}^3 \); then, up to automorphism of \( \mathbb{P}_0 \), we can assume that \( C \) is the twisted cubic curve \( \bar{V}(x_0x_3 - x_1x_2, x_1^2 - x_0x_2, x_2^2 - x_1x_3) \). Suppose that \( B \) doesn’t pass through the point \((0 : 0 : 0 : 1) \in C \), so that the intersection \( B \cap C \) is contained in the chart \( U = \{x_0 \neq 0\} \subseteq \mathbb{P}^3 \) with coordinates \((1 : u : v : w) \). Then \( C \vert_U = V(w - u^2, w - u^3) \), so if \( B \) is given by a quartic \( f(x_0, x_1, x_2, x_3) \in H^0(\mathcal{O}_{\mathbb{P}^3}(4)) \), the intersection \( B \cap C \subseteq U \) is given by the vanishing of

\[
g(u) = f(1, u, u^2, u^3).
\]

Now an argument as in the case \( k = 6 \) shows that \( \mathcal{M}_{16} \) is unirational.

6.9. \( k = 9 \). Let \( Q \subseteq \mathbb{P}^4 \) be a quadric containing a plane \( \pi \subseteq \mathbb{P}^4 \). We assume that \( Q \) is the cone over a smooth quadric in \( \mathbb{P}^3 \), so that it has a unique singular point, the vertex \( P \). Let \( K \subseteq \mathbb{P}^4 \) be a smooth cubic containing a conic and a line \( C, L \subseteq \pi \) with \( C \cap L \) consisting of two points. If \( P \notin K \) and \( X = Q \cap K \) is a complete intersection, then \( X \) is a smooth K3 surface. By construction \( X \) contains \( C \) and \( L \), so that

\[
\begin{pmatrix}
6 & 1 & 2 \\
1 & -2 & 2 \\
2 & 2 & -2
\end{pmatrix} \cong U \oplus \langle -18 \rangle \hookrightarrow \text{NS}(X).
\]

The isomorphism follows from the fact that the elliptic fibration induced by \( E := H - L - C \) has a section \( S_0 := 3H - 4L - 2C \). Moreover the previous embedding is primitive. If it weren’t, its saturation in \( \text{NS}(X) \) would be the only non-trivial overlattice of \( U \oplus \langle -18 \rangle \), which is \( U \oplus \langle -2 \rangle \). This is however impossible, since it is easy to check that \( |E| \) has infinitely many sections, while any elliptic fibration on \( U \oplus \langle -2 \rangle \) has only one section.

Conversely, let \( X \) be a K3 surface with \( \text{NS}(X) \) isometric to the previous lattice, with basis \( \{H, L, C\} \). The divisor \( H \) is ample, and actually very ample by [Sai74, Theorem 5.2], since there is no isotropic vector \( E \) with \( EH = 2 \). The image \( \varphi(X) \) is the complete intersection of a quadric \( Q \) and a cubic \( K \). Moreover \( \varphi(L) \) and \( \varphi(C) \) are a line and a conic respectively. Since \( \varphi(L) \) and \( \varphi(C) \) meet at two points, their union is contained in a plane \( \pi \subseteq \mathbb{P}^4 \). The quadric \( Q \) contains \( \varphi(L) \cup \varphi(C) \) if and only if it contains the plane \( \pi \), so \( Q \) must be singular. Generically \( Q \) will be the cone over a smooth quadric in \( \mathbb{P}^3 \).

Now fix the plane \( \pi = V(x_3, x_4) \subseteq \mathbb{P}^4 \). Up to an automorphism of \( \pi \), we can assume that \( L = V(x_2, x_3, x_4) \) and \( C = V(x_2^2 - x_0x_1, x_3, x_4) = \{(u^2 : v^2 : uv : 0 : 0) \mid (u : v) \in \mathbb{P}^1\} \). \( Q \) contains the plane \( \pi \) if and only if it is given by an equation

\[
f_2(x_0, x_1, x_2, x_3, x_4) = x_3l_1 + x_4l_2
\]

(11)
for some linear forms $l_1, l_2 \in H^0(\mathcal{O}_{\mathbb{P}^4}(1))$. The cubic $K$ contains the line $L$ if and only if it is given by an equation,

$$f_3(x_0, x_1, x_2, x_3, x_4) = x_2q_1 + x_3q_2 + x_4q_3$$

(12)

for some quadratic forms $q_1, q_2, q_3 \in H^0(\mathcal{O}_{\mathbb{P}^4}(2))$. Moreover $K$ contains the conic $C$ if and only if

$$f_3(u^2, v^2, uv, 0, 0) \equiv 0$$

is zero as a polynomial in $(u : v)$. This imposes linear conditions on the coefficients of $q_1, q_2, q_3$. We denote by $\mathcal{Q}_{18}$ the set of quadratic forms $q_1, q_2, q_3$ satisfying such linear conditions. Then we have a dominant rational map

$$\mathcal{P}_{18} = \{(l_1, l_2) \in H^0(\mathcal{O}_{\mathbb{P}^4}(1))^2\} \times \mathcal{Q}_{18} \longrightarrow \mathcal{M}_{18}$$

sending the quadric $Q = V(f_3)$ and the cubic $K = V(f_3)$ defined as in (11) and (12) to the isomorphism class of the (smooth complete) intersection $Q \cap K$.

6.10. $k = 10$. Let $X \subseteq \mathbb{P}^3$ be a smooth quartic surface containing two disjoint lines $L_1, L_2$. Then $X$ is a K3 surface with

$$\begin{pmatrix} 4 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{pmatrix} \approx U \oplus \langle -20 \rangle \hookrightarrow \text{NS}(X).$$

Since $U \oplus \langle -20 \rangle$ has no non-trivial overlattices (cf. [Nik79, Proposition 1.4.1]), the previous embedding is primitive. A geometric way to see that $X$ is elliptic is to take the pencil of hyperplanes containing $L_1$. The linear system $|H - L_1|$ consists of planar cubic curves, and defines a genus one fibration on $X$. Since $(H - L_1)L_2 = 1$, $L_2$ is a section of this fibration. Conversely, let $X$ be a K3 surface with $\text{NS}(X)$ isometric to the previous lattice, and denote by $\{H, L_1, L_2\}$ the corresponding basis. Since there are no isotropic vectors $E$ with $EH = 2$, [Sai74, Theorem 5.2] shows that the linear system $|H|$ induces an embedding $\varphi : X \hookrightarrow \mathbb{P}^3$, sending $L_1$ and $L_2$ onto disjoint lines contained in the quartic $\varphi(X)$.

Now, up to automorphism of $\mathbb{P}^3$, we can fix $L_1 = V(x_0, x_1)$ and $L_2 = V(x_2, x_3)$. A quartic surface $X = V(f) \subseteq \mathbb{P}^3$ contains $L_1$ and $L_2$ if and only if the coefficients of $f$ of the terms only in $x_0, x_1$ and only in $x_2, x_3$ are zero. It follows that $\mathcal{M}_{20}$ is unirational.

6.11. $k = 11$. Consider a projective subspace $\pi \subseteq \mathbb{P}^5$ of dimension 3, a twisted cubic $C_3 \subseteq \pi$ and a conic $C_2 \subseteq \pi$ with $C_2 \cap C_3$ consisting of three points. Let $X = Q_1 \cap Q_2 \cap Q_3 \subseteq \mathbb{P}^5$ be a smooth complete intersection of three smooth quadrics containing the union $C_2 \cup C_3$. Then $X$ is a smooth K3 surface with

$$\begin{pmatrix} 8 & 2 & 3 \\ 2 & -2 & 3 \\ 3 & 3 & -2 \end{pmatrix} \approx U \oplus \langle -22 \rangle \hookrightarrow \text{NS}(X).$$

Conversely, let $X$ be a K3 surface with $\text{NS}(X)$ isometric to the previous lattice, with basis $\{H, C_2, C_3\}$. $H$ is very ample by [Sai74, Theorem 5.2], since there is no isotropic vector $E$ with $EH = 2$, thus the linear system $|H|$ induces an embedding $\varphi : X \hookrightarrow \mathbb{P}^5$. The image $\varphi(X)$ is a smooth complete intersection of three quadrics $Q_1, Q_2, Q_3$ containing the conic $\varphi(C_2)$ and the twisted cubic $\varphi(C_3)$.

Now fix the projective subspace $\pi = V(x_4, x_5) \subseteq \mathbb{P}^5$ of dimension 3, and consider $C_3 = \{(u^3 : u^2v : uv^2 : v^3 : 0 : 0) \mid (u : v) \in \mathbb{P}^1\}$. If $P$ is any plane contained in $\pi$, the intersection
If the coefficients of the terms of $f$ with $\varphi$ is any line disjoint from $C$, we conclude that $C_u = C$ is a polynomial in $(u : v)$.

Therefore the incidence variety

$$\mathcal{P}_{22} := \{ (P, g, f_1, f_2, f_3) \in |O_\pi(1)| \times |O_P(2)| \times |O_\varphi(2)| : C_2 = V(g) \supseteq P \cap C_3, V(f_i) \supseteq C_2 \cup C_3 \}.$$ 

A quadric $Q = V(f) \subseteq \mathbb{P}^5$ contains $C_3$ if and only if

$$f(u^3, u^2v, uv^2, v^3, 0, 0) \equiv 0$$

is zero as a polynomial in $(u : v)$, and this imposes linear conditions on the coefficients of $f$. Similarly, imposing that $Q$ contains any conic $C_2 = V(g) \subseteq P$, forces other linear conditions on the coefficients of $f$. This shows that $\mathcal{P}_{22}$ is a projective bundle over the variety

$$\mathcal{Z} := \{ (P, g) \in |O_\pi(1)| \times |O_P(2)| : C_2 = V(g) \supseteq P \cap C_3 \}.$$ 

By the discussion above, this is a projective bundle over $|O_\pi(1)| \cong \mathbb{P}^3$, so $\mathcal{Z}$ (and thus $\mathcal{P}_{22}$) is rational. There exists a dominant rational map

$$\mathcal{P}_{22} \dashrightarrow \mathcal{M}_{22}$$

sending $(P, g, f_1, f_2, f_3)$ to the isomorphism class of the (smooth complete) intersection $V(f_1) \cap V(f_2) \cap V(f_3)$. We conclude that $\mathcal{M}_{22}$ is unirational.

6.12. $k = 13$. Let $X \subseteq \mathbb{P}^3$ be a smooth quartic surface containing a line $L$ and a smooth conic $C$, with $L$ and $C$ disjoint. Then $X$ is a K3 surface with

$$\begin{pmatrix} 4 & 2 & 1 \\ 2 & -2 & 0 \\ 1 & 0 & -2 \end{pmatrix} \cong U \oplus \langle -26 \rangle \hookrightarrow \text{NS}(X).$$

Conversely, let $X$ be a K3 surface with $\text{NS}(X)$ isometric to the previous lattice, and denote by $\{H, C, L\}$ the corresponding basis. $H$ is very ample by [Sai74, Theorem 5.2], since there is no isotropic vector $E$ with $EH = 2$. Therefore the linear system $|H|$ induces an embedding $\varphi : X \hookrightarrow \mathbb{P}^3$, sending $C$ and $L$ onto a smooth conic and a line contained in the quartic $\varphi(X)$, with $\varphi(C) \cap \varphi(L) = \emptyset$.

Now, up to automorphism of $\mathbb{P}^3$, we can fix $C = V(x_0, x_1^3 - x_2x_3)$. If $L \in \text{Gr}(1, 3)$ is any line disjoint from $C$, a quartic surface $X = V(f) \subseteq \mathbb{P}^3$ contains $L$ if and only if the coefficients of the terms of $f$ satisfy some linear conditions. Moreover $X$ contains $C = \{(0 : uv : u^2 : v^2) | (u : v) \in \mathbb{P}^1\}$ if and only if

$$f(0, uv, u^2, v^2) \equiv 0$$

as a polynomial in $(u : v)$. This also imposes linear conditions on the coefficients of $f$. Therefore the incidence variety

$$\mathcal{P}_{26} := \{ (L, f) \in \text{Gr}(1, 3) \times |O_{\mathbb{P}^3}(4)| : V(f) \supseteq L \cup C \}$$

is a projective bundle over the rational variety $\text{Gr}(1, 3)$, hence $\mathcal{P}_{26}$ is rational. We have a dominant rational map

$$\mathcal{P}_{26} \dashrightarrow \mathcal{M}_{26}$$

sending the pair $(L, f)$ to the isomorphism class of the quartic surface $X = V(f)$. We conclude that $\mathcal{M}_{26}$ is unirational.
6.13. \(k = 16\). We consider smooth quartics \(X \subseteq \mathbb{P}^3\) containing a twisted cubic \(C\) and a line \(L\) meeting at two points. Then \(X\) is a smooth K3 surface with
\[
\begin{pmatrix}
4 & 3 & 1 \\
3 & -2 & 2 \\
1 & 2 & -2
\end{pmatrix} \approx U \oplus \langle -32 \rangle \hookrightarrow \text{NS}(X).
\]
An easy check shows that the embedding is not primitive if and only if the class \(C + L\) is divisible in \(\text{NS}(X)\). The divisor \(C + L\) has square zero and it is reduced, thus it is primitive in \(\text{NS}(X)\), and hence the embedding is primitive. Conversely, an argument as above using [Sai74, Theorem 5.2] shows that every K3 surface \(X\) with \(\text{NS}(X)\) isometric to the previous lattice is such a quartic surface.

Now, up to automorphism of \(\mathbb{P}^3\), we can fix \(C = \{(u^3 : u^2v : uv^2 : v^3) \mid (u : v) \in \mathbb{P}^1\}\). A quartic \(X = V(f) \subseteq \mathbb{P}^3\) contains \(C\) if and only if
\[f(u^3, u^2v, uv^2, v^3) \equiv 0\]
as a polynomial in \((u : v)\). This imposes linear conditions on the coefficients of \(f\). Moreover, if \(P_1, P_2 \in C\) are any points on \(C\), and \(L = \overline{P_1P_2} \subseteq \mathbb{P}^3\) is the line through them, \(X\) contains \(L\) if and only if the coefficients of \(f\) satisfy other linear conditions. This shows that the incidence variety
\[\mathcal{P}_{32} := \{(P_1, P_2, f) \in \text{Sym}^2(C) \times |\mathcal{O}_{\mathbb{P}^3}(4)| : V(f) \supseteq C \cup \overline{P_1P_2}\}\]
is a projective bundle over \(\text{Sym}^2(C) \cong \mathbb{P}^2\), thus it is rational. There exists a dominant rational map
\[\mathcal{P}_{32} \dashrightarrow \mathcal{M}_{32}\]
sending \((P_1, P_2, f)\) to the isomorphism class of the quartic surface defined by \(f\). We conclude that \(\mathcal{M}_{32}\) is unirational.

6.14. \(k = 17\). We consider a quartic surface \(X' \subseteq \mathbb{P}^3\) containing a twisted cubic curve \(C\) and a singular point \(P\) of type \(A_1\). Its desingularization \(X\) is a smooth K3 surface with
\[
\begin{pmatrix}
4 & 3 & 0 \\
3 & -2 & 0 \\
0 & 0 & -2
\end{pmatrix} \approx U \oplus \langle -34 \rangle \hookrightarrow \text{NS}(X).
\]
Conversely, an argument as above using [Sai74, Theorem 5.2] shows that every K3 surface \(X\) with \(\text{NS}(X)\) isometric to the previous lattice is the desingularization of such a quartic surface.

We fix \(C = \{(u^3 : u^2v : uv^2 : v^3) \mid (u : v) \in \mathbb{P}^1\}\). As above, a quartic \(X = V(f) \subseteq \mathbb{P}^3\) contains \(C\) if and only if the coefficients of \(f\) satisfy some linear conditions. Moreover, if \(P \in \mathbb{P}^3\) is any point not in \(C\), imposing that \(X\) has a singularity at \(P\) forces other linear conditions on the coefficients of \(f\). Therefore the variety
\[\mathcal{P}_{34} := \{(P, f) \in \mathbb{P}^3 \times |\mathcal{O}_{\mathbb{P}^3}(4)| : f\text{ is singular at }P, V(f) \supseteq C\} \rightarrow \mathbb{P}^3\]
is a projective bundle over \(\mathbb{P}^3\), hence it is rational. There exists a dominant rational map
\[\mathcal{P}_{34} \dashrightarrow \mathcal{M}_{34}\]
sending the pair \((P, f)\) to the isomorphism class of the desingularization of the quartic surface defined by \(f\). We conclude that \(\mathcal{M}_{34}\) is unirational.
6.15. $k = 19$. We consider smooth quartics $X \subseteq \mathbb{P}^3$ containing a twisted cubic curve $C$ and a line $L$ meeting at one point $P$. Then $X$ is a smooth K3 surface with

$$\begin{pmatrix} 4 & 3 & 1 \\ 3 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \cong U \oplus \langle -38 \rangle \hookrightarrow \text{NS}(X).$$

Conversely, an argument as above using [Sai74, Theorem 5.2] shows that every K3 surface $X$ with $\text{NS}(X)$ isometric to the previous lattice is such a quartic surface.

We fix the twisted cubic curve $C = \{(u^3 : u^2v : uv^2 : v^3) \mid (u : v) \in \mathbb{P}^1\}$. We consider the incidence variety

$$\mathcal{P}_{38} := \{(P, L, f) \in C \times \text{Gr}(1, 3) \times |\mathcal{O}_{\mathbb{P}^3}(4)| \mid P \in L, V(f) \supseteq C \cup L\},$$

where $\text{Gr}(1, 3)$ is the Grassmanian of lines in $\mathbb{P}^3$. An argument as in the case $k = 17$ shows that $\mathcal{P}_{34}$ is a projective bundle over the variety

$$\mathcal{Z} := \{(P, L) \in C \times \text{Gr}(1, 3) \mid P \in L\},$$

which in turn is a $\mathbb{P}^2$-bundle over $C \cong \mathbb{P}^1$. This shows that $\mathcal{P}_{38}$ is rational. There exists a dominant rational map

$$\mathcal{P}_{38} \dashrightarrow \mathcal{M}_{38}$$

sending the triple $(P, L, f)$ to the isomorphism class of the quartic surface defined by $f$. We conclude that $\mathcal{M}_{38}$ is unirational.

6.16. $k = 21$. Let $X = Q \cap K \subseteq \mathbb{P}^4$ be a smooth complete intersection of a quadric $Q$ and a cubic $K$ containing two conics $C_1, C_2$ meeting at one point. Then $X$ is a smooth K3 surface with

$$\begin{pmatrix} 6 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix} \cong U \oplus \langle -42 \rangle \hookrightarrow \text{NS}(X).$$

Conversely, let $X$ be a K3 surface with $\text{NS}(X)$ isometric to the previous lattice, with basis $\{H, C_1, C_2\}$. $H$ is ample, and actually very ample by [Sai74, Theorem 5.2], since there is no isotropic vector $E$ with $EH = 2$. Therefore $X$ is a smooth complete intersection of a quadric and a cubic containing two conics $C_1, C_2$ meeting at one point.

Now let $\pi_1, \pi_2 \subseteq \mathbb{P}^4$ be two planes meeting at one point $P$. Up to a change of coordinates, $\pi_1 = V(x_3, x_4), \pi_2 = V(x_0, x_1)$, $P = (0 : 0 : 1 : 0 : 0)$. Two conics $C_1 \subseteq \pi_1, C_2 \subseteq \pi_2$ intersect if and only if they pass through $P$. Since a quadric $Q = V(f) \subseteq \mathbb{P}^4$ (respectively, a cubic $K = V(g) \subseteq \mathbb{P}^4$) contains $C_1 \cup C_2$ if and only if the coefficients of $f$ (respectively, of $g$) satisfy some linear conditions, we have that the incidence variety

$$\mathcal{P}_{42} := \{(h_1, h_2, f, g) \in |\mathcal{O}_{\pi_1}(2)| \times |\mathcal{O}_{\pi_2}(2)| \times |\mathcal{O}_{\mathbb{P}^4}(2)| \times |\mathcal{O}_{\mathbb{P}^4}(3)| : h_1(P) = h_2(P) = 0, V(f) \cap V(g) \supseteq V(h_1) \cup V(h_2)\}$$

is a projective bundle over the rational variety

$$\mathcal{Z} := \{(h_1, h_2) \in |\mathcal{O}_{\pi_1}(2)| \times |\mathcal{O}_{\pi_2}(2)| : h_1(P) = h_2(P) = 0\},$$

thus $\mathcal{P}_{42}$ is rational. We conclude that $\mathcal{M}_{42}$ is unirational.
6.17. Let $X = Q \cap K \subseteq \mathbb{P}^4$ be a smooth complete intersection of a quadric $Q$ and a cubic $K$ containing a conic $C_2$ and a twisted cubic $C_3$ with $C_2 \cap C_3 = \emptyset$. Then $X$ is a smooth K3 surface with
\[
\begin{pmatrix} 6 & 2 & 3 \\ 2 & -2 & 0 \\ 3 & 0 & -2 \end{pmatrix} \cong U \oplus \langle -50 \rangle \hookrightarrow \text{NS}(X).
\]
An argument as in the case $k = 9$ shows that the embedding is primitive. Conversely, let $X$ be a K3 surface with $\text{NS}(X)$ isometric to the previous lattice, with basis $\{H, C_2, C_3\}$. $H$ is very ample by [Sai74, Theorem 5.2], since there is no isotropic vector $E$ with $EH = 2$. Therefore $X$ is a smooth complete intersection of a quadric and a cubic containing a conic $C_2$ and a twisted cubic $C_3$ with $C_2 \cap C_3 = \emptyset$.

Up to a change of coordinates, $C_3 = \{(0 : u^3 : u^2v : uv^2 : v^3) \mid (u : v) \in \mathbb{P}^1\}$. If $\pi \subseteq \mathbb{P}^4$ is a plane, a general conic $C_2$ in $\pi$ does not intersect $C_3$. The incidence variety
\[
\mathcal{P}_{50} := \{ (\pi, h, f, g) \in \text{Gr}(2, 4) \times |O_\pi(2)| \times |O_{\mathbb{P}^4}(2)| \times |O_{\mathbb{P}^4}(3)| : V(f) \cap V(g) \supseteq V(h) \cup C_3 \}
\]
is a projective bundle over the variety
\[
\mathcal{Z} := \{ (\pi, h) \in \text{Gr}(2, 4) \times |O_\pi(2)| \},
\]
which in turn is a projective bundle over the rational variety $\text{Gr}(2, 4)$. Thus $\mathcal{P}_{50}$ is rational. We conclude that $\mathcal{M}_{50}$ is unirational.

6.18. Let $X \subseteq \mathbb{P}^3$ be a smooth quartic surface containing two disjoint twisted cubics $C_1, C_2$. Then $X$ is a smooth K3 surface with
\[
\begin{pmatrix} 4 & 3 & 3 \\ 3 & -2 & 0 \\ 3 & 0 & -2 \end{pmatrix} \cong U \oplus \langle -52 \rangle \hookrightarrow \text{NS}(X).
\]
The embedding is primitive, as $U \oplus \langle -52 \rangle$ has no non-trivial overlattices (cf. [Nik79, Proposition 1.4.1]). An argument as above using [Sai74, Theorem 5.2] shows that a general K3 surface in $\mathcal{M}_{52}$ is a smooth quartic surface.

We fix $C_1 = \{(u^3 : u^2v : uv^2 : v^3) \mid (u : v) \in \mathbb{P}^1\}$. The variety $\mathcal{T} = \text{SL}(4, \mathbb{C})/\text{SL}(2, \mathbb{C})$ of twisted cubics in $\mathbb{P}^3$ is rational by [PS85]. An argument as in the case $k = 25$ shows that the incidence variety
\[
\mathcal{P}_{52} := \{ (C_2, f) \in \mathcal{T} \times |O_{\mathbb{P}^4}(4)| : V(f) \supseteq C_1 \cup C_2 \}
\]
is a projective bundle over $\mathcal{T}$, hence it is rational. We conclude that $\mathcal{M}_{52}$ is unirational.

6.19. Let $X = Q \cap K \subseteq \mathbb{P}^4$ be a smooth complete intersection of a quadric $Q$ and a cubic $K$ containing a rational normal curve $C$ of degree 4 and a line $L$ with $C \cap L = \emptyset$. Then $X$ is a smooth K3 surface with
\[
\begin{pmatrix} 6 & 4 & 1 \\ 4 & -2 & 0 \\ 1 & 0 & -2 \end{pmatrix} \cong U \oplus \langle -58 \rangle \hookrightarrow \text{NS}(X).
\]
An argument as above using [Sai74, Theorem 5.2] shows that a general K3 surface in $\mathcal{M}_{58}$ is such a sextic surface.
Up to a change of coordinates we can assume \( C = \{(u^4 : u^3v : u^2v^2 : uv^3 : v^4) \mid (u : v) \in \mathbb{P}^1\} \). An argument as in the case \( k = 25 \) shows that the incidence variety

\[
P_{58} := \{(L, f, g) \in \text{Gr}(1, 4) \times |\mathcal{O}_{\mathbb{P}^4}(2)| \times |\mathcal{O}_{\mathbb{P}^4}(3)| : V(f) \cap V(g) \supseteq C \cup L\}
\]

is a projective bundle over the rational variety \( \text{Gr}(1, 4) \), hence it is rational. We conclude that \( M_{58} \) is unirational.

6.20. \( k = 37 \). Let \( X = Q \cap K \subseteq \mathbb{P}^4 \) be a smooth complete intersection of a quadric \( Q \) and a cubic \( K \) containing a rational normal curve \( C_1 \) of degree 4 and a conic curve \( C_2 \) intersecting transversely at one point. Then \( X \) is a smooth K3 surface with

\[
\begin{pmatrix}
6 & 4 & 2 \\
4 & -2 & 1 \\
2 & 1 & -2
\end{pmatrix} \cong U \oplus \langle -74 \rangle \hookrightarrow \text{NS}(X)
\]

An argument as above using [Sai74, Theorem 5.2] shows that a general K3 surface in \( M_{74} \) can be embedded into \( \mathbb{P}^4 \) as such a complete intersection.

Up to a change of coordinates we can assume \( C_1 = \{(u^4 : u^3v : u^2v^2 : uv^3 : v^4) \mid (u : v) \in \mathbb{P}^1\} \). An argument as in the case \( k = 25 \) shows that the incidence variety

\[
P_{74} := \{(P, \pi, C_2, f, g) \in C_1 \times \text{Gr}(2, 4) \times |\mathcal{O}_{\mathbb{P}^4}(2)| \times |\mathcal{O}_{\mathbb{P}^4}(3)| : \pi \ni P, C_2 \ni P, V(f) \cap V(g) \supseteq C_1 \cup C_2\}
\]

is a projective bundle over the incidence variety

\[
Z_1 := \{(P, \pi, C_2) \in C_1 \times \text{Gr}(2, 4) \times |\mathcal{O}_{\mathbb{P}^4}(2)| : \pi \ni P, C_2 \ni P\}.
\]

The variety \( Z_1 \) is a projective bundle over

\[
Z_2 := \{(P, \pi) \in C_1 \times \text{Gr}(2, 4) : \pi \ni P\},
\]

which in turn is a \( \text{Gr}(1, 3) \)-bundle over \( \mathbb{P}^1 \), thus rational. It follows that \( P_{74} \) is rational, too. Therefore \( M_{74} \) is unirational.

6.21. The remaining cases. We are going to discuss the remaining cases at once, since the strategy will be the same for all of them. We consider the following isomorphisms of lattices:

\[
\begin{pmatrix}
8 & 2 & 3 \\
2 & -2 & 1 \\
3 & 1 & -2
\end{pmatrix} \cong U \oplus \langle -62 \rangle, \quad
\begin{pmatrix}
8 & 3 & 3 \\
3 & -2 & 0 \\
3 & 2 & -2
\end{pmatrix} \cong U \oplus \langle -68 \rangle, \quad
\begin{pmatrix}
8 & 3 & 3 \\
3 & -2 & 2 \\
3 & 2 & -2
\end{pmatrix} \cong U \oplus \langle -72 \rangle,
\]

\[
\begin{pmatrix}
8 & 3 & 3 \\
3 & -2 & 1 \\
3 & 1 & -2
\end{pmatrix} \cong U \oplus \langle -78 \rangle, \quad
\begin{pmatrix}
8 & 3 & 4 \\
3 & -2 & 3 \\
4 & 3 & -2
\end{pmatrix} \cong U \oplus \langle -82 \rangle, \quad
\begin{pmatrix}
8 & 1 & 5 \\
1 & -2 & 1 \\
5 & 1 & -2
\end{pmatrix} \cong U \oplus \langle -86 \rangle,
\]

\[
\begin{pmatrix}
8 & 3 & 4 \\
3 & -2 & 1 \\
4 & 1 & -2
\end{pmatrix} \cong U \oplus \langle -98 \rangle, \quad
\begin{pmatrix}
8 & 3 & 5 \\
3 & -2 & 3 \\
5 & 3 & -2
\end{pmatrix} \cong U \oplus \langle -118 \rangle, \quad
\begin{pmatrix}
8 & 3 & 5 \\
3 & -2 & 1 \\
5 & 1 & -2
\end{pmatrix} \cong U \oplus \langle -122 \rangle,
\]

\[
\begin{pmatrix}
8 & 3 & 5 \\
3 & -2 & 2 \\
5 & 2 & -2
\end{pmatrix} \cong U \oplus \langle -128 \rangle, \quad
\begin{pmatrix}
8 & 4 & 5 \\
4 & -2 & 1 \\
5 & 1 & -2
\end{pmatrix} \cong U \oplus \langle -146 \rangle, \quad
\begin{pmatrix}
8 & 5 & 5 \\
5 & -2 & 2 \\
5 & 2 & -2
\end{pmatrix} \cong U \oplus \langle -200 \rangle,
\]
We will closely follow the approach for the case $k = 11$. Let $X \subseteq \mathbb{P}^5$ be a smooth complete intersection of three quadrics containing two rational normal curves $C_1, C_2$ with the appropriate degrees and intersection number, depending on the case above. We denote by $H$ the hyperplane class. The embedding of the parameter space $\mathbb{P}^1$ through $0$ for $1 \leq n \leq 3$ points of $\mathbb{P}^d$ in general position is unirational.

Conversely, a general K3 surface $X$ with $\text{NS}(X)$ isomorphic to one of the previous lattices is the smooth complete intersection of three quadrics in $\mathbb{P}^5$ containing rational normal curves with suitable degrees and intersection number. This follows from [Sai74, Theorem 5.2], by showing that $H$ is very ample in each case.

We now study the geometry of such K3 surfaces to prove the unirationality of the corresponding moduli spaces. For each case, we will denote by $C_1, C_2$ two rational normal curves of degree $d \in \{1, \ldots, 5\}$, $e \in \{3, 4, 5\}$ respectively, intersecting at $n \in \{0, \ldots, 3\}$ points. Up to automorphism of $\mathbb{P}^5$ we fix the curve $C_2$ which spans a linear subspace $\pi_2 \subseteq \mathbb{P}^5$ of dimension $e$. First we choose a set of points $P_1, \ldots, P_n \in C_2$, which will be the points of intersection of $C_1$ and $C_2$. Then we choose another linear subspace $\pi_1 \subseteq \mathbb{P}^5$ of dimension $d \geq n$ containing $P_1, \ldots, P_n$, and a rational normal curve $C_1 \subseteq \pi_1$ of degree $d$, passing through $P_1, \ldots, P_n$.

We will need the following:

**Lemma 6.1.** The parameter space $\mathcal{C}_{d,n}$ of rational normal curves in $\mathbb{P}^d$ of degree $d$ passing through $0 \leq n \leq 3$ points of $\mathbb{P}^d$ in general position is unirational.

*Proof.* Rational normal curves of degree $d$ are parametrized by $d + 1$ linearly independent forms $A_j \in |\mathcal{O}_{\mathbb{P}^1}(d)|$, inducing the embedding

$$
\varphi_A : \mathbb{P}^1 \dashrightarrow \mathbb{P}^d,

(u : v) \mapsto (A_0(u, v) : \ldots : A_d(u, v)).
$$

This shows that the parameter space $\mathcal{C}_{d,0}$ of rational normal curves is unirational, as there exists a dominant rational map

$$
|\mathcal{O}_{\mathbb{P}^1}(d)|^{d+1} \dashrightarrow \mathcal{C}_{d,0}.
$$

Let $P_1, \ldots, P_n \in \mathbb{P}^d$ be $n$ points in general position. The curve $C = \varphi_A(\mathbb{P}^1)$ passes through $P_1, \ldots, P_n$ if and only if there exist $R_1, \ldots, R_n \in \mathbb{P}^1$ mapped to $P_1, \ldots, P_n$ under $\varphi_A$. Since $n \leq 3$, we can suppose that $\{R_1, \ldots, R_n\}$ is a subset of $\{(1 : 0), (0 : 1), (1 : 1)\}$ up to automorphism of $\mathbb{P}^1$.

Thus $C$ passes through $P_1, \ldots, P_n$ if and only if

$$
P_i = (A_0(R_i) : \ldots : A_d(R_i))
$$

for $1 \leq i \leq n$. These equations define a linear subspace $\tilde{\mathcal{C}}_{d,n}$ of $|\mathcal{O}_{\mathbb{P}^1}(d)|^{d+1}$. There exists a dominant rational map

$$
\tilde{\mathcal{C}}_{d,n} \dashrightarrow \mathcal{C}_{d,n},
$$

hence the space $\mathcal{C}_{d,n}$ is unirational. \qed
We consider the following incidence varieties:
\[ \mathcal{P}_{2k} := \{(P_1, \ldots, P_n, \pi_1, C_1, f_1, f_2, f_3) \in \text{Sym}^n(C_2) \times \text{Gr}(d, 5) \times C_{d,n} \times |\mathcal{O}_P^5(2)|^3 : P_1, \ldots, P_n \in C_1 \subseteq \pi_1, V(f_1) \cap V(f_2) \cap V(f_3) \supseteq C_1 \cup C_2 \}, \]
\[ \mathcal{P}'_{2k} := \{(P_1, \ldots, P_n, \pi_1, A, f_1, f_2, f_3) \in \text{Sym}^n(C_2) \times \text{Gr}(d, 5) \times \tilde{C}_{d,n} \times |\mathcal{O}_P^5(2)|^3 : P_1, \ldots, P_n \in C_1 = \varphi_A(\mathbb{P}^1) \subseteq \pi_1, V(f_1) \cap V(f_2) \cap V(f_3) \supseteq C_1 \cup C_2 \}, \]
\[ Z_1 := \{(P_1, \ldots, P_n, \pi_1, A) \in \text{Sym}^n(C_2) \times \text{Gr}(d, 5) \times \tilde{C}_{d,n} : \pi_1 \ni P_1, \ldots, P_n \}, \]
\[ Z_2 := \{(P_1, \ldots, P_n, \pi_1) \in \text{Sym}^n(C_2) \times \text{Gr}(d, 5) : \pi_1 \ni P_1, \ldots, P_n \}. \]

They are related as shown in the diagram below:
\[
\begin{array}{ccc}
\mathcal{P}'_{2k} & \xrightarrow{\psi_1} & Z_1 \\
\downarrow \psi_1 & & \downarrow \psi_2 \\
\mathcal{P}_{2k} & & Z_2 \\
\downarrow \psi_3 & & \downarrow \psi_3 \\
\mathcal{M}_{2k} & & \text{Sym}^n(C_2)
\end{array}
\]

The three maps \( \psi_1, \psi_2 \) and \( \psi_3 \) are the obvious forgetful maps. The map \( p_1 \) is induced by \( \tilde{C}_{d,n} \to C_{d,n} \) in the proof of Lemma [6.1] while \( p_2 \) sends \( (P_1, \ldots, P_n, \pi_1, C_1, f_1, f_2, f_3) \) to the isomorphism class of the (smooth complete) intersection \( V(f_1) \cap V(f_2) \cap V(f_3) \).

In order to prove the unirationality of \( \mathcal{M}_{2k} \) we show that \( \mathcal{P}'_{2k} \) is rational. The variety \( \text{Sym}^n(C_2) \cong \mathbb{P}^n \) is rational and \( Z_2 \) is also rational, since it is a \( \text{Gr}(d-n, 5-n) \)-bundle over \( \text{Sym}^n(C_2) \). Then \( Z_1 \) is a projective bundle over \( Z_2 \) with fiber isomorphic to \( \tilde{C}_{d,n} \) in the proof of Lemma [6.1] Finally an argument as in the case \( k = 25 \) shows that \( \mathcal{P}'_{2k} \) is a projective bundle over \( Z_1 \) due to the rationality of \( C_1 \) and \( C_2 \). In conclusion, it follows that \( \mathcal{P}'_{2k} \) is rational. Therefore \( \mathcal{M}_{2k} \) is unirational, since \( p_1 \) and \( p_2 \) are dominant rational maps.

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