Consequences of Approximate $S_3$ Symmetry of the Neutrino Mass Matrix

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Assuming that the neutrino mass matrix is dominated by a term with the permutation symmetry $S_3$ it is possible to explain neutrino data only if the masses are quasi-degenerate. A sub-dominant term with an approximate $\mu - \tau$ symmetry leads to an approximate tri-bimaximal form. Experimental consequences are discussed.

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Early solar neutrino data suggested that one neutrino eigenstate could be 

$$ S = \frac{1}{\sqrt{3}} (\nu_e + \nu_\mu + \nu_\tau) \ . \quad (1) $$

This led to the consideration of an $S_3$ symmetry \[1\]. Today the MSW solution to the solar neutrino problem has the higher-energy neutrinos emerging from the sun in a state given to a good approximation by $S$. Here we consider the possibility that the neutrino mass matrix is dominated by a term with $S_3$ symmetry leading to $S$ as an eigenstate. We then consider possible perturbations that violate the symmetry.

Our assumption is that neutrino mass is due to new physics not directly related to the origin of the masses of other particles. A large number of papers \[2\] have presented detailed models based on $S_3$ symmetry. Here we do not consider a model but simply try to relate possible symmetries of the new physics to observations. The most general Majorana mass matrix invariant under $S_3$ is

$$ M_0 = \begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix} \ . \quad (2) $$

The eigenstates are necessarily \[1\] a singlet given by $S$ and a degenerate doublet $D$ which can be chosen as

$$ D_a = \frac{\nu_e - \nu_\tau}{\sqrt{2}} , \quad (3a) $$

$$ D_b = \sqrt{\frac{2}{3}} \nu_e - \sqrt{\frac{1}{6}} (\nu_\mu + \nu_\tau) \ . \quad (3b) $$

The masses are

$$ m_s = A + 2B \ , \quad (4a) $$

$$ m_D = A - B \ . \quad (4b) $$

The eigenstates in Eq.(1) and Eq.(3) are those of the tri-bimaximal form of the mixing matrix discussed in many papers \[3\] as a fit to neutrino oscillation data. However, in the fit the largest mass splitting is that between $D_a$ and $D_b$ responsible for the atmospheric neutrino oscillation with smaller splitting between $S$ and $D_a$ associated with the solar neutrino oscillation. We assume that the breaking of the degeneracy is due to the perturbation that breaks $S_3$. In order that the $S_3$ term dominate we require that all three masses start out approximately equal by choosing

$$ B = -2A + b \ , \quad (5a) $$

with $b \ll B$ so that

$$ m_D \approx -m_s \approx 3A \ . \quad (5b) $$

The minus sign means that the state $S$ has the opposite CP eigenvalue from that of $D$. We have assumed here for simplicity that $A$ and $B$ are real; otherwise $D$ and $S$ would have a relative Majorana phase. The sub-dominant mass matrix $M_1$ that breaks $S_3$ has the result of raising the mass of one $D$ state above $m_s$ and leaving the mass of the other slightly below $m_s$. These mass values then correspond to what is called the "quasi-degenerate" case for neutrino masses.
We now assume that the perturbing matrix $M_1$, which is added to $M_0$, breaks $S_3$ but retains a $S_2$ symmetry between $\nu_\mu$ and $\nu_\tau$.

$$ M_1 = \begin{pmatrix} e & f & f \\ f & t & \epsilon \\ f & \epsilon & t \end{pmatrix} \quad (6) $$

As a result of the symmetry $D_a$ remains an eigenstate and the parameter known as $\theta_{13}$ vanishes. In addition to providing the mass splitting between $D_a$ and $D_b$, $M_1$ causes a small mixing of $D_b$ with $S$. The parameters $e$ and $f$ can be absorbed into $A$ and $B$ and so they are set to zero in what follows. Matrices of the form $M_0 + M_1$, with four parameters are discussed in many papers [4].

We now identify the states which start out as $(D_a, S, D_b)$ as $(3, 2, 1)$. The mass $m_2$ is understood to be positive although $m_S$ is negative (assuming $A$ is positive). The mass differences are

$$ m_3 - m_1 = \frac{2}{3} (t - 2\epsilon), \quad (7a) $$

$$ m_2 - m_1 = -(b + t + \epsilon). \quad (7b) $$

The small value of $m_2 - m_1$ required to fit the data involves the fine-tuning of the value of $b$. The resulting deviation of the factor $\frac{1}{\sqrt{3}}$ for $\nu_e$ in $S$ is given approximately by

$$ D = \frac{2}{3\sqrt{3}} \left( \frac{t + \epsilon}{6A} \right) = \frac{k}{\sqrt{3}} \left( \frac{m_3 - m_1}{2m_2} \right), \quad (8) $$

$$ k = \frac{t + \epsilon}{t - 2\epsilon}, $$

$$ \sin^2 \theta_{12} = \left( \frac{1}{\sqrt{3}} + D \right)^2. $$

Since by our assumption of a quasi-degenerate neutrino mass spectrum the mass ratio in Eq. [8] is small so that $D$ is predicted to be small. To obtain the doublet mass splitting without large parameters we choose $\frac{t}{\epsilon}$ to be negative. As $\frac{t}{\epsilon}$ varies from 0 to a large negative value $k$ varies from 1 to $-\frac{1}{2}$; for $\frac{t}{\epsilon} = -1$, $D = 0$ and we obtain the tri-bimaximal form. Choosing values for the mass-splittings fitted from oscillation data [3]

$$ m^2_3 - m^2_2 = 2.6 \times 10^{-3} eV^2, $$

$$ m^2_2 - m^1_2 = 8 \times 10^{-5} eV^2, \quad (9) $$

we give in Table 1 three sets of mass values. The largest values (like set 1) are limited by cosmology [4] whereas the smallest values (like set 3) are limited by the requirement that the magnitude of $M_1$ is smaller than $M_0$. For each of these we show in Fig. 1 the solar neutrino survival $\sin^2 \theta_{12}$ for the higher energy neutrinos for the LMA-MSW solution as a function of $\frac{t}{\epsilon}$. Note that the sign of the deviation from $\frac{1}{3}$ can be either positive or negative. We have shown the case of the "normal hierarchy" with $(m_3 - m_1)$ positive. In the case of the inverse hierarchy the curves are flipped about the $\sin^2 \theta_{12} = \frac{1}{3}$ axis. Assuming negligible Majorana phases the mass that enters the double beta-decay formula is

$$ m_{ee} = -\sin^2 \theta_{12} m_2 + \cos^2 \theta_{12} m_1 \approx (1 - 2\sin^2 \theta_{12}) m_2, \quad (10) $$
given the small difference between \( m_2 \) and \( m_1 \).

We finally consider a possible small violation of \( \mu - \tau \) symmetry by changing the 22 element in Eq.(6) to \( t + \delta \) and the 33 element to \( t - \delta \). The main effect is to mix \( D_a \) and \( D_b \), or the states now labeled 3 and 1. There is also a small mixing of 2 and 1 but this is suppressed by the "mass difference" \( 6A \). The important result is a non-zero value of \( \theta_{13} \), the \( \nu_e \) amplitude in state 3. Directly correlated with \( \theta_{13} \) there is a deviation of \( \theta_{23} \), the \( \nu_\mu \) amplitude in state 3, from \( \pi/4 \).

Starting with the tri-bimaximal mixing, corresponding to the limit \( \xi = -1 \), this correlation is given by

\[
\tan^2 \theta_{23} = 1 - 2\sqrt{2}S + 4S^2 ,
\]

\[
S = \sin \theta_{13} \left( \frac{1 + 2\lambda}{1 - \lambda} \right) ,
\]

\[
\lambda = \frac{m_3 - m_1}{m_2 + m_1} ,
\]

to order \( S^2 \). In Fig. 2, we show (\( \tan^2 \theta_{23} - 1 \)) as a function of \( \sin \theta_{13} \). Different values of \( \xi \) makes only small changes since they are proportional to \( \lambda D \). Given the small value of \( \theta_{13} \) from experiment, there is less than as 1 \% contribution of \( \theta_{13} \) to Eq.(10) for \( m_{ee} \).

In this paper we have looked at possible experimental signatures of the assumption that the physics yielding the neutrino mass matrix has a predominant \( S_3 \) symmetry. We further assume a sub-dominant term which breaks \( S_3 \) but has an \( S_2 \mu - \tau \) symmetry.

(1) The neutrino masses must be quasi-degenerate.

(2) \( \theta_{13} \), the \( \nu_e \) component in the atmospheric mixing, vanishes and the mixing is maximal.

(3) The high-energy solar neutrino survival, governed by the LMA-MSW solution, deviates only a little from \( \frac{\pi}{4} \) as illustrated in Fig. 1.

(4) In the absence of significant Majorana phases the mass \( m_{ee} \) governing double beta decay is approximately equal to \( m_3^2 \).

If we further allow a small term involving only \( \nu_e \) and \( \nu_\tau \) that violates the \( S_2 \) symmetry then there is a non-zero \( \theta_{13} \).

In this case the atmospheric mixing angle is no longer maximum and its value is directly correlated with \( \theta_{13} \), as shown in Eq.(11) and Fig. 2.

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\begin{table}
\centering
\caption{Three sets of mass values}
\begin{tabular}{ccc}
\hline
 & $m_1$ (eV) & $m_2$ (eV) & $m_3$ (eV) \\
1 & 0.1845 & 0.1847 & 0.1913 \\
2 & 0.1247 & 0.1250 & 0.1350 \\
3 & 0.0512 & 0.0520 & 0.0729 \\
\hline
\end{tabular}
\end{table}
FIG. 1: The solar neutrino survival $\sin^2 \theta_{12}$ for the higher energy neutrinos for the LMA-MSW solution as a function of $\epsilon/t$.

FIG. 2: Diagram with $\tan^2 \theta_{23} - 1$ v.s $\sin \theta_{13}$. 