Locality in Online, Dynamic, Sequential, and Distributed Graph Algorithms

Amirreza Akbari\textsuperscript{1}  Navid Eslami\textsuperscript{13}  Henrik Lievonen\textsuperscript{1}  Darya Melnyk\textsuperscript{12}
Joona Särkijärvi\textsuperscript{1}  Jukka Suomela\textsuperscript{1}

\textsuperscript{1}Aalto University, Finland
\textsuperscript{2}TU Berlin, Germany
\textsuperscript{3}Sharif University of Technology, Iran

July 14, 2023
Locality in Online, Dynamic, Sequential, and Distributed Graph Algorithms

Amirreza Akbari\textsuperscript{1} \hspace{0.5cm} Navid Eslami\textsuperscript{13} \hspace{0.5cm} Henrik Lievonen\textsuperscript{1} \hspace{0.5cm} Darya Melnyk\textsuperscript{12}

Joona Särkijärvi\textsuperscript{1} \hspace{0.5cm} Jukka Suomela\textsuperscript{1}

\textsuperscript{1}Aalto University, Finland

\textsuperscript{2}TU Berlin, Germany

\textsuperscript{3}Sharif University of Technology, Iran

July 14, 2023
Overview of Models

Online-LOCAL

LOCAL($T$) [Linial, 1992]

SLOCAL($T$) [Ghaffari et al., 2017]

Dynamic-LOCAL($T$) This work

Online-LOCAL($T$) This work

Online Graph Algorithms
Online Graph Algorithms
Online Graph Algorithms
Online Graph Algorithms
Online Graph Algorithms
Online Graph Algorithms
Local Advice: Online-LOCAL ($T = 1$)
Local Advice: Online-LOCAL ($T = 1$)
Local Advice: Online-LOCAL ($T = 1$)
Local Advice: Online-LOCAL ($T = 1$)
Local Advice: Online-LOCAL ($T = 1$)
Local Advice: Online-LOCAL ($T = 1$)
Local Advice: Online-LOCAL ($T = 1$)
Local Advice: Online-LOCAL ($T = 1$)
Local Advice: Online-LOCAL ($T = 1$)
Local Advice: Online-LOCAL ($T = 1$)
Local Advice: Online-LOCAL ($T = 1$)
Local Advice: Online-LOCAL ($T = 1$)
Local Advice: Online-LOCAL ($T = 1$)
Local Advice: Online-LOCAL ($T = 1$)
Local Advice: Online-LOCAL ($T = 1$)
Overview of Models

Online Graph Algorithms

Online-LOCAL ($T$) [Linial, 1992]

SLOCAL ($T$) [Ghaffari et al., 2017]

Dynamic-LOCAL ($T$) This work

For LCL problems in paths, cycles and rooted trees
Overview of Models

Online Graph Algorithms

Online-LOCAL($T$)
This work

SLOCAL($T$)
[Linial, 1992]

Dynamic-LOCAL($T$)
This work

For LCL problems in paths, cycles and rooted trees
Overview of Models

LOCAL($T$) [Linial, 1992]

Online-LOCAL($T$)
This work

Online Graph Algorithms

For LCL problems in paths, cycles and rooted trees
Distributed Local Algorithms: LOCAL ($T = 1$)
Distributed Local Algorithms: LOCAL \( (T = 1) \)
Distributed Local Algorithms: LOCAL ($T = 1$)
Distributed Local Algorithms: LOCAL ($T = 1$)
Distributed Local Algorithms: LOCAL \((T = 1)\)
Distributed Local Algorithms: LOCAL ($T = 1$)
Distributed Local Algorithms: LOCAL ($T = 1$)
Overview of Models

Online-LOCAL($T$) [Linial, 1992]

SLOCAL($T$) [Ghaffari et al., 2017]

Dynamic-LOCAL($T$) This work

For LCL problems in paths, cycles and rooted trees
Overview of Models

- **LOCAL**($T$) [Linial, 1992]
- **Online-LOCAL**($T$) [This work]
- **Online Graph Algorithms**

For LCL problems in paths, cycles, and rooted trees.
Overview of Models

Online Graph Algorithms

LOCAL($T$) [Linial, 1992]

SLOCAL($T$) [Ghaffari et al., 2017]

Dynamic-LOCAL($T$) This work

Online Graph Algorithms

For LCL problems in paths, cycles and rooted trees
Sequential Local Algorithms: SLOCAL (T = 1)
Sequential Local Algorithms: SLOCAL ($T = 1$)
Sequential Local Algorithms: SLOCAL (T = 1)
Sequential Local Algorithms: SLOCAL ($T = 1$)
Sequential Local Algorithms: SLOCAL ($T = 1$)
Sequential Local Algorithms: SLOCAL ($T = 1$)
Sequential Local Algorithms: SLOCAL ($T = 1$)
Sequential Local Algorithms: SLOCAL \((T = 1)\)
Sequential Local Algorithms: SLOCAL ($T = 1$)
Sequential Local Algorithms: SLOCAL \( (T = 1) \)
Overview of Models

Online Graph Algorithms

**LOCAL(T)**
[Linial, 1992]

**SLOCAL(T)**
[Ghaffari et al., 2017]

**Online-LOCAL(T)**
This work

For LCL problems in paths, cycles and rooted trees
Overview of Models

Online Graph Algorithms

LOCAL($T$) [Linial, 1992]

SLOCAL($T$) [Ghaffari et al., 2017]

Dynamic-LOCAL($T$) [This work]

For LCL problems in paths, cycles and rooted trees
Overview of Models

SLOCAL($T$) [Ghaffari et al., 2017]

LOCAL($T$) [Linial, 1992]

Dynamic-LOCAL($T$) [Linial, 1992]

Online-LOCAL($T$) This work

Online Graph Algorithms

For LCL problems in paths, cycles and rooted trees
Overview of Models

- **Online Graph Algorithms**
  - **Online-LOCAL** ($T$)
    - This work
  - **SLOCAL** ($T$)
    - [Ghaffari et al., 2017]
  - **LOCAL** ($T$)
    - [Linial, 1992]
  - **Dynamic-LOCAL** ($T$)
    - This work
  - **Online-LOCAL** ($T$)
    - This work
  - **Online Graph Algorithms**
Dynamic Local Algorithms: Dynamic-LOCAL ($T = 1$)
Dynamic Local Algorithms: Dynamic-LOCAL ($T = 1$)
Dynamic Local Algorithms: Dynamic-LOCAL ($T = 1$)
Dynamic Local Algorithms: Dynamic-LOCAL ($T = 1$)
Dynamic Local Algorithms: Dynamic-LOCAL ($T = 1$)
Overview of Models

Online Graph Algorithms

- **Online-LOCAL(T)**
  - This work

**LOCAL(T)**
- [Linial, 1992]

**SLOCAL(T)**
- [Ghaffari et al., 2017]

**Dynamic-LOCAL(T)**
- This work

**Online Graph Algorithms**

- **Online-LOCAL(T)**
  - This work
Overview of Models

SLOCAL(\(T\))
[Ghaffari et al., 2017]

LOCAL(\(T\))
[Linial, 1992]

Dynamic-LOCAL(\(T\))
This work

Online-LOCAL(\(T\))
This work

Online Graph Algorithms
Overview of Models

Online Graph Algorithms

SLOCAL($T$)  
[Ghaffari et al., 2017]

LOCAL($T$)  
[Linial, 1992]

Dynamic-LOCAL($T$)  
This work

Online-LOCAL($T$)  
This work

Online Graph Algorithms

For LCL problems in paths, cycles and rooted trees
Overview of Models

Online Graph Algorithms

Online-LOCAL($T$)  
[This work]

LOCAL($T$)  
[Linial, 1992]

SLOCAL($T$)  
[Ghaffari et al., 2017]

Dynamic-LOCAL($T$)  
[This work]

Online-LOCAL($T$)  
[This work]

Online Graph Algorithms

For LCL problems in paths, cycles and rooted trees
Overview of Models

- **Online Graph Algorithms**
  - **Online-LOCAL** ($T$)
    - This work
  - **SLOCAL** ($T$)
    - [Ghaffari et al., 2017]
  - **LOCAL** ($T$)
    - [Linial, 1992]
  - **Dynamic-LOCAL** ($T$)
    - This work
  - **Online-LOCAL** ($T$)
    - This work
  - **Online Graph Algorithms**
Overview of Models

- **Online Graph Algorithms**
  - **Online-LOCAL**\( (T) \)
    - This work
  - **SLOCAL**\( (T) \)
    - [Ghaffari et al., 2017]
  - **LOCAL**\( (T) \)
    - [Linial, 1992]
  - **Dynamic-LOCAL**\( (T) \)
    - This work

* For LCL problems in paths, cycles and rooted trees
Overview of Models

- **Online Graph Algorithms**
  - **Online-LOCAL** ($T$)
    - This work
  - **SLOCAL** ($T$)
    - [Ghaffari et al., 2017]
  - **LOCAL** ($T$)
    - [Linial, 1992]
  - **Dynamic-LOCAL** ($T$)
    - This work

- For LCL problems in paths, cycles and rooted trees
Locally Checkable Labeling (LCL) Problems
Locally Checkable Labeling (LCL) Problems
Locally Checkable Labeling (LCL) Problems
Locally Checkable Labeling (LCL) Problems
Locally Checkable Labeling (LCL) Problems
Locally Checkable Labeling (LCL) Problems
Locally Checkable Labeling (LCL) Problems
Locally Checkable Labeling (LCL) Problems
Locally Checkable Labeling (LCL) Problems

- Vertex Coloring
- Maximal Independent Set
- Edge Coloring
- Minimal Dominating Set
- Maximal Matching
- Sinkless Orientation
### Locality of Graph Coloring: Paths and Rooted Trees

| colors | competitive ratio | LOCAL | SLOCAL | online-LOCAL |
|--------|--------------------|-------|--------|--------------|
| 2      | 1                  | $\Theta(n)$ | $\Theta(n)$ | $\Theta(n)$ |
| 3      | 1.5                | $\Theta(\log^* n)$ | $O(1)$ | $O(1)$ |
| 4      | 2                  | $\Theta(\log^* n)$ | $O(1)$ | $O(1)$ |

\[\ldots\]

[Linial, 1992] [Cole and Vishkin, 1986]
### Locality of Graph Coloring: Paths and Rooted Trees

| colors | competitive ratio | LOCAL  | SLOCAL | online-LOCAL |
|--------|--------------------|--------|--------|--------------|
| 2      | 1                  | $\Theta(n)$ | $\Theta(n)$ | $\Theta(n)$ |
| 3      | 1.5                | $\Theta(\log^* n)$ | $O(1)$ | $O(1)$ |
| 4      | 2                  | $\Theta(\log^* n)$ | $O(1)$ | $O(1)$ |
| ...    |                    |        |        |              |

[Linial, 1992] [Cole and Vishkin, 1986]
| colors | competitive ratio | LOCAL | SLOCAL | online-LOCAL |
|--------|-------------------|-------|--------|--------------|
| 2      | 1                 | $\Theta(n)$ | $\Theta(n)$ | $\Theta(n)$ |
| 3      | 1.5               | $\Theta(\log^* n)$ | $O(1)$ | $O(1)$ |
| 4      | 2                 | $\Theta(\log^* n)$ | $O(1)$ | $O(1)$ |
| ...    |                   |       |        |              |

[Linial, 1992] [Cole and Vishkin, 1986]
### Locality of Graph Coloring: Grids

[Brandt et al., 2017]

| colors | competitive ratio | LOCAL   | SLOCAL   | online-LOCAL |
|--------|-------------------|---------|----------|--------------|
| 2      | 1                 | $\Theta(n^{1/2})$ | $\Theta(n^{1/2})$ | $\Theta(n^{1/2})$ |
| 3      | 1.5               |         |          |              |
| 4      | 2                 | $\Theta(\log^* n)$ | $O(1)$ | $O(1)$ |
| 5      | 2.5               | $\Theta(\log^* n)$ | $O(1)$ | 0           |
## Locality of Graph Coloring: Grids

### Table: Locality of Graph Coloring

| colors | competitive ratio | LOCAL   | SLOCAL  | online-LOCAL |
|--------|--------------------|---------|---------|--------------|
| 2      | 1                  | $\Theta(n^{1/2})$ | $\Theta(n^{1/2})$ | $\Theta(n^{1/2})$ |
| 3      | 1.5                |         |         |              |
| 4      | 2                  | $\Theta(\log^* n)$ | $O(1)$    | $O(1)$       |
| 5      | 2.5                | $\Theta(\log^* n)$ | $O(1)$    | 0            |
| ...    |                    |         |         |              |

[Brandt et al., 2017]
| colors | competitive ratio | LOCAL       | SLOCAL      | online-LOCAL |
|--------|-------------------|-------------|-------------|--------------|
| 2      | 1                 | $\Theta(n^{1/2})$ | $\Theta(n^{1/2})$ | $\Theta(n^{1/2})$ |
| 3      | 1.5               |             |             |              |
| 4      | 2                 | $\Theta(\log^* n)$ | $O(1)$      | $O(1)$       |
| 5      | 2.5               | $\Theta(\log^* n)$ | $O(1)$      | 0            |
| ...    |                   |             |             |              |

[Brandt et al., 2017]
### Localities of Graph Coloring: Grids

| colors | competitive ratio | LOCAL | SLOCAL  | online-LOCAL |
|--------|-------------------|-------|---------|--------------|
| 2      | 1                 | \(\Theta(n^{1/2})\) | \(\Theta(n^{1/2})\) | \(\Theta(n^{1/2})\) |
| 3      | 1.5               |       |         |              |
| 4      | 2                 | \(\Theta(\log^* n)\) | \(O(1)\) | \(O(1)\)   |
| 5      | 2.5               | \(\Theta(\log^* n)\) | \(O(1)\) | 0            |
| ...    |                   |       |         |              |

[Brandt et al., 2017]
Locality of Graph Coloring: Grids

| colors | competitive ratio | LOCAL      | SLOCAL | online-LOCAL |
|--------|-------------------|------------|--------|--------------|
| 2      | 1                 | $\Theta(n^{1/2})$ | $\Theta(n^{1/2})$ | $\Theta(n^{1/2})$ |
| 3      | 1.5               | $\Theta(n^{1/2})$ | $\Omega(n^{1/10})$ |               |
| 4      | 2                 | $\Theta(\log^* n)$ | $O(1)$   | $O(1)$       |
| 5      | 2.5               | $\Theta(\log^* n)$ | $O(1)$   | 0            |
| ...    |                   |            |        |              |

[Brandt et al., 2017]
## Locality of Graph Coloring: Grids

| colors | competitive ratio | LOCAL | SLOCAL | online-LOCAL |
|--------|--------------------|-------|--------|--------------|
| 2      | 1                  | $\Theta(n^{1/2})$ | $\Theta(n^{1/2})$ | $\Theta(n^{1/2})$ |
| 3      | 1.5                | $\Theta(n^{1/2})$ | $\Omega(n^{1/10})$ | $O(\log n)$ |
| 4      | 2                  | $\Theta(\log^* n)$ | $O(1)$ | $O(1)$ |
| 5      | 2.5                | $\Theta(\log^* n)$ | $O(1)$ | 0 |
| ...    |                    |       |        |              |

[Brandt et al., 2017]
### Locality of Graph Coloring: Unrooted Trees

| colors | competitive ratio | online graph algorithms | online-LOCAL |
|--------|-------------------|-------------------------|--------------|
| 2      | 1                 | impossible              | Θ(n)         |
| 3      | 1.5               | impossible              | O(log n)     |
| ...    |                   |                         |              |
| Ω(\log n) | Ω(\log n)     | possible                | 0            |
### Locality of Graph Coloring: Unrooted Trees

| colors | competitive ratio | online graph algorithms | online-LOCAL |
|--------|-------------------|-------------------------|--------------|
| 2      | 1                 | impossible              | $\Theta(n)$  |
| 3      | 1.5               | impossible              | $O(\log n)$ |
| $\ldots$ | $\Omega(\log n)$ | $\Omega(\log n)$     | possible $0$ |
### Locality of Graph Coloring: Unrooted Trees

| colors \ competitive ratio | online graph algorithms | online-LOCAL |
|---------------------------|-------------------------|-------------|
| 2 \ 1                     | impossible              | $\Theta(n)$ |
| 3 \ 1.5                   | impossible              | $O(\log n)$|
| ...                       |                         |             |
| $\Omega(\log n)$         | $\Omega(\log n)$       | possible    |
|                           |                         | 0           |
## Locality of Graph Coloring: Unrooted Trees

| colors | competitive ratio | online graph algorithms | online-LOCAL |
|--------|-------------------|-------------------------|--------------|
| 2      | 1                 | impossible              | $\Theta(n)$  |
| 3      | 1.5               | impossible              | $O(\log n)$  |
| ...    |                   |                         |              |
| $\Omega(\log n)$ | $\Omega(\log n)$ | possible               | 0            |
• Can we interpolate between $O(1)$ colors and $O(\log n)$ locality and $\Omega(\log n)$ colors and $O(1)$ locality in online-LOCAL?

• Can bipartite graphs be 3-colored with $o(\log n)$ locality?

• Is dynamic-LOCAL closer to LOCAL or online-LOCAL?

• How to handle randomized models?

https://arxiv.org/abs/2109.06593