Extended-cycle integrals of modular functions for badly approximable numbers

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Abstract

Cycle integrals of modular functions are expected to play a role in real quadratic analogue of singular moduli. In this paper, we extend the definition of cycle integrals of modular functions from real quadratic numbers to badly approximable numbers. We also give explicit representations of values of extended-cycle integrals for some cases.

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1 Introduction

The elliptic modular \( j \)-function \( j(z) \) is an \( SL_2(\mathbb{Z}) \)-invariant holomorphic function on the upper half-plane \( \mathbb{H} := \{ z = x + y \sqrt{-1} \in \mathbb{C} \mid x, y \in \mathbb{R}, y > 0 \} \). It plays an essential role in the complex multiplication theory. Indeed, its special values at imaginary quadratic points generate the Hilbert class fields of imaginary quadratic fields ([8, Corollary 11.34]).

On the other hand, for real quadratic fields, any definitive construction of the Hilbert class fields or ray class fields is still unknown except for a few cases, that is [17, 18]. With this motivation, Kaneko [11] introduced the “value” of \( j(z) \), written \( \text{val}(w) \) derived from Dedekind’s notation, at any real quadratic number \( w \) which is defined by

\[
\text{val}(w) := \frac{1}{2 \log \varepsilon} \int_{\gamma w \rightarrow z_0} j(z) \left( \frac{1}{z - w} - \frac{1}{z - w'} \right) dz,
\] (1.1)
where \( z_0 \in \mathbb{H} \) is any point, \( w' \) is the non-trivial Galois conjugate of \( w \), and \( \gamma_w = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \) is the unique element of \( \text{SL}_2(\mathbb{Z})_w \) such that \( \text{SL}_2(\mathbb{Z})_w = \{ \pm \gamma_w^n \mid n \in \mathbb{Z} \} \) and \( \varepsilon_w := cw + d > 1 \). The motivation for adding the factor \( \frac{1}{z-w'} - \frac{1}{z-w} \) in the integrand is that this factor coincides with the line element \( y^{-1} \sqrt{dx^2 + dy^2} \) on the upper half-plane on the Heegner geodesic from \( w \) to \( w' \) (this is proved in Lemma 2.3 later). The integral is independent of \( z_0 \) since the integrand is holomorphic and invariant under \( \gamma_w \) (this follows from Lemma 2.2 (or [16, Lemma 2.5]) stated later). We can prove that \( \varepsilon_w \) is the minimal unit greater than 1 with the norm 1 in the order \( \mathcal{O}_w \) in the real quadratic field \( \mathbb{Q}(w) \) whose discriminant coincides with the discriminant of \( w \).

This value can be written as \( \text{val}(w) = \tilde{\text{val}}(w) / \tilde{\alpha}(w) \), where

\[
\tilde{\text{val}}(w) := \int_{\text{SL}_2(\mathbb{Z})_w \setminus S_{w,w}} j(z) ds, \quad \tilde{\alpha}(w) := \int_{\text{SL}_2(\mathbb{Z})_w \setminus S_{w',w}} ds = 2 \log \varepsilon_w, \quad (1.2)
\]

are cycle integrals, \( ds := y^{-1} \sqrt{dx^2 + dy^2} \), and \( S_{w',w} \) is the Heegner geodesic in \( \mathbb{H} \) from \( w' \) to \( w \).

In the last decade, it turned out that the values of \( \text{val} \) have interesting properties. They are related to a mock modular form of weight 1/2 for \( \Gamma_0(4) \) studied by Duke–Imamoglu–Tóth [10], which is a real quadratic analogue of Zagier’s work on traces of singular moduli in [19]. They are also related to locally harmonic Maass forms studied by Matsusaka [13, Theorem 1.1], Mono [14, 15], and Lageler–Schwagenscheidt [12]. Bengoechea–Imamoglu [5, Theorems 1, 2] showed some boundness for the values of \( \text{val} \) at Markov quadratics which is conjectured by Kaneko [11] and some kind of continuity for the values of \( \text{val} \). Bengoechea [3] proved asymptotic boundness for the values of \( \text{val} \) at Markov quadratics. Bengoechea–Imamoglu [4, Theorem 1.1] also showed some kind of continuity for the values of \( \text{val} \) at Markov quadratics which is conjectured by Kaneko [11]. The author [16, Theorem 1.1] generalized it to the values of \( \text{val} \) at real quadratic numbers whose periods of continued fraction expansions have long cyclic parts. It revealed that the continuity of \( \text{val} \) differs from Euclidean topology. However, the definition of the continuity of \( \text{val} \) is still missing. We would expect a topological space \( H \) containing all real quadratic numbers and a continuous function \( ? : H \to \mathbb{C} \) such that the diagram

\[
\begin{array}{ccc}
H & \xrightarrow{?} & \mathbb{C} \\
\downarrow_{\text{val}} & & \\
\{ \text{real quadratic numbers} \} & \xrightarrow{？} & \mathbb{C}
\end{array}
\]

commutes.

In this paper, we attempt to choose \( H \) as a set of badly approximable numbers and give some partial continuity of \( \text{val} \) on \( H \) in the sense that a sequence of values of \( \text{val} \) converges to a value of \( \text{val} \). Here, badly approximable numbers are defined as irrational numbers \( x \) whose Lagrange number

\[
L(x) := \sup \{ L > 0 \mid \exists p_n, q_n \in \mathbb{Z} \text{ such that } (p_n/q_n)^2 = 1, q_n \to \infty, |x - p_n/q_n| \leq 1/Lq_n^2 \},
\]
This condition is equivalent to that the regular continued fraction expansion of \( x \)

\[
x = [k_1, k_2, k_3, \ldots] := k_1 + \frac{1}{k_2 + \frac{1}{k_3 + \frac{1}{\ddots}}}, \quad k_1 \in \mathbb{Z}, \quad k_2, k_3, \ldots \in \mathbb{Z}_{>0},
\]

has bounded coefficients, that is, the set \( \{k_1, k_2, k_3, \ldots\} \) is bounded ([9, Chapter VII, Theorem 3.4]). For example, real quadratic irrational numbers are badly approximable since they have periodic continued fraction expansions.

We define \( \text{val}(x) \) and related values for a badly approximable number \( x = [k_1, k_2, k_3, \ldots] \) as

\[
\text{val}(x) := \lim_{z \to x, z \in \mathbb{H}} \frac{1}{d_{\text{hyp}}(z_0, z)} \int_{S_{z_0, z}} j(z) \, ds,
\]

\[
\hat{\text{val}}(x) := \lim_{n \to \infty} \frac{1}{n} \sum_{z_0} \gamma_n := \begin{pmatrix} k_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} k_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} k_{2n} & 1 \\ 1 & 0 \end{pmatrix} \in \text{SL}_2(\mathbb{Z}),
\]

\[
\hat{1}(x) := \lim_{n \to \infty} \frac{1}{n} \sum_{z_0} \gamma_n := \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} \cdots \begin{pmatrix} 1 \end{pmatrix} \in \mathbb{Z}^\times,
\]

\[
\hat{\varepsilon}_x := \lim_{n \to \infty} c_n^{1/n},
\]

where \( z_0 \in \mathbb{H}, S_{z_0, z} \) is the geodesic in \( \mathbb{H} \) from \( z_0 \) to \( z \in \mathbb{H} \cup \mathbb{R} \),

\[
d_{\text{hyp}}(z_0, z) := \int_{S_{z_0, z}} ds,
\]

\( \gamma_n := \begin{pmatrix} k_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} k_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} k_{2n} & 1 \\ 1 & 0 \end{pmatrix} \in \text{SL}_2(\mathbb{Z}), \)

and \( c_n \in \mathbb{Z}_{>0} \) be the denominator of the rational number \( [k_1, k_2, \ldots, k_{2n}] \). We can also prove that if \( x \) is a real quadratic number, then the limits \( \text{val}(x), \hat{\text{val}}(x), \hat{1}(x), \) and \( \hat{\varepsilon}_x \) converge, \( \text{val}(x) \) coincides with the values defined in (1.1), and we can write

\[
\hat{\text{val}}(x) = \frac{\text{val}(x)}{r_x}, \quad \hat{1}(x) = \frac{1}{r_x}, \quad \hat{\varepsilon}_x = \varepsilon_x^{1/r_x},
\]

where \( r_x \) is half the length of the minimum even period of the continued fraction expansion of \( x \). Thus, \( \hat{\varepsilon}_x \) for a badly approximable number \( x \) is an analog of fundamental units of real quadratic orders.

In Sect. 3 later, we will introduce more general representations of \( \text{val}(x), \hat{\text{val}}(x), \hat{1}(x), \) and \( \hat{\varepsilon}_x \). We will prove in Theorem 3.1 that these limit values are bounded, but we do not know if the limit values are singleton.

Our purpose in this paper is to give uncountable infinitely many badly approximable numbers \( x \) such that the limits \( \text{val}(x), \hat{\text{val}}(x), \hat{1}(x), \) and \( \varepsilon_x \) converge. We also explicitly represent values of extended \( \text{val} \) function at badly approximable numbers with infinitely long cyclic parts in their continued fractions.

To explain the main results, we introduce some notation for even words. For an even number \( 2r \in 2\mathbb{Z}_{>0} \) and positive integers \( k_1, \ldots, k_{2r} \), we call a tuple \( W = (k_1, \ldots, k_{2r}) \) an even word. When \( r = 0 \), we call it the empty word and denote it by \( \emptyset \). For even words \( W_1 = (k_1^{(1)}, \ldots, k_{2r_1}^{(1)}) \), \( W_n = (k_1^{(n)}, \ldots, k_{2r_n}^{(n)}) \), define their product

\[
W_1 \ldots W_n := (k_1^{(1)}, \ldots, k_{2r_1}^{(1)}, \ldots, k_1^{(n)}, \ldots, k_{2r_n}^{(n)}).
\]
For an infinite sequence of even words \( W_1 = (k_1^{(1)}, \ldots, k_2^{(1)}), \ldots, \), define
\[
[W_1 \ldots W_n \ldots] := [k_1^{(1)}, \ldots, k_2^{(1)}, k_2^{(2)}, \ldots, k_2^{(n)}, \ldots].
\]

For a non-empty even word \( W \), define
\[
N(W) := \max\{n \in \mathbb{Z}_{>0} \mid \text{there exists an even word } W_1 \text{ such that } W = W_1^n\}.
\]

Our first main result is as follows.

**Theorem 1.1** Let \( V_0, \ldots, V_k \) be even words, \( W_1, \ldots, W_k \) be non-empty even words, and \( \{a_{i,n}\}_{n=1}^{\infty}, \ldots, \{a_{k,n}\}_{n=1}^{\infty} \) be sequences in \( \mathbb{Z}_{\geq 0} \) such that \( a_{i,n} \rightarrow \infty \) and \( 2^{-n}a_{i,n} \rightarrow 0 \) when \( n \) goes to infinity. Let \( w_i := [W_i]^\ast \), \( k' := \#\{0 \leq i \leq k \mid V_i \neq \emptyset\} \),
\[
A_n := k'n + \sum_{1 \leq i \leq k} \sum_{1 \leq j \leq n} a_{i,j}, \quad a_i := \lim_{n \rightarrow \infty} \frac{1}{A_n} \sum_{1 \leq j \leq n} a_{i,j},
\]
and
\[
U_n := V_1 W_1^{\alpha_1} V_2 W_2^{\alpha_2} \cdots V_k W_k^{\alpha_k}, \quad x := [U_1 U_2 \ldots U_n \ldots].
\]

Then the limits \( \text{val}(x), \hat{\text{val}}(x), \hat{\text{I}}(x), \) and \( \hat{\text{I}}_x \) converge and it holds
\[
\hat{\text{I}}(x) = 2\log \hat{\text{I}}_x > 0 \quad \text{and} \quad \text{val}(x) = \hat{\text{val}}(x)/\hat{\text{I}}(x).
\]
Moreover, it also holds
\[
\text{val}(x) = \frac{a_1N(W_1)\hat{\text{val}}(w_1) + \cdots + a_kN(W_k)\hat{\text{val}}(w_k)}{a_1N(W_1)\hat{\text{I}}(w_1) + \cdots + a_kN(W_k)\hat{\text{I}}(w_k)}, \quad (1.3)
\]

**Remark 1.2** Murakami [16, Theorem 1.1] proved that the right-hand side in (1.3) equals to the limit of \( \text{val}([T_n^\ast]) \).

Next, we consider the value of \( \text{val} \) function at the Thue–Morse word. It is a typical word which is repetition-free. We fix two even words \( V \) and \( W \). Let \( \{V, W\}^* \) be the set of even words generated by \( V, W \) of finite length. Let \( \{V, W\}^\omega \) be the set of even words generated by \( V, W \) of infinite length. We define the monoid homomorphism \( h: \{V, W\}^* \rightarrow \{V, W\}^* \) by \( h(V) := VW, h(W) := WV \). In this setting, the word
\[
V_h := \lim_{n \rightarrow \infty} h^n(V) = VWVVVV\cdots \in \{V, W\}^\omega,
\]
is called the Thue–Morse word. It is known that \( V_h \) is cube-free (for example, see [6, Sect. 3]), and thus it is not a word which satisfies the conditions in Theorem 1.1.

The value of \( \text{val} \) function at the Thue–Morse word is calculated as follows.

**Theorem 1.3** Let \( x := [V_h] \) and \( w_n := [\hat{h}^n(V)] \). Then \( \hat{\text{val}}(x) \) converges if and only if
\[
\lim_{n \rightarrow \infty} \hat{\text{val}}(w_{2n})/2^{2n} \text{ converges. Moreover, we have}
\]
\[
\hat{\text{val}}(x) = \lim_{n \rightarrow \infty} \frac{1}{2^{2n}} \hat{\text{val}}(w_{2n}), \quad \hat{\text{I}}(x) = \lim_{n \rightarrow \infty} \frac{1}{2^{2n}} \hat{\text{I}}(w_{2n}), \quad \text{val}(x) = \lim_{n \rightarrow \infty} \frac{1}{2^{2n}} \text{val}(w_{2n}).
\]

**Remark 1.4** The definitions of \( \hat{\text{val}}(x) \) and \( \hat{\text{I}}(x) \) are like the Birkoff average in ergodic theory. Khinchin–Lévy’s theorem ([7, Chapter 7, §4, Theorem 4]) in ergodic theory asserts that for almost all \( x \in [0, 1] \) with respect to the measure
\[
d\mu := \frac{1}{\log 2} \frac{dx}{1 + x},
\]
on $[0, 1]$, the limit defining $\hat{\varepsilon}_x$ converges and we have
\[
\log \hat{\varepsilon}_x = -2 \int_0^1 \log(x) d\mu = \frac{\pi^2}{6 \log 2}.
\]

Although ergodic theory makes it possible to study values like the Birkhoff average at almost all points, it does not tell us about a value at each point, for example, a badly approximable number. We can state that this paper studies a range which ergodic theory does not deal with.

This paper will be organized as follows. In Sect. 2, we give renewal definitions of $\text{val}(w), \widehat{\text{val}}(w)$, and $\widehat{\varepsilon}_x$ for real quadratic numbers $w$, which are suitable to generalize for badly approximable numbers. In Sect. 3, we state some fundamental properties of the limits $\text{val}(x), \widehat{\text{val}}(x), \widehat{\varepsilon}_x$ for each badly approximable number $x$. In Sect. 4, we describe $\text{val}(x)$ in terms of the limit value along a geodesic and study its properties. In Sect. 5, we also study basic properties of $\widehat{\text{val}}(x)$ and $\widehat{\varepsilon}_x$ which are defined using the continued fraction expansion of $x$. In Sect. 6, we study $\varepsilon_x$ and a relation between it and $\widehat{\varepsilon}_x$.

Finally we prove Theorem 1.1 in Sect. 7 and Theorem 1.3 in Sect. 8.

2 A renewal definition of $\text{val}$ for real quadratic numbers

In this section, we give renewal definitions of $\text{val}$ for real quadratic numbers $w$, which are suitable to generalize for badly approximable numbers. To begin with, we list some basic notations and facts.

2.1 Basic notation and properties for geodesics

As introduced in Sect. 1, we will denote $ds$ by the line element of the upper half plane $\mathbb{H}$. This is invariant under $\text{SL}_2(\mathbb{R})$. We denote by $\infty := \lim_{t \to +\infty} t \sqrt{-1}$ the point at infinity. Let $\mathbb{P}^1(\mathbb{R}) := \mathbb{R} \cup \{\infty\}$. For two points $x', x \in \mathbb{H} \cup \mathbb{P}^1(\mathbb{R})$, we denote by $S_{x', x}$ the geodesic in $\mathbb{H}$ from $x'$ to $x$. For two points $z, z' \in \mathbb{H}$, define hyperbolic distance between them by
\[
d_{\text{hyp}}(z, z') := \int_{S_{z, z'}} ds.
\]

The following lemma is fundamental.

**Lemma 2.1** For two points $z, z' \in \mathbb{H}$, the following statements hold.

(i) For any $\sigma \in \text{SL}_2(\mathbb{R})$, it holds $d_{\text{hyp}}(\sigma(z), \sigma(z')) = d_{\text{hyp}}(z, z').$

(ii) ([2, Theorem 7.2.1 (ii)])
\[
cosh d_{\text{hyp}}(z, z') = 1 + \frac{|z - z'|^2}{2 \Im(z)\Im(z')}.
\]

For $x', x \in \mathbb{P}^1(\mathbb{R})$, define a 1-form
\[
\eta_{x', x}(z) := \left( \frac{1}{z - x} - \frac{1}{z - \infty} \right) dz.
\]

Here, let $1/(z - \infty) := 0$ if $x = \infty$ or $x' = \infty$.

By a direct calculation, we have the following lemma.

**Lemma 2.2** ([16, Lemma 2.5]) For any $x, x' \in \mathbb{P}^1(\mathbb{R})$ and $\gamma \in \text{SL}_2(\mathbb{R})$, we have
\[
\gamma^* \eta_{x', x} = \eta_{\gamma^{-1}x', \gamma^{-1}x'}
\]
where $\gamma^* \eta_{x', x}$ is the pullback of $\eta_{x', x}$ via $\gamma : \mathbb{H} \to \mathbb{H}$.
We need the following lemma to express \( \text{val}(w) \) as a quotient of two cycle integrals.

**Lemma 2.3** For two points \( x, x' \in \mathbb{P}^1(\mathbb{R}) \) with \( x \neq x' \), we have \( ds = \eta_{x,x}' \) on \( S_{x,x}' \).

**Proof** Let

\[
\sigma := \begin{cases}
(x, x')/(x - x') & \text{if } x \neq \infty, x' \neq \infty, \\
1/1/(x - x') & \text{if } x = \infty, \\
1/x' & \text{if } x' = \infty,
\end{cases}
\]

be a matrix in \( \text{SL}_2(\mathbb{R}) \). Since \( \sigma^* \eta_{x,x} = \eta_{0,\infty} \) and \( \sigma^* ds = ds \), it is enough to show when \( x = \infty \) and \( x' = 0 \). On the geodesic \( S_{0,\infty} = \{ t \sqrt{-1} \mid t \in \mathbb{R}_{>0} \} \), we have \( ds = dt/t = \eta_{0,\infty} \).

\[ \square \]

2.2 Basic notation and properties for real quadratic numbers

For a real quadratic number \( w \), let \( w' \) be the non-trivial Galois conjugate of \( w \), \( \text{SL}_2(\mathbb{Z})_w \) be the stabilizer of \( w \) in \( \text{SL}_2(\mathbb{Z}) \), and \( \gamma_w = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \) be the unique element of \( \text{SL}_2(\mathbb{Z})_w \) such that \( \text{SL}_2(\mathbb{Z})_w = \{ \pm \gamma^n_n \mid n \in \mathbb{Z} \} \) and \( d - c > 1 \). It holds \( \gamma_w w' = w' \). Thus, the Heegner geodesic \( S_{w, w'} \) is invariant under \( \gamma_w \). It also holds \( \gamma^*_w \eta_{w,w'} = \eta_{w,w'} \) by Lemma 2.2.

2.3 Basic notation and properties for continued fractions and words

For the convenience of the reader, we recall notations for even words prepared in Sect. 1.

Each real number can be represented by the unique continued fraction

\[
[k_1, k_2, k_3, \ldots] := k_1 + \frac{1}{k_2 + \frac{1}{k_3 + \frac{1}{\ddots}}},
\]

where \( k_1 \) is an integer and \( k_2, k_3, \ldots \) are positive integers. For positive integers \( k_1, \ldots, k_i \), we set a periodic continued fraction

\[
[k_1, \ldots, k_r, k_{r+1}, \ldots, k_s] := [k_1, \ldots, k_r, k_{r+1}, \ldots, k_s, k_{r+1}, \ldots, k_s, k_{r+1}, \ldots, k_s, \ldots].
\]

For a real number, being quadratic (resp. rational) is equivalent to having a periodic (resp. finite) continued fraction expansion ([1, Theorem 1.17]).

For an even number \( 2r \in 2\mathbb{Z}_{\geq 0} \) and positive integers \( k_1, \ldots, k_{2r} \), we call a tuple \( W = (k_1, \ldots, k_{2r}) \) an even word. In the case when \( r = 0 \), we call it the empty word and denote it by \( \emptyset \). For even words \( W_n = (k_1^{(n)}, \ldots, k_{2r}^{(n)}) \), define their product

\[
W_1 \cdots W_n := (k_1^{(1)}, k_{2r}^{(1)}, \ldots, k_1^{(n)}, k_{2r}^{(n)}).
\]

For an infinite sequence of even words \( W_1 = (k_1^{(1)}, \ldots, k_{2r}^{(1)}), \ldots, W_n = (k_1^{(n)}, \ldots, k_{2r}^{(n)}) \), define

\[
W_1 \cdots W_n \cdots := [k_1^{(1)}, k_{2r}^{(1)}, \ldots, k_1^{(n)}, k_{2r}^{(n)}].
\]
For the empty word $\emptyset$ and a non-empty even word $W = (k_1, \ldots, k_{2r})$, define

\[
\gamma_{\emptyset} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \text{SL}_2(\mathbb{Z}),
\]

\[
\gamma_W := \begin{pmatrix} k_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} k_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} k_{2r} & 1 \\ 1 & 0 \end{pmatrix} \in \text{SL}_2(\mathbb{Z}),
\]

$$|W| := r,$$

\[N(W) := \max\{n \in \mathbb{Z}_{>0} \mid \text{there exists an even word } W_1 \text{ such that } W = W_1^n\}.
$$

Moreover, for an even word $V = (l_1, \ldots, l_{2s})$ and a non-empty even word $W = (k_1, \ldots, k_{2r})$, define

$$[V \overline{W}] := [l_1, \ldots, l_{2s}, k_1, \ldots, k_{2r}, k_1, \ldots, k_{2r}, \ldots].$$

The matrix $\gamma_W$ has the following properties.

**Lemma 2.4** ([16, Lemma 2.3]). The following properties hold.

(i) For even words $W_1, \ldots, W_n$, we have $\gamma_{W_1 \cdots W_n} = \gamma_{W_1} \cdots \gamma_{W_n}$.

(ii) For a reduced real quadratic number $w$, there exists the minimal even word $W$ such that $w = [\overline{W}]$. Then we have $N(W) = 1$ and $\gamma_W = \gamma_w$. Generally, for a non-empty even word $W$ and a real quadratic number $w = [\overline{W}]$, we have $\gamma_W = \gamma_w^{N(W)}$.

(iii) For a real quadratic number $w$, there exist even words $V$ and $W$ such that $w = [V \overline{W}]$. Then $\gamma_V \gamma_W \gamma_V^{-1} = \gamma_w^{N(W)}$.

The following lemma follows from a direct calculation.

**Lemma 2.5** For an even word $V = (k_1, k_2, \ldots, k_{2r})$, let $V' = (k_{2r}, k_{2r-1}, \ldots, k_1)$ and $\delta := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then, it holds $\gamma_V^{-1} = \delta^{-1} \gamma_{V'} \delta$.

### 2.4 Kaneko’s val

Let $j: \mathbb{H} \to \mathbb{C}$ be the elliptic modular $j$-function. For a real quadratic number $w$, define

\[
\tilde{\text{val}}(w) := \int_{z_0}^{w} j(z) \eta_{w', w}(z), \quad \tilde{1}(w) := \int_{z_0}^{w} \eta_{w', w}(z), \quad \text{val}(w) := \frac{\tilde{\text{val}}(w)}{\tilde{1}(w)},
\]

where $z_0 \in \mathbb{H}$ is any point. The following lemma implies that these values are the same as those defined in (1.2).

**Lemma 2.6** For a real quadratic number $w$, any point $z_0 \in S_{w', w}$ and any positive integer $n$, it holds

\[
n\tilde{\text{val}}(w) = \int_{z_0}^{w} j(z) \eta_{w', w}(z) = \int_{z_0}^{w} j(z) ds = n \int_{\text{SL}_2(\mathbb{Z}) \backslash \mathcal{S}_{w', w}} j(z) ds,
\]

\[
n\tilde{1}(w) = \int_{z_0}^{w} \eta_{w', w}(z) = \int_{z_0}^{w} ds = n \int_{\text{SL}_2(\mathbb{Z}) \backslash \mathcal{S}_{w', w}} ds = d_{\text{hyp}}(z_0, \gamma_w^n z_0).
\]
Proof. The first equality follows from
\[ \int_{\gamma_{wz_0}} j(z) n_{w',w}(z) = \sum_{0 \leq i < n} \int_{\gamma_{i+1}} j(z) n_{w',w}(z) = \sum_{0 \leq i < n} \int_{\gamma_{wz_0}} j(\gamma_{wz}^i) n_{w',w}(z) = \sum_{0 \leq i < n} \int_{\gamma_{wz_0}} j(z) n_{w',w}(z) = n \tilde{\text{val}}(w). \]

Here we used Lemma 2.2. The second equality follows from Lemma 2.3 and the third equality follows from the definition of $\gamma_w$ and the fact that $\gamma_{wz_0}$ lies on the geodesic $S_{z_0,w}$.

The following lemma is fundamental.

Lemma 2.7 ([16, Proposition 2.8]). For a real quadratic number $w$, it holds
\[ \overline{\text{val}}(w) = 2 \log \varepsilon_w. \]

By the above lemmas, we obtain the following expression of $\text{val}(w)$, suitable for generalizing badly approximable numbers.

Proposition 2.8. For a real quadratic number $w$ and any point $z_0 \in S_{w',w}$, it holds
\[ \text{val}(w) = \lim_{z \to w} \frac{1}{d_{\text{hyp}}(z_0, z)} \int_{S_{z_0,z}} j(z) ds. \]

Proof. For any positive number $d \in \mathbb{R}_{>0}$, there exists the unique point $z_d \in S_{z_0,w}$ such that $d_{\text{hyp}}(z_0, z_d) = d$. Let $l := d_{\text{hyp}}(z_0, \gamma_{wz_0}) = \overline{\text{val}}(w)$. It suffices to show that
\[ \lim_{n \to \infty} \frac{1}{nl + d} \int_{S_{z_0,\gamma_{wz_0}}} j(z) ds = \text{val}(w), \]
for any number $0 \leq d < l$. Since $\gamma_{wz}^n z_d \in S_{z_0,w}$ and $d_{\text{hyp}}(z_d, \gamma_{wz_0}) = nl$ by Lemma 2.6, we have $z_{nl+d} = \gamma_{w}^n z_d$. Thus, we obtain
\[ \lim_{n \to \infty} \frac{1}{nl + d} \int_{S_{z_0,\gamma_{wz_0}}} j(z) ds = \lim_{n \to \infty} \frac{1}{nl + d} \left( \int_{S_{z_0,z_d}} + \int_{S_{z_d,\gamma_{wz_0}}} \right) j(z) ds \]
\[ = \lim_{n \to \infty} \frac{1}{nl + d} \int_{S_{z_0,z_d}} j(z) ds + n \tilde{\text{val}}(w) \]
\[ = \text{val}(w), \]
by Lemma 2.6.

To generalize $\tilde{\text{val}}(w)$, $\overline{\text{val}}(w)$, and $\varepsilon_w$ for badly approximable numbers, we need some notation. For a real quadratic number $w = [V W]$, define
\[ \tilde{\text{val}}(w) = \frac{N(W)}{|W|} \tilde{\text{val}}(w), \quad \overline{\text{val}}(w) = \frac{N(W)}{|W|} \overline{\text{val}}(w), \quad \varepsilon_w = \varepsilon_w^{N(W)/|W|}. \quad (2.1) \]

These values are independent of a choice of an even word $W$ and have the following expression.
Proposition 2.9 Let $w$ be a real quadratic number. Let $r \geq 0$ and $s \geq 1$ be integers and $W_1, \ldots, W_{r+s}$ be non-empty even words such that $w = [W_1 \ldots W_r W_{r+1} \ldots W_{r+s}]$. For an integer $n > r+s$, let $W_n := W_{r+s}$ where $0 \leq l < s$ is an integer such that $n \equiv r + l \mod s$. For a positive integer $i$, let $\gamma_i := \gamma_{W_i}$. Let $z_0 \in \mathbb{H}$ be a point. Then, we have

$$
\hat{\text{val}}(w) = \lim_{n \to \infty} \frac{1}{n} \int_{z_0}^{y_1 \ldots y_n z_0} j(z) \eta_{w',w}(z),
$$

$$
\hat{I}(w) = \lim_{n \to \infty} \frac{1}{n} \int_{z_0}^{y_1 \ldots y_n z_0} \eta_{w',w}(z).
$$

Proof Let $N := N(W_{r+1} \ldots W_{r+s})$. For a positive integer $n$, let

$$
a_n := \int_{y_1 \ldots y_n z_0}^{y_1 \ldots y_n z_0} j(z) \eta_{w',w}(z).
$$

Since

$$
a_1 + \cdots + a_n = \frac{1}{n} \int_{z_0}^{y_1 \ldots y_n z_0} j(z) \eta_{w',w}(z), \quad \hat{\text{val}}(w) = \frac{N}{s} \hat{\text{val}}(w),
$$

it suffices to show that

$$
\lim_{n \to \infty} \frac{a_1 + \cdots + a_{r+l+ns}}{r + l + ns} = \frac{N}{s} \hat{\text{val}}(w),
$$

(2.2)

for any integer $0 \leq l < s$.

Since $y_{m+s} = y_m$ for any integer $m > r + s$, it holds

$$
y_1 \cdots y_{r+l+ms} = (y_1 \cdots y_r y_{r+l+1} \cdots y_{r+l+s}(y_1 \cdots y_{r+l})^{-1})^s y_1 \cdots y_{r+l} = y_w^{N_0} y_1 \cdots y_{r+l}
$$

by Lemma 2.4 (iii). Let $z_0' := y_1 \cdots y_{r+l} z_0$. Then, we have

$$
a_{r+l} + \cdots + a_{r+l+ns} = \int_{y_1 \ldots y_{r+l+ns} z_0}^{y_1 \ldots y_{r+l+ns} z_0} j(z) \eta_{w',w}(z)
$$

$$
= \int_{z_0'}^{z_0'} j(z) \eta_{w',w}(z)
$$

$$
= N \hat{\text{val}}(w),
$$

by Lemma 2.4 (i) and (ii) and Lemma 2.6. Thus, we obtain (2.2). The second equality is similarly proved. \qed

Remark 2.10 In Proposition 2.9, the first finite term of a continued fraction does not contribute to the left-hand side. Thus, the right-hand side in Proposition 2.9 depends only on the period in the continued fraction.

3 Fundamental properties of the extended val function

With reference to Propositions 2.8 and 2.9, we now define $\text{val}(x)$ for a badly approximable number $x$ as follows. Let $W_1, W_2, \ldots$ be an infinite sequence of even words such that $x = [W_1 W_2 \ldots]$ and the set $\{W_1, W_2, \ldots\}$ is finite. We remark that if $x < 1$, then the first entry of $W_1$ is not positive. Let $c_n \in \mathbb{Z}_{>0}$ be the denominator of the rational number $[W_1 \cdots W_n]$ and put

$$
L := \lim_{n \to \infty} \frac{|W_1| + \cdots + |W_n|}{c_n}.
$$
Take \( x' \in \mathbb{P}^1(\mathbb{R}) \backslash \{x\} \) and \( z_0 \in \mathbb{H} \). We define the limits \( \text{val}(x), \widehat{\text{val}}(x), \widehat{\Gamma}(x) \), and \( \widehat{\varepsilon}_x \) by

\[
\text{val}(x) := \lim_{z \in S_{z_0} \rightarrow x} \frac{1}{d_{\text{hyp}}(z_0, z)} \int_{S_{z_0}} j(z) ds,
\]

\[
\widehat{\text{val}}(x) := \lim_{n \rightarrow \infty} \frac{1}{|W_1| + \cdots + |W_n|} \int_{z_0}^{\gamma_{W_1 \ldots W_n} z_0} j(z) \eta_{x', x}(z),
\]

\[
\widehat{\Gamma}(x) := \lim_{n \rightarrow \infty} \frac{1}{|W_1| + \cdots + |W_n|} \int_{z_0}^{\gamma_{W_1 \ldots W_n} z_0} \eta_{x', x}(z),
\]

\[
\widehat{\varepsilon}_x := \lim_{n \rightarrow \infty} c_n^{1/n}.
\]

In the later sections, we will prove the following.

**Theorem 3.1** Let \( x \) be a badly approximable number and \( W_1, W_2, \ldots \) be an infinite sequence of even words such that \( x = [W_1 W_2 \ldots] \) and the set \( \{W_1, W_2, \ldots\} \) is finite.

Take any points \( x' \in \mathbb{P}^1(\mathbb{R}) \backslash \{x\} \) and \( z_0 \in \mathbb{H} \). Then the following statements hold.

(i) The limit values \( \text{val}(x), \widehat{\text{val}}(x), \widehat{\Gamma}(x) \), and \( \widehat{\varepsilon}_x \) defined in each of (3.1), (3.2), (3.3), and (3.4) are bounded. Moreover, if they converge, then they are independent of \( x', z_0 \) and \( W_1, W_2, \ldots \). Further, they are \( \text{SL}_2(\mathbb{Z}) \)-invariant.

(ii) If \( x \) is a real quadratic number, then \( \text{val}(x), \widehat{\text{val}}(x), \widehat{\Gamma}(x) \), and \( \widehat{\varepsilon}_x \) converge and coincide with the values defined in (1.1) and (2.1).

(iii) If \( \widehat{\text{val}}(x) \) and \( \widehat{\Gamma}(x) \) converge, then for any point \( x' \in \mathbb{P}^1(\mathbb{R}) \backslash \{x\} \) and a sequence \( \{x_n\}_{n=1}^{\infty} \) of real numbers with \( z_n := \gamma_{W_1 \ldots W_n} (x_n + \sqrt{-1}) \in S_{\infty, x'} \), we have

\[
\widehat{\text{val}}(x) = \lim_{n \rightarrow \infty} \frac{1}{|W_1| + \cdots + |W_n|} \int_{z_0}^{z_n} j(z) \eta_{x', x}(z),
\]

\[
\widehat{\Gamma}(x) = \lim_{n \rightarrow \infty} \frac{1}{|W_1| + \cdots + |W_n|} \int_{z_0}^{z_n} \eta_{x', x}(z).
\]

Here we define \( 1/(z - \infty) := 0 \) if \( x' = \infty \).
(iv) The limit $\hat{\epsilon}_x$ converges if and only if $\hat{1}(x)$ converges. Moreover, we have $\hat{1}(x) = 2 \log \hat{\epsilon}_x$.

(v) It holds $\hat{\epsilon}_x \geq (3 + \sqrt{5})/2$. In particular, $\hat{1}(x) > 0$.

(vi) If $\hat{\text{val}}(x)$ and $\hat{1}(x)$ converge, then $\text{val}(x)$ converges and we have $\text{val}(x) = \hat{\text{val}}(x)/\hat{1}(x)$.

Here we remark that Theorem 3.1 (vi) follows from Theorem 3.1 (iv) and (v).

4 The extended val function as the limit value along a geodesic

In this section, we prove Theorem 3.1 (i) and (ii) for $\text{val}(x)$. We start with the following properties of badly approximable numbers. Let $\pi : \mathbb{H} \to \text{SL}_2(\mathbb{Z})\mathbb{H}$ be the natural projection.

**Proposition 4.1** ([9, Chapter VII, Theorem 3.4], [9, Chapter VII, Lemma 3.5]). For an irrational number $x \in \mathbb{R} \setminus \mathbb{Q}$, the following statements are equivalent.

(i) $x$ is badly approximable.

(ii) $L(x) := \sup \{ L > 0 \mid \exists p_n, q_n \in \mathbb{Z} \text{ s.t. } (p_n, q_n) = 1, q_n \to \infty, |x - p_n/q_n| \leq 1/Lq_n^2 \} < \infty$.

(iii) $h(x) := \sup \{ t > 0 \mid \exists z_n \in S_{2t,x} \text{ s.t. } z_n \to x, \pi(z_n) \in \pi(\{ z \in \mathbb{H} \mid \text{Im}(z) = t \}) \} < \infty$.

(iv) For any point $z_0 \in \mathbb{H}$, the closure $\pi(S_{2t,x}) \subset \text{SL}_2(\mathbb{Z})\mathbb{H}$ is compact.

The constant $L(x)$ is called the Lagrange number of $x$ ([1, Definition 1.7]). Usually, badly approximable numbers are defined as irrational numbers with finite Lagrange numbers.

The following is a key fact in this section.

**Theorem 4.2** (Theorem 3.1 (i) and (ii) for $\text{val}(x)$). Let $x$ be a badly approximable number. Take any point $z_0 \in \mathbb{H}$. Then the following statements hold.

(i) The limit value

$$\text{val}(x) := \lim_{z \to x} \frac{1}{d_{\text{hyp}}(z_0, z)} \int_{S_{z_0,z}} j(z) ds,$$

where $j(z)$ is a suitable function.

Fig. 2 The path of integrals for the limit values in Theorem 3.1 (iii)
is bounded.

(ii) For any matrix $\gamma \in SL_2(\mathbb{Z})$, we have $\text{val}(\gamma x) = \text{val}(x)$.

(iii) If the limit $\text{val}(x)$ converges, then it is independent of $z_0$.

(iv) For a real quadratic number $x$, $\text{val}(x)$ converges and coincides with the value defined in (1.1).

**Proof** To begin with, we remark that (iv) follows from Proposition 2.8 and (iii).

We will prove (i). Since $x$ is badly approximable, the closure $\pi(S_{x,z_0}) \subseteq SL_2(\mathbb{Z}) \setminus \mathbb{H}$ is compact by Proposition 4.1. Thus, we can choose $K > 0$ such that $|j(z)| \leq K$ on $S_{2q,x}$. Since

$$\left| \frac{1}{d_{\text{hyp}}(z_0, z)} \int_{S_{q,x}} j(z) ds \right| \leq \frac{1}{d_{\text{hyp}}(z_0, z)} \int_{S_{q,x}} K ds = K,$$

the limit is bounded.

The second claim (ii) follows from the fact that $j(z)$ is $SL_2(\mathbb{Z})$-invariant.

As for the third claim (iii), for each point $z_0 \in \mathbb{H}$, we consider

$$\text{val}(x, z_0) := \lim_{z \to x} \frac{1}{d_{\text{hyp}}(z_0, z)} \int_{S_{q, x}} j(z) ds.$$

We will show that this value is independent of $z_0$ in two steps.

**Step 1** We will show that $\text{val}(x, z_0) = \text{val}(x, z_1)$ for any point $z_1 \in S_{q,x}$. For $i \in \{0, 1\}$, let

$$a_i(z) := \int_{S_{q, x}} j(z) ds, \quad b_i(z) := \int_{S_{q, x}} ds.$$

Clearly,

$$\lim_{z \to x} b_i(z) = \infty.$$

Since

$$a_0(z) - a_1(z) = \int_{S_{q, z_1}} j(z) ds, \quad b_0(z) - b_1(z) = \int_{S_{q, z_1}} ds,$$

is bounded, we have

$$\text{val}(x, z_0) = \lim_{z \to x} \frac{a_0(z)}{b_0(z)} = \lim_{z \to x} \frac{a_1(z) + O(1)}{b_1(z) + O(1)} = \lim_{z \to x} \frac{a_1(z)}{b_1(z)} = \text{val}(x, z_1).$$

**Step 2** We will show that $\text{val}(x, z_0) = \text{val}(x, z_1)$ for any points $z_0, z_1 \in \mathbb{H}$. This part is due to Matsusaka. Take a matrix $\sigma \in SL_2(\mathbb{R})$ such that $\sigma^{-1}x = \infty$. Let $v_0$ be any positive number. By Step 1, we can replace $z_i$ by $\mathbb{H}(z_i) + v_0 \sqrt{-1}$. Thus, we may assume $v_0 = \Im(\sigma^{-1}z_0) = \Im(\sigma^{-1}z_1)$. Let $u_i := \Re(\sigma^{-1}z_i)$ for $i \in \{0, 1\}$. Then $\sigma^{-1}S_{u_i,x}$ is a subgeodesic of $S_{u_i,\infty}$. Since $ds = y^{-1} \sqrt{dx^2 + dy^2} = y^{-1} dy$ on $S_{u_i,\infty}$, we have

$$\text{val}(x, z_i) = \lim_{z \to x} \frac{1}{d_{\text{hyp}}(z_0, z)} \int_{S_{q, x}} j(z) ds$$

$$= \lim_{v \to \infty} \frac{1}{d_{\text{hyp}}(\sigma(u_i + v \sqrt{-1}), \sigma(u_i + v_0 \sqrt{-1}))} \int_{v_0} j(\sigma(u_i + y \sqrt{-1})) \frac{dy}{y}.$$
Since

\[ d_{\text{hyp}}(\sigma(u_i + v \sqrt{-1}), \sigma(u_i + u_0 \sqrt{-1})) = d_{\text{hyp}}(v \sqrt{-1}, u_0 \sqrt{-1}) = \int_{v_0}^{v} \frac{dy}{y} = \log \frac{v}{v_0}, \]

we obtain

\[ \text{val}(x, z_0) - \text{val}(x, z_1) = \lim_{v \to \infty} \frac{1}{\log v/v_0} \int_{v_0}^{v} \left( j(\sigma(u_0 + y \sqrt{-1})) - j(\sigma(u_1 + y \sqrt{-1})) \right) \frac{dy}{y}. \]

For two points \( z, z' \in \mathbb{H} \), the hyperbolic distances on \( \mathbb{H} \) and SL\(_2\)(\( \mathbb{Z} \)) \( \setminus \mathbb{H} \) are respectively defined by

\[ d_{\text{hyp}}(z, z') := \text{length}(S_{zz'}), \quad d_{\text{SL}_2(\mathbb{Z}) \setminus \mathbb{H}}(\pi(z), \pi(z')) := \min \{ d_{\text{hyp}}(\gamma z, z') \mid \gamma \in \text{SL}_2(\mathbb{Z}) \}. \]

They define the hyperbolic distances on \( \mathbb{H} \) and SL\(_2\)(\( \mathbb{Z} \)) \( \setminus \mathbb{H} \) respectively. By Lemma 2.1, we have

\[ \cosh d_{\text{hyp}}(\sigma(u_0 + y \sqrt{-1}), \sigma(u_1 + y \sqrt{-1})) = \cosh d_{\text{hyp}}(u_0 + y \sqrt{-1}, u_1 + y \sqrt{-1}) \]

\[ = 1 + \frac{|u_0 - u_1|^2}{2y^2}. \]

Thus, we have

\[ \lim_{y \to \infty} d_{\text{hyp}}(\sigma(u_0 + y \sqrt{-1}), \sigma(u_1 + y \sqrt{-1})) = 0. \]

Since \( x \) is badly approximable, \( \pi(S_{x,z_0}) \cup \pi(S_{x,z_1}) \subset \text{SL}_2(\mathbb{Z}) \setminus \mathbb{H} \) is compact by Proposition 4.1. Thus, \( j(z) \) is uniformly continuous on \( \pi(S_{x,z_0}) \cup \pi(S_{x,z_1}) \) with respect to the metric \( d_{\text{SL}_2(\mathbb{Z}) \setminus \mathbb{H}} \). Hence for any \( \varepsilon > 0 \), there exists \( v_0 > 0 \) such that

\[ |j(\sigma(u_0 + y \sqrt{-1})) - j(\sigma(u_1 + y \sqrt{-1}))| < \varepsilon, \]

for any \( y > v_0 \). Then we have

\[ |\text{val}(x, z_0) - \text{val}(x, z_1)| \leq \lim_{v \to \infty} \frac{1}{\log v/v_0} \int_{v_0}^{v} \varepsilon \frac{dy}{y} = \varepsilon, \]

that is, \( \text{val}(x, z_0) = \text{val}(x, z_1) \). \( \square \)
5 The extended $\text{val}$ function as the limit value along a continued fraction

In this section, we prove Theorem 3.1 (i), (ii), and (iii) for $\text{val}(x)$ and $\tilde{1}(x)$.

To prove Theorem 3.1 (ii), we prepare several lemmas.

**Lemma 5.1** For a badly approximable number $x = [k_1, k_2, \ldots]$, let $\gamma_n := \gamma(k_1, \ldots, k_{2n})$. Then for any point $x' \in \mathbb{P}^1(\mathbb{R})$, the set $\{\gamma_n^{-1}x' \mid n \in \mathbb{Z}_{>0}\}$ is bounded.

**Proof** Let $h(x)$ be the number defined in Proposition 4.1. By Proposition 4.1, we have

$$\gamma_n^{-1}S_{x',x} \cap \{z \in \mathbb{H} \mid \text{Im}(z) \geq h(x) + 1\} = \emptyset,$$

for all sufficiently large $n$. Thus, the radius of $\gamma_n^{-1}S_{x',x}$ is less than $h(x) + 1$. Since coefficients of the continued fraction expansion of $x$ are bounded, $\{\gamma_n^{-1}x' \mid n \in \mathbb{Z}_{>0}\}$ is bounded. Thus, $\{\gamma_n^{-1}x' \mid n \in \mathbb{Z}_{>0}\}$ is also bounded. □

**Lemma 5.2** For an even word $W$, a point $z_0 \in \mathbb{H}$, and sequences $\{x_n\}, \{x_n'\}, \{y_n\}, \{y_n'\}$, we have

$$\int_{z_0}^{\gamma_W z_0} j(z) \left( \eta_{x_n,x_n}(z) - \eta_{y_n,y_n}(z) \right) = O(|x_n - y_n| + |x_n' - y_n'|).$$

**Proof** Since

$$\left| \frac{1}{z - x_n} - \frac{1}{z - y_n} \right| \leq \frac{|x_n - y_n|}{\min\{\text{Im}(z_0), \text{Im}(\gamma_W z_0)\}},$$

on the contour $\{tzz_0 + (1 - t)\gamma_W z_0 \mid 0 \leq t \leq 1\}$, it holds

$$\left| \int_{z_0}^{\gamma_W z_0} j(z) \left( \frac{1}{z - x_n} - \frac{1}{z - y_n} \right) \right| \leq \frac{|x_n - y_n|}{\min\{\text{Im}(z_0), \text{Im}(\gamma_W z_0)\}} \left| \int_{z_0}^{\gamma_W z_0} j(z)dz \right| = O(|x_n - y_n|).$$

Similar argument shows

$$\left| \int_{z_0}^{\gamma_W z_0} j(z) \left( \frac{1}{z - x_n'} - \frac{1}{z - y_n'} \right) \right| = O(|x_n' - y_n'|).$$

□
A key result in this section is the following.

**Theorem 5.3** (Theorem 3.1(i) for \( \widehat{\text{val}}(x) \) and \( \widehat{I}(x) \)). Let \( x \) be a badly approximable number and \( W_1, W_2, \ldots \) be an infinite sequence of even words such that \( x = [W_1 W_2 \ldots] \) and the set \( \{W_1, W_2, \ldots\} \) is finite. Take a point \( z_0 \in \mathbb{H} \). Then the following statements hold.

(i) The sequences defining the limits

\[
\widehat{\text{val}}(x) := \lim_{n \to \infty} \frac{1}{n} \sum_{1 \leq i \leq n} \left( W_i + \cdots + W_n \right) \int_{z_0}^{\gamma W_{i-1} z_0} j(z) \eta_{\infty,x}(z),
\]

\[
\widehat{I}(x) := \lim_{n \to \infty} \frac{1}{n} \sum_{1 \leq i \leq n} \left( W_i + \cdots + W_n \right) \int_{z_0}^{\gamma W_{i-1} z_0} \eta_{\infty,x}(z)
\]

is bounded.

(ii) The limits \( \widehat{\text{val}}(x) \) and \( \widehat{I}(x) \) are independent of \( z_0 \).

(iii) The limits \( \widehat{\text{val}}(x) \) and \( \widehat{I}(x) \) are independent of \( \{W_n \mid n \geq 1\} \).

(iv) The limits \( \widehat{\text{val}}(x) \) and \( \widehat{I}(x) \) are \( SL_2(\mathbb{Z}) \)-invariant.

**Proof** For the first claim (i), let

\[
L := \lim_{n \to \infty} \frac{\left| W_1 \right| + \cdots + \left| W_n \right|}{n},
\]

\[
a_n(z_0) := \int_{\gamma W_1 z_0}^{\gamma W_n z_0} j(z) \eta_{\infty,x}(z),
\]

\[
K := \max \{ \left| j(z) \right| : \left| \text{Re}(z) \right| \leq 1/2, \left| z \right| \geq 1, \mathcal{N}(z) \leq h(x) \},
\]

\[
R := \min \{ \text{Im}(z_0), \text{Im}(\gamma W_n z_0) : n \in \mathbb{Z}_{>0} \},
\]

\[
R' := \max \{ \left| z_0 - \gamma W_n z_0 \right| : n \in \mathbb{Z}_{>0} \},
\]

\[
M := \sup \{ \left| \gamma W_{i-1} x z_0 \right|, \left| \gamma W_{i-1} \right| : n \in \mathbb{Z}_{>0} \}
\]

Here \( K, R \) and \( R' \) are positive real numbers and \( L \) converges since the set \( \{W_1, W_2, \ldots\} \) is finite. Since \( x \) is badly approximable, \( M \) is a positive real number by Lemma 5.1. We can write

\[
\widehat{\text{val}}(x) = \frac{1}{L} \lim_{n \to \infty} \frac{a_1(z_0) + \cdots + a_n(z_0)}{n}.
\]

For any positive integer \( n_0 \), we have

\[
\left| a_1(z_0) + \cdots + a_n(z_0) \right| \leq \sum_{1 \leq i \leq n} \int_{z_0}^{\gamma W_i z_0} \left| j(z) \eta_{\gamma W_{i-1} z_0} \right| dz
\]

\[
\leq \sum_{1 \leq i \leq n} K \left| \gamma W_i z_0 \right| \left| \gamma W_{i-1} \right| \left| z \right| \leq \frac{KM R'}{R^2} n,
\]

and thus the sequence \( \{(a_1(z_0) + \cdots + a_n(z_0))/n\}_{n \geq 1} \) is bounded.
For (ii), pick any other point $z'_0 \in \mathbb{H}$. By Lemma 2.2, we have
\[
(a_1(z_0) + \cdots + a_n(z_0)) - (a_1(z'_0) + \cdots + a_n(z'_0))
\]
\[
= \left( \int_{\gamma_{W_1, W_{n+1}}}^{\gamma_{W_1, W_n}} j(z) \eta_{\infty, x}(z) \right. - \left. \int_{\gamma_{W_1, W_n}}^{\gamma_{W_1, W_{n+1}}} j(z) \eta_{\infty, x}(z) \right)
\]
\[
= \int_{\gamma_{W_1, W_n}}^{\gamma_{W_1, W_{n+1}}} j(z) \left( \eta_{\gamma_{W_1, W_n}, \infty, x}(z) - \eta_{\infty, x}(z) \right)
\]
by Lemma 2.2. Since the most right-hand side is bounded by Lemmas 5.1 and 5.2, we have
\[
\lim_{n \to \infty} \frac{a_1(z_0) + \cdots + a_n(z_0)}{n} = \lim_{n \to \infty} \frac{a_1(z'_0) + \cdots + a_n(z'_0)}{n}.
\]
As for (iii), let $x = [k_1, k_2, \ldots]$, $V_n := (k_{2n-1}, k_{2n})$, and $L_n := |W_1| + \cdots + |W_n|$. Since $\gamma_{W_1, W_n} = \gamma_{V_1, V_{2n}}$, we have
\[
\frac{1}{L_n} \int_{z_0}^{\gamma_{W_1, w_n}} j(z) \eta_{\infty, x}(z) = \frac{1}{L_n} \int_{z_0}^{\gamma_{V_1, V_n}} j(z) \eta_{\infty, x}(z),
\]
and thus the values $\widehat{\text{val}}(x)$ and $\widehat{T}(x)$ are independent of $|W_n| \mid n \geq 1$.

Finally, we prove (iv). Since $\text{SL}_2(\mathbb{Z})$-equivalent real numbers have the same continued fraction expansions except for the first few terms, it suffices to show $\widehat{\text{val}}(x) = \widehat{\text{val}}(\gamma^{-1}x)$ in the case when $x = [W_1 W_2 \ldots], \gamma = \gamma_{W_1, W_k}$. For $n > k$, it holds
\[
\int_{z_0}^{\gamma_{W_1, W_n}} j(z) \eta_{\infty, x}(z) - \int_{z_0}^{\gamma_{W_1, W_{n+1}}} j(z) \eta_{\infty, \gamma^{-1} x}(z)
\]
\[
= \int_{z_0}^{\gamma_{W_1, W_n}} j(z) \eta_{\infty, x}(z) - \int_{z_0}^{\gamma_{W_1, W_{n+1}}} j(z) \eta_{\infty, x}(z)
\]
\[
= \int_{z_0}^{\gamma_{W_1, W_n}} j(z) \eta_{\infty, x}(z) + \int_{z_0}^{\gamma_{W_1, W_n}} j(z) \eta_{\infty, x}(z),
\]
by Lemma 2.2. In the last line, the first integral is bounded by the following Lemma 5.4 and the second integral is also bounded. Thus, we obtain the claim. \(\square\)

In Theorem 5.3 (i), we consider the differential form $\eta_{\infty, x}$. In fact, we can replace it by $\eta_{x', x}$ for any point $x' \in \mathbb{P}^1(\mathbb{R}) \setminus \{x\}$. To prove it, we prepare the following lemma.

**Lemma 5.4** In the setting of Theorem 5.3, let $\{x'_n\}_{n=1}^{\infty} \subset \mathbb{R}$ be a bounded sequence such that $x$ is not an accumulation point. Then the sequence
\[
\int_{z_0}^{\gamma_{W_1, W_n}} j(z) \frac{dz}{z - x'_n},
\]
is bounded.

**Proof** Let
\[
K := \max \{|j(z)| \mid |\text{Re}(z)| \leq 1/2, |z| \leq 1, \Im(z) \leq h(x)\}.
\]
Then
\[ \left| \int_{z_0}^{\gamma_{w_1 \cdots w_n z_0}} \frac{dz}{z - x'_n} \right| \leq K \int_{S_{ip} \gamma_{w_1 \cdots w_n z_0}} \frac{|dz|}{|z - x'_n|} \]
\[ \leq K \int_{S_{ip, x}} \frac{|dz|}{|z - x'_n|}. \]

By Lemma 5.4, we can reformulate the definition of \( \hat{\text{val}}(x) \) in Theorem 5.3 (i) as follows.

**Proposition 5.5** In the setting of Theorem 5.3, let \( \{x'_n\}_{n=1}^\infty \subset \mathbb{R} \) be a bounded sequence such that \( x \) is not an accumulation point. Then

\[ \hat{\text{val}}(x) = \lim_{n \to \infty} \frac{1}{|W_1| + \cdots + |W_n|} \int_{z_0}^{x_n} j(z) \eta_{x', x}(z), \]

holds.

Finally, we prove Theorem 3.1 (iii).

**Theorem 5.6** (Theorem 3.1 (iii)). In the setting of Theorem 5.3, for any point \( x' \in \mathbb{P}^1(\mathbb{R}) \setminus \{x\} \) and a sequence \( \{x_n\}_{n=1}^\infty \) of real numbers such that \( z_n := \gamma_{W_1 \cdots W_n} (x_n + \sqrt{-1}) \in S_{\infty, x} \) and \( z_n \to x \), we have

\[ \hat{\text{val}}(x) = \lim_{n \to \infty} \frac{1}{|W_1| + \cdots + |W_n|} \int_{z_0}^{x_n} j(z) \eta_{x', x}(z), \]

\[ \hat{I}(x) = \lim_{n \to \infty} \frac{1}{|W_1| + \cdots + |W_n|} \int_{z_0}^{x_n} \eta_{x', x}(z). \]

**Proof** We prove only the first equality. Let \( \gamma_n := \gamma_{W_1 \cdots W_n} \). By Proposition 5.5, we have

\[ \hat{\text{val}}(x) = \lim_{n \to \infty} \frac{1}{|W_1| + \cdots + |W_n|} \int_{z_0}^{\gamma_n \sqrt{-1}} j(z) \eta_{x', x}(z). \]

Take a point \( z_0 \in S_{x', x} \) such that \( \pi(z_0) \in \pi([z \in \mathbb{H} | 1 \leq \text{Im}(z) \leq h(x)]) \). Then we have

\[ \hat{\text{val}}(x) = \lim_{n \to \infty} \frac{1}{|W_1| + \cdots + |W_n|} \int_{z_0}^{\gamma_n \sqrt{-1}} j(z) \eta_{x', x}(z). \]

Since \( x_n + \sqrt{-1} \in \gamma_n^{-1}(S_{x', x}) \), we have \( |x_n| \leq \max\{|\gamma_n^{-1}x|, |\gamma_n^{-1}x'|\} \). Thus, \( |x_n| \) is bounded by Lemma 5.1. Since

\[ |d_{hyp}(z_0, z_0) - d_{h_{hyp}}(\gamma_n \sqrt{-1}, z_0)| \leq d_{hyp}(z_0, \gamma_n \sqrt{-1}) \leq d_{hyp}(x_n + \sqrt{-1}, \sqrt{-1}) = 1 + \frac{x_n^2}{2}, \]

by the triangle inequality and Lemma 2.1 (ii), \( d_{hyp}(z_n, z_0) - d_{h_{hyp}}(\gamma_n \sqrt{-1}, z_0) \) is bounded. Let

\[ K := \max \left\{ |j(z)|, |\text{Re}(z)| \leq \frac{1}{2}, |z| \leq 1, \text{Im}(z) \leq h(x) \right\}. \]

Then

\[ \left( \int_{z_0}^{x_n} - \int_{z_0}^{\gamma_n \sqrt{-1}} \right) j(z) \eta_{x', x}(z) = \int_{\gamma_n \sqrt{-1}}^{x_n + \sqrt{-1}} j(z) \eta_{y^{-1}x', y^{-1}x}(z) \]
\[ \leq K \int_{\gamma_n \sqrt{-1}}^{x_n + \sqrt{-1}} \frac{|y_n^{-1}x - y_n^{-1}x'|}{(\min(\text{Im}(x_n + \sqrt{-1}), \text{Im}(\sqrt{-1})))^2} dz \]
\[ \leq K \left| y_n^{-1}x - y_n^{-1}x' \right| x_n. \]
is bounded by Lemma 5.1. Thus, we obtain
\[ \hat{\text{val}}(x) = \lim_{n \to \infty} \frac{1}{|W_1| + \ldots + |W_n|} \int_{z_0}^{z_n} j(z) n(x', z). \]

6 Elementary units for badly approximable numbers

In this section, we consider an analogue of elementary units for badly approximable numbers and prove Theorem 3.1 (iv) and (v).

**Theorem 6.1** (Theorem 3.1 (iv) and (v)). Let \( x \) be a badly approximable number and \( W_1, W_2, \ldots \) be an infinite sequence of even words such that \( x = [W_1 W_2 \ldots] \) and the set \( \{W_1, W_2, \ldots\} \) is finite. Let \( c_n \in \mathbb{Z}_{>0} \) be the denominator of a rational number \( [W_1 \cdots W_n] \) and

\[ L := \lim_{n \to \infty} \frac{|W_1| + \ldots + |W_n|}{n}. \]

Then the limit
\[ \varepsilon_x := \lim_{n \to \infty} \frac{c_n^{L_n}}{n}, \]
converges if and only if \( \hat{1}(x) \) converges. Moreover, it holds \( \hat{1}(x) = 2 \log \varepsilon_x \) and \( \varepsilon_x \geq (3 + \sqrt{5})/2 \).

Here we call \( \varepsilon_x \) the elementary units for each badly approximable number \( x \).

**Proof of Theorem 6.1** Let
\[ \gamma_n := \gamma_{W_1 \cdots W_n} = \left( \begin{array}{*} \ast \ast \end{array} \right) \left( \begin{array}{c} c_n \\ d_n \end{array} \right). \]

Here \( c_n \) is a denominator of the rational number \( \gamma_n(\infty) = [W_1 \cdots W_n] \). Pick any point \( z_0 \in \mathbb{H} \). Then we have
\[ \hat{1}(x) = \lim_{n \to \infty} \frac{1}{n} \int_{z_0}^{-1} \frac{1}{z - x} dz = \lim_{n \to \infty} \frac{1}{n} \left[ -\log(z - x) \right]_{z_0}^{z_n}. \]

Here, we can choose any branch of the logarithm. Regardless of how it is chosen, it will not affect the convergence of the limit since \( \gamma_n z_0 \) converges to \( x \) as \( n \to \infty \) and the imaginary part of \( \log \) is bounded. For similar reasons, this limit value is equal to

\[ \lim_{n \to \infty} \frac{-1}{n} \log |\gamma_n z_0 - x|. \]

Since for \( \gamma = \left( \begin{array}{*} \ast \ast \end{array} \right) \left( \begin{array}{c} c \\ d \end{array} \right) \) it holds
\[ \gamma \tau - \gamma z = \frac{\tau - z}{(ct + d)(cz + d)}, \]

the above limit value is equal to
\[ \lim_{n \to \infty} \frac{-1}{n} \log \left| \frac{z_0 - \gamma_n^{-1} x}{(c_n z_0 + d_n)(c_n \gamma_n^{-1} x + d_n)} \right|. \]
This is equal to
\[ \lim_{n \to \infty} \frac{1}{n} \log |(c_n z_0 + d_n)(c_n \gamma_n^{-1} x + d_n)|, \]
since \( x \) is badly approximable and \( \{\gamma_n^{-1} x \mid n \in \mathbb{Z}_>0\} \) is bounded by Lemma 5.1. Since \( \gamma_n^{-1} \to \frac{-d_n}{c_n} \) is bounded by Lemma 5.1
\[ \{ c_n z_0 + d_n \mid n \in \mathbb{Z}_>0 \}, \]
is bounded for any point \( z \in \mathbb{H} \cup \mathbb{R} \). Thus, we obtain
\[ \widehat{1}(x) = \lim_{n \to \infty} \frac{2}{n} \log |c_n z + d_n|. \]
By substituting \( z = 0 \), we have
\[ \widehat{1}(x) = \lim_{n \to \infty} \frac{2}{n} \log |d_n|. \]
Since \( d_n/c_n \) is bounded, we obtain
\[ \widehat{1}(x) = \lim_{n \to \infty} \frac{2}{n} \log c_n = 2 \log \varepsilon_x. \]

Next, we show the last inequality. Let \( (k_n)_{n=1}^{\infty} \) be the sequence of positive integers such that \( x = [k_1, k_2, \ldots], \phi := \frac{1+\sqrt{5}}{2}, \) and
\[ \left( \begin{array}{cc} * & * \\ c_n & d_n \end{array} \right) := \left( \begin{array}{cc} k_1 & 1 \\ 1 & 0 \end{array} \right) \cdots \left( \begin{array}{cc} k_{2n} & 1 \\ 1 & 0 \end{array} \right), \quad \left( \begin{array}{cc} * & * \\ c_n(\phi) & d_n(\phi) \end{array} \right) := \left( \begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array} \right)^{2n}. \]
Then, we have \( c_n \geq c_n(\phi), \varepsilon_x := \lim_{n \to \infty} \sqrt[n]{c_n} \) and \( \varepsilon_\phi := \lim_{n \to \infty} \sqrt[n]{c_n(\phi)} \). Thus, we obtain \( \varepsilon_x \geq \varepsilon_\phi \). By the following Example 6.2, we obtain the claim. \( \square \)

**Example 6.2** For the case when \( x = \phi := (1 + \sqrt{5})/2 = [1, 1, \ldots] \), since
\[ \frac{[1, \ldots, 1]}{2n} = \frac{F_{2n+1}}{F_{2n}}, \]
where \( F_n \) denotes the \( n \)-th Fibonacci number, we have
\[ \varepsilon_\phi = \lim_{n \to \infty} \sqrt[n]{F_{2n}} = \lim_{n \to \infty} \sqrt[n]{\frac{\phi^{2n} - (-\phi)^{-2n}}{\sqrt{5}}} = \phi^2 = \frac{3 + \sqrt{5}}{2}. \]

In Example 7.2 later, we will give examples of \( \widehat{1}(x) \) and \( \varepsilon_x \) for some badly approximable numbers.

**7 An explicit computation of values of extended val function**

In this section, we prove Theorem 1.1. The most important point of the proof below is the following lemma which is based on the proof of [4, Theorem 4.3] and is a generalization of [16, Lemma 3.3].

**Lemma 7.1** (Repetition frequency estimation). Let \( \{a_{1,n}\}_{n=1}^{\infty}, \ldots, \{a_{k,n}\}_{n=1}^{\infty} \) be sequences in \( \mathbb{Z}_>0 \) such that \( a_{1,n} \to \infty \) and \( 2^{-\alpha} a_{1,n} \to 0 \) as \( n \to \infty \). Let \( V \) be a non-empty even word and put \( v := [V] \). Let \( V_n \) and \( W_n \) be non-empty even words such that \( V_n \) and \( V_{n+1}, W_n \).
and $W_{n+1}$ coincide in the first $n$ terms respectively. Let $\{x_n\}_{n=1}^{\infty}$ and $\{x'_n\}_{n=1}^{\infty}$ be sequences of real numbers whose continued fraction expansions are

$$x_n = [V^{a_n}V_n \ldots], \quad x'_n = [0, W_n \ldots],$$

respectively. Then

$$I_n := \int_{z_0}^{y_n z_0} f(z) \eta_{x'_n x_n}(z) - a_n N(V) \text{val}(v),$$

is a Cauchy sequence. In particular, we have

$$\int_{z_0}^{y_n z_0} f(z) \eta_{x'_n x_n}(z) = a_n N(V) \text{val}(v) + O(1) \quad \text{as} \quad n \to \infty.$$

**Proof**

Take positive integers $m < n$ with $a_m < a_n$. We have

$$I_n - I_m = \int_{z_0}^{y_n z_0} f(z) \left( \sum_{0 \leq i < a_n} \eta_{-V_i^m x'_n V_i^m x_n}^V (z) \right) \left( \sum_{0 \leq i < a_m} \eta_{-V_i^m x'_m V_i^m x_m}^{V'} (z) + (a_n - a_m) \eta_{V'} (z) \right),$$

where $V'$ is the non-trivial Galois conjugate of $v$. Let

$$S_1(m, n; z) := \sum_{0 \leq i < a_m} \left( \frac{1}{z - y_{-V_i^m x_n}^V} - \frac{1}{z - y_{-V_i^m x_m}^V} \right),$$

$$S'_1(m, n; z) := \sum_{0 \leq i < a_m} \left( \frac{1}{z - y_{-V_i^m x'_n}^V} - \frac{1}{z - y_{-V_i^m x'_m}^V} \right),$$

$$S_2(m, n; z) := \sum_{a_m \leq i < a_n} \left( \frac{1}{z - y_{-V_i^m x_n}^V} - \frac{1}{z - v'} \right),$$

$$S'_2(m, n; z) := \sum_{a_m \leq i < a_n} \left( \frac{1}{z - y_{-V_i^m x'_n}^V} - \frac{1}{z - v'} \right).$$

Then we can write

$$I_n - I_m = \int_{z_0}^{y_n z_0} f(z) (-S_1(m, n; z) + S'_1(m, n; z) + S_2(m, n; z) - S'_2(m, n; z)) \, dz.$$

Let $\delta := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in (k_1, k_2, \ldots, k_r)$, and $V' := (k_{2r}, k_{2r-1}, \ldots, k_1)$ be the inverse word of $V$. For each $0 \leq i < a_m$, since

$$y_{-V_i^m x_n}^V = [V^{-i} V_n \ldots], \quad y_{-V_i^m x_m}^V = [V^{-i} V_m \ldots],$$

$$y_{-V_i^m x'_n}^V = [0, (V')^i W_n \ldots], \quad y_{-V_i^m x'_m}^V = [0, (V')^i W_m \ldots],$$

by Lemma 2.5, we have

$$\left| y_{-V_i^m x_n}^V - y_{-V_i^m x_m}^V \right| \leq 2^{2-m}, \quad \left| y_{-V_i^m x'_n}^V - y_{-V_i^m x'_m}^V \right| \leq 2^{2-m}.$$ 

For each $a_m \leq i < a_n$, since

$$y_{-V_i^m x_n}^V = [V^{-i} V_n \ldots], \quad y_{-V_i^m x'_n}^V = [0, (V')^i W_n \ldots],$$

$$y_{-V_i^m x'_m}^V = [0, (V')^i W_m \ldots],$$

$$y_{-V_i^m x'_m}^V = [0, (V')^i W_m \ldots],$$
we have
\[ |y_{v}^{−a_{m}}x_{n} − ν| ≤ 2^{−a_{m}}, |y_{v}^{−a_{m}}x'_{n} − ν'| ≤ 2^{−a_{m}}. \]

By Lemma 5.2, we obtain
\[ |I_{n} − I_{m}| = \sum_{0 \leq \ell < a_{m}} O(2^{−m}) + \sum_{a_{m} \leq \ell < a_{m}} O(2^{−a_{m}}) = O(a_{m}2^{−m}) + O((a_{n} − a_{m})2^{−a_{m}}). \]

Since this converges to 0 as \( m \to \infty \) by the assumption, \( \{I_{n}\} \) is a Cauchy sequence. □

**Proof of Theorem 1.1** Keep the notation in Theorem 1.1 and let
\[ L := \lim_{n \to \infty} \frac{\sum_{k=0}^{n} U_{k}}{(A_{n})^{k}}. \]

It suffices to show that
\[ L \text{val}(x) = a_{1}N(W_{1} \text{val}(w_{1})) + \cdots + a_{k}N(W_{k} \text{val}(w_{k})). \]

Let \( \delta := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) and \( U'_{i} \) and \( V'_{i} \) are the inverse words of \( U_{i} \) and \( V_{i} \) respectively. Take any real number \( x' < −1 \) and for positive integers \( n \) and \( 1 \leq i \leq k \), let
\[ x_{n} := y_{U_{1}U_{n+1} \cdots U_{n}} = [U_{n}U_{n+1} \cdots U_{n}], \]
\[ x'_{n} := y_{U_{1}U_{n+1} \cdots U_{n}} = −[0, U'_{n-1} \cdots U'_{1} \delta x'_{1}], \]
\[ x_{L_{n}} := y_{V_{1}w_{2}w_{3} \cdots w_{n}} = [V_{1}w_{1}a_{m} \cdots V_{k}w_{k}a_{m} \cdots U_{n+1}U_{n+2} \cdots], \]
\[ x'_{L_{n}} := y_{V_{1}w_{2}w_{3} \cdots w_{n}} = −[0, (W'_{i-1})^{a_{1} \cdots a_{m}}V'_{i-1} \cdots (W'_{i})^{a_{1} \cdots a_{m}}V'_{i}U'_{i-1} \cdots U'_{1} \delta x'_{1}]. \]

Each second equality follows from Lemma 2.5. Since \( x' < −1 \), the far right-hand sides in the second and fourth rows are continued fraction expansions. Then we have
\[ L \text{val}(x) = \lim_{n \to \infty} \frac{1}{A_{n}} \sum_{1 \leq m \leq n} \int_{z_{0}}^{y_{U_{1}U_{m} \cdots U_{m}}} j(z)η_{x'_{m},x_{m}}(z) \]
\[ = \lim_{n \to \infty} \frac{1}{A_{n}} \sum_{1 \leq m \leq n} \int_{z_{0}}^{y_{U_{1} \cdots U_{m}}} j(z)η_{x'_{m},x_{m}}(z) \]
\[ = \lim_{n \to \infty} \frac{1}{A_{n}} \sum_{1 \leq m \leq n} \sum_{1 \leq i \leq k} \left( \int_{z_{0}}^{y_{V_{i}w_{i}a_{m}} j(z)η_{x'_{m},x_{m}}(z) \right) \]
\[ + \int_{z_{0}}^{y_{V_{i} \cdots w_{i}a_{m}}} j(z)η_{x'_{m},x_{m}} \left( y_{V_{i} \cdots w_{i}a_{m}} \right) \right). \]

Since we have
\[ \left| x_{L_{m}} − y_{V_{i}w_{i}} \right| ≤ 2^{−a_{1} \cdots a_{m}}, \left| x'_{L_{m}} − y_{V_{i}w_{i}}' \right| ≤ 2^{−a_{1} \cdots a_{m}}, \]
by continued fraction expansion, it follows from Lemma 5.2 that
\[ \int_{z_{0}}^{y_{V_{i}w_{i}}} j(z) \left( η_{x'_{L_{m}},x_{L_{m}}} \left( z \right) − η_{y_{V_{i}w_{i}}'} \left( z \right) \right) = O(2^{−a_{1} \cdots a_{m}} + 2^{−a_{i} \cdots a_{m}}). \]
Since
\[ \gamma_{V_1^{-1}x_{i,m}} = [W_i^{a_{i,m}} \cdots W_k^{a_{k,m}} U_{n+1} U_{n+2} \cdots], \]
\[ \gamma_{V_1^{-1}x'_{i,m}} = [-0, V'_i (W'_{i-1})^{a_{i-1,m}} V'_i \cdots (W'_1)^{a_{1,m}} V'_1 U'_{n-1} \cdots U'_1, \delta x'], \]
we have
\[ \int_{z_0}^{W_i^{a_{i,m}} z_0} j(z) \eta_{\gamma_{V_1^{-1}x_{i,m}}} (z) = a_{i,m} N(W_i) \hat{\text{val}}(w_i) + O(1), \]
by Lemma 7.1. Thus, we obtain
\[ \hat{\text{Lval}}(x) = \lim_{n \to \infty} \frac{1}{A_n} \sum_{1 \leq m \leq n} \sum_{1 \leq i \leq k} a_{i,m} N(W_i) \hat{\text{val}}(w_i) \]
\[ = \sum_{1 \leq i \leq k} a_i N(W_i) \hat{\text{val}}(w_i). \]
By performing similar calculations with \( j(z) \) replaced by the constant function 1, we also obtain
\[ \hat{\text{L1}}(x) = \sum_{1 \leq i \leq k} a_i N(W_i) \hat{\text{1}}(w_i). \]

\[ \square \]

Example 7.2
(i) In the situation in Theorem 1.1, let \( k = 2, W_1 = (1, 1), W_2 = (2, 2), a_{1,n} = a_{1,n} = n \). Then, we have \( x = [1, 1, 2, 2, 1, 1, 1, 2, 2, 2, \ldots, (1, 1)^n, (2, 2)^n, \ldots] \) and
\[ w_1 = \frac{1 + \sqrt{5}}{2}, \quad w_2 = 1 + \sqrt{2}, \quad N(W_1) = N(W_2) = 1, \quad A_n = n(n+1), \quad a_1 = a_2 \]
\[ = \frac{1}{2}, \quad L = 1. \]
Moreover, we have
\[ \hat{\text{1}}(w_1) = \hat{\text{1}}(w_1) = 4 \log \left( \frac{1 + \sqrt{5}}{2} \right), \quad \hat{\text{1}}(w_2) = \hat{\text{1}}(w_2) = 4 \log \left( 1 + \sqrt{2} \right), \]
by Lemma 2.7 and Theorem 3.1 (i). Thus, we obtain
\[ \hat{\text{1}}(x) = \frac{\hat{\text{1}}(w_1) + \hat{\text{1}}(w_1)}{2} = 2 \log \left( \frac{1 + \sqrt{5}}{2} \right) + 2 \log \left( 1 + \sqrt{2} \right). \]

(ii) In the above situation, we replace \( a_{1,n} \) and \( a_{1,n} \) by \( a_{1,n} = 2n-1 \) and \( a_{1,n} = \lfloor tn^2 \rfloor - \lfloor t(n-1)^2 \rfloor \), where \( t \) is an arbitrary positive number. Then, we have
\[ A_n = n^2 + \lfloor tn^2 \rfloor, \quad a_1 = \frac{1}{1+t}, \quad a_2 = \frac{t}{1+t}, \quad L = 1. \]
Thus, we obtain
\[ \hat{\text{1}}(x) = \frac{4}{1+t} \left( \log \left( \frac{1 + \sqrt{5}}{2} \right) + t \log \left( 1 + \sqrt{2} \right) \right). \]
8 The value of \( \text{val} \) function at the Thue–Morse word

In this section, we calculate the value of \( \text{val} \) function at the Thue–Morse word and prove Theorem 1.3.

We fix two even words \( V \) and \( W \). Let \( h: \{V, W\}^* \to \{V, W\}^* \) be a monoid homomorphism such that \( h(V) := VW \), \( h(W) := WV \) and \( V_h := \lim_{n \to \infty} h^n(V) \) be the Thue–Morse word. For a word \( U = U_1 \cdots U_n \in \{V, W\}^* \) with \( U_1, \ldots, U_n \in \{V, W\} \), let \( U' := U_n \cdots U_1 \).

To begin with, we give a simple representation of the Thue–Morse word.

Lemma 8.1 Let \( \tau: \{V, W\}^* \to \{V, W\}^* \) be a monoid homomorphism such that \( \tau(V) := W \) and \( \tau(W) := V \). Then the following statements hold.

(i) For any positive integer \( n \), we have \( h^n(V) = h^{n-1}(V) \tau h^{n-1}(V) \).

(ii) \( h^{2n}(V) = h^{2n}(V) \).

Proof The first claim (i) follows from the facts that \( h \circ \tau = \tau \circ h \) and \( h^2(U) = h(U) \tau h(U) \) for a word \( U \in \{V, W\}^* \) such that \( h(U) = U \tau(U) \). The last claim (ii) follows from the fact that \( h^2(U) = h^2(U) \) for a word \( U \in \{V, W\}^* \) such that \( h(U) = U \tau(U) \) and \( U' = U \).

Proof of Theorem 1.3 Let \( V_h := h^n(V) \). Since \( [V_h]' = -[0, V_h] \) by Lemma 8.1 (ii), we have

\[
\text{val}([V_h]) = \lim_{n \to \infty} \frac{1}{2^{2n}} \int_{[0, V_h]} j(z) \eta_{-}[0, V_h] V_h(z), \quad \text{val}([\nabla V]) = \int_{[0, V]} j(z) \eta_{-}[0, V] V(z).
\]

By performing similar calculations with \( j(z) \) replaced by the constant function 1, we also obtain the same equations for \( \tilde{1}([V_h]) \) and \( \tilde{1}([\nabla V]) \). Here the first \( 2^{2n} \) terms of \( V_h \) are \( V_n \) by Lemma 8.1 (i). Thus, we obtain the claim.

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Data availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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