Exact law for compressible pressure-anisotropic magnetohydrodynamic turbulence: a link between fluid cascade and instabilities

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We derive a first exact law for compressible pressure-anisotropic magnetohydrodynamic turbulence. For a gyrotrropic pressure tensor, we study the double-adiabatic case and show the presence of new flux and source terms in the exact law, reminiscent of the firehose and mirror instabilities. The Hall term is shown to bring ion-scale corrections to the exact law without affecting explicitly the pressure terms. In the pressure isotropy limit we recover all known results obtained for isothermal and polytropic closures. The incompressible limit of the gyrotrropic system leads to a generalization of the Politano and Pouquet’s law where a new incompressible source term is revealed and reflects exchanges of the magnetic and kinetic energies with the no-longer-conserved internal energy. We highlight the possibilities offered by the new laws to investigate potential links between turbulence cascade and instabilities widely observed in laboratory and astrophysical plasmas.

Introduction.— In recent years there has been a growing interest in deriving Von-Karman-Howarth-Monin (vKHM) equations that describe turbulent energy cascade in magnetized plasmas. Those equations present the double advantage of being fully nonlinear and of linking the turbulent energy cascade (or dissipation) rate to measurable fields. The cascade rate, is used to estimate energy dissipation from spacecraft data taken in the solar wind (SW) and the planetary plasma environments. Efforts were thus put in generalizing the laws to more realistic conditions met in those plasmas at the cost of increasing complexity. Two main lines of research are pursued: one aiming at extending the range of the described scales, from magnetohydrodynamics (MHD), to Hall-MHD and two-fluids; the second by incorporating density fluctuations described within isothermal or polytropic closures or gravitational effects to study star formation in the interstellar medium.

Despite these important improvements, a key missing ingredient that none of the existing models can describe, is the presence of pressure anisotropy (with respect to the background magnetic field $B_0$). Indeed, while the existing laws do consider the presence of a background magnetic field, which allows one to study energy transfers along the parallel and perpendicular directions to $B_0$, they however all assume a scalar pressure, which is unrealistic to describe most of magnetized collisionless astrophysical (or laboratory) plasmas where ion and electron pressure anisotropies are frequently reported from particle measurements.

In order to include pressure anisotropy in fluid modeling of magnetized plasmas, Chew et al. introduced the double-adiabatic closure (known also as CGL). One of the main changes to the dynamics of the plasma brought up by pressure anisotropy in CGL-MHD equations is the presence of instabilities, which in the linear limit coincide with the firehose when $a_p > 1$ and the mirror when $\beta_p a_p > 1 < \beta_p^2$ ($\beta_p$ is the ratio of the parallel thermal to magnetic pressure, $a_p = T_\perp/T_\parallel$ is the ratio between the proton perpendicular and parallel temperatures) These instabilities (or their kinetic counterparts) were shown to constrain part of the dynamics of the SW and are thought to operate in laboratory devices, clusters of galaxies and black holes’ accretion disks. However, the interplay between turbulence and instabilities remains an unsettled question although some hints were already reported. These include driving of sub-ion scale turbulence, influencing the scaling of the high frequency magnetic energy spectra in the SW, or the link between unstable plasmas and high energy cascade rates as measured in the near-Earth space, which remains to date not fully understood.

It is the goal of this work to fill the existing gap by providing a self-consistent (fluid) theoretical framework to investigate the potential coupling between plasma turbulence and instabilities.

Theoretical model.— We use the classical MHD equations but assume a (symmetric) pressure tensor rather than a scalar one,

$$\partial_t \rho = -\nabla \cdot (\rho \mathbf{v}) ,$$

$$\partial_t (\rho \mathbf{v}) = \nabla \cdot (\rho \mathbf{v} \mathbf{v} - \rho \mathbf{v} - \nabla P_m) + \mathbf{d}_k + \mathbf{f} ,$$

$$\partial_t (\rho \mathbf{v}_A) = \nabla \cdot (\rho \mathbf{v}_A \mathbf{v} - \rho \mathbf{v} \mathbf{v}_A) + \rho \nabla \cdot \mathbf{v}_A - \frac{1}{2} \rho \mathbf{v}_A \nabla \cdot \mathbf{v} + \mathbf{d}_m ,$$

where $\rho$ is the mass density, $\mathbf{v}$ the velocity field, $\mathbf{v}_A$ the Alfvén velocity, and $P_m = P + P_M$ is the total pressure tensor, i.e., the sum of the pressure tensor $\nabla \cdot$ and the magnetic pressure tensor $P_M = P_M \nabla = (\rho v_A^2/2)\nabla$, $\mathbf{d}_k$ the kinetic viscous dissipation, $\mathbf{d}_m$ the magnetic diffusivity, $\mathbf{f}$ a stationary homogeneous external force assumed to act on the largest scales.

Since we want to derive the exact law for the total
energy of the system, equations (1)-(3) are complemented by that of the (specific) internal energy $u$, which reads

$$\partial_t u = -\nabla \cdot (u \mathbf{v}) + u \nabla \cdot \mathbf{v} - \frac{\mathcal{P}}{\rho} : \nabla \mathbf{v},$$  \hspace{1cm} (4)$$

since $\mathcal{P}$ and $P_{\alpha}$ are symmetrical tensors, i.e., $P_{ij} = P_{ji}$, the dual product between two of such tensors $\mathcal{P}$ and $\mathbf{A}$ obeys $\mathcal{P} : \mathbf{A} = P_{ij} A_{ij} = P_{ji} A_{ji}$. Equation (1), valid for any symmetric pressure tensor when the heat flux is neglected, can be derived from thermodynamical considerations [33] or from the moments of the Vlasov-Maxwell equations [50]. For a scalar pressure, i.e., $\mathcal{P} = P \rho^2$, we recover the equation of the internal energy used in [34].

Equations (1)-(4) will be used in the following section to derive the exact law of interest.

**General exact law for compressible pressure-anisotropic MHD turbulence.**— Following the standard approach used in statistical theories of fully developed turbulence [4, 9], we define the spatial increment (or scale) $\ell$ connecting two points $\mathbf{x}$ and $\mathbf{x}'$ as $\mathbf{x}' = \mathbf{x} + \ell$ and introduce the notations, $\xi(\mathbf{x}) \equiv \xi$, its conjugate (i.e., taken at the position $\mathbf{x}'$) $\xi(\mathbf{x}') \equiv \xi'$ and the incremental quantity $\delta \xi = \xi' - \xi$. These definitions impose that $\partial_\mathbf{x}' \xi' = \partial_\mathbf{x} \xi = 0$, while the hypothesis of space homogeneity implies the relations $\langle \nabla \xi' \rangle = \nabla \xi \cdot \langle \xi \rangle$ and $\langle \nabla^2 \rangle = -\nabla^2 \langle \xi \rangle$, where $\partial_\mathbf{x}$ denotes the derivative operator along the increment vector $\ell$ and $\langle \xi \rangle$ an ensemble average.

We consider the mean correlation function of the total energy $R_{tot} = \langle R + R' \rangle / 2$ with $\langle R \rangle = \langle \rho \mathbf{v} \cdot \mathbf{v}' / 2 + \rho \mathbf{v}_A \cdot \mathbf{v}_A' / 2 + \rho u' \rangle = \langle R_k + R_B + R_a \rangle$ a correlation function taken at the point $\mathbf{x}$ and $\mathbf{x}'$ its conjugate. We remark that if $\mathbf{x} = \mathbf{x}'$, $R_{tot} = \langle E \rangle = \langle \rho v^2 / 2 + \rho v_A^2 / 2 + \rho u \rangle$, i.e. the mean total energy of the system.

Using the property $\partial_\mathbf{x} \langle \xi \rangle = \langle \partial_\mathbf{x} \xi \rangle$, the equations (1)-(4) written at the independent positions $\mathbf{x}$ then $\mathbf{x}'$ and multiplied by the appropriate variables (e.g., equation (2) multiplied by $\mathbf{v}'$) and the space homogeneity assumption (see [34] for more details), we obtain the temporal evolution of the kinetic, $\langle R_k \rangle$, the magnetic, $\langle R_B \rangle$, and the internal energy, $\langle R_a \rangle$, correlators:

$$2 \partial_t \langle R_k \rangle = -\nabla \cdot \langle \rho v \cdot \mathbf{v}' \delta \mathbf{v} + \rho v A \cdot \mathbf{v}_A \delta \mathbf{v}_A - \rho v A' \cdot \mathbf{v}_A' \rangle + \nabla \cdot \langle \mathcal{P}_{\alpha} - \rho \mathcal{P}_{\alpha} \cdot \mathbf{v}' \rangle \cdot \mathbf{v}' + 2 \rho v \cdot \mathbf{v}_A \delta \mathbf{v}_A + \mathcal{F} + \mathcal{D}_k, \hspace{1cm} (5)$$

$$2 \partial_t \langle R_B \rangle = -\nabla \cdot \langle \rho v A \cdot \mathbf{v}_A' \delta \mathbf{v} + \rho v A' \cdot \mathbf{v}_A \delta \mathbf{v}_A \rangle - \nabla \cdot \langle \mathcal{P}_{\alpha} \rangle \cdot \mathbf{v}_A - \rho \mathbf{v}_A \delta \mathbf{v}_A + \mathcal{D}_m, \hspace{1cm} (6)$$

$$\partial_t \langle R_a \rangle = -\nabla \cdot \langle \rho u \cdot \mathbf{v}' \delta \mathbf{v} + \rho u A' \cdot \mathbf{v}_A' \rangle + \mathcal{F}' + \mathcal{D}_k + \mathcal{D}_b + \mathcal{D}_m + \mathcal{D}_m', \hspace{1cm} (7)$$

where the terms depending on the forcing, the kinetic, and magnetic dissipation are regrouped respectively in $\mathcal{F}$, $\mathcal{D}_k$ and $\mathcal{D}_m$. Then the temporal evolution of $R_{tot}$ is the sum of the relations (5)-(7) and of their conjugates (written at position $\mathbf{x}'$). By recognizing the developed form of the structure functions $\langle \delta (\rho v) \cdot \delta \mathbf{v} \rangle$, $\langle \delta (\rho v A) \cdot \delta \mathbf{v}_A \rangle$, $\langle \delta (\rho v A) \cdot \delta \mathbf{v}_A \rangle$, $\langle \delta (\rho v A) \cdot \delta \mathbf{v}_A \rangle$, $\langle \delta (\rho v A) \cdot \delta \mathbf{v}_A \rangle$, we obtain the following exact law valid in the inertial range:

$$4 \partial_t R_{tot} = \nabla \cdot \left( \langle \delta (\rho v) \cdot \delta \mathbf{v} + \delta (\rho v A) \cdot \delta \mathbf{v}_A + 2 \rho \delta u \rangle \delta \mathbf{v} - \langle \delta (\rho v A) \cdot \delta \mathbf{v} + \delta (\rho v) \cdot \delta \mathbf{v}_A \rangle \delta \mathbf{v}_A - \delta \rho \mathcal{P} \cdot \delta \mathbf{v} \right)$$

$$+ \left( \langle \rho v \cdot \delta \mathbf{v} + \frac{1}{2} \rho v A \cdot \delta \mathbf{v}_A - \frac{1}{2} \delta (\rho v A) \cdot \delta \mathbf{v}_A + 2 \rho \delta u \rangle \right) \delta \mathbf{v}_A' - 2 \rho \delta \left( \frac{\mathcal{P}_{\alpha}}{\rho} \right) : \nabla \delta \mathbf{v}'$$

$$+ \left( \rho v' \cdot \delta \mathbf{v} - \frac{1}{2} \rho v A' \cdot \delta \mathbf{v}_A + \frac{1}{2} \delta (\rho v A') \cdot \delta \mathbf{v}_A - 2 \rho' \delta u \rangle \delta \mathbf{v}_A + 2 \rho' \delta \left( \frac{\mathcal{P}_{\alpha}}{\rho} \right) : \nabla \delta \mathbf{v} \right)$$

$$+ \left( \langle -2 \rho v \cdot \delta \mathbf{v}_A - \rho v A \cdot \delta \mathbf{v} + \delta (\rho v A) \cdot \delta \mathbf{v}_A \rangle \delta \mathbf{v}_A' + \left( 2 \rho v' \cdot \delta \mathbf{v}_A + \rho v A' \cdot \delta \mathbf{v} - \delta (\rho v) \cdot \delta \mathbf{v}_A \right) \delta \mathbf{v} \right)$$

$$+ \left( \delta \rho \mathcal{P}_{\alpha} \cdot \delta \mathbf{v} - \rho \delta \left( \frac{\mathcal{P}_{\alpha}}{\rho} \right) : \nabla \delta \mathbf{v} \right) + \mathcal{F} + \mathcal{D}_k + \mathcal{D}_b + \mathcal{D}_m + \mathcal{D}_m'$$

From this relation and following the usual assumptions used in fully developed homogeneous turbulence, namely infinite kinetic and magnetic Reynolds numbers, stationary state, balance between forcing (at the largest scales) and dissipation (at the smallest ones) [6, 23, 50], we obtain the following exact law valid in the inertial range:
\[-4\varepsilon_{\text{MHD}} = \nabla_\rho F_{\text{MHD}} + S_{\text{MHD}} + S'_{\text{MHD}} \text{ with}
\]
\[
\begin{aligned}
F_{\text{MHD}} &= \left( \delta (\rho \nu) \cdot \delta \nu + \delta (\rho \nu A) \cdot \delta \nu A + 2\delta \rho \delta u \right) \delta \nu - \left( \delta (\rho \nu A) \cdot \delta \nu + \delta (\rho \nu) \cdot \delta \nu A \right) \delta \nu A - \delta \rho \delta \left( \frac{P_\rho}{\rho} \right) \cdot \delta \nu, \\
S_{\text{MHD}} &= \left( \rho \nu \cdot \delta \nu + \frac{1}{2} \rho \nu A \cdot \delta \nu A - \frac{1}{2} \nu A \cdot \delta (\rho \nu A) + 2\rho \delta u \right) \nabla' \nu' - 2\delta \left( \frac{P_\rho}{\rho} \right) \cdot \nabla' \nu', \\
S'_{\text{MHD}} &= \text{conjugate} (S_{\text{MHD}}),
\end{aligned}
\]

where \(\varepsilon_{\text{MHD}}\) is the classical mean energy dissipation rate by unit mass assumed to be equal to the injection rate due to the forcing, i.e. \(F + F' \simeq 4\varepsilon_{\text{MHD}}\), and to the cascade rate in the inertial range due to nonlinearities. The exact law \([29] \) is the first main result of this paper. It is valid for any MHD flow with a (symmetric) pressure tensor when the heat flux is neglected. Its extension to Hall-MHD flows is given in the Appendix.

As in other compressible exact laws, we can recognize the terms introduced by \([29] \): \(F_{\text{MHD}}\) is the flux terms (increment derivative \(\nabla_\rho \cdot \)) \(S_{\text{MHD}}\) and its conjugate \(S'_{\text{MHD}}\) are generally known as source terms (see below about the physical meaning of this terminology) where terms in \(\langle \nabla \nu \rangle\) reflect the role of velocity dilatation, terms in \(\langle \nabla \cdot \nu A \rangle\) involve the (compressible) Alfvén speed dilatation and terms in \(\langle \nabla \rho \rangle\) contain density dilatation. Note that some hybrid and the \(\beta\)-dependent terms introduced in \([29] \) are hidden in the new structure function \(\left( \delta \rho \delta \left( \frac{P_\rho}{\rho} \right) \cdot \delta \nu \right)\) and the terms in \(\langle \nabla \rho \rangle\). For a scalar pressure we recover all the compressible laws derived for polytropic or isothermal flows \([34] \) (see Appendix).

**Compressible MHD Exact law with a gyroscopic pressure.**— The gyroscopic exact law can be readily obtained from relation \([9] \) by imposing the pressure tensor decomposition \(\overline{P} = (P_\perp + P_\parallel T) + (P_\parallel - P_\perp) \mathbf{b} \mathbf{b}^\top\), with \(\mathbf{b} = \nu A / |\nu A|\) the magnetic field direction \([50] \). These definitions yield the following form of the total pressure \(\overline{P} = (P_\perp + P_\parallel T) + (P_\parallel - P_\perp) \mathbf{b} \mathbf{b}^\top\). Using the pressure tensor equation, one can define the internal energy density as \(\rho u = \frac{1}{2} \overline{P} \cdot T = \frac{1}{2} P_\parallel + P_\perp \mathbf{b} \mathbf{b}^\top\). To highlight the terms in the exact law \([9] \) that can be linked to known (linear) plasma instabilities \([50] \), we further introduce the parameters \(\beta_\parallel = \frac{P_\parallel}{P_\perp}\) and \(a_\parallel = \frac{P_\parallel}{P_\perp} = T_\perp/T_\parallel\). Injecting these relations in equation \([9] \) yields the new gyroscopic-MHD exact law, which is the second main result of this paper:

\[-4\varepsilon_{\text{GYR}} = \nabla_\rho F_{\text{GYR}} + S_{\text{GYR}} + S'_{\text{GYR}} \text{ with}
\]
\[
\begin{aligned}
F_{\text{GYR}} &= \left( \delta (\rho \nu) \cdot \delta \nu + \delta (\rho \nu A) \cdot \delta \nu A - \delta (\rho \nu A) \cdot \delta \nu A - \delta (\rho \nu) \cdot \delta \nu A \right) \\
&\quad + \left( \delta \rho \delta \left( \frac{\nu A^2}{2} \right) \beta_\parallel [1 + a_\parallel] - 1 \right) \delta \nu - \delta \rho \delta \left( \frac{\nu A^2}{2} \beta_\parallel [1 + a_\parallel] \nu A \nu A^2 \cdot \delta \nu A \right), \\
S_{\text{GYR}} &= \left( \rho \nu \cdot \delta \nu + \frac{1}{2} \rho \nu A \cdot \delta \nu A - \frac{1}{2} \nu A \cdot \delta (\rho \nu A) + \rho \delta \left( \frac{\nu A^2}{2} \beta_\parallel \right) \right) \nabla' \nu' - \delta \rho \delta \left( \beta_\parallel [1 - a_\parallel] \nu A \nu A \cdot \nabla' \nu' \right), \\
S'_{\text{GYR}} &= \text{conjugate} (S_{\text{GYR}}),
\end{aligned}
\]

Equation \([10] \) shows the presence of new terms brought in by pressure anisotropy, which reveals how the turbulent cascade can be impacted by the so-called mirror and firehose instabilities. For instance, the terms proportional to \(1 - a_\parallel\) will have either positive or negative contribution to the cascade rate depending on the sta-
bility condition $a_p > 1$ or $a_p < 1$. In case of a positive (resp. negative) contribution to the cascade rate, pressure anisotropy can be seen as a source of “free energy” (resp. a sink) that can reinforce (resp. diminish) the turbulence cascade. Furthermore, if the pressure anisotropy terms dominate the cascade then the instability would impact both the value of the energy cascade rate and its “sense” (direct vs. inverse). Equation (10), which can be used on simulation and spacecraft data, may thus provide a solid theoretical explanation of the results reported in [39, 11] and to the overall prominent role of the instabilities (non necessarily linear) in controlling part of the dynamics in astrophysical plasmas [45, 46, 50, 61].

Note that in relation (10) the parameters $\beta_\parallel$ and $\alpha_p$ that depend on the pressure components $P_\parallel$ and $P_\perp$ are not yet determined since this relation derives from the internal energy equation (4), which constrains the sum of the two pressure components but not the individual ones. The latter can be determined by further introducing any closure equation compatible with the definition of the internal energy $\rho u = \frac{1}{\rho} \frac{\partial}{\partial t} \mathbf{F}$ for each pressure component as done in the CGL-MHD theory.

**Exact law for the CGL-MHD system.**— The CGL-MHD closure equations written in their conservative form [10] read

$$
\frac{d}{dt} \left( \frac{P_\parallel B^2}{\rho^3} \right) = 0, \quad \text{and} \quad \frac{d}{dt} \left( \frac{P_\perp}{\rho B} \right) = 0, \quad (11)
$$

where $d/dt$ is the total time derivative. Equations (11) lead to the integrated form of the pressures and, consequently, to the forms of the parameters $\beta_\parallel = 2C_\parallel \frac{\rho}{V_A^2}$ and $\alpha_p = C_p \frac{|V_A|^3}{\rho^{1/2}}$, where the constants $C_\parallel$ and $C_p$ guaranty the homogeneity. Injecting these integrated relations in equation (10) yields the new CGL-MHD exact law, which is the third result of this paper:

$$
-4\varepsilon_{\text{CGL}} = \nabla \cdot \mathbf{F}_{\text{CGL}} + S_{\text{CGL}} + S'_{\text{CGL}} \text{with}
$$

$$
\mathbf{F}_{\text{CGL}} = (\delta (\rho V) \cdot \delta \rho \delta \mathbf{V} + \delta (\rho V_A) \cdot \delta \rho \delta \mathbf{V} - \delta (\rho V_A) \cdot \delta \rho \delta \mathbf{V} - \delta (\rho V_A) \cdot \delta \rho \delta \mathbf{V} A \cdot \delta \rho \delta \mathbf{V} A)
$$

$$
+ \left\{ \begin{array}{l}
\left[ (\delta (\rho V_A) \cdot \delta \rho \delta \mathbf{V} A) + \delta (\rho V_A) \cdot \delta \rho \delta \mathbf{V} A) \cdot \delta \rho \delta \mathbf{V} A + \delta (\rho V_A) \cdot \delta \rho \delta \mathbf{V} A + \delta \rho \delta \mathbf{V} A \right]
\end{array} \right\}
$$

$$
S_{\text{CGL}} = \left\{ \begin{array}{l}
\left[ (\delta (\rho V_A) \cdot \delta \rho \delta \mathbf{V} A) + \delta (\rho V_A) \cdot \delta \rho \delta \mathbf{V} A + \delta \rho \delta \mathbf{V} A + \delta \rho \delta \mathbf{V} A \right]
\end{array} \right\}
$$

$$
S'_{\text{CGL}} = \text{conjugate} (S_{\text{CGL}}).
$$

Note that in the isotropic limit $P_\parallel = P_\perp$ one finds the adiabatic (monoatomic) case with a polytropic index $\gamma = 5/3$ and $\rho u = 3P/2$.

**The incompressible MHD with a gyrotropic pressure: a generalization of the Politano and Pouquet’s law.**— In the incompressible limit, i.e. $\rho = \rho_0$ and $\nabla \cdot \mathbf{V} = 0$, equation (10) becomes:

$$
4\varepsilon_{\text{IGY}} = 4\varepsilon_{\text{PP98}} + \rho_0 \left( \delta (\beta_\parallel [1 - a_p] V_A V_A) \cdot \delta (\nabla \mathbf{V}) \right), \quad (13)
$$

where $\varepsilon_{\text{IGY}}$ stands for the cascade rate of incompressible gyrotropic model and $-4\varepsilon_{\text{PP98}} = \rho_0 \nabla \cdot \left( (\delta \mathbf{V} \cdot \delta \nabla + \delta V_A \cdot \delta \mathbf{V}) \right) \delta \mathbf{V} - 2\delta \mathbf{V} \cdot \delta \nabla \cdot \delta \mathbf{V} A) \right)$ is the so-called Politano and Pouquet’s law [18], hereafter PP98. Interestingly, we evidence in equation (13) the presence of a new source term brought in by the anisotropy of the pressure tensor, which is written as a contraction of two increment tensors. Equation (13) is the fourth result of this paper. It generalizes PP98 to incompressible plasmas with a gyrotropic pressure and the notion of source terms. Indeed, so far the terminology of “source” terms introduced in [24] reflects compression (resp. dilatation) of the plasma that can sustain (resp. oppose) the cascade in the inertial range [33]. Here we evidence a new source term in the incompressible gyrotropic limit that is not tied to plasma contraction/dilatation, but to pressure anisotropy. It reflects the exchange between the no longer-conserved internal energy (unlike in incompressible pressure-isotropic flows [34]) with the sum of the magnetic and kinetic energies as can be seen in equation (14) where we have $-\frac{\mathbf{F}}{\rho_0} : \nabla \mathbf{V} \neq 0$. This leads us to propose the following generalization of the notion of source: for compressible
isentropic flows with a gyrotropic pressure tensor, the cascade of the kinetic and magnetic energy can be opposed/sustained by compression/dilatation of the fluid and by pressure anisotropy, the latter being relevant even in incompressible flows. For weakly compressible plasmas (e.g., SW), this result implies that the first order correction to the PP98 law would not come from density fluctuations, but rather from (incompressible) pressure anisotropy.

Similarly to the compressible gyrotropic case discussed above, the parameters $\beta_1$ and $a_\rho$ remain undetermined. To determine the pressure components $P_\parallel$ and $P_\perp$ the internal energy equation \ref{eq:9} (with now $\rho = \rho_0$) is complemented by a new equation coming from imposing the incompressibility condition $\nabla \cdot v = 0$ on the momentum equation \ref{eq:2} (with $\rho = \rho_0$, and $d_k = f = 0$ for simplicity), as done for incompressible isotropic hydrodynamics \ref{eq:4} or Hall-MHD \ref{eq:62}. This yields the generalized pressure balance equation for incompressible gyrotropic pressure tensor, namely,

$$\nabla \cdot \left( \rho v_A v_A - \rho v v - P_e \right) = 0,$$

Solving equations \ref{eq:4} (with $\rho = \rho_0$) and \ref{eq:14} allows one to close the new incompressible gyrotropic MHD system proposed here and to self-consistently determine $P_\parallel$ and $P_\perp$. However, for nearly incompressible plasmas such as the SW, the exact law \ref{eq:13} can be directly applied to spacecraft data when $P_\parallel$ and $P_\perp$ are accessible to measurements assuming equation \ref{eq:14} to hold.

Note finally that the new model of incompressible gyrotropic (whose exact law is given by equation \ref{eq:13}) admits the oblique firehore instability as a linear solution, which is the unstable version of the known shear Alfvén mode \ref{eq:50}.

**Conclusion.**— We derived new general exact laws for homogeneous MHD and Hall-MHD turbulent flows that go beyond the pressure isotropy assumption, which make them more realistic to study strong turbulence in magnetized plasmas. By considering the specific case of a CGL closure, we showed that the new law involves new flux and source terms that potentially can reflect the impact of plasma instabilities on the turbulent cascade. In the limit of incompressible MHD with a gyrotropic pressure we provided a generalization of the Politano and Pouquet’s law \ref{eq:17} to pressure anisotropic plasmas, where a new incompressible source term is revealed and highlights a fundamental difference between pressure isotropic and anisotropic plasmas: internal energy is not conserved in the latter and pressure anisotropy can act as a source of free-energy to supply the turbulent cascade with an additional energy. This work thus paves the road to new and more rigorous (albeit fluid) studies of the interplay between turbulent (fluid) cascade and plasma instabilities, both in numerical simulations and spacecraft observations when the full pressure tensor is accessible to measurements.

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**APPENDIX**

**The exact law for scalar pressure.**— When considering a (total) scalar pressure $P = P^T$ the MHD exact law \ref{eq:9} takes the form:

$$-4\varepsilon_{MHD} = \nabla \cdot F_{MHD} + S_{MHD} + S'_{MHD}$$

with

\[
\begin{align*}
F_{MHD} &= \left\langle \left( \delta (\rho v) \cdot \delta v + \delta (\rho v_A) \cdot \delta v_A + 2\delta \rho \delta u \right) \delta v - \left( \delta (\rho v_A) \cdot \delta v + \delta (\rho v) \cdot \delta v_A \right) \delta v_A - \delta \rho \delta \left( \frac{P}{\rho} \right) \delta v \right\rangle \\
S_{MHD} &= \left\langle \left( \rho v \cdot \delta v + \frac{1}{2} \rho v_A \cdot \delta v_A - \frac{1}{2} v_A \cdot \delta (\rho v_A) + 2\rho \delta u - 2\rho \delta \left( \frac{P}{\rho} \right) \right) \nabla \cdot \delta v \right\rangle \\
S'_{MHD} &= \left\langle \left\langle \left( \delta \rho \frac{P}{\rho} \cdot \nabla - \delta \rho \left( \frac{P}{\rho} \right) \cdot \nabla \right) \cdot \frac{\nabla \rho}{\rho} \right\rangle \right\rangle \quad \text{conjugate (S'_{MHD})}.
\end{align*}
\]

One can notice in relation \ref{eq:15} the presence of a new flux term that was not recognized as such in the previous models derived for scalar pressure \ref{eq:26} \ref{eq:29} \ref{eq:34}:
It is worth noting that the $\beta$-dependent term introduced by [22] is hidden in this line since $P_*/P = 1 + P_M/P = 1 + \beta^{-1}$. After some other manipulations, we obtain the general exact law for isentropic flows derived in [34]:

$$-4\varepsilon_{\text{MHD}} = \nabla \cdot (\rho \nu \cdot \delta \nu \delta v + \delta(\rho v_A) \cdot \delta v_A \delta v + 2\delta \rho \delta u \delta v - \delta(\rho v_A) \cdot \delta v_A \delta v_A)$$

$$+ \nabla \cdot \left( \left( 1 + \frac{\rho}{\rho'} \right) (P + P_M) \nu' - \left( 1 + \frac{\rho}{\rho'} \right) (P' + P_M') \nu + \rho' \nu u' - \rho u' \right)$$

$$+ \left( \nabla' \cdot \rho \nu \cdot \delta v - \delta \nu_A \cdot \delta v_A - \frac{1}{2} \rho' \nu_A' \cdot \nu_A - \frac{1}{2} \rho \nu_A \cdot \nu_A' + 2 \rho \left( \delta u - \frac{P'}{\rho'} \right) \right)$$

$$+ \left( \nabla' \cdot \left( -\rho' \nu \cdot \delta v - \rho' \nu_A \cdot \delta v_A - \frac{1}{2} \rho' \nu_A \cdot \nu_A' - \frac{1}{2} \rho \nu_A' \cdot \nu_A - 2 \rho' \left( \delta u - \frac{P'}{\rho'} \right) \right) \right)$$

$$- \left( \nabla' \cdot \left( 2 \rho \nu \cdot \delta v_A - \rho \nu' \cdot \nu_A + \rho \nu_A \cdot \nu' \right) - \nabla' \cdot \left( 2 \rho \nu' \cdot \delta v_A + \rho \nu \cdot \nu_A' - \rho \nu_A' \cdot \nu' \right) \right)$$

$$- \left( \frac{P_M}{P} \nabla' \cdot (\rho \nu u') + \frac{P_M}{P} \nabla' \cdot (\rho u' \nu') \right).$$

(17)

It is worth recalling that this exact law is an extension of all scalar pressure models such as the isothermal and polytropic, which can be obtained by introducing the adequate state equation in relation (17) (i.e., specifying the relation between the pressure $P$ and the density $\rho$) that are compatible with the isentropic hypothesis [34].

**Extension to pressure anisotropic Hall-MHD.**—The extension of the previous MHD model to Hall-MHD flows can be readily obtained by noticing that the only change to the original model is to introduce the Hall term in equation (3), while the internal energy equation remains unchanged. Therefore, the changes to the exact law (9) will occur through the sole terms that depend on the current density, which were already derived in [30] for compressible isothermal MHD, without impacting pressure terms. The final exact law for Hall-MHD thus writes

$$-4\varepsilon_{\text{HMH}} = -4\varepsilon_{\text{MHD}} + 2d_i \nabla \cdot \rho J_c \times \nu_A - \delta(J_c \times \nu_A)$$

$$- \frac{d_i}{2} \left( \left( \delta \rho \nu_A \cdot \nu_A \right) \nabla \cdot J_c - \left( \delta \rho \nu_A' \cdot \nu_A \right) \nabla' \cdot J_c' \right) + d_i \left( \left( \delta \rho J_c \cdot \nu_A' \right) \nabla \cdot J_A - \left( \delta \rho J_c' \cdot \nu_A \right) \nabla' \cdot J_A' \right).$$

(18)

where $\varepsilon_{\text{MHD}}$ is given by equation (9), $d_i$ is the ion inertial length, and $J = \rho J_c$ is the current density in Alfvénic units. Note that in the CGL-Hall-MHD the pressure equations do not write in a conservative form as those of the CGL-MHD (see equation (11)) [50]. This prevents us from obtaining a reduced form of the exact law for the CGL-Hall-MHD as that of the CGL-MHD. Nevertheless, the exact law [18] is applicable to any CGL-Hall-MHD simulation data since the closure equations of the latter are compatible with the internal energy (equation (4)) used to derive the law [18] above.

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