Alternative Technique for ”Complex” Spectra Analysis

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The choice of a suitable random matrix model of a complex system is very sensitive to the nature of its complexity. The statistical spectral analysis of various complex systems requires, therefore, a thorough probing of a wide range of random matrix ensembles which is not an easy task. It is highly desirable, if possible, to identify a common mathematical structure among all the ensembles and analyze it to gain information about the ensemble- properties. Our successful search in this direction leads to Calogero Hamiltonian, a one-dimensional quantum Hamiltonian with inverse-square interaction, as the common base. This is because both, the eigenvalues of the ensembles, and, a general state of Calogero Hamiltonian, evolve in an analogous way for arbitrary initial conditions. The varying nature of the complexity is reflected in different form of the evolution parameter in each case. A complete investigation of Calogero Hamiltonian can then help us in the spectral analysis of complex systems.

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Recent statistical studies in various branches of theoretical physics, ranging from Calogero model of 1-d fermionic system, random matrix (RM) model of disordered systems, matrix models of random surfaces to non-linear sigma model of quantum chaotic systems have revealed the presence of a common mathematical structure. The connecting-web of these various models with each other is well-described in. However, so far, the connection of RM model with other models was established only for standard Gaussian ensembles (SGE), that is, Gaussian ensembles invariant under unitary transformation. This was achieved by showing that distribution of the eigenvalues of the ensemble is governed by a Fokker-Planck (F-P) equation similar to that of Dyson’s ”Brownian” motion model. Through the reduction of F-P equation to the Schrodinger equation, the latter model is already known to be connected to Calogero Hamiltonian and thereby to various other models. In this paper, we explore RM models with non-invariant distributions, and, following the same route as in the case of SGE, connect them to Calogero Hamiltonian. This gives us a new technique to analyze the spectral behavior of the quantum operators of complex systems.

The connection between Complex systems and Calogero Hamiltonian seems to be wide-ranging. The eigenvalue dynamics of Hermitian operators, for example, Hamiltonians of complex quantum systems such as chaotic systems, disordered systems seems to have an intimate connection with the particle-dynamics of Calogero-Moser (CM) Hamiltonian. The latter describes the dynamics of particles interacting via pairwise inverse square interaction and confined to move along a real line.

\[
\hat{H} = -\sum_i \frac{\partial^2}{\partial \mu_i^2} + \frac{1}{4} \sum_{i<j} \frac{\beta(\beta - 2)}{(\mu_i - \mu_j)^2} - \sum_i V(\mu_i)
\]  

(1)

here \(\mu_i\) is the position of the \(i^{th}\) particle and \(V(\mu_i)\) is the confining potential. Similarly the level-dynamics of the unitary operators e.g. time-evolution operator is connected to Calogero-Sutherland (CS) Hamiltonian.

\[
\hat{H} = -\sum_i \frac{\partial^2}{\partial \mu_i^2} + \frac{\beta(\beta - 2)}{16} \sum_{i\neq j} \csc^2 \left( \frac{\mu_i - \mu_j}{2} \right) - \frac{\beta^2}{48} N(N^2 - 1)
\]  

(2)

where particles are confined to move in a circle thus mimicking the similar confinement of eigenvalues due to unitary nature of the operator. The morphological transition caused by the interacting steps on a miscut crystal surface can also be modeled by the CS Hamiltonian. Here the complexity is thermodynamic in nature. It is already well-known that the parametric dispersion of the eigenvalues of the quantum system, with non-integrable classical limit, is described by a set of equations similar to the equations of motions of particles, in time, of classical Calogero Hamiltonian. This analogy extends also to the statistical properties in the two cases. The parametric-evolution of the distribution \(P_{N\beta}(\epsilon_1, \ldots, \epsilon_N; \tau)\) of the eigenvalues \(\epsilon_i\) of a Hamiltonian \(\hat{H} = H_0 + \tau H_1\) (of size \(N\), with perturbation \(H_1\) taken from a SGE corresponds to the time-evolution of the distribution \(P_{N\beta}(r_1, \ldots, r_N; t)\) of positions \(r_i's\) of the particles and both the static as well as dynamical correlators of the eigenvalues turn out to be similar to those of the particles in CM Hamiltonian (with \(V(\mu) \propto \mu^2\) in eq.(1)) Here \(\beta\) refers to the generic symmetry-class of the complex systems and therefore the transformation properties of associated RM models (known as GOE, GUE and GSE for \(\beta = 1, 2\) and 4). In limit \(\tau \to \infty\), the eigenvalues attain an equilibrium distribution, known as Wigner-Dyson, which coincides with the probability distribution of \(N\)-particle coordinates \(P_{N\beta}(r; t \to \infty)\) of the
ground state of the CM Hamiltonian \[H_0\]. Similar analogies can also be made between evolution of the eigenvalues of unitary operators \(U = U_0e^{i\tau M}\), with \(M\) taken from SGE, and CS Hamiltonian \[H\]. This is equivalent to saying that the stationary and non-stationary states of CSM Hamiltonian correspond to the eigenvalue distribution of the systems subjected to random perturbations, strong (\(\tau \to \infty\)) and weak (finite \(\tau\)) respectively, and thereby to equilibrium and non-equilibrium distribution of SGE. In this paper, we indicate towards a novel connection between the CM and RM model: a non-stationary state (finite \(t\)) of CM Hamiltonian can also be mapped to the eigenvalue distribution of a generalized Gaussian ensemble (GGE); the correspondence is established by identifying a parameter \(Y\) for GGE, equivalent to time \(t\) for CM Hamiltonian. This mapping can then be used to obtain the information about various spectral properties of GGEs.

In recent past, RM ensembles have been quite often used to model the physical systems with complicated interactions \[\[12\]\]. The logic which could be given in support of the model is that the missing information about the detailed nature of the interactions can be mimicked by randomizing the associated generators of motion, that is, by taking their matrix-representations as random matrices. However as the specific details of the complexity of an operator should be reflected in the associated RM model, the distribution of the matrix elements can be of various types. For example, for a Hamiltonian with chaotic classical limit (least predictability of the long-term dynamics), the distribution can be chosen as Gaussian (the least information ensemble), with distribution parameters to be determined by the associated quantum dynamics. The corresponding RM model will thus belong to a generalized Gaussian ensemble with matrix elements distribution given by \(P(H) \propto e^{-f_1(H)f_2(H)}\) (with \(f_1\) and \(f_2\) arbitrary functions and \(H\) as a typical matrix).

The SGEs, with matrix elements distribution given by \(P(H) \propto e^{-\tau H^2}\) are special cases of GGEs and many of their properties are already known. The various features of GGEs have, however, remained unknown so far. The purpose of this paper is to suggest a technique to fill in this gap in our information. As for SGE, the nature of matrix elements in GGE too depends on the exact symmetry conditions of the Hamiltonian and is again indicated by parameter \(\beta\), with \(\beta = 1, 2, 4\) for a generic matrix element to be real, complex or quaternion \[\[12\]\]. Here we discuss, in detail, the properties of GG ensemble of complex hermitian matrices \(\beta = 2\); the GG ensemble of real-symmetric matrices \(\beta = 1\) has been discussed elsewhere \[\[14\]\]. We also probe briefly the non-Gaussian ensembles which can serve as good models for complex systems with various conditions on the associated quantum dynamics.

We proceed as follows. Our technique is based on the statistical evolution of the eigenvalues of a GG ensemble with respect to a change in their distribution parameters. This requires a prior information about the effect of a small change in the matrix element on eigenvalues and eigenvectors; the related study is given in section I.A. These results are then used to obtain, as described in section I.B, the distribution of eigenvalues \(P(\mu, Y)\) of a matrix \(H\) taken from a Gaussian ensemble, non-invariant under unitary transformation. The evolution of the eigenvalues is governed by a partial differential equation which, after certain parametric redefinitions, turns out to be formally the same as the F-P equation for the Brownian motion of particles in Wigner-Dyson (WD) gas \[\[12\]\]. The section II contains the details of the reduction of the F-P equation to the Schrödinger equation of CM Hamiltonian and a mapping of their respective correlators. The section III deals with the application of our technique to some important physical processes e.g. localization and a brief discussion of our technique applied to a few other important matrix ensembles is given in section IV.. We conclude in section VI which is followed by the appendices containing the proofs of some of the results given in the main text of the paper.

I. EIGENVALUE DISTRIBUTION OF GENERALIZED GAUSSIAN ENSEMBLES

A. The Change of Eigenvalues and Eigenfunctions

The eigenvalue equation of a complex Hermitian matrix \(H\) is given by \(HU = U\Lambda\) with \(\Lambda\) as the matrix of eigenvalues \(\lambda_n\) and \(U\) as the eigenvector matrix, unitary in nature. As obvious, a slight variation of the matrix elements of \(H\) will, in general, lead to variation of both the eigenvalues as well as the eigenvectors and associated rates of change can be obtained as follows;

As \(\lambda_n = \sum_{i,j} U_{ni}H_{ij}U_{nj}^*\), the rate of change of \(\lambda_n\) with respect to \(H_{kl}\) (with \(s\) referring to real, \(s = 1\), and imaginary, \(s = 2\), parts of \(H_{kl}\)) can be given

\[
\frac{\partial \lambda_n}{\partial H_{kl,s}} = \frac{\partial H_{kl,s}}{\partial H_{kl,s}} \frac{\partial \lambda_n}{\partial H_{kl,s}} = \frac{i^{s-1}}{g_{kl}} [U_{ln}U_{kn}^* - (-1)^s U_{ln}^* U_{kn}].
\]

where \(g_{kl} = 1 + \delta_{kl}\). This can further be used to obtain the following relations (Appendix A)

\[
\sum_{k \leq l} \sum_{s=1}^{2} \frac{\partial \lambda_n}{\partial H_{kl,s}} H_{kl,s} = \sum_{k,l} H_{kl} U_{ln} U_{kn}^* = \lambda_n
\]
and
\[
\sum_{k \leq l} g_{kl} \sum_{s=1}^{2} \frac{\partial \lambda_s}{\partial H_{kl,s}} \frac{\partial \lambda_m}{\partial H_{kl,s}} = 2 \delta_{mn}
\] (5)

For our analysis later, we also require the information about the second order change of an eigenvalue with respect to a matrix element and, therefore, the rate of change of one of the eigenvector components with respect to \( H_{kl} \). This is given as follows (Appendix B),
\[
\frac{\partial U_{pm}}{\partial H_{kl,s}} = i \delta_{il} \sum_{m \neq n} \frac{1}{\lambda_n - \lambda_m} U_{pm} (U_{kn}^* U_{ln} + (-1)^s U_{ln}^* U_{kn})
\] (6)

and now by using eqs.(3,6), One can show that (Appendix C)
\[
\sum_{k \leq l} g_{kl} \sum_{s=1}^{2} \frac{\partial^2 \lambda_s}{\partial H^2_{kl,s}} = 4 \sum_{m} \frac{1}{\lambda_n - \lambda_m}
\] (7)

For the real-symmetric case, the corresponding relations can be obtained by using \( U^+ = U^T \) (as eigenvector matrix is now orthogonal) in eqs.(3-7) and taking \( H_{ij;2} = 0 \) for all values of \( i, j \) (see [8]).

B. The Evolution Equation For the Eigenvalues

Let us consider an ensemble of complex Hermitian matrices \( H \), with matrix elements \( H_{kl} = H_{kl;1} + i H_{kl;2} (1 - \delta_{kl}) \) distributed as Gaussians with arbitrary variances and mean-values; the variances of real and imaginary parts of a single matrix element also need not be same. Thus we choose the distribution \( \rho(H) \) of matrix \( H \) to be following:
\[
\rho(H, y, b) = C \exp \left( - \sum_{s=1}^{2} \sum_{k \leq l} \alpha_{kl,s} (H_{kl,s} - b_{kl,s})^2 \right)
\] (8)

with \( C = \prod_{k \leq l} \prod_{s=1}^{2} \frac{\sqrt{\alpha_{kl,s}}}{\pi} \) as the normalization constant, \( y \) as the set of the coefficients \( y_{kl,s} = \alpha_{kl,s} g_{kl} = \frac{\alpha_{kl,s}}{\sum_{m} \alpha_{kl,m}} \) and \( b \) as the set of all \( b_{kl,s} \). Note that such a choice leads to a non-random complex Hamiltonian \( (H_{kl} = b_{kl;1} + b_{kl;2} \alpha_{kl;2}) \) in limit \( \alpha_{kl;1}, \alpha_{kl;2} \to \infty \) and therefore can model various real physical situations such as switching of disorder in a non-random Hamiltonian e.g. metal-insulator transitions.

Let \( P(\mu, y, b) \) be the probability of finding eigenvalues \( \lambda_i \) of \( H \) between \( \mu_i \) and \( \mu_i + d\mu_i \) at a given \( y \) and \( b \),
\[
P(\mu, y, b) = \int \prod_{i=1}^{N} \delta(\mu_i - \lambda_i) \rho(H, y, b) dH
\] (9)

As the \( \alpha \)-dependence of \( P \) in eq.(9) enters only through \( \rho(H) \) and \( \frac{\partial \rho}{\partial \alpha_{kl,s}} = \left[ (2\alpha_{kl,s})^{-1} - (H_{kl,s} - b_{kl,s})^2 \right] \rho = (2\alpha_{kl,s})^{-1} \left[ \rho + (H_{kl,s} - b_{kl,s}) \frac{\partial \rho}{\partial H_{kl,s}} \right] \) with \( \frac{\partial \rho}{\partial b_{kl,s}} = -\frac{\partial \rho}{\partial H_{kl,s}} \), a derivative of \( P \) with respect to \( \alpha_{kl,s} \) can be written as follows
\[
\frac{\partial P}{\partial \alpha_{kl,s}} = \frac{P}{2\alpha_{kl,s}} + \frac{1}{2\alpha_{kl,s}} \int \prod_{i=1}^{N} \delta(\mu_i - \lambda_i) \frac{\partial \rho}{\partial H_{kl,s}} dH + \frac{1}{2\alpha_{kl,s}} \int \prod_{i=1}^{N} \delta(\mu_i - \lambda_i) \frac{\partial \rho}{\partial b_{kl,s}} dH
\] (10)

The second integral in eq.(10) is equal to \( b_{kl,s} \frac{\partial P}{\partial b_{kl,s}} \). The first integral can also be simplified by using integration by parts followed by a use of the equality \( \prod_{i=1}^{N} \delta(\mu_i - \lambda_i) \frac{\partial \rho}{\partial H_{kl,s}} = -\sum_{n=1}^{N} \frac{\partial}{\partial \mu_n} \prod_{i=1}^{N} \delta(\mu_i - \lambda_i) \frac{\partial \rho}{\partial H_{kl,s}} \)
\[
\int \prod_{i=1}^{N} \delta(\mu_i - \lambda_i) H_{kl,s} \frac{\partial \rho}{\partial H_{kl,s}} dH = - \int \prod_{i=1}^{N} \delta(\mu_i - \lambda_i) H_{kl,s} \rho dH - \int \prod_{i=1}^{N} \delta(\mu_i - \lambda_i) \rho dH = I_{kl,s} - P
\] (11)
where

$$I_{kl:s} = \sum_{n=1}^{N} \frac{\partial}{\partial \mu_n} \int \prod_{i=1}^{N} \delta(\mu_i - \lambda_i) \frac{\partial \lambda_n}{\partial H_{kl:s}} \rho \ dH$$  \hspace{1cm} (12)$$

Substitution of eq.(11) in eq.(10) then gives

$$2\alpha_{kl:s} \frac{\partial P}{\partial \alpha_{kl:s}} = I_{kl} + b_{kl:s} \frac{\partial P}{\partial \theta_{kl:s}}$$  \hspace{1cm} (13)$$

Our aim is to find a function $Y$ of the coefficients $\alpha_{kl:s}$'s and $b_{kl:s}$'s such that the evolution of $P(\mu, Y)$ in terms of $Y$ satisfies a F-P equation similar to that of Dyson's Brownian motion model (Wigner-Dyson gas) [5,12]. For this purpose, we consider the sum $2 \sum_{k \leq t} (\gamma - g_{kl} \alpha_{kl:s}) \alpha_{kl:s} \frac{\partial P}{\partial \alpha_{kl:s}}$ where $\gamma$ is an arbitrary parameter and thereby obtain following relation

$$2 \sum_{s=1}^{2} \sum_{k \leq t} (\gamma - y_{kl:s}) \left[ 2 \frac{\partial y_{kl:s}}{\partial y_{kl:s}} - b_{kl:s} \frac{\partial y_{kl:s}}{\partial \theta_{kl:s}} \right] = 2 \sum_{s=1}^{2} \left[ \gamma \sum_{k \leq t} I_{kl:s} - \sum_{k \leq t} y_{kl:s} I_{kl:s} \right]$$  \hspace{1cm} (14)$$

As shown in Appendix D, the first term on the right hand side of eq.(14) can further be simplified,

$$\sum_{s=1}^{2} \sum_{k \leq t} I_{kl:s} = \sum_{n} \frac{\partial}{\partial \mu_n} (\mu_n P)$$  \hspace{1cm} (15)$$

The second term can similarly be rewritten as follows (Appendix E):

$$\sum_{s} \sum_{k \leq t} y_{kl:s} I_{kl:s} = - \sum_{n} \frac{\partial}{\partial \mu_n} \left[ \frac{\partial P}{\partial \mu_n} + \sum_{m \neq n} \frac{\beta}{\mu_m - \mu_n} \right] P - \sum_{k \leq t} b_{kl:s} \frac{\partial P}{\partial \theta_{kl:s}}$$  \hspace{1cm} (16)$$

where $\beta = 2$. Using both the equalities (15) and (16) in eq.(14), we obtain the desired F-P equation

$$\frac{\partial P}{\partial Y} = \gamma \sum_{n} \frac{\partial}{\partial \mu_n} (\mu_n P) + \sum_{n} \frac{\partial}{\partial \mu_n} \left[ \frac{\partial P}{\partial \mu_n} + \sum_{m \neq n} \frac{\beta}{\mu_m - \mu_n} \right] P$$  \hspace{1cm} (17)$$

Here the left hand side of above equation, summing over all $y_{kl:s}$ and $b_{kl:s}$, has been rewritten as $\frac{\partial P}{\partial Y}$ with $Y$ given by the condition that

$$\frac{\partial P}{\partial Y} = 2 \sum_{s} \sum_{k \leq t} y_{kl:s} (\gamma - y_{kl:s}) \frac{\partial P}{\partial y_{kl:s}} - \gamma \sum_{s} \sum_{k \leq t} b_{kl:s} \frac{\partial P}{\partial \theta_{kl:s}}.$$  \hspace{1cm} (18)$$

By using the orthogonality of eigenvectors and following the same steps, it can be proved for real-symmetric case too (now $\beta = 1$) [14]. It is worth noting that the eq.(17) is same as the evolution equation for the eigenvalues of Brownian ensembles. It is also similar to the one governing the transitions between any two universality classes of SGE caused by a random perturbation of strength $\tau$ (with $\tau \to Y$) [34].

**C. How to Obtain the Complexity parameter $Y$:**

The variable $Y$, a function of relative values of the coefficients $\alpha_{kl:s}$'s and $b_{kl:s}$'s, is a measure of the degree and nature of the complexity of a system and can therefore be referred as the "complexity parameter". For the case discussed here (eq.(18)), $Y$ can be obtained by the following method.

We define $M = 2N^2$ variables $(Y_{11},..,Y_{M})$ as the functions of all $y_{kl:s}$'s and $b_{kl:s}$'s such that the condition given by eq.(18) (where $Y \equiv Y_1$) is satisfied. This is indeed possible by using the orthogonal (Jacobi) coordinate transformation between variables \( Y_i \) for $i = 1,...,M$ and $\{y_{kl:s}, b_{kl:s}\} \leq k, l, \leq N; s = 1,2$ defined by following rule,
\[ Y_i = \sum_{j=1}^{M} a_{ij} X_j \quad \text{for} \quad i = 1 \rightarrow M \] (19)

where \( X_j \equiv \frac{1}{2} \text{ln} \frac{|y_{kl;s}}{|y_{kl;s}| - \gamma} + c_j \) for \( j \leq N^2 \) and \( X_j \equiv -\frac{1}{\gamma} \text{ln}|b_{kl;s}| + c_j \) for \( j > N^2 \) with \( c_j \) as arbitrary constants of integration. Here coefficients \( a_{ij} \) must satisfy the relation \( \sum_{j=1}^{M} a_{ij} = \delta_{i1} \) which is a necessary condition for orthogonality but not sufficient to get the right form for \( \frac{\partial}{\partial Y_i} \). With \( D \) being the functional derivative of \( Y_i \)'s with respect to \( X_j \)'s, we also need the elements \( D_{1j} \) of its inverse to be unity. One way to achieve this is to set all adjuncts of the matrix elements \( \frac{\partial}{\partial X_j} \) equal. Now by choosing \( a_{ij} \) also equal, \( a_{ij} = M^{-1} \), we are left with \( M \) conditions for \( a_{ij} \), \( i \neq 1 \), which can easily be fulfilled.

The form of \( Y = \sum_{j} a_{1j} X_j \), fulfilling condition (18), can therefore be given as

\[ Y = \frac{1}{2N^2} \sum_{k \leq l} \sum_{s=1}^{2} \left[ \frac{1}{2} \text{ln} \frac{|y_{kl;s}}{|y_{kl;s} - \gamma}| - \frac{1}{\gamma} \text{ln}|b_{kl;s}| \right] + C \] (20)

with \( C = M^{-1}\sum c_j \).

As obvious, this method is applicable only for the case when the prefactor associated with a derivative of \( P \) with respect to a variable \( r \) in eq.(18) depends only on \( r \) (\( r \) can be any one of the \( y_{kl;s} \) or \( b_{kl;s} \)). Our studies on the ensembles more complicated than eq.(8) show that the prefactors can also depend on a combination of various \( r \) variables. This requires a more general method to obtain \( Y \) which can also be used for the case discussed here (Appendix F).

### D. Determination of \( P(\mu, Y) \)

The eq.(17) describes an evolution of the eigenvalues of GGE due to changing distribution parameters of the ensemble which can be solved, in principle, to obtain \( P(\mu, Y) \) for arbitrarily chosen initial values of the parameters. If the ensemble corresponding to initial set of the parameters is referred as \( H_0 \), an integration over \( H_0 \) would lead to \( P(\mu, Y) \), free of initial conditions. In fact, it can be shown that

\[ P(\mu; Y) = (4\pi Y)^{-N^2/2} \int \exp \left[ -\frac{1}{4Y} tr(\mu - U^+ \mu_0 U)^2 \right] f(\mu_0)|\Delta(\mu_0)|^{\beta_0} d\mu_0 dU \] (21)

where \( \mu_0 \) is the set of eigenvalues of the initial matrix \( H_0 \), with \( \beta_0 \) given by its symmetry conditions, and \( U \) is the integral over unitary (or orthogonal) space of matrices.

To show that eq.(21) is indeed a solution of eq.(17), we study a general case. Consider a partial differential equation for a function \( F(A; t) \) defined in the matrix space of \( N \times N \) Hermitian matrices \( A \)

\[ \frac{\partial F}{\partial t} = [\nabla_A^2 F + \nabla.(AF)] \] (22)

where \( \nabla_A^2 = \sum_i \frac{\partial^2}{\partial A_{ii}^2} + \frac{1}{2} \sum_{i<j} \frac{\partial^2}{\partial A_{ij}^2} \) and \( \nabla.(AF) = \sum_{i<j} \frac{\partial}{\partial A_{ij}} (A_{ij} F) \) (23)

with the initial condition \( F(A; 0) = f(A) \). This equation is known to have a unique solution (see page 174 of [12]),

\[ F(A; t) = \int K(A, B, t) f(B) dB \] (24)

where \( K(A, B, t) = (4\pi t)^{-N^2/2} \exp \left[ -\frac{1}{4t} tr(A - B)^2 \right] \) (25)

where \( B \) is a \( N \times N \) hermitian matrix. Depending on the nature of both \( A \) and \( B \), we can choose a special class of eigenvector matrices \( U_A \) and \( U_B \) (for \( A \) and \( B \) real-symmetric, complex hermitian or symplectic, \( U_A \) and \( U_B \) are orthogonal, unitary and symplectic matrices respectively) such that

\[ A = U_A^* a U_A \quad \text{and} \quad B = U_B^* b U_B \] (26)

where \( a = [a_{ij} \delta_{ij}] \), \( b = [b_{ij} \delta_{ij}] \) are diagonal matrices with \( a_i \) and \( b_i \) as the eigenvalues of \( A \) and \( B \) respectively and \( U^* = U^+ \) or \( U^T \) or \( U^R \) depending on whether \( U \) is an eigenvector matrix for a complex Hermitian, real symmetric
or symplectic matrix \([12]\). Let \(\beta_A\) and \(\beta_B\) give the number of components of a typical matrix elements in \(A\) and \(B\) respectively. Changing the variables from matrix elements to the \(N\) eigenvalues and \(\beta N (N - 1)/2\) angle (i.e. eigenvector) parameters on which \(U_B\) depends, we have

\[
dB = |\Delta(b)|^{\beta_b} \, db \, dU_B \quad \text{with} \quad \Delta(b) = \prod_{i \leq j} (b_i - b_j) \tag{27}
\]

The substitution of these relations in eq.(24) gives us

\[
F(A; t) = (2\pi t)^{-N^2/2} \int \exp \left[ -\frac{1}{2t} tr(a - U^*bU)^2 \right] f(b, U_B) |\Delta(b)|^{\beta_b} db dU_B \tag{28}
\]

where \(U = U_B U_A^*\) and \(U^* = U_A U_B^*\). Now if \(f(b, U_B)\) is independent of \(U_B\) then \(F(A; t)\) is also independent of \(U_A\). This helps us to rewrite the eq.(28) as follows,

\[
F(a; t) = (4\pi t)^{-N^2/2} \int G(a, b, t) f(b) |\Delta(b)|^{\beta_b} db \tag{29}
\]

where

\[
G(a, b, t) = \int \exp \left[ -\frac{1}{4t} tr(a - U^*bU)^2 \right] dU
\]

Here the integral is over the group \(U\) of orthogonal, unitary and symplectic matrices respectively. Further the Laplacian \(\nabla_A^2\) can also be written in terms of eigenvalues and angle parameters of \(A\) (see appendix A.5 of \([12]\))

\[
\nabla^2(A) = \frac{1}{|\Delta(a)|^{\beta_a}} \sum_i \frac{\partial}{\partial a_i} |\Delta(a)|^{\beta_a} \frac{\partial}{\partial a_i} + \nabla^2_{U_a} \cdot \tag{30}
\]

By the substitution of eq.(30) in eq.(22) and using independence of \(F(a; t)\) of \(U_A\), one can rewrite eq.(22) as follows,

\[
\frac{\partial F(a; t)}{\partial t} = \frac{1}{|\Delta(a)|^{\beta_a}} \sum_i \frac{\partial}{\partial a_i} \left[ |\Delta(a)|^{\beta_a} \frac{\partial F(a; t)}{\partial a_i} \right] + \sum_i \frac{\partial}{\partial a_i} (a_i F) \tag{31}
\]

with \(F(a; t)\) given by eq.(29). Now by using the equality \(\sum_i \frac{\partial^2}{\partial a_i^2} |\Delta(a)|^{\beta_a} = 0\), eq.(31) can be reduced in the following form:

\[
\frac{\partial F}{\partial t} = \sum_i \frac{\partial}{\partial a_i} (a_i F) + \sum_i \frac{\partial}{\partial a_i} \left[ \frac{\partial}{\partial a_i} + \sum_{j \neq i} \frac{\beta_a}{a_j - a_i} \right] F \tag{32}
\]

which is similar to eq.(17) with \(a_i 
rightarrow \mu_i\), \(t \nrightarrow Y\), \(\gamma = 1\) and \(F \nrightarrow P\). The joint probability density \(P\) can therefore be obtained by evaluating the integral (29). However, so far, the integration could be performed only for the unitary group of matrices \([11, 12]\).

E. Steady State, Level Density and Correlations

The steady state of eq.(17), \(P(\mu, \infty) \equiv P_\infty = |\Delta(\mu)|^{\beta} e^{-\pi \sum_i \mu_i^2}\), corresponds to \(Y - Y_0 \nrightarrow \infty\) (with \(Y_0\) as the complexity parameter of initial ensemble) which can be achieved by two ways (for finite \(Y_0\) values). The first is when almost all \(y_{k;1} \nrightarrow \gamma\) and \(y_{k;2} \nrightarrow \infty\) (for finite \(b_{k;1}\) and \(b_{k;2}\) values) which results in a GOE steady state. The second is when almost all \(y_{k;1} \nrightarrow \gamma\), \(y_{k;2} \nrightarrow \gamma\), resulting in a GUE. This indicates that, in the steady state limit, system tends to belong to one of the SGEs. The eq.(17) can, therefore, describe a transition from a given initial ensemble (with \(Y = Y_0\)) to either GOE or GUE with \(Y = Y_0\) as the transition parameter. The non-equilibrium states of this transition, given by non-zero finite values of \(Y - Y_0\), are various Gaussian ensembles corresponding to varying values of the coefficients \(y_{k;1}\) and \(b_{k;2}\) respectively. For example, the choice of the initial ensemble as GOE (almost all \(y_{k;1} = \gamma, y_{k;2} \nrightarrow \infty\) initially) and a decrease of \(y_{k;2}\) (from \(\infty \nrightarrow \gamma\) while keeping \(y_{k;1}\) fixed) leads to GOE \nrightarrow GUE transition with intermediate ensembles as those of complex Hermitian matrices. Similarly Poisson \nrightarrow GUE transition
can be brought about by choice of the initial ensemble as Poisson (almost all $y_{kl;1}, y_{kl;2} \to \infty$ for $k \neq l$, $y_{kk;1} = \gamma$, $y_{kk;2} = \gamma$ and $b_{kl;s} = 0$ for all $k, l, s$ values) and by varying both $y_{kl;1}$ and $y_{kl;2}$ up to $\gamma$. As clear from above, $\gamma$ fixes the variance of the final ensemble and an arbitrariness in $\gamma$ leaves the latter arbitrary. This however does not affect the statistical properties of the intermediate ensembles.

The eq.(21) for $P(\mu, Y)$ can be used to obtain $n$th order density correlator $R_{n}(\mu_{1},..\mu_{n};Y)$, defined by $R_{n} = \frac{N!}{(N-n)!} \int P(\mu,Y)d\mu_{n+1}..d\mu_{N}$. $(R_{n}$ can also be expressed in the form $<\nu(\mu_{1},Y)..\nu(\mu_{n},Y)>$ with $\nu(\mu, Y) = N^{-1}\sum_{j} \delta(\mu - \mu_{j})$ as the density of eigenvalues and $<..>$ implying the ensemble average). Here note that the analogy of eq.(17) to that of Dyson’s Brownian ensembles implies same form of $N_{R}$ (ground state). The F-P operator can now be hermiticized through a similarity transformation $\psi_{0} = \psi$ by $\lambda_{0} = \frac{<\nu(\mu_{1},Y)>}{<\nu(\mu_{2},Y)>}$ ($P_{0} = 1$; mean level spacing) [3],

$$\frac{\partial R_{n}}{\partial \lambda} = \sum_{j} \frac{\partial^{2} R_{n}}{\partial r_{j}^{2}} - \beta \sum_{j \neq k} \frac{\partial}{\partial r_{j}} \left( \frac{R_{n}}{r_{j} - r_{k}} \right) - \beta \sum_{j} \frac{\partial}{\partial r_{j}} \int_{-\infty}^{\infty} \frac{R_{n+1}}{r_{j} - r_{k}}$$

(33)

(here for simplification, $\gamma$ is chosen to be unity). As can be seen from the above equation, the transition for $R_{n}$ occurs on the scales determined by $Y \approx D_{2}$ and a smooth transition can be brought only in terms of the parameter $\lambda$, obtained by rescaling $Y$ by $D_{2}$. On the other hand, for $R_{1}$, the corresponding scale is given by $Y \approx N D_{2}$. This implies, therefore, during the transition in $R_{n}$, the density $R_{1}$ remains nearly unchanged; this fact is very helpful in unfolding the correlators $R_{n}$. For $n = 1$ and in large $N$-limit, above equation reduces to the Dyson-Pastur equation [13] for the level density $<\nu(\mu_{1},Y)> \equiv N^{-1}R_{1}$

$$\frac{\partial <\nu(\mu)>}{\partial Y} = -\beta \frac{\partial}{\partial \mu} \left( \sum_{m} P \int d\mu' <\nu(\mu')> \right) <\nu(\mu)>$$

(34)

which results in a semi-circular form for $\nu$: $\nu(r) = \frac{2}{\pi \eta^{2}}(\eta^{2} - r^{2})^{1/2}$ with $\eta^{2} = 4N(1+Y^{2})$ [7]. The application of super-symmetry (SUSY) technique [3] to ensemble (8) gave a similar result (also see section 4.3 of [12]).

II. CONNECTION TO CALOGERO HAMILTONIAN

A similarity transformation followed by a Wick rotation converts the F-P equation into a self-adjoint form [8]. This can be seen as follows. The F-P equation, in general, can be expressed in a form

$$\frac{\partial |P_{Y}>}{\partial Y} = -P|P_{Y}>$$

(35)

where $P$ is a F-P operator with non-negative eigenvalues. Here $|P_{Y}>$ is a general state of operator $P$ at "time" $Y$ and its projection in eigenvalue space can be obtained by the usual operation $P(\mu, Y) \equiv <\mu|P_{0}>$ (with $\mu$ as set of the eigenvalues). Let $P(\mu, Y_{0}) \equiv <\mu|P_{0}$ be the equilibrium probability. One can further define a vector $<0| = \int d\mu <\mu| .. satisfying $<0|P = 0$ thus implying the conservation of probability in "time" $Y$ in this state (the ground state). The F-P operator can now be hermiticized through a similarity transformation $S^{-1}PS = H$ where $S$ is Hermitian and invertible operator depending only on the eigenvalues. Thus the ground state condition must be given by $HS|0> = 0$ (as $P^{\dagger}|0> = 0$). Let the effect of similarity transformation on the state $|P_{Y}>$ and $|P_{0}$ be expressed by $|\psi_{0}> = S^{-1}|P_{Y}>$ and $|\psi_{0}> = S^{-1}|P_{0}$ respectively. The similarity transformation of eq.(35) will then give the desired form $\frac{\partial |\psi_{0}>}{\partial Y} = -H|\psi_{0}>$; the ground state $|\psi_{0}>$ must now satisfy the condition $H|\psi_{0}> \equiv 0$. The comparison of the two different forms of the ground state condition gives $|\psi_{0}> = S|0>$ and therefore $|P_{0} = S^{-1}|0>$. In the case of F-P equation (17), $H$ turns out to be CM Hamiltonian (eq.(1) with $r_{i} \rightarrow \mu_{i}$) and has well-defined eigenstates and eigenvalues [1][19]. As well-known, the particles in CM system undergo an integrable dynamics, thus implying a similar motion for the eigenvalues too. Here $H$ being a generic member of GGE, this result is valid for all systems with interactions complicated enough to be modeled by GGE.

The "state" $\psi$ or $P(\mu, Y|H_{0})$ can be expressed as a sum over the eigenvalues and eigenfunctions which on integration over the initial ensemble $H_{0}$ leads to the joint probability distribution $P(\mu, Y)$ and thereby static (at a single parameter value) density correlations $R_{n}$. The above correspondence can also be used to map the multi-parametric correlations to multi-time correlations of the of CM Hamiltonian. For example, the parametric correlation $<Q_{a}(Y)Q_{b}(0)>$,
for a classical variable $Q(Y)$ with $|Q, S| = 0$ can be mapped to the corresponding ground state correlation of CM hamiltonain $< \psi_0|Q_a(Y)e^{-YH}Q_b(0)|\psi_0 >$. This follows because

$$< Q_a(Y)Q_b(0) > = \int Q_aQ_bP(\mu, Y)d\mu = \int < \mu|Q_aQ_b|P_Y > d\mu$$

(36)

now as the evolution of $|P_Y >$ with respect to $Y$ is given by $|P_Y >= Se^{-YH}S^{-1}|P_0 >$, one has

$$< Q_a(Y)Q_b(0) >= < 0|Q_aSe^{-YH}S^{-1}Q_b|P_0 >= < \psi_0|Q_ae^{-YH}Q_b|\psi_0 >$$

(37)

III. APPLICATION TO PHYSICAL PROBLEMS

The given ensemble (8), referred here as "G", is represented by a point $Y$ in the parametric-space consisting of distribution parameters and various transition curves may pass through this point. The question therefore arises which curve should be chosen for the studies of the properties of $G$? The answer is the one which does not leave any arbitrariness behind and if there are more than one such curve, each one of them should give same answer for various fluctuation measures of $G$. This criteria for the right choice are based on the symmetry properties of ensemble $G$, that is, the nature of all $\alpha_{kl}$ and $b_{kl}$ with end-points (the final and initial ensemble, referred here as "F" and "O" respectively) chosen in such a way that the values corresponding to $G$ occur during the variation of distribution parameters from one end to the other. Further the chosen transition should preferably be the one whose properties are already known and can therefore tell us about $G$. For many GGE described by eq.(8), above criteria is satisfied by choosing $F$ as a SGE with variance $< F_1^2 >= 2 < F_2^2 >= \gamma^{-1}$, $\gamma \leq \min\{y_{kl}[G]\}$, $k, l = 1, 2, ..., N$, $s = 1, 2$, and O as an ensemble with each $\alpha_{kl}[O]$ given by the maximum value taken by the functional form of the corresponding $\alpha_{kl}[G]$. However, as explained in following examples, O can also be chosen as some other ensemble. For example, If $G$ is an ensemble of real-symmetric matrices $H$ represented by $\rho(H) \propto \exp[-\sum_{k \leq 1} \alpha_{kl}H^2_{kl}]$ with finite but different values for all $\alpha_{kl}$, the Poisson $\rightarrow$ GOE curve is more suitable for its study rather than GOE $\rightarrow$ GUE. Here the GOE ensemble is described by $< F_{11}^2 >= 2 < F_{21}^2 >= \gamma^{-1}$ with $\gamma$ as the minimum value among given $y_{kl}[G]$. However if various $\alpha_{kl}$ in the above example can also take complex values, the ensemble can now be chosen on any one of the curves, namely, Poisson $\rightarrow$ GUE or GOE $\rightarrow$ GUE. Here now GUE can be chosen as $< O_{11}^2 >= 2 < O_{ij}^2 >= 2 < O_{ij}^2 >= \gamma^{-1}$.

The GOE for the second curve can be chosen as the one with $< O_{11}^2 >= 2 < O_{ij}^2 >= q^{-1}$ and $< O_{ij}^2 >= 0$ with $q = \max\{y_{ij}[G]\}$. Similarly, for Poisson $\rightarrow$ GUE curve, the initial ensemble may be taken as one with $< O_{1i}^2 >= \gamma^{-1}$ (or $q^{-1}$) and $< O_{ij}^2 >= 0$ for $i \neq j$. The reason for the choice of the two transitions is due to availability of the results for their 2-point correlation $R_2$ [3].

For Poisson $\rightarrow$ GUE

$$R_2(r; \Lambda) - R_2(r; \infty) = \frac{4}{\pi} \int_0^\infty dx \int_{-1}^1 dz \cos(2\pi rx) \exp[-8\pi^2 \Lambda x(1 + x + 2z\sqrt{x})] \left( \frac{\sqrt{(1 - z^2)(1 + 2z\sqrt{x})}}{1 + x + 2z\sqrt{x}} \right)$$

(38)

and for GOE $\rightarrow$ GUE

$$R_2(r; \Lambda) - R_2(r; \infty) = -\frac{1}{\pi^2} \int_0^\pi dx \int_0^\infty dz \sin(rx) \exp[2\Lambda(x^2 - y^2)] \frac{\sin(yr)}{y}$$

(39)

where $R_2(r, \infty) = 1 - \frac{\sin^2(\pi r)}{(\pi r)^2}$ (the GUE limit).

It is obvious therefore that if $\Lambda_1$ and $\Lambda_2$ are the parameter values for the ensemble "G" on Poisson $\rightarrow$ GUE and GOE $\rightarrow$ GUE curves respectively, one should have $R_{2,P\rightarrow U}(r; \Lambda = \Lambda_1) = R_{2,O\rightarrow U}(r; \Lambda = \Lambda_2)$. This would require an intersection of two curves in the $R_2 - \Lambda$ space which however is possible. This is because the GOE can occur as an intermediate point in Poisson $\rightarrow$ GUE transition. The GOE $\rightarrow$ GUE curve can also appear as a part of the Poisson $\rightarrow$ GUE curve; thus the choice of two different initial ensembles here corresponds only to two different origins of dynamics on the same curve.

The parameter $\gamma$, which determines $Y$ as well as the variances of $F$, enters in calculation at step given by eq.(14) and can be chosen arbitrarily. As suggested by eq.(17), the choice of different $\gamma$-values corresponds to different $Y$-values as well as the transition curves with end-points of same nature but different variances; this, however, would not imply different properties for the ensemble $G$ (Appendix G). Similarly the F-P equation is although independent of the choice of the initial ensemble, the latter is required for determination of the correlations of $G$. The possibility of an
arbitrary choice of $\Omega$ may seem to imply a certain arbitrariness left in the correlation of $G$. However the choice of two different initial ensembles corresponds only to the two different origins of the dynamics approaching to the same point in the parametric space.

It will be clarified by the examples given below.

### A. Anderson Transition

Using above method, the transition parameter for a metal-insulator transition as a result of increasing disorder can exactly be calculated. To see this, let us consider the case of a d-dimensional disordered lattice, of size $L$, in tight-binding approximation. Here, in the configuration space representation of the Hamiltonian, a $N \times N$ matrix of size $N = L^d$, the diagonal matrix elements will be site-energies $\epsilon_i$. The hopping is generally assumed to connect only the $z$ nearest-neighbors with amplitude $t$ so that the electron kinetic energy spread or bandwidth is $zt$. This therefore results in sparse form of the matrix $H$. We first consider the case of $L \rightarrow D$ transition brought about by decreasing diagonal disorder only. In this case, site-energies $\epsilon_i$ are taken to be independent random variables with probability-density $p(\epsilon_i)$. In the Anderson model \(^1\) of metal-insulator (MI) transition, $p(\epsilon)$ was taken to be a constant $W^{-1}$ between $-W/2$ to $W/2$. Various physical arguments and approximations used in this case led to conclusion that here all the states are localized for $W > 4Kt\ln\left(\frac{W}{2}\right)$ with $K$ as a function of $z$ and $d$.

However, as well-known now, MI transition does not depend on nature of $p(\epsilon)$ and latter can also be chosen as Gaussian; the type of $p(\epsilon)$ affects only the critical point of the transition. The $p(H)$, for any intermediate state of MI transition brought about by diagonal disorder, can therefore be chosen as in eq.(8) with $\alpha_{kl} \to \infty$, $b_{kl} = -t$ for $k \neq l$, $\alpha_{kk} = \alpha$ and $b_{kk} = 0$ for all $k$ which results in $Y = \frac{1}{N^2} \left[ \frac{N}{2} \ln \frac{2\alpha}{\alpha - \gamma} - \gamma^{-1}K\ln t \right] + C$. Here $K$ is total number of the sites connected and depends on the dimensionality $d$ of the system. The system can initially be considered in an insulator regime where all the eigenvectors become localized on individual sites of the lattice (strong disorder limit). This results in a diagonal form of the matrix $H$ with the eigenvalues independent from each other. The insulator limit can therefore be modeled by ensemble (8) with $\alpha_{kl} \to \infty$ for $k \neq l$, $\alpha_{kk} = \alpha_0$ (for all $k$-values) and $b_{kl} \to 0$ (for all $k,l$), giving, $Y_0 = \frac{1}{N^2} \ln \frac{2\alpha_0}{\alpha_0 - \gamma} + C$ (as $K=0$ in the insulator regime). The decrease of the diagonal disorder, that is, an increase of $\alpha_{kk}$ from $\alpha_0$ to some finite values (while $\alpha_{kl}$, $k \neq l$, remains infinite throughout the transition) will ultimately lead to metal regime with fully delocalized wavefunctions. The eigenvalue distribution of $H$ in the regime can be well-modeled by the SGE; let it be described by $\alpha_M(>\alpha_0)$. Thus for the study of transition in this case we should choose $\gamma = 2\alpha_M$. The transition parameter can now be given as follows, with the mean level spacing $D \propto \frac{1}{\sqrt{N}}$,

$$\Lambda = \frac{Y - Y_0}{D^2} = \frac{1}{4} \left[ \ln \frac{\alpha}{\alpha_0} - \frac{\alpha}{\alpha_0(\alpha - \gamma)} - \frac{K}{N\alpha_M} \ln t \right]$$

As obvious from the above, the transition is governed by relative values of the disorder and the hopping. Here $\Lambda \to 0$ leads to fully localized regime which corresponds to following condition on $\alpha$ and $t$

$$\ln \frac{\alpha}{\alpha_0} + \frac{\alpha - \alpha_0}{\alpha_M} = \frac{K}{N\alpha_M} \ln t$$  \hspace{1cm} (41)

The eq.(40) gives, therefore, the condition for the critical region or mobility edge ($\frac{K}{N\alpha_M} \to \text{finite as } N \rightarrow \infty$). As $\frac{\alpha - \alpha_0}{\alpha_M} \ll 1$ even for large $\alpha$-values, the condition is always satisfied if $\frac{K}{N\alpha_M} \to 0$. This explains the localization of all the states in infinitely long wires (or strictly 1-d systems where $K << N$) even for very weak disorder. With increasing dimensionality $d$, connectivity $K$ of the lattice and thereby the possibility of $|\alpha| >> 0$ and the delocalized states increases. The $\Lambda$ can similarly be calculated when off-diagonal disorder is also present.

### B. 1-D, Quasi 1-D, Periodic 1-D disordered and Chaotic systems

In 1-D geometry of a solid state system e.g a chain of $N$ interacting sites, in tight binding approximation, the long-range random hopping leads to a banded structure of the matrix, known as random banded matrix (RBM) \(^2\)\(^3\). Here the effectively non-zero, randomly distributed, matrix elements are confined within a band with its width governed by the range of hopping. The 1-D periodic geometry e.g. a chain of interacting sites joined into a ring leads to a periodic RBM in which all non-zero matrix elements belong to three regions: a band along the
main diagonal, the upper right corner and the lower left one \(\Gamma\). A real disordered wire has finite cross-section (referred as quasi 1-D geometry) and therefore allows for propagating modes with different transverse quantization numbers frequently referred as transverse channels. This case can again be modeled by RBMs with band-width given by number of transverse channels \(23,22\). In the case of dynamical systems too, exhibiting strong chaos in classical limit, the generic structure of Hamiltonian matrix in some basis is banded and matrix elements can be assumed to be pseudo-random \(24\). For example, the Hamiltonian of quantum kicked rotor turns out to be a RBM in momentum basis \(21\).

In all these cases, nature of the disorder or associated randomness decides the nature of the distribution of the matrix elements. The physical properties of such systems can therefore be analyzed by studying the distribution of the eigenvalues of associated RBMs. The most studied type of RBM is that with the zero mean value of all matrix elements and variance given by \(\langle H_{nm}^2 \rangle = v^2 a |n-m|/b\) where \(a(r)\) is some function satisfying the condition \(\lim_{r \to \infty} a(r) = 0\) and determines the shape of the band \(22\). For large but finite size of the matrix \(N >> b >> 1\), its statistical properties were shown (by SUSY method) to be determined by the scaling parameter \(b^2/N\) with the transition parameter scaling as \(N f(\frac{b}{N}) \approx 1\).

The transition parameter for the RBM can also be calculated by our method. Let us first consider the simplest case i.e. Rosenzweig-Porter where all the off-diagonal matrix elements are distributed with same variance which is different from the diagonal ones. Let us take \(\alpha_{ij} G = 2(1 + \mu)\) and \(\alpha_{ii} G = 1\) with \(\mu \geq 0\); thus \(\min \{y_{ij} G\} = 2\) and we can choose \(\gamma = 2\). This GGE can therefore be mapped to a Brownian ensemble, with \(Y - Y_0 = -\frac{N-1}{4N} \ln |1 - \frac{1}{1+\mu}| \approx \frac{1}{4\mu}\) for \(\mu > 1\), appearing in a Poisson \(\to\) GOE transition where the initial matrix elements distribution is given by \(P(H_0) \propto e^{-\sum H_i^2}\) and the final, stationary state, obtained for large \(\Lambda\), values, is \(P(H) \propto e^{-\frac{1}{2} Tr H^2}\). Now as \(R_1 \approx \sqrt{N} \[21,25\], the \(D^2 \approx 1/N\) and therefore \(\Lambda \approx \frac{N}{\mu}\) which implies that the GGE will have an eigenvalue statistics very different from that of Poisson or GOE only if \(\mu \approx cN\) (c a finite constant). For \(\mu > cN\), \(\Lambda \to 0\) and for \(\mu < cN\), \(\Lambda \to \infty\) for \(N \to \infty\) and thus the GGE behaves like a Poisson ensemble in the first case and like a GOE in the second; this result is in agreement with the one obtained, in \(24\), by using NLSM technique. (Note in ref. \[23\], \(D\) is taken as \(D \propto 1/N\), which gives \(\Lambda \approx \frac{N}{2\pi}\) and therefore GOE and Poisson ensemble result for \(\mu < cN^2\) and \(\mu > cN^2\) respectively).

Consider the ensemble with exponential decay of the variances away from the diagonal i.e \(\alpha_{kl} = e^{|k-l|/b}, k \leq l, 1 << b << N\). Thus, again \(\gamma = 2\) and the final ensemble is a SGE with \(P(H) = e^{-\frac{1}{2} Tr H^2}\) and therefore \(Y = -\frac{1}{4N} \sum_{i \leq j=1}^{N} \ln |1 - \gamma y_{ij}^{-1}| + C\). Here the initial ensemble is that of the diagonal matrices with a Poisson distribution of the eigenvalues which corresponds to \(y_{ii}[O] = 2\) and \(y_{ij;i \neq j}[O] \to \infty\) (this being maximum value of \(y_{kl}[G]\) giving \(Y_0 = -\frac{1}{4N} \sum_{i=1}^{N} \ln |1 - \gamma y_{ii}^{-1}[O]| + C\). Thus \(Y - Y_0 = -\frac{1}{4N} \sum_{r=1}^{N} (N - r) \ln |1 - 2e^{-r/b}| - \frac{1}{4\mu}\). As \(R_1 \approx \sqrt{N}\), the transition parameter for infinite system \(N \to \infty\) turn out to be \(\Lambda = Y_0/2\) (see \[24\]) which reconfirms that, in infinite systems, the transition is governed only by the band-width \(b \[22,21\].

Another case of importance is the ensemble with power law decay of variances \(H_{ij} = \tilde{G}_{ij} a(|i-j|)\) with \(\tilde{G}\) a typical member of SGE \((\tilde{G}_0^2 > = 2 < \tilde{G}_0^\sigma > = v^2)\) and \(a(r) = 1\) and \((b/\sigma)^\sigma\) for \(r \leq b\) and \(r > b\) \((b >> 1)\) respectively (known as PRBM model with P stands for power) \[29\]. This corresponds to \(y_{ij} = \frac{1}{v^2 a^2(|i-j|)}\) and therefore \(\gamma = \min \{y_{ij}\} = \frac{1}{\sigma^2}\). Again as for the exponential case, the choice of initial and the final ensemble remains the same. Now as \(y_{ij} = y_{r} = \gamma(\frac{b}{N})^{2\sigma}\) (with \(r = |i-j|\)), we get

\[
\Lambda = D^{-2}(Y - Y_0) = -\frac{N}{4N} \sum_{r=b+1}^{N} (N-r) \ln \left(1 - \left(\frac{b}{r}\right)^{2\sigma}\right)
\]

\[
\approx \frac{N}{4} \sum_{j=1}^{N} \frac{1}{j} \int_{b/N}^{1} dx (1-x)^{-2j}\sigma = \frac{N}{4} \sum_{j=1}^{N} \frac{1}{j} \left[ \frac{1}{2(1-2\sigma)(1-\sigma)} \left(\frac{b}{N}\right)^{2\sigma} - \frac{1}{(1-2\sigma)N} + \frac{b}{2(1-j\sigma)} \left(\frac{b}{N}\right)^2 \right]
\]

Thus, for large \(N\)-values and \(\sigma < 1/2\), \(\Lambda(\propto N^{1-2\sigma})\) is sufficiently large and the eigenvalue statistics approaches SG limit. At \(\sigma = 1/2\), the statistics is governed by the parameter \(b^2/N\) instead of \(N\) only. For \(\sigma = 1\), the non-zero, finite \(\Lambda\)-value \((\Lambda \propto b\ even\ when\ N \to \infty)\) leads to an eigenvalue statistics intermediate between that of SGE or Poisson. For \(\sigma > 3/2\) with \(N \to \infty\), \(\Lambda \to 0\) therefore, the eigenvalue statistics approaches Poisson limit, \(\Lambda\) being very small. All these results are in agreement with those obtained in \[20\] by SUSY technique.
Another type of RBMs often encountered in atomic and nuclear systems is those with the non-zero mean value of all matrix elements and with variance given by $< H_{ij}^2 > = \nu^2 a(n - m)/b$; the transition parameter for them can also be obtained as for the above cases [23,28,43,24,30].

C. Quantum Hall Case

A Quantum Hall system without disorder has all the states degenerate within a single Landau level. The introduction of disorder leads to a broadening of the levels (also termed as diagonal disorder) as well as random hopping between them (off-diagonal disorder) and a competition between the two leads to a $L \rightarrow D$ transition. Note this is different from the Anderson model where the $L \rightarrow D$ transition is caused by the competition between diagonal disorder and non-random hopping (bandwidth) [20]. The $N \times N$ Hamiltonian matrix in presence of disorder therefore belongs to an ensemble far more complicated than eq.(8), known as random Landau matrix, as now various matrix elements are no longer independently distributed: $\rho(H, y, b) = C \exp[- \sum_{s=1,\ldots,1}^2 \sum_{k,l,k,l} \alpha_{kl}s H_{kl}s - \sum_i' \sum_j' \sum_{i,j} b_{ijkl}s H_{ij}s)]$ with $C$ as the normalization constant and $y$ and $b$ as the sets of inverse of variances $y_{kl}s = \alpha_{kl}s y_{kl}$ and coefficients $b_{ijkl}s$ respectively with $g_{kl} = 1 + \delta_{kl}$. Here $\sum_i' \sum_j' \sum_{i,j} b_{ijkl}s$ will imply that the summation is over all possible pairs of indices $(i, j)$ such that the pair $(i, j) \neq (k, l)$ or $(l, k)$ [31]. In this case too, one can show that the eigenvalue distribution $P$ satisfies eq.(17) but the condition for the determination of $Y$ is no longer given by eq.(18); the details will be presented elsewhere.

D. Critical Ensemble and Multifractality of Eigenvectors

Recent studies of some metal-insulator transitions revealed that the energy level statistics in the critical region is universal and different from both Wigner-Dyson as well as Poisson statistics. The eigenfunctions associated with the critical statistics show multifractal characteristics [22,23]. The level number variance $\Sigma^2(N)$ is believed to be an important indicator of this critical behaviour with its asymptotic linearity in the mean number of levels $N$ [33]: $\Sigma^2(N) = < (\delta N)^2 > = \chi N, \chi < 1$. The critical statistics, therefore, governs the spectral fluctuations that are weaker than for the Poisson statistics ($\Sigma^2(N) = N$) but much stronger than for the Wigner-Dyson statistics, ($\Sigma^2(N) = \ln N$). Later on remarkable similarities were found between the spectral statistics of a number of dynamical systems e.g pseudointegrable billiards and the critical statistics near the mobility edge [30]. However such a critical region being inaccessible either perturbatively or semiclassically, a different tool was required to probe into it. This led to the suggestion of a RM modelling of this region [34]. The $N \times N$ matrices in this model are Hermitian and matrix elements are Gaussian distributed with zero mean and the variance given by

$$< (H_{ij})^2 > = \left[1 + \left(\frac{|i-j|}{B}\right)^{2\sigma}\right]^{-1}$$

(44)

Using SUSY technique, it has been shown [20] that for large $B$-values ($B >> 1$), this ensemble behaves like a SGE for $\sigma < 1$ and as a Poisson for $\sigma > 1$. The case $\sigma = 1$ is believed to be of special relevance as it supports critical statistics and multifractal eigenstates; the application of SUSY technique gives $R_2(r) \approx 1 - \frac{1}{\pi^2} \frac{\sin^2(\pi r)}{\sin^2(\pi r/4)}$ and $\Sigma^2(N) \approx \chi N$ [37,38,34].

The existence of the ensembles with critical statistics is indicated by our technique too. The $N$-dependence of the transition parameter $\Lambda$, entering through $Y$ and the mean level-spacing $D$, causes the transition to reach the equilibrium in limit $N \rightarrow \infty$ for finite, non-zero $Y$-values. In some cases, however, the $N$-dependence of $Y$ may be such that it balances the one due to $D$, thus resulting in an $N$-independent $\Lambda$ (as shown in section III.A,B) and therefore critical statistics. As can be seen from eq.(20), $\Lambda$ for the ensemble, given by eq.(44), is also $N$-independent for $\sigma = 1$; here the ensemble appears as an intermediate point between Poisson $\rightarrow$ GUE transition with $Y - Y_0 = \frac{\Lambda}{2\pi^2} \sum_{r=1}^N (N - r) \ln(1 + (\frac{r}{2})^{2\sigma})$ and $\Lambda$ behaves as in the case of PRBM model discussed above, showing criticality for $\sigma = 1$. The correlation $R_2$ for the ensemble (44) can therefore be given by eq.(38) which for large $\Lambda$-values (for all $r$), can be approximated as follows [40,38]:

$$R_2(r, \Lambda) = 1 + \frac{\Lambda}{\pi^2\Lambda^2 + r^2} + \frac{1}{2\pi^2r^2}[\cos(2\pi r)e^{-2\pi r} - 1]$$

(45)

$$= 1 + \frac{1}{\pi^2\Lambda} + \frac{1}{2\pi^2r^2}[e^{-2\pi r} - 2e^{-2\pi r}\sin^2(\pi r) - 1]$$

(46)
\[
\approx 1 + \frac{1}{\pi^2 \Lambda} - \frac{\sin^2 \pi r}{\pi^2 y^2 r^2} \approx 1 - \frac{6}{\pi^2 \Lambda} \sin^2(\pi r) (r^2/\Lambda) \quad \text{for} \quad r << \sqrt{\Lambda} \tag{47}
\]

which is similar to the result given by SUSY technique. However, for \( \Lambda >> r >> \sqrt{\Lambda} \), our method gives \( 1 - R_2(r, \Lambda) \approx \frac{\Lambda}{\pi^2 \Lambda + r^2} + \frac{1}{\pi^2 r^2} \) while SUSY technique gives \( 1 - R_2 \) as an exponentially decaying function.

As obvious from eq.(47), \( R_2 \) approaches GUE limit as \( \Lambda \to \infty \) but, for finite \( \Lambda \)-values, it is very different from both Poisson as well as GUE. This indicates that the ensembles with distribution parameters giving rise to a finite \( \Lambda \) do not reach stationarity even for infinite size of their matrices, and, their properties being very different from those of the equilibrium ensembles, can be referred as "critical". However in our technique, as shown in previous sections, the difference between various GG ensembles (within same stationarity limits) manifest itself only in different \( \Lambda \)-values, leaving the functional form of various statistical measures unaffected. Thus RP model as well as ensemble (44), both being GGEs and lying on Poisson \to GUE curve, would follow similar formulations for various statistical measures; For example, \( R_2 \) for both of them is given by eq.(47) although with different formulas for \( \Lambda \) and both can show the critical behavior. However a contradiction arises when one considers the Number variance statistics \( \Sigma_1^2(r) \) which can be expressed in terms of \( R_2(r) \) [12],

\[
\Sigma_1^2(r; \Lambda) = r - 2 \int_0^r (r - s)(1 - R_2(s)) \, ds \tag{48}
\]

and therefore should show a similar behavior, as a function of \( \Lambda \), for both (RP model and ensemble (44)). But a detailed study of RP model by SUSY technique [12] suggests that although it shows critical statistics for \( \mu = cN \), it can not support linear nature of \( \Sigma_1^2 = \chi r \) with \( \chi < 1 \). As claimed by this study, the difference in \( \Sigma_1^2(r) \) behavior arises due to difference in large-\( r \), \( (\Lambda >> r >> \sqrt{\Lambda}) \), behavior of \( R_2(r) \) in the two cases.

As our technique is equally well-applicable to both these systems, multifractality should exist in either both or none of them. Note that the multifractal nature of an ensemble is so far believed to be indicated by its \( \Sigma_1^2 \)-behavior. But the latter is not yet clearly understood for RP model (see [33,41]) and therefore question of multifractality is still not fully settled. Also note that the earlier results for both models are obtained by SUSY technique using saddle point approximation at various stages which may also be misleading. It is also possible that (i) the seeming multifractality of ensemble (44) is the erroneous conclusion of various approximations, or (ii) \( \Sigma_1^2(r) \approx \chi r \) is not always a correct indicator of multifractality and therefore its absence in RP model.

We believe that the \( \Sigma_1^2(r) \)-behavior is a bigger suspect [33,1]. Our belief has its roots in the direct applicability of our technique to Anderson model too. Here also the ensemble for \( H \) is located between Poisson \to GUE (for a time-reversal breaking disorder) with corresponding \( R_2 \)-behavior given by eq.(38). Thus for finite \( \Lambda \)-values corresponding to critical region, the eigenvalue statistics is different from Poisson or GUE. But again for \( \Sigma_1^2 \) obtained by using eq.(38), \( \Sigma_1^2(r) \neq \chi r \) with \( \chi < 1 \) and therefore if it is indeed an indicator of multifractality of eigenfunctions, our technique would suggest its absence in Anderson model. However the existence of multifractality among the eigenfunction of Anderson Hamiltonian is experimentally confirmed.

Our results indicate that multifractality will either be a common feature of all the Gaussian ensembles with finite \( \Lambda \)-values in limit \( N \to \infty \) or it does not exist in any of them. Thus the questions related to critical statistics, the correct criteria for multifractality and its analysis by SUSY technique require further probing.

**IV. OTHER CASES**

**A. A perturbed Hamiltonian with GG type perturbation**

The intimate connection of RMT \to CM Hamiltonian continues also for system \( H = H_0 + xV \) with a random perturbation \( V \) drawn from a GGE (i.e \( \rho(V, y, b) = C \exp(-\sum s=1 \sum k l \alpha_{kl,s} (V_{kl,s} - b_{kl,s})^2) \)). In this case, the eigenvalues evolve due to changing strength of perturbation too. To obtain the desired evolution equation, therefore, one needs to consider the sum \( \partial P / \partial x + \sum s \sum k \lesssim \ell (\gamma - y_{kl,s}) \left[ 2 y_{kl,s} \partial P / \partial y_{kl,s} - b_{kl,s} \partial P / \partial b_{kl,s} \right] \) which leads to following equality

\[
\frac{\partial P}{\partial x} + \sum s \sum k \lesssim \ell (\gamma - y_{kl,s}) \left[ 2 y_{kl,s} \frac{\partial P}{\partial y_{kl,s}} - b_{kl,s} \frac{\partial P}{\partial b_{kl,s}} \right] = \sum s \sum k \lesssim \ell I_{kl,s} - \sum s \sum k \lesssim \ell y_{kl,s} I_{kl,s} \tag{49}
\]

where \( I_{kl,s} \) is still given by same form as eq.(12) but with \( H \) replaced by \( V \). As the right hand side of eq.(49) is same as that of eq.(14), one again obtains obtains the evolution equation (17) but now \( Y \) is given by the condition
that \( \frac{\partial P}{\partial \sigma} = \frac{\partial P}{\partial \sigma} + \sum_s \sum_{k \leq l} (\gamma - y_{kl:s}) \left[ 2y_{kl:s} \frac{\partial P}{\partial y_{kl:s}} - b_{kl:s} \frac{\partial P}{\partial b_{kl:s}} \right] \). Proceeding just as in section I.C, \( Y \) can be shown to be given by the following relation:

\[
Y = \frac{1}{2N^2 + 1} \left[ x + \sum_{k \leq l} \sum_{s=1}^{2} \left( \frac{1}{2} \ln \left| y_{kl}^{(s)} \right| - \gamma^{-1} \ln \left| b_{kl:s} \right| \right) \right] + C
\]  

(50)

Again the steady state is achieved for \( Y \rightarrow \infty \) which corresponds to \( x \rightarrow \infty \) and \( y_{kl:s} \rightarrow \gamma \); the steady state solution for \( P \) is given by \( \prod_{\mu} |\mu_i - \mu_j|^{\beta e^{-\gamma}} + \sum_\nu \nu^2 \). Here only \( x \rightarrow \infty \) (with \( \frac{\partial P}{\partial \sigma} = 0 \) and \( H = xV \)) no longer represents a steady state, as in the case when \( V \) belongs to SGE, but represents a transition state with \( \frac{\partial P}{\partial \gamma} \neq 0 \). Note from eq.(50) that \( Y \rightarrow \infty \) as \( x \rightarrow \infty \), seemingly implying that the equilibrium is reached and therefore \( H \) belongs to SGE. But, as obvious from \( H = H_0 + xV \), in limit \( x \rightarrow \infty \), \( H = xV \) and therefore \( H \) must be a GG matrix. This contradiction is a result of the error made in not ensuring the mean spacing of \( H \) same as \( H_0 \) and \( V \).

[8] Here, to ensure the latter, we need to use a modified Hamiltonian, given by \( H = e^{-\gamma}H_0 + \sqrt{\frac{1-e^{-2\gamma}}{\gamma}}V \) with \( \tau = -N^{-1} \ln \cos(x/N) \) (same as before in large-\( N \) limit). The effect of this modification on F-P equation (17) is that now \( \frac{\partial P}{\partial r} = \frac{\partial P}{\partial r} + \frac{\partial P}{\partial \gamma} \sum_{k \leq l} (\gamma - y_{kl:s}) \left[ 2y_{kl:s} \frac{\partial P}{\partial y_{kl:s}} - \gamma^{-1} b_{kl:s} \frac{\partial P}{\partial b_{kl:s}} \right] \) and the coefficient \( \gamma \) of the drift term is now replaced by \( N^{-1} \gamma \) (see eq.(13) of [42]). The \( Y \) can now be obtained by the second method given in section I.C.

**B. Non-Gaussian Ensembles**

As mentioned before, the RM models of complex systems can, in general, be non-Gaussian, e.g. \( \rho(H) = C \exp^{-\sum_{k\leq l} f(H_{kl})} \) with \( f \) as an arbitrary function and it is not an easy task to obtain the correlations in this case. However this case can be analyzed by our method if \( f \) is a well behaved function and can be expanded in a Taylor's series. To understand this, let us consider an ensemble of real-symmetric matrices \( H \) with distribution of a more general nature e.g. \( f \) as a polynomial of \( H \) with degree 2, \( f(x) = \sum_{r=1}^{M} \gamma_{kl}(r)x^{2r} \) with \( C \) as the normalization constant and variances for the diagonal and off-diagonal matrix elements chosen to be arbitrary.

To obtain an evolution equation in this case, we now consider the sum \( 2 \sum_{r=1}^{M} r \sum_{k \leq l} (\gamma - y_{kl}(1)) y_{kl}(r) \frac{\partial P}{\partial y_{kl}(r)} \) (with \( P = C \hat{P} \) and \( y_{kl}(r) = y_{kl} \gamma_{kl}(r) \)) where the derivative of \( \hat{P} \) with respect to \( \gamma_{kl}(1) \) can be shown to be following (with \( \rho = C \hat{P} \))

\[
\frac{\partial \hat{P}}{\partial \gamma_{kl}(1)} = \frac{1}{2\gamma_{kl}(1)} \left[ \prod_{i=1}^{N} \delta(\mu_i - \lambda_{kl}) H_{kl} \frac{\partial \hat{P}}{\partial H_{kl}} + \sum_{r=2}^{M} r \gamma_{kl}(r) \int \prod_{i=1}^{N} \delta(\mu_i - \lambda_{kl}) \frac{\partial \hat{P}}{\partial \gamma_{kl}(r)} \partial H \right] - \sum_{r=2}^{M} r \gamma_{kl}(r) \left[ \prod_{i=1}^{N} \delta(\mu_i - \lambda_{kl}) \frac{\partial \hat{P}}{\partial \gamma_{kl}(r)} \partial H \right] \]  

(51)

Now as \( \frac{\partial \hat{P}}{\partial \gamma_{kl}(r)} = -H_{kr}^{2r-1} \rho \) and \( \frac{\partial \hat{P}}{\partial H_{kl}} = -2 M \sum_{r=1}^{M} r \gamma_{kl}(r) H_{kl}^{2r-1} \rho \), the second integral in eq.(51) being equal to \( \frac{\partial P}{\partial \gamma_{kl}(r)} \), eq.(11) can be rearranged to show that \( 2 \sum_{r=1}^{M} r \gamma_{kl}(r) \frac{\partial P}{\partial y_{kl}(r)} = I_{kl} \) with \( I_{kl} \) given by eq.(12), (but without subscript \( s \) on quantities). The required evolution equation in this case, can be obtained from the following equality:

\[
2 \sum_{r=1}^{M} \sum_{k \leq l} r (\gamma - y_{kl}(1)) y_{kl}(r) \frac{\partial \hat{P}}{\partial y_{kl}(r)} = \gamma \sum_{k \leq l} I_{kl} - \sum_{k \leq l} y_{kl}(1) I_{kl}
\]

(52)

where, again, \( \sum_{k \leq l} I_{kl} = \sum_{n} \frac{\partial}{\partial \mu_n} (\mu_n \hat{P}) \) and

\[
\sum_{k \leq l} y_{kl}(1) I_{kl} = - \sum_{n} \frac{\partial}{\partial \mu_n} \left[ \frac{\partial \hat{P}}{\partial \mu_n} + \sum_{m \neq n} \frac{\beta}{\mu_m - \mu_n} \right] \hat{P} + \sum_{k \leq l} J_{kl}
\]

(53)

with \( J_{kl} \) now given by following relation:

\[
J_{kl} = - \sum_{n} \frac{\partial}{\partial \mu_n} \sum_{k \leq l} \int \prod_{i=1}^{N} \delta(\mu_i - \lambda_{kl}) \frac{\partial \lambda_n}{\partial H_{kl}} \left[ \sum_{r=2}^{M} r y_{kl}(r) H_{kl}^{2r-1} \right] \rho \partial H
\]

(54)

\[
g_{kl} \sqrt{\frac{1-e^{-2\gamma}}{\gamma}} \sum_{s=1}^{M} s \frac{\partial \hat{P}}{\partial \gamma_{kl}(s)} \left[ (2r+1) + 2 \sum_{s=1}^{M} s y_{kl}(s) \frac{\partial \hat{P}}{\partial y_{kl}(s)} \right]
\]

(55)

13
However, as obvious from eq.(13), $P$ show that $Z$ even for systems with interactions too intricate to be modeled by SGE. So far, the probing of GGE is carried out only known. This also indicates that a thorough knowledge of the properties of SGE or CSM will be highly advantageous. This greatly reduces the degree of difficulty of the original problem as many of the properties of SGE are already known. This also indicates that a thorough knowledge of the properties of SGE or CSM will be highly advantageous even for systems with interactions too intricate to be modeled by SGE. So far, the probing of GGE is carried out only

C. Block-Diagonal Ensembles

The eq.(7) and, therefore, evolution equation (17) of $P(\mu,Y)$ is no longer valid if the matrix $H$ is in a block-diagonal form. This is because the eigenvalues belonging to different blocks don’t repel each other, are not correlated and undergo an evolution independent of the other block. For this case, the evolution of eigenvalues in each block can be considered separately, leading to one F-P equation similar to eq.(17) for each block. A detailed discussion of this case in given in [14].

V. AN ALTERNATIVE EVOLUTION EQUATION FOR THE EIGENVALUES

In section I.B, the eq.(17) governing the evolution of the eigenvalues was obtained by using the relation (14). However, as obvious from eq.(13), $P$ also satisfies the relation

$$\sum_{k\leq l} \left[ 2y_{kl;s} \frac{\partial P}{\partial y_{kl;s}} - b_{kl;s} \frac{\partial P}{\partial \bar{b}_{kl;s}} \right] = \sum_{k\leq l} I_{kl;s} \tag{56}$$

and, therefore, one can define a function $Z(y_{kl;s}, b_{kl;s})$ such that

$$\frac{\partial P}{\partial Z} = \sum_n \frac{\partial}{\partial \mu_n} (\mu_n P) \tag{57}$$

Here $Z$ is given by the condition $\frac{\partial P}{\partial Z} = \sum_{k\leq l} \left[ 2y_{kl;s} \frac{\partial P}{\partial y_{kl;s}} - b_{kl;s} \frac{\partial P}{\partial \bar{b}_{kl;s}} \right]$ which can be solved (as in section II) to show that $Z = \frac{1}{2N} \ln \prod_{k\leq l} \prod_{s=1}^{2} [y_{kl;s} b_{kl;s} - 2] + C$.

The eq.(57) also describes the evolution of eigenvalues for the same ensemble (3). But now the "time"-scale is such that the eigenvalues seem to be drifting only, hiding the repulsion between them. Again the steady state of eq.(57) is given by $|Z - Z_0| \to \infty$ and the final ensemble as Poisson (with finite, non-zero variances for diagonal matrix elements and zero variances for the off-diagonal ones). The ensemble G will now appear as an intermediate point in a transition from some initial ensemble $\to$ Poisson ensemble and, in principle, the transition can be used for the analysis of $G$. For example, the critical parameter for Anderson transition (same model as used in section III) can be obtained by taking the initial state "O" as metal with energy level distribution described by a GUE ($<O_{ii}^2> = \sigma_M^{-2}$, $<O_{ij}^2> = 0$) and all $<O_{ij}> = t_M$ which gives $Z_0 = \frac{1}{2N} \ln [Nna_M - 2Kln(t_M)] + C$ The critical region will therefore occur as an intermediate point in the GUE $\to$ Poisson transition with transition parameter $\lambda = D^{-2}(Z - Z_0) = \frac{1}{2N} \ln [Nna_M - 2Kln(t_M)]$. As obvious, the increase of diagonal disorder ($\frac{\sigma}{a_M} < 1$) for a fixed hopping rate ($t = t_M$) will ultimately lead to Poisson statistics, implying localization of states; note here the transition occurs backwards in "time" $\lambda$. However the results for correlations associated with SGE $\to$ Poisson transition are not known which leaves eq.(17) as a better tool to analyze the properties of GGEs. The eq.(17) has one more advantage over eq.(58): the reduction of former to CSM Hamiltonian reveals the underlying universality of statistical formulation among various complex systems.

VI. CONCLUSION

In this paper, we have described a new method to analyze the statistical properties of the RM model of complex systems. Our technique is based on the exact reduction of spectral analysis in the general case to the one in SGE. This greatly reduces the degree of difficulty of the original problem as many of the properties of SGE are already known. This also indicates that a thorough knowledge of the properties of SGE or CSM will be highly advantageous even for systems with interactions too intricate to be modeled by SGE. So far, the probing of GGE is carried out only
by SUSY technique which requires a saddle point approximation at various steps and is not easily applicable, even approximately, to cases where our technique can be used for exact probing. Note the main term in GGEs responsible for the correspondence with CSM Hamiltonian is due to the repulsion between eigenvalues. As the mathematical origin of this term lies in the transformation from matrix space to eigenvalue space which is same for all the hermitian ensembles (belonging to same symmetry class), the correspondence with CSM Hamiltonian should exist for almost all of them irrespective of the distribution of their matrix elements. As discussed in section III, our study also confirms the conjecture regarding the one parameter scaling of localization and provides the formula for relevant parameter.

The reduction technique presented here raises some basic questions. Why the reparametrization of the spectral properties of different RM ensembles results in a similar mathematical formulation for them? In other words, why the eigenvalues of quantum operators associated with complex systems evolve in a similar ordered way (like equations of motion for Calogero particles) notwithstanding the varied nature of their complexity? The reason may lie in the following. The eigenvalues and eigenfunctions of a Hamiltonian evolve due to a change in either degree or nature of its complexity. The evolution of the eigenfunction is chaotic in the sense that the overlapping between the eigenfunctions, associated with two Hamiltonians even with slightly different complexity, decreases rapidly in time (page 2 of [3].) However an eigenvalue of an operator is its average value in the state described by the associated eigenfunction and an ordered evolution of the former will, in general, imply an ordered change in the average behavior of the latter. Furthermore, it seems that the eigenvalues and eigenfunctions, on an average, are not able to view the fine subtilties of the varied nature of complexity and therefore are not affected too drastically to loose correlations even when nature of the complexity changes. Note, for a small change in the interactions, this result is not surprising and used as the base for the perturbation theory. But the results in this paper imply that the eigenvalues (and physics based on them) even after a violent change in the interactions remain correlated in the parametric space. Thus it seems that certain physical properties, based on average behavior of eigenvalues and eigenfunctions, of one complex system are related to the physics of another, very different in nature of the interactions.

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APPENDIX A: PROOF OF EQS.(3,4,5)

The use of the eigenvalue equation $HU = U\lambda$, with $U$ as a unitary matrix and $\lambda$ the eigenvalue matrix, leads to following:

\[
\sum_j H_{ij} U_{jn} = \lambda_n U_{in} \quad \text{and} \quad \sum_i H_{ij} U^*_{in} = \lambda_n U^*_{jn}
\]  

(A1)

where $H_{ij} = H_{ij:1} + iH_{ij:2}$. Differentiating both sides of above equation with respect to $H_{kl:s}$ (with $s = 1$ or 2), we get

\[
\sum_j \frac{\partial U_{jn}}{\partial H_{kl:s}} H_{ij} + \sum_j U^*_{jn} \frac{\partial H_{ij}}{\partial H_{kl:s}} = \lambda_n \frac{\partial U_{in}}{\partial H_{kl:s}} + \frac{\partial \lambda_n}{\partial H_{kl:s}} U_{in}
\]  

(A2)

Now as $\sum_i U^*_{in} U_{im} = \delta_{nm}$, multiplying both the sides by $U^*_{in}$ followed by a summation over all $i$’s, we get the following

\[
\frac{\partial \lambda_n}{\partial H_{kl:s}} = \sum_{i,j} U^*_{in} \frac{\partial H_{ij}}{\partial H_{kl:s}} U_{jn}
\]  

(A3)

which further gives

\[
\frac{\partial \lambda_n}{\partial H_{kl:s}} = i^{s-1} \frac{1}{g_{kl}} [U_{ln} U^*_{kn} - (-1)^s U^*_{in} U_{kn}]
\]  

(A4)

This can further be used to show that
\[
\sum_{k \leq l} \sum_{s=1}^{2} \frac{\partial \lambda_n}{\partial H_{kl,s}} H_{kl,s} = \sum_{k \leq l} \frac{1}{g_{kl}} \left[ U_{ln} U_{kn} \sum_s i^{s-1} H_{kl,s} + U_{ln}^* U_{kn} \sum_s i^{s-1} (-1)^{s+1} H_{kl,s} \right] \\
= \sum_{k \leq l} \frac{1}{g_{kl}} \left[ H_{kl} U_{ln} U_{kn}^* + H_{kl}^* U_{ln}^* U_{kn} \right] \\
= \sum_{k \leq l} \frac{1}{g_{kl}} H_{kl} U_{ln} U_{kn}^* + \sum_{k \geq l} \frac{1}{g_{kl}} H_{lk}^* U_{ln}^* U_{ln} \\
= \sum_{k,l} H_{kl} U_{ln} U_{kn}^* = \lambda_n
\]

where eq.(A8) is obtained from eq.(A7) by using Hermitian properties of \( H (H^*_{lk} = H_{lk}) \). By using eq.(A4), One can also show that

\[
\sum_{k \leq l} g_{kl} \sum_{s=1}^{2} \frac{\partial \lambda_n}{\partial H_{kl,s}} \frac{\partial \lambda_m}{\partial H_{kl,s}} = \sum_{k \leq l} \sum_{s=1}^{2} i^{2(s-1)} \frac{1}{g_{kl}} \left[ U_{ln} U_{kn}^* - (-1)^s U_{ln}^* U_{kn} \right] \left[ U_{lm} U_{km}^* - (-1)^s U_{lm}^* U_{km} \right] \\
= \sum_{k \leq l} \frac{2}{g_{kl}} \left[ U_{ln} U_{kn}^* U_{km} U_{lm}^* + U_{ln}^* U_{km} U_{lm} U_{ln}^* \right] \\
= 2 \sum_{k,l} U_{kn}^* U_{km} U_{lm} = \sum_k U_{kn} U_{km} \sum_l U_{lm} U_{ln}^* = 2\delta_{mn}
\]

where eq.(A11) follows from eq.(A10) by writing \( \sum_{k \leq l} U_{kn}^* U_{km} U_{lm} = \sum_{k \geq l} U_{kn}^* U_{km} U_{lm}^* \) and the last equality in eq.(A11) is due to unitary nature of \( U \).

**APPENDIX B: PROOF OF EQ.(6)**

Multiplying both the sides of eq.(A2) by \( U_{im}^* (m \neq n) \) followed by a summation over all \( i \)'s, we get the following

\[
\sum_{j} U_{jm}^* \frac{\partial U_{jn}}{\partial H_{kl,s}} = \frac{1}{\lambda_n - \lambda_m} \sum_{i,j} U_{im}^* \frac{\partial H_{ij}}{\partial H_{kl,s}} U_{jn} \\
\]

a multiplication of both the sides by \( U_{rn} \) followed by a summation over all \( m \)'s then gives

\[
\frac{\partial U_{rn}}{\partial H_{kl,s}} = i^{s-1} \frac{1}{g_{kl}} \sum_{m \neq n} \frac{U_{rn}}{\lambda_n - \lambda_m} \left( U_{kn} U_{ln} - (-1)^s U_{ln}^* U_{kn} \right) \\
\]

**APPENDIX C: PROOF OF EQ.(7)**

\[
\sum_{k \leq l} g_{kl} \sum_{s=1}^{2} \frac{\partial^2 \lambda_n}{\partial H_{kl,s}} = \sum_{k \leq l} \sum_{s=1}^{2} i^{s-1} \frac{1}{g_{kl}} \frac{\partial}{\partial H_{kl,s}} \left[ U_{ln} U_{kn}^* - (-1)^s U_{ln}^* U_{kn} \right] \\
= \sum_{k \leq l} \sum_{s=1}^{2} i^{s-1} \left[ \frac{\partial U_{ln}^*}{\partial H_{kl,s}} U_{ln} + \frac{\partial U_{ln}}{\partial H_{kl,s}} U_{kn}^* + (-1)^{s+1} \frac{\partial U_{ln}^*}{\partial H_{kl,s}} U_{kn} + (-1)^{s+1} \frac{\partial U_{kn}}{\partial H_{kl,s}} U_{ln}^* \right] \\
\]

Now by using eq.(B2) and its complex conjugate in eq.(C2) and by summing over \( s \), we get

\[
\sum_{k \leq l} g_{kl} \sum_{s=1}^{2} \frac{\partial^2 \lambda_n}{\partial H_{kl,s}} = 4 \sum_{k \leq l} \frac{1}{g_{kl}} \sum_m \frac{1}{\lambda_n - \lambda_m} \left[ U_{kn} U_{km} U_{ln} U_{ln}^* + U_{kn} U_{kn} U_{ln} U_{ln}^* \right] \\
= 4 \sum_{k,l} \sum_m \frac{1}{\lambda_n - \lambda_m} \left[ U_{kn} U_{km} U_{ln} U_{ln}^* \right] \\
= 4 \sum \frac{1}{\lambda_n - \lambda_m} \left[ \sum U_{kn} U_{km}^* \right] \left[ \sum U_{ln} U_{ln}^* \right]
\]
Now by using the unitary relation \( \sum_j U_{jm}^* U_{jm} = 1 \), one obtains the desired relation (7).

**APPENDIX D: PROOF OF EQ.(15)**

The eq.(12) gives us the following,

\[
\sum_{k \leq l}^{2} I_{kl:s} = \sum_{n} \frac{\partial}{\partial \mu_n} \int \prod_{i} \delta(\mu_i - \lambda_i) \left[ \sum_{k \leq l}^{2} \frac{\partial \lambda_n}{\partial H_{kl:s}} H_{kl:s} \right] \rho \mathrm{d}H \tag{D1}
\]

The use of eq.(A8) will further simplify it in following form

\[
\sum_{k \leq l}^{2} I_{kl:s} = \sum_{n} \frac{\partial}{\partial \mu_n} \int \prod_{i} \delta(\mu_i - \lambda_i) \lambda_n \rho \mathrm{d}H \tag{D2}
\]

\[
= \sum_{n} \frac{\partial}{\partial \mu_n} (\mu_n P) \tag{D3}
\]

**APPENDIX E: PROOF OF EQ.(16)**

For each \( s \)-value, we have the following relation

\[
\sum_{k \leq l} y_{kl:s} I_{kl:s} = \sum_{n=1}^{N} \frac{\partial}{\partial \mu_n} \sum_{k \leq l} g_{kl} \alpha_{kl:s} \int \prod_{i=1}^{N} \delta(\mu_i - \lambda_i) \frac{\partial \lambda_n}{\partial H_{kl:s}} H_{kl:s} \rho \mathrm{d}H \tag{E1}
\]

\[
= - \sum_{n=1}^{N} \frac{\partial}{\partial \mu_n} \sum_{k \leq l} g_{kl} \int \prod_{i=1}^{N} \delta(\mu_i - \lambda_i) \frac{\partial \lambda_n}{\partial H_{kl:s}} \left[ \frac{\partial}{\partial H_{kl:s}} - 2 \alpha_{kl:s} \frac{\partial H_{kl:s}}{H_{kl:s}} \right] \rho \mathrm{d}H \tag{E2}
\]

\[
= - \sum_{n=1}^{N} \frac{\partial}{\partial \mu_n} \sum_{k \leq l} g_{kl} \int \prod_{i=1}^{N} \delta(\mu_i - \lambda_i) \frac{\partial \lambda_n}{\partial H_{kl:s}} \frac{\partial \rho}{\partial H_{kl:s}} \mathrm{d}H + \sum_{k \leq l} J_{kl:s} \tag{E3}
\]

where eq.(E3) is obtained by using the equality: \( \frac{\partial \rho}{\partial H_{kl:s}} = -2 \alpha_{kl:s} (H_{kl:s} - b_{kl:s}) \rho \) and \( J_{kl:s} \) is given by eq.(E9).

By integrating eq.(E3) further by parts, one obtains

\[
\sum_{k \leq l}^{2} y_{kl:s} I_{kl:s} = \sum_{s} \sum_{n} \frac{\partial}{\partial \mu_n} \sum_{k \leq l} \int \left( \frac{\partial}{\partial H_{kl:s}} \prod_{i} \delta(\mu_i - \lambda_i) \right) \frac{\partial \lambda_n}{\partial H_{kl:s}} \rho \mathrm{d}H \tag{E4}
\]

\[
+ \sum_{s} \sum_{n} \frac{\partial}{\partial \mu_n} \sum_{k \leq l} \int \prod_{i} \delta(\mu_i - \lambda_i) \frac{\partial^2 \lambda_n}{\partial H_{kl:s}} \rho \mathrm{d}H + \sum_{k \leq l} \sum_{s} J_{kl:s} \tag{E5}
\]

\[
= - \sum_{n} \frac{\partial}{\partial \mu_n} \sum_{m} \frac{\partial}{\partial \mu_m} \int \prod_{i} \delta(\mu_i - \lambda_i) \left[ \sum_{k \leq l} \frac{g_{kl}}{2} \frac{\partial \lambda_n}{\partial H_{kl:s}} \frac{\partial \lambda_n}{\partial H_{kl:s}} \right] \rho \mathrm{d}H \tag{E6}
\]

\[
- \sum_{n} \frac{\partial}{\partial \mu_n} \int \prod_{i} \delta(\mu_i - \lambda_i) \left[ \sum_{m \neq n} \frac{2}{\lambda_m - \lambda_n} \right] \rho(H) \mathrm{d}H + \sum_{k \leq l} \sum_{s} J_{kl:s} \tag{E7}
\]

\[
= - \sum_{n} \frac{\partial^2 P}{\partial \mu_n^2} - \sum_{n} \frac{\partial}{\partial \mu_n} \left[ 2 \sum_{m \neq n} \frac{P}{\mu_m - \mu_n} \right] + \sum_{k \leq l} \sum_{s} J_{kl:s} \tag{E8}
\]

where \( J_{kl:s} \) can be obtained as follows:
\[ J_{kl,s} = y_{kl,s} b_{kl,s} \sum_{n=1}^{N} \frac{\partial}{\partial \mu_n} \int \prod_{i=1}^{N} \delta(\mu_i - \lambda_i) \frac{\partial \lambda_n}{\partial H_{kl,s}} \rho dH \]

\[ = -y_{kl,s} b_{kl,s} \int \prod_{i=1}^{N} \delta(\mu_i - \lambda_i) \frac{\partial \rho}{\partial H_{kl,s}} dH \]

\[ = y_{kl,s} b_{kl,s} \int \prod_{i=1}^{N} \delta(\mu_i - \lambda_i) \frac{\partial \rho}{\partial b_{kl,s}} dH = -y_{kl,s} b_{kl,s} \frac{\partial P}{\partial b_{kl,s}} \]

(E9) 

(E10) 

(E11) 

(E12)

where in eq.(A32), the equality \[ \frac{\partial \rho}{\partial b_{kl,s}} = 2\alpha_{kl,s} (H_{kl,s} - b_{kl,s}) \rho = -\frac{\partial \rho}{\partial H_{kl,s}} \]
is used. A substitution of eq.(E12) in eq.(E8) now leads to the eq.(16).

APPENDIX F: A GENERAL METHOD TO OBTAIN \( Y \)

Let us consider a transformation of \( M = 2N^2 \) coordinates \( \{r_j\} \) to another set of \( M \) coordinates \( \{Y_i\} \), where \( r_j \)'s are various coefficients \( y_{kl,s} \) (total \( N^2 \)) and \( b_{kl,s} \) (total \( N^2 \)). The \( Y_i \)'s should be chosen such that the right hand side of the eq.(18), summing over all \( y_{kl,s} \)'s and \( b_{kl,s} \)'s can be rewritten as

\[ \sum_{i=1}^{M} \frac{\partial P}{\partial Y_i} = \sum_{k=1}^{M} 2(\gamma - y_{kl,s}) y_{kl,s} \frac{\partial P}{\partial y_{kl,s}} - \gamma \sum_{k=1}^{M} b_{kl,s} \frac{\partial P}{\partial b_{kl,s}} = \sum_{j=1}^{M} g_j(r_1, r_2, \ldots, r_M) \frac{\partial P}{\partial r_j} \]

(F1)

where, for our case, \( g_i(r_1, \ldots, r_M) = 2(\gamma - r_i) r_i \) if \( r_i \) is one of the \( y_{kl,s} \), and, \( g_i(r_1, \ldots, r_M) = -\gamma r_i \) if \( r_i \) is one of the \( b_{kl,s} \).

Now, as we want \( \sum_{i=1}^{M} \frac{\partial P}{\partial Y_i} = \frac{\partial P}{\partial Y_1} \), with \( Y_1 = Y \), this imposes following conditions on the functions \( Y_i \)'s (as can be shown by using the theory of partial differentiation)

\[ \frac{\partial P}{\partial Y_1} = \sum_{i=1}^{M} \sum_{j=1}^{M} g_j(r_1, r_2, \ldots, r_M) \frac{\partial P}{\partial Y_i} \frac{\partial Y_i}{\partial r_j} \]

(F2)

and therefore

\[ \sum_{j=1}^{M} g_j(r_1, r_2, \ldots, r_M) \frac{\partial Y_i}{\partial r_j} = \delta_{1i} \]

(F3)

According to theory of partial differential equations \[E\], the general solution of linear PDE \[ \sum_{i=1}^{M} P_i(x_1, x_2, \ldots, x_M) \frac{\partial^2}{\partial x_i^2} = R \] is \( F(u_1, u_2, \ldots, u_n) = 0 \) where \( F \) is an arbitrary function and \( u_i(x_1, x_2, \ldots, x_n, Z) = c_i \) (a constant), \( i = 1, 2, \ldots, n \) are independent solutions of the following equation

\[ \frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \ldots = \frac{dx_k}{P_k} = \ldots = \frac{dx_M}{P_M} = \frac{dZ}{R} \]

(F4)

Thus the general solution of eq.(F3) for each \( Y_i \) is given by a relation \( F_j(u_{1j}, u_{2j}, \ldots, u_{Mj}) = 0 \) where function \( F_j \) is arbitrary and \( u_{ij}(r_1, r_2, \ldots, r_M, Y_j) = c_{ij}, \ (i = 1, 2, \ldots, M) \) (with \( c_{ij} \)'s as constants) are independent solution of the equation

\[ \frac{dr_1}{g_1} = \frac{dr_2}{g_2} = \ldots = \frac{dr_k}{g_k} = \ldots = \frac{dr_M}{g_M} = \frac{dY_j}{\delta_{ij}} \]

(F5)

The above set of equations can be solved for various \( Y_j \) to obtain \( F_j \). For \( Y_1 \), we get the relations \( Y_1 - \frac{1}{2} \log_{\gamma} \frac{r}{|r_i - \gamma|} = c_{i1} \ (i = 1, \ldots, M/2), \ Y_1 + \frac{1}{2} \log|r_i| = c_{i1} \ (i = 1 + M/2, \ldots, M) \), and therefore \( F_1 \) satisfies the relation
\[ F_1(Y_1 - \frac{1}{2}\log |r_1 - \gamma|, \ldots, Y_1 - \frac{1}{2}\log |r_{M/2} - \gamma|), Y_1 + \gamma^{-1}\log |r_{M/2}|, \ldots, Y_1 + \gamma^{-1}\log |r_M|) = 0. \]

The function \( F_1 \) being arbitrary here, this relation can also be expressed in the following form:

\[ Y_1 = \frac{1}{M} \left[ \frac{1}{2} \sum_{i=1}^{M/2} \log \frac{r_i}{|r_i - \gamma|} - \frac{1}{\gamma} \sum_{i=M/2+1}^{M} \log |r_i| \right] + C \quad (F6) \]

where \( C \) is another arbitrary function of constants: for example \( C \equiv C(\frac{1}{2}\log |r_1 - \gamma| + \gamma^{-1}\log |r_{M/2}|, \frac{1}{2}\log |r_2 - \gamma| + \gamma^{-1}\log |r_M|, \ldots, \frac{1}{2}\log |r_{M-1} - \gamma| + \gamma^{-1}\log |r_M|) \).

Similarly the variables \( Y_i, i > 1 \), can be obtained however their knowledge is not required for our analysis.

**APPENDIX G:**

The choice of \( \gamma \) is based only on the requirement that \( y_{kl}(O) > y_{kl}(G) > \gamma \) for all \( k, l \). Thus \( \gamma \) can take any value such that \( \gamma \leq \min y_{kl}(G) \). Let us consider two such possibilities for \( \gamma, \gamma = \gamma_1 \) and \( \gamma = \gamma_2 \) and try to evaluate properties of \( G \) on these curves referred as \( T1 \) and \( T2 \) respectively. Let the value of \( Y \) for \( G \) on these curves be \( Y_1 \) and \( Y_2 \) where

\[ Y_1 = \frac{1}{2N^2} \sum_{k \leq l} \sum_{s=1}^{2} \left[ \frac{1}{2} \ln \frac{y_{kl,s}^{y}}{|y_{kl,s}^{y} - \gamma_1|} - \frac{1}{\gamma_1} \ln b_{kl,s}^{y} \right] + C \quad (G1) \]

\[ Y_2 = \frac{1}{2N^2} \sum_{k \leq l} \sum_{s=1}^{2} \left[ \frac{1}{2} \ln \frac{y_{kl,s}^{y}}{|y_{kl,s}^{y} - \gamma_2|} - \frac{1}{\gamma_2} \ln b_{kl,s}^{y} \right] + C \quad (G2) \]

However \( Y_1 \) can also be written as follows

\[ Y_1 = \frac{1}{2N^2} \sum_{k \leq l} \sum_{s=1}^{2} \left[ \frac{1}{2} \ln \frac{y_{kl,s}^{y}}{|y_{kl,s}^{y} - \gamma_2|} - \frac{1}{\gamma_2} \ln b_{kl,s}^{y} \right] + C \quad (G3) \]

Now as \( y_{kl,s}^{y} = y_{kl,s}^{y_1} \neq y_{kl,s}^{y_2} \), this implies that \( Y_1 \) would correspond to a point, different from \( Y_2 \), on the transition curve \( T2 \) and therefore would give properties for the ensemble \( G \) different from those given by \( Y_2 \). This conclusion is, however, erroneous and is a result of the rescaling applied only to one point \( Y_1 \) on the transition curve \( T1 \). To get the right answer, the whole curve \( T1 \) should be rescaled which would require a rescaling of the end-points too and therefore changed distances on the rescaled curve (call it \( T1' \)). Thus the point \( Y_1 \) will appear at the same location on \( T1' \)-curve, relative to end-points, where \( Y_2 \) appears on \( T2 \)-curve and therefore both will imply the same properties for the ensemble \( G \).

[1] F.Calogero, J. Math. Phys., 10, 2191, 2197 (1969).
[2] T.Guhr, G.A. Muller-Groeling and H.A. Weidenmuller, Phys. Rep. V299, 189, (1998).
[3] B.D.Simon, P.A.Lee, B.L.Altshuler, Phys.Rev.Lett, 72, 64,(1994).
[4] K.B.Efetov, Adv. Phys. 32, 53, (1983).
[5] F.Dyson, J. Math. Phys. 3, 1191 (1962).
[6] F.Haake, Quantum Signature of Chaos, Springer, Berlin (1991).
[7] O.Narayan and B.S.Shastry, Phys. Rev. Lett., 71, 2106, (1993).
[8] B.S.Shastry, Proceedings of 16th Taniguchi International Symposium on the theory of Condensed Matter: "Correlated effects in low dimensional systems", October 23-29, 1993, Shima, Japan, (Springer Verlag 1994, Eds: N.Kawakami and A.Okiji).
[9] B.Sutherland, J. Math. Phys., 12, 246, (1971); 12, 252, (1971); Phys. Rev. A. 4, 2019, (1971); 5, 1372, (1972).
[10] S.M.Bhattacharjee and S.Mukherjee, Phys. Rev. Lett. 83, (1999); S.M.Bhattacharjee, Phys. Rev. Lett, 76, (1996); M.Lassig, Phys. Rev. Lett, 77, 526, (1996).
[11] P.Pechukas, Phys. rev. Lett., 51, 943, (1983); S.Wojciechowski, Phys. Lett., 111A, 101, 1985; T.Yukawa, Phys. Rev. Lett., 54, 1883, (1985); K.Nakamura and M.Lakshmanan, Phys. Rev. Lett., 57, 1661, (1986).
[12] M.L.Mehta, Random Matrices (Academic Press, Boston) (1991).
[13] A.Pandey, Chaos, Soliton and Fractals, 5, (1995). A.Pandey and P.Shukla, J. Phys. A, (1991).
[14] P.Shukla, to appear in Physica E, 2000 and Physica A, 2000.
[15] Snedden,Elements of Partial Differential Equations, McGraw-Hill, (1988).
[16] C.Itzykson and J.B.Zuber, J. Math. Phys. 21, 411 (1980).
[17] J.B.French, V.K.B.Kota, A.Pandey and S.Tomsovic, Ann. Phys. (N.Y.), 181, (1988).
[18] M.Kus, M.Lewenstein and F.Haake, Phys. rev. A, 44, 2800, (1994).
[19] Z.N.C. Ha, Phys. rev. Lett. 73, 1574 (1994) and Nucl. Phys. B 435 604 (1995).
[20] P.W.Anderson, Phys. Rev. 19, 1492, (1958).
[21] F.M.Izrailev, Phys. rep. 196, 299 (1990).
[22] Y.V.Fyodorov and A.D. Mirlin, Int. J. Mod. Phys. B, 8, 3795 (1994).
[23] Y.V.Fyodorov and A.D. Mirlin, Phys. Rev. Lett., 67, 2405 (1991).
[24] T.Prosen and M. Robnik, J.Phys.A 26, 1105, (1993).
[25] A.Altland, M.Janssen and B.Shapiro, Phys. Rev. E 56, 1471 (1997).
[26] A.D.Mirlin, Y.V.Fyodorov, F-M.Dittes, J.Quezada and T.H.Seligman, Phys. Rev. E 54 3221 (1996).
[27] V.V.Flambaum, A.A.Gribakina, G.F.Gribakin and M.G.Kozolov, Phys. Rev. A 50 267 (1994).
[28] E.Wigner, Ann. Math. 62, 548 (1955); 65, 203 (1957).
[29] Y.V.Fyodorov, O.A.Chubykalo, F.M.Izrailev and G.Casati, Phys. Rev. Lett. 76 1603 (1996).
[30] D.L.Sheplyansky, Phys. Rev. Lett. 73, 2607 (1994).
[31] B.Huckestein, Rev. Mod. Phys. 67, 1995.
[32] B.Huckestein and L.Schweitzer, Phys. Rev. Lett. 72, 713, (1994).
[33] V.E.Kravtsov, I.V.Lerner, B.L.Altshuler and A.G.Aronov, Phys. Rev. Lett. 72, 888 (1994)
[34] V.E.Kravtsov and K.A.Muttalib, Phys. Rev. Lett. 79, 1913, (1997).
[35] J.T.Chalker, V.E.Kravtsov and I.V.Lerner, Pisma Zh. Eksp. Teor. Fiz. 64, 355 (1996).
[36] E.B.Bogomolny, U.Gerland and C.Schmit, Phys. Rev. E59, R1315 (1999).
[37] K.A.Muttalib, Y.Chen, M.E.H.Ismail and V.N.Nicopoulos, Phys. Rev. Lett. 71, 471 (1993).
[38] M.Moshe, H.Neuberger and B.Shapiro, Phys. Rev. Lett. 73, 1497 (1994).
[39] K.M.Frahm, T.Guhr, A.Muller-Groeling, Ann. Phys. (N.Y.) 270, 292 (1998).
[40] H.Kunz and B.Shapiro, Phys. Rev. E, 58, 400, 1998.
[41] F.Evers and A.D.Mirlin, cond-mat/0001086
[42] P.Shukla, Phys. Rev. E 59, 5207, 1999.
[43] Y.V.Fyodorov, O.A.Chubykalo, F.M.Izrailev and G.Casati, Phys. Rev. Lett. 76 1603 (1996).
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