New generalisation of Jacobi’s derivative formula

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Abstract

A stream of new theta relations is obtained. They follow from the general Thomae formula, which is a new result giving expressions for theta derivatives (the zero values of the lowest non-vanishing derivatives of theta functions with singular half-period characteristics) in terms of branch points and the period matrix of a hyperelliptic Riemann surface. The new theta relations contain (i) linear relations on the vector space of first-order theta derivatives which are arranged in gradients, (ii) relations between second-order theta derivatives and symmetric bilinear forms on the vector space of the gradients, (iii) relations between third-order theta derivatives and symmetric trilinear forms on the vector space of the gradients and (iv) a conjecture regarding higher-order theta derivatives. It is shown how the Schottky identity (in the hyperelliptic case) is derived from the obtained relations.

Keywords  Theta constants · First-, second- and third-order theta derivatives · The general Thomae formula · Hyperelliptic curve · The Schottky identity

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1 Introduction

The recently discovered general Thomae formula [3] gives rise to completely new relations between theta constants and theta derivatives. The zero values of high-order derivatives of theta functions with singular half-period characteristics are expressed in terms of theta constants and the zero values of first-order derivatives of the theta functions with non-singular odd characteristics. In the hyperelliptic case, explicit formulae are obtained for the zero values of second- and third-order derivatives of the theta functions with characteristics of multiplicities 2 and 3. Also formulae for the zero values of higher-order derivatives are suggested. On the other hand, known relations
can be derived from these new relations, for example, the Schottky relation and its
generalisations.

We start with a brief review of known generalisations of Jacobi’s derivative formula

\[ \theta'\left[ \begin{array}{l} 1 \\ 1 \end{array} \right] = -\pi \theta\left[ \begin{array}{l} 0 \\ 0 \end{array} \right] \theta\left[ \begin{array}{l} 1 \\ 0 \end{array} \right]. \]  

(1)

Its first generalisation is known as the Rosenhain formulae, see [25] and a revision
in [4, 13.14 and 13.15], which expresses the zero values of determinants of Jacobian
matrices in terms of theta constants in genus two. Later Frobenius [13] gave similar
formulas for genera three and four, and Riemann examined cases up to genus seven
[24].

This type of generalisation in the hyperelliptic case is known as the Riemann–Jacobi
derivative formula. Let \( \delta_i, i = 1, \ldots, g \) denote odd characteristics, and \( \varepsilon_j, j = 0, \ldots, 
\)
g + 1 denote even characteristics as described in [5, Theorem 3.4, pp. 1015–1016]. Here
and in what follows \( g \) denotes the genus of a curve. The Riemann–Jacobi derivative
formula has the form

\[
\det \left\| \partial_v \theta[\delta_1], \partial_v \theta[\delta_2], \ldots, \partial_v \theta[\delta_g] \right\| = \pm \pi^g \theta[\varepsilon_0] \theta[\varepsilon_1] \cdots \theta[\varepsilon_{g+1}],
\]

where \( \partial_v \theta[\delta_i] \) denotes the gradient of \( \theta[\delta_i] \). Riemann and Weil posed the following
questions, see [18, p. 170]. Let \( D_{\text{odd}} \) denote \( \det \left\| \partial_v \theta[\delta_1], \partial_v \theta[\delta_2], \ldots, \partial_v \theta[\delta_g] \right\| \).

(W-1) Is \( D_{\text{odd}} \) always a polynomial in the theta constants with integral rational coef-
ficients?

(W-2) Can \( D_{\text{odd}} \) be expressed as a quotient of two such polynomials?

In [16], it was shown that \( D_{\text{odd}} \) is always a rational function of the theta constants
with rational coefficients. In [17], a general theory (including the non-hyperelliptic
case) was developed which gives the answer to the question when an expression for
\( D_{\text{odd}} \) becomes a polynomial.

Another well-developed direction of generalisation of (1) relates to the elliptic theta
functions with rational characteristics\(^1\). In [22, p. 117] Mumford posed the question
about generalisation of Jacobi’s formula to the form

\[
\frac{\partial}{\partial v} \theta\left[ \begin{array}{l} a \\ b \end{array} \right](v; \tau) \bigg|_{v=0} = \{ \text{cubic polynomial in } \theta\left[ \begin{array}{l} c \\ d \end{array} \right]'s \} 
\]

(2)

for all \( a, b \in \mathbb{Q} \), where \( c, d \in \mathbb{Q} \). Are there similar generalisations of Jacobi’s formula
with higher-order derivatives? In [11] the following theta constant identity was derived:

\[^1\] The theta function with a rational characteristic is defined for all \( v \in \mathbb{C} \) and \( \tau \in \mathbb{C}, \Im \tau > 0 \) by

\[
\theta\left[ \begin{array}{l} a \\ b \end{array} \right](v; \tau) = \sum_{n \in \mathbb{Z}} \exp \left( i\pi(n + a)^2 \tau + 2i\pi(n + a)(v + b) \right).
\]

where \( a, b \in \mathbb{Q} \).
2 With a positive integer $N$, an $N$-th-order theta function with characteristic $[\varepsilon] = [\varepsilon']$, $\varepsilon, \varepsilon' \in \mathbb{R}$, is such a function $\varphi$ which satisfies the following functional equations

\[
\varphi(v + 1) = e^{i\pi \varepsilon} \varphi(v), \quad \varphi(v + \tau) = e^{-i\pi i(\varepsilon' + 2Nv + N\tau)} \varphi(v), \quad \forall v \in \mathbb{C}.
\]
the vector of Riemann constants in genus 2 is expressed in terms of theta constants. A proof is based on the addition formula for sigma function and relations between \( \wp \) functions. This result arises as a particular case in [3, Eqs (31), (30)], where it is derived with the help of other technique.

The present paper is a continuation and development of [3], where the general Thomae formula in hyperelliptic case is obtained. This new formula expresses theta derivative of an arbitrary order in terms of branch points and period matrix of a hyperelliptic Riemann surface. By an \( m \)-th order theta derivative, we call the zero value of order \( m \) partial derivative of theta function with a singular characteristic of multiplicity \( m \). Zero values of first-order derivatives of theta functions with odd non-singular characteristics are called first-order theta derivatives or simply theta derivatives. Zero values of theta functions with even non-singular characteristics are called theta constants as usual.

Applying the second Thomae formula, one can derive relations between theta derivatives and theta constants from the general Thomae formula. Furthermore, these relations show a clear structure. Let \( V \) be the vector space of theta derivatives, namely:

\[
V = \text{span}\{ \partial_v \theta[\varepsilon] \mid \varepsilon \text{ are odd non-singular characteristics of half-periods}\},
\]

where \( \partial_v \theta[\varepsilon] \) is a gradient of \( \theta[\varepsilon](v; \tau) \) at \( v = 0 \). Every vector \( \partial_v \theta[\varepsilon] \) has \( g \) components. And \( V \) has dimension \( g \) indeed, that follows from linear relations between gradients of theta functions with different odd non-singular characteristics, see Sect. 3.1. Second-order theta derivatives are expressed through symmetric bilinear forms on \( V \), see Sect. 3.2. Third-order theta derivatives are represented as symmetric trilinear forms on \( V \), see Sect. 3.3, and so on. All these relations can be viewed as generalisations of Jacobi’s derivative formula since they relate high-order theta derivatives and theta constants, similar to (1). These basic relations can be used to produce further relations. As an example, the Schottky identity is derived below.

The paper is organised as follows. Section 2 contains the minimal background: definitions and notation regarding theta functions, Thomae theorems with corollaries and some auxiliary lemmas. Section 3 is devoted to the main result: deriving relations between theta derivatives and theta constants. Relations with first-, second- and third-order theta derivatives are described in detail, and a conjecture about higher-order theta derivatives is made. In Sect. 4, Schottky-type relations are derived from the relations with second-order theta derivatives. A variety of explicit relations can be found in appendices.

2 Preliminaries

2.1 Hyperelliptic curve and its Jacobian

In the paper, we deal with hyperelliptic curves \( C \) defined by

\[
0 = f(x, y) = -y^2 + \prod_{j=1}^{2g+1} (x - e_j),
\]
where \([e_j, 0]\) are finite branch points and \((x, y) \in \mathbb{C}^2\). In what follows, for the sake of brevity notation, \(e_i\) is employed both for branch point \((e_i, 0)\) and its \(x\)-coordinate. All branch points are distinct, so the curves are non-degenerate. One more branch point is located at infinity and serves as the base point.

A homology basis is adopted from Baker [2, p. 303]. Imagine a continuous path through all branch points which ends at infinity. The branch points are denoted by \(e_1, e_2, \ldots, e_{2g+1}\) along the path, infinity is denoted by \(e_0\). Cuts are made between points \(e_{2k-1}\) and \(e_{2k}\) with \(k\) from 1 to \(g\) and between \(e_{2g+1}\) and \(e_0\). With \(k\) running from 1 to \(g\), an \(a_k\)-cycle encircles the cut \((e_{2k-1}, e_{2k})\) clockwise, and \(b_k\)-cycle comes out from the cut \((e_{2k-1}, e_{2k})\) and enters into the cut \((e_{2g+1}, e_0)\). These cycles constitute a canonical homology basis on a curve (4).

Standard holomorphic differentials are employed, see, for example [2, p. 306],

\[
du_{2n-1} = \frac{x^{g-n} dx}{-2y}, \quad n = 1, \ldots, g.
\] (5)

All together, they are denoted by \(du = (du_1, du_3, \ldots, du_{2g-1})'\). The differentials are labelled by Sato weights in subscripts. The weight shows the exponent of the first term in the expansion of the corresponding integral \(u_{2n-1}\) about infinity in the local parameter \(\xi\) which is introduced by \(x = \xi^{-2}\).

Integrals of the holomorphic differentials along the canonical homology cycles give first kind periods

\[
\omega = (\omega_{ij}) = \left( \int_{a_j} du_i \right), \quad \omega' = (\omega'_{ij}) = \left( \int_{b_j} du_i \right).
\]

The both first kind periods form the matrix \((\omega, \omega')\) of non-normalised periods. Normalised period matrix is \((1_g, \tau)\), where \(1_g\) denotes the identity matrix of size \(g\), and \(\tau = \omega^{-1}\omega'\). Matrix \(\tau\) is symmetric with a positive imaginary part: \(\tau^t = \tau, 3\tau > 0\), that is \(\tau\) belongs to the Siegel upper half-space. Normalised holomorphic differentials are defined by

\[
dv = \omega^{-1} du,
\]

and Abel’s map \(\mathcal{A}\) with respect to these differentials is

\[
\mathcal{A}(P) = \int_\infty^P dv, \quad P = (x, y) \in \mathcal{C}.
\] (6)

Abel’s map of a positive divisor \(D = \sum_{i=1}^nP_i\) on \(\mathcal{C}\) is defined by

\[
\mathcal{A}(D) = \sum_{i=1}^n \int_\infty^{P_i} dv.
\] (7)

\(\mathcal{A}\) maps a genus \(g\) curve to its Jacobian variety \(\text{Jac}(\mathcal{C}) = \mathbb{C}^g \setminus \mathcal{P}\), which is the quotient space of \(\mathbb{C}^g\) by the lattice \(\mathcal{P}\) of periods. The lattice is constructed from columns of
normalised period matrix $(1_g, \tau)$. We denote points of Jacobian by $v$ with coordinates $(v_1, v_2, \ldots, v_g)^t$. The map is one-to-one on the non-special locus of $g$-th symmetric power of the curve.

2.2 Theta functions

The Riemann theta function related to a curve $C$ is defined on its Jacobian $\text{Jac}(C)$ by Fourier series

$$\theta(v; \tau) = \sum_{n \in \mathbb{Z}^g} \exp \left( i\pi n^t \tau n + 2i\pi n^t v \right), \quad (8)$$

see, for example [22, p. 118].

Values of theta function at half-periods are of importance. Relations between these values are an objective of the paper. Therefore, it is suitable to introduce \textit{theta function with half-period characteristic} as in [23, Def 3 p. 4], namely

$$\theta[\epsilon](v; \tau) = \exp \left( i\pi (\epsilon''/2) \tau (\epsilon'/2) + 2i\pi (v + \epsilon/2)^t \epsilon'/2 \right) \times \theta(v + \epsilon/2 + \tau \epsilon'/2; \tau), \quad (9)$$

where $g$-component vectors $\epsilon$ and $\epsilon'$ consist of entries 0 and 1. Characteristic $[\epsilon]$ is a $2 \times g$ matrix of the form

$$[\epsilon] = \begin{bmatrix} \epsilon'' \\ \epsilon' \end{bmatrix}.$$

Each branch point $e$ of a hyperelliptic curve (4) is identified with a half-period

$$\mathcal{A}(e) = \int_{\infty}^e \left( v + \frac{\epsilon}{2} + \frac{\tau \epsilon'}{2} \right) dv$$

where vectors $\epsilon$ and $\epsilon'$ form characteristic $[\epsilon]$, see [1, § 202, pp. 300–301].

One can add characteristics by the rule: $[\epsilon] + [\delta] = ([\epsilon] + [\delta]) \mod 2$. A characteristic $[\epsilon]$ is odd whenever $\epsilon' \epsilon' \mod 2 = 1$, and even whenever $\epsilon' \epsilon' \mod 2 = 0$. Theta function with characteristic has the same parity as its characteristic.

2.3 Characteristics in hyperelliptic case

A method of constructing characteristics in the hyperelliptic case is employed from [5, p. 1012]. It is based on the definition (10) of a half-period characteristic with the help of Abel’s map (6). We denote characteristics of branch points $e_k$ by $[\epsilon_k]$; and $[\epsilon_0] = 0$. Guided by the picture of canonical homology cycles, we obtain

$$\mathcal{A}(e_{2g+1}) = \mathcal{A}(e_0) - \sum_{k=1}^g \int_{e_{2k-1}}^{e_{2k}} dv \quad [\epsilon_{2g+1}] = [\begin{array}{c} 0 \ldots 0 \\ 1 \ldots 1 \end{array}].$$
\[ A(e_{2g}) = A(e_{2g+1}) + \int_{e_{2g}}^{e_{2g+1}} dv = [\varepsilon_{2g}] = [00...01]_{11...11}, \]
\[ A(e_{2g-1}) = A(e_{2g}) + \int_{e_{2g-1}}^{e_{2g}} dv \quad [\varepsilon_{2g-1}] = [00...01]_{11...10}, \]

for \( k \) from \( g - 1 \) to 2

\[ A(e_{2k}) = A(e_{2k+1}) + \int_{e_{2k}}^{e_{2k+1}} dv = [\varepsilon_{2k}] = \left[00...010...0\overset{k-1}{11...10...0}\right], \]
\[ A(e_{2k-1}) = A(e_{2k}) + \int_{e_{2k}}^{e_{2k-1}} dv \quad [\varepsilon_{2k-1}] = [00...010...0]_{11...100...0}, \]

and finally

\[ A(e_2) = A(e_3) + \int_{e_2}^{e_3} dv \quad [\varepsilon_2] = [10...0]_{10...0}, \]
\[ A(e_1) = A(e_2) + \int_{e_2}^{e_1} dv \quad [\varepsilon_1] = [10...0]_{00...0}. \]

Characteristic \([K]\) of the vector of Riemann constants equals the sum of all odd characteristics of branch points, and there are \( g \) such characteristics according to [1, § 200 p. 297, § 202 p. 301], and [8, VII.1.2 p. 305]. Actually,

\[ [K] = \sum_{k=1}^{g} [\varepsilon_{2k}]. \]

### 2.4 Characteristics and partitions

Let \( \mathcal{I} \cup \mathcal{J} \) be a partition of the set of indices of all branch points \( \{e_0, e_1, \ldots, e_{2g+1}\} \) of a curve. Denote by \([\varepsilon(\mathcal{I})] = \sum_{i \in \mathcal{I}} [\varepsilon_i]\) the characteristic of

\[ \mathcal{A}(\mathcal{I}) = \sum_{i \in \mathcal{I}} A(e_i) = \varepsilon(\mathcal{I})/2 + \tau\varepsilon'(/2). \quad (11) \]

Introduce also characteristic \([\mathcal{I}_m] = [\varepsilon(\mathcal{I}_m)] + [K] \) of \( \mathcal{A}(\mathcal{I}_m) + K \) which corresponds to a partition \( \mathcal{I}_m \cup \mathcal{J}_m \). Here \( K \) denotes the vector of Riemann constants, and \([K]\) denotes its characteristic. Note that \([\mathcal{J}_m]\) represents the same characteristic as \([\mathcal{I}_m]\).

According to [12, p. 13] (for more details, see [1, § 202 p. 301]) there is one-to-one correspondence between \( 2^{2g} \) characteristics and \( 2^{2g} \) partitions \( \mathcal{I}_m \cup \mathcal{J}_m \) with \( \mathcal{I}_m = \{i_1, \ldots, i_{g+1-2m}\} \) and \( \mathcal{J}_m = \{j_1, \ldots, j_{g+1+2m}\} \) of the set of \( 2g + 2 \) indices of branch points of a hyperelliptic curve, where \( m \) is between 0 and \((g + 1)/2\). Here,
[z] means the integer part of z. Number m is called multiplicity. Thus, characteristics of 2g + 2 branch points of a hyperelliptic curve (4) serve as a basis to construct all 2^{2g} half-period characteristics.

Characteristics \([I_m]\) of even multiplicity m are even, and of odd m are odd. According to the Riemann vanishing theorem, \(\theta(v + A(I_m) + K)\) vanishes to order m at \(v = 0\), see \([12, \text{p}.13]\). Characteristics of multiplicity 0 are called non-singular even characteristics, there are \(\binom{2g+1}{g}\) such characteristics. There exist \(\binom{2g+2}{g-1}\) characteristics of multiplicity 1, which are called non-singular odd. The number of characteristics of multiplicity \(m > 1\) is \(\binom{2g+2}{g+1-2m}\).

Below characteristic \([K]\) is also denoted by \([\emptyset]\) since it corresponds to partition \(\emptyset \cup \{1, 2, \ldots, 2g + 1\}\), which is always unique. At that \(\theta(K)(v)\) vanishes to the maximal order \([(g + 1)/2] = 0\), see, for example \([8, \text{§ VII.1.4} \text{p}.306]\). The maximal order is greater than 1 if \(g > 2\). If the genus \(g\) of a curve is odd, the mentioned partition contains the empty set: \(I_{(g+1)/2} = \emptyset\), because the omitted index of infinity belongs to \(J_{(g+1)/2}\). It means that \(\theta(K)(v)\) is the only theta function with characteristic of the maximal multiplicity. If \(g\) is even, part \(I_{g/2} = \emptyset\) of the partition actually contains the omitted index of infinity. In this case, \(\theta(K)(v)\) is not the only theta function with characteristic of the maximal multiplicity, there exist \(2g + 2\) such functions. Vanishing to the order equal to multiplicity is the distinctive property of hyperelliptic curves, see \([8, \text{§ VII.1.5}\text{, 7–8 p}.308–309]\).

The index of infinity is usually omitted, and it belongs to the part \(I_m\) or \(J_m\) with the number of indices less than \(g + 1 - 2m\) or \(g + 1 + 2m\), respectively. Notation \(I_m\) is used for the part of less cardinality.

Characteristics of multiplicity 0 are usually described by the partitions whose part \(I_0\) contains \(g\) non-zero indices and the omitted infinity index. Partitions corresponding to characteristics of multiplicity 1 can be obtained from \(I_0 \cup J_0\) by moving two indices from \(I_0\) to \(J_0\). These two indices can be of finite points the both or one index of a finite point and the index of infinity. The former case is described by dropping two indices from \(I_0\), so cardinality of the obtained \(I_1\) becomes \(g - 2\), then \(J_1\) contains \(g + 3\) indices and the infinity index is located in \(I_1\). In the latter case, only one index drops from \(I_0\), so the cardinality of \(I_1\) is \(g - 1\), and the infinity index moves into \(J_1\) which has cardinality \(g + 2\). (Note that computing cardinality of a set we always omit the zero index of infinity.) The same occurs in higher multiplicities. Regarding characteristics \([I_m]\), it is convenient to distinguish between ones with the index of infinity in part \(J_m\) (card \(I_m = g + 1 - 2m\)), and ones with the index of infinity in part \(I_m\) (cardinality of \(J_m\) is \(g - 2m\)).

In what follows, we also use notation \(B^{[r]}\) for a set of cardinality \(r\).

### 2.5 Notations

In what follows \(\partial_{v_i}\) stands for a partial derivative with respect to variable \(v_i\), and argument \(\tau\) of a theta function is usually omitted, so

\[
\partial_{v_i} \theta[\varepsilon](v) \equiv \frac{\partial}{\partial v_i} \theta[\varepsilon](v; \tau),
\]
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\[
\frac{\partial^2 v_i, v_j}{\partial v_i \partial v_j} \theta[\varepsilon](v) \equiv \frac{\partial^2}{\partial v_i \partial v_j} \theta[\varepsilon](v; \tau),
\]

etc.

The gradient of a theta function is denoted by \( \partial_v \theta[\varepsilon](v) \).

The theta constant with characteristic \([I_0]\) corresponding to a partition \(I_0 \cup J_0\) is denoted by \( \theta[I_0] \). The lowest non-vanishing derivative of \( \theta[I_m](v; \tau) \) at \( v = 0 \) is a tensor of order \( m \) denoted by \( \partial^m_v \theta[I_m] \). Such a tensor consists of zero values of partial derivatives \( \partial v_{i_1} \cdots \partial v_{i_m} \theta[I_m] \) of order \( m \) with respect to all combinations constructed from \( g \) components of \( v \). These zero values of derivatives are called \( m \)-th-order theta derivatives, as mentioned above. The following notation is used:

\[
\theta[I_0] \equiv \theta[I_0](0; \tau),
\]

\[
\partial_v \theta[I_1] \equiv \frac{\partial}{\partial v_i} \theta[I_1](0; \tau),
\]

\[
\frac{\partial^2}{\partial v_i \partial v_j} \theta[I_2] \equiv \frac{\partial^2}{\partial v_i \partial v_j} \theta[I_2](0; \tau),
\]

etc.

Another notation like \( \theta\{i_1, i_2, i_3\} \) is used for \( \theta[I] \) with \( I = \{i_1, i_2, i_3\} \) in the case when \( I \) is described by its elements. Recall that theta constants are actually functions of normalised period matrix \( \tau \) also called the Riemann period matrix.

Throughout the paper, representation of characteristics in terms of partitions is used, because this makes clear which order of vanishing theta functions have at \( v = 0 \). Representation of characteristics in terms of matrices with entries 0 and 1, which is used in Appendices, is derived from characteristics of branch points given in Sect. 2.3.

Let \( \{e_i \mid i \in I\} \) be a collection of branch points (here and below, we use only \( x \)-coordinates to specify branch points) corresponding to a partition \( I \cup J \). By \( s_n(I) \), an elementary symmetric polynomial of degree \( n \) in \( \{e_i \mid i \in I\} \) is denoted, at that elementary symmetric polynomials \( s_n \) are defined by

\[
\sum_{n \geq 0} t^n s_n = \prod_{i \in I} (1 + e_i t).
\]

And \( \Delta(I) \) denotes the Vandermonde determinant in \( \{e_i \mid i \in I\} \), namely:

\[
\Delta(I) = \prod_{i, j \in I, i > j} (e_i - e_j).
\]

By \( \Delta \), the Vandermonde determinant over the set of all finite branch points of the curve is denoted.

2.6 First Thomae formula

The first Thomae theorem and its corollaries below are given in the form proposed in [5, p. 1014]
First Thomae theorem Let $I_0 \cup J_0$ with $I_0 = \{i_1, \ldots, i_g\}$ and $J_0 = \{j_1, \ldots, j_{g+1}\}$ be a partition of the set $\{1, 2, \ldots, 2g+1\}$ of indices of finite branch points of a hyperelliptic curve (4), and $[I_0]$ denotes the non-singular even characteristic corresponding to $A(I_0) + K$. Then

$$\theta[I_0] = \epsilon \left( \frac{\det \omega}{\pi^g} \right)^{1/2} \Delta(I_0)^{1/4} \Delta(J_0)^{1/4},$$

(12)

where $\epsilon$ satisfies $\epsilon^8 = 1$, then $\Delta(I_0)$ and $\Delta(J_0)$ denote the Vandermonde determinants built from $\{e_i \mid i \in I_0\}$ and $\{e_j \mid j \in J_0\}$.

For a proof see [12, Proposition 3.6, p.46].

FTT Corollary 1 Let $I = \{i_1, \ldots, i_{g-1}\}$ and $J = \{j_1, \ldots, j_{g-1}\}$ be two disjoint sets picked out from $2g+1$ indices of finite branch points of a hyperelliptic curve (4), let $k, m, n$ be the remaining indices. Then the following formula is valid:

$$e_k - e_m \frac{e_k - e_n}{e_k - e_m} = \epsilon \frac{\theta([n] \cup I)^2 \theta([n] \cup J)^2}{\theta([m] \cup I)^2 \theta([m] \cup J)^2},$$

(13)

where $\epsilon^4 = 1$.

FTT Corollary 2 Let $I_0 = \{i_1, \ldots, i_g\}$ and $J_0 = \{j_1, \ldots, j_{g+1}\}$ form a partition of $2g+1$ indices of finite branch points of a hyperelliptic curve (4). Choose $i_k, i_l \in I_0$, and $j_n, j_m \in J_0$. Then

$$\prod_{j \in J_0} \frac{(e_{ik} - e_j)}{(e_{ik} - e_{il})^2 \prod_{i \in I_0, i \neq i_k} (e_{ik} - e_i)} = \pm \frac{\theta[I_0(i \rightarrow j)]^4 \theta[I_0(l \rightarrow m)]^4 \theta[J_0(j \rightarrow i)]^4 \theta[J_0(m \rightarrow l)]^4}{\theta[I_0(i \rightarrow j)]^4 \theta[I_0(l \rightarrow m)]^4 \theta[J_0(j \rightarrow i)]^4 \theta[J_0(m \rightarrow l)]^4},$$

(14)

where $J_0(j)$ stands for $J_0 \setminus \{j\}$, and $I_0(i \rightarrow j)$ denotes that index $i$ is replaced by $j$ in $I_0$, the same refers to $J_0(j \rightarrow i)$.

Note that the right-hand side does not depend on the choice of $j_n, j_m \in J_0$.

2.7 Second Thomae formula

The second Thomae theorem is also taken from [5, p.1015].

Second Thomae theorem Let $I_1 \cup J_1$ with $I_1 = \{i_1, \ldots, i_{g-1}\}$ and $J_1 = \{j_1, \ldots, j_{g+2}\}$ be a partition of the set $\{1, 2, \ldots, 2g+1\}$ of indices of finite branch points of a hyperelliptic curve (4), and $[I_1]$ denote the non-singular odd characteristic corresponding to $A(I_1) + K$. Then
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\[ \frac{\partial}{\partial v_n} \theta[\mathcal{I}_n](v) \bigg|_{v=0} = \epsilon \left( \frac{\det \omega}{\pi^g} \right)^{1/2} \Delta(\mathcal{I}_1)^{1/4} \Delta(\mathcal{J}_1)^{1/4} \sum_{j=1}^{g} (-1)^{j-1}s_{j-1}(\mathcal{I}_1)\omega_{j,n}, \]  

where \( \epsilon \) satisfies \( \epsilon^8 = 1 \), then \( \Delta(\mathcal{I}_1) \) and \( \Delta(\mathcal{J}_1) \) denote the Vandermonde determinants built from \( \{e_i \mid i \in \mathcal{I}_1\} \) and \( \{e_j \mid j \in \mathcal{J}_1\} \).

This result is nicely presented in the matrix form

\[
\begin{pmatrix}
\frac{\partial}{\partial v_1} \\
\frac{\partial}{\partial v_2} \\
\vdots \\
\frac{\partial}{\partial v_g}
\end{pmatrix} \theta[\mathcal{I}_1](v) \bigg|_{v=0} = \epsilon \left( \frac{\det \omega}{\pi^g} \right)^{1/2} \Delta(\mathcal{I}_1)^{1/4} \Delta(\mathcal{J}_1)^{1/4} \omega^* 
\begin{pmatrix}
s_0(\mathcal{I}_1) \\
-s_1(\mathcal{I}_1) \\
\vdots \\
(-1)^{g-1}s_{g-1}(\mathcal{I}_1)
\end{pmatrix}.
\]

\[
2.8 \text{ General Thomae formula}
\]

Proposed and proven in [3].

**Theorem 1** (The general Thomae theorem) Let \( \mathcal{I}_m \cup \mathcal{J}_m \) with \( \mathcal{I}_m = \{i_1, \ldots, i_{g+1-2m}\} \) and \( \mathcal{J}_m = \{j_1, \ldots, j_{g+1+2m}\} \) be a partition of the set \( \{0, 1, \ldots, 2g+1\} \) of indices of all branch points of a hyperelliptic curve \( \mathcal{C} \), and \( \mathcal{I}_m \) denotes the singular characteristic of multiplicity \( m \) corresponding to \( \mathcal{A}(\mathcal{I}_m) + \mathcal{K} \). Then for any \( n_1, \ldots, n_m \in \{1, \ldots, g\} \) and any set \( \mathcal{K} \subset \mathcal{J}_m \) of cardinality \( q = 2m - 1 \) or \( 2m \), the following relation holds:

\[
\frac{\partial}{\partial v_{n_1}} \cdots \frac{\partial}{\partial v_{n_m}} \theta[\mathcal{I}_m](v) \bigg|_{v=0} = \epsilon \left( \frac{\det \omega}{\pi^g} \right)^{1/2} \Delta(\mathcal{I}_m)^{1/4} \Delta(\mathcal{J}_m)^{1/4}
\times \prod_{p_1,\ldots,p_m \in \mathcal{K}} \prod_{i=1}^{m} \sum_{\substack{j=1 \\
\text{all different}}\part_{j=1}^{g} (-1)^{j-1}s_{j-1}(\mathcal{I}_m \cup \mathcal{K}(p_i))\omega_{j,n_i},
\]

where \( \mathcal{K}(p_i) = \mathcal{K}\setminus\{p_i\} \), \( \epsilon \) satisfies \( \epsilon^8 = 1 \), then \( \Delta(\mathcal{I}_m) \) and \( \Delta(\mathcal{J}_m) \) denote the Vandermonde determinants built from \( \{e_i \mid i \in \mathcal{I}_m\} \) and \( \{e_j \mid j \in \mathcal{J}_m\} \), and \( s_j(\mathcal{I}) \) denotes the elementary symmetric polynomial of degree \( j \) in \( \{e_i \mid i \in \mathcal{I}\} \) at that index 0 of infinity is omitted when it occurs in sets. The relation does not depend on the choice of \( \mathcal{K} \).
Lemma 1 Under the assumptions of the general Thomae theorem (Theorem 1) with $I_0 = I_m \cup K$, the following holds:

$$\frac{\partial^{v_1} \cdots \partial^{v_m} \theta[I_m]}{\theta[I_0]} = \epsilon \prod_{\kappa \in K} \left( \prod_{j \in J_0} (e_{\kappa} - e_j) \right)^{1/4} \prod_{i \in I_m} (e_{\kappa} - e_i)^{1/4} \prod_{j \in J_0} (e_{\kappa} - e_j)^{1/4} \sum_{\text{all different}} \prod_{i=1}^m \sum_{j=1}^g (-1)^{j-1} s_{j-1} (I_m \cup K(p)) \omega_{j, i} \prod_{k \in K \setminus \{p_1, \ldots, p_m\}} (e_{p_i} - e_k)$$

(18)

Proof Formula (18) follows immediately from (17) and the first Thomae formula in the form

$$\left( \frac{\det \omega}{\pi^g} \right)^{1/2} \Delta(I_m)^{1/4} \Delta(J_m)^{1/4} = \epsilon \prod_{\kappa \in K} \left( \prod_{j \in J_0} (e_{\kappa} - e_j) \right)^{1/4} \prod_{i \in I_m} (e_{\kappa} - e_i)^{1/4} \prod_{j \in J_0} (e_{\kappa} - e_j)^{1/4} \sum_{\text{all different}} \prod_{i=1}^m \sum_{j=1}^g (-1)^{j-1} s_{j-1} (I_m \cup K(p)) \omega_{j, i} \prod_{k \in K \setminus \{p_1, \ldots, p_m\}} (e_{p_i} - e_k),$$

where $I_m = I_0 \setminus K$ and $J_m = J_0 \cup K$, and the following relations are applied

$$\Delta(I_0) = \Delta(K) \Delta(I_m) \prod_{\kappa \in K} \prod_{i \in I_m} (e_{\kappa} - e_i),$$

$$\Delta(J_m) = \Delta(K) \Delta(J_0) \prod_{\kappa \in K} \prod_{j \in J_0} (e_{\kappa} - e_j).$$

\[ \square \]

Also recall the following Theorem proven in [3].

Theorem 2 For hyperelliptic curves of genera $g \geq 3$, rank of every matrix of second-order theta derivatives equals three, that is

$$\text{rank} \left( \partial^2 \theta[I_2] \right) = 3.$$ 

Therefore, $\det \left( \partial^2 \theta[I_2] \right) = 0$ in genera $g > 3$.

Remark 1 We assume that branch points in all factors $(e_i - e_l)$ are ordered in such a way that $i > l$, we call this the normal order. Such an order was introduced by Baker [2, p. 307, 346], in the case of real values of $e_k$, all factors are guaranteed to be positive. This allows to avoid the multiplier $\epsilon$ which arises in Thomae formulas and their corollaries since $\epsilon$ is the same for all relations in the class of a fixed multiplicity. The same is true for curves with branch points not necessary real, but arbitrary complex. Once a homology basis as described in Sect. 2.1 is chosen, the normal order of branch points is defined. Note that period matrices reflect this order.
In fact, the first Thomae formula displays the same multiplier \( \epsilon = 1 \) for all partitions \( \mathcal{I}_0 \cup \mathcal{J}_0 \) if the arguments of \( \Delta(\mathcal{I}_0)^{1/4} \) and \( \Delta(\mathcal{J}_0)^{1/4} \) are computed as follows:

\[
\arg \Delta(\mathcal{I})^{1/4} = \frac{1}{4} \sum_{i > I, i, i \in \mathcal{I}} \arg(e_i - e_i),
\]

where \( \mathcal{I} \) is \( \mathcal{I}_0 \) or \( \mathcal{J}_0 \), the normal order is applied, and the range of \( \arg \) is \((-\pi, \pi] \). The normal order leads to the same 8-th root of unity in FTT Corollary 1 if one takes the square root from both sides of the equality, and FTT Corollary 2 if one takes the 4th root from both sides of the equality.

To underline the fact that elements \( \{e_k\} \) are normally ordered the operator of order \( \text{Ord} \) is introduced. Then the first Thomae formula and FTT Corollaries 1 and 2 read as

\[
\theta[\mathcal{I}_0] = \left(\frac{\det \omega}{\pi^g}\right)^{1/2} \text{Ord} \Delta(\mathcal{I}_0)^{1/4} \text{Ord} \Delta(\mathcal{J}_0)^{1/4},
\]

\[
(\text{Ord } e_k - e_m/e_k - e_n)^{1/2} = \frac{\theta([n] \cup \mathcal{I})\theta([n] \cup \mathcal{J})}{\theta([m] \cup \mathcal{I})\theta([m] \cup \mathcal{J})},
\]

\[
(\text{Ord } \prod_{j \in \mathcal{J}_0}(e_{ik} - e_j)/(e_i - e_j)^2 \prod_{i \in \mathcal{I}_0}(e_{ik} - e_i) )^{1/4} = \frac{\theta[\mathcal{I}_0^{(ik \rightarrow jn)}] \theta[\mathcal{I}_0^{(ik \rightarrow jm)}] \theta[\mathcal{J}_0^{(jn \rightarrow im)}]}{\theta[\mathcal{I}_0^{(ik, ii \rightarrow jn, jm)}] \theta[\mathcal{J}_0^{(jn \rightarrow jm)}] \theta[\mathcal{J}_0^{(jn)}]}.
\]

The second Thomae formula also displays the same multiplier for all partitions \( \mathcal{I}_1 \cup \mathcal{J}_1 \), namely: \( \epsilon = 1 \) if the genus \( g \) of a curve is even, and \( \epsilon = -1 \) if \( g \) is odd, at that expression (19) is used for computing arguments of \( \Delta(\mathcal{I}_1)^{1/4} \) and \( \Delta(\mathcal{J}_1)^{1/4} \). That is, the second Thomae formula acquires the form

\[
\frac{\partial}{\partial v_n} \theta[\mathcal{I}_1](v) \bigg|_{v=0} = (-1)^g \left(\frac{\det \omega}{\pi^g}\right)^{1/2} \text{Ord} \Delta(\mathcal{I}_1)^{1/4} \text{Ord} \Delta(\mathcal{J}_1)^{1/4} \times \sum_{j=1}^g (-1)^{j-1} s_{j-1}(\mathcal{I}_1) \omega_{j,n}.
\]

The general Thomae formula acquires the form

\[
\frac{\partial}{\partial v_{n_1}} \cdots \frac{\partial}{\partial v_{n_m}} \theta[\mathcal{I}_m](v) \bigg|_{v=0} = \epsilon \left(\frac{\det \omega}{\pi^g}\right)^{1/2} \text{Ord} \Delta(\mathcal{I}_1)^{1/4} \text{Ord} \Delta(\mathcal{J}_1)^{1/4} \times \sum_{p_1, \ldots, p_m \in \mathcal{K}} \prod_{i=1}^m (-1)^{j-1} s_{j-1}(\mathcal{I}_m \cup \mathcal{K}(p_i)) \omega_{j,n_i}.
\]
Note that Ord is not applied to factors \((e_{p_i} - e_k)\) in the denominator under the product. The general Thomae formula displays the same multiplier \(\epsilon\) in each class of relations with a fixed multiplicity. Actually, \(\epsilon = -1\) at multiplicity \(m = 2\), and \(\epsilon = -(-1)^g\) at \(m = 3\). It seems reasonable to guess that \(\epsilon = (-1)^{g\frac{m+\lfloor m/2 \rfloor}{m}}\) at multiplicity \(m\), whereas further investigations are required.

2.9 Verification

Formulas (12′)–(15′) as well as all the relations presented below had been verified by direct computation of the left- and right-hand sides for all possible partitions. Curves with real and complex branch points had been used. Period matrices \(\omega, \omega'\) had been computed explicitly, as well as matrix \(\tau\) for each curve. Hyperelliptic curves of genera 3, 4, 5 and 6 (four curves with different sets of branch points in each genus) had been taken.

3 Generalisation of Jacobi’s formula

This section consists of three subsections, corresponding to the cases of multiplicities 1, 2 and 3. In the case of multiplicity 1 relations between gradient vectors \(\partial v\) with different characteristics are found (Proposition 1), and the question about rank of a collection of such vectors is elucidated (Propositions 2, 3, Conjecture 1, and Theorem 3). Then a representation of second-order theta derivatives \(\partial^2 v\) (the Hesse matrix) in terms of theta constants and first-order theta derivatives is proposed (Theorem 4). A similar result is obtained in the case of multiplicity 3 (Theorem 5) and can be generalised to higher multiplicities (Conjecture 2).

3.1 First-order theta derivatives

Proposition 1 Let \(\mathcal{I}_0 \cup \mathcal{J}_0\) with \(\mathcal{I}_0 = \{i_1, \ldots, i_g\}\) and \(\mathcal{J}_0 = \{j_1, \ldots, j_{g+1}\}\) be a partition of the set \(\{1, 2, \ldots, 2g + 1\}\) of indices of finite branch points of a genus \(g\) hyperelliptic curve. Then with arbitrary \(K_2, K_2 \in \mathcal{I}_0, K_1 < K_2\) and any \(j_m, j_k \in \mathcal{J}_0\)

\[
\partial v_n \theta[\mathcal{I}_0 \setminus \{K_1, K_2\}] = \left(\theta[\mathcal{I}_0^{(K_1, K_2 \rightarrow j_m, j_k)}] \theta[\mathcal{J}_0^{(j_m)}] \theta[\mathcal{J}_0^{(j_k)}]\right)^{-1} \times \left(\theta[\mathcal{I}_0^{(K_1 \rightarrow j_m)}] \theta[\mathcal{I}_0^{(K_2 \rightarrow j_k)}] \theta[\mathcal{J}_0^{(j_m, j_k \rightarrow K_2)}] \partial v_n \theta[\mathcal{I}_0^{(K_2)}] \right.
\]
\[
\left. - \theta[\mathcal{I}_0^{(K_2 \rightarrow j_m)}] \theta[\mathcal{I}_0^{(K_2 \rightarrow j_k)}] \theta[\mathcal{J}_0^{(j_m, j_k \rightarrow K_1)}] \partial v_n \theta[\mathcal{I}_0^{(K_1)}]\right),
\]

where \(\mathcal{I}_0^{(K_1, K_2 \rightarrow j_m, j_k)}\) denotes the set \(\mathcal{I}_0\) with indices \(K_1, K_2\) replaced by \(j_m, j_k\), etc. The equality does not depend on the choice of \(j_m, j_k\).
Proof Formula (18) from Lemma 1 with \( m = 1 \) and \( \mathcal{K} = 2 \) gives
\[
\frac{\partial_v \theta[\mathcal{I}_0 \setminus \{\kappa_1, \kappa_2\}]}{\theta[\mathcal{I}_0]} = \epsilon \left( \frac{\prod_{j \in \mathcal{J}_0} (e_{\kappa_1} - e_j) (e_{\kappa_2} - e_j)}{\prod_{t \in \mathcal{I}_0 \setminus \{\kappa_1, \kappa_2\}} (e_{\kappa_1} - e_t) (e_{\kappa_2} - e_t)} \right)^{1/4} \\
\times \left( \sum_{j=1}^{g} (-1)^{j-1} s_{j-1} (\mathcal{I}_0^{(k_1)})^j \omega_j n \right) + \sum_{j=1}^{g} (-1)^{j-1} s_{j-1} (\mathcal{I}_0^{(k_2)})^j \omega_j n \right). \tag{21}
\]
Then second Thomae theorem in the form
\[
\sum_{j=1}^{g} (-1)^{j-1} s_{j-1} (\mathcal{I}_0^{(p)})^j \omega_j n = \tilde{\epsilon} \left( \frac{\prod_{t \in \mathcal{I}_0^{(p)} (e_p - e_t)}^{1/4}}{\prod_{j \in \mathcal{J}_0} (e_p - e_j)}^{1/4} \right) \frac{\partial_v \theta[\mathcal{I}_0^{(p)}]}{\theta[\mathcal{I}_0]} \tag{22}
\]
is applied to the right-hand side of (21), so
\[
\partial_v \theta[\mathcal{I}_0 \setminus \{\kappa_1, \kappa_2\}] = \text{Ord} \left\{ \left( \prod_{j \in \mathcal{J}_0} (e_{\kappa_1} - e_j) \right)^{1/4} \frac{\partial_v \theta[\mathcal{I}_0^{(k_1)}]}{\theta[\mathcal{I}_0]_0} \right\} \left( - (e_{\kappa_1} - e_{\kappa_2})^2 \prod_{t \in \mathcal{I}_0^{(k_2)}} (e_{\kappa_1} - e_t) \right)^{1/4} \\
- \text{Ord} \left\{ \left( \prod_{j \in \mathcal{J}_0} (e_{\kappa_1} - e_j) \right)^{1/4} \frac{\partial_v \theta[\mathcal{I}_0^{(k_2)}]}{\theta[\mathcal{I}_0]_0} \right\} \left( - (e_{\kappa_2} - e_{\kappa_1})^2 \prod_{t \in \mathcal{I}_0^{(k_1)}} (e_{\kappa_1} - e_t) \right)^{1/4}.
\]
Recall that \( \tilde{\epsilon} \) for all characteristics of multiplicity, 1 is the same if branch points in all factors \((e_i - e_j)^{1/4}\) are ordered normally, that is \( i > j \). Using FTT Corollary 2 with the normal order in all factors \((e_i - e_j)\), relation (20) is obtained. At that relation, (20) does not depend on the choice of \( j_m, j_k \) due to FTT Corollary 1.

Example 1 In genus 2 case let \( \mathcal{I}_0 = \{\kappa_1, \kappa_2\} \), \( \kappa_1 < \kappa_2 \) and \( \mathcal{J}_0 = \{j_1, j_2, j_3\} \) form a partition of the set \( \{1, 2, 3, 4, 5\} \) of finite branch points. Then (20) reads as
\[
\partial_v \theta^\phi = \left( \theta^{[j_1, j_2]} \theta^{[j_1, j_3]} \theta^{[j_2, j_3]} \right)^{-1} \left( \theta^{[k_2, j_1]} \theta^{[k_2, j_3]} \theta^{[k_2, j_2]} \partial_v \theta^{[k_1]} \right) \\
- \theta^{[k_1, j_1]} \theta^{[k_1, j_2]} \theta^{[k_1, j_3]} \partial_v \theta^{[k_2]}, \tag{23}
\]
where \( \theta^{[i, j]} \) denotes the theta constant with characteristic corresponding to point \( \mathcal{A}([i, j]) + K \). There exist 10 partitions \( \mathcal{I}_0 \cup \mathcal{J}_0 \) of this type and 10 right-hand side expressions all equal to \( \partial_v \theta^\phi \). Some relations are given in Appendix A1, where characteristics of theta functions are presented in two forms: in terms of partitions and in the standard form.

Remark 2 Relations (23) are mentioned in [5, p. 1021] as ‘derived from addition theorems (e.g. Baker 1897, p. 342)’. Here, an alternative proof is proposed.
Example 2} In genus 3 case let $I_0 = \{\iota, \kappa_1, \kappa_2\}$ with $\kappa_1 < \kappa_2$. For each $\iota$ one obtains $(2^g) = 15$ relations of the form

$$\partial_{\iota} \theta^{[\iota]} = \left(\theta^{[\iota, j_1]} \theta^{[j_2, j_3, j_4]} \theta^{[j_1, j_3, j_4]} \right)^{-1} \times \left(\theta^{[\iota, \kappa_2, j_1]} \theta^{[\iota, \kappa_2, j_2]} \theta^{[\iota, \kappa_2, j_3]} \theta^{[\iota, \kappa_2, j_4]} \right) \partial_{\iota} \theta^{[\iota, \kappa_1]} - \theta^{[\iota, \kappa_1, j_1]} \theta^{[\iota, \kappa_1, j_2]} \theta^{[\iota, \kappa_1, j_3]} \theta^{[\iota, \kappa_1, j_4]} \partial_{\iota} \theta^{[\iota, \kappa_2]} \right). \quad (24)$$

Some relations with $\partial_{\iota} \theta^{[\iota]}$ are given in Appendix A2, where characteristics of theta functions are presented both in terms of partitions and in the standard form.

Proposition 1 implies an obvious observation: every vector $\partial_{\iota} \theta(I_0 \setminus \{\kappa_1, \kappa_2\})$ is a linear combination of $\partial_{\iota} \theta(I_0 \setminus \{\kappa_1\})$ and $\partial_{\iota} \theta(I_0 \setminus \{\kappa_2\})$. Note that the intersection of $I_0 \setminus \{\kappa_1, \kappa_2\}$, $I_0 \setminus \{\kappa_1\}$, and $I_0 \setminus \{\kappa_2\}$ has cardinality $g - 2$. Presumably, every three $g$-component vectors $\partial_{\iota} \theta[B_1]$, $\partial_{\iota} \theta[B_2]$, $\partial_{\iota} \theta[B_3]$ such that $B_1 \cap B_2 \cap B_3$ is of cardinality $g - 2$ contain only two linearly independent vectors.

Proposition 2} Let $\mathcal{I} \cup \{\kappa_1, \kappa_2, \kappa_3\} \cup \mathcal{J}$ with $\mathcal{I} = \{i_1, \ldots, i_{g-2}\}$ and $\mathcal{J} = \{j_1, \ldots, j_g\}$ be a partition of the set $\{1, 2, \ldots, 2g + 1\}$ of indices of finite branch points of a genus $g$ hyperelliptic curve, and $\kappa_1 < \kappa_2 < \kappa_3$. Then theta derivative vectors $\partial_{\iota} \theta[\mathcal{I} \cup \{\kappa_1\}]$, $\partial_{\iota} \theta[\mathcal{I} \cup \{\kappa_2\}]$, and $\partial_{\iota} \theta[\mathcal{I} \cup \{\kappa_3\}]$ are linearly dependent, and the following relation between these three holds

$$\theta[\mathcal{I} \cup \{\kappa_1\}] \theta[\mathcal{I} \cup \{\kappa_2\}] \theta[\mathcal{I} \cup \{\kappa_3\}] = 0. \quad (25)$$

Proof is given in Appendix B.

In the context of Proposition 2 with a partition $\mathcal{I} \cup \{\kappa_1, \kappa_2, 0\} \cup \mathcal{J}$, where 0 is the index of infinity, $\mathcal{I} = \{i_1, \ldots, i_{g-2}\}$ and $\mathcal{J} = \{j_1, \ldots, j_{g+1}\}$, relation (20) acquires the form

$$\theta[\mathcal{I} \cup \{\kappa_2\}] \theta[\mathcal{I} \cup \{\kappa_1\}] \theta[\mathcal{I} \cup \{0\}]$$

One should take into account that complement partitions produce the same characteristic, for example $[\mathcal{I} \cup \{\kappa_2, j, 0\}] = [\mathcal{J}^{[j \rightarrow \kappa_1]}]$. Assertions of Proposition 1 and Proposition 2 are combined into

**Proposition 2’** Let $\mathcal{I} \cup \{\kappa_1, \kappa_2, \kappa_3\} \cup \mathcal{J}$ with $\mathcal{I} = \{i_1, \ldots, i_{g-2}\}$ and $\mathcal{J} = \{j_1, \ldots, j_{g+1}\}$ be a partition of the set $\{0, 1, \ldots, 2g + 1\}$ of indices of all branch points of a genus $g \geq 2$ hyperelliptic curve, and $\kappa_1 < \kappa_2 < \kappa_3$, taking into account that the index of infinity is 0. Then theta derivative vectors $\partial_{\iota} \theta[\mathcal{I} \cup \{\kappa_1\}]$, $\partial_{\iota} \theta[\mathcal{I} \cup \{\kappa_2\}]$, and
\( \partial_v \theta[\mathcal{I} \cup \{ \kappa_3 \}] \) are linearly dependent, and the following relation between these three holds:

\[
\begin{align*}
\theta[\mathcal{J}(j_m \to \kappa_1)] &\theta[\mathcal{J}(j_m \to \kappa_2)] \theta[\mathcal{J}(j_m, j_n \to \kappa_2, \kappa_3)] \partial_v \theta[\mathcal{I} \cup \{ \kappa_1 \}] \\
- \theta[\mathcal{J}(j_m \to \kappa_1)] &\theta[\mathcal{J}(j_m \to \kappa_2)] \theta[\mathcal{J}(j_m, j_n \to \kappa_1, \kappa_3)] \partial_v \theta[\mathcal{I} \cup \{ \kappa_2 \}] \\
+ \theta[\mathcal{J}(j_m \to \kappa_1)] &\theta[\mathcal{J}(j_m \to \kappa_3)] \theta[\mathcal{J}(j_m, j_n \to \kappa_1, \kappa_2)] \partial_v \theta[\mathcal{I} \cup \{ \kappa_3 \}] = 0,
\end{align*}
\]

where \( \mathcal{J}(j_m, j_n \to \kappa_1, \kappa_2) \) denotes the set \( \mathcal{J} \) with indices \( j_m, j_n \) replaced by \( \kappa_1, \kappa_2 \) etc.

Introduce the co-lexicographic order of sets of indices. Only sets of equal cardinality are compared (the index of infinity should not be omitted). Two sets are compared by highest indices, if the highest indices are equal then a smaller ones should be compared, and so on.

The result of Proposition 2' can be extended to the case of three linearly independent vectors of theta derivatives.

**Proposition 3**  Let \( \mathcal{I} \cup \{ \kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5 \} \cup \mathcal{J} \) with \( \mathcal{I} = \{ i_1, \ldots, i_g-3 \} \) and \( \mathcal{J} = \{ j_1, \ldots, j_g \} \) be a partition of the set \( \{ 0, 1, \ldots, 2g+1 \} \) of indices of all branch points of a genus \( g \geq 3 \) hyperelliptic curve. Then any collection of three theta derivative vectors of the form \( \partial_v \theta[\mathcal{I} \cup \{ \kappa_1, \kappa_2 \}], \partial_v \theta[\mathcal{I} \cup \{ \kappa_1, \kappa_3 \}], \) and \( \partial_v \theta[\mathcal{I} \cup \{ \kappa_2, \kappa_3 \}] \) is linearly independent. And vector \( \partial_v \theta[\mathcal{I} \cup \{ \kappa_4, \kappa_5 \}] \) is spanned by these three, at that the following relation holds:

\[
\begin{align*}
\theta[\mathcal{J}(j_m \to \kappa_1, \kappa_2)] &\theta[\mathcal{J}(j_m \to \kappa_1, \kappa_3)] \theta[\mathcal{J}(j_m, j_n \to \kappa_2, \kappa_3)] \partial_v \theta[\mathcal{I} \cup \{ \kappa_1, \kappa_2 \}] \\
- \theta[\mathcal{J}(j_m \to \kappa_2, \kappa_3)] &\theta[\mathcal{J}(j_m \to \kappa_1, \kappa_3)] \theta[\mathcal{J}(j_m, j_n \to \kappa_1, \kappa_3)] \partial_v \theta[\mathcal{I} \cup \{ \kappa_1, \kappa_3 \}] \\
+ \theta[\mathcal{J}(j_m \to \kappa_1, \kappa_3)] &\theta[\mathcal{J}(j_m \to \kappa_2, \kappa_3)] \theta[\mathcal{J}(j_m, j_n \to \kappa_1, \kappa_2)] \partial_v \theta[\mathcal{I} \cup \{ \kappa_2, \kappa_3 \}] \\
- \theta[\mathcal{J}(j_m \to \kappa_2, \kappa_3)] &\theta[\mathcal{J}(j_m \to \kappa_1, \kappa_2)] \theta[\mathcal{J}(j_m, j_n \to \kappa_1, \kappa_3)] \partial_v \theta[\mathcal{I} \cup \{ \kappa_3, \kappa_4, \kappa_5 \}] = 0.
\end{align*}
\]

where \( \mathcal{J}(j_m, j_n \to \kappa_1, \kappa_2, \kappa_3) \) denotes the set \( \mathcal{J} \) with indices \( j_m, j_n \) replaced by \( \kappa_1, \kappa_2, \kappa_3 \) etc. Every set \( \mathcal{I} \cup \{ \kappa_1, \kappa_2 \} \) is arranged in the ascending order, and \( \kappa_1 < \kappa_2 < \kappa_3 < \kappa_4 < \kappa_5 \) is supposed, the index of infinity is 0.

**A brief proof.** According to Proposition 2' vector \( \partial_v \theta[\mathcal{I} \cup \{ \kappa_4, \kappa_5 \}] \) is a linear combination of \( \partial_v \theta[\mathcal{I} \cup \{ \kappa_4, \kappa_1 \}] \) and \( \partial_v \theta[\mathcal{I} \cup \{ \kappa_4, \kappa_2 \}] \). Next, \( \partial_v \theta[\mathcal{I} \cup \{ \kappa_4, \kappa_1 \}] \) is a linear combination of \( \partial_v \theta[\mathcal{I} \cup \{ \kappa_1, \kappa_2 \}] \) and \( \partial_v \theta[\mathcal{I} \cup \{ \kappa_1, \kappa_3 \}] \), and \( \partial_v \theta[\mathcal{I} \cup \{ \kappa_4, \kappa_2 \}] \) is a linear combination of \( \partial_v \theta[\mathcal{I} \cup \{ \kappa_1, \kappa_2 \}] \) and \( \partial_v \theta[\mathcal{I} \cup \{ \kappa_2, \kappa_3 \}] \). This leads to three relations of the form (27) which reduce to (28). A more detailed proof is given in Appendix C.

Therefore, a collection of vectors of the form \( \partial_v \theta[\mathcal{I} \cup \{ \kappa_1, \kappa_j \}] \) with card \( \mathcal{I} = g-3 \) and \( i, j \in \{ 1, 2, 3, 4, 5 \} \) contains three linear independent vectors. One can choose \( \partial_v \theta[\mathcal{I} \cup \{ \kappa_1, \kappa_2 \}], \partial_v \theta[\mathcal{I} \cup \{ \kappa_1, \kappa_3 \}], \) and \( \partial_v \theta[\mathcal{I} \cup \{ \kappa_2, \kappa_3 \}] \) as a basis. Then \( \partial_v \theta[\mathcal{I} \cup \{ \kappa_4, \kappa_5 \}] \) is spanned by all three vectors due to the cardinality of the intersection of the four sets: \( \mathcal{I} \cup \{ \kappa_1, \kappa_2 \}, \mathcal{I} \cup \{ \kappa_1, \kappa_3 \}, \mathcal{I} \cup \{ \kappa_2, \kappa_3 \} \) and \( \mathcal{I} \cup \{ \kappa_4, \kappa_5 \} \) is \( g-3 \). Any other vector from the collection is spanned by only two basis vectors due to the intersection of the sets defining characteristics has cardinality \( g-2 \). Indeed, if \( \{ \kappa_i, \kappa_j \} \) is not \( \{ \kappa_4, \kappa_5 \} \) then one of indices \( i, j \) belongs to \( \{ 1, 2, 3 \} \). For example, vector \( \partial_v \theta[\mathcal{I} \cup \{ \kappa_1, \kappa_4 \}] \) is
linearly dependent on $\mathcal{I} \cup \{\kappa_1, \kappa_2\}$, $\mathcal{I} \cup \{\kappa_1, \kappa_3\}$, according to Proposition 2’, since the intersection of sets $\mathcal{I} \cup \{\kappa_1, \kappa_2\}$, $\mathcal{I} \cup \{\kappa_1, \kappa_3\}$ and $\mathcal{I} \cup \{\kappa_1, \kappa_4\}$ has cardinality $g - 2$.

One can choose another linearly independent set, say $\partial_v \theta[\mathcal{I} \cup \{\kappa_2, \kappa_3\}]$, $\partial_v \theta[\mathcal{I} \cup \{\kappa_2, \kappa_5\}]$, and $\partial_v \theta[\mathcal{I} \cup \{\kappa_3, \kappa_5\}]$. Then the vector spanned by all these three vectors is $\partial_v \theta[\mathcal{I} \cup \{\kappa_1, \kappa_4\}]$, and the following relation holds:

$$\theta(\mathcal{J}(\kappa_2 \rightarrow \kappa_3)) \theta(\mathcal{J}(\kappa_4 \rightarrow \kappa_5)) \partial_v \theta[\mathcal{I} \cup \{\kappa_2, \kappa_3\}]$$

$$- \theta(\mathcal{J}(\kappa_2 \rightarrow \kappa_4)) \theta(\mathcal{J}(\kappa_3 \rightarrow \kappa_5)) \partial_v \theta[\mathcal{I} \cup \{\kappa_2, \kappa_4\}]$$

$$+ \theta(\mathcal{J}(\kappa_3 \rightarrow \kappa_5)) \theta(\mathcal{J}(\kappa_4 \rightarrow \kappa_5)) \partial_v \theta[\mathcal{I} \cup \{\kappa_3, \kappa_5\}] - \theta(\mathcal{J}(\kappa_2 \rightarrow \kappa_3)) \theta(\mathcal{J}(\kappa_3 \rightarrow \kappa_5)) \theta(\mathcal{J}(\kappa_4 \rightarrow \kappa_5)) \partial_v \theta[\mathcal{I} \cup \{\kappa_2, \kappa_3\}] = 0.$$

**Remark 3** Note that summands in relations (25)–(29) have alternating signs when all vectors $\partial_v \theta[\mathcal{I} \cup \{\kappa_i, \kappa_k\}]$ are arranged in the ascending order of sets $\mathcal{I} \cup \{\kappa_i, \kappa_k\}$. We suppose that $\kappa_1 < \kappa_2 < \kappa_3 < \kappa_4 < \kappa_5$.

The above is generalised in the following

**Theorem 3** Let $\{\partial_v \theta[\mathcal{B}_n]\}$ be a collection of $g$-component vectors, where $\mathcal{B}_n$ denotes a subset of the set $\{0, 1, \ldots, 2g + 1\}$ of indices of all branch points of a genus $g$ hyperelliptic curve, such that $[\mathcal{B}_n]$ is a characteristic of multiplicity 1.

If $\mathcal{B}_n$ has cardinality $g - n$, then the collection of vectors has rank not greater than $n$. The rank equals $n$ if and only if for every $n$ from $\tau$ to 2 there exists at least one subcollection $\{\partial_v \theta[\mathcal{B}_{k_1}], \ldots, \partial_v \theta[\mathcal{B}_{k_{n+1}}]\}$ of $n + 1$ vectors such that $\bigcap_{i=1}^{n+1} \mathcal{B}_{k_i}$ is of cardinality $g - n$.

**Proof** As stated in Propositions 2’, a collection of $g$-component vectors $\{\partial_v \theta[\mathcal{B}_n]\}$ with $\bigcap_n \mathcal{B}_n = \mathcal{B}^{[g-2]}$, where $\mathcal{B}^{[g-2]}$ denotes a set of cardinality $g - 2$, contains at most two linearly independent vectors. Every $\mathcal{B}_n$ in the collection contains $\mathcal{B}^{[g-2]}$, and is obtained from $\mathcal{B}^{[g-2]}$ by joining one index (due to multiplicity 1 of the corresponding characteristic, see Sect. 2.4 for a detailed explanation). Any pair of vectors $\partial_v \theta[\mathcal{B}_{n_1}]$ and $\partial_v \theta[\mathcal{B}_{n_2}]$ such that $\mathcal{B}_{n_1} = \mathcal{B}^{[g-2]} \cup \{\kappa_1\}$ and $\mathcal{B}_{n_2} = \mathcal{B}^{[g-2]} \cup \{\kappa_2\}$ with $\kappa_1, \kappa_2 \notin \mathcal{B}^{[g-2]}$ serves as a spanning set (or a basis) of the collection, and any other theta derivative $\partial_v \theta[\mathcal{B}_n]$ from the collection is a linear combination of these two as given by (27).

Let $\{\partial_v \theta[\mathcal{B}_n]\}$ be a collection of $g$-component vectors such that $\bigcap_n \mathcal{B}_n$ equals a set $\mathcal{B}^{[g-3]}$ of cardinality $g - 3$. Then every $\mathcal{B}_n$ is obtained from $\mathcal{B}^{[g-3]}$ by joining two indices. As stated in Proposition 3, vectors $\partial_v \theta[\mathcal{B}_{n_1}]$, $\partial_v \theta[\mathcal{B}_{n_2}]$ and $\partial_v \theta[\mathcal{B}_{n_3}]$ such that $\mathcal{B}_{n_1} = \mathcal{B}^{[g-3]} \cup \{\kappa_1, \kappa_2\}$, $\mathcal{B}_{n_2} = \mathcal{B}^{[g-3]} \cup \{\kappa_1, \kappa_3\}$ and $\mathcal{B}_{n_3} = \mathcal{B}^{[g-3]} \cup \{\kappa_2, \kappa_3\}$ with $\kappa_1, \kappa_2, \kappa_3 \notin \mathcal{B}^{[g-3]}$ form a linearly independent set. So the collection containing such three vectors has rank 3, and all other vectors of the collection are linear combinations of these three as given by (28). Note that a collection of rank 2 (if two spanning vectors are sufficient) necessarily has $\bigcap_n \mathcal{B}_n = \mathcal{B}^{[g-2]}$, where $\mathcal{B}^{[g-2]}$ is of cardinality $g - 2$.

Next, consider a collection $\{\partial_v \theta[\mathcal{B}_n]\}$ of $g$-component vectors with $\bigcap_n \mathcal{B}_n$ equal to a set $\mathcal{B}^{[g-4]}$ of cardinality $g - 4$. Then every $\mathcal{B}_n$ is obtained from $\mathcal{B}^{[g-4]}$ by joining three indices, let $\mathcal{K}_{i,j,k} = \{\kappa_i, \kappa_j, \kappa_k\}$ denote a set of indices joined to $\mathcal{B}^{[g-4]}$, that is $\mathcal{B}_n = \mathcal{B}^{[g-4]} \cup \mathcal{K}_{i,j,k}$,
$B^{[g-4]} \cup \mathcal{K}_{i,j,k}$. Among all vectors of the collection at most four constitute a spanning set, for example with $\mathcal{K}_{1,2,3}, \mathcal{K}_{1,2,4}, \mathcal{K}_{1,3,4}, \mathcal{K}_{2,3,4}$. Indeed, by Proposition 2’ a vector $\partial_\eta \theta[B^{[g-4]} \cup \mathcal{K}_{i,j,k}]$ is spanned by two: $\partial_\eta \theta[B^{[g-4]} \cup \mathcal{K}_{1,j,k}]$ and $\partial_\eta \theta[B^{[g-4]} \cup \mathcal{K}_{2,j,k}]$. Then the former is spanned by $\partial_\eta \theta[B^{[g-4]} \cup \mathcal{K}_{1,k}]$ and $\partial_\eta \theta[B^{[g-4]} \cup \mathcal{K}_{3,k}]$, and the latter by $\partial_\eta \theta[B^{[g-4]} \cup \mathcal{K}_{1,k}]$ and $\partial_\eta \theta[B^{[g-4]} \cup \mathcal{K}_{2,k}]$. By Proposition 3 three vectors $\partial_\eta \theta[B^{[g-4]} \cup \mathcal{K}_{1,2,k}], \partial_\eta \theta[B^{[g-4]} \cup \mathcal{K}_{1,3,k}]$ and $\partial_\eta \theta[B^{[g-4]} \cup \mathcal{K}_{2,3,k}]$ form a spanning set for any vector $\partial_\eta \theta[B^{[g-4]} \cup \mathcal{K}_{l,m,k}]$ with $l, m \notin \{1, 2, 3, 4\}$. Similarly to the proof of Proposition 3 one can prove that vectors $\partial_\eta \theta[B^{[g-4]} \cup \mathcal{K}_{1,2,3}], \partial_\eta \theta[B^{[g-4]} \cup \mathcal{K}_{1,2,4}], \partial_\eta \theta[B^{[g-4]} \cup \mathcal{K}_{1,3,4}], \partial_\eta \theta[B^{[g-4]} \cup \mathcal{K}_{2,3,4}]$ form a spanning set for any $\partial_\eta \theta[B^{[g-4]} \cup \mathcal{K}_{i,j,k}]$ with $i, j, k \notin \{1, 2, 3, 4\}$.

Note that a collection $\{\partial_\eta \theta[B_n]\}$ with an intersection $\cap_n B_n$ of cardinality $g - 4$ does not necessarily contain four linearly independent vectors. Suppose that all vectors but one of the collection are spanned by two vectors, say $\partial_\eta \theta[B^{[g-4]} \cup \mathcal{K}_{1,2,3}]$ and $\partial_\eta \theta[B^{[g-4]} \cup \mathcal{K}_{1,2,4}]$, that is the intersection of the corresponding partitions is $B^{[g-4]} \cup \{\kappa_1, \kappa_2\}$ of cardinality $g - 2$. These two vectors with the remaining one form a basis. At the same time, by the assumption, the intersection of all partitions in the collection is of cardinality $g - 4$. Thus, the collection has only three spanning vectors, so the rank is 3 (less than 4). In this case, there is no subcollection with an intersection of cardinality $g - 3$.

Now, let $\{\partial_\eta \theta[B_n]\}$ be a collection of $g$-component vectors such that $\cap_n B_n = B^{[g-1]}$ is of cardinality $g - 1$. Every $B_n$ is obtained from $B^{[g-1]}$ by joining $1 - \tau$ indices, due to multiplicity 1 of characteristic $[B_n]$. By induction, one can prove that a maximal spanning set can be composed from $\tau$ vectors of the form $\partial_\eta \theta[B^{[g-1]} \cup \mathcal{K}^{(\tau)}_2]$, where $\mathcal{K}$ is a set of $\tau$ indices, say $\{\kappa_1, \kappa_2, \ldots, \kappa_\tau\}$, and $\mathcal{K}^{(\tau)}_2 = \mathcal{K} \setminus \{\kappa\}$. However, collections with smaller number of spanning vectors also exist.

**Conjecture 1** Let $\mathcal{I} \cup \mathcal{B} \cup \mathcal{J}$ with $\mathcal{I} = \{i_1, \ldots, i_{g-1}\}, \mathcal{B} = \{\kappa_1, \ldots, \kappa_{2\tau-1}\}$, and $\mathcal{J} = \{j_1, \ldots, j_{g+3-\tau}\}$ be a partition of the set $\{0, 1, \ldots, 2g + 1\}$ of indices of branch points of a genus $g$ hyperelliptic curve. Let $\mathcal{K} = \{\kappa_1, \kappa_2, \ldots, \kappa_\tau\} \subset \mathcal{B}$, and $\mathcal{K}^{(\tau)}$ denote $\mathcal{K} \setminus \{\kappa\}$. Then vectors $\{\partial_\eta \theta[\mathcal{I} \cup \mathcal{K}^{(\tau)}_2]\}_l$ are linearly independent, and vector $\partial_\eta \theta[\mathcal{I} \cup (\mathcal{B} \setminus \mathcal{K})]$ is spanned by these $\tau$ vectors, at that the following relation holds:

$$
\sum_{l=1}^\tau (-1)^{l-1} \theta[\mathcal{J}^{(j_l)} \cup \mathcal{K}^{(\tau)}_2] \theta[\mathcal{J}^{(j_l)} \cup \mathcal{K}^{(\tau)}_2] \theta[\mathcal{J}^{(j_{m-l})} \cup (\mathcal{B} \setminus \mathcal{K}^{(\tau)}_2)] \partial_\eta \theta[\mathcal{I} \cup \mathcal{K}^{(\tau)}_2] \\
+ (-1)^\tau \theta[\mathcal{J}^{(j_l)} \cup (\mathcal{B} \setminus \mathcal{K})] \theta[\mathcal{J}^{(j_l)} \cup (\mathcal{B} \setminus \mathcal{K})] \theta[\mathcal{J}^{(j_{m-l})} \cup \mathcal{K}] \partial_\eta \theta[\mathcal{I} \cup (\mathcal{B} \setminus \mathcal{K})] = 0,
$$

(30)

where $\mathcal{J}^{(j)} = \mathcal{J} \setminus \{j\}$ and $\mathcal{J}^{(j_{m-l})} = \mathcal{J} \setminus \{j_{m-l}\}$. All sets $\{\mathcal{I} \cup \mathcal{K}^{(\tau)}_2\}$ and $\mathcal{I} \cup \mathcal{J}$ are arranged in the ascending order, and $\kappa_1 < \kappa_2 < \cdots < \kappa_{2\tau-1}$ is supposed.

**Remark 4** Under the assumptions of Conjecture 1, any vector $\partial_\eta \theta[\mathcal{I} \cup \mathcal{M}]$ with $\mathcal{M}$ consisting of $\tau - 1$ indices from $\mathcal{B}$ and $\mathcal{M} \neq \mathcal{B} \setminus \mathcal{K}$ is spanned by $\tau - \gamma$ vectors, where $\gamma$ is the cardinality of $\mathcal{M} \cap \mathcal{K}$. 

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3.2 Second-order theta derivatives

Let $I_0 \cup J_0$ be a partition of the set of $2g + 1$ finite branch points with $I_0 = \{i_1, \ldots, i_g\}$, and $J_0 = \{j_1, \ldots, j_{g+1}\}$, where index 0 of infinity belongs to $I_0$ but omitted. Characteristics of multiplicity 2 arise when 3 or 4 indices drop from $I_0$, see Sect. 2.4 for a detailed explanation. Let $K$ denote the set of indices which drop, then $I_2 = I_0 \setminus K$, and $J_2 = J_0 \cup K$.

With the help of the general and the second Thomae formulas, the following decomposition of the Hesse matrix $\partial^2 \theta[I_2]$ can be found.

**Theorem 4** Let $I_0 = \{i_1, \ldots, i_g\}$ and $J_0 = \{j_1, \ldots, j_{g+1}\}$ form a partition of the set $\{1, 2, \ldots, 2g + 1\}$ of indices of finite branch points of a genus $g$ hyperelliptic curve, and $I_2 = I_0 \setminus K$, where $K$ is a set of cardinality $t = 3$ or 4. Then

$$\partial^2_{v_n_1,v_n_2} \theta[I_2] = \frac{1}{\theta[I_0]} \sum_{k,l=1}^t R_{k,l} \partial_{v_n_1} \theta[I_0^{(p_k)}] \partial_{v_n_2} \theta[I_0^{(p_l)}],$$

(31)

or in the matrix form

$$\partial^2_v \theta[I_2] = \frac{1}{\theta[I_0]} \partial_v \theta[I_1] \hat{R} \partial_v \theta[I_1],$$

(32)

where $\partial_v \theta[I_1]$ denotes a $t \times g$ matrix with $(i, j)$-entry equal to $\partial_{v_j} \theta[I_0^{(p_i)}]$, $p_i \in K$, and $t \times t$ matrix $\hat{R}$ has vanishing diagonal entries $R_{k,k} = 0$, and off-diagonal entries are defined as follows with arbitrary $j_n, j_m \in J_0$ and $k, l$ running from 1 to $t$:

- **when** $t = 3$, so that $K \setminus \{p_k, p_l\} = \{q\}$

$$R_{k,l} = (-1)^{k+l} \frac{\theta[I_0^{(p_k \rightarrow j_n, j_m)}] \theta[I_0^{(j_n, j_m \rightarrow q)}] \theta[I_0^{(q \rightarrow j_m)}] \theta[I_0^{(q \rightarrow j_n)}]}{\theta[I_0^{(j_n)}] \theta[I_0^{(j_m)}] \theta[I_0^{(p_k)}] \theta[I_0^{(p_l)}] \theta[I_0^{(q \rightarrow j_n, j_m)}]},$$

(33)

- **when** $t = 4$, so that $K \setminus \{p_k, p_l\} = \{q_1, q_2\}$

$$R_{k,l} = (-1)^{k+l} \frac{\theta[I_0^{(p_k \rightarrow j_n, j_m)}] \theta[I_0^{(q_1, q_2 \rightarrow j_n \cdot j_m)}]}{\theta[I_0^{(j_n, j_m \rightarrow p_k)}] \theta[I_0^{(j_n, j_m \rightarrow p_l)}]} \frac{\theta[I_0^{(j_n)}] \theta[I_0^{(j_m)}]}{2} \prod_{q \in K \setminus \{p_k, p_l\}} \frac{\theta[I_0^{(q \rightarrow j_m)}] \theta[I_0^{(q \rightarrow j_n)}]}{\theta[I_0^{(p_k \rightarrow j_n, j_m)}] \theta[I_0^{(p_l \rightarrow j_n, j_m)}]}.$$

(34)

**Remark 5** Matrix $\hat{R}$ has one of the following forms:
• in the case of three dropped indices (card $K = 3$)

$$\hat{R} = \begin{pmatrix} 0 & R_{1,2} & R_{1,3} \\ R_{1,2} & 0 & R_{1,3} \\ R_{1,3} & R_{2,3} & 0 \end{pmatrix}, \quad (35)$$

• in the case of four dropped indices (card $K = 4$)

$$\hat{R} = \begin{pmatrix} 0 & R_{1,2} & R_{1,3} & R_{1,4} \\ R_{1,2} & 0 & R_{2,3} & R_{2,4} \\ R_{1,3} & R_{2,3} & 0 & R_{3,4} \\ R_{1,4} & R_{2,4} & R_{3,4} & 0 \end{pmatrix}, \quad (36)$$

the sign of $R_{k,l}$ is $(-1)^{k+l}$ in the case of ascending order of indices in $K$.

**Proof** In the case of multiplicity, $m = 2$ formula (18) acquires the form

$$\frac{\partial^2 v_{n_1,n_2}}{\partial [I_2]} = \epsilon \prod_{\kappa \in K} \left( \prod_{j \in J_0} (e_{\kappa} - e_j) \right)^{1/4} \times \sum_{p_1,p_2 \in K, p_1 \neq p_2} \frac{\sum_{j=1}^{g} (-1)^{j-1}s_j (p_1) \omega_{jn_1} \sum_{j=1}^{g} (-1)^{j-1}s_j (p_2) \omega_{jn_2}}{\prod_{\kappa \in K \setminus \{p_1,p_2\}} (e_{p_1} - e_{\kappa})(e_{p_2} - e_{\kappa})},$$

(37)

which holds when $K$ contains 3 or 4 indices. Applying the second Thomae theorem in the form (22), obtain

$$\frac{\partial^2 v_{n_1,n_2}}{\partial [I_0]} = \sum_{p_1,p_2 \in K, p_1 \neq p_2} \frac{\epsilon_{p_1,p_2} (e_{p_1} - e_{p_2})^{1/2}}{\prod_{\kappa \in K \setminus \{p_1,p_2\}} (e_{p_1} - e_{\kappa})^{1/2} (e_{p_2} - e_{\kappa})^{1/2}} \times \prod_{\kappa \in K \setminus \{p_1,p_2\}} \left( \prod_{j \in J_0} (e_{\kappa} - e_j) \right)^{1/4} \frac{1}{\theta[I_0]^{2}} \partial_{\nu_{n_1}} \theta[I_0^{(p_1)}] \partial_{\nu_{n_2}} \theta[I_0^{(p_2)}],$$

(38)

where $\epsilon_{p_1,p_2}^8 = 1$. Next, consider separately the cases of $t$ equal to 3 and 4.

Let $t = 3$, so $K = \{\kappa_1, \kappa_2, \kappa_3\}$. For a pair $\{\kappa_1, \kappa_2\} \in K$ with the help of FTT Corollaries 1 and 2 with the normal order of elements $\{e_k\}$ one finds the multiple at

$$\partial_{\nu_{n_1}} \theta[I_0^{(\kappa_1)}] \partial_{\nu_{n_2}} \theta[I_0^{(\kappa_2)}],$$

which is
Then apply FTT Corollaries 1 and 2 with the normal order. In particular, with a pair
where $p_\kappa \leq \kappa$ holds
and $\epsilon_\kappa$ at $\kappa$
the multiple of $\theta_I$ gets the form

$$\theta[I_{0}(p_1, p_2 \to j_n, j_m)] \theta[I_{0}(p_3 \to j_n)] \theta[I_{0}(p_3 \to j_m)]$$

where $p_1, p_2, p_3$ denote different elements of $\kappa$ such that $\{p_3\} = \kappa \setminus \{p_1, p_2\}$. In fact, at $\kappa_1 \leq \kappa_2 \leq \kappa_3$ the multiplier $\epsilon_{p_1, p_2}$ can be taken as follows: $\epsilon_{\kappa_1, \kappa_2} = \epsilon_{\kappa_2, \kappa_3} = -1$, and $\epsilon_{\kappa_1, \kappa_3} = 1$.

Let $\kappa = 4$, so $\kappa = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4\}$. Denote $\mathcal{I}_2 \cup \{\kappa_1, \kappa_2\} = I_0^{(3, k_4)}$ and so on. Then apply FTT Corollaries 1 and 2 with the normal order. In particular, with a pair

$$\theta[I_{0}^{(3)}] \theta[I_{0}^{(2)}]$$

gets the form

$$\theta[I_{0}(\kappa_1, \kappa_2 \to j_n, j_m)] \theta[I_{0}(\kappa_3, \kappa_4 \to j_n, j_m)] \theta[I_{0}(\kappa_3 \to j_m)] \theta[I_{0}(\kappa_4 \to j_n)]$$

where the following relations are used

$$\left(\text{Ord} \ \frac{(e_{k_1} - e_{k_2})}{(e_{k_3} - e_{k_2})}\right)^{1/2} = \frac{\theta[I_{0}(\kappa_1, \kappa_2 \to j_n, j_m)] \theta[I_{0}(\kappa_3 \to j_m)]}{\theta[I_{0}(\kappa_2, \kappa_3 \to j_n, j_m)]},$$

$$\left(\text{Ord} \ \frac{(e_{k_1} - e_{k_2})}{(e_{k_3} - e_{k_2})}\right)^{1/2} = \frac{\theta[I_{0}(\kappa_1, \kappa_2 \to j_n, j_m)]}{\theta[I_{0}(\kappa_2, \kappa_3 \to j_n, j_m)]},$$

$$\left(\text{Ord} \ \frac{(e_{k_1} - e_{k_2})}{(e_{k_3} - e_{k_2})}\right)^{1/2} = \frac{\theta[I_{0}(\kappa_1, \kappa_2 \to j_n, j_m)]}{\theta[I_{0}(\kappa_2, \kappa_3 \to j_n, j_m)]},$$

$$\left(\text{Ord} \ \frac{(e_{k_1} - e_{k_2})}{(e_{k_3} - e_{k_2})}\right)^{1/2} = \frac{\theta[I_{0}(\kappa_1, \kappa_2 \to j_n, j_m)]}{\theta[I_{0}(\kappa_2, \kappa_3 \to j_n, j_m)]}.$$
New generalisation of Jacobi’s derivative formula

\[
\left( \text{Ord} \frac{\prod_{j \in \mathcal{J}_0} (e_{k_3} - e_j)}{(e_{k_3} - e_{k_1})^2 \prod_{i \in \mathcal{J}_0^{(k_3)}} (e_{k_3} - e_i)} \right)^{1/4} = \frac{\theta[\mathcal{T}_0^{(k_3 \rightarrow j_n)}] \theta[\mathcal{T}_0^{(k_3 \rightarrow j_m)}] \theta[\mathcal{J}_0^{(j_n \rightarrow j_m)}]}{\theta[\mathcal{T}_0^{(k_1, k_3 \rightarrow j_n, j_m)}] \theta[\mathcal{J}_0^{(j_m \rightarrow j_n)}]}
\]

and \( j_m, j_n \) are chosen the same in all the relations. Finally,

\[
\partial^2_{v_{n_1}, v_{n_2}} \theta[n I_2] = \frac{\epsilon}{\theta[I_0] \theta[\mathcal{J}_0^{(j_n)}]^2 \theta[\mathcal{J}_0^{(j_m)}]^2} \times \sum_{p_1, p_2 \in K, p_1 \neq p_2} (-1)^{p_1 + p_2} \frac{\theta[\mathcal{T}_0^{(p_1, p_2 \rightarrow j_n, j_m)}] \theta[\mathcal{T}_0^{(p_1, p_4 \rightarrow j_n, j_m)}] \theta[\mathcal{T}_0^{(p_2, p_4 \rightarrow j_n, j_m)}]}{\theta[I_0^{(p_1, p_4 \rightarrow j_n, j_m)}] \theta[I_0^{(p_2, p_4 \rightarrow j_n, j_m)}]}
\]

where \( \{p_3, p_4\} = K \setminus \{p_1, p_2\} \) for each pair \( \{p_1, p_2\} \). In fact, the multiplier \( \epsilon \) equals 1 and remains the same for all partitions \( \mathcal{I}_2 \cup \mathcal{J}_2 \) in all genera. This accords with the statement of Remark 1 that multiplier \( \epsilon \) in the general Thomae formula for second-order theta derivatives is the same in all genera.

\[\square\]

Remark 6 Theorem 4 provides expressions for second-order theta derivatives in terms of theta constants and first-order theta derivatives. This can be considered as a generalisation of Jacobi’s derivative formula. Each entry of \( \partial^2 \theta[I_2] \) with \( I_2 = I_0 \setminus K \) equals a symmetric bilinear form with matrix \( \hat{R} \), namely:

\[
\partial^2_{v_{n_1}, v_{n_2}} \theta[I_2] = \partial_{v_{n_1}} \theta[I_1] \hat{R} \partial_{v_{n_2}} \theta[I_1],
\]

where in the case of \( K = \{i_1, i_2, i_3\} \subset I_0 \)

\[
\partial_{v_n} \theta[I_1] = (\partial_{v_n} \theta[I_0^{(i_1)}], \partial_{v_n} \theta[I_0^{(i_2)}], \partial_{v_n} \theta[I_0^{(i_3)}])^t
\]

or in the case of \( K = \{i_1, i_2, i_3, i_4\} \subset I_0 \)

\[
\partial_{v_n} \theta[I_1] = (\partial_{v_n} \theta[I_0^{(i_1)}], \partial_{v_n} \theta[I_0^{(i_2)}], \partial_{v_n} \theta[I_0^{(i_3)}], \partial_{v_n} \theta[I_0^{(i_4)}])^t
\]

Note that first-order theta derivatives \( \{\partial_v \theta[I_0^{(i)}]\}_{i \in K} \) involved into (44) form a set of linearly independent vectors.
Example 3 In the case of genus 3 hyperelliptic curve, there is a unique partition \( \mathcal{I}_2 = \emptyset \) of multiplicity 2, which is obtained by \( \binom{4}{1} = 35 \) ways from partitions \( \mathcal{I}_0 \cup \mathcal{J}_0 \) of 7 indices. With \( \mathcal{I}_0 = \{i_1, i_2, i_3\}, i_1 < i_2 < i_3 \), and \( \mathcal{J}_0 = \{j_1, j_2, j_3, j_4\} \), and \( j_m = j_1, j_n = j_2 \) relation (32) reads as

\[
\frac{\partial^2 \theta}{\partial v_{i_1} \partial v_{i_2}} \delta^0 = \frac{1}{\theta\theta_{i_1,i_2,i_3}\theta\theta_{j_1,j_2,j_3,j_4}} \left( \theta_{i_1,i_2,i_3,j_1,j_2,j_3,j_4} \right)
\times \left( - \theta_{i_1,i_2,j_1} \theta_{i_1,i_2,j_3,j_4} \theta_{i_1,i_2,j_1,j_2,j_1} \left( \theta_{i_1,i_2,j_1,j_2,j_1} \theta_{i_1,i_2,j_1,j_2,j_1} \right)^{-1} \right)
\times \left( \partial_{v_{i_1}} \theta_{i_1,i_1,i_1} \partial_{v_{i_2}} \theta_{i_1,i_1,i_1} + \partial_{v_{i_2}} \theta_{i_1,i_1,i_1} \partial_{v_{i_1}} \theta_{i_1,i_1,i_1} \right)
\times \left( \theta_{i_1,i_2,j_1} \theta_{i_1,i_2,j_3,j_4} \theta_{i_1,i_2,j_1,j_2,j_1} \left( \theta_{i_1,i_2,j_1,j_2,j_1} \theta_{i_1,i_2,j_1,j_2,j_1} \right)^{-1} \right)
\times \left( \partial_{v_{i_1}} \theta_{i_1,i_1,i_1} \partial_{v_{i_2}} \theta_{i_1,i_1,i_1} + \partial_{v_{i_2}} \theta_{i_1,i_1,i_1} \partial_{v_{i_1}} \theta_{i_1,i_1,i_1} \right) \times \left( \partial_{v_{i_1}} \theta_{i_1,i_1,i_1} \partial_{v_{i_2}} \theta_{i_1,i_1,i_1} + \partial_{v_{i_2}} \theta_{i_1,i_1,i_1} \partial_{v_{i_1}} \theta_{i_1,i_1,i_1} \right).
\]

(45)

Evidently, second-order theta derivatives are represented by expressions quadratic in first-order theta derivatives.

Moreover, each entry \( \partial_{v_{i_1}} \partial_{v_{i_2}} \theta^0 \) is represented by 35 expressions produced from different \( \mathcal{I}_0 \), and these expressions are all equal to each other. Some relations of this form are given in Appendix D1.

Example 4 In the case of genus 4 among half-period characteristics of multiplicity 2, there is one whose partition is obtained by dropping 4 indices from \( \mathcal{I}_0 \), that is \( \mathcal{I}_2 = \emptyset \). Other characteristics of multiplicity 2 are obtained by dropping 3 indices from \( \mathcal{I}_0 \), then \( \mathcal{I}_2 = \{i \} \).

Let \( \mathcal{I}_0 = \{i_1, i_2, i_3, i_4\} \) with \( i_1 < i_2 < i_3 < i_4 \), and \( \mathcal{J}_0 = \{j_1, j_2, j_3, j_4, j_5\} \). By dropping indices \( \{i_1, i_2, i_3\} = \mathcal{K} \) one gets a representation for \( \partial^2 \theta_{\{i\}} \) of the form (32) (here \( j_m = j_1, j_n = j_2 \))

\[
\frac{\partial^2 \theta}{\partial v_{i_1} \partial v_{i_2}} \theta_{\{i\}} = \frac{1}{\theta\theta_{i_1,i_2,i_3,i_4,j_1,j_2,j_3,j_4,j_5}} \left( \theta_{i_1,i_2,i_3,i_4,j_1,j_2,j_3,j_4,j_5} \right)
\times \left( - \theta_{i_1,i_2,j_1} \theta_{i_1,i_2,j_3,j_4} \theta_{i_1,i_2,j_1,j_2,j_1} \left( \theta_{i_1,i_2,j_1,j_2,j_1} \theta_{i_1,i_2,j_1,j_2,j_1} \right)^{-1} \right)
\times \left( \partial_{v_{i_1}} \theta_{i_1,i_1,i_1} \partial_{v_{i_2}} \theta_{i_1,i_1,i_1} + \partial_{v_{i_2}} \theta_{i_1,i_1,i_1} \partial_{v_{i_1}} \theta_{i_1,i_1,i_1} \right)
\times \left( \theta_{i_1,i_2,j_1} \theta_{i_1,i_2,j_3,j_4} \theta_{i_1,i_2,j_1,j_2,j_1} \left( \theta_{i_1,i_2,j_1,j_2,j_1} \theta_{i_1,i_2,j_1,j_2,j_1} \right)^{-1} \right)
\times \left( \partial_{v_{i_1}} \theta_{i_1,i_1,i_1} \partial_{v_{i_2}} \theta_{i_1,i_1,i_1} + \partial_{v_{i_2}} \theta_{i_1,i_1,i_1} \partial_{v_{i_1}} \theta_{i_1,i_1,i_1} \right) \times \left( \partial_{v_{i_1}} \theta_{i_1,i_1,i_1} \partial_{v_{i_2}} \theta_{i_1,i_1,i_1} + \partial_{v_{i_2}} \theta_{i_1,i_1,i_1} \partial_{v_{i_1}} \theta_{i_1,i_1,i_1} \right). \]

(46)

Next, let \( \mathcal{I}_0 = \{i_1, i_2, i_3, i_4\} \) with \( i_1 < i_2 < i_3 < i_4 \), and \( \mathcal{J}_0 = \{j_1, j_2, j_3, j_4, j_5\} \). A representation for \( \partial^2 \theta^0 \) is obtained from (32) when all four indices of \( \mathcal{I}_0 \) are
dropped. Examples of formulas for these second derivative theta constants are given in Appendix D2.

**Remark 7** Each entry in (32) has many representations depending on a choice of $K$ which is subtracted from $I_0$ and a choice of $j_n$ and $j_m$. All these expressions are equivalent due to Propositions 3 and 2’. In the case of three dropped indices, this is stated by the following

**Proposition 4** Let $I = \{i_1, i_2, \ldots, i_{g-3}\}$, $J = \{j_1, j_2, \ldots, j_g\}$, and $I \cup J \cup \{p_1, p_2, p_3, p_4\}$ be a partition of the set $\{1, 2, \ldots, 2g+1\}$ of indices of finite branch points. Then the right-hand sides of (32) with two partitions $(I \cup \{p_1, p_2, p_3\}) \cup (J \cup \{p_4\})$, and $(I \cup \{p_1, p_2, p_4\}) \cup (J \cup \{p_3\})$ are equal.

In the case of four dropped indices, the following holds:

**Proposition 5** Let $I = \{i_1, i_2, \ldots, i_{g-4}\}$, $J = \{j_1, j_2, \ldots, j_g\}$, and $I \cup J \cup \{p_1, p_2, p_3, p_4, p_5\}$ be a partition of the set $\{1, 2, \ldots, 2g+1\}$ of indices of finite branch points. Then the right-hand sides of (32) with two partitions $(I \cup \{p_1, p_2, p_3, p_4\}) \cup (J \cup \{p_5\})$, and $(I \cup \{p_1, p_2, p_3, p_5\}) \cup (J \cup \{p_4\})$ are equal.

### 3.3 Third-order theta derivatives

Recall that $I_0 \cup J_0$ denotes a partition of the set of $2g+1$ finite branch points with $I_0 = \{i_1, \ldots, i_g\}$, and $J_0 = \{j_1, \ldots, j_{g+1}\}$. A characteristic $[I_3]$ of multiplicity 3 corresponds to $I_3$ obtained from $I_0$ by dropping 5 or 6 indices. Let $K$ denote the set of indices which drop, then $I_3 = I_0 \setminus K$, and $J_3 = J_0 \cup K$.

Third-order theta derivatives are expressed in terms of theta constants and first-order theta derivatives as shown in the following

**Theorem 5** Let $I_0 = \{i_1, \ldots, i_g\}$ and $J_0 = \{j_1, \ldots, j_{g+1}\}$ form a partition of the set $\{1, 2, \ldots, 2g+1\}$ of indices of finite branch points of a genus $g$ hyperelliptic curve, and $I_3 = I_0 \setminus K$, where $K$ is a set of cardinality $t = 5$ or 6. Then

$$
\partial^3_{v_{n_1}, v_{n_2}, v_{n_3}} \theta[I_3] = \frac{-1}{\theta[I_0]^2} \sum_{k_1, k_2, k_3 = 1}^t R_{k_1, k_2, k_3} \partial_{v_{n_1}} \theta[I_0^{(p_{k_1})}] \partial_{v_{n_2}} \theta[I_0^{(p_{k_2})}] \partial_{v_{n_3}} \theta[I_0^{(p_{k_3})}],
$$

(47)

where $I_0^{(p)} = I_0 \setminus \{p\}$, and $R_{k_1, k_2, k_3}$ form a symmetric tensor of order 3 with vanishing diagonal entries $R_{k,k,k} = R_{k,k,l} = R_{k,l,k} = R_{l,k,k} = 0$, and off-diagonal entries defined as follows with arbitrary $j_n, j_m \in J_0$.
• when \( t = 5 \), so that \( \{q_1, q_2\} = K \backslash \{p_{k_1}, p_{k_2}, p_{k_3}\} \)

\[
R_{k_1, k_2, k_3} = (-1)^{p_1 + p_2 + p_3} \frac{\theta[\mathcal{I}_0^{(p_{k_1} \cdot p_{k_2} \rightarrow j_n \cdot j_m)}]}{\theta[\mathcal{J}_0^{(j_m)}] \theta[\mathcal{J}_0^{(j_n)}]} \prod_{q \in \mathcal{K} \backslash \{p_{k_1}, p_{k_2}, p_{k_3}\}} \frac{\theta[\mathcal{J}_0^{(q \rightarrow j_m)}] \theta[\mathcal{J}_0^{(q \rightarrow j_n)}]}{\theta[\mathcal{I}_0^{(p_{k_1} \cdot q \rightarrow j_n \cdot j_m)}] \theta[\mathcal{I}_0^{(p_{k_2} \cdot q \rightarrow j_n \cdot j_m)}] \theta[\mathcal{I}_0^{(p_{k_3} \cdot q \rightarrow j_n \cdot j_m)}]},
\]  

(48)

• when \( t = 6 \), so that \( \{q_1, q_2, q_3\} = K \backslash \{p_{k_1}, p_{k_2}, p_{k_3}\} \).

\[
R_{k_1, k_2, k_3} = (-1)^{p_1 + p_2 + p_3} \frac{1}{\theta[\mathcal{J}_0^{(j_m)}] \theta[\mathcal{J}_0^{(j_n)}]} \prod_{q \in \mathcal{K} \backslash \{p_{k_1}, p_{k_2}, p_{k_3}\}} \frac{\theta[\mathcal{J}_0^{(q \rightarrow j_m)}] \theta[\mathcal{J}_0^{(q \rightarrow j_n)}]}{\theta[\mathcal{I}_0^{(p_{k_1} \cdot q \rightarrow j_n \cdot j_m)}] \theta[\mathcal{I}_0^{(p_{k_2} \cdot q \rightarrow j_n \cdot j_m)}] \theta[\mathcal{I}_0^{(p_{k_3} \cdot q \rightarrow j_n \cdot j_m)}]},
\]  

(49)

**Proof** With \( m = 3 \) formula (18) gets the form

\[
\frac{\partial^3 v_{i_1, i_2, i_3}}{\partial \mathcal{I}_3 \partial [\mathcal{I}_0]} = \epsilon \prod_{\kappa \in \mathcal{K}} \left( \frac{\prod_{j \in \mathcal{J}_0} (e_k - e_j)}{\prod_{i \in \mathcal{I}_3} (e_k - e_i)} \right)^{1/4} \times \sum_{p_1, p_2, p_3 \in \mathcal{K}} \frac{\sum_{j=1}^g (-1)^{p_{k_1} - 1} s_{j-1}^{(p_1)} \omega_{j n_1} \sum_{j=1}^g (-1)^{p_{k_2} - 1} s_{j-1}^{(p_2)} \omega_{j n_2} \sum_{j=1}^g (-1)^{p_{k_3} - 1} s_{j-1}^{(p_3)} \omega_{j n_3}}{\prod_{\kappa \in \mathcal{K} \backslash \{p_{k_1}, p_{k_2}, p_{k_3}\}} (e_{p_1} - e_k) (e_{p_2} - e_k) (e_{p_3} - e_k)},
\]  

(50)

where \( \mathcal{K} \) consists of 5 or 6 indices. After substitution of (22) the right-hand side of (50) transforms into

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\[
\frac{1}{\theta[\mathcal{I}_0]^3} \sum_{p_1, p_2, p_3 \in \mathcal{K} \text{ all different}} \frac{\epsilon_{p_1, p_2, p_3} \Delta\{e_{p_1}, e_{p_2}, e_{p_3}\}}{\left( \prod_{k \in \mathcal{K} \setminus \{p_1, p_2, p_3\}} \prod_{i=1}^3 (e_{p_i} - e_k) \right)^{1/2}} \times \prod_{k \in \mathcal{K} \setminus \{p_1, p_2, p_3\}} \left( \frac{\prod_{j \in \mathcal{J}_0} (e_k - e_j)}{\prod_{i \in \mathcal{I}_3 \cup \{p_1, p_2, p_3\}} (e_k - e_i)} \right)^{1/4} \times \partial_{v_{n_1}} \theta[\mathcal{I}_0^{(p_1)}] \partial_{v_{n_2}} \theta[\mathcal{I}_0^{(p_2)}] \partial_{v_{n_3}} \theta[\mathcal{I}_0^{(p_3)}],
\]

(51)

where \( \epsilon_{p_1, p_2, p_3} = 1 \), and \( \Delta\{e_{p_1}, e_{p_2}, e_{p_3}\} \) denotes the Vandermonde determinant built from elements \( \{e_{p_1}, e_{p_2}, e_{p_3}\} \).

First, let the cardinality of \( \mathcal{K} \) be 5, so \( \mathcal{K} = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5\} \). For a set of three \( \{\kappa_1, \kappa_2, \kappa_3\} \in \mathcal{K} \), applying FTT Corollaries 1 and 2 with normal ordering, one finds the coefficient of \( \partial_{v_{n_1}} \theta[\mathcal{I}_0^{(\kappa_1)}] \partial_{v_{n_2}} \theta[\mathcal{I}_0^{(\kappa_2)}] \partial_{v_{n_3}} \theta[\mathcal{I}_0^{(\kappa_3)}] \):

\[
\text{Ord} \left( (e_{k_2} - e_{k_1})(e_{k_3} - e_{k_1})(e_{k_3} - e_{k_2})(e_{k_4} - e_{k_1})(e_{k_4} - e_{k_3})(e_{k_5} - e_{k_3}) \right)^{1/2} \times \left( \prod_{j \in \mathcal{J}_0} (e_{k_4} - e_j) \right)^{1/4} \times \left( \prod_{i \in \mathcal{I}_3} (e_{k_5} - e_i) \right)^{1/4}
\]

\[
= \frac{\theta[\mathcal{I}_0^{(\kappa_1, k_2 \rightarrow j_n, j_m)}] \theta[\mathcal{I}_0^{(\kappa_2, k_3 \rightarrow j_n, j_m)}] \theta[\mathcal{I}_0^{(\kappa_4, k_5 \rightarrow j_n, j_m)}]}{\theta[\mathcal{I}_0^{(\kappa_1, k_4 \rightarrow j_n, j_m)}] \theta[\mathcal{I}_0^{(\kappa_2, k_4 \rightarrow j_n, j_m)}] \theta[\mathcal{I}_0^{(\kappa_5, k_5 \rightarrow j_n, j_m)}]} \times \frac{\theta[\mathcal{I}_0^{(\kappa_4 \rightarrow j_n)}] \theta[\mathcal{I}_0^{(\kappa_5 \rightarrow j_n)}]}{\theta[\mathcal{I}_0^{(\kappa_2, k_5 \rightarrow j_n, j_m)}] \theta[\mathcal{I}_0^{(\kappa_3, k_5 \rightarrow j_n, j_m)}]} \theta[\mathcal{I}_0^{(\kappa_1 \rightarrow j_m)}] \theta[\mathcal{I}_0^{(\kappa_2 \rightarrow j_m)}] \theta[\mathcal{I}_0^{(\kappa_3 \rightarrow j_m)}] \theta[\mathcal{I}_0^{(\kappa_4 \rightarrow j_m)}] \theta[\mathcal{I}_0^{(\kappa_5 \rightarrow j_m)}] \theta[\mathcal{I}_0^{(j_n \rightarrow j_m)}],
\]

where relations similar to (42) are used. Finally, with an arbitrary pair \( j_n, j_m \in \mathcal{J}_0 \) and elements of \( \mathcal{K} \) denoted by \( p_1, p_2, p_3, q_1, q_2 \) the following holds

\[
\partial_{v_{n_1}, v_{n_2}, v_{n_3}} \theta[\mathcal{I}_3] = \frac{\epsilon}{\theta[\mathcal{I}_0]^2 \theta[\mathcal{J}_0^{(j_m)}]^2 \theta[\mathcal{J}_0^{(j_n)}]^2} \sum_{p_1, p_2, p_3 \in \mathcal{K} \text{ all different}} (-1)^{p_1+p_2+p_3}
\]

\[
\times \theta[\mathcal{I}_0^{(p_1, p_2 \rightarrow j_n, j_m)}] \theta[\mathcal{I}_0^{(p_1, p_3 \rightarrow j_n, j_m)}] \theta[\mathcal{I}_0^{(p_2, p_3 \rightarrow j_n, j_m)}] \theta[\mathcal{I}_0^{(q_1, q_2 \rightarrow j_n, j_m)}] \theta[\mathcal{I}_0^{(q_1 \rightarrow j_n, j_m)}] \theta[\mathcal{I}_0^{(q_2 \rightarrow j_n, j_m)}] \theta[\mathcal{I}_0^{(p_1, q_1 \rightarrow j_n, j_m)}] \theta[\mathcal{I}_0^{(p_2, q_2 \rightarrow j_n, j_m)}] \theta[\mathcal{I}_0^{(p_3, q_2 \rightarrow j_n, j_m)}] \times \partial_{v_{n_1}} \theta[\mathcal{I}_0^{(p_1)}] \partial_{v_{n_2}} \theta[\mathcal{I}_0^{(p_2)}] \partial_{v_{n_3}} \theta[\mathcal{I}_0^{(p_3)}].
\]

(52)

Next, let \( \mathcal{K} = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5, \kappa_6\} \). Then the coefficient of \( \partial_{v_{n_1}} \theta[\mathcal{I}_0^{(p_1)}] \partial_{v_{n_2}} \theta[\mathcal{I}_0^{(p_2)}] \partial_{v_{n_3}} \theta[\mathcal{I}_0^{(p_3)}] \) in (51) is expressed in terms of theta constants with the help of FTT Corollaries 1 and 2 with normal ordering, namely:
\[
\text{Ord} \left( (e_{k_2} - e_{k_1})(e_{k_3} - e_{k_1})(e_{k_1} - e_{k_2})(e_{k_5} - e_{k_4}) \right)^{1/2} \\
\prod_{i=4}^{6} (e_{k_i} - e_{k_2})(e_{k_i} - e_{k_3})^{1/2} \\
\times \left( (e_{k_6} - e_{k_4})(e_{k_6} - e_{k_5}) \right)^{1/2} \prod_{i=4}^{6} \frac{(\prod_{j \in \mathcal{I}_0}(e_{k_i} - e_j))^{1/4}}{(e_{k_i} - e_{k_1})^2 \prod_{i \in \mathcal{I}_0}(e_{k_i} - e_i))^{1/4}} \\
= \theta[I_0^{(k_1, k_2 \rightarrow j_n, j_m)}] \theta[I_0^{(k_1, k_3 \rightarrow j_n, j_m)}] \theta[I_0^{(k_2, k_3 \rightarrow j_n, j_m)}] \\
\times \theta[I_0^{(k_1, k_2 \rightarrow j_n, j_m)}] \theta[I_0^{(k_3, k_6 \rightarrow j_n, j_m)}] \prod_{i=1}^{3} \theta[\mathcal{J}_0^{(j_n, j_m \rightarrow k_i)}] \\
\times \prod_{i=4}^{6} \theta[I_0^{(k_i \rightarrow j_n, j_m)}] \theta[I_0^{(k_i, k_1 \rightarrow j_n, j_m)}] \theta[I_0^{(k_i, k_2 \rightarrow j_n, j_m)}] \theta[I_0^{(k_i, k_3 \rightarrow j_n, j_m)}].
\]

Finally,

\[
\partial_{\nu_1, \nu_2, \nu_3}^3 \theta[I_3] = \frac{\epsilon}{\theta[I_0]^2 \theta[\mathcal{J}_0^{(j_m)}]^3 \theta[\mathcal{J}_0^{(j_n)}]^3} \sum_{p_1, p_2, p_3 \in \mathcal{K}} (-1)^{p_1 + p_2 + p_3} \\
\times \theta[I_0^{(p_1, p_2 \rightarrow j_n, j_m)}] \theta[I_0^{(p_1, p_3 \rightarrow j_n, j_m)}] \theta[I_0^{(p_2, p_3 \rightarrow j_n, j_m)}] \\
\times \theta[I_0^{(q_1, q_2 \rightarrow j_n, j_m)}] \theta[I_0^{(q_1, q_3 \rightarrow j_n, j_m)}] \theta[I_0^{(q_2, q_3 \rightarrow j_n, j_m)}] \\
\times \theta[I_0^{(j_n, j_m \rightarrow p_2)}] \theta[I_0^{(j_n, j_m \rightarrow p_3)}] \\
\times \left( \prod_{q \in \mathcal{K} \setminus \{p_1, p_2, p_3\}} \theta[I_0^{(p_1, q \rightarrow j_n, j_m)}] \theta[I_0^{(p_2, q \rightarrow j_n, j_m)}] \theta[I_0^{(p_3, q \rightarrow j_n, j_m)}] \right) \\
\times \partial_{\nu_1} \partial_{\nu_2} \partial_{\nu_3} \theta[I_0^{(p_1)}] \theta[I_0^{(p_2)}] \theta[I_0^{(p_3)}],
\]

where \(\{q_1, q_2, q_3\} = \mathcal{K} \setminus \{p_1, p_2, p_3\}\) for each set of three \(\{p_1, p_2, p_3\}\). In fact, the multiplier \(\epsilon\) equals \(-1\), and remains the same for all partitions \(I_3 \cup J_3\) in all genera. This accords with the statement of Remark 1 that multiplier \(\epsilon\) in the general Thomae formula for third-order theta derivatives is alternating as well as \(\epsilon\) in the second Thomae formula. \qed

**Remark 8** Theorem 5 gives a representation of third-order theta derivatives in terms of theta constants and first-order theta derivatives which also can be considered as a generalisation of Jacobi’s derivative formula. Note that (47) is a symmetric trilinear form with tensor \(\hat{R}\) of order 3 on the space of vectors \(\partial_{\nu} \theta[I_1]\) of the form

\[
\partial_{\nu} \theta[I_1] = (\partial_{\nu} \theta[I_0^{(i_1)}], \partial_{\nu} \theta[I_0^{(i_2)}], \partial_{\nu} \theta[I_0^{(i_3)}], \partial_{\nu} \theta[I_0^{(i_4)}], \partial_{\nu} \theta[I_0^{(i_5)}])^t,
\]

where \(\{i_1, i_2, i_3, i_4, i_5\} = \mathcal{K}\) such that \(I_3 = I_0 \setminus \mathcal{K}\), or

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Remark 9 There exist many representations of the same third-order theta derivative produced with different choices of $K$ involved into (47) form a set of linearly independent vectors.

Remark 9 There exist many representations of the same third-order theta derivative produced with different choices of $K$ and $j_n, j_m \in J_3 \setminus K$, all of them are equivalent, cf. Remark 7.

Based on Theorems 4 and 5, the following generalisation to arbitrary multiplicity m arises.

Conjecture 2 Let $I_0 = \{i_1, \ldots, i_g\}$ and $J_0 = \{j_1, \ldots, j_{g+1}\}$ form a partition of the set $\{1, 2, \ldots, 2g+1\}$ of indices of finite branch points of a genus g hyperelliptic curve, and $I_m = I_0 \setminus K$, where $K$ is a set of cardinality $t = 2m - 1$ or $2m$. Then

$$\partial_v \theta[I_m] = \frac{\varepsilon}{\theta[I_0]^{m-1}} \sum_{k_1, \ldots, k_m=1}^t R_{k_1, \ldots, k_m} \prod_{i=1}^m \partial_v \theta[I_0^{(p_{k_i})}], \quad (54)$$

where $I_0^{(p)} = I_0 \setminus \{p\}$, and $R_{k_1, \ldots, k_m}$ form a symmetric tensor of order $m$ with vanishing diagonal entries, that is entries with two coinciding indices equal zero, and off-diagonal entries are defined as follows with arbitrary $j_n, j_m \in J_0$

- when $t = 2m - 1$, so that $\{q_1, \ldots, q_{m-1}\} = K \setminus \{p_{k_1}, \ldots, p_{k_m}\}$,

  $$R_{k_1, \ldots, k_m} = \frac{(-1)^{p_{k_1} + \cdots + p_{k_m}}}{\theta[I_0(J_{m})] \theta[I_0(J_{n})]} \prod_{l>1}^{m-1} \theta[I_0^{(p_{k_l} \rightarrow j_m \rightarrow j_n)}] \prod_{l=1}^m \theta[I_0^{(p_{k_l} \rightarrow j_n \rightarrow j_m)}], \quad (55)$$

- when $t = 2m$, so that $\{q_1, \ldots, q_m\} = K \setminus \{p_{k_1}, \ldots, p_{k_m}\}$,

  $$R_{k_1, \ldots, k_m} = \frac{(-1)^{p_{k_1} + \cdots + p_{k_m}}}{\theta[I_0(J_{m})] \theta[I_0(J_{n})]} \prod_{l>1}^{m-1} \theta[I_0^{(p_{k_l} \rightarrow j_m \rightarrow j_n)}] \prod_{l=1}^m \theta[I_0^{(j_n \rightarrow j_m \rightarrow p_{k_l})}] \prod_{l=1}^m \theta[I_0^{(p_{k_l} \rightarrow j_n \rightarrow j_m)}], \quad (56)$$
The multiplier $\epsilon$ equals 1 or $-1$, and this value is the same for all partitions $\mathcal{I}_m \cup \mathcal{J}_m$ with fixed $m$.

4 Schottky-type relation

Returning to second-order theta derivatives, we find out a relation similar to the Schottky identity.

Proposition 6 Let $\mathcal{I}_0 = \{i_1, \ldots, i_{g-4}, p_1, p_2, p_3, p_4\}$, $\mathcal{J}_0 = \{j_1, \ldots, j_{g+1}\}$, and $\mathcal{I}_0 \cup \mathcal{J}_0$ form a partition of the set $\{1, 2, \ldots, 2g+1\}$ of indices of finite branch points of a genus $g \geq 4$ hyperelliptic curve. With an arbitrary choice of the partition and an arbitrary pair $j_m, j_n \in \mathcal{J}_0$ the following relation holds

$$
0 = \theta[\mathcal{I}_0^{(p_1,p_2 \to j_m,j_n)}]^{8} \theta[\mathcal{I}_0^{(p_3,p_4 \to j_m,j_n)}]^{8} \\
+ \theta[\mathcal{I}_0^{(p_1,p_3 \to j_m,j_n)}]^{8} \theta[\mathcal{I}_0^{(p_2,p_4 \to j_m,j_n)}]^{8} \\
+ \theta[\mathcal{I}_0^{(p_1,p_4 \to j_m,j_n)}]^{8} \theta[\mathcal{I}_0^{(p_2,p_3 \to j_m,j_n)}]^{8} \\
-2 \left( \theta[\mathcal{I}_0^{(p_1,p_2 \to j_m,j_n)}]^{4} \theta[\mathcal{I}_0^{(p_3,p_4 \to j_m,j_n)}]^{4} \theta[\mathcal{I}_0^{(p_1,p_3 \to j_m,j_n)}]^{4} \theta[\mathcal{I}_0^{(p_2,p_4 \to j_m,j_n)}]^{4} \\
+ \theta[\mathcal{I}_0^{(p_1,p_3 \to j_m,j_n)}]^{4} \theta[\mathcal{I}_0^{(p_2,p_4 \to j_m,j_n)}]^{4} \theta[\mathcal{I}_0^{(p_1,p_4 \to j_m,j_n)}]^{4} \theta[\mathcal{I}_0^{(p_2,p_3 \to j_m,j_n)}]^{4} \\
+ \theta[\mathcal{I}_0^{(p_1,p_4 \to j_m,j_n)}]^{4} \theta[\mathcal{I}_0^{(p_2,p_3 \to j_m,j_n)}]^{4} \theta[\mathcal{I}_0^{(p_1,p_3 \to j_m,j_n)}]^{4} \theta[\mathcal{I}_0^{(p_2,p_4 \to j_m,j_n)}]^{4} \right).
$$

(57)

Proof This follows immediately from the fact that

$$
\det \hat{R} = 0,
$$

(58)

where $\hat{R}$ is $4 \times 4$ matrix with entries defined by (34) with the set $\mathcal{K} = \{p_1, p_2, p_3, p_4\}$ of dropped indices. To prove (58), we recall another representation of the entries $R_{k,l}$ of matrix $\hat{R}$, namely the representation in terms of branch points, which is given by (41) where $\kappa_1, \kappa_2$ is the pair of indices $\{p_k, p_l\}$ from $\{p_1, p_2, p_3, p_4\}$, and $\kappa_3, \kappa_4 = \mathcal{K}\setminus\{p_k, p_l\}$. We suppose that $p_1 < p_2 < p_3 < p_4$, and indices of branch points in factors $(e_l - e_k)$ are ordered normally.

Substituting the mentioned representation (41) for $R_{k,l}$ into

$$
\det \hat{R} = R_{1,2}^2 R_{3,4}^2 + R_{1,3}^2 R_{2,4}^2 + R_{1,4}^2 R_{2,3}^2 \\
-2 \left( R_{1,2} R_{3,4} R_{1,3} R_{2,4} + R_{1,2} R_{3,4} R_{1,4} R_{2,3} + R_{1,3} R_{2,4} R_{1,4} R_{3,4} \right)
$$

one obtains

$$
c \det \hat{R} = a_1^4 + a_2^4 + a_3^4 - 2(a_1^2 a_2^2 + a_1^2 a_3^2 + a_2^2 a_3^2)
$$

(59)
with a constant multiple $c$ and
\begin{align*}
a_1 &= (e_{p_2} - e_{p_1})(e_{p_4} - e_{p_2}), \\
a_2 &= (e_{p_3} - e_{p_1})(e_{p_4} - e_{p_2}), \\
a_3 &= (e_{p_4} - e_{p_1})(e_{p_3} - e_{p_2}).
\end{align*}

The right-hand side of (59) equals
\[(a_1 + a_2 + a_3)(a_1 + a_2 - a_3)(a_1 - a_2 + a_3)(a_1 - a_2 - a_3) = 0,
\]
due to the identity
\[a_1 - a_2 + a_3 = 0. \qedhere\]

\textbf{Remark 10} By straightforward computation, one can check that all $3 \times 3$ minors of $\hat{R}$ do not vanish. Therefore, in the case of four dropped indices matrix $\hat{R}$ in representation (32) of $\partial_2^2 \theta[I_2]$ has rank 3. By the Cauchy–Binet formula, this implies that the Hesse matrix $\partial_2^2 \theta[I_2]$ also has rank 3 that coincides with the statement of Theorem 2.

The reader can notice that (57) has the form of the Schottky invariant $J$, see [26, p.341–342]. According to [26], \textit{in the Abelian group of all half-period characteristics every syzygetic group of rank 3 (that is of order $2^3$) possesses three constants $\{r_1, r_2, r_3\}$ such that}
\[\sqrt{r_1} \pm \sqrt{r_2} \pm \sqrt{r_3} = 0 \quad (61)\]
\[\text{with a particular choice of signs, and the constant} \]
\[J = r_1^2 + r_2^2 + r_3^2 - 2r_1r_2 - 2r_1r_3 - 2r_2r_3 \quad (62)\]
is \textit{independent of the way of grouping characteristics.}

Recall that two characteristics $[\alpha]$ and $[\beta]$ are called syzygetic or azygetic according as $(\alpha^t \beta' - \beta^t \alpha') \mod 2 = 0$ or 1. Characteristics in a set are called syzygetic (azygetic) if the characteristics are pairwise syzygetic (azygetic). After [1, ch. XVII], we denote a syzygetic group (called Göpel group) by $(P)$. With the help of this group and another three characteristics $A_1$, $A_2$ and $A_3$, one produces three coset spaces (Göpel systems) $A_i + (P) = (A_i P), \ i = 1, 2, 3$. Each coset has the property that every three of its characteristics are syzygetic [1, ch. XVII p.490–491]. Then three constants in the Schottky invariant are defined as follows, see [26, Eq. (3) p.340],
\[r_i = \prod_{A \in (A_i P)} \theta[A]. \quad (63)\]

In [19, Lemma 3 p.538], Igusa gives a more accurate statement: \textit{We choose an even azygetic triplet $\{A_1, A_2, A_3\}$ in the space of all characteristics, a group $(P)$ of rank 3 such that elements of $(A_1 P), (A_2 P), (A_3 P)$ are all even and put (63) for $1 \leq i \leq 3$;
then \( J \) as in (62) depends neither on the triplet \( \{ A_1, A_2, A_3 \} \) nor on \( (P) \). In fact, group \( (P) \) is not required to be syzygetic, but all characteristics of \( (A_1, A_2, A_3) \) should be even. However, in the hyperelliptic case these characteristics should be even non-singular; otherwise, the products vanish, and relation (61) becomes trivial.

Relation (57) has the form of the Schottky invariant \( J \) with \( \{ r_i \} \) produced by a group \( (P) \) of rank 1 (and order 2) consisting of zero characteristics \([\varepsilon_0]\) and characteristic \([\mathcal{A}(\mathcal{K})]\) of point \( \mathcal{A}(\mathcal{K}) \in \mathcal{K} \) with \( \mathcal{K} = \{ p_1, p_2, p_3, p_4 \} \subset \mathcal{I}_0 \). Three characteristics which give rise to the coset spaces are the following:

\[
\begin{align*}
A_1 &= [\mathcal{I}_0(p_1, p_2 \rightarrow j_m, j_n)], \\
A_2 &= [\mathcal{I}_0(p_1, p_3 \rightarrow j_m, j_n)], \\
A_3 &= [\mathcal{I}_0(p_1, p_4 \rightarrow j_m, j_n)]
\end{align*}
\tag{64}
\]

with the arbitrary choice of \( j_m, j_n \in \mathcal{J}_0 \). Since constants \( r_i \) have degree 8, the products \( \prod_{A \in (A, P)} \theta[A] \) are taken to the 4-th power. Relation (57) holds in any genus and proves that the Schottky invariant \( J \) produced by a subgroup \( (P) \) of rank 1 vanishes in the hyperelliptic case.

**Remark 11** From (60) with the help of FTT Corollary 1, one can produce products of \( 2^3 \) theta functions as in (63) and construct the true Schottky invariant. Let a group \( (P) \) be generated by three characteristics, one of which is necessarily \([\mathcal{A}(\mathcal{K})]\). The two others can be taken in the form \([\mathcal{A}([q_1, j_m])]\) and \([\mathcal{A}([q_2, j_n])]\), where \( q_1, q_2 \in \mathcal{J}_0 \setminus \{ j_m, j_n \} \), \( q_1 \not= q_2 \). One or two of the elements \([\mathcal{A}([q_1, j_m])]\) and \([\mathcal{A}([q_2, j_n])]\) can be replaced by \([\mathcal{A}([\tau_1, q_1])]\) and \([\mathcal{A}([\tau_2, q_2])]\) with \( \tau_1, \tau_2 \in \mathcal{I}_0 \setminus \mathcal{K}, \tau_1 \not= \tau_2 \) and \( q_1, q_2 \in \mathcal{J}_0 \setminus \{ j_m, j_n \}, q_1 \not= q_2 \). This choice of generators of \( (P) \) guarantees that all elements of \((A_1, P), (A_2, P), (A_3, P)\) with \( A_1, A_2 \) and \( A_3 \) defined by (64) are even non-singular. The idea of extension of group \((P)\) comes from the fact that the ratio of differences of branch points on the left-hand side of (13) depends only on three indices and can be obtained from different ratios of theta constants.

Then constants \( \{ r_1, r_2, r_3 \} \) are constructed by (63). With \( (P) \) generated by \([\mathcal{A}([p_1, p_2, p_3, p_4])], [\mathcal{A}([q_1, j_m])], [\mathcal{A}([q_2, j_n])], \) and \( A_1 = [\mathcal{I}_0(p_1, p_2 \rightarrow j_m, j_n)], \) where \( \mathcal{I}_0 \supset \mathcal{K} \), the following product over characteristics from \( (A_1, P) \) is obtained:

\[
\begin{align*}
r_1 &= \theta[\mathcal{I}_0(p_1, p_2 \rightarrow j_m, j_n)] \theta[\mathcal{I}_0(p_3, p_4 \rightarrow j_m, j_n)] \theta[\mathcal{I}_0(p_1, p_2 \rightarrow q_1, j_n)] \theta[\mathcal{I}_0(p_3, p_4 \rightarrow q_1, j_n)] \\
&\times \theta[\mathcal{I}_0(p_1, p_2 \rightarrow j_m, q_2)] \theta[\mathcal{I}_0(p_3, p_4 \rightarrow j_m, q_2)] \theta[\mathcal{I}_0(p_1, p_2 \rightarrow q_1, q_2)] \theta[\mathcal{I}_0(p_3, p_4 \rightarrow q_1, q_2)].
\end{align*}
\tag{65}
\]

Similar products for \( r_2 \) and \( r_3 \) are taken over characteristics from \((A_2, P)\) and \((A_3, P)\), where \( A_2 \) and \( A_3 \) are given by (64). It is straightforward to verify that

\[
\sqrt{r_1} - \sqrt{r_2} + \sqrt{r_3} = 0
\]

with \( p_1 < p_2 < p_3 < p_4 \), due to \( \sqrt{r_i} = a_i \), where \( \{ a_1, a_2, a_3 \} \) are defined by (60). Thus, \( J = 0 \).

This way of obtaining the Schottky relation is applicable to arbitrary genus in the hyperelliptic case. A comparison with examples of the Schottky relations from [6] and Springer
and [7] is given in Appendix E1. At the same time, the present result differs from the proposed in [7] generalisation on the base of equality of ratios of the Schottky and the Riemann theta constants (in genera $g - 1$ and $g$) and provides another way of generalisations of the Schottky relation, see Appendices E2 and E3.

**Remark 12** In [19, Lemma 3 p. 538], the three characteristics $A_1, A_2$ and $A_3$ are specified to be an azygetic triplet. So we can deduce that $[\mathcal{I}_0^{(p_1, p_2 \rightarrow j_m, j_n)}], [\mathcal{I}_0^{(p_1, p_3 \rightarrow j_m, j_n)}], [\mathcal{I}_0^{(p_1, p_4 \rightarrow j_m, j_n)}]$, as defined by (64), form an azygetic triplet. Note that these characteristics are even non-singular by the construction.

### 5 Conclusion and discussion

The relations obtained in the present paper shed light on the heap of theta derivative identities. From Sect. 3.1, one can see that among all first-order theta derivatives, there exists a minimal set of gradient vectors serving as a basis. As shown in Subsections 3.2 and 3.3, all higher-order theta derivatives are expressed in terms of first-order theta derivatives and theta constants.

The results are summarised as follows. Let $\mathcal{I}_0 \cup \mathcal{J}_0$ with $\mathcal{I}_0 = \{i_1, \ldots, i_g\}$ and $\mathcal{J}_0 = \{j_1, \ldots, j_{g+1}\}$ be a partition of the set $\{1, 2, \ldots, 2g + 1\}$ of indices of finite branch points of a genus $g$ hyperelliptic curve. Among all gradient vectors $\partial_v \theta$ of null values of theta functions with half-period characteristics, a basis of $g$ linear independent vectors can be chosen. For example, vectors $\{\partial_v \theta[\mathcal{I}_0^{(\kappa_i)}] \mid \kappa_i \in \mathcal{I}_0, \ i = 1, \ldots, g\}$, where $\mathcal{I}_0^{(\kappa_i)} = \mathcal{I}_0 \setminus \{\kappa\}$, form such a basis. All other gradients $\partial_v \theta[\mathcal{I}_1]$ with characteristics of multiplicity 1 are expressed as linear combinations of the basis vectors, and coefficients have the form of ratios of theta constants (see Propositions 1, 2’, 3 and Conjecture 1). Also the question about the rank of a collection of such vectors is elucidated (Theorem 3).

Next, second-order theta derivatives are expressed as symmetric bilinear forms on the vector space of gradients (Theorem 4). Third-order theta derivatives are expressed as symmetric trilinear forms on the vector space of gradients (Theorem 5). Conjecture 2 extends this result to higher-order theta derivatives. All these expressions are called here generalisations of Jacobi’s derivative formula.

However, Jacobi’s derivative formula (1) does not appear as a particular case in this stream of relations. Jacobi’s derivative formula arises as genus 1 case of the Riemann–Jacobi derivative formula. And the latter follows from the second Thomae formula. On the other hand, new relations are derived from the general Thomae formula for higher-order theta derivatives, which express higher-order theta derivatives through first-order theta derivatives and theta constants. So the obtained relation allow to decrease the order of theta derivatives similar to Jacobi’s derivative formula.

The proposed relations serve as the basic material to produce further relations, known and unknown. This is an objective for a future research. As an example, the Schottky relation in the hyperelliptic case is derived (Proposition 6 and Remark 11). It is obtained in genus 4 and generalised to an arbitrary genus higher than 4. This result is in a good correspondence with the examples given in [6] and [7]. Equalities of ratios of the Schottky and the Riemann theta constants are confirmed in genera 3-4 and 4-5,
see Appendix E. In addition, a way of constructing the related Schottky and Riemann theta constants (in genera $g - 1$ and $g$, respectively) is clarified and explained in terms of partitions corresponding to characteristics.

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The datasets generated during the current study are available in the arXiv.

Appendix A: First-order theta derivative relations

A1: Genus 2

Below, a list of relations between vectors of first-order theta derivatives in genus 2 is presented (for brevity, the notation $\theta^{[\kappa_1, \kappa_2]}$ for $\theta[[\kappa_1, \kappa_2]](0; \tau)$ is used)

\[
\partial_v \theta^0 = \theta^{\{2,3\}} \theta^{\{2,4\}} \theta^{\{2,5\}} \partial_v \theta^{(1)} - \frac{\theta^{\{1,3\}} \theta^{\{1,4\}} \theta^{\{1,5\}}}{\theta^{\{3,4\}} \theta^{\{3,5\}} \theta^{\{4,5\}}} \partial_v \theta^{(2)} = \frac{\theta^{\{1,4\}} \theta^{\{1,5\}} \theta^{\{4,5\}}}{\theta^{\{1,2\}} \theta^{\{1,3\}} \theta^{\{2,5\}}} \partial_v \theta^{(3)} = \frac{\theta^{\{1,5\}} \theta^{\{2,5\}} \theta^{\{3,5\}}}{\theta^{\{1,2\}} \theta^{\{1,3\}} \theta^{\{2,3\}}} \partial_v \theta^{(4)} = \frac{\theta^{\{1,4\}} \theta^{\{2,4\}} \theta^{\{3,4\}}}{\theta^{\{1,2\}} \theta^{\{1,3\}} \theta^{\{2,3\}}} \partial_v \theta^{(5)},
\]

and the same with the standard representation of characteristics

\[
\partial_v \theta_{[10]} = (\theta_{[10]}) (\theta_{[00]}) (\theta_{[10]})^{-1} (\theta_{[01]} \theta_{[01]} \theta_{[01]} \partial_v \theta_{[01]} - \theta_{[00]} \theta_{[00]} \theta_{[00]} \partial_v \theta_{[00]} - \theta_{[01]} \theta_{[01]} \theta_{[01]} \partial_v \theta_{[01]})
\]

Note that the matrices of characteristics are constructed from the homology basis given in Sect.2.3. Any two vectors of first-order theta derivatives could serve as a basis, and all other vectors are expressed in terms of these two.
Appendix B: Proof of Proposition 2

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\[
\begin{align*}
\frac{\partial v_\theta^{[1]}}{\partial \theta} &= \frac{\partial (1.3,4)\theta (1.3,5)\theta (3.6,7)}{\partial (1.4,5)\theta (5.6,7)\theta (4.6,7)} \frac{\partial v_\theta^{[1.2]}}{\partial v_\theta} - \frac{\partial (1.2,4)\theta (1.2,5)\theta (2.6,7)}{\partial (1.4,5)\theta (5.6,7)\theta (4.6,7)} \frac{\partial v_\theta^{[1.3]}}{\partial v_\theta} \\
&= \frac{\partial (1.2,5)\theta (2.6,7)\theta (5.6,7)}{\partial (1.4,2)\theta (4.6,7)} \frac{\partial v_\theta^{[1.3]}}{\partial v_\theta} - \frac{\partial (1.2,5)\theta (2.6,7)\theta (5.6,7)}{\partial (1.4,2)\theta (4.6,7)} \frac{\partial v_\theta^{[1.4]}}{\partial v_\theta} \\
&= \frac{\partial (1.5,2)\theta (1.5,3)\theta (5.6,7)}{\partial (1.2,3)\theta (2.6,7)\theta (3.6,7)} \frac{\partial v_\theta^{[1.4]}}{\partial v_\theta} - \frac{\partial (1.2,3)\theta (2.6,7)\theta (5.6,7)}{\partial (1.5,2)\theta (1.5,3)\theta (5.6,7)} \frac{\partial v_\theta^{[1.5]}}{\partial v_\theta} \\
&= \frac{\partial (1.6,2)\theta (1.6,3)\theta (6,4,7)}{\partial (1.2,3)\theta (2.4,7)\theta (3,4,7)} \frac{\partial v_\theta^{[1.5]}}{\partial v_\theta} - \frac{\partial (1.6,2)\theta (1.6,3)\theta (6,4,7)}{\partial (1,7,2)\theta (1,7,3)\theta (7,4,5)} \frac{\partial v_\theta^{[1.6]}}{\partial v_\theta} \\
&= \frac{\partial (1.2,3)\theta (2.4,5)\theta (3,4,5)}{\partial (1,2,3)\theta (3,4,5)\theta (3,4,5)} \frac{\partial v_\theta^{[1.6]}}{\partial v_\theta} - \frac{\partial (1,2,3)\theta (2,4,5)\theta (3,4,5)}{\partial (1,2,3)\theta (2,4,5)\theta (3,4,5)} \frac{\partial v_\theta^{[1.7]}}{\partial v_\theta},
\end{align*}
\]

and the same with the standard representation of characteristics

\[\begin{align*}
\partial v_\theta^{[011]} &= \theta (1,11)\theta (1,01)\theta (0,01) - \theta (1,11)\theta (0,11)\theta (1,01) \\
&= \theta (1,11)\theta (0,01)\theta (0,01) - \theta (1,11)\theta (0,01)\theta (0,01) \\
&= \theta (1,11)\theta (0,01)\theta (0,01) - \theta (1,11)\theta (0,01)\theta (0,01) \\
&= \theta (1,11)\theta (0,01)\theta (0,01) - \theta (1,11)\theta (0,01)\theta (0,01) \\
&= \theta (1,11)\theta (0,01)\theta (0,01) - \theta (1,11)\theta (0,01)\theta (0,01).
\end{align*}\]

Appendix B: Proof of Proposition 2

Proof Applying (20) to decompositions of \(\partial v_\theta[I]\) into three pairs: 1) \(\partial v_\theta[I \cup \{k_1\}]\) and \(\partial v_\theta[I \cup \{k_2\}]\), 2) \(\partial v_\theta[I \cup \{k_1\}]\) and \(\partial v_\theta[I \cup \{k_3\}]\), 3) \(\partial v_\theta[I \cup \{k_2\}]\) and \(\partial v_\theta[I \cup \{k_3\}]\), one obtains three equalities:

\[\begin{align*}
\partial v_\theta[I] &= (\theta [\{j_m, j_n\}]\theta [\mathcal{J}^{(j_m \to k_3)}]\theta [\mathcal{J}^{(j_n \to k_3)}])^{-1} \\
&\quad \times \left\{ \theta [I \cup \{k_2, j_m\}]\theta [I \cup \{k_2, j_n\}]\theta [\mathcal{J}^{(j_m, j_n \to k_2, k_3)}] \partial v_\theta[I \cup \{k_1\}] \\
&\quad - \theta [I \cup \{k_1, j_m\}]\theta [I \cup \{k_1, j_n\}]\theta [\mathcal{J}^{(j_m, j_n \to k_1, k_3)}] \partial v_\theta[I \cup \{k_2\}] \right\} \\
&= (\theta [\{j_m, j_n\}]\theta [\mathcal{J}^{(j_m \to k_2)}]\theta [\mathcal{J}^{(j_n \to k_2)}])^{-1} \\
&\quad \times \left\{ \theta [I \cup \{k_3, j_m\}]\theta [I \cup \{k_3, j_n\}]\theta [\mathcal{J}^{(j_m, j_n \to k_2, k_3)}] \partial v_\theta[I \cup \{k_1\}] \\
&\quad - \theta [I \cup \{k_1, j_m\}]\theta [I \cup \{k_1, j_n\}]\theta [\mathcal{J}^{(j_m, j_n \to k_1, k_3)}] \partial v_\theta[I \cup \{k_2\}] \right\}.
\end{align*}\]
Indeed, extract the two relations and solve them for \( \partial_v \theta[I \cup \{k_1\}] \), \( \partial_v \theta[I \cup \{k_2\}] \), and \( \partial_v \theta[I \cup \{k_3\}] \), and these two relations are equivalent, so one gets (25) from Proposition 2. Indeed, extract the two relations and solve them for \( \partial_v \theta[I \cup \{k_3\}] \)

\[
\begin{aligned}
\partial_v \theta[I \cup \{k_3\}] &= \frac{\theta[J(j_n \to k_1)] \theta[J(j_m \to k_3)] \theta[J(j_m \to j_n, k_2, k_3)]}{\theta[J(j_n \to k_3)] \theta[J(j_m \to k_3)] \theta[J(j_m \to j_n, k_1, k_2)]} \\
&- \frac{\theta[J(j_n \to k_3)] \theta[J(j_m \to k_3)] \theta[I \cup \{k_1, j_m\}] \theta[I \cup \{k_3, j_m\}]}{\theta[J(j_n \to k_3)] \theta[J(j_m \to k_3)] \theta[I \cup \{k_1, j_m\}] \theta[I \cup \{k_2, j_m\}]} \partial_v \theta[I \cup \{k_1\}] \\
&+ \frac{\theta[J(j_n \to k_3)] \theta[J(j_m \to k_3)] \theta[I \cup \{k_1, j_m\}] \theta[I \cup \{k_2, j_m\}]}{\theta[J(j_n \to k_3)] \theta[J(j_m \to k_3)] \theta[I \cup \{k_1, j_m\}] \theta[I \cup \{k_2, j_m\}]} \partial_v \theta[I \cup \{k_2\}] \\
&+ \frac{\theta[J(j_n \to k_3)] \theta[J(j_m \to k_3)] \theta[I \cup \{k_1, j_m\}] \theta[I \cup \{k_1, j_m\}]}{\theta[J(j_n \to k_3)] \theta[J(j_m \to k_3)] \theta[I \cup \{k_1, j_m\}] \theta[I \cup \{k_2, j_m\}]} \partial_v \theta[I \cup \{k_1\}] \\
&- \frac{\theta[J(j_n \to k_3)] \theta[J(j_m \to k_3)] \theta[I \cup \{k_2, j_m\}] \theta[I \cup \{k_3, j_m\}]}{\theta[J(j_n \to k_3)] \theta[J(j_m \to k_3)] \theta[I \cup \{k_2, j_m\}] \theta[I \cup \{k_2, j_m\}]} \partial_v \theta[I \cup \{k_2\}].
\end{aligned}
\]
And this relation is an identity, since the expressions in parentheses vanish identically due to FTT Corollary 1, namely

\[
\frac{\theta[J(J_{j_{2}\rightarrow k_{2}})]^{2}\theta[I \cup \{ k_{3} , j_{m} \}]}{\theta[J(J_{j_{1}\rightarrow k_{1}})]^{2}\theta[I \cup \{ k_{1} , j_{m} \}]} = \frac{\theta[J(J_{j_{2}\rightarrow k_{2}})]^{2}\theta[I \cup \{ k_{2} , j_{m} \}]}{\theta[J(J_{j_{1}\rightarrow k_{1}})]^{2}\theta[I \cup \{ k_{1} , j_{m} \}]} = \frac{\theta[J(J_{j_{3}\rightarrow k_{3}})]^{2}\theta[I \cup \{ k_{3} , j_{m} \}]}{\theta[J(J_{j_{2}\rightarrow k_{2}})]^{2}\theta[I \cup \{ k_{2} , j_{m} \}]} \equiv \frac{e_{k_{2}} - e_{k_{1}}}{e_{k_{3}} - e_{k_{2}}} ,
\]

Thus,

\[
\frac{\theta[J(J_{j_{2}\rightarrow k_{2}})]^{2}\theta[I \cup \{ k_{3} , j_{m} \}]}{\theta[J(J_{j_{1}\rightarrow k_{1}})]^{2}\theta[I \cup \{ k_{1} , j_{m} \}]} - \frac{\theta[J(J_{j_{2}\rightarrow k_{2}})]^{2}\theta[I \cup \{ k_{2} , j_{m} \}]}{\theta[J(J_{j_{1}\rightarrow k_{1}})]^{2}\theta[I \cup \{ k_{1} , j_{m} \}]} + 1 = \frac{e_{k_{2}} - e_{k_{1}}}{e_{k_{3}} - e_{k_{2}}} - \frac{e_{k_{3}} - e_{k_{1}}}{e_{k_{3}} - e_{k_{2}}} + 1 = 0 ,
\]

Therefore, (66) contains one relation between vectors \( \partial_{v}\theta[I \cup \{ k_{1} \}] \), \( \partial_{v}\theta[I \cup \{ k_{2} \}] \), and \( \partial_{v}\theta[I \cup \{ k_{3} \}] \), and every two of the vectors are linearly independent. The relation simplifies to the form

\[
\theta[J(J_{j_{2}\rightarrow k_{2}})]^{2}\theta[I \cup \{ k_{3} , j_{m} \}]=\theta[J(J_{j_{2}\rightarrow k_{2}})]^{2}\theta[I \cup \{ k_{2} , j_{m} \}]=\theta[J(J_{j_{3}\rightarrow k_{3}})]^{2}\theta[I \cup \{ k_{3} , j_{m} \}]=0 .
\]

As an example, in genus 2, one finds

\[
\theta^{[1,4]}\theta^{[1,5]}\theta^{[2,3]}\partial_{v}\theta^{[1]} - \theta^{[2,4]}\theta^{[2,5]}\theta^{[1,3]}\partial_{v}\theta^{[2]} + \theta^{[3,4]}\theta^{[3,5]}\theta^{[1,2]}\partial_{v}\theta^{[3]} = 0 ,
\]

and in genus 3

\[
\theta^{[2,6,7]}\theta^{[5,2,7]}\theta^{[3,4,7]}\partial_{v}\theta^{[1,2]} - \theta^{[3,6,7]}\theta^{[5,3,7]}\theta^{[2,4,7]}\partial_{v}\theta^{[1,3]}
\]

\( \Box \) Springer
\[ +\theta^{[4,6,7]}\theta^{[5,4,7]}\theta^{[2,3,7]}\partial_v\theta^{[1,4]} = 0. \]

Recall that \( \theta^{[i,j]} \) stands for \( \theta([i,j])(0;\tau) \), as well as \( \partial_v\theta^{[i]} \) stands for \( \text{grad}_v\theta^{[i]}(v;\tau)|_{v=0} \).

**Appendix C: Proof of Proposition 3**

**Proof** The set \( \mathcal{I} = \{i_1, \ldots, i_{g-3}\} \) corresponds to a characteristic of multiplicity 2, then joining two indices from \( \{\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5\} \) results into a partition \( \mathcal{I} \cup \{\kappa_i, \kappa_j\} \) corresponding to a characteristic of multiplicity 1. Write down equality (27) with the following gradients:

(i) \( \partial_v\theta[\mathcal{I} \cup \{\kappa_1, \kappa_2\}] \), \( \partial_v\theta[\mathcal{I} \cup \{\kappa_1, \kappa_3\}] \) and \( \partial_v\theta[\mathcal{I} \cup \{\kappa_1, \kappa_5\}] \) with the common set \( \mathcal{I} \cup \{\kappa_1\} \) denoted by \( \mathcal{I} \) in (27),

(ii) \( \partial_v\theta[\mathcal{I} \cup \{\kappa_1, \kappa_2\}] \), \( \partial_v\theta[\mathcal{I} \cup \{\kappa_2, \kappa_3\}] \) and \( \partial_v\theta[\mathcal{I} \cup \{\kappa_2, \kappa_5\}] \) with the common set \( \mathcal{I} \cup \{\kappa_2\} \),

(iii) \( \partial_v\theta[\mathcal{I} \cup \{\kappa_1, \kappa_3\}] \), \( \partial_v\theta[\mathcal{I} \cup \{\kappa_2, \kappa_3\}] \) and \( \partial_v\theta[\mathcal{I} \cup \{\kappa_4, \kappa_5\}] \) with the common set \( \mathcal{I} \cup \{\kappa_5\} \),

and obtain three relations, namely:

\[
\theta[\mathcal{J}(j_m \to \kappa_2,\kappa_4)]\theta[\mathcal{J}(j_m \to \kappa_3,\kappa_4)] \theta[\mathcal{J}(j_m, j_n \to \kappa_3,\kappa_4,\kappa_5)] \partial_v\theta[\mathcal{I} \cup \{\kappa_1, \kappa_2\}] \\
- \theta[\mathcal{J}(j_m, j_n \to \kappa_3,\kappa_4)] \theta[\mathcal{J}(j_m, j_n \to \kappa_2,\kappa_3,\kappa_4,\kappa_5)] \partial_v\theta[\mathcal{I} \cup \{\kappa_1, \kappa_3\}] \\
+ \theta[\mathcal{J}(j_m, j_n \to \kappa_2,\kappa_3,\kappa_4,\kappa_5)] \theta[\mathcal{J}(j_m, j_n \to \kappa_2,\kappa_3,\kappa_4,\kappa_5)] \partial_v\theta[\mathcal{I} \cup \{\kappa_1, \kappa_5\}] = 0,
\]

\[
\theta[\mathcal{J}(j_m \to \kappa_1,\kappa_4)] \theta[\mathcal{J}(j_m, j_n \to \kappa_1,\kappa_4,\kappa_5)] \partial_v\theta[\mathcal{I} \cup \{\kappa_1, \kappa_2\}] \\
- \theta[\mathcal{J}(j_m, j_n \to \kappa_1,\kappa_4)] \theta[\mathcal{J}(j_m, j_n \to \kappa_1,\kappa_4,\kappa_5)] \partial_v\theta[\mathcal{I} \cup \{\kappa_2, \kappa_3\}] \\
+ \theta[\mathcal{J}(j_m, j_n \to \kappa_1,\kappa_4,\kappa_5)] \theta[\mathcal{J}(j_m, j_n \to \kappa_1,\kappa_4,\kappa_5)] \partial_v\theta[\mathcal{I} \cup \{\kappa_2, \kappa_5\}] = 0,
\]

\[
\theta[\mathcal{J}(j_m, j_n \to \kappa_2,\kappa_4)] \theta[\mathcal{J}(j_m, j_n \to \kappa_2,\kappa_3,\kappa_4,\kappa_5)] \partial_v\theta[\mathcal{I} \cup \{\kappa_1, \kappa_3\}] \\
- \theta[\mathcal{J}(j_m, j_n \to \kappa_2,\kappa_3,\kappa_4,\kappa_5)] \theta[\mathcal{J}(j_m, j_n \to \kappa_2,\kappa_3,\kappa_4,\kappa_5)] \partial_v\theta[\mathcal{I} \cup \{\kappa_2, \kappa_5\}] \\
+ \theta[\mathcal{J}(j_m, j_n \to \kappa_2,\kappa_4,\kappa_5)] \theta[\mathcal{J}(j_m, j_n \to \kappa_2,\kappa_4,\kappa_5)] \partial_v\theta[\mathcal{I} \cup \{\kappa_2, \kappa_5\}] = 0.
\]

Next, eliminate vectors \( \partial_v\theta[\mathcal{I} \cup \{\kappa_1, \kappa_5\}] \) and \( \partial_v\theta[\mathcal{I} \cup \{\kappa_2, \kappa_5\}] \) from the relations, and come to

\[
\left( -\theta[\mathcal{J}(j_m \to \kappa_1,\kappa_3)] \theta[\mathcal{J}(j_m \to \kappa_1,\kappa_3)] \theta[\mathcal{J}(j_m \to \kappa_2,\kappa_4)] \theta[\mathcal{J}(j_m \to \kappa_2,\kappa_4)] \\
+ \theta[\mathcal{J}(j_m \to \kappa_2,\kappa_4)] \theta[\mathcal{J}(j_m \to \kappa_2,\kappa_3)] \theta[\mathcal{J}(j_m \to \kappa_1,\kappa_4)] \theta[\mathcal{J}(j_m \to \kappa_1,\kappa_4)] \right) \\
\times \frac{\theta[\mathcal{J}(j_m, j_n \to \kappa_2,\kappa_3,\kappa_4,\kappa_5)]}{\theta[\mathcal{J}(j_m, j_n \to \kappa_2,\kappa_3,\kappa_4,\kappa_5)]} \partial_v\theta[\mathcal{I} \cup \{\kappa_1, \kappa_2\}] \\
+ \theta[\mathcal{J}(j_m \to \kappa_1,\kappa_3)] \theta[\mathcal{J}(j_m \to \kappa_1,\kappa_3)] \theta[\mathcal{J}(j_m, j_n \to \kappa_2,\kappa_4,\kappa_5)] \partial_v\theta[\mathcal{I} \cup \{\kappa_1, \kappa_3\}] \\
- \theta[\mathcal{J}(j_m \to \kappa_2,\kappa_3)] \theta[\mathcal{J}(j_m \to \kappa_2,\kappa_3)] \theta[\mathcal{J}(j_m, j_n \to \kappa_2,\kappa_4,\kappa_5)] \partial_v\theta[\mathcal{I} \cup \{\kappa_2, \kappa_3\}] \\
+ \theta[\mathcal{J}(j_m \to \kappa_4,\kappa_5)] \theta[\mathcal{J}(j_m \to \kappa_4,\kappa_5)] \theta[\mathcal{J}(j_m, j_n \to \kappa_1,\kappa_2,\kappa_3)] \partial_v\theta[\mathcal{I} \cup \{\kappa_4, \kappa_5\}] = 0.
\]
With the help of FTT Corollary 1 the following holds:

\[
\begin{align*}
&\theta[\mathcal{J}(j_m \mapsto k_1, k_3)] \theta[\mathcal{J}(j_m \mapsto k_1, k_3)] \theta[\mathcal{J}(j_m \mapsto k_2, k_4)] \\
&\quad + \theta[\mathcal{J}(j_m \mapsto k_2, k_3)] \theta[\mathcal{J}(j_m \mapsto k_3, k_4)] \theta[\mathcal{J}(j_m \mapsto k_1, k_2)] \\
&\quad + \theta[\mathcal{J}(j_m \mapsto k_3, k_4)] \theta[\mathcal{J}(j_m \mapsto k_1, k_3)] \theta[\mathcal{J}(j_m \mapsto k_1, k_2)] + 1 \\
&= \theta[\mathcal{J}(j_m \mapsto k_1, k_3)] \theta[\mathcal{J}(j_m \mapsto k_1, k_3)] \theta[\mathcal{J}(j_m \mapsto k_1, k_3)] \theta[\mathcal{J}(j_m \mapsto k_1, k_3)] \theta[\mathcal{J}(j_m \mapsto k_1, k_3)] \\
&\quad \times \theta[\mathcal{J}(j_m \mapsto k_1, k_3)] \theta[\mathcal{J}(j_m \mapsto k_1, k_3)] \theta[\mathcal{J}(j_m \mapsto k_1, k_3)] \theta[\mathcal{J}(j_m \mapsto k_1, k_3)] \\
&\quad \times \theta[\mathcal{J}(j_m \mapsto k_1, k_3)] \theta[\mathcal{J}(j_m \mapsto k_1, k_3)] \theta[\mathcal{J}(j_m \mapsto k_1, k_3)] \theta[\mathcal{J}(j_m \mapsto k_1, k_3)] + 1 \\
&= -\frac{(e_{k_4} - e_{k_5}) (e_{k_3} - e_{k_1})}{(e_{k_2} - e_{k_1}) (e_{k_4} - e_{k_3})} + 1 = 0,
\end{align*}
\]

and (67) simplifies to

\[
\begin{align*}
&-\theta[\mathcal{J}(j_m \mapsto k_1, k_2)] \theta[\mathcal{J}(j_m \mapsto k_1, k_2)] \theta[\mathcal{J}(j_m \mapsto k_1, k_2)] \theta[\mathcal{J}(j_m \mapsto k_1, k_2)] \\
&\quad + \theta[\mathcal{J}(j_m \mapsto k_1, k_2)] \theta[\mathcal{J}(j_m \mapsto k_1, k_2)] \theta[\mathcal{J}(j_m \mapsto k_1, k_2)] \theta[\mathcal{J}(j_m \mapsto k_1, k_2)] + \theta[\mathcal{J}(j_m \mapsto k_1, k_2)] \theta[\mathcal{J}(j_m \mapsto k_1, k_2)] \theta[\mathcal{J}(j_m \mapsto k_1, k_2)] \theta[\mathcal{J}(j_m \mapsto k_1, k_2)] \\
&= 0,
\end{align*}
\]

which coincides with (28).

Linear independence of any three vectors in (28), say \(\partial_1 \theta[\mathcal{I} \cup \{k_1, k_2\}]\), \(\partial_2 \theta[\mathcal{I} \cup \{k_1, k_3\}]\), and \(\partial_3 \theta[\mathcal{I} \cup \{k_2, k_3\}]\), follows from the fact that there is no relation between these theta derivatives due to Proposition 2', according to which a relation between three theta derivatives exists only if characteristics of these theta derivatives are such that the intersection of the corresponding sets of indices has cardinality \(g - 2\). Instead, intersection \(\mathcal{I}\) of the sets corresponding to characteristics \([\mathcal{I} \cup \{k_1, k_2\}], [\mathcal{I} \cup \{k_1, k_3\}], [\mathcal{I} \cup \{k_2, k_3\}]\) has cardinality \(n - 3\).

\[\square\]

**Appendix D: Second derivative theta relations**

**D1: Genus 3**

Formula (45) gives 35 representations (excluding different representations of ratios of theta constants following from FTT Corollary 1) for each entry of \(\partial_v^2 \theta^0\), all the relations are equivalent. In particular, with \(\mathcal{I}_0 = \{1, 2, 3\}, j_1 = 6, j_2 = 5\)

\[
\partial_{v_{j_1}, v_{j_2}}^2 \theta^0 = (\theta[1, 2, 3] \theta[4, 5, 7] \theta[4, 6, 7])^{-1}
\]
\[ \theta^{[111]} = \left( \theta^{[101]} \theta^{[010]} \theta^{[001]} \right)^{-1} \]

or with characteristics in the standard form

\[ \theta^{[101]} = \left( \theta^{[111]} \theta^{[010]} \theta^{[001]} \right)^{-1} \]

and with \( T_0 = \{1, 2, 4\}, j_1 = 6, j_2 = 5 \)

\[ \theta^{[101]} = \left( \theta^{[111]} \theta^{[010]} \theta^{[001]} \right)^{-1} \]

or with characteristics in the standard form

\[ \theta^{[101]} = \left( \theta^{[111]} \theta^{[010]} \theta^{[001]} \right)^{-1} \]
D2: Genus 4

There exist 9 characteristics of multiplicity 2 of the form \( \{\ell\} \), and the corresponding second-order theta derivatives are given by the formula (46). Let \( I_0 = \{1, 2, 3, 4\} \) and \( j_1 = 5, j_2 = 6 \), then one obtains with \( \ell = 1 \)

\[
\partial_{v_{n_1}, v_{n_2}}^2 \theta^{(1)} = \frac{1}{\theta^{(1,2,3,4)} \theta^{(6,7,8,9)} \theta^{(5,7,8,9)}} \times 
\left(- \theta^{(1,4,5,6)} \theta^{(4,7,8,9)} \theta^{(1,2,3,5)} \theta^{(1,2,3,6)} (\theta^{(1,2,5,6)} \theta^{(1,3,5,6)})^{-1} \times 
\left( \partial_{v_{n_1}} \theta^{(1,3,4)} \partial_{v_{n_2}} \theta^{(1,2,4)} + \partial_{v_{n_2}} \theta^{(1,3,4)} \partial_{v_{n_1}} \theta^{(1,2,4)} \right) 
\right.
\left. + \theta^{(1,3,5,6)} \theta^{(3,7,8,9)} \theta^{(1,2,4,5)} \theta^{(1,2,4,6)} (\theta^{(1,2,5,6)} \theta^{(1,4,5,6)})^{-1} \times 
\left( \partial_{v_{n_1}} \theta^{(1,3,4)} \partial_{v_{n_2}} \theta^{(1,2,3)} + \partial_{v_{n_2}} \theta^{(1,3,4)} \partial_{v_{n_1}} \theta^{(1,2,3)} \right) 
\right.
\left. - \theta^{(1,2,5,6)} \theta^{(2,7,8,9)} \theta^{(1,3,4,5)} \theta^{(1,3,4,6)} (\theta^{(1,3,5,6)} \theta^{(1,4,5,6)})^{-1} \times 
\left( \partial_{v_{n_1}} \theta^{(1,2,4)} \partial_{v_{n_2}} \theta^{(1,2,3)} + \partial_{v_{n_2}} \theta^{(1,2,4)} \partial_{v_{n_1}} \theta^{(1,2,3)} \right) 
\right)
\]

or in the standard form

\[
\partial_{v_{n_1}, v_{n_2}}^2 \theta^{(1)} = \left( \theta^{(1,11,1)} \theta^{(1,10,1)} \theta^{(1,10,1)} \right)^{-1} \times 
\left(- \theta^{(0,1,1)} \theta^{(1,11,1)} \theta^{(1,10,1)} \theta^{(1,10,1)} (\theta^{(1,11,1)} \theta^{(1,11,1)})^{-1} \times 
\left( \partial_{v_{n_1}} \theta^{(0,1,1)} \partial_{v_{n_2}} \theta^{(1,10,1)} + \partial_{v_{n_2}} \theta^{(0,1,1)} \partial_{v_{n_1}} \theta^{(1,10,1)} \right) 
\right.
\left. + \theta^{(1,11,1)} \theta^{(1,10,1)} \theta^{(1,10,1)} \theta^{(1,10,1)} (\theta^{(1,11,1)} \theta^{(1,11,1)})^{-1} \times 
\left( \partial_{v_{n_1}} \theta^{(0,1,1)} \partial_{v_{n_2}} \theta^{(1,10,1)} + \partial_{v_{n_2}} \theta^{(0,1,1)} \partial_{v_{n_1}} \theta^{(1,10,1)} \right) 
\right.
\left. - \theta^{(1,11,1)} \theta^{(1,10,1)} \theta^{(1,10,1)} \theta^{(1,10,1)} (\theta^{(1,11,1)} \theta^{(1,11,1)})^{-1} \times 
\left( \partial_{v_{n_1}} \theta^{(0,1,1)} \partial_{v_{n_2}} \theta^{(1,10,1)} + \partial_{v_{n_2}} \theta^{(0,1,1)} \partial_{v_{n_1}} \theta^{(1,10,1)} \right) 
\right)
\]

and with \( \ell = 2 \)

\[
\partial_{v_{n_1}, v_{n_2}}^2 \theta^{(2)} = \frac{1}{\theta^{(1,2,3,4)} \theta^{(6,7,8,9)} \theta^{(5,7,8,9)}} \times 
\left(- \theta^{(2,4,5,6)} \theta^{(4,7,8,9)} \theta^{(1,2,3,5)} \theta^{(1,2,3,6)} (\theta^{(1,2,5,6)} \theta^{(2,3,5,6)})^{-1} \times 
\left( \partial_{v_{n_1}} \theta^{(2,3,4)} \partial_{v_{n_2}} \theta^{(1,2,4)} + \partial_{v_{n_2}} \theta^{(2,3,4)} \partial_{v_{n_1}} \theta^{(1,2,4)} \right) 
\right.
\left. + \theta^{(2,3,5,6)} \theta^{(3,7,8,9)} \theta^{(1,2,4,5)} \theta^{(1,2,4,6)} (\theta^{(1,2,5,6)} \theta^{(2,4,5,6)})^{-1} \times 
\left( \partial_{v_{n_1}} \theta^{(2,3,4)} \partial_{v_{n_2}} \theta^{(1,2,3)} + \partial_{v_{n_2}} \theta^{(2,3,4)} \partial_{v_{n_1}} \theta^{(1,2,3)} \right) 
\right)
\]
\[-\theta^{1,2,5,6}\theta^{1,7,8,9}\theta^{2,3,4,5}\theta^{2,3,4,6}(\theta^{2,3,5,6}\theta^{2,4,5,6})^{-1}
\times(\partial_{v_{n_1}}\theta^{1,2,4}\partial_{v_{n_2}}\theta^{1,2,3} + \partial_{v_{n_2}}\theta^{1,2,4}\partial_{v_{n_1}}\theta^{1,2,3})
\]

or in the standard form

\[
\partial^2_{v_{n_1},v_{n_2}}\theta^{1011} = (\theta^{1111}\theta^{1101}\theta^{1011}\theta^{1111})^{-1}
\times(-\theta^{0011}\theta^{1011}\theta^{1001}\theta^{1101})(\theta^{1111}\theta^{1111}\theta^{1111})^{-1}
\times(\partial_{v_{n_1}}\theta^{1011}\partial_{v_{n_2}}\theta^{1011} + \partial_{v_{n_2}}\theta^{1011}\partial_{v_{n_1}}\theta^{1011})
\]

\[
+\theta^{1111}\theta^{1101}\theta^{1011}\theta^{1001}(\theta^{1111}\theta^{1111})(\theta^{1111}\theta^{0011})^{-1}
\times(\partial_{v_{n_1}}\theta^{1011}\partial_{v_{n_2}}\theta^{1011} + \partial_{v_{n_2}}\theta^{1011}\partial_{v_{n_1}}\theta^{1011})
\]

\[
-\theta^{1111}\theta^{1011}\theta^{1101}\theta^{1001}(\theta^{1111}\theta^{1111})(\theta^{0011}\theta^{1111})^{-1}
\times(\partial_{v_{n_1}}\theta^{0011}\partial_{v_{n_2}}\theta^{1011} + \partial_{v_{n_2}}\theta^{0011}\partial_{v_{n_1}}\theta^{1011}).
\]

Let \(I_0 = \{i_1, i_2, i_3, i_4\}, J_0 = \{j_1, j_2, j_3, j_4, j_5\}\) and \(j_m = j_1, j_n = j_2\). Second-order theta derivatives with characteristic \(\emptyset\) are given by the following formula

\[
\partial^2_{v_{n_1},v_{n_2}}\emptyset = (\theta^{[i_1,i_2,i_3,i_4]}(\theta^{[j_2,j_3,j_4,j_5]}\emptyset^{[j_1,j_3,j_4,j_5]}))^{-1}
\]

\[
\times(-\emptyset^{[i_2,i_3,i_4]}(\emptyset^{[j_1,j_2,j_3,j_4,j_5]}\emptyset^{[j_1,j_3,j_4,j_5]})\emptyset^{[i_1,i_2,i_3,i_4]}(\emptyset^{[i_1,i_2,j_3,j_4,j_5]}\emptyset^{[i_1,i_2,j_3,j_4,j_5]})
\]

\[
\times(\partial_{v_{n_1}}\emptyset^{[i_2,i_3,i_4]}\partial_{v_{n_2}}\emptyset^{[i_1,i_2,i_4]} + \partial_{v_{n_2}}\emptyset^{[i_2,i_3,i_4]}\partial_{v_{n_1}}\emptyset^{[i_1,i_2,i_4]})
\]

\[
+\emptyset^{[i_1,i_2,i_3,i_4]}(\emptyset^{[i_1,i_2,j_3,j_4,j_5]}\emptyset^{[i_1,i_2,j_3,j_4,j_5]})\partial_{v_{n_2}}\emptyset^{[i_1,i_2,i_3,i_4]}(\emptyset^{[i_1,i_2,j_3,j_4,j_5]}\emptyset^{[i_1,i_2,j_3,j_4,j_5]})
\]

\[
\times(\partial_{v_{n_1}}\emptyset^{[i_1,i_2,i_3]}\partial_{v_{n_2}}\emptyset^{[i_1,i_2,i_3]} + \partial_{v_{n_2}}\emptyset^{[i_1,i_2,i_3]}\partial_{v_{n_1}}\emptyset^{[i_1,i_2,i_3]})
\]

\[
-\emptyset^{[i_1,i_2,i_3,i_4]}(\emptyset^{[i_1,i_2,j_3,j_4,j_5]}\emptyset^{[i_1,i_2,j_3,j_4,j_5]})\partial_{v_{n_2}}\emptyset^{[i_1,i_2,i_3,i_4]}(\emptyset^{[i_1,i_2,j_3,j_4,j_5]}\emptyset^{[i_1,i_2,j_3,j_4,j_5]})
\]

\[
\times(\partial_{v_{n_1}}\emptyset^{[i_1,i_2,i_3]}\partial_{v_{n_2}}\emptyset^{[i_1,i_2,i_3]} + \partial_{v_{n_2}}\emptyset^{[i_1,i_2,i_3]}\partial_{v_{n_1}}\emptyset^{[i_1,i_2,i_3]})
\]

\[
-\emptyset^{[i_1,i_2,i_3,i_4]}(\emptyset^{[i_1,i_2,j_3,j_4,j_5]}\emptyset^{[i_1,i_2,j_3,j_4,j_5]})\partial_{v_{n_2}}\emptyset^{[i_1,i_2,i_3,i_4]}(\emptyset^{[i_1,i_2,j_3,j_4,j_5]}\emptyset^{[i_1,i_2,j_3,j_4,j_5]})
\]

\[
\times(\partial_{v_{n_1}}\emptyset^{[i_1,i_2,i_3]}\partial_{v_{n_2}}\emptyset^{[i_1,i_2,i_3]} + \partial_{v_{n_2}}\emptyset^{[i_1,i_2,i_3]}\partial_{v_{n_1}}\emptyset^{[i_1,i_2,i_3]})
\]

\[
+\emptyset^{[i_1,i_2,i_3,i_4]}(\emptyset^{[i_1,i_2,j_3,j_4,j_5]}\emptyset^{[i_1,i_2,j_3,j_4,j_5]})\partial_{v_{n_2}}\emptyset^{[i_1,i_2,i_3,i_4]}(\emptyset^{[i_1,i_2,j_3,j_4,j_5]}\emptyset^{[i_1,i_2,j_3,j_4,j_5]})
\]

\[
\times(\partial_{v_{n_1}}\emptyset^{[i_1,i_2,i_3]}\partial_{v_{n_2}}\emptyset^{[i_1,i_2,i_3]} + \partial_{v_{n_2}}\emptyset^{[i_1,i_2,i_3]}\partial_{v_{n_1}}\emptyset^{[i_1,i_2,i_3]})
\]

\[
+\emptyset^{[i_1,i_2,i_3,i_4]}(\emptyset^{[i_1,i_2,j_3,j_4,j_5]}\emptyset^{[i_1,i_2,j_3,j_4,j_5]})\partial_{v_{n_2}}\emptyset^{[i_1,i_2,i_3,i_4]}(\emptyset^{[i_1,i_2,j_3,j_4,j_5]}\emptyset^{[i_1,i_2,j_3,j_4,j_5]})
\]

\[
\times(\partial_{v_{n_1}}\emptyset^{[i_1,i_2,i_3]}\partial_{v_{n_2}}\emptyset^{[i_1,i_2,i_3]} + \partial_{v_{n_2}}\emptyset^{[i_1,i_2,i_3]}\partial_{v_{n_1}}\emptyset^{[i_1,i_2,i_3]}).
\]
New generalisation of Jacobi's derivative formula

\[ \partial_{x_1, x_2} \theta \phi^{[1, j_1, j_2] \phi^{[3, j_3, j_4] \phi^{[4, j_5, j_6]}} \]

\[ \times \partial_{x_1, x_2} \theta^{[1, j_1, j_2] \phi^{[3, j_3, j_4] \phi^{[4, j_5, j_6]}} \]

\[ \times \left( \partial_{x_1} \theta^{[1, j_1, j_2]} + \partial_{x_2} \theta^{[1, j_1, j_2]} \partial_{x_1} \theta^{[1, j_1, j_2]} \right) \]

For example with \( I_0 = \{1, 2, 3, 4\}, j_1 = 5, j_2 = 6 \) one gets

\[ \partial_{x_1, x_2}^2 \theta = \left( \phi^{[1, 2, 3, 4]} \phi^{[6, 7, 8, 9]} \phi^{[5, 7, 8, 9]} \right)^{-1} \]

or with characteristics in the standard form

\[ \partial_{x_1, x_2}^2 \theta_{1011} = \left( \phi^{[1, 111]} \phi^{[1, 010]} \phi^{[1, 011]} \right)^{-1} \]
Appendix E: Examples of the Schottky relations

Here, we analyse some examples of the Schottky invariants presented in the literature and propose some new examples derived on the base of ideas from Remark 11.

E1: Examples from papers of Farkas & Rauch

In genus 4 hyperelliptic case, the Schottky relation (61) is given by formula (R2) [6, p. 685], the same in [7, p.459]. In our notation and choice of homology basis, it gets the form

$$\times (\partial_{\nu_1} \theta_{1011} \partial_{\nu_2} \theta_{1011} + \partial_{\nu_2} \theta_{0111} \partial_{\nu_1} \theta_{1011})
- \theta_{1011} \theta_{0111} \theta_{1011} \theta_{1011}
- \theta_{1011} \theta_{0111} \theta_{1011} \theta_{1011}
\times (\partial_{\nu_1} \theta_{0111} \partial_{\nu_2} \theta_{1001} + \partial_{\nu_2} \theta_{0111} \partial_{\nu_1} \theta_{1001})
+ \theta_{0111} \theta_{0111} \theta_{0111} \theta_{0111}
+ \theta_{1011} \theta_{1011} \theta_{1011} \theta_{1011}
\times (\partial_{\nu_1} \theta_{0111} \partial_{\nu_2} \theta_{0111} + \partial_{\nu_2} \theta_{0111} \partial_{\nu_1} \theta_{0111})
- \theta_{0111} \theta_{0111} \theta_{0111} \theta_{0111}
- \theta_{0111} \theta_{0111} \theta_{0111} \theta_{0111}
\times (\partial_{\nu_1} \theta_{0111} \partial_{\nu_2} \theta_{0111} + \partial_{\nu_2} \theta_{0111} \partial_{\nu_1} \theta_{0111})
\times (\partial_{\nu_1} \theta_{0111} \partial_{\nu_2} \theta_{0111} + \partial_{\nu_2} \theta_{0111} \partial_{\nu_1} \theta_{0111})$$.

The signs are chosen according to the order of sets of indices. The initial syzygetic group \((P)\) of rank 3 is generated by 3 elements

\[
P_1 = [\varepsilon_1] + [\varepsilon_2] \quad \{1, 2\}
\]
\[
P_2 = [\varepsilon_1] + [\varepsilon_2] + [\varepsilon_3] \quad \{1, 2, 3\}
\]
\[
P_3 = [\varepsilon_1] + [\varepsilon_2] + [\varepsilon_3] + [\varepsilon_4] + [\varepsilon_5] \quad \{1, 2, 3, 4, 5\},
\]
given with the corresponding sets of indices. One can add sets by the union operation, taking into account that an index occurring twice drops (two equal elements cancel each other out). Recall that characteristic \([I]\) of theta constant corresponds to the set \(I + R\), where \(R = \{2, 4, 6, 8\}\) is associated with the vector of Riemann constants. For example, the set of indices in characteristic \([\{1, 4, 6, 8\}\) is obtained as \(\{1, 2\} + \{2, 4, 6, 8\}\). In such notation it becomes obvious that all theta constants in (68) have non-singular even characteristics, because all sets corresponding to characteristics...
contain $4 = g$ indices. Thus, characteristics of theta constants in the first summand of (68) correspond to the following elements of group $(P)$ (in the same order)

$$P_0 = 0, \quad P_1, \quad P_2, \quad P_1 P_2, \quad P_3, \quad P_1 P_3, \quad P_2 P_3, \quad P_1 P_2 P_3.$$ 

Here the notation of [1, ch. XVII] is adopted, that is $P_1 P_2$ denotes the sum of characteristics $P_1$ and $P_2$. Three cosets, which give rise to three products of theta constants in the Schottky invariant, are produced by the following azygetic triplet of characteristics

$$A_1 = 0 = [(2, 4, 6, 8)] \quad \emptyset$$
$$A_2 = [\varepsilon_6] + [\varepsilon_9] = [(2, 4, 6, 9)] \quad \{8, 9\}$$
$$A_3 = [\varepsilon_7] + [\varepsilon_8] = [(2, 4, 6, 7)] \quad \{7, 8\}.$$

A concise proof of (68) follows from FTT Corollary 1:

$$\left( \frac{\theta[2,4,6,8] \theta[1,3,5,8] \theta[1,3,5,7] \theta[2,4,7,9] \theta[1,4,7,9] \theta[2,3,5,7] \theta[2,3,5,8] \theta[1,4,6,8]}{\theta[2,4,6,7] \theta[1,3,5,7] \theta[1,3,5,8] \theta[2,4,8,9] \theta[1,4,8,9] \theta[2,3,5,8] \theta[2,3,5,7] \theta[1,4,6,7]} \right) \times \left( \frac{\theta[2,5,6,8] \theta[1,3,4,8] \theta[1,3,4,7] \theta[2,5,7,9] \theta[1,5,7,9] \theta[2,3,4,7] \theta[2,3,4,8] \theta[1,5,6,8]}{\theta[2,5,6,7] \theta[1,3,4,7] \theta[1,3,4,8] \theta[2,5,8,9] \theta[1,5,8,9] \theta[2,3,4,8] \theta[2,3,4,7] \theta[1,5,6,7]} \right)^{1/2}
- \left( \frac{\theta[2,4,6,9] \theta[1,3,5,9] \theta[1,3,5,7] \theta[2,4,7,8] \theta[1,4,7,8] \theta[2,3,5,7] \theta[2,3,5,9] \theta[1,4,6,9]}{\theta[2,4,6,7] \theta[1,3,5,7] \theta[1,3,5,8] \theta[2,4,8,9] \theta[1,4,8,9] \theta[2,3,5,8] \theta[2,3,5,7] \theta[1,4,6,7]} \right) \times \left( \frac{\theta[2,5,6,9] \theta[1,3,4,9] \theta[1,3,4,7] \theta[2,5,7,8] \theta[1,5,7,8] \theta[2,3,4,7] \theta[2,3,4,9] \theta[1,5,6,9]}{\theta[2,5,6,7] \theta[1,3,4,7] \theta[1,3,4,9] \theta[2,5,8,9] \theta[1,5,8,9] \theta[2,3,4,9] \theta[2,3,4,7] \theta[1,5,6,7]} \right)^{1/2}
- 1 = \frac{(e_9 - e_7)(e_8 - e_6)}{(e_9 - e_8)(e_7 - e_6)} - \frac{(e_8 - e_7)(e_9 - e_6)}{(e_9 - e_8)(e_7 - e_6)} - 1 = 0.$$ 

In the context of Remark 11, the group $(P)$ is generated by three even pairwise syzygetic characteristics: $[(2, 4, 7, 9)] = [\mathcal{A}((6, 7, 8, 9))]$, $[(1, 4, 6, 8)] = [\mathcal{A}((1, 2))]$, $[(2, 5, 6, 8)] = [\mathcal{A}((4, 5))]$. In this case $\mathcal{I}_0 = \{6, 7, 8, 9\}$ and three cosets are produced by the even azygetic triplet: $A_1 = [\mathcal{I}_0^{(7,9 \to 2,4)}]$, $A_2 = [\mathcal{I}_0^{(7,8 \to 2,4)}]$, $A_3 = [\mathcal{I}_0^{(8,9 \to 2,4)}]$.

At the same time, ratios from [6, Theorem 1, p. 680] connect theta constants in genera $g - 1$ and $g$. In the context of the above example, one gets the following equalities of ratios in genera 3 and 4

$$\frac{(\theta[2,4,6])^2}{\theta[2,4,6,8] \theta[1,4,6,8]} = \frac{(\theta[3,5,7])^2}{\theta[2,4,7,9] \theta[1,5,7,9]} = \frac{(\theta[2,5,7])^2}{\theta[2,4,7,9] \theta[1,4,7,9]} = \frac{(\theta[3,4,6])^2}{\theta[2,5,6,8] \theta[1,5,6,8]} = \frac{(\theta[2,4,7])^2}{\theta[2,4,6,9] \theta[1,4,6,9]} = \frac{(\theta[3,5,6])^2}{\theta[2,5,7,8] \theta[1,5,7,8]} = \cdots \quad (69)$$
And similar relations between theta constants in genera 4 and 5

\[
\frac{\left(\theta^{[2,4,6,8]}\right)^2}{\theta^{[2,4,6,8,10]}\theta^{[1,4,6,8,10]}} = \frac{\left(\theta^{[1,4,6,8]}\right)^2}{\theta^{[2,3,6,8,10]}\theta^{[1,3,6,8,10]}} = \frac{\left(\theta^{[2,5,7,9]}\right)^2}{\theta^{[2,4,7,9,11]}\theta^{[1,4,7,9,11]}} = \frac{\left(\theta^{[1,5,7,9]}\right)^2}{\theta^{[2,3,7,9,11]}\theta^{[1,3,7,9,11]}} = \cdots
\]

(70)

As stated in [6, p. 685], a genus 3 curve is obtained from a genus 4 curve with 9 branch points by dropping the first two branch points, that is the genus 3 curve has branch points at \(e_3, e_4, e_5, e_6, e_7, e_8\) and \(e_9\) which now are labelled by indices from 1 to 7. Then the following relation

\[
\theta^{[2,4,6]}\theta^{[3,5,7]}\theta^{[2,5,7]}\theta^{[3,4,6]}
\]

\[-\theta^{[2,4,7]}\theta^{[3,5,6]}\theta^{[2,5,6]}\theta^{[3,4,7]} - \theta^{[2,4,5]}\theta^{[3,6,7]}\theta^{[2,6,7]}\theta^{[3,4,5]} = 0, \quad (71)
\]

is derived from (68) with the help of (69). Alternatively, relation (71) can be proven using FTT Corollary 1.

Also, the following extension of the Schottky relation (68) to genus 5 is proposed in [6,7]

\[
\left(\theta^{[2,4,6,8,10]}\theta^{[1,4,6,8,10]}\theta^{[2,5,7,9,11]}\theta^{[1,5,7,9,11]}
\right) \times \theta^{[2,4,7,9,11]}\theta^{[1,4,7,9,11]}\theta^{[2,5,6,8,10]}\theta^{[1,5,6,8,10]} \left(\theta^{[2,5,7,9,11]}\theta^{[1,5,7,9,11]}\right)^{1/2}
\]

\[-\left(\theta^{[2,4,6,8,11]}\theta^{[1,4,6,8,11]}\theta^{[2,5,7,9,10]}\theta^{[1,5,7,9,10]}
\right) \times \theta^{[2,4,7,9,10]}\theta^{[1,4,7,9,10]}\theta^{[2,5,6,8,11]}\theta^{[1,5,6,8,11]} \left(\theta^{[2,5,7,10,11]}\theta^{[1,5,7,10,11]}\right)^{1/2}
\]

\[-\left(\theta^{[2,4,6,8,9]}\theta^{[1,4,6,8,9]}\theta^{[2,5,7,10,11]}\theta^{[1,5,7,10,11]}
\right) \times \theta^{[2,4,7,10,11]}\theta^{[1,4,7,10,11]}\theta^{[2,5,6,8,9]}\theta^{[1,5,6,8,9]} \left(\theta^{[2,5,7,10,11]}\theta^{[1,5,7,10,11]}\right)^{1/2}
\]

\[-\left(\theta^{[2,4,6,7,8]}\theta^{[1,4,6,7,8]}\theta^{[2,5,9,10,11]}\theta^{[1,5,7,9,10,11]}
\right) \times \theta^{[2,4,7,9,10,11]}\theta^{[1,4,7,9,10,11]}\theta^{[2,5,6,7,8]}\theta^{[1,5,6,7,8]} \left(\theta^{[2,5,7,10,11]}\theta^{[1,5,7,10,11]}\right)^{1/2} = 0,
\]

(72)

which has the form

\[
\sqrt{r_1} - \sqrt{r_2} - \sqrt{r_3} - \sqrt{r_4} = 0.
\]

Here, the group \((P)\) of rank 3 is generated by the same 3 elements

\[
P_1 = [e_1] + [e_2] \quad \{1, 2\}
\]

\[
P_2 = [e_1] + [e_2] + [e_3] \quad \{1, 2, 3\}
\]
New generalisation of Jacobi’s derivative formula

\[ P_3 = [\varepsilon_1] + [\varepsilon_2] + [\varepsilon_3] + [\varepsilon_4] + [\varepsilon_5] \quad \{1, 2, 3, 4, 5\}, \]

and four products \( \{r_1, r_2, r_3, r_4\} \) are constructed with the help of four characteristics azygetic in triplets, namely:

\[
\begin{align*}
A_1 &= 0 = \{(2, 4, 6, 8, 10]\} \\
A_2 &= [\varepsilon_{10}] + [\varepsilon_{11}] = \{(2, 4, 6, 8, 11]\} \quad \{10, 11\} \\
A_3 &= [\varepsilon_9] + [\varepsilon_{10}] = \{(2, 4, 6, 8, 9]\} \quad \{9, 10\} \\
A_4 &= [\varepsilon_7] + [\varepsilon_{10}] = \{(2, 4, 6, 7, 8]\} \quad \{7, 10\}.
\end{align*}
\]

The way of producing (72) is explained in [7, p. 457–458].

**E2: Schottky relations in genus 5**

With the help of results proposed in the present paper, the standard Schottky relation can be obtained in an arbitrary genus. In particular, in genus 5 let \( \mathcal{I}_0 = \{6, 8, 9, 10, 11\} \), \( \mathcal{K} = \{6, 9, 10, 11\} \) and \( A_1, A_2, A_3 \) be as above, that is

\[
\begin{align*}
A_1 &= [\mathcal{I}_0^{(9,11\rightarrow 2,4)}] = \{(2, 4, 6, 8, 10]\} \\
A_2 &= [\mathcal{I}_0^{(9,10\rightarrow 2,4)}] = \{(2, 4, 6, 8, 11]\} \quad \{10, 11\} \\
A_3 &= [\mathcal{I}_0^{(10,11\rightarrow 2,4)}] = \{(2, 4, 6, 8, 9]\} \quad \{9, 10\}.
\end{align*}
\]

Then relation (57) follows from the Schottky relation

\[ \sqrt{r_1} - \sqrt{r_2} - \sqrt{r_3} = 0, \quad (73) \]

which holds with the following products, derived from the group \( (P) \) of rank 1 generated by element \( \mathcal{K} \),

\[
\begin{align*}
\sqrt{r_1} &= (\theta[\mathcal{I}_0^{(9,11\rightarrow 2,4)}] \theta[\mathcal{I}_0^{(6,10\rightarrow 2,4)}])^4 = (\theta[2,4,6,8,10] \theta[2,4,8,9,11])^4, \\
\sqrt{r_2} &= (\theta[\mathcal{I}_0^{(9,10\rightarrow 2,4)}] \theta[\mathcal{I}_0^{(6,11\rightarrow 2,4)}])^4 = (\theta[2,4,6,8,11] \theta[2,4,8,9,10])^4, \\
\sqrt{r_3} &= (\theta[\mathcal{I}_0^{(10,11\rightarrow 2,4)}] \theta[\mathcal{I}_0^{(6,9\rightarrow 2,4)}])^4 = (\theta[2,4,6,8,9] \theta[2,4,8,10,11])^4.
\end{align*}
\]

On the other hand, one can construct a group \( (P) \) of rank 3 generated by elements \( \mathcal{K}, \{1, 2\}, \{4, 5\} \) that results into

\[
\begin{align*}
r_1 &= \theta[2,4,6,8,10] \theta[1,4,6,8,10] \theta[2,5,6,8,10] \theta[1,5,6,8,10] \\
&\times \theta[2,4,8,9,11] \theta[1,4,8,9,11] \theta[2,4,8,9,11] \theta[1,5,8,9,11], \\
r_2 &= \theta[2,4,6,8,11] \theta[1,4,6,8,11] \theta[2,5,6,8,11] \theta[1,5,6,8,11] \\
&\times \theta[2,4,8,9,10] \theta[1,4,8,9,10] \theta[2,5,8,9,10] \theta[1,5,8,9,10], \\
r_3 &= \theta[2,4,6,8,9] \theta[1,4,6,8,9] \theta[2,5,6,8,9] \theta[1,5,6,8,9]
\end{align*}
\]
or with characteristics in the standard form

\[
\begin{align*}
    r_1 &= \theta[00000]\theta[01000]\theta[01100]\theta[01100]\theta[10000]\theta[00100]\theta[01000]\theta[01000]\theta[10001]\theta[10001], \\
    r_2 &= \theta[00001]\theta[01001]\theta[01101]\theta[01101]\theta[10000]\theta[00101]\theta[01001]\theta[01001]\theta[10001]\theta[10001], \\
    r_3 &= \theta[00001]\theta[10001]\theta[01101]\theta[01101]\theta[10000]\theta[00101]\theta[01001]\theta[01001]\theta[10001]\theta[10001].
\end{align*}
\]

The latter products \( \{r_1, r_2, r_3\} \) also satisfy (73).

On the base of Remark 11, one can produce a group \((P)\) containing odd and even characteristics which are syzygetic in pairs. Let characteristics \(A_1, A_2, A_3\) be constructed by (64), then all characteristics in \((A_1 P), (A_2 P), (A_3 P)\) are even non-singular. Then the Schottky relation (73) holds.

For example, with the same \( I_0 = \{6, 8, 9, 10, 11\} \) and \( K = \{6, 9, 10, 11\} \) as above, let \( j_m = 3, j_m = 5 \) and group \((P)\) be generated by elements

\[
\begin{align*}
P_1 &= [\varepsilon_1] + [\varepsilon_3] & \{1, 3\} \\
P_2 &= [\varepsilon_4] + [\varepsilon_5] & \{4, 5\} \\
P_3 &= [\varepsilon_6] + [\varepsilon_9] + [\varepsilon_{10}] + [\varepsilon_{11}] & \{6, 9, 10, 11\} = K
\end{align*}
\]

With an azygetic triplet of characteristics

\[
\begin{align*}
    A_1 &= [Z_0^{9,11\rightarrow3,5}] = \{[3, 5, 6, 8, 10]\} & \{2, 3, 4, 5\} \\
    A_2 &= [Z_0^{9,10\rightarrow3,5}] = \{[3, 5, 6, 8, 11]\} & \{2, 3, 4, 5, 10, 11\} \\
    A_3 &= [Z_0^{10,11\rightarrow3,5}] = \{[3, 5, 6, 9]\} & \{2, 3, 4, 5, 9, 10\}
\end{align*}
\]

one obtains the following products of theta constants

\[
\begin{align*}
    r_1 &= \theta[3,5,6,8,10]\theta[1,5,6,8,10]\theta[3,4,6,8,10]\theta[1,4,6,8,10] \times \theta[3,5,8,9,11]\theta[1,5,8,9,11]\theta[3,4,8,9,11]\theta[1,4,8,9,11], \\
    r_2 &= \theta[3,5,6,8,11]\theta[1,5,6,8,11]\theta[3,4,6,8,11]\theta[1,4,6,8,11] \times \theta[3,5,8,9,10]\theta[1,5,8,9,10]\theta[3,4,8,9,10]\theta[1,4,8,9,10], \\
    r_3 &= \theta[3,5,6,8,9]\theta[1,5,6,8,9]\theta[3,4,6,8,9]\theta[1,4,6,8,9] \times \theta[3,5,8,10,11]\theta[1,5,8,10,11]\theta[3,4,8,10,11]\theta[1,4,8,10,11],
\end{align*}
\]

or with characteristics in the standard form

\[
\begin{align*}
    r_1 &= \theta[01000]\theta[01100]\theta[10000]\theta[10000]\theta[00000]\theta[10000]\theta[01001]\theta[01000]\theta[01000]\theta[10010]\theta[10010], \\
    r_2 &= \theta[00000]\theta[01101]\theta[11000]\theta[00000]\theta[10001]\theta[00001]\theta[10011]\theta[10010]\theta[01011]\theta[01011], \\
    r_3 &= \theta[01000]\theta[01001]\theta[10001]\theta[00000]\theta[10000]\theta[01000]\theta[01001]\theta[11011]\theta[00011]\theta[00010]\theta[00010].
\end{align*}
\]
This collection of products \(\{r_1, r_2, r_3\}\) satisfies (73).

**E3: Schottky relations from group of rank 4**

From (68) with the help of (70), one can derive a new version of the Schottky relation on the base of a group \((P)\) of rank 4, namely:

\[
\begin{align*}
    r_1 &= (\theta^{2,4,6,8,10} \theta^{1,4,6,8,10} \theta^{2,3,6,8,10} \theta^{1,3,6,8,10} \\
         &\quad \times \theta^{2,4,7,9,11} \theta^{1,4,7,9,11} \theta^{2,3,7,9,11} \theta^{1,3,7,9,11} \\
         &\quad \times \theta^{2,4,6,9,11} \theta^{1,4,6,9,11} \theta^{2,3,6,9,11} \theta^{1,3,6,9,11} \\
         &\quad \times \theta^{2,4,7,8,10} \theta^{1,4,7,8,10} \theta^{2,3,7,8,10} \theta^{1,3,7,8,10})^{1/2}, \\
    r_2 &= (\theta^{2,4,6,8,11} \theta^{1,4,6,8,11} \theta^{2,3,6,8,11} \theta^{1,3,6,8,11} \\
         &\quad \times \theta^{2,4,7,9,10} \theta^{1,4,7,9,10} \theta^{2,3,7,9,10} \theta^{1,3,7,9,10} \\
         &\quad \times \theta^{2,4,6,9,10} \theta^{1,4,6,9,10} \theta^{2,3,6,9,10} \theta^{1,3,6,9,10} \\
         &\quad \times \theta^{2,4,7,8,11} \theta^{1,4,7,8,11} \theta^{2,3,7,8,11} \theta^{1,3,7,8,11})^{1/2}, \\
    r_3 &= (\theta^{2,4,6,8,9} \theta^{1,4,6,8,9} \theta^{2,3,6,8,9} \theta^{1,3,6,8,9} \\
         &\quad \times \theta^{2,4,7,10,11} \theta^{1,4,7,10,11} \theta^{2,3,7,10,11} \theta^{1,3,7,10,11} \\
         &\quad \times \theta^{2,4,6,10,11} \theta^{1,4,6,10,11} \theta^{2,3,6,10,11} \theta^{1,3,6,10,11} \\
         &\quad \times \theta^{2,4,7,8,9} \theta^{1,4,7,8,9} \theta^{2,3,7,8,9} \theta^{1,3,7,8,9})^{1/2},
\end{align*}
\]

or in the standard form

\[
\begin{align*}
    r_1 &= (\theta^{00000} \theta^{00001} \theta^{10000} \theta^{00110} \theta^{01000} \theta^{01100} \theta^{01010} \theta^{10010} \theta^{11000} \theta^{11010} \\
         &\quad \times \theta^{00100} \theta^{00101} \theta^{01010} \theta^{01110} \theta^{01100} \theta^{10100} \theta^{10000} \theta^{10110} \theta^{11100} \theta^{11110})^{1/2}, \\
    r_2 &= (\theta^{00001} \theta^{00000} \theta^{10000} \theta^{00111} \theta^{01000} \theta^{01111} \theta^{01101} \theta^{10111} \theta^{11011} \theta^{11111} \\
         &\quad \times \theta^{00111} \theta^{00110} \theta^{01110} \theta^{01100} \theta^{01111} \theta^{10110} \theta^{10101} \theta^{11101} \theta^{11110} \theta^{11100})^{1/2}, \\
    r_3 &= (\theta^{00000} \theta^{00001} \theta^{10001} \theta^{00100} \theta^{01010} \theta^{01101} \theta^{01001} \theta^{10011} \theta^{10101} \theta^{11011} \\
         &\quad \times \theta^{00100} \theta^{00101} \theta^{01100} \theta^{01101} \theta^{01110} \theta^{10110} \theta^{10100} \theta^{11100} \theta^{11110} \theta^{11101})^{1/2}.
\end{align*}
\]

The new collection of products \(\{r_1, r_2, r_3\}\) again satisfies (73). In a similar way, one can increase the rank of group \((P)\) in the Schottky relation.

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