Horospherical limit points of $S$-arithmetic groups

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Abstract. Suppose $\Gamma$ is an $S$-arithmetic subgroup of a connected, semisimple algebraic group $G$ over a global field $Q$ (of any characteristic). It is well known that $\Gamma$ acts by isometries on a certain CAT(0) metric space $X_S = \prod_{v \in S} X_v$, where each $X_v$ is either a Euclidean building or a Riemannian symmetric space. For a point $\xi$ on the visual boundary of $X_S$, we show there exists a horoball based at $\xi$ that is disjoint from some $\Gamma$-orbit in $X_S$ if and only if $\xi$ lies on the boundary of a certain type of flat in $X_S$ that we call “$Q$-good.” This generalizes a theorem of G. Avramidi and D. W. Morris that characterizes the horospherical limit points for the action of an arithmetic group on its associated symmetric space $X$.

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1. Introduction

Definition 1.1 ([6, Defn. B]). Suppose the group $\Gamma$ acts by isometries on the CAT(0) metric space $X$, and fix $x \in X$. A point $\xi$ on the visual boundary of $X$ is a horospherical limit point for $\Gamma$ if every horoball based at $\xi$ intersects the orbit $x \cdot \Gamma$. Notice that this definition is independent of the choice of $x$. Also note that if $\Lambda$ is a finite-index subgroup of $\Gamma$, then $\xi$ is a horospherical limit point for $\Lambda$ if and only if it is a horospherical limit point for $\Gamma$.

In the situation where $\Gamma$ is an arithmetic group, with its natural action on its associated symmetric space $X$, the horospherical limit points have a simple geometric characterization:

Theorem 1.2 (Avramidi-Morris [1 Thm. 1.3]). Let

- $G$ be a connected, semisimple algebraic group over $Q$,
- $K$ be a maximal compact subgroup of the Lie group $G(\mathbb{R})$,
- $X = K \backslash G(\mathbb{R})$ be the corresponding symmetric space of noncompact type (with the natural metric induced by the Killing form of $G(\mathbb{R})$), and
- $\Gamma$ be an arithmetic subgroup of $G$.

Then a point $\xi \in \partial X$ is not a horospherical limit point for $\Gamma$ if and only if $\xi$ is on the boundary of some flat $F$ in $X$, such that $F$ is the orbit of a $Q$-split torus in $G(\mathbb{R})$.  

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This note proves a natural generalization that allows \( \Gamma \) to be \( S \)-arithmetic (of any characteristic), rather than arithmetic. The precise statement assumes familiarity with the theory of Bruhat-Tits buildings \([12]\), and requires some additional notation.

**Notation 1.3.**

1. Let
   - \( Q \) be a global field (of any characteristic),
   - \( G \) be a connected, semisimple algebraic group over \( Q \),
   - \( S \) be a finite set of places of \( Q \) (containing all the archimedean places if the characteristic of \( Q \) is 0),
   - \( G_v = G(Q_v) \) for each \( v \in S \), where \( Q_v \) is the completion of \( Q \) at \( v \),
   - \( K_v \) be a maximal compact subgroup of \( G_v \), for each \( v \in S \), and
   - \( Z_S \) be the ring of \( S \)-integers in \( Q \).
2. Adding the subscript \( S \) to any symbol other than \( Z \) denotes the Cartesian product over all elements of \( S \). Thus, for example, we have \( G_S = \prod_{v \in S} G_v = \prod_{v \in S} G(Q_v) \).
3. For each \( v \in S \), let
   
   \[
   X_v = \begin{cases} 
   \text{the symmetric space } K_v \backslash G(Q_v) & \text{if } v \text{ is archimedean}, \\
   \text{the Bruhat-Tits building of } G(Q_v) & \text{if } v \text{ is nonarchimedean}.
   \end{cases}
   \]

   Thus, \( G_v = G(Q_v) \) acts properly and cocompactly by isometries on the CAT(0) metric space \( X_v \). So \( G_S \) acts properly and cocompactly by isometries on the CAT(0) metric space \( X_S = \prod_{v \in S} X_v \).

**Definition 1.4.** We say a family \( \{Y_t\}_{t \in \mathbb{R}} \) of subsets of \( X_S \) is uniformly coarsely dense in \( X_S/G(Z_S) \) if there exists \( C > 0 \), such that, for every \( t \in \mathbb{R} \), each \( G(Z_S) \)-orbit in \( X_S \) has a point that is at distance \(< C \) from some point in \( Y_t \).

See Definition \([3.2]\) for the definition of a \( Q \)-good flat in \( X_S \).

**Theorem 1.5** (cf. \([1\text{ Cor. 4.5}]\)). For a point \( \xi \) on the visual boundary of \( X_S = \prod_{v \in S} X_v \), the following are equivalent:

1. \( \xi \) is a horospherical limit point for \( G(Z_S) \).
2. \( \xi \) is not on the boundary of any \( Q \)-good flat.
3. There does not exist a parabolic \( Q \)-subgroup \( P \) of \( G \), such that \( P_S \) fixes \( \xi \), and \( P(Z_S) \) fixes some (or, equivalently, every) horosphere based at \( \xi \).
4. The horospheres based at \( \xi \) are uniformly coarsely dense in \( X_S/G(Z_S) \).
5. The horoballs based at \( \xi \) are uniformly coarsely dense in \( X_S/G(Z_S) \).
6. \( \pi(B) = X_S/G(Z_S) \) for every horoball \( B \) based at \( \xi \), where \( \pi : X_S \to X_S/G(Z_S) \) is the natural covering map.

**Remarks 1.6.**

- \([1 \leftrightarrow 6]\) is a restatement of Definition \([1.1]\).
- \([1 \Rightarrow 5]\) is obvious, because horoballs are bigger than horospheres.
- \([5 \Rightarrow 1]\) is well known (see, for example, \([1\text{ Lem. 2.3(\( \Leftarrow \))}]\)).

The remaining implications \([1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4]\) are proved in the following sections, by fairly straightforward adaptations of arguments in \([1]\).
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2. Proof of \((3 \Rightarrow 4)\)

\((3 \Rightarrow 4)\) of Theorem 1.5 is the contrapositive of the following result.

Proposition 2.1 (cf. [1 Thm. 4.3]). If the horospheres based at \(\xi\) are not uniformly coarsely dense in \(X_S/G(Z_S)\), then there is a parabolic \(Q\)-subgroup \(P\) of \(G\), such that

1. \(P_S\) fixes \(\xi\), and
2. \(P(Z_S)\) fixes some (or, equivalently, every) horosphere based at \(\xi\).

Proof. We modify the proof of [1 Thm. 4.3] to deal with minor issues, such as the fact that \(G_S\) is not (quite) transitive on \(X_S\). To avoid technical complications, assume \(G\) is simply connected. We begin by introducing yet more notation:

(I) Let \(\Gamma = G(Z_S)\).

(x) Let \(x \in X_S\). If \(v \in S\) is a nonarchimedean place, then we choose \(x\) so that its projection to \(X_v\) is a vertex.

(γ) Let \(\gamma : \mathbb{R} \to X_S\) be a geodesic with \(\gamma(0) = x\) and \(\gamma(+\infty) = \xi\). Let \(\gamma^+ : [0, \infty) \to X\) be the forward geodesic ray of \(\gamma\). For each \(v \in S\), let \(\gamma_v\) be the projection of \(\gamma\) to \(X_v\), so \(\gamma_v\) is a geodesic in \(X_v\).

(F\(_S\)) For each \(v \in S\), choose a maximal flat (or “apartment”) \(F_v\) in \(X_v\) that contains \(\gamma_v\). Then \(F_S\) is a maximal flat in \(X_S\) that contains \(\gamma\).

(A\(_S\)) For each \(v \in S\), there is a maximal \(Q_v\)-split torus \(A_v\) of \(G(Q_v)\), such that \(A_v\) acts properly and cocompactly on the Euclidean space \(F_v\) by translations. Then \(A_S\) acts properly and cocompactly on \(F_S\) (by translations).

(C\(_S\)) For each \(v \in S\), choose a compact subset \(C_v\) of \(F_v\), such that \(C_v A_v = F_v\). Then \(C_S A_S = F_S\).

(A\(_x\)) Let \(A_\gamma = \{ a \in A_S \mid C_S a \cap \gamma \neq \emptyset \}\) and \(A_\gamma^+ = \{ a \in A_S \mid C_S a \cap \gamma^+ \neq \emptyset \}\).

(F\(_\perp\), \(A\(_\perp\)\)) Let \(F_\perp\) be the (codimension-one) hyperplane in \(F_S\) that is orthogonal to the geodesic \(\gamma\) and contains \(x\). Let

\[ A_\perp = \{ a \in A_S \mid C_S a \cap F_\perp \neq \emptyset \}. \]

\((P_v^\xi, N_v)\) For each \(v \in S\), let

\[ P_v^\xi = \{ g \in G(Q_v) \mid \{ a g a^{-1} \mid a \in A_\gamma^+ \} \text{ is bounded} \}, \]

so \(P_v^\xi\) is a parabolic \(Q_v\)-subgroup of \(G(Q_v)\) that fixes \(\xi\). The Iwasawa decomposition [12 §3.3.2] allows us to choose a maximal horospherical subgroup \(N_v\) of \(G(Q_v)\) that is contained in \(P_v^\xi\) and is normalized by \(A_v\), such that \(F_v N_v = X_v\).

\((P_v, M_v, T_v, M_v^*)\) By applying the \(S\)-arithmetic generalization of Ratner’s Theorem that was proved independently by Margulis-Tomanov [7] and Ratner [11] (or, if \(\text{char } Q \neq 0\), by applying a theorem of Mohammadi [8 Cor. 4.2]), we obtain an \(S\)-arithmetic analogue of [11 Cor. 2.13]. Namely, for some parabolic \(Q\)-subgroup \(P\) of \(G\), if we let \(P_v = P(Q_v)\) for each \(v \in S\), and let \(P_v = M_v T_v U_v\) be the Langlands decomposition over \(Q_v\) (so
$T_v$ is the maximal $Q_v$-split torus in the center of the reductive group $M_v T_v$, and $U_v$ is the unipotent radical), then we have

$$N_S \subseteq M_S^* U_S \quad \text{and} \quad M_S^* U_S \Gamma \subseteq N_S \Gamma,$$

where $M_v^*$ is the product of all the isotropic almost-simple factors of $M_v$.

Since $N_v \subseteq P_v$ for every $v$ (and $P_S$ is parabolic), we have $U_S \subseteq N_S$ and $A_S \subset P_S$ (cf. proof of [11, Lem. 2.10]). Therefore, since all maximal $Q_v$-split tori of $P_v$ are conjugate [2 Thm. 20.9(ii), p. 228], and $M_v^* T_v$ contains a maximal $Q_v$-split torus, there is no harm in assuming $A_S \subseteq M_S^* T_S$, by replacing $M_S^* T_S$ with a conjugate. Let $A_S^M = A_S \cap M_S = A_S \cap M_S^*$. Note that $N_v$ is in the kernel of every continuous homomorphism from $P_v^\xi$ to $\mathbb{R}$. Since $P_v^\xi$ acts continuously on the set of horospheres based at $\xi$, and these horospheres are parametrized by $\mathbb{R}$, this implies that $N_v$ fixes every horosphere based at $\xi$. Then, since $F_S N_S = X_S$, we see that, for each $a \in A_S$, the set $F_\perp a N_S$ is the horosphere based at $\xi$ through the point $xa$. By the definition of $A_\perp$, this implies that the horosphere is at bounded Hausdorff distance from

$$H_a = xa A_\perp N_S.$$

(Also note that every horosphere is at bounded Hausdorff distance from some $H_a$, since $A_S$ acts cocompactly on $F_S$.) We have

$$a A_\perp N_S \Gamma \supseteq a A_\perp \cdot N_S \Gamma \supseteq a A_\perp \cdot M_S^* U_S \Gamma.$$  

We claim that $F_\perp A_S^M$ is not coarsely dense in $F_S$. Indeed, suppose, for the sake of a contradiction, that the set is coarsely dense. Then $A_\perp A_S^M$ is coarsely dense in $A_S$, which means there is a compact subset $K_1$ of $A_S$, such that $A_S = K_1 A_\perp A_S^M$. Also, the Iwasawa decomposition [12 §3.3.2] of each $G(Q_v)$ implies there is a compact subset $K_S$ of $G_S$, such that $K_S a S N_S = G_S$. Then, for every $a \in A_\gamma$, we have

$$K_S K_1 \cdot a A_\perp M_S^* U_S = K_S a (K_1 A_\perp M_S^*) U_S \supseteq K_S a A_S M_S^* U_S \supseteq K_S a S N_S = G_S.$$  

Since the compact set $K_S K_1$ is independent of $a$, this (together with (2.2)) implies that the sets $H_a$ are uniformly coarsely dense in $X/\Gamma$. This contradicts the fact that the horospheres based at $\xi$ are not uniformly coarsely dense.

Since $F_\perp$ is a hyperplane of codimension one in $F_S$ (and $A_S^M$ is a group that acts by translations), the claim proved in the preceding paragraph implies $F_\perp = F_\perp A_S^M \supseteq x A_S^M$. This means that $\gamma$ is orthogonal to the convex hull of $x A_S^M$.

On the other hand, we know that $M_S$ centralizes $T_S$. Therefore, $M_S$ fixes the endpoint $\xi_T$ of any geodesic ray $\gamma_T$ in the convex hull of $x T_S$. So $M_S$ acts (continuously) on the set of horospheres based at $\xi_T$. However, $M_S$ is the almost-direct product of compact groups and semisimple groups over local fields, so it has no nontrivial homomorphism to $\mathbb{R}$. (For the semisimple groups, this follows from the truth of the Kneser-Tits Conjecture [10 Thm. 7.6].) Since the horospheres are parametrized by $\mathbb{R}$, we conclude that $M_S$ fixes every horosphere based at $\xi_T$. Hence $A_S^M$ also fixes these horospheres. So $x A_S^M$ is contained in the horosphere through $x$, which means the convex hull of $x A_S^M$ must be perpendicular to the convex hull of $x T_S$. Since $A_S^M T_S$ has finite index in $A_S$, the conclusion of the preceding paragraph now implies that $\gamma$ is contained in the convex hull of $x T_S$, so $C_{G_S}(T_S)$ fixes $\xi$.

We also have

$$P_S = M_S T_S U_S = C_{G_S}(T_S) U_S \subseteq C_{G_S}(T_S) N_S.$$
Since $C_{G_S}(T_S)$ and $N_S$ each fix the point $\xi$, we conclude that $P_S$ fixes $\xi$. This completes the proof of (1).

From here, the proof of (2) is almost identical to the proof of [1, Thm. 4.3(2)]. $\square$

3. Proof of $\mathbf{(2 \Rightarrow 3)}$

$\mathbf{(2 \Rightarrow 3)}$ of Theorem 1.5 is the contrapositive of Proposition 3.3 below.

Notation 3.1. Suppose $T$ is a torus that is defined over $Q$. Let

1. $X_Q^*(T)$ be the set of $Q$-characters of $T$, and
2. $T_S^{(1)} = \{ g \in T_S \mid \prod_{v \in S} \|\chi(g_v)\|_v = 1, \forall \chi \in X_Q(T) \}$.

Definition 3.2. Suppose $F$ is a flat in $X_S$ (not necessarily maximal). We say $F$ is $Q$-good if there exists a $Q$-torus $T$, such that

- $T$ contains a maximal $Q$-split torus of $G$,
- $T$ contains a maximal $Q_v$-split torus $A_v$ of $G_v$ for every $v \in S$,
- $F$ is contained in the maximal flat $F_S$ that is fixed by $A_S$, and
- $F$ is orthogonal to the convex hull of an orbit of $T_S^{(1)}$ in $F_S$.

Remark 3.3. $Q$-good flats are a natural generalization of $Q$-split flats. Indeed, the two notions coincide in the setting of arithmetic groups. Namely, suppose

- $Q$ is an algebraic number field,
- $S$ is the set of all archimedean places of $Q$,
- $T$ is a maximal $Q$-split torus in $G$, and
- $H = \text{Res}_{Q/Q} G$ is the $Q$-group obtained from $G$ by restriction of scalars.

Then $T_S$ can be viewed as the real points of a $Q$-torus in $H(\mathbb{R})$, and $T_S^{(1)}$ is the group of real points of the $Q$-anisotropic part of $T_S$. Thus, in this setting, the $Q$-good flats in the symmetric space of $G_S$ are naturally identified with the $Q$-split flats in the symmetric space of $H(\mathbb{R})$.

Proposition 3.4 (cf. [1 Prop. 4.4]). If there is a parabolic $Q$-subgroup $P$ of $G$, such that $P_S$ fixes $\xi$, and $P(Z_S)$ fixes every horosphere based at $\xi$, then $\xi$ is on the boundary of a $Q$-good flat in $X_S$.

Proof. Choose a maximal $Q$-split torus $R$ of $P$. The centralizer of $R$ in $G$ is an almost direct product $RM$ for some reductive $Q$-subgroup $M$ of $P$.

Choose a $Q$-torus $L$ of $M$, such that $L(Q_v)$ contains a maximal $Q_v$-split torus $B_v$ of $M(Q_v)$ for each $v \in S$. (This is possible when $\text{char } Q = 0$ by [10 Cor. 3 of §7.1, p. 405], and the same proof works in positive characteristic, because a theorem of A. Grothendieck tells us that the variety of maximal tori is rational [3 Exp. XIV, Thm. 6.1, p. 334], [10, Thm. 7.9].) Let $T = RL$ and $A_v = R(Q_v)B_v$, so that $T$ is a $Q$-torus that contains the maximal $Q$-split torus $R$ as well as the maximal $Q_v$-split torus $A_v$ for all $v \in S$.

Let $F_S$ be the maximal flat corresponding to $A_S$, and choose some $x \in F_S$. Since $P_S$ fixes $\xi$, there is a geodesic $\gamma = \{\gamma_t\}$ in $F$, such that $\lim_{t \to \infty} \gamma_t = \xi$ (and $\gamma_0 = x$).

Now $T(Z_S)$ is a cocompact lattice in $T_S^{(1)}$ (because the “Tamagawa number” of $T$ is finite: see [10 Thm. 5.6, p. 264] if $\text{char } Q = 0$; or see [9 Thm. IV.1.3] for the general case), and, by assumption, $T(Z_S)$ fixes the horosphere through $x$. This implies that all of $T_S^{(1)}$ fixes this
horosphere, so \( xT^{(1)}_S \) is contained in the horosphere. Therefore, the convex hull of \( xT^{(1)}_S \) is perpendicular to the geodesic \( \gamma \), so \( \gamma \) is a \( Q \)-good flat. \( \square \)

4. Proof of \((1) \Rightarrow (2)\)

\((1) \Rightarrow (2)\) of Theorem 1.5 is the contrapositive of the following result.

**Proposition 4.1** (cf. \([1\text{ Prop. 3.1}]\) or \([6\text{ Thm. A}]\)). If \( \xi \) is on the boundary of a \( Q \)-good flat, then \( \xi \) is not a horospherical limit point for \( G(Z_S) \).

**Proof.** Let:

- \( \mathcal{F} \) be a \( Q \)-good flat, such that \( \xi \) is on the boundary of \( \mathcal{F} \).
- \( \gamma \) be a geodesic in \( \mathcal{F} \), such that \( \lim_{t \to \infty} \gamma(t) = \xi \).
- \( T, A_S, \) and \( F_S \) be as in Definition 3.2.
- \( x = \gamma(0) \in F_S \).
- \( F_S \) be considered as a real vector space with Euclidean inner product, by specifying that the point \( x \) is the zero vector.
- \( C_x \) be a compact set, such that \( C_xA_S = F_S \) (and \( x \in C_x \)).
- \( \gamma^\perp \) be the orthogonal complement of the 1-dimensional subspace \( \gamma \) in the vector space \( F_S \).
- \( \gamma^+_A = \{ a \in A_S \mid C_x a \cap \gamma^\perp \neq \emptyset \} \).
- \( \gamma_A(t) \in A_S \), such that \( \gamma(t) \in C_x \gamma_A(t) \), for each \( t \in \mathbb{R} \).
- \( R \) be a maximal \( Q \)-split torus of \( G \) that is contained in \( T \).
- \( \Phi \) be the system of roots of \( G \) with respect to \( R \).
- \( \alpha^S : T_S \to \mathbb{R}^+ \) be defined by \( \alpha^S(g) = \prod_{v \in S} \| \alpha(g_v) \|_v \) for \( \alpha \in \Phi \) (where \( \| \cdot \|_v \circ \alpha \) is extended to be defined on all of \( T(Q_v) \) by making it trivial on the \( Q \)-anisotropic part).
- \( \hat{\alpha}^S : F_S \to \mathbb{R} \) be the linear map satisfying \( \hat{\alpha}^S(xa) = \log \alpha^S(a) \) for all \( a \in A_S \).
- \( \alpha^F : F_S \to \mathbb{R} \), such that \( \langle \alpha^F, y \rangle = \hat{\alpha}^S(y) \) for all \( y \in F_S \).
- \( \Phi^+ = \{ \alpha \in \Phi \mid \hat{\alpha}^S(\gamma(t)) > 0 \text{ for } t > 0 \} \).
- \( \Delta \) be a base of \( \Phi \), such that \( \Phi^+ \) contains \( \Phi^+ \).
- \( \Delta^{++} = \Delta \cap \Phi^{++} \).
- \( \mathcal{P}_\alpha = R_\alpha M_\alpha N_\alpha \) be the parabolic \( Q \)-subgroup corresponding to \( \alpha \), for \( \alpha \in \Delta \), where
  - \( \alpha \) is the one-dimensional subtorus of \( R \) on which all roots in \( \Delta \setminus \{ \alpha \} \) are trivial,
  - \( M_\alpha \) is reductive with \( Q \)-anisotropic center, and
  - the unipotent radical \( N_\alpha \) is generated by the roots in \( \Phi^+ \) that are not trivial on \( R_\alpha \).

Given any large \( t \in \mathbb{R}^+ \), we know \( \hat{\alpha}^S(\gamma(t)) \) is large for all \( \alpha \in \Delta^{++} \). By definition, we have \( T^{(1)}_S = \bigcap_{\alpha \in \Delta} \ker \alpha^S \). Since \( \gamma \) is perpendicular to the convex hull of \( x \cdot T^{(1)}_S \), this implies that \( \gamma(t) \) is in the span of \( \{ \alpha^F \}_{\alpha \in \Delta} \). Also, for \( \alpha \in \Delta \), we have

\[ \langle \alpha^F, \gamma(t) \rangle = \hat{\alpha}^S(\gamma(t)) \geq 0. \]

There is no harm in renormalizing the metric on \( X_S \) by a positive scalar on each irreducible factor (cf. \([1\text{ Rem. 5.4}]\)). This allows us to assume \( \langle \alpha^F, \beta^F \rangle \leq 0 \) whenever \( \alpha \neq \beta \) (see Lemma 4.2 below). Therefore, for any \( b \in \gamma^+_A \), there is some \( \alpha \in \Delta \), such that \( \hat{\alpha}^S(x\gamma_A(t)b) \) is large (see Lemma 4.3 below). This means \( \hat{\alpha}^S(\gamma_A(t)b) \) is large.
Since conjugation by the inverse of $\gamma_A(t)b$ contracts the Haar measure on $(N_\alpha)_S$ by a factor of $\alpha^k(\gamma_A(t)b)^k$ for some $k \in \mathbb{Z}^+$, and the action of $N_S$ on $(N_\alpha)_S$ is volume-preserving, this implies that, for any $g \in \gamma_A(t)bN_S$, conjugation by the inverse of $g$ contracts the Haar measure on $(N_\alpha)_S$ by a large factor. Since $N_\alpha(Z_S)$ is a cocompact lattice in $(N_\alpha)_S$ (because the “Tamagawa number” of $N_\alpha$ is finite: see [10] Thm. 5.6, p. 264) if char $Q = 0$; or see [9] Thm. IV.1.3) for the general case), this implies there is some nontrivial $h \in N_\alpha(Z_S)$, such that \( \|ghg^{-1} - e\| \) is small. We conclude that $\xi$ is not a horospherical limit point for $G(Z_S)$ (cf. [1] Lem. 2.5(2)).

□

Lemma 4.2. Assume the notation of the proof of Proposition 4.1. The metric on $X_S$ can be renormalized so that we have $\langle \alpha^F \mid \beta^F \rangle \leq 0$ for all $\alpha, \beta \in \Delta$ with $\alpha \neq \beta$.

Proof. When $v$ is archimedean, the Killing form provides a metric on $X_v$. We now construct an analogous metric when $v$ is nonarchimedean. To do this, let $\Phi_v$ be the root system of $G$ with respect to the maximal $Q_v$-split torus $A_v$, let $t \oplus \bigoplus_{\alpha \in \Phi_v} g_\alpha$ be the corresponding weight-space decomposition of the Lie algebra of $G_v$, choose a uniformizer $\pi_v$ of $Q_v$, let $\mathcal{X}(A_v)$ be the group of co-characters of $A_v$, and define a $\mathbb{Z}$-bilinear form $\langle \mid \rangle_v : \mathcal{X}(A_v) \times \mathcal{X}(A_v) \to \mathbb{R}$ by

$$
\langle \varphi_1 \mid \varphi_2 \rangle_v = \sum_{\alpha \in \Phi_v} v(\alpha(\varphi_1(\pi_v))).
$$

This extends to a positive-definite inner product on $\mathcal{X}(A_v) \otimes \mathbb{R}$ (and the extension is also denoted by $\langle \mid \rangle_v$). It is clear that this inner product is invariant under the Weyl group, so it determines a metric on $X_v$ [12] §2.3. By renormalizing, we may assume that the given metric on $X_v$ coincides with this one.

Let $E$ be the $Q$-anisotropic pair of $T$. Then it is not difficult to see that $\mathcal{X}(E) \otimes \mathbb{R}$ is the orthogonal complement of $\mathcal{X}(E(Q_v)) \otimes \mathbb{R}$, with respect to the inner product $\langle \mid \rangle_v$ (cf. [1] Lem. 2.8). Since every $Q$-root annihilates $E(Q_v)$, this implies that the $F_v$-component $\alpha^F_v$ of $\alpha^F$ belongs to the convex hull of $x R(Q_v)$, for every $\alpha \in \Phi$.

From [4] Cor. 5.5], we know that the Weyl group over $Q$ is the restriction to $R$ of a subgroup of the Weyl group over $Q_v$. So the restriction of $\langle \mid \rangle_v$ to $\mathcal{X}(R) \otimes \mathbb{R}$ is invariant under the $Q$-Weyl group. Assume, for simplicity, that $G$ is $Q$-simple, so the invariant inner product on $\mathcal{X}(E) \otimes \mathbb{R}$ is unique (up to a positive scalar). (The general case is obtained by considering the simple factors individually.) This means that, after passing to the dual space $\mathcal{X}^*(R) \otimes \mathbb{R}$, the inner product $\langle \mid \rangle_v$ must be a positive scalar multiple $c_v$ of the usual inner product (for which the reflections of the root system $\Phi$ are isometries), so $\langle \alpha^F \mid \beta^F \rangle_v = c_v \langle \alpha \mid \beta \rangle$ for all $\alpha, \beta \in \Delta$. Since it is a basic property of bases in a root system that $\langle \alpha \mid \beta \rangle \leq 0$ whenever $\alpha \neq \beta$, we therefore have

$$
\langle \alpha^F \mid \beta^F \rangle = \sum_{v \in S} \langle \alpha^F_v \mid \beta^F_v \rangle_v = \sum_{v \in S} c_v \langle \alpha \mid \beta \rangle = \sum_{v \in S} (> 0)(\leq 0) \leq 0.
$$

□

Lemma 4.3 ([1] Lem. 2.6). Suppose

1. $v, v_1, \ldots, v_n \in \mathbb{R}^k$, with $v \neq 0$,
2. $v$ is in the span of $\{v_1, \ldots, v_n\}$,
3. $\langle v \mid v_i \rangle \geq 0$ for all $i$,
4. $\langle v_i \mid v_j \rangle \leq 0$ for $i \neq j$, and
5. $T \in \mathbb{R}^+$.

Then, for all sufficiently large $t \in \mathbb{R}^+$ and all $w \perp v$, there is some $i$, such that $\langle tv + w \mid v_i \rangle > T$. 


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