Singular perturbation for abstract elliptic equations and application
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ABSTRACT

Boundary value problem for complete second order elliptic equation is considered in Banach space. The equation and boundary conditions involve a small and spectral parameter. The uniform $L_p$–regularity properties with respect to space variable and parameters are established. Here, the explicit formula for the solution is given and behavior of solution is derived when the small parameter approaches zero. It used to obtain singular perturbation result for abstract elliptic equation

Key Word: Singular perturbation; Semigroups of operators, Boundary value problems; Differential-operator equations; Maximal $L_p$ regularity; Operator-valued multipliers

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1. Introduction, notations and background

It is well known that differential equations with small parameter play important role in modeling of physical processes. Differential-operator equations (DOEs) with parameter have also significant applications in nonlinear analysis. DOEs are studied in [1, 2], [4–7], [9–14], [16–24] and the references therein. Main aim of this paper is to show the uniform separability properties of boundary value problems (BVPs) for elliptic DOE with parameters

$$-\varepsilon u^{(2)}(t, \varepsilon) + Au(t, \varepsilon) + Bu^{(1)}(t, \varepsilon) + \lambda u(t, \varepsilon) = f(t),$$

(1.1)

where $A, B$ are linear operators in a Banach space $E$, $\varepsilon$ is a small and $\lambda$ is a complex parameter. Particularly, the sharp coercive $L_p$ estimates for solution of (1.1) are obtained uniformly with respect to small and spectral parameter. Finally, these results are used in the singular perturbation problem, i.e. to study the behavior of solution $u(t, \varepsilon)$ of (1.1) and convergence of $u(t, \varepsilon)$ as $\varepsilon \to 0$ to the corresponding solution of the Cauchy problem for abstract parabolic equation

$$Bu^{(1)}(t) + Au(t) = f(t),$$

(1.2)

$$u(0) = u_0.$$
The treatment of the singular perturbation problem for parabolic equation is due to Fattorini [7, Ch.VI] (see also the references therein). The singular perturbation problem for abstract hyperbolic equation

\[ \varepsilon u^{(2)}(x, \varepsilon) + Au(x, \varepsilon) = f(x, \varepsilon), \]  

was first considered by Kisynski [12] in the case where \( A \) is a self adjoint, positive definite operator on a Hilbert space. Latter, Sova [15] study the problem under the assumptions that \( A \) is the generator of a strongly continuous cosine function.

Then in [6] the same problem considered for the complete hyperbolic equation

\[ \varepsilon u^{(2)}(x, \varepsilon) + Au(x, \varepsilon) + Bu^{(1)}(x, \varepsilon) = 0. \]

In contrast to these results, in this paper the singular perturbation elliptic problem (1.1) is considered and we show that the solution \( u(x, \varepsilon) \) of the equation (1.1) converge in \( L^p(0,1;E) \) as \( \varepsilon \to 0 \) to the corresponding solution of the equation (1.2) uniformly with respect to spectral parameter \( \lambda \). Moreover, the solution \( u(\varepsilon, x) \) of the elliptic BVP (1.1) converge in \( E \) as \( \varepsilon \to 0 \) to the corresponding solution of the Cauchy problem (1.2) uniformly with respect to spectral parameter \( \lambda \). This result allow to investigate the spectral properties of the parameter dependent elliptic BVP (1.1). Since the Banach space \( E \) is arbitrary and \( A \) is a possible linear operator, by chosing the spaces \( E \) and operators \( A \) we can obtained different results about singular perturbation properties numerous classes of elliptic, quasielliptic equations and its system which occur in a wide variety of physical systems. Let we choose \( E = L^2(0,1) \) in (1.1) and \( A \) to be differential operator with generalized Wentzell-Robin boundary condition defined by

\[ D(A) = \{ u \in W^2_{\text{loc}}(0,1), \quad Au(j) = 0, \ j = 0, 1 \}, \]

\[ Au = \varepsilon u^{(2)} + bu^{(1)} \]

where \( a \) is positive and \( b \) is a real-valued functions. Assume \( B \) is a integral operator defined by

\[ Bu = \int_0^1 K(y, \tau) u(y, \tau) d\tau, \]

here, \( K = K(y, \tau) \) is complex valued bounded function.

Then, we get the \( L^p(\Omega) \) -separability and singular perturbation properties of the Wentzell-Robin type BVP for elliptic equation with integral term

\[ -\varepsilon \frac{\partial^2 u}{\partial t^2} + a \frac{\partial^2 u}{\partial y^2} + b \frac{\partial u}{\partial y} + \int_0^1 K(y, \tau) \frac{\partial}{\partial t} u(t, y, \tau) d\tau + \lambda u(t, y) = f(t, y), \]

\[ \sum_{i=0}^{m_1} \varepsilon^{\tau} \alpha_i u^{(i)}(0, y, \varepsilon) = f_1, \quad \sum_{i=0}^{m_2} \varepsilon^{\tau} \beta_i u^{(i)}(T, y, \varepsilon) = f_2 \] for a.e. \( y \in (0,1) \),

\[ \sum_{i=0}^{m_1} \varepsilon^{\tau} \alpha_i u^{(i)}(0, y, \varepsilon) = f_1, \quad \sum_{i=0}^{m_2} \varepsilon^{\tau} \beta_i u^{(i)}(T, y, \varepsilon) = f_2 \] for a.e. \( y \in (0,1) \),

\[ \sum_{i=0}^{m_1} \varepsilon^{\tau} \alpha_i u^{(i)}(0, y, \varepsilon) = f_1, \quad \sum_{i=0}^{m_2} \varepsilon^{\tau} \beta_i u^{(i)}(T, y, \varepsilon) = f_2 \] for a.e. \( y \in (0,1) \),

\[ \sum_{i=0}^{m_1} \varepsilon^{\tau} \alpha_i u^{(i)}(0, y, \varepsilon) = f_1, \quad \sum_{i=0}^{m_2} \varepsilon^{\tau} \beta_i u^{(i)}(T, y, \varepsilon) = f_2 \] for a.e. \( y \in (0,1) \),
\(a(j) u_{yy}(t, j, \varepsilon) + b(j) u_y(t, j, \varepsilon) = 0, j = 0, 1,\) for a.e. \(t \in (0, T),\) \hspace{1cm} (1.5)

where \(m_k \in \{0, 1\}, \alpha_i, \beta_i\) are complex numbers, \(\varepsilon\) is a positive, \(\lambda\) is a complex parameter, \(L_p(\Omega), p = (p, 2)\) denotes mixed Lebesgue space and \(\Omega = (0, T) \times (0, 1).\)

Note that, the regularity properties of Wentzell-Robin type BVP for elliptic equations were studied e.g. in [8] and the references therein.

We start by giving the notation and definitions to be used in this paper.

Let \(E\) be a Banach space and \(L_p(\Omega; E)\) denotes the space of strongly measurable \(E\)-valued functions that are defined on the measurable subset \(\Omega \subset \mathbb{R}^n\) with the norm

\[\|f\|_{L_p(\Omega; E)} = \left(\int_\Omega \|f(x)\|_E^p \, dx\right)^{\frac{1}{p}}, 1 \leq p < \infty.\]

The Banach space \(E\) is called \(UMD\)-space (see e.g. [3]) if the Hilbert operator

\[(Hf)(x) = \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} \frac{f(y)}{x-y} \, dy\]

is bounded in \(L_p(R; E)\) for \(p \in (1, \infty).\) \(UMD\) spaces include e.g. \(L_p, l_p\) spaces and Lorentz spaces \(L_{pq}\) for \(p, q \in (1, \infty)\) and Morrey spaces (see e.g. [15]).

Let \(C\) be the set of the complex numbers and

\[S_\varphi = \{\lambda; \ \lambda \in C, |\arg \lambda| \leq \varphi \cup \{0\}, 0 \leq \varphi < \pi.\]

Let \(B(E)\) denote the space of all bounded linear operators in \(E\) and \(R(\lambda, A)\) denotes the resolvent of operator \(A.\)

A linear operator \(A\) is said to be \(\varphi\)-positive in a Banach space \(E\) with bound \(M > 0\) if \(D(A)\) is dense on \(E\) and

\[\|R(-\lambda, A)\|_{B(E)} \leq M (1 + |\lambda|)^{-1}\]

for any \(\lambda \in S_\varphi, 0 \leq \varphi < \pi.\) Sometimes \(A + \lambda I\) will be denoted by \(A + \lambda I\) or \(A_\lambda,\)

where \(I\) denotes an identity operator in \(E.\) It is known [22, §1.15.1] that there exist the fractional powers \(A^\theta\) of a positive operator \(A.\) Let \(E(A^\theta)\) denote the space \(D(A^\theta)\) with norm

\[\|u\|_{E(A^\theta)} = \left(\|u\|^p + \|A^\theta u\|^p\right)^{\frac{1}{p}}, 1 \leq p < \infty, 0 < \theta < \infty.\]

Let \(E_1\) and \(E_2\) be two Banach spaces. \((E_1, E_2)_{\theta, p}\) for \(0 < \theta < 1, 1 \leq p \leq \infty\) denotes the interpolation spaces obtained from \(\{E_1, E_2\}\) by the \(K\)-method [22, §1.3.2].

\(S(R^n; E)\) is the Schwartz class, i.e. the space of all \(E\)-valued rapidly decreasing smooth functions on \(R^n\) and \(F\) denotes the Fourier transformation.
the map $u \to \Lambda u = F^{-1}\Psi (\xi) Fu, u \in S(\mathbb{R}^n; E_1)$ is well defined and extends to a bounded linear operator

$$\Lambda : L_p(\mathbb{R}^n; E_1) \to L_p(\mathbb{R}^n; E_2)$$

then a function $\Psi \in C(\mathbb{R}^n; B(E_1, E_2))$ is called a Fourier multiplier from $L_p(\mathbb{R}^n; E_1)$ to $L_p(\mathbb{R}^n; E_2)$.

The set of all multipliers from $L_p(\mathbb{R}^n; E_1)$ to $L_p(\mathbb{R}^n; E_2)$ will be denoted by $M_p^p(E_1, E_2)$. For $E_1 = E_2 = E$ it denotes by $M_p^p(E)$. Most important facts on Fourier multipliers and some related reference can be found e.g. in [22, §2.2.4] and [5, 23].

Let

$$\Phi_h = \{\Psi_h \in M_p^p(E_1, E_2), h \in Q\}$$

be a collection of multipliers in $M_p^p(E_1, E_2)$ dependent on the parameter $h$. We say that $W_h$ is a uniform collection of multipliers if there exists a positive constant $M$ independent on $h \in Q$ such that

$$\|F^{-1}\Psi_h Fu\|_{L_p(\mathbb{R}^n; E_2)} \leq M \|u\|_{L_p(\mathbb{R}^n; E_1)}$$

for all $h \in Q$ and $u \in S(\mathbb{R}^n; E_1)$.

Let $\mathbb{N}$, $\mathbb{R}$ denote the sets of natural and real numbers, respectively. A set $G \subset B(E_1, E_2)$ is called $R$-bounded (see e.g. [5, 23]) if there is a positive constant $C$ such that for all $T_1, T_2, ..., T_m \in G$ and $u_1, u_2, ..., u_m \in E_1, m \in \mathbb{N}$

$$\int_{E_2} \left\| \sum_{j=1}^{m} r_j (y) T_j u_j \right\|_{E_2} dy \leq C \int_{E_2} \left\| \sum_{j=1}^{m} r_j (y) u_j \right\|_{E_1} dy,$$

where $\{r_j\}$ is a sequence of independent symmetric $\{-1, 1\}$-valued random variables on $\Omega$. The smallest $C$ for which the above estimate holds is called a $R$-bound of the collection $G$ and denoted by $R(G)$.

Let $G_h$ be subset of $B(E_1, E_2)$ depending on the parameter $h \in Q$. Here, $G_h$ is called uniform $R$-bounded in $h$ if there is a constant $C$ independent on $h \in Q$, such that

$$\sup_{h \in Q} R(G_h) \leq C.$$

**Definition 1.** A Banach space $E$ is said to be a space satisfying a multiplier condition if, for any $\Psi \in C^{(1)} (\mathbb{R}; B(E))$ the $R$-boundedness of the set

$$\left\{ \xi \frac{d}{d\xi} \Psi (\xi) : \xi \in \mathbb{R} \setminus \{0\}, j = 0, 1 \right\}$$

implies that $\Psi$ is a Fourier multiplier, i.e. $\Psi \in M_p^p(E)$ for any $p \in (1, \infty)$.

Note that $UMD$ spaces satisfies the multiplier condition (see e.g. [5, 23]). If

$$\sup_{h \in Q} R \left( \left\{ |\xi|^j D^j \Psi_h (\xi) : \xi \in \mathbb{R} \setminus \{0\}, j = 0, 1 \right\} \right) \leq K$$
then \( \Psi \) is called a uniform collection of Fourier multipliers.

The \( \varphi \)-positive operator \( A \) is said to be \( R \)-positive in a Banach space \( E \) if the set
\[
\left\{ \xi (A + \xi)^{-1} : \xi \in S_\varphi \right\}, \quad 0 \leq \varphi < \pi
\]
is \( R \)-bounded.

Let \( E_0 \) and \( E \) be two Banach spaces. \( E_0 \) is continuously and densely embedded into \( E \). Let \( m \) be a positive integer, \( W^m_p(a, b; E_0, E) \) denotes the collection of \( E \)-valued functions \( u \in L_p(a, b; E_0) \) that have the generalized derivatives \( u^{(m)} \in L_p(a, b; E) \) with the norm
\[
\| u \|_{W^m_p(a, b; E_0, E)} = \| u \|_{L_p(a, b; E_0)} + \| u^{(m)} \|_{L_p(a, b; E)} < \infty.
\]

For \( E_0 = E \) it denotes by \( W^m_p(\Omega; E) \).

Let \( \varepsilon \in (0, \varepsilon_0] \) be a parameter for some positive bounded numbers \( \varepsilon_0 \). We define in \( W^m_p(a, b; E_0, E) \) the following parameterized norm
\[
\| u \|_{W^m_p,\varepsilon(a, b; E_0, E)} = \| u \|_{L_p(a, b; E_0)} + \| \varepsilon u^{(m)} \|_{L_p(a, b; E)}.
\]

From [20] we obtain:

**Theorem A1.** Assume the following conditions are satisfied:

1. \( E \) is a Banach space satisfying the uniform multiplier condition for \( p \in (1, \infty) \):
2. \( 0 \leq \mu \leq 1 - \frac{j}{m}, \quad j = 1, 2, \ldots, m - 1 \);
3. \( A \) is an \( R \)-positive operator in \( E \) with \( 0 \leq \varphi < \pi \).

Then:

(a) the embedding
\[
D^j W^m_p(a, b; E(A), E) \subset L_p\left(a, b; E\left(A^{1 - \frac{\mu}{m}}\right)\right)
\]
is continuous and there exists a positive constant \( C_\mu \) such that
\[
\varepsilon^{\frac{1}{m}} \| u^{(j)} \|_{L_p\left(\Omega; E\left(A^{1 - \frac{\mu}{m}}\right)\right)} \leq C_\mu \left[h^\mu \| u \|_{W^m_p(\Omega; E(A), E)} + h^{-(1-\mu)} \| u \|_{L_p(a, b; E)}\right]
\]
for all \( u \in W^m_p(a, b; E(A), E) \):

(b) If \( A^{-1} \in \sigma(\infty) \) and \( 0 < \mu \leq 1 - \frac{j}{m} \) then the embedding
\[
D^j W^m_p(a, b; E(A), E) \subset L_p\left(a, b; E\left(A^{1 - \frac{\mu}{m}}\right)\right)
\]
is compact.

**Theorem A2.** Suppose all conditions of Theorem A1 satisfied and \( 0 < \mu < 1 - \frac{j}{m} \). Then the embedding
\[
D^j W^m_p(a, b; E(A), E) \subset L_p\left(a, b; E(\Omega), E\left(\frac{\mu}{m}\right)\right)
\]

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is continuous and there exists a positive constant $C_\mu$ such that for all $u \in W^m_p(a, b; E(A), E)$ the uniform estimate holds

$$
\varepsilon^{\theta_j} \left\| u^{(j)} \right\|_{L_p\left(a, b; E(A), E \right)} \leq C_\mu \left[ h^\mu \left\| u \right\|_{W^m_p(a, b; E(A), E)} + h^{-(1-\mu)} \left\| u \right\|_{L_p(a, b; E)} \right].
$$

In a similar way as [22, §1.7.7, Theorem 2] and [24, § 10.1] we obtain, respectively:

**Theorem A.3.** Let $m, j$ be integer numbers, $0 \leq j \leq m - 1$, $\theta_j = \frac{p+1}{pm}$ and $x_0 \in [0, b]$.

Then the transformation $u \to u^{(j)}(x_0)$ is bounded linear from $W^m_p(0, b; E_0, E)$ onto $(E_0, E)_{\theta_j, p}$ and the following inequality holds

$$
\varepsilon^{\theta_j} \left\| u^{(j)}(x_0) \right\|_{(E_0, E)_{\theta_j, p}} \leq C \left( \left\| \varepsilon u^{(m)} \right\|_{L_p(0, b; E)} + \left\| u \right\|_{L_p(0, b; E_0)} \right).
$$

**Theorem A.4.** Let $m, j$ be integer numbers, $0 \leq j \leq m - 1$, $\theta_j = \frac{p+1}{pm}$ and $x_0 \in [0, b]$.

Then the transformation $u \to u^{(j)}(x_0)$ is bounded linear from $W^m_p(0, b; E)$ into $E$ and the following inequality holds

$$
\varepsilon^{\theta_j} \left\| u^{(j)}(x_0) \right\|_E \leq C \left( \left\| h^{1-\theta_j} \left\| u^{(m)} \right\|_{L_p(0, b; E)} + h^{-\theta_j} \left\| u \right\|_{L_p(0, b; E)} \right). \right.
$$

From [4, Theorem 2.1] we obtain

**Theorem A.5.** Let $E$ be a Banach space, $A$ be a $\varphi$-positive operator in $E$ with bound $M$, $0 \leq \varphi < \pi$. Let $m$ be a positive integer, $p \in (1, \infty)$ and $\alpha \in \left( \frac{1}{2p}, \frac{1}{2p} + m \right)$. Then, for $\lambda \in S_\varphi$, the operator $-A^\frac{1}{2}$ generates a semigroup $e^{-tA^\frac{1}{2}}$ which is holomorphic for $x > 0$. Moreover, there exists a positive constant $C$ (depending only on $M, \varphi, m, \alpha$ and $p$) such that for every $u \in (E, E(\lambda^m))_{\frac{1}{2p}}$, $\lambda \in S_\varphi$,

$$
\int_0^\infty \left\| A^\alpha e^{-xA^\frac{1}{2}} u \right\|^p dx \leq M_0 \left[ \left\| u \right\|^p_{(E, E(\lambda^m))_{\frac{1}{2p}}} + |\lambda|^{\alpha p - \frac{1}{2}} \left\| u \right\|^p_E \right]. (1.3)
$$

Consider the nonlocal BVP for parameter dependent differential operator-equation

$$
-\varepsilon u^{(2)}(x, \varepsilon) + (A + \lambda) u(x, \varepsilon) = 0,
$$

$$
\sum_{i=0}^{m_k} \varepsilon^{\alpha_i} \left[ \alpha_k \varepsilon^{(i)}(0, \varepsilon) + \beta_k \varepsilon^{(i)}(1, \varepsilon) \right] = f_k, k = 1, 2.
$$
where \( f_k \in E \), \( \sigma_i = \frac{i}{n} + \frac{1}{2} \), \( p \in (1, \infty) \), \( m_k \in \{0, 1\} \); \( \alpha_{ki}, \beta_{ki} \) are complex numbers; \( \varepsilon \) is a positive and \( \lambda \) is a complex parameter; \( A \) is a linear operator in \( E \). Let

\[
E_k = (E (A), E)_{\theta_k, p}, \quad \theta_k = \frac{m_k}{2} + \frac{1}{2p}.
\]

**Condition 1.** Let \( \alpha_k = \alpha_{k,m_k}, \beta_k = \beta_{k,m_k} \). Suppose

\[
d = (-1)^{m_1} \alpha_1 \beta_2 - (-1)^{m_2} \alpha_2 \beta_1 \neq 0,
\]

and

\[
\sum_{j=1}^{2} \sum_{i=0}^{2} |\alpha_{i,j}| + |\beta_{i,j}| < |d|.
\]

From [17, Theorem 2] we obtain

**Theorem A.** Let the Condition 1 hold and \( 0 < \varepsilon \leq \varepsilon_0 \). Assume \( E \) is a Banach space satisfying the uniform multiplier condition for \( p \in (1, \infty) \) and \( A \) is a \( R \)-positive operator in \( E \) for \( 0 \leq \varphi < \pi \). Then problem (1.3) has a unique solution \( u \in W_p^2 (0, 1; E) \) for \( f_k \in E_k, \theta_k = \frac{m_k}{2} + \frac{1}{2p}, \) \( p \in (1, \infty), \lambda \in S_\varphi \) with large enough \( |\lambda| \) and the coercive uniform estimate holds

\[
\sum_{i=0}^{2} \varepsilon^\varphi |\lambda|^{1-\varphi} \|u^{(i)}\|_{L_p(0,1;E)} + \|Au\|_{L_p(0,1;E)} \leq M \sum_{k=1}^{2} \left( \|f_k\|_{E_k} + |\lambda|^{1-\frac{\varphi}{2}} \|f_k\|_{E} \right).
\]

2. Abstract elliptic equation with parameters

Consider the BVP for DOE with parameters

\[
(L_\varepsilon + \lambda) u = -\varepsilon u^{(2)} (x, \varepsilon) + Au (x, \varepsilon) + Bu^{(1)} (x, \varepsilon) + \lambda u (x, \varepsilon) = f (x), \quad x \in (0, T), \quad (2.1)
\]

\[
L_1 u = \sum_{i=0}^{m_1} \varepsilon^\varphi \alpha_i u^{(i)} (0, \varepsilon) = f_1 (\varepsilon), \quad L_2 u = \sum_{i=0}^{m_2} \varepsilon^\varphi \beta_i u^{(i)} (T, \varepsilon) = f_2 (\varepsilon), \quad (2.2)
\]

where \( m_k \in \{0, 1\}, \alpha_i, \beta_i \) are complex numbers; \( \varepsilon \) is a positive and \( \lambda \) is a complex parameter; \( A \) and \( B \) are linear operators in \( E \) and \( u (x) = u (x, \varepsilon) \) is a solution of (2.1) \(-\) (2.2).

First all of, consider the problem (2.1) \(-\) (2.2) with \( f_k = 0 \), i.e. consider the homogenous problem

\[
-\varepsilon u^{(2)} (x, \varepsilon) + Bu^{(1)} (x, \varepsilon) + (A + \lambda) u (x, \varepsilon) = 0, \quad x \in (0, T), \quad (2.3)
\]

\[
\sum_{i=0}^{m_k} \varepsilon^\varphi \alpha_i u^{(i)} (0, \varepsilon) = f_1 (\varepsilon), \quad \sum_{i=0}^{m_k} \varepsilon^\varphi \beta_i u^{(i)} (T, \varepsilon) = f_2 (\varepsilon), \quad (2.4)
\]
where
\[ f_k = f_k(\varepsilon) \in E_k = (E(A), E)_{\theta_k, p} \text{ for all } \varepsilon > 0, \]
\[ \theta_k = \frac{m_k}{2} + \frac{1}{2p}, \quad m_k \in \{0, 1\}, \quad k = 1, 2, \quad p \in (1, \infty). \]

Let \[ d = \alpha_0 \beta_1 - \beta_0 \alpha_1, \quad X = L_p(0, T; E), \quad Y = W_p^2(0, T; E(A), E). \]

**Condition 2.1.** Assume the following conditions are satisfied:

1. Assume \( E \) is a Banach space satisfying the uniform multiplier condition for \( p \in (1, \infty) \);
2. \( A \) is a \( \mathbb{R} \)-positive operator in \( E \) for \( 0 \leq \varphi < \pi \) and \( d \neq 0 \);
3. \( B \) is a bounded operator, \( (A + B)^{-\frac{\theta}{2}} \in B(E) \) and
   \[ \|B\|_{B(E)} < \sup_{t \in [0, \infty]} \|A(A + t)^{-1}\|_{B(E)}. \]

**Theorem 2.1.** Assume the Condition 2.1 hold. Then problem (2.3) − (2.4) has a unique solution \( u \in Y \) for \( f_k \in E_k, \lambda \in S_\varphi \) with large enough \( |\lambda| \).

Moreover, the coercive estimate holds
\[ \sum_{i=0}^2 \varepsilon^{\frac{\varphi}{2} - \frac{1}{p}} |\lambda|^{1 - \frac{\theta}{2}} \left\| u^{(i)}(., \varepsilon) \right\|_X + \|Au\|_X \leq M \sum_{k=1}^2 \left( \|f_k\|_{E_k} + |\lambda|^{1 - \theta_k} \|f_k\|_E \right) \]
uniformly with respect to \( \varepsilon \) and \( \lambda \).

**Proof:** By definition of positive operator, \( 4\varepsilon A \) is \( \varphi \)-positive uniformly in \( \varepsilon \in (0, 1] \). Then for \( |\arg \lambda| \leq \varphi, |\arg \mu| \leq \varphi_1 \) and \( \varphi + \varphi_1 < \pi \) we have the estimate
\[ \left\| (4\varepsilon A \lambda + \mu)^{-1} \right\| \leq \frac{M_0}{|\mu|} \]
where \( A_\lambda = A + \lambda \) and \( M_0 \) depend only on \( \varphi \). By perturbation theory of positive operators and semigroups (see e.g. [14, § 1.3] and [7, § 3]) there exists the analytic semigroups
\[ U_\lambda(x, \varepsilon) = \exp - \left\{ \varepsilon^{-1} x A_{\lambda}^1 \right\}. \]

Moreover, by virtue of Condition 2.1 and in view of the same perturbation theory, the following semigroups
\[ U_{1, \lambda}(x, \varepsilon) = \exp - \{ xQ_{1, \lambda}(\varepsilon) \}, \quad U_{2, \lambda}(x, \varepsilon) = \exp - \{ xQ_{2, \lambda}(\varepsilon) \} \]
are holomorphic for $x > 0$ and strongly continuous for $x \geq 0$, where

$$Q_{1,\lambda}(\varepsilon) = \frac{1}{2\varepsilon} \left[ B + (B^2 + 4\varepsilon A\lambda)^{\frac{1}{2}} \right], \quad Q_{2,\lambda}(\varepsilon) = \frac{1}{2\varepsilon} \left[ B - (B^2 + 4\varepsilon A\lambda)^{\frac{1}{2}} \right].$$

(2.6)

Let firstly, show that the function $u(x, \varepsilon) = U_{1,\lambda}(x, \varepsilon) g_1 + U_{2,\lambda}(x, \varepsilon) g_2$ is a solution of the equation (2.3) belonging $\mathbb{Y}$ for

$$g_1, g_2 \in (E(A), E)^{\frac{1}{p}}_p.$$ 

Indeed, by properties of continuous semigroups it is clear to see that operator functions $U_{1,\varepsilon}(x)$ and $U_{2,\varepsilon}(x)$ are solution of (2.3). From (2.6) we get

$$\frac{d^2u}{dx^2} = Q_{1,\lambda}(\varepsilon) U_{1,\lambda}(x, \varepsilon) g_1 + Q_{2,\lambda}(\varepsilon) U_{2,\lambda}(x, \varepsilon) g_2,$$

$$Au(x, \varepsilon) = A[U_{1,\lambda}(x, \varepsilon) g_1 + U_{2,\lambda}(x, \varepsilon) g_2].$$

Then

$$||u||_Y = ||Au||_X + ||u(2)||_X \leq$$

$$\left( \int_0^T \|AU_{1,\lambda}(x, \varepsilon) g_1\|^p_E dx \right)^{\frac{1}{p}} + \left( \int_0^T \|AU_{2,\lambda}(x, \varepsilon) g_2\|^p_E dx \right)^{\frac{1}{p}} + \left( \int_0^T \|Q_{1,\lambda}(\varepsilon, \lambda) U_{1,\lambda}(x, \varepsilon) g_1\|^p_E dx \right)^{\frac{1}{p}} + \left( \int_0^T \|Q_{2,\lambda}(\varepsilon) U_{2,\lambda}(x, \varepsilon) g_2\|^p_E dx \right)^{\frac{1}{p}}.$$

(2.7)

By properties of positive operators and by Theorem A_5 we have

$$\left( \int_0^T \|AU_{1,\lambda}(x, \varepsilon) g_1\|^p_E dx \right)^{\frac{1}{p}} + \left( \int_0^T \|AU_{2,\lambda}(x, \varepsilon) g_2\|^p_E dx \right)^{\frac{1}{p}} \leq \left(1 + \|AA_\lambda^{-1}\|_{B(E)}\right) \left( \int_0^T \|A\lambda U_{1,\lambda}(x, \varepsilon) g_1\|^p_E dx \right)^{\frac{1}{p}} +$$

$$\left( \int_0^T \|A\lambda U_{2,\lambda}(x, \varepsilon) g_2\|^p_E dx \right)^{\frac{1}{p}} \leq C_0 \|U_{1,\lambda}(x, \varepsilon) V_{\lambda^{-1}}(x)\|_{B(E)}$$

$$\left[ \left( \int_0^T \|A\lambda V_{\lambda}(x) g_1\|^p_E dx \right)^{\frac{1}{p}} + \left( \int_0^T \|A\lambda V_{\lambda}(x) g_2\|^p_E dx \right)^{\frac{1}{p}} \right] \leq$$

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C_0 N_0 M_0 \sum_{k=1}^{2} \left( \|g_k\|_{(E(A), E)} \right)^{2/p} + |\lambda|^{1-\frac{1}{p}} \|g_k\|_E),

where $M_0$ is a constant in (1.3) and

$$C_0 = \left( 1 + \|AA^{-1}\|_{B(E)} \right), \quad N_0 = \|U_{1, \lambda} (x, \varepsilon) V_{\lambda}^{-1} (x)\|_{B(E)} \quad \text{for } \lambda \in S (\varphi).$$

In a similar way, we get the uniform estimate

$$\left( \int_0^T \|Q_{1, \lambda}^2 (\varepsilon, \lambda) U_{1, \lambda} (x, \varepsilon) g_1\|_E^p \, dx \right)^{1/p} + \left( \int_0^T \|Q_{2, \lambda}^2 (\varepsilon) U_{2, \lambda} (x, \varepsilon) g_2\|_E^p \, dx \right)^{1/p} \leq M_1 \sum_{k=1}^{2} \left( \|g_k\|_{(E(A), E)} \right)^{2/p} + |\lambda|^{1-\frac{1}{p}} \|g_k\|_E. \quad (2.9)$$

From (2.7), (2.8) and (2.9) we obtain that

$$u (\varepsilon, \cdot) \in W^2_p (0, T; E (A), E) \quad \text{for } g_1, g_2 \in (E (A), E) \frac{1}{2}, p.$$

Without loss of generality assume $m_1 = m_2 = 1$. A function

$$u (x, \varepsilon) = U_{1, \lambda} (x, \varepsilon) g_1 + U_{2, \lambda} (x, \varepsilon) g_2$$

satisfies the boundary conditions (2.4) if

$$(\varepsilon \alpha_1 Q_{1, \lambda} (\varepsilon) + \alpha_0) g_1 + (\varepsilon \alpha_1 Q_{2, \lambda} (\varepsilon) + \alpha_0) g_2 = f_1, \quad (2.10)$$

$$(\varepsilon \beta_1 Q_{1, \lambda} (\varepsilon) + \beta_0) g_1 + (\varepsilon \beta_1 Q_{2, \lambda} (\varepsilon) + \beta_0) g_2 = f_2.$$

The main operator-determinant of the algebraic equation (2.10) (with respect to $g_1$ and $g_2$) can be expressed as

$$D_{\lambda}(\varepsilon) = \varepsilon^2 \alpha_1 \beta_1 Q_{1, \lambda}(\varepsilon) Q_{2, \lambda}(\varepsilon) + \varepsilon \alpha_1 \beta_0 Q_{1, \lambda}(\varepsilon) + \alpha_0 \varepsilon \beta_1 Q_{2, \lambda}(\varepsilon) + \alpha_0 \beta_0 = \varepsilon (\alpha_1 \beta_0 - \alpha_0 \beta_1) Q_{1, \lambda}(\varepsilon) + \varepsilon (\alpha_0 \beta_1 - \beta_0 \alpha_1) Q_{2, \lambda}(\varepsilon) = \varepsilon d [Q_{2, \lambda}(\varepsilon) - Q_{1, \lambda}(\varepsilon)].$$

Since $d \neq 0,$

$$[Q_{2, \lambda}(\varepsilon) - Q_{1, \lambda}(\varepsilon)] = \frac{1}{\varepsilon} Q_{\lambda}(\varepsilon) = \frac{1}{\varepsilon} (B^2 + 4 \varepsilon A_{\lambda})^{1/2},$$

where

$$Q_{\lambda}(\varepsilon) = (B^2 + 4 \varepsilon A_{\lambda})^{1/2}.$$

It is clear to see that $Q_{\lambda}(\varepsilon)$ has a bounded inverse $Q_{\lambda}^{-1}(\varepsilon)$. Hence, $D_{\lambda}(\varepsilon)$ has a bounded inverse

$$D_{\lambda}^{-1}(\varepsilon) = -d^{-1} Q_{\lambda}^{-1}(\varepsilon) \quad (2.11)$$
for \( \varepsilon > 0 \) and \( \lambda \in S(\varphi) \). So, the system (2.10) has a unique solution

\[
g_1 = D_{1,\lambda}(\varepsilon) D_{\lambda}^{-1}(\varepsilon), \quad g_2 = D_{2,\lambda}(\varepsilon) D_{\lambda}^{-1}(\varepsilon),
\]

where

\[
D_{1,\lambda}(\varepsilon) = \left| \begin{array}{c} f_1 \\ f_2 \end{array} \right|^{\varepsilon \alpha_1 Q_{2,\lambda}(\varepsilon) + \alpha_0}_{\varepsilon \beta_1 Q_{2,\lambda}(\varepsilon) U_{2,\lambda}(1, \varepsilon) + \beta_0 U_{2,\lambda}(1, \varepsilon)} = [\varepsilon \beta_1 Q_{2,\lambda}(\varepsilon) U_{2,\lambda}(1, \varepsilon) + \beta_0 U_{2,\lambda}(1, \varepsilon)] f_1 - [\varepsilon \alpha_1 Q_{2,\lambda}(\varepsilon) + \alpha_0] f_2,
\]

\[
D_{2,\lambda}(\varepsilon) = \left| \begin{array}{c} \varepsilon \alpha_1 Q_{1,\lambda}(\varepsilon) + \alpha_0 \\ \varepsilon \beta_1 Q_{1,\lambda}(\varepsilon) U_{1,\lambda}(\varepsilon, 1) + \beta_0 U_{1,\lambda}(\varepsilon, 1) \end{array} \right|^{f_1}_{f_2} = [\varepsilon \alpha_1 Q_{1,\lambda}(\varepsilon) + \alpha_0] f_2 - [\varepsilon \beta_1 Q_{1,\lambda}(\varepsilon) U_{1,\lambda}(\varepsilon, 1) + \beta_0 U_{1,\lambda}(\varepsilon, 1)] f_1.
\]

From (2.7) and (2.11) we get the following representation of solution (2.3) – (2.4):

\[
u(x, \varepsilon) = D_{\lambda}^{-1}(\varepsilon) [U_{1,\lambda}(x, \varepsilon) D_{1,\lambda}(\varepsilon) + U_{2,\lambda}(x, \varepsilon) D_{2,\lambda}(\varepsilon)] = (2.13)
\]

\[
D_{\lambda}^{-1}(\varepsilon) \{ U_{1,\lambda}(x, \varepsilon) U_{2,\lambda}(1, \varepsilon) [\varepsilon \beta_1 Q_{2,\lambda}(\varepsilon) + \beta_0] - U_{2,\lambda}(x, \varepsilon) U_{1,\lambda}(\varepsilon, 1) [\varepsilon \beta_1 Q_{1,\lambda}(\varepsilon) + \beta_0] f_1 + D_{\lambda}^{-1}(\varepsilon) \{ U_{2,\lambda}(x, \varepsilon) [\varepsilon \alpha_1 Q_{1,\lambda}(\varepsilon) + \alpha_0] - U_{1,\lambda}(x, \varepsilon) [\varepsilon \alpha_1 Q_{2,\lambda}(\varepsilon) + \alpha_0] f_2 \}.
\]

Due to uniform boundedness of \( D_{\lambda}^{-1}(\varepsilon) \) from (2.7) we obtain

\[
\sum_{i=0}^{2} \varepsilon^{\frac{1}{2}} |\lambda|^{1-\frac{i}{2}} \| u(i) \|_{X} + \| Au \|_{X} \leq C \sum_{i=0}^{2} \varepsilon^{\frac{1}{2}} |\lambda|^{1-\frac{i}{2}} (2.14)
\]

\[
\left\{ \sum_{k=1}^{2} \| \varepsilon U_{3-k,\lambda}(1, \varepsilon) Q_{3-k,\lambda}(\varepsilon) Q_{k,\lambda}(\varepsilon) U_{k,\lambda}(x, \varepsilon) f_1 \|_{X} + \sum_{k=1}^{2} \| \varepsilon Q_{3-k,\lambda}(\varepsilon) Q_{k,\lambda}(\varepsilon) U_{k,\lambda}(x, \varepsilon) f_2 \|_{X} + \sum_{k=1}^{2} \| Q_{k,\lambda}(\varepsilon) U_{k,\lambda}(x, \varepsilon) f_2 \|_{X} + \sum_{k=1}^{2} \| Q_{k,\lambda}(\varepsilon) U_{k,\lambda}(x, \varepsilon) f_1 \|_{X} + \sum_{k=1}^{2} \| U_{3-k,\lambda}(1, \varepsilon) Q_{3-k,\lambda}(\varepsilon) A U_{k,\lambda}(x, \varepsilon) f_1 \|_{X} + \sum_{k=1}^{2} \| A U_{k,\lambda}(x, \varepsilon) f_2 \|_{X} \right\}.
\]

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By [4, Lemma 2.6], we have
\[
\left\| A^\alpha A^\beta \right\|_{B(E)} \leq C (1 + |\lambda|)^{\alpha - \beta}, \quad 0 \leq \alpha \leq \beta, \tag{2.15}
\]
\[
\left\| A^\alpha U_{k,\lambda}(x, \varepsilon) \right\|_{B(E)} \leq C e^{-2\varepsilon^{-1}x|\lambda|^2}, \quad \text{for } \alpha \in \mathbb{R}, \, x \geq x_0 > 0, \, \lambda \in S(\varphi).
\]
By properties of positive operators, from (2.6) and (2.15) for \( u \in D\left(A^{1/2}\right) \) we get
\[
Q_\lambda(\varepsilon) = Q_\lambda(\varepsilon) A^{-\frac{1}{2}}_\lambda A^{\frac{1}{2}}_\lambda, \quad \left\| Q_\lambda(\varepsilon) u \right\|_E \leq \\
\left\| Q_\lambda(\varepsilon) A^{\frac{1}{2}}_\lambda \right\|_{B(E)} \left\| A^{\frac{1}{2}}_\lambda u \right\|_E \leq C \left\| A^{\frac{1}{2}}_\lambda u \right\|_E. \tag{2.16}
\]
Moreover, by virtue of analytic semigroups theory, for all \( u \in E \) we have
\[
\left\| U_{k,\lambda}(x, \varepsilon) u \right\|_E \leq C \left\| U_{\lambda}(x, \varepsilon) u \right\|_E, \quad k = 1, 2.
\]

By chance of variable, by estimates (2.14) – (2.16) and by virtue of Theorem 1.5 we obtain
\[
\sum_{i=0}^{2} \varepsilon^{\frac{2}{p} - \frac{1}{p}} |\lambda|^{1 - \frac{1}{p}} \left\| u^{(i)} \right\|_X + \left\| Au \right\|_X \leq M_1 \left\| A^{1 - \frac{1}{2p}}_\lambda \right\|_{B(E)} \sum_{i=0}^{2} \varepsilon^{\frac{2}{p} - \frac{1}{p}} |\lambda|^{1 - \frac{1}{p}} \left\| \right\|_X
\]
\[
\sum_{k=1}^{2} \left\| A^{1 - \frac{1}{2p}}_\lambda U_{k}(x, \varepsilon) f_k \right\|_X \leq M \sum_{k=1}^{2} \left( \left\| f_k \right\|_{E_k} + |\lambda|^{1 - \theta_k} \left\| f_k \right\| \right).
\]

**Remark 2.1.** It is clear to see that the solution of the problem (2.3) – (2.4) depends on \( \varepsilon \), i.e. \( u = u(x, \varepsilon) \). Hence, it is interesting to investigate behavior of solution when \( \varepsilon \rightarrow 0 \) and to have the smoothness properties of the solution with respect to parameter \( \varepsilon \). From Theorem 3.1 we obtain the following result

**Corollary 2.1.** Assume all conditions of Theorem 2.1 are satisfied. Then the solution \( u \) of the problem (2.3) – (2.4) satisfies the following:

1. (a) \( \varepsilon^{\frac{1}{p}} u(x, \varepsilon) = O\left( \sum_{k=1}^{2} A^{\frac{1}{2}}_\lambda f_k \right) \) when \( \varepsilon \rightarrow 0 \);
2. (b) \[
\varepsilon^{\frac{2}{p} - \frac{1}{p}} |\lambda|^{\frac{1}{2}} \left\| \frac{du}{dx} \right\|_X + \varepsilon^{3 - \frac{1}{p}} \left\| \frac{d^2 u}{dx^2} \right\|_X \leq C \sum_{k=1}^{2} \left( \left\| f_k \right\|_{E_k} + |\lambda|^{1 - \theta_k} \left\| f_k \right\| \right). \tag{2.17}
\]

**Proof.** The part (a) is obtained from the representation of solution (2.13). By differentiating both parts of (2.13) with respect to \( \varepsilon \) and by using Theorem 1.5, the part (b) is obtained.

**Theorem 2.2.** Assume the Condition 2.1 hold. Then the operator \( u \rightarrow \{(L_\varepsilon + \lambda)u, L_1 u, L_2 u\} \) is an isomorphism from \( Y \) onto \( X \times E_1 \times E_2 \) for \( |\arg \lambda| \leq \varphi, \, 0 \leq \varphi < \pi \) with large enough \( |\lambda| \). Moreover, the uniform coercive estimate holds:
\[ \sum_{j=0}^{2} \varepsilon^j |\lambda|^{1-\frac{j}{2}} \|u^{(j)}\|_X + \|Au\|_X \leq C \left[ \|f\|_X + \sum_{k=1}^{2} (\|f_k\|_{E_k} + |\lambda|^{1-\theta_k} \|f_k\|_E) \right]. \]

(2.18)

**Proof.** We have proved the uniqueness of solution of (2.1)–(2.2) in Theorem 2.1. Let us define

\[ \bar{f}(x) = \begin{cases} f(x) & \text{if } x \in [0,T] \\ 0 & \text{if } x \notin [0,T] \end{cases}. \]

We now show that problem (2.1)–(2.2) has a solution \( u \in Y \) for all \( f \in X \), \( f_k \in E_k \) and \( u = u_1 + u_2 \), where \( u_1 \) is the restriction on \([0,1]\) of the solution of the equation

\[ (L_\varepsilon + \lambda) u = \bar{f}(x), \quad x \in \mathbb{R} = (-\infty, \infty) \]

(2.19)

and \( u_2 \) is a solution of the problem

\[ (L_\varepsilon + \lambda) u = 0, \quad L_k u = f_k - L_k u_1. \]

(2.20)

By applying the Fourier transform we get that, the solution (2.19) can be given by

\[ u(x) = F^{-1} \Phi(\lambda, \varepsilon, \xi) F\bar{f} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} \Phi(\lambda, \varepsilon, \xi) (F\bar{f})(\xi) d\xi, \]

where

\[ \Phi(\lambda, \varepsilon, \xi) = (A - i\xi B + \varepsilon \xi^2 + \lambda)^{-1}, \]

here \( i \) is the complex unity. It follows from the above expression that

\[ \sum_{j=0}^{2} \varepsilon^j \|\lambda|^{1-\frac{j}{2}} \|u^{(j)}\|_{L_p(R;E)} + \|Au\|_{L_p(R;E)} = \]

(2.21)

\[ \sum_{j=0}^{2} \varepsilon^j \|\lambda|^{1-\frac{j}{2}} \|F^{-1} \xi^j \Phi(\lambda, \varepsilon, \xi) F\bar{f}\|_{L_p(R;E)} + \|F^{-1} A \Phi(\lambda, \varepsilon, \xi) F\bar{f}\|_{L_p(R;E)}. \]

Let us show that operator-functions

\[ \Psi(\lambda, \varepsilon, \xi) = A \Phi(\lambda, \varepsilon, \xi), \sigma(\lambda, \varepsilon, \xi) = \sum_{j=0}^{2} \varepsilon^j \|\lambda|^{1-\frac{j}{2}} \xi^j \Phi(\lambda, \varepsilon, \xi) \]

are Fourier multipliers in \( L_p(R;E) \). Actually, due to positivity of \( A \) and by assumption (2) we have

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\[ \| \Phi (\lambda, \varepsilon, \xi) \|_{B(E)} \leq M \left( 1 + |\xi^2 + \lambda| \right)^{-1} \leq C_1, \quad (2.22) \]

\[ \| \Psi (\lambda, \varepsilon, \xi) \|_{B(E)} = \| A \Phi (\lambda, \varepsilon, \xi) \| \leq C_2. \]

It is clear to observe that

\[ \xi \frac{d}{d\xi} \Phi (\lambda, \varepsilon, \xi) = -(-iB + 2 \varepsilon \xi) \Phi^2 (\lambda, \varepsilon, \xi). \]

Due to \( R \)-positivity of the operator \( A \) and by assumption (2) the sets

\[ \{- (iB + 2 \varepsilon \xi) \Phi^2 (\lambda, \varepsilon, \xi) : \xi \in \mathbb{R} \setminus \{0\}\}, \{A \Phi (\lambda, \varepsilon, \xi) : \xi \in \mathbb{R} \setminus \{0\}\} \]

are \( R \)-bounded. Then in view of the Kahane’s contraction principle and from the product properties of the collection of \( R \)-bounded operators (see e.g. [4] Lemma 3.5, Proposition 3.4) we obtain

\[ \sup_{\lambda, \varepsilon} R \left\{ \xi^i \frac{d}{d\xi} \Psi (\lambda, \varepsilon, \xi) : \xi \in \mathbb{R} \setminus \{0\} \right\} \leq M_1, i = 0, 1. \quad (2.23) \]

Namely, the \( R \)-bound of the above sets are independent on \( \varepsilon \) and \( \lambda \). Next, let us consider \( \sigma (\lambda, \varepsilon, \xi) \). It is clear to see that

\[ \| \sigma (\lambda, \varepsilon, \xi) \|_{B(E)} \leq C |\lambda| \sum_{j=0}^2 \left( \varepsilon^{\frac{j}{2}} |\xi|^{-\frac{1}{2}} |\lambda|^{-\frac{j}{2}} \right)^j \| \Phi (\lambda, \varepsilon, \xi) \|_{B(E)}. \quad (2.24) \]

Then by using the well known inequality \( y^j \leq C (1 + y^m) \), \( y \geq 0, j \leq m \) for \( y = \left( \varepsilon^{\frac{j}{2}} |\xi|^{-\frac{1}{2}} |\lambda|^{-\frac{j}{2}} \right)^j \) and \( m = 2 \) we get the uniform estimate

\[ \left| \sum_{j=0}^2 \varepsilon^{\frac{j}{2}} |\lambda|^{-\frac{j}{2}} \xi^j \right| \leq C \left( 1 + \varepsilon |\xi^2 |\lambda|^{-1} \right). \quad (2.25) \]

From (2.24) and (2.25) we have the uniform estimate

\[ \| \sigma (\lambda, \varepsilon, \xi) \|_{B(E)} \leq C |\lambda| \left( 1 + \varepsilon |\xi^2 + |\lambda|^{-1} \right) \left( 1 + \varepsilon |\xi^2 + |\lambda| \right)^{-1} \leq C. \quad (2.26) \]

Due to \( R \)-positivity of the operator \( A \), the set

\[ \{ (|\lambda| + \varepsilon |\xi^2 | \Phi (\lambda, \varepsilon, \xi) : \xi \in \mathbb{R} \setminus \{0\}\} \]

is \( R \)-bounded. Then from (2.26) and by Kahane’s contraction principle we obtain

\[ \sup_{\lambda, \varepsilon} R \left\{ \xi^i \frac{d}{d\xi} \sigma (\lambda, \varepsilon, \xi) : \xi \in \mathbb{R} \setminus \{0\} \right\} \leq M_2, i = 0, 1. \quad (2.27) \]

By multiplier theorem (see e.g [23]) from estimates (2.23) and (2.27) it follows that \( \Psi \) and \( \sigma \) are uniform collection of multipliers in \( L_p (\mathbb{R}; E) \). Then,
by using the equality (2.21) we obtain that problem (2.19) has a solution $u \in W^2_p(R; E(A), E)$ and the uniform estimate holds

$$\sum_{j=0}^{2} \varepsilon_{j}^{\frac{1}{2}} |\lambda|^{1+\frac{1}{2}} \left\| u^{(j)} \right\|_{L_p(R; E)} + \| Au \|_{L_p(R; E)} \leq C \| f \|_{L_p(R; E)}.$$ (2.28)

Let $u_1$ be the restriction of $u$ on $(0, T).$ Then the estimate (2.28) implies that $u_1 \in Y.$ By virtue of Theorem A_3 we get

$$u_1^{(m_k)}(\cdot) \in (E(A); E)_{\theta_{k, p}}, \quad k = 1, 2.$$ Hence, $L_k u_1 \in E_k.$ Thus, by Theorem 3.1 problem (2.20) has a unique solution $u_2 \in Y$ for sufficiently large $|\lambda|$ and

$$\sum_{j=0}^{2} \varepsilon_{j}^{\frac{1}{2}} |\lambda|^{1+\frac{1}{2}} \left\| u_1^{(j)} \right\|_{X} + \| Au_1 \|_{X} \leq C \sum_{k=1}^{2} \left( \| f_k \|_{E_k} + |\lambda|^{1-\theta_k} \| f_k \|_{E} \right).$$ (2.29)

Moreover, from (2.28) we obtain

$$\sum_{j=0}^{2} \varepsilon_{j}^{\frac{1}{2}} |\lambda|^{1+\frac{1}{2}} \left\| u_1^{(j)} \right\|_{X} + \| Au_1 \|_{X} \leq C \| f \|_{X}.$$ (2.30)

Therefore, in virtue of Theorem A_3 and by estimate (2.30) we have

$$\varepsilon_{\theta_k} \left\| u_1^{(m_k)}(\cdot) \right\|_{E_k} \leq C \| u_1 \|_{W^2_{p, r}(0, T; E(A), E)} \leq C \| f \|_{L_p(0, T; E)}.$$ (2.31)

In virtue of Theorem A_4 for $\lambda = \mu^2$, $u \in W^2_p(0, T; E)$ we obtain

$$|\mu|^{2-m_k} \varepsilon_{\theta_k} \left\| u^{(m_k)}(\cdot) \right\|_{E} \leq C \left( |\mu|^{\frac{1}{2}} \varepsilon u^{(2)} \right)_{X} + |\mu|^{2+\frac{1}{2}} \| u \|_{X}.$$ (2.32)

Hence, from estimates (2.29), (2.31) and (2.32) we have

$$\sum_{j=0}^{2} \varepsilon_{j}^{\frac{1}{2}} |\lambda|^{1+\frac{1}{2}} \left\| u_2^{(j)} \right\|_{X} + \| Au_2 \|_{X} \leq C \left( \| f \|_{X} + \sum_{k=1}^{2} \left( \| f_k \|_{E_k} + |\lambda|^{1-\theta_k} \| f_k \|_{E} \right) \right).$$ (2.33)

Finally, from (2.30) and (2.33) we obtain (2.18).
3. Singular perturbation problem for abstract elliptic equation

Consider the problem (1.2), i.e. the following Cauchy problem for abstract parabolic equation

\[ Bu'(t) + Au(t) = f_0(t), \quad t \in (0, T), \]
\[ u(0) = u_0, \]  
where \( A, B \) are linear operators in a Banach space \( E \).

The problem (2.1) – (2.2) can be regarded as the singular perturbation problem for (3.1) – (3.2).

In this section we prove the following result:

**Theorem 3.1.** Let the Condition 2.1 hold and the operator \( -AB^{-1} \) generates analytic semigroup in \( E \). Moreover, assume:

1. \( f_1(\varepsilon) \in E, f_2(\varepsilon) \in D(A), f_1(\varepsilon) \to u_1(\varepsilon) \to 0 \) in \( E(A) \) as \( \varepsilon \to 0 \);
2. \( f(\varepsilon, .) \in L_p(0, T; E) \) and \( f(\varepsilon, .) \to f_0(\varepsilon) \) in \( X \) as \( \varepsilon \to 0 \).

Then:

(a) the solution of the equation (2.1) for \( \lambda = 0 \) converges to the corresponding solution of (3.1) in \( X \) as \( \varepsilon \to 0 \);
(b) the solution of (2.1) – (2.2) converges to the corresponding solution of (3.1) – (3.2) in \( E \) as \( \varepsilon \to 0 \) uniformly in \( t \) on compact intervals of \( (0, T) \).

**Proof.** By virtue of Theorem 2.2, there is a unique solution of (2.1) – (2.2) expressed as

\[ u(t, \varepsilon) = M(t, \varepsilon) f_1(\varepsilon) + N(t, \varepsilon) f_2(\varepsilon) + r_{[0,T]} F^{-1} \Phi = \tilde{f}(\xi), \]  

where

\[ M(t, \varepsilon) = D^{-1}(\varepsilon) \{ U_1(t, \varepsilon) U_2(T, \varepsilon) [\xi \beta_1 Q_2(\varepsilon) + \beta_0] - U_2(t, \varepsilon) U_1(T, \varepsilon) [\xi \beta_1 Q_1(\varepsilon) + \beta_0] \}, \]
\[ N(t, \varepsilon) = D^{-1}(\varepsilon) \{ U_2(t, \varepsilon) [\xi \alpha_1 Q_1(\varepsilon) + \alpha_0] - U_1(t, \varepsilon) [\xi \alpha_1 Q_2(\varepsilon) + \alpha_0] \}, \]
\( \tilde{f} \) is a zero extension of \( f \) on \( \mathbb{R} \setminus [0, T] \), \( r_{[0,1]} \) is a restriction operator from \( \mathbb{R} \) to \([0, T]\),

\[ U_1(x, \varepsilon) = \exp \{ xQ_1(\varepsilon) \}, \quad U_2(x, \varepsilon) = \exp \{ xQ_2(\varepsilon) \}, \]

\( D^{-1}(\varepsilon), Q_1(\varepsilon), Q_2(\varepsilon) \) are denote \( D^{-1}_\lambda(\varepsilon), Q_{1,\lambda}(\varepsilon), Q_{2,\lambda}(\varepsilon) \) for \( \lambda = 0 \), respectively and

\[ \Phi(\xi, \varepsilon) = (A - i\xi B + \varepsilon \xi^2)^{-1}, \]

Let us show that the solution \( u(\varepsilon, .) \) of (2.1) – (2.2) approaches to the corresponding solution of (3.1) – (3.2) in \( E \) under conditions (H1) and (H1). Since \( A \) and \( B \) are close operators, it is clear to see that

\[ \Phi_0(\xi) = (A - i\xi B)^{-1} \]
is a Fourier transform of \( Bu' (t) + Au (t) \) and from (3.1) we get that
\[
\overline{u} (t) = \overline{r[0, T]} F^{-1} \Phi_0 (\xi) F \overline{f_0} (\xi)
\]
is a solution of the equation (3.1), where under Condition 2.1 \( \Phi_0 (\xi) \) is uniformly bounded in \( \xi \in \mathbb{R} \). The operator functions \( \Phi (\xi, \varepsilon), \Phi_0 (\xi) \) are uniform bounded and are multipliers in \( L_p (\mathbb{R}; E) \) (see the proof of Theorem 2.2). It is clear to see that
\[
\Phi (\xi, \varepsilon) \rightarrow \Phi_0 (\xi) \text{ in } B (E) \quad (3.5)
\]
as \( \varepsilon \rightarrow 0 \) uniformly in \( \xi \) and \( \lambda \). Moreover, we get
\[
\begin{align*}
\left\| \Phi (\xi, \varepsilon) F \overline{f} (\xi, \varepsilon) - \Phi_0 (\lambda, \xi) F \overline{f_0} (\xi) \right\|_E & \leq \\
\left\| \Phi (\xi, \varepsilon) F \overline{f} (\xi, \varepsilon) - \Phi (\xi, \varepsilon) F \overline{f_0} (\xi) \right\|_E + \\
\left\| \Phi (\xi, \varepsilon) F \overline{f_0} (\xi) - \Phi_0 (\xi) F \overline{f_0} (\xi) \right\|_E.
\end{align*}
\] (3.6)

Since \( \overline{f} (\xi, \varepsilon) \rightarrow \overline{f_0} (\xi) \) in \( E \) as \( \varepsilon \rightarrow 0 \) for a.e. \( \xi \in \mathbb{R} \), \( \Phi (\xi, \varepsilon) \) is bounded in \( E \) for all \( \xi \in \mathbb{R} \) and the Fourier transform \( F \) is continuous in \( X \). Then we get
\[
\left\| \Phi (\xi, \varepsilon) F \overline{f} (\xi, \varepsilon) - \Phi (\xi, \varepsilon) F \overline{f_0} (\xi) \right\|_E \rightarrow 0 \quad (3.7)
\]
as \( \varepsilon \rightarrow 0 \) for a.e. for \( \xi \in \mathbb{R} \).

By the same reason and due to \( \Phi (\xi, \varepsilon) \rightarrow \Phi_0 (\xi) \) in \( B (E) \) as \( \varepsilon \rightarrow 0 \) uniformly in \( \lambda \) and \( \xi \), we have
\[
\left\| \Phi (\xi, \varepsilon) F \overline{f_0} (\xi) - \Phi_0 (\xi) F \overline{f_0} (\xi) \right\|_E \rightarrow 0. \quad (3.8)
\]

Then due to boundedness of \( F^{-1} \) from (3.5) – (3.8) we obtain
\[
\left\| F^{-1} \Phi (\xi, \varepsilon) F \overline{f} (\xi, \varepsilon) - F^{-1} \Phi_0 (\xi) F \overline{f_0} (\xi) \right\|_X \rightarrow 0
\]
as \( \varepsilon \rightarrow 0 \), i.e.,
\[
r_{[0, 1]} F^{-1} \Phi (\xi, \varepsilon) F \overline{f} (\xi, \varepsilon) \rightarrow r_{[0, 1]} F^{-1} \Phi_0 (\xi) F \overline{f_0} (\xi) \quad \text{in } X. \quad (3.9)
\]

We have proved the assertion (a). Now, let us show the assertion (b). Indeed, known that (see e.g. [1, §3], [2, §1.5], [14, §4.2]) there is a unique solution of the Cauchy problem (3.1) – (3.2) for \( f \in L_p (0, T; E) \) expressed as
\[
u (t) = U_{0, \lambda} (t) u_0 + \int_0^t U_{0, \lambda} (t - \tau) f_0 (\tau) d\tau,
\]
where \( U_{0, \lambda} (t) \) is an analytic semigroup in \( E \) generated by the operator
\[
-A_0 (\lambda) = -A_\lambda B^{-1}.
\]
Due to uniform boundedness of $D^{-1}(\varepsilon)$ and by estimates of analytic semigroups from (3.4) we obtain

\[
\|M(t, \varepsilon)f_1\|_E \leq C \left\{ \|U_2(1, \varepsilon)\|_{B(E)} \|U_1(t, \varepsilon)Q(\varepsilon)f_1\|_E + \right.
\]
\[
\|U_1(t, \varepsilon)f_1\|_E + \|U_1(1, \varepsilon)\|_{B(E)} \|U_2(t, \varepsilon)Q(\varepsilon)f_1\|_E + \right.
\]
\[
\left. \|U_2(t, \varepsilon)f_1\|_E \right\} \leq C_1 \exp \left\{ -\varepsilon^{-1}\omega t \right\} \|f_1\|_E, \tag{3.12}
\]

for $f_1 \in E$ where,

\[
Q = Q(\varepsilon) = (B^2 + 4\varepsilon A)^{\frac{1}{2}}, \quad \omega > 0.
\]

From (3.4) in a similar way, for $f_2 \in E$ we get

\[
\|N(t, \varepsilon)f_2\|_E \leq C \left\{ \|U_1(t, \varepsilon)Q(\varepsilon)f_2\|_E + \|U_2(t, \varepsilon)Q(\varepsilon)f_2\|_E \right. + \|U_1(t, \varepsilon)f_2\|_E \right\} \leq C_0 \|f_2\|_E. \tag{3.13}
\]

From (3.12) and (3.13) we have

\[
\lim_{\varepsilon \to 0} \|M(t, \varepsilon)\|_{B(E)} = 0, \quad \lim_{\varepsilon \to 0} \|N(t, \varepsilon)\|_{B(E)} = 0. \tag{3.14}
\]

Let us show that

\[
K[N(., \varepsilon) - U_0(.)]v = U_0 * [N(., \varepsilon) - \varepsilon^{-1}BM(., \varepsilon)]A_0v \tag{3.15}
\]

for all $v \in D(A_0)$, where $K$ is a uniform bounded operator in $E$.

Indeed, the Laplace transform of $U_0(., \varepsilon)$, $U_1(., \varepsilon)$, $U_2(., \varepsilon)$ gives the resolvent $R(s, A_0)$, $R(s, B + Q)$, $R(s, B + Q)$, respectively. Hence, by using the linearity and convolution properties of the Laplace transform, (3.15), (3.4) and (2.6) it sufficient to show

\[
KD^{-1}[(\varepsilon\alpha_1Q_1 + \alpha_0)R(s, Q_2) - (\varepsilon\alpha_2Q_2 + \alpha_0)R(s, Q_1)] - K\lambda R(s, A_0) =
\]
\[
A_0R(s, A_0) \left\{ D^{-1}[(\varepsilon\alpha_1Q_1 + \alpha_0)R(s, Q_2) - \right. \]
\[
(\varepsilon\alpha_1Q_2 + \alpha_0)R(s, Q_1)] - \varepsilon^{-1}BD^{-1}[(\varepsilon\beta_1Q_2 + \beta_0)U_2(\varepsilon, T)R(s, Q_1) + \right. \]
\[
(\varepsilon\beta_1Q_1 + \beta_0)U_1(\varepsilon, T)R(s, Q_2)) \right\}. \tag{3.16}
\]

Indeed, by using (2.6), the resolvent equation, the exponential properties of strongly continuous semigroups we get that there is a bounded operator $K$ in $E$ that (3.16) is satisfied. Hence, from (3.4) and (3.13) for $v \in D(A)$ we get

\[
\|[N(., \varepsilon) - U_0(.)]v\|_E \leq C_1 \exp \left\{ -\varepsilon^{-1}\omega t \right\} \|A_0v\|_E + \right.
\]
\[
C_2 \exp \left\{ -\varepsilon^{-1}\omega t \right\} \|U_0(\cdot)\|_{B(E)} \|A_0v\|_E. \tag{3.17}
\]

Then from (3.3), (3.4) and (3.17) for $f_1 \in E, f_2 \in D(A)$ we deduced

\[
\|u(., \varepsilon) - u(.)\|_E \leq \|M(., \varepsilon)f_1\|_E + \|N(., \varepsilon)f_2 - U_0(.)u_0\|_E + \right.
\]

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\[ \| f(\cdot, \varepsilon) - f_0(\cdot) \|_E \leq C_1 \exp\left\{ -\varepsilon^{-1}\omega t \right\} \| f_1 \|_E + C_2 \exp\left\{ -\varepsilon^{-1}\omega t \right\} \| f_2 \|_E + \| f(\varepsilon, \cdot) - f_0(\cdot) \|_E, \] 

By conditions (H_1) and (H_2) we get

\[ \exp\left\{ -\varepsilon^{-1}\omega t \right\} \to 0 \quad \text{as} \quad \varepsilon \to 0 \]

uniformly with respect to \( t \) on all compact \( \sigma \subset (0,T) \). Then from (3.18) we obtain the assertion.

4. Wentzell-Robin type mixed problem for elliptic equation

Consider the BVP (1.4) - (1.5). For \( \mathbf{p} = (p,2) \) and \( L_\mathbf{p}(\Omega) \) will denote the space of all \( \mathbf{p} \)-summable scalar-valued functions with mixed norm. Analogously, \( W^2_\mathbf{p}(\Omega) \) denotes the Sobolev space with corresponding mixed norm, i.e., \( W^2_\mathbf{p}(\Omega) \) denotes the space of all functions \( u \in L_\mathbf{p}(\Omega) \) possessing the derivatives \( \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2} \in L_\mathbf{p}(\Omega) \) with the norm

\[ \| u \|_{W^2_\mathbf{p}(\Omega)} = \| u \|_{L_\mathbf{p}(\Omega)} + \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{L_\mathbf{p}(\Omega)} + \left\| \frac{\partial^2 u}{\partial y^2} \right\|_{L_\mathbf{p}(\Omega)}. \]

**Condition 4.1** Assume:

1. \( K(\cdot, \cdot) \in C([0, T] \times [0,1]) \);
2. \( a \) is positive, \( b \) is a real-valued functions on \( (0,1) \);
3. \( a(\cdot) \in C(0,1) \) and

\[ \exp\left( - \int_{\frac{x}{2}}^{x} b(t) a^{-1}(t) \, dt \right) \in L_1(0,1). \]

In this section, we present the following result:

**Theorem 4.1.** Suppose the Condition 4.1 hold. Then:

(a) For \( f \in L_\mathbf{p}(\Omega), p, p_1 \in (1, \infty) \) problem (1.4) - (1.5) has a unique solution \( u \in W^2_\mathbf{p}(\Omega) \) and the following uniform coercive estimate holds

\[ \sum_{i=0}^{2} \varepsilon^{\frac{i}{2}} \| \lambda^{1-\frac{i}{2}} \frac{\partial^i u}{\partial x^4} \|_{L_\mathbf{p}(\Omega)} + \| \frac{\partial^2 u}{\partial y^2} \|_{L_\mathbf{p}(\Omega)} + \| u \|_{L_\mathbf{p}(\Omega)} \leq C \left[ \| f \|_{L_\mathbf{p}(\Omega)} + \| f_1 \|_{L_{p_1}(0,1)} + \| f_2 \|_{W^2_\mathbf{p}(0,1)} \right]; \]
(b) the solution of the equation (1.4) for $\lambda = 0$ converges to the corresponding solution of the following equation

$$- \left( a \frac{\partial^2 u}{\partial y^2} + b \frac{\partial u}{\partial y} \right) + \int_0^1 K(y, \tau) \frac{\partial}{\partial \tau} u(t, y, \tau) \, d\tau = f(t, y),$$

in $L_p(\Omega)$ as $\varepsilon \to 0$;

(c) the solution of (1.4) − (1.5) converges to the corresponding solution of the following mixed problem

$$- \left( a \frac{\partial^2 u}{\partial y^2} + b \frac{\partial u}{\partial y} \right) + \int_0^1 K(y, \tau) \frac{\partial}{\partial \tau} u(t, y, \tau) \, d\tau = f(t, y),$$

$$u(0, y) = 0 \text{ for a.e. } y \in (0, 1),$$

in $L_p(0, 1)$ as $\varepsilon \to 0$ uniformly in $t$ on compact intervals of $(0, T)$.

**Proof.** Let $E = L_2(0, 1)$. It is known [5] that $L_2(0, 1)$ is an $UMD$ space. Consider the operator $A$ defined by

$$D(A) = W^2_2(0, 1; A (j) u = 0, j = 0, 1), \quad Au = -a \frac{\partial^2 u}{\partial y^2} + b \frac{\partial u}{\partial y}.$$

Therefore, the problem (1.4) − (1.5) can be rewritten in the form of (2.2), where $u(t) = u(t, .), \ f(t) = f(t, .)$ are functions with values in $E = L_2(0, 1)$. By virtue of [8] the operator $A$ generates analytic semigroup in $L_2(0, 1)$. Then in view of Hill-Yosida theorem (see e.g. [22, § 1.13]) this operator is sectorial in $L_2(0, 1)$. Since all uniform bounded set in Hilbert apace is an $R$-bounded (see [3]), i.e. we get that the operator $A$ is $R$-sectorial in $L_2(0, 1)$. Then from Theorem 2.2 and Theorem 3.1 we obtain the assertion.

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