ON THE MATHIEU CONJECTURE FOR $SU(2)$

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Abstract. We study the Mathieu Conjecture for $SU(2)$ using the matrix elements of its unitary irreducible representations. We state a conjecture for the particular case $SU(2)$ implying the Mathieu Conjecture for $SU(2)$.

1. Introduction

Conjecture 1.1 (Mathieu [6]). Let $G$ be a compact connected Lie group and let $f$ be complex-valued $G$-finite function on $G$ such that $\int_G f^P(g) \, dg = 0$ for every $P \in \mathbb{N}_{\geq 0}$. Then for any complex-valued $G$-finite function $h$ on $G$ we have $\int_G f^P(g) h(g) \, dg = 0$ for $P \gg 0$.

The Mathieu Conjecture 1.1 dates back to 1997 and is closely related to the Jacobian conjecture, since it actually implies the Jacobian conjecture, see [6]. See van den Essen [2], Smale [7] for more information on the history of the Jacobian conjecture. The Mathieu Conjecture 1.1 was proved for abelian compact groups by Duistermaat and Van der Kallen [1] in 1998. We study the Mathieu Conjecture 1.1 for the case $G = SU(2)$. Using explicit formulas for the Haar measure and known representation theoretic properties of $SU(2)$ we make the Mathieu Conjecture 1.1 more explicit. In particular, we use the fact that $SU(2)$-finite functions are finite linear combinations of matrix elements of finite dimensional irreducible representations of $SU(2)$ and that the matrix elements behave well under a subgroup $K \cong U(1)$ according to suitable characters. Note that the Mathieu Conjecture 1.1 is linear in the $G$-finite function $h$, but not in the $G$-finite function $f$. By the Peter-Weyl theorem, any $SU(2)$-finite function is the finite linear combination of matrix elements of irreducible representations. After recalling the necessary results on $SU(2)$ in Section 2, we show in Section 3 the validity of the Mathieu Conjecture 1.1 for $f$ a single matrix element or a sum of two matrix elements. For the sum of three matrix elements there is a partial result. These considerations lead to Conjecture 4.1 and Theorem 4.2 shows that this conjecture implies the Mathieu Conjecture 1.1 for $SU(2)$. Conjecture 4.1 describes the condition $\int_{SU(2)} f(g)^P \, dg = 0$ for all $P > 0$ in terms of a support condition on the characters of the abelian subgroup $U(1)$ of $SU(2)$ acting from the left and right on the individual matrix elements occurring in $f$.

We note that the Mathieu Conjecture 1.1 for bi-$K$-invariant functions is settled by Francoise et al. [3, Cor. 4.1], since the bi-$K$-invariant $SU(2)$-finite functions are the polynomials on $[-1, 1]$.

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2. SU(2)

We briefly recall some required notions of SU(2). Details can be found in e.g. [8], [9]. Let
\[ k(\phi) = \begin{pmatrix} e^{\frac{i\theta}{2}} & 0 \\ 0 & e^{-\frac{i\theta}{2}} \end{pmatrix} \text{ and } a(\theta) = \begin{pmatrix} \cos \frac{\theta}{2} & i \sin \frac{\theta}{2} \\ i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} \]
be elements of SU(2), then any element \( g \in SU(2) \) can be expressed in terms of Euler angles \( g = k(\phi)a(\theta)k(\psi) \) with \( \phi \in [0, 2\pi), \theta \in (0, \pi), \psi \in [-2\pi, 2\pi) \). In terms of the Euler angles the Haar integral is, cf [8, III, §6.1, (5)],
\[
\int_{SU(2)} f(g) \, dg = \frac{1}{16\pi^2} \int_0^{2\pi} \int_0^\pi \int_{-\pi}^{\pi} F(\phi, \theta, \psi) \sin \theta \, d\psi \, d\theta \, d\phi,
\]
where \( F(\phi, \theta, \psi) = f(k(\phi)a(\theta)k(\psi)) \). Denote the subgroup \( K \cong U(1) \) generated by \( k(\phi) \). For a function \( f \) transforming by a non-trivial \( K \)-character under left- or right multiplication by \( K \), we have \( \int_{SU(2)} f(g) \, dg = 0 \) by (2.1). The subgroup generated by \( a(\theta) \) is the group SO(2).

The finite-dimensional irreducible representations are labeled by the spin \( \ell \in \frac{1}{2}\mathbb{N} \) and are of dimension \( 2\ell + 1 \). The standard basis for the representation space is labeled as \( \{ -\ell, -\ell + 1, \ldots, \ell \} \), and the corresponding matrix elements \( t^\ell_{m,n} \) are \( SU(2) \)-finite functions, and any \( SU(2) \)-finite function is a finite linear combination of matrix elements of irreducible finite-dimensional representations. The matrix-elements \( t^\ell_{m,n} \) behave well according to left and right action by \( K \), cf. [8 III, §3.3, (3)]
\[
t^\ell_{m,n}(k(\phi)g) = e^{-im\phi} t^\ell_{m,n}(g), \quad t^\ell_{m,n}(gk(\psi)) = e^{-im\psi} t^\ell_{m,n}(g).
\]
In particular, \( t^0_{0,0}(g) = 1 \), and the algebra of bi-\( K \)-invariant \( SU(2) \)-finite functions consists of finite linear combinations of \( t^\ell_{0,0}, \ell \in \mathbb{N} \). For \( \ell \in \mathbb{N} \) we have \( t^\ell_{0,0}(a(\theta)) = P_\ell(\cos \theta) \), cf. [8 III, §3.9, (5)] where \( P_\ell \) is the Legendre polynomial in its standard normalisation \( P_\ell(1) = 1 \), [4, §4.5], [5, §1.8.3], which is real-valued on \([-1, 1]\). The Legendre polynomials are orthogonal on \([-1, 1]\) with respect to the uniform measure: \( \int_{-1}^{1} P_m(x)P_n(x) \, dx = \delta_{m,n}2/(2n + 1) \). Moreover, the Schur orthogonality relations are, [8 III, §6.2, (1)]
\[
\int_{SU(2)} t^\ell_{m,n}(g)t^{\ell_2}_{n,q}(g) \, dg = \frac{1}{2\ell_1 + 1} \delta_{\ell_1,\ell_2} \delta_{m,p} \delta_{n,q},
\]
which in case \( m = n = p = q = 0 \) give the orthogonality for the Legendre polynomials.

3. The Mathieu Conjecture for SU(2) for simple \( f \)

We start using some simple observations related to the condition in the Mathieu Conjecture for \( G = SU(2) \). Firstly, by the Schur orthogonality relations (2.3)
\[
\int_{SU(2)} t^\ell_{m,n}(g) \, dg \neq 0 \iff \ell = 0.
\]
Secondly, by the left and right \( K \)-behaviour of the matrix elements (2.2) and the Haar measure in Euler angles (2.1) we see
\[
\int_{SU(2)} (t^{\ell_1}_{m_1,n_1})^{\alpha_1}(g) \cdots (t^{\ell_k}_{m_k,n_k})^{\alpha_k}(g) \, dg \neq 0 \implies \sum_{i=1}^k \alpha_i m_i = 0 = \sum_{i=1}^k \alpha_i n_i
\]
for \( \alpha_i \in \mathbb{N}, \ell_i \in \frac{1}{2}\mathbb{N} \) and \( m_i, n_i \in \{-\ell_i, \ldots, \ell_i\} \).

**Lemma 3.1.** \( \int_{SU(2)} (t_{m,n}^{\ell})^P \, dg = 0 \) for all integer \( P > 0 \) if and only if \( m \neq 0 \) or \( n \neq 0 \).

**Proof.** The implication \( \Leftarrow \) follows from (3.2). To prove the other implication, we observe that for \( \ell \in \mathbb{N} \)

\[
\int_{SU(2)} (t_{0,0}^0)^2 \, dg = \frac{1}{2} \int_0^\pi (P_t(\cos \theta))^2 \sin \theta \, d\theta = \frac{1}{2} \int_{-1}^1 (P_t(x))^2 \, dx > 0. \]

Now we can verify the Mathieu Conjecture [1.1] in the case \( f \) consists of one matrix element.

**Proposition 3.2.** The Mathieu Conjecture [1.1] is true for \( G = SU(2) \) with \( f \) a single matrix element \( f = t_{m,n}^{\ell} \).

**Proof.** Since all non-negative powers of \( f \) integrate to zero, Lemma [3.1] shows that \( m \neq 0 \) or \( n \neq 0 \), so in particular \( \ell \neq 0 \). Let \( h = t_{a,b}^0 \). We assume \( m \neq 0 \), the case \( n \neq 0 \) being similar.

By (3.2) we see that \( Pm + a \neq 0 \) implies \( \int_{SU(2)} (f(g))^P h(g) \, dg = 0 \), which is the case for \( P > |a|/|m| \).

The same strategy can also be employed to deal with \( f = A_1 t_{m_1,n_1}^{\ell_1} + A_2 t_{m_2,n_2}^{\ell_2} \), where \( A_i \in \mathbb{C} \), assuming \( A_1 \neq 0 \neq A_2 \) and \((\ell_1, m_1, n_1) \neq (\ell_2, m_2, n_2)\). Note

\[
\int_{SU(2)} (f(g))^P \, dg = \sum_{\alpha=0}^P \binom{P}{\alpha} A_1^\alpha A_2^{P-\alpha} \int_{SU(2)} (t_{m_1,n_1}^{\ell_1})^\alpha(g)(t_{m_2,n_2}^{\ell_2})^{P-\alpha}(g) \, dg. \quad (3.3)
\]

**Lemma 3.3.** Let \( f \) be as above with at least one of \((m_1, m_2, n_1, n_2)\) non-zero, then

\[
\exists P > 0 : \int_{SU(2)} (f(g))^P \, dg \neq 0 \iff \det \begin{pmatrix} m_1 & m_2 \\ n_1 & n_2 \end{pmatrix} = 0 \wedge m_1 m_2 \leq 0 \wedge n_1 n_2 \leq 0.
\]

**Remark 3.4.** Note that the condition in Lemma [3.3] means that \((0,0)\) is on the line segment from \((m_1, n_1)\) to \((m_2, n_2)\).

**Proof.** \( \Rightarrow \): Since at least one term in the right hand side of (3.3) has to be non-zero, (3.2) shows that \( m_1 \alpha + m_2 (P - \alpha) = 0 = n_1 \alpha + n_2 (P - \alpha) \), which gives the result.

\( \Leftarrow \): Note that \( \dim \ker \begin{pmatrix} m_1 & m_2 \\ n_1 & n_2 \end{pmatrix} = 1 \). Pick a solution \( (\alpha, \beta) \in \mathbb{N}^2 \) to \( m_1 \alpha + m_2 \beta = 0 = n_1 \alpha + n_2 \beta \), and put \( M = \alpha + \beta \). Then

\[
\int_{SU(2)} (f(g))^M \, dg = \binom{\alpha + \beta}{\alpha} A_1^\alpha A_2^\beta \int_{SU(2)} (t_{m_1,n_1}^{\ell_1})^\alpha(g)(t_{m_2,n_2}^{\ell_2})^\beta(g) \, dg, \quad (3.4)
\]

using (3.2), since for \( \gamma \neq 0 \)

\[
\binom{m_1}{n_1} \binom{m_2}{n_2} \binom{\alpha + \gamma}{\beta - \gamma} = \gamma \binom{m_1}{n_1} \binom{m_2}{n_2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

since the kernel is one-dimensional. The integrand on the right hand side of (3.4) is a bi-\(K\)-invariant function, so that by (2.1) we can restrict to the integral over \( g = a(\theta), \theta \in [0, \pi] \).

By [8, III, §3,(3),(4)] the integrand in \( a(\theta) \) is real-valued. In case the integral is non-zero
we are done. Otherwise, we put $P = 2M$, and then in the same way there is again at most one non-zero integral in the right hand side of (3.3), namely for $(2\alpha, 2\beta)$. The integral can be restricted to $SO(2)$ as before. Since this is the integral of a square, since the function $(t_{m_1,n_1}^\alpha(x))(t_{m_2,n_2}^\beta(x))$ is real, the integral is non-zero. \hfill $\square$

Proposition 3.5. The Mathieu Conjecture $\mathbb{C}$ is true for $G = SU(2)$ with $f$ a sum of two matrix element $f = A_1t_{m_1,n_1}^\ell + A_2t_{m_2,n_2}^\ell$, where $A_1 \neq 0 \neq A_2$ and $(\ell_1, m_1, n_1) \neq (\ell_2, m_2, n_2)$.

Proof. It suffices to take $h = t_{a,b}^\ell$ and to assume that $\int_{SU(2)}(f(g))^P dg = 0$ for all $P > 0$. We need to show that $\int_{SU(2)}(f(g))^P t_{a,b}^\ell(g) dg$ vanishes for sufficiently large $P$.

First assume that all not all of $m_i$’s and $n_i$’s are zero, then by Lemma 3.3 we have $m_1m_2 > 0$ or $n_1n_2 > 0$ or $\det \begin{pmatrix} m_1 & m_2 \\ n_1 & n_2 \end{pmatrix} \neq 0$. Consider the last case, then by (3.2), (3.3) we see that $\int_{SU(2)}(f(g))^P t_{a,b}^\ell(g) dg$ can only be non-zero if

$$m_1\alpha + m_2\beta = -a, \quad n_1\alpha + n_2\beta = -b, \quad \alpha + \beta = P, \quad \alpha, \beta \in \mathbb{N}.$$ 

The first two equations have a unique solution $(\alpha_0, \beta_0) \in \mathbb{Q}^2$. In case $(\alpha_0, \beta_0) \in \mathbb{N}^2$, we see that for all $P > \alpha_0 + \beta_0$ the integral is zero. In case $m_1m_2 > 0$, we consider $m_1\alpha + m_2\beta + a = 0$. In case $\text{sgn}(m_1) = \text{sgn}(a)$, we have no solution $(\alpha, \beta) \in \mathbb{N}^2$, so that integral is zero using (3.2), (3.3). In case $\text{sgn}(m_1) = -\text{sgn}(a)$, we see that the integral is zero for $P > |a|/\min(|m_1|, |m_2|)$. The case $n_1n_2 > 0$ is dealt with analogously.

In case $m_1 = m_2 = n_1 = n_2 = 0$, $f$ is a bi-$K$-invariant function, and

$$\int_{SU(2)}(f(g))^P dg = \frac{1}{2} \int_{-1}^1 (A_1P_{\ell_1}(x) + A_2P_{\ell_2}(x))^P dx.$$ 

By Boyarchenko’s result, see [3, Cor. 4.1], there is no polynomial $f$ such that $\int_{-1}^1(f(x))^P dx = 0$ for all $P > 0$, so the Mathieu Conjecture $\mathbb{C}$ is trivially valid in this case. \hfill $\square$

The fact that at most one term in the binomial expansion leads to a non-zero integral is typical for $f$ a linear combination of two matrix elements. For a combination of three matrix-elements it gets more complicated.

Proposition 3.6. Let $f = \sum_{i=1}^3 A_i t_{m_i,n_i}^\ell$ with $A_i \neq 0$ for all $i$ and let $(\ell_i, m_i, n_i)$ be mutually different. Assume that $M = \begin{pmatrix} 1 & 1 & 1 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{pmatrix}$ has $\text{rank}(M) \neq 2$. Then the Mathieu Conjecture $\mathbb{C}$ is valid for $f$.

Proof. The analogue of (3.5) is the trinomial expansion

$$\int_{SU(2)} f^P(g) dg = \sum_{\alpha_1 + \alpha_2 + \alpha_3 = P}^{P} \left( \prod_{i=1}^{3} A_i^{\alpha_i} \int_{SU(2)} \prod_{i=1}^{3} (t_{m_i,n_i}^\ell)^{\alpha_i}(g) dg. \right)$$ 

As before, it suffices to consider the case $h = t_{a,b}^\ell$. We have to consider the cases $\text{rank}(M) = 1$ and $\text{rank}(M) = 3$. In the first case $m_i = m$ and $n_i = n$ for all $i$, and the integral in (3.5) is
zero if \( m \neq 0 \) or \( n \neq 0 \) by (3.2). In case \( m \neq 0 \) we see that
\[
\int_{SU(2)} f^P(g) t^P_{a,b}(g) \, dg = \sum_{a_1+\alpha_2+\alpha_3=P} P_{\alpha_1, \alpha_2, \alpha_3} \prod_{i=1}^3 A^\alpha_i \int_{SU(2)} \prod_{i=1}^3 (t_{m,n_i}^i)^\alpha_i (g) t^P_{a,b}(g) \, dg
\]
(3.6)
can only be non-zero if \( Pm + a = 0 \), so that for \( P > |a|/|m| \) the integral is zero. The case \( n \neq 0 \) is analogous. In case \( m = n = 0 \), we see that the condition in the Mathieu Conjecture is not valid using [3, Cor. 4.1] as in the proof of Proposition 3.5.

In case rank\((M) = 3\), \( M \) is invertible with \( M^{-1} \) having rational entries. In particular, for each \( P \in \mathbb{N} \) there is at most one term in the right hand side of (3.5) which can be non-zero, namely for \( \alpha P = (\alpha_1, \alpha_2, \alpha_3) = M^{-1} \begin{pmatrix} P \\ 0 \\ 0 \end{pmatrix} \) under the additional condition \( \alpha P \in \mathbb{N}^3 \). Assuming that this is the case, we see that, analogous to the proof of Proposition 3.5 \( \int_{SU(2)} f^P(g) \, dg \neq 0 \).

So we need to consider the case that \( \alpha P \notin \mathbb{N}^3 \) for all \( P > 0 \). Then the integral in (3.6) can only be non-zero in case
\[
M^{-1} \begin{pmatrix} P \\ -a \\ -b \end{pmatrix} = \frac{1}{\det(M)} \begin{pmatrix} P \\ m_2n_3 - m_3n_2 \\ m_3n_1 - m_1n_3 \\ m_1n_2 - m_2n_1 \end{pmatrix} - a \begin{pmatrix} n_2 - n_3 \\ n_3 - n_1 \\ n_1 - n_2 \end{pmatrix} - b \begin{pmatrix} m_3 - m_2 \\ m_1 - m_3 \\ m_2 - m_1 \end{pmatrix} \in \mathbb{N}^3
\]

Since \( \alpha P \) corresponds to the first term, i.e. \( a = b = 0 \), and \( \alpha P \notin \mathbb{N}^3 \) for all \( P > 0 \) we have \( \det(M)^{-1} (m_i n_{i+1} - m_{i+1} n_i) < 0 \) for some \( i \in \{1, 2, 3\} \) with convention \( m_4 = m_1 \), \( n_4 = n_1 \). Then for \( P > (|a||n_i - n_{i+1}| + |b||m_i - m_{i+1}|)/|m_i n_{i+1} - m_{i+1} n_i| \) the \( i \)-th coefficient is negative, so that the integral in (3.6) is zero.

Remark 3.7. In case rank\((M) = 1\) the convex hull \( C \) of \( (m_i, n_i)_{i=1}^3 \) equals \( \{(m, n)\} \), and in case rank\((M) = 3\) we have \((0, 0) \in C\) if and only if \( \exists \alpha \in \mathbb{Q}^3_{\geq 0} \) with \( M \alpha = (1, 0, 0)^t \).

From the proof of Proposition 3.5 we see that \( \int_{SU(2)} f(g)^P \, dg = 0 \) for all \( P > 0 \) precisely when \((0, 0) \notin C\) in the cases rank\((M) \neq 2\). In case rank\((M) = 2\) the integral in (3.5) can have more than one non-zero term, and we have no control on possible cancellations. However, one expects that these cancellations cannot occur for all multiples of \( P \) as well. The techniques of Francoise et al. [3] might be useful in this regard considering it as polynomial identities in the \( A_i \)’s.

4. AN ALTERNATIVE CONJECTURE FOR THE MATHIEU CONJECTURE FOR SU(2)

Consider an arbitrary \( SU(2) \)-finite function \( f = \sum_{i=1}^k A_i t_{m_i, n_i}^i \) with \( A_i \neq 0 \) for every \( 1 \leq i \leq k \), then applying the multinomial finite theorem shows that if
\[
\int_{SU(2)} (f(g))^P \, dg = \sum_{\alpha_i \in \mathbb{N}^k, \sum_{i=1}^k \alpha_i = P} P_{\alpha_1, \ldots, \alpha_k} \prod_{i=1}^k A_i^\alpha_i \int_{SU(2)} \prod_{i=1}^k (t_{m_i, n_i}^i)^\alpha_i (g) \, dg \neq 0 \quad (4.1)
\]
for some $P > 0$, then for some $(\alpha_1, \ldots, \alpha_k)$ we have $\sum_{i=1}^k \frac{\alpha_i}{P} m_i = \sum_{i=1}^k \frac{\alpha_i}{P} n_i = 0$ by (3.2), so $(0,0)$ is in the convex hull $C$ of $((m_i, n_i))_{i=1}^k$ over $\mathbb{Q}$.

**Conjecture 4.1.** For any $SU(2)$-finite function $f = \sum_{i=1}^k A_i t_{m_i, n_i}^\ell$, $A_i \neq 0$ for all $1 \leq i \leq k$, we have that $\int_{SU(2)} (f(g))^P dg = 0$ for all $P \in \mathbb{N}_{>0}$ if and only if $(0,0)$ is not contained in the closed convex hull of $((m_i, n_i))_{i=1}^k$.

Lemma 3.1, Remarks 3.4, 3.7 support Conjecture 4.1.

**Theorem 4.2.** Assume Conjecture 4.1 holds, then the Mathieu Conjecture 1.1 for $SU(2)$ holds.

**Proof.** It suffices to show that $\int_{SU(2)} (f(g))^P t_{a,b}^\ell(g) dg = 0$ for $P$ sufficiently large assuming that $(0,0)$ is not contained in the closed convex hull $C$ of $((m_i, n_i))_{i=1}^k$. Using (4.1) we see that $\int_{SU(2)} (f(g))^P t_{a,b}^\ell(g) dg$ can only be non-zero if $(-\frac{a}{P}, -\frac{b}{P}) \in C$. Since $(0,0) \not\in C$, we see that for $P$ sufficiently large this is not the case and the integral is zero. □

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