Some remarks about solenoids

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Abstract
A basic family of solenoids is discussed, especially from the point of view of analysis on metric spaces.

Contents

I Basic examples 2
1 A Cartesian product 2
2 A nice metric 4
3 Another Cartesian product 5
4 Another metric 6
5 $r$-adic integers 7
6 A nice mapping 9
7 A nice mapping, continued 10
8 Haar measure on $Y$ 12
9 Continuous functions on $Y$ 13
10 Concluding remarks 14

II More complicated versions 15
11 The $r$-adic metric 15
12 $r$-adic integers 16
13 Haar measure on $Y_0 \cong \mathbb{Z}_r$ 18
14 Quotients of $\mathbb{R}$  
15 Metrics on $X$  
16 A nice mapping  
17 A nice mapping, continued  
18 Another metric on $Y$  
19 Haar measure on $Y$  

III Another perspective  
20 Ultrametrics  
21 $r$-adic absolute values  
22 Coherent sequences  
23 Topological equivalence  
24 Comparisons  
25 Solenoids  
26 Filtrations  
27 Directed systems  
References  

Part I  
Basic examples  

1 A Cartesian product  

Let $r \geq 2$ be an integer, and consider the Cartesian product  

\begin{equation}
X = \prod_{l=0}^{\infty} (\mathbb{R}/r^l \mathbb{Z}).
\end{equation}

More precisely, the real line $\mathbb{R}$ is a commutative group with respect to addition, $r^l \mathbb{Z}$ is the discrete subgroup of $\mathbb{R}$ consisting of integer multiples of $r^l$, and $\mathbb{R}/r^l \mathbb{Z}$ is the corresponding quotient group. The quotient $\mathbb{R}/r^l \mathbb{Z}$ may also be
considered as a compact Hausdorff topological space and a 1-dimensional smooth manifold in the usual way. The quotients \( \mathbb{R}/r^l\mathbb{Z} \) may actually be considered as Lie groups, because the group operations are given by smooth mappings. These Lie groups are all isomorphic to each other, and to the multiplicative group of complex numbers with modulus equal to 1. The Cartesian product \( X \) consists of the sequences \( x = \{x_l\}_{l=0}^{\infty} \) with \( x_l \in \mathbb{R}/r^l\mathbb{Z} \), and is a compact Hausdorff space with respect to the product topology. Of course, \( X \) is a commutative group as well, where the group operations are defined coordinatewise. It is easy to see that the group operations on \( X \) are continuous with respect to the product topology on \( X \), so that \( X \) is a topological group.

Because \( r^l+1\mathbb{Z} \subseteq r^l\mathbb{Z} \), there is a natural homomorphism from \( \mathbb{R}/r^l+1\mathbb{Z} \) onto \( \mathbb{R}/r^l\mathbb{Z} \) for each \( l \geq 0 \). An element \( x = \{x_l\}_{l=0}^{\infty} \) of \( X \) is said to be a coherent sequence if \( x_l \) is the image in \( \mathbb{R}/r^l\mathbb{Z} \) of \( x_{l+1} \in \mathbb{R}/r^{l+1}\mathbb{Z} \) for each \( l \). Note that the set \( Y \) of coherent sequences in \( X \) is a closed subgroup of \( X \) with respect to the topology and group structure described in the previous paragraph.

Let \( q_l \) be the usual quotient mapping from \( \mathbb{R} \) onto \( \mathbb{R}/r^l\mathbb{Z} \) for each \( l \). Consider the mapping \( q \) from \( \mathbb{R} \) into \( X \) defined by

\[
q(a) = \{q_l(a)\}_{l=0}^{\infty}
\]

for each \( a \in \mathbb{R} \). This is a continuous homomorphism from \( \mathbb{R} \) into \( X \) with trivial kernel. Observe that \( q(a) \) is a coherent sequence in \( X \) for each \( a \in \mathbb{R} \), because \( q_l \) is the same as the composition of \( q_{l+1} \) with the natural mapping from \( \mathbb{R}/r^{l+1}\mathbb{Z} \) onto \( \mathbb{R}/r^l\mathbb{Z} \) for each \( l \). Thus \( q(\mathbb{R}) \subseteq Y \), and in fact \( q(\mathbb{R}) \) is dense in \( Y \), so that \( Y \) is the same as the closure of \( q(\mathbb{R}) \) in \( X \). To see this, let \( x \in Y \) and a positive integer \( L \) be given, and choose \( a \in \mathbb{R} \) such that \( q_L(a) = x_L \). The coherence condition implies that \( q_l(a) = x_l \) for each \( l \leq L \), and hence that \( q(a) \) is arbitrarily close to \( x \) with respect to the product topology on \( X \), as desired. It follows that \( Y \) is connected, since it is the closure of the connected set \( q(\mathbb{R}) \).

Let \( \pi_l \) be the \( l \)th coordinate projection of \( X \) onto \( \mathbb{R}/r^l\mathbb{Z} \), so that

\[
\pi_l(x) = x_l
\]

for each \( x \in X \) and \( l \geq 0 \). Thus \( \pi_l \) is a continuous homomorphism from \( X \) onto \( \mathbb{R}/r^l\mathbb{Z} \), and similarly the restriction of \( \pi_l \) to \( Y \) defines a continuous homomorphism from \( Y \) onto \( \mathbb{R}/r^l\mathbb{Z} \) for each \( l \). By construction, the restriction of \( \pi_l \) to \( Y \) is the same as the composition of the restriction of \( \pi_{l+1} \) to \( Y \) with the natural homomorphism from \( \mathbb{R}/r^{l+1}\mathbb{Z} \) onto \( \mathbb{R}/r^l\mathbb{Z} \). If \( y \in Y \) is in the kernel of \( \pi_0 \), then it follows that

\[
\pi_l(y) \in \mathbb{Z}/r^l\mathbb{Z}
\]

for each \( l \geq 0 \). In particular, the kernel of the restriction of \( \pi_0 \) to \( Y \) is totally disconnected.
2 A nice metric

Let \( \phi \) be the standard isomorphism between \( \mathbb{R}/r^l \mathbb{Z} \) and the unit circle \( T \) in the complex plane \( \mathbb{C} \). Thus

\[
\phi_l(q_l(a)) = \exp(2 \pi i r^{-l} a)
\]

for every \( a \in \mathbb{R} \), where \( \exp z \) is the usual complex exponential function on \( \mathbb{C} \).

Remember that

\[
|\exp(i t)| = 1
\]

for every \( t \in \mathbb{R} \), where \( |\zeta| \) denotes the modulus of \( \zeta \in \mathbb{C} \). Note that

\[
d_l(x_l, y_l) = |\phi_l(x_l) - \phi_l(y_l)|
\]

defines a metric on \( \mathbb{R}/r^l \mathbb{Z} \), and that the topology on \( \mathbb{R}/r^l \mathbb{Z} \) determined by this metric is the same as the quotient topology corresponding to the standard topology on \( \mathbb{R} \). This is the same as saying that \( \phi_l \) is a homeomorphism from \( \mathbb{R}/r^l \mathbb{Z} \) onto \( T \) with respect to the topology on \( T \) induced by the standard Euclidean metric on \( \mathbb{C} \).

If \( x, y \in X \), then put

\[
d(x, y) = \max_{l \geq 0} r^{-l} |\phi_l(x_l) - \phi_l(y_l)|.
\]

Of course,

\[
|\phi_l(x_l) - \phi_l(y_l)| \leq |\phi_l(x_l)| + |\phi_l(y_l)| = 2,
\]

which implies that

\[
\lim_{l \to \infty} r^{-l} |\phi_l(x_l) - \phi_l(y_l)| = 0
\]

for every \( x, y \in X \). This ensures that the maximum in (2.4) is always attained.

It is easy to see that \( d(x, y) \) satisfies the requirements of a metric on \( X \). In particular, the triangle inequality for \( d(x, y) \) can be verified using the triangle inequality for (2.3) for each \( l \). The topology on \( X \) corresponding to \( d(x, y) \) is the same as the product topology discussed in the previous section. More precisely,

\[
d(x, y) < t
\]

for some positive real number \( t \) if and only if

\[
r^{-l} |\phi_l(x_l) - \phi_l(y_l)| < t
\]

for each \( l \geq 0 \) such that \( 2r^{-l} \geq t \). Thus (2.7) only involves finitely many coordinates of \( x \) and \( y \) for any given \( t > 0 \), which implies that open subsets of \( X \) with respect to \( d(x, y) \) are also open with respect to the product topology. Conversely, one can show that open subsets of \( X \) with respect to the product topology also open with respect to \( d(x, y) \), because (2.7) implies that any finite number of coordinates of \( x \) and \( y \) are arbitrarily close to each other when \( t \) is sufficiently small. Of course, we are using the fact that (2.3) determines the
quotient topology on $\mathbb{R}/r^l \mathbb{Z}$ corresponding to the standard topology on $\mathbb{R}$ for each $l \geq 0$ here.

If $x_l, y_l, z_l \in \mathbb{R}/r^l \mathbb{Z}$, then

$$
(2.9) \quad d_l(x_l + z_l, y_l + z_l) = |\phi_l(x_l + z_l) - \phi_l(y_l + z_l)|
$$

$$
= |\phi_l(x_l) \phi_l(z_l) - \phi_l(y_l) \phi_l(z_l)|
$$

$$
= |\phi_l(x_l) - \phi_l(y_l)| |\phi_l(z_l)|
$$

$$
= |\phi_l(x_l) - \phi_l(y_l)| = d_l(x_l, y_l).
$$

This shows that $d_l(x_l, y_l)$ is invariant under translations on $\mathbb{R}/r^l \mathbb{Z}$ for each $l \geq 0$. It follows that

$$
(2.10) \quad d(x + z, y + z) = d(x, y)
$$

for every $x, y, z \in X$, so that $d(x, y)$ is also invariant under translations on $X$.

3 Another Cartesian product

Consider the Cartesian product

$$
(3.1) \quad X_0 = \prod_{l=1}^{\infty} (\mathbb{Z}/r^l \mathbb{Z}).
$$

Thus the elements of $X_0$ are sequences $x = \{x_l\}_{l=1}^{\infty}$ such that $x_l \in \mathbb{Z}/r^l \mathbb{Z}$ for each $l$. We can identify $X_0$ with a subset of $X$, because $\mathbb{Z}/r^l \mathbb{Z} \subseteq \mathbb{R}/r^l \mathbb{Z}$ for each $l \geq 1$, and by extending $x = \{x_l\}_{l=1}^{\infty}$ to $l = 0$ by taking $x_0 = 0$ in $\mathbb{R}/\mathbb{Z}$. Note that $X_0$ corresponds to a closed subgroup of $X$ with respect to coordinatewise addition in this way. The topology on $X_0$ induced by the product topology on $X$ is the same as the product topology on $X_0$ that corresponds to taking the discrete topology on $\mathbb{Z}/r^l \mathbb{Z}$ for each $l$. Actually, $r^l \mathbb{Z}$ is an ideal in the ring of integers for each $l$, so that each quotient $\mathbb{Z}/r^l \mathbb{Z}$ may be considered as a commutative ring. It follows that $X_0$ is a commutative ring with respect to coordinatewise addition and multiplication as well. It is easy to see that multiplication on $X_0$ is continuous with respect to the product topology, so that $X_0$ is a topological ring.

As before, there is a natural ring homomorphism from $\mathbb{Z}/r^{l+1} \mathbb{Z}$ onto $\mathbb{Z}/r^l \mathbb{Z}$ for each $l \geq 1$, because $r^{l+1} \mathbb{Z} \subseteq r^l \mathbb{Z}$. An element $x = \{x_l\}_{l=1}^{\infty}$ of $X_0$ is said to be a coherent sequence if $x_l$ is the image in $\mathbb{Z}/r^l \mathbb{Z}$ of $x_{l+1} \in \mathbb{Z}/r^{l+1} \mathbb{Z}$ for each $l$. Thus $x$ is a coherent sequence in $X_0$ if and only if the corresponding element of $X$ is a coherent sequence in the sense of Section 1. Equivalently, the set $Y_0$ of coherent sequences in $X_0$ can be identified with the subset of $X$ which is the intersection of the set $Y$ of coherent sequences in $X$ with the subset of $X$ identified with $X_0$. Note that $Y_0$ is a closed subring of $X_0$.

Let $\overline{q}_l$ be the natural quotient mapping from $\mathbb{Z}$ onto $\mathbb{Z}/r^l \mathbb{Z}$ for each $l \geq 1$. This is the same as the restriction of the quotient mapping $q_l : \mathbb{R} \to \mathbb{R}/r^l \mathbb{Z}$.
from Section 1 to \( \mathbb{Z} \), although now \( \tilde{q} \) is a ring homomorphism from \( \mathbb{Z} \) onto \( \mathbb{Z}/r^l \mathbb{Z} \). Similarly, let \( \tilde{q} \) be the mapping from \( \mathbb{Z} \) into \( X_0 \) defined by

\[
\tilde{q}(a) = \{\tilde{q}(a_n)\}_{n=1}^\infty
\]

for each \( a \in \mathbb{Z} \). This is a ring homomorphism from \( \mathbb{Z} \) into \( X_0 \) with trivial kernel, and which is the same as the restriction of the embedding \( q : \mathbb{R} \to X \) defined in Section 1 to \( \mathbb{Z} \) when we identify \( X_0 \) with a subset of \( X \) as before. In particular, \( \tilde{q}(a) \) is a coherent sequence in \( X_0 \) for each \( a \in \mathbb{Z} \), for the same reasons as before.

One can also check that \( \tilde{q}(\mathbb{Z}) \) is dense in \( Y_0 \), so that \( Y_0 \) is the same as the closure of \( \tilde{q}(\mathbb{Z}) \) in \( X_0 \) with respect to the product topology. Of course, \( X_0 \) is obviously totally disconnected, and so \( Y_0 \) is too.

Let \( \pi_0 \) be the \( l = 0 \) coordinate projection of \( X \) onto \( \mathbb{R}/\mathbb{Z} \), as in Section 1. The kernel of the restriction of \( \pi_0 \) to \( Y \) consists of the coherent sequences \( y = \{y_l\}_{l=0}^\infty \) in \( X \) such that \( y_0 = 0 \) in \( \mathbb{R}/\mathbb{Z} \). Because of the coherence condition, this implies that \( y_l \in \mathbb{Z}/r^l \mathbb{Z} \) for each \( l \geq 1 \). Thus the kernel of the restriction of \( \pi_0 \) to \( Y \) corresponds exactly to the subset of \( X \) identified with \( Y_0 \).

\section{Another metric}

Let \( x \) and \( y \) be distinct elements of the set \( X_0 \) defined in the previous section, and let \( l(x, y) \) be the smallest positive integer \( l \) such that \( x_l \neq y_l \). Equivalently, \( l(x, y) \) is the largest positive integer \( l \) such that \( x_j = y_j \) for every \( j < l \). Put

\[
\rho(x, y) = r^{-l(x, y)+1}.
\]

If \( x = y \), then we put \( \rho(x, y) = 0 \), which corresponds to taking \( l(x, y) = +\infty \) in (4.1). Of course,

\[
l(x, y) = l(y, x)
\]

for every \( x, y \in X_0 \), which implies that

\[
\rho(x, y) = \rho(y, x).
\]

Similarly,

\[
l(x, z) \geq \min(l(x, y), l(y, z))
\]

for every \( x, y, z \in X_0 \), and hence

\[
\rho(x, z) \leq \max(\rho(x, y), \rho(y, z)).
\]

It follows that \( \rho(x, y) \) defines an ultrametric on \( X_0 \), which means that \( \rho(x, y) \) is a metric on \( X_0 \) that satisfies the stronger ultrametric version (4.5) of the triangle inequality.

It is easy to see that the topology on \( X_0 \) determined by \( \rho(x, y) \) is the same as the product topology corresponding to the discrete topology on each factor \( \mathbb{R}/r^l \mathbb{Z} \) in (3.1). We also have that

\[
l(x + z, y + z) = l(x, y)
\]
for every \( x, y, z \in X_0 \), so that

\[
\rho(x + z, y + z) = \rho(x, y).
\]

Thus \( \rho(x, y) \) is invariant under translations on \( X_0 \).

We would like to compare this metric with the one in Section 2. As before, \( x, y \in X_0 \) may be identified with elements of \( X \), by taking \( x_0 = y_0 = 0 \) in \( \mathbb{R}/\mathbb{Z} \).

In this case, (2.4) reduces to

\[
d(x, y) = \max_{l \geq 1} r^{-l} |\phi_l(x_l) - \phi_l(y_l)|.
\]

We may as well suppose that \( x \neq y \), since otherwise \( d(x, y) = \rho(x, y) = 0 \), so that (4.8) reduces further to

\[
d(x, y) = \max_{l \geq l(x, y)} r^{-l} |\phi_l(x_l) - \phi_l(y_l)|.
\]

In particular,

\[
d(x, y) \geq 2 r^{-l(x, y)} = 2 r^{-1} \rho(x, y),
\]

by (2.5).

In the other direction, we can take \( l = l(x, y) \) in (4.9), to get that

\[
d(x, y) \geq r^{-l(x, y)} |\phi_l(x_l(x, y)) - \phi_l(x_l(x, y))|.
\]

Under these conditions, \( x_l(x, y) \) and \( y_l(x, y) \) are distinct elements of \( \mathbb{Z}/r^{l(x, y)} \mathbb{Z} \), and hence

\[
|\phi_l(x_l(x, y)) - \phi_l(y_l(x, y))| \geq |\exp(2 \pi i r^{-l(x, y)}) - 1|.
\]

If \( x, y \in Y_0 \), so that \( x \) and \( y \) are coherent sequences, then \( x_l(x, y) \) and \( y_l(x, y) \) are distinct elements of \( \mathbb{Z}/r^{l(x, y)} \mathbb{Z} \) which are equal module \( r^{l(x, y)} - 1 \mathbb{Z} \), and

\[
|\phi_l(x_l(x, y)) - \phi_l(y_l(x, y))| \geq |\exp(2 \pi i r^{-1}) - 1|.
\]

Combining this with (4.11), we get that

\[
d(x, y) \geq r^{-1} |\exp(2 \pi i r^{-1}) - 1| \rho(x, y)
\]

for every \( x, y \in Y_0 \).

### 5 \( r \)-Adic integers

Let \( a \) be a nonzero integer, and let \( l(a) \) be the largest nonnegative integer \( l \) such that \( a \) is an integer multiple of \( r^l \). If \( b \) is another nonzero integer, then it is easy to see that

\[
l(a + b) \geq \min(l(a), l(b))
\]

and

\[
l(ab) \geq l(a) + l(b).
\]
The $r$-adic absolute value $|a|_r$ of $a$ is defined by

\[(5.3) \quad |a|_r = r^{-l(a)}.\]

Of course, we put $|a|_r = 0$ when $a = 0$, which corresponds to taking $l(a) = +\infty$. Thus we get that

\[(5.4) \quad |a + b|_r \leq \max(|a|_r, |b|_r)\]

and

\[(5.5) \quad |ab|_r \leq |a|_r |b|_r\]

for all integers $a$, $b$. The $r$-adic metric on $\mathbb{Z}$ is defined by

\[(5.6) \quad \delta_r(a, b) = |a - b|_r.\]

It is easy to see that this defines a metric on $\mathbb{Z}$, and more precisely an ultrametric on $\mathbb{Z}$, since

\[(5.7) \quad \delta_r(a, c) \leq \max(\delta_r(a, b), \delta_r(b, c))\]

for every $a, b, c \in \mathbb{Z}$, by (5.4).

Let $a$, $b$ be integers, and let $\tilde{q}(a)$, $\tilde{q}(b)$ be their images in $X_0$, as in Section 3. We would like to check that

\[(5.8) \quad \rho(\tilde{q}(a), \tilde{q}(b)) = \delta_r(a, b),\]

where $\rho(x, y)$ is the ultrametric on $X_0$ defined in Section 4. To do this, it suffices to show that

\[(5.9) \quad l(\tilde{q}(a), \tilde{q}(b)) - 1 = l(a - b),\]

where $l(x, y)$ is defined for $x, y \in X_0$ as in the previous section. Thus $l(\tilde{q}(a), \tilde{q}(b))$ is the smallest positive integer $l$ such that $\tilde{q}(a) \neq \tilde{q}(b)$, which is the same as saying that $l(\tilde{q}(a), \tilde{q}(b)) - 1$ is the largest nonnegative integer $k$ such that $\tilde{q}_j(a) = \tilde{q}_j(b)$ for every $j \leq k$. Remember that $\tilde{q}_j$ is the natural quotient homomorphism from $\mathbb{Z}$ onto $\mathbb{Z}/r^j \mathbb{Z}$, so that $\tilde{q}_j(a) = \tilde{q}_j(b)$ exactly when $a - b$ is an integer multiple of $r^j$. It follows that $l(\tilde{q}(a), \tilde{q}(b)) - 1$ is the same as the largest nonnegative integer $k$ such that $a - b$ is an integer multiple of $r^k$, which is also the same as $l(a - b)$, as desired. Note that we could have reduced to the case where $b = 0$ at the beginning of the argument, because $\tilde{q}$ is a homomorphism from $\mathbb{Z}$ into $X_0$, and because of the translation-invariance of the metrics involved.

A sequence $s(1) = \{x_1(1)\}_{i=1}^{\infty}$, $s(2) = \{x_1(2)\}_{i=1}^{\infty}$, $s(3) = \{x_1(3)\}_{i=1}^{\infty}$, \ldots of elements of $X_0$ converges to an element $x = \{x_1\}_{i=1}^{\infty}$ of $X_0$ with respect to the product topology discussed in Section 3, or equivalently with respect to the ultrametric $\rho(\cdot, \cdot)$, if and only if for each positive integer $n$ we have that $x_1(n) = x_1$ for all sufficiently large $l$, depending on $n$. Similarly, if $s(1), s(2), s(3), \ldots$ is a Cauchy sequence in $X_0$ with respect to $\rho(\cdot, \cdot)$, then it is easy to see that $x_1(n)$ is eventually constant in $l$ for each $n$, and hence that $s(1), s(2), s(3), \ldots$ converges in $X_0$. This shows that $X_0$ is complete as a metric space with respect to $\rho(\cdot, \cdot)$, which could also be derived from the compactness of $X_0$. It follows that $Y_0$ is complete as a metric space with respect to $\rho(\cdot, \cdot)$ too, because $Y_0$ is a closed subset of $X_0$. 

8
Thus $Y_0$ can be identified with the completion $\mathbb{Z}_r$ of $\mathbb{Z}$ with respect to the $r$-adic metric, since $\tilde{q}$ is an isometric embedding of $\mathbb{Z}$ onto a dense subset of $Y_0$, and $Y_0$ is complete with respect to $\rho(\cdot, \cdot)$. In particular, the ring structure on $Y_0$ defined by coordinatewise addition and multiplication corresponds to the ring structure on $\mathbb{Z}_r$ obtained by extending addition and multiplication on $\mathbb{Z}$ to $\mathbb{Z}_r$ by continuity. The completion $\mathbb{Z}_r$ of $\mathbb{Z}$ with respect to the $r$-adic metric is known as the ring of $r$-adic integers, especially when $r = p$ is a prime number. In this case, equality holds in (5.2) and (5.5), and the $p$-adic absolute value and metric can be defined on the field $\mathbb{Q}$ of rational numbers. The completion $\mathbb{Q}_p$ of $\mathbb{Q}$ with respect to the $p$-adic metric is known as the field of $p$-adic numbers, and $\mathbb{Z}_p$ is the same as the closure of $\mathbb{Z}$ in $\mathbb{Q}_p$.

6 A nice mapping

Consider the mapping $A$ from $\mathbb{R} \times Y_0$ into $Y$ defined by

$$(6.1) \quad A(a, x) = q(a) + x.$$  

Remember that $q$ maps $\mathbb{R}$ into $Y$ as in Section 1, and that we identify $x = \{x_l\}_{l=1}^{\infty} \in Y_0$ with an element of $Y$ by setting $x_0 = 0$ in $\mathbb{R}/\mathbb{Z}$. Thus (6.1) is defined by taking the sum of $q(a)$ and $x$ as elements of $Y$ as a subgroup of $X$ as a commutative group with respect to coordinatewise addition. More precisely, $q$ is a homomorphism of $\mathbb{R}$ into $Y$ with respect to addition, and hence $A$ is a homomorphism from $\mathbb{R} \times Y_0$ into $Y$ with respect to coordinatewise addition on $\mathbb{R} \times Y_0$.

Suppose that $(a, x) \in \mathbb{R} \times Y_0$ is in the kernel of $A$, so that $q(a) + x = 0$ in $Y$. In particular, the $l = 0$ coordinate of $q(a) + x$ is equal to 0 in $\mathbb{R}/\mathbb{Z}$, which implies that $q_0(a) = 0$ in $\mathbb{R}/\mathbb{Z}$, because $x \in Y_0$. It follows that $a \in \mathbb{Z}$, and that $x = \tilde{q}(-a)$ in $Y_0$ in the notation of Section 3. Conversely, if $a \in \mathbb{Z}$ and $x = \tilde{q}(-a)$ in $Y_0$, then $A(a, x) = 0$.

Let $y = \{y_l\}_{l=0}^{\infty}$ be any element of $Y$. If $y_0 = 0$, then $y$ can be identified with an element of $Y_0$, and $y$ is in the image of $A$. Otherwise, we can choose $a \in \mathbb{R}$ such that $q_0(a) = y_0$ in $\mathbb{R}/\mathbb{Z}$, so that the $l = 0$ coordinate of $y - q(a) \in Y$ is equal to 0. This implies that $y - q(a)$ corresponds to an element of $Y_0$, and hence that $y = q(a) + (y - q(a))$ is in the image of $A$.

Remember that $Y$ and $Y_0$ are equipped with topologies induced by the product topologies on $X$ and $X_0$, respectively. It is easy to see that $A$ is continuous as a mapping from $\mathbb{R} \times Y_0$ into $Y$, where $\mathbb{R} \times Y_0$ is equipped with the product topology associated to the standard topology on $\mathbb{R}$ and the topology on $Y_0$ just mentioned. This uses the fact that $q_l : \mathbb{R} \rightarrow \mathbb{R}/r^l \mathbb{Z}$ is continuous for each $l$. One can also check that $A$ is a local homeomorphism with respect to these topologies. Continuous local inverses for $A$ can be given as in the previous paragraph, using the fact that $q_0 : \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z}$ is a local homeomorphism.
7 A nice mapping, continued

Consider the metric on $\mathbb{R} \times Y_0$ defined by

\[
D((a, x), (b, y)) = \max(|a - b|, \rho(x, y)).
\]

Here $|a|$ is the ordinary absolute value of a real number $a$, so that $|a - b|$ is the standard metric on the real line, and $\rho(x, y)$ is the ultrametric on $X_0$ defined in Section 4. Thus the topology on $\mathbb{R} \times Y_0$ determined by (7.1) is the same as the product topology associated to the standard topology on $\mathbb{R}$ and the usual topology on $Y_0$. We would like to look more precisely at the behavior of the mapping $A : \mathbb{R} \times Y_0 \to Y$ defined in the previous section with respect to this metric on $\mathbb{R} \times Y_0$ and the metric $d(\cdot, \cdot)$ on $Y$ discussed in Section 2.

Note that the derivative of $\exp(it)$ is equal to $i \exp(it)$, which has modulus equal to 1 for each $t \in \mathbb{R}$. Using this, one can check that

\[
|\exp(iu) - \exp(iv)| \leq |u - v|
\]

for every $u, v \in \mathbb{R}$, by expressing $\exp(iu) - \exp(iv)$ as an integral of $i \exp(it)$. If $|u - v| \leq \pi$, for instance, then we have that

\[
|\exp(iu) - \exp(iv)| \geq c_1 |u - v|
\]

for a suitable constant $c_1 > 0$, i.e., $2/\pi$.

Let $a$ and $b$ be real numbers, and let $\phi_l$, $d_l$, and $d$ be as in Section 2. Thus

\[
|\phi_l(q_l(a)) - \phi_l(q_l(b))| = |\exp(2\pi ir^{-l}a) - \exp(2\pi ir^{-l}b)|
\]

\[
\leq 2\pi |r^{-l}| |a - b|
\]

for each $l \geq 0$, by (7.2). This implies that

\[
d(q_l(a), q_l(b)) \leq 2\pi |a - b|.
\]

Now let $x, y \in Y_0$ be given as well. If $x = y$, then

\[
d(A(x, x), A(y, y)) = d(q_l(a) + x, q_l(b) + y) = d(q_l(a), q_l(b))
\]

\[
\leq 2\pi |a - b| = 2\pi D((a, x), (b, y)),
\]

using the translation-invariance of $d$ in the second step, and (7.5) in the third. Suppose instead that $x \neq y$, and let $l(x, y)$ be as in Section 4. Remember that $x$ and $y$ are identified with elements of $Y$ by putting $x_0 = y_0 = 0$ in $\mathbb{R}/\mathbb{Z}$. If $0 \leq j < l(x, y)$, then $x_j = y_j$ by the definition of $l(x, y)$, and hence

\[
d_j(q_j(a) + x_j, q_j(b) + y_j) = d_j(q_j(a), q_j(b))
\]

by translation-invariance. This implies that

\[
\max_{0 \leq j < l(x, y)} r^{-j} d_j(q_j(a) + x_j, q_j(b) + y_j) \leq 2\pi |a - b|,
\]
as before. If $j \geq l(x, y)$, then we have that
\begin{equation}
(7.9) \quad r^{-j} d_j(q_j(a) + x_j, q_j(b) + y_j) \leq 2 r^{-l(x, y)},
\end{equation}
because $d_j \leq 2$ automatically, as in (2.5). This implies that
\begin{equation}
(7.10) \quad \max_{j \geq l(x, y)} r^{-j} d_j(q_j(a) + x_j, q_j(b) + y_j) \leq 2 r^{-1} \rho(x, y),
\end{equation}
by the definition (4.1) of $\rho(x, y)$. Combining (7.8) and (7.10), we get that
\begin{equation}
(7.11) \quad d(A(a, x), A(b, y)) \leq 2 \pi D((a, x), (b, y)),
\end{equation}
since $r^{-1} \leq 1 \leq \pi$. This also holds when $x = y$, as in (7.6), which amounts to taking $l(x, y) = +\infty$ in this argument.

To get an estimate in the other direction, let us restrict our attention to $a, b \in \mathbb{R}$ such that
\begin{equation}
(7.12) \quad |a - b| \leq 1/2,
\end{equation}
for instance. Note that
\begin{equation}
(7.13) \quad d(A(a, x), A(b, y)) \geq d_0(q_0(a) + x_0, q_0(b) + y_0) = d_0(q_0(a), q_0(b)),
\end{equation}
by taking $l = 0$ in the definition (2.4) of $d$, and remembering that $x_0 = y_0 = 0$. Of course,
\begin{equation}
(7.14) \quad d_0(q_0(a), q_0(b)) = |\phi_0(q_0(a)) - \phi_0(q_0(b))| = |\exp(2 \pi i a) - \exp(2 \pi i b)|,
\end{equation}
so that
\begin{equation}
(7.15) \quad d(A(a, x), A(b, y)) \geq 2 \pi c_1 |a - b|
\end{equation}
when $a, b$ satisfy (7.12), by (7.3). In particular, we can combine this with (7.5) to get that
\begin{equation}
(7.16) \quad d(q(a), q(b)) \leq c_1^{-1} d(A(a, x), A(b, y))
\end{equation}
when $a, b$ satisfy (7.12).

Using translation-invariance and then the triangle inequality, we get that
\begin{equation}
(7.17) \quad d(x, y) = d(q(a) + x, q(a) + y) \leq d(q(a) + x, q(b) + y) + d(q(a), q(b)).
\end{equation}
Combining this with (7.16), it follows that
\begin{equation}
(7.18) \quad d(x, y) \leq d(A(a, x), A(b, y)) + c_1^{-1} d(A(a, x), A(b, y))
\end{equation}
when $a, b$ satisfy (7.12). We also know that $\rho(x, y)$ is bounded by a constant times $d(x, y)$, as in (4.14), so that
\begin{equation}
(7.19) \quad \rho(x, y) \leq r |\exp(2 \pi i r^{-1}) - 1|^{-1} (1 + c_1^{-1}) d(A(a, x), A(b, y))
\end{equation}
when $a, b$ satisfy (7.12). This together with (7.15) shows that
\begin{equation}
(7.20) \quad D((a, x), (b, y)) \leq c_2(r) d(A(a, x), A(b, y))
\end{equation}
when $a, b$ satisfy (7.12), where $c_2(r)$ is a positive real number that depends only on $r$.

Note that the comparison between $d(A(a, x), A(b, y))$ and $D((a, x), (b, y))$ in this section would be a bit simpler if we replaced $\rho(x, y)$ in (7.1) with $r^{-1} \rho(x, y)$, to get the metric

$$D'((a, x), (b, y)) = \max(|a - b|, r^{-1} \rho(x, y))$$

on $\mathbb{R} \times Y_0$. Similarly, the comparison between $d(x, y)$ and $\rho(x, y)$ for $x, y \in Y_0$ in Section 4 may be considered as a better comparison between $d(x, y)$ and $r^{-1} \rho(x, y)$. However, the original definition $\rho(x, y)$ has the advantage that it corresponds exactly to the $r$-adic metric on $\mathbb{Z}_r$, as in Section 5.

### 8 Haar measure on $Y$

It is well known that every locally compact commutative topological group has a nonnegative Borel measure which is invariant under translations, finite on compact sets, and positive on nonempty open sets, known as Haar measure. This measure is unique up to multiplication by a positive real number, at least under some additional regularity conditions, which are not necessary for the groups under consideration here. Of course, Lebesgue measure satisfies the requirements of Haar measure on the real line as a locally compact commutative topological group with respect to addition, and similarly for the unit circle. Alternatively, one can start with a nonnegative linear functional on the space of continuous real or complex-valued functions with compact support on the group which is invariant under translations and strictly positive for nonnegative continuous functions that are positive somewhere on the group. The Riesz representation theorem then leads to a nonnegative Borel measure on the group with the required properties.

Let us begin with $Y_0$, which we have identified with a closed subgroup of $Y$, and which is isomorphic as a topological group to the group $\mathbb{Z}_r$ of $r$-adic integers with respect to addition, as in Section 5. If we normalize Haar measure on $\mathbb{Z}_r$ so that the measure of $\mathbb{Z}_r$ is equal to 1, then it is easy to see that the measure of $r^l \mathbb{Z}_r$ has to be equal to $r^{-l}$ for each nonnegative integer $l$. This is because $\mathbb{Z}_r/r^l \mathbb{Z}_r$ is isomorphic to $\mathbb{Z}/r^l \mathbb{Z}$, so that $\mathbb{Z}_r$ is the union of $r^l$ pairwise-disjoint translates of $r^l \mathbb{Z}_r$. One can also define the Haar integral of a continuous function on $\mathbb{Z}_r$ directly as a limit of Riemann sums, using this partition of $\mathbb{Z}_r$ into translates of $r^l \mathbb{Z}_r$ for each $l \geq 0$.

If $f$ is a continuous real or complex-valued function on $Y$, then one can first integrate $f$ over $Y_0$ and its translates in $Y$, to get a continuous function $f_0$ on $Y$ that is constant on $Y_0$ and its translates in $Y$. Thus $f_0$ is basically the same as a continuous function on the unit circle, which can be integrated over $\mathbb{T}$ in the usual way. It is easy to see that translations of $f$ on $Y$ correspond to translations of the function on $\mathbb{T}$ associated to $f_0$ in a simple way, so that this defines a translation-invariant integral of continuous functions on $Y$ with the appropriate positivity properties. Equivalently, one can average $f$ over translates of the
subgroup of $Y_0 \cong \mathbb{Z}_r$ that corresponds to $r^l \mathbb{Z}_r$ for any nonnegative integer $l$, to get a continuous function $f_l$ on $Y$ that is basically the same as a continuous function on $\mathbb{R}/r^l \mathbb{Z}$. One can then take the average of the resulting function on $\mathbb{R}/r^l \mathbb{Z}$ to get a translation-invariant average of $f$ on $Y$ that does not depend on $l$.

Note that Haar measure on $\mathbb{Z}_r$ is Ahlfors regular of dimension 1 with respect to the $r$-adic metric on $\mathbb{Z}_r$, in the sense that the measure of a ball of radius $t > 0$ is bounded from above and below by constant multiples of $t$, at least when $t$ is less than or equal to the diameter of $\mathbb{Z}_r$, which is 1. More precisely, the closed balls in $\mathbb{Z}_r$ of radius $r^{-l}$ are the same as the translates of $r^l \mathbb{Z}_r$, which have measure equal to $r^{-l}$. Of course, Lebesgue measure on the real line is also Ahlfors regular of dimension 1 with respect to the standard metric on $\mathbb{R}$, and Haar measure on the unit circle is Ahlfors regular of dimension 1 with respect to the standard metric on $\mathbb{T}$ as well. Similarly, one can check that Haar measure on $Y$ is Ahlfors-regular of dimension 2 with respect to the metric $d$ defined in Section 2.

9 Continuous functions on $Y$

Let $\pi_l$ be the $l$th coordinate projection from $X$ onto $\mathbb{R}/r^l \mathbb{Z}$ for each nonnegative integer $l$, as in Section 1. Thus the restriction of $\pi_l$ to $Y$ defines a continuous homomorphism from $Y$ onto $\mathbb{R}/r^l \mathbb{Z}$ for each $l$. In particular, if $g$ is a continuous real or complex-valued function on $\mathbb{R}/r^l \mathbb{Z}$, then the restriction of $g \circ \pi_l$ to $Y$ is a continuous function on $Y$. These are the same as the continuous functions on $Y$ that are constant on the translates of the subgroup of $Y_0 \cong \mathbb{Z}_r$ corresponding to $r^l \mathbb{Z}_r$.

If $f$ is any continuous real or complex-valued function on $Y$, then $f$ can be approximated uniformly on $Y$ by functions of this type, as $l \to \infty$. One way to see this is to average $f$ over the translates of the subgroups of $Y_0$ corresponds to $r^l \mathbb{Z}_r$ in $\mathbb{Z}_r$, with respect to Haar measure on $\mathbb{Z}_r$. These averages will converge uniformly to $f$ as $l \to \infty$, because of the uniform continuity of $f$ on $Y$, and since $Y$ is a compact metric space. Alternatively, $f(q(a))$ is a continuous function on the real line such that

$$\lim_{l \to \infty} f(q(r^l)) = f(q(0)),$$

since $q(r^l) \to q(0)$ as $l \to \infty$ in $Y$. This permits one to approximate the restriction of $f(q(a))$ to $[0, r^l]$ by a continuous periodic function with period $r^l$. The latter corresponds exactly to a continuous function on $\mathbb{R}/r^l \mathbb{Z}$, whose composition with $\pi_l$ defines a continuous function on $Y$ as before. One can again use the uniform continuity of $f$ to show that $f$ is uniformly approximated on $Y$ by functions like these as $l \to \infty$.

Remember that a character on a locally compact commutative topological group is a continuous homomorphism from that group into the unit circle $\mathbb{T}$, as a group with respect to multiplication of complex numbers. It is well known that the characters on $\mathbb{T}$ are the mappings of the form $z \mapsto z^n$, where $n$ is an integer.
Equivalently, the characters on $\mathbb{R}/r^l\mathbb{Z}$ are given by the integer powers of the mappings $\phi_l$ defined in Section 2. The composition of any character on $\mathbb{R}/r^l\mathbb{Z}$ with $\pi_l$ defines a character on $Y$, since the restriction of $\pi_l$ to $Y$ is a continuous homomorphism from $Y$ onto $\mathbb{R}/r^l\mathbb{Z}$. If $k$ is an integer greater than or equal to $l$, then $r^k\mathbb{Z}$ is a subgroup of $r^l\mathbb{Z}$, which leads to a continuous homomorphism from $\mathbb{R}/r^k\mathbb{Z}$ onto $\mathbb{R}/r^l\mathbb{Z}$. The composition of a character on $\mathbb{R}/r^l\mathbb{Z}$ with this homomorphism leads to a character on $Y$ by composition with $\pi_k$. Thus the characters on $Y$ coming from those on $\mathbb{R}/r^l\mathbb{Z}$ are contained in the characters on $Y$ coming from those on $\mathbb{R}/r^k\mathbb{Z}$ when $l \leq k$.

Conversely, every character on $Y$ comes from one on $\mathbb{R}/r^l\mathbb{Z}$ in this way. To see this, it suffices to check that every character on $Y$ is constant on one of the subgroups of $Y_0 \cong \mathbb{Z}_r$ corresponding to $r^l\mathbb{Z}_r$ for some $l$. Note that every neighborhood of the additive identity element 0 in $Y$ contains these subgroups for sufficiently large $l$. A character on $Y$ maps small neighborhoods of 0 in $Y$ to small neighborhoods of 1 in $T$, and hence maps these subgroups into small neighborhoods of 1 in $T$ when $l$ is sufficiently large. However, the trivial subgroup $\{1\}$ of $T$ is the only subgroup contained in a suitable neighborhood of 1 in $T$, which implies that characters on $Y$ are constant on these subgroups when $l$ is sufficiently large, as desired.

10 Concluding remarks

The solenoid $Y$ seems to be an interesting example of a somewhat exotic “space of homogeneous type”, in the sense of [6, 7]. Of course, the local geometry of $Y$ is essentially that of a product of an interval with a Cantor set, but the global structure is more complicated, since $Y$ is connected in particular. In addition, $Y$ has the structure of a compact commutative topological group, and the geometry on $Y$ is compatible with this. It should also be mentioned that for some questions in analysis, one is probably better off looking at $Y$ as a topological group, without using this type of geometry. More precisely, one can approximate $Y$ by $\mathbb{R}/r^l\mathbb{Z}$, and functions on $Y$ be functions on $\mathbb{R}/r^l\mathbb{Z}$, as in the previous section.

By way of comparison, one might consider the ordinary product $W = T \times \mathbb{Z}_r$ of the unit circle and the $r$-adic integers. This is also a compact commutative topological group, where the group operations are defined coordinatewise, and one can get a natural translation-invariant metric on $W$ by taking the maximum of the usual metrics on $T$ and $\mathbb{Z}_r$ in their respective coordinates. Note that characters on $W$ are given by products of characters on $T$ and $\mathbb{Z}_r$. In this case, Haar measure on $W$ is given by the product of the Haar measures on $T$ and $\mathbb{Z}_r$, and is Ahlfors regular of dimension 2 in particular.

Let $M$ be a metric space which is the product of a closed interval in the real line with the standard metric and another metric space which is Ahlfors regular of some positive dimension. As in Theorem 4.12 in Section 4.4 of [33], one can use arguments like those in [14] to show that metric doubling measures are absolutely continuous, with density given by an $A_\infty$ weight. This is basically
the same as absolute continuity properties of quasisymmetric mappings from $M$ into another Ahlfors regular metric space of the same dimension, which is the Hausdorff dimension. This type of argument is essentially local, and hence works as well for spaces like $Y$. Of course, the global structure of $Y$ is important for the global behavior of quasisymmetric mappings on $Y$ too.

Part II
More complicated versions

11 The $r$-adic metric

Let $r = \{r_j\}_{j=1}^{\infty}$ be a sequence of integers with $r_j \geq 2$ for each $j$, and put

\begin{equation}
R_l = \prod_{j=1}^{l} r_j
\end{equation}

when $l \geq 1$, and $R_0 = 1$. In particular, if $r$ is a constant sequence, so that $r_j = r_1$ for each $j$, then

\begin{equation}
R_l = r_1^l
\end{equation}

for each $l$. Note that $R_l \geq 2^l$ for each $l \geq 0$, and hence that $R_l \to \infty$ as $l \to \infty$. If $r_j \leq C$ for some $C \geq 2$ and each $j \geq 1$, then $R_l \leq C^l$ for each $l \geq 0$.

Let $\mathbf{Z}$ be the ring of integers, as usual. If $x \in \mathbf{Z}$ and $x \neq 0$, then let $l(x)$ be the largest nonnegative integer $l$ such that $x$ is an integer multiple of $R_l$, and put $l(0) = +\infty$. Observe that

\begin{equation}
l(x + y) \geq \min(l(x), l(y))
\end{equation}

and

\begin{equation}
l(xy) \geq \max(l(x), l(y))
\end{equation}

for every $x, y \in \mathbf{Z}$. If $r$ is a constant sequence, then we get that

\begin{equation}
l(xy) \geq l(x) + l(y)
\end{equation}

for every $x, y \in \mathbf{Z}$. If $r$ is a constant sequence and $r_1$ is a prime number, then

\begin{equation}
l(xy) = l(x) + l(y)
\end{equation}

for every $x, y \in \mathbf{Z}$.

The $r$-adic absolute value of $x \in \mathbf{Z}$ is defined by

\begin{equation}
|x|_r = 1/R_{l(x)}
\end{equation}

when $x \neq 0$, and $|0|_r = 0$. Thus

\begin{equation}
|x + y|_r \leq \max(|x|_r, |y|_r)
\end{equation}
and
\[(11.9) \quad |xy|_r \leq \min(|x|_r, |y|_r)\]
for every \(x, y \in \mathbb{Z}\), by (11.3) and (11.4). If \(r\) is a constant sequence, then
\[(11.10) \quad |xy|_r \leq |x|_r |y|_r\]
for every \(x, y \in \mathbb{Z}\), by (11.5). Similarly, if \(r\) is a constant sequence and \(r_1\) is a prime number, then
\[(11.11) \quad |xy|_r = |x|_r |y|_r\]
for every \(x, y \in \mathbb{Z}\), by (11.6). In this case, \(|x|_r\) is the same as the usual \(p\)-adic absolute value of \(x\), with \(p = r_1\).

The \(r\)-adic metric on \(\mathbb{Z}\) is defined by
\[(11.12) \quad \delta_r(x, y) = |x - y|_r.\]

It is easy to see that this satisfies the requirements of a metric on \(\mathbb{Z}\), and is in fact an ultrametric on \(\mathbb{Z}\), since
\[(11.13) \quad \delta_r(x, z) \leq \max(\delta_r(x, y), \delta_r(y, z))\]
for every \(x, y, z \in \mathbb{Z}\), by (11.8). By construction, this ultrametric is invariant under translations on \(\mathbb{Z}\), in the sense that
\[(11.14) \quad \delta_r(x + z, y + z) = \delta_r(x, y)\]
for every \(x, y, z \in \mathbb{Z}\). One can check that addition and multiplication on \(\mathbb{Z}\) define continuous mappings from \(\mathbb{Z} \times \mathbb{Z}\) into \(\mathbb{Z}\) with respect to the topology associated to this ultrametric, using the corresponding product topology on \(\mathbb{Z} \times \mathbb{Z}\). More precisely, this follows from (11.8), (11.9), and standard arguments.

### 12 \(r\)-Adic integers

As usual, one can take the completion of \(\mathbb{Z}\) as a metric space with the ultrametric \(\delta_r(x, y)\) to get the \(r\)-adic integers \(\mathbb{Z}_r\). If \(r\) is a constant sequence and \(r_1 = p\) is a prime number, then this reduces to the usual construction of the \(p\)-adic integers \(\mathbb{Z}_p\). For any \(r\), addition and multiplication can be extended to \(\mathbb{Z}_r\), as well as the \(r\)-adic absolute value and metric, and with the same properties as before. In particular, \(\mathbb{Z}_r\) is a commutative topological ring with respect to the topology determined by the extension of the \(r\)-adic metric. It is easy to see that \(\mathbb{Z}\) and hence \(\mathbb{Z}_r\) are totally bounded with respect to the \(r\)-adic metric, which implies that \(\mathbb{Z}_r\) is compact, since it is complete.

Alternatively, consider the Cartesian product
\[(12.1) \quad X_0 = \prod_{l=1}^{\infty} (\mathbb{Z}/R_l \mathbb{Z}).\]
Thus the elements of $X_0$ are sequences $x = \{x_l\}_{l=1}^{\infty}$ with $x_l$ in the quotient ring $\mathbb{Z}/R_l\mathbb{Z}$ for each $l$. Addition and multiplication of elements of $X_0$ can be defined coordinatewise, so that $X_0$ is a commutative ring. Using the product topology on $X_0$ corresponding to the discrete topology on $\mathbb{Z}/R_l\mathbb{Z}$ for each $l$, $X_0$ becomes a compact Hausdorff topological space as well. It is easy to see that addition and multiplication on $X_0$ define continuous mappings from $X_0 \times X_0$ into $X_0$, so that $X_0$ is a topological ring.

If $x, y \in X_0$ and $x \neq y$, then let $l(x, y)$ be the smallest positive integer $l$ such that $x_l \neq y_l$. Equivalently, $l(x, y) - 1$ is the largest nonnegative integer such that $x_l = y_l$ when $l \leq l(x, y) - 1$. Put

$$\rho(x, y) = 1/R_{l(x, y)} - 1.$$  \hfill (12.2)

If $x = y$, then we put $l(x, y) = +\infty$ and $\rho(x, y) = 0$. Thus

$$l(x, y) = l(y, x)$$  \hfill (12.3)

and

$$\rho(x, y) = \rho(y, x)$$  \hfill (12.4)

for every $x, y \in X_0$. One can also check that

$$l(x, z) \geq \min(l(x, y), l(y, z))$$  \hfill (12.5)

for every $x, y, z \in X_0$, so that

$$\rho(x, z) \leq \max(\rho(x, y), \rho(y, z)).$$  \hfill (12.6)

This implies that $\rho(x, y)$ defines an ultrametric on $X_0$, and it is easy to see that the topology on $X_0$ corresponding to $\rho(x, y)$ is the same as the product topology mentioned in the previous paragraph. Note that

$$l(x + z, y + z) = l(x, y)$$  \hfill (12.7)

for every $x, y, z \in X_0$, and hence that

$$\rho(x + z, y + z) = \rho(x, y),$$  \hfill (12.8)

so that $\rho(x, y)$ is invariant under translations on $X_0$. Because $X_0$ is compact, it is complete as a metric space, which can also be verified directly from the definitions.

Let $\tilde{q}_l$ be the standard quotient map from $\mathbb{Z}$ onto $\mathbb{Z}/R_l\mathbb{Z}$ for each $l \geq 1$, which is a ring homomorphism. Put

$$\tilde{q}(a) = \{\tilde{q}_l(a)\}_{l=1}^{\infty}$$  \hfill (12.9)

for each $a \in \mathbb{Z}$, which is a ring homomorphism from $\mathbb{Z}$ into $X_0$ with trivial kernel. If $a, b \in \mathbb{Z}$, then it is easy to see that

$$l(\tilde{q}(a), \tilde{q}(b)) - 1 = l(a - b),$$  \hfill (12.10)
and hence that
\[(12.11) \quad \rho(\tilde{q}(a), \tilde{q}(b)) = |a - b|_\infty = \delta_r(a, b).\]
Thus \(\tilde{q}\) is an isometric embedding of \(\mathbb{Z}\) with the \(r\)-adic metric into \(X_0\) with the ultrametric \(\rho(x, y)\), which implies that the completion \(\mathbb{Z}_r\) of \(\mathbb{Z}\) with respect to the \(r\)-adic metric can be identified with the closure of \(\tilde{q}(\mathbb{Z})\) in \(X_0\), since \(X_0\) is complete.

As usual, there is a natural ring homomorphism from \(\mathbb{Z}/R_{l+1} \mathbb{Z}\) onto \(\mathbb{Z}/R_l \mathbb{Z}\) for each \(l \geq 0\), because \(R_{l+1} \mathbb{Z} \subseteq R_l \mathbb{Z}\). An element \(x = \{x_i\}_{i=1}^\infty\) of \(X_0\) is said to be a coherent sequence if \(x_i\) is the image in \(\mathbb{Z}/R_l \mathbb{Z}\) of \(x_{i+1} \in \mathbb{Z}/R_{l+1} \mathbb{Z}\) for each \(l \geq 0\). In particular, \(\tilde{q}(a)\) is a coherent sequence for each \(a \in \mathbb{Z}\), because \(\tilde{q}\) is the same as the composition of \(\tilde{q}_{l+1}\) with the natural homomorphism from \(\mathbb{Z}/R_{l+1} \mathbb{Z}\) onto \(\mathbb{Z}/R_l \mathbb{Z}\) for each \(l\). The set \(Y_0\) of coherent sequences is a closed subring of \(X_0\), and one can check that \(\tilde{q}(\mathbb{Z})\) is dense in \(Y_0\), so that \(Y_0\) is the same as the closure in \(X_0\) of \(\tilde{q}(\mathbb{Z})\). Thus the completion \(\mathbb{Z}_r\) of \(\mathbb{Z}\) with respect to the \(r\)-adic metric can be identified with \(Y_0\).

Let \(n\) be a positive integer, and let \(Y_n\) be the set of coherent sequences \(x = \{x_i\}_{i=1}^\infty\) such that \(x_n = 0 \in \mathbb{Z}/R_n \mathbb{Z}\). This implies that \(x_i = 0 \in \mathbb{Z}/R_l \mathbb{Z}\) when \(l \leq n\), because of coherence. It is easy to see that \(Y_n\) is a closed subring of \(X_0\) which is an ideal in \(Y_0\). If \(a \in \mathbb{Z}\), then \(\tilde{q}(a) \in Y_n\) if and only if \(a \in R_n \mathbb{Z}\), and \(Y_n\) is the same as the closure in \(X_0\) of \(\tilde{q}(R_n \mathbb{Z})\).

Equivalently, \(Y_n\) is the kernel of the homomorphism from \(Y_0\) into \(\mathbb{Z}/R_n \mathbb{Z}\) that sends a coherent sequence \(x = \{x_i\}_{i=1}^\infty\) to its \(n\)th term \(x_n\). More precisely, this is a homomorphism from \(Y_0\) onto \(\mathbb{Z}/R_n \mathbb{Z}\), because its composition with \(\tilde{q}: \mathbb{Z} \to Y_0\) is the quotient homomorphism \(\tilde{q}_n\) from \(\mathbb{Z}\) onto \(\mathbb{Z}/R_n \mathbb{Z}\). Thus the quotient \(Y_0/Y_n\) is isomorphic as a commutative ring to \(\mathbb{Z}/R_n \mathbb{Z}\).

### 13 Haar measure on \(Y_0 \cong \mathbb{Z}_r\)

Let \(\mu_0\) be Haar measure on \(Y_0\), normalized so that
\[(13.1) \quad \mu_0(Y_0) = 1.\]
Note that \(Y_n\) is both relatively open and closed in \(Y_0\) for each \(n \geq 1\). It is easy to see that
\[(13.2) \quad \mu_0(Y_n) = 1/R_n\]
for each \(n \geq 1\), because \(Y_0/Y_n \cong \mathbb{Z}/R_n \mathbb{Z}\), so that \(Y_0\) is the union of \(R_n\) pairwise-disjoint translates of \(Y_n\).

With respect to the restriction of the ultrametric \(\rho(x, y)\) on \(Y_0\) to \(Y_n\), \(Y_n\) is the same as the closed ball in \(Y_0\) centered at 0 and with radius \(1/R_n\), and every closed ball in \(Y_0\) with radius \(1/R_n\) is a translate of \(Y_n\). Thus (13.2) may be considered as a very precise form of Ahlfors regularity of dimension 1 for radii of the form \(1/R_n\). In particular, if the original sequence of \(r_j\)’s bounded, then it is easy to see that \(Y_0\) is Ahlfors regular of dimension 1. Of course, the \(r_j\)’s are bounded when they are all equal to each other, in which case \(Y_0\) enjoys additional self-similarity properties. However, if the \(r_j\)’s are not bounded, then
\( \mu_0 \) is not even a doubling measure on \( Y_0 \), and \( Y_0 \) does not satisfy a doubling condition as a metric space.

Even if the \( r_j \)'s are not bounded, the fact that the metric on \( Y_0 \) is an ultrametric implies that any two balls in \( Y_0 \) are either disjoint, or one of the balls is contained in the other. Given any collection of balls in \( Y_0 \), one can take the maximal balls in the collection to get a sub-collection of pairwise-disjoint balls with the same union. One can also look at this in terms of martingales, using the partitions of \( Y_0 \) obtained from the translations of \( Y_n \) for each \( n \). Thus one can get the usual estimates for the Hardy–Littlewood maximal function on \( Y_0 \), for instance, even when the \( r_j \)'s are not bounded.

If \( H^1(E) \) denotes the one-dimensional Hausdorff measure of a set \( E \subseteq Y_0 \) with respect to the restriction of the ultrametric \( \rho(x, y) \) on \( X_0 \) to \( Y_0 \), then it is easy to see that

\[
H^1(Y_n) \leq 1/R_n, \quad (13.3)
\]

for each \( n \geq 0 \), by considering coverings of \( Y_n \) by translates of \( Y_k \) when \( k \geq n \). One can get the opposite inequality by comparing other coverings of \( Y_n \) with these, so that

\[
H^1(Y_n) = 1/R_n \quad (13.4)
\]

for each \( n \). Of course, Hausdorff measure of any dimension is automatically invariant under translations on \( Y_0 \), because the ultrametric \( \rho(x, y) \) is invariant under translations. It follows that the normalized Haar measure \( \mu_0 \) on \( Y_0 \) is the same as one-dimensional Hausdorff measure on \( Y_0 \).

### 14 Quotients of \( \mathbb{R} \)

Consider the Cartesian product

\[
X = \prod_{l=0}^{\infty} (\mathbb{R}/R_l \mathbb{Z}). \quad (14.1)
\]

Here \( \mathbb{R}/R_l \mathbb{Z} \) refers to the quotient of \( \mathbb{R} \) as a commutative group by the subgroup \( R_l \mathbb{Z} \), which is a commutative group as well. Thus \( X \) is a commutative group too, where the group operations are defined coordinatewise. Using the usual quotient topology on \( \mathbb{R}/R_l \mathbb{Z} \) for each \( l \), \( X \) becomes a compact Hausdorff topological space with respect to the product topology. It is easy to see that the group operations on \( X \) are continuous with respect to this topology, so that \( X \) is a topological group. We can identify the Cartesian product \( X_0 \) in (12.1) with a subset of \( X \) by extending each \( x = \{x_l\}_{l=1}^{\infty} \) in \( X_0 \) to \( l = 0 \) by taking \( x_0 = 0 \) in \( \mathbb{R}/\mathbb{Z} \). This is very natural, since \( R_0 = 1 \) and hence \( \mathbb{Z}/R_0 \mathbb{Z} \) is the trivial group.

More precisely, \( X_0 \) corresponds to a closed subgroup of \( X \) in this way, and the topology induced on \( X_0 \) by the one on \( X \) is the same as the topology on \( X_0 \) defined in Section 12. The group structure on \( X_0 \) as a subgroup of \( X \) is the same as the additive group structure on \( X_0 \) as a commutative ring, as before.

Because \( R_{l+1} \mathbb{Z} \) is a subgroup of \( R_l \mathbb{Z} \), there is a natural homomorphism from \( \mathbb{R}/R_{l+1} \mathbb{Z} \) onto \( \mathbb{R}/R_l \mathbb{Z} \) for each \( l \geq 0 \). An element \( x = \{x_l\}_{l=0}^{\infty} \) of \( X_0 \) is said to
be a coherent sequence if \( x_i \) is the image of \( x_{i+1} \in \mathbb{R}/R_{l+1} \mathbb{Z} \) in \( \mathbb{R}/R_l \mathbb{Z} \) for each \( l \). As usual, the set \( Y \) of coherent sequences in \( X \) is a closed subgroup of \( X \).

This coherence condition reduces to the previous one for elements of \( X_0 \), so that the set \( Y_0 \) of coherent sequences in \( X_0 \) corresponds exactly to the intersection of \( X_0 \) with \( Y \) in \( X \). Note that \( Y_0 \) corresponds to a closed subgroup of \( Y \).

Let \( q_l \) be the usual quotient mapping from \( \mathbb{R} \) onto \( \mathbb{R}/R_l \mathbb{Z} \) for each \( l \geq 0 \), and put

\[
q(a) = \{q_l(a)\}_{l=0}^\infty
\]

for each \( a \in \mathbb{R} \). This defines a continuous homomorphism from \( \mathbb{R} \) into \( X \) with trivial kernel, whose restriction to \( \mathbb{Z} \) corresponds exactly to the mapping \( \tilde{q} \) in (12.9). As before, \( q(a) \) is a coherent sequence in \( X \) for each \( a \in \mathbb{R} \), because \( q_l \) is the same as the composition of \( q_{l+1} \) with the natural homomorphism from \( \mathbb{R}/R_{l+1} \mathbb{Z} \) onto \( \mathbb{R}/R_l \mathbb{Z} \) for each \( l \). Thus \( q(\mathbb{R}) \subseteq Y \), and one can check that \( q(\mathbb{R}) \) is dense in \( Y \), so that the closure of \( q(\mathbb{R}) \) in \( X \) is equal to \( Y \). This implies that \( Y \) is connected, while \( Y_0 \) is totally disconnected.

Let \( \pi_n \) be the \( n \)th coordinate projection of \( X \) onto \( \mathbb{R}/R_n \mathbb{Z} \), so that

\[
\pi_n(x) = x_n
\]

for each \( x = \{x_l\}_{l=0}^\infty \in X \) and nonnegative integer \( n \). This is a continuous group homomorphism from \( X \) onto \( \mathbb{R}/R_n \mathbb{Z} \) for each \( n \geq 0 \), and we are especially interested in the restriction of \( \pi_n \) to \( Y \). Note that \( \pi_n \) maps \( Y \) onto \( \mathbb{R}/R_n \mathbb{Z} \) for each \( n \), because \( q(\mathbb{R}) \subseteq Y \) and \( \pi_n \circ q = q_n \) maps \( \mathbb{R} \) onto \( \mathbb{R}/R_n \mathbb{Z} \). The kernel of the restriction of \( \pi_n \) to \( Y \) corresponds exactly to the subgroup \( Y_n \) of \( Y_0 \) defined in Section 12 for each \( n \geq 0 \). By construction, the restriction of \( \pi_n \) to \( Y \) is equal to the composition of the restriction of \( \pi_{n+1} \) to \( Y \) with the natural homomorphism from \( \mathbb{R}/R_{n+1} \mathbb{Z} \) onto \( \mathbb{R}/R_n \mathbb{Z} \) for each \( n \).

## 15 Metrics on \( X \)

Let \( \phi_l \) be the standard isomorphism from \( \mathbb{R}/R_l \mathbb{Z} \) onto the unit circle, so that

\[
\phi_l(q_l(a)) = \exp(2 \pi i R_{-1} a)
\]

for each \( a \in \mathbb{R} \). Thus

\[
d_l(x_l, y_l) = |\phi_l(x_l) - \phi_l(y_l)|
\]

defines a metric on \( \mathbb{R}/R_l \mathbb{Z} \), which determines the same topology on \( \mathbb{R}/R_l \mathbb{Z} \) as the usual quotient topology. Note that this metric is invariant under translations on \( \mathbb{R}/R_l \mathbb{Z} \), and that

\[
d_l(x_l, y_l) \leq |\phi_l(x_l)| + |\phi_l(y_l)| = 2
\]

for every \( x_l, y_l \in \mathbb{R}/R_l \mathbb{Z} \). If \( x_l, y_l \) are distinct elements of \( \mathbb{Z}/R_l \mathbb{Z} \) for some \( l \geq 1 \), then

\[
d_l(x_l, y_l) \geq |\exp(2 \pi i R_{-1}) - 1|.
\]
If in addition the images of \( x_l \) and \( y_l \) in \( \mathbb{Z}/R_{l-1} \mathbb{Z} \) are equal, then we get that

\[
(15.5) \quad d_l(x_l, y_l) \geq |\exp(2\pi i r_l^{-1}) - 1|, 
\]

which is stronger than (15.4).

Let \( t = \{t_l\}_{l=0}^{\infty} \) be a sequence of positive real numbers that converges to 0 in \( \mathbb{R} \), and put

\[
(15.6) \quad d(x, y) = \max_{l \geq 0} t_l \, d_l(x_l, y_l) 
\]

for each \( x, y \in X \). It is easy to see that the maximum is always attained under these conditions, and that \( d(x, y) \) defines a translation-invariant metric on \( X \) for which the corresponding topology is the product topology mentioned earlier. In particular, the restriction of \( d(x, y) \) to \( x, y \in Y \) defines a translation-invariant metric on \( Y \), for which the corresponding topology is the same as the one induced by the product topology on \( X \). Similarly, if we identify \( X_0, Y_0 \) with subsets of \( X \), then the restriction of \( d(x, y) \) to these subsets determine metrics on \( X_0, Y_0 \) for which the corresponding topologies are the same as before. Let us compare this with the ultrametric \( \rho(x, y) \) on \( X_0 \) defined in Section 12.

Let \( x = \{x_l\}_{l=0}^{\infty}, y = \{y_l\}_{l=0}^{\infty} \in X_0 \) be given, which can be identified with elements of \( X \) by taking \( x_0 = y_0 = 0 \) in \( \mathbb{R}/\mathbb{Z} \), as usual. We may as well suppose that \( x \neq y \), since otherwise \( d(x, y) = \rho(x, y) = 0 \). If \( l(x, y) \) is the smallest positive integer \( l \) such that \( x_l \neq y_l \), as in Section 12, then we get that

\[
(15.7) \quad d(x, y) = \max_{l \geq l(x, y)} t_l \, d_l(x_l, y_l) \leq 2 \max_{l \geq l(x, y)} t_l. 
\]

If the \( t_l \)'s are monotone decreasing, then this reduces to

\[
(15.8) \quad d(x, y) \leq 2 t_{l(x, y)}. 
\]

If we take \( t_0 = 1 \) and \( t_l = 1/R_{l-1} \) when \( l \geq 1 \), then we get that

\[
(15.9) \quad d(x, y) \leq 2/R_{l(x, y)} - 1 = 2 \rho(x, y). 
\]

In the case where \( r = \{r_l\}_{j=1}^{\infty} \) is a constant sequence, so that \( R_l = r_l^j \) for each \( l \geq 0 \), this is the same as taking \( t_0 = 1 \) and \( t_l = r_l - l \) when \( l \geq 1 \). Although \( t_l = r_l - l \) may be appealing in some ways, this slightly different choice for \( t_l \) has other advantages.

In the other direction, we can take \( l = l(x, y) \) in (15.6), to get that

\[
(15.10) \quad d(x, y) \geq t_{l(x, y)} \, d_l(x_l, y_l) \geq t_{l(x, y)} |\exp(2\pi i R^{-1}_{l(x, y)}) - 1|, 
\]

using (15.4) in the second step. If \( x, y \in Y_0 \), then we can use (15.5) instead of (15.4) to get that

\[
(15.11) \quad d(x, y) \geq t_{l(x, y)} |\exp(2\pi i r_{l(x, y)}^{-1}) - 1|. 
\]

More precisely, \( x_{l(x, y)} = y_{l(x, y)} \) by definition of \( l(x, y) \), which implies that the images of \( x_{l(x, y)} \) and \( y_{l(x, y)} \) in \( \mathbb{Z}/R_{l(x, y)} - 1 \mathbb{Z} \) are the same when \( x, y \in Y_0 \).
by coherence. If we take $t_0 = 1$ and $t_l = 1/R_{l-1}$ when $l \geq 0$, as before, then (15.11) becomes

\begin{equation}
(15.12) \quad d(x, y) \geq |\exp(2\pi i r_{l(x,y)}^{-1}) - 1| \rho(x, y).
\end{equation}

This implies that

\begin{equation}
(15.13) \quad d(x, y) \geq c_0 \rho(x, y)
\end{equation}

for some $c_0 > 0$ and every $x, y \in Y_0$ when the $r_j$'s are bounded.

16 A nice mapping

Consider the mapping $A : \mathbb{R} \times Y_0 \to Y$ defined by

\begin{equation}
(16.1) \quad A(a, x) = q(a) + x,
\end{equation}

where $q : \mathbb{R} \to Y$ is as in (14.2), and $x = \{x_l\}_{l=1}^\infty \in Y_0$ is identified with an element of $Y$ by setting $x_0 = 0$ in $\mathbb{R}/\mathbb{Z}$, as usual. More precisely, the sum $q(a) + x$ uses the group structure on $Y$ as a subgroup of $X$ as in Section 14. Note that $A$ is a homomorphism from $\mathbb{R} \times Y_0$ into $Y$ with respect to coordinatewise addition on $\mathbb{R} \times Y_0$, because $q$ is a homomorphism from $\mathbb{R}$ into $Y$.

If $(a, x)$ is in the kernel of $A$, then $q(a) + x = 0$ in $Y$, and hence $q_0(q(a)) = 0$ in $\mathbb{R}/\mathbb{Z}$, since $x_0 = 0$ by construction. Thus $a \in \mathbb{Z}$, which implies that $x = \tilde{q}(-a)$ in $Y_0$, where $\tilde{q} : \mathbb{Z} \to Y_0$ is as in (12.9). Conversely, if $a \in \mathbb{Z}$ and $x = \tilde{q}(-a)$ in $Y_0$, then $A(a, x) = 0$.

Let us check that $A$ maps $\mathbb{R} \times Y_0$ onto $Y$. If $y$ is any element of $Y$, then we can first choose $a \in \mathbb{R}$ such that $y_0 = q_0(a)$ in $\mathbb{R}/\mathbb{Z}$. Hence the $l = 0$ component of $x = y - q(a)$ is equal to 0, so that $x$ corresponds to an element of $Y_0$, and $y = A(a, x)$, as desired.

It is easy to see that $A$ is continuous as a mapping from $\mathbb{R} \times Y_0$ into $Y$, using the standard topology on $\mathbb{R}$, the topologies already discussed on $Y_0$ and $Y$, and the corresponding product topology on $\mathbb{R} \times Y_0$. One can also check that $A$ is a local homeomorphism, where local inverses for $A$ can be given in terms of local inverses for $q_0 : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$, as in the previous paragraph.

17 A nice mapping, continued

Consider the metric on $\mathbb{R} \times Y_0$ defined by

\begin{equation}
(17.1) \quad D((a, x), (b, y)) = \max(|a - b|, \rho(x, y)),
\end{equation}

where $\rho(x, y)$ is the ultrametric defined on $X_0$ as in Section 12. Note that this is a translation-invariant metric on $\mathbb{R} \times Y_0$, and that the topology on $\mathbb{R} \times Y_0$ determined by this metric is the same as the product topology associated to the standard topology on $\mathbb{R}$ and the usual topology on $Y_0$. Throughout this section, we let $d(x, y)$ be the metric on $X$ in Section 15, with $t_0 = 1$ and $t_l = 1/R_{l-1}$
when \( l \geq 1 \). We would like to look at the behavior of the mapping \( A \) from the 
previous section with respect to (17.1) on \( \mathbb{R} \times Y_0 \) and \( d(x, y) \) on \( Y \).

Remember that

\[
| \exp(i u) - \exp(i v) | \leq |u - v|
\]

for every \( u, v \in \mathbb{R} \), and that

\[
| \exp(i u) - \exp(i v) | \geq c_1 |u - v|
\]

for a suitable constant \( c_1 > 0 \) when \( |u - v| \leq \pi \). Thus

\[
| \phi_t(q_t(a)) - \phi_t(q_t(b)) | = | \exp(2 \pi i R_t^{-1} a) - \exp(2 \pi i R_t^{-1} b) | \leq 2 \pi R_t^{-1} |a - b|
\]

for every \( a, b \in \mathbb{R} \), and hence

\[
d(q(a), q(b)) \leq 2 \pi |a - b|.
\]

This only uses the fact that \( t_{ij} \leq 1 \) for each \( l \geq 0 \).

Now let \( x, y \in Y_0 \) be given. If \( x = y \), then

\[
d(A(a, x), A(b, y)) = d(q(a) + x, q(b) + y) = d(q(a), q(b)),
\]

by translation-invariance, so that

\[
d(A(a, x), A(b, y)) \leq 2 \pi |a - b| = 2 \pi D((a, x), (b, y)),
\]

by (17.5). Otherwise, suppose that \( x \neq y \), and let \( l(x, y) \) be as in Section 12. Remember that \( x = \{ x_i \}_{i=1}^{\infty} \) and \( y = \{ y_j \}_{j=1}^{\infty} \) are extended to \( l = 0 \) by putting \( x_0 = y_0 = 0 \) in \( \mathbb{R}/\mathbb{Z} \), and that \( x_j = y_j \) when \( j < l(x, y) \). This implies that

\[
d_j(q_j(a) + x_j, q_j(b) + y_j) = d_j(q_j(a), q_j(b))
\]

when \( j < l(x, y) \), by translation-invariance, so that

\[
\max_{0 \leq j < l(x, y)} t_j d_j(q_j(a) + x_j, q_j(b) + y_j) \leq 2 \pi |a - b|,
\]

as before. If \( j \geq l(x, y) \), then \( t_j = 1/R_{j-1} \), and

\[
t_j d_j(q_j(a) + x_j, q_j(b) + y_j) \leq 2/R_{j-1} \leq 2/R_{l(x, y)-1},
\]

since \( d_j \leq 2 \) automatically. Thus

\[
\max_{j \geq l(x, y)} t_j d_j(q_j(a) + x_j, q_j(b) + y_j) \leq 2 \rho(x, y),
\]

and hence

\[
d(A(a, x), A(b, y)) \leq 2 \pi D((a, x), (b, y)).
\]

To get an estimate in the other direction, we restrict our attention to \( a, b \in \mathbb{R} \) 
that satisfy

\[
|a - b| \leq 1/2,
\]

23
for instance. Observe that
\begin{equation}
    d(A(a,x),A(b,y)) \geq d_0(q_0(a) + x_0, q_0(b) + y_0) = d_0(q_0(a), q_0(b)),
\end{equation}
by taking \( l = 0 \) in the definition of \( d \), and remembering that \( t_0 = 1 \) and \( x_0 = y_0 = 0 \). By the definitions of \( d_0 \) and \( \phi_0 \), we have that
\begin{equation}
    d_0(q_0(a), q_0(b)) = | \exp(2 \pi i a) - \exp(2 \pi i b) |,
\end{equation}
and hence that
\begin{equation}
    d(A(a,x),A(b,y)) \geq 2 \pi c_1 |a - b|,
\end{equation}
when \( a, b \in \mathbb{R} \) satisfy (17.13), because of (17.3). Combining this with (17.5), we get that
\begin{equation}
    d(q(a),q(b)) \leq c_1^{-1} d(A(a,x),A(b,y))
\end{equation}
when \( a, b \in \mathbb{R} \) satisfy (17.13). We also have that
\begin{equation}
    d(x,y) = d(q(a) + x, q(a) + y) \leq d(q(a) + x, q(b) + y) + d(q(a), q(b)),
\end{equation}
by translation-invariance and the triangle inequality, so that
\begin{equation}
    d(x,y) \leq (1 + c_1^{-1}) d(A(a,x),A(b,y))
\end{equation}
when \( a, b \in \mathbb{R} \) satisfy (17.13). If the \( r_j \)'s are bounded, then (15.13) holds for some \( c_0 > 0 \), and we get that
\begin{equation}
    c_0 \rho(x,y) \leq (1 + c_1^{-1}) d(A(a,x),A(b,y))
\end{equation}
when \( a, b \in \mathbb{R} \) satisfy (17.13). Combining this with (17.16), we get that
\begin{equation}
    D((a,x),(b,y)) \leq c_2 d(A(a,x),A(b,y))
\end{equation}
for a suitable constant \( c_2 \) when \( a, b \in \mathbb{R} \) satisfy (17.13) and the \( r_j \)'s are bounded.

18 Another metric on \( Y \)

Even if the \( r_j \)'s are not bounded, we can simply choose a metric on \( Y \) which approximates \( D \) on \( \mathbb{R} \times Y_0 \). The easiest way to do that is to take
\begin{equation}
    \Delta(u,v) = \inf \{ D(A(a,x),A(b,y)) : (a,x),(b,y) \in \mathbb{R} \times Y_0, \ A(a,x) = u, A(b,y) = v \}
\end{equation}
for each \( u, v \in Y \). Note that
\begin{equation}
    \Delta(u,v) = \Delta(v,u)
\end{equation}
for every \( u, v \in Y \), and that \( \Delta(u,v) \) is translation-invariant on \( Y \), because \( D \) is symmetric and translation-invariant on \( \mathbb{R} \times Y_0 \). If \( (a,x) \in \mathbb{R} \times Y_0 \) satisfies \( A(a,x) = u \), then
\begin{equation}
    \Delta(u,v) = \inf \{ D(A(a,x),A(b,y)) : (b,y) \in \mathbb{R} \times Y_0, \ A(b,y) = v \}
\end{equation}
for every $v \in Y$, since one can use the translation-invariance of $D$ on $\mathbb{R} \times Y_0$ to simultaneously translate representatives of $u$ and $v$ in $\mathbb{R} \times Y_0$ to reduce to the case where $u$ is represented by $(a, x)$. Similarly, one can fix a representative $(b, y) \in \mathbb{R} \times Y_0$ of $v$, and express $\Delta(u, v)$ as the infimum of $D(A(a, x), A(b, y))$ over all representatives $(a, x) \in \mathbb{R} \times Y_0$ of $u$. By fixing a representative for $u$ or $v$ in $\mathbb{R} \times Y_0$, it is clear that these infima are attained. In particular, $\Delta(u, v) = 0$ if and only if $u = v$, because of the analogous property of $D$ on $\mathbb{R} \times Y_0$.

In order to show that $\Delta(u, v)$ defines a metric on $Y$, it remains to check that the triangle inequality holds, so that

\[(18.4) \quad \Delta(u, w) \leq \Delta(u, v) + \Delta(v, w)\]

for every $u, v, w \in Y$. To do this, it suffices to verify that

\[(18.5) \quad \Delta(u, w) \leq D((a, x), (b, y)) + D((b', y'), (c, z))\]

for every $(a, x), (b, y), (b', y'), (c, z) \in \mathbb{R} \times Y_0$ such that $A(a, x) = u$, $A(b, y) = A(b', y') = v$, and $A(c, z) = w$. The main point is to use translation-invariance of $D$ on $\mathbb{R} \times Y_0$ to reduce to the case where $(b, y) = (b', y')$, as in the previous paragraph. In this case, we get that

\[(18.6) \quad \Delta(u, w) \leq D((a, x), (c, z)) \leq D((a, x), (b, y)) + D((b, y), (c, z))\]

because of the triangle inequality for $D$ on $\mathbb{R} \times Y_0$, as desired. Observe also that

\[(18.7) \quad \Delta(A(a, x), A(b, y)) \leq D(A(a, x), A(b, y))\]

for every $(a, x), (b, y) \in \mathbb{R} \times Y_0$, by construction.

If $x, y \in Y_0$, then we can identify $x$ and $y$ with elements of $Y$ in the usual way, by putting $x_0 = y_0 = 0$ in $\mathbb{R}/\mathbb{Z}$. Let us check that

\[(18.8) \quad \Delta(x, y) = \rho(x, y)\]

By definition, $\Delta(x, y)$ is equal to the infimum of $D(A(a, x), A(b, y))$ over $a, b \in \mathbb{R}$ such that $q_0(a) = q_0(b) = 0$, which is to say that $a, b \in \mathbb{Z}$. If $a = b$, then

\[(18.9) \quad D((a, x), (b, y)) = \rho(x, y),\]

and hence $\Delta(x, y) \leq \rho(x, y)$. Otherwise, if $a \neq b$, then

\[(18.10) \quad \rho(x, y) \leq 1 \leq |a - b| \leq D((a, x), (b, y)),\]

and (18.8) follows.

Note that

\[(18.11) \quad \Delta(u, v) \leq 1\]

for every $u, v \in Y$, since one can always choose $(a, x), (b, y) \in \mathbb{R} \times Y_0$ such that $A(a, x) = u$, $A(b, y) = v$, and $|a - b| \leq 1/2$. Let us check that

\[(18.12) \quad D((a, x), (b, y)) \leq 2 \Delta(A(a, x), A(b, y))\]
for every \((a, x), (b, y) \in \mathbb{R} \times Y_0\) such that \(|a - b| \leq 1/2\). It suffices to show that
\[
D((a, x), (b, y)) \leq 2 D((a, x), (b', y'))
\]
whenever \((b', y') \in \mathbb{R} \times Y_0\) satisfies \(A(b', y') = A(b, y)\) and \((b', y') \neq (b, y)\). In this case, \(|b' - b| \geq 1\), because \(b' - b \in \mathbb{Z}\) and \(b' \neq b\). This implies that \(|a - b'| \geq 1/2\), and hence that
\[
D((a, x), (b, y)) \leq 1 \leq 2 |a - b'| \leq 2 D((a, x), (b, y)),
\]
as desired.

19 Haar measure on \(Y\)
Let \(\mu\) be Haar measure on \(Y\), normalized so that
\[
\mu(Y) = 1.
\]
Also let \(p_n\) be the restriction of the coordinate projection \(\pi_n\) in (14.3) to \(Y\) for each nonnegative integer \(n\), so that
\[
p_n(x) = x_n
\]
for every \(x = \{x_i\}_{i=0}^\infty \in Y\). Thus \(p_n\) is a continuous homomorphism from \(Y\) onto \(\mathbb{R}/\mathbb{R}_n \mathbb{Z}\) with kernel equal to \(Y_n\) for each \(n \geq 0\). If \(E\) is a Borel measurable subset of \(\mathbb{R}/\mathbb{R}_n \mathbb{Z}\), then \(p_n^{-1}(E)\) is a Borel measurable subset of \(Y\), and
\[
\mu(p_n^{-1}(E)) = |E|/\mathbb{R}_n.
\]
Here \(|E|\) denotes the measure of \(E\) as a subset of \(\mathbb{R}/\mathbb{R}_n \mathbb{Z}\) that comes from Lebesgue measure on \(\mathbb{R}\) in the obvious way, so that \(|\mathbb{R}/\mathbb{R}_n \mathbb{Z}| = 1. If the \(r_j\)'s are bounded, then one can check that \(\mu\) is Ahlfors regular of dimension 2 on \(Y\) with respect to the appropriate metrics discussed previously. Even if the \(r_j\)'s are not bounded, the subsets of \(Y\) of the form \(p_n^{-1}(E)\) with \(E\) a Borel set in \(\mathbb{R}/\mathbb{R}_n \mathbb{Z}\) define a nice filtration on \(Y\), with the corresponding martingales on \(Y\).

Part III
Another perspective

20 Ultrametrics
A metric \(d(x, y)\) on a set \(M\) is said to be an ultrametric if
\[
d(x, z) \leq \max(d(x, y), d(y, z))
\]
for every \( x, y, z \in M \), which is stronger than the usual triangle inequality. The discrete metric on any set is an ultrametric, for instance.

As another class of examples, let \( X_1, X_2, X_3, \ldots \) be a sequence of nonempty sets, and let \( X = \prod_{j=1}^{\infty} X_j \) be their Cartesian product, consisting of all sequences \( x = \{x_j\}_{j=1}^{\infty} \) such that \( x_j \in X_j \) for each \( j \). If \( x, y \in X \), then let \( n(x,y) \) be the largest nonnegative integer such that \( x_j = y_j \) for each \( j \leq n(x,y) \), with \( n(x,y) = +\infty \) when \( x = y \). Equivalently, \( n(x,y) + 1 \) is the smallest positive integer \( j \) such that \( x_j \neq y_j \) when \( x \neq y \). Of course, \( n(x,y) = n(y,x) \) for every \( x, y \in X \), and it is easy to see that

\[
\text{(20.2)} \quad n(x,y) = n(y,x)
\]

for every \( x, y \in X \), and it is easy to see that

\[
\text{(20.3)} \quad n(x,z) \geq \min(n(x,y), n(y,z))
\]

for every \( x, y, z \in X \). Let \( t = \{t_j\}_{j=0}^{\infty} \) be a decreasing sequence of positive real numbers that converges to 0. Put

\[
\text{(20.4)} \quad d(x,y) = t_{n(x,y)}
\]

for each \( x, y \in X \) with \( x \neq y \), and \( d(x,y) = 0 \) when \( x = y \), which corresponds to taking \( t_\infty = 0 \). It is easy to see that \( d(\cdot, \cdot) \) is an ultrametric on \( X \), using (20.2) and (20.3). The topology on \( X \) determined by this ultrametric is the same as the product topology corresponding to the discrete topology on \( X_j \) for each \( j \). In particular, \( X \) is compact with respect to this topology when \( X_j \) is a finite set for each \( j \).

If \( d(\cdot, \cdot) \) is a metric on any set \( M \), then the open ball in \( M \) centered at a point \( x \in M \) and with radius \( r > 0 \) is defined as usual by

\[
\text{(20.5)} \quad B(x,r) = \{y \in M : d(x,y) < r\}.
\]

If \( y \in B(x,r) \), then \( t = r - d(x,y) > 0 \), and one can check that

\[
\text{(20.6)} \quad B(y,t) \subseteq B(x,r),
\]

using the triangle inequality. However, if \( d(\cdot, \cdot) \) is an ultrametric on \( M \), then

\[
\text{(20.7)} \quad B(y,r) \subseteq B(x,r)
\]

for every \( y \in B(x,r) \). This implies that

\[
\text{(20.8)} \quad B(x,r) = B(y,r)
\]

when \( d(x,y) < r \), since the same argument can be applied with the roles of \( x \) and \( y \) reversed.

Similarly, the closed ball in \( M \)

\[
\text{(20.9)} \quad \overline{B}(x,r) = \{y \in M : d(x,y) \leq r\},
\]
with center \( x \in M \) and radius \( r \geq 0 \) is a closed set in \( M \) with respect to the topology determined by \( d(\cdot, \cdot) \), for any metric \( d(\cdot, \cdot) \) on \( M \). If \( d(\cdot, \cdot) \) is an ultrametric on \( M \), then
\[
\overline{B}(y, r) \subseteq \overline{B}(x, r)
\]
for every \( y \in \overline{B}(x, r) \), which implies that \( \overline{B}(x, r) \) is also an open set in \( M \). As before, one can apply this with the roles of \( x \) and \( y \) reversed, to get that
\[
\overline{B}(x, r) = \overline{B}(y, r)
\]
when \( d(x, y) \leq r \).

If \( d(\cdot, \cdot) \) is an ultrametric on \( M \), \( x, y, z \in M \), and \( d(y, z) \leq d(x, y) \), then
\[
d(x, z) \leq \max(d(x, y), d(y, z)) = d(x, y).
\]

If \( d(y, z) < d(x, y) \), then
\[
d(x, y) \leq \max(d(x, z), d(y, z))
\]
implies that
\[
d(x, y) \leq d(x, z).
\]
Hence
\[
d(x, y) = d(x, z)
\]
when \( d(y, z) < d(x, y) \), by (20.12).

Let \( d(\cdot, \cdot) \) be a metric on a set \( M \) again, and put
\[
V(x, r) = \{ y \in M : d(x, y) > r \}
\]
for each \( x \in M \) and \( r > 0 \), which is the same as \( M \setminus \overline{B}(x, r) \). If \( y \in V(x, r) \), then \( t = d(x, y) - r > 0 \), and one can check that
\[
B(y, t) \subseteq V(x, r),
\]
using the triangle inequality. If \( d(\cdot, \cdot) \) is an ultrametric on \( M \), then
\[
B(y, d(x, y)) \subseteq V(x, r)
\]
for every \( y \in V(x, r) \), by (20.15).

Similarly, put
\[
W(x, r) = \{ y \in M : d(x, y) \geq r \}
\]
for each \( x \in M \) and \( r > 0 \), which is the same as \( M \setminus B(x, r) \). If \( d(\cdot, \cdot) \) is an ultrametric on \( M \), then
\[
B(y, r) \subseteq B(y, d(x, y)) \subseteq W(x, r)
\]
for every \( y \in W(x, r) \), by (20.15). In particular, this implies that \( W(x, r) \) is an open set, so that \( B(x, r) \) is a closed set in \( M \).
21 $r$-Adic absolute values

Let $r = \{r_j\}_{j=1}^{\infty}$ be a sequence of positive integers, with $r_j \geq 2$ for each $j$. Put

$$R_l = \prod_{j=1}^{l} r_j \quad (21.1)$$

for each positive integer $l$ and $R_0 = 1$, and let $t = \{t_l\}_{l=0}^{\infty}$ be a decreasing sequence of real numbers that converges to 0. One can take $t_l = 1/R_l$ for each $l$, for instance, and in any case one might at least take $t_0 = 1$. If $r_j = (r_1)^j$ for each $j \geq 1$, then $R_l = (r_1)^l$ for each $l \geq 0$, and $t_l = 1/R_l = (r_1)^{-l}$ is especially nice. In particular, if $p$ is a prime number, $r_j = p$ for every $j \geq 1$, and $t_l = p^{-l}$ for each $l \geq 0$, then this reduces to the usual situation for $p$-adic numbers.

Let $a$ be an integer, and let $n(a)$ be the largest nonnegative integer $l$ such that $a$ is an integer multiple of $R_l$, with $n(0) = +\infty$. Thus

$$n(-a) = n(a) \quad (21.2)$$

for each integer $a$, and it is easy to see that

$$n(a + b) \geq \min(n(a), n(b)) \quad (21.3)$$

and

$$n(ab) \geq \max(n(a), n(b)) \quad (21.4)$$

for any two integers $a, b$. If $r_j = r_1$ for each $j \geq 1$, then

$$n(ab) \geq n(a) + n(b) \quad (21.5)$$

for all integers $a, b$. If $r_j = p^j$ for some prime number $p$ and every $j \geq 1$, then

$$n(ab) = n(a) + n(b) \quad (21.6)$$

for each $a, b$.

The $r$-adic absolute value of an integer $x$ is defined by

$$|a|_r = t_{n(a)} \quad (21.7)$$

when $a \neq 0$ and $|0|_r = 0$, which corresponds to (21.7) with $t_\infty = 0$. Note that

$$|a|_r \leq t_0 \quad (21.8)$$

for every integer $x$, and that

$$|-a|_r = |a|_r \quad (21.9)$$

by (21.2). Similarly,

$$|a + b|_r \leq \max(|a|_r, |b|_r) \quad (21.10)$$

and

$$|ab|_r \leq \min(|a|_r, |b|_r) \quad (21.11)$$
for every $a$ and $b$, by (21.3) and (21.4). Suppose that $r_j = r_1$ for each $j \geq 1$ and $t = \{t_l\}_{l=0}^{\infty}$ is submultiplicative, in the sense that

\begin{equation}
(21.12) \quad t_{k+l} \leq t_k t_l
\end{equation}

for every $k, l \geq 0$. Under these conditions, we get that

\begin{equation}
(21.13) \quad |a b|_r \leq |a|_r |b|_r
\end{equation}

for every $a$ and $b$, by (21.5). If $r_j = p$ for some prime number $p$ and every $j$, and if $t_l = (t_1)^l$ for each $l \geq 0$, then

\begin{equation}
(21.14) \quad |a b|_r = |a|_r |b|_r
\end{equation}

for every $a$ and $b$, by (21.6). In particular, $|a|_r$ reduces to the usual $p$-adic absolute value $|a|_p$ of $a$ when $p$ is a prime number, $r_j = p$ for each $j \geq 1$, and $t_l = p^{-l}$ for every $l \geq 0$.

The $r$-adic metric on the set $\mathbb{Z}$ of integers is defined by

\begin{equation}
(21.15) \quad d_r(a, b) = |a - b|_r.
\end{equation}

It is easy to see that this defines an ultrametric on $\mathbb{Z}$, using (21.9) and (21.10). Note that the topology determined on $\mathbb{Z}$ by (21.15) depends only on $r$, and not on $t$. Thus we shall sometimes refer to this topology simply as the $r$-adic topology on $\mathbb{Z}$. If $r_j = p$ for some prime number $p$ and every $j \geq 1$, and if $t_l = p^{-l}$ for each $l \geq 0$, then the $r$-adic metric on $\mathbb{Z}$ reduces to the usual $p$-adic metric.

### 22 Coherent sequences

Let us continue with the same notation and hypotheses as before, and put

\begin{equation}
(22.1) \quad X_l = \mathbb{Z}/R_l \mathbb{Z}
\end{equation}

for each positive integer $l$. Thus $X_l$ is a commutative ring with $R_l$ elements for each $l$. Consider the Cartesian product

\begin{equation}
(22.2) \quad X = \prod_{l=1}^{\infty} X_l,
\end{equation}

which is also a commutative ring with respect to coordinatewise addition and multiplication. Note that $X$ is a compact Hausdorff topological space, with respect to the product topology corresponding to the discrete topology on $X_l$ for each $l$. It is easy to see that addition and multiplication on $X$ are continuous with respect to the product topology, so that $X$ is a topological ring.

Let $q_l$ be the usual quotient homomorphism from $\mathbb{Z}$ onto $\mathbb{Z}/R_l \mathbb{Z}$ for each $l \geq 1$. If we put

\begin{equation}
(22.3) \quad q(a) = \{q_l(a)\}_{l=1}^{\infty}
\end{equation}
for each $a \in \mathbf{Z}$, then $q$ defines a ring homomorphism from $\mathbf{Z}$ into $X$. It is easy to see that the kernel of this homomorphism is trivial, since $R_l \to \infty$ as $l \to \infty$.

Observe that
\[
(22.4) \quad n(q(a), q(b)) = n(a - b)
\]
for every $a, b \in \mathbf{Z}$, where $n(q(a), q(b))$ is as in Section 20, and $n(a - b)$ is as in Section 21. This implies that
\[
(22.5) \quad d(q(a), q(b)) = d_r(a, b)
\]
for every $a, b \in \mathbf{Z}$, where $d(\cdot, \cdot)$ is defined on $X$ as in (20.4), and $d_r(a, b)$ is the $r$-adic metric on $\mathbf{Z}$, as in (21.15). Of course, it is important to use the same sequence $t = \{t_i\}_{i=0}^{\infty}$ in both cases. In particular, $q$ is a homeomorphism from $\mathbf{Z}$ onto its image in $X$, where $\mathbf{Z}$ is equipped with the topology determined by the $r$-adic metric, and $X$ is equipped with the product topology mentioned earlier.

There is a natural ring homomorphism from $X_{l+1}$ onto $X_l$ for each $l$, because $R_{l+1} = r_{l+1} R_l$, and hence $R_{l+1} \mathbf{Z} \subseteq R_l \mathbf{Z}$. Equivalently, this homomorphism maps $q_{l+1}(a) \in \mathbf{Z}/R_{l+1} \mathbf{Z}$ to $q_l(a) \in \mathbf{Z}/R_l \mathbf{Z}$ for each $a \in \mathbf{Z}$. An element $x = \{x_i\}_{i=1}^{\infty}$ of $X$ is said to be a coherent sequence if $x_i$ is the image of $x_{l+1}$ under this homomorphism from $X_{l+1}$ onto $X_l$ for each $l$. Thus $q(a)$ is a coherent sequence for each $a \in \mathbf{Z}$. It is easy to see that the set of coherent sequences forms a sub-ring of $X$, and also a closed set in $X$ with respect to the product topology.

In fact, the set of coherent sequences in $X$ is the same as the closure of $q(\mathbf{Z})$ in $X$ with respect to the product topology. To see this, it suffices to check that every coherent sequence $x \in X$ can be approximated by elements of $q(\mathbf{Z})$ with respect to the product topology on $X$. Of course, for any $x \in X$ and positive integer $k$, there is an $a \in \mathbf{Z}$ such that $q_k(a) = x$. If $x$ is a coherent sequence, then it follows that $q_j(a) = x_j$ for every $j \leq k$, which implies that $x$ can be approximated by elements of $q(\mathbf{Z})$ with respect to the product topology, as desired, by taking $k \to \infty$.

It is easy to check directly that $X$ is complete with respect to the metric $d(x, y)$ in (20.4), and hence that the set of coherent sequences is also complete with respect to this metric. Because $q$ is an isometric embedding of $\mathbf{Z}$ into $X$, as in (22.5), the completion of $\mathbf{Z}$ with respect to the $r$-adic metric can be identified with the set of coherent sequences in $X$. Let us refer to this completion as the ring $\mathbf{Z}_r$ of $r$-adic integers. Thus $\mathbf{Z}_r$ is a compact commutative topological ring which contains $\mathbf{Z}$ as a dense sub-ring, and the $r$-adic metric extends to $\mathbf{Z}_r$ in a natural way. The $r$-adic absolute value function also extends to $\mathbf{Z}_r$ in a natural way, and satisfies properties like those in the previous section.

Note that the identification of $r$-adic integers with coherent sequences does not depend on the choice of sequence $t = \{t_i\}_{i=0}^{\infty}$ in the definition of the $r$-adic absolute value function and metric. Different choices of $t$ lead to topologically-equivalent translation-invariant $r$-adic metrics on $\mathbf{Z}$ anyway, which have the same Cauchy sequences, and isomorphic completions. If $p$ is a prime number, $r_j = p$ for each $j \geq 1$, and $t_l = p^{-l}$ for each $l \geq 0$, then this description of $\mathbf{Z}_r$ is equivalent to the usual ring $\mathbf{Z}_p$ of $p$-adic integers.
23 Topological equivalence

Let \( r = \{ r_j \}_{j=1}^\infty \) and \( r' = \{ r'_j \}_{j=1}^\infty \) be sequences of integers, with \( r_j, r'_j \geq 2 \) for each \( j \). As before, put

\[
R_l = \prod_{j=1}^l r_j, \quad R'_l = \prod_{j=1}^l r'_j
\]

when \( l \geq 1 \), and \( R_0 = R'_0 = 1 \). If for each \( k \geq 1 \) there is an \( l \geq 1 \) such that \( R'_l \) is an integer multiple of \( R_k \), then put \( r \prec r' \). If \( r \prec r' \) and \( r' \prec r \), then put \( r \sim r' \). The relation \( r \prec r' \) is clearly reflexive and transitive, which implies that \( r \sim r' \) is an equivalence relation.

Note that \( \{ R_l \}_{l=1}^\infty \) automatically converges to 0 with respect to the \( r \)-adic topology on \( \mathbb{Z} \), and similarly that \( \{ R'_l \}_{l=1}^\infty \) automatically converges to 0 with respect to the \( r' \)-adic topology on \( \mathbb{Z} \). It is easy to see that \( \{ R'_l \}_{l=1}^\infty \) converges to 0 with respect to the \( r \)-adic topology on \( \mathbb{Z} \) if and only if \( r \prec r' \). If \( r \prec r' \), then the \( r \)-adic topology on \( \mathbb{Z} \) is weaker than the \( r' \)-adic topology on \( \mathbb{Z} \). Conversely, if the \( r \)-adic topology on \( \mathbb{Z} \) is weaker than the \( r' \)-adic topology, then \( \{ R'_l \}_{l=1}^\infty \) converges to 0 with respect to the \( r \)-adic topology, and hence \( r \prec r' \). In this case, every coherent sequence with respect to \( r' \) determines a coherent sequence with respect to \( r \), which leads to a continuous ring homomorphism from \( \mathbb{Z}_{r'} \) into \( \mathbb{Z}_r \). One can also check that this mapping is surjective, because \( \mathbb{Z} \) is dense in \( \mathbb{Z}_r \) and \( \mathbb{Z}_{r'} \) is compact. It follows that \( r \sim r' \) if and only if the \( r \)-adic and \( r' \)-adic topologies on \( \mathbb{Z} \) are the same, in which event \( \mathbb{Z}_r \) and \( \mathbb{Z}_{r'} \) are isomorphic as topological rings.

Let \( p \) be a prime number, and let \( c_{r,l}(p) \) be the number of factors of \( p \) in \( R_l \) for each positive integer \( l \). Thus \( c_{r,l}(p) \leq c_{r,l+1}(p) \) for each \( l \), and we put

\[
c_r(p) = \sup_{l \geq 1} c_{r,l}(p),
\]

which is either a nonnegative integer or \( +\infty \). If \( c_r(p) \) is defined in the same way, then \( r \prec r' \) if and only if \( c_r(p) \leq c_{r'}(p) \) for every prime number \( p \), and hence \( r \sim r' \) if and only if \( c_r(p) = c_{r'}(p) \) for every prime number \( p \). Because the number of factors in \( R_l \) is strictly increasing, either \( c_r(p) = +\infty \) for some \( p \), or \( c_r(p) > 0 \) for infinitely many \( p \). Conversely, if \( c(p) \) is any function on the set of prime numbers such that \( c(p) \) is a nonnegative integer or \( +\infty \) for each \( p \), and either \( c(p) = +\infty \) for some \( p \) or \( c(p) > 0 \) for infinitely many \( p \), then \( c(p) = c_r(p) \) for some \( r \) as before.

Consider the Cartesian product

\[
\left( \prod_{0 < c_r(p) < \infty} (\mathbb{Z}/p^{c_r(p)} \mathbb{Z}) \right) \times \left( \prod_{c_r(p) = +\infty} \mathbb{Z}_p \right),
\]

where more precisely one takes the product over the prime numbers \( p \) such that \( 0 < c_r(p) < \infty \) and \( c_r(p) = +\infty \), respectively. This is a commutative ring with respect to coordinatewise addition and multiplication. This is also a compact topological ring with respect to the product topology corresponding
to the discrete topology on $\mathbb{Z}/p^{c_r(p)}\mathbb{Z}$ when $0 < c_r(p) < \infty$, and the $p$-adic topology on $\mathbb{Z}$ when $c_r(p) = \infty$. As before, there is a natural continuous ring homomorphism from $\mathbb{Z}$ onto each of the factors, which one can get using coherent sequences. This leads to a continuous ring homomorphism from $\mathbb{Z}$ into (23.3). The image of $\mathbb{Z}$ is dense in (23.3) with respect to the product topology, by the Chinese remainder theorem. This implies that $\mathbb{Z}$ maps onto (23.3), because $\mathbb{Z}$ is compact. The kernel of this homomorphism from $\mathbb{Z}$ onto (23.3) is trivial, and this isomorphism is a homeomorphism as well.

24 Comparisons

Let $X_1, X_2, X_3, \ldots$ be a sequence of nonempty finite sets, each of which has at least two elements, and let $X = \prod_{j=1}^{\infty} X_j$ be their Cartesian product, as in Section 20. Also let $\{t_j\}_{j=0}^{\infty}$ be a strictly decreasing sequence of positive real numbers that converges to 0, which leads to an ultrametric $d(x, y)$ on $X$ as in (20.4). As before, the topology on $X$ determined by $d(x, y)$ is the same as the product topology corresponding to the discrete topology on each $X_j$. In particular, $X$ is a compact Hausdorff space, and more precisely a topological Cantor set.

Suppose that $\{\tilde{t}_j\}_{j=1}^{\infty}$ is another strictly decreasing sequence of positive real numbers that converges to 0, and let $\tilde{d}(x, y)$ be the corresponding ultrametric on $X$, as in (20.4). Thus $d(x, y)$ and $\tilde{d}(x, y)$ determine the same topology on $X$, and in fact they determine the same collections of open and closed balls in $X$. This is a very strong geometric property, and indeed this collection of balls has a lot of interesting structure. The nesting of these balls leads to very simple covering lemmas, which lead in turn to maximal function estimates.

Now let $\pi$ be a one-to-one mapping of the set $\mathbb{Z}_+$ of positive integers onto itself. This can be used to rearrange the initial sequence of $X_j$’s to get a new sequence $X_{\pi(1)}, X_{\pi(2)}, X_{\pi(3)}, \ldots$ of sets, and thus a new Cartesian product $X^\pi = \prod_{j=1}^{\infty} X_{\pi(j)}$. Of course, there is also a natural one-to-one correspondence between $X$ and $X^\pi$, which sends $x = \{x_j\}_{j=1}^{\infty} \in X$ to $\{x_{\pi(j)}\}_{j=1}^{\infty} \in X^\pi$. This mapping is a homeomorphism between $X$ and $X^\pi$, with respect to the product topologies on $X$ and $X^\pi$ corresponding to the discrete topologies on the $X_j$’s. However, this type of mapping can still change the geometric structures being considered in significant ways.

Let $r = \{r_j\}_{j=1}^{\infty}$ be a sequence of integers with $r_j \geq 2$ for each $j$ again, and let $t = \{t_i\}_{i=0}^{\infty}$ be a strictly decreasing sequence of positive real numbers that converges to 0. As before, this leads to an $r$-adic absolute value and metric on $\mathbb{Z}$, and on the corresponding completion $\mathbb{Z}_r$. If $\{\tilde{t}_i\}_{i=0}^{\infty}$ is another strictly decreasing sequence of positive real numbers converging to 0, then one gets another $r$-adic absolute value function and metric, but the same topology and completion $\mathbb{Z}_r$. One also gets the same collections of open and closed balls in $\mathbb{Z}$ and $\mathbb{Z}_r$, as in the previous situation. If $r' = \{r'_j\}_{j=1}^{\infty}$ is another sequence with $r \sim r'$, as in the previous section, then one gets the same topology on $\mathbb{Z}$ and an isomorphic completion $\mathbb{Z}_{r'}$, but the corresponding geometry can be affected significantly.
25 Solenoids

Let \( r = \{ r_j \}_{j=1}^{\infty} \) be a sequence of integers with \( r_j \geq 2 \) for each \( j \), and let \( R_l \) be as in (21.1). This leads to the ring \( Z_r \) of \( r \)-adic integers, as before, which contains \( Z \) as a dense sub-ring. We can also consider \( Z \) as a discrete subgroup of the real line \( \mathbb{R} \) with respect to addition, with the corresponding quotient \( \mathbb{R}/Z \) as a compact commutative topological group. Of course, \( \mathbb{R} \times Z_r \) is a locally compact commutative topological group with respect to coordinatewise addition and the product topology, using the standard topology on \( \mathbb{R} \). The group \( Z \) of integers with respect to addition is a subgroup of both \( \mathbb{R} \) and \( Z_r \), and hence

\[
A = \{(a, a) : a \in \mathbb{Z}\}
\]

(25.1)

is a subgroup of \( \mathbb{R} \times Z_r \). More precisely, \( A \) is a closed subgroup of \( \mathbb{R} \times Z_r \), because \( Z \) is a closed subgroup of \( \mathbb{R} \), so that the quotient group

\[
(\mathbb{R} \times Z_r)/A
\]

(25.2)

is also a topological group with respect to the quotient topology. The canonical quotient mapping from \( \mathbb{R} \times Z_r \) onto (25.2) is a local homeomorphism, and it is easy to see that (25.2) is compact, because \( Z_r \) is compact. One can also check that the image of \( \mathbb{R} \times \{0\} \) in (25.2) under the canonical quotient mapping from \( \mathbb{R} \times Z_r \) onto (25.2) is dense in (25.2), because \( Z \) is dense in \( Z_r \). This implies that (25.2) is connected, because \( \mathbb{R} \) is connected, and the closure of a connected set is connected.

There is also a nice description of (25.2) in terms of coherent sequences. Put

\[
Y_l = \mathbb{R}/R_l \mathbb{Z}
\]

(25.3)

for each \( l \geq 0 \), considered as a compact commutative topological group. Thus

\[
Y = \prod_{l=0}^{\infty} Y_l,
\]

(25.4)

is also a compact commutative topological group, with respect to coordinatewise addition and the product topology. There is a natural group homomorphism from \( Y_{l+1} \) onto \( Y_l \) for each \( l \geq 0 \), because \( R_{l+1} \mathbb{Z} \subseteq R_l \mathbb{Z} \). This homomorphism is also continuous, and in fact a local homeomorphism. An element \( y = \{ y_l \}_{l=0}^{\infty} \) of \( Y \) is said to be a coherent sequence if \( y_l \) is the image of \( y_{l+1} \) under this homomorphism for each \( l \geq 0 \). The set of coherent sequences in \( Y \) is a closed subgroup of \( Y \), and there is a natural isomorphism between \( Z_r \) and the coherent sequences \( y = \{ y_l \}_{l=0}^{\infty} \) in \( Y \) such that \( y_0 = 0 \) in \( Y_0 = \mathbb{R}/\mathbb{Z} \). There is a natural homomorphism from \( \mathbb{R} \) into \( Y \), whose \( l \)-th coordinate is the canonical quotient homomorphism from \( \mathbb{R} \) onto \( Y_l \) for each \( l \geq 0 \), and it is easy to see that this homomorphism sends real numbers to coherent sequences in \( Y \). This leads to a homomorphism from \( \mathbb{R} \times Z_r \) into the group of coherent sequences in \( Y \), which adds the images of elements of \( \mathbb{R} \) and \( Z_r \) in \( Y \). One can check that the kernel of
each isomorphic as a topological group to its analogue with one can combine the two metrics to get a translation-invariant metric on $\mathbb{R}$. Using this, one can get a translation-invariant quotient metric on $(25.2)$. Suppose that $r' = \{r'_j\}_{j=1}^{\infty}$ is another sequence of integers with $r'_j \geq 2$ for each $j$ such that $r \sim r'$, in the sense of Section 23. It is easy to see that $(25.2)$ is isomorphic as a topological group to its analogue with $r'$ instead of $r$, because of the isomorphism between $\mathbb{Z}_{r'}$ and $\mathbb{Z}_r$, which is the identity mapping on their common subgroup $\mathbb{Z}$. One can also look at this in terms of coherent sequences. However, this type of isomorphism may be rather complicated with respect to the corresponding geometries.

26 Filtrations

Remember that a filtration on a probability space $(X, \mathcal{A}, \mu)$ is an increasing sequence $\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \mathcal{B}_3 \cdots$ of $\sigma$-subalgebras of $\mathcal{A}$. This can be used to define conditional expectation operators, martingales, and so on. As a basic class of examples, suppose that $(X_j, \mathcal{A}_j, \mu_j)$ is a probability space for each positive integer $j$, and let

$$X = \prod_{j=1}^{\infty} X_j$$

be their product, with the corresponding product $\sigma$-algebra $\mathcal{A}$ and probability measure $\mu$. Let $\mathcal{B}_l$ be the collection of subsets of $\prod_{j=1}^{\infty} X_j$ that correspond to a product of a measurable subset of $\prod_{j=l}^{\infty} X_j$ with $\prod_{j=l+1}^{\infty} X_j$, for each positive integer $l$. It is easy to see that this defines a filtration on $X$. If $\pi$ is a one-to-one mapping from $\mathbb{Z}_+$ onto itself, then one can also consider the product $X^\pi = \prod_{j=1}^{\infty} X_{\pi(j)}$ as a probability space, with the analogous filtration. The mapping from $x = \{x_j\}_{j=1}^{\infty} \in X$ to $\{x_{\pi(j)}\}_{j=1}^{\infty} \in X^\pi$ defines an isomorphism between $X$ and $X^\pi$ as probability spaces, but this isomorphism may be rather complicated in terms of the corresponding filtrations. In particular, one might take $X_j$ to be a finite set with at least two elements for each $j$, where every subset of $X_j$ is measurable, and where $\mu_j$ assigns equal weight to each element of $X_j$. The resulting filtration on $(26.1)$ is closely related to the type of ultrametrics on $X$ discussed earlier.

Now let $r = \{r_j\}_{j=1}^{\infty}$ be a sequence of integers with $r_j \geq 2$ for each $j$, and let $R_l$ be as in $(21.1)$. Because $\mathbb{Z}_r$ is a compact commutative topological group, there is a natural translation-invariant Borel probability measure on $\mathbb{Z}_r$, given by Haar measure. If $r' = \{r'_j\}_{j=1}^{\infty}$ is another sequence of integers with $r'_j \geq 2$ for each $j$ and $r \sim r'$, then $\mathbb{Z}_r$ is isomorphic to $\mathbb{Z}_{r'}$ as a topological ring and hence as a topological group, and Haar measure on $\mathbb{Z}_r$ corresponds to Haar measure on $\mathbb{Z}_{r'}$ under this isomorphism.
Of course, there is a natural homomorphism from $\mathbb{Z}$ onto $\mathbb{Z}/R_l\mathbb{Z}$ for each $l$. This extends to a continuous homomorphism from $\mathbb{Z}_r$ onto $\mathbb{Z}/R_l\mathbb{Z}$ for each $l$, basically by construction. Let $\mathcal{B}_l$ be the collection of subsets of $\mathbb{Z}_r$ that can be expressed as the inverse image of a subset of $\mathbb{Z}/R_l\mathbb{Z}$ under the homomorphism from $\mathbb{Z}_r$ onto $\mathbb{Z}/R_l\mathbb{Z}$ just mentioned. This is a $\sigma$-subalgebra of the Borel sets in $\mathbb{Z}_r$ for each $l$, which defines a filtration on $\mathbb{Z}_r$. This filtration is closely related to the $r$-adic geometry on $\mathbb{Z}_r$.

Similarly, (25.2) is a compact commutative topological group, and thus has a natural translation-invariant Borel probability measure, given by Haar measure. Remember that there is a natural continuous homomorphism from (25.2) onto (25.3) for each nonnegative integer $l$, as in the previous section. Let $\mathcal{C}_l$ be the collection of subsets of (25.2) which can be expressed as the inverse image of a Borel subset of (25.3) under the homomorphism from (25.2) onto (25.3) just mentioned. This is a $\sigma$-subalgebra of the Borel sets in (25.2) for each $l$, which defines a filtration on (25.2). In effect, this filtration is part of the geometry of (25.2), which is specified by $r$.

27 Directed systems

Let $(A, \preceq)$ be a partially-ordered set which is a directed system, so that for every $a, b \in A$ there is a $c \in A$ such that $a, b \preceq c$. Also let $(X, A, \mu)$ be a probability space, and for each element $a$ of the directed set $A$, let $\mathcal{B}_a$ be a $\sigma$-subalgebra of the measurable sets in $X$. Suppose that these $\sigma$-algebras are compatible with the ordering on $A$, in the sense that

\begin{equation}
\mathcal{B}_a \subseteq \mathcal{B}_b
\end{equation}

when $a, b \in A$ satisfy $a \preceq b$. This includes the usual notion of a filtration on $X$, and some aspects of martingales still work in this setting, as in [41]. However, standard results about maximal functions do not always hold, for instance.

Let $I$ be an infinite set, and let $(X_j, A_j, \mu_j)$ be a probability space for each $j \in I$. Consider the product $X = \prod_{j \in I} X_j$, with the usual product $\sigma$-algebra $A$ of measurable sets and probability measure $\mu$. If $K$ is a nonempty finite subset of $I$, then let $\mathcal{B}_K$ be the $\sigma$-subalgebra of measurable subsets of $X$ that correspond to a product of measurable subset of $\prod_{j \in K} X_j$ with $\prod_{j \notin K} X_j$. Equivalently, $\mathcal{B}_K$ consists of the inverse images of measurable subsets of $\prod_{j \in K} X_j$ under the obvious coordinate projection from $X$ onto $\prod_{j \in K} X_j$. If $L$ is another finite subset of $I$ that contains $K$, then $\mathcal{B}_K \subseteq \mathcal{B}_L$. The collection of nonempty finite subsets of $I$ is a directed system with respect to inclusion, and this defines a compatible family of $\sigma$-subalgebras of measurable sets in $X$. In this situation, the usual arguments about maximal functions do not work, even when $I$ is countable. In particular, there are problems with pointwise convergence.

Now let $\preceq$ be the partial ordering on $\mathbb{Z}_+$ where $a \preceq b$ when $b$ is an integer multiple of $a$. Of course, $\mathbb{Z}_+$ is a directed system with respect to this ordering. If $r = \{r_j\}_{j=1}^\infty$ is a sequence of integers with $r_j \geq 2$ for each $j$, and if $R_l$ is as in (21.1), then the set of $R_l$’s is linearly-ordered with respect to $\preceq$, and hence
is a directed system. Let $E$ be an infinite subset of $\mathbb{Z}_+$, and suppose that $E$ is also a directed system with respect to this ordering. Let $p$ be a prime number, and let $c_E(p)$ be the supremum of the nonnegative integers $k$ for which there is an $R \in E$ that is an integer multiple of $p$. Thus $c_E(p)$ is either a nonnegative integer or $+\infty$ for each $p$. Because $E$ is infinite, either $c_E(p) = +\infty$ for some $p$, or $c_E(p) > 0$ for infinitely many $p$. Using $E$, we get a translation-invariant topology on $\mathbb{Z}$, for which a local base for the topology at 0 is given by the sets $R\mathbb{Z}$ with $R \in E$. As usual, $\mathbb{Z}$ is a topological ring with respect to this topology. If $r$ is as before and $c_r(p) = c_E(p)$ for every prime number $p$, then the topology on $\mathbb{Z}$ corresponding to $E$ is the same as the $r$-adic topology discussed previously.

Consider the Cartesian product

$$X_E = \prod_{R \in E} (\mathbb{Z}/R\mathbb{Z}).$$

(27.2)

This is a compact commutative ring with respect to coordinatewise addition and multiplication, and using the product topology associated to the discrete topology on $\mathbb{Z}/R\mathbb{Z}$ for each $R \in E$. There is a natural homomorphism from $\mathbb{Z}$ into $X_E$, defined by the canonical quotient mappings from $\mathbb{Z}$ onto $\mathbb{Z}/R\mathbb{Z}$ for each $R \in E$. It is easy to see that the kernel of this homomorphism is trivial, because $E$ is infinite. This homomorphism is also a homeomorphism from $\mathbb{Z}$ onto its image in $X_E$ with respect to the topology on $\mathbb{Z}$ associated to $E$ as in the preceding paragraph.

Let $x = \{x_R\}_{R \in E}$ be an element of $X_E$, so that $x_R \in \mathbb{Z}/R\mathbb{Z}$ for each $R \in E$. If $R_1, R_2 \in E$ and $R_1 \preceq R_2$, then $R_2 \mathbb{Z} \subseteq R_1 \mathbb{Z}$, and we get a natural ring homomorphism from $\mathbb{Z}/R_2 \mathbb{Z}$ onto $\mathbb{Z}/R_1 \mathbb{Z}$. Let us say that $x \in X_E$ is coherent if the $x_{R_1}$ is the image of $x_{R_2}$ under the natural homomorphism from $\mathbb{Z}/R_2 \mathbb{Z}$ onto $\mathbb{Z}/R_1 \mathbb{Z}$ for every $R_1, R_2 \in E$ such that $R_1 \preceq R_2$. The set of coherent elements of $X_E$ forms a closed sub-ring of $X_E$. The natural homomorphism from $\mathbb{Z}$ into $X_E$ maps $\mathbb{Z}$ into the set of coherent elements of $X_E$, and in fact the set of coherent elements of $X_E$ is the same as the closure of the image of $\mathbb{Z}$ in $X_E$. Let $\mathbb{Z}_E$ be the set of coherent elements of $X_E$. If $r = \{r_j\}_{j=1}^\infty$ is as before and $c_r(p) = c_E(p)$ for every prime number $p$, then $\mathbb{Z}_E$ can be identified with $\mathbb{Z}_r$.

There is a natural homomorphism from $\mathbb{Z}_E$ onto $\mathbb{Z}/R\mathbb{Z}$ for each $R \in E$, which is the restriction to $\mathbb{Z}_E$ of the coordinate mapping from $X_E$ onto $\mathbb{Z}/R\mathbb{Z}$. Let $\mathcal{B}_R$ be the collection of subsets of $\mathbb{Z}_E$ that can be expressed as the inverse image of a subset of $\mathbb{Z}/R\mathbb{Z}$ under the mapping just defined, for each $R \in E$. This is a $\sigma$-subalgebra of the Borel sets in $\mathbb{Z}_E$. If $R_1, R_2 \in E$ satisfy $R_1 \preceq R_2$, then it is easy to see that $\mathcal{B}_{R_1} \subseteq \mathcal{B}_{R_2}$. Thus we get a family of $\sigma$-subalgebras of the Borel sets in $\mathbb{Z}_E$ indexed by $E$ and compatible with the ordering on $E$.

Similarly,

$$Y_E = \prod_{R \in E} (\mathbb{R}/R\mathbb{Z})$$

(27.3)

is a compact commutative topological group with respect to coordinatewise addition and the product topology associated to the usual quotient topology on
for each \( R \in E \). There is a natural continuous group homomorphism from \( \mathbb{R} \) as a commutative topological group with respect to addition into \( Y_E \), defined by the canonical quotient mapping from \( \mathbb{R} \) onto \( \mathbb{R}/R\mathbb{Z} \) for each \( R \in E \). If \( R_1, R_2 \in E \) and \( R_1 \prec R_2 \), then \( R_2 \mathbb{Z} \subseteq R_1 \mathbb{Z} \), which leads to a continuous group homomorphism from \( \mathbb{R}/R_2 \mathbb{Z} \) onto \( \mathbb{R}/R_1 \mathbb{Z} \). As usual, an element \( y = \{y_R\}_{R \in E} \) of \( Y \) is said to be coherent if \( y_{R_1} \) is the image of \( y_{R_2} \) under the natural mapping from \( \mathbb{R}/R_2 \mathbb{Z} \) onto \( \mathbb{R}/R_1 \mathbb{Z} \) for every \( R_1, R_2 \in E \) with \( R_1 \leq R_2 \). The set of coherent elements of \( Y_E \) is a closed subgroup of \( Y_E \), which can be identified with the quotient of \( \mathbb{R} \times \mathbb{Z}_E \) by the image of \( \mathbb{Z} \) under the obvious diagonal embedding. Let \( C_R \) be the collection of subsets of the group of coherent elements of \( Y_E \) that can be expressed as the inverse image of a Borel set in \( \mathbb{R}/R\mathbb{Z} \) under the corresponding coordinate mapping, for each \( R \in E \). If \( R_1, R_2 \in E \) and \( R_1 \prec R_2 \), then \( C_{R_1} \subseteq C_{R_2} \), as before. This defines a family of \( \sigma \)-subalgebras of the Borel sets in the group of coherent elements of \( Y_E \) indexed by \( E \) which is compatible with the ordering on \( E \).

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