Multi-scale Renormalisation Group Improvement
of the Effective Potential

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Abstract

Using the renormalisation group and a conjecture concerning the perturbation series for the effective potential, the leading logarithms in the effective potential are exactly summed for $O(N)$ scalar and Yukawa theories.
1. Introduction

The effective potential (EP) is a very useful tool for the study of spontaneous symmetry breaking in field theory. However in many applications the perturbative loop expansion is inadequate even if the couplings are “small”. This is due to the presence of logarithmic terms, like $\ln(\phi/\mu)$, in the perturbation series which restrict the range of $\phi$ values (here $\phi$ is some generic scalar field, and $\mu$ is the renormalisation scale) where perturbation theory is credible. In applications where one needs to survey the EP for a wide range of $\phi$ values (eg. vacuum stability analyses in the Standard Model [1,2,3]), these logarithms must be dealt with. Of course, one can simply let the parameters run, and calculate the EP at $\mu = \phi$. Provided these running couplings remain perturbative at this scale, one can drastically extend the scope of perturbation theory. An alternative way of thinking about renormalisation group (RG) improvement, is to view it as a reorganization of the perturbation series, in which the first term is the sum of all the leading logarithms, the second term represents the sub-leading logarithms, and so on. The leading logarithms are terms of the form $h^n \ln(M_1(\phi)/\mu) \ln(M_2(\phi)/\mu)...\ln(M_n(\phi)/\mu)$, and represent the most “dangerous” logarithmic terms at each order in perturbation theory (note that the tree potential is counted as a leading logarithm). The sub-leading logarithms are proportional to $h^{n+1} \ln(M_1/\mu)...\ln(M_n/\mu)$. In dimensional regularisation, the most divergent $n$-loop terms one encounters (in a model with a single mass scale $m$) are proportional to $h^n (m/\mu)^{\epsilon}/\epsilon^n$. When these terms are expanded in powers of $\epsilon$ finite terms proportional to $h^n \ln^n(m/\mu)$ are generated (in a theory with two scales $m_1$ and $m_2$ terms of the form $h^n \ln^p(m_1/\mu)\ln^{n-p}(m_2/\mu)$ will appear). Therefore it is only the most divergent pieces of the Feynman diagrams that contribute to the leading logarithms.

Renormalisation group improved potentials were first considered in the con-

† We retain the factors of $h$ in all equations so that the reader can easily distinguish leading from sub-leading contributions.
text of massless models by Coleman and Weinberg [4]. Recently \[2,5,6\] it has been demonstrated that this treatment also works in the massive case provided one takes into account the running of the vacuum energy (or cosmological constant). However, when there is more than one mass scale present it is less clear how to proceed; no choice of \( \mu \) will kill all the logarithms. This is not usually important provided that the logarithms of the scale ratio's are “small”. However, in certain cases of interest these logarithms are large (for example \( \ln(M_{\text{GUT}}/M_{\text{electroweak}}) \approx 30 \)).

Even in situations where these logarithms are not so large the scope of perturbation theory is still reduced, for example consider a two scale model with \( m_1 > m_2 \), then perturbative credibility requires that the \( \lambda_i \ln(m_1/m_2) \) be “small” in addition to the usual requirement that just the couplings \( \lambda_i \) are small. If we could fully sum the multi-scale leading logarithms we should have an approximation that is useful despite the existence of widely differing scales. In ref. [7] it was argued that the decoupling theorem could be used to obtain approximations to the multi-scale leading logarithms within the \( \overline{\text{MS}} \) scheme. Alternatively, in ref. [8] it was found that some of the problems associated with RG improvement are absent in a modified mass-dependent scheme, although the RG equation is difficult to work with in such schemes. In this paper it is suggested that it may be possible to exactly sum the leading logarithms in a general theory, using a mass-independent renormalisation scheme. Explicit formulae are presented in the case of \( O(N) \) scalar and Yukawa models. The results presented here depend on an unproven conjecture, however the conjecture has been checked to two loops.

The outline of this paper is as follows. In section 2, we review the leading logarithms calculation for ordinary massive \( \phi^4 \) theory (the result is later used as a boundary condition for the full \( O(N) \) calculations). In section 3 we present our method and calculation of the leading logarithms in the \( O(N) \) scalar \( \phi^4 \) theory. We apply our method to the more complicated Yukawa model in section 4, and in section 5 we conclude with a discussion of the general validity of the method.
2. Massive $\phi^4$ Theory

Consider massive $\phi^4$ theory in four dimensions defined by the Lagrangian

$$L = \frac{1}{2}(\partial_{\mu}\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{24}\phi^4 - \Lambda.$$  \hspace{1cm} (2.1)

Here $\Lambda$ is a “cosmological constant” term. Assuming the EP, $V(\phi)$, is independent of the renormalisation scale, $\mu$, for fixed values of the bare parameters, one obtains the following RG equation

$$\mathcal{D}V = 0,$$  \hspace{1cm} (2.2)

where

$$\mathcal{D} = \frac{\partial}{\partial\mu} + \beta_\lambda \frac{\partial}{\partial\lambda} + \beta_{m^2} \frac{\partial}{\partial m^2} - \gamma_\phi \frac{\partial}{\partial\phi} + \beta_\Lambda \frac{\partial}{\partial\Lambda}. $$  \hspace{1cm} (2.3)

Here $\beta_\lambda$, $\beta_{m^2}$ and $\beta_\Lambda$ are the coupling constant, mass squared and cosmological constant beta functions, respectively, and $\gamma$ is the anomalous dimension. The tree potential can be read off from the Lagrangian,

$$V^{(0)} = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{24}\phi^4 + \Lambda,$$  \hspace{1cm} (2.4)

and the one-loop potential is easily calculated [4,9,10], the result in $\overline{\text{MS}}$ reads

$$V^{(1)} = \frac{\hbar(m^2 + \frac{1}{2}\lambda\phi^2)^2}{4(4\pi)^2}\left(\ln\frac{m^2 + \frac{1}{2}\lambda\phi^2}{\mu^2} - \frac{3}{2}\right).$$  \hspace{1cm} (2.5)

The one-loop RG functions are

$$\beta^{(1)}_\lambda = \frac{3\hbar\lambda^2}{(4\pi)^2}, \hspace{1cm} \beta^{(1)}_{m^2} = \frac{\hbar m^2\lambda}{(4\pi)^2}, \hspace{1cm} \beta^{(1)}_\Lambda = \frac{\hbar m^4}{2(4\pi)^2}, \hspace{1cm} \gamma^{(1)} = 0.$$  \hspace{1cm} (2.6)

Applying the method of characteristics to eq. (2.2)

$$V(\lambda, m^2, \mu, \Lambda, \phi) = V(\bar{\lambda}, \bar{m}^2, \bar{\mu}, \bar{\Lambda}, \bar{\phi}),$$  \hspace{1cm} (2.7)

where the “running” parameters satisfy

$$\hbar \frac{d\bar{\mu}}{dt} = \bar{\mu}, \hspace{1cm} \hbar \frac{d\bar{\lambda}}{dt} = \beta_\lambda(\bar{\lambda}), \hspace{1cm} \hbar \frac{d\bar{m}^2}{dt} = \beta_{m^2}(\bar{m}^2, \bar{\lambda}),$$

$$\hbar \frac{d\bar{\Lambda}}{dt} = \beta_\Lambda(\bar{\lambda}, \bar{m}^2), \hspace{1cm} \hbar \frac{d\bar{\phi}}{dt} = -\gamma(\bar{\lambda})\bar{\phi},$$  \hspace{1cm} (2.8)
and \( \mu(t = 0) = \mu, \lambda(t = 0) = \lambda, \bar{m}^2(t = 0) = m^2, \Lambda(t) = \Lambda, \bar{\phi}(t = 0) = \phi \). The idea of RG improvement is that via a judicious choice of \( t \), one can evaluate the right hand side of eq. (2.7) perturbatively even if large logarithms render the left hand side non-perturbative. This method yields approximations to the EP which are useful for a much wider range of \( \phi \) values than the conventional loop expansion.

The obvious choice would be to choose \( t \) so as to remove all the logarithms on the right hand side of eq. (2.7) (note that this is only possible because there is only one kind of logarithm, namely \( \ln[(m^2 + \frac{1}{2}\lambda\phi^2)/\mu^2] \), in the perturbation series). That is, \( t \) is chosen so that

\[
\frac{\bar{m}^2(t) + \frac{1}{2}\bar{\lambda}(t)\bar{\phi}(t)^2}{\bar{\mu}(t)^2} = 1. \quad (2.9)
\]

While eq. (2.9) seems the most natural choice, it is awkward to work with (even in the one-loop approximation). A rather less implicit choice is given by

\[
t = \frac{\hbar}{2}\ln \frac{m^2 + \frac{1}{2}\lambda\phi^2}{\mu^2}. \quad (2.10)
\]

Note that this choice does not kill the logarithms on the right hand side of eq. (2.7), however it allows one to explicitly sum the leading (and subleading, etc.) logarithms in the EP [2,5,6]. With this choice of \( t \), \( \bar{\mu}^2(t) = \mu^2 \exp(2t/\hbar) = m^2 + \frac{1}{2}\lambda\phi^2 \) which is independent of \( \mu \); all the \( \mu \) dependence is carried by \( \bar{\lambda}(t), \bar{m}^2(t), \Lambda(t) \) and \( \bar{\phi}(t) \). Solving eqs. (2.8) using the one-loop RG functions,

\[
\bar{\lambda}(t) = \lambda \left( 1 - \frac{3\lambda t}{(4\pi)^2} \right)^{-1} + \mathcal{O}(\hbar),
\]

\[
\bar{m}^2(t) = m^2 \left( 1 - \frac{3\lambda t}{(4\pi)^2} \right)^{-1/3} + \mathcal{O}(\hbar), \quad \bar{\phi}(t) = \phi + \mathcal{O}(\hbar), \quad (2.11)
\]

\[
\Lambda(t) = \Lambda - \frac{m^4}{2\lambda} \left[ \left( 1 - \frac{3\lambda t}{(4\pi)^2} \right)^{1/3} - 1 \right] + \mathcal{O}(\hbar).
\]
Inserting these into the right hand side of eq. (2.7), one finds

\[
V = \frac{\lambda}{24} \phi^4 \left( 1 - \frac{3\lambda t}{(4\pi)^2} \right)^{-1} + \frac{1}{2} m^2 \phi^2 \left( 1 - \frac{3\lambda t}{(4\pi)^2} \right)^{-1/3} - \frac{m^4}{2\lambda} \left( 1 - \frac{3\lambda t}{(4\pi)^2} \right)^{1/3} \\
+ \frac{m^4}{2\lambda} + \Lambda + \mathcal{O}(\bar{h}).
\]

With the choice of \( t \) given by (2.10), eq. (2.12) gives the sum of the leading logarithms in the EP (which was first obtained by Kastening [5]). The \( \mathcal{O}(\bar{h}) \) term represents the sub-leading, sub-sub-leading, etc. contributions to the EP. In general, to perform the leading (sub-leading, ...) logarithms expansion, one must expand the right hand side of eq. (2.7) in powers of \( \bar{h} \) but retaining all orders in \( t \). However, in this paper we just concentrate on the leading logarithms summation, which (in the single mass scale case) just amounts to substituting the one-loop running parameters into the tree level potential.

### 3. O(N) symmetric \( \phi^4 \) theory

Consider the theory defined by the Lagrangian

\[
\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{24} \phi^4 - \Lambda,
\]

where \( \phi^2 = \phi_i \phi_i \quad (i = 1, ..., N) \), and \( \phi_i \) is an \( N \)-component scalar field. Although this model has \( N \) scalar fields we can exploit the \( O(N) \) invariance to write the EP, \( V(\phi_i) \), as a function of \( \phi \) only. The tree-level potential is simply

\[
V^{(0)} = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{24} \phi^4 + \Lambda,
\]

note that this is independent of \( N \), which will be exploited when summing the leading logarithms. The one-loop potential reads

\[
(4\pi)^2 V^{(1)} = \frac{\hbar}{4} H^2 \left( \ln \frac{H}{\mu^2} - \frac{3}{2} \right) + \frac{\hbar}{4} (N - 1) G^2 \left( \ln \frac{G}{\mu^2} - \frac{3}{2} \right),
\]
where
\[ H = m^2 + \frac{1}{2} \lambda \phi^2, \quad G = m^2 + \frac{1}{6} \lambda \phi^2. \tag{3.4} \]

The two-loop potential is also known for this model [11]. Note that for \( N \neq 1 \) we have two distinct logarithms in the perturbation series, so unlike the \( N = 1 \) case no choice of \( \mu \) will remove all the logarithms. Of course we can write the second logarithm in terms of the first
\[ \ln \frac{G}{\mu^2} = \ln \frac{H}{\mu^2} + \ln \frac{G}{H}, \tag{3.5} \]

then provided \( \ln(H/G) \) is “small” we can sum up the \( \ln(H/\mu^2) \) terms in the same fashion as the \( N = 1 \) case [12]. Here we will consider summation of both logarithms. On physical grounds, one could argue that this is not really necessary, since although the EP contains two logarithms, there is really only one physical scale in the theory. For if \( m^2 > 0 \), we have \( N \) particles of the same mass, and if \( m^2 < 0 \) we have one massive particle and \( N - 1 \) massless Goldstone bosons. However, if we can sum up the two logarithms in this simple model we should be able to apply the method to cases where there is more than one physical scale (such as the Yukawa model treated in the next section).

Consider the sum of the leading logarithms
\[
L = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{24} \phi^4 + \Lambda + \frac{\hbar H^2}{4(4\pi)^2} \ln \frac{H}{\mu^2} + \frac{\hbar(N - 1)G^2}{4(4\pi)^2} \ln \frac{G}{\mu^2}
+ \frac{\hbar^2(\lambda \phi)^2 H}{8(4\pi)^4} \ln^2 \frac{H}{\mu^2} + \frac{\hbar^2 \lambda (N^2 - 1)G^2}{24(4\pi)^4} \ln^2 \frac{G}{\mu^2}
+ \frac{\hbar^2(N - 1)(\lambda \phi)^2}{72(4\pi)^4} \left( (2G - H) \ln^2 \frac{G}{\mu^2} + 2H \ln \frac{H}{\mu^2} \ln \frac{G}{\mu^2} \right)
+ \frac{\hbar^2 \lambda H^2}{8(4\pi)^4} \ln^2 \frac{H}{\mu^2} + \frac{\hbar^2 \lambda (N - 1)HG}{12(4\pi)^4} \ln \frac{H}{\mu^2} \ln \frac{G}{\mu^2} + O(\hbar^3), \tag{3.6}
\]

(the two-loop terms were obtained in [11]). Note that the \(-3/2\) terms in \( V^{(1)} \) are
not included, since they are counted as sub-leading logarithms which we do not attempt to sum here. Using the RG alone it is impossible to compute $L$ exactly (this is because the operator $\mu \partial / \partial \mu$ cannot distinguish between $\ln(H/\mu^2)$ and $\ln(G/\mu^2)$). Now the crucial point is to notice that if we set $\mu^2 = G$ in eq. (3.6), $L$ reduces to the $N = 1$ case (at least to the two-loop level). Note that this is not true for the sub-leading logarithms, but seems to hold for the leading logarithms.

We know (exactly) what $L$ is for $N = 1$ (eq. (2.12)), thus we assume

$$L(\mu^2 = G) = \frac{\lambda}{24} \phi^4 \left( 1 - \frac{3\lambda h}{2(4\pi)^2} \ln \frac{H}{G} \right)^{-1} + \frac{1}{2} m^2 \phi^2 \left( 1 - \frac{3\lambda h}{2(4\pi)^2} \ln \frac{H}{G} \right)^{-1/3}$$

$$- \frac{m^4}{2\lambda} \left( 1 - \frac{3\lambda h}{2(4\pi)^2} \ln \frac{H}{G} \right)^{1/3} + \frac{m^4}{2\lambda} + \Lambda.$$  \hspace{1cm} (3.7)

So if we solve the RG for the $O(N)$ EP using the $N = 1$ formula as a boundary condition at $\mu^2 = G$ we should be able to compute $L$ exactly.

The one-loop RG functions for this model are

$$\beta^{(1)}_\lambda = \frac{N + 8}{3(4\pi)^2} h \lambda^2, \hspace{1cm} \beta^{(1)}_m = \frac{N + 2}{3(4\pi)^2} h \lambda m^2, \hspace{1cm} \beta^{(1)}_\Lambda = \frac{N h m^4}{2(4\pi)^2}, \hspace{1cm} \gamma^{(1)} = 0,$$  \hspace{1cm} (3.8)

which gives the following one-loop running couplings

$$\bar{\lambda}(t) = \lambda \left( 1 - \frac{(N + 8)\lambda t}{3(4\pi)^2} \right)^{-1},$$

$$\bar{m}^2(t) = m^2 \left( 1 - \frac{(N + 8)\lambda t}{3(4\pi)^2} \right)^{-\frac{N+2}{N+8}}, \hspace{1cm} \bar{\phi}(t) = \phi,$$  \hspace{1cm} (3.9)

$$\bar{\Lambda}(t) = \Lambda + \frac{3N m^4}{2(N - 4)\lambda} \left[ \left( 1 - \frac{(N + 8)\lambda t}{3(4\pi)^2} \right)^{-\frac{N-4}{N+8}} - 1 \right].$$
Now choose
\[ t = \frac{\bar{h}}{2} \ln \frac{G}{\mu^2}, \quad (3.10) \]

which means that \( t = 0 \) corresponds to \( G = \mu^2 \), and insert eqs. (3.9) into eq. (2.7). The RG “improved” potential obtained using the one-loop running parameters together with the boundary condition eq. (3.7) is

\[
V_{\text{improved}} = \frac{\bar{\lambda}}{24} \phi^4 \left( 1 - \frac{3\bar{\lambda}h}{2(4\pi)^2} \ln \frac{\bar{H}}{G} \right)^{-1} + \frac{1}{2} \bar{m}^2 \phi^2 \left( 1 - \frac{3\bar{\lambda}h}{2(4\pi)^2} \ln \frac{H}{G} \right)^{-1/3} \\
- \frac{\bar{m}^4}{2\lambda} \left( 1 - \frac{3\bar{\lambda}h}{2(4\pi)^2} \ln \frac{\bar{H}}{G} \right)^{1/3} + \frac{\bar{m}^4}{2\lambda} + \bar{\Lambda},
\]

(3.11)

where \( \bar{H} = \bar{m}^2 + \frac{1}{2} \bar{\lambda} \bar{\phi}^2 \) and \( \bar{G} = \bar{m}^2 + \frac{1}{6} \bar{\lambda} \bar{\phi}^2 \). This formula is rather unwieldy, however, as we are only interested in the leading logarithms, it can be simplified somewhat. Consider the \( \hbar \ln \bar{H}/\bar{G} \) terms in eq. (3.11). We can write

\[
\hbar \ln \frac{\bar{H}}{G} = \hbar \ln \frac{H}{G} + \hbar \ln \frac{\bar{H}}{H} - \hbar \ln \frac{\bar{G}}{G},
\]

(3.12)

Now the point is that the second and third terms on the right hand side of eq. (3.12) do not contribute to the leading logarithms, since if \( \hbar \ln[(\bar{m}^2 + \frac{1}{2} \bar{\lambda} \bar{\phi}^2)/H] \) is expanded in powers of \( t \), all the terms will be of the form \( \hbar^{n+1} \ln^n(G/\mu^2) \) which are sub-leading logarithms. Thus, in the leading logarithmic approximation we are entitled to make the replacement \( \ln(\bar{H}/\bar{G}) \to \ln(H/G) \) in eq. (3.11), ie.

\[
\left( 1 - \frac{3\bar{\lambda}(t)\hbar}{2(4\pi)^2} \ln \frac{\bar{H}}{G} \right) \to \left( 1 - \frac{3\bar{\lambda}(t)\hbar}{2(4\pi)^2} \ln \frac{H}{G} \right)
\]

\[
= \left( 1 - \frac{(N - 1)\lambda t}{3(4\pi)^2} - \frac{3\lambda h}{2(4\pi)^2} \ln \frac{H}{\mu^2} \right) \left( 1 - \frac{(N + 8)\lambda t}{3(4\pi)^2} \right)^{-1}.
\]

(3.13)
Inserting eqs. (3.9) and (3.13) into eq. (3.11), the leading logarithms in $O(N)$ $\phi^4$ theory sum to

$$L = \frac{\lambda}{24} \phi^4 \left( 1 - \frac{3 \lambda s}{(4\pi)^2} - \frac{(N-1) \lambda t}{3(4\pi)^2} \right)^{-1}$$

$$+ \frac{1}{2} m^2 \phi^2 \left( 1 - \frac{3 \lambda s}{(4\pi)^2} - \frac{(N-1) \lambda t}{3(4\pi)^2} \right)^{-1/3} \left( 1 - \frac{(N+8) \lambda t}{3(4\pi)^2} \right)^{-\frac{2}{3}}$$

$$- \frac{m^4}{2\lambda} \left[ \left( 1 - \frac{3 \lambda s}{(4\pi)^2} - \frac{(N-1) \lambda t}{3(4\pi)^2} \right)^{1/3} \left( 1 - \frac{(N+8) \lambda t}{3(4\pi)^2} \right)^{-\frac{4}{3}} \right]$$

$$- \frac{4N-1}{N-4} \left( 1 - \frac{(N+8) \lambda t}{3(4\pi)^2} \right)^{-\frac{N-4}{N-8}} + \frac{3N}{N-4} + \Lambda,$$

where

$$s = \frac{\hbar}{2} \ln \frac{H}{\mu^2} \quad \text{and} \quad t = \frac{\hbar}{2} \ln \frac{G}{\mu^2}.$$ 

Note that this result must be considered as a conjecture, since its derivation relied on eq. (3.7) which is not proven here. Although we are unable to prove eq. (3.14), the reader can easily verify that it is correct at the tree, one-loop and two-loop level.

If $m^2 < 0$ then the tree level minimum of the EP is given by $\phi^2 = -6m^2/\lambda$ i.e. at $G = 0$. It is clear from eq. (3.6), that some of the leading logarithms are not well behaved in the limit $G \downarrow 0$. In particular the two-loop contribution proportional to

$$-H \ln^2 \frac{G}{\mu^2} + 2H \ln \frac{H}{\mu^2} \ln \frac{G}{\mu^2},$$

diverges as $G \downarrow 0$. In fact, these divergences in $V^{(2)}$ are cancelled by infrared divergences in the non-logarithmic part of $V^{(2)}$ [11]. Of course, just because individual terms in $L$ are divergent in this limit, this does not imply that $L$ as a whole is
also divergent in this limit. If we let \( t \to -\infty \) in eq. (3.14), then the first two terms vanish, however the behaviour of the remaining terms is more complicated.

If \( N \leq 4 \) then \( L \) diverges, but if \( N > 4 \) we are left with a peculiar finite term

\[
L(t \to -\infty) = -\frac{3Nm^4}{2(N-4)\lambda} + \Lambda, \quad N > 4. \tag{3.15}
\]

Note that this is independent of \( \hbar \), yet is not (except in the limit \( N \to \infty \)) equal to the classical vacuum energy density, which is given by

\[
\rho_{\text{classical}} = -\frac{3m^4}{2\lambda} + \Lambda, \tag{3.16}
\]

in the case \( m^2 < 0 \).

An alternative approximation to the loop or leading logarithms expansion is the large \( N \) expansion \([10]\). The first order term in this approximation amounts to summing up the leading terms in \( N \) at each order in perturbation theory.

\[
V_1 = \frac{\lambda}{24} \phi^4 + \frac{1}{2} m^2 \phi^2 + \frac{\hbar NG^2}{4(4\pi)^2} \left( \ln \frac{G}{\mu^2} - \frac{3}{2} \right) + \frac{\hbar^2 N^2 \lambda G^2}{24(4\pi)^4} \left( \ln \frac{G}{\mu^2} - 1 \right)^2 + \text{higher order terms}, \tag{3.17}
\]

where the term proportional to \( N \) is just the leading contribution to eq. (3.3), and the \( N^2 \) term is due to the two-loop “figure of eight” graph. Note that only bubble graphs with no \( H \) propagators contribute to \( V_1 \). There is an exact (although implicit) expression for \( V_1 \) \([13]\), which in \( \overline{\text{MS}} \) reads

\[
\frac{\partial V_1}{\partial \phi} = \chi \phi, \tag{3.18}
\]

where \( \chi \) satisfies the gap equation

\[
\chi = G + \frac{\hbar \lambda N \chi}{6(4\pi)^2} \left( \ln \frac{\chi}{\mu^2} - 1 \right). \tag{3.19}
\]

In fact, it is possible to derive the above expression using the RG; one simply notes that all contributions beyond the tree level to \( V'_1 = \partial V_1 / \partial \phi \) will be proportional to the one-loop tadpole with a \( G \) propagator, or \( V'_1 - V'_{\text{tree}} \propto G(\ln(G/\mu^2) - 1) \), so if we
set \( \mu^2 = G/e \) all loop terms in \( V_1' \) vanish! ie. \( V_1'(\mu^2 = G/e) = V_{\text{tree}}' = \phi G \). If we solve the RG for \( V' \); \( V'(\lambda, m^2, \phi, \mu) = V'(\bar{\lambda}(t), \bar{m}^2(t), \phi, \bar{\mu}(t)) \), now choose \( t \) such that \( \bar{\mu}^2(t) = G(t)/e \) (this choice is just the “gap” equation (3.19)) so that \( V' = \bar{V}'(t) = \phi G(t) \), and since \( V_1 \) is made up of products of one-loop graphs we only need use the one-loop RG functions, ie. \( \beta_\lambda = \frac{1}{3} \hbar N \lambda^2/(4\pi)^2, \beta_{m^2} = \frac{1}{3} \hbar N \lambda m^2/(4\pi)^2, \gamma = 0 \) where non-leading terms in \( N \) have been dropped. If we take the large \( N \) and the leading logarithmic approximation (which amounts to making the less implicit choice \( t = \frac{1}{2} \hbar \ln(G/\mu^2) \) when solving the RG) one finds that

\[
L_1 = \left[ \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{24} \phi^4 \right] \left( 1 - \frac{N \lambda t}{3(4\pi)^2} \right)^{-1} + \Lambda, \tag{3.20}
\]

which agrees with eq. (3.14) in the large \( N \) limit. The effective potential has been computed to the next order in the large \( N \) expansion [14], however the formula is not very managable (although it may be possible to extract just the leading logarithms and compare with eq. (3.14)).

To summarize, we have an expression for the leading logarithms sum with the following properties:

i) For \( N = 1 \), it reduces to the known result.

ii) In the large \( N \) limit, it reduces to the known result.

iii) It is correct through to two-loops.

iv) It has the correct \( \ln \mu^2 \) dependence.

Property i) is a consequence of the proposed boundary conditions eq.(3.7), while property iv) is a just a statement that the improved potential was constructed using the one-loop running parameters ie. eqs. (3.9). The reader might enquire whether it is possible to write down an alternative improved potential with the above properties. The answer is yes, but the result will look unnatural. For example, the first entry in eq. (3.14) (the \( \phi^4 \) term) could be replaced in the

\[\text{† In general the RG for } V' \text{ is } D V' = \gamma V' \text{ and so } V' = \bar{V}' \text{ only if } \gamma = 0.\]
following way
\[
\frac{\lambda \phi^4}{24} \left( 1 - \frac{3\lambda_s}{(4\pi)^2} - \frac{(N-1)\lambda t}{3(4\pi)^2} \right)^{-1} \rightarrow \frac{\lambda \phi^4}{24} \left( 1 - \frac{3\lambda_s}{(4\pi)^2} - \frac{(N-1)\lambda t}{3(4\pi)^2} \right)^{-1} \left( 1 - \frac{(N-1)\lambda^3(s-t)^3}{(4\pi)^6} \right)^p,
\]
without affecting the above properties (here \( p \) is just a constant).

4. \( O(N) \) Yukawa theory

Here we repeat the calculation of the previous section for the \( O(N) \) Yukawa model. This is defined by the Lagrangian
\[
L = \frac{1}{2} (\partial_{\mu} \phi \partial^{\mu} \phi) - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{24} \phi^4 - \Lambda + \bar{\psi}_i (i\sigma^\mu - g\phi) \psi_i,
\] (4.1)
where \( \psi_i \quad (i = 1, .., N) \) is a \( N \)-component Dirac field. We have \( N \) massless (Dirac) fermions interacting with a scalar field \( \phi \) via an \( O(N) \) invariant Yukawa coupling (here \( \phi \) is an \( O(N) \) singlet). The tree level potential is
\[
V^{(0)} = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{24} \phi^4,
\] (4.2)
and the one-loop potential is given by
\[
V^{(1)}(\phi) = \frac{hH^2}{4(4\pi)^2} \left( \ln \frac{H}{\mu^2} - \frac{3}{2} \right) - \frac{NhF^2}{\mu^2} \left( \ln \frac{F}{\mu^2} - \frac{3}{2} \right),
\] (4.3)
where
\[
H = m^2 + \frac{1}{2} \lambda \phi^2, \quad F = g^2 \phi^2.
\] (4.4)
Once again we have two distinct logarithms to sum. The one loop RG functions are
\[
\kappa \beta^{(1)}_\lambda = 3\lambda^2 + 8N\lambda g^2 - 48N g^4, \quad \kappa \beta^{(1)}_g = (2N + 3)g^3,
\]
\[
\kappa \beta^{(1)}_{m^2} = (\lambda + 4Ng^2)m^2, \quad \kappa \beta^{(1)}_\Lambda = \frac{1}{2}m^4,
\]
\[
\kappa \gamma^{(1)} = 2Ng^2,
\] (4.5)
where $\kappa = (4\pi)^2/h$. The one-loop running parameters are (see for example [15])

$$
\bar{\phi}(t) = \phi B(t)^{\frac{2N}{4N+6}}, \quad \bar{g}^2(t) = g^2/B(t),
$$

$$
\bar{\lambda}(t) = g^2 a(\lambda - bg^2) B(t)^{\frac{3a}{4N+6} - 1} - b(\lambda - ag^2) B(t)^{\frac{3b}{4N+6} - 1},
$$

$$
\bar{m}^2(t) = m^2 B(t)^{-\frac{4N}{4N+6}} \left[ \frac{(\lambda - bg^2) B(t)^{\frac{3a}{4N+6}} - (\lambda - ag^2) B(t)^{\frac{3b}{4N+6}}}{(a - b)g^2} \right]^{-\frac{1}{4N+6}},
$$

$$
\bar{\Lambda}(t) = \Lambda + \frac{1}{2(4\pi)^2} \int_0^t dt' \bar{m}^2(t'),
$$

where

$$
B(t) = \left( 1 - \frac{(4N + 6) g^2 t}{(4\pi)^2} \right), \quad (4.7)
$$

and $a$ and $b$ are the roots of the quadratic equation

$$
3y^2 + (4N - 6)y - 48N = 0. \quad (4.8)
$$

In order to sum the leading logarithms in this theory we solve the RG with suitable boundary conditions at $\mu^2 = F$. We assume that at $\mu^2 = F$ the leading logarithms reduce to the $N = 0$ form (eq. (2.12)). That is we take

$$
L(\mu^2 = F) = \frac{\lambda}{24} \phi^4 \left( 1 - \frac{3\lambda h}{2(4\pi)^2} \ln \frac{H}{F} \right)^{-1} + \frac{1}{2} m^2 \phi^2 \left( 1 - \frac{3\lambda h}{2(4\pi)^2} \ln \frac{H}{F} \right)^{-1/3}
$$

$$
- \frac{m^4}{2\lambda} \left( 1 - \frac{3\lambda h}{2(4\pi)^2} \ln \frac{H}{F} \right)^{1/3} + \frac{m^4}{2\lambda} + \Lambda.
$$

Choosing

$$
t = \frac{\hbar}{2} \ln \frac{F}{\mu^2}, \quad (4.9)
$$

Choosing

$$
t = \frac{\hbar}{2} \ln \frac{F}{\mu^2}, \quad (4.10)
$$
and proceeding in the same way as the $O(N)$ scalar calculation in the previous section, the leading logarithms sum to

$$L = \frac{\tilde{\lambda}(t)}{24} \tilde{\phi}^4(t) \left( 1 - \frac{3(s - t)\tilde{\lambda}(t)}{(4\pi)^2} \right)^{-1} + \frac{1}{2} \bar{m}^2(t)\tilde{\phi}^2(t) \left( 1 - \frac{3(s - t)\tilde{\lambda}(t)}{(4\pi)^2} \right)^{-1/3}$$

$$- \frac{\bar{m}^4(t)}{2\lambda(t)} \left( 1 - \frac{3(s - t)\tilde{\lambda}(t)}{(4\pi)^2} \right)^{1/3} + \frac{\bar{m}^4(t)}{2\lambda(t)} + \frac{1}{2(4\pi)^2} \int_0^t dt' \bar{m}^4(t') + \Lambda,$$

where

$$s = \frac{\hbar}{2} \ln \frac{H}{\mu^2} \quad \text{and} \quad t = \frac{\hbar}{2} \ln \frac{F}{\mu^2}.$$

As in the $O(N)$ scalar case we discarded non-leading logarithms. At this point it is appropriate to compare this calculation with the work of ref. [7]. In their treatment of the Yukawa model they considered two cases:

i) $m^2 << F$.

ii) $m^2 >> F$.

In case i) they noted that since ln($H/F$) can be considered small, one is entitled to write ln($H/\mu^2$) = ln($F/\mu^2$) + ln($H/F$), and sum up the ln($F/\mu^2$) terms in the usual way (ie. using the method reviewed in section 2).

In case ii) their treatment had similarities to the calculation presented here. Their improved potential was obtained by solving the RG with boundary conditions specified at $\mu^2 = F$. They also noted that one could not use the tree potential as a boundary condition, and as here they employed “improved” boundary conditions. Their boundary condition for the leading logarithms read

$$L(\mu) = \frac{\tilde{\lambda}}{24} \tilde{\phi}^4 + \frac{1}{2} \bar{m}^2\tilde{\phi}^2 + \bar{\Lambda} \quad \text{at} \quad \mu^2 = F,$$

(4.12)
where

\[ \tilde{\lambda} = \lambda + \frac{3\lambda^2 h}{2(4\pi)^2} \ln \frac{m^2}{\mu^2}, \quad \tilde{m}^2 = m^2 + \frac{\lambda m^2 h}{2(4\pi)^2} \ln \frac{m^2}{\mu^2}, \quad \tilde{\phi} = \phi \]

\[ \tilde{\Lambda} = \Lambda - \frac{m^4 h}{4(4\pi)^2} \ln \frac{m^2}{\mu^2}. \]  

(4.13)

The choice of boundary condition was motivated by ideas from effective field theory, where the scalar particle is treated as a very heavy particle of mass \( m \). The effects of the heavy particle lead to a shift of the parameters of the low-energy theory. However, it is arguable that eq. (4.12) is not really valid for \( \mu^2 = F \), since in the regime \( m^2 >> F \) terms of the form \( h^2 \ln^2 (m^2/F) \), \( h^3 \ln^3 (m^2/F) \), etc. (which are neglected in eq. (4.13)) will be large, and should be included in any calculation of the leading logarithms. Although the boundary condition used in ref. [7] seems to be questionable for \( H >> F \), it must be emphasized that the formula presented here (eq. (4.11)) may also break down at some higher order in perturbation theory. This is because our boundary condition (eq. (4.9)) is a conjecture, which does hold at the tree, one-loop and two-loop level. It is plausible (but not certain) that it survives to all orders in perturbation theory.

5. Discussion

We have presented a calculation of the leading logarithms of the EP for two simple theories. These theories are perhaps the simplest renormalisable theories where the perturbation series contains more than one logarithm. However, our calculations were based on an unproven conjectured property of the leading logarithms, which means they could break down at some power of Planck’s constant. Moreover, we were unable to (even in principle) sum the sub-leading logarithms,
since we have no corresponding conjecture concerning the sub-leading logarithms (in contrast to the single scale case, where it is known how to sum the sub-leading logarithms, and has been done explicitly for $\phi^4$ theory [5]). But it could be that there is a more systematic procedure that will reproduce eqs. (3.14) and (4.11), without the need for such conjectures. One possibility would be to introduce extra renormalisation scales [16], which lead to several RG equations. In this approach it may be possible to use one of the “partial” RG equations to obtain improved boundary conditions for the conventional RG equation (although the beta functions will depend on the ratios of the scales, beyond one loop). Alternatively, it may be possible to justify the boundary conditions used here by a careful analysis of the properties of the Feynman diagrams contributing to the EP.

Despite the rather unsystematic nature of our procedure it is quite easy to extend it to theories with more than two logarithms. For example, consider $N$ scalars interacting with $M$ fermions via an $O(M)$ invariant Yukawa coupling (assume that in the absence of the Yukawa term we also have the usual $O(N)$ invariant action for the scalars). Now the EP for this theory will have three logarithms ($\ln(H/\mu^2)$, $\ln(G/\mu^2)$ and $\ln(F/\mu^2)$). As a boundary condition for the RG we could use eq. (3.14) at $\mu^2 = F$. However things may not be so simple in gauge theories, since one does not have total freedom to vary the number of massive gauge bosons and scalars. This is because we must have sufficient Goldstone bosons to “feed” the vectors with masses.

In the case of more realistic theories (such as the Standard Model) one is forced to use numerical methods, since the running parameter equations can be very complicated (even at one-loop). The standard model effective potential has several logarithms (five, if one neglects all Yukawa couplings, except that involving the top quark), but in this case it is not always necessary to sum them all separately. In refs. [1,2,3] where bounds for the masses of the top and Higgs were obtained via the assumption of electroweak vacuum stability, the form of the EP was required in the region $\phi >> M_Z$. In this regime one certainly has large logarithms to sum,
fortunately for these large $\phi$ values, the differences between the five logarithms can be considered small (i.e. all the logarithms are well approximated by $t = \bar{h} \ln(\phi/\mu)$). Thus, in this case there is only one logarithm to sum up, which can be done in the usual way. However, many extensions of the standard model do possess additional scales. In such cases one could apply the methods described here by solving the RG in the usual way but with improved boundary conditions. These improved boundary conditions (analogous to eqs. (3.7) and (4.9)) could be determined numerically.

Acknowledgements

I am grateful to B. Dolan, J. Gracey, D. R. T. Jones, F. Krahe, L. O’Raifeartaigh and C. Stephens for useful discussions. Thanks also to the referee for a useful suggestion.

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