Matrix elements of vertex operators and fermionic limit of spin Calogero-Sutherland system

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Abstract
We present a construction of an integrable model as a projective type limit of spin Calogero-Sutherland model with $N$ fermionic particles, where $N$ tends to infinity. It is implemented in the multicomponent fermionic Fock space. Explicit formulas for limits of Dunkl operators and the Yangian generators are presented by means of fermionic fields.

Keywords: Dunkl operators, fermionic fields, Calogero-Sutherland system, Fock space

1. Introduction

Recent researches on limits of Calogero-Sutherland (CS) systems [15, 16, 18] followed the cornerstones of the theory of symmetric functions, where the ring $\Lambda$ of symmetric functions on the one side is regarded as the projective limit of the spaces of symmetric polynomials in finite number of variables [11], and on the other side is a representation of the Heisenberg algebra $H$ and thus can be related to the Fock space. Operators, forming the Hamiltonians of the finite CS system can be rewritten in a way admitting a translation in the Fock space.

These constructions heavily use an equivariant family of Heckman–Dunkl [6] operators as a counterpart of the Lax operator in integrable systems. The procedure requires a creation of an auxiliary space of functions symmetric over all variables except one [15, 16] or as a variant of polynomials in one variable with coefficients being symmetric functions of all variables [14, 18]. These spaces are created by the vertex operator

$$\Phi_+(z) = \exp \sum_{n \neq 0} z^n \frac{\partial}{\partial p_n}$$
which use in the finite system is nothing more than the Taylor expansion. Here $p_n = \sum x_k^n$ are Newton polynomials. Instead of taking trace of the Lax operator here one sums up the images of all powers of Dunkl operators. The latter is equivalent to the integration with the weight function $\varphi_-(z)$ being the negative part of the bosonic field

$$\varphi_-(z) = \sum_{n>0} \frac{p_n}{z^n}.$$  

The spin CS system has more symmetries, which constitute the representation of the Yangian $\text{Y}(\mathfrak{gl}_s)$. The integrals of motion of the system can be chosen as coefficients of the related quantum determinant. We provide a description of finite CS system in section 2. This limiting bosonic construction system was generalized in [9] to spin CS system using the language of polysymmetric functions, that is polynomials, symmetric on the groups of variables. See also [14].

In [8] analogous ideas were applied to CS system of fermionic particles. The finite object is the same—the ring $\Lambda$ of symmetric functions, but its finite reincarnations are different: these are antisymmetric polynomials. Then the Vandermonde $\prod_{i<j}(x_i - x_j)$ brings the appearance of the full vertex operator

$$\Psi(z) = z^{r_0} \exp \left( \sum_{n>0} \frac{p_n}{n^n} \right) \exp \left( \sum_{n>0} z^n \frac{\partial}{\partial p_n} \right),$$

in the creation of the auxiliary space. The weight function of the integration is now

$$\Psi^*(z) = z^{-r_0} \exp \left( \sum_{n>0} \frac{p_n}{n^n} \right) \exp \left( \sum_{n>0} z^n \frac{\partial}{\partial p_n} \right).$$

The fermionic limit of spin CS system was studied by Uglov. He suggested two constructions of integrable systems in a Fock space starting from finite-dimensional spin CS model. The first one [19] realizes a projective type limit, the second [20] deals with inductive limit. In this paper we suggest and develop another approach, which leads to the limiting integrable system closely related to [19], but realized by free fermionic fields. We start from the fermionic Fock space $F^s$, generated by $s$ fermion fields $\Psi_c(z), c = 1, \ldots, s$ and define the phase space of the finite-dimensional spin CS system as the space of matrix coefficients

$$\pi_N (|v\rangle) = \langle 0|\Psi(z_N)\Psi(z_2)\cdots\Psi(z_1)|v\rangle, \quad |v\rangle \in F^s,$$

where $\Psi(z) = \sum_{c=1}^s \Psi_c(z) \otimes e_c$ and $e_c \in C^s$ are basic vectors of $C^s$. Then we systematically construct the pullback with respect to the maps $\pi_N$ of all operation required for the construction of the Yangian action on the finite-dimensional spin CS system. Finally this gives the Yangian action on the Fock space $F^s$, which is a pullback of the Yangian action on finite-dimensional CS system. In particular, this includes the construction of commuting Hamiltonians in $F^s$. Note the importance of the polynomial property of the total zero mode in the constructed Yangian action on the Fock space $F^s$, which we prove by using projective properties of the Yangian action in the phase spaces of CS models, described in section 4.

The same language can be applied for the consideration of bosonic limit of CS system constructed previously in [9]. In this case the bosonic projection map,

$$\bar{\pi}_N (|v\rangle_+) = \langle 0|\Phi(z_N)\Phi(z_2)\cdots\Phi(z_1)|v\rangle_+, \quad |v\rangle_+ \in \tilde{\Lambda}^s,$$
actually coincides with that used in [2, 3]. We include this case in section 3, since it can clarify both the expositions of [9] and of the main ideas of the present work given in section 5. In section 6 we discuss the connections of our approach with the results of Uglov [19].

2. Spin Calogero-Sutherland system

The phase space of the quantum spin Calogero-Sutherland (CS) system consists of functions with values in vector space \((C^s) \otimes \mathbb{N}\) while the dependence on spin in the Hamiltonian

\[
H_{\text{CS}} = -\sum_{i=1}^{N} \left( \frac{\partial}{\partial q_i} \right)^2 + \sum_{i,j=1}^{N} \frac{\beta (\beta - K_{ij})}{\sin^2(q_i - q_j)}
\]

is implicit [7]. Here \(K_{ij}\) is the coordinate exchange operator of particles \(i\) and \(j\). After conjugating by the function \(\prod_{i<j} |\sin(q_i - q_j)|^\beta\) which represents the degenerated vacuum state, and passing to the exponential variables \(x_i = e^{2\pi i q_i}\) and the parameter \(\alpha = \beta^{-1}\) more common in mathematical literature, we arrive after simple rescaling to the effective Hamiltonian

\[
H = \alpha \sum_{i=1}^{N} \left( x_i \frac{\partial}{\partial x_i} \right)^2 + \sum_{i<j} x_i x_j \left( x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j} \right) - 2 \sum_{i<j} \frac{x_i x_j}{x_i - x_j} (1 - K_{ij}),
\]

which we restrict to the spaces \(\Lambda^N_\pm\) of total invariants or respectively skew invariants of the symmetric group \(S_N\) in the space \(V^\otimes \mathbb{N}\).

\[
\Lambda^N_\pm = (V^\otimes \mathbb{N})^{(\pm)}.
\] (2.1)

Here\(^4\) \(V = C[z] \otimes \mathbb{C}\). The (skew)invariants are taken with respect to the diagonal action of the symmetric groups, \(\sigma_{ij} \mapsto K_{ij} P_{ij}\), where \(K_{ij}\) is as above and \(P_{ij}\) is the permutation of \(i\)th and \(j\)th tensor copy of the vector space \(C^s\).

Further we use the Heckman–Dunkl operators \(D_i^{(N)} : V \otimes \Lambda^N_{\pm} \rightarrow V \otimes \Lambda^N_{\pm}\) in the form suggested by Polychronakos [17]:

\[
D_i^{(N)} = \alpha x_i \frac{\partial}{\partial x_i} + \sum_{j \neq i} \frac{x_i}{x_i - x_j} \left( 1 - K_{ij} \right).
\] (2.2)

These operators satisfy the relations

\[
K_{ij} D_i^{(N)} = D_j^{(N)} K_{ij},
\]

\[
[D_i^{(N)}, D_j^{(N)}] = (D_j^{(N)} - D_i^{(N)}) K_{ij},
\]

which coincide with the relations of the degenerate affine Hecke algebra \(H_N\). By Drinfeld duality [5], this representation of degenerate affine Hecke algebra transforms to the representation of the Yangian \(Y(gl_i)\) in \(\Lambda^N_\pm\), see [1, 10]

\[
t_{ab}(u) = \delta_{ab} + \sum_{i \neq u} E_{abj} D_i^{(N)},
\] (2.3)

\(^4\)We also use the notation \(V = V(z)\) when we need to specify the name of the variable.
Here $E_{abj}$ describes the action of $\mathfrak{gl}_s$ on $i$th tensor component,

$$E_{abj} \left( \cdots \otimes (x^a \otimes x^b) \otimes \cdots \right) = \delta_{bc} \left( \cdots \otimes (x^a \otimes x^d) \otimes \cdots \right).$$

and $t_{ab}(u)$, $a, b = 1, \ldots, s$,

$$t_{ab}(u) = \delta^{ab} + \sum_{i=0}^{\infty} t_{abj} u^{-i-1}$$

are generating functions of the generators $t_{abj}$ of the Yangian $\mathcal{Y}(\mathfrak{gl}_s)$. The defining relations of $\mathcal{Y}(\mathfrak{gl}_s)$ are [13]

$$[t_{ab}(u), t_{cd}(v)] = t_{bc}(u) t_{ad}(v) - t_{bc}(v) t_{ad}(u) - \frac{t_{abcd}(u)v - t_{abcd}(v)u}{u - v}.$$  \hspace{1cm} (2.5)

Then the higher Hamiltonians of spin CS system can be chosen as coefficients of the quantum determinant

$$q \det t(u) = \sum_{\sigma \in \mathcal{S}_m} (-1)^{\text{sgn}(\sigma)} t_{\sigma(1),1}(u) t_{\sigma(2),2}(u - 1) \cdots t_{\sigma(m),m}(u - m + 1).$$

which generate the center of the $\mathcal{Y}(\mathfrak{gl}_s)$ [4, 13].

The main goal of the present paper is to construct the limit of the above Yangian action when $N$ tends to infinity. In particular, we get the limits of the above commuting family of Hamiltonians.

3. Bosonic limit

In this section we observe the results of [9] using slightly different language.

Denote by $\Lambda^{(s)}$ the free unital associative commutative algebras generated by the elements

$$p_{a,k}, \quad a = 1, \ldots, s, \quad k = 1, 2, \ldots$$

The ring $\Lambda^{(s)}$ can be viewed as the ring of polysymmetric functions, that is the projective limit of polynomial functions on the variables $x_{11}, x_{12}, \ldots, x_{1n}, x_{21}, x_{22}, \ldots, x_{2n}, \ldots, x_{sn}$, symmetric on each group of variables $x_{a1}, \ldots, x_{an}, a = 1, \ldots, s$. Here $p_{a,k}$ corresponds to the Newton sums $x_{a1}^k + x_{a2}^k + \cdots$. Denote by $\hat{\Lambda}^{(s)}$ the free unital associative commutative algebras generated by the elements

$$\hat{p}_{a,k}, \quad a = 1, \ldots, s, \quad k = 0, 1, 2, \ldots$$

We have $\hat{\Lambda}^{(s)} \supset \Lambda^{(s)}$. Additional ‘zero modes’ $\hat{p}_{a,0}$ will further serve to count the numbers of variables in each group.

Let $\mathcal{H}^{\ell}$ be the Heisenberg algebra with generators $a_{c,k}, c = 1, \ldots, s, k = 0, 1, \ldots$ and $(q_{c})^{\pm1}$, which satisfy the relations

$$[a_{c,k}, a_{d,l}] = k \delta_{cd} \delta_{k,-l}, \quad q_c a_{d,k} = (a_{d,k} + \delta_{cd} \delta_{k,0}) q_c.$$  \hspace{1cm} (3.6)

The space $\hat{\Lambda}^{(s)}$ is a representation of the Heisenberg algebra $\mathcal{H}^{\ell}$, where

$$a_{c,k} \mapsto p_{c,-k}, \quad k \leq 0, \quad a_{c,k} \mapsto k \frac{\partial}{\partial p_{c,k}}, \quad k > 0, \quad q_c \mapsto e^{\frac{q_c}{\hbar}}.$$
The unit of the ring $\hat{\Lambda}^{(s)}$ is then identified with the vacuum vector $|0\rangle_+$, so that
\begin{equation}
 a_{c,k}|0\rangle_+ = 0, \quad c = 1, \ldots, s, \quad k > 0, \quad q_{c}|0\rangle_+ = |0\rangle_+, \quad c = 1, \ldots, s.
\end{equation}

Denote by $+|0\rangle$ the vector of the dual space, which satisfies the relations
\begin{equation}
 +|0\rangle a_{c,k} = 0, \quad c = 1, \ldots, s, \quad k \leq 0.
\end{equation}

For $c = 1, \ldots, s$ denote by $\varphi_c^{-}(z)$ the series
\begin{equation}
 \varphi_c^{-}(z) = \sum_{n \leq 0} a_{c,n} z^n
\end{equation}
and by $e^+_c$ the linear operator $C^s \to C$ given by the relation
\begin{equation}
 e^+_c = \delta_{bc}.
\end{equation}

Define linear operators
\begin{align*}
 \Phi_c(z) &= \exp\left(\sum_{n>0} \frac{a_{c,n}}{n} z^n\right) q_c: \quad \hat{\Lambda}^{(s)} \to \hat{\Lambda}^{(s)} \otimes \mathbb{C}[z], \quad c = 1, \ldots, s, \\
 \Phi(z) &= \sum_c c_{\Phi_c(z)} \otimes e^+_c: \quad \hat{\Lambda}^{(s)} \to \hat{\Lambda}^{(s)} \otimes V, \\
 \Phi^*(z) &= \sum_c \varphi_c^{-}(z) \cdot \Phi_c^{-1}(z) \otimes e^+_c: \quad \hat{\Lambda}^{(s)} \otimes V \to \hat{\Lambda}^{(s)} \otimes \mathbb{C}[z].
\end{align*}

For instance, for any $|v\rangle_+ \in \hat{\Lambda}^{(s)}$
\begin{equation}
 \Phi^*(z)(|v\rangle_+ \otimes z^k \otimes e_c) = z^k \varphi_c^{-}(z) \Phi_c^{-1}(z)|v\rangle_+,
\end{equation}
where
\begin{equation}
 \Phi_c^{-1}(z) = q_c^{-1} \exp\left(\sum_{n>0} \frac{a_{c,n}}{n} z^n\right).
\end{equation}

For any $|v\rangle_+ \in \hat{\Lambda}^{(s)}$ consider the matrix element $\hat{\pi}_N(|v\rangle_+ \in V^\otimes N$, 
\begin{equation}
 \hat{\pi}_N(|v\rangle_+ = +|0\rangle(\Phi_1(z_N) \otimes 1^\otimes (N-1)) \cdots (\Phi_{s}(z_2) \otimes 1)\Phi_{1}(z_1)|v\rangle_+.
\end{equation}

which we shortly denote by
\begin{equation}
 \hat{\pi}_N(|v\rangle_+ = +|0\rangle(\Phi(z_N) \Phi(z_2) \cdots \Phi(z_1)|v\rangle_+.
\end{equation}

In components,
\begin{equation}
 \hat{\pi}_N(|v\rangle_+ = \sum_{c_1, \ldots, c_N = 1}^{s} +|0\rangle(\Phi_{c_N}(z_N) \cdots \Phi_{c_1}(z_1)|v\rangle_+ \cdot e_{c_N} \otimes \cdots \otimes e_{c_1}.
\end{equation}

The commutativity
\begin{equation}
 \Phi_{b}(z_1)\Phi_{c}(z_2) = \Phi_{c}(z_2)\Phi_{b}(z_1),
\end{equation}
\begin{equation}
\Phi_{c}(z_1)\Phi_{c}(z_2) = \Phi_{c}(z_2)\Phi_{c}(z_1).
\end{equation}
implies that the matrix element (3.10) belongs to the space \( \Lambda^{\otimes N+} \). Indeed,

\[
\sigma_{ij} \left( \sum_{c_1,\ldots,c_N=1}^j + \langle 0 | \cdots | \Phi_{c_j}(z_j) \cdots \Phi(z_1) \cdots | v \rangle \cdots \otimes e_{c_j} \otimes \cdots \otimes e_i \otimes \cdots \right)
\]

\[
= \sum_{c_1,\ldots,c_N=1}^j + \langle 0 | \cdots | \Phi_{c_j}(z_j) \cdots | v \rangle \cdots \otimes e_{c_j} \otimes \cdots \otimes e_i \otimes \cdots
\]

In the last equality we change the indices of summation \( c_i \) by \( c_j \).

Our goal is to pull back the Yangian action \((2.3)\) in \( \Lambda_+^{\otimes N} \) through the map \( \bar{\pi}_N \). The dissection of the relation \((2.3)\) shows that the application of each Yangian generator to a vector \(|v\rangle_+ \in \Lambda_+^{\otimes N} \) can be decomposed into several steps. First we present the symmetric tensor \(|v\rangle_+ \in \Lambda_+^{\otimes N} \) as an element of \((C[x_1] \otimes C^\circ) \otimes \Lambda_+^{\otimes N-1} \) for each tensor component, producing an equivariant family of vectors, which can be completely described by the element of \( V \otimes \Lambda_+^{\otimes N-1} \sim (C[x_1] \otimes C^\circ) \otimes \Lambda_+^{\otimes N-1} \)—the decomposition of \(|v\rangle_+ \) over the first tensor component. Then we apply the power of Heckman operator \( D_i^{(N)} \) to the \( i \)th vector of this equivariant family and get another equivariant family. The last step is the symmetrization—the sum of all members of the equivariant family.

Denote by \( t_N : \Lambda_+^{\otimes N} \to V \otimes \Lambda_+^{\otimes N-1} \) the decomposition of the symmetric tensor \( v \) over the first tensor component,

\[
t_N \left( \sum_k f_{1k}(z) \otimes \cdots \otimes f_{Nk}(x_N) \right) = \sum_k f_{1k}(z) \otimes (f_{2k}(x_2) \otimes \cdots \otimes f_{Nk}(x_N)). \tag{3.12}
\]

Here, \( f_{1k}(z) \) and \( f_{jk}(x_k), j > 1 \) are \( C^\circ \) valued polynomials.

**Lemma 3.1.** We have the following equality of linear maps \( \hat{\Lambda}^{\otimes} \to \Lambda_+^{\otimes N} :\)

\[
(\bar{\pi}_{N-1} \otimes 1) \Phi(z) = t_N \bar{\pi}_N. \tag{3.13}
\]

**Proof.** Applying both sides of \((3.13)\) to a vector \(|v\rangle_+ \in \hat{\Lambda}^{\otimes} \) we get the tautology: both sides are equal to

\[
+ \langle 0 | \Phi(x_N) \Phi(x_{N-1}) \cdots \Phi(x_2) \Phi(z) | v \rangle_+.
\]

\(\blacksquare\)

For each tensor \( u \in V \otimes \Lambda_+^{\otimes N-1} \), symmetric with respect to diagonal permutations of all tensor factors except the first, denote by \( E_N(u) \) its total symmetrization

\[
E_N(u) = \sum_{j=1}^N \sigma_j(u), \tag{3.14}
\]

where \( \sigma_j = K_j P_j \) is the permutation of \( i \)th and \( j \)th tensor factors. On the other hand, for each \( F(z) \in \Lambda^{\otimes} \otimes V \) define the element \( S(F(z)) \in \hat{\Lambda}^{\otimes} \) as the formal integral

\[
S(F(z)) = \frac{1}{2\pi i} \oint \frac{dz}{z} \Phi^*(z) F(z), \tag{3.15}
\]
which counts zero term of the Laurent series. The following lemma establishes the map \( S \) as the pullback of the finite symmetrization. This is the crucial point of the construction.

**Lemma 3.2.** For each \( F(z) \in \hat{\Lambda}^{(s)} \otimes V \) and any natural \( N \) we have the equality of elements of \( \Lambda_{s,N} \),

\[
E_N(\pi_{N-1} \otimes 1)F(z) = \pi_N S(F(z)).
\] (3.16)

**Proof.** Let \( F(z) \) have the form

\[
F(z) = \sum_{c=1}^{s} F_c(z) \otimes e_c, \quad F_c(z) \in \hat{\Lambda}^{(s)} \otimes C[z].
\]

Consider first the LHS of (3.16). This is the symmetrization (3.14) of the tensor

\[
\sum_{c_1,\ldots,c_N=1}^{s} \langle 0 | \Phi_{c_N}(x_N) \cdots \Phi_{c_2}(x_2) F_{c_1}(x_1) \cdot e_{c_N} \otimes \cdots \otimes e_{c_1},
\]

which can be written by means of proper changes of summation indices as the sum

\[
\sum_{k=1}^{N} \sum_{c_1,\ldots,c_N=1}^{s} \langle 0 | \Phi_{c_N}(x_N) \cdots \Phi_{c_{k+1}}(x_{k+1}) \Phi_{c_{k-1}}(x_{k-1}) \cdots \Phi_{c_2}(x_2) \cdots F_{c_k}(x_k) \cdot e_{c_{N}} \otimes \cdots \otimes e_{c_1}.
\]

Inserting in each summand the corresponding product

\[
1 = \Phi_{c_k}(x_k) \Phi^{-1}_{c_k}(x_k)
\]

and using the commutativity (3.11) we rewrite it as

\[
\sum_{k=1}^{N} \sum_{c_1,\ldots,c_N=1}^{s} \langle 0 | \prod_{j=1}^{N} \Phi_{c_j}(x_j) \cdot \Phi^{-1}_{c_k}(x_k) F_{c_k}(x_k) \cdot e_{c_{N}} \otimes \cdots \otimes e_{c_1}
\]

\[
= \sum_{k=1}^{N} \sum_{c_1,\ldots,c_N=1}^{s} \frac{1}{2\pi i} \int_{\|\gamma\|} \langle 0 | \prod_{j=1}^{N} \Phi_{c_j}(x_j) \cdot \frac{\Phi^{-1}_{c_k}(z) F_{c_k}(z)}{z - x_k} \cdot e_{c_{N}} \otimes \cdots \otimes e_{c_1}, \quad (3.17)
\]

The RHS of (3.16) is

\[
\frac{1}{2\pi i} \sum_{j=1}^{N} \langle 0 | \prod_{j=1}^{N} \Phi(x_j) \cdot \oint \frac{dz}{z} \Phi^{*}(z) F(z).
\]

In components it looks as

\[
\frac{1}{2\pi i} \sum_{a=1}^{s} \sum_{c_1,\ldots,c_N=1}^{s} \langle 0 | \prod_{j=1}^{N} \Phi_{c_j}(x_j) \cdot \oint \frac{dz}{z} \varphi_a^{-1}(z) F_a(z) \cdot e_{c_{N}} \otimes \cdots \otimes e_{c_1}, \quad (3.18)
\]

The normal ordering of the above matrix elements assumes due to (3.8) the move of all \( \varphi_a^{-1}(z) \) to the left vacuum using the relation

\[
\Phi_{c}(x) \varphi_a^{-1}(z) = \left( \varphi_a^{-1}(z) + \frac{\delta_{ac}}{1 - z} \right) \Phi_{c}(x), \quad (3.19)
\]
which follows from (3.6). In particular, the formal integral in (3.18) can be regarded as a contour integral, where the contour \( C \) of integration encloses all points \( x_j \). Since
\[ 
\frac{1}{2\pi i} \oint_{C_j \subset C} + (0) \varphi_j^-(z) = 0,
\]
we arrive to the expression
\[ 
\sum_{\ell=1}^{N} \sum_{c_1, \ldots, c_N=1}^{s} \frac{1}{2\pi i} \oint_{C_j \subset C} + (0) \prod_{j=1}^{N} \Phi_{c_j}(x_j) \int_{\ell=1}^{N} \frac{dz}{z} \Phi^{-1}_{c_j}(z) F_{c_j}(z) \cdot e_{c_N} \otimes \ldots \otimes e_{c_1},
\]
which is identical to (3.17).

We now apply statements of lemmas 3.1 and 3.2 for the construction of a pullback of the Dunkl operator.

Let \( \mathcal{D} : \hat{\Lambda}^{(i)} \otimes V \to \hat{\Lambda}^{(i)} \otimes V \) be the linear map, such that for any \( F(z) \in \hat{\Lambda}^{(i)} \otimes V \)
\[ 
\mathcal{D} F^{(i)}(z) = \alpha \frac{d}{dz} F^{(i)}(z) + \frac{z}{2\pi i} \int \frac{d\xi}{\xi^2(1 - \xi^2) \Phi^{(2)}(\xi) \Phi^{(2)}(z) F^{(i)}(\xi)}
\]
Here the upper index \( (i) \), \( i = 1, 2 \) indicates in which tensor copy of \( \mathbb{C}^i \) the corresponding vector lives or an operator acts. In components,
\[ 
\mathcal{D}(F_{a}(z) \otimes e_{a}) = \left( \alpha \frac{d}{dz} F_{a}(z) + \frac{z}{2\pi i} \sum_{c=1}^{s} \frac{d\xi}{\xi^2(1 - \xi^2) \Phi^{(2)}(\xi) \Phi^{(2)}(z) F_{c}(\xi)} \right) \otimes e_{a}
\]
We state that the operator \( \mathcal{D} \) is the pullback of the equivariant family of Heckman operators \( \mathcal{D}^{(N)} \).

**Proposition 3.1.** For any \( F(z) \in \hat{\Lambda}^{(i)} \otimes V \) we have
\[ 
(\#_{N-1} \otimes 1) \mathcal{D}(F(x_1)) = \mathcal{D}^{(N)}(\#_{N-1} \otimes 1) F(x_1)
\]

**Proof.** The only nontrivial part is the pullback of the difference part of the Heckman operator. The difference part \( \mathcal{D}^{(N)} \) of Heckman operator \( \mathcal{D}^{(N)} \) in the space \( V(x_1) \otimes \Lambda^{N-1}_{\omega} \), where \( V(x_1) = \mathbb{C}[x_1] \otimes \mathbb{C}^s \), can be described as the composition of three operations. First we include into \( V(x_1) \otimes V(x_2) \otimes \Lambda^{N-2}_{\omega} \) by means of \( 1 \otimes t_{N-1} \), then apply the operator \( \frac{1 - K_{21}}{x_1 - x_2} \) and finally sum up over all the variables except \( x_1 \) by means of the summation \( E_{N-1} \),
\[ 
\mathcal{D}^{(N)} = E_{N-1} \circ \frac{1 - K_{21}}{x_1 - x_2} \circ 1 \otimes t_{N-1}.
\]
The pullback of the inclusion \( 1 \otimes t_{N-1} \) is \( \Phi^{(2)}(x_2) \) due to lemma 3.1, the pullback of \( E_{N-1} \) is \( \frac{1}{x_1} \frac{d}{dx_1} \Phi^{(2)}(x_2) \), the pullback of the operator \( \frac{1 - K_{21}}{x_1 - x_2} \) is this very operator \( \frac{1 - K_{21}}{x_1 - x_2} \). We see that the pullback of the difference operator \( \mathcal{D}^{(N)} \) has the form
\[ 
\mathcal{D} F^{(1)}(x_1) = \frac{x_1}{2\pi i} \int \frac{dx_2}{x_2} \Phi^{(2)}(x_2) \frac{\Phi^{(2)}(x_2) F^{(1)}(x_1) - \Phi^{(2)}(x_1) F^{(1)}(x_2)}{x_1 - x_2}.
\]
Any matrix element of the ratio inside the integral is a polynomial on \( x_1 \) and \( x_2 \) and can be equally decomposed into a series either in the region \( |x_1| < |x_2| \) or in the region \( |x_1| > |x_2| \). In the region \( |x_1| < |x_2| \) in the first integral,
\[ 
\frac{x_1}{2\pi i} \int \frac{dx_2}{x_2} \Phi^{(2)}(x_2) \frac{\Phi^{(2)}(x_2) F^{(1)}(x_1)}{x_1 - x_2} = \sum_{c=1}^{s} \frac{x_1}{2\pi i} \int \frac{dx_2}{x_2} \Phi^{(2)}(x_2) \frac{F^{(1)}(x_1)}{x_1 - x_2}
\]
we have only negative powers of $x_2$ and this integral vanish. Thus we get (3.20).

Let $E_{ab} \in \text{End } C^s$, be the matrix unit, $E_{ab}(e_i) = \delta_{bc} e_a$. Denote by $\mathcal{E}_{ab}$, the operator $1 \otimes 1 \otimes E_{ab} : \Lambda^{(s)} \otimes V \to \Lambda^{(s)} \otimes V$:

$$\mathcal{E}_{ab} F(z) = F_b(z) \otimes e_a.$$  

For $a,b = 1, \ldots, s$ and $n = 1, \ldots, s$ set

$$T_{ab,n} = \frac{(-1)^n}{2\pi i} \oint \frac{dz}{z} \Phi^*(z) \mathcal{E}_{ab} D^n \Phi(z). \quad (3.23)$$

Summarizing the statements above we get the following result [9].

**Proposition 3.2.** The operator $T_{ab,n}$, see (3.23) is the pullback of the Yangian generator $t_{ab,n}$, see (2.3) and (2.4):

$$\mathcal{p}_N T_{ab,n} = t_{ab,n} \mathcal{p}_N \text{ for any } \mathcal{p}_N \in \mathbb{N}.$$  

In particular, the operators (3.23) form level zero representation of the Yangian $Y(\mathfrak{gl}_s)$ in $\hat{\Lambda}^{(s)}$. Here we use the property

$$\cap_{N \in \mathbb{N}} \ker \mathcal{p}_N = 0 \quad (3.24)$$

of the ring of symmetric functions which we assume to be known.

### 4. Projective properties of Yangian action

1. In the previous section we explored the Yangian action in the phase space of bosonic spin CS system, using equivariant family of non-commuting Heckman–Dunkl operators. The alternative multiplicative presentation of the same action, using the commutative family of Dunkl operators, suits for analysis of projective properties of Yangian actions in phase spaces of CS models. Such an analysis was done by Uglov in [19], but our description differs from that of [19].

There are two natural commuting families in degenerated affine Hecke algebra $H_N$. The first one is formed by commuting operators $\varepsilon_i$:

$$\varepsilon_i = \alpha x_i \frac{\partial}{\partial x_i} + \sum_{j < i} \frac{x_j}{x_i - x_j} (1 - K_{ij}) + \sum_{i < j} \frac{x_i}{x_i - x_j} (1 - K_{ij}) + (i - 1).$$

They commute, and satisfy the relations

$$K_{i,i+1} \varepsilon_i = \varepsilon_{i+1} K_{i,i+1} - 1. \quad (4.1)$$

Another family is formed by the elements

$$d_i = \alpha x_i \frac{\partial}{\partial x_i} + \sum_{j < i} \frac{x_j}{x_i - x_j} (1 - K_{ij}) + \sum_{i < j} \frac{x_i}{x_i - x_j} (1 - K_{ij}) + (N - i). \quad (4.2)$$

The elements $d_i$ satisfy relations

$$K_{i,i+1} d_i = d_{i+1} K_{i,i+1} + 1, \quad (4.3)$$

**5** Here and further we omit upper index in $\varepsilon_i^{(N)}$ and in $d_i^{(N)}$ assuming their dependence of $N$ variables.
and are related to $\varepsilon_i$ as
\[
d_i = K_0 \varepsilon_{N-i} K_0,
\]
where $K_0(x_i) = x_{N-i}$ represents the permutation of coordinates, associated to the longest element of the symmetric group. Heckman operators $D_i$, see (2.2), which we use, are related to the above families by the relations
\[
D_i = K_1 \varepsilon_i K_1 = K_i N d_N K_N, \quad D_i = \varepsilon_i - \sum_{j<i} K_{ij} = d_i - \sum_{j>i} K_{ij}.
\]

Let
\[
T(u) = \sum_{a,b=1}^i E_{ab} \otimes t_{ab}(u) \in \text{End}(\mathbb{C}^r) \otimes Y(gl_r)[u^{-1}]
\]
be the generating matrix of Yangian generators. Then the prescription
\[
\xi_a : T(u) \rightarrow 1 + \frac{f^{(0)}_{01}}{u - a}
\]
describes the evaluation homomorphism $\xi_a : Y(gl_r) \rightarrow U(gl_r)$. Here, $f^{(0)} = \sum_{a,b=1}^r E_{ab}^{(0)} \otimes E_{ab}^{(1)}$. The upper index in (4.6) specifies the tensor component. Since Yangian is the Hopf algebra and the operators $\varepsilon_k : \mathbb{C}[z]^{\otimes N} \rightarrow \mathbb{C}[z]^{\otimes N}$ commute, the assignment
\[
T(u) \mapsto T_y(u) = \left( 1 + \frac{f^{(0)}_{01}}{u \pm \varepsilon_1} \right) \cdots \left( 1 + \frac{f^{(0)}_{0N}}{u \pm \varepsilon_N} \right)
\]
determines a representation of $Y(gl_r) \in V^{\otimes N} \simeq (\mathbb{C}^r)^{\otimes N} \otimes \mathbb{C}[z_1, \ldots, z_N]$. It is known [1, 5], that this Yangian action preserves the subspace $U_{sN}^+ = \sum_{i=1}^{N-1} (s_{i+1} \mp 1) V^{\otimes N}$ and thus equips the space
\[
\tilde{\Lambda}_{sN}^+ = V^{\otimes N} / U_{sN}^+
\]
of $S_N$—(skew)coinvariants of $V^{\otimes N}$ with the structure of $Y(gl_r)$—module. Conjugation of the RHS of (4.7) by means the longest element $w_0 = K_0 P_0$ of the symmetric group $S_N$ gives another presentation of the Yangian action in $\tilde{\Lambda}_{sN}^+$:
\[
T_y(u) = \left( 1 + \frac{f^{(0)}_{0N}}{u \pm d_N} \right) \cdots \left( 1 + \frac{f^{(0)}_{01}}{u \pm d_1} \right).
\]

Arakawa proved [1, proposition 5], that in antisymmetric case
\[
T_y(u) \equiv 1 + \sum_{i=1}^N \frac{f^{(0)}_{0i}}{u - D_i} \mod U_{sN}^+.
\]

Analogously, in the symmetric case one has
\[
T_y(u) \equiv 1 + \sum_{i=1}^N \frac{f^{(0)}_{0i}}{u + D_i} \mod U_{sN}^+.
\]

Note that the latter presentations can be equivalently used in the spaces $\Lambda_{sN}^+$ of $S_N$—(skew)invariants, since the RHS of (4.10) commutes with the total action of the symmetric group.
2. The rings $\Lambda_N^+ (\equiv \Lambda_{sN}^+)$ in the notations (2.1)) of scalar symmetric functions form the projective system with respect to the maps

$$\omega_N^+ : \Lambda_N^+ \to \Lambda_N^{N-1}, \quad \omega_N^+ f(x_1, \ldots, x_N) = f(x_1, \ldots, x_{N-1}, 0). \quad (4.11)$$

Analogously, the spaces $\Lambda_N^- (\equiv \Lambda_{sN}^-)$ of scalar skew-symmetric functions form the projective system with respect to the maps

$$\omega_N^- : \Lambda_N^- \to \Lambda_N^{-1}, \quad \omega_N^- f(x_1, \ldots, x_N) = (x_1 \ldots x_{N-1})^{-1} f(x_1, \ldots, x_{N-1}, 0). \quad (4.12)$$

The latter can be generalized to the spin case. Regard an element $f$ of $\Lambda_{sN}^-$ as ($\mathbb{C}^s$-$\otimes N$ valued function

$$\omega_N^- (f) = (x_1 \ldots x_{N-s})^{-1} \left( (\otimes_i e_i^+ \otimes \otimes_j e_j^-) \right) f(x_1, \ldots, x_{N-s}, 0 \ldots 0), \quad (4.13)$$

which coincides with (4.12) in case $s = 1$. In components,

$$\omega_N^- (x_1^a_1 e_1 \otimes \ldots \otimes x_N^a_N e_N) = \delta_{aN,0} \ldots \delta_{aN+s+1,0} \delta_{aN,s} \ldots \delta_{aN+s+1,s} x_1^{a_1 - 1} \ldots \otimes x_{N-s}^{a_{N-s} - 1} e_{N-s}.$$

One can see that $\omega_N$ is a linear map from $\Lambda_{sN}^-$ to $\Lambda_{sN-s}$.

For the analysis of compatibility of transfer matrices with projection maps (4.11) and (4.12) we use Dunkl operators $d_i^{(N)}$ (now we use the upper index to distinguish the number of variables on which this operator acts).

Consider first the scalar case $s = 1$. Set

$$A_N^{(N)} = \alpha \frac{\partial}{\partial x_N} + \sum_{j < N} \frac{1}{x_N - x_j} (1 - K_{Nj}),$$

$$A_i^{(N)} = \frac{1}{x_j - x_N} (1 - K_{Nj}),$$

$$B_i^{(N)} = d_i^{(N-1)} + 1, \quad i < N. \quad (4.14)$$

Note that operators $A_i^{(N)}$ and $B_i^{(N)}$ transform polynomials to polynomials and

$$[x_N, B_i^{(N)}] = 0 \quad (4.15)$$

The following statement is straightforward result of the analysis of (4.2):

**Lemma 4.1.** The Dunkl operators $d_i^{(N)}$ admit the decomposition

$$d_i^{(N)} = x_N A_i^{(N)} $$

$$d_i^{(N)} = x_N A_i^{(N)} + B_i^{(N)} = x_N A_i^{(N)} + d_i^{(N-1)} + 1, \quad i < N. \quad (4.17)$$
Relations (4.8), (4.16), (4.17) and (4.14) imply the compatibility relations of transfer matrices for \( N \) and \( N - 1 \) variables in scalar case.

**Proposition 4.1.** (i) In scalar symmetric case we have the following identity of operators from \( \tilde{\Lambda}_{N}^{N}[u^{-1}] \rightarrow \tilde{\Lambda}_{N}^{N-1}[u^{-1}] \):

\[
\omega_{N}^{+} T_{N}(u) = \frac{u + 1}{u} T_{N-1}(u + 1) \omega_{N}^{+}; \tag{4.18}
\]

(ii) In scalar skewsymmetric case the following identity of operators from \( \tilde{\Lambda}_{N}^{N}[u^{-1}] \rightarrow \tilde{\Lambda}_{N-1}^{N}[u^{-1}] \) holds:

\[
\omega_{N}^{-} T_{N}(u) = \frac{u + 1}{u} T_{N-1}(u - \alpha - 1) \omega_{N}^{-}. \tag{4.19}
\]

**Proof.** Due to (4.16), any power of the operator \( d_{N}^{(N)} \) is divisible by \( x_{N} \) so that the application of \( \omega_{N}^{\pm} \) to \( \left( 1 + \frac{\alpha u}{u + \alpha} \right) \) reduces to the multiplication by the scalar operator \( \frac{u + 1}{u} \). Next, in symmetric case for any \( i < N \) due to (4.17) and (4.15) the action of any power of \( d_{N}^{(N)} \) modulo ideal generated by \( x_{N} \) differs from the action of \( d_{N-1}^{(N-1)} \) by shift by 1. This gives (4.18). In skewsymmetric case the action of \( x_{i} y_{i} \) on the product \( (x_{1} \cdots x_{N-1})^{-1} \), see (4.13), gives additional shift by \( \alpha \). So we have (4.19). \[\square\]

Iterating the relations (4.18) we see, that in symmetric case the renormalized transfer matrices

\[
\tilde{T}_{N}(u) = \frac{u - N}{u} T_{N}(u - N) \tag{4.20}
\]

are compatible with projection maps \( \omega_{N}^{+} \),

\[
\omega_{N}^{+} \tilde{T}_{N}(u) = \tilde{T}_{N-1}(u) \omega_{N}^{+}. \tag{4.21}
\]

In antisymmetric case we can use

\[
\tilde{T}_{N}(u) = f_{N}(u) T_{N}(u + \gamma N), \quad \omega_{N}^{-} \tilde{T}_{N}(u) = \tilde{T}_{N-1}(u) \omega_{N}^{-}, \tag{4.22}
\]

where \( \gamma = \alpha + 1 \) and

\[
f_{N}(u) = \prod_{k=1}^{N} \frac{u + k \gamma}{u + k \gamma + 1}. \tag{4.23}
\]

The statement of proposition 4.1 can be generalized to skewsymmetric spin case.

**Proposition 4.2.** The following identities of operators from \( C^{s} \otimes \tilde{\Lambda}_{N}^{N,s}[u^{-1}] \rightarrow C^{s} \otimes \tilde{\Lambda}_{N-1}^{N,s}[u^{-1}] \) holds:

\[
\omega_{N}^{-} T_{N}(u) = \frac{u + 1}{u} T_{N-1}(u - \alpha - s) \omega_{N}^{-}. \tag{4.24}
\]

**Proof.** The proof of proposition 4.2 distinguishes from the proof of proposition 4.1 in two details. Set

\[
T_{N}(u) = T'(u) T''(u),
\]
where

\[ T'(u) = \left( 1 + \frac{I_{N_1}}{u - d_{N_1}^{N_3}} \right) \ldots \left( 1 + \frac{I_{(N - 1)s + 1}}{u - d_{(N - 1)s + 1}^{N_3}} \right), \]

\[ T''(u) = \left( 1 + \frac{I_{(N - 1)s}}{u - d_{(N - 1)s}^{N_3}} \right) \ldots \left( 1 + \frac{I_1}{u - d_1^{N_3}} \right). \]

Then modulo the ideal generated by \( x_{(N - 1)s + 1}, \ldots, x_{N_1} \) the action of each element \( d_k^{(N)} \) in \( T'(u) \) differs from that of \( d_k^{(N - 1)s} \) in \( T_{(N - 1)s}(u) \omega_N^N \) by \( s + \alpha \). This explains the shift of the spectral parameter. On the other hand the each \( d_k^{(N)} \) in \( T'(u) \) can be presented in a form

\[ d_k^{(N)} = \sum_{j=1}^s x_{(N - 1)s + j} A_{jk}, \quad k > (N - 1)s, \]

where \( A_{jk} \) transform polynomials to polynomials. Thus the computation of the action of \( \omega_N^N T'(u) \) reduces to the identity

\[ \left( 1 + \frac{I_{N_1}}{u} \right) \ldots \left( 1 + \frac{I_{(N - 1)s + 1}}{u - s + 1} \right) e_1 \otimes \ldots \otimes e_s \equiv \frac{u + 1}{u} e_1 \otimes \ldots \otimes e_s, \quad (4.25) \]

in the space \( (\text{End } C^* \otimes \Lambda^2)^{s} \), equivalent to the computation of the \( q \)-determinant of Yangian matrix [13]: We demonstrate (4.25) for \( s = 2 \), renaming tensor indices and assuming final projection to \( \Lambda^{s(N)} \) : 

\[
\left( 1 + \sum_{i,j=1} E_{ij}^{(0)} \otimes E_{ij}^{(1)} \right) \left( 1 + \sum_{i,j=1} E_{ij}^{(0)} \otimes E_{ij}^{(2)} \right) \left( 1 + \frac{\sum_{i,j=1} E_{ij}^{(0)} \otimes E_{ij}^{(1)}}{u - 1} \right) e_1 \otimes e_2 \\
\equiv \left( 1 + \frac{E_{11}^{(0)} \otimes E_{11}^{(1)}}{u} + \frac{E_{22}^{(0)} \otimes E_{22}^{(1)}}{u} + \frac{E_{22}^{(0)} \otimes E_{22}^{(2)}}{u - 1} \right) e_1 \otimes e_2 \\
\equiv \left( 1 + \frac{E_{11}^{(0)} \otimes E_{11}^{(1)}}{u} + \frac{E_{22}^{(0)} \otimes E_{22}^{(1)}}{u - 1} - \frac{E_{22}^{(0)} \otimes E_{22}^{(2)}}{u(u - 1)} \right) \otimes e_1 \otimes e_2 \\
\equiv \left( 1 + \frac{E_{11}^{(0)} \otimes E_{11}^{(1)}}{u} \otimes e_1 \otimes e_2 = \frac{u + 1}{u} \otimes e_1 \otimes e_2. \right)
\]

Set \( \gamma = \alpha + s \) and

\[ \tilde{T}_N(u) = f_N(u) T_N(u + \gamma N), \quad (4.26) \]

where now

\[ f_N(u) = \prod_{k=1}^{N} \frac{u + k\gamma}{u + k\gamma + 1} = \frac{\Gamma \left( \frac{u}{\gamma} + N + 1 \right) \Gamma \left( \frac{u + 1}{\gamma} + 1 \right)}{\Gamma \left( \frac{u + 1}{\gamma} + N + 1 \right) \Gamma \left( \frac{u}{\gamma} + 1 \right)}, \]

treated as asymptotical series in \( u^{-1} \). Then \( \tilde{T}_N(u) \) satisfy compatibility conditions

\[ \omega_N^N \tilde{T}_N(u) = \tilde{T}_{(N - 1)s}(u) \omega_N^N \]

then

\[ \tilde{T}_N(u) \equiv \frac{u + 1}{u} \otimes e_1 \otimes e_2. \]
and form a projective system of transfer matrices.

5. Fermionic limit

Let $\mathcal{H}^\ell_s$ be the algebra of $s$ free fermion fields. It is generated by the elements $\psi_{n\ell}$ and $\psi^*_{n\ell}$, where $n \in \mathbb{Z}$ and $\ell = 1, \ldots, s$, which subject the relations

$$
\psi_{n\ell} \psi_{m\ell} + \psi_{m\ell} \psi_{n\ell} = 0, \quad \psi_{n\ell}^* \psi_{m\ell}^* + \psi_{m\ell}^* \psi_{n\ell}^* = 0,
\psi_{n\ell} \psi_{m\ell}^* + \psi_{m\ell}^* \psi_{n\ell} = \delta_{nm} \delta_{\ell, -m}.
$$

(5.1)

The algebra $\mathcal{H}^\ell_s$ is graded with

$$
\deg \psi_{n\ell} = \deg \psi^*_{n\ell} = -n.
$$

(5.2)

The algebra $\mathcal{H}^\ell_s$ admits a family of commuting automorphisms $\hat{Q}_s \psi$, $s = 1, \ldots, s$ given by the relations

$$
\hat{Q}_s (\psi_{n\ell}) = \psi_{b,n+\delta_c}, \quad \hat{Q}_s (\psi^*_{n\ell}) = \psi^*_{b,n+\delta_c}.
$$

(5.3)

Let $\mathcal{F}^\ell$ be the left representations of $\mathcal{H}^\ell_s$, generated by the vacuum state $\langle 0 \rangle$, and $\mathcal{F}^\ell_r$ be the right $\mathcal{H}^\ell_s$-module generated by the vacuum state $\langle 0 \rangle$, such that

$$
\langle 0 | 0 \rangle = 1
$$

and

$$
\psi_{n\ell} | 0 \rangle = \psi^*_{n\ell} | 0 \rangle = 0 \quad c = 1, \ldots, s, \quad n \geq 0, \quad m > 0,
\langle 0 | \psi_{n\ell} = \langle 0 | \psi^*_{n\ell} = 0 \quad c = 1, \ldots, s, \quad n < 0, \quad m \leq 0.
$$

(5.4)

We use the following fermionic normal ordering rule:

$$
\hat{Q}^*_{s} \psi_{n\ell} \psi_{m\ell} = \begin{cases} 
\psi^*_{n\ell} \psi_{m\ell}, & m \geq 0 \\
-\psi_{m\ell} \psi^*_{n\ell}, & m < 0.
\end{cases}
$$

(5.5)

It is compatible with relations (5.4).

The automorphisms (5.3) define invertible linear maps $Q_{s}$ and $Q_{s}^{-1}$ of the Fock space to itself which are compatible with these automorphisms and anticommute for different indices $c_1$ and $c_2$:

$$
Q^{-1}_{s}(x | 0 \rangle) = Q^{-1}_{s}(x) \psi_{c,0} | 0 \rangle, \quad Q_{s}(x | 0 \rangle) = Q_{s}(x) \psi_{c,-1} | 0 \rangle,
\langle 0 | Q^{-1}_{s} = \langle 0 | \psi^*_{c,1}, \quad \langle 0 | Q_{s} = \langle 0 | \psi_{c,0}.
$$

(5.6)

so that for any $x \in \mathcal{H}^\ell_s$ and $|v\rangle \in \mathcal{F}^\ell$ we have

$$
\hat{Q}_{s}(x) | v \rangle = Q_{s}(x) Q_{s}^{-1} | v \rangle.
$$

(5.7)

Indeed, for $|v\rangle = y | 0 \rangle$ the RHS of (5.7) equals

$$
Q_{s}(x) Q_{s}^{-1} | v \rangle = Q_{s}(y \psi_{c,0} | 0 \rangle) = Q_{s}(x) y \psi^*_{c,1} \psi_{c,-1} | 0 \rangle
$$

$$
= Q_{s}(x) y (1 - \psi_{c,-1} \psi^*_{c,1}) | 0 \rangle = Q_{s}(x) y | 0 \rangle = \hat{Q}_{s}(x) | v \rangle.
$$
In the following we use the distinguished product of $s$ such maps and automorphisms

$$\hat{Q} := \hat{Q}_s \cdots \hat{Q}_1, \quad Q = Q_s \cdots Q_1.$$  \hspace{1cm} (5.8)

In particular,

$$\langle 0 | Q^{-1} = \langle 0 | \psi^*_{s,1} \cdots \psi^*_{1,1}, \quad Q^{-1} | 0 \rangle = \psi^*_{s,0} \cdots \psi^*_{1,0} | 0 \rangle,$$

$$\langle 0 | Q = \langle 0 | \psi_{s,0} \cdots \psi_{1,0}, \quad Q | 0 \rangle = \psi_{s,-1} \cdots \psi_{1,-1} | 0 \rangle.$$  \hspace{1cm} (5.9)

Denote by $H^1_+ (z)$ the space of Laurent series

$$\sum_{n \in \mathbb{Z}} h_n z^n \in H^1_+ (z),$$  \hspace{1cm} (5.10)

where each coefficient $h_n$ is a series

$$h_n = \sum_k a^1_k + \sum_{kl} a^{21}_k a^{22}_l + \sum_{klm} a^{31}_k a^{32}_l a^{33}_m + \ldots,$$

where $a^i_j$ are either $\psi^*_{c,n}$ or $\psi_{c,n}$ for some $c$ and $n$, such that the matrix coefficient $\langle \xi | h_n | v \rangle$ is well defined for any $\xi \in \mathcal{F}_-^\omega$ and $v \in \mathcal{F}_-^\omega$. We can assume for instance all series $h_n$ to be fermionic normal ordered according to (5.5), $h_n = [h_n]$. We also use the notation $\mathcal{F}_-^\omega (z)$ for the space $\mathcal{F}_-^\omega \otimes \mathbb{C}[z, z^{-1}]$.

Let $\Psi_c (z)$ and $\Psi^*_c (z)$ be the following elements of $H^1_+ (z)$,

$$\Psi_c (z) = \sum_{n \in \mathbb{Z}} \psi_{c,n} z^n, \quad \Psi^*_c (z) = \sum_{n \in \mathbb{Z}} \psi^*_{c,n} z^n.$$  \hspace{1cm} (5.11)

The field $\Psi_c (z)$ is of total degree zero, and the field $\Psi^*_c (z)$ is of total degree $-1$, once we set $\deg z = 1$. The relations (5.4) imply the commutativity

$$\Psi_c (x) \Psi_d (y) + \Psi_d (y) \Psi_c (x) = \Psi^*_c (x) \Psi^*_d (y) + \Psi^*_d (y) \Psi^*_c (x) = 0,$$  \hspace{1cm} (5.12)

and normal ordering rules

$$\Psi_c (x) \Psi^*_d (y) = \left\{ \begin{array}{ll} \Psi_c (x) \Psi^*_d (y), & x < y, \\ \frac{\delta_{d,\bar{c}} \Psi^*_c (x) \Psi_d (y)}{y-x}, & x = y, \\ \frac{\delta_{d,\bar{c}} \Psi^*_c (x) \Psi_d (y)}{y-x}, & x > y. \end{array} \right.$$  \hspace{1cm} (5.13)

which imply the relation

$$\frac{1}{2\pi i} \int_{z \in \mathbb{C}^w} \Psi_c (w) \Psi^*_c (z) dz = \frac{1}{2\pi i} \int_{z \in \mathbb{C}^w} \Psi^*_c (w) \Psi_c (z) dz = 1.$$  \hspace{1cm} (5.14)

One can also see that

$$\hat{Q} \left( \Psi_c (z) \right) = z \Psi_c (z), \quad \hat{Q} \left( \Psi^*_c (z) \right) = z^{-1} \Psi^*_c (z), \quad c = 1, \ldots, s.$$  \hspace{1cm} (5.15)

\footnote{For instance the monomial $z^c \left( \sum_{c,d} \psi_{c,d} \psi_{d,c} \right) \notin H^1_+ (z).}$
Boson–fermion correspondence says that the space $F^s$ is a representation of the affine Lie algebra $\hat{\mathfrak{gl}}_s$ of level one. The degree $-n$ generators $E_{ab,n}$ of $\hat{\mathfrak{gl}}_s$, where $a, b = 1, \ldots, s$ and $n \in \mathbb{Z}$ satisfy the relations

$$[E_{ab,n}, E_{cd,m}] = \delta_{bc}E_{ad,n+m} - \delta_{ad}E_{cb,n+m} + n\delta_{n,-m}\delta_{ad}\delta_{bc}.$$ 

They are presented in $\operatorname{End} F^s$ by operators

$$E_{ab,n} = \sum_{k+l=n} \hat{\psi}_{al}^* \psi_{bk},$$

where $\hat{;}$ means fermionic normal ordering (5.5). The generators $a_{b,n} := E_{bb,n}$ form the Heisenberg algebra $H^s$,

$$[a_{b,n}, a_{c,m}] = n\delta_{b,c}\delta_{n,-m},$$

so that

$$a_{c,k}|0\rangle = 0, \quad \langle 0|a_{c,l} = 0, \quad c = 1, \ldots, s, \quad k \geq 0, \quad l \leq 0.$$ 

On the other side of boson–fermion correspondence we have the relations:

$$\Psi_c(z) = z^{a_{c,0}} \exp \left( \sum_{n < 0}^{\infty} \frac{a_{c,n}}{n} z^n \right) \exp \left( \sum_{n > 0}^{\infty} \frac{a_{c,n}}{n} z^n \right) Q_c \Psi_c^*(z) = z^{-a_{c,0}} \exp \left( -\sum_{n < 0}^{\infty} \frac{a_{c,n}}{n} z^n \right) \exp \left( -\sum_{n > 0}^{\infty} \frac{a_{c,n}}{n} z^n \right) Q_c^{-1}.$$ 

The element

$$a_0 = \sum_{c=1}^{s} a_{c,0} = \sum_{c=1}^{s} \sum_{k \in \mathbb{Z}} \psi_{ck}^* \psi_{c,-k},$$

is central in $\hat{\mathfrak{gl}}_s$ and satisfies the relation

$$Q_0Q^{-1} = a_0 + s.$$ 

The Fock space $F^s$ admits the orthogonal decomposition into direct sum of eigenspaces of operator $a_0$,

$$F^s = \bigoplus_{N \in \mathbb{Z}} F^s_N, \quad \text{where} \quad F^s_N = \{ |v\rangle \in F^s : a_0 |v\rangle = N |v\rangle \}.\quad (5.17)$$

The relation (5.17) implies that

$$Q F^s_N = F^s_{N-1},$$

In the following we use the notation $\tau_N$ for the projection of $F^s$ to $F^s_N$ parallel to other eigenspaces of $a_0$:

$$\tau_N |v\rangle = \delta_{N,k} |v\rangle \quad \text{for} \quad |v\rangle \in F^s_k.$$ 

(5.19)
Let $\Psi(z)$ and $\Psi^*(z)$ be the following elements of $\mathcal{H}^e(z) \otimes \mathbb{C}^t$ and $\mathcal{H}^e(z) \otimes \mathbb{C}^{t*}$ correspondingly,

$$
\Psi(z) = \sum_c \Psi_c(z) \otimes e_c \quad \Psi^*(z) = \sum_c \Psi^*_c(z) \otimes e^*_c.
$$

(5.20)

The field $\Psi(z)$ defines the map from $\mathcal{F}^t$ to $\mathcal{F}^t(z) \otimes \mathbb{C}^t$,

$$
\Psi(z)|v\rangle = \sum_c \Psi_c(z)|v\rangle \otimes e_c,
$$

which we denote by the same symbol $\Psi(z)$. The field $\Psi^*(w)$ defines a map from $\mathcal{H}^e(z) \otimes \mathbb{C}^t$ to $\mathcal{H}^e(z,w)$,

$$
\Psi^*(w) \left( \sum_c F_c(z) \otimes e_c \right) = \sum_c \Psi^*_c(w) F_c(z),
$$

where $\mathcal{H}^e(z,w)$ is defined in the same way as $\mathcal{H}^e(z)$ (5.10). Here we regard $e^*_c$ as the linear map $e^*_c : \mathbb{C}^t \rightarrow \mathbb{C}$, such that $e^*_c(e_d) = \delta_{cd}$.

For any $|v\rangle \in \mathcal{F}^t$ consider the matrix element

$$
\pi_N(|v\rangle) = \langle 0| (\Psi(z_N) \otimes 1^\otimes (N-1)) \cdots (\Psi(z_2) \otimes 1) \Psi(z_1)|v\rangle,
$$

which we shortly denote by

$$
\pi_N(|v\rangle) = \langle 0| \Psi(z_N) \Psi(z_2) \cdots \Psi(z_1)|v\rangle.
$$

(5.21)

In components,

$$
\pi_N(|v\rangle) = \sum_{c_1, \ldots, c_N} \langle 0| \Psi_{c_N}(z_N) \cdots \Psi_{c_1}(z_1)|v\rangle \cdot e_{c_1} \otimes \cdots \otimes e_{c_N}.
$$

The commutativity (5.12) and the properties of the left vacuum (5.4) imply that the matrix element (5.21) belongs to the space $\Lambda^{\otimes N}$. Note that the map $\pi_N$ factors through the projection $\tau_N$ (5.19),

$$
\pi_N = \pi_N \tau_N
$$

and equals zero for any $\mathcal{F}_M$ with $M \neq N$.

We are going now to construct the pullback through the maps $\pi_N$ of the components of the Yangian generators.

Denote by $i_N : \Lambda^{\otimes N} \rightarrow \mathbb{C}[z] \otimes \Lambda^{\otimes N-1}$ the decomposition of the antisymmetric tensor $\nu$ over the first tensor component, given by the relation (3.12). Denote by $\pi_{N-1,1} : (\mathcal{H}^e(z) \otimes \mathbb{C}^t) \otimes \mathcal{F}^t \rightarrow \mathbb{C}[x_2, \ldots, x_N, z, \varepsilon^{-1}] \otimes \mathbb{C}^{t^{\otimes N}}$ the map defined as

$$
\pi_{N-1,1}(F(z) \otimes |v\rangle) = \langle 0| \Psi(x_N) \cdots \Psi(x_2) F(z)|v\rangle.
$$

Lemma 5.1. We have the following equality of linear maps $\mathcal{F}^t \rightarrow \Lambda^{\otimes N}$:

$$
\pi_{N-1,1} \Psi(z) = i_N \pi_N.
$$

(5.22)

Proof. This is again a tautology like in the proof of lemma 3.1. □
For each polynomial tensor $C'[z] \in V \otimes \Lambda^1 \otimes \cdots \otimes \Lambda^{N-1}$, antisymmetric with respect to diagonal permutations of all tensor factor except the first, denote by $A_N(u)$ its total (nonnormalized) antisymmetrization

$$A_N(u) = u - \sum_{j=2}^{N} \sigma_j(u). \tag{5.23}$$

On the other hand, for each $F(z) \in \mathcal{H}_1^e(z) \otimes C'$ define the element $A(F(z)) \in \mathcal{H}$ as the integral

$$A(F(z)) = \frac{1}{(2\pi i)^2} \int_{\mathbb{C}} dz \int_{w \geq z} du \frac{\Psi'(u) F(z)}{u - z}. \tag{5.24}$$

**Remark.** The integral over $z$ is actually formal. The form (5.24) indicates the following. Assume that $F(z)$ depends on a parameter $w$. Then the contour $C$ of integration over $z$ should not enclose the point $z = w$. One can always assume the condition $|w| > |z|$.

Let an element $F(z) \in \mathcal{H}_1^e(z) \otimes C'$ satisfies the following conditions:

1. $\pi_{N,1}(F(z) \otimes |v\rangle)$ is a polynomial on $z$ for any $N \in \mathbb{N}, \ v \in \mathcal{F}^s$ \tag{5.25}
2. deg $F(z) = 0$. \tag{5.26}

Here we assume that deg $e_c = 0$ for any $e_c \in C'$.

The following lemma establishes the map $A$ as the pullback of the finite antisymmetrization. This is the crucial point of the construction.

**Lemma 5.2.** For each $F(z) \in \mathcal{H}_1^e(z) \otimes C'$ satisfying the conditions (5.25) and (5.26), any $|v\rangle \in \mathcal{F}^s$ and any natural $N \in \mathbb{N}$ we have the equality of elements of $\Lambda^N$:

$$A_N \pi_{N-1,1}(F(z) \otimes |v\rangle) = \pi_N A(F(z)|v\rangle). \tag{5.27}$$

**Proof.** Let $F(z)$ has the form

$$F(z) = \sum_{c=1}^{s} F_c(z) \otimes e_c, \quad F_c(z) \in \mathcal{H}_1^e(z).$$

Consider first the LHS of (5.27). This is the antisymmetrization (5.23) of the tensor

$$\sum_{c_1, \ldots, c_N = 1}^{s} \langle 0|\Psi_{c_N}(x_N) \cdots \Psi_{c_2}(x_2)\Psi_{c_1}(x_1)|v\rangle \cdot e_{c_1} \otimes \cdots \otimes e_{c_N},$$

which can be written by means of proper changes of summation indices as the sum

$$\sum_{k=1}^{N} (-1)^{k+1} \sum_{c_1, \ldots, c_N = 1}^{s} \langle 0|\Psi_{c_N}(x_N) \cdots \Psi_{c_{k+1}}(x_{k+1})\Psi_{c_{k+1}}(x_{k+1}) \cdots \Psi_{c_1}(x_1)F_{c_k}(x_k)|v\rangle$$

$$\cdot e_{c_1} \otimes \cdots \otimes e_{c_N}.$$

Using the relation

$$\int_{z \in \mathbb{C}} dz \frac{F_{c_k}(z)}{x_k - z} = -F_{c_k}(x_k),$$

we have
we rewrite the LHS of (5.27) as
\[
\sum_{k=1}^{N} (-1)^{s} \sum_{c_{1}, \ldots, c_{N}=1}^{s} (0) \Psi_{c_{1}}(x_{N}) \cdots \Psi_{c_{s+1}}(x_{k+1}) \Psi_{c_{s+1}}(x_{k-1}) \cdots \Psi_{c_{1}}(x_{1}) \cdot \left(2 \pi i \sum_{z \in \mathbb{C}} \frac{F_{c_{k}}(z)}{x_{k} - z} \right) \cdot \langle u | v \rangle .
\]

Using (5.14), we insert the integral:
\[
- \frac{1}{2\pi i} \int_{u \in \mathbb{C}} \Psi_{c_{1}}(x_{1}) \Psi_{c_{s}}(u) \, du = 1
\]
into each summand of the \(k\)th group. Then the LHS of (5.27) takes the form
\[
- \frac{1}{(2\pi i)^2} \sum_{k=1}^{N} \left( \sum_{c_{1}, \ldots, c_{N}=1}^{s} \int_{z \in \mathbb{C}} \frac{\Psi_{c_{1}}(x_{N}) \cdots \Psi_{c_{s+1}}(x_{k+1}) \cdots \Psi_{c_{1}}(x_{1}) \Psi_{c_{s}}(u) F_{c_{k}}(z)}{u - z} \, |v\rangle \cdot e_{c_{1}} \otimes \cdots \otimes e_{c_{N}} \right)
\]
\[
= - \frac{1}{(2\pi i)^2} \sum_{k=1}^{N} \left( \sum_{c_{1}, \ldots, c_{N}=1}^{s} \int_{z \in \mathbb{C}} \frac{\Psi_{c_{1}}(x_{N}) \cdots \Psi_{c_{s+1}}(x_{k+1}) \cdots \Psi_{c_{1}}(x_{1}) \Psi_{c_{s}}(u) F_{c_{k}}(z)}{u - z} \, |v\rangle \cdot e_{c_{1}} \otimes \cdots \otimes e_{c_{N}} \right)
\]
\[
(5.28)
\]
Now in each summand we move the contour of integration for \(z\) close to the point \(x_{k}\), crossing the singularity at \(z = u\). Then the integral in every such summand transforms into the sum of two integrals,
\[
- \int_{z \in \mathbb{C}} \frac{\Psi_{c_{1}}(x_{N}) \cdots \Psi_{c_{s+1}}(x_{k+1}) \cdots \Psi_{c_{1}}(x_{1}) \Psi_{c_{s}}(u) F_{c_{k}}(z)}{u - z} \, |v\rangle
\]
\[
= - \int_{z \in \mathbb{C}} \frac{\Psi_{c_{1}}(x_{N}) \cdots \Psi_{c_{s+1}}(x_{k+1}) \cdots \Psi_{c_{1}}(x_{1}) \Psi_{c_{s}}(u) F_{c_{k}}(z)}{u - z} \, |v\rangle
\]
\[
+ \int_{z \in \mathbb{C}} \frac{\Psi_{c_{1}}(x_{N}) \cdots \Psi_{c_{s+1}}(x_{k+1}) \cdots \Psi_{c_{1}}(x_{1}) \Psi_{c_{s}}(u) F_{c_{k}}(z)}{u - z} \, |v\rangle .
\]
\[
(5.29)
\]
In the first integral, see the middle line of (5.29), after the change of the order of integration we observe its vanishing due to condition (i) of (5.25): there is no singularity of the integral at any point \(z = x_{j}\). We now conclude that the LHS of (5.27) equals to the double integral
\[
\frac{1}{(2\pi i)^2} \sum_{k=1}^{N} \left( \sum_{c_{1}, \ldots, c_{N}=1}^{s} \int_{z \in \mathbb{C}} \frac{\Psi_{c_{1}}(x_{N}) \cdots \Psi_{c_{s+1}}(x_{k+1}) \cdots \Psi_{c_{1}}(x_{1}) \Psi_{c_{s}}(u) F_{c_{k}}(z)}{u - z} \, |v\rangle \cdot e_{c_{1}} \otimes \cdots \otimes e_{c_{N}} \right)
\]
or
\[
\frac{1}{(2\pi i)^2} \sum_{k=1}^{N} \left( \sum_{c_{1}, \ldots, c_{N}=1}^{s} \int_{z \in \mathbb{C}} \frac{\Psi_{c_{1}}(x_{N}) \cdots \Psi_{c_{s+1}}(x_{k+1}) \cdots \Psi_{c_{1}}(x_{1}) \Psi_{c_{s}}(u) F_{c_{k}}(z)}{u - z} \, |v\rangle \cdot e_{c_{1}} \otimes \cdots \otimes e_{c_{N}} \right),
\]
\[
(5.30)
\]
In Fourier modes (5.32) looks as

\[
\sum_{n<0} \psi_{\lambda,n}^* f_{-n} \pm \sum_{n>0} f_{-n} \psi_{\lambda,n}^*.
\]
The first sum vanishes due to (5.4). By assumption, \( \deg F_{ij}(z) = 0 \) thus \( \deg f_n = -n \). We then see that in the second term all \( f_n \) have positive degree and if we assume them to be normal ordered they contain in each summand either \( \psi^a_{\alpha n} \) or \( \psi^a_{\alpha n} \) with \( n < 0 \) at their left end. Thus \((0)f_{-n} = 0 \) for \( n > 0 \) and the integral (5.32) vanishes. \( \square \)

Define an operator \( \mathcal{D} : \mathcal{H}^l_-(z) \otimes \mathbb{C}' \to \mathcal{H}^l_-(z) \otimes \mathbb{C}' \) by the relation

\[
\mathcal{D} F(z) = \alpha z \frac{d}{dz} F(z) + \frac{z}{(2\pi i)^2} \int_{|w|<|z|} dw \int_{w<z} du \frac{\Psi^{(2)}(u)}{u - w} F^{(1)}(z) - \frac{\Psi^{(2)}(z)}{u - w} F^{(1)}(w).
\]

Here upper indices (1) and (2) indicate tensor components where corresponding operators act. In components,

\[
\mathcal{D} F_i(z) \otimes e_c = \alpha z \frac{d}{dz} F_i(z) \otimes e_c + \frac{z}{(2\pi i)^2} \sum_{b=1}^{s} \int_{|w|<|z|} dw \int_{w<z} du \frac{\Psi^{(2)}(u)}{u - w} F^{(1)}(z) - \frac{\Psi^{(2)}(z)}{u - w} F^{(1)}(w) \otimes e_c.
\]

By means of lemma 5.2 we now can identify the operator \( \mathcal{D} \) as a pullback of the equivariant family of Heckman operators \( \mathcal{D}^N \) acting in the space of partially antisymmetric tensors.

**Proposition 5.1.** For any \( F(z) \in \mathcal{H}^l_-(z) \otimes \mathbb{C}' \) satisfying the condition (5.25) and (5.26), \( |v\rangle \in \mathcal{F}' \) and \( N \in \mathbb{N} \) we have the equality

\[
\pi_{N-1,1}(\mathcal{D} F(x_1) \otimes |v\rangle) = \mathcal{D}^N_1 \pi_{N-1,1}(F(x_1) \otimes |v\rangle).
\]

**Proof.** First we note that once the element \( F(z) \in \mathcal{H}^l_-(z) \otimes \mathbb{C}' \) satisfies the conditions (5.25) and (5.26), the same is true for the divided difference

\[
\frac{\Psi^{(2)}(w) F^{(1)}(z) - \Psi^{(2)}(z) F^{(1)}(w)}{z - w}.
\]

The property (5.25) is valid because both the differential and difference derivatives preserve the polynomial property. The property (5.26) is evident: the difference derivatives are homogeneous of degree zero. We thus can use lemma 5.2. Now the rest of the proof is identical to the proof of proposition 3.1. \( \square \)

Note that the application of the operator \( \mathcal{D} \) to some \( F(z) \in \mathcal{H}^l_-(z) \otimes \mathbb{C}' \), which satisfies the conditions (5.25) and (5.26), preserves these conditions by the same reasons of homogeneity and preservation of polynomial spaces by both difference and differential derivatives. This gives rise to the formulas for pullback of sum of powers of Dunkl operators.

Let \( E_{ab} \in \text{End} \, \mathbb{C}' \), be the matrix unit, \( E_{ab}(e_c) = \delta_{bc} e_a \). Denote by \( \mathcal{E}_{ab} \), the operator \( 1 \otimes E_{ab} : \mathcal{H}^l_-(z) \otimes \mathbb{C}' \to \mathcal{H}^l_-(z) \otimes \mathbb{C}' \):

\[
\mathcal{E}_{ab} F(z) = F_{ab}(z) \otimes e_a.
\]

For \( a, b = 1, \ldots, s \) and \( n = 1, \ldots \) define the element \( T_{ab,n} \in \mathcal{H}^l_- \) by the relation

\[
T_{ab,n} = \mathcal{A} \mathcal{E}_{ab} \mathcal{D}^n \Psi(z) = \frac{1}{2\pi i} \int_{z<0} dz \, T_{ab,n}.
\]

(5.37)
Here $T_{ab,n}$ is the $n$th order density defined by the formula:

$$T_{ab,n} = \frac{1}{2\pi i} \int_{u \neq z} du \frac{\Psi^*(u) E_{ab} D^n \Psi(z)}{u - z}.$$  \hfill (5.38)

In appendix we give expressions for the first densities for $n = 0, 1, 2$ in terms of normal ordering fermionic fields and in terms of generators of the affine Lie algebra $\hat{\mathfrak{gl}}_s$, see (5.15).

Summarizing the statements above we establish the operator $T_{ab,n}$ as the pullback of the Yangian generator $I_{ab,n}$ in $\Lambda^{(s)}_{N\otimes N}$.

**Proposition 5.2.** For any $|v\rangle \in F^s$ and $N \in \mathbb{N}$ we have the equality

$$\pi_N(T_{ab,n}|v\rangle) = I_{ab,n}\pi_N|v\rangle.$$  

We now reformulate projective properties of the Yangian action in the phase space of finite-dimensional CS system, see proposition 4.2, in terms of the constructed operators in the Fock space.

**Lemma 5.3.** For any $|v\rangle \in F^s$ we have the equality

$$\pi_N(Q|v\rangle) = \omega_{N,s} \cdot \pi_{N,s}(|v\rangle).$$  \hfill (5.39)

**Proof.** The LHS of (5.39) reads

$$\pi_N(Q|v\rangle) = \langle 0|\Psi(x_N) \cdots \Psi(x_2)\Psi(x_1)|v\rangle$$

$$= (x_1 \cdots x_N)^{-1} \cdot \langle 0|Q\Psi(x_N) \cdots \Psi(x_2)\Psi(x_1)|v\rangle$$

$$= (x_1 \cdots x_N)^{-1} \cdot \langle 0|\psi_{s,0} \cdots \psi_{1,0} \Psi(x_N) \cdots \Psi(x_2)\Psi(x_1)|v\rangle.$$  

The last line is precisely the RHS of (5.39). Indeed,

$$\omega_{N,s} \cdot \pi_{N,s}(|v\rangle) = \omega_{N,s} \cdot \sum_{c_1,\ldots,c_N=1}^s \langle 0|\psi_{c_1,0}(x_N) \cdots \psi_{c_2,0}(x_2)\psi_{c_1,0}(x_1)|v\rangle \cdot e_{c_1} \otimes \ldots \otimes e_{c_N}$$

$$= (x_1 \cdots x_N)^{-1} \cdot \sum_{c_1,\ldots,c_N=1}^s \langle 0|\psi_{s,0} \cdots \psi_{1,0} \Psi(x_N) \cdots \Psi(x_2)\Psi(x_1)|v\rangle$$

$$\times e_{c_1} \otimes \ldots \otimes e_{c_N}. \quad \Box$$

Denote by $T(u)$ the generating matrix of operators $T_{ab,n}$

$$T(u) = \sum_{a,b=1}^s E_{ab} \otimes T_{ab}(u) \in \text{End}(C^s) \otimes \mathcal{H}^s [u^{-1}],$$

where

$$T_{ab}(u) = \delta_{ab} + \sum_{n \geq 0} T_{ab,n} u^{-n}.$$  

Proposition 4.2 and lemma 5.3 imply
The following identity holds
\[ QT(u)Q^{-1} = \frac{u+1}{u} T(u - \alpha - s). \] (5.40)

The equality (5.40) can be regarded as a recurrence relation which expresses each \( T_{ab,n} \) via \( \text{Ad}_Q(T_{ab,k}) \) for \( k < n \). In particular, this means that
\[ (\text{Ad}_Q - 1)^{n+2}(T_{ab,n}) = 0, \] (5.41)
and thus any operator \( A = T_{ab,n} \) can be presented as a polynomial of degree \( n + 1 \)
\[ A = A_0 + a_0 A_1 + \cdots + a_0^{n+1} A_{n+1}, \quad \text{QA}_jQ^{-1} = A_j, \] (5.42)
where each \( A_j \) is an element of \( \mathcal{H} \) of zero charge. In its turn, the presentation (5.42) implies

**Proposition 5.3.** The operators \( T_{ab,n} \) satisfy Yangian relations (2.5)

In particular, the coefficients of the quantum determinant \( q \text{det} T(u) \) form a commutative family which can be regarded as the limits of the higher Hamiltonians of CS system. Indeed, due (5.42), each Yangian relation is polynomial over \( a_0 \) and thus it is enough to verify it on subspaces \( F^N_s \), which are eigenspaces of \( a_0 \) with eigenvalue \( N \) for \( N \) big enough. But here in the projection \( \pi_N \) we deal with polynomials of any desirable degree where the relation becomes nontrivial.

Analogously to finite-dimensional case, see (4.22), the transfer matrix \( T(u) \) can be renormalized by means of central operator \( a_0 \) in such a way that the new transfer matrix will commute with \( Q \) and thus acts in equal way in each sector \( F^N_s \) of the Fock space. Here we set
\[ \bar{T}(u) = f(u, a_0)T \left( u + a_0 \frac{\gamma}{s} \right), \] (5.43)
where \( \gamma = \alpha + s \) and
\[ f(u, b) = \frac{\Gamma \left( \frac{u}{2} + \frac{b}{2} + 1 \right) \Gamma \left( \frac{u+1}{2} + 1 \right)}{\Gamma \left( \frac{u+1}{2} + \frac{b}{2} + 1 \right) \Gamma \left( \frac{u}{2} + 1 \right)}. \] (5.44)

**6. Connection with Uglov construction**

Denis Uglov presented two construction of the Yangian action in fermionic space \( F^s \), starting from Yangian action in the phase space of fermionic spin CS system. The paper [19] deals with projective type construction while the work [20] develops an inductive limit approach. Our paper has a close connection with [19].

Initial data of the present work and that of [19] are the same: finite-dimensional representations of Yangians realized via polynomial representations of degenerated affine Hecke algebra. In this setting Uglov uses projective properties of these action and renormalize the transfer matrices of Yangian action in order to form the projective system. He identifies zero charge subspace of \( F^s \) with the projective limit of \( \Lambda^\infty \) and defines the Yangian action on \( F^0_s \) via this identification. The action is extended to other sectors by means of natural identifications of these sectors with \( F^0_s \). The resulting Yangian action is given by implicit formula, analogous (but not the same!) to (4.26), and actually coincides with renormalized finite-dimensional action on stable wedge. Using this description and representation theory of degenerate affine Hecke
algebra, Uglov suggested precise decomposition of the Fock space into direct sum of Yangian irreducibles.

Our construction can be regarded as a free field counterpart of Uglov investigations. However, there are certain differences in two approaches. First, we use different projections from the Fock space to spin CS phase space \( \Lambda^{L,N} \). They differ by the power of \( Q \). Namely, Uglov projection \( \tilde{\pi}_{N_s} : \mathcal{F}_s^0 \to \Lambda^{L,N_s} \) can be expressed via (5.21) by the relation

\[
\tilde{\pi}_{N_s} = \pi_{N_s} Q^{-N_s}.
\]

The use of our projections allows to lift the initial action to the whole Fock space without renormalization. However, after renormalization (5.43) both actions should coincide.

Second, in his normalization procedure [19, proposition 10.2] Uglov lost the shift of spectral parameter which led to disagreement in final results. This disagreement does not affect the decomposition into Yangian irreducibles, but changes the parameters of irreducibles. Namely, one should twist Uglov irreducible components by certain automorphisms of the Yangian. Surely, after the mentioned changes Uglov decomposition can be equally used in our interpretation of the model.

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Appendix

Here we present the expressions for the first densities \( T_{ab,n}(z) \) (5.38) \( n = 0, 1, 2 \) for the Yangian generators. There will be given two types of expressions for each density, the first answer is a normal ordered combination of fermionic fields \( \Psi_e(z) \), \( \Psi^*_d(z) \), the second is not normal ordered, it is given in terms the affine Lie algebra \( \hat{\mathfrak{gl}}_l \) generators.

Now we introduce several notations. Denote by \( T_{ab,n}(z) \) a coefficient of \( \alpha^l \) in \( T_{ab,k+l}(z) \):

\[
T_{ab,n}(z) = \sum_{l=0}^{n} \alpha^l T_{ab,n-l}(z).
\]

Denote by \( E_{ab}(z) \) a generating functions for the elements of the affine Lie algebra \( \hat{\mathfrak{gl}}_l \) :

\[
E_{ab}(z) = \sum_n E_{ab,n} z^n = \frac{z \Psi^*_a(z) \Psi_b(z)}{z - E_{ab}(z)}.
\]

For a formal series \( f(z) = \sum_{n \in \mathbb{Z}} f_n z^n \) we denote by \( f(z)_+ \) the series

\[
f(z)_+ = \sum_{n \geq 0} f_n z^n = \int_{u \geq 0} \frac{f(u)}{u - z} \, du,
\]

\[
f(z)_- = \sum_{n < 0} f_n z^n = \int_{u < 0} \frac{f(u)}{z - u} \, du.
\]

For \( n = 0 \) we simply have

\[
T_{ab,0}(z) = T_{ab,0}^0(z) = \frac{1}{z} E_{ab}(z).
\]
For \( n = 1 \)
\[
\mathcal{T}_{\text{d},1}(z) = \alpha \mathcal{T}_{\text{d},1}^{0,1}(z) + \mathcal{T}_{\text{d},1}^{1,0}(z).
\]

We distinguish the answers for diagonal \( \mathcal{T}_{\text{d},n}(z) \) and nondiagonal part \( \mathcal{T}_{\text{d},n}(z) \), where \( a \neq b \). Firstly we present the expressions for nondiagonal elements \( a \neq b \) as normal ordered combination of fermionic fields:

\[
\mathcal{T}_{ab}^{0,1}(z) = \frac{1}{2} \sum_{c=1}^{s} z \Psi^*_a(z) \Psi_b(z) \left( \Psi_c(z) \Psi_d(z) \right)_+ + \frac{1}{2} \sum_{c=1}^{s} z \Psi^*_a(z) \Psi_b(z) \left( \Psi_c(z) \Psi_d(z) \right)_- + (s + 1) \Psi^*_a(z) \left( \frac{\partial}{\partial z} \Psi_b(z) \right)_+ - \Psi_a(z) \left( \frac{\partial}{\partial z} \Psi_b(z) \right)_-.
\]

The bosonic answer has the recurrent form, we express it from \( \mathcal{T}_{ab}^{0,0}(z) \) (A.1):

\[
\mathcal{T}_{ab}^{0,1}(z) = \int_{w \in \mathbb{C}} \frac{dw}{w - z} E_{a\in}(z) \mathcal{T}_{ab}^{0,0}(w),
\]

\[
\mathcal{T}_{ab}^{1,0}(z) = \sum_{c=1}^{s} \int_{w \in \mathbb{C}, |w| < |z|} \frac{dw}{w - z} E_{a\in}(z) \mathcal{T}_{ab}^{0,0}(w) + \sum_{c=1}^{s} \int_{w \in \mathbb{C}, |w| > |z|} \frac{zdw}{w(z - w)} \mathcal{T}_{ab}^{0,0}(z) E_{a\in}(w)
\]

\[
- \int_{w \in \mathbb{C}, |w| < |z|} \frac{zdw}{w(w - z)} \mathcal{T}_{ab}^{0,0}(z) E_{a\in}(w).
\]

For diagonal elements in case \( n = 1 \) we present the answers in the same way, firstly as a normal ordered combination of fermionic fields:

\[
\mathcal{T}_{aa}^{0,1}(z) = \frac{1}{2} \Psi^*_a(z) \frac{\partial}{\partial z} \Psi_a(z);
\]

\[
\mathcal{T}_{aa}^{1,0}(z) = \frac{1}{2} \sum_{b=1}^{s} \Psi^*_a(z) \Psi_b(z) \left( \Psi_a(z) \Psi_b(z) \right)_+ + \frac{1}{2} \sum_{b=1}^{s} \Psi^*_a(z) \Psi_b(z) \left( \Psi_a(z) \Psi_b(z) \right)_- - \frac{1}{2} \sum_{b=1}^{s} \Psi_b(z) \left( \frac{\partial}{\partial z} \Psi_a(z) \right)_+ + (s + 1) \Psi^*_a(z) \left( \frac{\partial}{\partial z} \Psi_a(z) \right)_- - \Psi_a(z) \left( \frac{\partial}{\partial z} \Psi_a(z) \right)_+.
\]

Then the recurrent answer from previous densities in terms the affine Lie algebra \( \tilde{\mathfrak{gl}}_s \) generators:

\[
\mathcal{T}_{aa}^{0,1}(z) = \frac{1}{2} \int_{w \in \mathbb{C}} \frac{dw}{w - z} E_{a\in}(z) \mathcal{T}_{aa}^{0,0}(w) + \frac{1}{2} \int_{w \in \mathbb{C}} \frac{zdw}{(w - z)^2} \mathcal{T}_{aa}^{0,0}(w),
\]

\[
\mathcal{T}_{aa}^{1,0}(z) = \frac{1}{2} \int_{w \in \mathbb{C}} \frac{dw}{w - z} E_{a\in}(z) \mathcal{T}_{aa}^{0,0}(w) + \frac{1}{2} \int_{w \in \mathbb{C}} \frac{zdw}{(w - z)^2} \mathcal{T}_{aa}^{0,0}(w),
\]
\[ T_{ab}^{(1)}(z) = \sum_{c=1}^{s} \int_{w_0 \subset B \mid w \in \subset B} \frac{dw}{(z-w)} E_{ac}(z)T_{ca}^{(0)}(w) + \sum_{c=1}^{s} \int_{w_0 \subset B \mid w \in \subset B} \frac{zdw}{w(z-w)} T_{cc}^{(0)}(z)E_{cc}(w) - \]

\[ - \sum_{c=1}^{s} \left( \int_{w_0 \subset B \mid w \in \subset B} \frac{dw}{(w-z)} E_{cc}(z)T_{cc}^{(0)}(w) - T_{cc}^{(0)}(z) \right) + \int_{w_0 \subset B \mid w \in \subset B} \frac{zdw}{(w-z)^2} T_{cc}^{(0)}(w) - T_{cc}^{(0)}(z). \]

For \( n = 2 \)

\[ T_{ab,2}(z) = \alpha^2 T_{ab,2}^{(0)}(z) + \alpha T_{ab,1}^{(1)}(z) + T_{ab}^{(2)}(z) \]

We split \( T_{ab,1}^{(1)}(z) \) into two summands:

\[ T_{ab,1}^{(1)}(z) = (T_{ab,1}^{(1)})'(z) + (T_{ab,1}^{(1)})''(z). \]

Here \( (T_{ab,1}^{(1)})'(z) \) means that firstly we apply \( \frac{\partial}{\partial z} \) and then the difference part of the Dunkl operator, \( (T_{ab,1}^{(1)})''(z) \) backwards.

\[ T_{ab,1}^{(2)}(z) = \varepsilon^2 \Psi(z) \left( \frac{\partial}{\partial z} \right)^2 \Psi(z); \]

\[ (T_{ab,1}^{(1)})'(z) = \sum_{c=1}^{s} \varepsilon \Psi'(z) \left( z \frac{\partial}{\partial z} \Psi(z) \right) \left( \Psi_\varepsilon'(z) \Psi_\varepsilon(z) \right) - \sum_{c=1}^{s} \varepsilon \Psi'(z) \left( z \frac{\partial}{\partial z} \Psi(z) \right) \left( \Psi_\varepsilon'(z) z \frac{\partial}{\partial z} \Psi(z) \right) \]

\[ + \left( s + \frac{1}{2} \right) \Psi'(z) \left( z \frac{\partial}{\partial z} \right)^2 \Psi(z) + \left( z \frac{\partial}{\partial z} \Psi(z) \right) \left( \Psi'(z) z \frac{\partial}{\partial z} \Psi(z) \right) \]

\[ - \frac{1}{2} \Psi'(z) \left( z \frac{\partial}{\partial z} \Psi(z) \right) \cdot \]

\[ (T_{ab,1}^{(1)})''(z) = \sum_{c=1}^{s} \varepsilon \Psi'(z) \left( z \frac{\partial}{\partial z} \right) \left( z \frac{\partial}{\partial z} \right) \left( \Psi_\varepsilon'(z) \Psi_\varepsilon(z) \right) - \sum_{c=1}^{s} \varepsilon \Psi'(z) \left( z \frac{\partial}{\partial z} \right) \left( z \frac{\partial}{\partial z} \right) \left( \Psi_\varepsilon'(z) z \frac{\partial}{\partial z} \Psi(z) \right) \]

\[ + \left( s + 1 \right) \Psi'(z) \left( z \frac{\partial}{\partial z} \right)^2 \Psi(z) - \frac{1}{2} \Psi'(z) \left( z \frac{\partial}{\partial z} \Psi(z) \right) + \left( z \frac{\partial}{\partial z} \right) \left( \Psi'(z) z \frac{\partial}{\partial z} \Psi(z) \right) \]

\[ - \left( z \frac{\partial}{\partial z} \Psi(z) \right) \left( z \frac{\partial}{\partial z} \Psi'(z) z \frac{\partial}{\partial z} \Psi(z) \right) + \left( z \frac{\partial}{\partial z} \Psi(z) \right) \left( \Psi'(z) z \frac{\partial}{\partial z} \Psi(z) \right) \cdot \]

The recurrent formula from previous densities in terms the affine Lie algebra \( \widehat{\mathfrak{gl}}_s \) generators:

\[ T_{ab,2}^{(2)}(z) = \int_{w_0 \subset B \mid w \in \subset B} \frac{dw}{(z-w)^2} E_{ac}(z)T_{ca}^{(0)}(w) \]

\[ (T_{ab,1}^{(1)})'(z) = \sum_{c=1}^{s} \int_{w_0 \subset B \mid w \in \subset B} \frac{dw}{(z-w)} E_{ac}(z)T_{ca}^{(0)}(w) + \sum_{c=1}^{s} \int_{w_0 \subset B \mid w \in \subset B} \frac{zdw}{w(z-w)} T_{cc}^{(0)}(z)E_{cc}(w) \]

\[ (T_{ab,1}^{(1)})''(z) = \sum_{c=1}^{s} \int_{w_0 \subset B \mid w \in \subset B} \frac{dw}{(z-w)} E_{ac}(z)T_{ca}^{(0)}(w) + \sum_{c=1}^{s} \int_{w_0 \subset B \mid w \in \subset B} \frac{zdw}{w(z-w)} T_{cc}^{(0)}(z)E_{cc}(w) \]
For diagonal elements in case $n = 2$ we have more complicated formulas:

\[
\mathcal{T}_{aa}^{0,1}(z) = \sum_{\ell = 1}^{s} z \Psi_{\ell}(z) \left( z \frac{\partial}{\partial z} \right)^{2} \Psi_{\ell}(z),
\]

\[
(T_{aa}^{0,1})' = \sum_{\ell = 1}^{s} z \Psi_{\ell}(z) \left( z \frac{\partial}{\partial z} \right) \left( \Psi_{\ell}(z) \frac{\partial}{\partial z} \Psi_{\ell}(z) \right) + \sum_{\ell = 1}^{s} z \Psi_{\ell}(z) \left( \Psi_{\ell}(z) z \frac{\partial}{\partial z} \Psi_{\ell}(z) \right) - \frac{1}{2} \sum_{\ell = 1}^{s} \frac{\partial}{\partial z} \Psi_{\ell}(z) \left( z \frac{\partial}{\partial z} \Psi_{\ell}(z) \right) + \frac{1}{2} \sum_{\ell = 1}^{s} \frac{\partial}{\partial z} \Psi_{\ell}(z) \left( z \frac{\partial}{\partial z} \Psi_{\ell}(z) \right)
\]

\[
(T_{aa}^{0,1})'' = \sum_{\ell = 1}^{s} z \Psi_{\ell}(z) \left( z \frac{\partial}{\partial z} \right) \left( \Psi_{\ell}(z) \frac{\partial}{\partial z} \Psi_{\ell}(z) \right) + \sum_{\ell = 1}^{s} z \Psi_{\ell}(z) \left( z \frac{\partial}{\partial z} \Psi_{\ell}(z) \right) + \frac{1}{2} \sum_{\ell = 1}^{s} \frac{\partial}{\partial z} \Psi_{\ell}(z) \left( z \frac{\partial}{\partial z} \Psi_{\ell}(z) \right) - \frac{1}{2} \sum_{\ell = 1}^{s} \frac{\partial}{\partial z} \Psi_{\ell}(z) \left( z \frac{\partial}{\partial z} \Psi_{\ell}(z) \right)
\]

\[
\mathcal{T}_{aa}^{0,2}(z) = \sum_{\ell = 1}^{s} \sum_{\ell' = 1}^{s} \Psi_{\ell}(z) \left( z \frac{\partial}{\partial z} \right) \left( \Psi_{\ell'}(z) \frac{\partial}{\partial z} \Psi_{\ell'}(z) \right) + \sum_{\ell = 1}^{s} \sum_{\ell' = 1}^{s} \Psi_{\ell}(z) \left( z \frac{\partial}{\partial z} \Psi_{\ell}(z) \right) \left( \Psi_{\ell'}(z) \frac{\partial}{\partial z} \Psi_{\ell'}(z) \right) - \frac{1}{2} \sum_{\ell = 1}^{s} \sum_{\ell' = 1}^{s} \frac{\partial}{\partial z} \Psi_{\ell}(z) \left( z \frac{\partial}{\partial z} \Psi_{\ell}(z) \right) \left( z \frac{\partial}{\partial z} \Psi_{\ell'}(z) \right) + \frac{1}{2} \sum_{\ell = 1}^{s} \sum_{\ell' = 1}^{s} \frac{\partial}{\partial z} \Psi_{\ell}(z) \left( z \frac{\partial}{\partial z} \Psi_{\ell}(z) \right) \left( z \frac{\partial}{\partial z} \Psi_{\ell'}(z) \right)
\]
\begin{align*}
\frac{2}{3} \int \frac{zd}\langle w - z \rangle^2 T_{aa}^{0,1}(w) + \frac{1}{3} T_{aa}^{0,1}(z) \\
(T_{aa}^{1,1})'(z) = \sum_{c=1}^{s} \int \frac{d}\langle z - w \rangle E_{ac}(z) T_{aa}^{0,1}(w) + \sum_{c=1}^{s} \int \frac{zd}\langle w(z - w) \rangle T_{aa}^{0,1}(z) E_{cc}(w) \\
- \sum_{c=1}^{s} \int \frac{d}\langle w(z - w) \rangle E_{ac}(z) T_{aa}^{0,1}(w) - T_{aa}^{0,2}(z) \\
(T_{aa}^{1,1})''(z) = \int \frac{zd}{\langle w(z - w) \rangle} \left( T_{aa}^{0,1}(z) + z \frac{\partial}{\partial z} T_{aa}^{0,0}(z) \right) \\
- T_{aa}^{0,1}(z) E_{bb}(w) - \sum_{b=1}^{s} \int \frac{zd}{\langle w(z - w) \rangle} E_{ab}(z) E_{bb}(w) \\
- \sum_{b=1}^{s} \int \frac{zd}{\langle w(z - w) \rangle} E_{bb}(w) + \frac{zd}{\langle w(z - w) \rangle} T_{aa}^{0,0}(z) + z \frac{\partial}{\partial z} T_{aa}^{0,0}(z) \\
- \int \frac{zd}{\langle w(z - w) \rangle} E_{aa}(w) - \int \frac{zd}{\langle w(z - w) \rangle} E_{bb}(w) \\
+ \sum_{b=1}^{s} \int \frac{zd}{\langle w(z - w) \rangle} \left( T_{aa}^{0,0}(z) + z \frac{\partial}{\partial z} T_{aa}^{0,0}(z) \right) E_{bb}(w) \\
+ \int \frac{zd}{\langle w(z - w) \rangle} E_{bb}(w) - \sum_{b=1}^{s} \int \frac{zd}{\langle w(z - w) \rangle} E_{ab}(w) E_{bb}(w) \\
+ \int \frac{zd}{\langle w(z - w) \rangle} E_{aa}(w) - \int \frac{zd}{\langle w(z - w) \rangle} E_{bb}(w) \\
+ 2 \int \frac{zd}{\langle w(z - w) \rangle} E_{aa}(w) - 2 \int \frac{zd}{\langle w(z - w) \rangle} T_{aa}^{0,1}(w) + \int \frac{zd}{\langle w(z - w) \rangle} E_{aa}(w) + T_{aa}^{0,2}(z).
\end{align*}

The density \( T_{11}^{2,0}(\cdot) \) has a cumbersome form and we do not present it here. In scalar case \((s = 1)\) the matrix coefficient \( T_{11}^{1,0} \) is a polynomial in zero mode of the scalar bosonic field:

\[ T_{11}^{2,0} = \frac{1}{6} \left( 2a_0^1 - 3a_0^2 + a_0 \right). \]

In scalar case the same is for higher orders: the matrix coefficient \( T_{11}^{n,0} \) is a polynomial of degree \((n + 1)\) in zero mode of the scalar bosonic field \([12]\).
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