Localized States Emerging from Singular and Nonsingular Flat Bands in a Frustrated Fractal-Like Photonic Lattice

Haissam Hanafi,* Philip Menz,* and Cornelia Denz

Singular and nonsingular flat bands in a Sierpinski fractal-like photonic lattice are reported. It is demonstrated that the lowest two bands, being isolated and degenerate due to geometrical frustration, are nonsingular and thus can be spanned by a complete set of compact localized states. These states are experimentally proven to propagate diffractionless in the photonic lattice. The results reveal the interplay between geometrical frustration, degenerate flat bands, and compact localized states in a single photonic lattice, and pave the way to photonic spin liquid ground states.

1. Introduction

Periodic systems, described by the Bloch formalism and governed by Schrödinger-like equations, are the focus of various scientific fields. These systems possess a band structure that describes the energy levels of allowed and forbidden quantum mechanical states. One striking feature in such a band structure is the presence of entirely flat bands that are dispersionless over the whole Brillouin zone. The energy of the states occupying such a band is independent of their momentum. This makes flat band systems an ideal testbed to explore strong correlation physics as, for example, the fractional quantum Hall effect,[1] or Wigner crystallization.[2] In the context of the Bloch formalism, the flat band eigenstates of the tight-binding Hamiltonian are degenerate, having the same eigenvalue. Flat bands are usually described by real space eigenfunctions that are compact and localized, the so-called compact localized states (CLSs).[3] Localization is called compact when it has a strictly vanishing amplitude outside a finite region of the lattice. Such single-particle states originate from destructive interference of the wave function after the hopping process described by the Hamiltonian. The destructive interference leads zero amplitude at some key lattice sites forming the local symmetry partition and effectively confines the wavefunction to a characteristic trapping prison. The frozen amplitude distribution inside the prison does not allow any dynamics of the prisoner beyond the trapping cell making the kinetic information solely quenched.[4] This immobility of the CLSs can be seen as a consequence of the divergent effective mass tensor and the singularity in the density of states profile of the flat band. This gives rise to anomalous behavior in the transport and the response of the system.[5]

Compact localized states have been used to construct systematic flat band generators. Starting from the size and form of a CLSs, a family of lattice Hamiltonians hosting the corresponding flat band can be obtained in 1D[6] and 2D.[7] It is therefore possible to classify flat bands through the real space properties of their CLSs. In 2019 Rhim et al. proposed a momentum space classification of flat bands into singular and nonsingular ones based on the presence or absence of immovable discontinuities in their Bloch wave function.[8] This classification can also be interpreted as the incompleteness or completeness of the set of CLSs in spanning the whole flat band. There is no immovable discontinuity of its Bloch wave function for a nonsingular flat band. A discontinuity is immovable when there exists no local gauge choice which makes the Bloch wave function continuous by shifting the position of the singular point.[8] A nonsingular flat band can be isolated from the dispersive bands, and the CLSs form a complete set. On the other hand, if the band is singular, it possesses an immovable discontinuity that originates from touching with at least one other dispersive band at the singular point. In this case, the CLSs turn out to be incomplete, that is, missing states which manifest nontrivial real-space topology must be complemented to form a complete set. These states are known as noncontractible loop states (NLSs). They are localized and compact in one direction, while they extend infinitely in the other direction and cannot be contracted by adding CLSs.[9]

Up to now, singular and nonsingular flat bands have been realized mainly in separate, tailored photonic lattices that explicitly exhibit the state to be investigated. In a photonic lattice the band structure is a spatial diffraction relation. In this kind of system, CLSs and NLSs of singular flat bands have been shown in the Lieb lattice,[10–12] or the kagome lattice.[13,14] While CLSs of nonsingular flat bands have been demonstrated to arise in 1.5D lattices,[15] a driven graphene lattice,[16] or due to Aharonov–Bohm caging in a diamond lattice.[17] Only recently first efforts were made to combine the two classes of flat bands in an optically induced lattice in a photorefractive crystal.[18] However, up to now, no experimental realization has proved the coexistence of these two topological classes of...
singular and nonsingular flat bands in a single lattice by analyzing the completeness conditions of the sets of CLSs. Here, we propose and experimentally demonstrate, for the first time to our knowledge, a fractal-like lattice that exhibits multiple flat bands fabricated by femtosecond direct laser writing on a large-scale fused silica chip. Crucially, we deliver the fundamental proof of singularity or nonsingularity of these bands by deriving and experimentally demonstrating their relative incomplete and complete sets of CLSs. The fractal-like lattice used for this purpose was described theoretically in 2018 by Pal et al.\[19\] and is constructed taking a first-generation Sierpinski gasket as the unit cell (Figure 1a). Note that its band structure hosts three flat bands. We reveal the two degenerate flat bands at the lowest energy to be nonsingular, whereas the flat band at zero energy to be singular (Figure 1d). The double degeneracy of the isolated flat bands in the Sierpinski fractal-like lattice is a feature this lattice shares with the kagome-3 model. This theoretical model was proposed in early studies on singular flat bands,\[9,20\] emphasizing the frustrated configuration inherent to the kagome lattice in general.\[21\] Geometrical frustration is best known in antiferromagnetic interactions when the lattice geometry inhibits the formation of a simple, ordered spin configuration.\[22\] The resulting degenerate manifold of ground states can be interpreted as a spin liquid. In our photonic system, we demonstrate that the degenerate flat bands of the Sierpinski fractal-like lattice are nonsingular, similar to those in the kagome-3 model. In contrast, the zero-energy flat band turns out to be singular. Additionally, we explain how the geometrical frustration of the lattice is linked to multiple degenerate flat bands. For the experimental realization, we chose a photonic system based on an array of evanescently coupled waveguides known as a photonic lattice.\[23,24\] Photonic lattices represent a highly effective platform for studying many intriguing phenomena related to periodic structures.\[25–28\] They have the advantage of allowing to access the flat band signature as light localization. The light propagation in such a lattice is governed by a Schrödinger-like paraxial wave equation, with the band structure representing a spatial diffraction relation.\[29\] Compared to an atomic lattice where the transport dynamics of electrons are observed, a photonic lattice is characterized by the light field evolution in propagation direction describing photon transport. The role of the energy in the dispersion relation is taken by the propagation constant $\beta$. Accordingly, electron transport, which we refer to as hopping in electronic systems, in a photonic lattice is replaced by evanescent coupling of light between neighboring waveguides.\[30\] As for possible applications, diffractionless photonic flat band states have, for example, been proposed for distortion free image transmission,\[31\] and lasing.\[32\] Fractal geometries in particular have been proposed as prominent candidates for slow light.\[33\]

2. The Sierpinski Fractal-Like Lattice

2.1. Band Structure and CLSs

In a photonic lattice, the realization of the kagome-3 model is not trivial since it assumes equal coupling strength for nearest and next-nearest neighbors, whereas in a photonic lattice the coupling is proportional to the waveguide separation. However,
the Sierpinski fractal-like lattice shows doubly degenerate flat bands as the kagome-3 model does and includes even richer features. The Sierpinski fractal-like lattice is obtained by a “decoration” of the kagome lattice with additional lattice sites. Decorating lattices is a well-known technique in the context of localization and flat bands.\cite{34,35} Adding three lattice sites to the triangular kagome unit cell in between existing ones, and spacing them uniformly to obtain equal coupling, leads to the Sierpinski gasket unit cell shown in Figure 1a.

In order to calculate the band structure of the lattice, we chose a tight-binding approximation.\cite{36} We assume equal hopping amplitude between nearest neighbors, and set it to 1 (together with the lattice constant \(d = 1\)) for simplicity. We then obtain a \(6 \times 6\) Hamiltonian in momentum space whose dimension reflects the size of the unit cell (Equations (S1)–(S3), Supporting Information). By solving the eigenvalue problem, we obtain the band structure shown in Figure 1d consisting of six bands. The lowest two of them, located at \(\beta = -2\), are degenerated, isolated from the dispersive bands, and completely flat over the whole Brillouin zone. There is a third flat band at \(\beta = 0\) that touches two other dispersive bands at \(k = 0\) forming a spin-1 Dirac cone like in the kagome with staggered flux,\cite{37} or in the Lieb lattice.\cite{38} To obtain the CLSs of the flat bands, we perform an inverse Fourier transform of the corresponding eigenvectors of the Hamiltonian (see Equation (S5), Supporting Information). To validate the found CLSs, we prove that they fulfill a Schrödinger equation of the form

\[
(\beta - \varepsilon_i)\psi_i = \sum_j \tau_{ij} \psi_j
\]

where \(\varepsilon_i\) is the on-site potential at the \(i\)th site, \(\tau_{ij}\) is the hopping amplitude between neighboring sites, and \(\psi_i\) is the wave function amplitude at the \(i\)th site.\cite{19} In our case, we assume all sites to have equal potential and shift this value to zero and consider only nearest neighbors with a hopping amplitude \(\tau_{ij} = 1\).

The CLS from the flat band at \(\beta = 0\) is shown in Figure 2a1. It takes the form of a truncated triangle and corresponds to an extension of the normal kagome CLS. Due to the decoration of the kagome unit cell, the flat band is shifted from \(\beta = -2\) to \(\beta = 0\).\cite{35} Like in the kagome lattice, \(N\) translated copies of the CLS are linearly dependent due to the discontinuity of the eigenvector at the center of the Brillouin zone. The discontinuity is immovable, that is, the band touching is singular. Therefore, the flat band cannot be fully described by the CLSs, and some NLSs exist. The NLSs of the Sierpinski fractal-like lattice can be derived considering NLSs of the normal kagome lattice. In the kagome lattice, the NLSs consist of lines of lattice sites with the same amplitude but alternating opposite phase on the lattice axes.\cite{9} In the Sierpinski lattice, they essentially remain the same, only being decorated (see Figure S3, Supporting Information). Like in the kagome lattice, the NLSs are stable only under periodic boundary conditions as they extend infinitely in one direction. In systems with finite open boundary conditions, they cannot be obtained (unlike in the case of the Lieb lattice.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure2.png}
\caption{CLSs and experimental demonstration of their diffraction-free propagation in the Sierpinski fractal-like lattice. a1) CLS at \(\beta = 0\); intensity (a2) and phase (a3) of the CLS light field at \(\beta = 0\) after propagation in the lattice; intensity (a4) and phase (a5) of a light field with the same amplitude of the CLS at \(\beta = 0\) but with flat phase after propagation in the lattice. b1) reindeer CLS-1; (b2)–(b5) same as (a2)–(a5) but for the reindeer CLS-1. c1) reindeer CLS-2; (c2)–(c5) same as (a2)–(a5) but for the reindeer CLS-2.}
\end{figure}
where designed boundaries allow this.\cite{22} However, to realize the NLSs, the lattice must be constructed with a specific periodic boundary geometry (Corbino geometry), or the mode must be spanned on a closed contour along the boundary of a finite lattice, realizing a robust boundary mode (RBM).\cite{24}

At first sight, the situation seems similar for the flat bands at $\beta = -2$. The two eigenvectors are discontinuous at two distinct points in the Brillouin zone (see Equation (S4), Supporting Information). The CLSs we obtain by inverse Fourier transform do not form a complete set, that is, they cannot wholly span the flat band (note that previously for $\beta = -2$ only one CLS was proposed, which has a less compact form and also does not form a complete set).\cite{23} Thus, additional NLSs are needed, which intuitively can be identified to be similar to those at $\beta = 0$, but with alternating opposite phases (see Figure S3, Supporting Information). However, the two flat bands at $\beta = -2$ are isolated from the dispersive bands, that is, there is no band touching (besides from the complete degeneracy itself). Therefore, these bands are nonsingular, and thus a complete set of CLSs can be found. In fact, it turns out that, exactly as in the kagome-3 model, it is possible to find two nonsingular eigenvectors and their related CLSs, which make the supposed NLSs contractible. The key point is that the flat bands are completely degenerate. Hence, it is possible to mix the two eigenvectors by linearly combining them to obtain nonsingular ones that do not have discontinuities. As before, from these new eigenvectors, we obtain an expression for the CLSs (see Equation (S7), Supporting Information). These CLSs are shown in Figure 2 b1,c1, respectively. Due to their appearance we name this states reindeer CLS-1 and reindeer CLS-2. Since this set of CLSs is complete, it is possible to construct the supposed NLSs and any other state on the flat band by a linear combination of them.

**2.2. Geometrical Frustration, Fractal Self-Similarity, and Degenerate Flat Bands**

Most of the early work on singular flat bands and NLSs focused on lattices exhibiting strong geometrical frustration.\cite{9} It was, however, demonstrated that this is not a necessary condition.\cite{12,39} The NLSs are a direct manifestation of the immovable discontinuities of the Bloch functions generated by the band touching.\cite{8} We extend these findings and demonstrate that flat bands of a geometrically frustrated lattice can be singular and nonsingular. Note that there is a relation between degenerate flat bands and geometrical frustration in the fractal-like lattice. This optical analog of geometrical frustration does not involve spins. Rather, it arises because the destructive interference interactions leading to compact localization are, to a certain degree, incompatible with the underlying lattice geometry. This can be seen in the form of the reindeer CLSs: if we try to construct a CLS into the first-generation Sierpinski triangle unit cell, we cannot simultaneously obtain destructive interference in all three corners of the triangle (lattice sites A, B, and C). If we choose destructive interference in B and C, we end up with the reindeer CLS-1 as the smallest possible CLS; if instead we want to achieve destructive interference in A and B, we get the reindeer CLS-2.

Geometrical frustration is not solely responsible for the degeneracy of the flat bands; also the fractal self-similarity of the unit cell plays a role. Here we choose a first generation $n = 1$ Sierpinski gasket with a generator side length $b = 2$ as the unit cell.\cite{39} For higher generations, the number of nonsingular degenerate flat bands at $\beta = -2$ increases with the number of lattice sites per unit cell $N_n$ as $N_n^3$.\cite{41} Due to the fractal nature of the lattice, the CLSs of those flat bands retain their size and shape in a self-similar manner for higher generation of the Sierpinski gasket. This is not the case for all flat bands of such a lattice. In fact, CLSs at other $\beta$s are localized over self-similar clusters of increasing size for higher generations of the Sierpinski gaskets.\cite{42} Thus, we conclude that geometrical frustration and fractal geometry lead to a high degeneracy of flat bands and dispersionless localized states.

**3. Experimental Realization of the Sierpinski Fractal-Like Photonic Lattice**

We experimentally realize the previously discussed localized flat band states of the Sierpinski fractal-like lattice by fabricating a sample composed of 181 single-mode waveguides arranged in $5 \times 5$ unit cells with a rhombic termination. The single-mode waveguides are induced by femtosecond direct laser writing in a fused silica glass chip (Ultrapure Synthetic Fused Silica SQ0).\cite{43-45} The induction laser emits light at a central wavelength of 1030 nm, with a pulse duration of 250 fs at a repetition rate of 125 kHz. In order to ensure circular and homogeneous single-mode waveguides over the entire extent of the lattice, we apply a slit beam shaping method in combination with a spherical aberration correction.\cite{46} The induction laser beam is hereby shaped by a single phase-only spatial light modulator (SLM). A white light microscope image of the front-facet of the fabricated lattice is shown in Figure 1b. A lattice constant of $d = 66 \mu$m (waveguide separation of 22 \mu m) is chosen. This ensures sufficient coupling during z-propagation in the 2cm sample to distinguish clearly between diffracting and nondiffracting states, while also being large enough to prevent next-nearest neighbor coupling disturbing the flatness of the bands.

We excite the flat band states by using the experimental system depicted in Figure 1c. The probing continuous wave (cw) laser beam operates at a wavelength of 532 nm, and is modulated in amplitude and phase with a phase-only SLM to generate the different CLS-light fields.\cite{47} To ensure high coupling of the CLS-light fields into the photonic lattice, we monitor the front-facet of the lattice and align in to the back-reflection. We record the output by imaging the lattice back-facet onto a camera using a microscope objective. The phase of the light field is recorded by interference with a tilted plane wave and reconstructed via a digital holographic method.\cite{48} In principle, thanks to the flexibility of these fabrication and probing systems, our experimental procedure is not limited to certain orders of the Sierpinski fractal, so higher generations are possible. The only limitations are the physical size of the glass sample and the finite resolution of the spatial light modulator.

We now experimentally demonstrate flat bands of singular and nonsingular nature in a single photonic lattice. First, we investigate the CLS of the flat band located at $\beta = 0$. The light field exciting the CLS is composed of Gaussian spots of...
alternating opposite phase arranged in a truncated triangle and grouped two-by-two (Figure 2 a1). We compare the propagation of this light field in the photonic lattice with a light field with the same intensity distribution but equal phase at all excited lattice sites. This state will excite not only the flat band, but also diffracting ones. The outcome should therefore be clearly distinguishable from the CLS one. The propagated lights fields, which we measure at the back-facet of the lattice, are depicted in Figure 2 a2–a5. In Figure 2 a2,a3 we observe the light field exciting the CLS propagating without diffraction, in a robust and localized way over the whole lattice length of 2 cm. In contrast, as shown in Figure 2 a4,a5, a light field having a flat phase strongly diffracts, and the initial light field distribution can no longer be recognized in the output. The non-flat band state diffracts anomalously during propagation in the lattice because it is partly composed of Bloch modes belonging to convex regions of the band structure. That is why in Figure 2 a4 there is intensity at the inner lattice sites. For a longer propagation distance, the CLS would stay localized in its trapping prison as in Figure 2 a2, while its diffracting counterpart would diffract further eventually spreading over the entire lattice.

The same clear results are obtained for the two degenerated flat bands at $\beta = -2$ and the corresponding reindeer CLSs. The light fields corresponding to the CLSs of the degenerate flat bands (Figure 2 b1,c1) propagate diffraction free, and the initial state is preserved (Figure 2 b2,b3,c2,c3). On the other hand, for a light field with the same intensity distribution but equal phase at all excited lattice sites, we observe strong diffraction into the photonic lattice (Figure 2 b4,b5,c4,c5). Overall, the observed diffractionless propagation of the CLSs reflects our theoretical expectations nearly perfectly. We can quantify this by evaluating how much of the total output intensity is found in the originally excited waveguides after propagation. For the CLS at $\beta = 0$ shown in Figure 2 a2 we obtain a value of $\approx 96\%$. For the reindeer CLSs shown in Figure 2 b2,c2 we obtain values of around 98% and 96%, respectively. The reindeer CLSs are a complete set spanning the whole flat band. To demonstrate this, we prove that the supposed NLSs are contractible, as they can be constructed by a linear superposition of reindeer CLSs. In Figure 3, we show a linear superposition of two horizontally-shifted reindeer CLSs that are either out of phase a3, or in phase c3. By adding more reindeer CLSs, the zigzag or straight lines can be extended indefinitely. We demonstrate that these line segments remain intact during propagation in the lattice (Figure 3 b1,b2, d1,d2), while the line segments with flat phase diffract during propagation (Figure 3 b3,b4, d3,d4). Though within this paper we only demonstrate linear superposition of one type of reindeer CLS, the findings hold true also for the other reindeer CLSs, and the associated line segments (see Figure S8, Supporting Information).

4. Conclusion

In conclusion, we have demonstrated singular and nonsingular flat bands in a first-generation Sierpinski fractal-like
photonic lattice fabricated by femtosecond direct laser writing on a large-scale fused silica chip. We explained that the Sierpinski fractal-like lattice is constructed by a decoration of the kagome lattice and has three flat bands. We revealed that one flat band is singular as in the kagome lattice, while the other two are degenerate, isolated and therefore nonsingular as in the kagome-3 model.[9,20] We theoretically derived the compact localized states of the singular and nonsingular flat bands, and experimentally proved them to propagate diffractionless in the photonic lattice. We confirm the nonsingularity of the flat bands by showing theoretically and experimentally that with the right choice of CLSs, the reindeer-CLSs, every supposed NLS is a superposition of them. Our results validate the classification of flat bands according to the band-crossing singularity of the respective Bloch wave function.[8] Our findings allow a general insight into the nature of degenerate flat band systems, and pave the way to photonic lattice realizations based on frustrated interactions, including photonic analogues of spin liquid phenomena.

Supporting Information

Supporting Information is available from the Wiley Online Library or from the author.

Acknowledgements

Open access funding enabled and organized by Projekt DEAL.

Conflict of Interest

The authors declare no conflict of interest.

Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Keywords

compact localized states, flat bands, fractal, geometrical frustration, photonic lattices

Received: November 19, 2021
Revised: February 18, 2022
Published online: April 3, 2022

[1] T. Neupert, L. Santos, C. Chamon, C. Mudry, Phys. Rev. Lett. 2011, 106, 236804.
[2] C. Wu, D. Bergman, L. Balents, S. Das Sarma, Phys. Rev. Lett. 2007, 99, 070401.
[3] B. Sutherland, Phys. Rev. B 1986, 34, 5208.
[4] A. Nandy, Eur. Phys. J. B 2019, 92, 213.
[5] A. Nandy, A. Mukherjee, Physics Lett. A 2019, 383, 2318.
[6] W. Maimaiti, A. Andreanov, H. C. Park, O. Gendelman, S. Flach, Phys. Rev. B 2017, 95, 115135.
[7] W. Maimaiti, A. Andreanov, S. Flach, Phys. Rev. B 2021, 103, 165116.
[8] J. W. Rhim, B. J. Yang, Phys. Rev. B 2019, 99, 045107.
[9] D. L. Bergman, C. Wu, L. Balents, Phys. Rev. B 2008, 78, 125104.
[10] R. A. Vicenzo, C. Cantillano, L. Morales-Inostroza, B. Real, C. Mejia-Cortes, S. Weimann, A. Szameit, M. I. Molina, Phys. Rev. Lett. 2015, 114, 245503.
[11] S. Mukherjee, A. Spracklen, D. Choudhury, N. Goldman, P. Ohberg, E. Andersson, R. R. Thomson, Phys. Rev. Lett. 2015, 114, 245504.
[12] S. Xia, A. Ramachandran, S. Xia, D. Li, X. Liu, L. Tang, Y. Hu, D. Song, J. Xu, D. Leykam, S. Chen, Phys. Rev. Lett. 2018, 121, 263902.
[13] Y. Zong, S. Xia, L. Tang, D. Song, Y. Hu, Y. Pei, J. Su, Y. Li, Z. Chen, Opt. Express 2016, 24, 8877.
[14] J. Ma, J. W. Rhim, L. Tang, S. Xia, H. Wang, X. Zheng, S. Xia, D. Song, Y. Hu, Y. Li, B. J. Yang, D. Leykam, S. Chen, Phys. Rev. Lett. 2020, 124, 183901.
[15] E. Travkin, F. Diebel, C. Denz, Appl. Phys. Lett. 2017, 111, 011104.
[16] A. Crespi, G. Corrielli, G. D. Valle, R. Osellame, S. Longhi, New J. Phys. 2013, 15, 013012.
[17] S. Mukherjee, M. Di Liberto, P. Ohberg, R. R. Thomson, N. Goldman, Phys. Rev. Lett. 2018, 121, 075502.
[18] Y. Xie, L. Song, W. Yan, S. Xia, L. Tang, D. Song, J.-W. Rhim, Z. Chen, APL Photonics 2021, 6, 116104.
[19] B. Pal, K. Saha, Phys. Rev. B 2018, 97, 195101.
[20] L. Balents, M. P. Fisher, S. M. Girvin, S. M. Girvin, Phys. Rev. B 2002, 65, 224412.
[21] A. Mielke, J. Phys. A: Math. Gen. 1992, 25, 4335.
[22] R. Moessner, A. P. Ramirez, Phys. Today 2006, 59, 24.
[23] D. Leykam, S. Flach, APL Photonics 2018, 3, 070901.
[24] L. Tang, D. Song, S. Xia, S. Xia, J. Ma, W. Yan, Y. Hu, J. Xu, D. Leykam, S. Chen, Nanophotonics 2020, 9, 1161.
[25] N. K. Efremidis, S. Sears, D. N. Christodoulides, J. W. Fleischer, M. Segev, Phys. Rev. E 2002, 66, 046602.
[26] B. Terhalke, T. Richter, A. S. Desyatnikov, D. N. Neshev, W. Krolikowski, F. Kaiser, C. Denz, Y. S. Kivshar, Phys. Rev. Lett. 2008, 101, 013903.
[27] I. L. Garanovich, S. Longhi, A. A. Sukhorukov, Y. S. Kivshar, Phys. Rep. 2012, 518, 1.
[28] H. Hanafi, presented at OSA Nonlinear Optics, Virtual event, August, 2021.
[29] S. Longhi, Laser Photonics Rev. 2009, 3, 243.
[30] Z.-Q. Zhang, P. Sheng, Phys. Rev. B 1994, 49, 83.
[31] S. Xia, Y. Hu, D. Song, Y. Zong, L. Tang, Z. Chen, Optics Lett. 2016, 41, 1435.
[32] S. Longhi, Optics Lett. 2019, 44, 287.
[33] A. Nandy, A. Chakrabarti, Phys. Rev. A 2016, 93, 013807.
[34] H. Tasaki, Phys. Rev. Lett. 1992, 69, 1608.
[35] R. G. Dias, J. D. Gouveia, Sci. Rep. 2015, 5, 16852.
[36] P. R. Wallace, Phys. Rev. 1947, 71, 622.
[37] D. Green, L. Santos, C. Chamon, Phys. Rev. B 2010, 82, 075104.
[38] F. Diebel, D. Leykam, C. Kroesen, C. Denz, A. S. Desyatnikov, Phys. Rev. Lett. 2016, 116, 185302.
[39] W. Yan, H. Zhong, D. Song, Y. Zhang, S. Xia, L. Tang, D. Leykam, Z. Chen, Adv. Opt. Mater. 2020, 8, 1902174.
[40] T. Stošić, B. Stošić, S. Milošević, H. E. Stanley, Phys. Rev. E 1994, 49, 1009.
[41] E. Domany, S. Alexander, D. Bensimon, L. P. Kadanoff, Phys. Rev. B 1983, 28, 3110.
[42] A. Nandy, B. Pal, A. Chakrabarti, J. Phys.: Condens. Matter 2015, 27, 125501.
[43] T. Pertsch, U. Peschel, F. Lederer, J. Burghoff, M. Will, S. Nolte, A. Tünnermann, Optics Lett. 2004, 29, 468.
[44] K. Itoh, W. Watanabe, S. Nolte, C. B. Schaffer, MRS Bull. 2006, 31, 620.
[45] A. Szameit, S. Nolte, J. Phys. B: At., Mol. Opt. Phys. 2010, 43, 163001.
[46] B. P. Cumming, S. Debbarma, B. Luther-Davis, M. Gu, Opt. Express 2013, 21, 19135.
[47] J. A. Davis, D. M. Cottrell, J. Campos, M. J. Yzuel, I. Moreno, Appl. Opt. 1999, 38, 5004.
[48] T. Kreis, J. Opt. Soc. Am. A 1986, 3, 847.