PERSISTENCE PROPERTIES FOR THE GENERALIZED CAMASSA-HOLM EQUATION

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Abstract. In present paper, we study the Cauchy problem for a generalized Camassa-Holm equation, which was discovered by Novikov. Our purpose here is to establish persistence properties and some unique continuation properties of the solutions of this equation in weighted spaces.

1. Introduction. In this paper, we consider the following Cauchy problem of the generalized Camassa-Holm equation

\[ u_t - u_{txx} = \partial_x (2 - \partial_x)(1 + \partial_x)u^2, \]  
\[ u(0, x) = u_0(x) \]  

Recently, Novikov in [16] proposed the following integrable quasi-linear scalar evolution equation of order 2

\[ (1 - \epsilon^2 \partial_x^2)u_t = \partial_x (2 - \epsilon \partial_x)(1 + \epsilon \partial_x)u^2, \]  

where \( \epsilon \neq 0 \) is a real constant. It was shown in [16] that Eq. (3) possesses a hierarchy of local higher symmetries and the first non-trivial one is \( u_\tau = \partial_x[(1 - \epsilon \partial_x u)]^{-1} \). Letting \( v(t, x) = u(\epsilon t, \epsilon x) \), then one can transform Eq. (1) into the equivalent Eq. (1).

Eq. (1) belongs to the following class [16]

\[ (1 - \partial_x^2)u_t = F(u, u_x, u_{xx}, u_{xxx}) \]  

which has attracted much attention on the Cauchy problem of the possible integrable members of (4). The well-known integrable member of (4) is the Camassa-Holm equation[3]. It is a well-known integrable equation describing the velocity

2010 Mathematics Subject Classification. 35B30, 35G25.

Key words and phrases. Persistence properties, Camassa-Holm equation, local well-posedness.

The first author is supported by NSF of China (Grant No: 11671055), NSF of Chongqing (Grant No: cstc2018jcyj AX0273) and Key project of science and technology research program of Chongqing Education Commission (Grant No: KJZD-K20180140). The second author is supported by NSF of China (Grant No: 11771062). The third author is supported by NSF of China (Grant No: 11601053).
dynamics of shallow water waves. This equation spontaneously exhibits emergence of singular solutions from smooth initial conditions. It has a bi-Hamilton structure [12] and is completely integrable [3, 5]. In particular, it possesses an infinity of conservation laws and is solvable by its corresponding inverse scattering transform. After the birth of the Camassa-Holm equation, many works have been carried out to probe its dynamic properties. Such as, the Camassa-Holm equation has travelling wave solutions of the form $ce^{-|x-ct|}$, called peakons, which describes an essential feature of the travelling waves of largest amplitude (see [6, 8, 7]). It is shown in [10, 4, 9] that the inverse spectral or scattering approach is a powerful tool to handle the Camassa-Holm equation and analyze its dynamics.

In [14], Yin and Li first establish the local existence and uniqueness of strong solutions for the Cauchy problem (1)-(2). Then, they prove the solution depends continuously on the initial data. Finally, they derive a blow-up criterion and present a global existence result for the equation. Recently, Mi et. al. [15] study non-uniform dependence and well-posedness for the Cauchy problem (1)-(2) in both the periodic and the nonperiodic case, respectively.

To our best knowledge, persistence properties of the Cauchy problem for (1)-(2) has not been studied yet. In this paper we first show a persistence property of the strong solutions to (1)-(2). The analysis of the solutions in weighted spaces is useful to obtain information on their spatial asymptotic behavior.

We give the definition for admissible weight function.

**Definition 1.1.** An admissible weight function for Eq. (1) is a locally absolutely continuous function $\phi: \mathbb{R} \to \mathbb{R}$ such that, for some $A>0$ and a.e. $x \in \mathbb{R}$, $|\phi'(x)| \leq A|\phi(x)|$, and that is $v$-moderate for some sub-multiplicative weight function $v$ satisfying $\inf_{\mathbb{R}} v > 0$ and

$$\int_{\mathbb{R}} \frac{v(x)}{e^{|x|}} dx < \infty$$  \hspace{1cm} (5)$$

Our main results are stated as follows.

**Theorem 1.1.** Let $T > 0$, $s > 3/2$, and $2 \leq p < \infty$. Let also $u \in C((0, T], H^s(\mathbb{R}))$ be a strong solution of the Cauchy problem (1)-(2), such that $u|_{t=0} = u_0$ satisfies $u_0 \phi \in L^p(\mathbb{R})$ and $(\partial_x u_0) \phi \in L^p(\mathbb{R})$, where $\phi$ is an admissible weight function for Eq. (1). Then, for all $t \in [0, T]$, we have the estimate

$$\|u(t)\|_p + \|\partial_x u(t)\|_p \leq (\|u_0\|_p + \|\partial_x u_0\|_p) e^{CMt}$$

for some constant $C > 0$ depending only on $v, \phi$ (through the constants $A, C_0, \inf_{\mathbb{R}} v$, and $M \equiv \sup_{t \in [0, T]} (\|u(t)\|_\infty + \|\partial_x u(t)\|_\infty < \infty)$.

The basic example of the application of Theorem 1.1 is obtained by taking the standard weights $\phi = \phi_{a, b, c, d}(x) = e^{a|x|^b} (1 + |x|)^c \log(e + |x|^d)$ with the following conditions:

$$a \geq 0, \quad c, d \in \mathbb{R}, \quad a \leq b \leq 1, \quad ab < 1.$$ 

The restriction $ab < 1$ guarantees the validity of condition (5) for a multiplicative function $v(x) \geq 0$. Indeed, for $a < 0$ one has $\phi(x) \to 0$ as $|x| \to \infty$: the conclusion
of Theorem 1.1 remains true but it is not interesting in this case, we are interested in the following two special persistence properties:

**Remark 1.1.** (1) Power weights: Take $\phi = \phi_{0,0,c,0}$ with $c > 0$, and choose $p = \infty$. In this case Theorem 1.1 states that the condition

$$|u_0(x)| + |\partial_x u_0(x)| \leq C(1 + |x|)^{-c}$$

implies the uniform algebraic decay in $[0, T]$:

$$|u(x,t)| + |\partial_x u(x,t)| \leq C(1 + |x|)^{-c}$$

(2) Exponential weights: Choose $\phi = \phi_{a,1,c,d}$ if $x \geq 0$ and $\phi(x) = 1$ if $x \leq 0$ with $0 \leq a < 1$. It is easy to see that such a weight satisfies the admissibility conditions of Definition 1.1. Let further $p = \infty$ in Theorem 1.1. Then one deduces that Eq. (1) preserves the pointwise decay $O(e^{-ax})$ as $x \to +\infty$ for any $t > 0$. Similarly, we have persistence of the decay $O(e^{-ax})$ as $x \to -\infty$. Some similar results on persistence of strong solutions of the Camassa-Holm equation (or generalized Camassa-Holm equation) can be found in [2, 13, 17].

Clearly, the limit case $\phi = \phi_{1,1,c,d}$ is not covered by Theorem 1.1. In the following theorem however we may choose the weight $\phi = \phi_{1,1,c,d}$ with $c < 0$, $d \in \mathbb{R}$, and $\frac{1}{|c|} < p \leq \infty$, or more generally when $(1 + |\cdot|)^c \log(e + |\cdot|)^d \in L^p(\mathbb{R})$. See Theorem 1.2 below, which covers the case of such fast growing weights. In other words, we want to establish a variant of Theorem 1.1 that can be applied to some $V$-moderate weights $\phi$ for which condition (5) does not hold. Instead of assuming (5), we now put the weaker condition

$$ve^{-|x|} \in L^p(\mathbb{R}),$$

where $2 \leq p \leq \infty$.

**Theorem 1.2.** Let $2 \leq p \leq \infty$ and $\phi$ be a $V$-moderate weight function as in Definition 1.1 satisfying condition (6) instead of (5). Let also $u_{|t=0} = u_0$ satisfy

$$u_0 \phi \in L^p(\mathbb{R}), \quad u_0 \phi^\frac{d}{2} \in L^2(\mathbb{R})$$

and

$$(\partial_x u_0) \phi \in L^p(\mathbb{R}), \quad (\partial_x u_0) \phi^\frac{d}{2} \in L^2(\mathbb{R})$$

Let also $u \in C([0,T], H^s(\mathbb{R}))$, $s > 3/2$, be the strong solution of the Cauchy problem (1)-(2), emanating from $u_0$. Then,

$$\sup_{t \in [0,T]} \left( \|u(t)(x)\phi\|_{L^p} + \|\partial_x u(t)\phi\|_{L^p} \right)$$

and

$$\sup_{t \in [0,T]} \left( \|u(t)(x)\phi^{\frac{d}{2}}\|_{L^3} + \|\partial_x u(t)\phi^{\frac{d}{2}}\|_{L^3} \right)$$

are finite.

Choosing $\phi(x) = \phi_{1,1,0,0}(x) = e^{|x|}$ and $p = \infty$ in Theorem 1.2, it follows that if $|u_0(x)|$ and $|\partial_x u_0(x)|$ are both bounded by $ce^{-|x|}$, then the strong solution satisfies

$$|u(x,t)| + |\partial_x u(x,t)| \leq C^{-|x|}$$

uniformly in $[0,T]$. In the following result we compute the spatial asymptotic profiles of solutions with exponential decay. As a further consequence we may infer that the peakon-like decay $O(e^{-|x|})$ mentioned above is the fastest possible decay for a
nontrivial solution \( u \) of the Cauchy problem (1)-(2) to propagate for a nontrivial solution \( u \).

**Theorem 1.3.** Let the initial data \( u_0 \in H_s(s > 3/2) \) and satisfy that
\[
\sup_{x \in \mathbb{R}} e^{x/(m+1)} (1 + |x|)^{(m+1)} \log (e + |x|)^d (|u_0(x)| + |(\partial_x u_0)(x)|) < \infty, \tag{8}
\]
with some \( d > 1/(m+1) \). Then the condition (8) uniformly in the time interval \([0, T]\) for the strong solution \( u \in C([0, T], H^s) \) of the Cauchy problem (1)-(2) starting from \( u_0 \). Moreover, suppose that the functions \( \Phi(t) \) and \( \Psi(t) \) satisfy
\[
c_1 \leq \Phi(t) \leq c_2, \quad c_1 \leq \Psi(t) \leq c_2,
\]
with some constants \( c_1, c_2 > 0 \) independent of \( t \), then the following asymptotic profiles hold:
\[
\begin{cases}
    u(x,t) = u_0(x) + e^{-x}[\Phi(t) + \epsilon(x,t)], & \text{with } \lim_{x \to +\infty} \epsilon(x,t) = 0, \\
    u(x,t) = u_0(x) - e^x[\Psi(t) + \epsilon(x,t)], & \text{with } \lim_{x \to -\infty} \epsilon(x,t) = 0
\end{cases} \tag{10}
\]
for all \( t \in [0, T] \).

The paper is organized as follows. In Section 2, we obtain a persistence result on solutions and give the proofs of Theorems 1.1-1.3.

2. **Persistence properties of solutions.** In this section, we shall discuss the persistence properties for a generalized Camassa-Holm equation (1)-(2) in weighted \( L^p \) spaces. First, using the Green function \( G(x) := e^{-|x|}, x \in \mathbb{R} \), and the identity \((1 - \partial_x^2)^{-1} f = G * f \) for all \( f \in L^2 \), we can rewrite the Cauchy problem (1)-(2) as follows:
\[
\begin{cases}
    u_t - 2u u_x = P(D) (u^2)_x + u^2 \\
    u(x,0) = u_0(x)
\end{cases} \tag{11}
\]
with the operator \( P(D) := \partial_x (1 - \partial_x^2)^{-1} \).

Next, for the convenience of the readers, we present some standard definitions. In general a weight function is simply a non-negative function. A weight function \( v : \mathbb{R}^n \to \mathbb{R} \) is called sub-multiplicative if \( v(x + y) \leq v(x)v(y) \), for all \( x, y \in \mathbb{R}^n \). Given a sub-multiplicative function \( v \), a positive function \( \phi \) is \( v \)-moderate if and only if \( \exists C_0 > 0 : \phi(x + y) \leq C_0 v(x) \phi(y) \), for all \( x, y \in \mathbb{R}^n \). If \( \phi \) is \( v \)-moderate for some sub-multiplicative function \( v \), then we say that \( \phi \) is moderate. This is the usual terminology in time-frequency analysis papers [1]. Let us recall the most standard examples of such weights. Let
\[
\phi(x) = \phi_{a,b,c,d}(x) = e^{a|x|^b} (1 + |x|^c) \log (e + |x|)^d.
\]
We have (see [2]) the following conditions:

(i) For \( a, c, d \geq 0 \) and \( 0 \leq b \leq 1 \) such a weight is sub-multiplicative.

(ii) If \( a, c, d \in \mathbb{R} \) and \( 0 \leq b \leq 1 \), then \( \phi \) is moderate. More precisely, \( \phi_{a,b,c,d} \) is \( \phi_{a,b,c,d} \)-moderate for \( |a| \leq \alpha, |b| \leq \beta, |c| \leq \gamma \) and \( |d| \leq \delta \).

The elementary properties of sub-multiplicative and moderate weights can be found in [2]. Now, we prove Theorem 1.1.

**Proof of Theorem 1.1.** Assume that \( \phi \) is \( v \)-moderate satisfying the above conditions. Our first observation is that the first equation in (11) can be rewritten as:
\[
u_t - 2u u_x - \partial_x G * ((u^2)_x + u^2) = 0 \tag{12}
\]
with the kernel \( G(x) = \frac{1}{2}e^{-|x|} \).
On the other hand, from the assumption $u \in C([0, T], H^s)$, $s > 3/2$, we get

\[ M \equiv \sup_{t \in [0, T]} (\|u(t)\|_\infty + \|\partial_x u(t)\|_\infty < \infty). \]

For any $N \in \mathbb{Z}^+$ let us consider the $N$-truncations of $\phi(x) : f(x) = f_N(x) = \min\{\phi, N\}$. Then $f : \mathbb{R} \to \mathbb{R}$ is a locally absolutely continuous function such that $\|f\|_\infty \leq N$, $|f'(x)| \leq A|f(x)|$ a.e. on $\mathbb{R}$. In addition, if $C_1 = \max\{C_0, \alpha^{-1}\}$, where $\alpha = \inf_{x \in \mathbb{R}} v(x) > 0$, then $f(x + y) \leq C_1 v(x)f(y)$, $\forall x, y \in \mathbb{R}$. Moreover, as shown in [2], the $N$-truncations $f$ of a $v$-moderate weight $\phi$ are uniformly $v$-moderate with respect to $N$.

We start considering the case $2 \leq p < \infty$. Multiply Eq. (12) by $|uf|^p(u)\partial_x u$ and integrate to obtain

\[ \int_{\mathbb{R}} |uf|^{p-2}(uf)(\partial_t uf)dx - 2 \int_{\mathbb{R}} |uf|^p \partial_x u dx \]

\[ - \int_{\mathbb{R}} |uf|^{p-2}(uf) (f \cdot G * ((u^2)_x + u^2)) = 0 \quad (13) \]

Note that the estimates

\[ \int_{\mathbb{R}} |uf|^{p-2}(uf)(\partial_x uf)dx = \frac{1}{p} \frac{d}{dt} \|uf\|^p_{L^p} = \|uf\|_{L^p}^{p-1} \frac{d}{dt} \|uf\|_{L^p} \]

and

\[ \left| 2 \int_{\mathbb{R}} |uf|^p \partial_x u dx \right| \leq 2\|\partial_x u\|_{L^\infty} \|uf\|_{L^p}^p \leq 2M \|uf\|_{L^p}^p \]

are true. Moreover, we get

\[ \left| \int_{\mathbb{R}} |uf|^{p-2}(uf) (f \cdot G * ((u^2)_x + u^2)) \right| \leq \|uf\|_{L^p}^{p-1} \|f\partial_x G \cdot ((u^2)_x + u^2)\|_{L^p} \]

\[ \leq C \|uf\|_{L^p}^{p-1} \|\partial_x G\|_{L^1} \|f((u^2)_x + u^2)\|_{L^p} \]

\[ \leq CM \|uf\|_{L^p}^{p-1} \left( \|uf\|_{L^p} + \|f\partial_x u\|_{L^p} \right) \]

In the first inequality we used Hölder’s inequality, and in the second inequality we applied Proposition 3.1 and 3.2 in [2], and in the last we used condition (5). Here, $C$ depends only on $v$ and $\phi$. Form (13) we can obtain

\[ \frac{d}{dt} \|uf\|_{L^p} \leq C_1 M \|uf\|_{L^p} + C_2 M \|f\partial_x u\|_{L^p} \quad (14) \]

Next, we will give estimates on $u_x f$. Differentiating (12) with respect to $x$-variable, next multiplying by $f$ produces the equation

\[ f\partial_t [\partial_x u] - 2uf\partial_x^2 u - 2f(\partial_x u)^2 \partial_x^2 G * ((u^2)_x + u^2) = 0 \]

Multiply this equation by $|f\partial_x u|^{p-2}(f\partial_x u)$ with $p \in \mathbb{Z}^+$, integrate the result in the $x$-variable, and note that

\[ \int_{\mathbb{R}} |f\partial_x u|^{p-2}(f\partial_x u)\partial_t ([\partial_x u \partial_t] f) dx = \|f\partial_x u\|_{L^p}^{p-1} \frac{d}{dt} \|f\partial_x u\|_{L^p} \]

and

\[ \int_{\mathbb{R}} |f\partial_x u|^{p-2}(f\partial_x u) f\partial_x^2 G * ((u^2)_x + u^2) dx \leq \|f\partial_x u\|_{L^p}^{p-1} \|f\partial_x^2 G \cdot ((u^2)_x + u^2)\|_{L^p} \]

\[ \leq CM \|f\partial_x u\|_{L^p}^{p-1} \left( \|uf\|_{L^p} + \|f\partial_x u\|_{L^p} \right) \]
For the second order derivative term, we have
\[
\left| \int f \partial_x u |^{p-2} (f \partial_x u) f (\partial_x u)^2 dx \right| \leq \| \partial_x u f \|_{L^p}^{p-1} \| u_x f u_x \|_{L^p} \leq M \| u_x f \|_{L^p}^{p-1} \| u_x f \|_{L^p},
\]
where the inequality $| \partial_x f | \leq Af(x)$, for a.e. $x$, is applied. Thus, it follows that
\[
\frac{d}{dt} \| \partial_x u \|_{L^p} \leq C_3 M \| u f \|_{L^p} + C_4 M \| f \partial_x u \|_{L^p}. \tag{15}
\]

Now, together the inequalities (14) with (15) and then integrating yield,
\[
\| u(t) f \|_{L^p} + \| \partial_x u(t) \|_{L^p} \leq \left( \| u_0 \|_{L^p} + \| \partial_x u_0 \|_{L^p} \right) \exp(C Mt) \text{ for all } t \in [0, T].
\]

Since $f(x) = f_N(x) \uparrow \phi(x)$ as $N \to \infty$ for a.e. $x \in \mathbb{R}$. Recalling that $u_0 \phi \in L^p(\mathbb{R})$ and $\partial_x u_0 \phi \in L^p(\mathbb{R})$, we get
\[
\| u(t) \phi \|_{L^p} + \| \partial_x u \phi \|_{L^p} \leq \left( \| u_0 \phi \|_{L^p} + \| \partial_x u_0 \phi \|_{L^p} \right) \exp(C Mt) \text{ for all } t \in [0, T].
\]

At last, we treat the case $p = \infty$. We have $u_0, \partial_x u_0 \in L^2 \cap L^\infty$ and $f(x) = f_N(x) \in L^\infty$, hence, we have
\[
\| u(t) f \|_{L^\infty} + \| \partial_x u(t) \|_{L^\infty} \leq \left( \| u_0 \|_{L^\infty} + \| \partial_x u_0 \|_{L^\infty} \right) \exp(C Mt), q \in [2, \infty]. \tag{16}
\]

The last factor in the right-hand side is independent of $q$. Since $\| f \|_{L^p} \to \| f \|_{L^\infty}$ as $p \to \infty$ for any $f \in L^\infty \cap L^2$, implies that
\[
\| u(t) f \|_{L^\infty} + \| \partial_x u(t) \|_{L^\infty} \leq \left( \| u_0 \|_{L^\infty} + \| \partial_x u_0 \|_{L^\infty} \right) \exp(C Mt).
\]

The last factor in the right-hand side is independent of $N$. Now taking $N \to \infty$ implies that estimate (16) remains valid for $p = \infty$.

\textbf{Proof of Theorem 1.2.} Arguing as in the proof of Theorem 1.1, we can easily get,
\[
\frac{d}{dt} \| u f \|_{L^p} \leq M \| u f \|_{L^p} + \| f \partial_x G * ((u^2)_x + u^2) \|_{L^p}, \text{ for } p \leq \infty, \tag{17}
\]
and
\[
\frac{d}{dt} \| \partial_x u \|_{L^p} \leq CM \| f \partial_x u \|_{L^p} + \| f \partial_x^2 G * ((u^2)_x + u^2) \|_{L^p}, \text{ for } p \leq \infty, \tag{18}
\]
where $f(x) = f_N(x) \max \{ \phi(x), N \}$. Next, we estimate $\| f \partial_x G * ((u^2)_x + u^2) \|_{L^p}$ and $\| f \partial_x^2 G * ((u^2)_x + u^2) \|_{L^p}$. Note that $\phi^\frac{1}{2}$ is a $v^\frac{1}{2}$-moderate weight such that $\left( \phi^\frac{1}{2} \right)^{(x)} \leq \frac{4}{3} \phi^\frac{1}{2}(x)$. Moreover, $\inf \{ v^\frac{1}{2} > 0 \}$. By condition (6), $v^\frac{1}{2} e^{-|x|/(2)} \in L^{2p}(\mathbb{R})$, hence Hölder’s inequality implies that $v^\frac{1}{2} e^{-|x|} \in L^1(\mathbb{R})$. Then Theorem 1.1 applies with $p = 2$ to the weight $\phi^\frac{1}{2}$ yielding
\[
\| u(t) \phi^\frac{1}{2} \|_{L^2} + \| \phi^\frac{1}{2} \partial_x u(t) \|_{L^2} \leq \left( \| u_0 \|_{L^2} + \| \phi^\frac{1}{2} \partial_x u_0 \|_{L^2} \right) \exp(C Mt).
\]
Therefore
\[ \|f \partial_x G \ast (u^2 + (u^2)_x)\|_{L^p} \leq C \| (\partial_x G) v\|_{L^p} \| \phi (u^2 + (u^2)_x)\|_{L^1}, \]
\[ \leq C_0 \exp(2CMt), \]
where the constants \( C_0 \) depend only on \( \phi, b \) and the initial data.

Similarly, recall that \( \partial_x G \leq \frac{1}{2} e^{-|x|} \) and \( \partial_x^2 G = G - \delta, \)
\[ \|f \partial_x^2 G \ast (u^2 + (u^2)_x)\|_{L^p} \leq C_1 \exp(2CMt) + CM(\|uf\|_{L^p} + \|f \partial_x u\|_{L^p}) \]
Plugging the two last estimates in (17)-(18), and summing we obtain
\[ \frac{d}{dt} (\|uf\|_{L^p}\|f \partial_x u\|_{L^p}) \leq K_1 M (\|uf\|_{L^p}\|f \partial_x u_0\|_{L^p} + K_0 \exp(2CMt) \]
Integrating and finally letting \( N \to \infty \) yields the conclusion in the case \( 2 \leq p < \infty. \)
The constants throughout the proof are independent of \( p. \) Therefore, for \( p = \infty \) one can rely on the result established for finite exponents \( q \) and then let \( q \to \infty. \)
The rest argument is fully similar to that of Theorem 1.1.

**Proof of Theorem 1.3.** Conservation of (8) follows from Theorem 1.2 with \( p = \infty \) and \( \phi(x) = e^{x^2/2}(1 + |x|)^{1/2} \log(e + |x|)^d \) Integration of (12) yields
\[ u(x,t) = u_0(x) + 2 \int_0^t uu_x(x,s)dx - \int_0^t \partial_x G \ast ((u^2(x,s))_x + u^2(x,s)) ds = 0 \]
(19)
Using the fact that condition in Theorem 1.1 is conserved uniformly in \([0,T], \) we get
\[ \left| \int_0^t uu_x(x,s)dx \right| \leq Cte^{-|x|}(1 + |x|)^{-1} \log(e + |x|)^{-2d}. \]
(20)

Next, we show that the last term in (19) can be included inside the lower order terms of the asymptotic profiles (10). The validity of the following equality is true
\[ \int_0^t \partial_x G \ast ((u^2(x,s))_x + u^2(x,s)) ds = \partial_x G \ast H(x,t), \]
(21)
with \( H(x,t) = \frac{1}{2} \int_0^t ((u^2(x,s))_x + u^2(x,s)) ds \geq 0 \) for \( t \in (0,T]. \) It is easy to see that the function \((1 + |\cdot|)^{-1/2} \log(e + |\cdot|)^d \) belongs to \( L^2(\mathbb{R}) \) From the given condition and Theorem 1.1 with \( p = 2 \) and \( \phi(x) = e^{x^2/2}, \) we know that \( \int_{-\infty}^{\infty} e^{y|f(y,t)dy < \infty}. \)
Now, we denote
\[ \Phi = \frac{1}{2} \int_{-\infty}^{\infty} e^{y}H(y,t)dy, \quad \Psi = \frac{1}{2} \int_{-\infty}^{\infty} e^{-y}H(y,t)dy. \]
By the arguments in [2], this achieves the asymptotic representation of \( u. \)

**Acknowledgments.** We would like to thank the referees very much for their valuable comments and suggestions.
REFERENCES

[1] A. Aldroubi and K. Grochenig, Nonuniform sampling and reconstruction in shift-invariant spaces, SIAM Rev., 43 (2001), 585–620.

[2] L. Brandolese, Breakdown for the Camassa-Holm equation using decay criteria and persistence in weighted spaces, Int. Math. Res. Not., IMRN, 2012 (2012), 5161–5181.

[3] R. Camassa and D. D. Holm, An integrable shallow water equation with peaked solitons, Phys. Rev. Lett., 71 (1993), 1661–1664.

[4] A. Constantin, On the inverse spectral problem for the Camassa-Holm equation, J. Funct. Anal., 155 (1998), 352–363.

[5] A. Constantin, On the scattering problem for the Camassa-Holm equation, Proc. R. Soc., Lond. A, 457 (2001), 953–970.

[6] A. Constantin, The trajectories of particles in Stokes waves, Invent. Math., 166 (2006), 523–535.

[7] A. Constantin and J. Escher, Analyticity of periodic traveling free surface water waves with vorticity, Ann. of Math., 173 (2011), 559–568.

[8] A. Constantin and J. Escher, Particle trajectories in solitary water waves, Bull. Amer. Math. Soc., 44 (2007), 423–431.

[9] A. Constantin, V. S. Gerdjikov and R. I. Ivanov, Inverse scattering transform for the Camassa-Holm equation, Inverse Problems, 22 (2006), 2197–2207.

[10] A. Constantin and H. P. McKean, A shallow water equation on the circle, Comm. Pure Appl. Math., 52 (1999), 949–982.

[11] A. Constantin and W. A. Strauss, Stability of peakons, Comm. Pure Appl. Math., 53 (2000), 603–610.

[12] A. S. Fokas and B. Fuchssteiner, Symplectic structures, their Bäcklund transformation and hereditary symmetries, Phys. D, 4 (1981/82), 47–66.

[13] A. Himonas, G. Misiolek, G. Ponce and Y. Zhou, Persistence properties and unique continuation of solutions of the Camassa-Holm equation, Comm. Math. Phys., 271 (2007), 511–522.

[14] J. L. Li and Z. Y. Yin, Well-posedness and global existence for a generalized Degasperis-Procesi equation, Nonlinear Anal. RWA., 28 (2016), 72–90.

[15] Y. S. Mi, Y. Liu, B. L. Guo and T. Luo, The Cauchy problem for a generalized Camassa-Holm equation, J. Differential Equations, 266 (2019), 6739–6770.

[16] V. Novikov, Generalization of the Camassa-Holm equation, J. Phys. A, 42 (2009), 342002, 14 pp.

[17] S. M. Zhou, Persistence properties for a generalized Camassa-Holm equation in weighted $L^p$ spaces, J. Math. Anal. Appl., 410 (2014), 932–938.

Received for publication February 2019.

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