RESTRICTING SCHUBERT CLASSES TO SYMPLECTIC GRASSMANNIANS USING SELF-DUAL PUZZLES

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Abstract. Given a Schubert class on $\text{Gr}(k, V)$ where $V$ is a symplectic vector space of dimension $2n$, we consider its restriction to the symplectic Grassmannian $\text{SpGr}(k, V)$ of isotropic subspaces. Pragacz gave tableaux formulæ for positively computing the expansion of these $H^*(\text{Gr}(k, V))$ classes into Schubert classes of the target when $k = n$, which corresponds to expanding Schur polynomials into $Q$-Schur polynomials. Coşkun described an algorithm for their expansion when $k \leq n$. We give a puzzle-based formula for these expansions, while extending them to equivariant cohomology. We make use of a new observation that usual Grassmannian puzzle pieces are already enough to do some 2-step Schubert calculus.

1. Introduction

1.1. Grassmannian duality of puzzles. The Littlewood-Richardson coefficients $c_{\lambda \mu}^\nu$, where $\lambda, \mu, \nu$ are (for now) partitions, satisfy a number of symmetries, one of which is $c_{\lambda \mu}^\nu = c_{\mu \lambda}^\nu$. One origin of L-R coefficients is as structure constants in the product in $H^*(\text{Gr}(k, V))$ of Schubert classes on the Grassmannian of $k$-planes in $V$. In that formulation, the Grassmannian duality homeomorphism $\text{Gr}(k, V) \cong \text{Gr}((\dim V) - k, V^*)$, $(U \leq V) \mapsto (U^\perp \leq V^*)$ induces an isomorphism of cohomology rings and a correspondence of Schubert bases, giving the symmetry above. This symmetry is not at all manifest in tableaux-based computations of the $\{c_{\lambda \mu}^\nu\}$, but it is in the “puzzle” rule of [1], which is based on the puzzle pieces

$$
\begin{array}{c}
\begin{tikzpicture}
\draw[fill=white] (0,0) circle (0.5);
\draw[fill=white] (1,0) circle (0.5);
\draw[fill=black] (0,1) circle (0.5);
\draw[fill=black] (1,1) circle (0.5);
\end{tikzpicture}
\end{array}
\quad
\begin{array}{c}
\begin{tikzpicture}
\draw[fill=white] (0,0) circle (0.5);
\draw[fill=white] (1,0) circle (0.5);
\draw[fill=black] (0,1) circle (0.5);
\draw[fill=black] (1,1) circle (0.5);
\end{tikzpicture}
\end{array}
\quad
\begin{array}{c}
\begin{tikzpicture}
\draw[fill=white] (0,0) circle (0.5);
\draw[fill=black] (1,0) circle (0.5);
\draw[fill=white] (0,1) circle (0.5);
\draw[fill=black] (1,1) circle (0.5);
\end{tikzpicture}
\end{array}
\quad
\begin{array}{c}
\begin{tikzpicture}
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\draw[fill=white] (0,1) circle (0.5);
\draw[fill=black] (1,1) circle (0.5);
\end{tikzpicture}
\end{array}
\quad
\begin{array}{c}
\begin{tikzpicture}
\draw[fill=white] (0,0) circle (0.5);
\draw[fill=black] (1,0) circle (0.5);
\draw[fill=white] (0,1) circle (0.5);
\draw[fill=black] (1,1) circle (0.5);
\end{tikzpicture}
\end{array}
\end{array}
$$

Specifically, the dual of a puzzle is constructed by flipping it left-right and simultaneously exchanging all $0 \leftrightarrow 1$ (in particular, 10-labels again become 10s). The duals of the puzzles counted by $c_{\lambda \mu}^\nu$ are exactly those counted by $c_{\mu \lambda}^{\nu r}$. This prompts the question: what do self-dual puzzles count? One might expect it is something related to an isomorphism $V \cong V^*$ e.g. a bilinear form, and indeed our main theorems I, II, and III interpret self-dual puzzles as computing the restrictions of Schubert classes on $\text{Gr}(k, 2n)$ to the symplectic Grassmannian $\text{SpGr}(k, 2n)$. (We will address elsewhere the minimal modifications necessary to handle the orthogonal case.) For $k = n$, there was already a tableaux-based formula for these restrictions in [2] which is less simple.

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In particular, [2] provides a cohomological interpretation of algebraic results of Stembridge [3] about expanding Schur functions into Schur P- and Q-functions.
to state than Theorem [LA]. This is perhaps another effect of tableaux being less suited to Grassmannian duality than puzzles are.

1.2. Restriction from $\text{Gr}(n, 2n)$. Let $V$ be a vector space over $\mathbb{C}$ equipped with a symplectic form, so the Grassmannian $\text{Gr}(k, V)$ of $k$-planes contains the subscheme

$$\text{SpGr}(k, V) := \{ L \leq V : \dim L = k, \ L \leq L^\perp \}$$

where $\perp$ means perpendicular with respect to the symplectic form. Then the inclusion $\iota: \text{SpGr}(k, V) \hookrightarrow \text{Gr}(k, V)$ induces a pullback $\iota^*: H^*(\text{Gr}(k, V)) \to H^*(\text{SpGr}(k, V))$ in cohomology. As both cohomology rings possess bases consisting of Schubert classes $\{ S_\lambda \}$, one can ask about expanding $\iota^*(S_\lambda)$ in the basis of $\text{SpGr}(k, V)$’s Schubert classes $\{ S_\nu \}$.

Let $\dim V = 2n$ (necessarily even, since $V$ is symplectic), and for the simplest version of the theorem assume $k = n$. Then the Schubert classes on $\text{Gr}(n, V)$ are indexed by the $\binom{2n}{n}$ binary strings with $n$ 0s and $n$ 1s, whereas the Schubert classes on $\text{SpGr}(n, V)$ are indexed by the $2^n$ binary strings of length $n$ (with more detail on this indexing in [2]).

**Theorem 1A.** Let $S_\lambda$ be a Schubert class on $\text{Gr}(n, 2n)$, indexed by a string $\lambda$ with content in $0^n1^n$, and $S_\nu$ a Schubert class on $\text{SpGr}(n, 2n)$, indexed by a length $n$ binary string. Then the coefficient of $S_\nu$ in $\iota^*(S_\lambda)$ is the number of self-dual puzzles with $\lambda$ on the Northwest side, $\nu$ on the left half of the South side (both $\lambda$ and $\nu$ read left to right), and equivariant pieces only allowed along the axis of reflection.

**Example 1.** For $\lambda = 0101$, a self-dual puzzle with $\lambda$ on the Northwest side has to be of the form $\begin{array}{ccc} & \mu \end{array}$ for some $\mu$.

So, it will appear in the usual calculation of $S_{0101} \in H_0^5(\text{Gr}(2, 4))$, which involves three puzzles. Only one of these puzzles is self-dual, and its only equivariant piece is on the centerline. From this we compute $\iota^*(S_{0101}) = S_{01}$ in $H^*(\text{SpGr}(2, 4))$.

A surprising aspect of Theorem [LA] is that equivariant pieces appear in this nonequivariant calculation, albeit only down the centerline. If we allow them elsewhere (self-dually occurring in pairs), the puzzles compute the generalization of Theorem [LA] to equivariant cohomology. We leave this statement until Theorem [LC] in §1.3 because it requires some precision about the locations of the symplectic Schubert varieties.

1.3. Interlude: puzzles with 10s on the South side. To generalize Theorem [LA] to $\text{SpGr}(k, 2n)$, we need strings that index its $\binom{2n}{k}$ many Schubert classes. We do this using the third edge label, 10: consider strings $\nu$ of length $n$ with $(n - k)$ 10s, the rest a mix of 1s and 0s.

Before considering self-dual puzzles with Southside 10s, we mention a heretofore unobserved capacity of the puzzle pieces from [1], available once we allow for Southside 10s. It turns out they are already sufficient to compute certain products in the cohomology of

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$\text{Note that the general 2-step problem has received a puzzle formula }$[1]$\text{, but using many more puzzle pieces than we use here. The problem of multiplying classes from different Grassmannians was studied already in }$[2].$
2-step flag manifolds! The only necessary new idea is to allow the previously internal label 10 to appear on the South side.

**Theorem 2.** Let \( 0 \leq j \leq k \leq n \), and let \( \lambda, \mu \) be \( 0,1 \)-strings with content \( 0^j1^{n-j}, 0^k1^{n-k} \) respectively, defining equivariant Schubert classes \( S_\lambda, S_\mu \) on \( \text{Gr}(j; \mathbb{C}^n), \text{Gr}(k; \mathbb{C}^n) \) respectively. Let \( \pi_\lambda, \pi_\mu \) be the respective projections of the 2-step flag manifold \( \text{Fl}(j, k; \mathbb{C}^n) \) to those Grassmannians. Let \( \nu \) be a string in the ordered alphabet \( 0,1,1 \), with content \( \lambda, \mu, \nu \), made from the puzzle pieces in \( \mathcal{P}(\lambda) \) of \( \text{fug} \) (the usual product over the equivariant pieces).

Then the coefficient of \( S_\nu \) in the product \( \pi_\lambda^* \{ S_\lambda \} \pi_\mu^* \{ S_\mu \} \in H^*_T(\text{Fl}(j, k; \mathbb{C}^n)) \) is the usual sum from \( \mathcal{P} \) over puzzles \( P \) with boundary labels \( \lambda, \mu, \nu \), made from the puzzle pieces in \( \mathcal{P}(\lambda) \) of \( \text{fug} \) (the usual product over the equivariant pieces).

**Example 3.** If \( \lambda = 101, \mu = 100 \), then their pullbacks give \( \pi_\lambda^* \{ S_{101} \} = S_{10,0,1}, \pi_\mu^* \{ S_{100} \} = S_{1,0,10} \), with product \( (y_1 - y_2)S_{10,0,1} + S_{1,10,0} \) (note: to compare strings to permutations requires inversion, as in Proposition 4).

1.4. **Restriction from** \( \text{Gr}(k, 2n), k < n \).

**Theorem 1B.** Let \( \lambda \) be a string with content \( 0^k1^{2n-k} \), whereas \( \nu \) is of length \( n \) with \( (n-k) \) \( 10 \)s, the rest a mix of \( 1 \)s and \( 0 \)s. Consider the puzzles from Theorem 2, where we allow 10 labels to appear on the South side.

Then as before, in \( H^*(\text{SpGr}(k, 2n)) \), the coefficient of \( S_\nu \) in \( t^*(\lambda) \) is the number of self-dual puzzles with \( \lambda \) on the Northwest side, \( \nu \) on the left half of the South side (both \( \lambda \) and \( \nu \) read left to right), and equivariant pieces only allowed along the axis of reflection.

**Example 4.** In the remainder of the paper we work with the left halves \( \mathcal{B} \) of self-dual puzzles, since the centerline and right half can be inferred. The half-puzzles pictured here (really for equivariant Theorem \( 1C \) to come) show \( t^*(S_{110101}) = (y_2 - y_3)S_{10,1,0} + S_{10,1,1} + S_{1,10,0} \).

In this extended abstract we only include the proof of Theorem 1B (containing Theorem 1A) in ordinary and equivariant cohomology (see Theorem 1C); the K-version will appear elsewhere.

The proof is based on the “quantum integrability” of \( R \)-matrices, and closely follows that of [6]; the principal new feature is the appearance of \( K \)-matrices. The “reflection equation” \( RKRK = KRKR \) (more precisely, equation (4) in Lemma 7) is standard; however since the approach of [3] requires not just \( R \)-matrices but the trivalent \( U \)-matrix we needed here the possibly novel “\( K \)-fusion equation” (5) in Lemma 7.

2. **The groups, flag manifolds, and cohomology rings**

While all symplectic structures on \( \mathbb{C}^{2n} \) are \( \text{GL}_{2n} \)-equivalent, in order to compute in \( T^n \)-equivariant cohomology we need to be more specific about our choice of form: we take the Gram matrix to be antidiagonal. This is so that if \( B_{\pm} \) are the upper/lower triangular Borel subgroups of \( \text{GL}_{2n} \), then \( B_{\pm} \cap \text{Sp}_{2n} \) will again be opposite Borel subgroups.
Consider $\text{Gr}(k, 2n) = \{0 \leq V \leq \mathbb{C}^{2n} | \dim V = k\} \cong \text{GL}_{2n}/P$ where $k \leq n$ and $P$ is the parabolic subgroup of block type $(k, 2n-k)$ containing $B = B_+$. Then $P_{\text{Sp}_{2n}} = P \cap \text{Sp}_{2n}$ is a parabolic for the Lie subgroup $\text{Sp}_{2n}$ and the symplectic Grassmannian is $\text{SpGr}(k, 2n) = \{0 \leq V < \mathbb{C}^{2n} | \dim V = n, V \leq V^\perp\} \cong \text{Sp}_{2n}/P_{\text{Sp}_{2n}}$. Let $T^{2n} := B_- \cap B_+$ be the diagonal matrices in $\text{GL}_{2n}$, and $T^n := \text{Sp}_{2n} \cap T^{2n}$. Note that $(\text{GL}_{2n}/P)^T = (\text{GL}_{2n}/P)^T^{2n}$ since there exist $x \in T^n$ with no repeated eigenvalues. The following diagram of spaces commutes.

$\{\nu : \exists a \text{ s.t. } \text{content}(\nu) = 0^a(10)^{n-k}1^{k-a}\} \xrightarrow{\iota} (\text{SpGr}(k, 2n))^T \xleftarrow{\iota^*} \text{SpGr}(k, 2n)$

$\{\lambda : \text{content}(\lambda) = 0^k1^{2n-k}\} \xrightarrow{\text{coord}} (\text{Gr}(k, 2n))^T \xleftarrow{\iota} \text{Gr}(k, 2n)$

Throughout the paper we freely interchange between strings $\lambda, \nu$ and left cosets $W_0 \backslash W$ to index $\langle G/P \rangle^T$. The map $\iota$ takes a sequence $\nu$ first to its double $\nu\nu$ where $\nu$ is $\nu$ reflected and its 0s and 1s are switched; after that, all 10s in $\nu\nu$ are turned into 1s, e.g.

$0,10,1,0,10 \leftrightarrow 0,10,1,0,10,1,0,10,1 \leftrightarrow 0,1,1,0,1,1,1,0,1,1$

The bijective map $\text{coord}$ takes a $0,1$-sequence $\lambda$ to the coordinate $k$-plane that uses the coordinates in the 0 positions of $\lambda$ (so, 1, 4, 8 in the above example). Note that $\text{coord}(\iota(\lambda)) \in \text{SpGr}(k, 2n)$ by the anti-diagonality we required of the Gram matrix.

The right-hand square, and the inclusion $T^n \hookrightarrow T^{2n}$, induce the ring homomorphisms

$$
\begin{align*}
\text{H}^*_T(\text{Gr}(k, 2n)) & \xrightarrow{f_1} \text{H}^*_T(\text{Gr}(k, 2n)) & \xrightarrow{f_2 = \iota^*} \text{H}^*_T(\text{SpGr}(k, 2n)) \\
\downarrow g_1 & \quad & \downarrow g_2 \\
\text{H}^*_T(\text{Gr}(k, 2n)^T) & \xrightarrow{h_1} \text{H}^*_T(\text{Gr}(k, 2n)^T) & \xrightarrow{h_2} \text{H}^*_T(\text{SpGr}(k, 2n)^T)
\end{align*}
$$

and since each $g_i$ is injective (see e.g. [13]), we can compute along the bottom row, the proof technique used in [6] and [15]. On each of our flag manifolds, we define our Schubert classes as associated to the closures of orbits of $B_-$ or $B_- \cap \text{Sp}_{2n}$.

3. Scattering diagrams and their tensor calculus

In the statement and proof of Theorem [13] we work with half-puzzles, i.e., labeled half-triangles $2n\mathcal{A}$ of size $2n$, tiled with the triangle and rhombus puzzle pieces described in [17] as well as half-rhombus puzzle pieces obtained by cutting the existing self-dual ones vertically in half. As discussed earlier, a half-puzzle can be considered as half of a self-dual puzzle with all three sides of length $2n$. In our notation, a “rhombus” can also be made of a $\Delta$ and a $\nabla$ triangle glued together.

To linearize the puzzle pictures and relate them back to the restriction of cohomology classes, we consider the puzzle labels $\{0,10,1\}$ as indexing bases for three spaces $\mathbb{C}^3_{G_1}, \mathbb{C}^3_{K_3}, \mathbb{C}^3_{B_3}$ (Green, Red, Blue). In our scattering diagrams below, each coloured edge will carry its corresponding vector space.

(1) Take an unlabeled size $2n$ half-puzzle triangle $2n\mathcal{A}$ tiled by rhombi, half-rhombi (on the East) and triangles (on the South) as before, with assigned “spectral parameters” $y_1, \ldots, y_n, -y_n, \ldots, -y_1$ on the Northwest side.
Consider the dual graph picture of strands, oriented downwards. Each rhombus corresponds to a crossing of two strands, each half-rhombus to a bounce off the East wall and negates the spectral parameter, and each triangle to a trivalent vertex with all parameters equal.

We also colour the Southeast-pointing strands green, Southwest-pointing red, and South-pointing blue.

We let \( a \) and \( b \) denote two spectral parameters from Step 1. We assign the following linear maps

- to each crossing of two strands with left and right parameters \( a \) and \( b \), and colours \( C \) and \( D \), a linear map \( R_{CD}(a - b) : \mathbb{C}_C^3 \otimes \mathbb{C}_D^3 \rightarrow \mathbb{C}_B^3 \otimes \mathbb{C}_C^3 \);
- to each wall-bounce of a colour \( C \) strand with parameter \( a \), a map \( K_C(a) : \mathbb{C}_C^3 \rightarrow \mathbb{C}_C^3 \);
- to each trivalent vertex with incoming green and red strand parameters both \( a \), a map \( U(a) : \mathbb{C}_G^3 \otimes \mathbb{C}_R^3 \rightarrow \mathbb{C}_B^3 \).

Connecting two strands corresponds to composing the corresponding maps, so the whole \( 2n \) corresponds to a linear map \( \Phi : (\mathbb{C}_G^3)^{\otimes n} \rightarrow (\mathbb{C}_B^3)^{\otimes n} \).

**Definition 5 (The \( R \)-, \( K \)-, and \( U \)-matrices).** In terms of the bases of \( \mathbb{C}_G^3, \mathbb{C}_R^3, \mathbb{C}_B^3 \) indexed by \( \{0, 10, 1\} \), the above sparse matrices can be written compactly as follows (where a labeled diagram corresponds to the coefficient of the map in those basis elements):

\[
R_{CC}(a - b) = \begin{cases} 
1 & \text{if } (i, j) = (k, l), \\
(a - b) & \text{if } (i, j, k, l) \in \{(1, 0, 0, 1), (10, 0, 0, 10), (1, 10, 10, 1)\}
\end{cases}
\]

where \( C \in \{R, G, B\} \) and the two strands are any identical colour

\[
R_{GR}(a - b) = \begin{cases} 
b - a & \text{if } (i, j, k, l) = (0, 1, 1, 0), \\
1 & \text{if } (i, j, k, l) \in I_{GR}
\end{cases}
\]

\[
I_{GR} = \{0^4, 1^4, 0^21(10), 0(10)1^2, 0(10)^20, 10^21, 1^2(10)0, (10)10^2, (10)1^2(10)\}
\]

\[
U_{GR}(a) = \begin{cases} 
1 & \text{if } (i, j, k) \in \{(0, 0, 0), (0, 1, 0), (1, 0, 10), (1, 1, 1), (10, 1, 0)\}
\end{cases}
\]

\[
K_G(a) = \begin{cases} 
1 & \text{if } (i, j) \in \{(1, 0), (0, 1)\}
\end{cases}
\]

\[
K_R(a) = \begin{cases} 
1 & \text{if } i = j, \\
2a & \text{if } (i, j) = (1, 0)
\end{cases}
\]

\[
K_B(a) = \begin{cases} 
1 & \text{if } (i, j) = (1, 0)
\end{cases}
\]

The subscripts \( R, G, B \) on the maps indicate the colours of the incoming edges (listed clockwise). For each map, the matrix entries which are not listed are zero.
Note that if we take the corresponding bases with lexicographic ordering, with alphabet ordered as \( \{0,1,0,1\} \), then the matrices for \( R_{CC} \) and \( K_B \) are upper-triangular. See [8] for these \( R \)-matrices and [8, §3] for their representation-theoretic origins.

**Definition 6.** With the above notation, let \( P \) be a half-puzzle with boundary labels \( \lambda \) where \( \lambda \in 0^k1^{2n-k} \) and \( \nu \in (10)^{n-k}(0,1)^k \). The fugacity of \( P \) is

\[
\text{fug}(P) := \text{the (}\nu,\lambda\text{) matrix entry of } \Phi.
\]

**Lemma 7.** The matrices defined in (4) satisfy the following identities:

i) The Yang-Baxter equation.

\[
\begin{align*}
(1) \quad &u_1 & u_2 & u_3 \\
& u_2 & u_3 & u_1 \\
& u_3 & u_1 & u_2
\end{align*}
\]

\[
\begin{align*}
(2) \quad &u_1 & u_2 & u_3 \\
& u_2 & u_3 & u_1 \\
& u_3 & u_1 & u_2
\end{align*}
\]

For example, the linear map form of the Northwest equation is

\[
(R_{GR}(u_2 - u_3) \otimes \text{Id}) \circ (\text{Id} \otimes R_{GR}(u_1 - u_3)) \circ (R_{GR}(u_1 - u_2) \otimes \text{Id})
\]

\[
= (\text{Id} \otimes R_{GR}(u_1 - u_2)) \circ (R_{GR}(u_1 - u_3) \otimes \text{Id}) \circ (\text{Id} \otimes R_{RR}(u_2 - u_3))
\]

ii) Swapping of two trivalent vertices.

\[
\begin{align*}
(3) \quad &u_1 & u_2 & u_3 \\
& u_2 & u_3 & u_1 \\
& u_3 & u_1 & u_2
\end{align*}
\]

\[
R_{BB}(u_1 - u_2) \circ (U(u_1) \otimes U(u_2)) \circ (\text{Id} \otimes R_{GR}(u_2 - u_1) \otimes \text{Id})
\]

\[
= (U(u_2) \otimes U(u_1)) \circ (\text{Id} \otimes R_{GR}(u_1 - u_2) \otimes \text{Id}) \circ (R_{GG}(u_1 - u_2) \otimes R_{RR}(u_1 - u_2))
\]

iii) The reflection equation.

\[
\begin{align*}
(4) \quad &u_1 & u_2 & u_3 \\
& u_2 & u_3 & u_1 \\
& u_3 & u_1 & u_2
\end{align*}
\]

\[
R_{RR}(u_1 - u_2) \circ (\text{Id} \otimes K_G(u_1)) \circ R_{GR}(u_1 + u_2) \circ (\text{Id} \otimes K_G(u_2))
\]

\[
= (\text{Id} \otimes K_G(u_2)) \circ R_{GR}(u_1 + u_2) \circ (\text{Id} \otimes K_G(u_1)) \circ R_{GG}(u_1 - u_2)
\]

In linear map terms, the left equation says

\[
R_{RR}(u_1 - u_2) \circ (\text{Id} \otimes K_G(u_1)) \circ R_{GR}(u_1 + u_2) \circ (\text{Id} \otimes K_G(u_2))
\]

\[
= (\text{Id} \otimes K_G(u_2)) \circ R_{GR}(u_1 + u_2) \circ (\text{Id} \otimes K_G(u_1)) \circ R_{GG}(u_1 - u_2)
\]
iv) K-fusion.

\[ \begin{align*}
K_B(u_1) & \circ U_{GR}(u_1) \circ (\text{Id} \otimes K_G(-u_1)) \\
& = U_{GR}(-u_1) \circ (\text{Id} \otimes K_G(u_1)) \circ R_{GG}(2u_1)
\end{align*} \]

4. AJS/Billey formulæ as scattering diagrams

We first discuss the general AJS/Billey formula for restricting an equivariant Schubert class to a torus-fixed point, and then consider the special cases of types A and C. Let \( G \) be an algebraic group and fix a pinning \( G \supseteq B \supseteq T \), with \( W_G = N(T)/T \). Let \( B^- \) denote the opposite Borel and \( P \supseteq B \) a parabolic, with Weyl group \( W_P \). Since we need to index classes using strings, or “permutations with ambiguous values” in \( W_P \setminus W \), their inverses in \( W/W_P \) are “permutations with ambiguous positions” which is the more usual way to index \( G/P \) Schubert classes. As such, to compare to the standard result below we need to invert our permutations when indexing Schubert classes.

**Proposition 8.**

1. (\cite[I.10]{AJS}) For the Schubert class \( S_n := \left[ \left. B - \pi^{-1} B / B \right] \right] \in H^*_T(G/B), \)

\[ \pi, \sigma \in W, \quad \text{and} \quad Q = (q_1, \ldots, q_k) \text{ a reduced word in simple reflections with } \prod Q = \sigma^{-1}, \text{ the AJS/Billey formula tells us that} \]

\[ S_{\pi|\sigma} = \sum_{R \subseteq Q} \prod_{i=1}^k (\alpha_{-q_i}^{-q_i} \mathbb{C}) \cdot 1 = \sum_{R \subseteq Q} \prod_{i \in R} \beta_i \in H^*_T(pt) \]

where \( \beta_i := q_1 q_2 \cdots q_{i-1} \cdot \alpha_{q_i} \) and the summation is over reduced subwords \( R \) of \( Q \).

2. To compute a point restriction \( S_{\lambda|\mu} \) on \( G/P \), where \( \lambda, \mu \in W_P \setminus W \), we use lifts \( \tilde{\lambda}, \tilde{\mu} \in W \) such that \( \tilde{\lambda} \) is the shortest length representative of \( \lambda \), and observe that \( S_{\lambda|\mu} = S_{\tilde{\lambda}|\tilde{\mu}} \).

Below we give a diagrammatic description of the formula from Proposition 8 in the cases when \( (G, W_G) = (GL_{2n}, S_{2n}) \) or \( (Sp_{2n}, S_n \ltimes (Z/2Z)^n) \) using the tensor calculus setting of \cite{BB}. We first introduce some notation.

Consider \( \sigma \in W_G \) (generated by simple reflections \( \{s_i^G\} \)) and a reduced word for \( \sigma \), \( Q_\sigma = (q_1, \ldots, q_k) \) (where \( q_i = s_{p_i} \) for some \( p_i \)). We can associate to it a wiring diagram \( D(Q_\sigma) \) by assigning to a simple reflection \( s_i^G \), \( 1 \leq i \leq m_G - 1 \), for \( (m_{GL_{2n}}, m_{Sp_{2n}}) = (2n, n) \), and to the generators \( \{s_i^{Sp_{2n}}\} \) the diagrams below. Then, a word in simple reflections corresponds to a concatenation of such diagrams.

\[ s_i^G \mapsto \ldots \begin{array}{cccc}
1 & i & i + 1 & m_G \\
\end{array} \ldots, \text{ for } 1 \leq i \leq m_G - 1 \quad \text{and} \quad s_i^{Sp_{2n}} \mapsto \ldots \begin{array}{cccc}
1 & \cdots & n \\
\end{array} \]

Each wire in a wiring diagram is also assigned an spectral parameter. For \( G = GL_{2n} \), they are given by \( y_1, \ldots, y_{2n} \) (which we need to later specialize to \( y_1, \ldots, y_n, -y_n, \ldots, -y_1 \) as in the maps \( f_1, h_1 \) in \cite{BB}), and for \( G = Sp_{2n} \) by \( y_1, \ldots, y_n \).

In the context of the tensor calculus from \cite{BB} the wiring diagram \( D(Q_\sigma) \) can be interpreted as a scattering diagram, i.e., giving a map \( (C^3)^{\otimes m_G} \rightarrow (C^3)^{\otimes m_G} \); we replace each crossing with \( R_{RR} \) for \( G = GL_{2n} \) (and \( C = R \)) or with \( R_{BB} \) for \( Sp_{2n} \) (and \( C = B \), and replace each...
bounce with $K_B$ which also negates the spectral parameter. For instance, take $G = Sp_6$ and $Q_\sigma = s_2 s_3 s_1$, then

$$D(Q_\sigma) = (R_{BB}(y_1 - y_3) \otimes \text{Id}) \circ (\text{Id} \otimes K_B(y_2)) \circ (\text{Id} \otimes R_{BB}(y_2 - y_3)) : C_B^3 \to C_B^3$$

**Proposition 9.** Let $\lambda, \mu$ be strings in $0, 10, 1$ as in §2, which we identify with cosets $W_P \setminus W$ where $W$ is of type $C$ and $P$ is maximal, or of type $A$ and $P$ is maximal or submaximal. Let $\omega_{Gr} = 0 \ldots 0 1 \ldots 1 \in 0^k 1^{2n-k}$ for $G/P = \text{Gr}(k, \mathbb{C}^{2n})$, $\omega_{SpGr} = 0 \ldots 0 10 \ldots 10 \in 0^k (10)^{n-k}$ for $G/P = \text{SpGr}(k, \mathbb{C}^{2n})$, or $\omega_{Fl} = 0 \ldots 0 1 \ldots 1 \in 0^l (10)^{k-l-1} 1^{2n-k}$ for $G/P = \text{Fl}(j, k; \mathbb{C}^{2n})$. Make a wiring diagram as just explained, using a reduced word for the shortest lift $\tilde{\mu}^{-1}$; interpret it as a scattering diagram map, using the $R_{BB} (= R_{RR})$ matrix for crossings and (in type $C$) $K_B$ for bounces. Then $S_{\lambda|\mu}$ is the $(\omega_{G/P}, \lambda)$ matrix entry of the resulting product.

The essentially routine rewriting of Proposition 8 to give Proposition 9 will appear elsewhere. The principal thing one checks is that $R_{BB}$ is the correct $R$-matrix for three labels.

In view of Proposition 9, for $\lambda, \mu, \nu \in W_P \setminus W$ as above, we denote

$$\nu_{\lambda \mu} := \text{the } (\nu, \lambda) \text{ matrix entry for the scattering diagram map}$$

$$\text{coming from a reduced word for } \tilde{\mu}^{-1}. \text{By the proposition, when } \nu = \omega_{G/P} \text{ this gives } S_{\lambda|\mu}. \text{In view of Proposition 8, for } \lambda, \mu, \nu \in W_P \setminus W \text{ as above, we denote}$$

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$$\text{coming from a reduced word for } \tilde{\mu}^{-1}. \text{By the proposition, when } \nu = \omega_{G/P} \text{ this gives } S_{\lambda|\mu}. \text{In view of Proposition 8, for } \lambda, \mu, \nu \in W_P \setminus W \text{ as above, we denote}$$

5. PROOF OF THEOREM 1B

The proof of Theorem 2 is very much as in §3 and will appear elsewhere. Theorem 1A is the $k = n$ special case of Theorem 1B. In fact, we give a more general puzzle rule for equivariant cohomology in Theorem 1C which in particular implies Theorem 1B.

**Theorem 1C.** For every $S_{\lambda} \in H^*_T(\text{Gr}(k, 2n))$, where $\lambda \in 0^k 1^{2n-k}$, and $t^*$ as in §2

$$t^*(S_{\lambda}) = \sum_{\nu \in (10)^{n-k}[0,1]^k} \left( \sum_P \{ \text{fug}(P) \mid P \text{ is a puzzle with boundary } \begin{array}{c} \lambda \end{array} \} \right) S_{\nu}$$

As explained in §3 it suffices to check Theorem 1C’s equality at each $T^n$-fixed point $\sigma \in (10)^{n-k}[0,1]^k$ of $\text{SpGr}(k, 2n)$. To do so, we first prove several preliminary results in the language of §3.

**Proposition 10.** Given $\sigma \in (10)^{n-k}[0,1]^k$, fixing the Northwest and South boundaries to be strings of length $2n$ and $n$ respectively, one has

$$\sigma$$

$$= \sigma$$
Proof. If suffices to consider $\tilde{\sigma}$ a simple reflection. For the purposes of illustration, we set $n = 4$ and demonstrate the equality in the case of an $s_i$ where $i < n$, as well as for $s_n$.

Lemma 11. For $\omega = \omega_{\text{SpGr}}$ as in Proposition 9 and $\lambda \in \omega^k \cdot 1^{2n-k}$, we have $\frac{\lambda}{\omega} = \delta_{\lambda, \omega}$. 

Proof. This is a straightforward consequence of Definition 5 when considering the $(\omega, \lambda)$ matrix entry of the product of $R$, $K$, and $U$-matrices making up the half-puzzle. Alternatively, note that this is half of a classical triangular self-dual puzzle with NW, NE, S boundaries labelled by $\lambda$, $\lambda$, $\omega$ $\omega$, and so the result follows from [6, Proposition 4].

Lemma 12. a) [4, Proposition 4] Type A. Let $\sigma \in 0^k \cdot 1^{2n-k}$ and $\lambda$ be a string of length $2n$: 

If $\lambda_{\text{Gr}} = \omega_{\text{Gr}} = 0$ for $\omega_{\text{Gr}}$ as in Proposition 6 then $\lambda$ consists only of 0s and 1s (no 10s).

b) Type C. Let $\sigma \in (10)^{n-k}(0, 1)^k$ and $\lambda$ be a string of length $n$: If $\lambda_{\text{SpGr}} = \omega_{\text{SpGr}} = 0$ for $\omega_{\text{SpGr}}$ as in Proposition 9 then $\lambda$ has the same number of 10s as $\omega_{\text{SpGr}}$.

Proof. To prove part b), recall that $\lambda_{\text{SpGr}} = \omega_{\text{SpGr}}$ is the $(\omega_{\text{SpGr}}, \lambda)$ matrix entry for the composition of $R_{BB}$ and $K_B$ maps. From Definition 5, we see that both of these maps preserve the number of 10s in a string, hence so will compositions of these maps. 

\[ \square \]
Proof of Theorem 1C. In \( H^*_T(\text{pt}) \), we have the following equality

\[
\lambda \omega \Gamma_{\text{Gr}} \left( \sigma \right) (L^{11}) = \sum_{\mu} \lambda \omega \Gamma_{\text{SpGr}} \left( \sigma \right) (L^{12}) = \omega \Gamma_{\text{SpGr}} \left( \sigma \right) (P^{10}) = \lambda \omega \Gamma_{\text{SpGr}} \left( \sigma \right) (L^{12}) = \sum_{\nu} \lambda \omega \Gamma_{\text{SpGr}} \left( \sigma \right) (L^{12})
\]

The left side corresponds to \( \iota^*(S_\lambda) \big|_\sigma \) by Proposition 9. In the second and fourth equality, the strings \( \mu \) and \( \nu \) have content \( 0^k 1^{2n-k} \) and \( (10)^{n-k} \{0, 1\}^k \) respectively, and all other terms of the sum vanish. \( \square \)

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