A Comparative Study of Two Dimensional Legendre & Chebyshev Wavelet with an Extended case

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Abstract

In this paper, we present an extended case of Two Dimensional Legendre & Chebyshev wavelet to solve a system of PDE’s and a Comparative Study of these two wavelets. In this article, we construct an operational matrix from two Dimensional Chebyshev and Legendre wavelets which is then used to convert PDE into some algebraic equations and hence we solve by the method of collocation. A new operational matrix was derived and it is utilized to convert PDE with BVP to a set of algebraic equations. Some examples are given to evaluate the effectiveness of the newly found approximation technique and compared with the existing result

Keywords. TDLW, TDCW, Error in TDLW & TDCW.

1. Introduction

Wavelet analysis plays a vital role in decomposing images and sounds into waves of different periods called wavelets. Wavelets are used in localizing vibrations of both sound and image signal in detail. Multiresolution analysis is the heart of wavelet analysis, which is the decomposition of a function of different levels of resolutions. In the field of mathematics, the development of wavelets are relatively very recent compared the other domains wavelets is a simple tool in mathematics with a huge variety of utmost applications. On the other hand, wavelets appeal to engineers and scientists of several backgrounds. We take the wave equation in our example. The wavelets equations is an important second order linear PDE for the observation of waves such as water, Seismic waves, sound or light waves are the examples of mechanical waves[12]. It arises in the field of acoustics, fluid dynamics and electromagnetics. In the literature, the problems based on Vibrating string of musical instruments are studied by d ‘Alembert, Bernoulli, L. Euler & J.L Lagrange[1] [2]. We take Klein –Gordan equation in our example which is a relativistic energy- momentum wave equations, its solutions are of quantum scalar or pseudo scalar field. This equation explains all the spinless particles with zero, negative and positive charges. Klein-Gordan equation plays a key role in the quantum field theory. [3] [4] “Wavelets”- Very famous topic of discussions among many scientists of science, Engineering [9] [11]. Wavelets are the new basis for the representations of functions and a few considered it as a new technique to define the analysis of time frequency and the other hand a few thought wavelets was a new mathematical subject. Wavelets acts as a versatile tool with very rich content in the mathematics and high potential for applications.[6]Wavelets are really a special type of oscillatory which has compact support that plays the platform for most of the domains. The Klien-Gordan equation is a relativistic version of the schrodinger equation which describes the behaviour of spinless particles[3] [4] [5]. For full reconciliation of quantum mechanics with relativity quantum field theory is required, in which the K.G. Equations remerges as the equations obeyed by the component of all free quantum
field. In quantum field then, the solution of the free version of the original equations still plays a vital role. They are needed to build the Hilbert Space and to express quantum field by spanning sets of Hilbert space of wave functions.

We compare our both the new ideas with the exact value for some cases with proper tabulation and hence we conclude our comparative analysis of our method followed by references. We compare the results obtained in the previous two cases as we discussed in the research paper [13], [14].

Two-dimensional Legendre wavelet [9], [10]

Two-dimensional Legendre Wavelet can be explained as a product of one dimensional Legendre Wavelet as given below.

\[
\psi_{n,n'}^L(t, x) = \begin{cases} 
\psi_{n,m}^L(t) \psi_{n',m'}^L(x), & \text{if } \frac{n-1}{2^k} \leq t < \frac{n+1}{2^k} \\
\psi_{n,m}^L(x) \psi_{n',m'}^L(t), & \text{otherwise}
\end{cases}
\]

\[
\psi_{n,m}^L(t) = \sqrt{\frac{m+\frac{1}{2}}{2^n}} \, \psi_m(2^n t - m)
\]

\[
\psi_{n,m}^L(x) = \sqrt{\frac{m'+\frac{1}{2}}{2^n}} \, \psi_{m'}(2^{n'} x - m')
\]

and \( m=0,1,2,\ldots,M-1, \) \( m'=0,1,2,\ldots,M'-1, \) \( n=2n-1, \)

\( n'=2n'-1, \) \( n=1,2,3,\ldots,2^{k-1}, \) \( n'=1,2,3,\ldots,2^{k-1}, \) in which, \( p_m \) and \( p_{m'} \)

are Legendre functions of order \( m \) and \( m' \) defined in the interval \([0,1]\) and the two-dimensional Legendre Wavelets is a orthonormal set in \( \Omega = [0,1] \times [0,1]\).

Operational matrices [9],[10]

Legendre wavelets operational matrix of differentiation with respect to variable \( t \).

Let \( \psi^L(t, x) \) be two-dimensional Legendre wavelet vector defined and its derivative matrix as follows:

\[
\frac{\partial}{\partial t} \psi^L(t, x) = D_t^L \psi^L(t, x)
\]

\( D_t^L = D_t^L \otimes I \) is the matrix of order \( 2^{k-1} \times 2^{k-1} \) \( MM' \) Where \( I \) is the identity matrix. Legendre wavelet operational matrix of differentiation with respect to variable \( x \).

Extended case of Two-Dimensional Chebyshev and Legendre wavelet Operational Matrices [9], [10], [11]

Extended case of TDLW:

For \( K=3, n=1, m=0,1,2 \)
Extended case of TDCW

*For* $K = 3, \ n = 1, \ m = 0, 1, 2$

\[
\begin{align*}
\frac{1}{2} & \\
\frac{\sqrt{3}}{2} (2x - 5) & \\
\frac{\sqrt{5}}{2} (12x^2 - 60x + 74) & \\
\frac{\sqrt{3}}{2} (2t - 5) & \\
\frac{3}{2} (2t - 5) (2x - 5) & \\
\frac{\sqrt{15}}{2} (2t - 5)(12x^2 - 60x + 74) & \\
\frac{\sqrt{5}}{2} (12t^2 - 60t + 74) & \\
\frac{\sqrt{15}}{2} (12t^2 - 60t + 74)(2x - 5) & \\
\frac{5}{2} (12t^2 - 60t + 74)(12x^2 - 60x + 74) & \\
\end{align*}
\]

\[
\begin{align*}
\frac{8}{\pi} & \\
\frac{8\sqrt{2}}{\pi} (8x - 1) & \\
\frac{8\sqrt{2}}{\pi} (128x^2 - 32x + 1) & \\
\frac{8\sqrt{2}}{\pi} (8t - 1) & \\
\frac{16}{\pi} (8x - 1) (8t - 1) & \\
\frac{16}{\pi} (128x^2 - 32x + 1) & \\
\frac{8\sqrt{2}}{\pi} (128t^2 - 32t + 1) & \\
\frac{16}{\pi} (128t^2 - 32t + 1)(8x - 1) & \\
\frac{16}{\pi} (128x^2 - 32x + 1)(128t^2 - 32t + 1) & \\
\end{align*}
\]

\[u(x,t) = (f_{1010} f_{1011} f_{1012} f_{1110} f_{1111} f_{1112} f_{1210} f_{1211} f_{1212})\]

\[u(x,t) = (f_{1010} f_{1011} f_{1012} f_{1110} f_{1111} f_{1112} f_{1210} f_{1211} f_{1212})\]
2. Numerical examples

Example: 1

Consider the 2-Dimensional PDE. [13]
\[
\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}
\]
\[u(0,t) = \cos t \quad \text{and} \quad \frac{\partial u(0,t)}{\partial x} = 0\]

and \[u(x,0) = \cos x \quad \text{and} \quad \frac{\partial u(x,0)}{\partial t} = 0\]

The exact solution is \[u(x,t) = \cos x \cos t\].

We apply the Two-Dimensional Legendre wavelet (TDLW) and Two-Dimensional Chebyshev wavelet (TDCW) [13] [14] we obtain Nine algebraic equations and solving those equations with the Collocation points and solving them using Matlab, we arrive Nine unknowns. We multiply those nine values with the formulated Operational matrices of TDLW and TDCW and get a polynomial containing \(x\) and \(t\). Here we compare both the wavelets

Case-(i)

\(K = 0, \hat{n} = 3, M = 3, M’=3\) then \(u(x,t)\) is a \(9 \times 1\) vector (TDLW)

\(K = 1, n = 1, m = 0,1,2\), then we get \(u(x,t)\) is a \(9 \times 1\) vector (TDCW).

Therefore the results are furnished below.

| X  | t  | Exact Solution | Approximate Solutions | Error in TDLW | Error in TDCW |
|----|----|----------------|-----------------------|---------------|---------------|
|    |    |                | TDLW                  |               |               |
| 0.1| 0.5| 0.87319        | 0.49678               | 0.999903      | 0.37641       | -0.126713     |
| 0.2| 0.5| 0.86008        | 0.49678               | 0.999903      | 0.3633        | -0.139823     |
| 0.3| 0.5| 0.83838        | 0.49678               | 0.999903      | 0.3416        | -0.161523     |
| 0.4| 0.5| 0.80830        | 0.49678               | 0.999903      | 0.31152       | -0.191603     |
| 0.5| 0.5| 0.77015        | 0.49678               | 0.999903      | 0.27337       | -0.229753     |
| 0.6| 0.5| 0.72430        | 0.49678               | 0.999903      | 0.22752       | -0.275603     |
| 0.7| 0.5| 0.67121        | 0.49678               | 0.999903      | 0.17443       | -0.328693     |
| 0.8| 0.5| 0.61141        | 0.49678               | 0.999903      | 0.11463       | -0.388493     |
| 0.9| 0.5| 0.54551        | 0.49678               | 0.999903      | 0.04873       | -0.454393     |
Case (ii):

$u(x,t)$ which is a $9 \times 1$ matrix for $K = 1, M = 3, n = 3, M' = 3$. (TDLW)

For $n = 1, k = 2, m = 0, 1, 2$ we obtain a $9 \times 1$ matrix (TDCW)

Thus we obtain

| X  | t   | Exact Solution | Approximate Solutions | Error in TDLW | Error in TDCW |
|----|-----|----------------|-----------------------|---------------|---------------|
|    |     |                | TDLW                 |               |               |
| 0.1| 0.5 | 0.87319        | 0.99929              | -0.1261       | -0.12674      |
| 0.2| 0.5 | 0.86008        | 0.99929              | -0.13921      | -0.13985      |
| 0.3| 0.5 | 0.83838        | 0.99929              | -0.16091      | -0.16155      |
| 0.4| 0.5 | 0.80830        | 0.99929              | -0.19099      | -0.19163      |
| 0.5| 0.5 | 0.77015        | 0.99929              | -0.22914      | -0.22978      |
| 0.6| 0.5 | 0.72430        | 0.99929              | -0.27499      | -0.27563      |
| 0.7| 0.5 | 0.67121        | 0.99929              | -0.32808      | -0.32872      |
| 0.8| 0.5 | 0.61141        | 0.99929              | -0.38788      | -0.38852      |
| 0.9| 0.5 | 0.54551        | 0.99929              | -0.45378      | -0.45442      |

Case (iii): For $n = 1, K = 3, m = 0, 1, 2$ (TDCW) Extended case

| X  | t   | Exact Solutions | Approximate Solutions | Error in TDCW |
|----|-----|----------------|-----------------------|---------------|
|    |     |                | TDCW                 |               |
| 0.1| 0.5 | 0.87319        | 1.82622               | -0.95303      |
| 0.2| 0.5 | 0.86008        | 4.70723               | -3.84715      |
| 0.3| 0.5 | 0.83838        | 7.58824               | -6.74986      |
| 0.4| 0.5 | 0.80830        | 10.46925              | -9.66095      |
| 0.5| 0.5 | 0.77015        | 13.35026              | -12.58011     |
| 0.6| 0.5 | 0.72430        | 16.23127              | -15.50697     |
| 0.7| 0.5 | 0.67121        | 19.13052              | -18.45931     |
| 0.8| 0.5 | 0.61141        | 19.44683              | -18.83542     |
| 0.9| 0.5 | 0.54551        | 24.87431              | -24.3288      |
Example: 2 Consider the Klein-Gordon Equation [14]

\[ u(x,0) = 1 + \sin x, \ u_t(x,0) = 0 \] and the boundary \n\[ u_n - u_{xx} = u; \ 0 < x < 1 \] subject to the initial conditions \n\[ u(0,t) = 0, \ u(1,t) = 0, \ t > 0 \] with the exact solution is, \[ u(x,t) = \sin x + \cosh t \]

Case (i):

| X  | t  | Exact Solution | Approximate Solution | Error in TDLW | Error in TDCW |
|----|----|----------------|----------------------|---------------|---------------|
| 0.1| 0.5| 1.22745        | -1.3992              | 1.49014       | 2.62665       | -0.26269      |
| 0.2| 0.5| 1.32629        | -1.4906              | 1.83512       | 2.81689       | -0.50883      |
| 0.3| 0.5| 1.42314        | -1.4101              | 1.95103       | 2.83324       | -0.52789      |
| 0.4| 0.5| 1.51701        | -1.3292              | 1.72664       | 2.84641       | -0.20963      |
| 0.5| 0.5| 1.60705        | -1.2488              | 2.14078       | 2.85858       | -0.53373      |
| 0.6| 0.5| 1.69226        | -1.1682              | 2.55493       | 2.86046       | -0.86267      |
| 0.7| 0.5| 1.77184        | -1.0876              | 2.96907       | 2.85944       | -1.19723      |
| 0.8| 0.5| 1.84498        | 1.0071               | 3.38322       | 0.83788       | -1.53824      |
| 0.9| 0.5| 1.91095        | -0.9264              | 3.79736       | 2.83735       | -1.88641      |

Case (ii):

| X  | t  | Exact Solution | Approximate Solution | Error in TDLW | Error in TDCW |
|----|----|----------------|----------------------|---------------|---------------|
| 0.1| 0.5| 1.22745        | -1.93540             | -0.424084     | 3.16285       | 1.651534      |
| 0.2| 0.5| 1.32629        | -1.76920             | 1.174875      | 3.09549       | 0.151415      |
| 0.3| 0.5| 1.42314        | -1.60292             | 2.543355      | 3.026064      | -1.120215     |
| 0.4| 0.5| 1.51701        | -1.43664             | 3.681355      | 2.95365       | -2.164345     |
| 0.5| 0.5| 1.60705        | -1.27037             | 4.588870      | 2.87742       | -2.98182      |
| 0.6| 0.5| 1.69226        | -1.10409             | 5.265912      | 2.796354      | -3.573652     |
| 0.7| 0.5| 1.77184        | -0.77154             | 5.712469      | 2.54338       | -3.940629     |
| 0.8| 0.5| 1.84498        | -1.33375             | 5.928540      | 3.17873       | -4.08356      |
| 0.9| 0.5| 1.91095        | -0.60526             | 5.914142      | 2.516213      | -4.003192     |

Case (iii): (Extended Case)

| X  | t  | Exact Solutions | Approximate Solutions | Error in TDCW |
|----|----|----------------|----------------------|---------------|
| 0.1| 0.5| 1.22745        | 1.982087             | -0.754637     |
| 0.2| 0.5| 1.32629        | 1.968100             | -0.641810     |
| 0.3| 0.5| 1.42314        | 1.695910             | -0.272770     |
| 0.4| 0.5| 1.51701        | 1.940480             | -0.423470     |
| 0.5| 0.5| 1.60705        | 1.926628             | -0.319578     |
| 0.6| 0.5| 1.69226        | 1.912771             | -0.220511     |
| 0.7| 0.5| 1.77184        | 1.898915             | -0.127075     |
| 0.8| 0.5| 1.84498        | 1.885058             | -0.040078     |
| 0.9| 0.5| 1.91095        | 1.871202             | 0.039748      |
Example: 3
Consider another system of PDE. [13]

\[ u_x + u_t + u = (x - t)^2 \quad \text{with conditions} \]

\[ u(x, 0) = x^2 ; \quad u(0, t) = t^2 \]

The Exact solution is \( u(x, t) = (x - t)^2 \)

**Case (i):**

| X  | t  | Exact Solutions | Approximate Solutions | Error in TDCW |
|----|----|-----------------|-----------------------|---------------|
|    |    | TDCW            |                       |               |
| 0.5 | 0.5 | 0.0000          | 0.269080              | -0.269080     |
| 0.5 | 0.25| 0.0625          | 0.287809              | -0.225309     |
| 0.5 | 0.85| 0.1225          | 0.187264              | -0.064764     |
| 0.25| 0.5 | 0.0625          | 0.269082              | -0.206582     |
| 0.25| 0.25| 0.0000          | 0.231630              | -0.231630     |
| 0.25| 0.85| 0.3600          | 0.175451              | 0.184549      |
| 0.85| 0.5 | 0.1225          | 0.287809              | -0.165309     |
| 0.85| 0.25| 0.3600          | 0.006913              | 0.353087      |
| 0.85| 0.85| 0.0000          | -0.105445             | 0.105445      |

**Case (ii):**

| X  | t  | Exact Solutions | Approximate Solutions | Error in TDCW |
|----|----|-----------------|-----------------------|---------------|
|    |    | TDCW            |                       |               |
| 0.1 | 0.5 | 0.1600          | 0.402336              | -0.242336     |
| 0.2 | 0.5 | 0.0900          | 0.218992              | -0.128992     |
| 0.3 | 0.5 | 0.0400          | 0.035648              | 0.004352      |
| 0.4 | 0.5 | 0.0100          | -0.147696             | 0.157696      |
| 0.5 | 0.5 | 0.0000          | -0.331040             | 0.331040      |
| 0.6 | 0.5 | 0.0100          | -0.514384             | 0.524384      |
| 0.7 | 0.5 | 0.0400          | -0.697728             | 0.737728      |
| 0.8 | 0.5 | 0.0900          | 0.881072              | -0.791072     |
| 0.9 | 0.5 | 0.1600          | 0.081464              | 0.078536      |

**Case (iii): (Extended case)**

| X  | t  | Exact Solutions | Approximate Solutions | Error in TDCW |
|----|----|-----------------|-----------------------|---------------|
|    |    | TDCW            |                       |               |
| 0.125 | 0.125 | 0.000000        | 2.697400              | -2.697400     |
| 0.125 | 0.2133| 0.00779         | 2.697400              | -2.689610     |
| 0.125 | 0.0366| 0.00718         | 2.697400              | -2.690220     |
| 0.2133 | 0.125 | 0.00779         | 0.742600              | -0.734810     |
| 0.2133 | 0.2133| 0.000000        | 0.742600              | -0.742600     |
| 0.2133 | 0.0366| 0.03122         | 0.742600              | -0.711380     |
| 0.0366 | 0.125 | 0.00781         | 0.024750              | -0.016940     |
| 0.0366 | 0.2133| 0.03122         | 0.024750              | 0.006470      |
| 0.0366 | 0.0366| 0.000000        | 0.024750              | -0.024750     |
3. Conclusion
We have the TDCW and TDLW with some examples. However, as we extend the case, we are not getting the accurate value and the refinement is very less. We have noticed in ODE if we increase the scale we will get a better refinement, but it is not the case here in PDE., we have compared the error of our proposed method by comparing it with exact value and gave a graph with Matlab coding. Our examples give the proof of our comparative technique among the TDCW,TDLW and the exact values.

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