ABSTRACT. The colouring defect of a cubic graph, introduced by Steffen in 2015, is the minimum number of edges that are left uncovered by any set of three perfect matchings. Since a cubic graph has defect 0 if and only if it is 3-edge-colourable, this invariant can measure how much a cubic graph differs from a 3-edge-colourable graph. Our aim is to examine the relationship of colouring defect to oddness, an extensively studied measure of uncolourability of cubic graphs, defined as the smallest number of odd circuits in a 2-factor. We show that there exist cyclically 5-edge-connected snarks (cubic graphs with no 3-edge-colouring) of oddness 2 and arbitrarily large colouring defect. This result is achieved by means of a construction of cyclically 5-edge-connected snarks with oddness 2 and arbitrarily large girth. The fact that our graphs are cyclically 5-edge-connected significantly strengthens a similar result of Jin and Steffen (2017), which only guarantees graphs with cyclic connectivity at most 3. At the same time, our result improves Kochol’s original construction of snarks with large girth (1996) in that it provides infinitely many nontrivial snarks of any prescribed girth $g \geq 5$, not just girth at least $g$.

1. Introduction

The colouring defect of a cubic graph $G$, denoted by $df(G)$, is the smallest number of edges left uncovered by any set of three perfect matchings of $G$. For brevity, we usually drop the adjective “colouring” and speak of the defect of a cubic graph. Clearly, the defect of a 3-edge-colourable cubic graph is 0, but in general it can be arbitrarily large. Defect thus can be regarded as a measure of uncolourability of a cubic graph.

The concept of defect was introduced by Steffen as $\mu_3(G)$ in [21], where he also established its fundamental properties. Among other things he proved that every 2-connected cubic graph which is not 3-edge-colourable – a snark – has defect at least three. Another notable result of [21] states that the defect of a snark is at least as large as one half of its girth. Since there exist snarks of arbitrarily large girth [9], there exist snarks of arbitrarily large defect.

The defect of a cubic graph was further examined by Jin and Steffen in [6] and was also discussed in the survey of uncolourability measures by Fiol et al. [3, pp. 13–14]. Jin and Steffen [6] studied the relationship of defect to other measures of uncolourability, in particular its relationship to oddness. The oddness of a cubic graph $G$, denoted by $\omega(G)$, is the minimum number of odd circuits in a 2-factor of $G$; it is correctly defined for any bridgeless cubic graph. In [6, Corollary 2.4], Jin and Steffen proved that $df(G) \geq 3\omega(G)/2$ and investigated the extremal case where $df(G) = 3\omega(G)/2$ in detail. The inequality implies that with increasing oddness the difference between defect and oddness becomes arbitrarily large.

Measures of uncolourability are particularly interesting for nontrivial snarks, those which are cyclically 4-edge-connected and have girth at least 5. The reason is that several important conjectures in graph theory, such as the cycle double
cover conjecture, Fulkerson’s conjecture, and others, would have their minimal counterexamples in this class [5, 12]. Note that nontrivial snarks with arbitrarily large oddness were constructed in [10, 11, 20]. In this context it is natural to ask whether the difference between defect and oddness remains arbitrarily large when oddness is fixed even for nontrivial snarks. To this end, Jin and Steffen [6, Theorem 3.4] proved that for any given oddness $\omega > 0$ and any $d \geq 3\omega/2$ there exists a bridgeless cubic graph with oddness $\omega$ and defect at least $d$. However, their construction produces graphs with cyclic connectivity not exceeding 3.

Our main result, Theorem 5.1, improves the result of Jin and Steffen for $\omega = 2$ by establishing the existence of cyclically 5-edge-connected snarks with oddness 2 and arbitrarily large defect. This result is achieved through a construction of cyclically 5-edge-connected snarks with oddness 2 and arbitrarily large girth. Our construction strengthens the original construction by Kochol [9] in that it provides infinitely many nontrivial snarks of any prescribed girth $g \geq 6$, not just girth at least $g$. Snarks with arbitrarily large defect, oddness 2, and cyclic connectivity 4 can be constructed in a similar manner. Note that the existence of nontrivial snarks of arbitrarily large girth was not confirmed until 1996, when Kochol [9] disproved a conjecture by Jaeger and Swart [4, Conjecture 2].

A detailed study of colouring defect, focused on snarks with defect 3, is carried out in our companion papers [7, 8], in which the basic properties of defect and structures related to it are discussed in a greater detail.

2. PRELIMINARIES

All graphs in this paper are finite and for the most part cubic (3-valent). Multiple edges and loops are permitted. We use the term circuit to mean a connected 2-regular graph. The length of a shortest circuit in a graph is its girth. By a $k$-cycle we mean a circuit of length $k$.

A graph $G$ is said to be cyclically $k$-edge-connected if the removal of fewer than $k$ edges from $G$ cannot create a graph with at least two components containing circuits. An edge cut $S$ in $G$ that separates two circuits from each other is cycle-separating. It is not difficult to see that the set of edges of a cubic graph leaving a shortest circuit is cycle-separating in all connected cubic graphs other than the complete bipartite graph $K_{3,3}$, the complete graph $K_4$, and the graph consisting of two vertices and three parallel edges. An edge cut of a cubic graph consisting of independent edges is always cycle-separating. Conversely, a cycle-separating edge cut of minimum size is independent. A cycle-separating edge cut that separates a shortest cycle from the rest of $G$ is called trivial.

Large graphs are typically constructed from smaller building blocks called multipoles. Similarly to graphs, each multipole $M$ has its vertex set $V(M)$, its edge set $E(M)$, and an incidence relation between vertices and edges. Each edge of $M$ has two ends, and each end may, but need not be, incident with a vertex of $M$. An end of an edge that is not incident with a vertex is called a free end or a semiedge. An edge with exactly one free end is called a dangling edge. An isolated edge is an edge whose both ends are free. All multipoles considered in this paper are cubic; it means that every vertex is incident with exactly three edge-ends. An $n$-pole $M$ is a multipole with $n$ free ends. If its free ends are $s_1, s_2, \ldots, s_n$, we write $M = M(s_1, s_2, \ldots, s_n)$. 
Free ends of a multipole can be distributed into pairwise disjoint sets, called *connectors*. Connectors of multipoles are usually matched and their free ends are subsequently identified in a straightforward manner to produce cubic graphs. An \((n_1, n_2, \ldots, n_k)\)-pole is an \(n\)-pole with \(n = n_1 + n_2 + \cdots + n_k\) whose semiedges are distributed into \(k\) connectors \(S_1, S_2, \ldots, S_k\), each \(S_i\) being of size \(n_i\). A *dipole* is a multipole with two connectors, while a *tripole* is a multipole with three connectors. An *ordered* multipole is the one where each connector is endowed with a linear order.

An *edge colouring* of a multipole \(M\) is a mapping from the edge set of \(M\) to a set of colours such that any two edge-ends incident with the same vertex carry distinct colours. A \(k\)-edge-colouring is a colouring where the set of colours has \(k\) elements. A cubic graph \(G\) is *colourable* if it admits a 3-edge-colouring. A 2-connected cubic graph which does not admit a 3-edge-colouring is called a *snark*.

In the study of snarks it is useful to take the colours 1, 2, and 3 to be the nonzero elements of the group \(\mathbb{Z}_2 \times \mathbb{Z}_2\). Specifically, one can identify a colour with its binary representation: \(1 = (0, 1)\), \(2 = (1, 0)\), and \(3 = (1, 1)\). The condition that the three colours meeting at any vertex \(v\) are all distinct then becomes equivalent to requiring that the sum of colours at \(v\) is \(0 = (0, 0)\). In other words, identifying the colours with the elements of \(\mathbb{Z}_2 \times \mathbb{Z}_2 - \{0\}\) turns a 3-edge-colouring to a nowhere-zero \(\mathbb{Z}_2 \times \mathbb{Z}_2\)-flow.

Recall that an *A-flow* on a multipole \(M\) is a function \(\sigma : E(M) \to A\), with values in an abelian group \(A\), together with an orientation of \(M\), such that Kirchhoff’s law is fulfilled: at each vertex of \(M\) the sum of all incoming values equals the sum of all outgoing ones. An *A-flow* \(\sigma\) is *nowhere-zero* if \(\sigma(e) \neq 0\) for each edge \(e\) of \(G\).

The following well-known statement is a direct consequence of Kirchhoff’s law. Roughly speaking, it tells us that the total outflow from any nonempty set of vertices equals 0.

**Lemma 2.1 (Parity Lemma).** Let \(M = M(s_1, s_2, \ldots, s_n)\) be a \(n\)-pole endowed with a 3-edge-colouring \(\sigma\). Then

\[
\sum_{i=1}^{n} \sigma(s_i) = 0.
\]

Equivalently, the number of free ends of \(M\) carrying any fixed colour has the same parity as \(n\).

Our definition of a snark leaves the concept as wide as possible since more restrictive definitions could lead to overlooking certain important phenomena that occur among snarks. In this manner we follow works of Cameron et al. [1], Nedela and Škoviera [16], Steffen [19], and others, rather than a common approach where snarks are required to be cyclically 4-edge-connected and have girth at least 5, see for example [3]. In this paper, such snarks are called *nontrivial*. The problem of nontriviality of snarks has been widely discussed in the literature, see for example [1, 16, 19]. Here we adopt a systematic approach to nontriviality of snarks proposed by Nedela and Škoviera [16] based on the concept of removability of certain sets of vertices or subgraphs. We say that an induced subgraph \(H\) of a snark \(G\) is *non-removable* if \(G - V(H)\) is colourable; otherwise, \(H\) is *removable*. It is an easy consequence of Parity Lemma that circuits of length at most 4 in snarks are removable.
3. Arrays of perfect matchings and the defect of a snark

In order to formalise our discussion of colouring defect it is convenient to define a 3-array of perfect matchings in a cubic graph $G$, briefly a 3-array of $G$, as an arbitrary collection $\mathcal{M} = \{M_1, M_2, M_3\}$ of three not necessarily distinct perfect matchings of $G$. Since every proper 3-edge-colouring can be regarded as an array whose members are the three colour classes, 3-arrays can be viewed as approximations of 3-edge-colourings. An edge of $G$ that belongs to at least one of the perfect matchings of the array $\mathcal{M} = \{M_1, M_2, M_3\}$ will be considered to be covered. An edge will be called uncovered, simply covered, doubly covered, or triply covered if it belongs, respectively, to zero, one, two, or three members of $\mathcal{M}$.

Given a graph $G$, it is a natural task to maximise the number of covered edges in a 3-array of $G$, or equivalently, to minimise the number of uncovered ones. A 3-array that leaves the minimum number of uncovered edges will be called optimal. The number of edges left uncovered by an optimal 3-array is the colouring defect of $G$, denoted by $\text{df}(G)$.

Let $\mathcal{M} = \{M_1, M_2, M_3\}$ be a 3-array of a cubic graph $G$. One way to describe $\mathcal{M}$ is based on regarding the indices 1, 2, and 3 as colours. Since the same edge may belong to more than one member of $\mathcal{M}$, an edge of $G$ may receive from $\mathcal{M}$ more than one colour. To each edge $e$ of $G$ we can therefore assign the list $\varphi(e)$ of all colours in lexicographic order it receives from $\mathcal{M}$. In this way $\mathcal{M}$ gives rise to a mapping

$$\varphi: E(G) \rightarrow \{\emptyset, 1, 2, 3, 12, 13, 23, 123\}$$

where $\emptyset$ denotes the empty list. Such a mapping determines a 3-array of $G$ if and only if each number from $\{1, 2, 3\}$ occurs precisely once on the edges around any vertex. Moreover, $\varphi$ is a proper edge colouring if and only if $G$ has no triply covered edge with respect to $\mathcal{M}$. For more details, see [7].

Another important structure associated with a 3-array is its core. The core of a 3-array $\mathcal{M} = \{M_1, M_2, M_3\}$ of $G$ is the subgraph of $G$ induced by all the edges of $G$ that are not simply covered; we denote it by $\text{core}(\mathcal{M})$. The core will be called optimal whenever $\mathcal{M}$ is optimal. Given a 3-array $\mathcal{M}$, let $E_i = E_i(\mathcal{M})$ denote the set of all edges of $G$ that belong to precisely $i$ perfect matchings of $\mathcal{M}$, where $0 \leq i \leq 3$. The edge set of $\text{core}(\mathcal{M})$ thus coincides with $E_0(\mathcal{M}) \cup E_2(\mathcal{M}) \cup E_3(\mathcal{M})$. It is worth mentioning that if $G$ is 3-edge-colourable and $\mathcal{M}$ consists of three disjoint perfect matchings, then $\text{core}(\mathcal{M})$ is empty. If $G$ is not 3-edge-colourable, then the core must be nonempty for every 3-array $\mathcal{M}$ of $G$.

Figure 1 shows the Petersen graph endowed with a 3-array whose core is the “outer” 6-cycle. The core is in fact optimal.

The following proposition, much of which was proved by Steffen in [21, Lemma 2.2] and [6, Lemma 2.1] lists the most fundamental properties of cores.

**Proposition 3.1.** Let $\mathcal{M} = \{M_1, M_2, M_3\}$ be an arbitrary 3-array of perfect matchings of a snark $G$. Then the following hold:

(i) Every component of $\text{core}(\mathcal{M})$ is either an even circuit or a subdivision of a cubic graph. If $G$ has no triply covered edge, then $\text{core}(\mathcal{M})$ is a set of disjoint even circuits, and vice-versa.
(ii) Every 2-valent vertex of core(\(\mathcal{M}\)) is incident with one doubly covered edge and one uncovered edge, while every 3-valent vertex is incident with one triply covered edge and two uncovered edges.

(iii) \(|E_0(\mathcal{M})| = |E_2(\mathcal{M})| + 2|E_3(\mathcal{M})|\).

(iv) \(G - E_0(\mathcal{M})\) is 3-edge-colourable.

Proposition 3.1 (i) implies that the smallest possible cores are the 2-cycle and the 4-cycle. However, Parity Lemma implies that circuits of length at most four are removable, so neither of them can occur as a core. Consequently, the following important fact holds.

**Corollary 3.2** ([21]). The defect of every snark has value at least three.

Following Steffen [21] we say that the core of a 3-array \(\mathcal{M}\) of a cubic graph \(G\) is **cyclic** if each component of core(\(\mathcal{M}\)) is a circuit. By Proposition 3.1 (ii), the core is cyclic if and only if \(G\) has no triply covered edge. The well-known conjecture of Fan and Raspaud [2] suggests that every bridgeless cubic graph has three perfect matchings \(M_1, M_2,\) and \(M_3\) with \(M_1 \cap M_2 \cap M_3 = \emptyset\). Equivalently, the conjecture states that every bridgeless cubic graph has a 3-array with a cyclic core. The conjecture is trivially true for 3-edge-colourable graphs. Mácajová and Škoviera [13] proved this conjecture to be true for cubic graphs with oddness 2. We emphasise that neither the conjecture nor the proved facts suggest anything about optimal cores.

## 4. Oddness, Girth and Colouring Defect

In this section we discuss relationships between several measures of uncolourability of cubic graphs (in the sense of the survey [3]), with particular emphasis on oddness and defect. Most of the inequalities proved here are known, however, the proofs which we offer are cleaner and more transparent. The main result of this paper, Theorem 5.1 (to be proved in the next section), relates oddness, defect and – implicitly – girth. Its proof uses one of the inequalities established in present section.

Let \(G\) be a bridgeless cubic graph. The **resistance** of \(G\), denoted by \(\rho(G)\), is the smallest number of edges whose removal from \(G\) yields a 3-edge-colourable graph. It is well known that \(\rho(G) \leq \omega(G)\) and that \(\rho(G) = 2\) if and only if \(\omega(G) = 2\), see [19, Lemma 2.5]. The **density** \(\text{dn}(G)\) of \(G\) is the minimum number of common edges that two perfect matchings in \(G\) can have. This invariant was introduced by Steffen in [22] and denoted by \(\gamma_2(G)\) in [6]. Jin and Steffen in [6, Theorem 2.2] proved that

\[
\omega(G) \leq 2 \text{dn}(G) \leq \text{df}(G) - 1,
\]

(1)
if \( G \) is not 3-edge-colourable. As a consequence, if \( \text{df}(G) = 3 \), then \( \text{dn}(G) = 1 \) and \( \omega(G) = 2 \).

We prove (I) starting with the inequality on the left-hand side.

**Proposition 4.1.** If \( G \) is a bridgeless cubic graph, then \( \omega(G) \leq 2\text{dn}(G) \).

**Proof.** Let \( M_1 \) and \( M_2 \) be any two perfect matchings of \( G \). Take the 2-factor \( F_1 \) complementary to \( M_1 \), and assume that it has \( c \) odd circuits. Since every set with an odd number of vertices sends out an edge of \( M_2 \), each odd circuit of \( F_1 \) is incident with at least one edge from \( M_1 \cap M_2 \). If \( E' \) denotes the set of all edges of \( M_1 \cap M_2 \) incident with an odd circuit of \( F_1 \), then clearly \( |E'| \geq c/2 \geq \omega/2 \), where \( \omega = \omega(G) \).

Consequently, \( |M_1 \cap M_2| \geq |E'| \geq \omega/2 \) for each pair \( M_1 \) and \( M_2 \) of perfect matchings of \( G \), and so \( \text{dn}(G) = \min_{M_1,M_2} |M_1 \cap M_2| \geq \omega/2 \). \( \square \)

Now we are ready for the inequality on the right-hand side of (I).

**Proposition 4.2.** If \( G \) is a snark, then

\[
\text{df}(G) \geq 2\text{dn}(G) + 1.
\]

**Proof.** Let \( \mathcal{M} = \{M_1, M_2, M_3\} \) be an optimal 3-array of \( G \). Since \( G \) is a snark, \( \text{core}(\mathcal{M}) \) is nonempty. We claim that \( \text{core}(\mathcal{M}) \) contains at least one doubly covered edge. Suppose not. Then \( \text{core}(\mathcal{M}) \) consists of uncovered and triply covered edges, which implies that \( M_1 = M_2 = M_3 \). Pick an uncovered edge \( e \) and take a perfect matching \( M'_1 \) containing \( e \); it is well known that such a perfect matching always exists [18]. Clearly, the 3-array \( \{M'_1, M_2, M_3\} \) has fewer uncovered edges than \( \mathcal{M} \), so \( \mathcal{M} \) was not optimal. Thus, if \( \mathcal{M} \) is optimal, there exist indices \( i \neq j \) such that \( |M_i \cap M_j| - |E_3| \geq 1 \); without loss of generality we may assume that \( |M_1 \cap M_2| - |E_3| \geq 1 \). By applying Proposition 3.1 (iii) we obtain

\[
\begin{align*}
\text{df}(G) &= |E_2| + 2|E_3| = \left( \sum_{i \neq j} |M_i \cap M_j| \right) - |E_3| \geq |M_1 \cap M_3| + |M_2 \cap M_3| + 1 \\
&\geq 2\text{dn}(G) + 1,
\end{align*}
\]

as required. \( \square \)

Proposition 3.1 (iv) implies that \( \text{df}(G) \geq \rho(G) \) for every bridgeless cubic graph \( G \). Jin and Steffen [6] Corollary 2.4] proved the following stronger result.

**Theorem 4.3.** For every bridgeless cubic graph \( G \) one has

\[
\text{df}(G) \geq 3\omega(G)/2.
\]

**Proof.** Let \( \mathcal{M} = \{M_1, M_2, M_3\} \) be an optimal 3-array of \( G \), and for \( i \in \{1, 2, 3\} \) let \( F_i \) be the 2-factor \( F_i \) complementary to \( M_i \). Our aim is to estimate the number of odd circuits in each \( F_i \) and then use the estimate to bound the oddness of \( G \).

For each \( i \in \{1, 2, 3\} \) we partition the set of odd circuits of \( F_i \) into three subsets \( C^1_i, C^2_i, \) and \( C^3_i \) as follows:

(i) \( C^1_i \) will consist of all odd circuits of \( F_i \) contained in \( \text{core}(\mathcal{M}) \) in which all edges are uncovered;

(ii) \( C^2_i \) will consist of all odd circuits of \( F_i \) contained in \( \text{core}(\mathcal{M}) \) which contain at least one doubly covered edge; and

(iii) \( C^3_i \) will consist of all odd circuits not contained in \( \text{core}(\mathcal{M}) \).
Observe that the edges leaving a circuit \( C \) from \( C_1 \) are all triply covered. In other words, \( C_1 = C_2 = C_3 \), so for simplicity we write \( C_1 = C \). A vertex of \( G \) incident with a triply covered edge will be called special. Next, each circuit \( C \) from any \( C_i \) consists of uncovered edges and doubly covered edges, and since \( C \) is odd, at least two uncovered edges of \( C \) must be adjacent. It follows that each \( C \in C_i \) has at least one special vertex.

At first we derive a bound on \( |C_i^2| \). Pick an arbitrary circuit \( C \in C_i^3 \); since \( C \) is odd, it contains an edge of core(\( M \)). Consider a component of the intersection of circuit \( C \) and core(\( M \)), which must be a path \( P \subseteq C \). Let \( u \) and \( v \) be the endvertices of \( P \), and let \( e \) and \( f \) be the edges of \( M_i \) incident with \( u \) and \( v \), respectively. By Proposition 4.4, \( e \) and \( f \) cannot be simply covered, because each of them is adjacent to an edge of core(\( M \)) \( \cap C \) and to a simply covered edge of \( C \). As both \( e \) and \( f \) are covered, they must be doubly covered and hence belong to \( E_2 \cap M_i \). In other words, each \( C \in C_i^3 \) produces at least two edges from \( E_2 \cap M_i \).

Form an auxiliary graph \( X_i \) with bipartition \( \{ C_i^3, E_2 \cap M_i \} \), where \( C \in C_i^3 \) is joined to \( e \in E_2 \cap M_i \) whenever \( e \) is incident with \( C \). As previously explained, \( \deg(C) \geq 2 \) for each \( C \in C_i^3 \) while \( \deg(e) \leq 2 \) for each \( e \in E_2 \cap M_i \). Hence, counting the edges of \( X_i \) in two ways yields

\[
2|C_i^3| \leq \sum_{C \in C_i^3} \deg_X(C) = |E(X_i)| = \sum_{e \in E_2 \cap M_i} \deg_X(e) \leq 2|E_2 \cap M_i|.
\]

It follows that \( |C_i^3| \leq |E_2 \cap M_i| \) and hence

\[
\sum_{i=1}^{3} |C_i^3| \leq 2|E_2|.
\]

Now we bound \( |C^1| \). Let \( S \) denote the set of special vertices of \( G \) with regard to \( M \). By Proposition 3.1(ii), \( |S| = 2|E_3| \). Since each vertex in a circuit from \( C^1 \) is special, each circuit from \( C^1 \) has at least three special vertices. Moreover, any two circuits from \( C^1 \) are disjoint. Therefore

\[
|C^1| \leq |S|/3 = 2|E_3|/3.
\]

Lastly, we deal with \( C_i^2 \). Every circuit from \( C_i^2 \) contains at least one pair of adjacent uncovered edges, and therefore at least one special vertex. We further show that any two circuits from \( C_i^1 \cup C_i^2 \cup C_i^3 \) are disjoint. Suppose that this is false and two circuits \( C \in C_i^2 \) and \( D \in C_j^2 \) have a nonempty intersection. Then there exists an edge \( e \) in \( C \cap D \), which is adjacent to two edges \( f \) and \( g \), such that \( f \) lies in \( C \) but not in \( D \) and \( g \) lies in \( D \) but not in \( C \). Since \( f \) is contained in \( C \), it is uncovered or doubly covered. At the same time, \( f \) leaves \( D \in C_j^2 \), so it is simply or triply covered, which is clearly impossible. Therefore \( C \cap D = \emptyset \). Hence

\[
\sum_{i=1}^{3} |C_i^2| \leq |S| = 2|E_3|.
\]

Summing up, if we denote the number of odd circuits in the 2-factor \( F_i \) by \( \omega_i \), from (2)-(4) we obtain

\[
3\omega(G) \leq \sum_{i=1}^{3} \omega_i = 3|C^1| + \sum_{i=1}^{3} |C_i^2| + \sum_{i=1}^{3} |C_i^3| \leq 4|E_3| + 2|E_2|.
\]

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By Proposition 3.1 (iii), the right-hand side of (5) equals $2 \text{df}(G)$, and the theorem follows. □

The next result is due to Steffen [21, Corollary 2.5].

**Proposition 4.4.** For every snark $G$ one has $\text{df}(G) \geq \lceil \text{girth}(G)/2 \rceil$.

**Proof.** Let $\mathcal{M}$ be a optimal 3-array and let $H$ be its core. Since each vertex of $H$ is either 2-valent or 3-valent, $H$ contains a cycle $K$. Let $q$ be the length of $K$. By Proposition 3.1 (ii), at least $\lceil q/2 \rceil$ edges of $K$ are left uncovered. Hence,

$$\text{df}(G) \geq \lceil q/2 \rceil \geq \lceil \text{girth}(G)/2 \rceil,$$

as claimed. □

5. **Main result**

In this section we show that there exist nontrivial snarks with oddness 2 and arbitrarily large defect. As a consequence, nontrivial snarks with oddness 2 are split into infinitely many subclasses according to their defect.

The proof makes use of the method of superposition, introduced by Kochol in [9], whose main idea is to ‘inflate’ a given snark $G$ into a large cubic graph $\tilde{G}$ by substituting vertices of $G$ with ‘fat vertices’ (tripoles), called *supervertices*, and edges of $G$ with ‘fat edges’ (dipoles), called *superedges*. Under suitable conditions the inflated graph $\tilde{G}$ is a snark. For formal definitions and a detailed description of the method we refer the interested reader to the original paper [9] or to a recent paper [14]. Our proof is self-contained.

**Theorem 5.1.** There exist nontrivial snarks of oddness 2 with arbitrarily large defect.

**Proof.** To prove the theorem we modify the construction of snarks of arbitrarily large girth due to Kochol [9, Section 4] in such a way that a specified pair $\{u, v\}$ of adjacent vertices of the resulting graph $\tilde{G}$ will be non-removable. This fact will guarantee that $\omega(\tilde{G}) = 2$ while $\text{df}(\tilde{G})$ may take an arbitrarily large value, according to Proposition 4.4.

The key ingredient of our construction is a $(3,3)$-pole $F = F_g$ that contains no cycles of length smaller than $g$ for any prescribed $g \geq 6$. It is represented in Figure 3(b) together with a partial 3-edge-colouring of its edges; the subgraphs indicated in Figure 3(b) as $M_g$ are copies of a 5-pole obtained from a suitable cubic graph $L_g$ of girth $g$ by removing a path of length 2. The $(3,3)$-pole $F_g$ will serve as a superedge in our construction. It will be built in several steps.

![Figure 2](image-url)
We start the construction of $F_9$ by taking three copies of the Petersen graph, denoted by $P_1$, $P_2$, and $P_3$. In each $P_i$ with $i \in \{1, 3\}$ we choose a set $\{u_i, v_i, w_i\}$ of three vertices at distance 2 from each other, and in $P_2$ we choose two edges $x_1x_2$ and $x_3x_4$ such that their endvertices, if taken from distinct edges, are again at distance 2 from each other. (For example, one can take $\{u_i, v_i, w_i\} = \{v_0, v_2, v_4\}$, $x_1x_2 = e_0$, and $x_3x_4 = e_3$, see Figure 2.) It is important that $\{u_i, v_i, w_i\}$, with $i \in \{1, 3\}$, and $\{x_1, x_2, x_3, x_4\}$ are decycling sets, which means that the removal of any of them from the Petersen graph leaves an acyclic subgraph. We construct a new graph $K$ from $P_1 \cup (P_2 - \{x_1x_2, x_3x_4\}) \cup P_3$ as follows: for $i \in \{1, 3\}$ we create a new vertex $z_i$ by identifying $v_i$ with $x_i$, and new vertex $z_i + 1$ by identifying $w_i$ with $x_i + 1$, thereby producing four 5-valent vertices $z_1, z_2, z_3, z_4$. The result is shown in Figure 3(a).

Set $W = \{z_1, z_2, z_3, z_4\}$ and $U = W \cup \{u_1, u_3\}$. Note that $U$ is a decycling set for $K$. We show that $K$ admits no nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow. Suppose to the contrary that

$\sigma$ is a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow on $K$. For $i \in \{1, 2, 3, 4\}$ let $r_{i1}$ and $r_{i2}$ denote the edges joining the vertex $z_i$ of $K$ to vertices of $P_2 - \{x_1, x_2, x_3, x_4\}$; see Figure 3(a). To derive a contradiction we first prove that $\sigma(r_{11}) \neq \sigma(r_{12})$. If $\sigma(r_{11}) = \sigma(r_{12})$, then $\sigma(r_{21}) = \sigma(r_{22})$ as well, because the outflow from every nonempty set of vertices is 0, by the Kirchhoff law. This in turn implies that the sum of flow values on the three edges incident with $z_1$ and different from $r_{11}$ and $r_{12}$ must be 0 as well. Similarly, the sum of flow values on the three edges incident with $z_2$ and different from $r_{21}$ and $r_{22}$ must be 0 as well.

![Figure 3. Main ingredients of the construction of $\tilde{G}$.](image-url)
$r_{22}$ must be 0. It follows that $\sigma$ induces a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow on $P_1$, which is a contradiction. Therefore $\sigma(r_{11}) \neq \sigma(r_{12})$. By analogous arguments we can show that $\sigma(r_{i1}) \neq \sigma(r_{i2})$ for each $i \in \{1,2,3,4\}$. Moreover, the fact that the outflow from every nonempty set of vertices is 0 also implies that $\sigma(r_{11}) + \sigma(r_{12}) = \sigma(r_{21}) + \sigma(r_{22})$ and $\sigma(r_{31}) + \sigma(r_{32}) = \sigma(r_{41}) + \sigma(r_{42})$. Thus if we take the induced valuation on $P_2 = \{x_1x_2, x_3x_4\}$ and assign the value $\sigma(r_{11}) + \sigma(r_{12})$ to the edge $x_1x_2$ and the value $\sigma(r_{31}) + \sigma(r_{32})$ to the edge $x_3x_4$, we obtain a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow on $P_2$. This is again a contradiction, so $K$ admits no nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow.

If we substitute every vertex $s \in W$ in $K$ with a copy of a cubic 5-pole $M$, identifying the dangling edges of $\bar{M}$, and each of the vertices $v$ incident with three dangling edges with a copy of the (3, 5)-pole $\xi$, which consists of a single vertex $z$ incident with three dangling edges and of two additional isolated edges; the edges incident with the vertex contribute to all three connectors of $Z$ while each isolated edge contributes to two different connectors of size 3, see Figure 1. Finally, we join the connectors of each copy of $\xi$ to a connector of a copy of $M_\xi$ and a connector of a copy of $Z$, and connect the copies of $M_\xi$ between themselves and to the remaining vertices of $P$ in such a way that a cubic graph $G$ arises.

Next we prove that $G$ is a snark. If $G$ was 3-edge-colourable, then it would admit a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow $\xi$. Recall that $G$ is obtained from the Petersen graph $P$ by substituting each of the edges $e_1, e_2, e_4,$ and $e_5$ with a copy of the (3, 3)-pole $F_g$.¹
Now we define an edge valuation $\xi_*$ on $P$ in $\mathbb{Z}_2 \times \mathbb{Z}_2 - \{0\}$ as follows. For each edge $e$ of $P$ different from the four previously mentioned edges set $\xi_*(e) = \xi(e)$; for each of the remaining four edges set $\xi_*(e)$ to be the total flow through the corresponding copy of $F_g$. Since $\xi$ fulfils the Kirchhoff law, so does $\xi_*$. Thus $\xi_*$ is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow. However, $F_g$ is a proper dipole, so $\xi_*$ is a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow on the Petersen graph, which is absurd. Therefore $\tilde{G}$ is a snark. (For a more detailed argument see [9, Theorem 4].)

By inspecting Figure 3(a) it is easy to see that every edge cut in the graph $K$ has at least three edges. Recall that $L_g$ is vertex-transitive of girth $g \geq 6$, and therefore it is cyclically $g$-edge-connected by Theorem 17 of [13]. Since $M_g$ arises from $L_g$ by removing a path of length 2, every edge cut of $F_g$ has at least three edges, too. Now, let us look at the cycle-separating edge cuts in $\tilde{G}$. If such a cut disconnects at least two copies of $F_g$, then it has at least six edges. If, on the other hand, it disconnects exactly one copy of $F_g$, then the cut contains at least three edges in the copy of $F_g$ and at least two edges outside of $F_g$. Finally, if the cut does not intersect any of the copies of $F_g$, then it has at least five edges because the Petersen graph is cyclically 5-edge-connected, and each copy of $M_g$ is connected and is separated from the rest by five edges. Summing up, every cycle-separating edge cut in $\tilde{G}$ has at least five edges, in other words, $\tilde{G}$ is cyclically 5-edge-connected. In particular, $\tilde{G}$ is a nontrivial snark.

We further show that $\text{girth}(\tilde{G}) = g$. If we take into account the fact that the dipole $F_g$ contains no cycles of length smaller than $g$ and that $\{v_0, v_1, v_3, v_4\}$ is a decycling set of $P$, we can conclude that $\text{girth}(\tilde{G}) \geq g$. However, Theorem 4.8 in [17] states that there exist infinitely many arc-transitive cubic graphs $X$ of any given girth $g \geq 6$. It follows that such graphs may have arbitrarily large diameter, and hence infinitely many of them contain at least two disjoint $g$-cycles. Therefore $M_g$ can be constructed in such a way that it still contains a $g$-cycle. Summing up, $\text{girth}(\tilde{G}) = g$. From Proposition 4.4 we now infer that $\text{df}(\tilde{G}) \geq g/2$.

Observe that our construction does not determine the snark $\tilde{G}$ uniquely, because the order of semiedges in the connectors is irrelevant for the result. We take this advantage to show that the identification of the free ends of semiedges in connectors can be performed in such a way that $\tilde{G} - \{u, v\}$ is 3-edge-colourable. For this purpose we first extend the partial 3-edge-colouring of $F_g$ shown in Figure 3(b) to the entire edge set. Recall that $F_g$ was created from a bipartite cubic graph $L_g$ by removing a path of length 2, so $F_g$ is colourable. To make the extension of the partial colouring possible we need to be more specific about how the five dangling edges of $M_g$ are joined to the five dangling edges of $K - s$ for every vertex $s \in W$. To this end, it is sufficient to realise that by Parity Lemma every 3-edge-colouring of $M_g$ induces the colour vector $aaabc$ where $a$, $b$, and $c$ are the three nonzero elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ in some order. Although the edges receiving the lonely colours $b$ and $c$ may not be chosen arbitrarily, we can always attach a coloured copy of $M_g$ to $K - s$, possibly after permuting the colours, in such a way that the colours of the corresponding edges match. Hence, $F_g$ admits a 3-edge-colouring where each connector receives colours 1, 1, and 2 as shown in Figure 3(b). Finally, we insert the coloured copies of $F_g$ and $M_g$ into $P$, possibly after permuting the colours, in such a way that the colours of the edges in the joined connectors again match. The result is a defective edge colouring of $\tilde{G}$ where the Kirchhoff law fails only at the vertices $u$ and $v$, see...
Figure 4. The resulting snark $\tilde{G}$ of girth $g$

Figure 4. Thus $\tilde{G} - \{u, v\}$ is 3-edge-colourable, and consequently, the resistance of $\tilde{G}$ equals 2. It follows that $\omega(\tilde{G}) = 2$, as claimed. This completes the proof. \hfill $\square$

The following interpretation of the previous proof is also important, as one can see in our paper [7].

**Theorem 5.2.** There exist nontrivial snarks with arbitrarily large girth that contain a non-removable pair of adjacent vertices.

Another benefit of the construction presented in the proof of Theorem 5.1 is a strengthening of the original Kochol’s construction [9].

**Theorem 5.3.** For every $g \geq 5$ there exist infinitely many cyclically 5-connected snarks whose girth equals $g$.

*Proof.* Snarks constructed in the proof of Theorem 5.1 satisfy the statement for every even $g \geq 6$. If $g \geq 7$ is odd, we modify the construction by taking $L_g$ from the infinitely many graphs of girth $g$ constructed in Theorem 4.8 in [17]. Since we do not care whether $L_g$ is 3-edge-colourable or not, we do not require $L_g$ to be bipartite. Otherwise, the construction proceeds as in the proof of Theorem 5.1. Finally, if $g = 5$, there are several available constructions of infinitely many cyclically 5-connected snarks of girth 5, for example rotation snarks or permutation snarks constructed in Theorem 5.1 and Example 6.4 of [14], respectively. \hfill $\square$

It is also possible to construct snarks with cyclic connectivity 4 and girth $g$ for each $g \geq 5$. The construction is similar to that described in the proof of Theorem 5.1 except that one has to use the method of Section 5 of [9] instead of Section 4.
6. Final remarks

We believe that nontrivial snarks with any given oddness $\omega$ and colouring defect $d$ exist for each pair $d \geq 3\omega/2$ (which is the restriction posed by Theorem 4.3). Constructing such snarks would be worthwhile as it would provide a complete generalisation of Theorem 3.4 of Jin and Steffen [6] to nontrivial snarks. Unfortunately, our construction, which only deals with $\omega = 2$, does not easily generalise to larger values of $\omega$.

Another possibility is to consider an analogous (but weaker) problem where oddness is replaced with resistance. Recall that $\omega(G) \geq \rho(G)$ for every bridgeless cubic graph $G$. It follows from Theorem 4.3 that $df(G) \geq 3\rho(G)/2$, and we may ask whether for any given $\rho \geq 2$ and $d \geq 3\omega/2$ there exists a nontrivial snark with resistance $\rho$ and colouring defect $d$. We think that the answer is “yes”.

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