Dipolar Excitons, Spontaneous Phase Coherence, and Superfluid-Insulator Transition in Bi-layer Quantum Hall Systems at $\nu = 1$

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The spontaneous interlayer phase coherent (111) state of bi-layer Quantum Hall system at filling factor $\nu = 1$ may be viewed as a condensate of interlayer particle-hole pairs or excitons. We show in this paper that when the layers are biased in such a way that these excitons are very dilute, they may be viewed as point-like bosons. We calculate the exciton dispersion relation, and show that the exciton-exciton interaction is dominated by the dipole moment they carry. In addition to the phase coherent state, we also find a Wigner Crystal/Glass phase in the presence/absence of disorder which is an insulating state for the excitons. The position of the phase boundary is estimated and the properties of the superfluid-insulator type transition between these two phases is discussed. We also discuss the relation between these “dipolar” excitons and the “dipolar” composite fermions studied in the context of half-filled Landau level.

The (111) State. We begin by considering the Halperin (111) state \[ \Psi_{111}(z_i, z_{[k]}) = \prod_{i<j} (z_i - z_j) \prod_{k<l} (z_{[k]} - z_{[l]}) \prod_{i,k} (z_i - z_{[k]}), \] (1)

where $z_j = x_j + iy_j$ and $z_{[k]} = x_{[k]} + iy_{[k]}$ are the complex coordinates of the $j$th electron in layer 1 and $k$th electron in layer 2 respectively, and $N_1$ and $N_2$ are the numbers of electrons in the upper and lower layers, which are separately conserved in the absence of tunneling (assumed in this paper). The total number of electrons $N = N_1 + N_2$ equals the Landau level degeneracy. We have neglected the common exponential factors in the wave function. The analytic form of the wave function guarantees that it contains no weight outside the lowest Landau level (LLL). Unlike most other multicomponent wave functions proposed by Halperin \[ \Psi_{111} \], the (111) wave function is a valid state for $\nu = 1$ for arbitrary relative population in the two layers, an indication of interlayer coherence and broken symmetry. In the special case of interlayer distance $d = 0$, states with different $N_1$ are degenerate. For finite $d$, the two layers are equally populated in the ground state to minimize Coulomb charging energy in the absence of biasing voltage: $N_1 = N_2 = N/2$; experimentally one can adjust $N_1$ and $N_2$ by applying a finite biasing voltage. In order to see clearly the pairing between electrons in one layer and holes in the other,
we make a particle-hole transformation \[10\] in the lower layer and express the \((111)\) wave function in terms of coordinates of electrons in layer 1 and holes in layer 2:

\[
\Psi_{111}(z, \xi) \propto \mathbf{A} \prod_{i=1}^{N_1} e^{z_i \xi_i^*/2} = \text{Det}[M_{ij}],
\]

where \(\mathbf{A}\) is an antisymmetrizer, \(M_{ij} = e^{z_i \xi_j^*/2}\), and \(\xi_i\) is the complex coordinate of the \(i\)th hole in layer 2. Notice that at \(\nu = 1\), the number of particles in one layer is always equal to the number of holes in the other, and that the wave function in the LLL are analytic functions of \(\xi^*\) since holes carry opposite charges. One can easily show that the states \(\langle 1 \rangle\) and \(\langle 2 \rangle\) are equivalent by particle-hole transform \([16]\) back to the wave function in terms of coordinates of electrons in both layers. The state \(\langle 2 \rangle\) has a simple and clear physical interpretation. It indicates that every electron in the upper layer is paired up with a hole in the other layer, with the pairing wave function

\[
\Psi_\mu(z, \xi) = e^{z \xi^*/2}.
\]

This is the closest pair that can be formed in the LLL and in fact represent \(\delta(z - \xi)\) projected onto the LLL \([17]\). It has angular momentum zero and hence \(S\) wave symmetry. We thus have shown that the spontaneous interlayer coherence can also be viewed as \(S\) wave pairing between particles in one layer and holes in the other.

**Exciton Dispersion Relation.** Now consider the limit that \(N_1 = 1\) and \(N_2 = N - 1\). In this case there is a single particle-hole pair or exciton, and Eq. \((3)\) is the *exact* ground state wave function of the system that describes a zero-momentum exciton, which we label as \(\{0\}\). The state that describes a single exciton with finite momentum \(k\) can be constructed:

\[
|k\rangle = e^{ikR}|0\rangle,
\]

where \(R\) is the guiding center coordinate \([13]\) of the electron in layer 1. The state \(|k\rangle\) takes the same form as a single spin-wave of a spin-polarized filled Landau level \([19]\), which is an *exact* eigenstate of the system in both contexts. The \(N^2\) allowed \(k\)'s exhaust all the possible states for a single particle-hole pair. We can calculate the dispersion relation of the single exciton:

\[
E(k) = \langle k | H | k \rangle - \langle 0 | H | 0 \rangle
\]

\[
= \frac{1}{2\pi} \int_0^\infty q V^E(q) e^{-q^2 \ell^2/2} \left[ 1 - J_0(q \ell) \right] d\ell,
\]

where \(\ell\) is the magnetic length and \(V^E(q)\) is the Fourier transform of the interlayer Coulomb interaction; neglecting layer thickness we have \(V^E(q) = 2\pi \epsilon^2 e^{-qd}/(\epsilon q)\), where \(\epsilon\) is the dielectric constant. In the long-wave length limit the exciton has a quadratic dispersion:

\[
E(k) \approx \frac{k^2}{2m(d)},
\]

where the inverse effective mass for the exciton is

\[
\frac{1}{m(d)} = \frac{e^2 \ell}{2\epsilon} \int_0^\infty x^2 e^{-xd/\ell - x^2/2} dx.
\]

It is worth noting that the origin of the exciton dispersion (or “kinetic energy”) is solely due to electron-electron/hole interaction, since the kinetic energy of the electrons has been quenched by the strong magnetic field; the momentum of the exciton \(k\) is actually a measure of how closely bound the electron-hole pair is.

**Exciton-Exciton Interaction.** When \(N_1\) is more than 1, we expect that the electrons in layer 1 form pairs with the exactly same number of holes in layer 2 as in the \((111)\) state, and there will be \(N_1\) excitons in the system. It is very tempting to treat the excitons as point-like bosons and map the problem onto that of an interacting boson system. There is, however, a caveat. When there are more than one excitons, the naive number of possible states for excitons are bigger than the dimensionality of the Hilbert space for the electron system. For example, when \(N_1 = 2\), the dimensionality of the electron Hilbert space is \(N^2(N - 1)^2/4\), while when there are \(N^2\) independent states for a single boson, the number of states for a pair of bosons would be \(N^2(N^2 + 1)/2\), roughly a factor of two bigger. This factor of two can be understood as due to the ambiguity in the way electrons and holes form pairs, as illustrated in Fig. 1. Thus the multiple exciton/boson states are overcomplete and obviously the overcompleteness problem worsens very quickly as \(N_1\) increases. We believe, however, mapping onto interacting bosons is a valid approach when \(\nu_1 = N_1/N \ll 1\) so that the excitons are very dilute, and if we concentrate only on low-energy properties of the system. This is because at low-energies the excitons are very closely bound while the inter-exciton distance is very large; thus the ambiguity illustrated in Fig. 1 disappears because the “unnatural” description involves very loosely bound excitons that cost large amount energy.

To complete the description of the system in terms of the bosonic excitons, we calculate the exciton-exciton interaction matrix elements. Consider the unsymmetrized basis states for two excitons which are normalized:

![Layer 1 and Layer 2 illustration](image-url)
neous interlayer phase coherence, and is a superfluid (SF) the excitons Bose condense; this phase exhibits sponta-

The Hamiltonian in this case is

\[ H = H_0 - \Delta H, \]

\[ H_0 = \sum_{i<j} V^A(r_i - r_j), \]

\[ \Delta H = \sum_{j>2} (\Delta V(r_1 - r_j) + \Delta V(r_2 - r_j)). \]

Here \( V^A \) is the interlayer electron-electron interaction, whose Fourier transform is: \( V^A(q) = 2\pi e^2/eq \), and \( \Delta V = V^A - V^E \) is the difference between intra- and inter-layer interactions. We find for small \( k \)’s,

\[ \langle k'_1 k'_2 | H | k_1 k_2 \rangle = \text{const.} \times \langle k'_1 k'_2 | k_1 k_2 \rangle + O(k^2) \]

\[ + \frac{2\delta_{k_1+k_2, k'_1+k'_2}}{A} \left[ \Delta V_{k_1-k_1} - \frac{1}{N} \sum_q \Delta V_q e^{-q^2\ell^2/2} \right], \]

where \( A = 2\pi N\ell^2 \) is the area of the system. In Eq. [13], the first term is due to the non-orthogonality of the basis states which has no physical consequence, and the remaining terms describe the interactions of the pair of excitons. If we neglect terms of higher order in \( k \), we find the effective interaction between the excitons (in momentum space),

\[ \tilde{V}_k = 2\Delta V_k - \frac{2}{N} \sum_q \Delta V_q e^{-q^2\ell^2/2}, \]

includes two terms; the first term is a repulsive dipole-dipole interaction that decays as \( 1/R^3 \) at large distance \( R \); the second term that is \( k \)-independent describes a local attraction that softens the dipole-dipole interaction at short distances. It can be shown that \( \tilde{V}_k \) does not vanish so that the overall interaction is repulsive. The dipole-dipole in-

Possible Phases. As argued above, for small \( \nu_1 = N_1/N \) these excitons may be viewed as point-like bosons. Such a dilute Bose gas with repulsive interaction supports two phases. At small \( d, 1/m(d) \) is large while \( V \) is small, so the kinetic energy dominates the physics and the excitons Bose condense; this phase exhibits spontaneous interlayer phase coherence, and is a superfluid (SF) phase for the excitons. In particular, in the limit \( d \to 0 \), \( \Delta V \) and \( \tilde{V} \) vanish, there is no interaction among the ex-

tons and they all condense into the zero momentum state, which is precisely what the (111) state describes; it is the exact ground state of the system in this limit. The hallmark of this phase is a linear Goldstone mode whose velocity we estimate using a weak-coupling Bogli-

ubov theory [20],

\[ v_g \approx \sqrt{\frac{\nu_1 \tilde{V}_{k=0}}{2\pi e^2 m(d)}}. \]

On the other hand, for large \( d, 1/m(d) \) is small and \( \tilde{V} \) is large; the repulsive interaction dominates and the excitons form a Wigner crystal (WC). Upon introducing weak disorder, the WC is pinned and becomes a Wigner glass; thus the WC phase is an insulating phase for the excitons. The fact that such a WC phase must exist at large \( d \) can also be understood from a different viewpoint. Consider the limit \( d \to \infty \) where the two layers decouple. It is widely expected that for small enough \( \nu_1 \) the electrons in layer 1 form a WC [21]. The holes in layer 2 will thus form its own WC with identical structure. Thus for large but finite \( d \), the two WC’s lock together and electrons and holes pair up; this is exactly the exciton WC.

It should be emphasized however, that both of these two phases are incompressible and the system exhibit \( \nu = 1 \) quantum Hall effect, even though the neutral exciton systems are gapless. This is because excessive charge in the system will induce unpaired electrons or holes, which cost a finite amount of energy. Thus the WC phase is an explicit example of the phase with quantum Hall effec-

t but no interlayer phase coherence discussed in Ref. [10], in which a broken translational symmetry was antici-

pated.

Experimentally, one can easily distinguish between these two phases by performing tunneling or drag mea-

surements. In tunneling, one should see a zero-bias peak in the SF phase, and Coulomb gap-like behavior in the WC phase; the Goldstone mode velocity \( v_g \) can be mea-

ured by applying a parallel magnetic field as in Ref. [22]. In drag measurement, one should find that a driving cur-

rent in one layer always induces the same Hall voltage in both layers [23] in the SF phase, while in the WC phase all the current flow in layer 2 while layer 1 cannot carry any current due to pinning; all current going through the sample flow in layer 2 and is carried by the background \( \nu = 1 \) electron gas in which the holes are embedded.

Superfluid-Insulator Phase Transition. The critical layer spacing separating these two phases may be esti-

mated by comparing the (approximate) energies of the corresponding states. For the (111) state that represents the SF phase, there is no exciton “kinetic” energy and the interaction energy per exciton is \( E_{SF} \approx \nu_1 \tilde{V}_{k=0}/(4\pi e^2) \); while in the WC phase the interaction energy is neg-
ligible since the dipolar repulsion vanishes faster than the kinetic energy in the low density limit; the latter is estimated using the uncertainty principle to be $E_W \sim \nu_1/(4\pi^2m(d))$. Comparing the two we obtain a very crude estimate of the phase boundary to be $d^* \sim 0.6\ell$.

This superfluid-insulator transition is expected to be of second-order [23], at which the conductivity of the excitons, $\sigma_{xx}^*$, is universal [24]. $\sigma_{xx}^*$ can be measured in a drag experiment; if the electric fields are in the $\hat{x}$ direction in both layers, then $\sigma_{xx}^* = (J_{1x} - J_{2x})/(E_{1x} - E_{2x})$ at the critical point. It should be noted that the universality class of this transition is different from the one in superconductors because i) the excitons are neutral while Cooper pairs are charged; and ii) due to the absence of time-reversal symmetry, potential scattering of the excitons resembles that of scattering off a random flux [24].

Relation with Dipolar Composite Fermions. The dipolar excitons discussed here have close family relations with the dipolar composite fermions (DCF) [26,27] studied in the context of half-filled Landau level, and in particular, a related model of repulsive bosons at filling factor $\nu = 1$ [27,28]. In our case the exciton is made of an electron in layer 1 and a hole in layer 2; the latter is a vortex in the $\nu = 1$ background in layer 2. Thus the exciton is an electron-vortex pair, just like the DCF. The fact that the electron and vortex live in separate layers makes the dipolar nature even more transparent. The fundamental difference is that here the exciton is made of two fermionic objects and is therefore a bosonic particle. In the case of half-filled Landau level the overcompleteness problem similar to the one discussed here severely limits the usefulness of the DCFs as elementary particles [27]; nevertheless it may be useful to study the low density limit of the DCFs (which is unphysical in that context) as in this work, where this problem can be avoided.

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