Using relativistic conformal hydrodynamics coupled to the linear $\sigma$ model we study the evolution of matter created in heavy-ion collisions. We focus the study on the influence of the dynamics of the chiral fields on the charged-hadron elliptic flow $v_2$ for a temperature-independent as well as for a temperature-dependent $\eta/s$ that is calculated from kinetic theory. We find that $v_2$ is not very sensitive to the coupling of chiral fields to the hydrodynamic evolution, but the temperature dependence of $\eta/s$ plays a much bigger role on this observable.

Keywords: Heavy Ion Collisions; Relativistic Fluid Dynamics; Linear Sigma Model.

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1. Introduction

In any attempt of describing experimental results of ultrarelativistic heavy ion collisions at the Relativistic Heavy Ion Collider (RHIC) and the Large Hadron Collider (LHC) via a viscous fluid dynamic model, crucial physics input reflecting the properties of the flowing nuclear matter must be supplemented, like the equation of state (EOS) and transport coefficients such as the shear ($\eta$) and bulk ($\zeta$) viscosities.

The EOS of nuclear matter created at RHIC and LHC can be obtained from Lattice QCD simulations and is fairly known in the low chemical potential regime of heavy ion collisions. In contrast, the temperature dependence of the transport coefficients describing the flow of this matter is not precisely known. Although it is expected that for the Quark–Gluon Plasma (QGP) $\eta$ and $\zeta$ will depend strongly on temperature, usually in hydrodynamic simulations temperature-independent values

\[ \frac{\eta(T)}{s(T)} \] AND COLLECTIVE FLOW IN CHIRAL HYDRODYNAMICS
for these coefficients are assumed throughout the entire evolution. The impact of the temperature dependence of transport coefficients on momentum anisotropies as obtained from fluid dynamic models has been investigated only recently2–4.

The comparison of hydrodynamic models to data would allow, in principle, to pinpoint the location of the QCD phase transition or of a crossover from hadronic matter to the QGP. In this context, in addition to the hydrodynamic degrees of freedom related to energy-momentum conservation, degrees of freedom associated with order parameters of broken continuous symmetries must be considered as well since they are all coupled to each other (see e.g. Refs. 5,6).

We study, in the context of second–order dissipative relativistic fluid dynamics – see Refs. 1,7,8,9,10 for details, the influence of the long–wavelength modes of chiral fields on the expansion of the fireball created in Au+Au collisions at \( \sqrt{s_{NN}} = 200 \) GeV. In particular, we aim at quantifying the effect of the coupled evolution of chiral degrees of freedom on the flow asymmetry characterized by \( v_2 \), when shear finite viscosity is taken into account within a simple microscopic model for the chiral condensate.

2. Conformal fluid dynamics coupled to the linear \( \sigma \) model

In this section we describe the chiral–hydrodynamic model used to compute \( v_2 \). The quark fluid is described by second–order conformal fluid dynamics9,10, while the chiral dynamics is obtained from the linear \( \sigma \) model (LSM). Further details on the model and its numerical implementation can be found in Ref.6.

The classical equations of motion of the LSM are given by

\[
D_\mu D^\mu \phi_a + \frac{\delta U}{\delta \phi_a} = -g \rho_a
\]

where \( \phi = (\sigma, \vec{\pi}) \) and

\[
\rho_a = g \phi_a d_q \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{p^2 + g^2 \sum_\alpha \phi_\alpha^2}} f_0(p^\mu, x^\mu)
\]

\( f_0(p^\mu, x^\mu) \) is the Fermi–Dirac function and \( U \) is the potential exhibiting chiral symmetry breaking. The quarks act as a thermal bath, leading to an effective potential \( V_e(\phi_a, T) = U(\phi_a) - d_q T \int \frac{d^3 p}{(2\pi)^3} \ln[1 + e^{-E/T}] \)

where \( d_q = 24 \) and \( E = \sqrt{p^2 + m^2_\eta} \) with \( m^2_\eta = g^2 \sum_\alpha \phi_\alpha^2 \).

The classical equations of motion of the chiral fields are solved selfconsistently together with the conservation equations for the quark fluid

\[
(D_\epsilon = -(\epsilon + p) \nabla \mu u^\mu + \Pi^{\mu\nu} \sigma_{\mu\nu} + g(\rho_s D\sigma + \tilde{\rho}_{ps} \cdot D\vec{\pi}))
\]

\[
(\epsilon + p) D u^i = \frac{1}{3}(g^{ij} \partial_j \epsilon - u^i u^\alpha \partial_\alpha \epsilon) - \Delta_i^j D_\beta \Pi^\alpha\beta + g(\rho_s \nabla_i \sigma + \tilde{\rho}_{ps} \cdot \nabla_i \vec{\pi}).
\]
where \((\rho_s, \vec{\rho}_p) = \rho_a\), \(D = u_\mu D^\mu\) is the comoving time derivative, \(\nabla_\mu = \Delta_{\mu\alpha} D^\alpha\) is the spatial gradient, and \(\sigma_{\mu\nu}\) is shear tensor. The evolution equation for \(\Pi^{\mu\nu}\) contains terms up to second order in velocity gradients and reads

\[
\partial_\tau \Pi^{\mu\nu} = -\frac{4}{3u^\tau} \Pi^{\mu\nu} \nabla_\mu u^\nu - \frac{1}{\tau_\pi u^\tau} \Pi^{\mu\nu} + \frac{\eta}{\tau_\pi u^\tau} \sigma^{\mu\nu} - \lambda_1 \frac{\eta}{2\tau_\pi^2 u^\tau} \Pi^{\mu<\nu>\gamma\alpha} + \frac{u^\mu \Pi^{\rho\nu} + u^\rho \Pi^{\mu\nu}}{u^\tau} D u^\mu - \frac{u^\nu}{u^\tau} \partial_\nu \Pi^{\mu\nu}
\]

where \(\eta\) is the shear viscosity and \((\tau_\pi, \lambda_1)\) are second–order transport coefficients.

From Eqs. ?? it is seen that the evolution of the chiral fields affects the evolution of the quark fluid through the sources in the energy–momentum conservation equations; in turn, the quark fluid affects the evolution of the chiral fields through the densities \(\rho_s\) and \(\vec{\rho}_p\), that depend on the local values of the fluid dynamic variables \(T\) and \(u^\nu\).

In order to compute the temperature dependence of the shear viscosity \(\eta(T)\) in the LSM, we adapt the method described in Ref.\(^{11}\) and employ the linearized Boltzmann equation in the relaxation time approximation (see Ref.\(^{6}\)). The shear viscosity is given by

\[
\eta = \frac{4\tau}{5T} \int \frac{d^3p}{(2\pi)^3} p^4 \left( f_0(1 - f_0) \right)
\]

where \(\tau = \tau(T)\) is the collision time calculated from the averaged cross sections \(\bar{\sigma}\) for quark–quark and quark–antiquark scattering processes including \(1/N_c\) next to leading order corrections as

\[
\tau^{-1} = 6 f_0 \left( \bar{\sigma}_{uu\rightarrow uu} + \bar{\sigma}_{ud\rightarrow ud} + \bar{\sigma}_{u\bar{u}\rightarrow u\bar{u}} + \bar{\sigma}_{u\bar{u}\rightarrow ud} \right) + \bar{\sigma}_{ad\rightarrow ud}
\]

We refer the reader to Ref.\(^{11}\) for details on the calculation of the \(\bar{\sigma}'s\).

The speed of sound \(c_s^2\) and \(\eta/s\) of the LSM with \(g = 3.2\) is shown in Fig. 1 together with the \(c_s^2\) corresponding to Lattice QCD\(^{12}\) (shown for comparison). The value of \(g = 3.2\) is chosen because it leads to a smooth crossover with a drop in \(c_s^2\) near the critical temperature \(T_c\) which is comparable to the one obtained from Lattice QCD simulations. For larger (smaller) values of \(g\), the drop in \(c_s^2\) near \(T_c\) is significantly larger (smaller), as shown in Fig. 2. It is seen that decreasing the value of \(g\) leads to a softening of the crossover and an increase in \(T_c\), while increasing it leads to a decrease in \(T_c\) and a steeper variation of \(c_s^2\) with temperature. We note that a first order phase transition is obtained if \(g \sim 3.8\), but we will not consider this case since Lattice QCD simulations\(^{12}\) indicate the QGP–hadron transition is a smooth crossover (see Fig. 1).

\(^{a}\)Strictly speaking, there is no \(T_c\) since we are dealing with a crossover and not with a phase transition. However, we will still refer to a \(T_c\) as an approximate critical temperature around which \(c_s^2\) varies rapidly.
For the second-order transport coefficients we take
\[ \tau_\pi = 2(2 - \ln 2) \frac{\eta}{sT} \] and
\[ \lambda_1 = \frac{\eta}{sT} \] corresponding to the \( \mathcal{N} = 4 \) Super Yang–Mills theory. We use a 13 fm ×

![Graph 1](image1)

**Fig. 1.** Speed of sound of the LSM and that of the model EOS of Laine and Schröder\textsuperscript{12} connecting a high-order weak-coupling perturbative QCD calculation to a hadron resonance gas through a crossover (upper panel); and \( \eta/s(T) \) in the LSM with \( g = 3.2 \) (lower panel).

![Graph 2](image2)

**Fig. 2.** Square of the sound speed \( c_s^2 \) of the LSM as a function of temperature, for three values of the chiral coupling: \( g = 3, 3.2, 3.4 \).
13 fm transverse plane, set the impact parameter to $b = 7$ fm and the initialization time to $\tau_0 = 1$ fm/c. As initial conditions we use $u^x = u^y = 0$ and $\Pi^{xx} = \Pi^{yy} = \Pi^{xy} = 0$. These are typical values for the parameters used as input in hydrodynamic simulations of heavy ion collisions at RHIC, and were used in Refs. 7, 6. The initial energy density profile is obtained from Glauber’s model, with a temperature $T_i = 330$ MeV at the center of the fireball. As a reasonable ansatz we take $\bar{\pi}(\tau_0, \vec{r}) = 0$ and $\sigma(\tau_0, \vec{r}) = f_\pi[1 - e^{-r/r_0^2}]$ with $r_0 = 9$ fm as initial values for the chiral fields. We use the isothermal Cooper-Frye freeze–out prescription with a freeze-out temperature $T_F = 130$ MeV.

3. Charged–hadron elliptic flow

We will now discuss the results obtained for the charged–hadron elliptic flow $v_2$ computed from the chiral–hydrodynamic model described above.

Fig. 3 shows $v_2$ calculated taking or not taking into account the source terms in the hydrodynamic equations, for either the temperature–dependent computed from kinetic theory or a temperature–independent $\eta/s$ which is chosen to be the averaged value throughout the evolution of the fireball. The results shown were obtained with a value $g = 3.2$ in the LSM, which corresponds to a smooth crossover between hadronic matter and the QGP as suggested by Lattice QCD calculations (see e.g. Ref. 12).

It is seen from Fig. 3 that both for a temperature–dependent and a temperature–
independent $\eta/s$, the elliptic flow does not depend strongly on the chiral sources. More precisely, we see that for a temperature–independent $\eta/s$, $v_2$ turns out to be practically independent of the dynamics of the chiral sources for $p_T < 3$ GeV. In contrast, for the temperature–dependent $\eta/s$ there are small but visible differences in the elliptic flow calculated with or without the chiral sources. Specifically, $v_2$ is slightly larger at low $p_T$ and saturates at a smaller value of $p_T$ when the chiral fields are taken into account as sources for the hydrodynamic variables.

In order to determine the impact of the value of $g$ on the behavior of $v_2$ with $p_T$, we also calculated $v_2$ for a temperature–dependent $\eta/s$ and different values of $g$ close to $g = 3.2$. The results are shown in Fig. 4. It is seen that when the value of $g$ increases, $v_2$ decreases (this is due to the fact that for larger values of $g$, we find that the average value of $\eta/s$ throughout the hydrodynamic evolution becomes smaller). Comparing the values of $v_2$ calculated taking or not taking into account the chiral fields as sources for different values of $g$, it is seen that the influence of chiral dynamics on $v_2$ becomes smaller as $g$ increases. This supports our earlier conclusion that the evolution of the chiral fields have only a small impact on $v_2$.

It seems clear that there are difficulties in extracting (an average value of) $\eta/s$ from data on $v_2$. Our results show that uncertainties associated with the dependence of $\eta/s$ on temperature lead to appreciable changes in the curve of $v_2$ versus $p_T$, much larger than the changes stemming from the coupling of fluid dynamic variables to evolving classical chiral fields. The uncertainty associated with the temperature–dependence of $\eta/s$ should be added to the theoretical uncertainty that comes e.g. from the initial conditions (for example using Color Glass Condens-
sate or Glauber initial conditions) and the freeze–out process, that according to recent studies add up to an overall uncertainty which can be roughly estimated in 0.1 (see e.g. Refs. [17],[18]).

4. Conclusions

We have found that the values of \( v_2 \) do not depend strongly on the evolution of the chiral fields. Specifically, for a temperature–independent \( \eta/s \) this dependence is negligible for \( p_T < 3 \text{ GeV} \), while for a temperature–dependent \( \eta/s \) it is appreciable but still small even at small \( p_T \).

In line with the results of Refs. [2],[13], our results show that not knowing precisely the temperature–dependence of \( \eta/s \) leads to significant uncertainties in attempts of extracting this ratio from data on \( v_2 \), in addition to the uncertainties that stem from the initial conditions and the freeze–out process, among others sources.

It is worth noting that despite the coupling of chiral sources to the hydrodynamic evolution would add further uncertainties to the values of \( \eta/s \) extracted from data, they are not very big and do not affect in a significant way the possible extraction of \( \eta/s \) values from charged–hadron elliptic flow data.

Future work includes improvements such as the consideration of bulk viscosity in the hydrodynamic equations and fluctuations of the chiral fields. This latter effect would act as noise sources in the classical equations of motion [13],[14],[15].

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