Weak-universal critical behavior and quantum critical point
of the exactly soluble spin-1/2 Ising-Heisenberg model with
the pair XYZ Heisenberg and quartic Ising interactions

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Abstract. Spin-1/2 Ising-Heisenberg model with XYZ Heisenberg pair interaction and two different Ising quartic interactions
is exactly solved with the help of the generalized star-square transformation, which establishes a precise mapping equivalence
with the corresponding eight-vertex model on a square lattice generally satisfying Baxter’s zero-field (symmetric) condition.
The investigated model exhibits a remarkable weak-universal critical behavior with two marked wings of critical lines along
which critical exponents vary continuously with the interaction parameters. Both wings of critical lines merge together at
a very special quantum critical point of the infinite order, which can be characterized through diverging critical exponents.
The possibility of observing reentrant phase transitions in a close vicinity of the quantum critical point is related to a relative
strength of the exchange anisotropy in the XYZ Heisenberg pair interaction.

Keywords: Ising-Heisenberg model, eight-vertex model, quantum critical point, weak universality, reentrant phase transitions
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INTRODUCTION

Exactly soluble quantum spin models belong to the most fascinating topics to deal with in the area of modern
equilibrium statistical mechanics [1, 2, 3]. It should be pointed out, however, that quantum effects usually compete
with a cooperative nature of spontaneous long-range ordering and thus, it is quite intricate to find an exactly solvable
model that simultaneously exhibits both spontaneous long-range order as well as obvious macroscopic features of
quantum origin. On the other hand, it is a competition between quantum and cooperative phenomena that is an essential
ingredient for observing a quite remarkable and unexpected behavior of low-dimensional quantum spin models.

The hybrid Ising-Heisenberg models on decorated planar lattices, whose nodal sites are occupied by the classical
Ising spins and decorating sites by the quantum Heisenberg ones, belong to the simplest rigorously solved quantum
spin models that exhibit a spontaneous long-range ordering with apparent quantum manifestations. It is worthwhile to
remark, moreover, that the Ising-Heisenberg planar models [4, 5, 6, 7, 8, 9, 10, 11, 12, 13] where a finite cluster of the
Heisenberg spins interacts with either two or three nodal Ising spins are in principle tractable by the use of generalized
decoration-iteration or star-triangle transformations [14, 15, 16], which establish a precise mapping equivalence
between them and the spin-1/2 Ising model on the corresponding undecorated planar lattice [1, 15, 17, 18, 19, 20].
Among other matters, the exact solutions for this special class of the Ising-Heisenberg planar models might serve in
evidence that these rigorously solvable models exhibit a strong-universal critical behavior, which can be characterized
by critical exponents from the standard Ising universality class. Contrary to this, the more interesting weak-universal behavior [21] of the critical exponents has been recently announced for two Ising-Heisenberg planar models [22, 23],
where a finite cluster of the Heisenberg spins interacts with four nodal Ising spins. The spin-1/2 Ising-Heisenberg
model with the pair XYZ Heisenberg interaction and two quartic Ising interactions [23] has surprisingly turned out
to be the fully exactly solvable model due to a validity of the precise mapping equivalence with Baxter’s zero-field
(symmetric) eight-vertex model [1, 24, 25]. The main purpose of this work is to examine in detail how the weak-
universal critical behavior of this exactly soluble model depends on a spatial anisotropy in two quartic Ising interactions
and on the exchange anisotropy of the XYZ Heisenberg pair interaction, whose effect have not been dealt with in our
preceding work [23] for the most general case.
where $\beta$ is the inverse temperature. The partition function into the following product:

$Z_{\text{HIM}} = \sum_{\sigma} \prod_p \text{Tr}_p \exp(-\beta \mathcal{H}_p) = \sum_{\{\sigma\}} \prod_p \omega_p(\sigma^z_{p1}, \sigma^z_{p2}, \sigma^z_{p3}, \sigma^z_{p4}),$  \hspace{1cm} (2)

where $\beta = 1/(k_B T)$, $k_B$ is Boltzmann’s constant, and $T$ is the absolute temperature. The summation $\sum_{\{\sigma\}}$ to emerge in Eq. (2) is carried out over all possible configurations of the Ising spins, the product runs over all octahedron unit cells and the symbol Tr$_p$ stands for a trace over spin degrees of freedom of the Heisenberg spin pair from the $p$th octahedron. In the latter step of our calculation we have used a straightforward diagonalization of the Hamiltonian $\mathcal{H}_p$ of the $p$th octahedron in order to obtain the relevant trace over spin degrees of freedom of the Heisenberg spin pair.

This paper is so organized. In the following section, we will describe the hybrid Ising-Heisenberg model and recall basic steps of the exact mapping procedure to the zero-field eight-vertex model. The most interesting results for the ground-state and finite-temperature phase diagrams, which are supported by a detailed analysis of critical exponents, are subsequently presented in the next section. Finally, some concluding remarks are mentioned along with a brief summary of the most important scientific achievements in the last section.

**FIGURE 1.** The elementary unit cell of the spin-1/2 Ising-Heisenberg model. Full (empty) circles denote positions of the Heisenberg (Ising) spins, thick solid line represents the pairwise $XYZ$ Heisenberg interaction between the apical Heisenberg spins and both types of broken lines connect spins involved in the quartic Ising interactions. Thin solid lines connecting four Ising spins are guides for eyes only.
This procedure yields some effective Boltzmann’s weight $\omega_p$, which explicitly depends merely on four Ising spins $\sigma_{p1}$, $\sigma_{p2}$, $\sigma_{p3}$ and $\sigma_{p4}$ from the basal plane of the $p$th octahedron. In addition, the explicit form of the effective Boltzmann’s factor $\omega_p$ immediately implies a possibility of performing the generalized star-square transformation

$$\omega_p(\sigma_{p1}, \sigma_{p2}, \sigma_{p3}, \sigma_{p4}) = 2 \exp\left[ \frac{\beta}{4} (J_z + J_1 \sigma_{p1} \sigma_{p3} + J_2 \sigma_{p2} \sigma_{p4}) \right] \cosh\left[ \frac{\beta}{4} (J_x - J_y) \right]$$

+ 2 \exp\left[ -\frac{\beta}{4} (J_z + J_1 \sigma_{p1} \sigma_{p3} + J_2 \sigma_{p2} \sigma_{p4}) \right] \cosh\left[ \frac{\beta}{4} (J_x + J_y) \right] = R_0 \exp(\beta R_1 \sigma_{p1} \sigma_{p3} + \beta R_2 \sigma_{p2} \sigma_{p4} + \beta R_4 \sigma_{p1} \sigma_{p3} \sigma_{p2} \sigma_{p4}), \quad (3)$$

which substitutes the effective Boltzmann’s weight $\omega_p$ by the equivalent expression containing two pair ($R_1$ and $R_2$) and one quartic ($R_4$) interaction between four nodal Ising spins from an elementary square face of the square lattice. Of course, the algebraic transformation (3) must satisfy the ‘self-consistency’ condition, which means that it must hold independently of spin states of four Ising spins involved therein. It can be easily verified that a substitution of all sixteen possible spin configurations of the nodal Ising spins gives just four independent equations, which unambiguously determine so far not specified mapping parameters $R_0$, $R_1$, $R_2$, and $R_4$

$$R_0 = \left( \frac{\omega_0 \omega_1 \omega_2 \omega_3}{\omega_0 \omega_5} \right)^{1/4}, \quad \beta R_1 = \ln \left( \frac{\omega_0 \omega_7 \omega_8}{\omega_0 \omega_5} \right), \quad \beta R_2 = \ln \left( \frac{\omega_0 \omega_6}{\omega_0 \omega_7} \right), \quad \beta R_4 = 4 \ln \left( \frac{\omega_0 \omega_7 \omega_8}{\omega_0 \omega_5} \right). \quad (4)$$

that can be expressed in terms of four different Boltzmann’s weights $\omega_i$ ($i = 1, 3, 5, 7$)

$$\omega_1(\pm, \pm, \pm, \pm) = 2 \exp\left[ \frac{\beta}{4} \left( J_z + J_1 + J_2 - \frac{J_3}{4} \right) \right] \cosh\left[ \frac{\beta}{4} (J_x - J_y) \right] + 2 \exp\left[ -\frac{\beta}{4} \left( J_z + J_1 + J_2 - \frac{J_3}{4} \right) \right] \cosh\left[ \frac{\beta}{4} (J_x + J_y) \right]. (5)$$

$$\omega_3(\pm, \mp, -, \mp) = 2 \exp\left[ \frac{\beta}{4} \left( J_z - J_1 + J_2 - \frac{J_3}{4} \right) \right] \cosh\left[ \frac{\beta}{4} (J_x - J_y) \right] + 2 \exp\left[ -\frac{\beta}{4} \left( J_z - J_1 + J_2 - \frac{J_3}{4} \right) \right] \cosh\left[ \frac{\beta}{4} (J_x + J_y) \right]. (6)$$

$$\omega_5(\mp, \mp, \mp, \mp) = 2 \exp\left[ \frac{\beta}{4} \left( J_z + J_1 - J_2 - \frac{J_3}{4} \right) \right] \cosh\left[ \frac{\beta}{4} (J_x - J_y) \right] + 2 \exp\left[ -\frac{\beta}{4} \left( J_z + J_1 - J_2 - \frac{J_3}{4} \right) \right] \cosh\left[ \frac{\beta}{4} (J_x + J_y) \right]. (7)$$

$$\omega_7(\mp, \mp, \mp, \mp) = 2 \exp\left[ \frac{\beta}{4} \left( J_z + J_1 - J_2 + \frac{J_3}{4} \right) \right] \cosh\left[ \frac{\beta}{4} (J_x - J_y) \right] + 2 \exp\left[ -\frac{\beta}{4} \left( J_z + J_1 - J_2 + \frac{J_3}{4} \right) \right] \cosh\left[ \frac{\beta}{4} (J_x + J_y) \right]. (8)$$

Note that the effective Boltzmann’s weights (5)–(8) are assigned to eight spin configurations of four nodal Ising spins explicitly specified in round brackets (the symbol $\pm$ denotes the spin state $\sigma_{p4}^{\alpha} = \pm 1/2$, $\alpha = 1, 2, 3, 4$), as well as, another eight spin configurations, which can be obtained from them under the reversal of all four nodal Ising spins.

A substitution of the generalized star-square transformation (3) with appropriately chosen mapping parameters (4) into Eq. (2) then straightforwardly leads to a precise mapping equivalence between the spin-1/2 Ising-Heisenberg model and the zero-field eight-vertex model. As a result of this procedure, one actually obtains a simple mapping relationship

$$Z_{8-\nu}^{\beta R_{1H}}(\beta, J_1, J_y, J_z, J_1, J_2) = R_0^{2N} Z_{8-\nu}(\beta, R_1, R_2, R_4), \quad (9)$$

which connects the partition function of the investigated Ising-Heisenberg model with the partition function of the spin-1/2 Ising model on two interpenetrating square lattices that are coupled together by means of the quartic interaction ($N$ is the total number of the Ising spins). It should be mentioned that the latter model is nothing but one of many alternative definitions of the zero-field eight-vertex model [24, 25], which is reformulated in the Ising spin representation following the ideas of Wu [26], Kadanoff and Wegner [27]. Besides, the physical meaning of the mapping parameters $R_1$, $R_2$ and $R_4$ becomes quite evident from the mapping transformation (3). The parameters $R_1$ and $R_2$ denote the effective pair interactions in two different interpenetrating Ising square lattices and the mapping parameter $R_4$ determines the effective quartic interaction that couples together both Ising square lattices. Last but not least, the mapping parameter $R_0$ (multiplicative factor in Eq. (9)) in fact represents the partition function of the Heisenberg spin pair in the effective field produced by the four enclosing nodal Ising spins.

It becomes quite clear from the mapping relationship (9) that the Ising-Heisenberg model becomes critical just if the corresponding zero-field eight-vertex model becomes critical as well. As a matter of fact, the partition function $Z_{8-\nu}^{\beta R_{1H}}$ will exhibit a non-analyticity if and only if the corresponding partition function $Z_{8-\nu}$ will exhibit the same kind of
the non-analytic behavior, because the parameter \( R_0 \) is analytic in the whole region of interaction parameters. In this regard, the respective critical points of the Ising-Heisenberg model can be found from the relevant critical condition of the zero-field eight-vertex model \([1, 24, 25]\)

\[
\omega_1 + \omega_2 + \omega_3 + \omega_4 = 2\max\{\omega_1, \omega_3, \omega_5, \omega_7\}, \tag{10}
\]

which determines phase transitions of the Ising-Heisenberg model on assumption that the effective Boltzmann’s weights \([5, 8]\) are substituted into this critical condition. It should be also mentioned that the critical exponents of the zero-field eight-vertex model \([1, 24, 25]\) continuously change with the parameter \( \mu = 2\arctan(\omega_5/\omega_3)^{1/2} \) by following the formulas

\[
\alpha = \alpha' = 2 - \frac{\pi}{\mu}, \quad \beta = \frac{\pi}{16\mu}, \quad \nu = \nu' = \frac{\pi}{2\mu}, \quad \gamma = \gamma' = \frac{7\pi}{8\mu}, \quad \delta = 15, \quad \eta = \frac{1}{4}. \tag{11}
\]

Consequently, the relations \((11)\) will also govern changes of the critical exponents of the Ising-Heisenberg model when the effective Boltzmann’s weights \([5, 8]\) are used for a calculation of the parameter \( \mu \).

**RESULTS AND DISCUSSION**

In this part, let us proceed to a discussion of the most interesting results obtained for critical properties of the spin-1/2 Ising-Heisenberg model with the pair and quartic interactions. Before doing so, however, it is worthy of notice that some special cases of this model system have already been investigated by the present authors in our earlier paper \([23]\), where the critical behavior of the particular case with two identical quartic interactions \( J_1 = J_2 \) and the more symmetric XXZ Heisenberg interaction \( J_1 = J_2 \neq J_3 \) was explored in detail by assuming both ferromagnetic as well as antiferromagnetic pair interaction. Exact results for this more symmetric version of the Ising-Heisenberg model indicate that the model with the antiferromagnetic pair interaction exhibits less significant changes of both critical temperatures as well as critical exponents than the model with the ferromagnetic pair interaction. As a matter of fact, it has been demonstrated that only the Ising-Heisenberg model with the ferromagnetic pair interaction shows a unusual quantum critical point of the infinite order, which characterizes a remarkable singular behavior of the critical exponents near the isotropic limit of the Heisenberg pair interaction \( J_1 = J_2 = J_3 \). With this background, the main purpose of the present work is to shed light on how this strange weak-universal critical behavior will change by introducing a spatial anisotropy in two quartic Ising-type interactions \( J_1 \neq J_2 \) or by assuming the most general form of the XYZ exchange anisotropy in the Heisenberg pair interaction.

It is worthwhile to remark that the investigated Ising-Heisenberg model possesses a rather high symmetry, because all four different Boltzmann’s weights \([5, 8]\) are mutually interchangeable under the transformations \( J_1 \rightarrow -J_1 \) and/or \( J_2 \rightarrow -J_2 \). This means, among other matters, that one may further restrict both quartic interaction parameters to positive values, since the transformations \( J_1 \rightarrow -J_1 \) and \( J_2 \rightarrow -J_2 \) merely cause rather trivial changes of the nodal Ising spins \((\sigma_{p1}^z, \sigma_{p2}^z, \sigma_{p3}^z, \sigma_{p4}^z) \rightarrow (\pm \sigma_{p1}^z, \pm \sigma_{p2}^z, \sigma_{p3}^z, \sigma_{p4}^z) \) and \((\sigma_{p1}^x, \sigma_{p2}^x, \sigma_{p3}^x, \sigma_{p4}^x) \rightarrow (\sigma_{p1}^x, \pm \sigma_{p2}^x, \sigma_{p3}^x, \sigma_{p4}^x)\), respectively. Owing to this fact, let us further assume that all the interaction terms \( J_x, J_y, J_z, J_x, J_y \) and \( J_z \) entering the Hamiltonian \([1]\) are positive and moreover, the symmetry of the Hamiltonian allows us to consider \( J_x, J_y, J_z \geq 0 \) without loss of the generality. For easy reference, the number of free parameters is lowered by introducing the following set of dimensionless parameters: \( k_B T/J \), marks the dimensionless temperature, \( J_1/J \) and \( J_2/J \) are proportional to a relative strength of the quartic Ising-type interactions, and finally, the parameters \( J_x/J_x \) and \( J_y/J_y \) determine a relative strength of the exchange anisotropy in the XYZ Heisenberg pair interaction. The former anisotropy parameter \( J_x/J \) determines a difference in the exchange interactions along a quantization \( z \)-axis and its perpendicular \( x \)-axis, whereas the latter anisotropy parameter determines a difference between the stronger and weaker exchange interaction in the \( xy \)-plane.

First, let us take a closer look at the ground-state behavior. It can be readily understood that the ground-state spin arrangement will always correspond to the lowest-energy eigenstate that enters into the greatest Boltzmann’s weight among the four Boltzmann’s weights given by Eqs. \([5, 8]\). In the zero temperature limit, the greatest Boltzmann’s weight is \( \omega_j \) if \( J_j < J_i \) or \( \omega_i \) if \( J_j > J_i \). This observation would suggest that the ground-state spin arrangement will basically change whenever the weaker exchange interaction \( J_j \) in the \( xy \)-plane exceeds the exchange interaction \( J_z \) along the quantization axis. One actually finds that the lowest-energy eigenstate is either

\[
|I\rangle = \prod_p (|+, +, +, +\rangle \sigma_p \frac{1}{\sqrt{2}}(|+, +\rangle + |-, -\rangle)_{S_p}, \tag{12}
\]
if \( J_x < J_z \), or,

\[
|\text{II} \rangle = \prod_p |+,\pm,\pm,\pm\rangle_{\sigma_p} \frac{1}{\sqrt{2}} (|+,\pm\rangle + |\pm,+,\pm\rangle)_{\text{Is},p},
\]

(13)

if the reverse inequality holds. In Eqs. (12)–(13), the former ket vector unambiguously determines the states of four nodal Ising spins from an elementary square face of the \( p \)th octahedron and the latter ket vector unambiguously specifies the relevant state of the Heisenberg spin pair. It should be also noticed that another equivalent representations of these eigenstates can be obtained from the eigenvectors (12)–(13) under the reversal of all four nodal Ising spins and consequently, the phases \(|\text{I}\rangle\) and \(|\text{II}\rangle\) are both four-fold degenerate.

Let us now make a few comments on spin arrangements appearing in both ground-state phases. In the phase \(|\text{I}\rangle\), the Ising spins placed on a square lattice either exhibit a perfect ferromagnetic or antiferromagnetic long-range order, which is accompanied with the entangled spin state \((|+,+\rangle + |\pm,\pm\rangle)/\sqrt{2}\) consisting of both ferromagnetic states of the Heisenberg spin pairs. On the other hand, the Heisenberg spin pairs reside in the phase \(|\text{II}\rangle\) the entangled spin state \((|+,\pm\rangle + |\pm,\pm\rangle)/\sqrt{2}\) consisting of both antiferromagnetic states and the Ising spins prefer a superantiferromagnetic long-range order with the ferromagnetic alignment in a horizontal direction and the antiferromagnetic alignment in a vertical direction, or vice versa. To compare with, is might quite useful to mention that the phase \(|\text{II}\rangle\) preserves its character even if the less general case of the XXZ exchange anisotropy is assumed, whereas the spin arrangement of the phase \(|\text{I}\rangle\) drastically changes disentanglement of both ferromagnetic states of the Heisenberg spin pairs

\[
|\text{I}'\rangle = \prod_p |+,\pm,\pm,\pm\rangle_{\sigma_p} |\pm,\pm\rangle_{\text{Is},p}.
\]

(14)

In the particular limit \( J_x = J_z \), the phase \(|\text{I}'\rangle\) actually becomes macroscopically degenerate as the Heisenberg spin pairs may choose independently of each other one of two ferromagnetic states \(|\pm,\pm\rangle_{\text{Is},p}\). Hence, it follows that the phase \(|\text{I}'\rangle\) will exhibit a rather high macroscopically degeneracy of the order \( 4 \times 2^N \), which is proportional to the total number of the Heisenberg spin pairs (remember that \( N \) simultaneously marks the total number of the Ising spins, the total number of the Heisenberg spin pairs, as well as, the total number of elementary unit cells).

Now, we will turn our attention to a detailed study of finite-temperature phase diagrams. One should recall that phase transition lines of the spin-1/2 Ising-Heisenberg model with the pair and quartic interactions can be straightforwardly calculated from the critical condition of the corresponding zero-field eight-vertex model by substituting the effective Boltzmann’s weights (5)–(8) into Eq. (10). It is worthwhile to remember, moreover, that the greatest Boltzmann’s weight is either \( \omega_1 \) or \( \omega_3 \). For both particular cases, the critical condition can uniquely be expressed as

\[
\sinh \left[ \frac{\beta}{k_B} (J_1 + J_2) \right] = \pm \cosh \left[ \frac{\beta}{k_B} (J_1 - J_2) \right] \exp \left[ -\frac{\beta J_0}{k_B} \right] \left[ \cosh \left[ \frac{\beta}{k_B} (J_1 + J_2) \right] - \exp \left[ -\frac{\beta J_0}{k_B} \right] \right],
\]

(15)

where \( \beta = 1/(k_B T_c) \), \( T_c \) is the critical temperature and the plus (minus) sign applies for a situation when \( \omega_1 > \omega_3 \) (\( \omega_1 < \omega_3 \)).

By exploiting the critical condition (15), let us examine first how a spatial anisotropy in two quartic Ising-type interactions influences the critical behavior of the model under investigation. To see this effect, the critical temperature is plotted in figure 2 against the anisotropy parameter \( J_x/J_z \) for the particular case of the XXZ exchange anisotropy (i.e. \( J_z = J_y = J_x \)) when a strength of the stronger quartic interaction is fixed \((J_1/J_x = 1.0)\) and a strength of the weaker quartic interaction varies. As one can see, the phase diagram consists of two marked wings of critical lines that separate both spontaneously long-range ordered phases \(|\text{I}\rangle\) and \(|\text{II}\rangle\), since both critical lines merge together at the ground-state boundary \( J_x = J_y = J_z \) between these two phases. It is noteworthy that the left (right) wing represents a line of critical points of the phase \(|\text{I}\rangle\) (the phase \(|\text{II}\rangle\)), which can be obtained as a numerical solution of the critical condition (15) by assuming the plus (minus) sign therein. It becomes quite evident from figure 2 that one may also found reentrant phase transitions slightly above the ground-state boundary between the phases \(|\text{I}\rangle\) and \(|\text{II}\rangle\), i.e. under the constraint \( J_x = J_y > J_z \). In this parameter region, the reentrant phase transitions from the spontaneously ordered phase \(|\text{II}\rangle\) to the disordered phase and from the disordered phase to the spontaneously ordered phase \(|\text{I}\rangle\) take place because the free energy of the phase \(|\text{I}\rangle\) decreases much more rapidly with the increase of temperature than the free energy of the phase \(|\text{II}\rangle\) due to the much higher entropy gain of the phase \(|\text{I}\rangle\). Another interesting fact to observe here is that a spatial anisotropy in the quartic Ising-type interactions does not qualitatively affect the critical behavior of the investigated
closely resembles the one formerly discussed by the analysis of the particular case with the exchange anisotropy and that of another one quartic Ising-type interaction. It can be clearly seen from this figure that the finite-temperature phase diagram of this more general case quite closely resembles the one formerly discussed by the analysis of the corresponding zero-field eight-vertex model. Of course, it is well known that the anisotropy in exchange anisotropy model system. As it can be easily understood from Eqs. (4) and (7)–(8), the spatial anisotropy in the quartic Ising-type interactions \( J_1 \neq J_2 \) merely causes an anisotropy in the effective pair interactions \( R_1 \neq R_2 \) of two Ising square lattice coupled together through the effective quartic interaction \( R_4 \) when speaking in the language of the Ising spin representation of the corresponding zero-field eight-vertex model. Of course, it is well known that the anisotropy in those pair interactions does not fundamentally affect the overall critical behavior of the zero-field eight-vertex model.

For the sake of comparison, we have displayed in figure 2 the critical temperature as a function of the exchange anisotropy \( J_x/J_z \) for one illustrative example of the less symmetric XYZ exchange anisotropy \( (J_x/J_z = 0.5) \) when a strength of the stronger quartic interaction is fixed \( (J_1/J_z = 1.0) \) and a strength of the weaker quartic interaction varies. It can be clearly seen from this figure that the finite-temperature phase diagram of this more general case quite closely resembles the one formerly discussed by the analysis of the particular case with the XXZ exchange anisotropy. Actually, one still finds two wings of the critical lines that separate both spontaneously long-range ordered phases \( |I \rangle \) and \( |II \rangle \), whereas the zero-temperature transition between these phases moves towards the higher values of the exchange anisotropy \( J_z/J_x \) as it occurs just if the weaker exchange interaction in the \( xy \)-plane \( J_y \) overwhelms the one \( J_z \) along the quantization axis. However, the most obvious difference between the phase diagrams shown in figures 2 and 3 is that the latter phase diagram does not imply an existence of the reentrant phase transitions near the ground-state boundary between the phases \( |I \rangle \) and \( |II \rangle \). This specific feature can be attributed to the fact that the macroscopical degeneracy of the phase \( |I \rangle \) is entirely lifted whenever the most general case of the XYZ exchange anisotropy is assumed. Hence, it follows that both wings of critical lines tend to zero temperature with an infinite gradient and thus, there does not emerge reentrant phase transitions near the ground-state boundary between the phases \( |I \rangle \) and \( |II \rangle \). To provide an independent check of the aforementioned scenario, the critical temperature is plotted in figure 4 against the exchange anisotropy \( J_y/J_x \) for one selected value of the quartic Ising-type interactions \( (J_1/J_z = J_2/J_z = 1.0) \) at four different strengths of the exchange anisotropy \( J_z/J_x \). Figure 4 apparently supports all previous statements, since the critical lines obviously have vertical tangents at the zero-temperature transition between the phases \( |I \rangle \) and \( |II \rangle \), which is generally shifted towards the higher values of the anisotropy parameter \( J_y/J_x \) upon strengthening the exchange anisotropy in the \( xy \)-plane (i.e. when the ratio \( J_y/J_x \) is lowered). Altogether, it could be concluded that the spatial anisotropy in the quartic Ising-type interactions has a less significant impact on the overall critical behavior compared to the exchange anisotropy of the Heisenberg pair interaction. Therefore, our subsequent analysis will be mainly concentrated on how the exchange anisotropy in the Heisenberg pair interaction influences the most essential features of the critical behavior.

At this place, let us make a few remarks on possible changes of the critical behavior and critical exponents, which can be induced upon varying the anisotropy parameters \( J_x/J_z \) and \( J_y/J_z \). For this purpose, the figures 5, 6 and 7 display typical changes of the critical temperature and the critical exponent \( \alpha \) with the anisotropy parameter \( J_x/J_z \) for the Ising-Heisenberg model with a fixed strength of the quartic Ising-type interactions \( (J_1/J_z = J_2/J_z = 1.0) \) at three different values of the exchange anisotropy \( J_y/J_x = 1.0, 0.9 \) and \( 0.5 \), respectively. In these figures, solid lines scaled
FIGURE 3. The dimensionless critical temperature as a function of the anisotropy parameter \( J_x/J_z \) for the Ising-Heisenberg model with the XYZ exchange anisotropy \( (J_x \neq J_z) \) when a strength of the one quartic Ising-type interaction is fixed \( (J_1/J_z = 1.0) \) and that of another one quartic Ising-type interaction \( J_2/J_z \) varies.

FIGURE 4. The dimensionless critical temperature as a function of the anisotropy parameter \( J_x/J_z \) for the spin-1/2 Ising-Heisenberg model with a fixed strength of the quartic Ising-type interactions \( (J_1/J_z = J_2/J_z = 1.0) \) at four different values of the exchange anisotropy \( J_y/J_x \).

with respect to left axes depict a variation of the critical temperature with the exchange anisotropy \( J_x/J_z \), while broken lines scaled with respect to right axes show in a semilogarithmic scale the relevant changes of the critical exponent \( \alpha \). The figures on the left show the overall finite-temperature phase diagrams, whereas the figures on the right show in an enlarged scale the most striking part of these phase diagrams in a close vicinity of the ground-state boundary between the phases \( |\text{I}\rangle \) and \( |\text{II}\rangle \).

It is quite obvious from figures 3 and 4 that the critical exponent \( \alpha \) varies continuously along the line of critical points, because its value changes with the interaction parameters involved in the Hamiltonian (1) according to the relation (11) through the respective changes of the parameter \( \mu \). It should be pointed out, moreover, that the relevant changes of the critical exponent bring also insight into a nature of phase transitions as the order of phase transitions is proportional to \( r = 2 - \alpha \) (see for instance pp. 16–17 in Reference [1]). From this point of view, the displayed critical lines turn out to be lines of rather smooth continuous phase transitions, since the respective variations of the critical exponent \( \alpha \) are restricted to the range \( \alpha \in (-\infty, 0) \) and consequently, the order of phase transitions is \( r > 2 \). However, the most striking finding to emerge from figures 3 and 4 is that the critical exponent \( \alpha \) exhibits a very special singular behavior in
FIGURE 5. The changes of the critical exponent $\alpha$ along the critical line of the spin-1/2 Ising-Heisenberg model with a fixed relative strength of the quartic Ising-type interactions $J_1/J_z = J_2/J_z = 1.0$ and the exchange anisotropy $J_y/J_x = 1.0$. The solid line, which is scaled with respect to the left axis, shows the critical temperature as a function of the exchange anisotropy $J_x/J_z$. The broken line, which is scaled with respect to the right axis, displays in a semilogarithmic scale the relevant changes of the critical exponent $\alpha$ along this critical line. Figure 5b) shows a detail of the phase diagram near the zero-temperature transition.

FIGURE 6. The same as in figure 5 but for the exchange anisotropy $J_y/J_x = 0.9$.

a neighborhood of the ground-state boundary between the phases $|I\rangle$ and $|II\rangle$ where $\alpha \to -\infty$. In this respect, the zero-temperature phase transition between the phases $|I\rangle$ and $|II\rangle$ might be regarded as a phase transition of the infinite order and hence, this special critical point in fact represents a quite remarkable quantum critical point. Another interesting finding stems from a direct comparison of the figures 5b)–7b). In agreement with the aforedescribed analysis, the reentrant phase transitions occur in a vicinity of the zero-temperature transition between the phases $|I\rangle$ and $|II\rangle$ by considering the XXZ exchange anisotropy (figure 5b), while the reentrant phenomenon obviously vanishes when considering the sufficiently strong XYZ exchange anisotropy (figure 7b). A disappearance of the reentrant transitions occurs on behalf of the XYZ exchange anisotropy, which generally lifts a macroscopical degeneracy of the phase $|I\rangle$, as it has been reasoned by the ground-state analysis. Owing to this fact, the critical lines of both four-fold degenerate phases $|I\rangle$ and $|II\rangle$ should always tend to zero temperature with the infinite gradient at the ground-state boundary between them whenever the XYZ exchange anisotropy is considered. The quite interesting situation thus appears if there is a small but non-zero exchange anisotropy between both interactions in the xy-plane (see figure 6b). In this
particular case, both critical lines meet at the ground-state boundary between the phases $|I\rangle$ and $|II\rangle$ with an infinite gradient but they curve in the same direction at small enough temperatures, which gives rise to another type of the reentrant phenomenon [28]. Naturally, the greater a difference between the exchange interactions $J_x$ and $J_y$ is, the smaller is a temperature range where the reentrant phenomenon may be observed.

**CONCLUSIONS**

In this work, the critical properties of the spin-1/2 Ising-Heisenberg model with the pair $XYZ$ Heisenberg interaction and two quartic Ising-type interactions have been studied in particular within the exact mapping technique based on the generalized star-square transformation. This algebraic transformation establishes an exact mapping equivalence between the proposed Ising-Heisenberg model and the Baxter’s zero-field (symmetric) eight-vertex model, which has been subsequently used for obtaining several interesting exact results for the ground-state and finite-temperature phase diagrams, phase transitions and critical phenomena. The most interesting finding to emerge from the present study is an exact evidence of the quantum critical point, which appears whenever the weaker exchange interaction in the $xy$-plane equals the exchange interaction $J_z$ along the quantization axis, i.e. whenever $J_z = \min\{J_x, J_y\}$. It turns out that the critical exponents exhibit a peculiar singular behavior in a close vicinity of this quantum critical point, which consequently represents a phase transition of the infinite order.

The main emphasis of the present paper was to provide a deeper insight into how a spatial anisotropy in two quartic Ising-type interactions and the exchange anisotropy in the $XYZ$ Heisenberg pair interaction influence the striking weak-universal critical behavior. It has been shown that a spatial anisotropy in two quartic Ising-type interactions has less pronounced effect on the critical behavior, which changes quantitatively rather than qualitatively upon varying the anisotropy in two quartic Ising-type interactions. On the other hand, the exchange anisotropy in the $XYZ$ Heisenberg pair interaction significantly changes the overall critical behavior, namely, it enables to change a location of the quantum critical point as well as an appearance (or disappearance) of the reentrant phase transitions.

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