Transverse Oscillations of a Cantilever Rod of Rectangular Cross Section and Variable Thickness

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Abstract. A numerical-analytical algorithm for calculating (a type of accelerated convergence method) of eigenvalues and modes of transverse vibrations of a rod of variable cross-section is proposed. The algorithm is implemented as a software package in the Maple symbolic computations language. The use of the package was demonstrated by the example of solving problems of determining the eigen forms and natural frequencies of rods with different dependences of the cross section on the longitudinal distance.

1. Statement the problem

There is a rod cross-section, the width of the rod \( b = b_0 = \text{const} \), thickness \( h(x) \) varies along the axis of symmetry of the rod of the law \( h(x) = h_1 + (h_0 - h_1)F\left(\frac{x}{l}\right) \), where \( l \) – the length of the rod. At the point \( x = 0 \): \( F = 0 \), and at the point \( x = l \): \( F = 1 \). The rod is rigidly fixed at the point \( x = 0 \) and free at the point \( x = l \). The transverse oscillations of such a rod are described by a differential equation

\[
\frac{\partial^2 u}{\partial x^2} + \rho S \frac{\partial^2 u}{\partial t^2} = 0 \tag{1}
\]

Here \( E \) is the Young's modulus of the material of which the rod is made from, \( \rho \) is material density.

\[
I = \frac{1}{12} bh^3 = \frac{b}{12} h_1 + (h_0 - h_1)F\left(\frac{x}{l}\right)^3
\]

is the moment of cross section relative to the axis perpendicular to the axis of symmetry; \( S = bh = b h_1 + (h_0 - h_1)F\left(\frac{x}{l}\right)\) is the cross-sectional area.

Console console conditions are written as follows

\[
u(0,t) = \frac{\partial u}{\partial t}(0,t) = 0, \quad \frac{\partial^2 u}{\partial x^2}(l,t) = \frac{\partial^2 u}{\partial x^2}(l,t) = 0 \tag{2}
\]

We seek periodic in \( t \) solutions of equation (1) satisfying the boundary conditions (2) in the form \( u(x,t) = T(t)U(x) \). After substitution of this expression into equation (1) and separation of variables we proceed to the Sturm – Liouville problem.
\[
\frac{d^2}{dx^2} \left( EI \frac{d^2 U}{dx^2} \right) - \rho S \omega^2 U = 0
\]

\[
U|_{x=0} = U'|_{x=0} = 0, \quad U''|_{x=0} = U'''|_{x=0} = 0
\]

Sturm – Liouville problem is formulated as follows: find those values of the parameter \( \omega^2 \) for which there are non-trivial solutions of the problem (3). Once found, these values of the parameter (hereinafter referred to as own numbers), are themselves solutions \( U(x) \) (hereinafter called eigenfunctions or waveforms). Finally, integrating the equation

\[
T^* + \omega^2 T = 0
\]

we find the dependence of the desired solution

\[
T(t) = T \left( \frac{t}{\lambda} \right) \quad \text{and} \quad F = \left( \frac{t}{\lambda} \right)^2.
\]

We introduce a new independent variable \( x = zl \) and parameter \( \lambda = \frac{12 \rho l}{E h^4} \alpha \). The boundary value problem (3) is transformed into

\[
\frac{d^2}{dz^2} \left[ 1 + \frac{h_0}{h_1} F(z) \right] \frac{d^2 U}{dz^2} \left[ 1 + \frac{h_0}{h_1} F(z) \right] U = 0
\]

\[
U(0) = U'(0) = 0, \quad U''(1) = U'''(1) = 0
\]

Note that in the second case under \( F = z^2 \), the problem (3) can be solved analytically by Bessel functions of real and complex argument. However, obtaining numerical results is associated with solving cumbersome transcendental equations [1]. Next, to solve the problem, a numerical-analytical method of accelerated convergence [2] is proposed, which can be implemented for a wide class of functions \( F(z) \).

2. Sagitary function and method of accelerated convergence

We introduce the notation \( p = \left[ 1 + \left( \frac{h_0}{h_1} \right) z \right] \), \( r = \left[ 1 + \left( \frac{h_0}{h_1} \right) z \right] \), and new functions

\[
U(z) = Q_1(z), \quad Q_2(z), \quad Q_3(z), \quad Q_4(z),
\]

which are defined as

\[
\frac{dQ_1}{dz} = Q_2, \quad \frac{dQ_2}{dz} = Q_3, \quad \frac{dQ_3}{dz} = Q_4,
\]

\[
\frac{dQ_4}{dz} = -2p' \frac{Q_1}{p} - \frac{p''}{p} Q_3 + \lambda \frac{r}{p} Q_1
\]

The resulting system of equations (6) is equivalent to the differential equation of the boundary value problem (5). For system (6), we formulate the Cauchy problem I with initial conditions

\[
Q_1(0) = 0, \quad Q_2(0) = 0, \quad Q_3(0) = 1, \quad Q_4(0) = 0
\]

For functions \( Q_5, \ Q_6, \ Q_7, \ Q_8 \), we formulate the Cauchy problem II

\[
\frac{dQ_5}{dz} = Q_6, \quad \frac{dQ_6}{dz} = Q_7, \quad \frac{dQ_7}{dz} = Q_8
\]

\[
\frac{dQ_8}{dz} = -2p' \frac{Q_5}{p} - \frac{p''}{p} Q_7 + \lambda \frac{r}{p} Q_5
\]

\[
Q_5(0) = 0, \quad Q_6(0) = 0, \quad Q_7(0) = 0, \quad Q_8(0) = 1
\]
Note that the functions \( Q_1, Q_2, Q_3, Q_4 \) satisfy the initial conditions (7) and (9), and, therefore, they satisfy the boundary conditions at the point \( z = 0 \) for problem (5). Let \( \lambda' = \lambda' \), where \( \lambda' \) is the upper bound for the first eigenvalue. Substituting \( \lambda' \) in systems (6) and (8), we solve Cauchy problems I and II. We find numerically the functions \( Q_1(z, \lambda') \) and \( Q_3(z, \lambda') \). We compose a two-parameter family of functions.

\[
U(z, \lambda') = c_1 Q_1(z, \lambda') + c_2 Q_3(z, \lambda'), \quad \text{where } c_1 \text{ and } c_2 = \text{const}.
\]

We define the point \( z = \xi \) from the conditions

\[
U' (\xi, \lambda') = U''(\xi, \lambda') = 0 \quad \text{or} \quad c_1 Q_1(\xi, \lambda') + c_2 Q_3(\xi, \lambda') = 0
\]

(10)

In order that there be a non-trivial solution of the system (10) it is necessary that its determinant turns to zero

\[
\Delta(\xi, \lambda') = Q_3(\xi, \lambda') Q_4(\xi, \lambda') - Q_2(\xi, \lambda') Q_1(\xi, \lambda') = 0
\]

(11)

For a given value of \( \lambda' \), the position of point \( \xi \) is determined from equality (11). Accordingly, we find \( c_2 = -Q_3(\xi)/Q_1(\xi) \) and also the function \( U \) from system (10), putting \( c_1 = 1 \),

\[
U = Q_1(\xi, \lambda') - \frac{Q_3(\xi, \lambda')}{Q_1(\xi, \lambda')} Q_2(\xi, \lambda')
\]

(12)

The constructed function \( U \) (11) satisfies equation (5), and at the point \( z = 0 \), the boundary conditions of problem (5), and at the point \( z = \xi \), the boundary conditions (10).

We introduce a function

\[
Q_4(z, \lambda') = Q_1(z, \lambda') Q_3(z, \lambda') - Q_2(z, \lambda') Q_1(z, \lambda')
\]

(13)

that has the following property

\[
Q_4(0, \lambda') = 1
\]

(14)

For a given value of \( \lambda' \), we can determine the roots of the function \( Q_4 \). If the estimation \( \lambda' \) exceeds the first eigenvalue \( \lambda_1 \), then the value is \( \xi > 1 \), and vice versa. Thus, the search for a true eigenvalue \( \lambda_1 \) can be carried out using the shooting algorithm, selecting as a target function \( Q_4(\xi, \lambda') \) which was named sagitary (shooting) function. General statements about the convergence of the algorithm were formulated and proved earlier [2]. Using the sagitary function, it is possible to organize a shooting algorithm to find your eigen-numbers, starting from \( \lambda_1 \), then \( \lambda_2 \), etc. The disadvantages of the method include a large number (~ 10) of iterations required for calculating the eigenvalues with high accuracy.

3. The method of accelerated convergence

We set for definiteness \( \lambda_1' > \lambda_1 \), let us integrate the above equations and find the sagitary function \( Q_4 \). For a given value \( \lambda_1' \), we define the point \( \xi \), where \( Q_4(\xi, \lambda') = 0 \). We introduce the proximity criterion of the values \( \lambda_1' \) and \( \lambda_1 \) as follows

\[
1 - \xi = \varepsilon \ll 1
\]

(15)
Further, for the given \( \lambda^*_1 \), we define a function \( U(z, \lambda^*_1) \) by the formula (12). This function is an eigenfunction of the boundary value problem (4) on the interval \( 0 \leq z < \xi \) and corresponds to an eigenvalue \( \lambda^*_1 \). Further, by estimating the eigenvalue \( \lambda^*_1 \), the first approximation of the eigenvalue was constructed using the formula

\[
\lambda^{(1)}_1 = \lambda^*_1 - e \lambda^*_1 r(\xi) \frac{U^2(\xi, \lambda^*_1)}{\|U\|^2}
\]

(16)

where \( \|U\|^2 = \int_0^\xi r(z)U^2(z, \lambda^*_1)dz \).

The found approximate value \( \lambda^{(1)}_1 \) is substituted into the Cauchy problems I and II and the sagitary function. All previous calculations are repeated. The new position of the point \( \xi_1 \) and the value of the parameter \( e_1 \) are found, by which the eigenfunction \( U(z, \lambda^{(1)}_1) \) and the refined eigenvalue are determined.

\[
\lambda^{(2)}_1 = \lambda^{(1)}_1 - e_1 \lambda^{(1)}_1 r(\xi_1) \frac{U^2(\xi_1, \lambda^{(1)}_1)}{\|U\|^2}
\]

(17)

This process is repeated from 1 to 5 times, until a given accuracy of the solution is achieved. Note that \( \xi_1 = c_1 \xi^2 \), \( \xi_2 = c_2 \xi^4 \) etc. (For details, see [2]). Since \( e \ll 1 \) in formulas (15), (16) the value \( \xi \) in the calculations can be set equal to one. The described algorithm was implemented as a software package written in the symbolic computation system Maple. The package allows solving the boundary value problem (5) with any predetermined accuracy.

4. Numerical results

Table 1 shows the dimensionless natural frequencies \( \omega^{(i)}_i \), \( i = 1,2,3,4 \) found by the method of accelerated convergence for the case \( p(z) = [1 + (\frac{h}{h_1} + 1)z] \), \( r = 1 + (\frac{h}{h_1} - 1)z \).

| \( \frac{h}{h_1} \) | \( \omega^{(1)}_1 = \sqrt{\lambda^{(1)}_1} \) | \( \omega^{(2)}_1 = \sqrt{\lambda^{(2)}_1} \) | \( \omega^{(3)}_1 = \sqrt{\lambda^{(3)}_1} \) | \( \omega^{(4)}_1 = \sqrt{\lambda^{(4)}_1} \) |
|---|---|---|---|---|
| 0.1 | 4.6307 | 11.9308 | 32.8331 | 58.9171 |
| 0.2 | 4.2925 | 15.742 | 36.8846 | 68.1164 |
| 0.3 | 4.0817 | 16.6253 | 40.5879 | 76.1821 |
| 0.4 | 3.9343 | 17.4879 | 44.0248 | 83.5541 |
| 0.5 | 3.8238 | 18.3179 | 47.2648 | 90.4505 |
| 0.6 | 3.7371 | 19.1138 | 50.36 | 96.9954 |
| 0.7 | 3.6668 | 19.8806 | 53.3222 | 103.267 |
| 0.8 | 3.6083 | 20.6202 | 56.1923 | 109.3184 |
| 0.9 | 3.5587 | 21.3146 | 58.9799 | 115.1873 |
| 1 | 3.516 | 22.0345 | 61.6972 | 120.3019 |
Table 2 shows the dimensionless natural frequencies \( \omega_i \), \( i = 1,2,3,4 \) found by the method of accelerated convergence for the case \( p(z) = \left[1+\left(\frac{h_0}{h_1}+1\right)z^2\right]^3, r = 1+\left(\frac{h_0}{h_1}-1\right)z^2 \).

| \( h_0 / h_1 \) | \( \omega_1^{(i)} = \sqrt{\lambda_1} \) | \( \omega_2^{(i)} = \sqrt{\lambda_2} \) | \( \omega_3^{(i)} = \sqrt{\lambda_3} \) | \( \omega_4^{(i)} = \sqrt{\lambda_4} \) |
|-------|----------------|----------------|----------------|----------------|
| 0.1   | 5.1065         | 22.0345        | 61.6972        | 120.3019       |
| 0.2   | 4.7628         | 21.7424        | 60.0537        | 117.2509       |
| 0.3   | 4.4967         | 21.4460        | 58.3595        | 113.4642       |
| 0.4   | 4.2822         | 21.1461        | 56.6084        | 109.5201       |
| 0.5   | 4.1043         | 20.8474        | 54.7931        | 105.3905       |
| 0.6   | 3.9532         | 20.5563        | 52.9060        | 101.0385       |
| 0.7   | 3.8227         | 20.2255        | 50.9404        | 96.4138        |
| 0.8   | 3.7083         | 20.0620        | 48.8972        | 91.4480        |
| 0.9   | 3.6068         | 19.9475        | 46.8110        | 86.0549        |
| 1.0   | 3.5160         | 20.1117        | 44.8942        | 80.2310        |

From the obtained results it is obvious that the first natural frequencies \( \omega_1^{(i)} \) and \( \omega_2^{(i)} \) decreasing functions of the parameter \( h_0 / h_1 \) (Fig. 1). The calculations were carried out with an accuracy of seven significant digits. In the limiting case \( h_0 / h_1 = 0.02 \), the value of the function \( p(z) \) changed by four orders of magnitude (about 1 to \( 10^{-4} \)), nevertheless, the method of accelerated convergence turned out to be applicable for solving the Sturm–Liouville problem.
5. References
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