Consistency-based weight form and its application in multi-attribute group decision making

Wen-zhan Dai¹, Hao Nan¹
¹School of Electronic and Information Engineering, Zhejiang Gongshang University, Hangzhou, Zhejiang, 310018, Chinese

Abstract. Due to the complexity of the multi-attribute group decision-making problem, how to determine the weight of different experts on different attributes is a difficult point in current multi-attribute decision-making problem. The authors believe that weights of experts should be reflected in the decision-making process, rather than subjectively determined before the decision. If an expert’s decision-making results can better express the collective will, this expert should be given greater weight. Based on this idea, this paper proposes a weight determination method based on the similarity of individual and collective wishes of experts. First, the improved traditional projection method is proposed. The weight is determined by the degree of agreement between the experts’ individual evaluation vector and the ideal evaluation vector, so that the experts’ weights are more consistent with their performance in the decision-making process. Secondly, combining the new weight form with the ideal point method, a multi-attribute group decision-making method based on the agreement degree is proposed. Finally, the validity of the proposed multi-attribute group decision-making method based on the new weight form is verified by a case simulation.

1. Introduction
The essence of the decision-making problem is the process of ranking the alternatives according to the known conditions, so as to select the relatively best one. For a multi-attribute decision-making problem, due to the complexity of the problem, expert group decision-making is often used. Each expert is given a weight, and then each expert independently evaluates the different attributes of the decision-making object, and finally forms a collective evaluation of the decision-making object. The weights of the experts have a fundamental effect on the decision result.

Wang[2] introduced the dominant particle structure to measure the support of each pair of sorting. Yue[3] used interval intuitionistic fuzzy numbers to establish ideal group decisions and to measure the reliability of individual decisions. Liu[4] used clustering to determine the weights of decision makers in complex multi-attribute large group decision-making problems. Yue[5] calculated the weights of decision makers through positive and negative ideal solutions. Yue[6] calculated the weights of decision makers by projecting onto ideal vectors. Zheng[7] determined the weight by combining the subjective weighting method and the objective weighting method. Based on the theory of probabilistic language terminology (PLTS), Bai[8] proposed a possibility formula that can rank PLTSs. Zhang[9] proposed a projection model to calculate the projection of the solution to the positive ideal solution and the negative ideal solution.

There are many methods to calculate the weight, but the unsolved problem is that the decision-making group is composed of experts in different fields, and the different research directions and abilities of each expert determine the different weight of different experts in group decision-making. And a expert should have different weights for different attributes. This paper proposes a form of
decision maker weights that can contain more information to better distinguish the relative weights of decision makers for different attributes.

2. New weight form and problem description
In a multi-attribute group decision problem, \( A = \{A_1, A_2, \cdots, A_m\} \) is the set of options, \( B = \{B_1, B_2, \cdots, B_n\} \) is the attributes set, \( M = \{M_1, M_2, \cdots, M_t\} \) is the experts set, the weights of the attributes is \( \omega = (\omega_1, \omega_2, \cdots, \omega_n) \sum_{j=1}^{n} \omega_j = 1, \omega_j \geq 0, j = 1, 2, \cdots, n \). Experts weight matrix for attribute \( B \) is \( \lambda \in R^{t \times n} \),

\[
\lambda = \begin{pmatrix}
\lambda_1^{(1)} & \lambda_2^{(1)} & \cdots & \lambda_n^{(1)} \\
\lambda_1^{(2)} & \lambda_2^{(2)} & \cdots & \lambda_n^{(2)} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^{(t)} & \lambda_2^{(t)} & \cdots & \lambda_n^{(t)}
\end{pmatrix}.
\]

Vector \( \lambda_j = (\lambda_j^{(1)}, \lambda_j^{(2)}, \cdots, \lambda_j^{(t)})^T \) is weights vector of \( t \) experts for attribute \( B_j \), \( \sum_{k=1}^{t} \lambda_j^{(k)} = 1, \lambda_j^{(k)} \geq 0, k = 1, 2, \cdots, t \).

This form of weights not only reflects the different authority of experts in decision-making problem, but also reflects the different weights of the same expert for different attributes. At the same time, the expert's weight for a particular attribute is determined by the expert compared to other experts, and will not be affected by his weight for other attributes. The next step is to find an objective and scientific method to calculate the weight in this weight form.

3. Improved projection method and scheme sorting

3.1. Improved projection method
Let the first expert's original evaluation matrix for attributes be \( X_k \in R^{m \times n} \),

\[
X_k = \begin{pmatrix}
A_1 & A_2 & \cdots & A_n \\
B_1 & B_2 & \cdots & B_n \\
\end{pmatrix}
\]

\[
x_{ij}^{(k)}
\]

is the original evaluation value of expert \( M_k \) on the \( B_j \) attribute of plan \( A_i, k = 1, 2, \cdots, t ; i = 1, 2, \cdots, m ; j = 1, 2, \cdots, n \).

In order to unify the dimension and facilitate comparison, it is first necessary to standardize the original evaluation value \( x_{ij}^{(k)} \) to generate a standardized evaluation value \( \bar{x}_{ij}^{(k)} \).

For benefit attributes:

\[
\bar{x}_{ij}^{(k)} = \frac{x_{ij}^{(k)}}{\max_i (x_{ij}^{(k)})}
\]

For cost attributes:

\[
\bar{x}_{ij}^{(k)} = \frac{x_{ij}^{(k)}}{\min_i (x_{ij}^{(k)})}
\]

Get the normalized evaluation matrix \( R_k \in R^{m \times n} \):

\[
R_k = \begin{pmatrix}
A_1 & A_2 & \cdots & A_n \\
B_1 & B_2 & \cdots & B_n \\
\end{pmatrix}
\]

\[
X_k = \begin{pmatrix}
\bar{x}_{11}^{(k)} & \bar{x}_{12}^{(k)} & \cdots & \bar{x}_{1n}^{(k)} \\
\bar{x}_{12}^{(k)} & \bar{x}_{12}^{(k)} & \cdots & \bar{x}_{2n}^{(k)} \\
\vdots & \vdots & \ddots & \vdots \\
\bar{x}_{1t}^{(k)} & \bar{x}_{12}^{(k)} & \cdots & \bar{x}_{tn}^{(k)}
\end{pmatrix}.
\]
Define the ideal vector of each expert under attribute $B_j$ as $\beta_j$:

$$
\beta_j = (\beta_{j1}, \beta_{j2}, \cdots, \beta_{jm})^T = \frac{1}{t} \left( \sum_{k=1}^{t} r_{ij}^{(k)}, \sum_{k=1}^{t} r_{ij}^{(k)}, \cdots, \sum_{k=1}^{t} r_{ij}^{(k)} \right)^T, \quad k = 1, 2, \ldots, t ; j = 1, 2, \ldots, n \quad (4)
$$

Obviously, the vector $\beta_j$ is obtained by averaging the evaluation values of experts, which reflects the collective will.

The projection of the evaluation vector $r = \mathbf{r}_j^{(k)}$ on the ideal vector $\beta = \beta_j$ is $P_\beta(r)$:

$$
P_\beta(r) = |r| \cos (r, \beta) = \frac{r \cdot \beta}{|r||\beta|} = \frac{\sum_{i=1}^{m} r_{ij}^{(k)} \beta_{ij}}{k = 1, 2, \ldots, t ; i = 1, 2, \ldots, m ; j = 1, 2, \ldots, n} \quad (5)
$$

The projection $P_\beta(r)$ reflects the degree of agreement between the evaluation vector $\mathbf{r}_j^{(k)}$ of the expert $M_k$ and the ideal vector $\beta_j$. Generally, the larger $P_\beta(r)$, the greater the weight of the expert. But through further research, it was found there is a prerequisite for this conclusion.

For example: there are three vectors $\alpha^{(1)} = (1,2,3), \alpha^{(2)} = (2,4,6), \alpha^{(3)} = (3,6,9)$. Their ideal vector is $\beta = (2,4,6)$. Obviously, the vector $\alpha^{(2)}$ is closest to the ideal vector. But directly use equation (5), the projection value of vector $\alpha^{(3)}$ is the largest. As another example, $\alpha^{(4)} = (14,4,2)$ matches the ideal vector less, but the projection of $\alpha^{(4)} = (14,4,2)$ on the ideal vector is the same as the projection of $\alpha^{(2)} = (2,4,6)$ on the ideal vector. Therefore, the direct application of equation (5) is insufficient.

The reason is found: because some element values in the individual evaluation vector are larger than the values of the corresponding elements of the ideal vector, and the values of some other elements are smaller than the values of the corresponding elements of the ideal vector. So that the degree of deviation offsets each other. Therefore, there is a prerequisite for the correct application of formula (5), that is, the values of all elements in the individual evaluation vector are either greater than the values of the corresponding elements of the ideal vector, or they are all smaller than the values of the corresponding elements in the ideal vector. For example, the first value $\alpha_{1}^{(4)} = 14$ of $\alpha^{(4)} = (14,4,2)$ is greater than the first value $\beta_{1} = 2$ of the ideal vector, and $\alpha_{3}^{(4)} = 2$ is less than $\beta_{3} = 6$, the deviations they produce cancel each other, resulting in $\alpha^{(4)} \cdot \beta = \alpha^{(2)} \cdot \beta$. To avoid the above problems, the following equation (7) is proposed:

$$
\mathbf{r}_j^{(i)} = \beta_j - \left| \mathbf{r}_j^{(k)} - \beta_j \right| \quad k = 1, 2, \ldots, t ; i = 1, 2, \ldots, m ; j = 1, 2, \ldots, n \quad (7)
$$

All individual evaluation values are converted into values with the same deviation less than the ideal vector, and then these processed evaluation values are substituted into the following equation (8) to get $P_\beta(\mathbf{r}^*)$:

$$
P_\beta(\mathbf{r}^*) = |\mathbf{r}^*| \cos (\mathbf{r}^*, \beta) = \frac{\mathbf{r}^* \cdot \beta}{|\mathbf{r}^*| |\beta|} = \frac{\sum_{i=1}^{m} r_{ij}^{(k)} \beta_{ij}}{k = 1, 2, \ldots, t ; i = 1, 2, \ldots, m ; j = 1, 2, \ldots, n} \quad (8)
$$

In order to increase the weight differentiation, the concept of the difference value $d_j^{(k)}$ between the projection value $P_\beta(\mathbf{r}^*)$ and the ideal vector $\beta$ is introduced:

$$
d_j^{(k)} = \left| P_\beta(\mathbf{r}^*) - |\beta| \right| \quad k = 1, 2, \ldots, t ; j = 1, 2, \ldots, n \quad (10)
$$

The weight $\lambda_j^{(k)}$ of expert $M_k$ to attribute $B_j$ is:

$$
\lambda_j^{(k)} = \frac{d_j^{(k)}}{\sum_{k=1}^{t} d_j^{(k)}} \quad k = 1, 2, \ldots, t ; j = 1, 2, \ldots, n \quad (11)
$$
And $\lambda_j^{(k)} \geq 0$, $\sum_{k=1}^{t} \lambda_j^{(k)} = 1$.

### 3.2. Scheme sorting

In the previous section, expert weights were obtained by processing the experts evaluation matrices. In this section, this paper will combine the obtained expert weights to sort the schemes to select the optimal decision result. The ideal point method is the ranking method used in this paper. Using equation (12) and (13), the ideal point $+_r_j^{(k)}$ and ideal vector $+_r_j$ of the decision maker $M_k$ for the attribute $B_j$ can be obtained respectively:

$$+_r_j^{(k)} = \max_i \left( r_i^{(k)} \right) \quad k = 1,2,\ldots, t \ ; \ i = 1,2,\ldots, m \ ; \ j = 1,2,\ldots, n \ .$$  \hspace{1cm} (12)

$$+_r_j = \left( +_r_j^{(1)}, +_r_j^{(2)},\ldots, +_r_j^{(t)} \right) \quad j = 1,2,\ldots, n \ .$$  \hspace{1cm} (13)

Using equation (14), after synthesizing the opinions of all decision makers, can get the distance between the evaluation value of each attribute of each scheme and the ideal point,

$$s_i = \sqrt{\sum_{k=1}^{t} \lambda_j^{(k)} \left( r_{ij}^{(k)} - +_r_j^{(k)} \right)^2}$$

Using equation (15), get the total distance between each scheme and the ideal solution:

$$S_i = \sum_{j=1}^{n} \omega_j \sqrt{\sum_{k=1}^{t} A_j^{(k)} \left( r_{ij}^{(k)} - +_r_j^{(k)} \right)^2} = \sum_{j=1}^{n} \omega_j S_{ij}$$

Comparing the size of $S_1, S_2, \ldots, S_m$, the smaller the distance from the ideal solution is, the better the scheme is.

### 4. Example analysis

#### 4.1. Example

This is an example of air conditioning system selection [10]. A city plans to build a city library, and government departments need to consider which air-conditioning system to install in the library. The builder provided 5 alternatives ($A_1, A_2, A_3, A_4, A_5$), the three decision makers were $M_1, M_2, M_3$ and evaluated these 5 alternatives based on 8 attributes. The 8 attributes are (purchase cost (yuan)), (operating cost (yuan)), (effect (0-1 scale)), (noise level (decibel)), (convenience of maintenance (0-1 scale)), (reliability (percentage scale)), (flexibility (0-1 scale)), (safety (0-1 scale)). The attribute $B_1, B_2, B_3$ are cost attributes, and the remaining attributes are benefit attributes. The weight vectors of these 8 attributes are $\omega = (0.11, 0.09, 0.11, 0.12, 0.18, 0.16, 0.10, 0.13)$, the evaluation tables given by the three experts are shown in Tables 1 to 3:

|      | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ | $B_6$ | $B_7$ | $B_8$ |
|------|-------|-------|-------|-------|-------|-------|-------|-------|
| $A_1$ | 5.00  | 7.00  | 0.80  | 36.00 | 0.60  | 94.00 | 0.50  | 0.80  |
| $A_2$ | 2.00  | 6.00  | 0.50  | 72.00 | 0.50  | 76.00 | 0.80  | 0.60  |
| $A_3$ | 6.00  | 6.00  | 0.70  | 64.00 | 0.70  | 84.00 | 0.90  | 0.70  |
| $A_4$ | 7.00  | 5.00  | 0.80  | 42.00 | 0.80  | 91.00 | 0.70  | 0.70  |
| $A_5$ | 5.00  | 7.00  | 0.70  | 54.00 | 0.60  | 96.00 | 0.60  | 0.90  |

|      | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ | $B_6$ | $B_7$ | $B_8$ |
|------|-------|-------|-------|-------|-------|-------|-------|-------|
| $A_4$ | 6.00  | 7.00  | 0.80  | 38.00 | 0.40  | 97.00 | 0.60  | 0.50  |
| $A_2$ | 3.00  | 6.00  | 0.70  | 76.00 | 0.70  | 72.00 | 0.70  | 0.60  |
| $A_3$ | 6.00  | 5.00  | 0.60  | 70.00 | 0.90  | 83.00 | 0.70  | 0.70  |
Table 3 Evaluation matrix given by the third expert.

|    | B₁ | B₂ | B₃ | B₄ | B₅ | B₆ | B₇ | B₈ |
|----|----|----|----|----|----|----|----|----|
| A₁ | 4.00 | 7.00 | 0.90 | 43.00 | 0.70 | 95.00 | 0.50 | 0.60 |
| A₂ | 5.00 | 8.00 | 0.60 | 75.00 | 0.50 | 82.00 | 0.90 | 0.80 |
| A₃ | 7.00 | 6.00 | 0.70 | 78.00 | 0.90 | 87.00 | 0.95 | 0.70 |
| A₄ | 8.00 | 7.00 | 1.00 | 49.00 | 0.80 | 96.00 | 0.70 | 0.90 |
| A₅ | 6.00 | 5.00 | 0.80 | 62.00 | 0.90 | 93.00 | 0.60 | 0.90 |

Calculation results:

| j | j = 1 | j = 2 | j = 3 | j = 4 | j = 5 | j = 6 | j = 7 | j = 8 |
|---|-------|-------|-------|-------|-------|-------|-------|-------|
| k = 1 | 0.280 | 0.375 | 0.367 | 0.422 | 0.371 | 0.593 | 0.458 | 0.252 |
| k = 2 | 0.430 | 0.380 | 0.257 | 0.307 | 0.368 | 0.173 | 0.198 | 0.433 |
| k = 3 | 0.290 | 0.245 | 0.376 | 0.270 | 0.261 | 0.235 | 0.345 | 0.315 |

The distance between the five options and the ideal point:

\[ S_1 = 0.24, S_2 = 0.2803, S_3 = 0.2291, S_4 = 0.1746, S_5 = 0.2034 \]

Therefore scheme \( A_4 \) is better than \( A_5 \) better than \( A_3 \) better than \( A_1 \) better than \( A_2 \).

In order to further illustrate the effectiveness of the algorithm in this paper, the weight calculation methods in reference [8] and reference [6] are used to calculate this example. The obtained decision maker weights and final results are sorted as follows:

Expert weights under the projection method [8]:

\[ \lambda_1 = 0.334 \quad \lambda_2 = 0.329 \quad \lambda_3 = 0.337 \]

Scheme sorting under projection method [8]:

\[ A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2 \]

Expert weights under TOPSIS method [6]:

\[ \lambda_1 = 0.338 \quad \lambda_2 = 0.335 \quad \lambda_3 = 0.327 \]

Scheme sorting under TOPSIS method [6]:

\[ A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2 \]

4.2. Result analysis

Through the analysis of experimental data, the advantages of this algorithm are as follows:

- Compared with the weight results of other algorithms, it is obvious that the expert weight form in this paper can provide more detailed information. Under the premise of scientific methods, the form of expert weights proposed in this paper will surely be able to obtain more accurate decision ranking results;
- The ranking results of the relative importance of the three experts in this example are shown in Table 5 (if the expert weight form is the form proposed in this paper, by accumulating the weights under each attribute, the result can measure the relative importance of the three experts). It can be seen that the expert ranking results of the projection method [8] are different from the other two, and the reason for this difference is that the evaluation matrix provided by the third expert has a high overall score (the reason is explained in Section 2). According to the weights ranking result, the algorithm in this paper has made up for the defect of the original projection method, making the ranking results closer to reality.
Table 5 Relative importance ranking of three experts.

| Algorithm   | Sort results |
|-------------|--------------|
| Projection method | $\lambda_3 > \lambda_1 > \lambda_2$ |
| TOPSIS method   | $\lambda_1 > \lambda_2 > \lambda_3$ |
| This paper method | $\lambda_1 > \lambda_2 > \lambda_3$ |

Acknowledgments
Foundation item: National Natural Science Foundation of China (N0: 61374022)

References
[1] Emrah Koksalmis, Özgür Kabak. (2019) Deriving decision makers’ weights in group decision making: An overview of objective methods[J]. Information Fusion, 49.
[2] Wang, B., Liang, J., Qian, Y. (2015) Determining decision makers’ weights in group ranking: A granular computing method. Int. J. Mach. Learn. Cybern., 6 pp. 511-521.
[3] Yue, Z. (2011) Deriving decision maker’s weights based on distance measure for interval-valued intuitionistic fuzzy group decision making[J]. Expert Systems With Applications, 38(9).
[4] Liu, B., Shen, Y., Chen, Y., Chen, X., Wang, Y. (2015) A two-layer weight determination method for complex multi-attribute large-group decision-making experts in a linguistic environment[J]. Information Fusion, 23.
[5] Yue, Z. (2010) A method for group decision-making based on determining weights of decision makers using TOPSIS[J]. Applied Mathematical Modelling, 35(4).
[6] Yue, Z. (2012) Approach to group decision making based on determining the weights of experts by using projection method[J]. Applied Mathematical Modelling, 36(7).
[7] Zheng, G., Jing, Y., Huang, H., Gao, Y. (2009) Application of improved grey relational projection method to evaluate sustainable building envelope performance[J]. Applied Energy, 87(2).
[8] Bai, C., Zhang, R., Qian, L., Wu, Y. (2017) Comparisons of probabilistic linguistic term sets for multi-criteria decision making[J]. Knowledge-Based Systems, 119.
[9] Zhang, X., Gou, X., Xu, Z., et al. (2019) A projection method for multiple attribute group decision making with probabilistic linguistic term sets. Int. J. Mach. Learn. & Cyber. 10, 2515–2528.
[10] Xu, Y., Li, D. (2010) Approach to reaching consensus in multiple attribute group decision making[J]. Control and decision making, 25(12):1810-1814+1820.