Semiconformal symmetry- A new symmetry of the spacetime manifold of the general relativity

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Abstract. In this paper we have introduced a new symmetry property of spacetime which is named as semiconformal curvature collineation, and its relationship with other known symmetry properties has been established. This new symmetry property of the spacetime has also been studied for non-null and null electromagnetic fields.

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1 Introduction

In recent years general relativists have been much interested in symmetries of spacetime in general relativity. Such interest is due to the need to simplify the Einstein’s field equations in the search for their exact solutions. These geometrical symmetries of the spacetime are often defined by the vanishing of the Lie derivative of certain tensors with respect to a vector (this vector may be time-like, space-like or null). The symmetries in general theory of relativity have been introduced by Katzin, Levine and Davis in the papers ([24] and [25]). These symmetries which are also known as collineations, were further studied by Ahsan ([3] - [8]), Ahsan and Hussain ([13]), Ahsan and Siddiqui ([14]), Ahsan and Ali ([9] - [12]) and Ali and Ahsan ([15]-[17]) among many others. However, in this paper our study is focused on these symmetries which can be used as simplifying assumptions in the exact solution of Einstein’s field equations but solving EFE by our findings will be the next target. Main objective of this paper is to give new symmetry in mathematical approach and analyse it on parameters of the well established literature on symmetries of spacetime manifolds.

As a special subgroup of the conformal transformation group, Ishii([23]) defined a rank four tensor $L_{bcd}^h$ that remains invariant under conharmonic transformation for a $n$-dimensional Riemannian differentiable manifold $(M^n, g)$ of dimension $n \geq 4$, as follows:

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\[ L_{bcd}^h = R_{bcd}^h + \frac{1}{n-2} (\delta_c^h R_{bd}^h - \delta_d^h R_{bc}^h + g_{bd} R_{c}^h - g_{bc} R_{d}^h), \quad (1.1) \]

where \( R_{bcd}^h, R_{bd}^h \) are Riemann curvature tensor and Ricci tensor respectively. The geometric properties of conharmonic curvature tensor have been discussed by Shaikh and Hui (\cite{33}), while the relativistic significance of this tensor has been investigated by Abdussattar and Dwivedi (\cite{1}) and Siddiqui and Ahsan (\cite{31}). In 2016, J. Kim (\cite{27}) introduced curvature like tensor field which remain invariant under conharmonic transformation. He named this new tensor as semiconformal curvature tensor and denoted it by \( P_{bcd}^h \). For a Riemannian manifold \( M^n \) with metric \( g \), this tensor is defined as (see also \cite{28})

\[ P_{bcd}^h = -(n-2) B C_{bcd}^h + [A + (n-2) B] L_{bcd}^h, \quad (1.2) \]

provided the constants \( A \) and \( B \) are not simultaneously zero, \( C_{bcd}^h \) is conformal curvature tensor defined as

\[ C_{bcd}^h = R_{bcd}^h + \frac{1}{n-2} (\delta_c^h R_{bd}^h - \delta_d^h R_{bc}^h + g_{bd} R_{c}^h - g_{bc} R_{d}^h) + \frac{R}{(n-1)(n-2)} (\delta_d^h g_{bc} - \delta_c^h g_{bd}), \quad (1.3) \]

where \( R_{ab} \) is the Ricci tensor and \( R \) is the scalar curvature.

For a special substitution \( A = 1 \) and \( B = \frac{-1}{n-2} \), the semiconformal curvature tensor reduces to conformal curvature tensor, while for \( A = 1 \) and \( B = 0 \), it reduces to conharmonic curvature tensor. The semiconformal curvature \( P_{bcd}^h \) satisfy the following symmetry properties

\[ P_{hbcd} = -P_{bhd} = -P_{hbdc} = P_{cdhb}, \quad (1.4) \]

and

\[ P_{hbd} + P_{chbd} + P_{bchd} = 0. \quad (1.5) \]

In this paper we define a new symmetry in terms of semiconformal curvature tensor and study its relationship with other symmetries of the spacetime. We call this new symmery as semiconformal curvature collineation. Section 2 contains some known results that are required for our investigation. In sections 3 and 4, the relationship between semiconformal curvature collineation and the other symmetry properties for a general Riemannian space and for a Riemannian space with vanishing Ricci Tensor, respectively, have been established. Finally in section 5, the semiconformal curvature collineation has been studied for non-null and null electromagnetic fields.
2 Preliminaries

A geometrical symmetry of the spacetime is often defined in terms of the Lie derivative of a tensor. These symmetries are also known as collineations. The literature on such collineations is very large and still expanding with results of elegance. Here we shall mention only those symmetry assumptions that are necessary for our study and we have (cf. [20], [34], [36])

**Definition 2.1. Motion (M)** A spacetime is said to admit motion if there exists a vector field $\xi^a$ such that

$$\eta_{ab} \equiv \mathcal{L}_\xi g_{ab} = \xi_{a;b} + \xi_{b;a} = 0.$$  \hspace{1cm} (2.1)

Equation (2.1) is known as Killing equation and the vector $\xi^a$ is known as a Killing vector field.

**Definition 2.2. Affine Collineation (AC)** A spacetime is said to be an affine collineation if there is a vector field $\xi^a$ such that

$$\mathcal{L}_\xi \left\{ \begin{array}{c} c \\ ab \end{array} \right\} \equiv \frac{1}{2} g^{cd}(\eta_{d;ab} + \eta_{db,a} - \eta_{ab,d}) = 0,$$  \hspace{1cm} (2.2)

where $\left\{ \begin{array}{c} c \\ ab \end{array} \right\}$ is the Christoffel symbol of second kind. Hence the necessary and sufficient condition for an AC [from (2.2)] is

$$\eta_{ab;\cdot} = 0.$$  \hspace{1cm} (2.3)

It may be noted, from equations (2.2) and (2.3), that every M is AC.

**Definition 2.3. Conformal Motion (Conf M)** A spacetime is said to admit a conformal motion if there exist a vector field $\xi^a$ such that

$$\mathcal{L}_\xi \mathcal{R}_{ab} = 0,$$  \hspace{1cm} (2.4)

where $\mathcal{R}$ ([34]) is defined by

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†Indices takes the values from 1, 2, 3, ..., $n$ and the summation convention is used. Covariant differentiation is indicated by a semicolon (;) and partial differentiation by a comma (,).
Accordingly we have

\[ \eta_{ab} = 2\phi g_{ab}, \]  
(2.6)

where \( \phi \) is scalar and we may express in the following form

\[ \phi = \frac{1}{4} \xi_{;d}. \]  
(2.7)

**Definition 2.4. Projective Collineation (PC)** A spacetime is said to admit projective collineation if there exist a vector \( \xi^a \) such that

\[ \mathcal{L}_\xi \prod_{bc}^a = 0, \]  
(2.8)

where the projective connection is defined as for \( n = 4 \)

\[ \prod_{bc}^a = \left\{ a \begin{array}{c} \frac{1}{5} \left( \delta_{,b}^c h_{;c} + \delta_{,c}^b h_{;b} \right) \right. \right\}, \]  
(2.9)

From equations (2.8) and (2.9), we get

\[ \mathcal{L}_\xi \left\{ a \begin{array}{c} \frac{1}{5} \left( \delta_{,b}^c \sigma_{c}^e + \delta_{,c}^b \sigma_{b}^e \right) \right. \right\}, \]  
(2.10)

where

\[ \sigma_{;c} = \frac{1}{5} \xi_{;mc} \quad \text{and} \quad \sigma_{;b} = \frac{1}{5} \xi_{;mb}. \]  
(2.11)

Further, for every projective collineation, we have ([24])

\[ \mathcal{L}_\xi W_{bcd}^h = 0, \]  
(2.12)

where the Weyl projective curvature tensor for \( n = 4 \) is given by

\[ W_{bcd}^h = R_{bcd}^h - \frac{1}{3}(\delta_{;d}^h R_{bc} - \delta_{;c}^h R_{bd}). \]  
(2.13)

From equations (2.10), (2.11) and (2.12) it follows that every AC is PC.

**Definition 2.5. Conformal Collineation (Conf C)** A spacetime is said to admit a conformal collineation if there exist a vector \( \xi^a \) such that
\[ \mathcal{L}_\xi \left\{ \begin{array}{c} a \\ b \end{array} \right\} = \delta^a_b \phi_{,c} + \delta^c_b \phi_{,b} - g_{bc}g^{am} \phi_{,m}, \quad (2.14) \]

where \( \phi = \frac{1}{4} \xi_d \).

Equations (2.6) and (2.14) may be expressed as ([24])

\[ \eta_{abc} = 2 \phi_{,c} g_{ab}, \quad (2.15) \]

and that every Conf C must satisfy (for explanation c.f., [36])

\[ \mathcal{L}_\xi C^h_{bcd} = 0, \quad (2.16) \]

where \( C^h_{bcd} \) is conformal curvature tensor, which from equation (1.3) for \( n = 4 \), is given by

\[ C^h_{bcd} = R^h_{bcd} + \frac{1}{2} (\delta^h_d R_{bd} - \delta^h_b R_{cd} + g_{bd} R^h_c - g_{bc} R^h_d) + \frac{R}{6} (\delta^h_d g_{bc} - \delta^h_c g_{bd}). \quad (2.17) \]

**Definition 2.6. Curvature Collineation (CC)** A spacetime is said to admit curvature collineation if there is a vector field \( \xi^a \) such that

\[ \mathcal{L}_\xi R^h_{bcd} = 0, \quad (2.18) \]

where Riemann curvature tensor is defined as ([20])

\[ R^h_{bcd} = \left\{ \begin{array}{c} h \\ bd \end{array} \right\}_{,c} - \left\{ \begin{array}{c} h \\ bd \end{array} \right\}_{,d} + \left\{ \begin{array}{c} m \\ bd \end{array} \right\}_{,mc} - \left\{ \begin{array}{c} m \\ bc \end{array} \right\}_{,md}. \quad (2.19) \]

**Definition 2.7. Ricci Collineation (RC)** A spacetime is said to admit Ricci collineation if there is a vector field \( \xi^a \) such that

\[ \mathcal{L}_\xi R_{ab} = 0, \quad (2.20) \]

where \( R_{ab} \) is the Ricci tensor.

**Definition 2.8. Maxwell collineation (MC)** The electromagnetic field inherits the symmetry property of spacetime such that

\[ \mathcal{L}_\xi F_{ab} = F_{abc} \xi^c + F_{ac} \xi^c_b + F_{bc} \xi^c_a = 0, \quad (2.21) \]

where \( F_{ab} \) is the electromagnetic field tensor. A point transformation that leave \( F_{ab} \) invariant, i.e., equation (2.21) is satisfied, is called a Maxwell collineation ([18]).
3 Semiconformal Symmetry

For \( n = 4 \), the semiconformal curvature tensor, from equation (1.2), is given by

\[
P^h_{bcd} = -2BC^h_{bcd} + [A + 2B]L^h_{bcd},
\]

where \( C^h_{bcd} \) and \( L^h_{bcd} \) are the conformal and conharmonic curvature tensor respectively.

We now define a new symmetry for the spacetime manifold of general relativity as

**Definition 3.1. Semiconformal Curvature Collineation (Semiconf CC)** A spacetime \( V_4 \) is said to admit a semiconformal curvature collineation if there exists a vector field \( \xi^a \) such that

\[
\mathcal{L}_\xi P^h_{bcd} = 0,
\]

where \( P^h_{bcd} \) is semiconformal curvature tensor is defined in equation (1.2).

**Definition 3.2. Special Semiconformal Curvature Collineation (S Semiconf CC)** A semiconformal curvature collineation with the following condition

\[
\sigma_{,bc} = 0,
\]

is called a special semiconformal curvature collineation.

where \( \sigma_{,bc} = \frac{1}{4} \xi^d \xi_{dbc} \) and \( \xi^a \) is Killing vector field.

We also define

**Definition 3.3. Conharmonic Curvature Collineation (Conh CC)** A spacetime \( V_4 \) is said to admit a conharmonic curvature collineation if there exists a vector \( \xi^a \) such that

\[
\mathcal{L}_\xi L^h_{bcd} = 0,
\]

where conharmonic curvature tensor \( L^h_{bcd} \) is defined by ([2], [31])

\[
L^h_{bcd} = R^h_{bcd} + \frac{1}{2}(\delta^h_c R_{bd} - \delta^h_d R_{bc} + g_{bd} R^h_c - g_{bc} R^h_d).
\]
\[ \mathcal{L}_\xi M^h_{bcd} = 0, \]  \hspace{1cm} (3.6)  

where concircular curvature tensor \( M^h_{bcd} \) is defined by \([2], [14]\)

\[ M^h_{bcd} = R^h_{bcd} - \frac{R}{12} (\delta^h_d g_{bc} - \delta^h_c g_{bd}). \]  \hspace{1cm} (3.7)  

**Main results of the section 3.**

**Theorem 3.1.** The necessary and sufficient condition for a semiconformal curvature collineation (Semiconf CC) to be a curvature collineation (CC) is that

\[ \phi_{,bc} = 0, \]

where \( \phi = \frac{1}{4} \xi^{cd} \) is a scalar function.

**Theorem 3.2.** The necessary and sufficient condition for a semiconformal curvature collineation to be a Weyl projective curvature collineation is that

\[ \sigma_{,bc} = 0. \]

where \( \sigma = \frac{1}{5} \xi^{m}_{..m} \).

**Theorem 3.3.** A spacetime \( V_4 \) admits semiconformal curvature collineation along a vector field \( \xi^a \) provided that \( \xi^a \) is Killing.

### 4 Semiconformal symmetry in empty spacetime

The Einstein field equations are given by

\[ R_{bc} - \frac{1}{2} g_{bc} R = - k T_{bc} \]  \hspace{1cm} (4.1)  

where \( R_{bc} \) is the Ricci tensor, \( g_{bc} \) is the metric tensor, \( T_{bc} \) is the energy momentum tensor, \( R \) is the scalar curvature tensor and \( k \) is the constant.

Multiplying by \( g^{bc} \) and using \( g^{bc} g_{bc} = 4 \), equation (4.1) takes the form

\[ R = k T. \]  \hspace{1cm} (4.2)  

from the equations (4.1) and (4.2), we get

\[ R_{bc} = k (T_{bc} - \frac{1}{2} g_{bc} T) \]  \hspace{1cm} (4.3)
If \( T_{bc} = 0 \), then \( T = g^{bc}T_{bc} = 0 \), equation (4.3) yields

\[
R_{bc} = 0,
\]  

(4.4)
these equations are the field equations for empty spacetime.

**Main results of the section 4.**

**Theorem 4.1.** In an Empty spacetime \( V^0_4 \) Lie derivative of semiconformal curvature tensor is proportional to Lie derivative of Riemann curvature tensor.

**Theorem 4.2.** In empty spacetime \( V^0_4 \) the Lie derivatives of semiconformal curvature and Weyl projective curvature tensors are proportional.

## 5 Semiconformal collineation and electromagnetic fields

It is known that in general relativity, the electromagnetism can be described through Maxwell’s equation

\[
F_{[abc]} = 0, \quad F_{;b}^{ab} = J^a,
\]  

(5.1)
where the skew-symmetric tensor \( F_{ab} \) represents the electromagnetic field tensor and \( J^a \) the current density. Moreover, we have defined the Einstein field equations in equation (4.1) and in presence of matter in equation (4.3)

The energy - momentum tensor for an electromagnetic field is given by

\[
T_{ab} = -F_{ac}F^c_b + \frac{1}{4}g_{ab}F_{pq}F^{pq},
\]  

(5.2)
which is symmetric tensor. Equation (5.2) leads to \( T^a_a = T = 0 \) and thus the Einstein equation for a purely electromagnetic distribution is given by

\[
R_{ab} = kT_{ab},
\]  

(5.3)
The geometrical symmetry defined by equation (2.21) along with the symmetry given by equation (2.1) has been the subject of interest for quite some time. Thus for example, for non-null electromagnetic fields Woolley ([35]) has shown that if equation (2.1) holds then \( F_{ab} \) satisfies \( \mathcal{L}_\xi F_{ab} = k(\alpha)F_{ab} \) form some constant \( k(\alpha) \), \( \alpha = 1, 2, \ldots, \ldots, r \); while Michalski and Wainwright ([29]) have shown that \( \mathcal{L}_\xi g_{ab} = 0 \Rightarrow \mathcal{L}_\xi F_{ab} = 0 \) for non-null fields. On the other hand, for non-null fields, Duggal ([19]) has proved the converse part under certain conditions. Maxwell collineations have also been studied
by Ahsan and Ahsan and Husain ([13], [3], [5]). It is seen that for null electromagnetic fields neither MC is a consequence of Motion nor Motion is a consequence of Maxwell collineation. Moreover, using Newman-Penrose formalism, Ahsan ([5]) has obtained the conditions under which a null electromagnetic field may admit Maxwell collineation and Motion. The concept of Maxwell collineation was further extended as Maxwell Inheritance (MI) by Ahsan and Ahsan ([8]), who applied this concept to (i) the spacetime solution corresponding to strong gravitational waves propagating in generalized electromagnetic universes and (ii) the algebraically general twist-free solution of Einstein-Maxwell equation for non-radiative electromagnetic fields.

In 1986 Khlebnikov ([26]) has obtained the solutions for Einstein-Maxwell equations corresponding to the strong gravitational waves in the generalized electromagnetic universe. He used the technique of N.P. formalism ([30]) to obtain the solution in non-radiative electromagnetic fields. In his solution he took the first real null tetrad vector $l^a$ as geodetic and shear-free and the tetrad as parallelly propagated along $l^a$ also proved that the solution does not admit the Maxwell collineation. As another example we can refer the twist-free algebraically general solution given by Tariq and Tupper ([32]). For this solution of Einstein-Maxwell equations in non-null electromagnetic fields together with the condition of coupling theorem Tariq and Tupper proved that their solutions also do not admit the Maxwell collineation.

Motivated by above discussions, in this section, we shall investigate the role of semiconformal curvature collineation to the non-null and null electromagnetic fields.

**Lemma 5.1.** ([29]) In a non-null electromagnetic field, the Lie derivative of electromagnetic field tensor $F_{ab}$ with respect to a vector field $\xi$ vanishes, if $\xi$ is Killing vector.

**Main results of section 5.**

**Theorem 5.1.** A non-null electromagnetic field admits semiconformal curvature collineation along a Killing vector field.

**Theorem 5.2.** A non-null electromagnetic field admits semiconformal curvature collineation if it admits Maxwell collineation.

**Theorem 5.3.** A null electromagnetic field admits semiconformal curvature collineation along a propagation (polarization) vector if propagation (polarization) vector is killing and expansion-free.

Note: Proof of the main results will be uploaded after publication of the article.
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