Double Trace Flows and Holographic RG in dS/CFT correspondence

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Abstract

If there is a dS/CFT correspondence, time evolution in the bulk should translate to RG flows in the dual euclidean field theory. Consequently, although the dual field is expected to be non-unitary, its RG flows will carry an imprint of the unitary time evolution in the bulk. In this note we examine the prediction of holographic RG in de Sitter space for the flow of double and triple trace couplings in any proposed dual. We show quite generally that the correct form of the field theory beta functions for the double trace couplings is obtained from holography, provided one identifies the scale of the field theory with $(i|T|)$ where $T$ is the ‘time’ in conformal coordinates. For $dS_4$, we find that with an appropriate choice of operator normalization, it is possible to have real $n$-point correlation functions as well as beta functions with real coefficients. This choice leads to an RG flow with an IR fixed point at negative coupling unlike in a unitary theory where the IR fixed point is at positive coupling. The proposed correspondence of $Sp(N)$ vector models with de Sitter Vasiliev gravity provides a specific example of such a phenomenon. For $dS_{d+1}$ with even $d$, however, we find that no choice of operator normalization exists which ensures reality of coefficients of the beta-functions as well as absence of $n$-dependent phases for various $n$-point functions, as long as one assumes real coupling constants in the bulk Lagrangian.

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1 Introduction

The dS/CFT correspondence [1-3] proposes that quantum gravity in asymptotically de Sitter space is dual to a Euclidean conformal field theory which lives on $I^+$ or $I^-$. Specifically, it has been proposed that the partition function of the CFT deformed by single trace operators (which equals the generating functional for correlators of the CFT) is the Bunch-Davies wavefunctional obtained by performing the bulk path integral with Dirichlet boundary conditions on $I^+$ and Bunch-Davies condition in the infinite past. Unlike in AdS/CFT [5-8], the meaning of this correspondence is not completely clear, particularly because of the difficulty in defining observables in de Sitter space [1]. While these issues are obviously important, one can nevertheless perform computation in the dS bulk where gravity is treated semiclassically [3]. Keeping this in view, in this note we will address the question: if a dS/CFT correspondence does exist, what does it say about the dual field theory?
To begin with, the dual field theory cannot be unitary in the usual sense [3, 9]. The symmetry group of the putative $d$-dimensional Euclidean CFT, $SO(d + 1, 1)$, is the isometry group of both $dS_{d+1}$ and Euclidean $AdS_{d+1}$. If the CFT is unitary, one would expect that the dual is a bulk theory living in Euclidean $AdS_{d+1}$. Thus, the CFT dual to $dS_{d+1}$ is non-unitary. On the other hand, there is a unitary time evolution in the $dS_{d+1}$ bulk (examples of which we will consider explicitly below); if the holographic correspondence is true, this will clearly imply some constraints on the dual field theory. In this note, we will explore these constraints on the RG flow of double and triple trace deformations in the dual field theory. For double trace couplings, the story for AdS is well known [15, 17, 19]: for a relevant deformation with positive coupling, the theory flows into a IR fixed point, in complete agreement with the prediction of the dual large-N field theory.

We will calculate the beta function for the double and triple trace couplings of a proposed CFT dual to de Sitter space using the holographic renormalization group techniques of [20] and [21] (for previous work on the subject, see [22]-[29]). We will show that the beta function has the same structure as that expected from general field theory considerations, along with holographically determined coefficients. In particular the coefficient of the quadratic term of the double trace beta function equals the normalization of the two point function; similar statements are true for the triple trace beta function. For $dS_4$, we find that the specific choice of operator normalization which leads to real $n$-point correlation functions [9] also leads to beta functions with real coefficients. This leads to a beta function whose quadratic term differs in sign from that in Euclidean $AdS_4$, so that the IR fixed point now appears at negative rather than positive coupling. The recent proposal of a duality between $Sp(N)$ vector models in three Euclidean dimensions and Vasiliev theory in $dS_4$ [9]-[14] provides a specific realization of the above result.

For $dS_{d+1}$ with even $d$, however, we find, first of all, that no choice of operator normalization exists which ensures absolute reality of the $n$-point functions; furthermore, any choice of operator normalization which ensures reality of coefficients of the beta-functions forces us to have $n$-point functions with very specific $n$-dependent complex phases, $\langle O_1 \cdots O_n \rangle \sim i^{(n-2)(1-d)/2}$ as explained in Section 7. These assertions are proved in Section 7 under the general condition of real coupling constants in the bulk Lagrangian. It is important to note that the reality of the coefficients of the bulk Lagrangian, which is tied to the unitarity of the bulk field theory, plays a crucial role here.
2 The main result

In this section we first derive the field theory beta function at leading order of $1/N$. We then summarize our findings for the holographic beta function.

2.1 Field theory: 2-pt function vs. double trace beta-function

Consider the two-point function of an operator $O(x)$ in a $d$-dimensional Euclidean CFT:

$$\langle O(k_1)O(k_2)\rangle_0 = \mathcal{G}_0(k)(2\pi)^d\delta(k_1 + k_2), \quad \mathcal{G}_0(k) = bk^{-2\nu}, \quad 2\nu \equiv d - 2\Delta \quad (2.1)$$

where $O$ is a scalar operator of dimension $\Delta$. The exponent of $k$ follows from dimensional analysis; the subscript 0 implies that the correlator is computed in the unperturbed CFT. The constant $b$ denotes the normalization of the operator $O$.

In the following we will assume that, for large central charge $c$ of the CFT, the leading contribution to the $2n$-point function of $O$ has a factorized form (similar to Wick’s theorem):

$$\langle O(k_1)O(k_2)....O(k_{2n})\rangle = \left[\sum_{\text{permutations}} \langle O(k_{i_1})O(k_{i_2})\rangle...\langle O(k_{i_{2N-1}})O(k_{i_{2N}})\rangle\right] + ... \quad (2.2)$$

where the ... terms at the end denote $O(1/c)$ corrections. Well-known CFT’s with such properties are conformal large $N$ gauge theories with $O$ a single trace operator (or conformal large $N$ vector theories with $O$ some appropriate bilinear of vectors).

This has the following consequences:

1. The dimension of the “double trace” operator $O^2$ is $2\Delta$. \footnote{In the context of (A)dS/CFT, we will consider alternative quantization, where $O$ will be identified with $O_-$, as in (3.15). In that case, $\Delta = \Delta_-$ (see (3.6)), and the value of $\nu$ follows the usual definition. Among other things, the choice of alternative quantization ensures that the double trace flow is relevant.}

2. Under a double trace deformation ($f_0$ is a bare coupling)

$$S = S_0 + \frac{f_0}{2} \int d^dxO(x)^2 \quad (2.3)$$

\footnote{For more general examples, see, e.g. [10].}

\footnote{We will call $O$ and $O^2$ “single trace” and “double trace” operators, respectively, by analogy with large $N$ gauge theories; however, at least for the purposes of this section, this only implies the factorization property \footnote{2.2}.}
the Green’s function (2.1) changes to

$$G_f(k) = G_0(k) - f_0 G_0(k)^2 + ... = \frac{G_0(k)}{1 + f_0 G_0(k)} \tag{2.4}$$

We will derive the same equation in (4.4) from a dS bulk dual.

The above Green’s function implies the following ‘running coupling constant’

$$f(k) = \frac{f_0}{1 + f_0 G_0(k)} \tag{2.5}$$

Let us define a dimensionless renormalized coupling $\lambda(\mu)$ by the relation

$$f(\mu) = \lambda(\mu) \mu^{2\nu} \tag{2.6}$$

By using the above equations, we get

$$\lambda(\mu) = \frac{f_0}{\mu^{2\nu} + f_0 b}$$

Since $f_0$ is a bare coupling, it should not depend on $\mu$. By differentiating the above with respect to $\mu$, we get

$$\mu \frac{d\lambda(\mu)}{d\mu} = -2\nu \lambda + 2 \nu b \lambda^2 \tag{2.7}$$

At this stage the constant $b$ is arbitrary and is not necessarily real; holography allows us to determine the value of $b$, as in (3.18) (for a dS/CFT) where $b$ is complex and (3.21) (for AdS/CFT) where $b$ is real and positive. For unitary theories, on general grounds, $b$ must be real and positive and we have the well-known result that the theory flows to a IR fixed point at positive coupling.

Note that we have arrived at (2.7) with minimal assumptions about the CFT and about the operator $O$ (essentially its scaling and factorization).

### 2.2 Bulk dual

Let us now assume that our CFT has a bulk dual. The $SO(d+1,1)$ conformal symmetry implies that the bulk must be either $AdS_{d+1}$ or $dS_{d+1}$. A double trace deformation then translates to modified boundary conditions for the dual bulk field [15]. For $\nu > 0$ in

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7This is easy to derive by expanding $\exp[-S] = \exp[-S_0](1 - S_{int} + \frac{1}{2} S_{int}^2 - ...)$, and using (2.2).

8We define the running coupling $f(k)$ by $G_f(k) = G_0(k) - f(k) G_0(k)^2$ (thus $f(k)$ represents the Dyson Schwinger sum of an infinite number of Feynman diagrams in the middle expression of (2.4)).

9Note that $f(k)$ is of dimension $2\nu \equiv d - 2\Delta$ since $O^2$ is of dimension $2\Delta$. 

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the deformation has to be around alternative quantization. Following the procedure of integrating out geometry devised in [20] and [21] we will derive the beta-function of the field theory from bulk Schrodinger equations. For AdS the time in the Schrodinger equation is euclidean and identified with the radial coordinate, which is identified with the RG scale of the field theory: this derivation is already contained in [20][21]. For dS, bulk evolution is in real time, and the precise relationship of time with the field theory scale is less clear. If T denotes the bulk time in inflationary coordinates (which in our convention is negative), we will find that the beta function (2.7) is again reproduced, provided we identify (−iT) with the RG scale of the dual theory.

We will find below that, the equation (2.4) is reproduced holographically both in the case of AdS and dS (see (4.3) and (4.4)). Further, with the above holographic identification of the field theory cut-off, the beta-function (2.7) is reproduced exactly in both cases. For dS, unlike in AdS we cannot demand that b > 0 or even real in the field theory. However, for dSd it was argued in [9] that the only way to ensure real n point functions is to have b real and negative. This is the normalization used in [3] as well. This leads to the conclusion that the IR fixed point of the dual theory is at negative coupling. This is consistent with the conjecture of [9]: indeed a calculation of the beta function of Sp(N) field theory leads to the same beta function (this has been calculated to one loop in [33]).

However, for dSd+1 with even d, as explained at the end of the Introduction, reality of b is only possible if one allows for specific n-dependent complex phases of the n-point correlation functions (see Section 7 for details).

3 Holographic dictionaries

3.1 dS/CFT dictionary

We will consider the inflationary patch of dSd+1 with a metric

\[ ds^2 = \frac{L_{dS}^2}{T^2} [-dT^2 + d\vec{x}^2] \]

with \(-\infty \leq T \leq 0\) We will consider a massive minimally coupled scalar in this geometry with the action

\[ S_\epsilon = S_{gr} + \frac{1}{2G_N} \int_{-\infty}^{\epsilon} dT \int d^d x \left( \frac{L_{dS}}{T} \right)^{d+1} \left[ \left( \frac{-T}{L_{dS}} \right)^2 (\partial_T \phi)^2 - (\nabla \phi)^2 \right] - m^2 \phi^2 \]

where \(S_{gr}\) is the gravity action and \(\epsilon\) is a cutoff. In the following we will consider the dynamics of the scalar - we will therefore drop the gravity part. We will work in a probe
approximation and ignore the backreaction on gravity. A bulk wavefunction can be now defined by the path integral

\[ \Psi[\phi_0(\vec{x}), \epsilon] = \int_{\phi(\epsilon, \vec{x}) = \phi_0(\vec{x})} \mathcal{D}\phi(T, \vec{x}) \exp (iS_\epsilon) \]  (3.3)

where the field satisfies Bunch-Davies conditions at \( T = -\infty \). \( S_\epsilon \) is the action obtained by integrating from \( T = -\infty \) to \( T = \epsilon \), and \( \epsilon < 0 \).

In the following we will use a notation

\[ \rho \equiv \sqrt{\frac{L^{d-1}}{G_N}} \]  (3.4)

The dS/CFT correspondence as interpreted in \cite{3, 9, 14} then claims that this wavefunctional is related to the partition function of a dual CFT in the presence of a source. More precisely, in the standard quantization of the CFT

\[ \langle \exp \left[ \int d^d x \, \phi(\vec{x}) \mathcal{Z}(\epsilon) \mathcal{O}_+(\vec{x}) \right] \rangle_{st} = \Psi[\phi_0(\vec{x}), \epsilon], \quad \mathcal{Z}(\epsilon) = \rho \frac{\gamma}{\sqrt{\epsilon}} (-i\epsilon)^{-\Delta_-} \]  (3.5)

where

\[ \Delta_\pm = d/2 \pm \nu, \quad \nu = \sqrt{d^2/4 - m^2L_{ds}^2} \]  (3.6)

Here \( \mathcal{Z}(\epsilon) \) is a normalization factor used to define the GKPW relation \( 3.5 \). The important part of this factor is the numerical coefficient \( \gamma \) which we treat \( a \ priori \) to be complex. This constant is taken to be \( \gamma = 1 \) in \cite{4, 9, 14}. We will come back to a detailed discussion of this coefficient later. Note that the factor \((-i\epsilon)\) is naturally identified with the field theory UV cutoff \cite{9}.

We will be concerned with the semiclassical limit where the functional integral on the right hand side of \( 3.3 \) can be evaluated by saddle point. The classical solution which satisfies the Bunch-Davies condition at \( T = -\infty \) and the specified boundary condition at \( T = \epsilon \) is given, in momentum space, by

\[ \phi(T, k) = \left( \frac{T}{\epsilon} \right)^{d/2} \frac{H^{(2)}_\nu(-kT)}{H^{(2)}_\nu(-k\epsilon)} \phi_0(\vec{k}) \]  (3.7)

where \( \nu \) is given by \( 3.6 \). This leads to the following on-shell action

\[ iS_{on} = -\frac{i}{2G_N} \int [dk] L_{ds}^{d-1} \left( \frac{\Delta_-}{(-\epsilon)^d} - \frac{k\epsilon H^{(2)}_{\nu-1}(-k\epsilon)}{(-\epsilon)^d H^{(2)}_\nu(-k\epsilon)} \right) \phi_0(\vec{k})\phi_0(-\vec{k}) \]  (3.8)
At late times \( k|\epsilon| \ll 1 \)
\[
\begin{align*}
    iS_{\text{on}} &= -i \frac{\rho^2}{2} \int [dk] \left( \frac{\Delta_-}{(-\epsilon)^d} - \frac{\Gamma(1 - \nu)}{\Gamma(2 - \nu) 2(-\epsilon)^{d-2}} \right) \phi_0(k) \phi_0(-k) \\
    &+ \frac{\rho^2}{2} \int [dk] \phi_0(k) \phi_0(-k)(-i\epsilon)^{-2\Delta-} H(k)
\end{align*}
\] (3.9)
where
\[
H(k) = (i)^{d-1} C_1(\nu) k^{2\nu}, \quad C_1(\nu) \equiv -2\nu \frac{\Gamma(1 - \nu)}{\Gamma(1 + \nu) 2^{-2\nu}}
\] (3.10)

In the semiclassical limit the wavefunction is then
\[
\Psi[\phi_0(\vec{x}), \epsilon] \sim \exp[iS_{\text{on}}]
\] (3.11)

It may be easily checked that at early times \( k|\epsilon| \gg 1 \) this reproduces the ground state of a bunch of harmonic oscillators with “coordinates” \( \chi_\epsilon(k) = (-\epsilon) \frac{i d}{2} \phi(k) \). At late times \( k|\epsilon| \ll 1 \) we need to remove the divergent piece by holographic renormalization and define the wavefunction by
\[
\Psi[\phi_0(\vec{x}), \epsilon] \sim \exp[iS'_{\text{on}}]
\] (3.12)
where
\[
iS'_{\text{on}} = \frac{L_{dS}^{-1}}{2 G_N} \int [dk] \phi_0(k) \phi_0(-k)(-i\epsilon)^{-2\Delta-} H(k)
\] (3.13)
is the finite part of the on-shell action. The divergent first term in (3.9) has to be removed by addition of a counterterm to the action. Using (3.5) the two point correlator of the dual operator \( O^+ \), is given by
\[
\langle O^+(k) O^+(-k) \rangle_{\text{st}} = G_{\text{st}}(k) = \gamma H(k) = \gamma i^{d-1} C_1(\nu) k^{2\nu}
\] (3.14)

We will be interested in alternative quantization. The generating functional for correlators in the appropriate CFT in this case is obtained by extending the corresponding prescription in AdS [16],
\[
\langle \exp \left[ \int d^d x J(\vec{x}) O^-(\vec{x}) \right] \rangle_{\text{alt}} = \int D\phi_0(\vec{x}) \langle \exp \left[ \int d^d x \phi_0(\vec{x}) Z(\epsilon) O^+(\vec{x}) \right] \rangle_{\text{st}} \exp \left[ Z(\epsilon) \int d^d x \frac{J(\vec{x})}{2\nu} \phi_0(\vec{x}) \right]
\] (3.15)

In the semiclassical approximation we may replace the generating functional of standard quantization by the wavefunction (3.12). Performing the \( \phi_0 \) integral leads to a two point correlator in alternative quantization
\[
G_{\text{alt}}(k) = \frac{\delta^2}{\delta J(k) \delta J(-k)} \langle e^{\int d^d x J(\vec{x}) O^-(\vec{x})} \rangle_{\text{alt}} = -\frac{1}{(2\nu)^2 G_{\text{st}}(k)}
\] (3.16)
This inverse relation between the Green’s function is exactly the same as in AdS/CFT [16].

Combining (3.16), (3.14) and (3.10) we get

\[
\langle O_-(k)O_-( -k) \rangle_{\text{alt}} = G_{\text{alt}}(k) = \frac{i^{1-d}}{\gamma} C(\nu) k^{-2\nu}, \quad C(\nu) \equiv \frac{2^{2\nu} \Gamma(1 + \nu)}{(2\nu)^2 \Gamma(1 - \nu)}
\]  

(3.17)

Comparing with (2.1), we get the following holographically determined value of \( b \):

\[
b_{dS} = \frac{i^{1-d}}{\gamma} C(\nu)
\]  

(3.18)

In case of \( dS_4 \), Ref. [9] chose \( \gamma = 1 \) in keeping with the reality of the \( n \)-point functions, which was also reproduced by a CFT calculation using \( SP(N) \). However, in this paper we are dealing with \( dS_{d+1} \) for arbitrary \( d \) and will keep \( \gamma \) arbitrary and in principle complex. We will come back to the important issue of the phase of \( \gamma \) (equivalently of \( Z \)) and its relation to the phases of the \( n \)-point functions and beta-function coefficients in detail in Section 7.

The relationship (3.16) can be inverted to rewrite the Bunch-Davies wavefunction in terms of the generating functional in alternative quantization,

\[
\Psi[\phi_0(\vec{x}), \epsilon] = \int \mathcal{D}J(\vec{x}) \exp \left[ -Z(\epsilon) \int d^d x \frac{J(\vec{x})}{2\nu} \phi_0(\vec{x}) \right] \langle \exp \left[ \int d^d x J(\vec{x}) O_-(\vec{x}) \right] \rangle_{\text{alt}}
\]  

(3.19)

### 3.2 The formulae for AdS

It will be useful to record the corresponding well known formulae in euclidean AdS space. The GKPW prescription for the generating functional for correlators in standard quantization reads

\[
\langle \exp \left[ \int d^d x (\epsilon)^{-\Delta} \phi_0(\vec{x}) \tilde{Z}(\epsilon) O_+(\vec{x}) \right] \rangle_{st} = Z[\phi_0(\vec{x}), \epsilon] \tilde{Z}(\epsilon) \equiv \frac{\rho}{\sqrt{\gamma}} (\epsilon)^{-\Delta}
\]  

(3.20)

where we of course need to replace \( L_{dS} \rightarrow L_{AdS} \). There are no factors of \( i \) in the formulae, the rescaling factor \( \gamma \) has to be real, the Hankel functions are replaced by Modified Bessel functions and the quantity in square brackets in (3.10) is the boundary Green’s function in \( AdS_{d+1} \) leading to the proportionality constant

\[
b_{AdS} = \frac{1}{\gamma} C(\nu)
\]  

(3.21)

where \( C(\nu) \) is defined in (3.17). Since everything needs to be real, (3.20) requires \( \gamma \) to be real and positive, leading to a real positive \( b_{AdS} \). Finally, the analog of (3.19) for \( AdS \) may be obtained by replacing \((-i\epsilon) \rightarrow \epsilon\).
4 Double Trace deformations

In the following we will be interested in the deformation of the CFT dual to alternative quantization in $dS_{d+1}$ by a double trace operator. The Euclidean field theory action is given by (2.3). As argued in Sec 2.1 to leading order in large $N$, the dimension of $O^2$ is then $2\Delta$. We require the perturbation to be relevant, which means that the CFT action $S_0$ must correspond to alternative quantization (see also footnote 4), ensuring that $2\Delta = 2\Delta_-$ (see (3.6)). The generating function for correlators in the presence of the deformation may be now written using a Hubbard-Stratanovich transformation,

$$\langle \exp \left[ \int d^d x J(\vec{x}) O(\vec{x}) \right] \rangle^{f_0}_{alt} = \int D\sigma \exp \left[ \frac{1}{2f_0} \int d^d x \sigma(\vec{x})^2 \right] \langle \exp \left[ \int d^d x (J(\vec{x}) + \sigma(\vec{x})) O(\vec{x}) \right] \rangle^{alt}$$  \hspace{1cm} (4.1)

where the notation $\langle ... \rangle^{f_0}_{alt}$ denotes correlations in presence of the double trace deformation (2.3). Using (3.13) and performing the integral over $\phi_0$ this leads to the prediction that the deformed CFT has a Green’s function

$$G_f(k) = \frac{G_{alt}(k)}{1 + f_0 G_{alt}(k)}$$  \hspace{1cm} (4.4)

This relation can be of course obtained directly from the large-N field theory (2.3) (see Eq. (2.4)). The holographic derivation of this formula is a consistency check on the above dS/CFT prescription.

5 Holographic RG

We now adapt the holographic renormalization group procedure developed in [20, 21] to de Sitter space. we rewrite the right hand side of (3.3) by introducing a floating cutoff at $T = l$,

$$\Psi[\phi_0(\vec{x}), \epsilon] = \int D\tilde{\phi}(\vec{x}) \Psi_{IR}[\tilde{\phi}, l] \Psi_{UV}[\tilde{\phi}, \phi_0]$$  \hspace{1cm} (5.1)

where

$$\Psi_{IR}[\tilde{\phi}] = \Psi[\tilde{\phi}(\vec{x}), l]$$  \hspace{1cm} (5.2)
and
\[ \Psi_{UV}[\tilde{\phi}, \phi_0] = \int_{\phi(l, \vec{x})}\mathcal{D}\phi(T, \vec{x}) \exp \left( i \int^l dT L \right) \]
(5.3)
where \( L \) is the Lagrangian.

The idea is now to obtain an effective action of the dual theory at a finite cutoff \( l \) by extending the dS/CFT relationship (3.19) for \( \Psi_{IR}[\tilde{\phi}, l] \),
\[ \langle e^{-S_{eff}(l)} \rangle_{alt} = \int \mathcal{D}\tilde{\phi}(\vec{x}) \int \mathcal{D}J(\vec{x}) \Psi_{UV}[\tilde{\phi}, \phi_0] \exp \left[ -Z(l) \int d^d x J(\vec{x}) \frac{\partial \tilde{\phi}}{\partial(\partial l)} \right] \langle \exp \left[ \int d^d x J(\vec{x}) \mathcal{O}_-(\vec{x}) \right] \rangle_{alt} \]
where \( Z(l) \) is defined as in (3.5), with \( \epsilon \) replaced by \( l \). This relates the parameters in \( \Psi_{UV} \) to couplings in the effective action. The expression for \( \langle e^{\int d^d x J(\vec{x}) \mathcal{O}_-(\vec{x})} \rangle_{alt} \) in terms of bulk quantities in (3.12) and (3.15) are valid in the \( \epsilon \to 0 \) limit. When we use these expressions for finite \( l \), there is a freedom of choosing counterterms [31,32]. We will stick to the counterterm implied in (3.12), and comment on the implications of this freedom later.

From the definition (5.3), \( \Psi_{UV} \) satisfies a Schrodinger equation with the Hamiltonian derived from the Lagrangian,
\[ iG_N \frac{\partial}{\partial(\partial l)} \Psi_{UV}(\tilde{\phi}, l) = H(l)\Psi_{UV}(\tilde{\phi}, l) \]
(5.4)
which give flow equations for the parameters in \( \Psi_{UV} \) and hence couplings in the effective action. The negative sign in the left hand side of (5.4) comes because time evolution corresponds to decreasing \( l \), which appears as the lower limit of integration in (5.3).

For the free scalar field we are considering the hamiltonian at some time slice \( T \) is given by
\[ H(T) = \frac{1}{2} \int d^d x \left[ -G^2_N \left( \frac{-T}{L_{ds}} \right)^{d-1} \left( \frac{\partial^2}{\partial^2 \phi} \right) + \left( \frac{L_{ds}}{-T} \right)^{d-1} (\nabla \phi)^2 + \left( \frac{L_{ds}}{-T} \right)^{d+1} m^2 \phi^2 \right] \]
(5.5)
In the semiclassical limit \( G_N \ll L_{ds}^{d-1} \) the Schrodinger equation reduces to a Hamilton-Jacobi equation. For a wavefunction
\[ \Psi_{UV} = \exp[iK] \]
(5.6)
the Hamilton-Jacobi equation is given by
\[ \frac{1}{2} \left[ G^2_N \left( \frac{-l}{L_{ds}} \right)^{d-1} \left( \frac{\partial K}{\partial \phi} \right)^2 + \left( \frac{L_{ds}}{-l} \right)^{d-1} (\nabla \phi)^2 + \left( \frac{L_{ds}}{-l} \right)^{d+1} m^2 \phi^2 \right] + G_N \frac{\partial K}{\partial(-l)} = 0 \]
(5.7)
Consider now a general quadratic form for $K$

$$K = \frac{1}{G_N} \left( \frac{L_{dS}}{-l} \right)^d \int d^d x \left[ -\frac{1}{2L_{dS}} g(l) \tilde{\phi}^2 + h(l) \tilde{\phi} + c(l) \right] \quad (5.8)$$

Note that the parameters in (5.8) depend on the cutoff $l$. The flow equations for these parameters follow from substituting (5.8) in (5.7). For consistency we really need to replace these parameters by space-dependent parameters (e.g. $g(x)$). However as shown in [20] and [30] the flow equations for the zero momentum modes of these couplings decouple from the non-zero momentum modes. With this understanding,

$$\beta_g = -(-il) \frac{\partial g}{\partial (-il)} = -g^2 - dg - m^2 L_{dS}^2$$

$$\beta_h = -(-il) \frac{\partial h}{\partial (-il)} = -h(g + d) \quad (5.9)$$

As is clear from the discussion of [32] and [31], the freedom of choosing different counterterms at finite $l$ modifies the last term in the first equation of (5.9). We have written the equations (5.9) using $(-il)$ as a cutoff scale. This is a natural choice (as will be discussed further below).

The zeroes of $\beta_g$ are at $g = \pm -\Delta$ and alternative quantization means we have to expand the coupling as

$$g = g_- + \delta g \quad (5.10)$$

The beta function for $\delta g$ is given by

$$\beta_{\delta g} = -(-il) \frac{\partial \delta g}{\partial (-il)} = -2\nu(\delta g) - (\delta g)^2 \quad (5.11)$$

To relate this flow equations to beta functions of the dual field theory we need to establish a relationship between $g, f$ and the couplings of the field theory. This may be done by substituting (5.8) in (5.4) and performing the integrals over $J(\vec{x})$ and $\tilde{\phi}(\vec{x})$ by saddle point method. This leads to a field theory effective action

$$S_{eff} = \frac{f^2}{2} \int d^d x \; O_-^2 + j \int d^d x \; O_- + c \quad (5.12)$$

where

$$j = -2\nu \sqrt{\frac{L_{dS}^{d+1}}{G_N} \gamma} (i)^{d+1} (-il)^{-\Delta+} h(l) \quad (5.13)$$

$$f = (i)^{d+1} (-il)^{-2\nu(2\nu)^2} \gamma = -(2\nu)^2 C(\nu) \frac{1}{b_{dS}} (-il)^{-2\nu} g \quad (5.14)$$
and $c$ is a constant independent of the operator $O$. In the above we have used the expression for $b_{dS}$ in (3.18).

The fixed point values of the parameter $g$ simply corresponds to the minimal counterterm in the bulk action. The field theory couplings, which are defined as departures from a CFT have to be related to the departure from the fixed point.

The couplings $f, j$ and hence $\delta f$ and $\delta j$ have the appropriate dimensions $2\nu$ and $\Delta_+$ respectively, as is clear from the powers of $l$ which appear in (5.14). The beta functions of the field theory are, however, those of dimensionless couplings. In the field theory this is done by multiplying by an appropriate power of the cutoff or renormalization scale, as in (2.6). In the holographic setup this requires specifying a relationship between the cutoff in the bulk with a UV cutoff on the boundary. As is quite clear from all the formulae above, it is natural to identify $(-il)$ as the renormalization scale $\mu$ of the field theory. Let us identify the field theory renormalization scale $\mu$ to be $a^{1/(2\nu)}$ times the holographic cut-off scale $1/(-il)$, for some positive constant $a$. With this choice, we have the following identification of the dimensionless coupling of the field theory $\lambda$ with the departure from the fixed point,

$$\lambda a \left( (-il)^{-1} \right)^{2\nu} \equiv \delta f = -(2\nu)^2 C(\nu) \frac{1}{b_{dS}} (-il)^{-2\nu} \delta g$$

where we have used (2.6). Making the convenient choice $a = (2\nu)^3 C(\nu)$ (which gives a specific choice of the field theory renormalization scale), we get

$$\delta g = -2\nu b_{dS} \lambda$$

(5.16)

Substituting this in (5.11) finally leads to a beta function for $\lambda$

$$\beta_\lambda = -2\nu \lambda + 2\nu b_{dS} \lambda^2 = -2\nu \lambda + 2\nu \frac{i^{1-d}}{\gamma} C(\nu) \lambda^2$$

(5.17)

This is the same as the general field theory answer, (2.7).

As we have remarked above and will discuss in detail in Section 7, the requirement that there are no relative phases between various $n$-point functions of the dual field theory implies that $b_{dS} \sim i^{d-1}$. This implies, in turn, that for even $d$ we have purely imaginary $b_{dS}$ and hence a complex beta function.

\[
\text{It is clear from our discussion in Section 7 that under no circumstance can the n-point functions be all real.}
\]
5.1 Results in AdS

For comparison let us recall the results of the above analysis in euclidean AdS. In this case the range of the radial coordinate is $0 \leq z \leq \infty$. The radial evolution equation satisfied by $\Psi_{UV,AdS}$ is

$$G_N \frac{\partial}{\partial(l)} \Psi_{UV,AdS}(\tilde{\phi}, l) = -H_{AdS}(l) \Psi_{UV,AdS}(\tilde{\phi}, l)$$

(5.18)

where

$$H(l) = \frac{1}{2} \int d^d x \left[ -G_N^2 \left( \frac{l}{L_{AdS}} \right)^{d-1} \frac{\delta^2}{\delta \tilde{\phi}^2} + \left( \frac{L_{AdS}}{l} \right)^{d-1} (\nabla \tilde{\phi})^2 + \left( \frac{L_{AdS}}{l} \right)^{d+1} m^2 \tilde{\phi}^2 \right]$$

(5.19)

With the form

$$\Psi_{UV,AdS} = \exp \left[ \frac{1}{G_N} \left( \frac{L_{AdS}}{l} \right)^d \int d^d x \left[ - \frac{1}{2L_{AdS}} g'(l) \tilde{\phi}^2 + h'(l) \tilde{\phi} + c'(l) \right] \right]$$

(5.20)

which leads to the flow equation

$$l \frac{\partial g'}{\partial l} = -(g')^2 - dg' + m^2 L_{AdS}^2$$

(5.21)

The expressions for the fixed points are changed appropriately, but the flow equation for the departure from the fixed point $\delta g'$ is, instead of (5.11)

$$\beta_{\delta g'} = -l \frac{\partial \delta g'}{\partial l} = -2\nu(\delta g') + (\delta g')^2$$

(5.22)

Finally the relationship between the field theory dimensionless coupling and $\delta g'$ is

$$\lambda = \gamma (2\nu)^2 \delta g' = (2\nu)^2 \frac{\Gamma(1+\nu)}{\Gamma(1-\nu)} \frac{1}{b_{AdS}} \delta g'$$

(5.23)

which leads once again to a beta function of the expected form (2.7)

6 Beta function of Triple and Higher trace couplings

In this section we will discuss a generalization of the above methods to derive the holographic beta-function of triple and higher trace couplings (in perturbation theory). We will be brief, emphasizing mainly the new features.

For concreteness, we will focus on triple trace couplings, of the form $O^3_3$; however, the generalization to higher trace operators is straightforward. Triple trace operators are
induced in a holographic RG, as we will see, when the dual scalar field theory has a cubic coupling

\[\Delta S \sim \int_{-\infty}^{\epsilon} dT \int d^d x \left( \frac{L_{ds}}{-T} \right)^{d+1} \left[ -\frac{r}{3} \phi^3 \right] \]  

in additional to the quadratic action \((3.2)\). The Hamilton-Jacobi equation \((5.7)\) gets modified by the addition of a cubic term

\[
\left( \frac{L_{ds}}{-l} \right)^{d+1} \frac{2r}{3} \phi^3
\]

to the term inside the square bracket. It is easy to see that a quadratic ansatz for the kernel \(K\) such as \((5.8)\) will not satisfy such a Hamilton-Jacobi equation. Let us, therefore, take \(K\) to be cubic, viz. of the form

\[
K = \rho^2 L e^{-d} \int d^d x \left( -\frac{g}{2L} \tilde{\phi}^2 + h \tilde{\phi} + cL + \frac{A}{L^2} \tilde{\phi}^3 \right)
\]  

\((6.2)\)

By repeating the steps leading to \((5.9)\), and equating the coefficients of \(\tilde{\phi}, \tilde{\phi}^2\) and \(\tilde{\phi}^3\) in the Hamilton-Jacobi equation\(^{11}\) we now get the following cut-off dependence of the couplings in \((6.2)\)

\[
\begin{align*}
\beta_g &= -g^2 - d g - \bar{m}^2 - 2hA, \quad \bar{m} = mL_{ds}, \\
\beta_A &= (-3g - d)A + 3\bar{r}, \quad \bar{r} = rL_{ds}^3, \\
\beta_h &= (-g - d)h
\end{align*}
\]  

\((6.3)\)

Note that this generalizes \((5.9)\), and reduces to it for \(A = 0\). It is easy to find the following UV fixed point (near which \(\beta_g\) is negative):

\[
h_c = 0, \quad g_c = -\Delta, \quad A_c = 3\bar{r}/(d - 3\Delta)
\]  

\((6.4)\)

The linearized beta-functions for the deformations \(\delta h, \delta g\) and \(\delta A\) (measured from this fixed point) are

\[
\begin{align*}
\beta_{\delta g} &= -2\nu \delta g, \quad \beta_{\delta A} = (3\Delta - d)\delta A, \\
\beta_{\delta h} &= (\Delta - d)\delta h
\end{align*}
\]  

\((6.5)\)

\(^{11}\)Our approach here is perturbative; the Hamilton-Jacobi analysis generates \(\tilde{\phi}^4\) terms. We imagine them to be taken care of by higher couplings, and focus here on couplings up to cubic order. It is straightforward, although cumbersome, to write more general beta-functions involving arbitrary Wilso-
How does one read off the field theory beta-functions from these? We can, once again, use (5.4), and show that it leads to a field theory with the following effective action

$$S_{\text{eff}} = \int d^d x \left( \frac{f}{2} O_-^2 + j O_0 + \frac{B}{3} O_-^3 + c \right)$$  \hspace{1cm} (6.6)$$

where

$$j = -(i)^{d+1} (-il)^{-\Delta + h(l) 2\nu} \sqrt{\frac{L_{d+1}^d}{G_N}}$$

$$f = (i)^{d+1} (-il)^{-2\nu (2\nu)^2} g(l) \gamma$$

$$B = -(i)^{d+1} (-il)^{3\Delta - d (2\nu)^3} A(l) \gamma \sqrt{\frac{\gamma G_N}{L_{d+1}^d}}$$  \hspace{1cm} (6.7)$$

which generalizes the equation (5.14) encountered for double trace couplings. The beta-function for the field theory couplings $f, j, B$ can easily be read off from the above identifications (6.7) with the bulk couplings $g, h, A$ and their beta-functions (6.3) or (6.5).

The beta function for the dimensionless cubic trace coupling $(\delta \bar{B})$, which measures the deviation from the fixed point, turns out to be,

$$\beta_{\delta \bar{B}} = -3\Delta \delta \bar{B} + \frac{i^{1-d}}{(2\nu)^2 \gamma} \delta \bar{f} \delta \bar{B} = -3\Delta \delta \bar{B} + 3 b_{dS} \delta \bar{f} \delta \bar{B} \frac{\Gamma(1 - \nu)}{2^{2\nu (2\nu) \Gamma(1 + \nu)}}$$  \hspace{1cm} (6.8)$$

where, $\delta \bar{f}$ is the deviation of the dimensionless double trace coupling from the fixed point.

One can easily check that the field theory beta-functions have the correct form. E.g., $\beta_{\delta \bar{B}}$ includes a term $\propto \delta \bar{f} \delta \bar{B}$; to see this from a field theory reasoning, one needs to simply note that the three-point function $\langle O_0(x) O_0(y) O_0(z) \rangle$ has a perturbative expansion of the schematic form $\delta \bar{B} \int d^d w G_0(x - w) G_0(y - w) G_0(z - w) + \delta \bar{f} \delta \bar{B} \int d^d w d^d w' G_0(x - w) G_0(y - w) G_0(z - w') G_0(w - w')$ (where we have shown only the first two terms). Using large $N$ methods, one can organize such perturbation expansions [15, 17, 19].

Significantly, the beta function for $A$ does not have an $A^2$ term (in field theory terms, $\beta_{\delta \bar{B}}$ does not have a $\delta \bar{B}^2$ term), and is in fact the same as in $AdS$. For the special case where $d = 3, \Delta_- = 1$ (which implies $\bar{m}^2 = 2, \Delta_+ = 2, \nu = 1/2$) the linearized beta-function indicates correctly the fact that the cubic coupling is marginal [12]. This is consistent with the known field theory result for vector models that a $[(\vec{\phi})^2]^3$ coupling acquires a nontrivial beta function only due to $1/N$ corrections. Our holographic result shows that this is a general result in large-$N$ field theories.

\footnote{The fixed point value of $A$ is infinite for these values, as can be seen from (6.4). However, as remarked earlier, the fixed point value of holographic couplings is non-universal as they are affected by the choice of holographic counterterms. The linearized beta-functions (6.5) are free of such non-universalities.}
7 Complex Phases

Here we focus on the structure of complex phases of the $n$-point correlation functions of the field theory. As seen in [9] even with interactions present in the bulk, the overall factor in $iL_{\text{on-shell}}$ is $i^{d-1}$. This implies the following schematic relations for leading order contributions to the first few $n$-point correlation functions,

$$\begin{align*}
i^{d-1} &= Z^2 \langle OO \rangle = \gamma^{-1} b_{dS} \\
r_3i^{d-1} &= Z^3 \langle OOO \rangle \\
r_4i^{d-1} &= Z^4 \langle OOOO \rangle
\end{align*}$$

(7.1)

In these equations, we display only those quantities which possibly contain complex phases. The quantity $Z$ is defined in (3.5); since $-i\epsilon$ has been identified with a real cut-off of the field theory, i.e. $-i\epsilon \propto 1/\Lambda_{\text{UV}}$, $Z$ is essentially equal to $1/\sqrt{\gamma}$ so far as keeping track of complex phases is concerned. Similarly, we have written $\langle OO \rangle \propto b_{dS}$. The left hand sides of the above set of equations are obtained from the bulk; e.g. the LHS of the top equation displays the complex phase of (3.13). The couplings $r_3, r_4$ represent cubic, quartic, etc. couplings of the scalar Lagrangian (e.g. $r_3$ is the same as $r$ in (6.1)).

In keeping with unitarity of the bulk field theory, we will assume that these coefficients are all real. The right hand sides of equations (7.1) are obtained by the GKPW prescription, i.e. by expanding $\Psi[\phi_0(x), \epsilon]$ in (3.5) in powers of $\phi_0(x)$. Now, if we require that there is no relative phase between the correlation functions, i.e., the phase of $\langle O_1 \cdots O_n \rangle$ is the phase of $\langle O_1 \cdots O_{n+1} \rangle$, then we must have $Z$ real. Recalling that $Z(\epsilon) \propto 1/\sqrt{\gamma}$ (where the proportionality constant is positive), the reality of $Z$ implies that $\gamma$ is real. Thus, so far as keeping track of complex phases is concerned, it can be taken to be 1. It then follows immediately that $b_{dS}$ is complex for even $d$ leading to complex beta functions.

Alternatively if we want to require the beta function to be always real, i.e., $b_{dS}$ to be real, then we must choose the phase of $Z$ to be $i^{(d-1)/2}$. However since the phases of the left hand sides of (7.1) are all equal, this will now imply the following relative complex phase,

$$\langle O_1 \cdots O_{n+1} \rangle = i^{(1-d)/2} \langle O_1 \cdots O_n \rangle$$

(7.2)

In particular, since in this choice $\langle OO \rangle$ is real, we get that the phase of $\langle O_1 \cdots O_n \rangle$ is $i^{(n-2)(1-d)/2}$. This clearly shows that we cannot have both the beta function as real and the absence of $n$-dependent phases.
8 Note added

While this paper was in the final stages of its preparation, [34] appeared on the archive, which contains a discussion of the effect of double trace deformations of the free $Sp(N)$ theory in $2+1$ dimensions.

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