Relation to a property of the angular momentum zero space of states of four fermions in an angular momentum $j = 9/2$ shell unexpectedly found to be stationary for any rotationally invariant two-body interaction

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The existence of states with angular momenta $I = 4$ and 6 of four fermions in an angular momentum $j = 9/2$ shell that are stationary for any rotationally invariant two-body interaction despite the presence of other states with the same angular momentum, the Escuderos-Zamick states, is shown to be equivalent to the invariance to any such interaction of the span of states generated from $I = 0$ states by one-body operators. This invariance is verified by exact calculation independently of previous verifications of the equivalent statement. It explains the occurrence of the Escuderos-Zamick states for just $I = 4$ and 6. The action of an arbitrary interaction on the invariant space and its orthogonal complement is analyzed, leading to a relation of the Escuderos-Zamick energy levels to levels with $I = 10$ and 12. Aspects of the observed spectra of $^{96}$Ru, $^{96}$Pd, and $^{54}$Ni are discussed in the light of this relation.

I. INTRODUCTION

Escuderos and Zamick found in a numeric study of the system of four nucleons in an angular momentum $j = 9/2$ shell of a semi-magic nucleus that for each angular momentum $I = 4$ and 6, the states in one irreducible module of the angular momentum algebra, briefly a multiplet, are stationary for any rotationally invariant two-body interaction [1], that is, solvable in Talmi’s terminology [2], despite the presence of other multiplets with the same $I$. It follows that they have definite seniority $v$ [2], which gives rise, in certain nuclei, to particular patterns of transition rates in E2 decay and single-nucleon transfer [3,4]. The solvability of the Escuderos-Zamick states was subsequently confirmed in exact calculations by Van Isacker and Heinzé [4,5], and Qi, Xu and Liotta [6]. These calculation are case-by-case examinations of the individual instances of two-body and total angular momentum, which led the authors of [4] to conclude that “a simple, intuitive reason for [the solvability] is still lacking”. I show below that the existence of the Escuderos-Zamick states is equivalent to a property of the space of $I = 0$ states of the system. The verification of this property again leads to an examination of several cases one by one. The equivalence explains, however, that the solvable multiplets occur for exactly $I = 4$ and 6.

Throughout this paper, $j = 9/2$. Let $\Phi_0$ denote the space of $I = 0$ states of the four-fermion system, and let $a_m$ be the annihilator of a fermion in the state $|jm\rangle$ in the conventional notation [2]. One can then define a space

$$\Phi_4 = \text{span}_{m,m'} a_m^\dagger a_{m'} \Phi_0.$$  

The property to be verified below and shown there to be equivalent to the existence of the Escuderos-Zamick states is the following. $\Phi_4$ is invariant to any rotationally invariant two-body interaction. To see how this explains the appearance of solvable multiplets for just $I = 4$ and 6, note that the tensor operators $T_{IM_I} = \sum_{m,m'} (-)^{j-m'} \langle jm-m' |IM_I|a_m^\dagger a_{m'} \rangle$, where $\langle j_1m_1j_2m_2jm\rangle$ is the vector coupling coefficient [2], form a basis for the span of operators $a_m^\dagger a_{m'}$. The subspace of $\Phi_4$ carrying quantum numbers $I, M_I$ is $T_{IM_I} \Phi_0$. Now consider Table I obtained by a straightforward count of $m$-combinations. Since $\Phi_0$ is 2-dimensional, $T_{IM_I} \Phi_0$ has dimension 2, at most. Angular momenta $I = 4$ and 6 are the only ones allowing more than 2 linearly independent multiplets in the four-fermion system, exactly 3 in both cases. It may be verified by direct calculation, and also follows from a general result in Sec III that in each case, $T_{IM_I} \Phi_0$ is exactly 2-dimensional. If $\Phi_4$ is invariant to a Hermitian and rotationally invariant operator $V$, then so is also $T_{IM_I} \Phi_0$, and so is then also its 1-dimensional orthogonal complement within the space of states with quantum numbers $I, M_I$. This means that the states in the orthogonal complement are eigenstates of $V$.

The proof of equivalence is completed in Sec. III and the verification of the invariance of $\Phi_4$ in Sec. V. Analyzing the actions of an arbitrary $V$ on $\Phi_4$ and its orthogonal complement $\Phi_4^\perp$ reveals remarkable regularities, one of which leads to a rule for relative level spacings that is accessible to experimental verification and so far lacks fundamental explanation. This analysis is the topic of the Secs. VI and VII followed by my conclusion in Sec. VIII. A detail of my formalism is discussed and one other observed regularity explained in two appendices.
II. ANALYSIS

Below, $I_E = 4$ or 6. Important for the following is also the space $\Phi_3$ of states with $I = j$ of three $j = 9/2$ fermions, which is spanned by single multiplets $\Phi_{3v}$ with $v = 1$ and 3. For each $v$, at most one multiplet with a given $I$ can be formed by adding a $j = 9/2$ fermion to the states in $\Phi_{3v}$. This multiplet can be written $P_I \text{span}_m a^\dag_m \Phi_{3v}$, where $P_I$ is the projection onto angular momentum $I$. It may be verified by direct calculation, and also follows from the general result in Sec III that for each $I_E$, these two multiplets are independent. The space $\Phi_{I_E \nu} = P_{I_E} \text{span}_m a^\dag_m \Phi_{31}$ necessarily has $v = 2$. By Table I its orthogonal complement $\Phi_{I_E \beta}$ within $P_{I_E} \text{span}_m a^\dag_m \Phi_3$ then has $v = 4$. The Escudero-Zamick multiplet $\Phi_{I_E \alpha}$ is, by definition, the orthogonal complement of $\Phi_{I_E \beta}$ within the space of states with $I = I_E$ and $v = 4$ of the four fermions. It may be characterized also among such states by vanishing parentage by $\Phi_{33}$. Evidently, it is also the orthogonal complement of $P_{I_E} \text{span}_m a^\dag_m \Phi_3$ within the space of states with $I = I_E$.

In the remainder of this paper, $V$ denotes any rotationally invariant two-body interaction. Because $V$ acts as a scalar on the irreducible module $\Phi_{I_E \alpha}$, the states in $\Phi_{I_E \alpha}$ being eigenstates of $V$ is equivalent to $\Phi_{I_E \alpha}$ being invariant to $V$. By Hermiticity of $V$ and conservation of angular momentum, this is, in turn, equivalent to $P_{I_E} \text{span}_m a^\dag_m \Phi_3$ being invariant to $V$. The space $\text{span}_m a^\dag_m \Phi_3$ cannot contain states with $I > 2j$. For every $I \leq 2j$ except $I = 4$ and 6, it may be verified by direct calculation, and also follows from the general result in Sec III that $P_{I_E} \text{span}_m a^\dag_m \Phi_3$ exhausts the space of states of the four fermions with angular momentum $I$ and thus is invariant to any rotationally invariant operator. Invariance of both spaces $P_{I_E} \text{span}_m a^\dag_m \Phi_3$ to $V$ is then equivalent to $\Phi_4 = \text{span}_m a^\dag_m \Phi_3$ being invariant to $V$. In summary, the existence of the Escudero-Zamick states is equivalent to $\Phi_4$ being invariant to any $V$.

To establish the equivalence stated in the introduction, it remains to show that $\Phi_4$ can be written in the form (1). To this end, notice $\text{span}_m a^\dag_m \phi_0 \subset \Phi_3$. It may be verified by direct calculation, and also follows from the general result in Sec III that the left hand side exhausts $\Phi_3$ so that $\text{span}_m a^\dag_m \phi_0 = \Phi_3$. This evidently leads to the expression (1). The remainder of this paper is dedicated to a proof (independent of the proofs in (3), (1), (2)) of the equivalent statement that $\Phi_4$ as given by (1) is actually invariant to any $V$, and analyses of the actions of an arbitrary $V$ on $\Phi_4$ and its orthogonal complement.

III. SPACES $\Phi_0$ AND $\Phi_4$

The structure of multi-fermion states in the $j = 9/2$ shell is conveniently described in terms of creation operators

$$\alpha_m^+ = \sqrt{(j+m)!/(j-m)!} \alpha_m^+, \quad \text{(2)}$$

corresponding to unnormalized single-fermion states. In terms of the usual complex coordinates $(I_0, I_\pm)$ of the total angular momentum $I$, these operators obey

$$[I_0, \alpha_m^+] = m \alpha_m^+, \quad [I_+, \alpha_m^+] = \left\{ \begin{array}{ll} \alpha_{m+1}^+, & m < j, \\
0, & m = j, \end{array} \right. \quad \text{(3)}$$

A state of four $j = 9/2$ fermions can be expanded on the states

$$|m_1 m_2 m_3 m_4\rangle = (\prod_{i=1}^4 \alpha_m^+)|\rangle \quad \text{(4)}$$

with $j \geq m_1 > m_2 > m_3 > m_4 \geq -j$, where $|\rangle$ is the vacuum. The eigenspaces of $I_0$ with eigenvalues $M_I$ are spanned by the states with $\sum_i m_i = M_I$. The space $\Phi_0$ is the subspace of the $M_I = 0$ space characterized by $J_\perp \Phi_0 = 0$. Since there are 18 states $|m_1 m_2 m_3 m_4\rangle$ with $M_I = 0$ and 16 with $M_I = 1$ (in accordance with the total multiplicities for $I \geq 0$ and 1 in Table I), this constraint can be expressed by a homogeneous system of 16 linear equations in 18 expansion coefficients. The equations turn out independent in accordance with the dimension 2 of $\Phi_0$. Two linearly independent solution are

$$\phi_0 = \left| \frac{9}{22} \frac{-5}{-2} - \frac{9}{22} \right\rangle - \left| \frac{9}{22} \frac{-5}{-2} - \frac{5}{22} \right\rangle - \left| \frac{5}{22} \frac{-5}{-2} - \frac{9}{22} \right\rangle - \left| \frac{5}{22} \frac{-5}{-2} - \frac{5}{22} \right\rangle$$

$$+ \left| \frac{2}{2} \frac{-3}{-2} - \frac{2}{2} \right\rangle - \left| \frac{2}{2} \frac{-3}{-2} + \frac{2}{2} \right\rangle - \left| \frac{2}{2} \frac{-3}{-2} + \frac{2}{2} \right\rangle - \left| \frac{2}{2} \frac{-3}{-2} + \frac{2}{2} \right\rangle$$

$$- \left| \frac{1}{2} \frac{-1}{-2} - \frac{1}{2} \right\rangle - \left| \frac{1}{2} \frac{-1}{-2} - \frac{1}{2} \right\rangle - \left| \frac{1}{2} \frac{-1}{-2} + \frac{1}{2} \right\rangle - \left| \frac{1}{2} \frac{-1}{-2} + \frac{1}{2} \right\rangle$$

$$\phi_1 = - \left| \frac{9}{22} \frac{-5}{-2} + \frac{9}{22} \frac{-3}{-2} - \frac{9}{22} \right\rangle + \left| \frac{9}{22} \frac{-5}{-2} + \frac{5}{22} \frac{-3}{-2} - \frac{9}{22} \right\rangle - \left| \frac{9}{22} \frac{-5}{-2} + \frac{5}{22} \frac{-3}{-2} - \frac{9}{22} \right\rangle - \left| \frac{9}{22} \frac{-5}{-2} + \frac{5}{22} \frac{-3}{-2} - \frac{9}{22} \right\rangle$$

$$- \left| \frac{5}{22} \frac{-3}{-2} - \frac{9}{22} \right\rangle - \left| \frac{5}{22} \frac{-3}{-2} - \frac{9}{22} \right\rangle - \left| \frac{5}{22} \frac{-3}{-2} - \frac{9}{22} \right\rangle - \left| \frac{5}{22} \frac{-3}{-2} - \frac{9}{22} \right\rangle$$

$$- \left| \frac{3}{22} \frac{-1}{-2} - \frac{5}{22} \right\rangle - \left| \frac{3}{22} \frac{-1}{-2} - \frac{5}{22} \right\rangle - \left| \frac{3}{22} \frac{-1}{-2} + \frac{5}{22} \right\rangle - \left| \frac{3}{22} \frac{-1}{-2} + \frac{5}{22} \right\rangle$$

$$- \left| \frac{1}{2} \frac{-1}{-2} + \frac{3}{22} \right\rangle - \left| \frac{1}{2} \frac{-1}{-2} + \frac{3}{22} \right\rangle - \left| \frac{1}{2} \frac{-1}{-2} + \frac{3}{22} \right\rangle - \left| \frac{1}{2} \frac{-1}{-2} + \frac{3}{22} \right\rangle$$

(5)

Here, $\phi_0$ evidently has $v = 0$. The state $\phi_0 + 2\phi_1$ is orthogonal to $\phi_0$ and thus has $v = 4$. In the expansion of $\phi_1$, the coefficients of $\left| \frac{3}{22} \frac{-5}{-2} + \frac{3}{22} \frac{-3}{-2} - \frac{9}{22} \right\rangle$, $\left| \frac{3}{22} \frac{-5}{-2} + \frac{3}{22} \frac{-3}{-2} - \frac{9}{22} \right\rangle$, $\left| \frac{3}{22} \frac{-5}{-2} + \frac{3}{22} \frac{-3}{-2} - \frac{9}{22} \right\rangle$, $\left| \frac{3}{22} \frac{-5}{-2} + \frac{3}{22} \frac{-3}{-2} - \frac{9}{22} \right\rangle$, and $\left| \frac{3}{22} \frac{-5}{-2} + \frac{3}{22} \frac{-3}{-2} - \frac{9}{22} \right\rangle$ have equal coefficients except for a sign. The ratios of coefficients of $|m_1 m_2 m_3 m_4\rangle$ and $|m_4, -m_3, -m_2, -m_1\rangle$ are
\[ \Pi_i^4 (j - m_i!)^i (j + m_i)! \] so that the corresponding ratios in the basis of states \( \left\{ \Pi_i^4 a_m^\dagger \right\} \) equal one, as required by the symmetry under half-turn rotations about axes perpendicular to the quantization axis.

Since \( \Phi_4 \) is rotationally invariant, its invariance to \( V \) is equivalent to invariance of its \( M_I = 0 \) subspace

\[ \Phi_{40} = \text{span}_m n_m \Phi_0, \quad (6) \]

where \( n_m = a_m^\dagger a_m \). This space is spanned by the 20 states \( n_m \phi_i \) with \( m = j, j - 1, \ldots, -j \) and \( i \in \{0, 1\} \). Each of these states is obtained by selecting in the expansion \( (5) \) of \( \phi_i \) the terms where the orbit \( |jm\rangle \) is occupied. Not all of them are linearly independent. Thus evidently \( n_m \phi_0 = n_m \phi_0 \). Further, \( I_0 \phi_0 = 0 \) is a linear relation among the 10 states \( n_m \phi_1 \). There remain 14 states, which turn out linearly independent. This number coincides with the total multiplicity for \( I \leq 2j \), excepting the Escuderos-Zanick multiplets. The \( M_I = 0 \) state in every remaining multiplet thus belongs to \( \Phi_{40} \). Consequently, every such multiplet is contained in \( \Phi_4 \). This requires, in turn, that the multiplets \( P_l \) span \( a_m^\dagger \Phi_3 \) with \( v = 1 \) and 3 be independent, that for \( I \leq 2j \) and \( I \neq 4, 6 \), the space \( P_l \) span \( a_m^\dagger \Phi_3 \) exhausts the space of states with angular momentum \( l \), and that equality hold in the inclusion span \( a_m \Phi_0 \subset \Phi_3 \), all of which was used in Secs. [I] and [II].

I choose in \( \Phi_{40} \) a basis \( \{ \psi_i \} i = 1, \ldots, 14 \), where \( \{ \psi_i \} i = 1, \ldots, 5 \) are the states \( n_m \phi_0 \) with \( m = j, j - 1, \ldots, 1/2 \) in this order, and \( \{ \psi_i \} i = 6, \ldots, 14 \) are the states \( n_m \phi_1 \) with \( m = j, j - 1, \ldots, -j + 1 \) in this order. By Hermiticity and angular momentum conservation, \( \Phi_{40} \) is invariant to \( V \) if and only if its orthogonal complement \( \Phi_{40}^\perp \) within the \( M_I = 0 \) space is so. This space is spanned by the states

\[ \begin{align*}
\chi_1 &= 14\left[ \frac{2}{3^5} \psi_0 + 6 \frac{2}{3} \psi_1 \right] + 2 \frac{2}{159} \left(-768\frac{2}{3^5} + 231\frac{2}{3} \psi_1 \right) \\
&\quad + 927\frac{2}{3} \psi_1 - 72\frac{2}{3} \psi_0 \\
&\quad + 159\frac{2}{3} \psi_1 - 553\frac{2}{3} \psi_0 \\
&\quad + 56\frac{2}{3} \psi_1 - 640\frac{2}{3} \psi_0 \\
&\quad - 712\frac{2}{3} \psi_1 - 840\frac{2}{3} \psi_0 \\
&\quad + 16\left[ \frac{1}{2^5} \psi_2 + 6 \frac{1}{2^3} \psi_3 \right] + 16\frac{1}{2^{1071}} \left( 516\frac{1}{2^5} + 105\frac{1}{2^5} \right) \\
&\quad + 16\frac{1}{2^{1071}} \left(-171\frac{1}{2^5} + 450\frac{1}{2^5} \right) \\
&\quad + 16\frac{1}{2^{1071}} \left( 345\frac{1}{2^5} + 1225\frac{1}{2^5} \right) \\
&\quad + 16\frac{1}{2^{1071}} \left( 364\frac{1}{2^5} + 430\frac{1}{2^5} \right) \\
&\quad + 16\frac{1}{2^{1071}} \left( 880\frac{1}{2^5} + 966\frac{1}{2^5} \right) \\
&\quad \left( \psi_4 \right) \\
&\quad + 25\frac{1}{2^{1071}} \left( 345\frac{1}{2^5} + 1225\frac{1}{2^5} \right) \\
&\quad + 25\frac{1}{2^{1071}} \left(-171\frac{1}{2^5} + 450\frac{1}{2^5} \right) \\
&\quad + 16\frac{1}{2^{1071}} \left( 364\frac{1}{2^5} + 430\frac{1}{2^5} \right) \\
&\quad + 16\frac{1}{2^{1071}} \left( 880\frac{1}{2^5} + 966\frac{1}{2^5} \right) \\
&\quad \left( \psi_5 \right) \\
\end{align*} \]

It is straightforward to check \( \psi_i^\dagger \chi_k = 0 \) for every \( i, k \) using \( \|m_1 m_2 m_3 m_4 \| = \Pi_i (j + m_i)!/(j - m_i)! \). For example,

\[ \begin{align*}
\psi_6^\dagger \chi_1 &= \phi_1^\dagger n_{9/2} \chi_1 = (-14) \frac{3}{7} \cdot 14 \\
&\quad + \frac{2}{153} \left( (-5) \cdot (-786) + 5 \cdot 231 \right) \\
&\quad + 1 \cdot 927 + (-7) \cdot (-72) = 0.
\end{align*} \]

(7)
IV. INTERACTION $V$ AND INvariance OF $\Phi_4$

Every $V$ is a linear combination of five basic interactions $V_J$, where $J = 0, 2, \ldots, 2j - 1$. They can be chosen in the form

$$V_J = \frac{1}{2} \sum_{M=-J}^{J} P_{JM}^J P_{JM}$$

(9)

with

$$P_{JM} = c_J \sum_{m_1+m_2=M} \langle jm_1jm_2|JM\rangle a_{m_1}a_{m_2},$$

(10)

where $c_J$ is a positive constant. I set

$$c_J \langle jm_1jm_2|JM\rangle = \left( \frac{2J}{J+M} \right)^{-1/2} \sqrt{\frac{(j+m_1)!(j+m_2)!}{(j-m_1)!(j-m_2)!}} c_{m_1m_2}$$

(11)

so that by (2),

$$P_{JM}^J = \left( \frac{2J}{J+M} \right)^{-1/2} \sum_{m_1+m_2=M} c_{m_1m_2}^J a_{m_1}^\dagger a_{m_2}.$$  

(12)

The definition (11) implies $c_{m_1m_2}^J = 0$ for $|m_1 + m_2| > J$. It follows from $[I_+P_{JM}^J]_\pm = 0$, (12), (9), and (11) that $c_{m_1m_2}^J$ can be chosen such that $c_{m_1m_2}^J = (-1)^{m_1} \sqrt{n}$ for $m_1 + m_2 = J$. From $[I_-P_{JM}]_\pm = \sqrt{(J+M)(J+M+1)} P_{JM-1}^J$ for $M > -J$, (12), and (3), one gets the recursion relation

$$(j-m_1-m_2)c_{m_1m_2}^J = (j-m_1)(j+m_1+1)c_{m_1+1,m_2}^J + (j-m_2)(j+m_2+1)c_{m_1,m_2+1}^J,$$

(13)

which then determines $c_{m_1m_2}^J$ for $J \leq m_1 + m_2 < J$. (Continuation of the recursion in fact results in $c_{m_1m_2}^J = 0$ for $m_1 + m_2 < -J$. Terms in (13) with $m_1$ or $m_2$ equal to $j$, which involve undefined values of $c_{m_1m_2}^J$, are just omitted.) All $c_{m_1m_2}^J$ turn out integral, which is explained in Appendix A. From the definition (11) and symmetries of the vector coupling coefficients (2), one gets

$$c_{m_1m_2}^J = -c_{m_2m_1}^J,$$

$$c_{m_1m_2}^J = \frac{(j-m_1)!(j-m_2)!}{(j+m_1)!(j+m_2)!} c_{m_1,m_2}^J,$$

(14)

whence by (12) and (2) follows

$$P_{JM} = \left( \frac{2J}{J+M} \right)^{-1/2} \sum_{m_1+m_2=M} c_{m_1m_2}^J (j+m_1)!(j+m_2)! a_{m_2} a_{m_1}$$

$$= \left( \frac{2J}{J+M} \right)^{-1/2} \sum_{m_1+m_2=M} c_{m_1m_2}^J (j-m_1)!(j-m_2)! a_{m_2} a_{m_1}$$

(15)

in terms of annihilation operators

$$a_m = \sqrt{\frac{(j-m)!}{(j+m)!}} a_{m},$$

(16)

obeying

$$\{a_m, a_{m'}^\dagger\} = \delta_{m,m'}.$$

(17)

It follows that the action of $V_J$ on a basic state $|m_1m_2m_3m_4\rangle$ can be described by the following operation $V_{jm}^J$. If $m_p + m_q - m$ is outside the range of $m_i$, then $V_{jm}^J|m_1m_2m_3m_4\rangle = 0$. Otherwise replace $m_p$ and $m_q$ by $m$ and $m_p + m_q - m$. If this results in two $m$’s being equal, $V_{jm}^J|m_1m_2m_3m_4\rangle = 0$. Otherwise reorder, if necessary, the $m$’s to decreasing order and multiply the state by the sign of the permutation. Finally multiply the state by $\left( \frac{2J}{J + m_p + m_q} \right)^{-1/2} c_{m_p,m_q-m} c_{m-q,m_p}$.

$$V_J|m_1m_2m_3m_4\rangle = \sum_{1 \leq p < q \leq 4, m} w_{pq}^J |m_1m_2m_3m_4\rangle.$$  

(18)

The state $V_J \chi_i$, is obtained by applying this formula to each term in the expansion (7) of $\chi_i$. I did this calculation for every $J, i$ and found that in every case, $V_J \chi_i$ is a linear combination of $\{\chi_i | i = 1, \ldots, 4\}$. This proves that $\Phi_J \psi_0$, and in turn, $\Phi_40, \Phi_4$, and the orthogonal complement $\Phi_4^\perp$ of the latter, are invariant to every $V$. For completeness, I also verified directly that every $V_J \psi_i$ is a linear combination of $\{\psi_i | i = 1, \ldots, 14\}$.

V. ACTION OF $V$ ON $\Phi_4^0$

The expansion of $V_J \chi_i$, on states $\chi_k$ may be expressed by a matrix $V^{k,J} = \langle \psi_k^{J} | k, = 1, \ldots, 4 \rangle$ defined by

$$V_J \chi_i = \sum_{k=1}^{4} \psi_{k}^{J} \chi_{k}.$$  

(19)

These matrices are given by

$$V^{1,0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$V^{1,2} = \frac{-1071}{16} V^{1,2},$$

$$= \begin{pmatrix} -38134 & 15068 & 294390 & -18468 \\ 90027 & -161100 & 94950 & 59778 \\ 0 & 5148 & -42909 & 28917 \\ -207088 & 422480 & -1012475 & -50895 \end{pmatrix},$$

$$V^{1,4} = \frac{51}{64} V^{1,4},$$

$$= \begin{pmatrix} -800010 & 170280 & 249300 & -18468 \\ 90027 & -649050 & 94950 & 59778 \\ 0 & 5148 & -90285 & 28917 \\ -207088 & 422480 & -1012475 & -50895 \end{pmatrix},$$

$$V^{1,6} = \frac{187}{128} V^{1,6},$$

$$= \begin{pmatrix} 578227 & 170280 & 249300 & -18468 \\ 90027 & 753527 & 94950 & 59778 \\ 0 & 5148 & 475524 & 28917 \\ -207088 & 422480 & -1012475 & -50895 \end{pmatrix},$$

$$V^{1,8} = 12\sqrt{\frac{5}{16}} V^{1,8},$$

$$= \begin{pmatrix} 647892 & 170280 & 249300 & -18468 \\ 90027 & 804942 & 94950 & 59778 \\ 0 & 5148 & 545139 & 28917 \\ -207088 & 422480 & -1012475 & -50895 \end{pmatrix}.$$  

(20)
where the entries are determined by the condition that the entries in each $V^{I, J}$ be coprime integers. That the pairing force $V_0$ kills the $v = 4$ space $\Phi_{40}$ is no surprise. The matrices $\bar{V}^{\perp, J}$ exhibit the remarkable similarity

\begin{equation}
\bar{V}^{\perp, 2} = \bar{V}^{\perp, 4} + 481950 = \bar{V}^{\perp, 6} - 896427 = \bar{V}^{\perp, 8} - 966042
\end{equation}

with multiplication of the scalars by the unit matrix understood. Like the invariance of the entire $\Phi_{40}$, this regularity lacks fundamental explanation. It follows that the interactions $\bar{V}_j$ represented on $\Phi_{40}$ by these matrices also act identically on $\Phi_{40}$ except for these scalar terms. Any linear combination of these interactions, that is, an arbitrary $V$, then acts on $\Phi_{40}$ as a linear combination of any one of them and a scalar. This applies, in particular, to the two-body interaction $I^2 - j(j + 1)N$, where $N = \sum m n_m$. Because $N$ acts on the four-body space as the scalar 4, conversely then every $V_j$ acts on $\Phi_{40}$ as a linear combination of $I^2$ and a scalar. This explains, in particular, the zeros in the third row and first column of every $V^{\perp, J}$. For no $|m_1 m_2 m_3 m_4\rangle$ in the expansion (7) of $\chi_1$, the expansion of $I^2 |m_1 m_2 m_3 m_4\rangle$ on states $|m_1 m_2 m_3 m_4\rangle$ indeed contains $(\frac{9}{2} - \frac{1}{2} - \frac{3}{2} - \frac{1}{2})$. Therefore $\chi_3$ cannot appear in the expansion of $I^2 \chi_1$ on states $\chi_1$.

One arrives at a prediction that might be tested experimentally. To the extent of validity of the $j = 9/2$ shell model, the spacings of the energy levels with angular momenta $I$ and $I'$ must have the ratio of $I(I+1) - I'(I'+1)$. The nucleus $^{94}$Ru has a closed neutron major shell and 4 protons in the $1g_{9/2}$ subshell. The yrast $I = 4, 6, 10,$ and 12 levels (with tentative assignments $I = 10$ and 12) have excitation energies 2186.6, 2498.0, 3991.2, and 4716.6 MeV [8]. The states with $I = 10$ and 12 are expected to have fairly pure $1g_{9/2}$ configurations while, according to Das et al. [9], both multiplets with $I = 4$ and 6 could be mixtures of those labeled $\gamma$ and $\alpha$ in Sec. [11] due to perturbation by configurations outside the proton $1g_{9/2}$ shell. The pure Escuderos-Zamick energy levels should then be close to the observed yrast levels. Extrapolation by the spacing rule from $I = 10$ and 12 gives excitation energies 2571.9 and 2918.9 MeV, somewhat above the yrast levels. A similar analysis for $^{96}$Pd, with 4 holes in the $1g_{9/2}$ shell (and tentative assignments of the angular momenta concerned), predicts Escuderos-Zamick levels at 2237.9 and 2616.8 MeV, closer to the yrast levels at 2099.01 and 2424.19 MeV. Interpreting the second observed $I = 4$ and 6 levels in $^{74}$Ni [10], with a closed proton major shell and 4 holes in the neutron $1g_{9/2}$ subshell, as Escuderos-Zamick levels leads to the prediction of the $I = 10$ and 12 levels at 4287 and 5577 MeV.

Since the operator $I^2$ acts on the $M_I = 0$ space as $I^2 I_+$, the matrix $C$ representing its action on $\Phi_{40}$ is easily calculated by (7), (4), and (3). By comparison with (20), one finds in the notation of (21) that

\begin{equation}
\frac{1}{3213} \bar{V}^{\perp, 2} + 156 = \frac{1}{3213} \bar{V}^{\perp, 4} + 306
= \frac{1}{3213} \bar{V}^{\perp, 6} - 123 = \frac{1}{3213} \bar{V}^{\perp, 8} - \frac{434}{3} = C.
\end{equation}

No simple expression in terms of $J$ seems to reproduce these displacements. For $I = 4$ and 6, Van Isacker and Heinze calculated the ratios $r_I^1 = \mu_1^I / \nu_I$, where $\mu_1^I$ and $\nu_I$ are the eigenvalues of $V_I$ in the four-fermion system and a two-fermion state with $I = J [3, 4]$. From (9)–(11), one gets

\begin{equation}
\nu_I = c_1 = \left(\frac{2J}{I}\right)^{-1} \sum_{m=-J}^{m=+J} (c_I^{m,-m})^2.
\end{equation}

My calculations confirm the values $r_I^1 = \frac{68}{33}, 1, \frac{13}{11}, \frac{114}{91}$ and $r_I^2 = \frac{10}{11}, \frac{12}{13}, \frac{13}{14}, \frac{33}{43}$ for $J = 2\ldots8$ reported in [3, 4], and further provide $r_{10} = \frac{23}{31}, \frac{98}{143}, \frac{283}{490}, \frac{1673}{2964}$ and $r_{12} = 0, \frac{25}{143}, \frac{93}{55}, \frac{248}{65}$.

VI. ACTION OF $V$ ON $\Phi_{40}$

Like in (19), the action of $V_j$ on the states $\psi_i$ may be expressed by matrices $V^{J, i}$. They are
Some patterns leap to the eye. The entries in \( V^0 \) are easily understood. Thus \( V_0(m_1m_2 m_3 m_4) \) vanishes unless one of the two of the forms a pair \( m, -m \), in which case the remaining two do the same. Further, \( V_0(m_1 m_2 -m_2 -m_1) = (1 - m_1 m_2 + 1) (n_{m_1} + n_{m_2}) \), so the states \( n_{m_1} \phi_0 \), \( n_{m_2} \phi_0 \) do not contribute to any \( V_0 \). This expression follows from \( V_0(m, -m) = 2 \sum (1 - m_1 m_2 + 1) (n_{m_1} + n_{m_2}) \phi_0 \), where \( |m_1 m_2 m_3 m_4| \), and the observation that the coefficient of \( |n_{m_1} m_{-m_2 -m_1}| \) in the expansion of \( \phi_0 \) is \( (1 - m_1 m_2 + 1) \). By using it in combination with the expansions \( \phi_0 \), it is, in fact, straightforward to reconstruct every entry in \( V^0 \), and in particular, the simple pattern in its upper left \( 5 \times 5 \) submatrix. Notice to this end that the last eight terms in the expansion of \( \phi_0 \) do not contribute to \( V_0 \).

For a general \( J \), one notices in the upper right \( 5 \times 8 \) submatrix of \( V^2 \) equal contributions to \( V_0 m \pm \phi_0 \) from any \( n_{m \phi_0} \phi_0 \). This is an immediate consequence of \( n_{m \phi_0} \phi_0 \) and the symmetry under half-turn rotations about an axis perpendicular to the quantization axis. The same pattern is seen in the parts of the sixth rows just below, which display contributions to \( V_0 m \phi_0 \) from \( n_{m \phi_0} \phi_0 \), for \( m \neq \pm j \), and again the reason is the symmetry under half-turn rotations about an axis perpendicular to the quantization axis. Such a rotation thus leads to both the replacement of \( m \) by \( -m \) and the omission of \( n_{-j \phi_0} \) instead of \( n_{-j \phi_0} \) in the selection of the states \( \phi_0 \).
But by $\sum_m mn_m \phi_1 = J_0 \phi_1 = 0$, the state $n_{-j} \phi_1$ equals $n_j \phi_1$ plus a linear combination of states that are common to both the original and the new basis. Therefore in the original basis, the contribution of $n_j \phi_1$ to $V_j n_m \phi_1$ equals its contribution to $V_j n_m \phi_1$. It follows further that when the lower right $9 \times 9$ submatrix of $V^J$ is written
\[ v_{jm',m}^{1J} = v_{jm,m}^{1J} - \frac{m'}{J} v_{jm,m}^{1J}, \] (25)
This is verified by inspection. Similar patterns occur when a state $n_m \phi_1$ other than $n_{-j} \phi_1$ is omitted in the selection of the states $\psi_i$.

It is trivial by $n_m \phi_0 = n_{-m} \phi_0$ that the state $n_m \phi_0$ contributes equally to $V_j n_m \phi_0$. Therefore when $v_{mm'}^{0J}$, denotes the entries in the upper left $5 \times 5$ submatrix of $V^J$ with indices referring to the basic states $n_m \phi_0$, and $v_{mm'}^{0J}$, those of its neighboring $5 \times 5$ submatrix to the right with indices referring to the basic states $n_m \phi_0$ and $n_{m'} \phi_1$, then for both $i = 0$ and 1 one has
\[ \frac{1}{2} V_j n_m \phi_1 = \sum_{m'>0} v_{mm'}^{0J} n_{m'} \phi_0 + \text{linear combination of } n_m \phi_1 \] (26)
with $n_m = n_m + n_{-m}$. The operators $n_m$ can be expressed by the tensor operators $T_{10}$ defined in the introduction,
\[ n_m = (-)^{J-m} 2 \sum_{\text{even } I} \langle jm | m| I0 \rangle T_{10}. \] (27)

Since the states $T_{10} \psi_i$ span the subspace of $\Phi_{40}$ with angular momentum $I$, which is invariant to $V^J$, one can write
\[ V_j T_{10} \psi_i = \sum_{q'} w_{q'q}^{jm} T_{10} \psi_{q'}. \] (28)

By combining the equations (26), (28) and the definition of $T_{10}$ in the introduction one obtains
\[ (J + m_1 + m_2) c_{m_1 m_2}^J = c_{m_1-1,m_2}^J + c_{m_1,m_2-1}^J. \] (A1)

First notice that the algorithm for $c_{m_1 m_2}^J$ described in Sec. IV ensures the proportionality (11) so that $P_{J,M}^I$ given by (10) is a tensor operator. Besides (13),
\[ (I_z, P_{J,M}^I) = \sqrt{(J-M)(J+M+1)} P_{J,M+1}^I, \] (12), and (3) give
\[ (J + m_1 + m_2) c_{m_1 m_2}^J = c_{m_1-1,m_2}^J + c_{m_1,m_2-1}^J. \] (A1)

With
\[ d_{m_1 m_2}^J = (J - m_1 - m_2)! c_{m_1 m_2}^J, \] (A2)
the recursion relations (13) and (A1) take the forms
\[ d_{m_1 m_2}^J = (j - m_1)(j + m_1 + 1) d_{m_1-1,m_2}^J, \] (A3)
\[ (J + m_1 + m_2) d_{m_1 m_2}^J = d_{m_1,m_2+1}^J + (j - m_2)(j + m_2 + 1) d_{m_1,m_2+1}^J, \] (A4)

(Again, terms with undefined values of $d_{m_1 m_2}^J$ are omitted.) Setting $m_1 = j$ so that only the second term occurs on the right in (A3), and using also $d_{j,-j}^J = 1$,
one gets for $J - j \geq m_2 \geq -j$, by repeated application of (A3), an expression for $d_{jm_2}^j$ as a product of two products of $J - j - m_2$ consecutive integers. Hence $d_{jm_2}^j$ is divisible by $(J - j - m_2)!^2$ and, all the more, by $(J - j - m_2)!$. It then follows by induction by means of (A4) that $(J - m_1 - m_2)!$ divides $d_{m_1 m_2}^j$ for every $m_1, m_2$ with $m_1 + m_2 \geq 0$. Then by (A2), $c_{m_1 m_2}^j$ is integral for $m_1 + m_2 \geq 0$. For $m_1 + m_2 < 0$ one can now apply the second equation in (A4). In this case, $j + m_1 < j - m_2$ and $j + m_2 < j - m_1$, so the first factor on the right, and hence $c_{m_1 m_2}^j$, are integral.

**Appendix B: Explanation of (30)**

The first equation in (30) can be written

$$\phi_0 = -\frac{1}{2} P^{\dagger 2} |\rangle$$

(B1)

with

$$P^{\dagger} = \sum_{m > 0} (-)^{j-m} a_m^\dagger a_{-m}.$$  

(B2)

Hence

$$n_m \phi_0 = a_m^\dagger a_m \phi_0$$

$$= -a_m^\dagger P^{\dagger} [a_m, P^{\dagger} |\rangle] = (-)^{j-m} P^{\dagger} a_m^\dagger a_{-m} |\rangle$$

(B3)

and

$$V_j n_m \phi_0 = \frac{1}{2} \sum_M P^{\dagger}_M P_{JM} a_m^\dagger a_{-m}$$

$$= (-)^{j-m} \frac{1}{2} \sum_M P^{\dagger}_M (P^{\dagger}_M P_{JM} + [P_{JM}, P^{\dagger}] a_m^\dagger a_{-m}).$$

(B4)

Since $P^{\dagger}_M a_m^\dagger a_{-m} |\rangle \propto |\rangle$, the first term in the parentheses contributes to $V_j n_m \phi_0$ a term proportional to $P^{\dagger}_M P^{\dagger}$, which has $\nu \leq 2$. I proceed by calculating the commutator $[P_{JM}, P^{\dagger}]$.

For convenience, I omit for now the factor $c_j$ in (10). It is reentered at the end of this appendix. I then have

$$[P_{JM}, P^{\dagger}] = \frac{1}{2} \sum_{m_1 m_2} (-)^{j-m}$$

$$\langle jm_1 j m_2 |JM| a_{m_2} a_{m_1}, a^\dagger_{m_2} a^\dagger_{m_1} \rangle$$

$$= \sum_{m_1 m_2} \langle jm_1 j m_2 |JM \rangle$$

$$((-)^{j-m_1} a_{m_2} a^\dagger_{-m_1} + (-)^{j-m_2} a^\dagger_{-m_2} a_{m_1})$$

$$= \sum_{m_1 m_2} (-)^{j-m_2} \langle jm_1 j m_2 |JM | a^\dagger_{-m_2} a_{m_1} \rangle$$

$$= 2 \sum_{m_1 m_2} (-)^{j-m_2} \langle jm_1 j m_2 |JM | a^\dagger_{-m_2} a_{m_1} \rangle$$

$$- \sum_{m} (-)^{j+m} \langle jm - m |JM \rangle$$

$$= 2 \sum_{m_1 m_2} (-)^{j-m_2} \langle jm_1 j m_2 |JM | a^\dagger_{-m_2} a_{m_1} \rangle$$

$$+ \delta j_0 \sqrt{2 j + 1}.$$  

(B5)

Since $P^{\dagger}_0 \propto P^{\dagger}$, the last term in this expression is seen by comparison with (A2) to contribute in (B4) a term proportional to $n_m \phi_0$.

Further,

$$a_{-m_2} a_{m_1} a^\dagger_{-m_1} |\rangle = (\delta_{m_1, m} - \delta_{m_1, -m}) a_{-m_2} a^\dagger_{-m_1} |\rangle,$$

so the first term in the expression (B5) contributes to the sum in (B4) terms

$$2 a_m^\dagger \sum_{M m_2} (-)^{j-m_2} \langle j-m j m_2 |JM | P^{\dagger}_M a^\dagger_{-m_2} \rangle$$

same with $-m$ instead of $m$.  

(B7)

Here (11),

$$\xi = \sum_{M m_2} (-)^{j-m_2} \langle j-m j m_2 |JM | P^{\dagger}_M a^\dagger_{-m_2} \rangle$$

$$= -\sqrt{2 j + 1} \sum_{M m_2} (j-m) a_{-m_2} P^{\dagger}_M |\rangle$$

(B8)

is a member with $M = -m$ of the space $\Phi_3$ defined in Sec. (11) With standard relative phases within its multiplet, one therefore has

$$\xi = (-)^{j+m} a_m (\zeta \phi_0 + \eta \phi_1),$$

(B9)

where $\zeta$ and $\eta$ do not depend on $m$. Totally, one arrives at

$$V_j n_m \phi_0 = \eta (n_m + n_{-m}) \phi_1 + v \leq 2 \text{ state},$$

(B10)

which is equivalent to (30) with $\gamma_J = \eta c_j^2$. 
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