Hardness of Permutation Pattern Matching

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Abstract

Permutation Pattern Matching (or PPM) is a decision problem whose input is a pair of permutations \( \pi \) and \( \tau \), represented as sequences of integers, and the task is to determine whether \( \tau \) contains a subsequence order-isomorphic to \( \pi \). Bose, Buss and Lubiw proved that PPM is NP-complete on general inputs.

We show that PPM is NP-complete even when \( \pi \) has no decreasing subsequence of length 3 and \( \tau \) has no decreasing subsequence of length 4. This provides the first known example of PPM being hard when one or both of \( \pi \) and \( \sigma \) are restricted to a proper hereditary class of permutations.

This hardness result is tight in the sense that PPM is known to be polynomial when both \( \pi \) and \( \tau \) avoid a decreasing subsequence of length 3, as well as when \( \pi \) avoids a decreasing subsequence of length 2. The result is also tight in another sense: we will show that for any hereditary proper subclass \( \mathcal{C} \) of the class of permutations avoiding a decreasing sequence of length 3, there is a polynomial algorithm solving PPM instances where \( \pi \) is from \( \mathcal{C} \) and \( \tau \) is arbitrary.

We also obtain analogous hardness and tractability results for the class of so-called skew-merged patterns.

From these results, we deduce a complexity dichotomy for the PPM problem restricted to \( \pi \) belonging to \( \text{Av}(\rho) \), where \( \text{Av}(\rho) \) denotes the class of permutations avoiding a permutation \( \rho \). Specifically, we show that the problem is polynomial when \( \rho \) is in the set \( \{1, 12, 21, 132, 213, 231, 312\} \), and it is NP-complete for any other \( \rho \).

1 Introduction

A permutation of size \( n \) is a bijection from the set \( \{1, 2, \ldots, n\} \) to itself. We represent a permutation \( \pi \) of size \( n \) by the sequence \( \pi = \pi(1), \pi(2), \ldots, \pi(n) \). We let \( S_n \) denote the set of permutations of size \( n \). When writing out short permutations explicitly, we usually omit the punctuation and write, e.g., 312 instead of 3, 1, 2. We let \([n]\) denote the set \( \{1, 2, \ldots, n\} \).

We say that a permutation \( \tau = \tau(1), \tau(2), \ldots, \tau(n) \in S_n \) contains a permutation \( \pi = \pi(1), \pi(2), \ldots, \pi(k) \in S_k \), which we denote by \( \pi \preceq \tau \), if \( \tau \) has a subsequence \( \tau(i_1), \tau(i_2), \ldots, \)

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\( \tau(i_k) \) whose elements have the same relative order as the elements of \( \pi \), that is, for any \( a, b \in [k] \) we have \( \tau(i_a) < \tau(i_b) \) if and only if \( \pi(a) < \pi(b) \). We call such a subsequence \( \tau(i_1), \tau(i_2), \ldots, \tau(i_k) \) an occurrence of \( \pi \) in \( \tau \). If \( \tau \) does not contain \( \pi \), we say that \( \tau \) avoids \( \pi \) or that \( \tau \) is \( \pi \)-avoiding.

In this paper, we study the computational complexity of determining for a given pair of permutations \( \pi \) and \( \tau \) whether \( \tau \) contains \( \pi \). In the literature, this problem is known as Permutation Pattern Matching, or PPM.

| Permutation Pattern Matching (PPM) |
|-----------------------------------|
| **Instance:** Permutations \( \pi \in S_k \) and \( \tau \in S_n \). |
| **Question:** Does \( \tau \) contain \( \pi \)? |

In the context of PPM, the permutation \( \pi \) is usually called the pattern, and \( \tau \) is called the text. When dealing with instances of PPM, we always assume that the pattern is at most as long as the text, that is, \( k \leq n \).

Observe that PPM can be solved by a simple brute-force algorithm in time \( O(\binom{n}{k} \cdot k) \). Thus, if the pattern \( \pi \) were fixed, rather than being part of the input, the problem would trivially be polynomial-time solvable.

Bose, Buss and Lubiw [7] have shown that PPM is NP-complete. This general hardness result has motivated the study of parameterized and restricted variants of PPM.

Guillemot and Marx [13] have shown that PPM can be solved in time \( 2^{O(k^2 \log k)} \cdot n \); in particular, PPM is fixed-parameter tractable with \( k \) considered as parameter. The complexity of the Guillemot–Marx algorithm was later reduced to \( 2^{O(k^2)} \cdot n \) by Fox [12].

A different parameterization was considered by Bruner and Lackner [8], who proved that PPM can be solved in time \( O(1.79^{\text{run}(\tau)} \cdot kn) \), where \( \text{run}(\tau) \) is the number of increasing and decreasing runs in \( \tau \); here an increasing run in \( \tau(1), \tau(2), \ldots, \tau(n) \) is a maximal consecutive increasing subsequence of length at least 2, and decreasing runs are defined analogously. The Bruner–Lackner algorithm shows that PPM is fixed-parameter tractable with respect to the parameter \( \text{run}(\tau) \). Moreover, since \( \text{run}(\tau) \leq n \) for any permutation \( \tau \in S_n \), the Bruner–Lackner algorithm solves PPM in time \( 1.79^n \cdot n^{O(1)} \). It is the first algorithm to improve upon the bound of \( 2^n \cdot n^{\Theta(1)} \) achieved by the straightforward brute-force approach.

Another algorithm for PPM was described by Albert, Aldred, Atkinson and Holton [3], and later a similar approach was analyzed by Ahal and Rabinovich [2]. Ahal and Rabinovich have proved that PPM can be solved in time \( n^{O(\text{tw}(G_\pi))} \), where \( G_\pi \) is a graph that can be associated to the pattern \( \pi \), and \( \text{tw}(G_\pi) \) denotes the treewidth of \( G_\pi \). We shall give the precise definitions and the necessary details of this approach in Section 3.

Apart from the above-mentioned algorithms, which all solve general PPM instances, various researchers have obtained efficient solutions for instances of PPM where \( \pi \) or \( \tau \) satisfy some additional restrictions. The most natural way to formalize such restrictions is to use the concept of permutation class, which we now introduce.

A permutation class is a set \( C \) of permutations with the property that for every \( \sigma \in C \), all the permutations contained in \( \sigma \) belong to \( C \) as well.

It is often convenient to describe a permutation class \( C \) by specifying the minimal permutations not belonging to \( C \). For a set of permutations \( F \), we let \( \text{Av}(F) \) denote the class of permutations that avoid all the permutations in \( F \). Note that for any permutation class \( C \) there is a unique (possibly infinite) antichain of permutations \( F \) such that \( C = \text{Av}(F) \). The set \( F \) is the basis of \( C \).
A principal permutation class is a permutation class whose basis has a single element. A proper permutation class is a permutation class whose basis is nonempty, or in other words, a permutation class that does not contain all permutations. For a recent overview of the structural theory of permutation classes, we refer the interested reader to the survey by Vatter [23].

When dealing with specific sets $F$, we often omit nested braces and write, e.g., $\text{Av}(321)$ or $\text{Av}(2413,3142)$, instead of $\text{Av}([321])$ or $\text{Av}([2413,3142])$, respectively.

In this paper, we focus on the complexity of PPM when one or both of the inputs are restricted to a particular proper permutation class. Following the terminology of Albert et al. [5], we consider, for a permutation class $C$, these two restricted versions of PPM:

**$C$-Permutation Pattern Matching ($C$-PPM)**

*Instance:* A pattern $\pi \in C$ of size $k$ and a text $\tau \in C$ of size $n$.

*Question:* Does $\tau$ contain $\pi$?

**$C$-Pattern Permutation Pattern Matching ($C$-Pattern PPM)**

*Instance:* A pattern $\pi \in C$ of size $k$ and a text $\tau \in S_n$.

*Question:* Does $\tau$ contain $\pi$?

Clearly, any instance of $C$-PPM is also an instance of $C$-Pattern PPM, and in particular, $C$-PPM is at most as hard as $C$-Pattern PPM.

Bose, Buss and Lubiw [7] have shown that $C$-Pattern PPM is polynomially tractable when $C$ is the class $\text{Av}(2413,3142)$ of the so-called separable permutations. Other algorithms for $\text{Av}(2413,3142)$-Pattern PPM were given by Ibarra [15], by Albert et al. [3], and by Yugandhar and Saxena [24].

An even more restricted case of $C$-Pattern PPM deals with monotone increasing patterns, that is, $C = \text{Av}(21)$. In this case, $C$-Pattern PPM reduces to finding the longest increasing subsequence in a given text. This is an old algorithmic problem [22], and can be solved in time $O(n \log \log n)$ [9, 18].

A natural generalization is to consider instances of PPM where patterns and texts can be partitioned into a bounded number of monotone sequences. For integers $r, s \geq 0$, we say that a permutation $\pi$ is an $(r,s)$-permutation if $\pi$ can be partitioned into $r$ increasing and $s$ decreasing (possibly empty) subsequences. We let $C_{r,s}$ denote the class of all $(r,s)$-permutations. In particular, $\text{Av}(21) = C_{1,0}$, and more generally, $\text{Av}(k(k-1)\ldots 1) = C_{k-1,0}$. The $(1,1)$-permutations are also known as skew-merged permutations, and it is not hard to see that $C_{1,1} = \text{Av}(2143,3412)$; see Atkinson [6]. Kézdy, Snevily and Wang [16] have shown that for any $r, s \geq 0$, the basis of the class $C_{r,s}$ is finite; however, they also pointed out that the basis of $C_{2,1}$ has more than 100 permutations.

Guillemot and Vialette [14] have shown that $C_{2,0}$-PPM is polynomial-time solvable. A different, faster algorithm for $C_{2,0}$-PPM has been described by Albert et al. [5]. By a similar approach, Albert et al. [5] have also obtained a polynomial algorithm for $C_{1,1}$-PPM.

It was an open problem to determine whether there is any proper permutation class $C$ for which $C$-Pattern PPM (or even $C$-PPM) is NP-complete [1, 5]. Our first main result solves this problem.

**Theorem 1.1.** It is NP-complete to decide, for a pattern $\pi \in C_{2,0}$ and a text $\tau \in C_{3,0}$, whether $\pi$ is contained in $\tau$. 
Consequently, $C_{2,0}$-Pattern PPM as well as $C_{3,0}$-PPM are NP-complete. These results are tight in the sense that both $C_{1,0}$-Pattern PPM and $C_{2,0}$-PPM are polynomial-time solvable, as mentioned above.

We obtain similar results when $C_{2,0}$ and $C_{3,0}$ are replaced with $C_{1,1}$ and $C_{2,1}$, respectively. In fact, here we can be even more restrictive. Let $C_{1,1}^x$ be the class $C_{1,1} \cap \text{Av}(3142)$.

**Theorem 1.2.** It is NP-complete to decide, for a pattern $\pi \in C_{1,1}^x$ and a text $\tau \in C_{2,1}$, whether $\pi$ is contained in $\tau$.

This result again implies that $C_{1,1}^x$-Pattern PPM and $C_{2,1}$-PPM are NP-complete, in contrast with the polynomiality of $C_{1,0}$-Pattern PPM, $C_{1,1}$-PPM and $C_{2,0}$-PPM.

Combining Theorems 1.1 and 1.2 with previously known polynomial cases of $C$-Pattern PPM, we obtain a complexity dichotomy of $C$-Pattern PPM for principal classes $C$.

**Theorem 1.3.** Let $\alpha$ be a permutation. The problem $\text{Av}(\alpha)$-Pattern PPM is polynomial-time solvable for $\alpha \in \{1, 12, 21, 132, 213, 231, 312\}$ and NP-complete for any other $\alpha$.

We also obtain new tractability results, which show that the NP-hardness results for $C_{2,0}$-Pattern PPM and $C_{1,1}^x$-Pattern PPM are tight in an even stronger sense than suggested above.

**Theorem 1.4.** If $C$ is a proper subclass of $C_{2,0}$ then $C$-Pattern PPM can be solved in polynomial time.

**Theorem 1.5.** If $C$ is a proper subclass of $C_{1,1}^x$ then $C$-Pattern PPM can be solved in polynomial time.

In Section 2, we give the proof of Theorem 1.1. Then, in Section 3, we prove Theorem 1.4. In the subsequent section, we explain how these proofs can be adapted to patterns from the class $C_{1,1}^x$ (and texts from $C_{2,1}$), and deduce Theorems 1.2, 1.3 and 1.5.

## 2 Hardness of $\text{Av}(321)$-Pattern PPM

Our goal is to show that for a pattern $\pi \in C_{2,0}$ and a text $\tau \in C_{3,0}$, it is NP-complete to decide whether $\pi$ is contained in $\tau$. Since the problem is clearly in NP, we focus on proving its NP-hardness. We take inspiration from the NP-hardness proof given by Bose, Buss and Lubiw [7] for general permutations and adapt it to the proper classes $C_{2,0}$ and $C_{3,0}$. We proceed by reduction from the classical NP-complete problem 3-SAT, whose input is a 3-CNF formula $\Phi$, and the goal is to determine whether $\Phi$ is satisfiable.

We first introduce several auxiliary notions. Let $\alpha = \alpha(1), \alpha(2), \ldots, \alpha(n)$ be a permutation of length $n$. For a pair of elements $\alpha(i), \alpha(j)$, we say that $\alpha(i)$ is above $\alpha(j)$ (and $\alpha(j)$ is below $\alpha(i)$), if $\alpha(i) > \alpha(j)$. Likewise, $\alpha(i)$ is left of $\alpha(j)$ (and $\alpha(j)$ is right of $\alpha(i)$) if $i < j$.

For disjoint sets $X, Y \subseteq \{\alpha(1), \alpha(2), \ldots, \alpha(n)\}$ we say that $X$ is above $Y$ if every element of $X$ is above all the elements of $Y$, and similarly for the other directions.

If $p = (\alpha(i), \alpha(j))$ is a pair of elements of $\alpha$ and $\alpha(k)$ is another element, we say that $\alpha(k)$ is sandwiched by $p$ from below if $i < k < j$ and $\alpha(k)$ is above $p$; see Figure 1, left. Similarly, we say that $\alpha(k)$ is sandwiched by $p$ from the left if $\alpha(i) < \alpha(k) < \alpha(j)$ and $\alpha(k)$ is to the right of $p$; see Figure 1, right. Analogously, we also define sandwiching from the right or from
Figure 1: Left: an element $\alpha(k)$ sandwiched from below by a pair of elements $p = (\alpha(i), \alpha(j))$. Right: $\alpha(k)$ sandwiched from the left by $p$.

Figure 2: A staircase of two steps. The shaded increases are the bends.

above. More generally, a set $A$ of elements of $\alpha$ is \textit{sandwiched from below by} $p$ if each element of $A$ is sandwiched from below by $p$, and similarly for the other directions.

A pair of elements $(\alpha(i), \alpha(j))$ is an \textit{increase} in $\alpha$ if $i < j$ and $\alpha(i) < \alpha(j)$; in other words, an increase is an occurrence of the permutation 12 in $\alpha$.

A \textit{staircase of $k$ steps} is a sequence $S$ of $2k$ disjoint increases

$$S = (q_1, p_1, q_2, p_2, \ldots, p_{k-1}, q_k, p_k),$$

with the following properties (see Figure 2):

- For every $i \in [k]$, $p_i$ is sandwiched by $q_i$ from the left, and for $i > 1$, $q_i$ is sandwiched by $p_{i-1}$ from below.

- For every $i \in [k-1]$, $q_{i+1}$ is to the right and above $q_i$, and $p_{i+1}$ is to the right and above $p_i$. In particular, the elements of $q_1, q_2, \ldots, q_k$, as well as the elements $p_1, p_2, \ldots, p_k$ form an increasing sequence of length $2k$.

We call $q_i$ the \textit{ith outer bend} of $S$, and $p_i$ the \textit{ith inner bend}. Additionally, we call $q_1$ the \textit{base} of $S$ and $p_k$ the \textit{top} of $S$.

A \textit{double staircase of $k$ steps} is a pair $(S, S')$ of disjoint staircases of $k$ steps, with $S = (q_1, p_1, q_2, p_2, \ldots, p_{k-1}, q_k, p_k)$ and $S' = (q'_1, p'_1, q'_2, p'_2, \ldots, p'_{k-1}, q'_k, p'_k)$, satisfying the following properties (see Figure 3):
• For each $i \geq 1$, the increase $p'_i$ is above and to the right of $p_i$, and also below (and necessarily to the right of) $q_{i+1}$. For $i < k$, $p'_i$ is also below and to the left of $p_{i+1}$.

• Similarly, for $i \geq 1$, $q'_i$ is above and to the right of $q_i$, and also to the left (and necessarily above) $p_i$. For $i < k$, $q'_i$ is below and to the left of $q_{i+1}$.

An $m$-fold staircase of $k$ steps is an $m$-tuple $S = (S_1, S_2, \ldots, S_m)$ of staircases such that for each $i, j \in [m]$ with $i < j$, the pair $(S_i, S_j)$ is a double staircase of $k$ steps. The $i$th inner bend of $S$ is the union of the $i$th inner bends of the staircases $S_1, S_2, \ldots, S_m$; the outer bends, the base and the top of $S$ are defined analogously.

Notice that an $m$-fold staircase avoids the pattern 321. Moreover, the union of its outer bends, as well as the union of its inner bends each form an increasing subsequence.

2.1 The Reduction

We now describe the reduction from 3-SAT. Let $\Phi$ be a given 3-CNF formula. Suppose that $\Phi$ has $v$ variables $x_1, x_2, \ldots, x_v$ and $c$ clauses $K_1, K_2, \ldots, K_c$. We will assume, without loss of generality, that each clause contains exactly three literals, and no variable appears in a single clause more than once. We will construct two permutations $\pi = \pi(\Phi) \in C_{2,0}$ and $\tau = \tau(\Phi) \in C_{3,0}$, such that $\Phi$ is satisfiable if and only if $\tau$ contains $\pi$.

The overall structure of $\pi$ and $\tau$ is depicted in Figure 4.

The pattern $\pi$ is the disjoint union of a $v$-fold staircase $X = (X_1, X_2, \ldots, X_v)$ of $2c + 1$ steps with an increasing sequence $A$ (called the anchor of $\pi$) of length $M$, where $M$ is a sufficiently large value to be specified later. Moreover, the sequence $A$ is below $X$, to the right of the base of $X$, and to the left of the first inner bend of $X$. This determines $\pi$ uniquely up to the value of $M$, and we may observe that $\pi$ avoids 321. We will say that the staircase $X_i$ represents the variable $x_i$ of $\Phi$.

We now describe the text $\tau$. As a starting point, we first build a permutation $\overline{\tau}$ and then explain how to modify $\overline{\tau}$ to obtain $\tau$. 
The permutation $\overline{\tau}$ is the disjoint union of a $2v$-fold staircase $Y = (X_1^T, X_1^F, X_2^T, X_2^F, \ldots, X_v^T, X_v^F)$ of $2c + 1$ steps, and an increasing sequence $A'$ (the anchor of $\overline{\tau}$) of length $M$. The sequence $A'$ is below the $2v$-fold staircase, to the right of the base of $Y$, and to the left of the first outer bend of $Y$.

Each of the $2v$ staircases in $Y$ will represent one of the $2v$ possible literals, with $X_i^T$ representing the literal $x_i$ and $X_i^F$ representing $\neg x_i$.

We now modify $\overline{\tau}$ to obtain the actual text $\tau$. The modification proceeds in two steps.

In the first step, we change the relative position of the bases of the individual staircases in $Y$. For every $i = 1, 2, \ldots, v$, let $b_i^T$ and $b_i^F$ be the respective bases of $X_i^T$ and $X_i^F$. In $\overline{\tau}$, the two bases together form an occurrence of 1234. We modify their relative horizontal position by moving $b_i^T$ to the right of $b_i^F$, so that in $\tau$ the two bases will form an occurrence of 3412.

The relative position of $b_i^T$ and $b_i^F$ to the remaining elements remains unchanged; in other words, the four elements of $b_i^T \cup b_i^F$ occupy the same four rows and the same four columns as before.

Before we describe the second step of the construction of $\tau$, we introduce the following notion: suppose $S = (q_1, p_1, q_2, p_2, \ldots, q_k, p_k)$ is a staircase of $k$ steps, and let $q_i$, $p_i$ and $q_{i+1}$ be three consecutive bends of $S$, with $i > 1$. A bypass of $q_i, p_i, q_{i+1}$ in $S$ is a sequence of three increases $q'_i, p'_i, q'_{i+1}$ disjoint from $S$ and satisfying these properties:

- The sequence $S'$ obtained from $S$ by replacing $q_i$ with $q'_i$, $p_i$ with $p'_i$ and $q_{i+1}$ with $q'_{i+1}$ is again a staircase with $k$ steps, and
- $q_i$ is above and to the left of $q'_i$, $p_i$ is above and to the right of $p'_i$, and $q_{i+1}$ is sandwiched by $q'_{i+1}$ from the right (see Figure 5).

Note that these conditions determine the relative positions of the elements of $S \cup S'$ uniquely.

We call the pair $(q_i, q'_i)$ the fork of the bypass, and the pair $(q_{i+1}, q'_{i+1})$ the merge of the bypass.

**Lemma 2.1.** Let $S = (q_1, p_1, q_2, p_2, \ldots, q_k, p_k)$ be a staircase of $k$ steps, let $i \in \{2, 3, \ldots, k-1\}$ be arbitrary, and let $S'$ be obtained from $S$ by replacing the three bends $q_i, p_i, q_{i+1}$ with a bypass $q'_i, p'_i, q'_{i+1}$. Let $S'' = (q'_1, p'_1, q''_2, p'_2, \ldots, q''_{k}, p''_k)$ be a staircase of $k$ steps such that $p''_{i-1} = q_{i-1}$, and moreover, each of the $4k$ elements of $S''$ also belongs to $S \cup S'$. Then the sequence of bends $(q''_i, p''_i, q''_{i+1}, p''_{i+1})$ is equal either to $(q_i, p_i, q_{i+1}, p_{i+1})$ or to $(q'_i, p'_i, q'_{i+1}, p'_{i+1})$. 

Figure 4: An overview of the general structure of $\pi$ (left) and $\tau$ (right).
Proof. By assumption, \( p_{i-1} = p''_{i-1} \). There are four elements of \( S \cup S' \) (namely \( q_i \cup q'_i \)) sandwiched from below by \( p''_{i-1} \), and two of them must form \( q''_i \). However, since \( q_i \) and \( q'_i \) are the only two increases in \( q_i \cup q'_i \), we have either \( q''_i = q_i \) or \( q''_i = q'_i \). In either case, there is a unique increase sandwiched by \( q''_i \) from the left, and this increase must therefore be \( p''_i \), and by the same argument, both \( q''_{i+1} \) and \( p''_{i+1} \) are determined uniquely. The lemma follows. \( \square \)

We continue by the second step of the construction of the text \( \tau \). The general approach is to modify certain parts of the staircases by adding bypasses whose structure depends on the clauses of the formula \( \Phi \). Recall that \( \Phi \) has \( c \) clauses \( K_1, K_2, \ldots, K_c \), and that the \( 2v \)-fold staircase \( Y \) has \( 2c + 1 \) steps.

Let \( Q_m \) and \( P_m \) denote the \( m \)-th outer bend and the \( m \)-th inner bend of \( Y \), respectively. For every \( t \in [c] \), we will associate to the clause \( K_t \) the sequence of three bends \( Q_{2t}, P_{2t}, Q_{2t+1} \) of \( Y \).

Let \( K_t \) be a clause of \( \Phi \) of the form \( (L_i \lor L_j \lor L_k) \), with \( L_i \in \{ x_i, \neg x_i \} \), \( L_j \in \{ x_j, \neg x_j \} \), and \( L_k \in \{ x_k, \neg x_k \} \), for some \( i < j < k \). Let \( X^+ \in \{ X^T_i, X^F_i \} \) denote the staircase representing the literal \( L_i \), and let \( X^- \in \{ X^T_i, X^F_i \} \setminus \{ X^+ \} \) be the staircase representing the other literal containing \( x_i \). We define \( X^+_j, X^-_j, X^+_k, \) and \( X^-_k \) analogously.

We will now add bypasses to (some of) the staircases \( X^+_1, X^+_2, X^-_2, X^-_3, \ldots, X^+_v, X^-_v \) into the three bends associated to \( K_t \), by inserting new bends into \( Q_{2t}, P_{2t}, Q_{2t+1} \), and sometimes changing the relative position of existing bends. The choice of the relative positions of these bends is the key aspect of our reduction.

We first describe how to modify \( Q_{2t} \). We will replace \( Q_{2t} \) by a so called fork gadget \( \tilde{Q}_{2t} \) containing the \( (2t) \)-th outer bends of all the staircases \( X^F_t \) and \( X^T_t \) and also the corresponding bends of their bypasses, if any. The gadget \( \tilde{Q}_{2t} \) is a union of three disjoint increasing sequences, which we call the top level, the middle level and the bottom level; see Figure 6. As the names suggest, the top level is above the middle one, and the middle one is above the bottom one.

The top level contains, in left-to-right order, the bend of \( X^-_j \), the bends of \( X^+_{j+1}, X^-_{j+1}, \ldots, X^+_{k-1}, X^-_{k-1} \), followed by the bend of \( X^+_k \), and finally the bends of \( X^T_{k+1}, X^F_{k+1}, \ldots, X^T_v, X^F_v \).

![Figure 5: A bypass.](image-url)
Figure 6: The fork gadget (left) and the corresponding merge gadget (right) for the clause \((x_i \lor \neg x_j \lor x_k)\). Each light square indicates the positions of two bends on the two staircases representing a variable other than \(x_i, x_j\) and \(x_k\). Each dark square indicates the position of one bend on a staircase representing one of the literals \(x_i, \neg x_i, x_j, \neg x_j, x_k, \neg x_k\).

The middle level contains, left to right, the bends of \(X_{-i}, X_{+i}, X_{T_i}, X_{F_i+1}, X_{T_{i+1}}-1, X_{F_{i+1}}-1, X_{-j}, X_{T_j+1}, X_{F_j+1}, X_{T_{j+2}}, X_{F_{j+2}}, \ldots, X_{-k}, X_{T_k-1}, X_{F_k-1}, X_{-j}, X_{T_j+1}, X_{F_j+1}, X_{T_{j+2}}, X_{F_{j+2}}, \ldots, X_{T_k-1}, X_{F_k-1}, X_{-j}, X_{+j}\).

The bottom level contains the bends of \(X_{-i}, X_{+i}, X_{T_i+1}, X_{F_i+1}, X_{T_{i+1}}, X_{F_{i+1}}, X_{+j}, X_{-j}, X_{T_j+1}, X_{F_j+1}, X_{T_{j+2}}, X_{F_{j+2}}, \ldots, X_{T_k-1}, X_{F_k-1}, X_{-j}, X_{+j}\).

Note that \(Q_{2t}\) does not contain the permutation 321: indeed, any occurrence of 321 would have to intersect all three levels, since each level is an increasing sequence. However, the only elements of the top level that have any element of the bottom level to the right of them are the two elements of the bend of \(X_{-j}\), and neither of these two elements belongs to an occurrence of 321.

We then replace the inner bend \(P_{2t}\) by a single increasing sequence \(\overset{\rightarrow}{P}_{2t}\) containing the first inner bend of all the staircases and bypasses emerging from \(Q_{2t}\). Note that the vertical position of these staircases is already determined by \(\overset{\rightarrow}{Q}_{2t}\).

Finally, we replace \(Q_{2t+1}\) by a merge gadget \(\overset{\leftarrow}{Q}_{2t+1}\), in which all the bypasses that forked in \(Q_{2t}\) will be merged. Note that the horizontal relative position of the bends in \(\overset{\leftarrow}{Q}_{2t+1}\) is determined uniquely by \(\overset{\rightarrow}{P}_{2t}\), while their relative vertical position is fixed by \(\overset{\rightarrow}{P}_{2t+1}\). Essentially, \(\overset{\rightarrow}{Q}_{2t+1}\) is similar to a transpose of \(\overset{\rightarrow}{Q}_{2t}\) along the North-East diagonal, except that a fork of a bypass forms an occurrence of 3412 in \(\overset{\rightarrow}{Q}_{2t}\), whereas the corresponding merge forms an occurrence of 3124 in \(\overset{\rightarrow}{Q}_{2t+1}\). In particular, we may again see that \(Q_{2t+1}\) is 321-free.

Performing the above-described modifications for each \(t \in \{1, 2, \ldots, c\}\), we obtain the text \(\tau\). We let \(\overset{\rightarrow}{Y}\) denote the part of \(\tau\) that was obtained by replacing the bends of \(Y\) by the corresponding gadgets; in other words, \(\overset{\rightarrow}{Y}\) contains all the elements of \(\tau\) except the anchor.

We let \(\overset{\rightarrow}{X}_i\) (or \(\overset{\leftarrow}{X}_i\)) denote the subpermutation obtained as the union of \(X_i\) (or \(X_i\)) and all the bypasses added to it.

It remains to determine the value of \(M\), i.e., the length of the anchors of \(\pi\) and \(\tau\). We
choose $M$ to be equal to $|\tilde{Y}| + 1$, that is, the elements of the anchor $A'$ will outnumber the remainder of $\tau$.

### 2.2 Correctness

We now verify that the construction has the desired properties.

**Proposition 2.2.** Let the formula $\Phi$, pattern $\pi$ and text $\tau$ be as described in the previous subsection. Then $\Phi$ is satisfiable if and only if $\tau$ contains $\pi$.

**Proof.** Suppose that $\Phi$ is satisfiable. Fix any satisfying assignment, and represent it by a function $\phi: \{1, 2, \ldots, v\} \to \{T, F\}$, where $\phi(i) = T$ if and only if the variable $x_i$ is true in the chosen satisfying assignment. We obtain a copy of $\pi$ in $\tau$ as follows. The elements of the anchor $A$ of $\pi$ will be mapped to the anchor $A'$ of $\tau$.

It remains to map the $v$-fold staircase $X = (X_1, X_2, \ldots, X_v)$ to $\tilde{Y}$. We will show that it is possible to map each $X_i$ into $\tilde{X}_i^{\phi(i)}$. To obtain such a mapping, we need to decide, for every bypass appearing in $\tilde{X}_i^{\phi(i)}$, whether to map $X_i$ to the bends of the bypass or to the bends of $X_i^{\phi(i)}$. The decision can be made for each bypass independently, but we need to make sure that for each gadget $\tilde{Q}_m$ in $\tilde{Y}$, the bends mapped into $\tilde{Q}_m$ will form an increasing sequence.

It can be verified by a routine case analysis that such a choice is always possible. To see this, suppose that $K_i$ is a clause $(L_i \lor L_j \lor L_k)$ whose literals contain variables $x_i, x_j$ and $x_k$, respectively, with $i < j < k$. Assume for instance, that the assignment $\phi$ satisfies $L_i$ but not $L_j$ and $L_k$. Then the $(2t)$th outer bends of $X_i, X_j$ and $X_k$ must be mapped to the bends of $\tilde{X}_i^+, \tilde{X}_j^-$ and $\tilde{X}_k^-$ in $\tilde{Q}_{2t}$. The bends of $\tilde{X}_i^+$ and $\tilde{X}_k^-$ are unique, and to preserve monotonicity, we need to choose the bend of $\tilde{X}_j^-$ in the bottom level of $\tilde{Q}_{2t}$, i.e., the bypass of $X_j^-$. Then all the staircases $X_1, X_2, \ldots, X_j$ may be mapped to bends in the bottom level of $\tilde{Q}_{2t}$, while $X_{j+1}, X_{j+2}, \ldots, X_v$ may be mapped to bends in the middle level, preserving monotonicity.

Notice that the position of the bends in $\tilde{Q}_{2t+1}$ is the transpose along the North-East diagonal of their position in $Q_{2t}$, and in particular, the bends will form an increasing sequence in $\tilde{Q}_{2t+1}$ as well.

We conclude that if $\phi$ is a satisfying assignment of $\Phi$, then $\pi$ occurs in $\tau$. For the converse, suppose that $\pi$ has an occurrence in $\tau$. Since the anchor $A$ of $\pi$ is longer than $\tilde{Y}$, at least one element of $A$ must be mapped to an element of $A'$. In particular, all the elements to the left and above $A$ are mapped to elements to the left and above $A'$. In other words, the base of $X$ gets mapped to a subset of the base of $\tilde{Y}$.

Recall that the base of $\tilde{Y}$ is an increasing sequence of $v$ blocks of size 4, where the $i$th block is the union of the base $b_i^{T}$ of $X_i^T$ with the base $b_i^{F}$ of $X_i^F$. Recall also, that each of these $v$ blocks is order isomorphic to 3412, and in particular, the longest increasing subsequence of each block has size 2, and there are exactly two such subsequences, namely $b_i^{T}$ and $b_i^{F}$.

This implies that any increasing subsequence of length $2v$ of the base of $\tilde{Y}$ contains exactly one of $b_i^{T}$ and $b_i^{F}$ for each $i$. In particular, in an occurrence of $\pi$ in $\tau$, the base of $X_i$ is mapped either to $b_i^{T}$ or to $b_i^{F}$. Fix an occurrence of $\pi$ in $\tau$ and use it to define a truth assignment $\phi: [v] \to \{T, F\}$, so that the base of $X_i$ is mapped to $b_i^{\phi(i)}$.

We claim that the assignment $\phi$ satisfies $\Phi$. To see this, we first argue that for every $i \in [v]$, the elements of $X_i$ are mapped to elements of $\tilde{X}_i^{\phi(i)}$, and more specifically, each (inner or outer) bend on $X_i$ is mapped either to the corresponding bend of $X_i^{\phi(i)}$ or to the
corresponding bend of a bypass of $X_i^{\phi(i)}$. We have already seen that this is the case for the base of $X_i$. To show that it holds for the remaining bends also, we may proceed by induction and simply note that the only elements sandwiched from below by an inner bend in $\tilde{X}_i^{\phi(i)}$ are the elements of the subsequent outer bend of $\tilde{X}_i^{\phi(i)}$, or perhaps a pair of outer bends forming the fork of a bypass. An analogous property holds for outer bends as well. Using Lemma 2.1, we may then conclude that the bends of $X_i$ map to corresponding bends in $\tilde{X}_i^{\phi(i)}$.

To see that $\phi$ is satisfying, assume for contradiction that there is a clause $K_t$ whose literals involve the variables $x_i, x_j$ and $x_k$, and neither of these literals is satisfied by $\phi$. It follows that inside the gadget $\tilde{Q}_{2t}$, the bends of $X_i$, $X_j$ and $X_k$ must map to the bends of $\tilde{X}_i^-$, $\tilde{X}_j^-$ and $\tilde{X}_k^-$. However, no three such bends form an increasing sequence in $\tilde{Q}_{2t}$, whereas the corresponding bends form an increasing sequence in $\pi$. This contradiction completes the proof of the proposition.

Proof of Theorem 1.1. Clearly, the problem is in the class NP. It is easy to observe that in our reduction, $\pi$ belongs to $C_{2,0}$. To see that $\tau$ belongs to $C_{3,0}$, it suffices to note that the gadgets $\tilde{Q}_m$ used in the construction of $\tau$ all avoid $321$, and that the base of $\tilde{Y}$ avoids $321$ as well. Note also that the base of $\tilde{Y}$ is to the left and below all the gadgets $\tilde{Q}_m$. Clearly, $\pi$ and $\tau$ can be constructed from $\Phi$ in polynomial time, and the correctness of the reduction follows from Proposition 2.2.

3 Patterns from a proper subclass of $\text{Av}(321)$

In this section we prove Theorem 1.4. We rely on a result by Ahal and Rabinovich [2], who showed that for patterns with bounded “treewidth”, the PPM problem can be solved in polynomial time. Our main contribution is in showing that patterns in $\text{Av}(321)$ of large “treewidth” contain a large universal pattern, containing all patterns in $\text{Av}(321)$ of a given size. To show this, we use the grid minor theorem by Robertson and Seymour [11, 19].

3.1 Permutations and treewidth

The following definition was introduced by Ahal and Rabinovich [2, Definition 2.3]. For a $k$-permutation $\pi$, a graph $G_\pi$ is defined as follows; see Figure 7. The vertices of $G_\pi$ are the numbers $1, 2, \ldots, k$, interpreted as the elements of $\pi = \pi(1), \pi(2), \ldots, \pi(k)$. Two vertices $\pi(i), \pi(j)$ are connected by an edge if $|i - j| = 1$ or $|\pi(i) - \pi(j)| = 1$. We say that an edge between $\pi(i)$ and $\pi(j)$ is red if $|\pi(i) - \pi(j)| = 1$, and it is blue if $|i - j| = 1$. Note that an edge can be both red and blue. Clearly, the edges of each color form a Hamiltonian path in $G_\pi$.

We note that in our definition, $G_\pi$ is a graph, whereas Ahal and Rabinovich [2] defined $G_\pi$ as a multigraph. Also, we label the vertices of $G_\pi$ by their value rather than position in $\pi$.

Let $\text{tw}(G)$ be the treewidth of a graph $G$. The main result of Ahal and Rabinovich [2] can be stated in the following form.

**Theorem 3.1** ([2, Theorem 2.10, Proposition 3.6]). For $\pi \in S_k$ and $\tau \in S_n$, the problem whether $\pi$ is contained in $\tau$ can be solved in time $O(kn^{2\text{tw}(G_\pi)+2})$. 

The $r \times r$ grid is the graph with vertex set $[r] \times [r]$ where vertices $(i, j)$ and $(i', j')$ are joined by an edge if and only if $|i - i'| + |j - j'| = 1$. See Figure 8, left.

Robertson and Seymour [19] proved that for every $r$, every graph of sufficiently large treewidth contains the $r \times r$ grid as a minor. Recently, Chekuri and Chuzhoy [10] showed that a treewidth polynomial in $r$ is sufficient. The upper bound has been further improved by Chuzhoy [11].

**Theorem 3.2** ([11]). There is a function $f : \mathbb{N} \to \mathbb{N}$ satisfying $f(r) \leq r^{10}(\log r)^{O(1)}$ such that every graph of treewidth at least $f(r)$ contains the $r \times r$ grid as a minor.

Since grids have vertices of degree 4, it is more convenient to consider their subgraphs of maximum degree 3, called walls. Let $r \geq 2$ be even. An elementary wall of height $r$ [21] is obtained from the $(r + 1) \times (2r + 2)$ grid by removing two opposite corners $(1, 2r + 2)$ and $(r + 1, 1)$, edges $\{(i, 2j), (i + 1, 2j)\}$ for $i$ odd and $1 \leq j \leq r + 1$, and edges $\{(i, 2j - 1), (i + 1, 2j - 1)\}$ for $i$ even and $1 \leq j \leq r + 1$. That is, an elementary wall of height $r$ is a planar graph of maximum degree 3, which can be drawn as a “wall” with $r$ rows of $r$ “bricks”, where each “brick” is a face of size 6. See Figure 8, right. A subdivision of an elementary wall of height $r$ is called a wall of height $r$ or simply an $r$-wall. It is well known that if $H$ is a graph of maximum degree 3, then a graph $G$ contains $H$ as a minor if and only if $G$ contains a subdivision of $H$ as a subgraph. Therefore, a graph containing the $(2r + 2) \times (2r + 2)$ grid as a minor also contains an $r$-wall as a subgraph.

### 3.3 Universal patterns in $\text{Av}(321)$

Let $\omega$ be a finite sequence of $n$ distinct positive integers. The reduction of $\omega$ is an $n$-permutation obtained from $\omega$ by replacing the $i$th smallest element by $i$, for every $i \in [n]$.
For every \( k \in \mathbb{N} \), we define the \( k \)-track as the \( k^2 \)-permutation that is the reduction of the sequence \((1, 2k, 3, 2k + 2, 5, 2k + 4, \ldots, k^2 - 1, k^2 + 2k - 2)\) if \( k \) is even, and the reduction of \((1, 2k, 3, 2k + 2, 5, 2k + 4, \ldots, k^2 - k - 1, k^2 + k - 2, k^2 - k + 1, k^2 - k + 3, \ldots, k^2 + k - 1)\) if \( k \) is odd. See Figure 9. The \( k \)-track clearly avoids 321 since it is a union of two increasing sequences. In Lemma 3.4 we will show that the \( k \)-track is a universal pattern for all 321-avoiding \( k \)-permutations.

We use the stair-decomposition of 321-avoiding permutations introduced by Guillemot and Vialette [14] and independently, in a slightly different way, by Albert et al. [4]. Let \( \pi \in \text{Av}(321) \) be a \( k \)-permutation. A stair-decomposition of \( \pi \) is a partition of \([k]\), regarded as the set of elements of \( \pi \), into possibly empty subsets \( B_1, B_2, \ldots, B_m \), for some \( m \), such that

- each \( B_i \) forms an increasing subsequence in \( \pi \),
- \( B_{2i} \) is above \( B_{2i-1} \) for each \( i \leq \lfloor m/2 \rfloor \),
- \( B_{2i+1} \) is to the right of \( B_{2i} \) for each \( i \leq \lceil m/2 \rceil - 1 \), and
- \( B_{i+2} \) is above and to the right of \( B_i \) for each \( i \leq m - 2 \).

The subsets \( B_i \) are called the blocks of the stair-decomposition. Sometimes it will be convenient to refer to blocks \( B_0 \) or \( B_{m+1} \), which we define as empty sets.

The \( k \)-track has a stair-decomposition into \( k \) blocks \( A_1, A_2, \ldots, A_k \), each containing exactly \( k \) elements; see Figure 9. Moreover, for every \( i \leq k/2 \), the subset \( A_{2i-1} \cup A_{2i} \) forms a vertical alternation in the \( k \)-track, that is, \( A_{2i} \) is above \( A_{2i-1} \) and the elements from \( A_{2i-1} \) alternate from left to right with the elements from \( A_{2i} \) in the \( k \)-track. Similarly, for every \( i \leq (k-1)/2 \), the subset \( A_{2i} \cup A_{2i+1} \) forms a horizontal alternation in the \( k \)-track, that is, \( A_{2i+1} \) is to the right of \( A_{2i} \) and the elements of \( A_{2i} \) alternate from bottom to top with the elements of \( A_{2i+1} \) in the \( k \)-track.

Guillemot and Vialette [14] state the following lemma without proof. Albert et al. [4] give an algorithm to compute the stair-decomposition but do not include verification of correctness.

**Lemma 3.3.** Every 321-avoiding \( k \)-permutation has a stair-decomposition with at most \( k \) blocks.

**Proof.** Let \( \pi \in \text{Av}(321) \) be a \( k \)-permutation. We define a stair-decomposition of \( \pi \) by a greedy algorithm. Let \( B_1 \) be the longest interval \([i]\) whose elements form an increasing subsequence
in \( \pi \). Let \( \rho_1 \) be the subsequence of \( \rho_0 = \pi \) formed by the elements in \([k] \setminus B_1\). Now let \( B_2 \) be the subset of \([k] \setminus B_1\) whose elements form the maximal increasing prefix of \( \rho_1 \). Let \( \rho_2 \) be the subsequence of \( \rho_1 \) obtained by removing the elements of \( B_2 \). We continue analogously. Suppose that \( \rho_{2i} \) and \( B_1, B_2, \ldots, B_{2i} \) have been defined. Then let \( B_{2i+1} \) be the maximal down-set of \([k] \setminus \bigcup_{j=1}^{2i} B_j\) forming an increasing subsequence in \( \rho_{2i} \), and let \( \rho_{2i+1} \) be the subsequence of \( \rho_{2i} \) obtained by removing the elements of \( B_{2i+1} \). Finally, let \( B_{2i+2} \) be the subset of \([k] \setminus \bigcup_{j=1}^{2i+1} B_j\) whose elements form the maximal increasing prefix of \( \rho_{2i+1} \), and let \( \rho_{2i+2} \) be the subsequence of \( \rho_{2i+1} \) obtained by removing the elements of \( B_{2i+2} \). We continue until \( \rho_{2i} \) or \( \rho_{2i+1} \) is empty and we denote by \( m \) the largest index such that \( B_m \) is nonempty. Clearly \( m \leq k \).

Now we verify that \( B_1, B_2, \ldots, B_m \) is indeed a stair-decomposition of \( \pi \). The facts that \( B_{2i} \) and \( B_{2i+1} \) are above \( B_{2i-1} \), and that \( B_{2i+2} \) and \( B_{2i+1} \) are to the right of \( B_{2i} \) for every \( i \) follow directly from the construction.

For every \( i \), the block \( B_{2i+1} \) is to the right of \( B_{2i-1} \) since the set of elements above and to the left of \( \max(B_{2i-1}) \) forms an increasing subsequence in \( \rho_{2i-2} \); a decreasing pair would form a forbidden pattern 321 with \( \max(B_{2i-1}) \). Finally, for every \( i \), the block \( B_{2i+2} \) is above \( B_{2i} \) since the set of elements below and to the right of \( \max(B_{2i}) \) forms an increasing subsequence in \( \rho_{2i-1} \); a decreasing pair would induce a forbidden pattern 321 with \( \max(B_{2i}) \). \( \square \)

Albert et al. [4, Proposition 6] proved that each 321-avoiding permutation of size \( k \) is contained in an \( m \)-track for some \( m \leq 2^k \). Using similar ideas, we observe the following stronger fact.

**Lemma 3.4.** Let \( \pi \in \text{Av}(321) \) be a \( k \)-permutation, and let \( B_1, B_2, \ldots, B_m \) be its stair-decomposition. Let \( q = \max\{k,m\} \). Let \( \tau_q \) be the \( q \)-track, and let \( A_1, A_2, \ldots, A_q \) be its stair-decomposition into blocks of size \( q \). Then \( \pi \) has an occurrence in \( \tau_q \) in which the elements of \( B_i \) are mapped into \( A_i \) for every \( i = 1, 2, \ldots, q \).

**Proof.** For \( m = 1 \) the claim is trivial, so we assume that \( m \geq 2 \).

For \( i \leq \lfloor m/2 \rfloor \), let \( \prec_2i-1 \) be the left-to-right linear order of the elements from \( B_{2i-1} \cup B_{2i} \) in \( \pi \). Similarly, for \( i < \lfloor m/2 \rfloor - 1 \), let \( \prec_2i \) be the bottom-to-top linear order of the elements from \( B_{2i} \cup B_{2i+1} \) in \( \pi \) (that is, \( \prec_2i \) is the restriction of the standard linear order \( \prec \) of the integers). We claim that there is a linear order \( \prec_\pi \) on \([k] \) that simultaneously extends all the orders \( \prec_1, \prec_2, \ldots, \prec_{m-1} \). We show this by induction on \( m \). For \( m = 2 \) we can take \( \prec_\pi \) as \( \prec_1 \). Now let \( m \geq 3 \) and assume that there is a linear order \( \prec' \) extending the union \( \prec_1 \cup \prec_2 \cup \cdots \cup \prec_{m-2} \). This implies that there is an injective function \( f: [k] \setminus B_m \to \mathbb{R} \) satisfying \( a \prec' b \Leftrightarrow f(a) < f(b) \). Since \( \prec_{m-2} \) and \( \prec_{m-1} \) intersect at a linear order on \( B_{m-1} \), which is a restriction of both \( \prec \) and \( \prec' \), we have \( a \prec_{m-1} b \Leftrightarrow a \prec' b \Leftrightarrow f(a) < f(b) \) for every \( a, b \in B_{m-1} \). Clearly, we can extend the function \( f \) to the elements of \( B_m \) so that \( a \prec_{m-1} b \Leftrightarrow f(a) < f(b) \) for every \( a, b \in B_{m-1} \cup B_m \). The order \( \prec_\pi \) can now be defined as \( a \prec_\pi b \Leftrightarrow f(a) < f(b) \) for all \( a, b \in [k] \).

With the order \( \prec_\pi \) at hand, we define an embedding of \( \pi \) to the \( q \)-track as follows. Let \( b_1, b_2, \ldots, b_k \) be the permutation of \([k] \) satisfying \( b_1 \prec_\pi b_2 \prec_\pi \cdots \prec_\pi b_k \). Similarly, for every \( i \in [q] \), let \( a_{i,1}, a_{i,2}, \ldots, a_{i,k} \) be the permutation of \( A_i \) satisfying \( a_{i,1} < a_{i,2} < \cdots < a_{i,k} \). For every \( j \in [k] \), let \( i(j) \) be the index such that \( b_j \in B_{i(j)} \). Then the map sending \( b_j \) to \( a_{i(j),j} \) for each \( j \in [k] \), is an embedding of \( \pi \) to the \( q \)-track. \( \square \)
3.4 Patterns in Av(321) of large “treewidth”

Using the results from previous subsections, Theorem 1.4 will follow from the following theorem.

**Theorem 3.5.** There is a function \( g : \mathbb{N} \to \mathbb{N} \) satisfying \( g(k) \leq O(k^{3/2}) \) with the following property. If \( \pi \in \text{Av}(321) \) is a permutation such that \( G_\pi \) contains a \( g(k) \)-wall, then \( \pi \) contains the \( k \)-track.

We will need the following lemma providing an upper bound on the treewidth of \( G_\rho \) for 321-avoiding permutations \( \rho \).

**Lemma 3.6.** If \( \rho \in \text{Av}(321) \) has a stair-decomposition with \( m \) blocks, then the treewidth of \( G_\rho \) is at most \( 2m \).

We show that, in fact, the pathwidth of \( G_\rho \) is at most \( 2m \). Kinnersley [17] proved that the pathwidth of a graph \( G \) is equal to the vertex separation number \( \text{vs}(G) \), defined as follows. If \( G \) is a graph with \( n \) vertices, \( L = (v_1, v_2, \ldots, v_n) \) is a linear ordering of the vertices and \( i \in [n] \), let \( V(G, L, i) \) be the set of vertices \( v_j \) such that \( j < i \) and \( G \) has an edge \( v_j v_k \) with \( k \geq i \). Then let

\[
\text{vs}(G) = \min_L \max_i |V(G, L, i)|.
\]

**Proof of Lemma 3.6.** Let \( \rho \in \text{Av}(321) \) be an \( n \)-permutation that has a stair-decomposition with blocks \( B_1, B_2, \ldots, B_m \). Let \( <_\rho \) be the linear ordering of the elements of \( \rho \) defined in the proof of Lemma 3.4. We claim that for each \( i \in [n] \) we have \( |V(G_\rho, <_\rho, i)| \leq 2m \).

The graph \( G_\rho \) is a union of two Hamiltonian paths, the blue path \( (\pi(1), \pi(2), \ldots, \pi(n)) \) formed by the blue edges and the red path \( (1, 2, \ldots, n) \) formed by the red edges. The blue path can be decomposed into \( \lceil m/2 \rceil \) \( <_\rho \)-increasing paths induced by the subsets \( B_{2j-1} \cup B_{2j} \) and \( \lfloor m/2 \rfloor - 1 \) remaining blue edges. Similarly, the red path can be decomposed into \( 1 + \lceil m/2 \rceil \) \( <_\rho \)-increasing paths induced by the subsets \( B_{2j} \cup B_{2j+1} \) and \( \lfloor m/2 \rfloor \) remaining red edges. All together, \( G_\rho \) has a decomposition into \( 2m <_\rho \)-monotone paths, and each such path contributes at most one vertex to each \( V(G_\rho, <_\rho, i) \).

Let \( \pi \in \text{Av}(321) \) be a permutation with a stair-decomposition \( B_1, B_2, \ldots, B_m \). Let \( xy \) be an edge of \( G_\pi \) such that \( x \in B_i, y \in B_j \) and \( i \leq j \). We say that \( xy \) is *good* if at least one of the following conditions is satisfied:

- \( i = j \),
- \( xy \) is blue, \( i \) is odd and \( j = i + 1 \), or
- \( xy \) is red, \( i \) is even and \( j = i + 1 \).

Otherwise \( xy \) is *bad*.

We introduce the following notation: if \( B_1, B_2, \ldots, B_m \) is a sequence of sets and \( i \in [m] \) an integer, then \( B_{<i} \) denotes the set \( \bigcup_{j=1}^{i-1} B_j \). The sets \( B_{\leq i}, B_{>i} \) and \( B_{\geq i} \) are defined analogously.

**Lemma 3.7.** Let \( \pi \in \text{Av}(321) \) be a permutation with a stair-decomposition \( B_1, B_2, \ldots, B_m \). Then

1) \( G_\pi \) has at most \( m - 1 \) bad edges,
2) for each \( i \in [m] \) there is at most one vertex \( x \in B_i \) incident to a bad edge \( xy \) with \( y \in B_{> i} \), and

3) for each \( i \in [m] \) there is at most one vertex \( x \in B_i \) incident to a bad edge \( yx \) with \( y \in B_{< i} \).

Proof. If \( xy \) is a bad edge with \( x \in B_i \) and \( y \in B_{> i} \), then \( x \) must be the topmost (or equivalently rightmost) vertex of \( B_i \). This shows part 2), and part 3) is analogous.

For part 1), we observe that every bad blue edge starts at the rightmost vertex of some pair of blocks \( B_{2i-1}, B_{2i} \), and for each such pair with \( 2i < m \) there is at most one such edge. Similarly, every bad red edge starts at the topmost vertex of some pair of blocks \( B_{2i}, B_{2i+1} \), and for each such pair with \( 2i + 1 < m \) there is at most one such edge. \(\square\)

The following lemma is the first step towards the proof of Theorem 3.5. The goal is to find “many” vertex-disjoint paths in \( G_x \) between two “distant” blocks of the stair-decomposition.

**Lemma 3.8.** Let \( k \) be an integer. Let \( G \) be a graph whose vertex set is partitioned into a sequence of (possibly empty) sets \( B_1, B_2, \ldots, B_m \), such that for every \( i \in [m - k + 1] \) the subgraph of \( G \) induced by the set \( \bigcup_{j=i}^{i+k-1} B_j \) has treewidth less than \( 10k \). If \( G \) contains an \( r \)-wall for some \( r \geq 300k^{3/2} \), then there is an integer \( b \leq m - k \) such that \( G \) contains 10k vertex-disjoint paths connecting the set \( B_{\leq b} \) to the set \( B_{\geq b+k} \).

Proof. Define \( q = 10k \) and \( s = 6\lceil \sqrt{k} \rceil \). We will show that \( G \) contains \( s^2/2 > 10k \) vertex-disjoint paths satisfying the properties of the lemma. Let \( W \) be an \( r \)-wall in \( G \).

For each \( i, j \in [s] \), define a \( q \)-wall \( W_{i,j} \) as a subgraph of \( W \), so that the walls \( W_{i,j} \) are arranged roughly in an \( s \times s \) lattice, every pair of these \( q \)-walls is separated by at least \( 2s + 2 \) bricks of \( W \), and each \( W_{i,j} \) is separated by at least \( s^2 + 1 \) bricks from the boundary of \( W \). This is possible as \( qs + (2s + 1)(s - 1) + 2s^2 + 2 \leq qs + 4s^2 < r \).

Let \( I_{i,j} \) be the set of indices \( l \) such that \( W_{i,j} \) intersects \( B_l \). Since every \( q \)-wall contains the \( q \times q \) grid as a minor, its treewidth is at least \( q = 10k \). By the assumption, the \( q \)-wall is not contained in any union of \( k \) consecutive blocks \( B_l \), and in particular, the minimum and the maximum of \( I_{i,j} \) differ by at least \( k \).

Let \( M_{i,j} \) be the median of \( I_{i,j} \). Let \( M \) be the median of the multiset \( \{M_{i,j} : i, j \in [s]\} \). Let \( c = M - k/2 \) and \( d = M + k/2 \). The numbers \( c, d \) are chosen so that every set \( I_{i,j} \) with \( M_{i,j} \leq M \) contains a number \( c_{i,j} \leq c \) and every set \( I_{i,j} \) with \( M_{i,j} \geq M \) contains a number \( d_{i,j} \geq d \). Let \( C, D \subseteq [s]^2 \) be sets of size \( s^2/2 \) forming a partition of \( [s]^2 \) such that \( M_{i,j} \leq M \) if \((i, j) \in C \) and \( M_{i,j} \geq M \) if \((i, j) \in D \). From every \( q \)-wall \( W_{i,j} \) such that \((i, j) \in C \), choose a vertex \( w_{i,j} \in B_{c_{i,j}} \cap W_{i,j} \). Similarly, from every \( q \)-wall \( W_{i,j} \) such that \((i, j) \in D \), choose a vertex \( w'_{i,j} \in B_{d_{i,j}} \cap W_{i,j} \).

Let \( w_1, w_2, \ldots, w_{s^2/2} \) be a relabeling of the vertices \( w_{i,j} \) corresponding to the lexicographic order of the pairs \((j, i)\). We claim that there is a relabeling \( w'_1, w'_2, \ldots, w'_{s^2/2} \) of the vertices \( w'_{i,j} \) and \( s^2/2 \) vertex-disjoint paths \( P_1, P_2, \ldots, P_{s^2/2} \) in \( W \) where each \( P_i \) starts at \( w_i \) and ends at \( w'_i \). We note that with a stronger assumption on the distances between the subwalls \( W_{i,j} \), a more general result by Robertson and Seymour [20] would imply the existence of such paths for any relabeling of the vertices \( w'_{i,j} \).

We choose the relabeling \( w'_1, w'_2, \ldots, w'_{s^2/2} \) of the vertices \( w'_{i,j} \) so that it again corresponds to the lexicographic order of the pairs \((j, i)\).
We construct the paths as follows; see Figure 10. Let \( t \in [s^2/2] \) and let \( i, j, i', j' \in [s] \) be such that \( w_t = w_{i,j} \) and \( w'_t = w'_{i',j'} \). The path \( P_t \) is composed from nine horizontal or nearly vertical subpaths \( P_{t,1}, P_{t,2}, \ldots, P_{t,9} \). Here by a nearly vertical path in \( W \) going upwards (downwards) we mean a path consisting of subpaths between branching vertices of \( W \) with internal vertices of degree 2, where the directions of the subpaths periodically alternate as up, right, up, left (down, right, down, left, respectively). The path \( P_{t,1} \) starts at \( w_t \) and goes up or to the right to the closest branching vertex in \( W \). The path \( P_{t,2} \) continues directly to the right to the \( 2(i+1) \)th branching vertex outside of \( W_{i,j} \). The path \( P_{t,3} \) is nearly vertical continuing upwards to the \( 2(t+1) \)th row of \( W \), and it is followed by a horizontal path \( P_{t,4} \) ending at the \( 2t \)th branching vertex from the right of \( W \). Then \( P_{t,5} \) is nearly vertical continuing downwards, to the \( 2t \)th row from the bottom of \( W \).

The remaining subpaths of \( P_t \) are defined symmetrically. Starting at \( w'_t \), the path \( P_{t,9} \) goes down or to the left to the closest branching vertex, followed by \( P_{t,8} \) that goes directly to the left to the \( 2(s+1) - i' \)th branching vertex outside of \( W_{i',j'} \). Then \( P_{t,7} \) is nearly vertical continuing downwards to the \( 2t \)th row of \( W \) from the bottom, and \( P_{t,6} \) continues directly to the right to the \( 2t \)th branching vertex from the right, which is also the endpoint of \( P_{t,5} \).

Since all the paths connect the set \( B_{\leq c} \) to the set \( B_{\geq d} \) with \( d \geq c + k \), the lemma follows.

The next lemma shows how to find the \( k \)-track in \( G_\pi \), given many vertex disjoint paths between two distant blocks of the stair-decomposition.

**Lemma 3.9.** Let \( \pi \in \text{Av}(321) \) be a permutation with a stair-decomposition \( B_1, B_2, \ldots, B_m \). Suppose that there is an odd index \( a \in [m - k + 1] \) such that the subgraph of \( G_\pi \) induced by \( \bigcup_{j=a}^{a+k-1} B_j \) contains \( 3k - 2 \) vertex-disjoint paths \( Q_1, Q_2, \ldots, Q_{3k-2} \), each of them connecting a vertex in \( B_a \) to a vertex in \( B_{a+k-1} \), and each of them containing only good edges. Then \( \pi \) contains the \( k \)-track \( \tau_k \), and moreover, it has an occurrence of \( \tau_k \) in which the elements from the \( j \)th block of \( \tau_k \) are mapped into \( B_{a+j-1} \).

**Proof.** Truncating the paths \( Q_i \) if necessary, we may assume that each of them has exactly one vertex in \( B_a \) and exactly one vertex in \( B_{a+k-1} \). Suppose without loss of generality that
the paths are numbered in such a way that the endpoint of $Q_t$ in $B_a$ is to the left of the endpoint of $Q_{t+1}$ in $B_a$ for every $t \in [3k-3]$.

For every $i, j \in [k]$, let $v_{i,j}$ be the last vertex on $Q_{2(i-1)+j}$ that lies in $B_{a-1+j}$. Let $\sigma$ be the pattern formed by these $k^2$ elements $v_{i,j}$ in $\pi$. We claim that $\sigma$ is the $k$-track. Clearly, the sets $B_a, B_{a+1}, \ldots, B_{a+k-1}$ induce a stair decomposition of $\sigma$, with each block having exactly $k$ elements. It remains to verify that consecutive blocks induce vertical or horizontal alternations in $\sigma$. More precisely, we want to show that for each $j \in [k-1]$, we have

$$v_{1,j} < v_{1,j+1} < v_{2,j} < v_{2,j+1} < \cdots < v_{k,j} < v_{k,j+1}$$

where $<j$ is the horizontal left-to-right order on $B_{a-1+j} \cup B_{a+j}$ for $j$ odd and the vertical bottom-to-top order on $B_{a-1+j} \cup B_{a+j}$ for $j$ even.

We show (1) only for $j = 1$ since the other cases are analogous and follow by induction on $j$, using the induction assumption that $v_{1,j}, v_{2,j}, \ldots, v_{k,j}$ is an increasing subsequence in $\pi$.

By symmetry, it is sufficient to prove that $v_{1,1} < v_{1,2}$; in other words, that $v_{1,1}$ is to the left of $v_{1,2}$ in $\pi$. This will follow from the fact that all vertices of $Q_1$ in $B_{a+1}$ are to the left of all vertices of $Q_2$ in $B_{a+1}$. Indeed, the neighbor $v'$ of $v_{1,1}$ in $Q_1$ is at a position adjacent to $v_{1,1}$ in $\pi$, so every element in $B_{a+1}$ that is to the right of $v'$ is also to the right of $v_{1,1}$.

Let $G$ be the subgraph of $G_\pi$ consisting of the vertices $B_a \cup B_{a+1} \cup \cdots \cup B_{a+k-1}$ and all the good edges among these vertices. Clearly, $G$ contains all the paths $Q_1, Q_2, \ldots, Q_{3k-2}$.

We observe that $G$ has the following planar drawing; see Figure 11. For each $j \in [k]$, draw the vertices of $B_{a-1+j}$ on the vertical line with $x$-coordinate $j$, ordered from bottom to top according to the order $<j$. Then draw all the edges as straight-line segments.

Let $\Omega = \{(x, y) \in \mathbb{R}^2; 1 \leq x \leq k\}$, which is a convex region containing all the vertices and edges of the drawing. All the vertices of $B_a$ and $B_{a+k-1}$ are on the boundary of $\Omega$. Each of the paths $Q_t$ connects a vertex in $B_a$ with a vertex in $B_{a+k-1}$, and all the inner vertices of $Q_t$ are in $B_{a+1} \cup B_{a+2} \cup \cdots \cup B_{a+k-2}$. Therefore, each path $Q_t$ divides $\Omega$ into two regions: the region below $Q_t$ and the region above $Q_t$. In particular, $Q_2$ is contained in the region above $Q_1$.

Suppose for contradiction that there are vertices $u_1 \in Q_1 \cup B_{a+1}$ and $u_2 \in Q_2 \cup B_{a+1}$ such

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure11.png}
\caption{Left: a subgraph $G$ of $G_\pi$ consisting of good edges. Right: a planar drawing of $G$.}
\end{figure}
that $u_2$ is below $u_1$ on the line with $x$-coordinate 2; see Figure 12. Let $Q_1'$ be the subpath of $Q_1$ between $v_{1,1}$ and $u_1$. Since $u_2$ is in the region above $Q_1$, the vertex $v'$, which is the neighbor of $v_{1,1}$ on $Q_1$, is below $u_2$. Consequently, $u_2$ is inside the bounded region formed by $Q_1'$ and the segment $u_1v_{1,1}$. Since $Q_1$ and $Q_2$ do not cross, it follows that $Q_2$ crosses the segment $u_1v_{1,1}$ twice. But this means that there are at least two vertices on $Q_2$ in $B_a$, which is a contradiction.

**Proof of Theorem 3.5.** We shall prove that the theorem holds for the function $g(k) = 300k^{3/2}$. Suppose that $\pi \in \text{Av}(321)$ is a permutation such that $G_\pi$ contains a $g(k)$-wall. Let $B_1, B_2, \ldots, B_m$ be a stair-decomposition of $\pi$. By Lemma 3.6, the subgraph of $G_\pi$ induced by any $k$ consecutive blocks of the decomposition has treewidth at most $2k$. We may therefore apply Lemma 3.8 to the graph $G_\pi$ and the sets $B_1, B_2, \ldots, B_m$, to conclude that there is an index $b \in [m - k]$ and a set of vertex-disjoint paths $P_1, P_2, \ldots, P_{10k}$, each connecting $B_{\leq b}$ to $B_{\geq b + k}$.

Let $a$ be the smallest odd integer greater than or equal to $b$. Note that $G_\pi$ has at most two edges connecting a vertex in $B_{\leq a}$ to a vertex in $B_{> a}$, since $B_{> a}$ is above and to the right of $B_{\leq a}$. By the same argument, there are at most two edges between $B_{\leq a + k - 1}$ and $B_{> a + k - 1}$. We remove from $P_1, P_2, \ldots, P_{10k}$ all the paths using any of these edges, and observe that each of the remaining paths contains a vertex from $B_a$ as well as from $B_{a + k - 1}$. Truncating these paths if necessary, we may assume that all their vertices are in the set $\bigcup_{j=a+1}^{a+k-1} B_j$. By Lemma 3.7, this set induces at most $k - 1$ bad edges in $G_\pi$. Removing from our set of paths any path containing a bad edge, we are left with at least $10k - 4 - k + 1 > 3k - 2$ paths from $B_a$ to $B_{a + k - 1}$ which only contain good edges. By Lemma 3.9, $\pi$ contains the $k$-track, as claimed.

**3.5 Proof of Theorem 1.4**

Let $\mathcal{C}$ be a proper subclass of $\mathcal{C}_{2,0}$. Let $s$ be the smallest positive integer such that there is a 321-avoiding $s$-permutation that is not contained in $\mathcal{C}$. By Lemma 3.4, the $s$-track is not contained in $\mathcal{C}$ either. By Theorem 3.5, for some function $g(s) \leq O(s^{3/2})$, there is no permutation $\pi \in \mathcal{C}$ such that $G_\pi$ contains a $g(s)$-wall. Consequently, no such graph $G_\pi$ contains the $(2g(s) + 2)\times (2g(s) + 2)$ grid as a minor. By Theorem 3.2, for every permutation $\pi \in \mathcal{C}$, the treewidth of the graph $G_\pi$ is at most $f(2g(s) + 2)) \leq s^{28.5}(\log s)^{O(1)}$. Finally, Theorem 3.1 implies that for $\pi \in S_k \cap \mathcal{C}$ and $\tau \in S_n$, the problem whether $\pi$ is contained in $\tau$ can be solved in time $kn s^{28.5}(\log s)^{O(1)}$.  

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4 Skew-merged patterns

Recall that $C_{1,1}^\times$ is the class $Av(2143, 3412, 3142)$. Our goal is to prove Theorems 1.2 and 1.5. To this end, we will first show that permutations from $C_{1,1}^\times$ admit a decomposition which is analogous to the stair-decomposition for the class $C_{2,0}$. This will allow us to show that the proofs of Theorems 1.1 and 1.4 can be straightforwardly adapted into proofs of Theorems 1.2 and 1.5, respectively.

Definition 4.1. Let $\pi \in S_k$ be a permutation. A spiral decomposition of $\pi$ is a partition of $\pi$ into a sequence $B_1, B_2, \ldots, B_m$, where each $B_i$ is a possibly empty subset of elements of $\pi$, satisfying the following properties:

(a) If $i$ is odd then $B_i$ is a decreasing subsequence, and if $i$ is even then $B_i$ is an increasing subsequence of $\pi$.

(b) For $i \in [m]$, let $r_i \in \{0, 1, 2, 3\}$ be the remainder of $i$ modulo 4. See the right part of Figure 13. If $r_i = 0$ then $B_i$ is above $B_{i-1}$ and $B_{>i}$ is above and to the left of $B_{i-1}$. If $r_i = 1$ and $i > 1$ then $B_i$ is to the left of $B_{i-1}$ and $B_{>i}$ is below and to the left of $B_{i-1}$. If $r_i = 2$ then $B_i$ is below $B_{i-1}$ and $B_{>i}$ is below and to the right of $B_{i-1}$. If $r_i = 3$ then $B_i$ is to the right of $B_{i-1}$ and $B_{>i}$ is above and to the right of $B_{i-1}$.

The subsequences $B_1, B_2, \ldots, B_m$ are the blocks of the spiral decomposition.

Lemma 4.2. A permutation $\pi$ belongs to the class $C_{1,1}^\times$ if and only if $\pi$ has a spiral decomposition. Moreover, every permutation in $C_{1,1}^\times$ has a spiral decomposition with no four consecutive empty blocks.

Proof. Suppose first that $\pi$ is in $C_{1,1}^\times$. In particular, $\pi$ is skew-merged, and therefore it can be partitioned into an increasing subsequence $I$ and a decreasing subsequence $D$.

We now define an infinite sequence $B_1, B_2, \ldots$, where each $B_i$ is a possibly empty subset of $\pi$. The $B_i$ will be defined inductively. For an $i \in \mathbb{N}$, suppose that $B_1, B_2, \ldots, B_{i-1}$ have already been defined. Define $I_i$ as $I \setminus (B_1 \cup B_2 \cup \cdots \cup B_{i-1})$ and $D_i$ as $D \setminus (B_1 \cup B_2 \cup \cdots \cup B_{i-1})$.

Let $r_i \in \{0, 1, 2, 3\}$ be the remainder of $i$ modulo 4. To define $B_i$, we distinguish four cases:

- If $r_i = 0$, then $B_i = \{x \in I_i; x$ is to the right of $D_i\}$.
- If $r_i = 1$, then $B_i = \{x \in D_i; x$ is above $I_i\}$.
- If $r_i = 2$, then $B_i = \{x \in I_i; x$ is to the left of $D_i\}$.
- If $r_i = 3$, then $B_i = \{x \in D_i; x$ is below $I_i\}$.

Let $B_m$ be the last nonempty block in the sequence $B_1, B_2, \ldots$ defined above (if all the blocks are empty, set $m = 0$). We now show that $B_1, B_2, \ldots, B_m$ is a spiral decomposition of $\pi$. Clearly, $B_i$ is a decreasing sequence for odd $i$ and an increasing sequence for even $i$. Let $B_\infty$ be the set of elements of $\pi$ not belonging to $B_i$ for any integer $i$.

Let us verify that the blocks have the correct mutual position. Let $B_{>i}$ be the set of elements of $\pi$ not belonging to $B_1 \cup B_2 \cup \cdots \cup B_i$; in particular, $B_\infty$ is a subset of $B_{>i}$. We
Suppose that \( j \) is even, the other case being analogous. Note that \( I_i = I_{i-1} \) and \( D_i = D_{i-1} \setminus B_{i-1} \). Since \( B_{i-1} \) is below \( I_{i-1} \) by definition and \( B_i \) is a subset of \( I_{i-1} \), we see that \( B_i \) above \( B_{i-1} \).

To see that \( B_{>i} \) is to the left and above \( B_{i-1} \), notice that \( B_{>i} \) is the disjoint union of \( D_i \) and \( I_i \setminus B_i \). Clearly \( D_i \) is above and to the left of \( B_{i-1} \), and \( I_i \) is above \( B_{i-1} \) by definition of \( B_{i-1} \). Moreover, every element of \( I_i \setminus B_i \) must be to the left of at least one element of \( D_i \) by definition of \( B_i \), and therefore also to the left of \( B_{i-1} \). This shows that \( B_{>i} \) is to the left and above \( B_{i-1} \).

To prove that \( B_1, B_2, \ldots, B_m \) is a spiral decomposition, it remains to show that each element of \( \pi \) is in \( B_1 \cup B_2 \cup \cdots \cup B_m \). Suppose for contradiction that this is not the case, that is, suppose that \( B_\infty \) is not empty. Let \( b, l, t, r \) be the bottommost, leftmost, topmost and rightmost element of \( B_\infty \), respectively.

Let \( M \) be the smallest multiple of 4 larger than \( m \). Note that \( B_{>M} = B_{>m} = B_\infty \), since all the blocks following \( B_m \) are empty. Consider the four empty blocks \( B_M, B_{M+1}, B_{M+2}, \) and \( B_{M+3} \). We may deduce that \( r \) belongs to \( D_M \), for otherwise it would be to the right of \( D_M \) and would belong to \( B_M \). By the same argument, \( t \) belongs to \( I_M \), \( l \) belongs to \( D_M \), and \( b \) belongs to \( I_M \). It follows that \( l, b, t, r \) form an occurrence of the pattern 3142, contradicting the assumption that \( \pi \) is in \( C_{1,1}^\times \).

We have thus shown that each permutation in \( C_{1,1}^\times \) has a spiral decomposition. The argument actually shows that we may find a spiral decomposition with no four consecutive empty blocks.

We now show that every permutation with a spiral decomposition is in \( C_{1,1}^\times \). Let \( \pi \) be a permutation with a spiral decomposition \( B_1, B_2, \ldots, B_m \). It easily follows from Definition 4.1 that the union of the odd-numbered blocks is a decreasing sequence and the union of the even-numbered ones is an increasing sequence. In particular, \( \pi \) is skew-merged.

It remains to show that \( \pi \) avoids 3142. Suppose for contradiction that \( \pi \) has a subsequence \( S = (l, b, t, r) \) forming an occurrence of 3142. Since there is only one way to partition 3142 into a decreasing and an increasing subsequence, we know that \( l \) and \( r \) belong to odd-numbered blocks, and \( t \) and \( b \) to even-numbered ones.

Let \( B_j \) be the first block of the spiral decomposition containing at least one element of \( S \). Suppose that \( j \) is even, the other case being analogous. Then \( B_j \) contains \( t \) or \( b \), and both \( l \) and \( r \) belong to \( B_{>j} \). From part (b) of Definition 4.1 we may deduce that either \( B_{>j} \) is to the left of \( B_j \) (if \( j \equiv 0 \pmod{4} \)), or \( B_{>j} \) is to the right of \( B_j \) (if \( j \equiv 2 \pmod{4} \)). This contradicts the fact that both \( l \) and \( r \) belong to \( B_{>j} \) and at least one of \( t \) and \( b \) is in \( B_j \). \( \square \)

### 4.1 The twirl

We now introduce an operation called the twirl, whose main purpose is to transform a permutation \( \pi \) with a given stair-decomposition \( B_1, B_2, \ldots, B_m \) into a permutation \( \pi^* \) with a spiral decomposition \( B_1^*, B_2^*, \ldots, B_m^* \). Intuitively, each block \( B_i^* \) will be obtained by rotating or reflecting \( B_i \), and then rearranging the blocks into a spiral.

Formally, given a permutation \( \pi \) with a stair-decomposition \( B_1, B_2, \ldots, B_m \), the **twirl** of \( \pi \) (with respect to \( B_1, B_2, \ldots, B_m \)) is a permutation \( \pi^* \) with a spiral decomposition \( B_1^*, B_2^*, \ldots, B_m^* \), determined by these rules:
Figure 13: Left: a permutation $\pi \in C_{2,0}$ with a stair-decomposition into six blocks. The top-right corner of each block is highlighted. Right: the result of twirling $\pi$—a spiral decomposition whose blocks are rotated and reflected versions of the blocks of $\pi$, with the highlighted corners now pointing towards the center of the spiral.

- For each $i$, the block $B^*_i$ is obtained by rotating or reflecting $B_i$, in such a way that rows of $B_i$ get mapped to rows of $B^*_i$ and similarly for columns. Moreover, the top-right corner of $B_i$ will be mapped to the corner of $B^*_i$ that is nearest to the center of the spiral; see Figure 13.

- For every pair of consecutive blocks $B_i$ and $B_{i+1}$, let $x \in B_i$ and $y \in B_{i+1}$ be a pair of elements, and let $x^*$ and $y^*$ be the corresponding elements of $B^*_i$ and $B^*_{i+1}$, respectively. Then the mutual position of $x^*$ and $y^*$ corresponds to the mutual position of $x$ and $y$, in the following sense:
  
  - if $i \equiv 0 \mod 4$, then $x^*$ is above $y^*$ if and only if $x$ is below $y$,
  - if $i \equiv 1 \mod 4$, then $x^*$ is to the right of $y^*$ if and only if $x$ is to the right of $y$,
  - if $i \equiv 2 \mod 4$, then $x^*$ is above $y^*$ if and only if $x$ is above $y$,
  - if $i \equiv 3 \mod 4$, then $x^*$ is to the right of $y^*$ if and only if $x$ is left of $y$.

Note that the properties above, together with the fact that $B^*_1, B^*_2, \ldots, B^*_m$ form a spiral decomposition, determine $\pi^*$ uniquely. We say that $\pi^*$ is the twirl of $\pi$ and $\pi$ is the untwirl of $\pi^*$ (with respect to given decompositions $B_1, B_2, \ldots, B_m$ and $B^*_1, B^*_2, \ldots, B^*_m$, respectively).

We will also need to apply this transform in situations when the blocks of the decomposition do not necessarily induce a monotone subsequence. We therefore generalize the two notions of decomposition appropriately. We say that a partition of a permutation $\pi$ into a sequence of blocks $B_1, B_2, \ldots, B_m$ forms a relaxed stair-decomposition if it satisfies all the properties of stair-decomposition except possibly for the property that each $B_i$ forms an increasing subsequence. Similarly, a relaxed spiral decomposition satisfies all the properties of a spiral decomposition, except perhaps the property that each block is a monotone subsequence.
The twirl transform can be applied to a permutation with a relaxed stair-decomposition to obtain a permutation with a relaxed spiral decomposition.

### 4.2 Proof of Theorem 1.2

We want to show that PPM is NP-hard for instances where the pattern belongs to $C_{1,1}^\times$ and the text to $C_{2,1}$. Our idea is to use the pattern $\pi \in C_{2,0}$ and the text $\tau \in C_{3,0}$ constructed from a 3-SAT formula $\Phi$ in the proof of Theorem 1.1, and twirl them into a pattern $\pi^* \in C_{1,1}$ and text $\tau^* \in C_{2,1}$. We will then show that $\Phi$ is satisfiable if and only if $\tau^*$ contains $\pi^*$.

For the twirl to be well defined, we need to specify a stair-decomposition of $\pi$ and a relaxed stair-decomposition of $\tau$. Recall that $\pi$ is a disjoint union of an increasing sequence $A$ (the anchor) and a $v$-fold staircase $X$ with $2c + 1$ steps. The staircase $X$ itself is a sequence of bends $Q_1, P_1, Q_2, P_2, \ldots, Q_{2c+1}, P_{2c+1}$, with $Q_i$ being the $i$th outer bend and $P_i$ the $i$th inner bend of $X$. The sequence $A, Q_1, P_1, Q_2, P_2, \ldots, Q_{2c+1}, P_{2c+1}$ is clearly a stair-decomposition of $\pi$, and we obtain $\pi^*$ by twirling $\pi$ with respect to this decomposition.

For $\tau$, we consider the relaxed stair-decomposition whose blocks are the anchor $A'$, followed by the modified bends $\tilde{Q}_1, \tilde{P}_1, \tilde{Q}_2, \tilde{P}_2, \ldots, \tilde{Q}_{2c+1}, \tilde{P}_{2c+1}$ of the modified staircase $\tilde{Y}$. Let $\tau^*$ be the permutation obtained by twirling $\tau$ with respect to this decomposition.

Note that the union of the odd-numbered blocks of $\tau^*$ is a decreasing sequence while the union of the even-numbered blocks can be decomposed into at most two increasing sequences. It follows that $\tau^*$ is in $C_{2,1}$.

We easily check that $\tau^*$ contains $\pi^*$ if and only if $\Phi$ is satisfiable, by following the same reasoning as in the proof of Theorem 1.1. This completes the proof of Theorem 1.2.

### 4.3 Proof of Theorem 1.3

Let $F$ denote the set of permutations $\{1, 12, 21, 132, 213, 231, 312\}$. Recall that Theorem 1.3 states that $Av(\alpha)$-Pattern PPM is polynomial for each $\alpha$ in $F$ and NP-complete for any other $\alpha$.

First of all, we note that for each $\alpha \in F$, the class $Av(\alpha)$ is a subclass of the class $Av(2413, 3142)$ of separable permutations. Since $Av(2413, 3142)$-Pattern PPM is polynomial by a result of Bose, Buss and Lubiw [7], $Av(\alpha)$-Pattern PPM is polynomial as well.

On the other hand, if $\alpha$ contains $321$ as a subpermutation, then $Av(\alpha)$ contains $Av(321)$ as a subclass. Since $Av(321)$-Pattern PPM is NP-hard by Theorem 1.1, $Av(\alpha)$-Pattern PPM is NP-hard as well. By symmetry, $Av(\alpha)$-Pattern PPM is also hard when $\alpha$ contains $123$.

There are only four permutations that do not belong to $F$ and do not contain a monotone subsequence of length $3$, namely 2143, 3412, 2413, and 3142. Theorem 1.2 implies that $Av(\alpha)$-Pattern PPM is hard for any $\alpha \in \{2143, 3412, 3142\}$. By symmetry, the problem is hard for $\alpha = 2413$ as well. This completes the proof of Theorem 1.3.

### 4.4 Proof of Theorem 1.5

Our goal is to show that for every proper subclass $C$ of $C_{1,1}^\times$, the $C$-Pattern PPM problem is polynomial. We will again use the twirl to adapt the proof of Theorem 1.4 that we gave in Section 3.

As in the proof of Theorem 1.4, our goal is to show that for permutations $\pi$ taken from any proper subclass of $C_{1,1}^\times$, the graphs $G_\pi$ have bounded treewidth, or more precisely, that every
permutation \( \pi \in C_{1,1}^\times \) whose graph \( G_\pi \) has sufficiently large treewidth contains a universal pattern.

Let \( \pi \) be a permutation with a stair-decomposition \( B_1, B_2, \ldots, B_m \), and let \( \sigma \) be a permutation with a stair-decomposition \( C_1, C_2, \ldots, C_k \) such that \( k \leq m \). We say that \( \sigma \) has a block-preserving occurrence in \( \pi \) (with respect to the decompositions \( B_1, B_2, \ldots, B_m \) and \( C_1, C_2, \ldots, C_k \)) if it has an occurrence in \( \pi \) with the property that for each \( i \leq k \) the elements of \( C_i \) are mapped to the elements of \( B_i \). We also use the same terminology when dealing with spiral decompositions.

Notice that block-preserving occurrences are preserved by the twirl operation, as formalized by the next observation.

**Observation 4.3.** Let \( \pi \) be a permutation with a stair-decomposition \( B_1, B_2, \ldots, B_m \), and let \( \sigma \) be a permutation with a stair-decomposition \( C_1, C_2, \ldots, C_k \) such that \( k \leq m \). Let \( \pi^* \) with spiral decomposition \( B_1^*, B_2^*, \ldots, B_m^* \) and \( \sigma^* \) with spiral decomposition \( C_1^*, C_2^*, \ldots, C_k^* \) be obtained by twirling \( \pi \) and \( \sigma \), respectively. Then \( \sigma \) has a block-preserving occurrence in \( \pi \) if and only if \( \sigma^* \) has a block-preserving occurrence in \( \pi^* \).

Let \( \tau_k \) be the \( k \)-track permutation, and let \( A_1, A_2, \ldots, A_k \) be its stair-decomposition into blocks of size \( k \), introduced in Section 3. The \( k \)-spiral is the permutation \( \tau_k^* \) obtained by twirling \( \tau_k \) with respect to this stair-decomposition. When dealing with \( k \)-spirals, we always assume they have a spiral decomposition obtained by twirling the stair decomposition \( A_1, A_2, \ldots, A_k \).

**Lemma 4.4.** Every \( k \)-permutation \( \pi \in C_{1,1}^\times \) with a spiral decomposition into \( m \) blocks has a block-preserving occurrence in the \( q \)-spiral, for \( q = \max\{k, m\} \).

**Proof.** This follows directly from Lemma 3.4, via Observation 4.3. \( \square \)

Since each \( n \)-permutation in \( C_{1,1}^\times \) has a spiral decomposition with at most \( 4n \) blocks, we conclude that each \( n \)-permutation in \( C_{1,1}^\times \) is contained in the \( 4n \)-spiral.

For a permutation \( \pi = \pi(1), \pi(2), \ldots, \pi(n) \), the reverse-complement of \( \pi \) is the permutation \( n + 1 - \pi(n), n + 1 - \pi(n - 1), \ldots, n + 1 - \pi(1) \). Intuitively, the diagram of the reverse-complement is obtained from the diagram of \( \pi \) by a rotation of 180 degrees. Notice that \( C_{1,1}^\times \) is closed under reverse-complements. Consequently, each \( n \)-permutation in \( C_{1,1}^\times \) is contained in the reverse-complement of the \( 4n \)-spiral.

The notion of good edges and bad edges in \( G_\pi \), which we introduced in Section 3, can be extended to spiral decompositions. Let \( \pi \in C_{1,1}^\times \) be a permutation with a spiral decomposition \( B_1, B_2, \ldots, B_m \). Let \( xy \) be an edge of \( G_\pi \), such that \( x \in B_i \) and \( y \in B_j \), with \( i \leq j \). Then the edge \( xy \) is good if it satisfies at least one of the following properties:

- \( i = j \)
- \( xy \) is blue, \( i \) is odd and \( j = i + 1 \), or
- \( xy \) is red, \( i \) is even and \( j = i + 1 \).

Otherwise \( xy \) is bad.

The twirl operation clearly preserves the good edges.
Observation 4.5. Let $\pi$ be a permutation with a stair-decomposition $B_1, B_2, \ldots, B_m$ and let $\pi^*$ be the permutation with spiral decomposition $B_1^*, B_2^*, \ldots, B_m^*$ obtained by twirling $\pi$. Let $x$ and $y$ be two elements of $\pi$ and $x^*$ and $y^*$ the corresponding elements of $\pi^*$. Then $xy$ is a good edge of $G_\pi$ if and only if $x^*y^*$ is a good edge of $G_{\pi^*}$.

Lemma 4.6. If a permutation $\pi \in C_{1,1}^X$ has a spiral decomposition with $m$ blocks, then $G_\pi$ has at most $4m$ bad edges.

Proof. It is easy to see that if $xy$ is a bad edge in $G_\pi$, then both $x$ and $y$ are either leftmost or rightmost elements of their blocks in the spiral decomposition of $\pi$. Since every vertex of $G_\pi$ has degree at most 4 and at most $2m$ vertices are incident to bad edges, $G_\pi$ has at most $4m$ bad edges.

Lemma 4.7. If $\pi \in C_{1,1}^X$ has a spiral decomposition with $m$ blocks, then the treewidth of $G_\pi$ is at most $6m$.

Proof. Untwirl $\pi$ into a permutation $\pi^* \in \text{Av}(321)$. By Lemma 3.6, the graph $G_{\pi^*}$ has treewidth at most $2m$. By Observation 4.5, every good edge of $G_{\pi^*}$ is also an edge of $G_{\pi^*}$. Therefore, by Lemma 4.6, $G_{\pi^*}$ can be obtained from $G_{\pi^*}$ by inserting at most $4m$ new bad edges, and possibly also removing some bad edges of $G_{\pi^*}$. Inserting an edge into a graph may increase its treewidth by at most 1, and removing an edge cannot increase the treewidth. Therefore, the treewidth of $G_\pi$ is at most $6m$, as claimed.

To prove Theorem 1.5, we first prove the following claim, which is the $C_{1,1}^X$-analogue of Theorem 3.5, and is proved by a similar argument.

Theorem 4.8. Let $\pi \in C_{1,1}^X$ be a permutation such that $G_\pi$ contains a $g(k)$-wall, where $g(k) = 300k^{3/2}$. Then $\pi$ contains the $k$-spiral or the reverse-complement of the $k$-spiral.

Proof. Let $\pi \in C_{1,1}^X$ have a spiral decomposition $B_1, B_2, \ldots, B_m$. By Lemma 4.7, the subgraph of $G_\pi$ induced by $k$ consecutive blocks of the decomposition has treewidth at most $6k$. We may invoke Lemma 3.8, to obtain $10k$ vertex-disjoint paths from $B_{<a}$ to $B_{>b+k}$ for some $b$. Let $a$ be the smallest odd integer greater than or equal to $b$. By discarding at most four paths containing an edge from $B_{<a}$ to $B_{>a}$ or an edge from $B_{<a+k-1}$ to $B_{>a+k-1}$, truncating the remaining paths to only contain vertices from $\bigcup_{j=a}^{a+k-1} B_j$, and discarding at most $4k$ paths containing bad edges, we end up with at least $10k - 4 - 4k = 6k - 4 > 3k - 2$ paths from $B_a$ to $B_{a+k-1}$ which only contain good edges.

Untwirl $\pi$ into a permutation $\pi^* \in \text{Av}(321)$ with a stair-decomposition $B_1^*, B_2^*, \ldots, B_m^*$. Since the untwirl preserves good edges, we know by Lemma 3.9 that $\pi^*$ contains the $k$-track $\tau_k$, and the occurrence of $\tau_k$ is block-preserving with respect to the stair-decomposition in which the first $a-1$ blocks of $\tau_k$ are empty and the remaining blocks have size $k$. By Observation 4.3, we conclude that $\pi$ contains the $k$-spiral (if $a \equiv 1 \mod 4$) or the reverse-complement of the $k$-spiral (if $a \equiv 3 \mod 4$).

Theorem 4.8 implies Theorem 1.5 in exactly the same way as Theorem 3.5 implies Theorem 1.4. Theorem 1.5 is thus proved.
5 Final remarks and open problems

We have shown that for every proper subclass $C$ of $\text{Av}(321)$, the $C$-Pattern PPM problem is polynomial-time solvable, and the same is true for proper subclasses of $C_{1,1}^x = \text{Av}(2143, 3412, 3142)$. The proofs are based on the fact that for permutations in such a class $C$, the associated graph $G_\pi$ has bounded treewidth, which, by results of Ahal and Rabinovich [2], implies that $C$-Pattern PPM is polynomial.

In fact, in all the cases when $C$-Pattern PPM is known to be polynomial, the class $C$ has bounded treewidth of $G_\pi$. We may therefore ask whether in fact the boundedness of treewidth can precisely characterize the polynomial cases of $C$-Pattern PPM.

**Problem 5.1.** Is the $C$-Pattern PPM problem NP-hard for every class $C$ for which the treewidth of $G_\pi$ is unbounded?

For the closely related decision problem $C$-PPM, we have obtained NP-hardness when $C$ is the class $C_{3,0} = \text{Av}(4321)$. This gives the first known example of a proper permutation class $C$ for which the problem is hard. We have subsequently shown that $C_{2,1}$-PPM is NP-hard as well. It follows that $\text{Av}(\rho)$-PPM is NP-hard for any permutation $\rho$ not belonging to the finite set $C_{3,0} \cap C_{3,1} \cap C_{2,1} \cap C_{1,2}$, and in particular, $\text{Av}(\rho)$-PPM is hard for every $\rho$ of size at least 10. However, a precise characterization of the polynomial cases of $C$-PPM is currently out of our reach, even for principal classes $C$.

**Problem 5.2.** For which permutations $\rho$ can $\text{Av}(\rho)$-PPM be solved in polynomial time?

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