Improvement of an Approximated Self-Improving Sorter and Error Analysis of Its Estimated Entropy

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Abstract

The self-improving sorter proposed by Ailon et al.\cite{1} consists of two phases: a relatively long training phase and rapid operation phase. In this study, we have developed an efficient way to further improve this sorter by approximating its training phase to be faster but not sacrificing much performance in the operation phase. It is very necessary to ensure the accuracy of the estimated entropy when we test the performance of this approximated sorter. Thus we further developed a useful formula to calculate an upper bound for the 'error' of the estimated entropy derived from the input data with unknown distributions. Our work will contribute to the better use of this self-improving sorter for huge data in a quicker way.

1 Introduction

The widely used sorting algorithms such as QuickSort and MergeSort do not utilize the information of the input data.\cite{3, 5} So Ailon et al.\cite{1} proposed an algorithm named as self-improving sorter and this sorter sorts the input data based on the distributions which tends to make the sorting process faster.

Generally, the self-improving sorter consists of two phases: training phase and operation phase. In the training phase, an array $V = \{v_0, v_1, \ldots, v_n\}$, called Vlist, is first built by sampling input data, $I = \{x_0, x_1, \ldots, x_n\}$, for certain times. Then, build an optimal binary search tree for each $x_i$ according to the frequency of $x_i$ falling into a certain bucket $(v_i', v_{i'+1})$. By Knuth's\cite{6} method, this step (building $n$ optimal binary search trees with $n$ nodes) takes $O(n^3)$ comparisons. In the operation phase, the sorter first uses optimal binary trees, built in the training phase, to locate each $x_i$ into the corresponding bucket $(v_i', v_{i'+1})$ belonged to, then use a standard sorting algorithm like QuickSort to sort within each bucket, and Ailon et al.\cite{1} have shown that this step theoretically needs $O(n + H(\pi(I)))$ comparisons and

$$n + H(\pi(I)) = O(\sum_i H_Y^V)$$

Since the training phase is extremely computationally expensive, it is crucial to approximate this process when we apply the self-improving sorter to huge data.
To address this issue, we need to propose an approximated self-improving sorter which can reduce the training phase complexity from $O(n^3)$ to $O(n^{1+\epsilon})$ and storage space from $O(n^2)$ to $O(n^{1+\epsilon})$ for any constant $\epsilon \in (0, 1)$ by implementing a new algorithm which can build a pruned nearly optimal binary search tree within $O(n^\epsilon)$ comparisons, and we called it pruned nearly self-improving sorter.

Next, we need to test its practical performance and compare it with the original self-improving sorter, especially the relationships between its operation phase complexity and $O(\sum_i H^V_i)$. Since the distribution of the input data is unknown, estimation of the entropy is the only choice. Based on this fact, it is crucial to ensure the accuracy of the estimated entropy. Therefore, we have developed a new formula to calculate an upper bound for the expectation of square error between $\sum_i \hat{H}^V_i$ and $\sum_i H^V_i$:

$$An\ upper\ bound\ of\ E\{(\sum_i \hat{H}^V_i - \sum_i \hat{H}^V_i)^2\}$$

In summary, we make the following contributions:

- We introduce a new approximation for the self-improving sorter and the algorithm for building this new sorter reducing the complexity in training phase from $O(n^3)$ to $O(n^{1+\epsilon})$ and storage space from $O(n^2)$ to $O(n^{1+\epsilon})$ for any constant $\epsilon \in (0, 1)$, which makes the self-improving sorter more suitable for huge data.

- We have developed an upper bound for $E\{(\sum_i \hat{H}^V_i - \sum_i \hat{H}^V_i)^2\}$, which can be used to ensure the accuracy of the estimation of the entropy.

2 The approximated self-improving sorter

The self-improving sorter proposed by Ailon et al.[1] used the optimal binary search tree to learn the distribution of the input data. In order to accelerate the training process, we have replaced the optimal binary search tree with the pruned nearly optimal binary search tree. In detail, we got rid of the nodes below $O(\epsilon \log n)$ to reduce the size of the tree from $O(n)$ to $O(n^\epsilon)$ for any constant $\epsilon \in (0, 1)$. To build the pruned nearly optimal binary search tree, we extended Fredman’s algorithm[4] which could build the nearly optimal binary tree with the complexity of $O(n)$. After modification, our algorithm could complete the job with the complexity of $O(n^\epsilon)$. 

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Given: The cumulative frequency array $F[i] := \sum_{j=0}^{i} w_j$.

Algorithm: Build(r,q,d)

if $d \geq \epsilon \log n$ then // Prune the tree if the depth exceed the threshold
  return;
end

$m \leftarrow \frac{r+q}{2}$;

if $F[m] \geq \frac{F[n]}{2}$ then // If the root is on the left side
  $i \leftarrow 1$;
  repeat // Locate the interval that the root belongs to
  $i \leftarrow i + 1$;
  until $F[r + 2^i] \geq \frac{F[n]}{2}$;
  $k \leftarrow \text{BinarySearch}(r + 2^{i-1}, r + 2^i)$; // Find the root among the interval
  Add($k$);
  // Recursively build the left and right subtrees
  Build($r, r + 2^{i-1}, d + 1$);
  Build($r + 2^i + 1, q, d + 1$);
  return;
else // If the root is on the right side
  $i \leftarrow 1$;
  repeat // Locate the interval that the root belongs to
  $i \leftarrow i + 1$;
  until $F[q - 2^i] \leq \frac{F[n]}{2}$;
  $k \leftarrow \text{BinarySearch}(q - 2^{i-1}, q - 2^i)$; // Find the root among the interval
  Add($k$);
  // Recursively build the left and right subtrees
  Build($q - 2^{i-1}, q - 2^i, d + 1$);
  Build($q - 2^i + 1, q - 2^{i-1}, d + 1$);
  return;
end

Notation:

- $n :=$ the total number of nodes.
- $k_i :=$ the $i$'s node of the tree, and each node is the bucket $(v_i, v_{i+1}]$.
- $w_i :=$ the weight of node $i$, i.e. , the frequency of $x$'s falling into the bucket $(v_i, v_{i+1}]$.
- Add($k$):= add node $k$ to the tree.
- Build($r, q, d$) := find the root among the interval $[r, q]$, and the current depth is $d$.

Analysis: With the initial call 'Build(0, n - 1, 0)', the nearly optimal binary search tree could be built and pruned at the same time, and the complexity is:

$$T(m, d) = \begin{cases} 
\max\{O(\log k) + T(m - k, d + 1) + T(k - 1, d + 1) : k \leq \frac{m}{2}\}, & \text{if } d < \epsilon \log n \\
0, & \text{if } d \geq \epsilon \log n
\end{cases}$$

, indicating that $T(n, 0) = O(n^\epsilon)$. Therefore, our algorithm could achieve the complexity of $O(n^\epsilon)$. 

3
3 Error analysis of the estimated entropy

To examine the relationships between \( \sum_i^n H_i^V \) and the complexity of our pruned nearly self-improving sorter in the operation phase, it is necessary to ensure the estimated \( \sum_i^n \hat{H}_i^V \) is close to the unknown \( \sum_i^n H_i^V \) where \( \sum_i^n H_i^V \) is estimated by the following plug-in method:

- Sample the input sequence \( I = (x_1, x_2, ..., x_n) \) with length \( n \) for \( N \) times.
- Let \( p_{i,j} = \Pr(x_i \in (v_{j-1}, v_j)) \), then \( \hat{p}_i^j = \frac{\sum_{x_i \in (v_{j-1}, v_j)} \hat{p}_i}{N} \).
- \( \sum_i^n \hat{H}_i^V = \sum_i^n \sum_j^K \hat{p}_i^j \), where \( K \) is the length of the \( V \)list.

We developed the following theorem to ensure and control the accuracy of the estimated entropy.

**Theorem.** Let \( N \) be the sampling time, \( K \) be the length of the \( V \)list. Then

\[
E\{\sum_i^n \hat{H}_i^V - \sum_i^n H_i^V\}^2 \leq \frac{n(\log N)^2}{N} + \frac{n^2(K - 1)^2}{N^2}
\]

**Proof.**

\[
E\{\sum_i^n \hat{H}_i^V - \sum_i^n H_i^V\}^2 = E\{\sum_i^n \hat{H}_i^V - E\{\sum_i^n \hat{H}_i^V\} + E\{\sum_i^n \hat{H}_i^V\} - \sum_i^n H_i^V\}^2
\]
\[
= E\{\sum_i^n \hat{H}_i^V - E\{\sum_i^n \hat{H}_i^V\}\}^2 + E\{\sum_i^n \hat{H}_i^V\} - \sum_i^n H_i^V\}^2
\]
\[
+ 2E\{\sum_i^n \hat{H}_i^V - E\{\sum_i^n \hat{H}_i^V\}\}(E\{\sum_i^n \hat{H}_i^V\} - \sum_i^n H_i^V)\}
\]
\[
= E\{\sum_i^n \hat{H}_i^V - E\{\sum_i^n \hat{H}_i^V\}\}^2 + E\{\sum_i^n \hat{H}_i^V\} - \sum_i^n H_i^V\}^2
\]
\[
+ 2(E\{\sum_i^n \hat{H}_i^V\} - E\{\sum_i^n \hat{H}_i^V\})E\{\sum_i^n \hat{H}_i^V\} - \sum_i^n H_i^V)\}
\]
\[
= E\{\sum_i^n \hat{H}_i^V - E\{\sum_i^n \hat{H}_i^V\}\}^2 + E\{\sum_i^n \hat{H}_i^V\} - \sum_i^n H_i^V\}^2
\]
\[
= \text{Var}\{\sum_i^n \hat{H}_i^V\} + \text{Var}\{\sum_i^n \hat{H}_i^V\} - \sum_i^n H_i^V\}^2
\]
\[
= \sum_i \text{Var}\{\hat{H}_i^V\} + \text{Var}\{\sum_i^n \hat{H}_i^V - \sum_i^n H_i^V\}^2
\]

The last line of the above equations can be deduced from the independence assumption for each \( \hat{H}_i^V \). Next, the two terms in the last line are to be handled separately.

Before stepping into the details, we first fix the \( V \)list, that is, all of the following probabilities are conditioned on \( V \)list, then we take the expectation concerning the random \( V \)list. The reason we do this is that if \( V \)list is fixed, then \( \hat{p}_i^j \) is nothing but a Bernoulli random variable with mean \( p_{i,j}^j \) and variance \( \frac{p_{i,j}^{(1-p_{i,j})}}{N} \).

For the former of the two terms, we use a special case of a theorem of Antos and Kontoyiannis\cite{2} recalling here only what we need.
Theorem. ([Antos and Kontoyiannis[2]]) Let $N$ be the sampling time. Then

$$\text{Var}\{\hat{H}\} \leq \frac{(\log N)^2}{N}$$

This theorem implies

$$\sum_{i}^{n} \text{Var}\{\hat{H}_i^V|\text{V list}\} \leq \frac{n(\log N)^2}{N}$$

For the latter term, we need the following proposition.

**Proposition.** For any $\hat{p}_i^j \in [0, 1]$, then

$$\hat{p}_i^j \log(\hat{p}_i^j) \leq \frac{(\hat{p}_i^j)^2}{p_i^j} + \hat{p}_i^j \log p_i^j - \hat{p}_i^j$$

This proposition implies

$$E\{\hat{p}_i^j \log \hat{p}_i^j|\text{V list}\} \leq \frac{E\{(\hat{p}_i^j)^2|\text{V list}\}}{p_i^j} + E\{\hat{p}_i^j|\text{V list}\} \log p_i^j - E\{\hat{p}_i^j|\text{V list}\}$$

$$= \frac{1 - p_i^j}{N} + p_i^j \log p_i^j$$

Sum each $p_i^j$ up,

$$\sum_{i}^{n} E\{\hat{H}_i^V|\text{V list}\} = \sum_{i}^{n} \sum_{j}^{K} E\{\hat{p}_i^j \log \hat{p}_i^j|\text{V list}\}$$

$$\leq \sum_{i}^{n} \sum_{j}^{K} \frac{1 - p_i^j}{N} + \sum_{i}^{n} \sum_{j}^{K} p_i^j \log p_i^j$$

$$= \frac{n(K - 1)}{N} + \sum_{i}^{n} H_i^V$$

When we combine the above equation with the Jensen’s inequality[7] which tell us

$$E\{\hat{p}_i^j \log \hat{p}_i^j|\text{V list}\} \geq p_i^j \log p_i^j$$

, then we can get

$$\left(\sum_{i}^{n} E\{\hat{H}_i^V|\text{V list}\} - \sum_{i}^{n} H_i^V\right)^2 \leq \frac{n^2(K - 1)^2}{N^2}$$

Finally, we can prove the theorem we have proposed

$$\sum_{i}^{n} \text{Var}\{\hat{H}_i^V\} + \left(\sum_{i}^{n} E\{\hat{H}_i^V\} - \sum_{i}^{n} H_i^V\right)^2 = E\{\sum_{i}^{n} \text{Var}\{\hat{H}_i^V|\text{V list}\} + \left(\sum_{i}^{n} E\{\hat{H}_i^V|\text{V list}\} - \sum_{i}^{n} H_i^V\right)^2\}$$

$$\leq E\{\frac{n(\log N)^2}{N} + \frac{n^2(K - 1)^2}{N^2}\}$$

$$= \frac{n(\log N)^2}{N} + \frac{n^2(K - 1)^2}{N^2}$$

$\square$
4 Experiment

To examine the actual performance of our pruned nearly self-improving sorter, we carried out an experiment to see the following metrics:

- $T_c :=$ training complexity
- $O_c :=$ operation complexity
- $\hat{E} := \sum_i \hat{H}_i$, estimated entropy
- $Ub := \frac{n(\log N)^2}{N} + \frac{n^2(K-1)^2}{N^2}$, upper bound of the error of the estimated entropy

Then, our pruned nearly self-improving sorter was compared with nearly self-improving sorters, full self-improving sorters and QuickSort. In the same time, the pruned nearly optimal binary search tree in our pruned nearly self-improving sorter was also partially visualized.

The experiments were conducted on three input distributions with different entropies, as shown in Table 1-3. (Remark: for the purpose of comparing different sorters, we choose QuickSort and $F$ as benchmarks in $O_c$ and $T_c$ respectively and denote them as 1 while others are represented as ratios of QuickSort or $F$.)

| $n$ | $E$   | $\sqrt{Ub/E}$ | $\text{sorter}$ | $O_c$ | $T_c$ |
|-----|-------|----------------|-----------------|-------|-------|
| 200 | 1057.09 | 0.0021         | $Qs$            | 1     | /     |
|     |        |                | $F$             | 0.9981| 1     |
|     |        |                | $N$             | 1.0068| 0.0065|
|     |        |                | $P_{0.8}$       | 1.0960| 0.0061|
|     |        |                | $P_{0.4}$       | 1.0797| 0.0034|
| 400 | 2391.50 | 0.0017         | $Qs$            | 1     | /     |
|     |        |                | $F$             | 0.9534| 1     |
|     |        |                | $N$             | 0.9612| 0.0033|
|     |        |                | $P_{0.8}$       | 0.9802| 0.0029|
|     |        |                | $P_{0.4}$       | 1.0304| 0.0014|
| 600 | 3830.63 | 0.0016         | $Qs$            | 1     | /     |
|     |        |                | $F$             | 0.9166| 1     |
|     |        |                | $N$             | 0.9255| 0.0022|
|     |        |                | $P_{0.8}$       | 0.9295| 0.0021|
|     |        |                | $P_{0.4}$       | 0.9872| 0.0008|

Table 1: $x_i \sim \text{Uniform}(0,1)$

Diagram 1: Visualization of part of the tree for $x_{100} \sim \text{Uniform}(0,1)$ in $P_{0.4}$, when $n = 200$
Table 2: $x_i \sim \chi^2(5 \cdot (i \mod 40), 1)$

| $n$ | $E$     | $\sqrt{Ub/E}$ | $sorter$ | $Oc$ | $Tc$ |
|-----|---------|----------------|----------|------|------|
| 200 | 765.72  | 0.0029         | $Qs$     | 1    | /    |
|     |         |                | $F$      | 0.4655| 1    |
|     |         |                | $N$      | 0.4721| 0.0067|
|     |         |                | $P_{0.8}$| 0.4737| 0.0040|
|     |         |                | $P_{0.4}$| 0.4927| 0.0032|
| 400 | 1808.15 | 0.0023         | $Qs$     | 1    | /    |
|     |         |                | $F$      | 0.4817| 1    |
|     |         |                | $N$      | 0.4894| 0.0033|
|     |         |                | $P_{0.8}$| 0.4912| 0.0018|
|     |         |                | $P_{0.4}$| 0.5262| 0.0013|
| 600 | 2956.35 | 0.0020         | $Qs$     | 1    | /    |
|     |         |                | $F$      | 0.4728| 1    |
|     |         |                | $N$      | 0.4762| 0.0021|
|     |         |                | $P_{0.8}$| 0.4791| 0.0012|
|     |         |                | $P_{0.4}$| 0.5157| 0.0008|

(103.8,104.6]  
(93.9,94.8]   
(87.1,88.0]  (98.9,99.9]  (107.8,108.7]  (120.7,121.9]]

Diagram 2: Visualization of part of the tree for $x_{100} \sim \chi^2(100, 1)$ in $P_{0.4}$, when $n = 200$

Table 3: $x_i \sim N(3 \cdot (i \mod 40), 1)$

| $n$ | $E$     | $\sqrt{Ub/E}$ | $sorter$ | $Oc$ | $Tc$ |
|-----|---------|----------------|----------|------|------|
| 200 | 388.61  | 0.0057         | $Qs$     | 1    | /    |
|     |         |                | $F$      | 0.1807| 1    |
|     |         |                | $N$      | 0.1848| 0.0055|
|     |         |                | $P_{0.8}$| 0.1853| 0.0028|
|     |         |                | $P_{0.4}$| 0.1875| 0.0024|
| 400 | 1049.21 | 0.0039         | $Qs$     | 1    | /    |
|     |         |                | $F$      | 0.1943| 1    |
|     |         |                | $N$      | 0.1959| 0.0026|
|     |         |                | $P_{0.8}$| 0.1967| 0.0011|
|     |         |                | $P_{0.4}$| 0.2053| 0.0010|
| 600 | 1816.70 | 0.0034         | $Qs$     | 1    | /    |
|     |         |                | $F$      | 0.2016| 1    |
|     |         |                | $N$      | 0.2042| 0.0018|
|     |         |                | $P_{0.8}$| 0.2048| 0.0007|
|     |         |                | $P_{0.4}$| 0.2131| 0.0006|
Diagram 3: Visualization of part of the tree for $x_{100} \sim N(60, 1)$ in $P_{0.4}$, when $n = 200$

According to the results of the experiment, our pruned nearly self-improving sorter takes less training time without sacrificing much in the operation phase. Intuitively speaking, the reason is, if the input data has high entropy, the optimal binary search tree of each $x_i$ is more likely to be balanced, so once the data fall into the pruned region, the later binary search process acts similarly to the full optimal binary search tree; if the input data has low entropy, the data will more likely focus on some buckets, so the top part of the tree captures most of the information.

Previous work has shown that for self-improving sorter with pruned real optimal binary search trees, the operation phase has the following property

$$O_c = O(E^{-1}(\sum_i H_i^V))$$

, it is natural to ask whether our pruned nearly self-improving sorter has such property. To test this property, we observe the change of the ratio $R$ as $n$ increases where the ratio $R_o$ for different sorters is defined as follows

$$R_o := \begin{cases} 
\frac{O_c}{E}, & \text{for F} \\
\frac{O_c}{E}, & \text{for N} \\
\frac{O_c}{\epsilon^{-1} E}, & \text{for } P_\epsilon
\end{cases}$$

if $R_o$ decreases as $n$ increases, we can experimentally say that our pruned nearly self-improving sorter has this property. But we do not know the ratio $R_o$ since we assume the distributions are unknown. However, since the ratio of the upper bound and the estimated entropy is tiny, which implies the error of the estimated entropy is small, it is reasonable to replace $E$ with $\hat{E}$, thus we replace $R_o$ with

$$\hat{R}_o := \begin{cases} 
\frac{O_c}{\hat{E}}, & \text{for F} \\
\frac{O_c}{\hat{E}}, & \text{for N} \\
\frac{O_c}{\epsilon^{-1} \hat{E}}, & \text{for } P_\epsilon
\end{cases}$$

Moreover, to test the property in the training phase:

- $T_c = O(n^3)$ for F
- $T_c = O(n^2)$ for N
- $T_c = O(n^{1+\epsilon})$ for $P_\epsilon$

we also define the ratio $R_t$:

$$\hat{R}_t := \begin{cases} 
\frac{T_c}{n^3}, & \text{for F} \\
\frac{T_c}{n^2}, & \text{for N} \\
\frac{T_c}{n^{1+\epsilon}}, & \text{for } P_\epsilon
\end{cases}$$

and observe the change of the ratio $R_t$ as $n$ increases.
Table 4: The ratio $R_o$ and $R_t$ for different distributions

| sorter | $n$ | $R_o$ | $R_t$ | $R_o$ | $R_t$ | $R_o$ | $R_t$ |
|--------|-----|-------|-------|-------|-------|-------|-------|
| $F$    | 200 | 1     | 1     | 1     | 1     | 1     | 1     |
|        | 400 | 0.9511| 0.9854| 0.9230| 0.9881| 0.8267| 0.9935|
|        | 600 | 0.9277| 0.9817| 0.8945| 0.9826| 0.7603| 0.9914|
| $N$    | 200 | 1     | 1     | 1     | 1     | 1     | 1     |
|        | 400 | 0.9512| 0.9766| 0.9249| 0.9708| 0.8145| 0.9539|
|        | 600 | 0.9293| 0.9682| 0.8889| 0.9536| 0.7524| 0.9166|
| $P_{0.8}$ | 200 | 1     | 1     | 1     | 1     | 1     | 1     |
|        | 400 | 0.9677| 0.9723| 0.9253| 0.9613| 0.8157| 0.8935|
|        | 600 | 0.9308| 0.9526| 0.8914| 0.9487| 0.7530| 0.8105|
| $P_{0.4}$ | 200 | 1     | 1     | 1     | 1     | 1     | 1     |
|        | 400 | 0.9510| 0.9142| 0.9530| 0.9863| 0.8414| 0.8692|
|        | 600 | 0.9245| 0.8794| 0.9225| 0.9250| 0.7743| 0.8243|

$distributions$  $x_{100} \sim Uniform(0,1)$  $x_i \sim \chi^2(5 \cdot (i \mod 40), 1)$  $x_{100} \sim N(60,1)$

5 Conclusions and future works

The pruned nearly self-improving sorter we proposed takes less time to train and less storage space compared with the previous self-improving sorters. And our upper bound for the error of the estimated entropy is not only applicable for self-improving sorter but also for more general discrete entropy estimation with unknown distribution. With this upper bound, the experiment results showed that our pruned nearly self-improving sorter tends to achieve similar complexity with that of the original self-improving sorter. Since the theoretical properties of this approximated sorter has not been well studied, the future work can be focused on the theoretical analysis for its performance.

References

[1] Nir Ailon, Bernard Chazelle, Kenneth L Clarkson, Ding Liu, Wolfgang Mulzer, and C Seashadri. Self-improving algorithms. *SIAM Journal on Computing*, 40(2):350–375, 2011.

[2] András Antos and Ioannis Kontoyiannis. Convergence properties of functional estimates for discrete distributions. *Random Structures & Algorithms*, 19(3-4):163–193, 2001.

[3] Coenraad Bron. Merge sort algorithm [m1]. *Communications of the ACM*, 15(5):357–358, 1972.

[4] Michael L Fredman. Two applications of a probabilistic search technique: Sorting $x + y$ and building balanced search trees. pages 240–244, 1975.

[5] Charles AR Hoare. Quicksort. *The Computer Journal*, 5(1):10–16, 1962.

[6] Donald E. Knuth. Optimum binary search trees. *Acta informatica*, 1(1):14–25, 1971.

[7] Tristan Needham. A visual explanation of jensen’s inequality. *The American mathematical monthly*, 100(8):768–771, 1993.