Marginal Independence Models

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ABSTRACT
We impose rank one constraints on marginalizations of a tensor, given by a simplicial complex. Following work of Kirkup and Sullivant, such marginal independence models can be made toric by a linear change of coordinates. We study their toric ideals, with emphasis on random graph models and independent set polytopes of matroids. We develop the numerical algebra of parameter estimation, using both Euclidean distance and maximum likelihood, and we present a comprehensive database of small models.

CCS CONCEPTS
• Mathematics of computing → Probabilistic representations; Maximum likelihood estimation; Mathematical software; Computations on polynomials.

KEYWORDS
marginal independence, tensor rank, simplicial complex, toric variety, matroid, independent set polytope, maximum likelihood degree, Euclidean distance degree

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1 INTRODUCTION
This article concerns a class of statistical models for discrete random variables that was introduced by Kirkup in [14] and studied further by Sullivant in [21, Section 4.3.2]. Let $X_1, X_2, \ldots, X_n$ be random variables, where $X_i$ has the finite state space $\{d_i\} = \{1, 2, \ldots, d_i\}$. A joint distribution for these random variables is a tensor $P = (p_{i_1 \cdots i_n})$ of format $d_1 \times d_2 \times \cdots \times d_n$ whose entries are nonnegative real numbers that sum to 1. These distributions are elements in the probability simplex $\Delta_{D-1}$, where $D = d_1d_2 \cdots d_n$. As is customary in algebraic statistics [1, 3, 20, 21], we identify this simplex with the projective space $\mathbb{P}^{D-1}$. Furthermore, by a “model” we will mean either a subvariety of $\mathbb{P}^{D-1}$ or its intersection with $\Delta_{D-1}$, depending on context. The latter is a semialgebraic set.

For any subset $\sigma$ of $[n] = \{1, \ldots, n\}$, we write $P_\sigma$ for the corresponding marginalization. Thus $P_\sigma$ is a tensor with $\prod_{i \in \sigma} d_i$ entries. These are obtained from the entries $p_{i_1 \cdots i_n}$ of $P$ by summing out the indices not in $\sigma$. For instance, if $n = 5$ and $\sigma = \{2, 3\}$ then $P_\sigma$ is the $d_2 \times d_3$ matrix whose entry in row $j$ and column $k$ equals $P_{ijk\lambda} = \sum_{l=1}^{d_1} p_{ijklm}$. Let $\Sigma$ be any simplicial complex with vertex set $[n]$. We assume $\{i\} \in \Sigma$ for all $i \in [n]$. The marginal independence model $M_\Sigma$ is the set of distributions $P \in \Delta_{D-1}$ such that $P_\sigma$ has rank 1 for all $\sigma \in \Sigma$. The random variables $\{X_i : i \in \sigma\}$ are completely independent for $\sigma \in \Sigma$. Since the rank 1 constraint is the vanishing of the $2 \times 2$ minors of all flattening of $P$, we see that $M_\Sigma$ is a variety defined by quadratic equations in the tensor space $\mathbb{P}^{D-1}$. The following model, pairwise independence for three variables, appears in [21, Section 4.3.2].

Example 1 (3-cycle). Let $n = 3$ and $\Sigma = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1\}, \{2\}, \{3\}, \emptyset\}$. The model $M_\Sigma$ comprises $d_1 \times d_2 \times d_3$ tensors whose three marginalizations are matrices of rank 1. This is an irreducible variety of dimension $|\Sigma| (d_j - 1) + \sum_{\sigma \in \Sigma} (d_j - 1)$ in the tensor space $\mathbb{P}^{D-1}$.

The natural constraints that describe this variety are the various $2 \times 2$ determinants

$$p_{1i} p_{kj} - p_{1j} p_{ki}, \quad p_{1i} p_{kj} - p_{3i} p_{kj},$$

and $p_{1i} p_{3j} - p_{3i} p_{kj}$. These do not generate the prime ideal of $M_\Sigma$. See [14, Section 6] for extraneous components.

The binary model (all $d_i = 2$), which serves as running example in [14], has dimension 4 and degree 5 in $\mathbb{P}^{32}$. Its ideal is generated by five quadrics and is Gorenstein. The ternary model (all $d_i = 3$) has dimension 14 and degree 43 in $\mathbb{P}^{92}$. For this model, Kirkup concluded on [14, page 453] that “a free resolution cannot be computed”. This is no longer true. Using the toric representation in Theorem 10, we easily find the Macaulay2 Betti diagram for $M_3$:

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|----|----|----|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 55 | 303 | 920 | 2309 | 4740 | 6700 | 5892 | 3348 | 1082 | 228 | 61 | 7 |
| 2 | 55 | 303 | 711 | 759 | 285 | . | . | . | . | . | . | . |
| 3 | . | 209 | 1508 | 4455 | 6699 | 6029 | 3276 | 887 | 30 | . | . | . |
| 4 | . | . | . | . | . | . | . | . | . | . | . | . |
| 5 | . | . | . | . | . | . | . | . | . | . | . | . |

Note that the ideal is perfect (Cohen-Macaulay), in accordance with [14, Conjecture 28].

The toric representation is due to Sullivant [21, Section 4.3.2], who proves it for $n = 3$ and states that it “will also work for many other marginal independence models”. Our Sections 2 and 3 develop this in detail. We show that each model $M_\Sigma$ is a toric variety after a linear change of coordinates. This reaffirms that “the world is toric” [18, Section 8.3]. The new coordinates were called Möbius parameters by Drton and Richardson [7]. These authors also treat marginal independence but their set-up is based on bidirected graphs and it

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differs from ours. Most of the models in [7] are not in the class we study here; see Example 17.

This paper is organized as follows. In Section 2 we introduce Möbius coordinates \(q_\alpha\), and we show that these have desirable properties. For instance, for \(2 \times 2 \times 2\) tensors, they are \(q = p_{+++}, q_1 = p_{++1}, q_2 = p_{+1+}, q_3 = p_{+11}, q_{12} = p_{1+1}, q_{13} = p_{11+}, q_{23} = p_{++1}\). In these coordinates, the prime ideal for the binary 3-cycle in Example 1 is the toric ideal

\[
\langle q_1q_2 - q_{13}, q_3 - q_{13}, q_2q_3 - q_{23} \rangle = \text{Pfaffians}_4
\]

In Section 3 we show that every marginal independence model \(M_\Sigma\) is toric in the Möbius coordinates \(q_\alpha\). In particular, we identify the associated polytope \(P_\Sigma\) and model matrix \(A_\Sigma\). Section 4 is devoted to the minimal generators of the toric ideals \(I_{\Sigma_\alpha}\). We start from the matroid case, where the simplicial complex \(\Sigma\) consists of all independent sets. We are especially interested in graphic matroids, where \(\Sigma\) is the set of all subforests in a graph with \(n\) edges. This leads to a new class of random graph models, featured in Example 18.

In Section 5 we turn to the statistical task of parameter estimation. Given any empirical distribution \(U\), we seek to compute a distribution \(P\) in the model \(M_\Sigma\) that best explains the data \(U\). We study this from the perspectives of maximum likelihood (ML) and Euclidean distance (ED), with emphasis on exact solutions to the corresponding optimization problems.

In Section 6 we present a complete list of all small marginal independence models with binary states. This includes all simplicial complexes for \(n \leq 6\) and all matroids for \(n \leq 7\).

2 MöBIUS COORDINATES AND THE SEGRE VARIETY

Let \(R^D\) be the space of real tensors of format \(d_1 \times \cdots \times d_n\) and \(R^{D-1}\) the corresponding projective space. Following [21, Section 4.3.2], we define an \(R\)-linear map \(\varphi : R^D \rightarrow R^D, P \mapsto Q\). The entries \(q_{\ell_1\ell_2\cdots\ell_n}\) of \(Q\) are called Möbius coordinates. Here, \(\ell_j \in \{1, 2, \ldots, d_j - 1, +\}\). The Möbius coordinate \(q_{\ell_1\ell_2\cdots\ell_n}\) is simply the linear form \(P_{\ell_1\ell_2\cdots\ell_n}\) in the probability coordinates. That linear form is an entry of the marginal tensor \(P_\varphi\) where \(\varphi = \{ j : \ell_j \neq + \}\).

Example 2 \((n = 3, all d_j = 3)\). The tensor \(Q\) has entries indexed by \(\{1, 2, +\}\)^3, for instance

\[
\begin{align*}
q_{121} &= p_{121}, \\
q_{12+} &= p_{121} + p_{122} + p_{123}, \\
q_{+1+} &= p_{+11} + p_{1+1} + p_{11+} + p_{+12} + p_{121} + p_{211} + p_{212} + p_{311} + p_{312} + p_{313}.
\end{align*}
\]

The Möbius coordinate \(q_{+++}\) is the sum of all 27 probability coordinates \(p_{\ell_1\ell_2\ell_3}\). The map \(\varphi\) is represented by an upper-triangular \(27 \times 27\) matrix with entries in \(\{0, 1\}\). All 27 diagonal entries are 1. Precisely 98 of the 351 entries above the diagonal are 1. The ordering of the basis which gives the upper-triangular form can be found under the “ternary 3-cycle” example on this project’s MathRepo page referred to below.

The map \(\varphi\) is invertible because it is represented by an upper triangular matrix with all diagonal entries equal to one. Moreover, all entries in the inverse matrix are \(+1, -1\) or \(0\). This is precisely the process of Möbius inversion for a poset on \(D\) elements, which is the direct product of the posets \(\{1, 2, \ldots, d_j - 1, +\}\), where \(+\) is below the other \(d_j - 1\) elements, which are incomparable.

A nonzero tensor \(P\) in \(R^D\) is said to have rank one if its entries factorize as follows:

\[
p_{i_1i_2\cdots i_n} = \lambda \theta^{(1)}_{i_1} \theta^{(2)}_{i_2} \cdots \theta^{(n)}_{i_n} \quad \text{for } i_1 \in [d_1], i_2 \in [d_2], \ldots, i_n \in [d_n].
\]

Here \(\lambda\) is an additional homogenization parameter, and the other parameters satisfy

\[
\theta^{(j)}_{i_j} = 1 - \sum_{k=1}^{d_j-1} \theta^{(j)}_{k} \quad \text{for } j \in [n].
\]

Let \(S\) denote the set of rank one tensors in \(R^D\). The image of \(S\) in \(R^{D-1}\) is isomorphic to \(R^{d_1 - 1} \times R^{d_2 - 1} \times \cdots \times R^{d_n - 1}\). The total number of free parameters in the parametrization (3) is

\[
dim(S) = d_1 + d_2 + \cdots + d_n - n + 1.
\]

In what follows, we refer to the affine cone \(S\) in the tensor space \(R^D\) as the Segre variety.

The homogeneous prime ideal \(I_S\) of the Segre variety is a toric ideal. It is known to be generated by quadratic binomials. Namely, these ideal generators are the \(2 \times 2\) minors of all the flattenings of the tensor \(P\). For more information see [18, Sections 8.2 and 9.2]. The parametrization \(\varphi\) corresponds dually to the map of polynomial rings \(\varphi^* : R[Q] \rightarrow R[P]\), where each unknown \(q_{\ell_1\ell_2\cdots\ell_n}\) is mapped to the corresponding linear form \(P_{\ell_1\ell_2\cdots\ell_n}\). We further identify the two index sets \(\times\) and \(\times\) by simply sending \(d_j\) to \(+\). We write \(I_S\) for the toric ideal in \(R[Q]\) that is obtained from the toric ideal \(I_S\) in \(R[P]\) from the resulting identification of labels. With this notation, we have the following key lemma.

Lemma 3. The inverse image of the toric ideal \(I_S\) in \(R[P]\) under the ring map \(\varphi^* : R[Q] \rightarrow R[P]\) is the toric ideal \(I_{\varphi} = \langle q_{+++} \rangle\) in \(R[Q]\). In other words, the linear change of coordinates \(\varphi^*\) preserves the Segre variety \(S\). In particular, \(S\) is still a toric variety, even when written in Möbius coordinates.

Example 4 \((n = 2, d_1 = d_2 = 2)\). Consider the most simple case, namely \(2 \times 2\) matrices. The objects of interest are the principal ideals \(I_{\varphi} = (p_{11}p_{22} - p_{12}p_{21})\) and \(I_S = (q_{11}q_{++} - q_{+1}q_{++})\). The image of \(I_S\) under the ring map \(\varphi^* : R[Q] \rightarrow R[P]\) is generated by the quadratic form

\[
p_{11}p_{++} - p_{++}p_{11} = \quad p_{11}p_{12} + p_{12}p_{21} - p_{11}p_{21} - p_{12}p_{22} = \quad p_{11}p_{21} - p_{12}p_{22},
\]

This identity shows that \(\varphi^*(I_S) = I_{\varphi}\), so the first assertion in Lemma 3 holds. △
PROOF OF LEMMA 3. We consider the one-to-one parametrization of $S$ given in (3) and (4). The image of $\varphi'(q_{i_1i_2\ldots i_n}) = p_{i_1i_2\ldots i_n}$ under the associated ring map equals $\lambda \prod_{j=1}^{n} \theta_{ij}^{(j)}$. Here some of the indices $i_j$ take the value +. This means that we are summing over all states in $[d_j]$. The hypothesis (4) ensures that $\theta_{ij}^{(j)} = 1$ whenever $i_j = +$. Hence, we conclude that

$$q_{i_1i_2\ldots i_n} \mapsto \lambda \prod_{j, i_j \in [d_j-1]} \theta_{ij}^{(j)}$$

(6)

under the composition of $\varphi$ with the parametrization (3) of the Segre variety $S$. By definition, the kernel of the ring map (6), which is an ideal in $\mathbb{R}[Q]$, is the inverse image of $I_S$ under $\varphi$.

The products on the right of (6) are monomials in the free parameters, which are counted in (5). The exponent vectors of these monomials are the vertices of a polytope affinely isomorphic to $\Delta_{d_1-1} \times \Delta_{d_2-1} \times \cdots \times \Delta_{d_n-1}$. Here we are using the identification of labels that was described prior to Lemma 3. Hence the kernel of (6) equals the toric ideal $I_S$.

Of special interest in statistics is the case of binary random variables, where $d_1 = \cdots = d_n = 2$. Here each Möbius coordinate is indexed by a string $(i_1, i_2, \ldots, i_n)$ in $\{1, +\}^n$. Following [7, Section 3], we represent this string by a subset of $[n]$, namely the set of indices $j$ where $i_j = +$. Thus, in the binary case, Möbius coordinates are indexed by subsets of $[n]$.

Example 5 (Binary 3-cycle). Let $n = 3$. The eight Möbius coordinates $q_A$ are indexed by subsets $A \subseteq [3]$, with the extra convention $q = q_{\emptyset}$. The toric ideal $I_S$ of the Segre variety $S$ is generated by nine quadratic binomials $p_{00} p_{01} - p_{01} p_{11} - p_{10} p_{11}$ in these unknowns. Now consider the 3-cycle model $M_{\Sigma}$ in Example 1. We write the prime ideal of this model in Möbius coordinates, and we find that it is generated by the $4 \times 4$ subpfaffians of the skew-symmetric $5 \times 5$ matrix shown in (2). This is the toric ideal obtained from $I_S$ by eliminating $q_{123}$. △

Example 6 (Binary bowtie). Let $n = 5$. The 32 Möbius coordinates $q_A$ are indexed by subsets $A \subseteq [5]$. Here $I_S$ is generated by 28 quadratic binomials. Let $\Sigma$ be the graph with edges 12, 13, 23, 14, 15, 145. As a simplicial complex, $\Sigma$ has 12 faces, namely six edges, five vertices, and the empty set. We now eliminate all 20 unknowns $q_B$ with $B \notin \Sigma$ from the Segre ideal $I_S$. The resulting toric ideal in 12 Möbius coordinates $q_C$ with $C \in \Sigma$ defines $M_{\Sigma}$. This ideal has codimension 6 and degree 15. It is minimally generated by 14 quadrics and one cubic, namely $q_{12} q_{15} q_{21} - q_{14} q_{15} q_{23}$. This is the smallest marginal independence model whose ideal requires a cubic generator. In Section 4 we shall see binomials of higher degree. △

3 TORIC REPRESENTATION

Fix positive integers $d_1, d_2, \ldots, d_n$, the Segre variety $S$, and a simplicial complex $\Sigma$ on $[n]$. The model $M_{\Sigma}$ consists of all probability tensors $P$ with marginals $P_\sigma$ of rank one for all $\sigma \in \Sigma$. Let $L_\Sigma$ denote the linear subspace of all tensors $P \in \mathbb{R}^D$ whose marginals $P_\sigma$ are zero for all $\sigma \in \Sigma$. This subspace and our marginal independence model are related as follows:

**Lemma 7.** A tensor lies in the model $M_{\Sigma}$ if and only if it is a sum of a rank one tensor and a tensor in $L_\Sigma$. In symbols, we have the following rational parametrization of our variety:

$$M_{\Sigma} = S + L_\Sigma.$$  (7)

**Proof.** This is [14, Theorem 1], and the argument is as follows. Consider the 1-marginals $P_1 \in \mathbb{R}^{d_1}, \ldots, P_n \in \mathbb{R}^{d_n}$. Their tensor product $P = P_1 \otimes \cdots \otimes P_n$ is a rank one tensor in $\mathbb{R}^D$. Assuming without loss of generality that the entries of $P$ sum to 1, this construction yields also $P' = P_i$ for all $i \in [n]$. Assume now $P \in M_{\Sigma}$, so $P_\sigma$ has rank one for all $\sigma \in \Sigma$. By independence this implies $P'_\sigma = P_\sigma$ for all $\sigma \in \Sigma$. Therefore, the difference $P - P'$ lies in the linear space $L_\Sigma$.

We now connect the decomposition (7) to the Möbius coordinates from Section 2. The Möbius coordinate $q_{i_1i_2\ldots i_n}$ is said to be relevant if the set $\{ j : i_j \in [d_j - 1] \}$ is a face of $\Sigma$. Otherwise, $q_{i_1i_2\ldots i_n}$ is called irrelevant. In the binary case, the relevant Möbius coordinates are those indexed by the faces of $\Sigma$, and the irrelevant ones are indexed by the nonfaces.

**Lemma 8.** The linear space $L_\Sigma$ is the common zero set of all relevant Möbius coordinates $q_{i_1i_2\ldots i_n}$. Hence its dimension equals the number of irrelevant Möbius coordinates, which is

$$\dim(L_\Sigma) = \sum_{r \in \Sigma, j \in r} (d_j - 1).$$  (8)

**Proof.** This result appears in [11, Theorem 2.6], albeit in a slightly different context. For completeness, we offer a proof. We identify each Möbius coordinate with its image under $\varphi^\star$, which is a sum of $p$-coordinates. With this, each relevant Möbius coordinate $q_\sigma$ is among the entries of the marginal tensor $P_\sigma$ where $\sigma = \{ j : i_j \in [d_j - 1] \}$. Since $\sigma \in \Sigma$, the linear form $q_\sigma$ vanishes on $L_\Sigma$. Now, it remains to consider entries $p_\sigma$ of $P_\sigma$ whose indices involve $d_j$ for some $j \in \sigma$. Each such entry is an alternating sum of relevant Möbius coordinates supported on faces contained in $\sigma$. In particular, $p_\sigma$ vanishes whenever all relevant Möbius coordinates vanish. This proves the first assertion. The second assertion is obtained by counting irrelevant coordinates according to their support $r$. △

Example 9. The expression (8) is a polynomial in $d_1, \ldots, d_n$, which often simplifies greatly. Consider the bowtie graph in Example 6. The sum (8) over all 20 nonfaces of $\Sigma$ becomes

$$d_1 d_2 d_3 (d_4 d_5 - d_1 d_2 - d_1 d_3 - d_2 d_3 - d_2 d_4) - d_3 d_4 - d_4 d_5 + d_1 + 3d_3 + d_4 + d_5 - 2.$$  Of course, this expression evaluates to 20 for $d_1 = \cdots = d_5 = 2$. The 20 coordinates $q_B$ referred to in Example 6 are irrelevant while the 12 coordinates $q_C$ are relevant. △

We now present the Toric Representation Theorem for marginal independence models.

**Theorem 10.** The variety $M_{\Sigma}$ is irreducible, and its prime ideal is toric in Möbius coordinates. It is obtained from the Segre ideal $I_S$ by eliminating all Möbius coordinates that are irrelevant. Viewed modulo the linear space $L_\Sigma$, the model $M_{\Sigma}$ is the toric variety given by the monomial parametrization (6) where $q_{i_1i_2\ldots i_n}$ runs over all relevant Möbius coordinates.
Proof. This was shown for $n = 3$ by Sullivant in [21, Section 4.3.2]. The proof for general $n$ is analogous. In our set-up, it consists of Lemmas 3, 7 and 8. Equation (7) says that $M_S$ is the cone over a projection of the Segre variety $S$. This projection has the center $L_S$, when viewed in $\mathbb{E}^{D-1}$. Algebraically, this corresponds to passing to the relevant Möbius coordinates, by Lemma 8. Lemma 3 furnishes the remaining step. Since the Segre variety $S$ is toric in Möbius coordinates, so is its projection onto the subset of relevant coordinates.

Corollary 11. The marginal independence model has the dimension expected from (7), i.e.

$$\dim(M_S) = \dim(S) + \dim(L_S) = \sum_{i=1}^{n} (d_i - 1) + \sum_{\tau \in \Sigma} \prod_{j \in \tau} (d_j - 1).$$

In particular, in the binary case, this dimension equals $n$ plus the number of nonfaces of $\Sigma$.

Proof. The hypothesis $(i) \in \Sigma$ for all $i$ ensures that the projection with center $L_S$ is birational on the Segre variety $S$. Indeed, all model parameters in (6) can be recovered rationally from the relevant Möbius coordinates: the quadratic monomials $\lambda \theta_j^{(i)}$ occur for all $i \in [n]$ and $j \in [d_i - 1]$, and the special parameter $\lambda$ occurs for the empty face in $\Sigma$.

In toric algebra [18, Chapter 8], one represents a toric variety by $\mathcal{A}_\Sigma$. It has $1 + \sum_{i=1}^{n} (d_i - 1)$ rows, indexed by the model parameters $\lambda$ and $\theta_j^{(i)}$. Its columns are indexed by the relevant Möbius coordinates. To be precise, the columns of $\mathcal{A}_\Sigma$ are the 0-1 exponent vectors of the relevant monomials seen on the right in (6). The convex hull of these vectors is the model polytope $\mathcal{P}_\Sigma = \text{conv}(\mathcal{A}_\Sigma)$. This is a full-dimensional subpolytope of the product of simplices $\Delta_{d_1-1} \times \cdots \times \Delta_{d_n-1}$ that is associated with $S$.

Example 12 (Binary 3-cycle). For Example 1 with $d_i = 2$, we obtain the $4 \times 7$ matrix

$$\mathcal{A}_\Sigma = \begin{pmatrix}
\lambda & q_1 & q_2 & q_3 & q_4 & q_5 & q_6 \\
\theta_1^{(i)} & 1 & 1 & 1 & 1 & 1 & 1 \\
\theta_2^{(i)} & 0 & 1 & 0 & 0 & 1 & 1 \\
\theta_3^{(i)} & 0 & 0 & 0 & 1 & 0 & 1
\end{pmatrix}$$

The toric ideal $I_{\mathcal{A}_\Sigma}$ is given in (2). The 3-dimensional polytope $\mathcal{P}_\Sigma = \text{conv}(\mathcal{A}_\Sigma)$ arises from the regular 3-cube by slicing off one vertex. By contrast, the binary hierarchical model for the 3-cycle, in the sense of [11], corresponds to a 6-dimensional polytope with 8 vertices.

Example 13 (Binary matroid models). Consider any matroid on $[n]$, and let $\Sigma$ be its complex of independent sets. Then $\mathcal{P}_\Sigma$ is the independent set polytope of the matroid.

4 IDEAL GENERATORS

We are interested in the toric ideal $I_{\mathcal{A}_\Sigma}$ of the marginal independence model $\mathcal{M}_\Sigma$ in Möbius coordinates $q_\Sigma$. For binary models arising from matroids (Example 13), we conjecture that $I_{\mathcal{A}_\Sigma}$ is generated by quadrics. This is closely related to a famous conjecture due to Neil White [18, Conjecture 13.16] which asserts quadratic generation for the toric ideal of the matroid base polytope. Here we consider the polytope of all independent sets. White’s conjecture has been proved for many special cases, e.g. for graphic matroids by Blasiak [2], for rank three matroids by Kashiwabara [13], and up to saturation in [15].

For our binary matroid models, the toric ideal $I_{\mathcal{A}_\Sigma}$ is normal (the proof is analogous to the base polytope case; see [18, Theorem 13.8]) and hence it is Cohen-Macaulay. In general, the Cohen-Macaulay property for simplicial complexes was asserted in [14, Conjecture 28], along with the expected dimension for $M_\Sigma$. We proved the second part of Kirkup’s conjecture in Corollary 11, but the first part remains open. Presently, we know no ideal $I_{\mathcal{A}_\Sigma}$ that fails to be Cohen-Macaulay. In particular, we do not yet know how to transfer the counterexamples in [10] to our setting. This issue is subtle because non-normal models exist (see [16] and Remark 15).

Our next result states that the toric ideals $I_{\mathcal{A}_\Sigma}$ can have minimal generators of arbitrarily high degree. This is based on a construction that extends the bowtie in Example 6.

Theorem 14. Let $\Sigma$ be the 1-dimensional simplicial complex associated with the graph

![Graph]

Then the toric ideal $I_{\mathcal{A}_\Sigma}$ for the binary model $\mathcal{M}_\Sigma$ has a minimal generator of degree $n - 2$.

Proof. Consider the following binomial in the Möbius coordinates associated with the edges:

\[ n \text{ even: } f = q_n - n q_1 q_2 q_3 \cdots q_{n-5} q_n - 3 q_n - 3 n - 2 \]

\[ q_1 n - 1 q_1 q_2 q_3 ... q_{n-6} q_n - 4 q_n - 4 n - 2. \]

\[ n \text{ odd: } f = q_n - n q_1 q_2 q_3 \cdots q_{n-6} q_n - 4 q_n - 4 n - 2 \]

\[ q_1 n - 1 q_1 q_2 q_3 q_5 ... q_{n-5} q_n - 3 q_n - n - 2. \]

The binomial $f$ has degree $n - 2$. Each of its two monomials uses the indices $1, 2, \ldots, n - 4$ twice, and it uses the indices $n - 3, n - 2, n - 1, n$ once. Hence $f$ vanishes under the monomial map in (6), which sends $q_{ij}$ to $\lambda \theta_i^{(j)} \theta_j^{(i)}$, and this means that $f$ lies in the toric ideal $I_{\mathcal{A}_\Sigma}$. We claim that $f$ occurs in every minimal generating set of $I_{\mathcal{A}_\Sigma}$.

In the language of algebraic statistics [1], $f$ is indispensable for the Markov basis of $\mathcal{A}_\Sigma$, i.e. the corresponding fiber of the nonnegative integer linear map given by $\mathcal{A}_\Sigma$ has only two elements.

This can be shown by means of a hypergraph technique. Namely, since $\mathcal{A}_\Sigma$ is 0/1, it is an incidence matrix of the parameter hypergraph $\mathcal{H}_\Sigma$ of the model. Defined in [19, Section 3.1], vertices of $\mathcal{H}_\Sigma$ are $(\lambda) \cup \{\theta_i^{(j)}\}_{i=1}^{n}$, and edges are of size 1, 2, and 3, representing monomial images of $q_{\emptyset}, q_i$, and $q_{ij}$, respectively. A pair

[266]}
of edge multisets $\mathcal{E} := \mathcal{E}_{\text{blue}} \cup \mathcal{E}_{\text{red}}$ in $\mathcal{H}_2$ is said to be balanced if $\deg_{\mathcal{E}_{\text{blue}}}(v) = \deg_{\mathcal{E}_{\text{red}}}(v)$ for each vertex $v$. Denote by $f_\mathcal{E} := f_{\mathcal{E}_{\text{red}}} - f_{\mathcal{E}_{\text{blue}}}$ a binomial supported on $\mathcal{E}$; by [9, Proposition 3.1], every binomial in $I_{\mathcal{H}_2}$ is of this form. [9, Proposition 4.1] states that if there exists no splitting set of the balanced edge set $\mathcal{E}$ supporting the binomial, then the binomial is indispensable. A splitting set of $\mathcal{E}$ is a set $S$ of edges in the hypergraph such that $\mathcal{E} + S$ can be decomposed into two other balanced sets overlapping on $S$, a condition that in $I_{\mathcal{H}_2}$ translates to $f_\mathcal{E}$ being generated by smaller binomials supported on subsets $\Gamma_1$ and $\Gamma_2$. Precisely, $\mathcal{E} + S = \Gamma_1 \cup \Gamma_2$; $S = \Gamma_{\text{red}} \cap \Gamma_{\text{blue}}$; and the two new sets respect coloring: $\Gamma_{\text{red}}, \Gamma_{\text{red}} \subset \mathcal{E}_{\text{red}} \cup S$, and similarly for blue.

For our $f$, a splitting set $S$ must correspond to proper faces of $\Sigma$, and due to the color-balancing requirement, the only choices that could split $\mathcal{E}$ are the middle vertices of the graph, that is, the size-2 edges $\{\lambda_i(1)\}$ for $1 \leq i \leq n - 4$ in the hypergraph. If any hypergraph edge $\{\lambda_i(1)\} \in \mathcal{S}$, then $\Gamma_1$ represents the collection to the left of vertex $i$. Then $\deg_{\mathcal{E}_{\text{blue}}}(<\lambda_i>) \geq 1 + \delta_i \deg_{\mathcal{E}_{\text{red}}}(<\lambda_i>)$ in $\Gamma_1$, requiring the addition of the singleton $\{\lambda_i\}$ to $\Gamma_{\text{blue}}$. In Möbius coordinates, this means $q_0$ is required in the support to make the binomial supported on $\Gamma_1$ homogeneous and thus living in $I_{\mathcal{H}_2}$. But this addition violates the splitting set definition, and as it is the only way to color-balance, a splitting set does not exist.

**Remark 15.** For $n \geq 7$, both terms of the indispensable binomial $f$ contain the square of a coordinate $q_a$. From this one can infer that $I_{\mathcal{H}_2}$ is not normal. We refer to [16] for a systematic construction of simplicial complexes $\Sigma$ whose binary model $I_{\mathcal{H}_2}$ is not normal.

Here is another example with a high-degree generator which we found quite interesting.

**Example 16** (Sextic for $n = 6$). Fix the binary model for the 2-dimensional complex with facets $320$. The ideal $I_{\mathcal{H}_2}$ is minimally generated by 161 quadrics and one sextic, namely $q_{123}^2 q_{145}^2 q_{245} - q_{124} q_{156} q_{236} q_{345}$. All 161 quadrics are squarefree.

We next offer a brief comparison to the graphical set-up of Drton and Richardson in [7].

**Example 17.** We fix the bidirected graph in the article [7] to be the 5-cycle. This represents the model defined by $\{1 \pm 34, 2 \pm 45, 3 \pm 15, 4 \pm 12, 5 \pm 23\}$. Its ideal is also toric in Möbius coordinates, by [7, Theorem 1]. A saturation step reveals that it is generated by 25 quadrics:

\[
\begin{align*}
q_{134} q - q_{134}, & \quad q_{245} q - q_{245}, \quad q_{135} q - q_{135}, \\
q_{124} q - q_{124}, & \quad q_{253} q - q_{253}, \quad q_{134} q - q_{134}, \quad q_{245} q - q_{245}, \\
q_{354} q - q_{354}, & \quad q_{145} q - q_{145}, \quad q_{254} q - q_{254} \end{align*}
\]

That bidirected 5-cycle model has dimension 21 and degree 83. In the simplex $\Delta_{11}$, it is naturally sandwiched between two of our models. It contains $M_{\Sigma}$ for $\Sigma = [134, 245, 135, 124, 235]$, which has dimension 16, degree 68, and 84 quadrics. And, it is contained in the model $M_{\Sigma}$ for $\Sigma' = [13, 24, 35, 14, 25]$, which has dimension 26, degree 12, and 10 quadrics.

The original motivation for this project came from algebraic models for random graphs. Marginal independence for binary random variables offers a natural class of such models. Let $n = \binom{4}{2}$ and consider undirected simple graphs on $s$ vertices, with edges labeled by $[n]$. These graphs are in bijection with subsets of $[n]$. Probability distributions on graphs are points in the simplex $\Delta_{2s-1}$. Example 13 introduces a new model for marginal independence of edges in a graph. This model is associated with the binary graph matroid of the complete graph $K_s$, and $\Sigma$ is the simplicial complex of all forests.

**Example 18** (Random graph model). We consider graphs on $s = 4$ nodes $a, b, c, d$. The $n = 6$ possible edges are labeled $1: ab, 2: ac, 3: ad, 4: bc, 5: bd, 6: cd$. The associated simplicial complex $\Sigma$ is 2-dimensional and it has $1 + 6 + 15 + 16 = 38$ simplices. These simplices are the 38 forests with vertices $a, b, c, d$. The facets of $\Sigma$ are the 16 spanning trees in the complete graph $K_4$. Explicitly, they are the triples in [6] other than the cycles $124, 135, 236, 456$. Each of the $2^6 = 64$ graphs with vertex set $\{a, b, c, d\}$ has a certain probability $p_{i_1, \cdots, i_6}$, according to our model. Being in $M_{\Sigma}$ means that the edges in any spanning tree are chosen independently. However, choices of edges are no longer independent when a cycle is formed.

The random graph model $M_{\Sigma}$ has dimension 32 and lives in $\Delta_{32}$. In the 32 Möbius coordinates, it is given by a toric ideal $I_{\mathcal{H}_2}$ of codimension 31 and degree 320. Here $\mathcal{P}_{\Sigma} = \text{conv}(\mathcal{H}_2)$ is the independent set polytope of the graph matroid of $K_4$, which has dimension 6 and volume 320. The ideal $I_{\mathcal{H}_2}$ is generated by 358 quadratic binomials, including

\[
\begin{align*}
q_{12} q - q_{12}, & \quad q_{13} q - q_{13}, \\
q_{34} q - q_{34}, & \quad q_{14} q - q_{14}, \\
q_{25} q - q_{25}, & \quad q_{26} q - q_{26}
\end{align*}
\]

Each of these is a quadratic constraint in the probability coordinates $p_{i_1, \cdots, i_6}$. These are the probabilities of the 64 graphs.

## 5 Parameter Estimation

Estimating model parameters from data is a fundamental task in statistics. In our setting, the data is a tensor $U$ of format $d_1 \times \cdots \times d_n$ whose entries $u_{i_1, \cdots, i_n}$ are nonnegative integers, indicating the number of times each given joint state was observed. The empirical distribution $\hat{U} := \frac{1}{|U|} U$ is a rational point in $\Delta_{2s-1}$. We seek a distribution $\hat{P}$ in the model $M_{\Sigma}$ that best explains the data, in an appropriate statistical sense. We wish to estimate the model parameters $(\lambda, \theta)$ that map to $\hat{P}$.

We examine three different paradigms for parameter estimation. These are Maximum Likelihood (ML) degree, and affine (aED) and projective (pED) Euclidean distance degree:

\[
\begin{align*}
\max \sum u_{*}(p_*) & \text{ s.t. } p \in M_{\Sigma}, \\
\min \sum (u_* - p_*)^2 & \text{ s.t. } p \in M_{\Sigma} \text{ and } \sum p_* = 1.
\end{align*}
\]

All three optimization problems are meaningful for data analysis, and also for algebraic geometry. Recall that $M_{\Sigma}$ is an affine cone in $\mathbb{P}^D$, encoding a projective variety in $\mathbb{P}^{D-1}$. Problem (pED) asks for the point in that affine cone which is closest to $U$. In the algebraic
approach we compute all critical points in $\mathbb{C}^D$. In problem (aED) we add the further constraint that the tensor entries $\hat{p}_{ij}$ must sum to 1. Thus (aED) refers to the Euclidean distance problem for an affine variety in the hyperplane $\Sigma: \sum \hat{p}_{ij} = 1$ namely the Zariski closure of the statistical model obtained by intersecting the cone $M_{\Sigma}$ with the simplex $\Delta_{D-1}$. The subtle distinction between the projective case and the affine case is discussed in [6, Section 6].

Problem (ML) is likelihood inference, which is ubiquitous in statistics. For a geometric introduction see [12]. The log-likelihood function is invariant, up to an additive constant, under scaling the tensor $P$, and there is no need to distinguish between affine and projective.

The intrinsic algebraic complexity of a polynomial optimization problem is the number of complex critical points, assuming the data tensor $\mathbf{U}$ is generic. For the problems above, that number is called the ML degree, the aED degree and the pED degree, as in [6, 12].

We performed this computation for a wide range of marginal independence models $M_{\Sigma}$. In what follows we discuss our methodology. Our findings are summarized in the next section, along with pointers to a comprehensive database.

We begin with the Euclidean distance problems (aED) and (pED). Our varieties are toric and hence rational, given both parametrically, by Lemma 7, and via an implicit representation, by Theorem 10. The former leads to an unconstrained optimization problem whose decision variables are the model parameters $(\lambda, \theta)$. Its critical equations are simply those in [6, equation (2.4)]. The latter leads to a constrained optimization problem whose decision variables are the tensor entries $p_{ijk}; i,j,k \leq n$; the critical equations are given in [6, equation (2.1)].

We solve the unconstrained problem using the Julia package HomotopyContinuation.jl due to Breiding and Timme [5]. To illustrate this package, we show the code for $n = 3$: using HomotopyContinuation

\begin{verbatim}
@var q1 q2 q3 q12 q13 q23 q123 
uw00, uw01, uw10, uw11, uw00, uw11 = [1//2, 1//4, 1//8, 1//16, 1//32, 1//64, 1//128, 1//256 ] 
diffs = [uw00 - z*(q123), uw00 - z*(q12-q123), uw10 - z*(q13-q123), uw11 - z*(q123-q12-q13), uw00 - z*(q12-q123), uw10 - z*(q13-q123), uw11 - z*(q123-q12-q13) ]
dist = sum([d^2 for d in diffs])
\end{verbatim}

This sets up the objective function $\text{dist}$ in M"{o}bius parameters. We next specify the model:

\begin{verbatim}
model = [q12=>q1*q2, q13=>q1*q3, q23=>q2*q3, q123=>q1*q2*q3]
dist = subs(dist, model...) # projective ED vars = variables(dist) 
eqns = differentiate(dist, vars) R = solve(eqns)
\end{verbatim}

Example 19 ($n = 3$). We fix three binary random variables. The Segre variety $\mathbf{Z} = M_{\Sigma}$ is the set of $2 \times 2 \times 2$ tensors of rank 1. This variety has ML degree 1 and aED degree 17. Its pED degree is 6, by [6, Example 8.2]. The three solutions for the data $(u_{111}, u_{121}, u_{122}, u_{211}, u_{221}, u_{222}) = (2^{-1}, 2^{-2}, 2^{-3}, 2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}, 2^{-8})$ are displayed in Table 1. The last row shows the rational numbers $\hat{p}_{ijk} = u_{ijk} u_{i+k} u_{i+j}$. The solutions for aED and ML sum to 1, but for pED it does not. For pED there are three other real critical points.

The three solutions are close to each other in $\mathbb{R}^8$. However, for an algebraist they are very different. To appreciate this, consider the respective minimal polynomials for the solution coordinate $x = \hat{p}_{111}$ in Figure 1. All three polynomials are irreducible in $\mathbb{Z}[x]$, but their sizes vary considerably.

\begin{table}[h]
\centering
\begin{tabular}{|l|l|l|l|l|l|l|l|l|}
\hline
 & Deg & # Real & $\hat{p}_{111}$ & $\hat{p}_{112}$ & $\hat{p}_{121}$ & $\hat{p}_{122}$ & $\hat{p}_{211}$ & $\hat{p}_{212}$ & $\hat{p}_{221}$ & $\hat{p}_{222}$ \\
\hline
aED & 17 & 1 & 0.50038 & 0.25053 & 0.12563 & 0.06290 & 0.03224 & 0.01614 & 0.00809 & 0.00405 \\
pED & 6 & 4 & 0.49995 & 0.25003 & 0.12508 & 0.06255 & 0.03161 & 0.01581 & 0.00790 & 0.00395 \\
ML & 1 & 1 & 0.49610 & 0.25096 & 0.12645 & 0.06397 & 0.03307 & 0.01673 & 0.00843 & 0.00426 \\
\hline
\end{tabular}
\caption{The degrees and approximate real solutions to the three estimation problems for the data in Example 19.}
\end{table}

Figure 1: Minimal polynomials for the exact solution coordinate $x = \hat{p}_{111}$ of the ML, pED and aED estimators in Table 1.
This solves the pED problem in Example 19. If we now run \( \mathcal{C} = \text{certify(eqns, R)} \), then this proves correctness, in the sense of [4]. By deleting entries of model1, we can specify other complexes \( \Sigma \). For instance, for the 3-cycle, delete \([q123=>q1*q2*q3] \). For aED we use the line
\[
\text{model} = [q12=>q1*q2, q13=>q1*q3, q23=>q2*q3, q123=>q1*q2*q3, z=>1] \quad \# \text{affine ED}
\]
Results of this computation for all binary models up to \( n = 5 \) are presented in Section 6. Symbolic computation was used to independently verify a range of small cases.

Our experiments suggest the following conjecture for all marginal independence models:

**Conjecture 20.** Given any nonnegative tensor \( U \) whose entries sum to 1, the aED problem has precisely one real critical point, namely the point \( \hat{P} \in M_{\Sigma} \cap \Delta_{D-1} \) that is closest to \( U \).

For maximum likelihood estimation, most statisticians use local hill-climbing methods, such as Iterative Conditional Fitting [7, Section 5]. By contrast, we here use the global numerical tools of nonlinear algebra [4, 5]. The objective function for ML is entered as follows:

\[
\text{loglike} = \sum[u_i \cdot \log(p_i) \text{ for } i = 1: \text{length(u)}]
\]

Following [20], we solve the rational function equations given by the gradient of loglike. We avoid the numerator polynomials, which are much too large. The introduction of [20] discusses numerical ML by stating that “a key idea is to refrain from clearing denominators”.

Example 19 suggests that the ED degrees exceed the ML degree. However, this is incorrect. The Segre variety is an exception. Almost all models \( M_{\Sigma} \) have a larger ML degree.

**Example 21** \((n = 5)\). Fix the bowtie in Example 6 and consider a data tensor \( U \in \Delta_{13} \). We estimate the 26 model parameters, in order to find the best fit \( \hat{P} \in M_{\Sigma} \). Solving the critical equations for Euclidean distance is much faster than for maximum likelihood. We see this from the degrees, which reveal the number of paths to which are much too large. The introduction of [20] discusses numerical ML by stating that “a key idea is to refrain from clearing denominators”.

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**Example 21** \((n = 5)\). Fix the bowtie in Example 6 and consider a data tensor \( U \in \Delta_{13} \). We estimate the 26 model parameters, in order to find the best fit \( \hat{P} \in M_{\Sigma} \). Solving the critical equations for Euclidean distance is much faster than for maximum likelihood. We see this from the degrees, which reveal the number of paths to be tracked in HomotopyContinuation.jl. The aED degree equals 113 and the pED degree equals 113. We do not yet know the ML degree, but monodromy loops show that it exceeds one million. An easier model is the claw graph \( \Sigma = \{12, 13, 14, 15\} \), whose ML degree equals 14693. The aED degree and the pED degree are only 5 and 2, respectively.

**Remark 22.** Some of our largest ML degrees still have a small upward margin of error. Those are indicated in Table 2 with a trailing “+”. All the numbers we report are certified lower bounds, thanks to [4], just like [20, Proposition 5].

## 6 CLASSIFICATION OF SMALL MODELS

Every \( n \)-way tensor has an associated simplicial complex, namely the index sets of all marginalizations that have rank \( \leq 1 \). Conversely, every simplicial complex \( \Sigma \) is realized by some tensor. Thus, the space of tensors \( \mathbb{R}^D \) has a natural stratification, with cells indexed by all simplicial complexes on \([n]\). Our model \( M_{\Sigma} \) is the closure of the cell indexed by \( \Sigma \). The map from simplicial complexes to marginal independence models is inclusion-reversing:

\[
\Sigma \subset \Sigma' \quad \text{if and only if} \quad M_{\Sigma} \supset M_{\Sigma'}.
\]

The geometry of this stratification is important for Bayesian model selection [21, Chapter 17].

In this section we present the classification of all small models. The number of unlabeled simplicial complexes on \( n = 1, 2, 3, 4, 5, 6, 7 \) vertices equals 1, 2, 5, 20, 180, 16143, 489996795. Exhaustive computations are thus limited to \( n \leq 6 \). Further, we restrict ourselves to the binary case \( d_1 = \cdots = d_n = 2 \). We denote each complex by its list of facets. For instance, \( \Sigma = \{12, 13, 23\} \) is the 3-cycle in Example 1. The complete list for \( n = 4 \) is shown in Table 2. The last three columns confirm that the ML degree exceeds the ED degrees for most models.

The first three columns in Table 2 describe \( M_{\Sigma} \) as a projective toric variety: its dimension, its degree, and the number of minimal generators, here all quadratic. Similar lists of all models for \( n = 3, 4, 5, 6 \) and samples of code in Macaulay2 and Julia can be found at

https://mathrepo.mis.mpg.de/MarginalIndependence.

For each model, the repository gives both parametrization and implicit representation, in machine-readable form, so a reader can experiment with these in a computer algebra system.

Concerning the minimal generators of \( I_{\mathcal{M}_{\Sigma}} \), we record the following result from our data.

**Proposition 23.** For 178 of the 180 models \( \Sigma \) with \( n = 5 \), the ideal \( I_{\mathcal{M}_{\Sigma}} \) is generated by quadrics. The two exceptions are the bowtie and the complex \( \Sigma = \{123, 124, 134, 145, 234, 235\} \) whose ideal has degree 73 and requires 97 quadrics and 1 cubic. For 14104 of the 16143 models with \( n = 6 \), the ideal \( I_{\mathcal{M}_{\Sigma}} \) is generated by quadrics. Of the others, 1930 ideals require cubics, 104 require quartics, four require quintics, and only one (that in Example 16) requires a sextic.

Here is the ideal with the most non-quadratic generators among those in Proposition 23:

**Example 24** \((\mathbb{R}^2 \times \mathbb{R}^2 \text{ with } n=6)\). Fix \( \Sigma = \{123, 124, 135, 146, 156, 236, 245, 256, 345, 346\} \). This is the minimal triangulation of the real projective plane. Its toric ideal \( I_{\mathcal{M}_{\Sigma}} \) is generated by 209 quadrics, 10 cubics and 15 quartics. This model has codimension 25 and degree 275.

We do not yet have complete results for the algebraic degrees of the estimation problems. Computing the ML degree seems to be out of reach for \( n = 5 \). The ED degrees are better behaved. The data provides the following result.

**Proposition 25.** Among all 180 binary models for \( n = 5 \), the largest aED degree equals 1457, namely for \( \Sigma = \{1234, 125, 135, 145, 235, 245, 345\} \); it has pED degree 1077. The largest pED degree equals 1247, namely for the complex \( \Sigma \) given by all \( 10 \) triangles; it has aED degree 1425.

We now turn to marginal independence models defined by matroids (Example 13). This includes our random graph models (Example 18). Matroid models are attractive because they connect independence in linear algebra and independence in statistics. For realizable matroids, the binary model \( M_{\Sigma} \) assigns a probability
18. Given any sample of models of special interest, we also computed beyond those limits.

Example 26 (Fano matroid). The finite vector space \( V = \mathbb{F}_2^3 \) has \( n = 7 \) non-zero vectors. The complex \( \Sigma \) consists of all 28 bases and their subsets, and \( M_{\Sigma} \) is a model for random subsets of \( V \setminus \{0\} \). The number of relevant Möbius coordinates is \( 1 \times 7 + 21 + 28 = 57 \). The ideal \( I_{\mathcal{M} \Sigma} \) has codimension 49 and degree 1207. It is generated by 868 quadrics; cf. [13].

We examined all loopless matroids on \( n \leq 7 \) elements. Their number is 305, up to isomorphism. Their toric ideals \( I_{\mathcal{M} \Sigma} \) are all generated by quadrics. The ED and ML degree computations do not become simpler in the matroid case. In fact, the highest ML degree in Table 2 is attained for the uniform matroid \( U_{5,4} \) and the highest aED degree for \( \Omega_{5,4} \).

The matroids appear among models in our database, where they are especially marked. Their ED degrees are available for all models up to \( n = 5 \), and the ML degrees for all models up to \( n = 4 \). For some models of special interest, we also computed beyond those limits.

Example 27 (\( n = 6 \)). Consider the random graph model in Example 18. Given any sample of 4-verext graphs, we compute its best fit in the model. Using HomotopyContinuation.jl, we determined that the aED degree equals 1981, while the pED degree equals 3512.

We conclude with several directions for future research. First, consider the statistical results in [7]. It would be worthwhile to identify all singularities of \( M_{\Sigma} \), with a view towards [7, Corollary 3]. Also, the Iterative Conditional Fitting algorithm [7, Section 5] should be developed for our models. Naturally, it is desirable to determine whether multiple local optima inside \( \Delta_{D-1} \) can occur, for the various models, under the three estimation paradigms.

For any given matroid, our model strictly contains the model of weak probabilistic representations due to Matúš [17]. This expresses higher conditional and functional (in)dependences. Among its points are tensors encoding linear representations of the matroid over certain finite fields, but not all tensors in the Matúš models arise in this way. It would be interesting to study these models in our setting. Example 26 is a point of departure.

One class of models to be examined in depth is the \( r \)-way marginal independence model. This is the \( M_{\Sigma} \) given by the uniform matroid of rank \( r \) on \( \{n\} \), so \( \Sigma \) is the collection of all subsets of \( \{n\} \) with at most \( r \) elements. A tensor is in \( M_{\Sigma} \) if and only if all its \( r \)-dimensional marginals have rank \( \leq 1 \). What can be said about parameter estimation for these models?

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