Abstract

We construct triangular hyperbolic polyhedra whose links are generalized 4-gons. The universal cover of such a polyhedron is a hyperbolic building, whose apartments are hyperbolic planes tessellated by regular triangles with angles $\pi/4$. The fundamental groups of the polyhedra are hyperbolic, torsion free, with property (T).

Résumé

Immeubles hyperboliques triangulaires. On construit des polyèdres hyperboliques dont les links en chaque sommet sont des 4-gones généralisées. Leurs revêtements universels sont des immeubles dont les appartements sont des plans hyperboliques pavés par des triangles réguliers d’angles $\pi/4$. Les groupes fondamentaux de nos polyèdres sont hyperboliques, sans torsion et ont la propriété (T).

1. Introduction

Hyperbolic torsion free groups with property (T) have uncountably many nonisomorphic quotient groups $(\Gamma_\alpha)_{\alpha \in I}$ which are simple and with infinitely many conjugacy classes (see [8,10,11]). Such groups exist: the random group of Gromov [9], cocompact lattices of $\text{Sp}(1,n)$ etc.

We give new examples of groups of this kind which are explicitly presented by generators and relations.

A polyhedron is a two-dimensional complex which is obtained from several oriented $p$-gons by identification of corresponding sides. Let us take a sphere of a small radius at a point of the polyhedron. The intersection of the sphere with the polyhedron is a graph, which is called the link at this point.

In this Note we construct polyhedra whose links at vertices are generalized 4-gons and whose faces are regular hyperbolic triangles with angles $\pi/4$. The universal covering of such a polyhedron is a hyperbolic building, see [6]. Moreover, with the metric introduced in [1, p. 165] it is a complete metric space of non-positive curvature in the sense of Alexandrov and Busemann [7]. It follows from [2] that the fundamental groups of our polyhedra satisfy the property...
(T) of Kazhdan. (Another relevant reference is [15].) So, our groups, which are explicitly presented by generators and relations, are hyperbolic, torsion free and they have property (T).

**Definition 1.1.** Let $\mathcal{P}(p, m)$ be a tessellation of the hyperbolic plane by regular polygons with $p$ sides, with angles $\pi/m$ at each vertex where $m$ is an integer. A hyperbolic building is a polygonal complex $X$, which can be expressed as the union of subcomplexes called apartments such that:

1. Every apartment is isomorphic to $\mathcal{P}(p, m)$.
2. For any two polygons of $X$, there is an apartment containing both of them.
3. For any two apartments $A_1, A_2 \in X$ containing the same polygon, there exists an isomorphism $A_1 \rightarrow A_2$ fixing $A_1 \cap A_2$.

Our construction gives new examples of hyperbolic triangular buildings with regular triangles as chambers. Examples of hyperbolic buildings with right-angled triangles were constructed by Bourdon in [3]. His construction has been generalized by Świątkowski in [12].

2. **Polygonal presentation and construction of polyhedra**

Recall that a generalized $m$-gon is a connected, bipartite graph of diameter $m$ and girth $2m$, in which each vertex lies on at least two edges. A graph is bipartite if its set of vertices can be partitioned into two disjoint subsets such that no two vertices in the same subset lie on a common edge. The vertices of one subset we will call black vertices, and the vertices of the other subset the white ones, denoted by $y_i$, $i \in \mathbb{Z}_+$. The diameter is the maximum distance between two vertices and the girth is the length of a shortest circuit.

We recall also the definition of a polygonal presentation introduced in [14]:

**Definition 2.1.** Suppose we have $n$ disjoint connected bipartite graphs $G_1, G_2, \ldots, G_n$. Let $P_i$ and $Q_i$ be the sets of black and white vertices respectively in $G_i$, $i = 1, \ldots, n$; let $P = \bigcup P_i$, $Q = \bigcup Q_i$, $P_i \cap P_j = \emptyset$, $Q_i \cap Q_j = \emptyset$ for $i \neq j$ and let $\lambda$ be a bijection $\lambda : P \rightarrow Q$.

A set $K$ of $k$-tuples $(x_1, x_2, \ldots, x_k)$, $x_i \in P$, will be called a polygonal presentation over $P$ compatible with $\lambda$ if

1. $(x_1, x_2, x_3, \ldots, x_k) \in K$ implies that $(x_2, x_3, \ldots, x_k, x_1) \in K$;
2. given $x_1, x_2 \in P$, then $(x_1, x_2, x_3, \ldots, x_k) \in K$ for some $x_3, \ldots, x_k$ if and only if $x_2$ and $\lambda(x_1)$ are incident in some $G_i$;
3. given $x_1, x_2 \in P$, then $(x_1, x_2, x_3, \ldots, x_k) \in K$ for at most one $x_3 \in P$.

If there exists such $K$, we will call $\lambda$ a basic bijection.

The polygonal presentations with $k = 3$, $n = 1$, and $G_1$ a generalized $3$-gon have been listed in [4,5].

We can associate a polyhedron $K$ on $n$ vertices with each polygonal presentation $K$ as follows: for every cyclic $k$-tuple $(x_1, x_2, x_3, \ldots, x_k)$ we take an oriented $k$-gon on the boundary of which the word $x_1 x_2 x_3 \cdots x_k$ is written. To obtain the polyhedron we identify the corresponding sides of our polygons, respecting orientation.

**Lemma 2.2** [14]. A polyhedron $K$ which corresponds to a polygonal presentation $K$ has graphs $G_1, G_2, \ldots, G_n$ as vertex-links.

Now we construct two polygonal presentations with $k = 3$ and $n = 1$, but for which the graph $G_1$ is a generalized $4$-gon. We denote the elements of $P$ by $x_i$ and the elements of $Q$ by $y_i$, $i = 1, 2, \ldots, 15$. Let $T_1$ and $T_2$ be the two following sets of triples, and in both cases define the basic bijection $\lambda : P \rightarrow Q$ by $\lambda(x_i) = y_i$ for all $i = 1, 2, \ldots, 15$.

$$T_1: \{ (x_1, x_2, x_7), (x_1, x_8, x_11), (x_1, x_{14}, x_5), (x_2, x_4, x_{13}), (x_{12}, x_4, x_2),$$

$$ (x_4, x_9, x_3), (x_6, x_3, x_1), (x_{14}, x_6, x_3), (x_{12}, x_{10}, x_5), (x_{13}, x_{15}, x_5),$$

$$ (x_{12}, x_9, x_6), (x_{11}, x_{10}, x_7), (x_{14}, x_{13}, x_7), (x_9, x_{15}, x_8), (x_{11}, x_{15}, x_{10}) \} ,$$

$$T_2: \{ (x_1, x_2, x_7), (x_1, x_8, x_11), (x_1, x_{14}, x_5), (x_2, x_4, x_{13}), (x_{12}, x_4, x_2),$$

$$ (x_4, x_9, x_3), (x_6, x_3, x_1), (x_{14}, x_6, x_3), (x_{12}, x_{10}, x_5), (x_{13}, x_{15}, x_5),$$

$$ (x_{12}, x_9, x_6), (x_{11}, x_{10}, x_7), (x_{14}, x_{13}, x_7), (x_9, x_{15}, x_8), (x_{11}, x_{15}, x_{10}) \} .$$
We can draw the bipartite graph $G_1$ for $T_1$ (Fig. 1). For every triple $(x_i, x_j, x_k)$ in $T_1$ the points $y_i$ and $x_j$ as well as $y_j$ and $x_k$ and also $y_k$ and $x_i$ have to be incident in the graph. For $T_2$ we obtain a similar graph, only with a different labeling of the points.

Let us check that these sets are desired polygonal presentations. Remark, that the smallest thick generalized 4-gon can be presented in the following way: its 'points' are pairs $(i, j)$, where $i, j = 1, \ldots, 6, i \neq j$ and 'lines' are triples $(i_1, j_1), (i_2, j_2), (i_3, j_3)$ of those pairs, where $i_1, i_2, i_3, j_1, j_2$ and $j_3$ are all different. We mark pairs $(i, j)$, where $i, j = 1, \ldots, 6, i \neq j$ by $x_1$ to $x_{15}$. Now one can check by direct examination, that the graph $G_1$ is really the smallest thick generalized 4-gon. (See [13] for classification of generalized quadrangles.)

**Definition 2.3.** Let $K_1$ and $K_2$ be two polygonal presentations with $k = 3$, $n = 1$, and for which the graph $G_1$ is a generalized 4-gon. Then $K_1$ and $K_2$ are equivalent, if there exists an automorphism of the generalized 4-gon which transforms the 4-gon of $K_1$ to the 4-gon of $K_2$.

In our case there is no such automorphism transforming $T_1$ to $T_2$, since in $T_1$ no element appears twice in one triple, but in $T_2$ there are triples of the form $(x_i, x_j, x_k)$. Thus the polygonal presentations $T_1$ and $T_2$ are not equivalent.

For polygonal presentation $T_i$, $i = 1, 2$, take 15 oriented regular hyperbolic triangles with angles $\pi/4$, write words from the presentation on their boundaries and glue together sides with the same letters, respecting orientation. The result is a hyperbolic polyhedron with one vertex and 15 faces and its universal covering is a triangular hyperbolic building. The fundamental group $\Gamma_i$, $i = 1, 2$, of the polyhedron acts simply transitively on vertices of the building. The group $\Gamma_i$, $i = 1, 2$, has 15 generators and 15 relations, which come naturally from the polygonal presentation $T_i$, $i = 1, 2$.

For the first homology groups we get $H_1(\Gamma_1) = \mathbb{Z}/162\mathbb{Z}$ and $H_1(\Gamma_2) = \mathbb{Z}/9\mathbb{Z}$.

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