ON SOME PROPERTIES OF QUANTUM DOUBLES OF FINITE GROUPS

PAVEL ETINGOF

1. Introduction

In this paper we prove two results about quantum doubles of finite groups over the complex field. The first result is the integrality theorem for higher Frobenius-Schur indicators for wreath product groups \( S_N \ltimes A^N \), where \( A \) is a finite abelian group. A proof of this result for \( A = 1 \) appears in [IMM]. The second result is a lower bound for the largest possible number of irreducible representations of the quantum double of a finite group with \( \leq n \) conjugacy classes. This answers a question asked to me by Eric Rowell.

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2. Integrality of higher Frobenius-Schur indicators for the quantum doubles of wreath product groups \( S_N \ltimes A^N \)

2.1. The result. Let \( H \) be a semisimple Hopf algebra over \( \mathbb{C} \), and \( \Lambda \) a left integral of \( H \) such that \( \varepsilon(\Lambda) = 1 \). Let \( V \) be an irreducible \( H \)-module. Then the higher Frobenius-Schur indicators \( \nu_n(V) \) are defined by the formula ([KSZ]):

\[
\nu_n(V) = \text{Tr}_V(\Lambda_1 \ldots \Lambda_n),
\]

where \( \Lambda_1 \otimes \ldots \otimes \Lambda_n = \Delta_n(\Lambda) \) (using Sweedler’s notation with implied summation). These are algebraic integers lying in \( \mathbb{Q}(e^{2\pi i/n}) \), which are known to be integers if \( H \) is a group algebra.

An interesting question is when \( \nu_n(V) \) are integers.

Here we prove the following theorem.

Let \( A \) be a finite abelian group.

**Theorem 2.1.** For any \( N \) and any irreducible representation \( V \) of the quantum double \( D(S_N \ltimes A^N) \), the higher Frobenius-Schur indicators \( \nu_n(V) \) are integers.

In the case \( A = 1 \), a proof of this theorem appears in [IMM].

**Remark.** For basics on the quantum double, see [Ka], Section IX.

2.2. Proof of Theorem 2.1. Let us start with deriving an explicit formula for \( \nu_n(V) \) (which is well known to experts, see e.g. [KSZ]). Let \( G \) be a finite group. For \( D(G) \), one has

\[
\Lambda = |G|^{-1}\left( \sum_{g \in G} g \right) \delta_1,
\]

where \( \delta_g(x) := 1 \) if \( g = x \) and \( \delta_g(x) = 0 \) otherwise. So

\[
\Lambda_1 \ldots \Lambda_n = |G|^{-1} \sum_g \sum_{x_1 \ldots x_n = 1} g \delta_{x_1} g \delta_{x_2} \ldots g \delta_{x_n} = |G|^{-1} \sum_g \sum_{x_1 \ldots x_n = 1} g^n \delta_{g^{1-n} x_1} g^{n-1} \ldots \delta_{x_n} =
\]

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Thus,
\[ \nu_n(V) = |G|^{-1} \sum_{g} \sum_{y: g^n = (y^{-1}g)^n} g^n \delta_y. \]

Let us now recall the structure of irreducible representations of \( D(G) \). Such a representation \( V \) is attached to a conjugacy class \( C \) in \( G \) and an irreducible representation \( W \) of the centralizer \( Z_y \) of an element \( y \in C \), and has the form \( V = \oplus_{u \in C} V_u \), where \( V_u = \delta_u V \), and \( V_y = W \). So the formula for \( \nu_n \) can be rewritten as
\[ \nu_n(V) = |G|^{-1} \sum_{u \in C} \sum_{g \in G: g^n = (u^{-1}g)^n} \text{Tr}_{V_u}(g^n). \]

(note that \( g^n \) is automatically in \( Z_u \) as it commutes with \( g \) and \( u^{-1}g \)). It is clear that the summands corresponding to all \( u \in C \) are the same, so we get
\[ \nu_n(V) = \frac{|C|}{|G|} \sum_{g \in G: g^n = (y^{-1}g)^n} \text{Tr}_W(g^n) = \frac{|C|}{|G|} \sum_{g, h \in G: g^n = h^n, gh^{-1} = y} \text{Tr}_W(g^n). \]

Let us multiply this formula by \( \text{Tr}_{W^*}(z) \) and sum over \( W \). To keep track of the dependence on \( W \), we will write \( V(W) \) instead of \( V \). By orthogonality of characters, we get
\[ \sum_{W \in \text{Irr } Z_y} \nu_n(V(W)) \text{Tr}_{W^*}(z) = \#\{(g, h) \in G^2 : g^n = h^n = z, gh^{-1} = y\}. \]

Thus, we see that integrality of \( \nu_n(V) \) for all \( V \) is equivalent to the statement that for any \( y \) the function \( f_y : Z_y \to \mathbb{Z} \) given by the formula
\[ f_y(z) = \#\{(g, h) \in G^2 : g^n = h^n = z, gh^{-1} = y\} \]

is a virtual character of \( Z_y \). This, in turn, is equivalent to saying that for any \( s \) which is coprime to \( |G| \), one has \( f_y(z) = f_y(z^s) \).

So the theorem follows from the following proposition.

**Proposition 2.2.** Let \( G = S_N \ltimes A^N \). Then for any \( s \) which is coprime to \( |G| \), one has \( f_y(z) = f_y(z^s) \).

The rest of the subsection is the proof of this proposition.

First of all, \( g, h, y \) commute with \( z \). So if we write \( z \) as \((z_1, ..., z_q)\) where \( z_j \) comprises all cycles of a fixed type in \( z \) (i.e. fixed length and monodromy\(^1\)), then we’ll get \( g = (g_1, ..., g_q) \), \( h = (h_1, ..., h_q) \), \( y = (y_1, ..., y_q) \) accordingly (where \( g_j, h_j, y_j \) are some elements commuting with \( z_j \)). Thus, \( f_y(z) = f_{g_1}(z_1)...f_{g_q}(z_q) \). This shows that we may assume that \( z \) has only cycles of some fixed length \( m \) with fixed monodromy \( u \in A \).

So we can assume that \( N = mr \) and \( z = (c, c, ..., c) \in (S_m \ltimes A^m)^r \subset G \), where \( c = (12...m)(u, 0, 0, 0) \). In this case, the centralizer \( Z_z \) is \( S_r \ltimes B^r \), where \( B \) is the central extension of \( Z/mZ \) by \( A \) which is generated in \( S_m \ltimes A^m \) by \( c \) and the diagonal copy of \( A \). So we have \( g = \tau \cdot (a_1, ..., a_r) \), \( h = \theta \cdot (b_1, ..., b_r) \), \( y = \sigma \cdot (k_{\theta^{-1}(1)}, ..., k_{\theta^{-1}(r)}) \), for \( \tau, \theta, \sigma \in S_r \subset S_{mr} \).

\(^1\)If \( z = c \cdot (a_1, ..., a_n) \in S_m \ltimes A^n \), where \( c \) is a cycle of length \( n \) and \( a_i \in A \), then by the monodromy of \( z \) we mean the element \( a_1...a_n \) in \( A \). It is invariant with respect to conjugation.
(diagonal copy in $S_r^m \subset S_{mr}$), $\sigma = \tau \theta^1$, and $a_i, b_i, k_i \in B$. Then the equation $gh^{-1} = y$ says that

$$a_j - b_j = k_j,$$

and the equations $g^n = h^n = z^s$ say that $\tau^n = \theta^n = 1$ (so all the cycles of $\tau$ and $\theta$ have lengths dividing $n$), and for any cycle $K = (i_1, ..., i_{d(K)})$ of $\tau$, one has

$$\frac{n}{d(K)} (a_{i_1} + ... + a_{i_{d(K)}}) = sc,$$

while for any cycle $K = (j_1, ..., j_{d(K)})$ of $\theta$, one has

$$\frac{n}{d(K)} (b_{j_1} + ... + b_{j_{d(K)}}) = sc.$$

Let $\ell$ be the least common multiple of $n/d(K)$ for all cycles $K$. If equations (2.4), (2.5) have a solution, then $c = \ell \overline{c}$ for some (possibly non-unique) $\overline{c} \in B$.

Thus, equations (2.4) and (2.5) can be rewritten as

$$a_{i_1} + ... + a_{i_{d(K)}} = \frac{\ell d(K)}{n} \overline{c} + v(K), \quad \frac{n}{d(K)} v(K) = 0$$

for any cycle $K = (i_1, ..., i_{d(K)})$ of $\tau$, and

$$b_{j_1} + ... + b_{j_{d(K)}} = \frac{\ell d(K)}{n} \overline{c} + v(K), \quad \frac{n}{d(K)} v(K) = 0$$

for any cycle $K = (j_1, ..., j_{d(K)})$ of $\theta$.

We can eliminate $a_j$ by setting $a_j = b_j + k_j$, so we get:

$$b_{i_1} + ... + b_{i_{d(K)}} = k_{i_1} + ... + k_{i_{d(K)}} + \frac{\ell d(K)}{n} \overline{c} + v(K), \quad \frac{n}{d(K)} v(K) = 0$$

for any cycle $K = (i_1, ..., i_{d(K)})$ of $\tau$, and

$$b_{j_1} + ... + b_{j_{d(K)}} = \frac{\ell d(K)}{n} \overline{c} + v(K), \quad \frac{n}{d(K)} v(K) = 0$$

for any cycle $K = (j_1, ..., j_{d(K)})$ of $\theta$.

For given $\tau, \theta$, equations (2.3), (2.8), (2.9) are linear inhomogeneous equations in $b_j, v(K)$, whose associated homogeneous equations don’t depend on $s$. So the number of solutions, if it is not zero, does not depend on $s$, and our job is just to show that the solvability of equations (2.3), (2.8), (2.9) does not depend on $s$.

Now we will need the following (well known) lemma.

**Lemma 2.10.** Let $B$ be a finite abelian group. Let $P = (P_i, 1 \leq i \leq l_P)$ and $Q = (Q_j, 1 \leq j \leq l_Q)$ be two set partitions of $\{1, ..., r\}$. Let $p_i, 1 \leq i \leq l_P, q_j, j = 1, ..., l_Q$ be elements of $B$. Then the system of equations

$$\sum_{k \in P_i} b_k = p_i, 1 \leq i \leq l_P; \quad \sum_{k \in Q_j} b_k = q_j, 1 \leq j \leq l_Q;$$

...
has a solution \((b_1, \ldots, b_r) \in B^r\) if and only if for every subset \(S \subset \{1, \ldots, r\}\) compatible with both \(P\) and \(Q\), one has
\[
\sum_{i:P \in S} p_i = \sum_{j:Q \in S} q_j.
\]

Proof. Clearly, the condition is necessary, since both sides equal \(\sum_{i \in S} b_i\). For sufficiency, it suffices to consider the case when \((P, Q)\) is indecomposable, i.e. any nonempty \(S\) is the whole \(\{1, \ldots, r\}\). In this case, we just have one equation
\[
\sum_{i=1}^{l_P} p_i = \sum_{j=1}^{l_Q} q_j.
\]

Let \(1_S\) be the characteristic function of a subset \(S\). Our job is to show that if \(a_i, b_j \in B^*\) are such that
\[
\sum a_i 1_{P_i} = \sum b_j 1_{Q_j},
\]
then \(a_i = b_j = \phi\) for some \(\phi \in B^*\) and all \(i, j\).

To this end, let \(f = \sum a_i 1_{P_i} - \sum b_j 1_{Q_j}\). Then \(f(x) = a_{i(x)} - b_{j(x)}\), where \(x \in P_{i(x)}\) and \(x \in Q_{j(x)}\). So if \(f = 0\) then \(a_{i(x)} = b_{j(x)}\). Consider the bipartite graph with vertices \(i\) and \(j\), where \(i\) is connected to \(j\) if \(P_i \cap Q_j \neq \emptyset\). Since \((P, Q)\) is indecomposable, this graph is connected. So we see that if \(f = 0\) then \(a_i = b_j = \phi\) for all \(i, j\), as desired. \(\square\)

Now, the lemma easily implies the proposition. Indeed, let \(P, Q\) be the partitions of \(\{1, \ldots, r\}\) into cycles of \(\tau\) and \(\theta\). Then the lemma implies that the condition of solvability of equations (2.3), (2.8), (2.9) in \(b_i\) is a system of linear equations in the variables \(v(K)\), which is independent on \(s\), as desired (the terms which depend on \(s\) cancel).

3. Bounds on the number of irreducible representations of the quantum double of a finite group

3.1. The result. The goal of this section is to establish bounds for the number of irreducible representations of the quantum double of a finite group, answering a question asked to me by E. Rowell.

Let \(f(n)\) be the maximal number of irreducible representations of the quantum double \(D(G)\) (over \(\mathbb{C}\)) of a finite group \(G\) with \(\leq n\) conjugacy classes (or, equivalently, irreducible representations). Note that \(f(n)\) is well defined since by Landau’s theorem (1903), the set of such groups is finite.

**Theorem 3.1.** There are positive constants \(C_1, C_2\) such that
\[
C_1 n^{1/3} \log(n) \leq \log f(n) \leq C_2 n \log^7(n).
\]

The proof, given in the next subsections, readily follows from the results in the literature.

**Remark.** We don’t know if the upper bound holds for semisimple Hopf algebras or fusion categories.

3.2. The upper bound. By a result of Pyber [P] improved by Keller [K], there is \(C > 0\) such that \(n \geq C \log |G|/(\log \log |G|)^7\). for sufficiently large \(n\). On the other hand, \(f(n) \leq |G|^2\). This implies the statement.
3.3. **The lower bound.** Let \( p \) be a prime. Consider the group \( G \) of changes of variable 
\[ x \rightarrow x + a_2x^2 + \ldots + a_{p+1}x^{p+1} \] 
modulo \( x^{p+2} \) (this kind of groups is considered in [KLG]).
This is a nilpotent group of order \( p^p \), so its exponential and logarithm map are well defined,
and \( \text{Lie}(G) \) is the Lie algebra spanned over \( \mathbb{F}_p \) by vector fields 
\[ L_i := x^{i+1}\partial_x, \quad i = 1, \ldots, p, \]
with commutator modulo \( x^{p+2} \) (so we have \( [L_i, L_j] = (j - i)L_{i+j} \), where \( L_k := 0 \) if \( k > p \)).
Conjugacy classes correspond to adjoint orbits in this Lie algebra, which are easily seen to be the orbit of \( 0 \) and the orbits of \( aL_i + bL_{2i} \), where \( a \neq 0 \), and \( i \) runs from \( 1 \) to \( p \). So the number of conjugacy classes of \( G \) is \( \leq p^3 \).

On the other hand, consider the representations of \( D(G) \). They are parametrized by conjugacy classes of \( G \) and irreducible representations of centralizers \( Z_g \). Take the conjugacy class of \( g = \exp(L_{(p+1)/2}) \). Then \( \text{Lie}(Z_g) \) is spanned by \( L_{(p+1)/2}, \ldots, L_p \), which commute, so \( Z_g \) is a vector space over \( \mathbb{F}_p \) of dimension \((p+1)/2\). This group has \( p^{p+1}/2 \) irreducible representations.

So we see that \( \log f(p^3) \geq \frac{p+1}{2} \log p \).

Now for large \( n \), find a prime \( p \) such that \( \frac{1}{2} n^{1/3} \leq p \leq n^{1/3} \). Then since by definition \( f \) is increasing, we find that \( \log f(n) \geq C_1 n^{1/3} \log(n) \) for some constant \( C_1 \), as desired.

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