The Implications of Naturalness in Effective Field Theory on the Masses of Resonances

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\textbf{ABSTRACT}

Many years ago Weinberg formulated a definition of “naturalness” for effective theories: if an effective theory is to make sense, coefficients must not change too much when the cutoff scale is changed by a factor of order 1. As an example, we consider simple field theories in which an $O(N)$ symmetry spontaneously breaks to $O(N - 1)$. We show that in these theories Weinberg’s criterion for a natural effective theory may be applied directly to the $S$-matrix; it implies that the scale of new physics, beyond the Goldstone bosons, may not be too large: there is always a particle or a cut of mass below or about $4\pi f/\sqrt{N}$. We discuss the range of convergence of the expansion of the chiral Lagrangian. It appears to be impossible to construct an underlying theory of the type considered here that fails to satisfy Weinberg’s criterion.

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1. Introduction

The origin of the hadron masses in QCD has always been something of a mystery. While one can easily see why the pions are nearly massless — they are approximate Goldstone Bosons — the precise value of the $\rho$ mass is difficult to connect to the QCD lagrangian. In recent papers\cite{1}, it was shown that the upper bound, $M$, for the mass of the lightest hadron in a QCD-like theory is reduced as the number of light flavors $N_f$ is increased:

$$M \approx \frac{4\pi f_\pi}{\sqrt{N_f}},$$

(1.1)

where $f_\pi$ is the pion decay constant. In theories in which the couplings between the hadrons are weak (QCD in the limit of a large number $N_c$ of colors is an example of such a theory) the masses of the hadrons may not come close to saturating this bound. However, the real world hadrons are strongly coupled to each other and the bound (1.1) is nearly saturated.

The relationship (1.1) was derived by considering the chiral Lagrangian — an effective field theory of pions. The scale at which the chiral expansion breaks down is roughly $M$, if the low energy theory is “natural”\cite{2}, and the scale of breakdown of chiral perturbation theory should be identified with a non-analytic structure in the $S$ matrix, such as a resonance\cite{1}. The notion of a natural effective theory was first formulated by Weinberg\cite{3}. An effective theory is natural if the sizes of the coefficients are stable against $O(1)$ changes in the the size of the cutoff. Equivalently, a coefficient in the effective lagrangian must not be small compared to its $\beta$ function.

In this paper we will explore the significance of requiring that the low energy reduction of a model have a natural effective theory (NET) in the sense of Weinberg. We will consider models in which the symmetry breaking pattern is $O(N) \rightarrow O(N - 1)$, rather than the $SU(N_f) \times SU(N_f) \rightarrow SU(N_f)$ of QCD-like theories. While such models are neither useful as models of hadrons nor as symmetry breaking sectors for the electroweak theory\cite{2}, they do have the merit that some of them become simple in the limit of large $N$.

By constructing toy models of this type, we show that the NET criterion actually implies something about the $S$-matrix. One need not even consider the effective Lagrangian. We will show that in a model in which there are an arbitrary number of resonances in

\footnote{In this context the word “light” means small in mass compared to $M$.}

\footnote{other than for $N = 4$, which is the standard model!}
Goldstone boson scattering, the lightest resonance is always lighter than $M$. We then generalize to a model in which the Goldstone boson scattering has a branch cut as well as a pole and we find again that the mass of the new physics, the cut or the pole, will be less than $M$.

In the models considered in this work, the underlying theories become sick if the couplings are chosen so as to violate the bound. Thus it appears to be impossible to construct a theory that does not satisfy the NET criterion. Whether this continues to be so in more complex, realistic theories remains an open question.

The plan of this paper is as follows. In section 2 we discuss the properties of $O(N)$ models in general. In section 3 we discuss their behavior in the limit of large $N$. We will consider a subclass of large-$N$ $O(N)$ models, in which all of the physics appears in the $s$-channel of Goldstone boson scattering. In section 4 we discuss the domain of convergence of the effective field theory and the relationship of this to the existence of poles in scattering amplitudes. In section 5 we discuss the meaning of the NET criterion in such models. Section 6 illustrates the NET criterion by applying it to the various models mentioned above. We then make some concluding remarks in section 7.

2. The $O(N)$ Model

A model in which the pattern of symmetry breaking is $O(N) \to O(N-1)$ will possess $N-1$ Goldstone bosons, which form a vector representation of the unbroken $O(N-1)$. These particles (which we shall refer to as pions, $\pi$) are the excitations of the degrees of freedom in the directions corresponding to the different vacua of the $O(N)$ theory. Consider elastic scattering of two pions. The most general form of the amplitude consistent with Bose symmetry, crossing, and the unbroken $O(N-1)$ symmetry is

$$a(\pi^i \pi^j \to \pi^k \pi^l) = A(s, t, u)\delta^{ij}\delta^{kl} + A(t, s, u)\delta^{ik}\delta^{jl} + A(u, s, t)\delta^{il}\delta^{jk},$$

where the function $A$ is symmetric in its last two arguments.

It is frequently desirable to express (2.1) in terms of the three invariant “isospin” channels of the unbroken $O(N-1)$. Regarding the initial state as a two index tensor in $i$ and $j$, we see that it may be decomposed into its trace times the identity tensor, an antisymmetric tensor, and a symmetric traceless tensor. We will refer to these three
invariant amplitudes as isospin 0, 1, and 2 respectively, in analogy to the case of $N = 4$, the case of ordinary pions. One obtains

$$a_0 = (N - 1)A(s, t, u) + A(t, s, u) + A(u, s, t)$$
$$a_1 = A(t, s, u) - A(u, s, t)$$
$$a_2 = A(t, s, u) + A(u, s, t) .$$

(2.2)

Note that the scalar singlet channel is enhanced by a factor of $N$ while the other two are not\textsuperscript{3}. These isospin amplitudes may in turn be decomposed into partial waves

$$a_I(\ell)(s) = \frac{1}{64\pi} \int_{-1}^{1} a_I(s, \cos \theta) P_\ell(\cos \theta) d \cos \theta ,$$

(2.3)

where $I = 0, 1, 2$, and $P_\ell$ is the Legendre polynomial of order $\ell$.

The interactions of the pions at low energies may be described by the chiral Lagrangian in the standard fashion \cite{4}. The chiral Lagrangian is an expansion in numbers of derivatives. The lowest order term has two:

$$L_2 = \frac{1}{2} \partial^\mu \pi_i \partial_\mu \pi_i + \frac{1}{2} \partial^\mu (\sqrt{f^2 - \pi^2}) \partial_\mu (\sqrt{f^2 - \pi^2}) .$$

(2.4)

Here $f$ is the decay constant of the pion. This Lagrangian $L_2$ is non-linear and expanding the square root in powers of $\pi/f$ gives vertices with arbitrary (even) numbers of pions.

Using this lowest order Lagrangian, we may calculate the low-energy behavior of the scattering amplitudes of pions. Since $L_2$ is unique, the result is independent of the exact details of the full theory. One finds

$$A(s, t, u) = \frac{s}{f^2} ,$$

(2.5)

and therefore,

$$a_{00} = \frac{(N - 2)s}{32\pi f^2}$$
$$a_{11} = \frac{s}{96\pi f^2}$$
$$a_{20} = -\frac{s}{32\pi f^2} .$$

(2.6)

\textsuperscript{3} This is an amusing contrast to the more realistic case of an $SU(N_f)_L \times SU(N_f)_R \rightarrow SU(N_f)_V$ pattern of symmetry breaking. In that case, the analogue of $a_{11}$, the channel with the quantum numbers of the $\rho$ meson, is enhanced by a factor of $N_f$.
To go beyond the threshold behavior of the pions — to higher order in $s$, $t$, and $u$ in the function $A$ above — one must go beyond the tree-level computations using $\mathcal{L}_2$ and include the calculation of loop diagrams. Since the Lagrangian above is non-renormalizable, there will be counterterms with more than two derivatives. Accordingly, to go up in energy one must also include the higher order terms in the chiral Lagrangian. Unfortunately, the coefficients of these terms in the Lagrangian depend in general on the physics of the full theory and are not universal.

Since different full theories will have different high energy behavior, it is not possible to say very much about the spectrum at high energies. Nonetheless, as we shall show in the following sections, the NET criterion does place some constraints on the physics, independent of particular theories.

3. The Limit of Large $N$

An interesting perspective on the problem is obtained if one contemplates theories in which there exists a good large-$N$ limit. By this we mean that it is possible to let $N \to \infty$ while keeping finite the scattering amplitudes and the masses of resonances. We note that in any such theory, equation (2.2) implies that the function $A$ must go to zero at least as quickly as $1/N$. Therefore (2.5) implies that $f$ must go to infinity as least as fast as $\sqrt{N}$. If the large-$N$ limit is to be non-trivial, then $f \to \infty$ exactly as fast as $\sqrt{N}$, and only $a_0(s)$ is nonvanishing. One can turn this around: the ratio of the resonance masses to $f$ must go like $1/\sqrt{N}$.

Note that this argument would apply to the $SU(N)_L \times SU(N)_R \to SU(N)_V$ linear $\sigma$-model as well, in contradiction to the claims of [3]. Any model with a well-defined $N \to \infty$ limit in which there is a low-energy amplitude enhanced by a factor of $N$ — no matter what the symmetry group or the pattern of symmetry breaking — will have resonances whose masses go like $f/\sqrt{N}$.

For simplicity, let us restrict ourselves for the remainder of the paper to $O(N) \to O(N-1)$ theories in which $A(s, t, u)$ is a function of $s$ only. We refer to these as $s$-channel $O(N)$ models. An example is the $O(N)$ linear $\sigma$-model. Such theories can frequently be

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4 Of course what one really wants to understand is whether the masses of resonances in QCD-like theories satisfy a bound like (1.1). For fixed number of colors, QCD does not have an $N_f \to \infty$ limit, and this argument does not strictly apply.
solved exactly in the limit of large $N$. However, that $A(s, t, u)$ depends only on $s$ does not follow automatically from the large $N$ limit — it is a very restrictive condition.\footnote{It is easy to see that there are theories that have a well-defined $N \to \infty$ limit that are not of this form. For example, consider a scalar field theory with the pions contained in an $O(N)$ vector, and include an additional symmetric tensor. When the pions exchange the tensor field in the $t$ channel the diagram makes a contribution to the scattering amplitude proportional to $M^2/(t - M^2)$, a non-trivial function of $t$. This model is not soluble even in the limit $N \to \infty$, because the tensor counts the same way as a gluon in the $N_c \to \infty$ limit of QCD and all planar diagrams contribute.}

We will now show that in an $s$-channel $O(N)$ model the most general form of the function $A$ is \footnote{Ref \cite{6} obtained the same form for the scattering amplitude by considering the chiral Lagrangian for $s$-channel $O(N)$ models.}

$$A(s) = \frac{1}{f^2} F(s; \mu) - \frac{N}{32\pi^2} \log \frac{\mu^2}{s} ,$$

(3.1)

where $F(s; \mu)$ is an arbitrary function, analytic in a neighborhood of $s = 0$ and with

$$F(0) = 1 .$$

(3.2)

Here $\mu$ is the scale at which the theory is renormalized.\footnote{Ref \cite{6} obtained the same form for the scattering amplitude by considering the chiral Lagrangian for $s$-channel $O(N)$ models.}

In the limit $N \to \infty$, one has

$$a_{00}(s) = \frac{N}{32\pi} A(s) .$$

(3.3)

General principles of field theory tell us that $a_{00}$ is an analytic function of $s$, everywhere except at values of $s$ corresponding to physical states. Below the energy of the lowest massive particle, the only state for two pions to scatter into is a multipion state. Since we know that $f^2$ goes like $N$, we deduce from (2.4) that the vertex for $2 \to 4$ pions is suppressed by a factor of $1/N^2$ — and we can easily see that any diagram for $2 \to 4$ pions is suppressed by exactly the same factor. There are $N^2$ four-pion states, and therefore the loop computation of the four-pion contribution to $A$ has two factors of $1/N^2$ at the vertices, but only one factor of $N^2$ from the intermediate state. Therefore, the four-pion contribution to $A$ is a factor of $N$ smaller than the two-pion loop and is negligible. Similarly, all other intermediate $2n$ pion states are suppressed for $n \geq 2$. From this we see that $a_{00}$ satisfies elastic unitarity in the $N \to \infty$ limit, and so it lies on the Argand
circle for \( s \) real, positive, and less than the lightest massive multi-particle threshold \( M^2_m \). Therefore

\[
\text{Im} \frac{1}{a_{00}(s)} = -1 \quad s \text{ real, } 0 < s < M^2_m ,
\]

(3.4)

or

\[
\text{Im} \frac{1}{A(s)} = -\frac{N}{32\pi} s \text{ real, } 0 < s < M^2_m
\]

(3.5)

except possibly at a set of isolated points where \( a_{00} \) vanishes.

On the other hand, for \( s \) real and negative, there is no way to cut an \( s \)-channel diagram to yield an on-shell intermediate state. Thus \( A(s) \) and hence \( a_{00}(s) \) must be real. There should be no poles on the negative real axis.

Now consider the function

\[
G(s; \mu) = \frac{1}{A(s)} + \frac{N}{32\pi^2} \log \left( \frac{\mu^2}{-s} \right)
\]

(3.6)

In fig. 1 we show a region of the complex \( s \) plane. The amplitude \( a_{00} \) has a branch cut at zero, and possibly also at some point further out the \( x \) axis. These are marked with \( x \)'s. Also, \( a_{00} \) may have zeros at some points, which we indicate by dots in the figure, and this will yield simple poles in \( G \). As we have seen, for \( s \) real and negative, \( a_{00} \), and hence \( G \), is real. Therefore, \( G(s^*) = G(s)^* \) inside the circle excluding the positive real axis. We showed above that the imaginary part of the function \( 1/A \) went to \( N/(32\pi) \) whenever we approach the positive real axis from above, except at the points at which \( a_{00}(s) \) vanishes. Therefore the imaginary part of \( G \) goes to zero when we approach the positive real axis, whether from above or below. Thus, the imaginary part of \( G \) is continuous across the \( s > 0 \) axis, except at the isolated points at which \( a_{00} \) vanishes. By using the Schwartz reflection theorem on the line segments between the poles, we see that the function \( G \) is analytic everywhere inside the circle except at the points at which \( a_{00} \) vanishes.

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7 In the models we consider below, there are tachyonic poles at large negative real \( s \), but they are an indication of the sickness of the theories at high energy. In such a case, we focus attention on \(|s|\) much less than the tachyon mass.

8 We are considering the case where \( a_{00} \) has no zeros off the real axis. A simple extension of this argument will allow us to handle the more general case in which \( a_{00} \) has zeros off the axis as well.
We know that there is one point at which $a_{00}$ vanishes and $G$ has a pole, namely $s = 0$. However, the form of the pole is determined by the low-energy theorem given in (2.3). If we now define

$$F(s; \mu) = \frac{s}{f^2} G(s; \mu),$$

then (3.2) enforces the low-energy theorem.\(^9\)

4. Convergence of the Effective Theory

It is of great interest to understand the range of convergence of the effective theory. Convergence of the expansion in the effective lagrangian is intimately related to the existence of poles in the scattering amplitudes. This discussion will make precise the arguments of ref [1].

To begin, we need to understand the relationship between the form of the amplitude given in (3.1) above and the effective Lagrangian describing the pions. As mentioned in the previous section, the only diagrams contributing to $A(s)$ for the pion scattering process at low energy are those that iterate the two-pion loop in the $s$ channel. Diagrams with more complicated topologies, or those that exchange the two-pion loop in the $t$ or $u$ channels are subleading in $N$. Let us suppose that the tree-level vertex for $\pi^i \pi^j \rightarrow \pi^k \pi^\ell$ scattering in the bare low-energy effective Lagrangian for an $s$-channel $O(N)$ model is

$$iT_0(s)\delta^{ij}\delta^{k\ell} + iT_0(t)\delta^{ik}\delta^{j\ell} + iT_0(u)\delta^{i\ell}\delta^{jk}.$$  

To compute pion scattering we simply iterate this vertex with the pion loop. To do this we need to compute the loop integral

$$i\overline{B}(p^2; \mu) = \frac{1}{2} \mu^\ell \int \frac{d^d \ell}{(2\pi)^d} \frac{i}{\ell^2 + i\epsilon} \frac{i}{(\ell + p)^2 + i\epsilon}. \quad (4.1)$$

This integral is divergent, but the infinity can be absorbed into the tree level vertex $T_0(s)$, yielding $T(s; \mu)$. Therefore, we only need to consider the finite parts of this integral.\(^10\)

$$B(s; \mu) = \frac{1}{32\pi^2} \log \left( -\frac{s + i\epsilon}{\mu^2} \right) = -\frac{1}{32\pi^2} \log \left( \frac{\mu^2}{-s} \right). \quad (4.2)$$

\(^9\) In $O(N)$ models in which $A$ is not only a function of $s$, the argument given here may be applied to the isospin zero amplitudes of any angular momentum $\ell$. In that case, the function $1/G(s; \mu)$ defined by (3.6) has a zero of order at least $\ell$ at $s = 0$. We thank Sidney Coleman for pointing this out.

\(^10\) The renormalization prescription used here is slightly different from $\overline{MS}$. The subtraction is $2/(4 - d) - \gamma_E + \log 4\pi - 2$. 
In sum, this procedure yields

\[ A(s) = \frac{T(s; \mu)}{1 + NT(s; \mu)B(s; \mu)} . \]  

(4.3)

Therefore, the relationship between the function \( F \) defined in the previous section and the tree-level chiral Lagrangian used here is

\[ F(s; \mu) = \frac{s}{f^2 T(s; \mu)} . \]  

(4.4)

What then is the procedure used in chiral perturbation theory to calculate pion scattering? First off, the chiral Lagrangian itself is a power series in derivatives. This means that the tree-level vertex is an expansion:

\[ T(s; \mu) = \frac{s}{f^2} \sum_{i=0}^{\infty} t_i(\mu) \left( \frac{Ns}{f^2} \right)^i , \]  

(4.5)

where \( t_0 = 1 \) in order to satisfy the low-energy theorem. On the other hand, this expansion is not the only one used in a chiral Lagrangian calculation. For example, if one uses the four-derivative vertices in the chiral Lagrangian (the next terms beyond (2.3) given above), then one must include the one-loop diagrams as well, since the four-derivative vertices are one–loop counterterms. A computation of the amplitude using the chiral lagrangian including diagrams of up to \( \ell \) loops is presented in the form

\[ NA_\ell(s) = \sum_{0 < n < m \leq \ell + 1} c_{mn}(\mu) \left( \frac{Ns}{f^2} \right)^m \left( \frac{1}{32\pi^2} \log \left( \frac{\mu^2}{-s} \right) \right)^n . \]  

(4.6)

We recognize this as the expansion of (4.3) in powers of \( s \) and \( \log(\mu / -s) \), truncated in a particular way. Indeed, we may work out the coefficients \( c_{ij}(\mu) \) by simply expanding the \( T \) and \( B \) out of the denominator of (4.3):

\[ c_{mn}(\mu) = \sum_{i_0 \ldots i_{m-n-1}} \left( \frac{(n+1)!}{i_0! \ldots i_{m-n-1}!} \right)^{m-n-i} \prod_{j=0}^{i_j} t_j^i(\mu) \]  

(4.7)

where \( j \) and \( i_j \) are non-negative integers restricted by

\[ \sum_{j=0}^{m-n-1} i_j = n + 1 \quad \text{and} \quad \sum_{j=0}^{m-n-1} j i_j = m - n - 1 . \]  

(4.8)
We see now that there are two distinct but related questions. First, what is the radius of convergence of the expansion for $T(s; \mu)$, and second, for which values of $s$ does

$$\lim_{\ell \to \infty} A_\ell(s)$$

exist?

To address the first question we introduce the quantity $m_l$, defined by the property that $m_l^2$ is the smallest positive\textsuperscript{11} value of $s$ such that the real part of the amplitude vanishes. We say that $m_l$ is the mass of the lightest resonance, since it corresponds to the energy of the top of the lightest experimentally observed bump in pion scattering. However, $m_l^2$ is not the same as the real part of the value of $s$ at which the pole in $a_{00}$ appears. Below, we will denote this latter value of $s$ as $m_p e^{-i\theta_p}$. If the particle corresponding to $m_l$ is relatively narrow, then we expect that $m_l$ and $m_p$ are more or less the same.

Let us suppose that $m_l$ is lighter than any other non-analytic structure in the pion scattering amplitude, such as a multi-particle cut, other than that from the two-pion loop. For $s$ smaller than or equal to $m_l^2$, we know that there are no poles in $T$ off the real axis, since a pole represents a particle, and, by hypothesis, there are no non-pionic multi-particle states for the pole to decay into. Now we choose $\mu = m_l$ in (4.3), and we note that $B(m_l^2; m)$ is pure imaginary for any $m$. Thus, for $A(m_l^2)$ to be pure imaginary, we see that $T(s; m_l)$ must have a pole at $s = m_l^2$. This is in fact the closest pole of $T(s; m_l)$ to $s = 0$, and we see that $m_l^2$ is the radius of convergence of the expansion for $T(s; \mu)$ when $\mu = m_l$.

For other values of $\mu$, we may use the renormalization group to deduce $t_i(\mu)$ and hence the radius of convergence. However, since the coefficients of the chiral Lagrangian depend only logarithmically on $\mu$, we expect that for reasonable values of $\mu$, neither exponentially large nor small, the radius of convergence of $T(s; \mu)$ is always about $m_l^2$.

Now we must address the more difficult question of the convergence of the chiral expansion itself, i.e. the existence of the limit in (4.3). We first note that there is some region near $s = 0$ for it converges. The power series for $T(s; m_l)$ is absolutely convergent in the region $|s| < m_l^2$, furthermore $T(s; m_l)$ goes to zero as fast as $s$ near $s = 0$. Therefore for $s$ in the region $R$ of the complex $s$ plane,

$$R = \left\{ s : \sum_{n=0}^{\infty} \left| t_n(\mu) \left( \frac{Ns}{f^2} \right)^n \right| < 1 \right\},$$

\textsuperscript{11} We are once again assuming that any tachyons are far away from the region of physical interest.
the expansion in (4.6) is a sum of terms which would be absolutely convergent even without
the restriction placed by chiral perturbation theory on the order of their summation i.e. if
we removed the $n < m \leq \ell + 1$ under the summation sign. Therefore, chiral perturbation
theory must converge in $R$.

The chiral perturbation series (4.6) is really an expansion of a function defined on
multiple sheets. Thus, the question of convergence of (4.3) is considerably more difficult
than that of $T$. Furthermore, because of the logarithm terms in the chiral expansion, the
usual theorems for convergence of a series do not apply. The region $R$ specified in (4.10)
could be smaller than the real region of convergence of (4.3).

There are, however, a few educated guesses we can make about the region of existence
of (4.9). First of all, one sees that when $s = m_l^2$, the chiral expansion is not absolutely
convergent. In fact, it seems unlikely that it can be convergent at all when $|s| > m_l^2$,
because the $c_{n0}$ terms by themselves are growing in magnitude — $c_{n0}(\mu)$ is just $t_n(\mu)$.
Admittedly the logarithm part of the $c_{nm}$ terms can have an arbitrary phase, but it seems
very unlikely that the radius of convergence of (4.9) can be substantially larger than $m_l$
on any sheet.

Doubtless the region of convergence of (4.9) looks something like that shown in fig. 2.
The dot is the location of the lightest physical pole in pion scattering, at $s = m_p^2 e^{-i\theta_p}$.
This pole is, as usual, on the second sheet. The X shows the location of $m_l$, the pole in tree
level scattering. The form of the region of convergence is some sort of gentle spiral, furthest
from $s = 0$ along the negative real axis. As the logarithm gets a nonzero imaginary part,
it grows in magnitude for fixed $|s|$, and so as we change the argument of $s$ away from the
negative real axis the radius of convergence is reduced. In the direction of $\theta_p$, the radius
of convergence is exactly equal to $m_p^2$, a number which is always smaller than or about the
same as $m_l^2$.

5. The Natural Effective Theory Criterion in $s$-channel $O(N)$ Models

Weinberg established a useful definition for the “naturalness” of an effective theory.
As stated above, the chiral Lagrangian is an expansion in derivatives. Each term in the
expansion serves as a counterterm for the loop diagrams with vertices from lower order
terms. Weinberg’s NET criterion states that the renormalized coefficients in the higher
order chiral Lagrangian may not be too small. The argument is as follows. Consider a
coefficient of a term in $\mathcal{L}^{2n}$. In an $s$-channel $O(N)$ model, it is precisely $t_n(\mu)$. Since it is
a counterterm in a loop diagram, \( t_n(\mu) \) is a running parameter. Now suppose we change the scale of the cutoff of the effective theory (or \( \mu \) if we are using a dimensional regulator) by a factor of order 1. Then \( t_n(\mu) \) changes. If we pick a particular value of \( \mu \), it would be “unnatural” to find that \( t_n(\mu) \) is very much smaller than, for example, \( t_n(e\mu) \) (where \( e \) is the base of the logarithm). Note that in this usage the word “natural” does not have precisely the same meaning as usual.

The standard naive dimensional analysis [3] estimate of the minimum size of \( t_k(\mu) \) is \((16\pi^2)^{-k}\), because a chain of \( k \) bubbles has \( k \) factors of \( \mathcal{B} \). Thus one expects that the radius of convergence of \( T(s; \mu) \), shown in the last section to be \( m_r^2 \), should be no larger than about \( 16\pi^2 f^2/N \). We will see this in explicit examples below.

Interestingly, we may apply the NET criterion directly to the \( S \)-matrix, (3.1), and it takes on an especially simple form. As we saw above, the scale \( \mu \) in the logarithm may be interpreted as the renormalization point of the infinite pion loop integral. Therefore, we may ask what happens when we shift \( \mu \) by a factor of \( e \).

Since the function \( F \) is analytic near \( s = 0 \), we may write

\[
F(s; \mu) = \sum_{k=0}^{\infty} f_k(\mu) \left( \frac{N s}{f^2} \right)^k.
\]  

(5.1)

The factors of \( N \) in the numerator above are there because, as we have seen, \( f^2 \) counts like a factor of \( N \) relative to other dimensionful scales in the \( N \to \infty \) limit. If there are any terms in the expansion for \( F \) that are not enhanced by a factor of \( N^k \), one is not entitled to keep them as \( N \to \infty \).

We note an interesting feature of (5.1). Since \( A(s) \) is independent of \( \mu \), we see from (3.1) that only \( f_1 \) has any dependence on \( \mu \) — the others are all constant. Also, (3.2) implies that \( f_0 = 1 \). It is important to realize that the reason the \( f_k \) may be independent of \( \mu \) is that they are not coefficients of the chiral Lagrangian. They are derived from the \( t_n(\mu) \)'s by a resummation.

We may now apply the NET criterion. Pick a value of \( \mu^2 \), some generic value appropriate for a scattering process. Since the value of \( f_1(\mu) \) would have changed by \( 1/(16\pi^2) \) if we had picked \( \mu \to e\mu \), we conclude that \(|f_1(\mu)| \gtrsim 1/(16\pi^2)\).

Therefore, we see that the NET criterion for the effective theory, which is formulated in terms of the \( \beta \) functions of coefficients of the effective lagrangian, actually puts constraints directly on the \( S \)-matrix of the full theory. We will show below that this implies that new physics, such as a pole or a branch cut, must enter at a scale below or about \( 4\pi f/\sqrt{N} \).
6. Some Examples

In this section we consider three examples of \( s \)-channel \( O(N) \) models in order to show how the NET criterion implies that new physics always enters at a scale at or below \( 4\pi f/\sqrt{N} \).

6.1. The Linear \( \sigma \) Model

We will first consider the \( O(N) \) linear \( \sigma \) model in the limit of large \( N \), the simplest example of an \( s \)-channel \( O(N) \) model. This model may be solved exactly in the \( N \to \infty \) limit\(^8\)\(^9\). The result is that there is a single Higgs resonance in the \( \text{Re} \ s > 0 \) region on the second sheet. Unfortunately, the model also contains a tachyon, so it cannot be considered an example of an exactly soluble non-trivial field theory in four dimensions. We regard the tachyon as a sign of the fundamental sickness of the theory, and so we must restrict attention to scales well below the tachyon mass. If one demands that the mass of the Higgs resonance is well below the mass of the tachyon, then the Higgs mass is bounded and one finds that it is less than \( 4\pi f/\sqrt{N} \). Equivalently, if one defines the model with a momentum space cutoff rather than a dimensional regulator, then there is never a tachyon below the cutoff. In this case, if we demand that the running coupling constant be finite at the cutoff, then whenever the Higgs resonance is below the cutoff, it is always lighter than \( 4\pi f/\sqrt{N} \). For our purposes we regard the tachyon as being analogous to the momentum cutoff.

In ref \(^9\), the relationship between \( m_l \) and \( m_p \) was considered. The tree level vertex is

\[
T_0(s) = \frac{s}{f^2} \left( \frac{m^2}{m^2 - s} \right),
\]

and therefore

\[
F(s; \mu) = 1 - \left( \frac{f^2}{Nm^2(\mu)} \right) \left( \frac{Ns}{f^2} \right),
\]

where \( m(\mu) \) is a parameter with dimensions of mass. As expected, only the \( f_1 \) term depends on \( \mu \). The quantity \( m_l \) is the lightest mass such that the real part of \( a_{00} \) vanishes, so as before we choose \( \mu = m_l \) and solve for vanishing \( F \). Thus,

\[
0 = F(m_l^2; m_l) = 1 - \frac{f^2}{Nm^2(m_l)} \frac{Nm_l^2}{f^2},
\]

where

\[
T_0(s) = \frac{s}{f^2} \left( \frac{m^2}{m^2 - s} \right),
\]

(6.1)

and therefore

\[
F(s; \mu) = 1 - \left( \frac{f^2}{Nm^2(\mu)} \right) \left( \frac{Ns}{f^2} \right),
\]

(6.2)

where \( m(\mu) \) is a parameter with dimensions of mass. As expected, only the \( f_1 \) term depends on \( \mu \). The quantity \( m_l \) is the lightest mass such that the real part of \( a_{00} \) vanishes, so as before we choose \( \mu = m_l \) and solve for vanishing \( F \). Thus,

\[
0 = F(m_l^2; m_l) = 1 - \frac{f^2}{Nm^2(m_l)} \frac{Nm_l^2}{f^2},
\]

(6.3)
so we see that \( m^2(m_l) = m_t^2 \). We may find the relationship between \( m_p \) and \( m_l \) by setting \( \mu = m_l \) in (6.2) and looking for a value of \( s = m_p^2 e^{-i\theta_p} \) such that \( A(s) \) has a pole. One finds the following simultaneous equations [9]:

\[
\begin{align*}
\cos \theta_p &= \left( \frac{m_p^2}{m_t^2} \right) \left( \frac{N m_t^2}{16 \pi^2 f^2} \right) \log \frac{m_p^2}{m_t^2} \\
\sin \theta_p &= \frac{1}{2} \left( \frac{m_p^2}{m_t^2} \right) \left( \frac{N m_t^2}{16 \pi^2 f^2} \right) (\theta_p + \pi) 
\end{align*}
\]

(6.4)

Note that there is always a solution at \( \theta_p = -\pi \). This is the tachyon. More interesting is “Higgs remnant” pole below the real axis on the second sheet, at positive \( \theta_p \). Whenever \( m_t \) is bigger than about \( e m_l \), one finds that \( m_t^2 \lesssim 16 \pi^2 f^2 / N \) and \( \theta_p \lesssim \pi / 2 \). The ratio \( m_p^2 / m_t^2 \) is always less than 1, ranging from very close to 1 when the theory is weakly coupled (\( m_t \) is small), to about .4 when the model ceases to make sense.

It is instead possible to derive the bound on the resonance mass rather simply from the NET criterion: the NET criterion applied at \( m_l \) is

\[ |f_1(m_l)| = \left( \frac{f^2}{N m^2(m_l)} \right) \gtrsim \frac{1}{16 \pi^2} , \]

(6.5)

so

\[ \frac{16 \pi^2 f^2}{N} \gtrsim m^2(m_l) = m_t^2 . \]

(6.6)

We see that the NET criterion directly implies that there is an upper bound on the mass of the resonance.

Derived in this way, the bound on the resonance mass is at least in principle independent of the existence of the tachyon (or the cutoff). As we stated above, this same bound is always obeyed in the \( O(N) \) linear \( \sigma \) model, whether or not it satisfies the NET criterion, so long as the tachyon mass, the cutoff, is less than the mass of the resonance. This is not too surprising. This model is sick at energy scales above the tachyon mass because the effective potential becomes unbounded below. In this model we may understand the naturalness criterion directly as a statement that we stay far from the sick part of the theory. Still, it is instructive that the naturalness of the effective Lagrangian implies something immediately about the \( S \)-matrix, without consideration of the effective potential.
6.2. Arbitrary Numbers of Poles

One might ask whether it is possible to evade the bound of the previous example by having more than one pole. Perhaps the existence of many poles at a higher mass is adequate to satisfy the NET criterion. To address this question we now discuss the $s$-channel $O(N)$ model in which the tree level expression for $A$ has multiple resonances but no branch cuts, other than the two-pion cut\footnote{We may make such a model by taking including in the Lagrangian several $O(N)$ singlets, and adding terms that cause them to mix with the $N^{th}$ component of $\phi$.}. We will show that any such model must have at least one resonance whose mass is less than $4\pi f/\sqrt{N}$.

We take $F^{-1}$ to be a sum of propagators:

$$\frac{1}{F(s; \mu)} = \sum_i a_i(\mu) \frac{m_i^2(\mu)}{m_i^2(\mu) - s}, \quad (6.7)$$

where the $a_i(\mu)$ are the strengths of the various resonances into $A$. Since the model has no tachyons or ghosts at tree level, we may assume that at any reasonable value of $\mu$, for which the theory is well defined, all the $a_i$ and $m_i^2$ are positive. Note that (3.2) implies that $\sum a_i(\mu) = 1$.

In this case, $F$ is no longer a simple linear function of $s$. In fact it is a rational function, and the power series expansion no longer terminates. We may compute $f_1$:

$$f_1(\mu) = \frac{f^2}{N} \frac{d}{ds} F(s; \mu) \bigg|_{s=0} = -\frac{f^2}{N} F^2(s; \mu) \frac{d}{ds} F^{-1}(s; \mu) \bigg|_{s=0} = -\sum_i a_i(\mu) \frac{f^2}{Nm_i^2(\mu)}. \quad (6.8)$$

We now show that the mass of the lightest resonance must be less than $4\pi f/\sqrt{N}$. Again, note that $1/F(s; m_\ell)$ must have a pole at $s = m_\ell$. From the expression (3.7), we conclude that there must be some $i$ such that $m_i(m_\ell) = m_\ell$. Without loss of generality, we may assume that it is $i = 1$. We now see that $m_1^2(m_\ell)$ is the smallest of the $m_i^2(m_\ell)$. Suppose that $m_2^2(m_\ell)$ were lower than $m_1^2(m_\ell) = m_\ell$. Start from from $s = m_\ell^2$ and reduce $s$. Then as we lower $s$ to zero, there must either be a point where $m_2^2(\sqrt{s}) = s$, or $m_2^2(\sqrt{s}) = -s$. If the first condition holds, then we see that the function $F(s; \sqrt{s})$ has a zero at a smaller value of $s$ than $m_\ell^2$, and thus the amplitude was pure imaginary at a lower
mass than $m_\ell$. The second condition implies that $F(-s, \sqrt{s})$ has a zero, and so there is a tachyon at a mass scale lower than $m_\ell$. In this case we reject the model.

The NET criterion at $\mu = m_\ell$ states

$$\frac{1}{16\pi^2} \lesssim |f_1(m_\ell)| = \sum_i a_i(m_\ell) \frac{f^2}{Nm_i^2(m_\ell)} \leq \sum_i a_i(m_\ell) \frac{f^2}{Nm_i^2(m_\ell)} = \frac{f^2}{Nm_i^2(m_\ell)} ,$$

so

$$m_i^2 = m_i^2(m_\ell) \lesssim \frac{16\pi^2 f^2}{N} .$$

Therefore the inclusion of many poles does not permit us to evade the bound on the mass of the lightest of them.

6.3. Cuts

Having concluded that in $s$-channel $O(N)$ models with arbitrary numbers of poles naturalness implies that there is always a light resonance, we next turn to a model with a two-particle branch cut. In this model too there is an upper bound on the mass of the new physics.

The model we consider has an $O(N) \times O(L)$ symmetry, which breaks to $O(N-1) \times O(L)$. The particles $\phi$ transform as a vector under the $O(N)$ symmetry, and $\psi$ is a vector under $O(L)$. The Lagrangian is

$$\mathcal{L} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi + \frac{1}{2} \partial^\mu \psi \partial_\mu \psi - \frac{\lambda_a}{N} (\phi^2 - v^2)^2 - \frac{\lambda_b}{N} (\psi^2)^2 - \frac{\lambda_c}{N} (\phi^2 - v^2)(\psi^2 - v^2) - \frac{1}{2} M_\psi^2 \psi^2 .$$

We will solve for the elastic two-to-two particle scattering amplitudes in the $N \to \infty$ and $L \to \infty$ limit, with $L/N$ held fixed.

So long as $v^2 > 0$ and $M_\psi^2 > 0$, one of the $\phi$'s will get a vacuum expectation value (VEV), while $\langle \psi \rangle = 0$. We will take $\langle \phi_N \rangle = f$, and $\langle \phi_i \rangle = 0$. The broken symmetry generators are those $O(N)$ generators $T$ such that $T^{iN} \neq 0$. We will refer to the first $N-1$ $\phi$'s as pions, and the $N^{th}$ as $\sigma$.

The conditions on the coupling constants such that the tree-level potential is bounded below are

$$\lambda_a + \lambda_b > 0$$

$$\lambda_a \lambda_b - \lambda_c^2 > 0$$

from which it follows that both $\lambda_a$ and $\lambda_b$ are positive. We will assume that these conditions hold as well for the renormalized couplings $\lambda(\mu)$, at any renormalization point $\mu$ under
consideration. Unfortunately, this model suffers from the same problem as the linear $\sigma$ model: at sufficiently large values of $\phi$ the potential becomes unbounded below.

First consider the diagrams that renormalize the VEV of $\phi_N$. At tree level one has the relationship $\langle \phi_N \rangle = v$, but the “tadpole” diagrams with an external $\sigma$ will shift the VEV, $f$, relative to the parameter in the Lagrangian, $v$. Accordingly, we may eliminate any diagram with a tadpole sub-diagram, so long as we consistently replace $v$ with $f$.

Next consider the diagrams renormalizing the $\phi$ or $\psi$ propagators. At leading $N$ these are “cactus” diagrams. Their effect is to shift the $\psi$ mass. There is no $\psi$ or $\pi$ wavefunction renormalization. We eliminate any diagram with a cactus sub-diagram, and replace $M_\psi$, the mass parameter in the Lagrangian, with $m_\psi$, the physical $\psi$ mass.

We may now derive the $\pi^i \pi^i \rightarrow \pi^j \pi^j$, $i \neq j$ scattering amplitude as follows. An arbitrary diagram for this process is a simple chain of $\sigma$ propagators, $\psi$ loops, and $\pi$ loops (fig. 3). Consider first the sum of all chains of any number of $\psi$ loops attached together, with no intervening $\sigma$’s or $\pi$’s. Having constructed this quantity, we may sum arbitrary alternations of it with $\sigma$’s. If we attach such chains to external $\pi\pi$ states, and include the direct $\pi\pi\pi\pi$ vertex, then we have at this stage all $\pi^i \pi^i \rightarrow \pi^j \pi^j$ diagrams in which the scattering proceeds without internal pion loops. Therefore, to get the entire $\pi^i \pi^i \rightarrow \pi^j \pi^j$ process, we simply iterate this scattering with the pion loop.

To do this we need the loop integral for the massive $\psi$’s as well as for the pions, and thus we introduce the massive loop function$^{13}$

$$J(s; \mu) = \frac{1}{32\pi^2} \left( 2 + \int_0^1 dx \log \left( \frac{m_\psi^2 - x(1-x)s - i\epsilon}{\mu^2} \right) \right). \quad (6.13)$$

With this definition we find that

$$\frac{1}{Na_{\pi^i \pi^i \rightarrow \pi^j \pi^j}(s)} = \frac{f^2}{N s} \left[ 1 - \frac{Ns}{f^2} \frac{1 - \lambda_b \frac{J(s)}{f^2}}{\lambda_a + (\lambda_c^2 - \lambda_a \lambda_b) \frac{J(s)}{f^2}} \right] + B(s). \quad (6.14)$$

(Here we have suppressed the $\mu$ dependence of the quantities $\lambda_a$, $\lambda_b$, $\lambda_c$, $B(s)$, and $J(s)$. The VEV $f$ and the function $A$ are physical quantities, independent of $\mu$.) The quantity in square brackets may be defined as $F(s; \mu)$, and so we see that this amplitude satisfies elastic unitarity so long as $J$ is real, which it is for $s < 4m_\psi^2$.

At energies above twice the $\psi$ mass, the $a_{\pi\pi \rightarrow \pi\pi}$ no longer satisfies elastic unitarity, because we may now produce a $\psi\psi$ final state. Accordingly, the function $J$, and hence

$^{13}$ The subtraction here is the same as for the function $B$ above.
\( A \) has a branch cut at \( s = 4m^2_\psi \). To get \( a_{\pi^+\pi^-\rightarrow\psi^+\psi^-} \), we use following procedure. A diagram for this process is an arbitrary string of pion loops, \( \sigma \) propagators, and \( \psi \) loops. First consider diagrams without \( \pi \) loops. These may be summed by a procedure exactly analogous to that described above. Now if the diagram has a pion loop, we divide it in half at its last \( \pi \) loop. Such a diagram is therefore the product of a diagram without a pion loop, a factor of \( B \), and an arbitrary \( \pi \pi \rightarrow \pi \pi \) scattering. We find in sum:

\[
\frac{a_{\pi^+\pi^-\rightarrow\pi^+\pi^-}(s)}{a_{\pi^+\pi^-\rightarrow\psi^+\psi^-}(s)} = \frac{\lambda_a}{\lambda_c} + \left( \frac{\lambda_c - \lambda_a \lambda_b}{\lambda_c} \right) \frac{L}{N} J(s) .
\]

(6.15)

Lastly, the calculation for \( a_{\psi^+\psi^-\rightarrow\psi^+\psi^-} \) is precisely analogous to that of \( a_{\pi^+\pi^-\rightarrow\pi^+\pi^-} \):

\[
\frac{1}{Na_{\psi^+\psi^-\rightarrow\psi^+\psi^-}(s)} = \frac{\lambda_a \left( \frac{f^2}{N} + sB(s) \right) - s}{\lambda_b s - (\lambda_a \lambda_b - \lambda_c^2) \left( \frac{f^2}{N} + sB(s) \right)} + \frac{L}{N} J(s) .
\]

(6.16)

We may now derive three renormalization group equations for the three coupling constants. First, note that since all amplitudes are independent of \( \mu \), the quantity on the right hand side of (6.15) must be independent of \( \mu \). Since

\[
\frac{\partial J(s; \mu)}{\partial \log \mu} = \frac{\partial B(s; \mu)}{\partial \log \mu} = -\frac{1}{16\pi^2}
\]

is independent of \( s \), we conclude that

\[
\frac{\partial}{\partial \log \mu} \left( \frac{\lambda_a \lambda_b}{\lambda_c} - \lambda_c \right) = 0 .
\]

(6.18)

We refer to this particular invariant combination of coupling constants as \( \gamma \). The invariance of the entire right hand side of (6.15) implies that

\[
\frac{\partial}{\partial \log \mu} \lambda_a \lambda_c - \gamma \frac{L}{N} \frac{\partial}{\partial \log \mu} J(s) = 0 .
\]

(6.19)

Lastly, the invariance of the \( \pi \pi \rightarrow \pi \pi \) scattering amplitude in (6.14) implies

\[
\frac{\partial}{\partial \log \mu} \lambda_b \lambda_c - \gamma \frac{\partial}{\partial \log \mu} B(s) = 0 .
\]

(6.20)

We may rewrite these equations as

\[
\frac{\partial}{\partial \log \mu} \left( \frac{1}{\lambda_a} \right) = -\frac{1}{16\pi^2} \left( 1 + \frac{L}{N} \frac{\lambda_c^2}{\lambda_a^2} \right)
\]

\[
\frac{\partial}{\partial \log \mu} \left( \frac{1}{\lambda_b} \right) = -\frac{1}{16\pi^2} \left( \frac{L}{N} + \frac{\lambda_a^2}{\lambda_b^2} \right)
\]

\[
\frac{\partial}{\partial \log \mu} \left( \frac{1}{\lambda_c} \right) = -\frac{1}{16\pi^2} \left( \frac{N}{\lambda_c} \lambda_b + \lambda_a \frac{\lambda_b}{\lambda_c} \right).
\]

(6.21)
The solutions to these equations are

\[
\begin{align*}
\frac{\lambda_a}{\lambda_c} &= \frac{\gamma}{16\pi^2} \frac{L}{N} \log \frac{\Lambda_a}{\mu}, \\
\frac{\lambda_b}{\lambda_c} &= \frac{\gamma}{16\pi^2} \log \frac{\Lambda_b}{\mu}, \\
\frac{1}{\lambda_c} &= \frac{\gamma}{(16\pi^2)^2} \frac{L}{N} \log \frac{\Lambda_a}{\mu} \log \frac{\Lambda_b}{\mu} - \frac{1}{\gamma}.
\end{align*}
\]  

(6.22)

Therefore, this model can be parametrized by two dimensionful quantities, \( \Lambda_a \) and \( \Lambda_b \), and one dimensionless combination of couplings, \( \gamma \).

The content of the NET criterion is

\[
\frac{1}{16\pi^2} \leq |f_1(\mu)| = \frac{1 - \lambda_b(\mu) \frac{L}{N} J(0; \mu)}{\lambda_a(\mu) + (\lambda_c(\mu))^2 - \lambda_a(\mu) \lambda_b(\mu)) \frac{L}{N} J(0; \mu)}
\]  

(6.23)

The content the NET criterion is therefore that this equation must hold for any scale \( \mu \) in which the theory is used to calculate scattering. In particular, we expect that it holds both at \( \mu = m_l \) and at \( \mu = m_\psi \). We may plug in the solutions to the renormalization group equations and the definition of \( J \) to find

\[
\Gamma \leq \left( \log \frac{\Lambda_a}{m_\psi} - 1 \right) \left( \log \frac{\Lambda_b}{\mu} - 1 \right)
\]  

(6.24)

where \( \Gamma = ((16\pi^2)^2 N) / (\gamma^2 L) \). For this to hold we must either have \( \log \Lambda_a / m_\psi \) and \( \log \Lambda_b / \mu \) both greater than 1 or both less than zero. The latter possibility is excluded, because then \( \lambda_a \) and \( \lambda_b \) themselves are negative, and the potential is unbounded below at a small scale, \( \mu \) or \( m_\psi \).

Now we will solve for \( m_l \), using \( F(m_l^2; m_l) = 0 \). Using the form of \( F \) given in (6.14), we find that the equation for \( m_l \) is

\[
\frac{Nm_l^2}{f^2} = \lambda_a(m_l) + \frac{\lambda_c^2(m_l) J(m_l^2; m_l)}{1 - \lambda_b(m_l) J(m_l^2; m_l)}.
\]  

(6.25)

We now want to know what the natural scale for \( m_l \) is.

We now plug the solutions of the renormalization group equations into (6.25):

\[
\frac{Nm_l^2}{f^2} = 16\pi^2 \frac{\log \frac{\Lambda_a}{m_l} - 16\pi^2 J(m_l^2; m_l)}{\log \frac{\Lambda_a}{m_l} \log \frac{\Lambda_b}{m_l} - \Gamma - \log \frac{\Lambda_b}{m_l} 16\pi^2 J(m_l^2; m_l)}.
\]  

(6.26)
We write this as
\[
\frac{m_l^2}{(16\pi^2 f^2/N)} = \left( \log \frac{\Lambda_b}{m_l} - \frac{\Gamma}{\log \frac{\Lambda_a}{m_l} - 16\pi^2 J(m_l^2; m_l)} \right)^{-1}.
\] (6.27)

We now show that this equation has a root in the range \(0 < m_l^2 < 16\pi^2 f^2/N\). The left hand side grows linearly with \(m_l^2\) and reaches one when \(m_l^2 = 16\pi^2 f^2/N\). We will show below that the right hand side has infinite derivative at \(m_l^2 = 0\), and it is bounded by one for \(m_l < 2m_\psi\). Therefore, when \(m_l^2\) is just above zero the right hand side is bigger than the left hand side, but it must fall below the left hand side before \(m_l^2 = 16\pi^2 f^2/N\), so long as \((2m_\psi)^2 > 16\pi^2 f^2/N\).

All that remains is to show the two assertions of the previous paragraph. First, we note that we may write
\[
16\pi^2 J(m_l^2, m_l) = \log \left( \frac{m_\psi}{m_l} \right) + g \left( \frac{m_l^2}{4m_\psi^2} \right). \tag{6.28}
\]
Here \(g(x)\) is a real analytic function for \(x < 1\). The function \(g\) is monotonically decreasing as \(x\) increases: \(g(0) = 1\) and \(g(1) = 0\). Now the right hand side of (6.27) is
\[
\left( \log \frac{\Lambda_b}{m_l} - \frac{\Gamma}{\log \frac{\Lambda_a}{m_l} - 16\pi^2 J(m_l^2; m_l)} \right)^{-1} = \left( \log \frac{\Lambda_b}{m_l} - \frac{\Gamma}{\log \frac{\Lambda_a}{m_l} - g \left( \frac{m_l^2}{4m_\psi^2} \right)} \right)^{-1}.
\] (6.29)

Since \(g\) is analytic as \(m_l \to 0\), the whole expression is well approximated by \((\log \Lambda_b/m_l)^{-1}\), and the derivative blows up.

To show the second assertion, we note that
\[
\left( \log \frac{\Lambda_b}{m_l} - \frac{\Gamma}{\log \frac{\Lambda_a}{m_l} - g \left( \frac{m_l^2}{4m_\psi^2} \right)} \right)^{-1} \leq \left( \log \frac{\Lambda_b}{m_l} - \frac{\Gamma}{\log \frac{\Lambda_a}{m_\psi} - 1} \right)^{-1}.
\] (6.30)

so long as \(m_l < 2m_\psi\). The right hand side is less than one if
\[
\left( \log \frac{\Lambda_a}{m_\psi} - 1 \right) \left( \log \frac{\Lambda_a}{m_l} - 1 \right) > \Gamma.
\] (6.31)
This must hold, because it is just the NET criterion with $\mu = m_l$.

In this section we have considered three explicit examples of $s$-channel $O(N)$ models. A general $s$-channel $O(N)$ model is never too much more complicated - one can simultaneously add multiple poles and branch cuts. In the models of this section either there is a resonance at a mass $m_l$ below $4\pi f/\sqrt{N}$, or there is a branch cut before that scale. Some new physics must always enter — the addition of multiple poles or of a branch cut did not permit us to evade the bound.

7. Conclusion

In this paper we have considered a type of strongly interacting field theory. We have seen that the tree-level chiral lagrangian for the pions converges only out to the mass of the lightest resonance, and that the chiral loop computations converge to a scale of this order or smaller.

In addition, we have shown that the natural effective theory criterion implies something about the $S$-matrix of the full theory. In the cases considered here, we have shown that there must always be new physics, a resonance or a cut, at or below $4\pi f/\sqrt{N}$. This bound holds independent of the precise form of the full theory or of the detailed consideration of the effective potential.

The theories considered here are really only toy models. Because of their simplicity, it was directly possible to interpret the natural effective theory criterion as the condition that keeps us far from the sick part of the theory, as indicated either by an explicit momentum space cutoff or the tachyon. Accordingly, in the cases considered here, it was not possible to construct any theory with an unnatural effective theory. The open question is, therefore, whether the NET criterion is a general property of field theory, or if there are models that somehow evade it and have resonances that are heavy compared to the naive scale.

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References

[1] R. S. Chivukula, M. Dugan, and M. Golden, Phys. Rev. D 47 (1993) 47; R. S. Chivukula, M. Dugan, and M. Golden, Phys. Lett. B292 (1992) 435.

[2] M. Soldate and R. Sundrum, Nucl. Phys. B340 (1990) 1

[3] S. Weinberg, Physica 96A (1979) 327; see also H. Georgi and A. Manohar, Nucl. Phys. B234 (1984) 189.

[4] S. Coleman, J. Wess, and B. Zumino, Phys. Rev. 177 (1969) 2239; C. Callan, S. Coleman, J. Wess, and B. Zumino, ibid 177 (1969) 2246.

[5] T. Appelquist and J. Terning, Phys. Rev. D 47 (1993) 3075.

[6] A. Dobado and J. R. Pelaez, Phys. Lett. B286 (1992) 136

[7] see, for example, W. Rudin, Real and Complex Analysis, McGraw-Hill, New York 1974, p260

[8] S. Coleman, R. Jackiw, and H. D. Politzer, Phys. Rev. D 10 (1974) 2491.

[9] M. B. Einhorn, Nucl. Phys. B246 (1984) 75; R. Casalbuoni, D. Dominici, and R. Gatto, Phys. Lett. 147B (1984) 419.
Figure Captions

Fig. 1. Analyticity of $a_{00}(s)$. The dots indicate zeros of $a_{00}$, which correspond to poles in $G$. The X’s indicate the branch points. We show that $G$ is analytic in the circle except for simple poles at the dots.

Fig. 2. The region of convergence of (4.9) for complex $s$. The dot shows the location of the lightest physical pole (on the second sheet), and the X shows the location of $m_L$. The dashed line is the region of convergence on the second sheet.

Fig. 3. A typical diagram contributing to $\pi\pi \rightarrow \pi\pi$ scattering.
