A sharp recovery condition for sparse signals with partial support information via orthogonal matching pursuit

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Abstract

This paper considers the exact recovery of \( k \)-sparse signals in the noiseless setting and support recovery in the noisy case when some prior information on the support of the signals is available. This prior support consists of two parts. One part is a subset of the true support and another part is outside of the true support. For \( k \)-sparse signals \( x \) with the prior support which is composed of \( g \) true indices and \( b \) wrong indices, we show that if the restricted isometry constant (RIC) \( \delta_{k+b+1} \) of the sensing matrix \( A \) satisfies

\[
\delta_{k+b+1} < \frac{1}{\sqrt{k-g+1}},
\]

then orthogonal matching pursuit (OMP) algorithm can perfectly recover the signals \( x \) from \( y = Ax \) in \( k-g \) iterations. Moreover, we show the above sufficient condition on the RIC is sharp. In the noisy case, we achieve the exact recovery of the remainder support (the part of the true support outside of the prior support) for the \( k \)-sparse signals \( x \) from \( y = Ax + v \) under appropriate conditions. For the remainder support recovery, we also obtain a necessary condition based on the minimum magnitude of partial nonzero elements of the signals \( x \).

Keywords: Orthogonal matching pursuit, Partial support information, Restricted isometry constant, Sensing matrix.

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1 Introduction

Compressive sensing has been a very active area of recent research in signal processing, applied mathematics and statistics \cite{5,8,14,19} and \cite{22}. A central aim of compressive sensing is to reconstruct sparse signals from inaccurate and incomplete measurements. In compressive sensing, one considers the following model:

\[ y = Ax + v, \]  

where \( y \in \mathbb{R}^m \) is a measurement vector, the matrix \( A \in \mathbb{R}^{m \times n} (m \ll n) \) is a known sensing matrix, the vector \( x \in \mathbb{R}^n \) is an \( k \)-sparse signal and \( v \in \mathbb{R}^m \) is a vector of measurement errors. In particular, \( v = 0 \) in the noiseless setting. Denote the support of the vector \( x \) by \( T = \text{supp}(x) = \{i|x_i \neq 0\} \) and the size of its support with \(|T| = |\text{supp}(x)|\). If \(|\text{supp}(x)| \leq k\), \( x \) is called \( k \)-sparse. The goal is to recover the unknown \( k \)-sparse signal \( x \) from \( y \) and \( A \) in the model (1.1) using fast and efficient algorithms.

In order to analyze the \( \ell_1 \)-minimization, Candès and Tao \cite{6} introduced a commonly used framework: the restricted isometry property (RIP).

**Definition 1.1.** A matrix \( A \) satisfies the RIP of order \( k \) if there exists a constant \( \delta_k \in [0,1) \) such that

\[ (1 - \delta_k)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta_k)\|x\|_2^2 \]  

holds for all \( k \)-sparse signals \( x \). And the smallest constant \( \delta_k \) is called the restricted isometry constant (RIC).

In this paper, we focus on a kind of sparse signals which have some prior support information (possibly erroneous). The recovery of such sparse signals with a strong dependence on their prior supports has been introduced in several contributions and possess practical and analytical interests in many setups \cite{12,13,17} and \cite{23}. For example, this type of signals occurs in video compression or dynamic magnetic resonance imaging where the supports of the sought vectors commonly evolve slowly with time.

Compressed sensing has previously been studied under different conditions for recovering sparse signal in the presence of prior support information. To make good use of prior support information of the signals, the following weighted \( \ell_1 \) minimization has been introduced

\[ \min_{x \in \mathbb{R}^n} \|x\|_{1,w} \text{ subject to } \|y - Ax\|_2 \leq \epsilon, \]  

(1.3)
where $w \in [0, 1]^n$ and $\|x\|_{1,w} = \sum_{i=1}^{n} w_i |x_i|$. The main idea of the weighted $\ell_1$ minimization (1.3) is to choose appropriately the weight vector $w$ such that in this weighted objective function, the entries of $x$ which are expected to be large are penalized less. In particular, the weighted $\ell_1$ minimization (1.3) reduces to the standard $\ell_1$ minimization by taking $w = 1$. The weighted $\ell_1$ minimization method (1.3) has now been well studied and achieved a complete theoretical system under various models on the weight vector $w$. For example, in the literature [9, 15, 13], the authors have previously studied the recovery of signals with prior support information and obtained different conditions to guarantee recovery of these signals via the weighted $\ell_1$ minimization which only applies a single weight. Chen and Li [3] show that a sharp sufficient recovery condition based on a high order RIP guarantees stable and robust recovery of signals via the weighted $\ell_1$ minimization (1.3) in bounded $\ell_2$ and Dantzig selector noise settings. And the authors not only point out that the sufficient recovery condition is weaker than that of the standard $\ell_1$ minimization method but also point out that the weighted $\ell_1$ minimization method gives better upper bounds on the reconstruction error, as the accuracy of prior support estimate is at least 50%. Lastly, Needell et al. [17] and Chen et al. [4] consider the sparse signal recovery with disjoint prior supports via the weighted $\ell_1$ minimization method (1.3) using arbitrarily many distinct weights and obtain the recovery condition and associated recovery guarantees.

It is well known that the standard OMP algorithm as a greedy algorithm is one of the most effective algorithms in sparse signal recovery because of its implementation simplicity and competitive recovery performance. Modifications of the standard OMP algorithm have also been studied for recovering the sparse signals under a partially known support. As we know, recovering sparse signals with some prior support information by using OMP algorithm and its modifications is much fewer than by using weighted $\ell_1$ minimization. Tropp and Gilbert [21] first demonstrate theoretically and empirically sparse signal recovery from prior information via a modified OMP algorithm. In [20], for the noiseless setting the authors derive a simple recovery guarantee based on the mutual coherence of the matrix $A$ and the number of true and wrong indices in prior support $T_0$ for the sparse signal recovery via the OMP$_{T_0}$ algorithm in Table 1. Karahanoglu and Erdogan [12] show that

$$\delta_{k+b+1} < \frac{1}{\sqrt{k-g+1}}$$

is sufficient to ensure the sparse signal recovery from $y = Ax$ via the OMP$_{T_0}$, where
\( T = \text{supp}(x) \) with \(|T| = k\), \(|T \cap T_0| = g\) and \(|T^c \cap T_0| = b\). However, the above condition on RIP is not optimal. On the other hand, there is no result considering support \( T \setminus T_0 \) recovery via the OMP_{T_0} algorithm in the noisy case.

In this paper, we consider optimal sufficient conditions and some necessary condition of the recovery of any \( k \)-sparse signal by the OMP_{T_0} algorithm in the noiseless and noisy cases. We consider any \( k \)-sparse signal \( x \) with the prior support \( T_0 \), where the support \( T = \text{supp}(x) \), \(|T \cap T_0| = g\) is the number of true indices and \(|T^c \cap T_0| = b\) is the number of wrong indices. For the noiseless case, it is shown that

\[
\delta_{k+b+1} < \frac{1}{\sqrt{k-g+1}}
\]

ensures the OMP_{T_0} algorithm exactly recover the \( k \)-sparse signal \( x \) in \( k-g \) iterations. Moreover, we point out that our condition is sharp in the following sense: there exist a sensing matrix \( A \) with \( \delta_{k+b+1} = \frac{1}{\sqrt{k-g+1}} \), a \( k \)-sparse signal \( \hat{x} \) and the prior support \( T_0 \) satisfying \(|\text{supp}(\hat{x}) \cap T_0| = g < k\) and \(|(\text{supp}(\hat{x}))^c \cap T_0| = b\) such that the OMP_{T_0} algorithm may fail to recover the \( k \)-sparse signal \( \hat{x} \) in \( k-g \) iterations. For the noisy case, we show that if the sensing matrix \( A \) satisfies \( \delta_{k+b+1} < \frac{1}{\sqrt{k-g+1}} \) and \( \|v\|_2 \leq \varepsilon \), then the OMP_{T_0} algorithm exactly recovers the remainder support \( T \setminus T_0 \) and obtains the estimated signal \( \hat{x} \) of the \( k \)-sparse signal \( x \) in \( k-g \) iterations provided that

\[
\min_{i \in T \setminus T_0} |x_i| > \max \left\{ \frac{\sqrt{2(1 + \delta_{k+b+1})\varepsilon}}{1 - \sqrt{k-g+1} \delta_{k+b+1}}, \frac{2\varepsilon}{\sqrt{1 - \delta_{k+b+1}}} \right\}.
\]

Further, we obtain the upper bounds of \( \|x - \hat{x}\|_2 \) and \( \max_{i \in T \setminus T_0} |\hat{x}_i| \), and the lower bound of \( \min_{i \in T \setminus T_0} |\hat{x}_i| \). At last, we obtain a necessary condition for exactly recovering the remainder support \( T \setminus T_0 \) of the \( k \)-sparse signal \( x \) based on the minimum magnitude of elements of \( x_{T \setminus T_0} \). That is, if the sensing matrix \( A \) satisfies the RIP of order \( k+b+1 \) with \( 0 \leq \delta_{k+b+1} < 1 \) and the OMP_{T_0} algorithm exactly recovers the remainder support \( T \setminus T_0 \), then

\[
\min_{i \in T \setminus T_0} |x_i| > \frac{\sqrt{1 - \delta_{k+b+1} \varepsilon}}{1 - \sqrt{k-g+1} \delta_{k+b+1}}.
\]

The rest of the paper is organized as follows. In Section 2, we give some notations that will be used throughout this paper, some significant lemmas and the proofs of them. The main results on the exact recovery of \( k \)-sparse signals in the noiseless case and their proofs are given in Section 3. Section 4 considers the exact recovery of the remainder support \( T \setminus T_0 \) in the noisy setting. In Section 5, we discuss the validity of our sufficient conditions comparing with previous results.
2 Notations and preliminaries

Let us now define basic notations. Boldface lowercase letters and boldface uppercase letters respectively denote column vectors and matrices in the real field \( \mathbb{R} \). \( \langle \cdot, \cdot \rangle \) refers to the inner product between vectors and \( \| \cdot \|_p \) with \( p = 1, 2 \) stands for \( \ell_p \) norm. \( [n] \) denotes the index set \( \{1, 2, \ldots, n\} \). Let \( \Gamma \subseteq [n] \) be an index set and \( \Gamma^c \subseteq [n] \) be the complementary set of \( \Gamma \). \( x_{\Gamma} \in \mathbb{R}^{|\Gamma|} \) denotes the vector composed of components of \( x \in \mathbb{R}^n \) indexed by \( i \in \Gamma \). Define \( \tilde{x}_\Gamma \in \mathbb{R}^n \) by

\[
(\tilde{x}_\Gamma)_i = \begin{cases} 
    x_i, & i \in \Gamma; \\
    0, & i \in \Gamma^c,
\end{cases}
\]

where \( i \in [n] \). Let the matrix transpose of the matrix \( A \) be \( A' \). \( A_i \) with \( i \in [n] \) denotes the \( i \)-th column of \( A \). Denote by \( A_{\Gamma} \) a submatrix of \( A \) corresponding to \( \Gamma \) which consists of all columns of \( A \) with index \( i \in \Gamma \). Let \( e_i \in \mathbb{R}^n \) be the \( i \)-th coordinate unit vector.

Let \( A_{\Gamma}^\dagger \) denote the pseudo-inverse of \( A_{\Gamma} \). When \( A_{\Gamma} \) is full column rank (\( |\Gamma| \leq m \)), then \( A_{\Gamma}^\dagger = (A_{\Gamma}'A_{\Gamma})^{-1}A_{\Gamma}' \). Moreover, \( P_T = A_T A_T^\dagger \) and \( P_T^\perp = I - P_T \) represent two orthogonal projection operators, where \( P_T \) projects a given vector orthogonally onto the spanned space by all columns of \( A_{\Gamma} \), \( P_T^\perp \) projects onto its orthogonal complement and \( I \) is identity mapping.

The frame of the OMP\(_T_0\) algorithm is formally listed in Table 1.

| Table 1: The OMP\(_T_0\) algorithm |
|-------------------------------------|
| **Input** | measurements \( y \in \mathbb{R}^m \), sensing matrix \( A \in \mathbb{R}^{m \times n} \), sparse level \( k \), the number of correct indices \( g \), prior support \( T_0 \). |
| **Initialize** | iteration count \( t = 0 \), estimated support set \( \Lambda_0 = T_0 \), residual vector \( r^{(0)} = y - P_{\Lambda_0} y \). |
| **While** | stopping criterion is not met \( t = t + 1 \). |
| (Identification step) | \( j_t = \arg \max_i |\langle r^{(t-1)}, Ae_i \rangle| \). |
| (Augmentation step) | \( \Lambda_t = \Lambda_{t-1} \cup \{j_t\} \). |
| (Estimation step) | \( x^{(t)} = \min_u \| y - A_{\Lambda_t} u \|_2 \). |
| (Residual update step) | \( r^{(t)} = y - A_{\Lambda_t} x^{(t)} \). |
| **End** |
| **Output** | the estimated signal \( \hat{x}_{\Lambda_t} = x^{(t)} \), \( \hat{x}_{\Lambda_t^c} = 0 \). |

It is clear that the OMP\(_T_0\) algorithm reduces to the standard OMP algorithm as \( T_0 = \emptyset \). In
Herzet et al. give a rigorous definition of “success” for the OMP\(_{T_0}\) algorithm, which matches the classical “\(k\)-step” analysis of the standard OMP algorithm.

**Definition 2.1.** [20] The OMP\(_{T_0}\) algorithm with \(y\) defined in (1.1) as input succeeds if and only if it selects indices in \(T \setminus T_0\) during the first \(k - g\) iterations.

The authors [20] also proposed the OMP\(_{T_0}\) algorithm can be understood as a particular instance of the standard OMP algorithm, in which indices in the prior support \(T_0\) have been identified during the first \(g + b\) iterations. And any condition which guarantees the success of the OMP\(_{T_0}\) algorithm in the sense of Definition 2.1 ensures the success of the standard OMP algorithm in \(k + b\) iterations provided that the indices in the prior support \(T_0\) are selected during the first \(g + b\) iterations.

For each iteration of the OMP\(_{T_0}\) algorithm, the solution of the minimization problem

\[
\min_u \| y - A_\Lambda x \|_2
\]

is

\[
x^{(t)} = \arg \min_u \| y - A_\Lambda x \|_2 = A_{\Lambda_t}^\dagger y
\]

by the least-square method. Further, by the definition

\[
A_{\Lambda_t}^\dagger = (A_{\Lambda_t}' A_{\Lambda_t})^{-1} A_{\Lambda_t}'
\]

and some simple calculations, one has

\[
r^{(t)} = y - A_\Lambda x^{(t)}
\]

\[
= y - A_\Lambda A_{\Lambda_t}^\dagger y
\]

\[
= A_T x_T - A_\Lambda A_{\Lambda_t}^\dagger A_T x_T + v - A_\Lambda A_{\Lambda_t}^\dagger v
\]

\[
= A_T x_{T \setminus \Lambda_t} + A_\Lambda x_{\Lambda_t} - A_\Lambda A_{\Lambda_t}^\dagger (A_{T \setminus \Lambda_t} x_{T \setminus \Lambda_t} + A_\Lambda x_{\Lambda_t}) + (I - P_{\Lambda_t}) v
\]

\[
= A_T x_{T \setminus \Lambda_t} - A_\Lambda A_{\Lambda_t}^\dagger A_{T \setminus \Lambda_t} x_{T \setminus \Lambda_t} + (I - P_{\Lambda_t}) v
\]

\[
= A_T x_{T \setminus \Lambda_t} + (I - P_{\Lambda_t}) v
\]

(2.1)

where

\[
z_{T \cup \Lambda_t} = \begin{pmatrix} x_{T \setminus \Lambda_t} \\ -A_{\Lambda_t}^\dagger A_{T \setminus \Lambda_t} x_{T \setminus \Lambda_t} \end{pmatrix}.
\]

(2.2)

It is clear that if \(T \setminus \Lambda_t \neq \emptyset\) then \(z_{T \cup \Lambda_t} \neq 0\). And \(r^{(t)} = P_{\Lambda_t}^\perp y\), which implies the residual \(r^{(t)}\) is orthogonal to the columns of \(A_{\Lambda_t}\).

To analyze the main results of this paper, we establish the following important lemma.
Lemma 2.2. Let the support $T = \text{supp}(x)$ with $|T| = k$ and the prior support $T_0$ satisfy $|T \cap T_0| = g < k$ and $|T^c \cap T_0| = b$. Suppose the sensing matrix $A$ satisfies the RIP of order $k + b + 1$ and $\Lambda_t \subseteq T \cup T_0$ for $0 \leq t < k - g$ in the OMP$_{T_0}$ algorithm, then

$$
\max_{i \in T \setminus \Lambda_t} |\langle A e_i, A_{T \cup \Lambda_t} z_{T \cup \Lambda_t} \rangle| - \max_{i \in (T \cup \Lambda_0)^c} |\langle A e_i, A_{T \cup \Lambda_t} z_{T \cup \Lambda_t} \rangle| \\
\geq \frac{1}{\sqrt{k - g - t}} \left(1 - \sqrt{k - g - t + 1}\delta_{k+b+1}\right) \|z_{T \cup \Lambda_t}\|_2.
$$

(2.3)

Proof. For simplicity, let

$$
\alpha_1^{(t)} = \max_{i \in T \setminus \Lambda_t} |\langle A e_i, A_{T \cup \Lambda_t} z_{T \cup \Lambda_t} \rangle| = \|A'_{T \setminus \Lambda_t} A_{T \cup \Lambda_t} z_{T \cup \Lambda_t}\|_\infty
$$

(2.4)

and

$$
\beta_1^{(t)} = \max_{i \in (T \cup \Lambda_0)^c} |\langle A e_i, A_{T \cup \Lambda_t} z_{T \cup \Lambda_t} \rangle| = |\langle A e_i, A_{T \cup \Lambda_t} z_{T \cup \Lambda_t} \rangle| = |\langle A e_i, A_{T \cup \Lambda_t} z_{T \cup \Lambda_t} \rangle|
$$

(2.5)

where $i_t = \arg\max_{i \in (T \cup \Lambda_t)^c} |\langle A e_i, A_{T \cup \Lambda_t} z_{T \cup \Lambda_t} \rangle|$. Based on the definition of $\alpha_1^{(t)}$ in (2.3), one obtains that

$$
\langle A z_{T \cup \Lambda_t}, A z_{T \cup \Lambda_t} \rangle = \langle A_{T \cup \Lambda_t} z_{T \cup \Lambda_t}, A_{T \cup \Lambda_t} z_{T \cup \Lambda_t} \rangle \\
= \langle z_{T \cup \Lambda_t}, A'_{T \cup \Lambda_t} A_{T \cup \Lambda_t} z_{T \cup \Lambda_t} \rangle \\
\leq \|z_{T \cup \Lambda_t}\|_2 \|A'_{T \cup \Lambda_t} A_{T \cup \Lambda_t} z_{T \cup \Lambda_t}\|_2 \\
(1) \leq \|z_{T \cup \Lambda_t}\|_2 \|A'_{T \setminus \Lambda_t} A_{T \cup \Lambda_t} z_{T \cup \Lambda_t}\|_2 \\
\leq \sqrt{k - g - t}\|z_{T \cup \Lambda_t}\|_2 \|A'_{T \setminus \Lambda_t} A_{T \cup \Lambda_t} z_{T \cup \Lambda_t}\|_\infty \\
= \sqrt{k - g - t}\|z_{T \cup \Lambda_t}\|_2 \alpha_1^{(t)}
$$

(2.6)

where (1) follows from

$$
A'_{\Lambda_t} A_{T \cup \Lambda_t} z_{T \cup \Lambda_t} = A'_{\Lambda_t} \left( A_{T \setminus \Lambda_t} x_{T \setminus \Lambda_t} - A_{\Lambda_t} A_{\Lambda_t}^t A_{T \setminus \Lambda_t} x_{T \setminus \Lambda_t} \right) \\
= A'_{\Lambda_t} A_{T \setminus \Lambda_t} x_{T \setminus \Lambda_t} - A'_{\Lambda_t} A_{\Lambda_t} \left(A'_{\Lambda_t} A_{\Lambda_t} \right)^{-1} A'_{\Lambda_t} A_{T \setminus \Lambda_t} x_{T \setminus \Lambda_t} \\
= 0.
$$

Let $s = -\frac{\sqrt{k - g - t + 1} - 1}{\sqrt{k - g - t}}$ and

$$
\hat{s} = \begin{cases} 
+\|z_{T \cup \Lambda_t}\|_2 s, & \langle A z_{T \cup \Lambda_t}, A e_{i_t} \rangle \geq 0, \\
-\|z_{T \cup \Lambda_t}\|_2 s, & \langle A z_{T \cup \Lambda_t}, A e_{i_t} \rangle < 0,
\end{cases}
$$
Further, based on (2.6), (2.5) and some simple calculations we derive that

\[
0 \leq s, \quad T_t(1) = \sqrt{k - g - t + 1} - 1 < 1
\]

and

\[
\frac{2\tilde{s}_{it}}{1 - s^2} = \begin{cases} 
-\sqrt{k - g - t}\|z_{T \cup \Lambda_t}\|_2, & \langle A\tilde{z}_{T \cup \Lambda_t}, Ae_{it} \rangle \geq 0; \\
\sqrt{k - g - t}\|z_{T \cup \Lambda_t}\|_2, & \langle A\tilde{z}_{T \cup \Lambda_t}, Ae_{it} \rangle < 0.
\end{cases}
\]

Further, based on (2.6), (2.5) and some simple calculations we derive that

\[
(1 - s^4)\sqrt{k - g - t}\|z_{T \cup \Lambda_t}\|_2(\alpha_1^{(t)} - \beta_1^{(t)}) \\
\geq (1 - s^4)\left(\langle A\tilde{z}_{T \cup \Lambda_t}, A\tilde{z}_{T \cup \Lambda_t} \rangle - \sqrt{k - g - t}\|z_{T \cup \Lambda_t}\|_2\langle Ae_{it}, A_{T \cup \Lambda_t}z_{T \cup \Lambda_t} \rangle\right) \\
\geq (1 - s^4)\left(\langle A\tilde{z}_{T \cup \Lambda_t}, A\tilde{z}_{T \cup \Lambda_t} \rangle - \sqrt{k - g - t}\|\tilde{z}_{T \cup \Lambda_t}\|_2\langle Ae_{it}, A\tilde{z}_{T \cup \Lambda_t} \rangle\right) \\
= \|A(\tilde{z}_{T \cup \Lambda_t} + \tilde{s}_{it}e_{it})\|_2^2 - \|A(s^2\tilde{z}_{T \cup \Lambda_t} - \tilde{s}_{it}e_{it})\|_2^2. \tag{2.7}
\]

Because \(0 \leq t < k - g\), the sensing matrix \(A\) satisfies the RIP of order \(k + b + 1\), \(|T \cup \Lambda_t| = k + b\) and \(i_t \in (T \cup \Lambda_t)^c\), we obtain that

\[
\|A(\tilde{z}_{T \cup \Lambda_t} + \tilde{s}_{it}e_{it})\|_2^2 - \|A(s^2\tilde{z}_{T \cup \Lambda_t} - \tilde{s}_{it}e_{it})\|_2^2 \\
\geq (1 - \delta_{k+b+1})(\|\tilde{z}_{T \cup \Lambda_t} + \tilde{s}_{it}e_{it}\|_2^2 - (1 + \delta_{k+b+1})(s^2\|\tilde{z}_{T \cup \Lambda_t} - \tilde{s}_{it}e_{it}\|_2^2) \\
= (1 - \delta_{k+b+1})(1 + s^2)\|\tilde{z}_{T \cup \Lambda_t}\|_2^2 - (1 + \delta_{k+b+1})(s^4 + s^2)\|\tilde{z}_{T \cup \Lambda_t}\|_2^2 \\
= \|\tilde{z}_{T \cup \Lambda_t}\|_2^2(1 + s^2)(1 - \delta_{k+b+1} - (1 + \delta_{k+b+1})s^2) \\
= \|\tilde{z}_{T \cup \Lambda_t}\|_2^2(1 + s^2)^2\left(\frac{1 - s^2}{1 + s^2} - \delta_{k+b+1}\right). \tag{2.8}
\]

From the definition of \(s\), it follows that

\[
1 - s^2 = \frac{1 - \sqrt{k - g - t + 1}}{\sqrt{k - g - t + 1}} = \frac{1}{\sqrt{k - g - t + 1}}.
\]

Therefore, by (2.7), (2.8) and the above equality we have that

\[
\alpha_1^{(t)} - \beta_1^{(t)} \geq \frac{(1 + s^2)^2\left(\frac{1 - s^2}{1 + s^2} - \delta_{k+b+1}\right)}{(1 - s^4)\sqrt{k - g - t}}\|\tilde{z}_{T \cup \Lambda_t}\|_2 \\
= \frac{1}{\sqrt{k - g - t}}\left(1 - \sqrt{k - g - t + 1}\delta_{k+b+1}\right)\|\tilde{z}_{T \cup \Lambda_t}\|_2.
\]
3 An optimal exact recovery condition in noiseless case

In this section, we establish the exact recovery results in Theorem 3.1 and Theorem 3.2. If \( j_t \in T \setminus \Lambda_{t-1} \) \((1 \leq t \leq k - g)\) in the \(t\)-th iteration, the OMP\(_T\) algorithm makes a success, i.e.,

\[
\max_{i \in T \setminus \Lambda_{t-1}} |\langle Ae_i, r^{(t-1)} \rangle| > \max_{i \in (T \cup T_0)^c} |\langle Ae_i, r^{(t-1)} \rangle| \text{ in the } t\text{-th iteration. Theorem 3.1 presents a condition to ensure the exact recovery of all } k\text{-sparse signals via the OMP\(_T\) algorithm in } k-g \text{ iterations. And we show that our condition is sharp in Theorem 3.2.}

**Theorem 3.1.** Let \( x \in \mathbb{R}^n \) be a \( k\)-sparse signal in \( y = Ax \), \( T \) be the support of \( x \) with \(|T| = k\) and \( T_0 \) be a prior support of \( x \) satisfying \( 0 \leq |T \cap T_0| = g < k \) and \( |T^c \cap T_0| = b \). Suppose the sensing matrix \( A \) satisfies the RIP of order \( k+b+1 \) with

\[
\delta_{k+b+1} < \frac{1}{\sqrt{k-g+1}}
\]

Then the OMP\(_T\) algorithm exactly recovers the signal \( x \) in \( k-g \) iterations.

**Proof.** We first prove that under the condition \( \delta_{k+b+1} < \frac{1}{\sqrt{k-g+1}} \), the OMP\(_T\) algorithm succeeds in the sense of Definition 2.1 by the inductive method. For the first iteration, \( \Lambda_0 = T_0 \) and \( r^{(0)} = A_{T \cup T_0} z_{T \cup T_0} \). By Lemma 2.2 with \( t = 0 \) and \( \delta_{k+b+1} < \frac{1}{\sqrt{k-g+1}} \), we have that

\[
\max_{i \in T \setminus T_0} |\langle Ae_i, A_{T \cup T_0} z_{T \cup T_0} \rangle| - \max_{i \in (T \cup T_0)^c} |\langle Ae_i, A_{T \cup T_0} z_{T \cup T_0} \rangle| \\
\geq \frac{1}{\sqrt{k-g}} (1 - \sqrt{k-g+1} \delta_{k+b+1}) \|z_{T \cup T_0}\|_2 > 0
\]

which means that \( \max_{i \in T \setminus T_0} |\langle Ae_i, r^{(0)} \rangle| > \max_{i \in (T \cup T_0)^c} |\langle Ae_i, r^{(0)} \rangle| \). Then the OMP\(_T\) algorithm selects a correct index \( j_1 \in T \setminus T_0 \) in the first iteration. Suppose that the OMP\(_T\) algorithm has performed \( t \) \((1 \leq t < k - g)\) iterations successfully, that is, \( \Lambda_t \setminus T_0 \subseteq T \setminus T_0 \). For the \((t+1)\)-th iteration, from the equality (2.1) with \( v = 0 \), Lemma 2.2 and \( \delta_{k+b+1} < \frac{1}{\sqrt{k-g+1}} \) it follows that

\[
\max_{i \in T \setminus \Lambda_t} |\langle Ae_i, r^{(t)} \rangle| - \max_{i \in (T \cup \Lambda_t)^c} |\langle Ae_i, r^{(t)} \rangle| \\
= \max_{i \in T \setminus \Lambda_t} |\langle Ae_i, A_{T \cup \Lambda_t} z_{T \cup \Lambda_t} \rangle| - \max_{i \in (T \cup \Lambda_t)^c} |\langle Ae_i, A_{T \cup \Lambda_t} z_{T \cup \Lambda_t} \rangle| \\
\geq \frac{1}{\sqrt{k-g-t}} (1 - \sqrt{k-g-t+1} \delta_{k+b+1}) \\
\geq \frac{1}{\sqrt{k-g}} (1 - \sqrt{k-g+1} \delta_{k+b+1}) \\
> 0,
\]
which implies that the OMP\(_{T_0}\) algorithm make a success in the \((t + 1)\)-th iteration, i.e., \(j_{t+1} \in T \setminus \Lambda_t \subseteq T \setminus T_0\). Therefore, if \(\delta_{k+b+1} < \frac{1}{\sqrt{k-g+1}}\) then the OMP\(_{T_0}\) algorithm succeeds by the Definition 2.1.

It remains to prove \(x = \hat{x}\), where \(\hat{x}\) is the estimated signal of \(x\) in Table 1. As the OMP\(_{T_0}\) algorithm has performed \(k - g\) iterations successfully, we have that \(\Lambda_{k-g} = T \cup T_0\) and

\[
\hat{x}_{\Lambda_{k-g}} = x^{(k-g)} = A_{\Lambda_{k-g}}^\dagger y
\]

\[
\overset{(1)}{=} (A_{\Lambda_{k-g}} A_{\Lambda_{k-g}})A_{\Lambda_{k-g}}^T x_T
\]

\[
= (A_{\Lambda_{k-g}} A_{\Lambda_{k-g}})^{-1} A_{\Lambda_{k-g}}^T A_{\Lambda_{k-g}} x_{\Lambda_{k-g}} - (A_{\Lambda_{k-g}} A_{\Lambda_{k-g}})A_{\Lambda_{k-g}}^T x_{\Lambda_{k-g}} \setminus T
\]

\[
\overset{(2)}{=} x_{\Lambda_{k-g}}
\]

where (1) and (2) respectively follows from the facts that the matrix \(A\) satisfies the RIP of order \(k + b + 1\), which means \(A_{\Lambda_{k-g}}\) is full column rank, and \(x_{\Lambda_{k-g}} \setminus T = 0\). We have completed the proof of the theorem.

**Remark 1.** For any integers \(b\) and \(g\), the condition \(\delta_{k+b+1} < \frac{1}{\sqrt{k-g+1}}\) is weaker than the sufficient condition \(\delta_{k+b+1} < \frac{1}{\sqrt{k-g+1}}\) in [11].

Next, we show that the condition \(\delta_{k+b+1} < \frac{1}{\sqrt{k-g+1}}\) is optimal in the following theorem.

**Theorem 3.2.** Let \(k\) be any given positive integer, \(0 \leq g < k\) and \(b\) be any given nonnegative integer. There exist a \(k\)-sparse signal \(\bar{x}\) with \(|T| = |\text{supp}(\bar{x})| = k\), a prior support \(T_0\) fulfilling \(|T \cap T_0| = g\) and \(|T^c \cap T_0| = b\) and a matrix \(A\) satisfying

\[
\delta_{k+b+1} = \frac{1}{\sqrt{k-g+1}}
\]

such that the OMP\(_{T_0}\) algorithm may fail.

**Proof.** For given integers \(k > 0\), \(b \geq 0\) and \(0 \leq g < k\), let \(A \in \mathbb{R}^{(k+b+1) \times (k+b+1)}\) be

\[
A = \begin{pmatrix}
0 & \cdots & 0 & \frac{1}{\sqrt{(k-g+1)(k-g)}} & 0 & \cdots & 0 \\
\frac{\sqrt{k-g}}{k-g+1} I_{k-g} & \vdots & \vdots & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \frac{1}{\sqrt{(k-g+1)(k-g)}} & I_{g+b+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & \cdots & 1
\end{pmatrix},
\]

(3.1)
Then the eigenvalues \( I_{k-g} \) and \( I_{g+b+1} \) are unitary matrices. Then

\[
\begin{pmatrix}
\frac{k-g}{k-g+1} & I_{k-g} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
\frac{1}{k-g+1} & \cdots & \frac{1}{k-g+1} & 0 & \cdots & 0 \\
\end{pmatrix}
\]

By elementary transformation of determinant, one can verify that

\[
\left| A' A - \lambda I_{k+b+1} \right| = (-1)^{k+1} \left( \frac{k-g}{k-g+1} - \lambda \right) \left( \frac{k-g}{k-g+1} - \lambda \right)^{k-g-1} (1 - \lambda)^{g+b} \left( 1 + \frac{1}{k-g+1} - \lambda \right)
\]

Moreover, by definition of the RIP and Remark 1 in [7], the matrix \( A \) in (3.1) satisfies the RIP with

\[
\delta_{k+b+1} = \max\{1 - \lambda_{\min}(A' A), \lambda_{\max}(A' A) - 1\} = \max\{1 - \lambda_{k+b}, \lambda_{k+b+1} - 1\} = \frac{1}{\sqrt{k-g+1}}.
\]
Consider $k$-sparse signal $\bar{x} = (1, \ldots, 1, 0, \ldots, 0)' \in \mathbb{R}^{k+b+1}$ and the prior support $T_0 = \{k - g, \ldots, k + b\}$. For the first iteration, 

$$
\begin{align*}
\mathbf{r}^{(0)} &= \mathbf{A}_{T_\setminus T_0} \bar{x}_{T_\setminus T_0} - \mathbf{A}_{T_0} \left( \mathbf{A}^T_{T_0} \mathbf{A}_{T_0} \right)^{-1} \mathbf{A}^T_{T_0} \mathbf{A}_{T_\setminus T_0} \bar{x}_{T_\setminus T_0} \\
&= \left( \sqrt{\frac{k - g}{k - g + 1}}, \ldots, \sqrt{\frac{k - g}{k - g + 1}}, 0, \ldots, 0 \right)' \in \mathbb{R}^{k+b+1}.
\end{align*}
$$

In fact, 

$$
\mathbf{A}_{T_\setminus T_0} \bar{x}_{T_\setminus T_0} = \left( \sqrt{\frac{k - g}{k - g + 1}}, \ldots, \sqrt{\frac{k - g}{k - g + 1}}, 0, \ldots, 0 \right)' \in \mathbb{R}^{k+b+1},
$$

and $\mathbf{A}^T_{T_0} \mathbf{A}_{T_\setminus T_0} \bar{x}_{T_\setminus T_0} = 0 \in \mathbb{R}^{g+b}$.

For $i \in T \setminus T_0$, we have 

$$
|\langle \mathbf{A}e_i, \mathbf{r}^{(0)} \rangle| = \frac{k - g}{k - g + 1}.
$$

For $i \in (T \cup T_0)^c = \{k + b + 1\}$, it follows immediately that 

$$
|\langle \mathbf{A}e_i, \mathbf{r}^{(0)} \rangle| = \frac{k - g}{k - g + 1}.
$$

It is obvious that 

$$
\max_{i \in T \setminus T_0} \langle \mathbf{A}e_i, \mathbf{r}_0 \rangle = \max_{i \in (T \cup T_0)^c} \langle \mathbf{A}e_i, \mathbf{r}_0 \rangle \text{ which implies the OMP}_{T_0} \text{ algorithm may fail to identify one index of the subset } T \setminus T_0 \text{ in the first iteration. So the OMP}_{T_0} \text{ algorithm may fail for the given matrix } \mathbf{A}, \text{ the } k\text{-sparse signal } \bar{x} \text{ and the prior support } T_0.
$$

\[\square\]

4 Analysis on the remainder support $T \setminus T_0$ recovery in noisy case

In this section, we respectively establish sufficient conditions and a necessary condition for the exact remainder support $T \setminus T_0$ recovery of the $k$-sparse signal $\mathbf{x}$ with the prior support $T_0$ in the model (1.1) with $\mathbf{v} \neq \mathbf{0}$ via the OMP$_{T_0}$ algorithm within $k - g$ iterations. In such case, since the exact reconstruction of the $k$-sparse signal $\mathbf{x}$ cannot be guaranteed, we use the upper bound of $\|\mathbf{x} - \hat{\mathbf{x}}\|_2$ as a performance measure of the OMP$_{T_0}$ algorithm and obtain the upper bound. In order to recover the whole support $T$, we investigate the upper bound of $\max_{i \in T \setminus T_0} |x_i|$ and the lower bound of $\min_{i \in T \setminus T_0} |x_i|$. Here, we only consider $l_2$ bounded noise, i.e., $\|\mathbf{v}\|_2 \leq \varepsilon$. 

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4.1 Sufficient conditions for the remainder support $T \setminus T_0$ recovery

In Theorem 4.1, our conditions are in terms of the RIP of order $k + b + 1$ and the minimum magnitude of the entries of $x_{T \setminus T_0}$. The upper bounds of $\max_{i \in T \setminus T_0} |x_i|$ and $\|x - \hat{x}\|_2$ and the lower bound of $\min_{i \in T \setminus T_0} |x_i|$ are obtained in Theorem 4.2.

**Theorem 4.1.** Let $x$ be a $k$-sparse signal in the model (1.1), $T$ be the support of the signal $x$ with $|T| = k$ and $T_0$ be a prior support of the signal $x$ such that $|T \cap T_0| = g < k$ and $|T^c \cap T_0| = b$. Suppose $\|v\|_2 \leq \varepsilon$ and the sensing matrix $A$ satisfies

$$\delta_{k+b+1} < \frac{1}{\sqrt{k - g + 1}}. \quad (4.1)$$

Then the OMP$_{T_0}$ algorithm with the stopping rule $\|r^{(t)}\|_2 \leq \varepsilon$ exactly recovers the remainder support $T \setminus T_0$ of the signal $x$ in $k - g$ iterations provided that

$$\min_{i \in T \setminus T_0} |x_i| > \max \left\{ \frac{\sqrt{2(1 + \delta_{k+b+1})\varepsilon}}{1 - \sqrt{k - g + 1}\delta_{k+b+1}}, \frac{2\varepsilon}{\sqrt{1 - \delta_{k+b+1}}} \right\}. \quad (4.2)$$

**Proof.** The proof consists of two parts. In the first part we show that the OMP$_{T_0}$ algorithm selects indices of the remainder support $T \setminus T_0$ in each iteration under conditions (4.1) and (4.2). In the second part we prove that the OMP$_{T_0}$ algorithm exactly performs $|T \setminus T_0| = k - g$ iterations with the stopping rule $\|r^{(t)}\|_2 \leq \varepsilon$.

Part I: By mathematical induction method, suppose first that the OMP$_{T_0}$ algorithm performed $t$ ($1 \leq t < k - g$) iterations successfully, that is, $\Lambda_t \subseteq T \cup T_0$ and $j_1, \ldots, j_t \in T \setminus T_0$. Then by the OMP$_{T_0}$ algorithm in Table 1, we need to show $j_{t+1} \in T \setminus \Lambda_t$ which means the OMP$_{T_0}$ algorithm makes a success in the $(t + 1)$-th iteration. By the fact that $r^{(t)}$ is orthogonal to each column of $A_{\Lambda_t}$, we only need to prove that

$$\max_{i \in T \setminus \Lambda_t} |\langle A e_i, r^{(t)} \rangle| > \max_{i \in (T \cup T_0)^c} |\langle A e_i, r^{(t)} \rangle| \quad (4.3)$$

for the $(t + 1)$-th iteration.

From (2.1), one has that

$$\max_{i \in T \setminus \Lambda_t} |\langle A e_i, r^{(t)} \rangle| = \max_{i \in T \setminus \Lambda_t} |\langle A e_i, A \hat{z}_{T \cup \Lambda_t} + P_{\Lambda_t}^\perp v \rangle|$$

$$\geq \max_{i \in T \setminus \Lambda_t} |\langle A e_i, A \hat{z}_{T \cup \Lambda_t} \rangle| - \max_{i \in T \setminus \Lambda_t} |\langle A e_i, P_{\Lambda_t}^\perp v \rangle| \quad (4.4)$$

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and
\[
\max_{i \in (T \cup T_0)^c} |\langle Ae_i, r^{(t)} \rangle| = \max_{i \in (T \cup T_0)^c} |\langle Ae_i, A\hat{z}_{T \cup \Lambda_t} + P_{\Lambda_t}^\perp v \rangle| \\
\leq \max_{i \in (T \cup T_0)^c} |\langle Ae_i, A\hat{z}_{T \cup \Lambda_t} \rangle| + \max_{i \in (T \cup T_0)^c} |\langle Ae_i, P_{\Lambda_t}^\perp v \rangle|.
\] (4.5)

Therefore, by (4.4) and (4.5), it suffices to prove that
\[
\max_{i \in T \setminus \Lambda_t} |\langle Ae_i, A\hat{z}_{T \cup \Lambda_t} \rangle| - \max_{i \in (T \cup T_0)^c} |\langle Ae_i, A\hat{z}_{T \cup \Lambda_t} \rangle| \\
> \max_{i \in T \setminus \Lambda_t} |\langle Ae_i, P_{\Lambda_t}^\perp v \rangle| + \max_{i \in (T \cup T_0)^c} |\langle Ae_i, P_{\Lambda_t}^\perp v \rangle|.
\] (4.6)

One first gives a lower bound on the left-hand side of (4.6). From Lemma 2.2, the definition of \(z_{T \cup T_0}\) in (2.2) and the induction assumption \(j_1, \cdots, j_t \in T \setminus T_0\) which implies \(|T \setminus \Lambda_t| = k - g - t\), it follows that
\[
\max_{i \in T \setminus \Lambda_t} |\langle Ae_i, A\hat{z}_{T \cup \Lambda_t} \rangle| - \max_{i \in (T \cup T_0)^c} |\langle Ae_i, A\hat{z}_{T \cup \Lambda_t} \rangle| \\
\geq \frac{1}{\sqrt{k - g - t}} \left(1 - \sqrt{k - g - t + 1} \delta_{k+b+1}\right) \|\hat{z}_{T \cup \Lambda_t}\|_2 \\
> \frac{1}{\sqrt{k - g - t}} \left(1 - \sqrt{k - g - t + 1} \delta_{k+b+1}\right) \|x_{T \setminus \Lambda_t}\|_2 \\
\geq \frac{1}{\sqrt{k - g - t}} \left(1 - \sqrt{k - g - t + 1} \delta_{k+b+1}\right) \sqrt{k - g - t} \min_{i \in T \setminus \Lambda_t} |x_i| \\
\geq \left(1 - \sqrt{k - g + 1} \delta_{k+b+1}\right) \min_{i \in T \setminus T_0} |x_i|.
\] (4.7)

One now gives an upper bound on the right-hand side of (4.6). There exist the indices \(i^{(t)} \in T \setminus \Lambda_t\) and \(i_1^{(t)} \in (T \cup T_0)^c\) satisfying
\[
\max_{i \in T \setminus \Lambda_t} |\langle Ae_i, P_{\Lambda_t}^\perp v \rangle| = |\langle Ae_{i^{(t)}}, P_{\Lambda_t}^\perp v \rangle| \\
\text{and}
\]
\[
\max_{i \in (T \cup T_0)^c} |\langle Ae_i, P_{\Lambda_t}^\perp v \rangle| = |\langle Ae_{i_1^{(t)}}, P_{\Lambda_t}^\perp v \rangle|.
\]
respectively. Therefore, we obtain that

\[
\max_{i \in T \setminus \Lambda_t} |\langle Ae_i, P_{\Lambda_t}^\perp v \rangle| + \max_{i \in (T \cap T_0)^c} |\langle Ae_i, P_{\Lambda_t}^\perp v \rangle| \\
= |\langle Ae_{i(t)}, P_{\Lambda_t}^\perp v \rangle| + |\langle Ae_{i'(t)}, P_{\Lambda_t}^\perp v \rangle| \\
= \| A' \{i(t), i'(t)\} P_{\Lambda_t}^\perp v \|_1 \\
\leq \sqrt{2} \| A' \{i(t), i'(t)\} P_{\Lambda_t}^\perp v \|_2 \\
\overset{(1)}{\leq} \sqrt{2(1 + \delta_{k-g+1})} \| P_{\Lambda_t}^\perp v \|_2 \\
\overset{(2)}{\leq} \sqrt{2(1 + \delta_{k-g+1})} \varepsilon
\]

where (1) follows from \( A \) fulfilling the RIP with order \( k - g + 1 \) \((g < k)\) and (2) is because the fact

\[ \| P_{\Lambda_t}^\perp v \|_2 \leq \| P_{\Lambda_t}^\perp \|_2 \| v \|_2 \leq \| v \|_2 \leq \varepsilon. \]

By (4.1) and (4.2), there is

\[
\left(1 - \sqrt{k - g + 1 + \delta_{k+b+1}}\right) \min_{i \in T \setminus T_0} |x_i| > \sqrt{2(1 + \delta_{k-g+1})} \varepsilon.
\]

It is obvious that (4.6) holds by the above inequality. Then the OMP\(_{T_0}\) algorithm selects one index from the subset \( T \setminus \Lambda_t \) in the \((t + 1)\)-th iteration. In conclusion, we have shown that the OMP\(_{T_0}\) algorithm selects one index from \( T \setminus T_0 \) in each iteration.

Part II: We prove that the OMP\(_{T_0}\) algorithm performs exactly \( k - g \) iterations. It remains to show that \( \| r^{(t)} \|_2 > \varepsilon \) for \( 0 \leq t < k - g \) and \( \| r^{(k-g)} \|_2 \leq \varepsilon \).

Since the OMP\(_{T_0}\) algorithm selects an index of \( T \setminus T_0 \) in each iteration under the conditions (4.1) and (4.2), \( \Lambda_{k-g} = T \cup T_0 \) which means \( P_{\Lambda_{k-g}} A_T x_T = 0 \). Moreover,

\[ \| r^{(k-g)} \|_2 = \| P_{\Lambda_{k-g}} A_T x_T + P_{\Lambda_{k-g}}^\perp v \|_2 = \| P_{\Lambda_{k-g}}^\perp v \|_2 \leq \| v \|_2 \leq \varepsilon. \]
For $0 \leq t < k - g$, we have that $\Lambda_t \subseteq T \cup T_0$, $(T \cup T_0) \setminus \Lambda_t \neq \emptyset$ and
\[
\|r(t)\|_2 = \|A_{T \cup \Lambda_t} z_{T \cup \Lambda_t} + (I - P_{\Lambda_t}) v\|_2 \\
\geq \|A_{T \cup \Lambda_t} z_{T \cup \Lambda_t}\|_2 - \|P_{\Lambda_t}^\perp v\|_2 \\
\geq \sqrt{1 - \delta_{k+b+1}} \|z_{T \cup \Lambda_t}\|_2 - \varepsilon \\
\geq \sqrt{1 - \delta_{k+b+1}} \|x_{T \setminus \Lambda_t}\|_2 - \varepsilon \\
\geq \sqrt{1 - \delta_{k+b+1}} \min_{T \setminus T_0} |x_i| - \varepsilon \\
\geq \varepsilon
\]
where (1) is because $A$ satisfies the RIP with order $k + b + 1$ and $\|P_{\Lambda_t}^\perp v\|_2 \leq \varepsilon$ and (2) is because of (4.2). We have completed the proof.

**Theorem 4.2.** Let $x$ be a $k$-sparse signal in the model (1.1) with $\|v\|_2 \leq \varepsilon$. $T$ be the support of $x$ with $|T| = k$ and $T_0$ be a prior support of $x$ such that $|T \cap T_0| = g < k$ and $|T^c \cap T_0| = b$. If $\delta_{k+b+1} < \frac{1}{\sqrt{k-g+1}}$,
\[
\min_{i \in T} |x_i| > \max \left\{ \frac{\sqrt{2(1 + \delta_{k+b+1})\varepsilon}}{1 - \sqrt{k-g + 1}\delta_{k+b+1}}, \frac{2\varepsilon}{\sqrt{1 - \delta_{k+b+1}}} \right\}. \quad (4.9)
\]
and the stopping rule $\|r(t)\|_2 \leq \varepsilon$, then
\[
\min_{i \in T \cap T_0} |\hat{x}_i| > \frac{\varepsilon}{\sqrt{1 - \delta_{k+b+1}}}, \quad \max_{i \in T_0 \setminus T} |\hat{x}_i| \leq \frac{\varepsilon}{\sqrt{1 - \delta_{k+b+1}}}
\]
and
\[
\|x - \hat{x}\| \leq \frac{\varepsilon}{\sqrt{1 - \delta_{k+b+1}}},
\]
where $\hat{x}$ is the estimated signal of $x$ in Table 1.

**Proof.** It is obvious that the condition (4.2) is satisfied by (4.9). From Theorem 4.1 the condition $\delta_{k+b+1} < \frac{1}{\sqrt{k-g+1}}$ and the the lower bound (4.9) ensure the OMP$_{T_0}$ algorithm with the stopping rule $\|r(t)\|_2 \leq \varepsilon$ exactly stops after performing $k - g$ iterations successfully, which implies $\Lambda_{k-g} = T \cup T_0$. For the OMP$_{T_0}$ algorithm in Table 1, there exists
\[
x_{(k-g)} = \arg\min_u \|y - A_{\Lambda_{k-g}} u\| = A_{T \cup T_0}^\dagger y = A_{T \cup T_0}^\dagger (A_{T \cup T_0} x_{T \cup T_0} + v) = x_{T \cup T_0} + \omega
\]
where
\[
\omega = (A'_{T \cup T_0}A_{T \cup T_0})^{-1}A'_{T \cup T_0}v.
\]
Furthermore, we have that
\[
\hat{x}_i = \begin{cases} 
  x_i + \omega_i, & i \in T, \\
  \omega_i, & i \in T_0 \setminus T, \\
  0, & i \in (T_0 \cup T)^c,
\end{cases}
\]
and
\[
\sqrt{1 - \delta_{k+b+1}} \|\omega\|_2 \leq \|A_{T \cup T_0}\|_2 = \|P_{T \cup T_0}v\|_2 \leq \|v\| \leq \varepsilon.
\]
Therefore, by (4.9) and the above equalities and inequality, we obtain that
\[
\min_{i \in T \cap T_0} |\hat{x}_i| \geq \min_{i \in T} (|x_i| - |\omega_i|) > \frac{\varepsilon}{\sqrt{1 - \delta_{k+b+1}}},
\]
\[
\max_{i \in T_0 \setminus T} |\hat{x}_i| = \max_{i \in T_0 \setminus T} |\omega_i| \leq \|\omega\|_2 \leq \frac{\varepsilon}{\sqrt{1 - \delta_{k+b+1}}}
\]
and
\[
\|x - \hat{x}\|_2 \leq \frac{1}{\sqrt{1 - \delta_{T \cup T_0}}} \|A(x - \hat{x})\|_2
\]
\[
= \frac{1}{\sqrt{1 - \delta_{T \cup T_0}}} \|A_{T \cup T_0}x_{T \cup T_0} - A_{T \cup T_0}x^{(k-g)}\|_2
\]
\[
= \frac{1}{\sqrt{1 - \delta_{T \cup T_0}}} \|A_{T \cup T_0}\omega\|_2
\]
\[
\leq \frac{\varepsilon}{\sqrt{1 - \delta_{k+b+1}}}.
\]

4.2 A necessary condition for the remainder support $T \setminus T_0$ recovery

In this subsection, we derive a necessary condition on the minimum magnitude of the components of $x_{T \setminus T_0}$ for the exact recovery of the remainder support $T \setminus T_0$.

**Theorem 4.3.** Let $x$ be a $k$-sparse signal in the model (1.1), $T$ be the support of $x$ with $|T| = k$ and $T_0$ be a prior support of the $x$ such that $|T \cap T_0| = g < k$ and $|T^c \cap T_0| = b$. Suppose $\|v\|_2 \leq \varepsilon$ and the sensing matrix $A$ satisfies the RIP of order $k + b + 1$ with...
0 \leq \delta_{k+b+1} < 1$. If the OMP\textsubscript{$T_0$} algorithm exactly recovers the remainder support $T \setminus T_0$ of the signal $x$ in $k - g$ iterations, then

\[
\min_{T \setminus T_0} |x_i| > \frac{\sqrt{1 - \delta_{k+b+1} \varepsilon}}{1 - \sqrt{k - g + 1} \delta_{k+b+1}}. \tag{4.10}
\]

**Proof.** The proof below roots in [24]. However, some essential modifications are necessary in order to adapt the results to sparse signals $x$ with the prior support $T_0$. Using proofs by contradiction, we show the theorem. We construct a linear model of the form $y = Ax + v$, where the sensing matrix $A$ and the error vector $v$ respectively satisfy the RIP of order $k + b + 1$ with $0 \leq \delta_{k+b+1}(A) = \delta_{k+b+1} < 1$ and $\|v\|_2 \leq \varepsilon$, and $x$ is a $k$-sparse signal with the prior support $T_0$ and satisfies

\[
\min_{T \setminus T_0} |x_i| \leq \theta := \frac{\sqrt{1 - \delta_{k+b+1} \varepsilon}}{1 - \sqrt{k - g + 1} \delta_{k+b+1}}, \tag{4.11}
\]

such that the OMP\textsubscript{$T_0$} algorithm may fail to exactly recover the remainder support $T \setminus T_0$ of the signal $x$ within $k - g$ iterations.

It is well known that there exist the unit vectors $\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(k-g-1)} \in \mathbb{R}^{k-g}$ such that the matrix

\[
\begin{pmatrix}
\xi^{(1)} & \xi^{(2)} & \cdots & \xi^{(k-g-1)} & \frac{1}{\sqrt{k-g}} \mathbf{1}_{k-g}
\end{pmatrix} \in \mathbb{R}^{(k-g) \times (k-g)}
\]

is orthogonal, which implies $\langle \xi^{(i)}, \xi^{(j)} \rangle = 0$ and $\langle \xi^{(i)}, \mathbf{1}_{k-g} \rangle = 0$ for $i, j = 1, \ldots, k - g - 1$ and $i \neq j$, where $\mathbf{1}_{k-g} = (1, \ldots, 1)' \in \mathbb{R}^{k-g}$. Let the matrix

\[
U' = \begin{pmatrix}
\xi^{(1)} & \cdots & \xi^{(k-g-1)} & \frac{1}{\sqrt{(k-g)(\eta^2+1)}} \mathbf{1}_{k-g} & 0_{(k-g) \times (g+b)} & \frac{\eta \mathbf{1}_{k-g}}{\sqrt{(k-g)(\eta^2+1)}} \\
0 & \cdots & 0 & 0 & \vdots & I_{g+b} & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & \frac{\eta}{\sqrt{\eta^2+1}} & 0 & \cdots & 0 & -\frac{1}{\sqrt{\eta^2+1}}
\end{pmatrix}, \tag{4.12}
\]

where

\[
\eta = \frac{\sqrt{k - g + 1} - 1}{\sqrt{k - g}}.
\]

Then $U$ is also an orthogonal matrix.

Let $D \in \mathbb{R}^{(k+b+1) \times (k+b+1)}$ be a diagonal matrix with

\[
d_{ii} = \begin{cases}
\sqrt{1 - \delta_{k+b+1}}, & i = k - g, \\
\sqrt{1 + \delta_{k+b+1}}, & i \neq k - g.
\end{cases} \tag{4.13}
\]
and the sensing matrix \(A = DU\), then \(A' A = U' D^2 U\). In the following, we show that 
\(\delta_{k+b+1}(A) = \delta_{k+b+1}\). For any \(x \in \mathbb{R}^{k+b+1}\), setting \(\hat{\nu} = Ux\), we have that

\[
\|Ax\|_2^2 = \langle Ax, Ax \rangle = x' A' Ax = (Ux)' D' D (Ux)
\]

\[
= \hat{\nu}' D^2 \hat{\nu} = (1 + \delta_{k+b+1}) \|\hat{\nu}\|_2^2 - 2\delta_{k+b+1} \hat{\nu}_{k-g}^2
\]

\[
\leq (1 + \delta_{k+b+1}) \|\hat{\nu}\|_2^2 \overset{(1)}{=} (1 + \delta_{k+b+1}) \|x\|_2^2
\]

and

\[
\|Ax\|_2^2 = \langle Ax, Ax \rangle = (1 - \delta_{k+b+1}) \|\hat{\nu}\|_2^2 + 2\delta_{k+b+1} \sum_{1 \leq i \leq k+b+1, i \neq k-g} \hat{\nu}_i^2
\]

\[
\geq (1 - \delta_{k+b+1}) \|\hat{\nu}\|_2^2 \overset{(2)}{=} (1 - \delta_{k+b+1}) \|x\|_2^2
\]

where (1) and (2) result of the fact that \(U\) is an orthogonal matrix. Then, based on the definition \[1,1\] we have \(\delta_{k+b+1}(A) \leq \delta_{k+b+1}\). It remains to prove that the matrix \(A = DU\) satisfies \(\delta_{k+b+1}(A) \geq \delta_{k+b+1}\). Let the vector

\[
\hat{x} = ((\xi^{(1)})', 0, \cdots, 0)' \in \mathbb{R}^{k+b+1},
\]

then \(\hat{x}\) is \((k + b + 1)\)-sparse and \(\|\hat{x}\|_2^2 = 1\). By the definitions of \(D\) and \(A\), we obtain that

\[
\|A\hat{x}\|_2^2 = (U\hat{x})' D^2 U \hat{x} = e_1' D^2 e_1 = 1 + \delta_{k+b+1} = (1 + \delta_{k+b+1}) \|\hat{x}\|_2^2.
\]

So \(\delta_{k+b+1}(A) \geq \delta_{k+b+1}\). In conclusion, \(\delta_{k+b+1}(A) = \delta_{k+b+1}\).

Let the original signal

\[
x = \left(\begin{array}{c}
\theta, \cdots, \theta, 1, \cdots, 1, 0, \cdots, 0, 0, 0
\end{array}\right)' \in \mathbb{R}^{k+b+1},
\]

where \(\theta\) is defined in \[4.11\]. Then the signal \(x\) is \(k\)-sparse with the support \(T = \{1, 2, \cdots, k\}\), the prior support \(T_0 = \{k-g+1, \cdots, k+b\}\) and satisfies \[4.11\]. It is not hard to prove that \(A_{T \setminus T_0} = DU_{T \setminus T_0}\). Moreover, by some simple calculations we derive that

\[
A_{T \setminus T_0} x_{T \setminus T_0} = DU_{T \setminus T_0} x_{T \setminus T_0}
\]

\[
= D \left(0, \cdots, 0, \sqrt{\frac{k-g}{\eta^2 + 1}}, 0, \cdots, 0, \sqrt{\frac{k-g}{\eta^2 + 1}}\right)'
\]

\[
= \sqrt{1 + \delta_{k+b+1}} \left(0, \cdots, 0, \sqrt{\frac{(k-g)(1 - \delta_{k+b+1})}{(\eta^2 + 1)(1 + \delta_{k+b+1})}} \theta, 0, \cdots, 0, \sqrt{\frac{k-g}{\eta^2 + 1}}\right)'.
\]

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and

$$A' A_{\mathcal{T} \setminus T_0} x_{T \setminus T_0} = \left( \mu, \ldots, \mu, 0, \ldots, 0, -\frac{2\eta}{\eta^2 + 1} \sqrt{k - g \delta_{k+b+1}} \theta \right)'$$  \hspace{1cm} (4.14)

where $\mu = \frac{(1-\delta_{k+b+1}) + (1+\delta_{k+b+1}) \eta^2}{\eta^2 + 1} \theta$. Similarly, let the error vector

$$v = D^{-1} U (0, \ldots, 0, -\sqrt{1 - \delta_{k+b+1} \varepsilon})'$$

$$= D^{-1} (0, \ldots, 0, -\sqrt{1 - \delta_{k+b+1} \eta \varepsilon} g \theta, 0, \ldots, 0, \sqrt{1 - \delta_{k+b+1} \varepsilon})'$$

then $\|v\|_2 \leq \varepsilon$,

$$A' v = U' D D^{-1} U (0, \ldots, 0, -\sqrt{1 - \delta_{k+b+1} \varepsilon})'$$

$$= (0, \ldots, 0, -\sqrt{1 - \delta_{k+b+1} \varepsilon})'.$$  \hspace{1cm} (4.15)

By (4.14) and (4.15), it is clear that

$$A_{T_0}' A_{T \setminus T_0} x_{T \setminus T_0} = 0, \quad A_{T_0}' v = 0.$$  

Therefore, using (2.1) and the above equality, we obtain that

$$r^{(0)} = A_{T \setminus T_0} x_{T \setminus T_0} - A_{T_0} (A_{T_0}' A_{T_0})^{-1} A_{T_0}' A_{T \setminus T_0} x_{T \setminus T_0} + v - A_{T_0} (A_{T_0}' A_{T_0})^{-1} A_{T_0}' v$$

$$= A_{T \setminus T_0} x_{T \setminus T_0} + v.$$  

Therefore, we have that

$$\langle A e_i, r^{(0)} \rangle = \left\{ \begin{array}{ll}
\frac{(1-\delta_{k+b+1}) + (1+\delta_{k+b+1}) \eta^2}{\eta^2 + 1} \theta, & i \in T \setminus T_0 \\
-\frac{2\eta}{\eta^2 + 1} \sqrt{k - g \delta_{k+b+1}} \theta - \sqrt{1 - \delta_{k+b+1} \varepsilon}, & i = k + b + 1
\end{array} \right.$$  

$$= \left\{ \begin{array}{ll}
(1 - \frac{1}{\sqrt{k-g+1}} \delta_{k+b+1}) \theta, & i \in T \setminus T_0 \\
-\frac{1}{\sqrt{k-g+1}} \delta_{k+b+1} \theta - \sqrt{1 - \delta_{k+b+1} \varepsilon}, & i = k + b + 1.
\end{array} \right.$$  

From (4.11), it follows that

$$\max_{i \in T \setminus T_0} |\langle A e_i, r^{(0)} \rangle| = \max_{i \in (T \cup T_0)^c} |\langle A e_i, r^{(0)} \rangle|,$$

which means the OMP_{T_0} algorithm may choose a wrong index $k + b + 1$ in the first iteration. That is, the remainder support $T \setminus T_0$ of the signal $x$ may not be exactly recovered in $k - g$ iterations by the OMP_{T_0} algorithm. We completed the proof.  \hfill \Box
5 Discussion

In this section, we shall focus exclusively the discussions on the validity of our sufficient condition. In section 3, for any \( k \)-sparse signals \( x \) with \( |T| = |\text{supp}(x)| = k \) from \( y = A x \) and the prior support \( T_0 \) satisfying \( |T \cap T_0| = g < k \) and \( |T_0 \setminus T| = b \), we have established the condition based on the RIC \( \delta_{k+b+1} < \frac{1}{\sqrt{k-g+1}} \) to guarantee the exact recovery of the signal \( x \) via the OMP\(_T\) algorithm in \( k - g \) iterations and proved the upper bound of RIC depending on \( g \) is sharp. It is known from Theorem III.1 in \([16]\) that if \( A \) satisfies the condition \( \delta_{k+1} < \frac{1}{\sqrt{k+1}} \) then the standard OMP algorithm will recover any \( k \)-sparse signals \( x \) from \( y = A x \) in \( k \) iterations. Moreover, the author \([16]\) also show that the condition \( \delta_{k+1} < \frac{1}{\sqrt{k+1}} \) is sharp. In order to state the validity of the sharp condition in this paper, we need to compare the two bounds

\[
\delta_{k+b+1} < \frac{1}{\sqrt{k-g+1}} \quad (5.1)
\]

and

\[
\delta_{k+1} < \frac{1}{\sqrt{k+1}}. \quad (5.2)
\]

Since \( \delta_{k+b+1} \geq \delta_{k+1} \) and \( \frac{1}{\sqrt{k-g+1}} \geq \frac{1}{\sqrt{k+1}} \), it is impossible to compare these two sharp conditions directly. Intuitively, when \( b \) is very small and \( g \) is large, we expect that the sharp condition (5.1) to be weaker than the condition (5.2). For example, taking \( b = 0 \) and \( 0 < g < k \), the condition (5.1) is weaker than the condition (5.2). Now, we establish exact comparison of these two bounds of \( \delta_{k+b+1} \) in (5.1) and \( \delta_{k+1} \) in (5.2) for some particular cases in the following theorem.

**Theorem 5.1.** For any positive integers \( c \geq 3 \), assume that \( k > 2c^2 - 1 \), \((1 - \frac{1}{c^2})(k + 1) \leq g < k \) and \( 1 \leq b \leq (c - 2)\lceil \frac{k}{2} \rceil \), then the condition \( \delta_{k+b+1} < \frac{1}{\sqrt{k-g+1}} \) in this paper is weaker than the sufficient condition \( \delta_{k+1} < \frac{1}{\sqrt{k+1}} \) [16].

**Proof.** By \( g \geq (1 - \frac{1}{c^2})(k + 1) \), we derive that

\[
\frac{c}{\sqrt{k+1}} \leq \frac{1}{\sqrt{k-g+1}} \quad (5.3)
\]

Since \( 1 \leq b \leq (c - 2)\lceil \frac{k}{2} \rceil \), we have \( k + b + 1 \leq c\lceil \frac{k}{2} \rceil \). Then, \( \delta_{k+b+1} \leq \delta_{c\lceil \frac{k}{2} \rceil} \). Therefore, from \( \delta_{c\ell} < c \cdot \delta_{2\ell} \) for any positive integers \( c \) and \( r \) (seeing Corollary 3.4 in [18]), the fact
\(k + 1 \geq 2\lceil \frac{k}{2} \rceil\) with \(k \geq 2\), \(\delta_{k+1} < \frac{1}{\sqrt{k+1}}\) and the inequality (5.5), it follows that

\[
\delta_{k+b+1} \leq \delta_{\lceil \frac{k}{2} \rceil} < c\delta_{2\lceil \frac{k}{2} \rceil} \leq c\delta_{k+1} < \frac{c}{\sqrt{k+1}} \leq \frac{1}{\sqrt{k-g+1}},
\]

which implies the condition \(\delta_{k+b+1}\) in this paper is weaker than the sufficient condition \(\delta_{k+1} < \frac{1}{\sqrt{k+1}}\). We complete the proof of the theorem.

\[\square\]

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