Inviscid limit of stochastic damped 2D Navier–Stokes equations

Hakima Bessaih\(^1\) and Benedetta Ferrario\(^2\)

\(^1\) Department of Mathematics, University of Wyoming, Dept. 3036, 1000 East University Avenue, Laramie WY 82071, USA
\(^2\) Università di Pavia, Dipartimento di Matematica, via Ferrata 1, 27100 Pavia, Italy

E-mail: bessaih@uwyo.edu and benedetta.ferrario@unipv.it

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Abstract
We consider the inviscid limit of the stochastic damped 2D Navier–Stokes equations. We prove that, when the viscosity vanishes, the stationary solution of the stochastic damped Navier–Stokes equations converges to a stationary solution of the stochastic damped Euler equation and that the rate of dissipation of enstrophy converges to zero. In particular, this limit obeys an enstrophy balance. The rates are computed with respect to a limit measure of the unique invariant measure of the stochastic damped Navier–Stokes equations.

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1. Introduction

In this paper, we are interested in the equations of motion of incompressible fluids in a bounded domain of \(\mathbb{R}^2\). In particular, we consider the Euler or Navier–Stokes equations damped by a term proportional to the velocity. Damping terms in two-dimensional turbulence studies have been considered to model pumping due to friction with boundaries. Numerical studies of two-dimensional turbulence employ devices to remove the energy that piles up at the large scales, and damping is the most common such device. We refer to [7,21] for a physical motivation of the model and to [1,25,26] for a mathematical analysis of the deterministic damped Navier–Stokes equations and to [4,5] for the stochastic damped Euler equations.

These stochastic damped equations are given by

\[
\frac{du}{dt} + [-\nu \Delta u + (u \cdot \nabla)u + \gamma u + \nabla p] \, dt = dw, \\
\nabla \cdot u = 0.
\]
The non-negative coefficients $\nu$ and $\gamma$ are called kinematic viscosity and sticky viscosity, respectively. The unknowns are the velocity $u$ and the pressure $p$. Suitable boundary conditions have to be considered; in this paper the spatial domain is a box and periodic boundary conditions are assumed.

For a fixed $\gamma > 0$, if $\nu > 0$ these are called the stochastic damped Navier–Stokes equations, whereas if $\nu = 0$ they are the stochastic damped Euler equations. If $\gamma = 0$ and $\nu = 0$, we refer to [3,9,10,12,22,27,33] for an analysis of the existence and/or uniqueness of solutions and to [15] where some dissipation of enstrophy arguments are discussed in Besov spaces.

Turbulence theory investigates the behaviour of certain quantities as the viscosity $\nu$ vanishes. In particular, in the two-dimensional setting one is interested in understanding what happens to the balance equation of energy and enstrophy (in the stationary regime) as the viscosity vanishes. Bernard [2] suggested that there is no anomalous dissipation of enstrophy in damped and driven Navier–Stokes equations; Constantin and Ramos [11] proved that there is no anomalous dissipation neither of energy nor of enstrophy as $\nu \to 0$ for the deterministic damped Navier–Stokes equations in the whole plane. Some similar questions were suggested by Kupiainen [30] for the stochastic case. Therefore we address the same problem when the forcing term is of white noise type. Tools from stochastic analysis are very useful to investigate the same problem studied in [11], giving a rigorous meaning to the averages of velocity and vorticity. Indeed, using stochastic PDE’s allows us to express the stationary regime by means of an invariant measure, whereas in the deterministic setting the stationary regime is described by taking time averages on the infinite time interval.

In this paper, we shall prove that in the stationary regime system (1) has no anomalous dissipation neither of energy nor of enstrophy as $\nu \to 0$. However, we shall be working in a finite two-dimensional spatial domain and not in the whole plane; this answers one of the questions posed by Kupiainen in [30] about the behaviour of the stochastic damped Navier–Stokes equations on a torus for vanishing viscosity.

As far as the content of the paper is concerned, in section 2 we introduce some functional spaces, the equations in their vorticity formulation and the assumptions on the noise term. We also introduce the classical properties of the nonlinear term associated with these equations. Section 3 is devoted to the well posedness of the stochastic 2D damped Navier–Stokes equations, where some uniform estimates are computed. Starting from a known result of existence and uniqueness of the invariant measure, we provide a balance law for the enstrophy. In section 4, we deal with the stochastic 2D damped Euler equation. In particular the Itô formula is computed rigorously for the $L^2$-norm of the vorticity that will be used in the following section. The vanishing viscosity limit is studied in section 5 and stationary solutions are constructed by means of a tightness argument providing a balance relation for these stationary solutions. Using these results, we provide a proof of no anomalous dissipation of enstrophy and energy for the stochastic damped 2D Navier–Stokes equations.

2. Notations and hypothesis

Let the spatial domain $D$ be the square $[-\pi, \pi]^2$; periodic boundary conditions are assumed. A basis of the space $L^2(D)$ with periodic boundary conditions is $\{e_k\}_{k \in \mathbb{Z}^2}$, $e_k(x) = \frac{1}{\sqrt{\pi}} e^{ik \cdot x}$, whereas a basis for the space of periodic vector fields which are square integrable and divergence free is $\{\frac{1}{\sqrt{\pi}} e_k\}_{k \in \mathbb{Z}^2}$, being $k^\perp = (-k_2, k_1)$. Actually we consider $k \neq (0,0)$, that is we consider velocity fields with vanishing mean value.
Let \( \mathbb{Z}_k^2 = \mathbb{Z}^2 \setminus \{(0,0)\} \), and \( \mathbb{Z}_k^2 = \{(k_1, k_2) \in \mathbb{Z} : k_1 > 0 \} \cup \{(k_0, k_2) \in \mathbb{Z}^2 : k_2 > 0\} \). Given \( x = (x_1, x_2) \in \mathbb{R}^2 \) we denote by \(|x|\) its norm: \(|x| = \sqrt{x_1^2 + x_2^2}\). Given \( y = \Re y + i \Im y \in \mathbb{C} \) we denote by \(|y|\) its absolute value and by \(\overline{y}\) its complex conjugate: \(|y| = \sqrt{(\Re y)^2 + (\Im y)^2}, \overline{y} = \Re y - i \Im y\).

For any \( \alpha \in \mathbb{R} \) we define the Hilbert space
\[
H^\alpha = \left\{ f = \sum_{k \in \mathbb{Z}_k^2} f_k e_k(x) : \sum_{k \in \mathbb{Z}_k^2} |f_k|^2 |k|^{2\alpha} < \infty \right\}
\]
with scalar product
\[
\langle f, g \rangle_{H^\alpha} = \sum_{k \in \mathbb{Z}_k^2} |k|^{2\alpha} f_k \overline{g_k};
\]
we set
\[
||f||_{H^\alpha}^2 = \sum_{k \in \mathbb{Z}_k^2} |k|^{2\alpha} |f_k|^2.
\]

For a vector \( f = (f_1, f_2) \) we set
\[
||f||_{H^\alpha}^2 = ||f_1||_{H^\alpha}^2 + ||f_2||_{H^\alpha}^2.
\]
In particular, for scalar functions we have \( ||f||_{H^0}^2 = ||f||_{L^2(\Omega)}^2 \) and \( ||f||_{H^1}^2 = ||\nabla f||_{L^2(\Omega)}^2 \).

The space \( H^\alpha \) is compactly embedded in the space \( H^b \) if \( \alpha > b \).

Moreover, we consider the Banach spaces \( W^{1,q}(\Omega) (1 \leq q \leq \infty) \) endowed with the norm
\[
||f||_{W^{1,q}(\Omega)}^q = ||f||_{L^q(\Omega)}^q + ||\nabla f||_{L^q(\Omega)}^q
\]
where \( ||\cdot||_{L^q} \) is the \( L^q(\Omega) \)-norm.

Given a separable Banach space \( X \), for \( \alpha > 0 \) and \( p \geq 1 \) we define the Banach space
\[
W^{\alpha,p}(0,T;X) = \left\{ f \in L^p(0,T;X) : \int_0^T \int_0^T \frac{||f(t) - f(s)||_X^p}{|t-s|^{1+\alpha p}} \, dt \, ds < \infty \right\}
\]
and we set
\[
||f||_{W^{\alpha,p}(0,T;X)}^p = \int_0^T ||f(t)||_X^p \, dt + \int_0^T \int_0^T \frac{||f(t) - f(s)||_X^p}{|t-s|^{1+\alpha p}} \, dt \, ds.
\]

Let \((\Omega,F,P)\) be a complete probability space, with expectation denoted by \(\mathbb{E}\). We assume that the stochastic forcing term in (1) is of the form
\[
w = w(t,x) = \sum_{k \in \mathbb{Z}_k^2} \sqrt{q_k} \beta_k(t) \frac{k^\perp}{|k|} e_k(x).
\]

Here \( \{\beta_k\}_{k \in \mathbb{Z}_k^2} \) is a sequence of independent complex-valued standard Brownian motions on \((\Omega,F,P)\), i.e. \( \beta_k(t) = \Re \beta_k(t) + i \Im \beta_k(t) \) with \( \{\Re \beta_k\} \cup \{\Im \beta_k\}_{k \in \mathbb{Z}_k^2} \) a sequence of independent standard real Brownian motions; moreover, we set \( \beta_{-k} = -\overline{\beta_k} \) and \( q_{-k} = q_k \) for any \( k \in \mathbb{Z}_k^2 \).

Therefore
\[
w(t,x) = 2 \sum_{k \in \mathbb{Z}_k^2} \sqrt{q_k} \frac{k^\perp}{|k|} \left[ \Re \beta_k(t) \cos(k \cdot x) - \Im \beta_k(t) \sin(k \cdot x) \right].
\]

In the 2D setting it is convenient to introduce the (scalar) vorticity
\[
\xi = \nabla^\perp \cdot u \equiv \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}.
\]

3
System (1) corresponds to
\[ \frac{d\xi}{dt} + \left[ -\nu \Delta \xi + \gamma \xi + u \cdot \nabla \xi \right] = dw^{\text{curl}} \]
\[ \xi = \nabla \perp \cdot u \]  
(2)

obtained by taking the curl of both sides of the first equation of (1). Periodic boundary conditions have to be added to this system. The noise is \( w^{\text{curl}}(t, x) = -2 \sum_{k \in \mathbb{Z}^2} \sqrt{q_k} |k| |\Im \beta_k(t) \cos(k \cdot x) + \Re \beta_k(t) \sin(k \cdot x)| \). Let us define
\[ Q := \sum_{k \in \mathbb{Z}^2} |k|^2 q_k. \]
(3)

Classical results are
\[ E \| w^{\text{curl}}(t) \|_{L^2}^2 = 2 t Q \quad \forall t \geq 0 \]  
(4)
\[ E \| w^{\text{curl}} \|_{W^{\alpha, p}(0, T; H^0)}^p \leq C(\alpha, p)(T^{1+p/2} + 1)(Q)^{p/2} \]  
(5)
for any \( \alpha \in (0, 1/2) \), \( p \geq 2 \), and the Burkholder–Davies–Gundy inequality
\[ E \left( \sup_{0 \leq t \leq T} \int_0^t \langle |\xi(s)|^{p-2} \xi(s), dw^{\text{curl}}(s) \rangle_{L^2} \right) \leq C(p) \sqrt{Q} E \left( \int_0^T \| \xi(s) \|_{L^2}^{2(p-1)} \right) ds. \]  
(6)

For this latter inequality we have used that \( \sup_{x \in D} |e_k(x)| = 1/(2\pi) \) for all \( k \).

Knowing the vorticity \( \xi \), we recover the velocity \( u \) by solving the elliptic equation
\[ -\Delta u = \nabla \perp \xi. \]  
(7)

This means that if \( \xi(x) = \sum_k \xi_k e_k(x) \), then \( u(x) = -i \sum_k \frac{k}{|k|^2} \xi_k e_k(x) \).

We present basic properties of the bilinear term \( u \cdot \nabla \xi \) in the 2D setting. These are classical results in the analysis of incompressible fluids (see e.g. [34]).

**Lemma 2.1.** There exists a positive constant \( C \) such that
\[ \left| \int_D (u \cdot \nabla) v \cdot \psi \, dx \right| \leq C \| u \|_{L^r} \| v \|_{L^r} \| \psi \|_{H^1} \]  
(8)

for all divergence free vectors with the regularity specified in the r.h.s., and for any \( a > 1 \)
\[ \left| \int_D u \cdot \nabla \xi \, \phi \, dx \right| \leq C \| u \|_{H^a} \| \xi \|_{H^0} \| \phi \|_{H^a}, \]  
(9)
\[ \left| \int_D u \cdot \nabla \xi \, \phi \, dx \right| \leq C \| u \|_{H^a} \| \xi \|_{H^{a \rightarrow 0}} \| \phi \|_{H^0} \]  
(10)

for all functions with the regularity specified in the r.h.s..

**Proof.** The key relationship for (8) is
\[ \int_D [u \cdot \nabla] v \cdot \psi \, dx = - \int_D [u \cdot \nabla] \psi \cdot v \, dx \]
assuming sufficient regularity for \( u, v, \psi \); this is obtained by integrating by parts. Then, we obtain the estimate by Hölder inequality and this is extended by density to vectors with the specified regularity. For (9) we use Hölder inequality and the continuous embedding \( H^a \subset L^\infty(D) \) for \( a > 1 \). Similarly, we obtain the latter estimate. \( \square \)
Lemma 2.2. Let \( \xi = \nabla \cdot u \). We have
\[
\int_D [u \cdot \nabla \xi] \phi \, dx = - \int_D [u \cdot \nabla \phi] \xi \, dx \quad \forall \xi, \phi \in H^1
\] (11)
and for any \( q > 1 \)
\[
q \int_D [u \cdot \nabla \xi] \xi^{q-1} \psi \, dx = - \int_D [u \cdot \nabla \psi] \xi^q \, dx \quad \forall \xi \in L^{2q}, \quad \psi \in H^1.
\] (12)
Moreover,
\[
\int_D [u \cdot \nabla \xi] \xi^{q-1} \, dx = 0 \quad \forall \xi \in L^{2q}.
\] (13)

Proof. The two first relationships (11)–(12) are easily obtained by integrating by parts, where in (12) the proof is done first with smooth functions and then by density it is extended on the spaces specified; note that for \( q > 1 \), if \( \xi \in L^{2q} \) then \( u \in W^{1,2q} \subset L^\infty \) and the r.h.s. of (12) is meaningful (see [31]). Eventually, (13) is the particular case of (12) for \( \psi = 1 \); for a similar result see also [9].

□

3. The stochastic damped Navier–Stokes equations

The well posedness of the stochastic damped 2D Navier–Stokes equations
\[
d\xi^v + [-\nu \Delta \xi^v + u^v \cdot \nabla \xi^v + \gamma \xi^v] \, dt = dw^{\text{curl}},
\xi^v = \nabla \perp \cdot u^v
\] (14)
is very similar to the case when \( \gamma = 0 \). Here, we assume periodic boundary conditions with period box \([-\pi, \pi]^2\).

The proof of existence of a unique solution for square summable initial vorticity is the same as the proof for square summable initial velocity that can be found in [17], where the proof is performed for \( \gamma = 0 \). Similar proofs can also be found in [3, 9] with some uniform estimates with respect to the viscosity \( \nu \). Here, we point out the peculiar estimate (16) for \( \gamma > 0 \), useful in the analysis of the limit as \( \nu \to 0 \).

Theorem 3.1. Let \( \gamma, \nu > 0, \ p \geq 2 \). Assume
\[
\mathbb{E} \| \xi^v(0) \|_{L^p}^p < \infty, \quad Q < \infty.
\]
Then, there exists a process \( \xi^v \) with paths in \( C([0, \infty), L^p) \cap L^2_{\text{loc}}(0, \infty; H^1) \) \( P \)-a.s., which is a Feller Markov process in \( L^p \) and is the unique solution for (14) with initial data \( \xi^v(0) \).
Moreover, there exist two positive constants \( C(p, T) \) and \( C(p) \), independent of \( \nu \), such that
\[
\mathbb{E} \sup_{0 \leq t \leq T} \| \xi^v(t) \|_{L^p}^p \leq C(p, T)
\] (15)
for any finite \( T \), and
\[
\sup_{0 \leq t < \infty} \mathbb{E} \| \xi^v(t) \|_{L^p}^p \leq C(p).
\] (16)
In particular, the constants depend also on \( \gamma, Q, \mathbb{E} \| \xi^v(0) \|_{L^p}^p \).

Proof. The proof of the existence of solutions, which is quite classical requires some Galerkin approximation of \( \xi^v \), say \( \xi^{v,n} \), for which \( a \text{ priori} \) estimates are proved uniformly in \( n \). Using a subsequence of \( \xi^{v,n} \) which converges in the weak or weak-star topologies of appropriate
spaces, one can then prove that there exists a solution to (14). The proof of uniqueness and Feller property is standard and hence omitted.

Let \( v > 0, x \in D \) and \( r \in [0, T] \), Itô formula for \( |\xi^v(t, x)|^p \) gives
\[
d|\xi^v(t, x)|^p = p|\xi^v(t, x)|^{p-2}\xi^v(t, x) \, d\xi^v(t, x) + \frac{1}{2} p(p - 1)|\xi^v(t, x)|^{p-2} Q \, dt
\]
hence
\[
d|\xi^v(t, x)|^p + p|\xi^v(t, x)|^{p-2}\xi^v(t, x)[-v \Delta \xi^v(t, x) + \nabla \cdot \nabla \xi^v(t, x) + \gamma \xi^v(t, x)] \, dt
- p(p - 1)|\xi^v(t, x)|^{p-2} Q \, dt = p|\xi^v(t, x)|^{p-2} \xi^v(t, x) \, dw^{\text{curl}}(t, x)
\]
Integrating on the spatial domain \( D \), using (13) and then Hölder inequality
\[
d\|\xi^v(t)\|_{L^p}^p + p v (p - 1) \| \xi^v(t) \|_{L^p}^{p-2} \| \nabla \xi^v(t) \|_{L^p}^2 \, dt
- Q v (p - 1) \| \xi^v(t) \|_{L^p}^{p-2} \| \xi^v(t) \|_{L^p}^{p-2} \, dt
\]
Integrating over the finite time interval \( (0, s) \) we obtain
\[
\|\xi^v(s)\|_{L^p}^p + p v (p - 1) \| \xi^v(0) \|_{L^p}^{p-2} \| \xi^v(s) \|_{L^p}^2 \| \nabla \xi^v(s) \|_{L^p}^2 \, ds
- Q v (p - 1) \| \xi^v(s) \|_{L^p}^{p-2} \| \xi^v(s) \|_{L^p}^{p-2} \, ds
\]
Taking expectation in (19) and collecting all the estimates we obtain
\[
\frac{1}{2} \mathbb{E} \sup_{0 \leq s \leq T} \| \xi^v(s) \|_{L^p}^p \leq \mathbb{E} \| \xi^v(0) \|_{L^p}^p + Q \mathbb{C}(p) \sup_{0 \leq r \leq s} \| \xi^v(r) \|_{L^p}^{p-2} \, ds
\]
\[
\leq \mathbb{E} \| \xi^v(0) \|_{L^p}^p + \epsilon \int_0^T \mathbb{E} \sup_{0 \leq r \leq s} \| \xi^v(r) \|_{L^p}^p \, ds + C(\epsilon, p, Q) T
\]
for any $\epsilon > 0$, by Young inequality. Using Gronwall lemma we obtain (15). Taking expectation in (18) and using (15), we also obtain that

$$\nu(p - 1)\mathbb{E}\int_0^T \|\xi^\nu(s)\|^2_{H^p} ds + \nu^2 \mathbb{E}\int_0^T \|\xi^\nu(s)\|^2_{L^p} ds \leq C\left(T, Q, E\|\xi^\nu(0)\|^2_{L^p}\right).$$

For $p = 2$ this gives in particular

$$\mathbb{E}\int_0^T \|\nabla\xi^\nu(s)\|^2_{H^0} ds \leq C\left(T, Q, E\|\xi^\nu(0)\|^2_{L^2}\right).$$

Going back to estimate (18) and taking expectation, we have

$$\mathbb{E}\|\xi^\nu(s)\|^p_{L^p} + \nu \mathbb{E}\int_0^s \mathbb{E}\|\xi^\nu(r)\|^p_{L^p} dr \leq \mathbb{E}\|\xi^\nu(0)\|^p_{L^p} + \frac{\nu}{2} \mathbb{E}\int_0^s \mathbb{E}\|\xi^\nu(r)\|^p_{L^p} dr + C(\gamma, p, Q)s. \quad (21)$$

Hence

$$\mathbb{E}\|\xi^\nu(s)\|^p_{L^p} \leq \mathbb{E}\|\xi^\nu(0)\|^p_{L^p} e^{-\gamma ps/2} + \frac{2C(\gamma, p, Q)}{\gamma p} (1 - e^{-\gamma ps/2})$$

for any $s \in [0, \infty)$. This implies (16). \hfill \Box

**Remark 3.2.** The solution $\xi^\nu$ is a process whose paths are a.s. in $C([0, \infty), H^0) \cap L^2_{loc}(0, \infty; H^1)$ at least; therefore it solves system (14) in the following sense: for all $t \in [0, \infty)$ and $\phi \in H^a$ with $a > 1$, we have

$$\int_D \xi^\nu(t, x) \phi(x) dx + v \int_D \int_0^t \nabla\xi^\nu(s, x) \cdot \nabla \phi(x) dx ds$$

$$+ \int_D \int_0^t u^\nu(s, x) \cdot \nabla\xi^\nu(s, x) \phi(x) dx ds + \gamma \int_D \int_0^t \xi^\nu(s, x) \phi(x) dx ds$$

$$= \int_D \xi^\nu(0, x) \phi(x) dx + \int_D \mathcal{w}^\nu^\text{curl}(t, x) \phi(x) dx$$

$P-a.s.$

The trilinear term is well defined thanks to (7) and (9).

Moreover, let us denote by $\xi^\nu(\cdot; \eta)$ the solution with initial data $\eta$ and by $B_b(L^p)$, $C^a_b(L^p)$ the spaces of Borel bounded functions, respectively continuous and bounded functions, $\phi : L^p \rightarrow \mathbb{R}$. To say that the solution is a Feller process in $L^p$ (the $p$ depends on the assumption on the initial vorticity) means that the Markov semigroup $P_t^\nu : B_b(L^p) \rightarrow B_b(L^p)$, defined as

$$(P_t^\nu(\phi)(\eta) = \mathbb{E}\left[\phi(\xi^\nu(t; \eta))\right],$$

actually maps $C^a_b(L^p)$ into itself.

We finally recall what is an invariant measure $\mu^\nu$:

$$\int P_t^\nu \phi d\mu^\nu = \int \phi d\mu^\nu \quad \forall t \geq 0, \phi \in B_b(L^p).$$

The Feller property is important to prove the existence of invariant measures by means of Krylov–Bogoliubov method (see, e.g., [13]).
For any $\gamma > 0$ one can prove existence and uniqueness of the invariant measure for system (14), following the lines of the proofs for the 2D Navier–Stokes equation (the case $\gamma = 0$). Indeed, the Krylov–Bogoliubov method provides a way to prove the existence of an invariant measure; this applies for a wide class of noises. On the other side, uniqueness is a more delicate question. We just recall the best result of uniqueness of the invariant measure, proved by Hairer and Mattingly [23]. They assume that the noise acts on first few modes, i.e.

$$
\exists \mathcal{Z} \text{ finite : } q_k \neq 0 \forall k \in \mathcal{Z}, \quad q_k = 0 \forall k \notin \mathcal{Z}
$$

where $\mathcal{Z}$ has to be chosen in such a way that

- it contains at least two elements with different norms
- the integer linear combinations of elements of $\mathcal{Z}$ generates $\mathbb{Z}^2$.

Actually the kind and the number of forced modes, i.e. the elements of $\mathcal{Z}$, is chosen independently of the viscosity.

We summarize the result.

**Theorem 3.3.** Let $\gamma > 0$ and $2 \leq p < \infty$. If (22) holds, then for any $\nu > 0$ system (14) has a unique invariant measure $\mu^\nu$. Moreover it is ergodic, i.e.

$$
\lim_{T \to \infty} \frac{1}{T} \int_0^T \varphi(\xi^\nu(t)) \, dt = \int \varphi \, d\mu^\nu \quad \text{in } L^2(\Omega)
$$

for any $\varphi \in C_b(L^p)$ and initial vorticity in $L^p$. Finally

$$
\nu(p-1) \left\| |\xi|^{p-2} \nabla \xi \right\|_{L^2}^2 \, d\mu^\nu(\xi) + \gamma \int \left\| |\xi|^{p-2} \nabla \xi \right\|_{L^p}^p \, d\mu^\nu(\xi) = (p-1) Q \int \left\| |\xi|^{p-2} \nabla \xi \right\|_{L^p}^p \, d\mu^\nu(\xi).
$$

The latter equality comes from (17). Note that this invariant measure $\mu^\nu$ is independent of $p$, since the assumption on the noise is independent of $p$ when working on the torus.

**Remark 3.4.**

(i) Working on the torus simplifies the assumption on the covariance of the noise; indeed, in section 2 we used that $2\pi \| e_k \|_{L^\infty}^2 = 1$. On the other hand, when $D$ is a smooth bounded domain in $\mathbb{R}^2$ one can associate with the evolution equations (14) the slip boundary condition coupled with a null vorticity on the boundary. In that case, the assumption on the noise has to be modified as $\sum_{k \in \mathbb{Z}^2} |k|^2 q_k \| e_k \|_{L^\infty}^2 < \infty$.

(ii) For other conditions granting the uniqueness of the invariant measure see e.g. [8, 13, 14, 16, 18, 20, 24, 29, 32]. Anyway, our results hold when the noise is such that the evolution of system (14) is well defined for initial vorticity in $L^p$. In this case, we have that (24) is meaningful.

(iii) Actually theorem 3.1 tells us that for theorem 3.3 to hold it is enough that the initial vorticity is in $L^2$. Indeed, if $\xi^\nu(0) \in L^2$ then for any $t_0 > 0$ we have $\xi^\nu(t_0) \in H^1$; since $H^1 \subset L^p$ for any $2 \leq p < \infty$ we obtain that $\xi^\nu(t_0) \in L^p$, hence the results of theorem 3.1 hold on the time interval $[t_0, \infty]$ for any $p \in [2, \infty]$, when $\xi^\nu(0) \in L^2$. This implies that, as far as the asymptotic behaviour is concerned, the support of the invariant measure $\mu^\nu$ is contained in $\bigcap_{2 \leq p < \infty} L^p$.

Now, we fix the family of the unique invariant measures, as given in theorem 3.3, and consider the limit of vanishing viscosity.
Corollary 3.5. Let $\gamma > 0$. Then the family of invariant measures $\{\mu^\nu\}_{\nu>0}$ is tight in $H^{-s}$ for any $s>0$; in particular there exists a sequence $\{\mu^\nu_n\}$ (with $\nu_n \to 0$ as $n \to \infty$) and a measure $\mu^0$ in $H^{-s}$ such that

$$\mu^\nu \rightharpoonup \mu^0 \quad \text{weakly in } H^{-s}$$

as $n \to \infty$.

Proof. From (24) with $p=2$ we have

$$\int \|\xi\|^2_{H^p} \, d\mu^\nu(\xi) \leq \frac{Q}{\gamma}$$

uniformly in $\nu \in (0, \infty)$. Then, using that $H^0$ is compactly embedded in $H^{-s}$ we obtain tightness by means of the Chebyshev inequality. \hfill \Box

4. The stochastic damped Euler equations

When $\nu = 0$, we deal with the stochastic damped Euler equations

$$d\xi^0 + [u^0 \cdot \nabla \xi^0 + \gamma \xi^0] \, dt = dw^{\text{curl}}$$

$$\xi^0 = \nabla \perp \cdot u^0$$

with periodic boundary conditions, as before. We always consider $\gamma > 0$.

We show an enstrophy balance equation for any solution $\xi^0$ whose paths are in $L^\infty(0, T; L^p)$ a.s., assuming $p \geq 4$.

We can write Itô formula for $\|\xi^0(t)\|^2_{H^p}$, $\xi^0$ being a distributional solution of the damped Euler equations; this has been done in [28] in a particular setting. Krylov’s paper suggests that one can write Itô formula for a distributional solution when all the terms appearing in it make sense. In our case, the fact that the trilinear term disappears comes from (13) with $q = 2$. We use an approximation procedure to prove this result rigorously.

Proposition 4.1. Let $p \geq 4$. We are given a process $\xi^0$ whose paths are a.s. in $L^\infty_{\text{loc}}(0, \infty; L^p)$ and which is a solution of equation (25) in the following sense:

$$\int_0^t \int_D \xi^0(t, x) \phi(x) \, dx - \int_0^t \int_D \xi^0(s, x) u^0(s, x) \cdot \nabla \phi(x) \, dx \, ds + \gamma \int_0^t \int_D \xi^0(s, x) \phi(x) \, dx \, ds$$

$$= \int_0^t \int_D \xi^0(0, x) \phi(x) \, dx + \int_0^t w^{\text{curl}}(t, x) \phi(x) \, dx \quad \text{a.s.}$$

for all $t \in [0, \infty)$ and $\phi \in W^{1, p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Then for any $t > 0$ we have

$$\|\xi^0(t)\|^2_{H^p} + \gamma \int_0^t \|\xi^0(s)\|^2_{H^p} \, ds = \|\xi^0(0)\|^2_{H^p} + t Q + \int_0^t \langle \xi^0(s), dw^{\text{curl}}(s) \rangle$$

a.s.

Proof. We need to approximate the solution $\xi^0$ by smoother solutions. Working pathwise, we can follow the lines of [15].

Let $\varphi$ be a smooth mollifier and $\varphi_\epsilon(x) = \epsilon^{-2} \varphi(\epsilon^{-1} x)$. Setting

$$\xi^0_\epsilon = \varphi_\epsilon \ast \xi^0, \quad \xi^0_\epsilon = \nabla \perp \cdot u^0_\epsilon, \quad w^{\text{curl}}_\epsilon = \varphi_\epsilon \ast w^{\text{curl}}$$

we have that

$$d\xi^0_\epsilon + [u^0_\epsilon \cdot \nabla \xi^0_\epsilon + \gamma \xi^0_\epsilon] \, dt = \nabla \cdot \sigma_\epsilon \, dt + dw^{\text{curl}}_\epsilon$$

(27)
where
\[ \sigma_\epsilon = u_0^0 \xi_\epsilon^0 - (u_0^0 \xi^0) \cdot \varphi_\epsilon. \]

Since \( \xi_\epsilon^0 \) is smoother than \( \xi^0 \), we write Itô formula for \( \| \xi_\epsilon^0(t) \|_{H^0}\), keeping in mind (13) we obtain
\[
\| \xi_\epsilon^0(t) \|_{H^0}^2 + \gamma \int_0^t \| \xi_\epsilon^0(s) \|_{H^0}^2 \, ds = \| \xi_\epsilon^0(0) \|_{H^0}^2 + \int_0^t \langle \nabla \cdot \sigma_\epsilon(s), \xi_\epsilon^0(s) \rangle \, ds + tQ_\epsilon + \int_0^t \langle \xi_\epsilon^0(s), dw^{\text{curl}}_\epsilon(s) \rangle, \quad \text{a.s.} \tag{28}
\]
where \( Q_\epsilon \) is the trace of the covariance operator of \( w_\epsilon \).

The limit as \( \epsilon \to 0 \) provides (26); the crucial point is that (pathwise)
\[
\int_0^t \langle \nabla \cdot \sigma_\epsilon(s), \xi_\epsilon^0(s) \rangle \, ds \to 0. \tag{29}
\]
From lemma 2 and corollary 1 of [15], we have that for any \( p \geq 2 \)
\[
\| \nabla \cdot \sigma_\epsilon(s) \|_{L^p} \leq C \| u_0^0(s) \|_{W^{1,p}} \| \xi^0(s) \|_{L^p} \tag{30}
\]
for a constant \( C \) independent of \( \epsilon \) and \( s \), and
\[
\lim_{\epsilon \to 0} \| \nabla \cdot \sigma_\epsilon(s) \|_{L^{p/2}} = 0. \tag{31}
\]
Hence,
\[
\left| \int_0^t \langle \nabla \cdot \sigma_\epsilon(s), \xi_\epsilon^0(s) \rangle \, ds \right| \leq C \| \xi^0 \|_{L^\infty(0,T;L^p)} \int_0^t \| \nabla \cdot \sigma_\epsilon(s) \|_{L^{p/2}} \, ds
\]
and thanks to (30)–(31) we obtain (29) by the dominated convergence theorem.

Since \( t \mapsto \int_0^t \langle \xi_\epsilon^0(s), dw^{\text{curl}}_\epsilon(s) \rangle \) is a continuous martingale, we can deduce that
\[
\int_0^t \langle \xi_\epsilon^0(s), dw^{\text{curl}}_\epsilon(s) \rangle \to \int_0^t \langle \xi^0(s), dw^{\text{curl}}(s) \rangle \quad \text{a.s.}
\]
and \( Q_\epsilon \to Q \). Thus, the limit of the balance (28) gives (26). \( \square \)

**Corollary 4.2.** Under the assumptions of proposition 4.1, we have that \( \xi^0 \in L^\infty_{\text{loc}}(0,\infty;L^p) \cap C([0,\infty);H^0) \) a.s..

### 5. The vanishing viscosity limit

We are going to prove that the stochastic damped Euler equations (25) have a stationary solution whose marginal at fixed time is a measure \( \mu^0 \) obtained as in corollary 3.5, and that the following balance equation holds:
\[
\gamma \int_0^t \| \xi \|_{H^0}^2 \, d\mu^0(\xi) = Q;
\]
moreover, considering the limit in the balance equation (24) with \( p = 2 \) we prove that
\[
\lim_{\nu \to 0} \nu \int \| \nabla \xi \|_{H^0}^2 \, d\mu^\nu(\xi) = 0.
\]
This means that in the limit of vanishing viscosity, the damped stochastic equations (14) have no dissipation of enstrophy.

However, instead of dealing with invariant measures, we deal with stationary processes (see remark 5.3). Heuristically, we expect that there exists a stationary solution for the stochastic...
damped Euler system (25), due to a balance between the energy injected by the noise term and the dissipation of the damping term. More rigorously, in [5] it has been shown that the damped Euler equation with a multiplicative noise has a stationary solution; there, the crucial estimate (16), that holds for \( \gamma > 0 \) and \( \nu \geq 0 \), was used. The proof is even easier with an additive noise; indeed, estimate (16) on the finite-dimensional approximating Galerkin system gives the existence of an invariant measure by means of Krylov–Bogoliubov technique and we recover the existence of a stationary solution for (25).

Here, we want to investigate the properties for vanishing viscosity; in particular the limit in the balance equation (24) with \( p = 2 \), that is

\[
\nu \int \| \nabla \xi \|_{L^2}^2 \, d\mu^v(\xi) + \gamma \int \| \xi \|_{H^0}^2 \, d\mu^v(\xi) = Q.
\] (32)

Keeping in mind corollary 3.5, we consider the stationary stochastic process \( \xi^v \) whose law at any fixed time is the measure \( \mu^v \) of theorem 3.3, and take the limit of vanishing viscosity. We have

**Proposition 5.1.** Let \( s > 0 \). There exists a sequence \( \{ \xi^v \} \) of stationary processes, solving (14), and converging, as \( \nu \to 0 \), in \( L^2_{loc}(0, \infty; H^{-s}) \cap C([0, \infty); H^{-2-s}) \) (a.s.) to a process, which solves the damped Euler system (25). Moreover, for any finite \( p \geq 2 \) the paths of the limit process belong (a.s.) to \( C([0, \infty); L^p) \cap C(0, \infty; L^p) \) and the limit process is a stationary process in \( L^p \). The marginal at any fixed time of this limit process is a measure \( \mu^0 \), as defined in corollary 3.5.

**Proof.** The proof is based on two steps: first we show that the sequence of the laws of \( \xi^v \), \( v > 0 \), is tight; then we pass to the limit in a suitable way and obtain that the limit process is a weak solution of system (25). Note that we find a weak solution to system (25) (in the probabilistic sense), whereas system (14) has a unique strong solution.

Actually, the tightness and the convergence of the stationary processes have already been done in [5] for the damped Navier–Stokes equations with a multiplicative noise; but there the analysis involved the velocity instead of the vorticity. For the reader’s convenience we recall the basic steps of the proof; the details can be found in [3, 5].

Writing equation (14) in the integral form

\[
\xi^v(t) = \xi^v(0) + \nu \int_0^t \Delta \xi^v(s) \, ds - \int_0^t u^v(s) \cdot \nabla \xi^v(s) \, ds - \gamma \int_0^t \xi^v(s) \, ds + w^{\text{curl}}(t),
\]

by usual estimations and bearing in mind estimate

\[
\sup_{0 \leq t < \infty} E\| \xi^v(t) \|_{L^1}^2 \leq C(4)
\]

from theorem 3.1 (so we estimate \( \sup_{0 \leq t < \infty} E\| \xi^v(t) \|_{L^2}^2 \)), one obtains that there exist constants \( C \) and \( C(p) \) such that

\[
E\| \int_0^t \Delta \xi^v(s) \, ds \|_{W^{1,2}(0,T;W^{-1})}^2 \leq C
\]

\[
E\| \int_0^t u^v(s) \cdot \nabla \xi^v(s) \, ds \|_{W^{1,2}(0,T;H^{-1})}^2 \leq C \quad \text{by (11) and (10)}
\]

\[
E\| \xi^v(s) \, ds \|_{W^{1,2}(0,T;H^s)}^2 \leq C
\]

\[
E\| u^{\text{curl}} \|_{L^p(0,T;H^s)}^p \leq C(\nu) \quad \text{by (3)}
\]
for some (and all) \( s > 0, \alpha \in (0, \frac{1}{2}) \) and \( p \geq 2 \). Therefore
\[
\sup_{\nu \in (0,1)} E \| x_\nu \|^2_{W^{s+2}(0,T;H^{-2-s})} < \infty.
\]
On the other hand, we already know from theorem 3.1 that
\[
\sup_{\nu > 0} E \| x_\nu \|^2_{L^2(0,T;H^\alpha)} < \infty.
\]
Using that the space \( L^2(0,T;H^0) \cap W^{a,2}(0,T;H^{-2-s}) \) is compactly embedded in \( L^2(0,T;H^{-s}) \) (see, e.g., [19]), it follows that the sequence of laws of processes \( x_\nu \) (\( 0 < \nu < 1 \)) is tight in \( L^2(0,T;H^\alpha) \). On the other hand, using that both the spaces \( W^{a,2}(0,T;H^{-2-s}) \) and \( W^{\alpha,p}(0,T;H^{-2-s}) \) with \( \alpha p > 1 \) are compactly embedded in \( C([0,T];H^{-2-2s}) \), we obtain tightness in \( C([0,T];H^{-2-2s}) \).

Let us endow \( L^2_{loc}(0,\infty;H^{-s}) \) by the distance
\[
d_2(\xi, \zeta) = \sum_{n=1}^{\infty} 2^{-n} \min(\| \xi - \zeta \|_{L^2(0,\nu;H^{-s})}, 1)
\]
and \( C([0,\infty);H^{-2-2s}) \) by the distance
\[
d_\infty(\xi, \zeta) = \sum_{n=1}^{\infty} 2^{-n} \min(\| \xi - \zeta \|_{C([0,\nu];H^{-2-2s})}, 1).
\]
We have that the sequence \( \{x_\nu\} \) is tight in \( L^2_{loc}(0,\infty;H^{-s}) \cap C([0,\infty);H^{-2-2s}) \).

From Prokhorov and Skorohod theorems follows that there exists a basis \((\tilde{\xi}, \tilde{\zeta}, \tilde{P})\) on this basis, \( L^2_{loc}(0,\infty;H^{-s}) \cap C([0,\infty);H^{-2-2s}) \)-valued random variables \( \tilde{\xi}_0, \tilde{\xi}_\nu \) such that \( \mathcal{L}(\tilde{\xi}_\nu) = \mathcal{L}(\tilde{\xi}_0) \) on \( L^2_{loc}(0,\infty;H^{-s}) \cap C([0,\infty);H^{-2-2s}) \), and
\[
\lim_{n \to \infty} \tilde{\xi}_{\nu_n} = \tilde{\xi}_0 \quad \text{in} \quad L^2_{loc}(0,\infty;H^{-s}) \cap C([0,\infty);H^{-2-2s}), \tilde{P}-a.s.
\]
for a subsequence with \( \lim_{n \to \infty} \nu_n = 0 \).

The fact that the process \( \tilde{\xi}_0 \) solves system (25) is classical. Indeed, considering \( s = \frac{1}{2} \) we have that \( \tilde{\xi}_0 \to \xi_0 \) in \( L^2_{loc}(0,\infty;H^{-1/2}) \); this means, according to (7), that \( \tilde{u}_0 \to \tilde{\xi}_0 \) in \( L^2_{loc}(0,\infty;H^{1/2}) \). Since \( H^{1/2}(D) \subset L^4(D) \), we obtain by estimates similar to (8) that the quadratic term \( [\tilde{u}_0 \cdot \nabla] \tilde{u}_0 \) converges weakly to \( [\tilde{u}_0 \cdot \nabla] \tilde{u}_0 \), i.e.
\[
\int_D \int_0^t [\tilde{u}_0 \cdot \nabla] \tilde{u}_0 \cdot \psi \ dx \, dt \to \int_D \int_0^t [\tilde{u}_0 \cdot \nabla] \tilde{u}_0 \cdot \psi \ dx \quad \tilde{P}-a.s.
\]
for all \( t \) finite and \( \psi \in [H^3]^2 \). For this it is enough to write
\[
\int_D [([\tilde{u}_0 \cdot \nabla] \tilde{u}_0 \cdot \psi - [\tilde{u}_0 \cdot \nabla] \tilde{u}_0 \cdot \psi)] \ dx
\]
\[
= \int_D [([\tilde{u}_0 - \bar{u}_0] \cdot \nabla] \tilde{u}_0 \cdot \psi \ dx + \int_D [\tilde{u}_0 \cdot \nabla] (\tilde{u}_0 - \bar{u}_0) \cdot \psi \ dx.
\]
In addition, \( \tilde{\xi}_0 \) and \( \tilde{\xi}_\nu \) have the same law; then \( \tilde{\xi}_\nu \) is a stationary process. By the convergence \( \tilde{P}-a.s. \) in \( C([0,\infty);H^{-2-2s}) \) we obtain that also \( \tilde{\xi}_0 \) is a stationary process in \( H^{-2-2s} \).

Finally, from (15) we have that for \( 2 \leq p < \infty \)
\[
\tilde{\xi}_0 \in L^\infty_{loc}(0,\infty;L^p) \quad \tilde{P}-a.s.
\]
Then, for \( T < \infty \) almost each path \( \tilde{\xi}_0 \in C([0,T];H^{-2-2s}) \) \( \cap L^\infty(0,T;L^p) \); thus it is weakly continuous in \( L^p \), i.e. we have for any \( \phi \in L^p \) (\( \frac{1}{p} + \frac{1}{p} = 1 \))
\[
\lim_{t \to 0} \int_D \tilde{\xi}_0(t) \phi \ dx = \int_D \tilde{\xi}_0(0) \phi \ dx \quad \tilde{P}-a.s.
\]
and for any $t \in [0, T]$
$$\|\tilde{\xi}^0(t)\|_{L^p} \leq \|\tilde{\xi}^0\|_{L^\infty(0,T;L^p)} \quad \tilde{P}\text{-a.s.}$$
(see [34, p 263]).

Hence, for every $t \in [0, T]$, the mapping $\tilde{\omega} \mapsto \tilde{\xi}^0(t, \tilde{\omega})$ is well defined from $\tilde{\Omega}$ to $L^p$ and it is weakly measurable. Since $L^p$ is a separable Banach space, it is strongly measurable (see [35, p 131]). Therefore, it is meaningful to speak about the law of $\tilde{\xi}^0(t)$ in $L^p$. The stationarity of $\tilde{\xi}^0$ in $L^p$ has to be understood in this sense.

By taking suitable subsequences we have that $\mu^0$ is the law of $\tilde{\xi}^0(t)$ for any time $t$. □

Let us denote by $\tilde{\xi}^0$ the stationary process solving (25), as given in proposition 5.1. We have

**Proposition 5.2.** For any time $t$
$$\gamma \tilde{E}\|\tilde{\xi}^0(t)\|^2_{H^0} = Q. \quad (34)$$

**Proof.** We have that for any $p \geq 2$ the paths of the process $\tilde{\xi}^0$ are in $L^\infty_{\text{loc}}(0, \infty; L^p)$ a.s. Therefore the balance equation (26) holds; taking expectation and using stationarity we obtain (34). □

Equation (34) can be rewritten as
$$\gamma \int \|\xi\|^2_{H^0} d\mu^0(\xi) = Q.$$

**Remark 5.3.** At this point, we are not able to prove that $\mu^0$ is an invariant measure for system (25). In fact, the transition semigroup associated with (25) can not be defined in $H^0$: existence of a solution holds for initial vorticity in $H^0$ but uniqueness requires stronger assumptions (see [4, 6]). But to obtain the Feller and Markov properties in a space smaller than $H^0$ is not trivial.

Now we have our main result

**Theorem 5.4.** For any $\gamma > 0$, we have
$$\lim_{\nu \to 0} \nu \int \|\nabla \xi\|^2_{H^0} d\mu^\nu(\xi) = 0. \quad (35)$$

**Proof.** Let us write the balance equation (32) in terms of the stationary process $\xi^\nu$, at any fixed time $t$:
$$\nu E\|\nabla \xi^\nu(t)\|^2_{H^0} + \gamma \tilde{E}\|\xi^\nu(t)\|^2_{H^0} = Q. \quad (36)$$
We consider the weak limit as in proposition 5.1. Choosing $p = 4$ in (16) of theorem 3.1 we have
$$\tilde{E}\|\xi^\nu(t)\|^4_{L^4} \leq C(4).$$
This bound implies
$$\xi^\nu(t) \rightharpoonup \tilde{\xi}^0(t) \quad \text{weakly in } L^4(\tilde{\Omega} \times D);$$
for the limit we have
$$\tilde{E}\|\tilde{\xi}^0(t)\|^4_{L^4} \leq \liminf_{\nu \to 0} \tilde{E}\|\xi^\nu(t)\|^4_{L^4} \leq C(4). \quad (37)$$
Then
\[
\limsup_{\nu \to 0} \nu \mathbb{E} \| \nabla \xi^\nu(t) \|_{H^0}^2 = Q - \gamma \liminf_{\nu \to 0} \mathbb{E} \| \xi^\nu(t) \|_{H^0}^2 \leq Q - \gamma \mathbb{E} \| \xi^0(t) \|_{H^0}^2
\]
by (36)
\[
= 0 \quad \text{by (34).} \tag{38}
\]
This gives (35). □

From this result we obtain the convergence of the mean enstrophy.

**Corollary 5.5.** For any \( \gamma > 0 \), we have
\[
\lim_{\nu \to 0} \int \| \xi \|_{H^0}^2 \, d\mu^\nu(\xi) = \int \| \xi \|_{H^0}^2 \, d\mu^0(\xi). \tag{39}
\]

**Proof.** We consider the limit as \( \nu \to 0 \) in (32); then use (35) and (34). □

**Remark 5.6.** All the results proved for the enstrophy \( \xi \) can be repeated and hence hold for the velocity \( u \); norms of one order less of regularity are involved and therefore the proofs are even easier. This means in particular that for the stochastic damped 2D Navier–Stokes equations, there is no anomalous dissipation of energy as \( \nu \to 0 \) and energy balance equation holds for \( \nu > 0 \) and also \( \nu = 0 \).

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