AN EXAMPLE OF A NON-AMENABLE DYNAMICAL SYSTEM WHICH IS BOUNDARY AMENABLE

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ABSTRACT. It is shown that the action of $\text{SL}(3, \mathbb{Z})$ on the Stone-Čech boundary of $\text{SL}(3, \mathbb{Z})/\text{SL}(2, \mathbb{Z})$ is amenable. This confirms a prediction by Bekka and Kalantar.

1. A DYNAMICAL PRESENTATION OF $\text{SL}(3, \mathbb{Z})/\text{SL}(2, \mathbb{Z})$ AND BOUNDARY AMENABILITY

Bekka and Kalantar discovered in [3] a deep relationship between properties of a subgroup $H$ of a discrete group $G$ and the ideal structure of the image of the full group $C^*$-algebra of $G$ under the corresponding quasi-regular representation $\lambda_{G/H}$. In particular they isolated two classes of subgroups (among others), namely the class $\text{Sub}_{\text{s}}(G)$ of subgroups with the spectral gap property, for which $\lambda_{G/H}(C^*(G))$ contains a minimal (non-trivial, closed) ideal and the class $\text{Sub}_{\text{w-par}}(G)$ of weakly parabolic subgroups, for which $\lambda_{G/H}(C^*(G))$ contains a maximal ideal. In [3] Remark 7.6 the authors speculate about the existence of a group $G$ and a subgroup $H \in \text{Sub}_{\text{s}}(G) \cap \text{Sub}_{\text{w-par}}(G)$ for which the two ideals coincide.

In the following we show that an example of such a situation is provided by the copy of $\text{SL}(2, \mathbb{Z})$ inside $\text{SL}(3, \mathbb{Z})$ given by

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & \text{SL}(2, \mathbb{Z})
\end{pmatrix}.
$$

The proof relies on a dynamical presentation of $\text{SL}(3, \mathbb{Z})/\text{SL}(2, \mathbb{Z})$ as an orbit of the diagonal action of $\text{SL}(3, \mathbb{Z})$ on itself. The boundary of this orbit sits in a suitable boundary piece introduced in [4], from which it follows that this action is boundary amenable. Hence the $C^*$-simplicity of $\text{SL}(3, \mathbb{Z})$ gives the desired example.

We retain the notation of [3], [4] and refer to [1], [5] for the definition of amenable action and related topics.

If $Z$ is a discrete countable space, we denote by $\Delta_\beta Z$ its Stone-Čech compactification and by $\partial_\beta Z = \Delta_\beta Z \setminus Z$ its Stone-Čech boundary. Recall that $\Delta_\beta Z$ is the set of ultrafilters on $Z$ with the topology generated by the basis of open sets of the form $U_E := \{ \omega \in \Delta_\beta Z \mid E \in \omega \}$, for $E \subset Z$; the Stone-Čech boundary $\partial_\beta Z$ is the set of free ultrafilters. Every function $f$ from $Z$ to a compact Hausdorff space admits a unique continuous extension to the Stone-Čech compactification given by $f(\omega) = \lim_{z \to \omega} f(z)$. In particular group actions on $Z$ extend to $\Delta_\beta Z$.

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The following is an application of the results contained in [3]. We recall that a group $G$ is $C^*$-simple if its reduced group $C^*$-algebra $C^*_r(G)$ is simple.

**Proposition 1.1.** Let $G$ be a discrete countable group and $H \in \text{Sub}_{sg}(G)$. If $G$ is $C^*$-simple and the action of $G$ on the Stone-Čech boundary of $G/H$ is amenable, then $\lambda_{G/H}(C^*(G))$ contains the compact operators as unique non-trivial closed ideal.

**Proof.** In virtue of [3] Theorem 7.4, the ideal $I_{min}$ does not contain non-trivial ideals and $I_{min}$ is the only ideal in $\lambda_{G/H}(C^*(G))$ since $G$ is $C^*$-simple. In particular $C^*(\lambda_{G/H}(G))/I_{min}$ contains the compact operators as unique non-trivial closed ideal. \hfill \Box

In order to apply the above result for the case at hand, we proceed in showing amenability of the action of $\text{SL}(3, \mathbb{Z})$ on the Stone-Čech boundary of $\text{SL}(3, \mathbb{Z})/\text{SL}(2, \mathbb{Z})$.

Let $\Gamma$ be a discrete countable group acting on a compact Hausdorff space $X$. We denote by $\mathcal{P}(\Gamma)$ the set of probability measures on $\Gamma$ endowed with the action of $\Gamma$ given by $g\mu = \mu \circ g^{-1}$. We recall from [1] Definition 5.1 that an approximate invariant continuous mean for the action is a net $\mu_\lambda : X \to \mathcal{P}(\Gamma)$ satisfying $\lim_{\lambda} \sup_{x \in X} \|g\mu_\lambda(x) - \mu_\lambda(gx)\|_1 = 0$ for every $g \in \Gamma$, where $\| \cdot \|_1$ denotes the $l^1$-norm. The action is called amenable if it admits an approximate invariant continuous mean. If a set $Z$ is endowed with an action of $\Gamma \times \Gamma$, we will refer to the diagonal action of $\Gamma$ as to the action of $\Gamma$ on $Z$ given by $(\gamma, z) \mapsto (\gamma, \gamma) \cdot z$.

**Lemma 1.2.** Let $\Gamma$ be a discrete countable group, $X$ a compact $\Gamma \times \Gamma$-space and $Y$ a compact $\Gamma$-space. Suppose the $\Gamma$-action on $Y$ is amenable and there is a continuous map $\phi : X \to Y$ such that $\phi((g, h)x) = g\phi(x)$ for every $(g, h) \in \Gamma \times \Gamma$ and every $x \in X$. Then the diagonal action of $\Gamma$ on $X$ is amenable.

**Proof.** Let $\mu_\lambda : Y \to \mathcal{P}(\Gamma)$ be an approximate invariant continuous mean. Then $\text{diag}(\mu_\lambda \circ \phi)$ is an approximate invariant continuous mean for the diagonal action of $\Gamma$ on $X$, where $\text{diag} : \mathcal{P}(\Gamma) \to \mathcal{P}(\text{diag}(\Gamma))$ is given by $\text{diag}(\mu)(g, g) = \mu(g)$. \hfill \Box

**Theorem 1.3.** The action of $\text{SL}(3, \mathbb{Z})$ on the Stone-Čech boundary of $\text{SL}(3, \mathbb{Z})/\text{SL}(2, \mathbb{Z})$ is amenable.

**Proof.** Consider the diagonal action of $\text{SL}(3, \mathbb{Z})$ on itself, which is given by $(g, x) \mapsto gxg^{-1}$ for $g, x \in \text{SL}(3, \mathbb{Z})$. The stabilizer of the point $z_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ is a diagonal copy of $\text{SL}(2, \mathbb{Z})$. Hence the orbit $Z$ of $z_0$ is a dynamical system isomorphic to $\text{SL}(3, \mathbb{Z})/\text{SL}(2, \mathbb{Z})$ and we can identify the Stone-Čech boundary of $\text{SL}(3, \mathbb{Z})/\text{SL}(2, \mathbb{Z})$ with the closed subset of $\partial_{\beta}\text{SL}(3, \mathbb{Z})$ given by $\partial_{\beta}Z = \{ \omega \in \partial_{\beta}\text{SL}(3, \mathbb{Z}) \mid F \cap Z \neq \emptyset \ \forall F \in \omega \}$. Let $M_3(\mathbb{R})_{\leq 1}$ be the compact set of $3 \times 3$ real matrices of norm at most $1$ and $\phi : \text{SL}(3, \mathbb{R}) \to M_3(\mathbb{R})_{\leq 1}$ be given by $\phi(g) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.
Every $z \in Z$ satisfies $z = z^{-1}$ (since $z_0^{-1} = z_0$), it follows that for every diverging sequence $\{z_n\} \subset Z$ for which $\phi(z_n)$ converges we have $(\lim_n \phi(z_n))^2 = \lim_n \phi(\phi(z_n)) = 0$. Let $\omega \in \partial_2(Z) \subset \partial_3 SL(3, Z)$. For every $g \in SL(3, Z)$ let $s_i(g), i = 1, 2, 3$ be the eigenvalues of $\sqrt{g^*g}$, ordered in such a way that $s_1(g) \geq s_2(g) \geq s_3(g)$ (cf. [4] Lemma 3.10). Consider the functions $f_{1,2} : g \mapsto s_1(g)/s_2(g)$ and $f_{2,3} : g \mapsto s_2(g)/s_3(g)$. We keep the same notation for their extensions to functions from $\Delta_3(SL(3, Z))$ to $[0, \infty]$. Suppose that $f_{1,2}(\omega) = c < \infty$. Then there is a sequence $(g_n)_n$ in $SL(3, Z)$ with $\lim_n f_{1,2}(g_n) = c$ and $\lim_n \phi(g_n)^2 = 0$. Following [4] Lemma 6.1 for every $n \in \mathbb{N}$ let $s_i := s_i(g_n)$ for $i \in \{1, 2, 3\}$ and let $s_n$ be the diagonal matrix with entries $(s_n)_{i,i} = s_{i,n}$; we can write

$$g_n = a_n s_n b_n = a_n \begin{pmatrix} s_{1,n} & 0 & 0 \\ 0 & s_{2,n} & 0 \\ 0 & 0 & s_{3,n} \end{pmatrix} b_n,$$

where $a_n, b_n \in SO(3, \mathbb{R})$. Up to taking a subsequence, we can suppose that $a_n \to a$, $b_n \to b$ in $SO(3, \mathbb{R})$, thus also $\lim_n \phi(s_n)$ exists and

$$\lim_n \phi(a_n s_n b_n) = \lim_n a_n \phi(s_n) b_n = a \lim_n \phi(s_n) b.$$

The condition $\lim_n f_{1,2}(g_n) = c$ entails

$$\text{rank}(\phi(\omega)) = \text{rank}(\lim_n s_n/\|s_n\|)) = 2,$$

but then we should have $(\lim_n \phi(g_n))^2 \neq 0$, which is impossible. In the same way, if $f_{2,3}(\omega) = c < \infty$, then there is a sequence $g_n$ in $SL(3, Z)$ such that $\phi(\omega) = \lim_n \phi(g_n)$ and $f_{2,3}(\omega) = \lim_n f_{2,3}(g_n)$. But then $\lim_n \phi(g_n^{-1})$ has rank 2, which again is impossible. Hence $\partial_3 Z \subset \partial_0 SL(3, Z)$ in virtue of [4] Lemma 6.1, where $\partial_0 SL(3, Z)$ is the boundary piece associate to the subgroup $P_0 \subset SL(3, \mathbb{R})$ of upper triangular matrices. Since the action of $SL(3, Z)$ on $SL(3, \mathbb{R})/P_0$ is amenable, it follows from [4] Lemma 3.10 that the left action of $SL(3, Z)$ on $C(\partial_0 SL(3, Z))^{SL(3, Z)}$ is amenable; it follows from Lemma 1.2 that the diagonal action of $SL(3, Z)$ on $\partial_0 SL(3, Z)$ is amenable (the action of $SL(3, Z) \times SL(3, Z)$ on both $\partial_0 SL(3, Z)$ and $\partial_3 SL(3, Z)$ is given by $(g_1, g_2) \cdot \omega = g_1 \omega g_2^{-1}$). Hence the diagonal action of $SL(3, Z)$ on $\partial_3 Z$ is amenable. □

Corollary 1.4. The $C^*$-algebra $\lambda_{SL(3, Z)/SL(2, Z)}(C^*(SL(3, Z)))$ contains the compact operators as unique non-trivial closed ideal.

Proof. In [3] Example 7.7 it is shown that $SL(2, Z) \subset Sub_{sg}(SL(3, Z)) \cap Sub_{\omega-par}(SL(3, Z))$. The $C^*$-simplicity of $SL(3, Z)$ is proved in [2]. Hence the result follows from an application of Theorem 1.3 and Proposition 1.1. □

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