Filamentation instability in a quantum plasma

A. Bret

ETSI Industriales, Universidad de Castilla-La Mancha, 13071 Ciudad Real, Spain

(Dated: February 1, 2008)

The growth rate of the filamentation instability triggered when a diluted cold electron beam passes through a cold plasma is evaluated using the quantum hydrodynamic equations. Compared with a cold fluid model, quantum effects reduce both the unstable wave vector domain and the maximum growth rate. Stabilization of large wave vector modes is always achieved, but significant reduction of the maximum growth rate depends on a dimensionless parameter that is provided. Although calculations are extended to the relativistic regime, they are mostly relevant to the non-relativistic one.

I. INTRODUCTION

There is currently a growing interest in quantum plasmas in connection with many areas of physics like small electronic devices, astrophysics or laser plasma interaction (see Refs. [1, 2] and references therein). As far as plasma instabilities are concerned, the quantum theory of the two-stream instability has been elaborated in Refs. [3, 4, 5, 6] while quantum effects on the filamentation instability are still to evaluate. The purpose of this work is to deal with this issue which could be relevant for some astrophysical settings, or the Fast Ignition Scenario for Inertial Confinement Fusion [7] where a relativistic electron beam is supposed to reach the partially degenerate dense core of a pre-compressed target.

We thus consider an infinite and homogenous cold relativistic electron beam of velocity $V_b$, relativistic factor $\gamma_b$ and density $n_b$ entering a cold plasma of electronic density $n_p$. Ions form a fixed neutralizing background of density $n_b + n_p$ and the beam prompts a return current in the plasma with velocity $V_p$ such as $n_p V_p = n_b V_b$. In a first attempt to quantify quantum effects for the filamentation instability, we implement the quantum fluid formalism relying on the conservation equations for the beam ($j = b$) and the plasma ($j = p$),

$$\frac{\partial n_j}{\partial t} + \nabla \cdot (n_j v_j) = 0$$

and the force equation with a Bohm potential term [2],

$$\frac{\partial p_j}{\partial t} + (v_j \cdot \nabla) p_j = -q \left( \frac{E + v_j \times B}{e} \right) + \frac{\hbar^2}{2m} \nabla \left( \frac{\nabla^2 \sqrt{n_j}}{\sqrt{n_j}} \right),$$

where $q > 0$ and $m$ are the charge and mass of the electron, $n_j$ the density of species $j$, and $p_j$ its momentum. Quantum corrections are clearly contained within the Bohm potential, which can be obtained from the moments of the non-relativistic Wigner function [2]. The present calculations are here extended to the relativistic regime, but a correct treatment of quantum effects in that limit should rely on the moments of a relativistic Wigner function such as the one described in Refs. [8, 9]. Nevertheless the modified force equation [2] has to our knowledge no simple relativistic counterpart yet, and present relativistic results make sense physically. This is why we found relevant to deal with the relativistic regime with the present equations. Although the following theory is only correct in the non-relativistic case, it could also be interesting to compare future relativistic results with the present ones.

Assuming quantities perturbed according to $\exp(ik \cdot r - i\omega t)$, we find the linearized equations,

$$n_{j1} = n_{j0} \frac{k \cdot v_{j1}}{\omega - k \cdot v_{j0}},$$

and

$$im\gamma_j (k \cdot v_{j0} - \omega) \left( v_{j1} + \frac{\gamma_j^2}{c^2} (v_{j0} \cdot v_{j1}) v_{j0} \right) = -q \left( \frac{E_1 + v_{j0} \times B_1}{e} \right) - i\frac{\hbar \gamma_j^2}{4m n_j} k,$$

Assuming quantities perturbed according to $\exp(ik \cdot r - i\omega t)$, we find the linearized equations,
where subscript 0 and 1 stand for the equilibrium and perturbed quantities, and $\gamma_j$ for the relativistic factor of the unperturbed beam and return current. Note that the diluted beam hypothesis $n_b \ll n_p$ implies a non-relativistic return current with $\gamma_p = 1$.

We now choose to align the beam velocity along the $z$ axis and $k$ along the $x$ axis, and calculate the dispersion equation following a quite standard procedure [10]. Expressing the current $J$ in terms of the electric field $E$ and inserting it into a combination of Maxwell’s equations,

$$\frac{\epsilon^2}{\omega^2} \mathbf{k} \times (\mathbf{k} \times \mathbf{E}) + \mathbf{E} + \frac{4i\pi}{\omega} \mathbf{J} = 0,$$

(5)

allows for the derivation of the dielectric tensor. Its determinant finally gives the dispersion equation

$$x^4 Z^2 \alpha^2 (\gamma_b - 1)^2 \left( x^2 - x^2 \gamma_b - \Theta Z^4 \right)^2 = \left[ 1 + \frac{1}{\Theta Z^4 - x^2} + \frac{\alpha}{\Theta Z^4 - x^2 \gamma_b} \right] \times \left[ x^2 - 1 - \frac{x}{\gamma_b^3} - \frac{Z^2}{\beta^2} - \frac{\alpha^2 Z^2}{\Theta Z^4 - x^2} + \frac{\alpha Z^2}{\Theta Z^4 - x^2 \gamma_b} \right],$$

(6)

in terms of,

$$x = \frac{\omega}{\omega_p}, \quad Z = \frac{kV_b}{\omega_p}, \quad \beta = \frac{V_b}{c}, \quad \alpha = \frac{n_b}{n_p},$$

(7)

where $\omega_p$ is the electronic plasma frequency, and

$$\Theta = \left( \frac{\hbar \omega_p}{2mc^2} \right)^2 \equiv \Theta_c \beta^4,$$

(8)

with $\Theta_c = (\hbar \omega_p/2mc^2)^2$. Although this quantity remains small even in a very dense plasma with $\Theta = 1, 3.10^{-7}$ assuming $n_p = 10^{26}$ cm$^{-3}$ and $V_b \sim c$, we will check in the sequel that quantum effects may not be negligible for the filamentation instability.

II. QUANTUM EFFECTS ANALYSIS

Let us start plotting on Figure 11 the growth rate in terms of $Z$ for a diluted beam passing through a dense plasma in the non-relativistic and relativistic regimes. We make the comparison between the calculations for $\Theta_c = 0$ and $\Theta_c = 1, 3.10^{-7}$. In both cases, it is straightforward that although the maximum growth rate is not modified (and we will see why later), quantum effects introduce a cut off at large $Z$ whereas the growth rate just saturates in the classical limit $\Theta = 0$.

The cut off value of $Z$ can be calculated exactly in terms of $\Theta$ noting that a vanishing growth rate implies that Eq. (6) be verified with $x = 0$ since the real part of the root is 0 for the filamentation instability. The resulting expression can be solved exactly in terms of $Z$ and gives the cut off wave vector $Z_m$,

$$Z_m^2 = \frac{\beta^2 (\alpha + \gamma_b^3)}{2\gamma_b^3} \left( \frac{4\alpha (1+\alpha) \gamma_b^6}{\Theta^2 \beta^2 (\alpha + \gamma_b^3)^2} - 1 \right).$$

(9)

Accounting for $\alpha \ll 1$, we find

$$Z_m^2 \sim \frac{\beta^2}{2} \left( \frac{4\alpha}{\Theta \beta^2} - 1 \right).$$

(10)

It turns out that the parameters $4\alpha/\Theta \beta^2$ measures the strength of quantum effects as far as $Z_m$ is concerned. For the sake of physical interpretation, let us consider the most common case where this parameter is large with respect to 1, and set $\beta \sim 1$. Equation (10) can be developed and yields

$$K_m \sim \sqrt{\frac{2mc\hbar}{\hbar}},$$

(11)
FIG. 1: Growth rate $\delta$ in $w_p$ units of the filamentation instability for a diluted non-relativistic (upper graph) and relativistic (lower graph) beam. The plasma density in both cases is $n_p = 10^{26}$ cm$^{-3}$ yielding $\Theta_c = 1.3 \times 10^{-7}$. The range of unstable wave vectors is dramatically reduced whereas the most unstable mode and its growth rate are only weakly affected. Parameter $\Omega$ given by Eq. (11) is here 307 and 439 respectively.

which is the inverse of the beam quantum wavelength. Simply put, perturbations which scale length is smaller than the beam quantum wavelength are relaxed and do not grow. Also remarkable, the $\gamma_b$ dependence is lost between Eqs. [9] and [10] because of the diluted beam hypothesis, so that it can be said that to a large extent, quantum corrections to the instability domain in the relativistic regime do not depend on the beam energy as long as they are described with the correction term inserted in Eq. (2).

While quantum effects on $Z_m$ are necessarily important since $Z_m = \infty$ in the classical case, effects on the largest growth rate are even more relevant for the physics of the system. If we recall that in the classical limit, the growth rate $\delta$ saturates for $Z \gtrsim \beta$ with $\delta \sim \beta \sqrt{\alpha/\gamma_b}$ [11], the occurrences of $\Theta$ in dispersion equation [6] indicate that quantum effects near the most unstable wave vectors are negligible if

$$\Theta Z^4 |_{Z \sim \beta} \ll \frac{x^2}{x \sim \beta \sqrt{\alpha/\gamma_b}},$$

$$\Theta Z^4 |_{Z \sim \beta} \ll \frac{\gamma_b x^2}{x \sim \beta \sqrt{\alpha/\gamma_b}}. \quad \text{(12)}$$

These two conditions are clearly met if the first one is satisfied,

$$\Theta \beta^4 \ll \beta^2 \frac{\alpha}{\gamma_b} \ll \frac{\alpha}{\Theta \gamma_b \beta^2} \gg 1. \quad \text{(13)}$$

We here recover the $\alpha/\Theta \beta^2$ parameter already evidenced when dealing with the largest unstable wave vector, but we find it must dominate the beam relativistic factor if quantum effects are to be negligible with respect to the most unstable mode. Therefore, depending on the inequality [13], we will find situations where quantum effects stabilize
the instability at large $Z$ while changing, or not, the most unstable mode. The parameter

$$\Omega = \frac{\alpha}{\Theta \gamma_b \beta^2} = \frac{\alpha \beta^2}{\Theta c \gamma_b},$$  \hspace{1cm} (14)$$

is thus the most relevant one from the physical point of view because a change in the fastest growing mode brings more physical consequences than the stabilization of the non-dominant large $Z$ modes. $\Omega \gg 1$ implies few quantum effects for the most unstable mode whereas $\Omega \ll 1$ means strong quantum effects.

The physical interpretation of $\Omega$ can be elaborated after the one we used for $Z_m$ in Eqs. (10,11). We found that wavelengths larger than the beam quantum wavelength are stabilized. Let us now construct the ratio of $k_m$ as given by Eq. (11) with the plasma skin depth $\omega_p/c$ wave number, where the instability reaches its maximum in the classical regime. We take the fourth power of this quantity and consider the relativistic beam regime and find,

$$\left( \frac{k_m}{\omega_p/c} \right)^4 = \frac{n_p}{n_b} \left( \frac{2mc^2}{\hbar \omega_p} \right)^2 = \frac{\alpha}{\Theta c},$$  \hspace{1cm} (15)$$

We thus check that once added a relativistic contraction factor $1/\gamma_b$, the parameter $\Omega$ can be constructed from the ratio of the plasma skin depth $\lambda_s$ to the beam quantum wavelength $k_m^{-1}$. When $\lambda_s$ is much smaller than $k_m^{-1}$, the perturbations which should give rise to the most unstable modes are “blurred” by quantum effects, resulting in a tamed instability.

In the non-relativistic regime, $\Omega = \alpha / \beta^2 / \Theta_c$ can be traced back, after some manipulations exploiting the identity $n_b/n_p = \omega_b^2/\omega_p^2 = V_p/V_b$, to

$$\Omega = 4 \left( \frac{c/\omega_p}{\lambda_c} \right)^2,$$  \hspace{1cm} (16)$$

where $\lambda_c = h/mV_b$ is the de Broglie length of a beam electron. Quantum effects are here important for the maximum growth rate when the beam electron de Broglie length is much larger than the plasma skin depth.

We picture on Figure 2 the region of the $(\alpha, \beta$ or $\gamma_b)$ plan where $\Omega \gg 1$ for two different plasma densities. From Eq. (14) and Fig. 2, we infer that quantum effects tend to be important for beams with $\beta \ll 1$ or $\gamma_b \gg 1$, that is,
the non-relativistic and ultra-relativistic limits. The smallest beam densities are required in the intermediate regime for $\gamma_b = \sqrt{2}$ with $\alpha \ll 2\Theta_c$.

Figure 1 already displays two systems with $\Omega = 307$ and $439 \gg 1$. As expected, according to the previous analysis, the maximum growth rates are weakly affected by quantum effects whereas the ranges of the instability are reduced. Figure 3 pictures now one non-relativistic and one relativistic situations with small $\Omega$ parameters. The maximum growth rate is significantly reduced in both cases so that quantum effects should have easily observable consequences. Noteworthily, the relativistic scenario requires a much higher plasma density than the non-relativistic one, where $\Omega$ can be lowered down through the $\beta^2$ factor at the numerator. Quantum effects in the relativistic regime should thus be expected mostly in astrophysical settings, as already noted in the context of quantum magnetohydrodynamics [2] for example.

III. DISCUSSION AND CONCLUSION

We have implemented the quantum fluid equations to evaluate quantum corrections to the filamentation instability. Quantum effects stabilize the instability for large wave vectors and can significantly reduce the most unstable mode if parameters $\Omega$ defined by Eq. (14) is much smaller than unity. Under such conditions, quantum effect can reduce the filamentation instability. We could check that there is eventually no quantum cancelation of the instability. Indeed, a numerical exploration of the regime $\Omega \ll 1$ shows that the maximum growth rate behaves like $\sqrt{\Omega}$ in this limit. Given the requirements on the parameters for quantum corrections to become significant, the necessity of a quantum treatment may be restricted to some astrophysical scenarios when relativistic beams are involved. Even in the conditions of the Fast Ignition Scenario for Inertial Confinement Fusion, cold fluid quantum corrections are only about 10%, as $n_p = 10^{26}$ cm$^{-3}$, $n_b = 10^{21}$ cm$^{-3}$ and $\gamma_b = 5$ yields $\Omega = 159$. Nevertheless, a correct relativistic treatment should rely on the moments of the relativistic Wigner function, so that present results are only valid in the non-relativistic case while their relevance for the relativistic regime may be only qualitative.

Let us add that it would be very interesting to investigate the joined effects of temperature and quantum degeneracy. In the classical relativistic limit, it is well known that kinetic effects can literally suppress filamentation [12, 13]. Since...
quantum effects can also have some stabilizing function, a quantum kinetic treatment of the problem could well unravel even more pronounced stabilizing quantum effects than those investigated here.

In the same way, it has been known for long that filamentation instability can be significantly reduced, if not completely suppressed, in a magnetized plasma [14]. While various quantum effects have been recently investigated in the case of a magnetized plasma [15, 16, 17], the behavior of the filamentation instability in such settings should be a subject of much interest.

IV. ACKNOWLEDGEMENTS

This work has been achieved under projects FTN 2003-00721 of the Spanish Ministerio de Educación y Ciencia and PAI-05-045 of the Consejería de Educación y Ciencia de la Junta de Comunidades de Castilla-La Mancha.

[1] H. Ren, Z. Wu, and P. Chu, Phys. Plasmas 14, 062102 (2007).
[2] F. Haas, Phys. Plasmas 12, 062117 (2005).
[3] D. Anderson, B. Hall, M. Lisak, and M. Marklund, Phys. Rev. E 65, 046417 (2002).
[4] F. Haas, G. Manfredi, and M. Feix, Phys. Rev. E 62, 2763 (2000).
[5] F. Haas, G. Manfredi, and J. Goedert, Braz. J. Phys. 33, 128 (2003).
[6] J. G. F. Haas, L. G. García and G. Manfredi, Phys. Plasmas 10, 3858 (2003).
[7] M. Tabak, J. Hammer, M. E. Glinsky, W. L. Krueer, S. C. Wilks, J. Woodworth, E. M. Campbell, M. D. Perry, and R. J. Mason, Phys. Plasmas 1, 1626 (1994).
[8] I. Bialynicki-Birula, P. Górnicki, and J. Rafelski, Phys. Rev. D 44, 1825 (1991).
[9] G. Shin, J. Korean Phys. Soc. 29, 571 (1996).
[10] A. Bret and C. Deutsch, Phys. Plasmas 13, 042106 (2006).
[11] A. Bret, M.-C. Firpo, and C. Deutsch, Phys. Rev. E 70, 046401 (2004).
[12] L. O. Silva, R. A. Fonseca, J. W. Tonge, W. B. Mori, and J. M. Dawson, Phys. Plasmas 9, 2458 (2002).
[13] A. Bret and C. Deutsch, Phys. Plasmas 13, 022110 (2006).
[14] J. R. Cary, L. E. Thode, D. S. Lemons, M. E. Jones, and M. A. Mostrom, Phys. Fluids 24, 1818 (1981).
[15] P. Shukla and L. Stenflo, Phys. Lett. A 357, 229 (2006).
[16] P. Shukla and L. Stenflo, J. Plasma Phys. 72, 605 (2006).
[17] P. Shukla, S. Ali, L. Stenflo, and M. Marklund, Phys. Plasmas 13, 112111 (2006).