RESEARCH ARTICLE

Multicolor containers, extremal entropy, and counting

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Funding information
This research was supported by the Swedish Research Council.

Abstract
In breakthrough results, Saxton-Thomason and Balogh-Morris-Samotij developed powerful theories of hypergraph containers. In this paper, we explore some consequences of these theories. We use a simple container theorem of Saxton-Thomason and an entropy-based framework to deduce container and counting theorems for hereditary properties of \(k\)-colorings of very general objects, which include both vertex- and edge-colorings of general hypergraph sequences as special cases. In the case of sequences of complete graphs, we further derive characterization and transference results for hereditary properties in terms of their stability families and extremal entropy. This covers within a unified framework a great variety of combinatorial structures, some of which had not previously been studied via containers: directed graphs, oriented graphs, tournaments, multigraphs with bounded multiplicity, and multicolored graphs among others. Similar results were recently and independently obtained by Terry.

KEYWORDS
entropy, extremal graph theory, hereditary property, hypergraph container method

1 | INTRODUCTION

1.1 Notation and basic definitions

Given a natural number \(r\), we write \(A^r\) for the collection of all subsets of \(A\) of size \(r\). We denote the powerset of \(A\) by \(2^A\) and the collection of nonempty subsets of \(A\) by \(2^A \setminus \{\emptyset\}\). An \(r\)-uniform hypergraph, or \(r\)-graph, is a pair \(G = (V, E)\), where \(V = V(G)\) is a set of vertices and \(E = E(G) \subseteq V^r\) is a set of
r-edges. We write “graph” for “2-graph” and, when there is no risk of confusion, “edge” for “r-edge.” We let \( e(G) := |E(G)| \) denote the size of \( G \) and \( v(G) := |V(G)| \) denote its order.

A subgraph of an r-graph \( G \) is an r-graph \( H \) with \( V(H) \subseteq V(G) \) and \( E(H) \subseteq E(G) \). We use \( H \leq G \) to denote that \( H \) is a subgraph of \( G \). Given a set of vertices \( A \subseteq V(G) \), the subgraph of \( G \) induced by \( A \) is \( G[A] := (A, E(G) \cap A^2) \). A set of vertices \( A \) is independent in \( G \) if the subgraph it induces contains no edges. The degree of a vertex \( v \in V(G) \) is the number of r-edges of \( G \) containing \( v \). Finally an isomorphism between r-graphs \( G_1 \) and \( G_2 \) is a bijection \( \phi : V(G_1) \to V(G_2) \) which sends edges to edges and nonedges to nonedges.

Let \( [n] := \{1, 2, \ldots, n\} \). A property \( \mathcal{P} \) of (labeled) r-graphs is a sequence \( (\mathcal{P}_n)_{n \in \mathbb{N}} \), where \( \mathcal{P}_n \) is a collection of r-graphs on the labeled vertex set \( [n] \). Hereafter, we do not distinguish between a property \( \mathcal{P} \) and the class of all r-graphs belonging to \( \mathcal{P}_n \) for some \( n \in \mathbb{N} \). An r-graph property is symmetric if it is closed under relabeling of the vertices, that is, under permutations of the vertex set \( [n] \). A symmetric r-graph property is monotone (decreasing) if for every r-graph \( G \in \mathcal{P} \), every subgraph \( H \) of \( G \) is isomorphic to an element of \( \mathcal{P} \). A symmetric r-graph property is hereditary if for every r-graph \( G \in \mathcal{P} \) every induced subgraph \( H \) of \( G \) is isomorphic to an element of \( \mathcal{P} \). Note that every monotone property is hereditary, but that the converse is not true. For example, the property of not containing a 4-cycle as an induced subgraph is hereditary but not monotone.

In order to encode certain combinatorial objects of interest, such as directed graphs, we will consider a weaker notion of symmetry.

**Definition 1.1** Let \( m, n \in \mathbb{N} \) with \( m \leq n \). An order-preserving map from \( [m] \) to \( [n] \) is a function \( \phi : [m] \to [n] \) such that \( \phi(i) \leq \phi(j) \) whenever \( i \leq j \). Given \( e \in [m]^2 \), we write \( \phi(e) \) for the set \( \phi(e) = \{\phi(v) : v \in e\} \).

Given r-graphs \( G_1 \) on \( [m] \) and \( G_2 \) on \( [n] \), we say that \( G_2 \) contains \( G_1 \) as an ordered subgraph if there is an order-preserving isomorphism from \( G_1 \) to an \( m \)-vertex subgraph \( H \) of \( G_2 \). We further say that \( G_2 \) contains \( G_1 \) as an induced ordered subgraph if the \( m \)-vertex subgraph \( H \) in question is an induced subgraph of \( G_2 \).

An r-graph property \( \mathcal{P} \) is said to be order-hereditary if for every \( G \in \mathcal{P}_n \) and every order-preserving injection \( \phi : [m] \to [n] \), the graph \( G_{\phi} = ([m], \{e : \phi(e) \in E(G)\}) \) is a member of \( \mathcal{P}_m \).

Clearly, every hereditary property is order-hereditary, but the converse is not true. As an example, consider the property \( \mathcal{P} \) of not containing an increasing path of length 2, that is, the collection of graphs on \( [n] \) \((n \in \mathbb{N})\) not containing vertices \( i < j < k \) such that \( ij \) and \( jk \) are both edges. This is order-hereditary, but not symmetric—and, as we shall see in Section 4, is much larger than the symmetric monotone property of not containing a path of length 2.

We use standard Landau notation throughout this paper, which we recall here. Given functions \( f, g : \mathbb{N} \to \mathbb{R} \), we have \( f = O(g) \) if there exists a constant \( C > 0 \) such that \( \limsup_{n \to \infty} f(n)/g(n) \leq C \). If \( \lim_{n \to \infty} f(n)/g(n) = 0 \), then we write \( f = o(g) \). We write \( f = \Omega(g) \) and \( f = \omega(g) \) to denote \( g = O(f) \) and \( g = o(f) \) respectively. If we have both \( f = O(g) \) and \( f = \Omega(g) \), we say that \( f \) and \( g \) are of the same order and denote this by \( f = \Theta(g) \). We also use \( f \ll g \) and \( g \gg g \) as alternatives to \( f = o(g) \) and \( f = \omega(g) \), respectively. Finally, we say that a sequence of events \( A_n \) occurs with high probability (whp) if \( \lim_{n \to \infty} \mathbb{P}(A_n) = 1 \).

### 1.2 Background: speeds of hereditary graph properties

The problem of counting and characterizing graphs in a given hereditary property \( \mathcal{P} \) has a long and distinguished history. The speed \( n \mapsto |\mathcal{P}_n| \) of a graph property was introduced in 1976 by Erdős,
Kleitman, and Rothschild [27]. Together with the structural properties of a “typical” element of $P_n$, it has received extensive attention from the research community.

Early work focused on the case where $P = \text{Forb}(F)$, the monotone decreasing property of not containing a fixed graph $F$ as a subgraph. We refer to the graphs in $\text{Forb}(F)$ as $F$-free graphs. The Turán number of $F$, a function of $n$ denoted by $\text{ex}(n, F)$, is the maximum number of edges in an $F$-free graph on $n$ vertices. Clearly, any subgraph of an $F$-free graph is also $F$-free. This gives the following lower bound on the number of $F$-free graphs on $n$ labeled vertices:

$$\text{Forb}(F)_n \geq 2^{\text{ex}(n,F)}.$$  

Erdős, Kleitman, and Rödl [27] showed that if $F = K_t$, the complete graph on $t$ vertices, then the exponent in this lower bound is asymptotically tight:

$$\text{Forb}(K_t)_n \leq 2^{(1+o(1)) \text{ex}(n,K_t)}.$$  

Their work was generalized by Erdős, Frankl, and Rödl [26] to the case of arbitrary forbidden subgraphs $F$ and by Prömel and Steger [49], who considered the property $\text{Forb}^*(F)$ of not containing $F$ as an induced subgraph. Finally, Alekseev [1] and Bollobás-Thomason [17] independently determined the asymptotics of the logarithm of the speed for any hereditary property in terms of its coloring number, which we now define.

**Definition 1.2** For each $r \in \mathbb{N}$ and $v \in \{0, 1\}^r$, let $H(r, v)$ be the collection of all graphs $G$ such that $V(G)$ may be partitioned into $r$ disjoint sets $A_1, \ldots, A_r$ such that for each $i$, $G[A_i]$ is an empty graph if $v_i = 0$ and a complete graph if $v_i = 1$. The coloring number $\chi_c(P)$ of a hereditary property is defined to be

$$\chi_c(P) := \sup \{ r \in \mathbb{N} : H(r, v) \subseteq P \text{ for some } v \in \{0, 1\}^r \}.$$  

**Theorem 1.3** (Alekseev-Bollobás-Thomason Theorem) If $P$ is a hereditary property of graphs with $\chi_c(P) = r$, then

$$\lim_{n \to \infty} \frac{\log_2 |P_n|}{\binom{n}{2}} = 1 - \frac{1}{r}.$$  

Subsequently, the rate of convergence of $\log_2 |P_n|/(\binom{n}{2})$ and the structure of typical graphs were investigated by Balogh, Bollobás, and Simonovits [10, 11] for monotone properties, and by Alon, Balogh, Bollobás, and Morris [3] for hereditary properties.

There has also been interest in the speed of monotone properties in other discrete structures. Kohayakawa, Nagle, and Rödl [41], Ishigami [37], Dotson and Nagle [23], and Nagle, Rödl, and Schacht [47] investigated the speed of hypergraph properties, while in a series of papers Balogh, Bollobás, and Morris [7–9] studied the speed of properties of ordered graphs, oriented graphs and tournaments. Many of these results relied on the use of graph and hypergraph regularity lemmas. See the survey of Bollobás [16] for an overview of the state of the area before the breakthroughs discussed in the next subsection.

### 1.3 Background: transference and containers

Recently, there has been great interest in transference theorems, in which central results of extremal combinatorics are shown to also hold in “sparse random” settings. These results are motivated by, inter
alia, the celebrated Green-Tao theorem on arithmetic progressions in the primes [34] and the KŁR conjecture of Kohayawa, Łuczak, and Rödl [40] and its applications.

Very roughly, the KŁR conjecture says the following: let \( H \) be an arbitrary graph, and let \( p = p(n) \) be such that whp the Erdős-Rényi random graph \( G_{n,p} \) contains many more copies of \( H \) than edges (so that in particular it cannot be made \( H \)-free by deleting a negligible proportion of the edges). Then all but an exponentially small proportion of the graphs obtained by replacing each vertex \( x \in V(H) \) by an \( n \)-set \( V_x \) and each edge \( xy \in E(H) \) by a “sparse” \((\varepsilon, p)\)-regular bipartite graph between \( V_x \) and \( V_y \) contain a “canonical” copy of \( H \) (a copy taking exactly one vertex from each of the parts \((V_x)_{x \in V(H)}\)).

This implies in particular that whp the partitions obtained from applications of a sparse regularity lemma to a sufficiently dense subgraph of an Erdős-Rényi random graph satisfy “sparse” analogs of the embedding lemmas for Szemerédi regularity partitions. See [22] for a more rigorous statement and discussion of the conjecture and its applications.

In major breakthroughs a little over five years ago, Conlon and Gowers [21] and independently Schacht [55] proved very general transference results, yielding important corollaries of the KŁR conjecture. Their work was soon followed by another dramatic breakthrough: Balogh, Morris, and Samotij [13] and independently Saxton and Thomason [53], building on work of Kleitman-Winston [39] and of Sapozhenko [51, 52] for graphs, developed powerful theories of hypergraph containers.

These container theories imply that hereditary properties of graphs and hypergraphs can be “covered” by “small” families of “containers,” which are themselves “almost in the property.” We discuss containers with more precision and details in Section 2. As an application of their theories, Balogh-Morris-Samotij and Saxton-Thomason gave both new proofs of known counting/characterization results and many new counting/characterization results for hereditary properties, and in addition a spate of transference results. In particular Balogh-Morris-Samotij and Saxton-Thomason settled the KŁR conjecture in full generality—see the excellent ICM survey of Conlon [20] for an in-depth discussion of some of the recent groundbreaking progress made by researchers in the area.

1.4 Background: entropy and graph limits

A parallel but separate development at the intersection of extremal combinatorics and discrete probability has been the rise of theories of limit objects for sequences of discrete structures. One of several approaches to limits of sequences of graphs (see [5] for a description of other approaches and the links between them) is the theory of left convergence and graphons, which is developed at length in the monograph of Lovász [44].

Recently, Hatami, Janson, and Szegedy [35] defined and studied the entropy of a graphon. They used this notion to recover Theorem 1.3 and to describe the typical structure of a graph in a hereditary property. In a separate paper [31] (see also [30]), Strömberg and a subset of the authors of the present article follow the Hatami-Janson-Szegedy approach to recover some of the main results in this paper using the entropy of decorated graphons rather than multicolor containers as their main tool. We discuss this briefly in Section 5.2.

Let us however note here that the Hatami-Janson-Szegedy notion of entropy can be viewed as a graphon analog of the classical notion of the entropy of a discrete random variable, which first appeared in Shannon’s foundational paper [56]. Using entropy to count objects is an old and celebrated technique in discrete probability—see for example Galvin [33] for an exposition of the applications of entropy to counting. This provides a natural motivation for the arguments in this paper.
1.5 Contributions of this paper

In this paper, we explore consequences of the container theories of Balogh-Morris-Samotij and Saxton-Thomason. We use existing container theorems, together with ideas of Saxton-Thomason and Balogh-Wagner and an entropy-based framework, to deduce container and counting theorems for general hereditary properties of $k$-colorings of very general objects (set-sequences equipped with embeddings, or see, see Definition 3.5). As special cases relevant in many applications, our results cover vertex- and edge-colorings of graph and hypergraph sequences; examples of such sequences include hypercube graphs, multipartite graphs, and grid graphs among others.

In the case of sequences of complete graphs, we further derive characterization and transference results for order-hereditary properties in terms of their stability families and extremal entropy. Among other structures of interest, these latter results cover $k$-colored graphs, directed graphs, oriented graphs, tournaments, and multipartite graphs with bounded multiplicity.

As we restrict ourselves to the study of “dense” properties, our container statements and their corollaries are (we believe) simple and easy to apply (albeit weaker than the full strength of the Balogh-Morris-Samotij and Saxton-Thomason container theorems), which we hope may be useful to other researchers. In particular, the corollaries (counting/characterization/transference) are very general “assumption-free” statements, which do not require checking any codegree condition or even any knowledge of container theory.

We give a number of examples and applications. First, we give a very short proof of the Alekseev-Bollobás-Thomason theorem. Second, we solve a problem of Kühn, Osthus, Townsend, and Zhao [43] on $H$-free digraphs. Third, we prove a counting result for multigraphs in which no triple of vertices supports more than 4 edges (this is a special case of recent and much more general results of Mubayi and Terry [45, 46]). Fourth, we prove counting and stability results for 3-colored graphs with no rainbow triangle, which is a special case of a problem of Erdős and Rothschild [24]. Fifth, we determine the asymptotic number of induced subgraphs of the hypercube graph containing no 4-cycle.

Our main tools are a container theorem of Saxton-Thomason for linear hypergraphs and the adoption of an entropy-based framework. We should like to emphasize here once more the intellectual debt this paper owes to the pioneering work of Balogh-Morris-Samotij and Saxton-Thomason: our work relies on theirs in a crucial way, and many of our ideas exist already in their papers in an embryonic form, which we explore further. The usefulness of our exploration is vindicated by the fact that some of the applications of containers to other discrete structures which we treat are new, and are not well understood by the mathematical community at the time of writing.

For example, finding a suitable container theorem for digraphs was a problem raised by Kühn, Osthus, Townsend, and Zhao [43], which we resolve in the present paper. The “twist” in our approach is that, following the earlier approaches of Saxton and Thomason [53] and Balogh and Wagner [14], our “containers” are, in essence, collection of random digraph models, rather than the collections of digraphs as had been used previously. Explicitly, given a digraph property $\mathcal{P}$, we derive the existence of a collection $\mathcal{T}$ of random digraph models in which the state of each pair of vertices is independent of the rest and such that (i) every digraph in $\mathcal{P}$ occurs with strictly positive probability as the outcome of some random digraph model from $\mathcal{T}$, (ii) outcomes of random graph models from $\mathcal{T}$ are either digraphs in $\mathcal{P}$ or digraphs close to $\mathcal{P}$ in edit distance, and (iii) $|\mathcal{T}|$ is small. As we show, the maximum entropy of a random digraph model satisfying (ii) then determines the speed of $\mathcal{P}$. This connection with random graphs is explored in greater detail in [31], where containers and graph limits are studied in parallel.

There is a significant overlap between the container, enumeration, and stability results presented here and those obtained independently by Terry [58] (see the discussion below), although the emphases and arguments of our two papers are quite different. Since the first version of this paper was written,
Terry has further explored interesting related questions on the different possible speeds in general multicolor properties [59], going beyond the case of “dense” properties we consider in the present work.

1.6 Structure of the paper

Section 2 gathers together our main results on multicolor containers for colorings of $K_n$. Section 2.1 contains our key definitions of templates and entropy. In Section 2.2, we state and derive our first multicolor container theorem (Theorem 2.6), and in Section 2.3 we introduce entropy density and prove a supersaturation result that is key to several of our applications. In Section 2.4, we use these tools to obtain container theorems for general order-hereditary properties (Corollary 2.14) and prove a general counting result (Corollary 2.15). Finally in Sections 2.5 and 2.6 we obtain general characterization and transference results (Theorems 2.18 and 2.23, respectively).

As indicated above, the results of Sections 2.2-2.5 are very similar to those recently obtained by Terry [58]. In particular, Terry’s Theorems 2, 3, 6, and 7 correspond to our Proposition 2.10, Corollary 2.15, Theorem 2.6, and Lemma 2.11, respectively, while Terry’s Theorem 5 is very similar to our Theorem 2.18. Both Terry’s results and our own hold for $r$-uniform hypergraphs (see Section 3.3). Terry’s results extend even further to any finite relational language, which we do not cover in this paper.

In Section 3, we extend our main results to a number of other discrete structures. Section 3.1 describes how our theorems apply to tournaments, oriented graphs and directed graphs; as mentioned earlier, this addresses an issue raised in [43]. In Section 3.2 we extend our main results to obtain container theorems for colorings of very general objects, namely set-sequences equipped with embeddings (or see, see Definition 3.5). This class of objects includes many examples of interest, including both edge- and vertex-colorings of sequences of graphs and hypergraphs, as well as other structures such as sequences of posets or groups. We restate the main results on containers for ssee-s in the more familiar terms of graph and hypergraph sequences in Section 3.3, and then illustrate their implications in Section 3.4 by deriving general counting results for hereditary properties of vertex- and edge-colorings of hypercube graphs.

Section 4 is dedicated to applications of our results to a variety of concrete examples. Among other things, we give a short proof of the Alekseev-Bollobás-Thomason theorem and prove counting and/or characterization results for some hereditary properties of directed graphs, multigraphs with bounded multiplicities, 3-colored graphs and hypercube graphs.

We end this paper in Section 5 with an open problem on the possible structure of entropy maximizers in the multicolor setting and a brief discussion of the links between the container and the graph limit approaches to counting, characterization, and transference.

2 MULTICOLOR CONTAINERS

2.1 Key definitions: templates and entropy

Let $K_n$ denote the complete graph $([n], [n]^{(2)})$. We study $k$-colorings of (the edges of) $K_n$, that is to say, we work with the set of coloring functions $c : E(K_n) \rightarrow [k]$. Denote by $[k]^{K_n}$ the set of all $k$-colorings of $K_n$. Such colorings are of interest as they allow us to encode many important combinatorial structures. Note that each color $i$ induces a graph $c^i$ on $[n]$, where $c^i = ([n], c^{-1}(i))$. An ordinary graph $G$ may thus be viewed as a 2-coloring of $E(K_n)$, with $G = c^1$ and its complement $\overline{G} = c^2$. Similarly, an oriented
graph $\tilde{G}$ may be viewed as a 3-coloring of $E(K_n)$, in which each edge $ij$ with $i < j$ is colored 2 if $\tilde{ij} \in D$, 3 if $\tilde{ji} \in D$ and 1 otherwise. See Section 3.1 for more examples in this vein.

Write $\binom{K_n}{K_m}$ for the collection of order-preserving injections $\phi : [m] \to [n]$. Given $\phi \in \binom{K_n}{K_m}$ and a coloring $c \in [k]^{K_n}$, we write $c_{|\phi}$ for the subcoloring of $c$ induced by $\phi$, defined by $c_{|\phi}(ij) = c(\phi(i)\phi(j))$ for all $ij \in [m]^2$. Further, we say $c' \in [k]^{K_n}$ is a subcoloring of $c \in [k]^{K_n}$ if there exists $\phi \in \binom{K_n}{K_m}$ with $c' = c_{|\phi}$. Our main object of study in this section will be order-hereditary properties of $[k]^{K_n}$.

**Definition 2.1** An order-hereditary property of $k$-colorings is a sequence $P = (P_0)_{n \in \mathbb{N}}$, such that:

(i) $P_n$ is a family of $k$-colorings of $K_n$,
(ii) for every $m \leq n$, $c \in P_n$ and $\phi \in \binom{K_n}{K_m}$, $c_{|\phi} \in P_m$.

A key tool in extending container theory to $k$-colored graphs will be the notion of a template. This was first introduced in the context of container theory by Saxton and Thomason in [53, Section 2.4] (in the case $k = 2$, under the name of “2-colored multigraphs”), and later by Balogh and Wagner in [14, Section 4] (in the case of general $k$, and simply called “containers” in that paper).

**Definition 2.2** (Template) A template for a $k$-coloring of $K_n$ is a function

$$t : E(K_n) \to 2^{[k]} \setminus \{\emptyset\},$$

associating to each edge $e$ of $K_n$ a nonempty list of colors $t(e) \subseteq [k]$; we refer to $t(e)$ as the palette available at $e$. We write $\left(2^{[k]} \setminus \{\emptyset\}\right)^{K_n}$ for the family of all $k$-coloring templates of $K_n$.

Given a template $t \in \left(2^{[k]} \setminus \{\emptyset\}\right)^{K_n}$, we say that a $k$-coloring $c \in [k]^{K_n}$ realizes $t$ if $c(e) \subseteq t(e)$ for every edge $e \in E(K_n)$. We write $\langle t \rangle$ for the collection of realizations of $t$ and $c \in \langle t \rangle$ as a shorthand for “$c$ is a realization of $t$.”

In other words, a template $t$ gives, for each edge of $K_n$, a palette of permitted colors, and $\langle t \rangle$ is the set of $k$-colorings of $K_n$ that respect the template. We observe that a $k$-coloring of $K_n$ may itself be regarded as a template, albeit with only one color allowed at each edge. We extend our notion of subcoloring to templates in the natural way.

**Definition 2.3** Let $t \in \left(2^{[k]} \setminus \{\emptyset\}\right)^{K_n}$ and let $\phi \in \binom{K_n}{K_m}$. The subtemplate of $t$ induced by $\phi$ is the template $t_{|\phi} \in \left(2^{[k]} \setminus \{\emptyset\}\right)^{K_m}$ defined by $t_{|\phi}(ij) = t(\phi(i)\phi(j))$.

Given $m \leq n$ and $k$-coloring templates $t$, $t'$ for $K_m$, $K_n$ respectively, we say that $t$ is a subtemplate of $t'$, which we denote by $t \leq t'$, if there exists $\phi \in \binom{K_n}{K_m}$ such that $t(e) \subseteq t'_{\phi}(e)$ for every $e \in E(K_m)$.

Our notion of subtemplates can be viewed as the template analog of the notion of an order-preserving subgraph for graphs on a linearly ordered vertex set.

Templates enable us to generalize the notion of containers to the $k$-colored setting.

**Definition 2.4** Given a family of $k$-colorings $F$ of $E(K_n)$, a container family for $F$ is a collection $T = \{t_1, t_2, \ldots, t_m\}$ of $k$-coloring templates such that for every template $t \in \left(2^{[k]} \setminus \{\emptyset\}\right)^{K_n}$ with $\langle t \rangle \subseteq F$, there exists $t_i \in T$ with $t \leq t_i$.

In particular, if $T$ is a container family for $F$ then every coloring from $F$ is a realization of some template from $T$ (so the set of realizations from $T$ “contains” $F$).
Next we define the key notion of the *entropy* of a template.

**Definition 2.5**  
The *entropy* of a $k$-coloring template $t$ is  
\[
\text{Ent}(t) := \log_k \prod_{e \in E(K_n)} |t(e)|.
\]

For any template $t$ we have $0 \leq \text{Ent}(t) \leq \binom{n}{2}$, and the number of distinct realizations of $t$ is exactly $|\langle t \rangle| = k^{\text{Ent}(t)}$. Observe also that zero-entropy templates correspond to $k$-colorings of $K_n$, and that if $t$ is a subtemplate of $t'$ then $\text{Ent}(t) \leq \text{Ent}(t')$. There is a direct correspondence between our notion of entropy and that of Shannon entropy in discrete probability: given a template $t$, we can define a $t$-random coloring $c_t$, by choosing for each $e \in E(K_n)$ a color $c_t(e)$ uniformly at random from $t(e)$. The entropy of $t$ defined above is precisely the $k$-ary Shannon entropy of the discrete random variable $c_t$.

The notion of $t$-random coloring allows us to view our templates as, in essence, random graph models, and their realizations as random graph outcomes. Using templates/the associated random colorings as containers is key to making multicolor containers work. This idea, due to Saxton and Thomason [53] and Balogh and Wagner [14], will allow us to overcome the obstacles to a container theorem for a particular digraph problem of Kühn, Osthus, Townsend, and Zhao [43] (see Section 4.3).

### 2.2 Container families

Let $N \in \mathbb{N}$ be fixed and let $F$ be a nonempty collection of $k$-colorings of $E(K_N)$. Let $\text{Forb}(F)$ be the collection of all $k$-colorings $c$ of $K_n$, $n \in \mathbb{N}$, such that for all $\phi \in \binom{K_n}{K_N}$, $c|\phi \not\in F$. More succinctly, $\text{Forb}(F)$ is the collection of all $k$-colorings avoiding $F$, which, clearly, is an order-hereditary property of $k$-colorings.

**Theorem 2.6**  
Let $N \in \mathbb{N}$ be fixed and let $F$ be a nonempty collection of $k$-colorings of $E(K_N)$. For any $\varepsilon > 0$, there exist constants $C_0$, $n_0 > 0$, depending only on $(\varepsilon, k, N)$, such that for any $n \geq n_0$ there exists a collection $T_n$ of $k$-coloring templates for $K_n$ satisfying:

(i) $T_n$ is a container family for $(\text{Forb}(F))_n$;

(ii) for each template $t \in T_n$, there are at most $\varepsilon \binom{n}{N}$ pairs $(\phi, c)$ with $\phi \in \binom{K_n}{K_N}$, $c \in F$ and $c \not\in \langle t|\phi \rangle$;

(iii) $\log_k |T_n| \leq \frac{C_0}{n^{1/\left(\frac{\log_2 2}{2} - 1\right)}} \cdot \varepsilon \binom{n}{2}$.

In other words, the theorem says that we can find a small (property (iii)) collection of templates, that together cover $\text{Forb}(F)_n$ (property (i)), and whose realizations are close to lying in $\text{Forb}(F)_n$ (property (ii)).

We shall deduce Theorem 2.6 from a hypergraph container theorem of Saxton and Thomason. Say that an $r$-graph $H$ is linear if each pair of distinct $r$-edges of $H$ meets in at most 1 vertex. Saxton and Thomason proved the following:

**Theorem 2.7** (Saxton-Thomason [Theorem 1.2 in [54]])  
Let $r \geq 2$ and let $0 < \delta < 1$. There exists $d_0 = d_0(r, \delta)$ such that if $G$ is a linear $r$-graph of average degree $d \geq d_0$, then there exists a collection $C$ of subsets of $V(G)$ satisfying:

1. if $I \subseteq V(G)$ is an independent set, then there exists $C \subseteq C$ with $I \subseteq C$;
2. $e(G[C]) < \delta e(G)$ for every $C \subseteq C$;
3. $|C| \leq 2^{\beta e(G)}$, where $\beta = (1/d)^{1/(2r-1)}$. 

In the proof of Theorem 2.6 and elsewhere, we shall use the following standard Chernoff bound: if $X \sim \text{Binom}(n, p)$, then for any $\delta \in [0, 1],$

$$\mathbb{P}(\left| X - np \right| \geq \delta np) \leq 2e^{-\frac{\delta^2 np}{4}}. \quad (2.1)$$

**Proof of Theorem 2.6**  
If $N = 2$, then $\mathcal{F}$ just gives us a list of forbidden colors, say $F \subseteq [k]$. Then $(\text{Forb}(\mathcal{F}))_n$, is exactly the collection of all realizations of the template $t$ assigning to each edge $e$ of $K_n$ the collection $[k] \setminus F$ of colors not forbidden by $\mathcal{F}$. Thus in this case our result trivially holds, and we may therefore assume $N \geq 3$ in the rest of the proof.

We define a hypergraph $H$ from $\mathcal{F}$ and $K_n$ as follows. Set $r = \binom{N}{2}$. We let the vertex set of $H$ consist of $k$ disjoint copies of $E(K_n)$, one for each of our $k$ colors: $V(H) = E(K_n) \times [k]$; this idea, allowing us to apply Theorem 2.7, first appeared in work of Saxton and Thomason [53, Section 2.4] (in the 2-color case) and of Balogh and Wagner [14, Section 4] (in the $k$-color case).

For every order-preserving embedding $\phi \in \binom{K_n}{k}$ and every $k$-coloring $c \in \mathcal{F}$, we add to $H$ an $r$-edge $e_{\phi,c}$, where

$$e_{\phi,c} = \{ (\phi(i)\phi(j), c(\langle ij \rangle)) : i, j \in [N]^2 \}.$$ 

This gives us an $r$-graph $H$. Let us give bounds on its average degree. Since $\mathcal{F}$ is nonempty, for every $N$-set $A \subseteq [n]$, there are at least 1 and at most $k\binom{n}{2}$ colorings $c$ of $A^2$ which are order-isomorphic to an element of $\mathcal{F}$. It follows that:

$$\frac{n^N}{N^N} \leq \binom{n}{N} \leq e(H) \leq k\binom{n}{2}\left(\frac{N}{n}\right)^N \leq k\binom{n}{2}\left(\frac{en}{N}\right)^N. \quad (2.2)$$

Thus $e(H)$ is of order $n^N$ and the average degree in $H$ is of order $n^{N-2}$ (since $v(H) = k\binom{n}{2}$), which tends to infinity as $n \to \infty$. We are almost in a position to apply Theorem 2.7, with one caveat: the hypergraph $H$ we have defined is in no way linear. Following Saxton-Thomason [54], we circumvent this difficulty by considering a random sparsification of $H$. We note that other approaches are possible, by using the original container theorems of [13,53] and computing co-degrees; this would avoid the need for sparsification and potentially give better bounds on $|\mathcal{F}_n|$, but make other aspects of our proof—and our later generalizations—less transparent. Since we are focussing on “dense” properties in this paper and easily applicable general statements, we do not pursue this here.

Let $\varepsilon_1 \in (0, 1)$ be a constant to be specified later and let

$$p = \varepsilon_1 / \left(12k\binom{n}{3}^{-3}\left(\frac{N}{3}\right)\left(\frac{n-3}{N-3}\right)\right). \quad (2.3)$$

Note that by (2.2) we have

$$pe(H) \geq p\binom{n}{N} = \Omega(n^3). \quad (2.4)$$

We shall keep each $r$-edge of $H$ independently with probability $p$, and delete it otherwise, to obtain a random subgraph $H'$ of $H$. Standard probabilistic estimates will then show that with positive probability the $r$-graph $H'$ is almost linear, has large average degree and respects the density of $H$. More precisely, we show:
Lemma 2.8  Let $p$ be as in (2.3), let $H'$ be the random subgraph of $H$ defined above and consider the following events:

- the event $F_1$ that $e(H') \geq \frac{pe(H)}{2}$;
- the event $F_2$ that $H'$ has at most $\frac{\ell_1}{4} p\binom{n}{3}$ pairs of edges $(e, e')$ with $|e \cap e'| \geq 2$;
- the event $F_3$ that for all $S \subseteq V(H)$ with $e(H[S]) \geq \epsilon_1 e(H)$, we have $e(H'[S]) \geq \frac{\ell_1}{2} e(H')$.

There exists $n_1 = n_1(\epsilon_1, k, N) \in \mathbb{N}$ such that for all $n \geq n_1$, $F_1 \cap F_2 \cap F_3$ occurs with strictly positive probability.

The proof of our lemma follows that of [54, Lemma 3.3] with minor modifications.

Proof  By (2.2), we have $\mathbb{E}e(H') = pe(H) \geq p\binom{n}{3}$. Applying the Chernoff bound (2.1) with $\delta = 1/2$, we get that the probability that $F_1$ fails in $H'$ is at most

$$\mathbb{P}(e(H') < \frac{1}{2}pe(H)) \leq 2e^{-\frac{\ell_1}{36}e(H)} = e^{-\Omega(n^3)},$$

where the last equality follows from (2.4). Next consider the pairs of $r$-edges $(e, e')$ in $H$ with $|e \cap e'| \geq 2$, which we refer to hereafter as overlapping pairs. Let $Y_H$ and $Y_{H'}$ denote the number of overlapping pairs in $H$ and $H'$ respectively. Since one needs at least 3 vertices to support 2 distinct edges in $K_n$, $Y_H$ is certainly bounded above by the number of ways of choosing an $N$-set $A \subseteq [n]$, a 3-set $B$ from $A$ and an $(N-3)$-set $A'$ from $[n] \setminus B$ (thereby making an overlapping pair of $N$-sets $(A, A' \cup B)$) and assigning an arbitrary $k$-coloring to the edges in $A^{(2)} \cup (A' \cup B)^{(2)}$. Thus,

$$Y_H \leq \binom{n}{N} \binom{N}{3} \binom{n-3}{N-3} k^{2(\binom{3}{2})-3}$$

and

$$\mathbb{E}(Y_H) = p^2 Y_H \leq p^2 \binom{n}{N} \binom{N}{3} \binom{n-3}{N-3} k^{2(\binom{3}{2})-3} = \frac{\epsilon_1}{12} p \binom{n}{N}.$$  

Applying Markov’s inequality, we have with probability at least $\frac{2}{3}$ that $Y_{H'} \leq \frac{\ell_1}{4} p\binom{n}{3}$ (and thus $F_2$) holds.

Finally, consider a set $S \subseteq V(H)$ with $e(H[S]) \geq \epsilon_1 e(H)$. Applying the Chernoff bound (2.1) with $\delta = 1 - 1/\sqrt{2}$ and the lower bound (2.4) for $pe(H)$, we get

$$\mathbb{P}\left(e(H'[S]) \leq \frac{1}{\sqrt{2}} \mathbb{E}e(H'[S])\right) \leq 2e^{-\frac{(1-\frac{1}{\sqrt{2}})^2\mathbb{E}e(H'[S])}{4}} = e^{-\Omega(\epsilon_1 e(H))} = e^{-\Omega(n^3)}. \quad (2.5)$$

Moreover, by (2.1) with $\delta = \sqrt{2} - 1$ and (2.4) again,

$$\mathbb{P}\left(e(H') \geq \sqrt{2}\mathbb{E}e(H')\right) \leq 2e^{-\frac{(\sqrt{2} - 1)^2 pe(H')}{4}} = e^{-\Omega(n^3)}. \quad (2.6)$$

Say that a nonempty set $S \subseteq V(H)$ is bad if $e(H[S]) \geq \epsilon_1 e(H)$ and $e(H'[S]) \leq \frac{\ell_1}{2} e(H')$. By (2.5), (2.6), and the union bound, the probability that $F_3$ fails, that is, that there exists some bad $S \subseteq V(H)$, is at most

$$\mathbb{P}(\exists \text{ bad } S) \leq \mathbb{P}(e(H') \geq \sqrt{2}\mathbb{E}e(H')) + \sum_S \mathbb{P}(e(H'[S]) \leq \frac{1}{\sqrt{2}} \mathbb{E}e(H'[S])) \leq 2^{\binom{k}{2}} e^{-\Omega(n^3)} = e^{-\Omega(n^3)}.$$
Therefore with probability at least $2/3 - o(1)$ the events $F_1, F_2$, and $F_3$ all occur, and in particular they must occur simultaneously with strictly positive probability for all $n \geq n_1 = n_1(\epsilon_1, k, N)$. □

By Lemma 2.8, for any $\epsilon > 0$ and $\epsilon_1 = k^{-\ell_2(2)}\epsilon$ fixed and any $n \geq n_1(\epsilon_1, k, N)$, there exists a sparsification $H'$ of $H$ for which the events $F_1, F_2$, and $F_3$ from the lemma all hold. Deleting one $r$-edge from each overlapping pair in $H'$, we obtain a linear $r$-graph $H''$ with average degree $d$ satisfying

$$d = \frac{re(H'')}{\nu(H'')} \geq \frac{(\frac{n}{2})^2}{k(\frac{n}{2})^2} \left( \epsilon(H') - Y_{H''} \right) \geq \frac{(\frac{n}{2})}{k(\frac{n}{2})} \left( 1 - \frac{\epsilon_1}{4} \right)p \left( \frac{n}{N} \right) = \Omega \left( \frac{n^N}{n^{2n^N-3}} \right) = \Omega(n). \quad (2.7)$$

We are now in a position to apply the container theorem for linear $r$-graphs, Theorem 2.7, to $H''$. Let $\delta = \delta(\epsilon_1)$ satisfy $0 < \delta < \epsilon_1/4$ and let $d_0 = d_0(\delta, r)$ be the constant in Theorem 2.7. For $n \geq n_2(k, N, \delta)$ sufficiently large, we have $d \geq d_0$. Thus there exists a collection $C$ of subsets of $V(H'') = V(H)$ satisfying conclusions 1-3 of Theorem 2.7.  

For each $C \subseteq C$, we obtain a template $t = t(C)$ for a partial $k$-coloring of $K_n$ as follows: for each edge $e$, we are given a palette $t(e) = \{ i \in [k] : (e, i) \in C \}$ of available colors (note that some edges may have the empty palette). Set

$$\mathcal{T} := \{ t(C) : C \subseteq C, t(e) \neq \emptyset \text{ for all } e \in E(K_n) \}$$

to be the family of templates from $(2^k \setminus \{ \emptyset \})^{K_n}$ which can be constructed in this way. We claim that the template family $\mathcal{T}$ satisfies conclusions (i)-(iii) of Theorem 2.6.

Indeed, by definition of $H$, any template $t'$ with $\langle t \rangle \subseteq \mathcal{P}_n$ gives rise to an independent set $I$ in the $r$-graph $H$ and hence its subgraph $H''$, namely $I = \{ (e, i) : i \in [k], e \in E(K_n), i \in t(e) \}$. Thus there exist $C \subseteq C$ with $I \subseteq C$, giving rise to a proper template $t \in \mathcal{T}$ with $t' \leq t$. Conclusion (i) is therefore satisfied by $\mathcal{T}$.

Further for each $C \subseteq C$, conclusion 2 of Theorem 2.7 and the event $F_2$ together imply

$$e(H'[C]) \leq e(H''[C]) + (e(H') - e(H'')) \leq \delta e(H'') + \frac{\epsilon_1}{4} e(H') < \frac{\epsilon_1}{2} e(H').$$

Together with the fact that $F_3$ holds, this implies $e(H[C]) < \epsilon_1 e(H)$, which by (2.2) and our choice of $\epsilon_1$ is at most $\epsilon_1 k^{\ell_2(2)} \binom{n}{\frac{\epsilon_1}{N}} \epsilon \binom{n}{\frac{1}{N}}$. In particular, by construction of $H$, we have that for each $t = t(C) \in \mathcal{T}$ there are at most $\epsilon \binom{n}{\frac{1}{N}}$ pairs $(\phi, c)$ with $\phi \in \binom{K_n}{\binom{K_n}{c}}$, $c \in \mathcal{P}$ and $c \in \langle t|\phi \rangle$. This establishes (ii).

Finally by conclusion 3 of Theorem 2.7 and our bound (2.7) on the average degree $d$ in $H''$, we have

$$|\mathcal{T}| \leq |C| \leq 2^{|d|d^{|C^2|}} = k^{\Omega(n^{-1/2d-1})} \binom{C}{d},$$

so that there exist constants $C_0, n_3 > 0$ such that for all $n \geq n_3$ sufficiently large,

$$\log_k \log_k \mathcal{T} \leq \frac{C_0}{n/(\binom{C}{2})} \binom{n}{\frac{1}{2}}$$

and (iii) is satisfied. This establishes the statement of Theorem 2.6 for $n \geq n_0 = \max(n_1, n_2, n_3)$. □

### 2.3 | Extremal entropy and supersaturation

In this section we derive the two ingredients needed in virtually all applications of containers, namely the existence of a limiting “entropy density” and a supersaturation result.
**Definition 2.9** Let \( P \) be an order-hereditary property of \( k \)-colorings with \( P_n \neq \emptyset \) for every \( n \in \mathbb{N} \). For every \( n \in \mathbb{N} \), we define the *extremal entropy* of \( P \) to be

\[
\text{ex}(n, P) = \max \{ \text{Ent}(t) : t \text{ is a } k\text{-coloring template for } K_n \text{ with } \langle t \rangle \subseteq P_n \}.
\]

Note that this definition generalizes the concept of the Turán number: if \( k = 2 \), \( F \) is a graph and \( P = \text{Forb}(F) \), then \( \text{ex}(n, P) = \text{ex}(n, F) \).

**Proposition 2.10** If \( P \) is an order-hereditary property of \( k \)-colorings with \( P_n \neq \emptyset \) for every \( n \in \mathbb{N} \), then the sequence \( \left( \text{ex}(n, P) / \binom{n}{2} \right)_{n \in \mathbb{N}} \) is nonincreasing and tends to a limit \( \pi(P) \in [0, 1] \) as \( n \to \infty \).

**Proof** This is similar to the classical proof of the existence of the Turán density. As observed after Definition 2.5, \( 0 \leq \text{Ent}(t) \leq \binom{n}{2} \) for any \( k \)-coloring template \( t \) of \( K_n \), so that \( \text{ex}(n, P) / \binom{n}{2} \in [0, 1] \). It is therefore enough to show that \( \left( \text{ex}(n, P) / \binom{n}{2} \right)_{n \in \mathbb{N}} \) is nonincreasing. Let \( t \) be any \( k \)-coloring template for \( K_{n+1} \) with \( \langle t \rangle \subseteq P_{n+1} \). For any \( \phi \in \binom{K_{n+1}}{K_n} \), consider \( t|_{\phi} \). Since \( P \) is order-hereditary, \( \langle t \rangle \subseteq P_{n+1} \) implies \( \langle t|_{\phi} \rangle \subseteq P_n \). By averaging over all choices of \( \phi \), we have:

\[
\frac{\text{Ent}(t)}{\binom{n+1}{2}} = \frac{1}{\binom{n+1}{2}} \log_k \left( \prod_{e \in [n+1]^2} |t(e)| \right) = \frac{1}{\binom{n+1}{2}} \log_k \left( \prod_{\phi \in \binom{K_{n+1}}{K_n}} \prod_{e \in [n]^2} |t|_{\phi}(e) | \right) ^{1/n-1}
\]

\[
= \frac{1}{n+1} \frac{1}{\binom{n}{2}} \sum_{\phi \in \binom{K_{n+1}}{K_n}} \text{Ent}(t|_{\phi})
\]

\[
\leq \frac{1}{n+1} \frac{1}{\binom{n}{2}} (n+1) \text{ex}(n, P).
\]

Thus \( \text{ex}(n+1, P) / \binom{n+1}{2} \leq \text{ex}(n, P) / \binom{n}{2} \) as required and we are done.

We call the limit \( \pi(P) \) the *entropy density* of \( P \). Observe that the entropy density gives a lower bound on the *speed* \( |P_n| \) of the property \( P \): for all \( n \in \mathbb{N} \),

\[
k^{\pi(P)} \leq k^{\text{ex}(n, P)} \leq |P_n|.
\]

We shall show (Theorem 2.12) that the exponent in this lower bound is asymptotically tight.

**Lemma 2.11** (Supersaturation) Let \( N \in \mathbb{N} \) be fixed and let \( P \) be a nonempty collection of \( k \)-colorings of \( K_N \). Set \( P = \text{Forb}(P) \). For every \( \varepsilon \) with \( 0 < \varepsilon < 1 \), there exist constants \( n_0 \in \mathbb{N} \) and \( C_0 > 0 \) such that for all \( n \geq n_0 \) and every template \( t \in \left( 2^{[k]} \setminus \{ \emptyset \} \right)^{K_n} \) with

\[
\text{Ent}(t) > (\pi(P) + \varepsilon) \binom{n}{2},
\]

there are at least

\[
C_0 \varepsilon \binom{n}{N}
\]

pairs \( \phi, c \) with \( \phi \in \binom{K_n}{K_N} \), \( c \in P \) and \( c \in \langle t|_{\phi} \rangle \).
Proof. Given a template \( t' \in (2^{[k]} \setminus \{\emptyset\})^{K_n} \), for some \( m \geq N \), let \( B(t') \) denote the collection of pairs \((\phi, c)\) with \( \phi \in (K_n)_{K_n}^c \), \( c \in \mathcal{F} \) and \( c \in \langle t' \rangle_{\phi} \).

By Proposition 2.10, there exists \( n_0 \geq N \) such that for all \( t' \in (2^{[k]} \setminus \{\emptyset\})^{K_n} \) with \( \text{Ent}(t') > \left( \pi(\mathcal{P}) + \frac{\epsilon}{2} \right) \binom{n_0}{2} \), we must have \( |B(t')| \geq 1 \). Let \( t \in (2^{[k]} \setminus \{\emptyset\})^{K_n} \) for some \( n \geq n_0 \), and suppose \( \text{Ent}(t) > \left( \pi(\mathcal{P}) + \epsilon \right) \binom{n}{2} \). Let \( X \) denote the number of sets \( \phi \in (K_n)_{K_n}^c \) such that \( \text{Ent}(t\phi) > \left( \pi(\mathcal{P}) + \frac{\epsilon}{2} \right) \binom{n_0}{2} \). By summing \( \text{Ent}(t\phi) \) over all \( \phi \in (K_n)_{K_n}^c \), we have

\[
(\pi(\mathcal{P}) + \epsilon) \binom{n}{2} \binom{n - 2}{n_0 - 2} < \text{Ent}(t) \binom{n - 2}{n_0 - 2} = \sum_{\phi} \text{Ent}(t\phi) \leq (n_0) \left( \pi(\mathcal{P}) + \frac{\epsilon}{2} \right) \left( \binom{n_0}{2} \right) + X \left( \binom{n_0}{2} \right),
\]

implying \( X > \frac{\epsilon}{2} \binom{n}{n_0} \). On the other hand, summing \( |B(t\phi)| \) over all \( \phi \in (K_n)_{K_n}^c \), we have

\[
|B(t)| \frac{n - N}{n_0 - N} = \sum_{\phi} |B(t\phi)| \geq X \times \frac{\epsilon}{2} \binom{n}{n_0},
\]

so that

\[
|B(t)| > \frac{1}{2} \binom{n_0}{n} \epsilon \binom{n}{N}.
\]

This proves the lemma with \( C_0 = \binom{2(n_0)}{N}^{-1} \).

\[\square\]

2.4 Speed of order-hereditary properties

In this section, we relate the speed of an order-hereditary property to its extremal entropy density and obtain container and counting theorems for arbitrary order-hereditary properties (ie, properties defined by a possibly infinite set of forbidden colorings).

Theorem 2.12. Let \( N \in \mathbb{N} \) be fixed and let \( \mathcal{P} \) be a nonempty collection of \( k \)-colorings of \( E(K_N) \). Set \( \mathcal{P} = \text{Forb}(\mathcal{P}) \). For all \( \epsilon > 0 \), there exists \( n_0 \) such that for all \( n \geq n_0 \) we have

\[
k^{\pi(\mathcal{P})} \leq |P_n| \leq k^{\pi(\mathcal{P}) + \epsilon} \binom{n}{2}.
\]

Proof. Inequality (2.8) already established the lower bound on the speed \( n \mapsto |P_n| \). For the upper bound, we apply multicolor containers. By Theorem 2.6 for any \( \eta > 0 \) there exists \( n_1 \in \mathbb{N} \) such that for all \( n \geq n_1 \), there exists a collection of templates \( \mathcal{T}_n \) from \( (2^{[k]} \setminus \{\emptyset\})^{K_n} \) such that (i) \( \mathcal{T}_n \) is a container family for \( \mathcal{P}_n \), (ii) for each \( t \in \mathcal{T}_n \) there are at most \( \eta \binom{n}{N} \) pairs \((\phi, c)\) with \( \phi \in (K_n)_{K_n}^c \), \( c \in \mathcal{P} \) and \( c \in \langle t \rangle_{\phi} \), and (iii) \( |\mathcal{T}_n| \leq k^{\eta} \binom{n}{2} \).

Provided we pick \( \eta > 0 \) sufficiently small, Lemma 2.11 implies there exists \( n_2 \in \mathbb{N} \) such that for all \( n \geq n_2 \), if \( t \in (2^{[k]} \setminus \{\emptyset\})^{K_n} \) and there are at most \( \eta \binom{n}{N} \) pairs \((\phi, c)\) with \( \phi \in (K_n)_{K_n}^c \), \( c \in \mathcal{P} \), and \( c \in \langle t \rangle_{\phi} \), then \( \text{Ent}(t) \leq \left( \pi(\mathcal{P}) + \frac{\epsilon}{2} \right) \binom{n}{2} \).
Thus choosing \( \eta = \eta(\epsilon) > 0 \) sufficiently small (in particular less than \( \epsilon/2 \)) and \( n_0 \geq \max(n_1, n_2) \), we have that for \( n \geq n_0 \) every template \( t \in \mathcal{T}_n \) has entropy at most \( \left( \pi(P) + \frac{\epsilon}{2} \right) \binom{\mathcal{C}}{2} \), whence we may at last bound above the number of realizations of templates from \( \mathcal{T}_n \), and hence the speed of \( P \): for \( n \geq n_0 \),

\[
|P_n| \leq |\mathcal{T}_n| k^{\max\{\text{Ent}(t) \colon t \in \mathcal{T}_n\}} \leq k^{\eta(\epsilon)(\pi(P)+\frac{\epsilon}{2})}\binom{\mathcal{C}}{2} \leq k^{\pi(P)\binom{\mathcal{C}}{2}+\epsilon\binom{\mathcal{C}}{2}}.
\]

**Theorem 2.13** (Approximation of general order-hereditary properties)  
Let \( P \) be an order-hereditary property of \( k \)-colorings with \( P_n \neq \emptyset \) for every \( n \in \mathbb{N} \). Let \( \epsilon > 0 \) be fixed. There exist constants \( N \) and \( n_0 \in \mathbb{N} \) and a family \( \mathcal{F} \) of \( k \)-colorings of \( E(K_N) \) such that for all \( n \geq n_0 \), we have

(i) \( P_n \subseteq \text{Forb}(\mathcal{F})_n \), and

(ii) \( |\text{Forb}(\mathcal{F})_n| \leq |P_n| k^{\binom{\mathcal{C}}{2}}. \)

**Proof**  
For every \( n \in \mathbb{N} \), let \( \mathcal{F}_n \) denote the collection of \( k \)-colorings of \( E(K_n) \) which are not in \( P_n \) (and thus, as \( P \) is order-hereditary, do not appear as subcolorings of elements of \( P_{n'} \) for any \( n' \geq n \)). Set \( Q^n = \text{Forb}(\bigcup_{m \leq n} \mathcal{F}_m) \) to be the order-hereditary property of \( k \)-colorings which avoids exactly the same \( k \)-colorings on at most \( n \) vertices as \( P \). By construction we have a chain of inclusions

\[
Q^1 \supseteq Q^2 \supseteq \cdots \supseteq Q^n \supseteq \cdots \supseteq P.
\]

Consequently, the sequence of entropy densities \( (\pi(Q^n))_{n \in \mathbb{N}} \) is nonincreasing and bounded below by \( \pi(P) \). We claim that \( \lim_{n \to \infty} \pi(Q^n) = \pi(P) \). Indeed, suppose this was not the case. Then there exists \( \eta > 0 \) such that \( \pi(Q^n) > \pi(P) + \eta \) for all \( n \in \mathbb{N} \). For each \( n \in \mathbb{N} \), let \( t^n \) be a template such that \( t^n \not\subseteq (Q^n) \) and \( \text{Ent}(t^n) = \text{ex}(n, Q^n) = \text{ex}(n, P) \). Since the sequence \( \left( \text{ex}(m, Q^n)/\binom{m}{2} \right)_{m \in \mathbb{N}} \) is nonincreasing (by Proposition 2.10), we have that for every \( n \in \mathbb{N} \),

\[
\pi(P) + \eta < \pi(Q^n) \leq \text{Ent}(t^n)/\binom{n}{2} = \text{ex}(n, P)/\binom{n}{2},
\]

contradicting Proposition 2.10. Thus \( \lim_{n \to \infty} \pi(Q^n) = \pi(P) \), as claimed. In particular there exists \( N \in \mathbb{N} \) for which \( \pi(Q^N) < \pi(P) + \epsilon/2 \).

By (2.8), \( k^{\pi(P)\binom{\mathcal{C}}{2}} \leq |P_n| \). On the other hand, by Theorem 2.12 (applied to the property \( Q^N \) with parameter \( \epsilon/2 \)) there exists \( n_0 \in \mathbb{N} \) with \( n_0 \geq N \) such that for all \( n \geq n_0 \) we have:

\[
|\!(Q^n)\!| \leq k^{\pi(Q^n) + \frac{\epsilon}{2}}\binom{\mathcal{C}}{2} < k^{\pi(P) + \epsilon}\binom{\mathcal{C}}{2} \leq |P_n| k^{\epsilon\binom{\mathcal{C}}{2}}.
\]

Observing that for \( n \geq n_0 \) we have \( (Q^n) = \text{Forb}(\mathcal{F}_n) \supseteq P_n \) we see that the triple \((N, n_0, \mathcal{F}_N)\) satisfies the conclusion of the theorem.

**Corollary 2.14** (Containers for order-hereditary properties)  
Let \( P \) be an order-hereditary property of \( k \)-colorings with \( P_n \neq \emptyset \) for every \( n \in \mathbb{N} \), and let \( \epsilon > 0 \), \( m \in \mathbb{N} \) be fixed. There exists \( n_0 \) such that for any \( n \geq n_0 \) there exists a collection \( \mathcal{T}_n \subseteq \{2^{[k]} \setminus \{\emptyset\}\}^{k_n} \) satisfying:

(i) \( \mathcal{T}_n \) is a container family for \( P_n \);

(ii) for each template \( t \in \mathcal{T}_n \), \( \text{Ent}(t) \leq (\pi(P) + \epsilon)\binom{n}{2} \);

(iii) for each template \( t \in \mathcal{T}_n \), there are at most \( \epsilon\binom{n}{m} \) pairs \((\phi, c)\) where \( \phi \in \binom{K_\epsilon}{K_\epsilon}, c \not\in P_m \) and \( c \in (t_\phi); \)

(iv) \( |\mathcal{T}_n| \leq k^{\epsilon\binom{\mathcal{C}}{2}}. \)
Fix $\varepsilon > 0$. Let $\mathcal{F}_n$ and $Q^n$ be defined as in the proof of Theorem 2.13. As shown in that proof, there exists some $N$ such that $Q^n \supseteq \mathcal{P}$ and $\pi(Q^n) < \pi(\mathcal{P}) + \frac{\varepsilon}{2}$. Without loss of generality we may take $N > m$. Note $(Q^n)_n = \text{Forb}(\mathcal{F}_n)_n$ for all $n \geq N$.

Pick $\delta > 0$ sufficiently small. By Theorem 2.6 applied to Forb($\mathcal{F}_n$), there exists $n_1 \geq N$ such that, for all $n \geq n_1$, there is a collection of templates $\mathcal{T}_n \subseteq (2^{|K|} \setminus \{\emptyset\})^{K_m}$ satisfying (a) $\mathcal{T}_n$ is a container family for $(Q^n)_n$, (b) for each $t \in \mathcal{T}_n$ there are at most $\delta \binom{n}{N}$ pairs $(\phi, c)$ with $\phi \in \binom{K_n}{K_m}$, $c \in \mathcal{F}_n$ and $c \in \langle t|\phi \rangle$, and (c) $|\mathcal{T}_n| \leq k^{\delta(n)}$.

Property (a) implies that $\mathcal{T}_n$ is a container family for $\mathcal{F}_n$, establishing part (i) of the corollary. Property (c) implies part (iv), provided we pick $\delta < \varepsilon$. For part (iii), let $t \in \mathcal{T}_n$ and consider a pair $(\psi, c)$ with $\psi \in \binom{K_n}{K_m}$, $c \in \mathcal{F}_m$ and $c \in \langle t|\psi \rangle$. Since $\mathcal{P}$ is order-hereditary, for every $\phi \in \binom{K_n}{K_m}$ extending $\psi$ (i.e., with $\phi(i) = \psi(i)$ for all $i \in [m]$, there exists $c' \in \mathcal{F}_n$ with $c' \in \langle t|\phi \rangle$. As we know by property (c) that there are at most $\delta \binom{n}{N}$ such pairs $(\phi, c')$ for any $t \in \mathcal{T}_n$, we have that

$$\left|\{(\psi, c) : \psi \in \binom{K_n}{K_m}, c \in \mathcal{F}_m, c \in \langle t|\psi \rangle\}\right| \leq \binom{n}{N} \delta \binom{n}{N}.$$ 

implying part (iii), provided we pick $\delta < \varepsilon$. Finally for part (ii) we use supersaturation: there exists $n_0 \geq n_1$ such that if $\delta$ is sufficiently small, $t \in (2^{|K|} \setminus \{\emptyset\})^{K_m}$ with $n \geq n_0$ and there are at most $\delta \binom{n}{N}$ pairs $(\phi, c) \in \binom{K_n}{K_m} \times \mathcal{F}_n$ with $c \in \langle t|\phi \rangle$, then

$$\text{Ent}(t) \leq \left(\pi(\text{Forb}(\mathcal{F}_n)) + \frac{\varepsilon}{2}\right) \binom{n}{2} = \left(\pi(Q^n) + \frac{\varepsilon}{2}\right) \binom{n}{2} < (\pi(\mathcal{P}) + \varepsilon) \binom{n}{2}.$$ 

Thus for $n \geq n_0$ and $\delta > 0$ sufficiently small, property (c) implies part (ii) and we are done.

**Corollary 2.15** (Speed of order-hereditary properties) Let $\mathcal{P}$ be an order-hereditary property of $k$-colorings with $\mathcal{P}_n \neq \emptyset$ for every $n \in \mathbb{N}$ and let $\varepsilon > 0$ be fixed. There exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$k^{\pi(\mathcal{P})} \leq |\mathcal{P}_n| \leq k^{\left(\pi(\mathcal{P}) + \varepsilon\right)}. $$

**Proof** The lower bound is given by inequality (2.8). For the upper bound, we apply Corollary 2.14 (with parameter $\varepsilon/2$) to obtain for all $n \geq n_0$ a family of templates $\mathcal{T}_n$ satisfying properties (i), (ii), and (iv). Thus for $n \geq n_0$,

$$|\mathcal{P}_n| \leq |\mathcal{T}_n|^{\max\{\text{Ent}(t) : t \in \mathcal{T}_n\}} \leq k^{\left(\frac{n}{2}\right)} k^{\pi(\mathcal{P})} k^{\left(\frac{n}{2}\right)} = k^{\left(\pi(\mathcal{P}) + \varepsilon\right)}.$$ 

**2.5 Stability and characterization of typical colorings**

**Definition 2.16** (Edit distance) The edit distance $\rho(t, t')$ between two $k$-coloring templates $t, t'$ of $K_n$ is the number of edges $e \in E(K_n)$ on which $t(e) \neq t'(e)$. Further, if $S$ is a family of $k$-coloring templates and $t$ is a $k$-coloring template of $K_n$, we define the edit distance between them to be

$$\rho(S, t) := \min\{\rho(s, t) : s \in S \cap (2^{|K|} \setminus \{\emptyset\})^{K_n}\}$$

if $S$ contains at least one element of $(2^{|K|} \setminus \{\emptyset\})^{K_n}$, and $\binom{n}{2}$ otherwise.
Given a family of \(k\)-coloring templates \(S\), we let \(\langle S \rangle\) denote the collection of all realizations from \(S\). We further let \(\rho((S), c)\) denote the edit distance between \(\langle S \rangle\) and a \(k\)-coloring \(c\).

**Definition 2.17** (Stability) Let \(P\) be an order-hereditary property of \(k\)-colorings of \(K_n\). A family \(S\) of \(k\)-coloring templates is said to be a stability family for \(P\) if the following holds:

For all \(\varepsilon > 0\), there exist \(\delta > 0\) and \(m, n_0 \in \mathbb{N}\) such that for all \(n \geq n_0\), every \(t \in (2^{|k|} \setminus \{\emptyset\})^k\) with \(c\)-colorings \(\in \mathbb{P}_n\) with \(c\) a \(k\)-coloring template for \(K_n\) and suppose \(c\) is a \(k\)-coloring of \(K_n\) which lies in \(P\) and \(e\) is any edge of \(K_n\), the coloring \(\tilde{c}\) obtained from \(c\) by changing the color of \(e\) to \(i\) also lies in \(P\).

**Theorem 2.18** (Characterization of typical colorings in stable order-hereditary properties) Let \(P\) be an order-hereditary property of \(k\)-colorings and suppose \(S\) is a stability family for \(P\). Then typical elements of \(P\) are close in the edit distance to realizations from \(S\).

Explicitly, for all \(\varepsilon > 0\), there exists \(n_0 \in \mathbb{N}\) such that for all \(n \geq n_0\) there are at most \(\varepsilon|P_n|\) colorings \(c \in \mathbb{P}_n\) with \(\rho((S), c) < \varepsilon\).

**Proof** Let \(\varepsilon > 0\), and let \(\delta, m, n_0\) be as given by Definition 2.17 applied to \(S\) and \(P\) with parameter \(\varepsilon\). Apply Corollary 2.14 to \(P\) with parameters \(\varepsilon_1 > 0, m, n \in \mathbb{N}\), with \(\varepsilon_1 < \delta\), to get (for all \(n\) sufficiently large) a container family \(T_n\) for \(P_n\).

Now remove from \(T_n\) any template \(t\) with \(\operatorname{Ent}(t) < (\pi(P) - \delta)^n\) and let \(T'_n \subseteq T_n\) denote the resulting subfamily. By Corollary 2.14 parts (i) and (iv), Proposition 2.10, and (2.8), the number of elements of \(P_n\) which are not realizable from a template \(t \in T'_n\) is then at most

\[
|T_n|k^{(\pi(P) - \delta)^n} \leq k^{(\pi(P) + \varepsilon_1 - \delta)^n} \leq k^{\varepsilon_1 n} \leq |P_n|k^{(\varepsilon_1 - \delta)^n}.
\]

Since \(\varepsilon_1 < \delta\), the right hand side is less than \(\varepsilon|P_n|\) for all \(n\) sufficiently large.

Now let \(c\) be a member of \(P_n\) which is realizable from a template \(t \in T'_n\). Since \(t \in T'_n\), we have \(\operatorname{Ent}(t) \geq (\pi(P) - \delta)^n\). Also since \(\varepsilon_1 < \delta\), Corollary 2.14 part (iii) implies that \(t\) satisfies condition (ii) from Definition 2.17, and so there is a template \(s \in S\) such that \(\rho(s, t) < \varepsilon^n\). Since \(c\) realizes \(t\), this readily implies that \(\rho((S), c) < \varepsilon^n\).

### 2.6 Transference

**Definition 2.19** (Multicolor monotonicity) An order-hereditary property \(P\) of \(k\)-colorings is monotone with respect to color \(i \in [k]\), or \(i\)-monotone, if whenever \(c\) is a \(k\)-coloring of \(K_n\) which lies in \(P\) and \(e\) is any edge of \(K_n\), the coloring \(\tilde{c}\) obtained from \(c\) by changing the color of \(e\) to \(i\) also lies in \(P\).

**Definition 2.20** (Meet of two templates) Given two \(k\)-coloring templates \(t, t'\) of \(K_n\) which have at least one realization in common, we denote by \(t \wedge t'\) the template with \((t \wedge t')(e) = t(e) \cap t'(e)\) for every \(e \in E(K_n)\); we refer to \(t \wedge t'\) as the meet of \(t\) and \(t'\). More generally, given a set \(S\) of \(k\)-coloring templates of \(K_n\) and \(t \in (2^{|k|} \setminus \{\emptyset\})^k\) we denote by \(S \wedge t\) the collection \(\{s \wedge t : s \in S\}\).

**Definition 2.21** (Complete, random, and constant templates) Let \(T_n = [k]^K\) denote the complete \(k\)-coloring template for \(K_n\), that is, the unique template allowing all \(k\) colors on every edge. Given a
fixed color \(i \in [k]\) and \(p \in [0, 1]\), we define the \(p\)-random template \(T_{n,p} = T_{n,p}(i)\) to be the random template for a \(k\)-coloring of \(K_n\) obtained by letting

\[
T_{n,p}(e) = \begin{cases} [k] & \text{with probability } p, \\ \{i\} & \text{otherwise}, \end{cases}
\]

independently for each edge \(e \in E(K_n)\). Finally, we define \(E_n = E_n(i)\) to be the \(i\)-monotone template with \(E_n(e) = \{i\}\) for each \(e \in E(K_n)\).

The \(p\)-random template \(T_{n,p}\) is a \(k\)-coloring analog of the celebrated Erdős-Rényi binomial random graph \(G_{n,p}\), while the zero-entropy template \(E_n\) is a \(k\)-coloring analog of the empty graph. Just as extremal theorems for the graph \(K_n\) can be reproved in the sparse random setting of \(G_{n,p}\), so also extremal entropy results for \(i\)-monotone properties in \(T_n\) can be transferred to the \(T_{n,p}(i)\) setting.

**Definition 2.22 (Relative entropy)** Let \(\mathcal{P}\) be a property of \(k\)-colorings of complete graphs and let \(t \in (2^k \setminus \{\emptyset\})^{K_n}\) with \(\langle t \rangle \cap \mathcal{P}_n \neq \emptyset\). We define the extremal entropy of \(\mathcal{P}\) relative to \(t\) to be:

\[
ex(t, \mathcal{P}) := \max\{\text{Ent}(t') : v(t') = n, t' \leq t, \langle t' \rangle \subseteq \mathcal{P}_n\}.
\]

Note that this extends the notion of extremal entropy introduced in Definition 2.5: \(\text{ex}(n, \mathcal{P}) = \text{ex}(T_n, \mathcal{P})\). Our next theorem states that for \(p\) not too small, whp the extremal entropy of an \(i\)-monotone property \(\mathcal{P}\) relative to a \(p\)-random template \(T_{n,p}(i)\) is \(p \left( \text{ex}(n, \mathcal{P}) + o(n^2) \right)\).

**Theorem 2.23 (Transference)** Let \(i \in [k]\) be fixed. Let \(\mathcal{P}\) be an order-hereditary, \(i\)-monotone property of \(k\)-colorings of complete graphs defined by forbidden colorings of \(E(K_N)\) for some \(N \geq 2\). Let \(p = p(n) \gg n^{-1/2(\varepsilon^2) - 1}\) and let \(T\) denote an instance of the \(p\)-random template \(T_{n,p}(i)\). For any \(\varepsilon > 0\), whp

\[
p \left( \text{ex}(n, \mathcal{P}) - \varepsilon n^2 \right) \leq \text{ex}(T, \mathcal{P}) \leq p \left( \text{ex}(n, \mathcal{P}) + 2\varepsilon n^2 \right). \tag{2.9}
\]

**Proof** Let \(\varepsilon > 0\) be fixed, and let \(C_0^{2.6}\) and \(C_0^{2.11}\) denote the constants in Theorem 2.6 and Lemma 2.11 respectively.

Applying Theorem 2.6 with parameter \(\varepsilon_1 = C_0^{2.11}\varepsilon\) followed by Lemma 2.11 gives for all \(n \geq n_0 = n_0(\varepsilon)\) sufficiently large a container family \(\mathcal{T}_n\) for \(\mathcal{P}_n\) such that \(\log_k |\mathcal{T}_n| \leq C_0^{2.6} n^{-1/2(\varepsilon^2) - 1} \binom{n}{2}\) and every template \(t \in \mathcal{T}_n\) has entropy at most \(\text{ex}(n, \mathcal{P}) + \varepsilon \binom{n}{2}\) (property (ii) of Theorem 2.6 together with Lemma 2.11).

Now let \(T\) be an instance of the random \(k\)-coloring template \(T_{n,p}\). As \(\mathcal{T}_n\) is a container family for \(\mathcal{P}_n\), \(\mathcal{T}_n \land T\) is a container family for \(\mathcal{P}_n \land T\). In particular,

\[
\text{ex}(T, \mathcal{P}) \leq \max\{\text{Ent}(t \land T) : t \in \mathcal{T}_n\}. \tag{2.10}
\]

We claim that whp the right-hand side is at most \(p \left( \text{ex}(n, \mathcal{P}) + 2\varepsilon n^2 \right)\). Indeed for each \(t \in \mathcal{T}_n\) we have

\[
\text{Ent}(t \land T) = \sum_{e \in E(K_n)} \log_k |t \land T(e)| = \sum_{e : T(e) = [k]} \log_k |t(e)|.
\]

Partition the edges of \(K_n\) into \(k\) sets, \(A_1, A_2, \ldots, A_k\), where \(A_i = \{e : |t(e)| = i\}\). Set \(A_{i,T} := A_i \cap \{e : T(e) = [k]\}\). By the equation above, each edge in \(A_{i,T}\) contributes \(\log_k i \leq 1\) to the entropy of \(t \land T\).
A simple union bound together with an application of the Chernoff bound (2.1) then yields:

\[ \mathbb{P} \left( \text{Ent}(t \land T) > p \text{Ent}(t) + p\varepsilon n^2 \right) \leq \mathbb{P} \left( \exists i : \ |A_i,t| > p |A_i| + \frac{2\varepsilon}{k^2} n^2 \right) \leq 2ke^{-\frac{c_1p^2}{2n^2}} \tag{2.11} \]

Bringing together (2.10), our bound on the entropy of templates from \( T_n \) and the inequality (2.11), we get

\[ \mathbb{P} \left( \text{ex}(T, \mathcal{P}) > p(\text{ex}(n, \mathcal{P}) + 2\varepsilon n^2) \right) \leq \mathbb{P} \left( \exists t \in T_n : \ \text{Ent}(t \land T) > p \text{Ent}(t) + p\varepsilon n^2 \right) \leq |T_n|2ke^{-\frac{c_1p^2}{2n^2}}. \]

Since \( \log_2 |T_n| = O\left( n^{\frac{2-\varepsilon}{(\zeta^2)-1}} \right) \), the expression above is of order \( o(1) \) if \( p \gg n^{-1/(\zeta^2)-1} \). Thus for such values of \( p \), whp \( \text{ex}(T, \mathcal{P}) \leq p(\text{ex}(n, \mathcal{P}) + 2\varepsilon n^2) \). This establishes the upper bound in (2.9).

For the lower bound, consider a maximum entropy template \( t_* \) for \( P_n \). Applying the Chernoff bound (2.1), we have

\[ \mathbb{P} \left( \text{Ent}(t_* \land T) < p(\text{ex}(n, \mathcal{P}) - \varepsilon n^2) \right) \leq 2e^{-\frac{c_2^2p}{2n^2}}, \]

which is \( o(1) \) for \( \varepsilon > 0 \) fixed and \( p \gg n^{-2} \). Thus certainly for \( p \gg n^{-\frac{1}{(\zeta^2)-1}} \) we have whp

\[ \text{ex}(T, \mathcal{P}) \geq \text{Ent}(t_* \land T) \geq p(\text{ex}(n, \mathcal{P}) - \varepsilon n^2). \]  

\[ \square \]

**Remark 2.24** The bound on \( p \) required in Theorem 2.23 is not best possible in general (see [20]). This bound can be improved by using the more powerful container theorems of [13,53] rather than the simple hypergraph container theorem, Theorem 2.7. However we do not pursue such improvements further here.

We note Theorem 2.23 can be extended to cover general order-hereditary properties.

**Corollary 2.25** (Transference for general order-hereditary properties) Let \( \mathcal{P} \) be an order-hereditary, \( i \)-monotone property of \( k \)-colorings of complete graphs. Let \( p = p(n) \) be a sequence of probabilities satisfying \( \log(1/p) = o(\log n) \). For any fixed \( \varepsilon > 0 \), whp

\[ p \left( \text{ex}(n, \mathcal{P}) - \varepsilon n^2 \right) \leq \text{ex}(T_{n,p}(i), \mathcal{P}) \leq p \left( \text{ex}(n, \mathcal{P}) + 4\varepsilon n^2 \right). \]

**Proof** Let \( \varepsilon > 0 \) be fixed. As in Theorem 2.13, approximate \( \mathcal{P} \) from above by some property \( \mathcal{Q} \) defined by forbidden colorings on at most \( N \) vertices and satisfying \( \mathcal{P} \subseteq \mathcal{Q} \) and \( \pi(\mathcal{Q}) \leq \pi(\mathcal{P}) + \varepsilon \). For \( n \) sufficiently large,

\[ \text{ex}(n, \mathcal{Q}) \leq \pi(\mathcal{Q})\left( \frac{n}{2} \right) + \varepsilon \left( \frac{n}{2} \right) \leq \pi(\mathcal{P})\left( \frac{n}{2} \right) + \varepsilon n^2 \leq \text{ex}(n, \mathcal{P}) + 2\varepsilon n^2. \]

Applying Theorem 2.23 to \( \mathcal{Q} \), and noting that our condition on \( p \) ensures \( p \gg n^{-\left(1/(\zeta^2)-1\right)} \) for all \( N \in \mathbb{N} \), we obtain the desired upper bound: whp

\[ \text{ex}(T_{n,p}(i), \mathcal{P}) \leq \text{ex}(T_{n,p}(i), \mathcal{Q}) \leq p \left( \text{ex}(n, \mathcal{Q}) + 2\varepsilon n^2 \right) \leq p \left( \text{ex}(n, \mathcal{P}) + 4\varepsilon n^2 \right). \]
For the lower bound, let $t^{\star}$ be a maximum entropy template for $\mathcal{P}_n$. Applying the Chernoff bound as in the proof of Theorem 2.23, we have that whp

$$
\text{ex}(T_n,p(i), P) \geq \text{Ent}(t^{\star} \land T_n, p(i)) \geq p \left( \text{ex}(n, P) - \varepsilon n^2 \right).
$$

\[\square\]

### 3 Containers for Other Discrete Structures

Our container results so far allow us to compute the speed of (dense) order-hereditary properties of $k$-colorings of $K_n$, as well as to characterize typical colorings and (in the $i$-monotone case) to transfer extremal entropy results to the sparse random setting. However, the container theory of Saxton-Thomason and Balogh-Morris-Samotij is robust enough to cover $k$-colorings of many other interesting discrete structures, which is what we explore in this section.

As we show, all that is required for (the existence of) a container theorem is, in essence, a sufficiently rich notion of substructure: provided we have a sequence of $r$-graphs $(H_n)_{n \in \mathbb{N}}$ such that $e(H_n) \to \infty$ as $n \to \infty$ and for each $N \in \mathbb{N}$ we have “many” embeddings of $H_N$ into $H_n$, we can derive a container theorem. In this regard, container theory is somewhat reminiscent of the versatile theory of flag algebras developed by Razborov [50], which can treat any class of discrete structures with a sufficiently rich notion of substructure.

This section is organized as follows: first in Section 3.1, we outline how our $k$-coloring extensions of the container theorems of Balogh-Morris-Samotij and Saxton-Thomason can be applied to tournaments, oriented graphs, and directed graphs; next in Section 3.2 we derive container theorems for very general discrete structure, namely set-sequences equipped with embeddings (or see, see Definition 3.5); in Section 3.3, we record the consequences of our results for sequences of graphs and hypergraphs, which are the special cases most relevant in applications; in Section 3.4, we illustrate our results by obtaining general counting theorems for $k$-coloring properties of hypercube graphs.

#### 3.1 Tournaments, oriented graphs and directed graphs

Tournaments, oriented graphs, and directed graphs are important objects of study in discrete mathematics and computer science, with a number of applications both to other branches of mathematics and to real-world problems. As we show below, we can encode these structures within our framework of order-hereditary properties for $k$-colorings of $K_n$, which immediately gives container, supersaturation, counting, characterization, and transference theorems for these objects. We note containers had not been successfully applied to the directed graph setting before (see Section 4.3 for a discussion, or the remark after Corollary 3.4 in Kühn, Osthus, Townsend, and Zhao [43]).

Formally, a **directed graph**, or **digraph**, is a pair $D = (V, E)$, where $V = V(D)$ is a set of vertices and $E = E(D) \subseteq V \times V$ is a collection of ordered pairs from $V$. By convention, we write $\vec{i}j$ to denote $(i, j) \in E$. Note that we could have both $\vec{i}j \in E(D)$ and $\vec{j}i \in E(D)$, in which case we say that $ij$ is a **double edge** of $D$.

An **oriented graph**, or **orgraph**, is a digraph $\vec{G}$ in which for each pair $ij \in V(\vec{G})^{(2)}$ at most one of $\vec{i}j$ and $\vec{j}i$ lies in $E(\vec{G})$. A **tournament** $\vec{T}$ is a digraph in which for each pair $ij \in V(\vec{T})^{(2)}$ exactly one $\vec{i}j$ and $\vec{j}i$ lies in $E(\vec{T})$; alternatively, a tournament can be viewed as an orientation of the edges of the complete graph.

A **monotone (decreasing)** property of digraphs/orgraphs is a property of digraphs/orgraphs which is closed with respect to taking subgraphs (ie, closed under the deletion of vertices
and oriented edges). A hereditary property of digraphs/orgraphs/tournaments is a property of digraphs/orgraphs/tournaments which is closed with respect to taking induced subgraphs.

**Observation 3.1** Tournaments, oriented graphs, and directed graphs on the labeled vertex set \([n]\) can be encoded as 2-, 3-, and 4-colorings of \(K_n\). Moreover, under this encoding, hereditary properties of tournaments, oriented graphs, and directed graphs correspond to order-hereditary properties of 2-, 3-, and 4-colorings of \(K_n\) respectively.

**Proof** Given a directed graph \(D\) on \([n]\), we define a coloring \(c\) of \(E(K_n)\) by setting for each pair \(ij \in [n]^{(2)}\) with \(i < j\)

\[
c(ij) := \begin{cases} 
1 & \text{if neither of } \vec{i}j, \vec{j}i \text{ lies in } E(D), \\
2 & \text{if } \vec{i}j \in E(D), \vec{j}i \notin E(D), \\
3 & \text{if } \vec{i}j \notin E(D), \vec{j}i \in E(D), \\
4 & \text{if both of } \vec{i}j, \vec{j}i \text{ lie in } E(D).
\end{cases}
\]

The digraph part of Observation 3.1 is immediate from this coloring and our definition of order-hereditary properties. Tournaments for their part correspond to colorings of \(E(K_n)\) with the palette \{2, 3\} and oriented graphs to colorings with the palette \{1, 2, 3\}.

**Remark 3.2** Monotone properties of digraphs/orgraphs give rise to 1-monotone order-hereditary properties of 4-/3-colorings of \(K_n\), and so are covered by our transference results. However, there are some theoretical subtleties to bear in mind in the digraph case: a monotone digraph property has monotonicity "away from color 4" as well as monotonicity "towards color 1." Thus there could be interesting and natural alternative models to the \(p\)-random template \(T = T_{n,p}\) to study in connection with transference for digraph properties. Instead of letting \(T(e) = [4]\) with probability \(p\) and \{1\} with probability \(1 - p\) for each edge \(e\), one could consider more general distributions on the collection of subsets of \([4]\) containing the color 1.

In addition, we should make it clear that there are examples of 1-monotone properties of 4-colorings of \(K_n\) which do not correspond to monotone properties of digraphs. A nice example of such a property suggested by one of the referees is the property of having every pair in color either 1 or 4. This is 1-monotone, but does not give rise to a monotone property of digraphs in our encoding.

**Corollary 3.3** If \(\mathcal{P}\) is a hereditary property of digraphs/orgraphs/tournaments defined by forbidden configurations on at most \(N\) vertices and \(k = 4/3/2\) is the corresponding number of colors from Observation 3.1, then the conclusions of Theorem 2.6, Lemma 2.11, Theorems 2.12, 2.18, and 2.23 hold for \(\mathcal{P}\).

**Corollary 3.4** If \(\mathcal{P}\) is a hereditary property of digraphs/orgraphs/tournaments and \(k = 4/3/2\) is the corresponding number of colors from Observation 3.1, then the conclusions of Corollaries 2.14 and 2.15, Theorem 2.18, and Corollary 2.25 hold for \(\mathcal{P}\).

In particular, we have general container, counting, stability, and transference results for hereditary properties of digraphs, orgraphs, and tournaments. As mentioned earlier, this overcomes an obstruction to the extension of containers to the digraph setting.
3.2 | Other host structures: set-sequences equipped with embeddings (ssee-s)

The work in Section 2 was concerned with k-coloring properties of the sequence of complete graphs $K = (K_n)_{n \in \mathbb{N}}$. However, there are many other interesting natural graph sequences we might wish to study. Examples of such sequences include:

- $P = (P_n)_{n \in \mathbb{N}}$, the sequence of paths on $[n]$, $P_n = ([n], \{i(i + 1) : 1 \leq i \leq n - 1\})$;
- $\text{Grid} = (P_n \times P_n)_{n \in \mathbb{N}}$, the sequence of $n \times n$ grids $P_n \times P_n$ obtained by taking the Cartesian product of $P_n$ with itself, or, more generally for $(a, b) \in \mathbb{N}^2$ the sequence of rectangular grids $\text{Grid}(a, b) = (P_{an} \times P_{bn})_{n \in \mathbb{N}}$ with vertex-set $\{(x, y) : 1 \leq x \leq an, 1 \leq y \leq bn\}$ and edge-set $\{(x, y)(x', y') : |x - x'| + |y - y'| = 1\}$;
- $B_b = (B_{b,n})_{n \in \mathbb{N}}$, the sequence of $b$-branching trees with $n$ generations from a single root;
- $K_q = (K_q(n))_{n \in \mathbb{N}}$, the sequence of complete balanced $q$-partite graphs on $qn$ vertices;
- $Q = (Q_n)_{n \in \mathbb{N}}$, the sequence of $n$-dimensional discrete hypercube graphs $Q_n = ([0, 1]^n$, $\{xy : x_i = y_i$ for all but exactly one index $i\})$.

Outside of extremal combinatorics, the sequences $Q$ and $B_2$ are of central importance in theoretical computer science and discrete probability (they represent $n$-bit sequences and binary search trees respectively), while the sequence $\text{Grid}$ has been extensively studied in the context of percolation theory, in particular with respect to crossing probabilities.

Each of the graph sequences above comes equipped with a natural notion of “substructure”—subpaths of a path, subgrids of a grid, subtrees of a branching tree, $q$-partite subgraphs of a $q$-partite graph, subcubes of a hypercube—and of “embeddings” of earlier terms of the sequence into later ones, which leads to a natural notion of an (order-) hereditary property.

As we shall see in this subsection, the container theory of Balogh-Morris-Samotij and Saxton-Thomason is powerful enough to cover the case of $k$-colorings of any graph sequence $G$ with a “sufficiently rich” notion of substructure. More generally, we shall derive container theorems for $k$-coloring properties of some very general structures (good ssee, defined below) which cover $k$-colorings of vertices and $k$-coloring of edges of “good” hypergraph sequences (and many other structures besides) as special cases. Roughly speaking, a hypergraph sequence $G$ is (edge-) good if it is rich in embeddings—for any $N$ fixed and $n \geq N$ there must be many almost disjoint ways of embedding $G_N$ into $G_n$ relative to the number of edges.

Our main results in this subsection are a container theorem for hereditary $k$-colorings of “good” set-sequences (Theorem 3.18), and, modulo some easily checkable technical conditions, the accompanying counting results (Theorems 3.20 and 3.21). Cases of interest covered by our result include $k$-colorings of $K_n$, $\text{Grid}$ and both vertex- and edge-$k$-colorings of $Q$. A final observation to make before we give our definitions and results is that, as we shall show in Section 4.7, some form of our “goodness” assumption is necessary—the sequence $P$, for instance, has too few embeddings to be “good,” and we give an example of a hereditary $k$-coloring property for $P$ for which the statement of Theorem 3.18 fails.

**Definition 3.5** (Ssee, embeddings) A set-sequence equipped with embeddings, or ssee, is a sequence $V = (V_n)_{n \in \mathbb{N}}$ of sets $V_n$, together with for every $N \leq n$ a collection $\binom{V_n}{V_N}$ of injections $\phi : V_N \to V_n$.

We refer to the members of $\binom{V_n}{V_N}$ as embeddings of $V_N$ into $V_n$.

Ssee may seem rather abstract, so let us immediately give some examples.
Example 3.6 Let $V$ denote a sequence $(V_n)_{n \in \mathbb{N}}$ of partially ordered sets, with embeddings $(V_n)_{V_n}$ consisting of all order-preserving injections from $V_N$ to $V_n$. Then $V$ is an ssee.

Example 3.7 Consider a sequence of graphs $G = (G_n)_{n \in \mathbb{N}}$ on linearly ordered vertex sets. We can obtain an ssee from $G$ by taking as our set-sequences the vertex-sets $V_n = V(G_n)$ and setting $(V_n)_{V_n}$ to be the collection of all order-preserving injections $\phi : V_N \to V_n$ such that $\phi(x)\phi(y) \in E(G_n)$ if and only if $xy \in E(G_N)$—in other words, the collection of all order-preserving embeddings of $G_N$ into $G_n$.

Example 3.8 Consider again a sequence of graphs $G = (G_n)_{n \in \mathbb{N}}$ on linearly ordered vertex-sets. We can obtain another ssee from $G$ by taking the sets of our sequence to be the edge-sets $E(G_n)$ and setting $(E(G_n))_{E(G_n)}$ to be the collection of injections $\psi : E_N \to E_n$ arising from order-preserving embeddings $\phi : V(G_N) \to V(G_n)$ (ie, such that $\psi(e) = \{\phi(x) : x \in e\}$).

Example 3.9 Consider a sequence of permutations $\sigma_n \in S_n$. Let $V_n = [n]$, and let $(V_n)_{V_n}$ denote the collection of order-preserving injections $\phi : [N] \to [n]$ such that $\sigma_n(\phi(i)) < \sigma_n(\phi(j))$ whenever $\sigma_N(i) < \sigma_N(j)$. This constitutes an ssee.

Example 3.10 Consider a sequence of groups $((\Gamma_n, +_n))_{n \in \mathbb{N}}$. For every $n$, let $V_n$ be a nonempty subset of $\Gamma_n$, and let $(V_n)_{V_n}$ denote the collection of injections $\phi : V_N \to V_n$ which preserve the group actions, that is, such that $\phi(x +_N y) = \phi(x) +_n \phi(y)$ for all $x, y \in V_N$ with $x +_N y \in V_N$. This constitutes an ssee.

A more concrete form of the last example, which has already been extensively studied using containers, is that of subsets of $(\mathbb{Z}, +)$, which are connected to many problems in additive combinatorics—see the original papers of Balogh, Morris, and Samotij [13] and of Saxton and Thomason [53].

Having thus set the scene with some motivational examples of ssee-s, we now turn to the main business of this section, namely generalizing Theorem 2.6 to the ssee setting. To do this, we need notions of colorings, templates and extremal entropy relative to a set.

Definition 3.11 (Colorings, templates, and entropy relative to a set) Let $V$ be a set. A $k$-coloring template of $V$ is a function $t : V \to 2^{[k]} \setminus \{\emptyset\}$, while a $k$-coloring of $V$ is a function $c : V \to [k]$. We denote the set of all $k$-coloring templates of $V$ and the set of all $k$-colorings of $V$ by $\left(2^{[k]} \setminus \{\emptyset\} \right)^V$ and $[k]^V$ respectively.

Given a template $t \in \left(2^{[k]} \setminus \{\emptyset\} \right)^V$, we write $\langle t \rangle$ for the collection of realizations of $t$, that is, the collection of $k$-colorings $c \in [k]^V$ such that $c(e) \in t(e)$ for every $e \in V$. The entropy of a $k$-coloring template $t$ of $V$ is

$$\text{Ent}(t) := \sum_{e \in V} \log_k |t(e)|.$$

Observe that $0 \leq \text{Ent}(t) \leq |V|$ and $|\langle t \rangle| = k^{\text{Ent}(t)}$.

Definition 3.12 (Extremal entropy relative to an ssee) Let $V = (V_n)_{n \in \mathbb{N}}$ be an ssee. A $k$-coloring property of $V$ is a sequence $P = (P_n)_{n \in \mathbb{N}}$, where $P_n$ is a collection of $k$-colorings of $V_n$. The extremal entropy of $P$ relative to $V$ is

$$\text{ex}(V, P)_n = \text{ex}(V_n, P_n) := \max \left\{ \text{Ent}(t) : t \in \left(2^{[k]} \setminus \{\emptyset\} \right)^{V_n}, \langle t \rangle \subseteq P_n \right\}.$$
Definition 3.13 (Hereditary properties for an ssee) Let \( V = (V_n)_{n \in \mathbb{N}} \) be an ssee. Given an embedding \( \phi \in \binom{V_n}{V} \) and a template \( t \in \left( 2^{[k]} \setminus \{\emptyset\} \right)^{V_n} \), we denote by \( t|_{\phi} \) the \( k \)-coloring template for \( V_N \) induced by \( \phi \),
\[
t|_{\phi}(x) = t(\phi(x)) \quad \forall x \in V_N.
\]

A hereditary \( k \)-coloring property for an ssee \( V \) is a \( k \)-coloring property \( P = (P_n)_{n \in \mathbb{N}} \) such that for all \( n \geq N \), \( c \in P_n \) and \( \phi \in \binom{V_n}{V} \), we have \( c|_{\phi} \in P_N \).

Remark 3.14 This is a common generalization of the notion of hereditary and order-hereditary properties for graphs: by choosing one’s embeddings appropriately when building an ssee from a graph sequence, we can encode either kind of property as an ssee hereditary property.

We are now in a position to state what a “sufficiently rich” notion of substructure means.

Definition 3.15 (Intersecting embeddings) Let \( V = (V_n)_{n \in \mathbb{N}} \) be an ssee. Let \( N_1, N_2 \leq n \). An \( i \)-intersecting embedding of \((V_{N_1}, V_{N_2})\) into \( V_n \) is a function \( \phi : V_{N_1} \sqcup V_{N_2} \to V_n \) such that:

(i) the restriction of \( \phi \) to \( V_{N_1} \) lies in \( \binom{V_n}{V_{N_1}} \), and the restriction of \( \phi \) to \( V_{N_2} \) lies in \( \binom{V_n}{V_{N_2}} \);

(ii) \( |\phi(V_{N_1}) \cap \phi(V_{N_2})| = i \).

We denote by \( I_i((V_{N_1}, V_{N_2}), V_n) \) the number of \( i \)-intersecting embeddings of \((V_{N_1}, V_{N_2})\) into \( V_n \), and set
\[
I(N, n) := \sum_{i < i \leq |V_n|} I_i((V_N, V_N), V_n).
\]

Definition 3.16 (Good ssee) A ssee \( V \) is good if it satisfies the following conditions:

(i) \( |V_n| \to \infty \) (“the sets in the sequence become large”);

(ii) for all \( N \in \mathbb{N} \) with \( |V_n| > 1 \), \( \binom{V_n}{V_{N}} \gg |V_n| \) (“on average, vertices in \( V_n \) are contained in many embedded copies of \( V_N \)’s”);

(iii) for all \( N \in \mathbb{N} \) with \( |V_n| > 1 \), \( \left( |V_n|I(N, n)\right) \bigg/ \left| \binom{V_n}{V_{N}} \right|^2 \to 0 \) as \( n \to \infty \) (“most pairs of embeddings of \( V_N \) into \( V_n \) share at most one vertex”).

Remark 3.17 Condition (iii) can be interpreted as an “average co-degree condition” in a certain hypergraph, namely \( H \) in the proof of Theorem 3.18. Thus our “goodness” condition is related to the more usual ‘co-degree conditions’ found in the the container theorems of Balogh-Morris-Samotij [13] and Saxton-Thomason [53].

Let \( V \) be an ssee. Given a collection \( \mathcal{P} \) of \( k \)-colorings of \( V_N \), denote by \( \text{Forb}_V(\mathcal{P}) \) the order-hereditary property of \( k \)-colorings of \( V \) not containing an embedding of a coloring in \( \mathcal{P} \), that is
\[
\text{Forb}_V(\mathcal{P}) = \left\{ c \in [k]^{V_n} : \forall \phi \in \binom{V_n}{V_N}, c|_{\phi} \not\in \mathcal{P} \right\}.
\]

Our main result in this subsection is that if \( V \) is a good ssee, then we have a container theorem for \( \text{Forb}_V(\mathcal{P}) \). As before, we say that a template family \( \mathcal{T}_n \) is a container family for a family of colorings \( \mathcal{P}_n \) if for every \( c \in \mathcal{P}_n \) there is \( t \in \mathcal{T}_n \) with \( c \in \langle t \rangle \).
Theorem 3.18  Let $\mathbf{V}$ be a good ssee, and let $k, N \in \mathbb{N}$. Let $\mathcal{F}$ be a nonempty collection of $k$-colorings of $V_N$ and let $\mathcal{P} = \text{Forb}_V(\mathcal{F})$. For any $\varepsilon > 0$, there exists $n_0 > 0$ such that for any $n \geq n_0$ there exists a collection $\mathcal{T}_n$ of $k$-coloring templates for $V_n$ satisfying:

(i) $\mathcal{T}_n$ is a container family for $\mathcal{P}_n$;

(ii) for each template $t \in \mathcal{T}_n$, there are at most $\varepsilon \binom{V_n}{V_r}$ pairs $(\phi, c)$ with $\phi \in \binom{V_n}{V_r}$, $c \in \mathcal{F}$ and $c \in \langle t_{\phi} \rangle$;

(iii) $|\mathcal{T}_n| \leq k^{\varepsilon |V_n|}$.

Proof  We follow in the main the proof of Theorem 2.6. Let $\mathbf{H}$ be a nonempty collection of forbidden colors and the single template $t = (\{k\} \setminus \mathcal{F})^{V_2}$ is a container for $\mathcal{P}_n$ lying entirely inside $\mathcal{P}_n$.

First we modify the construction of the hypergraph $H = H(\mathcal{F}, n)$ in the proof of Theorem 2.6 as follows:

- we set $r = |V_N|$ (rather than $\binom{N}{2}$);
- we let $V(H) = V_n \times [k]$ (rather than $E(K_n) \times [k]$);
- for every $\phi \in \binom{V_n}{V_r}$, and every coloring $c \in \mathcal{F}$, we add to $E(H)$ the $r$-edge

$$e_{\phi, c} = \{ (\phi(v), c(v)) : v \in V_N \}.$$ 

As before, we bound $e(H)$; since $\mathcal{F}$ is nonempty, we have the following analog of (2.2):

$$\left| \binom{V_n}{V_N} \right| \leq e(H) \leq k^{\varepsilon |V_n|} \left| \binom{V_n}{V_r} \right|.$$  (3.1)

Just as before, our problem is that $H$ may be far from linear, so that we cannot apply Theorem 2.7 directly. Here unlike in Theorem 2.6 we have two cases to consider.

Observe that $I(N, n)$ is exactly the number of $r$-edges in $H$ which meet in at least two vertices, that is, the “bad” pairs that make $H$ nonlinear, henceforth referred to as overlapping pairs. If $I(N, n) \leq \varepsilon \left| \binom{V_n}{V_r} \right| / 2$, then we can delete at most $\varepsilon e(H) / 2 r$-edges from $H$ to make $H$ linear. This leaves us with a linear $r$-graph with average degree

$$d \geq |V_N| \left( \left| \binom{V_n}{V_m} \right| - I(N, n) \right) / |V_n| \geq |V_N|(1 - \varepsilon / 2) \frac{\left| \binom{V_n}{V_r} \right|}{|V_n|},$$

which by the goodness condition (ii) tends to infinity as $n \to \infty$. From there, Theorem 3.18 follows easily from Theorem 2.7 applied with parameter $\delta = \varepsilon / 2k^{\varepsilon |V_n|}$: we obtain a collection $\mathcal{C}$ of sets which together cover all the independent sets in $H$ (property 1 of Theorem 2.7), each containing at most

$$\varepsilon \left| \binom{V_n}{V_N} \right| / 2 + \delta e(H) \leq \varepsilon \left| \binom{V_n}{V_N} \right|$$

$r$-edges (by property 2, our choice of $\delta$ and inequality (3.1)), with $\log_k |\mathcal{C}| \leq O\left( |V_n|d^{-\frac{1}{2|V_N|-1}} \right)$ (property 3). For $n$ sufficiently large, $\log_k |\mathcal{C}|$ is less than $\varepsilon |V_n|$ (since $d \gg 1$). The family $\mathcal{C}$ then gives us our desired family of templates $\mathcal{T}_n$ here just as it did in the proof of Theorem 2.6.
We therefore consider the more interesting case where \( I(N, n) > \varepsilon |(V_n, V_N)|/2 \)—this is the case where on average embeddings of \( V_N \) in \( V_n \) are involved in \( \Omega(1) \) overlapping pairs. Here we need a \( V \)-analog of our random sparsification lemma, Lemma 2.8. The goodness of \( V \) is exactly what is needed for the proof to go through as before.

Pick \( \varepsilon_1 > 0 \) sufficiently small so that

\[
24\varepsilon_1 k |V_n| < \varepsilon, \quad \varepsilon_1 < 1/6
\]  

and

\[
p = \varepsilon_1 \frac{|(V_n, V_N)|^2}{I(N, n)} < 1.
\]

Keep each \( r \)-edge of \( H \) independently with probability \( p \), and delete it otherwise, to obtain a random subgraph \( H' \) of \( H \).

**Lemma 3.19** Let \( p \) be as in (3.3), let \( H' \) be the random subgraph of \( H \) defined above, and consider the following events:

- the event \( F_1 \) that \( e(H') \geq \frac{p |(V_n, V_N)|}{2} \);
- the event \( F_2 \) that \( H' \) has at most \( 3p^2 I(N, n) = 3\varepsilon_1 p |(V_n, V_N)| \) pairs of \( r \)-edges \((e, e')\) with \(|e \cap e'| \geq 2\);
- the event \( F_3 \) that for all \( S \subseteq V(H) \) with \( e(H[S]) \geq 24\varepsilon_1 e(H) \), we have \( e(H'[S]) \geq 12\varepsilon_1 e(H') \).

There exists \( n_1 \in \mathbb{N} \) such that for all \( n \geq n_1 \), \( F_1 \cap F_2 \cap F_3 \) occurs with strictly positive probability.

**Proof** We follow the proof of Lemma 2.8. By (3.1), the definition of \( p \) and the goodness conditions (iii) and (i) for \( V \) we have

\[
pe(H) \geq p |(V_n, V_N)| = \varepsilon_1 \frac{|(V_n, V_N)|^2}{I(N, n)} \gg |V_n| \gg 1.
\]

Together with the Chernoff bound (2.1), inequality (3.4) implies

\[
\mathbb{P}(F_1 \text{ does not hold}) \leq 2 \exp \left( -p |(V_n, V_N)|/16 \right) = o(1).
\]

By Markov’s inequality applied to the number \( Y_{H'} \) of pairs of \( r \)-edges \((e, e')\) with \(|e \cap e'| \geq 2\) (i.e., the number of overlapping pairs in \( H' \)),

\[
\mathbb{P}(F_2 \text{ does not hold}) = \mathbb{P}(Y_{H'} \geq 3\mathbb{E}Y_{H'}) \leq \frac{1}{3}.
\]

Finally, consider a set \( S \subseteq V(H) \) with \( e(H[S]) \geq 24\varepsilon_1 e(H) \). Applying the Chernoff bound (2.1) with \( \delta = 1 - 1/\sqrt{2} \) and the lower bound (3.4) for \( pe(H) \) we get

\[
\mathbb{P} \left( e(H'[S]) \leq \frac{1}{\sqrt{2}} \mathbb{E}e(H'[S]) \right) \leq 2e^{-\left(1 - \frac{1}{\sqrt{2}} \frac{\mathbb{E}e(H[S])}{4}\right)} = e^{-\Omega(\rho e(H))} = e^{-\omega(|V_n|)}.
\]
Moreover, by (2.1) and (3.4) again,

$$\mathbb{P}\left( e(H') \geq \sqrt{2Ee(H')} \right) \leq 2e^{-\left(\sqrt{2-1}\frac{\mu(H)}{r}\right)} = e^{-o(|V_n|)}.$$  

(3.8)

Say that a nonempty set $S \subseteq V(H)$ is bad if $e(H[S]) \geq 24\varepsilon_1 e(H)$ and $e(H'[S]) \leq 12\varepsilon_1 e(H')$. By (3.7), (3.8), and the union bound, the probability that $F_3$ fails, that is, that there exists some bad $S \subseteq V(H)$, is at most

$$\mathbb{P}(\exists \text{ bad } S) \leq \mathbb{P}(e(H') \geq \sqrt{2Ee(H')}) + \sum_S \mathbb{P}(e(H'[S]) \leq \frac{1}{\sqrt{2}}Ee(H'[S])) \leq 2^{k|V_n|}e^{-o(|V_n|)} = o(1).$$  

(3.9)

Putting (3.5), (3.6), and (3.9) together we have that $F_1$, $F_2$, and $F_3$ hold simultaneously with probability at least $2/3 - o(1)$, which is strictly positive for $n$ sufficiently large.

With this sparsification lemma, we can now finish the proof in exactly the same way as we did in Theorem 2.6.

By Lemma 3.19, for any $\varepsilon > 0$, any $\varepsilon_1 > 0$ satisfying (3.2) and (3.3) and all $n$ sufficiently large, there exists a sparsification $H'$ of $H$ for which the events $F_1$, $F_2$, and $F_3$ from the lemma all hold. Deleting one $r$-edge from each overlapping pair in $H'$, we obtain a linear $r$-graph $H''$ with average degree $d$ satisfying

$$d = \frac{r e(H'')}{V(H'')} = \frac{|V_n|}{k|V_n|} \left( e(H') - Y_{H'} \right) \geq \frac{|V_n|}{k|V_n|} \left( \frac{1}{2} - 3\varepsilon_1 \right)p \left( \frac{V_n}{V_n} \right) \gg 1,$$  

(3.10)

where in the first inequality we used the fact that $F_1$ and $F_2$ hold, and in the last two inequalities we used the bounds $\varepsilon_1 < 1/6$ from (3.2) and the lower bound on $p(V_n/V_n)$ from (3.4).

Apply Theorem 2.7 to $H''$ with parameter $\delta = 6\varepsilon_1$ and let $d_0 = d_0(\delta, r)$ be the constant in Theorem 2.7. Equation (3.10) tells us that for $n$ sufficiently large we have $d \geq d_0$. Thus there exists a collection $C$ of subsets of $V(H'') = V(H)$ satisfying conclusions 1-3. of Theorem 2.7. For each $C \in C$, we obtain a template $t = t(C)$ for a partial $k$-coloring of $V_n$, assigning to each $v \in V_n$ a (possibly empty) palette $t(v) = \{i \in [k] : (v, i) \in C\}$ of available colors. Set

$$\mathcal{T} := \{t(C) : C \in C, t(v) \neq \emptyset \text{ for all } v \in V_n\}$$

to be the family of templates from $\left(\binom{[k]}{\emptyset}\right)_{V_n}$ which can be constructed in this way. We claim that the template family $\mathcal{T}$ satisfies the conclusions (i)-(iii) of Theorem 3.18.

Indeed, by definition of $H$, any template $t'$ with $\langle t' \rangle \subseteq P_n$ gives rise to an independent set $I$ in the $r$-graph $H$ and hence its subgraph $H''$, namely $I = \{(v, i) : i \in [k], v \in V_n, i \in t(v)\}$. Thus there exists $C \in C$ with $I \subseteq C$, giving rise to a proper template $t \in \mathcal{T}$ with $t' \leq t$. Conclusion (i) is therefore satisfied by $\mathcal{T}$.

Further for each $C \in C$, conclusion 2 of Theorem 2.7 and the event $F_1$ and $F_2$ together imply

$$e(H'[C]) \leq e(H''[C]) + (e(H') - e(H'')) < \delta e(H'') + 6\varepsilon_1 e(H') = 12\varepsilon_1 e(H').$$

Since $F_3$ holds in $H'$, this implies $e(H[C]) < 24\varepsilon_1 e(H)$, which by our choice of $\varepsilon_1$ satisfying (3.2) and our upper bound (3.1) on $e(H)$ is at most $\varepsilon \binom{V_n}{V_n}$. In particular, by construction of $H$, we have that
for each $t = t(C) \in \mathcal{T}$ there are at most $\varepsilon \binom{n}{N}$ pairs $(\phi, c)$ with $\phi \in \binom{V_n}{V_N}$, $c \in \mathcal{P}$ and $c \in \langle t|\phi \rangle$. This establishes (ii).

Finally by conclusion 3. of Theorem 2.7 and our bound (3.10) on the average degree $d$ in $H''$, we have

$$|\mathcal{T}| \leq |\mathcal{C}| \leq 2^{\varepsilon(d|V_n|} = \exp\left(O\left(\left|V_n\right|\left(I(N, n) / \left(V_n \right)^2\right)^{1/2|V_n|^{-1}} |V_n|\right)^|V_n|\right) = k^{o(V_n)},$$

which means that (iii) is satisfied. This concludes the proof of the theorem.

Theorem 3.18 gives us container theorems for hereditary properties of $k$-colorings of $ssee$-s defined by a finite family of forbidden coloring. To obtain the standard counting applications of containers for a given $ssee$ $V$, we need two more ingredients, namely (a) the existence of an entropy density function for $V$ (ie, an analog of Proposition 2.10) and (b) a supersaturation theorem for $V$ (ie, an analog of Lemma 2.11).

These ingredients are obtained on a more ad hoc basis than the general container theorem, Theorem 3.18—the proofs have to be tailored to $V$ to a greater extent—though in many of the most interesting cases the same arguments as those we used in Section 2.3 will work with only trivial modifications. In Section 3.4 we shall illustrate this by giving a complete treatment of the case of hypercube graphs.

Provided we can obtain (a) and (b), we have as an immediate corollary of Theorem 3.18 the following:

**Theorem 3.20**  Let $V$ be a good $ssee$ and let $k, N \in \mathbb{N}$. Let $\mathcal{P}$ be a nonempty collection of $k$-colorings of $V_N$ and let $\mathcal{P} = \text{Forb}_V(\mathcal{P})$. Suppose that the following hold:

(a) $\pi(\mathcal{P}) := \lim_{n \to \infty} \exp(V_n, \mathcal{P}) / |V_n|$ exists;

(b) for all $\varepsilon > 0$ there exist $\delta > 0, n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then every $t \in \left(2^k \setminus \{\emptyset\}\right)^{V_n}$ with at most $\delta|\left(V_n\right)^{V_n}|$ pairs $(\phi, c)$ with $\phi \in \binom{V_n}{V_N}$, $c \in \mathcal{P}$ and $c \in \langle t|\phi \rangle$ must have entropy at most $\text{Ent}(t) \leq (\pi(\mathcal{P}) + \varepsilon)|V_n|$.

Then

$$|\mathcal{P}_n| = k^{(\varepsilon(\pi(\mathcal{P}) + o(1)))|V_n|}.$$  

**Proof**  This is identical to the deduction of Theorem 2.12 from Theorem 2.6, Proposition 2.10, and Lemma 2.11, by using Theorem 3.18 and assumptions (a) and (b) to replace these three ingredients.

To obtain counting results for general order-hereditary properties we need a little more.

**Theorem 3.21**  Let $V$ be a good $ssee$ and let $k \in \mathbb{N}$. Suppose that the following hold:

(a) $\exp(V_n, \mathcal{P}) / |V_n|$ is nonincreasing (and in particular tends to a limit $\pi(\mathcal{P})$) for all hereditary properties $\mathcal{P}$ of $k$-colorings of $V$;

(b) for all $N$ and all nonempty families $\mathcal{P} \subseteq [k]^{V_N}$ we have supersaturation for $\text{Forb}_V(\mathcal{P})$:

$$\forall \varepsilon > 0, \exists \delta > 0, n_0 \in \mathbb{N} \text{ such that if } n \geq n_0 \text{ then for every } t \in \left(2^k \setminus \{\emptyset\}\right)^{V_n} \text{ with at most } \delta|\left(V_n\right)^{V_n}| \text{ pairs } (\phi, c) \text{ with } \phi \in \binom{V_n}{V_N}, c \in \mathcal{P} \text{ and } c \in \langle t|\phi \rangle \text{ we have } \text{Ent}(t) \leq (\pi(\mathcal{P}) + \varepsilon)|V_n|.$$
Then for any hereditary property $\mathcal{P}$ of $k$-colorings of $V$,

$$|\mathcal{P}_n| = k^{\left(\pi(\mathcal{P}) + o(1)\right)}|V_n|.$$ 

**Proof** The monotonicity in (a) and Theorem 3.20 allow us to apply the proof of Theorem 2.13 and obtain an ssee-version of our approximation of arbitrary hereditary $k$-coloring properties by properties defined by finite families of forbidden colorings. We then deduce the claimed counting result in exactly the same way as we deduced Corollary 2.15 from Theorems 2.12 and 2.13. □

**Remark 3.22** While we do not pursue this here, one can also use Theorem 3.18 to derive stability and transference results by following the proofs of Theorems 2.18 and 2.23. In the latter case, the lower bound on $p$ required for transference is

$$p \gg \left(\frac{|V_n|I(N,n)}{|(V_n)|^{1/(2|V_n|-1)}}\right)^{1/(2|V_n|-1)}$$

provided $I(N,n) = \Omega\left(\binom{V_n}{V_n}^2\right)$, as opposed to $p \gg n^{1/(2^2-1)}$ when we were coloring $E(K_n)$.

### 3.3 Graph and hypergraph sequences

As an ssee is quite an abstract object, we feel it is helpful to clearly state the implications of Theorem 3.18 in the language of graphs and hypergraphs. Recall from Example 3.8 that we may obtain an ssee $V$ from a sequence of $l$-graphs $G$ on linearly ordered vertex sets by taking $V_n = E(G_n)$ and taking as our set of embeddings $(V_n)$ all maps $\psi : E(G_N) \to E(G_n)$ arising from order-preserving embeddings $\phi$ from $G_N$ into $G_n$. We now restate the main definitions and results of the previous section (Definition 3.16, Theorem 3.18) in terms of hypergraph sequences:

**Definition 3.23** (Good $l$-graph sequence) Let $l \geq 2$. An $l$-graph sequence $G$ is **good** if it satisfies the following conditions:

1. $e(G_n) \to \infty$ ("the graphs in the sequence become large");
2. for all $N \in \mathbb{N}$ with $e(G_N) > 1$, $(G_n) \gg e(G_n)$ as $n \to \infty$ ("on average, edges of $G_n$ are contained in many embeddings of $G_N$’s");
3. for all $N \in \mathbb{N}$ with $e(G_N) > 1$, $e(G_n)l(N,n)/\binom{G_n}{G_n}^2 \to 0$ as $n \to \infty$ ("most pairs of edges in $G_n$ are contained in relatively few embeddings of $G_N$’s").

**Theorem 3.24** Let $l \geq 2$ and let $G$ be a good $l$-graph sequence. Let $k, N \in \mathbb{N}$. Let $\mathcal{P}$ be a nonempty collection of $k$-colorings of $G_N$ and let $\mathcal{P} = \text{Forb}_k(\mathcal{P})$. For any $\varepsilon > 0$, there exists $n_0 > 0$ such that for any $n \geq n_0$ there exists a collection $\mathcal{T}_n$ of $k$-coloring templates for $G_n$ satisfying:

1. $\mathcal{T}_n$ is a container family for $\mathcal{P}_n$;
2. for each template $t \in \mathcal{T}_n$, there are at most $\varepsilon|\binom{G_n}{G_n}|$ pairs $(\phi, c)$ with $\phi \in \binom{G_n}{G_n}$, $c \in T$ and $c \in \langle t, \phi \rangle$;
3. $|\mathcal{T}_n| \leq k^{\varepsilon|G_n|}$.
In certain applications one maybe interested in vertex-colorings of l-graph sequences, rather than the edge-colorings considered above. As described in Example 3.7, we may obtain an ssee V from a sequence of l-graphs G by taking V_n = V(G_n) and taking as our collection of embeddings (G_n) all (order-preserving) embeddings from G_N to G_n. In this case, Definition 3.5 and Theorem 3.18 become:

**Definition 3.25** (Vertex-good l-graph sequences) Let l ≥ 2. An l-graph sequence G is vertex-good if it satisfies the following conditions:

(i) v(G_n) → ∞ (“the graphs in the sequence become large”);
(ii) for all N ∈ ℤ with |V_N| > 1, (G_n) ≥ v(G_n) (“on average, vertices in G_n are in many embeddings of G_N”);
(iii) for all N ∈ ℤ with |V_N| > 1, v(G_n)I(N, n)/(G_n)^2 → 0 as n → ∞ (“most pairs of embeddings of G_N share at most one vertex”).

**Theorem 3.26** Let l ≥ 2 and let G be a vertex-good l-graph sequence. Let k, N ∈ ℤ. Let P be a nonempty collection of k-colorings of V(G_N) and let P = Forb_G(P). For any ε > 0, there exists n_0 > 0 such that for any n ≥ n_0 there exists a collection T_n of k-coloring templates for V(G_n) satisfying:

(i) T_n is a container family for P_n,
(ii) for each template t ∈ T_n, there are at most ε(G_n)| pairs (φ, c) with φ ∈ (G_n), c ∈ P and c ∈ t(φ);
(iii) |T_n| ≤ k^{εv(G_n)}.

Furthermore, as discussed in Section 3.2, and as more specifically stated in Theorem 3.20, if P = Forb_G(P) admits an entropy density π(P) (condition (a)) and the supersaturation property (condition (b)) then Theorems 3.24 and 3.26 immediately give a counting result, with |P| = k^{(π(P) + ε(1))v(G_n)} in the k-edge-coloring case and |P| = k^{(π(P) + ε(1))v(G_n)} in the k-vertex-coloring case.

However rather than state edge- and vertex-coloring versions of the counting result for ssee, Theorem 3.20, we give an illustrative example in the next subsection by providing counting results for hereditary properties of edge- and vertex-colorings of hypercubes. This will hopefully make the situation clearer than an abstract theorem, avoid repetition, and in addition show how one goes about checking in practice that a sequence of graphs is good/vertex-good, has an entropy density function and satisfies the supersaturation property.

### 3.4 Colorings of hypercube graphs

In this subsection we show the sequence Q of hypercube graphs is good, has an entropy density function and satisfies the supersaturation property with respect to both edge- and vertex-colorings. Using our results from Section 3.2, we immediately deduce for hereditary properties of edge- and vertex-colorings of hypercubes.

In both settings, let (G_n) denote the collection of injections φ : V(Q_n) → V(Q_n) obtained by selecting an N-set A = {a_1, …, a_N} ⊆ [n] with a_1 < a_2 < … < a_N and a vector v ∈ {0, 1}^[n]\{A}, and setting

\[ φ(x_i) = \begin{cases} v_i & \text{if } i \notin B \\ x_i & \text{if } i = a_j \in B. \end{cases} \]
In other words, we have one embedding \( \phi \) for each copy of \( Q_N \) in \( Q_n \). An edge/vertex-coloring property of hypercubes \( P \) is then called hereditary if for every \( c \in P_n \) and \( \phi \in \left( \frac{Q_n}{Q_{n'}} \right) \) we have \( c_{|\phi} \in P_N \).

3.4.1 Edge-colorings of hypercubes

**Proposition 3.27** (Goodness of hypercube graphs) The sequence \( Q \) is good.

**Proof** Let \( N > 1 \). We have \( e(Q_n) = n2^{n-1} \) and \( \left( \frac{Q_n}{Q_n} \right) \) \( = \left( \frac{n}{N} \right) \) \( = \Omega(nNn) \gg e(Q_n) \), establishing parts (i) and (ii) of Definition 3.23. For part (iii), noting that two \( N \)-dimensional subcubes with at least two edges in common must meet in an \( i \)-dimensional subcube for some \( i \) such that \( 2 \leq i \leq N \), we have

\[
I(N, n) = \frac{1}{2} \left| \left( \frac{Q_n}{Q_{n'}} \right) \right| \sum_{2 \leq i \leq N} \binom{N}{i} 2^{N-i} \left( \frac{n}{N-i} \right) = O\left( \frac{Q_n}{Q_{n'}} \right)^{n^{N-2}}.
\]

It follows that \( I(N, n)e(Q_n)/\left| \left( \frac{Q_n}{Q_{n'}} \right) \right|^2 = O(1/n) = o(1) \) as required. \( \blacksquare \)

**Proposition 3.28** (Entropy density for edge-colorings of hypercubes) If \( P \) is a hereditary property of \( k \)-edge-colorings of \( Q \), then the sequence \( \text{ex}(Q_n, P)/2^{n-1}n \) is nonincreasing and tends to a limit \( \pi(P) \) as \( n \to \infty \).

**Proof** Let \( t \) be an extremal entropy template for \( P \) in \( Q_{n+1} \). By averaging over all \( \phi \in \left( \frac{Q_{n+1}}{Q_n} \right) \), we have

\[
n \text{ex}(Q_{n+1}, P) = n \text{Ent}(t) = \sum_{\phi} \text{Ent}(t_{|\phi}) \leq 2(n+1) \text{ex}(Q_n, P),
\]

whence \( \text{ex}(Q_n, P)/2^{n-1}n \) is nonincreasing in \([0, 1]\), and hence tends to a limit as \( n \to \infty \). \( \blacksquare \)

**Proposition 3.29** (Supersaturation for edge-colorings of hypercubes) Let \( N \in \mathbb{N} \) be fixed and let \( P \) be a nonempty collection of \( k \)-colorings of \( Q_N \). Set \( P = \text{Forbidden}(P) \). For every \( \varepsilon \) with \( 0 < \varepsilon < 1 \), there exist constants \( n_0 \in \mathbb{N} \) and \( C_0 > 0 \) such that for all \( n \geq n_0 \) and every template \( t \in \left( \frac{2^k}{\{0\}} \right)^{Q_n} \) with

\[
\text{Ent}(t) > (\pi(P) + \varepsilon)e(Q_n),
\]

there are at least

\[
C_0\varepsilon\left| \left( \frac{Q_n}{Q_{n'}} \right) \right|
\]

pairs \( (\phi, c) \) with \( \phi \in \left( \frac{Q_n}{Q_{n'}} \right) \), \( c \in P \) and \( c \in \langle t_{|\phi} \rangle \).

**Proof** We follow the proof of Lemma 2.11, modifying it as needed to fit the hypercube setting. Given a template \( t' \in \left( \frac{2^k}{\{0\}} \right)^{Q_m} \), for some \( m \geq N \), let \( B(t') \) denote the collection of pairs \( (\phi, c) \) with \( \phi \in \left( \frac{Q_n}{Q_{n'}} \right), c \in P \), and \( c \in \langle t'_{|\phi} \rangle \).

By Proposition 3.28, there exists \( n_0 \geq N \) such that for all \( t' \in \left( \frac{2^k}{\{0\}} \right)^{Q_{n_0}} \) with \( \text{Ent}(t') > \left( \pi(P) + \varepsilon \right)e(Q_{n_0}) \), we must have \( |B(t')| \geq 1 \). Let \( t \in \left( \frac{2^k}{\{0\}} \right)^{Q_n} \) for some \( n \geq n_0 \), and suppose \( \text{Ent}(t) > (\pi(P) + \varepsilon)e(Q_n) \). Let \( X \) denote the number of embeddings \( \phi \in \left( \frac{Q_n}{Q_{n'}} \right) \) such that
Ent(tϕ) > (π(P) + ε) e(Qn). By summing Ent(tϕ) over all ϕ ∈ (Qn), we have

\[ (π(P) + ε) e(Qn) \left( \frac{n-1}{n_0 - 1} \right) < \text{Ent}(t) \left( \frac{n-1}{n_0 - 1} \right) = \sum_ϕ \text{Ent}(tϕ) \]

≤ \left( \frac{n}{n_0} \right) 2^{n-n_0} \left( π(P) + \frac{ε}{2} \right) e(Qn) + Xe(Qn),

implying X > \frac{ε}{2} \left( \frac{Q_n}{Q_{n_0}} \right). On the other hand, summing |B(tϕ)| over all ϕ ∈ (Qn) yields:

\[ |B(t)| \left( \frac{n-N}{n_0 - N} \right) = \sum_ϕ |B(tϕ)| ≥ X > \frac{ε}{2} \left( \frac{Q_n}{Q_{n_0}} \right), \]

so that

\[ |B(t)| > \frac{ε}{2} \left( \frac{Q_n}{Q_{n_0}} \right) = \frac{1}{2} \left( \frac{|Q_n|}{Q_N} \right) ε \left( \frac{Q_n}{Q_N} \right). \]

This proves the lemma with C0 = \left( \frac{2(|Q_n|)}{|Q_N|} \right)^{-1}.

**Corollary 3.30 (Counting for hypercube graph colorings)** If P is a hereditary property of k-edge-colorings of Q, then |Pn| = k^{(π(P)+o(1))} 2^{n-1} n.

**Proof** Propositions 3.27, 3.28, and 3.29 tell us that the hypotheses of Theorem 3.21 are satisfied; applying it yields the desired counting result.

### 3.4.2 Vertex-colorings of hypercubes

**Proposition 3.31 (Vertex-goodness of hypercube graphs)** The sequence Q is vertex-good

**Proof** Let N > 1. We have v(Qn) = 2^n > 1 and \(|Q_n| = \left( \frac{n}{N} \right) 2^{n-N} \geq v(Qn)|, so parts (i) and (ii) of Definition 3.25 are satisfied. Part (iii) is a simple calculation: two copies of QN in Qn sharing at least two vertices must intersect in an i-dimensional subcube for some i such that 1 ≤ i ≤ N. Thus

\[ \frac{v(Qn)|I(N,n)|}{\left( \frac{Q_n}{Q_N} \right)^2} = \frac{2^{n-1}}{2^{n-N} \binom{n}{N} i} \sum_{i=1}^{N} \binom{N}{i} \left( \frac{n-N}{N-i} \right) 2^{N-i} = O\left( \frac{1}{n} \right), \]

which tends to 0 as n → ∞ as required.

**Proposition 3.32 (Entropy density for vertex-colorings of hypercubes)** If P is a hereditary k-vertex-coloring property of hypercubes, then the sequence \( \text{ex}(Q_n, P) / 2^n \) is nonincreasing and tends to a limit \( π(P) \) as n → ∞.

**Proof** Let t be a k-vertex-coloring template for Qn+1 which is entropy-extremal for P. By averaging over the 2(n+1) distinct embeddings \( ϕ ∈ (Q_{n+1}) \),

\[ (n+1) \text{ex}(Q_{n+1}, P) = (n+1) \text{Ent}(t) = \sum_ϕ \text{Ent}(tϕ) \leq 2(n+1) \text{ex}(Q_n, P), \]

whence \( \text{ex}(Q_n, P) / 2^n \) is nonincreasing in [0, 1] and converges to a limit as required.
\textbf{Proposition 3.33} (Supersaturation for vertex-colorings of hypercubes) \textit{Let }$N \in \mathbb{N}$\textit{ be fixed and let }$\mathcal{P}$\textit{ be a nonempty collection of }$k$\textit{-vertex-colorings of }$Q_N$. \textit{Let }$\mathcal{P} = \text{Forb}_0(\mathcal{P})$\textit{ be the hereditary }$k$\textit{-vertex-coloring property of being }$F$\textit{-free.}

\textit{For every }$\varepsilon$\textit{ with }$0 < \varepsilon < 1$, \textit{there exist constants }$n_0 \in \mathbb{N}$\textit{ and }$C_0 > 0$\textit{ such that for all }$n \geq n_0$\textit{ and every template }$t \in (2^{[k]} \setminus \{\emptyset\})^{V(Q_n)}$\textit{ with}

$$\text{Ent}(t) > (\pi_\mathcal{P}(t) + \varepsilon)2^n,$$

\textit{there are at least}

$$C_0 \varepsilon \left| \binom{Q_n}{Q_N} \right|$$

\textit{pairs }$(\phi, c)$\textit{ with }$\phi \in \binom{Q_n}{Q_N}$, $c \in \mathcal{P}$\textit{ and }$c \in \langle t_{|\phi} \rangle$\textit{.}

\textbf{Proof} \textit{We follow the proof of Proposition 3.29, modifying it as needed to fit the vertex-coloring setting. Given a template }$t' \in (2^{[k]} \setminus \{\emptyset\})^{V(Q_n)}$\textit{ for some }$m \geq N$, \textit{let }$B(t')$\textit{ denote the collection of pairs }$(\phi, c)$\textit{ with }$\phi \in \binom{Q_n}{Q_N}$, $c \in \mathcal{P}$\textit{ and }$c \in \langle t_{|\phi} \rangle$\textit{.}

\textit{By Proposition 3.32, there exists }$n_0 \geq N$\textit{ such that for all }$t' \in (2^{[k]} \setminus \{\emptyset\})^{V(Q_m)}$\textit{ with }$\text{Ent}(t') > \left( \pi(\mathcal{P}) + \frac{\varepsilon}{2} \right) v(Q_{n_0})$, \textit{we must have }$|B(t')| \geq 1$. \textit{Let }$t \in (2^{[k]} \setminus \{\emptyset\})^{V(Q_n)}$\textit{ for some }$n \geq n_0$, \textit{and suppose }$\text{Ent}(t) > (\pi(\mathcal{P}) + \varepsilon) v(Q_n)$\textit{. Let }$X$\textit{ denote the number of }$\phi \in \binom{Q_n}{Q_{n_0}}$\textit{ such that }$\text{Ent}(t_{|\phi}) > \left( \pi(\mathcal{P}) + \frac{\varepsilon}{2} \right) v(Q_{n_0})$. \textit{By summing }$\text{Ent}(t_{|\phi})$\textit{ over all }$\phi \in \binom{Q_n}{Q_{n_0}}$\textit{ we have}

$$(\pi(\mathcal{P}) + \varepsilon) v(Q_n) \binom{n}{n_0} < \text{Ent}(t) \binom{n}{n_0} = \sum_{\phi} \text{Ent}(t_{|\phi}) \leq \binom{n}{n_0} 2^{n-n_0} \left( \pi(\mathcal{P}) + \frac{\varepsilon}{2} \right) v(Q_{n_0}) + X v(Q_{n_0}),$$

\textit{implying }$X > \frac{\varepsilon}{2} \binom{Q_n}{Q_{n_0}}$. \textit{On the other hand, summing }$|B(t_{|\phi})|$\textit{ over all }$\phi \in \binom{Q_n}{Q_{n_0}}$\textit{ yields:}

$$|B(t)| \binom{n-N}{n_0-N} = \sum_{\phi} |B(t_{|\phi})| \geq X > \frac{\varepsilon}{2} \left| \binom{Q_n}{Q_N} \right|,$$

\textit{so that}

$$|B(t)| > \frac{\varepsilon}{2} \frac{|\binom{Q_n}{Q_{n_0}}|}{|\binom{Q_n}{Q_N}|} = \frac{1}{2|\binom{Q_n}{Q_N}|} \varepsilon \left| \binom{Q_n}{Q_N} \right|.$$ 

This proves the lemma with $C_0 = \left( 2|\binom{Q_n}{Q_{n_0}}| \right)^{-1}$.

From there, the counting result is immediate:

\textbf{Corollary 3.34} \textit{If }$\mathcal{P}$\textit{ is a hereditary property of }$k$\textit{-vertex-colorings of }$Q_n$, \textit{then}

$$|\mathcal{P}_n| = k^{(\pi_{\mathcal{P}}(\mathcal{P}) + o(1))2^n}.$$ 

\textbf{Proof} \textit{Propositions 3.31, 3.32, and 3.33 tell us that the hypotheses of Theorem 3.21 are satisfied; applying it yields the desired counting result.}
4 | EXAMPLES AND APPLICATIONS

4.1 | Order-hereditary versus hereditary

Here we include a quick example stressing the essential difference between hereditary and order-hereditary properties. We identify graphs with \{0, 1\}-colorings of \(K_n\) in the usual way; as the properties we consider in this example are in fact monotone, templates will consist of pairs \(e \in E(K_n)\) with \(t(e) = \{0, 1\}\) and entropy 1 and pairs \(e\) with \(t(e) = \{0\}\) and entropy 0. We can thus represent the templates simply as the graph of edges with entropy 1.

Let \(\mathcal{P}_1\) be the hereditary property of graphs on \([n]\) of having maximum degree 2, and let \(\mathcal{P}_2\) be the order-hereditary property of graphs on \([n]\) of not having any triples of vertices \(i < j < k\) with \(ij, jk\) both being edges. Clearly, \(\mathcal{P}_1\) is the collection of all matchings on \([n]\), with \(\text{ex}(n, \mathcal{P}_1) = \lceil n/2 \rceil\) and maximal matchings as the extremal entropy templates. The speed of this property is known as the Hosoya index or Z-index of \(K_n\), and is equal to the \(n\)th telephone number, which is of order \(n^{n/2+o(n)}\) [19]. On the other hand, \(\text{ex}(n, \mathcal{P}_2) = \lfloor n^2/4 \rfloor\).

4.2 | Graphs

In this subsection, we give a short proof of the Alekseev-Bollobás-Thomason theorem (Theorem 1.3). Our argument is similar to the proof in [17]. However, container theory allows us to avoid using the Regularity Lemma, which simplifies the argument.

We shall use the Erdős-Stone theorem [29].

Theorem 4.1 (Erdős-Stone theorem) Let \(r \geq 2, m \geq 1\) and \(\varepsilon > 0\). There exists \(n_0(r, m, \varepsilon)\) such that if \(G\) is a graph of order \(n \geq n_0\) and

\[
e(G) \geq (1 - \frac{1}{r} + \varepsilon) \binom{n}{2},
\]

then \(G\) contains a copy of \(K_{\frac{r+1}{r}}(m)\).

Recall the family \(\mathcal{H}(r, v)\) from Definition 1.2 and observe that \(\mathcal{H}(r, v)\) is a hereditary property of graphs. Write \(\mathcal{H}(r, v)_l\) for the collection of all members of \(\mathcal{H}(r, v)\) in which each of the \(r\) parts contains at most \(l\) vertices.

Lemma 4.2 Let \(\mathcal{P}\) be a hereditary property of graphs, let \(r \geq 2\), let \(\ell^r \geq 1\) and let \(\varepsilon > 0\). There exists a constant \(n_0 \in \mathbb{N}\) such that if \(n \geq n_0\) and \(t \in (2^{[k]} \setminus \{\emptyset\})^{K_n}\) with \(t \subseteq \mathcal{P}_n\) and

\[
\text{Ent}(t) \geq (1 - \frac{1}{r} + \varepsilon) \binom{n}{2},
\]

then \(\mathcal{H}(r + 1, v)_\ell \subseteq \mathcal{P}\) for some \(v \in \{0, 1\}^{r+1}\).
Proof \quad Recall that we may identify graphs with colorings of $E(K_n)$ by colors from $[2]$. Let $t$ be as above. Let $G$ be the graph with vertex set $[n]$ and $E(G) = \{e \in E(K_n) : t(e) = \{1, 2\}\}$. By Ramsey’s theorem, for every $\ell'$, there exists $m$ such that any 2-coloring of $E(K_m)$ contains a monochromatic copy of $K_{\ell'}$. Our assumption on $\text{Ent}(t)$ and the Erdős-Stone theorem imply that if $n$ is sufficiently large, then $G$ contains a copy $K$ of $K_{r+1}(m)$.

Let $t'$ denote the restriction of $t$ to $V(K)$ and let $V_1$, ..., $V_{r+1}$ denote the classes of $V(K)$. Now we construct a vector $v \in \{0, 1\}^{r+1}$. By our choice of $m$, for each $i$, either $\{e \in E(K[V_i]) : 1 \in t(e)\}$ or $\{e \in E(K[V_i]) : 2 \in t(e)\}$ contains a copy of $K_{\ell'}$. In the former case set $v_i = 0$, and otherwise set $v_i = 1$. In either case, we let $U_i$ denote the vertex set of the monochromatic copy of $K_{\ell'}$.

Let $H \in H(r + 1, v)_{\ell'}$ and let $W_1$, ..., $W_{r+1}$ be a partition of $V(H)$ such that for each $i$, $W_i$ is a clique if $v_i = 1$ and an independent set if $v_i = 0$. Because $|W_i| \leq \ell'$ for each $i$, we may embed $W_i$ into $U_i \subseteq V_i$ arbitrarily. It follows that there is a realization $c$ of $t'$ such that $H$ is a subgraph of (the graph corresponding to) $c(K_n)$. Since $H$ was arbitrary and $\mathcal{P}$ is hereditary, it follows that $H(r + 1, v)_{\ell'} \subseteq \langle t' \rangle \subseteq \mathcal{P}$, as desired.

\textbf{Proof of Theorem 1.3.} First, by the definition of $\chi_{\ell'}(\mathcal{P})$, there exists $v \in \{0, 1\}^r$ such that $H(r, v) \subseteq \mathcal{P}$. By considering the graphs in $H(r, v)$ such that each clique or independent set has size $\lceil n/r \rceil$ or $\lfloor n/r \rfloor$, we see that

\[ |\mathcal{P}_n| \geq |H(r, v)_n| \geq 2^{\left(1 - 1/r + o(1)\right)\left(\ell' \right)}. \]

Second, suppose for a contradiction that for some $\varepsilon > 0$, there exist infinitely many $n$ such that

\[ |\mathcal{P}_n| \geq 2^{(1 - 1/r + \varepsilon)\left(\ell' \right)}. \] (4.1)

Corollary 2.15 implies that there exists $n_0$ such that for all $n \geq n_0$ for which (4.1) holds, there exists a template $t \in \langle 2^{|[r]|} \setminus \{\emptyset\} \rangle_{\ell'}$ such that $\langle t \rangle \subseteq \mathcal{P}_n$ and

\[ \text{Ent}(t) \geq (1 - \frac{\varepsilon}{r} + \frac{1}{2})\left(\frac{n}{2}\right). \]

It follows from Lemma 4.2 that for each $\ell' \geq 1$, there exists $v \in \{0, 1\}^{r+1}$ such that $H(r + 1, v)_{\ell'} \subseteq \mathcal{P}$. In particular, there is some $v \in \{0, 1\}^{r+1}$ such that $H(r + 1, v)_{\ell'} \subseteq \mathcal{P}$ for infinitely many $\ell'$, and thus for all $\ell'$. However, this contradicts the definition of $\chi_{\ell'}(\mathcal{P})$. The theorem follows.

\textbf{4.3 \quad Digraphs}

As mentioned earlier, hereditary properties for tournaments, orgraphs, and digraphs have received significant attention from the extremal combinatorics research community, see [16]. In a recent paper, Kühn, Osthus, Townsend, and Zhao [43] determined the typical structure of certain families of oriented and directed graphs. As part of their argument, they proved a container theorem and, using it, a counting theorem for $H$-free orgraphs and $H$-free digraphs, where $H$ is a fixed orgraph with at least two edges (Theorem 3.3 and Corollary 3.4 in [43]).

They went on to observe that their results did not extend to the case where $H$ is a digraph, giving the specific example when $H = DK_3$, the double triangle ($[3]$, $[3] \times [3]$). Their approach considered the extremal weight achievable in an $H$-free digraph where double edges receive a different weight from single edges. In the case of $DK_3$, they observed that the extremal weight did not predict the correct count of $DK_3$-free digraphs, showing that their container theorem failed to generalize in its given form.
to the digraph case. Giving some vindication to our entropy-based approach to containers, we use our theorems to determine the speed of the digraph property \( \text{Forb}(DF) \) of not containing any \( DF \). More generally, given a graph \( F \), let \( DF \) be the digraph obtained by replacing each edge of \( F \) with a directed edge in each direction.

**Theorem 4.3**  Let \( P = \text{Forb}(DF) \) be the digraph property of not containing any \( DF \). Then

\[
\text{ex}(n, P) = (1 - \log_4 3) \text{ex}(n, F) + \log_4 3 \left( \frac{n}{2} \right),
\]

where \( \text{ex}(n, F) \) is the Turán number of \( F \).

**Proof**  We use the correspondence between digraphs and 4-colorings of \( K_n \) from Observation 3.1. Let \( t \) be an \( n \)-vertex 4-coloring template for \( P \) with maximal entropy. The monotonicity of \( \text{Ent}(t) \) and the maximality of \( \text{Ent}(t) \) imply that all edges \( e \) of \( K_n \) have \( t(e) = [4] \) or \( t(e) = [3] \). As \( P \) is exactly the property of having no copy of \( F \) in color 4, at most \( \text{ex}(n, F) \) edges can have full entropy (entropy 1), with the rest having entropy \( \log_4 3 \). This gives the upper bound on \( \text{ex}(n, P) \).

For the lower bound, consider an \( F \)-free graph \( G \) on \( [n] \) with \( e(G) = \text{ex}(n, F) \). Let \( t \) be the 4-coloring template with \( t(e) = [4] \) if \( e \in E(G) \) and \( t(e) = [3] \) otherwise. Clearly every realization of \( t \) contains no copy of \( F \) in color 4, and hence lies in \( P \). The entropy of \( t \) exactly matches the upper bound we established above, concluding the proof of the theorem.

**Corollary 4.4**  Let \( F \) be a graph with chromatic number \( r \). Then there are

\[
| \text{Forb}(DF)_n | = 3 \left( \frac{1}{r-1} \right) 4^{1 - \frac{1}{r-1}} \left( \frac{n}{2} \right)^2 + o(n^2)
\]

digraphs on \( [n] \) not containing any copy of \( DF \).

**Proof**  The Erdős-Stone-Simonovits theorem \([28, 29]\) implies that for every graph \( F \) with chromatic number \( r \), \( \text{ex}(n, F) = (1 - \frac{1}{r-1}) \left( \frac{n}{2} \right)^2 + o(n^2) \). Together with theorem 4.3 this implies \( \pi(\text{Forb}(DF)) = (1 - \frac{1}{r-1}) + \frac{1}{r-1} \log_4 3 \). The result is then immediate from Corollary 2.15.

Furthermore, we can characterize typical graphs in \( P = \text{Forb}(DF) \). Suppose \( F \) has chromatic number \( r \). Let \( S_n = S_n(F) \) denote the collection of \( t \in (2^{[4]} \setminus \{\emptyset\})^K_n \) obtained by taking a balanced \( (r-1) \)-partition \( \bigsqcup_{i=1}^{r-1} A_i \) of \([n]\) and setting \( t(e) = [4] \) for all edges \( e \) between distinct parts \( A_i, A_j \) with \( i \neq j \), and letting \( t(e) = [3] \) for all other edges. The celebrated Erdős-Simonovits stability theorem \([57]\) applied to the graph \( F \) immediately implies the following result:

**Proposition 4.5**  Let \( P = \text{Forb}(DF) \) and \( S_n \) be as above. For every \( \epsilon > 0 \), there exists \( \delta > 0 \) and \( n_0 \) such that if \( n \geq n_0 \) and \( t \in (2^{[4]} \setminus \{\emptyset\})^K_n \) satisfies

(i) \( \text{Ent}(t) \geq (\pi(P) - \delta) \left( \frac{n}{2} \right) \), and

(ii) there are at most \( \delta \left( \frac{n}{\chi(F)} \right) \) monochromatic copies of \( F \) in color 4 which can be found in the realizations of \( t \),

then \( \rho(S_n, t) \leq \epsilon \left( \frac{n}{2} \right) \).

Applying Theorem 2.18 then yields:
Corollary 4.6  Let $P = \text{Forb}(DF)$ and $S_n$ be as above. For every $\varepsilon > 0$ there exists $n_0 > 0$ such that for all $n \geq n_0$, all but $\varepsilon |P_n|$ colorings in $P$ are within edit distance $\varepsilon \binom{n}{2}$ of a realization from $S_n$.

Equivalently, for all but an $\varepsilon$-proportion of $DF$-free digraphs $D$ on $[n]$, there exists a digraph $H$ that is obtained by taking a balanced $(r-1)$-partition $\bigsqcup_{i=1}^{r-1} A_i$ of $[n]$, setting double edges between distinct parts $V_i, V_j$ and placing quasirandom tournaments inside each of the parts, and a subdigraph $H'$ of $H$ such that $\rho(D, H') \leq \varepsilon \binom{n}{2}$.

The case $F = K_3$ (which has chromatic number $r = 3$) in the results above resolves the problem identified by Kühn, Osthus, Townsend, and Zhao.

4.4  Multigraphs

In this subsection we study multigraphs with bounded edge multiplicities, viewed as weightings of the edges of $K_n$ by nonnegative integers. An $n$-vertex multigraph $G$ in which all edge multiplicities are at most $d$ can be encoded as a $(d+1)$-colorings of $E(K_n)$, with each edge colored by its multiplicity. In this way, the problem of counting such multigraphs is placed in our framework of counting $k$-colorings.

Let $\mathcal{P}$ be the property of multigraphs that no triple of vertices supports more than 4 edges (counting multiplicities). Clearly no edge of such a multigraph can have weight more than 4. We shall determine the speed of $\mathcal{P}_n$. As always, we do this by first proving an extremal result (which in this case is quite easy), with the counting result then following immediately from an application of Corollary 2.15.

Similar extremal problems for multigraphs were previously considered by Bondy and Tuza [18] and Füredi and Kündgen [32]. However, the crucial difference is that, as far as counting results are concerned, we need to determine the asymptotically extremal entropy, rather than the asymptotically extremal total number of edges that was studied in [32]. Indeed, in our problem, there exist configurations which are extremal with respect to the number of edges but not with respect to entropy—see Examples 4.7 and 4.9.

Very recently in a pair of papers, Mubayi and Terry [45, 46] study our problem in much greater generality, determining the extremal entropy, number and typical structure of multigraphs in which no $s$ vertices support more than $q$ edges for a very large class of pairs $(s, q)$. Our work in this subsection is thus a special case of their much more general results.

Example 4.7  Consider a balanced bipartition $V_1 \sqcup V_2$ of $[n]$ and let $G_1$ be the multigraph assigning weight 2 to every edge from $V_1$ to $V_2$ and weight 0 to every other edge. Let also $t_1$ be the associated template, assigning color list $\{0, 1, 2\}$ to every edge from $V_1$ to $V_2$ and color list $\{0\}$ to every other edge.

Clearly $G_1 \in \langle t_1 \rangle \subseteq \mathcal{P}$. The total edge weight of $G_1$ is $\left\lfloor \frac{n^2}{2} \right\rfloor$, and the entropy of $t_1$ is $\log_5(3) \left\lfloor \frac{n^2}{4} \right\rfloor$.

It is not hard to show that the total edge weight of $G_1$ is extremal; this is an easy exercise on proof by induction, and follows from results of Bondy-Tuza [18] and Füredi-Kündgen [32].

Proposition 4.8  If $G$ is a multigraph in $\mathcal{P}_n$, for some $n \geq 3$, then $e(G) \leq \left\lfloor \frac{n^2}{2} \right\rfloor$.

The total edge weight of $G_1$ is thus maximal; however, the entropy of the associated template $t_1$ is not. Indeed we can construct a different edge-extremal construction with strictly larger entropy.
Example 4.9  Let $M$ be a maximal matching in $[n]$ and let $G_2$ be the multigraph assigning weight 2 to every edge in $M$ and weight 1 to every other edge. Let also $t_2$ be the associated template, assigning color list $\{0, 1, 2\}$ to every edge of $M$ and color list $\{0, 1\}$ to every other edge.

As before, we have $G_2 \in \langle t_2 \rangle \subseteq P$ and $e(G_2) = \lceil \frac{n^2}{2} \rceil$. However,

$$\text{Ent}(t_2) = \log_5(2) \left( \frac{n}{2} \right) + \log_5 \left( \frac{3}{2} \right) \left\lceil \frac{n}{2} \right\rceil = \log_5(\sqrt{2})n^2 + o(n^2) > \log_5(3^{1/4})n^2 \geq \text{Ent}(t_1).$$

It is straightforward to show that $t_2$ is indeed an entropy-extremal template for $P$; this is a special case of recent and much more general results of Mubayi and Terry [46].

Theorem 4.10  For all $n \geq 3$, $\text{ex}(n, P) = \log_5(2) \left( \frac{n}{2} \right) + \log_5 \left( \frac{3}{2} \right) \left\lceil \frac{n}{2} \right\rceil$.

Proof  By induction on $n$. The base cases $n = 3, 4$ are again easily checked by hand. For $n \geq 4$, consider a template $t$ for a 5-coloring of $E(K_{n+1})$ with $\langle t \rangle \subseteq P_{n+1}$, with colors from $\{0, 1, 2, 3, 4\}$ corresponding to edge weights. Suppose $\text{Ent}(t) \geq \log_5(2) \left( \frac{n+1}{2} \right) + \log_5 \left( \frac{3}{2} \right) \left\lfloor \frac{n+1}{2} \right\rceil$. We claim that we must in fact have equality. By the inductive hypothesis it is enough to show that we can find a pair of vertices $u_1u_2$ such that the sum of the entropies of the edges incident to $u_1$ or $u_2$ is at most $2(n-1) \log_5(2) + \log_5(3)$. By monotonicity of the property $P$, we may assume that for every edge $e$ if $i < j$ and $j \in t(e)$ then $i \notin t(e)$. Thus the possible entropies for a single edge are 0 (weight zero), $\log_5(2)$ (weight 0 or 1), $\log_5(3)$ (weight 0, 1 or 2), and so on.

Suppose $G$ contains an edge $u_1u_2$ with entropy at least $\log_5(4)$. Then $3 \in t(u_1u_2)$, and thus for every other vertex $v$, the combined weight of $u_1v, u_2v$ in any realization of $t$ must be at most one, so that $\log_5 |t(u_1v)| + \log_5 |t(u_2v)| \leq \log_5(2)$. Thus the total entropy of the edges incident to $u_1$ or $u_2$ is at most $\log_5(5) + (n-1) \log_5(2) < 2(n-1) \log_5(2)$. We may therefore assume that every edge $u_1u_2$ has entropy at most $\log_5(3)$ in $t$, and, given the bound we are trying to prove, that there is some edge with entropy exactly $\log_5(3)$. Then $2 \in t(u_1u_2)$, and for every other vertex $v$ the pairs $u_1v, u_2v$ can have combined weight at most 2 in every realization of $t$. In particular,

$$\log_5 |t(u_1v)| + \log_5 |t(u_2v)| \leq \max\{\log_5(3) + \log_5(1), \log_5(2) + \log_5(2)\} = 2 \log_5(2).$$

Thus the total entropy of the edges incident to $u_1$ or $u_2$ is at most $2(n-1) \log_5(2) + \log_5(3)$, as required, and

$$\text{Ent}(t) \leq \log_5(2) \left( \frac{n+1}{2} \right) + \log_5 \left( \frac{3}{2} \right) \left\lceil \frac{n+1}{2} \right\rceil.$$  

We may thereby deduce a counting result for $P$:

Corollary 4.11  There are $2^{\langle t \rangle + o(n^2)}$ multigraphs on $[n]$ for which no triple of vertices supports more than 4 edges (counting multiplicities).

Proof  Immediate from Theorem 4.10 and Corollary 2.15.

Remark 4.12  With a little more work, it can be shown that $t_2$ and its isomorphic copies constitute a strong stability template for $P$ and that typical members of $P$ are close to realizations of $t_2$—and thus far from realizations of $t_1$, despite the fact that $t_1$ was constructed from an edge-extremal graph. This also follows from considerably more general (and more difficult) stability results for multigraphs in
which no $s$-set spans more than $q$ edges, which was obtained by Mubayi and Terry [46], and shows how different the extremal problems for the total number of edges and for the entropy are in this setting.

4.5 | 3-Colored graphs

Let $P$ denote the set of 3-colored graphs with no rainbow triangle, where a triangle is called rainbow if it has an edge in each of the three colors $\{1, 2, 3\}$. We use our multicolor container results to count the number of graphs in $P$ and to characterize typical elements of $P$. This is related to the multicolor Erdős-Rothschild problem [24], which has received significant attention, see for example, Alon, Balogh, Keevash, and Sudakov’s proof of a conjecture of Erdős and Rothschild in [4], as well as the recent work of Benevides, Hoppen, and Sampaio [15], Pikhurko, Staden, and Yilma [48] and Hoppen, Lefmann, and Odermann [36].

**Theorem 4.13 (Extremal entropy)** Let $P$ denote the set of 3-colored graphs with no rainbow triangle. For all $n \geq 3$,

$$\text{ex}(n, P) = (\log_3 2) \binom{n}{2}.$$  

Furthermore, the unique extremal templates $t$ are obtained by choosing a pair of colors $\{c_1, c_2\}$ from $\{1, 2, 3\}$ and setting $t(e) = \{c_1, c_2\}$ for every $e \in E(K_n)$.

**Proof** Our theorem shall follow from the following observation and a straightforward averaging argument.

**Observation 4.14** Suppose $\langle t \rangle \subseteq P$ and $e = \{v_1, v_2\}$ is some edge of $K_n$. Then rainbow $K_3$-freeness implies the following:

(i) if $|t(e)| = 3$, then for all $x \in V(K_n) \setminus e$ and $i \in \{1, 2\}$, we have $|t(xv_i)| = 1$;
(ii) if $|t(e)| = |t(f)| = 2$ and $t(e) \neq t(f)$, then $e \cap f = \emptyset$;
(iii) if $|t(e)| = 2$ and $c$ is the color missing from $t(e)$, then for every $x \in V(K_n) \setminus e$, either $t(xv_1) = t(xv_2) = \{c\}$ or $c$ is missing from both $t(xv_1)$ and $t(xv_2)$.

In particular, for any 3-set $A \subseteq [n]$, we have $\text{Ent}(t|_A) \leq 3 \log_3 2$, with equality attained if and only if all three edges of $A$ are assigned the same pair of colors $\{c_1, c_2\}$ by $t$.

Now, suppose $t$ is a template with $\text{Ent}(t) \geq (\log_3 2) \binom{n}{2}$. The average entropy of $t|_A$ over all 3-sets $A \subseteq [n]$ is:

$$\frac{1}{\binom{n}{3}} \sum_A \text{Ent}(t|_A) = \frac{1}{\binom{n}{3}} (n - 2) \text{Ent}(t) \geq 3 \log_3 2. \quad (4.2)$$

Our previous bound on the entropy inside triangles then tells us that we must have equality everywhere in (4.2) and that $t$ must have entropy $3 \log_3 2$ inside every 3-set $A$. In particular, all edges $e$ must have $|t(e)| = 2$. Finally by (ii) in Observation 4.14, there exists a pair of colors $\{c_1, c_2\}$ such that $t(e) = \{c_1, c_2\}$ for all edges $e \in E(K_n)$. This concludes the proof of the theorem.

**Corollary 4.15 (Counting)** For all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$3 \cdot 3^{(\log_3 2)^3} - 3 \leq |\mathcal{P}_n| \leq 3^{(\log_3 2)^3 + \varepsilon^3}.$$
Proof The lower bound is the number of colorings of $E(K_n)$ such that each edge receives one of a prescribed pair of colors. For the upper bound, Theorem 4.13 gives $\pi(P) = \log_3 2$, and the result then follows from Corollary 2.15.

We note that the stronger bound $|P_n| \leq 3^{(\log_3 2) \binom{n}{2} + \Theta(n \log n)}$ was proved in [15]. With a bit more case analysis, we can obtain the following stability result—see the Appendix for a proof.

**Theorem 4.16 (Stability)** The family of templates $S = \bigcup_n \{\{1, 2\}_K, \{1, 3\}_K, \{2, 3\}_K\}$ is a stability family for $P$. That is, for all $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon) > 0$ and $n_0 = n_0(\delta) \in \mathbb{N}$ such that the following holds: if $t$ is a 3-coloring template on $n \geq n_0$ vertices satisfying

(i) $\text{Ent}(t) \geq (\log_3 2 - \delta) \binom{n}{2}$, and
(ii) there at most $\delta \binom{n}{3}$ rainbow triangles in $K_n$ which can be realized from $t$,

then there exists a pair of colors $\{c_1, c_2\} \in [3]^2$ such that $t(e) = \{c_1, c_2\}$ for all but at most $\varepsilon \binom{n}{2}$ edges of $K_n$.

**Corollary 4.17 (Typical colorings)** Almost all 3-colored graphs with no rainbow triangles are almost 2-colored: for every $\varepsilon > 0$ there exists $n_0$ such that for all $n \geq n_0$ at most $3^{O(\varepsilon)}$ (rainbow $K_3$)-free 3-colorings of $K_n$ have at least $\varepsilon \binom{n}{2}$ edges in each of the colors $\{1, 2, 3\}$.

Proof Instant from Theorems 2.18 and 4.16.

We note that there are (many) examples of rainbow $K_3$-free 3-colored graphs in which all three colors are used. Indeed, consider a balanced bipartition $[n] = A \sqcup B$. Color the edges from $A$ to $B$ Red, and then arbitrarily color the edges internal to $A$ Red or Blue and the edges internal to $B$ Red or Green. The resulting 3-coloring has no rainbow $K_3$, and by randomly coloring the edges inside $A$ and $B$ we can in fact ensure that all three colors are used on at least $(1 + o(1)) \frac{n^2}{16}$ edges.

### 4.6 Hypercubes

Let $P = (P_n)_{n \in \mathbb{N}}$ be the collection of all induced subgraphs of $Q_n$, $n \in \mathbb{N}$, with no copy of the square, or 4-cycle, $Q_2$. Clearly, this may be viewed as a hereditary property of 2-vertex-colorings of $Q_n$. We have $\pi_v(P) \geq \frac{2}{3}$, as may be seen for example by removing every third layer of $Q_n$, that is, taking as our construction the family of all $x \in Q_n$ with $\sum_i x_i \equiv 0 \mod 3$, which clearly contains no $Q_2$. Kostochka [42] and, later and independently, Johnson and Entringer [38] showed that this lower bound is tight:

$$\pi_v(P) = \frac{2}{3}.$$ 

By Corollary 3.34 this immediately implies the following counting result:

**Corollary 4.18** There are $|P_n| = 2^{\left(\frac{2}{3} + o(1)\right)2n}$ $Q_2$-free induced subgraphs of $Q_n$.

In a different direction, let $Q = (Q_n)_{n \in \mathbb{N}}$ be the collection of all subgraphs of $Q_n$, $n \in \mathbb{N}$, with no copy of $Q_2$. This may be viewed as a hereditary property of 2-edge-colorings of $Q_n$. A long-standing conjecture of Erdős [25] states that the edge-Turán density (entropy density relative to $Q$) of this property is $\pi(Q) = 1/2$. The lower bound is obtained by deleting all edges between layer $2i$ and layer $2i + 1$ for $0 \leq i \leq \lfloor n/2 \rfloor$. The best upper bound to date is 0.603 ... from applications of flag
algebras due to Baber [6] and Balogh, Hu, Lidický, and Liu [12]. By Corollary 3.30 we have the following:

**Corollary 4.19** There are at most

$$|Q_n| = 2^{(0.604+o(1))2^{n-1}n}$$

$Q_2$-free subgraphs of $Q_n$. Further, if Erdős’s conjecture on $\pi(Q)$ is true, then there are

$$|Q_n| = 2^{\left(\frac{1}{3}+o(1)\right)2^{n-1}n}$$

$Q_2$-free subgraphs of $Q_n$.

### 4.7 A nonexample: a graph sequence with too few embeddings

Let $P = (P_n)_{n \in \mathbb{N}}$ be the sequence of paths on $[n]$ introduced in Section 3.2. An easy calculation reveals that $P$ fails to satisfy the “goodness” condition introduced in Definition 3.23, and is therefore not covered by Theorem 3.24. There is a good reason for this: the conclusion Theorem 3.24 does not hold for $P$ (or, more generally, for sequences of “tree-like” graphs).

Let $P$ be the order-hereditary property of 3-colorings of $P$ of not having two consecutive edges in the same color. It is easy to see that $|P_n| = 3 \cdot 2^{n-2} = 3^{\log_3(2) - O(1)}$. On the other hand, the extremal entropy of $P_n$ is only about $n \log_3 \sqrt{2}$.

**Theorem 4.20** For any $n \geq 3$, $\text{ex}(P_n, P) = [(n - 1)/2] \log_3 2$.

**Proof** If $f$ and $f'$ are consecutive edges and $t$ is a 3-coloring template with $t \subseteq P$ then $t(f) \cap t(f') = \emptyset$, from which it follows that $\log_3(|t(f)|) + \log_3(|t(f')|) \leq \log_3 2$. Further there can be no edge $f$ with $t(f) = [3]$, since otherwise we would have a realization of $t$ with two consecutive edges of the same color. Partitioning the path $P_n$ into disjoint pairs of consecutive edges and at most one single edge, we get $\text{Ent}(t) \leq [(n - 1)/2] \log_3 2$ as desired. For the lower bound, consider the template $t$ defined by setting $t(\{2i + 1, 2i + 2\}) = \{2\}$ and $t(\{2i, 2i + 1\}) = \{3\}$ for $0 \leq i \leq [(n - 1)/2]$. This has the correct entropy and all of its realizations clearly lie in $P$.

Now, $(P_n, P) = n - 3$, and it is easy to see that we have supersaturation of sorts for $P$: if $t$ is a template with $\text{Ent}(t) \geq n \log_3(\sqrt{2} + \varepsilon) + \varepsilon n$, there are at least $\Omega(\varepsilon n) = \Omega(\varepsilon (P_n, P))$ pairs of consecutive edges which can be made monochromatic in some realization of $t$. In particular, templates having $o(n)$ such pairs must have entropy at most $\log_3(\sqrt{2})n + o(n)$. A collection of $3^{o(n)}$ such templates can thus cover at most $2^{n/2}3^{o(n)} = o(2^n) = o(|P_n|)$ colorings—in particular, it cannot form a container family for $P_n$. This shows that the analog of Theorem 3.24 does not hold for the graph sequence $P$, and that some form of the “goodness” assumption in the statement of that theorem is necessary, as we claimed.

### 5 CONCLUDING REMARKS

#### 5.1 Entropy maximization in the multic和平 setting

In the 2-color setting, the rough structure of entropy maximizers for hereditary properties is well-understood, via the choice number $\chi_c$: given a hereditary property $P$ with $\chi_c(P) = r$, partition the vertex sets into $r$ equal parts and define a template by giving the $r$-partite edges full entropy (ie, free
choice of their color) and the other edges zero entropy (ie, fix their color). In particular, Theorem 1.3 implies that the set of possible entropy densities for hereditary properties is \( \{0, 1/2, 2/3, 3/4, \ldots\} \cup \{1\} \).

By contrast, it is less clear what the set of possible values of entropy densities or the possible rough structure of entropy maximizers should be in the \( k \)-colored setting for \( k \geq 3 \). We are only aware of one partial result in this area: Alekseev and Sorochan [2] showed that if \( P \) is a hereditary property of \( k \)-colored graphs, then either \( \pi(P) = 0 \) or \( \pi(P) \geq (1/2) \log_k(2) \). Moreover, the examples in Section 4 suggest that the possible structures of entropy maximizers are much more varied than in the case \( k = 2 \).

**Problem 5.1** Let \( k \in \mathbb{N} \) with \( k \geq 3 \). Determine the set of possible entropy densities of hereditary properties of \( k \)-colorings of \( K_n \) and the rough structure of entropy maximizers.

### 5.2 Containers and the entropy of graph limits

In a forthcoming paper [31] (see also [30] for a preliminary version of these results), Johanna Strömberg and a subset of the authors of the present paper relate the container theorems to work of Hatami-Janson-Szegedy on the entropy of graph limits [35]. The multicolor container theorems in the present paper are used to obtain generalizations of Hatami-Janson-Szegedy’s results to the setting of decorated graph limits. In the other direction, a second proof of those generalizations is obtained by working directly in the world of decorated graphons and using tools from analysis; it is further shown that these analytic results can then be used to recover some of the main combinatorial applications of containers, namely counting and characterization (the case of transference is more delicate) for hereditary properties of multicolored graphs. There thus appear to be significant links—or at least similarities—between the applications of the rich and currently quite distinct theories of graph limits and of hypergraph containers. The general “abstract” container results obtained in this paper and those obtained by Terry [58] may thus be seen as first steps towards an elucidation of those links.

**ACKNOWLEDGMENTS**

We are extremely grateful to two anonymous referees for their careful work and scholarship. Not only did they help us greatly improve the readability and exposition of this paper, but they pointed us to some crucial references which we had missed, in particular the work of Balogh-Wagner and Mubayi-Terry, and corrected our misunderstandings of some recent work in the area, in particular the KŁR conjecture. Their comments on the notation and organization of the paper were particularly appreciated, and led us to formulate a more general version of Theorem 3.18 than in our initial manuscript.

Victor Falgas-Ravry is grateful for an AMS-Simons award which allowed him to invite Andrew Uzzell to visit him and Kelly O’Connell at Vanderbilt University in November 2015, when the last stages of this research were carried out, and to the Swedish Research Council (Vetenskapsrådet) for a grant supporting his research. The authors would also like to thank Caroline Terry for helpful remarks about [58] and Daniel Toundykov for Russian language assistance.

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How to cite this article: Falgas-Ravry V, O’Connell K, Uzzell A. Multicolor containers, extremal entropy, and counting. Random Struct Alg. 2019;54:676–720. https://doi.org/10.1002/rsa.20777
APPENDIX

Proof of Theorem 4.16  Fix $\varepsilon > 0$. Let $\delta > 0$ and $n_0 \in \mathbb{N}$ be sufficiently small and sufficiently large constants respectively, to be specified later. Let $n \geq n_0$ and let $t \in (2^{|3|} \setminus \{0\})^{K_r}$ satisfy conditions (i) and (ii) in the statement of the theorem. Our proof is a (lengthy) exercise in stability analysis—essentially, we shall prove an approximate version of Observation 4.14, and then run through the proof of Theorem 4.13 replacing each “for all pairs” by a “for almost all pairs.”

By Lemma 2.11, there exists an absolute constant $C_0 = C_0(P)$ such that for all $\eta > 0$ there exists $n_1(\eta, P)$ such that for all $n \geq n_1$, if $t \in (2^{|3|} \setminus \{0\})^{K_r}$ can realize at most $\eta \binom{n}{3}$ rainbow triangles, then $\text{Ent}(t) \leq (\pi(P) + C_0 \eta) \binom{n}{2}$.

Let $e_3'$ be the number of edges $e = \{u, v\} \in E(K_n)$ for which there are at least $\delta n$ vertices $x \in V(K_n) \setminus \{u, v\}$ for which $|t([x, u])| + |t([x, v])| > 2$. For each such edge $e$ and each such vertex $x$, there is at least one rainbow triangle which can be realized inside $e \cup \{x\}$. Each such triangle is counted at most 3 times, so that in total we must have at least $\frac{e_3' \delta n}{3} < \delta \binom{n}{3}$ rainbow triangles, and in particular we must have $e_3' < \frac{\delta}{2} n^2$.

Now let $e_3''$ denote the number of edges $e = \{u, v\}$ for which there are at most $\delta n$ vertices $x$ with $|t([x, u])| + |t([x, v])| > 2$. We shall choose $\delta$ sufficiently small to ensure that (a) $1 - 200C_0 \delta > 2/3$ and (b) $\delta < \frac{2\pi(P)}{200(C_0 + 2)}$ (we can certainly do that since the value of the constant $C_0$ does not depend on $\delta$).

Suppose $n > 3n_1(2\delta)$. We claim that $e_3'' < 200(C_0 + 1)\delta n^2$. Indeed suppose not. Then we can find a set $E_3''$ of at least $200(C_0 + 1)\delta n^2 / 2n = 100(C_0 + 1)\delta n$ : $\{u, v\}$ pairwise vertex-disjoint edges $e = \{u, v\}$ with $|t(e)| = 3$ and $|t([x, u])| + |t([x, v])| = 2$ for all but at most $\delta n$ vertices $x$. Remove from $K_n$ the pairs of vertices $e = \{u, v\}$ from $E_3''$ one by one. This leaves us with a graph on $n' = n - 2cn$ vertices, which by (a) and our assumption on $n$ is strictly greater than $n_1(2\delta)$.

Let $t'$ denote the subtemplate of $t$ induced by the remaining vertices. Clearly $t'$ can realize at most $\delta \binom{n'}{3}$ rainbow triangles, which by (a) is at most $2\delta \binom{n}{3}$. Now Lemma 2.11 and the fact that $n' > n_1(2\delta)$ implies that

$$\text{Ent}(t') \leq (\pi(P) + C_0 2\delta) \binom{n'}{2} \leq \pi(P) \binom{n}{2} + \frac{C_0 2\delta}{2} n^2 - 2c(1 - c)\pi(P)n^2. \quad (A.1)$$

On the other hand, each of the edges $e$ from $E_3''$ we removed decreased the entropy by at most $\delta n$, so we have the following lower bound on $\text{Ent}(t')$:

$$\text{Ent}(t') \geq \text{Ent}(t) - c\delta n^2 \geq \pi(P) \binom{n}{2} - (c\delta + \delta) \binom{n}{2}. \quad (A.2)$$

Bringing the two bounds (A.1) and (A.2) together and cancelling terms as appropriate, we get

$$-c\delta - \frac{\delta}{2} \leq \frac{C_0 2\delta}{2} - 2c(1 - c)\pi(P).$$

Rearranging yields

$$c \left(2(1 - c)\pi(P) - \frac{\delta}{2} \right) \leq \frac{\delta}{2}(1 + 2C_0).$$

Since $c = 100(C_0 + 1)$, this contradicts our assumption (b) on $\delta$. It follows that $e_3'' < 200(C_0 + 1)\delta n^2$, as claimed. Thus in total, there are at most $e_3' + e_3'' = (\delta/2 + 200(C_0 + 1)\delta)n^2 := C_2 \delta n^2$ edges $e$ with
\[ |t(e)| = 3. \] We now move on to bounding the number \( e_1 \) of edges \( e \) with \(|t(e)| = 1\). We have
\[
\left( \pi(P) - \delta \right) \binom{n}{2} \leq \text{Ent}(t) \leq \pi(P) \binom{n}{2} - e_1 + e'_3 + e''_3,
\]
which together with our bound on \( e'_3 + e''_3 \) implies that
\[
e_1 < \frac{1}{\pi(P)} \left( \frac{1}{2} + C_2 \right) \delta n^2.
\]
In particular, all but at most
\[
\left( \frac{1}{\pi(P)} \left( \frac{1}{2} + C_2 \right) + C_2 \right) \delta n^2 \defeq C_3 \delta n^2
\]
edges \( e \) have \(|t(e)| = 2\).

Finally we turn to the edges assigned two colors by \( t \). For each pair of colors \( A \in [3]^{(2)} \), let \( V_A \) denote the collection of vertices incident to at least \( \delta^{1/3} n \) edges that are assigned \( A \) by \( t \). For any \( A \neq B \), each vertex in \( V_A \cap V_B \) gives rise to at least \( \delta^{2/3} n^2 \) distinct rainbow triangles, whence
\[
\frac{|V_A \cap V_B|}{3} \leq \delta \binom{n}{3},
\]
implying \( |A \cap V_B| \leq \delta^{1/3} n/2 \). Suppose we had \(|V_A|\) and \(|V_B|\) both greater than \( (\sqrt{C_3} + 3)\delta^{1/3} n \) for some color pairs \( A \neq B \), and let \( C \) denote the third color pair from \([3]\). Then all but at most \( \delta^{1/3} n \) vertices in \( A \) are incident to at most \( 2\delta^{1/3} n \) edges whose \( t \)-color assignment is \( B \) or \( C \). In particular such vertices \( a \) must be incident to at least \( |V_B| - |V_B \cap V_A| - 2\delta^{1/3} n \) edges \( ab \) with \( b \in V_B \setminus V_A \) and \( t(a,b) \notin \{A, B, C\} \). This gives at least
\[
\left( |V_A| - \delta^{1/3} n \right) \left( |V_B| - |V_B \cap V_A| - 2\delta^{1/3} n \right) \geq (\sqrt{C_3} + 1)\delta^{1/3} n \sqrt{C_3} \delta^{1/3} n > C_3 \delta n^2
\]
edges \( e \) with \(|t(e)| \neq 2\), a contradiction. It follows that there is at most one color pair, say \( A \), with \(|V_A| \geq (\sqrt{C_3} + 3)\delta^{1/3} n \). Let \( B, C \) denote the two other color pairs, and \( e_B, e_C \) the number of edges \( e \) with \( t(e) = B \) and \( t(e) = C \) respectively. By the definition of \( V_B \), we have
\[
e_B \leq |V_B| n/2 + (n - |V_B|)\delta^{1/3} n/2 < \frac{(\sqrt{C_3} + 4)\delta^{1/3} n}{2} n^2.
\]
and similarly \( e_C \leq (\sqrt{C_3} + 4)\delta^{1/3} n^2/2 \). We have thus shown that all but at most \((C_3 \delta + (\sqrt{C_3} + 4)\delta^{1/3}) n^2 \) edges \( e \in E(K_n) \) have \( t(e) \neq A \). Picking \( \delta = \delta(e) \) sufficiently small (and \( n_0 \geq 3n_1(2\delta) \)), this is less than \( \epsilon \binom{n}{2} \), proving the theorem. 

\[ \blacksquare \]