Nonlinear analysis of unsaturated soils using a meshfree method

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ABSTRACT

A fully coupled algorithm based on a recently developed meshfree method, called smoothed point interpolation method, is introduced for hydro-mechanical analysis of unsaturated porous media considering hydraulic hysteresis and material nonlinearity. A simple node selection scheme is adopted which ensures the non-singularity of the moment matrix in constructing the polynomial point interpolation shape functions. An effective stress based framework based on the work of Khalili et al. (2008) is followed in this study, and a hysteretic water retention model is taken into account which enables the evolution of water retention curve (WRC) with volumetric changes (Pasha et al., 2017). An elastoplastic constitutive model is employed within the context of bounding surface plasticity theory for predicting the nonlinear behavior of soil skeleton in unsaturated porous media. The applicability of the model is verified through several numerical examples.

Keywords: unsaturated soils, bounding surface plasticity, meshfree method, point interpolation method

1 INTRODUCTION

Research on the mechanics of unsaturated soils were sporadic for several decades, until seminal works by Bishop (1959) and Bishop et al. (1960) followed by a series of experiments by Bishop and Donald (1961) introduced the effective stress concept for unsaturated soils. Loret and Khalili (2000) later showed that besides using a proper definition of effective stress, a suitable plasticity model for soils should also be invoked for successful application of an effective stress based framework. Several elastoplastic constitutive models have so far been suggested for numerical analysis of unsaturated soils (Zienkiewicz et al., 1984; Dafalias, 1986; Hashiguchi and Chen, 1998). Among all, the bounding surface plasticity is perhaps the simplest and most efficient approach (Habte, 2006). Hence, this plasticity framework is adopted in this study to model the behaviour of soil skeleton.

The majority of the numerical models developed for simulation of flow and deformation in unsaturated porous media are in the framework of the finite element method (FEM) (Schrefler et al., 1993; Sheng et al. 2003; Khoei and Mohammadnejad, 2011); however, FEM has some intrinsic difficulties that were partly overcome by another class of numerical methods, referred to as meshfree methods (MMs) (Liu and Zhang, 2013). In MMs, a series of nodes define the problem domain and a predefined connectivity of the nodes is avoided. Unlike the FEM, the numerical operations are not heavily tied to the background mesh in MMs and therefore their performance is not heavily dependent on the quality of the background mesh.

To date, a limited number of studies have adopted MMs to evaluate different aspects of coupled hydro-mechanical behaviour of deforming unsaturated porous media (Khoshghalb and Khalili, 2013; Iranmanesh et al., 2018). Khoshghalb and Khalili (2013) adopted the radial point interpolation method (PIM) for flow-deformation analysis of unsaturated elastic porous media based on the effective stress approach including the hydraulic hysteresis. The PIM, however, lacks mathematical rigor due to the problem related to discontinuity of the approximation function in the problem domain.

Smoothed point interpolation methods (SPIMs) are a recently developed category of MMs (Liu and Zhang, 2008) that overcomes the abovementioned shortcomings. The SPIMs are developed through application of the generalised gradient smoothing operation to the PIM. The SPIM have been proved to perform efficiently and accurately in many engineering applications. In SPIMs, the numerical operations at each point of interest can go beyond the confinement of the element hosting the point of interest, and the connectivity of the nodes can be defined with more versatility compared to the standard FEM. The SPIMs have been so far exploited in several fields of engineering, albeit for only dry or fully saturated

https://doi.org/10.3208/jgssp.v07.072 450
porous media.

In this paper, an SPIM is developed for fully coupled hydro-mechanical analysis of unsaturated porous media based on effective stress theory. The developments are based on the work of Khalili et al. (2008), ameliorated by considering a void ratio dependent hysteretic water retention model. In the proposed model, the problem domain is discretized using triangular elements, and edge-based smoothing domains are then constructed using the edges of the triangular cells. The polynomial PIM which results in shape functions with Kronecker delta property for easy imposition of Dirichlet boundary conditions is considered for approximation of the independent variables. The model is verified, and its application is demonstrated through numerical examples.

2 GOVERNING EQUATIONS

Governing equations for solid skeleton and fluid phases are derived based on the theory of mixtures using effective stress concept for unsaturated porous media. The couplings between the phases are stabilized through the effective stress concept, SWRC and its volume change dependency.

2.1 Deformation model

Combining momentum balance of solid-fluid mixture, effective stress equation, stress-strain and strain-displacement relationships for small strains of the solid skeleton, the incremental form of the deformation model can be obtained as follows,

\[
L_d^T \{ \bar{D} \bar{\sigma} \} + \bar{b} = 0 \quad (1)
\]

where

\[
L_d = \begin{bmatrix}
\frac{\partial}{\partial x_2} & 0 \\
0 & \frac{\partial}{\partial x_2}
\end{bmatrix}
\]

(2)

with \(x_1\) and \(x_2\) being the space coordinates in a two-dimensional setting. Bold face letters indicate matrices and vectors throughout this paper. \(\bar{D}\) is the tangent elastoplastic constitutive matrix and \(\bar{u}\) is the solid phase displacement vector. In this paper, pore fluid pressures (\(P_w\) and \(P_a\)) are assumed positive in compression following the soil mechanics convention, whereas for the solid skeleton, compression is taken as negative, and tension is taken as positive, according to the continuum mechanics sign convention. \(\delta = [1 \; 1 \; 0]^T\) in equation (1) is the Kronecker delta vector, \(\psi\) is the incremental effective stress parameter, \(\bar{b}\) is the vector of body force per unit mass, and \(\rho\) is the average density of the mixture.

2.2 Flow model

The flow model which describes the flow of water and air through unsaturated porous media is obtained by combining the equation of linear momentum balance for each fluid phase with the mass balance equation for the same phase. Ignoring the mass exchange between the phases due to evaporation and condensation, and considering the definition of compressibility of barometric fluids and ideal gas law, following equation for the flow of fluid phase \(\pi\) through porous media is obtained,

\[
\frac{1}{\rho_{\pi}} \div \left[ \bar{\rho}_\pi k_{\pi} \bar{k} \left( \bar{P}_\pi + \bar{P}_a \bar{g} \right) \right] - n_{\pi} c_{\pi} \bar{P}_\pi \frac{\bar{v}_\pi}{\bar{v}} = 0 \quad (3)
\]

where \(\bar{P}_\pi = w_{\pi} a\) represents water and air phases; \(k_{\pi}\) is the relative permeability of phase \(\pi\); \(c_{\pi}\) is the intrinsic permeability matrix of the medium. For isotropic materials, \(k = kI\) where \(k\) is the intrinsic permeability of the material (i.e., \(k_x = k_y = k\)) and \(I\) is the 2x2 identity matrix. \(\mu_{\pi}\) is the dynamic viscosity of fluid phase \(\pi\); \(\rho_{\pi}\) is the density of phase \(\pi\); \(n_{\pi}\) is the density of phase \(\pi\); \(\bar{v}_{\pi}\) is the volumetric content of fluid phase \(\pi\) \((\bar{n}_{\pi} = \frac{\bar{v}_{\pi}}{\bar{v}})\); \(\bar{g}\) is the gravity acceleration vector and \(c_{\pi}\) is the coefficient of compressibility for the fluid phase \(\pi\).

2.3 Effective stress parameters

The effective stress parameter used in Biot’s effective stress equation for unsaturated soils, specifies the relative contribution of the pore air and pore water pressures to the effective stress. In this study, the effective stress parameter proposed by Khalili and Zargarbashi (2010) is used to consider the effect of hydraulic hysteresis on the changes in effective stress parameter when suction reversal occurs.

2.4 Constitutive coefficients

To capture the dependency of the model parameters on suction and volume change, the constitutive relationships can be expressed relating the pore water and pore air volumetric deformations to changes in volumetric strain and suction. Following the approach presented in Khalili et al. (2008), and considering zero compressibility for the solid skeleton, the fully coupled flow equations can be obtained as,

\[
\frac{1}{\rho_w} \div \left( \bar{\rho}_w k_{uw} \left( \bar{P}_w + \bar{P}_a \bar{g} \right) \right) - \psi \div (\bar{u}) - a_{11} \bar{p}_w + a_{12} \bar{P}_a = 0 \quad (4)
\]

\[
\frac{1}{\rho_a} \div \left( \bar{\rho}_a k_{ua} \left( \bar{P}_a + \bar{P}_w \bar{g} \right) \right) - (1 - \psi) \div (\bar{u}) + a_{21} \bar{P}_w - a_{22} \bar{P}_a = 0 \quad (5)
\]

where

\[
a_{11} = c_{uw} n_w + a_{12}, \; a_{22} = c_{ua} n_a + a_{21}
\]

2.5 Void ratio dependent Water Retention Curve

A critical step in modelling the behaviour of unsaturated soils is the determination of the soil water retention capacity at various suction and hydraulic loading conditions (i.e., main drying, main wetting, or
suction reversals) at a given density state. In this study, a void ratio dependent WRC similar to that presented in Pasha et al. (2017) is adopted. The salient feature of this model is that while it captures the volume change dependency of all branches of the WRC, it does not introduce any additional parameter into the formulation.

2.6 Coefficient of permeability

The coefficients of permeability of both water and air phases are assumed functions of void ratio according to the widely used Kozeny–Carman model (Scheidegger, 1958), and the degree of saturation according to Brooks and Corey (1964).

3  EDGE-BASED SMOOTHED POINT INTERPOLATION METHOD

3.1 Function approximation

The polynomial point interpolation method (PIM) (Liu and Gu, 2001) is considered in this study for determination of nodal shape functions. The PIM nodal shape functions possess Kronecker delta property, facilitating the implementation of the essential boundary conditions.

3.2 Construction of the smoothing domains

To discretise the system of equations, the generalised smoothed Galerkin (GS-Galerkin) weak formulation is adopted allowing the use of discontinuous approximation functions (Liu, 2008). In the GS-Galerkin approach, the problem domain is divided into \( n_{SD} \) smoothing domains on top of the \( n_c \) triangular background cells in a non-overlapping and no-gap fashion, so that

\[
\Omega = \bigcup_{i=1}^{n_{SD}} \Omega_i^{SD}
\]

and

\[
\Omega_i^{SD} \cap \Omega_j^{SD} = \emptyset, \quad \forall i \neq j
\]

where \( \Omega_i^{SD} \) represents the \( i \)th smoothing domain and \( \Omega \) is the whole domain.

Permissible smoothing domains can be constructed in different ways resulting in different numerical solution schemes with different features. In this study, the edge-based smoothing domain construction approach has been adopted since it has been shown to result in a numerical procedure possessing excellent efficiency and accuracy (Liu and Zhang, 2013).

3.3 Strain field construction

In the SPIM, a constant smoothed strain \( \tilde{\varepsilon}_k \) is constructed over each smoothing domain \( \Omega_k^{SD} \) by an integral representation as follows (Liu and Zhang, 2008)

\[
\tilde{\varepsilon}_k = \frac{1}{A_k^{SD}} \int_{\Omega_k^{SD}} L_n \mathbf{u}(x) d\Gamma
\]

where \( A_k^{SD} = \int_{\Omega_k^{SD}} d\Omega \) is the area of the \( k \)th smoothing domain. \( L_n \) is the matrix of unit outward normal and

\( \mathbf{u} \) is the displacement field.

Substituting the equations of displacement and fluid pressures at the point of interest \( \mathbf{x} \) into equation (9), the smoothed strain for the \( k \)th smoothing domain is obtained as

\[
\tilde{\varepsilon}_k = \left( \frac{1}{A_k^{SD}} \int_{\Omega_k^{SD}} L_n \Phi_k^i(x) d\Gamma \right) u^g = \sum_{i=1}^{2q} \left[ \begin{array}{c} b_{i1} \\ b_{i2} \end{array} \right] \left[ \begin{array}{c} \{ u_i \} \\ \{ \dot{u}_i \} \end{array} \right] = \hat{B}_i u^g
\]

with

\[
\Phi_k^i(x) = \begin{bmatrix} \varphi_1(x) & \varphi_2(x) & \ldots & \varphi_q(x) \\ \varphi_1(x) & \varphi_2(x) & \ldots & \varphi_q(x) \end{bmatrix}_{2q \times 2q}
\]

\[
\hat{B}_i = \begin{bmatrix} b_{i1} & 0 & b_{i2} & 0 & b_{i3} & 0 & \ldots & 0 & b_{i2q} \\ 0 & b_{i1} & 0 & b_{i2} & b_{i3} & \ldots & 0 & b_{i2q} \end{bmatrix}_{3 \times 2q}
\]

\[
u^g = \left[ u_{i1} \quad u_{i2} \quad u_{i3} \quad \ldots \quad u_{iq} \quad u_{i2q} \right]^{T}
\]

and

\[
\tilde{\varepsilon}_k = \frac{1}{A_k^{SD}} \int_{\Omega_k^{SD}} \Phi_k^i(x) n_i(x) d\Gamma, \quad i, j = 1, 2, \ldots, 4
\]

where \( q \) is the total number of supporting nodes of all the Gauss points on the boundaries of the \( k \)th smoothing domain. Employing the Gauss integration scheme, the integration in equation (14) can be further simplified to a summation form considering the linear segments of the smoothing domain boundaries.

4  NUMERICAL ALGORITHM

4.1 Discretisation

The governing equations are discretised in space by introducing the GS-Galerkin approach to equations (1), (4) and (5), and neglecting the soil self-weight. Moreover, three-point time discretisation scheme (Khoshghalb et al., 2011) is used for temporal discretisation. The fully coupled discretised governing equations is then obtained as follows

\[
K^t_{ij} \mathbf{u}^t - \chi^t Q P^t_w - (1 - \chi^t) Q P^t_w + (K_{i+j}^{t+\Delta t} P^t_{i+j} - K_{i-j}^{t+\Delta t} P^t_{i-j}) = \psi^{t+\Delta t} Q (P^t_w + \Delta P^t_{w} - P^t_w) - (1 - \psi^{t+\Delta t}) Q (P^t_{i+j} + \Delta P^t_{i+j} - P^t_{i-j}) = F_{i+j}^{t+\Delta t}
\]

\[
\psi^{t+\Delta t} Q (A U^t + \Delta C U^{t-\Delta t}) + (1 - \psi^{t+\Delta t}) Q (A U^t - \Delta C U^{t-\Delta t}) = \Delta t H_{i+j} P^t_{i+j} + \Delta t C P^t_{i-j} + \Delta t A P^t_{i+j} - \Delta t B P^t_{i-j} + \Delta t A P^t_{i-j} - \Delta t B P^t_{i+j} + \Delta t C P^t_{i-j}
\]

where

\[
\Delta t = \frac{t_{n+1} - t_n}{k}, \quad k = \text{time step size}
\]

\[
K^t_{ij} = \frac{1}{A_k^{SD}} \int_{\Omega_k^{SD}} K_n \mathbf{u}(x) d\Gamma
\]

\[
K_{i+j}^{t+\Delta t} = \frac{1}{A_{i+j}^{SD}} \int_{\Omega_{i+j}^{SD}} K_n \mathbf{u}(x) d\Gamma
\]

\[
K_{i-j}^{t+\Delta t} = \frac{1}{A_{i-j}^{SD}} \int_{\Omega_{i-j}^{SD}} K_n \mathbf{u}(x) d\Gamma
\]

\[
P^t_w = \frac{1}{A_k^{SD}} \int_{\Omega_k^{SD}} P_n \mathbf{u}(x) d\Gamma
\]

\[
F_{i+j}^{t+\Delta t} = \frac{1}{A_k^{SD}} \int_{\Omega_k^{SD}} F_n \mathbf{u}(x) d\Gamma
\]

\[
Q = \frac{1}{A_k^{SD}} \int_{\Omega_k^{SD}} Q_n \mathbf{u}(x) d\Gamma
\]

\[
\psi^{t+\Delta t} = \frac{1}{A_k^{SD}} \int_{\Omega_k^{SD}} \psi_n \mathbf{u}(x) d\Gamma
\]

\[
\Delta t = \frac{t_{n+1} - t_n}{k}
\]

\[
t_n = \text{current time step}
\]

\[
k = \text{time step size}
\]
\[
\alpha^i_{t=t+\Delta t} s (\Delta P_w^{t+\Delta t} a - \beta \Delta P_a^{t+\Delta t}) - \alpha^i_{t=t+\Delta t} s (\Delta P_w^{t+\Delta t} a - \beta \Delta P_a^{t+\Delta t}) = \\
\Delta \Delta P_a^{t+\Delta t} a
\]

where \( \alpha \) is a constant time step growth factor expanding the time interval in each step.

### 4.2 Solution algorithm

Adopting the modified Newton-Raphson iterative procedure and expanding the equations with the first-order truncated Taylor series, the vector of nodal displacements and pore fluid pressures corrections at iteration \( i+1 \) is obtained as

\[
\begin{bmatrix}
\Delta u_{i+1, t+\Delta t} \\
\Delta P_w_{i+1, t+\Delta t} \\
\Delta P_a_{i+1, t+\Delta t}
\end{bmatrix} = \{J(t, t+\Delta t)^{-1}\}
\begin{bmatrix}
\Delta u_i \\
\Delta P_w_i \\
\Delta P_a_i
\end{bmatrix}
\]

in which \( J \) is the Jacobian matrix and \( \{J(t, t+\Delta t)^{-1}\} \) indicates the value of \( \{J\} \) at the \( i \)th iteration at time \( t \). The stresses are then obtained from the updated solution and used to form the residual vector for the next iteration or as the final stresses for the current time step if the convergence is reached in the current iteration.

### 5 NUMERICAL EXAMPLES

Two examples are investigated in this section to examine the proposed model. One-dimensional consolidation of unsaturated elastic porous media is studied first. The effect of hydraulic hysteresis on hydro-mechanical response of unsaturated porous media is investigated and the results are compared to those of an FEM solution. The next example involves a series of plain strain compression (PSC) tests incorporating the bounding surface plasticity model detailed in Khalili et al. (2008) and the numerical results are compared to the results from another study.\( g = 9.8 \text{m/s}^2 \) is assumed in all examples.

#### 5.1 1D consolidation problem

A one-dimensional consolidation problem whose FEM solution is available (Shahbodagh-Khan et al., 2015) is adopted in this example. The problem involves a 100m long unsaturated soil column as illustrated in Figure 1. Drainage is only allowed on the upper boundary of the soil, and other boundaries are considered impervious. The displacement boundary conditions are also shown in Figure 1, along with the triangular background mesh used for the simulations.

The material properties and fluid hypothetical mechanical properties are adopted from Shahbodagh-khan et al. (2015). A linearly increasing distributed surcharge, is applied on the soil surface which reaches a maximum value of \( \sigma_{\text{max}} = 100 \text{kPa} \) at \( t = 100 \text{s} \). A series of simulations are carried out considering different initial degrees of saturation. In the first set of the analyses, the hydraulic hysteresis ignored, i.e., \( s_{w0} = s_{ew} = 10 \text{kPa} \). The WRC is also assumed void ratio independent throughout the analyses, in accordance with Shahbodagh-Khan et al. (2015).

Figure 1 Schematic representation of the soil column and its associated mesh and boundary conditions.

Figure 2 shows the surface settlement of the soil column versus time for dry, saturated, and three unsaturated cases with different initial states of suction to air entry value ratios of 1.5, 2.0, and 4.0. Also shown in Figure 2 are the results of the simulations by Shahbodagh-khan et al. (2015) using an FEM model. As seen in Figure 2, the SPIM results in terms of vertical settlement are in excellent accordance with the results of the FEM simulation in all cases studied.

To verify the implementation of the hydraulic hysteresis model included in the SPIM of this study, and also to highlight the effect of hydraulic hysteresis on the results of one dimensional consolidation of unsaturated soils, another simulation is carried out assuming and initial suction of \( s_0 = 20 \text{kPa} \), an air entry value of
\[ s_{ae} = 10 \text{kPa}, \] and an air expulsion value of \[ s_{ex} = 5 \text{kPa}, \] implying the existence of hydraulic hysteresis in this case. The results in terms of the surface settlement versus time obtained from the SPIM are shown in Figure 3. Also included in this figure are the results obtained using an FE model. As can be seen from Figure 3, the numerical results of this study are in perfect agreement with the benchmark solution. Figure 3 shows that taking account of hydraulic hysteresis markedly reduces the rate of consolidation. This essentially happens due to the change in the hydraulic path from the main path in the non-hysteretic model to the scanning path in the hysteretic model, which leads to larger pore pressure generations and lower suction during the analysis when hydraulic hysteresis is included in the model (Shahbodagh-Khan et al., 2015).

5.2 Plain strain compression problem

The bounding surface plasticity model implemented in the SPIM developed in this study is examined in this example which concerns a series of plane strain compression (PSC) tests conducted on unsaturated Bourke silt from the Bourke region of New South Wales, Australia as reported in Peric et al. (2014). Model parameters are selected the same as those presented in Peric et al. (2014). The background mesh adopted in the numerical analyses along with the displacement boundary conditions are illustrated in Figure 4. The tests involve application of a vertical compressive strain to the top of the medium at the constant rate of \( 10^{-6} \text{s}^{-1} \). In the numerical simulations, a stationary WRC with no hydraulic hysteresis is considered in accordance with the assumptions in Peric et al. (2014). Drained plain strain tests with initial net stresses of 100 kPa and initial suctions of 50, 150, and 250 kPa are carried out.

The results of the drained analyses are summarised in Figure 5 and 6. The variations of the deviatoric stress versus the axial strain are illustrated in Figure 5. Figure 6 shows how the volumetric strain is generated with loading. The SPIM results are compared to the numerical results presented in Peric et al. (2014) which were already verified using a series of conventional triaxial compression tests performed on Bourke silt by Uchaipichat and Khalili (2009). Perfect agreements are observed between the results of this study and the reference solutions.

Figure 3 Effect of hydraulic hysteresis on the surface settlement of the soil column.

Figure 4 Background mesh and displacement boundary conditions for the PSC problem.

Figure 5 Variations of the deviatoric stress versus axial strain in the drained PSC analyses for different initial suctions with the initial net stress of 100 kPa.

Figure 6 Volumetric strain versus axial strain in the drained analysis for an initial suction of 50 kPa and initial net stress of 100 kPa.
6 CONCLUSION

A computational framework based on the SPIM was introduced for flow and deformation analysis of unsaturated porous media. The deformation and flow models were developed based on the principle of effective stress, and momentum and mass conservation of the phases. A hysteretic water retention model was implemented which takes into account the evolution of hydraulic hysteresis in unsaturated porous media. The proposed model was verified against different reference solutions from the literature.

ACKNOWLEDGEMENTS

The authors would like to thank Dr Babak Shahbodagh for providing the results of his numerical simulations for one of the examples studied in this work.

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