RIEMANNIAN SUBMERSIONS WITH DISCRETE SPECTRUM

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ABSTRACT. We prove some estimates on the spectrum of the Laplacian of the total space of a Riemannian submersion in terms of the spectrum of the Laplacian of the base and the geometry of the fibers. When the fibers of the submersions are compact and minimal, we prove that the total space is discrete if and only if the base is discrete. When the fibers are not minimal, we prove a discreteness criterion for the total space in terms of the relative growth of the mean curvature of the fibers and the mean curvature of the geodesic spheres in the base. We discuss in particular the case of warped products.

1. INTRODUCTION

Let $M$ be a complete Riemannian manifold and $\triangle = \text{div} \circ \text{grad}$ be the Laplace-Beltrami operator acting on the space of smooth functions on $M$ with compact support. The operator $\triangle$ is essentially self-adjoint, thus it has a unique self-adjoint extension, to an unbounded operator, denoted by $\triangle$, whose domain is the set of functions $f \in L^2(M)$ so that $\triangle f \in L^2(M)$. Recall that the spectrum of a self-adjoint operator $A$, denoted by $\sigma(A)$, is formed by all $\lambda \in \mathbb{R}$ for which $A - \lambda I$ is not injective or the inverse operator $(A - \lambda I)^{-1}$ is unbounded, [7]. In this paper we are going to study the spectrum of $-\triangle$, (the operator $\triangle$ is negative), and we refer to $\sigma(-\triangle)$ as the spectrum of $M$ and in this case only, we denote by $\sigma(M)$. It is important (in our study) to distinguish the various types of elements of the spectrum of $M$ in order to have a better understanding of the relations between $M$ and $\sigma(M)$. This way, it is said that the set of all eigenvalues of $\sigma(M)$ is the point spectrum $\sigma_p(M)$, while the discrete spectrum $\sigma_d(M)$ is the set of all isolated$^1$ eigenvalues of finite multiplicity. The essential spectrum $\sigma_{\text{ess}}(M) = \sigma(M) \setminus \sigma_d(M)$ is the complement of the discrete spectrum.

There is a vast literature studying the spectrum of complete Riemannian manifolds, among that, we point out geometric restrictions implying that the spectrum is purely continuous ($\sigma_p(M) = \emptyset$), see [9], [10], [12], [16], [21], [23] or implying that the spectrum if discrete ($\sigma_{\text{ess}}(M) = \emptyset$), see [1], [4], [11], [15], [17], [18].

$^1$Isolated in the sense that for some $\varepsilon > 0$ one has that $(\lambda - \varepsilon, \lambda + \varepsilon) \cap \sigma(\triangle^M) = \lambda$. Date: January 5th, 2010.

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In [1], Baider studied the essential spectrum of warped product manifolds \( W = X \times_\gamma Y \), where \( \gamma : X \to \mathbb{R} \) is a positive smooth function. The Laplace-Beltrami operator \( \Delta W \) restricted to \( C^\infty_0(X) \otimes C^\infty_0(Y) \) has this form 
\[
\Delta W = A_0 \otimes 1_Y + \gamma^{-2} \otimes (\Delta Y),
\]
where \( A_0 \) is an elliptic operator, symmetric relative to the density \( \gamma^n dX \) with the same symbol as \( \Delta X \). Baider showed that if \( \sigma_{\text{ess}}(Y) = \emptyset \) then
\[
\sigma_{\text{ess}}(W) = \emptyset \iff \sigma_{\text{ess}}(A_1) = \emptyset,
\]
where \( A_1 = A_0 + \lambda^*(Y)\gamma^{-2} \) and \( \lambda^*(Y) = \inf \sigma(\Delta Y) \). When \( \gamma \equiv 1 \) then \( W = X \times Y \) and \( A_1 = \Delta X + \lambda^*(Y) \). In general, one can not substitute \( \sigma_{\text{ess}}(A_1) = \emptyset \) by \( \sigma_{\text{ess}}(X) = \emptyset \). There are examples of warped manifolds \( \mathbb{R}^n \times_\gamma S^1 \), with discrete spectrum, therefore \( \sigma_{\text{ess}}(A_1) = \emptyset \), see [1], but the spectrum \( \sigma_{\text{ess}}(\mathbb{R}^n) = [0, \infty) \). Riemannian manifolds whose Laplacian has empty essential spectrum are sometimes called discrete in the literature.

In this paper we consider Riemannian submersions \( \pi : M \to N \) and we prove some spectral estimates relating the (essential) spectrum of \( M \) and \( N \). Riemannian submersions were introduced in the sixties by B. O'Neill and A. Gray (see [13, 19, 20]) as a tool to study the geometry of a Riemannian manifold with an additional structure in terms of certain components, that is, the fibers and the base space. When \( M \) (and thus also \( N \)) is compact, estimates on the eigenvalues of the Laplacian of \( M \) have been studied in [5], under the assumption that the mean curvature vector of the fibers is basic, i.e., \( \pi \)-related to some vector field on the basis. We will consider here the non compact case, assuming initially that the fibers are minimal.

An important class of examples are Riemannian homogeneous spaces \( G/K \), where \( G \) is a Lie group endowed with a bi-invariant Riemannian metric and \( K \) is a closed subgroup of \( G \), see [19] for details. The projection \( G \to G/K \) is a Riemannian submersions with totally geodesic fibers, and with fibers diffeomorphic to \( K \).

Another important class of examples of manifolds that can be described as the total space of Riemannian submersions with minimal fibers are the homogeneous 3-dimensional Riemannian manifolds with isometry group of dimension four, see [22]. This class includes the special linear group \( SL(2, \mathbb{R}) \) endowed with a family of left-invariant metrics, which is the total space of Riemannian submersions with base given by the hyperbolic spaces, and fibers diffeomorphic to \( S^1 \).

Given a Riemannian submersion \( \pi : M \to N \) with compact minimal fibers, we prove that
\[
\sigma_{\text{ess}}(M) = \emptyset \iff \sigma_{\text{ess}}(N) = \emptyset,
\]
see Theorem 1. This result coincides with Baider’s result when \( M = X \times Y \) is a product manifold, \( Y \) is compact, \( N = X \) and \( \pi : X \times Y \to X \) is the projection on the first factor.

**Theorem 1.** Let \( \pi : M \to N \) be a Riemannian submersion with compact minimal fibers. Then
A few remarks on this result are in order. First, we observe that for
the inequality \(\inf \sigma_{\text{ess}}(M) \leq \inf \sigma_{\text{ess}}(N)\), Lemma 3.7, we need only the
compactness of the fibers with uniformly bounded volume, meaning that
\(0 < c^2 \leq \text{vol}(F_p) \leq C^2\) for all \(p \in N\). Second, the example of [1] shows
that the assumption of minimality of the fibers is necessary in Theorem 1. In
fact, one has examples of Riemannian submersions having compact fibers
with discrete base and non-discrete total space, or with discrete total space
but not discrete base, see Example 4.2.

In the second part of the paper we study the essential spectrum of the
total space when the minimality assumption on the fiber is dropped. In this
case, we prove that a sufficient condition for the discreteness of the total
space is that the growth of the mean curvature of the fibers at infinity is
controlled by the growth of the mean curvature of the geodesic spheres in
the base manifold. In order to state our result, let us introduce the following
terminology. The cut locus \(\text{cut}(p)\) of a point \(p\) in a Riemannian \(n\)-manifold
is said to be thin, if its \((n-1)\)-Hausdorff measure zero,
\(H^{n-1}(\text{cut}(p)) = 0\).

\textbf{Theorem 2.} Let \(\pi : M \rightarrow N\) be a Riemannian submersion with compact
fibers, and assume that \(N\) has a point \(x_0\) with thin cut locus. If the function
\(h : M \rightarrow \mathbb{R}\) defined by
\[ h(q) = (\nabla^N \rho_{p_0})_{\pi(q)} + g^N((\nabla^N \rho_{p_0})_{\pi(q)}, \text{d} \pi_q(H_q)) \]
is proper then \(\sigma_{\text{ess}}(M) = \emptyset\). Here \(\rho_{p_0}\) is the distance function in \(N\) to \(p_0\).

The Theorem 2 can be interpreted geometrically in terms of the mean
curvature of the geodesic spheres in the base and the mean curvature of the
fibers. Namely, the Laplacian of the distance function \(\rho_{p_0}(p)\) is exactly the
value of the mean curvature of the geodesic sphere \(S_p = \rho_{p_0}^{-1}(\rho_{p_0}(p))\) at
the point \(p\). Thus, assumption says that the sum of the mean curvature of the
geodesic balls in \(N\) and the mean curvature of the fibers must diverge at
infinity.

Theorem 1 is proved in Section 3 and Theorem 2 in Section 4. An alter-
native statement of Theorem 2 can be given in terms of radial curvature, see
Corollary 4.2. There are two basic ingredients for the proof of our results.

\begin{itemize}
  \item The Decomposition Principle, that relates the fundamental tone of
    the complement of compact sets with the infimum of the essential
    spectrum, see Proposition 3.2;
  \item Two estimates of the fundamental tones of open sets in terms of
    the divergence of vector fields, proved recently in [2] and [3], see
    Propositions 3.4 and 3.5.
\end{itemize}
2.1. Preliminaries. Given manifolds $M$ and $N$, a smooth surjective map $\pi: M \to N$ is a submersion if the differential $d\pi(q)$ has maximal rank for every $q \in M$. If $\pi : M \to N$ is a submersion, then for all $p \in N$ the inverse image $F_p = \pi^{-1}(p)$ is a smooth embedded submanifold of $M$, that will be called the fiber at $p$. If $M$ and $N$ are Riemannian manifolds, then a submersion $\pi : M \to N$ is called a Riemannian submersion if for all $p \in N$ and all $q \in F_p$, the restriction of $d\pi(q)$ to the orthogonal subspace $T_qF_p$ is an isometry onto $T_pM$.

Given $p \in N$ and $q \in F_p$, a tangent vector $\xi \in T_qM$ is said to be vertical if it is tangent to $F_p$, and it is horizontal if it belongs to the orthogonal space $(T_qF_p)^\perp$. Let $\mathcal{D} = (T\mathcal{F})^\perp \subset TM$ denote the smooth rank $k$ distribution on $M$ consisting of horizontal vectors. The orthogonal distribution $\mathcal{D}^\perp$ is clearly integrable, the fibers of the submersion being its maximal integral leaves. Given $\xi \in TM$, its horizontal and vertical components are denoted respectively by $\xi^h$ and $\xi^v$. The second fundamental form of the fibers is a symmetric tensor $S^\mathcal{F}: \mathcal{D}^\perp \times \mathcal{D}^\perp \to \mathcal{D}$, defined by

$$S^\mathcal{F}(v,w) = (\nabla^M_vw)^h,$$

where $W$ is a vertical extension of $w$ and $\nabla^M$ is the Levi–Civita connection of $M$.

For any given vector field $X \in \mathfrak{X}(N)$, there exists a unique horizontal $\widetilde{X} \in \mathfrak{X}(M)$ which is $\pi$-related to $X$, this is, for any $p \in N$ and $q \in F_p$, then $d\pi_q(\widetilde{X}_q) = X_p$, called horizontal lifting of $X$. A horizontal vector field $\widetilde{X} \in \mathfrak{X}(M)$ is called basic if it is $\pi$-related to some vector field $X \in \mathfrak{X}(N)$.

If $\widetilde{X}$ and $\widetilde{Y}$ are basic vector fields, then these observations follows easily.

(a) $g^M(\widetilde{X},\widetilde{Y}) = g^N(X,Y) \circ \pi.$

(b) $[\widetilde{X},\widetilde{Y}]^h$ is basic and it is $\pi$-related to $[X,Y]$.

(c) $(\nabla^M_X\widetilde{Y})^h$ is basic and it is $\pi$-related to $\nabla^N_XY$,

where $\nabla^N$ is the Levi-Civita connection of $g^N$.

Let us now consider the geometry of the fibers. First, we observe that the fibers are totally geodesic submanifolds of $M$ exactly when $S^\mathcal{F} = 0$. The mean curvature vector of the fiber is the horizontal vector field $H$ defined by

$$H(q) = \sum_{i=1}^k S^\mathcal{F}(q)(e_i,e_i) = \sum_{i=1}^k (\nabla^M_{e_i}e_i)^h$$

where $(e_i)_{i=1}^k$ is a local orthonormal frame for the fiber through $q$. Observe that $H$ is not basic in general. For instance, when $n = 1$, i.e., when the fibers are hypersurfaces of $M$, then $H$ is basic if and only if all the fibers have constant mean curvature. The fibers are minimal submanifolds of $M$ when $H \equiv 0$. 
2.2. **Differential operators.** Let \( \pi : M \to N \) be a Riemannian submersion. Besides the natural operations of lifting a vector or vector fields in \( N \) to horizontal vectors and basic vector fields one has that functions on \( N \) can be lifted to functions on \( M \) that are constant along the fibers. Such operations preserves the regularity of the lifted objects. One can also (locally) lift curves in the base \( \gamma : [a, b] \to N \) to horizontal curves \( \tilde{\gamma} : [a, c) \to M \) with the same regularity as \( \gamma \) with arbitrary initial condition on the fiber \( F_{\gamma(a)} \).

We will need formulas relating the derivatives of \( \pi \)-related objects in \( M \) and \( N \). Let us start with divergence of vector fields.

**Lemma 2.1.** Let \( \tilde{X} \in X(M) \) be a basic vector field, \( \pi \)-related to \( X \in X(N) \).

The following relation holds between the divergence of \( \tilde{X} \) and \( X \) at \( p \in N \) and \( q \in F_p \).

\[
\text{div}^M(\tilde{X})_q = \text{div}^N(X)_p + g^M(\tilde{X}_q, H_q)
\]

In particular, if the fibers are minimal, then \( \text{div}^M(\tilde{X}) = \text{div}^N(X) \).

**Proof.** Formula (2.2) is obtained by a direct computation of the left-hand side, using a local orthonormal frame \( e_1, \ldots, e_k, e_{k+1}, \ldots, e_{k+n} \) of \( TM \), where \( e_1, \ldots, e_k \) are basic fields. The equality follows using equalities (a) and (c) in Subsection 2.1, and formula (2.1) for the mean curvature. \( \square \)

Given a smooth function \( f : N \to \mathbb{R} \), denote by \( \tilde{f} = f \circ \pi : M \to \mathbb{R} \) its lifting to \( M \). It is easy to see that the gradient \( \text{grad}^M \tilde{f} \) of \( \tilde{f} \) is the horizontal lifting of the gradient \( \text{grad}^N f \). If we denote with a tilde \( \tilde{X} \) the horizontal lifting of a vector field \( X \in X(N) \), then the previous statement can be written as

\[
\text{grad}^M \tilde{f} = \text{grad}^N f.
\]

Now, given a function \( f : M \to \mathbb{R} \), one can define a function \( f_{av} : N \to \mathbb{R} \) by averaging \( f \) on each fiber

\[
f_{av}(p) = \int_{F_p} f \, d\mathcal{F}_p,
\]

where \( d\mathcal{F}_p \) is the volume element of the fiber \( \mathcal{F}_p \) relative to the induced metric. We are assuming that this integral is finite. As to the gradient of the averaged function \( f_{av} \), we have the following lemma.

**Lemma 2.2.** Let \( p \in N \) and \( v \in T_pN \) and denote by \( V \) the smooth normal vector field along \( \mathcal{F}_p \), defined by the property \( d\pi_q(V_q) = v \) for all \( q \in \mathcal{F}_p \). Then, for any smooth function \( f : M \to \mathbb{R} \)

\[
g^N(\text{grad}^N f_{av}(p), v) = \int_{\mathcal{F}_p} \left[ g^M(\text{grad}^M f, V) + f \cdot g^M(H, V) \right] \, d\mathcal{F}_q.
\]
Proof. A standard calculation as in the first variation formula for the volume functional of the fibers. Notice that when $f \equiv 1$, then $f_{av}$ is the volume function of the fibers, and (2.4) reproduces the first variation formula for the volume. □

Observe that, in (2.4), the gradient $\text{grad}^M f$ need not be basic or even horizontal\(^2\). An averaging procedure is available also to produce vector fields $X_{av}$ on the basis out of vector fields $X$ defined in the total space. If $X \in \mathfrak{X}(M)$, let $X_{av} \in \mathfrak{X}(N)$ be defined by

$$(X_{av})_p = \int_{\mathcal{F}_p} d\pi_q(X_q) \, d\mathcal{F}_p(q).$$

Observe that the integrand above is a function on $\mathcal{F}_p$ taking values in the fixed vector space $T_p N$. If $X \in \mathfrak{X}(M)$ is a basic vector field, $\pi$-related to the vector field $X_* \in \mathfrak{X}(N)$, then $(X_{av})_p = \text{vol}(\mathcal{F}_p) \cdot (X_*)_p$, where $\text{vol}$ denotes the volume. Using the notion of averaged field, equality (2.4) can be rewritten as

$$\text{grad}^N(f_{av}) = \left(\text{grad}^M f + f \cdot H\right)_{av}.$$  

Remark 2.3. From the above formula it follows easily that the averaged mean curvature vector field $H_{av}$ vanishes at the point $p \in N$ if and only if $p$ is a critical point of the function $z \mapsto \text{vol}(\mathcal{F}_z)$ in $N$. This happens, in particular, when the leaf $\mathcal{F}_p$ is minimal. When all the fibers are minimal, or more generally when the averaged mean curvature vector field $H_{av}$ vanishes identically, then the volume of the fibers is constant.

Corollary 2.4. Let $\pi : M \to N$ be a Riemannian submersion with compact minimal fibers $\mathcal{F}$. Let $h \in L^2(N)$. If $f \in C_0^\infty(M)$ such that $f_{av} = 0$ for all $q \in N$ then

$$\int_M \tilde{h} \Delta^M f \, dM = 0. \tag{2.5}$$

Proof. Suppose first that $h$ is smooth. By the Divergence Theorem, Fubini’s Theorem for Riemannian submersions and 2.4 we have

$$\int_M \tilde{h} \Delta^M f \, dM = - \int_M g^M(\text{grad}^M \tilde{h}, \text{grad}^M f) \, dM$$

$$= - \int_N \int_{\mathcal{F}_q} g^M(\text{grad}^M \tilde{h}, \text{grad}^M f) \, d\mathcal{F}_q dN$$

$$= - \int_N g^N(\text{grad}^M \tilde{h}, \text{grad}^N f_{av}) \, dN$$

$$= 0.$$

\(^2\)In fact, a gradient is basic if and only if it is horizontal.
If \( h \in L^2(N) \) there exists a sequence of smooth functions \( h_k \in C^\infty(N) \) converging to \( h \) with respect to the \( L^2 \)-norm. On the other hand
\[
\left| \int_M \tilde{h} \Delta^M f \, dM \right| = \left| \int_M (\tilde{h}_k - \tilde{h}) \Delta^M f \, dM \right|
\leq \int_M |\tilde{h}_k - \tilde{h}| |\Delta^M f| \, dM
\leq \left( \int_M |\tilde{h}_k - \tilde{h}|^2 \, dM \right)^{1/2} \cdot \left( \int_M |\Delta^M f|^2 \, dM \right)^{1/2}
= \|\Delta^M f\|_{L^2(M)} \cdot \left( \int_N \int_{F_q} |h_k - h|^2 \, dF_q \, dM \right)^{1/2}
= \text{vol}(F_q)^{1/2} \cdot \|\Delta^M f\|_{L^2(M)} \cdot \|h_k - h\|_{L^2(N)}
\]
Since \( h_k \to h \) in \( L^2(N) \) then 2.5 holds. Observe that we used that the volume of the minimal fibers is constant, see Remark 2.3.

3. Spectral estimates in Riemannian submersions

3.1. Generalities on the Laplace-Beltrami operator. Let \( \Omega \subset M \) be an open set in complete Riemannian manifold. The fundamental tone of \( \Omega \) is defined by
\[
\lambda^*(\Omega) = \inf \frac{\int_\Omega |\nabla f|^2}{\int_\Omega f^2},
\]
where the infimum is taken over all smooth non zero functions \( f \in C^\infty_0(\Omega) \).

The fundamental tone has the following monotonicity property: if \( A \subset B \) are open subsets then \( \lambda^*(A) \geq \lambda^*(B) \). If \( \Omega \) is an open subset of \( M \) with compact closure, or if \( \lambda^*(\Omega) \notin \sigma_{\text{ess}}(M) \), then \( \lambda^*(\Omega) \) coincides with the first eigenvalue \( \lambda_1(\Omega) \) of the Dirichlet problem
\[
\begin{cases}
\Delta^M u + \lambda u = 0 \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega.
\end{cases}
\]
When \( \Omega = M \) then \( \lambda^*(M) = \inf \sigma(M) \).

The Laplace-Beltrami operator in \( N \) of a smooth function \( f : N \to \mathbb{R} \) and the Laplace-Beltrami operator in \( M \) of its extension \( \tilde{f} = f \circ \pi \) are related by the following formula.

**Lemma 3.1.** Let \( f : N \to \mathbb{R} \) be a smooth function and set \( \tilde{f} = f \circ \pi \). Then, for all \( p \in N \) and all \( q \in F_p \):
\[
(\Delta^M \tilde{f})_q = (\Delta^N f)_p + g^M((\nabla^M \tilde{f})_q, H_q)
= (\Delta^N f)_p + g^N((\nabla^N f)_p, d\pi_q(H_q)).
\]
(3.1)
The proof follows easily from (2.2) applied to the vector fields \( X = \nabla^M \tilde{f} \) and \( X_* = \nabla^N f \), using (2.3).
3.2. **Decomposition Principle.** Let \( K \subset M \) be a compact set of the same dimension of \( M \). The Laplace-Beltrami operator \( \triangle \) of \( M \) acting on the space \( C^0_0(M \setminus K) \) of smooth compactly supported functions of \( M \setminus K \) has a self-adjoint extension, denoted by \( \triangle' \). The Decomposition Principle [11] says that \( \sigma_{\text{ess}}(M) = \sigma_{\text{ess}}(M \setminus K) \). On the other hand, 

\[
0 \leq \lambda^*(M \setminus K) = \inf \sigma(M \setminus K) \leq \inf \sigma_{\text{ess}}(M \setminus K) = \sigma_{\text{ess}}(M),
\]

thus \( \mu = \sup \{ \lambda^*(M \setminus K), K \subset M \text{ compact} \} \leq \sigma_{\text{ess}}(M) \). We will show that \( \inf \sigma_{\text{ess}}(M) \leq \mu \). To that we will suppose that \( \mu < \infty \), otherwise there is nothing to prove. Let \( K_1 \subset K_2 \subset \cdots \) be a sequence of compact sets with \( M = \cup_{i=1}^{\infty} K_i \). We have that

\[
\lambda^*(M) \leq \lambda^*(M \setminus K_1) \leq \lambda^*(M \setminus K_2) \leq \cdots \leq \mu.
\]

Given \( \varepsilon > 0 \), there exists \( f_1 \in C^\infty_0(M \setminus K_1) \) with \( \| f_1 \|_{L^2} = 1 \) and \( \int_M |\text{grad} f_1|^2 \leq \lambda^*(M \setminus K_1) + \varepsilon < \mu + \varepsilon \). This is \( \langle (\triangle - \mu - \varepsilon) f_1, f_1 \rangle_{L^2} < 0 \).

We can suppose that \( \text{supp} \ f_1 \subset (K_2 \setminus K_1) \). There exists \( f_2 \in C^\infty_0(M \setminus K_2) \) with \( \| f_2 \|_{L^2} = 1 \) and \( \int_M |\text{grad} f_2|^2 \leq \lambda^*(M \setminus K_2) + \varepsilon < \mu + \varepsilon \). This is equivalent to \( \langle (\triangle - \mu - \varepsilon) f_2, f_2 \rangle_{L^2} < 0 \). Moreover, \( \int_M f_1 f_2 = 0 \) since \( \text{supp} \ f_1 \cap \text{supp} \ f_2 = \emptyset \). This way, we obtain an orthonormal sequence \( \{ f_k \} \subset C^\infty_0(M) \) such that \( \langle (\triangle - \mu - \varepsilon) f_k, f_k \rangle_{L^2} < 0 \). By 3.3 we have that \( \langle \lambda^*(M \setminus K_n) \neq \emptyset \rangle \) and \( \inf \sigma_{\text{ess}}(M) \leq \mu \). This proves the following proposition.

**Proposition 3.2.** The infimum of the essential spectrum is characterized by

\[
\inf \sigma_{\text{ess}}(M) = \sup \{ \lambda^*(M \setminus K) : K \text{ compact subset of } M \}.
\]

In particular, \( \sigma_{\text{ess}}(M) \) is empty if and only if given any compact exhaustion \( K_1 \subset K_2 \subset \cdots \subset K_n \subset \cdots \) of \( M \), the limit \( \lim_{n \to \infty} \lambda^*(M \setminus K_n) \) is infinite.

Let \( H \) be a Hilbert space and \( A: D \subset H \to H \) be a densely defined self-adjoint operator. Given \( \lambda \in \mathbb{R} \), we write \( A \geq \lambda \) if \( \langle Ax, x \rangle \geq \lambda \|x\|^2 \) for all \( x \in D \). By the Spectral Theorem for (unbounded) self-adjoint operators, we have that \( A \geq \lambda \) iff \( \sigma(A) \subset [\lambda, +\infty) \). Let us write \( A > -\infty \) if there exists \( \lambda_* \in \mathbb{R} \) such that \( A \geq \lambda_* \).

**Lemma 3.3.** Let \( A: D \subset H \to H \) be a self-adjoint operator with \( A > -\infty \), and let \( \lambda \in \mathbb{R} \) be fixed. Assume that for all \( \varepsilon > 0 \) there exists an infinite dimensional subspace \( G_{\varepsilon} \subset D \) such that \( \langle Ax, x \rangle < (\lambda + \varepsilon) \|x\|^2 \) for all \( x \in G_{\varepsilon} \). Then,

\[
\sigma_{\text{ess}}(A) \cap (-\infty, \lambda] \neq \emptyset.
\]

This lemma is well known, see [8] but for sake of completeness we present here its proof.

**Proof.** First we will show that \( \sigma(A) \cap (-\infty, \lambda] = \sigma(A) \cap [\lambda_*, \lambda] \neq \emptyset \).

Take \( \varepsilon_k = 1/k, k \geq 1 \). By our hypothesis there exists \( x_k \neq 0 \) such that \( \langle Ax_k, x_k \rangle < (\lambda + 1/k) \|x_k\|^2 \), and thus \( \sigma(A) \cap [\lambda_*, \lambda + 1/k] \neq \emptyset \) for all \( k \geq 1 \). Since \( \sigma(A) \) is closed, it follows \( \sigma(A) \cap (-\infty, \lambda] \neq \emptyset \).
suppose that $\sigma(A) \cap (-\infty, \lambda] \not\subset \sigma_{\text{ess}}(A)$, otherwise there is nothing to prove. Thus

$$(\sigma(A) \setminus \sigma_{\text{ess}}(A)) \cap (-\infty, \lambda] = \{\lambda_1, \ldots, \lambda_n\}$$

is a finite set of eigenvalues of $A$ of finite multiplicity. Denote by $H_i \subset D$ the $\lambda_i$-eigenspace of $A$, $i = 1, \ldots, n$, and set $X = \bigoplus_i H_i \subset D$. This is clearly an invariant subspace of $A$. Since $X$ has finite dimension, then $D = X \oplus X_1$ where $X_1 = X^\perp \cap D$ is also invariant by $A$. Denote by $A_1$ the restriction of $A$ to the Hilbert space $X_1$ which is still self-adjoint. Clearly, $\sigma(A_1) = \sigma(A) \setminus \{\lambda_1, \ldots, \lambda_n\}$ and $\sigma_{\text{ess}}(A_1) = \sigma_{\text{ess}}(A)$. In particular, we have $\sigma(A_1) \cap (-\infty, \lambda] \subset \sigma_{\text{ess}}(A_1)$. Using the infinite dimensionality of the space $G_\varepsilon$, it is now easy to see that the assumptions of our lemma hold for the operator $A_1$, and the first part of the proof applies to obtain

$$\sigma_{\text{ess}}(A) \cap (-\infty, \lambda] \neq \emptyset.$$  \hfill \Box

Let us recall from [2] and [3] the following estimates for the fundamental tone of open sets of Riemannian manifolds.

**Proposition 3.4.** Let $\Omega \subset M$ be an open set of a Riemannian manifold. Then

$$(3.3) \quad \lambda^*(\Omega) \geq \frac{1}{4} \sup_X \left[ \frac{\inf_{\Omega} \text{div}(X)}{\sup_{\Omega} \|X\|} \right]^2,$$

where the supremum is taken over all smooth vector fields $X$ in $\Omega$ satisfying

$$\inf_{\Omega} \text{div}(X) > 0, \quad \text{and} \quad \sup_{\Omega} \|X\| < +\infty.$$

**Proposition 3.5.** Let $\Omega \subset M$ be an open set of a Riemannian manifold. Given any smooth vector field $X \in X(\Omega)$ then

$$(3.4) \quad \lambda^*(\Omega) \geq \inf_{\Omega} \left[ \text{div}(X) - |X|^2 \right].$$

Equality in (3.4) holds when $\lambda^*(\Omega) \in \sigma(\Omega) \setminus \sigma_{\text{ess}}(\Omega)$, by considering the field $X = -\nabla(\log f)$, where $f$ is the positive eigenfunction associated to $\lambda^*(\Omega)$.

**Remark 3.6.** Propositions (3.4) and (3.5) hold for vector fields $X$ of class $C^1(\Omega \setminus F) \cap L^\infty(\Omega)$ such that $\text{div}(X) \in L^1(\Omega)$, where $F \subset M$ is a closed subset with $(n-1)$-Hausdorff measure $H^{n-1}(F \cap \Omega) = 0$, see [3, Lemma 3.1].

**3.3. Fundamental tones estimates of Riemannian submersions.** Let $M$ and $N$ be connected Riemannian manifolds and $\pi: M \to N$ be a Riemannian submersion. Denote by $\Delta^M$ and $\Delta^N$ the Laplacian operator on functions of $(M, g^M)$ and of $(N, g^N)$ respectively. We want to compare the fundamental tones of open subsets $\Omega \subset N$ with the fundamental tones of its lifting $\tilde{\Omega} = \pi^{-1}(\Omega)$.
Lemma 3.7. Assume that the fibers of \( \pi : M \to N \) are compact. Let \( \Omega \) be an open subset of \( N \), and denote by \( \tilde{\Omega} \) the open subset of \( M \) given by the inverse image \( \pi^{-1}(\Omega) \). Then

\[
\inf_{p \in \Omega} \text{vol}(F_p) \cdot \lambda^*(\tilde{\Omega}) \leq \sup_{p \in \Omega} \text{vol}(F_p) \cdot \lambda^*(\Omega).
\]

In particular, if the fibers are minimal, then

\[
\lambda^*(\tilde{\Omega}) \leq \lambda^*(\Omega).
\]

Moreover, if \( \inf_{p \in \Omega} \text{vol}(F_p) > 0 \) and \( \sup_{p \in \Omega} \text{vol}(F_p) < \infty \) then

\[
\inf_{p \in \Omega} \text{vol}(F_p) \cdot \inf_{p \in \Omega} \sigma_{\text{esa}}(M) \leq \sup_{p \in \Omega} \text{vol}(F_p) \cdot \inf_{p \in \Omega} \sigma_{\text{esa}}(N).
\]

Proof. Let \( \varepsilon > 0 \) and choose \( f_\varepsilon \in C^\infty_0(\Omega) \) such that

\[
\int_\Omega |\text{grad}^N f_\varepsilon|^2 < (\lambda^*(\Omega) + \varepsilon) \int_\Omega f_\varepsilon^2.
\]

Let us consider the function \( \tilde{f}_\varepsilon = f_\varepsilon \circ \pi \). By the assumption that the fibers of \( \pi \) are compact, \( \tilde{f}_\varepsilon \) has compact support in \( M \). Using Fubini’s Theorem for submersions we have

\[
\int_{\tilde{\Omega}} |\text{grad}^M \tilde{f}_\varepsilon|^2 \, dM = \int_{\tilde{\Omega}} \left( \int_{F_p} |\text{grad}^N f_\varepsilon|^2 \, dF_p \right) \, dN = \int_{\Omega} \text{vol}(F_p) \cdot |f_\varepsilon|^2 \, dN.
\]

Thus

\[
\int_{\tilde{\Omega}} |\text{grad}^M \tilde{f}_\varepsilon|^2 \, dM = \inf_{p \in \Omega} \text{vol}(F_p) \cdot \int_{\Omega} |f_\varepsilon|^2 \, dN.
\]

Similarly, using (2.3), we have

\[
\int_{\Omega} |\text{grad}^M \tilde{f}_\varepsilon|^2 \, dM = \int_{\Omega} |\text{grad}^N f_\varepsilon|^2 \, dM = \int_{\Omega} \left( \int_{F_p} |\text{grad}^N f_\varepsilon|^2 \, dF_p \right) \, dN = \int_{\Omega} \text{vol}(F_p) \cdot |\text{grad}^N f_\varepsilon|^2,
\]

thus

\[
\int_{\tilde{\Omega}} |\text{grad}^M \tilde{f}_\varepsilon|^2 \, dM \leq \sup_{p \in \Omega} \text{vol}(F_p) \cdot \int_{\Omega} |\text{grad}^N f_\varepsilon|^2.
\]
Lemma 3.8. Using (3.8), (3.9) and (3.11), we then obtain

\[
\inf_{p \in \Omega} \text{vol}(\mathcal{F}_p) \cdot \lambda^*(\tilde{\Omega}) \leq \inf_{p \in \Omega} \text{vol}(\mathcal{F}_p) \cdot \frac{\int_{\Omega} |\text{grad}^M \tilde{f}_\varepsilon|^2}{\int_{\Omega} |\tilde{f}_\varepsilon|^2} \\
\leq \sup_{p \in \Omega} \text{vol}(\mathcal{F}_p) \cdot \frac{\int_{\Omega} |\text{grad}^N \tilde{f}_\varepsilon|^2}{\int_{\Omega} |\tilde{f}_\varepsilon|^2} \\
< \sup_{p \in \Omega} \text{vol}(\mathcal{F}_p) \cdot [\lambda^*(\Omega) + \varepsilon].
\]

(3.12)

This proves (3.5). If all the fibers are minimal (or more generally if the averaged mean curvature vector field \(H_{\text{av}}\) vanishes identically on \(N\), see Remark 2.3), then the volume of the fibers is constant, and inequality (3.6) follows from (3.5). To prove the inequality (3.7) we pick a compact subset \(K \subset M\) and set \(\tilde{K} = \pi^{-1}(K)\) and let \(\tilde{K} = \pi^{-1}(K_0)\). The set \(\tilde{K}\) is compact by the assumption that the fibers of \(\pi\) are compact. Let \(\Omega = N \setminus \tilde{K}\) and \(\tilde{\Omega} = \pi^{-1}(\Omega) = M \setminus \tilde{K}\). Clearly, \(\tilde{\Omega} \subset M \setminus K\) and thus \(\lambda^*(\tilde{\Omega}) \geq \lambda^*(M \setminus K)\). Hence, using (3.5) we get

\[
\lambda^*(M \setminus K) \leq \lambda^*(\tilde{\Omega}) \leq \frac{\sup_{p \in \Omega} \text{vol}(\mathcal{F}_p)}{\inf_{p \in \Omega} \text{vol}(\mathcal{F}_p)} \lambda^*(\Omega) \leq \frac{\sup_{p \in \Omega} \text{vol}(\mathcal{F}_p)}{\inf_{p \in \Omega} \text{vol}(\mathcal{F}_p)} \inf \sigma_{\text{ess}}(N).
\]

Taking the supremum over all compact subset \(K \subset M\) in the left-hand side, we obtain the desired inequality. \(\Box\)

Now consider the case that the fibers of the submersion \(\pi : M \to N\) are compact and minimal.

**Lemma 3.8.** Let \(\pi : M \to N\) be a Riemannian submersion with compact and minimal fibers \(\mathcal{F}\). Then for every open subset \(\Omega \subset N\), denoting by \(\tilde{\Omega}\) the inverse image \(\pi^{-1}(\Omega)\), one has that

\[
\lambda^*(\tilde{\Omega}) = \lambda^*(\Omega).
\]

(3.13)

**Proof.** In view of (3.6), it suffices to show the inequality \(\lambda^*(\tilde{\Omega}) \geq \lambda^*(\Omega)\). To this aim, we will use the estimate in (3.4). We observe initially that it suffices to prove the inequality when \(\Omega\) is bounded. Namely, the general case follows from \(\lambda^*(\Omega) = \lim_{n \to \infty} \lambda^*(\Omega_n)\), by considering an exhaustion of \(\Omega\) by a sequence of bounded open subsets \(\Omega_n\). Note that \(\Omega\) is bounded if and only if \(\tilde{\Omega}\) is bounded, by the compactness of the fibers. Let \(f\) be the first eigenfunction of the problem \(\Delta^N u + \lambda u = 0\) in \(\Omega\) with Dirichlet boundary conditions, that can be assumed to be positive in \(\tilde{\Omega}\).

Set \(X = -\text{grad}^N (\log f)\), so that \(\text{div}^N (X) - |X|^2 = \lambda_1(\Omega)\) is constant in \(\Omega\). If \(\tilde{X}\) is the horizontal lifting of \(X\), then clearly \(|\tilde{X}| = |X|\) for all \(q \in \tilde{\Omega}\). Moreover, by Lemma 2.1, since \(H = 0\), \(\text{div}^M (\tilde{X})_q = \text{div}^N (X)_{\pi(q)}\).
Using (3.4), we then obtain:

$$\lambda^*(\tilde{\Omega}) \geq \inf_{\tilde{\Omega}} \left[ \text{div}^M(\tilde{X}) - |\tilde{X}|^2 \right] = \inf_{\Omega} \left[ \text{div}^N(X) - |X|^2 \right] = \lambda^*(\Omega).$$

This proves Lemma (3.8).

**Corollary 3.9.** Assume that the fibers of $\pi$ are compact and minimal. Then, $\sigma_{\text{ess}}(M) = \emptyset$ if and only if $\sigma_{\text{ess}}(N) = \emptyset$.

The above result applies in particular to Riemannian coverings, yielding the following

**Corollary 3.10.** If $M$ is a finite covering of $N$, then $\sigma_{\text{ess}}(M) \neq \emptyset$ if and only if $\sigma_{\text{ess}}(N) \neq \emptyset$.

3.4. **Proof of Theorem 1.** The item ii. of Theorem 1 follows from this Lemma 3.8. For if we take a sequence of compact sets $K_1 \subset K_2 \subset \cdots$ with $N = \bigcup_{i=1}^{\infty} K_i$. Likewise we have $M = \bigcup_{i=1}^{\infty} \tilde{K}_i$, where $\tilde{K}_i = \pi^{-1}(K_i)$. By the proof of (3.2) we have that $\inf \sigma_{\text{ess}}(N) = \lim_{i \to \infty} \lambda^*(N \setminus K_i)$ and $\inf \sigma_{\text{ess}}(M) = \lim_{i \to \infty} \lambda^*(M \setminus \tilde{K}_i)$. However, $\lambda^*(N \setminus K_i) = \lambda^*(M \setminus \tilde{K}_i)$, by the Lemma (3.8). Before we prove item i. we need the following lemma.

**Lemma 3.11.** Let $\pi: M \to N$ be a Riemannian submersion with compact minimal fibers $\mathcal{F}$. If $f \in L^2(N)$ and $\Delta^N f \in L^2(N)$ then $\tilde{f} \in L^2(M)$ and $\Delta^M \tilde{f} = \tilde{\Delta}^N f \in L^2(M)$. In other words, if $f \in \text{Dom}(\Delta^N)$ then $\tilde{f} \in \text{Dom}(\Delta^M)$.

**Proof.** Let $\tilde{f} = f \circ \pi$ be the lifting of $f$. By Fubini’s Theorem we have

$$\int_M \tilde{f}^2 dM = \int_N \left( \int_{\mathcal{F}_p} f^2 d\mathcal{F}_p \right) dN = \text{vol}(\mathcal{F}_p) \int_N f^2 dN < \infty.$$

This shows that $\tilde{f} \in L^2(M)$. To show that $\Delta^M \tilde{f} \in L^2(M)$ we will show that $\Delta^M \tilde{f} = \tilde{\Delta}^N f$. Every $\varphi \in C^\infty_0(M)$ can be decomposed as $\varphi = \varphi_1 + \varphi_2$ where $\varphi_1$ is constant along the fibers $\mathcal{F}$ and $(\varphi_2)_{av} = 0$, see [5]. Moreover, $\varphi_1$ and $\varphi_2$ has compact support. Observe that we can define $\psi: N \to \mathbb{R}$ by $\psi(\pi(p)) = \varphi_1(p)$ so that $\varphi_1 = \tilde{\psi}$. By the Lemma 3.1 we have that $\Delta^M \varphi_1(p) = \Delta^N \tilde{\psi}(\pi(p))$ for every $p \in M$. By Corollary 2.4
\[ \int_M \tilde{f} \Delta^M \varphi_2 dM = 0, \text{ therefore} \]
\[ \int_M \tilde{f} \Delta^M \varphi \, dM = \int_M \tilde{f} \Delta^M \varphi_1 dM \]
\[ = \int_N \left( \int_{\mathcal{F}_p} f \Delta^M \varphi_1 d\mathcal{F}_p \right) dN \]
\[ = \int_N (f \Delta^N \psi \int_{\mathcal{F}_p} d\mathcal{F}_p) dN \]
\[ = \text{vol}(\mathcal{F}_p) \int_N f \Delta^N \psi dN \]
\[ = \text{vol}(\mathcal{F}_p) \int_N \psi \Delta^N f dN \]
\[ = \int_N \left( \int_{\mathcal{F}_p} \psi \Delta^N f d\mathcal{F}_p \right) dN \]
\[ = \int_M \tilde{\psi} \Delta^N \varphi dM \]
\[ = \int_M \varphi_1 \Delta^N \varphi dM \]

□

To show that \( \sigma_p(N) \subset \sigma_p(M) \) we take \( \lambda \in \sigma_p(N) \) and \( f \in L^2(N) \) with \( -\Delta^N f = \lambda f \) in distributional sense. This implies that \( -\Delta^N \tilde{f} = \lambda \tilde{f} \).

By Lemma 3.11, \( -\Delta^M \tilde{f} = \lambda \tilde{f} \) showing that \( \lambda \in \sigma_p(M) \). To show that \( \sigma_{\text{ess}}(N) \subset \sigma_{\text{ess}}(M) \) we take \( \mu \in \sigma_{\text{ess}}(N) \). There exists an orthonormal sequence of functions \( f_k \in \text{Dom}(\Delta) \) such that \( \| -\Delta^N f_k - \mu f_k \|_{L^2(N)} \to 0 \) as \( k \to \infty \). By Lemma 3.11, we have that \( \tilde{f}_k \in \text{Dom}(\Delta^M) \). Now
\[ \| -\Delta^M \tilde{f}_k - \mu \tilde{f}_k \|_{L^2(M)}^2 = \int_M \| -\Delta^M \tilde{f}_k - \mu \tilde{f}_k \|^2 \, dM \]
\[ = \int_N \int_{\mathcal{F}_q} \| -\Delta^N f_k - \mu f_k \|^2 d\mathcal{F}_q \, dN \]
\[ = \text{vol}(\mathcal{F}_q) \int_N \| -\Delta^N f_k - \mu f_k \|^2 \, dN \]
\[ = \text{vol}(\mathcal{F}_q) \| -\Delta^N f_k - \mu f_k \|^2_{L^2(N)} \to 0 \]

This shows that \( \mu \in \sigma_{\text{ess}}(M) \), the proof of Theorem 1 is concluded.

**Corollary 3.12.** Let \( G \) be a Lie group endowed with a bi-invariant metric. Then, \( \sigma_{\text{ess}}(G) \) is empty if and only if for some (hence for any) compact subgroup \( K \subset G \), the Riemannian homogeneous space \( G/K \) has empty essential spectrum.
\textbf{Proof.} Apply Theorem 1 to the Riemannian submersion $G \mapsto G/K$, which has minimal and compact fibers. \hfill \Box

Other interesting examples of applications of Theorem 1 arise from non compact Lie groups.

\textbf{Example.} Consider the $2 \times 2$ special linear group $\text{SL}(2, R)$. There exists a 2-parameter family of left-invariant Riemannian metrics $g_{\kappa, \tau}$, with $\kappa < 0$ and $\tau \neq 0$, for which $(\text{SL}(2, R), g_{\kappa, \tau}) \to \mathbb{H}^2(\kappa)$ is a Riemannian submersion with geodesic fibers diffeomorphic to the circle $S^1$. An explicit description of these metrics can be found, for instance, in [24]. Endowed with these metrics, $\text{SL}(2, R)$ is one of the eight homogeneous Riemannian 3-geometries, as classified in [22], and its isometry group has dimension 4.

\textbf{Proposition 3.13.} For all $\kappa < 0$ and $\tau \neq 0$,

$$\sigma(\text{SL}(2, R), g_{\kappa, \tau}) = \sigma_{\text{ess}}(\text{SL}(2, R), g_{\kappa, \tau}) = \left[-\frac{\kappa}{4}, +\infty\right].$$

\textbf{Proof.} It is known that the spectrum $\sigma([H(\kappa)]) = \sigma_{\text{ess}}(H(\kappa)) = \left[-\frac{\kappa}{4}, +\infty\right)$, see [8]. By Lemma 3.8

$$\lambda^*([\text{SL}(2, R), g_{\kappa, \tau}] = \lambda^*([H(\kappa)]) = -\frac{\kappa}{4},$$

hence $\sigma([\text{SL}(2, R), g_{\kappa, \tau}] \subset \left[-\frac{\kappa}{4}, +\infty\right)$. On the other hand, by Theorem 1

$$\left[-\frac{\kappa}{4}, +\infty\right) = \sigma_{\text{ess}}(H(\kappa)) \subset \sigma_{\text{ess}}([\text{SL}(2, R), g_{\kappa, \tau}]).$$

This proves the proposition. \hfill \Box

\section{4. MEAN CURVATURE OF GEODESIC SPHERES VERSUS MEAN CURVATURE OF THE FIBERS. PROOF OF THEOREM 2.}

We will now drop the minimality and the compactness assumption on the fibers, however, we will make some assumptions on the curvature of the base and the fibers of the submersion. Assume that $(N, g^N)$ has a pole $p_0$ or more generally has a point $p_0$ with thin cut locus, see the Introduction. For $p \in N \setminus \{p_0\}$, let $\gamma_p : [0, 1] \to N$ be the unique affinely parameterized geodesic in $(N, g^N)$ such that $\gamma_p(0) = p_0$ and $\gamma_p(1) = p$. The radial curvature function of $(N, g^N, p_0)$, denoted by $\kappa_{p_0} : N \to \mathbb{R}$, is defined by $\kappa_{p_0}(p) = \max_{\sigma} \sec(\sigma)$, where $\sec$ is the section curvature and the maximum is taken over all 2-planes $\sigma \subset T_p N$ containing the direction $\gamma_p'(1)$. Finally, let us denote by $\rho_{p_0} : N \to [0, +\infty)$ the distance function in $N$ given by $\rho_{p_0}(p) = \text{dist}_N(p, p_0)$.

We are now ready for

\textbf{Proof of Theorem 2.} Assume first that $\pi : M \to N$ is a Riemannian submersion satisfying the following assumptions:

(a) $(N, g^N)$ has a pole $p_0$,

(b) the function $h(q) = (\Delta^N \rho_{p_0})_{\pi(q)} + g^N((\text{grad}^N \rho_{p_0})_{\pi(q)}, d\pi q(H_q))$ is proper.
Consider the lifting \( \tilde{\rho}_p : M \to \mathbb{R} \) defined by \( \tilde{\rho}_p = \rho_{p_0} \circ \pi \). Then by Lemma 3.1 we have that \( h = \Delta^M \tilde{\rho}_p \). Moreover, by (2.3),

\[
|\text{grad}^M \tilde{\rho}_p| = |\text{grad}^N \rho_{p_0}| \equiv 1.
\]

If \( K_1 \subset K_2 \subset \cdots \subset K_n \subset \cdots \) is a compact exhaustion of \( M \), then by (3.3) applied to \( X = \text{grad}^M \tilde{\rho}_p \)

\[
(4.1) \quad \lambda^* (M \setminus K_n) \geq \frac{1}{4} \left( \inf_{M \setminus K_n} h \right)^2.
\]

Since \( h \) is proper, the right-hand side in the above inequality tends to \(+\infty\) as \( n \to \infty \), thus, by Proposition 3.2, \( \sigma_{\text{ess}}(M) = \emptyset \).

If \( p_0 \) has thin cut locus, the same proof above holds, since \( X = \text{grad}^M \tilde{\rho}_p \) satisfies the Proposition 3.4 and therefore 4.1, see Remark 3.6. \( \square \)

**Corollary 4.1.** If the fibers of \( \pi : M \to N \) are compact and if the function \( l(p) = \Delta^N \rho_{p_0} (p) - \max_{q \in F_p} \| H_q \| \) is proper then \( \sigma_{\text{ess}}(M) = \emptyset \).

A different statement can be obtained in terms of radial curvature.

**Corollary 4.2.** Assume that \( G : [0, +\infty[ \to \mathbb{R} \) is a smooth function such that:

\[
(4.2) \quad \kappa_{p_0}(p) \leq -G(\rho_{p_0}(p))
\]

for all \( p \in N \). Denote by \( \psi : [0, +\infty[ \to \mathbb{R} \) the solution of the Cauchy problem:

\[
\psi''(t) = G(t) \psi(t), \quad \psi(0) = 0, \quad \psi'(0) = 1,
\]

and set \( \ell(t) = (n-1) \psi'(t)/\psi(t) \), \( t > 0 \). If

\[
(4.3) \quad \lim_{q \to \infty} \left[ \ell\left( \rho_{p_0}(\pi(q)) \right) + g^N \left( (\nabla^N \rho_{p_0})_{\pi(q)}, d\pi_q(H_q) \right) \right] = +\infty
\]

then \( \sigma_{\text{ess}}(M) = \emptyset \).

**Proof.** Using the Hessian Comparison Theorem [14, Chapter 2], under the assumption (4.2) one has:

\[
(4.4) \quad \text{Hess}(\rho^N) \geq \frac{\psi'}{\psi} \cdot \left( g^N - d\rho_{p_0} \otimes d\rho_{p_0} \right).
\]

Considering an orthogonal basis of \( T_pN \) of the form \( \{ \nabla^N \rho_{p_0}, e_1, \ldots, e_{n-1} \} \), where \( \{ e_1, \ldots, e_{n-1} \} \) is an orthonormal basis of \( T_pS_p \), and taking the trace of the symmetric bilinear forms in the two sides of (4.4), we get

\[
(4.5) \quad \Delta^N \rho_{p_0} \geq (n-1) \frac{\psi'}{\psi}.
\]

It is clear that (4.3) implies that \( h(q) = \Delta^N \rho_{p_0} + g^N \left( \text{grad}^N \rho_{p_0}, d\pi_q(H_q) \right) \) is proper. \( \square \)
4.1. Warped products. Let \((N, g^N)\) and \((F, g^F)\) be Riemannian manifolds and let \(\psi : N \to \mathbb{R}^+\) be a smooth function. The warped product manifold \(M = N \times \psi F\) is the product manifold \(N \times F\) endowed with the Riemannian metric \(g^N + \psi^2 g^F\). It is immediate to see that the projection \(\pi : M \to N\) onto the first factor is a Riemannian submersion, with fiber \(\mathcal{F}_p = \{p\} \times F\).

Among Riemannian submersions, warped products are characterized by the following properties:

- the horizontal distribution is integrable, and its leaves are totally geodesic;
- the fibers are totally umbilical.

For warped products, the results of the paper can be stated in a more explicit form in terms of the warping function \(f\). The mean curvature of the fibers are given by

\[
H = -\dim(F) \frac{\text{grad} \tilde{\psi}}{\tilde{\psi}},
\]

where \(\tilde{\psi}\) is the lifting of \(\psi\).

**Proposition 4.3.** Let \(M = N \times \psi F\) be a warped product, with \(F\) compact.

(a) If \(\sigma_{\text{ess}}(N) \neq \emptyset\), and \(0 < \inf_N \psi \leq \sup_N \psi < +\infty\).

Then \(\sigma_{\text{ess}}(M) \neq \emptyset\).

(b) If \((N, g^N)\) has a pole \(p_0\) and the function

\[
\Delta^N \rho_{p_0} - \frac{1}{\psi} g^N(\nabla^N \rho_{p_0}, \nabla^N \psi)
\]

is proper, then \(\sigma_{\text{ess}}(M) \neq \emptyset\).

**Proof.** Part (a) follows from Proposition 3.7, observing that the volume of the fiber \(\mathcal{F}_p = \{p\} \times F\) equals \(\psi(p) \dim(F) \text{vol}(F)\).

Part (b) follows from Theorem 2 and formula (4.6). \(\square\)

4.2. Example. Let \(\mathbb{R}^n = [0, \infty) \times \mathbb{S}^{n-1}/\sim\) endowed with the smooth metric \(ds^2 = dt^2 + f(t)^2 d\theta^2\), where \(f(0) = 0\), \(f'(0) = 1\). The equivalence relation \(\sim\) is the following

\[(t, \theta) \sim (s, \alpha) \iff t = s = 0 \text{ or } t = s > 0 \text{ and } \theta = \alpha.\]

The radial sectional curvatures \(K^\text{rad}\) along the geodesic issuing the origin \(0 = \{0\} \times \mathbb{S}^{n-1}/\sim\) is given by \(K^\text{rad}(t, \theta) = -\frac{f''(t)}{f(t)}\). Let us consider \(W = \mathbb{R}^n \times \mathbb{S}^1\) with metric \(dw^2 = ds^2 + g^2(\rho(x)) d\theta^2\), where \(\rho\) is the distance function to the origin in \((\mathbb{R}^n, ds^2)\) and \(g : [0, \infty) \to (0, \infty)\) is a smooth function. Choosing \(f(t) = te^{t^2}\) we have \(K^\text{rad}(t, \theta) = -4t^3 - 6t\), thus \(\lim_{t \to \infty} K^\text{rad}(t, \theta) = -\infty\). By Donnelly-Li’s Theorem [11], the spectrum of \((\mathbb{R}^n, ds^2)\) is discrete. Choosing \(g(t) = e^{t-t^2}\) An easy computation yields that

i. The volume \(\text{vol}(W, dw^2) = \infty\).
ii. The limit $\mu = \limsup_{r \to \infty} \frac{\log(\text{vol}(B_W(r)))}{r} < \infty$, where $B_W(r)$ is the geodesic ball centered at a point $p = (0, \xi) \in W$ and radius $r$.

The items i. and ii. imply by Brooks’ Theorem [6] that $\sigma_{\text{ess}}(W) \neq \emptyset$. This gives an example of a Riemannian submersion $\pi: (W, dw^2) \to (\mathbb{R}^n, ds^2)$ where the base space is discrete but the total space is not, while the fiber is compact but not minimal.

An example of a Riemannian submersion $\pi: (\mathbb{R}^n \times S^1, dw^2) \to (\mathbb{R}^n, ds^2)$ where the total space is discrete but the base space is not, while the fiber is compact but not minimal is presented in [1, Proposition 4.3].

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