On relationship between canonical momentum and geometric momentum

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(Dated: May 6, 2016)

Abstract

Decompositing of $N + 1$-dimensional gradient operator in terms of Gaussian normal coordinates $(\xi^0, \xi^\mu)$, ($\mu = 1, 2, 3, ..., N$) and making the canonical momentum $P_0$ along the normal direction $\mathbf{n}$ to be hermitian, we obtain $i\hbar (\mathbf{n} \partial_0 - M_0)$ with $M_0$ denoting the mean curvature vector on the surface $\xi^0 = \text{const}$. The remaining part of the momentum operator lies on the surface, which is identical to the geometric one.

Mathematics Subject Classification 2010: 81Q10, 81Q12, 47E05, 70G10

PACS numbers: 03.65.-w Quantum mechanics, 04.60.Ds Canonical quantization

Keywords: momentum operator, canonical momentum, geometric momentum, Gaussian normal coordinates, differential geometry, hypersurface.
I. INTRODUCTION

Exploration of the proper form and the physical meaning of momentum in curvilinear coordinates attracts constant attention since the birth of quantum mechanics. In this paper, we show that canonical momenta \( P_\xi \) associated with its conjugate canonical positions, or coordinates, \( \xi \), are closely related to mean curvatures of the surface \( \xi = \text{const} \). So, the geometric momenta that are under extensive studies and applications are closely related to natural decompositions of the momentum operator in gaussian normal, curvilinear in general, coordinates.

In next section II, we study a simple but illuminating example: how the momentum operators in 3D spherical polar coordinates \( (r, \theta, \varphi) \) are related to three mean curvature vectors. In section III, we present a theorem for the general case. In final section IV, a brief conclusion is given.

II. AN EXAMPLE: MEAN CURVATURES AND SPHERICAL POLAR COORDINATES

The gradient operator in the 3D cartesian coordinate system \( \nabla_{\text{cart}} \equiv e_x \partial_x + e_y \partial_y + e_z \partial_z \) can be expressed in the 3D spherical polar coordinates \( (r, \theta, \varphi) \),

\[
\nabla_{\text{sp}} = e_r \frac{\partial}{\partial r} + e_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + e_\varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi}.
\]

(1)

The momentum operator can thus be written in the following way,

\[
P \equiv -ih\nabla_{\text{cart}} = -ih\nabla_{\text{sp}} = \{e_r, P_r\} + \frac{1}{r} \{e_\theta, P_\theta\} + \frac{1}{r \sin \theta} \{e_\varphi, P_\varphi\},
\]

(2)

where \( \{A, B\} \equiv (AB + BA)/2 \) and,

\[
\{e_r, P_r\} = e_r P_r, P_r = -ih(\frac{\partial}{\partial r} + \frac{1}{r}),
\]

(3a)

\[
\{e_\theta, P_\theta\} = e_\theta P_\theta + ih e_r^2/2, P_\theta = -ih(\frac{\partial}{\partial \theta} + \frac{1}{2} \cot \theta),
\]

(3b)

\[
\{e_\varphi, P_\varphi\} = e_\varphi P_\varphi + ih \frac{1}{2} (e_r \sin \theta + e_\theta \cos \theta), P_\varphi = -ih \frac{\partial}{\partial \varphi}.
\]

(3c)

On one hand, these spherical polar coordinates have three mutually orthogonal surfaces given by \( r = \text{const}., \theta = \text{const.} \) and \( \varphi = \text{const.} \) respectively. They are, respectively, a
spherical surface of radius $r$, a cone of polar angle $\theta$, and a flat plane alone azimuthal angle $\varphi$. These (curved) surfaces have three mean curvature vectors, respectively,

$$M_r = -\frac{e_r}{r}, \quad M_\theta = -\frac{e_\theta}{2r \tan \theta}, \quad M_\varphi = 0, \quad (r \neq 0).$$

(4)

On the other hand, if looking closely into the canonical momenta multiplied by their vector coefficients, $e_r, e_\theta/r$ and $e_\varphi/(r \sin \theta)$, respectively, we find,

$$e_r P_r = -i\hbar e_r \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) = -i\hbar(e_r \frac{\partial}{\partial r} - M_r),$$

(5a)

$$e_\theta P_\theta = -i\hbar \frac{e_\theta}{r} \left( \frac{\partial}{\partial \theta} + \frac{1}{2} \cot \theta \right) = -i\hbar \left( \frac{e_\theta}{r} \frac{\partial}{\partial \theta} - M_\theta \right),$$

(5b)

$$\frac{e_\varphi}{r \sin \theta} P_\varphi = -i\hbar \frac{e_\varphi}{r \sin \theta} \frac{\partial}{\partial \varphi} = -i\hbar \left( \frac{e_\varphi}{r \sin \theta} \frac{\partial}{\partial \varphi} - M_\varphi \right).$$

(5c)

Now, the mean curvature vectors exhibit themselves respectively in the brackets, which result from making the derivative $-i\hbar \partial_\xi (\xi = r, \theta, \varphi)$ Hermitian operators multiplied, respectively, by the vector coefficients $e_\xi/H_\xi$ with $H_\xi$ denoting Lamé coefficients for orthogonal curvilinear coordinates, where the Lamé coefficients are defined by $d\textbf{x} \equiv dx e_x + dy e_y + dz e_z = \sum H_\xi d\xi e_\xi$.

For the spherical polar coordinates $(r, \theta, \varphi)$, three Lamé coefficients are $H_r = 1$, $H_\theta = r$, and $H_\varphi = r \sin \theta$. So, Eq. (2) has three ways of decomposition in the following,

$$P = -i\hbar \nabla_{\text{sp}} = e_r P_r + \Pi_r = \frac{e_\theta}{r} P_\theta + \Pi_\theta = \frac{e_\varphi}{r \sin \theta} P_\varphi + \Pi_\varphi,$$

(6)

where, $\Pi_r, \Pi_\theta$ and $\Pi_\varphi$ are so-called the geometric momenta for the corresponding surfaces, though the last one $M_\varphi = 0$ is trivial,

$$\Pi_r \equiv -i\hbar \left( e_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + e_\varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} + M_r \right),$$

(7a)

$$\Pi_\theta \equiv -i\hbar \left( e_r \frac{\partial}{\partial r} + e_\varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} + M_\theta \right),$$

(7b)

$$\Pi_\varphi \equiv -i\hbar \left( e_r \frac{\partial}{\partial r} + e_\theta \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} + M_\varphi \right).$$

(7c)

They are special cases of the general form of the geometric momentum, \cite{9-20}

$$\Pi \equiv -i\hbar \left( r^\xi \partial_\xi + M \right), \quad \text{or} \quad \Pi \equiv -i\hbar \left( r^\xi \partial_\xi + \frac{M}{2} \right)$$

(8)

where $M = M \textbf{n}$ with $\textbf{n}$ denoting the unit normal vector for a surface and $M$ standing for the mean curvature. In the first equation of (8), the mean curvature $M$ is usually defined by the true average of the two principal curvatures and usually applies for the 2D surface, whereas the second one uses another convention in which $M$ is defined as sum of all principal curvatures. In the rest part of this paper, we will use the latter convention.
Theorem: In the \((N + 1)D\) Euclidean space \(\mathbb{R}^{N+1}\), we can define the usual Cartesian coordinates whose corresponding momentum is \(P = -i\hbar \nabla_{\text{cart}}\) as usual, which can also be expressed in terms of the curvilinear coordinates \((\xi^0, \xi^\mu), (\mu = 1, 2, \ldots N)\). Assuming that the curvilinear coordinates take the form of gaussian normal coordinates that have a metric that satisfies conditions \(g^{00} > 0\) does not depend on \(\xi^0\) and \(g_{0\mu} = 0\). There is a mean-curvature dependent decomposition of the momentum in the following,

\[ P \equiv -i\hbar \nabla_{\text{cart}} = -i\hbar \left( \frac{n}{\sqrt{g^{00}}} \frac{\partial}{\partial \xi^0} - \frac{M_0}{2} \right) + \Pi_0, \quad (9) \]

where \(-M_0\) is the mean curvature vector, and \(\Pi_0\) defines the geometric momentum of the surface \(\xi^0 = \text{const.}\) whose unit normal vector is denoted by \(n\), \(\Pi_0 = -i\hbar \left( r^\mu \partial_\mu + \frac{M_0}{2} \right)\).

The proof is straightforward. The coordinate transformation from the cartesian ones \(x \equiv (x_1, x_2, x_3, \ldots x_{N+1})\) to the gaussian normal ones \((\xi^0, \xi^\mu), (\mu = 1, 2, 3, \ldots, N)\) are,

\[ x_i = x_i(\xi^0, \xi^\mu), \text{ and } \xi^0 = \xi^0(x), \xi^\mu = \xi^\mu(x). \quad (10) \]

The line element \(d\mathbf{x} \cdot d\mathbf{x}\) is \(d\mathbf{x} \cdot d\mathbf{x} = dx_i dx_i = g^{00} \partial_0^2 + g^{\mu\nu} \partial_\mu \partial_\nu\), and the determinant of the metric matrix \(g_{\mu\nu}\) is then \(g = |g_{\mu\nu}|\). The gradient operator is in the gaussian normal coordinates,

\[ \nabla_{gn} = \frac{n}{\sqrt{g^{00}}} \partial_0 + r^\mu \partial_\mu. \quad (11) \]

This gradient operator contains no mean curvature. The mean-curvature dependence becomes evident in quantum momentum in the following.

First, we assume \(g^{00} = 1\). The canonical momentum operators \((P_0, P_\mu)\) associated with canonical positions \((\xi^0, \xi^\mu)\) are, respectively, given by,

\[ -i\hbar \partial_0 \rightarrow P_0 = -i\hbar \frac{1}{g^{1/2}} \partial_0 g^{1/2}, \quad -i\hbar \partial_\mu \rightarrow P_\mu = -i\hbar \frac{1}{g^{1/2}} \partial_\mu g^{1/2}. \quad (12) \]

To note that \(\xi^0(x) = \text{const.}\) forms a surface whose mean curvature \(M_0\) is simply,

\[ \frac{1}{g^{1/2}} \partial_0 g^{1/2} = -M_0, \text{ and } M_0 = M_0 n. \quad (13) \]

We have then,

\[ P = -i\hbar (n \partial_0 + r^\mu \partial_\mu) = -i\hbar \left( n \partial_0 - \frac{M_0}{2} + r^\mu \partial_\mu + \frac{M_0}{2} \right) = -i\hbar n P_0 + \Pi_0. \quad (14) \]
Secondly, for \( g^{00} \) takes any positive values, we can easily prove,

\[
-\imath \hbar \frac{\partial_0}{\sqrt{g^{00}}} \rightarrow P_0 = -\imath \hbar \frac{1}{g^{1/2}} \frac{\partial_0}{\sqrt{g^{00}}} g^{1/2} = -\imath \hbar \left( \frac{\partial_0}{\sqrt{g^{00}}} - \frac{M_0}{2} \right).
\]

The decomposition (14) remains the same.

For clearly see that \( \Pi_0 \) is really lying on the surface \( \xi^0 = \text{const.} \), we can verify the orthogonal relation \( n \cdot \Pi_0 + \Pi_0 \cdot n = 0 \). Q.E.D.

Thus, we understand why there are three mean curvature vectors associated with 3D spherical polar coordinates. This is because any one coordinate is normal to other two. Moreover, for an \( ND \) surface in \( R^{N+1} \), any point in the neighborhood of the surface can be clearly specified by the gaussian normal coordinates \( r(\xi^\mu) + \xi^0 n(\xi^\mu) \) with \( n(\xi^\mu) \) is the unit normal vector on point \( \xi^\mu \) of the surface, and we can define the geometric momentum on the surface.\(^{16,20}\)

IV. CONCLUSIONS

Many come across a fact that the canonical momenta in the orthogonal curvilinear coordinates are closely related to the mean curvature vectors of some properly defined curved surfaces, and the mean curvature vectors are geometric invariant, rather than the Christoffel symbols that give different values from one set of coordinates to another. We demonstrate that a decomposition of momentum operator in Gaussian normal coordinates straightforwardly leads to a natural appearance of the mean curvature vectors. Once the canonical momentum along the normal becomes hermitian, the remaining part of the momentum is lying on the surface, which is the geometric momentum which recently attracts much attention.

ACKNOWLEDGMENTS

This work is financially supported by National Natural Science Foundation of China under Grant No. 11175063.

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