On 2D gravity coupled to $c \leq 1$ matter in Polyakov light-cone gauge

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Abstract

A new formulation of $c \leq 1$ matter coupled to 2D gravity is proposed. This model, being closely analogous to one in the Polyakov light-cone gauge, possesses well defined global properties which allow to calculate correlation functions. As an example, the three point correlation functions of discrete states are found.

Since the seminal works of Polyakov, Knizhnik and Zamolodchikov [1, 2], there has been much progress in understanding the continuum field theory approach to 2D gravity (see e.g. [3] and refs. therein). Majority of efforts has been devoted to the study of the coupling of conformal matter to gravity in the conformal gauge. The reason why it is useful lies in the facts that it is the standard gauge for conformal field theory and its properties on the Riemann surfaces are well known. At the same time, the properties of the Polyakov gauge are little known which restricts the applications of such a gauge. In this letter I will present results of a new formulation of $c \leq 1$ matter coupled to 2D gravity motivated by the hope that this theory, being closely analogous to one in the Polyakov gauge, possesses well defined global properties which permit to calculate correlation functions.

The starting point is the $G/G$ topological model for $G=SL(2)$ [4]. The Hilbert space of the model decomposes into holomorphic and anti-holomorphic sectors. For my purposes, I need only one of them. Let me choose the holomorphic sector. It has $\hat{sl}_2 \oplus \hat{sl}_2 \oplus \hat{sl}_2$ as the symmetry algebra. The corresponding currents form the following OP algebras

$$ J_\alpha^\alpha(z_1)J_\beta^\beta(z_2) = \frac{k_\alpha}{2} q^{\alpha\beta} \frac{1}{(z_1 - z_2)^2} + \frac{f^{\alpha\beta}}{(z_1 - z_2)} J_\gamma^\gamma(z_2) + O(1) ,$$

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where $q^{\alpha\beta}$ is the Killing metric of $\mathfrak{sl}_2$, $f^{\alpha\beta}$, are its structure constants and $a$ is a parameter labelling algebras in the direct sum. One can always choose a basis of $\mathfrak{sl}_2$ such that $q^{00} = 1$, $q^{++} = q^{-+} = 2$, $f^{0+} = f^{-0} = 1$, $f^{+0} = 2$; $\alpha, \beta, \gamma = 0, +, -$. The levels are given by

\[ k_1 = k, \quad k_2 = -k - 4, \quad k_3 = 4. \] (2)

It should be noted that $J^{\alpha}_3$ are expressed in terms of the first order fermionic systems (ghosts) of weights (1.0). Explicitly

\[ J^{\alpha}_3(z) = f^{\alpha\beta\gamma} : b^{\beta}(z) c^{\gamma}(z) :. \] (3)

The stress-energy tensor is a sum of Sugawara terms of $J^{\alpha}_1$ and $J^{\alpha}_2$ currents and the usual contribution of the ghost systems:

\[ T(z) = \frac{1}{k + 2} q^{\alpha\beta} : J^{\alpha}_1(z) J^{\beta}_1(z) : - \frac{1}{k + 2} q^{\alpha\beta} : J^{\alpha}_2(z) J^{\beta}_2(z) : + q^{\alpha\beta} : b^{\alpha}(z) \partial c^{\beta}(z) :. \] (3)

It is easy to see that the Virasoro algebra generated by this stress-energy tensor has zero central charge.

As to the algebraic structure and physical states of the model I refer to refs. [4, 7].

In general, when given representations of a chiral algebra (symmetry algebra), in order to define fields of quantum field theory, one needs a construction attaching representations to a point. In [5] Feigin and Malikov proposed the improved construction for the case of $\hat{\mathfrak{sl}}_2$ (see also [3]). The point is that a module should be attached to a pair $(x, z)$. The coordinate $z$ is a point on a curve. As to $x$, it must be taken as a point on $\mathbb{CP}^1$. Note that in physics $x$ is called as isotopic coordinate. The generators of $\mathfrak{sl}_2$ are given by

\[ S^{-}_j = \frac{\partial}{\partial x}, \quad S^{0}_j = -x \frac{\partial}{\partial x} + j, \quad S^{+}_j = -x^2 \frac{\partial}{\partial x} + 2 j x. \] (4)

Here $j$ is the weight of representation.

The chiral currents are turned into a form (current)

\[ J(x, z) = J^+(z) - 2x J^0(z) - x^2 J^-(z). \] (5)

It is easily shown that the Operator Product (OP) expansion of $J(x, z)$ is

\[ J(x_1, z_1) J(x_2, z_2) = -k \frac{x_{12}^2}{z_{12}^2} - 2 \frac{x_{12}}{z_{12}} J(x_2, z_2) - \frac{x_{12}^2}{z_{12}} \frac{\partial}{\partial x_2} J(x_2, z_2) + O(1), \] (6)

\[ \text{Note that in my conventions } |0\rangle^{gh} = c_0^+ |0\rangle \otimes c_0^+ |0\rangle \otimes c_0^+ |0\rangle. \]
where \( z_{ij} = z_i - z_j \), \( x_{ij} = x_i - x_j \).

The primary fields are defined via their OP expansions with the current

\[
J(x_1, z_1) \Phi^j(x_2, z_2) = -2j \frac{x_{12}}{z_{12}} \Phi^j(x_2, z_2) - \frac{x_{12}^2}{z_{12}} \frac{\partial}{\partial x_2} \Phi^j(x_2, z_2) + O(1) \quad .
\]

It should be noted that in the general case the primary fields are non-polynomial in \( x \).

Furthermore, \( J(x, z) \) is not primary.

It is now straightforward to use this machinery in the case at hand. Let \( J_1(x, z) \) and \( J_2(\bar{x}, z) \) be the corresponding forms of the algebras in the direct sum. The primary fields at ghost number zero are given by

\[
\Phi^{j_1, j_2}(x, \bar{x}, z) = \Phi^{j_1}(x, z) \Phi^{j_2}(\bar{x}, z) \quad .
\]

Here \( \Phi^{j_1}(x, z)(\Phi^{j_2}) \) is primary with respect to \( J_1(x, z)(J_2) \).

The idea that the SL(2)/SL(2) model is connected to the minimal models coupled to gravity was put forward in ref.[7], in a study of some "numerological" correspondences and partition functions. This discusses mainly the conformal gauge.

Let me now clarify some points in my framework. Setting \( x = \bar{x} = z \), which corresponds to the quantum hamiltonian reduction of \( \hat{sl}_2 \oplus \hat{sl}_2 \) to \( \text{Vir} \oplus \text{Vir} \), one immediately obtains the minimal model coupled to gravity, more correctly its holomorphic sector in the conformal gauge. In this case the first \( \text{Vir} \) describes the matter sector. The second \( \text{Vir} \) corresponds to the Liouville(gravity) sector. It is straightforward to see that, under \( x = \bar{x} = z \), \( J^-_1(z) \) and \( J^-_2(z) \) are constrained. It leads to the following stress-energy tensors

\[
T_a(z) = \frac{1}{k + 2} g_{\alpha \beta} : J^\alpha_a(z) J^\beta_a(z) : -\partial J^0_a(z) \quad .
\]

In terms of fields the reduction is given by

\[
\Phi^{j_{n,m}}(x, \bar{x}, z) \Phi^{-1-j_{n,m}}(\bar{x}, z) \rightarrow \phi_{n,m}(z) \exp \beta_{n,m} \varphi(z) \quad .
\]

In the above, \( j_{n,m} \) take values defined by the Kac-Kazhdan list. Namely

\[
j^-_{n,m} = \frac{n - 1}{2} j_- + \frac{m - 1}{2} j_+ \quad \text{or} \quad j^+_{n,m} = -\frac{n + 1}{2} j_- - \frac{m}{2} j_+ \quad ,
\]

with \( j_- = 1 \), \( j_+ = -k - 2 \), \( k \in \mathbb{Q} \), \( \{n, m\} \in \mathbb{N} \). As to the right-hand side it is the primary field \( \phi_{n,m}(z) \) of the minimal conformal theory dressed by the Liouville exponent (see e.g. [3] and refs. therein).

\(^2\)In fact, in the case of integer levels one doesn’t need the \( x \) variable, so the ghosts don’t lead to an additional isotopic coordinate.

\(^3\)Note that \( \text{Vir} \) means the Virasoro algebra.

\(^4\)For the rational level \( k \) the weights given in (11) are called admissible.
It is surprising that there exists another construction which represents an analog of the minimal conformal matter coupled to gravity in the Polyakov gauge. Let me explain how this idea can be implemented. In contrast to the previous case set $x = z$. From this it follows that only $J_1^-(z)$ is constrained. As a result one has $Vir \oplus \hat{sl}_2$ as the symmetry algebra. The stress-energy tensors are those given in (9). It is worth noting that they take such form due to entirely different reasons, namely the quantum hamiltonian reduction and decomposition (5) respectively.

For the primary fields one obtains

$$\Phi^{j_{n.m}}(x, z) \Phi^{j_{n.m}}(\bar{x}, z) \to \phi_{n.m}(x)\Phi^{j_{n.m}}(\bar{x}, x)$$ \hspace{1cm} (12)

where $j_{n.m}$’s are from the Kac-Kazhdan list. It should be stressed that $\Phi^{j_{n.m}}(\bar{x}, x)$ is primary with respect to $\hat{sl}_2$ but not with respect to $T$ given by eq.(9).

Now let me show that the proposed construction provides all features of the minimal models coupled to 2D gravity in the Polyakov gauge.

It is easy to check that a condition $\epsilon^{tot} = 0$ is equivalent to a relation for the levels $k_1 + 2 = -k_2 - 2$ given by eq.(2). The same is also true for the conformal gauge where it automatically leads to a relation between background charges of the matter and Liouville sectors [3].

The KPZ scaling law [1, 2], determining the $\hat{sl}_2$ weights of the primary (spinless) field $\phi_{n.m}$ interacting with gravity is satisfied by setting $j_1 = j_2$ for the primary fields (8). By the way, in the case of the conformal gauge a proper Liouville exponent is reproduced by $j_1 = -j_2 - 1$.

Moreover the residual $\hat{sl}_2$ algebra assumes the Knizhnik-Zamolodchikov (KZ) equation for the correlators of the primary fields $\Phi^{j_{n.m}}(\bar{x}, x)$. Explicitly

$$-(k + 2) \frac{\partial}{\partial x_i} \langle \Phi^{j_1}(\bar{x}_1, x_1) \ldots \Phi^{j_N}(\bar{x}_N, x_N) \rangle = \sum_{i \neq l} \frac{q_{\alpha \beta}S^{\alpha}_{j_i}S^{\beta}_{j_l}}{x_i - x_l} \langle \Phi^{j_1}(\bar{x}_1, x_1) \ldots \Phi^{j_N}(\bar{x}_N, x_N) \rangle,$$ \hspace{1cm} (13)

where $S^{\alpha}_{j_i}$ are the generators of $sl_2$ (4) i.e. the differential operators with respect to $\bar{x}_i$. Note that the term $\partial J^0_2(x)$ modifying the stress-energy tensor doesn’t affect the KZ equation because the current $J^0_2(x)$ has no log $x$ terms in its mode expansion.

In contrast to the Polyakov gauge where a global structure of 2d world sheet is unclear in the case at hand one has a well-defined $\mathbb{CP}^1 \times \mathbb{CP}^1$ structure. It allows to solve the KZ equation for the admissible representations by the methods of conformal field theory (see for instance [11]).

Of course in the above I have not said anything specific about the BRST analysis of physical states. In order to find them one must solve the BRST cohomology problem. I refer to the paper by Marcus and Oz for more details [12].
Now let me give an explicit calculation of correlation functions. My aim is to find the three point functions of operators

\[ \mathcal{O}_{n,m} = \int d\mu_{n,m}(x, \bar{x}; k) \phi_{n,m}(x) \Phi^{j_{n,m}}(\bar{x}, x) . \]  

(14)

Here \( \mu_{n,m}(x, \bar{x}; k) \) represents a measure which will be defined later. \( \phi_{n,m}(x) \Phi^{j_{n,m}} \) are the primary fields (12). Having set notations as above one gets

\[ \langle \mathcal{O}_{n_1,m_1} \mathcal{O}_{n_2,m_2} \mathcal{O}_{n_3,m_3} \rangle = \prod_{i=1}^{3} \int d\mu_{n_i,m_i}(x_i, \bar{x}_i; k) \phi_{n_i,m_i}(x_i) \Phi^{j_{n_i,m_i}}(\bar{x}_i, x_i) \]  

(15)

The integrand is factorized into two pieces:

\[ \langle \phi_{n_1,m_1}(x_1) \phi_{n_2,m_2}(x_2) \phi_{n_3,m_3}(x_3) \rangle \langle \Phi^{j_{n_1,m_1}}(\bar{x}_1, x_1) \Phi^{j_{n_2,m_2}}(\bar{x}_2, x_2) \Phi^{j_{n_3,m_3}}(\bar{x}_3, x_3) \rangle . \]  

(16)

These correlators are standard, and I find

\[ \langle \phi_{n_1,m_1}(x_1) \phi_{n_2,m_2}(x_2) \phi_{n_3,m_3}(x_3) \rangle = C^{(n_1,m_1)}_{(n_2,m_2)(n_3,m_3)} \prod_{i<l} \frac{1}{(x_{il})^{\gamma_{il}(\Delta_0)}} , \]  

(17)

\[ \langle \Phi^{j_{n_1,m_1}}(\bar{x}_1, x_1) \Phi^{j_{n_2,m_2}}(\bar{x}_2, x_2) \Phi^{j_{n_3,m_3}}(\bar{x}_3, x_3) \rangle = \tilde{C}^{(n_1,m_1)}_{(n_2,m_2)(n_3,m_3)} \prod_{i<l} (\bar{x}_{il})^{\gamma_{il}(\Delta)} . \]  

(18)

with \( y_{nm} = y_n - y_m \), \( \gamma_{12}(y) = y_1 + y_2 - y_3 \), \( \gamma_{13}(y) = y_1 + y_3 - y_2 \), \( \gamma_{23}(y) = y_2 + y_3 - y_1 \) and

\[ \Delta_0 = \frac{j(j + 1)}{k + 2} - j , \quad \Delta = -\frac{j(j + 1)}{k + 2} . \]

Moreover, \( C(\tilde{C}) \) are the square roots of the structure constants for the minimal models and SL(2) CFT respectively.

Substituting (17) and (18) into (15) one arrives at

\[ \langle \mathcal{O}_{n_1,m_1} \mathcal{O}_{n_2,m_2} \mathcal{O}_{n_3,m_3} \rangle = C^{(n_1,m_1)}_{(n_2,m_2)(n_3,m_3)} \tilde{C}^{(n_1,m_1)}_{(n_2,m_2)(n_3,m_3)} \prod_{i=1}^{3} \int d\mu_{n_i,m_i}(x_i, \bar{x}_i; k) \prod_{i<l} (\bar{x}_{il})^{\gamma_{il}(\Delta)}(x_{il})^{\gamma_{il}(\Delta)} . \]  

(19)

Let me now consider the euclidian domain \( \bar{x}_i = x_i^* \), where the star denotes the complex conjugation. The correlation function is rewritten as

\[ \langle \mathcal{O}_{n_1,m_1} \mathcal{O}_{n_2,m_2} \mathcal{O}_{n_3,m_3} \rangle = C^{(n_1,m_1)}_{(n_2,m_2)(n_3,m_3)} \tilde{C}^{(n_1,m_1)}_{(n_2,m_2)(n_3,m_3)} \prod_{i=1}^{3} \int d\mu_{n_i,m_i}(x_i, x_i^*; k) \prod_{i<l} (x_{il})^{2\gamma_{il}(\Delta)} . \]  

(20)
The factor $C\tilde{C}$ can be found from the explicit expressions of the structure constants [13, 11], after some simple but tedious algebra. Unfortunately this is not the case for the integral at generic weights $j_{n,m}$. However, if $m$ is equal to 1, then $j_{n,1}^+$ is an integer or half-integer. At these values of the weights the primary fields form SU(2) multiplets [6].

The integrand is the generating function for the Clebsch-Gordan coefficients of SU(2). As to the measure, one can consider the limit $k \to \infty$. In this limit it is the standard SU(2) invariant measure. Explicitly

$$d\mu_n(x, x^*) = \frac{d^2x}{(1 + |x|^2)^{n+1}}.$$  \hspace{1cm} (21)

The integral in eq.(20) reduces to

$$I(n_1, n_2, n_3) = \prod_{i=1}^3 \int \frac{d^2x}{(1 + |x|^2)^{n_i+1}} \prod_{i<l} |x_{il}|^{\gamma_{il}(n-1)}.$$  \hspace{1cm} (22)

It assumes a remarkably simple form

$$I(n_1, n_2, n_3) = \Gamma(\sigma + \frac{1}{2}) \prod_{i=1}^3 \frac{\Gamma(\sigma - n_i + \frac{1}{2})}{\Gamma(n_i + 1)},$$  \hspace{1cm} (23)

where $\sigma = \frac{n_1}{2} + \frac{n_2}{2} + \frac{n_3}{2}$.

The integral has been calculated for some cases, by using the following normalization

$$\int d\mu_n(x, \bar{x}) |x|^m = \Gamma(m + 1)\Gamma(n - m)/\Gamma(n + 1),$$

and then the general form (22) has been guessed.

Using the expressions for the structure constants and result (23) the three point function of the operators (14) with $j_{n,m} = j_{n,1}$ can be found in the form

$$\langle O_{n_1}^{-} O_{n_2}^{-} O_{n_3}^{-}\rangle = \left(\frac{1}{n_1 n_2 n_3}\right)^{\frac{1}{2}}.$$  \hspace{1cm} (24)

The non-trivial $n$-dependence cancels out, and I end up only with leg factors.

Finally, normalize the correlation functions in the same way as in [14] one gets

$$\langle O_{n_1}^{-} O_{n_2}^{-} O_{n_3}^{-}\rangle_{\text{norm}} = n_1 n_2 n_3.$$  \hspace{1cm} (25)

This formula agrees with both the matrix model result and conformal gauge one [14, 3]. Note that the operators $O_{n,1}^{-}$ correspond to $\Phi_{n,1}^{-}$ operators in the conformal gauge. Thus they represent discrete states in the Polyakov light-cone gauge.

To summarize, the main point in this letter is the well defined structure of 2d world sheet. It allows to avoid a question on a global fixing of the Polyakov gauge. Moreover all properties of $c \leq 1$ matter coupled to 2D gravity in the Polyakov gauge are retained. So
one gets better control of the model. The construction reminds one of an idea by Schwarz that there exists a well defined gauge so that the theory has the same properties as in the Polyakov gauge.

Let me also mention some interesting features of the construction together with open problems.

(i) One is on a ”world sheet-isotopic” transmutation. Indeed, starting with the SL(2)/SL(2) model, and defining $z$ as the world sheet coordinate and $x, \bar{x}$ as the isotopic ones, one arrives at a rather amusing picture: $x, \bar{x}$ become the world sheet coordinates of the model.

(ii) The chiral sector of the SL(2)/SL(2) theory reduces to the chiral sector of the minimal models coupled to gravity in the conformal gauge under the quantum hamiltonian reduction. On the other hand it is possible to reduce the same sector to the full theory for the Polyakov gauge. So, one can imagine that this gauge provides a ”minimal” description of the model.

(iii) In order to calculate the correlation functions for $O_{n,m}$ operators one needs SL(2) invariant measure $d\mu$ which depends on $k$ in a rather nontrivial way as well as integrands. In fact integrands for the four point (etc.) correlation functions are known only for the simplest case of the free fermions where they were found due to the path integral methods. Unlike the minimal models there is no general principle for combining the conformal blocks in the model. The obvious origin of this trouble is that the number of conformal blocks in the minimal models and SL(2) conformal field theory are different. The problem is to find measure and integrands.

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References

[1] A.Polyakov, Mod.Phys.Lett. A2 (1987) 893; in Les Houches 1988: Two-dimensional quantum gravity. Superconductivity at high $T_c$.

[2] V.Knizhnik, A.Polyakov and A.Zamolodchikov, Mod.Phys.Lett. A3 (1988) 819.

[3] J.Ambjørn, in Les Houches 1990: Quantization of Geometry, Preprint NBI-HE-94-53;
L.Alvarez-Gaume and C.Gomez, Topics in Liouville Theory, Lectures at the Trieste Spring School, 1991, Preprint CERN-TH.6175/91;
F.David, in Les Houches 1992: Simplicial Quantum Gravity and Random Lattices, Preprint Saclay T93/028.

[4] K.Gawedzki and A.Kupianen, Phys.Lett. B215 (1988) 119, Nucl.Phys. B320 (1989) 649.
[5] B. Feigin and F. Malikov, Fusion algebra at a rational level and cohomology of nilpotent subalgebras of $\mathfrak{sl}_2$, [hep-th/9310003].

[6] V. A. Fateev and A. B. Zamolodchikov, *Sov. J. Nucl. Phys.* **43** (1986) 657.

[7] O. Aharony, O. Ganor, J. Sonnenschein, S. Yankielowicz and N. Sochen, *Nucl. Phys.* **B399** (1993) 527;
H. Hu and M. Yu, On the Equivalence of Non-critical Strings and $G_k/G_k$ Topological Field Theories, Preprint AS-ITP-92-23; *Nucl. Phys.* **B391** (1993) 389.

[8] P. Furlan, A. Ch. Ganchev, R. Panov and V. B. Petkova, *Phys. Lett.* **B267** (1991) 63;
*Nucl. Phys.* **B394** (1993) 665.

[9] V. G. Kac and D. A. Kazhdan, *Adv. Math.* **34** (1979) 97.

[10] V. G. Kac and M. Wakimoto, *Proc. Natl. Acad. Sci. USA* **85** (1988) 4956.

[11] O. Andreev, *Phys. Lett.* **B363** (1995) 166.

[12] N. Marcus and Y. Oz, *Nucl. Phys.* **B392** (1993) 281

[13] Vl. S. Dotsenko and V. A. Fateev, *Phys. Lett.* **B157** (1985) 291.

[14] P. Di Francesco and D. Kutasov, *Phys. Lett.* **B261** (1991) 385;
M. Gouilian and M. Li, *Phys. Rev. Lett.* **66** (1991) 2051;
Vl. S. Dotsenko, *Mod. Phys. Lett.* **A6** (1991) 3601.

[15] A. S. Schwarz, Private communication.

[16] A. Bilal and I. Kogan, Gravitationally dressed conformal field theory and emergence of logarithmic operators, [hep-th/9407151]. *Nucl. Phys.* **B449** (1995) 569.