The Parameterized Complexity of Domination-type Problems and Application to Linear Codes

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Abstract

We study the parameterized complexity of domination-type problems. \((\sigma, \rho)-\text{domination}\) is a general and unifying framework introduced by Telle: a set \(D\) of vertices of a graph \(G\) is \((\sigma, \rho)-\text{dominating} if for any \(v \in D\), \(|N(v) \cap D| \in \sigma\) and for any \(v \notin D\), \(|N(v) \cap D| \in \rho\). We mainly show that for any \(\sigma\) and \(\rho\) the problem of \((\sigma, \rho)-\text{domination}\) is \(W[2]\) when parameterized by the size of the dominating set. This general statement is optimal in the sense that several particular instances of \((\sigma, \rho)-\text{domination}\) are \(W[2]\)-complete (e.g. DOMINATING SET). We also prove that \((\sigma, \rho)-\text{domination}\) is \(W[2]\) for the dual parameterization, i.e. when parameterized by the size of the dominated set. We extend this result to a class of domination-type problems which do not fall into the \((\sigma, \rho)-\text{domination}\) framework, including CONNECTED DOMINATING SET. We also consider problems of coding theory which are related to domination-type problems with parity constraints. In particular, we prove that the problem of the minimal distance of a linear code over \(F_q\) is \(W[2]\) for both standard and dual parameterizations, and \(W[1]\)-hard for the dual parameterization.

To prove \(W[2]\)-membership of the domination-type problems we extend the Turing-way to parameterized complexity by introducing a new kind of non deterministic Turing machine with the ability to perform ‘blind’ transitions, i.e. transitions which do not depend on the content of the tapes. We prove that the corresponding problem SHORT BLIND MULTI-TAPE NON-DETERMINISTIC TURING MACHINE is \(W[2]\)-complete. We believe that this new machine can be used to prove \(W[2]\)-membership of other problems, not necessarily related to domination.

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1 Introduction

Domination-type problems. Domination problems are central in graph theory. Telle [16] introduced the notion of \((\sigma, \rho)-\text{domination}\) as a unifying framework for many problems of domination: for any two sets of integers \(\sigma\) and \(\rho\), a set \(D\) of vertices of a graph \(G\) is \((\sigma, \rho)-\text{dominating} if for any vertex \(v \in D\), \(|N(v) \cap D| \in \sigma\) and for any vertex \(v \notin D\), \(|N(v) \cap D| \in \rho\). Among others, dominating sets, independent sets, and perfect codes are some particular instances of \((\sigma, \rho)-\text{domination}\). When \(\sigma, \rho \in \{\text{ODD}, \text{EVEN}\}\) (where \(\text{EVEN} := \{2n, n \in \mathbb{N}\}\) and \(\text{ODD} := \mathbb{N} \setminus \text{EVEN}\)), \((\sigma, \rho)-\text{domination}\) is strongly related to problems in coding

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theory like finding the minimal distance of a linear code \[13\]. Despite its genericity, the \((\sigma,\rho)\)-domination framework does not capture all the variants of domination, for instance connected dominating set (i.e. a dominating set which induces a connected subgraph) do not fall into the \((\sigma,\rho)\)-domination framework.

**Parameterized complexity of domination-type problems.** Most of the domination-type problems are NP-hard \[10\], however some of them are fixed parameter tractable. We assume the reader familiar with parameterized complexity and the W-hierarchy, otherwise we refer to \[5\]–\[8\]. The parameterized complexity of domination-type problems has been intensively studied \[11\]–\[14\] since the seminal paper by Downey and Fellows \[5\]. For instance, **Dominating Set** is known to be W[2]-complete \[5\], whereas **Independent Set** and **Perfect Code** are W[1]-complete \[5, 2\] (see figure 1 for a list of domination-type problems with their parameterized complexity). Another example is **Total Dominating Set** which is known to be W[2]-hard \[1\] but was not known to belong to any W[t] class. Parameterized complexity of domination-type problems with parity constraints – and as a consequence the parameterized complexity of the corresponding problems in coding theory – has been studied in \[9\]: **Odd Set** and Weight Distribution are W[1]-hard and W[2], whereas **Even Set** and Minimal Distance are W[2]. Additionally to these particular cases of domination-type problems, general results reveal how the parameterized complexity of \((\sigma,\rho)\)-domination depends on the choice of \(\sigma\) and \(\rho\). For instance, Golovach, Kratochvíl, and Suchý \[11\] have proved that when \(\sigma \subseteq \mathbb{N}\) and \(\rho \subseteq \mathbb{N}^+\) are nonempty finite sets then the decision problem associated with \((\sigma,\rho)\)-domination is W[1]-complete.

In parameterized complexity, the choice of the parameter is decisive. For all the problems mentioned above the standard parameterization is considered, i.e. the parameter is the size of the dominating set. Domination-type problems have also been studied according to the dual parameterization, i.e. the parameter is the size of the dominated set. With the dual parameterization, the problem associated with \((\sigma,\rho)\)-domination is FPT when \(\sigma\) and \(\rho\) are either finite or cofinite \[14\]. As a consequence, **Independent Set**, **Dominating Set** and **Perfect Code** are FPT for the dual parameterization. With parity constraints (i.e. \(\sigma,\rho \in \{\text{ODD}, \text{EVEN}\}\)), the problem associated with \((\sigma,\rho)\)-domination has been proved to be W[1]-hard \[9\] for the dual parameterization. Attention was also paid to the parameterized complexity of \((\sigma,\rho)\)-domination when parameterized by the tree-width of the graph \[3\]–\[17\].

**Our results.** The main result of the paper is that for any \(\sigma\) and \(\rho\) recursive sets, \((\sigma,\rho)\)-domination belongs to W[2] for the standard parameterization.

This general statement, with no particular constraint on \(\sigma\) and \(\rho\), is optimal in the sense that problems of \((\sigma,\rho)\)-domination are known to be W[2]-hard for some particular instances of \(\sigma\) and \(\rho\) (e.g. **Dominating Set**). We also prove that for any \(\sigma\) and \(\rho\), \((\sigma,\rho)\)-domination belongs to W[2] for the dual parameterization. For several particular instances of \(\sigma\) and \(\rho\), the W[2]-membership was unknown: the standard parameterization of **Total Dominating Set** was not known to belong to W[2], and the dual parameterization of \((\sigma,\rho)\)-domination for \(\sigma,\rho \in \{\text{ODD}, \text{EVEN}\}\) was not known to be W[2] neither.

Moreover, we prove that two particular \((\sigma,\rho)\)-domination problems, namely **Strong Stable Set** and **Induced r-Regular Subgraph**, are W[1]-complete for the standard parameterization.

We also consider problems that do not fall into the \((\sigma,\rho)\)-domination framework. For any property \(P\) and any set \(\rho\) of integers, \(D\) is a \((P,\rho)\)-dominating set in a graph \(G\) if the subgraph induced by \(D\) satisfies the property \(P\) and for any vertex \(v \notin D\), \(|N(v) \cap D| \in \rho\). Connected dominating set corresponds to \(\rho = \mathbb{N}^+\) and \(P\) the property that the graph is connected. We prove that the standard parameterization of \((P,\rho)\)-domination is W[2] for
any $P$ and $\rho$ recursive. As a consequence, Connected Dominating Set is $W[2]$-complete. We also prove that another domination problem, Digraph Kernel, is $W[2]$-complete.

Finally, regarding problems in linear coding theory, we show that the dual parameterization of Weight Distribution and Minimum Distance are both $W[2]$. We also consider extensions of these two problems to the field $F_q$, instead of $F_2$, and show that Weight Distribution over $F_q$ is $W[1]$-hard and $W[2]$ for both standard and dual parameterizations; and that Minimum Distance over $F_q$ is $W[2]$ for standard parameterization, and $W[1]$-hard and $W[2]$ for dual parameterization.

Our contributions are summarized in Figure 1.

Our approach: extending the Turing way to parameterized complexity. The Turing way to parameterized complexity [3] consists in solving a problem with a Turing machine of a special kind to prove that the problem belongs to some class of the W-hierarchy. For instance, if a problem can be solved by a Non Deterministic Turing machine in a number of steps which only depends on the parameter then the problem is $W[1]$. $W[1]$-membership of Perfect Code has been proved using such a Turing machine [2]. When the problem is solved by a multi-tape non-deterministic machine in a number of steps which only depends on the parameter, it proves that the problem is $W[2]$. To prove the $W[2]$ membership of $(\sigma, \rho)$-domination for any $\sigma$ and $\rho$, we introduce an extension of the multi-tape non deterministic Turing machine by allowing ‘blind’ transitions, i.e. transitions which do not depend on the symbols pointed out by the heads. We crucially show that the extra capability of doing blind transitions does not change the computational power of the machine in terms of parameterized complexity by proving that the problem Short Blind Multi-Tape Turing Machine is $W[2]$-complete. We believe that this machine can be used to prove $W[2]$-membership of other problems, not necessarily related to domination-type problems.

The paper is organized as follows: the next section is dedicated to the introduction of the blind multi-tape Truing machine and the proof that the corresponding parameterized problem is $W[2]$-complete. In section 3, several results on the parameterized complexity of $(\sigma, \rho)$-domination are given. In section 4, the parameterized complexity of domination-type problems which do not fall in the $(\sigma, \rho)$-domination are given. Finally, section 5 is dedicated to problems in coding theory which are related to domination-type problems with parity conditions.

2 Blind Multi-Tape Non-Deterministic Turing Machine

A blind Turing machine is a Turing Machine with the capability to do ‘blind’ transitions, i.e. transitions which do not depend on the symbol under the head. Blind transitions are of interest in the multitape case since a single blind transition can be seen as a shortcut for up to $|\Sigma|^m$ transitions, where $\Sigma$ is the alphabet and $m$ the number of tapes. For the description of the transitions of a Blind $m$-Tape Turing Machine $M = (\Sigma, Q, \Delta)$, we introduce a neutral symbol ‘$\_\$’ and define the transitions as: $\Delta \subseteq \Sigma^m \times Q \times \Sigma^m \times Q \times [-1, 1]^m$, where $\Sigma = \Sigma \cup \{\_\}$. A neutral symbol on the left part means that the transition can be applied whatever the symbol of the alphabet on the corresponding tape is, and a neutral symbol on the right part means that the symbol on the tape is kept. For instance $(\_, \_, q, a, q, 0, 0)$ is a blind transition of a 2-tape machine which, whatever the symbols under the heads are, changes the

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1 When parameterized by the size of the dominating set, the parameterised complexity of $(\sigma, \text{ODD})$-domination (resp. $(\sigma, \text{EVEN})$-domination) for $\sigma \in \{\text{ODD}, \text{EVEN}\}$ can be derived from the parameterised complexity of OddSet (resp. EvenSet) which has been proved in [11].
| Name                          | Formulation                                      | Standard                  | Dual                   |
|-------------------------------|--------------------------------------------------|---------------------------|------------------------|
| Dominating Set                | \((N,N^+)\)                                      | \(W[2]\)-Complete \[5\]  | FPT \[11\]            |
| Independent Set               | \(\{0\} \times N\)                              | \(W[1]\)-Complete \[6\]  | FPT \[11\]            |
| Perfect Code                  | \(\{0\} \times \{1\}\)                         | \(W[1]\)-Complete \[6, 2\]| FPT \[11\]            |
| Strong Stable Set             | \(\{0\} \times \{0,1\}\)                       | \(W[1]\)-complete \(W[1]\)\) | FPT \[11\]            |
| Induced r-Regular Subgraph    | \(\{r\} \times N\)                              | \(W[1]\)-complete \(W[1]\)-hard \[15\]\) | FPT \[11\] |
| Total Dominating Set          | \((N^+,N^+)\)                                    | \(W[2]\)-complete \(W[2]\)-hard \[1\]\) | FPT \[11\] |
| \((\sigma,ODD)\)-Dominating Set, \(\sigma\in\{ODD,EVEN\}\) | \(W[1]\)-hard, \(W[2]\)\) \[9\] | \(W[1]\)-hard \[11\], \(W[2]\) | \(W[2]\) |
| \((\sigma,EVEN)\)-Dominating Set, \(\sigma\in\{ODD,EVEN\}\) | \(W[2]\)\[9\] | \(W[1]\)-hard \[11\], \(W[2]\) | \(W[2]\) |
| \((\sigma,\rho)\)-Dominating Set, when \(\sigma,\rho\) recursive | \(W[2]\) | \(W[2]\) | \(W[2]\) |

**Other Domination Problems**

| Connected Dominating Set      | \(W[2]\)-complete \(W[2]\)-hard \[10\]\)         | Unknown                  |
| Digraph Kernel                | \(W[2]\)-complete \(W[2]\)-hard \[12\]\)         | Unknown                  |

**Problems in Coding Theory**

| Weight Distribution          | \(W[1]\)-hard, \(W[2]\) \[9\] | \(W[1]\)-hard \[11\], \(W[2]\) | \(W[2]\) |
| Minimum Distance             | \(W[2]\) \[9\] | \(W[1]\)-hard \[11\], \(W[2]\) | \(W[2]\) |
| Weight Distribution On \(\mathbb{F}_q\), \((q\) a parameter) | \(W[1]\)-hard, \(W[2]\) | \(W[1]\)-hard, \(W[2]\) | \(W[2]\) |
| Minimum Distance On \(\mathbb{F}_q\), \((q\) a parameter) | \(W[2]\) | \(W[1]\)-hard, \(W[2]\) | \(W[2]\) |

Figure 1: Overview of the parameterized complexity of some domination-type problems and some problems in coding theory. The ‘standard’ column corresponds to a parameterization by the size of the dominating set (or the Hamming weight for the problems in coding theory). The ‘Dual’ column corresponds to the dual parameterization, e.g. the size of the dominated set for domination-type problems. Our contributions, depicted in bold font, improve the results indicated in parenthesis.

An internal state \(q\) into \(q'\) and writes ‘\(a\)’ on both tapes; \(\langle m, q, m, q, 1^m \rangle\) (where \(\sigma^m\) stands for \(\sigma, \ldots, \sigma, m\) times) is a blind transition of a \(m\)-tape machine which moves all the \(m\) heads to the write without modifying the content of the tapes.

The parametric computation problem associated with the Blind Multi-Tape Non-Deterministic Turing Machines is defined as:

**Short Blind Multi-Tape Non-Deterministic Turing Machine Computation**

Input: A blind \(m\)-tape non-deterministic Turing Machine \(M\) and a word \(w\) on the alphabet, An integer \(k\).

Parameter: \(k\).

Question: Is there a computation of \(M\) on \(w\) that reaches an accepting state in at most \(k\) steps?

We show that the ability to perform blind transitions does not give any extra power to the Turing machine in terms of parameterized complexity: **Short Blind Multi-Tape Non-Deterministic Turing Machine Computation** is, like **Short Multi-Tape Non-Deterministic Turing Machine Computation** \[3\], complete for \(W[2]\).
Theorem 1. Short Blind Multi-Tape Non-Deterministic Turing Machine Computation is complete for W[2].

The rest of the section is dedicated to the proof of Theorem 1. The hardness for W[2] comes from the non-blind case which has been proven to be complete for W[2] [3]. The proof of W[2]-membership consists, like in the non-blind case [3], in a reduction to Weighted Weft-2 Circuit Satisfiability [7]. This problem consists in deciding whether a weft-2 mixed-type boolean circuit of depth bounded by a function of the parameter $k$, accepts some input of Hamming weight $k$. A mixed type circuit is composed of 'small' gates of fan-in $\leq 2$ and 'large' AND and/or OR gates of unbounded fan-in. The weft of the circuit is the maximum number of unbounded fan-in gates on an input/output path.

First, we transform $M$ into a machine which accepts its input in (exactly) $k + 2$ steps iff $M$ accepts its input in at most $k$ steps. To this end, all accepting states of $M$ are merged into a fresh non-accepting state $q_a$; moreover an accepting state $q_A$ is added as well as two blind transitions: $\langle \_m, q_a, \_m, 0^m \rangle$ and $\langle \_m, q_a, \_m, q_A, 0^m \rangle$.

In the following, a weft-2 mixed circuit $C$ is constructed in such a way that the accepted inputs correspond to the sequences of $k$ transitions of a machine $M$ from the initial state to the accepting state. The set $\Delta$ of the transitions of $M$ are indexed by $j \in [1, |\Delta|]$. The symbols $\Sigma$ are indexed by $s \in [0, |\Sigma|]$, where 0 is the index of the blank symbol and $|\Sigma|$ is the index of the neutral symbol \_\_. Let $x[i, j]$ for $i \in [1, k], j \in [1, |\Delta|]$ be the input wires of the circuit. $x[i, j]$ is true if and only if the $i^{th}$ transition of the sequence is the transition indexed by $j$. The following gates encode some information about the transitions of $M$: $\forall i \in [1, k], \forall q \in [1, |Q|], \forall s \in [0, |\Sigma|], \forall t \in [1, m], \forall d \in \{-1, 0, 1\}$,

- $\tau_o(i, q)$ outputs true iff the initial state on the $i^{th}$ transition is $q$:
  $\tau_o(i, q) := \sum_{j \in J_q} x[i, j]$, where $J_q = \Delta \cap (\Sigma^m \times \{q\} \times \Sigma^m \times Q \times [-1, 1]^m)$

- $\tau_n(i, q)$ outputs true iff the final state on the $i^{th}$ transition is $q$:
  $\tau_n(i, q) := \sum_{j \in J_q} x[i, j]$, where $J'_q = \Delta \cap (\Sigma^m \times Q \times \Sigma^m \times \{q\} \times [-1, 1]^m)$

- $\sigma_o(i, s, t)$ outputs true iff either the symbol read by the $i^{th}$ transition on tape $t$ is $s$, or the transition does not read symbol on tape $t$ in the 'blind' case $s = |\Sigma|$:
  $\sigma_o(i, s, t) := \sum_{j \in J_q} x[i, j]$, where $J_{s, t} = \Delta \cap (\Sigma^t \times \{s\} \times \Sigma^{m-t} \times Q \times \Sigma^m \times Q \times [-1, 1]^m)$

- $\sigma_n(i, s, t)$ outputs true iff either the symbol written by the $i^{th}$ transition on tape $t$ is $s$, or the transition rewrites the symbol on tape $t$ in the 'blind' case $s = |\Sigma|$:
  $\sigma_n(i, s, t) := \sum_{j \in J_q} x[i, j]$, where $J'_{s, t} = \Delta \cap (\Sigma^m \times Q \times \Sigma^t \times \{s\} \times \Sigma^{m-t} \times Q \times [-1, 1]^m)$

- $\mu(i, d, t)$ outputs true iff the head of $t$ has a movement $d$ on the $i^{th}$ transition:
  $\mu(i, d, t) := \sum_{j \in J_q} x[i, j]$, where $J_{d, t} = \Delta \cap (\Sigma^m \times Q \times \Sigma^m \times Q \times [-1, 1]^t \times \{d\} \times [-1, 1]^m-t)$

Notice that most of these gates require an unbounded fan-in OR gates in general.

The following gates encode the position of the heads and all the symbols in every cell of the tapes. These gates guarantee the correctness of the transition sequence. $\forall i \in [1, k], \forall t \in [-k, k], \forall t \in [1, m], \forall s \in [0, |\Sigma| - 1]$,

- $\beta(i, l, t)$ outputs true iff the head of tape $t$ is at position $l$ before step $i$. Since the transition sequence is of length $k$, $l$ is in the interval $[-k, k]$. The gate is defined as:
  $\beta(0, l, t) := \begin{cases} 1 & \text{if } l = 0 \\ 0 & \text{if } l \neq 0 \end{cases}$
  $\beta(i, l, t) := \beta(i-1, l, t) \cdot \mu(i-1, 0, t) + \beta(i-1, l-1, t) \cdot \mu(i-1, 1, t) + \beta(i-1, l+1, t) \cdot \mu(i-1, -1, t)$

- $\sigma(i, l, s, t)$ outputs true iff the cell $l$ of tape $t$ contains the symbol $s$ before step $i$. Let $w$
be the input word of the machine, located on tape 1.

\[ \sigma(0, l, s, t) := \begin{cases} 1 & \text{if } (s = 0) \land (t \neq 1 \lor 0 \lor l \geq |w|) \\ 0 & \text{otherwise} \end{cases} \]

\[ \sigma(i, l, s, t) := \neg \beta(i - 1, l, t) \cdot \sigma(i - 1, l, s, t) + \beta(i - 1, l, t) \cdot \sigma_n(i - 1, s, t) + \beta(i - 1, l, t) \cdot \sigma_n(i - 1, |\Sigma|, t) \cdot \sigma(i - 1, l, s, t) \]

One can see in the definition of \( \sigma(i, l, s, t) \) for \( i > 0 \) that there are three different cases: either the head was not pointing out the cell \( l \), so the symbol remains unchanged; or the head was pointing out the cell \( l \), and the symbol has been written in the previous step; or the head was on the cell but the transition was blind, so the symbol was already \( s \).

Notice that these gates have a bounded fan-in, and that the recursion is on the number of transitions, so their depth is bounded by the parameter \( k \). Notice also that there is a polynomial number of such gates since there are \( k \cdot 2k \cdot m, \mu \) gates and \( k \cdot 2k \cdot |\Sigma| \cdot m, \sigma \) gates.

All the information about the computation path has been encoded, the remaining gates check the validity of these transition sequences:

\( E := E_0 \cdot E_1 \cdot E_2 \cdot E_3 \cdot E_4 \) is the final gate of the circuit. As a consequence, for any input accepted by the circuit, the following conditions \( E_0, \ldots, E_4 \) must be satisfied:

\( E_0 = \neg x[1,-1] \) ensures that \( x[1,-1] \) is the constant 0, so \( \neg x[1,-1] \) is the constant 1.

\( E_1 \) ensures that for every \( i \), at most one wire among the block \( x[i,1] \ldots x[i,|\Delta|] \) is true, which means that at each step at most one transition is performed. \( E_1 \) is defined as:

\[ E_1 := \prod_{i=1}^{k} \prod_{j=0}^{|\Sigma|-1} \prod_{j'=0, j' \neq j}^{|\Sigma|-1} (\neg \sigma[i, j] + \sigma[i, j']) \]

\( E_2 \) ensures that the initial state of each step is equal to the final state of the previous step. \( E_2 \) is defined as:

\[ E_2 := \prod_{i=2}^{k} \prod_{q=1}^{Q} (\neg \tau_n(i - 1, q) + \tau_o(i, q)) \]

Notice that this formula is an encoding of \( \forall i \in [1, k], \forall q \in [1, |Q|], \tau_o(i - 1, q) \Rightarrow \tau_o(i, q) \)

\( E_3 \) ensures that the symbol read by a transition on a tape is either the one pointed out by the head or any symbol when the transition is blind. It is defined as:

\[ E_3 := \prod_{l=1}^{k} \prod_{s=1}^{m} \prod_{t=0}^{|\Sigma|-1} (\neg \beta(i, l, t) + \neg \sigma(i, l, s, t) + \sigma_o(i, s, t) + \sigma_o(i, |\Sigma|, t)) \]

Notice that this formula is an encoding of \( \forall i \in [1, k], \forall l \in [-k, k], \forall s \in [0, |\Sigma|], \forall t \in [1, m], (\beta(i, l, t) \land \sigma(i, l, s, t)) \Rightarrow (\sigma_o(i, s, t) \lor \sigma_o(i, |\Sigma|, t)) \).

\( E_4 \) ensures that the initial state on the first step is \( q_o \), the initial state of \( M \) of index 0, and that the last state is the accepting state \( q_A \) of index \( |Q| - 1 \). So \( E_4 \) is defined as:

\[ E_4 := \tau_o(0, 0) \cdot \tau_n(k - 1, |Q| - 1) \]

All the \( E_i, i \in [0, 4] \) gates are independent, so every input-output path passes through at most one of these unbounded fan-in gates. Since it is also the case for the gates encoding the transitions and that the \( \sigma \) and \( \beta \) gates are bounded fan-in gates, then the weft of this circuit is 2. Since the only recursive gates have a depth bounded by the parameter, the depth of this circuit is bounded by the parameter. Notice also that the number of gates is polynomial in \( |M| \). This circuit outputs true if and only if \( M \) has an accepting computation path of length \( k \) on the word \( w \), i.e. if and only if \( M \) has an accepting computation path of length at most \( k \) on \( w \). Therefore, \textit{Short Blind Multi-Tape Non-Deterministic Turing Machine Computation} belongs to \( W[2] \).

### 3 Parameterized complexity of \((\sigma, \rho)\)-Domination

In this section, we prove the central result of the paper: for any \( \sigma \) and \( \rho \), \((\sigma, \rho)\)-domination belongs to \( W[2] \) for both \textit{standard} and \textit{dual} parameterizations, i.e. when the parameter
is the size of the dominating set or the size of the dominated set. To this end, we show that for any $\sigma$, $\rho$, the problem of $(\sigma, \rho)$-domination can be decided using a blind multi-tape Turing machine. The standard and dual parameterizations of $(\sigma, \rho)$-domination problems are: $(\sigma, \rho)$-Dominating Set of Size at Most $k$:

Input: A graph $G = (V, E)$, an integer $k$.
Parameter: $k$.

Question: Is there a $(\sigma, \rho)$-dominating set $D \subseteq V$ such that $|D| \leq k$?

$(\sigma, \rho)$-Dominating Set of Size at Least $n - k$:

Input: A graph $G = (V, E)$ with $n = |V|$, an integer $k$.
Parameter: $k$.

Question: Is there a $(\sigma, \rho)$-dominating set $D \subseteq V$ such that $|D| \leq n - k$?

**Theorem 2.** For any recursive sets of integers $\sigma$ and $\rho$, $(\sigma, \rho)$-Dominating Set of Size at Most $k$ belongs $W[2]$.

Proof. Given $\sigma$, $\rho$, $k$ and a graph $G = \{\{v_1, \ldots, v_n\}, E\}$, we consider the following $(n+1)$-tape Turing machine $M$ which decides whether $G$ has a $(\sigma, \rho)$-dominating set of size at most $k$. $M$ works in 3 phases (see an example in Figure 2): (1) a subset $D$ of size at most $k$ is non-deterministically chosen and written on the first tape, moreover the first $k$ cells of the following $n$ tapes – one tape for each vertex of the graph – are filled with 0s and 1s such that the $i^{th}$ cell of each tape is 1 iff $i \in \rho$; (2) The content of the tapes associated with the vertices in $D$ is removed and replaced by the characteristic vector of $\sigma$, i.e. the $i^{th}$ cell is 1 iff $i \in \sigma$. At the end of this second phase, all heads are located on the leftmost non-blank symbol; (3) For each vertex $v$ in $D$, the heads of all the tapes associated with a neighbor of $v$ move to the right. At the end of this third phase, for every $v \in D$ (resp. $v \in \overline{D}$), the head of the tape associated with $v$ reads 1 iff $|N(v) \cap D| \in \sigma$ (resp. $|N(v) \cap D| \in \rho$), so $D$ is a $(\sigma, \rho)$-dominating set iff all heads but the first one read a symbol 1.

The actual description of the blind $(n+1)$-tape non-deterministic Turing machine is as follows: $M = (\Sigma, Q, Q_1, Q_A, \Delta)$, where $\Sigma = \{\square, 0, 1, v_1, \ldots, v_n\}$, $Q = \{q_{i,s} | i \in [1,n], s \in [0,k]\} \cup \{q_1^{\text{ext}} | s \in [1,n+k+1]\} \cup \{q_{i,s}^{\text{sig}} | i \in [1,n], s \in [0,k]\} \cup \{q_1^{\text{sig}}, q_0^{\text{end}}, q_\rho^{\text{end}}, q_{\text{read}}, q_A\}$, $Q_1 = \{q_{1,0}\}$ and $Q_A = \{q_A\}$. The initial word $w$ is the empty word, so at the beginning every cell contains the blank symbol $\square$. The transitions are:

**Phase 1 – Initialisation of $D$ and $p$:**

- $(\square^n, q_{i,s}, v_j, 1^n, q_{j+1,s+1}, 1^n) \quad i \in [1,n], j \in [i,n]$,
- $(\square^n, q_{i,s}, v_j, 0^n, q_{j+1,s+1}, 1^n) \quad i \in [1,n], j \in [i,n]$,
- $(\square^n, q_{i,s}, 1^n, q_{j+1,s+1}, 0^n) \quad i \in [1,n+1], j \in [i,n]$,
- $(\square^n, q_{i,s}, 0^n, q_{j+1,s+1}, 0^n) \quad i \in [1,n+1], j \in [i,n]$,
- $(\square^n, q_{i,s}, q_{0^n}, q_{j+1,s+1}, 0^n) \quad i \in [1,n+1], j \in [i,n]$,
- $(\square^n, q_{i,s}, q_{1^n}, q_{j+1,s+1}, -1^n) \quad i \in [1,n+1], j \in [i,n]$,
- $(\square^n, q_{i,s}, q_{0^n}, q_{j+1,s+1}, -1^n) \quad i \in [1,n+1], j \in [i,n]$,
- $(\square^n, q_{i,s}, q_{1^n}, 0^n, q_{j+1,s+1}, 0^n) \quad i \in [1,n]$,
- $(\square^n, q_{i,s}, q_{1^n}, 0^n, q_{j+1,s+1}, -1^n) \quad i \in [1,n]$,
- $(\square^n, q_{i,s}, q_{0^n}, q_{j+1,s+1}, 0^n) \quad i \in [1,n]$,
- $(\square^n, q_{i,s}, q_{0^n}, q_{j+1,s+1}, -1^n) \quad i \in [1,n]$,
- $(\square^n, q_{i,s}, q_{1^n}, q_{j+1,s+1}, 0^n) \quad i \in [1,n]$,
- $(\square^n, q_{i,s}, q_{1^n}, q_{j+1,s+1}, -1^n) \quad i \in [1,n]$,
- $(\square^n, q_{i,s}, q_{0^n}, q_{j+1,s+1}, 0^n) \quad i \in [1,n]$,
- $(\square^n, q_{i,s}, q_{0^n}, q_{j+1,s+1}, -1^n) \quad i \in [1,n]$,
- $(\square^n, q_{i,s}, q_{1^n}, q_{j+1,s+1}, 0^n) \quad i \in [1,n]$,
- $(\square^n, q_{i,s}, q_{1^n}, q_{j+1,s+1}, -1^n) \quad i \in [1,n]$,
- $(\square^n, q_{i,s}, q_{0^n}, q_{j+1,s+1}, 0^n) \quad i \in [1,n]$,
- $(\square^n, q_{i,s}, q_{0^n}, q_{j+1,s+1}, -1^n) \quad i \in [1,n]$.

The state $q_{d,s}$ means that $d-1$ vertices among $v_1, \ldots, v_{i-1}$ have already been written on the first tape, and $s$ symbols have been written on the other tapes.

**Phase 2 – Initialisation of $\sigma$:**

- $(v_i^n, q_{i,s}, v_i^n, q_{i,s}, 0^n) \quad i \in [1,n]$,
- $(v_i^n, q_{i,s}, v_i^n, q_{i,s}, -1^n) \quad i \in [1,n], s \in \sigma \cap [0,k]$,
- $(v_i^n, q_{i,s}, v_i^n, q_{i,s}, 0^n) \quad i \in [1,n], s \in \sigma \cap [0,k]$,
- $(v_i^n, q_{i,s}, v_i^n, q_{i,s}, -1^n) \quad i \in [1,n], s \in \sigma \cap [0,k]$,
- $(v_i^n, q_{i,s}, v_i^n, q_{i,s}, 0^n) \quad i \in [1,n], s \in \sigma \cap [0,k]$,
- $(v_i^n, q_{i,s}, v_i^n, q_{i,s}, -1^n) \quad i \in [1,n], s \in \sigma \cap [0,k]$,
The Parameterized Complexity of Domination-type Problems and Application to Linear Codes

Figure 2 Computation of (\{0\}, N^+)-Dominating Set Of Size At Most 2 on a blind multitape Turing machine (see proof of Theorem 2). (a) Input graph; (b) State of the machine at the end of phase (1). The candidate set D is on the first tape, the other tapes are initialized according to \( \rho \); (c) End of phase (2): the tapes associated with vertices in D are now initialized according to \( \sigma \); (d) End of phase (3): all heads (underlined symbols) read 1, so \( \{v_1, v_4\} \) is a (\{0\}, N^+)-dominating set.

\[
\begin{align*}
\langle v_{i-1}, n, q^\text{sig}, v_i, i-1, 0, -n-1, q^\text{ret}_i, +1, 0 \rangle & \quad i \in [1, n], \text{ if } k \in \sigma \\
\langle v_{i-1}, n, q^\text{ret}, v_i, i-1, q^\text{ret}_{i-1}, 0, -1 \rangle & \quad i \in [1, n], s \in [1, k] \\
\langle v_{i-1}, n, q^\text{sig}, v_i, i-1, n, q^\text{sig}_i, 0, 0 \rangle & \quad i \in [1, n] \\
\langle \square, n, q^\text{sig}, \square, n, q^\text{sig}, 0, 0 \rangle & \\
\langle v_{i-1}, n, q^\text{read}, v_i, i-1, q^\text{read}, -1, 0 \rangle & \quad i \in [1, n] \\
\langle \square, n, q^\text{read}, \square, n, q^\text{read}, +1, 0 \rangle &
\end{align*}
\]

The state \( q^\text{sig}_{i+1} \) means that the first \( s \) symbols of the characteristic vector of \( \sigma \) have been written on the tape associated with the vertex \( v_i \).

Phase 3: Neighborhood Checking
\[
\begin{align*}
\langle v_{i-1}, n, q^\text{read}, v_i, i-1, q^\text{read}, +1, d_1, \ldots, d_n \rangle & \quad i \in [1, n], \text{ with } d_e = 1 \text{ if } v_e \in N(v_i) \text{ and 0 otherwise.} \\
\langle \square, 1, q^\text{read}, \square, 1, q_4, 0, 0 \rangle &
\end{align*}
\]

The number of transitions is polynomial in \( |G| \) and the acceptance is made at most \( 2(k+1) + k(2k+2) \) steps if a (\( \sigma, \rho \))-dominating set of size at most \( k \) exists. As a consequence, (\( \sigma, \rho \))-Dominating Set Of Size At Most \( k \) belongs to W[2]. Notice that the use of blind transitions in the third phase is crucial. Indeed, a naïve simulation of any of these blind transitions uses \( 2^n \) non-blind transitions since the transition should be applicable for any of the \( 2^n \) possible configurations read by the heads of the machine.

\[ \blacktriangleright \text{Theorem 3. For any recursive sets of integers } \sigma \text{ and } \rho, (\sigma, \rho)\text{-Dominating Set Of Size At Least } n - k \text{ belongs to W[2].} \]

\[ \text{Proof. To decide whether a given graph } G \text{ has a (} \sigma, \rho \text{)-dominating set of size at least } n - k, \text{ we slightly modify the blind Turing machine used in the proof of Theorem 2 in such a way that at the end of the second phase, the first tape contains the description of a set } D \text{ of size at most } k, \text{ and for any } v \in D (\text{resp. } v \notin D), \text{ the } i^{th} \text{ cell of the tape associated with } v \text{ is 1 if } \delta(v)-i \in \rho \text{ (resp. } \delta(v)-i \in \sigma) \text{ and 0 otherwise, where } \delta(v) \text{ is the degree of } v. \text{ Thus, the machine reaches the accepting state if there exists a set } D \text{ of size at most } k \text{ such that } \forall v \in D, \delta(v) - |N(v) \cap D| \in \rho \text{ and } \forall v \in V \setminus D, \delta(v) - |N(v) \cap D| \in \sigma. \text{ Since for any } v \in V, |N(v) \cap (V \setminus D)| = \delta(v) - |N(v) \cap D|, V \setminus D \text{ is a (} \sigma, \rho \text{)-dominating set of size at least } n - k. \]

Notice that one can prove in a similar way that for any recursive \( \sigma \) and \( \rho \), the problems (\( \sigma, \rho \))-dominating Set Of Size Exactly \( k \) and (\( \sigma, \rho \))-dominating Set Of Size Exactly \( n-k \) are both W[2].
**Strong Stable Set** \( (\{0\}, \{0, 1\})\)-Domination:

Input: A graph \( G = (V, E) \), an integer \( k \).

Parameter: \( k \).

Question: Is there an independent set \( S \subseteq V \) of size \( k \) such that \( \forall v \in V \setminus S, |N(v) \cap S| \leq 1 \)?

**Theorem 4. Strong Stable Set is complete for \( W[1] \).**

**Proof.** The \( W[1]\)-membership is an application of Theorem 4 in [11]. We prove the hardness by a reduction from \textsc{Independent Set} which is complete for \( W[1] \) [3]. Given an instance \((G, k)\) of \textsc{Independent Set}, we consider the instance \((G', k)\) of \textsc{Strong Stable Set} where \( G' = (V \cup E, E') \) with \( E' = \{(u, v) \in V \times E \mid e \text{ is incident to } u \in G\} \cup (E \times E) \). By construction \( G' \) consists in a stable set \( V \) and a clique \( E \), the edges between these two sets representing the edges of \( G \). If \( S \) is an independent set in \( G \), then, by construction, \( S \) is a strong stable set in \( G' \). Let \( S \) be a strong stable set in \( G' \) of size \( k \). Since \( E \) is a clique, \( |S \cap E| \in \{0, 1\} \). If \( |S \cap E| = 0 \), then \( S \subseteq V \) and for any \( u, v \in S \), they have no common neighbor in \( G' \) so there is no edge between \( u \) and \( v \) in \( G \), so \( S \) is an independent set in \( G \). Otherwise, if \( |S \cap E| = 1 \) then \( \forall u \in S \cap V \) and \( u \) is isolated in \( G' \), so there are at least \( k - 1 \) isolated vertices in \( G \), so there is an independent set of size \( k \) in \( G \).

---

**Induced \( r \)-Regular Sub-Graph** \( (\{r\}, \mathbb{N})\)-Dominating Set:

Input: A graph \( G = (V, E) \) with \( n = |V| \), an integer \( k \).

Parameter: \( k \).

Question: Is there a subset \( D \subseteq V \) such that \(|D| \leq k \) and \( \forall u \in D, |N(u) \cap D| = r \)?

**Theorem 5. Induced \( r \)-Regular Sub-Graph is complete for \( W[1] \).**

**Proof.** The hardness for \( W[1] \) is proved in [15]. We prove the membership by reducing to \textsc{Short Non-Deterministic Turing Machine Computation}, which is complete for \( W[1] \) [3]. The principle is to non-deterministically choose a set \( D \subseteq V \) of size \(|D| \leq k \), and then check that \( D \) is \( r \)-regular by counting the neighbourhood in \( D \) of each vertex. Given a graph \( G = (\{v_1, \ldots, v_n\}, E) \), let \( M = (\Sigma, Q, Q_I, Q_A, \Delta) \) be a one-tape non-deterministic Turing machine with \( \Sigma = \{\square, v_1, \ldots, v_n\}, Q = \{q_{i,s} \mid i \in [1,n+1], s \in [0,k]\} \cup \{q_{i,s}^1 \mid i \in [1,n], l \in [0,r]\} \cup \{q_{i,s}^{\text{cr}} \mid i \in [1,n], q \in \{0,1, \ldots, r\}\} \cup \{q_{i,s}^{\text{cl}} \mid i \in [1,n], l \in [0,r]\} \cup \{q_{i,s}^{\text{br}} \mid i \in [1,n], l \in [0,r]\} \cup \{q_{i,s}^{\text{new}} \mid i \in [1,n]\}, Q_I = \{q_{1,0}\} \) and \( Q_A = \{q_{A}\} \). The transitions are:

- \( (\square, q_{i,s}, v_j, q_{j,s+1+r,0}^1, +1) \) \( i \in [1,n], s \in [1,k-1], j \in [i,n] \)
- \( (\square, q_{i,s}, v_j, q_{j,s+1,0}^1, 0) \) \( i \in [1,n+1], s \in [1,k-1] \)
- \( (\square, q_{i,k}, v_j, q_{i,s+1,0}^1, 0) \) \( i \in [1,n], l \in [i,n] \)
- \( (\square, q_{i,k}, v_j, q_{i,s+1,0}^1, -1) \) \( i \in [1,n+1] \)
- \( (v_j, q_{j,s}^{\text{cr}}, v_j, q_{j,s+1+r,0}^1, +1) \) \( i \in [1,n], j \in [1,n], l \in [0,r], \text{ if } v_i v_j \in E \)
- \( (v_j, q_{j,s}^{\text{cr}}, v_j, q_{j,s+1,0}^1, +1) \) \( i \in [1,n], j \in [1,n], l \in [0,r], \text{ if } v_i v_j \notin E \)
- \( (\square, q_{j,s}^{\text{cr}}, v_j, q_{j,s+1,0}^1, -1) \) \( i \in [1,n], l \in [0,r] \)
- \( (v_j, q_{j,s}^{\text{cl}}, v_j, q_{j,s+1,0}^1, -1) \) \( i \in [1,n], j \in [1, i-1] \cup [i + 1, n], l \in [0,r] \)
- \( (v_i, q_{i,s}^{\text{br}}, v_j, q_{j,s+1,0}^1, -1) \) \( i \in [1,n], l \in [0,r] \)
- \( (v_j, q_{j,s}^{\text{br}}, v_j, q_{j,s+1,0}^1, -1) \) \( i \in [1,n], j \in [1, n], l \in [0,r], \text{ if } v_i v_j \in E \)
- \( (\square, q_{j,s}^{\text{br}}, v_j, q_{j,s+1,0}^1, +1) \) \( i \in [1,n] \)
- \( (v_j, q_{j,s}^{\text{br}}, v_j, q_{j,s+1,0}^1, +1) \) \( i \in [1,n], j \in [1, n], l \in [0,r], \text{ if } v_i v_j \notin E \)
- \( (v_i, q_{i,s}^{\text{cl}}, v_j, q_{j,s+1,0}^1, -1) \) \( i \in [1,n] \)
- \( (v_i, q_{i,s}^{\text{br}}, v_j, q_{j,s+1,0}^1, -1) \) \( i \in [1,n] \)
- \( (\square, q_{i,s}^{\text{new}}, v_j, q_{j,s+1,0}^1, +1) \) \( i \in [1,n] \)
The size of the machine is polynomial in $|G|$, if there exists a set $D$ of size at most $k$ which induces a $r$-regular subgraph, then the accepting state is reached in $k^2 + k$ steps.

4 Other Domination Problems

Some natural domination problems cannot be described in terms of $(\sigma, \rho)$-domination like CONNECTED DOMINATING SET. In this section, we show that the proof of the $(\sigma, \rho)$-domination W[2] membership (Theorem 2) can be generalized to $(P, \rho)$-domination where $P$ is no longer a domination constraint but any recursive property. It implying that CONNECTED DOMINATING SET known to be hard for W[2] is actually complete for W[2]. We also show that this technique can be applied to digraph problems with the example of DIGRAPH KERNEL.

$(P, \rho)$-DOMINATING SET OF SIZE AT MOST $k$:  
Input: A graph $G = (V, E)$, an integer $k$. 
Parameter: $k$. 
Question: Is there a subset $D \subseteq V$ such that $|D| \leq k$ and: 
- the sub-graph of $G$ induced by $D$ satisfies the property $P$; 
- $\forall v \in V \setminus D, |N(v) \cap D| \in \rho$ ?

▶ Theorem 6. If $\rho$ is a recursive set of integers and $P$ is a recursive property, then $(P, \rho)$-DOMINATING SET of Size at Most $k$ belongs to W[2].

Proof. We use the blind multitape Turing machine of theorem 2 with $\sigma = N$ which outputs a $(N, \rho)$-dominating set if it exists, then we compose this machine with a machine which decides whether such a set $D$ induces a subgraph which satisfies the property $P$. Since the subgraph is of size $O(k^2)$ and $P$ is recursive, the computation time of the second machine is $f(k)$ for some function $f$.

DIGRAPH KERNEL:  
Input: A directed graph $G = (V, A)$, an integer $k$. 
Parameter: $k$. 
Question: Is there a kernel of $D$ of size at most $k$? A kernel is an independent set $S$ such that for every vertex $x \in V \setminus S$, there is $y \in S$ such that $(x, y) \in A$.

▶ Theorem 7. DIGRAPH KERNEL is complete for W[2].

Proof. The hardness for W[2] has been proved in [12]. The proof of the membership is very similar to the W[2] membership of $(\sigma, \rho)$-DOMINATING SET (Theorem 2). The machine and the initialization are the same, with $\sigma = \{0\}$ and $\rho = N^+$. In the third phase, only the heads of the tapes associated with incoming neighbors move to the right.

5 Problems in Coding Theory

Parameterized complexity of problems in coding theory, in particular MINIMAL DISTANCE and WEIGHT DISTRIBUTION, have been studied in [8]. We prove that the dual parameterization of these problems are W[2]. Moreover, we consider extensions of these problems to linear codes over $\mathbb{F}_q$:  
MINIMAL DISTANCE OVER $\mathbb{F}_q$:  
Input: Two integers $q$ and $k$, a $m \times n$ matrix $H$ with entries in $\mathbb{F}_q$. 
Parameters: $q$, $k$. 
Question: Is there a non-empty set of at most $k$ columns that sum to the all-zero vector?
Weight Distribution Over $\mathbb{F}_q$:
Input: Two integers $q$ and $k$, a $m \times n$ matrix $H$ with entries in $\mathbb{F}_q$.
Parameters: $q$, $k$.
Question: Is there a set of $k$ columns of $H$ that sum to the all-zero vector?

**Theorem 8.** Weight Distribution Over $\mathbb{F}_q$ is hard for $W[1]$ and belongs to $W[2]$, and Minimal Distance Over $\mathbb{F}_q$ belongs to $W[2]$.

Proof. Since Weight Distribution is a particular case of Weight Distribution Over $\mathbb{F}_q$ with $q = 2$, Weight Distribution Over $\mathbb{F}_q$ is hard for $W[2]$. For the $W[2]$ membership, we show that the problem can be decided by a blind $(m + 1)$-tape Turing Machine $M = (\Sigma, Q, Q_f, \Delta)$. The first tape is associated with the set of columns and each remaining tape is associated with a row of $H$. The alphabet is $\Sigma = \{\square, 0, 1\} \cup \{h_i|i \in [1, n]\}$ and the states are $Q = \{q_{s,i} | i \in [1, n + 1], s \in [0, k \ast q]\} \cup \{q_{s}^e | s \in [1, k \ast q + 1]\} \cup \{q_{s,n} | i \in [1, n], s \in [0, q - 1]\} \cup \{q_{\text{read}, Q_A}, Q_f = \{q_{1,0}\}$ and $Q_A = \{q_A\}$. The initial word $w$ is the empty word. The transitions are separated in two phases:

**Phase 1: Initialisation:** First, $k$ columns of $H$ are non-deterministically chosen on the first tape, while the remaining tapes are initialized with $k$ times the pattern $10^{q-1}$ (i.e. 1 followed by $q - 1$ times 0), such that the $i^{th}$ cell is 1 iff $i \equiv 0 \mod q$.

\[
\begin{align*}
(\square, m, q_k, h_j, 1^m, q_{j+1, i+1}, 1, 1^m) & \quad i \in [1, n], j \in [0, k - 1], j \in [i, n], if s \equiv 0[q] \\
(\square, m, q_k, h_j, 0^m, q_{j+1, i+1}, 1, 1^m) & \quad i \in [1, n], j \in [0, k - 1], j \in [i, n], if s \not\equiv 0[q] \\
(\square, m, q_k, 1^m, q_{j+1}, 0, 1^m) & \quad i \in [1, n + 1], s \in [k, k \ast q], if i \equiv 0[q] \\
(\square, m, q_k, 0^m, q_{j+1}, 0, 1^m) & \quad i \in [1, n + 1], s \in [k, k \ast q], if i \not\equiv 0[q] \\
(\square, m, q_k, 1^m, q_{i, k+q}, 1^m, q_{i+1}, 1, 1^m) & \quad i \in [1, n + 1] \\
(\ldots, m, q_{i, k+q}, 1^m, q_{i+1}, 1, 1^m) & \quad s \in [1, k] \\
(\ldots, m, q_{i, k+q}, 1^m, q_{i+1}, 0, 1^m) & \quad s \in [k + 1, k \ast q + 1] \\
(\ldots, m, q_{i, k+q}, 1^m, q_{i+1}, 0, 1^m) & \quad s \in [0, q - 2], with \forall i \in [1, m], \\
& d_i = +1 if H_{i,i} > i, 0 otherwise.
\end{align*}
\]

**Phase 2: Recognition:** In order to check that the sum of those columns is the all-zero vector on $\mathbb{F}_q$, for any column $i$ in the chosen set, and for any row $l$, the head of the tape associated with row $l$ moves $H_{l,i}$ times to the right, using blind transitions.

\[
\begin{align*}
(h_i, \ldots, m, q_{\text{read}}, h_i, \ldots, n, q_{n,i+1}, 1, 1^m) & \quad i \in [1, n] \\
(\ldots, m, q_{\text{read}}, \ldots, m, q_{\text{read}}, d_i, \ldots, d_m) & \quad i \in [1, n], s \in [0, q - 2], with \forall i \in [1, m], \\
& d_i = +1 if H_{i,i} > i, 0 otherwise.
\end{align*}
\]

The size of the transition table is polynomial in $|H|$. The machine accepts iff at the end all the heads are on position $i$ such that $i \equiv 0[q]$, i.e. iff the sum of all columns is the all-zero vector. The accepting state is reached in at most $3kq$ steps. The proof of $W[2]$ membership for Minimum Distance is the same, except that $D$ is chosen of size at most $k$.

**Dual Minimum Distance:**
Input: A $m \times n$ matrix $H$ with value on $\mathbb{F}_2$, an integer $k$.
Parameter: $k$.
Question: Is there a non-empty set of at least $n-k$ columns that sum to the zero vector?

**Dual Weight Distribution:**
Input: A $m \times n$ matrix $H$ with entries in $\mathbb{F}_2$, an integer $k$.
Parameter: $k$.
Question: Is there a set of $n-k$ columns of $H$ that sum to the all-zero vector?

**Theorem 9.** Dual Minimum Distance and Dual Weight Distribution are $W[2]$. 
Proof. With the dual parameterization, the set $S$ of chosen columns is of size $n - k$, thus $k$ is the maximum size of the set of non chosen columns $S'$. In the proof of theorem 8, $S$ is non-deterministically chosen and all tapes represent the sum on each row of the matrix. The total sum of the row can be done in polynomial time. In consequence, a reduction to Short Blind Multi-Tape Non-Deterministic Turing Machine Computation can be done using a similar process. $S'$ is non-deterministically chosen on the first tape. In each of the $m$ remaining tapes, one foe each row, $\forall p \in [0, k - 1]$ the cell $p$ on the tape $j$ contains the parity of $(\sum_{i=1}^{n} H_{i,j}) - p$ ($0$ for even, $1$ for odd). Therefore, after reading $S'$ and moving to the right the heads of the tapes associated with the neighbours, the set sums to the all-zero vector iff all the heads except the first point on 0. The proof that Dual Weight Distribution is $W[2]$ is the same except that the size of $S'$ is fixed to $k$. ▷

One can show, by combining the proofs of theorem 8 and 9, that Dual Weight Distribution Over $F_q$ and Dual Minimum Distance Over $F_q$ belong to $W[2]$.

6 Conclusion

We have demonstrated several results on the parameterized complexity of domination-type problems, including that for any $\sigma$ and $\rho$, $(\sigma, \rho)$-domination is $W[2]$ for both standard and dual parameterizations. To this end, we have extended the Turing way to parameterized complexity with a new way to prove $W[2]$ membership using ‘blind’ Turing machines. We believe that this machine can be used to prove $W[2]$ membership of other problems, not necessarily related to domination.

Several questions remain open. First, the long-standing question regarding the $W[1]$-hardness of Minimum Distance remains open [9]. Moreover, several problems related to domination with parity constraints, like Weight Distribution, are $W[1]$-hard and $W[2]$, are they complete for one of these two classes, or intermediate? It is interesting to notice that, for the dual parameterization, the difference between Minimal Distance and Weight Distribution seems to vanish in the sense that both problems are $W[1]$-hard, but the completeness for $W[1]$ or $W[2]$ is also open. In fact, no problem of $(\sigma, \rho)$-domination is known to be $W[2]$-complete for the dual parameterization, thus one can wonder if such a problem exists or if for any $\sigma$ and $\rho$, $(\sigma, \rho)$-domination is $W[1]$ for the dual parameterization?

References

1. H.L. Bodlaender and D. Kratsch. A note on fixed parameter intractability of some domination-related problems. Private communication, 1994.
2. M. Cesati. Perfect Code is $W[1]$-complete. Inf. Proc. Let., 81:163–168, 2002.
3. M. Cesati. The Turing way to parameterized complexity. Journal of Computer and System Sciences, 67:654–685, 2003.
4. M. Chapelle. Parameterized Complexity of Generalized Domination Problems on Bounded TreeWidth Graphs. Computing Research Repository, abs/1004.2, 2010.
5. R. G. Downey and M. R. Fellows. Fixed-Parameter Tractability and Completeness I: Basic Results. Siam Journal on Computing, 24:873–921, 1995.
6. R. G. Downey and M. R. Fellows. Fixed-Parameter Tractability and Completeness II: On Completeness for $W[1]$. Theoretical Computer Science, 141:109–131, 1995.
7. R. G. Downey and M. R. Fellows. Threshold Dominating Sets and an Improved Characterization of $W[2]$. Theoretical Computer Science, 209:123–140, 1998.
8 R. G. Downey, M. R. Fellows, and U. Stege. Parameterized complexity: A framework for systematically confronting computational intractability. In *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, 1999.

9 R. G. Downey, M. R. Fellows, A. Vardy, and G. Whittle. The Parametrized Complexity of Some Fundamental Problems in Coding Theory. *Siam J. on Computing*, 29:545–570, 1999.

10 M. R. Fellows. Blow-Ups. Win/Win’s, and Crown Rules: Some New Directions in FPT. In *Workshop on Graph-Theoretic Concepts in Computer Science*, pages 1–12, 2003.

11 P.A. Golovach, J. Kratochvil, and O. Suchy. Parameterized complexity of generalized domination problems. *Discrete Applied Mathematics*, 160(6):780–792, 2009.

12 G. Gutin, T. Kloks, C-M. Lee, and A. Yeo. Kernels in planar digraphs. *Journal of Computer and System Sciences*, 71:174–184, 2005.

13 M. Halldórsson, J. Kratochvil, and J. A. Telle. Mod-2 independence and domination in graphs. *Graph-Theoretic Concepts In Computer Science*, 1665:101–109, 1999.

14 T. Kloks and L. Cai. Parameterized tractability of some (efficient) Y-domination variants for planar graphs and t-degenerate graphs. 2000.

15 H. Moser and D. M. Thilikos. Parameterized complexity of finding regular induced subgraphs. *Journal of Discrete Algorithms*, 7:181–190, 2009.

16 J. A. Telle. Complexity of Domination-Type Problems in Graphs. *Nordic Journal of Computing*, 1:157–171, 1994.

17 J. A. Telle and A. Proskurowski. Algorithms for Vertex Partitioning Problems on Partial k-Trees. *Siam Journal on Discrete Mathematics*, 10:529–550, 1997.