ADDENDUM TO THE PAPER “HYPERSURFACES WITH ISOMETRIC REEB FLOW IN COMPLEX HYPERBOLIC TWO-PLANE GRASSMANNIANS”

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Abstract. We classify all of real hypersurfaces $M$ with Reeb invariant shape operator in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2 U_m)$, $m \geq 2$. Then it becomes a tube over a totally geodesic $SU_{2,m-1}/S(U_2 U_{m-1})$ in $SU_{2,m}/S(U_2 U_m)$ or a horosphere whose center at infinity is singular and of type $JN \in \mathfrak{J}N$ for a unit normal vector field $N$ of $M$.

INTRODUCTION

Let us introduce a paper due to Suh [9] for the classification of all real hypersurfaces with isometric Reeb flow in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2 U_m)$ as follows:

Theorem A. Let $M$ be a connected orientable real hypersurface in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2 U_m)$, $m \geq 3$. Then the Reeb flow on $M$ is isometric if and only if $M$ is an open part of a tube around some totally geodesic $SU_{2,m-1}/S(U_2 U_{m-1})$ in $SU_{2,m}/S(U_2 U_m)$ or a horosphere whose center at infinity is singular and of type $JN \in \mathfrak{J}N$ for a unit normal vector field $N$ of $M$.

A tube around $SU_{2,m-1}/S(U_2 U_{m-1})$ in $SU_{2,m}/S(U_2 U_m)$ is a principal orbit of the isometric action of the maximal compact subgroup $SU_{1,m+1}$ of $SU_{m+2}$, and the orbits of the Reeb flow corresponding to the orbits of the action of $U_1$. The action of $SU_{1,m+1}$ has two kinds of singular orbits. One is a totally geodesic $SU_{2,m-1}/S(U_2 U_{m-1})$ in $SU_{2,m}/S(U_2 U_m)$ and the other is a totally geodesic $\mathbb{C}H^m$ in $SU_{2,m}/S(U_2 U_m)$.

When the shape operator $A$ of $M$ in $SU_{2,m}/S(U_2 U_m)$ is Lie-parallel along the Reeb vector field $\xi$, that is $\mathcal{L}_\xi A = 0$, we say that the shape operator is Reeb invariant. The purpose of this addendum is, by Theorem A, to give a complete classification of real hypersurfaces in $SU_{2,m}/S(U_2 U_m)$ with Reeb invariant shape operator as follows:

Main Theorem. Let $M$ be a connected orientable real hypersurface in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2 U_m)$, $m \geq 3$. Then the shape operator on $M$ is Reeb invariant if and only if $M$ is an open part of a tube around
some totally geodesic $SU_{2,m-1}/S(U_2\cdot U_{m-1})$ in $SU_{2,m}/S(U_2\cdot U_m)$ or a horosphere whose center at infinity is singular and of type $JN\in\mathfrak{q}N$ for a unit normal vector field $N$ of $M$.

Moreover, related to the invariancy of shape operator, by using the result of Main Theorem, we have the following two corollaries.

**Corollary 1.** There does not exist any connected orientable real hypersurface in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2\cdot U_m)$, $m \geq 3$, with $\mathcal{F}$-invariant shape operator.

**Corollary 2.** There does not exist any connected orientable real hypersurface in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2\cdot U_m)$, $m \geq 3$, with invariant shape operator.

In previous corollaries, if the shape operator $A$ of $M$ in $SU_{2,m}/S(U_2\cdot U_m)$ satisfies a property of $\mathcal{L}_X A = 0$ on a distribution $\mathcal{F}$ defined by $\mathcal{F} = \mathcal{C}^\perp \cup \mathcal{Q}^\perp$ (or for any tangent vector field $X$ on $M$, resp.), then it is said to be $\mathcal{F}$-invariant (or invariant, resp.).

We use some references [1], [2], [3], [4], and [5] to recall the Riemannian geometry of complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2\cdot U_m)$. And some fundamental formulas related to the Codazzi and Gauss equations from the curvature tensor of complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2\cdot U_m)$ will be recalled (see [6], [7], and [8]). In this addendum we give an important Proposition 1.1 and prove our Main Theorem in section 1. Lastly, we give a brief proof for Corollaries 1 and 2 by using our Main Theorem.

### 1. Proof of the Main Theorem

In order to give a complete proof of our Main Theorem in the introduction, we need the following Key Proposition. Then by virtue of Theorem A we give a complete proof of our main theorem.

**Proposition 1.1.** Let $M$ be a real hypersurface in noncompact complex two-plane Grassmannian $SU_{2,m}/S(U_2\cdot U_m)$, $m \geq 3$. If the shape operator on $M$ is Reeb invariant, then the shape operator $A$ commutes with the structure tensor $\phi$.

**Proof.** First note that

\[
(\mathcal{L}_\xi A)X = \mathcal{L}_\xi (AX) - A\mathcal{L}_\xi X
= \nabla_\xi (AX) - \nabla_{AX} \xi - A(\nabla_\xi X - \nabla_X \xi)
= (\nabla_\xi A)X - \nabla_{AX} \xi + A\nabla_X \xi
= (\nabla_\xi A)X - \phi A^2 X + A\phi AX
\]

for any vector field $X$ on $M$. Then the assumption $\mathcal{L}_\xi A = 0$, that is, the shape operator is Reeb invariant if and only if $(\nabla_\xi A)X = \phi A^2 X - A\phi AX$. 

On the other hand, by the equation of Codazzi in [9] and the assumption of Reeb invariant, we have

\[(\nabla_X A)\xi = \phi A^2 X - A\phi AX\]

\[+ \frac{1}{2} \left[ \phi X + \sum_{\nu=1}^{3} \left\{ \eta_{\nu}(\xi) \phi_{\nu} X - \eta_{\nu}(X) \phi_{\nu} \xi + 3\eta_{\nu}(\phi X)\xi_{\nu} \right\} \right].\]

Now, let us take an orthonormal basis \(\{e_1, e_2, \cdots, e_{4m-1}\}\) for the tangent space \(T_xM, x \in M\), for \(M\) in \(SU_{2,m}/SU_{2,m}\). Then the equation of Codazzi gives

\[(\nabla_{e_i}A)X - (\nabla_X A)e_i = -\frac{1}{2} \left[ \eta(e_i)\phi X - \eta(X)\phi e_i - 2g(\phi e_i, X)\xi \right.\]

\[+ \sum_{\nu=1}^{3} \left\{ \eta_{\nu}(e_i)\phi_{\nu} X - \eta_{\nu}(X)\phi_{\nu} e_i - 2g(\phi_{\nu} e_i, X)\xi_{\nu} \right\}\]

\[+ \sum_{\nu=1}^{3} \left\{ \eta(e_i)\eta_{\nu}(\phi X) - \eta(X)\eta_{\nu}(\phi e_i) \right\} \xi_{\nu},\]

from which, together with the fundamental formulas mentioned in [9], we know that

\[\sum_{i=1}^{4m-1} g((\nabla_{ei}A)X, \phi e_i)\]

\[= (2m - 1)\eta(X) + \frac{1}{2} \sum_{\nu=1}^{3} \left\{ g(\phi_{\nu} \xi, \phi_{\nu} X) - \eta_{\nu}(X) \text{Tr}(\phi_{\nu}) \right\}\]

\[+ \frac{1}{2} \sum_{\nu=1}^{3} \left\{ g(\phi_{\nu} \phi X, \phi_{\nu} \xi) + \eta(X)g(\phi_{\nu} \xi, \phi_{\nu} \xi) \right\}\]

\[= (2m + 1)\eta(X) - \frac{1}{2} \sum_{\nu=1}^{3} \eta_{\nu}(X)\text{Tr}\phi_{\nu} - \sum_{\nu=1}^{3} \eta_{\nu}(\xi)\eta_{\nu}(X),\]

where the following formulas are used in the second equality

\[\sum_{\nu=1}^{3} g(\phi_{\nu} \xi, \phi_{\nu} X) = 3\eta(X) - \sum_{\nu=1}^{3} \eta_{\nu}(\xi)\eta_{\nu}(X),\]

\[\sum_{\nu=1}^{3} g(\phi_{\nu} \phi X, \phi_{\nu} \xi) = \sum_{\nu=1}^{3} \eta(X)\eta^2_{\nu}(\xi) - \sum_{\nu=1}^{3} \eta_{\nu}(\xi)\eta_{\nu}(X),\]

\[\sum_{\nu=1}^{3} \eta(X)g(\phi_{\nu} \xi, \phi_{\nu} \xi) = 3\eta(X) - \sum_{\nu=1}^{3} \eta^2_{\nu}(\xi)\eta(X).\]

Now let us denote by \(U\) the vector \(\nabla_{\xi} \xi = \phi A\xi\). Then using the equation \((\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi\) given in [9] and taking the derivative to the vector field \(U\) gives

\[\nabla_{ei} U = \eta(\xi) Ae_i - g(Ae_i, A\xi)\xi + \phi(\nabla_{e_i} A)\xi + \phi A(\nabla_{e_i} \xi).\]
Then naturally its divergence can be given by

\[
\text{div } U = \sum_{i=1}^{4m-1} g(\nabla_{e_i}U, e_i)
\]

(1.4)

\[
= h\eta(A\xi) - \eta(A^2\xi) - \sum_{i=1}^{4m-1} g((\nabla_{e_i}A)\xi, \phi e_i) - \sum_{i=1}^{4m-1} g(\phi Ae_i, A\phi e_i),
\]

where \(h\) denotes the trace of the shape operator of \(M\) in \(SU_{2,m}/SU_2U_m\). Now we calculate the squared norm of the tensor \(\phi A - A\phi\) as follows:

\[
\| \phi A - A\phi \|^2 = \sum_i g((\phi A - A\phi)e_i, (\phi A - A\phi)e_i)
\]

(1.5)

\[
= \sum_{i,j} g((\phi A - A\phi)e_i, e_j)g((\phi A - A\phi)e_i, e_j)
\]

\[
= \sum_{i,j} \left\{ g(\phi Ae_j, e_i) + g(\phi Ae_i, e_j) \right\} \left\{ g(\phi Ae_j, e_i) + g(\phi Ae_i, e_j) \right\}
\]

\[
= 2 \sum_{i,j} g(\phi Ae_j, e_i)g(\phi Ae_i, e_i) + 2 \sum_{i,j} g(\phi Ae_j, e_i)g(\phi Ae_i, e_j)
\]

\[
= 2 \div U - 2h\eta(A\xi) + 2 \sum_j g((\nabla_{e_j}A)\xi, \phi e_j) + 2\text{Tr}A^2,
\]

where \(\sum_i\) (respectively, \(\sum_{i,j}\)) denotes the summation from \(i = 1\) to \(4m - 1\) (respectively, from \(i, j = 1\) to \(4m - 1\)) and in the final equality we have used (1.4). From this, together with the formula (1.3), it follows that

\[
\text{div } U = \frac{1}{2} \| \phi A - A\phi \|^2 - \text{Tr}A^2
\]

(1.6)

\[
+ h\eta(A\xi) - 2(m + 1) + \frac{1}{2} \sum_{\nu=1}^{3} \eta_{\nu}(\xi)\text{Tr}\phi_{\nu} + \sum_{\nu=1}^{3} \eta_{\nu}^2(\xi).
\]

From (1.6), together with the assumption of Reeb invariant shape operator, we want to show that the structure tensor \(\phi\) commutes with the shape operator \(A\), that is, \(\phi A = A\phi\).

Let us take the inner product (1.1) with the Reeb vector field \(\xi\). Then we have

\[
g((\nabla_X A)\xi, \xi) = -g(A\phi AX, \xi) + \frac{1}{2} \sum_{\nu=1}^{3} \left\{ \eta_{\nu}(\xi)g(\phi_{\nu}X, \xi) + 3\eta_{\nu}(\phi X)g(\xi_{\nu}, \xi) \right\}
\]

(1.7)

\[
= -g(A\phi AX, \xi) + 2 \sum_{\nu=1}^{3} \eta_{\nu}(\xi)\eta_{\nu}(\phi X)
\]

\[
= g(AX, U) + 2 \sum_{\nu=1}^{3} \eta_{\nu}(\xi)\eta_{\nu}(\phi X).
\]
On the other hand, by applying the structure tensor $\phi$ to the vector field $U$, we have
\[ \phi U = \phi^2 A\xi = -A\xi + \eta(A\xi)\xi = -A\xi + \alpha\xi, \]
where the function $\alpha$ denotes $\eta(A\xi)$. From this, differentiating and using the formula $(\nabla_X\phi)Y = \eta(Y)AX - g(AX,Y)\xi$ gives
\[ (\nabla_X A)\xi = g(AX,U)\xi - \phi(\nabla_X U) - A\phi AX + (X\alpha)\xi + \alpha\phi AX. \]
Taking the inner product (1.8) with $\xi$ and using $U = \phi A\xi$ gives
\[ g((\nabla_X A)\xi, \xi) = g(AX,U) - g(A\phi AX, \xi) + (X\alpha) = 2g(AX,U) + (X\alpha). \]
Then, together with (1.7), it follows that
\[ g(AX,U) - 2\sum_{\nu=1}^{3} \eta_\nu(\xi)\eta_\nu(\phi X) + (X\alpha) = 0. \]
Substituting (1.1) and (1.9) into (1.8) gives
\[ \sum_{\nu=1}^{3} \eta_\nu(\xi)\eta_\nu(\phi X) + 3\eta_\nu(\phi X)\xi_\nu = \frac{1}{2}(m - 1) - \text{Tr}A^2 + \eta(A^2\xi) + h - \alpha^2 + \sum_{\nu=1}^{3} \eta_\nu^2(\xi) + \frac{1}{2} \sum_{\nu=1}^{3} \eta_\nu(\xi)\text{Tr}(\phi_\nu), \]
where in the first equality we have used the notion of \( \text{div} \). Then it follows that

\[
(1.12) \quad \text{div} U = -2(m + 1) - \text{Tr} A^2 + \alpha h + \sum_{\nu=1}^{3} \eta_{\nu}^2(\xi) + \frac{1}{2} \sum_{\nu=1}^{3} \eta_{\nu}(\xi) \text{Tr}(\phi \phi_{\nu}),
\]

where we have used \( \| U \|^2 = \eta(A^2 \xi) = \alpha^2 \) in (1.11).

Now if we compare (1.6) with the formula (1.12), we can assert that the squared norm \( \| \phi A - A \phi \|^2 \) vanishes, that is, the structure tensor \( \phi \) commutes with the shape operator \( A \). This completes the proof of our proposition.

Hence by Proposition 1.1 we know that the Reeb flow on \( M \) is isometric. From this, together with Theorem A we give a complete proof of our Main Theorem in the introduction.

\[\square\]

Remark 1.1. It can be easily checked that the shape operator of real hypersurfaces \( M \) in \( SU_{2,m}/S(U_2 U_m) \) is Reeb invariant, that is, \( \mathcal{L}_{\xi} A = 0 \) when \( M \) is locally congruent to a tube around some totally geodesic \( SU_{2,m-1}/S(U_2 U_{m-1}) \) in \( SU_{2,m}/S(U_2 U_m) \) or a horosphere whose center at infinity is singular and of type \( JN \subset 3N \) for a unit normal vector field \( N \) of \( M \). So the converse of our main theorem naturally holds.

2. Proof of Corollaries

From the definitions of three kinds of the invariancy of the shape operator \( A \) defined on \( M \) in the Introduction, namely invariant, \( \mathcal{F} \)-invariant and Reeb invariant shape operator, the notion of Reeb invariant is the most weakest condition. Thus from our Main Theorem, we assert that if a real hypersurface \( M \) in \( SU_{2,m}/S(U_2 U_m) \), \( m \geq 3 \), has \( \mathcal{F} \)-invariant (or invariant) shape operator, then \( M \) is locally congruent to a tube around some totally geodesic \( SU_{2,m-1}/S(U_2 U_{m-1}) \) in \( SU_{2,m}/S(U_2 U_m) \) or a horosphere whose center at infinity is singular.

Conversely, if we check whether a tube \( M_r \) of radius \( r \) around the totally geodesic \( SU_{2,m-1}/S(U_2 U_{m-1}) \) in \( SU_{2,m}/S(U_2 U_m) \) and a horosphere \( \mathcal{H} \) in \( SU_{2,m}/S(U_2 U_m) \) whose center at infinity is singular have the \( \mathcal{F} \)-invariant (or invariant) shape operator, then it does not hold. In fact, we get a contradiction for the case \( (\mathcal{L}_{\xi_3} A)\xi_3 \). From such a view point, we can assert that the shape operator \( A \) of \( M_r \) (or \( \mathcal{H} \), respectively) satisfy neither the property of \( \mathcal{F} \)-invariant nor invariant shape operator.

Summing up these discussion, we give a complete proof of our Corollaries in the introduction.

\[\square\]

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