A stochastic approach to the solution to the differential equation of heat transfer in the atmosphere

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Abstract. In this work, a model of heat transfer in the atmosphere is proposed. This model is based on a stochastic interpretation of the velocity vector components. Histograms of the wind speed distribution averaged over a relatively short time interval are obtained and analyzed. The proposed model is formulated based on empirical distributions. Explicit expressions for the first and the second-moment functions solving the heat transfer equation with random coefficients are presented. A function that estimates errors resulting from replacing a random coefficient in an equation with its mathematical expectation is also obtained. An example that demonstrates the effectiveness of the proposed approach in the case of a Gaussian distribution of the horizontal component of wind speed is presented. In this case, the first and second-moment functions in the frame of the proposed model are presented.

1. Introduction

Moisture in the atmosphere can be in different aggregative states. The formation of clouds and precipitation are connected with the transfer of moisture from one state to another. When forecasting precipitation and dangerous meteorological phenomena, the simulation of condensation and moisture transfer processes should be done at the first stage. In this regard, we note some works devoted to this topic [1-12].

Forecasting models of hazardous phenomena connected with increased water vapor content, in turn, directly include the equation of heat transfer and moisture condensation. The main factor of phase transitions of moisture in the atmosphere is the temperature which is associated with heat transfer.

The main source of heat for our planet and its components (mainly the atmosphere) is solar radiation, most of which reaches the Earth's surface. The troposphere receives heat mainly from the surface of the Earth. The following processes play a major role in the heat transfer from the surface to the atmosphere, and inside the atmosphere: a) convective and turbulent heat exchange; b) radiation and absorption processes; c) phase conversions of the water; d) molecular heat transfer [1, 2].

The air is in continuous motion. Together with the moving particles (masses) of air, the heat
content of these particles is also transferred. The resulting heat flux consists of two components: convective and turbulent. Turbulent heat flow is caused by transfer rate pulsations. Convective heat flux is due to the ordered motion of the air at an average speed. At the same time, the convective heat flux is a transfer of heat mainly in the horizontal direction and this suggestion is often used in models at the micro level [1, 2].

The heat transfer equations are widespread in hydrodynamic modeling. Mainly, the hydrodynamic models deal with averaged values of meteorological parameters. In meso- and macro-level predictive models, the averaged meteorological parameters in hydrodynamic equations are certainly justified. However, in the micro-level models, such simplifications are not quite correct, and significant fluctuations in the wind field have to be taken into account.

With non-periodic temperature changes, the process of heat transfer above the boundary layer (for relatively short time intervals) is considered adiabatically [1, 2]. In this case, the equation for the heat gain is:

\[
\frac{\partial U}{\partial t} + u \frac{\partial U}{\partial x} + v \frac{\partial U}{\partial y} = \frac{(\gamma_v - \gamma)}{\rho g} \tilde{w},
\]

where \(U\) is the air temperature; \(u, v\) are the projections of the particle velocity vector on the axis of the local coordinate system; \(\gamma_v\) is the dry adiabatic temperature gradient; \(\gamma\) is the vertical temperature gradient; \(g\) is the acceleration of gravity; \(\rho\) is the air density; \(\tilde{w}\) is an analogue of the vertical velocity in the \(p\)-coordinate system.

Expression (1) is inhomogeneous and is used in conjunction with the hydrodynamic system of equations for the motion of air masses [1-3]. In the frame of hydrodynamic models, the averaged values of the components of the velocity vector are used in equation (1). Such an approach seems to be quite justified when modeling large and medium-scale processes with a characteristic spatial scale (more than 10 kilometers). However, real meteorological values (for example, wind direction and speed), as is shown by observations, are under random chaotic perturbations, which should be taken into account considering the small-scale phenomena. The proposed approach takes into account the turbulent properties of the atmosphere using stochastic methods [13-17]. In this regard, we note the cycle of works devoted to various aspects of stochastic differential equations [18-20], and, first of all, the fundamental monograph [16], in which the main results in the field of differential equations with random parameters are systematized. In this case, it seems natural to treat the projection of the velocity vector as a random process. Figure 1 shows examples of plots corresponding to changes in the values of the projection of the velocity vector at averaging intervals of 15, 60 and 180 s (results for the projection of the velocity vector for 21 minutes are presented). Note that the spatial changes in the wind speed in the surface layer are also quite significant.

2. Statement of the problem

The proposed approach allows taking into account the turbulent properties of the atmosphere leading to processes of chaotic air pulsations. Thus, it seems natural to use stochastic methods [13-17] in the frame of this approach. In this case, the projection of the velocity vector can be treated as a random process, and the distribution law for this random process has to be determined. To simplify the calculations, vertical motions of the air are not taken into account, the \(x\)-axis is chosen in the direction of the predominant transfer of air mass and the notation \(f(x) = \frac{(\gamma_v - \gamma)}{\rho g} \tilde{w}\) is introduced. The function \(f(x)\) is considered deterministic. Thus, equation (1) will be rewritten in the form:

\[
\frac{\partial U}{\partial t} + \varepsilon(t) \frac{\partial U}{\partial x} = f(x),
\]

where \(\varepsilon(t)\) is a random process associated with the wind speed projection.

We use the deterministic initial condition:

\[
U(t_0, x) = U_0(x).
\]

The process is considered on the entire number axis \((-\infty < x < \infty)\).
Thus, this work is focused on the solution to the problem (2), where components of the velocity vector are instantaneous characteristics of the turbulent motion and are treated as random processes. The solution to this problem will allow taking into account the turbulent properties of the atmosphere at the model level.

3. **Statistical characteristics of horizontal atmospheric motions**

For further calculations, we have to establish the distribution law of the velocity vector projection. To determine the distribution law a series of experiments is carried out. In these experiments, we obtained instant data of wind direction and speed. Following the obtained data, the values of the projection of the velocity vector on the corresponding axis were calculated. Further, distribution histograms were constructed. As an example, a few histograms for different time moments are shown in Figures 2 – 4.

**Figure 1.** a) averaged values of the velocity vector projection for averaging intervals of 15 s; b) averaged values of the projection of the velocity vector for averaging intervals of 60 s; c) averaged values of the velocity vector projection for averaging intervals of 180 s.
At the initial stage of histogram analysis, estimates of the coefficients of asymmetry and kurtosis were determined. Based on the obtained values of the coefficients, a hypothesis on the normal distribution of the velocity vector projection was assumed. To approve this assumption, the Pearson method and the Kolmogorov method were used, and two levels of significance were introduced as $\alpha = 0.05$ and 0.1. The numerical test showed that the hypothesis was not rejected in 60% of cases. In other cases, this hypothesis was not accepted.
4. The solution to the heat transfer equation

We assume that a random process is determined to utilize the characteristic functional [13–15]:

$$\phi (v) = M \left[ \exp \int_0^t e(\tau) v(\tau) d\tau \right],$$  \hspace{1cm} (4)

where function $v$ belongs to space $L_1(T)$ of functions summable on the segment $T$ with the norm $\|v\| = \int_0^t |v(\tau)| d\tau$; $T$ is the period in which the process is studied $[0, t]$; $M$ is the mathematical expectation by the distribution function $e(\tau)$. In this work, we determine the first and second-moment functions of the solution to equation (2) with the deterministic initial condition (3).

One of the methods solving this problem is related to a transition to an equivalent deterministic equation based on the approach developed in [13–15] and connected to the technique of variational derivatives.

Multiplying equation (2) by $\exp(\int_0^t e(\tau) v(\tau) d\tau)$, and determining the expectation of the resulting equality, we can write:

$$M \left[ \frac{\partial U}{\partial t} \exp(\int_0^t e(\tau) v(\tau) d\tau) \right] = M \left[ -e(t) \frac{\partial U}{\partial x} \exp(\int_0^t e(\tau) v(\tau) d\tau) \right] + f(x) M \left[ \exp(\int_0^t e(\tau) v(\tau) d\tau) \right].$$  \hspace{1cm} (5)

For further calculations we introduce the mapping:

$$y(t, x, v) = M \left[ U(t, x) \exp(\int_0^t e(\tau) v(\tau) d\tau) \right],$$  \hspace{1cm} (6)

where $t \geq t_0; x \in R; v(t) \in L_1(T)$.

Taking into account the mapping (6), the resulting relation (5) is written in the form:

$$\frac{\partial y(t, x, v)}{\partial t} = i \frac{\delta}{\delta v(t)} y(t, x, v) + f(x) \phi(v).$$  \hspace{1cm} (7)

The initial condition for the equation (7), taking into account the mapping (6), takes the form:

$$y(t_0, x, v) = U_0(x) \phi(v),$$

where $U(t_0, x) \left[ \exp(\int_0^t e(\tau) v(\tau) d\tau) \right] = U_0(x) M \left[ \exp(\int_0^t e(\tau) v(\tau) d\tau) \right].$

The approach describing the solution to a differential equation in the form (7) with the initial condition (8) is described in [13–15]. Using this approach, the mathematical expectation of the solution to the problem (2) with the initial condition (3) takes the form:

$$M \left[ U(t, x) \right] = U_0(x) F_{\xi}^{-1} \left[ \phi(\xi_0 (0, t)) \right] + f(x) * F_{\xi}^{-1} \left[ \int_0^t \phi(\xi_0 (\tau, t)) d\tau \right],$$ \hspace{1cm} (9)

where $*$ is the convolution sign function by $x$; $F_{\xi}^{-1}$ is an inverse Fourier transform by $\xi$; $\xi$ is the characteristic dual to $x$; $\chi$ is the function dependent on three variables $\chi(\tau, w, t)$ the following way:

$$\chi(\tau, w, t) = \text{sign} (w - \tau) w \text{ at belonging to a segment with ends } \tau, t \ chi(w, t) = 0 \text{ and otherwise}.$$

The second moment function of the solution to the equation (2) is determined by the expression:

$$M \left[ U(t, x) U(y, x) \right] = U_0(x) F_{\xi}^{-1} \left[ \phi(\xi_0 (0, t)) + i \xi_0 (0, y) \right] + f(x) * F_{\xi}^{-1} \left[ \int_0^t \phi(\xi_0 (\tau, t)) + i \xi_0 (0, y) d\tau \right].$$ \hspace{1cm} (10)

The dispersion function $D[U(t, x)] = M[U^2(t, x)] - (M[U(t, x)])^2$ for the solution to the equation (2) is:
\[ D[U(t, x)] = U_0(x) * F^{-1}_\xi \left[ \phi(2 \xi \mathcal{X}(0, t)) \right] + f(x) * F^{-1}_\xi \left[ \int_0^t \phi(2 \xi \mathcal{X}(\tau, t)) d\tau \right] \]

\[ - U_0(x) * F^{-1}_\xi \left[ \phi(2 \xi \mathcal{X}(0, t)) \right] - f(x) * F^{-1}_\xi \left[ \int_0^t \phi(2 \xi \mathcal{X}(\tau, t)) d\tau \right]^2. \]  

\text{(11)}

Expressions (9), (10) determine the first and second-moment functions of the solution to the differential equation (2) with the initial condition (3). These expressions are obtained for a random process with the characteristic function in the form (4). The characteristic function of the random process will also change depending on the distribution law. At the next stage, we consider how statistical characteristics of the solution to the equation (2) change (taking into account the definition of the distribution law for the velocity vector projection) in the case of a Gaussian distribution.

4.1. Solution to the heat transfer equation in the case of a Gaussian distribution

One of the most important and common in technical and other applications is a Gaussian random process. The series of experiments mentioned above did not always correspond to the characteristics of this distribution (let us recall that the corresponding hypothesis was not rejected in 60% of cases and was rejected by 40% at a significant statistical level). Nevertheless, the identification of quantitative characteristics of the solution to the heat transfer equation with random coefficients seems to be quite important in the case where the corresponding coefficients are treated as a Gaussian random process.

In the case of a Gaussian distribution the characteristic function has the form:

\[ \phi(t) = \exp \left\{ \int t M [\phi(t)] d\tau - \frac{1}{2} \int t^2 b(t, t_2) d\tau_1 d\tau_2 \right\}. \]

\text{(12)}

where \( b(t_1, t_2) = M [\phi(t_1) \phi(t_2)] - M [\phi(t_1)] M [\phi(t_2)] \) is the covariance function.

To simplify further cumbersome calculations, we introduce the notation:

\[ A = \int M [\phi(t)] d\tau; \quad A_1 = \int t b(t_1, t_2) d\tau_1 d\tau_2; \quad B = \int M [\phi(t)] d\tau; \quad B_1 = \int t b(t_1, t_2) d\tau_1 d\tau_2; \]

\[ C = \int \frac{\partial b(t_1, t_2)}{\partial t_1} d\tau_1 d\tau_2; \quad D = \int M [\phi(t)] d\tau; \quad D_1 = \int t b(t_1, t_2) d\tau_1 d\tau_2; \quad E = \int \frac{\partial b(t_1, t_2)}{\partial t_1} d\tau_1 d\tau_2; \]

\[ E_1 = \int M [\phi(t)] d\tau; \quad G = \int \frac{\partial b(t_1, t_2)}{\partial t_1} d\tau_1 d\tau_2; \quad G_1 = \int b(t_1, t_2) d\tau_1 d\tau_2; \]

\[ K = \int \frac{\partial b(t_1, t_2)}{\partial t_1} d\tau_1 d\tau_2; \quad K_1 = \int M [\phi(t)] d\tau; \quad N = \int b(t_1, t_2) d\tau_1 d\tau_2; \quad N_1 = \int b(t_1, t_2) d\tau_1 d\tau_2; \]

\[ S = \int b(t_1, t_2) d\tau_1 d\tau_2. \]

Using the above-introduced notation, the mathematical expectation \( M[U(t, x)] \) of the solution to the equation (2) with the initial condition (3), is:

\[ M[U(t, x)] = U_0(x) \frac{1}{\sqrt{2\pi B_1}} \exp \left\{ \frac{(x - B)^2}{2B_1} \right\} + f(x) \frac{1}{\sqrt{2\pi B_1}} \exp \left\{ \frac{(x - B)^2}{2B_1} \right\} d\tau. \]

\text{(13)}

The second moment function of the solution to the equation (2) with a Gaussian distribution of a random process is written as:

\[ M[U(t, x)U(\tau, x_1)] = U_0(x) \frac{1}{\sqrt{\pi B_1}} \exp \left\{ \frac{(x - B)^2}{2B_1} \right\} \]

\[ \times U_0(x_1) \frac{1}{\sqrt{\pi A_1}} \exp \left\{ \frac{(x_1 - A)^2}{2A_1} \right\} \exp \left\{ i(x_1 - A) \frac{C}{A_1} \exp \left\{ \frac{C}{2A_1} \right\} (x_1) \right\}. \]
In this case, the dispersion function is:

\[
D \left[ U(t, x) \right] = \left( U_0(x) + \frac{1}{2 \pi} \int_0^x f(x) \right) \frac{1}{2 \pi} \int_0^{x_1} \left( \exp \left[ \frac{-i}{2} \left( K_i + B \right) \right] - \frac{1}{2} x^2 \left( D_i + 2 N_i + G_i \right) \right) d \tau \left( x \right)
\]

\[
+ f(x) \cdot \frac{1}{2 \pi} \int_0^x \left( \delta \left( x, A \right) \right) \exp \left[ -i x^2 \left( B + E_i \right) - \frac{1}{2} x^2 \left( B_i + 2 N_i + G_i \right) \right] d \tau \left( x \right)
\]

\[
+ \left( U_0(x) + \frac{1}{2 \pi} \int_0^{x_1} \left( \exp \left[ \frac{-i}{2} \left( K_i + B \right) \right] - \frac{1}{2} x^2 \left( D_i + 2 N_i + G_i \right) \right) d \tau \left( x \right) \right) \left( U_0(x) + \frac{1}{2 \pi} \int_0^x f(x) \right) \frac{1}{2 \pi} \int_0^{x_1} \left( \exp \left[ \frac{-i}{2} \left( K_i + B \right) \right] - \frac{1}{2} x^2 \left( D_i + 2 N_i + G_i \right) \right) d \tau \left( x \right) \right)
\]

\[
- \left( U_0(x) + \frac{1}{2 \pi} \int_0^x f(x) \right) \frac{1}{2 \pi} \int_0^{x_1} \left( \exp \left[ \frac{-i}{2} \left( K_i + B \right) \right] - \frac{1}{2} x^2 \left( D_i + 2 N_i + G_i \right) \right) d \tau \left( x \right) \right)^2
\]

Let us estimate the effect of random factors on the behavior of the system described by equations (2), (3). To do this, we replace the equation (2) by a deterministic equation

\[
\frac{\partial U}{\partial t} + M \left[ \dot{e} \right] \frac{\partial U}{\partial x} = f(x)
\]

It is easy to check that

\[
U_\theta(x) = U_0(x - \int_0^t f(x) \delta t) + \int_0^t f(x) \delta t\]

is the solution to this equation with the initial condition (3). Then, the function

\[
I(x) = \left| U_\theta(x) - M \left[ U(t, x) \right] \right|
\]

estimates the error between deterministic and stochastic cases.

Note, that when \( B_1(t) \to 0 \) expression \( \frac{1}{\sqrt{2 \pi B_1}} \exp \left\{ \frac{-(x-B)^2}{2 B_1} \right\} \) turns into the Dirac \( \delta \)-function and the corresponding function \( I(x) \to 0 \). Thus, \( B_1(t) \to 0 \) the expression (13) turns into the expression for a solution to the deterministic equation. This proves the following statement: if a random process corresponding to the horizontal component of the wind speed has a "quickly" decreasing autocorrelation function, then an approximate solution can be obtained by replacing the wind speed with its average value.
5. Conclusions
This article proposes a new model of heat transfer taking into account the effect of random factors, such as uncontrolled fluctuations of the wind speed due to turbulent processes. It is shown that at small time intervals the components of the velocity vector are random variables with a normal distribution law. In the corresponding transport equations, the velocity vector projections are treated as random processes with a given characteristic functional. Analytic expressions for the first and second-moment functions, as well as for the dispersion function are obtained, taking into account the turbulent (stochastic) properties of the atmosphere. As an example, the case of a Gaussian random process is considered. In this case, the explicit expressions for solution and moment functions are obtained. Analytic expressions that make it possible to estimate the error resulting from replacing a random coefficient in an equation with its mathematical expectation are presented.

Possible applications of the proposed approach to the problems of meteorology are primarily focused on the forecasting of weather phenomena on a small spatial-temporal scale (at distances up to several kilometers and time intervals up to several hours). At these intervals, the counting of random components in transport models seems to be quite justified, since uncontrolled fluctuations of the wind speed can make significant changes in the distribution of meteorological quantities. Let us note also that possible applications of the proposed method are not limited by meteorology since transport phenomena are widespread in various technological processes.

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