On a Proof of the Convergence Speed of a Second-order Recurrence Formula in the Arimoto-Blahut Algorithm

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Abstract

In [8] (Nakagawa et al., IEEE Trans. IT, 2021), we investigated the convergence speed of the Arimoto-Blahut algorithm. In [8], the convergence of the order $O(1/N)$ was analyzed by focusing on the second-order nonlinear recurrence formula consisting of the first- and second-order terms of the Taylor expansion of the defining function of the Arimoto-Blahut algorithm. However, in [8], an infinite number of inequalities were assumed as a “conjecture,” and proofs were given based on the conjecture. In this paper, we report a proof of the convergence of the order $O(1/N)$ for a class of channel matrices without assuming the conjecture. The correctness of the proof will be confirmed by several numerical examples.

1 Introduction

In [8], we showed that in the Arimoto-Blahut algorithm, the speed at which the input probability distribution $\lambda^N$ converges to the capacity achieving distribution $\lambda^*$ includes exponential convergence and convergence of the order $O(1/N)$. In [8], the convergence of the order $O(1/N)$ was analyzed by focusing on the second-order nonlinear recurrence formula consisting of the first- and second-order terms of the Taylor expansion of the defining function of the Arimoto-Blahut algorithm. However, in [8], an infinite number of inequalities were assumed as a “conjecture” and proofs were given based on the conjecture. In this paper, we report a proof of convergence of the order $O(1/N)$ for a class of channel matrices without assuming the conjecture. A key idea of the proof is to examine the continuous graph obtained by connecting the discrete graph of a sequence of numbers defined by the second-order recurrence formula with a line segment. The correctness of the proof will be confirmed by several numerical examples.

However, this paper still assumes some finite number of inequalities. It is a future challenge to reduce these assumptions as much as possible.

2 Arimoto-Blahut algorithm

2.1 Channel matrix and channel capacity

Consider a discrete memoryless channel $X \rightarrow Y$ with the input source $X$ and the output source $Y$. Let $\mathcal{X} = \{x_1, \ldots, x_m\}$ be the input alphabet and $\mathcal{Y} = \{y_1, \ldots, y_n\}$ be the output alphabet. The conditional probability that the output symbol $y_j$ is received when the input symbol $x_i$ is

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transmitted is denoted by \( P^i_j = P(Y = y_j | X = x_i), i = 1, \ldots, m, j = 1, \ldots, n \), and the row vector \( P^i \) is defined by \( P^i = (P^i_1, \ldots, P^i_n), i = 1, \ldots, m \). The channel matrix \( \Phi \) is defined by

\[
\Phi = \left( \begin{array}{cccc}
P^1_1 & \cdots & P^1_n \\
\vdots & \ddots & \vdots \\
P^m_1 & \cdots & P^m_n
\end{array} \right).
\]

We assume that for any \( j (j = 1, \ldots, n) \), there exists at least one \( i (i = 1, \ldots, m) \) with \( P^i_j > 0 \). This assumption means that there are no useless output symbols.

The set of input probability distributions on the input alphabet \( \mathcal{X} \) is denoted by \( \Delta(\mathcal{X}) \equiv \{ \lambda = (\lambda_1, \ldots, \lambda_m) | \lambda_i \geq 0, i = 1, \ldots, m, \sum_{i=1}^m \lambda_i = 1 \} \). The interior of \( \Delta(\mathcal{X}) \) is denoted by \( \Delta(\mathcal{X})^o \equiv \{ \lambda = (\lambda_1, \ldots, \lambda_m) \in \Delta(\mathcal{X}) | \lambda_i > 0, i = 1, \ldots, m \} \). Similarly, the set of output probability distributions on the output alphabet \( \mathcal{Y} \) is denoted by \( \Delta(\mathcal{Y}) \equiv \{ Q = (Q_1, \ldots, Q_n) | Q_j \geq 0, j = 1, \ldots, n, \sum_{j=1}^n Q_j = 1 \} \), and its interior \( \Delta(\mathcal{Y})^o \) is similarly defined as \( \Delta(\mathcal{X})^o \).

Let \( Q = \lambda \Phi \) be the output distribution for an input distribution \( \lambda \in \Delta(\mathcal{X}) \) and write its components as \( Q_j = \sum_{i=1}^m \lambda_i P^i_j, j = 1, \ldots, n \). Then, the mutual information is defined by \( I(\lambda, \Phi) = \sum_{i=1}^m \sum_{j=1}^n \lambda_i P^i_j \log \left( \frac{P^i_j}{Q_j} \right) \), where \( \log \) is the natural logarithm. The channel capacity \( C \) is defined by

\[
C = \max_{\lambda \in \Delta(\mathcal{X})} I(\lambda, \Phi).
\]

The unit of \( C \) is nat/symbol.

The Kullback-Leibler divergence \( D(Q\|Q') \) for two probability distributions \( Q = (Q_1, \ldots, Q_n), Q' = (Q'_1, \ldots, Q'_n) \) is defined [7] by

\[
D(Q\|Q') = \sum_{j=1}^n Q_j \log \frac{Q_j}{Q'_j}.
\]

An important proposition for investigating the convergence speed of the Arimoto-Blahut algorithm is the Kuhn-Tucker condition on the input distribution \( \lambda = \lambda^* \) that achieves the maximum of (2).

**Theorem A** (Kuhn-Tucker condition [6]) In the maximization problem (2), a necessary and sufficient condition for the input distribution \( \lambda^* = (\lambda^*_1, \ldots, \lambda^*_m) \in \Delta(\mathcal{X}) \) to achieve the maximum is that there is a certain constant \( \tilde{C} \) with

\[
D(P^i\|\lambda^* \Phi) \begin{cases} 
= \tilde{C}, & \text{for } i \text{ with } \lambda^*_i > 0, \\
\leq \tilde{C}, & \text{for } i \text{ with } \lambda^*_i = 0.
\end{cases}
\]

In (4), \( \tilde{C} \) is equal to the channel capacity \( C \).

Since the Kuhn-Tucker condition is a necessary and sufficient condition, all the information about the capacity-achieving input distribution \( \lambda^* \) can be derived from this condition.

### 2.2 Definition of the Arimoto-Blahut algorithm

A sequence \( \{\lambda^N = (\lambda_1^N, \ldots, \lambda_m^N)\}_{N=0,1,\ldots} \subset \Delta(\mathcal{X}) \) of input distributions is defined by the Arimoto-Blahut algorithm as follows [2], [4]. First, let \( \lambda^0 = (\lambda_1^0, \ldots, \lambda_m^0) \) be an initial distribution taken in \( \Delta(\mathcal{X})^o \), i.e., \( \lambda_i^0 > 0, i = 1, \ldots, m \). Then, the Arimoto-Blahut algorithm is
given by the recurrence formula
\[ \lambda_{i}^{N+1} = \frac{\lambda_{i}^{N} \exp D(P^i \| \lambda^{N} \Phi)}{\sum_{i' = 1}^{m} \lambda_{i'}^{N} \exp D(P^{i'} \| \lambda^{N} \Phi)} , \quad N = 0, 1, \ldots, i = 1, \ldots, m. \]  

(5)

2.3 Function from \( \Delta(\mathcal{X}) \) to \( \Delta(\mathcal{X}) \)

Let \( F_i(\lambda) \) be the defining function of the Arimoto-Blahut algorithm (5), i.e.,
\[ F_i(\lambda) = \frac{\lambda_{i} \exp D(P^i \| \lambda \Phi)}{\sum_{i' = 1}^{m} \lambda_{i'} \exp D(P^{i'} \| \lambda \Phi)} , \quad i = 1, \ldots, m. \]  

(6)

Define \( F(\lambda) \equiv (F_1(\lambda), \ldots, F_m(\lambda)) \). Then, \( F(\lambda) \) is a differentiable function from \( \Delta(\mathcal{X}) \) to \( \Delta(\mathcal{X}) \), and (5) is represented by \( \lambda_{N+1} = F(\lambda^N) \).

In this paper, for the analysis of the convergence speed, we assume \( \text{rank } \Phi = m \).

Lemma 1 \[8\] The capacity-achieving input distribution \( \lambda^* \) is the fixed point of the function \( F(\lambda) \). That is, \( F(\lambda^*) = \lambda^* \).

The sequence \( \{\lambda^N\}_{N=0,1,\ldots} \) of the Arimoto-Blahut algorithm converges to the fixed point \( \lambda^* \), i.e., \( \lambda^N \to \lambda^* \), \( N \to \infty \). We will investigate the convergence speed by using the Taylor expansion of \( F(\lambda) \) about \( \lambda = \lambda^* \).

2.4 Convergence speed of \( \lambda^N \to \lambda^* \)

Now, we define two kinds of convergence speeds for investigating \( \lambda^N \to \lambda^* \).

- Exponential convergence
  \( \lambda^N \to \lambda^* \) is the exponential convergence if
  \[ \|\lambda^N - \lambda^*\| < K \cdot (\theta)^N , \quad K > 0 , \quad 0 \leq \theta < 1 , \quad N = 0, 1, \ldots, \]  
  where \( \|\lambda\| \) denotes the Euclidean norm, i.e., \( \|\lambda\| = (\lambda_1^2 + \cdots + \lambda_m^2)^{1/2} \), and \( (\theta)^N \) denotes \( \theta \) to the power \( N \).

- Convergence of the order \( O(1/N) \)
  \( \lambda^N \to \lambda^* \) is the convergence of the order \( O(1/N) \) if
  \[ \lim_{N \to \infty} N \cdot (\lambda_i^N - \lambda_i^*) = K_i \neq 0 , \quad i = 1, \ldots, m. \]  

(9)

2.5 Type of index

Now, we classify the indices \( i (i = 1, \ldots, m) \) in the Kuhn-Tucker condition (4) in more detail into the following 3 types.

\[ D(P^i \| \lambda^* \Phi) \begin{cases} = C , & \text{for } i \text{ with } \lambda_i^* > 0 \text{ [type-I]}, \\ = C , & \text{for } i \text{ with } \lambda_i^* = 0 \text{ [type-II]}, \\ < C , & \text{for } i \text{ with } \lambda_i^* = 0 \text{ [type-III]}. \end{cases} \]  

(10)
Let us define the sets of indices as follows:

all the indices : $\mathcal{I} \equiv \{1, \ldots, m\}$, \hspace{1cm} (11)
type-I indices: $\mathcal{I}_1 \equiv \{1, \ldots, m_1\}$, \hspace{1cm} (12)
type-II indices: $\mathcal{I}_{II} \equiv \{m_1 + 1, \ldots, m_1 + m_2\}$, \hspace{1cm} (13)
type-III indices: $\mathcal{I}_{III} \equiv \{m_1 + m_2 + 1, \ldots, m\}$. \hspace{1cm} (14)

We have $|\mathcal{I}| = m$, $|\mathcal{I}_1| = m_1$, $|\mathcal{I}_{II}| = m_2$, $|\mathcal{I}_{III}| = m - m_1 - m_2 \equiv m_3$, $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_{II} \cup \mathcal{I}_{III}$ and $m = m_1 + m_2 + m_3$. For any channel matrix, $\mathcal{I}_1$ is not empty and $|\mathcal{I}_1| = m_1 \geq 2$, but $\mathcal{I}_{II}$ and $\mathcal{I}_{III}$ may be empty for some channel matrices.

### 2.6 Related Works

The previous studies [2], [3], [11] investigate the exponential convergence. In [2], [3], [11], it is proved that if the capacity achieving input distribution $\lambda^* = (\lambda^*_1, \ldots, \lambda^*_m)$ satisfies $\lambda^*_i > 0$, $i = 1 \ldots, m$, then the convergence of $\lambda^N \to \lambda^*$ is exponential. In this case, all indices are type-I and there are no type-II indices nor type-III indices. When there are no type-II indices, the speed of convergence is determined by the Jacobian matrix of the defining function $F(\lambda)$ of the Arimoto-Blahut algorithm, and the analysis is not very difficult.

On the other hand, in [8], we showed that when there are type-II indices, the convergence speed of the Arimoto-Blahut algorithm is not determined by the Jacobian matrix alone, and the Hessian matrix must also be examined. Then, the convergence of the order $O(1/N)$ was investigated using the second-order nonlinear recurrence formula obtained by the Jacobian and Hessian matrices.

### 3 Taylor expansion of $F(\lambda)$ about $\lambda = \lambda^*$

The convergence speed of the Arimoto-Blahut algorithm will be investigated by the Taylor expansion of $F(\lambda)$ about the fixed point $\lambda = \lambda^*$.

The Taylor expansion of the function $F(\lambda) = (F_1(\lambda), \ldots, F_m(\lambda))$ about $\lambda = \lambda^*$ is

$$F(\lambda) = F(\lambda^*) + (\lambda - \lambda^*)J(\lambda^*) + \frac{1}{2!}(\lambda - \lambda^*)H(\lambda^*)^{\text{t}}(\lambda - \lambda^*) + o\left(\|\lambda - \lambda^*\|^2\right), \quad (15)$$

where $^t\lambda$ is the transpose of $\lambda$. In (15), $J(\lambda^*)$ is the Jacobian matrix of $F(\lambda)$ in $\lambda = \lambda^*$, i.e.,

$$J(\lambda^*) = \left(\frac{\partial F_i}{\partial \lambda_{t'}}\bigg|_{\lambda=\lambda^*}\right)_{t', i = 1, \ldots, m}. \quad (16)$$

Further, in (15), $H(\lambda^*) \equiv (H_1(\lambda^*), \ldots, H_m(\lambda^*))$, where $H_i(\lambda^*)$ is the Hessian matrix of $F_i(\lambda)$ at $\lambda = \lambda^*$, i.e.,

$$H_i(\lambda^*) = \left(\frac{\partial^2 F_i}{\partial \lambda_{t'} \partial \lambda_{t''}}\bigg|_{\lambda=\lambda^*}\right)_{t', i'' = 1, \ldots, m}, \quad (17)$$

and $(\lambda - \lambda^*)H(\lambda^*)^{\text{t}}(\lambda - \lambda^*)$ is an abbreviated expression of the $m$-dimensional row vector

$$((\lambda - \lambda^*)H_1(\lambda^*)^{\text{t}}(\lambda - \lambda^*), \ldots, (\lambda - \lambda^*)H_m(\lambda^*)^{\text{t}}(\lambda - \lambda^*)) \cdot (18)$$
In (15), put \( \lambda = \lambda^N, \mu^N \equiv \lambda^N - \lambda^* \), then from \( F(\lambda^*) = \lambda^* \) and \( F(\lambda^N) = \lambda^{N+1} \), we have
\[
\mu^{N+1} = \mu^N J(\lambda^*) + \frac{1}{2!} \mu^N H(\lambda^*)^t \mu^N + o(\|\mu^N \|^2). \tag{19}
\]

From \( \lambda^N \to \lambda^* \) we have \( \mu^N \to 0 \). The convergence of \( \mu^N \to 0, N \to \infty \) is analyzed based on the Taylor expansion (19). We write the components of \( \mu^N \) as \( \mu^N = (\mu_1^N, \ldots, \mu_m^N) \), then \( \mu_i^N = \lambda_i^N - \lambda_i^*, i = 1, \ldots, m \).

### 3.1 Eigenvalues of the Jacobian matrix \( J(\lambda^*) \)

The Jacobian matrix has the following form [8]:
\[
J(\lambda^*) \equiv \begin{pmatrix} J^I & O & O \\ \ast & J^II & O \\ \ast & O & J^III \end{pmatrix}, \tag{20}
\]
\[
J^I \equiv (\partial F_i / \partial \lambda_{i'} |_{\lambda=\lambda^*})_{i,i' \in I_I} \in \mathbb{R}^{m_1 \times m_1}, \\
J^II \equiv (\partial F_i / \partial \lambda_{i'} |_{\lambda=\lambda^*})_{i,i' \in I_{II}} \in \mathbb{R}^{m_2 \times m_2}, \\
J^III \equiv (\partial F_i / \partial \lambda_{i'} |_{\lambda=\lambda^*})_{i,i' \in I_{III}} \in \mathbb{R}^{m_3 \times m_3},
\]

\( O \) denotes the all-zero matrix of appropriate size, 
\( \ast \) denotes an appropriate matrix.

Let \( J(\lambda^*) \) have eigenvalues \( \{\theta_1, \ldots, \theta_m\} \equiv \{\theta_i | i \in I_I\} \), then from (20), the eigenvalues of \( J^I, J^II, J^III \) are \( \{\theta_i | i \in I_I\}, \{\theta_i | i \in I_{II}\} \{\theta_i | i \in I_{III}\} \), respectively. For each eigenvalue, the following evaluations are obtained.

**Theorem 1** [8] The eigenvalues of \( J^I \) satisfy \( 0 \leq \theta_i < 1, i \in I_I \). The eigenvalues of \( J^II \) satisfy \( \theta_i = 1, i \in I_{II} \), more precisely, \( J^II = I \) (the identity matrix) \( \in \mathbb{R}^{m_2 \times m_2} \). The eigenvalues of \( J^III \) satisfy \( 0 < \theta_i < 1, i \in I_{III} \).

### 4 Analysis of the convergence speed of the order \( O(1/N) \)

From Theorem 1, when there is no type-II index, i.e., \( I_{II} = \emptyset \), all the eigenvalues of \( J(\lambda^*) \) satisfy \( 0 \leq \theta_i < 1 \), hence the speed of convergence of the recurrence formula \( \lambda^{N+1} = F(\lambda^N) \) is determined only by \( J(\lambda^*) \), and it is an exponential convergence [2, 3, 8, 11].

In contrast, when \( I_{II} \neq \emptyset \), the maximum eigenvalue of \( J(\lambda^*) \) is 1, so the convergence speed is not determined by \( J(\lambda^*) \) alone, and the Hessian matrix \( H(\lambda^*) \) must be analyzed.

Therefore, in the following, we assume
\[
I_{II} \neq \emptyset, \tag{21}
\]
and consider the convergence speed of the order \( O(1/N) \).

Now, we consider the types-I and -III indices together and reorder the indices so that \( I_I \cup I_{III} = \{1, \ldots, m'\} \) and \( I_{II} = \{m'+1, \ldots, m\} \). We have \( m_2 = m - m' \) and \( |I_{II}| = m_2 \). We define \( \mu^N_{I_{III}} \equiv (\mu_1^N, \ldots, \mu_m^N) \) by \( \mu^N_{II} \equiv (\mu_{m'+1}^N, \ldots, \mu_m^N) \).

In the Jacobian matrix (20), by changing the order of \( J^II \) and \( J^III \), we have
\[
J(\lambda^*) = \begin{pmatrix} J^I & O & O \\ \ast & J^III & O \\ \ast & O & J^II \end{pmatrix}. \tag{22}
\]
Then, by defining
\[ J' \equiv \begin{pmatrix} J^I & O \\ * & J^{	ext{III}} \end{pmatrix}, \] (23)
we have
\[ J(\lambda^*) = \begin{pmatrix} J' & O \\ * & J^{	ext{II}} \end{pmatrix}. \] (24)

By Theorem 1, the eigenvalues of \( J(\lambda^*) \) are \( \theta_i \), \( i = 1, \ldots, m \), and then the eigenvalues of \( J' \) are \( \theta_i \), \( i = 1, \ldots, m' \) with \( 0 \leq \theta_i < 1 \), and \( J^{	ext{II}} = I \in \mathbb{R}^{m_2 \times m_2} \).

### 4.1 Second-order recurrence formula of \( \mu^N_{\text{II}} \)

Now, let \( a_i \) be a right eigenvector of \( J(\lambda^*) \) for \( \theta_i \) and define
\[ A \equiv (a_1, \ldots, a_m) \in \mathbb{R}^{m \times m}. \] (25)

\( J(\lambda^*) \) is diagonalizable under the assumption that \( \theta_i \neq \theta_{i'} \) for \( i \in I_1 \) and \( i' \in I_{\text{III}} \) (see Lemma 5 in [8]). Then, by choosing the eigenvectors \( a_1, \ldots, a_m \) appropriately, we can make \( A \) a regular matrix (see [10], p.161, Example 4).

For \( i = m' + 1, \ldots, m \), define
\[ e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^m. \] (26)

Then, because \( J^{	ext{II}} = I \), we can take
\[ a_i = e_i, \ i = m' + 1, \ldots, m. \] (27)

Therefore, we have
\[ A = \begin{pmatrix} a_1, \ldots, a_{m'}, e_{m'+1}, \ldots, e_m \end{pmatrix} \] (28)
\[ = \begin{pmatrix} a_{11} & \ldots & a_{m'1} & \vdots & \vdots \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{1m'} & \ldots & a_{m'm'} & \ddots & \ddots \\ a_{1,m'+1} & \ldots & a_{m',m'+1} & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ a_{1m} & \ldots & a_{m'm} & 0 & 1 \end{pmatrix} \] (29)
\[ \equiv \begin{pmatrix} A_1 & O \\ A_2 & I \end{pmatrix}, \] (30)

where
\[ A_1 \equiv \begin{pmatrix} a_{11} & \ldots & a_{m'1} \\ \vdots & \ddots & \vdots \\ a_{1m'} & \ldots & a_{m'm'} \end{pmatrix} \in \mathbb{R}^{m' \times m'}, \] (31)
\[ A_2 \equiv \begin{pmatrix} a_{1,m'+1} & \ldots & a_{m',m'+1} \\ \vdots & \ddots & \vdots \\ a_{1m} & \ldots & a_{m'm} \end{pmatrix} \in \mathbb{R}^{m_2 \times m'}. \] (32)
Because $A$ is regular, $A_1$ is also regular by (30). Then, under the assumptions in [8], we have
\[ \mu_{\mathit{i, III}}^N = -\mu_{\mathit{II}}^N A_2 A_1^{-1}. \] (33)

See Eq. (99) in [8].

**Remark 1** $A, A_1, A_2$ depend on the choice of eigenvectors, but $A_2 A_1^{-1}$ does not depend on their choice.

Consider the following recurrence formula consisting of the first- and second-order terms of the Taylor expansion (19):
\[ \mu_{\mathit{N+1}} = \mu_{\mathit{N}}(\lambda^*) + \frac{1}{2!} \mu_{\mathit{N}}^2 H(\lambda^*) \mu_{\mathit{N}}. \] (34)

From (33), (34) and the Hessian matrix which is calculated in [8], the following recurrence formula for $\mu_{\mathit{II}}^N$ is obtained.

**Theorem 2** (Theorem 7 in [8]) $\mu_{\mathit{II}}^N = (\mu_{m'+1}^N, \ldots, \mu_{m}^N)$ satisfies a second-order recurrence formula of the following form:
\[ \mu_{\mathit{i+1}}^N = \mu_{\mathit{i}}^N + \mu_{\mathit{i}}^N \sum_{i' = m'+1}^{m} r_{ii'} \mu_{i'}^N, \ i = m'+1, \ldots, m, \] (35)

where $r_{ii'}$ are determined by $J(\lambda^*), A_1, A_2,$ and $H(\lambda^*)$.

The purpose of the study is to show that the sequence $\{\mu_{\mathit{i}}^N\}_{N=0,1,\ldots}$ defined by (35) converges to 0 at the speed of the order $O(1/N)$ and to calculate $\lim_{N \to \infty} N \mu_{\mathit{i}}^N$. To do so, we will reformulate (35) in a canonical form and we make some assumptions.

### 4.2 Canonical form of (35) and some assumptions

About (35), define $R = (r_{ii'}), i', i = m'+1, \ldots, m, \ 1 \equiv (1, \ldots, 1) \in \mathbb{R}^{m_2}, (m_2 = m - m').$ Now, we assume the following assumptions (i)-(iv) on $R$:

(i) $r_{ii'} \leq 0, \ i, i' = m' + 1, \ldots, m,$

(ii) $R$ is a regular matrix,

(iii) for $\sigma = -1(t R)^{-1}$ with $\sigma = (\sigma_{m'+1}, \ldots, \sigma_{m}),$ we have $\sigma_i > 0, \ i = m' + 1, \ldots, m,$

(iv) $-r_{ii'} \sigma_i > 1/2, \ i = m' + 1, \ldots, m.$

Let us define $\nu_{\mathit{i}}^N \equiv \mu_{\mathit{i}}^N/\sigma_i, \ i = m'+1, \ldots, m,$ $q_{ii'} \equiv -r_{ii'} \sigma_{i'}, \ i, i' = m'+1, \ldots, m,$ then
\[ \nu_{\mathit{i+1}}^N = \nu_{\mathit{i}}^N - \nu_{\mathit{i}}^N \sum_{i' = 1}^{m} q_{ii'} \nu_{i'}^N, \ i = m'+1, \ldots, m, \] (36)
\[ q_{ii'} \geq 0, \ i, i' = m' + 1, \ldots, m, \] (37)
\[ \sum_{i' = m'+1}^{m} q_{ii'} = 1, \ i = m'+1, \ldots, m, \] (38)
\[ q_{ii} > 1/2, \ i = m'+1, \ldots, m. \] (39)
From (37) and (38), $q_i \equiv (q_{i,m+1}, \ldots, q_{im})$ is a probability vector. We call (36)-(39) a canonical form of (35) under the assumptions (i)-(iv).

In [8], in addition to the above assumptions (i)-(iv), we assumed, through a lot of numerical examples, the inequalities
\[ \nu^N_{m+1} \geq \cdots \geq \nu^N_m, \quad \exists N_0 \geq 0, \forall N \geq N_0, \tag{40} \]
by reordering the indices in $I_{II}$ if necessary. In [8], we called (40) a conjecture. However, these are an infinite number of inequalities and cannot be checked numerically. In this paper, we will prove the convergence speed of $\mu^N \to 0$ without assuming (40), but assuming only (i)-(iv).

5 Second-order simultaneous recurrence formulas

(36)-(39) are complicated, so for simplicity, we will change the variable names and indices in (36)-(39) as follows, and add an initial condition.

Consider the following second-order simultaneous recurrence formulas for $h$ variables $z_1^N, \ldots, z_h^N$.

For $i = 1, \ldots, h$,
\[ z_{i}^{N+1} = z_i^N - z_i^N \sum_{i'=1}^{h} q_{i'i}^N z_{i'}^N, \tag{41} \]
initial condition: $0 < z_i^0 \leq 1/2$, \tag{42}
$q_i \equiv (q_{i1}, \ldots, q_{ih})$ is a probability vector, \tag{43}
diagonal dominance condition: $q_{ii} > 1/2$. \tag{44}

Now, let us define
\[ w_i^N \equiv \sum_{i'=1}^{h} q_{i'i}^N z_{i'}^N, \quad N = 0, 1, \ldots, i = 1, \ldots, h. \tag{45} \]

Then, by (45), we can write (41) as
\[ z_i^N = z_i^N - z_i^N w_i^N, \quad i = 1, \ldots, h. \tag{46} \]

For each $N \geq 0$, let us define $\bar{i}(N)$ as the index that achieves $\max(z_1^N, \ldots, z_h^N)$ and put $\bar{Z}^N \equiv z^N_{\bar{i}(N)}$, i.e.,
\[ \bar{Z}^N = z^N_{\bar{i}(N)} = \max(z_1^N, \ldots, z_h^N), \tag{47} \]
Similarly, for $N \geq 0$, define $\underline{i}(N)$ as the index that achieves $\min(z_1^N, \ldots, z_h^N)$ and put $\underline{Z}^N \equiv z^N_{\underline{i}(N)}$, i.e.,
\[ \underline{Z}^N = z^N_{\underline{i}(N)} = \min(z_1^N, \ldots, z_h^N). \tag{48} \]

5.1 Convergence speed of the Order $O(1/N)$

The purpose of the study is to prove the following Theorem 3 for $\{z_i^N\}$ that satisfies (41)-(44).

**Theorem 3** \[ \lim_{N \to \infty} N z_i^N = 1, \quad i = 1, \ldots, h. \]
To that end, we prove the following Theorem 4.

**Theorem 4** \( \lim_{N \to \infty} NZ^N = 1, \lim_{N \to \infty} NZ_N = 1. \)

Theorem 3 follows from Theorem 4 and the squeeze theorem.

Now, let us prove Theorem 4.

**Lemma 2** \( 0 < z_i^N \leq 1/2, \ N = 0, 1, \ldots, \ i = 1, \ldots, h. \)

**Proof:** We prove it by mathematical induction. For \( N = 0 \), the assertion holds by (42). Assuming that the assertion holds for \( N \) and noting \( 1/2 \leq 1 - \sum_{i'}^h q_{ii'} z_{i'}^N < 1 \) by (43), we have \( 0 < z_i^{N+1} \leq 1/2. \) \( \blacksquare \)

**Lemma 3** \( 0 < w_i^N \leq 1/2, \ N = 0, 1, \ldots, \ i = 1, \ldots, h. \)

**Proof:** The assertion holds by (45), (43) and Lemma 2. \( \blacksquare \)

**Lemma 4** For \( i = 1, \ldots, h \), the sequence \( \{z_i^N\}, \ N = 0, 1, \ldots \) is strictly decreasing.

**Proof:** Because \( z_i^N - z_i^{N+1} = z_i^N \sum_{i'}^h q_{ii'} z_{i'}^N > 0 \) holds by Lemma 2. \( \blacksquare \)

**Lemma 5** \( \lim_{N \to \infty} z_i^N = 0, \ i = 1, \ldots, h. \)

**Proof:** \( z_i^\infty \equiv \lim_{N \to \infty} z_i^N \geq 0 \) exists by Lemma 2 and Lemma 4. Then, \( z_i^\infty = z_i^\infty - z_i^\infty \sum_{i'}^h q_{ii'} z_{i'}^\infty \) holds by (41), hence we have \( z_i^\infty = 0 \) by (44). \( \blacksquare \)

We will use the following theorem in our proof.

**Theorem B** ([1], p.37, Exercise 2) If \( \lim_{N \to \infty} a_N = a \), then \( \lim_{N \to \infty} \frac{1}{N} (a_0 + a_1 + \cdots + a_{N-1}) = a. \)

**Lemma 6** \( \liminf_{N \to \infty} N\overline{Z}^N \geq 1. \)

**Proof:** We have

\[
\overline{Z}^{N+1} = \frac{z_i^{N+1}}{\overline{Z}(N+1)} \\
\geq \frac{z_i^{N+1}}{\overline{Z}(N)} \\
= z_i^N - \frac{q_{\overline{T}(N)} z_{i'}^N}{\overline{Z}(N)} \\
\geq z_i^N - \sum_{i'}^h q_{i'i'} z_{i'}^N \\
= \overline{Z}^N - \left(\overline{Z}^N\right)^2 \quad \text{(by (43)).}
\]
From (53),

\[
\frac{1}{N} \left( \frac{1}{Z_{N}} - \frac{1}{Z_{0}} \right) = \frac{1}{N} \sum_{l=0}^{N-1} \left( \frac{1}{Z_{l+1}} - \frac{1}{Z_{l}} \right)
\]

(54)

\[
\leq \frac{1}{N} \sum_{l=0}^{N-1} \frac{1}{1 - Z_{l}}.
\]

(55)

Therefore,

\[
\limsup_{N \to \infty} \frac{1}{NZ_{N}} \leq \limsup_{N \to \infty} \frac{1}{N} \sum_{l=0}^{N-1} \frac{1}{1 - Z_{l}}
\]

(56)

\[
= \lim_{N \to \infty} \frac{1}{N} \sum_{l=0}^{N-1} \frac{1}{1 - Z_{l}}
\]

(57)

\[
= 1,
\]

(58)

where (57) and (58) follow from Lemma 5 and Theorem B.

\[\square\]

**Lemma 7** \( \limsup_{N \to \infty} NZ_{N} \leq 1. \)

**Proof:** We have

\[
Z_{N+1} = z_{\frac{N+1}{2}} z_{\frac{N}{2}} (N+1) \leq z_{\frac{N+1}{2}} z_{\frac{N}{2}} (N+1)
\]

(59)

\[
\leq z_{\frac{N+1}{2}} z_{\frac{N}{2}} \sum_{l=1}^{h} \frac{N}{2} \frac{N}{2} = z_{\frac{N}{2}} z_{\frac{N}{2}}^N
\]

(60)

\[
\leq z_{\frac{N}{2}} z_{\frac{N}{2}} \sum_{l=1}^{h} \frac{N}{2} \frac{N}{2} = z_{\frac{N}{2}} z_{\frac{N}{2}}^N
\]

(61)

\[
= Z_{N} - (Z_{N})^2 \text{ (by (43)).}
\]

(62)

From (63),

\[
\frac{1}{N} \left( \frac{1}{Z_{N}} - \frac{1}{Z_{0}} \right) = \frac{1}{N} \sum_{l=0}^{N-1} \left( \frac{1}{Z_{l+1}} - \frac{1}{Z_{l}} \right)
\]

(64)

\[
\geq \frac{1}{N} \sum_{l=0}^{N-1} \frac{1}{1 - Z_{l}}.
\]

(65)

Therefore,

\[
\liminf_{N \to \infty} \frac{1}{NZ_{N}} \geq \liminf_{N \to \infty} \frac{1}{N} \sum_{l=0}^{N-1} \frac{1}{1 - Z_{l}}
\]

(66)

\[
= \lim_{N \to \infty} \frac{1}{N} \sum_{l=0}^{N-1} \frac{1}{1 - Z_{l}}
\]

(67)

\[
= 1,
\]

(68)
where (67) and (68) follow from Lemma 5 and Theorem B.

Lemma 8 For any \( i = 1, \ldots, h \) and \( N \geq 1 \), we have

\[
    z_i^N < \frac{2}{N}. \tag{69}
\]

Proof: We will prove it by mathematical induction. First, by the initial condition (42) and Lemma 4, we have

\[
    z_i^2 < z_i^1 < z_i^0 \leq \frac{1}{2}, \tag{70}
\]

hence (69) holds for \( N = 1, 2 \).

Next, assume that (69) holds for \( N \geq 2 \). Because the function \( z - (1/2)z^2 \) is monotonically increasing for \( 0 < z \leq 1 \),

\[
    z_i^{N+1} = z_i^N - q_{ii} z_i^N \sum_{i'=1}^{h} q_{ii'} z_i^{N'} \leq z_i^N - q_{ii} (z_i^N)^2 \leq z_i^N - \frac{1}{2} (z_i^N)^2 \tag{71}
\]

(by (44))

\[
    < z_i^N - \frac{1}{2} \left( \frac{2}{N} \right)^2 \tag{72}
\]

\[
    < \frac{2}{N+1}. \tag{73}
\]

thus, (69) holds for \( N + 1 \).

5.2 Connect \( z_i^N \) and \( z_i^{N+1} \) with a line segment

For \( t \geq 0 \), we define \( z_i(t) \) as the polyline obtained by connecting \( z_i^0, z_i^1, \ldots, z_i^N, \ldots \) with line segments, i.e.,

\[
    z_i(t) \equiv (N + 1 - t) z_i^N + (t - N) z_i^{N+1}, \text{ for } N \leq t \leq N + 1, \ i = 1, \ldots, h. \tag{74}
\]

In particular,

\[
    z_i(N) = z_i^N, \ z_i(N + 1) = z_i^{N+1}. \tag{75}
\]

Lemma 9 \( z_i(t) \) is strictly decreasing, \( i = 1, \ldots, h \).

Proof: The assertion holds by Lemma 4 and (76).

Now, let us define

\[
    w_i(t) \equiv \sum_{i'=1}^{h} q_{ii'} z_{i'}(t), \ 0 \leq t < \infty, \ i = 1, \ldots, h. \tag{76}
\]

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Next, for $t \geq 0$, let us define $\bar{t}(t)$ as the index that achieves $\max(z_1(t), \ldots, z_h(t))$ and put $Z(t) \equiv z_{\bar{t}(t)}(t)$, i.e.,

$$Z(t) = z_{\bar{t}(t)}(t) = \max(z_1(t), \ldots, z_h(t)). \quad (79)$$

Similarly, for $t \geq 0$, define $\bar{i}(t)$ as the index that achieves $\min(z_1(t), \ldots, z_h(t))$ and put $Z(t) \equiv z_{\bar{i}(t)}(t)$, i.e.,

$$Z(t) = z_{\bar{i}(t)}(t) = \min(z_1(t), \ldots, z_h(t)). \quad (80)$$

In particular,

$$Z(N) = Z^N, \quad Z(N) = Z^N. \quad (81)$$

**Lemma 10** There exists $L_1 > 0$ such that for $t \geq 0$,

$$Z(t) - Z(t) \leq L_1 \left( w_{\bar{t}(t)}(t) - w_{\bar{i}(t)}(t) \right). \quad (82)$$

**Proof:** Because $t$ is fixed, we write $\bar{t} \equiv \bar{t}(t)$ and $\bar{i} \equiv \bar{i}(t)$ for the sake of simplicity. Then, we have

$$w_{\bar{t}(t)}(t) - w_{\bar{i}(t)}(t) = w_{\bar{t}}(t) - w_{\bar{i}}(t) \quad (83)$$

$$= \sum_{i' = 1}^h q_{i'} z_{i'}(t) - \sum_{i' = 1}^h q_{i'} z_{i'}(t) \quad (84)$$

$$= q_{\bar{t}} z_{\bar{t}}(t) + \sum_{i' \neq \bar{t}} q_{i'} z_{i'}(t) - q_{\bar{i}} z_{\bar{i}}(t) - \sum_{i' \neq \bar{i}} q_{i'} z_{i'}(t) \quad (85)$$

$$\geq q_{\bar{t}} z_{\bar{t}}(t) + \sum_{i' \neq \bar{t}} q_{i'} z_{i'}(t) - q_{\bar{i}} z_{\bar{i}}(t) - \sum_{i' \neq \bar{i}} q_{i'} z_{i'}(t) \quad (86)$$

$$= q_{\bar{t}} Z(t) + (1 - q_{\bar{t}}) Z(t) - q_{\bar{i}} Z(t) - (1 - q_{\bar{i}}) Z(t) \quad (87)$$

$$= (q_{\bar{t}} + q_{\bar{i}} - 1) (Z(t) - Z(t)). \quad (88)$$

Here, put

$$\ell \equiv \min_{i, i'} (q_{ii} + q_{i'i'} - 1), \quad (89)$$

then $\ell > 0$ by the assumption (44). Hence, the assertion of the Lemma holds by putting $L_1 = \ell^{-1}$.  \hfill \blacksquare

**Lemma 11** $Z(t)$ and $Z(t)$ are strictly decreasing.

**Proof:** For every $i$, $z_i(t)$ is strictly decreasing by Lemma 9, hence for $t_1 < t_2$ we have $Z(t_1) = z_{\bar{t}(t_1)}(t_1) \geq z_{\bar{t}(t_2)}(t_1) > z_{\bar{t}(t_2)}(t_2) = Z(t_2)$. Similarly for $Z(t)$.  \hfill \blacksquare
Lemma 12 \( \sum_{N=0}^{\infty} \left( Z^N - Z^N \right) < \infty \) is equivalent to \( \int_0^{\infty} \{ Z(t) - Z(t) \} \, dt < \infty \).

Proof: For \( N \leq t \leq N + 1, \, N = 0, 1, \ldots \), we have by Lemma 11 and (81),

\[
Z^N \geq Z(t) \geq Z^{N+1},
\]

(90)

\[
Z^N \geq Z(t) \geq Z^{N+1}.
\]

(91)

By integrating (90) and (91) on the interval \([N, N + 1]\), we have

\[
Z^N = \int_N^{N+1} Z^N \, dt \geq \int_N^{N+1} Z(t) \, dt \geq \int_N^{N+1} Z^{N+1} \, dt = Z^{N+1},
\]

(92)

\[
Z^N = \int_N^{N+1} Z^N \, dt \geq \int_N^{N+1} Z(t) \, dt \geq \int_N^{N+1} Z^{N+1} \, dt = Z^{N+1}.
\]

(93)

Then by (92), (93), we have

\[
Z^N \geq \int_N^{N+1} Z(t) \, dt \geq Z^{N+1},
\]

(94)

\[-Z^{N+1} \geq -\int_N^{N+1} Z(t) \, dt \geq -Z^N.\]

(95)

Adding the both sides of (94) and (95), we have

\[
Z^N - Z^{N+1} \geq \int_N^{N+1} \{ Z(t) - Z(t) \} \, dt \geq Z^{N+1} - Z^N.
\]

(96)

Taking summation of the both sides of (96) for \( N = 0, 1, \ldots, N' \), we have, by a simple calculation,

\[
Z^0 + \sum_{N=0}^{N'} \left( Z^N - Z^N \right) \geq \sum_{N=0}^{N'} \int_N^{N+1} \{ Z(t) - Z(t) \} \, dt \geq \sum_{N=0}^{N'} \left( Z^N - Z^N \right) + Z^{N+1}.
\]

(97)

\[
\geq -Z^0 + \sum_{N=0}^{N'} \left( Z^N - Z^N \right) + Z^{N+1}.
\]

(98)

From (97), (98) and Lemma 2, we have

\[
\frac{1}{2} + \sum_{N=0}^{N'} \left( Z^N - Z^N \right) \geq \int_0^{N+1} \{ Z(t) - Z(t) \} \, dt \geq \frac{1}{2} + \sum_{N=0}^{N'} \left( Z^N - Z^N \right),
\]

(99)

hence the assertion of the Lemma holds. \( \blacksquare \)

Lemma 13 There exists \( L_2 > 0 \) such that for \( t > 1 \)

\[
-\frac{\dot{z}_i(t)}{z_i(t)} - \frac{L_2}{(t-1)^2} < w_i(t) < -\frac{\dot{z}_i(t)}{z_i(t)},
\]

(100)

where \( \dot{z}_i(t) \) denotes the derivative of \( z_i(t) \) by \( t \).
Proof: For $N < t < N + 1$,

\[
\dot{z}_i(t) = \text{the slope of } z_i(t) \tag{101}
\]

\[
= \frac{z_i^{N+1} - z_i^N}{(N + 1) - N} \tag{102}
\]

\[
= z_i^{N+1} - z_i^N \tag{103}
\]

\[
= -z_i^N \sum_{i' = 1}^{h} q_{ii'} z_{i'}^N \tag{104}
\]

\[
< -z_i(t) \sum_{i' = 1}^{h} q_{ii'} z_{i'}(t) \quad \text{(by (77) and Lemma 9)} \tag{105}
\]

\[
= -z_i(t) w_i(t), \tag{106}
\]

then, we have

\[
w_i(t) < -\frac{\dot{z}_i(t)}{z_i(t)} \tag{107}
\]

Next, we bound $w_i(t)$ from below. Because $z_i(t)$ is decreasing, we have $z_i(t) > z_i^{N+1}$ for $N < t < N + 1$. Therefore,

\[
z_i^N < z_i(t) + z_i^N - z_i^{N+1}, \quad i = 1, \ldots, h. \tag{108}
\]

Then, by (104) and (108),

\[
\dot{z}_i(t) > -\left\{z_i(t) + z_i^N - z_i^{N+1}\right\} \sum_{i' = 1}^{h} q_{ii'} \left\{z_{i'}(t) + z_{i'}^N - z_{i'}^{N+1}\right\} \tag{109}
\]

\[
= -z_i(t) w_i(t) - \left[z_i(t) \sum_{i' = 1}^{h} q_{ii'} \left(z_{i'}^N - z_{i'}^{N+1}\right) + \left(z_i^N - z_i^{N+1}\right) \sum_{i' = 1}^{h} q_{ii'} \left(z_{i'}(t) + z_{i'}^N - z_{i'}^{N+1}\right)\right]. \tag{110}
\]
hence,

\[
\begin{align*}
\frac{w_i(t)}{z_i(t)} & > -\frac{\dot{z}_i(t)}{z_i(t)} - \frac{1}{h}\sum_{i'=1}^{h} q_{ii'} (z_{i'}^N - z_{i'}^{N+1}) + \frac{z_i^N - z_i^{N+1}}{z_i(t)} \sum_{i'=1}^{h} q_{ii'} \left\{ z_{i'}(t) + z_{i'}^N - z_{i'}^{N+1} \right\} \\
\frac{w_i(t)}{z_i(t)} & > -\frac{\dot{z}_i(t)}{z_i(t)} - \frac{1}{h}\sum_{i'=1}^{h} q_{ii'} (z_{i'}^N - z_{i'}^{N+1}) + \frac{z_i^N - z_i^{N+1}}{z_i(t)} \sum_{i'=1}^{h} q_{ii'} \left\{ z_{i'}(t) + z_{i'}^N - z_{i'}^{N+1} \right\} \\
&= -\frac{\dot{z}_i(t)}{z_i(t)} - \frac{1}{h}\sum_{i'=1}^{h} q_{ii'} z_{i'}^N w_{i'}^N + \frac{w_i^N}{1 - w_i^N} \sum_{i'=1}^{h} q_{ii'} \left\{ z_{i'}(t) + z_{i'}^N w_{i'}^N \right\} \\
&> -\frac{\dot{z}_i(t)}{z_i(t)} - \frac{1}{h}\sum_{i'=1}^{h} q_{ii'} z_{i'}^N w_{i'}^N + \frac{2w_i^N}{1 - w_i^N} \sum_{i'=1}^{h} q_{ii'} \left\{ z_{i'}(t) + z_{i'}^N w_{i'}^N \right\} \\
&= -\frac{\dot{z}_i(t)}{z_i(t)} - \frac{L_2}{N^2} (\exists L_2 > 0, \text{ by Lemma 8}) \\
&> -\frac{\dot{\bar{z}}(t)}{\bar{z}(t)} - \frac{L_2}{(t-1)^2}. 
\end{align*}
\]

\ref{5.3} The values of \( t \) at which the slope of \( \bar{Z}(t) \) or \( \bar{Z}(t) \) changes

Let \( t_1, t_2, \ldots, t_k, \ldots \) be the values of \( t \) at which the slope of \( \bar{Z}(t) \) or \( \bar{Z}(t) \) changes among all \( 0 < t < \infty \), that is, \( \bar{Z}(t) \) or \( \bar{Z}(t) \) is discontinuous at \( t_1, t_2, \ldots, t_k, \ldots \). See Fig. 1. We put \( t_0 = 0 \) for convenience. Let \( \mathcal{T} \) be the collection of \( t_k \), i.e., \( \mathcal{T} = \{t_k\}_{k=0,1,2,\ldots} \).

When \( t \) is a positive integer, both the slope of \( \bar{Z}(t) \) and \( \bar{Z}(t) \) change, so \( t = 1, 2, \ldots \) are included in \( \mathcal{T} \). These values are shown in Fig. 1 by black squares. Furthermore, the values of \( t \) at which \( \bar{\bar{z}}(t) \) that achieves \( \bar{Z}(t) \) or \( \bar{\bar{z}}(t) \) that achieves \( \bar{Z}(t) \) changes are also included in \( \mathcal{T} \). These values are shown in Fig. 1 by black circles.
Figure 1: The values of $t$ at which the slope of $\bar{Z}(t)$ or $\underline{Z}(t)$ changes
Lemma 14 \[ \int_0^\infty \{ \overline{Z}(t) - \underline{Z}(t) \} \, dt < \infty. \]

**Proof:** By Lemma 10 and Lemma 13,
\[
\overline{Z}(t) - \underline{Z}(t) \leq L_1 \left\{ w_{\overline{\eta}(t)}(t) - w_{\underline{\eta}(t)}(t) \right\}
\]
\[
< L_1 \left\{ -\frac{\dot{\overline{\eta}}(t)}{\overline{\eta}(t)} + \frac{\dot{\underline{\eta}}(t)}{\underline{\eta}(t)} + \frac{L_2}{(t-1)^2} \right\}, \quad t \notin T
\]
\[
= L_1 \left\{ -\frac{\dot{\overline{Z}}(t)}{\overline{Z}(t)} + \frac{\dot{\underline{Z}}(t)}{\underline{Z}(t)} + \frac{L_2}{(t-1)^2} \right\}, \quad t \notin T.
\]

Because \( 2 \in T \), there exists an integer \( k^* > 1 \) with \( 2 = t_{k^*} \). Thus, by (119), for arbitrary integer \( K > k^* \)
\[
\sum_{k=0}^{K} \int_{t_k}^{t_{k+1}} \{ \overline{Z}(t) - \underline{Z}(t) \} \, dt
\]
\[
= \sum_{k=0}^{k^*-1} \int_{t_k}^{t_{k+1}} \{ \overline{Z}(t) - \underline{Z}(t) \} \, dt + \sum_{k=k^*}^{K} \int_{t_k}^{t_{k+1}} \{ \overline{Z}(t) - \underline{Z}(t) \} \, dt
\]
\[
< \int_0^{t_{k^*}} \{ \overline{Z}(t) - \underline{Z}(t) \} \, dt + L_1 \sum_{k=k^*}^{K} \left[ -\log \overline{Z}(t) + \log \underline{Z}(t) - \frac{L_2}{t} \right]_{t_k}^{t_{k+1}}
\]
\[
= \int_0^{2} \{ \overline{Z}(t) - \underline{Z}(t) \} \, dt
\]
\[
+ L_1 \sum_{k=k^*}^{K} \left\{ \log \frac{\overline{Z}(t_k)}{\underline{Z}(t_k)} - \log \frac{\overline{Z}(t_{k+1})}{\underline{Z}(t_{k+1})} + L_2 \left( \frac{1}{t_k - 1} - \frac{1}{t_{k+1} - 1} \right) \right\}
\]
\[
= \int_0^{2} \{ \overline{Z}(t) - \underline{Z}(t) \} \, dt
\]
\[
+ L_1 \left\{ \log \frac{\overline{Z}(t_{k^*})}{\underline{Z}(t_{k^*})} - \log \frac{\overline{Z}(t_{K+1})}{\underline{Z}(t_{K+1})} + L_2 \left( \frac{1}{t_{k^*} - 1} - \frac{1}{t_{K+1} - 1} \right) \right\}
\]
\[
< \int_0^{2} \overline{Z}(t) \, dt + L_1 \left\{ \log \frac{\overline{Z}(2)}{\underline{Z}(2)} - \log \frac{\overline{Z}(t_{K+1})}{\underline{Z}(t_{K+1})} + L_2 \left( \frac{1}{2} - \frac{1}{t_{K+1} - 1} \right) \right\}
\]
\[
< 2\overline{Z}(0) + L_1 \left\{ \log \frac{\overline{Z}(2)}{\underline{Z}(2)} + L_2 \right\}.
\]

The integral (120) has an upper bound (126) which does not depend on \( K \), hence we obtain the Lemma. \( \blacksquare \)

Lemma 15 \[ \sum_{N=0}^{\infty} \left( \overline{Z}^N - \underline{Z}^N \right) < \infty. \]

**Proof:** The assertion holds by Lemma 12 and Lemma 14. \( \blacksquare \)

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Lemma 16  The sequence \( \{ Z_N - Z_N' \}_{N=0,1,\ldots} \) is strictly decreasing.

Proof: We have
\[
Z_N^{N+1} - Z_N^{N+1} = z_{i(N+1)}^{N+1} - z_{i(N+1)}^{N+1}
\]
\[
= z_{i(N+1)}^N - z_{i(N+1)}^N \sum_{i'=1}^h q_{i(N+1)i'} z_{i'}^N - z_{i(N+1)}^N \sum_{i'=1}^h q_{i(N+1)i'} z_{i'}^N
\]
\[
< z_{i(N+1)}^N - z_{i(N+1)}^N \sum_{i'=1}^h q_{i(N+1)i'} z_{i'}^N
\]
\[
= z_{i(N+1)}^N \left( 1 - Z_N \right) - z_{i(N+1)}^N \left( 1 - Z_N \right)
\]
\[
\leq Z_N^N \left( 1 - Z_N \right) - Z_N^N \left( 1 - Z_N \right)
\]
\[
= Z_N^N - Z_N
\]

We will use the following theorem in our proof.

Theorem C [5], p.31  Let \( \{a_N\}_{N=0,1,\ldots} \) be a decreasing positive sequence. If \( \sum_{N=0}^\infty a_N \) is convergent, then \( Na_N \to 0, \ N \to \infty \).

Lemma 17  \( \lim_{N \to \infty} N (Z_N^N - Z_N^N) = 0. \)

Proof: The assertion holds by Lemma 15, Lemma 16 and Theorem C.

Theorem 4  \( \lim_{N \to \infty} NZ_N^N = 1, \ \lim_{N \to \infty} NZ_N^N = 1. \)

Proof: By Lemma 7 and Lemma 17, we have
\[
\limsup_{N \to \infty} NZ_N^N = \limsup_{N \to \infty} \left( NZ_N^N - NZ_N^N + NZ_N^N \right)
\]
\[
\leq \limsup_{N \to \infty} N \left( Z_N^N - Z_N^N \right) + \limsup_{N \to \infty} NZ_N^N
\]
\[
\leq 1,
\]
thus, together with Lemma 6, we have
\[
\lim_{N \to \infty} NZ_N^N = 1.
\]

Further, we have
\[
\lim_{N \to \infty} NZ_N^N = \lim_{N \to \infty} \left( NZ_N^N - NZ_N^N + NZ_N^N \right)
\]
\[
= - \lim_{N \to \infty} N(\overline{Z}_N^N - \overline{Z}_N^N) + \lim_{N \to \infty} NZ_N^N
\]
\[
= 1.
\]

As stated above, Theorem 4 implies Theorem 3, the goal of this study.
6 Numerical examples

Based on Example 7 in [8], consider the following channel matrix \( \Phi \):

\[
\Phi = \begin{pmatrix}
P^1 \\
P^2 \\
P^3 \\
P^4 \\
P^5
\end{pmatrix} = \begin{pmatrix}
0.6 & 0.1 & 0.1 & 0.1 & 0.1 \\
0.1 & 0.6 & 0.1 & 0.1 & 0.1 \\
P_3^3 & P_3^3 & P_3^3 & P_3^3 & P_3^5 \\
P_4^4 & P_4^4 & P_4^4 & P_4^4 & P_4^5 \\
P_5^5 & P_5^5 & P_5^5 & P_5^5 & P_5^5
\end{pmatrix},
\]

(140)

where \( P^3 \), \( P^4 \), and \( P^5 \) will be fixed later.

Let the output distribution \( Q^* \) be the midpoint of \( P^1 \) and \( P^2 \), i.e.,

\[
Q^* = (0.35, 0.35, 0.1, 0.1, 0.1),
\]

(141)

and define a real number \( C \) as

\[
D(P^1 \| Q^*) = D(P^2 \| Q^*) = 0.198121603 \equiv C.
\]

(142)

We will choose \( P^3 \), \( P^4 \), \( P^5 \) so that the above \( C \) is the channel capacity of \( \Phi \) and \( Q^* \) is the capacity achieving output distribution and \( \{3, 4, 5\} \) are type-II indices. That is, we will choose \( P^3 \), \( P^4 \), \( P^5 \) that satisfy the following conditions:

\[
D(P^i \| Q^*) = C, \quad i = 3, 4, 5,
\]

(143)

\[
P_j^i \geq 0, \quad j = 1, \ldots, 5, \quad i = 3, 4, 5, \quad \sum_{j=1}^5 P_j^i = 1, \quad i = 3, 4, 5,
\]

(144)

\[
\text{rank } \Phi = 5.
\]

(145)

See Fig. 2.

![Diagram](image)

Figure 2: How to choose \( P^3 \), \( P^4 \), \( P^5 \) that satisfy (143), (144), (145).
The channel matrix $\Phi$ of (140) is constructed using three solutions $P^3, P^1, P^2$. In this case, $C$ is the channel capacity of $\Phi$ and $Q^*$ is the capacity achieving output distribution. The capacity achieving input distribution $\lambda^*$ is

$$\lambda^* = (0.5, 0.5, 0, 0, 0).$$  \hfill (146)

Let us find the recurrence formula (35) for $\Phi$ of (140) obtained in this way. That is, find the matrix $R = (r_{ii'})_{i,i'=3,4,5}$.

### 6.1 Calculation of Jacobian matrix $J(\lambda^*)$

For $\Phi$ of (140), we see from (143), (146), that the type-I indices are $I_1 = \{1, 2\}$ and type-II indices are $I_2 = \{3, 4, 5\}$, and there are no type-III indices. Therefore, (20) and (24) are the same and they equal

$$J(\lambda^*) = \begin{pmatrix} J^1 & O \\ * & J^\Pi \end{pmatrix}.$$  \hfill (147)

From Theorem 1 in [8], we have

$$J(\lambda^*) = \begin{pmatrix} 1 + \lambda^*_1 D^*_{1,1} & \lambda^*_2 D^*_{1,2} & 0 & 0 \\ \lambda^*_1 D^*_{2,1} & 1 + \lambda^*_2 D^*_{2,2} & 0 & 0 \\ \lambda^*_1 D^*_{3,1} & \lambda^*_2 D^*_{3,2} & 1 & 0 \\ \lambda^*_1 D^*_{4,1} & \lambda^*_2 D^*_{4,2} & 0 & 1 \\ \lambda^*_1 D^*_{5,1} & \lambda^*_2 D^*_{5,2} & 0 & 0 \end{pmatrix},$$  \hfill (148)

where

$$D^*_{i,i'} = -\sum_{j=1}^5 \frac{P^i_j P^{i'}_j}{Q^*_j}, \ i, i' = 1, \ldots, 5.$$  \hfill (149)

See Lemma 4 in [8].

Let us compute $D^*_i$ of (149) for $\Phi$ of (140). Indeed, we have

$$D^*_{1,1} = -\frac{19}{14}, \ D^*_{1,2} = D^*_{2,1} = -\frac{9}{14}, \ D^*_{2,2} = -\frac{19}{14},$$  \hfill (150)

and further, for $i = 3, 4, 5$,

$$D^*_{1,i} = D^*_{i,1} = -1 - \frac{5}{7} P^i_1 + \frac{5}{7} P^i_2, \ D^*_{2,i} = D^*_{i,2} = -1 + \frac{5}{7} P^i_1 - \frac{5}{7} P^i_2.$$  \hfill (151)

Therefore, by substituting (150) and (151) into (148), we have

$$J(\lambda^*) = \begin{pmatrix} \frac{9}{28} & \frac{9}{28} & 0 & 0 & 0 \\ -\frac{9}{28} & \frac{9}{28} & 0 & 0 & 0 \\ -\frac{1}{2} - \frac{5}{14} P^3_1 + \frac{5}{14} P^3_2 & -\frac{1}{2} + \frac{5}{14} P^3_1 - \frac{5}{14} P^3_2 & 1 & 0 & 0 \\ -\frac{1}{2} - \frac{5}{14} P^4_1 + \frac{5}{14} P^4_2 & -\frac{1}{2} + \frac{5}{14} P^4_1 - \frac{5}{14} P^4_2 & 0 & 1 & 0 \\ -\frac{1}{2} - \frac{5}{14} P^5_1 + \frac{5}{14} P^5_2 & -\frac{1}{2} + \frac{5}{14} P^5_1 - \frac{5}{14} P^5_2 & 0 & 0 & 1 \end{pmatrix}.$$  \hfill (152)
Furthermore, for $i, i' = 3, 4, 5$, we have

$$D_{i,i'}^* = D_{i,i}' = -\left\{ \frac{20}{7} P_1^i P_1^{i'} + \frac{20}{7} P_2^i P_2^{i'} + 10 \left( P_3^i P_3^{i'} + P_4^i P_4^{i'} + P_5^i P_5^{i'} \right) \right\}.$$  \hspace{1cm} (153)

### 6.2 Eigenvalues and eigenvectors of $J(\lambda^*)$

From (152), the eigenvalues of $J(\lambda^*)$ are

$$(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5) = (0, 9/14, 1, 1, 1).$$  \hspace{1cm} (154)

Taking appropriate eigenvector $a_i$ for each eigenvalue $\theta_i$, then by (25)-(32), we have

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 2P_1^3 - 2P_2^3 & 1 & 0 & 0 \\ 1 & 2P_1^4 - 2P_2^4 & 0 & 1 & 0 \\ 1 & 2P_1^5 - 2P_2^5 & 0 & 0 & 1 \end{pmatrix},$$  \hspace{1cm} (155)

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$  \hspace{1cm} (156)

$$A_2 = \begin{pmatrix} 1 & 2P_3^3 - 2P_2^3 \\ 1 & 2P_1^4 - 2P_2^4 \\ 1 & 2P_1^5 - 2P_2^5 \end{pmatrix}.$$  \hspace{1cm} (157)

Then, from $\mu_{i,III}^N = -\mu_{i,II}^N A_2 A_1^{-1}$ of (33), we have

$$\left( \mu_1^N, \mu_2^N \right) = -\frac{1}{2} \left( \mu_3^N, \mu_4^N, \mu_5^N \right) \begin{pmatrix} 1 & 2P_3^3 - 2P_2^3 \\ 1 & 2P_1^4 - 2P_2^4 \\ 1 & 2P_1^5 - 2P_2^5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$  \hspace{1cm} (158)

thus,

$$\mu_1^N = \left( -\frac{1}{2} - P_1^3 + P_2^3 \right) \mu_3^N + \left( -\frac{1}{2} - P_1^4 + P_2^4 \right) \mu_4^N + \left( -\frac{1}{2} - P_1^5 + P_2^5 \right) \mu_5^N,$$  \hspace{1cm} (159)

$$\mu_2^N = \left( -\frac{1}{2} + P_1^3 - P_2^3 \right) \mu_3^N + \left( -\frac{1}{2} + P_1^4 - P_2^4 \right) \mu_4^N + \left( -\frac{1}{2} + P_1^5 - P_2^5 \right) \mu_5^N.$$  \hspace{1cm} (160)

### 6.3 On the Hessian matrix $H_i(\lambda^*)$, $i = 3, 4, 5$

For $i = 3, 4, 5$, we will calculate the Hessian matrices $H_3(\lambda^*), H_4(\lambda^*), H_5(\lambda^*)$. From Theorem 6 in [8], we have

$$H_3(\lambda^*) = \begin{pmatrix} 0 & 0 & D_{1,3}^* & 0 & 0 \\ 0 & 0 & D_{2,3}^* & 0 & 0 \\ D_{1,3}^* & D_{2,3}^* & 2D_{3,3}^* & D_{3,4}^* & D_{3,5}^* \\ 0 & 0 & D_{3,4}^* & 0 & 0 \\ 0 & 0 & D_{3,5}^* & 0 & 0 \end{pmatrix},$$  \hspace{1cm} (161)
\[
H_4(\lambda^*) = \begin{pmatrix}
0 & 0 & 0 & D_{1,4}^* & 0 \\
0 & 0 & 0 & D_{2,4}^* & 0 \\
0 & 0 & 0 & D_{3,4}^* & 0 \\
D_{1,4}^* & D_{2,4}^* & D_{3,4}^* & 2D_{4,4}^* & D_{4,5}^* \\
0 & 0 & 0 & D_{4,5}^* & 0 
\end{pmatrix},
\]
(162)

\[
H_5(\lambda^*) = \begin{pmatrix}
0 & 0 & 0 & 0 & D_{1,5}^* \\
0 & 0 & 0 & 0 & D_{2,5}^* \\
0 & 0 & 0 & 0 & D_{3,5}^* \\
D_{1,5}^* & D_{2,5}^* & D_{3,5}^* & D_{4,5}^* & 2D_{5,5}^* \\
\end{pmatrix}
\]
(163)

Then, for \(\mu^N = (\mu_1^N, \mu_2^N, \mu_3^N, \mu_4^N, \mu_5^N)\), we will calculate \((1/2)\mu^N H_i(\lambda^*)^T \mu^N\), \(i = 3, 4, 5\).

First, for \(i = 3\), from (159), (160), (161), we have

\[
\frac{1}{2} \mu^N H_3(\lambda^*)^T \mu^N = \mu_3^N (r_{33}\mu_3^N + r_{34}\mu_4^N + r_{35}\mu_5^N),
\]
(164)

where

\[
r_{33} = 1 + \frac{10}{7} (P_1^3 - P_2^3)^2 + D_{3,3}^*,
\]
(165)

\[
r_{34} = 1 + \frac{10}{7} (P_1^3 - P_2^3) (P_1^4 - P_2^4) + D_{3,4}^*,
\]
(166)

\[
r_{35} = 1 + \frac{10}{7} (P_1^3 - P_2^3) (P_1^5 - P_2^5) + D_{3,5}^*.
\]
(167)

Similarly, for \(i = 4\), from (159), (160), (162), we have

\[
\frac{1}{2} \mu^N H_4(\lambda^*)^T \mu^N = \mu_4^N (r_{43}\mu_3^N + r_{44}\mu_4^N + r_{45}\mu_5^N),
\]
(168)

where

\[
r_{43} = 1 + \frac{10}{7} (P_1^4 - P_2^4) (P_1^3 - P_2^3) + D_{3,4}^*,
\]
(169)

\[
r_{44} = 1 + \frac{10}{7} (P_1^4 - P_2^4)^2 + D_{4,4}^*,
\]
(170)

\[
r_{45} = 1 + \frac{10}{7} (P_1^4 - P_2^4) (P_1^5 - P_2^5) + D_{4,5}^*.
\]
(171)

For \(i = 5\), for \(i = 4\), from (159), (160), (163), we have

\[
\frac{1}{2} \mu^N H_5(\lambda^*)^T \mu^N = \mu_5^N (r_{53}\mu_3^N + r_{54}\mu_4^N + r_{55}\mu_5^N),
\]
(172)

where

\[
r_{53} = 1 + \frac{10}{7} (P_1^5 - P_2^5) (P_1^3 - P_2^3) + D_{3,5}^*,
\]
(173)

\[
r_{54} = 1 + \frac{10}{7} (P_1^5 - P_2^5) (P_1^4 - P_2^4) + D_{4,5}^*,
\]
(174)

\[
r_{55} = 1 + \frac{10}{7} (P_1^5 - P_2^5)^2 + D_{5,5}^*.
\]
(175)
Then, for
\[ R = \begin{pmatrix} r_{33} & r_{34} & r_{35} \\ r_{43} & r_{44} & r_{45} \\ r_{53} & r_{54} & r_{55} \end{pmatrix}, \] (176)
we will solve the equation
\[ \sigma = -1^t R^{-1}, \quad \sigma = (\sigma_3, \sigma_4, \sigma_5), \] (177)
in the assumption (iii). If \( R \) of (176) satisfies the assumptions (i)-(iv), we have, from Theorem 3,
\[ \lim_{N \to \infty} N \mu_i^N \]
\[ \begin{cases} \left( -\frac{1}{2} - P_1^3 + P_2^3 \right) \sigma_3 + \left( -\frac{1}{2} - P_1^4 + P_2^4 \right) \sigma_4 + \left( -\frac{1}{2} - P_1^5 + P_2^5 \right) \sigma_5, & i = 1, \\ \left( -\frac{1}{2} + P_1^3 - P_2^3 \right) \sigma_3 + \left( -\frac{1}{2} + P_1^4 - P_2^4 \right) \sigma_4 + \left( -\frac{1}{2} + P_1^5 - P_2^5 \right) \sigma_5, & i = 2, \\ \sigma_i, & i = 3, 4, 5. \end{cases} \] (178, 179, 180)

6.4 How to make a channel matrix of (140)

We will show here how to make a channel matrix of (140), i.e., how to choose probability distributions \( P^3, P^4 \) and \( P^5 \).

As \( \mathcal{Y} = \{y_1, \ldots, y_5\} \), take a probability distribution \( P \) from \( \Delta(\mathcal{Y}) \) whose components are non-negative integral multiples of 1/3. There are 35 such probability distributions \( P \) in total. Consider a half-line \( P_s = (1 - s)Q^* + sP, \ s \geq 0 \) starting from \( Q^* \) and passing through \( P \). Determine \( s \) by a numerical search so that \( D(P_s||Q^*) = C \). Let \( P^3 \equiv P_s \) be a probability distribution determined in this way. Similarly, 35 \( P^3 \) are obtained. In the same way, 35 \( P^4 \) and 35 \( P^5 \) are obtained because we choose so that \( P^3 \neq P^4 \neq P^5 \neq P^3 \). Then, using those \( P^3, P^4, P^5 \), we obtain 6545 channel matrices \( \Phi \) of (140).

Examining these 6545 \( \Phi \), we found that 109 of them satisfy the assumptions (i)-(iv). This is 109/6545 = 1.7% of the total. We had expected that 100% of the total would satisfy (i)-(iv), but in fact, it is a very small percentage.

We have already proved in Chapter 5 that the Arimoto-Blahut algorithm converges at the speed of the order \( O(1/N) \) for those 109 \( \Phi \) satisfying (i)-(iv), but in that proof, we used all the assumptions (i)-(iv). Thus, the present proof cannot be used for \( \Phi \) if one or more among (i)-(iv) are not satisfied. Nonetheless, it seems correct that for all \( \Phi \) the speed of convergence is the order \( O(1/N) \) by a lot of numerical experiments. So, we would like to prove it somehow, but we have no clue at this moment. This is an issue for the future.
6.5 Channel matrices that satisfy the assumptions (i)-(iv)

First, we will compare the numerical values $N\mu_i^N$ with the theoretical value $\sigma_i$ obtained in this paper for the following two channel matrices $\Phi^{(1)}, \Phi^{(2)}$ among 109 $\Phi$ that satisfy (i)-(iv).

**Example 1** We will show below the channel matrix $\Phi^{(1)}$ together with $R^{(1)}$ and $\sigma^{(1)}$ for $\Phi^{(1)}$, where $R^{(1)}$ and $\sigma^{(1)}$ are $R$ and $\sigma$ defined in section 4.2, respectively:

$$\Phi^{(1)} = \begin{pmatrix} 0.600 & 0.100 & 0.100 & 0.100 & 0.100 \\ 0.100 & 0.600 & 0.100 & 0.100 & 0.100 \\ 0.231 & 0.231 & 0.066 & 0.179 & 0.292 \\ 0.161 & 0.341 & 0.226 & 0.226 & 0.046 \\ 0.341 & 0.161 & 0.226 & 0.046 & 0.226 \end{pmatrix}, \quad (181)$$

$$R^{(1)} = \begin{pmatrix} -0.522 & -0.020 & -0.223 \\ -0.020 & -0.401 & -0.078 \\ -0.223 & -0.078 & -0.401 \end{pmatrix}, \quad (182)$$

$$\sigma^{(1)} = (-2.251, -2.532, 1.241, 2.161, 1.381). \quad (183)$$

We set the initial distribution as $\lambda^0 = (0.200, 0.200, 0.200, 0.200, 0.200)$. From (182), (183), we see that all the assumptions (i)-(iv) are satisfied.

Fig. 3 shows the values of $N\mu_i^N$ and $\sigma_i^{(1)}$ of (183). From the Fig. 3, $N\mu_i^N$ converges to a value close enough to $\sigma_i^{(1)}$ at $N = 1000$.

![Figure 3: Comparison of $N\mu_i^N$ and $\sigma_i^{(1)}$ of Example 1.](image-url)
Example 2 We will show the channel matrix $\Phi^{(2)}$ together with $R^{(2)}$ and $\sigma^{(2)}$ for $\Phi^{(2)}$:

$$
\Phi^{(2)} = \begin{pmatrix}
0.600 & 0.100 & 0.100 & 0.100 & 0.100 \\
0.100 & 0.600 & 0.100 & 0.100 & 0.100 \\
0.231 & 0.231 & 0.066 & 0.179 & 0.292 \\
0.161 & 0.341 & 0.226 & 0.226 & 0.046 \\
0.522 & 0.160 & 0.046 & 0.227 & 0.046
\end{pmatrix}, \quad (184)
$$

$$
R^{(2)} = \begin{pmatrix}
-0.522 & -0.020 & -0.020 \\
-0.020 & -0.401 & -0.126 \\
-0.020 & -0.126 & -0.221
\end{pmatrix}, \quad (185)
$$

$$
\sigma^{(2)} = (-4.415, -2.221, 1.723, 1.263, 3.650). \quad (186)
$$

We set the initial distribution as $\lambda^0 = (0.200, 0.200, 0.200, 0.200, 0.200)$. From (185), (186), we see that all the assumptions (i)-(iv) are satisfied.

Fig. 4 shows the values of $N\mu_i^N$ and $\sigma_i^{(2)}$ of (186). From the Fig. 4, $N\mu_i^N$ converges to a value close enough to $\sigma_i^{(2)}$ at $N = 1000$.

![Figure 4: Comparison of $N\mu_i^N$ and $\sigma_i^{(2)}$ of Example 2.](image)
6.5.1 Channel matrices that do not satisfy the assumptions (i)-(iv)

Next, we will show the numerical values $N\mu_i^N$ for the following two channel matrices $\Phi^{(3)}$, $\Phi^{(4)}$ among $6436 (= 6545 - 109)$ $\Phi$ that do not satisfy (i)-(iv).

**Example 3** We will show the channel matrix $\Phi^{(3)}$ together with $R^{(3)}$ and $\sigma^{(3)}$ for $\Phi^{(3)}$:

$$\Phi^{(3)} = \begin{pmatrix}
0.600 & 0.100 & 0.100 & 0.100 & 0.100 \\
0.100 & 0.600 & 0.100 & 0.100 & 0.100 \\
0.260 & 0.260 & 0.074 & 0.074 & 0.331 \\
0.231 & 0.231 & 0.066 & 0.179 & 0.292 \\
0.215 & 0.344 & 0.061 & 0.319 & 0.061 \\
\end{pmatrix},$$

$$R^{(3)} = \begin{pmatrix}
-0.594 & -0.493 & 0.099 \\
-0.493 & -0.522 & -0.160 \\
0.099 & -0.160 & -0.536 \\
\end{pmatrix},$$

$$\sigma^{(3)} = (-75.23, -145.2, 528.0, -579.6, 272.1).$$

We set the initial distribution as $\lambda^0 = (0.200, 0.200, 0.200, 0.200, 0.200)$.

From (188), (189), we see that the assumptions (i), (iii), (iv) are not satisfied.

Fig. 5 shows the values of $N\mu_i^N$. From the Fig. 5, $N\mu_i^N$ looks to converge to a constant value, but they are completely different from the values $\sigma_i^{(3)}$ of (189).

![Figure 5: $N\mu_i^N$ of Example 3.](image-url)
**Example 4** We will show the channel matrix $\Phi^{(4)}$ together with $R^{(4)}$ and $\sigma^{(4)}$ for $\Phi^{(4)}$:

\[
\Phi^{(4)} = \begin{pmatrix}
0.600 & 0.100 & 0.100 & 0.100 & 0.100 \\
0.100 & 0.600 & 0.100 & 0.100 & 0.100 \\
0.522 & 0.160 & 0.046 & 0.046 & 0.227 \\
0.522 & 0.160 & 0.046 & 0.227 & 0.046 \\
0.522 & 0.160 & 0.227 & 0.046 & 0.046 \\
\end{pmatrix},
\]

\[
R^{(4)} = \begin{pmatrix}
-0.221 & 0.108 & 0.108 \\
0.108 & -0.221 & 0.108 \\
0.108 & 0.108 & -0.221 \\
\end{pmatrix},
\]

\[
\sigma^{(4)} = (-550.8, -87.65, 212.8, 212.8, 212.8).
\]

We set the initial distribution as $\lambda^0 = (0.256, 0.144, 0.128, 0.206, 0.267)$.

From (191), (192), we see that the assumptions (i), (ii), (iv) are not satisfied. Fig. 6 shows the values of $N\mu_i^N$. This example requires a large number of iterations, $N = 100000$, to converge. However, the converged value is close to the value of $\sigma_i^{(4)}$ of (192).
7 Conclusion and future issues

In this study, we considered the convergence speed of the order $O(1/N)$, which is a slow convergence in the Arimoto-Blahut algorithm. We considered the second-order nonlinear recurrence formula consisting of the first- and second-order terms of the Taylor expansion of the defining function of the Arimoto-Blahut algorithm. The convergence of the recurrence formula was proved under the assumptions (i)-(iv). The correctness of the proof was confirmed by several numerical examples. However, the channel matrices satisfying all the assumptions (i)-(iv) were only a very small fraction of the total matrices examined, so further generalization of the proof obtained in this paper is a future issue.

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