Modified Dispersion Relations in Hořava–Lifshitz Gravity and Finsler Brane Models

Sergiu I. Vacaru *

University "Al. I. Cuza" Iaşi, Science Department,
54 Lascar Catargi street, Iaşi, Romania, 700107

December 16, 2011

Abstract

We study possible links between quantum gravity phenomenology encoding Lorentz violations as nonlinear dispersions, the Einstein–Finsler gravity models, EFG, and nonholonomic (non–integrable) deformations to Hořava–Lifshitz, HL, and/or Einstein’s general relativity, GR, theories. EFG and its scaling anisotropic versions formulated as Hořava–Finsler models, HF, are constructed as covariant metric compatible theories on (co) tangent bundle to Lorentz manifolds and respective anisotropic deformations. Such theories are integrable in general form and can be quantized following standard methods of deformation quantization, A–brane formalism and/or (perturbatively) as a nonholonomic gauge like model with bi–connection structure. There are natural warping/trapping mechanisms, defined by the maximal velocity of light and locally anisotropic gravitational interactions in a (pseudo) Finsler bulk spacetime, to four dimensional (pseudo) Riemannian spacetimes. In this approach, the HL theory and scenarios of recovering GR at large distances are generated by imposing nonholonomic constraints on the dynamics of HF, or EFG, fields.

Keywords: Hořava–Lifshitz gravity, Einstein–Finsler gravity, modified dispersion relation, Lorentz violation, Finsler brane.

PACS: 04.50.Kd, 04.90.+e, 04.60.-m

1 Introduction

There is a considerable recent interest in two directions of classical and quantum gravity and possible implications in cosmology and astrophysics:

*sergiu.vacaru@aei.mpg.de, sergiu.vacaru@uaic.ro, Sergiu.Vacaru@gmail.com
The first one is related to gravity models with anisotropic scaling between space and time at short distances which is usually referred to as the Hořava–Lifshitz, HL, theory \[1, 2, 3\]. Such theories with generic anisotropy are usually non–relativistic and ultra–violet complete; the local Lorentz invariance is violated/broken (LV) at short distances but constructed to reduce to the general relativity (GR) theory in the infrared limit.\[1\] One of the main features of this class of theories is that they can be elaborated in a "power–counting" renormalizable form (unlike Einstein gravity) at least if the so called detailed balance condition is respected. This can be understood, for instance, as a result of stochastic quantization in relation to topological massive gravity \[4, 5\]. The second direction consists from a series of models related to quantum gravity (QG) phenomenology also including LV effects and general relativistic and non–relativistic constructions with extra dimensions, generalized symmetries, and compactification or trapping scenarios etc (see, for instance, reviews \[6, 7, 8, 9, 10, 11\]).

The above mentioned classes of gravitational theories are characterized, in general, by LV and respective modified dispersion relations (MDR), local and/or global anisotropies, nonhomogeneous and, for certain models, they are defined by nonholonomic (non–integrable) constraints on the dynamics of gravitational and matter fields. For instance, the implications of violations of Poincaré symmetry for kinematic conditions and MDR at the "threshold" for some particle–creation interactions in HL–type theories are studied in \[16\]. One of the most important aspects to be understood is the way when such theories can be constructed in a general geometric form and to analyze possible applications.

During last decade, there have been published some series of works relating quantum phenomenology and, for instance, anisotropy and dark energy/matter problems to Finsler gravity models with LV, MDR and locally anisotropic spacetime configurations, see explicit constructions and references in \[17, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31\]. There is a study of possible links between anisotropic–scaling scenarios and Finsler spacetimes \[32\]. A surprising conclusion which can be drawn from such approaches is that we have to include certain Finsler type physical objects into various schemes of quantization of gravity and apply corresponding geometric methods in order to elaborate in a self–consistent form relativistic and non–relativistic models of QG.

---

1. The problem of reduction is still an open issue: the HL theory with global Hamiltonian does not reproduce GR in the infrared domain \[12\]. There are necessary certain further modifications of the theory because both projectable and non–projectable versions of the HL models seem to contain certain inconsistency \[13, 14, 15\].
There were proposed different ideas and elaborated explicit theoretical constructions related to Finsler geometry, generalizations and applications in modern physics (for particle and mathematical physics researches, we cite Refs. [33, 34, 35, 36, 37, 38]). For instance, Finsler gravity models can be derived in low energy limits from string gravity theories [39, 40] and brane gravity [41, 42, 35] and induced by noncommutative generalizations of Einstein gravity [43, 44, 45]. Various classes of commutative and noncommutative Finsler type geometries and gravity theories are induced via nonholonomically constrained Ricci flows of (pseudo) Riemannian metrics [46, 47, 48, 49]. Finsler variables can be introduced in GR and extra-dimension generalizations which allows us to formulate geometric methods of constructing exact (and vary general classes of) solutions in different gravity theories [50, 51, 52, 36, 44]. Re-writing the Einstein equations in the so-called almost Kähler – Finsler variables, it was possible to apply rigorous methods of deformation and A-brane quantization and nonholonomic gauge methods in order to elaborate quantum models of Einstein gravity and Lagrange–Finsler–Hamilton generalizations [53, 54, 55, 56, 57, 58].

It is our purpose to elaborate a modification of HL and GR theories which will include MDR defined, in general, for tangent bundles to Einstein spacetimes and nonholonomic/anisotropic modifications. Naturally such constructions can be performed in the framework of Finsler geometry and generalizations. We consider such an approach to be motivated because any type of nonlinear dispersion relations are canonically related to a Finsler generation function (usual constructions in special and general relativity theories are contained as particular (quadratic) cases). In this sense, QG models of HL and/or other types should be more realistically elaborated in terms of (pseudo) Finsler fundamental geometric objects; this seems to give a more realistic quantum theory the existing quantum versions of (pseudo) Riemannian geometry.

The paper is organized as follows. In section 2 we provide a brief summary of HL and GR theories and consider possible MDR for scaling anisotropies. We show that certain class of fundamental Finsler functions can be derived from HL theory and various types of QG models with MDR. We formulate the HF gravity as a theory generalizing HL models on Finsler spaces in section 3. Trapping mechanism for Finsler branes resulting in HL gravity (and for corresponding nonholonomic constraints, in GR theory) are

\footnote{there are thousands of papers and tens of monographs on Finsler geometry and applications - it is not possible to summarize in this paper and discuss all such "standard" and "nonstandard" theories in relation to modern gravity and analogous mechanical models}
studied in section 4. Finally, conclusions are provided in section 5. In Appendix, we summarize some technical details on diagonal solutions in HF gravity.

2 Finsler Geometry Induced by MDR in HL Gravity

The goal of this section is to show how fundamental Finsler geometric objects are induced from some general MDR and, in particular, for HL gravity: We outline in brief the HL gravity theory, analyze possible MDR and show how the fundamental Finsler generating function can be associated to such anisotropic configurations and nonlinear dispersions.

2.1 Preliminaries on the HL and GR theories

In standard form, the dynamical variables of HL gravity are the lapse function, \( N \), the shift function, \( \hat{N}^i \), and the spacelike metric, \( \hat{g}^{ij} \), in terms of which the metric is written as the Arnowitt–Deser–Misner, ADM, (1+3) splitting,

\[
d s^2 = g_{ij} dx^i dx^j = -N^2 dt^2 + g_{ij}^{\hat{\ }}(dx^{\hat{i}} + N^{\hat{i}} dt)(dx^{\hat{j}} + N^{\hat{j}} dt). \tag{1}
\]

The above metric \( g_{ij} = (N^2, g_{ij}^{\hat{\ }}) \) (we shall write in brief, \( hg = (N^2, \hat{g}) \)) is supposed to be invariant under the foliation–preserving diffeomorphisms of the HL theory, \( t' = t'(t) \) and \( \hat{x}^{\hat{i}} = \hat{x}^{\hat{i}}(t, x^{\hat{k}}) \), where indices \( i, i', j, j', ... = 1, 2, 3, 4 \), for \( x^i = (x^1 = t, x^{\hat{i}}) \) and \( \hat{x}^{\hat{i}}, \hat{x}^{\hat{j}}, \hat{x}^{\hat{j}'} , ... = 2, 3, 4 \). The theory is invariant under the anisotropic scaling symmetry

\[
t \to l^2 t, \ x^{\hat{i}} \to lx^{\hat{i}}, \text{ when for } z = 3, \ N \to l^{-2} N, \ N^{\hat{i}} \to l^{-2} N^{\hat{i}}, \ g_{ij}^{\hat{\ }} \to g_{ij}^{\hat{\ }}(2)
\]

(to elaborate a power–counting renormalizable theory of gravity in four dimensions, 4–d, is considered \( z = 3 \)). The projectability condition requires a homogeneous lapse function \( N = N(t) \) but admits general shift and 3–d metric, i.e. \( N^{\hat{i}}(x^k) = N^{\hat{i}}(t, x^{\hat{k}}) \) and \( g_{ij}^{\hat{\ }}(x^k) \to g_{ij}^{\hat{\ }}(t, x^{\hat{k}})\) \(^3\)

\(^3\)We have to elaborate a new system of notations which will be compatible with 3+1 splitting for ADM formalism and 4+4, or 2+2/3+2 / 4+3 nonholonomic splitting used in Finsler geometry, see details in \(^3\).

\(^4\)It is possible to consider a general nonhomogeneous lapse function but this may result in problems when attempting to quantize the model, see \(^2\) \([59]\).
The action for HL gravity is postulated as a sum of "kinetic", \( K \), and "potential" part, \( V \),
\[
H_L S = K S + V S,
\]
where
\[
K S = \frac{2}{\kappa^2} \int d^3 x \sqrt{|g|} N \left( K_{ij} \hat{K}^{ij} - \lambda K^2 \right)
\]
\[
V S = \int d^3 x \sqrt{|g|} N \left[ \frac{\kappa^2 \mu}{2 \varpi^2} \epsilon^{ijk} R_{ij} \hat{R}_{kl} R^{kl} - \frac{\kappa^2 \mu}{8} R_{ij} \hat{R}^{ij} \right]
\]
for some constants \( \kappa, \mu, \varpi, \Lambda \) and a dynamical constant \( \lambda \) running as the energy scale changes. The general covariance in GR imposes the condition \( \lambda = 1 \).

It is known that the important variation-interval of \( \lambda \) is between \( 1/3 \) (the ultra–violet, UV, limit) and 1 (the infra–red, IR, limit). In the above formulas,
\[
K_{ij} = \frac{1}{2N} \left( \frac{\partial g_{ij}}{\partial t} - \nabla_i N_j - \nabla_j N_i \right)
\]
is the extrinsic curvature with \( K = g^{ij} K_{ij} \); the Cotton tensor is defined
\[
C^{ij} = \frac{\epsilon^{ijk}}{\sqrt{|g|}} \nabla_k \left( R_{ij} - \frac{1}{4} R \delta^i_j \right),
\]
where such geometric objects are constructed for the Levi–Civita connection \( \nabla_k \) and \( R \) determined by the 3–d spacial metric \( g_{ij} \), for \( \delta^i_j \) being the Kronecker symbol and \( |g| \) computed as the determinant of 3–d metric. In this work, we shall consider a simple form of theory with field equations derived from (3) (for instance, we can also consider the "detailed balance" condition which reduces the number of terms in the potential).

In the infrared limit of (3) we can obtain the ADM form of the Einstein–Hilbert action if the speed of light, \( c \), gravitational constant, \( G \), and cosmological constant, \( \Lambda \), (all in GR) are defined, respectively,
\[
c = \frac{\kappa^2 \mu}{4} \sqrt{\frac{\Lambda}{1 - 3 \lambda}} \\
16\pi G = \frac{\kappa^4 \mu}{8} \sqrt{\frac{\Lambda}{1 - 3 \lambda}} \\
GR \Lambda = \frac{3 \kappa^4 \mu^2 \Lambda^2}{32(1 - 3 \lambda)}.
\]

\[\text{\footnote{The most possible general potential is analyzed in [60, 61]; we shall elaborate more simple constructions which do not change our basic conclusions on relation of MDR and Finsler geometry.}}\]
There is also a coefficient before the $R^2$ term, $\kappa^2 \mu^2 = 8(1 - 3\lambda)c^3/16\pi GA$. The GR theory can be considered as a "homogeneous" and locally isotropic version of HL gravity.

2.2 MDR in HL gravity

A stability analysis of HL gravity is performed, for instance, in Ref. [62]. The conclusion of that work is that the HL gravity in original form suffers from instabilities and fine-tuning which cannot be overcome by simple tricks such an analytic continuation, see also [12, 13]. We propose that the HL theory should be extended on tangent/cotangent bundle (with velocity type coordinates) in order to include MDR which will put the problem of stability of gravitational field equations with nonholonomic constraints in a different form.

Let us outline some typical dispersion relations in HL gravity. Under the so-called "detailed balance" conditions, there are possible the following variants (with Fourier transforms of type $\psi(t, x^\hat{i}) = \int (2\pi)^{3/2} \psi_p(t)e^{ip^\hat{i}x^\hat{i}}$).

- For scalar perturbations and considering a low-$p$ behavior, we acquire
  \[ \omega^2 = -\frac{9\kappa^4 \mu^2 \Lambda^2}{32(1 - 3\lambda)^2} < 0. \]
  Such a MDR induces instabilities at the IR for all values of $\lambda$ and both signs of $\Lambda$.

- For high-$p$, the dispersion relation is
  \[ \omega^2 = \kappa^4 \mu^2 \left( \frac{1 - \lambda}{1 - 3\lambda} \right)^2 p^4. \]

- Similar computations can be performed for tensor perturbations,
  \[ \omega^2 = c^2 p^2 + \kappa^4 \mu^2 \frac{p^4}{16} \pm \kappa^4 \mu^2 \frac{p^5}{4c^2} \pm \kappa^4 \mu^4 \frac{p^6}{4c^6}. \]

A perturbative analysis can be extended beyond detailed balance. Such extended relations can be written (using an additional parameter for the corresponding contribution to the action):

\footnote{which can be computed by perturbing the action \cite{3} up to second order of metric preserving the ADM 3+1 foliation preserving formalism around a flat background}
For the UV–behavior of scalar perturbations,
\[
\omega^2 = \frac{\kappa^2(1 - \lambda)^2}{16(1 - 3\lambda)^2} p^4 - \frac{3\kappa^2(1 - \lambda)}{2(1 - 3\lambda)} \eta p^6.
\]

Finally, we present the formula for tensor perturbations:
\[
\omega^2 = c^2 p^2 + \frac{\kappa^4 \mu^2}{16} p^4 \pm \frac{\kappa^4 \mu}{4\pi^2} p^5 + \left( \frac{\kappa^4}{4\pi^4} - \frac{\kappa^2 \eta}{2} \right) p^6.
\]

We conclude that HL theory with Minkovski background is characterized by corresponding MDR \(\omega(p, \kappa, \mu, \Lambda, \varpi, c, \lambda, \eta)\) depending nonlinearly on momentum variables and with critical behavior (up to instabilities, branching of dispersion relation etc) determined by the values of the fundamental constants of the theory. The formulas for nonlinear dispersions presented in this sections are typical ones which can be derived in various models of HL gravity or alternative theories (in different approaches, one can be considered only "even powers" of momenta, parametric deformations etc).

### 2.3 Fundamental Finsler functions and the HL theory

In a more general context, we can perform an analysis of propagation of light rays in HL and various classes of gravity theories with LV, see details, for instance, in Refs. [24, 34, 35]. For light rays propagating on HL space–time, the nonlinear dispersion relation\(^7\) between the frequency \(\omega\) and the wave vector \(k_i\), can be written in a general abstract form
\[
\omega^2 = c^2 p^2 + \frac{\kappa^4 \mu^2}{16} p^4 \pm \frac{\kappa^4 \mu}{4\pi^2} p^5 + \left( \frac{\kappa^4}{4\pi^4} - \frac{\kappa^2 \eta}{2} \right) p^6.
\]

Depending on explicit parametrizations, with \(k_i \rightarrow p_i \sim y^a\), we can include the above dispersion formulas for scalar and tensor perturbations, or for light propagation, into a formal expression of type (5). Such MDR can be derived from very general arguments for a large class quantum and classical, commutative and noncommutative, gravity and particle field theories with LV, see [6, 7, 8, 63, 64, 65, 9, 10, 16] (the coefficients \(q_{\hat{i}_1 \hat{i}_2 ... \hat{i}_r} \) are computed in explicit form for corresponding models).

\(^7\)we can consider such a relation in a fixed point \(x^k = x^k_0\), when \(g_{\hat{i} \hat{j}}(x_0) = g_{\hat{i} \hat{j}}\) and \(q_{\hat{i}_1 \hat{i}_2 ... \hat{i}_r} = q_{\hat{i}_1 \hat{i}_2 ... \hat{i}_r}(x_0)\).
In a series of works [17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 34, 35], there were analyzed various possibilities when MDR \(5\), or certain particular forms, can be naturally associated to nonlinear homogeneous quadratic elements (with \(F(x^i, \beta y^j) = \beta F(x^i, y^j)\), for any \(\beta > 0\)), when

\[
ds^2 = F^2(x^i, y^j) \\ \approx -(cdt)^2 + g^{-\frac{1}{2}}_{ij}(x^k)\hat{y}^i\hat{y}^j \left[ 1 + \frac{1}{r} \frac{q_{\hat{t}\hat{t}...\hat{r}}}{{g^{-\frac{1}{2}}_{ij}(x^k)}{\hat{y}^i}{\hat{y}^j}} \right] + O(q^2). (6)
\]

Such nonlinear metric elements are usually considered in Finsler geometry. A value \(F\) is considered to be a fundamental (generating) Finsler function usually satisfying the condition that the Hessian

\[
F g_{ij}(x^i, y^j) = \frac{1}{2} \frac{\partial F^2}{\partial y^i \partial y^j}
\]

is not degenerate.

For \(q_{\hat{t}\hat{t}...\hat{r}} \to 0\) and a corresponding re-definition of frames and co-ordinates, we can generate elements of type \(1\) for GR. The HL theory is with generic anisotropy and LV characterized by dispersion relations \(\omega(p^i, \kappa, \mu, \Lambda, \varpi, c, \lambda, \eta)\) considered in section 2.2. Our idea is to extend the Hořava constructions in a (pseudo) Finsler form on tangent bundles to Lorentz modified manifolds which will include nonlinear dispersion relations and parametric dependence of solutions with various stable and nonstable nonlinear properties. Such Finsler structures are determined naturally from perturbative properties and light/probing bodies propagations in HL gravity. A Finsler generalization of HL gravity can be constructed in metric compatible form following principles very similar to the Einstein and Einstein–Finsler gravity (EFG) [34, 35, 33], for metric compatible Finsler connections. The gravitational field equations for such a theory can be integrated in general form following methods [50, 51, 52] (with parametric dependence of solutions which allows us to consider stable and non–stable configurations). It is also possible to quantize certain classes of Hořava–Finsler (HF) gravity models following methods of deformation quantization, A–brane formalism, gauge like methods etc, see [53, 54, 55, 56, 57, 58].

---

8for instance, in the very special relativity, with corrections from string and/or non-commutative dynamics, with Higgs type induced Finsler structures etc
3 Hořava–Finsler Gravity

In this section, we provide a Finsler generalization of the HL theory (called the Hořava–Finsler, in brief, HF) which will include as some ”branch” configurations respective MDR on tangent bundle $T V$, where $V$ is a (pseudo) Riemannian spacetime in GR or its anisotropic modifications defined by a HL action [3].

3.1 Fundamental geometric objects for HF gravity

We shall label local coordinates on $T V$ in the form $u^\alpha = (x^i, y^a)$ (in brief $u = (x, y)$), where $x^i$ are local coordinates on $V$ and $y^a$ are fiber (velocity, or momentum type) coordinates. Indices $\alpha, \beta, ...$ will run values 1, 2, ..., 8.

Contrary to the case of (pseudo) Riemannian geometry (which is completely determined by its metric tensor), a fundamental Finsler metric (equivalently, generating function) $F^2(6)$ and/or its Hessian $F_{ij}(7)$ do not define completely a geometric/physical model on $T V$. We need certain additional assumptions in order to construct in a unique form a triple of fundamental geometric objects (a nonlinear connection, N–connection, structure, a metric structure on the total space and a linear connection which is adapted to a chosen N–connection structure, called a distinguished connection, in brief, a d–connection; in a canonical approach all such objects are induced in a unique way by fundamental Finsler function $F$), which are necessary for definition of a physical generalized spacetime/gravitational model using principles of Einstein–Finsler gravity (EFG) [34, 33].

3.1.1 N–connections induced by MDR and associated Finsler generating functions

A N–connection $N$ is defined as a Whitney sum

$$TTV = hTV \oplus vTV.$$  (8)

With respect to a local coordinate base, it is determined by its coefficients $N = \{N^a_i(x, y)\}$, i.e. $N = N^a_i dx^i \otimes \partial/\partial y^a$ [3]. There is a class of associated to

---

Following our notation conventions [33, 36], we use boldface symbols for spaces and geometric object on spaces endowed with N–connection structure. Because there are standard denotations using symbol $N$ both in ADM model of gravity and in Finsler geometry, we have to use $(N, N^a)$ for lapse and shifting functions and $N^a_i$ for the N–connection coefficients.

---
N–connection local bases, \( \mathbf{e}_\nu = (\mathbf{e}_i, e_a) \), and cobases, \( \mathbf{e}^\mu = (e^i, \mathbf{e}^a) \), when
\[
\mathbf{e}_i = \frac{\partial}{\partial x^i} - N^a_i(u) \frac{\partial}{\partial y^a} \quad \text{and} \quad e_a = \frac{\partial}{\partial y^a}.
\]
\[
e^i = dx^i \quad \text{and} \quad \mathbf{e}^a = dy^a + N^a_i(u) dx^i.
\]
Such a structure is, in general, nonholonomic (equivalently, anholonomic/ non–integrable) because, for instance, \([9]\) satisfy nontrivial nonholonomy relations of type
\[
[e_\alpha, e_\beta] = e_\alpha e_\beta - e_\beta e_\alpha = W^\gamma_{\alpha\beta} e_\gamma,
\]
with (antisymmetric) nontrivial anholonomy coefficients \(W^b_{ia} = \partial_a N^b_i\) and \(W^a_{ji} = \Omega^a_{ij}\) determined by the coefficients of curvature of N–connection \(\Omega^a_{ij} = e_j(N^a_i) - e_i(N^a_j)\). It should be emphasized here that there is a N–connection structure \(\mathbf{N} = \mathbf{N}\) which is canonically defined by \(F^{10}\). Under general (co) frame/coordinate transform, \(\mathbf{e}^a \rightarrow \mathbf{e}^{a'} = e^{a'} e^a\) and/or \(u^a \rightarrow u^{a'} = u^{a'}(u^a)\), preserving the splitting \([8]\), we get a corresponding transformation law \( cN^a_j \rightarrow N^a_j\), when \(\mathbf{N} = N^a_j(u) dx^i \otimes \frac{\partial}{\partial y^a}\) is given locally by a set of coefficients \(\{N^a_j\}\) (we shall omit priming, underlying etc of indices if that will not result in ambiguities)\([11]\).

3.1.2 Finsler metric structure on total tangent bundle

We can use the so–called Sasaki lift in order to construct on \(TV\) a metric structure completely determined by a fundamental Finsler function \(F(x, y)\),
\[
F_{\mathbf{g}} = (h F_{g_{ij}}, v F_{g_{ij}}) = F_{g_{ij}(x, y)}[ e^i \otimes e^j + (^*L)^2 F e^i \otimes F e^j],
\]
\[
e^i = dx^i \quad \text{and} \quad F e^a = dy^a + F N^a_i(u) dx^i.
\]

\(^{10}\) Considering \(L = F^2\) as a regular Lagrangian (i.e. with nondegenerate \(F g_{ij}\))

\(^{11}\) We can use any convenient (for constructing exact solutions of field equations, or geometric considerations) equivalent sets \(\mathbf{N} = \{N^a_j\}\), which under corresponding frame/coordinate transform can be parametrized in a form \(^c\mathbf{N} = ^c\mathbf{N} = \{ ^cN^a_j\}\). Here, we also emphasize that we can define conventionally a N–connection structure on any manifold (not only on tangent/vector bundles) by prescribing a fibered structure with conventional horizontal (h) and vertical (v) splitting, for instance, a nonholonomic 2+2 splitting in GR as we considered in \([50, 52]\).
where for canonical constructions $F = N$. In the above formula we consider a length constant $*l_P$ which can be just the Planck length $l_P$ in models of QG for the GR but it can be a different one for brane models. We have to consider such a value before the v–part of metric (12) in order to have the same dimensions for the h– and v–components of metric when coordinates have the dimensions $[x^i] = cm$ and $[y^i \sim dx^i/ds] = cm/cm$. In our further considerations, we shall include such a constant into $h$–coefficients of metrics if that will not result in ambiguities.

Under general frame transforms $e^{\alpha'}_{\alpha} = e^{\alpha'}_{\alpha} e^\alpha_{\alpha}$, the above Finsler metric can be represented in a general 4+4 form

$$Hg = (h_{ij}, v_{ab}) = Hg_{\alpha\beta}(x, y) e^\alpha \otimes e^\beta,$$

for arbitrary $N^a_i$ (we put the left label $H$ in order to emphasize that such a metric is induces by MDR and nonholonomic deformations from HL gravity). With respect to a coordinate co-basis $du^\beta = (dx^j, dy^b)$, when $\partial_\alpha = \partial/\partial u^\alpha = (\partial_i = \partial/\partial x^i, \partial_a = \partial/\partial y^a)$, both metrics can be transformed equivalently into

$$Hg = H g_{\alpha\beta}(u) du^\alpha \otimes du^\beta,$$

where

$$H g_{\alpha\beta} = \begin{bmatrix} g_{ij} + (*l_P)^2 h_{ab} N^a_i N^b_j & (*l_P)^2 h_{ae} N^e_j \\ (*l_P)^2 h_{be} N^e_i & (*l_P)^2 h_{ab} \end{bmatrix}.$$  

The values $N^a_i(u)$ should be not identified to certain gauge fields in a Kaluza–Klein theory on tangent bundle with the potentials depending on velocities if we do not consider compactifications on coordinates $y^a$. In Finsler like theories, a set $\{N^a_i\}$ defines a N–connection structure, with elongated partial derivatives (13).

We can invert the constructions for arbitrary (15) and/or (14) and introduce Finsler variables and define metric (12) by prescribing an arbitrary generating function $F$ on a manifold or bundle space.

### 3.1.3 The canonical distinguished Finsler connection

In order to perform self–consistent geometric constructions with h– and v–splitting, it was introduced the concept of distinguished connection (in brief, d–connection). A d–connection $D = (hD, vD)$ is defined as a linear one preserving under parallelism the N–connection structure on $V$. 

11
The N–adapted components $\Gamma^\alpha_{\beta\gamma}$ of a d–connection $\bf{D}$ are computed following equations $\bf{D}_\alpha e_\beta = \Gamma^\gamma_{\alpha\beta} e_\gamma$ and parametrized in the form $\Gamma^\gamma_{\alpha\beta} = \left( L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc} \right)$, where $\bf{D}_\alpha = (D_i, D_a)$, with $h\bf{D} = (L^i_{jk}, L^a_{bk})$ and $v\bf{D} = (C^i_{jc}, C^a_{bc})$ defining certain covariant, respectively, h– and v–derivatives.

The simplest way to perform computations with a d–connection $\bf{D}$ is to associate it with a N–adapted differential 1–form $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} e_\gamma$, and apply on $T\bf{V}$ the well known formalism of differential forms as in GR. For instance, the torsion of $\bf{D}$ is defined/computed

$$T^\alpha \equiv De^\alpha = de^\alpha + \Gamma^\alpha_{\beta\gamma} e_\beta \wedge e_\gamma. \quad (18)$$

With respect to a N–adapted basis, this torsion is stated by $T = \{ T^i_{\alpha\beta}, T^i_{ja}, T^a_{bi}, T^a_{bc} \}$, where the nontrivial coefficients are

$$T^i_{jk} = L^i_{jk} - L^i_{kj}, \quad T^i_{ja} = C^i_{ja}, \quad T^a_{ji} = -\Omega^a_{ji}, \quad T^a_{bc} = C^a_{bc} - C^a_{cb}. \quad (19)$$

There is a canonical d–connection $\bf{D}^\sim = \{ \bf{\hat{D}}^{\Gamma^\gamma_{\alpha\beta}} = (\bf{\hat{L}}^i_{jk}, \bf{\hat{L}}^a_{bk}, \bf{\hat{C}}^i_{jc}, \bf{\hat{C}}^a_{bc}) \}$, which is uniquely and completely defined by the coefficients of metric $\bf{g}$ (equivalently, $\bf{\hat{D}}^{\Gamma^\gamma_{\alpha\beta}}$ and/or $\bf{\hat{D}}^{\Gamma^\gamma_{\alpha\beta}}$) following the metric compatibility conditions that $\bf{\hat{D}}g = 0$ and the “pure” horizontal and vertical torsion coefficients are zero, i.e., $\bf{\hat{T}}^i_{jk} = 0$ and $\bf{\hat{T}}^a_{bc} = 0$,

$$\bf{\hat{L}}^i_{jk} = \frac{1}{2} g^{iv} (e_k g_{jv} + e_j g_{kv} - e_r g_{jk}), \quad (20)$$

$$\bf{\hat{L}}^a_{bk} = e_b(N^c_k) + \frac{1}{2} h^{ac} \left( e_k h_{bc} - h_{dc} e_b N^d_k - h_{db} e_c N^d_k \right),$$

$$\bf{\hat{C}}^i_{jc} = \frac{1}{2} g^{ik} e_c g_{jk}, \quad \bf{\hat{C}}^a_{bc} = \frac{1}{2} h^{ad} (e_c h_{bd} + e_c h_{cd} - e_d h_{bc}).$$

Such a d–connection contains nontrivial torsion components $\bf{\hat{T}}^i_{ja}, \bf{\hat{T}}^a_{ji}, \bf{\hat{T}}^c_{aj}$, i.e., in general, $\bf{\hat{T}} \neq 0$. It is very different from various types of torsions in Einstein–Cartan, gauge, string and other type gravity theories (for which

---

12 For any of type of metric parametrizations $\bf{\Gamma^\gamma_{\alpha\beta}}$, $\bf{\hat{D}}^{\Gamma^\gamma_{\alpha\beta}}$ and/or $\bf{\hat{D}}^{\Gamma^\gamma_{\alpha\beta}}$, we can construct the Levi–Civita connection $\nabla = \{ \Gamma^\gamma_{\alpha\beta} \}$ on $\bf{V}$ in a standard form. This connection is not used in Finsler geometry and generalizations because it is not compatible with a N–connection splitting; under parallel transports with $\nabla$, it is not preserved the Whitney sum $\bf{\hat{D}}g$. 

---

12
additional field equations are defined) because its N–adapted components are completely by the metric structure, which in its turn (in our model) is related to MDR in HL gravity - we do not need additional field equations for this type of torsions induced nonholonomically via the N–connection structure.

Via nonholonomic transforms, we can transform $H\hat{D}$ into the Cartan d–connection $H\tilde{D}$ in Finsler geometry which is also metric compatible and completely defined by the same metric structure. On spaces of even dimensions such connections contain the same physical information if a Finsler generating function $F$ on spacetime manifold. The metric compatibility play a crucial role in defining Finsler generalizations of gravity in an "almost standard form" following principles which are similar to those in GR (it is a more sophisticate task to elaborate viable physical models using metric noncompatible connections, for instance, the so–called Chern connection for Finsler geometry, see critical remarks and details in Refs. [37, 34, 33]).

3.1.4 Nonholonomic deformations relating HF and HL metrics

The Horava–Finsler (HF) gravity theory is a (pseudo) Finsler geometry model induced canonically on $TV$ by a Finsler generating function $F$ associated to MDE relations in "standard" HL gravity. Such a theory is determined by the data $[F : Fg = (hFg_{ij}, vFg_{ij}), FN, FD = H\hat{D}]$, where (up to frame transforms)

$$Fg_{ij}(x, y) \sim g_{ij}(x, y) = e^i_j(x, y)e^j_i(x, y) \text{HL} g_{ij}(x),$$

$13$ Any geometric/physical construction for $\hat{D}$ can be re–defined equivalently into a similar one with the Levi–Civita connection following formula

$$\Gamma^\gamma_{\alpha\beta} = \tilde{\Gamma}^\gamma_{\alpha\beta} + Z^\gamma_{\alpha\beta},$$

where the distortion tensor $Z^\gamma_{\alpha\beta}$ is given by nontrivial coefficients

$$Z^a_{jk} = -\tilde{\Gamma}^i_{jk}h^{ab} - \frac{1}{2} \Omega^a_{jk}, Z^b_{hk} = \frac{1}{2} \Omega^a_{jk}h_{cb}g^{ji} - \Xi^b_{hk},$$

$$Z^b_{kj} = -\Xi_{cd}^k \tilde{T}^c_{kd}, Z^c_{bk} = \frac{1}{2} \Omega^a_{jk}h_{cb}g^{ji} + \Xi^b_{hk} \tilde{\Gamma}^i_{jk}, Z^c_{jk} = 0,$$

$$Z^c_{ia} = -\Xi_{cd}^a \tilde{T}^d_{ja}, Z^a_{bc} = 0, Z^i_{ab} = \frac{g^ij}{2} \left[ T^c_{ja}h_{cb} + \tilde{T}^c_{ja}h_{ca} \right],$$

for $\Xi^b_{hk} = \frac{1}{2} (\delta^b_{h} \delta^k_{k} - g_{hk}g^{ih})$ and $\Xi_{cd}^a = \frac{1}{2} (\delta^a_{c} \delta^d_{d} + h_{cd}h^{ab})$.

$14$ the word standard is an approximation because up till present there are different versions of HL with, or not, detailed balance conditions, generalized forms etc

13
for $^{HL}g_{ij}(x)$ being a solution of gravitational field equations in HL gravity on $V$, derived from action $^3$. The values $e'^i_j(x,y)$ and $h_{bc}(x,y)$ have to be defined from certain solutions of gravitational field equations in HF gravity, see next section 3.2.

If the conditions (4) are imposed in HF gravity, we can state such limits that the model defines an **Einstein–Finsler gravity** theory (EFG) $^34$. This class of metric compatible Finsler gravity theories on $TV$ is defined by data $[F: F^g=(h^Fg_{ij},v^Fg_{ij}), F^N,F^D=\hat{D}]$, when $F^g_{ij}(x,y) \sim g_{ij}(x,y) = e'^i_j(x,y)E_{ij}(x,y)$, for $E_{ij}(x)$ being a solution of the Einstein equations in GR. Via nonholonomic frame transforms, the theory can be equivalently described in standard variables of GR with $[g, g^\nabla]$.

In explicit form, we have to elaborate a natural trapping/warped mechanisms defined by explicit solutions of (Finsler type) gravitational field equations which in classical limits for $^*l_P \to 0$, when HF / EFG $\to$ HL, or GR, determining QG corrections to gravitational and matter field interactions at different scales depending on the class of considered models and solutions, see below section 4.

### 3.2 Field equations in canonical Hořava–Finsler gravity

A canonical (pseudo) Finsler structure on $TV$ determined by MDR in HL gravity contains already all anisotropic properties which are contained in metrics $^1$ with scaling properties $^2$ included in the $h$–part of corresponding N–adapted metric $^14$ and/or $^12$. We elaborate a Hořava–Finsler gravity theory not just lifting formally the geometric objects and action $^3$ on $TV$ (geometrically such a procedure can be defined in a canonical way). There are not experimental data about matter fields and their energy–momentums on tangent/vector bundles. A "simple" approach is to develop a Finsler brane gravity model with general assumptions on matter field in the bulk and warping/trapping of matter on a 4–d base spacetime (see details in Refs. $^35$ and, for some yearly off–diagonal constructions with N–connection structure for generalized Rundall–Sundrum scenarios, $^11$ $^12$, and references therein). A well–defined trapping mechanism with effective (in general, anisotropically polarized) cosmological constant and maximal speed of light (as solutions for the bulk HF gravity) allows us to simplify substantially the constructions related to possible models of HF gravity which can transform in the quasi–classical limit into the HL or/and GR theories.

Using the canonical d–connection 1–form of type $^{17}$, with coefficients
we can compute the curvature of $\hat{H}\hat{D}$,

$$
H\hat{\nabla}_{\alpha\beta}^{\gamma} \quad (20)
$$

$(\hat{H}\hat{\nabla}_{\alpha\beta}^{\gamma} - \hat{H}\hat{\nabla}_{\beta \gamma}^{\alpha} - H\hat{\nabla}_{\gamma}^{\alpha} = \hat{R}_{\alpha\beta\gamma\delta}^{\gamma} e^{\gamma} \wedge e^{\delta} , \quad (21)\quad$(21)

see details in $\[34, 33\]$ where the formulas for all coefficients are given in explicit form. The Ricci d–tensor $\hat{\text{Ric}} = \{\hat{R}_{\alpha\beta}\}$ is defined by contracting respectively the components of curvature tensor, $\hat{R}_{\alpha\beta} = \hat{R}_{\alpha\beta} = \hat{R}_{\alpha\beta}^{\tau}$. The h–/v–components of this d–tensor, $\hat{R}_{\alpha\beta} = \{\hat{R}_{ij}, \hat{R}_{ia}, \hat{R}_{ai}, \hat{R}_{ab}\}$ are

$\hat{R}_{ij} := \hat{\nabla}_{ij}^{k}, \quad \hat{R}_{ia} := -\hat{\nabla}_{ika}, \quad \hat{R}_{ai} := \hat{\nabla}_{aib}, \quad \hat{R}_{ab} := \hat{\nabla}_{abc}$. $(22)\quad$

The scalar curvature of $\hat{H}\hat{D}$ is constructed by using the inverse to $g$

$\hat{\text{R}} := g^{\alpha\beta} \hat{R}_{\alpha\beta} = g^{ij} \hat{R}_{ij} + h^{ab} \hat{R}_{ab} = \hat{R} + \hat{S}, \quad (23)\quad$

where $\hat{R} = g^{ij} \hat{R}_{ij}$ and $\hat{S} = h^{ab} \hat{R}_{ab}$ are respectively the h– and v–components of scalar curvature.

The Einstein tensor for $\hat{H}\hat{D}$ is, by definition,

$$
\hat{H}\hat{\text{E}}_{\alpha\beta} := \hat{R}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \hat{\text{R}}. \quad (24)\quad$

We can postulate the gravitational field equation for the HF gravity on $TV$ in the form

$$
\hat{H}\hat{\text{E}}_{\alpha\beta} = \hat{\Upsilon}_{\beta\delta}, \quad (25)\quad$

for arbitrary sources $\hat{\Upsilon}_{\beta\delta}$ which can be, as a matter of principle, defined as certain lifts of energy–momentum tensors of matter fields in HL, or GR, theory. It should be emphasized here that the solutions of equations $(25)$, for ”projections” $TV \rightarrow V$, in general, do not transform trivially into solutions of HL gravity with action $(3)$. Certain warped/trapping scenarios can be constructed in such a form that nonholonomic deformations of exact solution in HF brane gravity are, in general, non–explicitly related to solutions in HL gravity. This is a consequence of nontrivial nonholonomic structure and generic nonlinear character of such locally anisotropic gravitational systems.

### 3.3 Magic splitting of gravitational HF filed equations

The gravitational field equations in HF gravity can be integrated in very general forms on $TV$ following the anholonomic deformation method summarized in Refs. $\[50, 51, 52\]$ (necessary ”velocity” type coordinated should be treated as certain ”extra” dimension to two/four dimensional base
space ones). The bulk of such solutions do not have obvious implications in modern physics. For simplicity, in this work we shall use a more restricted class of exact solutions in HF gravity which seem to be related to models of Finsler branes.

We parametrize the metric \([\mathbf{14}]\) in a form with three "shell" anisotropy (with a nonholonomic splitting 2+2 and 2+2+2),

\[
\mathbf{g} = g_{ij}(x)dx^i \otimes dx^j + h_{\alpha \beta}(x, 0) e^{\alpha} \otimes e^{\beta} + h_{1\alpha} 1_b(x, 0) e^{1\alpha} \otimes e^{1b} + h_{2\alpha} 2_b(x, 0, 1, y, 2y) e^{2\alpha} \otimes e^{2b},
\]

for local \(x = \{x^i\}, 0 = \{y^0\}, 1 = \{y^1\}, 2 = \{y^2\}\); the vertical indices and coordinates split in the form \(y = \{0, 1, 2, 3\}\), or \(y^a = [y^0, y^1, y^2, y^3]\); \(0 = (2, 0, 1, 2)\), \(1 = (2, 0, 1, 2)\), or \(u = (u^0, u^1, u^2, u^3)\). There is a "less" general ansatz of type \([26]\) (with Killing symmetry on \(y^8\), when the metric coefficients do not depend on variable \(y^8\); it is convenient to write \(y^3 = 0v\), \(y^5 = 1v, y^7 = 2v\) and express the \(N\)–coefficients via \(v–\)functions)

\[
\text{sol} \mathbf{g} = g_{ij}(x^k)dx^i \otimes dx^j + h_{\alpha \beta}(x^k, 0) e^{\alpha} \otimes e^{\alpha} + h_{1\alpha} 1_b(x^k, 0) e^{1\alpha} \otimes e^{1b} + h_{2\alpha} 2_b(x^k, 0, 1, y, 2y) e^{2\alpha} \otimes e^{2b},
\]

The HF gravitational field equations \([25]\) for the canonical d–connection \(\hat{\mathbf{D}}\) can be solved in general forms for ansatz \([27]\) and sources parametrized with respect \(N\)–adapted frames in the form

\[
\hat{\mathbf{Y}}_\beta^\alpha = \text{diag}[\hat{\mathbf{Y}}^1_1 = \hat{\mathbf{Y}}^2_2 = \hat{\mathbf{Y}}^3_2 = \hat{\mathbf{Y}}^4_4 = \hat{\mathbf{Y}}^5_5 = \hat{\mathbf{Y}}^6_6 = \hat{\mathbf{Y}}^7_7 = \hat{\mathbf{Y}}^8_8],
\]

when the coefficients are subjected to algebraic conditions (for vanishing \(N\)–coefficients, containing respectively the functions \([A, 2]\) determining sources
in the gravitational field equations)

\[ h^{i} \Lambda(x^i) = \tilde{\gamma}_1 + \tilde{\gamma}_6 + \tilde{\gamma}_8, \quad v^i \Lambda(x^i, v) = \tilde{\gamma}_2 + \tilde{\gamma}_6 + \tilde{\gamma}_8, \quad (29) \]

\[ 1^{\Lambda}(u^\alpha, y^5) = \tilde{\gamma}_2 + \tilde{\gamma}_4 + \tilde{\gamma}_8, \quad 2^{\Lambda}(u^{1\alpha}, y^7) = \tilde{\gamma}_2 + \tilde{\gamma}_4 + \tilde{\gamma}_6. \]

Introducing the coefficients of metric (27) into the formulas for d–connection (20 after tedious calculations (see details in [50][51])) we obtain

\[
\begin{align*}
\hat{R}_1 &= \hat{R}_2^5 = \frac{1}{2g_1g_2} \left[ g_{\cdot\cdot} - \frac{g_{1^*}g_{2^*}}{2g_1} - \frac{(g_2^2)^2}{2g_2} + g_{1''} - \frac{g_{1^*}g_{2^*}}{2g_2} \right] = - h^{i} \Lambda(x^i), \\
\hat{R}_3 &= \hat{R}_4^3 = - \frac{1}{2h_3h_4} \left[ h_{\cdot\cdot\cdot} - \frac{(h_4^*)^2}{2h_4} - \frac{h_5^* h_4^*}{2h_3} \right] = - v^i \Lambda(x^i, y^3),
\end{align*}
\]

\[
\begin{align*}
\hat{R}_{3k} &= \frac{w_k}{2h_4} \left[ h_{\cdot\cdot\cdot} - \frac{(h_4^*)^2}{2h_4} - \frac{h_5^* h_4^*}{2h_3} \right] + \frac{h_4^*}{2h_4} \left( \frac{\partial k h_3}{h_3} + \frac{\partial k h_4}{h_4} - \frac{\partial h_4^*}{2h_4} \right) = 0 \quad (32) \\
\hat{R}_{4k} &= \frac{h_4}{2h_3} n_{k^*} + \left( \frac{h_4 h_3^*}{h_3} - \frac{3}{2} h_4^* \right) \frac{n_k^*}{2h_3} = 0, \quad (33)
\end{align*}
\]

where certain differential derivatives are denoted in the form \( a^\bullet = \partial a/\partial x^1 \), \( a' = \partial a/\partial x^2 \), \( a^* = \partial a/\partial y^3 \), and (extra to 4–d “shell” equations)

\[
\begin{align*}
\hat{R}_5 &= \hat{R}_6 = - \frac{1}{2h_5 h_6} \left[ 2v_{1^*} h_{6^*} - 2v_{1^*} h_{6^*} - (\partial_{1^*} h_{6^*})^2 - \frac{2h_5}{2h_5} \right] \\
&= - 1^{\Lambda}(u^\alpha, y^5), \\
\hat{R}_7 &= \hat{R}_8 = - \frac{1}{2h_7 h_8} \left[ 2v_{2^*} h_{8^*} - 2v_{2^*} h_{8^*} - (\partial_{2^*} h_{8^*})^2 - \frac{2h_7}{2h_7} \right] \\
&= - 2^{\Lambda}(u^{1\alpha}, y^7),
\end{align*}
\]

\[
\begin{align*}
\hat{R}_5^{\alpha} &= \frac{1 w_{\alpha}}{2 h_6} \left[ 2v_{1^*} h_{6^*} - (\partial_{1^*} h_{6^*})^2 - \frac{2h_5}{2h_5} \right] \\
&+ \frac{1}{4h_6} \left( \frac{\partial_{\alpha} h_5}{h_5} + \frac{\partial_{\alpha} h_6}{h_6} \right) - \frac{2 h_5}{2h_5} = 0, \quad (34) \\
\hat{R}_6^{\alpha} &= \frac{h_6}{2h_5} 2v_{1^*} h_{6^*} - \alpha + \left( \frac{h_6}{h_5} \frac{\partial_{1^*} h_5}{h_5} - \frac{3}{2} \frac{\partial_{1^*} h_6}{h_6} \right) \frac{1 n_{\alpha}}{2h_5} = 0,
\end{align*}
\]
general solutions depend on integration functions depending on coordinates $y^i$. Integrating two times on conditions for such integration functions.

$v^{\alpha \beta}$

$w^i_v$ integrating two times on $y^i$ for a given $\Upsilon^i$ such way organized that the resulting in $[51, 34, 33]$.

For physical considerations, we have to consider well defined boundary conditions if certain partial derivatives are zero, or not), see detailed analysis, discussions possible applications in modern gravity and cosmology in [51, 34, 33].

Let us explain using the set of equations (30)–(33) the property of separation of equations for ansatz of type (27). For a HL model with given matter fields on $\mathbf{V}$, we construct the energy–momentum tensor $T_{ij}$. We can consider a nonholonomic lift on $T \mathbf{V}$ such way organized that the resulting in $\Upsilon^i = diag[Y^1_1 = Y^2_2 = \Upsilon^4_4(x^k, y^3), Y^3_3 = Y^4_4 = \Upsilon^2_2(x^k)]$ (using corresponding nonholonomic distributions and transforms, various types of physically motivated energy–momentum tensors can be parametrized in such a diagonal form with respect to $N$–adapted frames). Taking the value $\Upsilon^2(x^k)$, we can define $g_1(x^k)$ (or, inversely, $g_2(x^k)$) for a given $g_2(x^k)$ (or, inversely, $g_1(x^k)$) as an explicit, or non–explicit, solution of (30) by integrating two times on $h$–variables. Similarly, for a given $\Upsilon^4(x^k, y^3)$, we solve (31) by integrating one time on $y^3$ and defining $h_3(x^k, y^3)$ for a given $h_3(x^k, y^3)$ (or, inversely, by integrating two times on $y^3$ and defining $h_4(x^k, y^3)$ for a given $h_3(x^k, y^3)$).

After we determined the values $g_i(x^k)$ and $h_{\alpha \beta}(x^k, y^3)$, we can compute the coefficients of $N$–connection: The functions $w_j(x^k, y^3)$ are solutions of algebraic equations (32). Integrating two times on $y^3$, we find $n_j(x^k, y^3)$. The general solutions depend on integration functions depending on coordinates $x^k$. For physical considerations, we have to consider well defined boundary conditions for such integration functions.
4 Finsler Branes and Trapping to HL and GR

In this section, we analyze brane models when the 4–d Horava–Lifshitz theory is embedded into 8–d Finsler spaces with non–factorizable velocity type coordinates (experimentally, the light velocity is finite). We shall adapt to nonholonomic and/or scale anisotropic configurations some ideas and methods from Refs. [35, 66, 67, 68, 69, 70, 71] when various trapping/localizing mechanisms for various spins (0, 1/2, 1, 2) on the 4–d brane/observable spacetime were analyzed.

We have to consider warped Finsler geometries and analyze trapping mechanisms because there are not experimental data for Finsler like metrics depending on coordinates and velocities. Such dependencies can be always derived in various isotropic and anisotropic QG models with nonlinear dispersions. We expectations that brane trapping effects may allow us to detect QG and LV effects experimentally even at scales much large than the Planck one. On Finsler branes, we can consider that gravitons are allowed to propagate in the bulk of a Finsler spacetime with dependence of geometric/physical objects on velocity/ momentum coordinates.

4.1 An ansatz for generating HF–brane solutions

For constructing brane solutions in EFG, we use the ansatz for a class of metrics which via frame transform can be parametrized in the form

\[
g = \phi^2(y^5)[g_1(x^k) \ e_1 \otimes e_1 + g_2(x^k) \ e_2 \otimes e_2 \\
+ h_3(x^k, v) \ e_3 \otimes e_3 + h_4(x^k, v) \ e_4 \otimes e_4] \\
+ (\ast l_P)^2 [h_5(x^k, v, y^5) \ e_5 \otimes e_5 + h_6(x^k, v, y^5) \ e_6 \otimes e_6] \\
+ (\ast l_P)^2 [h_7(x^k, v, y^5, y^7) \ e_7 \otimes e_7 + h_8(x^k, v, y^5, y^7) \ e_8 \otimes e_8],
\]

where

\[
\begin{align*}
\mathbf{e}_3 &= dv + w_1 dx^i, \\
\mathbf{e}_4 &= dy^4 + n_1 dx^i, \\
\mathbf{e}_5 &= dy^5 + w_1 dx^i + w_3 dv + w_4 dy^4, \\
\mathbf{e}_6 &= dy^6 + n_1 dx^i + n_3 dv + n_4 dy^4, \\
\mathbf{e}_7 &= dy^7 + w_1 dx^i + w_3 dv + w_4 dy^4 + w_5 dy^5 + w_6 dy^6, \\
\mathbf{e}_8 &= dy^8 + n_1 dx^i + n_3 dv + n_4 dy^4 + n_5 dy^5 + n_6 dy^6,
\end{align*}
\]
for nontrivial N–connection coefficients

\[ N_i^3 = w_i(x^k, v), N_i^4 = n_i(x^k, v); \]
\[ N_i^5 = 1w_i(x^k, v, y^5), N_i^6 = 1w_3(x^k, v, y^5), N_i^7 = 1w_4(x^k, v, y^5); \]
\[ N_i^6 = 1n_i(x^k, v, y^5); N_3^6 = 1n_3(x^k, v, y^5), N_4^6 = 1n_4(x^k, v, y^5); \]
\[ N_i^7 = 2w_i(x^k, v, y^7), N_3^7 = 2w_3(x^k, v, y^7), N_4^7 = 2w_4(x^k, v, y^7), \]
\[ N_i^8 = 2n_i(x^k, v, y^7), N_3^8 = 2n_3(x^k, v, y^7), N_4^8 = 2n_4(x^k, v, y^7), \]
\[ N_5^8 = 2n_3(x^k, v, y^7), N_6^8 = 2n_4(x^k, v, y^7). \]

The local coordinates in the above ansatz (35) are labelled in the form \( x^i = (x^1, x^2), \) for \( i, j, ... = 1, 2; \) \( y^3 = v. \)

We can include solutions of HL gravity into (35) via polarization \( \eta– \)functions when

\[ g_i(x^k) = \eta_i(x^k, v) \circ g_i(x^k, v), h_a(x^k, v) = \eta_a(x^k, v) \circ h_a(x^k, v), \]
\[ N_i^3(x^k, v) = \eta_i^3(x^k, v) \circ w_i(x^k, v), N_i^4(x^k, v) = \eta_i^4(x^k, v) \circ n_i(x^k, v), \]

where "primary" data \( \{ g_i, h_a, w_i, n_i \} \) are defined for a solution of gravitational field equations derived from HL action (3), and gravitational polarizations \( \{ \eta_i, \eta_a, \eta_i^3, \eta_i^4 \} \) should be defined from the condition that the "target" data \( \{ g_i, h_a, w_i, n_i \} \) determine solutions of the system (30–33); the nontrivial \( \{ N_i^3, N_i^4, N_i^5, N_i^6, N_i^7, N_i^8, h_5, h_6, h_7, h_8 \} \) should be constructed as solutions of the system (34). For instance, we can take that some values with \( \circ \) are correspondingly given by solutions on extremal spherical and rotating black holes of Hořava gravity [72] and derive generic off–diagonal generalizations, for instance, with ellipsoidal configurations like we considered in Refs. [41] [44].

The purpose of this section is to construct and analyze physical implications of solutions of equations (25) and, in particular, (30)–(34) defined by ansatz (35) with, respectively, trivial and non–trivial N–connection coefficients (36). The diagonal scenario from HF to GR is outlined in brief in Appendix in a form very similar that for the diagonal transition from EFG to GR in Ref. [35]. In this work, we use the canonical d–connection instead of the Cartan d–connection.

### 4.2 Finsler brane solutions

One of the main goals of this work is to elaborate trapping scenarios for "true" Finsler like configurations with positively nontrivial N–connections as
solutions of nonholonomic gravitational equations \(^{(25)}\). The priority of such generic off–diagonal solutions is that they allows us to distinguish the QG phenomenology and effects with LV of (pseudo) Finsler type from that described by (pseudo) Riemannian ones (following analysis from Introduction section the last variant is less natural with very special types of nonlinear dispersions which must result in vanishing N–connection structures).

### 4.2.1 Separation of equations in HF models of brane gravity

We consider an ansatz \((35)\) multiplied to \(\phi^2(y^5)\) and with non–trivial N–connection coefficients \((36)\) and respective \(\eta\)–polarizations. We define the conditions when the coefficients generate exact solutions of \((25)\) for general sources of type \((29)\) and \((A.2)\). For such ansatz, the system of equations in HF gravity \((30)\)–\((34)\) (we label \(g_1 = g_2 = \epsilon_{\pm} e^{\psi(x^i)}\), for \(\epsilon_{\pm} = \pm 1\) transform into

\[
\epsilon_{\pm} \psi^{**}(x^i) + \epsilon_{\pm} \psi''(x^i) = 2 \, h \Lambda (x^i),
\]

\[
h^*_i(x^i, v) = 2 h_3(x^i, v) h_4(x^i, v) u \Lambda (x^i, v)/\hat{\phi}^*(x^i, v),
\]

\[
\partial_{y^\rho} h_6(u^\alpha, y^5) = 2 h_5(u^\alpha, y^5) h_6(u^\alpha, y^5) \, \Lambda(u^\alpha, y^5) / \partial_{y^\rho} \hat{\phi}(u^\alpha, y^5),
\]

\[
\partial_{y^\eta} h_8(u^\alpha_i, y^7) = 2 h_7(u^\alpha_i, y^7) h_6(u^\alpha_i, y^7) \, \Lambda(u^\alpha_i, y^7) / \partial_{y^\eta} \hat{\phi}(u^\alpha_i, y^7),
\]

and the solutions for N–connection coefficients,

\[
\beta(x^i, v) \, w_i(x^i, v) + \alpha_i(x^i, v) = 0,
\]

\[
1 \beta(u^\alpha_i, y^7) \, \gamma(u^\alpha_i, y^7) + 1 \alpha_i(u^\alpha_i, y^7) = 0,
\]

\[
2 \beta(u^\alpha_i, y^7) \, 2 w_i(u^\alpha_i, y^7) + 2 \alpha_i(u^\alpha_i, y^7) = 0,
\]

\[
n_i^{**}(x^i, v) + \gamma(x^i, v) n_i^*(x^i, v) = 0,
\]

\[
\partial_{y^\rho} \partial_{y^\sigma} \, n_{i \mu}(u^\alpha_i, y^7) + 1 \gamma(u^\alpha_i, y^7) \partial_{y^\rho} \, n_{i \mu}(u^\alpha_i, y^7) = 0,
\]

\[
\partial_{y^\eta} \partial_{y^\tau} \, 2 n_{i \mu}(u^\alpha_i, y^7) + 2 \gamma(u^\alpha_i, y^7) \partial_{y^\eta} \, 2 n_{i \mu}(u^\alpha_i, y^7) = 0,
\]

where

\[
\alpha_i = h_3^* \partial_i \hat{\phi}, \quad \beta = h_4^* \hat{\phi}^*, \quad \hat{\phi} = \ln \left| \frac{h_4^*}{\sqrt{|h_3 h_4|}} \right|, \quad \gamma = \left( \frac{|h_4|^{3/2}}{|h_3|} \right),
\]

\[
1 \alpha_{i \mu} = (\partial_{y^\rho} h_6) \partial_{y^\mu} \hat{\phi}, \quad 1 \beta = (\partial_{y^\rho} h_6)(\partial_{y^\eta} \hat{\phi}),
\]

\[
1 \hat{\phi} = \ln \left| \frac{\partial_{y^\rho} h_6}{\sqrt{|h_5 h_6|}} \right|, \quad 1 \gamma = \partial_{y^\rho} \left( \frac{|h_6|^{3/2}}{|h_5|} \right),
\]
\[ 2\alpha \ i_\mu = (\partial_{y^7} h_8) \partial_{i_\mu} \ 2\phi, \ 2\beta = (\partial_{y^7} h_8) (\partial_{y^7} 2\phi), \]
\[ 2\phi = \ln \left( \frac{\partial_{y^7} h_8}{\sqrt{|h_7 h_8|}} \right), \]
\[ 2\gamma = \partial_{y^7} \left( \frac{|h_8|^{3/2}}{|h_7|} \right), \]

for \( h_{3,4} \neq 0, \partial_{y^7} h_6 \neq 0, \partial_{y^7} h_8 \neq 0, \ b\Lambda \neq 0, \ v\Lambda \neq 0, \ b\Lambda \neq 0. \)

### 4.2.2 Exact solutions for HF brane models

The system of partial derivative equations (37) and (38) can be integrated in general form:

\[ g_1 = g_2 = e^{\pm \psi(x^k)}, \]

\[ h_4 = 0 h_4(x^k) \pm 2 \int \frac{\exp[2 \phi(x^i, v)]}{v\Lambda(x^i, v)}dv, \]

\[ h_3 = \pm \frac{1}{4} \left[ \sqrt{|h_4|^2} \right] \exp \left[ -2 \phi(x^i, v) \right] \]

\[ h_6 = 0 h_6(u^\alpha) \pm 2 \int \frac{\exp[2 \phi(u^\alpha, y^5)]}{1\Lambda(u^\alpha, y^5)}dy^5, \]

\[ h_5 = \pm \frac{1}{4} \left[ \sqrt{|\partial_{y^7} h_6(u^\alpha, y^5)|} \right] \exp \left[ -2 \phi(u^\alpha, y^5) \right], \]

\[ h_8 = 0 h_8(u^{1\alpha}) \pm \int \frac{\exp[2 \phi(u^{1\alpha}, y^7)]}{2\Lambda(u^{1\alpha}, y^7)}dy^7, \]

\[ h_7 = \pm \frac{1}{4} \left[ \sqrt{|\partial_{y^7} h_8(u^{1\alpha}, y^7)|} \right] \exp \left[ -2 \phi(u^{1\alpha}, y^7) \right], \]

and, for N-connection coefficients,

\[ w_i = -\partial_i \phi / \phi^*, \]

\[ n_k = n_k^{[0]}(x^i) + n_k^{[1]}(x^i) \int \left[ h_3/ \left( \sqrt{|h_4|} \right)^3 \right] dv, \]

\[ 1w_{\alpha} = -\partial_{\alpha}(1/\phi)/\partial_{y^7}(1/\phi), \]

\[ 1n_{\beta} = \left[ n_{\beta}^{[0]}(u^\alpha) + n_{\beta}^{[1]}(u^{1\alpha}) \int \left[ h_5/ \left( \sqrt{|h_6|} \right)^3 \right] dy^5 \right], \]

\[ 2w_{1\alpha} = -\partial_{1\alpha}(2/\phi)/\partial_{y^7}(2/\phi), \]

\[ 2n_{1\beta} = \left[ n_{1\beta}^{[0]}(u^{1\alpha}) + 2 n_{1\beta}^{[1]}(u^{1\alpha}) \int \left[ h_7/ \left( \sqrt{|h_8|} \right)^3 \right] dy^7. \]
The above presented classes of solutions with nonzero \( h^*_3, h^*_4, \partial_y^* h_5, \partial_y^* h_6, \partial_y^* h_7, \partial_y^* h_8 \) are determined by generating functions \( \hat{\phi}(x^i, v), \hat{\phi}^* \neq 0 \);

\[ \hat{\phi}(x^i, y^5), \partial_y^* 1 \hat{\phi} \neq 0, \quad 2 \hat{\phi}(x^i, y^5, y^7), \partial_y^* 2 \hat{\phi} \neq 0, \]

and integration functions \( n_k^{[0]}(x^i), n_k^{[1]}(x^i), \quad 1_n^{[0]}(u^\alpha), \quad 1_n^{[1]}(u^\alpha), \quad 2_n^{[0]}(u^\alpha), \quad 2_n^{[1]}(u^\alpha) \).

In order to construct explicit solutions, we have to choose and/or fix such functions following additional assumptions on symmetry of solutions, boundary conditions etc.

The coefficients (39) and (40) can be additionally constrained if we want to construct solutions for the Levi–Civita connection on \( TV \). By straightforward computations (see details in [50, 51, 52]), we can verify that all torsion coefficients vanish if

\[
\begin{align*}
    w^*_i &= \mathbf{e}_l \ln |h_4|, \quad \mathbf{e}_k w_i = \mathbf{e}_i w_k, \quad n^*_i = 0, \quad \partial_i n_k = \partial_k n_i, \\
    \partial_y^* (1 w_\alpha) &= \mathbf{e}_\alpha \ln |h_6|, \quad \mathbf{e}_\alpha 1 w_\beta = \mathbf{e}_\beta 1 w_\alpha, \\
    \partial_y^* (1 n_\alpha) &= 0, \quad \partial_\alpha 1 n_\beta = \partial_\beta 1 n_\alpha,
\end{align*}
\]

\[
\begin{align*}
    \partial_y^* (2 w_\alpha) &= 2 \mathbf{e}_\alpha \ln |h_8|, \quad 2 \mathbf{e}_\alpha 2 w_\beta = 2 \mathbf{e}_\beta 2 w_\alpha, \\
    \partial_y^* (2 n_\alpha) &= 0, \quad \partial_\alpha 2 n_\beta = \partial_\beta 2 n_\alpha.
\end{align*}
\]

Such conditions can be satisfied by imposing certain constraints on the considered classes of generating and integration functions. This class of generic off–diagonal solutions are important if we want to construct trapping configurations from the HF brane gravity to GR on \( V \), when the conditions (41) are imposed.

### 4.2.3 Remarks on (non) diagonal HF brane solutions on TV

The above solutions in HF gravity are still very general. It is not clear what physical meaning they may have and we must impose additional restrictions on some coefficients of metrics and sources in order to construct in explicit form certain Finsler brane configurations resulting in HL, or GR, theories and model a trapping mechanism with generic off–diagonal metrics.

There is a class of sources in HF gravity when for trivial \( N \)–connection coefficients (i.e. for zero values (36)) the sources \( \bar{\Upsilon}^{2\beta 2\delta} (28) \) transform into data (which in the diagonal limit we get sources for the gravitational equations for the Levi–Civita connection labeled with a left low bar) \( \bar{\Upsilon}^{2\beta 2\delta} (A.2) \), with nontrivial limits for \( \bar{\Upsilon}_\delta^\beta = \Lambda - M^{-(m+2)}K_1(y^5) \) and \( \bar{\Upsilon}_5^5 = \bar{\Upsilon}_6^6 = \).


\[ \Lambda - M^{-(m+2)}K_2(y^5), \] being preserved certain conditions of type \([29]\). The generating functions are taken in the form when

\[
h_5(x^i, y^5) = *l_P \frac{h(y^5)}{\phi^2(y^5)} q h_5(x^i, y^5),
\]

\[
h_6(x^i, y^5) = *l_P \frac{h(y^5)}{\phi^2(y^5)} q h_6(x^i, y^5),
\]

\[
h_7(x^i, y^5, y^7) = *l_P \frac{h(y^5)}{\phi^2(y^5)} q h_7(x^i, y^5, y^7),
\]

\[
h_8(x^i, y^5, y^7) = *l_P \frac{h(y^5)}{\phi^2(y^5)} q h_8(x^i, y^5, y^7),
\]

where the generating functions are parametrized in such a form that \(\phi^2(y^5)\) and \(h_5(y^5)\) are those for diagonal metrics and \(q h_5, q h_6, q h_7, q h_8\) are computed following formulas \([39]\) and \([40]\). This class of off–diagonal metrics are parametrized in the form

\[
g = g_1 dx^1 \otimes dx^1 + g_2 dx^2 \otimes dx^2 + h_3 e^3 \otimes e^3 + h_4 e^4 \otimes e^4 + (41)
\]

\[
( *l_P)^2 \frac{h}{\phi^2} [ q h_5 e^5 \otimes e^5 + q h_6 e^6 \otimes e^6 + q h_7 e^7 \otimes e^7 + q h_8 e^8 \otimes e^8 ],
\]

where

\[
e^3 = dy^3 + w_i dx^i, e^4 = dy^4 + n_i dx^i,
\]

\[
e^5 = dy^5 + w_i dx^i, e^6 = dy^6 + n_i dx^i,
\]

\[
e^7 = dy^7 + w_i dx^i, e^8 = dy^8 + n_i dx^i.
\]

Such off–diagonal parameterizations of metrics where considered in \([35]\) but the coefficients of the metric and N–connection where computed there for a different d–connection (for the Cartan d–connection).

Any solution of type \([41]\) describes an off–diagonal canonical nonholonomic trapping for 8–d (respectively, for corresponding classes of generating and integration functions, 5–, 6–, 7–d) to 4–d modifications of HL and/or GR with some corrections depending on bulk Finsler "fluctuations" and LV effects. There is a class of sources when for vanishing N–connection coefficients \([32]\) we get diagonal metrics of type \([35]\), considered in Appendix, but multiplied to a conformal factor \(\phi^2(y^5)\) when the \(h\)–coefficients are solutions of equations of type \([37]\).
With respect to local coordinate cobase $du^{2\alpha} = (dx^i, dy^a, dy^1a, dy^2a)$ a solution (41) is parametrized by an off–diagonal matrix $g_{\alpha\beta}$:

$$
B_{11} = w_1 h_3 + n_1 h_4 \quad B_{12} = w_1 h_3 + n_1 h_4 \quad B_{21} = n_2 h_4 \quad B_{22} = n_2 h_4
$$

$$
\begin{bmatrix}
B_{11} & B_{12} & w_1 h_3 & n_1 h_4 & 1 w_1 h_5 & 1 n_1 h_6 & 2 w_1 h_7 & 2 n_1 h_8 \\
B_{21} & B_{22} & w_2 h_3 & n_2 h_4 & 1 w_2 h_5 & 1 n_2 h_6 & 2 w_2 h_7 & 2 n_2 h_8 \\
w_1 h_3 & w_2 h_3 & h_3 & 0 & 0 & 0 & 0 & 0 \\
n_1 h_4 & n_2 h_4 & 0 & h_4 & 0 & 0 & 0 & 0 \\
1 w_1 h_5 & 1 w_2 h_5 & 0 & 0 & h_5 & 0 & 0 & 0 \\
1 n_1 h_6 & 1 n_2 h_6 & 0 & 0 & 0 & h_6 & 0 & 0 \\
2 w_1 h_7 & 2 w_2 h_7 & 0 & 0 & 0 & 0 & h_7 & 0 \\
2 n_1 h_8 & 2 n_2 h_8 & 0 & 0 & 0 & 0 & 0 & h_8
\end{bmatrix}
$$

where possible observable Finsler brane and LV contributions are distinguished by terms proportional to $(lP)^2$. The formulas for coefficients (39) and (40) computed (in this work) for the canonical d–connection describe an off–diagonal brane extension to a Finsler spacetime from the HL gravity with scaling anisotropy (we proved that there is a trapping mechanism encoded into such solutions relating HF and HL gravity models). For Finsler branes induced from a QG models with LV, based on "nonrenormalizable" GR (which we studied in the just mentioned our paper) the trapping scenario was modelled by the Cartan d–connection on a Finsler brane and the resulting configuration was certain one in "locally isotropic" GR theory.
5 Discussion and Conclusions

Summarizing the results of this paper (see also a series of partner works [34, 35, 45, 37]), we conclude that there are at least nine substantial arguments to consider that Finsler geometry and related geometric methods are of crucial importance in modern classical and quantum gravity, particle physics, cosmology and modifications:

1. The bulk of models of quantum gravity (QG) and related phenomenology are with Lorentz violation (LV) being characterized by corresponding modified dispersion relations (MDR). In its turn, such a MDR determines naturally a fundamental/generating Finsler function on (co)tangent space. This QG–LV–MDR–Finsler geometry scheme works for various models of QG with limits to, or warped/trapped, configurations derived in (super) string/brane/noncommutative/ analogous gravity/gauge gravity etc theories. The Hořava–Lifshitz (HL) theory with scaling anisotropy and MDR can be included into such a generalized Finsler gravity scheme.

2. Locally anisotropic structures and Finsler geometries are considered in analogous gravity, geometric mechanics and various models of condensed matter physics; certain important ideas and methods from physics of phase transitions are exploited in modern QG and phenomenology.

3. The ideas on restricted special relativity, scenarios with LV, modified/generalized Lorentz symmetries have straightforward relations to some special models of Finsler geometry, anisotropic symmetries and corresponding local/global transformation laws.

4. There are certain ideas and explicit constructions suggesting that various problems related to dark energy and dark matter physics, accelerating Universes, anisotropies etc can be cured by modifying the pseudo–Riemannian/ Lorentzian spacetime paradigm to (pseudo) Finsler spacetimes and generalizations.

5. Finsler like geometries are "canonically" generated/induced as exact solutions of nonholonomic Ricci flows of (pseudo) Riemannian metrics, and for various evolution scenarios with noncommutative and/or non-symmetric metrics, gravitational diffusion and stochastic processes, fractional derivatives and/or fractional dimensions, memory and self–organization etc.
6. Noncommutative generalizations of gravity theories can be modelled equivalently as complex Finsler like geometries.

7. Finsler configurations can be derived as exact solutions of gravitational field equations in GR, string, brane, gauge gravity theories.

8. Finsler geometry methods happen to be very efficient in elaborating a new geometric method (the so-called anholonomic deformation method) of constructing exact solutions in gravity, even for the general relativity (GR) theory. Such an approach allows us to generate very general classes of exact solutions of Einstein equations and generalizations (with generic off-diagonal metrics, linear and nonlinear connections and nonholonomic frame coefficients depending generally on all coordinates etc). Constraining nonholonomically certain general integral varieties of solutions with generalized connections, we obtain subvarieties for the Levi–Civita connections in GR. Such a method of constructing exact solutions can be applied in HL gravity.

9. Re–writing the Einstein gravity and/or certain generalizations in canonical Finsler variables, and then using almost Kähler equivalents, we can quantize various types of gravity theories using methods of deformation quantization, A–brane approach, nonholonomic canonical quantization etc. It seems that it is possible to renormalize gravity using the so–called bi–connection formalism and/or HL approach.

Following the above mentioned reasons, we consider that HL gravity should be extended in a form encoding also the physics of MDR, nonholonomic configurations and anisotropic configurations. In explicit form, we elaborated a model of Hořava–Finsler (HF) gravity following generalized relativity principles [34, 37, 33, 36]. We used metric compatible distinguished connections from Finsler geometry which allows us to formulate and study classical and quantum models following standard approaches with spinors, Dirac operators and vielbeins, metrics and connections as in GR but adapted to nonholonomic structures, in our approach, to nonlinear connections (N–connection).

The HF gravity theory is canonically formulated on tangent bundle. From a formal point of view, it is generally integrable and can be quantized/renormalized following standard methods. There are many open issues regarding HL and HF gravity models. Here, we emphasize the following. A sensible problem to be solved is that why classical limits do not contain anisotropies and dependencies on velocity/momentum type coordinates.
explicit form, we can apply certain ideas and methods from brane gravity which was studied intensively during last twelve years beginning Gogberashvili and Rundal–Sundrum works. Nevertheless, for Finsler branes and HF gravity, such constructions can not be applied in a straightforward form. Possible warping, trapping, compactification etc scenarios for Finsler spaces should encode, in general, a nontrivial N–connection structure. Technically, to construct such HF brane exact solutions with generic off–diagonal metrics is a very difficult task. One of the main results of this work is that we were able to solve and analyze such off–diagonal locally anisotropic trapping scenario from HF to HL and/or GR theories. Such a nonholonomic gravitational dynamics encode also in general form various types of MDR, parametric dependencies, possible generalized symmetries etc.

The length of this paper does not allow us to address the question of stability of Finsler brane solutions. In general, stable configurations can be constructed for diagonal solutions which survive for nonholonomically constrained off–diagonal ones (proofs are similar to those for extra dimensional brane solutions; we plan to study the problem in details in our further works). Hopeful, future work will concern various topics from QG with LV and Finsler geometry methods and possible applications in modern cosmology and astrophysics.

Acknowledgements: I’m grateful for support, hospitality and/or important discussions on generalized/modified Finsler gravity to G. Calcagni, E. Elizalde, M. Francavigla, C. Lämmerzahl, N. Mavromatos, S. Odintsov, D. Pavlov, V. Perlick, E. Radu, S. Sarkar, F. P. Schuller, L. Sindoni and P. Stavrinos. I thank E. Hatefi, A. Kobakhidze, F. Mercati and D. Orlando for some critical remarks and pointing additional references related to existing problems, present status and further developments of HL models. The research for this paper was partially supported by Romanian Government via Program IDEI, PN-II-ID-PCE-2011-3-0256.

A Holonomic Configurations for HF–branes

A trapping scenario from HF to GR with diagonal metrics can be constructed for a simplified ansatz \((35)\) with zero N–connection coefficients \((36)\) when \(h_5, h_7, h_8 = const\) and data \([g_i, h_a]\) define a trivial solution in GR and the local signature for metrics os of type \((+, −, −, ...)\).\(^\text{15}\) Such metrics can

\(^{15}\) The nonholonomic deformation method allows us to construct exact solutions with any signature we consider physically important; for Finsler brane configurations, we adapt the constructions form \(66\) \(67\) \(68\) \(69\) \(70\) \(71\) in nonholonomic form as in \(35\).
be written in the form
\[
g = \phi^2(y^5)\eta_{\alpha\beta}d\alpha \otimes d\beta - (l_P)^2 h(y^5)[d\gamma^5 \otimes d\gamma^5 + dy^6 \otimes dy^6 \pm dy^7 \otimes dy^7 \pm dy^8 \otimes dy^8],
\]
where \(\eta_{\alpha\beta} = \text{diag}[1,-1,-1,1]\) and \(\alpha,\beta,... = 1,2,3,4\); extra dimension indices will be considered of type \(^1\alpha = (\alpha,5,6)\) and \(^2\alpha = (\alpha,7,8)\). Indices of type \(^2\alpha,^2\beta,...\) will run values 1,2,3,4,5,.....,m where \(m \geq 2\). The fiber coordinates \(y^5,y^6,y^7,y^8\) are velocity type. We analyze here a toy model when sources are defined by a cosmological constant \(\Lambda\) and nonzero components of stress–energy tensor,

\[
\Upsilon^\beta_\delta = \Lambda - M^{-(m+2)}K_1(y^5), \quad \Upsilon^5_5 = \Upsilon^6_6 = \Lambda - M^{-(m+2)}K_2(y^5),
\]
for a fundamental mass scale \(M\) on \(T\mathbf{V}\), \(\dim T\mathbf{V} = 8\).

A metric (A.1) generates a solution of gravitational field equation in HF gravity if

\[
\phi^2(y^5) = 3\epsilon^2 + a(y^5)^2 \quad \text{and} \quad *l_P \sqrt{|h(y^5)|} = \frac{9\epsilon^4}{[3\epsilon^2 + (y^5)^2]^2}.
\]

In the above formulas we consider \(a\) as an integration constant and the width of brane is \(\epsilon\), with some fixed integration parameters when \(\frac{\partial^2 \phi}{\partial(y^5)^2} |_{y^5=e} = 0\) and \(*l_P \sqrt{|h(y^5)|} |_{y^5=0} = 1\); this states the conditions that on diagonal branes the Minkowski metric on \(T\mathbf{V}\) is 6-d, or 8-d.

The sources (A.2) are compatible with the field equations if

\[
K_1(y^5)M^{-(m+2)} = \Lambda + [3\epsilon^2 + (y^5)^2]^{-2}(2\frac{\phi m(\phi(m+2) - 3)}{3\epsilon^2}(y^5)^4 + 2(-2\frac{\phi m^2 + 2m + 6}{3\epsilon^2} + 3(m+3)(1 + \phi^2))(y^5)^2 - 6\epsilon^2 m(m - 3\phi + 2))
\]

\[
K_2(y^5)M^{-(m+2)} = \Lambda + [3\epsilon^2 + (y^5)^2]^{-2}(2\frac{\phi m - 1(\phi(m+2) - 4)}{3\epsilon^2} \times (y^5)^4 + 4(-\frac{\phi m^2 + m + 10}{3\epsilon^2} + 2(m+2)(1 + \phi^2))) \times (y^5)^2 - 6\epsilon^2 (m - 1)(m - 4\phi + 2))
\]

For Finsler branes, the width \(\epsilon^2 = 40M^4/3\Lambda\) is with extra velocity type coordinates and certain constants are related to \(*l_P\).
For the considered diagonal ansatz, the coefficients of the canonical d–
connection are the same as for the for the Levi–Civita connection when
\[
\nabla \chi_\alpha \chi^{\alpha} \chi^{\beta} = (\sqrt{|Fg|})^{-1} e^\chi_{\alpha} (\sqrt{|Fg|} \chi^{\alpha} \chi^{\beta}) + \Gamma^{\beta} \chi_{\alpha} \chi^{\gamma} \chi^{\alpha} \chi^{\gamma} = 0,
\]
(A.5)
which for the conditions (A.3) and (A.4) such a conservation law is satisfied if
\[
\frac{\partial \chi}{\partial \beta} = 4 (\chi_2 - \chi_1) \frac{\partial \ln |\phi|}{\partial \beta}.
\]
(A.6)

We constructed a metric (A.1) with coefficients subjected to conditions (A.3) – (A.6). Such a solution defines trapping solutions containing ”diagonal” extensions of GR to a 8–d TV and/or possible restrictions to 6–d and 7–d (to consider HF configurations, we have to include off–diagonal interactions). Such solutions provide also mechanisms of corresponding gravitational trapping for fields of spins 0, 1/2, 1, 2 (proofs are very similar to those presented in Refs. [66, 67, 68, 69, 70, 71]).

References

[1] P. Hořava, Membranes at quantum criticality, JHEP, 020 (2009) 0903

[2] P. Hořava, Quantum gravity at a Lifshitz point, Phys. Rev. D79 (2009) 084008

[3] P. Hořava, Spectral dimension of the Universe in quantum gravity at a Lifshitz point, Phys. Rev. Lett. 102 (2009) 161301

[4] D. Orlando and S. Reffert, On the renormalizability of Horava–Lifshitz–type gravities, Class. Quant. Grav. 26 (2009) 155021

[5] D. Orlando and S. Reffert, On the Perturbative Expansion around a Lifshitz Point, Phys. Lett. B 683 (2010) 62–68

[6] V. A. Kostelecky and J. D. Tasson, Matter–gravity couplings and Lorenz violation, Phys. Rev. D 83 (2011) 016012

[7] Zhi Xiao and Bo–Qiang Ma, Constraints on Lorentz invariance violation from gamma–ray burst GRB090510, Phys. Rev. D 80 (2009) 116005

[8] S. Liberati and L. Maccione, Lorentz violation; motivation and new constraints, Ann. Rev. Nucl. Part. Sci. 59 (2009) 245–267

30
[9] S. M. Carroll, J. A. Harvey, V. A. Kostelecky, C. D. Lane and T. Okamoto, Noncommutative field theory and Lorentz violation, Phys. Rev. Lett. 87 (2001) 141601

[10] C. P. Burgess, J. Cline, E. Filotas, J. Matias and G. D. Moore, Loop–Generated bounds on changes to the graviton dispersion relation, JHEP 0203 (2002) 043

[11] C. Barcelo, S. Liberati and M. Visser, Analogue gravity, Living Rev. Rel. 8 (2005) 12

[12] A. Kobakhidze, On the infrared limit of Hořava’s gravity with the global Hamiltonian constraint, Phys. Rev. D 82 (2010) 064011

[13] D. Blas, O. Pujolas and S. Sibiryakov, On the extra mode and inconsistency of Hořava Gravity, JHEP 0910 (2009) 029

[14] E. Elizalde, S. Nojiri, S. D. Odintso, D. Saez–Gomez, Unifying inflation with dark energy in modified F(R) Horava-Lifshitz gravity, Eur. Phys. J. C 70 (2010) 351–361

[15] S. Carloni, E. Elizalde and P. J. Silva, Matter couplings in Horava–Lifshitz and their cosmological applications, Class. Quant. Grav. 28 (2011) 195002

[16] G. Amelino–Camelia, L. Gualtieri and F. Mercati, Threshold anomalies in Hořava–Lifshitz–type theories, Phys. Lett. B 686 (2010) 283–287

[17] N. E. Mavromatos, Lorentz Invariance Violation from String Theory, arXiv: 0708.2250

[18] J. Ellis and. N. E. Mavromatos, Probes of Lorentz Violation, arXiv: 1111.1178

[19] N. E. Mavromatos, S. Sarkar and A. Vergou, Stringy Space–Time Foam, Finsler–like Metrics and Dark Matter Relics, Phys. Lett. B 696 (2011) 300-304

[20] S. Mignemi, Doubly special relativity and Finsler geometry, Phys. Rev. D 76 (2007) 047702

[21] F. Girelli, S. Liberati and L. Sindoni, Phenomenology of quantum gravity and Finsler geometry, Phys. Rev. D 75 (2007) 064015
[22] G. W. Gibbons, J. Gomis and C. N. Pope, General very special relativity is Finsler geometry, Phys. Rev. D 76 (2007) 081701

[23] L. Sindoni, The Higgs mechanism in Finsler spacetimes, Phys. Rev. D 77 (2008) 124009

[24] C. Lämmerzahl, D. Lorek and H. Dittus, Confronting Finsler spacetime with experiment, Gen. Rel. Grav. 41 (2009) 1345-1353

[25] A. P. Kouretsis, M. Stathakopoulos and P. C. Stavrinos, The general very special relativity in Finsler cosmology, Phys. Rev. D 79 (2009) 104011

[26] A. P. Kouretsis, M. Stathakopoulos and P. C. Stavrinos, Imperfect fluids, Lorentz violations and Finsler cosmology, Phys. Rev. D 82 (2010) 064035

[27] J. Skakala and M. Visser, Pseudo–Finslerian spacetimes and multirefrigerence, Int. J. Mod. Phys. D 19 (2010) 119–1146

[28] S. Weinfurther, T. P. Sotriou and M. Visser, Projectable Hořava–Lifshitz gravity in a nutshell, J. Phys. Conf. Ser. 222 (2010) 012054

[29] H. S. Yang, Emergent spacetime and the origin of gravity, JHEP 5 (2009) 012

[30] G. Calcagni, Detailed balance in Hořava–Lifshitz gravity, Phys. Rev. D 81 (2010) 044006

[31] D. Raetzel, S. Rivera and F. P. Schuller, Geometry of Physical Dispersion Relations, Phys. Rev. D 83 (2011) 0444047

[32] L. Sindoni, A Note on Particle Kinematics in Hořava–Lifshitz Scenarios, arXiv: 0910.1329

[33] S. Vacaru, Finsler and Lagrange Geometries in Einstein and String Gravity, Int. J. Geom. Methods. Mod. Phys. 5 (2008) 473-511

[34] S. Vacaru, Principles of Einstein–Finsler Gravity and Perspectives in Modern Cosmology, arXiv: 1004.3007

[35] S. Vacaru, Finsler Branes and Quantum Gravity Phenomenology with Lorentz Symmetry Violations, Class. Quant. Grav. 28 (2011) 215001
[36] S. Vacaru, P. Stavrinos, E. Gaburov and D. Gonta, *Clifford and Riemann-Finsler Structures in Geometric Mechanics and Gravity*, Selected Works, Differential Geometry – Dynamical Systems, Monograph 7 (Geometry Balkan Press, 2006); www.mathem.pub.ro/dgds/mono/va-t.pdf and arXiv: gr-qc/0508023

[37] S. Vacaru, Critical remarks on Finsler modifications of gravity and cosmology by Zhe Chang and Xin Li, Phys. Lett. B 690 (2010) 224-228

[38] R. Miron and M. Anastasiei, *The Geometry of Lagrange Spaces: Theory and Applications*, FTPH no. 59 (Kluwer Academic Publishers, Dordrecht, Boston, London, 1994)

[39] S. Vacaru, Locally anisotropic gravity and strings, Ann. Phys. (NY), 256 (1997) 39-61

[40] S. Vacaru, Superstrings in higher order extensions of Finsler super-spaces, Nucl. Phys. B, 434 (1997) 590-656

[41] S. Vacaru and D. Singleton, Warped solitonic deformations and propagation of black holes in 5D vacuum gravity, Class. Quant. Grav. 19 (2002) 3583-3602

[42] S. Vacaru and D. Singleton, Warped, anisotropic wormhole soliton configurations in vacuum 5D gravity, Class. Quant. Grav. 19 (2002), 2793-2811

[43] S. Vacaru, Gauge and Einstein gravity from non-Abelian gauge models on noncommutative spaces, Phys. Lett. B 498 (2001) 74-82

[44] S. Vacaru, Exact solutions with noncommutative symmetries in Einstein and gauge gravity, J. Math. Phys. 46 (2005) 042503

[45] S. Vacaru, Finsler black holes induced by noncommutative anholonomic distributions in Einstein gravity, Class. Quant. Grav. 27 (2010) 105003

[46] S. Vacaru, Nonholonomic Ricci flows: II. Evolution equations and dynamics, J. Math. Phys. 49 (2008) 043504

[47] S. Vacaru, Ricci flows and solitonic pp–waves, Int. J. Mod. Phys. A 21 (2006) 4899-4912

[48] S. Vacaru, Nonholonomic Ricci flows, exact solutions in gravity, and symmetric and nonsymmetric metrics, Int. J. Theor. Phys. 48 (2009) 579-606

33
[49] S. Vacaru, Spectral functionals, nonholonomic Dirac operators, and noncommutative Ricci flows, J. Math. Phys. 50 (2009) 073503

[50] S. Vacaru, Parametric nonholonomic frame transforms and exact solutions in gravity, Int. J. Geom. Meth. Mod. Phys. 4 (2007) 1285-1334

[51] S. Vacaru, On general solutions in Einstein and high dimensional gravity, Int. J. Theor. Phys. 49 (2010) 884-913

[52] S. Vacaru, On general solutions in Einstein gravity, Int. J. Geom. Meth. Mod. Phys. 8 (2011) 9–21

[53] S. Vacaru, Deformation quantization of almost Kähler models and Lagrange–Finsler spaces, J. Math. Phys. 48 (2007) 123509

[54] S. Vacaru, Deformation quantization of nonholonomic almost Kähler models and Einstein gravity, Phys. Lett. A 372 (2008) 2949-2955

[55] M. Anastasiei and S. Vacaru, Fedosov quantization of Lagrange–Finsler and Hamilton–Cartan spaces and Einstein gravity lifts on (co) tangent bundles, J. Math. Phys. 50 (2009) 013510

[56] S. Vacaru, Branes and quantization for an A–model complexification of Einstein gravity in almost Kähler variables, Int. J. Geom. Meth. Mod. Phys. 6 (2009) 873-909

[57] S. Vacaru, Einstein gravity as a nonholonomic almost Kähler geometry, Lagrange–Finsler variables, and deformation quantization, J. Geom. Phys. 60 (2010) 1289-1305

[58] S. Vacaru, Two–connection renormalization and nonholonomic gauge models of Einstein gravity, Int. J. Geom. Meth. Mod. Phys. 7 (2010) 713-744

[59] M. Li and Y. Pang, A trouble with Hořava–Lifshitz gravity, JHEP 0908 (2009) 015

[60] T. Sotiriou, M. Visser and S. Weinfurtner, Phenomenologically viable Lorentz–violating quantum gravity, Phys. Rev. Lett. 102 (2009) 251601

[61] T. P. Sotiriou, M. Visser and S. Weinfurtner, Quantum gravity without Lorentz invariance, JHEP 0910 (2009) 033

[62] C. Bogdanos and E. N. Saridakis, Perturbative Instabilities in Hořava Gravity, Class. Quant. Grav. 27 (2010) 075005
[63] S. Dimopoulos and G. Landsberg, Black holes at the large hadron collider, Phys. Rev. Lett. 87 (2001) 161602

[64] L. A. Anchordoqui, J. L. Feng, H. Goldberg and A. D. Shapere, Black holes from cosmic rays: Probes of extra dimensions and new limits on TeV–scale gravity, Phys. Rev. D 65 (2002) 124027

[65] G. Amelino–Camelia, Gravity–wave interferometers as probes of a low–energy effective quantum gravity, Phys. Rev. D 62 (2000) 024015

[66] P. Midodashvili, Brane in 6D and localization of matter fields, arXiv: hep-th/0308051

[67] M. Gogberashvili and P. Midodashvili, Brane–universe in six dimensions, Phys. Lett. B 515 (2001) 447–450

[68] M. Gogberashvili and P. Midodashvili, Localization of fields on a brane in six dimensions, Europhys. Lett. 61 (2003) 308–313

[69] M. Gogberashvili and D. Singleton, Nonsingular increasing gravitational potential for the brane in 5D, Phys. Lett. B 582 (2004) 95–101

[70] M. Gogberashvili and D. Singleton, Brane in 6D with increasing gravitational trapping potential, Phys. Rev. D 69 (2004) 026004

[71] D. Singleton, Gravitational trapping potential with arbitrary extra dimensions, Phys. Rev. D 70 (2004) 065013

[72] A. Ghodsi and E. Hatefi, Extremal rotating solutions in Hořava gravity, Phys. Rev. D 81 (2010) 044016