LYAPUNOV INEQUALITIES FOR THE PERIODIC BOUNDARY VALUE PROBLEM AT HIGHER EIGENVALUES

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Abstract. This paper is devoted to provide some new results on Lyapunov type inequalities for the periodic boundary value problem at higher eigenvalues. Our main result is derived from a detailed analysis on the number and distribution of zeros of nontrivial solutions and their first derivatives, together with the study of some special minimization problems, where the Lagrange multiplier Theorem plays a fundamental role. This allows to obtain the optimal constants. Our applications include the Hill’s equation where we give some new conditions on its stability properties and also the study of periodic and nonlinear problems at resonance where we show some new conditions which allow to prove the existence and uniqueness of solutions.

1. Introduction

Lyapunov type inequalities provide optimal necessary conditions for certain linear homogeneous boundary value problems to have nontrivial solutions. For instance, if the function $a$ satisfies

$$a \in L_T(\mathbb{R}, \mathbb{R}) \setminus \{0\}, \quad \int_0^T a(x) \, dx \geq 0$$

where $L_T(\mathbb{R}, \mathbb{R})$ denotes the set of $T$-periodic functions $a : \mathbb{R} \to \mathbb{R}$, such that $a|_{[0,T]} \in L^1(0, T)$, then it may be proved (see [8]) that if the periodic boundary value problem

$$u''(x) + a(x)u(x) = 0, \quad x \in (0, T), \quad u(0) - u(T) = u'(0) - u'(T) = 0$$

has nontrivial solutions, then

$$\int_0^T a^+(x) \, dx > 16/T$$

where $a^+(x) = \max\{a(x), 0\}$. This fact has a trivial consequence: if $a$ satisfies (1.1) and $\int_0^T a^+(x) \, dx \leq 16/T$, then the unique solution of (1.2) is the
trivial one. Moreover, this constant is optimal: for any constant \( k > 16/T \),
there is some function \( a \) satisfying (1.1) and \( \int_0^T a^+(x) \, dx \leq k \), such that (1.2) has nontrivial solutions (8).

This kind of result has been generalized in different ways. For example, in [2], [4] and [15], the authors provide, for each \( p \) with \( 1 \leq p \leq \infty \), optimal necessary conditions for boundary value problems similar to (1.2) to have nontrivial solutions, given in terms of the \( L^p \) norm of the function \( a^+ \). This includes the case of Dirichlet, Neumann or mixed boundary conditions. Also, some Lyapunov inequalities may be obtained for \( q \)-Laplacian operators ([15]) and for elliptic PDE ([3]).

Let us observe that the real number zero plays a fundamental role in the condition (1.1). The number zero is, precisely, the first (or principal) eigenvalue of the eigenvalue problem

\[
(1.4) \quad u''(x) + \lambda u(x) = 0, \quad x \in (0, T), \quad u(0) - u(T) = u'(0) - u'(T) = 0
\]

where where for \( c, d \) \in \( L^1(0, T) \), we write \( c \prec d \) if \( c(x) \leq d(x) \) for a.e. \( x \in [0, T] \) and \( c(x) < d(x) \) on a set of positive measure.

On the other hand, the set of eigenvalues of (1.4) is given by \( \lambda_0 = 0, \lambda_{2n-1} = \lambda_{2n} = (2n)^2 \pi^2 / T^2, \) \( n \in \mathbb{N} \) and it is clear that if for some sufficiently large \( n \in \mathbb{N} \), the function \( a \) satisfies \( \lambda_{2n-1} \prec a \), then the inequality \( \int_0^T a^+(x) \, dx \leq 16/T \) is not possible. To this respect, the first part of this paper deals with \( L_1 \)-Lyapunov inequality for the periodic problem (1.2) at higher eigenvalues. More precisely, if \( n \in \mathbb{N} \) is fixed, we introduce the set \( \Lambda_n \) as

\[
(1.5) \quad \Lambda_n = \{ a \in L_T(\mathbb{R}, \mathbb{R}) : \lambda_{2n-1} \prec a \text{ and (1.2) has nontrivial solutions} \}
\]

and we give an explicit expression for the number

\[
(1.6) \quad \gamma_{1,n} = \inf_{a \in \Lambda_n} \| a \|_{L^1(0,T)}
\]

In addition, we prove that this infimum is not attained. To the best of our knowledge these results are new if \( n \geq 1 \). In Remarks 1, 2 and 3 below we compare our results with others obtained by different authors.

Our main result is derived from a detailed analysis on the number and distribution of zeros of nontrivial solutions and their first derivatives, together with the study of some special minimization problems. We apply our method to the periodic and also to the antiperiodic boundary value problem.

In the last section of the paper we give some applications. In particular, we obtain in a very easy way a generalization of a result previously proved by Krein ([7]), on the so called \( n^{th} \) stability zone of the Hill’s equation. Also, we compare our results with those obtained by Borg in [1], about the absolute stability of the Hill’s equation with two parameters. Moreover, we use an especial continuation method to prove the positivity of some eigenvalues; this provides some new stability results (see Subsection 4.1 below).
Finally, we present some new conditions which allow to prove the existence and uniqueness of solutions for some nonlinear periodic and resonant problems.

2. The periodic problem

If $n \in \mathbb{N}$ is fixed, we introduce the set $\Lambda_n$ as

\[(2.1) \quad \Lambda_n = \{a \in L_T(\mathbb{R}, \mathbb{R}) : \lambda_{2n-1} < a \text{ and } (1.2) \text{ has nontrivial solutions } \}
\]

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(Remember that $\lambda_{2n-1} = \lambda_{2n} = 4n^2\pi^2/T^2$). If $a \in \Lambda_n$, and $u$ is any nontrivial solution of (1.2), then $u$ is not a constant function. In addition, $u$ must have a zero in the interval $[0, T]$. If $r \in [0, T]$ is such that $u(r) = 0$, the periodic and nontrivial function $v(x) = u(r + x)$ satisfies $v''(x) + a(r + x)v(x) = 0, x \in (0, T)$ and $\|a(r + \cdot) - \lambda_{2n-1}\|_{L^1(0, T)} = \|v'(\cdot)\|_{L^1(0, T)}$. Finally, since $a \in \Lambda_n$, $n \in \mathbb{N}$, it is clear that between two consecutive zeros of the function $u$ there must exists a zero of the function $u'$ and between two consecutive zeros of the function $u'$ there must exists a zero of the function $u$.

Previously to state and prove the main result in this section, we remember that for any function $a \in L_T(\mathbb{R}, \mathbb{R})$, the eigenvalues for

\[(2.2) \quad u''(x) + (\lambda + a(x))u(x) = 0, \quad x \in (0, T), \quad u(0) - u(T) = u'(0) - u'(T) = 0
\]

form a sequence $\lambda_n(a), n \in \mathbb{N} \cup \{0\}$, such that

\[(2.3) \quad \lambda_0(a) < \lambda_1(a) < \lambda_2(a) < \ldots < \lambda_{2n-1}(a) \leq \lambda_{2n}(a) < \ldots
\]

with $\lambda_0(a)$ simple and such that if $\phi_n$ is the corresponding eigenfunction to $\lambda_n(a)$, then $\phi_0$ has no zeros in $[0, T]$ and $\phi_{2n-1}$ and $\phi_{2n}$ have exactly $2n$ zeros in $[0, T]$ (see [4]). In particular, $\lambda_0 = \lambda_0(0) = 0$, $\lambda_{2n-1} = \lambda_{2n} = \lambda_{2n-1}(0) = \lambda_{2n}(0) = (2n)^2\pi^2/T^2, n \in \mathbb{N}$.

**Theorem 2.1.** Let $n \in \mathbb{N}$ and $a \in \Lambda_n$ be given and $u$ any nontrivial solution of (1.2) such that $u(0) = u(T) = 0$. If the zeros of $u$ in $[0, T]$ are denoted by $0 = x_0 < x_2 < \ldots < x_{2m} = T$ and the zeros of $u'$ in $(0, T)$ are denoted by $x_1 < x_3 < \ldots < x_{2m-1}$, then:

1. $x_{i+1} - x_i \leq \frac{T}{4n}, \forall i : 0 \leq i \leq 2m-1$. Moreover, at least one of these inequalities is strict.
2. $m$ is an even number and $m \geq 2(n+1)$. Any even value $m \geq 2(n+1)$ is possible.
3. $\|a - \lambda_{2n-1}\|_{L^1(x_i, x_{i+1})} \geq \frac{2n\pi}{T} \cot \left( \frac{2n\pi}{T} (x_{i+1} - x_i) \right), 0 \leq i \leq 2m - 1$.
4. $\beta_{1,n} \equiv \inf_{a \in \Lambda_n} \|a - \lambda_{2n-1}\|_{L^1(0,T)} = \frac{8\pi n(n+1)}{T} \cot \frac{n\pi}{2(n+1)}$

and $\beta_{1,n}$ is not attained.
(5) If \( a \in L^T(\mathbb{R}, \mathbb{R}) \) satisfies
\[
\lambda_{2n-1} < a, \quad \|a\|_{L^1(0,T)} \leq \gamma_{1,n}
\]
where
\[
\gamma_{1,n} = T\lambda_{2n-1} + \beta_{1,n},
\]
then
\[
\lambda_{2n}(a) < 0 < \lambda_{2n+1}(a)
\]

**Proof.** Let \( i \), \( 0 \leq i \leq 2m - 1 \), be given. Then, function \( u \) satisfies either the problem
\[
u''(x) + a(x)u(x) = 0, \quad x \in (x_i, x_{i+1}), \quad u(x_i) = 0, \quad u'(x_{i+1}) = 0,
\]
or the problem
\[
u''(x) + a(x)u(x) = 0, \quad x \in (x_i, x_{i+1}), \quad u'(x_i) = 0, \quad u(x_{i+1}) = 0.
\]
Let us assume the first case. The reasoning in the second case is similar. Note that \( u \) may be chosen such that \( u(x) > 0, \quad \forall \ x \in (x_i, x_{i+1}) \). Let us denote by \( \mu_1^i \) and \( \varphi_1^i \), respectively, the principal eigenvalue and eigenfunction of the eigenvalue problem
\[
u''(x) + \mu v(x) = 0, \quad x \in (x_i, x_{i+1}), \quad v(x_i) = 0, \quad v'(x_{i+1}) = 0.
\]
It is known that
\[
\mu_i^1 = \frac{\pi^2}{4(x_{i+1} - x_i)^2}, \quad \varphi_1^i(x) = \sin \frac{\pi(x - x_i)}{2(x_{i+1} - x_i)}
\]
Choosing \( \varphi_1^i \) as test function in the weak formulation of (2.9) and \( u \) as test function in the weak formulation of (2.11) for \( \mu = \mu_1^i \) and \( v = \varphi_1^i \), we obtain
\[
\int_{x_i}^{x_{i+1}} (a(x) - \mu_1^i)u\varphi_1^i(x) \, dx = 0.
\]
Then, if \( x_{i+1} - x_i > \frac{T}{4n} \), we have
\[
\mu_1^i = \frac{\pi^2T^2}{4(x_{i+1} - x_i)^2} < \frac{4n^2\pi^2}{T^2} = \lambda_{2n-1} \leq a(x), \quad \text{a.e. in } (x_i, x_{i+1}),
\]
which is a contradiction with (2.13). Consequently, \( x_{i+1} - x_i \leq \frac{T}{4n}, \forall \ i : 0 \leq i \leq 2m - 1 \). Also, since \( \lambda_{2n-1} < a \) in the interval \( (0,T) \), we must have \( \lambda_{2n-1} < a \) in some subinterval \( (x_j, x_{j+1}) \). If \( x_{j+1} - x_j = \frac{T}{4n} \), it follows \( \mu_1^j < a \) in \( (x_j, x_{j+1}) \) and this is again a contradiction with (2.13). These reasonings complete the first part of the theorem.

For the second one, let us observe that
\[
T = \sum_{i=0}^{2m-1} (x_{i+1} - x_i) < 2m \frac{T}{4n}
\]
In consequence, \( m > 2n \) and therefore \( m \geq 2n + 1 \). At this point, we claim that, for the periodic problem (1.2), \( m \) must be an even number. This implies \( m \geq 2(n+1) \).
To prove the claim, let us observe that \( u(x_0) = 0 \) and \( u'(x_0) \neq 0 \). Assume, for instance that \( u'(x_0) > 0 \). Then, we have
\[
\begin{align*}
&u(x_0) = 0, \quad u'(x_0) > 0; \quad u(x_1) > 0, \quad u'(x_1) = 0, \\
&u(x_2) = 0, \quad u'(x_2) < 0; \quad u(x_3) < 0, \quad u'(x_3) = 0, \\
&u(x_4) = 0, \quad u'(x_4) > 0; \quad u(x_5) > 0, \quad u'(x_5) = 0,
\end{align*}
\]
... 

Since \( u'(x_0)u'(x_{2m}) = u'(0)u'(T) > 0 \), \( m \) must be an even number. Also, note that for any given even and natural number \( q \geq 2(n + 1) \), function \( b(x) \equiv \lambda_q \) belongs to \( \Lambda_n \) and for function \( v(x) = \sin \frac{\pi x}{T} \), we have \( m = q \). In this way, we have proved the first two parts of the Theorem.

Continuing with the proof of the Theorem, if \( i \), with \( 0 \leq i \leq 2m - 1 \) is given and \( u \) satisfies \((2.9)\), then
\[
\int_{x_i}^{x_{i+1}} u^2(x) = \int_{x_i}^{x_{i+1}} a(x)u^2(x) = \int_{x_i}^{x_{i+1}} (a(x) - \lambda_{2n-1})u^2(x) + \int_{x_i}^{x_{i+1}} \lambda_{2n-1}u^2(x)
\]
Therefore,
\[
\int_{x_i}^{x_{i+1}} u^2(x) - \lambda_{2n-1} \int_{x_i}^{x_{i+1}} u^2(x) \leq \|a - \lambda_{2n-1}\|_{L^1(x_i,x_{i+1})}\|u^2\|_{L^\infty(x_i,x_{i+1})}
\]
Since \( u' \) has no zeros in the interval \((x_i,x_{i+1})\) and \( u(x_i) = 0 \), we have \( \|u^2\|_{L^\infty(x_i,x_{i+1})} = u^2(x_{i+1}) \). This proves
\[(2.15) \quad \|a - \lambda_{2n-1}\|_{L^1(x_i,x_{i+1})} \geq \frac{\int_{x_i}^{x_{i+1}} u^2(x) - \lambda_{2n-1} \int_{x_i}^{x_{i+1}} u^2}{u^2(x_{i+1})}.
\]
At this point, the following Lemma \((5, \text{Lemma 2.3.})\) may be useful.

**Lemma 2.2.** Assume that \( a < b \) and \( 0 < M \leq \frac{u^2}{(b-a)^2} \) are given real numbers. Let \( H = \{u \in H^1(a,b) : u(a) = 0, u(b) \neq 0\} \). If \( J : H \to \mathbb{R} \) is defined by
\[(2.16) \quad J(u) = \frac{\int_a^b u^2 - M \int_a^b u^2}{u^2(b)}
\]
and \( c \equiv \inf_{u \in H} J(u) \), then \( c \) is attained. Moreover
\[(2.17) \quad c = M^{1/2} \cot(M^{1/2}(b-a))
\]
and if \( u \in H \), then \( J(u) = c \iff u(x) = k \frac{\sin(M^{1/2}(x-a))}{\sin(M^{1/2}(b-a))} \) for some non zero constant \( k \).

By using this Lemma in \((2.15)\) with \( a = x_i, b = x_{i+1}, M = \lambda_{2n-1} \), we deduce the third part of the Theorem. In particular, this implies
\[(2.18) \quad \|a - \lambda_{2n-1}\|_{L^1(0,T)} \geq \frac{2n\pi}{T} \sum_{i=0}^{2m-1} \cot\left(\frac{2n\pi}{T}(x_{i+1} - x_i)\right)
\]
The right-hand side of (2.18) attains its minimum if and only if \( x_{i+1} - x_i = \frac{T}{2m}, \) \( 0 \leq i \leq 2m - 1 \) (see Lemma 2.5 in [5]) so that

\[
\beta_{1,n} \geq \frac{4n\pi}{T} m \cot \frac{n\pi}{m}.
\]

Taking into account that the function \( m \cot \frac{n\pi}{m} \) is strictly increasing with respect to \( m \), and that \( m \geq 2(n + 1) \), we deduce

\[
\beta_{1,n} \geq \frac{8\pi n(n + 1)}{T} \cot \frac{n\pi}{2(n + 1)}.
\]

In the next Lemma, we define a minimizing sequence for \( \beta_{1,n} \). To this respect, we construct, for each positive and sufficiently small number \( \varepsilon \), an appropriate \( T \)-periodic function \( u_\varepsilon \in C^2[0,T] \) in the following way:

1. \( u_\varepsilon \) is first explicitly defined in \([0,\varepsilon]\) as

\[
u_\varepsilon(x) = -\sin\left(\frac{2n\pi}{T}(x - \frac{T}{4(n+1)})\right) + h(x, \varepsilon)
\]

where \( h(x, \varepsilon) \) is such that \( a_\varepsilon(x) = \frac{-u''_\varepsilon(x)}{u_\varepsilon(x)} > \lambda_{2n-1}, \) \( \forall \ x \in [0, \varepsilon] \).

2. \( u_\varepsilon(x) = -\sin\left(\frac{2n\pi}{T}(x - \frac{T}{4(n+1)})\right), \) \( \forall \ x \in (\varepsilon, \frac{T}{4(n+1)}]. \) Then, \( a_\varepsilon(x) = \frac{-u''_\varepsilon(x)}{u_\varepsilon(x)} \equiv \lambda_{2n-1}, \) in the interval \((\varepsilon, \frac{T}{4(n+1)}].\)

3. \[
\lim_{\varepsilon \to 0^+} ||a_\varepsilon - \lambda_{2n-1}||_{L^1(0,\frac{T}{4(n+1)})} = \frac{2n\pi}{T} \cot \frac{n\pi}{2(n + 1)}
\]

4. In the interval \([0, \frac{2T}{4(n+1)}], \) the function \( u_\varepsilon \) is odd with respect to \( \frac{T}{4(n+1)} \).

5. In the interval \([\frac{2T}{4(n+1)}, T], \) the function \( u_\varepsilon \) is an even function with respect to \( \frac{T}{4(n+1)} \).

6. The function \( u_\varepsilon \) is a periodic function with period \( \frac{T}{n+1} \).

The details are given in the next Lemma.

**Lemma 2.3.** Let \( \varepsilon > 0 \) be sufficiently small. Let us define the function \( u_\varepsilon : [0, T] \to \mathbb{R} \) by

\[
u_\varepsilon(x) =
\begin{cases}
-\sin\left(\frac{2n\pi}{T}(x - \frac{T}{4(n+1)})\right) + \frac{2n\pi(x-\varepsilon)^3}{3\varepsilon^2} \cos\left(\frac{n\pi}{2(n+1)}\right), & \text{if } 0 \leq x \leq \varepsilon, \\
-\sin\left(\frac{2n\pi}{T}(x - \frac{T}{4(n+1)})\right), & \text{if } \varepsilon \leq x \leq \frac{T}{4(n+1)}, \\
-u_\varepsilon\left(\frac{T}{4(n+1)} - x\right), & \text{if } \frac{T}{4(n+1)} \leq x \leq \frac{2T}{4(n+1)}, \\
u_\varepsilon\left(\frac{4T}{4(n+1)} - x\right), & \text{if } \frac{2T}{4(n+1)} \leq x \leq \frac{4T}{4(n+1)}, \\
u_\varepsilon \text{ is extended to the interval } [0, T] \text{ as a } \frac{T}{n+1} \text{ - periodic function}.
\end{cases}
\]
Then $u_\varepsilon \in C^2[0,T]$, the function $a_\varepsilon(x) = -\frac{u''_\varepsilon(x)}{u'_\varepsilon(x)}$, $\forall x \in [0,T]$, $x \neq \frac{(2k-1)T}{4(n+1)}$, $1 \leq k \leq 2(n+1)$, belongs to $\Lambda_n$ and

$$\liminf_{\varepsilon \to 0^+} \|a_\varepsilon - \lambda_{2n-1}\|_{L^1(0,T)} = \frac{8\pi n(n+1)}{T} \cot \frac{n\pi}{2(n+1)}$$

(2.22)

Proof. We claim that for each $0 \leq i \leq 4n+3$, function $a_\varepsilon$ satisfies

$$\lambda_{2n-1} \prec a_\varepsilon, \text{ in the interval } \left(\frac{iT}{4(n+1)}, \frac{(i+1)T}{4(n+1)}\right)$$

(2.23)

and

$$\liminf_{\varepsilon \to 0^+} \|a_\varepsilon - \lambda_{2n-1}\|_{L^1(\frac{iT}{4(n+1)}, \frac{(i+1)T}{4(n+1)})} = \frac{2n\pi}{T} \cot \frac{n\pi}{2(n+1)}$$

(2.24)

It is trivial that from (2.23) and (2.24) we deduce (2.22). Moreover, taking into account the definition of the function $u_\varepsilon$, it is clear that it is sufficient to prove the claim in the case $i = 0$. Now, if $x \in (0, \frac{T}{4(n+1)})$, we can distinguish two cases:

1. $x \in (\varepsilon, \frac{T}{4(n+1)})$. Then $a_\varepsilon(x) = \frac{-u''_\varepsilon(x)}{u'_\varepsilon(x)} \equiv \lambda_{2n-1}$.
2. $x \in (0, \varepsilon)$. Then

$$a_\varepsilon(x) - \lambda_{2n-1} = \frac{-4x-\varepsilon}{\varepsilon^2} n\pi \cos \frac{n\pi}{2(n+1)} - 8 \frac{(x-\varepsilon)^3}{3\varepsilon^2} n\pi \cos \frac{n\pi}{2(n+1)} > 0$$

Therefore $a_\varepsilon \in \Lambda_n$. Moreover, if $\varepsilon \to 0^+$, then

$$\frac{-8 \frac{(x-\varepsilon)^3}{3\varepsilon^2} n\pi \cos \frac{n\pi}{2(n+1)}}{-\sin \left(\frac{2n\pi}{T} \left(x - \frac{T}{4(n+1)}\right)\right) + 2 \frac{(x-\varepsilon)^3}{3\varepsilon^2} \frac{n\pi}{T} \cos \frac{n\pi}{2(n+1)}} \to 0,$$

uniformly if $x \in (0, \varepsilon)$.

Finally, since

$$\lim_{\varepsilon \to 0^+} \int_0^\varepsilon \left[ \frac{-4x-\varepsilon}{\varepsilon^2} n\pi \cos \frac{n\pi}{2(n+1)} - \frac{2n\pi}{T} \left(x - \frac{T}{4(n+1)}\right) \right] = 0$$

and

$$-\sin \left(\frac{2n\pi}{T} \left(x - \frac{T}{4(n+1)}\right)\right) \to \sin \frac{n\pi}{2(n+1)},$$

uniformly in $x \in (0, \varepsilon)$ when $\varepsilon \to 0^+$, we deduce

$$\liminf_{\varepsilon \to 0^+} \|a_\varepsilon - \lambda_{2n-1}\|_{L^1(0, \frac{T}{4(n+1)})} = \liminf_{\varepsilon \to 0^+} \frac{n\pi}{T} \cos \frac{n\pi}{2(n+1)} \int_0^\varepsilon (\varepsilon-x) = \frac{2n\pi}{T} \cos \frac{n\pi}{2(n+1)}$$

which is (2.24) for the case $i = 0$. □

In the next Lemma we prove that the infimum $\beta_{1,n}$ is not attained. The key point in the proof is the optimality property of the different inequalities which have been obtained previously.

Lemma 2.4. $\beta_{1,n}$ is not attained.
Proof. Let \( a \in \Lambda_n \) be such that \( \|a - \lambda_{2n-1}\|_{L^1(0,T)} = \beta_{1,n} \). Let \( u \) be any nontrivial solution of (1.2) associated to the function \( a \). As previously, we denote the zeros of \( u \) by \( 0 = x_0 < x_2 < \ldots < x_{2m} = T \) and the zeros of \( u' \) by \( x_1 < x_3 < \ldots < x_{2m-1} \). By using (2.4) and Lemma 2.2, we have

\[
\beta_{1,n} = \|a - \lambda_{2n-1}\|_{L^1(0,T)} = \sum_{i=0}^{2m-1} \|a - \lambda_{2n-1}\|_{L^1(x_i,x_{i+1})} \geq
\]

(2.25)

\[
\sum_{i=0}^{2m-1} J_i(u) \geq \sum_{i=0}^{2m-1} \frac{2n\pi}{T} \cot \frac{n\pi}{2(n+1)} \geq \frac{4\pi nm}{T} \cot \frac{n\pi m}{4(n+1)} \geq \beta_{1,n}
\]

where \( J_i(u) \) is given either by

\[
J_i(u) = \int_{x_i}^{x_{i+1}} u'^2 - \lambda_{2n-1} \int_{x_i}^{x_{i+1}} u^2 \frac{u^2(x)}{u^2(x_i)}, \quad \text{if } u(x_i) = 0
\]

or by

\[
J_i(u) = \int_{x_i}^{x_{i+1}} u'^2 - \lambda_{2n-1} \int_{x_i}^{x_{i+1}} u^2 \frac{u^2(x_i)}{u^2(x)}, \quad \text{if } u(x_{i+1}) = 0.
\]

Consequently, all inequalities in (2.25) transform into equalities. In particular we obtain that

\[
m = 2(n + 1), \quad x_{i+1} - x_i = \frac{T}{4(n + 1)}, \quad 0 \leq i \leq 4n + 3.
\]

Also, it follows

\[
J_i(u) = \frac{2n\pi}{T} \cot \frac{2n\pi}{T} \frac{T}{4(n + 1)}, \quad 0 \leq i \leq 4n + 3.
\]

From Lemma 2.2 we deduce that, up to some nonzero constants, function \( u \) fulfills in each interval \([x_i, x_{i+1}]\),

\[
u(x) = \frac{\sin \frac{2n\pi}{T}(x - x_i)}{\sin \frac{2n\pi}{T}(x_{i+1} - x_i)}, \quad \text{if } i \text{ is even},
\]

and \( u(x) = \frac{\sin \frac{2n\pi}{T}(x - x_{i+1})}{\sin \frac{2n\pi}{T}(x_i - x_{i+1})}, \quad \text{if } i \text{ is odd}.
\]

In particular, in the interval \([0, \frac{T}{4(n+1)}] = [x_0, x_1]\), \( u \) must be the function

\[
u(x) = \frac{\sin \frac{2n\pi}{T}(x)}{\sin \frac{2n\pi}{T}(x_{i+1})}
\]

which does not satisfy the condition \( u'(x_1) = 0 \). The conclusion is that \( \beta_{1,n} \) is not attained. \( \square \)
For the last part of the theorem, let us assume that the function \( a \) satisfies (2.6). Then, since \( \lambda_{2n-1} < a \), we trivially have \( \lambda_{2n}(a) < \lambda_{2n}(\lambda_{2n-1}) = 0 \).

To prove that \( \lambda_{2n+1}(a) > 0 \) we use a continuation method: let us define the continuous function \( g : [0, 1] \to \mathbb{R} \) by

\[
g(\varepsilon) = \lambda_{2n+1}(a_{\varepsilon}(\cdot))
\]

where \( a_{\varepsilon}(x) = \lambda_{2n-1} + \varepsilon(a(\cdot) - \lambda_{2n-1}) \). Then \( g(0) = \lambda_{2n+1}(\lambda_{2n-1}) = \lambda_{2n+1} - \lambda_{2n-1} > 0 \). Moreover, \( g(\varepsilon) \neq 0 \), \( \forall \varepsilon \in (0, 1] \). In fact, for each \( \varepsilon \in (0, 1] \) the function \( a_{\varepsilon}(x) \) satisfies \( \lambda_{2n-1} < a_{\varepsilon} \) and \( \|a_{\varepsilon}(\cdot) - \lambda_{2n-1}\|_{L^1(0, T)} \leq \beta_{1,n} \).

Consequently, we deduce from the previous parts of the Theorem that the number 0 is not an eigenvalue of the function \( a_{\varepsilon} \) for the periodic boundary conditions. As a consequence, \( \lambda_{2n+1}(a) = g(1) > 0 \) and the Theorem is proved.

\[\square\]

Remark 1. Let us observe that we can obtain similar results if, in the definition of the set \( \Lambda_n \) in (2.1), we consider \( n \in \mathbb{R}^+ \) instead of \( n \in \mathbb{N} \). Only some minor changes are necessary. From this point of view, if we consider \( \beta_{1,n} \) as a function of \( n \in (0, +\infty) \), then \( \lim_{n \to 0^+} \beta_{1,n} = \frac{1}{16} \), the constant of the classical \( L^1 \) Lyapunov inequality at the first eigenvalue which was obtained in (8) by using methods of optimal control theory. In fact, we can use similar reasonings to those of Theorem 2.1 if \( a \in \Lambda_0 \) where (2.26)

\[\Lambda_0 = \{ a \in L_T(\mathbb{R}, \mathbb{R}) \setminus \{0\} : 0 \leq \int_0^T a(x) \, dx \text{ and (1.2) has nontrivial solutions} \} \]

In this case \( m \geq 2 \) and any even value \( m \geq 2 \) is possible. Consequently

\[\beta_{1,0} = \inf_{a \in \Lambda_0} \|a^+\|_{L^1(0, T)} = \frac{16}{T} \tag{2.27}\]

Let us remark that the restriction

\[a \in L_T(\mathbb{R}, \mathbb{R}) \setminus \{0\} : \lambda_0 = 0 \leq \int_0^T a(x) \, dx \]

is more general that the restriction \( \lambda_0 < a \).

Remark 2. The case where \( T = 2\pi \) and function \( a \) satisfies the condition \( A \leq a(x) \leq B \), a.e. in \((0, 2\pi)\) where \( k^2 < A < (k + 1)^2 < B \) for some \( k \in \mathbb{N} \cup \{0\} \), has been considered in [14], where the authors also use optimal control theory methods. In this paper, the authors define the set \( \Lambda_{A,B} \) as the set of functions \( a \) such that \( A \leq a(x) \leq B \), a.e. in \((0, T)\) and (1.2) has nontrivial solutions. Then, by using the Pontryagin’s maximum principle they prove that the number

\[\beta_{A,B} = \inf_{a \in \Lambda_{A,B}} \|a\|_{L^1(0, T)} \]

is attained. In addition, they calculate \( \lim_{B \to +\infty} \beta_{A,B} \). However, if \( A \to k^2 \), it does not seem possible to deduce from [14] that the constant \( \beta_{1,k} \) (defined
in (2.5)) is not attained. In fact, to the best of our knowledge, this result is new. Moreover, our method, which combines a detailed analysis about the number and distribution of zeros of nontrivial solutions of (1.2) and their first derivatives, together with the use of some special minimization problems, can be used to unify other results obtained for different boundary conditions (see [5] for the Neumann and Dirichlet problem).

Remark 3. In [15], the author proves that if the function $a \in L_T(\mathbb{R}, \mathbb{R})$ satisfies

$$\lambda_{2n-1}(0) < a, \|a\|_{L^1(0,T)} \leq \frac{16(n+1)^2}{T},$$

then $\lambda_{2n}(a) < 0 < \lambda_{2n+1}(a)$. It is trivial that

$$\frac{16(n+1)^2}{T} < \gamma_{1,n}, \ \forall \ n \in \mathbb{N}$$

and that

$$\lim_{n \to \infty} T\gamma_{1,n} \frac{16(n+1)^2}{T} = \frac{\pi^2}{4}.$$ 

Therefore, the results given in Theorem 2.1 is more precise. Moreover, taking into account the definition of $\gamma_{1,n}$ (see (2.6)), if $\lambda_{2n-1} < a$, our result is optimal from the point of view of the nonexistence of nontrivial solution of (1.2). On the other hand, in [15] the author studies the case of Dirichlet, periodic and antiperiodic boundary conditions for one dimensional $q-$Laplacian operators, by using $L^p-$ norms of the function $a$ (see also [2] for Lyapunov inequalities at the first eigenvalue).

3. THE ANTIPERIODIC PROBLEM

We can do an analogous study for other boundary value problems. In fact, the key point of the method used in Theorem 2.1 is to have an optimal knowledge about the number and distribution of zeros of the function $u$ and its derivative $u'$, moreover of knowing the best value of the constant $m$. Thinking in the next section, we consider the anti-periodic boundary value problem

(3.1) $u''(x) + a(x)u(x) = 0, \ x \in (0,T), \ u(0) + u(T) = u'(0) + u'(T) = 0$

where $a \in L_T(\mathbb{R}, \mathbb{R})$. To this respect, it is very well known that for any function $a \in L_T(\mathbb{R}, \mathbb{R})$, the eigenvalues for

(3.2) $u''(x) + (\tilde{\lambda} + a(x))u(x) = 0, \ x \in (0,T), \ u(0) + u(T) = u'(0) + u'(T) = 0$

form a sequence $\tilde{\lambda}_n(a)$, $n \in \mathbb{N}$, such that

(3.3) $\tilde{\lambda}_1(a) \leq \tilde{\lambda}_2(a) < \ldots < \tilde{\lambda}_{2n-1}(a) \leq \tilde{\lambda}_{2n}(a) < \ldots$

such that if $\tilde{\phi}_n$ is the corresponding eigenfunction to $\lambda_n(a)$, then $\tilde{\phi}_{2n-1}$ and $\tilde{\phi}_{2n}$ have exactly $2n - 1$ zeros in $[0,T)$ (see [4]). In particular, the set of eigenvalues of

(3.4) $u''(x) + \lambda u(x) = 0, \ x \in (0,T), \ u(0) + u(T) = u'(0) + u'(T) = 0$
is given by \( \tilde{\lambda}_{2n-1}(0) = \tilde{\lambda}_{2n}(0) = (2n - 1)^2\pi^2/T^2 \), \( n \in \mathbb{N} \). We will denote \( \tilde{\lambda}_i = \tilde{\lambda}_i(0), \forall i \in \mathbb{N} \).

If \( n \in \mathbb{N} \) is fixed, we can introduce the set \( \tilde{\Lambda}_n \) as

\[
\tilde{\Lambda}_n = \left\{ a \in L_T(\mathbb{R}, \mathbb{R}) : \tilde{\lambda}_{2n-1} \prec a \text{ and } (3.1) \text{ has nontrivial solutions} \right\}
\]

The similar Theorem to Theorem 2.1 is the following one.

**Theorem 3.1.** Let \( n \in \mathbb{N} \) and \( a \in \tilde{\Lambda}_n \) be given and \( u \) any nontrivial solution of (3.1) such that \( u(0) = u(T) = 0 \). If the zeros of \( u \) in \([0, T]\) are denoted by \( x_0 < x_2 < \ldots < x_{2m} = T \) and the zeros of \( u' \) in \((0, T)\) are denoted by \( x_1 < x_3 < \ldots < x_{2m-1} \), then:

1. \( x_{i+1} - x_i \leq \frac{T}{2(2n-1)}, \forall i: 0 \leq i \leq 2m - 1 \). Moreover, at least one of these inequalities is strict.
2. \( m \) is an odd number and \( m \geq 2n + 1 \). Any odd value \( m \geq 2n + 1 \) is possible.
3. \( (3.6) \) \( \|a - \tilde{\lambda}_{2n-1}\|_{L^1(x_i, x_{i+1})} \geq \frac{(2n - 1)\pi}{T} \cot\left(\frac{(2n - 1)\pi}{T}\left(x_{i+1} - x_i\right)\right), 0 \leq i \leq 2m-1 \).
4. \( (3.7) \) \( \tilde{\beta}_{1,n} = \inf_{a \in \tilde{\Lambda}_n} \|a - \tilde{\lambda}_{2n-1}\|_{L^1(0,T)} = \frac{2\pi(2n - 1)(2n + 1)}{T} \cot\left(\frac{(2n - 1)\pi}{2(2n + 1)}\right) \)
   and \( \tilde{\beta}_{1,n} \) is not attained.
5. If \( a \in L_T(\mathbb{R}, \mathbb{R}) \) satisfies

\[
(3.8) \quad \tilde{\lambda}_{2n-1} \prec a, \|a\|_{L^1(0,T)} \leq \tilde{\gamma}_{1,n} = T\tilde{\lambda}_{2n-1} + \tilde{\beta}_{1,n},
\]

then

\[
(3.9) \quad \tilde{\lambda}_{2n}(a) < 0 < \tilde{\lambda}_{2n+1}(a)
\]

**Remark 4.** A similar theorem to the previous one may be proved if \( a \in \tilde{\Lambda}_0 \) where

\[
(3.10) \quad \tilde{\Lambda}_0 = \left\{ a \in L_T(\mathbb{R}, \mathbb{R}) : (3.1) \text{ has nontrivial solutions} \right\}
\]

In this case \( m \geq 1 \) and any even value \( m \geq 1 \) is possible. Consequently

\[
(3.11) \quad \tilde{\beta}_{1,0} = \inf_{a \in \tilde{\Lambda}_0} \|a^+\|_{L^1(0,T)} = \frac{4}{T}
\]

Let us remark that the restriction \( 0 \prec a \) which is natural for the periodic problem (1.2), is not necessary in this case (see Remark 4 in [2]).
4. Some applications

4.1. Stability of linear periodic equations. We begin with an application to the Lyapunov stability of the Hill’s equation

\[ u''(x) + a(x)u(x) = 0, \ a \in L_T(\mathbb{R}, \mathbb{R}). \]

To this respect, it is convenient to introduce the parametric equation

\[ u''(x) + (\mu + a(x))u(x) = 0, \ a \in L_T(\mathbb{R}, \mathbb{R}), \ \mu \in \mathbb{R}. \]

Remember that if \( \lambda_i(a), \ i \in \mathbb{N} \cup \{0\} \) and \( \tilde{\lambda}_i(a), \ i \in \mathbb{N} \), denote, respectively the eigenvalues of (4.1) for the periodic and antiperiodic problem, then it is known ([6], [7]) that

\[ \lambda_0(a) < \tilde{\lambda}_1(a) \leq \lambda_1(a) \leq \lambda_2(a) < \tilde{\lambda}_3(a) \leq \lambda_3(a) < \ldots \]

and that equation (4.2) is stable if

\[ \mu \in (\lambda_2n(a), \tilde{\lambda}_{2n+1}(a)) \cup (\tilde{\lambda}_{2n+2}(a), \lambda_{2n+1}(a)) \]

for some \( n \in \mathbb{N} \cup \{0\} \) and that equation (4.2) is unstable if

\[ \mu \in (-\infty, \lambda_0(a)) \cup (\lambda_{2n+1}(a), \lambda_{2n+2}(a)) \cup (\tilde{\lambda}_{2n+1}(a), \tilde{\lambda}_{2n+2}(a)) \]

for some \( n \in \mathbb{N} \cup \{0\} \). If \( \mu = \lambda_{2n+1}(a) \) or \( \mu = \lambda_{2n+2}(a) \), (4.2) is stable if and only if \( \lambda_{2n+1}(a) = \tilde{\lambda}_{2n+2}(a) \) and, finally, if \( \mu = \tilde{\lambda}_{2n+1}(a) \) or \( \mu = \tilde{\lambda}_{2n+2}(a) \), (4.2) is stable if and only if \( \tilde{\lambda}_{2n+1}(a) = \tilde{\lambda}_{2n+2}(a) \).

**Theorem 4.1.** Let \( a \in L_T(\mathbb{R}, \mathbb{R}) \) satisfying

\[ \exists \ p \in \mathbb{N}, \ \exists \ k \in [\frac{2\pi^2}{T^2}, \frac{(p+1)^2\pi^2}{T^2}]: \]

\[ k \leq a, \ ||a||_{L^1(0,T)} \leq kT + k^{1/2}2(p+1) \cot \frac{k^{1/2}T}{2(p+1)}. \]

Then \( \mu = 0 \) is in the \( n^{th} \) stability zone of the Hill’s equation (4.2).

**Proof.** If \( \frac{2\pi^2}{T^2} \equiv a \) or \( \frac{(p+1)^2\pi^2}{T^2} \equiv a \) or \( k \equiv a \), then (4.2) is trivially stable. Therefore, we can assume

\[ \exists \ p \in \mathbb{N}, \ \exists \ k \in (\frac{2\pi^2}{T^2}, \frac{(p+1)^2\pi^2}{T^2}): \]

\[ k < a, \ ||a||_{L^1(0,T)} \leq kT + k^{1/2}2(p+1) \cot \frac{k^{1/2}T}{2(p+1)}. \]

In this case, the proof is a combination of different ideas used in the previous two sections. Let us suppose, for instance, that \( p = 2n, \ n \in \mathbb{N} \). As in Theorem 2.1 \( \lambda_{2n}(a) < \lambda_{2n}(\lambda_{2n-1}) = 0 \). On the other hand, since \( k < a \), doing a similar reasoning to that in Theorem 2.1 but for the antiperiodic problem, we have that if \( u = u(0) = 0 \), then \( |x_{i+1} - x_i| \leq \frac{\pi}{2k^{1/2}T}. \) This implies the relation \( m > \frac{T^{k^{1/2}}}{2} \) in 4.1. But since we are now considering the antiperiodic problem (3.1), \( m \)
must be an odd number. Also \( p < \frac{T k^{1/2}}{\pi} < p + 1 \), and as \( p = 2n \), we deduce \( m \geq 2n + 1 \). Consequently,

\[
\|a - k\|_{L^1(0,T)} \geq k^{1/2} \sum_{i=0}^{2m-1} \cot(k^{1/2}(x_{i+1} - x_i)) \geq
\]

(4.8)

\[ k^{1/2}(\frac{k^{1/2}T}{2m}) \geq k^{1/2}(2n + 1) \cot(\frac{k^{1/2}T}{2(2n+1)}) \]

(see (2.19)). Moreover, this last constant is not attained. As in Theorem 2.1, if \( h : [0,1] \rightarrow \mathbb{R} \) is defined as \( h(\varepsilon) = \lambda_{2n+1} (k + \varepsilon(a(\cdot) - k)) \), we obtain \( h(0) > 0 \) and \( h(\varepsilon) \neq 0 \), \( \forall \varepsilon \in (0,1] \). Then, \( h(1) = \lambda_{2n+1}(a) > 0 \). As a consequence, \( \mu = 0 \in (\lambda_{2n}(a), \lambda_{2n+1}(a)) \) and the Theorem is proved. The proof is similar if \( p \) is an odd number.

Remark 5. The case where \( a(x) = \alpha + \beta \psi(x) \), with \( \psi \in L_T(\mathbb{R}, \mathbb{R}) \), \( \int_0^T \psi(x) \, dx = 0 \) and \( \int_0^T |\psi(x)| \, dx = 1/T \), was studied by Borg ([1]). Borg used the characteristic multipliers determined from Floquet’s theory. He deduced stability criteria for (4.2) by using the two parameters \( \alpha \) and \( \beta \). For a concrete function \( a \), this implies the use of the two quantities

\[
\frac{1}{T} \int_0^T a(x) \, dx, \quad \frac{1}{T} \int_0^T |a(\cdot) - \frac{1}{T} \int_0^T a(x) \, dx| \| \in L^1(0,T)
\]

It is clear that the results given in Theorem 4.1 are of a different nature (see [1] and the translator’s note in [9]). In fact, our results are similar to those obtained by Krein [9] by using a different procedure. However, Krein assumed \( k = \frac{p^2\pi^2}{T^2} \) and an strict inequality for \( \|a\|_{L^1(0,T)} \) in (4.6) (see Theorem 9 in [9]). By using Theorem 3.1 we can assume a non strict inequality in (4.6) since the constant \( \tilde{\beta}_{1,n} \) is not attained.

Finally, if for a given function \( a \in L_T(\mathbb{R}, \mathbb{R}) \) we know that \( a \) satisfies (4.7), the result given in Theorem 4.1 is more precise than Krein’s result since the function

\[
kT + k^{1/2}2(p + 1) \cot(\frac{k^{1/2}T}{2(p + 1)}), \quad k \in [\frac{p^2\pi^2}{T^2}, \frac{(p + 1)^2\pi^2}{T^2}]
\]

is strictly increasing.

Remark 6. The result obtained in previous Theorem uses \( L_1 \) Lyapunov inequalities. In a similar way, if one uses \( L_\infty \) Lyapunov inequalities, the following result may be proved (see [1], Chapter V, Theorem 5.5). Here we take \( T = \pi \) for simplicity:

If \( r \) and \( s \) are given real numbers and

(4.9)

\[ r^2 \leq a(x) \leq s^2 \]

then (4.1) is stable for all possible functions \( a(\cdot) \) satisfying (4.9) if and only if the interval \((r^2, s^2)\) does not contain the square of an integer.
In particular, concerning to the first stability zone, (4.1) is stable if
\[(4.10) \quad 0 \leq a(x) \leq 1\]
and for functions satisfying \(0 \leq a(x)\), this result is optimal in the following sense: for any positive number \(\varepsilon\) there is some function \(a(x)\) with \(a \in L_T(\mathbb{R}, \mathbb{R})\), satisfying \(0 \leq a(x) \leq 1 + \varepsilon\) and such that (4.1) is unstable.

We can exploit the results obtained in Theorem 2.1 to obtain new results on the stability properties of (4.1). This is the purpose of the next Theorem where function \(a\) can be uniformly greater than 1 in an appropriate interval \((0, x_0)\) as long as the length of the interval \((0, x_0)\) is sufficiently small.

**Theorem 4.2.** Let us choose \(T = \pi\). If function \(a\) fulfills
\[(4.11) \quad a \in L_\pi(\mathbb{R}, \mathbb{R}), \quad 0 \prec a, \quad \exists \alpha \in (0, \frac{\pi}{2}), \quad \exists x_0 \in \left(\frac{\pi}{2}(1 - \cos \alpha), \frac{\pi}{2}(1 + \cos \alpha)\right) :\]
\[\max\{x_0^2\|a\|_{L^\infty(0, x_0)}, (\pi - x_0)^2\|a\|_{L^\infty(x_0, \pi)}\} \leq \alpha^2,\]
then (4.1) is stable.

**Proof.** The proof is based on the following two lemmas. The first one is trivial but necessary. In the second Lemma we exploit the same idea as in the continuation method used in the proof of the last part of Theorem 2.1.

**Lemma 4.3.** Let \(\alpha \in (0, \frac{\pi}{2})\) and \(x_0 \in (0, \pi)\) be given. If \(a_{\alpha,x_0}\) is defined by
\[(4.12) \quad a_{\alpha,x_0}(x) = \begin{cases} \alpha^2 x_0^2, & \text{if } x \in (0, x_0), \\ \frac{\alpha^2}{(\pi - x_0)^2}, & \text{if } x \in (x_0, \pi), \end{cases}\]
then the antiperiodic boundary value problem
\[(4.13) \quad u''(x) + a_{\alpha,x_0}(x)u(x) = 0, \quad x \in (0, \pi), \quad u(0) + u(\pi) = u'(0) + u'(\pi) = 0,\]
has nontrivial solutions if and only if \(x_0 \in \left\{\frac{\pi}{2}(1 - \cos \alpha), \frac{\pi}{2}(1 + \cos \alpha)\right\}\).

**Proof.** Taking into account the formula (4.12) for the function \(a_{\alpha,x_0}\), any solution of (4.13) must be of the form
\[(4.14) \quad u(x) = \begin{cases} A \sin \frac{\alpha x}{x_0} + B \cos \frac{\alpha x}{x_0}, & 0 \leq x < x_0, \\ C \sin \frac{\alpha(x-x_0)}{\pi-x_0} + D \cos \frac{\alpha(x-x_0)}{\pi-x_0}, & x_0 < x \leq \pi. \end{cases}\]
Moreover, since \(a_{\alpha,x_0} \in L^\infty(0, \pi)\), any solution of (4.13) is, in fact, a \(C^1[0, \pi]\) function. Therefore, it must satisfies the four conditions:
\[u(0) + u(\pi) = 0, \quad u'(0) + u'(\pi) = 0,\]
\[\lim_{x \to x^-_0} u(x) = \lim_{x \to x^+_0} u(x), \quad \lim_{x \to x^-_0} u'(x) = \lim_{x \to x^+_0} u'(x).\]
These four conditions are equivalent to the system of equations

\( B + D = 0, \)

\[
\frac{\alpha}{x_0}A + \frac{\alpha}{\pi-x_0}C = 0,
\]

\[ A \sin \alpha + B \cos \alpha + C \sin \alpha - D \cos \alpha = 0, \]

\[
\frac{A_0}{x_0} \cos \alpha - \frac{B_0}{x_0} \sin \alpha - \frac{C_0}{\pi-x_0} \cos \alpha - \frac{D_0}{\pi-x_0} \sin \alpha = 0.
\]

The determinant of the previous system is equal to

\[
\frac{-4x_0^2 + 4x_0\pi - \pi^2 \sin^2 \alpha}{x_0^2(\pi - x_0)^2} \alpha^2
\]

and it is zero if and only if \( x_0 \in \{ \frac{\pi}{2}(1 - \cos \alpha), \frac{\pi}{2}(1 + \cos \alpha) \} \). The Lemma is proved.

**Lemma 4.4.** Let \( \alpha \in (0, \frac{\pi}{2}) \) be given and

\[
a \in L_\pi(\mathbb{R}, \mathbb{R}) \text{ such that } \exists x_0 \in \left( \frac{\pi}{2}(1 - \cos \alpha), \frac{\pi}{2}(1 + \cos \alpha) \right):
\]

\[
(4.15) \quad \max\{x_0^2\|a^+\|_{L^\infty(0,x_0)}, (\pi - x_0)^2\|a^+\|_{L^\infty(x_0,\pi)}\} \leq \alpha^2
\]

then \( \tilde{\lambda}_1(a) > 0. \)

**Proof.** Let \( a_{\alpha,x_0} \) be the function defined in (4.12). If we define the continuous function \( g : [0, 1] \to \mathbb{R}, \) as \( g(\varepsilon) = \tilde{\lambda}_1(\varepsilon^2a_{\alpha,x_0}), \) then \( g(0) = \tilde{\lambda}_1(0) = 1. \)

Moreover, for each \( \varepsilon \in (0, 1], \varepsilon^2a_{\alpha,x_0} = a_{\xi_0,x_0} \) and since \( \varepsilon \in (0, 1], \) we deduce from (4.15) that \( x_0 \notin \{ \frac{\pi}{2}(1 - \cos(\varepsilon \alpha)), \frac{\pi}{2}(1 + \cos(\varepsilon \alpha)) \}. \) From the previous Lemma we conclude that the unique solution of the antiperiodic problem

\[
(4.16) \quad u''(x) + a_{\xi_0,x_0}(x)u(x) = 0, \quad x \in (0, \pi), \quad u(0) + u(\pi) = u'(0) + u'(') = 0
\]

is the trivial one. Thus, \( g(\varepsilon) \neq 0, \forall \varepsilon \in (0, 1]. \) Consequently \( g(1) = \tilde{\lambda}_1(a_{\alpha,x_0}) > 0. \) Since \( a(x) \leq a_{\alpha,x_0}(x), \) \( x \in (0, \pi), \) we have \( \tilde{\lambda}_1(a) \geq \tilde{\lambda}_1(a_{\alpha,x_0}). \)

By using these two Lemmas, the proof of the Theorem is trivial since in the hypothesis (4.11) is included the condition \( 0 < a. \) This allows to prove that \( \mu = 0 \in (\lambda_0(a), \tilde{\lambda}_1(a)), \) the first stability zone of (4.12). \( \square \)

**Remark 7.** It is trivially deduced from the previous proof that the conclusion of Theorem (4.2) is true if we assume the hypothesis

\[
(4.17) \quad a \in L_\pi(\mathbb{R}, \mathbb{R}) \setminus \{0\}, \quad \int_0^T a \geq 0, \quad \exists \alpha \in (0, \frac{\pi}{2}), \quad \exists x_0 \in \left( \frac{\pi}{2}(1 - \cos \alpha), \frac{\pi}{2}(1 + \cos \alpha) \right):
\]

\[
\max\{x_0^2\|a^+\|_{L^\infty(0,x_0)}, (\pi - x_0)^2\|a^+\|_{L^\infty(x_0,\pi)}\} \leq \alpha^2,
\]

which is more general than (4.11).
Remark 8. Taking into account the inequality
\[ \frac{\pi}{2} (1 - \cos \alpha) < \alpha < \frac{\pi}{2} (1 + \cos \alpha), \quad \forall \, \alpha \in (0, \frac{\pi}{2}), \]
if we select in Theorem 4.2, \( x_0 \in \left( \frac{\pi}{2} (1 - \cos \alpha), \alpha \right) \), then the quantity \( \|a(\cdot)\|_{L^\infty(0,\pi)} \) can be greater than one. Consequently, the \( L^\infty \) criterion (4.10) for the stability of (4.1) can not be applied. On the other hand, for general \( \alpha \in (0, \frac{\pi}{2}) \), we may choose \( x_0 = x_0(\alpha) \) so that the quantity \( \alpha x_0 \) is as close as we want to \( \frac{2\alpha}{\pi (1 - \cos \alpha)} \). Since \( \lim_{\alpha \to 0^+} \frac{2\alpha}{\pi (1 - \cos \alpha)} = +\infty \), the conclusion is that the norm \( \|a\|_{L^\infty(0,x_0(\alpha))} \) may be arbitrary large as long as the interval \( (0, x_0(\alpha)) \) is sufficiently small.

Finally, in Theorem 4.2 the norm \( \|a(\cdot)\|_{L^1(0,\pi)} \) may be chosen as near as we want to \( \frac{\alpha^2}{x_0} + \frac{\alpha^2}{\pi - x_0} = \frac{\alpha^2}{x_0(\pi - x_0)} \) and if \( \alpha \to \frac{\pi}{2}^{-} \), we have \( \|a(\cdot)\|_{L^1(0,\pi)} \to \frac{\pi}{2} > \frac{\pi}{2} \). Consequently, the classical Lyapunov’s criterion for the stability of (4.1) can not be applied.

4.2. Nonlinear resonant problems. We finish this paper with some new results on the existence and uniqueness of solutions of nonlinear periodic b.v.p.

\[ u''(x) + f(x, u(x)) = 0, \quad x \in (0, T), \quad u(0) - u(T) = u'(0) - u'(T) = 0. \]  

Taking into account previous discussion (see Remark 1 and Remark 2 above), next theorem includes different situations which can not be studied from the results in [14], Theorem 6 and in [8], Theorem 6. The proof, which uses similar ideas to that given in [2] for the case of Neumann boundary conditions at the first two eigenvalues, combines the linear results of the previous sections with Schauder’s fixed point theorem. We omit the details.

Theorem 4.5. Let us consider (4.18) where:

1. \( f \) and \( f_u \) are Carathéodory function on \( \mathbb{R} \times \mathbb{R} \) and \( f(x + T, u) = f(x, u), \quad \forall \, (x, u) \in \mathbb{R} \times \mathbb{R}. \)

2. There exist functions \( \alpha, \beta \in L^\infty(0, T) \) satisfying
\[ \lambda_{2n-1} < \alpha(x) \leq f_u(x, u) \leq \beta(x), \quad \|\beta\|_{L^1(0,T)} \leq \gamma_{1,n} \]
where \( \gamma_{1,n} \) has been defined in (2.7). Then, problem (4.18) has a unique solution.

Remark 9. By using an example in [10], it may be seen that the restriction
\[ \lambda_{2n-1} < \alpha(x) \leq f_u(x, u) \]
in (4.19) cannot be replaced (in nonlinear problems) by the weaker condition
\[ \lambda_{2n-1} < f_u(x, u) \leq \beta(x) \]

The previous Theorem uses \( L^1 \) Lyapunov inequality. The last part of this section is dedicated to show how we can again use the idea of the
continuation method used in Theorem 2.1 and in Lemma 4.4 to obtain new results on the existence and uniqueness of solutions for resonant problems like (4.18), by using $L^\infty$ Lyapunov inequalities. In this sense, it is very well known (see [8]) that if, in addition to the first hypothesis of the previous Theorem, $f_u$ fulfills the nonresonance condition

$$\exists \, n \in \mathbb{N} \cup \{0\}, \lambda, \mu \in \mathbb{R} : \frac{(2n)^2\pi^2}{T^2} < \lambda \leq f_u(x, u) \leq \mu < \frac{(2(n+1))^2\pi^2}{T^2},$$

then (4.18) has a unique solution. In particular, if $n = 0$, (4.20) becomes

$$\exists \, \lambda, \mu \in \mathbb{R} : 0 < \lambda \leq f_u(x, u) \leq \mu < \frac{4\pi^2}{T^2}.$$

To obtain an strict generalization of these results, we return to the linear periodic problem (1.2). Again, we take $T = \pi$ for simplicity.

**Theorem 4.6.** If function $a$ satisfies

$$a \in L_\pi(\mathbb{R}, \mathbb{R}), 0 \prec a(x),$$

(4.22)

$$\exists \, x_0 \in (0, \pi) : \max\{x_0^2\|a\|_{L^\infty(0, x_0)}, (\pi - x_0)^2\|a\|_{L^\infty(x_0, \pi)}\} < \pi^2$$

then $\lambda_0(a) < 0 < \lambda_1(a)$.

**Proof.** The proof is based on the following Lemma, which points out an important qualitative difference with respect to the antiperiodic problem (see Lemma 4.3).

**Lemma 4.7.** Let $\alpha \in (0, \pi)$ and $x_0 \in (0, \pi)$ be given. If $a_{\alpha, x_0}$ is the function defined in (4.12), the periodic boundary value problem

$$u''(x) + a_{\alpha, x_0}(x)u(x) = 0, \quad x \in (0, \pi), \quad u(0) - u(\pi) = u'(0) - u'(|\pi|) = 0$$

(4.23) has only the trivial solution.

**Proof.** As in the proof of the Lemma 4.3, taking into account the formula (4.12) for the function $a_{\alpha, x_0}$, any solution of (4.23) must be of the form

$$u(x) = \begin{cases} 
A \sin \frac{\alpha x}{x_0} + B \cos \frac{\alpha x}{x_0}, & 0 \leq x < x_0, \\
C \sin \frac{\alpha(x-\pi)}{\pi-x_0} + D \cos \frac{\alpha(x-\pi)}{\pi-x_0}, & x_0 < x \leq \pi 
\end{cases}$$

(4.24)

Again, any solution of (4.23) is, in fact, a $C^1[0, \pi]$ function. Therefore, it must satisfies the four conditions:

$$u(0) - u(\pi) = 0, \quad u'(0) - u'(|\pi|) = 0,$$

$$\lim_{x \to x_0^+} u(x) = \lim_{x \to x_0^-} u'(x), \quad \lim_{x \to x_0^+} u'(x) = \lim_{x \to x_0^-} u'(x).$$
These four conditions are equivalent to the system of equations

\[ B - D = 0, \]

\[ \frac{\alpha}{x_0} A - \frac{\alpha}{\pi - x_0} C = 0, \]

\[ A \sin \alpha + B \cos \alpha + C \sin \alpha - D \cos \alpha = 0, \]

\[ \frac{A_0}{x_0} \cos \alpha - \frac{B_0}{x_0} \sin \alpha - \frac{C_0}{\pi - x_0} \cos \alpha - \frac{D_0}{\pi - x_0} \sin \alpha = 0 \]

The determinant of the previous system is equal to

\[ \frac{\pi^2 \alpha^2 \sin^2 \alpha}{x_0^2 (\pi - x_0)^2} \]

which is always different from zero. \(\square\)

Now, to prove the Theorem 4.6, let us define

\[ \alpha^2 = \max \{ x_0^2 \| a \|_{L^\infty(0,x_0)}, (\pi - x_0)^2 \| a \|_{L^\infty(x_0,\pi)} \} \]

Clearly \( a(x) \leq a_{\alpha,x_0}(x) \), \( x \in (0,\pi) \), and therefore \( \lambda_1(a) \geq \lambda_1(a_{\alpha,x_0}) \). On the other hand, if we define the continuous function \( g : [0,1] \to \mathbb{R} \), as \( g(\varepsilon) = \lambda_1(\varepsilon^2 a_{\alpha,x_0}) \), then \( g(0) = \lambda_1(0) > 0 \). Moreover, for each \( \varepsilon \in (0,1) \), \( \varepsilon^2 a_{\alpha,x_0} = a_{\varepsilon\alpha,x_0} \) and from the previous Lemma, we deduce that the unique solution of the periodic problem

\[ (4.25) \quad u''(x) + a_{\varepsilon\alpha,x_0}(x) u(x) = 0, \quad x \in (0,\pi), \quad u(0) - u(\pi) = u'(0) - u'((\pi) = 0 \]

is the trivial one. Therefore \( g(\varepsilon) \neq 0, \quad \forall \varepsilon \in (0,1) \). Consequently \( g(1) = \lambda_1(a_{\alpha,x_0}) > 0 \) and \( \lambda_1(a) > 0 \). \(\square\)

Remark 10. If in the previous Theorem we select \( x_0 \in (0,\pi/2) \), function \( a \) can satisfies \( \| a \|_{L^\infty(0,x_0)} = \pi^2/x_0^2 \) (which is a quantity greater than 4, see (4.21) for \( T = \pi \)) as long as \( \| a \|_{L^\infty(x_0,L)} \leq \pi^2/(T - x_0)^2 \).

By using the ideas of Theorem 2.1 we can obtain analogous results for the case of higher eigenvalues, i.e., in the case where \( \lambda_{2n-1} < a(x), \quad n \in \mathbb{N} \).

The corresponding nonlinear Theorem to Theorem 4.6 is the following one.

**Theorem 4.8.** Let us consider (4.18) where:

1. \( f \) and \( f_u \) are Caratheodory function on \( \mathbb{R} \times \mathbb{R} \) and \( f(x + T, u) = f(x, u), \quad \forall \ (x, u) \in \mathbb{R} \times \mathbb{R} \).
2. There exist functions \( \alpha, \beta \in L^\infty(0,T) \) satisfying

\[ (4.26) \quad 0 < \alpha(x) \leq f_u(x,u) \leq \beta(x). \]

3. \( (4.27) \quad \exists x_0 \in (0,\pi) : \max \{ x_0^2 \| \beta \|_{L^\infty(0,x_0)}, (\pi - x_0)^2 \| \beta \|_{L^\infty(x_0,\pi)} \} < \pi^2 \)

Then, problem (4.18) has a unique solution.
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