Cavity-assisted backaction cooling of mechanical resonators

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Abstract. We analyze the quantum regime of the dynamical backaction cooling of a mechanical resonator assisted by a driven harmonic oscillator (cavity). Our treatment applies to both optomechanical and electromechanical realizations and includes the effect of thermal noise in the driven oscillator. In the perturbative case, we derive the corresponding motional master equation using the Nakajima–Zwanzig formalism and calculate the corresponding output spectrum for the optomechanical case. Then we analyze the strong optomechanical coupling regime in the limit of small cavity linewidth. Finally, we consider the steady state covariance matrix of the two coupled oscillators for arbitrary input power and obtain an analytical expression for the final mechanical occupancy. This is used to optimize the drive’s detuning and input power for an experimentally relevant range of parameters that includes the resolved-sideband-limit ground state cooling regime.
1. Introduction

Recent progress in the emerging fields of nanoelectromechanical [1, 2] and optomechanical systems [3, 4] promises to enable quantum-limited control of a single macroscopic mechanical degree-of-freedom [5, 6]. This is relevant in the context of high precision measurements [7]–[11] like single spin magnetic resonance force microscopy (MRFM) [12, 13] and for fundamental studies of the quantum to classical transition [14]–[16]. A paradigmatic goal which has triggered a surge of activity is to prepare the eigenmode associated with an ultra long-lived mechanical resonance (angular frequency $\omega_m$ and $Q$-value $Q_m$) in its quantum ground state with high fidelity [17]–[29]. The considerable difficulty to achieve the desired combination $k_B T \ll \hbar \omega_m, Q_m \gg 1$ with state of the art micro-fabrication and cryogenic techniques [2] has naturally motivated ideas to use cooling schemes analogous to the laser-cooling of atoms [30]–[33].

In these schemes, the mechanical resonator’s displacement is coupled parametrically to an auxiliary high-frequency bosonic or fermionic resonator (pseudospin) that can act as a ‘cooler’ [34]. To drive the latter while monitoring its output allows detection of the mechanical displacement. Naturally there is a backaction force associated with this measurement process [35]–[37]. Due to the dissipative dynamics of the cooler for a negative detuning of the drive this force becomes anti-correlated with the Brownian motion resulting in net cooling. In turn the quantum fluctuations of the cooler—which in the atomic laser cooling manifest in the inherent stochastic nature of the spontaneous photon emissions—result in a quantum noise spectrum for this backaction force that sets a fundamental lower bound for the final temperature [31, 32]. Thus the structured reservoir afforded by the driven cooler and its environment provides an effective thermal bath for the mechanical resonator. The concomitant absorptions of motional quanta (cooling) correspond to Raman scattering processes in which a drive quanta is up converted, whereas emission events (heating) are associated with Raman processes in which a drive quanta is down converted (cf figure 1(b)).

A host of concrete realizations of the above generic scenario have been discussed in the literature. These range from electronic or electrical devices where the cooler is provided by a (superconducting) single electron transistor [20], a Cooper-pair box [38], an LC-circuit [3], or a quantum dot [39]; to optomechanical systems where this role is played by an optical cavity mode [4]. In the latter systems, which are equivalent to a Fabry–Pérot with a moving
mirror (cf figure 1(a)), the optical field couples parametrically to the mechanical motion via radiation pressure. This effect has already been thoroughly demonstrated experimentally [40] and harnessed to provide appreciable cooling, albeit in the classical regime [21, 22, 24, 27, 28]. Recent experiments involve both optical and microwave cavities [29]. In turn, completely analogous effects have been shown using a capacitively coupled radio frequency LC-circuit [25]. These systems share the common feature that the cooler is well approximated by a single harmonic oscillator which henceforth will be referred to as the ‘cavity’. While for optical cavities the vacuum constitutes an excellent approximation for the input when the drive is switched off, in the case of radio and microwave frequencies thermal noise in the cavity needs to be taken into account. A quantum treatment of the corresponding temperature limits has already been given which predicts that ground state cooling is possible when the mechanical oscillation frequency is larger than the cavity’s linewidth $\kappa$ [41]–[43]. Here we provide a rigorous analysis of the cooling dynamics that drives the mechanical resonator mode to a thermal state with a well-defined effective final temperature that for a finite $Q_m$ is imprinted on the cavity’s output. This is done in section 3, where we obtain the motional master equation and the corresponding output spectrum. Throughout our analysis we focus on the negative drive detunings that are relevant for cooling.

Figure 1. (a) Fabry–Pérot equivalent of a mechanical eigenmode (frequency $\omega_m/2\pi$ and $Q$-value $Q_m$) coupled to an optical cavity mode. (b) Level diagram of the two modes in a ‘shifted’ representation for perturbative optomechanical coupling $\eta$. Raman scattering processes induced by the latter can decrease (solid lines) or increase (dashed) the mechanical eigenmode’s quantum number $n$ ($\alpha$ is the steady state amplitude in the cavity mode and $|0\rangle_p, |1\rangle_p, \ldots$, its Fock states).

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The above picture is only valid when the optomechanical coupling defined by $g_m \equiv 2\eta|\alpha|\omega_m$ is perturbative (here $|\alpha|$ is the steady state amplitude in the cavity and $\eta$ characterizes the intrinsic nonlinearity). Thus, it breaks down when the resulting cooling rate becomes comparable to the cavity linewidth $\kappa$ or to the mechanical oscillation frequency $\omega_m$ [42]. In the Doppler limit $\omega_m \ll \kappa$ [31, 32], the system then either becomes unstable [44] (cf below equation (58)) or settles into a regime where back-action effects are mainly diffusive with no appreciable cooling. On the other hand, in the resolved sideband limit $\kappa \ll \omega_m$ as the optomechanical coupling exceeds the cavity linewidth the system enters into a strong coupling regime in which the motion hybridizes with the cavity fluctuations. The ensuing optomechanical normal modes are then cooled simultaneously. This phenomenon is analyzed in section 4 in the limit of small cavity linewidth where we find that the dynamics of each of the normal modes can be described by a master equation analogous to the one valid in the perturbative regime with a cooling rate given by half the cavity linewidth. Finally, in the same section we derive an analytical expression for the final (steady state) average mechanical occupancy (phonon number) valid for arbitrary optomechanical coupling and use it to optimize the parameters of the drive.

2. Optomechanical master equation

In optomechanical systems (and their electromechanical analogues), the cavity resonant frequency $\omega_p$ depends inversely on a characteristic length that is modified by the mechanical resonator’s displacement. The fact that $\omega_p \gg \omega_m$ allows for an adiabatic treatment of this effect which results in the aforementioned parametric coupling. In general the leading contribution to the latter is linear in the mechanical displacement—but situations in which it is instead quadratic can be readily engineered [27, 43]. The desired cooling dynamics is induced by a slightly detuned electromagnetic drive (angular frequency $\omega_L$), which for optomechanical systems corresponds to an incident laser and in the electromechanical case is afforded by a suitable external ac voltage. Thus the Hamiltonian describing the coupled system (in a rotating frame at $\omega_L$) is given by [45]

$$ H' = -\hbar \Delta'_L a_p^\dagger a_p + \hbar \eta \omega_m a_p^\dagger a_p (a_m + a_m^\dagger) + \frac{\hbar \Omega}{2} (a_p + a_p^\dagger) + \hbar \omega_m a_m^\dagger a_m. $$

(1)

Here $a_p$ ($a_m$) is the annihilation operator for the electromagnetic (mechanical) oscillator and $\Delta'_L$ is the detuning of the drive from $\omega_p$. Here we have defined $\Omega \equiv 2\sqrt{P}\kappa_{ex}/\hbar\omega_L$, where $P$ is the input power of the drive and $\kappa_{ex}$ is the cavity decay rate into the associated outgoing electromagnetic modes. The dimensionless parameter $\eta$, characterizing the nonlinear coupling between the cavity and the mechanical resonator, is given by $\eta = (\omega_p/\omega_m)(x_0/L)$ where $x_0$ is the zero point motion of the mechanical resonator mode and the characteristic length $L$ depends on the physical realization. In the optomechanical case, it corresponds to an effective optical cavity length while for electromechanical realizations [25] $L = 2dC_{tot}/C_C$, where $C_C \propto 1/d$ is the dynamical capacitance, $d$ is the distance between the corresponding electrodes and $C_{tot}$ is the total capacitance.

We treat the losses induced by the electromagnetic and mechanical baths within the rotating wave Born–Markov approximation using the standard Lindblad form Liouvillians [46]. It is important to note that the validity of a rotating wave approximation (RWA) in the environmental coupling responsible for the mechanical losses only amounts to $Q_m \gg 1$. New Journal of Physics 10 (2008) 095007 (http://www.njp.org/)
provided the optomechanical coupling is weak enough that there is no appreciable mixing between the annihilation and creation operators of the modes (cf section 4). Clearly if the latter is not satisfied the usual RWA will result in the unwarranted neglection of resonant terms. This can be borne out by comparing the corresponding displacement spectra and results in the condition \( \eta \alpha \omega_m \ll \max\{ \sqrt{\omega_m \kappa}, \omega_m \} \). Henceforth, we focus on parameters that satisfy it which include the most relevant regimes for cooling and ensure that the system has a stable steady state for the relevant detunings [44, 47]. Thus the evolution for the density matrix of the resonator–cavity system reads

\[
\dot{\rho} = -\frac{i}{\hbar} [\hat{H}', \rho] + \frac{\kappa}{2} n(\omega_p) \left( 2a_p^{\dagger} \rho a_p - a_p a_p^{\dagger} \rho - \rho a_p a_p^{\dagger} \right) + \frac{\kappa}{2} [n(\omega_p) + 1] \left( 2a_p^{\dagger} \rho a_p - a_p a_p^{\dagger} \rho - \rho a_p a_p^{\dagger} \right)
\]

\[
+ \frac{\gamma_m}{2} n(\omega_m) \left( 2a_m^{\dagger} \rho a_m - a_m a_m^{\dagger} \rho - \rho a_m a_m^{\dagger} \right)
\]

\[
+ \frac{\gamma_m}{2} [n(\omega_m) + 1] \left( 2a_m^{\dagger} \rho a_m - a_m a_m^{\dagger} \rho - \rho a_m a_m^{\dagger} \right). \tag{2}
\]

The total cavity decay rate \( \kappa \) has two contributions: (i) the rate at which photons are lost from the open port (where the driving field comes in) given by \( \kappa_{ex} \) and (ii) the ‘internal loss’ rate \( \kappa - \kappa_{ex} \) due to the other losses of the electromagnetic resonator (i.e. absorption inside the dielectric, scattering into other modes, etc). Naturally the thermal noise is determined by the Bose number \( n(\omega) \). At room temperature and for optical frequencies \( n(\omega_p) \) is negligible, however for the much lower radio and microwave frequencies characterizing electromechanical setups this quantity can be comparable to the final mechanical occupations achieved. Similarly \( \gamma_m = \omega_m / Q_m \) is the mechanical resonator’s natural linewidth and \( n(\omega_m) \) its mean occupation number at thermal equilibrium (i.e. in the absence of the drive). Mechanical damping, which plays a ubiquitous role in determining the temperature limits (cf equation (54)), is usually described in a phenomenological manner by introducing an Ohmic damping force [48] \( \sim -\gamma_m \hat{X}_m \) (where \( \hat{X}_m \equiv (a_m + a_m^{\dagger}) / \sqrt{2} \)). A microscopic derivation of mechanical dissipation due to phonon-tunneling into the supports of the mechanical resonator has recently been given by Wilson-Rae in [49], which identifies conditions under which such an Ohmic model is indeed justified and gives geometric upper bounds on the associated mechanical \( Q \)-values \( Q_m \).

To study the cooling process it proves useful to apply a canonical transformation of the form \( a_p \rightarrow a_p + \alpha, a_m \rightarrow a_m + \beta \) with the amplitudes \( \alpha, \beta \) chosen so that the linear terms in the transformed Liouvillian cancel out. This condition leads to the following coupled equations for the amplitudes

\[
\Omega - \left( \Delta_L + i \frac{\kappa}{2} \right) \alpha + \eta \omega_m \alpha (\beta + \beta^*) = 0,
\]

\[
(\omega_m - i \frac{\gamma_m}{2}) \beta + \eta \omega_m |\alpha|^2 = 0. \tag{3}
\]

We assume \( \eta \ll 1 \) and that the mechanical dissipation rate \( \gamma_m \) is much smaller than \( \omega_m \). To lowest order in the small parameters \( \eta \) and \( 1/Q_m \) we obtain \( \alpha \approx \Omega / (2 \Delta_L + i \kappa) \) and \( \beta \approx -\eta |\alpha|^2 \). Here \( |\alpha|^2 \) is the steady state occupancy of the cavity and \( \beta \) is the static shift of the mechanical amplitude due to the radiation pressure. The normal coordinates after the transformation are shifted so that the new amplitudes correspond to the deviation from the steady state equilibrium position. This transformation leaves the dissipative part of the Liouvillian invariant and transforms the Hamiltonian into:

\[
H = -\hbar \Delta_L a_p^{\dagger} a_p + \hbar \omega_m a_m^{\dagger} a_m + \hbar \eta \omega_m (a_p^{\dagger} a_p + \alpha^* a_p + \alpha a_p^{\dagger}) (a_m + a_m^{\dagger}), \tag{4}
\]
where we have introduced the effective detuning $\Delta'_L + 2\eta^2|\alpha|^2\omega_m \rightarrow \Delta_L$. It is interesting to note that if the bosonic cooler is replaced by a fermionic one so that $a_p \rightarrow \sigma_-$ we obtain a Hamiltonian that resembles the one describing a trapped ion in the Lamb–Dicke regime. As a result for perturbative $\eta$ (as analyzed in the next section) the cooling cycle (cf figure 1(b)) becomes analogous to the Lamb–Dicke regime of atomic laser-cooling [41].

3. Perturbative cooling

3.1. Master equation for mechanical motion

We first focus on the regime in which the input laser power $P$ is low enough so that the timescales over which the populations of the mechanical resonator’s Fock states evolve (leading to cooling or heating) are much slower than those associated with the losses of the cavity and with the free mechanical frequency. As will become clear below this requires $\eta^2|\alpha|^2 \ll (\kappa/\omega_m)^2$.

Here we also assume $[n(\omega_m) + 1]_{\gamma_m} \ll \kappa$, $\omega_m$ which must hold to allow for appreciable cooling, and $\eta^2 \ll 1$ (in optomechanical realizations $\eta \lesssim 10^{-4}$). Hence the electromagnetic degrees of freedom can be treated as a structured environment that affects the mechanical motion perturbatively. Along these lines the latter can be described by a Markovian master equation for its reduced density matrix [46].

To derive it we take the optomechanical master equation in the shifted representation, transform to an interaction picture for the resonator mode and adiabatically eliminate the cavity using the Nakajima–Zwanzig formalism [46, 50, 51]. The optomechanical coupling and the mechanical losses are treated perturbatively. We define the projection

$$ P \rho = \text{Tr}_p(\rho) \otimes \rho_p^{(\text{th})}, \quad Q \equiv \mathbb{1} - P, $$

with

$$ \dot{\rho}_p^{(\text{th})} = \frac{1}{n_p + 1} \sum_{n=0}^{\infty} \left[ \frac{n_p}{n_p + 1} \right]^n |n \rangle \langle n|, $$

where $n_p \equiv n(\omega_p)$ is the Bose number, and introduce the formal parameter $\zeta$ such that

$$ L(t) = \zeta^2 L_0 + \zeta L_1(\zeta^2 t) + L_2, $$

with

$$ L_0 \rho \equiv i \left[ \Delta_L a_p^\dagger a_p, \rho \right] + \frac{\kappa}{2} n(\omega_p) \left[ 2a_p^\dagger \rho a_p - a_p a_p^\dagger \rho - \rho a_p a_p^\dagger \right] + \frac{\kappa}{2} \left[ n(\omega_p) + 1 \right] \left[ 2a_p \rho a_p^\dagger - a_p^\dagger a_p \rho - \rho a_p^\dagger a_p \right], $$

$$ L_1(\zeta^2 t) \equiv e^{i \omega_m \zeta^2 t} L_1^{(+)} + e^{-i \omega_m \zeta^2 t} L_1^{(-)}, $$

$$ L_1^{(+)} \rho \equiv -i \left[ \eta \omega_m \left[ a_p^\dagger a_p + \alpha^* a_p + \alpha a_p^\dagger \right] a_m^\dagger, \rho \right], $$

$$ L_1^{(-)} \rho \equiv -i \left[ \eta \omega_m \left[ a_p^\dagger a_p + \alpha^* a_p + \alpha a_p^\dagger \right] a_m^\dagger, \rho \right], $$

$$ L_2 \equiv \frac{\gamma_m}{2} n(\omega_m) \left( 2a_m^\dagger \rho a_m - a_m a_m^\dagger \rho - \rho a_m a_m^\dagger \right) + \frac{\gamma_m}{2} \left[ n(\omega_m) + 1 \right] \left( 2a_m \rho a_m^\dagger - a_m^\dagger a_m \rho - \rho a_m^\dagger a_m \right). $$

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We note that $\mathcal{P}\rho$ is a stationary state of $\mathcal{L}_0$ for any $\rho$ implying that $\mathcal{L}_0\mathcal{P} = 0$, while $\mathcal{P}\mathcal{L}_0 = 0$ follows from trace preservation, so that we have

$$Q\mathcal{L}_0 Q = \mathcal{L}_0, \quad \mathcal{P}\mathcal{L}_0\mathcal{P} = Q\mathcal{L}_0\mathcal{P} = \mathcal{P}\mathcal{L}_0 Q = 0. \quad (13)$$

As will emerge from our derivation the basic idea is that the rates for cooling and heating set the relevant timescale (zeroth order in $1/\zeta$) which is widely separated from the mechanical oscillation period $2\pi/\omega_m$ and the cavity lifetime $1/\kappa$ (order $1/\zeta^2$). In fact, the asymptotic expansion for $\zeta \to \infty$ pursued below amounts to a controlled expansion in the ratio between the fast and the slow timescales. Given that we are interested in the behavior as $t \to \infty$, the initial condition is immaterial and we choose for simplicity one in the $\mathcal{P}$-manifold so that $Q\rho_0 = 0$. Subsequently, by explicitly integrating the differential equation for the irrelevant part $\mathcal{Q}\rho$, we obtain a closed equation for the relevant part $\mathcal{P}\rho$:

$$\mathcal{P}\dot{\rho} = \mathcal{P}\mathcal{L}(t)\mathcal{P}\rho + \mathcal{P}\mathcal{L}(t) \int_0^t d\tau T_+ \left[ e^{\int_0^\tau d\tau' Q\mathcal{L}(\tau') Q} \right] T_- \left[ e^{-\int_0^\tau d\tau' Q\mathcal{L}(\tau') Q} \right] \mathcal{L}(\tau) \mathcal{P}\rho(\tau), \quad (14)$$

where $T_+$ ($T_-$) is the time-ordering (anti-time-ordering) operator. For the purpose of analyzing the asymptotic limit $\zeta \to \infty$ of equation (14) we have

$$T_+ \left[ e^{\int_0^\tau d\tau' Q\mathcal{L}(\tau') Q} \right] T_- \left[ e^{-\int_0^\tau d\tau' Q\mathcal{L}(\tau') Q} \right] = e^{\tau^2 Q\mathcal{L}_0 (\tau - \tau)} \left[ \mathcal{1} + O \left( \frac{1}{\zeta} \right) \right], \quad (15)$$

which can be understood considering the corresponding Laplace transforms. Substitution of equations (13) and (15) into (14) and the change of variables $\tau' = \zeta^2 (t - \tau)$ then yields

$$\mathcal{P}\dot{\rho} = \mathcal{P} \left[ \zeta \mathcal{L}_1 (\zeta^2 t) + \mathcal{L}_2 \right] \mathcal{P}\rho + \mathcal{P} \mathcal{L}_1 (\zeta^2 t) Q \int_0^\infty d\tau e^{\zeta \tau} Q \mathcal{L}_1 (\zeta^2 t - \tau) \mathcal{P}\rho (t - \tau'/\zeta^2) + \cdots. \quad (16)$$

Here we have also used $Q^2 = Q$, $\mathcal{P}^2 = \mathcal{P}$ and $\mathcal{P}\mathcal{L}_2 = \mathcal{L}_2 \mathcal{P}$. The leading order in $1/\zeta$ of the evolution over the aforementioned relevant timescale will be determined by the limit as $\zeta \to \infty$ of the Laplace transform of the above. In the time domain all the fast rotating terms (frequencies of order $\zeta^2 \omega_m$) drop out and equation (16) reduces to

$$\mathcal{P}\dot{\rho} = \mathcal{P} \mathcal{L}_2 \mathcal{P}\rho + \left[ \mathcal{P} \mathcal{L}_1^{(+)} Q \int_0^\infty d\tau e^{(i \omega_m + \mathcal{L}_0)\tau} Q \mathcal{L}_1^{(-)} \mathcal{P}\rho + h.c. \right]. \quad (17)$$

We note that as $\mathcal{P} = \lim_{\zeta \to \infty} e^{\mathcal{L}_0 t}$ we have

$$\mathcal{L}_0 \rho = 0 \quad \Rightarrow \quad Q\rho = 0. \quad (18)$$

It follows that the restriction of $\mathcal{L}_0$ to the $Q$-manifold has only eigenvalues with negative real parts which allows us to establish

$$Q \int_0^\infty d\tau e^{(i \omega_m + \mathcal{L}_0)\tau} Q = Q (-i \omega_m - \mathcal{L}_0)^{-1} Q. \quad (19)$$

We focus on the parameter regime where $|\alpha|^2 \gg n_p$. The behavior of the cavity correlations that determine the second term in equation (17) implies that in this regime contributions arising from the cubic term in $\mathcal{L}_1^{(\pm)}$ are negligible compared to those generated by the quadratic term—for $n_p = 0$, the contributions of the former are higher order in $\eta$. The conditions that warrant this linearization of $\mathcal{L}_1^{(\pm)}$ for the case $n_p = 0$ will be discussed further in the next subsection.
The restriction of the quadratic term to the $P$-manifold vanishes and we obtain
\[ \text{Tr}_p \left\{ \mathcal{L}^{(+)}_1 Q \int_0^\infty d\tau e^{i(\omega_0 \tau - \mathcal{P})} \mathcal{L}^{(-)}_1 \mathcal{P} \rho \right\} \]
\[ \approx - \frac{g_m^2}{2} \left\{ G(\omega_m, n_p) \left[ a_m^\dagger a_m^\dagger - \mu \right] - G^*(-\omega_m, n_p) \left[ a_m^\dagger a_m^\dagger - \mu a_m a_m \right] \right\}. \]

Here, we have used equations (5), (10) and (11), and introduced the optomechanical coupling
\[ g_m \equiv 2\eta|\alpha|\omega_m, \]
the reduced density matrix for the mechanical mode $\mu \equiv \text{Tr}_p[\rho]$, and the cavity-quadratures’ correlations
\[ G(\omega, n_p) = \int_0^\infty d\tau e^{i\omega\tau} \text{Tr}_p \left\{ X_p(0)e^{i\tau}X_p(0)\rho_p^{(\text{th})} \right\}, \]
with $X_p(0) \equiv (a^\dagger a + a a^\dagger )/\sqrt{2}|\alpha|$. If we substitute equations (5), (12) and (20) into (17), trace out the cavity, and rearrange we finally obtain the following master equation:
\[ \dot{\mu} = -i \left[ (\omega_m + \Delta_m) a_m^\dagger a_m^\dagger, \mu \right] + \frac{1}{2} \left\{ \gamma_m (n(\omega_m) + 1) + A_-(n_p) \right\} (2a_m^\dagger a_m^\dagger - a_m^\dagger a_m^\dagger - \mu a_m a_m^\dagger - \mu a_m^\dagger a_m), \]
where the cooling (heating) rate $A_-(n_p)$ [$A_+(n_p)$] and the mechanical frequency shift $\Delta_m$ induced by the optomechanical coupling are given by
\[ A_\pm(n_p) = g_m^2 \gamma_m \{ G(\pm\omega_m, n_p) \}, \]
\[ \Delta_m = \frac{g_m^2}{2} \gamma_m \{ G(\omega_m, n_p) - G(-\omega_m, n_p) \}. \]

It is straightforward to calculate the necessary two-time correlations using the quantum regression theorem given the steady state moments:
\[ \langle a_p a_p^\dagger \rangle = n_p + 1, \quad \langle a_p^\dagger a_p^\dagger \rangle = n_p, \quad \langle a_p a_p \rangle = 0 \]
and that the evolution of the mean amplitude reads
\[ \langle a_p \rangle = (i\Delta_L - \kappa/2) \langle a_p \rangle. \]
Thus we obtain
\[ G(\omega, n_p) = \left[ G(\omega, 0) + G^*(\omega, 0) \right] n_p + G(\omega, 0) \]
with
\[ G(\omega, 0) = \frac{1}{-2i(\omega + \Delta_L + \kappa)}, \]
which substituted into equations (24) and (25) leads to
\[ A_\pm(n_p) = \left[ A_-(0) + A_+(0) \right] n_p + A_\pm(0), \]
\[ \Delta_m = g_m^2 \left[ \frac{\Delta_L - \omega_m}{4(\Delta_L - \omega_m)^2 + \kappa^2} + \frac{\Delta_L + \omega_m}{4(\Delta_L + \omega_m)^2 + \kappa^2} \right], \]
where the corresponding rates in the absence of thermal noise in the driven cavity are given by
\[ A_\pm(0) = g_m^2 \kappa \frac{\Delta_L}{4(\Delta_L^2 + \omega_m)^2 + \kappa^2}. \]
The above can be related to the input power and the frequency of the drive via

\[ g_m = 2\eta\omega_m \sqrt{\frac{P\kappa_{\text{ex}}}{\hbar\omega_L (\Delta'_L^2 + \kappa^2/4)}}, \]  

\[ \Delta_L = \Delta'_L + 2\eta^2\omega_m \sqrt{\frac{P\kappa_{\text{ex}}}{\hbar\omega_L (\Delta'_L^2 + \kappa^2/4)}}. \]  

The master equation (23) generalizes the one obtained in [41] by including thermal noise in the cavity input. As shown below, this effect is significant for determining the ultimate limit to which the mechanical resonator can be cooled for ratios \( \omega_p/\omega_m \) like the ones that characterize electromechanical realizations. Equation (23) has as its steady state a thermal state which defines the effective final temperature to which the mechanical resonator is cooled. The corresponding final occupancy reads \[ \Gamma_{1} \equiv \frac{\gamma_m}{\Gamma + \gamma_m} \left[ (2\tilde{n}_f + 1)n_p + \tilde{n}_f \right], \]  

where \( \tilde{n}_f \) is the quantum backaction contribution derived in [41, 42], namely

\[ \tilde{n}_f = -\frac{(\Delta_L + \omega_m)^2}{4\Delta_L\omega_m}. \]  

We note that in the ‘unshifted’ representation there is in addition a coherent shift of the resonator’s normal coordinate so that

\[ \langle a_m^+ a_m \rangle = n_f + \eta^2 \left[ \frac{P\kappa_{\text{ex}}}{\hbar\omega_L (\Delta'_L^2 + \kappa^2/4)} \right]^2. \]  

If we now consider the appreciable cooling limit \( \Gamma \gg \gamma_m \) and minimize with respect to the detuning we obtain

\[ \min\{n_f\} \approx \frac{\gamma_m k\sqrt{\omega_m^2 + \kappa^2/4}}{g_m^2\omega_m} n(\omega_m) + n_p + \left( n_p + \frac{1}{2} \right) \left( \sqrt{1 + \frac{\kappa^2}{4\omega_m^2}} - 1 \right), \]  

for the optimal detuning \( \Delta_L^{\text{opt}} = \sqrt{\omega_m^2 + \kappa^2/4} \). The first term corresponds to the ‘linear cooling’ limited by thermal noise. In turn, the second term shows that the final occupancy is necessarily bounded by the equilibrium thermal occupancy of the cooler. Finally, the last term for \( n_p = 0 \) corresponds to the fundamental temperature limit imposed by the quantum backaction which in the Doppler regime \( \kappa \gg \omega_m \) reduces to \( \kappa/4\omega_m \), precluding ground state cooling, whereas in the RSB regime \( \kappa \ll \omega_m \) it yields \( \kappa^2/16\omega_m^2 \) corresponding to occupancies well below unity [41, 42]. We note that for a given \( \kappa/\omega_m < \sqrt{32} \) occupancies below unity are only attainable within a finite detuning window \( |\Delta_L + 3\omega_m| \leq \sqrt{8\omega_m^2 - \kappa^2/4} \), as illustrated in figure 2.

3.2. Output power spectrum and temperature measurement

The spectrum of the cavity output constitutes a crucial observable to understand the back-action cooling. It allows the cooling cycle to be visualized as a frequency up-conversion process and for optomechanical realizations it provides an efficient way to measure the final temperature [41]. We focus on the latter for which \( n_p = 0 \) and consider the experimentally relevant case in which the output is measured in the same modes in which the coherent laser drive is fed [24].
Figure 2. Final (steady state) average phonon number \( n_f \) as a function of the normalized detuning \( \Delta L/\kappa \) for values of the ratio \( \kappa/\omega_m = 10, 5.7, 2.8 \) and 1.4 with \( \gamma_m = n_p = 0 \).

To calculate its spectrum we apply the standard input–output formalism [46] and treat the parameters \( \eta|\alpha|, \eta \) perturbatively along the lines of the previous subsection. For this purpose, it proves useful to consider the quantum Langevin equation for \( a_p \) associated with the Liouvillian (7), namely

\[
\dot{a}_p = \left( i\Delta_L - \frac{\kappa}{2} \right) a_p - i\eta \omega_m (a_m + a_m^\dagger) (a_p + \alpha) + \sqrt{\kappa \kappa_{ex}} \delta a_{in}(t) + \sqrt{\kappa - \kappa_{ex}} b_{in}(t),
\]

where \( \delta a_{in} \) and \( b_{in} \) correspond to the vacuum noise associated, respectively, with the laser mode and with the other cavity losses. Here we use the shifted representation (cf (4)) and also take into account that Hamiltonian (1) presupposes the standard time-dependent canonical transformation that maps the coherent input state associated with the laser into a classical field so that \( a_{in}(t) \rightarrow \delta a_{in}(t) + \langle a_{in}(t) \rangle \). If we assume that the term linear in \( \eta \) is a specified function of time and that the solution \( a_p^{(0)}(t) \) for \( \eta = 0 \) is known, equation (39) can be formally integrated to obtain

\[
a_p(t) = a_p^{(0)}(t) - i\eta \omega_m \int_0^t d\tau e^{i(\Delta_L - \kappa/2)(t-\tau)} \left[ a_m^\dagger(\tau) + a_m(\tau) \right] [a_p(\tau) + \alpha].
\]

This integral equation can be iterated to generate the following Dyson series type result

\[
a_p(t) = a_p^{(0)}(t) + \sum_{n=1}^{\infty} (-i\eta \omega_m)^n \int_0^t d\tau_n \int_0^{\tau_n} d\tau_{n-1} \cdots \int_0^{\tau_2} d\tau_1 e^{i(\Delta_L - \kappa/2)(t-\tau_1)}
\times \left[ a_m^\dagger(\tau_n) + a_m(\tau_n) \right] \cdots \left[ a_m^\dagger(\tau_1) + a_m(\tau_1) \right] [a_p^{(0)}(\tau_1) + \alpha].
\]

In turn, the output spectrum is given by

\[
S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau e^{-i(\omega - \omega_m)\tau} \langle a_{out}^\dagger(t + \tau) a_{out}(t) \rangle_{SS}.
\]
which in the shifted representation reads

\[
S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\tau e^{-i(\omega - \omega_L)\tau} \left\{ \left[ \sqrt{\kappa_\text{ex}} a_p^\dagger(t + \tau) + \delta a_m^\dagger(t + \tau) + \sqrt{\kappa_\text{ex}} \alpha + \langle a_m(0) \rangle^* \right] \times \left[ \sqrt{\kappa_\text{ex}} a_p(t) + \delta a_m(t) + \sqrt{\kappa_\text{ex}} \alpha + \langle a_m(0) \rangle \right] \right\}_{SS}.
\]

(43)

We now seek the lowest nontrivial order in \( \eta \). The output \( a_{\text{out}} \) has a \( c \) number part arising from the classical drive and the cavity steady state amplitude and an operator part corresponding to the fluctuations. Equation (43) directly implies that terms involving the \( c \) numbers only contribute to the ‘main line’ \( \propto \delta(\omega - \omega_L) \). Furthermore, as in the shifted representation the steady state of the electromagnetic modes is the vacuum \( |0\rangle \) it follows from equations (41) and (43) that the contribution of the cross-term with the operator part—which vanishes if the cubic term in equation (4) is omitted—is at most higher order in \( \eta \) and can be neglected relative to the contribution bilinear in the \( c \) numbers. The latter corresponds to the classical reflection coefficient that is straightforward to obtain considering the near-resonant scattering into modes (real or fictitious) responsible for the other losses \( \kappa - \kappa_\text{ex} \). Thus we arrive at

\[
S(\omega) \approx \frac{P}{\hbar \omega_L} \left[ 1 - \frac{(\kappa - \kappa_\text{ex}) \kappa_\text{ex}}{\Delta_L^2 + \kappa^2/4} \right] \delta(\omega - \omega_L) + \frac{\kappa_\text{ex} \Delta_m^2}{8\pi} \times \int_{-\infty}^{+\infty} d\tau e^{-i(\omega - \omega_L)\tau} \left\{ \int_{0}^{t + \tau} d\tau' e^{\left(-i\Delta_L - \kappa/2\right)(t + \tau - \tau')} \left[ a_m^\dagger(\tau') + a_m(\tau') \right] \right\}_{SS},
\]

(44)

where we have used equation (41) and \( a_p(0) = \delta a_m(0) = 0 \). Note that the correction is higher order in \( \eta \) for all frequencies. To calculate the steady state two-time average in equation (44) we adopt (as in the previous subsection) an interaction picture for the resonator mode (i.e. \( a_m(t) \rightarrow e^{-i\omega_m t} a_m(t) \)) and make the substitutions \( t + \tau - \tau_1 \rightarrow \tau_1', t - \tau_1 \rightarrow \tau_1 \) in the time integrals. In this representation, the mechanical mode operators evolve slowly compared with \( 1/\kappa \) so that, in line with our derivation of a Markovian master equation for the mechanical motion, we can factor them out of the time integrals whose upper limit can be extended to infinity. This Markovian approximation then yields

\[
\left\langle \int_{0}^{t + \tau} d\tau' e^{\left(-i\Delta_L - \kappa/2\right)(t + \tau - \tau')} \left[ a_m^\dagger(\tau') + a_m(\tau') \right] \int_{0}^{\tau'} d\tau_1 e^{i\left(\Delta_L - \kappa/2\right)(t - \tau_1)} \left[ a_m^\dagger(\tau_1) + a_m(\tau_1) \right] \right\}_{SS} \approx 4 \left\{ \left[ G^*(\omega_m, 0) e^{i\omega_m t} a_m^\dagger(t + \tau) + G^*(-\omega_m, 0) e^{-i\omega_m t} a_m(t + \tau) \right] \times \left[ G(-\omega_m, 0) e^{i\omega_m t} a_m^\dagger(t) + G(\omega_m, 0) e^{-i\omega_m t} a_m(t) \right] \right\}_{SS}
\]

(45)

where we have used equation (29). The two-time averages of the mechanical mode operators can now be calculated from the master equation (23) using the quantum regression theorem. The needed one-time averages satisfy

\[
\langle a_m \rangle = -\left( i\Delta_m + \frac{\gamma_m + \Gamma}{2} \right) \langle a_m \rangle,
\]

\[
\langle a_m a_m^\dagger \rangle_{SS} = n_f
\]

\[
\langle a_m^\dagger a_m \rangle_{SS} = n_f + 1
\]

\[
\langle a_m a_m \rangle_{SS} = \langle a_m^\dagger a_m^\dagger \rangle_{SS} = 0.
\]

(46)
Finally, a straightforward calculation leads from equations (44)–(46), (29) and (32) to the spectrum already given in [41], namely

\[
S(\omega) \approx \frac{P}{\hbar \omega_L} \left[ 1 - \frac{(\kappa - \kappa_{ex})\kappa_{ex}}{\Delta_L^2 + \kappa^2/4} \right] \delta(\omega - \omega_L) + \frac{\kappa_{ex}A_-(n_f + 1)}{\kappa \pi} \frac{(\gamma_{eff}/2)}{(\omega - \omega_L - \omega_m - \Delta_m)^2 + (\gamma_{eff}/4)},
\]

where we have introduced \( \gamma_{eff} = \gamma_m + \Gamma \) which is the total dissipation rate for the mechanical mode in the presence of the drive that determines the linewidths of the motional sidebands peaked at \( \omega_L \pm \omega_m \). One should note that the above corresponds to photon rate per unit frequency and that the weight of the motional sidebands relative to the main line is of order \( \eta^2 (2n_f + 1) \) (\( \eta^2 \lesssim 10^{-8} \) for typical systems). Here unlike the case of atomic laser cooling [31, 32] the mechanical dissipation induces an asymmetry in the weights of the sidebands that allows the final temperature to be retrieved directly from the steady state. The ‘blue’ sideband weighted by \( N_- = \frac{\kappa}{\kappa - \kappa_{ex}} A_-(n_f + 1) \) corresponds to the up-converted photons (anti-Stokes scattering) responsible for the cooling while the ‘red’ sideband weighted by \( N_+ = \frac{\kappa}{\kappa + \kappa_{ex}} A_+(n_f + 1) \) corresponds to the down-converted photons (Stokes scattering) that result in heating.

It is interesting to note that the formalism of this section does not presuppose linearizing around the steady state and neglecting the cubic term in Hamiltonian (4) accordingly, but rather the validity of such treatment emerges from a controlled procedure that would allow the necessary corrections to be incorporated if the intrinsic nonlinearity \( \eta \) were larger. A straightforward self-consistency criterion is to compare the steady state fluctuation of the cubic term with that of the quadratic one. An heuristic estimate for their ratio can be extracted from the total weight of the motional sidebands in equation (47) which together with the analysis in section 3.1 implies that the cubic term can be neglected provided the conditions \( \eta^2 (2n_f + 1) \omega_m^2 \ll \kappa^2 \) and \( |\alpha|^2 \gg \eta^2 \) are satisfied—for the typical parameters in cooling experiments these are always met.

4. Linearized theory for coupled optomechanical modes

4.1. Small cavity linewidth limit \( \kappa \ll g_m \)

The treatment in the previous section is only applicable when the cooling rate \( A_- \) given by equations (30) and (32) is much smaller than \( \kappa \) (note that the heating rate \( A_+ \) is always bounded by \( A_- \) for negative detuning). This condition for arbitrary negative detuning results in \( g_m^2 \ll \kappa^2 \) so that it follows (as expected) that the motional master equation is only warranted for small enough optomechanical coupling. In the Doppler regime [31, 32], \( \kappa \gg \omega_m \), when this is violated the aforementioned condition \( g_m \ll \frac{\kappa}{\kappa_{ex}} \omega_m \) underpinning the RWA for the mechanical losses will also fail. In contrast, in the resolved sideband regime (RSB) relevant for ground state cooling, \( \kappa \ll \omega_m \) implies that there is a wide parameter range of interest in which equation (2) remains valid while equation (23) fails. Here we consider this RSB regime beyond perturbative optomechanical coupling. As \( g_m \) becomes comparable to \( \kappa \) it becomes necessary to follow the coupled dynamics of both modes as described by equation (2) which for \( g_m > \kappa/2 \) exhibits normal mode splitting. Though for a Gaussian initial condition the approach to the steady state is always amenable to a straightforward description, in the intermediate regime \( g_m \sim \kappa \) there will be no simple analog of equation (23) that allows the cooling process to be visualized in terms

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of phonon jumps. In turn, deep in the strong coupling regime but away from the instability [42], i.e. for $\kappa \ll g_m \ll \omega_m$, the dynamics can be described by two decoupled master equations for the optomechanical normal modes analogous to equation (23).

Within the latter parameter range it is permissible to start from a Hamiltonian description including the optomechanical coupling and treat the losses (as described by $\gamma_m$, $\kappa$) perturbatively. We focus on the ‘resonant case’ $-\Delta_\ell = \omega_m$—which in the RSB regime can be shown to be optimal for minimizing the final occupancy—and consider the canonical transformation that diagonalizes Hamiltonian (4) for $n = 0$ with $g_m \neq 0$ (after performing the convenient rotation $a_\eta \rightarrow (\alpha/|\alpha|)a_\eta$). The latter is given by

$$a_{m/p} = \frac{1}{2\sqrt{2}} \left[ \left( \sqrt{\frac{\omega_m}{\omega_+}} + \sqrt{\frac{\omega_m}{\omega_+}} \right) a_+ \pm \left( \frac{\omega_m}{\omega_-} + \frac{\omega_m}{\omega_-} \right) a_- \right. + \left( \frac{\omega_m}{\omega_+} - \frac{\omega_m}{\omega_+} \right) a_+ \pm \left( \frac{\omega_m}{\omega_-} - \frac{\omega_m}{\omega_-} \right) a_- \right], \quad (48)$$

where the eigenfrequencies of the normal modes read

$$\omega_{\pm} = \omega_m (1 \pm g_m/\omega_m)^{1/2}. \quad (49)$$

If we now consider the expansion of the above in the small parameter $g_m/\omega_m$ the zeroth order of equation (49) yields a splitting given by $g_m$ while equation (48) reduces to the transformation that diagonalizes the rotating wave part of Hamiltonian (4) ―that results from neglecting the terms that involve $a_m a_\eta$ and $a_m^\dagger a_\eta^\dagger$. The latter transformation does not mix the annihilation and creation operators. For small but finite $g_m/\omega_m$ there will be small admixtures that for the purpose of analyzing the cooling dynamics will only be relevant insofar as they give rise to qualitatively new terms in the dissipative part of the Liouvillian―otherwise they can be shown to result in contributions of relative order $(g_m/\omega_m)^2$ for all values of the other parameters. Hence we have

$$a_{m/p} \approx \frac{1}{2\sqrt{2}} a_+ \pm \frac{1}{2\sqrt{2}} a_- - \frac{g_m}{4\sqrt{2}\omega_m} a_+ \pm \frac{g_m}{4\sqrt{2}\omega_m} a_- \quad (50)$$

To proceed we: (i) apply the transformation given by equation (48) to the master equation (2), (ii) transform the result to an interaction picture with respect to the (now diagonal) Hamiltonian, (iii) neglect all the resulting fast rotating terms which are $\propto e^{\pm 2\omega_m t}$ or $\propto e^{\pm 2i\omega_m t}$ (up to corrections higher order in $g_m/\omega_m$), and (iv) expand to lowest order in $g_m/\omega_m$ following the aforementioned ‘qualitative’ criterion. Naturally (iii) relies on the small cavity linewidth condition $\kappa \ll g_m$. Thus we obtain

$$\dot{\rho} = \sum_{\xi = \pm} \left[ \frac{\gamma_m}{4}[n(\omega_m) + 1] + \frac{\kappa g_m^2}{64\omega_m^2} \right] \left( 2a_\xi^\dagger \rho a_\xi - a_\xi^\dagger a_\xi^\dagger \rho - \rho a_\xi a_\xi^\dagger \right) \right. + \left. \left( \frac{\gamma_m}{4}[n(\omega_m) + 1] + \frac{\kappa g_m^2}{64\omega_m^2} \right) \left( 2a_\xi^\dagger \rho a_\xi - a_\xi^\dagger a_\xi^\dagger \rho - \rho a_\xi a_\xi^\dagger \right) \right]. \quad (51)$$

The only corrections in $(g_m/\omega_m)^2$ appear in the first term and correspond to the ‘high power limit’ of the heating induced by the quantum backaction of the cavity. Analogous heating terms $\propto \gamma_m$ are neglected given that they are comparable to corrections to the RWA treatment of the mechanical dissipation. It follows from equation (51) that the losses do not couple the normal
modes (annihilation operators \(a_{\pm}\)) so that the steady state is given by the tensor product of thermal states for each of them characterized by

\[
\langle a_{\pm}^\dagger a_{\pm}\rangle_{SS} = \frac{\gamma_m n(\omega_m) + \kappa n(\omega_p) + (\kappa g_m^2/16 \omega_m^2)}{\gamma_m + \kappa},
\]

(52)

(where we neglect higher order terms in \(g_m/\omega_m\)) to which the average occupancies converge with a cooling rate \((\kappa + \gamma_m)/2\). Hence equations (50) and (52) finally yield

\[
n_f \approx \frac{\gamma_m n(\omega_m) + \kappa n(\omega_p) + (\kappa g_m^2/16 \omega_m^2)}{\gamma_m + \kappa} + \frac{g_m^2}{16 \omega_m^2} + \frac{1}{2} \left( \frac{\kappa^2}{4 \omega_m^2} - 1 \right),
\]

(53)

where the corrections are higher order in the small parameters \((g_m/\omega_m)^2\) and \(\gamma_m/\kappa\). The first term corresponds to the heating associated with the mechanical dissipation and can be identified with the corresponding term in equation (35) showing a saturation of the usual linear cooling law. Similarly, the second and third terms can be identified with the corresponding contributions in equation (35) arising from the thermal noise in the cavity and the quantum backaction. Thus ground state cooling requires \(k_B T/\hbar Q_m \ll \kappa \ll \omega_m\) and \(\omega_p < k_B T/\hbar\). We note that equations (50) and (52) imply that \(\langle a_{\pm}^2\rangle_{SS} = 0\) so that the reduced state for the motion is also thermal in the small \(\kappa\) limit.

If we now compare equations (38) and (53) and consider minimizing them with respect to \(g_m\) within their respective ranges of validity, heuristic considerations imply that the following formula should always constitute a lower bound for the final mechanical occupancy optimized with respect to the parameters of the drive

\[
n_{TL} = \frac{\gamma_m n(\omega_m)}{\kappa} + n_p + \frac{1}{2} \left( \frac{\kappa^2}{4 \omega_m^2} - 1 \right).
\]

(54)

This will be borne out quantitatively in section 4.2 by comparing with the results of a treatment that is exact within the linearized theory.

### 4.2. Final occupancy for arbitrary ratio \(g_m/\kappa\) and optimal parameters

The approximate expressions (35) and (53) for the final mechanical occupancy that we have derived in the limits \(g_m \ll \kappa\) and \(g_m \gg \kappa\) provide a basic understanding of the requirements for ground state cooling and the expected order of magnitude for the optimum. Notwithstanding they have the drawback that they fail to settle which is the optimal input power as minimization with respect to \(g_m\) shifts this variable away from the domain where they are valid. In addition, given the experimental progress towards achieving ultra-cold states in these systems, it is clear that precise quantitative predictions for the steady state that results from a given input are highly desirable. To this effect we complement the above analysis by deriving an analytical expression for \(n_f\) valid for arbitrary values of the ratio \(g_m/\kappa\). The optomechanical master equation (2) directly implies that, when the cubic nonlinearity in Hamiltonian (4) is neglected, the time evolution for the ten independent second-order moments that determine the covariance matrix...
is given by a linear system of ordinary differential equations. This is determined by

\[
\frac{d}{dt} \left\{ a_m^+ a_m \right\} = -\gamma_m (a_m^+ a_m) + \gamma_m n(\omega_m) - \frac{i g_m}{2} \left( (a_p^+ + a_p) (a_m^+ - a_m) \right),
\]

\[
\frac{d}{dt} \left\{ a_p a_m \right\} = - \left( \frac{\kappa}{2} + \frac{\gamma_m}{2} - i \Delta_L + i \omega_m \right) (a_p a_m) - \frac{i g_m}{2} \left( (1 + a_m^+ a_m) + (a_p^+ a_p) + \{a_p^+, a_m\} \right),
\]

\[
\frac{d}{dt} \left\{ a_p^+ a_m^+ \right\} = - \left( \frac{\kappa}{2} + \frac{\gamma_m}{2} - i \Delta_L - i \omega_m \right) (a_p a_m^+) - \frac{i g_m}{2} \left( (a_m^+ a_m) + (a_p^+) a_p - \{a_p, a_m^+\} \right),
\]

\[
\frac{d}{dt} \left\{ a_p^+ a_p \right\} = - \kappa \{a_p, a_p\} - \frac{g_m}{2} \left( (a_p^+ - a_p) (a_m^+ + a_m) + \kappa \omega_p \right),
\]

\[
\frac{d}{dt} \left\{ a_m^+ \right\} = - (\gamma_m + 2 \omega_m) a_m^+ + i g_m \left\{ (a_p^+ + a_p) a_m \right\},
\]

\[
\frac{d}{dt} \left\{ a_p^+ \right\} = - (\kappa - 2 i \Delta_L) a_p^+ - i g_m \left\{ (a_m^+ + a_m) a_p \right\},
\]

and their Hermitian conjugates. The exact solution for the linear system of equations that results for the steady state covariance matrix then yields an analytical formula for the final occupancy of the mechanical resonator that is a sum of three independent contributions

\[
n_f = n_{f}^{(m)} + n_{f}^{(p)} + n_{f}^{(0)}. \tag{56}
\]

Here \(n_{f}^{(m)}\) and \(n_{f}^{(p)}\) arise, respectively, from the mechanical dissipation and the thermal noise in the cavity input, and \(n_{f}^{(0)}\) corresponds to the heating induced by the quantum backaction exerted by the cavity. The expressions for these different contributions are rather unwieldy for arbitrary mechanical dissipation \(\gamma_m\), but it is simple to realize that for appreciable cooling \(n_f \ll n(\omega_m)\) to be possible the conditions \(\gamma_m \ll \kappa, g_m, \omega_m\) are needed. Hence, in this regime of interest, a good approximation for \(n_f\) is obtained if one takes \(\gamma_m \to 0\) keeping \(\gamma_m n(\omega_m)\) finite. In this limit we obtain

\[
n_{f}^{(0)} = - \frac{1}{4 \omega_m \Delta_L} \left[ (\omega_m + \Delta_L)^2 + \frac{\kappa^2}{4} \right] + \frac{g_m R}{8 \omega_m} \left( \Delta_L^2 + \frac{\kappa^2}{4} \right),
\]

\[
n_{f}^{(m)} = - \frac{\gamma_m n(\omega_m) R}{\kappa \Delta_L g_m^2 \omega_m^2} \left[ \frac{\Delta_L g_m^2}{4} \left( \Delta_L^2 + \frac{\kappa^2}{4} - 4 \omega_m^2 \right) + \omega_m^2 \left( \Delta_L^2 + \frac{\kappa^2}{4} \right) \right] \left[ (\omega_m - \Delta_L)^2 + \frac{\kappa^2}{4} \right] \left[ (\omega_m + \Delta_L)^2 + \frac{\kappa^2}{4} \right] + \Delta_L g_m^2 \omega_m \left[ \omega_m^4 + 2 \Delta_L^2 \omega_m^2 - 3 \omega_m^2 \Delta_L^2 + \frac{\kappa^2}{4} + \frac{\kappa^4}{16} + \omega_m^2 \frac{\kappa^2}{4} \right],
\]

\[
n_{f}^{(p)} = - \frac{g_p R}{2 \Delta_L \omega_m} \left[ \Delta_L \left( \omega_m^3 g_m^2 \Delta_L^3 + \omega_m^3 \Delta_L + g_m^2 \Delta_L^2 / 2 + \omega_m^3 \right) + \omega_m^3 \left( 2 \omega_m \Delta_L^2 + \frac{g_m^2 \Delta_L^2}{2} + \frac{g_m^2 \Delta_L^2}{2} \right) \frac{\kappa^2}{4} + \omega_m^4 \frac{\kappa^4}{16} \right],
\]

with

\[
R = \frac{1}{\left[ \Delta_L \left( \omega_m \Delta_L + g_m^2 \right) + \omega_m \Delta_L^2 \right]^2}. \tag{58}
\]

It is straightforward to show that in the limits \(g_m \ll \kappa\) with \(\Delta_L < 0\), and \(\kappa \ll g_m \ll \omega_m\) with \(\Delta_L = -\omega_m\) we recover, respectively, the approximate expressions (35) and (53) (to lowest order in \(\gamma_m / \Gamma\)).
Figure 3. (a) Final (steady state) average phonon number minimized with respect to $g_m$ and $\Delta_L \left[ \min\{n_f\} \right]$ as a function of the thermal mechanical occupancy $n(\omega_m)$ (solid lines). The lower bound $n_{TL}$ given by equation (54) is shown for comparison (dashed lines). (b) Optimal normalized optomechanical coupling $g_m^{\text{opt}}/\omega_m$ for which the minimum mechanical occupancies shown in (a) are attained. In both (a) and (b), $Q_m = 5 \times 10^4$, the light blue lines correspond to $n_p = 0$ and $\kappa/\omega_m = 10$, the dark blue lines to $n_p = 0$ and $\kappa/\omega_m = 0.1$, and the red lines to $\kappa/\omega_m = 0.1$ and a finite $n_p$ determined by $\omega_p/\omega_m = 10^3$.

The analytical formulae given by equations (56) and (57) provide a quantitative basis for a precise analysis of the final temperatures that can be attained for arbitrary values of $g_m/\kappa$. In particular they can be readily minimized numerically with respect to the drive’s detuning and input power subject to the constraint imposed by the stability condition $R^{-1} > 0$ that emerges from the Routh–Hurwitz criteria [47]. This is the natural optimization problem that is posed by an experimental realization in which the natural linewidth of the cavity is fixed or hard to modify (admittedly sweeping this parameter within some range should be straightforward in electromechanical setups). We have performed this optimization using instead the exact expression that follows from equations (55), i.e. without the small $\gamma_m$ approximation, so that the transition to the trivial regime $\min\{n_f\} = n(\omega_m)$ is captured. Figure 3(a) compares this optimum (solid lines) as a function of the thermal equilibrium occupancy $n(\omega_m)$ with the lower bound $n_{TL}$ (dashed lines) furnished by equation (54) for representative values of the other parameters. The blue lines correspond to $n_p = 0$ (i.e. $\omega_p/\omega_m \to \infty$) for values of $\kappa/\omega_m$ in the Doppler regime (light blue) and in the RSB regime (dark blue). The red lines show the latter with $\omega_p/\omega_m = 10^3$ instead. There are three distinct regimes: (i) $n(\omega_m) < n_{TL}$ so that the optomechanical coupling only raises the temperature and $g_m^{\text{opt}} = 0$, (ii) the minimum occupancy is determined by the fundamental quantum backaction limit and is thus independent of the ambient temperature (i.e. constant as a function of $n(\omega_m)$), and (iii) the minimum occupancy is determined by classical noise and is thus linear in the ambient temperature. In this latter regime for finite $\omega_p/\omega_m$ there is a noticeable transition at which the minimum deviates from the $n_p = 0$ result as the cavity becomes thermally activated. One can also note that in this linear regime $n_{TL}$ is only reachable in the RSB limit. In turn figure 3(b) shows the corresponding optimal values for $g_m$. The saturation at high temperatures just reflects the fact that $(n_1^m + n_1^0)/n(\omega_m)$ becomes temperature independent in that limit and attains its minimum at a finite $g_m$ (cf (57)). Finally, figures 4(a) and (b) illustrate, respectively, the dependence of the optimum with the ratios $\kappa/\omega_m$

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and $\omega_p/\omega_m \gg 1$ as the ambient temperature is varied in the RSB regime. The fundamental quantum backaction limit results in a distinct shoulder. In figure 4(a) its lower edge traces a line with a slope consistent with the quadratic dependence on $\kappa/\omega_m$. It is clear from figure 4(b) that for ratios $\omega_p/\omega_m \lesssim 10^4$ and $n(\omega_m) \gtrsim 10^3$ (parameters that are relevant for electromechanical setups [25]) thermal noise in the cavity needs to be taken into account.

5. Conclusion

In summary, we have analyzed the cavity-assisted backaction cooling of a mechanical resonator in the quantum regime by deriving effective master equations for the relevant degrees of freedom both in the perturbative and strong coupling limits. These provide a description of the cooling dynamics that allows a simple lower bound to be established for the final occupancy (equation (54)) that can only be reached in the RSB regime. This bound implies that the ground state cooling is only possible when the cavity linewidth is much larger than the heating rate induced by the mechanical dissipation but much smaller than the mechanical oscillation frequency, and when the equilibrium thermal occupancy of the cavity is well below unity. In addition, we give an analytical expression for the final occupancy valid in all regimes of interest that allows for a straightforward optimization of the parameters of the drive. Finally, we analyze the dependence of this optimum on the ambient temperature, the cavity linewidth and the ratio of the cavity’s frequency to the mechanical frequency.

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