Report on Progress

Bootstrap and amplitudes: a hike in the landscape of quantum field theory

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Abstract
This article is an introduction to two research programs that are currently very active, the conformal bootstrap and scattering amplitudes. Rather than attempting full surveys, the emphasis is on common ideas and methods shared by these two seemingly very different programs. In both fields, mathematical and physical constraints are placed directly on the physical observables in order to explore the landscape of possible consistent quantum field theories (QFTs). We give explicit examples from both programs: the reader can expect to encounter boiling water, ferromagnets, pion scattering, and emergent symmetries on this journey into the landscape of local relativistic QFTs. The first part is written for a general physics audience. The second part includes further details, including a new on-shell bottom-up reconstruction of the $\mathbb{CP}^1$ model with the Fubini–Study metric arising from resummation of the $n$-point interaction terms derived from amplitudes.

The presentation is an extended version of a colloquium given at the Aspen Center for Physics in August 2019.

Keywords: amplitudes, conformal bootstrap, quantum field theory, conformal field theory, critical phenomena

((Some figures may appear in colour only in the online journal)

1. Introduction

What do boiling water and ferromagnetics have in common? At first sight, not much. However, near the critical point in their phase diagrams, water and ferromagnets exhibit a similar behavior: various physical quantities scale in the same way, with exactly the same critical exponents. This is an example of universality. Despite their widely different microscopics, systems in the same universality class can be described at the critical point by the same scale-invariant theory.

Another very different class of physics is explored by scattering experiments, such as at the Large Hadron Collider (LHC) at CERN. In order to compare with experimental data, particle theorists compute scattering amplitudes that encode the probability that a given initial state interacts and scatters to a particular final state. A simple example is the scattering of pions, $\pi$, the lightest hadron associated with the strong nuclear force. For the process $\pi\pi \rightarrow \pi\pi$, the amplitude $A_4(\pi\pi \rightarrow \pi\pi)$ encodes the probability of the process as a function of the center of mass energy and the scattering angle. Specifically, the (differential) scattering cross section is proportional to a phase-space integral over the norm-squared of the amplitude $|A_4|^2$. The theory that describes interacting massless pions is absolutely not scale-invariant and it is therefore completely differ-
ent from the type of theory that describes the critical point of boiling water and ferromagnets.

Despite their obvious differences, boiling water, ferromagnets, and pion scattering are part of a vastly broader class of physical systems that are explored using a set of powerful methods in modern theoretical physics. The basic idea is to ‘bootstrap’ the physical observables directly from physical and mathematical consistency constraints rather than calculating them from detailed microscopic descriptions. One then uses the observables—subject to desired properties and symmetries—to learn about the landscape of possible theoretical models that can give rise to such observables. A specific goal is to understand the structure of quantum field theories (QFTs).

QFT is a mathematical framework for theoretical physics. It has a plethora of applications and direct experimental relevance. QFT is relevant for particle physics, condensed matter systems, string theory, gravitational waves, and beyond. There is not one QFT but many. Some QFTs describe particles that are weakly interacting and one can use Lagrangian techniques to study them. Other QFTs are always strongly coupled; in those cases words such as ‘particles’ and their ‘interactions’ are not useful and they may have no Lagrangian descriptions. Some QFTs describe physics that depends heavily on the energy scale (or length scale) while other QFTs do not care a whit about scale.

The subject of QFT is incredibly rich. QFTs describe the critical points of water and ferromagnets as well as the scattering of pions. The set of consistent QFTs can be thought of as a landscape: an abstract landscape that is so vast and complex and interesting that theorists constantly venture into its unknowns to explore and discover new features, new connections, and new properties.

It can be hazardous to venture out on a hike into unknown terrain, so we consult maps in order to know the local topographical features of the landscape, such as the beautiful Rocky Mountains. The peaks of the mountains, the valleys, and the saddlepoints are the most prominent features and they guide our choice of path. Likewise, we wish to map out the landscape of QFTs. There are special places in the QFT landscape that can help us understand it better and navigate it. It is very useful to determine these special QFTs and understand their properties. Examples of such special points in the QFT landscape are known as conformal field theories (CFTs): the CFTs are the metaphorical peaks, ridges, and valleys of the QFT landscape.

Two modern approaches to explore the landscape of QFTs are:

- the conformal bootstrap program focused on the CFTs, and
- the scattering amplitudes program.

The goal of this article is to give colloquium-level introductions to these two highly-active research areas and describe how they share a common approach to physics that leads to powerful and novel results. It is my hope that this will be useful for researchers in other fields of physics and math, as well as for students. For those with more background in QFT, I have included two sections with technical details beyond the colloquium-level because I wanted to illustrate the ideas concretely and explicitly.

Overview. The presentation has two main parts: the first part—sections 2–5—is intended for a general physics audience with no prior knowledge of the subjects. The second part—sections 6 and 7—provides technical details that put more equations behind the words in the first part.

We begin with the description of the critical points of water and ferromagnets in section 2; we discuss critical exponents and scale invariance. It has been proposed that a 3D CFT describes the physics at the critical point. Before exploring that 3D theory further, we illustrate in section 3 the richness of the landscape of 4D relativistic QFTs by describing a few examples of QFTs and we then introduce CFTs. In section 4, we introduce the modern amplitudes program with focus on the ideas of using scattering amplitudes to explore the landscape of QFTs. Section 5 offers an introduction to the conformal bootstrap program with particular emphasis on what it teaches us about the critical points of boiling water and ferromagnets.

The presentations in sections 4 and 5 attempt to avoid the full technical detail, but for those who want to see more, please see sections 6 and 7. In particular, section 6 provides the full details of how to bootstrap a scalar model from very simple assumptions about the behavior of the scattering processes and we shall see how a global symmetry emerges from the construction. We show how the Lagrangian interaction terms can be reconstructed from the bootstrapped amplitudes and that they can be re-summed to the Fubini–Study metric. Section 7 presents technical details about the conformal bootstrap setup and as an example it is shown how the crossing relations require an interacting 4D CFT to have an infinite number of primary operators. We conclude in section 8 with very brief closing remarks.

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1 This article is based on a colloquium given August 8, 2019, at the Aspen Center for Physics, Aspen, Colorado, USA.
2. Water and magnets

Consider the phase diagram of water in figure 1. Under normal conditions of pressure at about 1 atm, water freezes at 0°C and boils at 100°C, so as the temperature is varied at constant pressure, water exhibits three phases: solid, liquid, and gas. As is well-known by people in mountainous regions and students in thermodynamics classes, the boiling point is lower at higher altitude. In Aspen, at about 8000 ft = 2440 m, the air pressure drops to around 0.75 atm and the boiling point of water is 92°C. So it takes a little longer to boil your pasta al dente.

The familiar solid–liquid and liquid–gas phase transitions of water involve latent heat and are called first order phase transitions. As pressure increases, the boiling point of water goes up and at high enough pressure, the phases of liquid and gas are no longer distinguishable.

The familiar solid–liquid and liquid–gas phase transitions of water involve latent heat and are called first order phase transitions. As pressure increases, the boiling point of water goes up and at high enough pressure, the phases of liquid and gas are no longer distinguishable. The liquid–gas transition curve in the phase diagram ends at a point called the critical point with $p_c \sim 217$ atm and $T_c \sim 374°C$. As the critical point is approached, the latent heat needed to transition between liquid and gas goes to zero and at the critical point the phase transition becomes continuous (also called second order).

Ok, so what? Well, near the critical point, something special happens: the liquid becomes milky in appearance as the fluctuations in the system become of the same order as the scale of fluctuations is governed by the critical point. The scale of fluctuations is captured by the correlation length $\xi$, which encodes how strongly coupled disparate parts of the system are to one another. As $T \to T_c$, the correlation length diverges:

$$\xi \to \infty.$$  \hspace{1cm} (1)

At large spatial separation $r$, the correlations in the system are proportional to $e^{-r/\xi}$, so an infinite correlation length means that every part of the system couples with equal strength to every other part. Not just nearest-neighbor friendliness here, everybody is coupled to everybody else. Moreover, when $\xi \to \infty$ there is no distinguishing scale in the system: it has become scale invariant. Physically, the phenomenon of scale invariance is observed as critical opalescence.

The dependence of physics with scale is captured by the renormalization group (RG), a framework developed by Kenneth Wilson (Nobel Prize 1982). Wilson’s seminal work from 1971 [1, 2] built upon the advances of Michael Fisher [3, 4], Benjamin Widom [5, 6], Leo Kadanoff [7] and others in the 1960s on scaling behavior, critical exponents, and block spin methods. Most relevant for us here is the scaling behavior of the correlation length $\xi$: as $T$ approaches $T_c$, the correlation length $\xi$ diverges as

$$\xi \sim |(T - T_c)/T_c|^{-\nu},$$ \hspace{1cm} (2)

where the number $\nu$ in equation (2) is an example of a critical exponent. Critical exponents characterize the approach to the critical point and play a key role in understanding critical systems. For the critical point of a liquid–vapor transition, the value of $\nu$ is

$$\nu \approx 0.63.$$ \hspace{1cm} (3)

Now let us momentarily switch gears and discuss a physically very different system, namely ferromagnets. They exhibit critical behavior at the Curie temperature $T_c$, which separates the ordered ferromagnetic phase at $T < T_c$ from the disordered nonmagnetic phase at $T > T_c$. As an example, the Curie temperature of iron is $T_c = 1043 K = 770°C$; for reference, iron melts at $1811 K = 1538°C$. At the critical point set by the Curie temperature, the correlation length between dipoles in the ferromagnet diverges just as in equation (2). And here comes the best part: the critical exponent $\nu$ for ferromagnets takes the same value as for water $\nu \approx 0.63$. This is quite amazing: these are different physical phenomena whose microscopics are totally unrelated. Nonetheless, at their critical points, these two systems—water and ferromagnets—behave alike. This is an example of universality.

There is a theoretical model, the 3D Ising model, that describes both water and ferromagnets near their critical points. For the ferromagnet, this is a model in which each site of a 3D lattice can either be spin up or down. For water, replace spin up/down with occupation number: site has a molecule or not. Block spin techniques [7] allow one to study such a lattice model at greater and greater length scales, which translates to lower energy scales. In the deep infrared, one approaches a fixed point of scale invariance (called the critical 3D Ising model) and finds that the correlation length diverges with a critical exponent $\nu \approx 0.63$.

It appears that the value of $\nu \approx 0.63$ has not been measured directly experimentally for the critical point of regular water $H_2O$, but it can be inferred from other measurements of critical exponents. There are, however, multiple other direct measurements of $\nu$ in other systems: small angle neutron scattering in heavy water $D_2O$ [8], light-scattering experiments in an electrolyte solution [9] as well as other systems in the same 3D Ising universality class, see for example table 7 in [10].

Now you may have started reading this article in the hope of learning about the landscape of QFTs and instead you have gotten an earful about the phase diagrams of water and ferromagnets. Fear not, there is a purpose behind the madness. Polyakov [11] conjectured that at critical points, the symmetries of the system are enhanced to conformal symmetry: they can be described by CFTs. We have started out with the critical continuous phase transitions in water and ferromagnets because these are real-world examples of the power of QFT and a case where the conformal bootstrap techniques are directly applicable. The point here is that by studying the 3D Ising model (henceforth referring to it with implicit understanding that it is at the fixed point) as a CFT using the techniques we describe in section 5, one can extract, to an incredible precision, information about the critical exponents of the system, such as $\nu$. 

\hspace{1cm}
3. Field theories: a brief survey

In this article we study relativistic QFT\(^2\). This means that we consider QFTs that are invariant under Poincaré symmetry: spatial rotations, Lorentz boosts, and spacetime translations. Moreover, we assume that the theories we study are local and unitary. Loosely speaking locality means that there is no action at a distance. Technically, this means that local fluctuations can be described in terms of local operators that depend on a single point of spacetime. Locality manifests itself on the physical observables in terms of what kind of singularity structures they are allowed to have. We discuss this further in section 4.

As stated in the introduction, there is not just one QFT but a vast and rich landscape of QFTs. To illustrate how different QFTs can be—and to set the stage for the later discussion—I will now briefly discuss some examples of relativistic QFTs.

3.1. Examples of relativistic QFTs

Here follows some key examples of 4D QFTs:

- **Quantum electrodynamics (QED)** describes the interaction of photons and electrons/positrons. The classical equations of motion are Maxwell’s equations for the photons and the Dirac equation for the electrons/positrons. At the quantum level, the strength of the coupling of photons to electrons/positrons depends on the energy scale. At atomic-level energies the effective coupling, the fine structure constant \(\alpha\), is \(\alpha \approx 1/137\). However, at energies around the mass of the weak force mediators \(W^\pm\) and \(Z\), which is about 90 times the proton mass, the strength of the coupling increases to around \(\alpha \approx 1/127\). Thus the coupling runs with scale: it approaches zero at very low energies where the theory becomes trivial, but at high energies it becomes stronger.

- **Quantum chromodynamics (QCD)** describes the strong nuclear force. More precisely, QCD is the theory that describes gluons, \(N_f\) flavors of quarks (with \(N_f = 6\) in Nature for the \(d, u, s, c, b, t\) quarks), and their interactions. The dynamics of gluons is captured by Yang–Mills theory. The coupling strength \(\alpha_s\) of QCD also depends on the energy scale. Famously, QCD in our world behaves oppositely to QED in that it becomes free (coupling goes slowly to zero) at high energies while it is strongly coupled and confining at low energies.

- **The standard model of particle physics** combines QED with three generations of leptons (electrons, muons, taus, and their antiparticles, as well as the neutrinos), QCD with six flavors of quarks, the electroweak force, and the Higgs mechanism to give an incredibly successful QFT in which one can calculate physical observables to high precision and compare with experimental data. An example is the multi-digit precision agreement between the theoretical predictions and experimental measurements of both the electron and muon Landé g-factor. The Higgs mechanism and spontaneously broken symmetry are key ingredients in the standard model and the discovery of the Higgs boson announced in 2012 was a tremendous success of both long-term experimental perseverance and the power of theoretical studies of QFTs.

In the standard model, the electromagnetic force and the weak nuclear force unify at the electroweak unification scale of 246 GeV. Furthermore, the running of the couplings are important for the grand unification models that seek to unify the electroweak force with the strong force at an even higher scale.

- **Gravity** is not included in the standard model. However, it is successfully described by its own field theory, namely general relativity. As a field theory, general relativity differs from those discussed above in that the gravitational coupling \(k = G^{1/2}\) is dimensionful. Specifically, Newton’s constant \(G\) has the dimension of (mass)\(^{-2}\). This is in contrast with the dimensionless fine structure constant \(\alpha\) of QED. As a consequence, the effective dimensionless coupling in gravity is \(E \kappa = E \sqrt{G}\), where \(E\) is the energy scale in the particular problem. Thus gravity is a weak force (i.e. perturbative) only at energy scales much smaller than \(\kappa^{-1} = G^{-1/2} \sim M_{\text{Planck}} \sim 10^{19}\) GeV. In a sense, this is like QED in that gravity is weak at low energies and strong at high energies. But gravity is actually much more complicated than QED. QED is a renormalizable theory meaning that it is a sensible predictive theory at the quantum level. Gravity, on the contrary, is nonrenormalizable and therefore, broadly speaking, it is nonpredictive at the quantum level at energies approaching the Planck scale.

It is useful to think of standard general relativity as a low-energy effective field theory (EFT): it gives a description of gravity that makes sense only at sufficiently low energies \(E \ll M_{\text{Planck}}\). Within this regime of validity it is, as we know well, a highly successful theory of gravitational phenomena. At energies above the Planck scale, it needs a new framework (sometimes called a UV completion) in order to make sense. String theory is a theoretical framework that, among many other properties, is the most promising candidate for a theory of quantum gravity.

- **EFTs** describe physical phenomena in an expansion in some small parameter(s). In the context here, we focus on low-energy EFTs. The idea is to work in a low-energy regime, where powers of energy–momentum are suppressed by a particular ‘cut-off’ scale \(\Lambda_{\text{UV}}\) above which the expansion in small \(E/\Lambda_{\text{UV}}\) is no longer valid. In terms of a Lagrangian, derivatives are (via Fourier transform) directly counting powers of energy–momentum, so in an EFT one includes higher-derivative terms with increasing suppression by powers of \(1/\Lambda_{\text{UV}}\). This means that the \(1/\Lambda_{\text{UV}}\)-expansion becomes a derivative expansion. The principle of EFTs is to include all Lagrangian terms that are allowed by symmetries of the system up to a

\(^2\) Non-relativistic QFT is an important subject too that is highly relevant in condensed matter contexts. The examples of water and ferromagnetics are non-relativistic, but at the critical points the symmetries are expected to be enhanced, as discussed at the end of section 2. Note that the analytic continuation from Lorentzian to Euclidean metric signature makes it possible to interpret a relativistic field theory as a statistical mechanical system.
given order in $1/\Lambda_{\text{UV}}$. The arbitrary coefficients in this expansion parameterize the (potentially unknown) UV physics.

As an example, general relativity (including the possibility of a cosmological constant) is the leading two-derivative interactions of a gravitational EFT in which higher-derivative corrections are suppressed by inverse powers of the Planck mass. The historical successes of general relativity—as well as the frequent use you make of it via the GPS built into your smart phone— tells you that EFTs are incredibly useful.

Physics beyond the standard model, such as proton decay or dark matter, is often encoded in terms of EFTs. The scale of the EFT is then associated with the scale of the new physics at higher energies.

- **Nonlinear sigma models (NLSM)** describe scalar fields that take values in some manifold, called a target space, and the model inherits the symmetries from this target space. An important class of NLSM are the EFTs that govern the low-energy dynamics of massless Goldstone bosons arising from spontaneously broken global symmetries. In these cases, the scale built into the EFT is associated with the symmetry-breaking scale.

As an example, QCD has an approximate chiral symmetry that is spontaneously broken at a scale of about 1 GeV (i.e. about the mass of the proton). The symmetry breaking gives rise to Goldstone bosons that are identified as the pions. While Goldstone bosons are exactly massless, pions in nature are not; they have mass (135–140 MeV) that is low compared to the mass. The historical successes of general relativity—as well as the frequent use you make of it via the GPS built into your smart phone—tells you that EFTs are incredibly useful.

### 3.2. Conformal field theories

CFTs are characterized by having, in addition to Poincaré symmetry, conformal boost symmetry. One way to describe it is that special conformal boosts are transformations that preserve angles. Another way is to consider the inversion transformation that sends a spacetime coordinate $x^\mu$ to $x^\mu/x^2$, where $x^2 = -t^2 + |\vec{x}|^2$ is the relativistic invariant spacetime distance. (We use conventions of $c = 1$ throughout.) Then a special conformal boost is what you get when you do inversion, followed by a translation, followed by another inversion.

Sounds a little complicated? Ok, let us try to get a little intuition. Inversion sends $x^2$ to $1/x^2$. Ignoring the time component of $x^\mu$ we then see that inversion trades long distance with short distance, and vice versa. In a relativistic theory, this then interchanges low energy and high energy. For a theory to have such a property\(^3\), it cannot have any preferred scales because inversion symmetry requires that the physics above and below a given scale has to be the same. For instance, if a particle has mass $m$, then for energies $E \geq 2mc^2$ such particles can be pair-produced, but for $E < 2mc^2$ they cannot. Thus physics is different above and below $2mc^2$ and hence masses cannot be allowed with inversion symmetry. Similarly no other dimensionful parameters are allowed. Hence, a relativistic theory with inversion symmetry is necessarily also scale invariant. This is the aspect of CFTs that is most relevant for us in this presentation: CFTs are scale invariant\(^4\).

Scale invariance is very different from our everyday experience. It is not the same to take an ice bath as it is to put your finger in boiling water (and neither is pleasant for very long). People age differently over the time scale of a year than they do over ten years. So it may seem like conformal symmetry, or even just scale invariance, is a property very different from what we encounter in our everyday world. That is true, nonetheless, scale invariance does happen in nature—and it can be found in the lab.

We already encountered an example of scale invariance in section 2. Recall that near the critical point of water (or the Curie point for the ferromagnet), the correlation length $\xi$ diverges so that all length scales become of equal relevance and the system becomes scale invariant at the critical point. There is no proof that it also becomes conformal, however, in section 5 we describe how modeling the critical point using CFT techniques (specifically here for the 3D Ising model) makes it possible to determine critical exponents such as $\nu$ in (2).

Other examples of CFTs arise in four-dimensional (4D) supersymmetric gauge theories. In section 3.1, we noted that gluons, the spin-1 massless particles of the strong nuclear force, are described by Yang–Mills theory. Yang–Mills theory is not a conformal theory, but the gluons can be coupled to other particles in such a way that the resulting theory is conformal. A useful tool in this context is supersymmetry, a symmetry that partners bosons and fermions into supermultiplets in which all particles must have the same mass and their interactions are quite restricted. The massless fermion superpartners of gluons are called gluinos and the QFTs describing them are called super Yang–Mills (SYM) theories. In models without gravity, the gluons can have either $N = 0, 1, 2$ or 4

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\(^3\) CFTs need not have inversion symmetry, for example chiral CFTs do not, but for the purpose of this introduction let us just consider CFTs that are invariant under inversion.

\(^4\) The reverse may in general not be true: scale invariance does not in general imply conformal invariance.
The massless $N = 4$ SYM theory in 4D is very special: its Lagrangian is completely fixed by supersymmetry and the couplings do not run at all with scale. Not only is $\mathcal{N} = 4$ SYM theory scale-invariant, it is in fact also conformally invariant, even at the quantum level. It is a theory that in many respects is considered ‘the simplest’ QFT due to its strong constraints from symmetries that allows calculational control that one often lacks in other theories. $\mathcal{N} = 4$ SYM theory appears in many areas of theoretical high energy physics, often as a ‘testing lab’ for developing new techniques and gaining insights into similar systems, for example QCD.

Coupling $\mathcal{N} = 2$ SYM supersymmetrically to matter multiplets can give rise to superconformal (supersymmetric and conformal) theories (SCFT). The subject of $\mathcal{N} = 2$ SCFTs is in itself very rich, with some areas of theoretical high energy physics, often as a ‘testing lab’ for developing new techniques and gaining insights into similar systems, for example QCD.

There is a multitude of other known CFTs and SCFTs. Some arise in condensed matter systems, others in string theory. Recall that (S)CFTs are the light beacons in the much vaster landscape of QFTs, so it is noteworthy that they themselves make up a rich landscape that is even far from fully explored.

### 3.3. QFT observables

The key observables in QFTs are correlation functions. Correlation functions are familiar from cosmology where they measure the correlations in an observable like the temperature between different locations in the map of the cosmic microwave background. They are familiar from condensed matter physics where we may be interested in whether interactions are short distance or long distance; in fact the correlation length $\xi$ we discussed in section 2 is the characteristic length associated with a two-point (connected) correlation function

$$
\langle \sigma(x) \sigma(y) \rangle \sim e^{-|x-y|/\xi}
$$

for $|x - y| \gg \xi$. It measures the correlations between interactions at different locations $x$ and $y$. In a lattice model, the field $\sigma(x)$ can be thought of as designating whether the site at position $x$ has spin up or down (say $\sigma(x) = \pm 1$). In general, the $n$-point correlation function measures the correlation between quantities at $n$ different spacetime locations.

One approach to correlation functions is to reduce them from their general form to an ‘on-shell’ form in momentum space; this gives the on-shell scattering amplitudes from which one can compute the observable scattering cross-sections. The amplitudes are the observables of interest in the scattering amplitudes program. Another approach to study correlation functions is to impose on them symmetries and mathematical consistency conditions: that is what is done in the conformal bootstrap program in the context of CFTs.

### 4. Introduction to the modern scattering amplitudes program

A scattering experiment consists of banging things together to see what comes out. For example, at the LHC protons are collided against protons at about 10 000 times their rest mass. At the microscopic level, the partons (gluons and quarks) inside the protons interact with each other in the collisions. A process representative for the physics we discuss here is the process of two gluons interacting to produce a new set of gluons, e.g.

$$
g + g \to g + g + g + g + g.
$$

The gluons are described by Yang–Mills theory. At sufficiently high energies (such as at the LHC) it is weakly coupled and it therefore makes sense to study the scattering of gluons and quarks perturbatively. Eventually the partons hadronize and form jets of mesons (like the pions) and baryons (like the proton). Here we focus on the high-energy part of the process that just involves gluons/quarks scattering inelastically to gluons/quarks.

The probability of a scattering process occurring is encoded in the scattering amplitude $A_n(i \to f)$, where $n$ is the total number of initial and final state particles in the process $i \to f$. It is related to the experimentally measurable observable, the differential scattering cross section, as

$$
\frac{d\sigma}{d\Omega} \sim \int |A_n(i \to f)|^2.
$$

The integral here is taken suitably over phase space. For a given initial state $i$, $d\sigma/d\Omega$ gives the probability of measuring the final state $f$ as a function of scattering angles and energies.

Amplitudes are traditionally calculated as the sum of Feynman diagrams constructed from the vertices and propagators of a theory described by some Lagrangian. The perturbative expansion in small couplings is organized diagrammatically in a loop-expansion where the leading order is tree-level, the first correction is the sum of one-loop diagrams, the next correction consists of the two-loop diagrams, etc. A general rule of thumb: the more loops and the more particles, the harder it is to calculate the amplitudes. For more particles, this is because the number of diagrams tends to grow combinatorially. For higher loops, the number of diagrams is a significant issue too, but not the only one; the evaluation of highly nontrivial integrals over energy–momentum running in the closed loops can be a challenging roadblock as well.

Amplitudes depend on ‘external data’: the momenta $p_i$ for each of the external particles and polarization vectors for spin 1 particles and spinor wavefunctions for fermions. The external momenta $p_i$ are subject to the requirements of being on-shell and satisfying momentum conservation, i.e. ($c = 1$)

$$
p_i^2 \equiv -E_i^2 + |\vec{p}_i|^2 = -m_i^2 \quad \text{and} \quad \sum_{\text{incoming}} p_i^\nu = \sum_{\text{outgoing}} p_i^\nu.
$$

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5 Why not $\mathcal{N} = 3$, you ask? Sure, $\mathcal{N} = 3$ is fine too. For a Lagrangian theory, CPT invariance (charge conjugation, parity, time-reversal) implies that $\mathcal{N} = 3$ SYM theory is equivalent to $\mathcal{N} = 4$ SYM.
Here $m_i$ denotes the mass of the $i$th particle; for gluons $m_i = 0$. Amplitudes with (7) satisfied and appropriate polarizations or spinor wavefunctions included for all external particles are called on-shell amplitudes.

To calculate the process (5) at the leading order in perturbation theory, one needs to add up all the tree Feynman diagrams with six external gluons built from the Feynman rules extracted from the Yang–Mills Lagrangian. There are cubic and quartic gluon self-interactions so one gets

$$A_6 = \text{[Feynman diagrams]} + 219 \text{ more diagrams.} \tag{8}$$

For scattering seven gluons one needs 2485 diagrams, and for eight gluons it would be 34 300 diagrams. Each diagram translates into a rather complicated mathematical expression via the Feynman rules. Calculating these amplitudes by hand using Feynman diagrams is not a great way to spend your time. And this is still only the leading order in perturbation theory, imagine the complications at loop-level!

One of the goals of the modern on-shell amplitudes program is indeed to come up with better and more efficient ways to calculate scattering amplitudes, but there are also several other avenues of progress.

4.1. Modern amplitudes program

Early modern approaches to scattering amplitudes were pioneered by Bern, Dixon, and Kosower, often driven by applications in particle phenomenology. This thrust continues as the field has broadened significantly in the past ~17 years and has attracted many more people to the field. We outline five directions in modern research on scattering amplitudes:

(a) New computational techniques. One goal of the amplitudes program is to develop new calculational techniques to facilitate more efficient computation of scattering amplitudes—and to perform calculations of amplitudes that may even be impossible using traditional Feynman diagrammaticks. At tree-level, examples of such new techniques are on-shell recursion relations (they go under names such as BCFW [12, 13], CSW expansion [14], all-line shifts [15, 16], soft shift recursion [17, 18], etc). Some on-shell recursions are quite general and others are more closely adapted to the field theory they are applied to. The general idea is to recycle lower-point amplitudes into higher-point ones. For example, in many theories, such as Yang–Mills theory and gravity, the three-particle scattering processes determines the four-particle scattering processes. Then three- and four-particle amplitudes can be recycled into five-particle scattering, etc. Sometimes the recursion relations can be solved exactly; there are for instance closed-form expressions for scattering of any number of gluons at tree-level [19].

The derivation of on-shell recursion relations exploits knowledge of the analytic structure of amplitudes. Tree amplitudes are rational functions of the external data and they have simple poles where physical particles can be exchanged. On these poles, unitarity guarantees that the amplitudes factorize into products of lower-point amplitudes. The information of the location of the poles in momentum space and the factorized form of their residues are the basis of the on-shell recursion relations.

Loop-amplitudes have a more complicated analytical structure. While they can have rational terms, they generally also involve more complicated analytical functions such as logarithms, polylogarithms, and worse. One powerful technique is the method of generalized unitarity (pioneered in [20] and since applied in countless contexts; see the review [21] and references therein). Here one uses the fact that the integrand of a loop amplitude is a rational function that develops poles for specific choices of the loop-momenta. On these poles, the integrand factorizes into amplitudes with fewer loops. At one-loop level, such ‘cuts’ (and their $d$-dimensional generalizations) can be used to determine the integrand from tree-level amplitudes. At two-loop level, the one-loop and tree amplitudes are recycled into determining the cuts, etc.

The fact that the loop-integrand is rational can be used to construct on-shell recursion relations at loop-level in certain models, such as for the planar limit of $\mathcal{N} = 4$ SYM [22].

(b) Mathematical structure and geometry. We have already alluded to how the mathematical structure of amplitudes inform the development of calculational tools. But on-shell amplitudes themselves also harbor hidden structures that cannot be inferred from the Lagrangian. For example, many amplitudes have analytic expressions that are much simpler than the Feynman diagram representations would suggest. What is responsible for such simplifications? Elucidating the mathematical structure of amplitudes, uncovering hidden symmetries, and reformulating the scattering problem in novel mathematical terms are other goals of the amplitudes program.

Recent ideas include representations of amplitudes in terms of contour integrals in Grassmannian spaces (which are spaces of $k$-planes in $n$-dimensional space) [23] and geometrizations such as polytopes [24, 25], amplituhedrons [26], associahedrons and more generally positive geometry [27]. The idea is that the amplitude is related to a volume form for a geometric object in some abstract mathematical space. The boundaries of the geometric object correspond to the location of poles. Different triangulations of the volume of this object can be mapped to different equivalent mathematical formulas for the amplitudes, one is the Feynman diagram representation, some correspond to the results of on-shell recursion relations, and others again are inherently different. This is explored at both tree- and loop-level.

Through the connection to interesting mathematical structures, there are now fruitful collaborations between the amplitude community and mathematicians on subjects such as positive geometry and cluster algebras.

(c) Exploring the space of QFTs: amplitude bootstrap. Traditionally, one starts with a Lagrangian, writes down
the Feynman rules, and uses them to compute the amplitudes. Any symmetries of the Lagrangian manifest themselves on the amplitudes as ‘Ward identities’. Here is an example: if the Lagrangian has a symmetry that gives charge conservation, the associated Ward identity says that the amplitude of any process that violates charge conservation has to vanish.

A new approach to QFTs is to turn this logic on its head and instead of starting with the Lagrangian, one takes the physical observables, the amplitudes in particular, as the starting point, impose constraints on particle spectrum and symmetries on the amplitudes and subject them to tests of mathematical consistency. This then allows one to explore the existence of QFTs with the assumed properties. In particular, it gives a systematic way to explore the landscape of possible theories with a set of specified symmetries. We describe this further in section 4.2 and in more detail in the technical section 6.

(d) Double-copy. In the mid-80s it was realized that tree-level closed string amplitudes can be written as sums of products of tree-level open-string amplitudes [28]—these are known as the Kawai–Lewellen–Tye relations. In the limit of infinite string tension, this becomes the field theory statement that the graviton tree amplitudes can be obtained as a sum of products of gluon scattering amplitudes. This relation is sometimes written

\[ \text{gravity} = (\text{gauge theory})^2 \tag{9} \]

and is by now referred to as an example of the double copy.

Starting in 2008, it became clear that there is more to this story. Bern et al [29] found that tree-level gauge theory amplitudes of gluon scattering could be written in a form where certain kinematic numerators obey the same Jacobi identities as the algebraic color-factors of the non-abelian gauge group of the theory. This is called color-kinematics duality. Moreover, if one replaces the color factors in this representation of the amplitude with the kinematic factors of gauge theory, remarkably the result is the gravity tree amplitude! This is the BCJ double copy and it has since been generalized to amplitudes of other field theories (e.g. [30]). BCJ conjectured (and it has been tested in multiple contexts) that a similar color-kinematics prescription and double-copy also holds at the level of the loop-integrand [31].

Since tree-level scattering represents the classical physics, it is natural to explore if there is a similar way to double copy solutions to the classical equations of motion. For example, the double-copy of a Coulomb-type solution in gauge theory to a black hole solution in general relativity as a weak-field expansion. This direction has thus attracted attention of researchers from other fields, such as theorists studying classical solutions in general relativity and supergravity and cosmology. For a recent review of the double-copy and its applications, see [32].

(e) Gravitational wave physics. With the recent detection of gravitational waves from black hole inspirals made by the LIGO detector, and the awarding of the 2017 Nobel Prize to the pioneers of experimental gravitational wave physics, the field of gravitational waves has received intense interest. Remarkably, amplitude techniques prove very useful here too. Starting with the Einstein equation, it is not hard to derive the linearized solution for a freely propagating gravitational wave. It is, however, a highly non-trivial matter to model the gravitational wave radiation resulting from the inspiral of two heavy objects such as black holes or neutron stars. Numerical breakthroughs have been an essential part of the study. There are also many other techniques, such as EFT formulations [33].

On the analytic side, one uses a post-Minkowskian or post-Newtonian expansion for a Hamiltonian with an effective potential; the corrections here are in terms of powers of Newton’s constant and orbital speed \( v/c \), respectively. The inspiral problem is an elliptical solution to this Hamiltonian: they are the bound states. At first sight this has absolutely nothing to do with scattering amplitudes. However, there are also hyperbolic solutions: a classical example of the hyperbolic problem is the famous deflection of light by a heavy object.

The deflection (i.e. hyperbolic) problem is basically a scattering process: two massive objects are in the initial state, they interact and then fly apart again after some exchange of energy–momentum. We can compute the scattering of massive particles under exchange of gravitons. The higher order graphs in such a calculation are in direct correspondence with the higher-order corrections in the effective Hamiltonian, so the effective Hamiltonian can be reconstructed from the scattering amplitudes and then used to study the bound state problem. If this were done with Feynman rules there would be limited calculational advantage. However, with the modern on-shell amplitude machinery, very promising progress has been made. At this stage, an example in this direction includes the calculation of the Hamiltonian for massive spinless binary systems to 3rd post-Minkowskian order (meaning order \( G^3 \) in the Newton coupling) [34–36]. There are also approaches [37, 38] that try to circumvent the construction of the effective potential and directly get gravitational wave information from the scattering amplitudes as well as related EFT approaches [39]. This is a rapidly developing field that has facilitated fruitful interactions between the community of general relativity theorists and the amplitude community.

There are many other very interesting developments in the field of scattering amplitudes. One is the system of scattering equations that has led to the so-called CHY construction of amplitudes that comes with its own formulation of the double-copy and has led to the realization of new examples of its application [30, 40, 41]. Another direction has been the connection between the universal soft behavior of gravitons, and the infinite-dimensional BMS symmetry in general relativity [42]. There are other types of amplitude bootstraps too, such as the integrability approach [43], the loop-amplitude bootstrap using cluster algebraic structures [44], and the S-matrix...
bootstrap [45]. The latter is an example of the fruitful overlap between the conformal bootstrap and amplitudes communities. Furthermore, phenomenologists are increasingly using amplitude techniques to study physics beyond the standard model, for example to organize higher-derivative operators in standard model effective field theory (SMEFT) [46, 47] (for EFT and SMEFT reviews, see [48, 49]). Finally, using basic properties of amplitudes, one can prove a number of interesting general theorems about QFTs such as [50–54]:

- There can be no theories in flat space with massless particles of spin greater than 2 interacting with gravity.
- There can only be one graviton field (i.e. only one massless spin-2 particle), it must self-interact and it must couple exactly the same way to any other particle (the equivalence principle).
- A spin 3/2 particle must couple supersymmetrically to the spin-2 graviton.
- Spin-1 massless fields can only self-interact if there is a Lie algebra structure with three-index antisymmetric structure constants, and
- A, \( N = 8 \) superconformal 3D theory requires the existence of fully four-index antisymmetric structure constants [55].

As this hopefully illustrates, the field of amplitudes concerns a diverse range of subjects. At this point, the annual amplitudes conference, now in its 12th year, attracts 150–200 international participants (and even more in its recent online Zoomplitudes version). We have highlighted here some general directions and current areas of interests, but of course this is in no way complete. The interested reader may want to consult the several newer reviews and textbooks on modern methods in amplitudes, see for example [53, 54, 56–58].

I am now switching gears to discuss in a little more detail one direction that shares ideology with the conformal bootstrap program. In the following, I outline the ideas, then in section 6 I provide a very detailed example of its implementation and results.

4.2. Amplitude bootstrap on the space of QFTs

Suppose someone asks:

‘Does there exist a local relativistic QFT with two massless real scalars such that every tree amplitude vanishes in the limit where a single momentum is taken soft? Is such a model unique? Must it have any particular symmetries, such as a symmetry that requires the scalar particles to be on equal footing?’

The vanishing soft limit means that the amplitude \( A_n \to 0 \) as \( p_i^0 \to 0 \) for any on-shell external momentum \( i = 1, 2, \ldots, n \). These soft limits are called Adler zeros and were first discovered in the context of pion scattering [59]. One intuition behind the vanishing soft limit is that it explores the nearby degenerate vacuum; a nice discussion is given in [60].

The physical context of this question is that such a model describes the low-energy dynamics of two massless Goldstone scalar particles arising from some spontaneous symmetry breaking. For any explicitly given symmetry breaking pattern of some symmetry group \( G \) to a subgroup \( H \), there are techniques for the systematic construction of a Lagrangian of the Goldstone modes [61–63]. But for more open-ended questions aimed at understanding the space of possible theories and any additional emergent symmetries they may have, the Lagrangian approach is limited and often complicated by field-redefinitions that can obscure symmetry properties.

A traditional ‘bottom-up’ approach to such a question is to try to write down a Lagrangian with kinetic terms for the two scalar fields and some local interactions that preserve the desired symmetries. Suppose we fail to construct a Lagrangian with these properties: does it mean that such a theory does not exist? Or did we miss out on some smart way to do this? Or suppose we did succeed in writing down a Lagrangian with these properties, is it unique? Or did we miss some other allowed interactions? Are there different ways to write the Lagrangian that nonetheless result in exactly the same observables? For example by being related by field redefinitions.

A modern approach to such questions is to start with the physical observables, namely the amplitudes. A clear advantage of this approach is that the amplitudes are independent of field redefinitions (and in case of gauge fields, the amplitudes are gauge invariant so gauge-choices and so on do not matter). The symmetries of the model manifest themselves on the amplitudes via Ward identities: linear relationships among amplitudes, valid either at all generic momenta or in certain momentum limits.

In the on-shell approach, one imposes on the amplitudes a set of assumptions about the particle spectrum of the model and its symmetries (exact or spontaneously broken) as well as mathematical consistency on the amplitudes. At tree-level, consistency conditions refer to properties like:

- Locality \( \Rightarrow \) correct simple poles corresponding to exchanges of physical particles; no spurious (unphysical) poles.
- Unitarity \( \Rightarrow \) the residues on the simple poles factorize into lower-point on-shell amplitudes.

One starts with the most general ansatz for the lowest-point amplitudes subject to the symmetries. As the lowest-point amplitudes in the model, they cannot have any physical poles since there are no lower-point amplitudes they can factorize into. So they must be polynomials in the external data (momenta, polarizations, etc) and each independent polynomial corresponds to an independent interaction term in an associated Lagrangian. Independence means under the use of momentum conservation and other algebraic identities; at the level of the Lagrangian, momentum conservation simply translates to integration-by-parts.

Next, fuse the lowest-point amplitudes together to make higher-point amplitudes, for example via a recursion relation of some valid form. The higher-point amplitude must have the required symmetries too. This may fix constants in the parametrization of the amplitudes. It may even set the amplitudes to zero. If all constants are set to zero by the mathematical consistency conditions, it means that there are no amplitudes that respect the requested symmetries and hence there can be no such non-trivial QFT. For if there were, it would produce non-vanishing scattering amplitudes. On the
other hand, if all imposed mathematical consistency checks are satisfied with some nonvanishing scattering amplitudes, then it is evidence that perhaps such a QFT may exist. It cannot be a definitive ‘yes’ because there could be further restrictions arising at higher-points and one would have to work harder to prove existence of a field theory.

What we have described here is the ‘amplitude bootstrap’. It is very powerful as a tool to rule out the existence of theories with too strong symmetry requirements, but cannot say ‘yes, it does exist’ without further input. As it turns out, this ability to answer ‘no’ or ‘maybe’ is one thing it has in common with the conformal bootstrap, as we shall see in section 5.

Let us illustrate the idea briefly using a combination of Lagrangian reasoning and on-shell amplitudes. We want a model of two massless scalars \( \phi_1 \) and \( \phi_2 \) such that the amplitudes vanish in the limit where any one of the particle momenta goes soft, i.e. \( p_i^\mu \to 0 \) for any one of the external momenta in the process. Any model with \( \phi^4 \)-type interaction would fail the criteria since the four-point amplitude is a constant and therefore does not vanish for any choice of momentum. What about an interaction term like \( \phi_1^4 \partial^2 \phi_2^2 = \phi_1^4 \partial\phi_2 \partial\phi_2 \phi_1 \phi_2 \)? Since \( \phi_1 \to i p_1 \) and \( p_2^\mu \to 0 \), this gives a four-point amplitude \( A_4(\phi_1 \phi_2 \phi_2 \phi_2) = 2p_1 \cdot p_2 = (p_1 + p_2)^2 \) (the momenta are labeled 1, 2, 3, 4 and related to the particles as the order in which they are given in the amplitude). This vanishes for any one of the momenta going to zero since the massless particles have \( p_i^2 = 0 \) and momentum conservation ensures \( (p_1 + p_2)^2 = (p_3 + p_4)^2 \).

So we are good, right? Not so fast! At six-point, the amplitude \( A_6(\phi_1 \phi_2 \phi_1 \phi_2 \phi_2 \phi_2) \) includes diagrams like

\[
\begin{align*}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{example_diagram.png}
\end{array}
\end{align*}
\]

where the solid lines indicates the \( \phi_1 \)-particles and the dashed line the \( \phi_2 \)-particles. In the limit where \( p_2^\mu \to 0 \), the diagram gives \( (p_1 + p_2)^2 = 2p_1 \cdot p_1 \) which is non-zero for generic momenta. So even though we had a four-particle interaction that did the work we wanted for four-point amplitudes, it failed at six-point. Now there are of course other diagrams that contribute too: for example, we can exchange (2 ↔ 3) and (2 ↔ 4) in (10). Together with (10), these then contribute

\[
2p_1 \cdot p_3 + 2p_2 \cdot p_4 + 2p_1 \cdot p_2 = (p_2 + p_3 + p_4)^2
\]

(11)
to the \( p_5 \to 0 \) soft limit of the amplitude. But there are also diagrams where line 1 is not on the same vertex as the soft line 5. They contribute

\[
2p_1 \cdot p_2 + 2p_1 \cdot p_3 + 2p_1 \cdot p_4 = 2p_1 \cdot (p_2 + p_3 + p_4).
\]

(12)

Thus the sum of the pole diagrams contribute the sum of (11) and (12)

\[
(p_1 + p_2 + p_3 + p_4)^2 = (p_3 + p_6)^2
\]

(13)

by momentum conservation. This is generically nonzero, so there is no way to get zero if only these four-point interactions are included. We need six-particle interactions in order to cancel the nonvanishing results from diagrams such as (10). This can, for example, be engineered from a Lagrangian interaction term of the form \( \phi_1^4 (\partial \phi_2)^2 \) whose coefficient is tuned such that its contribution to the six-point amplitudes exactly cancels that of the pole terms like (10) in the soft limit. Along with other terms needed at six-point order, the continued consistency with vanishing soft limits at higher points ends up fully dictating the model!

As for being on equal footing, it would appear from the above construction that \( \phi_1 \) and \( \phi_2 \) are not interchangeable. However, when one is more careful about setting up the problem, a symmetry between \( \phi_1 \) and \( \phi_2 \) does in fact emerge. This is shown in full detail in section 6.

For now, what I wanted to illustrate here was the logic of how the constraints are imposed on the amplitudes and how it allows us to learn about the structure of the model. In the more general approach, we parameterize all four-particle amplitudes in the most general way and then impose the soft constraints on the four- and higher-point amplitudes (section 6). This allow us to get ‘no, does not exist’ or ‘maybe’ as the answer to whether such a QFT exists. This is very similar to how the conformal bootstrap works, as we now explain.

5. Introduction to the modern conformal bootstrap

To set up a scattering problem, we start at time \( t = -\infty \) with initial state particles that are infinitely far apart and noninteracting (i.e. free). Then the particles come together, scatter through some (weak) interaction process, and end up far apart as free particles in the far future, \( t = +\infty \). These so-called initial and final asymptotic states of the far past and future are the in and out states of the perturbative scattering amplitudes.

In a CFT, there is no scale, so there is no sense of ‘far apart’ or ‘far into the past/future’. A CFT is scale invariant and has no asymptotic states. For that reason, a scattering amplitude is a priori not a good observable\(^6\). Moreover, if we wish to ask ‘what CFTs exist?’ we have to include also theories without weakly coupled limits, i.e. those in which we cannot talk about the ‘free’ particle states because the interactions are always strong.

The conformal bootstrap is a method to explore the landscape of CFTs without need for Lagrangians, weak couplings, or the concept of particles or scattering amplitudes; instead physical and mathematical consistency constraints are imposed on the observables in the CFT, namely the correlation functions of local operators.

5.1. Observables and CFT data

Correlation functions are the central observables in a CFT. So, what does that mean? Classical fields are familiar from 4D electromagnetism: at each point in space and time, the electric \( \vec{E}(\vec{x}, t) \) and magnetic \( \vec{B}(\vec{x}, t) \) fields give a vector-valued result for the strength and direction of the electromagnetic field. They are local fields in that they depend on a single point in space-time. Under Lorentz transformations, the electric and magnetic

\(^6\) Nonetheless, amplitudes in some CFTs, such as in \( \mathcal{N} = 4 \) SYM, play a central role in the amplitudes program. They can be understood as limits of amplitudes in a non-conformal QFT.
fields are mixed, and in relativistic contexts it is useful to combine them into the field strength $F_{\mu\nu}$, where in a given inertial frame the components of the electric and magnetic fields are $E_i = F_{i\mu} B_\mu = \sum_{k=1}^3 \epsilon_{ijk} F_{jk}$. Here $\epsilon_{ijk}$ is fully antisymmetric in its indices and $\epsilon_{123} = 1$. The field $F_{\mu\nu}$ is an example of an operator with non-zero spin. We can form other operators from the field strength, such as $F^2 = F_{\mu\nu} F^{\mu\nu}$ and $F^3$, etc. Examples of other operators are spin-0 scalar fields $\phi(x)$ and spin-1/2 fermion fields $\psi(x)$ and powers thereof, i.e. $\phi^2$ and $\psi^2$.

We can take derivatives of operators to get new ‘descendant’ operators, such as $\partial_\mu \phi$ or $\partial_\mu \partial_\nu \phi$, etc. These are examples of operators based on local fields, but more generally we consider local operators in a completely abstract sense. We denote such operators as $O_i$, where $i$ is a collective index that includes both operator type and any Lorentz indices it may have.

A property of operators that is important for our discussion is their scaling dimension. Under a scaling $x^\mu \to \lambda x^\mu$ of the spacetime coordinates, an operator scales homogeneously as

$$O_i(x) \to \lambda^{\Delta_i} O_i(\lambda x). \quad (14)$$

Consider a scalar field $\phi$ in a $d$-dimensional free theory. The action just has the kinetic term

$$S = \int d^d x \, \partial_\mu \phi \partial^\mu \phi. \quad (15)$$

The scalar field has mass-dimension $[\phi] = (d - 2)/2$ in order for the action to be dimensionless ($h = c = 1$). Performing a trivial change in integration variable $x \to \chi = \lambda x$, we see that we must also have $\Delta_\phi = (d - 2)/2$. As in this example, the scaling dimension $\Delta_i$ is the same as the mass-dimension of an operator $O_i$ in a free theory. However, in an interacting quantum theory, $\Delta_i$ receives quantum corrections and generally departs from the free value. Hence $\Delta_i$ is considered a real number in the following.

In a CFT, the symmetries are so powerful that the one-, two-, and three-point correlation functions are completely fixed up to a set of constants in the three-point correlator. One-point functions vanish $\langle O_i(x) \rangle = 0$. We write the two-point and three-point correlators of operators $O_i$ diagrammatically as

$$\langle O_i(x) O_j(y) \rangle = \sum_{\kappa} \frac{1}{2^\kappa} \rho_{\kappa} (x) \rho_{\kappa}(y) \rho_{\kappa}(z) = \sum_{\kappa} \frac{1}{2^\kappa} \rho_{\kappa}. \quad (16)$$

Operators with different scaling dimensions are orthogonal in the sense that their two-point functions vanish. If there are degenerate operators with the same operator dimension, they can be organized in a basis such that their two-point correlators vanish for distinct operators:

$$\langle O_i(x) O_j(y) \rangle = 0 \text{ for } i \neq j. \quad (17)$$

The unified constant in the three-point correlator is called the ‘OPE coefficient’. This comes from the idea of the operator product expansion (OPE) which states that in a local theory, the fluctuation generated by a product of operators in close proximity should be expressible as a sum of local operators. The coefficients of the operators in that sum are the OPE coefficients $c_{ijk}$; so as $y \to x$

$$O_j(x) O_k(y) \sim \sum_i c_{ijk} O_i(x). \quad (18)$$

(We defer discussion of some of the spacetime dependence to section 7.) The subscript $i, j, n, \ldots$ is some collective index that labels the operators in the CFT. The sum on the right is in principle over all operators in the theory, but some coefficients may be zero. For example, a fermionic operator would not appear in the OPE of two bosonic operators.

Multiplying (18) by $O_i(x)$ and taking the expectation value (think of the analogy to undergraduate quantum mechanics here) selects on the rhs the coefficient $c_{ijk}$ via the ‘orthogonality’ property of the two-point correlation function, $\langle O_i(x) O_j(y) O_k(z) \rangle$ is determined by the OPE coefficient $c_{ijk}$. In the above, we are completely glossing over the details of the dependence on the spacetime coordinates $x, y, z$; those who wish to see a more detailed account can find it in the technical review in section 7.

In order to explore the landscape of CFTs, we have to describe what characterizes a CFT. For the purpose here we will take a CFT to be determined by

- All operators $O_i$ with their spin $s (= 0, \frac{1}{2}, 1 \ldots)$ and scaling dimension $\Delta_i$, and
- The OPE coefficients $c_{ijk}$.

This is jointly called the CFT data: $\{\Delta_i, s_i, c_{ijk}\}$. This data defines a CFT assuming that it satisfies the mathematical consistency constraints of a CFT. We now proceed to discuss one such key constraint used in the conformal bootstrap.

The four-point correlators are not completely fixed by conformal symmetry, but we still have a certain handle on them. Suppressing the spacetime dependence, $\langle O_1 O_2 O_3 O_4 \rangle$ can be expressed via the OPE. For example, we can use the OPE on $O_1 O_2$ and $O_3 O_4$ or alternatively we could do with $O_1 O_3$ and $O_2 O_4$. The two expansions have to give the same result. Diagrammatically, we can illustrate this as

$$\langle O_1 O_2 O_3 O_4 \rangle = \sum_{\kappa} \frac{1}{2^\kappa} \rho_{\kappa} = \sum_{\kappa} \frac{1}{2^\kappa} \rho_{\kappa}. \quad (19)$$

This is called the crossing relation.

The idea of the conformal bootstrap is to assume some CFT data and then subject them to the constraints of the crossing relation. It sounds perhaps too simple that the equivalence of two infinite sums can lead to significant and powerful constraints on the CFT data, but this is nonetheless the case.

### 5.2. Bootstrap of the 3D Ising Model

Consider, as an example, a scalar field $\sigma(x)$ in a 3D CFT. It has spin 0 and its scaling dimension is classically equal to its mass-dimension: $\Delta_\sigma = \frac{d}{2}$. In a free theory, i.e. with no interactions, the operator $\epsilon \equiv (\sigma(x))^2$ has scaling dimension twice that of $\sigma(x)$ so $\Delta_\epsilon = 1$. Now suppose quantum corrections in some putative interactive CFT makes $\Delta_\epsilon = 0.53$. Now $\Delta_\epsilon$ no longer has to be $2\Delta_\sigma$, it has a quantum life of its own. But what
possible values can it take? Would it be realistic to think that a small quantum correction that increases $\Delta_\sigma$ from 0.5 to 0.53 allows $\Delta_\tau$ to become as big as, say, ?? Our intuition says that this is unreasonable: if the correction to $\Delta_\tau$ is small, then the correction to $\Delta_\sigma$ should also be small. Indeed, by analyzing the crossing relation (19) for the four-point correlator $\langle \sigma \sigma \sigma \sigma \rangle$, it can be shown numerically [64] that there exists no 3D CFT with $\Delta_\tau = 0.53$ and $\Delta_\sigma \gtrsim 1.45$.

The numerical implementation of the conformal bootstrap takes advantage of the fact that the crossing relation for $\langle \sigma \sigma \sigma \sigma \rangle$ can be reformulated into a statement that schematically looks like

$$\sum \tilde{v}_{\sigma \sigma O}^2 = 0,$$  \hspace{1cm} (20)

where $\tilde{v}_{\sigma \sigma O}$ represent vectors in a multi-dimensional abstract space (for more details, see section 7). In a unitary CFT, the OPE coefficients $c_{\sigma \sigma O}$ are real-valued so that means the coefficients in (20) are non-negative. Thus, consistency with the crossing relation requires that a sum of vectors with non-negative coefficients vanish. This is a non-trivial constraint. The key point then is that the functional form of the vectors $\tilde{v}_{\sigma \sigma O}$ is known and the only unknowns in (20) are the scaling dimensions $\Delta_O$ that the $\bar{v}_{\sigma \sigma O}$ depend on and the OPE coefficients $c_{\sigma \sigma O}$. So for given input, say $\Delta_\sigma = 0.55$ in 3D, one can scan over values of $\Delta_\sigma$ and test numerically if there exist solutions to (20). A ‘no’ means no: there is no 3D CFT with scalar operators of those scaling dimensions. This is how the bound $\Delta_\sigma \lesssim 1.45$ for $\Delta_\tau = 0.53$ was found in [64]. A ‘yes’ means maybe: the putative CFT is not ruled out, but it does not mean it exists. One has to study higher dimensional operators and a broader set of four-point correlators to find out if they are consistent with crossing too. So this analysis does not state whether or not there exists any 3D CFT with $\Delta_\tau = 0.53$ and $\Delta_\sigma < 1.45$.

When the crossing constraints are simultaneously applied to multiple correlators, the numerical bootstrap becomes even more powerful. For example, when applied [65] simultaneously to the three correlators $\langle \sigma \sigma \sigma \sigma \sigma \rangle$, $\langle \sigma \sigma \varepsilon \varepsilon \rangle$, and $\langle \varepsilon \varepsilon \varepsilon \varepsilon \rangle$, the conformal bootstrap with $\sigma \rightarrow -\sigma$ symmetry rules out any interacting 3D CFT with $\Delta_\sigma = 0.53$! While this is nice, there is an even more impressive and important result coming out of this analysis.

To back up, consider again the numerical bootstrap applied to a single $\langle \sigma \sigma \sigma \sigma \rangle$. As a function of $\Delta_\sigma$, a scan over possible values of $\Delta_\sigma$ gives a bound $\Delta_\sigma < f(\Delta_\sigma)$ for some function $f$. The plot is shown in figure 2. The white region is ruled out, the shaded region is not ruled out in this analysis. The key property to note here is the indication of a kink in the boundary curve near $\Delta_\sigma \approx 0.52$ and $\Delta_\tau \approx 1.42$ [64]. Those are in fact close to the values expected for the two lowest-dimension operators in the 3D Ising model.

Now beefing up the analysis to apply the numerical bootstrap to the three correlators simultaneously, it was found [65] that a small island around the expected ‘location’ of the 3D Ising model is cut out: see figure 3. This means that a large region of parameter space (white in the plot) is ruled out by the crossing constraints and there is a small shaded island-region not ruled out.

Precision numerics has made it possible to zoom in on this island and constrain it much further. The authors of [65, 66] used this to determine

$$\langle \Delta_\sigma, \Delta_\tau \rangle = (0.5181489(10), 1.412625(10)),$$  \hspace{1cm} (21)

Figure 2. Plot showing the bounds on the scaling dimensions $\Delta_\tau$, and $\Delta_\sigma$ of the two lowest-dimension operators in a 3D CFT with $\mathbb{Z}_2$ symmetry. The white region is excluded. This is based on numerical examination of the crossing constraints on a single correlator $\langle \sigma \sigma \sigma \sigma \rangle$. The kink in the boundary between the excluded and non-excluded regions occurs near the expected location of the 3D Ising model. Reprinted figure with permission from [64], Copyright (2012) by the American Physical Society.

Figure 3. Plot showing the excluded regions of scaling dimensions $\Delta_\tau$, and $\Delta_\sigma$ of the two lowest-dimension operators in a 3D CFT with $\mathbb{Z}_2$ symmetry. This is based on numerical examination of the crossing constraints on three correlators $\langle \sigma \sigma \sigma \sigma \rangle$, $\langle \sigma \sigma \varepsilon \varepsilon \rangle$, and $\langle \varepsilon \varepsilon \varepsilon \varepsilon \rangle$. Where the kink occurred in figure 2, there is now a small island of non-excluded theory-space, narrowing in on the 3D Ising model. Reprinted by permission from Springer Nature Customer Service Centre GmbH: Springer. [65] © 2014.
Figure 4. Comparing the multi-correlator bootstrap results for the 3D Ising model versus Monte Carlo. Reprinted by permission from Springer Nature Customer Service Centre GmbH: Springer. [66] © 2016.

which is higher precision than the available Monte Carlo results; see the comparison in figure 4. Moreover, the numerical bootstrap determines the scaling dimensions and OPE coefficients to high precision of several of the lowest dimension operators in the 3D Ising model, not just of \( \sigma \) and \( \epsilon \), see for example table II in [67]. This offers evidence that the 3D Ising model may be a CFT, as proposed by Polyakov [11].

Earlier in this presentation we mentioned the 3D Ising model: recall from section 2 that the critical point of water and ferromagnets is expected to be described by the 3D Ising model. The critical exponents of the approach to the critical point are directly related to the scaling dimensions \( \Delta_\sigma \) and \( \Delta_\epsilon \). For example, the critical exponent \( \nu \) of the correlation length in (2) is determined by \( \Delta_\epsilon \) as

\[
\nu = \frac{1}{d - \Delta_\epsilon}. \tag{22}
\]

Hence, for \( d = 3 \) and the numerical bootstrap value of \( \Delta_\epsilon \), one finds \( \nu = 0.629 \pm 0.005 \) [68]. Compare that with the experimental value of \( \nu = 0.63 \). It is clear that the formal theory exploration of the landscape of CFTs is relevant also for observable physics in our world.

The ideas behind the conformal bootstrap dates back about 50 years [69, 70] and it was developed fruitfully in the mid-80s for 2D CFTs [71]. The conformal bootstrap program saw a revival starting around 2008 when the bootstrap problem was phrased in terms of bounding operators and especially when it was realized that the condition (19) can be phrased a form that makes it particularly well-suited for numerical implementation. Initial work was driven by particle physics applications (see e.g. [72, 73]) and the recent application to the 3D Ising model was initiated in [64, 74]. Practically, this means that the crossing relation is rephrased a statement about whether a set of vectors can add to zero using only non-negative coefficients. Semidefinite programming techniques [75] have turned out to be powerful approaches to assess such questions numerically. By now, quite a number of analytical results have also been developed to further enhance the exploration of the CFT landscape.

Other examples: helium, QED\(_3\), SCFTs across dimensions. There have been a multitude of applications of—and other developments closely related to—the conformal bootstrap.

A very nice example is the set of 3D CFTs with \( O(N) \) global symmetry studied using the conformal bootstrap in [66, 76]. In particular, the \( O(2) \) model is expected to describe the \( \lambda \)-line superfluid transition in helium-4. The physics of this critical transition lies in a different universality class (sometimes also called the 3D XY universality class) from that of the liquid–vapor critical point that we described in section 2 for water and the Curie point of ferromagnets. The difference is manifest in the values of the critical exponents such as \( \nu \) associated with the divergence of the correlation length: for the liquid–vapor critical point \( \nu = 0.63 \ldots \), but for the \( \lambda \)-line transition it is approximately \( \nu = 0.67 \ldots \) Experimental results [77], Monte Carlo simulations [78, 79], and the conformal bootstrap methods [80] agree on the value of \( \nu \) to two-decimal places. The theoretical methods are in agreement beyond the leading digits within the uncertainties, with the conformal bootstrap giving the highest precision result [80, 81]

\[
\nu = 0.67175(10). \tag{23}
\]

However, there is an 8\( \sigma \) tension [78, 80, 81] between the theoretical values of \( \nu \) and the experimental one.

There are many other applications of the conformal bootstrap. Supersymmetric CFTs have been proposed to be relevant in the context of topological superconductors and for the description of a critical point on the surface of topological insulators. Another class of 3D CFTs arises as fixed points of 3D models with gauge fields. Two main classes have been examined: 3D Chern–Simons fields coupled conformally to
of section 4.2 and address it in full technical detail:

6. Technical: amplitude bootstrap

Lagrangian. First, it is convenient to combine the two real
introduction of [92].

…

6. Technical: amplitude bootstrap

In this section, we revisit the question posed in the beginning
of section 4.2 and address it in full technical detail:

‘Does there exist a local relativistic QFT with two mass-
less real scalars such that every tree amplitude vanishes
in the limit where a single momentum is taken soft?
Is such a model unique? Must it have any particular
symmetries, such as a symmetry that requires the scalar
particles to be on equal footing?’.

Let us begin by phrasing the question in the traditional
QFT language, i.e. in terms of fields and symmetries of a
Lagrangian. First, it is convenient to combine the two real
scalar fields \( \phi_1 \) and \( \phi_2 \) into a complex scalar
\[
Z = \phi_1 + i\phi_2, \quad \bar{Z} = \phi_1 - i\phi_2. \quad (24)
\]

A canonical kinetic term can be written \( \partial_\mu Z \partial^\mu Z \). In the
bottom-up Lagrangian approach, one assumes such a kinetic
term and then builds up interaction terms.

Secondly, the vanishing soft limit is related to (but not iden-
tical to) the Lagrangian having a shift symmetry \( Z \to Z + c + \cdots \),
where \( c \) is a complex-valued constant and the ellipses
stand for field-dependent terms. (For a more precise statement
of this relation, see section 6.5.) Models in which there is at
least one derivative on every scalar field trivially have such
a shift symmetry and also have amplitudes with vanishing
soft limits. For example interactions like \( (\partial Z)^4 \) gives
Feynman rules in which each term is a product of the four momenta
and therefore the vertex vanishes when any one of them is taken
soft. This trivial way of realizing the shift symmetry gives
contributions at \( O(p^4) \) to amplitudes, so the question here really is
if there are models that realize the symmetries at lower order
in the low-energy expansion. With no derivatives on the fields
in the interactions, the model cannot have a shift symmetry
or vanishing soft limits. A two-derivative theory gives
\( O(p^2) \) amplitudes and it could have four-point interaction such as
\( Z^2(\partial Z)^2 \) and similar with \( Z \)'s too. Such terms do not obvi-
ously have a shift symmetry or give amplitudes with vanishing
soft limits. So we have to add other interaction terms to
achieve this and the question is how much freedom there is to
do so.

Thirdly, the question of whether the two real scalars \( \phi_1 \)
and \( \phi_2 \) are on equal-footing translates to whether the model
has an \( SO(2) \) symmetry that acts on \( (\phi_1, \phi_2) \) as a rotation.
In the language of the complex scalar, this is equivalent to
asking if there is a \( U(1) \) symmetry that takes \( Z \to e^{i\alpha}Z \) and
\( Z \to e^{-i\alpha}Z \) for some constant \( \alpha \). We do not assume a \( U(1) \)
symmetry, but will see that it emerges in the leading-order
model.

Rather than attempting to build a Lagrangian whose Feyn-
man rules give amplitudes that vanish in the soft limit, the
modern on-shell amplitude approach starts with the amplitudes
to systematically determine what models can be ruled out and
map out which ones may exist.

6.1. Setup

We rephrase the problem in terms of on-shell amplitudes.

Variables. We consider scattering of \( n \) massless par-
ticles so the external four-momenta \( p_i^0 \), where \( i = 1, 2, 3, \ldots, n \),
are required to satisfy the on-shell condition \( p_i^2 = p_i^0 p_i^\mu = 0 \).
Momentum conservation requires \(^8\)
\[
\sum_{i=1}^n p_i^\mu = 0. \quad (25)
\]

The amplitude must depend on the external momenta in a
Lorentz invariant way, namely as dot-products \( p_i \cdot p_j = p_i^0 p_j^0 \),
where \( i \) is the particle label \( i = 1, 2, 3, \ldots, n \). Since \( p_i^2 = 0 \) for
all \( i \), we have
\[
s_{ij} \equiv (p_i + p_j)^2 = 2p_i \cdot p_j, \quad s_{ik} \equiv (p_i + p_j + p_k)^2 = 2p_i \cdot p_j + 2p_i \cdot p_k + 2p_j \cdot p_k, \text{ etc.} \quad (26)
\]

The Mandelstam variables \( s_{ij} \) are not all independent due to
the constraints of momentum conservation (25). For example,

\(^8\)The conservation of momentum is really that the sum of incoming momenta
must equal the sum of outgoing momenta. As a technical tool, crossing sym-
metry is often used to trade in-coming particles for outgoing ones, so that the
amplitude has all external particles on equal footing as outgoing. This helps
make the calculations a bit simpler and once a result is obtained, particles
can simply the ‘crossed’ back to let some of them be incoming again for the
purpose of computing the cross-section.
for $n = 4$, we have
\begin{align*}
n &= 4: s_{12} = s_{34}, \quad s_{13} = s_{24}, \\
&\quad s_{14} = s_{23}, \quad \text{and} \quad s_{12} + s_{13} + s_{14} = 0.
\end{align*}
(27)

Thus the scalar amplitudes are functions of Mandelstam variables subject to the constraints of momentum conservation.

**Analytic structure.** Tree amplitudes must be rational functions of Mandelstam variables. In a local theory of massless scalars, they can have simple poles (and no higher-order poles) at locations where $s_{ij,k} \to 0$ and the residue of such a pole is, by unitarity, a product of lower-point amplitudes. There can also be polynomial terms in the amplitude; in an $n$-point amplitude such polynomial terms can arise from $n$-point interactions in the Lagrangian.

**Bose symmetry.** Bose symmetry requires that the amplitude is invariant under exchanges of identical bosons. Specifically, $A_n(Z \ldots Z\ldots Z)$ must be symmetric under all permutations of the momenta of the $Z$’s, and likewise for those of the $\bar{Z}$’s.

**Vanishing single soft limits.** As any one of the external momenta is taken to zero, the amplitude has to vanish in the momentum:
\begin{align*}
A_n(Z \ldots Z\ldots Z) &\to 0 \quad \text{for any } p_i \to 0.
\end{align*}
(28)

As noted in section 4.2, this is called a vanishing soft theorem or an Adler zero [59].

**$U(1)$ symmetry.** The $U(1)$ symmetry can be understood as the statement that $Z$ particles have charge $+1$ and $\bar{Z}$ particles have charge $-1$. The associated Ward identity then says
\begin{align*}
A_n(Z \ldots Z\ldots Z) &= 0 \quad \text{for } n_\bar{Z} \neq n_Z.
\end{align*}
(29)

We do not assume $U(1)$ symmetry; we shall see it emerge in the leading order theory in the sense that it only has non-vanishing amplitudes with $n_\bar{Z} = n_Z$.

The goal is to construct the model subject to the constraints of vanishing soft limit at the lowest possible order in the energy–momentum expansion; higher order terms are considered in section 6.4. To streamline the discussion, we first look at amplitudes that violate the $U(1)$, then those that respect it.

### 6.2. Bootstrapping the model: $U(1)$-violating amplitudes

Amplitudes that do not obey the $U(1)$ symmetry are those with an unequal number of $Z$ and $\bar{Z}$’s.

- **Three-point.** Momentum conservation for 3 massless particles sets all Mandelstams to zero, e.g. $s_{12} = (p_1 + p_2)^2 = p_3^2 = 0$. So there can be no momentum-dependence in the three-particle scalar amplitudes, they have to be constants. For example, $A_3(ZZZ) = d_3$ and $A_3(\bar{Z}\bar{Z}Z) = d_5$. But this is at odds with the assumption of vanishing soft behavior (28).\footnote{The special kinematics associated with taking soft limits of a three-particle amplitude may appear tricky; however, a constant three-particle amplitude necessarily implies a divergence of the four-particle amplitudes via pole diagrams [52], so the requirement of a vanishing soft limit rules out the three-particle amplitudes.} We conclude that there can be no three-point amplitudes.

- **Four-point.** At four-points, the $U(1)$ violating amplitudes are $A_4(ZZZZ), A_4(\bar{Z}ZZZ)$, and their conjugates. They cannot have poles, since there are no three-particle amplitudes they could factor into (unitarity), so they have to be polynomial in the Mandelstam variables. Constant terms are excluded by the soft limit (28). At $O(p^2)$, the only Mandelstam polynomial compatible with the Bose symmetry of three or four identical scalars is $s_{12} + s_{13} + s_{23}$, but that vanishes by momentum conservation (27). Hence $A_4(ZZZZ)$ or $A_4(\bar{Z}ZZZ)$ start at $O(p^4)$.

- **Five-point.** In the absence of three-particle amplitudes, the five-particle amplitudes cannot have poles, so they must be polynomial in the Mandelstam variables. A constant is incompatible with the vanishing soft limit. At order $O(p^2)$, Bose symmetry requires $A_5(ZZZZZ)$ and $A_5(\bar{Z}ZZZZ)$ to be $\sum_{1 \leq i < j < 4} s_{ij}$ or $\sum_{1 \leq i < j < 5} s_{ij}$, but both of these are zero due to momentum conservation. The amplitude with three $Z$’s and two $\bar{Z}$’s is uniquely determined at $O(p^2)$ to be $A_5(ZZZZZ) = a_{345}$, but this does not vanish in the limit of taking (say) $p_3 \to 0$. So we must set $a = 0$ and hence five-point amplitudes are at least $O(p^4)$.

- **Six-point and above.** Just as in the five-point case, any amplitude with all $Z$’s or a single $Z$ vanishes at $O(p^2)$. With at least two of both $Z$ and $\bar{Z}$, there is a unique Bose symmetric Mandelstam polynomial at $O(p^2)$, namely $s_{12} - s_{23}$, where we have chosen the $Z$ particles to have momenta labels $i = 1, 2, \ldots, 3 n_Z$. However, this is nonvanishing in the soft limits of any $Z$ momenta.

Based on the above, we conclude that any $U(1)$-violating amplitude obeying the vanishing soft limit condition have to be at least $O(p^4)$. Such higher-order corrections are considered in section 6.4. The lesson here is that any complex scalar two-derivative (i.e. $O(p^2)$ amplitudes) model with vanishing soft limits must have $U(1)$ symmetry: this is an example of a—perhaps surprising—emergent symmetry.

### 6.3. Bootstrapping the model: $U(1)$-conserving amplitudes

Amplitudes with $U(1)$ symmetry have an equal number of $Z$ and $\bar{Z}$’s, so in particular they must be even-point.

- **Four-point $U(1)$ conserving.** As the lowest-point amplitude $A_4(ZZZZ)$ must be polynomial in the Mandelstam variables. A constant is excluded by the vanishing soft limit constraint, so we write as most general ansatz with the appropriate Bose symmetry:
\begin{align*}
A_4(ZZZZ) = \frac{a_1}{\Lambda^2} s_{13} + O(p^4).
\end{align*}
(30)

At order $O(p^2)$, the polynomial $s_{12} + s_{23} = s_{34} + s_{43}$ is also compatible with Bose symmetry, but by momentum conservation (27) is equal to $-s_{13}$. We have included a scale $\Lambda$ of mass-dimension 1 so that $a_1$ is a pure number. The amplitude vanishes in the soft limit of any one of the momenta, so there are no constraints on $a_1$.\footnote{The special kinematics associated with taking soft limits of a three-particle amplitude may appear tricky; however, a constant three-particle amplitude necessarily implies a divergence of the four-particle amplitudes via pole diagrams [52], so the requirement of a vanishing soft limit rules out the three-particle amplitudes.}
• **Six-point.** The argument in section 6.2 shows that any polynomial term at \(O(p^2)\) in a scalar amplitude with more than 4 external legs is non-vanishing in the single soft limit. This means that to achieve a vanishing soft limit beyond four-points, there must be cancellations among the pole-terms and the contact terms.

At six-points the pole terms must have residues that are products of two four-point amplitudes, for example on the 123-channel:

\[
\frac{1}{3 \, \frac{\not{p}}{\not{r}}} \, A_4(z_{12}z_{65}z_{34}z_{56}) = \frac{a_1^2 \, s_{13} s_{46}}{\Lambda^2 \, s_{123}} \, s_{123}.
\]

(31)

The most general contact terms can be parameterized as

\[
\frac{1}{3 \, \frac{\not{p}}{\not{r}}} \, = \, b_0 + b_1 \, s_{246} + O(p^3).
\]

(32)

There are no other independent polynomials with Bose symmetry at \(O(p^2)\). So we can write the six-point amplitude as the sum of all the pole diagrams and the possible contact terms:

\[
A_6(\mathcal{Z}\mathcal{Z}\mathcal{Z}\mathcal{Z}\mathcal{Z}) = \frac{a_1^2}{\Lambda^2} \left( \frac{s_{13} s_{46}}{s_{123}} + \frac{s_{13} s_{26}}{s_{143}} + \frac{s_{13} s_{24}}{s_{163}} \right) + (1 \leftrightarrow 5) + (3 \leftrightarrow 5)
\]

\[
+ \, \left( \frac{a_1^2}{\Lambda^2} s_{246} \right) + b_0 + b_1 \, s_{246} + O(p^3).
\]

(33)

In the soft limit \(p_k \to 0\), the first two pole terms in (33) vanish while the third one reduces to \(s_{24}\). Under the exchanges \((1 \leftrightarrow 5)\) and \((3 \leftrightarrow 5)\) two more \(s_{24}\)’s are generated. So we get

\[
A_6(\mathcal{Z}\mathcal{Z}\mathcal{Z}\mathcal{Z}\mathcal{Z}) \to 3 \frac{a_1^2}{\Lambda^2} s_{24} + b_0 + b_1 s_{24} + O(p^3).
\]

(34)

Therefore, in order to have vanishing soft limits (28) at six-points, we must have

\[
b_0 = 0 \quad \text{and} \quad b_1 = -3 \frac{a_1^2}{\Lambda^2}.
\]

(35)

Hence, the four-point and six-point amplitudes are completely fixed at order \(O(p^2)\) in terms of just one number, the coupling constant \(a_1\).

• **Eight-point.** The above pattern continues to higher-point amplitudes: at \(O(p^2)\) the whole model is uniquely fixed by the symmetry requirements in terms of a number, \(a_1\), and a single dimensionful scale \(\Lambda\).

Consider at eight-points, the \(p_k \to 0\) soft limit. Some diagrams directly vanish in this limit. Of diagrams that do not vanish, those that result in terms with poles directly vanish among themselves. To see this, consider the three diagrams with a \(1/s_{123}\) pole that do not vanish in the \(p_k \to 0\) limit:

\[
\frac{1}{3 \, \frac{\not{p}}{\not{r}}} \, \frac{1}{3 \, \frac{\not{p}}{\not{r}}} \Rightarrow \left( \frac{a_1^2 \, s_{13}}{s_{123}} \right) \left( \frac{a_1^2 \, s_{46}}{s_{123}} \right) \Rightarrow -3 \frac{a_1^2 \, s_{13} \, s_{46}}{\Lambda^2 \, s_{123}}.
\]

(36)

\[
\frac{1}{3 \, \frac{\not{p}}{\not{r}}} \, \frac{1}{3 \, \frac{\not{p}}{\not{r}}} \Rightarrow \left( \frac{a_1^2 \, s_{13}}{s_{123}} \right) \left( \frac{a_1^2 \, s_{56}}{s_{123}} \right) \Rightarrow -3 \frac{a_1^2 \, s_{13} \, s_{56}}{\Lambda^2 \, s_{123}}.
\]

(37)

\[
\frac{1}{3 \, \frac{\not{p}}{\not{r}}} \, \frac{1}{3 \, \frac{\not{p}}{\not{r}}} \Rightarrow \left( \frac{a_1^2 \, s_{13}}{s_{123}} \right) \left( \frac{a_1^2 \, s_{56}}{s_{123}} \right) \Rightarrow -3 \frac{a_1^2 \, s_{13} \, s_{56}}{\Lambda^2 \, s_{123}}.
\]

(38)

The three contributions (36)–(38) cancel, and it is clear why: this is guaranteed by the vanishing of the six-point in the soft limit.

Finally there are pole diagrams with nonvanishing soft limits that do not cancel among themselves but leave behind polynomial terms: for example

\[
\frac{1}{3 \, \frac{\not{p}}{\not{r}}} \, \frac{1}{3 \, \frac{\not{p}}{\not{r}}} \Rightarrow \left( \frac{a_1^2 \, s_{13}}{s_{123}} \right) \left( \frac{a_1^2 \, s_{56}}{s_{123}} \right) \Rightarrow -3 \frac{a_1^2 \, s_{13} \, s_{56}}{\Lambda^2 \, s_{123}}.
\]

(39)

There are six distinct such diagrams (from the six possible pairings of odd-numbered momenta 1357 on the lhs), so that means that

\[
A_6(\mathcal{Z}\mathcal{Z}\mathcal{Z}\mathcal{Z}\mathcal{Z}\mathcal{Z}) \to -18 \frac{a_1^2}{\Lambda^2} s_{246} \quad \text{as} \quad p_k \to 0.
\]

(40)

This can be canceled by an eight-point local contact term \(+18 \frac{a_1^2}{\Lambda^2} s_{246}\) in order to ensure the vanishing soft theorem.

• **Ten-point.** The pattern continues. The local terms that arise in the soft limit \(p_{10} \to 0\) come from the pole diagrams with the eight-point contact terms, so it gives \(18 \frac{a_1^2}{\Lambda^2} s_{246}\). There are \((5 \times 4) = 10\) such diagrams, so the \(p_{10}\) soft limit of all the pole terms can be canceled by a contribution \(-180 \frac{a_1^2}{\Lambda^2} s_{246} s_{10}\) from a local ten-point interaction.

To summarize, we have found that the assumptions have fixed the amplitudes in the theory completely in the leading order of the low-energy expansion: the lowest-point nonvanishing amplitudes are at order \(O(p^2)\)

\[
A_4(\mathcal{Z}\mathcal{Z}) = \frac{1}{\Lambda^2} s_{13},
\]

\[
A_6(\mathcal{Z}\mathcal{Z}\mathcal{Z}\mathcal{Z}) = \frac{1}{\Lambda^4} \left( \frac{s_{13} s_{46}}{s_{123}} + \frac{s_{13} s_{26}}{s_{143}} + \frac{s_{13} s_{24}}{s_{163}} \right) + (1 \leftrightarrow 5) + (3 \leftrightarrow 5) - 3 \frac{s_{24}}{s_{123}}.
\]

\[
A_8(\mathcal{Z}\mathcal{Z}\mathcal{Z}\mathcal{Z}\mathcal{Z}) = \frac{1}{\Lambda^6} (\text{pole terms} + 18 s_{246}),
\]

\[
A_{10}(\mathcal{Z}\mathcal{Z}\mathcal{Z}\mathcal{Z}\mathcal{Z}\mathcal{Z}) = \frac{1}{\Lambda^8} (\text{pole terms} - 180 s_{246} s_{10}).
\]

(41)

Here we have set the four-point coupling \(a_1 = 1\) without any loss of generality.
For those who love Lagrangians, one can retro-engineer the interaction terms based on the polynomial terms above to find

$$\mathcal{L} = -\partial_{\mu}Z\partial^{\mu}Z + \frac{1}{\Lambda^2}ZZ\partial_{\mu}Z\partial^{\mu}Z - \frac{3}{4\Lambda^2}Z^2\partial_{\mu}Z\partial^{\mu}Z + \frac{1}{2\Lambda^4}Z^3\partial_{\mu}Z\partial^{\mu}Z + \cdots,$$

(42)

where the dots stand for interactions with more than 10 fields. Here we have used that the (2\textsuperscript{n})-point matrix element of

$$Z^{n-1}\partial_{\mu}Z\partial^{\mu}Z = \frac{1}{n!}(\partial_{\mu}Z^n)(\partial^{\mu}Z^n)$$

(43)

is

$$\frac{1}{n!}C_{\text{odd}}(1)C_{\text{even}}(2)$$

using momentum conservation. This was used to normalize the interaction terms so they exactly reproduce the local terms in the amplitudes above. For example, for the ten-point term in (42), the overall numerical factor of the local contact term contribution is 

$$\frac{5}{16}(4!)^2 = -180.$$  

We can extend this reasoning to 2\textsuperscript{n}-points. Suppose the numerical coefficient of the 2\textsuperscript{n}-point interaction term in the Lagrangian is \(\alpha_n\); e.g. \(\alpha_4 = \frac{1}{4}\). Then from (43) and (44), the polynomial term it contributes to the amplitude is

$$\alpha_n[(n-1)!]^2Z_{2\text{even}}Z_{2\text{odd}}.$$  

On the other hand, the purpose of this term will be to cancel the local contribution from the soft limit of the (n choose 2) pole diagrams involving the 2\textsuperscript{(n-1)}-point local contribution, which has coefficient \(\alpha_{n-1}[(n-2)!]^2\). So we see that

$$\alpha_n[(n-1)!]^2 = \frac{(n-1)!}{2}(n-2)!\alpha_{n-1}.$$  

(45)

With \(\alpha_1 = -1\) (the kinetic term), we get \(\alpha_2 = 1\) and likewise we reproduce the numerical coefficients of other terms in (42). The recursive formula (45) is straightforward to solve and one finds

$$\alpha_n = (-1)^n\frac{n}{2n-1}. $$

(46)

These are exactly the series coefficients of \(1/(1 + \frac{1}{2}ZZ)^2\) expanded around zero! Thus, including the scale \(\Lambda\), we have discovered that the Lagrangian can be resummed to

$$\mathcal{L} = -\frac{\partial_{\mu}Z\partial^{\mu}Z}{(1 + 2\Lambda^2ZZ)^2} = -G_{ZZ}\partial_{\mu}Z\partial^{\mu}Z,$$

(47)

where \(G_{ZZ} = 1/(1 + \frac{1}{2\Lambda^2}ZZ)^2\) can be identified as the Fubini–Studie Kähler metric on \(\mathbb{C}P^1\). This two-scalar model is well-known, it is the \(\mathbb{C}P^1\) nonlinear sigma model (NLSM) [93]. It describes two real Goldstone modes arising from the spontaneous symmetry breaking of \(SU(2)\) to \(U(1)\). The real Goldstones are paired into the complex scalar \(Z\) that ‘lives’ in the symmetric coset space \(SU(2)/U(1) \sim \mathbb{C}P^1\). The coset has the \(U(1)\) symmetry: this is exactly the \(U(1)\) symmetry that emerged in our on-shell analysis.

At this point, the engaged reader may complain: but you said that the vanishing soft limits were associated with a shift symmetry \(Z \to Z + c + \ldots\)? This does not appear to be a symmetry of the Lagrangian (47). However, a Lagrangian is not unique but can take a different form upon field redefinitions; this cannot change the physical observables, the amplitudes. So the shift symmetry can be accompanied by a field redefinition and in fact the Lagrangian (47) is invariant under infinitesimal shift symmetry

$$Z \to Z + c + \bar{c} - \frac{1}{2\Lambda^2}Z^2, \quad \bar{Z} \to \bar{Z} + \bar{c} - \frac{1}{2\Lambda^2}Z^2.$$  

(48)

It is one of the appealing features of the on-shell amplitudes approach that it is independent of having to deal with redundancies such as those arising from field redefinitions (or gauge transformations).

The very simple amplitude analysis shows that at the leading two-derivative order, the model had to have \(U(1)\) symmetry; in that sense it is emergent! Recognizing the leading-order model as the \(\mathbb{C}P^1 \sim SU(2)/U(1)\) NLSM, it is clear why the \(U(1)\) had to be there. Next we discuss interactions beyond the leading order.

6.4. Beyond leading order

In the \(\mathbb{C}P^1\) NLSM, it is fairly easy to see that there is only one possible two-derivative operator at four-point: \(ZZ\partial_{\mu}Z\partial^{\mu}Z = \frac{1}{2}(\partial_{\mu}Z^2)(\partial_{\mu}Z^2).\) Any other way of arranging the derivatives on the four fields is equivalent to this using integration by parts and the leading order equations of motion (EOM) \(\Box Z = \partial_{\mu}\partial^{\mu}Z = 0\) and \(\Box Z = 0\). But what about higher derivative corrections? There are many ways of sprinkling four derivatives on the four fields, but how many are actually independent under integration by parts and use of the EOM? And with \(2k\)-derivatives? This is very easy to answer using the on-shell amplitude methods as we now demonstrate.

**Higher order corrections.** A \(2k\)-derivative term generates contributions to the amplitudes at \(O(p^2)\). So the question of the number of independent \(2k\)-derivative operators is simply rephrased as: how many Bose symmetric second order polynomials in the Mandelstam variables are independent under the relations of momentum conservation (translates to integration by parts) and on-shellness (translates to EOM)?

For the four-point \(U(1)\)-conserving \(O(p^2)\) case we find two such independent polynomials,

$$A_4^{(op)}(ZZZZ) = \frac{a_1}{\Lambda^8} s_{13} + a_2 s_{13} + a'_4 (s_{12}^2 + s_{14}^2) + O(p^6).$$

(49)

The two terms correspond to the two independent Lagrangian terms \(\partial_{\mu}Z\partial_{\nu}Z\partial^{\mu}Z\partial^{\nu}Z\) and \(ZZ\partial_{\mu}\partial_{\nu}Z\partial^{\mu}Z\partial^{\nu}Z\).

For the cases of \(2k\)-derivatives one similarly finds

$$\# \text{ independent } \partial^{\mu}Z^{2k} \text{ operators} \quad 1 \quad 2 \quad 4 \quad 6 \quad 8 \quad 10 \quad 12 \quad 2k$$

(50)

As \(k\) or \(n\) grows, it gets harder to determine by brute force the number of independent Mandelstam polynomials subject to
the given constraints. However, there are powerful mathematical tools, such as the Gröbner basis, for solving such problems and they have indeed been applied for these purposes [94].

At four-point, the matrix elements trivially satisfy the vanishing soft-limit condition, simply due to the special three-particle kinematics of the resulting limit. So one has to analyze more carefully at six-point level (and higher) whether cancellations can occur to ensure that the soft limit gives zero.

**Lowest order U(1)-violating operators.** At leading order, the model we consider has $U(1)$ symmetry, but at subleading orders, our formulation of the problem allows for $U(1)$-violating terms. We found in section 6.2 that at four-point, the $U(1)$-violating amplitudes start at $O(p^4)$. The explicit matrix elements at this order are

\[
A_4(ZZZZ) = \frac{d_4}{\Lambda} (s_{12}^2 + s_{13}^2 + s_{23}^2) + O(p^6),
\]

\[
A_6(ZZZZZ) = \frac{d_6}{\Lambda} (s_{12}^2 + s_{13}^2 + s_{23}^2) + O(p^8),
\]

and similarly for those with conjugate states.

At five-point, the all-$Z$ matrix element is nonvanishing at $O(p^4)$,

\[
A_5(ZZZZZ) = \frac{d_5}{\Lambda} \sum_{1 \leq i < j \leq 5} s_{ij}^2 + O(p^6),
\]

however, this amplitude does not vanish in the soft limit. Likewise, $A_5(ZZZZZ)$ and $A_5(ZZZZZ)$ have two and three independent matrix elements, respectively, but no linear combination of them vanishes in the soft limit. At $O(p^4)$, there are two independent terms in the five-point amplitude with vanishing soft limit:

\[
A_5(ZZZZZ) = \frac{e_1}{\Lambda} s_{45} \left[ (s_{14} + s_{15})^2 + (s_{24} + s_{25})^2 + (s_{34} + s_{35})^2 - 2s_{45}^2 \right] + \frac{e_2}{\Lambda} \left[ s_{14} s_{15} s_{24} s_{25} + s_{24} s_{25} s_{34} s_{35} \right].
\]

Both vanish trivially as $p_4$ or $p_5 \to 0$. When one of the $Z$ particles goes soft, say $p_4 \to 0$, the resulting four-particle kinematics $p_2 + p_3 + p_4 + p_5 = 0$ ensures that both expressions in $[\ldots]$ vanish. These $O(p^4)$ operators are subleading to the $O(p^3)U(1)$-violating ones from (51). There are of course many other operators one can construct. These examples simply illustrate the amplitude-based method.

6.5. **Postscript: coset story**

The context of the problem studied here is the low-energy physics of Goldstone bosons arising from spontaneous breaking of an internal symmetry group $G$ to a subgroup $H$. The number of Goldstone bosons is equal to the number of broken symmetry generators, i.e. $\dim(G/H) = \dim(G) - \dim(H)$. The scalars ‘live’ in the coset space $G/H$. When $G/H$ is a symmetric space and there are no cubic interactions\(^{10}\), it can be shown that the amplitudes of the Goldstone bosons vanish in the single scalar soft limit.

For specific symmetry breaking patterns $G \to H$, there are techniques for systematic construction of Lagrangians of the Goldstone modes [61–63]. But for more open-ended questions aimed at understanding the space of possible theories and any additional emergent symmetries they may have, the Lagrangian approach is limited.

In our particular example with two real Goldstone modes, there are two obvious candidate theories: (1) $SU(2)$ broken to $U(1)$, or (2) $U(1) \times U(1)$ completely broken. As we have seen, the vanishing soft limit criterion selects for the former. The example serves to illustrate how the bottom-up approach to construction of theories via amplitudes can have symmetries emerge that were not part of the input assumptions. There are examples where the emergence of symmetries are perhaps more surprising. This is, for example, the case with supersymmetric extensions of the $\mathbb{CP}^1$ model. When constructed from the on-shell amplitudes approach, one finds [18] that not only does the $N = 1$ supersymmetric $\mathbb{CP}^1$ model have the $U(1)$ symmetry under which the scalars $Z$ and $\bar{Z}$ are charged (and their fermions are uncharged), it also has a second global $U(1)$ symmetry under which the scalars and fermions in the same supermultiplet have the same charge.

We have presented a basic systematic way to study explore the space of field theories to illustrate the powerful amplitudes-based methods. On-shell methods have been used to explore NLSMs, and more broadly the space of EFTs, in a number ways, see for example [17, 18, 96].

7. **Technical: conformal bootstrap**

This section gives more a technical account of the conformal bootstrap, offering some details that was left out in the introduction of the method given in section 5. We start with some CFT background, then discuss the bootstrap method.

7.1. **CFT background**

As described in section 5, an operator $O(x)$ is characterized by its spin $s$ (how it transforms under the Lorentz group) and its scaling dimension $\Delta$ introduced via the homogeneous scaling property (14). Unitarity enforces a lower bound on the possible value of $\Delta$ for a given spin $s$. For a scalar operator, $(s = 0)$ in $d$-dimensions, this bound is

\[
\Delta \geq d/2 - 1.
\]

The bound is exactly saturated for a free scalar field, which has $\Delta = d/2 - 1$.

The correlation functions $\langle O_1(x_1)O_2(x_2)\ldots \rangle$, i.e. the vacuum expectation values of strings of local operators at different space time locations $x_i$, have to respect Poincaré symmetry; in particular translation invariance means that they depend on the spacetime coordinates only via the differences have vanishing soft limits [18], despite the $\mathbb{CP}^1$ target space being symmetric. Non-vanishing soft theorems for non-symmetric target spaces were studied recently in [95].
$x_i^m \equiv (x_i - x_j)^m$. In particular, this means that a one-point function $\langle O(x) \rangle$ must be a constant. By dimensionality, when $\Delta > 0$, it therefore has to vanish in a CFT since a conformal theory has no dimensionless constants.

One can use scale invariance to prove that a two-point correlation function $\langle O_1(x_1)O_2(x_2) \rangle$ vanishes unless $\Delta_1 = \Delta_2$. Moreover, the form of the correlation function is completely fixed by symmetries and one can organize the operators such that the two-point correlation functions of scalar operators take the form

$$\langle O_i(x_i)O_j(x_j) \rangle = \frac{\delta_{ij}}{|x_{ij}|^{2\Delta}}, \quad (55)$$

where $|x_{ij}|^2 = x_i^mu_j^v$. This expression has the correct scaling behavior under (14).

Conformal invariance fixes the structure of all three-point correlation functions, in particular the correlation function of three scalar operators takes the form

$$\langle O_i(x_i)O_j(x_j)O_k(x_k) \rangle = \frac{\lambda_{ijk}}{|x_{ij}|^{2\Delta_i - 2\Delta_j - 2\Delta_k} |x_{jk}|^{2\Delta_j - 2\Delta_k - 2\Delta_i} |x_{ki}|^{2\Delta_k - 2\Delta_i - 2\Delta_j}}. \quad (56)$$

The unfixed constants $\lambda_{ijk}$ are real in a unitary theory.

Now it feels like we are on a good roll with conformal invariance determining almost everything for us, but the fun stops at three-point. Or maybe we should say that the fun begins at four-points? Starting with four-point correlation functions, one can build conformal cross-ratios like

$$u = \frac{|x_{12}|^2|x_{34}|^2}{|x_{13}|^2|x_{24}|^2} \quad \text{and} \quad v = \frac{|x_{14}|^2|x_{23}|^2}{|x_{13}|^2|x_{24}|^2}, \quad (57)$$

which are scale-invariant. One can also check that they are invariant under inversion, hence since conformal boosts are inversion–translation–inversion, they are also conformally invariant. As far as scale invariance is concerned, a four-point correlation function can depend in arbitrary ways on $u$ and $v$; for example for four identical scalar operators with scaling dimension $\Delta$, the expression

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4) \rangle = \frac{1}{|x_{12}|^{2\Delta}|x_{34}|^{2\Delta}} g(u,v) \quad (58)$$

has the correct scaling for any function $g$ of the conformal cross-ratios (57). We could also have exchanged 2 ↔ 4 and written

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4) \rangle = \frac{1}{|x_{14}|^{2\Delta}|x_{23}|^{2\Delta}} g(v,u). \quad (59)$$

Note that $u \leftrightarrow v$ under 2 ↔ 4, hence the exchange of the arguments in the function $g$. The two expressions (58) and (59) must give rise to the same correlation function, so the function $g$ is in fact not completely arbitrary but must obey

$$g(u,v) = \left( \frac{u}{v} \right)^{\Delta} g(v,u). \quad (60)$$

This constraint plays a role in the following.

Let us now dive into a little more depth with the OPE expansion than we did in section 5. When $x^\mu$ is close to $y^\nu$, the product $O_i(x)O_j(y)$ of two local operators create a local fluctuation and as such it should be possible to describe it by some linear combination of local operators. In particular, in the limit $x^\mu \to 0$, the OPE expansion is

$$\langle O_i(x)O_j(0) \rangle \sim \sum_n c_{ijn} \left( x, \frac{\partial}{\partial x} \right) \langle O_n(0) \rangle. \quad (61)$$

The OPE functions $c_{ij}(x)$ depend on $x$ and can in general be expected to be divergent as $x \to 0$. Applied to the three-point correlation function $\langle O_i(x)O_j(0)O_k(x_k) \rangle$ as $x \to 0$, the OPE (61) reduces it to a (sum of) simple two-point correlators (55). If we compare that result with the expression (56) with $x_j = 0$ and expand around $x_j = 0$, we find

$$c_{ijk} \left( x, \frac{\partial}{\partial x} \right) = \frac{\lambda_{ijk}}{|x|^{\Delta_i - \Delta_j - \Delta_k}} \left( 1 + \alpha x^\mu \frac{\partial}{\partial x^\mu} + \cdots \right) \quad (62)$$

where the dots stand for terms with subleading powers in small $x$ with coefficients (like $\alpha$) that depend on the operator scaling dimensions $\Delta_i$, $\Delta_j$, and $\Delta_k$. This means that the OPE can be written

$$\langle O_i(x)O_j(0)O_k(x_k) \rangle = \sum_n \lambda_{ijk} c_{ijn} \left( x_{ij}, \frac{\partial}{\partial x_{ij}} \right) \langle O_n(x_k) \rangle, \quad (63)$$

where the sum is over primary operators $O_n$. One can think of primary operators as the operators that cannot be obtained as derivatives (descendants) of other operators. Because of their appearance in (63), the constants $\lambda_{ijk}$ in the three-point correlation functions (56) are called OPE coefficients.

One can apply the OPE to obtain expressions for higher-point correlation functions in terms of the so-called CFT data:

- A list $\{\Delta_i, s_i\}$ of all the operator scaling dimensions $\Delta_i$ and spin representations $s_i$ of the local primary operators of the theory.
- A list of all OPE coefficients $\lambda_{ijk}$.

Not any list of $\{\Delta_i, s_i\}$ and OPE coefficients defines a CFT; there are certain constraints that must be obeyed. At the core, the conformal bootstrap is about how to impose such a constraint and the remarkable mileage gained from it.

72. Bootstrap
Suppose we apply the OPE (63) twice in a four-point correlation function to get

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4) \rangle$$

$$= \sum_{c^a, c^b} \lambda_{a2b} \lambda_{3ab} C_{12a}(x_{12}, \partial_2) C_{34b}(x_{34}, \partial_4) \times \langle O^a(x_2)O^b(x_4) \rangle. \quad (64)$$

The sum is over primary operators and $a, b$ are collective indices that include both operators and the index structure associated with their spin.

Here we made a choice to pair 1 and 2 in the OPE and 3 with 4, but obviously there are two other possible choices.
These three choices have to be equivalent since they are simply different representations of the same correlation function. The requirement of equivalence is very nontrivial and gives rise to the crossing relations, sometimes also called OPE associativity. Pictorially we can illustrate the crossing relations for the 12 pairing and the 14 pairing as

$$\sum_{\sigma} \lambda_{12\sigma} \lambda_{34\sigma} \left( \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \right) = \sum_{\sigma'} \lambda_{12\sigma'} \lambda_{34\sigma'} \left( \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \right).$$ (65)

It is not hard to see that if the four-point correlation functions satisfy crossing relations, then it is also true of the n-point functions. For the purpose of this presentation, we consider a CFT to be defined as a set of CFT data \( \{ \Delta_i, s_i \} \) and \( \lambda_{ijk} \)'s that satisfy the crossing relations for four-point correlators\(^{11}\).

As described in section 5, the idea of the conformal bootstrap is to take a set of CFT data of a putative CFT with specified symmetries and apply the constraints of the crossing relations to find out if such a CFT may exist.

Take all four operators in the four-point correlator to be identical scalar operators \( \mathcal{O} \) with scaling dimension \( \Delta \). The expression (64) can then be written

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4) \rangle = \frac{1}{|x_{12}|^{2\Delta}|x_{34}|^{2\Delta}} \sum_{\sigma} \lambda_{12\sigma}^2 g_{\Delta_{\sigma}}(u,v),$$ (66)

where we have introduced the conformal blocks

$$g_{\Delta_{\sigma}}(u,v) = |x_{12}|^{2\Delta}|x_{34}|^{2\Delta} C_{12\sigma}(x_{12},\partial_2) \times C_{34\sigma}(x_{34},\partial_4) \frac{Y_{ab}}{|x_{24}|^{2\Delta}}$$ (67)

using

$$\langle \mathcal{O}_a(x_2)\mathcal{O}_b(x_4) \rangle = Y_{ab} \frac{x_{34}}{|x_{24}|^{2\Delta}}$$ (68)

in which \( Y_{ab} \) captures the appropriate index structure for operators with nonvanishing spin and is simply \( \delta_{ab} \) for scalar operators. (For the case here, only operators with even spin appear in the OPE.) Comparing with (58), we see that the OPE has decomposed the function \( g(u,v) \) as

$$g(u,v) = \sum_{\sigma} \lambda_{12\sigma}^2 g_{\Delta_{\sigma}}(u,v).$$ (69)

This sum over primary operators is called the conformal block decomposition.

Now we saw already in the previous subsection that the different ways of decomposing the four-point correlation function required the function \( g \) to satisfy (60), i.e. \( v^{\Delta} g(u,v) - u^{\Delta} g(v,u) = 0 \). This means that the crossing relation becomes the mathematical requirement

$$\sum_{\sigma} \lambda_{12\sigma}^2 \left( v^\Delta g_{\Delta_{\sigma}}(u,v) - u^\Delta g_{\Delta_{\sigma}}(v,u) \right) = 0.$$ (70)

The conformal blocks \( g_{\Delta_{\sigma}}(u,v) \) are fixed by conformal symmetry, for example via a differential equation derived from the quadratic Casimir (or by series expansion or by recursion relations). In 2D and 4D, they can be expressed in terms of hypergeometric functions \( _2F_1 \). Hence, the functional form of \( F_{\Delta_{\sigma}}(u,v) \) is known, and the only unknown ingredients in (70) are the scaling dimensions and the OPE coefficients. Thus for given input CFT data, the crossing relations (70) is a consistency condition. It is the central workhorse of the conformal bootstrap.

We mentioned in (20) that the bootstrap equation (70) has a geometric interpretation: for a given list of vectors \( \vec{v}_{\text{set}} \), which we now identify as \( F_{\Delta_{\sigma}}(u,v) \), does there exist real non-negative coefficients\(^{12}\) \( \lambda_{12\sigma}^2 \) such that the linear combination (70) vanishes? If one can show that for given assumptions on the conformal dimensions, all \( F_{\Delta_{\sigma}}(u,v) \)'s lie on one side of some plane then no CFT can exist with that data because it could never satisfy the crossing relations.

The basic approach in the (numerical) bootstrap can be described algorithmically:

- Make assumptions about the spectrum of the lowest dimension operators of a putative CFT: their scaling dimensions and spin.
- Test the crossing relation. Numerically, this can be done by searching for a functional \( \alpha \) such that \( \alpha(F_{\Delta_{\sigma}}(u,v)) > 0 \).

If such an \( \alpha \) exists, then the crossing relations can never be satisfied and no such CFT can exist. This means that the bootstrap algorithm is especially powerful for ruling theories out. If no \( \alpha \) is found, it does not prove that a theory exists, it is at best a ‘maybe’. Scanning through the space of possible scaling dimensions of the lowest dimension operators numerically using semidefinite programming techniques has resulted in powerful bounds on existence of CFTs in diverse dimensions, as we described for the 3D Ising model and other applications in section 5. We finish with one other example.

73. Example: infinite number of primary operators

As an example (borrowed from [72, 97]), see also the review [89]) of analytic bootstrap, we ask if there exist any CFTs with a finite number of primary operators?\(^{13}\) To answer this question, introduce complex variables \( z \) and \( \bar{z} \) such that the conformal cross-ratios (57) are

$$u = \bar{z}z \quad \text{and} \quad v = (1 - z)(1 - \bar{z}).$$ (71)

\(^{11}\) Other constraints may also be useful or needed, such as modular invariance in 2D or constraints arising from supersymmetry or the presence of boundaries, interfaces, and line operators [89].

\(^{12}\) The OPE coefficients were called \( c_{ijk} \) in section 5 because we glossed over the dependence on the spacetime coordinates.

\(^{13}\) We will not discuss 2D CFTs and their larger Virasoro algebra here.
Using the representation of the conformal blocks in terms of hypergeometric functions one can show that in the limit $z \to 0$, taken along the line $\bar{z} = z$, one has

\[ g_{\Delta_a \ell}(u, v) \to z^{\Delta_a} + \cdots \quad \text{and} \quad g_{\Delta_a \ell}(v, u) \to \log(z) + \cdots. \]  

(72)

This means that $g(u, v)$ is dominated by the operator with the smallest scaling dimension, namely the identity operator which has $\Delta_a = 0$, so that

\[ g(u, v) = \lambda^2_{O_1} \cdot 1 + \cdots. \]  

(73)

On the other hand, as $z \to 0$ along the line $\bar{z} = z$, the other side of the crossing relation (60) behaves as

\[ \left( \frac{u}{v} \right)^\Delta g(v, u) \to z^\Delta \sum \lambda^2_{O_i} \log(z) + \cdots \]  

(74)

Since $\Delta > 0$, this vanishes unless there are infinitely many terms in the sum. So in order for crossing to hold in the form (60), there must be an infinite number of primary operators in any CFT.

8. Concluding remarks

The amplitudes program and the conformal bootstrap share a common ‘philosophy’: at the center of the explorations are the physical observables, the on-shell amplitudes and the correlation functions, respectively. The overlap goes beyond that, for example, in the increasing use of common tools. There are also differences in that the conformal bootstrap is inherently nonperturbative in nature whereas much of the implementation of the amplitude bootstrap of field theories described here is perturbative; however, bridging these differences is one of the directions for future developments. Joint workshops, like the two summer programs on the conformal bootstrap and amplitudes in 2015 and 2019 at the Aspen Center for Physics contribute to the increased communication between the communities of researchers. And that will hopefully continue in the future.

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No new data were created or analysed in this study.

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