Error Constants for the Semi-Discrete Galerkin Approximation of the Linear Heat Equation

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Abstract
In this paper, we propose \(L^2(J; H^1_0(\Omega))\) and \(L^2(J; L^2(\Omega))\) norm error estimates that provide the explicit values of the error constants for the semi-discrete Galerkin approximation of the linear heat equation. The derivation of these error estimates shows the convergence of the approximation to the weak solution of the linear heat equation. Furthermore, explicit values of the error constants for these estimates play an important role in the computer-assisted existential proofs of solutions to semi-linear parabolic partial differential equations. In particular, the constants provided in this paper are better than the existing constants and, in a sense, the best possible.

Keywords
semi-discrete Galerkin approximation · Error constant · A priori error estimate · Best possible

Mathematics Subject Classification 65N15 · 65N30 · 35K05

1 Introduction

In this paper, we propose norm error estimates that provide explicit values of error constants for the semi-discrete Galerkin approximation of the linear heat equation.

Let \(\Omega \subset \mathbb{R}^N (N \in \mathbb{N})\) be a bounded Lipschitz domain. \(L^2(\Omega)\) denotes the real Hilbert space endowed with inner product \((u, v)_{L^2(\Omega)} := \int_{\Omega} u(x)v(x)dx\) and norm \(\|u\|_{L^2(\Omega)} := \sqrt{(u, u)_{L^2(\Omega)}}\) for \(u, v \in L^2(\Omega)\). The real Hilbert space \(H^1_0(\Omega)\) is endowed with inner product

\[ (u, v)_{H^1_0(\Omega)} := \int_{\Omega} u(x)v(x)dx + \int_{\partial \Omega} n \cdot \nabla u(x)v(x)ds \]
In particular, explicit values of function \( u \in W \) in \( H^1_0(\Omega) \) are dependent on shapes of the domain \( \Omega \). Let \( H^{-1}(\Omega) \) be the dual space of \( H^1_0(\Omega) \), and \( \langle \cdot, \cdot \rangle \) be the real dual product between \( H^{-1}(\Omega) \) and \( H^1_0(\Omega) \). We identify \( u \in H^1_0(\Omega) \) with \( u \in L^2(\Omega) \) and with \( u \in H^{-1}(\Omega) \) based on the Gelfand triple \( H^1_0(\Omega) \subset L^2(\Omega) = L^2(\Omega)^* \subset H^{-1}(\Omega) \) (all inclusions are dense with continuous injections), where \( L^2(\Omega)^* \) denotes the dual space of \( L^2(\Omega) \). Let \( A : H^1_0(\Omega) \rightarrow H^{-1}(\Omega) \) be defined by

\[
\langle Au, v \rangle = a(u, v) \quad \forall v \in H^1_0(\Omega).
\]

We also define as \( W = \{ u \in H^1_0(\Omega) \mid Au \in L^2(\Omega) \} \), where the regularities of the functions in \( W \) are dependent on shapes of the domain \( \Omega \); (see e.g., [4]).

For parameter \( h > 0 \), the function space \( V_h \) denotes a finite-dimensional subspace of \( H^1_0(\Omega) \). We define the Ritz projection \( R_h : H^1_0(\Omega) \rightarrow V_h \) as

\[
a(u - R_hu, vh) = 0 \quad \forall vh \in V_h.
\]

Assume that the constant \( C_h \) satisfies

\[
\|u - R_hu\|_{H^1_0(\Omega)} \leq C_h \|Au\|_{L^2(\Omega)} \quad \forall u \in W,
\]

where \( C_h \rightarrow 0 \) as \( h \rightarrow 0 \). Then, Aubin-Nitsche’s trick implies

\[
\|u - R_hu\|_{L^2(\Omega)} \leq C_h \|u - R_hu\|_{H^1_0(\Omega)} \quad \forall u \in H^1_0(\Omega).
\]

The estimates (2) and (3) derive very meaningful inequalities for the numerical analysis of elliptic partial differential equations (PDEs); (see e.g., [1]). In particular, explicit values of \( C_h \) play an important role in computer-assisted existential proofs of solutions to elliptic PDEs; (see e.g., [12]). Therefore, many estimates for obtaining the values have been proposed and applied to computer-assisted existential proofs of solutions to semi-linear elliptic PDEs; (see e.g., [6–8, 10, 11, 15, 17] and references therein).

In this paper, we propose two norm error estimates, which provide the best possible error constants using only \( C_h \) in (2) for the semi-discrete Galerkin approximation of the linear heat equation. Let \( J = (t_0, t_1) \) \((0 \leq t_0 < t_1 < \infty)\). For any function \( v : J \times \Omega \rightarrow \mathbb{R} \), we introduce the shortened form \( v(t) := v(t, \cdot) \) and \( \partial_t v(t) := (\partial_t v)(t, \cdot) \), where \( \partial_t \) denotes the weak derivative for \( t \in J \). For any real Hilbert space \( Y \), \( L^2(J; Y) \) is defined by the function space of Lebesgue integrable functions \( J \ni t \mapsto v(t) \in Y \) endowed with the norm \( \|v\|_{L^2(J; Y)} := \sqrt{\int_J \|v(s)\|^2_Y ds} \) for \( v \in L^2(J; Y) \). Let \( H^1(J; Y) \) denote the set of weak differentiable functions for \( J \) endowed with the norm \( \|v\|_{H^1(J; Y)} = \sqrt{\int_J \left( \|v(s)\|^2_Y + \|\partial_t v(s)\|^2_Y \right) ds} \) for \( v \in H^1(J; Y) \). The function space \( C^0([t_0, t_1]; L^2(\Omega)) \) is defined by the set of continuous functions as \( \{ v \mid t \mapsto v(t) \in L^2(\Omega) \} \). Let \( Z := H^1(J; H^{-1}(\Omega)) \cap L^2(J; H^1_0(\Omega)) \) be endowed with the norm \( \|v\|_Z = \sqrt{\|v\|^2_{H^1(J; H^{-1}(\Omega))} + \|v\|^2_{L^2(J; H^1_0(\Omega))}} \). Let \( w_0 \in L^2(\Omega) \) and \( f \in L^2(J; H^{-1}(\Omega)) \). We define the weak solution as the function \( w \in Z \) that satisfies the linear heat equation:

\[
\begin{cases}
\langle \partial_t w(t), v \rangle + a(w(t), v) = \langle f(t), v \rangle & \forall v \in H^1_0(\Omega), \text{ a.e. } t \in J \\
w(t_0) = w_0.
\end{cases}
\]
Let $V_{J,h} := H^1(J; V_h)$. We define the semi-discrete Galerkin approximation of (4) as the function $w_h \in V_{J,h}$ that satisfies
\begin{align}
\begin{cases}
\langle \partial_t w_h(t), v_h \rangle + a(w_h(t), v_h) = \langle f(t), v_h \rangle \quad \forall v_h \in V_h, \ a.e. \ t \in J \\
w_h(t_0) = \hat{w}_0,
\end{cases}
\end{align}
(5)
where $\hat{w}_0 \in V_h$ is any approximation of $w_0$ in (4). The error estimates for the semi-discrete Galerkin approximation have been proposed in, for example, $L^2(\Omega), H^1(\Omega), L^\infty(\Omega), L^2(J; H^1_0(\Omega))$, and $L^2(J; L^2(\Omega))$ norms; (see e.g., [16]). The regularities of $w_0$ and $f$ required for deriving the convergence of the semi-discrete Galerkin approximation $w_h$ to the weak solution $w$ have been studied. For instance, for $w_0 \in L^2(\Omega)$ and $f \in L^2(J; H^{-1}(\Omega))$, $\|w - w_h\|_Z \to 0$ as $h \to 0$ holds under some assumptions [2, Theorem 3.2 and 3.3]. In these studies, there is a case in which an $L^2(J; L^2(\Omega))$ norm error estimate of the form $\|w - w_h\|_{L^2(J; L^2(\Omega))} \leq E_h \|w - w_h\|_{L^2(J; H^1_0(\Omega))}$ is derived. The estimate of such a form is called the parabolic Aubin-Nitsche’s trick; (see e.g., [2, Theorem 3.5]).

By contrast, there are few results of studies for the explicit values of the error constants. Nakao et al. started pioneering studies with the constants and they have shown that for $w$ in (4) and $w_h$ in (5),
\begin{align}
\|w - w_h\|_{L^2(J; H^1_0(\Omega))} &\leq 2C_h \|f\|_{L^2(J; L^2(\Omega))} \\
\|w - w_h\|_{L^2(J; L^2(\Omega))} &\leq 4C_h \|w - w_h\|_{L^2(J; H^1_0(\Omega))},
\end{align}
(6) and (7)
where they assume that $t_0 = 0, w_0 = \hat{w}_0 = 0, f \in L^2(J; L^2(\Omega))$, and $\Omega$ is a bounded convex polygonal or polyhedral domain [14, Theorem 4, 5]. Furthermore, these estimates (6) and (7) have been applied to verified numerical computations for semi-linear parabolic PDEs [14]. Currently, following the estimates in (6) and (7), methods, which are related to verified numerical computations to semi-linear parabolic PDEs, have been proposed; (see e.g., [5,9,13] and references therein).

In this paper, we provide sharp $L^2(J; H^1_0(\Omega))$ and $L^2(J; L^2(\Omega))$ norm error estimates, which contribute to improving methods for computer-assisted proofs for semi-linear parabolic PDEs, assuming $w_0 \in L^2(\Omega), \hat{w}_0 \in V_h$, and a bounded Lipschitz domain $\Omega$. First, we derive an $L^2(J; H^1_0(\Omega))$ norm error estimate.

**Theorem 1** For $w$ and $w_h$ defined by (4) and (5) with $f \in L^2(J; L^2(\Omega))$, we have
\[\|w - w_h\|_{L^2(J; H^1_0(\Omega))} \leq \sqrt{\|w_0 - \hat{w}_0\|_{L^2(\Omega)}^2 + C_h^2 \|f\|_{L^2(J; L^2(\Omega))}^2 + \|w_0\|_{H^1_0(\Omega)}^2}.\]

**Corollary 1** follows immediately from Theorem 1 with $w_0 = \hat{w}_0 = 0$.

**Corollary 1** We use the same notation and assumptions as in Theorem 1 and assume that $w_0 = \hat{w}_0 = 0$ in (4) and (5). Then, we obtain
\[\|w - w_h\|_{L^2(J; H^1_0(\Omega))} \leq C_h \|f\|_{L^2(J; L^2(\Omega))}.\]

Next, we provide the parabolic Aubin-Nitsche’s trick as the following theorem:

**Theorem 2** For $w$ and $w_h$ defined by (4) and (5), we have
\[\|w - w_h\|_{L^2(J; L^2(\Omega))} \leq \sqrt{\|R_h A^{-1}(w_0 - \hat{w}_0)\|_{H^1_0(\Omega)}^2 + C_h^2 \|w - w_h\|_{L^2(J; H^1_0(\Omega))}^2}.\]
We define \( P_h : L^2(\Omega) \to V_h \) as
\[
(u - P_h u, v_h)_{L^2(\Omega)} = 0 \quad \forall v_h \in V_h.
\] (8)
Because \( R_h A^{-1} (w_0 - \hat{w}_0) = 0 \) when \( \hat{w}_0 = P_h w_0 \), Theorem 2 with \( \hat{w}_0 = P_h w_0 \) leads to Corollary 2.

**Corollary 2** We use the same notation and assumptions as in Theorem 2 and assume that \( \hat{w}_0 = P_h w_0 \) in (5). Then, we obtain
\[
\| w - w_h \|_{L^2(J;L^2(\Omega))} \leq C_h \| w - w_h \|_{L^2(J;H^1_0(\Omega))}.
\]
Assuming that \( t_0 = 0 \) and \( w_0 = \hat{w}_0 = 0 \), Corollaries 1 and 2 immediately yield sharper estimates than (6) and (7). Each of the constants derived by Corollaries 1 and 2 should be the best possible in the sense that we only use the error constant \( C_h \) for the Ritz projection in (2).

In this paper, we prove Theorem 1 in Sect. 2 and Theorem 2 in Sect. 3.

### 2 Proof of Theorem 1

We provide the proof of Theorem 1.

**Proof** For \( t \in J \), it follows from (5) with \( v_h = R_h (w - w_h)(t) \in V_h \) that
\[
\langle f(t) - Aw_h(t) - \partial_t w_h(t), R_h (w - w_h)(t) \rangle = \langle f(t), R_h (w - w_h)(t) \rangle - a(w_h(t), R_h (w - w_h)(t)) - \langle \partial_t w_h(t), R_h (w - w_h)(t) \rangle = 0.
\] (9)

From (4) with \( v = (w - w_h)(t) \),
\[
\langle \partial_t (w - w_h)(t), (w - w_h)(t) \rangle + a((w - w_h)(t), (w - w_h)(t)) = \langle f(t) - Aw_h(t) - \partial_t w_h(t), (w - w_h)(t) \rangle = \langle f(t) - Aw_h(t) - \partial_t w_h(t), (w - w_h)(t) \rangle = a(\partial_t (w - w_h)(t), (I - R_h)(w - w_h)(t)) = a((I - R_h)A^{-1}(f(t) - Aw_h(t) - \partial_t w_h(t)), (w - w_h)(t)) = a((I - R_h)A^{-1}(f(t) - \partial_t w_h(t)), (w - w_h)(t)),
\]
The equality (9) yields
\[
\langle \partial_t (w - w_h)(t), (w - w_h)(t) \rangle + a((w - w_h)(t), (w - w_h)(t)) = \langle f(t) - Aw_h(t) - \partial_t w_h(t), (w - w_h)(t) \rangle = a(\partial_t (w - w_h)(t), (I - R_h)(w - w_h)(t)) = a((I - R_h)A^{-1}(f(t) - Aw_h(t) - \partial_t w_h(t)), (w - w_h)(t)) = a((I - R_h)A^{-1}(f(t) - \partial_t w_h(t)), (w - w_h)(t)),
\]
where the last equality holds because \( (I - R_h)w_h(t) = 0 \) for \( w_h(t) \in V_h \). Because \( f(t) - \partial_t w_h(t) \in L^2(\Omega) \), it follows from (2) that
\[
\langle \partial_t (w - w_h)(t), (w - w_h)(t) \rangle + a((w - w_h)(t), (w - w_h)(t)) \leq \| (I - R_h)A^{-1}(f(t) - \partial_t w_h(t)) \|_{H^1_0(\Omega)} \| (w - w_h)(t) \|_{H^1_0(\Omega)} = C_h \| f(t) - \partial_t w_h(t) \|_{L^2(\Omega)} \| (w - w_h)(t) \|_{H^1_0(\Omega)}.
\] (10)
Note that \( w - w_h \in Z \subset C^0([t_0, t_1]; L^2(\Omega)) \) and

\[
\left( \frac{d k}{dt} \right)(t) = 2\langle \partial_t (w - w_h)(t), (w - w_h)(t) \rangle \quad t \in J
\]

are satisfied, where \( k(t) := \| (w - w_h)(t) \|_{L^2(\Omega)}^2 \); (see e.g., [3, Theorem 3 in Sect. 5.9]).

Integrating both sides of (10) on \( J \) yields,

\[
\frac{1}{2}\| (w - w_h)(t) \|_{L^2(\Omega)}^2 - \frac{1}{2}\| (w - w_h)(t_0) \|_{L^2(\Omega)}^2 + \| w - w_h \|_{L^2(J; H^1_0(\Omega))}^2
= \int_J \langle \partial_t (w - w_h)(s), (w - w_h)(s) \rangle ds + \int_J a((w - w_h)(s), (w - w_h)(s)) ds
\leq C_h \| f - \partial_t w_h \|_{L^2(J; L^2(\Omega))}^2 \| w - w_h \|_{L^2(J; H^1_0(\Omega))}^2.
\]

We consider an estimate of \( \| f - \partial_t w_h \|_{L^2(J; L^2(\Omega))}^2 \). Equation (5) with \( v_h = \partial_t w_h(t) \in V_h \) provides that

\[
\| \partial_t w_h(t) \|_{L^2(\Omega)}^2 + a(w_h(t), \partial_t w_h(t)) = (f(t), \partial_t w_h(t))_{L^2(\Omega)}
\]

holds. Integrating on \( J \) yields

\[
\| \partial_t w_h \|_{L^2(J; L^2(\Omega))}^2 + \int_J a(w_h(s), \partial_t w_h(s)) ds = \int_J (f(s), \partial_t w_h(s))_{L^2(\Omega)} ds.
\]

Because \( w_h \in H^1(J; V_h) \), we have

\[
\int_J a(w_h(s), \partial_s w_h(s)) ds
= \int_J \int_\Omega \nabla w_h(s, x) \cdot \nabla \partial_s w_h(s, x) dxds
= \int_J \frac{d}{ds} \left( \int_\Omega \nabla w_h(s, x)^2 dx \right) ds - \int_J \int_\Omega \nabla \partial_s w_h(s, x) \cdot \nabla w_h(s, x) dxds
= \int_J \left( \frac{dg}{ds} \right)(s) ds - \int_J a(w_h(s), \partial_s w_h(s)) ds,
\]

where \( g(t) := a(w_h(t), w_h(t)) = \| w_h(t) \|_{H^1_0(\Omega)}^2 \). Since \( w_h \in H^1(J; H^1_0(\Omega)) \subset C^0([t_0, t_1]; H^1_0(\Omega)) \); (see e.g., [3, Theorem 2 in Sect. 5.9]), we obtain

\[
\int_J a(w_h(s), \partial_s w_h(s)) ds = \frac{1}{2} \int_J \left( \frac{dg}{ds} \right)(s) ds
= \frac{1}{2} \left( \| w_h(t_1) \|_{H^1_0(\Omega)}^2 - \| w_h(t_0) \|_{H^1_0(\Omega)}^2 \right).
\]

From (12) and (13),

\[
\| f - \partial_t w_h \|_{L^2(J; L^2(\Omega))}^2
= \| f \|_{L^2(J; L^2(\Omega))}^2 - 2 \int_J (f(s), \partial_t w_h(s))_{L^2(\Omega)} ds + \| \partial_t w_h \|_{L^2(J; L^2(\Omega))}^2
= \| f \|_{L^2(J; L^2(\Omega))}^2 - 2 \| \partial_t w_h \|_{L^2(J; L^2(\Omega))}^2 - 2 \int_J a(w_h(s), \partial_s w_h(s)) ds + \| \partial_t w_h \|_{L^2(J; L^2(\Omega))}^2
= \| f \|_{L^2(J; L^2(\Omega))}^2 - \| w_h(t_1) \|_{H^1_0(\Omega)}^2 + \| w_h(t_0) \|_{H^1_0(\Omega)}^2 - \| \partial_t w_h \|_{L^2(J; L^2(\Omega))}^2
\leq \| f \|_{L^2(J; L^2(\Omega))}^2 + \| w_h(t_0) \|_{H^1_0(\Omega)}^2.
\]
It follows from (11), (14), and the additive geometric mean that
\[
\frac{1}{2} \|(w - w_h)(t_1)\|_{L^2(\Omega)}^2 + \|w - w_h\|_{L^2(J; H^1_0(\Omega))}^2 \\
\leq \frac{1}{2} \|(w - w_h)(t_0)\|_{L^2(\Omega)}^2 + C_h \sqrt{\|f\|_{L^2(J; L^2(\Omega))}^2 + \|w_h(t_0)\|_{H^1_0(\Omega)}^2} \|w - w_h\|_{L^2(J; H^1_0(\Omega))} \\
\leq \frac{1}{2} \|(w - w_h)(t_0)\|_{L^2(\Omega)}^2 + \frac{C_h^2}{2} \left( \|f\|_{L^2(J; L^2(\Omega))}^2 + \|w_h(t_0)\|_{H^1_0(\Omega)}^2 \right) \\
+ \frac{1}{2} \|w - w_h\|_{L^2(J; H^1_0(\Omega))}^2.
\]
Then,
\[
\|(w - w_h)(t_1)\|_{L^2(\Omega)}^2 + \|w - w_h\|_{L^2(J; H^1_0(\Omega))}^2 \\
\leq \|(w - w_h)(t_0)\|_{L^2(\Omega)}^2 + C_h \left( \|f\|_{L^2(J; L^2(\Omega))}^2 + \|w_h(t_0)\|_{H^1_0(\Omega)}^2 \right).
\]
Because \(w(t_0) = w_0\) and \(w_h(t_0) = \hat{w}_0\), this proof is complete. \(\square\)

3 Proof of Theorem 2

We provide notation and lemmas, that are used for proving Theorem 2. Because \(w - w_h \in Z \subset C^0([t_0, t_1]; L^2(\Omega))\), for \(t \in [t_0, t_1]\), we may define
\[
z(t) := A^{-1}(w - w_h)(t) \in H^1_0(\Omega), \quad z_h(t) := R_h A^{-1} (w - w_h)(t) \in V_h.
\]
We show Lemma 1, which is to be used to prove Theorem 2.

Lemma 1 The function \(z_h\) defined by (15) is in \(H^1(J; H^1_0(\Omega))\) and we have
\[
\partial_t z_h = R_h A^{-1} \partial_t (w - w_h).
\]

Proof We first verify that \(R_h A^{-1} \partial_t (w - w_h) \in L^2(J; H^1_0(\Omega))\). Since \(R_h A^{-1} : H^{-1}(\Omega) \to V_h\) is a bounded operator, we only have to show that \(\partial_t (w - w_h) \in L^2(J; H^{-1}(\Omega))\). We have \(\partial_t w_h \in L^2(J; V_h) \subset L^2(J; H^1_0(\Omega))\) because of \(w_h \in H^1(J; V_h)\). We can consider \(\partial_t w_h\) as \(\partial_t w_h \in L^2(J; H^{-1}(\Omega))\) and conclude that \(\partial_t (w - w_h) \in L^2(J; H^{-1}(\Omega))\). Therefore, we have \(R_h A^{-1} \partial_t (w - w_h) \in L^2(J; V_h) \subset L^2(J; H^1_0(\Omega))\).

Next, we show that \(z_h \in H^1(J; H^1_0(\Omega))\) and \(\partial_t z_h = R_h A^{-1} \partial_t (w - w_h)\). Let the function space \(C^\infty_0(J)\) be the set of infinitely differentiable functions with compact support on \(J\). For any \(\phi \in C^\infty_0(J)\), it follows that
\[
\int_J \partial_t z_h(s) \phi(s) ds = - \int_J z_h(s) \frac{d\phi}{ds}(s) ds \\
= - \int_J R_h A^{-1} (w - w_h)(s) \frac{d\phi}{ds}(s) ds \\
= - R_h A^{-1} \int_J (w - w_h)(s) \frac{d\phi}{ds}(s) ds.
\]
where the last equation is led by the boundedness of \( R_h A^{-1} : H^{-1}(\Omega) \rightarrow \mathcal{V}_h \); (see e.g., [18, Corollary 2 on Sect. 5 in Chapter V]). It follows from \( \partial_t (w - w_h) \in L^2(J; H^{-1}(\Omega)) \) and the boundedness of \( R_h A^{-1} : H^{-1}(\Omega) \rightarrow \mathcal{V}_h \) that

\[
\int J \partial_s z_h (s) \phi(s) ds = R_h A^{-1} \int J \partial_s (w - w_h)(s) \phi(s) ds
\]

\[
= \int J R_h A^{-1} \partial_s (w - w_h)(s) \phi(s) ds.
\]

Since \( R_h A^{-1} \partial_t (w - w_h) \in L^2(J; H^1_0(\Omega)) \), we have \( z_h \in H^1(J; H^1_0(\Omega)) \) and \( \partial_t z_h = R_h A^{-1} \partial_t (w - w_h) \).

\[ \square \]

Now, we prove Theorem 2.

**Proof** For \( t > t_0 \), substituting \( v = z_h(t) \) into (4) and \( v_h = z_h(t) \) in (5) yields

\[
\langle \partial_t (w - w_h)(t), z_h(t) \rangle + a((w - w_h)(t), z_h(t)) = \langle \partial_t w_h(t), z_h(t) \rangle - \langle \partial_t w_h(t), z_h(t) \rangle + a(w_h(t), z_h(t))
\]

\[
= \langle f(t), z_h(t) \rangle - \langle f(t), z_h(t) \rangle = 0.
\]

(16)

Because the bilinear form \( a \) is symmetric, it follows from (16) that for \( t > t_0 \),

\[
\| w(t) - w_h(t) \|_{L^2(\Omega)}^2 = \langle (w - w_h)(t), (w - w_h)(t) \rangle_{L^2(\Omega)}
\]

\[
= a(z(t), (w - w_h)(t))
\]

\[
= a((z - z_h)(t), (w - w_h)(t)) + a(z_h(t), (w - w_h)(t))
\]

\[
= a((z - z_h)(t), (w - w_h)(t)) + a((w - w_h)(t), z_h(t))
\]

\[
= a((z - z_h)(t), (w - w_h)(t)) - \langle \partial_t (w - w_h)(t), z_h(t) \rangle
\]

\[
= a((z - z_h)(t), (w - w_h)(t)) - a(R_h A^{-1} \partial_t (w - w_h)(t), z_h(t)).
\]

Because \( R_h A^{-1} \partial_t (w - w_h) = \partial_t z_h \) holds from Lemma 1, we obtain

\[
\| w(t) - w_h(t) \|_{L^2(\Omega)}^2 = a((z - z_h)(t), (w - w_h)(t)) - a(R_h A^{-1} \partial_t (w - w_h)(t), z_h(t))
\]

\[
= a((z - z_h)(t), (w - w_h)(t)) - a(\partial_t z_h(t), z_h(t)).
\]

(17)

Integrating both sides of (17) for \( t \in J \), we obtain

\[
\| w - w_h \|_{L^2(J; L^2(\Omega))}^2
\]

\[
= \int J a((z - z_h)(s), (w - w_h)(s)) ds - \int J a(\partial_s z_h(s), z_h(s)) ds
\]

\[
\leq \int J a((z - z_h)(s), (w - w_h)(s)) ds - \int J a(\partial_s z_h(s), z_h(s)) ds
\]

\[
\leq \int J a((z - z_h)(s), (w - w_h)(s)) ds - \int J a(\partial_s z_h(s), z_h(s)) ds
\]

\[
\leq \sqrt{\int J \| (z - z_h)(s) \|_{H^1_0(\Omega)}^2 ds} \| w - w_h \|_{L^2(J; L^2(\Omega))} - \int J a(\partial_s z_h(s), z_h(s)) ds
\]

\[ \square \]
\[
\begin{align*}
&= \sqrt{\int_{J} \| (I - R_h) A^{-1} (w - w_h)(s) \|^2_{H^1_0(\Omega)} ds \| w - w_h \|^2_{L^2(J; H^1_0(\Omega))} \\
&\quad - \int_{J} a(\partial_s z_h(s), z_h(s)) ds \\
&\leq C_h \| w - w_h \|^2_{L^2(J; L^2(\Omega))} - \int_{J} a(\partial_s z_h(s), z_h(s)) ds,
\end{align*}
\]

where because \((w - w_h)(t) \in L^2(\Omega) \ (t \in [t_0, t_1])\), the last inequality follows from (2). It follows from (13), where \(w_h\) is replaced by \(z_h\), that
\[
\begin{align*}
\| w - w_h \|^2_{L^2(J; L^2(\Omega))} &\leq C_h \| w - w_h \|^2_{L^2(J; L^2(\Omega))} - \int_{J} a(\partial_s z_h(s), z_h(s)) ds, \\
\end{align*}
\]

where the last inequality follows from the additive geometric mean. Therefore, we have
\[
\begin{align*}
\| w - w_h \|^2_{L^2(J; L^2(\Omega))} \leq \| z_h(t_0) \|^2_{H^1_0(\Omega)} + C_h \| w - w_h \|^2_{L^2(J; H^1_0(\Omega))}.
\end{align*}
\]

\[\Box\]

4 Conclusion

We proposed \(L^2(J; H^1_0(\Omega))\) and \(L^2(J; L^2(\Omega))\) norm error estimates that provide explicit values of the error constants for the semi-discrete Galerkin approximation of the linear heat equation (4) in Theorems 1 and 2, respectively. Furthermore, we derived Corollaries 1 and 2 as special cases of Theorems 1 and 2, respectively. The estimates in Corollaries 1 and 2 are sharper than those given by Nakao et al. [14]. Moreover, we showed that these constants coincide with \(C_h\) in (2). From this fact we believe that our error estimates should be, in a sense, the best possible. Therefore, our results contribute to the theoretical and numerical basis for computer-assisted existential proofs of solutions to semi-linear parabolic PDEs.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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