How to Delay Death and Look Further Into the Future if You Fall Into a Black Hole*

Alexei Toporensky and Sergei Popov

In this note, we present a pedagogical illustration of peculiar properties of motion in the vicinity of and inside black holes. We discuss how a momentary impulse can modify the lifetime of an object radially falling into a Schwarzschild black hole down to singularity. The well-known upper limit for a proper time spent within a horizon, in fact, requires an infinitely powerful kick. We calculate the proper time interval (perceived as the personal lifetime of a falling observer) till the contact with the singularity, as well as the time interval in the Lemaître frame (which reflects how far into the future of the outer world a falling observer can look), for different values of the kick received by the falling body. We discuss the ideal strategy to increase both time intervals by an engine with a finite power. This example is suitable for university seminars for undergraduate students specializing in general relativity and related astrophysical subjects.

1. Introduction

It is well known that even simple processes involving black holes (BHs) can produce apparent paradoxes. Many of them are discussed in the literature; others still wait for a detailed explanation. Analysis of such interesting problems, which involve non-trivial aspects of general relativity (GR) effects, can be very useful in pedagogical practice. Illuminating examples of interesting phenomena with clear physical explanations can help students to understand BH properties and motivate them for further studies of GR.

*Vol.28, No.5, DOI: https://doi.org/10.1007/s12045-023-1602-8

Dr. Alexei Toporensky works at the Sternberg Astronomical Institute and delivers seminars at the HSE University, Moscow. His main research interests are in the cosmology of the early Universe and theories of gravity.

Prof. Sergei Popov works at the Sternberg Astronomical Institute and at the Abdus Salam International Center for Theoretical Physics. Main scientific interests are related to the evolution of compact objects, especially neutron stars. He also actively participates in popular science activities and public outreach.
In this note, we apply and develop some results from [1] to illustrate how a fall time into a Schwarzschild BH (till the singularity is reached) is modified by a momentary kick. We used this example in seminars for masters-level students at the Lomonosov Moscow State University and HSE University. Discussions during these seminars demonstrated that the chosen topic helps shed light on important aspects not well-understood by many students.

The task is to derive and discuss the necessary conditions to make the fall time as long as possible. For a Schwarzschild black hole, the ideal strategy for maximizing the proper time can be easily derived from equations of motion. The metric in static coordinates \((t, r, \phi, \theta)\) has the form (we set the units so that \(c = 1\))

\[
ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\omega^2,
\]

where \(f = 1 - r_g/r\), \(r_g\) is the Schwarzschild radius \((r_g = 2GM)\) and the angular part of the metric is \(d\omega^2 = d\theta^2 + d\phi^2 \sin^2 \theta\).

We write the equations of motion for a particle in the space-time defined by (1) in such a form that the proper time \(\tau\) is present explicitly (the angular coordinates are chosen so that the motion to occur within the plane \(\theta = \frac{\pi}{2}\)). We write the equations of motion for a particle in the space-time defined by (1) in such a form that the proper time \(\tau\) is present explicitly (the angular coordinates are chosen so that the motion to occur within the plane \(\theta = \frac{\pi}{2}\)). This form of equations of motion can be obtained using the existence of two integrals of motion: energy \(E\) (since the metric is static) and angular momentum \(L\) (since the metric does not depend on \(\phi\)), see [1] for details.

\[
\dot{r} = \pm Z, \quad (2)
\]

\[
\dot{\theta} = \frac{\varepsilon}{f}, \quad (3)
\]

\[
m^2 \dot{\phi} = \frac{L}{r^2}. \quad (4)
\]

In the equations above a dot denotes differentiation with respect to the proper time \(\tau\). We denote \(\varepsilon = E/m\) and

\[
Z = \sqrt{\varepsilon^2 - f (1 + \frac{L^2}{m^2 r^2})}. \quad (5)
\]
The ‘+’ sign in (2) indicates outward motion, while the ‘−’ sign corresponds to inward motion. Since we are interested in a motion under horizon, we consider only the latter possibility in this article.

From (2), it follows that a proper time required for a travel from the horizon $r_g$ to $r_1 < r_g$ is equal to

$$\tau = \int_{r_1}^{r_g} \frac{dr}{Z} = -\int_{r_1}^{r_g} \frac{dr}{\sqrt{\epsilon^2 - f \left(1 + \frac{L^2}{m^2 r^2}\right)}}. \quad (6)$$

For an object which is radially free falling into a black hole with zero velocity at infinity, we have $\epsilon = 1$ and $L = 0$, so the free-fall time from the Schwarzschild radius $r_g$ to singularity is:

$$\tau_{ff} = -\int_{r_1}^{0} \frac{dr}{\sqrt{r_g^2/r}} = \frac{2}{3} r_g. \quad (7)$$

Is it possible to increase this time for an object equipped with a rocket engine? The known answer is: ‘Yes, but not much’. Since under the horizon, both integrals of motion enter equations in the quadratic form with a positive sign, the integral in (6) has its maximum value for $\epsilon = 0$, $L = 0$. This maximum value is equal to $(\pi/2)r_g$ for $r_1 = 0$. So, the best strategy for maximizing the proper time under a horizon is to reach this optimal trajectory with two integrals of motion being equal to zero. Note that this trajectory is geodesic since the motion is two-dimensional, and the two above-mentioned integrals of motion represent a full set of integrals fixing a particular geodesic.

The fact that the lifetime of an object with a jet engine is maximal at a geodesic trajectory (i.e., at a trajectory which requires the engine to be turned-off after the ideal trajectory is reached) is, in some sense, a counter-intuitive statement. Sometimes it is claimed that this result easily follows from the fact that the interval $ds$ along a curve is just a proper time $d\tau$ of an observer following this curve. So as a geodesic maximises $s$, it also maximises $\tau$. However, as it has been already discussed in [2], this is correct only if we consider a motion between two fixed space-time points. Only under this condition a geodesic is unique and
maximises $\tau$. However, a singularity is not a space-time point! Its appearance differs for different coordinate systems, and for the Gullstrand–Painlevé coordinates, which we use below, it is simply the line $r = 0$. Hence, we can not apply the above argument directly. Indeed, both $\varepsilon = 1$ and $\varepsilon = 0$ trajectories are geodesics, but the latter works obviously better.

However, we will see soon that if the problem is considered in a bit more realistic situation—with an engine of finite power—the final answer appears to be closer to intuitive feeling.

2. **Maximization of the Proper Fall Time Using a Single-pulse Engine**

We consider a free-fall that is perturbed by a single engine thrust. It is assumed that this event momentarily gives a peculiar velocity $V_p$ to the falling body. Afterwards, the free-fall continues till the object reaches the singularity.

Velocity $V_p$ can be illustrated in the following simple way. The ship with an engine is free falling, and just near it along the whole trajectory till the engine is turned on, there is another free falling object. $V_p$ is their relative velocity immediately after the momentary engine thrust.

The thrust can be given just once. We consider only situations when the thrusting happens below the horizon (for a more realistic modelling of a rocket behaviour near a BH horizon, see, for example, [3]). The task is to figure out the best strategy to maximize the time (in the falling body frame), till the singularity is reached, for a given $V_p$. We assume $\varepsilon = 1$, which is approximately true for an object with a small kinetic energy compared to $mc^2$ at a large distance from the BH. We also assume a radial fall.

Kinematics of the motion with respect to the frame with $\varepsilon = 1$, $L = 0$ (usually called the Lemaître frame) have been considered recently in detail [4, 5]. The natural coordinate system associated with such a fall is the Lemaître one:

$$ ds^2 = -dt^2 + (r_g/r)d\rho^2 + r^2(\rho, \tilde{r})d\omega^2, $$

(8)
with timelike coordinate $\tilde{t}$ and spacelike coordinate $\rho$, so that the metric in these coordinates (in contrast to static coordinates $t$ and $r$) is regular at a horizon and everywhere else except for the singular point $r = 0$. The Lemaître time is defined as \( d\tilde{t} = dt + dr \sqrt{1 - f/f} \) and approximately coincides with the static time $t$ in the large distance limit where $f \sim 1$. On the contrary, its behavior is very different for small $r$, in particular, it is finite at a horizon and inside a horizon. The former static coordinate $r$ is now a function, which for the Schwarzschild metric, is known to be $r = r_g^{1/3}((3/2)(\rho - \tilde{t}))^{2/3}$. This system is a synchronous one, and the coordinate $\rho$ is constant during the free fall for particles with $\varepsilon = 1$ and $L = 0$. However, sometimes it is better to keep the coordinate $r$ and use the metric in the following form (Gullstrand–Painlevé metric):

\[
ds^2 = -d\tilde{t}^2 + (dr + v d\tilde{t})^2 + r^2 d\omega^2,
\]

(9)

with $v = \sqrt{1 - f}$.

This form of metric is not so common in the pedagogical literature, though the famous textbook [6] is based mostly on it, where it is called a ‘rain frame’. The price of using $r$ is the appearance of the off-diagonal term. The coefficient of this term is easily recognizable as a free fall velocity with respect to the stationary frame $v = \sqrt{r_g/r}$. Unlike a coordinate velocity, the velocity with respect to a frame has a direct physical meaning since it is by definition the velocity, which is measured by a local observer belonging to the considered frame. In particular, this velocity is always subluminal. Under a horizon a stationary frame does not exist anymore, so $v$ loses its direct physical meaning becoming superluminal. Note, that $v$ characterizes the ‘rain frame’ itself. A velocity of a particle with respect to this frame will be introduced in the next paragraph.

The metric (9) has rather interesting properties. We mention here that the proper distance between two points at the same radius measured at $\tilde{t} = \text{const}$ hypersurface is simply the difference in $r$ coordinate [7]. Moreover, $\tilde{t} = \text{const}$ sections are flat. This property is rather useful when we try to visualize the structure of a BH,
demonstrating together regions outside and inside the horizon in a single picture. What is also useful, is that the rate of change of the radial coordinate \( r \) of any object with respect to the Lemaître time \( \tilde{t} \) can be decomposed as

\[
\frac{dr}{d\tilde{t}} = \nu - v_p, \tag{10}
\]

where \( v_p \) is the velocity of an object with respect to the Lemaître frame. So that, the second term in the right-hand side has a local meaning, while two other terms do not. It is important, that this relation (which resembles the Galilean summation rule for classical velocities) is valid independently of the values of \( \nu \) and \( v_p \) everywhere, even under the horizon where \( \nu \) exceeds the speed of light. Since the left-hand side of this equation is a coordinate (i.e., not physical!) velocity, this does not contradict special relativity.

The same situation is known for cosmology where the velocity of a distant object (which is a non-local entity, and, so, is not bounded from above) is decomposed into a sum of the Hubble and peculiar velocities (see, for example, an excellent discussion in \([8]\)). To avoid confusion, we repeat that in our notation the value \( V_p \) indicates the velocity caused by the engine, and, so, represents its power capability (i.e., it is a constant in a given set of conditions), while \( v_p \) indicates a changing with time velocity of any object with respect to the Lemaître frame and so can be considered as a variable peculiar velocity.

We need to also take into account the fact that for a radially falling object with the velocity \( v_p \) with respect to the Lemaître frame, the integral of motion can be written as:

\[
\varepsilon = \frac{1 - \nu v_p}{\sqrt{1 - v_p^2}}. \tag{11}
\]

The equation above also determines the integral of motion \( \varepsilon \) of the spaceship after the engine have been used. The values of both \( \nu \) and \( v_p \) should be taken at the point of the thrust. Since we assume that initially the spaceship have followed trajectory with
\( \varepsilon = 1 \), being at rest with respect to the Lemaître frame, the value \( v_p \) immediately after the thrust equals \( V_p \).

Now we return to the question of the proper time. We know that the proper time spent inside the horizon is maximized for \( \varepsilon = 0 \). However, we see from the above equation that reaching such a trajectory is not an easy task! It can be done only well inside a horizon since \( \varepsilon = 0 \) corresponds to \( v_p = 1/v \), so that \( r \) should be smaller than \( r_g V_p^2 \) to accomplish this maneuver. In particular, it cannot be made exactly at a horizon since it requires an infinitely powerful engine (\( \varepsilon = 0 \) corresponds to \( v_p = 1 \) at a horizon), so that the well-known upper limit for a proper time inside a horizon equal to \( (\pi/2) r_g \) is an unreachable limit, indeed.

The natural goal for the captain of a spaceship would be to maximize the proper time since the ship’s fate is understood. The captain can decide to wait till \( r_c = r_g V_p^2 \) is reached and then switch to the \( \varepsilon = 0 \) trajectory. The price required to be paid is the time interval from the initial point (which we assume to be at the horizon) to the critical point, located inside the horizon. During this period, the ship falls along the trajectory with \( \varepsilon = 1 \), i.e., far from the optimal one. Instead, the captain could decide to turn the engine on immediately, reaching the minimal possible value of \( \varepsilon \), and then to fall along this still not optimized trajectory. Finally, the jet firing can be executed at some \( r_{on} \) in between \( r_g \) and \( r_g V_p^2 \). To understand the best strategy in this setting, we need to make corresponding numerical calculations.

We start calculations at the horizon. Then the time of the fall is written as a sum of two integrals:

\[
\tau = - \left( \int_{r_0}^{r_{on}} \frac{dr}{\sqrt{r_g/r}} + \int_{r_{on}}^{0} \frac{dr}{\sqrt{r_g/r + (\varepsilon^2 - 1)}} \right). \tag{12}
\]

In this exercise, we neglect tidal effects which would cause an actual death of an observer before reaching the singularity (for stellar mass BHs even before horizon crossing!). So, to be realistic we think about supermassive BHs where tidal effects close
**Figure 1.** Falling from the horizon. A radial coordinate at the horizontal axis corresponds to the point where the engine is turned on. The vertical axis gives the time of falling from the horizon to the center of the BH. Different curves correspond to different values of $V_p$.

To the horizon are negligible. Of course, tidal distortion is always strong near the singularity, but it is easy to demonstrate, see e.g. [6], that effects are terribly strong just for a fraction of a second, which is much shorter than the considered time of a free fall into a supermassive BH. Thus, we safely integrate down to $r = 0$.

We can see that the earlier the engine is activated, the longer the proper time is reached (see Figure 1). Hence, it is better not to wait until the optimal trajectory becomes available but to react to the situation immediately. If the spaceship captain recognizes the bad luck before the point $r = r_s V_p^2$, then the best strategy (i.e., switching on the engine for full power as early as possible) can be considered as matching our ‘general’ intuition. Only if the critical point has already passed, the captain should deliberately switch off the engine at the trajectory with $\varepsilon = 0$ despite some fuel being left over.

In principle, the curves in Figure 1 can be prolonged even further to the left, i.e., to the region that corresponds to a free fall from outside the horizon. Our calculations show that curves there are still monotonic: earlier the kick given, larger is the proper time of a free fall. However, a radial kick outside the horizon is not the best option since the angular momentum enters (12) with the negative sign for $r > r_s$. So, it is better to deflect the trajectory
than to slow down the free fall. That is why we do not show this part of curves in our *Figure* 1.

### 3. Maximization of the Lemaître Time

Though maximization of the proper time has its obvious motivation, this is not the unique reasonable strategy for a falling observer. The other option is to maximize the Lemaître time. The reason for this strategy is to observe as many events as possible in the ‘outer’ world. Since the Lemaître time coincides with the static time $t$ for infinitely large $r$, the larger the Lemaître time needed to reach the singularity, the larger part of the history of the Universe surrounding the black hole in question witnessed.

Properties of the Lemaître free fall time are quite different from those considered in the previous section. To see this, we can calculate the interval of $\tilde{t}$ for a radial trajectory with $\varepsilon = 0$. It is possible to obtain an analytical result using the fact that at such trajectory $v_p = -1/v$. Then the integral

$$\tilde{t} = \int_0^{r_1} \frac{dr}{v - v_p},$$

which is giving us the free fall time from $r = r_1$ to singularity, can be expressed through $v$ only. After substitution $v = \sqrt{r_g/r}$ we get:

$$\tilde{t} = r_g \ln \left( \frac{\sqrt{r_g} + \sqrt{r_1}}{\sqrt{r_g} - \sqrt{r_1}} \right) - 2 \sqrt{r_gr_1}. \quad (14)$$

The equation evidently diverges if $r_1 \to r_g$—there is no upper limit for the Lemaître free fall time! Again, this infinite limit is unreachable since it is impossible to switch to the trajectory with $\varepsilon = 0$ exactly at the horizon. Note, however, that the ability to see a very remote future is limited only by the power of the observer’s engine.

We stop here for a moment and remind the reader that the ability to see an infinitely remote future during a free fall into a black hole is one of the most popular misconceptions in black hole physics.
Figure 2. Fall time from the horizon in the Lemaître frame. Different curves correspond to different values of $V_p$.

physics, which is often considered while teaching GR (see, for example, [9]). While crossing a horizon along a geodesics cannot help witness the remote future, a powerful engine ignited at a horizon can! An infinitely remote future still cannot be seen, but more the engine power available, the longer is the period of history of the outer Universe observed during the fall.

Equation (13) tells us also that the larger the $V_p$, the larger the Lemaître free fall time, so the seemingly paradoxical situation that in some cases it is better to switch the engine off, never occurs while maximizing $\tilde{t}$. The fact that the working engine can increase the coordinate time while decreasing the proper time has already been mentioned in [2] where the Eddington–Finkelstein time has been considered. The unreachable upper limit for the Lemaître time evidently corresponds to $V_p = 1$, so it is given by the integral

$$\tilde{t}_{\max} = \int_0^{r_1} \frac{dr}{\sqrt{r_g/r} - 1}. \quad (15)$$

Note, that the maximum possible proper time is given by

$$\tau_{\max} = \int_0^{r_1} \frac{dr}{\sqrt{(r_g/r) - 1}}. \quad (16)$$

So for any $r_1$, the maximum Lemaître time is larger that the max-
imum proper time left to a singularity. For a finite power engine with $V_p < 1$ we need the evolution equation for $v_p$ which can be obtained by reversing (11):

$$v_p = \frac{v - \varepsilon Z}{v^2 + \varepsilon^2}.$$  \hspace{1cm} (17)

Substituting (17) into (13) we can obtain the Lemaître time in the same way as we got the proper time in the previous section. The numerical results plotted in Figure 2 show that in order to maximize $\ell$, we should (as for maximizing $\tau$) switch on the engine—the sooner the better. This again matches our intuitive feeling that it is better to start to struggle without any delay. Moreover, now this struggle should use all available fuel independently of the observer’s position.

4. Conclusion

Our teaching experience shows that problems like the one about maximization of the fall time serve as a good motivator for students. However, during our seminars, students almost never provide the standard (correct) answer to the question about the ideal strategy to maximize the proper time (‘to reach the optimal trajectory and turn off the engine!’) before this problem is analyzed at a blackboard in detail. Indeed, it can be hard to accept the counter-intuitive approach when the usage of the engine after some point might be considered an act of sabotage. Luckily, the detailed analysis (see above) shows that it is necessary to have an unrealistically powerful engine to reach the optimal trajectory close to the horizon. However, a realistic strategy for a finite power engine is more intuitive and students typically provide such an answer—to use all the power as soon as possible. In this note, we have provided a detailed consideration, that this is the best approach when the optimal trajectory cannot be reached and the falling body has not passed the critical point ($r = v_p^2$), yet.

In this note, we have also demonstrated that maximization of the proper and Lemaître time might be done following different pro-

Maximization of the proper and Lemaître time might be done following different procedures.
cedures. If we already have $r < r_g V_p^2$, then the curiosity to know the remote future (i.e., maximizing the Lemaître time) has its price in decreasing the proper lifetime. Note, however, that since the integral defining the Lemaître time diverges at a horizon, any maneuver deep inside the horizon cannot increase it significantly. For example, consider a situation when a spaceship with the engine giving at most $V_p = 0.9$ is located already at $r = 0.49r_g$. The proper time maximization strategy requires to use the boost only to $v_p = 0.7$, which gives the proper time equal to $0.275(r_g/c)$. So, using the full power, a hypothetical observer would sacrifice $0.024(r_g/c)$ of his own proper time, instead increasing the Lemaître time for about 23% from $0.335(r_g/c)$ to $0.413(r_g/c)$.

The considerations of the present article assume that the initial kinetic energy of the spaceship is small with respect to its rest energy, so we use the Lemaître frame to describe the motion. However, in our science-fiction setup, we can easily imagine a situation when this condition is not satisfied. To deal with such a situation we need to use the kinematic with respect to a general radial free-falling frame. We leave this question for future work.

Acknowledgement

We thank Dr. Oleg Zaslavsky from Karazin National University (Kharkiv, Ukraine) and Dr. Tarun Deep Saini from the Indian Institute of Science (Bangalore, India) for discussions and useful comments. We also thank the referee for notes and suggestions, which helped to improve the paper.

Sergei Popov was supported by the Ministry of Science and higher Education of the Russian Federation under the contract 075-15-2020-778 in the framework of the Large Scientific Projects Program within the national project ‘Science’.

References

[1] A. V. Toporensky and O. B. Zaslavskii. Strategies of motion under the black hole horizon. *International Journal of Modern Physics D*, 29(3):2030003, January 2020.
[2] Geraint F. Lewis and Juliana Kwan. No Way Back: Maximizing Survival Time Below the Schwarzschild Event Horizon. PASA, 24(2):46–52, June 2007.

[3] Yu. V. Pavlov and O. B. Zaslavskii. The Oberth effect and relativistic rocket in the Schwarzschild background. arXiv e-prints, page arXiv:2111.09240, November 2021.

[4] A. V. Toporensky and O. B. Zaslavskii. Zero-momentum trajectories inside a black hole and high energy particle collisions. J. Cosmol. Astropart. Phys., 2019(12):063, December 2019.

[5] A. V. Toporensky and O. B. Zaslavskii. Flow and Peculiar Velocities for Generic Motion in Spherically Symmetric Black Holes. Gravitation and Cosmology, 27(2):126–135, April 2021.

[6] Edwin F. Taylor and John Archibald Wheeler. Exploring Black Holes: Introduction to General Relativity. Addison Wesley Longman, New York, 2000.

[7] Ronald Gautreau and Banesh Hoffmann. The Schwarzschild radial coordinate as a measure of proper distance. Phys. Rev. D, 17(10):2552–2555, May 1978.

[8] Tamara M. Davis and Charles H. Lineweaver. Expanding Confusion: Common Misconceptions of Cosmological Horizons and the Superluminal Expansion of the Universe. PASA, 21(1):97–109, January 2004.

[9] Andrei A. Grib and Yuri V. Pavlov. Methodological Notes: Is it possible to see the infinite future of the Universe when falling into a black hole? Physics Uspekhi, 52(3):257–261, March 2009.

Address for Correspondence
Alexei Toporensky
Sergei Popov
Email: polar@sai.msu.ru
segepol@gmail.com
Sternberg Astronomical Institute
Universitetski pr. 13
Moscow, 119234, Russia.