SPECTRAL STRUCTURE OF THE NEUMANN–POINCARÉ OPERATOR ON THIN DOMAINS IN TWO DIMENSIONS*

By

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Abstract. We consider the spectral structure of the Neumann–Poincaré operators defined on the boundaries of thin domains of rectangular shape in two dimensions. We prove that as the aspect ratio of the domains tends to ∞, or equivalently, as the domains get thinner, the spectra of the Neumann–Poincaré operators are densely distributed in $[-1/2, 1/2]$, the interval which contains the spectrum of Neumann–Poincaré operators.

1 Introduction and statement of results

For a bounded domain $\Omega$ with the Lipschitz continuous boundary in $\mathbb{R}^2$, the Neumann–Poincaré (abbreviated by NP) operator is the boundary integral operator on $\partial \Omega$ defined by

$$K_{\partial \Omega} \varphi(x) = \frac{1}{2\pi} \int_{\partial \Omega} \frac{\langle y - x, \nu_y \rangle}{|x - y|^2} \varphi(y) ds(y), \quad x \in \partial \Omega,$$

where p.v. stands for the Cauchy principal value and $ds$ the line element on $\partial \Omega$.

This operator naturally appears when solving the classical Dirichlet or Neumann problems using layer potentials. The NP operator is also called the double layer potential.

Even though $K_{\partial \Omega}$ is not self-adjoint on $L^2(\partial \Omega)$ unless $\partial \Omega$ is a disk or a ball, it can be realized as a self-adjoint operator on $H^{1/2}(\partial \Omega)$ (the $L^2$-Sobolev space of order 1/2) using the Plemelj’s symmetrization principle (see [4]). Thus the spectrum of the NP operator on $H^{1/2}(\partial \Omega)$, which is denoted by $\sigma(K_{\partial \Omega})$, consists of the essential spectrum and the pure point spectrum [6].

It has been proved lately in [5] that if a two-dimensional domain $\Omega$ has corners on its boundary, then $K_{\partial \Omega}$ has essential spectrum which is a connected interval

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symmetric with respect to 0, and the end points of the interval are completely determined by the smallest angle of the corners. In particular, if $\Omega$ is a rectangle, then the essential spectrum is the interval $[-1/4, 1/4]$. It is known that $\sigma(K_{\partial\Omega}) \setminus \{1/2\}$ is a closed subset of $(-1/2, 1/2)$. In recent work [3], a classification method to distinguish eigenvalues from the essential spectrum has been proposed and implemented numerically to investigate the existence of eigenvalues on various domains with corners. The numerical experiments reveal that on rectangles, more and more eigenvalues of the NP operator appear outside the interval $[-1/4, 1/4]$ of the essential spectrum as the aspect ratio of the rectangle gets larger. It is also proved that if the aspect ratio is large enough, there is at least one eigenvalue outside $[-1/4, 1/4]$. In this paper we improve this result drastically and prove that the spectra actually fill up the whole interval $(-1/2, 1/2)$ in some sense as the aspect ratio gets larger.

The two-dimensional domains to be considered in this paper are not just rectangles. The long sides are lines, but the short sides do not have to be lines, they can be curves. Since the NP operator is dilation invariant, we define planar thin domains as follows: for $R \geq 1$, let $\Omega_R$ be a rectangle-shaped domain whose boundary consists of three parts, say

\begin{equation}
\partial\Omega_R = \Gamma^+_R \cup \Gamma^-_R \cup \Gamma^s_R,
\end{equation}

where the top and bottom are

\begin{equation}
\Gamma^+_R = [-R, R] \times \{1\}, \quad \Gamma^-_R = [-R, R] \times \{-1\},
\end{equation}

and the side $\Gamma^s_R$ consists of the left and right sides, namely, $\Gamma^s_R = \Gamma^l_R \cup \Gamma^r_R$, where $\Gamma^l_R$ and $\Gamma^r_R$ are curves connecting points $(\mp R, 1)$ and $(\mp R, -1)$, respectively. We assume that $\Gamma^l_R$ and $\Gamma^r_R$ are of any but fixed shape independent of $R$. In other words, $\Gamma^l_R$ and $\Gamma^r_R$ are of the form $\Gamma^l_R = \Gamma^l - (R, 0)$ and $\Gamma^r_R = \Gamma^r + (R, 0)$, where $\Gamma^l$ and $\Gamma^r$ are curves connecting points $(0, 1)$ and $(0, -1)$. If both $\Gamma^l$ and $\Gamma^r$ are line segments, $\Omega_R$ is a rectangle. The boundary $\partial\Omega_R$ is assumed to be Lipschitz continuous. We say that the domain $\Omega_R$ is of the aspect ratio $R$ even if it is not necessarily a rectangle. It is worthwhile to emphasize that $\partial\Omega_R$ is allowed to be smooth in which case the associated NP operator is compact and has eigenvalues accumulating to 0.

The following theorem is the main result of this paper.

**Theorem 1.1.** If $\{R_j\}$ be an increasing sequence such that $R_j \to \infty$ as $j \to \infty$, then

\begin{equation}
\bigcup_{j=1}^{\infty} \sigma(K_{\partial\Omega_{R_j}}) = [-1/2, 1/2].
\end{equation}
Theorem 1.1 shows that if \( \partial \Omega_R \) is smooth, then more and more eigenvalues of the NP operator \( \mathcal{K}_{\partial \Omega_R} \) are approaching \( \pm 1/2 \) as \( j \to \infty \), that is, as the domains get longer (or equivalently, thinner). It actually shows that the totality of eigenvalues of \( \mathcal{K}_{\partial \Omega_R} \) are densely distributed in \([-1/2, 1/2]\) as \( R_j \to \infty \). It is rather surprising that (1.4) holds regardless of the choice of \( \{R_j\} \). If \( \Omega_R \) are rectangles, then the totality of eigenvalues of \( \mathcal{K}_{\partial \Omega_R} \) is dense outside the essential spectrum, namely, in \([-1/2, -1/4] \cup [1/4, 1/2]\). In particular, there are infinitely many eigenvalues outside the essential spectrum.

The characteristic feature of the set \( \Omega_R \) is that it tends to the infinite strip \( \mathbb{R} \times [-1, 1] \) as \( R \to \infty \). Furthermore, the NP operator on \( \partial \Omega_R \) behaves like the Poisson integral. Since the Fourier transform of the Poisson kernel is \( e^{-2\pi|\xi|^2} \), the Poisson integral has \([0, 1]\) as its essential spectrum. This is the key observation in proving Theorem 1.1.

This paper is organized as follows. In the next section we relate the NP operator on thin domain with the Poisson integral on the half space. Theorem 1.1 is proved in section 3. This paper ends with a discussion on some related problems.

2 Poisson integral and construction of test functions

To motivate the construction of test functions in this section, we define

\[
\phi(x) = \phi(x_1, x_2) := \begin{cases} 
  f(x_1) & \text{if } x \in \Gamma_R^+ \cup \Gamma_R^- \\
  0 & \text{if } x \in \Gamma_R^c \end{cases},
\]

for a given compactly supported function \( f \) on \([-R, R]\). From here on, let us put \( \mathcal{K}_R = \mathcal{K}_{\partial \Omega_R} \) for simplicity of notation. Note that

\[
\mathcal{K}_R[\phi](x_1, x_2) = -\frac{1}{2\pi} \int_{\mathbb{R}^d} \frac{x_2 - 1}{(x_1 - y_1)^2 + (x_2 - 1)^2} f(y_1) dy_1 + \frac{1}{2\pi} \int_{\mathbb{R}^d} \frac{x_2 + 1}{(x_1 - y_1)^2 + (x_2 + 1)^2} f(y_1) dy_1.
\]

Thus, if \((x_1, x_2) \in \Gamma_R^+ \cup \Gamma_R^-\), namely, if \( x_2 = 1 \) or \(-1 \), then

\[
\mathcal{K}_R[\phi](x_1, x_2) = \frac{1}{2\pi} \int_{\mathbb{R}^d} \frac{2}{(x_1 - y_1)^2 + 2^2} f(y_1) dy_1 = \frac{1}{2} P_2 * f(x_1),
\]

where \( P_t \) is the Poisson kernel on the half space, namely,

\[
P_t(x_1) = \frac{1}{\pi} \frac{t}{|x_1|^2 + t^2}, \quad t > 0.
\]
The argument above compels us to look for a function (or a distribution) \( h \) such that
\[
\lambda h - \frac{1}{2} P_2 * h = 0
\]
for a given number \( \lambda \in (0, 1/2] \). Since \( \hat{P}_t(\xi) = \exp(-2\pi t|\xi|) \), it amounts to
\[
\left( \lambda - \frac{1}{2} e^{-4\pi|\xi|} \right) \hat{h}(\xi) = 0.
\]
Here, the Fourier transform is defined by
\[
\hat{f}(\xi) = \mathcal{F}[f](\xi) := \int_{\mathbb{R}^1} e^{-2\pi i \xi x} f(x) dx.
\]
Let \( \xi_0 > 0 \) be such that
\[
\lambda - \frac{1}{2} e^{-4\pi|\xi_0|} = 0.
\]
Such a point \( \xi_0 \) exists since \( \lambda \in (0, 1/2] \). The relation (2.6) can be satisfied only when \( \hat{h} \) is supported on the set \( \{ |\xi| = |\xi_0| \} \). Thus \( \hat{h}(\xi) = \delta_{\xi_0}(\xi) \), the Dirac-delta function at \( \xi_0 \), is a good candidate.

We now construct the desired test functions \( f_R \) to be used in the next section. Let \( \psi \) be a function such that \( \hat{\psi} \) is a non-negative compactly supported smooth function such that
\[
\int_{\mathbb{R}} \hat{\psi}(\xi) d\xi = 1.
\]
Then, \( R \hat{\psi}(R(\xi - \xi_0)) \) converges weakly to \( \delta_{\xi_0}(\xi) \), namely, \( R \int_{\mathbb{R}} \hat{\psi}(R(\xi - \xi_0)) p(\xi) d\xi \) tends to \( p(\xi_0) \) for any compactly supported smooth function \( p \), as \( R \to \infty \). Define \( g_R \) by
\[
\hat{g}_R(\xi) = R \hat{\psi}(R(\xi - \xi_0)).
\]
Then one can see easily that
\[
g_R(x) = e^{2\pi i \xi_0 x} \psi(R^{-1} x).
\]
Let \( \chi \) be a smooth cut-off function such that \( \text{supp}(\chi) \subset [-1/2, 1/2] \) and \( \chi = 1 \) on \( [-1/4, 1/4] \). Define
\[
f_R(x) := \chi(R^{-1} x) g_R(x) = e^{2\pi i \xi_0 x} (\chi \psi)(R^{-1} x).
\]
In what follows, \( \| \cdot \|_{1/2} \) denotes the Sobolev 1/2-norm on \( \mathbb{R}^1 \), and \( A \lesssim B \) means that there is a constant independent of \( R \) such that \( A \leq CB \).
**Lemma 2.1.** For $\lambda \in (0, 1/2]$, let $f_R$ be the function defined by (2.10) where $\xi_0$ satisfies (2.7). We have that

\[(2.11) \quad R^{1/2} \lesssim \|f_R\|_{1/2}\]

and

\[(2.12) \quad \|\lambda f_R - \frac{1}{2} P_2 \ast f_R\|_{1/2} \lesssim R^{-1/2}.\]

**Proof.** Note that

\[(2.13) \quad \hat{f}_R(\xi) = R(\hat{\chi_0})(R(\xi - \xi_0)).\]

Thus,

\[
\|f_R\|_{1/2}^2 = \int_{\mathbb{R}} (1 + |\xi|)|\hat{f}_R(\xi)|^2 d\xi
\]

\[
= R^2 \int_{\mathbb{R}} (1 + |\xi|)|\hat{\chi_0})(R(\xi - \xi_0))|^2 d\xi
\]

\[
= R \int_{\mathbb{R}^1} \left(1 + \left|\frac{\xi}{R} + \frac{\xi_0}{R}\right|\right)|\hat{\chi_0})(R(\xi - \xi_0))|^2 d\xi \geq R \int_{\mathbb{R}^1} |\hat{\chi_0})(R(\xi - \xi_0))|^2 d\xi.
\]

hence we have (2.11).

On the other hand,

\[
\mathcal{F} \left(\lambda f_R - \frac{1}{2} P_2 \ast f_R\right)(\xi) = \left(\lambda - \frac{1}{2} e^{-4\pi|\xi|} R(\hat{\chi_0})(R(\xi - \xi_0))\right).
\]

Thus,

\[
\left\|\lambda f_R - \frac{1}{2} P_2 \ast f_R\right\|_{1/2}^2 = R^2 \int_{\mathbb{R}^1} \left(1 + |\xi|\right)\left(\lambda - \frac{1}{2} e^{-4\pi|\xi|} R(\hat{\chi_0})(R(\xi - \xi_0))\right)^2 d\xi
\]

\[
= R \int_{\mathbb{R}^1} \left(1 + \left|\frac{\xi}{R} + \frac{\xi_0}{R}\right|\right)\left|\lambda - \frac{1}{2} e^{-4\pi|\xi|} R(\hat{\chi_0})(R(\xi - \xi_0))\right|^2 d\xi.
\]

Let

\[
\left\|\lambda f_R - \frac{1}{2} P_2 \ast f_R\right\|_{1/2}^2 = R \left(\int_{|\xi| \leq \sqrt{R}} + \int_{|\xi| > \sqrt{R}}\right) =: I + II.
\]

If $|\xi| \leq \sqrt{R}$, we have from (2.7)

\[
\left|\lambda - \frac{1}{2} e^{-4\pi|\xi|} R(\hat{\chi_0} + \hat{\xi_0})\right| \lesssim \frac{|\xi|}{R},
\]

and hence

\[
I = R \int_{|\xi| \leq \sqrt{R}} \left(1 + \left|\frac{\xi}{R} + \frac{\xi_0}{R}\right|\right)\left|\lambda - \frac{1}{2} e^{-4\pi|\xi|} R(\hat{\chi_0} + \hat{\xi_0})\right|^2 d\xi
\]

\[
\lesssim R^{-1} \int_{\mathbb{R}^1} |\xi|^2 |\hat{\chi_0})^2 d\xi \lesssim R^{-1}.
\]
To estimate $II$, we observe that since $\chi \psi$ is a compactly supported smooth function,
\[ |\hat{(\chi \psi)}(\xi)| \lesssim (1 + |\xi|)^{-N} \]
for any $N$. Thus, we have
\[ II \lesssim R \int_{|\xi| > \sqrt{R}} (1 + |\xi|)^{1-2N} d\xi \lesssim R^{1-N}, \]
and we arrive at (2.12).

3 Proof of Theorem 1.1

We prove the following proposition.

**Proposition 3.1.** Let $\lambda \in (0, 1/2]$. There is a sequence $\varphi_R \in H^{1/2}(\partial\Omega_R)$ such that
\[ \lim_{R \to \infty} \frac{\| (\lambda I - K_R) [\varphi_R] \|_{H^{1/2}(\partial\Omega_R)} \| \varphi_R \|_{H^{1/2}(\partial\Omega_R)} }{\| \varphi_R \|_{H^{1/2}(\partial\Omega_R)} } = 0. \]

Theorem 1.1 is an immediate consequence of Proposition 3.1. In fact, if $\lambda \in (0, 1/2]$, but $\lambda \notin \bigcup_{j=1}^{\infty} \sigma(K_{R_j})$, then
\[ \text{dist} \left( \lambda, \bigcup_{j=1}^{\infty} \sigma(K_{R_j}) \right) > 0. \]
Thus, there is a constant $C$ independent of $j$ such that
\[ \| \varphi \|_{H^{1/2}(\partial\Omega_R)} \leq C \| (\lambda I - K_R) [\varphi] \|_{H^{1/2}(\partial\Omega_R)} \]
for all $\varphi \in H^{1/2}(\partial\Omega_R)$. Therefore, the existence of the sequence $\{ \varphi_R \}$ satisfying (3.1) implies that
\[ \lambda \in \bigcup_{j=1}^{\infty} \sigma(K_{R_j}). \]
Thus, $[0, 1/2] \subset \bigcup_{j=1}^{\infty} \sigma(K_{R_j})$. Since the spectrum of the NP operator in two dimensions is symmetric with respect to 0, we have
\[ [-1/2, 1/2] \subset \bigcup_{j=1}^{\infty} \sigma(K_{R_j}). \]
Since $\sigma(K_{R_j}) \subset [-1/2, 1/2]$, we arrive at (1.4).
Proof of Proposition 3.1. With the function $f_R$ defined in the previous section, define $\varphi_R$ on $\partial \Omega_R$ by

\begin{align}
\varphi_R(x) = \varphi_R(x_1, x_2) := \begin{cases} 
f_R(x_1) & \text{if } x \in \Gamma_R^+ \cup \Gamma_R^-, \\
0 & \text{if } x \in \Gamma_R^s.
\end{cases}
\end{align}

Then, (2.11) yields

\begin{align}
R^{1/2} \lesssim \|\varphi_R\|_{H^{1/2}(\partial \Omega_R)}.
\end{align}

We show that

\begin{align}
\| (\lambda I - \mathcal{K}_R)[\varphi_R] \|_{H^{1/2}(\partial \Omega_R)} \lesssim 1.
\end{align}

These two estimates yield (3.1).

To prove (3.4), choose a constant $C > 0$ so that

\begin{align*}
\Gamma_R^l \subset \{(x_1, x_2) : x_1 < -R + C\} \quad \text{and} \quad \Gamma_R^r \subset \{(x_1, x_2) : x_1 > R - C\}.
\end{align*}

Note that we can choose such a constant independently of $R$ if $R$ is sufficiently large. Let $\zeta_1(x_1, x_2) = \zeta_1(x_1)$ be a smooth function supported in $(-R + C, R - C)$ such that $\zeta_1 = 1$ on $[-R + 2C, R - 2C]$ assuming that $R$ is sufficiently large, and let $\zeta_2 := 1 - \zeta_1$. Then we have

\begin{align*}
\| (\lambda I - \mathcal{K}_R)[\varphi_R] \|_{H^{1/2}(\partial \Omega_R)} \leq \sum_{j=1}^2 \| \zeta_j(\lambda I - \mathcal{K}_R)[\varphi_R] \|_{H^{1/2}(\partial \Omega_R)}.
\end{align*}

Thanks to (2.2), we see that if $x \in \Gamma_R^+ \cup \Gamma_R^-$, then

\begin{align}
\mathcal{K}_R[\varphi_R](x) = \frac{1}{2}(P_2 * f_R)(x_1).
\end{align}

Thus, we have

\begin{align*}
\zeta_1(\lambda I - \mathcal{K}_R)[\varphi_R](x) = \zeta_1(x_1)\left(\lambda f_R(x_1) - \frac{1}{2}(P_2 * f_R)(x_1)\right).
\end{align*}

It then follows from (2.12) that

\begin{align}
\| \zeta_1(\lambda I - \mathcal{K}_R)[\varphi_R] \|_{H^{1/2}(\partial \Omega_R)} \lesssim R^{-1/2}.
\end{align}

We now estimate $\| \zeta_2(\lambda I - \mathcal{K}_R)[\varphi_R] \|_{H^{1/2}(\partial \Omega_R)}$. Let

\begin{align*}
\Gamma := \partial \Omega_R \cap \{(x_1, x_2) : x_1 < -R + C \text{ or } x_1 > R - C\}.
\end{align*}
Then the shape of \( \Gamma \) is independent of \( R \). Note that
\[
\zeta_2(\lambda I - K_R)[\phi_R] = \zeta_2 K_R[\phi_R]
\]
and
\[
\| \zeta_2(\lambda I - K_R)[\phi_R] \|_{H^{1/2}(\partial \Omega_R)} = \| \zeta_2 K_R[\phi_R] \|_{H^{1/2}(\Gamma)}.
\]
We use the following characterization of the space \( H^{1/2}(\Gamma) \) (see, e.g., [2]):
\[
\| h \|_{H^{1/2}(\Gamma)}^2 = \| h \|_{L^2(\Gamma)}^2 + \int_{\Gamma} \int_{\Gamma} \frac{|h(x) - h(z)|^2}{|x - z|^2} d\sigma(x) d\sigma(z).
\]
Let \( k_R(x, y) \) be the integral kernel of \( K_R \), namely,
\[
k_R(x, y) = \frac{1}{2\pi} \frac{\langle y - x, v_y \rangle}{|y - x|^2}.
\]
If \( x, z \in \text{supp}(\zeta_2) \) and \( y \in \text{supp}(\phi_R) \), then
\[
|x - z| \lesssim 1, \quad |x - y| \gtrsim R, \quad |z - y| \gtrsim R.
\]
Thus, \( |k_R(x, y)| \lesssim R^{-1} \). It therefore follows from (2.10) that
\[
\| \zeta_2 K_{\partial \Omega_R}[\phi_R] \|_{L^2(\Gamma)} \lesssim R^{-1} \int_{\mathbb{R}^1} |f_R(x_1)| dx_1 \lesssim \int_{\mathbb{R}^1} |(\chi \psi)(x_1)| dx_1 \lesssim 1.
\]
We also have
\[
|\zeta_2(x)k_R(x, y) - \zeta_2(z)k_R(z, y)| \lesssim R^{-1}|x - z| + R^{-2}|x - z| \leq R^{-1}|x - z|.
\]
Thus we have
\[
|\zeta_2(x)K_R[\phi_R](x) - \zeta_2(z)K_R[\phi_R](z)| \lesssim R^{-1}|x - z| \int_{\mathbb{R}^1} |f_R(x_1)| dx_1 \lesssim |x - z| \int_{\mathbb{R}^1} |(\chi \psi)(x')| dx' \lesssim |x - z|.
\]
It then follows that
\[
\int_{\Gamma} \int_{\Gamma} \frac{|\zeta_2(x)K_R[\phi_R](x) - \zeta_2(z)K_R[\phi_R](z)|^2}{|x - z|^2} d\sigma(x) d\sigma(z) \lesssim 1,
\]
which together with (3.8) implies
\[
\| \zeta_2(\lambda I - K_R)[\phi_R] \|_{H^{1/2}(\partial \Omega_R)} \lesssim 1.
\]
Thus, (4.4) follows. This completes the proof.
Discussion

The spectral property (1.4) on thin domains of rectangular shape is shared by NP operators on thin ellipses. In fact, if $E_j$, $j = 1, 2, \ldots$, is the ellipse defined by $x_1^2/a_j^2 + x_2^2/b_j^2 < 1$ and $\mathcal{K}_{\partial E_j}$ is the corresponding NP operator, where $a_j$ and $b_j$ are positive numbers such that $b_j < a_j$ for all $j$ and $b_j/a_j \to 0$ as $j \to \infty$, then

$$(3.9) \quad \bigcup_{j=1}^{\infty} \sigma(\mathcal{K}_{\partial E_j}) = [-1/2, 1/2].$$

We present a short proof of this fact below for the readers’ sake. It would be quite interesting to characterize the geometric properties of the family of thin domains which guarantee the spectral properties like (1.4) and (3.9).

There are at least two different kinds of thin domains in three dimensions: thin plate-like domains and thin cylinder-like domains. In the first case, it can be proved by modifying the proof of this paper that the same spectral property of the NP operators holds. However, it seems that the NP operators on the second kind of thin domains exhibit a completely different spectral structure. This investigation is in progress and the outcome will be reported in a forthcoming paper.

Let us now prove (3.9). It is known that the spectrum $\sigma(\mathcal{K}_{\partial E_j})$ of the NP operator on $E_j$ is $\{\pm \frac{1}{2} r_j^n : n = 1, 2, \ldots\}$, where

$$r_j = \frac{a_j - b_j}{a_j + b_j}$$

(see [1]). Thus,

$$\bigcup_{j=1}^{\infty} \sigma(\mathcal{K}_{\partial E_j}) = \left\{ \pm \frac{1}{2} r_j^n : j = 1, 2, \ldots, n = 1, 2, \ldots \right\}.$$

Let $t_j := -\log r_j$, and consider the set $S := \{nt_j : j = 1, 2, \ldots, n = 1, 2, \ldots\}$. Note that $t_j > 0$ and $t_j \to 0$ as $j \to \infty$ since $r_j \to 1$. One can easily show that for any pair of real numbers $0 \leq p < q$, there are $n$ and $j$ such that $p < nt_j < q$. Thus, $S$ is dense in $[0, \infty)$. We then infer that $\{r_j^n : j = 1, 2, \ldots, n = 1, 2, \ldots\}$ is dense in $[0, 1]$. So, (3.9) follows.

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