Discovering Domain Orders through Order Dependencies

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ABSTRACT

Much real-world data come with explicitly defined domain orders; e.g., lexicographic order for strings, numeric for integers, and chronological for time. Our goal is to discover implicit domain orders that we do not already know; for instance, that the order of months in the Chinese Lunar calendar is Corner ≺ Apricot ≺ Peach. To do so, we enhance data profiling methods by discovering implicit domain orders in data through order dependencies. We enumerate tractable special cases and proceed towards the most general case, which we prove is NP-complete. We show that the general case nevertheless can be effectively handled by a SAT solver. We also devise an interesting measure to rank the discovered implicit domain orders, which we validate with a user study. Based on an extensive suite of experiments with real-world data, we establish the efficacy of our algorithms, and the utility of the domain orders discovered by demonstrating significant added value in three applications (data profiling, query optimization, and data mining).

1 INTRODUCTION

Much real-world data come with explicitly defined orders; e.g., lexicographic for strings, numeric for integers and floats, and chronological for time. Our goal is to go a step further to discover potential domain orders that are not already known. We call these 
implicit orders. Consider Table 1, describing festivals in various countries. The timestamp column has an explicit chronological order. Given this explicit order, we show how to discover the implicit order of months in the Gregorian calendar, monthGreg (January ≺ February ≺ March, etc). Moreover, we will show how to find implicit orders based on other implicit orders. For instance, given the implicit order on monthGreg, we can find the implicit order of months in the traditional Chinese (Lunar) calendar, monthLun (Corner ≺ Apricot ≺ Peach ≺ Plum ≺ Pomegranate ≺ Lotus ≺ Orchid ≺ Osmanthus ≺ Chrysanthemum ≺ Dew ≺ Winter ≺ Ice).

Domain orders are useful in a number of important applications:

- Implicit orders can enhance data profiling methods to identify new data quality rules, such as order dependencies over implicitly ordered attributes. (See Section 7.3, Exp-7 and Exp-8.)
- The SQL standard includes an order-by clause to sort the output, and aggregation with respect to minimum and maximum values requires a domain order. By capturing relationships between ordered attributes, we can eliminate the necessity to sort if the query plan already produces results in a needed order. These applications have been investigated for explicit orders in [11, 26, 29]. In this work, we demonstrate the additional benefits of implicit orders. (See Section 7.3, Exp-9.)
- Implicit orders can improve the performance of machine learning techniques by turning categorical features into ordinal ones. One case of this is the splitting condition in decision trees. Similarly, implicit orders can produce concise data summaries, with ranges over ordered attributes instead of individual categories. We demonstrate this by enhancing a recent data summarization method [31] with newly discovered implicit orders. (See Section 7.3, Exp-10.)

1.1 Methodology

Manual specification does not scale as it requires domain experts [10, 13, 15, 22]. This motivates the need to discover implicit orders automatically. Integrity constraints (ICs) specify relationships between attributes in databases [1]. To discover implicit orders, we use order dependencies (ODs) which capture relationships between orders [9, 18]. Intuitively, an order dependency asserts that sorting a table according to some attribute(s) implies that the table is also sorted according to some other attribute(s). For instance, in Table 1, timestamp orders yearGreg. If the tax per country is a fixed or a progressive percentage of the profit, then sorting the table by country, profit results in the table also being sorted by country, tax. Hence, “country, profit order country, tax.” The order of attributes on the left- and right-hand sides in an OD matters, as in the SQL order-by clause, while the order of attributes in a functional dependency (FD) [4] does not, as in the SQL group-by clause.

An OD implies the corresponding FD, modulo lists and sets of attributes but, not vice versa; e.g., country, profit orders country, tax implies that country, profit functionally determines country, tax. Order compatibility (OC) [25] is a weaker version of an OD, without the implied FD. Two lists of attributes in a table are said to be order compatible if there exists an arrangement for the tuples in the database in which the tuples are sorted according to both of the lists of attributes. For instance, yearGreg, monthNum is order compatible with yearGreg, week, where the attribute monthNum (not included in Table 1) denotes the Gregorian month of the year in numeric format and week represents the week of the year. A corresponding FD does not hold: yearGreg, monthNum does not functionally determine yearGreg, week (there are multiple weeks in a month) and yearGreg, week does not functionally determine yearGreg, monthNum (a week may span two months).

When an OD or OC has a common prefix on its left- and right-side, we can “factor out” the common prefix to increase understandability and refer to it as the context. Intuitively, this means that the respective OD or OC holds within each partition group of data by the context. For instance, if country, profit orders country, tax, then given a partitioning of the data by country (i.e., the context), profit orders tax within each group (that is, for any given country).
When an OD or OC has no common prefix, we say it has an empty context; e.g., the OD of timestamp orders yearGreg has no common prefix, and thus an empty context.

Algorithms for OD and OC discovery from data [15, 22, 23, 30] use explicit domain orders. Let us say that they discover explicit-to-explicit (E/E) ODs. We discover implicit orders by extending the machinery of OD discovery. We first leverage explicitly known domain orders, where, say, the left-hand-side of a “candidate” OD is an explicit domain order and the right-hand side is a learned, implicit domain order. Call this an E/I OC. For instance, in the context of yearGreg, timestamp is order compatible with monthGreg*, where the star denotes implicit domain order over an attribute. Astonishingly, implicit domain orders can also be discovered from a “candidate” OD for both the left- and right-hand sides of the OD! Call this an I/I OC. For implicit domain order discovery through I/I ODs simply are E/I ODs and I/I ODs are conditional in the relative context of the left-hand side to the right. For example, in the context of yearGreg and yearLun, monthGreg* is order compatible with monthLun*.

1.2 Overview and Contributions

Our goal is to discover implicit domain orders. To do this, we define candidate classes for E/I ODs and I/I ODs, and we extend the discovery methods for these. To the best of our knowledge, this is the first attempt to discover implicit domain orders through ICs. The problem space can be factorized by the following dimensions.

- Whether there is a corresponding FD (thus, finding ODs rather than OCs).
- Whether the context is empty.
- When the context is non-empty, whether the discovered domain orders across different partition groups with respect to the context are to be considered as independent of each other, and so can be different (conditional), or they are to be considered the same across partition groups, and must be consistent (unconditional). In Table 1, the implicit order monthGreg* is unconditional; however, the implicit order ribbon* is conditional in the relative context of the country with respect to the size of the festival, with White < Blue < Red in Canada and White < Red < Blue in China.1
- Whether we are considering E/I ODs or I/I ODs.

Our key contributions are as follows.

1 Implicit Domain Orders (Section 2).

We formulate a novel data profiling problem of discovering implicit domain orders through a significant broadening of OD/OC discovery, and we parameterize the problem space.

We divide the problem space between explicit-to-implicit and implicit-to-implicit, which we present in Sections 3 and 4, respectively. We identify tractable cases, and then proceed towards the general case of I/I OC discovery, which we prove is NP-complete.

(2) E/I Discovery (Section 3).

For implicit domain order discovery through E/I ODs, we present efficient algorithms, taking polynomial time in the number of tuples to verify a given OD or OC candidate.

(3) I/I Discovery (Section 4).

For implicit domain order discovery through I/I ODs,
(a) we present a polynomial candidate verification algorithm when the context is empty,
(b) we present a polynomial candidate verification algorithm when the context is non-empty but taken as conditional,
(c) we prove that, for non-empty contexts taken as unconditional, the problem is NP-complete, and
(d) we show why the candidate set of conditional I/I ODs is always empty, although it is not necessarily empty for unconditional I/I ODs.

(4) Algorithmic Approaches (Sections 5 & 6).

(a) While the general case of implicit order discovery through I/I ODs is NP-hard, we show that the problem can be effectively handled by a SAT solver (Section 5). We implement our methods in a lattice-based framework that has been used to mine FDs and ODs from data [15, 22, 23].
(b) We propose an interestingness measure to rank the discovered orders and simplify manual validation (Section 6).

(5) Experiments (Section 7).

We mirror the sub-classes and approaches to discover implicit domain orders defined in Sections 3 and 4 and the algorithms in Sections 5 and 6 via experiments over real-world datasets.

(a) Scalability: In Section 7.1, we demonstrate scalability in the number of tuples and attributes, and the effectiveness of our method for handling NP-complete instances.
(b) Effectiveness: In Section 7.2, we validate the utility of the discovered orders.
(c) Applications: In Section 7.3, we demonstrate the usefulness of implicit orders in data profiling (by finding more than double the number of data quality rules by involving orders not found in existing knowledge bases), query optimization (by reducing query runtime by up to 30%), and data summarization (by increasing the information contained in the summaries by an average of 60%).

We review related work in Section 8 and conclude in Section 9.

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1This example is inspired by equine competitions (https://en.wikipedia.org/wiki/Horse_show#Awards).
Table 2: Notation

| Notation         | Description                                                                 |
|------------------|-----------------------------------------------------------------------------|
| \(\mathcal{D}, \mathcal{D}\) | Domain of ordered values, a partition                                        |
| \(\mathcal{T}_D^{\leq}\), \(\mathcal{T}_D^{<}\), \(\mathcal{P}_D^{\leq}\) | Strong total, weak total, and strong partial order                            |
| \(\mathcal{R}, \mathcal{R}, \mathcal{A}\) | Relational schema, table instance, and one attribute                         |
| \(\mathcal{A} \cup \mathcal{Y}, \{\}\) | Set of attributes, set union, and the empty set                              |
| \(\mathcal{X}, \mathcal{X}\) | List of attributes and arbitrary permutation                                 |
| \(\mathcal{X} \cup \{\}\) | List concatenation and empty list                                            |
| \(\mathcal{E}(\mathcal{t}, \mathcal{X})\) | Tuple and projections over attributes (cast to set)                          |
| \(\mathcal{E}(\mathcal{t}, \pi X, \gamma)\) | Partition group, partition, and sorted partition                             |
| \(\mathcal{X} \rightarrow \mathcal{Y}\) | Canonical OC                                                                 |
| \(\mathcal{X} \rightarrow \mathcal{Y}\) | Canonical QFD                                                                |
| \(\mathcal{P}_{\mathcal{A}}, \mathcal{A}\) | Derived and strongest derivable orders over \(\mathcal{A}\)                 |
| \(\mathcal{B}_G^{A,B}, \mathcal{B}_G^{A,B}\) | Initial and simplified bipartite graphs over \(A\) and \(B\)                 |

2 PRELIMINARIES

2.1 Domain Orders and Partitions

We first review definitions of order, as introduced in [12]. A glossary of relevant notation is provided in Table 2. If we can write down a sequence of a domain’s values to represent how they are ordered, this defines a strong total order over the values. For two distinct dates, for example, one always precedes the other in time.

**Definition 1.** (strong total order) Given a domain of values \(\mathcal{D}\), a strong total order is a relation \(<\) which, \(\forall x, y, z \in \mathcal{D}\), is
- transitive: if \(x < y\) and \(y < z\), then \(x < z\),
- connex: \(x < y\) or \(y < x\) and
- irreflexive: \(x \not< x\).

One may name a strong total order over \(\mathcal{D}\) as \(\mathcal{T}_D^{\leq}\). Then \(x < y \in \mathcal{T}_D^{<}\) asks whether \(x\) precedes \(y\) in the order.

We might know how groups of values are ordered, but not how the values within each group are ordered. This is a weak total order.

**Definition 2.** (weak total order) Given a domain of values \(\mathcal{D}\), a weak total order is a relation \(<\) defined over \(\mathcal{D}\). \(\mathcal{D} = \{D_1, \ldots, D_k\}\) and a strong total order \(\mathcal{T}_D^{\leq}\) over \(\mathcal{D}\) such that
\[
\mathcal{W}_D^{\leq} = \{a < b \mid a \in D_i \land b \in D_j \land D_i < D_j \in \mathcal{T}_D^{\leq}\}.
\]
A strong partial order defines order precedence for some pairs of items in the domain, but not all.

**Definition 3.** (strong partial order) Given a domain of values \(\mathcal{D}\), a strong partial order is a relation \(<\) which \(\forall x, y, z \in \mathcal{D}\) is
- asymmetric: if \(x < y\), then \(y \not< x\),
- transitive: if \(x < y\) and \(y < z\), then \(x < z\), and
- irreflexive: \(x \not< x\).

One may name a strong partial order over \(\mathcal{D}\) as \(\mathcal{P}_D^{\leq}\). Then \(x < y \in \mathcal{P}_D^{<}\) asks whether \(x\) precedes \(y\) in the order.

For any strong partial order \(\mathcal{P}_D^{\leq}\), there exists a strong total order \(\mathcal{T}_D^{\leq}\) such that \(\mathcal{P}_D^{\leq} \subseteq \mathcal{T}_D^{\leq}\). A nested order (lexicographic ordering) with respect to a list of attributes \(\mathcal{X}\) corresponds to the semantics of SQL’s order by, shown as \(s \preceq t\) or \(t \preceq s\) between tuples \(s\) and \(t\).

**Definition 4.** (partition) The partition group of a tuple \(t \in \mathcal{R}\) over an attribute set \(\mathcal{X} \subset \mathcal{R}\) is defined as \(\mathcal{E}(\mathcal{t}, \mathcal{X}) = \{s \in \mathcal{R} \mid s\mathcal{X} = t\mathcal{X}\}\).
can be written as \( X: \{ \} \rightarrow A \). This is equivalent to the OD \( X' \rightarrow XA \) in list notation. Call this an order functional dependency (OFD).

The canonical OD that states that A and B are order compatible within each partition group over the set of attributes X is denoted as \( X: A \prec B \). This is equivalent to the OC \( X' \rightarrow XA \).

The set \( X \) in this notation is called the OFD’s or OC’s context. ODs and canonical ODs constitute the canonical ODs, which we express using the notation \( X: A \rightarrow B \).

We are interested in ODs of the form \( X' \rightarrow X' \) as written in the canonical form as \( X': A \rightarrow B \). To discover such ODs, we limit the search to find canonical ODs and OFDs. This generalizes: \( X \rightarrow Y \iff X \rightarrow XY \) and \( X \sim Y \). These can be encoded into an equivalent set of ODs and OFDs [22, 23]. In the context of \( X \), all attributes in \( Y \) are constants. In the context of all prefixes of \( X \) and of \( Y \), the trailing attributes are order compatible. Thus, we can encode \( X \rightarrow Y \) based on the following polynomial mapping.

\[
R \models X \rightarrow Y \iff \forall a \in R \Rightarrow X: \{ \} \rightarrow A \text{ and } \forall j \in R \Rightarrow X: \{ Y \} \rightarrow J,
\]

This establishes a mapping of list-based ODs into equivalent set-based canonical ODs; i.e., the OD \( X' \rightarrow X' \) is logically equivalent to the pair of the OC \( X: A \prec B \) and OFD \( XA: \{ \} \rightarrow B \). This is because \( X' \), which is a common prefix for both the left and right side of this OD, can be factored out, making \( X': A \rightarrow B \) the only non-trivial OC that needs to hold. Thus, OD \( \equiv \) OC + OFD.

**Example 5.** In Table 1, \( \{\text{country, profit}\} \rightarrow \text{tax} (\text{ODF}) \) and \( \{\text{country}\} \rightarrow \text{profit} (\text{OC}) \). Hence, \( \{\text{country}\} \rightarrow \text{tax} (\text{OD}) \), as tax rates vary in different countries.

**Lattice Traversal.** To discover ODs, our algorithm starts with single attributes and proceeds to larger attribute sets by traversing a lattice of all possible sets of attributes in a level-wise manner. This search space is referred to as the lattice space. When processing an attribute set \( X \), the algorithm verifies \( (OFD) \)s of the form \( X' \rightarrow A \) for which \( A \in X \), and ODs of the form \( X \prec A \rightarrow B \) for which \( B \in X \) and \( A \neq B \). The set-based OD-discovery algorithm has exponential worst-time complexity in the number of attributes (to generate the candidate ODs), but linear complexity in the number of tuples (to verify each OD candidate) [22]. That the canonical ODs have a set-based representation rather than list-based means that the lattice is set-based, not list-based, making it significantly smaller.

**Problem Statement.** Given a dataset, we want to find implicit domain orders by extending the set-based OD discovery algorithm [22, 23] to E/I ODs and I/I ODs of the form \( X' \prec A: B \rightarrow A \rightarrow B \) and \( X \prec A, B \rightarrow A \sim B \), respectively. Since OD \( \equiv \) OFD + FD, we also want to find implicit domain orders via E/I ODs and I/I ODs, for which the OFD \( XA \rightarrow B \) (i.e., \( XA: \{ \} \rightarrow \) B) additionally holds.

**2.3 Discovery Framework**

Figure 1 illustrates the framework of our algorithm (iORDER). First, potential OD candidates are generated for one level of the lattice. These candidates are then pruned using the dependencies found in the previous levels of the lattice and the OC axioms. Next, the existence of an FD is checked for each candidate, and depending on whether an FD holds or not, different types of implicit ODs are examined using the algorithms described in Sections 3 through 5. Next, valid candidates and the strongest implicit OD types that were validated are stored (e.g., unconditional E/I ODs are preferred over conditional E/I ODs and unconditional I/I ODs). The candidates for the next level of the lattice are then generated, until the search for candidates is finished. In the final step, the discovered implicit orders are ranked based on their interestingness scores (Section 6).

**3 E/I DISCOVERY**

We begin with explicit-to-implicit (E/I) domain order discovery through ODs and ODs in which an attribute with an explicit order is used to find an implicit domain order over another attribute.

- We first define an implicit domain order with respect to the table and an explicit domain order on another attribute (Section 3.1).
- We then subdivide the problem of domain-order discovery via E/I ODs and ODs as follows:
  - with FDs, thus effectively via ODs (Section 3.2);
  - without corresponding FDs, thus effectively via ODs
    + with empty contexts (Section 3.3) and
    + with non-empty contexts (Section 3.4).

Thus, in Section 3.1, we define when two attributes can be co-ordered, given an explicit order on one, and define what the strongest derived order is. We then provide algorithms to determine when \( X: A \sim B \), and to compute the strongest order \( B^* \) when it does.

**3.1 Implicit Domain Orders**

For explicit-explicit OD discovery, say, for columns A and B, it suffices to check that the tuples of \( r \) can be ordered in some way that is consistent both with ordering the tuples of \( r \) with respect to column A’s explicit domain order and with respect to column B’s explicit domain order. That way of ordering the tuples of \( r \) is a witness that A and B can be “co-ordered”; it justifies that A \( \sim B \).

To define explicit-implicit order compatibility, we want to maintain this same concept: there is a way to order the tuples of \( r \) with respect to column A’s explicit domain order and for which the projection on B provides a valid order over B’s values. For E/I ODs with a non-empty context, \( X: A \sim B \), there must be a witness total order over \( r \) that is, within each partition group of \( X \), compatible with the explicit order of A and the order over B dictated by this is valid. This answers one of our two questions: whether the candidate OC of \( A \sim B \) holds over \( r \). The second question in this case, though, that we also need to answer is, what is that order \( B^* \)?

Such a witness order over \( r \) derives a total order (perhaps weak) over B. There may be more than one witness order over \( r \). Consider the OC monthNum ~ monthLun* over the first five tuples in Table 1. While the ordering \( \{t_1, t_2, t_3, t_4, t_5\} \) is a valid witness that gives the order Corner ~ Peach ~ Plum ~ Pomegranate ~ Winter over monthLun, so is the ordering \( \{t_1, t_3, t_2, t_4, t_5\} \), where the order of month values Peach and Plum is swapped. This indicates that
However, since \( \mathcal{P}_{\text{Orchard}} \) is valid with the implied domain order \( \text{Jan} < \text{Feb} \), we have three cases for \( \mathcal{E}/\mathcal{I} \) OC. The first case above is trivial. There exists exactly one \( \mathcal{E}/\mathcal{I} \) OC candidate \( B \), the empty partial order.

### 3.2 \( \mathcal{E}/\mathcal{I} \) OD, Empty Context

We first consider \( \mathcal{E}/\mathcal{I} \) ODs with an empty context; i.e., we are looking to establish whether there is a \( B \) with respect to \( A \) over the whole table \( r \) when there is a functional dependency from one side to the other of a candidate. When we have an explicit order over one side, we might discover an implicit order over the other side by finding an \( \mathcal{E}/\mathcal{I} \) OD between them.

Let our pair of attributes be \( A \) and \( B \), assume we have an explicit order over \( A \), and we want to discover an implicit order over \( B \) (i.e., \( B \)). We have three cases for FDs between the pair: (1) \( A \rightarrow B \) and \( B \rightarrow A \); (2) \( A \rightarrow B \) but \( B \not\rightarrow A \); or (3) \( B \rightarrow A \) but \( A \not\rightarrow B \). We devise efficient algorithms for each case.

The first case above is trivial. There exists exactly one implicit order over \( B \), which is a strong total order. To discover this order over \( B \), sort the table over \( A \), and project out \( B \). (If \( A \) is not a key of the table and may have duplicates, then \( B \) would too; eliminate these duplicates, which must be adjacent.) This is \( B^* \). This is unique with respect to \( A \) and is a strong total order.

**Example 7.** Let the attribute \( \text{monthNum} \) be added to Table 1 to denote the Gregorian month of the year in the numeric format. Then the FDs \( \text{monthNum} \rightarrow \text{monthGreg} \) and \( \text{monthGreg} \rightarrow \text{monthNum} \) hold. Thus, the \( \mathcal{E}/\mathcal{I} \) OC of \( \text{monthNum} \sim \text{monthGreg}^* \) is valid with the implied domain order \( \text{Jan} < \text{Feb} \sim \text{monthGreg}^* \) of \( \text{Jan} < \text{Apr} < \text{Jun} < \text{Aug} < \text{Oct} < \text{Dec} \).

For the second case, since \( B \rightarrow A \), this means some \( B \) values are associated with more than one \( A \) value. We can partition the tuples of \( r \) by \( B \). This can be done in \( O(|r|) \) via a hash. Scanning the partition, we find the minimum and maximum values of \( A \) within each \( B \)-value group. Then the \( B \)-value partition groups are sorted by their associated min-\( A \)'s. If \( |B| \ll |A| \approx |r| \), this is less expensive than sorting by \( A \). If the intervals of the values of \( A \) co-occurring with each value of \( B \) do not overlap, then this is \( B^* \). To formalize this, let us define the notion of an interval partitioning.

**Definition 10.** Let \( \mathcal{E}/\mathcal{I} \) OC \( \mathcal{E}/\mathcal{I} \) cases. The third case looks like the second case, except the explicit order known over \( A \) is now on the right-hand side of our FD, \( B \rightarrow A \). We can take a similar interval-partitioning approach as before. Sort the table \( r \) by \( A \). If \( |A| \ll |r| \), this is more efficient than fully sorting \( r \). This computes the sorted partition \( \mathcal{T}_A \). The \( A \) values partition the \( B \) values, since \( B \rightarrow A \) and \( \mathcal{T}_A \) orders these \( B \) values. Since there are multiple \( B \) values in some of the partition groups of \( \mathcal{T}_A \), given that \( A \rightarrow B \), this does not determine an order over \( B \) values within the same group. Thus the \( B^* \) implied by \( \mathcal{T}_A \) is not a strong total order, but it is a weak total order.

**Example 9.** Let the attribute \( \text{quarter} \) be added to Table 1 to denote the quarter year; i.e., Q1, Q2, Q3, and Q4. The FD of \( \text{monthGreg} \rightarrow \text{quarter} \) holds as the Gregorian months perfectly align with the quarters. Thus, the \( \mathcal{E}/\mathcal{I} \) OC quarter \( \sim \text{monthGreg}^* \) holds; \( \text{monthGreg}^\text{quarter} \) is a weak total order with \( \text{Jan} \sim \text{April} \sim \text{July} \sim \text{October} \). Between months within each quarter, we cannot infer any order.

Let the FD be \( A \rightarrow B \), \( m = |B| \) (the number of distinct values of \( B \)), and \( n = |r| \) (the number of tuples). In practice, it is common that \( m \ll n \).

**Lemma 1.** The runtime of discovering \( \mathcal{E}/\mathcal{I} \) ODs with empty context is \( O(m \ln m + n) \).

Proofs and pseudocode can be found in the appendix in Section 10.

### 3.3 \( \mathcal{E}/\mathcal{I} \) OC, Empty Context

We next consider \( \mathcal{E}/\mathcal{I} \) OCs with an empty context in the form of \( A \sim B^* \). Similar to the previous section, the goal is to verify whether
there is an order over the values of B with respect to the order over the values of A. Using the sorted partitions of $\tau_A$, we infer the order $b_i < b_j$ for every two distinct values of B which co-occur with two consecutive partition groups of A. Let $B^\pi$ denote the set of these inferred relations over B. We next check whether $B^\pi$ is a valid weak total order; if it is so, $A \sim B^r$ is a valid E/I OC and $B^* \equiv B^\pi$. Figure 2a demonstrates these steps.

**Theorem 2.** $A \sim B^r$ is valid iff $B^\pi$ is a weak total order.

**Example 10.** The E/I OC monthNum $\sim$ monthLun$^*$ does not hold in Table 1 since the lunar month Winter co-occurs both with December and January, which have numerical ranks of 6 and 1 in the table, resulting in monthLun$^{monthNum}$ being invalid.

In the following sections, $n, m,$ and $p$ denote the number of tuples, the number of distinct values of the candidate attribute(s) with implicit order, and the number of partition groups of the context.

**Lemma 2.** The runtime of discovering E/I OCs with empty context is $O(n + m^2)$, given an initial sorting of the values in the first level of the lattice.

### 3.4 E/I OD and E/I OC, Non-Empty Context

When the context is non-empty, say $X$, we first consider each partition group in $\pi_X$ independently. This is equivalent, with respect to each partition group, to the empty-context E/I OD and E/I OC discoveries above. For a candidate $X : A \sim B^r$, if either of the FDs $XA \rightarrow B$ or $XB \rightarrow A$ hold, we can use the algorithms in Section 3.2. Otherwise, we use the approach in Section 3.3. If an implicit order is discovered within each partition group, then the conditional E/I OC (or E/I OD) holds. To verify the unconditional case, we take the union of those orders—each of which represents a weak total order—by including the edge $(a, b)$ in the union graph iff this edge exists in at least one of the individual orders, and test whether this union graph represents a strong partial order (i.e., is cycle free). If so, we have established an unconditional $B^*$ in the context of $X$. These steps are included in Figure 2a. Note the third step on constructing the union order graph, which is not necessary for conditional E/I OCs.

**Theorem 3.** There exists an implicit domain order $B^\pi \leq B^r$ such that the E/I OC $X : A \sim B^r$ holds iff the union graph is cycle free.

**Example 11.** The E/I OC (yearGreg, yearLunar): monthNum $\sim$ monthLun$^{monthNum}$ holds unconditionally in Table 1 since the union graph is cycle free. Figure 3a shows the partial orders corresponding to this E/I OC for years (2020, 4718) and (2021, 4719), each derived from one partition group using the algorithm described in Section 3.3. (Note that the partition group for years (2021, 4718) is ignored since it only has one tuple.) Figure 3b shows the union order. Note that in the resulting union order, an edge is included if it belongs to at least one of the order graphs in Figure 3a; e.g., the edge (Pomeg., Lotus) is included while (Lotus, Osman) is not.

**Example 12.** In Table 1, the FD country, count $\rightarrow$ ribbon holds. Given the E/I OD candidate {country: count $\sim$ ribbon}, ($\pi_{ribbon}$)count is an interval partitioning within each partition group with respect to the context. However, the candidate implicit orders over ribbon—White $\prec$ Red $\prec$ Blue within China and White $\prec$ Blue $\prec$ Red within Canada—are not consistent, as the Blue and Red values are flipped, making this candidate hold only conditionally.

Building the graph data-structure for the union of the group orders (DAGs) is simple. This can be done by traversing the order from each partition group and adding each of their edges to the final graph if they have not been added yet. We then walk the resulting graph by depth-first search (DFS) to determine whether it is cycle free.

**Lemma 3.** The time complexity of discovering E/I OCs with non-empty context is $O(n \ln n + pm^2)$.

### 4 I/I DISCOVERY

A surprise for us was that domain orders can also be discovered even when no explicit domain orders are known!

- We first must extend what is meant by an implicit domain order as defined in Section 3.1: now it is two implicit domain orders that we seek to discover (Section 4.1).
- We then subdivide the problem of domain-order discovery via I/I OCs and ODs as follows:
  - candidates that have an empty context or that have a non-empty context that is treated as conditional (Section 4.2);
  - candidates that have a non-empty context that is treated as unconditional (Section 4.3); and
  - that have a corresponding FD (Section 4.4).

Thus, in Section 4.1, we define when two attributes, A and B, with a context $X$ can be co-ordered. We also define what strongest orders can be derived; i.e., $A^r \prec B^r$. The following sections then provide algorithms to determine when $X : A^r \prec B^r$, and to compute the strongest orders $A^r$ and $B^r$ when it does.

### 4.1 Pairs of Implicit Domain Orders

As in the explicit-implicit case, we have two questions to address: when does $X : A^r \prec B^r$ hold over $r$; and, if it does, what are strongest partial orders that we can derive for $A^r$ and $B^r$. Our criterion for whether $X : A^r \prec B^r$ holds over $r$ is the same as before: there exists some strong total order $T^r_s$ over the tuples in $r$, a witness, such that $A$ and $B$’s values projected into lists from $r$ ordered thusly represent strong total orders over $A$ and $B$’s values, respectively.

To determine the strongest derivable orders for $A^r$ and $B^r$ is not the same as before, however. We cannot define it as simply, as the intersection of all the projected orders. The reason is that there is never a single witness; witnesses come in pairs. Since we have no explicit order to anchor the choice, if we have a strong total order on $r$ that is a witness, then the reversal of that order is also a witness.
We first consider the cases of when the context is not empty but for which we treat the partition groups as independent \((\text{conditional})\). For the \(l/I\ OC\) candidate \(X: A^* \prec B^*\), our goal is to discover whether, within each partition group, there exist \(A^*\) and \(B^*\) such that they can be co-ordered. To do so, we build a bipartite graph, \(BG_{AB}\), over \(r\). In this, the nodes on the left represent the partition groups by \(A^*\)'s values in \(r\), \(\pi_X A\), and those on the right represent the partition groups by \(B^*\)'s values in \(r\), \(\pi_X B\). For each tuple \(t \in r\), there is an edge between \(a_t\) (left) and \(b_t\) (right).

**Definition 11.** \((3\text{-fan-out})\) A bipartite graph has a 3-fan-out iff it has a node that is connected to at least three other nodes.

It does not suffice to consider directly \(BG_{AB}\) to determine whether \(A^* \prec B^*\). This is because a node of degree one in the \(BG\) over \(r\) can never invalidate the \(l/I\ OC\). E.g., \(White\) has just degree one in both of the \(BGs\) in Figures 4a and 4b. These have to be excluded before we check the 3-fan-out property.

**Definition 12.** \((\text{Singletons and } BG')\) Call a node in a \(BG\) with degree one a singleton. Let \(BG'\) be the \(BG\) in which the singletons and their associated edges have been removed.

With \(BG'_{AB}\), we can test whether \(A^* \prec B^*\).

**Theorem 4.** \(A^* \prec B^*\) is valid over \(r\) iff both of the following two conditions are true for \(BG'_{AB}\) over \(r\):

1. it contains no 3-fan-out; and
2. it is acyclic.

The intuition behind the requirement of no 3-fan-outs is that there has to be a way to order the left values in an attribute on the left to order the right values in an attribute on the right such that none of the edges of \(BG'\) cross. Also, there is no order if there is a cycle. Figure 2b demonstrates these steps, where only the first two steps are required for conditional \(l/I\ OCs\).

**Example 4.** The \(BG'_{\text{size,ribbon}}\) over Table 1 and shown in Figure 4c has 3-fan-out: \(\text{Medium}\) connects to \(\text{White}\), \(\text{Blue}\), and \(\text{Red}\). Thus, the candidate \(l/I\ OC\) of \(\{\}\): \(\text{size}^* \prec \text{ribbon}^*\) is not valid.

Even though the \(l/I\ OC\) candidate \(\{\}\): \(\text{size}^* \prec \text{ribbon}^*\) over Table 1 does not hold, it does not mean that \(X: \text{size}^* \prec \text{ribbon}^*\) does not hold with respect to some context \(X\). The latter is a weaker statement.

**Example 15.** Consider Table 1 and the \(l/I\ OC\) of \{\}:

\(\text{size}^* \prec \text{ribbon}^*\). Figures 4a and 4b show the two \(BGs\) for China and Canada (the values of our context, country), respectively. Thus, there exists a co-order between size and ribbon over \(E(1\text{country})\) (that is, for country = ‘China’) and a co-order between size and ribbon over \(E(6\text{country})\) (that is, for country = ‘Canada’).

We next need to show how to extract a co-order once we know one exists. As with Sections 3.3 and 3.4, we may discover a partial order, this time both for \(left\) and \(right\), within each partition group with respect to the context. The partial order is of a specific type: we find a disjoint collection of chains. Each chain is a strong total order over its values. Note that the singleton elements (which were initially ignored in \(BG'\)) will be inserted into this order, creating the final order. Again, there is no specified direction in which to read each chain; i.e., what its polarity is.

If, for each partition group with respect to \(X\) over \(r\), \(BG'_{AB}\) over the partition group satisfies the conditions in Theorem 4, then the conditional \(l/I\ OC\) of \(X: A^* \prec B^*\) holds over \(r\). \(BG'_{AB}\) over each partition group yields a strong partial order—a disjoint collection of chains—for each of A and B. A walk of \(BG'_{AB}\) suffices to enumerate the chains, for both A and B.

**Example 16.** Consider the \(BG\) in Figure 4b over the \(l/I\ OC\) of (country): \(\text{size}^* \prec \text{ribbon}^*\) over values in Table 1. By iteratively zig-zagging from left to right in this bipartite graph, we obtain the chains \([\text{Small}, \text{Medium}, \text{Large}]\) and \([\text{White}, \text{Blue}, \text{Red}]\) over size and ribbon, respectively, over partition group \(E(6\text{country})\).

As in Section 3.4, an \(l/I\ OC\) with a non-empty context can be treated either as conditional or unconditional. Our discovered domain orders between partition groups with respect to the context may differ. For the conditional case, this is considered fine; e.g., in Table 1, the order of ribbon colors w.r.t. the festival size differs per country: in China, \(\text{White} < \text{Blue} < \text{Red}\); in Canada, \(\text{White} < \text{Blue} < \text{Red}\).
Example 17. In Table 1, the conditional I/I OC of \{country\}: size* ~ ribbon* holds as \(E(t_{country}) \models \text{size}^* \sim \text{ribbon}^*\) and \(E(t_{country}) \models \text{size} \sim \text{ribbon}^*\).

Lemma 4. The runtime of validating a conditional I/I OC with empty or non-empty context is \(O(n)\).

4.3 I/I OCs, Unconditional

To validate an I/I OC candidate with a non-empty context unconditionally and find implicit orders \(A^*\) and \(B^*\) that hold over the entire dataset is significantly harder. The implicit orders for left and for right discovered per partition group must be consistent and polarity choices must be made for them.

For example, the months in the Gregorian and lunar calendars are dependent in the context of the year types with respect to the I/I OC of \{yearGreg, yearLun\}: \text{monthGreg}^* \sim \text{monthLun}^*. In the lunar calendar, there are twelve months (sometimes, thirteen), with the new year starting a bit later than in the Gregorian calendar, with the lunar months overlapping the Gregorian months.

We prove that this is computationally hard. To do this, we show that to determine whether, for a collection of chains, there exists a polarization, a directional choice for each chain, such that the union of the chains so directed represent a strong partial order. This is a sub-problem for deciding the validity of an I/I OC; therefore, this establishes that our problem is hard.

Definition 13. (the chain polarity problem) For the Chain Polarity Problem (CPP), the structure is a collection of lists of elements. Each list is constrained such that no element appears twice in the list. A list can be interpreted as defining a total order over its elements; e.g., list \([a, b, c, d]\) infers \(a < b, b < c,\) and \(c < d\).

A polarization of the list collection is a new list collection in which, for each list in the original, the list or the reverse appears. The decision question for CPP is whether there exists a polarization of the CPP instance such that the union of the total orders represented by the polarization’s lists is a strong partial order.

Lemma 5. The Chain Polarization Problem is NP-Complete.

We prove Lemma 5 using a reduction from NAE-3SAT, which is a variation of 3SAT that requires that the three literals in each clause are not all equal to each other. Thus, our problem being NP-hard follows from Lemma 5 (as it is a superset of the CPP problem), and as a solution that an unconditional I/I OC is valid can be verified in polynomial time, it is NP-complete.

Theorem 5. The problem of validating a given unconditional I/I OC with non-empty context is NP-complete.

Example 18. Table 3 illustrates a NAE-3SAT instance with three clauses and its equivalent CPP instance with nine lists. In the CPP instance, \(t_1 < t_2\) in the partial order is interpreted as assigning proposition \(p_1\) as true, and \(f_1 < t_1\) as assigning it false. Also, the variables \(a_i, b_1\) and \(c_i\) ensure that there exists at least one true and one false assignment for the literals in each clause. This condition is satisfied as among the three lists generated for each clause, exactly one or two of them have to be reversed in order to avoid a cycle among \(a_i, b_i\) and \(c_i\). This translates to the corresponding literals having false assignment and the rest true assignments. Hence, any valid polarization for the lists in the CPP instance can be translated to a valid solution for the NAE-3SAT instance.

We illustrate our approach for validating I/I OC candidates by employing a high quality SAT solver in Section 5.

4.4 I/I ODs

Discovery of domain orders via I/I ODs with an empty context (or with a non-empty context but considered conditionally) is essentially impossible. While we can discover I/I ODs that hold over the data, we can only infer the empty order for the domains. The FD essentially masks any information that could be derived about the orders.

Theorem 6. If \(X \rightarrow A \rightarrow B\), then the conditional I/I OD candidate \(X^*: A^* \rightarrow B^*\) must be valid. Furthermore, there is a unique partial order that can be derived for \(A^*\) and for \(B^*\): the empty order.

Example 19. Consider the I/I OC festival^* \sim monthGreg^* and Table 1. Since the FD of festival \(\rightarrow\) monthGreg holds, the empty partial order is the implicit order over monthGreg.

However, in the case of a candidate I/I OC with a non-empty context considered unconditionally paired with an FD that holds also “only within a non-empty context”, it is possible for us to discover meaningful domain orders.

Example 20. Consider Table 4, which shows different versions of a software released in each year and month, and the unconditional I/I OD of \{year\}: month \(\leftrightarrow\) version#. The only valid strong partial orders over the values of month and version# are Jan < Feb < March and v99 < v100, or the reversals of these, respectively.

5 USING A SAT SOLVER FOR I/I OC’S

Given that discovering implicit domain orders via I/I OCs is NP-hard, we reduce it to an instance of the SAT problem to validate the candidate and then to establish valid strong partial orders. The first step is similar to the conditional case in Section 4.2: we derive bipartite graphs, \(BG_i\)’s, for the tuples from each partition group. Presence of cyclicity or 3-fan-out invalidates the candidate, as by Theorem 4. Thus next, we check each \(BG_i\) for cyclicity or 3-fan-out; this validates or invalidates the candidate in linear time.\(^2\)

To translate an instance of our I/I OC validation problem isomorphically into a SAT instance, we create two types of clauses: one

\(^2\)The constraint that each \(BG_i\) has no 3-fan-out restricts the size of the graph to be linear in the number of distinct values of the domain. Without this, the size of the graph could be quadratic in the number of distinct values of the domain.

Table 3: A NAE-3SAT instance and the reduced CPP instance.

| Clauses | Lists |
|---------|-------|
| \(p_1 \lor p_2 \lor \neg p_3\) | \([t_1, a_1, b_1, f_1], [t_2, b_1, c_1, f_2], [f_3, c_1, a_1, t_3]\) |
| \(\neg p_1 \lor p_2 \lor \neg p_3\) | \([t_1, a_2, b_2, f_2], [t_2, b_2, c_2, f_2], [f_3, c_2, a_2, t_3]\) |
| \(\neg p_1 \lor \neg p_2 \lor p_3\) | \([t_1, a_3, b_3, f_3], [f_3, b_3, c_3, f_3], [t_3, c_3, a_3, t_3]\) |

Table 4: Valid I/I OD.

| #  | year | month | version* |
|----|------|-------|----------|
| t₁ | 2018 | Jan   | v99      |
| t₂ | 2018 | Feb   | v100     |
| t₃ | 2019 | Jan   | v99      |
| t₄ | 2019 | March | v100     |
| t₅ | 2020 | Feb   | v99      |
| t₆ | 2020 | March | v100     |

Table 5: CPP to SAT.

| #  | C | A | B |
|----|---|---|---|
| t₁ | 1 | 1 | 1 |
| t₂ | 1 | 2 | 2 |
| t₃ | 1 | 3 | 3 |
| t₄ | 2 | 1 | 4 |
| t₅ | 2 | 2 | 5 |
| t₆ | 2 | 4 | 5 |
| t₇ | 2 | 4 | 6 |
Finally, we add the clauses $l_1 \leq l_j$ and $\forall 1 \leq i, j \leq k : r_{ij}$. Assigning true to a variable $l_{ij}$ indicates $l_i < l_j$, while assigning false means that either $l_i < l_j$, or the order between these values has not been discovered. Thus, for every two variables $l_{ij}$ and $l_{ji}$, $-(l_{ij} \land l_{ji}) \equiv (-l_{ij} \lor -l_{ji})$ is added, as these variables cannot both be true. The same applies for variables $r_{ij}$.

**No swaps.** For every pair of tuples $(l_u, r_t), (l_v, r_w) \in \text{BG}_1$ and $r_t \neq r_w$, $l_u \neq l_v$, and $\{l_u, r_t\} \lor \{l_v, r_w\}$. Note that the initial conditions $((-l_{u,v} \lor -l_{v,u}) \land (l_{u,v} \lor -r_{u,v}))$ were used to simplify these conditions.

**Transitivity.** Next, we add the following clauses to ensure that transitivity is satisfied: $\forall 1 \leq u, v, w \leq m \land u, v, w \text{ distinct: } (l_{u,v} \land l_{v,w} \implies l_{u,w})$. We add similar clauses for the RHS values.

**Theorem 7.** The unconditional $\text{I/O}$ candidate is valid iff the corresponding SAT instance is satisfiable.

If the SAT instance is satisfiable, to derive the final partial orders over the values of $A$ and $B$, we take the satisfying assignment and set $i < j$ for $A$ iff $l_{ij} = true$, and similarly for the values of $B$. To achieve a pair of strongest derivable orders, we remove the order over pairs of values which should not exist in the final order, while keeping the order graph valid. For every pair of distinct values $l_u, l_v \in \text{PG}_1$ where $l_u < l_v$, we keep them in the final order iff one of these conditions holds (and similarly for the RHS values): 1) the nodes $l_u$ and $l_v$ are in the same connected component in $\text{BG}_1$ and the path from $l_u$ to $l_v$ contains at least two nodes with degree two or larger; 2) $\exists \text{BG}_2, j \neq i$ and distinct values $v_r, v_s$ belonging to the same attribute, s.t. $v_r, v_s$ in $\text{BG}_1 \land v_r, v_s$ in the same connected component in $\text{BG}_1$, as $l_u$ and $l_v$, respectively (note that $v_r$ and $v_s$ could be the same as $l_u$ and $l_v$). Intuitively, either of these conditions would make it impossible to remove an order between two values through valid transpositions within the same witness class, as defined in Section 4.1. These steps for validating an unconditional $\text{I/O}$ candidate are included in Figure 2b.

**Example 21.** Consider Table 5 and the $\text{I/O}$ candidate $\{C\}: A^* \sim B^*$. The propositional variables for the reduction are $l_{1,2}, l_{1,3}, \ldots, l_{1,3}$ and $r_{2,1}, r_{2,2}, \ldots, r_{6,5}$. Initial clauses are generated for each pair of values to ensure that an order in both directions is not established, e.g., $(-l_{1,2} \lor -l_{2,1}) \land (-l_{2,3} \lor -l_{3,2})$. For the no swaps condition, the clauses $(l_{1,2} \lor \neg r_{2,1}) \land (l_{2,1} \lor \neg r_{1,2})$ are generated for tuples $l_1$ and $l_2$, and similarly for the rest of the pairs of tuples in $\text{BG}_1$ and $\text{BG}_2$. Finally, we add the clauses $l_{1,3}, l_{1,3} \land l_{2,2} \Rightarrow l_{1,2}$, and accordingly the remaining clauses to capture the transitivity.

Since this is a valid $\text{I/O}$, some strong partial order can be derived using the SAT variable assignments. To derive the strongest orders from this pair, the final orders between values such as $(2, 4)$ and $(1, 2)$ on the LHS are kept, as these pairs satisfy the first and second condition, respectively (values 2 and 4 exist in a connected component in $\text{BG}_2$ with two nodes with degree two along their path, and values 1 and 2 are present on the LHS of both $\text{BG}_1$ and $\text{BG}_2$). However, the relation between values 1 and 3 (and similarly 2 and 3) is removed since these values do not satisfy any of the conditions. This process is repeated for the RHS as well.

**Lemma 6. The cost of the SAT reduction is $O(n + pm^2 + m^3)$.**

Since $m$ tends to be small in practice (i.e., $m \ll |r|$) for meaningful cases and $p$ heavily depends on $m$ as well, this runtime is manageable in real-life applications (Section 7).

We also use an optimization based on the overlap between values in different partition groups to decrease the number of variables and clauses. Initially, we compute disjoint sets of values that have co-appeared in the same partition groups. While considering pairs or triples of values to generate variables or transitivity clauses, respectively, we consider these sets of attributes separately, which reduces the runtime of the algorithm, without affecting the correctness of the reduction. Furthermore, before reducing to a SAT instance, we consider pairwise BGs and check their compatibility, in order to falsify impossible cases as early as possible.

### 6 MEASURE OF INTERESTINGNESS

The space and the number of discovered implicit domain orders may be large in practice. Inspired by previous work [3, 22], to decrease the cognitive burden of human verification, we propose a measure of interestingness to rank the discovered domain orders based on how close each is to being a strong total order. We argue that by focusing on similarity to a strong total order, this measure is successful in detecting meaningful and accurate implicit orders, and demonstrate this in our experiments in Section 7.

Given a DAG $G$ representing a strong partial order, the pairwise interestingness measure is defined as $\text{pairwise}(G) = |\text{pairs}(G)| / (m^2)$, where $\text{pairs}(G) = \{(u, v) : u, v \in G \text{ and there is a path between } u \text{ and } v\}$, and $m$ is the number of vertices in $G$. The number of pairs of vertices that are connected demonstrates the quality of the found strong partial orders, while the binomial coefficient in the denominator is used to normalize. Based on this measure, a strong total order graph has the perfect score of 1, while a completely disconnected graph has a score of 0.

**Example 22.** Consider the order graph $G$ presented in Figure 3b. There are 23 pairs of connected vertices and $\binom{28}{2} = 28$ possible pairs. Thus, the pairwise score is $\text{pairwise}(G) = \frac{23}{28} \approx 0.82$.

For conditional implicit orders, we divide the number of pairs in each partition group over the total number of pairs possible among all the values in the attribute, and then compute their average. This is to prevent candidates with many partition groups with less interesting partial orders from achieving a high score. To achieve a score of 1, the partial order in each partition group needs to be strong total order over all the values in the attribute. Our algorithm for computing this measure may take quadratic time $O(m^2)$ in the number of vertices in the graph, which corresponds to the number of unique elements in the attribute. This is not significant, in practice (Section 7).
7 EXPERIMENTS

We implemented our implicit domain order discovery algorithm, named iORDER, on top of a Java implementation of the set-based E/E OD discovery algorithm [22, 23]. Furthermore, we use Sat4j, which is an efficient SAT solver library [16]. Our experiments were run on a machine with a Xeon CPU 2.4GHz with 64GB RAM. We use two integrated datasets from the Bureau of Transportation Statistics (BTS) and the North Carolina Board of Elections (NCSBE): 3

• Flight contains information about flights in the US with 1M tuples and 35 attributes (https://www.bts.gov).
• Voter contains data about voters in the US with 1M tuples and 35 attributes (https://www.ncsbe.gov).

We chose these datasets due to their size for scalability experiments and for having real-life attributes with interesting implicit orders.

7.1 Scalability

Exp-1: Scalability in |r|. We measure the running time of iORDER by varying the number of tuples (Figure 5). We use the Flight and Voter datasets with 10 attributes and up to 1M tuples, by showing data samples to users and asking them to mark attributes as potential candidates. In the absence of user annotation, the algorithm can be run multiple times over different subsets of attributes to capture potential implicit orders. Figure 5 shows a linear runtime growth as computation is dominated by the verification of OD candidates, and the non-linear factors in our algorithm tend to have less impact. Thus, iORDER scales well for large datasets.

Exp-2: Scalability in |R|. Next, we vary the number of attributes. We use the Flight and Voter datasets with 1K tuples (to allow experiments with a large number of attributes in reasonable time) and up to 35 attributes. Figure 6 illustrates that the running time increases exponentially with the number of attributes (the Y-axis is in log scale). This is because the number of implicit order candidates is exponential in the worst case. The Voter dataset requires more time for the same number of attributes due to a larger number of candidates.

Exp-3: NP-complete Cases in Practice. The most general case of implicit domain order discovery through unconditional I/O OCs is NP-complete (Section 4.3). However, the majority of observed cases took a short time. The cases reduced to SAT were on average solved in under 60 ms in Exp-1 and Exp-2, indicating that NP-complete cases are handled well in practice. In Exp-2, with varying the number of attributes and 1K tuples, on average, 33% of the total runtime was spent on reducing to, and solving, the SAT instances. However, in the corresponding Exp-1, with varying the number of tuples up to 1M tuples, this ratio was less than 1%. This is attributed to the large number of candidates in Exp-2 and the small number of tuples, so the linear steps of the algorithm do not dominate the runtime.

3These datasets can be accessed through https://cs.uwaterloo.ca/~mkaregar/datasets/.

7.2 Effectiveness

Exp-4: Effectiveness over lattice levels. Here, we measure the running time and the number of discovered implicit domain orders at different levels of the lattice (Figure 7). We report the results with 10 attributes over 1M tuples (from Exp-1) in the Flight and Voter datasets. Since the attribute lattice is diamond-shaped and nodes are pruned over time through axioms, the time to process each level first increases, up to level five, and decreases thereafter.

As most of the interesting implicit orders are found at the top levels with respect to a smaller context (as verified in Exp-5), we can prune the lower levels to reduce the total time. In the Flight and Voter datasets, approximately 85% and 73% of the orders are found in the first three levels, taking about 45% and 19% of the total time, respectively. In the Voter dataset, fewer implicit orders are found in the first levels of the lattice, creating fewer pruning opportunities. Therefore, more time is spent on validating candidates with larger contexts, explaining the runtime difference between the two datasets.

Exp-5: Interestingness of implicit orders. We argue that implicit domain orders found at upper levels of the lattice are the most interesting. Implicit orders found with respect to a context with more attributes contain more partition groups. Hence, an implicit order with respect to a less compact context may hold, but may not be as meaningful, due to overfitting. Figure 7 illustrates that the interestingness score drops from the fourth level on for the Flight dataset and from the third level on for the Voter dataset.

Figure 8 illustrates that our interestingness measure can reduce the number of implicit domain orders. In the Flight dataset, other than monthGreg∗ and monthLunar∗, we found a high-scoring order delayDesc*, which orders flight delay as Early < On-time < Short delay < Long delay, and is discovered through the E/I OD delay ~ delayDesc*. This order is interesting since long delays may result in fines on the airline, so detecting these instances is valuable. Implicit orders over distanceRangeKM were discovered through the I/I OC distanceRangeKM ~ distanceRangeKM*, as the categories for these attributes are overlapping (e.g., ranges 0 – 700 and 700 – 3000 miles overlap with the range of 1100 – 4800 kilometers). Another ordered attribute in this dataset is flightLength (Short-haul < Medium-haul < Long-haul), which was discovered through the E/I OD (airline): flightDuration ~ flightLength*. The non-empty context is due to different airlines using different ranges to define flight duration. In the Voter dataset, the attributes ageRange (12–17 < 18–24 < 25–74) and generation (e.g., Baby Boomer < Generation X < Millennial < Generation Z) are discovered through E/I ODs age ~ ageRange* and birthYear ~ generation*, respectively, as well as the I/I OD ageRange* ~ generation* as the categories are overlapping. Finally, an order over birthYearAbbr was detected using the conditional E/I OD (isCentenarian): birthYear ~ birthYearAbbr*. This is because for the people born in the early 1917 or before with isCentenarian = True we have ‘97 < ‘98 < ... < ‘16.
with our intuition since the Flight dataset has more attributes with explicit or implicit orders, and confirms the importance of a scalable approach to invalidate spurious orders as the number of tuples grows. To investigate the importance of different types of implicit OGs, we categorize the number of top-k orders found in the Voter dataset using each type of implicit OC in Table 7. It can be seen that all types of implicit OGS contribute to the most interesting orders found. A similar pattern was observed in the Flight dataset.

**Exp-8: Knowledge Base Enhancement.** As another application of implicit domain orders in data profiling, we now compare with an open-source manually curated knowledge-base: YAGO. We quantify the percentage of automatically discovered implicit domain orders by our algorithm among the top-5 (ranked by our pairwise measure of interestingness) that exist in YAGO. The existence of an implicit order in YAGO is evaluated by considering pairs of values within the ordered domain and verifying if there exist knowledge triples specifying the relationship between the two entities. The result is that only 20% of the top discovered orders exist in YAGO. As shown in Exp-5, the top orders indicated by our measure of interestingness tend to be meaningful. Thus, existing knowledge bases can be enhanced by our techniques, especially in instances where the discovered orders are domain-specific or in knowledge bases that focus on objects rather than concepts, where implicit orders are more common. This may be done by incorporating pairs of ordered values as knowledge triples, e.g., (Corner, Less than, Peach).

**Exp-9: Query Optimization.** We show that ODS derived from implicit orders can eliminate sort operations in query plans for queries issued by IBM customers.\(^4\) We use the 10GB TPC-DS benchmark in Db2®.

Q1 (Figure 9) employs a year abbreviation by removing the first two digits from the year and appending an apostrophe (e.g., 2002 becomes ’02), denoted by the attribute dYearAbbr. Our work on implicit orders can be combined with the notion of near-sortedness [2]. Since the OD sZip \(\leq\) dYearAbbr\(^+\) is found, the optimizer can then take advantage of the index on dDate, speeding up the sort operator in the plan to accomplish the order-by. Given that each dYearAbbr block fits in memory, it can be re-sorted in main memory on-the-fly by strType and wsWebName.

Next, consider Q2 (Figure 10). Let there be an index on sZip in the table store. The first five digits indicate the post office and the following four digits indicate the delivery route. The attribute sPostOffice denotes the string code of the post office. Since there is the OD sZip \(\leq\) sPostOffice\(^-\), the optimizer can choose to scan the index on sZip to accomplish the group-by on-the-fly, so no partitioning or sorting is required. Note that a clever programmer cannot rewrite the query manually with a group-by sZip since sPostOffice changes the partitioning of the group-by.

\(^4\)We thank IBM for providing us with access to these queries.
select wWarehouseCode from webSales, warehouse where wWarehouseSk = wWarehouseSk and wQuantity > 90 order by wWarehouseCode;

**Figure 13:** Warehouse code with order-by (Q5)

select count(*) as count over (partition by wsAbsoluteMonth) from webSales;

**Figure 14:** OLAP query (Q6).

select dYearAbbr, smType, wsWebName, sum(case when (ws_ship_date_sk - ws_sold_date_sk) <= 30) then 1 else 0 end) as "30 days", ... sum(case when (ws_ship_date_sk - ws_sold_date_sk) > 120) then 1 else 0 end) as "> 120 days" from webSales, warehouse, shipMode, webSite, dateDim where wWarehouseCode = wWarehouseCode and ... group by dYearAbbr, smType, wsWebName order by dYearAbbr, smType, wsWebName;

**Figure 9:** Year abbreviation variation with order-by (Q1).

select sPostOffice, count(distinct sZip) as cnt, sum(netProfit) as net, count(sStoreSk) as cnt, ... from storeSales, store where sStoreSk = sStoreSk group by sPostOffice;

**Figure 10:** Post office with group-by (Q2).

select iItemDesc, dDateHour as date when sum(wsSalesPrice) as total from webSales, item, dateDim where wIItemSk = iItemSk and iCategory = 'Children' and wsSoldDateSk = dDateSk group by iItemDesc, dDateHour order by dDateHour;

**Figure 11:** Using string conversion and concatenation (Q3).

select iItemDesc, iCategory, iClass, iCurrentPrice, sum(wsExtSalesPrice) as revenue from webSales, item, dateDim where wIItemSk = iItemSk and wsSoldDateSk = dDateSk and dYearAbbr in ('98, ..., '02), group by iItemId, iItemDesc, iCategory, iClass, iCurrentPrice order by iCategory, iClass, iItemId;

**Figure 12:** With a predicate when abbreviating year (Q4).

business intelligence reporting. The optimizer can leverage the OD dDate → dDateHour* to use the index on dDate.

Q4 (Figure 12) can have its where-clause rewritten as: dDate between date('1998-01-01') and date('2002-12-31'). This rewrite allows the use of an index on dDate. Since the OD dDate → dYearAbbr holds (with implicit order ‘98 < '99 < ... < '02), the rewritten query is semantically equivalent.

Q5 (Figure 13) is similar to Q2, but with a filter predicate in its where-clause. The attribute wWarehouseCode is derived by compressing the first ten characters in the attribute wWarehouseName. Using an index on wWarehouseName and the OD wWarehouseName → wWarehouseCode*, we can remove the sort operator.

Finally, Q6 (Figure 14) is an OLAP query that uses the partition-by clause over the attribute wsAbsoluteMonth, which is computed from the last two digits of the year multiplied by 100 and adding the value of the month in the numeric format. The query plan can employ the index on wsSoldDate if the optimizer detects the OD wsSoldDate → wsAbsoluteMonth*.

Table 8 shows the execution times in two modes: with and without implicit orders rewrites. The results for implicit order optimized queries are significantly better, with an average decrease in runtime of 30%. Our techniques optimize expensive operations such as sort, which are super-linear, and which dominate the execution costs as the dataset size increases.

### 8 RELATED WORK

Previous work investigated the properties of and relationships between sorted sets [6]. However, to the best of our knowledge, no algorithms for discovery of implicit domain orders were proposed.

Existing OD discovery algorithms require some notion of explicit order [14, 15, 22, 23] and can benefit from implicit orders to find “hidden” ODs that have not been feasible before. In our solution, we use the set-based OD discovery algorithm [22, 23] since other approaches cannot discover a complete set of non-trivial ODs. For example, the list-based approach in [15] is intentionally incomplete in order to prune the much larger list-based search space. A similar
approach, recently shown in [14] is also incomplete despite the authors’ claim of completeness: it omits ODs in which the same attributes are repeated in the left- and the right-hand side, such as [country, profit] ↦ [country, tax] and reports an OD only when both the corresponding OFD and OC hold. Thus, it leaves out cases when only an OFD or only an OC is true (e.g., OC week ~ month holds, but OFDs {week}: [ ] ↦ month and {month}: [ ] ↦ week do not hold over the tuples within a single year). Additionally, the algorithm recently presented in [5] is incomplete, as shown in [24].

The importance of sorted sets has been recognized for query optimization and data cleaning. In [11], the authors explored sorted sets for executing nested queries. Sorted sets created as generated columns (SQL functions and algebraic expressions) were used in predicates for query optimization [17, 28]. Relationships between sorted attributes have been also used to eliminate joins [27] and to generate interesting orders [21, 29]. A practical application of sorted sets to reduce the indexing space was presented in [7]. In [20], the authors proved that finding minimal-cost repairs with respect to ODs is NP-complete, and introduce an approach to greedy repair.

9 CONCLUSIONS

We devised the first techniques to discover implicit domain orders. In future work, we plan to study the discovery of approximate implicit orders through ODs that hold with some exceptions. While in this work, we discover implicit domain orders with respect to a single set-based OD, we plan to extend our framework to merge orders found for a given attribute with multiple set-based ODs. We will also address implicit order discovery in dynamic tables, as was recently done for explicit OD discovery [30].
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Algorithm 2 derives total orders over the values of A and B based on the disconnected subtrees in BG'A,B, as described in Section 4.2. Line 1 sets currentNode to a node with degree one. Note that since BG'A,B is acyclic, such a node must always exist in any subtree G of it. Lines 5 to 8 traverse the tree back and forth between the nodes of A and B. The zigzag chain is uniquely determined since BG'A,B has no 3-fan-out, and thus, the degree of each node is at most two. Depending on the side (A or B), each node is appended to one of the total orders in Lines 4 or 7, and these two orders are returned as the output of the algorithm. Using the original graph BG'A,B, the singleton nodes can easily be inserted into the derived total orders, resulting in the final weak total orders, A* and B*.

Algorithm 3 IIOC-SameConnectedComponent

Input: a connected subtree G in BG'A,B.
Output: total orders over the values of A and B in G.
1: set currentNode to an arbitrary node with degree 1
2: set A and B to empty lists
3: set visited to the set {currentNode}
4: append currentNode to the appropriate list A or B depending on which side it belongs to
5: while currentNode has an unvisited neighbor neighbor do
6: add neighbor to visited
7: append neighbor to the appropriate list A or B
8: set currentNode to neighbor
9: end while
10: return A, B

Algorithms 3 and 4 demonstrate the main steps to remove edges from the partial order output of the SAT solver in order to achieve a pair of strongest derivable orders, as described in Section 5. Each of these algorithms generates a set of pairs of values (i.e., nonremovables), which must be kept in the final strongest derivable order graph. Note that these algorithms describe the steps taken only for values on one side of the candidate, which can be repeated for the other side as well.

Algorithm 3 corresponds to the first condition described in Section 5. Lines 2 to 10 find pairs of values that satisfy this condition, i.e., pairs which exist within the same connected component of some PG_i such that there are at least two nodes with degree two or larger along their connecting path. This is done by running a DFS from every node l_i in the graph and detecting all nodes l_p which satisfy this condition. Algorithm 4 corresponds to the second condition described in Section 5, i.e., all pairs in a connected component of some PG_i, such that there exist two nodes v_r and v_s in the same connected component in PG_i which also exist in some other bipartite graph G_j. Line 2 creates a hashmap which, for each two values l_p and l_q, stores the bipartite graphs in which l_p and l_q are connected. Lines 3 to 7 store the co-occurrences of such pairs of values in the variable samePG. Lines 8 to 14 traverse over all bipartite graphs to detect BG_i's and values l_p and l_q such that l_p and l_q are connected in BG_i as well as in at least one other bipartite graph. Next, all pairs of connected values within such BG_i are added to the set nonremovables.

The remaining step is to compute the union of output sets of Algorithms 3 and 4 and to traverse over all the edges of the initial partial order graph (i.e., the output of the SAT solver) and remove those which do not exist in the union set, creating the final graph for a strongest derivable order.
Algorithm 4 IIOC-MutualPairsOfValuesInPGs

Input: set of bipartite graphs \( BG = \{BG_1, \ldots, BG_q\} \).

Output: set of pairs of values in one side of the candidate that must be kept in the final order

1: set nonremovable as an empty set
2: set samePG as a hash map from pairs of values in L to empty sets
3: for each \( BG_i \in BG \) do
4: for each \( l_i, l_j \in BG_i \) do
5: add \( i \) to samePG[\( l_i, l_j \)]
6: end for
7: end for
8: for each \( BG_k \in BG \) do
9: if there exist \( l_p, l_q \in BG_k \) such that samePG[\( l_p, l_q \)] \( \geq 2 \) then
10: for each \( l_a, l_b \in BG_k \) do
11: add the pair \( (l_a, l_b) \) to nonremovable
12: end for
13: end if
14: end for
15: return nonremovable

\((ap_{i+1}, b_q), \ldots, (ap_q, b_q)\) (tuples with duplicate values are represented as one tuple for simplicity). Therefore, \( E_i(b_k) = (ap_{i+1}, \ldots, ap_q) \) (assuming \( p = 0 \)). The total order \( T^\ast \), \( l_1 < \ldots < l_p \) is a valid witness and its projection on \( B \) results in total order \( b_1 < \ldots < b_q \). Since this is the only valid order (as it is enforced by the explicit order over the values of \( A \)), \( T^\ast \) is the only valid—and therefore the strongest derivable—implicit order over the values of \( B \).

If \((\pi_B)_A\) is not an interval partitioning, then there exist \( i, j \in [1, \ldots, k] \) such that \( i < j \) and \( \min(E_i(b_k)_A) < \min(E_j(b_k)_A) \) \( \leq \max(E_i(b_k)_A) \). Let \( t_i = \min(E_i(b_k)_A), b_i \), \( t_j = \max(E_j(b_k)_A), b_j \), and \( t_j = \min(E_j(b_k)_A), b_j \). If this E/I OD candidate is valid, then a total order \( T_i^\ast \) must exist that is: 1) compatible with the explicit order over the values of \( A \) and 2) its projection produces a valid total order over the values of \( B \). Given the first condition, in any total order \( T_i^\ast \), there must be \( I_j < I_j < I_j \), while the projection of this order would produce a relation over \( B \) with \( b_i \prec b_j \) and \( b_j \prec b_i \), which cannot be a valid total order. Therefore, by contradiction, this E/I OC candidate is not valid.

Let the FD be \( A \rightarrow B, m = |B| \) (the number of distinct values of \( B \)), and \( n = |r| \) (the number of tuples).

**Lemma 1.** The runtime of discovering E/I ODs with an empty context is \( O(m \ln m + n) \).

**Proof.** Let the attributes involved in the E/I OD candidate with an empty context be \( A \) and \( B \), and assume the FD \( A \rightarrow B \) holds. There are three cases (as enumerated in Section 3.2) that depend on which side has the implicit order and whether the FD \( B \rightarrow A \) holds. In all cases, we only need to sort \( m \) values (the number of distinct values of \( B \)), for a cost of \( m \ln m \). The other steps can be done in linear time using a hash table. Thus, the total runtime is \( O(m \ln m + n) \).

\[ \text{Algorithm 4 IIoC-MutualPairsOfValuesInPGs} \]

**Theorem 2.** \( A \sim B^+ \) is valid iff \( B^\ast \) is a weak total order.

**Proof.** Assume \( B^\ast \) is a valid weak total order and let \( T^\ast \) denote an arbitrary strong total order compatible with—that is, is a superset of—\( B^\ast \). To find a witness order \( T^\ast \) over the tuples of the database, sort the tuples based on the explicit order over \( A \) and break ties by \( T^\ast \). Considering the projected order of this witness over \( B \), the \( b_i \prec b_j \) derived within a partition group of \( A \) cannot result in a cycle, as they were initially enforced by \( T^\ast \). For the \( b_i \prec b_j \) derived from different partition groups of \( A \), it suffices to verify consecutive partition groups of \( A \) due to the transitivity of the order relation. Assume the relation \( b_i \prec b_j \) can be derived from two tuples in different partition groups \( \mathcal{PG}_p \) and \( \mathcal{PG}_q \) where, without loss of generality, \( p < q \). If \( q = p + 1 \), these partition groups are consecutive and \( b_i \prec b_j \) is derived using our algorithm. Otherwise, the partition group \( \mathcal{PG}_{p+1} \neq \mathcal{PG}_q \) contains some tuple with the B-value of \( b_j \) (note that \( b_k \) could be the same as \( b_i \) or \( b_j \)) since otherwise \( \mathcal{PG}_p \) and \( \mathcal{PG}_q \) would have been consecutive. Using our algorithm, the order \( b_i \prec b_k \) is derived (if \( b_i \neq b_k \)). Due to transitivity, it is now only required for \( b_k \prec b_j \) to be derived (if \( b_i \neq b_k \)), which is inductively derived from \( \mathcal{PG}_{p+1} \) and \( \mathcal{PG}_p \). Therefore, the validity of \( B^\ast \) implies the validity of the witness order over tuples.

Now we prove the other direction. Without loss of generality, let \( \mathcal{PG}_i \) denote the \( i \)-th sorted partition group of \( A \) and \( B \) the set of values of \( B \) that co-occur with \( \mathcal{PG}_i \). Assume \( A \sim B^+ \) is valid and that \( T^\ast \) is some witness order over the tuples. Based on the definition of a witness order, the partition groups of \( A \) and \( B \) need to be placed consecutively (and in ascending order for \( A \)) in \( T^\ast \). Since the projected order of \( T^\ast \) over \( B \) is a valid total order, for any two consecutive \( \mathcal{PG}_i \) and \( \mathcal{PG}_{i+1} \), \( B_i \) and \( B_{i+1} \) have at most one value in common. For each three consecutive partition groups \( \mathcal{PG}_{i-1}, \mathcal{PG}_i \), and \( \mathcal{PG}_{i+1} \), let \( B_{i-1} = B_i \cap B_{i+1}, B_{i+1} = B_i \setminus B_{i+1} \), and \( B_i = B_i \setminus \{B_i \setminus B_{i+1} \} \). Since \( T^\ast \) is a valid order, for any two \( i \neq j, B_i \cap B_j = \emptyset \). Therefore, by deriving pairs of ordered values from consecutive partition groups of \( A \), the resulting order graph corresponds to a valid weak total order where each partition over the values of \( B \) correspond to the value(s) within a set \( B_i \) or \( B_{i+1} \) for some \( i \) (in cases where \( B_i = \emptyset \) and \( B_{i-1} = B_{i+1} \), consecutive partitions with the same values can be considered as one).

In the remainder of this section, \( n, m, \) and \( p \) denote the number of tuples, the number of distinct values of the candidate attribute(s) with implicit order, and the number of partition groups of the context, respectively.

**Lemma 2.** The runtime of discovering E/I OCs with an empty context is \( O(n + m^2) \), given an initial sorting of the values in the first level of the lattice.

**Proof.** Consider the E/I OC candidate \( A \sim B^+ \). Since the attributes in the OD-discovery algorithm in Section 2 have been sorted in advance for the first level of the lattice, the sorted partition groups over the values of \( A \) can be created in \( O(n) \) time. A similar argument about the 3-fan-out rule for I/I OCs in Section 4.2 also applies to E/I OCs with an empty context as E/I OCs are more restrictive than I/I OCs. This ensures that when traversing consecutive partition groups of \( A \) and inferring relations of the form \( b_i \prec b_j \), the number of these relations will be bounded by \( m^2 \). Since the graph storing the partial order over the values of \( B \) will at most have size \( O(m^2) \) as well, the total runtime for an E/I OC candidate with an empty context is \( O(n + m^2) \).

**Theorem 3.** There exists an implicit domain order \( P^\ast \), such that the E/I OC \( X: A \sim B^+ \) holds iff the union graph is cycle free.
Proof
Let \( G_i \) denote the set of strongest derivable orders from within each partition group, and let \( G \) be the union graph generated from these graphs using the procedure described in Section 3.4. Each \( G_i \) corresponds to a valid partial order over the values of the attribute with the implicit order. First, assume \( G \) corresponds to a valid partial order; i.e., it is acyclic. The witness order over the tuples within each partition group represents a valid witness order over the entire dataset, since the union of all the derived relations over the values corresponds to a valid partial order.

Assume that \( G \) is cyclic and, without loss of generality, let \( \{(b_1, b_2), (b_2, b_3), \ldots, (b_{p-1}, b_p), (b_p, b_1)\} \) denote the set of edges involved in a cycle in this graph. Clearly, each edge \( (b_j, b_k) \) in this cycle must exist within some graph \( G_i \). Since the order graphs \( G_i \) correspond to the strongest derivable order over each partition group, an edge \( (b_j, b_k) \in G_i \) must be present in all witness total orders of the corresponding partition group. Therefore, for any witness order over the partition groups, and consequently over the entire dataset, \( b_j < b_k \) for any \( b_j \) and \( b_k \) will be derived, meaning that the partial order derived over the entire dataset cannot be valid. Thus, the unconditional \( E/I \) OC candidate with a non-empty context is invalid.

Lemma 3. The time complexity of discovering \( E/I \) OCs with a non-empty context is \( O(n \ln n + pm^2) \).

Proof
Given an \( E/I \) OC candidate with a non-empty context \( X \), the algorithms in Sections 3.2 or 3.3 run for each partition group over \( X \). The runtime of discovering an implicit order over each partition group is \( O(k \ln k + m^2) \), where \( k \) denotes the number of tuples in the partition group. Furthermore, the size of the order graph constructed from each partition group is \( O(m^2) \). Since creating the union graph using the individual orders and checking for cycles can be done in linear time in size of the graphs, the total runtime is \( O(pk \ln k + pm^2) \) which is bounded by \( O(n \ln n + pm^2) \).

Theorem 4. \( A^* \sim B^* \) is valid over \( r \) iff the following two conditions are true for \( BG'_{AB} \) over \( r \):

1. it contains no 3-fan-out; and
2. it is acyclic.

Proof
Assume that \( BG'_{AB} \) contains a 3-fan-out. Without loss of generality, let the tuples participating in this 3-fan-out be as follows (the minimal required tuples for a node \( a_i \) containing the 3-fan-out): \( t_1 = (a_1, b_1), t_2 = (a_2, b_2), t_3 = (a_3, b_3), t_4 = (a_4, b_1), t_5 = (a_4, b_2), \) and \( t_6 = (a_4, b_3) \). Assume that a witness total order \( T_w^* \) exists over the tuples, and, without loss of generality, \( t_4 < t_5 \times t_6 \). Since the projections of \( T_w^* \) over both \( A \) and \( B \) need to result in a valid total order, it must be that \( t_1 < t_2 \times t_3 \). However, it is impossible to find a valid placement for \( t_2 \), as any placement results in a cycle in the projected order over \( A \) or \( B \). Therefore, by contradiction, the candidate is invalid.

Assume \( BG'_{AB} \) contains a cycle. Let a cycle with length \( k \) be over tuples \( t_1 = (a_1, b_1), t_2 = (a_2, b_2), \ldots, t_{k-1} = (a_{k/2}, b_{k/2}), \) and \( t_k = (a_1, b_{1/2}) \), without loss of generality. Assume that a witness total order \( T_w^* \) exists over the tuples, and without loss of generality, \( t_1 < t_2 \). Then \( t_2 < t_3, \) and, inductively, \( t_j < t_{j+1} \) must hold. Thus, \( t_1 < t_2 < \ldots < t_k \). However, this results in \( a_1 < a_{k/2} \) and \( a_{k/2} < a_1 \), which makes a valid order over \( A \) impossible. Therefore, by contradiction, the candidate is invalid.

The other direction follows the correctness argument of Algorithm 2. Let \( T_w^* \) and \( T_{BG}^* \) denote the orders derived using Algorithm 2 (which are unique modulo polarity). Without loss of generality, let \( a_1 \preceq a_2 < \ldots < a_k \) and \( b_1 < b_2 < \ldots < b_k \) denote these total orders (the case where the length of one of the chains is longer by one value is resolved similarly). Without loss of generality, let \( a_k \) denote the first node in \( BG'_{AB} \) with degree 1. This implies that the tuples need to be of either of two patterns: 1) \((a_1, b_1); \) or 2) \((a_i, b_{i-1}) \) and \((a_i, b_i)\) for \( i > 1 \). The total order \( T_{BG}^* \) derived by sorting the tuples by \( T_{BG}^* \), and breaking ties by \( T_w^* \), results in a valid witness order. This is because any cycles in the derived orders over \( A' \) or \( B' \) would indicate the existence of a cycle (e.g., some \( a_i \) occurring with \( b_{i-2} \) or a 3-fan-out (e.g., some \( a_i \) occurring with three values \( b_{i-1}, b_1, b_{i+1} \)). For the singleton values removed in \( A' \) and \( B' \), they can easily be added to \( T_w^* \), where a tuple \((a_i, b_{i-1})\) can be added between the tuples \((a_i, b_1)\) and \((a_i, b_i),\) resulting in the valid final witness order \( T_{BG}^* \), implying the validity of this \( I/I \) OC candidate. □

Lemma 4. The runtime of validating a conditional \( I/I \) OC with an empty or a non-empty context is \( O(n) \).

Proof
To validate an \( I/I \) OC candidate with an empty context involves generating the BG, iterating over the tuples once, and then using DFS traversal to check for cycles and 3-fan-outs. This can be done in linear time in the number of tuples. Note that if the 3-fan-out condition holds, the size of BG is linear in the number of distinct values \( m \). Thus, deriving an order can be done in \( O(m) \) time, as it only requires traversing the bipartite graph a constant number of times, as shown in Algorithm 2. For non-empty contexts, validation of \( I/I \) OCs requires the above steps for each partition group. Thus, the overall runtime remains \( O(n) \). □

Lemma 5. The Chain Polarization Problem is NP-Complete.

Proof
The input size of a CPP instance may be measured as the sum of the lengths of its lists; let this be \( n \). Consider a pair explicitly implied by the list collection to be in the binary ordering relation if the pair of elements appears immediately adjacent in one of the lists. Thus, the number of explicitly implied pairs is bounded by \( n \).

CPP is in the class NP. An answer of yes to the corresponding decision question means there exists a polarization of the CPP instance that admits a strong partial order. Given such a polarization witness, its validity can be checked in polynomial time. The size of the polarization is at most \( n \). The set of explicitly implied ordered-pair pairs is at most \( n \). Computing the transitive closure over this set of pairs is then polynomial in \( n \). If no reflexive pair (e.g., \( a < a \)) is discovered, then there are no cycles in the transitive closure, and thus this represents a strong partial order. Otherwise, not.

CPP is NP-complete. The known NP-complete problem NAE-3SAT (Not-All-Equal 3SAT) can be reduced to CPP.

The structure of a NAE-3SAT instance is a collection of clauses. Each clause consists of three literals. A literal is a propositional variable or the negation thereof. A clause is interpreted as the disjunction of its literals, and the overall instance is interpreted as the logical formula which is the conjunction of its clauses. Since each
Without loss of generality, let $t_a$ correspond to the
consider an arbitrary partition group and, since $I/I_{OD}$
other order that can be derived for $X_2$ satisfies the NAE-3SAT instance such that, for each clause, at least
admits a strong partial order, then there is no truth assignment that
be a cycle in the ordering relation over the confounder elements,
corresponding lists must have been reversed. Otherwise, there will
been assigned false. For the propositional elements, let us interpret
for each:
add three lists. For clause $i$, we build a corresponding CPP instance as follows. For each clause,
the strongest derivable orders over $A^*$ and $B^*$, without
loss of generality, let the value of tuples $t_1$ to $t_{p_9}$ in the columns $A$ and $B$ be the following: $(a_1, b_1)$, $(a_2, b_1)$, ..., $(a_{p_9}, b_1)$, $(a_{p_9+1}, b_2)$, ..., $(a_{p_9+1}, b_2)$, ..., $(a_{p_9}, b_9)$ (tuples with duplicate values are represented as one tuple for simplicity). The total order $T^*_p$
$1 \times \ldots \times t_{p_9}$ over the tuples is a valid witness and its projection would result in total orders $a_1 \times \ldots \times a_9$ and $b_1 \times \ldots \times b_9$ over the values of $A$ and $B$, respectively. The same argument can be applied
to the rest of the partition groups, meaning that a conditional $E/I_{OD}$
candidate is always valid.

As for the strongest derivable orders over $A^*$ and $B^*$, without
loss of generality, let the initial witness order be $T^*_p$ and consider
transpositions of either of these types applied on it: 1) relocating the
tuple $t_i$, $p_{j+1} + 1 \leq i \leq p_j$, within the range $[p_{j-1} + 1, p_j]$; 2) relocating
the consecutive set of tuples $t_{p_{j-1}+1}, \ldots, t_{p_j}$ to any location
either between $t_{p_k}$ and $t_{p_{k+1}}$, or the beginning or end of $T^*_p$. Both
transpositions are valid since the resulting projected orders over the
values of $A$ and $B$ are valid. The intersection of all such transposed
witness orders for each side is the empty partial order. Therefore,
the strongest derivable orders over $A$ and $B$ for this partition group
(and, subsequently, for the other partition groups) are empty partial
orders.

Theorem 7. The unconditional $I/I_{OC}$ candidate is valid iff the
corresponding SAT instance is satisfiable.

Proof
Assume the SAT instance is satisfiable. A truth assignment to
the SAT instance variables that satisfies all the clauses can be translated
to two valid implicit orders for the $I/I_{OC}$ instance, by considering
the variables for each pair of values, where a true assignment to
$l_{u,v}$ or $l_{u,w}$ (or $r_{u,v}$ and $r_{u,w}$, accordingly) would result in the implied order $l_u < l_v$ or $l_u < l_w$, respectively. Exactly one of these two
variables is set to true, given the initial clauses generated for each
pair of variables. To derive a witness order $T^*_p$, it is enough to sort
the tuples in each partition group by the order derived over the values
of LHS, and break ties by the order derived over the RHS (or vice versa).
The projection of $T^*_p$ over the LHS and RHS attributes will result in
a valid total order, as all pairs of tuples within each partition group
were considered when generating the no swap clauses, meaning that
the projected order over the attributes is perfectly captured by these
clauses. Furthermore, transitivity ensures that the projected order is
acyclic, and as a result, is a valid order.

In the other direction, assume that the $I/I_{OC}$ candidate is valid
and let $T^*_p$ denote a valid witness order over the tuples. To derive
a solution to the SAT instance, the reverse of the previous algorithm
can be performed; i.e., the truth assignment for variables $l_{u,v}$ and
$l_{u,w}$ (or $r_{u,v}$ and $r_{u,w}$) can be set based on the order between values
in the projected order (if no order between the values exists, both
variables are set to false). Since $T^*_p$ is a valid witness, the projected
order over the attributes will be a valid order, meaning that the truth
assignments for the SAT variables can be uniquely determined, and
would satisfy the initial conditions (since for each two values $l_u$
and $l_v$, at most one of $l_u < l_v$ or $l_v < l_u$ holds in the projected
orders). Furthermore, this variable assignment would also satisfy the
no swaps and transitivity clauses, as the projected order over the
attributes is directly derived from pairs of tuples and is acyclic,
respectively. Therefore, the SAT instance is satisfiable.
Lemma 6. The cost of the SAT reduction is $O(n + pm^2 + m^3)$.

Proof
The SAT representation has $O(m^2)$ propositional variables. We initially also generate $O(m^2)$ clauses, one for each pair of variables, $v_{i,j}$ and $v_{j,i}$. Generating the no-swap clauses for each BG takes $O(m^2)$ time, as the number of edges in the bipartite graph derived from each partition group is $O(m)$, since the initial BG is acyclic and does not contain any 3-fan-outs. This makes the runtime of this step (and the number of generated clauses) $O(pm^2)$, where $p$ denotes the number of partition groups. Adding the transitivity clauses takes $O(m^3)$ time. This makes the total cost of the reduction to the SAT problem $O(n + pm^2 + m^3)$.

$\square$