Characterizations of extra-invariant spaces under the left translations on a Lie group

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Abstract
In the context of a connected, simply connected nilpotent Lie group, whose representations are square-integrable modulo the center, we find characterization results of extra-invariant spaces under the left translations associated with the range functions. Consequently, the theory is valid for the Heisenberg group \( \mathbb{H}^d \), a 2-step nilpotent Lie group.

Keywords Nilpotent Lie group · Translation invariant space · Range function · Plancherel transform and periodization operator · Heisenberg group

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1 Introduction
Let \( G \) be a connected, simply connected nilpotent Lie group with Lie algebra \( g \) and \( Z \) be the center of \( G \). Then, \( G \) is an \( SI/Z \) group if almost all of its irreducible representations are square-integrable (SI) modulo the center \( Z \). An irreducible representation \( \pi \) of \( G \) is called \( \textit{square integrable modulo the center} \) if it satisfies the condition:

\[
\int_{G/Z} |\langle \pi(g)u, v \rangle|^2 \; dg < \infty \quad \text{for all} \; u, v.
\]

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We briefly start by describing the left translation generated systems in \( L^2(G) \) as follows:

**Definition 1.1** Let \( \Lambda \) be a uniform lattice in the center \( Z \) of \( G \) and \( \Gamma \) be a discrete set lying outside the center \( Z \). A closed subspace \( W \) of \( L^2(G) \) is said to be \( \Gamma \Lambda \)-invariant if

\[
L_{\gamma \lambda} f \in W \quad \text{for all } \gamma \in \Gamma, \lambda \in \Lambda \text{ and } f \in W,
\]

where for each \( y \in G \), \( L_y f(x) = f(y^{-1} x) \) for \( x \in G \) and \( f \in L^2(G) \).

For a sequence of functions \( \mathcal{A} = \{ \varphi_k : k \in I \} \) in \( L^2(G) \), we define the \( \Gamma \Lambda \)-invariant space \( S^\Gamma \Lambda(\mathcal{A}) \) generated by \( \mathcal{A} \) with the action of \( \Gamma \Lambda \) as follows:

\[
S^\Gamma \Lambda(\mathcal{A}) = \text{span}^\Gamma \Lambda(\mathcal{A}), \quad \text{where } \mathcal{E}^\Gamma \Lambda(\mathcal{A}) = \{ L_{\gamma \lambda} \varphi : \gamma \in \Gamma, \lambda \in \Lambda, \varphi \in \mathcal{A} \}.
\]

If \( \mathcal{A} = \{ \varphi \} \), we denote \( S^\Gamma \Lambda(\mathcal{A}) \) as \( S^\Gamma \Lambda(\varphi) \). In general, \( \mathcal{E}^\Gamma \Lambda(\mathcal{A}) \) and \( S^\Gamma \Lambda(\mathcal{A}) \) are known as translation generated (TG) system and translation invariant (TI) space, respectively. Translation invariant spaces have enormous applications in sampling, approximation, wavelets, etc. Bownik in [5] characterized all \( \mathbb{Z}^d \)-invariant subspaces in \( L^2(\mathbb{R}^d) \) followed by the works of Ron and Shen in [12]. For the locally compact abelian group setup, the theory of TI spaces were studied in [6, 7, 11, 13]. Moving towards the non-abelian group setup, Currey et al. in [10] provided a characterization of all \( \Gamma \Lambda \)-invariant spaces using the range function for the \( SI/Z \) nilpotent Lie group.

Next, we define an invariance set in the center \( Z \) of the \( SI/Z \) nilpotent Lie group \( G \).

**Definition 1.2** For a given \( \Gamma \Lambda \)-invariant subspace \( W \), an invariance set \( \Theta \) is defined by

\[
\Theta = \{ \lambda \in Z : L_{\gamma \lambda} f \in W, \text{ for all } \gamma \in \Gamma \text{ and } f \in W \}.
\]

The set \( \Theta \) is a closed subgroup of \( Z \) containing \( \Lambda \). For this let us consider a net \( (\lambda_\alpha) \) in \( \Theta \) such that \( \lim_\alpha \lambda_\alpha = \lambda \), say. Then, we have \( \lim_\alpha \| L_{\gamma \lambda_\alpha} f - L_{\gamma \lambda} f \|_{L^2(G)} = 0 \) for \( f \in W \) and \( \gamma \in \Gamma \), and hence \( \lambda \in \Theta \), since \( W \) is a closed subspace. Therefore, \( \Theta \) is a closed set. Since \( \Theta \) is a semigroup of \( Z \) and the image of quotient map from \( Z \) to \( Z/\Lambda \) of \( \Theta \) is closed in \( Z/\Lambda \) and hence compact, therefore, the group property of \( \Theta \) follows from the fact that a compact semigroup of \( Z/\Lambda \) is a group.

This paper aims to characterize all \( \Gamma \Lambda \)-invariant spaces \( W \) to become \( \Gamma \Theta \)-invariant, where \( \Theta \) is a closed subgroup of \( Z \) and \( \Lambda \subset \Theta \). This is known as the extra-invariance of a translation invariant space \( W \). The current study of extra-invariance encompasses the non-abelian setup for a nilpotent Lie group, which is considered to be a high degree of non-abelian structure. Consequently, the theory is valid for the \( d \)-dimensional Heisenberg group \( \mathbb{H}^d \), a 2-step nilpotent Lie group. Shift-invariant spaces that are \( \frac{1}{n} \mathbb{Z} \)-invariant in \( L^2(\mathbb{R}) \) were completely characterized by Aldroubi et al. in [1] for the
one-dimensional Euclidean case, and by Anastasio et al. in [2, 3] for higher dimensions and locally compact abelian groups.

We find necessary and sufficient conditions under which a \( /Gamma_1/Lambda_1 \)-invariant space becomes \( /Gamma_1/Theta_1 \)-invariant in the context of nilpotent Lie group \( G \) whose representations are \( SI/Z \) type. The characterization results below are based on the Plancherel transform. Unlike the Euclidean and LCA group setup, the Plancherel transform of a function is operator-valued, so that the technique used in the Euclidean and LCA groups is restrained. We now state main results of the paper.

**Theorem 1.3** Let \( Lambda \) be a uniform lattice in the center \( Z \) of \( G \) and \( Gamma \) be a discrete set lying outside the center \( Z \) containing the identity element \( e \) such that \( J \) and \( Sigma \) are the Borel sections of \( Lambda^\perp/Theta^\perp \) and \( Z/Lambda^\perp \), respectively, where \( Theta \) is a closed subgroup of \( Z \) containing \( Lambda \). If \( W \) is a \( /Gamma_1/Lambda_1 \)-invariant subspace of \( L^2(G) \), then it is \( /Gamma_1/Theta_1 \)-invariant if and only if for each \( j \in J \), \( W \) contains \( V_j^\Theta \), where

\[
V_j^\Theta = \{ f \in L^2(G) : \hat{f} = \chi_{H_j^\Theta} \hat{g}, \text{ with } g \in W \}, \quad \text{ and } \quad H_j^\Theta := Sigma_j + j + Theta^\perp.
\]

In this case, the space \( W \) can be decomposed as the orthogonal direct sum of \( V_j^\Theta \)'s, i.e., \( W = \bigoplus_{j \in J} V_j^\Theta \).

As a consequence, we find the below characterization result for \( S^{/Gamma_1/Lambda_1}(mathcal{A}) \) to become \( /Gamma_1/Theta_1 \)-invariant using the associated range function.

**Theorem 1.4** In addition to the hypotheses of Theorem 1.3, let \( mathcal{A} \) be a sequence of functions in \( L^2(G) \). Then \( S^{/Gamma_1/Lambda_1}(mathcal{A}) \) is a \( /Gamma_1/Theta_1 \)-invariant if and only if the Plancherel transform followed by periodization \( T^{/Gamma_1/Lambda_1} \) satisfies

\[
T^{/Gamma_1/Lambda_1}(L_\gamma \varphi_j)(\sigma) \in J(\sigma) \text{ a.e. } \sigma \in Sigma, \text{ for all } j \in J \text{ and } \gamma \in Gamma,
\]

where \( J(\sigma) \) is the associated range function, \( J(\sigma) = \text{span}(T^{/Gamma_1/Lambda_1}(L_\gamma \varphi)(\sigma) : \varphi \in mathcal{A}, \gamma \in Gamma) \).

We further characterize this extra-invariance property using the dimension function. Given any \( /Gamma_1/Lambda_1 \)-invariant subspace \( W \) of \( L^2(G) \), we define the **dimension function** as

\[
\dim_W : Sigma \to \mathbb{N}_0 \cup \{\infty\} \text{ by } \dim_W(\sigma) := \dim(J_W(\sigma)) \text{ for a.e. } \sigma \in Sigma,
\]

where \( J_W(\sigma) \) is the range function associated with \( W \).

**Theorem 1.5** Under the standing hypotheses of Theorem 1.3, the \( /Gamma_1/Lambda_1 \)-invariant space \( W \) is \( /Gamma_1/Theta_1 \)-invariant if and only if the dimension function satisfies the following relation:

\[
\dim_W(\sigma) = \sum_{j \in J} \dim_{V_j^\Theta}(\sigma) \text{ a.e. } \sigma \in Sigma.
\]
The following result can be established easily for the $d$-dimensional Heisenberg group $\mathbb{H}^d$, a 2-step nilpotent Lie group, using Theorems 1.3, 1.4 and 1.5. The $d$-dimensional Heisenberg group, denoted by $\mathbb{H}^d$, is an example of $SI/Z$ group. The group $\mathbb{H}^d$ can be identified with $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$ under the group operation $(x, y, w) \cdot (x', y', w') = (x + x', y + y', w + w' + x \cdot y)$. Let $G$ be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. We identify $G$ with $\mathfrak{g} \cong \mathbb{R}^n$ due to the analytic diffeomorphism of the exponential map $\exp: \mathfrak{g} \rightarrow G$, where $n = \dim \mathfrak{g}$. To choose a basis for the Lie algebra $\mathfrak{g}$, we consider

**Theorem 1.6** Let $A, B \in GL(d, \mathbb{R})$ such that $AB^t \in \mathbb{Z}$ and let $N \in \mathbb{N}$. If $W$ is an $A\mathbb{Z}^d \times B\mathbb{Z}^d \times \mathbb{Z}$-invariant subspace of $L^2(\mathbb{H}^d)$, then it is $A\mathbb{Z}^d \times B\mathbb{Z}^d \times \mathbb{Z}^{1/N}$-invariant if and only if for each $n \in \mathbb{Z}_N := \{0, 1, 2, \ldots, N - 1\}$, $W$ contains $V_n^{1/N}$, where

$$V_n^{1/N} = \left\{ f \in L^2(G) : \hat{f} = \chi_{\mathcal{H}_n^{1/N}} g, \text{ with } g \in W \right\}, \text{ and } \mathcal{H}_n^{1/N} = [n, n + 1] + N\mathbb{Z}. \quad (1.3)$$

In this case, the space $W = \bigoplus_{n \in \mathbb{Z}_N} V_n^{1/N}$, and $\dim \mu(\xi) = \sum_{n \in \mathbb{Z}_N} \dim V_n^{1/N}(\xi)$ a.e. $\xi \in [0, 1)$.

As an application of the above results, the following consequence provides an estimate to measure the support of the Plancherel transform of a generator of $S^{\varGamma\Lambda}(\mathcal{A})$.

**Theorem 1.7** In addition to the hypotheses of Theorem 1.3, let $\mathcal{A} = \{\varphi_i\}_{i=1}^n \subset L^2(G)$ and $\Gamma$ be a finite set having cardinality $k$, i.e., $|\Gamma| = k$. If $S^{\varGamma\Lambda}(\mathcal{A})$ be $\Gamma\Theta$-invariant, the following inequality holds:

$$\mu(\delta \in D : \hat{\varphi}(\delta) \neq 0) \leq \sum_{m=0}^{nk} m \mu(\Sigma_m) \leq nk \quad \text{for all } i \in \{1, 2, \ldots, n\}, \quad (1.4)$$

where $D$ is the Borel section of $\hat{\mathcal{A}}/\Theta^\perp$, $\Sigma_m = \{\sigma \in \Sigma : \dim W(\sigma) = m\}$ and “0” is in the sense of the zero operator.

This paper is organized as follows: in Sect. 2, we discuss the irreducible representations of a nilpotent Lie group and the Plancherel transform for the $SI/Z$ group. Employing the Plancherel transform followed by a periodization, we establish the proof of our main results Theorem 1.3, 1.4, 1.5 and 1.7 in Sect. 3 by involving the range function associated with a $\Gamma\Lambda$-invariant space.

## 2 Irreducible representations of a nilpotent Lie group

Let $G$ be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. We identify $G$ with $\mathfrak{g} \cong \mathbb{R}^n$ due to the analytic diffeomorphism of the exponential map $\exp: \mathfrak{g} \rightarrow G$, where $n = \dim \mathfrak{g}$. To choose a basis for the Lie algebra $\mathfrak{g}$, we consider
the Jordan–Hölder series \((0) \subset g_1 \subset g_2 \subset \cdots \subset g_n = g\) of ideals of \(g\), such that \(\dim g_j = j\) for \(j = 0, 1, \ldots, n\) satisfying \(\operatorname{ad}(X)g_j \subset g_{j-1}\) for \(j = 1, \ldots, n\), for all \(X \in g\), where for \(X, Y \in g\), \(\operatorname{ad}(X)(Y) = [X, Y]\), the Lie bracket of \(X\) and \(Y\). Now, we pick \(X_j \in g_j \setminus g_{j-1}\) for each \(j = 1, 2, \ldots, n\) such that the collection \(\{X_1, X_2, \ldots, X_n\}\) is a Jordan–Hölder basis. Then, the map \(\mathbb{R}^n \rightarrow g \rightarrow G\) defined by \((x_1, x_2, \ldots, x_n) \mapsto \sum_{j=1}^n x_j X_j \mapsto \exp(\sum_{j=1}^n x_j X_j)\) is a diffeomorphism, and hence the Lebesgue measure on \(\mathbb{R}^n\) can be realized as a Haar measure on \(G\).

Note that the center \(Z\) of the Lie algebra \(g\) is non-trivial, and it maps to the center \(Z := \exp \mathfrak{z}\) of \(G\). The Lie group \(G\) acts on \(g\) and \(g^*\) by the adjoint action \(\operatorname{exp}(\text{Ad}(X)g) := x \exp(X)x^{-1}\) and the co-adjoint action \((\operatorname{Ad}^* (x))\ell)(X) = \ell(Ad(x^{-1})X)\), respectively, for \(x \in G, X \in g\), and \(\ell \in g^*\). The \(g^*\) denotes the vector space of all real-valued linear functionals on \(g\). For \(\ell \in g^*\) the stabilizer \(R_\ell = \{x \in G : (\operatorname{Ad}^* x)\ell = \ell\}\) is a Lie group with the associated Lie algebra \(r_\ell := \{X \in g : \ell[Y, X] = 0\}\) for all \(Y \in g\).

Our aim is to discuss Kirillov Theory \([9]\) to define the Plancherel transform for \(SI/Z\) group. Given any \(\ell \in g^*\) there exists a subalgebra \(h_\ell\) (known as polarizing or maximal subordinate subalgebra) of \(g\) which is maximal with respect to the property \(\ell[h_\ell, h_\ell] = 0\). Then the map \(X_\ell : \exp(h_\ell) \rightarrow \mathbb{T}\) defined by \(X_\ell(\exp X) = e^{2\pi i \ell(X)}\), \(X \in h_\ell\) is a character on \(\exp(h_\ell)\), and hence the representations induced from \(X_\ell\), \(\pi_\ell := \text{ind}_{\exp h_\ell}^{\exp g} X_\ell\), have the following properties:

(i) \(\pi_\ell\) is an irreducible unitary representation of \(G\).
(ii) Suppose \(h_\ell'\) is another subalgebra which is maximal with respect to the property \(\ell[h_\ell', h_\ell'] = 0\), then \(\text{ind}_{\exp h_\ell}^{\exp g} X_\ell \cong \text{ind}_{\exp h_\ell'}^{\exp g} X_{\ell'}\).
(iii) \(\pi_{\ell_1} \cong \pi_{\ell_2}\) if and only if \(\ell_1\) and \(\ell_2\) lie in the same co-adjoint orbit.
(iv) Suppose \(\pi\) is an irreducible unitary representation of \(G\), then there exists \(\ell \in g^*\) such that \(\pi \cong \pi_\ell\).

Therefore, there exists a bijection \(\lambda^* : g^*/\operatorname{Ad}^*(G) \rightarrow \hat{G}\) which is also a Borel isomorphism, where \(\hat{G}\) is the collection of all irreducible unitary representations of \(G\).

For an irreducible representation \(\pi \in \hat{G}\), let \(O_\pi\) denotes as the co-adjoint orbit corresponding to the equivalence class of \(\pi\). Then the orbital characterization for the \(SI/Z\) representation is: \(\pi\) is square integrable modulo the center if and only if for \(\ell \in O_\pi\), \(r_\ell = \mathfrak{z}\) and \(O_\pi = \mathfrak{z} + \mathfrak{z}^\perp\). If \(SI/Z \neq \phi\), then \(SI/Z = \hat{G}_{\text{max}}\), where the Borel subset \(\hat{G}_{\text{max}} \subset \hat{G}\) corresponds to co-adjoint orbits of maximal dimension which is co-null for Plancherel measure class. Hence when \(G\) is an \(SI/Z\) group, \(\hat{G}_{\text{max}}\) is parameterized by a subset of \(\mathfrak{z}^*\). If \(\pi \in \hat{G}_{\text{max}},\) then \(\dim O_\pi = n - \dim \mathfrak{z}\), since \(O_\pi\) is symplectic manifold, it is of even dimension, say, \(\dim O_\pi = 2d\). By Schur’s Lemma the restriction of \(\pi\) to \(Z\) is a character and hence it is a unique element \(\sigma = \sigma_\pi \in \mathfrak{z}^*\) (say) and \(\pi(z) = e^{2\pi i (\sigma, \log z)}I\), where \(I\) is the identity operator. It shows that \(O_\pi = \{l \in g^* : l|_\mathfrak{z} = \sigma\}\) and \(\pi \mapsto \sigma_\pi\) is injective.

Let \(G\) be an \(SI/Z\) group and \(\mathcal{W} = \{\sigma \in \mathfrak{z}^* : \text{Pf}(\sigma) \neq 0\}\) be a cross section for the co-adjoint orbits of maximal dimension, where the Pfaffian determinant \(\text{Pf} : \mathfrak{z}^* \rightarrow \mathbb{R}\) is given by \(\ell \mapsto \sqrt{|\det(\ell[X_i, X_j])|_{i,j=r\ldots n}}\). Then for a fixed \(\sigma \in \mathcal{W}\), \(p(\sigma) = \sum_{j=1}^d g_j(\sigma|_{g_j})\) is a maximal subordinate subalgebra for \(\sigma\) and the corresponding induced representation \(\pi_\sigma\) is realized naturally in \(L^2(\mathbb{R}^d)\), where \(n = r + 2d\) for some
d. For each \( \varphi \in L^1(G) \cap L^2(G) \), the Fourier transform of \( \varphi \) given by

\[
\hat{\varphi}(\sigma) = \int_G \varphi(x) \pi_\sigma(x) \, dx, \quad \sigma \in \mathcal{W},
\]
defines a trace-class operator on \( L^2(\mathbb{R}^d) \), with the inner product \( \langle A, B \rangle_{\mathcal{H}_S} = \text{tr}(B^* A) \). This space is denoted by \( \mathcal{H}_S(L^2(\mathbb{R}^d)) \). When \( d\sigma \) is suitable normalized:

\[
\| \varphi \|_2^2 = \int_\mathcal{W} \| \hat{\varphi}(\sigma) \|^2_{\mathcal{H}_S(L^2(\mathbb{R}^d))} |\text{Pf}(\sigma)| \, d(\sigma).
\]

The Fourier transform can be extended unitarily as \( \mathcal{F} \)-the Plancherel transform

\[
\mathcal{F} : L^2(G) \rightarrow L^2(\mathbb{S}^*, \mathcal{H}_S(L^2(\mathbb{R}^d)), |\text{Pf}(\sigma)|d\sigma), \quad \mathcal{F}f = \hat{f}.
\]

Note that the Plancherel transform \( \mathcal{F} \) satisfies the relation

\[
\mathcal{F}(L_\gamma f)(\sigma) = \pi_\sigma(\gamma) \mathcal{F}f(\sigma) \quad \text{for } \gamma \in G, \ a.e. \ \sigma \in \mathbb{S}^*, \ \text{and } f \in L^2(G),
\]

where the left translation operator \( L_\gamma \) on \( L^2(G) \) is given by \( L_\gamma f(x) = f(\gamma^{-1}x) \).

In case of \( \mathbb{H}^d \), \( \mathcal{W} = \mathbb{R}\setminus\{0\} \) and \( |\text{Pf}(\sigma)| = |\sigma|^d \). Then the Plancherel transform for \( \varphi \in L^1(\mathbb{H}^d) \cap L^2(\mathbb{H}^d) \) is defined by \( \mathcal{F}\varphi(\lambda) = \int_{\mathbb{H}^d} \varphi(x) \pi_\lambda(x) \, dx \) for \( \lambda \in \mathcal{W} = \mathbb{R}\setminus\{0\} \), where the Schrödinger representations \( \pi_\lambda \) on \( L^2(\mathbb{R}^d) \) is given by \( \pi_\lambda(x, y, w) f(v) = \pi_\lambda(x, y, w) f(v) = e^{2\pi i \lambda w} e^{-2\pi i \lambda y \cdot v} f(v - x), \ (x, y, w) \in \mathbb{H}^d, \ f \in L^2(\mathbb{R}^d). \)

### 3 Proof of the main results

Throughout the paper, let us assume that \( G \) be an \( SI/Z \) nilpotent Lie group with center \( Z \). From the Sect. 2, we consider the center \( Z \) identified with \( \mathbb{R}^{r} \) (\( r < n \)) for a chosen (ordered) basis \( \{X_1, X_2, \ldots, X_n\} \) of the corresponding Lie algebra \( \mathfrak{g} \), as follows:

\[
Z = \exp \mathbb{R}X_1 \exp \mathbb{R}X_2 \ldots \exp \mathbb{R}X_r.
\]

Also we write a set \( \mathcal{X} \) identified with \( \mathbb{R}^{2d} \) (\( n = r + 2d \)) as follows:

\[
\mathcal{X} = \exp \mathbb{R}X_{r+1} \exp \mathbb{R}X_{r+2} \ldots \exp \mathbb{R}X_n.
\]

The elements \( y = (y_1, y_2, \ldots, y_r) \in \mathbb{R}^r \) and \( x = (x_{r+1}, \ldots, x_n) \in \mathbb{R}^{2d} \) are identified by

\[
y = \exp y_1 X_1 \exp y_2 X_2 \ldots \exp y_r X_r, \quad \text{and} \quad x = \exp x_{r+1} X_{r+1} \exp x_{r+2} X_{r+2} \ldots \exp x_n X_n.
\]
The identification can be noted from the homeomorphism between \( \mathbb{R}^r \times \mathbb{R}^{2d} \) and \( G \) given by

\[
(y_1, y_2, \ldots, y_r, x_{r+1}, \ldots, x_n) \mapsto \exp y_1 X_1 \exp y_2 X_2 \ldots \exp y_r X_r \\
\exp x_{r+1} X_{r+1} \exp x_{r+2} X_{r+2} \ldots \exp x_n X_n.
\]

Further assume that \( \Lambda \) is a uniform lattice in \( Z \), means, it is a discrete closed subgroup of \( Z \) such that \( Z/\Lambda \) is compact. Then we have \( \hat{Z}/\Lambda \cong \Lambda^{\perp} \) and \( \hat{Z}/\Lambda^{\perp} \cong \hat{\Lambda} \) since the center \( Z \) becomes a locally compact abelian group. The dual group of \( Z \), denoted by \( \hat{Z} \), is also identical with \( \mathbb{R}^r \). The group \( \hat{Z} \) consists of continuous homomorphisms from \( Z \) to \( \mathbb{T} \), and the annihilator \( \Lambda^{\perp} \) is defined by

\[
\Lambda^{\perp} = \{ \lambda^* \in \hat{Z} : \lambda^*(\lambda) = 1, \forall \lambda \in \Lambda \}.
\]

The set \( \hat{Z} \) can be tiled by \( \Sigma \) with the tiling partner \( \Lambda^{\perp} \), where \( \Sigma \) is a measurable section of \( \hat{Z}/\Lambda^{\perp} \) having finite measure. The set \( \Sigma \) is a tiling set of \( \hat{Z} \), means, the collection \( \{ \Sigma + \lambda^* : \lambda^* \in \Lambda^{\perp} \} \) is a measurable partition of \( \hat{Z} \). For a measure space \( (X, \mu) \), a countable set \( \{ \Omega_j \}_j \) of subsets of \( X \) is tiling of \( X \) if \( \mu(X \setminus \bigcup \Omega_j) = 0 \), and \( \mu(\Omega_i \cap \Omega_j) = 0 \) when \( i \neq j \). A set \( T \) is a tiling partner of \( \Omega \) for \( X \) if there is a set \( \Omega \) in \( X \) such that the collection \( \{ \Omega + x : x \in T \} \) is a tiling of \( X \).

We also fix the set \( \Gamma \) in \( \mathcal{X} \) lying outside of the center \( Z \). Recall the definitions of \( \Gamma \Lambda \)-invariant subspace \( W \) from Definition 1.1, the \( \Gamma \Lambda \)-invariant space \( \mathcal{S}^{T\Lambda} (\mathcal{A}) \) from (1.1) generated by \( \mathcal{A} \) and invariance set \( \Theta \) from (1.2). Firstly we concentrate on the properties of \( \Theta \). We observe the tiling property of \( \Sigma \) with respect to \( \Theta^{\perp} \) in the following result.

**Proposition 3.1** For any section \( \mathcal{J} \) of \( \Lambda^{\perp}/\Theta^{\perp} \), the set \( \Theta^{\perp} + \mathcal{J} \) is a tiling partner of \( \Sigma \) for \( \hat{Z} \). That means, the collection \( \{ \mathcal{H}_j^{\Theta} \}_{j \in \mathcal{J}} \) is tiling of \( \hat{Z} \), where

\[
\mathcal{H}_j^{\Theta} := \Sigma + j + \Theta^{\perp}.
\]

**Proof** The set \( \Sigma \) is a tiling set of \( \hat{Z} \), means, the collection \( \{ \Sigma + \lambda^* : \lambda^* \in \Lambda^{\perp} \} \) is a measurable partition of \( \hat{Z} \). Since \( \hat{Z}/\Lambda \cong \Lambda^{\perp} \) and \( Z/\Lambda \) is compact, therefore \( \Lambda^{\perp} \) is discrete and countable, and hence \( \Theta^{\perp} \) is also discrete and countable follows from \( \Theta^{\perp} \subset \Lambda^{\perp} \). Hence the collection \( \{ \Theta^{\perp} + j : j \in \mathcal{J} \} \) is a tiling of \( \Lambda^{\perp} \) by considering a Borel section \( \mathcal{J} \) of \( \Lambda^{\perp}/\Theta^{\perp} \). Thus, the result follows by employing the fact \( \Lambda^{\perp} \) is a tiling partner of \( \Sigma \) for \( \hat{Z} \) and \( \mathcal{J} \) is a tiling partner of \( \Theta^{\perp} \) for \( \Lambda^{\perp} \). \( \square \)

**Example 3.1** For the Heisenberg group \( \mathbb{H}^d \), the uniform lattice \( \Lambda \) in the center \( \mathbb{R} \), is the set of all integers \( Z \). Since the only proper closed additive subgroups of \( \mathbb{R} \) containing \( Z \) are \( \frac{1}{N} Z \) for some natural number \( N \), we consider the extra-invariance set \( \Theta = \frac{1}{N} Z \). Then the annihilators of \( \Lambda \) and \( \Theta \) are \( \Lambda^{\perp} = Z \) and \( \Theta^{\perp} = N Z \), respectively. Note that the set \( \mathbb{R} \) can be tiled by a Borel section \( \Sigma \) = [0, 1) with the tiling partner \( \Lambda^{\perp} = Z \). By assuming the Borel section \( Z_N := \{ 0, 1, \ldots, N-1 \} \) of \( \Lambda^{\perp}/\Theta^{\perp} = Z/NZ \), the set \( nZ + Z_N \) is a tiling partner of [0, 1) for \( \mathbb{R} \), that means, the collection \( \{ \mathcal{H}_n^{\frac{1}{N} Z} \}_{n=0}^{N-1} \)
is tiling of \( \mathbb{R} \), where

\[
\mathcal{H}^1_n = [0, 1) + n + N\mathbb{Z} = \bigcup_{k \in \mathbb{Z}} [n, n + 1) + Nk.
\]

For the case of Heisenberg group \( \mathbb{H}^d \), we consider the set \( \Gamma = A\mathbb{Z}^d \times B\mathbb{Z}^d \) from outside of the center of \( \mathbb{H}^d \), where \( A, B \in GL(d, \mathbb{R}) \) such that \( AB^t \in \mathbb{Z} \).

In the present section our first goal is to prove Theorem 1.3 which characterizes invariant subspaces of \( L^2(G) \) with the action of \( \Theta \) from the center \( Z \) containing the uniform lattice \( \Lambda \). The following lemma plays a crucial role to establish the Theorem 1.3.

**Lemma 3.2** Let \( W \) be a \( \Gamma \Lambda/\Theta \)-invariant subspace of \( L^2(G) \) and let \( J \) be a Borel section of \( \Lambda / \Theta \). For each \( j \in J \), consider the space \( V_j^\Theta \) given by

\[
V_j^\Theta = \{ f \in L^2(G) : \hat{f} = \chi_{\mathcal{H}_j^\Theta} g \text{ for some } g \in W \},
\]

where \( \mathcal{H}_j^\Theta \) is defined in (3.1). If \( V_j^\Theta \subset W \), it is a \( \Gamma \Theta \)-invariant (and hence, \( \Gamma \Lambda \)-invariant) subspace of \( L^2(G) \).

For the proof of Lemma 3.2, we discuss the Plancherel transform followed by a periodization \( T^\Lambda \) and the range function \( J \) associated with a \( \Gamma \Lambda/\Lambda \)-translation generated space \( \mathcal{S}^{\Gamma \Lambda}(\mathcal{A}) \). First we state the Plancherel transform followed by a periodization \( T^\Lambda \), which is an operator-valued linear isometry. It is well known but for the sake of completion, we provide its proof with the approach of the composition of unitary maps. The map \( T^\Lambda \) intertwines left translation with a representation \( \tilde{\pi} \). For \( g \in G \) and \( h \in L^2(\Sigma, \ell^2(\Lambda^\perp, \mathcal{H}S(L^2(\mathbb{R}^d)))) \), the representation \( \tilde{\pi} \) is given by

\[
\tilde{\pi}(g) h(\sigma) = \tilde{\pi}_\sigma(g) h(\sigma) \quad \text{a.e. } \sigma \in \Sigma.
\]

The associated representation \( \tilde{\pi}_\sigma(g) \) on \( \ell^2(\Lambda^\perp, \mathcal{H}S(L^2(\mathbb{R}^d))) \) is given by

\[
\tilde{\pi}_\sigma(g) z(\lambda^*) = \pi_{\sigma + \lambda^*}(g) \circ z(\lambda^*), \quad \lambda^* \in \Lambda^\perp,
\]

where the sequence \( (z(\lambda^*)) \) lies in \( \ell^2(\Lambda^\perp, \mathcal{H}S(L^2(\mathbb{R}^d))) \), “\( \circ \)” denotes the composition of operators in \( \mathcal{H}S(L^2(\mathbb{R}^d)) \) and \( \pi_{\sigma + \lambda^*}(g) \) is the Hilbert-Schmidt operator defined on \( L^2(\mathbb{R}^d) \).

**Proposition 3.3** (i) There is a unitary map \( T^\Lambda : L^2(G) \to L^2(\Sigma, \ell^2(\Lambda^\perp, \mathcal{H}S(L^2(\mathbb{R}^d)))) \) given by

\[
T^\Lambda f(\sigma)(\lambda^*) = \mathcal{F} f(\sigma + \lambda^*) |Pf(\sigma + \lambda^*)|^{1/2},
\]

\( f \in L^2(G), \lambda^* \in \Lambda^\perp \) and a.e. \( \sigma \in \Sigma \).
(ii) The map $T^\Lambda$ satisfies the intertwining property of left translation with the representation $\tilde{\pi}$:

$$ T^\Lambda(L^\gamma f)(\sigma) = e^{2\pi i\langle \sigma, \lambda \rangle} \tilde{\pi}_\sigma(\gamma) T^\Lambda f(\sigma), $$

$$ f \in L^2(G), \quad \gamma \in \Gamma, \quad \lambda \in \Lambda \text{ and a.e. } \sigma \in \Sigma. \quad (3.3) $$

**Proof**  
(i) Since the set $\mathcal{W}$ is Zariski open in $\mathfrak{g}^*$ and $\text{Pf}(\sigma)$ is non vanishing on $\mathcal{W}$, the map

$$ \mathcal{W}_1 : L^2(\mathcal{W}, \mathcal{H}S(L^2(\mathbb{R}^d))), |\text{Pf}(\sigma)|d\sigma \to L^2(\mathfrak{g}^*, \mathcal{H}S(L^2(\mathbb{R}^d))), \quad h \mapsto h|\text{Pf}(\sigma)|^{1/2} $$

is unitary. Further note that $\Lambda^\perp$ is a tiling partner of $\Sigma$ for $\tilde{\mathcal{Z}} \cong \mathfrak{g}^*$, we can define a periodization map

$$ \mathcal{W}_2 : L^2(\mathfrak{g}^*, \mathcal{H}S(L^2(\mathbb{R}^d))) \to L^2\left(\Sigma, \ell^2(\Lambda^\perp, \mathcal{H}S(L^2(\mathbb{R}^d)))\right), \quad h \mapsto (h(\cdot + \lambda^*))_{\lambda^* \in \Lambda^\perp}. $$

It is also unitary by identifying the linear dual $\mathfrak{g}^*$ of $\mathfrak{g}$ with $\mathbb{R}^r$. Therefore we get a sequence of unitary maps as follows:

$$ L^2(G) \xrightarrow{\mathcal{U}_1} L^2(\mathcal{W}, \mathcal{H}S(L^2(\mathbb{R}^d))), |\text{Pf}(w)|dw \xrightarrow{\mathcal{U}_2} L^2(\mathfrak{g}^*, \mathcal{H}S(L^2(\mathbb{R}^d))) \xrightarrow{\mathcal{U}_3} L^2\left(\Sigma, \ell^2(\Lambda^\perp, \mathcal{H}S(L^2(\mathbb{R}^d)))\right), $$

where the unitary map $\mathcal{U}_1$ is the Plancherel transform $\mathcal{F}$.

For $h \in L^2(\mathcal{W}, \mathcal{H}S(L^2(\mathbb{R}^d))), |\text{Pf}(\sigma)|d\sigma$, we observe

$$ (\mathcal{U}_2 \mathcal{U}_1)h(\sigma) = (\mathcal{U}_1 h)(\sigma)|\text{Pf}(\sigma)|^{1/2} = \mathcal{F} h(\sigma)|\text{Pf}(\sigma)|^{1/2}, \quad \text{a.e. } \sigma \in \mathcal{W} $$

and then for a.e. $\sigma \in \Sigma$, $\lambda^* \in \Lambda^\perp$ and $f \in L^2(G)$, we have

$$ (\mathcal{U}_3 \mathcal{U}_2 \mathcal{U}_1) f(\sigma)(\lambda^*) = (\mathcal{U}_3(\mathcal{U}_2 \mathcal{U}_1) f(\sigma)) (\lambda^*) = (\mathcal{U}_2 \mathcal{U}_1) f(\sigma + \lambda^*) $$

$$ = \mathcal{F} f(\sigma + \lambda^*)|\text{Pf}(\sigma + \lambda^*)|^{1/2}. $$

Thus the result follows by choosing $T^\Lambda = \mathcal{U}_3 \mathcal{U}_2 \mathcal{U}_1$.

(ii) For $\lambda^* \in \Lambda^\perp$ and a.e. $\sigma \in \Sigma$, we get

$$ T^\Lambda(L^\gamma f)(\sigma)(\lambda^*) = \mathcal{F}(L^\gamma f)(\sigma + \lambda^*)|\text{Pf}(\sigma + \lambda^*)|^{1/2} $$

$$ = \pi_{\sigma + \lambda^*}(\gamma) \mathcal{F} f(\sigma + \lambda^*)|\text{Pf}(\sigma + \lambda^*)|^{1/2} $$

$$ = e^{2\pi i\langle \sigma, \lambda \rangle} \pi_{\sigma + \lambda^*}(\gamma) \mathcal{F} f(\sigma + \lambda^*)|\text{Pf}(\sigma + \lambda^*)|^{1/2} $$

$$ = e^{2\pi i\langle \sigma, \lambda \rangle} (\tilde{\pi}_\sigma(\gamma) T^\Lambda f(\sigma))(\lambda^*), $$

since $\mathcal{F}(L^\gamma f)(\sigma) = \pi_{\sigma}(\gamma) \mathcal{F} f(\sigma).$  \qed
Remark 3.4 For the case of Heisenberg group $\mathbb{H}^d$, the unitary map $T^\Lambda$ takes the form:

\[
T^\Lambda : L^2(\mathbb{H}^d) \rightarrow L^2([0, 1), \ell^2(\mathbb{Z}, L^2(\mathbb{R}^d))), \quad T^\Lambda f(\xi)(n) = \mathcal{F}f(\xi + n)\xi + n|^{d/2}
\]
a.e. $\xi \in [0, 1), n \in \mathbb{Z}$, $f \in L^2(\mathbb{H}^d)$, and the intertwining property of $T^\Lambda$ gives

\[
T^\Lambda(L_{ny}f)(\xi) = e^{2\pi i n\xi} \hat{\pi}_\xi(\gamma) T^\Lambda f(\xi), \quad \gamma \in A\mathbb{Z}^d \times B\mathbb{Z}^d \text{ such that } AB' \in \mathbb{Z}.
\]

Next, we discuss the range function $J$ for a $\Gamma \Lambda$-invariant space. The range function $J$ is a mapping $J : \Sigma \mapsto \{\text{closed subspaces of } \ell^2(\Lambda^\perp, \mathcal{H}(L^2(\mathbb{R}^d)))\}$. It is measurable if the projection map $P(\sigma) : L^2(G) \rightarrow J(\sigma)$ is weakly measurable, i.e., for each $a, b \in \ell^2(\Lambda^\perp, \mathcal{H}(L^2(\mathbb{R}^d)))$, the map $\sigma \mapsto \langle P(\sigma) a, b \rangle$ is measurable. The space $W$ can be expressed as follows: $W = \{\varphi(\sigma) \in J(\sigma) \text{ for a.e. } \sigma \in \Sigma \}$ and $\pi_\sigma(\Gamma) \subset J(\sigma)$. Also, there is a bijection $W \mapsto J$. We refer [4, 6, 8, 10, 12, 13] for more details about shift-invariant spaces and associated range functions for the abelian and non-abelian setups.

Proposition 3.5 The range function $J$ associated with the $\Gamma \Lambda$-invariant space $W = S^{T^\Lambda}(\mathscr{A})$ satisfies

\[
J(\sigma) = \text{span}\{T^\Lambda(L_\gamma \varphi)(\sigma) : \varphi \in \mathscr{A}, \gamma \in \Gamma\} \text{ a.e. } \sigma \in \Sigma. \quad (3.4)
\]

Proof From the intertwining relation (3.3), $T^\Lambda(L_{\gamma\lambda}\varphi)(\sigma) = e^{2\pi i \langle \sigma, \lambda \rangle} \hat{\pi}(\gamma) T^\Lambda \varphi(\sigma)$ for $\gamma \lambda \in \Gamma \Lambda$ and a.e. $\sigma \in \Sigma$, and hence $T^\Lambda(S^{T^\Lambda}(\mathscr{A}))$ is invariant under exponential and $\pi(\Gamma) T^\Lambda(S^{T^\Lambda}(\mathscr{A})) \subset T^\Lambda(S^{T^\Lambda}(\mathscr{A}))$. Therefore, we get the result by observing $T^\Lambda(S^{T^\Lambda}(\mathscr{A})) = M_J$, where the space $M_J$ is defined by

\[
M_J = \{f \in L^2(\Sigma, \ell^2(\Lambda^\perp, \mathcal{H}(L^2(\mathbb{R}^d))) : f(\sigma) \in J(\sigma) \text{ for a.e. } \sigma \in \Sigma\}, \quad (3.5)
\]

corresponding to the range function $J$ given in (3.4). For this, let us consider a function $g \in T^\Lambda(S^{T^\Lambda}(\mathscr{A}))$. Choose a sequence $(g_i)$ converging to $g$ such that $T^{\Lambda^{-1}} g_i \in \text{span}\{L_{\gamma\lambda} \varphi : \gamma \lambda \in \Gamma \Lambda, \varphi \in \mathscr{A}\}$. Then we have $g_i(\sigma) \in J(\sigma)$ in view of (3.3), and hence $g(\sigma) \in J(\sigma)$ since $J(\sigma)$ is closed. Therefore, $g \in M_J$, i.e., $T^\Lambda(S^{T^\Lambda}(\mathscr{A})) \subset M_J$. For the equality $T^\Lambda(S^{T^\Lambda}(\mathscr{A})) = M_J$, we need to show $T^\Lambda(S^{T^\Lambda}(\mathscr{A})) \perp \cap M_J = 0$. Choose a function $h \in T^\Lambda(S^{T^\Lambda}(\mathscr{A})) \perp \cap M_J$. Then for any $f \in \text{span}\{T^\Lambda(L_{\gamma\lambda}\varphi) : \gamma \in \Gamma, \varphi \in \mathscr{A}\}$ and $\lambda \in \Lambda$, we have $e^{2\pi i \langle \cdot, \lambda \rangle} f(\cdot) \in T^\Lambda(S^{T^\Lambda}(\mathscr{A}))$, and then we obtain

\[
0 = \int_{\Sigma} \langle e^{2\pi i \langle \cdot, \lambda \rangle} f(\sigma), h(\sigma) \rangle \, d\sigma = \int_{\Sigma} e^{2\pi i \langle \cdot, \lambda \rangle} \langle f(\sigma), h(\sigma) \rangle \, d\sigma.
\]

Hence all the Fourier coefficients of a scalar function given by $\sigma \mapsto \langle f(\sigma), h(\sigma) \rangle$ are zero. Thus $\langle f(\sigma), h(\sigma) \rangle = 0$ a.e. $\sigma \in \Sigma$ and $f(\sigma) \in J(\sigma)$, i.e., $h(\sigma) \in J(\sigma) \perp$ a.e. $\sigma \in \Sigma$. \hfill $\Box$

Employing the Proposition 3.5, we characterize a member of $S^{T^\Lambda}(\varphi)$ with the help of Plancherel transform.
Proposition 3.6 For $f \in S_{\Gamma \Lambda}(\varphi)$, the Plancherel transform of $f$ is given by

$$\mathcal{F}f(\omega) = \sum_{\gamma \in \Gamma} \beta_{\gamma}(\omega) \mathcal{F}(L_{\gamma} \varphi)(\omega) \quad a.e. \ \omega \in \mathbb{Z}^*,$$

(3.6)

where $\beta_{\gamma}$ is a $\Lambda^\perp$-periodic function. Conversely if $\beta_{\gamma}$ is a $\Lambda^\perp$-periodic function such that

$$\sum_{\gamma \in \Gamma} \beta_{\gamma}(\cdot) \mathcal{F}(L_{\gamma} \varphi)(\cdot) \in L^2(\mathbb{Z}^*; \mathcal{H}S(L^2(\mathbb{R}^d))),$$

then the function $f$ defined by (3.6) is a member of $S_{\Gamma \Lambda}(\varphi)$.

Proof Applying the Plancherel transform followed by periodization $T^\Lambda$ on a function $f \in S_{\Gamma \Lambda}(\varphi)$, we get

$$T^\Lambda f(\sigma) = T^\Lambda(Pf)(\sigma) = P(\sigma)T^\Lambda f(\sigma)$$

$$= \sum_{\gamma \in \Gamma} \frac{\langle T^\Lambda f(\sigma), T^\Lambda(L_{\gamma} \varphi)(\sigma) \rangle}{\|T^\Lambda(L_{\gamma} \varphi)(\sigma)\|^2} T^\Lambda(L_{\gamma} \varphi)(\sigma)$$

(3.7)

a.e. $\sigma \in \Sigma$, in view of Proposition 3.5 and commutativity of $P$ and $T^\Lambda$, where $P$ and $P(\sigma)$ are orthogonal projections on $S_{\Gamma \Lambda}(\varphi)$ and $J(\sigma)$, respectively. The above expression (3.7) can be written as $T^\Lambda f(\sigma) = \sum_{\gamma \in \Gamma} \beta_{\gamma}(\sigma) T^\Lambda L_{\gamma} \varphi(\sigma)$ for a.e. $\sigma \in \Sigma$, where the $\Lambda^\perp$-periodic function $\beta_{\gamma}$ is defined by

$$\beta_{\gamma}(\sigma) = \begin{cases} \frac{\langle T^\Lambda f(\cdot), T^\Lambda(L_{\gamma} \varphi)(\cdot) \rangle}{\|T^\Lambda(L_{\gamma} \varphi)(\cdot)\|^2}, & \sigma \in \Sigma_{\varphi} \\ 0, & \text{otherwise}, \end{cases}$$

where $\Sigma_{\varphi} = \{ \sigma \in \Sigma : \|T^\Lambda(L_{\gamma} \varphi)(\sigma)\|^2 \neq 0 \}$. The function $\beta_{\gamma}$ can be extended periodically on $\hat{\mathbb{Z}}$ since $\Lambda^\perp$ is a tiling partner of $\Sigma$ for $\hat{\mathbb{Z}}$. Also observe that for any $w \in \hat{\mathbb{Z}}$, there exist unique $\sigma \in \Sigma, \lambda^* \in \Lambda^\perp$ such that $w = \sigma + \lambda^*$, and hence from (3.7), we obtain

$$\mathcal{F}f(w)| Pf(\omega)| = \mathcal{F}f(\sigma + \lambda^*)| Pf(\sigma + \lambda^*)| = T^\Lambda f(\sigma)(\lambda^*)$$

$$= \sum_{\gamma \in \Gamma} \beta_{\gamma}(\sigma) T^\Lambda(L_{\gamma} \varphi)(\sigma)(\lambda^*)$$

$$= \sum_{\gamma \in \Gamma} \beta_{\gamma}(\sigma + \lambda^*) \mathcal{F}(L_{\gamma} \varphi)(\sigma)(\lambda^*)| Pf(\sigma + \lambda^*)|$$

$$= \sum_{\gamma \in \Gamma} \beta_{\gamma}(w) \mathcal{F}(L_{\gamma} \varphi)(w)| Pf(\omega)|.$$
The converse part follows from the above calculations by writing \( \mathcal{F}f(\cdot) = \sum_{\gamma \in \Gamma} \beta_\gamma(\cdot)\mathcal{F}(L_\gamma \varphi)(\cdot) \) in the form \( T^\wedge f(\tau) = \sum_{\gamma \in \Gamma} \beta_\gamma(\sigma)T^\wedge (L_\gamma \varphi)(\sigma) \), and noting \( T^\wedge f(\sigma) \in J(\sigma) \) gives \( f \in S'^{\Gamma \Lambda}(\varphi) \) from Proposition 3.5.

**Proof of Lemma 3.2** For \( j \in \mathcal{J} \), let us consider the space \( V_j^\Theta = \{ f \in L^2(G) : \widehat{f} = \chi_{\mathcal{H}_j^\Theta} \hat{g} \text{ for some } g \in W \} \), where \( \mathcal{H}_j^\Theta = \Sigma + j + \Theta^\perp \). To prove it as a \( \Gamma \Theta \)-invariant subspace of \( L^2(G) \), we first assume a sequence \( \{ \varphi_k \} \) in \( V_j^\Theta \) converging to \( \varphi \in L^2(G) \). Then \( \varphi \in W \) since \( V_j^\Theta \subset W \) and \( W \) is closed, and hence \( \varphi \in V_j^\Theta \). This follows by writing \( \widehat{\varphi} = \chi_{\mathcal{H}_j^\Theta} \varphi \) since \( \| \varphi_k - \varphi \| \to 0 \) implies \( \widehat{\varphi} \chi_{\mathcal{H}_j^\Theta} = 0 \) from

\[
\| \varphi_k - \varphi \|^2 \geq \| \langle \widehat{\varphi}_k - \widehat{\varphi} \rangle \chi_{\mathcal{H}_j^\Theta} \| \|^2 \geq \| \widehat{\varphi} \chi_{\mathcal{H}_j^\Theta} \|^2,
\]

where \( c \) denotes complement of a set. Therefore, \( V_j^\Theta \) is closed. Further, we observe that if \( f \in V_j^\Theta \), we have \( \widehat{f} = \chi_{\mathcal{H}_j^\Theta} \hat{g} \) for some \( g \in W \), and hence for \( \theta \in \Theta \) and \( \gamma \in \Gamma \), we can write \( e^{2\pi i (\omega, \theta)} \mathcal{F}(L_\gamma f)(\omega) = \chi_{\mathcal{H}_j^\Theta}(\omega) e^{2\pi i (\omega, \theta)} \mathcal{F}(L_\gamma g)(\omega) \) for \( \omega \in \mathcal{S}^* \) since \( \mathcal{F}(L_\gamma g)(\omega) = \pi_\omega(\gamma) \hat{g}(\omega) \). For the \( \Gamma \Theta \)-invariant it suffices to show \( e^{2\pi i (\omega, \theta)} \mathcal{F}(L_\gamma g)(\omega) \in \mathcal{F}(W) \) that gives \( e^{2\pi i (\omega, \theta)} \mathcal{F}(L_\gamma f)(\omega) \in \mathcal{F}(V_j^\Theta) \). Observe that \( e^{2\pi i (\omega, \theta)} \mathcal{F}(L_\gamma g)(\omega) \in \mathcal{F}(\mathcal{S}^{\Gamma \Lambda}(g)) \subset \mathcal{F}(W) \) due to the converse part of Proposition 3.6, provided \( t_\theta(\omega) := e^{2\pi i (\omega, \theta)} \) a.e. \( \omega \in \mathcal{H}_j^\Theta \), is a \( \Lambda^\perp \)-periodic function. Since \( e^{2\pi i (\cdot, \theta)} \) is \( \Theta^\perp \)-periodic, we have \( e^{2\pi i (\sigma + j, \theta)} = e^{2\pi i (\sigma + j + \theta^*, \theta)} \) for a.e \( \sigma \in \Sigma \), \( j \in \mathcal{J} \) and for every \( \theta^* \in \Theta^\perp \), and then for each \( \lambda^* \in \Lambda^\perp \), we define \( t_\theta(\sigma + \lambda^*) = e^{2\pi i (\sigma + j, \theta)} \) a.e. \( \sigma \in \Sigma \). Thus the function \( t_\theta \) is \( \Lambda^\perp \)-periodic on \( \Sigma \), can be extended to \( \mathcal{S}^* \) since \( \Lambda^\perp \) is tiling partner of \( \Sigma \) for \( \mathcal{S}^* \equiv \mathcal{Z} \).

**Proof of Theorem 1.3** For each \( j \in \mathcal{J} \), if \( V_j^\Theta \subset W \), then the space \( V_j^\Theta \) is \( \Gamma \Theta \)-invariant from Lemma 3.2, and hence the space \( \bigoplus_{j \in \mathcal{J}} V_j^\Theta \subset W \) is so. Since \( \{ \mathcal{H}_j^\Theta \}_{j \in \mathcal{J}} \) is a tiling of \( \mathcal{Z} \cong \mathcal{S}^* \), therefore any element \( f \in W \) can be written as \( \widehat{f}(\omega) = \sum_{j \in \mathcal{J}} \mathcal{H}_j^\Theta(\omega) \) a.e. \( \omega \in \mathcal{S}^* \), where \( \mathcal{H}_j^\Theta = \widehat{f} \chi_{\mathcal{H}_j^\Theta} \). By the definition of \( V_j^\Theta \), \( g_j \in V_j^\Theta \) for every \( j \in \mathcal{J} \) and hence \( f \in \bigoplus_{j \in \mathcal{J}} V_j^\Theta \). Therefore, \( W \) is \( \Gamma \Theta \)-invariant.

Conversely, let us assume that the \( \Gamma \Lambda \)-invariant space \( W \) is \( \Gamma \Theta \)-invariant. For \( V_j^\Theta \subset W \), we choose \( f \in V_j^\Theta \). Then, we have \( \widehat{f} = \chi_{\mathcal{H}_j^\Theta} \hat{g} \) for some \( g \in W \). Employing the Plancherel transform followed by periodization \( T^\Theta \) from \( L^2(G) \) to \( L^2(\mathcal{D}, \mathcal{L}^2(\Theta^\perp), \mathcal{H}S(L^2(\mathcal{R}^d))) \) given by \( T^\Theta f(\delta)(\theta^*) = \mathcal{F}(\delta + \theta^*)|\mathcal{P}f(\delta + \theta^*)|^{1/2} \subset L^2(G), \theta^* \in \Theta^\perp \) and a.e. \( \delta \in \mathcal{D} \), \( \mathcal{D} \) is the Borel section of \( \mathcal{Z}/\Theta^\perp \), we obtain

\[
T^\Theta f(\delta)(\theta^*) = \chi_{\mathcal{H}_j^\Theta}(\delta)T^\Theta g(\delta)(\theta^*),
\]

since \( \chi_{\mathcal{H}_j^\Theta} \) is \( \Theta^\perp \)-periodic due to the definition of \( \mathcal{H}_j^\Theta \) in (3.1). Then for a.e. \( \delta \in \mathcal{D} \), we have \( T^\Theta(L_e g)(\delta) \in J^\Theta(\delta) \), where \( J^\Theta(\delta) = \overline{\text{span}}(T^\Theta(L_\gamma g)(\delta) : \gamma \in \Gamma) \), and hence \( T^\Theta f(\delta) \in J^\Theta(\delta) \) for a.e. \( \delta \in \mathcal{D} \). Thus \( f \in S'^{\Gamma \Theta}(g) \subset W \) due to Proposition 3.5. □
Proof of Theorem 1.4 For \( j \in J \), let \( V_j^\Theta = \{ f \in L^2(G) : \widehat{f} = \chi_{H_j^\Theta} \hat{g} \text{ for some } g \in W \} \), and \( W_j = \{ f \in L^2(G) : \text{supp}(\widehat{f}) \subset H_j^\Theta \} \), where \( H_j^\Theta = \Sigma + j + \Theta \). Let \( P_j \) be the orthogonal projection on \( W_j \). Then

\[
P_j(S^{\Gamma \Lambda}(\mathcal{A})) = \{ f_j : \widehat{f_j} = \widehat{f} \chi_{H_j^\Theta}, f \in S^{\Gamma \Lambda}(\mathcal{A}) \} = V_j^\Theta,
\]

whose associated range function is \( J_{V_j^\Theta}(\sigma) = \text{span}(T_\Lambda(L_{\gamma \hat{g}^j})(\sigma) : \varphi \in \mathcal{A}, \gamma \in \Gamma, \hat{g}^j = \hat{\varphi} \chi_{H_j^\Theta} \) for a.e. \( \sigma \in \Sigma \). Therefore from Theorem 1.3, \( S^{\Gamma \Lambda}(\mathcal{A}) \) is a \( \Gamma \Theta \)-invariant if and only if \( V_j^\Theta \subset S^{\Gamma \Lambda}(\mathcal{A}) \) for each \( j \in J \). Further it is equivalent to \( J_{V_j^\Theta}(\sigma) \subset J(\sigma) \) for a.e. \( \sigma \in \Sigma \) and for all \( j \in J \), where \( J(\sigma) \) is the range function associated with \( S^{\Gamma \Lambda}(\mathcal{A}) \), see Proposition 3.5 and [6]. Thus the result follows. \( \square \)

Proof of Theorem 1.5 From Theorem 1.3, we have \( W = \bigoplus_{j \in J} V_j^\Theta \) if \( W \) is \( \Gamma \Theta \)-invariant. Then for a.e. \( \sigma \in \Sigma \), the range function satisfies \( J_W(\sigma) = \bigoplus_{j \in J} J_{V_j^\Theta}(\sigma) \), follows by observing the orthogonality of \( J_{V_j^\Theta}(\sigma) \) and \( J_{V_j^\Theta}(\sigma) \) for \( j \neq j' \), since \( \{ H_j^\Theta \}_{j \in J} \) is a tiling of \( \widehat{Z} \cong \mathbb{Z}^* \). Hence \( \dim_W(\sigma) = \sum_{j \in J} \dim_{V_j^\Theta}(\sigma) \) a.e. \( \sigma \in \Sigma \).

For the converse part, first observe that the \( \Gamma \Lambda \)-invariant space \( W \) is contained in \( \bigoplus_{j \in J} V_j^\Theta \). This follows by writing \( f \in W \) as \( \widehat{f}(\omega) = \sum_{j \in J} \hat{g}_j(\omega) \) a.e. \( \omega \in \mathbb{Z}^* \), where \( \hat{g}_j = \widehat{\chi_{H_j^\Theta}} \) since \( \{ H_j^\Theta \}_{j \in J} \) is a tiling of \( \widehat{Z} \cong \mathbb{Z}^* \). Then the range function satisfies \( J_W(\sigma) \subset \bigoplus_{j \in J} J_{V_j^\Theta}(\sigma) \), and hence we have \( J_W(\sigma) = \bigoplus_{j \in J} J_{V_j^\Theta}(\sigma) \) for a.e. \( \sigma \in \Sigma \), due to the condition \( \dim_W(\sigma) = \sum_{j \in J} \dim_{V_j^\Theta}(\sigma) \) a.e. \( \sigma \in \Sigma \). Therefore we get \( J_{V_j^\Theta}(\sigma) \subset J_W(\sigma) \) for each \( j \in J \), i.e. \( V_j^\Theta \subset W \) for all \( j \). Thus \( W \) is \( \Gamma \Theta \)-invariant follows from Theorem 1.3. \( \square \)

Proof of Theorem 1.7 For \( \theta^* \in \Theta \) and \( \varphi \in \mathcal{A} \), we first estimate the measure of the following expression:

\[
\mu \left( \{(\sigma, j) \in \Sigma \times J : \hat{\varphi}(\sigma + j + \theta^*) \neq 0 \} \right) \\
= \mu \left( \{(\sigma, j) \in \Sigma \times J : \pi_{\sigma + j + \theta^*}(\gamma) \hat{\varphi}(\sigma + j + \theta^*) \neq 0 \} \right) \text{ for any } \gamma \in \Gamma \\
= \mu \left( \{(\sigma, j) \in \Sigma \times J : F(L_{\gamma \varphi})(\sigma + j + \theta^*) \neq 0 \} \right) \\
= \int_{\Sigma} |S^J_\sigma| \, d\sigma,
\]

where the set \( S^J_\sigma := \{ j \in J : F(L_{\gamma \varphi})(\sigma + j + \theta^*) \neq 0 \} \) and \( |S^J_\sigma| \) denotes the cardinality of \( S^J_\sigma \). For a.e. \( \sigma \in \Sigma \), the set \( S^J_\sigma \) is contained in the set \( \{ j \in J : \dim_{V_j^\Theta}(\sigma) \neq 0 \} \) since \( \dim_{V_j^\Theta}(\sigma) = \dim J_{V_j^\Theta}(\sigma) \), where \( J_{V_j^\Theta}(\sigma) = \text{span}(T_\Lambda(L_{\gamma \hat{g}^j})(\sigma) : \varphi \in \mathcal{A}, \gamma \in \Gamma, \hat{g}^j = \hat{\varphi} \chi_{H_j^\Theta} \). Then, we have

\[
|S^J_\sigma| \leq |\{ j \in J : \dim_{V_j^\Theta}(\sigma) \neq 0 \}| \leq \sum_{j \in J} \dim_{V_j^\Theta}(\sigma) = \dim_W(\sigma) \text{ a.e. } \sigma \in \Sigma.
\]
Since the set \( \{ \Sigma + j + \theta^* \}_{j \in J, \theta^* \in \Theta} \) is a tiling set for \( \hat{\mathbb{Z}} \), therefore for a fixed \( \sigma \in \Sigma \) and \( j \in J \) there is a unique \( \theta^*_{\sigma,j} \in \Theta^\perp \) such that \( \sigma + j + \theta^*_{\sigma,j} \in D \), and hence we have

\[
\mu(\{ \delta \in D : \hat{\varphi}(\delta) \neq 0 \}) = \sum_{j \in J} \mu(\{ \sigma \in \Sigma : \hat{\varphi}(\sigma + j + \theta^*_{\sigma,j}) \neq 0 \}) \\
= \mu(\sigma, j) \in J \times \Theta\perp : \hat{\varphi}(\sigma + j + \theta^*_{\sigma,j}) \neq 0 \} \\
= \int_{\Sigma} |S_{\sigma}^J| \, d\sigma \\
\leq \int_{\Sigma} \sum_{j \in J} \dim_{V_{\sigma}^\Theta}(\sigma) \, d\sigma = \int_{\Sigma} \dim_{V_{\sigma}}(\sigma) \, d\sigma \\
= \sum_{m=0}^{nk} m \mu(\Sigma_m) \leq nk,
\]

where \( |\Gamma| = k \) and \( \Sigma_m = \{ \sigma \in \Sigma : \dim_{V_{\sigma}}(\sigma) = m \} \). Thus the result follows. \( \square \)

We have an immediate consequence for the singly generated system.

**Corollary 3.7** Let \( \Sigma \) and \( J \) be the Borel sections of \( \hat{\mathbb{Z}}/\Lambda\perp \) and \( \Lambda\perp/\Theta\perp \), respectively, such that the cardinality \( |J| \) of \( J \) and measure \( \mu(\Sigma) \) of \( \Sigma \) satisfies the relation:

\[
(|J| \mu(\Sigma) - k) > 0, \text{ where } k \text{ is the cardinality of a non-empty set } \Gamma.
\]

When the space \( \mathcal{S}_{\Gamma}^{\Lambda}(\varphi) \) becomes \( \Gamma\Theta \)-invariant, then the Plancherel transform \( \hat{\varphi} \) of \( \varphi \) satisfies the following relation:

\[
\mu(\{ \omega \in z^* : \hat{\varphi}(\omega) = 0 \}) \geq |\Theta\perp| (|J| \mu(\Sigma) - k) > 0.
\]

**Proof** Considering a Borel section \( D \) of \( \hat{\mathbb{Z}}/\Theta\perp \) and noting \( \hat{\mathbb{Z}} \cong z^* \), we have the following from Theorem 1.7:

\[
\mu(\{ \omega \in z^* : \hat{\varphi}(\omega) = 0 \}) = \sum_{\theta^* \in \Theta\perp} \mu(\{ \delta \in D + \theta^* : \hat{\varphi}(\delta) = 0 \}) \\
= \sum_{\theta^* \in \Theta\perp} \mu(\{ \delta \in D + \theta^* : \hat{\varphi}(\delta) \neq 0 \}) \\
= \sum_{\theta^* \in \Theta\perp} \mu(D) - \sum_{\theta^* \in \Theta\perp} \mu(\{ \delta \in D : \hat{\varphi}(\delta) \neq 0 \}) \\
= \sum_{\theta^* \in \Theta\perp} \sum_{j \in J} \mu(\Sigma + j) - \sum_{\theta^* \in \Theta\perp} \mu(\{ y \in D : \hat{\varphi}(y) \neq 0 \}) \\
\geq |\Theta\perp| |J| \mu(\Sigma) - k|\Theta\perp| = |\Theta\perp| |J| \mu(\Sigma) - k > 0.
\]

Thus the result follows. \( \square \)
Remark 3.8 For the Heisenberg group $\mathbb{H}^d$, the center $Z = \mathbb{R}$, the uniform lattice $\Lambda = \mathbb{Z}$ and the extra-invariance set $\Theta = \frac{1}{N}\mathbb{Z}$. Then the annihilators of $\Lambda$ and $\Theta$ are $\Lambda^\perp = \mathbb{Z}$ and $\Theta^\perp = N\mathbb{Z}$, respectively. Consider $\Sigma = \{0, 1\}$ and $J = \{0, 1, 2, \ldots, N - 1\}$ be the Borel sections of $\hat{\mathbb{Z}}/\Lambda^\perp = \mathbb{R}/\mathbb{Z}$ and $\Lambda^\perp/\Theta^\perp = \mathbb{Z}/N\mathbb{Z}$, and choose $\Gamma = \{0\}$. Then $D = \{0, N\}$ is the Borel section of $\hat{\mathbb{Z}}/\Theta^\perp$ and the estimate (1.4) mentioned in Theorem 1.7 can be written as $\mu(\{\xi \in [0, N) : \hat{\varphi}_i(\xi) \neq 0\}) \leq n$ for all $i \in \{1, 2, \ldots, n\}$, since the cardinality of $\Gamma$ is $k = 1$. For $N > 1$, when the space $S^\Gamma\Lambda(\varphi)$ becomes $\Gamma\Theta$-invariant, the measure of the set $\{\xi \in \mathbb{R} : \hat{\varphi}(\xi) = 0\}$ is infinite from Corollary 3.7.

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Declarations

Conflict of interest The authors have no competing interests to declare that are relevant to the content of this article.

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