Stable Equivalence of Knots on Surfaces and Virtual Knot Cobordisms

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Abstract

We introduce an equivalence relation, called stable equivalence, on knot diagrams and closed curves on surfaces. We give bijections between the set of abstract knots, the set of virtual knots, and the set of the stable equivalence classes of knot diagrams on surfaces. Using these bijections, we define concordance and link homology for virtual links. As an application, it is shown that Kauffman’s example of a virtual knot diagram is not equivalent to a classical knot diagram.

1 Introduction

Virtual knots were defined in [9] via diagrams. These capture the combinatorial structure of Gauss codes and provide interesting examples that contrast with classical knot theory. They were used in [3] to study invariants of finite type. The combinatorial nature of virtual knots, however, has caused difficulty in attempts to generalize classical invariants.

A bijective relation between virtual knots and certain knots on surfaces, called abstract knots was given [7]. In this paper, we give an alternate geometric interpretation of virtual knots, called stable equivalence of knots on surfaces. Our interpretation enables us to introduce notions of cobordisms for virtual knots, for example. In particular, we classify link homology of virtual links, and use sliceness to distinguish virtual knots from classical knots as applications.

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The paper is organized as follows. In Section 2, we define stable equivalence. Relations to abstract knots and virtual knots are established in Section 3. Cobordisms for virtual knots are defined and studied in Section 4. Applications are given in Sections 5.

2 Stable equivalence of knots on surfaces

Let \( D \) be the set of all pairs \((F, D)\) such that \( F \) is a compact oriented surface and \( D \) is a link diagram on \( F \). For two elements \((F_1, D_1)\) and \((F_2, D_2)\) of \( D \), by \((F_1, D_1) \sim (F_2, D_2)\) we mean that there exists a compact oriented surface \( F_3 \) and orientation-preserving embeddings \( f_1 : F_1 \to F_3, f_2 : F_2 \to F_3 \) such that \( f_1(D_1) \) and \( f_2(D_2) \) are related by Reidemeister moves on \( F_3 \) (Fig. 1).

\[ \begin{align*}
&\hspace{1cm} \raisebox{-0.5cm}{\includegraphics[width=0.2\textwidth]{moves.png}} \hspace{1cm} \\
&\hspace{1cm} \text{Figure 1: Reidemeister moves}
\end{align*} \]

**Definition 2.1** Stable Reidemeister equivalence on \( D \) is an equivalence relation on \( D \) generated by the relation \( \sim \); that is, two elements \((F_1, D_1)\) and \((F_2, D_2)\) of \( D \) are stably Reidemeister equivalent, denoted by \((F_1, D_1) \sim (F_2, D_2)\), if there exists a sequence \((F_1, D_1) = (F_1, D_1) \sim (F_2, D_2) \sim \cdots \sim (F_n, D_n) = (F', D')\).

For example, let \( F_1 \) and \( F_2 \) be a torus \( S^1 \times S^1 \) and let \( D_1 \) and \( D_2 \) be simple closed curves in the torus such that \( D_1 \) is null-homotopic and \( D_2 \) is not. It is easily seen that \((F_1, D_1) \sim (F_2, D_2)\) does not hold. However, \((F_1, D_1) \sim (F_2, D_2)\) does not hold. However, \((F_1, D_1) \sim (F_2, D_2)\). Consider an element \((F, D)\) in \( D \) such that \( F = S^1 \times [-1, 1] \) and \( D = S^1 \times \{0\} \). Then \((F_1, D_2) \sim (F, D) \sim (F_2, D_2)\).

We note that for an element \((F, D)\) in \( D \), a quandle \( Q(D) \) and a group \( G(D) \) are defined diagrammatically in the usual way in knot theory. These are preserved under stable Reidemeister equivalence.

Let \( C \) be the set of all pairs \((F, C)\) such that \( F \) is a compact oriented surface and \( C \) is generic closed curves on \( F \). (Generic means that \( C \) is immersed and the singularities are transverse double points.) By \((F_1, C_1) \sim (F_2, C_2)\) we mean that there exists a compact oriented surface \( F_3 \) and orientation-preserving embeddings \( f_1 : F_1 \to F_3, f_2 : F_2 \to F_3 \) such that \( f_1(C_1) \) and \( f_2(C_2) \) are homotopic in \( F_3 \).

**Definition 2.2** Stable equivalence on \( C \) is an equivalence relation on \( C \) generated by the relation \( \sim \).
A natural map
\[ \pi : \mathcal{D} \to \mathcal{C} \]
sending a knot diagram to its underlying immersed curve induces a map
\[ \pi_{\sim} : \mathcal{D}/_{\sim} \to \mathcal{C}/_{\sim} \]
The map \( \pi_{\sim} \) is well-defined since homotopy of curves is generated by Reidemeister-type moves — more precisely the projection of the Reidemeister moves.

3 Virtual knots and abstract knots

Definition 3.1 ([8, 9]) A virtual link diagram consists of generic closed curves in \( \mathbb{R}^2 \) such that each crossing is either a classical crossing with over- and under-arcs, or a virtual crossing without over or under information. Let \( \mathcal{VL} \) be the set of virtual link diagrams. The virtual Reidemeister equivalence is an equivalence relation on \( \mathcal{VL} \) generated by the Reidemeister moves depicted in Fig. 2. Put \( V\mathcal{L} = \mathcal{VL}/_{\sim}\ ), where \( \sim \) is the virtual Reidemeister equivalence. Each element of \( V\mathcal{L} \) is called a virtual link.

If the given set of curves of a diagram is connected (i.e., the diagram consists of a single component curve), then it is called a virtual knot diagram. The set of virtual knot diagrams are denoted by \( V\mathcal{K} \), and the set of equivalence classes are denoted by \( V\mathcal{K} = V\mathcal{K}/_{\sim} \), whose elements are called virtual knots.

It is known that there is a bijection between \( V\mathcal{K} \) and the set of Gauss codes (or Gauss diagrams) modulo Reidemeister moves defined in the Gauss code level, [3, 8, 9]. (Refer to [11, 12] for Gauss codes and Reidemeister moves on them.)

Figure 2: Virtual Reidemeister moves

Let \( \mathcal{AL} \) be the subset of \( \mathcal{D} \) consisting of \( (F, D) \) such that \( |D| \) is a deformation retract of \( F \), where \( |D| \) is the underlying immersed curve in \( F \). See Fig. [3] (5). For \( (F_1, D_1), (F_2, D_2) \in \mathcal{AL} \), by \( (F_1, D_1) \sim_{ae} (F_2, D_2) \) we mean that there exists a closed connected oriented surface \( F_3 \) and orientation-preserving embeddings \( f_1 : F_1 \to F_3 \), \( f_2 : F_2 \to F_3 \) such that \( f_1(D_1) \) and \( f_2(D_2) \) are related by Reidemeister moves on \( F_3 \).

Definition 3.2 ([4, 5, 6, 7]) An abstract link diagram is an element of \( \mathcal{AL} \). Abstract Reidemeister equivalence, denoted by \( \sim_{ae} \), is an equivalence relation on \( \mathcal{AL} \) generated by the relation \( \sim_{ae} \). Put \( A\mathcal{L} = \mathcal{AL}/_{\sim_{ae}} \), whose elements are abstract links.
Theorem 3.3 (1) There is a map (which we call skimming process)

\[ \phi : \mathcal{VL} \to \mathcal{AL} \]

that induces a bijection

\[ \phi : \mathcal{VL} \leftrightarrow \mathcal{AL}. \]

An abstract link diagram is regarded as a disk-band surface such that there is a usual crossing in each disk and a proper arc in each band, see Fig. 3 (4). Fig. 3 is an illustration of the skimming process, see [7] for the definition.

The inclusion map

\[ \iota : \mathcal{AL} \to \mathcal{D} \]

induces a map

\[ \iota_\sim : \mathcal{AL} \to \mathcal{D}/\sim. \]

Proposition 3.4 The map \( \iota_\sim : \mathcal{AL} \to \mathcal{D}/\sim \) is a bijection.

Proof. For \((F, D) \in \mathcal{D}\), let \(N(D)\) be a regular neighborhood of \(|D|\) in \(F\). Then \((N(D), D)\) is an abstract link diagram, which we denote by \(\text{Abs}(F, D)\). Since \(\text{Abs}(F, D) \sim (F, D)\), we see that the map \(\iota_\sim\) is surjective. Suppose that two abstract link diagrams \((F, D)\) and \((F', D')\) are stably Reidemeister equivalent. There exists a sequence \((F, D) = (F_1, D_1) \sim (F_2, D_2) \sim \cdots \sim (F_{n-1}, D_{n-1}) \sim (F_n, D_n) = (F', D')\). Then we have a sequence \((F, D) = (F_1, D_1) \sim (F_2, D_2) \sim \cdots \sim (F_{n-1}, D_{n-1}) \sim (F_n, D_n) = (F', D')\). Thus \((F, D) \sim (F', D')\) and the map \(\iota_\sim\) is injective. \(\Box\)

Now we see that \(\mathcal{VL}, \mathcal{AL}, \mathcal{D}/\sim\) and the set of Reidemeister equivalence classes of Gauss codes are mutually equivalent.

4 Link homology and concordance of virtual links

We recall the definition of knotted surface diagrams [2]. A knotted surface diagram \(K\) is a generically and properly mapped surface in a 3-manifold \(M\) such that the double point curves are given crossing information. Thus \(K\) has isolated branch and triple points and double curves. Along each double curve, one of the two sheets involved is over-sheet, the other is the under-sheet, and the under-sheet is broken (interior of small neighborhood
Definition 4.1 Let \((F_i, D_i), i = 0, 1\) be two elements of \(\mathcal{D}\) such that \(D_i\) consists of \(n\) components \(D^j_i, j = 1, \ldots, n\) for a positive integer \(n\). Then \((F_0, D_0)\) and \((F_1, D_1)\) are called virtually link-homologous if there exists a compact oriented 3-manifold \(M\) and a knotted surface diagram \(S\) in \(M\) with the following properties.

1. \(F_0 \cup -F_1 \subset \partial M\), where \(-F_1\) denotes \(F_1\) with its orientation reversed.
2. \(S\) is a knotted surface diagram of an oriented surface with \(n\) components \(S^j, j = 1, \ldots, n\), such that \(\partial S^j = D^j_0 \cup -D^j_1\) for all \(j\).

Definition 4.2 Two elements \((F_i, D_i)\) \((i = 0, 1)\) of \(\mathcal{D}\) as above, are called virtually link-concordant (or simply concordant if no confusion occurs) if there exists a compact oriented 3-manifold \(M\) and a knotted surface diagram \(S\) in \(M\) with the following properties.

1. \(F_0 \cup -F_1 \subset \partial M\).
2. \(S\) is a knotted surface diagram of an oriented surface with \(n\) components \(S^j, j = 1, \ldots, n\), such that \(\partial C^j = D^j_0 \cup -D^j_1\) and each \(S^j\) is an annulus.

Lemma 4.3 If two elements \((F, D)\) and \((F', D')\) of \(\mathcal{D}\) are stably equivalent, then they are virtually link-concordant, and hence virtually link-homologous.

Proof. It is sufficient to prove that if \((F, D) \sim (F', D')\) then \((F, D)\) and \((F', D')\) are virtually link-concordant. Let \(f : F \to G\) and \(f' : F' \to G\) be embeddings into a surface \(G\) such that \(f(D)\) and \(f(D')\) are related by Reidemeister moves in \(G\). Let \(M = G \times [0, 1]\) and regard \(f : F \to G \times \{0\}\) and \(f' : F' \to G \times \{1\}\), and identify \(F\) and \(F'\) with the subsets \(f(F)\) and \(f'(F')\) of \(M\) respectively, so that \(F \cup -F' \subset \partial M\). Reidemeister moves between \(f(F)\) and \(f'(F')\) in \(G\) yield a knotted surface diagram of an annulus in \(G \times [0, 1]\) such that the type I, II, and III moves correspond to branch points, minimal points of double point curves, and triple points (cf. 2), respectively. Hence the result follows. \(\blacksquare\)

Corollary 4.4 The virtual link-concordance and the virtual link-homology are well-defined for elements of \(\mathcal{D}/\sim\).

Lemma 4.5 Let \((F_0, D_0)\) and \((F_1, D_1)\) be elements of \(\mathcal{D}\). They are virtually link-homologous if and only if one is obtained from the other by a sequence of moves depicted in Fig. 3 together with Reidemeister moves.

Proof. Let \((F_0, D_0)\) and \((F_1, D_1)\) be virtually link-homologous via a knotted surface diagram \(S\) in a 3-manifold \(M\). In the following, we regard \(S\) as the underlying generic surface without crossing information for considerations of Morse critical points. Let
Figure 4: Moves for link-homology

$h : M \rightarrow [0, 1]$ be a smooth map such that $h(F_0) = 0$ and $h(F_1) = 1$. We may assume (after a small perturbation if necessary) that $h$ satisfies the following conditions.

1. $h$ is transverse at 0 and 1.
2. $h$ is generic on $M$, $\partial M$, and $S$, and on all the self intersections and singularities of $M$, $\partial M$, and $S$.

Thus $h$ has isolated Morse critical points on all the sets listed in (2), at distinct critical values. The singularities on $S$ gives Reidemeister moves, and the move listed in Fig. 4 bottom. Specifically, the type I, II, and III moves correspond to branch points, minimal/maximal points of double point curves, and triple points. The minimal/maximal points and saddle points of $S$ corresponds to bottom left and right, respectively, of Fig. 4. The Morse critical points as handle moves are listed in Fig. 4 top and middle. The critical points of $\partial M$ are maxima/minima (the top left entry) or saddle points (the top right). From the point of view of the boundary 1-manifold, they correspond to handles of indices 0/2 and 1, respectively. The critical points of $\text{Int}M$ are similar, and depicted in the second row left and right. Theorem follows as these exhaust generic singularities and critical points.

Similarly, we have

**Lemma 4.6** Let $(F, D)$ and $(F', D')$ be elements of $\mathcal{D}$. They are virtually link-concordant if and only if one is obtained from the other by a sequence of Reidemeister moves and moves depicted in Fig. 4, such that the moves satisfy the following condition: the sequence of moves form surface diagrams whose underlying surfaces are annuli.

**Definition 4.7** Let $(F, D)$ be an element of $\mathcal{D}$ such that $D$ is a link diagram with $n$ components, $D_j$, $j = 1, \ldots, n$. The **linking number** between the $j$ and $k$th components,
denoted by $\text{Link}(D_j, D_k)$ is the number of crossings between $D_j$ and $D_k$ where $D_j$ is over and $D_k$ is under-arc respectively, counted with signs.

This linking number is the same as the virtual linking number $\text{vlk}(D_j, D_k)$ in the sense of [3] under the correspondence between virtual links and links on surfaces via skimming process.

**Example 4.8** In the first row of Fig. 5, virtual pseudo-Hopf links are depicted. The images of them by the skimming process (in the second row) are abstract pseudo-Hopf links. They are positive if the crossings are positive (right figure); otherwise negative (left). The component containing the upper crossing is called an upper component and the other a lower component. Let $D = D_1 \cup D_2$ be a positive abstract pseudo-Hopf link such that $D_1$ is upper and $D_2$ is lower, then $\text{Link}(D_1, D_2) = 1$ and $\text{Link}(D_2, D_1) = 0$.

**Proposition 4.9** Virtual link-homology classes of the elements of $\mathcal{D}$ are completely classified by pairwise linking numbers.
Proof. By Lemma 4.5 and the definition, the linking numbers are invariants of link homology. We prove the converse. Since \((F, D) \in \mathcal{D}\) and \(\text{Abs}(F, D)\) are virtually link-homologous and have the same linking numbers, we may assume that \((F, D)\) is an abstract link diagram. Eliminate each crossing point of \(D\) as in Fig. 6, and we have a split sum of a trivial abstract link diagram and some abstract pseudo-Hopf links. We cap off each component of the trivial abstract link diagram. The remainder is a union of abstract pseudo-Hopf links.

![Figure 7: Canceling a pair of pseudo-Hopf links](image)

For each \(j\) and \(k\) with \(j \neq k\), collect abstract pseudo-Hopf links whose upper components come from \(D_j\), the \(j\)th component of \(D\), and the lower components come from \(D_k\). In this family, a pair of positive and negative abstract pseudo-Hopf links are canceled as in Fig. 7. (For simplicity, the figure is drawn in terms of virtual link diagrams. Apply the skimming process to obtain the moves in terms of the abstract pseudo-Hopf links.) So we have \(|\text{Link}(D_j, D_k)|\) copies of abstract pseudo-Hopf links whose signs are the same with the sign of \(\text{Link}(D_j, D_k)\).

Collect abstract pseudo-Hopf links whose upper and lower components come from the \(j\)th component of \(D\), for \(j = 1, \ldots, n\). If necessary, applying Reidemeister moves of type I, we may assume that the number of positive crossings of \(D_j\) and the negative crossings of \(D_j\) were the same. Then we can eliminate the abstract pseudo-Hopf links in this family as in Fig. 8. This implies the proposition. □

Lemma 4.10 For any \((F, D) \in \mathcal{D}\), there is an oriented 3-manifold \(M\) and an oriented surface diagram \(G\) in \(M\) such that \(\partial M = F\) and \(\partial G = D\).

Proof. Perform a smoothing at each crossing of \(D\) to obtain disjoint simple closed curves \(D'\) on \(F\). A smoothing is realized as a branch point. Specifically, regard \(D\) as lying on \(F \times \{0\}\) and \(D'\) on \(F \times \{1\}\), then there is an oriented knotted surface diagram \(S\) with a branch point corresponding to each smoothing, such that \(\partial S = D \cup -D'\). We cap off each component of \(D'\) by attaching a 2-handle. Then we have a desired \(M\) and \(G\). □

A classical link diagram \(D\) on \(\mathbb{R}^2\) is regarded as an element of \(\mathcal{D}\) by considering \((E, D)\), where \(E\) is a large 2-disk in \(\mathbb{R}^2\) containing \(D\) inside.
Corollary 4.11 Two classical link diagrams are virtually link-homologous if and only if the classical links represented by them are link-homologous in classical sense.

Proof. Link homology classes in classical sense are classified by linking numbers, whose definition match that of the linking numbers for elements of $\mathcal{D}$. The above proposition, then, implies this corollary. □

Definition 4.12 An abstractly spanning surface of $(F, D) \in \mathcal{D}$ is a surface $G$ as inLemma 4.11. The spanning genus of $(F, D) \in \mathcal{D}$, denoted by $\text{Span-g}(F, D)$, is the minimal genus of all abstractly spanning surfaces for $(F, D)$.

Lemma 4.13 If two elements $(F, D)$ and $(F', D')$ are stably equivalent, then $\text{Span-g}(F, D) = \text{Span-g}(F', D')$.

Proof. This is a consequence of Lemma 4.3. □

Remark 4.14 Other kinds of genera of interest are defined as follows. A closed realization of $(F, D) \in \mathcal{D}$ is an embedding of $F$ to a closed oriented surface $G$. The supporting genus of $(F, D) \in \mathcal{D}$, denoted by $\text{Supp-g}(F, D)$, is the minimal genus of such closed oriented surfaces $G$, cf. [4, 5, 6, 7]. The ground genus of $(F, D) \in \mathcal{D}$, denoted by $\text{Ground-g}(F, D)$, is the minimal of $\text{Supp-g}(F', D')$ such that $(F', D')$ is stably equivalent to $(F, D)$.

5 Slice curves on surfaces and Kauffman’s example

Virtual link-concordance on $\mathcal{D}$ is naturally defined similarly for $\mathcal{C}$ simply ignoring the crossing informations. Lemma 4.3 holds for $\mathcal{C}$ under such a definition, and thus the virtual link-concordance is well-defined for $\mathcal{C}/\sim$.

Definition 5.1 If $(F, D) \in \mathcal{D}$ or $(F, C) \in \mathcal{C}$ is virtually concordant to the unlink in the plane, then it is called slice.

By the remark before the definition, we have

Proposition 5.2 Sliceness for $\mathcal{C}$ is an invariant under stable equivalence: Suppose that $(F, C) \sim (F', C')$. Then $(F, C)$ is slice if and only if $(F', C')$ is slice.

In [1], a necessary condition for sliceness of immersed closed curves in a surface was given.

Theorem 5.3 ([1]) The pair $(F, C)$ in Fig. 8 is not slice.

Figure 8 is different from the example given in [1]. However it has the same Gauss code with that in [1] and hence it is not slice.

In [10], L. Kauffman gave two problems:
1. Is the virtual knot diagram in Fig. 9 (1) virtually Reidemeister equivalent to a classical knot diagram? (The quandle and the group are the same as those of a trivial knot diagram.)

2. Is the universe (Fig. 9 (2)) of the virtual knot irreducible?

Here a universe of a virtual knot diagram is a virtual knot diagram without information of over/under crossings for real crossings (do not confuse them with virtual crossings). Virtual Reidemeister moves for the universes of virtual knot diagrams are defined by ignoring over/under information for real crossings. The universe of a virtual knot is reducible if it is transformed into the universe of a classical knot diagram by virtual Reidemeister moves.

Proposition 5.4 The virtual knot in Fig. 9 (1) is not virtually Reidemeister equivalent to a classical knot diagram.

Proof. We have a map

\[ \rho = \pi \circ \iota \circ \phi : \mathcal{VL} \to \mathcal{AL} \to \mathcal{D} \to \mathcal{C} \]

which induces a map

\[ \rho = \pi_\sim \circ \iota_\sim \circ \phi : V\mathcal{L} \to A\mathcal{L} \to \mathcal{D}_\sim \to \mathcal{C}_\sim. \]
The virtual knot diagram in Fig. 9 (1) is mapped to \((F, C) \in \mathcal{C}\) in Fig. 8. This is not slice (by Theorem 5.3). On the other hand, any classical knot diagram, which is regarded as an element of \(\mathcal{D}\) by considering it is on a large 2-disk in \(\mathbb{R}^2\), is mapped to an element of \(\mathcal{C}\) which is slice. Since sliceness is invariant under stably equivalence on \(\mathcal{C}\), we see that the virtual knot is not virtually Reidemeister equivalent to a classical knot diagram. 

Alternate proofs are given in [13] and [14].

**Proposition 5.5** The universe in Fig. 9 (2) is not equivalent to the universe of a classical knot diagram.

**Proof.** The map

\[
\pi : \mathcal{VL} \to \mathcal{SVL}
\]

sending a virtual link diagram to its universe induces a map

\[
\pi_\sim : \mathcal{VL} \to \mathcal{SVL},
\]

where \(\mathcal{SVL}\) is the set of universes of virtual link diagrams and \(\mathcal{SVL}\) is the set of equivalence classes. It is not difficult to see that the map \(\rho : \mathcal{VL} \to \mathcal{C}/_{\sim}\) factors through \(\mathcal{SVL}\); namely, when we put \(f = \iota_{\sim} \circ \phi\), there is a map \(f'\) which makes the following diagram commutative.

\[
\begin{array}{ccc}
\mathcal{VL} & \xrightarrow{f} & \mathcal{D}/_{\sim} \\
\pi_\sim & & \downarrow \pi_\sim \\
\mathcal{SVL} & \xrightarrow{f'} & \mathcal{C}/_{\sim}.
\end{array}
\]

The universe in Fig. 9 (2) is not equivalent to the universe of a classical link diagram, because their images under \(f'\) are distinguished in \(\mathcal{C}/_{\sim}\) by sliceness. 

**References**

[1] J. S. Carter, *Closed curves that never extend to proper maps of disks*, Proc. Amer. Math. Soc. **113** (1991), 879–888.

[2] J.S. Carter and M. Saito, *Knotted surfaces and their diagrams*, the American Mathematical Society, 1998.

[3] M. Goussarov, M. Polyak and O. Viro, *Finite type invariants of classical and virtual knots*, to appear Topology. Preprint [http://xxx.lanl.gov/abs/math.GT98100073].

[4] N. Kamada, *On alternating simple formal presentation (in Japanese)*, talk at the regional conference held at Waseda University, Tokyo, Japan, in December 1993.

[5] N. Kamada, *Alternating link diagrams on compact oriented surfaces*, preprint (1995).

[6] N. Kamada, *The crossing number of alternating link diagrams of a surface*, talk at the international conference “Knots 96” held at Waseda University, Tokyo, Japan, in July 1996.
[7] N. Kamada and S. Kamada, *Abstract link diagrams and virtual knots*, J. Knot Theory Ramifications, to appear.

[8] L. H. Kauffman, *Virtual knots*, talks at MSRI Meeting in January 1997 and AMS Meeting at University of Maryland, College Park in March 1997.

[9] L. H. Kauffman, *Virtual Knot Theory*, European J. Combin. 20 (1999), 663–690.

[10] L. H. Kauffman, *Virtual Knot Theory*, talk at AMS Meeting, Washington D.C. in January 2000.

[11] D. E. Penney, *Establishing isomorphism between tame prime knots in* $E^3$, Pacific J. Math. 40 (1972), 675–680.

[12] D. E. Penney, *An algorithm for establishing isomorphism between tame prime knots in* $E^3$, Doctoral Dissertation, Tulane University, New Orleans (1965).

[13] J. Sawollek, *On Alexander-Conway polynomials for virtual knots and links*, preprint (available at [http://xxx.lanl.gov/abs/math.GT/9912173](http://xxx.lanl.gov/abs/math.GT/9912173)).

[14] D. S. Silver and S. G. Williams, *Alexander groups and virtual links*, preprint.