Possible failure of asymptotic freedom in two-dimensional $RP^2$ and $RP^3$ $\sigma$-models

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Abstract

We have simulated the two-dimensional $RP^2$ and $RP^3$ $\sigma$-models, at correlation lengths up to about 220 (resp. 30), using a Wolff-type embedding algorithm. We see no evidence of asymptotic scaling. Indeed, the data rule out the conventional asymptotic scaling scenario at all correlation lengths less than about $10^9$ (resp. $10^5$). Moreover, they are consistent with a critical point at $\beta \approx 5.70$ (resp. 6.96), only 2% (resp. 5%) beyond the largest $\beta$ at which we ran. Preliminary studies of a mixed $S^{N-1}/RP^{N-1}$ model (i.e. isovector + isotensor action) show a similar behavior when $\beta_T \to \infty$ with $\beta_V$ fixed $\lesssim 0.6$, while they are consistent with conventional asymptotic freedom along the lines $\beta_T/\beta_V$ fixed $\lesssim 2$. Taken as a whole, the data cast doubt on (though they do not completely exclude) the idea that $RP^{N-1}$ and $S^{N-1}$ $\sigma$-models lie in the same universality class.
The present work grows out of our study of Wolff-type embedding algorithms for general nonlinear $\sigma$-models \[1, 2\]. Our main result \[2\] was that a Wolff-type embedding algorithm can perform well — in the sense of having a dynamic critical exponent $z \ll 2$ — only if it is based on an involutive isometry of the target manifold $M$ whose fixed-point manifold has codimension 1. Such an involutive isometry exists only if $M$ is a sphere, a real projective space, or a discrete quotient of products of such spaces.

After extensive studies of the models where $M$ is a sphere $S^{N-1}$ (the so-called $N$-vector model), we considered next the models with $M = \mathbb{R}P^{N-1} = S^{N-1}/\mathbb{Z}_2$. We were thus led to re-examine the long-standing puzzle concerning the phase diagram of these models.

This interest is not purely academic. $\mathbb{R}P^2 \sigma$-models are used as models for nematic liquid crystals, where instead of spins one deals with the orientation of elongated molecules exhibiting a head-tail ($\sigma \to -\sigma$) symmetry.

On a $\mathbb{Z}^2$ lattice we shall consider (for the moment)

$$-\beta H_T(\{\sigma\}) = \frac{\beta}{2} \sum_{(xy)} (\sigma_x \cdot \sigma_y)^2$$

(T for “tensor”) with $\sigma_x \in S^{N-1}$. Clearly $H_T$ is invariant under the $\mathbb{Z}_2$ gauge transformations

$$\sigma_x \to \eta_x \sigma_x$$

with $\eta_x \in \mathbb{Z}_2 = \{\pm 1\}$. It follows that all non-gauge-invariant correlation functions vanish identically, for example

$$\langle \sigma^a_x \sigma^b_y \rangle = 0 \quad \text{if } x \neq y .$$

The fundamental correlation function is thus the isotensor

$$G_T(x, y) = \langle (\sigma_x \cdot \sigma_y)^2 \rangle - 1/N .$$

In the formal continuum limit, the discrete gauge symmetry plays no role. Locally the spaces $S^{N-1}$ and $\mathbb{R}P^{N-1}$ are the same, and one thus expects the model (1) to have the same continuum limit in the isotensor sector as the ordinary $N$-vector model.

At $N = \infty$, the model (1) undergoes a first-order transition \[4\]; but this transition involves the spontaneous breaking of the $\mathbb{Z}_2$ gauge symmetry, and
thus cannot persist to finite $N$. For finite $N$, some workers have found (by high-temperature expansions or Monte Carlo simulations) indications of a second-order phase transition, which could be related [3] to a condensation of $Z_2$-vortices in the field

$$A_\mu(x) = \text{sign} [\sigma_x \cdot \sigma_{x+\mu}] . \quad (5)$$

If there is not a phase transition, the model is presumably asymptotically free, and a renormalization-group calculation predicts that for $N > 2$ the correlation length $\xi$ and susceptibility $\chi$ behave as

$$\xi^{af}(\beta) \sim C_\xi \beta^{-1/(N-2)} e^{2\pi \beta/(N-2)} \quad (6)$$
$$\chi^{af}(\beta) \sim C_\chi \beta^{-(N+1)/(N-2)} e^{4\pi \beta/(N-2)} \quad (7)$$

up to corrections of order $1/\beta$.

By using the exact results for the mass gap for the $O(N)$-models [6] and the ratios of $\Lambda$-parameters in different schemes (which are extracted from first-order perturbation theory), it is even possible to predict what is the value of the constant $C_\xi$ for the exponential correlation length (inverse mass gap). In the following we shall make use of the so-called second-moment correlation length, which should differ by a few percent from the exponential correlation length.

In Fig. 1 we report the ratio of the measured correlation length to the value (6) predicted by asymptotic freedom, as a function of $\beta$ on lattices of sizes $32 \leq L \leq 512$, for both $RP^2$ and $RP^3$. The observed correlation length is smaller than the expected value by a factor of about $10^7$ [7] (resp. $10^4$). The simulations at the larger values of $\beta$ and $L$ clearly show that the asymptotic regime has not been reached, as the curves show a clear rise up.

In any case, our simulations on small lattices prove that if the conventional scenario is true, the asymptotic regime cannot be seen on lattices of size less than about $10^9$ (resp. $10^5$) — a result which is quite astonishing.

Let us assume, instead, that there is a phase transition at a finite $\beta = \beta_c$. Then we expect by finite-size scaling that

$$\xi(\beta) = L f_\xi(|\beta - \beta_c|^\nu L) \quad (8)$$
$$\chi(\beta) = L^{\gamma/\nu} f_\chi(|\beta - \beta_c|^\nu L) \quad (9)$$

where $L$ is the size of the lattice, and $\nu$ and $\gamma$ are, as usual, the critical exponents for the correlation length and the susceptibility. Thus we could
try to fix $\beta_c$ and $\nu$ by using the data for $\xi$ on different lattice sizes and by requiring that all the obtained values of $\xi/L$, when plotted as a function of $(1 - \beta/\beta_c)L^{1/\nu}$, fall on the same curve. Analogously, once $\beta_c$ and $\nu$ have been fixed, $\gamma$ can be found by imposing (9).

In Fig. 2 we have plotted the observed values of $\xi/L$ as a function of $(1 - \beta/\beta_c)L^{1/\nu}$ with $\beta_c = 5.70$ and $\nu = 2.0$ (see also [5]). The same analysis for $\chi$ fixes $\gamma \approx 3.3$. Also for $RP^3$ it is possible to fit the data by supposing a $\beta_c = 6.96$ and the same values of $\nu$ and $\gamma$ as for $RP^2$.

The quite large values of the critical exponents may suggest a phase transition of the Kosterlitz-Thouless type: in this case the behaviour given in (8) and (9) should be substituted by

$$\xi(\beta) = Lf_\xi\left(\exp\left[-A(1 - \beta/\beta_c)^{-1/2}\right]L\right)$$  \hspace{1cm} \text{(10)}$$

$$\chi(\beta) = L^{\gamma/\nu}f_\chi\left(\exp\left[-A(1 - \beta/\beta_c)^{-1/2}\right]L\right)$$  \hspace{1cm} \text{(11)}$$

where (10) should be used to determine the constants $A$ and $\beta_c$. In Fig. 3 we have plotted the observed values of $\xi/L$ as a function of $\exp(-A/|1 - \beta/\beta_c|^{1/2})L$ with $\beta_c = 5.85$ and $A = 1.73$. The same analysis for $\chi$ fixes $\gamma/\nu \approx 1.7$. Also for $RP^3$ it is possible to fit the data by supposing $\beta_c = 7.15$ and the values $A = 1.7$ and $\gamma/\nu = 1.65$.

A plot of $\chi/L^{\gamma/\nu}$ as a function of $\xi/L$ provides directly an estimate of $\gamma/\nu$, which is independent of the assumptions on the scaling behaviour. See Fig. 4.

Next we considered the Hamiltonian

$$-\beta H_{\text{mixed}}(\{\sigma\}) =$$

$$= \beta_V \sum_{(xy)} \sigma_x \cdot \sigma_y + \frac{\beta_T}{2} \sum_{(xy)} (\sigma_x \cdot \sigma_y)^2$$  \hspace{1cm} \text{(12)}$$

(V for “vector”) which interpolates between the models with target space $S^{N-1}$ and $RP^{N-1}$. The normalization of the two terms has been chosen in order to get $\beta = \beta_V + \beta_T$ as the coupling constant in the usual low-temperature expansion.

For $N = 4$, we made simulations along the lines with $r = \beta_T/\beta_V$ fixed and not too large (0.5, 1.0, 2.0). We found that the model behaves qualitatively in agreement with the conventional scenario of asymptotic freedom in the sense that
• the ratio $\xi/\xi^{af}$ is not very far from 1, although it becomes worse and worse when $r$ increases and at $r = 2.0$ it is around 0.4;

• the estimated values of $\beta_c$ which appear when we try the Ansätze (8)/(9) or (10)/(11) are relatively far from the actual values of $\beta$ which can be simulated effectively. But this gap decreases as $r$ increases;

• the ratio of the vectorial over the tensorial correlation length seems to approach a constant value in the continuum limit;

• the dynamic behaviour of the Wolff embedding algorithm is in the same universality class as for the $O(N)$ model, which means that the dynamic critical exponent is almost zero (critical slowing-down is eliminated).

On the other hand, in the region where $\beta_T \to \infty$ with $\beta_V$ fixed and small ($\lesssim 0.6$), the model looks very similar to the $RP^{N-1}$ case: the ratio $\xi/\xi^{af}$ is extremely small, and the estimated values of a transition point $\beta_c$ are very close to the largest values of $\beta$ at which we ran. The vectorial sector appears decoupled from the tensorial one: the vectorial correlation length and susceptibility appear to approach finite values even when the corresponding tensorial quantities start to diverge. Also from the dynamical point of view the behaviour is the same as in the $RP^{N-1}$ case, with a dynamic critical exponent for the tensorial susceptibility close to 1 [3].

But we have also seen, in our Monte Carlo runs for $N = 3$, that there is a crossover, near $\beta_V \approx 0.5$, where also the vectorial sector begins to become critical. Ising-like fluctuations seem to become important in this regime, so that the Wolff embedding algorithm is less efficient. We are currently investigating this region: it would be interesting to understand whether the apparent transition curve which starts on the axis $\beta_V = 0$ stops somewhere, and whether the Ising critical point which occurs at $(\beta_T = \infty, \beta_V = 0.44 \ldots)$ is isolated or belongs to a phase-separation curve extending to finite $\beta_T$. The phase diagram could appear like in Fig. [3], but are the phase transitions actually there? An unconventional scenario has been suggested [3] in which asymptotic freedom does not hold even in the pure vector model. Indeed, conjectured correlation inequalities, based on the remark that the couplings are both ferromagnetic, would suggest that the line where the tensorial correlation length diverges — if there is one — cannot stop somewhere in the middle of the phase diagram, and must at least intersect every ray with
$r = \beta_T / \beta_V$ fixed $> 0$. We are continuing to run our simulation, and hope to add other pieces of evidence to clarify these important open questions.

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Figure 1: Asymptotic freedom test for $RP^2$ and $RP^3$.

Figure 2: Scaling for the correlation length for $RP^2$.

Figure 3: Scaling of the Kosterlitz-Thouless type for the correlation length for $RP^2$.

Figure 4: $\chi/L^{1.7}$ vs. $\xi/L$ for $N = 3$ with $0 \leq \beta V \leq 0.5$.

Figure 5: A possible phase diagram.