THE WEIGHTS OF SIMPLE MODULES IN CATEGORY $\mathcal{O}$ FOR KAC–MOODY ALGEBRAS

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ABSTRACT. We give the first positive formulas for the weights of every simple highest weight module $L(\lambda)$ over an arbitrary Kac–Moody algebra. Under a mild condition on the highest weight, we also express the weights of $L(\lambda)$ as an alternating sum similar to the Weyl–Kac character formula.

To obtain these results, we show the following data attached to a highest weight module are equivalent: (i) its integrability, (ii) the convex hull of its weights, (iii) the Weyl group symmetry of its character, and (iv) when a localization theorem is available, its behavior on certain codimension one Schubert cells. We further determine precisely when the above datum determines the weights themselves. Moreover, we use condition (iv) to relate localizations of the convex hull of the weights with the introduction of poles of the corresponding $D$-module on certain divisors, which answers a question of Brion.

Many of these results are new even in finite type. We prove similar assertions for highest weight modules over a symmetrizable quantum group.

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1. Introduction

A fundamental problem in the representation theory of semisimple and Kac–Moody Lie algebras is to compute characters of simple highest weight modules. In all cases where they are known even conjecturally, the solutions proceed roughly by (i) expressing simple characters as linear combinations of Verma characters. The relevant block of Category $\mathcal{O}$ is then (ii) identified as a categorification of a module for the Hecke algebra. The two bases of simple and Verma characters correspond to two canonical bases for the module, whose change of basis is determined via the combinatorics of the Coxeter system.

The subtlety in steps (i) and (ii) is already apparent for regular integral blocks for affine Lie algebras. At noncritical levels, as in finite type, one takes bona fide Verma modules, and the relevant Hecke module is the regular representation. At the critical level, one replaces each Verma module with a quotient, the restricted Verma module, and the desired Hecke module is the periodic module. To our knowledge, there is not even a conjectural description of critical blocks beyond affine type along the lines of (i) and (ii).

Another issue of longstanding interest is to obtain manifestly positive formulas for simple characters. I.e., in step (i) the coefficients of Verma characters generically come with signs, as visible in the Weyl–Kac and Kazhdan–Lusztig character formulas, and one would like alternative formulae for simple characters without this issue. This problem is solved via crystal bases for regular dominant and regular anti-dominant highest weights, but is wide open in general.

With these motivations, in this paper we address the simpler question of determining not the character of a simple highest weight module but rather its weights. I.e., one can ask for the eigenvalues of the Cartan’s action, but not the exact multiplicities. We give multiple formulas for weights of an arbitrary simple highest weight module for any Kac–Moody algebra. In light of the above discussion, we emphasize first that the formulas hold in the cases where there are no predictions for characters in the spirit of (i) and (ii), and moreover are remarkably uniform across different levels. We emphasize second that the formulas are visibly positive.

After conversations with experts, it seems the formulas described in this paper likely did not appear earlier, even as folklore. This is rather remarkable, considering both the accessibility and naturality of the question and the transparency of its solution.

2. Statements of results

Throughout the paper, unless otherwise specified $\mathfrak{g}$ is a Kac–Moody algebra over $\mathbb{C}$ with fixed triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. We will write $I$ for the vertices of its Dynkin diagram, and denote its simple roots, simple coroots, and the
simple reflections in its Weyl group $W$ by

$$\alpha_i, \tilde{\alpha}_i, s_i, \quad \text{for } i \in I,$$

respectively. We will denote by $V$ a $\mathfrak{g}$-module of highest weight $\lambda \in \mathfrak{h}^*$, and by $L(\lambda)$ the simple module with highest weight $\lambda$.

2.1. **Three positive formulas for the weights of simple modules.** If for $\mu \in \mathfrak{h}^*$ we consider the corresponding weight space of $L(\lambda)$, namely

$$L(\lambda)_\mu := \{v \in L(\lambda) : hv = \langle \lambda, h \rangle v \text{ for all } h \in \mathfrak{h}\},$$

then our main result is several descriptions of the set of nonzero weight spaces of $L(\lambda)$, i.e.,

$$\text{wt } L(\lambda) := \{\mu \in \mathfrak{h}^* : L(\lambda)_\mu \neq 0\}. \quad (2.1)$$

In every case when the character of $L(\lambda)$ is known, an essential role is played by a subgroup of $W$ known as the integral Weyl group. To determine the weights of $L(\lambda)$, perhaps surprisingly we will need much less, namely the intersection of the integral Weyl group with the simple reflections. Accordingly, consider the subset of $I$ given by

$$I_\lambda := \{i \in I : \langle \tilde{\alpha}_i, \lambda \rangle \in \mathbb{Z}\}. \quad (2.2)$$

2.1.1. Our first formula uses the restriction of $L(\lambda)$ to the maximal standard Levi subalgebra whose action is integrable. To state it, write $l_\lambda$ for the standard Levi subalgebra of $\mathfrak{g}$ corresponding to $I_\lambda$. For $\nu \in \mathfrak{h}^*$, write $L_{l_\lambda}(\nu)$ for the simple $l_\lambda$-module of highest weight $\nu$, and as in (2.1) write $\text{wt } L_{l_\lambda}(\nu)$ for its set of nonzero weights.

**Proposition 2.3.** For all $\lambda \in \mathfrak{h}^*$, we have the equality

$$\text{wt } L(\lambda) = \bigcup_{\mu \in \mathbb{Z}_{\geq 0}^{\text{det}} \{\alpha_i : i \in I \setminus I_\lambda\}} \text{wt } L_{l_\lambda}(\lambda - \mu). \quad (2.4)$$

In words, the proposition asserts that (i) for every positive integral combination $\mu$ of the simple roots not indexed by $I_\lambda$, the weight space $L(\lambda)_{\lambda - \mu}$ is nonzero. After restriction to $l_\lambda$, such a weight space consists of highest weight vectors, and the proposition implies that (ii) all the weights of $L(\lambda)$ may be obtained as union of those of the $L_{l_\lambda}(\lambda - \mu)$. The latter are integrable $l_\lambda$-modules, and hence their weights are well known, cf. Proposition 4.3 below.
2.1.2. Our second formula shows the relationship between $\text{wt} L(\lambda)$ and its convex hull, which we denote by $\text{conv} L(\lambda)$. To state it, recall the standard partial order $\leq$ on $h^*$, where $\mu \leq \lambda$ if $\lambda - \mu$ is a (possibly empty) sum of simple roots.

**Proposition 2.5.** For all $\lambda \in h^*$, we have the equality

$$\text{wt} L(\lambda) = \text{conv} L(\lambda) \cap \{ \mu \in h^* : \mu \leq \lambda \}.$$  

(2.6)

In words, the proposition asserts that the left-hand side of Equation (2.6), which necessarily lies in the intersection of the two terms of the right-hand side, in fact coincides with it. The question of whether Proposition 2.5 holds, i.e., whether the weights of simple highest weight modules are no finer an invariant than their convex hull, was brought to our attention by Daniel Bump [9].

Moreover, we give an explicit description of $\text{conv} L(\lambda)$, and more generally of the convex hull of the weights of any highest weight module, in Proposition 4.10 below.

2.1.3. Our third formula describes the weights of $L(\lambda)$ using the group $W_{I_\lambda}$ generated by the simple reflections $s_i$, for $i \in I_\lambda$, i.e., the Weyl group of $I_\lambda$. We will also need the dominant integral weights of $I_\lambda$, namely

$$P^+_\lambda := \{ \mu \in h^* : \langle \mu, \check{\alpha}_i \rangle \in \mathbb{Z}^\geq 0, i \in I_\lambda \}.$$

**Proposition 2.7.** If $\lambda$ has finite stabilizer in $W_{I_\lambda}$, then we have the equality

$$\text{wt} L(\lambda) = W_{I_\lambda} \{ \mu \in P^+_\lambda : \mu \leq \lambda \}.$$  

(2.8)

We remind that the assumption that $\lambda$ has finite stabilizer is very mild, as recalled in Proposition 3.2(4). In words, the proposition states that the left-hand side of (2.8), which necessarily lies in the right-hand side by the integrability of $L(\lambda)$ as a $I_\lambda$-module, coincides with it.

2.1.4. Propositions 2.3, 2.5, and 2.7 are well known for integrable $L(\lambda)$, and Propositions 2.3 and 2.5 were proved by the second named author in finite type [27]. The remaining cases, in particular Proposition 2.7 in all types, are to our knowledge new.

Propositions 2.3, 2.5, and 2.7 are particularly striking in infinite type. When specialized to affine Kac–Moody algebras, the formulae are insensitive to whether the highest weight is critical or non-critical. In contrast, critical level modules behave very differently from noncritical modules, even at the level of characters. For symmetrizable Kac–Moody algebras, we similarly obtain weight formulae for highest weights on the critical hyperplanes. The authors are unaware of even conjectural formulae for the simple characters in this case. Finally, for non-symmetrizable Kac–Moody algebras, we recall that it is unknown how to compute weight space multiplicities even for integrable $L(\lambda)$. 
2.2. Three invariants of highest weight modules. The main input into our three formulas presented above is the equivalence of three invariants of general highest weight modules. In what follows, we denote the simple lowering operators by $f_i$, for $i \in I$.

**Theorem 2.9.** The following invariants of a highest weight module $V$ determine one another.

1. The integrability of $V$, i.e. the subset of the vertices of the Dynkin diagram defined as
   
   $$I_V := \{ i \in I : f_i \text{ acts locally nilpotently on } V \}.$$

2. The convex hull of the weights of $V$.
3. The stabilizer of the character of $V$ in the Weyl group $W$.

Explicitly, the convex hull in (2) is always that of the parabolic Verma module $M(\lambda, I_V)$ (cf. Equation (3.5) below), and the stabilizer in (3) is always the parabolic subgroup $W_{I_V}$.

To our knowledge, the equivalences of Theorem 2.9 are new even for $\mathfrak{g}$ of finite type. Moreover, when a localization theorem is available, we describe an equivalent geometric invariant of the perverse sheaf corresponding to $V$ in Proposition 2.14 below.

In the proof of Theorem 2.9, the difficult implication is that the natural map

$$M(\lambda, I_V) \otimes V_\lambda \to V,$$

where $V_\lambda$ is the highest weight line of $V$, induces an equality on convex hulls of weights.

For a general highest weight module $V$, its set of weights is a finer invariant than the datum of Theorem 2.9. For example, for $\mathfrak{g} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, the highest weight modules $M(0)$ and $V = M(0)/M(s_1s_2 \cdot 0)$, where $s_1, s_2$ are the simple reflections and $\cdot$ denotes the dot action, have different sets of weights but the same convex hull. In the following theorem, we determine precisely when the two invariants are equivalent.

To state it, we consider the potential integrability of $V$, i.e. the subset of the vertices of the Dynkin diagram given by

$$I_\lambda \setminus I_V.$$

Note these index the simple roots for which the corresponding $\mathfrak{sl}_2$ action is integrable on a nonzero quotient of $V$. The following theorem shows that the phenomenon of the above example, namely the potential integrability contains orthogonal roots, is the only obstruction to $\text{conv } V$ and $\text{wt } V$ being distinct invariants.
Theorem 2.10. Fix \( \lambda \in \mathfrak{h}^* \) and \( \tilde{I} \subset I_\lambda \). Every highest weight module \( V \) of highest weight \( \lambda \) and integrability \( \tilde{I} \) has the same weights if and only if \( I_\lambda \setminus \tilde{I} \) is complete, i.e.,

\[ \langle \hat{\alpha}_i, \alpha_j \rangle \neq 0, \quad \text{for all } i, j \in I_\lambda \setminus \tilde{I}. \]

The question of whether Theorem 2.10 holds was brought to our attention by James Lepowsky [34]. We are not aware of a precursor to Theorem 2.10 in the literature, even in finite type.

2.3. An alternating formula for the weights of simple modules. We now discuss an alternating formula for the weights of \( L(\lambda) \). We will first write the formula, and then explain the appearing notation.

Theorem 2.11. For \( \lambda \in \mathfrak{h}^* \) such that the stabilizer of \( \lambda \) in \( W_{I_\lambda} \) is finite, we have:

\[
\text{wt } L(\lambda) = \sum_{w \in W_{I_\lambda}} w \frac{e^\lambda}{\prod_{i \in \tilde{I}} (1 - e^{-\alpha_i})}. \tag{2.12}
\]

On the left-hand side of (2.12) we mean the ‘multiplicity-free’ character

\[
\sum_{\mu \in \mathfrak{h}^*: L(\lambda)_\mu \neq 0} e^\mu.
\]

On the right-hand side of (2.12), in each summand we take the ‘highest weight’ expansion of \( w(1 - e^{-\alpha_i})^{-1} \), i.e.:

\[
w \frac{1}{1 - e^{-\alpha_i}} := \begin{cases} 
1 + e^{-w\alpha_i} + e^{-2w\alpha_i} + \ldots, & \text{if } w\alpha_i > 0, \\
-e^{w\alpha_i} - e^{2w\alpha_i} - e^{3w\alpha_i} - \ldots, & \text{if } w\alpha_i < 0.
\end{cases}
\]

We now make three qualitative comments on the appearance of (2.12). First, as claimed in the introduction, each summand comes with signs but no multiplicities as in Kazhdan–Lusztig theory. Second, we note that the denominator in (2.12) is not the usual Weyl denominator, but instead a multiplicity-free variant which runs only over the simple roots, i.e. replaces \( n \) by \( n/\lfloor n/n \rfloor \). Finally, the group \( W_{I_\lambda} \) which indexes the sum is a parabolic subgroup of \( W \), rather than the integral Weyl group which appears in the known cases of the characters of simple highest weight modules.

Theorem 2.11 was previously known for integrable modules by work of Kass [26], Brion [7], Walton [42], Postnikov [39], and Schützer [40], where it closely resembles the usual Weyl–Kac character formula [43]. As pointed out to us by Michel Brion [8], in finite type Theorem 2.11 follows from a more general formula for exponential sums over polyhedra, cf. Remark 7.9 below. All other cases are to our knowledge new.
2.4. A geometric interpretation of integrability. In this section we use a topological manifestation of the datum in Theorem 2.9 to answer a question brought to our attention by Brion [8] in geometric representation theory.

To pose the question, let $\mathfrak{g}$ be of finite type, $G$ the corresponding simply-connected algebraic group, and $B$ the subgroup with Lie algebra $\mathfrak{h} \oplus \mathfrak{n}^+$. Fix $\lambda$ a regular dominant integral weight, and consider $\text{conv } L(\lambda)$, i.e., the Weyl polytope. For any $\check{I} \subset I$, if one intersects the tangent cones of $\text{conv } L(\lambda)$ at the vertices $w\lambda$, for $w \in W_{\check{I}}$, one obtains the convex hull of $M(\lambda, \check{I})$. Brion first asked whether an analogous formula holds for other highest weight modules. We answer this question affirmatively in the companion paper [10].

Brion further observed that this procedure of localizing $\text{conv } L(\lambda)$ in convex geometry is the shadow of a localization in complex geometry. More precisely, consider the flag variety $G/B$, and let $L_\lambda$ denote the line bundle with

$$H^0(G/B, L_\lambda) \cong L(\lambda).$$

As usual, for any $w \in W$ write $X_w$ for the closure of the Bruhat cell $BwB/B$, and write $w_o$ for the longest element of $W$. Let $\mathbb{D}$ denote the standard dualities on Category $\mathcal{O}$ and regular holonomic $D$-modules, i.e., the contragredient and Verdier dualities, respectively. Then for a union of Schubert divisors

$$Z = \bigcup_{i \in I \setminus \check{I}} X_{w_i w_o}$$

with complement $U$, we have

$$H^0(U, L_\lambda) \cong \mathbb{D}M(\lambda, \check{I}).$$

Thus in this case localization of the convex hull could be recovered as taking the convex hull of the weights of sections of $L_\lambda$ on an appropriate open set $U$. Brion raised in [8] the question of whether a similar result holds for more general highest weight modules.

We answer this affirmatively for a regular integral infinitesimal character. By translation, it suffices to examine the regular block $\mathcal{O}_0$.

**Theorem 2.13.** Let $\mathfrak{g}$ be of finite type, and $\lambda = w \cdot -2\rho$. Let $V$ be a $\mathfrak{g}$-module of highest weight $\lambda$, and write $\mathcal{V}$ for the corresponding $D$-module on $G/B$. For $\check{I} \subset I_V$, set

$$Z = \bigcup_{i \in I_V \setminus \check{I}} X_{w_i w_o},$$

with complement $U$, we have

$$H^0(U, \mathcal{V}_\lambda) \cong \mathbb{D}M(\lambda, \check{I}).$$
and write $U$ for its open complement in $G/B$. Then the $\mathfrak{g}$-module
\[ D^0(U, D\mathcal{V}) \]
has highest weight $\lambda$ and integrability $\hat{I}$.

Let us mention one ingredient in the proof of Theorem 2.13, namely that the datum of Theorem 2.9 has the following geometric manifestation.

**Proposition 2.14.** For $V$ and $\mathcal{V}$ as above, we have
\[ I_V = \{ i \in I : \text{the *-restriction of } \mathcal{V} \text{ to } C_{s_iw} \text{ is nonzero} \}. \]

### 2.5. Highest weight modules over symmetrizable quantum groups.

In the final section, we apply both the methods and the results from earlier to study highest weight modules over quantum groups, and obtain results similar to those discussed above.

### 2.6. A further problem – multiplicity-free Macdonald identities.

We conclude this section by calling attention to a problem suggested by our results. Namely, the statement of Theorem 2.11 includes the assumption that the highest weight has finite integrable stabilizer. However, the asserted identity otherwise tends to fail in interesting ways. For example, we found the following by a direct calculation (see also the extended abstract [11]).

**Proposition 2.15.** For $\mathfrak{g}$ of rank 2 and the trivial module $L(0)$, we have
\[ \sum_{w \in W} \frac{w}{(1 - e^{-\alpha_1})(1 - e^{-\alpha_2})} = 1 + \sum_{\alpha \in \Delta_{\text{im}}} e^\alpha, \tag{2.16} \]
where $\Delta_{\text{im}}$ denotes the set of negative imaginary roots.

It would be interesting to extend Theorem 2.11 to more general objects, e.g. the trivial representation, which may be thought of as multiplicity-free Macdonald identities. Indeed, Equation (2.16) suggests correction terms coming from imaginary roots, akin to [23].

**Remark 2.17.** Since the writing of this paper, such a multiplicity-free Macdonald identity was established in untwisted affine type by Niu–Zhang [38].

**Organization of the paper.** After introducing notation in Section 3 we prove in Section 4 the main result. In Sections 5–8 we develop different applications of the main result. Finally, in Section 9 we discuss extensions of the previous sections to quantized enveloping algebras.
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3. Preliminaries and notations

The contents of this section are mostly standard. We advise the reader to skim Subsection 3.3 and refer back to the rest only as needed.

3.1. Notation for numbers and sums. We write \( \mathbb{Z} \) for the integers, and \( \mathbb{R}, \mathbb{C} \) for the real and complex numbers respectively. For a subset \( S \) of a real vector space \( E \), we write \( \mathbb{Z} \geq 0 S \) for the set of finite linear combinations of \( S \) with coefficients in \( \mathbb{Z} \geq 0 \), and similarly \( \mathbb{Z} S, \mathbb{R} \geq 0 S \), etc.

3.2. Notation for Kac–Moody algebras, standard parabolic and Levi subalgebras. The basic references are [24] and [31]. In this paper we work throughout over \( \mathbb{C} \). Let \( I \) be a finite set, and \( A = (a_{ij})_{i,j \in I} \) a generalized Cartan matrix. Fix a realization \((\mathfrak{h}, \pi, \tilde{\pi})\), with simple roots \( \pi = \{\alpha_i\}_{i \in I} \subset \mathfrak{h}^* \) and coroots \( \tilde{\pi} = \{\tilde{\alpha}_i\}_{i \in I} \subset \mathfrak{h} \) satisfying \((\tilde{\alpha}_i, \alpha_j) = a_{ij}, \forall i,j \in I\).

Let \( g := g(A) \) be the associated Kac–Moody algebra generated by \( \{e_i, f_i : i \in I\} \) and \( \mathfrak{h} \), modulo the relations:

\[
\begin{align*}
[e_i, f_j] &= \delta_{ij}\tilde{\alpha}_i, \quad [h, e_i] = (h, \alpha_i)e_i, \quad [h, f_i] = -(h, \alpha_i)f_i, \quad [\mathfrak{h}, \mathfrak{h}] = 0, \quad \forall h \in \mathfrak{h}, \quad i, j \in I, \\
(ad e_i)^{1-a_{ij}}(e_j) &= 0, \quad (ad f_i)^{1-a_{ij}}(f_j) = 0, \quad \forall i, j \in I, \quad i \neq j.
\end{align*}
\]

Denote by \( \overline{g}(A) \) the quotient of \( g(A) \) by the largest ideal intersecting \( \mathfrak{h} \) trivially; these coincide when \( A \) is symmetrizable. When \( A \) is clear from context, we will abbreviate these to \( \overline{g} \).

In the following we establish notation for \( g \); the same apply for \( \overline{g} \) mutatis mutandis. Let \( \Delta^+, \Delta^- \) denote the sets of positive and negative roots, respectively. We write \( \alpha > 0 \) for \( \alpha \in \Delta^+ \), and similarly \( \alpha < 0 \) for \( \alpha \in \Delta^- \). For a sum of roots \( \beta = \sum_{i \in I} k_i \alpha_i \) with all \( k_i \geq 0 \), write \( \text{supp} \beta := \{i \in I : k_i \neq 0\} \). Write

\[
\begin{align*}
n^- &:= \bigoplus_{\alpha < 0} g_\alpha, \quad n^+ := \bigoplus_{\alpha > 0} g_\alpha.
\end{align*}
\]
Let $\leq$ denote the standard partial order on $\mathfrak{h}^*$, i.e. for $\mu, \lambda \in \mathfrak{h}^*$, $\mu \leq \lambda$ if and only if $\lambda - \mu \in \mathbb{Z}^{\geq 0}$. For any $J \subset I$, let $I_J$ denote the associated Levi subalgebra generated by $\{e_i, f_i : i \in J\}$ and $\mathfrak{h}$. For $\lambda \in \mathfrak{h}^*$ write $L_{I_J}(\lambda)$ for the simple $I_J$-module of highest weight $\lambda$. Writing $A_J$ for the principal submatrix $(a_{i,j})_{i,j \in J}$, we may (non-canonically) realize $\mathfrak{g}(A_J) =: \mathfrak{g}_J$ as a subalgebra of $\mathfrak{g}(A)$. Now write $\pi_J, \Delta^+_J, \Delta^-_J$ for the simple, positive, and negative roots of $\mathfrak{g}(A_J)$ in $\mathfrak{h}^*$, respectively (note these are independent of the choice of realization). Finally, we define the associated Lie subalgebras $u^+_J, u^-_J, n^+_J, n^-_J$ by:

$$u^+_J := \bigoplus_{\alpha \in \Delta^+ \setminus \Delta^+_J} \mathfrak{g}_\alpha, \quad n^+_J := \bigoplus_{\alpha \in \Delta^+_J} \mathfrak{g}_\alpha,$$

and $p_J := I_J \oplus u^+_J$ to be the associated parabolic subalgebra.

### 3.3. Weyl group, parabolic subgroups, Tits cone.

Write $W$ for the Weyl group of $\mathfrak{g}$, generated by the simple reflections $\{s_i, i \in I\}$, and let $\ell : W \to \mathbb{Z}^{\geq 0}$ be the associated length function. For $J \subset I$, let $W_J$ denote the parabolic subgroup of $W$ generated by $\{s_j, j \in J\}$.

Write $P^+$ for the dominant integral weights, i.e. $\{\mu \in \mathfrak{h}^* : (\alpha_i, \mu) \in \mathbb{Z}^{\geq 0}, \forall i \in I\}$. The following choice is non-standard. Define the real subspace $\mathfrak{h}^*_{\mathbb{R}} := \{\mu \in \mathfrak{h}^* : (\alpha_i, \mu) \in \mathbb{R}, \forall i \in I\}$. Now define the dominant chamber as $D := \{\mu \in \mathfrak{h}^* : (\alpha_i, \mu) \in \mathbb{R}^{\geq 0}, \forall i \in I\} \subset \mathfrak{h}^*_{\mathbb{R}}$, and the Tits cone as $C := \bigcup_{w \in W} wD$.

**Remark 3.1.** In [31] and [24], the authors define $\mathfrak{h}^*_{\mathbb{R}}$ to be a real form of $\mathfrak{h}^*$. This is smaller than our definition whenever the generalized Cartan matrix $A$ is non-invertible, and has the consequence that the dominant integral weights are not all in the dominant chamber, unlike for us. This is a superficial difference, but our convention helps avoid constantly introducing arguments like [31, Lemma 8.3.2].

We will also need parabolic analogues of the above. For $J \subset I$, define $\mathfrak{h}^*_{\mathbb{R}}(J) := \{\mu \in \mathfrak{h}^* : (\alpha_j, \mu) \in \mathbb{R}, \forall j \in J\}$, the $J$ dominant chamber as $D_J := \{\mu \in \mathfrak{h}^* : (\alpha_j, \mu) \in \mathbb{R}^{\geq 0}, \forall j \in J\}$, and the $J$ Tits cone as $C_J := \bigcup_{w \in W_J} wD_J$. Finally, we write $P^+_J$ for the $J$ dominant integral weights, i.e. $\{\mu \in \mathfrak{h}^* : (\alpha_j, \mu) \in \mathbb{R}^{\geq 0}, \forall j \in J\}$. The following standard properties will be used without further reference in the paper:

**Proposition 3.2.** For $\mathfrak{g}(A)$ with realization $(\mathfrak{h}, \pi, \bar{\pi})$, let $\mathfrak{g}(A^\ell)$ be the dual algebra with realization $(\mathfrak{h}^*, \bar{\pi}, \pi)$. Write $\Delta^+_J$ for the positive roots of the standard Levi subalgebra $\mathfrak{p}_J^* \subset \mathfrak{g}(A^\ell)$.

1. For $\mu \in D_J$, the isotropy group $\{w \in W_J : w\mu = \mu\}$ is generated by the simple reflections it contains.
The $J$ dominant chamber is a fundamental domain for the action of $W_J$ on the $J$ Tits cone, i.e., every $W_J$ orbit in $C_J$ meets $D_J$ in precisely one point.

$C_J = \{ \mu \in \mathfrak{h}_\mathbb{R}^* (J) : (\check{\alpha}, \mu) < 0 \text{ for at most finitely many } \check{\alpha} \in \check{\Delta}_J^+ \}$; in particular, $C_J$ is a convex cone.

Consider $C_J$ as a subset of $\mathfrak{h}_\mathbb{R}^* (J)$ in the analytic topology, and fix $\mu \in C_J$. Then $\mu$ is an interior point of $C_J$ if and only if the isotropy group \( \{ w \in W_J : w \mu = \mu \} \) is finite.

We also fix $\rho \in \mathfrak{h}^*$ satisfying $(\check{\alpha}_i, \rho) = 1$, $\forall i \in I$, and define the dot action of $W$ via $w \cdot \mu := w(\mu + \rho) - \rho$; this does not depend on the choice of $\rho$.

3.4. Representations, integrability, and parabolic Verma modules. Given an $\mathfrak{h}$-module $M$ and $\mu \in \mathfrak{h}^*$, write $M_\mu$ for the corresponding simple eigenspace of $M$, i.e., $M_\mu := \{ m \in M : hm = (h, \mu)m \forall h \in \mathfrak{h} \}$, and write $\text{wt} M := \{ \mu \in \mathfrak{h}^* : M_\mu \neq 0 \}$.

Let $V$ be a highest weight $\mathfrak{g}$-module with highest weight $\lambda \in \mathfrak{h}^*$. For $J \subset I$, we say $V$ is $J$ integrable if $f_j$ acts locally nilpotently on $V$, $\forall j \in J$. The following standard lemma may be deduced from \cite{31} Lemma 1.3.3 and the proof of \cite{31} Lemma 1.3.5.

Lemma 3.3.

1. If $V$ is of highest weight $\lambda$, then $V$ is $J$ integrable if and only if $f_j$ acts nilpotently on the highest weight line $V_\lambda$, $\forall j \in J$.

2. If $V$ is $J$ integrable, then $\text{ch} V$ is $W_J$-invariant.

Let $I_V$ denote the maximal $J$ for which $V$ is $J$ integrable, i.e.,

$$I_V = \{ i \in I : (\check{\alpha}_i, \lambda) \in \mathbb{Z}^{\geq 0}, f_i^{(\check{\alpha}_i, \lambda) + 1} V_\lambda = 0 \}.$$  \hspace{1cm} (3.4)

We will call $W_{I_V}$ the integrable Weyl group.

We next remind the basic properties of parabolic Verma modules over Kac–Moody algebras. These are also known in the literature as generalized Verma modules, e.g. in the original papers by Lepowsky (see \cite{33} and the references therein).

Fix $\lambda \in \mathfrak{h}^*$ and a subset $J$ of $I_\lambda = \{ i \in I : (\check{\alpha}_i, \lambda) \in \mathbb{Z}^{\geq 0} \}$. The parabolic Verma module $M(\lambda, J)$ co-represents the following functor from $\mathfrak{g}$-mod to Set:

$$M \rightsquigarrow \{ m \in M_\lambda : n^+ m = 0, f_j \text{ acts nilpotently on } m, \forall j \in J \}.$$  \hspace{1cm} (3.5)

When $J$ is empty, we simply write $M(\lambda)$ for the Verma module. From the definition and Lemma 3.3, it follows that $M(\lambda, J)$ is a highest weight module that is $J$ integrable.
More generally, for any subalgebra \( l_J \subset \mathfrak{s} \subset \mathfrak{g} \), the module \( M_\mathfrak{s}(\lambda, J) \) will corepresent the functor from \( \mathfrak{s} \)-mod to Set:

\[
M \leadsto \{ m \in M_\lambda : (\mathfrak{n}^+ \cap \mathfrak{s})m = 0, \ f_j \text{ acts nilpotently on } m, \forall j \in J \}.
\] (3.6)

We will only be concerned with \( \mathfrak{s} \) equal to a Levi or parabolic subalgebra. In accordance with the literature, in the case of “full integrability” we define \( L_{\text{max}}(\lambda) := M_\mathfrak{p}(\lambda, J) \) and \( L_{\text{max}}^+(\lambda) := M_\mathfrak{l}(\lambda, J) \) for \( \lambda \in P_J^+ \). Finally, write \( L_{l_J}(\lambda) \) for the simple quotient of \( L_{\text{max}}(\lambda) \).

**Proposition 3.7** (Basic formulae for parabolic Verma modules). Let \( \lambda \in P_J^+ \).

1. \( M(\lambda, J) \simeq M_\lambda/(f_j^\langle \alpha_j, \lambda \rangle + 1) M_\lambda, \forall j \in J \).
2. Given a Lie subalgebra \( l_J \subset \mathfrak{s} \subset \mathfrak{g} \), define \( \mathfrak{s}^+_J := l_J \oplus (\mathfrak{u}^+_J \cap \mathfrak{s}) \). Then:

\[
M_\mathfrak{s}(\lambda, J) \simeq \text{Ind}_{\mathfrak{s}_J^+}^{\mathfrak{s}_J} \text{Res}_{\mathfrak{s}_J^+}^{\mathfrak{s}_J} L_{l_J}(\lambda),
\] (3.8)

where we view \( l_J \) as a quotient of \( \mathfrak{s}^+_J \) via the short exact sequence:

\[
0 \rightarrow \mathfrak{u}^+_J \cap \mathfrak{s} \rightarrow \mathfrak{s}^+_J \rightarrow l_J \rightarrow 0.
\]

In particular, for \( \mathfrak{s} = \mathfrak{g} \):

\[
\text{wt } M(\lambda, J) = \text{wt } L_{l_J}(\lambda) - \mathbb{Z}^{\geq 0}(\Delta^+ \setminus \Delta_J^+).
\] (3.9)

We remark that some authors use the term inflation (from \( l_J \) to \( \mathfrak{s}^+_J \)) in place of \( \text{Res}_{\mathfrak{s}_J^+}^{\mathfrak{s}_J} \).

## 4. Three Invariants of Highest Weight Modules

In this section we prove the main result of the paper, Theorem 2.9, except for the perverse sheaf formulation, which we address in Section 8. The two tools we use are: (i) an “Integrable Slice Decomposition” of the weights of parabolic Verma modules \( M(\lambda, J) \), which extends a previous construction in [27] to representations of Kac–Moody algebras; and (ii) a “Ray Decomposition” of the convex hull of these weights, which is novel in both finite and infinite type for non-integrable highest weight modules.

### 4.1. The Integrable Slice Decomposition.

Our first goal will be to prove the following statement.

**Proposition 4.1** (Integrable Slice Decomposition).

\[
\text{wt } M(\lambda, J) = \bigcup_{\mu \in \mathbb{Z}^{\geq 0}(\pi \setminus \pi_J)} \text{wt } L_{l_J}(\lambda - \mu).
\] (4.2)

In particular, \( \text{wt } M(\lambda, J) \) lies in the \( J \) Tits cone (cf. Section 3.3).
In proving Proposition 4.1 and below, the following results will be of use to us:

**Proposition 4.3** (§11.2 and Proposition 11.3(a)). For \( \lambda \in P^+ \), \( \mu \in h^* \), say \( \mu \) is non-degenerate with respect to \( \lambda \) if \( \mu \leq \lambda \) and \( \lambda \) is not perpendicular to any connected component of \( \text{supp}(\lambda - \mu) \), cf. Section 3.2 for notation. Let \( V \) be an integrable module of highest weight \( \lambda \).

1. If \( \mu \in P^+ \), then \( \mu \in \text{wt} \ V \) if and only if \( \mu \) is non-degenerate with respect to \( \lambda \).
2. If the sub-diagram on \( \{ i \in I : (\alpha_i, \lambda) = 0 \} \) is a disjoint union of diagrams of finite type, then \( \mu \in P^+ \) is non-degenerate with respect to \( \lambda \) if and only if \( \mu \leq \lambda \).
3. \( \text{wt} \ V = (\lambda - Z_{\geq 0}^{\pi}) \cap \text{conv}(W \lambda) \).

Proposition 4.3 is explicitly stated in [24] for \( \mathfrak{g} \), but also holds for \( \mathfrak{g} \). To see this, note that \( \text{wt} \ V \subset (\lambda - Z_{\geq 0}^{\pi}) \cap \text{conv}(W \lambda) \), and the latter are the weights of its simple quotient \( L(\lambda) \), which is inflated from \( \mathfrak{g} \).

**Proof of Proposition 4.1.**

The disjointness of the terms on the right-hand side is an easy consequence of the linear independence of simple roots.

We first show the inclusion \( \supset \). Recall the isomorphism of Proposition 3.7:

\[
M(\lambda, J) \simeq M(\lambda)/(f_j^{(\alpha_j, \lambda)}\lambda + 1 M(\lambda), \forall j \in J).
\]

It follows that the weights of \( \ker(M(\lambda) \to M(\lambda, J)) \) are contained in \( \bigcup_{j \in J} \{ \nu \leq s_j \lambda \} \). Hence \( \lambda - \mu \) is a weight of \( M(\lambda, J), \forall \mu \in Z_{\geq 0}^{\pi_{I \setminus J}}. \) Any nonzero element of \( M(\lambda, J)_{\lambda - \mu} \) generates an integrable highest weight \( I_J \)-module. As the weights of all such modules coincide by Proposition 4.3(3), this shows the inclusion \( \supset \).

We next show the inclusion \( \subset \). For any \( \mu \in Z_{\geq 0}^{\pi_{I \setminus J}} \), the ‘integrable slice’

\[
S_\mu := \bigoplus_{\nu \in \lambda - \mu + Z_{\pi_J}} M(\lambda, J)_\nu
\]

lies in Category \( \mathcal{O} \) for \( I_J \), and is furthermore an integrable \( I_J \)-module. It follows that the weights of \( M(\lambda, J) \) lie in the \( J \) Tits cone.

Let \( \nu \) be a weight of \( M(\lambda, J) \), and write \( \lambda - \nu = \mu_J + \mu_{I \setminus J}, \mu_J \in Z_{\geq 0}^{\pi_J}, \mu_{I \setminus J} \in Z_{\geq 0}^{\pi_{I \setminus J}}. \) We need to show \( \nu \in \text{wt} \ L_{I_J}(\lambda - \mu_{I \setminus J}) \). By \( W_J \)-invariance, we may assume \( \nu \) is \( J \) dominant. By Proposition 4.3 it suffices to show that \( \nu \) is non-degenerate with respect to \( (\lambda - \mu_{I \setminus J}) \). To see this, using Proposition 3.7 write:

\[
\nu = \lambda - \mu_L - \sum_{k=1}^n \beta_k, \quad \text{where} \quad \lambda - \mu_L \in \text{wt} \ L_{I_J}(\lambda), \ \beta_k \in \Delta^+ \setminus \Delta_J^+, \ 1 \leq k \leq n.
\]

The claimed nondegeneracy follows from the fact that \( \lambda - \mu_L \) is nondegenerate with respect to \( \lambda \) and that the support of each \( \beta_k \) is connected. \( \square \)
As an immediate consequence of the Integrable Slice Decomposition 4.1, we present a family of decompositions of $\text{wt} M(\lambda, J)$, which interpolates between the two sides of Equation (4.2):

**Corollary 4.4.** For subsets $J \subset J' \subset I$, we have:

$$\text{wt} M(\lambda, J) = \bigcup_{\mu \in \mathbb{Z}^\geq 0 (\pi \setminus \pi_{J'})} \text{wt} M_{I_{\mu'}}(\lambda - \mu, J),$$

(4.5)

where $M_{I_{\mu'}}(\lambda, J)$ was defined in Equation (3.6).

As a second consequence, Proposition 4.3(2) and the Integrable Slice Decomposition 4.1 yield the following simple description of the weights of most parabolic Verma modules.

**Corollary 4.6.** Suppose $\lambda$ has finite $W_J$-isotropy. Then,

$$\text{wt} M(\lambda, J) = \bigcup_{w \in \mathcal{W}_J} \{ \nu \in P^+_J : \nu \leq \lambda \}.$$ 

(4.7)

Equipped with the Integrable Slice Decomposition, we provide a characterization of the weights of a parabolic Verma module that will be helpful in Section 5.

**Proposition 4.8.**

$$\text{wt} M(\lambda, J) = (\lambda + \mathbb{Z}\pi) \cap \text{conv} M(\lambda, J).$$

(4.9)

**Proof.** The inclusion $\subset$ is immediate. For the reverse $\supset$, note that $\text{conv} M(\lambda, J)$ lies in the $J$ Tits cone and is $W_J$ invariant. It then suffices to consider a point $\nu$ of the right-hand side which is $J$ dominant. Write $\nu$ as a convex combination:

$$\nu = \sum_{k=1}^n t_k (\lambda - \mu^k_{I \setminus J} - \mu^k_J),$$

where $t_k \in \mathbb{R}^\geq, \sum_k t_k = 1, \mu^k_{I \setminus J} \in \mathbb{Z}^\geq 0 \pi_{I \setminus J}, \mu^k_J \in \mathbb{Z}^\geq 0 \pi_J, 1 \leq k \leq n$. By Propositions 4.3 and 4.1, it remains to observe that $\nu$ is non-degenerate with respect to $\lambda - \sum_k t_k \mu^k_{I \setminus J}$, as a similar statement holds for each summand. □

4.2. **The Ray Decomposition.** Using the Integrable Slice Decomposition, we obtain the following novel description of $\text{conv} M(\lambda, J)$, which is of use in the proof of the main theorem and throughout the paper.

**Proposition 4.10** (Ray Decomposition).

$$\text{conv} M(\lambda, J) = \text{conv} \bigcup_{w \in W_J, i \in I \setminus J} w(\lambda - \mathbb{Z}^\geq 0 \alpha_i).$$

(4.11)

When $J = I$, by the right-hand side we mean $\text{conv} \bigcup_{w \in W} w\lambda$. 

Proof. The containment $\supset$ is straightforward using the definition of integrability and $W_I$ invariance. For the containment $\subset$, it suffices to show every weight of $M(\lambda, J)$ lies in the right-hand side. Note that the right-hand side contains $W_J(\lambda - Z^{\geq 0} \pi_{J \setminus I})$, so we are done by the Integrable Slice Decomposition \[4.11\] \[4.11\]

4.3. Proof of the main result. With the Integrable Slice and Ray Decompositions in hand, we now turn to our main theorem.

Theorem 4.12. Given a highest weight module $V$ and a subset $J \subset I$, the following statements are equivalent.

1. One has the equality of subsets of vertices of the Dynkin diagram $J = I_V$.
2. One has the equality of convex hulls $\text{conv} \ wt V = \text{conv} \ wt M(\lambda, J)$.
3. The stabilizer of $\text{conv} V$ in $W$ is $W_J$.

The connection to perverse sheaves promised in Section 3 is proven in Proposition 5.5.

Proof. To show (1) implies (2), note that $\lambda - Z^{\geq 0} \alpha_i \subset \text{wt} V, \forall i \in I \setminus I_V$. The implication now follows from the Ray Decomposition \[4.10\]. The remaining implications follow from the assertion that the stabilizer of $\text{conv} V$ in $W$ is $W_J$. Thus, it remains to prove the assertion. It is standard that $W_{I_V}$ preserves $\text{conv} V$. For the reverse, since (1) implies (2), we may reduce to the case of $V = M(\lambda, J)$. Suppose $w \in W$ stabilizes $\text{conv} M(\lambda, J)$. It is easy to see $\lambda$ is a face of $\text{conv} M(\lambda, J)$, hence so is $w\lambda$. However by the Ray Decomposition \[4.10\] it is clear that the only 0-faces of $\text{conv} M(\lambda, J)$ are $W_J(\lambda)$, so without loss of generality we may assume $w$ stabilizes $\lambda$.

Recalling that $W_J$ is exactly the subgroup of $W$ which preserves $\Delta^+ \setminus \Delta^+_J$, it suffices to show $w$ preserves $\Delta^+ \setminus \Delta^+_J$. Let $\alpha \in \Delta^+ \setminus \Delta^+_J$; then by Proposition 3.7 $\lambda - Z^{\geq 0} \alpha \subset \text{wt} M(\lambda, J)$, whence so is $\lambda - Z^{\geq 0} w(\alpha)$ by Proposition 4.8. This shows that $w(\alpha) > 0$; it remains to show $w(\alpha) \notin \Delta^+_J$. It suffices to check this for the simple roots $\alpha_i, i \in I \setminus J$. Suppose not, i.e. $w(\alpha_i) \in \Delta^+_J$ for some $i \in I \setminus J$. In this case, note $w(\alpha_i)$ must be a real root of $\Delta^+_J$, e.g. by considering $2w(\alpha_i)$. Thus the $w(\alpha_i)$ root string $\lambda - Z^{\geq 0} w(\alpha_i)$ is the set of weights of an integrable representation of $g_{-w(\alpha_i)} \oplus [g_{-w(\alpha_i)}, g_{w(\alpha_i)}] \oplus g_{w(\alpha_i)} \simeq sl_2$, which is absurd. \[4.11\]

Corollary 4.13. The stabilizer of $\text{ch} V$ in $W$ is $W_{I_V}$.

Corollary 4.14. For any $V$, $\text{conv} V$ is the Minkowski sum of $\text{conv} W_{I_V}(\lambda)$ and the cone $\mathbb{R}^{\geq 0} W_{I_V}(\pi_{J \setminus I_V})$.

Proof. By Theorem 4.12, we reduce to the case of $V$ a parabolic Verma module. Now the result follows from the Ray Decomposition \[4.10\] and Equation \[8.9\]. \[4.11\]
Remark 4.15. For \( g \) semisimple, the main theorem 4.12 and the Ray Decomposition \( 4.10 \) imply that the convex hull of weights of any highest weight module \( V \) is a \( W_{I_V} \)-invariant polyhedron. To our knowledge, this was known for many but not all highest weight modules by recent work \[27\], and prior to that only for parabolic Verma modules. We develop many other applications of the main theorem to the convex geometry of \( \text{conv} V \), including the classification of its faces and their inclusions, in the companion work \[10\].

For completeness, we record here the description of the extremal rays of the convex shape \( \text{conv} V \), which is only implicit in the work \[10\].

Corollary 4.16. The extremal rays in \( \text{conv} V \) are the rays
\[
w(\lambda - \mathbb{R} \geq 0 \alpha_i), \quad \text{for all } w \in W_{I_V} \text{ and } i \in I \setminus I_V. \quad (4.17)
\]

See also the recent preprint \[31\], where G.K. Teja employs a different argument to prove the same result.

Proof. By Theorem 4.12, we may assume \( V \) is a parabolic Verma module. We will first show that any extremal ray \( R \) of \( \text{conv} V \) must lie in the collection \( (4.17) \). To see this, by the Ray Decomposition \( 4.10 \) we may write an interior point \( r \) of \( R \) as a convex combination of points \( r_{w,i} \) of the rays in \( (4.17) \). If any \( r_{w,i} \) is an interior point of the corresponding ray of \( (4.17) \), we are done. If not, then \( R \) would be an extremal ray of \( \text{conv} W_{I_V}(\lambda) \). However, as each
\[
w\lambda, \quad \text{for } w \in W_{I_V},
\]
is a face of \( \text{conv} W_{I_V}(\lambda) \), this straightforwardly implies that \( \text{conv} W_{I_V}(\lambda) \) contains no extremal rays.

It remains to check that each element of \( (4.17) \) is extremal, i.e. is a face of \( \text{conv} V \). As \( W_{I_V} \) acts by automorphisms of \( \text{conv} V \), we may assume \( w = e \). For a fixed \( i \in I \setminus I_V \), it suffices to show that the ray
\[
\lambda - \mathbb{R} \geq 0 \alpha_i
\]
maximizes the restriction to \( \text{conv} V \) of a real functional \( \xi : \mathfrak{h}^* \to \mathbb{R} \). To see this, one may pick any \( \xi \) satisfying
\[
\xi(\alpha_i) = 0 \quad \text{and} \quad \xi(\alpha_j) > 0, \quad \text{for all } j \in I \setminus i,
\]
which completes the proof. \( \square \)

5. Three positive formulas for the weights of simple modules

We remind the three positive formulas for the weights of simple highest weight modules to be obtained in this section.
Proposition 5.1. Write \( l \) for the Levi subalgebra corresponding to \( I_\lambda \), and write \( L_l(\nu) \) for the simple \( l \) module of highest weight \( \nu \in \mathfrak{h}^* \). Then:

\[
\wt L(\lambda) = \bigsqcup_{\mu \in \mathbb{Z}^{\geq 0} \setminus \mathfrak{h} \lambda} \wt L_l(\lambda - \mu).
\]

(5.2)

Proposition 5.3.

\[
\wt L(\lambda) = \text{conv } L(\lambda) \cap \{ \mu \in \mathfrak{h}^* : \mu \leq \lambda \}.
\]

(5.4)

Proposition 5.5. Suppose \( \lambda \) has finite stabilizer in \( W_{I_\lambda} \). Then:

\[
\wt L(\lambda) = W_{I_\lambda} \{ \mu \in P^+_{I_\lambda} : \mu \leq \lambda \}.
\]

(5.6)

Remark 5.7. For an extension of Proposition 5.5 to arbitrary \( \lambda \), see the companion work [10].

Note that the three weight formulas above are immediate consequences of combining the following theorem with Propositions 4.1 and 4.8 and Corollary 4.6 respectively. Recall from Section 2.1 that for a highest weight module

\[
\text{Proposition 5.1.}
\]

Suppose \( \lambda \) is a highest weight line. Let \( l \) denote the Levi subalgebra corresponding to \( I_\lambda \). Then \( \ wt V = \ wt M(\lambda, I_V) \) if and only if

\[
\wt U(l) V_\lambda = \wt U(l) M(\lambda, I_V)_\lambda,
\]

i.e. \( \wt U(l) V_\lambda = \lambda - \mathbb{Z}^{\geq 0} \pi_{I_V} \). In particular, if \( V = L(\lambda) \), then \( \wt L(\lambda) = \wt M(\lambda, I_\lambda) \).

Proof. It is clear, e.g. by the Integrable Slice Decomposition 4.1 that if \( \wt V = \wt M(\lambda, I_V) \), then \( \wt I_V V = \lambda - \mathbb{Z}^{\geq 0} \pi_{I_V} \). For the reverse implication, by again using the Integrable Slice Decomposition and the action of \( I_V \), it suffices to show that \( \lambda - \mathbb{Z}^{\geq 0} \pi_{I_V} \subset \wt V \).

Suppose not, i.e. there exists \( \mu \in \mathbb{Z}^{\geq 0} \pi_{I_V} \) with \( \lambda - \mu = 0 \). We decompose \( I \setminus I_V = I^p \cup I^q \), and write accordingly \( \mu = \mu_p + \mu_q \), where \( \mu_p \in \mathbb{Z}^{\geq 0} \pi_{I^p} \) and \( \mu_q \in \mathbb{Z}^{\geq 0} \pi_{I^q} \). By assumption \( V_{\lambda - \mu_p} \) is nonzero, so by the PBW theorem, there exists \( \partial \in U(n_{I_V}^{\mu}) \) of weight \( -\mu_p \) such that \( \partial \cdot V_\lambda \) is nonzero. Now choose an enumeration of \( I^q = \{ i_k : 1 \leq k \leq n \} \), write \( \mu_q = \sum k m_k \alpha_{i_k} \), and consider the monomials

\[
E := \prod_k e^{m_k}_{i_k}, \quad F := \prod_k f^{m_k}_{i_k}.
\]

By the vanishing of \( V_{\lambda - \mu_p - \mu_q} \), \( \partial F \) annihilates the highest weight line \( V_\lambda \), whence so does \( E \partial F \). Since \([e_i, f_j] = 0 \forall i \neq j\), we may write this as \( \partial \prod_k e^{m_k}_{i_k} f^{m_k}_{i_k} \), and each factor \( e^{m_k}_{i_k} f^{m_k}_{i_k} \) acts on \( V_\lambda \) by a nonzero scalar, a contradiction. \( \square \)
Remark 5.9. Given the delicacy of Jantzen’s criterion for the simplicity of a parabolic Verma module (see [20, 21]), it is interesting that the equality on weights of $L(\lambda)$ and $M(\lambda, I_\lambda)$ always holds.

Remark 5.10. Since the writing of this paper, G.K. Teja has obtained in [41] a ‘minimal description’ of the set of weights of $L(\lambda)$ for arbitrary non-integrable $\lambda$—and more generally, for any parabolic Verma module as in Equation (3.9). Explicitly, Teja shows that

$$\text{wt } M(\lambda, J) = \text{wt } L_{\{j\}}(\lambda) - \mathbb{Z}^{\geq 0} \Delta_{J^c, 1},$$

where given a subset $J \subset I$ of simple roots, the set $\Delta_{J^c, 1}$ comprises the positive roots $\alpha_{i_1} + \cdots + \alpha_{i_n}$ with “$J^c$-height 1”, i.e., with a unique $k$ such that $i_k \notin J$. (In particular, for $J = I_\lambda$ we obtain $\text{wt } L(\lambda)$.) The ‘minimality’ of this description is the assertion that the set $\Delta_{J^c, 1}$ in the above formula cannot be reduced.

6. Weights and convex hulls of highest weight modules

In Section 5, we applied our main result to obtain formulas for the weights of simple modules. We now explore the extent to which this can be done for arbitrary highest weight modules.

**Theorem 6.1.** Fix $\lambda \in \mathfrak{h}^*$ and $J \subset I_\lambda$. Every highest weight module $V$ of highest weight $\lambda$ and integrability $J$ has the same weights if and only if $J^p := I_\lambda \setminus J$ is complete, i.e. $(\alpha_j, \alpha_{j'}) \neq 0, \forall j, j' \in J^p$.

We first sketch our approach. The subcategory of modules $V$ with fixed highest weight $\lambda$ and integrability $J$ is basically a poset modulo scaling. The source here is the parabolic Verma module $M(\lambda, J)$, and we are concerned with the possible vanishing of weight spaces as we move away from the source. The most interesting ingredient of the proof is the observation that there is a sink, which we call $L(\lambda, J)$, and hence the question reduces to whether $M(\lambda, J)$ and $L(\lambda, J)$ have the same weights.

**Lemma 6.2.** For $J \subset I_\lambda$, there is a minimal quotient $L(\lambda, J)$ of $M(\lambda)$ satisfying the equivalent conditions of Theorem 4.12.

**Proof.** Consider all submodules $N$ of $M(\lambda)$ such that $N_{\mathbb{Z}^{\geq 0} \alpha_i} = 0, \forall i \in I \setminus J$. There is a maximal such, namely their sum $N'$, and it is clear that $L(\lambda, J) = M(\lambda)/N'$ by construction. \qed

As the reader is no doubt aware, the inexplicit construction of $L(\lambda, J)$ here is parallel to that of $L(\lambda)$—in fact, $L(\lambda) = L(\lambda, I_\lambda)$. The existence of the objects $L(\lambda, J)$ was noted by the second named author in [27], but we are unaware of other appearances in the literature. In [27], the character of $L(\lambda, J)$ was not determined.
However, it turns out to be no more difficult than \( \text{ch } L(\lambda) \) under the following sufficient condition, which as we explain below is often satisfied:

**Proposition 6.3.** Fix \( \lambda \in \mathfrak{h}^* \), \( J \subset I_\lambda \). Suppose that \( \text{Ext}^1_O(L(\lambda), L(s_i \cdot \lambda)) \neq 0 \), \( \forall i \in I_\lambda \setminus J \). Then there is a short exact sequence:

\[
0 \rightarrow \bigoplus_{i \in I_\lambda \setminus J} L(s_i \cdot \lambda) \rightarrow L(\lambda, J) \rightarrow L(\lambda) \rightarrow 0. \tag{6.4}
\]

**Proof.** The choice of a nonzero class in each \( \text{Ext}^1_O(L(\lambda), L(s_i \cdot \lambda)), \) \( i \in I_\lambda \setminus J \) gives an extension \( E \) as in Equation (6.4). The space \( E_\lambda \) is a highest weight line, and we obtain an associated map \( M(\lambda) \simeq M(\lambda) \otimes E_\lambda \rightarrow E \). This is surjective, as follows from an easy argument using Jordan–Hölder content and the nontriviality of each extension. The consequent surjection \( E \rightarrow L(\lambda, J) \) is an isomorphism, by considering Jordan–Hölder content. \( \square \)

**Remark 6.5.** By identifying \( \text{Ext}^1_O(L(\lambda), L(s_i \cdot \lambda)) \) with \( \text{Hom}(N(\lambda), L(s_i \cdot \lambda)) \), where \( N(\lambda) \) is the maximal submodule of \( M(\lambda) \), one sees it is at most one dimensional.

Suppose \( \mathfrak{g} \) is symmetrizable, and that \( \lambda \) is a (i) noncritical weight which (ii) lies in the dot Weyl group orbit of a dominant or antidominant weight. If \( \lambda \) and \( s_i \cdot \lambda \) are distinct and lie in the same block, then we expect that the extension group

\[
\text{Ext}^1_O(L(\lambda), L(s_i \cdot \lambda))
\]

is nonzero. Indeed, by work of Fiebig \([13]\), we may assume that \( \lambda \) is integral. Let us sketch a proof in the case where \( \lambda \) is regular and lies in the orbit of an antidominant weight. By Kashiwara–Tanisaki localization \([25]\), we must equivalently show the nonvanishing of the extensions between the corresponding intersection cohomology sheaves on \( B \setminus G/N \). Using parity vanishing, the Cousin spectral sequence converging to

\[
\text{Ext}^*_O(L(\lambda), L(s_i \cdot \lambda))
\]
collapses on its first page. This reduces the nonvanishing assertion to only the Schubert strata indexed by \( s_i \cdot \lambda \) and \( \lambda \), where it is clear.

A similar argument applies if \( \lambda \) lies in the orbit of an antidominant weight and has a finite stabilizer. In particular, this covers all cases in finite type (where the result is well known, cf. \([18]\) Theorem 8.15(c)) for \( \lambda \) regular and see \([19]\) Corollary 1.3.5 for \( \lambda \) singular). A similar argument should apply for orbits containing dominant weights with finite stabilizers, although the convergence of the spectral sequence requires justification. One can also see this nonvanishing directly if \( \lambda \) is a regular dominant weight by using the last two terms of the BGG resolution, cf. the proof of Proposition 6.6 below.
Returning to Theorem 6.1 for \( \mathfrak{g} \) possibly non-symmetrizable, we first prove the following weaker form of Equation (6.4).

**Proposition 6.6.** Let \( \mathfrak{g} \) be of arbitrary type and \( \lambda \in P^+ \). Then:

\[
\text{wt } L(\lambda, J) = \text{wt } L(\lambda) \cup \bigcup_{i \in I \setminus J} \text{wt } L(s_i \cdot \lambda).
\]  

(6.7)

**Proof.** The restriction of \( \text{ch } M(\lambda) \) to the root string \( \lambda - \mathbb{Z}_{\geq 0} \alpha_i, i \in I \setminus J \) is entirely determined by the Jordan–Hölder content \( \left[ M(\lambda) : L(\lambda) \right] = \left[ M(\lambda) : L(s_i \cdot \lambda) \right] = 1 \). Using this, it suffices to exhibit a module with weights given by the right-hand side of (6.7).

Consider the end of the BGG resolution:

\[
\bigoplus_{i \in I} M(s_i \cdot \lambda) \rightarrow M(\lambda) \rightarrow L_{\text{max}}(\lambda) \rightarrow 0.
\]  

(6.8)

For each \( i \in I \), let \( N(s_i \cdot \lambda) \) denote the maximal submodule of \( M(s_i \cdot \lambda) \). Consider the quotient \( Q \) of \( M(\lambda) \) by the image of \( \bigoplus_{j \in J} M(s_j \cdot \lambda) \oplus \bigoplus_{i \in I \setminus J} N(s_i \cdot \lambda) \), under (6.8). We deduce:

\[
\text{ch } Q = \text{ch } L_{\text{max}}(\lambda) + \sum_{i \in I \setminus J} \text{ch } L(s_i \cdot \lambda),
\]

which implies that \( \text{wt } Q \), whence \( \text{wt } L(\lambda, J) \), coincides with the right-hand side of (6.7). \( \square \)

We now are ready to prove Theorem 6.1. By the above results, it suffices to determine when \( \text{wt } L(\lambda, J) = \text{wt } M(\lambda, J) \).

**Proof of Theorem 6.1.** It will be clarifying, though not strictly necessary for the argument, to observe the following compatibility of the construction of \( L(\lambda, J) \) and restriction to a Levi.

**Lemma 6.9.** Let \( I' \subset I \), \( l = l_{I'} \) the corresponding Levi subalgebra, and for \( J \subset I \) write \( J' = J \cap I' \). Then:

\[ L_l(\lambda, J') \simeq U(l) L_l(\lambda, J), \]

where \( L_l(\lambda, J') \) is the smallest \( l \)-module with integrability \( J' \), constructed as in Lemma 6.2.

**Proof.** Since the integrability of \( U(l) L(\lambda, J) \) is \( J' \), we have a surjection:

\[ U(l) L(\lambda, J) \rightarrow L_l(\lambda, J') \rightarrow 0. \]  

(6.10)

To see that Equation (6.10) is an isomorphism, we need to show the kernel \( K \) of \( M_l(\lambda, J') \rightarrow L_l(\lambda, J') \) is sent into the kernel of \( M(\lambda, J) \rightarrow L(\lambda, J) \) under the
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inclusion \( M_i(\lambda, J') \to M(\lambda, J) \). To do so, choose a filtration of \( K \):

\[ 0 \subseteq K_1 \subseteq K_2 \subseteq \cdots, \quad K = \bigcup_{m \geq 0} K_m, \]

where \( K_m/K_{m-1} \) is generated as a \( U(\mathfrak{l}) \)-module by a highest weight vector \( v_m \).

It suffices to prove by induction on \( m \) that the \( U(\mathfrak{g}) \)-module generated by \( K_m \) lies in the kernel of \( M(\lambda, J) \to L(\lambda, J), \forall m \geq 0 \). For the inductive step, suppose that \( M(\lambda, J)/U(\mathfrak{g})K_{m-1} \) has integrability \( J \). Then choosing a lift \( \tilde{v}_m \) of \( v_m \) to \( K_m \), observe that \( \tilde{v}_m \) is a highest weight vector in \( M(\lambda, J)/U(\mathfrak{g})K_{m-1} \) for the action of all of \( U(\mathfrak{g}) \), by the linear independence of simple roots. It is clear that \( U(\mathfrak{g})\tilde{v}_m \) has trivial intersection with the \( (\lambda - \sum_{j \in I} k_j \alpha_j) \)-weight spaces of \( M(\lambda, J)/U(\mathfrak{g})K_{m-1} \), for all \( i \in I \setminus J \), hence \( M(\lambda, J)/U(\mathfrak{g})K_m \) again has integrability \( J \).

We are now ready to prove Theorem 6.1. By Theorem 5.8, we have \( \text{wt } L(\lambda, J) = \text{wt } M(\lambda, J) \) if and only if \( \text{wt } J^p L(\lambda, J) = \text{wt } J^p M(\lambda, J) \). By Lemma 6.9, we may therefore assume that \( J = \emptyset \), and \( \lambda \in P^+ \), and hence \( J^p = I \).

By Proposition 6.6, we have:

\[
\begin{align*}
\text{wt } L(\lambda, \emptyset) &= \text{wt } L(\lambda) \cup \bigcup_{i \in I} \text{wt } L(s_i \cdot \lambda), \\
\text{wt } L(\lambda, I \setminus i) &= \text{wt } L(\lambda) \cup \text{wt } L(s_i \cdot \lambda), \quad \forall i \in I.
\end{align*}
\]

Combining the two above equalities, we have:

\[
\text{wt } L(\lambda, \emptyset) = \bigcup_{i \in I} \text{wt } L(\lambda, I \setminus i).
\]

For \( L(\lambda, I \setminus i) \) the set of potentially integrable directions is \( \{i\} \), so we may apply Theorem 5.8 to conclude:

\[
\text{wt } L(\lambda, \emptyset) = \bigcup_{i \in I} \text{wt } M(\lambda, I \setminus i).
\]

Thus, it remains to show that for \( \lambda \) dominant integral, \( \text{wt } M(\lambda) = \bigcup_{i \in I} \text{wt } M(\lambda, I \setminus i) \) if and only if the Dynkin diagram of \( \mathfrak{g} \) is complete.

Suppose first that the Dynkin diagram of \( \mathfrak{g} \) is not complete. Pick \( i, i' \in I \) with \( (\alpha_i, \alpha_{i'}) = 0 \). Then \( M(\lambda) \) and \( M(\lambda)/s_i s_{i'} \cdot M(\lambda) \) have distinct weights, as can be seen by a calculation in type \( A_1 \times A_1 \).

Suppose now that the Dynkin diagram of \( \mathfrak{g} \) is complete. By (6.11) it suffices to show:

\[
\text{wt } M(\lambda, I \setminus i) \supset \{\lambda - \sum_{j \in I} k_j \alpha_j : k_j \in \mathbb{Z}_{\geq 0}, k_i \geq k_j, \forall j \in I\}.
\]

But this follows using the assumption that every node is connected to \( i \), and the representation theory of \( \mathfrak{sl}_2 \).

\( \square \)
Remark 6.12. Theorem 6.1 can be proved without working directly with the objects \( L(\lambda, J) \), but instead using only the lower bound (6.7) for the weights:

\[
\text{wt } V \supset \text{wt } L(\lambda, I_V) = \text{wt } L(\lambda) \cup \bigcup_{i \in I_\lambda \setminus I_V} \text{wt } L(s_i \cdot \lambda).
\]

7. An alternating formula for the weights of simple modules

The main goal of this section is to prove the following:

Theorem 7.1. Let \( \lambda \in \mathfrak{h}^* \) be such that the stabilizer of \( \lambda \) in \( W_\lambda \) is finite. Then:

\[
\text{wt } L(\lambda) = \sum_{w \in W_\lambda} w e^{\lambda} \prod_{\alpha \in \pi} \left(1 - e^{-\alpha}\right).
\]  (7.2)

We remind that on the left hand side of (7.2) we mean the ‘multiplicity-free’ character \( \sum_{\mu \in \mathfrak{h}^*: L(\lambda)_{\mu} \neq 0} e^\mu \). On the right hand side of (7.2), in each summand we take the ‘highest weight’ expansion of \( w(1 - e^{-\alpha})^{-1} \), i.e.:

\[
w \frac{1}{1 - e^{-\alpha}} := \begin{cases} 
1 + e^{-w\alpha_i} + e^{-2w\alpha_i} + \cdots, & w\alpha_i > 0, \\
- e^{-w\alpha_i} - e^{-2w\alpha_i} - e^{-3w\alpha_i} - \cdots, & w\alpha_i < 0.
\end{cases}
\]  (7.3)

By Theorem 5.8 we will deduce Theorem 7.1 from the following:

Proposition 7.4. Fix \( \lambda \in \mathfrak{h}^* \), \( J \subset I_\lambda \) such that the stabilizer of \( \lambda \) in \( W_J \) is finite. Then:

\[
\text{wt } M(\lambda, J) = \sum_{w \in W_J} w e^{\lambda} \prod_{\alpha > 0} \left(1 - e^{-\alpha}\right). 
\]  (7.6)

Before proving Proposition 7.4, we note the similarity between the result and the following formulation of the Weyl–Kac character formula. This is due to Atiyah and Bott [2] for finite dimensional simple modules in finite type, but appears to be new in this generality:

Proposition 7.5. Fix \( \lambda \in \mathfrak{h}^* \), \( J \subset I_\lambda \). Then:

\[
\text{ch } M(\lambda, J) = \sum_{w \in W_J} w \prod_{\alpha > 0} (1 - e^{-\alpha})^{\text{dim } \mathfrak{g}_\alpha}. 
\]  (7.6)

The right hand side of Equation (7.6) is expanded in the manner explained above. We now prove Proposition 7.4 and then turn to Proposition 7.5.
Proof of Proposition 7.4. By expanding $w \prod_{\alpha \in \pi_{I \setminus J}} (1 - e^{-\alpha})^{-1}$, note the right hand side equals:

$$
\sum_{w \in W_J} \sum_{\mu \in \mathbb{Z}_{\geq 0}} \sum_{\pi \in \pi_{I \setminus J}} w \prod_{\alpha \in \pi_{I \setminus J}} (1 - e^{-\alpha}) \cdot e^{\lambda - \mu} \prod_{\alpha \in \pi_J} (1 - e^{-\alpha}).
$$

Therefore we are done by the Integrable Slice Decomposition 4.1 and the existing result for integrable highest weight modules with finite stabilizer [26]. □

Specialized to the trivial module in finite type, we obtain the following curious consequence, which roughly looks like a Weyl denominator formula without a choice of positive roots:

**Corollary 7.7.** Let $\mathfrak{g}$ be of finite type with root system $\Delta$, and let $\Pi$ denote the set of all $\pi$, where $\pi$ is a simple system of roots for $\Delta$. Then we have:

$$
\prod_{\alpha \in \Delta} (1 - e^{\alpha}) = \sum_{\pi \in \Pi} \prod_{\beta \notin \pi} (1 - e^{\beta}). \tag{7.8}
$$

**Remark 7.9.** When $\mathfrak{g}$ is of finite type, Theorem 7.1 follows from Proposition 2.5 and general formulae for exponential sums over polyhedra due to Brion [7] for rational polytopes and Lawrence [32] and Khovanskii–Pukhlikov [28] in general, cf. [3, Chapter 13]. For regular weights, one uses that the tangent cones are unimodular and hence the associated polyhedra are Delzant. For general highest weights one may apply a deformation argument due to Postnikov [39]. We thank Michel Brion for sharing this observation with us.

Moreover, it is likely that a similar approach works in infinite type, though a cutoff argument needs to be made owing to the fact that $\text{conv} V$ is in general locally, but not globally, polyhedral, cf. our companion work [10].

### 7.1. Some related results

We conclude this section with several related observations which are not needed in the remainder of the paper, beginning with Proposition 7.5.

#### 7.1.1. Parabolic Atiyah–Bott formula and the Bernstein–Gelfand–Gelfand–Lepowsky resolution

**Proposition 7.10.** Fix $\lambda \in \mathfrak{h}^*$, $J \subset I_\lambda$. Then:

$$
\text{ch} M(\lambda, J) = \sum_{w \in W_J} \frac{e^{w(\lambda + \rho) - \rho}}{\prod_{\alpha > 0} (1 - e^{-\alpha})^{\dim g}}. \tag{7.11}
$$
To our knowledge, even Proposition 7.10 does not appear in the literature in this generality. However, via Levi induction one may reduce to the case of $\lambda$ dominant integral, $J = I$, where it is a famed result of Kumar [29, 30]. Along these lines, we also record the BGGL resolution for arbitrary parabolic Verma modules, which may be of independent interest:

**Proposition 7.12.** Fix $M(\lambda, J)$ and $J' \subset J$. Then $M(\lambda, J)$ admits a BGGL resolution, i.e., there is an exact sequence:

$$\cdots \to \bigoplus_{w \in W_{J'}} M(w \cdot \lambda, J') \to \bigoplus_{w \in W_{J'}} M(w \cdot \lambda, J') \to M(\lambda, J') \to M(\lambda, J) \to 0.$$ (7.13)

Here $W_{J'}$ runs over the minimal length coset representatives of $W_{J'} \backslash W_J$. In particular, we have the Weyl–Kac character formula:

$$\text{ch} M(\lambda, J) = \sum_{w \in W_{J'}} (-1)^{\ell(w)} \text{ch} M(w \cdot \lambda, J').$$ (7.14)

This can be deduced by Levi induction from the result of Heckenberger and Kolb [15], which builds on [14, 30, 37].

**7.1.2. Generalization of a result of Kass.** In the paper of Kass [26] that proves Theorem 7.1 for integrable modules, the main result is a recursive formula for the character of an integrable module. After establishing notation, we will state the analogous result for parabolic Verma modules.

For $J \subset I$, if $\nu \in \mathfrak{h}^*$ lies in the $J$ dot Tits cone, write $\overline{\nu} = w_\nu \cdot \nu$ for the unique $J$ dot dominant weight in its $W_J$ orbit. For $\lambda \in P_J^+$, we write $\chi_\lambda := \text{ch} M(\lambda, J)$, and for $\nu$ lying in the dot $J$ Tits cone, we set:

$$\chi_{\nu} := \begin{cases} (-1)^{\ell(w_\nu)} \chi_{\overline{\nu}} & \text{if } \overline{\nu} \text{ is in the } J \text{ dominant chamber} \\ 0 & \text{otherwise.} \end{cases}$$

This makes sense because if $\overline{\nu}$ lies in the $J$ dominant chamber, it is regular dominant under the $J$ dot action, so there is a unique $w_\nu \in W_J$ such that $w_\nu \cdot \nu = \overline{\nu}$.

Define the elements $\langle \Phi \rangle \in \mathbb{Z}^{\geq 0} \pi$ and the integers $c_{\langle \Phi \rangle}$ via the expansion,

$$\prod_{\alpha \in \Delta^+ \backslash \pi} (1 - e^{-\alpha})^{\dim g_\alpha} =: \sum_{\langle \Phi \rangle} c_{\langle \Phi \rangle} e^{-\langle \Phi \rangle}.$$ (7.15)

Informally, this expansion counts the number of ways to write $\langle \Phi \rangle$ as sums of non-simple positive roots, with consideration of the multiplicity of the root and the parity of the sum.
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For a parabolic Verma module $M(\lambda, J)$, by abuse of notation identify its weights with the corresponding multiplicity-free character:

$$\text{wt } M(\lambda, J) = \sum_{\mu : M(\lambda, J)_{\mu} \neq 0} e^\mu.$$  

With this notation, we are ready to state the following:

**Proposition 7.15.** Let $M(\lambda, J)$ be a parabolic Verma module such that the stabilizer of $\lambda$ in $W_J$ is finite. Then:

$$\text{wt } M(\lambda, J) = \sum_{\langle \Phi \rangle} c_{\langle \Phi \rangle} \chi_{\lambda - \langle \Phi \rangle},$$  

(7.16)

Furthermore, for all $\langle \Phi \rangle$, we have (i) $\lambda - \langle \Phi \rangle \leq \lambda$, (ii) with equality if and only if $\Phi$ is empty.

In Equation (7.16), we expand denominators as ‘highest weight’ power series as explained in Equation (7.3). The arguments of [26] or [40] apply with suitable modification to prove Proposition 7.15.

**Remark 7.17.** As the remainder of the paper concerns only symmetrizable $g$, we now explain the validity of earlier results for $\mathfrak{g}$ (cf. Section 3.2), should $g$ and $\mathfrak{g}$ differ. The proofs in Sections 4–7 prior to Remark 7.9 apply verbatim for $g$. Alternatively, recalling that the roots of $g$ and $\mathfrak{g}$ coincide, it follows from Equation (3.9) and the remarks following Proposition 4.3 that the weights of parabolic Verma modules for $g, \mathfrak{g}$ coincide, as do the weights of simple highest weight modules. Hence many of the previous results can be deduced directly for $\mathfrak{g}$ from the case of $g$. These arguments for $\mathfrak{g}$ apply for any intermediate Lie algebra between $g$ and $\mathfrak{g}$.

8. A GEOMETRIC INTERPRETATION OF INTEGRABILITY

Throughout this section, $g$ is of finite type. We will answer the question of Brion [8] discussed in Section 2.4. We first recall notation. For $\lambda$ a dominant integral weight, consider $L_\lambda$, the line bundle on $G/B$ with $H^0(L_\lambda) \simeq L(\lambda)$ as $g$-modules. For $w \in W$, write $C_w := BwB/B$ for the Schubert cell, and write $X_w$ for its closure, the Schubert variety. To affirmatively answer Brion’s question for any regular integral infinitesimal character, it suffices by translation to consider the regular block $O_0$.

Let us recall the geometric side of the localization theorem. For an Artin stack $X$, we write $D(X)$ for its bounded derived category of $D$-modules with coherent cohomology. We write $N$ for the unipotent radical of $B$, and consider the associated category of $D$-modules on the quotient stack $B \backslash G/N$, i.e.

$$D(B \backslash G/N).$$  

(8.1)
As every object of (8.1) has regular holonomic cohomology, the Riemann–Hilbert correspondence gives an equivalence with the bounded constructible derived category of sheaves

$$D(B\setminus G/N) \simeq \text{Sh}(B\setminus G/N),$$

which exchanges the standard $t$-structure on D-modules with the perverse $t$-structure on constructible sheaves.

Let us next recall the form of Beilinson–Bernstein localization we will use. Consider the action of the abstract Cartan $T \simeq B/N$ on $B\setminus G/N$ by right translation. Note that all the objects of (8.1) are $T$-monodromic, i.e. lie in the full subcategory generated under colimits by $!$-pulling back along the projection

$$B\setminus G/N \to (B\setminus G/N)/T \simeq B\setminus G/B.$$  

In particular, they carry a canonical datum of weak $T$-equivariance. Accordingly, one may form the $T$-invariant sections of the underlying quasicoherent sheaf of the (left) D-module on $G/N$, i.e. one has a functor

$$\Gamma(G/B, -) : D(B\setminus G/N) \to \text{Vect},$$

where $\text{Vect}$ denotes the bounded derived category of vector spaces. The canonical embedding of $\mathfrak{g}$ into global vector fields on $G/N$ factors $\Gamma(G/B, -)$ as a composition

$$D(B\setminus G/N) \to \mathfrak{g} \text{-mod} \xrightarrow{\text{Oblv}} \text{Vect},$$

where $\mathfrak{g}$-mod denotes the bounded derived category of $\mathfrak{g}$-modules and $\text{Oblv}$ denotes the forgetful functor. Then the theorem of Beilinson–Bernstein [4, Corollary 3.3.3] asserts that the obtained functor to $\mathfrak{g}$-modules induces a $t$-exact equivalence with the bounded derived category of $\mathcal{O}_0$, i.e.

$$\Gamma(G/B, -) : D(B\setminus G/N) \simeq D^b(\mathcal{O}_0).$$  

Before turning the problem at hand, let us briefly comment on the interaction of the above equivalences with duality. For (8.2), we recall that the Riemann–Hilbert correspondence exchanges the standard duality on D-modules with Verdier duality for constructible sheaves, see for example [17, Corollary 4.6.5]. For (8.3), it is known that localization exchanges the standard duality on D-modules with the contragredient duality on $D^b(\mathcal{O})$, see for example Section 1 of [1]. In particular, we use the normalization of [1] for the contragredient duality. We refer to all three dualities by $D$ below.

Having established our conventions, we turn to Brion’s question. Let $V$ be a highest weight module with highest weight $w \cdot (-2\rho), w \in W$. Write $\mathcal{V}$ for the corresponding regular holonomic $D$-module on $G/B$, and $\mathcal{V}_{rh}$ for the corresponding perverse sheaf under the Riemann–Hilbert correspondence. Then $\mathcal{V}, \mathcal{V}_{rh}$ are supported on $X_w$. Note that for $i \in I_V$, $X_{s_iw}$ is a Schubert divisor of $X_w$. 
Theorem 8.4. Let \( V \) and \( \mathcal{V} \) be as above. For a subset \( J \subset I_V \), consider the corresponding union of Schubert divisors
\[
Z := \bigcup_{i \in I_V \setminus J} X_{s_i w},
\]
and denote its complement by \( \bar{U} := (G/B) \setminus Z \). Then the \( \mathfrak{g} \)-module \( D^H_0(\bar{U}, \mathcal{D} \mathcal{V}) \) is of highest weight \( w \cdot (-2\rho) \) and has integrability \( J \).

To prove Theorem 8.4, we will use the following geometric characterization of integrability, which was promised in Theorem 2.9.

Proposition 8.5. For \( J \subset I_\lambda \), consider the smooth open subvariety of \( X_w \) given on complex points by:
\[
\mathcal{U}_J := C_w \sqcup \bigcup_{i \in J} C_{s_i w}.
\]

Let \( V, \mathcal{V}_{rh} \) be as above, and set \( \mathcal{U} := \mathcal{U}_I \). Upon restricting to \( \mathcal{U} \) we have:
\[
\mathcal{V}_{rh}|_{\mathcal{U}} \cong j^! C_{\mathcal{U}_I} [\ell(w)].
\]

In particular, \( I_V = \{ i \in I : \text{the } * \text{-stalks of } \mathcal{V}_{rh} \text{ along } C_{s_i w} \text{ are nonzero} \} \).

Proof. For \( J \subset I_\lambda \), let \( P_J \) denote the corresponding parabolic subgroup of \( G \), i.e. with Lie algebra \( \mathfrak{t}_J + n^+ \). Then it is well known that the perverse sheaf corresponding to \( M(\lambda, J) \) is \( j^! C_{P_J w B / B} [\ell(w)] \).

The map \( M(\lambda, I_V) \to V \to 0 \) yields a surjection on the corresponding perverse sheaves. By considering Jordan–Hölder content, it follows this map is an isomorphism when restricted to \( \mathcal{U}_I \), as for \( y \leq w \) the only intersection cohomology sheaves \( IC_y := IC_{X_y} \) which do not vanish upon restriction are \( IC_{w}, IC_{s_i w}, i \in I_\lambda \). We finish by observing:
\[
j^*_I j^! C_{\mathcal{U}_I} [\ell(w)] \cong j^! C_{\mathcal{U}_I \cap P_I w B / B} [\ell(w)] = j^! C_{\mathcal{U}_I} [\ell(w)]. \quad \square
\]

We deduce the following \( D \)-module interpretation of integrability:

Corollary 8.8. Let \( \mathcal{V}, \mathcal{U} \) be as above. The restriction of \( D \mathcal{V} \) to \( \mathcal{U} \) is \( j_* \mathcal{O}_{\mathcal{U}_I} \). If we define \( I_V := \{ i \in I : \text{the fibers of } D \mathcal{V} \text{ are nonzero along } C_{s_i w} \} \), then \( I_V = I_I \). Here we mean fibers in the sense of the underlying quasi-coherent sheaf of \( D \mathcal{V} \).

Before proving Theorem 8.4, we first informally explain the idea. Proposition 8.5 says that for \( i \in I_{L(w-2\rho)} \), the action of the corresponding \( \mathfrak{sl}_2 \to \mathfrak{g} \) on \( V \) is not integrable if and only if \( \mathcal{V}_{rh} \) has a ‘pole’ on the Schubert divisor \( X_{s_i w} \). Therefore to modify \( V \) so that it loses integrability along \( X_{s_i w} \), we will restrict \( \mathcal{V}_{rh} \) to the complement and then extend by zero.
Proof of Theorem [8.4] For ease of notation, write \( X := X_w \). Write \( X = Z \sqcup U \), where \( Z \) is as above, and \( G/B = Z \sqcup \tilde{U} \). Note \( \tilde{U} \cap X_w = U \). Write \( j_U : U \to X_w, j_{U*} : \tilde{U} \to G/B \) for the open embeddings, and \( i_Z : Z \to X_w, i'_{Z} : Z \to G/B, i_{X_w} : X_w \to G/B \) for the closed embeddings.

We will study \( H^0(\tilde{U}, \mathcal{V}) \) by studying the behavior of \( \mathcal{V} \) on \( U \), i.e., before \( * \)-extending from \( X_w \). Formally, write \( \mathcal{V}_{rh} = i_{X*}i_X^*\mathcal{V}_{rh} =: i_{X*}\mathcal{P} \), where \( \mathcal{P} \) is perverse. The distinguished triangle:

\[
j_U j_U^* \to i_{Z*}i_Z^* \to +1
\]

(8.9)
gives the following exact sequence in perverse cohomology:

\[
0 \to \mathcal{P} \to \mathcal{P} \to \mathcal{P} \to 0.
\]

We now show in several steps that \( \mathcal{P} \) is a highest weight module with the desired integrability.

Step 1: \( \mathcal{P} \) is a highest weight module of highest weight \( \rho - 2\rho \).

By definition, we have a surjection \( j_{C_u}^! C_{C_w} [\ell(w)] \to \mathcal{P} \to 0 \). Since \( j_{C_u}^! C_{C_w} \) is supported off of the Schubert divisors, we have \( j_{C_u}^! j_{U*}^* \mathcal{P} \simeq j_{C_u}^! C_{C_w} \). By right exactness, we obtain \( j_{C_u}^! C_{C_w} [\ell(w)] \to \mathcal{P} \). We remark that similarly, \( \mathcal{P} \) is a highest weight module with the desired integrability.

Step 2: The integrability of \( \mathcal{P} \) is \( J \). In ‘ground to earth’ terms, by Proposition [8.3] we need to look at this sheaf on \( \mathcal{U} \), where by design it has the correct behavior. More carefully, we have:

\[
j_{\mathcal{U}}^* \mathcal{P} \simeq \mathcal{P} \simeq \mathcal{P} \simeq \mathcal{P} \simeq \mathcal{P}
\]

As \( \mathcal{U}_{I_v} \cap U = \mathcal{U}_J \), we are done by Proposition [8.3].

We now push our analysis off of \( X_w \). To do so, we use the standard isomorphism of distinguished triangles:

\[
j_{U*} j_{U*} i_{X*} \longrightarrow i_{X*} \longrightarrow i'_{Z*} i_Z^* i_{X*} \longrightarrow +1
\]

(8.10)

Here the middle vertical map is the identity, and the left and right vertical isomorphisms are induced by adjunction. Plainly, the isomorphism (8.10) comes from
an isomorphism of short exact sequences of functors for the abelian categories of sheaves of abelian groups.

Translating our analysis of $p\mathbb{H}^0 j_{U\dagger} j_U^* P$ into the corresponding statement for $\mathfrak{g}$-modules and using Equation (8.10), we obtain a surjection of highest weight modules:

$$\Gamma(G/B, H^0 j_{U\dagger} j_U^* V) \to V \to 0,$$

where the former module has integrability $J$. To finish the proof of Theorem 8.4, it remains only to identify the dual of $\Gamma(G/B, H^0 j_{U\dagger} j_U^* V)$ with sections of $D V$ on $\tilde{U}$. But using standard compatibilities of $D$, and of composition of derived functors, we obtain:

$$D \Gamma(G/B, H^0 j_{U\dagger} j_U^* V) \simeq \Gamma(G/B, D H^0 j_{U\dagger} j_U^* V) \simeq \Gamma(G/B, R^0 j_{U\dagger} j_U^* D V) \simeq \Gamma(\tilde{U}, D V).$$

□

If one thinks about the above proof, in fact all we used about $U$ (and $\tilde{U}$) was the intersection of $U$ with $\mathfrak{u}$. The following proposition shows our $U$ has the correct components in codimension $\geq 2$ to mimic another feature of Brion’s example:

**Proposition 8.11.** The construction of Theorem 8.4 sends parabolic Verma modules to parabolic Verma modules.

**Proof.** For $J \subset K \subset I$, we know that $M(w; -2\rho, K)$ corresponds to $j_! C_{P_K w B/B}$, where $P_K$ is the parabolic subgroup corresponding to $K \subset I$. Applying the construction (before taking perverse cohomology), we obtain $j_{U\dagger} j_U^* j_! C_{P_K w B/B} \simeq j_! C_{U \cap P_K w B/B}$. The claim follows from the identity $P_K w B/B \setminus Z = P_J w B/B$, i.e. the identity

$$W_K w \cup_{k \in K \setminus J} \{y \in W : y \leq s_k w\} = W_J w.$$

To see this identity, recall by [6, Exercise 2.26 and proof of Proposition 2.4.4] that the assignment $w_k w \mapsto w_k w_o$ is an isomorphism of posets $W_K w \simeq W_K$, where $w_o$ is the longest element of $W_K$. Multiplying the claimed identity on the right by $w^{-1}$, it is therefore equivalent to:

$$W_K \setminus \cup_{k \in K \setminus J} \{y \in W_K : y \geq s_k\} = W_J,$$

which is clear. □

While the results in this section concern $\mathfrak{g}$ of finite type, we expect and would be interested to see that similar results hold for $\mathfrak{g}$ symmetrizable.
9. Highest Weight Modules over Symmetrizable Quantum Groups

We now extend many results of the previous sections to highest weight modules over quantum groups \( U_q(\mathfrak{g}) \), for \( \mathfrak{g} \) a Kac–Moody algebra. Given a generalized Cartan matrix \( A \), as for \( \mathfrak{g} = \mathfrak{g}(A) \), to write down a presentation for the algebra \( U_q(\mathfrak{g}) \) via generators and explicit relations, one uses the symmetrizability of \( A \). When \( \mathfrak{g} \) is non-symmetrizable, even the formulation of \( U_q(\mathfrak{g}) \) is subtle and is the subject of recent research \([12]\). In light of this, we restrict to \( U_q(\mathfrak{g}) \) where \( \mathfrak{g} \) is symmetrizable.

9.1. Notation and preliminaries. We begin by reminding standard definitions and notation. Fix \( \mathfrak{g} = \mathfrak{g}(A) \) for \( A \) a symmetrizable generalized Cartan matrix. Fix a diagonal matrix \( D = \text{diag}(d_i)_{i \in I} \) such that \( DA \) is symmetric and \( d_i \in \mathbb{Z}^{> 0} \), \( \forall i \in I \). Let \((\mathfrak{h}, \pi, \hat{\pi})\) be a realization of \( A \) as before; further fix a lattice \( P^\vee \subset \mathfrak{h} \), with \( \mathbb{Z} \)-basis \( \alpha_i, \beta_i, i \in I, 1 \leq l \leq |I| - \text{rk}(A) \), such that \( P^\vee \otimes_{\mathbb{Z}} \mathbb{C} \simeq \hat{\mathfrak{h}} \) and \((\beta_i, \alpha_i) \in \mathbb{Z}, \forall i \in I, 1 \leq l \leq |I| - \text{rk}(A) \). Set \( P := \{ \lambda \in \mathfrak{h}^* : (P^\vee, \lambda) \subset \mathbb{Z} \} \) to be the weight lattice. We further retain the notations \( \rho, \mathfrak{h}, \mathfrak{h}, I_f, P^+_f, M(\lambda, J), I_\chi \) from previous sections; note we may and do choose \( \rho = (\alpha_i, \beta_j) = d_i a_{ij} \) for all \( i, j \in I \).

Let \( q \) be an indeterminate. Then the corresponding quantum Kac–Moody algebra \( U_q(\mathfrak{g}) \) is a \( \mathbb{C}(q) \)-algebra, generated by elements \( f_i, q^h, e_i, i \in I, h \in P^\vee \), with relations given in e.g. \([10]\) Definition 3.1.1. Among these generators are distinguished elements \( K_i = q^{d_i \alpha_i} \in q^{P^\vee} \). Also define \( U^\pm_q \) to be the subalgebras generated by the \( e_i \) and the \( f_i \), respectively.

A weight of the quantum torus \( \mathbb{T}_q := \mathbb{C}(q)[q^{P^\vee}] \) is a \( \mathbb{C}(q) \)-algebra homomorphism \( \chi : \mathbb{T}_q \to \mathbb{C}(q) \), which we identify with an element \( \mu_q \in (\mathbb{C}(q)^\times)^{|I| - \text{rk}(A)} \) given an enumeration of \( \alpha_i, i \in I \). We will abuse notation and write \( \mu_q(q^h) \) for \( \chi_{\mu_q}(q^h) \). There is a partial ordering on the set of weights, given by: \( q^{-\nu} \mu_q \leq \mu_q \) for all weights \( \nu \in \mathbb{Z}^{\geq 0} \). We will mostly be concerned with integral weights \( \mu_q = q^\mu \) for \( \mu \in P \), which are defined via: \( q^\mu(q^h) = q^{(h, \mu)} \), \( h \in P^\vee \).

Given a \( U_q(\mathfrak{g}) \)-module \( V \) and a weight \( \mu_q \), the corresponding weight space of \( V \) is:

\[
V_{\mu_q} := \{ v \in V : q^h v = \mu_q(q^h) v \ \forall h \in P^\vee \}.
\]

Denote by wt \( V \) the set of weights \( \{ \mu_q : V_{\mu_q} \neq 0 \} \). A \( U_q(\mathfrak{g}) \)-module is highest weight if there exists a nonzero weight vector which generates \( V \) and is killed by \( e_i, \forall i \in I \). For a weight \( \mu_q \), let \( M(\mu_q), L(\mu_q) \) denote the Verma and simple \( U_q(\mathfrak{g}) \)-modules of highest weight \( \mu_q \), respectively.

Writing \( \lambda_q \) for the highest weight of \( V \), the integrability of \( V \) equals:

\[
I_V := \{ i \in I : \dim \mathbb{C}(q)[f_i]V_{\lambda_q} < \infty \}. \quad (9.1)
\]
In this case, the parabolic subgroup $W_{I_V}$ acts on $\text{wt } V$ by
\[ s_i(\lambda_q)(q^h) := \lambda_q(q^{\delta_i})^{-\alpha_i(h)}\lambda_q(q^h), \quad h \in P^\vee. \]
The braid relations can be checked using by specializing $q$ to 1, cf. [35]. In particular, $w(q^\lambda) = q^{w \lambda}$ for $w \in W$ and $\lambda \in P$.

Given a weight $\lambda_q$ and $J \subset I_{L(\lambda_q)}$, the parabolic Verma module $M(\lambda_q, J)$ co-represents the functor:
\[ M \mapsto \{ m \in M_{\lambda_q} : e_i m = 0 \forall i \in I, \quad f_j \text{ acts nilpotently on } m, \forall j \in J \}. \]

Note $\lambda_q(q^{\delta_i}) = \pm q^{n_j}$, $n_j \geq 0$, $\forall j \in J$ cf. [22, Proposition 2.3]. Therefore:
\[ M(\lambda_q, J) \cong M(\lambda_q)/\langle f_j^{n_j+1} M(\lambda_q) \rangle, \forall j \in J \].

In what follows, we will specialize highest weight modules at $q = 1$, as pioneered by Lusztig [35]. Let $A_1$ denote the local ring of rational functions $f \in \mathbb{C}(q)$ that are regular at $q = 1$. Fix an integral weight $\lambda_q = q^\lambda \in q^P$ and a highest weight module $V$ with highest weight vector $v_{\lambda_q} \in V_{\lambda_q}$. Then the classical limit of $V$ at $q = 1$, defined to be $V_1 := A_1 \cdot v_{\lambda_q}/(q - 1)A_1 \cdot v_{\lambda_q}$, is a highest weight module over $U(g)$ with highest weight $\lambda$. Moreover the characters of $V$ and $V_1$ are “equal”, i.e. upon identifying $q^P$ with $P$.

9.2. Three invariants of highest weight modules. We begin by extending Theorem 2.9 to quantum groups.

Theorem 9.4. Let $V$ be a highest weight module with integral highest weight $\lambda_q$. The following data are equivalent:

1. $I_V$, the integrability of $V$.
2. $\text{conv } V^1$, the convex hull of the specialization of $V$.
3. The stabilizer of $\text{ch } V$ in $W$.

Proof. The equivalence of the three statements follows from their classical counterparts, using the equality of characters under specialization. \hfill \Box

9.3. Three positive formulas for the weights of simple modules. We now extend the results of Section 5 to quantum groups.

Theorem 9.5. Let $\lambda_q$ be integral. The following are equivalent:

1. $\text{wt } V = \text{wt } M(\lambda_q, I_V)$.
2. $\text{wt } V = q^{2\lambda_q(\pi_{I_V}^P)\lambda_q}$, where $I_V^P = I_{L(\lambda_q)} \setminus I_V$.

In particular, if $V$ is simple, or more generally $|I_V^P| \leq 1$, then $\text{wt } V = \text{wt } M(\lambda_q, I_V)$.

To prove Theorem 9.5, we will need the Integrable Slice Decomposition for quantum parabolic Verma modules.
Proposition 9.6. Let \( \lambda_q \) be an integral weight and \( M(\lambda_q, J) =: V \) a parabolic Verma module. Then:

\[
\text{wt} M(\lambda_q, J) = \bigcup_{\mu \in \mathbb{Z}^{\geq 0} \setminus \pi I} \text{wt} L_{1j}(q^{-\mu} \lambda_q),
\]

(9.7)

where \( L_{1j}(\nu) \) denotes the simple \( U_q(\mathfrak{l}_J) \)-module of highest weight \( \nu \). In particular, \( \text{wt} V^1 = \text{wt} M(\lambda_1, J) \).

Proof. It suffices to check the formula after specialization. Since the characters of \( V \) and \( V^1 \) are “equal”, \( I_V = I_{V^1} \). It therefore suffices to check the surjection \( M(\lambda_1, J) \to V^1 \) induces an equality of weights. By the Integrable Slice Decomposition 4.1, it suffices to show that \( q^{-Z_{\geq 0} \setminus \pi I} \lambda_q \subset \text{wt} M(\lambda_q, J) \). But this is clear from (9.3) by considering weights and using that \( f_j \lambda_q \) is a highest weight line, for all \( j \in J \).

Remark 9.8. If \( \mathfrak{g} \) is of finite type, then in fact \( V^1 \simeq M(\lambda_1, J) \). The proof uses [5, Chapter 2] and the PBW theorem, as well as arguments similar to the construction of Lusztig’s canonical basis [36] to define \( U_q(\mathfrak{u}_J) \). It would be interesting to know if the parabolic Verma module \( M(\lambda_q, J) \) specializes to \( M(\lambda_1, J) \) for all symmetrizable \( \mathfrak{g} \).

Proof of Theorem 9.5. By Proposition 9.6, (1) implies (2). For the converse, by the Integrable Slice Decomposition (9.7), it suffices to show that \( q^{-Z_{\geq 0} \setminus \pi I} \lambda_q \subset \text{wt} V \). This follows by mimicking the proof of Theorem 5.8.

As an application of Theorem 9.5, we obtain the following positive formulas for weights of simple modules \( \text{wt} L(\lambda_q) \).

Proposition 9.9. Suppose \( \lambda_q \) is integral. Write \( \mathfrak{l} \) for the Levi subalgebra corresponding to \( I_L(\lambda_q) \), and write \( L(\nu) \) for the simple \( U_q(\mathfrak{l}) \)-module with highest weight \( \nu \). Then:

\[
\text{wt} L(\lambda_q) = \bigcup_{\mu \in \mathbb{Z}^{\geq 0} \setminus \pi I} \text{wt} L(\mu \lambda_q).
\]

(9.10)

Proposition 9.11. Let \( \lambda_q \) be integral and \( V := L(\lambda_q) \). Then:

\[
\text{wt} V^1 = (\lambda + \mathbb{Z} \pi) \cap \text{conv} V^1.
\]

(9.12)

By specialization, this determines the weights of \( L(\lambda_q) \).

Proposition 9.13. Suppose \( \lambda_q \) is integral and has finite \( W_{I_L(\lambda_q)} \)-isotropy. Then:

\[
\text{wt} L(\lambda_q) = \bigcup_{w \in W_{I_L(\lambda_q)}} \{ q^\nu : \nu \in P^+_I(\lambda_q), \ q^\nu \leq \lambda_q \}.
\]

(9.14)
9.4. Weights and convex hulls of highest weight modules. The result and arguments of Section 6 apply without change to quantum groups.

**Theorem 9.15.** Fix $J \subset I_{L(\lambda_q)}$ and define $J^p := I_{L(\lambda_q)} \setminus J$. The following are equivalent:

1. $\text{wt } V = \text{wt } M(\lambda_q, J)$ for every $V$ with $I_V = J$.
2. The Dynkin diagram for $g_{J^p}$ is complete.

9.5. An alternating formula for the weights of simple modules. The main result of Section 7 follows from combining Theorems 7.1, 9.5, Proposition 9.6, and specializing.

**Theorem 9.16.** For all integral $\lambda_q$ with finite stabilizer in $W_{I_{L(\lambda_q)}}$, we have:

$$\text{wt } L(\lambda_q) = \sum_{w \in W_{I_{L(\lambda_q)}}} w \frac{\lambda_q}{\prod_{\alpha \in \pi (1 - q^{-\alpha})}}. \tag{9.17}$$

9.6. The case of non-integral weights. We now explain how to extend the above results in this section to other highest weights. We do so in two ways. First, we observe that with some modifications, Lusztig’s specialization method applies to any weight $\lambda_q$ such that $\lambda_q(q^h)$ is regular at $q = 1$ with value $1$, $\forall h \in P^\vee$. Calling such weights specializable, one can show that as before, a highest weight module with specializable highest weight $\lambda_q$ specializes to a highest weight $U(g)$-module with highest weight $\lambda_1 \in \mathfrak{h}^*$, given by:

$$(h, \lambda_1) := \lim_{q \to 1} \frac{\lambda_q(q^h) - 1}{q - 1} \bigg|_{q=1}, \quad h \in P^\vee.$$ 

Moreover, the following holds:

**Theorem 9.18.** The above results in this section all extend to $\lambda_q$ specializable.

Second, we extend many of the above results to generic highest weights, i.e. with finite integrable stabilizer. In particular, this covers all cases in affine type, the remaining cases in affine type being trivial modules. As before, we obtain:

**Theorem 9.19.** Fix a weight $\lambda_q$ and a highest weight module $V$ such that the stabilizer of its highest weight $\lambda_q$ in $W_{I_V}$ is finite. The following are equivalent:

1. $\text{wt } V = \text{wt } M(\lambda_q, I_V)$.
2. $\text{wt } I_{V}^p V = q^{-2\pi i \nu} \lambda_q$, where $I_{V}^p = I_{L(\lambda_q)} \setminus I_V$.

In particular, if $V$ is simple, or more generally $|I_{V}^p| \leq 1$, then $\text{wt } V = \text{wt } M(\lambda_q, I_V)$.

To prove Theorem 9.19, we will need the Integrable Slice Decomposition for quantum parabolic Verma modules.
Proposition 9.20. Let $M(\lambda_q, J)$ be a parabolic Verma module such that the stabilizer of $\lambda_q$ in $W_J$ is finite. Then:

$$\text{wt } M(\lambda_q, J) = \bigcup_{\mu \in \mathbb{Z}^{\geq 0}(\pi \setminus \pi_{IV})} \text{wt } L_{1_J}(q^{-\mu}\lambda_q),$$

(9.21)

where $L_{1_J}(\nu)$ denotes the simple $U_q(\mathfrak{sl}_n)$-module of highest weight $\nu$.

Proof. The inclusion $\supset$ follows from considering weights as in Proposition 4.1. The inclusion $\subset$ follows by using Proposition 4.3(2), which holds for quantum groups by specialization to $q = 1$. □

Proof of Theorem 9.19. By Proposition 9.20, (1) implies (2). For the converse, by the Integrable Slice Decomposition (9.21), it suffices to show that $q^{-\mathbb{Z}^{\geq 0}(\pi \setminus \pi_{IV})}\lambda_q \subset \text{wt } V$. This follows by mimicking the proof of Theorem 5.8. □

As an application of Theorem 9.19, Equation (9.10) holds on the nose for all $\lambda_q$ with finite stabilizer in $W_{1_L(\lambda_q)}$, and Equation (9.14) holds, rephrased as follows:

Proposition 9.22. Suppose $\lambda_q$ has finite $W_{1_L(\lambda_q)}$-isotropy. Then:

$$\text{wt } L(\lambda_q) = \bigcup_{w \in W_{1_L(\lambda_q)}} \{ \nu_q : \nu_q \leq \lambda_q, \quad \nu_q(q^{\alpha_i}) = \pm q^{n_i}, n_i \in \mathbb{Z}^{\geq 0}, \forall i \in I_{1_L(\lambda_q)} \}.$$

(9.23)

Finally, we extend the results of Section 6 to quantum groups, as the same proof applies.

Theorem 9.24. Fix $\lambda_q$ and $J \subset I_{1_L(\lambda_q)}$ such that the stabilizer of $\lambda_q$ in $W_J$ is finite, and define $J^p_q := I_{1_L(\lambda_q)} \setminus J$. The following are equivalent:

1. $\text{wt } V = \text{wt } M(\lambda_q, J)$ for every $V$ with $IV = J$.
2. The Dynkin diagram for $\mathfrak{g}_{J^p_q}$ is complete.

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