A TOWER OF RAMANUJAN GRAPHS AND A RECIPROCITY LAW OF GRAPH ZETA FUNCTIONS

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Abstract. Let \( l \) be an odd prime. We will construct a tower of connected regular Ramanujan graph of degree \( l + 1 \) from modular curves. This supplies an example of a collection of graphs whose discrete Cheeger constants are bounded by \( (\sqrt{l} - 1)^2/2 \) from below. We also show graph (or Ihara) zeta functions satisfy a certain reciprocity law.

Key words: a Ramanujan graph, the Cheeger constant, an expander, a graph zeta function, a modular curve, a Brandt matrix, a reciprocity law.

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1. Introduction

Let \( p \) be a prime satisfying \( p \equiv 1 \pmod{12} \) and let us fix an odd prime \( l \) different from \( p \). In [20] we have constructed a connected regular Ramanujan graph \( G_p^{(l)}(1) \) of degree \( l + 1 \) non-bipartite. The number of vertices \( G_p^{(l)}(1) \) is \((p - 1)/12\) and the Euler characteristic is

\[
\chi(G_p^{(l)}(1)) = \frac{(p - 1)(1 - l)}{24}.
\]

The graph \( G_p^{(l)}(1) \) is regarded as a graph of level one. In this paper we will construct a connected non-bipartite regular Ramanujan graph of degree \( l + 1 \) of a higher level.

In the following let \( p \) be a prime such that \( p \equiv 1 \pmod{12} \) and \( l \) an odd prime different from \( p \). Let \( \mathcal{N}_{p,l} \) be the set of square free positive integers such that every member \( N \) is prime to \( lp \). Then to each \( N \) of \( \mathcal{N}_{p,l} \), a connected non-bipartite regular Ramanujan graph \( G_p^{(l)}(N) \) of degree \( l + 1 \) will be assigned. Let \( \lambda_0(G_p^{(l)}(N)) \leq \lambda_1(G_p^{(l)}(N)) \leq \cdots \leq \lambda_{(N)-1}(G_p^{(l)}(N)) \) denote eigenvalues of the Laplacian of \( G_p^{(l)}(N) \). Since \( G_p^{(l)}(N) \) is connected \( \lambda_0(G_p^{(l)}(N)) = 0 \) and \( \lambda_1(G_p^{(l)}(N)) \) is positive. A relationship between the adjacency matrix and the Laplacian (cf.
(2)) shows that
\[
\rho_i(G_p(N)) := (l + 1) - \lambda_i(G_p(N))
\]
is an eigenvalue of the adjacency matrix.

**Theorem 1.1.** (1) For \(i \geq 1\),
\[
(\sqrt{l} - 1)^2 \leq \lambda_i(G_p(N)) \leq (\sqrt{l} + 1)^2, \quad \forall N \in \mathcal{N}_{p,l}.
\]
(2) Let \(M\) and \(N\) be elements of \(\mathcal{N}_{p,l}\) satisfying \(M | N\). Then \(G_p(N)\) is a covering of \(G_p(M)\) of degree \(\sigma_1(N/M)\) and
\[
\rho_1(G_p(N)) \geq \rho_1(G_p(M)), \quad \lambda_1(G_p(N)) \leq \lambda_1(G_p(M)).
\]
Here \(\sigma_1\) is the Euler function defined by
\[
\sigma_1(n) = \sum_{d | n} d.
\]

Our tower of Ramanujan graphs \(\{G_p(N)\}_{N \in \mathcal{N}_{p,l}}\) has an interesting geometric property. In order to explain further we recall the (discrete) Cheeger constant. In general let \(G\) be a connected \(d\)-regular graph of \(n\) vertices. The Cheeger constant \(h(G)\) of \(G\) is defined by
\[
h(G) = \min \left\{ \frac{|\partial S|}{|S|} : S \subset V(G), 0 < |S| \leq \frac{n}{2} \right\},
\]
where \(V(G)\) denotes the set of vertices and
\[
\partial S := \{ \{u, v\} \in GE(G) : u \in S, v \in V(G) \setminus S \}.
\]
Here \(GE(G)\) is the set of geometric edges (i.e. the set of unoriented edges, see §2) and \(|\cdot|\) denotes the cardinality. Then the smallest non-zero eigenvalue \(\lambda_1(G)\) of the Laplacian satisfies (2) \([21]\]
\[
\frac{\lambda_1(G)}{2} \leq h(G) \leq \sqrt{2d\lambda_1(G)}
\]
and the next corollary is an immediate consequence of **Theorem 1.1**.

**Corollary 1.1.** (A gap theorem)
\[
\frac{(\sqrt{l} - 1)^2}{2} \leq h(G_p(N)) \leq \sqrt{2(l + 1)(\sqrt{l} + 1)}
\]
for any \(N \in \mathcal{N}_{p,l}\).

In general the graph zeta function (or the Ihara zeta function) \(Z(G)(t)\) is defined for a finite connected graph \(G\). Although a priori \(Z(G)(t)\) is a power series of \(t\), the Ihara formula tells us that it is a rational function (see Fact 2.1). We will show that the zeta functions of our graphs satisfy a reciprocity law.
Theorem 1.2. (A reciprocity law) Let \( p \) and \( q \) be distinct primes satisfying \( p \equiv q \equiv 1 \pmod{12} \) and \( l \) an odd prime different from \( p \) and \( q \). Then

\[
\frac{Z(G_p^{(l)}(q))(t)}{Z(G_p^{(l)}(1))(t)^2} = \frac{Z(G_q^{(l)}(p))(t)}{Z(G_q^{(l)}(1))(t)^2}.
\]

In particular

\[
Z(G_p^{(l)}(q))(t) \equiv Z(G_q^{(l)}(p))(t) \mod \mathbb{Q}(t)^2.
\]

Here is an application of Theorem 1.1 to modular forms. As before let \( p \) be a prime satisfying \( p \equiv 1 \pmod{12} \) and \( N \) a square free positive integer prime to \( p \). Then the spaces of cusp forms \( S_2(\Gamma_0(pN)) \) and one of \( p \)-new forms \( S_2(\Gamma_0(pN))_{pN/N} \) of level \( pN \) (see §4, especially (21)) have decompositions

\[
S_2(\Gamma_0(pN)) = \bigoplus \alpha \mathbb{C}f_\alpha, \quad S_2(\Gamma_0(pN))_{pN/N} = \bigoplus \chi \mathbb{C}f_\chi,
\]

where \( f_\alpha \) and \( f_\chi \) are normalized Hecke eigenforms of character \( \alpha \) and \( \chi \) (cf. Theorem 4.1 and (22)). Using the result due to Alon-Boppana ([1], [2]) we will show the following.

Theorem 1.3. Let \( p \) be a prime satisfying \( p \equiv 1 \pmod{12} \) and \( l \) an odd prime different from \( p \). Let \( \{r_i\}_{i=1}^\infty \) be a set of mutually distinct primes not dividing \( lp \). Set

\[
N_k = \prod_{i=1}^k r_i
\]

and then

\[
\lim_{k \to \infty} \max \{a_l(f_\chi) : S_2(\Gamma_0(pN_k))_{pN_k/N_k} = \bigoplus \chi \mathbb{C}f_\chi\} = 2\sqrt{l},
\]

where \( a_l(f_\chi) \) denotes the \( l \)-th Fourier coefficient of \( f_\chi \). In particular

\[
\lim_{k \to \infty} \max \{a_l(f_\alpha) : S_2(\Gamma_0(pN_k)) = \bigoplus \alpha \mathbb{C}f_\alpha\} = 2\sqrt{l}.
\]

2. Basic facts of the zeta function of a graph

A (finite) graph \( G \) consists of a finite set of vertices \( V(G) \) and a finite set of oriented edges \( E(G) \), which satisfy the following property: there are end point maps,

\[
\partial_0, \quad \partial_1 : E(G) \to V(G),
\]

and an orientation reversal,

\[
J : E(G) \to V(G), \quad J^2 = \text{identity},
\]

such that \( \partial_i \circ J = \partial_{1-i} \) (\( i = 0, 1 \)). The quotient \( E(G)/J \) is called the set of geometric edges and is denoted by \( GE(G) \). We regard an element of \( e \in GE(G) \) as an unoriented edge and if its end-points are \( u \) and \( v \) we write \( e = \{u, v\} \). For \( x \in V(G) \) we set

\[
E_j(x) = \{e \in E(G) | \partial_j(e) = x\}, \quad j = 0, 1.
\]
Thus $JE_j(x) = E_{1-j}(x)$. Intuitively $E_0(x)$ (resp. $E_1(x)$) is the set of edges departing from (resp. arriving at) $x$. The degree of $x$, $d(x)$, is defined by

$$d(x) = |E_0(x)| = |E_1(x)|.$$ 

$E(G)$ is naturally divided into two classes, loops and passes. An edge $e \in E(G)$ is called a loop if $\partial_0(e) = \partial_1(e)$ and is called a pass otherwise. Let $2l(x)$ and $p(x)$ be the number of loops and passes starting from $x$, respectively (both $l(x)$ and $p(x)$ are positive integers). Note that, because of the involution $J$, if we replace ”departing” by ”arriving” these number does not change. By definition, it is clear that

$$d(x) = 2l(x) + p(x).$$

We set $q(x) := d(x) - 1$. Let $C_0(G)$ be the free $\mathbb{Z}$-module generated by $V(G)$ with vertices as the natural basis. We define endomorphisms $Q$ and $A$ of $C_0(G)$ by

$$Q(x) = q(x)x, \quad x \in V(G),$$

and

$$A(x) = \sum_{e \in E(G), \partial_0(e)=x} \partial_1(e), \quad x \in V(G),$$

respectively. Note that because of the involution $J$,

$$A(x) = \sum_{e \in E(G), \partial_1(e)=x} \partial_0(e).$$

The operator $A$ will be called the adjacency operator. We sometimes identify it with the representing matrix with respect to the basis $\{x\}_{x \in V(G)}$. Thus the $yx$-entry $A_{yx}$ of $A$ is the number of edges departing from $x$ and arriving at $y$. The orientation reversing involution $J$ implies

$$A_{xy} = A_{yx}.$$ 

Note that $A_{xx} = 2l(x)$ and $p(x) = \sum_{y \neq x} A_{yx}$. If $d(x) = k$ for all $x \in V(G)$, $G$ is called $k$-regular.

Connecting distinct vertices $x$ and $y$ by geometric $A_{xy}$-edges and drawing $\frac{1}{2} A_{xx}$-loops at $x$, the adjacency matrix $A$ determines an unoriented 1-dimensional simplicial complex. We call it the geometric realization of $G$, and denote it by $\widetilde{G}$ again. We say that $G$ is connected if the geometric realization is. The Euler characteristic $\chi(G)$ is equal to $|V(G)| - |GE(G)|$, hence if $G$ is connected, the fundamental group is a free group of rank $1 - |V(G)| + |GE(G)|$. For a later purpose, we summarize the relationship between a graph and its adjacency matrix.

**Proposition 2.1.** Let $A = (a_{ij})_{1 \leq i,j \leq m}$ be an $m \times m$-matrix satisfying the following conditions.

1. The entries $(a_{ij})_{ij}$ are non-negative integers and satisfy

$$a_{ij} = a_{ji}, \quad \forall i, j.$$
(2) $a_{ii}$ is even for every $i$.

Then there is a unique graph $G$ whose adjacency matrix is $A$. Moreover, $G$ is $k$-regular if and only if one of the following equivalent condition satisfied:

(a) \[
\sum_{i=1}^{m} a_{ij} = k, \quad \forall j
\]

(b) \[
\sum_{j=1}^{m} a_{ij} = k, \quad \forall i.
\]

In the following, a graph $G$ is always assumed to be connected. A path of length $m$ is a sequence $c = (e_1, \ldots, e_m)$ of edges such that $\partial_0(e_i) = \partial_1(e_{i-1})$ for all $1 < i \leq m$ and the path is reduced if $e_i \neq J(e_{i-1})$ for all $1 < i \leq m$. The path is closed if $\partial_0(e_1) = \partial_1(e_m)$, and the closed path has no tail if $e_m \neq J(e_1)$. A closed path of length one is nothing but a loop. Two closed paths are equivalent if one is obtained from the other by a cyclic shift of the edges. Let $\mathcal{C}(G)$ be the set of equivalence classes of reduced and tail-less closed paths of $G$. Since the length depends only on the equivalence class, the length function descends to the map:

\[
l : \mathcal{C}(G) \to \mathbb{N}, \quad l([c]) = l(c),
\]

where $[c]$ is the class determined by $c$. We define a reduced and tail-less closed path $C$ to be primitive if it is not obtained by going $r (\geq 2)$ times some another closed path. Let $\mathcal{P}(G)$ be the subset of $\mathcal{C}(G)$ consisting of the classes of primitive closed paths (which are reduced and tail-less by definition). The graph zeta function (or Ihara zeta function) of $G$ is defined to be

\[
Z(G)(t) = \prod_{[c] \in \mathcal{P}(G)} \frac{1}{1 - t^{l([c])}}.
\]

Although this is an infinite product, it is a rational function.

**Fact 2.1.** ([4], [10], [11], [19])

\[
Z(G)(t) = \frac{(1 - t^2)^\chi(G)}{\det[1 - At + Qt^2]},
\]

**Fact 2.2.** ([20]) Let $G$ be a $k$-regular graph with $m$ vertices. Then the Euler characteristic $\chi(G)$ is

\[
\chi(G) = \frac{m(2-k)}{2}.
\]

**Remark 2.1.** Note that the Euler characteristic does not depend on the number of loops.
Let $E_{or}(G) \subset E(G)$ be a section of the natural projection $E(G) \to GE(G)$. In other word we choose an orientation on geometric edges and make the geometric realization into an oriented one dimensional simplicial complex. Let $C_{1}(G)$ be the free $\mathbb{Z}$-module generated by $E_{or}(G)$. Then the boundary map 
\[ \partial : C_{1}(G) \to C_{0}(G) \]
is naturally defined. Let $\partial^{t}$ be the dual of $\partial$ and the Laplacian $\Delta$ of $G$ is defined to be $\Delta = \partial \partial^{t}$. It is known (and easy to check) that \[^{22}, \text{[10]}, \text{2}\]
\[ \Delta = 1 - A + Q. \]
Now let $G$ be a connected $k$-regular graph. Since $0$ is an eigenvalue of $\Delta$ with multiplicity one, (2) shows that $k$ is an eigenvalue of $A$ with multiplicity one. Because of semi-positivity of $\Delta$ we find that
\[ |\lambda| \leq k \] for any eigenvalue $\lambda$ of $A$
and that $-k$ is an eigenvalue of $A$ if and only if $G$ is bipartite \[^{22}, \text{Chapter 3}\].

**Definition 2.1.** Let $G$ be a $k$-regular graph. We say that it is Ramanujan, if all eigenvalues $\lambda$ of $A$ with $|\lambda| \neq k$ satisfy
\[ |\lambda| \leq 2\sqrt{k-1}. \]

See \[^{14}, \text{[15]}, \text{and [23]}\] for detailed expositions of Ramanujan graphs.

A map $f$ from a graph $G'$ to $G$ is defined to be a pair $f = (f_{V}, f_{E})$ of maps 
\[ f_{V} : V(G') \to V(G), \quad f_{E} : E(G') \to E(G) \]
satisfying 
\[ \partial_{i}f_{E} = f_{V}\partial_{i}, \quad i = 0, 1. \]
Suppose that $G$ and $G'$ are connected. If there is a positive integer $d$ such that 
$|f_{V}^{-1}(v)| = |f_{E}^{-1}(e)| = d$ for any $v \in V(G)$ and $e \in E(G)$, $f$ is mentioned as a covering map of degree $d$.

**3. A construction of a Ramanujan graph**

Let $p$ be a prime, and $B$ the quaternion algebra over $\mathbb{Q}$ ramified at two places $p$ and $\infty$. Let $R$ be a fixed maximal order in $B$ and \{\(I_{1}, \cdots, I_{n}\)\} be the set of left $R$-ideals representing the distinct ideal classes. We choose $I_{1} = R$ and say $n$ the class number of $B$. For $1 \leq i \leq n$, $R_{i}$ denotes the right order of $I_{i}$, and let $w_{i}$ the order of $R_{i}^\times/\{\pm1\}$. The product
\[ W = \prod_{i=1}^{n} w_{i} \]
is independent of the choice of $R$ and is equal to the exact denominator of $\frac{p-1}{12}$ (§ p.117) and Eichler’s mass formula states that

$$\sum_{i=1}^{n} \frac{1}{w_i} = \frac{p-1}{12}.$$ 

Let $\mathbb{F}$ be an algebraic closure of $\mathbb{F}_p$. There are $n$ distinct isomorphism classes $\{E_1, \ldots, E_n\}$ of supersingular elliptic curves over $\mathbb{F}$ such that $\text{End}(E_i) \cong R_i$. Now we assume that $p-1$ is divisible 12. Then $\frac{p-1}{12}$ is an integer and $W = \prod_{i=1}^{n} w_i = 1$, namely $w_i = 1$ for all $i$. Hence by Eichler’s mass formula

$$n = \frac{p-1}{12}.$$ 

We fix an odd prime $l$ different from $p$ and let $\mathcal{N}_{p,l}$ denote the set of square free positive integers prime to $lp$. For $N \in \mathcal{N}_{p,l}$, an enhanced supersingular elliptic curve of level $N$ is defined to be a pair $E = (E, C_N)$ of a supersingular elliptic curve $E$ and its cyclic subgroup $C_N$ of order $N$. A homomorphism $\phi$ from $E = (E, C_N)$ to $E' = (E', C'_N)$ is defined by a homomorphism $\phi : E \to E'$ satisfying

$$\phi(C_N) = C'_N.$$ 

Let $\Sigma_N$ be the set of isomorphism classes of enhanced supersingular elliptic curve of level $N$ defined over $\mathbb{F}$. Then the cardinality $\nu(N)$ of $\Sigma_N$ is

$$\nu(N) = \frac{(p-1)\sigma_1(N)}{12}, \quad \sigma_1(N) = \sum_{d|N} d.$$ 

Here $\sigma_1(N)$ counts the number of cyclic subgroups of $E$ of order $N$. Let $\text{Hom}(E_i, E_j)(l)$ denote the set of homomorphisms from $E_i$ to $E_j$ of degree $l$. We define the Brandt matrix $B_p^{(l)}(N)$ is defined to be a $\nu(N) \times \nu(N)$-matrix whose $(i, j)$-entry is

$$b_{ij} = \frac{1}{2} |\text{Hom}(E_j, E_i)(l)|.$$ 

**Proposition 3.1.** Let $N \in \mathcal{N}_{p,l}$. Then the Brandt matrix $B_p^{(l)}(N) = (b_{ij})_{1 \leq i, j \leq \nu(N)}$ satisfies the following.

1. Every entry is a non-negative integer and $B_p^{(l)}(N)$ is symmetric; $b_{ij} = b_{ji}$.
2. The diagonal entries $\{b_{ii}\}_i$ are even for all $i$.
3. For any $i = 1, \ldots, \nu(N)$,

$$\sum_{j=1}^{n} b_{ij} = l + 1.$$
Proof. By definition a homomorphism from \( E_i = (E_i, C_N) \) to \( E_j = (E_j, D_N) \) is a homomorphism \( \phi : E_i \rightarrow E_j \) of degree \( l \) satisfying
\[
\phi(C_N) = D_N.
\]
Being \( \hat{\phi} \) the dual of \( \phi \), \( \hat{\phi} \circ \phi = l \) and \( \hat{\phi}(D_N) = \hat{\phi}(\phi(C_N)) = C_N \). Hence taking the dual homomorphisms yields bijective correspondence
\[
I : \text{Hom}(E_i, E_j)(l) \rightarrow \text{Hom}(E_j, E_i)(l), \quad I(\phi) = \hat{\phi},
\]
which implies (1). In order to show the claim (2), it is sufficient to show that the action of \( I \) on \( \text{End}(E_i)(l)/\pm 1 \) has no fixed point. Let \( \phi \) be an element of \( \text{End}(E_i)(l)/\pm 1 \). Then \( \text{Ker} \phi \simeq \text{Ker} \hat{\phi} \simeq \mathbb{F}_l \) and there is a skew-symmetric nondegenerate pairing derived from the Weil paring \((\mathbb{I}^7 \text{ Remark 8.4})\)
\[
\text{Ker} \phi \times \text{Ker} \hat{\phi} \rightarrow \mu_l.
\]
Suppose that there were \( \phi \in \text{End}(E_i)(l)/\pm 1 \) fixed by \( I \). Then \( \hat{\phi} = \pm \phi \) and \( \text{Ker} \phi = \text{Ker} \hat{\phi} \), which contradicts to non-degeneracy of the pairing. The claim (3) follows from the following observation: Let \( E_j \) be the underlying supersingular elliptic curve of \( E_i \). Then by definition \( \sum_{i=1}^{n} b_{ij} \) is equal to the number of cyclic subgroups of \( E_i \) of order \( l \), which is \( l + 1 \).

By Proposition 2.1 there is a regular graph \( G^{(l)}_p(N) \) of degree \( l + 1 \) whose adjacency matrix is \( B^{(l)}_p(N) \). In Theorem 5.1 we will show that it is a connected non-bipartite Ramanujan graph.

**Theorem 3.1.** Let \( M \) and \( N \) be elements of \( \mathcal{N}_{p,l} \) such that \( M \) is a divisor of \( N \). Then there is a covering map
\[
\pi_{N/M} : G^{(l)}_p(N) \rightarrow G^{(l)}_p(M)
\]
of degree \( \sigma_1(N/M) \)

**Proof.** Since \( N \) is square free \( M \) and \( N/M \) are coprime. Thus a cyclic subgroup \( C_N \) is written by
\[
C_N = C_M \oplus C_{N/M}
\]
and we define
\[
(\pi_{N/M})_V : V(G^{(l)}_p(N)) \rightarrow V(G^{(l)}_p(M)), \quad (\pi_{N/M})_V(E, C_M \oplus C_{N/M}) = (E, C_M).
\]
Since the number of cyclic subgroups of \( E \) of order \( N/M \) is \( \sigma_1(N/M) \), \( |\pi_{N/M}^{-1}(v)| = \sigma_1(N/M) \) for any \( v \in V(G^{(l)}_p(M)) \). By definition an edge of \( G^{(l)}_p(N) \) from \( E = (E, C_M \oplus C_{N/M}) \) to \( E' = (E', C'_M \oplus C'_{N/M}) \) is a homomorphism \( f \) from \( E \) to \( E' \) satisfying
\[
f(C_M) = C'_M, \quad f(C_{N/M}) = C'_{N/M}.
\]
Forget the homomorphism of cyclic subgroups of order \( N/M \) and we have
\[
\text{Hom}(E, E')(l)/\{\pm 1\} \rightarrow \text{Hom}(\pi_{N/M}(E), \pi_{N/M}(E'))(l)/\{\pm 1\},
\]
which defines a map of the set of edges

$$(\pi_{N/M})_E : E(G_p^{(l)}(N)) \to E(G_p^{(l)}(M))$$

satisfying

$$\partial_i \circ (\pi_{N/M})_E = (\pi_{N/M})_V \circ \partial_i, \quad i = 0, 1.$$ 

One finds that this map has degree $\sigma_1(N/M)$. In fact let $g$ be an element of $\text{Hom}(\pi_{N/M}(E), \pi_{N/M}(E'))(l)$. Thus $g$ is a homomorphism from $E$ to $E'$ of degree $l$ satisfying $g(C_M) = C'_M$.

Let $C_{N/M}$ be a cyclic subgroup of $E$ of order $N/M$ and we set $C'_{N/M} = g(C_{N/M})$. Then we have a homomorphism of enhanced supersingular elliptic curve of level $N$

$$g : (E, C_M \oplus C_{N/M}) \to (E', C'_M \oplus C'_{N/M})$$

which defines an edge of $G_p^{(l)}(N)$. The number of cyclic subgroups of order $N/M$ (i.e. choices of $C_{N/M}$) is $\sigma_1(N/M)$ and the claim has been proved.

4. A spectral decomposition of the character group

For a positive integer $N$, let $S_2(\Gamma_0(N))$ denote the space of cusp forms of weight 2 for the Hecke congruence subgroup

$$\Gamma_0(N) := \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : \quad c \equiv 0 \ (\text{mod} \ N) \}.$$ 

Let $Y_0(N)$ be the modular curve which parametrizes isomorphism classes of a pair $E = (E, C_N)$ of an elliptic curve $E$ and its cyclic subgroup $C_N$ of order $N$. It is a smooth curve defined over $\mathbb{Q}$ and the set of $\mathbb{C}$-valued points is the quotient of the upper half plane by $\Gamma_0(N)$. Let $X_0(N)$ be the compactification of $Y_0(N)$. It is a smooth projective curve defined over $\mathbb{Q}$ and has the canonical model over $\mathbb{Z}$ which has been studied by [7] and [12] in detail. The space of cusp forms $S_2(\Gamma_0(N))$ is naturally identified with the space of holomorphic 1-forms $H^0(X_0(N), \Omega)$ and in particular with the cotangent space $\text{Cot}_0(J_0(N))$ at the origin of the Jacobian variety $J_0(N)$ of $X_0(N)$.

For a prime $p$ with $(p, N) = 1$, $X_0(N)$ furnishes the $p$-th Hecke operator defined by

$$(7) \quad T_p(E, C_N) := \sum_C (E/C, (C_N + C)/C),$$
where $C$ runs through all cyclic subgroup schemes of $E$ of order $p$. If $p$ is a prime divisor of $N$, an operator $U_p$ is defined by

\[(8) \quad U_p(E, C_N) := \sum_{C \neq D} (E/C, (C_N + C)/C)\]

where $D$ is the cyclic subgroup of $C_N$ of order $p$. By the functoriality, Hecke operators act on $J_0(N)$ and $\text{Cot}_0(J_0(N)) = S_2(\Gamma_0(N))$ and the resulting action coincides with the usual one on $S_2(\Gamma_0(N))$ (see [18]). We define the Hecke algebra as $T_0(N) := \mathbb{Z}[\{T_p\}_{(p,N)=1}, \{U_p\}_{p|N}]$, which is a commutative subring of $\text{End} J_0(N)$.

The effects of $T_p$ and $U_p$ on $f = \sum_{n=1}^{\infty} a_n q^n \in S_2(\Gamma_0(N))$ are

\[(9) \quad f|U_p = \sum_{n=1}^{\infty} a_{pn} q^n\]

and

\[(10) \quad f|T_p = \sum_{n=1}^{\infty} (a_{pn} + pa_{n/p}) q^n.\]

Here $a_{n/p} = 0$ if $n/p$ is not an integer.

**Definition 4.1.** For a positive integer $M$, we define a subalgebra $\mathcal{T}_0(N)^{(M)}$ of $\mathcal{T}_0(N)$ to be the omitting of Hecke operators from $\mathcal{T}_0(N)$ whose indices are prime divisors of $M$, that is

\[\mathcal{T}_0(N)^{(M)} = \mathbb{Z}[\{T_p\}_{(p,NM)=1}, \{U_p\}_{p|N(M), (p,M)=1}].\]

We call an algebraic homomorphism from $\mathcal{T}_0(N)^{(M)}$ to $\mathbb{C}$ as a character. If the image is contained in $\mathbb{R}$ it is referred as real.

Let $M$ be a positive integer and $f$ an element of $S_2(\Gamma_0(M))$. For a positive integer $d$ we set

\[f^{(d)}(z) = f(dz) \in S_2(\Gamma_0(dM)).\]

**Definition 4.2.** Let $N$ be a square free positive integer and $M$ a divisor of $N$. For a divisor $d$ of $N/M$ we define

\[S_2(\Gamma_0(M))^{(d)} = \{f^{(d)}(z) \mid f \in S_2(\Gamma_0(M))\} \subset S_2(\Gamma_0(N)).\]

The space of old forms of level $N$ is defined to be

\[S_2(\Gamma_0(N))_{\text{old}} = \sum_{M|N, M \neq N} \sum_{d|(N/M)} S_2(\Gamma_0(M))^{(d)} \subset S_2(\Gamma_0(N))\]

and the orthogonal complement of $S_2(\Gamma_0(N))_{\text{old}}$ for the Petersson product is called by the space of new forms and denoted by $S_2(\Gamma_0(N))_{\text{new}}$. 
Let $N$ be a square free positive integer and $q$ a prime not dividing $N$. Since the action of $T_q$ on $S_2(\Gamma_0(N))$ is self-adjoint for the Petersson product and since $S_2(\Gamma_0(N))_{\text{odd}}$ is stable by $T_q$, $S_2(\Gamma_0(N))_{\text{new}}$ is stable by $T_0(N)^{(N)}$. This implies that $S_2(\Gamma_0(N))_{\text{new}}$ admits a spectral decomposition by $T_0(N)^{(N)}$. We will show that $S_2(\Gamma_0(N))$ has an irreducible decomposition of multiplicity one by the action of the full Hecke algebra $T_0(N)$ (cf. Theorem 4.1). In proving the theorem, a key fact is the following, which is mentioned as multiplicity one (21).

**Fact 4.1.** Let $N$ be a positive integer (which may not be square free) and $f = \sum_{n=1}^{\infty} a_n q^n$ an element of $S_2(\Gamma_0(N))$. Suppose that $a_n = 0$ for all $n$ with $(n, t) = 1$, where $t$ is a fixed positive integer. Then $f \in S_2(\Gamma_0(N))_{\text{odd}}$.

This fact shows that the above spectral decomposition of $S_2(\Gamma_0(N))_{\text{new}}$ by $T_0(N)^{(N)}$ has multiplicity one. One finds that this yields an irreducible decomposition of $S_2(\Gamma_0(N))_{\text{new}}$ for the full Hecke algebra. In fact let $f \in S_2(\Gamma_0(N))_{\text{new}}$ be the normalized eigenform of $T_0(N)^{(N)}$ and $p$ a prime not dividing $N$. Since $T_p$ is selfadjoint for the Petersson product its eigenvalue is real number. Moreover $f$ is automatically a Hecke eigenform of the full Hecke algebra by the following reason. Let $\alpha$ be the character of $T_0(N)^{(N)}$ associated to $f$ and $q$ be a prime divisor of $N$. Since $T_0(N)$ is commutative $f|U_q$ is also a Hecke eigenform of $T_0(N)^{(N)}$ whose character is $\alpha$. By the multiplicity one, $f|U_q$ should be a multiple of $f$;

$$f|U_q = \alpha_q f.$$ 

Defining $\alpha(U_q) = \alpha_q$, we have a character $\alpha$ of $T_0(N)$ and $f$ is the normalized Hecke eigenform of character $\alpha$. Moreover since $N$ is square free $\alpha_q = \pm 1$ for $q \mid N$ (22 Lemma 3.2) and $\alpha$ is real character. Thus we have an irreducible decomposition as a $T_0(N)$-module

$$S_2(\Gamma_0(N))_{\text{new}} = \bigoplus_\alpha S_2(\Gamma_0(N))_{\text{new}}(\alpha)$$

by real characters and every irreducible component has dimension one. Here $S_2(\Gamma_0(N))_{\text{new}}(\alpha)$ denotes the isotypic component of $\alpha$

$$S_2(\Gamma_0(N))_{\text{new}}(\alpha) = \{ f \in S_2(\Gamma_0(N))_{\text{new}} \mid f|T = \alpha(T)f, \ \forall T \in T_0(N) \},$$

which is spanned by the normalized Hecke eigenform. By the definition of the space of new forms we have

$$(11) \quad S_2(\Gamma_0(N)) = \bigoplus_{M \mid N} (\bigoplus_{d \mid (N/M)} S_2(\Gamma_0(M))^{(d)}_{\text{new}}).$$

Fix a divisor $M$ of $N$ and let us consider the subspace

$$S_M = \bigoplus_{d \mid (N/M)} S_2(\Gamma_0(M))^{(d)}_{\text{new}}.$$

Being $N/M = l_1 \cdots l_m$ a prime decomposition, there is an isomorphism as vector spaces

$$(12) \quad S_M \simeq S_2(\Gamma_0(M))^{\oplus 2^m}_{\text{new}}.$$  

We will explicitly describe this isomorphism.
Proposition 4.1. Let $N$ be a square free positive integer and $M$ a divisor of $N$. Let $f \in S_2(\Gamma_0(M))_{\text{new}}$ be a normalized Hecke eigenform. Then for $\epsilon = (\epsilon_{i_1}, \cdots, \epsilon_{i_m})$ ($\epsilon_{i_1} = \pm$) there is a normalized Hecke eigenform $f_\epsilon$ of level $N$ satisfying the following conditions.

1) If $q$ a prime not dividing $N/M$

\[ a_q(f_\epsilon) = a_q(f). \]

2) \[ a_{t_i}(f_\epsilon) = \alpha_{i_1}^{\epsilon_{i_1}} \]

where

\[ \alpha_{i_1}^+ = \frac{a_{i_1}(f) + \sqrt{\Delta_i}}{2}, \quad \alpha_{i_1}^- = \frac{a_{i_1}(f) - \sqrt{\Delta_i}}{2}, \quad \Delta_i = a_{i_1}(f)^2 - 4l_{i_1}( < 0). \]

Moreover the $2^m$ complex numbers $\{\alpha_{i_1}^{(\pm)}, \cdots, \alpha_{i_m}^{(\pm)}\}$ are mutually different.

Proof. In general let $p$ be a prime and $F$ a square free positive integer prime to $p$. We have two degeneracy maps $\alpha_p, \beta_p : X_0(pF) \to X_0(F)$ defined by

\[ \alpha_p(E, C_p \oplus C_F) = (E, C_F), \quad \beta_p(E, C_p \oplus C_F) = (E/C_p, (C_p \oplus C_F)/C_p), \]

which induces linear maps

\[ \alpha_p^*, \beta_p^* : S_2(\Gamma_0(F)) \to S_2(\Gamma_0(pF)) \]

whose effects on $f = \sum_{n=1}^{\infty} a_nq^n \in S_2(\Gamma_0(F))$ are

\[ \alpha_p^*(f) = f = \sum_{n=1}^{\infty} a_nq^n, \quad \beta_p^*(f) = f^{(p)} = \sum_{n=1}^{\infty} a_nq^{pn}. \]

Let $T$ be $T_r$ ($r \nmid pF$) or $U_l$ ($l \mid F$). Then $T$ commutes with $\alpha_p$ and $\beta_p$ and

\[ S_2(\Gamma_0(F)) \oplus S_2(\Gamma_0(F)) \xrightarrow{\alpha_p^* + \beta_p^*} S_2(\Gamma_0(pF)) \]

Using (14) and (15) we will inductively construct $f_\epsilon$ by the number of prime divisors $m$. We set $M_m = Ml_1 \cdots l_m$ ($m \geq 1$) and $M_0 = M$. Suppose that we have constructed a desired normalized Hecke eigenform $f_\epsilon \in S_2(\Gamma_0(M_{m-1}))$ of character $\chi_\epsilon$. For a prime $r$ different from $l_m$, we let $T$ be $T_r$ or $U_r$ according to $r \nmid M_m$ or $r \mid M_{m-1}$, respectively. Then (15) implies

\[ S_2(\Gamma_0(M_{m-1})) \oplus S_2(\Gamma_0(M_{m-1})) \xrightarrow{\alpha_{i_m}^* + \beta_{i_m}^*} S_2(\Gamma_0(M_m)) \]

\[ S_2(\Gamma_0(M_{m-1})) \oplus S_2(\Gamma_0(M_{m-1})) \xrightarrow{\alpha_{i_m}^* + \beta_{i_m}^*} S_2(\Gamma_0(M_m)). \]
Hence
\[ \alpha_{l-m}^* (f_\ell) | T = \alpha_{l-m}^* (f_\ell | T) = \chi_\ell (T) \alpha_{l-m}^* (f_\ell) \]
and
\[ \beta_{l-m}^* (f_\ell) | T = \beta_{l-m}^* (f_\ell | T) = \chi_\ell (T) \beta_{l-m}^* (f_\ell). \]
Define a character
\[ \chi_{l-m}^* (T) : \mathcal{P}_0 (M_m) \rightarrow \mathbb{C} \]
by
\[ \chi_{l-m}^* (T) = \chi_\ell (T), \]
and \( \alpha_{l-m}^* (f_\ell) \) and \( \beta_{l-m}^* (f_\ell) \) are \( \mathcal{P}_0 (M_m) \)-eigenforms of the same character \( \chi_{l-m}^* \).
Let us investigate the action of \( U_{l-m} \). By (9), (10) and (14)
\[
\begin{pmatrix}
\alpha_{l-m}^* (f_\ell) | U_{l-m} \\
\beta_{l-m}^* (f_\ell) | U_{l-m}
\end{pmatrix}
= \begin{pmatrix}
a_{l-m}(f_\ell) & -l_m \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
\alpha_{l-m}^* (f_\ell) \\
\beta_{l-m}^* (f_\ell)
\end{pmatrix}.
\]
Use the assumption (1) and the characteristic polynomial of \( U_{l-m} \) is
\[ \Phi (t) = t^2 - a_{l-m} (f_\ell) t + l_m = t^2 - a_{l-m} (f) t + l_m. \]
Since \( f \) is a normalized \( \mathcal{P}_0 (M) \)-eigenform which is new, the discriminant \( \Delta_m = a_{l-m} (f)^2 - 4 l_m \) is negative (4). Therefore the eigenvalue of \( U_{l-m} \) are mutually distinct and contained in \( \mathbb{C} \setminus \mathbb{R} \).
Set
\[ \alpha_{l-m}^+ = \frac{a_{l-m} (f) + \sqrt{\Delta_m}}{2}, \quad \alpha_{l-m}^- = \frac{a_{l-m} (f) - \sqrt{\Delta_m}}{2} \]
and let \( f_\ell^+ \) and \( f_\ell^- \) be the corresponding normalized cusp form of level \( M_m \) satisfying
\[ f_\ell^+ | U_{l-m} = \alpha_{l-m}^+ f_\ell^+, \quad f_\ell^- | U_{l-m} = \alpha_{l-m}^- f_\ell^- \]
Extend \( \chi_{l-m}^* \) to a character \( \chi_{l-m}^+ \) and \( \chi_{l-m}^- \) of \( \mathcal{P}_0 (M_m) = \mathcal{P}_0 (M_m) [U_{l-m}] \) by
\[ \chi_{l-m}^+ (U_{l-m}) = \alpha_{l-m}^+, \quad \chi_{l-m}^- (U_{l-m}) = \alpha_{l-m}^- \]
Then \( f_\ell^+ \) and \( f_\ell^- \) are \( \mathcal{P}_0 (M_m) \)-eigenforms whose characters are \( \chi_{l-m}^+ \) and \( \chi_{l-m}^- \), respectively. Observe that \( \alpha_{l-m}^+ \) and \( \alpha_{l-m}^- \) are different from each of \( \{ \alpha_{l_i}^+, \alpha_{l_i}^- \}_{1 \leq i \leq m - 1} \), where
\[ \alpha_{l_i}^+ = \frac{a_{l_i} (f) + \sqrt{\Delta_i}}{2}, \quad \alpha_{l_i}^- = \frac{a_{l_i} (f) - \sqrt{\Delta_i}}{2}, \quad \Delta_i = a_{l_i} (f)^2 - 4 l_i. \]
In fact if \( \alpha_{l_m}^+ = \alpha_{l_i}^+ (1 \leq i \leq m - 1) \), comparing their real and imaginary part we conclude
\[ a_{l_m} (f) = a_{l_i} (f), \quad \Delta_m = \Delta_i \]
which implies \( l_m = l_i \). Thus we have constructed normalized \( 2^m \) Hecke eigenforms of level \( M_m \) from \( f \) whose characters are mutually different.
Proposition 4.1 yields a spectral decomposition of multiplicity one

\[(18) \quad S_M = \bigoplus \beta C f_\beta \]

where \(f_\beta\) is the normalized Hecke eigenform of character \(\beta\). Let \(M'\) be a divisor of \(N\) different from \(M\) and we consider the decomposition (18) for \(M'\),

\[(19) \quad S_{M'} = \bigoplus \beta C f_{\beta'} \]

The following lemma shows that every character \(\beta\) in (18) is different from each of \(\beta'\) in (19).

**Lemma 4.1.** Let \(f \in S_2(\Gamma_0(N_f))\) (resp. \(g \in S_2(\Gamma_0(N_g))\)) be a normalized Hecke eigenform. If there is a positive integer \(t\) such that \(a_l(f) = a_l(g)\) for any prime \(l\) with \(l \nmid t\), then \(f = g\).

**Proof.** Let \(K_f\) (resp. \(K_g\)) be the number field generated by Fourier coefficients of \(f\) and (resp. \(g\)) over \(\mathbb{Q}\) and let \(K\) be the minimal extension of \(\mathbb{Q}\) that contains \(K_f\) and \(K_g\). We fix a prime \(l\) satisfying \(l \nmid N_f N_g\) and that completely splits in \(K\). Corresponding to \(f\) and \(g\), there are absolutely irreducible representations \(\rho_{f,l} : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{Q}_l)\), \(\rho_{g,l} : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{Q}_l)\) of the conductor \(N_f\) and \(N_g\) respectively which satisfy

\[\det(t - \rho_{f,l}(Frob_q)) = t^2 - a_q(f)t + q, \quad (q, lN_f) = 1\]

and

\[\det(t - \rho_{g,l}(Frob_q)) = t^2 - a_q(g)t + q, \quad (q, lN_g) = 1.\]

([6] Theorem 3.1). Here \(Frob_q\) is the Frobenius at a prime \(q\). Let \(S\) be a finite set of primes. Since a semi-simple representation \(\rho_l : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{Q}_l)\) is determined by values \(Tr\rho_l(Frob_q)\) on the primes \(q \notin S\) at which \(\rho_l\) is unramified ([6] Proposition 2.6 (3)), the assumption implies that \(\rho_{f,l} = \rho_{g,l}\) and in particular \(N_f = N_g\). Now we deduce that \(f = g\) from **Fact 4.1**.

\[\square\]

**Remark 4.1.** Here is another way to see that any \(\beta\) in (18) is different from each of \(\beta'\) in (19). If necessary changing \(M\) and \(M'\), let \(r\) be a prime divisor of \(M'\) not dividing \(M\). By the construction \(\beta'(U_r) \in \mathbb{R}\) and \(\beta(U_r) \in \mathbb{C} \setminus \mathbb{R}\) and therefore \(\beta\) and \(\beta'\) are different.

For a character \(\alpha\) of \(T_0(N)\), let \(S_2(\Gamma_0(N))(\alpha)\) denote the isotypic component of \(\alpha\);

\[S_2(\Gamma_0(N))(\alpha) = \{ f \in S_2(\Gamma_0(N)) \mid f|T = \alpha(T)f, \quad \forall T \in T_0(N)\}.\]
Theorem 4.1. (Strong multiplicity one) Let $N$ be a square free positive integer. Then there is an isomorphism as $\mathbb{T}_0(N)$-modules

$$S_2(\Gamma_0(N)) = \bigoplus_\alpha S_2(\Gamma_0(N))(\alpha)$$

such that every irreducible component has dimension one and is spanned by the normalized Hecke eigenform $f_\alpha$. The index $\alpha$ in the decomposition runs through the set of closed points $\text{Spec}(\mathbb{T}_0(N))(\mathbb{C})$ and there is an isomorphism

$$\Phi : \mathbb{T}_0(N) \otimes \mathbb{C} \simeq \prod_{\alpha \in \text{Spec}(\mathbb{T}_0(N))(\mathbb{C})} \mathbb{C}$$

such that the composition with the projection $\pi_\alpha$ to the $\alpha$-factor is $\alpha$:

$$\pi_\alpha \circ \Phi = \alpha.$$

Proof. The previous argument and (11) show that $S_2(\Gamma_0(N))(\alpha)$ is a $\mathbb{C}$-linear space generated by a normalized Hecke eigenform $f_\alpha$ and we have an irreducible decomposition of multiplicity one

(20) $$S_2(\Gamma_0(N)) = \bigoplus_\alpha S_2(\Gamma_0(N))(\alpha).$$

The linear isomorphism

$$\text{Hom}_{\mathbb{C}}(\mathbb{T}_0(N), \mathbb{C}) \simeq S_2(\Gamma_0(N)), \quad \rho \mapsto \sum_{m=1}^{\infty} \rho(T_m)q^m$$

implies that $\{\alpha\}$ in the right hand side of (20) is the set of closed points $\text{Spec}(\mathbb{T}_0(N))(\mathbb{C})$ and $\{f_\alpha\}_{\alpha \in \text{Spec}(\mathbb{T}_0(N))(\mathbb{C})}$ is a basis of $S_2(\Gamma_0(N))$. Now the desired decomposition of $\mathbb{T}_0(N) \otimes \mathbb{C}$ is obvious.

Let $p$ be any prime (not necessary $p \equiv 1(\text{mod} 12)$) and $N$ a square free positive integer prime to $p$. We define the space of $p$-new forms $S_2(\Gamma_0(pN))_{pN/N}$ to be the orthogonal complement of $\alpha_\ast^p(S_2(\Gamma_0(N))) + \beta_\ast^p(S_2(\Gamma_0(N)))$ in $S_2(\Gamma_0(pN))$ for the Petersson inner product. Then (11) and (14) imply

(21) $$S_2(\Gamma_0(pN))_{pN/N} = \bigoplus_{M|N} \bigoplus_{d|(N/M)} S_2(\Gamma_0(pM))^{(d)}_{\text{new}}$$

and by Theorem 4.1 we have a decomposition of $\mathbb{T}_0(N)$-module of multiplicity one

(22) $$S_2(\Gamma_0(pN))_{pN/N} = \bigoplus_\chi \mathbb{C} f_\chi.$$

Here $f_\chi$ is a normalized Hecke eigenform whose character is $\chi$. Let $\mathbb{T}_0(pN)_{pN/N}$ be the restriction of $\mathbb{T}_0(N)$ to this space. Then the set of characters in (22) coincides with $\text{Spec}(\mathbb{T}_0(pN)_{pN/N})(\mathbb{C})$ and there is an isomorphism

(23) $$\Phi : \mathbb{T}_0(pN)_{pN/N} \otimes \mathbb{C} \simeq \prod_{\chi \in \text{Spec}(\mathbb{T}_0(pN)_{pN/N})(\mathbb{C})} \mathbb{C}$$
such that the composition with the projection $\pi_\chi$ to $\chi$-factor is $\chi$:

$$\pi_\chi \circ \Phi = \chi.$$  

Using [10] we will clarify a relationship between $S_2(\Gamma_0(pN))_{pN/N}$ and the Ramanujan graph $G_0^{(l)}(N)$.

By the functoriality $\alpha_p$ and $\beta_p$ induce a homomorphism

$$(24) \quad \alpha_p^*, \beta_p^* : J_0(N) \to J_0(pN)$$

and we define a subvariety

$$J_0(pN)_{p-\text{odd}} = \alpha_p^* J_0(M) + \beta_p^* J_0(N) \subset J_0(pN)$$

which is called as $p$-old subvariety. We define $p$-new subvariety to be the quotient

$$J_0(pN)_{pN/N} = J_0(pN)/J_0(pN)_{p-\text{odd}}.$$  

Now we consider the actions of Hecke operators. Let $T$ be $T_r$ ($r \nmid pN$) or $U_l$ ($l \mid N$). Then $T$ commutes with $\alpha_p$ and $\beta_p$ and

$$J_0(N) \times J_0(N) \xrightarrow{\alpha_p^* \times \beta_p^*} J_0(pN) \xrightarrow{T} J_0(pN).$$  

(25)

and $J_0(pN)_{p-\text{odd}}$ is $T_0(pN)^{(p)}$-stable. By [10] Remark 3.9 $J_0(pN)_{p-\text{odd}}$ is also preserved by $U_p$ and it is $T_0(pN) = T_0(pN)^{(p)}[U_p]$-stable. Therefore $J_0(pN)_{pN/N}$ admits the action of $T_0(pN)$ and the image of $T_0(pN)$ in $\text{End}(J_0(pN)_{pN/N})$ is temporary denoted by $T'$. Having identified the holomorphic cotangent space of $J_0(pN)_{pN/N}$ at the origin with $S_2(\Gamma_0(pN))_{pN/N}$ let us consider the representation of $\text{End}(J_0(pN)_{pN/N})$ on $S_2(\Gamma_0(pN))_{pN/N}$. Then the image of $T'$ in $\text{End}(S_2(\Gamma_0(pN))_{pN/N})$ is $T_0(pN)_{pN/N}$. Since representation of $\text{End}(J_0(pN)_{pN/N})$ on $S_2(\Gamma_0(pN))_{pN/N}$ faithful, $T'$ and $T_0(pN)_{pN/N}$ are isomorphic and we identify them.

It is known that the Néron model of $J_0(pN)_{pN/N}$ over $\text{Spec}\mathbb{Z}$ has purely toric reduction $T$ at $p$. Let us describe its character group. $X_0(pN)_{\Sigma}$ has two irreducible components $Z_F$ and $Z_V$, which are isomorphic to $X_0(N)_{\Sigma_p}$. Over $Z_F$ (resp. $Z_V$) the parametrized cyclic group $C_p$ of order $p$ is the kernel of the Frobenius $F$ (resp. the Verschiebung $V$). $Z_F$ and $Z_V$ transversally intersect at enhanced supersingular points of level $N$, that is $\Sigma_N = \{ E_1, \ldots, E_{\nu(N)} \}$. Set

$$X_N = \bigoplus_{i=1}^{\nu(N)} \mathbb{Z}E_i$$

and we adopt $\{ E_1, \ldots, E_{\nu(N)} \}$ as a base. We define the action of Hecke operators on $X_N$ by (7) and (8) and let $T$ denote a commutative subring of $\text{End}_\mathbb{Z}(X_N)$.
generated by Hecke operators. Let us consider the boundary map of the dual graph of $X_0(pN)_{pN}$,
\[ \partial : X_N \to \mathbb{Z}Z_F \oplus \mathbb{Z}Z_F, \quad \partial(E_i) = Z_F - Z_V. \]
Being $X_N^{(0)}$ the kernel of $\partial$, we have the exact sequence of Hecke modules
\[ 0 \to X_N^{(0)} \to X_N \xrightarrow{\partial} \mathbb{Z} \epsilon \to 0, \quad \epsilon = Z_F - Z_V. \]
For brevity let us write $E_i$ by $[i]$. Then
\[ \partial([i]) = \epsilon, \quad 1 \leq \forall i \leq n \]
and
\[ X_N^{(0)} = \{ \sum_{i=1}^{n} a_i[i] | a_i \in \mathbb{Z}, \sum_{i=1}^{n} a_i = 0 \}. \]
The restriction $T_0$ of $T$ to $X_N^{(0)}$ has the following description. By [16] Proposition 3.1, $X_N^{(0)}$ is the character group of the connected component of the torus $T$. By the Néron property, $T$ admits the action of $T_0(pN)_{pN/N}(= T')$ and the induced action on $X_N^{(0)}$ is $T_0$. Therefore $T_0$ is the image of $T_0(pN)_{pN/N}$ in End$_\mathbb{Z}(X_N^{(0)})$. Since the action of $T_0(pN)_{pN/N}$ on $X_N^{(0)}$ is faithful ([16] Theorem 3.10), $T_0$ and $T_0(pN)_{pN/N}$ are isomorphic and they will be identified from now on.

**Theorem 4.2.** Let $N$ be a square free positive integer. There is an isomorphism as $T_0(pN)_{pN/N}$-modules
\[ X_N^{(0)} \otimes \mathbb{C} \simeq S_2(\Gamma_0(pN))_{pN/N}. \]

**Proof.** As we have mentioned before, the action of $T_0(pN)_{pN/N}$ on $X_N^{(0)}$ is faithful ([16] Theorem 3.10). Since the characters $\{ \chi \}$ in (22) are mutually different and by (23) we see every irreducible component of (22) should appear as irreducible factor of $X_N^{(0)} \otimes \mathbb{C}$. Thus $S_2(\Gamma_0(pN))_{pN/N}$ is contained in $X_N^{(0)} \otimes \mathbb{C}$. On the other hand the rank of $X_N^{(0)}$ is equal to $\dim T = \dim J_0(pN)_{pN/N}$. Since the holomorphic cotangent space of $J_0(pN)_{pN/N}$ at the origin is $S_2(\Gamma_0(pN))_{pN/N}$, we have
\[ \dim X_N^{(0)} \otimes \mathbb{C} = \dim S_2(\Gamma_0(pN))_{pN/N}, \]
and the claim is proved.

Let us state a real version of Theorem 4.2. Since the character of a normalized Hecke-eigen newform is real, using (15) and (20), Theorem 4.2 yields an decomposition as a $T_0(pN)_{pN/N} \otimes \mathbb{R}$-module
\[ X_N^{(0)} \otimes \mathbb{R} = \bigoplus \gamma V(\gamma), \]
where
\[ V(\gamma) = \{ v \in X_N^{(0)} \otimes \mathbb{R} | T(v) = \gamma(T)v \quad \forall T \in T_0(pN)_{pN/N} \}. \]
Here $\gamma$ is the real character of $T_0(pN)_{pN/N}$, which is the restriction of the character of the normalized Hecke eigen newform $f_\gamma$ whose level $N_\gamma$ satisfies

$$N_\gamma = pM, \quad M|N.$$  

Lemma 4.1 shows that $\{\gamma\}$ are mutually different. Being $N/M = l_1 \cdots l_m$ the prime decomposition, we write

$$T_0(pN)_{pN/N} \otimes \mathbb{R} = (T_0(pN)_{pN/N}^{(N/M)} \otimes \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{R}[U_{l_1}, \ldots, U_{l_m}]$$

and $V(\gamma)$ is a $\mathbb{R}[U_{l_1}, \ldots, U_{l_m}]$-module. As we have seen in the proof of Proposition 4.1, the characteristic polynomial of $U_{l_i}$ is $P_{l_i}(U_{l_i}) = U_{l_i}^2 - a_{l_i}(f_\gamma)U_{l_i} + l_i$ and $\dim_{\mathbb{R}} V(\gamma) = 2^m$. Therefore

$$V(\gamma) \simeq \mathbb{R}[U_{l_1}, \ldots, U_{l_m}] / I,$$

where $I$ is an ideal of $\mathbb{R}[U_{l_1}, \ldots, U_{l_m}]$ generated by the polynomials $\{P_{l_i}(U_{l_i})\}_{i=1, \ldots, m}$. Viewing $\mathbb{R}f_\gamma$ as a $T_0(pN)_{pN/N}^{(N/M)} \otimes \mathbb{R}$-module, we write it by $\mathbb{R}f_\gamma^{(N/M)}$. Using (27) we see

$$V(\gamma) \simeq \mathbb{R}f_\gamma^{(N/M)} \otimes_{\mathbb{R}} (\mathbb{R}[U_{l_1}, \ldots, U_{l_m}] / I).$$

as $T_0(pN)_{pN/N} \otimes \mathbb{R}$-modules. Thus we have proved a real version of Theorem 4.2.

**Theorem 4.3. (Weak multiplicity one)** There is an irreducible decomposition

$$X_N^{(0)} \otimes \mathbb{R} = \oplus \gamma V(\gamma)$$

as a $T_0(pN)_{pN/N} \otimes \mathbb{R}$-module. Here $\{\gamma\}$ runs through the real characters of normalized Hecke eigen newforms $\{f_\gamma\}$, such that the level $N_{f_\gamma}$ of $f_\gamma$ satisfies $N_{f_\gamma} = pM$ where $M$ is a divisor of $N$. Being $N/M = l_1 \cdots l_m$ the prime decomposition, a $T_0(pN)_{pN/N} \otimes \mathbb{R}$-module $V(\gamma)$ is defined to be

$$V(\gamma) \simeq \mathbb{R}f_\gamma^{(N/M)} \otimes_{\mathbb{R}} (\mathbb{R}[U_{l_1}, \ldots, U_{l_m}] / I).$$

Here the action of $T_0(pN)_{pN/N} \otimes \mathbb{R}$ is defined via (27) and $I$ is an ideal generated by polynomials $\{P_{l_i}(U_{l_i})\}_{i=1, \ldots, m}$ where

$$P_{l_i}(U_{l_i}) = U_{l_i}^2 - a_{l_i}(f_\gamma)U_{l_i} + l_i.$$ 

Moreover the characters $\{\gamma\}$ are mutually different.

Let $l$ be an odd prime different from $p$. Remember that $N \in \mathcal{N}_{p,l}$ is the set of square free positive integers prime to $lp$.

**Theorem 4.4. (Monotonicity)** For $N \in \mathcal{N}_{p,l}$ let $\rho_l^1(N)$ be the largest eigenvalue of the Hecke operator $T_l$ of $X_N^{(0)} \otimes \mathbb{R}$. Then for $M, N \in \mathcal{N}_{p,l}$ such that $M|N$,

$$\rho_l^1(N) \geq \rho_l^1(M)$$
Proof. Theorem 4.2 (or Theorem 4.3) shows that, under the decomposition (22), \( p_1^l(N) \) is the maximum of \( l \)-th coefficients of Hecke eigenform \( \{f_\chi\}_\chi \). By (21) we find \( S_2(\Gamma_0(pM))_{pM/M} \) is contained in \( S_2(\Gamma_0(pN))_{pN/N} \) and the claim is obtained.

\[ \square \]

5. Properties of the graphs

Let \( p \) be a prime satisfying \( p \equiv 1(\text{mod } 12) \) and \( l \) be an odd prime different from \( p \). Let us take \( N \in \mathcal{N}_{p,l} \). For brevity we write \( E_i = (E_i, C_N) \) and let \( \Gamma_l \) be the set of cyclic subgroups of \( E_i \) of order \( l \). The bijective correspondence

\[ \text{Hom}(E_i, E_j)(l)/\pm 1 \simeq \Gamma_l, \ f \mapsto \text{Ker} f. \]

shows that the Brandt matrix \( B_p^{(l)}(N) \) is the representation matrix of \( T_l \). Since \( B_p^{(l)}(N) \) is symmetric, the eigenvalues are all real. It is easy to check that \( \epsilon = Z_F - Z_V \) (cf. (26)) satisfies

\[ T_l(\epsilon) = (l + 1)\epsilon \]

and since \( \partial \) in (26) commutes with \( T_l \), \( l + 1 \) is an eigenvalue of \( B_p^{(l)}(N) \). Let \( \delta \) be a corresponding eigenvector. Using the Eichler-Shimura relation and the Weil conjecture, Theorem 4.2 (or Theorem 4.3) implies that the modulus of other eigenvalues are less than or equal to \( 2\sqrt{l} \) and

\[ X_N \otimes \mathbb{R} = (X_N^{(0)} \otimes \mathbb{R}) \hat{\oplus} \mathbb{R}\delta, \]

where \( \hat{\oplus} \) denotes an orthogonal direct sum. Moreover if \( N \) is generic, Theorem 4.3 and this decomposition yield a spectral decomposition of \( X_N \otimes \mathbb{R} \) in terms of eigenspaces of \( T_l \). Theorem 4.2 implies that

\[ \text{det}[1 - B_p^{(l)}(N)t + lt^2] = (1 - t)(1 - lt)\text{det}[1 - T_lt + lt^2|S_2(\Gamma_0(pN))_{pN/N}]. \]

Theorem 5.1. For any \( N \in \mathcal{N}_{p,l} \), \( G_p^{(l)}(N) \) is a connected regular Ramanujan graph of degree \( l + 1 \) not bipartite.

Proof. By construction \( G_p^{(l)}(N) \) is a regular graph of degree \( l + 1 \). Let us investigate the eigenvalues of the adjacency matrix \( B_p^{(l)}(N) \). As we have seen, \( l + 1 \) is an eigenvalue of \( B_p^{(l)}(N) \) and the modulus of other eigenvalues are less than or equal to \( 2\sqrt{l} \). Thus \( G_p^{(l)}(N) \) is a Ramanujan graph. By the equation (1) (see also (2)), 0 is an eigenvalue of the Laplacian with multiplicity one and we see that \( G_p^{(l)}(N) \) is connected. In general a connected finite regular graph of degree \( d \) is bipartite if and only if \( \pm d \) are eigenvalues of the adjacency matrix \( \text{[22]} \). Therefore \( G_p^{(l)}(N) \) is not bipartite.

\[ \square \]
Now Theorem 1.1 is a direct consequence of the equation (1) (see also (2)), Theorem 4.4 and Theorem 5.1.

Proof of Theorem 1.2 Set \( N = q \) and we use the decomposition (21). Since \( S_2(\Gamma_0(p))^{(a)} \) is isomorphic to \( S_2(\Gamma_0(p)) \) as a \( T_0(pq)^{(pq)} \)-module, we see
\[
S_2(\Gamma_0(pq))_{pq/q} = S_2(\Gamma_0(pq))_{new} \oplus S_2(\Gamma_0(p))_{pq/q}^{\otimes^2}
\]
as \( T_0(pq)^{(pq)} \)-modules and
\[
\frac{\det(1 - B_p^{(l)}(q)t + lt^2)}{\det(1 - B_p^{(l)}(1)t + lt^2)^2} = \frac{\det(1 - Tt + lt^2 \mid S_2(\Gamma_0(pq))_{new})}{(1-t)(1-lt)} = \frac{\det(1 - B_q^{(l)}(p)t + lt^2)}{\det(1 - B_q^{(l)}(1)t + lt^2)^2}
\]
by (28). On the other hand Fact 2.2 implies,
\[
\chi(G_p^{(l)}(q)) - 2\chi(G_p^{(l)}(1)) = \left( \frac{(p-1)(q-1)(1-l)}{24} \right) = \chi(G_q^{(l)}(p)) - 2\chi(G_q^{(l)}(1))
\]
and the claim follows from Fact 2.1.

\[\square\]

Proof of Theorem 1.3 Let us recall the decomposition (22)
\[
S_2(\Gamma_0(pN))_{pN/N} = \oplus_x \mathbb{C} f_x,
\]
where \( f_x \) is a normalized Hecke eigenform. Then the second largest eigenvalue \( \rho_1^i(N) \) of \( B_p^{(l)}(N) \) is the maximum of \( \{ a_i(f_x) \}_x \) by Theorem 4.2 and satisfies \( \rho_1^i(N) \leq 2\sqrt{d} \) by Theorem 5.1. Let \( \{ r_i \}_{i=1}^\infty \) be the set of primes and \( N_k = \prod_{i=1}^k r_i \).

Then by Theorem 4.4, \( \rho_1^i(N_k) \) is monotone increasing for \( k \). In general let \( \{ G_i \} \) be an infinite family of connected \( d \)-regular graphs satisfying
\[
\lim_{i \to \infty} |V(G_i)| = \infty.
\]
Then it is known that
\[
\liminf_{i \to \infty} \rho_1^i(G_i) \geq 2\sqrt{d-1}
\]
by Alon and Boppana (\[2\]). We will use this fact. Since \( \{ G_{p_0}^{(l)}(N_k) \}_k \) is an infinite family of connected regular Ramanujan graphs of degree \( l + 1 \) with
\[
\lim_{k \to \infty} |V(G_{p_0}^{(l)}(N_k))| = \lim_{k \to \infty} \left( \frac{p-1}{12} \left( \prod_{r=1}^k (1+r_i) \right) \right) = \infty,
\]
we see
\[
\lim_{k \to \infty} \rho_1^i(N_k) = 2\sqrt{l},
\]
and
\[
\lim_{k \to \infty} \text{Max} \{ a_i(f_x) \mid S_2(\Gamma_0(pN_k))_{pN_k/N_k} = \oplus_x \mathbb{C} f_x \} = 2\sqrt{l}.
\]
Since \( S_2(\Gamma_0(pN_k))_{pN_k/N_k} \) is a subspace of \( S_2(\Gamma_0(pN_k)) \), the remaining claim immediately follows from this result and the decomposition in Theorem 4.1.

\[\square\]
The proof implies the following corollary.

**Corollary 5.1.** Let \( p \) be a prime satisfying \( p \equiv 1(\text{mod } 12) \) and \( l \) an odd prime with \( l \neq p \). Then for any set of mutually distinct primes \( \{r_i\}_{i=1}^{\infty} \) which are different from \( l \) and \( p \), there is a sequence of normalized Hecke eigenforms \( \{f_i\}_i \) of weight 2 such that \( f_i \in S_2(\Gamma_0(p r_1 \cdots r_i)) \) new and

\[
\lim_{i \to \infty} a_l(f_i) = 2\sqrt{l}.
\]

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