Distribution System Voltage Control under Uncertainties

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Abstract

Voltage control plays an important role in the operation of electricity distribution networks, especially with high penetration of distributed energy resources. These resources introduce significant and fast varying uncertainties. In this paper, we focus on reactive power compensation to control voltage in the presence of uncertainties. We adopt a probabilistic approach that accounts for arbitrary correlations between renewable resources at each of the buses and we use the linearized DistFlow equations to model the distribution network. We then show that this optimization problem is convex for a wide variety of probabilistic distributions. Compared to conventional per-bus chance constraints, our formulation is more robust to uncertainty and more computationally tractable. We illustrate the results using standard IEEE distribution test feeders.

I. INTRODUCTION

Voltage control is crucial to stable operations of power distribution systems, where it is used to maintain acceptable voltages at all buses under different operating conditions [1]. To control voltage, reactive power is traditionally regulated through tap-changing transformers and switched capacitors [2]. With recent advances in cyber-infrastructure for communication and control, it is also possible to utilize distributed energy resources (DERs, i.e., electric vehicles [3], PV panels [4], [5]) to provide voltage regulation. There exists an extensive literature in controlling voltage in a distribution network, some of them focus on centralized control [6], [7], while the others address distributed algorithm [1], [2], [8], [9].

In this paper, we focus on centralized control frameworks to regulate voltage through DERs, that a central controller constantly sends out regulation signals to local DERs in order to control voltage. One important issue to deploy voltage control via DERs is to deal with the uncertainty they bring to the power distribution network [10]. Since most distribution networks still do not have real-time communication capabilities, decisions made have to be valid for a set of conditions. For example, assume that information is exchanged between a central controller and the local DERs every five minutes. Then the control signal that sets the reactive power on the DERs must be valid for the next five minutes, subject to the fast variabilities and randomness in the renewables.

One way to handle uncertainty is through stochastic programming [11]. Stochastic formulation takes into account the probabilistic nature of the uncertainty. Chance constraint, which bounds the probability of a certain event, is introduced in these optimization problems. However, it is not obvious to characterize the feasible region of the chance constraint. Monte Carlo simulation is therefore adopted by many researchers, for example in [2], to approximate the optimal solution.

In this paper, we assume that the distribution of the uncertainty is known, and we adopt a stochastic approach to bound the probability that voltage stays within prescribed bounds. Unlike most of the existing literature, we propose to impose a single chance constraint on the whole system. This is different from the standard practice in literature where chance constraints are placed on every bus of the network [12], [13]. Putting constraints on each single bus simplifies the problem, but suggests
that the uncertainty at each bus is unrelated. However, the randomness across the buses can be well correlated in practice. In this work, we show how a single constraint can be used to capture uncertainties from all buses in the system using the linearized DistFlow model introduced in [14]. Another approach is to adopt a robust optimization framework, but it is often nontrivial to set the budget of uncertainty and solving the optimization problem maybe computationally challenging [15], [16].

We show that our proposed voltage control problem is tractable without sampling techniques. We validate this statement by proving that the proposed chance constraint depicts a convex feasible region and adapts a variety of distributions. The exact optimal solution can therefore be found using standard gradient descent techniques.

In all, we make the following contributions to voltage control in distribution systems:

- We consider voltage control problem with uncertainties in the system. The uncertainty is correlated across the system and is captured by a single constraint imposed onto the whole system.
- The proposed framework is convex, that it only adds one additional convex constraint to the various existing voltage control problems. Therefore there is no need to adopt an approximate algorithm such as Monte Carlo sampling.

The rest of the paper is organized as follows. Section II presents the modeling of the distribution network. Section III proceeds with the formulation of the voltage control problem and demonstrate the robustness of the proposed framework with an illustrating example. Section IV states that the problem can be efficiently solved, by proving that the introduced chance constraint is convex. Section V validates the statement by simulation results. Finally Section VI concludes the paper.

II. PRELIMINARY: DISTRIBUTION NETWORK

In this section we present the modeling of components in a radial distribution network in power systems. For interested readers, please refer to [17], [18] for more details.

A. Power flow model for radial networks

We consider a radial distribution network with \( N + 1 \) buses collected in the set \( \{0, 1, \ldots, N\} \). We also denote a line in the network by the pair \((i, k)\) of buses it connects. Bus 0 is the reference bus.

For each line \((i, k)\) in the network, its impedance is denoted by \( z_{ik} = r_{ik} + jx_{ik} \), where \( r_{ik} \) and \( x_{ik} \) is its resistance and reactance.

For each bus \( i \in \{0, 1, \ldots, N\} \), let \( V_i \) be the voltage magnitude at bus \( i \) and \( s_i = p_i + jq_i \) be the complex power injection, i.e., the generation minus consumption. In addition, the subset \( M_k \) denotes bus \( k \)'s neighboring buses that are further down from the feeder head. The DistFlow equations [14] model the distribution network flow for every line \((i, k)\) as:

\[
P_{ik} = \sum_{l \in M_k} P_{kl} = -p_k + r_{ik} \frac{P_{ik}^2 + Q_{ik}^2}{V_i^2}, \tag{1a}
\]

\[
Q_{ik} = \sum_{l \in M_k} Q_{kl} = -q_k + x_{ik} \frac{P_{ik}^2 + Q_{ik}^2}{V_i^2}, \tag{1b}
\]

\[
V_i^2 - V_k^2 = 2(r_{ik} P_{ik} + x_{ik} Q_{ik}) - (r_{ik}^2 + x_{ik}^2) \frac{P_{ik}^2 + Q_{ik}^2}{V_i^2}, \tag{1c}
\]

where \( P_{ik}, Q_{ik} \) are respectively the real and reactive power flow on line \((i, k)\). We let \( V_0 \) denote the voltage at the reference bus. In addition, the term \( \frac{r_{ik}^2 + Q_{ik}^2}{V_i^2} \) represents the loss in the power flow in line \((i, k)\).
B. Linear approximation of the flow model

Assuming the loss is negligible compared to line flow, a linear approximation of can be constructed. Following [14], we assume that the losses is negligible. We then assume that the voltage at each bus is close to 1. This enables us to approximate \( V_i^2 - V_k^2 \) by \( 2(V_i - V_k) \). We then obtain the linearized DistFlow model:

\[
P_{ik} - \sum_{l \in N_k} P_{kl} = -p_k, \tag{2a}
\]
\[
Q_{ik} - \sum_{l \in N_k} Q_{kl} = -q_k, \tag{2b}
\]
\[
V_i - V_k = r_{ik}P_{ik} + x_{ik}Q_{ik}. \tag{2c}
\]

From (2), we can write the voltage magnitude \( V = [V_1, \ldots, V_N]^T \) in terms of reactive power injection \( Q = [Q_1, \ldots, Q_N]^T \) and real power injection \( P = [P_1, \ldots, P_N]^T \), and the reference voltage \( V_0 \):

\[
V = RP + XQ + V_0 \tag{3}
\]

where \( R, X \in \mathbb{R}^{N \times N} \) are matrices with \( R_{ik} \) and \( X_{ik} \) as the element in \( i^{th} \) row and \( k^{th} \) column, respectively. The voltage profile at bus \( 1, \ldots, N \) is denoted by \( V \in \mathbb{R}^N \).

Following the findings in [1], we give the expressions of \( R_{ik} \) and \( X_{ik} \) in terms of line resistance \( r_{ik} \) and reactance \( x_{ik} \):

\[
R_{ik} = \sum_{(h,l) \in P_i \cap P_k} r_{hl}, \tag{4a}
\]
\[
X_{ik} = \sum_{(h,l) \in P_i \cap P_k} x_{hl}, \tag{4b}
\]

where \( P_i \) is the set of lines on the unique path from bus 0 to bus \( i \). Note that \( R \) and \( X \) are positive definite matrices [1].

III. Voltage regulation with reactive power injection

Suppose that bus 0 is assumed to be operating at nominal voltage level, i.e., \( V_0 = 1 \) p.u. (per unit). Rewrite \( V \) as the difference between the bus voltage and reference voltage \( V_0 \), then (3) is reduced to the following form:

\[
V = RP + XQ \tag{5}
\]

As renewables introduce uncertainty in the bus voltages, the voltage profile is reformulated into the following form:

\[
V = RP + XQ + \varepsilon \tag{6}
\]

where \( \varepsilon \) is the uncertainty with zero mean and covariance matrix \( \Sigma \). The covariance matrix \( \Sigma \) is not necessarily a diagonal matrix since the uncertainty can be highly correlated across buses. In Section [IV] we illustrate a variety of distributions that \( \varepsilon \) can possibly follow.

In order to maintain the voltage at each bus close to the nominal level, we propose to bound the probability that the voltage profile is within some bounds:

\[
\Pr\{\mathbf{V} \leq \mathbf{V} \} \leq \alpha, \tag{7}
\]

\(^1\)Here we do not have a factor 2 as in [1] because we approximate \( V_i^2 - V_k^2 \) by \( 2V_i - 2V_k \).
which is equivalent to write as:
\[
\Pr\{ \mathbf{v} \leq \mathbf{r} \mathbf{p} + \mathbf{x} \mathbf{q} + \mathbf{e} \leq \mathbf{v} \} \geq \alpha,
\]
(8)

where \( \mathbf{v} \) and \( \mathbf{v} \) are the bounds prescribed to the random voltage profile. They indicate how far the voltage profile can be away from a nominal level. The value of \( \alpha \) indicates the probability that event \( \mathbf{v} \leq \mathbf{r} \mathbf{p} + \mathbf{x} \mathbf{q} + \mathbf{e} \leq \mathbf{v} \) occurs.

A. Main Optimizaion Problem

In this paper we only consider reactive power regulation and assume that active load injection \( \mathbf{p} \) is determined exogenously and the controllable variable is the reactive power injection \( \mathbf{q} \). Denote \( \Pr\{ \mathbf{v} \leq \mathbf{r} \mathbf{p} + \mathbf{x} \mathbf{q} + \mathbf{e} \leq \mathbf{v} \} \) by \( g(\mathbf{q}) \), for a given tolerance level \( \alpha \), the centralized voltage regulation problem is then captured as the following:

\[
\min_{\mathbf{q}} ||\mathbf{q}||_2
\]
(9a)

\[
s.t. \ g(\mathbf{q}) \geq \alpha,
\]
(9b)

\[
\mathbf{q} \leq \mathbf{q} \leq \mathbf{t} \]
(9c)

where the box constraint on \( \mathbf{q} \) represents the limits of reactive power injection at each bus. These bounds can be interpreted as the capacity or availability of devices at each bus. In addition, note that looser bounds on \( \mathbf{v} \) and \( \mathbf{v} \) in \( g(\mathbf{q}) \) correspond to a larger \( \alpha \), whereas a tighter bound corresponds to a smaller \( \alpha \). In practice, the value for \( \mathbf{v} \) is usually taken as 0.05 p.u. or 0.1 p.u. The objective function in (9a) minimize the total action of the DERs, although it can be easily replaced by any time of convex function of \( \mathbf{q} \).

B. Per-Bus Constraints

Our approach is inherently different from the existing literature when dealing with chance constraints. In most existing literature with randomness in the distribution network, chance constraints are introduced as [12]:

\[
\Pr\{ \mathbf{v}_i \leq \mathbf{v}_i \leq \mathbf{v}_i \}
= \Pr\{ \mathbf{v}_i \leq \mathbf{r}_{i}^\top \mathbf{p} + \mathbf{x}_{i}^\top \mathbf{q} + \mathbf{e}_i \leq \mathbf{v}_i \} \geq \eta_i,
\]
(10)

where \( \mathbf{r}_{i}^\top \) and \( \mathbf{x}_{i}^\top \) extracts the \( i \)th row in respective matrices. The chance constraint at bus \( i \) is associated with prescribed tolerance \( \eta_i \). We assume that each bus has the same tolerance, i.e., \( \eta_i = \eta \). The optimization problem that incorporates per-bus chance constraint is in the following form:

\[
\min_{\mathbf{q}} ||\mathbf{q}||_2
\]
(11a)

\[
s.t. \ g_i(\mathbf{q}) \geq \eta, \forall i
\]
(11b)

\[
\mathbf{q} \leq \mathbf{q} \leq \mathbf{t}
\]
(11c)

where \( g_i(\mathbf{q}) = \Pr\{ \mathbf{v}_i \leq \mathbf{r}_{i}^\top \mathbf{p} + \mathbf{x}_{i}^\top \mathbf{q} + \mathbf{e}_i \leq \mathbf{v}_i \} \).

The solution to problems described by (11) with chance constraints on each bus \( i \) is discussed widely in literature, for example in [12], [13]. This per-bus framework is not the same as having a single constraint on the whole system, i.e., (11b) is different from (9b). Those two different frameworks do not return the same feasible set and the proposed framework,
which places a single constraint on the whole system, captures the coupling between buses and is therefore more realistic and applicable. In addition, as we show in Section IV, our framework is tractable, in the sense that the chance constraint in (9b) is (after some transformations) convex.

C. Four-bus Toy Example

Let us take an illustrating example with a line network with four buses, shown in Fig. 1.

![Fig. 1: An illustration of 4 bus line network.](image)

Suppose that the reactive power injections at bus 2 and 3 are limited to 0.2 p.u., we have a linear constraint as $-0.2 \leq Q_2 \leq 0.2$. Then our proposed framework is solving the following optimization problem:

$$
\min_{\mathbf{Q}} ||\mathbf{Q}||_2
$$

s.t.

$$
g_i(\mathbf{Q}) \geq \alpha, \\
-0.2 \leq Q_j \leq 0.2, \forall j \in \{2,3\}
$$

and its optimal solution is denoted by $\mathbf{Q}_a^*$. Here we take $\alpha$ to be 0.88.

The per-bus formulation is the following optimization problem:

$$
\min_{\mathbf{Q}} ||\mathbf{Q}||_2
$$

s.t.

$$
g_i(\mathbf{Q}) \geq \eta, \forall i \in \{1,2,3\} \\
-0.2 \leq Q_j \leq 0.2, \forall j \in \{2,3\}
$$

and its optimal solution of (13) by $\mathbf{Q}_p^*$. Detailed configuration of the line network is left in appendices for interested readers. Unlike the problem in (12), setting a “right” $\eta$ in (13) is not straightforward. Suppose we want to achieve the same level confidence as (12) where the system operates within the prescribed bounds with probability at least 0.88, then what is the right $\eta$ to take?

As suggested by previous studies [2], a natural candidate for $\eta$ is to set it equal to $\sqrt{0.88} = 0.958$ by thinking of each bus as independent to each other. A second candidate is simply set it at 0.88, the same as $\alpha$.

The main results of the four-bus line network are shown in Table I where $g(\mathbf{Q})$ denotes the probability $\Pr[\mathbf{V} \leq \mathbf{R} + \mathbf{X} \mathbf{Q} + \mathbf{e} \leq \mathbf{V}]$ and is the figure of merit we compare the solutions with. For the proposed framework $\mathbf{Q} = \mathbf{Q}_a^*$, and $\mathbf{Q} = \mathbf{Q}_p^*$ for the per-bus framework. The bound on the voltage deviation is denoted by $\mathbf{V} = [0.1 \ 0.1 \ 0.1]^T$, which means that voltage deviates no more than 10% of the nominal level. The uncertainty is assume to be multivariate Gaussian and its covariance matrix is shown in the appendices.

It turns out that assuming each bus as independent drives the optimization problem with the per-bus constraint infeasible. This is due to the fact that in order to achieve 0.88 as the joint probability, the per bus constraint requires a very small $\eta$. 

TABLE I: The value of $g(Q')$ under different frameworks.

| Framework          | $g(Q')$ |
|--------------------|---------|
| $Q^p_\alpha$, $\alpha = 0.88$ | 0.88    |
| $Q^p_\eta$, $\eta = \sqrt{0.88}$ | Infeasible |
| $Q^p_\eta$, $\eta = 0.88$ | 0.78    |

that is too strict. The second value of $\eta = 0.88$ makes the problem feasible, but at the cost of lowering the joint probability $g(Q)$ to be 0.78. This result shows the difficult of using the per bus constraint, where it is difficult to set the correct tolerance level in order to be not overly pessimistic or optimistic. Of course, one could vary $\eta$ and resolve (11b), but the procedure is cumbersome (especially for large networks) and the best $\eta$ does not guarantee that the overall probability of violation is the lowest.

In the following section, we elaborate on the framework based on (9) and show that it can be casted into a convex optimization problem.

IV. SOLVING THE OPTIMIZATION PROBLEM

The proposed optimization problem in (9) has a linear objective with a box constraint and a chance constraint. In this section, we present our main statement in Theorem 1 that the proposed problem is convex.

Theorem 1. If the uncertainty $\epsilon$ has log-concave probabilistic distribution, then the optimization problem in (9) is convex.

Theorem 1 states that even with a chance constraint whose feasible region is not obvious at first sight, it does not complicate the voltage control problem. On the contrary, the chance constraint preserves convexity of the optimization problem and is tractable.

In Theorem 1, we require that the uncertainty has a PDF that belongs to a certain class of functions, i.e., log concave functions. We now introduce the definition of log-concavity.

Definition 1. A non-negative function $f : \mathbb{R}^N \rightarrow \mathbb{R}^+$ is logarithmically concave (or log-concave for short) if its domain is a convex set, and if it satisfies the inequality:

$$f(\theta x + (1 - \theta)y) \geq f(x)^\theta f(y)^{1-\theta} f(\theta x + (1 - \theta)y) \geq f(x)^\theta f(y)^{1-\theta},$$

for all $x, y \in \text{dom}(f)$ and $0 < \theta < 1$. In short, if $\log(f(x))$ is concave, then $f(x)$ is log-concave.

Log concave functions enjoy many properties that lead to convexity. For example, the level set of a log-concave function is convex, which means that if $g(Q)$ is log concave, then its level set $\{Q : g(Q) \geq \alpha\}$ is a convex set [19]. Therefore, the chance constraint $g(Q) \geq \alpha$ can be easily carried over to other existing deterministic optimization frameworks and preserves the convexity of the problem.
What is more, to maximize over a log concave function, it is sufficient to take the logarithmic of the original function and apply gradient descent techniques. The local optimum is guaranteed to be the global optimum. This suggests that we can easily maximize over \( g(Q) \) without using approximate algorithms such as sampling.

Therefore, to prove Theorem 1, \( g(Q) \) is log concave and we introduce two lemmas that discuss log-concavity of certain functions. Lemma 1 states that the accumulated mass of a log concave probabilistic function over a convex set is log concave. Lemma 2 states that applying linear transformation to the variable in a log-concave function yields another log-concave function, given that the linear transformation has full row rank.

**Lemma 1.** Denote \( F(z) = \int_{\epsilon_{1}^{1}}^{\epsilon_{N}^{1}} f(\epsilon_{1}^{1}, \ldots, \epsilon_{N}^{1}) d\epsilon_{1}^{1} \ldots d\epsilon_{N}^{1} \). If the distribution \( f(\epsilon_{1}^{1}, \ldots, \epsilon_{N}^{1}) \) is log concave in \( \epsilon_{1}, \ldots, \epsilon_{N} \), then \( F(z) \) is log concave in \( z \).

Lemma 1 only requires the log concave distribution of \( \epsilon_{1}, \ldots, \epsilon_{N} \) to ensure that the function \( F(z) \) is log concave. A lot of commonly known distributions fall within the log-concave probabilistic distributions, for example, Gaussian distribution and Weibull distribution which is usually adopted to generate intermittent wind energy. Detailed proof is shown in appendices.

**Lemma 2.** Assume that \( F(z) \) is a log concave function, \( z, \theta \in \mathbb{R}^{N} \) and that \( z = A\theta + b \) with \( \theta \in \mathbb{R}^{M} \), \( A \in \mathbb{R}^{N \times M} \), \( b \in \mathbb{R}^{N} \). If \( A \) has rank \( N \), then \( g(\theta) = F(A\theta + b) \) is also a log concave function.

With Lemma 1 and Lemma 2, we now prove the statement in Theorem 1.

**Proof of Theorem 1.** We reformulate \( g(Q) \) into:

\[
g(Q) = \Pr\{0 \leq -V + RP + XQ + \epsilon \leq -V + \bar{V}\} \tag{15a}
\]

\[
= \Pr\{0 \leq Y(Q) + \epsilon \leq u\}, \tag{15b}
\]

where \( Y(Q) = -V + RP + XQ \) and \( -V + \bar{V} = u \).

We can scale row of the inequality inside (15) to make the expression easier to work with. Defining \( U \) as \( \text{diag}(u) \) and scale the inequality in (15c) by \( U^{-1} \). Then (15c) is equivalent to:

\[
\Pr\{0 \leq Y'(Q) + \epsilon' \leq 1\} \tag{16}
\]

\[
= \Pr\{-Y'(Q) \leq \epsilon' \leq 1 - Y'(Q)\}
\]

\[
\overset{(a)}{=} \Pr\{Y'(Q) - 1 \leq \epsilon' \leq Y'(Q)\}
\]

where \( Y'(Q) = U^{-1}Y(Q) \) and \( \epsilon' \sim \mathcal{N}(0, \Sigma') \) and \( \Sigma' = U^{-1}\Sigma U^{-1} \). The last equality \( (a) \) is due to the symmetry of \( \epsilon' \) with respect to \( 0 \).

Denote the probability density function (PDF) of multivariate Gaussian variable as \( f(z) = f(z_1, \ldots, z_N) \), \( g(Q) \) is finalized...
as the integral of the PDF of a multivariate Gaussian variable over an interval $[Y'(Q) - 1, Y'(Q)]$:

$$g(Q) \triangleq F(z)|_{z=Y'(Q)} = \int_{z-1}^{z} f(\varepsilon'_1, \ldots, \varepsilon'_N) d\varepsilon'_1 \ldots d\varepsilon'_N |_{z=Y'(Q)}.$$  

(17)

More generally, $F(z)$ denotes the probability mass in $[z-1, z]^N$ from $\varepsilon'$ where $\varepsilon'$ has density function $f(\cdot)$. From Lemma 1, we know that $F(z)$ is log concave, when $\varepsilon'$ has log-concave distribution.

Since $Y'(Q) = U^{-1}Y(Q) = AQ + b$, where $A$ is full rank and $b$ is a constant, Lemma 2 implies $g(Q)$ is log concave function.

\[ \square \]

V. SIMULATION

In this section, we validate the statement by IEEE standard test feeder. Here we use IEEE 13 bus feeder [20] as an example. The test feeder is shown in Fig. 2. In Fig. 2 we assume bus 1 is the reference bus. Besides, in this distribution test feeder, we assume there are no distributed generation, so the active power injection on each bus is negative. The line impedance is retrieved from [20]. In addition, we restrict the available reactive power regulation at the buses to be no more than 0.1 (p.u.).

The randomness presented in this system is assumed to be a multivariate Gaussian distribution with high correlation, i.e., most of the buses have some degree of correlation with the other buses. We solve (9) for $\alpha = 0.92$ and (11) for various $\eta$’s. The results are shown in Table II.

As can be seen from Table II a value of $\sqrt{0.92} = 0.962$ is too strict as an requirement to each bus, therefore the optimization with per-bus constraints is infeasible. We therefore adopt two other values, i.e., $\eta = 0.92 = \alpha$ and $\eta = 0.98 > \alpha$, to ensure
TABLE II: The value of $g(Q')$ under the proposed framework and per-bus framework.

| Framework | $Q^\alpha_x$ | $Q^\alpha_p$ | $Q^\eta_p$ | $Q^\eta_p$ |
|-----------|--------------|--------------|------------|------------|
| $g(Q')$   | 0.92         | Infeasible   | 0.66       | 0.89       |

feasibility. However, neither of these values achieves a satisfactory probably of valid system operation. This highlights the fact that as the number of buses grows in the network, the per-bus framework becomes difficult to use and extremely sensitive to the exact parameter choices. In contrast, the single constraint formulation in this paper is much more robust and stable.

VI. CONCLUSION

In this paper we adopt a stochastic framework to formulate voltage control problems with uncertainty. Compared to existing literature, we use a single chance constraint to capture the uncertainty in the distribution system. We show that this formulation is more realistic and less conservative than placing constraints on every bus in the distribution network. We also show that the proposed problem is tractable and the feasible region of the constraints is convex, therefore there is no need to adopt approximate algorithms such as Monte Carlo sampling to solve for the problem. Simulation results validate our statement by standard IEEE test feeders.

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[12] H. Zhang and P. Li, “Chance constrained programming for optimal power flow under uncertainty,” IEEE Transactions on Power Systems, no. 4, pp. 2417–2424, 2011.
A. Details of the four-bus line network

The parameters of the line impedance in the four-bus line network in Fig.1 are given as follows:

\[ R = \begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 0.1 & 0.3 & 0.3 \\ 0.1 & 0.3 & 0.5 \end{bmatrix}, \quad X = \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.2 & 0.39 & 0.39 \\ 0.2 & 0.39 & 0.7 \end{bmatrix} \]  

(18)

In addition, suppose that there are only active loads at each bus, therefore the active power injection is negative, i.e.,

\[ P = \begin{bmatrix} -0.1 \\ -0.2 \\ -0.3 \end{bmatrix}^T. \]

The uncertainty is assume to be multivariate Gaussian, i.e., \( \mathbf{\epsilon} \sim \mathcal{N}(\mathbf{0}, \Sigma) \). The covariance matrix \( \Sigma \) is given as:

\[ \Sigma = 0.002 \begin{bmatrix} 1 & 0.7 & 0 \\ 0.7 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \]

(19)

B. Proof of Lemma 1

Before proving Lemma 1 we introduce Lemma 3 that is necessary in the proof.

Lemma 3. A convolution over two log-concave probabilistic functions is log concave \([27]\).

Using Lemma 3 we now proceed to prove Lemma 1.

Proof to Lemma 1. Let \( f(\cdot) \) be the PDF of \( \mathbf{\epsilon}' \) that is defined on \( \mathbb{R}^N \). Let \( u(\cdot) \) be the PDF of a uniform distribution that is defined on \([0,1]^N\). It is easy to check that \( u(\cdot) \) is log concave.
Let us use $\ast$ to denote convolution operation. The convolution of $f$ and $u$ is:

$$(f \ast u)(zzz) = \int_{[0,1]^N} f(z - y)u(y)dy$$

$$= \int_{[0,1]^N} f(z - y)dy$$

$$= -\int_{[zz,zz-1]^N} f(w)dw$$

$$= \int_{[zz-1,zz]^N} f(w)dw$$

$$= F(zz)$$

(20)

From Lemma 3 and that $f$ and $g$ are both log concave, we conclude that $F(zz)$ is log concave in $zz$.

\[ \square \]

C. Proof of Lemma 2

Proof. Since log-concavity of a function $F(zz)$ is equivalent to $\log(F(zz))$ is concave, we have the second order condition on concavity:

$$\nabla^2 \log(F(zz)) \preceq 0,$$

(21)

which after some simplification yielding the following form:

$$F(zz)\nabla^2 F(zz) - \nabla F(zz)\nabla F(zz)^\top \preceq 0.$$

(22)

Let us denote $F(zz)\nabla^2 F(zz) - \nabla F(zz)\nabla F(zz)^\top$ by $H_F(zz)$.

To show that $g(y) = F(Ay + b)$ is log-concave, it is equivalent to show that $g(y)\nabla^2 g(y) - \nabla g(y)\nabla g(y)^\top \preceq 0$.

We first compute $\nabla g(y)$ as the following:

$$\nabla g(y) = A^\top \nabla F(zz)|_{zz = Ay + b},$$

(23)

and $\nabla^2 g(y)$ as the following, with the fact that $\nabla^2 (Ay + b) = 0$:

$$\nabla^2 g(y) = A^\top (\nabla^2 F(zz)|_{zz = Ay + b})A.$$

(24)

So $g(y)\nabla^2 g(y) - \nabla g(y)\nabla g(y)^\top$ is equivalent to $A^\top H_F(Ay + b)A$. Let us denote this new quantity as $H_g(y) = A^\top H_F(Ay + b)A$.

Since $H_F(zz) \preceq 0, \forall zz \in \mathbb{R}^N$, it suggests that $\forall w \in \mathbb{R}^N$ and $Ay + b \in \mathbb{R}^N$, we know:

$$w^\top H_F(Ay + b)w \leq 0.$$

(25)

Since $A$ has rank $N$, we know that $w = Az'$ spans the whole subspace $\mathbb{R}^N, \forall z' \in \mathbb{R}^M$. Therefore:

$$z'^\top A^\top H_F(Ay + b)A^\top z' \leq 0,$$

(26)

which is equivalent to:

$$z'^\top H_g(y)z' \leq 0, \forall z' \in \mathbb{R}^M.$$

(27)

This suggests that $H_g$ is negative semi definite, and therefore concludes the proof.

\[ \square \]
D. Log-concavity of multivariate Gaussian variable

We use a two dimensional multivariate Gaussian to validate Lemma 1 that $F(z)$ is a log concave function.

Suppose that $\epsilon' \sim \mathcal{N}(0, \Sigma)$ and $\Sigma = \begin{bmatrix} 0.9 & 0.5 \\ 0.5 & 0.6 \end{bmatrix}$. The visualization of $\log(F(z))$ is shown in Fig. 3.

![Fig. 3: plot of log $F(z)$ in 2-D, where $\epsilon' \sim \mathcal{N}(0, \Sigma)$.](image)

From Fig. 3 we observe that $F(z)$, after taking its log, is concave in $z$. Therefore we can apply various gradient descent algorithms to search for the global optimum.