AN INDEX RELATION FOR THE
QUILTED ATIYAH-FLOER CONJECTURE

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Abstract. Given a closed, connected, oriented 3-manifold with positive first Betti number, one can define an instanton Floer group as well as a quilted Lagrangian Floer group. Each of these is equipped with a chain level grading. We show that the gradings agree.

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1. Introduction

This is the second paper in a series [7] [9] that prove various aspects of the quilted Atiyah-Floer conjecture. The conjecture states that instanton Floer cohomology and quilted Lagrangian Floer cohomology are naturally isomorphic. These cohomologies are obtained from relatively $\mathbb{Z}_4$-graded chain complexes $(CF_{\text{inst}}, \partial_{\text{inst}})$ and $(CF_{\text{symp}}, \partial_{\text{symp}})$, respectively. There is a natural group isomorphism $\Psi : CF_{\text{inst}} \to CF_{\text{symp}}$ [7, Theorem 5.2], and the main result of the present paper, Corollary 4.2, states that $\Psi$ preserves the gradings.

In a series of papers [3] [4] [5] [6], Dostoglou and Salamon proved the quilted Atiyah-Floer conjecture in the special case of mapping tori. Motivated by their approach, in [7] we outline a proof strategy of the quilted Atiyah-Floer conjecture that applies to the much larger class of 3-manifolds $Y$ with positive first Betti number. The idea is that, when the underlying metric on $Y$ is suitably degenerated, one hopes to use an implicit function theorem to construct a bijection between solutions of the instanton equation on $\mathbb{R} \times Y$ and solutions of the holomorphic curve equation on a suitable representation variety of $Y$. In proving the main result of the present paper, we treat the linearized version of this problem and show that solutions of the linearized instanton equation are in bijection with solutions of the linearized holomorphic curve equation, assuming both linearized operators are surjective. Dropping the surjectivity assumption, this is
equivalent to showing that the linearized operators have the same Fredholm index; see Theorem 4.1. Motivated by the non-linear version, our proof uses an implicit function theoretic approach, and so provides a second proof of Dostoglou-Salamon’s index theorem for mapping tori [3].

In the next section we explain our set-up, referring the reader to [2] for more details. We also introduce our notation, and the relevant equations. In particular, we will see that the holomorphic curve equation is manifestly a boundary-value problem. This boundary phenomena arises due to critical points of a circle-valued Morse function on \( Y \). At the level of analysis, this forces us to juggle differential equations on 2-manifolds that are coupled with differential equations on 3-manifolds.

Section 3 develops estimates that allow us to treat this coupled problem in a relatively dimension-independent way. The first application of these estimates is in showing that there are perturbations for which non-degeneracy can be achieved simultaneously in both theories; see Corollary 3.8. Section 3 also serves as a 3-dimensional prototype for the more involved 4-dimensional analysis required to prove the index relation in Section 4. Indeed, most of the work in Sections 3 and 4 is in reproving standard elliptic estimates, but in terms of constants that are independent of the degenerating metric. Finally, we mention that our proof of the index theorem hinges on a compactness theorem, Theorem 4.6, which is a linear version of the results appearing in [9].

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2. Background on the quilted Atiyah-Floer conjecture

Throughout this paper \( Y \) will denote a fixed closed, connected, oriented 3-manifold equipped with a Morse function \( f : Y \to S^1 \) with non-empty, connected fibers. By the identification \( [Y, S^1] = H^1(Y, \mathbb{Z}) \), such an \( f \) exists if and only if \( Y \) has positive first Betti number. We assume \( f \) has been suitably perturbed so that its set of critical points are in bijection with its set of critical values \( \{c_0, c_1, \ldots, c_{(N-1)0}\} \subset S^1 = \mathbb{R}/C\mathbb{Z} \), for some \( C > 0 \). When this is the case, it follows that the number of critical values \( N \) must be even, and we can find regular values \( \{(r_j)_{j=0}^{N-1} \subset S^1 \) with \( c_{j(j+1)} \subset [r_j + \delta, r_{j+1} - \delta] \), for some fixed \( \delta > 0 \). Indices should be taken modulo \( N \), when appropriate. We may assume the circumference \( C \) is large enough to take \( \delta = 1/2 \).

Define

\[
\Sigma_j := f^{-1}(r_j - 1/2), \quad Y_{j(j+1)} := f^{-1}([r_j + 1/2, r_{j+1} - 1/2]),
\]

which are closed, connected, oriented surfaces and elementary cobordisms, respectively.

![Figure 1. An illustration of a broken circle fibration.](image)

Fix a metric \( g \) on \( Y \). We refer to \( g \), or its restriction to any submanifold of \( Y \), as the fixed metric. Note that there are no critical values between \( r_j - 1/2 \) and \( r_j + 1/2 \), so \( V := \nabla f/|\nabla f| \) is well-defined on \( f^{-1}([r_j - 1/2, r_j + 1/2]) \). The time-1 gradient flow of \( V \) provides an identification

\[
(1) \quad f^{-1}([r_j - 1/2, r_j + 1/2]) \cong I \times \Sigma_j,
\]
where we have set \( I := [0, 1] \). This also provides an identification of \( f^{-1}(t) \) with \( \Sigma_j \) for \( t \in [r_j - 1/2, r_j + 1/2] \). So the function \( f \) together with the metric \( g \) allow us to view \( Y \) as the composition of cobordisms

\[
Y_{01} \cup_{\Sigma_1} (I \times \Sigma_{1}) \cup_{\Sigma_2} Y_{12} \cup_{\Sigma_{2}} \ldots \cup_{\Sigma_{N-1}} Y_{(N-1)0} \cup_{\Sigma_{N-1}} (I \times \Sigma_{0}) \cup_{\Sigma_{N}} Y_{01}.
\]

The case \( N = 4 \) is illustrated in Figure 1. Note that this is cyclic in the sense that the cobordism \( I \times \Sigma_0 \) on the right is glued to the cobordism \( Y_{01} \) on the left. We set \( \Sigma_* := \bigsqcup_j \Sigma_j \) and \( Y_* := Y \setminus (I \times \Sigma_*) \), so

\[
Y = Y_* \cup_{\partial Y_*} (I \times \Sigma_*)
\]

We will refer to the connected components of the boundary \( \partial Y_* \) as the \textit{seams}, and we will use the letter \( t \) to denote the coordinate variable on the interval \( I \).

Over \( I \times \Sigma_* \) the metric \( g \) has the form \( dt^2 + g_\Sigma \), where \( g_\Sigma \) is a path of metrics on \( \Sigma_* \). To simplify the discussion, we assume that \( g \) has been chosen so that \( g_\Sigma \) is a constant path, which can always be achieved using the decomposition in (2) and a bump function. For small \( \epsilon > 0 \) define a new metric

\[
g_\epsilon := \begin{cases} dt^2 + \epsilon^2 g_\Sigma & \text{on } I \times \Sigma_* \\ \epsilon^2 g & \text{on } Y_.* \end{cases}
\]

Let \( S_1 \) denote the smooth structure on \( Y \) (i.e. the smooth structure in which \( g \) and \( f \) are smooth). We call this the \textit{standard smooth structure}. It is important to note that when \( \epsilon \neq 1 \), the metric \( g_\epsilon \) is not smooth in the standard smooth structure. However, there is a different smooth structure \( S_\epsilon = S_\epsilon(Y) \) in which \( g_\epsilon \) is smooth, and \( Y^\epsilon := (Y, S_\epsilon) \) is diffeomorphic to \( (Y, S_1) \); see [7].

We call \( S_\epsilon \) the \textit{\( \epsilon \)-dependent smooth structure}, and say that a function, form, connection, etc. on \( Y \) is \textit{\( \epsilon \)-smooth} if it is smooth with respect to \( S_\epsilon \).

We will use the metric \( ds^2 + g_\epsilon \) on \( \mathbb{R} \times Y \), and use \( * \epsilon \) to denote the associated Hodge star. Given a measurable subset \( S \subset \mathbb{R} \times Y \), we will write \( \| \cdot \|_{L_p(S), \epsilon} \) for the \( L^p \)-norm on \( S \) determined by \( ds^2 + g_\epsilon \). We will not typically keep track of the underlying vector bundle (e.g., we use the same symbol \( \| \cdot \|_{L_p(Y), \epsilon} \) to denote the norm on sections of \( TY \) as on sections of \( \Lambda^2 T^*Y \)). When \( S \) is clear from context, we will simply write \( \| \cdot \|_{\epsilon} \) and \( \langle \cdot, \cdot \rangle_{\epsilon} \) for \( \| \cdot \|_{L^2(S), \epsilon} \) and the \( L^2 \)-inner product \( \langle \cdot, \cdot \rangle_{L^2(S), \epsilon} \). We will use \( s^{\epsilon}_\ast \) to denote the Hodge star of the restriction \( (ds^2 + g_\epsilon)\mid_S \), and when \( \epsilon = 1 \) we drop \( \epsilon \) from the notation. For example, on \( \Sigma_* = \{(s, t)\} \times \Sigma_* \subset \mathbb{R} \times Y \) we have

\[
\langle (\mu, \nu) \rangle_{L^2(\Sigma_*), \epsilon} := \int_{\Sigma_*} \langle \mu \wedge *_{\epsilon} s \cdot \nu \rangle = \epsilon^{2-2k} \int_{\Sigma_*} \langle \mu \wedge *_{\Sigma_*} s \cdot \nu \rangle,
\]

where \( \mu, \nu \) are \( k \)-forms on \( \{(s, t)\} \times \Sigma_* \) with values in some vector bundle with fiber inner product \( \langle \cdot, \cdot \rangle \). We will often abuse notation and write \( *_{\Sigma_*} \) for \( *_{\epsilon} \).

We will use \( W^{k,p}(S, V) \) to denote the space of maps of Sobolev class \( W^{k,p} \) from a space \( S \) to a Banach space (or bundle) \( V \), and we write \( L^p(S, V) = W^{0,p}(S, V) \). For example, if \( f : \mathbb{R} \to W^{1,2}(Y) \), then \( \| f \|_{L^2(\mathbb{R}, W^{1,2}(Y))} = \int_{\mathbb{R}} \| f(s) \|_{W^{1,2}(Y)} ds \). We note that the usual Sobolev embedding statements for \( W^{k,p}(S, R) \) hold equally well for \( W^{k,p}(S, V) \).

2.1. \textbf{Gauge theory.} Let \( X \) be an oriented manifold. Given a fiber bundle \( F \to X \), we will use \( \Gamma(F) \) to denote the space of smooth sections. If \( F \) is a vector bundle, then we will write \( \Omega^k(X, F) := \Gamma(L^k T^*X \otimes F) \) for the space of \( k \)-forms with values in \( F \), and we set

\[
\Omega^*(X, F) := \bigsqcup_k \Omega^k(X, F).
\]
Now suppose $P \to X$ is a principal $G$-bundle, where $G$ is compact and $\mathfrak{g}$ is equipped with an Ad-invariant inner product $\langle \cdot, \cdot \rangle$. Define the adjoint bundle $P(\mathfrak{g}) := (P \times \mathfrak{g}) / G$, where $G$ acts diagonally, with the adjoint action in the second factor. This space is naturally a vector bundle over $X$ with fiber $\mathfrak{g}$. The Ad-invariance of the inner product on $\mathfrak{g}$ determines an inner product on the fibers of $P(\mathfrak{g})$, which we also denote by $\langle \cdot, \cdot \rangle$. Similarly, since the Lie bracket $\mathfrak{g}$ is Ad-invariant, it combines with the wedge product to determine a graded Lie bracket on the space $\Omega^\bullet(X, P(\mathfrak{g}))$ given by $\mu \otimes \nu \mapsto [\mu \wedge \nu]$.

Denote by

$$A(P) = \left\{ A \in \Omega^1(P, \mathfrak{g}) \left| (g_P)^* A = \text{Ad}(g^{-1}) A, \quad \forall g \in G \right. \right\}$$

the space of connections on $P$. Here $g_P$ (resp. $\xi_P$) is the image of $g \in G$ (resp. $\xi \in \mathfrak{g}$) under the map $G \to \text{Diff}(P)$ (resp. $\mathfrak{g} \to \text{Vect}(P)$) afforded by the group action. It follows that $A(P)$ is an affine space modeled on $\Omega^1(X; P(\mathfrak{g}))$, and we denote the affine action by $(V, A) \mapsto A + V$. In particular, $A(P)$ is a smooth (infinite dimensional) manifold with tangent space $\Omega^2(X, P(\mathfrak{g}))$. Each connection $A \in A(P)$ determines a covariant derivative $d_A : \Omega^\bullet(X, P(\mathfrak{g})) \to \Omega^{\bullet+1}(X, P(\mathfrak{g}))$ and a curvature (2-form) $F_A \in \Omega^2(X, P(\mathfrak{g}))$. These satisfy $d_A + V = d_A + [V \wedge \cdot]$ and $F_A + V = F_A + d_A V + [\xi \wedge V]$. We say that a connection $A$ is irreducible if $d_A$ is injective on 0-forms.

Given a metric on $X$, we can define the formal adjoint $d_A^* := -(-1)^{(n-k)(k-1)} \circ d_A^*$. Stokes’ theorem shows that this satisfies $(d_A^* V, W)_{L^2} = (V, d_A^* W)_{L^2}$ for all compactly supported $V, W \in \Omega^\bullet(X, P(\mathfrak{g}))$, where $\langle \cdot, \cdot \rangle_{L^2}$ is the $L^2$-inner product coming from the metric on $X$. For $c > 0$, denote by $\ast(c)$ be the Hodge star on $X$ associated to the metric on $X$ conformally scaled by $c^2$. Then on $k$-forms on $X$ we have $\ast(c) = c^{(\dim X)/2} \ast$. Consequently, $d_A^* A \circ c^{-2} d_A^*$ where $d_A^* A \circ c^{-2} d_A^*$ is the formal adjoint defined using $\ast(c)$.

A connection $A$ is flat if $F_A = 0$. We will denote the set of flat connections on $P$ by $A_{\text{flat}}(P)$. If $A$ is flat then im $d_A \subseteq \ker d_A$ and we can form the harmonic spaces

$$H_A^k := H_A^k(X, P(\mathfrak{g})) := \frac{\ker (d_A | \Omega^k(X, P(\mathfrak{g})))}{\text{im} (d_A | \Omega^{k-1}(X, P(\mathfrak{g})))}, \quad H_A^* := \bigoplus_k H_A^k.$$

We denote the associated projection by $\text{proj}_A : \Omega^k(X, P(\mathfrak{g})) \to H_A^k$. Suppose $X$ is compact with (possibly empty) boundary, and let $\partial : \Omega^\bullet(X, P(\mathfrak{g})) \to \Omega^\bullet(\partial X, P(\mathfrak{g}))$ denote the restriction. Then the Hodge isomorphism [17] Theorem 6.8] says

$$H_A^\ast \cong \ker (d_A + d_A^* + \partial \ast), \quad \Omega^\bullet(X, P(\mathfrak{g})) \cong H_A^\ast \oplus \text{im} (d_A) \oplus \text{im} (d_A^* | \ast),$$

for any flat connections $A$ on $X$, where the summands on the right are $L^2$-orthogonal. We will treat these isomorphisms as identifications. From the first isomorphism in (4) we see that $H_A^\ast$ is finite dimensional since $d_A + d_A^*$ is elliptic. Furthermore, it is clear that $\ast : H_A^k \to H_A^{\dim(X) - k}$ restricts to an isomorphism on the harmonic spaces.

**Example 2.1.** Suppose $X = \Sigma$ is a closed, oriented surface equipped with a metric. Then the pairing $\omega(\mu, \nu) := \int_\Sigma \langle \mu \wedge \nu \rangle$ is a symplectic form on the vector space $\Omega^1(X, P(\mathfrak{g}))$. Note that changing the orientation on $\Sigma$ replaced $\omega$ by $-\omega$. On surfaces, the Hodge star $\ast$ squares to -1 on 1-forms and so defines a complex structure on $\Omega^1(\Sigma, P(\mathfrak{g}))$. It follows that the triple $\left( \Omega^1(\Sigma, P(\mathfrak{g})), \ast, \omega \right)$ is Kähler. If $\alpha \in A(P)$ is flat, then $H_A^1 \subset \Omega^1(\Sigma, P(\mathfrak{g}))$ is a Kähler subspace.

Now suppose $X$ is 4-manifold. Then on 2-forms the Hodge star squares to the identity, and it has eigenvalues $\pm 1$. Denoting by $\Omega^\pm(X, P(\mathfrak{g}))$ the $\pm 1$ eigenspace of $\ast$, we have an $L^2$-orthogonal decomposition $\Omega^2(X, P(\mathfrak{g})) = \Omega^+(X, P(\mathfrak{g})) \oplus \Omega^-(X, P(\mathfrak{g}))$. The elements of $\Omega^-(X, P(\mathfrak{g}))$ are called anti-self dual 2-forms. A connection $A \in X$ is said to be anti-self dual (ASD) or an instanton if its curvature $F_A \in \Omega^-(X, P(\mathfrak{g}))$ is an anti-self dual 2-form; that is, if $F_A + \ast F_A = 0$. 


A gauge transformation is an equivariant bundle map \( U : P \rightarrow P \) covering the identity. The set of gauge transformations on \( P \) forms a Lie group, called the gauge group, and is denoted \( \mathcal{G}(P) \). This is naturally a Lie group with Lie algebra \( \Omega^0(X, P(\mathfrak{g})) \) under the map \( R \mapsto \exp(-R) \), where \( \exp : \mathfrak{g} \rightarrow G \) is the Lie-theoretic exponential.

The gauge group acts on the space of connections by pullback \((U, A) \mapsto U^*A\). In terms of a faithful matrix representation of \( G \) we can write this as \( U^*A = U^{-1}AU + U^{-1}dU \), where the concatenation appearing on the right is just matrix multiplication, and \( dU \) is the linearization of \( U \) when viewed as a \( G \)-equivariant map \( P \rightarrow G \). The infinitesimal action of \( \mathcal{G}(P) \) at \( A \in \mathcal{A}(P) \) is

\[
\Omega^0(X, P(\mathfrak{g})) \rightarrow \Omega^1(X, P(\mathfrak{g})), \quad R \mapsto -d_AR
\]

The gauge group also acts on the left on \( \Omega^*(X, P(\mathfrak{g})) \) by the pointwise adjoint action. The curvature of \( A \in \mathcal{A}(P) \) transforms under \( U \in \mathcal{G}(P) \) by \( F_{U^*A} = Ad(U^{-1})F_A \). This shows that \( \mathcal{G}(P) \) restricts to an action on \( \mathcal{A}_{\text{flat}}(P) \) and, in 4-dimensions, the instantons.

We will be interested in the case \( G = \text{PU}(r) \) for \( r \geq 2 \). We equip the Lie algebra \( \mathfrak{g} \cong \mathfrak{su}(r) \subset \text{End}(\mathbb{C}^r) \) with the inner product \( \langle \mu, \nu \rangle := -\kappa_r \text{tr}(\mu \cdot \nu) \); here \( \kappa_r > 0 \) is arbitrary, but fixed. On manifolds \( X \) of dimension at most 4, the principal \( \text{PU}(r) \)-bundles \( P \rightarrow X \) are classified, up to bundle isomorphism, by two characteristic classes \( t_2(P) \in H^2(X, \mathbb{Z}_r) \) and \( q_4(P) \in H^4(X, \mathbb{Z}) \). These generalize the 2nd Stiefel-Whitney class and 1st Pontryagin class, respectively, to the group \( \text{PU}(r) \); see [20].

Consider a principal \( \text{PU}(r) \)-bundle \( P \rightarrow X \) where we assume \( \dim(X) \leq 3 \). Then there are maps

\[
\eta : \mathcal{G}(P) \rightarrow H^1(X, \mathbb{Z}_r), \quad \deg : \mathcal{G}(P) \rightarrow H^3(X, \mathbb{Z})
\]

called the parity and degree. These detect the connected components of \( \mathcal{G}(P) \) in the sense that \( u \) can be connected to \( u' \) by a path if and only if \( \eta(u) = \eta(u') \) and \( \deg(u) = \deg(u') \). We denote by \( \mathcal{G}_0(P) \) the identity component of \( \mathcal{G}(P) \). See [8].

Suppose \( X = \Sigma \) is a closed, connected, oriented surface, and \( P \rightarrow \Sigma \) is a principal \( \text{PU}(r) \)-bundle with \( t_2(P|\Sigma) \in \mathbb{Z}_r \) a generator. Then all flat connections on \( P \) on irreducible. It can be shown that \( \mathcal{G}_0(P) \) acts freely on \( \mathcal{A}_{\text{flat}}(P) \) and

\[
M(P) := \mathcal{A}_{\text{flat}}(P)/\mathcal{G}_0(P)
\]

is a compact, simply-connected, smooth manifold with tangent space at \([\alpha] \in M(P)\) canonically identified with \( H^*_\alpha \). It follows from Example 2.1 that \( M(P) \) is a symplectic manifold, and any metric on \( \Sigma \) determines an almost complex structure \( J_{\Sigma} \) on \( M(P) \) that is compatible with the symplectic form; see [18] for more details regarding these assertions.

Now suppose \( Y_{ab} \) is an oriented elementary cobordism between closed, connected, oriented surfaces \( \Sigma_a \) and \( \Sigma_b \). Fix a \( \text{PU}(r) \)-bundle \( Q_{ab} \rightarrow Y_{ab} \) with \( t_2(Q_{ab}|\Sigma_a) \in \mathbb{Z}_r \) a generator. Then the flat connections on \( Q_{ab} \) are irreducible, and the quotient \( \mathcal{A}_{\text{flat}}(Q_{ab})/\mathcal{G}_0(Q_{ab}) \) is a finite-dimensional, simply-connected, smooth manifold. Restricting to the two boundary components induces an embedding \( \mathcal{A}_{\text{flat}}(Q_{ab})/\mathcal{G}_0(Q_{ab}) \hookrightarrow M(Q|\Sigma_a) \times M(Q|\Sigma_b) \), and we let \( L(Q_{ab}) \) denote the image. It follows that \( L(Q_{ab}) \subset M(Q|\Sigma_a)^- \times M(Q|\Sigma_b)^- \) is a smooth Lagrangian submanifold, where the superscript in \( M(Q|\Sigma)^- \) means that we have replaced the symplectic structure with its negative. See [18].

2.2. Set-up for Atiyah-Floer. Now we specialize the discussion of the previous section to the set-up of the quilted Atiyah-Floer conjecture. Let \( f : Y \rightarrow S^1 \) be as in the beginning of Section 2 and let \( d \in \mathbb{Z}_r \) be a generator. We fix a \( \text{PU}(r) \)-bundle \( Q \rightarrow Y \) with the property that \( t_2(Q) \) is Poincaré dual to \( d[\gamma] \in H_1(Y, \mathbb{Z}_r) \), where \( \gamma : S^1 \rightarrow Y \) is a section of \( f \). This determines \( Q \) uniquely up to bundle isomorphism. It follows that all flat connections on \( Q \) are irreducible. In [8], we show that there exists a (unique up to homotopy) degree 1 gauge transformation on \( Q \), and
we let $G_2 \subset G(Q)$ denote the subgroup generated by this gauge transformation and the identity component $G_0(Q)$.

As in (3), we will write $Q = Q_\ast \cup \partial (I \times P_\ast)$, where $Q_\ast := \cup_j Q_j(j+1)$ and $Q|_{Y_{j(j+1)}} := Q|_{Y_{j(j+1)}}$ are the restrictions of $Q$ over $Y_\ast$ and $Y_{j(j+1)}$, respectively; $P_\ast$ is defined similarly. Then by construction $t_2(Q_j(j+1)) = t_2(P_j)$, $Y_{j(j+1)}$ = $\partial \delta d_t$. There is a canonical smooth structure $S_t(Q)$ on $Q$ such that the projection $Q \to Y^\ast$ is smooth. We will write $Q' := (Q, S_t(Q))$ for $Q$ together with the smooth structure $S_t(Q)$.

Following [18], we can associate symplectic data to $Q \to Y$ as follows: Set

\begin{equation}
M := M(P_0)^{-} \times M(P_1) \times M(P_2)^{-} \times \ldots \times M(P_{N-1}).
\end{equation}

We denote by $J_j$ the almost complex structure on $M(P_j)$ coming from the metric, and we set $J := \sum_{j=0}^{N-1} (-1)^{j+1} \text{proj}_{M(P_j)}^M J_j$, where $\text{proj}_{M(P_j)} : M \to M(P_j)$ is the projection. Then $J$ is a complex structure on $M$ that is compatible with the symplectic form. We will be interested in tuples of maps $\mathcal{P} = (x_j)_j$, where $x_j : I \to M(P_j)$. Any such tuple naturally defines a map $x : I \to M$ by the formula

\begin{equation}
x(t) := (x_0(1-t), x_1(t), \ldots, x_{N-1}(1-t), x_{N-1}(t)).
\end{equation}

Then the symplectic data $M, J$ and $x$ uniquely determines the \emph{quilted} symplectic data $\mathcal{M} := (M_J)_j, \mathcal{P} := (J_j)_j$ and $\mathcal{Q} = (x_j)_j$, and vice-versa. (From the perspective of this paper, ‘quilted’ essentially means ‘$N$-tuples’.) Next, set

\begin{equation}
L_{(0)} := L(Q_0) \times L(Q_{23}) \times \ldots \times L(Q_{(N-2)(N-1)}) \subset M.
\end{equation}

We define $L_{(1)} \subset M$ analogously but using $L(Q_{j(j+1)})$ for even $j$. Then $L_{(0)}, L_{(1)} \subset M$ are Lagrangians. In the language of quilts, the $L_{(j)}$ are equivalent to the cyclic Lagrangian correspondence $L(Q) := (L(Q_{j(j+1)}))_j$.

Using the product structure of $\mathbb{R} \times Y$, one can write any connection $A \in \mathcal{A}(\mathbb{R} \times Q)$ as $A|_{\{s\} \times Y} = a(s) + p(s) ds$, where $a(s) \in \mathcal{A}(Q)$ and $p(s) \in \Omega^0(Y, Q(g))$. Then the curvature of $A$ can be written as $F_A|_{\{s\} \times Y} = F_a(s) = b_s \wedge ds$, where $b_s := \partial_s a(s) - a_a(s) p(s)$. One can repeat this discussion on $\mathbb{R} \times I \times \Sigma_{\ast}$, and write $A|_{\{s\} \times I \times \Sigma_{\ast}} = a(s, t) + \phi(s, t) ds + \psi(s, t) dt$. In general, we will use the following notation for connections, forms, etc. on $\mathbb{R} \times Y$:

| Connections | $\mathbb{R} \times Y$ | $\mathbb{R} \times Y_{\ast}$ and $\mathbb{R} \times Y$ | $\mathbb{R} \times I \times \Sigma_{\ast}$ |
|------------|----------------------|--------------------------------|----------------------------------|
| $F_A$      | $A + p ds$           | $\alpha + \psi dt$           | $\alpha + \psi ds + \phi dt$    |
| Curvature  | $F_a \wedge ds$      | $\gamma ds \wedge dt$       | $\gamma ds \wedge dt$          |
| 1-forms    | $V$                  | $\nu + \theta dt$           | $\nu + \rho ds + \theta dt$    |

In short, capital Latin letters are typically reserved for objects on 4-manifolds, lower case Latin letters for 3-manifolds, and lower case Greek letters for 2-manifolds.

We set $\nabla_s := \partial_s + [\cdot, \cdot]$. Similarly, on $\mathbb{R} \times I \times \Sigma_{\ast}$, we will write $\nabla_s := \partial_s + [\phi, \cdot]$ and $\nabla_s := \partial_s + [\psi, \cdot]$. Note that, in terms of the components of $F_A$, these satisfy the following commutation relation:

$$\nabla_s d_a - d_a \nabla_s = [b_s, \cdot]$$

where the operators are acting on forms on $Y$, and

$$\nabla_s d_a - d_a \nabla_s = [\beta_s, \cdot], \quad \nabla_t d_a - d_a \nabla_t = [\beta_t, \cdot], \quad \nabla_s \nabla_t - \nabla_t \nabla_s = [\gamma, \cdot]$$

where here the operators are acting on forms on $\Sigma_{\ast}$.
2.3. Symplectic geometry. Let \((M, \omega)\) be a compact symplectic manifold, equipped with a compatible almost complex structure \(J \in \text{End}(TM)\). The minimal Chern number of \((M, \omega)\) is defined to be

\[ N_M := \inf \{ k > 0 \mid c_1(M)(S) = k \text{ for some } S \in \pi_2(M) \}, \]

which we assume is finite. We say that \((M, \omega)\) is monotone if there is a constant \(\tau > 0\) such that \([\omega](S) = \tau c_1(M)(S)\) for all \(S \in \pi_2(M)\). Here \([\omega]\) denotes the cohomology class of the closed form \(\omega\), and \(\tau\) is called the monotonicity constant.

Fix a Lagrangian \(L \subset M\) and suppose we are given a map \(v : (\mathbb{D}^2, \partial \mathbb{D}^2) \to (M, L)\). Since the disk \(\mathbb{D}^2\) is contractible we can find a trivialization \(v^*TM \cong \mathbb{D}^2 \times T_{v(0)}M\) that restricts to a symplectomorphism on the fibers. Then restricting to the boundary provides a loop

\[ S^1 = \partial \mathbb{D}^2 \to \text{Lag}(T_{v(0)}M) \]

into the Grassmannian of Lagrangian subspaces of \(T_{v(0)}M\). It is well-known that, for any symplectic vector space \(V\), the space \(\pi_1(\text{Lag}(V)) = \mathbb{Z}\), has a distinguished generator \(\mu_V\) called the Maslov class. We declare \(I(v) \in \mathbb{Z}\) to be the degree of the map \((9)\) defined with respect to \(\mu_{T_{v(0)}M}\). It follows that \(I(v)\) is independent of all choices and depends only on the homotopy class of \(v\). We therefore obtain a map \(I : \pi_2(M, L) \to \mathbb{Z}\) called the Maslov index.

The next result is due to Viterbo. We include a proof for convenience. See also \([11]\) and \([15]\) for more sophisticated treatment of gradings, and \([19]\) for an extension to quilts.

**Theorem 2.2.** \([10]\) Let \(M\) be a compact symplectic manifold with minimal Chern number \(N_M < \infty\). Suppose \(n_M \geq 2\) is an integer dividing \(2N_M\). Let \(L_0, L_1 \subset M\) be Lagrangian submanifolds that are compact and simply-connected, and assume that the intersection \(L_0 \cap L_1\) is transverse. Then the finite set \(L_0 \cap L_1\) admits a relative \(\mathbb{Z}_{n_M}\)-grading \(\mu_{\text{symp}} : L_0 \cap L_1 \times L_0 \cap L_1 \to \mathbb{Z}_{n_M}\), called the Maslov-Viterbo index. In particular, this satisfies

\[ \mu_{\text{symp}}(x_0, x_1) + \mu_{\text{symp}}(x_1, x_2) = \mu_{\text{symp}}(x_0, x_2) \]

for all \(x_j \in L_0 \cap L_1\).

**Proof.** Given \(x^-, x^+ \in L_0 \cap L_1\), the assumptions on the connectivity ensure that there exists a map \(v : I \times I \to M\) with the following boundary conditions:

\[ v(s, 0) = x^-, \quad v(s, 1) = x^+, \quad v(j, t) \in L_j \quad \forall s, t \in I. \]

Since the square \(I \times I\) is contractible, the pullback \(v^*TM\) admits a symplectic trivialization \(\Psi : v^*TM \to I \times I \times \mathbb{C}^n\). The \(L_j\) intersect transversely at \(x^\pm\), so to simplify notation we may assume \(\Psi\) is chosen so that \(\Psi(T_x - L_1) = j\Psi(T_x - L_0)\) and \(\Psi(T_x + L_1) = j\Psi(T_x + L_0)\), where \(j\) is the complex structure on \(\mathbb{C}^n\). The Lagrangian boundary conditions of \(v\) determine a loop \(\bar{v} : \partial(I \times I) \to \text{Lag}(\mathbb{C}^n)\) in the Lagrangian grassmannian, defined by the formula

\[ \bar{v}(s, 0) = \Psi(T_{v(s, 0)}L_0), \quad \bar{v}(s, 1) = \Psi(T_{v(s, 1)}L_1) \]

\[ \bar{v}(0, t) = e^{-j\pi t/2}\Psi(T_x - L_0), \quad \bar{v}(1, t) = e^{j\pi t/2}\Psi(T_x + L_0). \]

Then we define \(\mu_{\text{symp}}(x^-, x^+)\) to be the mod \(n_M\)-reduction of the Maslov index \(\mu_{\mathbb{C}^n}(\bar{v}) \in \mathbb{Z}\).

It remains to check that this definition of the Maslov-Viterbo index is independent of the choice of \(v\). So suppose \(v' : I \times I \to M\) is a second map satisfying the boundary conditions \((10)\). We will show that the difference \(\mu_{\mathbb{C}^n}(\bar{v}) - \mu_{\mathbb{C}^n}(\bar{v}')\) is a multiple of \(2N_M\), and hence a multiple of \(n_M\). Clearly this difference depends only on the homotopy types (relative to the boundary conditions \((10)\)) of \(v, v'\). In particular, since the \(L_j\) are simply-connected, we may assume that these agree on the boundary \(v|_{\partial(I \times I)} = v'|_{\partial(I \times I)}\). Then these patch together to form a sphere \(u = v#v' : S^2 \to M\).
Consider the pullback $u^\ast \text{Lag}(M) \to S^2$, where $\text{Lag}(M) \to M$ is the bundle with fiber $\text{Lag}(T_m M)$. The maps $v, v'$ provide charts for this bundle, and let $\gamma : S^1 \to \text{Lag}(\mathbb{C}^n)$ denote the transition function given by the equatorial circle $S^1 \subset S^2$ where $v$ and $v'$ are glued together. The homotopy type of $\gamma$ is exactly the difference of the Maslov indices $\mu_{\mathbb{C}^n}(\gamma) = \mu_{\mathbb{C}^n}(\bar{v}) - \mu_{\mathbb{C}^n}(\bar{v}')$. The key observation is that $\gamma$ factors as a map

$$\gamma : S^1 \to \text{Sp}(2n) \to \text{Lag}(\mathbb{C}^n).$$

The second map in (11) sends $\Phi \in \text{Sp}(2n)$ to the Lagrangian $\Phi(\mathbb{R}^n)$, and is multiplication by 2 on the level of fundamental groups $\pi_1(\text{Sp}(2n)) = \mathbb{Z} \to \mathbb{Z} = \pi_1(\text{Lag}(\mathbb{C}^n))$. The first map in (11) corresponds to the equatorial transition function for the symplectic bundle $u^\ast TM \to S^2$. At the level of fundamental groups it is given by multiplication by the Chern number $c_1(u^\ast TM) [S^2]$. It is then immediate from the definition of the minimal Chern number $N_M$ that $\mu_{\mathbb{C}^n}(\gamma) = 2c_1(u^\ast TM) [S^2]$. Fix a Hamiltonian $H \in C^\infty(I \times M, \mathbb{R})$ that vanishes to all orders at $\partial(I \times M)$. Let $X^H : I \to \Gamma(TM)$ be the corresponding Hamiltonian vector field, and $\Phi := \Phi^H : \mathbb{R} \times M \to M$ the flow of $X^H$. Depending on context, we may write $X^H_{\mathbb{R}}$ or $X_t$ for $X^H(t) \in \Gamma(TM)$, and $\Phi_t$ for $\Phi(t, \cdot)$.

Suppose $L_0, L_1 \subset M$ are Lagrangians. Consider the path-space

$$\mathcal{P}(L_0, L_1) := \{ x : (I, 0, 1) \to (M, L_0, L_1) \}.$$ 

The tangent space $T_x \mathcal{P}(L_0, L_1)$ at a path $x$ can be identified with the space of vector fields $\xi$ along $x$ with Lagrangian boundary conditions $\xi(j) \in x(j)^\ast TL_j$, for $j = 0, 1$. Then $\mathcal{P}(L_0, L_1)$ admits a natural 1-form $\lambda_H$ defined by

$$\lambda_H : T_x \mathcal{P}(L_0, L_1) \to \mathbb{R}, \quad \xi \mapsto \int_0^1 \omega_{x(t)}(\partial_t x - X^H_t(x(t)), \xi(t)) \, dt.$$ 

This 1-form is closed, and if $M$ is monotone with monotonicity constant $\tau$, then its cohomology class is integral (after dividing out by the constant $\tau$). If $M$ is simply-connected and the $L_j$ are connected, then $\mathcal{P}(L_0, L_1)$ is path-connected. Then, by fixing a base-point $x_0 \in \mathcal{P}(L_0, L_1)$, we can integrate $\lambda_H$ to a circle-valued function $\mathcal{A}_H := \mathcal{A}_{L_0, L_1, H} : \mathcal{P}(L_0, L_1) \to \mathbb{R}/\tau \mathbb{Z}$, called the perturbed symplectic action. This sends a path $x \in \mathcal{P}(L_0, L_1)$ to

$$\mathcal{A}_H(x) := - \int_{x\times I} v^\ast \omega - \int_0^1 H(t, x(t)) \, dt \mod \tau \mathbb{Z}.$$ 

Here $v : I \times I \to M$ is a smooth map with $v(0, t) = x_0(t), v(1, t) = x(t)$ and $v(s, j) \in L_j$ for $j = 0, 1$. Monotonicity implies that $\mathcal{A}_H$ depends on the choice of $v$ only up to an element of $\tau \mathbb{Z}$. Similarly, $\mathcal{A}_H$ depends on the choice of basepoint $x_0$ up to an overall constant. By definition, we have $\lambda_H = d\mathcal{A}_H$ is the differential of $\mathcal{A}_H$, and so the formula for $\lambda_H$ shows that the critical points of $\mathcal{A}_H$ are the paths $x : (I, 0, 1) \to (M, L_0, L_1)$ with $\partial_t x = X^H_t(x(t))$. We denote by $\mathcal{I}_H(L_0, L_1)$ the set of critical points. We say that a critical point $x$ is non-degenerate if the Hessian $D_x$ of $\mathcal{A}_H$ at $x$ is injective. It follows from standard elliptic theory that $D_x : W^{1,2} \to L^2$ always has a closed image, and so $x$ is non-degenerate if and only if there is a constant $c_0 > 0$ such that

$$\|\eta\|_{W^{1,2}(I)} \leq c_0 \|D_x \eta\|_{L^2(I)}$$

for all sections $\eta : (I, j) \to (x^\ast TM, x(j)^\ast TL_j)$ with Lagrangian boundary conditions. Moreover, there is a canonical identification $\mathcal{I}_H(L_0, L_1) \cong \Phi^H_1(L_0) \cap L_1$, where $\Phi^H_1$ is the time-1 flow of $H$, and $x \in \mathcal{I}_H(L_0, L_1)$ is non-degenerate if and only if the associated intersection point is transverse. Consequently, we refer to the elements of $\mathcal{I}_H(L_0, L_1)$ as $H$-Lagrangian intersection points.
Now we specialize to the case of where $M$ is as in \[\text{(9)}\]. It is shown in \[\text{(1)}\] that $M$ is monotone with monotonicity constant $\tau = 2\pi^2 \kappa_{r^{-1}}$. In \[\text{(1)}\] and \[\text{[3, Corollary 6.3]}\], it is shown that $M$ has minimal Chern number $N_M = 2$. Let $L_{(0)}, L_{(1)} \subset M$ be as in \[\text{(9)}\]. Assume a Hamiltonian $H$ has been chosen so that all elements of $\mathcal{I}_H(L_{(0)}, L_{(1)})$ are non-degenerate. Then by Theorem \[\text{(2.2)}\] this set admits a natural relative $\mathbb{Z}_4$-grading. For our application, we will need to assume $H$ is of split-type. This means that

\[
H(t, p_0, \ldots, p_{N-1}) = \sum_{j=0}^{N-1} (-1)^{j+1} H_j(t, p_j)
\]

for some Hamiltonians $H_j \in C^\infty(I \times M(P_j), \mathbb{R})$. Note that the tuple $H = (H_j)_{j=0}^{N-1} \in \bigoplus_{j=0}^{N-1} C^\infty(I \times M(P_j), \mathbb{R})$ is uniquely determined by the split-type Hamiltonian $H$, and vice-versa.

In this paper we are interested in the Lagrangian Floer cohomology of the pair $L_{(0)}, L_{(1)} \subset M$. This is the cohomology of a chain complex $(CF^\bullet_{\text{symp}}, \partial_{\text{symp}})$, and arises when one applies the general framework of Morse theory to the perturbed symplectic action $A_H$. (Since everything in sight is of split-type the resulting cohomology is, more precisely, the \textit{quilted} Floer cohomology of the cyclic Lagrangian correspondence $L(Q)$.)

We begin by describing the group $CF^\bullet_{\text{symp}}$. This is freely generated by $\mathcal{I}_H(L_{(0)}, L_{(1)})$. The elements of this set are canonically identified with $G_2$-equivalence classes of connections $a \in \mathcal{A}(Q)$ satisfying

\[
\partial_t a(t) - d_a(t) \psi(t) - X^H_a(t) = 0, \quad F_a(t) = 0, \quad F_a|_{Y_\ast} = 0,
\]

where we have written $a_{\{t\} \times \Sigma_\ast} = a(t) + \psi(t) \, dt$, and the first two equations are equations on $\Sigma_\ast$; see \[\text{(7)}\]. We say that $a$ is a \textbf{representative} for $x \in \mathcal{I}_H(L_{(0)}, L_{(1)})$. Given any $\epsilon > 0$, we can use the gauge freedom to choose $a$ so that it is $\epsilon$-smooth.

Next, we want to express the non-degeneracy of $x \in \mathcal{I}_H(L_{(0)}, L_{(1)})$ in terms of $a$ and forms on $Y$. To do this, we introduce the following notation: There is a natural subbundle of $T\mathcal{A}_{\text{flat}}(P_\ast)$ whose fiber at a connection is the harmonic space. Given a map $\alpha : S \to \mathcal{A}_{\text{flat}}(P_\ast)$, from a space $S$, we will use $H_\alpha \to S$ to denote the pullback of this subbundle. In particular, suppose $a$ is an $H$-flat connection on $Q$. Then write $a_{\{t\} \times \Sigma_\ast} = \alpha + \psi \, dt$, where we view $\alpha$ as a map $I \to \mathcal{A}_{\text{flat}}(P_\ast)$ with components $a_j$. Then this determines a bundle $H_\alpha \to I$ with fiber

\[
(H_\alpha)_t = H^1_{\alpha(t-1)} \oplus H^1_{\alpha_0(t)} \oplus \cdots \oplus H^1_{\alpha_{N-2}(t-1)} \oplus H^1_{\alpha_{N-1}(t)}.
\]

Similarly, the restriction of $a$ to $Y_\ast$ provides a Lagrangian subspace $H_a_{\{0\}} \subset (H_a)_0$ (coming from the ‘even’ cobordisms $Y_{2i(2i+1)}$) and another Lagrangian $H_a_{\{1\}} \subset (H_a)_1$ (coming from the ‘odd’ cobordisms $Y_{(2i-1)2i}$). The bundle $H_{a,\{j\}}$ is the tangent space to $L_{(j)} \subset M$ at $\alpha(\{j\})$; see \[\text{(12)}\]. Suppose $x \in \mathcal{I}_H(L_{(0)}, L_{(1)})$, and $a \in \mathcal{A}(Q)$ is a representative. Then under this identification, $D_x$ corresponds to the operator $D_{0,a} : \Gamma(H_\alpha) \to \Gamma(H_\alpha)$ defined by

\[
D_{0,a} : \xi \mapsto e^\nabla \text{proj}_\alpha (\nabla_t \xi - dX_\alpha \xi).
\]

Here $dX_\alpha$ is the linearization at $a(t)$ of the vector field $X^H_\alpha$. In particular, $x$ is non-degenerate if and only if $D_{0,a}$ is injective when restricted to sections $\xi : I \to H_\alpha$ with Lagrangian boundary conditions in the $H_{a,\{j\}}$, and this is the case if and only if \[\text{(12)}\] holds with $D_{0,a}, \xi$ in place of $D_x, \eta$.

Now we address the boundary operator $\partial_{\text{symp}}$. This is defined by counting finite $H$-energy solutions $v : (\mathbb{R} \times I, \mathbb{R} \times \{j\}) \to (M, L_{(j)})$ to the $H$-holomorphic curve equation $\partial_t v + J(\partial_v - X^H(v)) = 0$. As above, any map $v : (\mathbb{R} \times I, \mathbb{R} \times \{j\}) \to (M, L_{(j)})$ can be lifted to a \textbf{representative} $A \in \mathcal{A}(\mathbb{R} \times Q)$. Then $v$ is an $H$-holomorphic curve if and only if $A$ satisfies

\[
\partial_t A - d_A \psi - X^A = 0, \quad F_A = 0, \quad F_A|_{Q_\ast} = 0,
\]
(15) \[ \partial_a \alpha - d_\alpha \phi + *_\Sigma (\partial_t \alpha - X_t(\alpha) - d_\alpha \psi)|_{\Sigma} = 0, \quad F_\alpha = 0, \quad F_\alpha|_{Y^*} = 0, \]

where the two equations on the left are equations on \( \Sigma \), and we are using the notation for the components of \( A \) laid out at the end of Section 2.1. The connection \( A \) is uniquely determined by \( v \) up to the action of \( \mathcal{G}_\Sigma \), and we call \( A \) an \( H \)-holomorphic curve representative. Since \( v \) has finite \( H \)-energy, it follows that there are \( H \)-Lagrangian intersection representatives \( a^\pm \) for which \( a^\pm = \lim_{s \to \pm \infty} A_{\{s\} \times Y} \). Note that the definition of \( \mathcal{D}_{0,a} \) given above makes sense when we take \( a = a(s) \) for \( s \in \mathbb{R} \) (i.e., \( a \) does not have to represent an \( H \)-Lagrangian intersection point). The linearization of the \( H \)-holomorphic curve equation then takes the form

\[ \mathcal{D}_{0,A} := \text{proj}_s \nabla_s + \mathcal{D}_{0,a}. \]

A natural domain for this operator is the space of \( W^{1,2} \)-sections of the bundle \( H_\alpha \to \mathbb{R} \times I \) with boundary conditions in the Lagrangian subbundle \( H_{(\alpha)} \subset (H_\alpha)_{\{j\}} \to \mathbb{R} \times \{j\} \). Similarly, a natural codomain is the space of \( L^2 \)-sections of \( H_\alpha \) (without any assumptions on the behavior at the boundary). When all \( H \)-Lagrangian intersection points are non-degenerate, then \( \mathcal{D}_{0,A} \) is Fredholm with this domain and codomain, so the index \( \text{Ind}(\mathcal{D}_{0,A}) \) is well-defined. Moreover, Floer [10] showed that the Fredholm index \( \text{Ind}(\mathcal{D}_{0,A}) \) is given by the Maslov-Viterbo index defined above. In particular, the index recovers the relative grading

\[ \text{Ind}(\mathcal{D}_{0,A}) \equiv \mu_{\text{symp}}([a^-], [a^+]) \mod 4, \]

where \( [a^\pm] \in \mathcal{I}_H(L_{(0)}, L_{(1)}) \) denotes the \( H \)-Lagrangian intersection point represented by \( a^\pm \). We also note that \( \mathcal{D}_{0,A} \) is onto exactly when the formal adjoint \( \mathcal{D}_{0,A}^* := -\text{proj}_s \nabla_s + \mathcal{D}_{0,a} \) is injective when restricted to the space of sections with Lagrangian boundary conditions.

**Remark 2.3.** We call the pair \((J,H)\) regular (for Lagrangian Floer theory) if \( \mathcal{D}_{0,A} \) is onto for all holomorphic curve representatives \( A \). In showing \((CF_{\text{symp}}^*, \partial_{\text{symp}})\) is a well-defined chain complex, one typically wants to know that \((J,H)\) is regular. This yields smooth moduli spaces of holomorphic curves and also helps rule out bubbling phenomena, both of which are used to show \( \partial_{\text{symp}}^2 = 0 \). In [12] we show that, given any \( J \), there is split-type Hamiltonian \( H \) for which \((J,H)\) is regular.

In this paper, we do not need to worry about smooth moduli spaces or bubbling. Nonetheless, there are several stages where we also assume that \( \mathcal{D}_{0,A} \) is onto. However, it is likely that this assumption can be removed.

Finally, we mention that in the same way we were able to lift \((J,H)\)-holomorphic curves in \( M \) to holomorphic curve representatives \( A \in \mathcal{A}(\mathbb{R} \times Q) \), we can also lift sections \( \eta \) of \( H_\alpha \to \mathbb{R} \times I \) that have Lagrangian boundary conditions to 1-forms \( V_0 \) on \( \mathbb{R} \times Y \). For example, on \( \mathbb{R} \times I \times \Sigma \),

\[ V_0|_{\mathbb{R} \times I \times \Sigma} := \eta + \rho \, ds + \theta \, dt, \]

where \( \rho \) and \( \theta \) are determined by the requirement that \( \nabla_s \eta - d_\alpha \rho \) and \( \nabla_t \eta - dX_\alpha \eta - d_\alpha \theta \) are \( \alpha \)-harmonic. The restriction of \( V_0 \) to \( \mathbb{R} \times Y^* \) is determined uniquely by the Lagrangian boundary conditions of \( \eta \). Moreover, \( V_0 \) is \( \epsilon \)-smooth if \( A \) is \( \epsilon \)-smooth (this can always be arranged).

### 2.4 Instanton Floer cohomology.

Let \( f : Y \to S^1 \), \( Q \to Y \) and \( \mathcal{G}_\Sigma \) be as at the end of Section 2.2. For each \( 0 \leq j \leq N - 1 \), let \( H_j : I \times \mathcal{A}(P_j) \to \mathbb{R} \) be a \( \mathcal{G}(P_j) \)-invariant map that vanishes to infinite order on \( \partial(I \times \mathcal{A}(P_j)) \) (so \( H_j \) induce a Hamiltonian on \( M(P_j) \) as in the previous section). Define a map \( H : \mathcal{A}(Q) \to \mathbb{R} \) by

\[ H(a) := \sum_j \int_I H_j(t, a|_{\{t\} \times Y^*}) \, dt. \]
The differential of $H$ is represented by a map $\tilde{X} : A(Q) \to \Omega^2(Y, Q(g))$ in the sense that $(dH)_a v = \int_J \langle \tilde{X}(a) \wedge v \rangle$ for all $v \in T_a A(Q)$. It follows that $\tilde{X}(a)|_{Y_j} = 0$ and $\tilde{X}(a)|_{(t) \times \Sigma_j} = dt \wedge X^H_j(a(t))$, where $X^H_j$ is the time-dependent Hamiltonian vector field on $A(P_j)$ associated to $H_j$.

The instanton Floer cohomology associated to this data is the homology of a chain complex $(CF_{\text{inst}}^*, \partial_{\text{inst}})$ that arises as the Morse cohomology of the $H$-perturbed Chern-Simons functional. Explicitly, $CF_{\text{inst}}^*$ is freely generated by the $G_2$-equivalence classes of $H$-flat connections $a$. By definition, these satisfy $F_a = \tilde{X}(a)$, and we denote the set of $H$-flat connections by $A_{\text{flat}}(Q, H)$. In [7] it is shown that the set $A_{\text{flat}}(Q, H)/G_2$ admits a natural relative $\mathbb{Z}_4$-grading, which we denote by $\mu_{\text{inst}}$. It follows that there is a canonical bijection

$$(18) \quad \Psi : A_{\text{flat}}(Q, H)/G_2 \xrightarrow{\cong} \mathcal{I}_H(L(0), L(1))$$

where the latter set is the relatively $\mathbb{Z}_4$-graded set defined in Section 2.3 Corollary 4.2 below, says that $\Psi$ respects the gradings.

We say an $H$-flat connection is non-degenerate if the Hessian of the $H$-perturbed Chern-Simons functional is injective, when taken modulo the gauge action. In terms of the metric $g$, this is equivalent to saying that the operator $D_{\epsilon,a} := \left( \begin{array}{cc} a_Y(Y) - dX_a & -d_a \\ -d_a^* & 0 \end{array} \right) : \Omega^1(Y, Q(g) \oplus \Omega^0(Y, Q(g)) \to \Omega^1(Y, Q(g) \oplus \Omega^0(Y, Q(g))$

is injective, where $d_a^*$ is the adjoint taken with respect to the $\epsilon$-dependent Hodge star, and $dX_a$ is the linearization of $\tilde{X}$ at $a$. The notion of non-degeneracy is gauge-invariant and independent of the choice of metric. Moreover, it follows that $a$ is a non-degenerate $H$-flat connection if and only if there is a constant $c_0 > 0$ with $\|V\|_{\Omega^1(Y), \epsilon} \leq c_0 \|D_{\epsilon,a} V\|_{L^2(Y), \epsilon}$, for all $\epsilon$-smooth $V = (v, r) \in \Omega^1(Y, Q(g) \oplus \Omega^0(Y, Q(g))$. We show in Proposition 6.4 that $c_0$ can be taken to be independent of $0 < \epsilon$ sufficiently small.

Now we address the boundary operator $\partial_{\text{inst}}$. This counts $G_2$-equivalence classes of finite $H$-energy solutions $A = a + pds$ to the $(\epsilon, H)$-instanton equation $\partial_a - d_ap - s_\epsilon^Y(F_a - \tilde{X}(a)) = 0$. We call solutions of this equation either $(\epsilon, H)$-instantons or $(\epsilon, H)$-ASD connections. Sometimes we simply write ‘$\epsilon$-ASD’ for ‘$(\epsilon, H)$-ASD’. The linearization of the $\epsilon$-ASD equation at $A = a + pds$ is the operator

$D_{\epsilon,A} := \nabla_s + D_{\epsilon,a}$

where on the right we view it as operating on the space of maps $\mathbb{R} \to \Omega^1(Y, Q(g) \oplus \Omega^0(Y, Q(g))$. That is, we can write an element $V \in T_A A(\mathbb{R} \times Q)$ as $V = v + r ds$, which we view as a path $(v, r) : \mathbb{R} \to \Omega^1(Y, Q(g) \oplus \Omega^0(Y, Q(g))$. Then on $\mathbb{R} \times Y_\bullet$, this operator takes the following form:

$D_{\epsilon,A} V|_{\mathbb{R} \times Y_\bullet} = \left( \begin{array}{cc} \nabla_s + \epsilon^{-1} \ast d_a & -d_a \\ -\epsilon^{-2} d_a^* & \nabla_s \end{array} \right) \left( \begin{array}{c} v \\ r \end{array} \right)$

where all Hodge stars are on $Y_\bullet$ and defined by the fixed metric. Similarly, the product structure of $\mathbb{R} \times I \times \Sigma_\bullet$ allows us to identify the space $T_A A(\mathbb{R} \times I \times P_\bullet)$ as the space of maps from $\mathbb{R} \times I$ into $\Omega^1(\Sigma_\bullet, P_\bullet(g)) \oplus \Omega^0(\Sigma_\bullet, P_\bullet(g)) \oplus \Omega^0(\Sigma_\bullet, P_\bullet(g))$. This is simply reflecting the decomposition $V|_{(s,t) \times \Sigma_\bullet} = \nu(s,t) + \rho(s,t) ds + \theta(s,t) dt$. In terms of this splitting, the linearized operator takes the following form:

$D_{\epsilon,A} V|_{\mathbb{R} \times I \times \Sigma_\bullet} = \left( \begin{array}{cc} \nabla_s + \ast (\nabla_t - dX^*_\alpha) & -d_\alpha \\ \epsilon^{-2} \ast d_\alpha & \nabla_s \\ \ast d_\alpha & \epsilon^{-2} d_\alpha^* \end{array} \right) \left( \begin{array}{c} \nu \\ \rho \\ \theta \end{array} \right)$

where all Hodge stars are on $\Sigma_\bullet$ and are defined by the fixed metric.
Suppose all $H$-flat connections are non-degenerate and $A$ is any $\epsilon$-smooth $W^{1,2}$-connection on $\mathbb{R} \times Q$ such that $\lim_{s \to \pm \infty} A(I(s) \times Y) = a^\pm$ for some $a^\pm \in \mathcal{A}_{\text{flat}}(Q, H)$. Then $\mathcal{D}_{\epsilon,A} : W^{1,2}_\epsilon \to L^2_\epsilon$ is Fredholm and its index recovers the grading

$$\text{Ind}(\mathcal{D}_{\epsilon,A}) = \mu_{\text{inst}}(a^-, a^+)\).$$

Note that this also says the index is independent of the choice of connection $A$ limiting to $a^\pm$. Furthermore, $A$ does not need to be an $H$-ASD connection for the above index formula to hold (for our computations below we will take $A$ to be an $H$-holomorphic curve representative). The desirable cases are when $\mathcal{D}_{\epsilon,A}$ is onto. It is a standard fact that $\mathcal{D}_{\epsilon,A}$ is onto if and only if its formal adjoint $\mathcal{D}_{\epsilon,A}^* := -\nabla_s + \mathcal{D}_{\epsilon,A}$ is injective. Moreover, we show in Proposition 4.4 below that if $A$ is an $H$-holomorphic curve representative and $\mathcal{D}_{0,A}$ is onto, then $\mathcal{D}_{\epsilon,A}$ is onto for all $\epsilon > 0$ sufficiently small; see also Remark 2.3.

**Remark 2.4.** As in Lagrangian Floer theory, one typically wants to know that $\mathcal{D}_{\epsilon,A}$ is onto for all $H$-instantons $A$ with finite $H$-energy. We say that $(g_\epsilon, H)$ is regular (for instanton Floer theory) when this is the case. It is a standard fact that given any fixed $\epsilon > 0$ there is always some $H$ for which $(g_\epsilon, H)$ is regular. Moreover, in [7] we show that given any sequence $\epsilon_n$, there is a single perturbation $H$ for which $(g_{\epsilon_n}, H)$ is regular for instanton Floer theory, and $(J, H)$ is regular for quilted Floer theory. However, due to Proposition 4.4 we do not need such a high-powered perturbation result in the present paper, since we can prove our results when $A$ is an $H$-holomorphic curve representative.

### 3. Simultaneous non-degeneracy

In this section we prove a quantitative version of the statement that non-degeneracy can be achieved simultaneously in each of the Floer theories using the same perturbation. Along the way we establish several estimates that will be used repeatedly.

We fix a principal $\text{PU}(r)$-bundle $Q$ over a broken circle fibration $f : Y \to S^1$ as in Section 2.2. We also fix a perturbation $H : \mathcal{A}(Q) \to \mathbb{R}$ as in [17].

#### 3.1. Elliptic properties of small curvature connections

This section establishes several $\epsilon$-dependent elliptic estimates for connections with small curvature on surfaces and cobordisms. The results are all quite standard when the connections are flat and $\epsilon = 1$. Our primary interest is in how these estimates vary with $\epsilon$.

Since all flat connections $a$ are irreducible, it follows that $d_a : \Omega^0(Y, Q(g)) \to \Omega^1(Y, Q(g))$ is injective for any $a \in \mathcal{A}_{\text{flat}}(Q)$. It is well-known that $d_a$ has closed range in suitable Sobolev completions and so there is a bound of the form

$$\|r\|_{L^2(Y)} \leq C\|d_ar\|_{L^2(Y)}$$

for all 0-forms $r$. It turns out that the constant $C$ can be taken to be independent of $a \in \mathcal{A}_{\text{flat}}(Q)$. The type of argument used to prove this statement will be used several times, so we sketch the details: If such a bound does not hold then one could find sequences $r_n$ of 0-forms and $a_n$ of flat connections with $\|r_n\|_{L^2} = 1$, but $\|d_{a_n}r_n\|_{L^2} \to 0$. The bound (19) is gauge invariant, so by Uhlenbeck compactness we may assume that the $a_n$ converge in $C^\infty$ to a flat connection $a_\infty$, after passing to a subsequence. This implies that the $r_n$ are bounded in $W^{1,2}$ (with the derivatives defined using $a_\infty$). So by passing to a further subsequence, we may assume the $r_n$ converge weakly in $W^{1,2}$ and hence strongly in $L^2$ to some $r_\infty$. This must satisfy $1 = \|r_\infty\|_{L^2}$ and $d_{a_\infty}r_\infty = 0$, which contradicts (19) applied to $a = a_\infty$ (at this point using the constant that depends on $a_\infty$).

The same type of Uhlenbeck compactness argument can be used to show that there is some $\epsilon_0 > 0$ such that if $a$ is any connection on $Q$ with $\|F_a\|_{L^\infty} < \epsilon_0$, then (19) holds for all 0-forms...
$r$, with a constant $C$ depending only on $Q \to Y$ and $\epsilon_0$. More generally, still, one can show that
there is an $\epsilon_0 > 0$ such that the bound $|19|$ holds for all connections $\alpha \in \mathcal{A}(Q)$ satisfying

$$
\| F_\alpha \|_{L^\infty(Y_\bullet)} < \epsilon_0, \quad \text{and} \quad \sup_{t \in I} \| F_\alpha(t) \|_{L^\infty(Y_\bullet)} < \epsilon_0.
$$

Here we have written $a|_{(t) \times \Sigma_\bullet} = \alpha(t) + \psi(t) \, dt$.

**Lemma 3.1.** There are constants $\delta, C > 0$ such that the following holds for each $j \in \{0, \ldots, N - 1\}$ and all connections $\alpha \in \mathcal{A}(P_j)$ with $\| F_\alpha \|_{L^\infty(Y_j)} < \delta$:

- There is an estimate

$$
\| \rho \|_{L^2(Y_j, \alpha)} \leq \epsilon C \| d_\alpha \rho \|_{L^2(Y_j)}
$$

for all 0-forms $\rho \in \Omega^0(Y_j, (P_j(g)))$ and all $\epsilon > 0$;

- The space

$$
H^1_\alpha := \ker (d_\alpha \oplus d^*_\alpha) \subseteq \Omega^1(Y_j, P_j(g))
$$

of $\alpha$-harmonic 1-forms has finite dimension equal to $\dim H^1_\alpha$, for any flat connection $\alpha$ on $P_j$. Furthermore, the space $H^1_\alpha$ equals the $L^2$-orthogonal complement of the image of $d_\alpha \oplus d^*_\alpha$: $H^1_\alpha = (\text{im } d_\alpha \oplus \text{im } d^*_\alpha)^\perp$, and so there is a direct sum decomposition

$$
\Omega^1(Y_j, P_j(g)) = H^1_\alpha \oplus (\text{im } d_\alpha \oplus \text{im } d^*_\alpha).
$$

In particular, the $L^2$-orthogonal projection

$$
\text{proj}_\alpha : \Omega^1(Y_j, P_j(g)) \rightarrow H^1_\alpha
$$

is well-defined. Moreover, there is an estimate

$$
\| \nu \|_{L^2(Y_j, \alpha)} \leq C \left( \| \text{proj}_\alpha \nu \|_{L^2(Y_j)} + \epsilon \| d_\alpha \nu \|_{L^2(Y_j)} + \epsilon \| d^*_\alpha \nu \|_{L^2(Y_j)} \right)
$$

for all 1-forms $\nu \in \Omega^1(Y_j, P_j(g))$ and all $\epsilon > 0$.

**Lemma 3.2.** There are constants $\delta, C > 0$ such that the following holds for each $j \in \{1, \ldots, N\}$ and all connections $\alpha \in \mathcal{A}(Q_{(j+1)})$ with $\| F_\alpha \|_{L^\infty(Y_{(j+1)})} < \delta$:

- There is an estimate

$$
\| \nu \|_{L^2(Y_{(j+1)}, \alpha)} \leq \epsilon C \| d_\alpha \nu \|_{L^2(Y_{(j+1)})}
$$

for all 0-forms $r \in \Omega^0(Y_{(j+1)}, (Q_{(j+1)}(g)))$ and all $\epsilon > 0$;

- The space

$$
H^1_\alpha := \ker (d_\alpha \oplus \ker d^*_\alpha) \subseteq \Omega^1(Y_{(j+1)}, Q_{(j+1)}(g))
$$

of $\alpha$-harmonic 1-forms has finite dimension equal to $\dim H^1_\alpha$, for any flat connection $\alpha$ on $Q_{(j+1)}$. Here $\partial$ denotes restriction to the boundary of $Y_{(j+1)}$. Furthermore, the space $H^1_\alpha$ equals the $L^2$-orthogonal complement of the image of $d_\alpha \oplus (d^*_\alpha|_{\partial \Sigma})$: $H^1_\alpha = (\text{im } d_\alpha \oplus (\text{im } d^*_\alpha|_{\partial \Sigma})^\perp$, and so there is a direct sum decomposition

$$
\Omega^1(Y_{(j+1)}, Q_{(j+1)}(g)) = H^1_\alpha \oplus (\text{im } d_\alpha \oplus (\text{im } d^*_\alpha|_{\partial \Sigma})).
$$

In particular, the $L^2$-orthogonal projection

$$
\text{proj}_\alpha : \Omega^1(Y_{(j+1)}, Q_{(j+1)}(g)) \rightarrow H^1_\alpha
$$
is well-defined. Moreover, there is an estimate

\begin{equation}
\|v\|_{L^2(Y_0)} \leq C\left(\|\text{proj}_a v\|_{L^2(Y_0)}, \epsilon + \epsilon \|d_a v\|_{L^2(Y_0)}, \epsilon + \epsilon \|d_a^\star v\|_{L^2(Y_0)}, \epsilon \right)
\end{equation}

for all 1-forms \( v \in \Omega^1(Y_0), Q_j(\epsilon) \) and all \( \epsilon > 0 \).

**Proof of Lemmas 3.1 and 3.2.** When \( \alpha \) or \( a \) is flat and \( \epsilon = 1 \), this is immediate from the Hodge decomposition \([3]\). The case for small curvature connections follows from an Uhlenbeck compactness argument similar to the one we gave above. See also \([5, \text{Lemma 7.6}] \) and \([9, \text{Lemmas 3.9, 3.10}] \). The extension to \( \epsilon > 0 \) is immediate from the conformal rescaling properties of forms on 2- and 3-manifolds. \( \square \)

**Remark 3.3.** Following our conventions, if \( a \) is a flat connection on \( Q \) then we would write \( \text{proj}_a : \Omega^1(Y, Q(\epsilon)) \rightarrow H^a_a \) for the \( L^2 \)-orthogonal projection to the harmonic space (this is often the trivial vector space). It is possible that this map could be confused with the map in \([22]\). To account for this, we adopt the following convention: Given a 1-form \( v \) on \( Y \), we will write \( \text{proj}_a v \) for the 1-form on \( Y_0 \) given over \( Y_0 \) by the restricted connection

\[ \text{proj}_a|_{Y_0} v|_{Y_0} : \Omega^1(Y_0), Q_j(\epsilon) \rightarrow H^a_a. \]

Similarly, writing \( a|_{I \times \Sigma_0} = a|_{I \times \Sigma_0} \) and \( v|_{I \times \Sigma_0} = v|_{I \times \Sigma_0} \), we will write \( \text{proj}_a|_{\Sigma} v \) or \( \text{proj}_a v \) for the 1-form on \( I \times \Sigma_0 \) given on \( I \times \Sigma_0 \) by \( \text{proj}_a|_{\Sigma} v \). Likewise, we can extend these to 1-forms \( V = (v, r) \) on \( Y \), by writing \( \text{proj}_a|_V V \) and \( \text{proj}_a|_V V \) for \( \text{proj}_a|_V v \) and \( \text{proj}_a|_V r \), respectively.

The following proposition will be useful for bounding surface derivatives in terms of derivatives on the full 3-manifold \( Y \).

**Proposition 3.4.** Given \( \epsilon_0 > 0 \), there are constants \( \epsilon_0, C > 0 \) such that the following holds for all \( 0 < \epsilon < \epsilon_0 \) and all \( \epsilon \)-smooth 1-forms \( v \) on \( Y \) with \( v|_{I \times \Sigma_0} = \nu|_{I \times \Sigma_0} \epsilon \partial_t \) :

\begin{equation}
\|v\|_{L^2(I \times \Sigma)} \epsilon + \|\nu\|_{L^2(I \times \Sigma), \epsilon} + \|\bar{\nu}\|_{L^2(I \times \Sigma), \epsilon} + \|\bar{\nu}\|_{L^2(I \times \Sigma), \epsilon} \epsilon + \|\nu\|_{L^2(I \times \Sigma), \epsilon} \epsilon + \|\nu\|_{L^2(I \times \Sigma), \epsilon} \epsilon
\end{equation}

\begin{equation}
\leq C\left( \|\text{proj}_a|_V v\|_{L^2(I \times \Sigma), \epsilon} + \|\text{proj}_a|_V v\|_{L^2(I \times \Sigma), \epsilon} + \|d_a v\|_{L^2(Y), \epsilon} + \|d_a^\star v\|_{L^2(Y), \epsilon} \right).
\end{equation}

Here \( a \) is any \( \epsilon \)-smooth connection on \( Y \) satisfying \( \|F_a\|_{L^\infty(Y_0)} \leq \epsilon_0 \) and \( \|F_a\|_{L^\infty(Y_0)} \leq \epsilon_0 \) where we have written \( a|_{I \times \Sigma_0} = \alpha + \psi_\delta \).

Combining this proposition with Lemmas 3.1 and 3.2 immediately gives the following corollary.

**Corollary 3.5.** Under the same hypotheses as Proposition 3.1, there are bounds

\[ \|r\|_{L^2(Y), \epsilon} \leq \epsilon C\|d_a r\|_{L^2(Y), \epsilon} \]

for all \( \epsilon \)-smooth \( r \in \Omega^0(Y, Q(\epsilon)) \) and

\[ \|v\|_{L^2(Y), \epsilon} \leq C\left( \|\text{proj}_a|_V v\|_{L^2(I \times \Sigma), \epsilon} + \|\text{proj}_a|_V v\|_{L^2(I \times \Sigma), \epsilon} + \|d_a v\|_{L^2(Y), \epsilon} + \|d_a^\star v\|_{L^2(Y), \epsilon} \right), \]

for all \( \epsilon \)-smooth \( v \in \Omega^1(Y, Q(\epsilon)) \).

**Proof of Proposition 3.4.** Fix a smooth bump function \( h : Y \rightarrow \mathbb{R} \) with \( h \equiv 1 \) on \( I \times \Sigma_0 \), and such that \( h \) vanishes outside a small neighborhood \( U \) of \( I \times \Sigma_0 \). We assume \( U \) is small enough to avoid the critical points of the function \( f : Y \rightarrow S^1 \), so the normalized gradient flow of \( f \) is well-defined on \( U \) (the gradient is defined using the fixed metric \( g \) on \( Y \)). For simplicity, we assume \( U = (-\delta_0, 1 + \delta_0) \times \Sigma_0 \) and that \( h \) depends only on the \( (-\delta_0, 1 + \delta_0) \)-coordinate \( t \). Then there is
some $C_0 > 0$ such that $|dh| = |\partial h| \leq C_0 h$. In general, all constants $C_0, C_1, \ldots$ depend only on $c_0$ and the topology of $Q \to Y$ (in particular, they are independent of $\epsilon, v$ and $a$). We will write $\lVert \cdot \rVert_\epsilon$ for $\lVert \cdot \rVert_{L^2(Y), \epsilon}$, and $(\cdot, \cdot)_\epsilon$ for the inner product inducing $\lVert \cdot \rVert_\epsilon$.

In components on $I \times \Sigma$, we can write $d_a v = d_a \nu - (\nabla \nu - d_a \theta) \wedge dt$ and $d^*_a v = d^*_a \nu - \nabla \nu$. Taking the norm of each of these and adding gives

\begin{equation}
\|d_a v\|_\epsilon^2 + \|d^*_a v\|_\epsilon^2 \geq \|h d_a v\|_\epsilon^2 + \|h d^*_a v\|_\epsilon^2
\end{equation}

\begin{align}
\geq & \|h d_a \nu\|_\epsilon^2 + \|h (\nabla \nu - d_a \theta)\wedge dt\|_\epsilon^2 + \|h (\nabla \nu - d^*_a \nu)\|_\epsilon^2 \\
& - 2 \langle h \nabla \nu, h d^*_a \nu \rangle_\epsilon - 2 \langle h \nabla \nu, h d_a \theta \rangle_\epsilon.
\end{align}

The last two terms involving the inner products are the only troublesome ones. We will show that they cancel, modulo some other terms that we can control using the first five terms. Use the commutation relation $\nabla_t d_a - d_a \nabla_t = [\beta_t, \cdot]$ and integration parts to obtain:

\begin{align}
- \langle h \nabla \nu, h d^*_a \nu \rangle_\epsilon &= - \langle h d_a \nabla \theta, h \nu \rangle_\epsilon \\
&= - \langle h^2 \nabla_t d_a \theta, \nu \rangle_\epsilon + \langle h [\beta_t, \theta], h \nu \rangle_\epsilon \\
&= - \langle \nabla_t (h^2 d_a \theta), \nu \rangle_\epsilon + \langle 2 h \partial_t h d_a \theta, \nu \rangle_\epsilon + \langle h [\beta_t, \theta], h \nu \rangle_\epsilon \\
&= (h d_a \theta, h \nabla \nu) + 2 \langle h d_a \theta, \partial_t h \nu \rangle_\epsilon + \langle h [\beta_t, \theta], h \nu \rangle_\epsilon.
\end{align}

Claim: $\langle h [\beta_t, \theta], h \nu \rangle_\epsilon \geq \frac{1}{\delta} \|h d_a \theta\|_\epsilon^2 - C_1 \|h \nu\|_\epsilon^2$.

We will prove this claim in a moment. For now, apply the claim to the last term in the last line of (30) and use the estimate

\begin{equation}
2ab \leq \delta a^2 + \frac{1}{\delta} b^2,
\end{equation}

with $\delta = 1/3$ for the second-to-last term in (30) to get:

\begin{align}
- \langle h \nabla \nu, h d^*_a \nu \rangle_\epsilon &\geq (h d_a \theta, h \nabla \nu) + 2 \langle h d_a \theta, \partial_t h \nu \rangle_\epsilon - 2 C_2 \|h \nu\|_\epsilon^2,
\end{align}

where $C_2 = C_1 + 3$. Returning to (29), we therefore have

\begin{equation}
\|d_a v\|_\epsilon^2 + \|d^*_a v\|_\epsilon^2 \geq \|h d_a \nu\|_\epsilon + \|h d^*_a v\|_\epsilon + \|h \nabla \nu\|_\epsilon + \frac{1}{\delta} \|h d_a \theta\|_\epsilon + \|h \nabla \nu\|_\epsilon - 2 C_2 \|h \nu\|_\epsilon^2.
\end{equation}

The last term is again the troublesome one, but we deal with it as follows: By Lemmas 3.1 and 3.2, we have a bound of the form

\begin{align}
- \|h \nu\|_\epsilon^2 &\geq - \|v\|_{L^2(I \times \Sigma), \epsilon}^2 - \|v\|_{L^2(Y_\star), \epsilon}^2 \\
&\geq - \epsilon^2 C_4 \left( \|d_a v\|_{L^2(I \times \Sigma), \epsilon}^2 + \|d^*_a v\|_{L^2(Y_\star), \epsilon}^2 \right) - \epsilon^2 C_5 \left( \|d_a v\|_{L^2(Y_\star), \epsilon}^2 + \|d^*_a v\|_{L^2(Y_\star), \epsilon}^2 \right) \\
&\quad - C_6 \left( \|\text{proj}_a v\|_{L^2(I \times \Sigma), \epsilon}^2 + \|\text{proj}_a v\|_{L^2(Y_\star), \epsilon}^2 \right).
\end{align}

For $\epsilon$ sufficiently small the derivative terms on $I \times \Sigma$ can be absorbed by the analogous derivative terms in (32). This verifies (28).

It therefore suffices to prove the claim. First note that we have a $C^0$ bound for $\beta_t$, but this is only in terms of the $\epsilon$-independent metric. In fact, in terms of the $\epsilon$-dependent metric, we have

\begin{equation}
\|\beta_t\|_{L^\infty(U), \epsilon} = \epsilon^{-1} \|\beta_t\|_{L^\infty(U)} \leq \epsilon^{-1} \|F_a\|_{L^\infty(Y)} \leq \epsilon^{-1} c_0,
\end{equation}

which obviously cannot be bounded independently of $\epsilon$. However, this pesky $\epsilon^{-1}$ can be absorbed using the additional $\epsilon$ in the estimate (20). To see this, we again use (31).
\[(h[\beta_t, \theta], h\nu)_\epsilon \geq -\frac{\delta}{2} \|h[\beta_t, \theta]\|_\epsilon^2 - \frac{1}{2\delta} \|h\nu\|_\epsilon^2 \]

\[\geq -\delta \|\beta\|_{L^\infty(U), \epsilon}^2 \|h\theta\|_\epsilon^2 - \frac{1}{2\delta} \|h\nu\|_\epsilon^2 \]

\[\geq -\delta \epsilon^2 \epsilon \|h\theta\|_\epsilon^2 - \frac{1}{2} \|h\nu\|_\epsilon^2 \]

\[\geq -\delta C_\gamma \|h\alpha\|_\epsilon^2 - \frac{1}{2} \|h\nu\|_\epsilon^2 ,\]

where \(C_\gamma = C^2\) and \(C\) is the constant from Lemma 3.1. (Note that since the metric on \(Y\) is being conformally scaled at the same rate as the metric on \(\Sigma\), Lemma 3.1 continues to hold on all surface fibers in \(U = \{\delta - 1 + \delta\} \times \Sigma\), and not just those over the unit interval \(I\).) The claim follows by taking \(\delta = 1/3C_\gamma\).

### 3.2. Linearized operators and non-degeneracy.

As usual, for a connection \(\alpha \in A(Q)\) we will write \(a|_{(t) \times \Sigma} = \alpha(t) + \psi(t) dt\) and we direct the reader to Remark 3.3 for the definition of the projections \(\text{proj}_{\alpha}\) and \(\text{proj}_{\alpha}^\perp\). We also remind the reader of the operators \(D_{\epsilon}, D_{0, \alpha}\) and \(D_{\epsilon, \alpha}\) from Sections 2.3 and 2.4.

#### Lemma 3.6.

Suppose \(\alpha \in A(Q)\) is \(\epsilon\)-smooth and is such that \(a|_{Y}\) and \(\alpha(t)\) are flat on \(Y\) and \(\Sigma\), respectively. Then there are constants \(\epsilon_0, C > 0\) such that

\[\|d_\alpha v\|_{L^2(Y), \epsilon} + \|d_\alpha^* v\|_{L^2(Y), \epsilon} + \|d_\alpha r\|_{L^2(Y), \epsilon}\]

\[\leq C(\|D_{\epsilon, \alpha} V\|_{L^2(Y), \epsilon} + \|\text{proj}_{\alpha}\|_{L^2(I \times \Sigma), \epsilon} + \|\text{proj}_{\alpha}^\perp\|_{L^2(Y), \epsilon})\]

for all \(0 < \epsilon < \epsilon_0\) and all \(\epsilon\)-smooth \(V = (v, r) \in \Omega^j(Y, Q(g)) \oplus \Omega^j(\Sigma, Q(g))\). The constants depend only on \(a\) and \(H\) and can be chosen to depend continuously in these variables in the \(C^1\) and \(C^2\)-topology, respectively.

#### Proof.

Throughout we let \(\|\cdot\|_\epsilon\) and \((\cdot, \cdot)_\epsilon\) denote the \(\epsilon\)-dependent \(L^2\)-norm and inner product on the 3-manifold \(Y\). Also, \(*\) will denote the \(\epsilon\)-dependent Hodge star on \(Y\). Set \(\tilde{V} = (\tilde{v}, \tilde{r}) := D_{\epsilon, \alpha} V\).

By definition, these satisfy

\[(33) \quad *_{\epsilon} d_\alpha v - *_{\epsilon} d\tilde{X}_\alpha v - d_\alpha r = \tilde{v}\]

\[(34) \quad -d_\alpha^* v = \tilde{r}\]

Apply \(d_\alpha\) to (33) to get \(d_\alpha d_\alpha^* v = -d_\alpha \tilde{r}\). Now apply \((\cdot, \cdot)_\epsilon\) to both sides and integrate by parts

\[\|d_\alpha^* v\|_\epsilon^2 = - (\tilde{r}, d_\alpha^* v)_\epsilon \leq \|\tilde{r}\|_\epsilon \|d_\alpha^* v\|_\epsilon .\]

Dividing by \(\|d_\alpha^* v\|_\epsilon\) proves the bound for \(d_\alpha^* v\).

The bounds for the remaining terms are similar albeit more involved: Apply \(d_\alpha^*\) to (33)

\[d_\alpha^* d_\alpha r = -d_\alpha^* \tilde{v} - *_{\epsilon} [F_\alpha \wedge v] - d_\alpha^* *_{\epsilon} d\tilde{X}_\alpha v.\]

Apply \((\cdot, r)_\epsilon\) and integrate by parts to get

\[\|d_\alpha r\|_\epsilon^2 = -(\tilde{v}, d_\alpha r)_\epsilon - (*_{\epsilon} [F_\alpha \wedge v], r)_\epsilon - (*_{\epsilon} d\tilde{X}_\alpha v, d_\alpha r)_\epsilon.\]

We can bound the curvature term as follows: Note that \(F_\alpha\) vanishes on \(Y\) and \(F_\alpha|_{I \times \Sigma} = -\beta_t \wedge dt\). Writing \(v|_{I \times \Sigma} = v + \theta dt\), it follows that the only non-vanishing term of \([F_\alpha \wedge v] \wedge dt\) is \([\beta_t \wedge v] \wedge dt\). Then for any \(\delta > 0\) we can use (31) to get
Proof of Corollary 3.8. The condition that a generalized intersection point \( x \in \mathcal{I}_H(L(Q)) \) be non-degenerate is equivalent to (36). Similarly, an \( H \)-flat connection is non-degenerate if \( D_{x,a} \) is injective. The ‘if’ part of the corollary then follows immediately from Proposition 3.7.

The ‘only if’ part follows essentially from the definitions: Write \( x = \Psi([a]) \), and suppose \( \xi \) is in the kernel of \( D_x \). Then \( \xi \) is a vector field along \( x \) with Lagrangian boundary conditions and so can
be represented by an $\epsilon$-smooth 1-form $v \in \Omega^1(Y, Q(g))$. It follows essentially from the definition of $\mathcal{D}_{0,a}$ that $\mathcal{D}_\alpha \xi = 0$ if and only if $\mathcal{D}_{0,a} v = 0$ (see also the proof of Proposition 3.7 below). Moreover, writing $V = (v, 0)$ one can check that $\Omega^1$-component of $\mathcal{D}_\alpha V$ is exactly $\mathcal{D}_{0,a} v$, so $\mathcal{D}_{0,a} V = 0$. By the injectivity of $\mathcal{D}_{0,a}$ it follows that $V = 0$, and hence $\xi = 0$. □

**Proof of Proposition 3.7.** In light of Lemma 3.6 it suffices to show

$$\|\text{proj}_{\alpha} v\|_{L^2(I \times \Sigma^*)} + \epsilon^{-1/2} \|\text{proj}_{\alpha} v\|_{L^2(Y^*)} \leq C \|\mathcal{D}_{c,a} V\|_{L^2(Y), \epsilon}.$$  

It follows from standard harmonic analysis that if $v$ is an $a|Y^*$-harmonic 1-form on $Y^*$ that restricts to zero on the boundary of $Y^*$, then $v = 0$. The next claim provides a quantitative version of this.

**Claim 1:** There is a constant $C_0 > 0$ such that $\|\text{proj}_{\alpha} v\|_{L^2(Y^*)} \leq C_0 \|\text{proj}_{\alpha} v\|_{W^{1,2}(I \times \Sigma^*)}$ for all forms $v$ on $Y$ of Sobolev class $W^{1,2}(I) \to L^2(I)$, and the identities

$$\|\text{proj}_{\alpha} v\|_{L^2(Y^*)} = \epsilon^{-1/2} \|\text{proj}_{\alpha} v\|_{L^2(Y^*)}, \quad \|\text{proj}_{\alpha} v\|_{L^2(I \times \Sigma^*)} = \|\text{proj}_{\alpha} v\|_{L^2(I \times \Sigma^*)},$$

prove the Proposition it suffices to show

$$\|\text{proj}_{\alpha} v\|_{W^{1,2}(I \times \Sigma^*)} \leq C \|\mathcal{D}_{c,a} V\|_{L^2(Y), \epsilon}.$$  

First we sketch the proof of (38): We will use (38) to bound $\|\text{proj}_{\alpha} v\|_{W^{1,2}(I \times \Sigma^*)}$ in terms of $\|\mathcal{D}_{0,a} \text{proj}_{\alpha} v\|_{L^2(I \times \Sigma^*)}$ plus a small error term (this is Claim 2 below). Write $v\big|_{I \times \Sigma^*} = \nu + \theta dt$ (by our conventions $\text{proj}_{\alpha} v = \text{proj}_{\alpha} v^\nu$). By the Hodge isomorphism on surfaces, we can write $\nu = \text{proj}_{\alpha} \nu + d_\alpha \xi + \ast d_\alpha \xi$. Then it follows that

$$\mathcal{D}_{0,a} \text{proj}_{\alpha} v = \text{proj}_{\alpha} \mathcal{D}_{c,a} V - \text{proj}_{\alpha} \omega,$$

where

$$\omega := \ast [\beta_t + \xi] - [\beta_t \wedge \xi] - \ast dX_\alpha + d_\alpha \xi.$$  

and, as we discuss below, $\mathcal{D}_{0,a} = \mathcal{D}_{0,\nu}$. Then $\|\mathcal{D}_{0,a} \text{proj}_{\alpha} v\|_{L^2(I \times \Sigma^*)}$ is controlled by $\|\mathcal{D}_{c,a} V\|_{L^2(I \times \Sigma^*)}$, and $\|\omega\|_{L^2(I \times \Sigma^*), \epsilon}$, and we will see that this last term is small as well. Here are the details.

We begin by translating (38) into a more useful form. As usual, write $a_j (t+1) = a(t+1)$, $a_j (t+1) = \alpha + \theta dt$ and $\alpha_j = a(t)$. Then the $i$th-component of $x(t) \in M = M(P_0) \times \ldots \times M(P_{N-1})$ is $[\alpha(i)]$ if $i$ is odd and $[\alpha(i - t)]$ if $i$ is even, where the brackets denote $G_0(P_i)$-equivalence class. Let $H_\alpha \to I$ denote the bundle $\{H_\alpha\}$. Then there is a canonical identification of sections

$$\Gamma(x^* TM) = \Gamma(H_\alpha)$$

Recall the operator $\mathcal{D}_{0,a} : \Gamma(H_\alpha) \to \Gamma(H_\alpha)$ is given by $\mathcal{D}_{0,a} \eta := \ast \text{proj}_{\alpha} (\nabla \eta - dX_\alpha \eta)$, where $\ast$ is the Hodge star on $\Sigma^*$. This is exactly the operator $\mathcal{D}_{0,a}$ under the identification (40) (we do not need to restrict to forms $\eta$ with Lagrangian boundary conditions for this to make sense).

**Claim 2:** There is a constant $C_1 > 0$ such that

$$\|\text{proj}_{\alpha} v\|_{W^{1,2}(I)} \leq C_1 \left(\|\mathcal{D}_{0,a} \text{proj}_{\alpha} v\|_{L^2(I)} + \|v - \text{proj}_{\alpha} v\|_{W^{1,2}(Y^*)}\right).$$

We will prove Claim 2 below, and proceed with the proof of the rest of the proposition. By Claim 2 we have
There are embeddings $\rho$ of harmonic forms on $Y$ equivalent, and that a harmonic form on $\Sigma$ is bounded by

$$\|\omega\|_{L^2(I \times \Sigma)} \leq C_4 \left( \|d_a \omega\|_{L^2(I \times \Sigma)} + \|d_a^* \omega\|_{L^2(I \times \Sigma)} \right)$$

where in the second inequality we are using (63) and the fact that the $W^{1,2}(Y)$-norm is equivalent to the norm

$$\|\omega\|_{L^2(Y)} + \|d_a \omega\|_{L^2(Y)} + \|d_a^* \omega\|_{L^2(Y)}$$

(see Lemma 5.2). To bound the $\omega$ term in the last line of (41), we note that the forms $\zeta$ and $\xi$ appearing in the definition of $\omega$ are determined uniquely by the equations $d_a^* d_a \xi = d_a^* \nu$ and $d_a^* d_a \xi = d_a \nu$. It follows that

$$\|\omega\|_{L^2(I \times \Sigma)} \leq C_4 \left( \|d_a \nu\|_{L^2(I \times \Sigma)} + \|d_a^* \nu\|_{L^2(I \times \Sigma)} + \|d_a^* \nu\|_{L^2(I \times \Sigma)} + \|d_a \nu\|_{L^2(I \times \Sigma)} \right)$$

which is the bound we want on $\omega$.

Putting this all together shows that $\|\omega\|_{L^2(I \times \Sigma)}$ is bounded by

$$C_5 \left( \|D_{\alpha,a} \nu\|_{L^2(Y), \epsilon} + \epsilon \|d_a \nu\|_{L^2(I \times \Sigma), \epsilon} + \epsilon \|d_a^* \nu\|_{L^2(I \times \Sigma), \epsilon} \right)$$

where in the last two lines we used Proposition 5.4 (assume $\epsilon \leq 1$), and then Lemma 5.6 (assume $\epsilon$ is less than the constant from this lemma). By Claim 1 again, we therefore have

$$\|\omega\|_{L^2(I \times \Sigma)} \leq C_7 \left( \|D_{\alpha,a} \nu\|_{L^2(Y), \epsilon} + \epsilon \|d_a^* \nu\|_{L^2(I \times \Sigma), \epsilon} \right)$$

Taking $\epsilon$ small enough so $\epsilon \leq 1$ proves the proposition.

Now we prove Claim 1. The proof uses the fact that all norms on finite-dimensional spaces are equivalent, and that a harmonic form on $Y$ is zero if and only if it restricts to the zero form on the boundary. More explicitly, we have

$$\|\omega\|_{L^2(Y)} \leq C_8 \|\omega\|_{L^2(Y)} \leq C_9 \sup_{t \in I} \|\omega\|_{L^2(I \times \Sigma(t))}$$

The norm in the ‘$C_8$’ line is the standard norm on the space of $W^{1,2}$ maps from $I$ into the Banach space $L^2(\Sigma)$ (so we are using the $W^{1,2}$-norm on the $I$-variables and the $L^2$-norm on the $\Sigma$-variables), and this inequality holds by the Sobolev embedding $W^{1,2}(I, W) \hookrightarrow C^0(I, W)$ in one-dimension, where $W$ is any Banach space.

Now we prove Claim 2. Consider the spaces

$$H_{a,(0)} := H_{a_0} + H_{a_2} + \cdots + H_{a_{(N-2)(N-1)}}$$

and $H_{a,(1)} := H_{a_2} + H_{a_4} + \cdots + H_{a_{(N-1)}}$ of harmonic forms on $Y$. (These are tangent spaces to the Lagrangians $L_{(0)}$ and $L_{(1)}$, respectively.) There are embeddings $\rho_j : H_{a,(j)} \rightarrow (H_a)_j$, for $j = 0, 1$, given by restricting to the boundary, where $(H_a)_j$ denotes the fiber over $t$ of the bundle $H_a \rightarrow I$. (These are the targets for Lagrangian boundary conditions for sections of $H_a \rightarrow I$.) For $j = 0, 1$, consider the $L^2$-orthogonal projection
of \( H_{a,j} \) onto the orthogonal complement of \( \rho_j(H_{a,(j)}) \). It will be notationally more convenient to take the direct sum of these projections:

\[
\text{proj}_{H_{a}^+} : (H_{a})_0 \oplus (H_{a})_1 \rightarrow (H_{a})_0 \oplus (H_{a})_1.
\]

Note that by Sobolev embedding in 1-dimension, the map

\[
W^{1,2}(H_{a}) \rightarrow L^2(H_{a}) \oplus (H_{a})_0 \oplus (H_{a})_1, \quad \eta \mapsto (D_{0,a} \eta, \text{proj}_{H_{a}^+}(\eta|_{t=0}, \eta|_{t=1}))
\]

is bounded (we take the norm on the harmonic spaces to be the \( L^2 \)-norm).

Subclaim: There is a bound of the form

\[
\|\eta\|_{W^{1,2}(I)} \leq C_{12} \left( \|D_{0,a} \eta\|_{L^2(I)} + \left| \text{proj}_{H_{a}^+}(\eta|, \eta) \right| \right)
\]

for all \( \eta \in W^{1,2}(H_{a}) \).

First note that the operator \( (43) \) is injective by \( (50) \), so to prove the subclaim it suffices to show that this operator has closed image. For future purposes, we present an argument that does not rely on the finite-dimensionality of \((H_{a})_0 \oplus (H_{a})_1\). The only fact about the space \((H_{a})_0 \oplus (H_{a})_1\) that we use is that there is a bounded extension map

\[
E : (H_{a})_0 \oplus (H_{a})_1 \rightarrow W^{1,2}(H_{a})
\]

that is a right inverse to the map that restricts to the boundary. This extension map can be defined as follows: Choose a trivialization \( H_{a} \cong I \times (H_{a})_0 \). The declare \( E(\eta_0, \eta_1) \) to be the section determined by the line segment connecting \( \eta_0, \eta_1 \in (H_{a})_0 \). Clearly this is a \( W^{1,2} \)-section (it is affine linear), and \( E \) is bounded.

To continue the proof of the subclaim, suppose we are given \( \eta \in W^{1,2}(H_{a}) \). Then we set \( L := E(\text{proj}_{H_{a}^+}(\eta|, \eta)) \). It follows that \( \eta - L \) has Lagrangian boundary conditions, so by \( (56) \) we have

\[
\|\eta\|_{W^{1,2}(I)} \leq \|\eta - L\|_{W^{1,2}(I)} + \|L\|_{W^{1,2}(I)} \\
\leq c_0 \|D_{0,a} (\eta - L)\|_{L^2(I)} + \|L\|_{W^{1,2}(I)} \\
\leq c_0 \|D_{0,a} \eta\|_{L^2(I)} + \|D_{0,a} \eta\|_{L^2(I)} + \|D_{0,a} \eta\|_{L^2(I)} + \|D_{0,a} \eta\|_{L^2(I)} + \|L\|_{W^{1,2}(I)}
\]

where \( \| \cdot \|_{op} \) is the operator norm. But \( E \) is bounded, so \( \|L\|_{W^{1,2}(I)} \leq \|E\|_{op} |\text{proj}_{H_{a}^+}(\eta|, \eta)| \), which proves the subclaim.

To complete the proof of Claim 2, it therefore suffices to show

\[
\left| \text{proj}_{H_{a}^+}(\eta|, \eta) \right| \leq C \left\| \eta - \text{proj}_{a|}v \right\|_{W^{1,2}(Y_\star)},
\]

where \( \eta = \text{proj}_{a|}v \). To see this, note that \( \text{proj}_{a|}(v|_{\partial Y_\star}) \) is exactly \( (\eta|_{t=0}, \eta|_{t=1}) \). On the other hand, \( \{\text{proj}_{a|}v\}|_{\partial Y_\star} \) is the Lagrangian part of \( (\eta|_{t=0}, \eta|_{t=1}) \). It follows that

\[
\text{proj}_{H_{a}^+}(\eta|, \eta) = \text{proj}_{a|} \left( \{v - \text{proj}_{a|}v\}|_{\partial Y_\star} \right).
\]

Recalling that the norm on the harmonic space is the \( L^2 \)-norm, we have

\[
\left| \text{proj}_{a|} \left( \{v - \text{proj}_{a|}v\}|_{\partial Y_\star} \right) \right| \leq \left\| \{v - \text{proj}_{a|}v\}|_{\partial Y_\star} \right\|_{L^2(\partial Y_\star)} \leq C_{13} \left\| v - \text{proj}_{a|}v \right\|_{W^{1,2}(Y_\star)}
\]

as desired. In the last line we use that restriction to the boundary extends to a bounded map \( W^{1,2}(Y_\star) \rightarrow L^2(\partial Y_\star) \). \( \square \)
4. The index relation

Fix a principal PU(r)-bundle $Q$ over a broken circle fibration $f : Y \to S^1$ as in Section 2.2 as well as a perturbation $H : \mathcal{A}(Q) \to \mathbb{R}$ as in (17). Suppose that $A_0$ is an $H$-holomorphic curve representative and $A$ is an $\epsilon$-ASD connection, both both converging, as $s \to \pm \infty$, to non-degenerate $H$-flat connections $a^\pm \in \mathcal{A}(Q)$. Recall that $\mathcal{D}_{0,A_0}$ and $\mathcal{D}_{\epsilon,A}$ are the linearizations of the $H$-perturbed holomorphic curve and $\epsilon$-ASD equations, respectively. The non-degeneracy assumption implies that these operators are Fredholm and so their Fredholm indices are well-defined. The goal of this section is to prove that these indices agree.

**Theorem 4.1. (Index Relation)** Suppose the $H$-flat connections $a^\pm$ are non-degenerate. Then

$$\text{Ind} (\mathcal{D}_{0,A_0}) = \text{Ind} (\mathcal{D}_{\epsilon,A}).$$

**Corollary 4.2.** Suppose $H$ has been chosen so all $H$-flat connections are non-degenerate. Then the map $\Psi$ from (18) respects the relative $\mathbb{Z}_4$-gradings: For all $a^\pm \in \mathcal{A}_{\text{flat}}(Q,H)$,

$$\mu_{\text{inst}} \left( [a^-], [a^+] \right) = \mu_{\text{symp}} (\Psi ([a^-]), \Psi ([a^+])) \mod 4.$$

The corollary follows immediately from the index relation. Moreover, the index relation also shows that the index of $\mathcal{D}_{\epsilon,A}$ is independent of $\epsilon$. This is not surprising since the index of a Fredholm operator is unchanged under compact perturbation. More generally, this index is independent of the choice of $A \in \mathcal{A}(\mathbb{R} \times Q)$, so long as $A$ converges to the same $a^\pm$ at $\pm \infty$.

**Theorem 4.1**. The index relation holds for all $\epsilon > 0$ sufficiently small.

In this case the index of each operator is just the dimension of the kernel. In Section 1.3 we construct a linear isomorphism $\mathcal{F} : \ker \mathcal{D}_{0,A_0} \to \ker \mathcal{D}_{\epsilon,A}$ from the space of linearized holomorphic strip representatives to the space of linearized $\epsilon$-ASD forms. The key estimate for defining $\mathcal{F}$ is Corollary 4.2, which we prove using a contradiction argument based on a compactness theorem for linearized ASD forms.

4.1. Elliptic estimates for $\mathcal{D}_{\epsilon,A}$. We continue to use the notation established in Section 2.2. In particular, we identify 1-forms $V = v + r \, ds$ on $\mathbb{R} \times Y$ with tuples $(v, r)$ of (paths of) 1- and 0-forms on $Y$. Set $\bar{V} = (\bar{v}, \bar{r}) := \mathcal{D}_{\epsilon,A} V$. By definition, these satisfy

\begin{align}
\nabla_s v + s_\epsilon d_a v - s_\epsilon d_{\bar{X}_a} v - d_a r &= \bar{v} \\
\nabla_s r - d_a^* v &= \bar{r}.
\end{align}

In what follows it will be convenient to use the following semi-norms for measurable $K \subset \mathbb{R}$:

$$
\| V \|^2_{W(\mathbb{R} \times Y), \epsilon} := \| \text{proj}_{|K} v \|^2_{L^2(\mathbb{R} \times Y), \epsilon} + \| \text{proj}_{|K} d_a v \|^2_{L^2(\mathbb{R} \times Y), \epsilon} + \| \nabla_s v \|^2_{L^2(\mathbb{R} \times Y), \epsilon} + \| d_a r \|^2_{L^2(\mathbb{R} \times Y), \epsilon} + \| d_a^* v \|^2_{L^2(\mathbb{R} \times Y), \epsilon} + \| \nabla_s r \|^2_{L^2(\mathbb{R} \times Y), \epsilon} + \| d_a r \|^2_{L^2(\mathbb{R} \times Y), \epsilon},
\$$

where the maps $\text{proj}_{|K}$ and $\text{proj}_{|K}$ denote the projections from Remark 3.3. The topology defined by these semi-norms is exactly the $W^{1,2}_t$-topology. We set $\| \cdot \|_{W, \epsilon} := \| \cdot \|_{W(\mathbb{R} \times Y), \epsilon}$, which is a norm equivalent to the $W^{1,2}_t$-norm and is induced by an inner product defined in the obvious way.

The following theorem provides an $\epsilon$-independent elliptic estimate. It applies to the operator $\mathcal{D}_{\epsilon,A}$ linearized at any connection $A$ that is sufficiently close to an $H$-holomorphic curve representative. For the purposes of proving the index theorem, it suffices to assume $A$ is a $H$-holomorphic curve representative, but for future applications it will be useful to have a statement for the case when $A$ is an $\epsilon$-instanton.
Theorem 4.3. Suppose $A$ is an $H$-holomorphic curve representative that limits to $H$-flat connections at $\pm \infty$. There are constants $\epsilon_0, C > 0$ such that

$$
\|V\|_{W(\mathbb{R} \times Y), \epsilon} \leq C \left( \|D_c \alpha V\|_{L^2(\mathbb{R} \times Y), \epsilon} + \|\text{proj}_a V\|_{L^2(\mathbb{R} \times Y), \epsilon} + \|\text{proj}_a V\|_{L^2(\mathbb{R} \times I \times \Sigma_a), \epsilon} \right),
$$

for all $0 < \epsilon < \epsilon_0$, and all $\epsilon$-smooth 1-forms $V$. The same result holds with $D_{c,A}$ replaced by $D^*_{c,A}$.

More generally, given any $c_0$, there are $\delta, \epsilon_0, C > 0$ such that the same conclusion holds for any $\epsilon$-smooth connection $A = a + p ds$ on $\mathbb{R} \times X$ satisfying the following:

$$
\|\partial_s a - d_a p\|_{L^\infty(\mathbb{R} \times Y)} + \epsilon^{-1}\|F_a\|_{L^\infty(\mathbb{R} \times Y)} \leq c_0, \quad \text{and} \quad \|F_a\|_{L^\infty(\mathbb{R} \times I \times \Sigma_a)} + \|F_a\|_{L^\infty(\mathbb{R} \times Y)} < \delta.
$$

Proof. We prove the theorem for $D_c \equiv D_{c,A}$. The result for $D^*_c$ is the same except one uses the self-dual equations instead of the anti-self dual equations (44) (equivalently, one can simply reverse the orientation on $Y$ and then follow the proof we give here). Throughout this proof, we use $\| \cdot \|_\epsilon$ and $(\cdot, \cdot)_\epsilon$ to denote the $L^2$-norm and inner product on the 4-manifold $\mathbb{R} \times Y$ defined with respect to the $\epsilon$-dependent metric. However, $\ast_\epsilon$ will denote the Hodge star on the 3-manifold $Y$. For example, if $x, y: \mathbb{R} \to \Omega^2(Y, Q(q))$ are paths of forms on $Y$, then $(x, y)_\epsilon = \int_{\mathbb{R} \times Y} ds \wedge (x(s) \wedge \ast_\epsilon y(s))$. In light of Proposition 3.3, Lemma 3.1 and Lemma 3.2 it suffices to show

$$
\|\nabla_s r\|_\epsilon^2 + \|d_a v\|_\epsilon^2 + \|d_a^* v\|_\epsilon^2 + \|\nabla_s r\|_\epsilon^2 + \|d_a r\|_\epsilon^2 \leq C \left( \|\tilde{V}\|_\epsilon^2 + \|V\|_\epsilon^2 \right).
$$

All constants $C, C_0, C_1, \ldots$ depend only on $H$ and $c_0$.

Bounds for the derivatives of $r$:

Apply $d_a^*$ to (44):

$$
d_a^* d_a r = -d_a^* \tilde{v} + d_a^* \nabla_s v - \ast_\epsilon [F_a \wedge v] + \ast_\epsilon d_a \left( d X_a v \right).
$$

We will be done with the bounds for the derivatives of $r$ if we can bound the last two terms. These are a little subtle, and they will appear several times below, so we will explain in a bit of detail how they can be bounded.

Claim 1: There is some $C_0 > 0$ such that for all $\delta > 0$, $\ast \epsilon d X_a v, d_a r)_\epsilon \leq \delta \|d_a r\|_\epsilon^2 + \frac{C_0}{\delta} \|v\|_\epsilon^2$.

We clearly have $(\ast \epsilon d X_a v, d_a r)_\epsilon \leq \frac{1}{2} \|\ast \epsilon d X_a v\|_\epsilon^2 + \delta \|d_a r\|_\epsilon^2$, for any $\delta > 0$. By definition, the 2-form $\tilde{X}(a) = X_t(a) \wedge dt$ vanishes on $Y_a$. Let $dX_a$ denote the linearization of $X_t$ at $a$. Then these linearizations are related by $dX_a v = (dX_a v) \wedge dt$, where $v = \nu + \theta \wedge dt$. This gives

$$
\ast \epsilon d X_a v = - \ast \epsilon d X_a v,
$$
where $*^\Sigma$ is the induced Hodge star on $\Sigma$. By conformal invariance of 1-forms on 2-manifolds, this star is independent of $\epsilon$, and so $-\star^\Sigma dX_\alpha \nu = -\star^\Sigma dX_\alpha$. For the same reason, the $L^2$-norm is independent of $\epsilon$ and so we have

$$\| \star d\bar{X}_a \nu \|^2_t = \| \star^\Sigma dX_\alpha \nu \|^2_t = \| dX_\alpha(\nu) \|^2_{L^2(\mathbb{R} \times I \times \Sigma)} \leq C_0' \| \nu \|^2_{L^2(\mathbb{R} \times I \times \Sigma)},$$

where $C_0'$ is the square of the operator norm of $dX_\alpha$. It follows from Uhlenbeck compactness that the constant $C_0'$ can be chosen to be independent of all connections $A$ satisfying the hypotheses of the theorem ($dX_\alpha$ depends continuously on $A$ and is gauge equivariant). This proves Claim 1.

**Claim 2:** There is some $C_1 > 0$ such that for all $\delta > 0$, $(\star b_\alpha - F_a, \star [v, r]) \leq \delta \| d_a r \|^2_2 + \frac{C_1}{\delta} \| v \|^2_2$.

This is where we use the bounds on $b_\alpha$ and $F_a$ in (40). First, we have

$$(48) \quad (\star b_\alpha - F_a, \star [v, r]) \leq (\star b_\alpha - F_a, \star [v, r])_{L^2(\mathbb{R} \times Y_\bullet), \epsilon} + (\star b_\alpha - F_a, \star [v, r])_{L^2(\mathbb{R} \times I \times \Sigma)}.$$ 

The first term on the right can be bounded as follows: Setting $S := \mathbb{R} \times Y_\bullet$, we have

$$(\star b_\alpha - F_a, \star [v, r]) \leq \frac{\delta}{\epsilon} \left[ (\| \star b_\alpha \|_{L^\infty(S), \epsilon} + \| F_a \|_{L^\infty(S), \epsilon}) \| [v, r] \|^2_{L^2(S), \epsilon} + \frac{1}{16} \| v \|^2_{L^2(S), \epsilon} \right]$$

$$= \delta \left( \| \star b_\alpha \|^2_{L^\infty(S), \epsilon} + \| F_a \|^2_{L^\infty(S), \epsilon} \right) \| [v, r] \|^2_{L^2(S), \epsilon} + \frac{1}{16} \| v \|^2_{L^2(S), \epsilon}$$

$$\leq \delta \epsilon^{-2} C_0' \| d_a r \|^2_{L^2(S), \epsilon} + \frac{1}{16} \| v \|^2_{L^2(S), \epsilon},$$

where $C_0'$ is the constant $C$ appearing in (23), and the penultimate line holds by (40). The bound for the second term in (48) is similar, and we leave it to the reader (one does not need to assume any additional conditions on the $L^\infty$-norm of $F_a$ over $\mathbb{R} \times I \times \Sigma$ as we did over $\mathbb{R} \times Y_\bullet$ in (40)).

This completes our analysis of (47), and hence the bounds of the derivatives of $r$.

### Bounds for the derivatives of $v$:

Apply $d_a$ to (44):

$$d_a d_a^* v = -d_a \tilde{v} + d_a \nabla_s r$$

$$= -d_a \tilde{v} + \nabla_s d_a r - [b_s, r]$$

$$= -d_a \tilde{v} - \nabla_s \tilde{v} + \nabla_s \tilde{v} + \star \nabla_s d_a v - \star \nabla_s (d\bar{X}_a v) - [b_s, r],$$

where we have used (44) in the third equality. Similarly, apply $\star d_a = d_a^* \star$ to (44),

$$d_a^* d_a v = -d_a^* \tilde{v} + \star \nabla_s d_a v + \star [F_a, r] + d_a^* (d\bar{X}_a v)$$

$$= d_a^* \tilde{v} - \star \nabla_s d_a v + \star [F_a, r] + \star [b_a \wedge v] + d_a^* (d\bar{X}_a v).$$

Add these together (the $\star \nabla_s d_a v$ terms cancel).

$$-\nabla^2_s v + d_a^* d_a v + d_a d_a^* v = -\nabla_s \tilde{v} + d_a^* \star \tilde{v} - d_a \tilde{v} + [\star F_a - b_s, r] + \star [b_a \wedge v] - \star \nabla_s (d\bar{X}_a v) + d_a^* (d\bar{X}_a v).$$

Now apply $(v, \cdot)$ and integrate by parts.
\[ \|\nabla v\|^2 + \|d_a v\|^2 + \|d_{\alpha}^* v\|^2 = \] \[ = (\bar{v}, \nabla_{\bar{v}}) + (\bar{v}, d_{\alpha}^* v) - (\bar{r}, d_{\alpha}^* v) + ([*_{\epsilon} F_a - b_s, r], v) + ([*_{\epsilon} [b_s \wedge v], v) + \] \[ + (\epsilon d_{X_a} v, \nabla_{d_{X_a} v}) + (\epsilon d_{X_a} v, d_a v) \epsilon. \]

Taking \( \delta = 1/2 \) allows us to move the derivative terms to the other side.

\[ \|\nabla v\|^2 + \|d_a v\|^2 + \|d_{\alpha}^* v\|^2 \leq 4C_2 \|\nabla v\|^2 + 2 ([*_{\epsilon} F_a - b_s, r], v) + 2 ([*_{\epsilon} [b_s \wedge v], v) + \] \[ + 2(\epsilon d_{X_a} v, \nabla_{d_{X_a} v}) + 2(d_{X_a} v, d_a v) \epsilon. \]

It remains to bound the last four terms. The analysis from Claims 1 and 2 above applies to all terms except the third term. This third term is addressed by the following claim.

**Claim 3:** For any \( \delta > 0 \) there is a constant \( C_3 > 0 \) such that

\[ \|[*_{\epsilon} [b_s \wedge v], v) \| \leq \delta (\|d_a v\|^2 + \|d_{\alpha}^* v\|^2) + C_3 \|v\|^2 \]

All integrals below are over \( \mathbb{R} \times Y \), unless indicated otherwise. Note that \( ([*_{\epsilon} [b_s \wedge v], v) = \int ds \wedge ([b_s \wedge v] \wedge v) \), and that this term is topological in the sense that the metric does not make an appearance. This gives us the freedom to work with respect to whichever metric we want. Break the integral up over the different pieces of \( Y \):

\[ \int_{\mathbb{R} \times I \times Y} ds \wedge ([b_s \wedge v] \wedge v) = \int_{\mathbb{R} \times I \times Y} ds \wedge ([b_s \wedge v] \wedge v) \]

First we estimate the integral over \( S' := \mathbb{R} \times I \times Y \). Write \( v = \nu + \theta dt \) and \( b_s = \beta_s + \gamma dt \).

\[ \int_{S'} ds \wedge ([b_s \wedge v] \wedge v) = 2 \int_{S'} ds \wedge dt \wedge ([b_s \wedge v], \theta) + \int_{S'} ds \wedge dt \wedge ([\gamma, v] \wedge v) \]

\[ \leq \delta^{-1} 4\|\beta_s\|^2 \|v\|^2 L_2(S') + \|\theta\|^2 \|v\|^2 L_2(S') + 2\|\gamma\| \|v\|^2 L_2(S') \]

\[ \leq \delta^{-1} 4\|\beta_s\|^2 \|v\|^2 L_2(S') + \delta C_4 \|d_\alpha \theta\|^2 L_2(S') + 2\|\gamma\| \|v\|^2 L_2(S') \]

\[ = \delta^{-1} 4\|\beta_s\|^2 \|v\|^2 L_2(S'), + \delta C_4 \|d_\alpha \theta\|^2 L_2(S'), + 2\|\gamma\| \|v\|^2 L_2(S'), \]

where \( C_4 \) is the constant from Lemma 3.1 and \( C_5 \) is the constant from Proposition 3.1. This is the desired bound for the portion of the integral lying over \( \mathbb{R} \times I \times Y \).

Now we seek to bound the portion of the integral over \( S := \mathbb{R} \times Y \). Write \( v|_{Y_\bullet} = \text{proj}_a v + w \), so \( w \) is orthogonal to the kernel of \( d_\alpha \oplus d_{\alpha}^* \). Then

\[ \int_S ds \wedge ([b_s \wedge v] \wedge v) = 2 \int_S ds \wedge ([b_s \wedge w] \wedge \text{proj}_a v) + \int_S ds \wedge ([b_s \wedge w] \wedge w) + \]

To bound the first term, use Lemma 3.2 and the fact that \( w \) has no \( a|_{Y_\bullet} \)-harmonic component:
\[ \int_S ds \wedge \langle [b_s \wedge w] \wedge \text{proj}_a v \rangle \leq \epsilon \delta^{-1} \|a_s\|_{L^\infty}^2 \|\text{proj}_a v\|_{L^2(S)}^2 + \delta \epsilon^{-1} \|w\|_{L^2(S)}^2 \]

\[ \leq \delta^{-1} c_0^2 \|v\|_{L^2(S)}^2 + \delta \epsilon^{-1} C_6 \left( \|d_a v\|_{L^2(S)}^2 + \|d_a^* v\|_{L^2(S)}^2 \right) \]

\[ = \delta^{-1} c_0^2 \|v\|_{L^2(S),\epsilon}^2 + \delta C_6 \left( \|d_a v\|_{L^2(S),\epsilon}^2 + \|d_a^* v\|_{L^2(S),\epsilon}^2 \right), \]

where \( C_6 \) is the constant from Lemma 3.2. The bound for the second term in (49) is similar.

The argument above used the exterior derivative to increase the degree of the form \( w \), thereby pushing it into the range where its norm scales favorably with respect to \( \epsilon \). Unfortunately, this argument breaks down for the last term in (49), since we cannot control the norm of a harmonic form in terms of its derivatives. We use the following alternative approach. First we have

\[ \int_S ds \wedge \langle [b_s \wedge \text{proj}_a v] \wedge \text{proj}_a v \rangle \leq c_0 \|\text{proj}_a v\|_{L^2(S)}^2. \]

Since the harmonic space if finite-dimensional, any two norms are equivalent. In particular, there is a constant \( C_7 \) with \( \|\text{proj}_a v\|_{L^2(S)}^2 \leq C_7 \|\text{proj}_a v\|_{L^2(S),\epsilon}^2 \). Write \( v|_{[\nu \times \Sigma^*]} = \nu + \theta dt \) and note that \( \|\text{proj}_a v\|_{L^2(S),\epsilon}^2 \leq ||v||_{L^2(S),\epsilon}^2 \). It follows from a straightforward contradiction argument that for any \( \delta > 0 \), there is a constant \( C_8 \) such that

\[ \|v\|_{L^2(S),\epsilon}^2 \leq \delta \left( \|\nabla_t v\|_{L^2(S)}^2 + \|d_a v\|_{L^2(S)}^2 + \|d_a^* v\|_{L^2(S)}^2 \right) + C_8 \|v\|_{L^2(S)}^2 \]

(e.g., integrate the bound in [12, Lemma 5.1.3] over \( \nu \), or see the proof of the claim appearing in the proof of Proposition 4.4). This brings us to a realm in which forms scale favorably in \( \epsilon \):

\[ \int_S ds \wedge \langle [b_s \wedge \text{proj}_a v] \wedge \text{proj}_a v \rangle \leq \delta c_0 C_7 \left( \|\nabla_t v\|_{L^2(S),\epsilon}^2 + \epsilon^2 \|d_a v\|_{L^2(S),\epsilon}^2 + \epsilon^2 \|d_a^* v\|_{L^2(S),\epsilon}^2 \right) + c_0 C_7 C_8 \|v\|_{L^2(S),\epsilon}^2. \]

By Proposition 5.3 again we can bound the derivatives in terms of the norms of \( v \), \( d_a v \) and \( d_a^* v \). This proves Claim 3, as well as the theorem.

\[ \square \]

**Proposition 4.4.** Fix a constant \( c_0 > 0 \) and an \( H \)-holomorphic curve representative \( A_0 \) limiting to non-degenerate \( H \)-flat connections \( a^\pm \) at \( \pm \infty \). Assume \( D_{0,A_0} \) is onto when restricted to sections with Lagrangian boundary conditions. Then there are constants \( \delta_0, \epsilon_0, C > 0 \) with the following significance: Suppose \( A \) is a second connection limiting to the same connections \( a^\pm \), satisfying (46) and with the additional assumption that \( \|A - A_0\|_{L^\infty(\Sigma \times Y)} < \delta_0 \). Then for any \( 0 < \epsilon < \epsilon_0 \),

\[ (50) \quad \|V\|_{W(\Sigma \times Y),\epsilon} \leq C \|D_{e,A}^* V\|_{L^2(\Sigma \times Y),\epsilon} \]

for all \( \epsilon \)-smooth \( V \), where \( D_{e,A}^* \) is the formal adjoint of \( D_{e,A} \). In particular, \( D_{e,A} \) is onto.

**Proof.** We prove this by a refinement of the argument given in the proof of Proposition 3.7. The refinement is due to the fact that we want to use norms that yield results that scale favorably in \( \epsilon \) (the \( \nu \)-parameter present in the proposition at hand causes problems when one tries to carry over the argument from Proposition 3.7 (directly)).

Write \( V = v + r ds \), \( D_{0,A_0} = D_{0,A_0,H} \) and \( D_{e,A} = D_{e,A,H} \). All constants depend only on \( A_0, H \) and \( c_0 \). In light of Theorem 4.3 (applied to the operator \( D_{e,A}^* \), together with the identities \( 1\text{-form}_{L^2(\Sigma,\epsilon)} = 1\text{-form}_{L^2(\Sigma)} \) and \( 1\text{-form}_{L^2(Y,\epsilon)} = \epsilon^{1/2} 1\text{-form}_{L^2(Y)} \), it suffices to show that for every \( \delta > 0 \) there is some constant \( C \) with

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\[
\|\text{proj}_{\alpha}\|^2_{L^2(\mathbb{R} \times I \times \Sigma_\star)} + \|\text{proj}_{\alpha}\|^2_{L^2(\mathbb{R} \times Y_\star)} \leq C\|D^*_{\alpha, A} V\|_{L^2(\mathbb{R} \times Y), \epsilon} \delta \left( \|V\|_{W(\mathbb{R} \times Y), \epsilon} + \|\text{proj}_{\alpha}\|^2_{L^2(\mathbb{R} \times Y_\star)} \right).
\]

By the proof of Claim 1 appearing in the proof of Proposition 3.2, there is a bound of the form \(\|\text{proj}_{\alpha}\|^2_{L^2(\mathbb{R} \times Y_\star)} \leq C_0\|\text{proj}_{\alpha}\|^2_{L^\infty(I, L^2(\Sigma_\star))}\). The embedding \(L^\infty(I) \hookrightarrow L^2(I)\) gives a second bound \(\|\text{proj}_{\alpha}\|^2_{L^2(I \times \Sigma_\star)} \leq C_1\|\text{proj}_{\alpha}\|^2_{L^\infty(I, L^2(\Sigma_\star))}\). Integrating these over \(\mathbb{R}\) gives

\[
\|\text{proj}_{\alpha}\|^2_{L^2(\mathbb{R} \times I \times \Sigma_\star)} + \|\text{proj}_{\alpha}\|^2_{L^2(\mathbb{R} \times Y_\star)} \leq (C_0 + C_1)\|\text{proj}_{\alpha}\|^2_{L^\infty(I, L^2(\mathbb{R} \times \Sigma_\star))}.
\]

So we seek to bound the right-hand side. The strategy at this point is to transfer the discussion over to the operator \(D^*_{0, A_0}\), and use the hypothesis that \(D^*_{0, A_0}\) is onto. The first step is to translate from projections with respect to \(A\) to projections with respect to \(A_0\). By [9, Proposition 3.12], the operator norm of \(\text{proj}_{\alpha} - \text{proj}_{\alpha_0}\) is controlled by \(\|\alpha - \alpha_0\|_{L^\infty}\), so we have

\[
\|\text{proj}_{\alpha}\|_{L^\infty(I, L^2(\mathbb{R} \times \Sigma_\star))} \leq \|\text{proj}_{\alpha_0}\|_{L^\infty(I, L^2(\mathbb{R} \times \Sigma_\star))} + C_2\|\alpha - \alpha_0\|_{L^\infty} \|V\|_{L^\infty(I, L^2(\mathbb{R} \times \Sigma_\star))} + C_3\|\alpha - \alpha_0\|_{L^\infty} \|\nabla V\|_{L^2(\mathbb{R} \times Y_\star)}
\]

where the last inequality holds by Proposition 3.4. By taking \(\|\alpha - \alpha_0\|_{L^\infty}\) sufficiently small, the second term in the last line is good, so we need to bound the first term.

By assumption, \(D^*_{0, A_0}\) is onto when restricted to forms with Lagrangian boundary conditions, so it follows that \(D^*_{0, A_0}\) is injective on forms with Lagrangian boundary conditions (see [13, Equation C.2.7]). This will be used to prove the following claim.

**Claim:** There is a constant \(C_5\) such that

\[
\|\text{proj}_{\alpha}\|_{L^\infty(I, L^2(\mathbb{R} \times \Sigma_\star))} \leq C_5 \left( \|D^*_{0, A_0} \text{proj}_{\alpha_0} V\|_{L^2(\mathbb{R} \times I \times \Sigma_\star)} + \|V - \text{proj}_{\alpha_0} V\|_{L^2(\mathbb{R}, W^{1, 2}(Y_\star))} \right)
\]

We defer the proof of the claim for the end. Summarizing the above, we have that \(\|\text{proj}_{\alpha}\|_{L^2(\mathbb{R} \times I \times \Sigma_\star)}\) and \(\|\text{proj}_{\alpha}\|_{L^2(\mathbb{R} \times Y_\star)}\) are bounded by

\[
C_1C_5 \left( \|D^*_{0, A_0} \text{proj}_{\alpha_0} V\|_{L^2(\mathbb{R} \times I \times \Sigma_\star)} + \|V - \text{proj}_{\alpha_0} V\|_{L^2(\mathbb{R}, W^{1, 2}(Y_\star))} \right) + \delta_0 C_1 C_4 \|V\|_{W(\mathbb{R} \times Y), \epsilon}.
\]

where \(\delta_0 \geq \|\alpha - \alpha_0\|_{L^\infty(\mathbb{R} \times I \times \Sigma_\star)}\) can be assumed to be as small as we need. Consequently, the last term is fine, so we focus on the first two, beginning with the second:

\[
\|V - \text{proj}_{\alpha_0} V\|_{L^2(\mathbb{R}, W^{1, 2}(Y_\star))} \leq C_6 \left( \|d_{\alpha_0} V\|_{L^2(\mathbb{R} \times Y_\star)} + \|d^*_{\alpha_0} V\|_{L^2(\mathbb{R} \times Y_\star)} \right)
\]

\[
\leq C_6 \left( \|d_{\alpha_0} V\|_{L^2(\mathbb{R} \times Y_\star)} + \|d^*_{\alpha_0} V\|_{L^2(\mathbb{R} \times Y_\star)}
\right.
\]

\[
+ \|a - a_0 \wedge V\|_{L^2(\mathbb{R} \times Y_\star)} + \|a - a_0 \wedge *V\|_{L^2(\mathbb{R} \times Y_\star)}
\]

\[
\leq C_6 \left( \|d_{\alpha_0} V\|_{L^2(\mathbb{R} \times Y_\star)} + \|d^*_{\alpha_0} V\|_{L^2(\mathbb{R} \times Y_\star)}
\right.
\]

\[
+ 4\|a - a_0\|_{L^\infty}\|V\|_{L^2(\mathbb{R} \times Y_\star)}
\]

\[
\leq (C_6 + C_7) \left( \|d_{\alpha_0} V\|_{L^2(\mathbb{R} \times Y_\star)}, \epsilon + \|d^*_{\alpha_0} V\|_{L^2(\mathbb{R} \times Y_\star)}, \epsilon
\right.
\]

\[
+ 4\|a - a_0\|_{L^\infty}\|\text{proj}_{\alpha_0} V\|_{L^2(\mathbb{R} \times Y_\star)}
\].
In the first and penultimate line we used Lemma 3.3 (with \( \epsilon = 1 \)), in the second line we used \( d_{a_0} = d_a + [a_0 - a \land \cdot] \), and in the last line we used the conformal scaling properties of \( L^2 \)-norms. This shows that \( \| \text{proj}_{a_0} v \|_{L^2(\mathbb{R} \times I \times \Sigma_\ast)} + \| \text{proj}_{a_0} v \|_{L^2(\mathbb{R} \times Y_\ast)} \) is bounded by

\[
C_1 C_5 \| D^\ast_{c,A_0} \text{proj}_{a_0} V \|_{L^2(\mathbb{R} \times I \times \Sigma_\ast)} + \left( \delta_0 + \epsilon^{1/2} \right) C_6 \| V \|_{W(\mathbb{R} \times Y_\ast)} + \delta_0 C_5 \| \text{proj}_{a_0} v \|_{L^2(\mathbb{R} \times Y_\ast)},
\]

where \( \delta_0 \geq \| A - A_0 \|_{L^\infty} \) sufficiently small. So we will be done if we can bound the \( D^\ast_{c,A_0} \)-term in terms of \( D^\ast_{c,A} \). We begin by relating it back to \( D^\ast_{c,A_0} \) using the identity

\[
\text{proj}_{a_0} D^\ast_{c,A_0} V - D^\ast_{c,A_0} \text{proj}_{a_0} V = \text{proj}_{a_0} \varpi,
\]

where

\[
\varpi := [\beta_{a_0} - \beta_{A_0}, \xi] + [\beta_{a_0} + \beta_{A_0}, \xi] + *dX_{a_0}(\ast d_{a_0} \xi)
\]

and we have written \( \beta_{A_0} \) and \( \beta_{A_0} \) for the \( ds \)- and \( dt \)-components of the curvature \( F_{A_0} \), and \( \nu = \text{proj}_{a_0} \nu + d_{a_0} \xi + \ast d_{a_0} \xi \). Note that \( \varpi \) can be bounded as follows:

\[
\| \varpi \|_{L^2(\mathbb{R} \times I \times \Sigma_\ast)} \leq C_9 \left( \| \varpi \|_{L^2(\mathbb{R} \times I \times \Sigma_\ast)} + \| \varpi \|_{L^2(\mathbb{R} \times I \times \Sigma_\ast)} \right)
\]

\[
\leq C_{10} \left( \| \varpi \|_{L^2(\mathbb{R} \times I \times \Sigma_\ast)} + \| \varpi \|_{L^2(\mathbb{R} \times I \times \Sigma_\ast)} \right)
\]

\[
= C_{10} \| \nu - \text{proj}_{a_0} \nu \|_{L^2(\mathbb{R} \times I \times \Sigma_\ast)}
\]

\[
= C_{10} \| \nu - \text{proj}_{a_0} \nu \|_{L^2(\mathbb{R} \times I \times \Sigma_\ast)}
\]

\[
\leq \epsilon^{1/2} C_{11} \left( \| \varpi \|_{L^2(\mathbb{R} \times I \times \Sigma_\ast)} + \| \varpi \|_{L^2(\mathbb{R} \times I \times \Sigma_\ast)} \right)
\]

\[
\leq \epsilon^{1/2} C_{12} \| \varpi \|_{L^2(\mathbb{R} \times I \times \Sigma_\ast)}
\]

The coefficient can be made as small as we want, so to finish the proof of the proposition, it therefore suffices to bound \( \| \text{proj}_{a_0} D^\ast_{c,A_0} V \|_{L^2(\mathbb{R} \times I \times \Sigma_\ast)} \) by \( \| \text{proj}_{a_0} D^\ast_{c,A} V \|_{L^2(\mathbb{R} \times I \times \Sigma_\ast)} \) plus a small error. We leave the details to the reader, but we note that the key point is that the operator norms of the operators \( \text{proj}_{a_1} - \text{proj}_{a_0} \) and \( \text{proj}_{a_0} (D_{c,A} - D_{c,A_0}) \) \( \| R \times I \times \Sigma_\ast \) are controlled by \( \| A - A_0 \|_{L^\infty} \leq \delta_0 \).

Now we prove the claim. If it does not hold then there is a sequence \( v_n : \mathbb{R} \to \Omega^1(Y, Q(g)) \) with

\[
\| \text{proj}_{a_0} v_n \|_{L^\infty(I, L^2(\mathbb{R} \times I \times \Sigma_\ast))} > n \left( \| D^\ast_{c,A_0} \text{proj}_{a_0} v_n \|_{L^2(\mathbb{R} \times I \times \Sigma_\ast)} + \| v_n - \text{proj}_{a_0} v_n \|_{L^2(\mathbb{R}, W^{1,2}(Y_\ast))} \right).
\]

By rescaling, we may suppose \( \| \text{proj}_{a_0} v_n \|_{L^\infty(I, L^2(\mathbb{R} \times I \times \Sigma_\ast))} = 1 \). It follows that

\[
\| D^\ast_{c,A_0} \text{proj}_{a_0} v_n \|_{L^2(\mathbb{R} \times I \times \Sigma_\ast)} + \| v_n - \text{proj}_{a_0} v_n \|_{L^2(\mathbb{R}, W^{1,2}(Y_\ast))} \to 0.
\]

By the elliptic properties of the operator \( D^\ast_{c,A_0} \), there is an a priori bound of the form

\[
\| \eta \|_{W^{1,2}(\mathbb{R} \times I, L^2(\Sigma_\ast))} \leq C \left( \| D^\ast_{c,A_0} \eta \|_{L^2(\mathbb{R} \times I \times \Sigma_\ast)} + \| \eta \|_{L^2(\mathbb{R} \times I \times \Sigma_\ast)} \right)
\]

for all sections \( \eta \) of \( H_{a_0} \to \mathbb{R} \times I \) (this can be proved using the same strategy as Theorem 4.3 except one no longer has to worry about obtaining \( \epsilon \)-independent bounds). It follows that the sequence \( \text{proj}_{a_0} v_n \) is uniformly bounded in \( W^{1,2}(\mathbb{R} \times I, L^2(\Sigma_\ast)) \). This implies that a subsequence converges weakly in \( W^{1,2}(\mathbb{R} \times I, L^2(\Sigma_\ast)) \) to a limiting section \( \eta_{\infty} \) of \( H_{a_0} \to \mathbb{R} \times I \), with \( D^\ast_{c,A_0} \eta_{\infty} = 0 \). Moreover, there is a compact inclusion \( W^{1,2}(\mathbb{R} \times I, L^2(\Sigma_\ast)) \hookrightarrow L^\infty(I, L^2(\mathbb{R} \times \Sigma_\ast)) \) since it can be written as a composition

\[
W^{1,2}(\mathbb{R} \times I, L^2(\Sigma_\ast)) \hookrightarrow W^{1,2}(I, L^2(\mathbb{R} \times \Sigma_\ast)) \hookrightarrow L^\infty(I, L^2(\mathbb{R} \times \Sigma_\ast))
\]

where the last inclusion is compact (even though \( \mathbb{R} \) is not compact) and the first is bounded. So we may assume this subsequence also converges strongly in \( L^\infty(I, L^2(\mathbb{R} \times \Sigma_\ast)) \). It follows

\[
\text{proj}_{a_0} v_n \to \text{proj}_{a_0} v_{\infty}
\]

weakly in \( L^2(\mathbb{R} \times I \times \Sigma_\ast) \), which is the conclusion.
that \( \| \eta_\infty \|_{L^\infty(I_L^2(\mathbb{R}^n \times \Sigma))} = 1 \). In particular, \( \eta_\infty \) is not zero. We will show that \( \eta_\infty \) has Lagrangian boundary conditions. This will be the desired contradiction since we have assumed \( D_{0,A_0}^* \) is injective on sections with Lagrangian boundary conditions.

Let \( H_{a_0} \to \mathbb{R} \times \partial I \) denote the Lagrangian subbundle of \( H_{a_0} \to \mathbb{R} \times \partial I \) given by the \( a_0 \)-th harmonic forms. Note that \( \eta_\infty \) has Lagrangian boundary conditions if and only if

\[
\inf \ell \| \eta_\infty \| - \ell \| L^2(K \times \partial I \times \Sigma) = 0,
\]

for each compact \( K \subset \mathbb{R} \), where the infimum is over all sections \( \ell \) of \( H_{a_0} \). For \( s \in \mathbb{R} \), let \( \ell_n(s) \) be the \( L^2 \)-orthogonal projection of \( \text{proj}_{D_{a_0}}v_n(s,\cdot) \) to the Lagrangian fiber \((H_{a_0})_s\). Then each \( \ell_n \) is a section of \( H_{a_0} \). By passing to a further subsequence, we may suppose the \( \ell_n \) converge strongly in \( L^2 \) on compact subsets to some \( \ell_\infty : \mathbb{R} \to H_{a_0} \). Then we have

\[
\inf \ell \| \eta_\infty \| - \ell \| L^2(K \times \partial I \times \Sigma) \leq \| \eta_\infty \| - \ell \| L^2(K \times \partial I \times \Sigma) \to \lim \| v_n - \text{proj}_{a_0}v_n \| L^2(K \times \partial I \times \Sigma).
\]

By the inclusion \( W^{1,2}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n) \), we also have

\[
\| v_n - \text{proj}_{a_0}v_n \|^2_{L^2(K \times \partial I \times \Sigma)} = \int_K \| v_n - \text{proj}_{a_0}v_n \|^2_{L^2(\partial I \times \Sigma)} \leq C_{13} \int_K \| v_n - \text{proj}_{a_0}v_n \|^2_{W^{1,2}(\mathbb{R}^n)} = \| v_n - \text{proj}_{a_0}v_n \|^2_{L^2(K \times \partial I \times \Sigma)}
\]

and we have already seen that this goes to zero. \( \square \)

The next result says that solutions of \( D_{0,A_0}V = 0 \) are almost solutions of \( D_{0,A_0}V = 0 \) when \( \epsilon \) is small. For simplicity, we assume the operators are both linearized at the same connected \( A_0 = a + p ds \) and that this is an \( H \)-holomorphic curve representative. Recall from \({\text{[15]}}\) that any section of \( H_{a} \to \mathbb{R} \times I \) with Lagrangian boundary conditions (the natural domain for \( D_{0,A} \)) can be canonically embedded into the space of 1-forms on \( \mathbb{R} \times Y \). In terms of this embedding, the natural \( W^{1,2} \)-norm on section of \( H_{a} \to \mathbb{R} \times I \) takes the form

\[
\| V_0 \|_{W^2(\mathbb{R} \times I,0)} := \| \text{proj}_a V \|^2_{L^2(\mathbb{R} \times I \times \Sigma)} + \| \text{proj}_a V \|^2_{L^2(\mathbb{R} \times I \times \Sigma)} + \| \text{proj}_a V \|^2_{L^2(\mathbb{R} \times I \times \Sigma)}
\]

where \( V_0 = v + r ds \). Of course, this definition makes sense for any 1-form \( V_0 \) on \( \mathbb{R} \times Y \), however it is no longer a norm on this larger space (e.g., there is no control over \( r_0 \) or \( v_0 \) for \( Y \)). Lastly we note the obvious bound \( \| V_0 \|_{W^2(\mathbb{R} \times I,0)} \leq \| V_0 \|_{W^2(\mathbb{R} \times Y)} \) for all 1-forms \( V_0 \).

**Lemma 4.5.** Assume \( A_0 \) is an \( H \)-holomorphic curve representative limiting to non-degenerate \( H \)-flat connections. Then there is a constant \( C > 0 \) such that

\[
\| D_{\epsilon,A_0} V_0 \|_{L^2(\Omega \times Y),\epsilon} \leq \epsilon^{1/2} C \| V_0 \|_{W^2(\Omega \times I),\epsilon}
\]

for all \( 0 < \epsilon \leq 1 \), all measurable \( \Omega \subset \mathbb{R} \) and all elements \( V_0 \in \ker D_{0,A_0}^* \) in the kernel.

**Proof.** Set \( D_0 := D_{0,A_0} \) and \( D_\epsilon := D_{\epsilon,A} \). We prove the lemma with \( \Omega = \mathbb{R} \), and the more general case is just a matter of notation. Let \( V_0 \in \ker D_0 \), and we use coordinates \( V_0 = (v,r) \) on \( \mathbb{R} \times Y \) and \( V_0 = (v,\rho,\theta) \) on \( \mathbb{R} \times I \times \Sigma \). Similarly, we write \( A_0 = a + p ds \) and \( A = a + \phi ds + \psi dt \) on \( \mathbb{R} \times I \times \Sigma \). Since \( V_0 \) is holomorphic, we have

\[
D_\epsilon V_0 \big|_{\mathbb{R} \times Y} = \begin{pmatrix} \nabla_s v - d_a r \\ -\nabla_s r \end{pmatrix}, \quad D_\epsilon V_0 \big|_{\mathbb{R} \times I \times \Sigma} = \begin{pmatrix} 0 \\ \nabla_s \rho - \nabla_t \theta \\ \nabla_t \rho + \nabla_s \theta \end{pmatrix}.
\]

This gives
for compact sets $K$ is uniformly bounded for each compact $K$. As usual, set $\alpha$. Proof. Suppose $\parallel A \parallel_{L^2(\mathbb{R} \times Y)}$ for every compact set $A$. (Compactness) Suppose that $D \rightarrow 0$ such that

$$\parallel D \parallel_{L^2(\mathbb{R} \times Y), \epsilon} \leq C \parallel A \parallel_{W(\mathbb{R} \times I), \epsilon} \leq C \parallel A \parallel_{W(\mathbb{R} \times I), \epsilon}. \tag{52}$$

This gives

where the first inequality holds because $\epsilon^2 \leq \epsilon \leq 1$. Since the limiting $H$-flat connections are non-degenerate, the space $\ker D_0$ is finite-dimensional and so any two norms are equivalent. In particular, there is some constant $C_1 > 0$ such that

$$\parallel D \parallel_{L^2(\mathbb{R} \times Y), \epsilon} \leq C_0 C_1 \parallel A \parallel_{W(\mathbb{R} \times I), \epsilon} \leq C_0 C_1 \parallel A \parallel_{W(\mathbb{R} \times I), \epsilon}. \tag{52}$$

4.2. Compactness. The next theorem is a compactness result that says elements in $\ker D_{t, A_0}$ converge to elements in $\ker D_{0, A_0}$, as $\epsilon$ approaches zero. Throughout this section we view elements of $\ker D_{0, A_0}$ as $1$-forms on $\mathbb{R} \times Y$ and $\mathbb{R} \times X$ as in [(19)].

**Theorem 4.6.** (Compactness) Suppose $A_0 = a_0 + p_0$ is an $H$-holomorphic curve representative limiting to non-degenerate $H$-flat connections at $\pm \infty$. Let $(\epsilon_n)_{n \in \mathbb{N}}$ be a sequence of positive numbers that converge to zero, and suppose that for each $n$ there is an $\epsilon_n$-smooth 1-form $V_n$ satisfying:

- $\lim_{n \to \infty} \parallel D_{\epsilon_n, A_0} V_n \parallel_{L^2(\mathbb{R} \times Y), \epsilon_n} = 0$;
- $\sup_n \parallel V_n \parallel_{W(\mathbb{R} \times Y), \epsilon_n} < \infty$;
- $\sup_n \parallel \text{proj}_{A_0} V_n \parallel_{W^{1,2}(K \times L^2(\Sigma_i))} < \infty$, for every compact $K \subset \mathbb{R}$, where $a_0 \parallel_{\mathbb{R} \times I} = a_0 + \psi_0 dt$.

Then there is some $V_\infty \in \ker D_{0, A_0}$ and a subsequence of $(V_n)_{n \in \mathbb{N}}$ (still denoted $(V_n)_{n \in \mathbb{N}}$), such that

$$\parallel \text{proj}_{A_0} V_n - V_\infty \parallel_{L^2(K \times I \times \Sigma_i)} + \parallel \text{proj}_{A_0} V_n - V_\infty \parallel_{L^2(K \times Y)} \to 0$$

for every compact set $K \subset \mathbb{R}$.

**Proof.** As usual, set $A_0 |_{\mathbb{R} \times I} = a_0 + \phi_0 ds + \psi_0 dt$ and $V_n |_{\mathbb{R} \times I} = \nu_n + p_0 ds + \theta_n dt$. We also set $D_0 := D_{0, A_0}$ and $D_\epsilon := D_{\epsilon, A_0}$. Let $v_0 : \mathbb{R} \times I \to M$ be the holomorphic curve associated to $A_0$, and view $\text{proj}_{A_0} \nu_n$ as a section of the pullback bundle $v_0^* TM \to \mathbb{R} \times I$. Recall that the standard fiber-norm on this bundle is exactly the $L^2(\Sigma_i)$-norm on $\text{proj}_{A_0} \nu_n$.

By assumption, we have that

$$\parallel \text{proj}_{A_0} \nu_n \parallel_{W^{1,2}(K \times I, L^2(\Sigma_i))} \leq C K$$

is uniformly bounded for each compact $K \subset \mathbb{R}$. This norm is independent of $\epsilon_n$, so this implies that a subsequence of the $\text{proj}_{A_0} \nu_n$ converges weakly in $W^{1,2}(\mathbb{R} \times I, L^2(\Sigma_i))$ and strongly in $L^2(K \times I \times \Sigma_i)$, for compact subsets $K \subset \mathbb{R}$. Let $\eta_\infty$ denote the limiting section. The strong $L^2$-convergence gives

$$\parallel \text{proj}_{A_0} \nu_n - \eta_\infty \parallel_{L^2(K \times I \times \Sigma_i)} \to 0$$

for compact sets $K \subset \mathbb{R}$. The weak $W^{1,2}$ convergence will be used to prove the following claim.

$$\parallel D \parallel_{L^2(\mathbb{R} \times Y), \epsilon} \leq C \parallel A \parallel_{W(\mathbb{R} \times I), \epsilon} \leq C \parallel A \parallel_{W(\mathbb{R} \times I), \epsilon}. \tag{52}$$
Claim: $\eta_\infty$ is holomorphic $D_0\eta_\infty = 0$.

To see this, write $\nu_n = \text{proj}_{\alpha_0}|\nu_n| + d_{\alpha_0}\zeta_n + *d_{\alpha_0}\xi_n$, where $*$ is the Hodge star on $\Sigma_*$. This gives

$$D_0\text{proj}_{\alpha_0}|\nu_n| = \text{proj}_{\alpha_0}|D_\xi V_n - \text{proj}_{\alpha_0}|\omega_n|,$$

where $\omega_n$ is defined similarly to $\varpi$ in (51). As with $\varpi$, the $L^2$-norm of $\omega_n$ is bounded by $\epsilon_n C\|V_n\|_{W(\mathbb{R} \times Y), \epsilon_n}$ for some constant $C$ depending on $A_0$ and $H$. We therefore have

$$\|D_0\text{proj}_{\alpha_0}|\nu_n|\|_{L^2(\mathbb{R} \times I \times \Sigma_*)} \leq \|D_\xi V_n\|_{L^2(\mathbb{R} \times Y), \epsilon_n} + \epsilon_n C\|V_n\|_{W(\mathbb{R} \times Y), \epsilon_n} \rightarrow 0. \quad (54)$$

To prove the claim it suffices to show that the $L^2$ section $D_0\eta_\infty$ vanishes on each compact set $K$ in the interior $\mathbb{R} \times (0, 1)$. Fix a bump function $h$ equal to 1 on $K$ and with support in $\mathbb{R} \times (0, 1)$. Then by the extremal case of Hölder’s inequality, there is some $L^2$ function $v$ with $\|hD_0\eta_\infty\|_{L^2(\mathbb{R} \times I \times \Sigma_*)} = (hD_0\eta_\infty, v)$, where the parentheses denote the $L^2$-pairing. Then we have

$$\|D_0\eta_\infty\|_{L^2(K \times \Sigma_*)} \leq \|hD_0\eta_\infty\|_{L^2(\mathbb{R} \times I \times \Sigma_*)} = (hD_0\eta_\infty, v) = (\eta_\infty, D_0^* (hv)).$$

This last term is exactly the duality pairing of $D_0^* (hv)$ in $W^{-1,2}$ with $\eta_\infty \in W^{1,2}$. Using the weak $W^{1,2}$ convergence of the $\text{proj}_{\alpha_0}\nu_n$ to $\eta_\infty$, this gives

$$\|D_0\eta_\infty\|_{L^2(K \times \Sigma_*)} \leq \lim_{n \rightarrow \infty} \|\text{proj}_{\alpha_0}\nu_n\|_{L^2(\mathbb{R} \times I \times \Sigma_*)} \leq \lim_{n \rightarrow \infty} \|D_0\text{proj}_{\alpha_0}\nu_n\|_{L^2(\mathbb{R} \times I \times \Sigma_*)} \rightarrow 0.$$

This limit is zero by (54), so we have proved the claim.

Our next goal is to show that $\eta_\infty$ has Lagrangian boundary conditions. The idea is that, for any $s \in \mathbb{R}$, the form $\text{proj}_{\alpha_0}|\nu_n(s)|_{\partial Y_*}$ (which lies in the Lagrangian subbundle by definition) can be made arbitrarily close to $(\eta_\infty(s, 0), \eta_\infty(s, 1))$. Since the Lagrangian subbundle is closed, we must have that $(\eta_\infty(s, 0), \eta_\infty(s, 1))$ lies in the Lagrangian subbundle as well. Strictly speaking, it may not be clear what this means since $\eta_\infty$ is only $W^{1,2}$ and so its pointwise evaluation may not make sense. However, we can make this precise as follows: First note that restriction to the boundary extends to a compact map $W^{1,2}(K \times I, L^2(\Sigma_*)) \rightarrow L^2(\partial K \times I \times \Sigma_*)$ for compact $K \subset \mathbb{R}$. By (53), this implies that

$$\|\text{proj}_{\alpha_0}\nu_n - \eta_\infty\|_{L^2(K \times \partial (I \times \Sigma_*)}) \rightarrow 0, \quad (55)$$

for compact $K \subset \mathbb{R}$, after possibly passing to a further subsequence. We claim that it suffices to show that

$$\left| \left\{ \text{proj}_{\alpha_0}\nu_n \right\}_{\partial (I \times \Sigma_*)} - \left\{ \text{proj}_{\alpha_0}\nu_n \right\}_{\partial Y_*} \right|_{L^2(\partial (I \times \Sigma_*)}) \rightarrow 0 \quad (56)$$

pointwise for each $s \in \mathbb{R}$; note that this restriction makes sense because the forms $\nu_n$ and $v_n$ are continuous. Once we have shown (56), it then follows from the triangle inequality applied to (55) and (56) that $\eta_\infty$ has Lagrangian boundary conditions almost everywhere, since the paths of Lagrangian vectors $s \mapsto \{\text{proj}_{\alpha_0}(v_n(s))\}_{\partial Y_*}$ are converging in $L^2$ to the path $s \mapsto \eta_\infty(0, \cdot)|_{\partial Y}$, and the Lagrangian tangent bundle is closed. By Lemmas B.4.6 and B.4.9 in [K], elliptic regularity holds for holomorphic curves with such boundary conditions and so $\eta_\infty$ has Lagrangian boundary conditions in the strong sense, as desired.

To prove we have Lagrangian boundary conditions, it therefore suffices to prove (56). Write $v_n|_{Y_*} = \text{proj}_{\alpha_0}|v_n| + d_{\alpha_0}x + *d_{\alpha_0}y$. Then

$$\left( \text{proj}_{\alpha_0}|\nu_n\right)|_{\partial (I \times \Sigma_*)} - \left( \text{proj}_{\alpha_0}|\nu_n\right)|_{\partial Y_*} = \text{proj}_{\alpha_0}( *d_{\alpha_0}y|_{\partial Y_*}).$$
These are vectors in the finite-dimensional $\alpha_0$-harmonic space, so any two norms are equivalent.

In particular, there is some $C_0 > 0$ such that

$$\|\eta\|_{L^2(\partial(I \times \Sigma_\bullet))} \leq C_0 \inf_{\tilde{x}} \|\tilde{x}\|_{W^{1,2}(\Sigma_\bullet)},$$

for all $\eta$ in the $\alpha_0$-harmonic space over $\partial(I \times \Sigma_\bullet)$, and where the infimum is taken over all smooth forms $\tilde{x}$ on $\Sigma_\bullet$ that restrict to $\eta$ on the boundary. This gives

$$\left\| \text{proj}_{\alpha_0} v_n - \text{proj}_{\alpha_0} v_n \right\|_{L^2(\partial(I \times \Sigma_\bullet))} = \|d_{\alpha_0} y\|_{W^{1,2}(\Sigma_\bullet)} \leq C_0 \|v_n - \text{proj}_{\alpha_0} v_n\|_{W^{1,2}(\Sigma_\bullet)}$$

$$= C_0 \left( \|d_{\alpha_0} v_n\|_{L^2(\Sigma_\bullet)} + \|d_{\alpha_0} v_n\|_{L^2(\Sigma_\bullet)} \right) = C_0 \left( \|d_{\alpha_0} v_n\|_{L^2(\Sigma_\bullet)} + \|d_{\alpha_0} v_n\|_{L^2(\Sigma_\bullet)} \right),$$

where we used Lemma 3.2 in the last line. Integrate this over your favorite compact set $K \subset \mathbb{R}$.

Now we use Theorem 4.3 to estimate the derivatives, obtaining

$$\left\| \text{proj}_{\alpha_0} v_n - \text{proj}_{\alpha_0} v_n \right\|_{L^2(K \times \partial(I \times \Sigma_\bullet))} \leq C_0 \left( \|D_{\alpha_0} y\|_{L^2(\Sigma_\bullet)} + \|v_n\|_{L^2(\Sigma_\bullet)} \right).$$

By the assumptions on $V_n$, the right-hand side is going to zero, which proves (56).

To wrap this up, we have just shown that $\eta_\infty$ has Lagrangian boundary conditions. This implies that there is some $\alpha_0$-harmonic form $h_\infty$ on $\Sigma_\bullet$ that agrees with $\eta_\infty$ at the seam. Then the data $\eta_\infty, h_\infty$ define a unique form $V_\infty$ in the kernel of $D_\alpha$. Note that the above computation also shows that $\text{proj}_{\alpha_0} v_n |_{\Sigma_\bullet} - h_\infty$ is converging to zero in $L^2(K \times \Sigma_\bullet)$, which therefore completes the proof of the theorem.

As an application of the compactness theorem above, we prove that $D_{\varepsilon,A_0}$ has a uniformly bounded right inverse when restricted to the image of the formal adjoint $D_{\varepsilon,A_0}^*$. 

**Corollary 4.7.** Let $A_0$ be an $H$-holomorphic curve representative limiting to non-degenerate $H$-flat connections, and assume $D_{0,A_0}$ is onto when restricted to sections with Lagrangian boundary conditions. Then there are constants $\varepsilon_0, C > 0$ such that

$$\|V\|_{W(\mathbb{R} \times Y), \varepsilon} \leq C\|D_{\varepsilon,A_0} V\|_{L^2(\mathbb{R} \times Y), \varepsilon}$$

for all $0 < \varepsilon < \varepsilon_0$ and $\varepsilon$-smooth $V$ in the image of $D_{\varepsilon,A_0}^*$. 

**Proof.** Write $D_\varepsilon := D_{\varepsilon,A_0}$ and $D_0 := D_{0,A_0}$. We will carry out a contradiction argument similar in spirit to the one in the compactness theorem above: If the corollary does not hold, then there is a sequence of positive numbers $\varepsilon_n \to 0$, and for each $n$ an $\varepsilon_n$-smooth form $V_n$ with $\|D_{\varepsilon_n}^* V_n\|_{W(\mathbb{R} \times Y), \varepsilon_n} = 1$, but

$$\|D_{\varepsilon_n} D_{\varepsilon_n}^* V_n\|_{L^2(\mathbb{R} \times Y), \varepsilon_n} \to 0.$$

By the compactness theorem applied to the sequence $D_{\varepsilon_n}^* V_n$, there is a limiting connection $V_\infty \in \ker D_0$ (with Lagrangian boundary conditions) such that $\text{proj}_{\alpha_0} D_{\varepsilon_n}^* V_n$ converges to $\text{proj}_{\alpha_0} V_\infty$ in $L^2$ on compact subsets of $\mathbb{R} \times I \times \Sigma_\bullet$. We claim that $\text{proj}_{\alpha_0} V_\infty$ lies in the image of $D_0$ (where we are considering $D_0$ as being restricted to the space of sections with Lagrangian boundary conditions). Consider the identity

$$\text{proj}_{\alpha_0} D_{\varepsilon_n}^* V_n - D_0^* \text{proj}_{\alpha_0} V_n = \text{proj}_{\alpha_0} \omega_n$$

where $\omega_n$ is as defined in (51) using the coordinates of $V_n$ in place of those of $V$. There we showed that $\|\omega_n\|_{L^2}$ goes to zero at the rate of $\varepsilon_n$, so this implies that
lies in the image of $D_0$, since this image is closed. If we can show that the sequence $\text{proj}_{\alpha_0}V_n$ converges to a form with Lagrangian boundary conditions, then we will have proved the claim. This follows from Proposition 4.4, which says

$$\|V_n\|_{W(\mathbb{R}^2 Y),\epsilon_n} \leq C\|D^*_{\epsilon_n}V_n\|_{L^2(\mathbb{R}^2 Y)},\epsilon_n \leq C.$$ 

Just as in the proof of the compactness theorem above, this implies that $\text{proj}_{\alpha_0}V_n$ is uniformly bounded in $W^{1,2}(\mathbb{R} \times I, L^2(\Sigma_*))$, and so a subsequence converges in $L^2$ to a form with Lagrangian boundary conditions, as desired.

At this point we have shown $\text{proj}_{\alpha_0}V_\infty$ lies in the kernel of $D_0$ (restricted to sections with Lagrangian boundary conditions), and lies in the image of $D_0^*$ (restricted to sections with Lagrangian boundary conditions). However, these two spaces are complementary (see [13 Equation C.2.7]) and so $\text{proj}_{\alpha_0}V_\infty = 0$. On the other hand, by Theorem 4.3 we have

$$0 \neq \lim_{n \to \infty} \|\text{proj}_{\alpha_0}D^*_{\epsilon_n}V_n\|_{L^2} = \|\text{proj}_{\alpha_0}V_\infty\|_{L^2},$$

which contradicts $\text{proj}_{\alpha_0}V_\infty = 0$.

4.3. Proof of the index relation. Here we prove Theorem 4.1. Let $H$ be a perturbation as in (17), and $A_0$ be an $H$-holomorphic curve representative limiting to non-degenerate $H$-flat connections at $\pm\infty$. Set $D_\epsilon := D_{\epsilon, A_0}$ and $D_0 := D_{0, A_0}$, and we assume $H$ has been chosen so that $D_{0, A_0}$ is onto, which is always possible. It follows from Proposition 4.4 that $D_\epsilon$ is onto when $\epsilon$ is sufficiently small. To prove the theorem, it suffices to show that there is an isomorphism $\ker D_0 \cong \ker D_\epsilon$.

We recall the norm $\|\cdot\|_{W,\epsilon} := \|\cdot\|_{W(\mathbb{R}^2 Y),\epsilon}$ defined above Theorem 4.3. We will also use $\|\cdot\|_{L,\epsilon} := \|\cdot\|_{L^2(\mathbb{R}^2 Y),\epsilon}$ to denote the usual $L^2$-norm defined with respect to the $\epsilon$-dependent metric. We denote by $L_\epsilon$ (resp. $W_\epsilon$) the completion of the space of compactly supported $\epsilon$-smooth $Q(g)$-valued 1-forms on $\mathbb{R} \times Y$ with respect to $\|\cdot\|_{L,\epsilon}$ (resp. $\|\cdot\|_{W,\epsilon}$). Then $D_\epsilon$ extends to a map of the form $D_\epsilon : W_\epsilon \to L_\epsilon$. Let $D_\epsilon^*$ denote the formal adjoint of $D_\epsilon$.

By Corollary 4.7, it follows that $D_\epsilon : W_\epsilon \to L_\epsilon$ restricts to a Banach space isomorphism of the form $D_\epsilon| : im D_\epsilon^* \to L_\epsilon$. Let $Q_\epsilon = (D_\epsilon|)^{-1}$ denote the inverse of this restriction. Then there is some constant $C_0 > 0$ (independent of $\epsilon$) with

$$\|Q_\epsilon V\|_{W,\epsilon} \leq C_0\|V\|_{L,\epsilon}$$

for all $V$ in the image of $D_\epsilon^*$. Note that $Q_\epsilon$ is a right inverse to $D_\epsilon$. Then we can define a bounded linear map

$$W_\epsilon \to W_\epsilon, \quad V \mapsto V - Q_\epsilon D_\epsilon V,$$

which has image $\ker D_\epsilon$. Identify $\ker D_0$ with its image in $W_\epsilon$ under the embedding (16). Then restricting (59) gives a map $F_\epsilon : ker D_0 \to ker D_\epsilon$. Our goal is to show this is an isomorphism.

For injectivity, suppose $F_\epsilon(V_0) = 0$ for some $V_0 \in ker D_0$. Then $V_0 = Q_\epsilon D_\epsilon V_0$ and so

$$V_0 = 0,$$

as desired.

For surjectivity, suppose $V_\epsilon \in ker D_\epsilon$. Then $V_\epsilon = Q_\epsilon D_\epsilon V_\epsilon$, so $V_\epsilon = Q_\epsilon D_\epsilon Q_\epsilon V_\epsilon = Q_\epsilon D_\epsilon D_\epsilon Q_\epsilon V_\epsilon$. Hence $V_\epsilon \in ker D_0$. Since $\ker D_0$ is an isomorphism, we have

$$V_\epsilon = Q_\epsilon D_\epsilon Q_\epsilon V_\epsilon,$$

as desired.
\[ \|V_0\|_{W,0} \leq \|V_0\|_{W,\epsilon} = \|Q, D_\epsilon V_0\|_{W,\epsilon} \leq C_0 \|D_\epsilon V_0\|_{L,\epsilon} \leq \epsilon^{1/2} C_1 \|V_0\|_{W,0}, \]

where the last two lines follow by (69) and Lemma 4.5 respectively. This implies \( V_0 = 0 \) when \( \epsilon^{1/2} C_1 < 1 \), since \( \|\cdot\|_{W,0} \) is a (non-degenerate) norm on \( \ker D_0 \).

For surjectivity, suppose \( F_\epsilon \) is not onto regardless of how small we take \( \epsilon \). Then for each \( n \) we can find a positive number \( \epsilon_n \), and a vector \( v_n \in W_{\epsilon_n} \) with \( \epsilon_n \to 0 \) and \( D_{\epsilon_n} v_n = 0 \), but \( v_n \notin \text{im} F_{\epsilon_n} \). We may assume \( \|v_n\|_{W,\epsilon_n} = 1 \) and \( v_n \) is orthogonal to the image of \( F_{\epsilon_n} \), where we are using the \( W_{\epsilon_n} \)-inner product to measure orthogonality. Then by the Compactness Theorem 4.6 there is a subsequence of the \( v_n \) (still denoted \( v_n \)), and a limiting connection \( V_\infty \in \ker D_0 \) such that

\[ \|V_n - V_\infty\|_{W,\epsilon_n} \to 0. \]

Then since \( v_n \) is orthogonal to the image of \( F_{\epsilon_n} \), we have

\[ 1 = \|v_n\|_{W,\epsilon_n} \leq \|v_n - F_{\epsilon_n} V_\infty\|_{W,\epsilon_n} \leq \|v_n - V_\infty\|_{W,\epsilon_n} + \|Q, D_{\epsilon_n} V_\infty\|_{W,\epsilon_n} \leq \|v_n - V_\infty\|_{W,\epsilon_n} + C_0 \|D_{\epsilon_n} V_\infty\|_{L,\epsilon_n} \leq \|v_n - V_\infty\|_{W,\epsilon_n} + \epsilon_n^{1/2} C_2 \|V_\infty\|_{W,\epsilon_n}. \]

The last two lines follow by (69) and Lemma 4.5 respectively. The right-hand side is going to zero, so this contradiction proves the theorem.

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