Abstract

The congestion of a curve is a measure of how much it zigzags around locally. More precisely, a curve $\pi$ is $c$-packed if the length of the curve lying inside any ball is at most $c$ times the radius of the ball, and its congestion is the maximum $c$ for which $\pi$ is $c$-packed. This paper presents a randomized $(288 + \varepsilon)$-approximation algorithm for computing the congestion of a curve (or any set of segments in constant dimension). It runs in $O(n \log^2 n)$ time and succeeds with high probability. Although the approximation factor is large, the running time improves over the previous fastest constant approximation algorithm [GSW20], which took $\tilde{O}(n^{4/3})$ time. We carefully combine new ideas with known techniques to obtain our new, near-linear time algorithm.

1. Introduction

In 2010, Driemel et al. [DHW12] provided a measurement of how “realistic” a curve is (there are several alternative definitions – see [DHW12] and references therein for details). Formally, a curve $\pi$ is $c$-packed if the total length of $\pi$ inside any ball is bounded by $c$ times the radius of the ball. The minimum $c$ for which the curve is $c$-packed is the congestion of the curve. Intuitively, if a curve has high congestion, then it zigzags back and forth around some locality. We expect that most real-world curves do not behave so pathologically – instead, we expect them to exhibit low congestion.

Curves with low congestion lend themselves to efficient algorithms. Notably, they can be efficiently approximated by simpler curves which nearly preserve Fréchet distances, and therefore Fréchet distances between them can be approximated in near-linear time [DHW12]. In general, if the congestion is $\Omega(n)$, then computing the Fréchet distance is more difficult. Indeed, assuming the Strong Exponential Time Hypothesis (SETH), even approximating Fréchet distance within a constant factor requires quadratic time [Bri14]. The decision version of the problem is also conjectured to be 3SUM-hard [Alt09]. Note that proving a direct connection between 3SUM-hardness and SETH is still an open problem [Wil15].

We would like to verify that a given curve is indeed $c$-packed for a low value of $c$. (Some algorithms for $c$-packed curves do not require knowing the value of the congestion – rather, their analyses show that if the curve is $c$-packed for some small $c$, then the algorithms run in near-linear time. However, verifying that curves are $c$-packed would increase our confidence that these algorithms are generally applicable.) This leads to the question of how quickly one can estimate or compute the congestion. In this paper, we present a constant-factor approximation algorithm for the congestion that runs in near-linear time.

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Previous work. As mentioned earlier, the concept of c-packedness was introduced in Driemel et al. [DHW12]. Computing the congestion exactly runs into the issue of minimizing sums of square roots. As such, an exact algorithm in the standard RAM model is unlikely. The work of Vigneron [Vig14] provided a \((1 + \varepsilon)\)-approximation algorithm which runs in \(O\left(\left(\frac{n}{\varepsilon}\right)^{d+2} \log^{d+2} \left(\frac{n}{\varepsilon}\right)\right)\) time. Gudmundson et al. offered a cubic-time algorithm for this problem [GKS13]. Aghamolaei et al. [AKGM20] gave a \((2 + \varepsilon)\)-approximation for the problem of approximating the congestion of planar curves, but unfortunately their running time seems to be at least quadratic.

Some of the previous work [GKS13] was also concerned with computing hotspots, which are small regions in the plane where many segments pass through. Computing regions of high congestion naturally leads to finding hotspots. However, one usually fixes the resolution of the desired hotspots (i.e. the radius of the balls being intersected) before searching them.

More recently, Gudmundsson et al. [GSW20] gave a \((6 + \varepsilon)\)-approximation in \(\tilde{O}(n^{4/3}/\varepsilon^4)\) time. As part of their algorithm, they showed that one can quickly find a set of \(O(n)\) squares whose congestions yield a constant approximation to the congestion of the curve. They also noted that the problem of finding the congestion of these squares with respect to the segments seems quite similar to the Hopcroft problem, as discussed below.

The Hopcroft problem. Given a set \(P\) of \(n\) points in the plane, and a set \(L\) of \(n\) lines in the plane, the question is whether there is a point of \(P\) that is incident to one of the lines of \(L\).

The challenge. There is an \(\Omega(n^{4/3})\) lower bound for Hopcroft’s problem due to Erickson [Eri95]. If we replace a point by a short segment, then deciding the packedness for sufficiently small squares centered at these points is equivalent to solving the Hopcroft problem. For this reason, Gudmunsson et al. [GSW20] deemed it unlikely that their approach, “or a similar approach, can lead to a considerably faster algorithm” for computing congestion.

Our result. Despite this difficulty, we present a randomized \(O(n \log^2 n)\) time algorithm that provides a constant approximation to the congestion of a set of segments in \(\mathbb{R}^d\). The algorithm bypasses the barrier presented by Hopcroft’s problem by observing that, when computing congestion, the generated instances for the Hopcroft problem have high congestion. In such scenarios, we do not need to compute the congestion of \(\Omega(n)\) disjoint cubes exactly.

Sketch of algorithm. The congestion of a square with respect to a curve \(\pi\) is the total length of \(\pi\) inside it, divided by its side length. Following [GSW20], we reduce the problem of approximating the congestion of a curve to that of computing the congestion of \(O(n)\) squares. Then we build a quadtree whose cells approximate these squares, so that it suffices to compute the congestion of the cells.

We can compute the congestion of a quadtree cell by finding all the segments of the curve which intersect it, then explicitly computing the total length of the intersections. However, naively searching for all the segments intersecting each of the squares takes \(O(n^2)\) time. To speed things up, we store each segment at some quadtree cell with comparable length. For a given cell, its short segments are those stored at descendants in the quadtree, and its long segments are those stored at ancestors. To find the congestion of a cell, we compute the sum of its long and short congestions – the congestions with respect to all of its long and short segments, respectively. We can quickly compute the short congestion of a cell by summing the lengths of short segments in the quadtree bottom-up.

Exactly computing the long congestion can be tricky. Fortunately, approximating the long congestion only requires counting the maximum number of long segments intersecting any cell. If every cell intersects
only a few long segments, then we can quickly enumerate all the intersections by searching the quadtree top-down. If there is a cell intersecting many long segments, then the algorithm performs an exponential search to guess how many intersections it has. The algorithm quickly verifies its guess by taking a random sample of input segments, enumerating its intersections with each quadtree cell, and using the counts to estimate the maximum number of intersections.

**Highlight.** This work breaks what was previously believed to be a lower bound on the running time for approximating congestion (i.e., \(\Omega(n^{4/3})\)). While most of the tools we use are standard, the way we combine them is highly non-trivial and offers new insights into the problem.

Since we lose constant factors in several places, the resulting approximation factor is quite bad compared to previous work. However, our algorithm runs in near-linear time, which is significantly faster. We find this result surprising, as it bypasses the lower bound for the closely related Hopcroft problem mentioned above.

**Paper organization.** We start in Section 2 by providing some definitions and background. In Section 3, we reduce the problem of computing the congestion to that of computing the congestions of a constant number of quadtrees. This in turn reduces to the problem of computing the congestion from long and short segments. The short congestion is handled in Section 3.3, while the main challenge of approximating the long congestion is addressed in Section 4. In Section 5, we put everything together. We conclude in Section 6 with a few remarks.

## 2. Preliminaries

For simplicity of exposition, we define the packedness for squares. This changes the value only by a constant factor.

**Definition 2.1.** Let \(\Box = \Box(p, r)\) denote the axis-parallel square centered at \(p\) with **side length** \(2r\). The square \(\Box\) can be interpreted as a ball in the \(L_\infty\) norm, and as such, its **radius** is \(r\).

**Definition 2.2.** For a segment \(s\), let \(\|s\|\) denote the length of \(s\). Similarly, for a set of segments \(S\), let \(\|S\| = \sum_{s \in S} \|s\|\) denote the total length of segments in \(S\).

For a square \(\Box = \Box(p, r)\), the conflict list of \(\Box\) (for a set \(S\) of segments) is the set of segments intersecting \(\Box\), that is

\[S \cap \Box = \{s \in S | s \cap \Box \neq \emptyset\}\,.

Let

\[S \cap \Box = \{s \cap \Box | s \in S \cap \Box\}\]

be the clipping of the segments of \(S \cap \Box\) to \(\Box\). The **congestion** of the square \(\Box\), with respect to \(S\), is \(c(\Box) = \|S \cap \Box\| / r\). The congestion of the segments \(S\) is \(c(S) = \max_{p, r} c(\Box(p, r))\).

Given a set of squares \(\Xi\), its **congestion** is \(c(\Xi, S) = \max_{\Box \in \Xi} c(\Box)\).

**Definition 2.3.** For a constant \(c > 0\), a set \(S\) of segments in \(\mathbb{R}^d\) is **\(c\)-packed** if, for any point \(p \in \mathbb{R}^d\) and any value \(r > 0\), the total length of the segments of \(S\) inside a square \(\Box = \Box(p, r)\) is at most \(cr\). That is, the congestion of \(S\) is at most \(c\).
Thus, the congestion of $S$ is the minimum $c$ for which $S$ is $c$-packed. We are interested in approximating $c(S)$. To this end, we follow Gudmundsson et al. [GSW20], who reduced the problem to querying the lengths of intersections between the curve and some squares. While Gudmundsson et al. state their result for a curve, it holds for any set of segments.

**Lemma 2.4 (Lemma 12 in [GSW20]).** Given a set $S$ of $n$ segments in the plane, and a parameter $\varepsilon \in (0, 1)$, one can compute, in $O(n \log n + n/\varepsilon^2)$ time, a set $G_S$ of $O(n/\varepsilon^2)$ axis-aligned squares, such that $c(S) \geq c(G_S, S) \geq c(S)/(6 + \varepsilon)$.

In what follows, let $G_S$ be the set of squares computed by Lemma 2.4 for $S$ (the value of $\varepsilon$ is determined later). To approximate $c(S)$, it suffices to approximate $c(S, G_S)$.

### 3. The algorithm: The long and short of it

#### 3.1. Reduction to quadtrees

**Definition 3.1.** For a real positive number $\tau$, let $G_\tau$ be the grid partitioning the plane into axis-parallel squares of side length $\tau$, with the two axes serving as separating lines in this grid. The number $\tau$ is the **width** or **side length** of $G_\tau$. A $\tau \times \tau$ square in this grid is a cell. Formally, for any $i, j \in \mathbb{Z}$, the grid cell at position $(i, j)$ is the intersection of the halfplanes $x \geq \tau i, x < \tau (i + 1), y \geq \tau j,$ and $y < \tau (j + 1)$.

We can (conceptually) construct an infinite quadtree over the unit square. Starting with $[0,1)^2$ as the root node, we partition each node into four congruent squares, which serve as its child nodes. This process continues infinitely onward, with each level of the quadtree containing smaller squares.

**Definition 3.2.** A square is a canonical square if it is contained inside the unit square, it is a cell in a grid $G_w$, and $w$ is a power of two. That is, the square corresponds to a node in the infinite quadtree defined over $[0,1)^2$. The grid generating a canonical square is a canonical grid.

We can shift the plane three times and perform this construction, yielding a collection of shifted quadtrees with the following property. For any segment in $[0,1)^2$, one of the shifted quadtrees has a cell that contains it and whose diameter is not much longer than the segment.

**Lemma 3.3 ([Cha98, CHJ19]).** Consider any two points $p, q \in [0,1)^d$, and let $T$ be the infinite quadtree of $[0,1)^d$. For $D = 2^\lceil d/2 \rceil$ and $i = 0, \ldots, D$, let $v_i = (i/(D + 1), \ldots, i/(D + 1))$. Then there exists an $i \in \{0, \ldots, D\}$, such that $p + v_i$ and $q + v_i$ are contained in a cell of $T$ with side length $\leq 2(D + 1)\|p - q\|$.

We can use this lemma to replace $G_S$ with the cells of three quadtrees. Then it is enough to compute the congestion for these quadtree cells.

**Definition 3.4.** For a finite quadtree $T$, let $c(T) = \max_{\square \in \text{cells}(T)} c(\square)$ be the congestion of $T$. Here, the maximum is taken over all the squares that correspond to cells in $T$.

**Lemma 3.5.** Given a set $S$ of $n$ segments, and a set $G_S$ of $m$ axis-aligned squares in the plane, one can compute three shifted quadtrees $T_1, T_2, T_3$, such that $c(G_S)/6\sqrt{2} \leq \max_{i=1,2,3} c(T_i) \leq c(S)$.

The construction takes $O((m + n) \log (m + n))$ time.
Proof: Let \( P \) be the set of points formed by the endpoints of segments in \( S \) and the corners of squares in \( G_S \). Shifting the canonical quadtree by the vectors \( v_i \) of Lemma 3.3, construct \( D + 1 = 3 \) shifted quadtrees on \( P \). (The algorithm computes the finite compressed quadtrees on \( P \), and not their somewhat impractical infinite versions). Let \( T_1, T_2, T_3 \) be the resulting quadtrees. The runtime follows from constructing \( O(1) \) quadtrees on \( O(m + n) \) points.

Consider a square \( \Box \in G_S \) with side length \( \tau \), and let \( p \) and \( q \) be an arbitrary pair of opposing vertices of \( \Box \). Then, in some shifted quadtree \( T_i \), there is a cell \( \Box' \in T_i \) with side length \( \tau' \leq 6 \| p - q \| \leq 6 \sqrt{2} \tau \) which contains both \( p \) and \( q \) (i.e., \( \Box \subseteq \Box' \)). We have that

\[
\mathcal{c}(\Box') = \frac{\| S \cap \Box' \|}{\| \Box' \|} \geq \frac{\| S \cap \Box \|}{6 \sqrt{2} \tau} \geq \mathcal{c}(\Box)/6 \sqrt{2}.
\]

We conclude that \( \max_{i=1,2,3} \max_{\Box \in T_i} \mathcal{c}(\Box) \geq \mathcal{c}(G_S)/6 \sqrt{2} \), as claimed.

This reduction implies that, to approximate the congestion of \( S \), the algorithm only needs to compute the congestion of the three quadtrees.

Remark 3.6. Throughout the algorithm, we use compressed quadtrees, rather than regular quadtrees. The use of compressed quadtrees is necessary to get an efficient runtime, but it does not affect the description of our algorithm significantly. (Intuitively, the compressed quadtree compresses paths and not cells, so one can easily ensure that all cells of interest appear in the compressed quadtree as regular cells). As such, from this point on, we use quadtree as a shorthand for a compressed quadtree, and we ignore the minor low-level technical details that arise because of the compression.

3.2. First steps towards approximating the congestion of a quadtree

Given a set \( S \) of \( n \) segments and a quadtree \( T \) of size \( O(n) \), we wish to compute the congestion of \( T \). The quadtree might be the result of shifting the plane, as in Lemma 3.3. To simplify the discussion, we assume that the shift is applied to the input segments, and the quadtree is the standard quadtree applied to this shifted input.

3.2.1. A naïve exact algorithm for the congestion of a quadtree

At each node \( v \in T \), the algorithm stores a conflict list \( L(v) \) – a list of all the segments of \( S \) intersecting the cell \( \Box_v \) associated with \( v \). It begins by storing \( S \) at the root of \( T \), then recursively traverses down the tree. At each parent node \( u \), the algorithm sends the list \( L(u) \) to all its children. Each child \( v \) finds all the segments which intersect its cell \( \Box_v \) and adds them to its own list. At the end of this process, the algorithm has computed the segments intersecting each quadtree node, and it can use them to compute the congestion. Overall, this algorithm takes \( O(n^2) \) time.

3.2.2. The long and the short of it

To speed up the naïve algorithm, we implement several strategies. The first is to register the segments directly in the cells at a suitable resolution of the quadtree.

Definition 3.7 (long/short threshold). Let \( \alpha > 0 \) be a fixed integer constant. We use \( \alpha \) to distinguish between short and long segments (the value of \( \alpha \) will be specified later).

A segment \( s \) is \( \alpha \)-long (resp., \( \alpha \)-short) for a square \( \Box = \Box(p, r) \) if it intersects the interior of \( \Box \) and \( \| s \| \geq \alpha r \) (resp., \( \| s \| < \alpha r \)). For a square \( \Box \) with radius \( r \), its set of \( \alpha \)-long (resp., \( \alpha \)-short) segments is denoted by \( S_{\geq \alpha}(\Box) \) (resp., \( S_{< \alpha}(\Box) \)).
Lemma 3.8. For each segment $s \in S$, let $G(s, \alpha)$ be the set of interior-disjoint canonical squares (of maximal size) for which $s$ is $\alpha$-long. There are at most $O(1 + \alpha)$ such squares, and they can be computed in $O(1 + \alpha)$ time.

Proof: Let $i = \lfloor \log_2(\|s\|/\alpha) \rfloor$, and $\tau = 2^i$. Since $\alpha \tau \leq \|s\| < 2\alpha \tau$, the segment $s$ intersects at most $2(\alpha + 1)$ horizontal lines of the grid $G_\tau$. This implies that $s$ can intersect at most $2(\alpha + 1) + 1$ grid cells of $G_\tau$. Computing the grid cells that intersect $s$ is a classical problem in graphics which can be solved in $O(1 + \alpha)$ time; see Figure 3.1.

3.2.3. Registering the segments

Given the set $S$ of $O(n)$ segments and the above quadtree $T$, the algorithm first computes the set of canonical squares $\Xi = \text{cells}(T) \cup \bigcup_{s \in S} G(s, \alpha)$; see Lemma 3.8. The algorithm then computes the (compressed) quadtree $T^+$ for $\Xi$. This can be done in $O(n \log n)$ time [Har11], as $|S| = n$, $|\text{cells}(T)| = O(n)$, and $|\Xi| = O((1 + \alpha)n) = O(n)$, as $\alpha$ is a constant. Each square of $\Xi$ is now present as a cell of a node of the computed quadtree.

The algorithm now stores every segment $s \in S$ in the cells of $G(s, \alpha)$. Each cell $□ \in G(s, \alpha)$ corresponds to a node $v$ in the quadtree, and the algorithm stores $s$ in $L(v)$. This takes $O(n \log n)$ time; see [Har11]. Thus, for every quadtree node $v$, the algorithm computes a list $L_{\text{long}}(v)$ of segments registered there. Segments registered in this list are long for the cell $□_v$ but short for cells in higher levels of the quadtree.

Propagating each segment up to the ancestor nodes, we also register each segment in a list $L_{\text{short}}(v)$ of another node $v$. Segments registered in this list are short for the cell $□_v$, but long for cells in lower levels of the quadtree. Since computing the registering cells for each segment takes $O(1)$ time, computing the lists $L_{\text{short}}(\cdot)$ and $L_{\text{long}}(\cdot)$ for all nodes in the tree takes $O(n)$ time, and the lists themselves have total length $O(n)$.

A segment is registered only once as a short or long segment on any path in the quadtree.

Definition 3.9. The $\alpha$-long congestion of $\square$ is $c_{\geq \alpha}(\square) = \|S_{\geq \alpha}(\square) \cap □\|/r$. Similarly the $\alpha$-short congestion of $\square$ is $c_{< \alpha}(\square) = \|S_{< \alpha}(\square) \cap □\|/r$.

Given a quadtree $T$, its $\alpha$-long congestion is the maximum $\alpha$-long congestion over all its cells: $c_{\geq \alpha}(T) = \max_{□ \in \text{cells}(T)} c_{\geq \alpha}(□)$. The $\alpha$-short congestion $c_{< \alpha}(T)$ is analogous.

For a node $v \in T^+$, let $\text{anc}(v)$ (resp., $\text{desc}(v)$) be the list of ancestors (resp. descendants) of $v$ in the tree $T^+$. Here $v \in \text{anc}(v)$ and $v \in \text{desc}(v)$. Consider a node $v \in T^+$, and let $□_v$ be its associated
square. We have that
\[ S_{\geq \alpha}(\Box_v) = \bigcup_{u \in \text{anc}(v)} (L_{\text{long}}(u) \cap \Box_v) \quad \text{and} \quad S_{< \alpha}(\Box_v) = \bigcup_{u \in \text{desc}(v)} L_{\text{short}}(u). \]

To summarize, we have described how to augment a quadtree \( T \) by adding more cells and registering short and long segments at the cells, yielding a new quadtree \( T^+ \). In what follows, we use this stored information to compute the long and short congestions of \( T^+ \) (which are at least the congestions of the sub-quadtree \( T \)).

### 3.3. Computing the congestion of the short segments

It is straightforward to compute the \( \alpha \)-short congestion of all cells in a quadtree by dynamic programming, as we describe next.

**Lemma 3.10.** Given a set \( S \) of \( n \) segments in the plane and a quadtree \( T^+ \) of size \( O(n) \), one can compute, in \( O(n \log n) \) time, the \( \alpha \)-short congestion for all the nodes of \( T^+ \), where \( \alpha \) is a constant.

**Proof:** We compute the \( \alpha \)-short congestion of a quadtree via dynamic programming. The algorithm finds the total length of short segments intersecting each leaf and propagates the values upward.

For every node \( v \in T^+ \), the algorithm computes the quantity \( \|L_{\text{short}}(v) \cap \Box_v\| \). Computing the value for node \( v \) requires time proportional to the total size of the list \( L_{\text{short}}(v) \), so doing it for all the nodes of the tree takes \( O(n) \) time overall. Next, the algorithm traverses the tree bottom-up. For each node along the way, it computes the total lengths of the intersecting short segments:

\[
\text{len}_{\text{short}}(v) = \sum_{u \in \text{desc}(v)} \|L_{\text{short}}(u) \cap \Box_u\| = \|L_{\text{short}}(v) \cap \Box_v\| + \sum_{u \text{ child of } v} \text{len}_{\text{short}}(u).
\]

It is easy to verify that \( \text{len}_{\text{short}}(v) = \|S_{< \alpha}(\Box_v) \cap \Box_v\| \). The lemma follows.

### 4. Approximating the \( \alpha \)-long congestion of all cells in a quadtree

Handling the long congestion is more challenging. To compute its exact value for a given square, we need to find the long segments intersecting the cell. We can compute the conflict list for each cell by pushing long segments downward from the conflict lists of its ancestors (as done in the naïve algorithm). Unfortunately, these lists can get quite long.

#### 4.1. A naïve algorithm for computing the long congestion

**Lemma 4.1.** One can compute the \( \alpha \)-long congestion of \( T^+ \) in \( O(n \log n + \rho n) \) time, where

\[
\rho = \max_{\Box \in T^+} |S_{\geq \alpha}(\Box)|.
\]

(4.1)

More generally, given a set \( R \subseteq S \) and a threshold \( t \), one can decide whether \( \rho(R) = \max_{\Box \in T^+} |R \cap S_{\geq \alpha}(\Box)| \leq t \), in \( O(n \log n + tn) \) time.
Proof: The algorithm starts with the precomputed lists $L_{\text{long}}(u)$ and traverses the tree top-down. At each node, it pushes the stored list down to the children. Each child $v$ selects the segments of the incoming list that intersect its cell, and takes the union of this filtered incoming list with $L_{\text{long}}(v)$. This yields the conflict list of $v$ of all $\alpha$-long segments that intersect it, denoted by $L_{\text{LONG}}(v)$. It then pushes this list down to its children, and so on.

For each node $v$ of the quadtree, it is now straightforward to compute the congestion of the segments of $L_{\text{LONG}}(v)$ (or the length of the list $L_{\text{LONG}}(v)$) for the cell of the node $v$. It follows that this computes the $\alpha$-long congestion for each node of the quadtree. Since the maximum length of the lists sent down is $\rho$, the claim follows.

The second algorithm works in a similar fashion, except that the algorithm propagates downwards only segments of $R$. If in any point in time the computed conflict list gets bigger than $t$, the algorithm bails out, returning that $\rho(R) > t$. \qed

4.2. Counting the segments

Instead of computing the $\alpha$-long congestion of a quadtree exactly, we content ourselves with a constant-factor approximation. It turns out that the $\alpha$-long congestion of a cell is bounded by the number of $\alpha$-long segments intersecting it.

Lemma 4.2. For a cell $\square = \square(p,r)$ of $T^+$, we have

$$\frac{c_{\geq \alpha}(\square)}{\sqrt{8}} \leq |S_{\geq \alpha}(\square)| \leq \frac{1 + \alpha}{\alpha} c(S).$$

Proof: The intersection of each segment of $S_{\geq \alpha}(\square)$ with $\square$ can have length at most $r\sqrt{8}$, so $c_{\geq \alpha}(\square) = \|\square \cap S_{\geq \alpha}(\square)\| \leq |S_{\geq \alpha}(\square)| \cdot r\sqrt{8}$, and the first inequality follows.

As for the second inequality, consider $\square' = \square(p,(1 + \alpha)r)$. By definition, each long segment $s \in S_{\geq \alpha}(\square)$ has length at least $\alpha r$ and intersects $\square$. The length of $s \cap \square'$ is at least $\alpha r$, see Figure 4.1. As such, we have

$$|S_{\geq \alpha}(\square)| \cdot \frac{\alpha r}{(1 + \alpha)r} \leq \frac{\|S_{\geq \alpha}(\square) \cap \square'\|}{(1 + \alpha)r} \leq c(\square') \leq c(S).$$

Consequently, to approximate the $\alpha$-long congestion of a cell, it suffices to upper bound the number of $\alpha$-long segments intersecting that cell.

Figure 4.1: An $\alpha$-long segment for a square $\square = \square(p,r)$ intersects the square $\square' = \square(p,(1 + \alpha)r)$ with a segment of length at least $\alpha r$ (here, $\alpha = 3$).
4.3. Sketch of algorithm: Exponential search for the long congestion

It remains to estimate the maximum number of $\alpha$-long segments intersecting any cell in the quadtree $T^+$. We begin by presenting an overview of our randomized algorithm for this task.

Conceptually, each cell $\square$ is associated with a conflict list $S_{\geq \alpha}(\square)$ storing its long segments. The task at hand is to estimate $\rho$ (see Lemma 4.1) – the maximum length of these conflict lists. However, these lists may be as long as $\Omega(n)$, so we will not explicitly compute them. By the registration algorithm of Section 3.2.3, the computed list $L_{\text{long}}(v)$ for each node $v$ stores the segments of $S$ which are $\alpha$-long for itself but not for its ancestors.

The query algorithm proceeds in a series of rounds. At each round, it computes a guess $t$ for $\rho$, and then verifies that the guess was not too large. This works by taking a random sample $R \subseteq S$ of the segments and traversing down the quadtree, computing $S_{\geq \alpha}(\square) \cap R$ for each of the squares $\square$. If at any time the algorithm finds that one of the sampled conflict lists is too long, it terminates the round and increases the guess $t$, as the guess $t$ was too small.

Remark 4.3. Using sampling and exponential search to estimate a quantity is an old idea. In the context of geometric settings, it was used before to estimate the maximum depth of nicely behaved regions, see [AH08].

4.4. The verification algorithm

Let $m = O(n)$ be the number of cells in $T^+$, and let $\delta \in (0,1)$ be a constant. At each round, the algorithm receives a number $t$ and checks whether $\rho \geq t$. The initial guess is

$$t = C \log m,$$

where $C$ is some arbitrary constant such that

$$C \geq \max \left( \frac{2(2 + \delta)}{\delta^2}, \frac{2(1 - \delta)}{(1 + \delta)\delta^2} \right).$$

After each failed round, the guess increases by a factor of $1 + \delta$, so the next value of $t$ is $\lceil (1 + \delta)t \rceil$.

At the first round, $t = O(\log n)$, so one can use the algorithm of Lemma 4.1 directly. If it finds some conflict list that contains more than $t$ segments at any point during the execution, it immediately reports that $t$ is too small and moves to the next round. Otherwise, the algorithm computes $\rho$ exactly and returns its value.

For larger values of $t$, the round begins by taking a random sample $R \subseteq S$ of the segments. Each segment is included in $R$ independently with probability

$$\xi = \frac{C \log m}{t}.$$  

The algorithm then calls the subroutine of Lemma 4.1 with $R$ as the list of segments and with $(1 + \delta)t\xi$ as the threshold. If the output reveals that $\rho(R) > (1 + \delta)t\xi$, then the guess $t$ is too small, and the algorithm continues to the next round.

If a round succeeds, the algorithm outputs that $\rho \leq (1 + \delta)^2t$. 

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4.5. Analysis

Running time. Since there are only \( n \) segments in \( S \), the algorithm always stops by round \( O(\log_{1+\delta} n) = O(\delta^{-1} \log n) = O(\log n) \), as \( \delta \) is a constant. The threshold used in each round is \( (1 + \delta)t^t = O(\log n) \). As such, the running time of each round is \( O(n \log n) \); see Lemma 4.1. We conclude that the overall running time is \( O(n \log^2 n) \).

Correctness of output. It remains to analyze the error probability and approximation quality of this algorithm. We need Chernoff’s inequality, stated next.

Theorem 4.4 (Chernoff bound). Suppose \( X_1, \ldots, X_n \) are independent random variables taking values in \( \{0, 1\} \). Let \( X = \sum_{i=1}^{n} X_i \) be their sum, and let \( \mu = \mathbb{E}[X] \). Then for any \( \delta \in [0, 1] \), we have \( \Pr[X \geq (1 + \delta)\mu] \leq \exp(-\mu\delta^2/(2 + \delta)) \) and \( \Pr[X \leq (1 - \delta)\mu] \leq \exp(-\mu\delta^2/2) \).

Lemma 4.5. The above randomized algorithm computes, in \( O(n \log^2 n) \) time, a number \( \hat{\rho} \), such that

\[
\hat{\rho} \leq \rho \leq \frac{(1 + \delta)^2}{1 - \delta} \hat{\rho},
\]

with high probability.

Proof: For each segment \( s \in S \), let \( X_s \) be the indicator variable that is one if \( s \in R \). For a square \( \square \), let \( X_\square = \sum_{s \in S_\rho(\square)} X_s \) be the number of long segments of \( \square \) which were sampled.

There are two ways the algorithm can fail. The first possibility is that it outputs an estimate which is too large. This happens when \( \rho < t \) and \( X_\square \geq (1 + \delta)t^\xi \) for some square \( \square \). For a fixed square \( \square \), let \( x = |S_\rho(\square)| \) be the length of the conflict list at \( \square \), and observe that \( \mu = \mathbb{E}[X_\square] = x^\xi \). Hence, the algorithm overestimates the list at this square exactly when \( X_\square \geq (1 + \delta)x^\xi \). By Chernoff’s inequality, this bad event happens with probability

\[
\Pr[X_\square \geq (1 + \delta)t^\xi] = \Pr[X_\square \geq (1 + \delta)\frac{t^\xi}{x^\xi}x^\xi] \leq \Pr[X_\square \geq \left(1 + \frac{t\delta}{x}\right)x^\xi] \\
\leq \exp\left(-x^\xi\left(\frac{t\delta}{x}\right)^2 \frac{1}{2 + \delta}\right) = \exp\left(-\frac{C \log m}{t^2} \frac{C \delta^2}{x^2(2 + \delta)}\right) \\
\leq \exp\left(-\frac{t\delta^2}{x(2 + \delta)} C \log m\right) = m^{-C\delta^2/(2+\delta)} \leq \frac{1}{n^{\Omega(1)}},
\]

by Eq. (4.3) and since we have picked \( C \) to be sufficiently large (see Eq. (4.2)). By the union bound over the \( O(n) \) nodes of the quadtree \( T^+ \), the probability that the algorithm fails is polynomially small.

The algorithm can also fail if it stops with a guess that is too small. This happens if \( \rho > (1 + \delta)t/(1 - \delta) \) and \( X_\square \leq (1 + \delta)t^\xi \) for all the cells \( \square \) in the quadtree. In this case, the algorithm terminates and outputs \( t/(1 + \delta) \), while the true value could be much higher. Specifically, the algorithm fails at a cell attaining the maximum conflict list – that is, a cell \( \square^* \) such that \( |S_\rho(\square^*)| = \rho \). Since \( t < \rho(1 - \delta)/(1 + \delta) \), and by Chernoff’s inequality,

\[
\Pr[X_\square^* < (1 + \delta)t^\xi] \leq \Pr[X_\square^* < (1 - \delta)\rho^\xi] \leq \exp\left(-\rho\delta^2/2\right) \\
= \exp\left(-\rho\frac{C \log m}{2t}\delta^2\right) \leq \frac{1}{n^{\Omega(1)}},
\]
again by picking $C$ to be sufficiently large.

By a union bound over the two cases, the probability that the algorithm fails at a given round is $1/n^{\Omega(1)}$. Thus, the probability that the algorithm fails in any of the $O(\log n)$ rounds is $O(\log n/n^{\Omega(1)})$. We conclude that the algorithm succeeds with high probability.

Suppose that the algorithm succeeds. Since it does not fail in the first case, it terminates in the first round such that $t/(1+\delta) \leq \rho$. Since it does not fail in the second case, $\rho < (1+\delta)t/(1-\delta)$.

Since the algorithm outputs $\hat{\rho} = t/(1+\delta)$,

$$\hat{\rho} = \frac{t}{1+\delta} \leq \rho \leq \frac{(1+\delta)t}{1-\delta} = \frac{(1+\delta)^2}{1-\delta} \hat{\rho}.$$ 

Namely, the approximation quality is $(1+\delta)^2/(1-\delta)$.

### 4.6. The result for long congestion

**Lemma 4.6.** Let $S$ be a collection of $n$ segments, and let $T^+$ be a quadtree with $O(n)$ cells. In $O(\delta^{-3}n \log^2 n)$ time, the above randomized algorithm computes a number $\overline{c_{\geq \alpha}(T^+)}$ such that, with high probability,

$$\frac{c_{\geq \alpha}(T^+)}{\sqrt{8}} \cdot \frac{\alpha}{1+\alpha} \cdot \frac{1-\delta}{(1+\delta)^2} \leq \overline{c_{\geq \alpha}(T^+)} \leq \nu(T^+).$$

**Proof:** Let $\nu(T^+) = \max_{\square \in T} |S_{\geq \alpha}(\square)|$ be the maximum length of a conflict list in the tree $T^+$. Using the algorithm of Lemma 4.5, compute an estimate $\hat{\nu}(T^+)$ for $\nu(T^+)$. Then let $\overline{c_{\geq \alpha}(T^+)} = \hat{\nu}(T^+) \cdot \alpha/(1+\alpha)$ be the desired approximation.

Lemma 4.5 implies that $\hat{\nu}(T^+) \leq \nu(T^+) \leq (1+\delta)^2/(1-\delta) \cdot \hat{\nu}(T^+)$. By Lemma 4.2,

$$\frac{c_{\geq \alpha}(T^+)}{\sqrt{8}} \leq \nu(T^+) \leq \frac{1+\alpha}{\alpha} \cdot \nu(S).$$

The result now follows directly, as

$$\frac{c_{\geq \alpha}(T^+)}{\sqrt{8}} \cdot \frac{1-\delta}{(1+\delta)^2} \cdot \frac{\alpha}{1+\alpha} \leq \nu(T^+) \cdot \frac{1-\delta}{(1+\delta)^2} \cdot \frac{\alpha}{1+\alpha} \leq \overline{c_{\geq \alpha}(T^+)},$$

and

$$\overline{c_{\geq \alpha}(T^+)} \leq \nu(T^+) \cdot \frac{\alpha}{1+\alpha} \leq \nu(S).$$

### 5. Putting everything together

We use the algorithms discussed previously to approximate the congestion of a quadtree $T$.

**Lemma 5.1.** Given a set $S$ of $n$ segments and a quadtree $T$ with $O(n)$ cells, one can compute, in $O(n \log^2 n)$ expected time, a number $\overline{\nu}(T)$ such that, with high probability,

$$\frac{c(T)}{4\sqrt{2} + \varepsilon} \leq \overline{\nu}(T) \leq \nu(S).$$
Proof: Recall that $c_{<\alpha}(T)$ and $c_{\geq \alpha}(T)$ denote the short and long congestions of $T$; see Definition 3.9. The algorithm augments $T$ to construct the quadtree $T^+$, as described previously. Using the algorithm of Lemma 3.10, it computes $c_{<\alpha}(T^+)$ exactly. Using Lemma 4.6, the algorithm computes an estimate $\hat{c}_{\geq \alpha}(T^+)$ for the $\alpha$-long congestion, such that

$$ \frac{c_{\geq \alpha}(T^+)}{\gamma} \leq \hat{c}_{\geq \alpha}(T^+) \leq c(S), $$

where

$$ \gamma = \frac{1 + \alpha}{\alpha} \cdot \frac{\sqrt{8(1 + \delta)^2}}{1 - \delta}. $$

Now, the algorithm estimates $c(T)$ by outputting

$$ \hat{c}(T) = \max \left( c_{<\alpha}(T^+), \hat{c}_{\geq \alpha}(T^+) \right). $$

For every square $\square$, we have $c(\square) = c_{<\alpha}(\square) + c_{\geq \alpha}(\square)$. Moreover, the augmented tree $T^+$ contains all the cells of $T$. Hence

$$ c(T) = \max_{\square \in \text{cells}(T)} (c_{<\alpha}(\square) + c_{\geq \alpha}(\square)) $$

$$ \leq \max_{\square \in \text{cells}(T^+)} (c_{<\alpha}(\square) + c_{\geq \alpha}(\square)) $$

$$ \leq 2 \max_{\square \in \text{cells}(T^+)} (c_{<\alpha}(\square), c_{\geq \alpha}(\square)) $$

$$ \leq 2 \max (c_{<\alpha}(T^+), c_{\geq \alpha}(T^+)) \leq 2\gamma \hat{c}(T^+). $$

Now, setting $\delta \in (0, 1)$ to be sufficiently small, and then setting $\alpha$ to be a sufficiently large constant, we can set $\gamma$ to arbitrarily close to $2\sqrt{2}$. Hence, we can ensure that $\hat{c}(T)$ is a $(4\sqrt{2} + \varepsilon)$-approximation to $c(T)$. \hfill \blacksquare

Finally, we arrive at our main result.

**Theorem 5.2.** Let $S$ be a set of $n$ segments in the plane. One can compute, in $O(n \log^2 n)$ time, a $O(1)$-approximation to $c(S)$. The algorithm is randomized and succeeds with high probability.

Proof: By Lemma 2.4, compute a set of $O(n/\varepsilon^2) = O(n)$ representative squares $G_S$, where $\varepsilon$ is a constant. Using Lemma 3.5 on $G_S$, compute the three shifted quadtrees $T_1, T_2, T_3$. By the algorithm of Lemma 5.1, compute an estimate $\hat{c}(T_i)$ for the congestion of each quadtree $T_i$, and output $\hat{c} = \max_{i=1,2,3} \hat{c}(T_i)$.

By Lemma 2.4, $c(S) \leq (6 + \varepsilon)c(G_S)$. We have shown that

$$ c(G_S) \leq 6\sqrt{2} \max_{i=1,2,3} \hat{c}(T_i), $$

and that $c(T_i) \leq (4\sqrt{2} + \varepsilon)\hat{c}(T_i^+)$ for $i = 1, 2, 3$. Moreover, each $c(T_i^+) \leq c(S)$, so $\hat{c}$ is feasible. Hence, $\hat{c}$ is a $(6 \cdot 6 \cdot 4 + \varepsilon) = (288 + \varepsilon)$-approximation to the congestion of $S$. \hfill \blacksquare

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6. Conclusions

We provided a near-linear time algorithm that computes a constant factor approximation to the congestion of a polygonal curve (i.e., the minimum $c$ such that the curve is $c$-packed). We consider the result to be quite surprising, even though the constant is undesirably large.

The new algorithm works verbatim in any constant dimension – our algorithm has not used planarity in any way. The quality of approximation deteriorates with the dimension $d$, but it is still a constant when $d$ is a constant. The running time remains $O(n \log^2 n)$.

Another important property of the new algorithm is that it does not require the input segments to form a curve. It is natural to conjecture that in the plane, if the input is a polygon curve that does not self intersect, then one should be able to $(1 + \varepsilon)$-approximate the congestion in near linear time. We leave this as an open problem for further research.

As mentioned above, the constant in the approximation quality of the new algorithm is not pretty. By $\varepsilon$-shifting the quadtrees (thus, using $O(1/\varepsilon^d)$ quadtrees in $d$ dimensions), one could reduce the approximation factor, probably by a factor of two. However, the resulting constant is still quite large. Reducing the constant further while keeping the running time near-linear is an interesting problem for future research.

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