EXPONENTIAL MIXING FOR SDES FORCED BY DEGENERATE LÉVY NOISES

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ABSTRACT. We modify the coupling method established in [44] and develop a technique to prove the exponential mixing of a 2D stochastic system forced by degenerate Lévy noises. In particular, these Lévy noises include \( \alpha \)-stable noises \((0 < \alpha < 2)\). This technique is promising to study the exponential mixing problem of stochastic Navier-Stokes and complex Ginzburg-Landau equations driven by degenerate kick noises only with some \( p \) moment ([29]).

Keywords: SDEs driven by degenerate \( \alpha \)-stable noises, coupling, exponential mixing.
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1. Introduction

We shall study in this paper the exponential ergodicity of degenerate stochastic evolution equation

\[
\begin{aligned}
dX_1(t) &= [-\lambda_1 X_1(t) + F_1(X(t))]dt + dz(t), \\
dX_2(t) &= [-\lambda_2 X_2(t) + F_2(X(t))]dt
\end{aligned}
\]

where \( X(t) := (X_1(t), X_2(t))^T \in \mathbb{R}^2 \) for all \( t \geq 0, \lambda_2, \lambda_1 > 0, F: \mathbb{R}^2 \to \mathbb{R}^2 \) is bounded and Lipschitz, i.e.

\[|F(x) - F(y)| \leq \|F\|_{Lip}|x - y|, \quad \forall \ x, y \in \mathbb{R}^2.\]

\( z(t) \) is a one dimensional Lévy process satisfying Assumption 2.1 below. We often simply write the above equation as the following form:

\[
\begin{aligned}
dX(t) &= [AX(t) + F(X(t))]dt + dZ_t,
\end{aligned}
\]

where \( A := \text{diag}\{-\lambda_1, -\lambda_2\} \) and \( Z_t = [z(t), 0]^T \).

The SDEs and SPDEs driven by Lévy noises have been intensively studied in recent years; e.g., see the papers [3, 41, 34, 32, 23, 38, 46], the book [33] and the references therein. Invariant measures and long-time asymptotics for stochastic systems with Lévy noises were also studied in a number of papers; e.g., see [41, 48, 35, 39, 13, 30, 25, 10, 46, 47]. However, there are not many results on ergodicity and exponential mixing (cf. [40, 47, 17, 39, 35]).
Since the end of the last century, the ergodicity of stochastic systems forced by degenerate noises has also been intensively studied, see [11, 12, 14, 15, 16] for the SPDEs with degenerate Wiener noises and [19, 20, 21, 42, 43, 44, 28] for those forced by kick noises. However, there seems no ergodicity result for the stochastic systems driven by degenerate Lévy jump noises. To our knowledge, this paper seems the first one in this direction.

The main novelty of the present paper is that we obtain the exponential ergodicity for a family of 2D SDEs driven by a large class of degenerate Lévy jump noises which include $\alpha$-stable noises ($0 < \alpha < 2$). Our approach is by modifying the coupling method established in [44]. This method is a powerful tool for handling the ergodicity problems of degenerate stochastic systems ([19, 20, 21, 42, 43, 44, 28]). In [19, 20, 21, 42, 43, 44], to get the exponential ergodicity, the authors had to assume that the kick noises come periodically and are bounded or with exponential moments. [28] studied polynomial mixing for the complex Ginzburg-Landau equation driven by a random kick noises at random times, under the assumption that the noises have all $p > 0$ moments. Clearly, all these assumptions in the above literatures ruled out the interesting Lévy noises only with some $p > 0$ moment such as $\alpha$-stable noises.

Let us also compare our result with those known for SDEs and SPDEs forced by Lévy noises. [35] established the exponential mixing for a family of SPDEs with a form similar to (1.2) under total variational norm, provided that the noises are non-degenerate $\alpha$-stable with $1 < \alpha < 2$. The non-degeneracy assumption and the regime of $\alpha \in (1, 2)$ are crucial to get the strong Feller property, which is the key point for applying the coupling or Lyapunov function technique. Comparing with [35], the two new points in the present paper are that our noises are degenerate and include all $\alpha$-stable noises. [17] established some nice criteria of the exponential mixing (under total variation norm) for a family of finite dimensional SDEs driven by jump noises which include some one dimensional equations driven by $\alpha$-stable noises.

We need to stress that our stochastic system (1.1) is two dimensional and that the Lévy jump noises are one dimensional. It is natural to ask whether one can extend our exponential ergodicity result to SPDEs forced by finite dimensional cylindrical Lévy jump noises (the noises are of course degenerate). Unfortunately, it seems our technique is not applicable even for the case of 3d SDEs driven by 2d Lévy jump noises. Let us point out the difficulty (very) roughly by the following toy models. Consider

\[
\begin{align*}
  dX_1(t) &= [-\lambda_1 X_1(t) + F_1(X(t))]dt + dz_1(t), \\
  dX_2(t) &= [-\lambda_2 X_2(t) + F_2(X(t))]dt + dz_2(t), \\
  dX_3(t) &= [-\lambda_3 X_3(t) + F_3(X(t))]dt
\end{align*}
\]
where $\lambda_1, \lambda_2, \lambda_3 > 0$, $F : \mathbb{R}^3 \to \mathbb{R}^3$ is bounded and Lipschitz, $z_1(t)$ and $z_2(t)$ are independent 1d Lévy jump processes. We assume that $\lambda_3$ is sufficiently large to make the dissipative term $-\lambda_3 X_3(t)$ dominate the third equation. For the first two equations, when $z_1(t)$ has a jump $\eta_1$ at some moment $\tau$, there is no jumps for $z_2(t)$ at $\tau$ almost surely. We can take the advantage of the jump $\eta_1$ to control the growth of some sample paths of $X_1(t)$ in a short time interval $[\tau, \tau + \delta)$ by coupling technique, and the probability of these paths are positive. However, due to the lack of the jump, the growth of almost all the sample paths of $X_2(t)$ can not be handled in $[\tau, \tau + \delta)$.

According to [29], the technique developed in the present paper is promising to handle the exponential mixing problem of stochastic Navier-Stokes and complex Ginzburg-Landau equations driven by degenerate kick noises only with some $p$ moment. These will hopefully be stressed in some future papers.

The structure of the paper is as follows. Section 2 introduces the notations and gives the main theorem. Section 3 contains some bounds about the solution of Eq. (1.1), which are used to estimate the stopping times in Section 5. The coupling Markov chain is introduced in Section 4, and used to prove the main theorem in Section 6.

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### 2. Notations and main results

Denote by $B_b(\mathbb{R}^2)$ the Banach space of bounded Borel-measurable functions $f : \mathbb{R}^2 \to \mathbb{R}$ with the supremum norm

$$\|f\|_0 := \sup_{x \in \mathbb{R}^2} |f(x)|.$$ 

Further denote by $L_b(\mathbb{R}^2)$ the Banach space of global Lipschitz bounded functions $f : \mathbb{R}^2 \to \mathbb{R}$ with the norm

$$\|f\|_1 := \sup_{x \in \mathbb{R}^2} |f(x)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$ 

Let $\mathcal{B}(\mathbb{R}^2)$ be the Borel $\sigma$-algebra on $\mathbb{R}^2$ and let $\mathcal{P}(\mathbb{R}^2)$ be the set of probabilities on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$. Recall that the total variation distance between two measures...
$\mu_1, \mu_2 \in \mathcal{P}(\mathbb{R}^2)$ is defined by
\[
\|\mu_1 - \mu_2\|_{TV} = \frac{1}{2} \sup_{f \in B(\mathbb{R}^2)} |\mu_1(f) - \mu_2(f)| = \sup_{f \in B(\mathbb{R}^2)} |\mu_1(\Gamma) - \mu_2(\Gamma)|.
\]
Given a random variable $X$, we shall use $\mathcal{L}(X)$ to denote the distribution of $X$.

2.1. Some preliminary of Lévy process ($[4]$). Let $(z(t))_{0 \leq t < \infty}$ be a one-dimensional purely jumping Lévy process. Recall that it has the characteristic function
\[
\mathbb{E} e^{i\xi z(t)} = e^{-t\psi(\xi)}, \quad t \geq 0,
\]
$\psi(\xi)$ is called the symbol of $z(t)$ with the following form
\[
\psi(\xi) = \int_{\mathbb{R} \setminus \{0\}} \left(1 - e^{i\xi y} + i\xi y 1_{\{|y| \leq 1\}}\right) \nu(dy)
\]
where $\nu$ is the Lévy measure and satisfies that
\[
\int_{\mathbb{R} \setminus \{0\}} 1 \wedge |y|^2 \nu(dy) < \infty.
\]

For $t > 0$ and $\Gamma \in B(\mathbb{R} \setminus \{0\})$, the Poisson random measure associated with $z(t)$ is defined by
\[
N(t, \Gamma) := \sum_{s \in (0, t]} 1_{\Gamma}(\Delta z(s))
\]
where $\Delta z(s) = z(s) - z(s-)$. One has
\[
(2.1) \quad z(t) = z_K(t) + z^K(t) \quad \forall \ K > 0,
\]
where
\[
(2.2) \quad z_K(t) := \int_{0<|x|\leq K} xN(t, dx), \quad z^K(t) := \int_{|x|\geq K} xN(t, dx).
\]
Now define $\Gamma_K := (-\infty, -K] \cup [K, \infty)$ and
\[
\gamma_K := \nu(\Gamma_K),
\]
it is clear that $\gamma_K < \infty$ and is a decreasing function of $K$. $N(t, \Gamma_K)$ is a Poisson random variable with intensity $\gamma_K t$, and can be constructed in the following way.

Let $\tilde{\tau}_1, \tilde{\tau}_2, \ldots, \tilde{\tau}_n, \ldots$ be a sequence of random times (more precisely, stopping times) such that
\[
\tilde{\tau}_1, \tilde{\tau}_2 - \tilde{\tau}_1, \ldots, \tilde{\tau}_n - \tilde{\tau}_{n-1}, \ldots
\]
are independent exponential random variables with parameter $\gamma_K$, i.e. $\mathbb{P}(\tilde{\tau}_n - \tilde{\tau}_{n-1} > s) \leq e^{-\gamma_K s}$ for $s > 0$, so $\tilde{\tau}_n$ satisfies the Gamma$(\gamma_K, n)$ distribution
\[
\mathbb{P}(\tilde{\tau}_n \in ds) = \frac{\gamma_K^n}{(n-1)!} s^{n-1} e^{-\gamma_K s} ds \quad (s \geq 0).
\]
Then \( N(t, \Gamma_K) \) is defined by
\[
N(t, \Gamma_K) := \sup\{n \in \mathbb{N}; \tilde{\tau}_n \leq t\}.
\]

It follows from (2.2) that
\[
(2.3) \quad z^K(t) = \sum_{k \geq 1} \eta_k \mathbbm{1}_{\tilde{\tau}_k \leq t},
\]
where \( \eta_k \) are independent random variable sequences with distribution
\[
(2.4) \quad \nu_K := \frac{1}{\gamma_K} \nu|_{\Gamma_K}.
\]

2.2. Assumptions. We shall assume the Lévy noises satisfy the following assumptions.

**Assumption 2.1.** We assume that
\[
(A1) \quad \sup_{0 \leq t < \infty} \mathbb{E}\left[\int_0^t e^{-\lambda(t-s)} dz(s)\right]^p < \infty \text{ for all } \lambda > 0 \text{ and } p \in (0, \alpha).
\]
\[
(A2) \quad \text{For some } K > 0, \nu_K \text{ has a density } p_K \text{ such that for all } z_1, z_2 \in \mathbb{R}
\]
\[
\int_{\mathbb{R}} |p_K(z - z_1) - p_K(z - z_2)| dz \leq \min \{ \beta_0, \beta_1 |z_1 - z_2|^{\beta_2} \}
\]
where \( 0 < \beta_0 < 2, \beta_1, \beta_2 > 0 \) are constants only depending on \( K \).
\[
(A3) \quad \gamma_K \geq 2\beta_2 \|F\|_{\text{Lip}}.
\]

**Remark 2.2.** The number '2' in '\( \gamma_K \geq 2\beta_2 \|F\|_{\text{Lip}} \)' of (A3) can be replaced by any number \( c > 1 \). We choose the special '2' to make the computation in sequel more simple. Roughly speaking, (A3) means that the process \((z(t))_{t \geq 0}\) has sufficiently many jumps bigger than \( K \).

The \( \alpha \)-stable process \((z(t))_{t \geq 0}\) (\( 0 < \alpha < 2 \)) with the Lévy measure \( \nu(dx) = \frac{1}{|x|^\alpha + 1}1_{|x| > 0}dx \) satisfies this assumption.

It is well known that \( z(t) \) has the characteristic function \( |\xi|^\alpha t \). Writing \( z_A(t) := \int_0^t e^{-\lambda(t-s)} dz_s \), one can easily check that
\[
\mathbb{E}[e^{i \xi z_A(t)}] = \exp \left\{ -|\xi|^\alpha \frac{1 - e^{-\alpha t}}{\alpha \lambda} \right\}
\]
(A1) follows immediately from (3.2) of [38].

Let us now check that the inequality in (A2) is true for all \( K > 0 \) (this is of course stronger than (A2) itself). It is easy to see that
\[
p_K(z - z_i) = \frac{\alpha K^\alpha}{2} \frac{1}{|z - z_i|^\alpha} 1_{|z - z_i| < \frac{K}{2}} (i = 1, 2).
\]
If \( |z_2 - z_1| \leq \frac{K}{4} \), we assume \( 0 \leq z_2 - z_1 \leq \frac{K}{4} \) without loss of generality and thus have
\[
z_1 - K \leq z_2 - K \leq z_1 + K \leq z_2 + K.
\]
It follows from the above relation that
\[ 
\int_{\mathbb{R}} |p_K(z - z_1) - p_K(z - z_2)| \, dz = 
\int_{-\infty}^{z_1 - K} \frac{\alpha K^\alpha}{2(z_1 - z)^{\alpha+1}} \, dz + \int_{z_1 - K}^{z_2 - K} \frac{\alpha K^\alpha}{2(z - z_2)^{\alpha+1}} \, dz 
+ \int_{z_1 + K}^{z_2 + K} \frac{\alpha K^\alpha}{2(z - z_1)^{\alpha+1}} \, dz + \int_{z_2 + K}^{\infty} \left[ \frac{\alpha K^\alpha}{2(z - z_2)^{\alpha+1}} - \frac{\alpha K^\alpha}{2(z - z_1)^{\alpha+1}} \right] \, dz 
= 2 \left[ 1 - \left( \frac{1}{1 + \frac{z_2 - z_1}{K}} \right)^\alpha \right]. 
\]

By the easy fact
\[ 
1 - \left( \frac{1}{1 + r} \right)^\alpha \leq ([\alpha] + 1) r \quad |r| < \frac{1}{2 + 2[\alpha]},
\]
we immediately get
\[ 
\int_{\mathbb{R}} |p_K(z - z_1) - p_K(z - z_2)| \, dz \leq \frac{2[\alpha] + 2}{K} |z_2 - z_1|.
\]

On the other hand, it is clear that
\[ 
\beta := \int_{\mathbb{R}} |p_K(z - z_1) - p_K(z - z_2)| \, dz < 2.
\]

It follows from the above two inequalities that (A2) is satisfied with \( \beta_0 = 1 + \frac{\beta}{2} \), \( \beta_1 = \frac{4}{K}, \beta_2 = 1 \).

Since \( (z(t))_{t \geq 0} \) is \( \alpha \)-stable noise, \( \gamma_K \to \infty \) as \( K \downarrow 0 \). Therefore, (A3) is clearly true.

2.3. **Main result.** Before giving the main theorem, let us first prove that the problem (1.1) is well-posed.

**Theorem 2.3.** For any initial data \( x \in \mathbb{R}^2 \), problem (1.1) has a unique strong solution \( (X^x(t))_{t \geq 0} \) with the form:
\[ 
(2.5) \quad X^x(t) = e^{At} x + \int_0^t e^{A(t-s)} F(X^x(s)) \, ds + \int_0^t e^{A(t-s)} dZ_s.
\]

Moreover, this solution satisfies the following properties:

(1) \( (X^x(t))_{t \geq 0} \) has a Càdlàg version in \( \mathbb{R}^2 \).

(2) \( (X^x(t))_{t \geq 0} \) forms an \( \mathbb{R}^2 \)-valued Markov process starting from \( x \).

**Proof.** The existence, uniqueness and Markov property of the strong solution have been proved in [38]. Since \( Z_t \) clearly has a Càdlàg version, \( \int_0^t e^{A(t-s)} dZ_s \) also has a Càdlàg one. The other two terms on the r.h.s. of (2.5) are both continuous, so \( (X^x(t))_{t \geq 0} \) is Càdlàg. \( \square \)
Let us denote by \((P_t)_{t \geq 0}\) the Markov semigroup associated with \((1.1)\), i.e.
\[ P_t f(x) := \mathbb{E} [f(X^x(t))], \quad f \in B_b(\mathbb{R}^2), \]
and by \((P^*_t)_{t \geq 0}\) the dual semigroup acting on \(\mathcal{P}(\mathbb{R}^2)\). Our main result is the following ergodic theorem which will be proven in the last section.

**Theorem 2.4.** Under Assumption 2.1, if \(\lambda_2 > 0\) is sufficiently large so that
\[
\lambda_2 > \frac{8 M (1 + 2 e^{\|F\|_{Lip} T})}{d(2 - \beta_0)} \|F\|_{Lip} - \|F\|_{Lip},
\]
where \(T > \frac{(p-1) \log 3}{p \lambda_1} \vee 0, 0 < d < \left(\frac{1}{3 \lambda_1}\right)^{\frac{1}{p+1}} e^{-\|F\|_{Lip} T} \) are both some fixed constants, \(M\) is a fixed constant defined in Theorem 2.3 below, then the system \((1.1)\) is ergodic and exponentially mixing under the weak topology of \(\mathcal{P}(\mathbb{R}^2)\). More precisely, there exists a unique invariant measure \(\mu \in \mathcal{P}(\mathbb{R}^2)\) so that for any \(p \in (0, \alpha)\) and any measure \(\tilde{\mu} \in \mathcal{P}(\mathbb{R}^2)\) with finite \(p^{th}\) moment, we have
\[
|\langle P^*_t \tilde{\mu}, f \rangle - \langle \mu, f \rangle| \leq C e^{-c t} \|f\|_1 \left(1 + \int_{\mathbb{R}^2} |x|^p \tilde{\mu}(dx)\right) \forall f \in L_b(\mathbb{R}^2),
\]
where \(C, c\) depend on \(p, \nu, K, \|F\|_{Lip}, \|F\|_0, \lambda\).

Let us briefly give the strategy of the coupling method we shall use (it is a modification of the method established in [14]):

(i) Take a sequence of stopping time \(\{\tau_k\}_{k \geq 0}\) with \(\tau_k = 0\) and \(\tau_k\) denoting the moment that the \(k\)-th jump comes (see the exact definition of \(\tau_k\) in Section 4).

(ii) For any \(x, y \in \mathbb{R}^2\), take two copies of processes \((X^x(t))_{t \geq 0}\) and \((X^y(t))_{t \geq 0}\), consider the corresponding embedded Markov chains \((X^x(\tau_k))_{k \geq 0}\) and \((X^y(\tau_k))_{k \geq 0}\). Using maximal coupling, we construct the coupling chain \((S^x,y(k))_{k \geq 0}\) with \(S^x,y(k) = (S^x(k), S^y(k))\) for all \(k \geq 0\). \((S^x(k))_{k \geq 0}, (S^y(k))_{k \geq 0}\) have the same distributions as those of \((X^x(\tau_k))_{k \geq 0}\) and \((X^y(\tau_k))_{k \geq 0}\) respectively.

(iii) Define
\[
\hat{\sigma} = \inf \{k > 0; |S^x(k)| + |S^y(k)| \leq M\},
\]
\[
\hat{\sigma} = \inf \left\{k > 0; |S^x(k) - S^y(k)| \geq \frac{\tau_k}{\lambda_2^2}\right\},
\]
the \(\hat{\sigma}\) is exactly defined in (6.1), but the above definition takes the essential part of (6.1). We show that \(\hat{\sigma}\) has exponential moment and \(\mathbb{P}(\hat{\sigma} = \infty) > 0\). Roughly speaking, the system \((S(k))_{k \geq 0}\) enters the \(M\)-radius ball exponentially frequently. As the system is in the ball, for some sample paths with
positive probability, $|S^x(k) - S^y(k)|$ converges to zero exponentially fast as long as $\lambda_2$ is sufficiently large.

For the simplicity of computation in sequel, from now on we assume

\[ \lambda_1 \leq \lambda_2. \]

Our method of course covers the regime $\lambda_1 > \lambda_2$, in which the dissipative term $AX(t)$ dominates the system. In this case, one can prove the exponential mixing by a quite easy argument (36).

3. SOME EASY ESTIMATES ABOUT THE SOLUTION

In this section, we prove some easy estimates about the solution $X(t)$ of problem (1.1), which will play an essential role for estimating some stopping times the sections later.

**Lemma 3.1.** The following statements hold:

1. For $x, y \in \mathbb{R}^2$, $p \in (0, \alpha)$, we have

\[
\mathbb{E}|X^x(t)|^p \leq (3p^{-1} \vee 1)e^{-\lambda_1 p t}|x|^p + C \quad \forall \ t \geq 0,
\]

\[
\mathbb{E}|X^x(t) - X^y(t)|^p \leq (3p^{-1} \vee 1)e^{-\lambda_1 p t}|x - y|^p + C \quad \forall \ t \geq 0,
\]

where $a \vee b := \max\{a, b\}$ for $a, b \in \mathbb{R}$ and $C$ depends on $p, \lambda, \|F\|_0, \nu$.

2. For $x, y \in \mathbb{R}^2$, we have

\[
|X^x(t) - X^y(t)| \leq e^{|F|_{\text{Lip}} |x - y|},
\]

\[
|X^x_2(t) - X^y_2(t)| \leq \left( e^{-\lambda_2 t} + \frac{|F|_{\text{Lip}}}{\lambda_2 + |F|_{\text{Lip}}} e^{|F|_{\text{Lip}} t} \right) |x - y|,
\]

for all $t \geq 0$.

**Proof.** Denote

\[ Z_A(t) := \int_0^t e^{A(t-s)}dZ_s, \]

note that $Z_A(t) = [z_A(t), 0]^T$ with

\[ z_A(t) = \int_0^t e^{-\lambda_1(t-s)}dz(s). \]

By (2.5) we have

\[
|X^x(t)| \leq |e^{At}x| + \left| \int_0^t e^{A(t-s)}F(X^x(s))ds \right| + |Z_A(t)|
\]

\[
\leq e^{-\lambda_1 t}|x| + \int_0^t e^{-\lambda_1(t-s)}ds\|F\|_0 + |Z_A(t)|,
\]
and
\[ |X^x(t) - X^y(t)| \leq |e^{At}(x - y)| + \left| \int_0^t e^{A(t-s)} [F(X^x(s)) - F(X^y(s))] ds \right| \]
\[ \leq e^{-\lambda_1 t}|x - y| + 2 \int_0^t e^{-\lambda_1 (t-s)} \|F\|_0 ds, \]

The first statement follows from the above inequality and (A1) of Assumption 2.1.

Let us now prove the second statement. It is easy to have
\[ X^x(t) - X^y(t) = e^{At}(x - y) + \int_0^t e^{A(t-s)} [F(X^x(s)) - F(X^y(s))] ds \]
which implies
\[ |X^x(t) - X^y(t)| \leq |x - y| + \int_0^t \|F\|_{Lip}|X^x(s) - X^y(s)| ds. \]

From this we immediately get the first inequality by Gronwall’s inequality. It follows from the first inequality that
\[ |X_2^x(t) - X_2^y(t)| \leq e^{-\lambda t}|x_2 - y_2| + \int_0^t e^{-\lambda (t-s)} \|F\|_{Lip}|X^x(s) - X^y(s)| ds \]
\[ \leq e^{-\lambda t}|x - y| + \int_0^t e^{-\lambda (t-s)} \|F\|_{Lip} e^{s} \|F\|_{Lip}|x - y| ds \]
This immediately implies the second inequality. \(\square\)

4. CONSTRUCTION OF THE COUPLING

In this section, let us construct a coupling Markov chain which will play an essential role for proving our ergodicity result. Let
\[ T > 0 \]
be a fixed number
\[ \tau := \inf \{ t > T : |\Delta z(t)| \geq K \}, \]
\(\tau\) is a stopping time with probability density
\[ \gamma_K \exp \{-\gamma_K (t - T)\} 1_{\{t > T\}}. \]
Define \(\tau_0 := 0\) and
\[ \tau_k := \inf \{ t > \tau_{k-1} + T : |\Delta z(t)| \geq K \} \quad \text{for all } k \geq 1. \]
It is easy to see that \(\{\tau_k\}_{k \geq 0}\) are a sequence of stopping times such that
\[ \{\tau_k - \tau_{k-1}\}_{k \geq 1} \] are independent and have the same density as \(\tau\).
Since the solution of problem (1.1) with the initial data \( X(0) = x \) has a Càdlàg version, \( X^x(\tau_1-) \) is well defined with the form:

\[
X^x(\tau_1-) = e^{A\tau_1}x + \int_0^{\tau_1} e^{A(t_1-s)}F(X^x(s))ds + \int_0^{\tau_1} e^{-A(t_1-s)}dZ_s,
\]

By (2.3) and strong Markov property of \( z(t) \), at the time \( \tau_1 \), there is only one jump \( \eta \) almost surely and \( \eta \) has the probability density \( \nu_K \) (see (2.4)). Therefore,

\[
X^x(\tau_1) = X^x(\tau_1-) + \eta[1, 0]^T \quad a.s.
\]

Denote by \( P_x^{(1)}(.) : \mathcal{B}(\mathbb{R}^2) \to [0, 1] \) the distribution of \( X^x(\tau_1-) \) for all \( x \in \mathbb{R}^2 \), and by \( P_x^{(2)}(.) : \mathcal{B}(\mathbb{R}^2) \to [0, 1] \) the distribution of \( \hat{x} + \eta[1, 0]^T \) for all \( \hat{x} \in \mathbb{R}^2 \). For any \( A \in \mathcal{B}(\mathbb{R}^2) \), define

\[
P_x(A) := \int_{\mathbb{R}^2} P_x^{(2)}(A)P_x^{(1)}(d\hat{x}),
\]

\((X^x(\tau_k))_{k \geq 0}\) is an \( \mathbb{R}^2 \)-valued Markov chain with transition probability \((P_x(.)\) \) \(x \in \mathbb{R}^2\).

Consider the two processes \((X^x(t))\) and \((X^y(t))\) starting from \( x \) and \( y \) respectively and the corresponding Markov chains \((X^x(\tau_k))_{k \geq 0}\) and \((X^y(\tau_k))_{k \geq 0}\).

For all \( \hat{x}, \hat{y} \in \mathbb{R}^2 \), denote by \( \mathcal{L}(\hat{x}_1+\eta) \) and \( \mathcal{L}(\hat{y}_1+\eta) \) the distributions of \( \hat{x}_1+\eta \) and \( \hat{y}_1+\eta \) respectively. Let \((\xi_x(\hat{x}_1, \hat{y}_1), \xi_y(\hat{x}_1, \hat{y}_1))\) be the maximal coupling of \( \mathcal{L}(\hat{x}_1+\eta) \) and \( \mathcal{L}(\hat{y}_1+\eta) \).

**Lemma 4.1.** We have

\[
\mathbb{P}(\xi_x(\hat{x}_1, \hat{y}_1) \neq \xi_y(\hat{x}_1, \hat{y}_1)) \leq \frac{1}{2} \min \{\beta_0, \beta_1|\hat{x}_1 - \hat{y}_1|^{\beta_2}\}
\]

where \(\beta_0, \beta_1, \beta_2\) are the constants in Assumption [2.1].

**Proof.** Since \((\xi_x(\hat{x}_1, \hat{y}_1), \xi_y(\hat{x}_1, \hat{y}_1))\) is the maximal coupling of \( \mathcal{L}(\hat{x}_1+\eta) \) and \( \mathcal{L}(\hat{y}_1+\eta) \),

\[
\mathbb{P}(\xi_x(\hat{x}_1, \hat{y}_1) \neq \xi_y(\hat{x}_1, \hat{y}_1)) = \|\mathcal{L}(\hat{x}_1+\eta) - \mathcal{L}(\hat{y}_1+\eta)\|_{TV}.
\]

Note that the distributions \( \mathcal{L}(\hat{x}_1+\eta) \) and \( \mathcal{L}(\hat{y}_1+\eta) \) have the densities \( p_K(z - \hat{x}_1) \) and \( p_K(z - \hat{y}_1) \) respectively, where \( p_K \) is defined in Assumption [2.1]. It is easy to see that

\[
\|\mathcal{L}(\hat{x}_1+\eta) - \mathcal{L}(\hat{y}_1+\eta)\|_{TV} \leq \frac{1}{2} \int_{\mathbb{R}} |p_K(z - \hat{x}_1) - p_K(z - \hat{y}_1)|dz,
\]

this, together with (A2) of Assumption [2.1] immediately implies the desired inequality. \( \square \)

Define

\[
\bar{\xi}_x(\hat{x}, \hat{y}) := \begin{bmatrix} \xi_x(\hat{x}_1, \hat{y}_1) \\ \xi_x(\hat{x}_2, \hat{y}_2) \end{bmatrix}, \quad \bar{\xi}_y(\hat{x}, \hat{y}) := \begin{bmatrix} \xi_y(\hat{x}_1, \hat{y}_1) \\ \xi_y(\hat{x}_2, \hat{y}_2) \end{bmatrix}.
\]
Since $\mathcal{L}(\xi_x(\hat{x}_1, \hat{y}_1)) = \mathcal{L}(\hat{x}_1 + \eta)$ and $\mathcal{L}(\xi_y(\hat{x}_1, \hat{y}_1)) = \mathcal{L}(\hat{y}_1 + \eta)$, we have

\begin{align}
\mathcal{L}(\xi_x(\hat{x}, \hat{y})) &= \mathcal{L}(\hat{x} + \eta[1,0]^T), \\
\mathcal{L}(\xi_y(\hat{x}, \hat{y})) &= \mathcal{L}(\hat{y} + \eta[1,0]^T).
\end{align}

Denote by

$$P^{(1)}_{(x,y)}(\cdot) : \mathcal{B}(\mathbb{R}^2 \times \mathbb{R}^2) \to [0,1]$$

the probability of $(X^x(\tau_1), X^y(\tau_1))$ for $(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2$, and by

$$P^{(2)}_{(\hat{x}, \hat{y})}(\cdot) : \mathcal{B}(\mathbb{R}^2 \times \mathbb{R}^2) \to [0,1]$$

the probability of $(\xi_x(\hat{x}, \hat{y}), \xi_y(\hat{x}, \hat{y}))$ for $(\hat{x}, \hat{y}) \in \mathbb{R}^2 \times \mathbb{R}^2$. For all $A \in \mathcal{B}(\mathbb{R}^2 \times \mathbb{R}^2)$, define

$$P_{(x,y)}(A) := \int_{\mathbb{R}^2 \times \mathbb{R}^2} P^{(2)}_{(\hat{x}, \hat{y})}(A) P^{(1)}_{(x,y)}(d\hat{x}, d\hat{y}).$$

Using the transition probability family $(P^{(1)}_{(x,y)}(\cdot))_{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2}$, we construct an $\mathbb{R}^2 \times \mathbb{R}^2$-valued Markov chain $\{S(k)\}_{k \geq 0}$ on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. For $(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2$, denote by $(S^x(k))_{k \geq 0}$ the chain starting from $(x,y)$ and by $(S^x(k))_{k \geq 0}$, $(S^y(k))_{k \geq 0}$ the two marginal chains, i.e. $S^x(k) = (S^x(k), S^y(k))$ for all $k \geq 0$.

**Proposition 4.2.** For all $(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2$, $\{S^x(k)\}_{k \geq 0}$ (or $\{S^y(k)\}_{k \geq 0}$) has the same distribution as $\{X^x(\tau_k)\}_{k \geq 0}$ (or $\{X^y(\tau_k)\}_{k \geq 0}$ respectively). Therefore, $\{S^x(k)\}_{k \geq 0}$ (or $\{S^y(k)\}_{k \geq 0}$) is an $\mathbb{R}^2$-valued Markov chain starting from $x$ (or $y$ respectively).

**Proof.** To prove the claim in the proposition, it suffices to show that for all $x \in \mathbb{R}^2$, $y \in \mathbb{R}^2$, $A \in \mathcal{B}(\mathbb{R}^2)$, we have

$$P_{(x,y)}(A \times \mathbb{R}^2) = P_x(A),$$

where $P_{(x,y)}(\cdot)$ and $P_x(\cdot)$ are the transition probabilities of $(S(k))_{k \geq 0}$ and $(X(\tau_k))_{k \geq 0}$ respectively.

Recall that $P^{(1)}_x(\cdot)$ is the distribution of $X^x(\tau_1)$ and that $P^{(2)}_x(\cdot)$ is the distribution of $\hat{x} + \eta[1,0]^T$. It is clear that

$$P^{(1)}_{(x,y)}(\cdot \times \mathbb{R}^2) = P^{(1)}_x(\cdot).$$

By (4.8), we have

$$P^{(2)}_{(\hat{x}, \hat{y})}(\cdot \times \mathbb{R}^2) = P^{(2)}_\hat{x}(\cdot).$$
It follows from the definitions of $P_{(x,y)}(\cdot)$ and $P_x(\cdot)$ that
\[
P_{(x,y)}(A \times \mathbb{R}^2) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} P_{(x,y)}^{(2)}(A \times \mathbb{R}^2) P_{(x,y)}^{(1)}(d\hat{x}, d\hat{y})
\]
\[
= \int_{\mathbb{R}^2} P_{x}^{(2)}(A) P_{x}^{(1)}(d\hat{x})
\]
\[
= \int_{\mathbb{R}^2} P_{\hat{x}}^{(2)}(A) P_{\hat{x}}^{(1)}(d\hat{x})
\]
\[
= P_x(A).
\]

\[\square\]

5. Some estimates of the coupling chain $(S^{x,y}(k))_{k \geq 0}$

Recall that $(S^{x,y}(k))_{k \geq 0}$ is a Markov chain on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ is not necessarily the same as $(\Omega, \mathcal{F}, \mathbb{P})$ on which $(X^x(t))_{t \geq 0}$ and $(X^y(t))_{t \geq 0}$ is located. Without loss of generality, we assume that
\[
(5.1) \quad (\Omega, \mathcal{F}, \mathbb{P}) = (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}).
\]

Otherwise we can introduce the product space $(\tilde{\Omega} \times \Omega, \tilde{\mathcal{F}} \times \mathcal{F}, \tilde{\mathbb{P}} \times \mathbb{P})$ and consider $(S^{x,y}(k))_{k \geq 0}$, $(X^x(t))_{t \geq 0}$ and $(X^y(t))_{t \geq 0}$ all together on this new space. However, this will make the notations unnecessarily complicated, for instance, we have to always use $\tilde{\mathbb{P}} \times \mathbb{P}$.

From now on, we always assume (5.1) and consider $(S^{x,y}(k))_{k \geq 0}$, $(X^x(t))_{t \geq 0}$ and $(X^y(t))_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$.

**Proposition 5.1.** For all $x, y \in \mathbb{R}^2$, we have
\[
\mathbb{P}\{|S^x(k+1) - S^y(k+1)| \geq \delta_k |S^x(k) - S^y(k)| |S^{x,y}(k)\}
\]
\[
\leq \min \{ \kappa |S^x(k) - S^y(k)|^{\beta_2}, \beta_0/2 \}
\]
for all $k \geq 0$, where
\[
\delta_k := e^{-\lambda_2(\tau_{k+1} - \tau_k)} + \frac{\|F\|_{Lip}(\tau_{k+1} - \tau_k)\|F\|_{Lip}}{\lambda_2 + \|F\|_{Lip}}, \quad \kappa := \beta_1 e^{\beta_2 \|F\|_{Lip} T}
\]
and $\beta_0, \beta_1, \beta_2$ are the constants in Assumption 2.7.

**Proof.** Since $\{S^{x,y}(k)\}_{k \geq 0}$ is a time-homogeneous Markov chain, it suffices to show the inequality for $k = 0$, i.e.
\[
\mathbb{P}\{|S^x(1) - S^y(1)| \geq \delta_0 |x - y|\} \leq \min\{\beta_0/2, \kappa |x - y|^{\beta_2}\}.
\]
By the construction of the Markov chain $\{S^{x,y}(k)\}_{k \geq 0}$, $S^{x,y}(1)$ has the same distribution as
\[
(\xi_x(X^x(\tau_1), X^y(\tau_1)), \xi_y(X^x(\tau_1), X^y(\tau_1)))
\]
For the notational simplicity, we shall write (5.4) by \((\xi_x, \xi_y)\) in shorthand. By (5.7),
\[
\bar{\xi}_x = \left[ \begin{array}{c} \xi_x \\ X_y^\frac{\beta}{2}(\tau_1^-) \end{array} \right], \quad \bar{\xi}_y = \left[ \begin{array}{c} \xi_y \\ X_x^\frac{\beta}{2}(\tau_1^-) \end{array} \right],
\]
where \(\xi_x\) and \(\xi_y\) are \(\xi(X^\tau_1(-), X_y^\tau_1(-))\) and \(\xi_y(X^\tau_1(-), X_y^\tau_1(-))\) in shorthand respectively. We have
\[
P(|\bar{\xi}_x - \bar{\xi_y}| > \delta_0|x - y|)
\leq P(|\xi_x - \xi_y| + |X^\tau_x(\tau_1) - X^\tau_y(\tau_1)| > \delta_0|x - y|)
\leq P(\xi_x \neq \xi_y) + P(\xi_x = \xi_y, |X^\tau_x(\tau_1) - X^\tau_y(\tau_1)| > \delta_0|x - y|)
\]
On the one hand, it follows from Lemma 4.1 that
\[
\mathbb{P}(\xi_x \neq \xi_y) = \mathbb{E}\left[\mathbb{P}(\xi_x \neq \xi_y | (X^\tau_1(\tau_1^-), X^\tau_1(\tau_1^-)))\right]
\leq \frac{1}{2} \min \left\{ \beta_0, \beta_1 \mathbb{E}\left|X^\tau_1(\tau_1^-) - X^\tau_1(\tau_1^-)\right|^2 \right\}
\]
This, together with (2) of Lemma 3.1 implies
\[
P(\xi_x \neq \xi_y) \leq \frac{1}{2} \min \left\{ \beta_0, \beta_1 \mathbb{E}\left[e^{\beta_2 \|F\|_L_{p,\tau_1}} \right] |x - y|^{\beta_2} \right\}
\leq \min \left\{ \beta_0/2, \kappa |x - y|^{\beta_2} \right\}
\]
where the last inequality is by (A2) of Assumption 2.1.

On the other hand, it follows from (2) of Lemma 3.1 that
\[
|X^\tau_x(\tau_1) - X^\tau_y(\tau_1)| \leq \delta_0|x - y| \quad \text{a.s.,}
\]
therefore,
\[
P\left( |X^\tau_x(\tau_1) - X^\tau_y(\tau_1)| > \delta_0|x - y| \right) = 0.
\]
Collecting (5.5)-(5.7), we immediately get the desired inequality. \(\square\)

Given \(M, d > 0\), define the stopping times
\[
\bar{\sigma}(x, y, M) := \inf \left\{ k > 0 : |S^x(k)| + |S^y(k)| \leq M \right\},
\]
\[
\sigma(x, y, d) := \inf \left\{ k > 0 : |S^x(k)| - |S^y(k)| \leq d \right\},
\]
we shall \(\bar{\sigma} = \bar{\sigma}(x, y, M), \sigma = \sigma(x, y, d)\) in shorthand if no confusions arise. Let us prove the following two theorems:

**Theorem 5.2.** For all \(p \in (0, \alpha)\), as \(T > T_0 := \frac{(p-1)\log 3}{p\lambda_1} \lor 0\), there exist positive constants \(M, \bar{\sigma}, C\) depending on \(p, \lambda, \|F\|_L, T, \nu\) so that
\[
\mathbb{E}(x, y)[e^{\bar{\sigma}(x, y, M)}] < C(1 + |x|^p + |y|^p)
\]
for all \(x, y \in \mathbb{R}^2\).
Theorem 5.3. There exists some constants $\vartheta, C > 0$ depending on $p, \lambda, \|F\|_0, \|F\|_{{Lip, d}}, K, \nu$, such that for all $p \in (0, \alpha)$ and $x, y \in \mathbb{R}^2$,
\begin{equation}
E_{(x,y)}[e^{\vartheta \sigma(x,y,d)}] \leq C (1 + |x|^p + |y|^p).
\end{equation}

Proof of Theorem 5.2. To prove the theorem, it suffices to show that for all $p \in (0, \alpha)$, as $T > T_0 := \frac{(p-1) \log 3}{p \lambda_1} \vee 0$, there exist some $M > 0$ depending on $p, \lambda, M$ and some $q \in (0, 1)$ depending on $p, \lambda, M$ such that
\begin{equation}
P_{(x,y)}(\tilde{\sigma} > k) \leq q^k (1 + |x|^p + |y|^p) \quad k \geq 1,
\end{equation}
for all $x, y \in \mathbb{R}^2$. Note that (5.11) immediately implies
\begin{equation}
P_{(x,y)}(\tilde{\sigma} = \infty) = 0.
\end{equation}

The proof of (5.11) is by the same argument as that in Lemma 6.5 of [35]. To apply that argument, we only need to show
\begin{equation}
E(|S^x(\tau_1)|^p + |S^y(\tau_1)|^p) \leq q^2 (|x|^p + |y|^p) + C.
\end{equation}
where $C$ depends on $\lambda, p, \|F\|_0, \nu$.

By Proposition 4.2, for all $p \in (0, \alpha)$ we have
\begin{equation}
E(|S^x(\tau_1)|^p + |S^y(\tau_1)|^p) = E|X^x(\tau_1)|^p + E|X^y(\tau_1)|^p
\end{equation}
which, together with the first statement of Lemma 3.1 implies
\begin{equation}
E(|S^x(\tau_1)|^p + |S^y(\tau_1)|^p) \leq \gamma K |x|^p + |y|^p + C
\end{equation}
where $C$ depends on $\lambda, \|F\|_0, p, \nu$. Therefore, to show (5.13), we only need to show that
\begin{equation}
(3^{p-1} \wedge 1)E[e^{-p \lambda_1 \tau_1}] < 1.
\end{equation}
When $p \leq 1$, (5.15) automatically holds for all $T \geq 0$. When $p > 1,$
\begin{equation}
3^{p-1}E[e^{-p \lambda_1 \tau_1}] = \frac{3^{p-1} \gamma K e^{-p \lambda_1 T}}{\gamma K + p \lambda_1} < 1
\end{equation}
as $T > T_0$. \hfill \square

Lemma 5.4. Let $x, y \in \mathbb{R}^2$ be such that $|x| + |y| \leq M$. As $\lambda_2$ satisfies (2.6), we have
\begin{equation}
P\{|S^x(1) - S^y(1)| > d\} < \frac{1}{2} + \frac{\beta_0}{4}
\end{equation}
where $\beta_0 \in (0, 2)$ is the constant defined in Assumption 2.7.
Proof. It is easy to have
\[ P\{|S^x(1) - S^y(1)| > d\} \leq P\{|S^x(1) - S^y(1)| > \delta_0|x - y|, \delta_0 \leq d/M\} + P\{\delta_0 > d/M\} \]
where \(\delta_0\) and \(\kappa\) are defined in Proposition \ref{prop:5.1}. This inequality, together with Proposition \ref{prop:5.1}, Markov inequality, implies
\[ P\{|S^x(1) - S^y(1)| > d\} \leq \beta_0/2 + M\mathbb{E}(x,y)\delta_0 \]
\[ \leq 1/2 + \beta_0/4, \]
where the last inequality is by (2.6). \(\square\)

Proof of Theorem \ref{thm:5.3}. To prove the theorem, it suffices to show that (5.16)
\[ P_{(x,y)}\{\sigma = \infty\} = 0, \]
and that for all \(p \in (0, \alpha)\), there exist some \(\gamma > 0\) and \(C > 0\) depending on \(p, \lambda, \|F\|_0, \|F\|_{Lip}, \nu, K\), so that (5.17)
\[ P_{(x,y)}\{\sigma = k\} \leq Ce^{-\gamma k}(1 + |x|^p + |y|^p) \]
for all \(k > 0\) and \(x, y \in \mathbb{R}^2\). Let us first prove (5.17) and then (5.16) in the following four steps.

Step 1. Write \(\tilde{\sigma}_0 = 0\), define
\[ \tilde{\sigma}_{k+1} = \inf\{j > \tilde{\sigma}_k + 1; |S^x(j)| + |S^y(j)| \leq M\} \]
for all integer \(k \geq 0\). Since \((S(k))_{k \geq 0}\) is a discrete time Markov chain, it is strong Markovian. Therefore, it follows from Theorem \ref{thm:5.2} that
\[ \mathbb{E}(S(\tilde{\sigma}_k)) e^{\tilde{\vartheta}(\tilde{\sigma}_{k+1} - \tilde{\sigma}_{k+1})} \leq C(1 + |S(\tilde{\sigma}_k)|^p) \leq C(1 + M^p). \]
where \(C, \tilde{\vartheta}\) depends on \(\lambda, p, M, T, \nu, \|F\|_0\). The above inequality, together with strong Markov property, implies
\[ \mathbb{E}(x,y)[e^{\tilde{\vartheta} \tilde{\sigma}_k}] = \mathbb{E}(x,y) \left[ e^{\tilde{\vartheta} \tilde{\sigma}_1} \mathbb{E}(\tilde{\sigma}_1) \left[ e^{\tilde{\vartheta} \tilde{\sigma}_2 - \tilde{\sigma}_1} \cdots \mathbb{E}(\tilde{\sigma}_{k-1}) \left[ e^{\tilde{\vartheta} \tilde{\sigma}_k - \tilde{\sigma}_{k-1}} \cdots \right] \right] \right] \]
\[ \leq C^k e^{\tilde{\vartheta} k} (1 + M^p)^{k-1} (1 + |x|^p + |y|^p). \]
Step 2. Given any \( k \in \mathbb{N} \), define
\[
\tilde{\rho}_k = \sup \{ j; \tilde{\sigma}_j \leq k \}.
\]
Clearly, \( \tilde{\rho}_{k+1} > k \) if \( \tilde{\rho}_k < \infty \). We have
\[
\mathbb{P}_{(x,y)}(\sigma = k) = \sum_{j=0}^{k} \mathbb{P}_{(x,y)}(\sigma = k, \tilde{\rho}_k = j)
\]
(5.20)
\[
\leq \sum_{j=0}^{k} \mathbb{P}_{(x,y)}(\tilde{\rho}_k = j) + \sum_{j=1}^{k} \mathbb{P}_{(x,y)}(\sigma = k, \tilde{\rho}_k = j)
\]
\[
= I_1 + I_2
\]
where \( l < k \) is some integer number to be chosen later.

Step 3. Let us estimate the above \( I_1 \) and \( I_2 \). By the definition of \( \tilde{\rho}_k \), Chebyshev inequality and strong Markov property, we have
\[
\mathbb{P}_{(x,y)}(\tilde{\rho}_k = j) \leq \mathbb{P}_{(x,y)}(\tilde{\sigma}_j \leq k/2, \tilde{\rho}_k = j) + \mathbb{P}_{(x,y)}(\tilde{\sigma}_j > k/2)
\]
\[
\leq \mathbb{P}_{(x,y)}(\tilde{\sigma}_j \leq k/2, \tilde{\sigma}_{j+1} > k) + \mathbb{P}_{(x,y)}(\tilde{\sigma}_j > k/2)
\]
\[
\leq \mathbb{E}_{(x,y)} \left[ \mathbb{P}_{S(\tilde{\sigma}_j)}(\tilde{\sigma}_{j+1} - \tilde{\sigma}_j > k/2) \right] + e^{-\tilde{\delta}k/2} \mathbb{E}_{(x,y)}[e^{\tilde{\delta}\tilde{\sigma}_j}]
\]
By (5.18) and (5.19), the above inequality implies
\[
\mathbb{P}_{(x,y)}(\tilde{\rho}_k = j) \leq Ce^{\tilde{\delta}}(1 + M^p)e^{-\tilde{\delta}k/2}
\]
\[
+ C'\tilde{e}^{\tilde{\delta}} (1 + M^p)^{-1}(1 + |x|^p + |y|^p)e^{-\tilde{\delta}k/2}
\]
Hence,
\[
I_1 \leq (Ce^{\tilde{\delta}})^{l+2}(1 + M^p)^{l+2}(1 + |x|^p + |y|^p)e^{-\tilde{\delta}k/2}.
\]
Now we estimate \( I_2 \). For \( j \in \mathbb{N} \), define
\[
A_j := \{ |S^x(\tilde{\sigma}_1 + 1) - S^y(\tilde{\sigma}_1 + 1)| > d, \ldots, |S^x(\tilde{\sigma}_j + 1) - S^y(\tilde{\sigma}_j + 1)| > d \}.
\]
By the definitions of \( \sigma \) and \( \tilde{\rho}_k \), strong Markov property, we have
\[
\mathbb{P}_{(x,y)}(\sigma = k, \tilde{\rho}_k = j) \leq \mathbb{P}_{(x,y)}(A_{j-1})
\]
\[
= \mathbb{P}_{(x,y)} \left\{ |S^x(\tilde{\sigma}_{j-1} + 1) - S^y(\tilde{\sigma}_{j-1} + 1)| > d, A_{j-2} \right\}
\]
\[
= \mathbb{E}_{(x,y)} \{ \mathbb{P}_u \{ |S^{u^x}(1) - S^{u^y}(1)| > d \} A_{j-2} \},
\]
where \( u = S^{x,y}(\tilde{\sigma}_{j-1}) \). Combining with Lemma 5.4, the above inequality implies
\[
\mathbb{P}_{(x,y)}(\sigma = k, \tilde{\rho}_k = j) \leq \left( \frac{1}{2} + \frac{\beta_0}{4} \right)^j \mathbb{P}_{(x,y)}(A_{j-2}) \leq \left( \frac{1}{2} + \frac{\beta_0}{4} \right)^{j-1}
\]
Hence,
\[(5.22)\]
\[I_2 \leq \left(\frac{1}{2} + \frac{\beta_0}{4}\right)^l\]

Take \(l = \varepsilon k\), it follows from the bounds of \(I_1\) and \(I_2\) that as \(\varepsilon > 0\) is sufficiently small, (5.17) holds.

**Step 4:** Let us now show (5.16). Define \(\hat{\rho}_\infty = \sup\{j; \hat{\sigma}_j < \infty\}\), it is clear that \(\hat{\rho}_{\hat{\rho}_\infty + 1} = \infty\) if \(\hat{\rho}_\infty < \infty\). For all \(j \in \mathbb{N} \cup \{0\}\), by strong Markov property and Theorem 5.2 we have
\[(5.23)\]
\[\mathbb{P}_{(x,y)}(\hat{\rho}_\infty = j) = \mathbb{E}_{(x,y)}(\mathbb{P}_{S(\hat{\sigma}_j)}(\hat{\sigma}_{j+1} - \hat{\sigma}_j = \infty)) = 0.\]
Hence,
\[\mathbb{P}_{(x,y)}(\hat{\rho}_\infty = \infty) = 1 \quad \forall \ x, y \in \mathbb{R}^2.\]
By a similar computation as estimating \(I_2\) in step 3, we have
\[\mathbb{P}_{(x,y)}(\sigma = \infty) = \mathbb{P}_{(x,y)}(\sigma = \infty, \hat{\rho}_\infty = \infty) \leq \mathbb{P}_{(x,y)}(A_j) \leq \left(\frac{1}{2} + \frac{\beta_0}{4}\right)^j \to 0\]
as \(j \to \infty\).

6. **Proof of the main theorem**

Define
\[(6.1)\]
\[\hat{\sigma}(x,y) := \inf\{k \geq 1; |S^x(k) - S^y(k)| > (\delta_0 \ldots \delta_{k-1})|x - y|\}\]
where \(\delta_j (j = 0, \ldots, k - 1)\) are defined in Proposition 5.1, we shall often write \(\hat{\sigma} = \hat{\sigma}(x,y)\) in shorthand.

**Lemma 6.1.** If \(|x - y| \leq d\) with \(0 < d < (\frac{1}{4\kappa})^{1/\beta_2}\) and \(\kappa\) defined in Proposition 5.7, we have

1. \(\mathbb{P}_{(x,y)}(\hat{\sigma} = \infty) > 1/2.\)
2. There exists some \(\epsilon, C > 0\) (possibly small) depending on \(d, \lambda, \|F\|_0, \|F\|_{Lip}, \alpha, K, \epsilon, \nu\) such that
\[\mathbb{E}_{(x,y)}[e^{\epsilon \hat{\sigma}} 1_{\{\hat{\sigma} < \infty\}}] \leq C.\]

**Proof.** For all \(k \geq 0\), define
\[B_k := \{|S^x(k+1) - S^y(k+1)| > \delta_k|S^x(k) - S^y(k)|\},\]
\[C_k := \{|S^x(j+1) - S^y(j+1)| \leq (\delta_0 \ldots \delta_j)|x - y|, \ 0 \leq j \leq k\},\]
it is easy to see that \( C_k \supset B_k^c \cap C_{k-1} \). It follows from Proposition 5.1 that
\[
\mathbb{P}_{(x,y)}(C_k) \geq \mathbb{P}_{(x,y)}(B_k^c \cap C_{k-1})
\]
\[
= \mathbb{E}_{(x,y)}\left[ \mathbb{P}\left( B_k^c \left| S^{x,y}(k) \right. \right) 1_{C_{k-1}} \right]
\]
\[
= \mathbb{E}_{(x,y)}\left\{ \left[ 1 - \mathbb{P}\left( B_k \left| S^{x,y}(k) \right. \right) \right] 1_{C_{k-1}} \right\}
\]
\[
\geq \mathbb{E}_{(x,y)}\left[ (1 - \kappa|S^x(k) - S^y(k)|^{\beta_2}) 1_{C_{k-1}} \right]
\]
\[
\geq \mathbb{P}_{(x,y)}(C_{k-1}) - \kappa \mathbb{E}_{(x,y)}\left[ (|\delta_0 \ldots \delta_{k-1}|^{\beta_2}) |x - y|^{\beta_2} \right]
\]
This inequality, together with (4.3), (4.4), (A3) of Assumption 2.1, implies that
\[
\mathbb{P}_{(x,y)}(C_k) \geq \mathbb{P}_{(x,y)}(C_{k-1}) - \kappa \theta^k |x - y|^{\beta_2}
\]
\[
\geq 1 - \frac{1}{\theta} \frac{1 - \theta^{k+1}}{1 - \theta} |x - y|^{\beta_2} > 1/2
\]
since \( \theta < 1/2 \) (see (2.7)) and \( d < \left( \frac{1}{\theta} \right)^{1/\beta_2} \). Let \( k \to \infty \), we get (1).

Defining \( D_k := \{|S^x(k+1) - S^y(k+1)| > (|\delta_0 \ldots \delta_k|) |x - y|\} \) for all \( k \geq 0 \), by a similar calculation as above we have
\[
\mathbb{P}_{(x,y)}(\hat{\sigma} = k) = \mathbb{P}_{(x,y)}(D_{k-1} \cap C_{k-2})
\]
\[
\leq \mathbb{P}_{(x,y)}(B_{k-1} \cap C_{k-2})
\]
\[
= \mathbb{E}_{(x,y)}\left[ \mathbb{P}\left( B_{k-1} \left| S^{x,y}(k-1) \right. \right) 1_{C_{k-2}} \right]
\]
\[
\leq \kappa \mathbb{E}_{(x,y)}\left[ (|\delta_0 \ldots \delta_{k-2}|^{\beta_2}) |x - y|^{\beta_2} \right]
\]
\[
\leq \frac{1}{2} \theta^{k-1} \leq \left( \frac{1}{\theta} \right)^k.
\]
This immediately implies (2). \( \square \)

Define
\[
\sigma^+(x, y, d) := \sigma + \hat{\sigma}(S^{x,y}(\sigma))
\]
where \( \sigma = \sigma(x, y, d) \) defined by (5.9). Further define
\[
\bar{\sigma}(x, y, d, M) := \sigma^+ + \hat{\sigma}(S^{x,y}(\sigma^+), M).
\]
where \( \sigma^+ = \sigma^+(x, y, d) \) and \( \bar{\sigma} \) is defined in (5.8).

The motivation for defining \( \bar{\sigma} \) is the following: we only know \(|S^x(\sigma^+) - S^y(\sigma^+)| \leq d\), but have no idea about the bound of \(|S^x(\sigma^+) - |S^y(\sigma^+)\|\). This bound is very important for iterating a stopping time argument as in Step 1 of the proof of Theorem 5.3. To this aim, we introduce (6.3) and thus have \(|S^{x,y}(\bar{\sigma})| \leq M\) for all \( x, y \in \mathbb{R}^2 \).

**Lemma 6.2.** Let \( 0 < d < \left( \frac{1}{\theta} \right)^{1/\beta_2} \) and \( p \in (0, \alpha) \). There exist some \( \gamma, C > 0 \) depending on \( d, \lambda, \|F\|_0, \|F\|_1, p, \nu, M, K \) such that
\[
\mathbb{E}_{(x,y)}[e^{\gamma \bar{\sigma}(x,y,d,M)} 1_{(\sigma(x,y,d,M) < \infty)}] \leq C(1 + |x|^p + |y|^p).
\]
Proof. Note that $\sigma < \infty$ a.s. by Theorem 5.3. By the strong Markov property we have
\[
\mathbb{E}_{(x,y)}[e^{\gamma \sigma^l(x,y,d)}1_{\{\sigma^l(x,y,d) < \infty\}}] = \mathbb{E}_{(x,y)}\left\{\mathbb{E}_u[e^{\gamma \tilde{\sigma}}1_{\{\tilde{\sigma} < \infty\}}] e^{\gamma \sigma^l(x,y,d)}\right\}
\]
where $\sigma = \sigma(x,y,d), \ u = S^{x,y}(\sigma), \ \tilde{\sigma} = \tilde{\sigma}(S^{x,y}(\sigma))$.

By (2) of Lemma 6.1 and Theorem 5.3 as $\gamma > 0$ is sufficiently small we immediately get
\[
\mathbb{E}_{(x,y)}[e^{\gamma \sigma^l(x,y,d)}1_{\{\sigma^l(x,y,d) < \infty\}}] \leq C (1 + |x|^p + |y|^p)
\]
where $C$ is some constant depending on $d, \lambda, \|F\|_0, \|F\|_{\text{Lip}}, p, \nu, \alpha, K$.

By strong Markov property and the above inequality, we have
\[
\mathbb{E}_{(x,y)}[e^{\gamma \tilde{\sigma}(x,y,d,M)}1_{\{\sigma(x,y,d,M) < \infty\}}] \leq \mathbb{E}_{(x,y)}\left\{\mathbb{E}_u\left[e^{\gamma \tilde{\sigma}(u,M)}\right] e^{\gamma \sigma^l(x,y,d)}1_{\{\sigma^l(x,y,d) < \infty\}}\right\}
\]
\[
\leq C \mathbb{E}_{(x,y)}\left[(1 + |u|^{p/2}) e^{\gamma \sigma^l(x,y,d)}1_{\{\sigma^l(x,y,d) < \infty\}}\right]
\]
where $u = S^{x,y}(\sigma^l)$ and $C$ depends on $M, \lambda, \|F\|_0, \|F\|_{\text{Lip}}, \nu, p, \alpha, K$. Note that $\mathbb{E}[S^{x,y}(\sigma^l)]^p = \mathbb{E}[X^x(\tau_{\sigma^l})]^p + \mathbb{E}[X^y(\tau_{\sigma^l})]^p \leq C(1 + |x|^p + |y|^p)$ from (1) of Lemma 3.1. The inequality (6.5), together with Hölder inequality and (6.4), immediately implies the desired inequality as $\gamma > 0$ is sufficiently small. \hfill \Box

Define $\tilde{\sigma}_0 = 0$, for all $k \geq 0$ we define
\[
\tilde{\sigma}_{k+1} = \tilde{\sigma}_k + \tilde{\sigma}(S^{x,y}(\tilde{\sigma}_k), d, M).
\]
Of course each $\tilde{\sigma}_k$ depends on $x, y, d, M$.

Lemma 6.3. Let $k \in \mathbb{N}$. For all $x, y \in \mathbb{R}^2$, we have
\[
\mathbb{P}_{(x,y)}(\tilde{\sigma}_k < \infty) \leq 1/2^k.
\]

Proof. Recall that $\sigma < \infty$ a.s., by the definition of $\tilde{\sigma}$, strong Markov property, (1) of Lemma 6.1 we have that for all $x, y \in \mathbb{R}^2$
\[
\mathbb{P}_{(x,y)}(\tilde{\sigma} = \infty) = \mathbb{E}_{(x,y)}\left[\mathbb{P}_{S^{x,y}(\sigma)}(\tilde{\sigma} = \infty)\right] > 1/2.
\]
This, together with strong Markov property, implies that as $\tilde{\sigma}_{k-1} < \infty$,
\[
\mathbb{P}_u(\tilde{\sigma}_k - \tilde{\sigma}_{k-1} = \infty) > 1/2,
\]
where $u = S^{x,y}(\tilde{\sigma}_{k-1})$. Hence
\[
\mathbb{P}_{(x,y)}(\tilde{\sigma}_k < \infty) = \mathbb{P}_{(x,y)}(\tilde{\sigma}_k < \infty, \tilde{\sigma}_{k-1} < \infty)
\]
\[
\leq \mathbb{E}_{(x,y)}\left[\mathbb{P}_u(\tilde{\sigma}_k - \tilde{\sigma}_{k-1} < \infty)1_{\{\tilde{\sigma}_{k-1} < \infty\}}\right] \leq \frac{1}{2}\mathbb{P}_{(x,y)}(\tilde{\sigma}_{k-1} < \infty) \leq \frac{1}{2^k}. \hfill \Box
\]
Proof of Theorem 2.4. The existence of invariant measures has been established in [36]. According to Section 2.2. of [44], the inequality (2.8) in the theorem implies the uniqueness of the invariant measure. So now we only need to show (2.8), by [44] again, it suffices to show that for all \( p \in (0, \alpha) \) we have

\[
|P_t f(x) - P_t f(y)| \leq C e^{-c t} \| f \|_p (1 + |x|^p + |y|^p) \quad \forall f \in L_b(\mathbb{R}^2),
\]

where \( C, c \) depend on \( p, \alpha, \beta, \nu, K, \| F \|_{Lip}, \lambda \). Let us prove (6.7) by the following five steps.

Step 1. Let \( l \in \mathbb{N} \) be some constant to be determined later. We easily have

\[
|\mathbb{E}[f(X^x(t))] - \mathbb{E}[f(X^y(t))]| \leq I_1 + I_2
\]

with

\[
I_1 := \left| \mathbb{E} \left\{ [f(X^x(t)) - f(X^y(t))] 1_{\{\tau_l > t\}} \right\} \right|,
\]

\[
I_2 := \left| \mathbb{E} \left\{ [f(X^x(t)) - f(X^y(t))] 1_{\{\tau_l \leq t\}} \right\} \right|.
\]

By (4.3), we have \( e^{\gamma \tau_{K/2}} = e^{\gamma K T/2} 2^l \), therefore,

\[
I_1 \leq 2 \| f \|_0 \mathbb{P}(\tau_l > t) \leq 2 \| f \|_0 (2 e^{\gamma K T/2})^{l} e^{-\gamma K T/2}.
\]

Step 2. Now we bound \( I_2 \). The strong Markov property implies

\[
I_2 \leq \sum_{j \geq l} \left| \mathbb{E} \left\{ [f(X^x(t)) - f(X^y(t))] 1_{\{\tau_j \leq t < \tau_{j+1}\}} \right\} \right| = \sum_{j = l}^{\infty} I_{2,j},
\]

where

\[
I_{2,j} := \left| \mathbb{E} \left\{ [g_j(X^x(\tau_j)) - g_j(X^y(\tau_j))] 1_{\{\tau_j \leq t\}} \right\} \right|
\]

with

\[
g_j(X^x(\tau_j)) = \mathbb{E}[f(X^x(t)) 1_{\{\tau_{j+1} > t\}} | X^x(\tau_j)]
\]

and similarly for \( g_j(X^y(\tau_j)) \).

Note that on the set \( \{\tau_j \leq t\} \),

\[
g_j(u_x) = \mathbb{E} \left[ f(X^{u_x}(t - \tau_j)) 1_{\{\tau_{j+1} > t\}} \right],
\]

\[
g_j(u_y) = \mathbb{E} \left[ f(X^{u_y}(t - \tau_j)) 1_{\{\tau_{j+1} > t\}} \right],
\]

where \( u_x = X^x(\tau_j), u_y = X^y(\tau_j) \). It follows from (2) of Lemma 3.1 that

\[
|g_j(u_x) - g_j(u_y)|
\]

\[
\leq \mathbb{E} \left[ |f(X^{u_x}(t - \tau_j)) - f(X^{u_y}(t - \tau_j))| 1_{\{\tau_{j+1} > t\}} \right]
\]

\[
\leq \| f \|_1 \mathbb{E} \left[ |X^{u_x}(t - \tau_j) - X^{u_y}(t - \tau_j)| 1_{\{\tau_{j+1} > t\}} \right]
\]

\[
\leq \| f \|_1 \mathbb{E} \left[ e^{\| F \|_{Lip} (\tau_{j+1} - \tau_j)} \right] |u_x - u_y|
\]

\[
\leq C \| f \|_1 |u_x - u_y|,
\]

where \( C > 0 \) is some constant. Letting \( l \to \infty \), we obtain the desired result.

End of proof.
where $C$ depends on $T, \alpha, \|F\|_{Lip}, K$.

By Proposition 6.2 and the easy fact $\|g\|_0 \leq \|f\|_0$, we have

\[
I_{2,j} \leq \|\mathbb{E} [g_j(X^x(\tau_j)) - g_j(X^y(\tau_j))] + \mathbb{E} \left\{ [g_j(X^x(\tau_j)) - g_j(X^y(\tau_j))] 1_{\{\tau_j > t\}} \right\} \|
\leq \|\mathbb{E} [g_j(S^x(j)) - g_j(S^y(j))] \| + 2\|f\|_0 \mathbb{P}(\tau_j > t)
\]

Let $m = [\varepsilon j]$ with $0 < \varepsilon < 1/2$ to be determined later, we further have

\[
I_{2,j} \leq J_0 + J_1 + J_2,
\]

where

\[
J_0 := 2\|f\|_0 \mathbb{P}(\tau_j > t),
J_1 := \|\mathbb{E} \left\{ [g_j(S^x(j)) - g_j(S^y(j))] 1_{\{\hat{\sigma}_m \leq j/2\}} \right\} \|
J_2 := \|\mathbb{E} \left\{ [g_j(S^x(j)) - g_j(S^y(j))] 1_{\{\hat{\sigma}_m > j/2\}} \right\} \|.
\]

Step 3. By a similar calculation as for $I_1$, we have

\[
J_0 \leq 2\|f\|_0 (2e^{\gamma \kappa T/2})^j e^{-\gamma \kappa t/2}.
\]

By the easy fact $\|g\|_0 \leq \|f\|_0$ and Lemma 6.3 we have

\[
(6.10) \quad J_1 \leq 2\|f\|_0 \mathbb{P}_{(x,y)} \{ \hat{\sigma}_m \leq j/2 \} \leq \frac{\|f\|_0}{2^{m-1}} \leq \frac{\|f\|_0}{2^{j/2-2}}.
\]

As for $J_2$, we have

\[
J_2 \leq \mathbb{E} \left\{ |g_j(S^x(j)) - g_j(S^y(j))| 1_{\{\hat{\sigma}_m > j/2\}} \right\}
\leq 2\|f\|_0 \mathbb{P}_{(x,y)} \left( \frac{j}{2} < \hat{\sigma}_m < \infty \right) + \mathbb{E} \left\{ |g_j(S^x(j)) - g_j(S^y(j))| 1_{\{\hat{\sigma}_m = \infty\}} \right\}.
\]

On the one hand, thanks to Lemma 6.2, it follows from a similar argument as (5.19) that

\[
(6.12) \quad \mathbb{P}_{(x,y)} \left( \frac{j}{2} < \hat{\sigma}_m < \infty \right) \leq C^m (1 + Mp)^{m-1} e^{-\gamma j/2} (1 + |x|^p + |y|^p).
\]

As $\varepsilon > 0$ is sufficiently small, we get

\[
(6.13) \quad \mathbb{P}_{(x,y)} \left( \frac{j}{2} < \hat{\sigma}_m < \infty \right) \leq e^{-\gamma j/4} (1 + |x|^p + |y|^p).
\]

Hence,

\[
J_{2,1} := 2\|f\|_0 \mathbb{P}_{(x,y)} \left( \frac{j}{2} < \hat{\sigma}_m < \infty \right) \leq 2\|f\|_0 e^{-\gamma j/4} (1 + |x|^p + |y|^p).
\]
On the other hand, 
\[
\mathbb{E} \left\{ |g_j(S^x(j)) - g_j(S^y(j))| \mathbf{1}_{\sigma_m = \infty} \right\} \\
= \sum_{i=0}^{m-1} \mathbb{E} \left\{ |g_j(S^x(j)) - g_j(S^y(j))| \mathbf{1}_{\sigma_i < j < \infty, \sigma_{i+1} = \infty} \right\} \\
+ \sum_{i=0}^{m-1} \mathbb{E} \left\{ |g_j(S^x(j)) - g_j(S^y(j))| \mathbf{1}_{\sigma_i / 2 < \sigma_i < \infty, \sigma_{i+1} = \infty} \right\} \\
=: J_{2,2} + J_{2,3}.
\]

By Lemma 6.2 and the same argument as above, it is easy to see that
\[
J_{2,2} \leq 2\|f\|_0 \sum_{i=0}^{m-1} \mathbb{P}(x,y) \left\{ j/2 < \bar{\sigma}_i < \infty \right\} \\
\leq 2\|f\|_0 e^{-\frac{2\delta}{\bar{\sigma}^2}} \sum_{i=0}^{m-1} C^{i+1}(1 + M^p)^i(1 + |x|^p + |y|^p).
\]

As \( \varepsilon > 0 \) is small enough, we have
\[
J_{2,2} \leq e^{-\frac{2\delta}{\bar{\sigma}^2}} \|f\|_0 (1 + |x|^p + |y|^p).
\]

**Step 4.** It remains to bound \( J_{2,3} \). Recall the definition of \( \sigma, \hat{\sigma}, \sigma^\dagger, \tilde{\sigma}, \bar{\sigma} \) and note that
\[
(6.14) \quad \bar{\sigma}_{i+1} = \bar{\sigma}_i + \sigma + \hat{\sigma} + \tilde{\sigma},
\]
with \( \sigma = \sigma(S^{x,y}(\bar{\sigma}_i), d), \hat{\sigma} = \hat{\sigma}(S^{x,y}(\bar{\sigma}_i + \sigma), \tilde{\sigma} = \tilde{\sigma}(S^{x,y}(\bar{\sigma}_i + \sigma + \hat{\sigma}, M) \). It is clear to see \( \bar{\sigma}_i + \sigma(S^{x,y}(\bar{\sigma}_i), d) \leq \bar{\sigma}_{i+1} \). Observe that
\[
J_{2,3} = J_{2,3,1} + J_{2,3,2},
\]
with
\[
J_{2,3,1} := \sum_{i=0}^{m-1} \mathbb{E} \left[ |g_j(S^x(j)) - g_j(S^y(j))| \mathbf{1}_{\sigma_i \leq j / 2, \sigma_i + \sigma > \frac{3}{4}, \sigma_{i+1} = \infty} \right];
\]
\[
J_{2,3,2} := \sum_{i=0}^{m-1} \mathbb{E} \left[ |g_j(S^x(j)) - g_j(S^y(j))| \mathbf{1}_{\sigma_i \leq j / 2, \sigma_i + \sigma \leq \frac{3}{4}, \sigma_{i+1} = \infty} \right].
\]
By strong Markov property, Theorem 5.3 and the clear fact \( |S(\bar{\sigma})| \leq M \) for all \( i \geq 1 \), as \( \varepsilon > 0 \) is sufficiently small we have

\[
J_{2,3,1} \leq 2 \|f\|_0 \sum_{i=0}^{m-1} \mathbb{E}_{(x,y)} \left[ \mathbb{P}_{u_i}(\sigma > j/4) \right] \\
\leq C \|f\|_0 e^{-\theta j/4} \left[ (m - 1)(1 + M^p) + (1 + |x|^p + |y|^p) \right] \\
\leq C \|f\|_0 e^{-\theta j/8} (1 + |x|^p + |y|^p)
\]

(6.15)

where \( u_i = \sigma^{x,y}(\bar{\sigma}_i) \) and \( C \) depends on \( d, \lambda, \|F\|_0, \|F\|_1, p, \nu, M \).

As for \( J_{2,3,2} \), recall (6.14) and note that \( \hat{\sigma} < \infty \) a.s. from (5.12), we have

\[
J_{2,3,2} = \sum_{i=0}^{m-1} \mathbb{E} \left\{ |g_j(S^x(j)) - g_j(S^y(j))| 1_{\{\sigma_i \leq j/2, \sigma_i + \sigma \leq \frac{\varepsilon}{12}, \sigma = \infty}\} \right\} \\
= \sum_{i=0}^{m-1} \mathbb{E} \left\{ |g_j(S^x(j)) - g_j(S^y(j))| 1_{\{\sigma_i \leq j/2, \sigma_i + \sigma \leq \frac{\varepsilon}{12}, \sigma = \infty\}} \right\}.
\]

(6.16)

It follows from the above equality, (6.9) and strong Markov property that

\[
J_{2,3,2} \leq C \|f\|_1 \sum_{i=0}^{m-1} \mathbb{E} \left[ |S^x(j) - S^y(j)| 1_{\{\sigma_i + \sigma \leq \frac{\varepsilon}{12}, \sigma = \infty\}} \right] \\
\leq C \|f\|_1 \sum_{i=0}^{m-1} \mathbb{E} \left[ |S^x(j) - S^y(j)| 1_{\{\sigma_i + \sigma \leq \frac{\varepsilon}{12}, \sigma = \infty\}} \right] \\
= C \|f\|_1 \sum_{i=0}^{m-1} \mathbb{E} \left[ \mathbb{E}_u \left( |S^x(j) - S^y(j)| 1_{\{\sigma = \infty\}} \right) 1_{\{\sigma_i + \sigma \leq \frac{\varepsilon}{12}\}} \right]
\]

where \( u = S^{x,y}(\bar{\sigma}_i + \sigma) \). Note that \( |u_x - u_y| < d \). By the definition (6.1) with \( \bar{\sigma} = \hat{\sigma}(S^{x,y}(\bar{\sigma}_i + \sigma)) \), the previous inequality and (2.1), we have

\[
J_{2,3,2} \leq C \|f\|_1 \sum_{i=0}^{m-1} \mathbb{E} \left[ \mathbb{E}_u(\delta_0 \ldots \delta_{j/4})|u_x - u_y| \right] \\
\leq C \|f\|_1 dm \left( \frac{\gamma K}{\gamma K + \lambda_2} e^{-\lambda_2 T} + \frac{\|F\|_{\text{Lip}}}{\|F\|_{\text{Lip}} + \lambda_2} \right)^{j/4} \\
\leq C \|f\|_1 dm \left( \frac{1}{2} \right)^{\frac{j}{\lambda_2}}
\]

As \( \varepsilon > 0 \) is sufficiently small, we have

\[
J_{2,3,2} \leq C \|f\|_1 \left( \frac{1}{2} \right)^{\frac{j}{\lambda_2}}.
\]

(6.17)
Step 5. Collecting the bounds for $J_{2,3,1}$, $J_{2,3,2}$, $J_{2,1}$, $J_{2,2}$, $J_{1}$ and $J_{0}$, we have that there exist some $\epsilon, C > 0$ depending on $p, \lambda, \|F\|_0, \|F\|_{Lip, \nu}, K$ such that

$$I_{2,j} \leq J_{0} + J_{1} + J_{2,1} + J_{2,2} + J_{2,3,1} + J_{2,3,2} \leq C \|f\|_1 e^{-\epsilon j} (1 + |x|^p + |y|^p).$$

Hence,

$$I_{2} \leq \sum_{j=1}^{\infty} I_{2,j} \leq C \|f\|_1 e^{-\epsilon l} (1 + |x|^p + |y|^p).$$

Combining the estimates of $I_{1}$ and $I_{2}$, choosing $l = [\delta t]$ with $\delta > 0$ sufficiently small, we immediately obtain the inequality (6.7). \qed

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