Plate theories: A-priori fulfillment of the local conditions by the consistent-approximation approach

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Classical plates theories, like Kirchhoff’s plate theory [1], are based on kinematical a-priori assumptions. Avoiding these assumptions, we derive from the three-dimensional theory of linear elasticity by means of Taylor-series expansions the quasi two-dimensional problem. This problem consists of infinitely many partial-differential equations (PDEs) written in infinitely many displacement coefficients. With the consistent approximation approach we arrive at solvable hierarchical plate theories. By using the modular structure of the displacements coefficients (modularity), we obtain from these generic-plate theories the complete-plate theories, whose results fulfill the strong form of the local equilibrium conditions and the Neumann boundary conditions on the upper and lower face of the plate (local conditions) a-priori. Furthermore, we show that every variable of the complete-plate theories can be calculated.

1 Introduction

We assume a homogeneous, linear elastic, cuboid solid for which the thickness h is much smaller than the in-plane dimensions a and b (cf. fig.1). The midplane of this plate continuum is the $(x_1, x_2)$-plane (with $x_3 = 0$). So, it is parallel to the upper and lower faces of the solid and perpendicular to the $x_3$-direction. We allow arbitrary surface loads $g_1 = g_1(x_1, x_2, −\frac{h}{2})$ and $g_2 = g_2(x_1, x_2, \frac{h}{2})$ on the upper and lower faces of the plate as well as volume loads $f_i$. For the plate continuum at hand we can establish the elastic and dual potentials $E_{pot}$ and $E_{dual}$ of the three-dimensional theory of linear elasticity.

Fig. 1: Plate continuum [2]

2 Generic-plate theories

By using Taylor-series expansions in thickness-direction for all variables and integrating over the thickness, we derive from both potentials the quasi two-dimensional problem. For isotropic and transverse-isotropic material, this problem can be split into a disc and a plate problem. In the following, we only deal with the plate problem. Because it is written in infinitely many PDEs with infinitely many displacement coefficients, we use the consistent-approximation approach to arrive at solvable plate theories. The main idea of this approach is to generate theories that contain only terms up to a certain energetic magnitude [3]. Because it decreases very fast for thin plates, this magnitude is characterized by the plate parameter $c = h/\sqrt{12}\alpha$. Following the consistent-approximation approach, a generic $N$th-order plate theory is given by considering all summands with $c^n$, $n ≤ 2N$.

3 Complete-plate theories

For the plate problem, the Taylor-series of the displacements $u_i$ are given by $u_{a} = a \left( u_{a}^{0} x_{3} + u_{a}^{3} x_{3}^{3} + u_{a}^{5} x_{3}^{5} + \ldots \right)$ and $u_{3} = a \left( u_{3}^{0} x_{3}^{0} + u_{3}^{2} x_{3}^{2} + u_{3}^{4} x_{3}^{4} + \ldots \right)$. With $x_{3} = x_{3}/a$ we introduced the dimensionless thickness coordinate and the $j u_{i}$
are called displacement coefficients. According to Kienzler & Schneider [3], these displacement coefficients itself can be split into the infinite sum

\[ j u_i = j u_i^0 + j u_i^1 + j u_i^2 + \ldots \]  

(1)

Here, with \( j u_i^0 \) we introduced the displacement-coefficient parts (dcps) of magnitude \( c^0 \). While the displacement coefficients change among the different orders of generic-plate theories, we can show that the dcps are unchangeable. We call this characteristic the modularity of the displacement coefficients. By using the modularity we can set up plate theories where all variables can be calculated (cf. next section). We call them complete-plate theories. A schema for the construction of them is provided.

4 Reducibility

Starting from the PDE systems of the complete-plate theories, the pseudo-reduction method leads to one main PDE, which is entirely written in the main variable \( (0 u_i^0) \), and several reduction PDEs that express the non-main variables in terms of the main variable. If there is a reduction PDE for each of the non-main variables, the PDE system is called fully reducible. Applying the pseudo-reduction method is related to solving a linear algebraic equation system. The determinant of the coefficient matrix \( K^N \) of the \( N \)th-order PDE system has to be unequal to zero for a fully reducible system. With \( \Delta \) as the Laplace operator it reads

\[
\text{Det} \left( K^N \right) = (-1)^{N+1} \left( \frac{2}{1-\nu} \right) c^2 \Delta \left( \frac{1-\nu}{1-2\nu} \right)^N \prod_{i=2}^{N+1} \left( \prod_{j=1}^{N} (i-j)^6 \right) 
\]

Here, \( \Delta \) is the Laplace operator and \( c^2 \) the square of the wave number. This determinant can be expressed as a product of factors, each of which is equal to zero for a fully reducible system.

5 Results

The results of the complete first and second-order plate theories coincide with the classical theories of Kirchhoff [1] and Reissner [4], respectively. Moreover the resulting stress field fulfills a-priori the local conditions. Here, we have to insert the modularity and compare only terms with the same magnitude \( c^{2n} \) (Neumann conditions) or the same combinations \( \xi^m c^{2n} \) (local equilibrium conditions).

6 Conclusion

In the talk we show that the displacement coefficients have a modular structure (modularity). Based on this, we build complete-plate theories. The proof that these theories are fully reducible is provided by using the determinants of the coefficient matrices of the corresponding PDE systems. It turns out that the results of the complete-plate theories are in good accordance with those of the classical theories and that they fulfill a-priori the local conditions. Note that this is the first time a plate theory satisfies these conditions without enforcing the conditions during the derivation of the theory.

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References

[1] G. Kirchhoff, Journal für die reine und angewandte Mathematik 40, 51–88 (1850).
[2] M. Meyer-Coors, R. Kienzler, and P. Schneider, Continuum Mechanics and Thermodynamics (2021).
[3] R. Kienzler and P. Schneider, International Journal of Solids and Structures 115-116, 14–26 (2017).
[4] E. Reissner, Journal of Mathematics and Physics 23(1–4), 184–191 (1944).