Operator product expansion in QCD in off-forward kinematics: Separation of kinematic and dynamical contributions

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based on

V.M. Braun, A.N. Manashov, JHEP 1201 (2012) 085

Newport News, 22.02.2012
- **Descendants of leading twist operators**

  \[ O_{\mu_1 \ldots \mu_n} = \bar{q} D_{\{\mu_1 \ldots D_{\mu_n}\}} q - \text{traces} \]

  \[ \partial^+ O_{\mu_1 \ldots \mu_n}, \quad \partial^\perp O_{\mu_1 \ldots \mu_n}, \quad \partial^- O_{\mu_1 \ldots \mu_n}, \ldots \]

  twist \(-2\) \quad twist \(-3\) \quad twist \(-4\)

- **Uses of conformal invariance for non-leading twist operators**

  V. M. Braun, A. N. Manashov and J. Rohrwild, *Nucl. Phys. B* 807 (2009) 89-137
  V. M. Braun, A. N. Manashov and J. Rohrwild, *Nucl. Phys. B* 826 (2010) 235-293

- **Inspired by recent studies of \( N = 4 \) SUSY YM theory, e.g.**

  N. Beisert and M. Staudacher, *Nucl. Phys. B* 670 (2003) 439-463
  N. Beisert, G. Ferretti, R. Heise and K. Zarembo, *Nucl. Phys. B* 717 (2005) 137-189
  A. Belitsky, S. Derkachov, G. Korchemsky and A. Manashov, *Nucl. Phys. B* 722 (2005) 191-221
Conformal symmetry and $SU(1, 1)$ scalar product

collinear conformal transformations

$$x_\mu = zn_\mu, \quad z \in \mathbb{R} \rightarrow z' = \frac{az + b}{cz + d}, \quad \Leftrightarrow \quad z \in \mathbb{C} \rightarrow z' = \frac{az + b}{bz + \bar{a}}$$

representations are labeled by conformal spin

$$\varphi(z) \rightarrow T^j \varphi(z) = \frac{1}{(bz + \bar{a})^{2j}} \varphi \left( \frac{az + b}{bz + \bar{a}} \right)$$

This is a unitary transformation with respect to the following scalar product:

$$\langle \phi, \psi \rangle_j = \frac{2j - 1}{\pi} \int_{|z|<1} d^2z \left(1 - |z|^2\right)^{2j-2} \bar{\phi}(z)\psi(z) \equiv \int_{|z|<1} D_j z \bar{\phi}(z)\psi(z), \quad ||\phi||^2 = \langle \phi, \phi \rangle$$

similar for several variables

$$\langle \phi, \psi \rangle_{j_1, j_2} = \int_{|z_1|<1} D_{j_1} z_1 \int_{|z_2|<1} D_{j_2} z_2 \bar{\phi}(z_1, z_2)\psi(z_1, z_2)$$
Generators

- Generators of infinitesimal $SU(1, 1)$ transformations

\[ S_+ = z^2 \partial_z + 2jz, \quad S_0 = z \partial_z + j, \quad S_- = -\partial_z \]

satisfy the usual $SL(2)$ algebra

\[ [S_+, S_-] = 2S_0, \quad [S_0, S_\pm] = \pm S_\pm \]

- For products of fields, e.g. $\phi(z_1)\phi(z_2)\ldots$

\[ S_+^{(j_1,j_2)} = S_+^{(j_1)} + S_+^{(j_2)} = z_1^2 \partial_{z_1} + z_2^2 \partial_{z_2} + 2j_1 z_1 + +2j_2 z_2, \]
\[ S_+^{(j_1,j_2,j_3)} = z_1^2 \partial_{z_1} + z_2^2 \partial_{z_2} + z_3^2 \partial_{z_3} + 2j_1 z_1 + +2j_2 z_2 + 2j_3 z_3, \]

- Hermiticity properties

\[ S_0^\dagger = S_0 \quad (S_+)^\dagger = -S_- \quad \langle \phi, S\psi \rangle = \langle S^\dagger \phi, \psi \rangle \]
Some properties

- **different powers are orthogonal**

\[ \langle z^n, z^{n'} \rangle = \delta_{nn'} |z^n|^2 , \quad \langle (z_1 - z_2)^n, (z_1 - z_2)^{n'} \rangle = \delta_{nn'} |z_{12}^n|^2 \]

- **reproducing (unit) operator**

\[
\phi(z) = \int_{|w|<1} D_j w \mathcal{K}_j(z, \bar{w}) \phi(w), \quad \mathcal{K}_j(z, \bar{w}) = \frac{1}{(1 - z\bar{w})^{2j}}
\]

- **Fourier**

\[
\rho_N \int \int D_1 z_1 D_2 z_2 (\bar{z}_1 - \bar{z}_2)^N e^{ip_1 z_1 + ip_2 z_2} = i^N (p_1 + p_2)^N C^3_N \left( \frac{p_1 - p_2}{p_1 + p_2} \right)
\]
Conformal operators vs. Light-ray operators

- **Light-ray operators**

  \[ O_+ (z_1, z_2) = \bar{\psi}_+ (z_1 n) \psi_+ (z_2 n) \]

- **(Local) conformal operators**

  \[ O_N (y) = (-\partial_+)^N \bar{\psi}_+ (y) C_N^{3/2} \left( \frac{\rightarrow D_+ - \leftarrow D_+}{\rightarrow D_+ + \leftarrow D_+} \right) \psi_+ (y) \]

→ Projecting \( O_N \) from \( O_+ (z_1, z_2) \):

\[
O_N (y) = (\partial_{z_1} + \partial_{z_2})^N C_N^{3/2} \left( \frac{\partial_{z_1} - \partial_{z_2}}{\partial_{z_1} + \partial_{z_2}} \right) O_{++} (z_1 n + y, z_2 n + y) \big|_{z_i=0}
\]

← Expanding \( O_+ (z_1, z_2) \) in \( O_N \):

\[
O_{++} (z_1, z_2) = \sum_N \kappa_N z_{12}^N \int_0^1 du (u \bar{u})^{N+1} O_N (z_{21}^u)
\]

\[
= \sum_N \kappa_N z_{12}^N \sum_k \frac{1}{k!} \int_0^1 du (u \bar{u})^{N+1} (z_{21}^u)^k \partial_+ O_N (0),
\]

where

\[ z_{12} = z_1 - z_2 \quad z_{21}^u = uz_1 + \bar{u}z_2 \quad \kappa_N = \frac{2(2N+3)}{(N+1)!} \]
Conformal operators vs. Light-ray operators (2)

- Alternatively:

\[
O_N = \rho_N \langle z_{12}^N, O_{++}(z_1, z_2) \rangle_{11} = \rho_N \int \int \mathcal{D}z_1 \mathcal{D}z_2 \bar{z}_{12}^N O_{++}(z_1, z_2) ~ |z_i|<1
\]

Proof:

\[
\delta_K O_N = \rho_N \int \int \mathcal{D}z_1 \mathcal{D}z_2 \bar{z}_{12}^N \delta_K O_{++}(z_1, z_2) \sim \int \int \mathcal{D}z_1 \mathcal{D}z_2 \bar{z}_{12}^N S_{+}^{(1,1)} O_{++}(z_1, z_2) ~ |z_i|<1
\]

\[
= - \int \int \mathcal{D}z_1 \mathcal{D}z_2 \left(S_{-}^{(1,1)} \bar{z}_{12}^N \right)^* O_{++}(z_1, z_2) = 0 ,
\]

\[
\rho_N = \frac{1}{2} (N + 1) (N + 2)!
\]
Conformal operators vs. Light-ray operators (3)

- Similar:

\[ \partial^k_+ O_N = \rho_N \left\langle z_{12}^N, (-S^{(1,1)}_-)^k O_{++}(z_1, z_2) \right\rangle_{11} = \rho_N \left\langle (S^{(1,1)}_+)^k z_{12}^N, O_{++}(z_1, z_2) \right\rangle_{11} \]

which we can write as

\[ O_{Nk} = \partial^k_+ O_N \]

- The functions \( \Psi_{Nk}^{t=2}(z_1, z_2) \) form an orthogonal basis

\[ \left\langle \Psi_{Nk}^{t=2}, \Psi_{N',k'}^{t=2} \right\rangle = \delta_{NN'} \delta_{kk'} \left\| \Psi_{Nk}^{t=2} \right\|^2, \]

so that one can represent the light-ray operator in the form

\[ O_{++}(z_1, z_2) = \sum_{N=0}^{\infty} \sum_{k=0}^{\infty} \rho_N^{-1} \left\| \Psi_{Nk}^{t=2} \right\|^{-2} \Psi_{Nk}^{t=2}(z_1, z_2) O_{Nk} \]
Conformal operators vs. Light-ray operators (4)

- **Calculation of** $\Psi_{Nk}^{t=2}(z_1, z_2)$:

\[
\exp[aS_+] \text{ corresponds to a finite conformal transformation } z \to z/(1 - az):
\]

\[
\exp[aS_+(j_1,j_2)] z_{12}^N = \frac{z_{12}^N}{(1 - az_1)^{N+2j_1} (1 - az_2)^{N+2j_2}}
\]

\[
= \frac{\Gamma[2N+2j_1+2j_2]}{\Gamma[N+2j_1] \Gamma[N+2j_2]} \int_0^1 dt \frac{z_{12}^N t^{N+2j_1-1} t^{N+2j_2-1}}{(1 - az_{21}^t)^{2N+2j_1+2j_2}}
\]

This gives:

\[
(S_+^{(j_1,j_2)})^k z_{12}^N = z_{12}^N \frac{\Gamma[2N+2j_1+2j_2+k]}{\Gamma[N+2j_1] \Gamma[N+2j_2]} \int_0^1 dt t^{N+2j_1-1} t^{N+2j_2-1} (z_{21}^t)^k
\]

- **The norm** $||\Psi_{Nk}^{t=2}||^2$ can be calculated recurrently:

\[
||\Psi_{Nk}^{t=2}||^2 = \langle S_+^k z_{12}^N, S_+^k z_{12}^N \rangle = -\langle S_- S_+^k z_{12}^N, S_+^{k-1} z_{12}^N \rangle = k(2N + k + 3) ||\Psi_{Nk-1}^{t=2}||^2
\]

One obtains

\[
||\Psi_{Nk}^{t=2}||^2 = ||z_{12}^N||^2 ||p_{Nk}^{-1}||^2, \quad p_{Nk} = \frac{1}{k!} \frac{\Gamma(2N+4)}{\Gamma(2N+4+k)}
\]
Spinor Representation

**Coordinates:**

\[ x_{\alpha\dot{\alpha}} = x_\mu \sigma^\mu_{\alpha\dot{\alpha}} = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} = \begin{pmatrix} x_+ & w \\ \bar{w} & x_- \end{pmatrix}, \quad \sigma^\mu = (1, \bar{\sigma}) \]

To maintain Lorentz–covariance, introduce two light-like vectors \( n^2 = \tilde{n}^2 = 0 \)

\[ n_{\alpha\dot{\alpha}} = \lambda_{\alpha} \bar{\lambda}_{\dot{\alpha}}, \quad \tilde{n}_{\alpha\dot{\alpha}} = \mu_{\alpha} \bar{\mu}_{\dot{\alpha}} \]

with auxiliary spinors \( \lambda \) and \( \mu \)

\[ x_{\alpha\dot{\alpha}} = z \lambda_{\alpha} \bar{\lambda}_{\dot{\alpha}} + \bar{z} \mu_{\alpha} \bar{\mu}_{\dot{\alpha}} + w \lambda_{\alpha} \bar{\mu}_{\dot{\alpha}} + \bar{w} \mu_{\alpha} \bar{\lambda}_{\dot{alpha}} \]

**Fields:**

\[ q = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^\dot{\beta} \end{pmatrix}, \quad \bar{q} = (\chi^\beta, \bar{\psi}_{\dot{\alpha}}), \]

\[ F_{\alpha\beta,\dot{\alpha}\dot{\beta}} = \sigma^\mu_{\alpha\dot{\alpha}} \sigma^\nu_{\beta\dot{\beta}} F_{\mu\nu} = 2 \left( \epsilon_{\dot{\alpha}\dot{\beta}} f_{\alpha\beta} - \epsilon_{\alpha\beta} \bar{f}_{\dot{\alpha}\dot{\beta}} \right) \]

\( f_{\alpha\beta} \) and \( \bar{f}_{\dot{\alpha}\dot{\beta}} \) transform according to \((1, 0)\) and \((0, 1)\) representations of Lorentz group.
“Plus” and “Minus” components

\[
\begin{align*}
\psi_+ &= \lambda^\alpha \psi_\alpha, & \chi_+ &= \lambda^\alpha \chi_\alpha, & f_{++} &= \lambda^\alpha \lambda^\beta f_{\alpha\beta}, \\
\bar{\psi}_+ &= \bar{\lambda}^{\dot{\alpha}} \bar{\psi}_{\dot{\alpha}}, & \bar{\chi}_+ &= \bar{\lambda}^{\dot{\alpha}} \chi_{\dot{\alpha}}, & \bar{f}_{++} &= \bar{\lambda}^{\dot{\alpha}} \bar{\lambda}^{\dot{\beta}} \bar{f}_{\dot{\alpha}\dot{\beta}}, \\
\psi_- &= \mu^\alpha \psi_\alpha, & \bar{\psi}_- &= \bar{\mu}^{\dot{\alpha}} \bar{\psi}_{\dot{\alpha}}, & f_{+-} &= \lambda^\alpha \mu^\beta f_{\alpha\beta}
\end{align*}
\]

similar for derivatives \(\partial_\mu \rightarrow \partial_{\alpha\dot{\alpha}}\)

\[
\begin{align*}
\partial_{++} &= 2\partial_z, & \partial_{--} &= 2\partial_{\bar{z}}, & \partial_{+-} &= 2\partial_w, & \partial_{-+} &= 2\partial_{\bar{w}}
\end{align*}
\]

- \(\psi_+, \chi_+, f_{++}\) and \(\bar{\psi}_+, \bar{\chi}_+, \bar{f}_{++}\) are defined as quasipartonic
Conformal basis for twist-four non-quasipartonic operators

Braun, Manashov, Rohrwild, Nucl. Phys. **B826** (2010) 235.

\[
\begin{align*}
Q_1(z_1, z_2, z_3) &= \bar{\psi} + (z_1)f_+ - (z_2)\psi_+ (z_3), \\
Q_2(z_1, z_2, z_3) &= \bar{\psi} + (z_1)f_+ + (z_2)\psi_-(z_3), \\
Q_3(z_1, z_2, z_3) &= \frac{1}{2} [D_+ + \bar{\psi} +] (z_1)f_+ + (z_2)\psi_+ (z_3),
\end{align*}
\]

and three similar operators with \( f \to \bar{f} \)

cf. in usual notation
\[
\begin{align*}
\bar{q}_L(z_1) \left[ F_{\mu_+} (z_2) + i \tilde{F}_{\mu_+} (z_2) \right] \gamma^\mu q_L (z_3) &= Q_2 (z_1, z_2, z_3) - Q_1 (z_1, z_2, z_3) \\
\bar{q}_L(z_1) \left[ F_{\mu_+} (z_2) - i \tilde{F}_{\mu_+} (z_2) \right] \gamma^\mu q_L (z_3) &= \bar{Q}_2 (z_1, z_2, z_3) - \bar{Q}_1 (z_1, z_2, z_3)
\end{align*}
\]

\[
\overrightarrow{Q}(z_1, z_2, z_3) = \begin{pmatrix} Q_1 (z_1, z_2, z_3) \\ Q_2 (z_1, z_2, z_3) \\ Q_3 (z_1, z_2, z_3) \end{pmatrix}
\]
Renormalization group equations

Braun, Manashov, Rohrwild, Nucl. Phys. \textbf{B826} (2010) 235.

The complete RG equation for twist-four operators has the following structure:

\[
\left( \mu \frac{\partial}{\partial \mu} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} \right) \begin{pmatrix} \overrightarrow{Q} \\ \overrightarrow{P} \end{pmatrix} = -\frac{\alpha_s}{2\pi} \begin{pmatrix} H_{QQ} & 0 & H_{QP} \\ 0 & H_{\overline{Q}\overline{Q}} & H_{\overline{Q}P} \\ H_{QP} & H_{\overline{Q}P} & H_{PP} \end{pmatrix} \begin{pmatrix} \overrightarrow{Q} \\ \overrightarrow{P} \end{pmatrix}
\]

• quasipartonic operators \( P(z_1, z_2, z_3, z_4) \) have autonomous evolution
• \( \overrightarrow{Q} \) and \( \overrightarrow{\bar{Q}} \) do not mix with each other → decoupling:

\[
\left( \mu \frac{\partial}{\partial \mu} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} \right) \begin{pmatrix} \overrightarrow{Q} \\ \overrightarrow{P} \end{pmatrix} = -\frac{\alpha_s}{2\pi} \begin{pmatrix} H_{QQ} & H_{QP} \\ 0 & H_{\overline{Q}P} \end{pmatrix} \begin{pmatrix} \overrightarrow{Q} \\ \overrightarrow{P} \end{pmatrix},
\]

\[
\left( \mu \frac{\partial}{\partial \mu} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} \right) \begin{pmatrix} \overrightarrow{\overline{Q}} \\ \overrightarrow{\overline{P}} \end{pmatrix} = -\frac{\alpha_s}{2\pi} \begin{pmatrix} H_{\overline{Q}\overline{Q}} & H_{\overline{Q}P} \\ 0 & H_{PP} \end{pmatrix} \begin{pmatrix} \overrightarrow{\overline{Q}} \\ \overrightarrow{\overline{P}} \end{pmatrix}.
\]

\( \leftrightarrow \) these two pairs of equations are equivalent
Expand light-ray $\vec{P}$ and $\vec{Q}$ in terms of multiplicatively renormalizable quasipartonic $P_{Np}$ and non-quasipartonic $Q_{Np}$ local operators:

$$
\vec{P}(z_1, z_2, z_3, z_4) = \vec{a}_{Np}(z_1, z_2, z_3, z_4)P_{Np},
$$
$$
\vec{Q}(z_1, z_2, z_3) = \sum_{Np} \vec{b}_{Np}(z_1, z_2, z_3)Q_{Np} + \sum_{Np} \vec{c}_{Np}(z_1, z_2, z_3)P_{Np}
$$

Substituting this expansion in the RG equation one finds that $b_{Np}(z_1, z_2, z_3)$ can be found as solutions of the integral equation that only involves $H_{QQ}$:

$$
H_{QQ} \vec{b}_{Np}(z_1, z_2, z_3) = \gamma_{Np} \vec{b}_{Np}(z_1, z_2, z_3),
$$

$\leftrightarrow$ can ignore quasipartonic operators altogether

We need a particular solution corresponding to the leading-twist anomalous dimension

$$
\gamma_N = C_F \left( 1 - \frac{2}{(N+1)(N+2)} + 4 \sum_{m=2}^{N+1} \frac{1}{m} \right) = 2C_F \left[ \psi(N+3) + \psi(N+1) - \psi(3) - \psi(1) \right]
$$

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Renormalization group equations (3)

\[ \mathbb{H}_{QQ} \rightarrow \Psi(z_1, z_2, z_3) = \left( \begin{array}{ccc} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{array} \right) \left( \begin{array}{c} \Psi_1^{(1,1,1)} \\ \Psi_2^{(1, \frac{3}{2}, \frac{1}{2})} \\ \Psi_3^{(\frac{3}{2}, \frac{3}{2}, 1)} \end{array} \right) = \gamma_N \rightarrow \Psi(z_1, z_2, z_3) \]

Unfortunately this Eq. appears to be too complicated to be solved directly. The way out is that \( \mathbb{H}_{QQ} \) turns out to be self-adjoint (hermitian) with respect to the following scalar product:

\[ \langle \langle \rightarrow \Phi, \rightarrow \Psi \rangle \rangle = 2 \langle \Phi_1, \Psi_1 \rangle_{111} + \langle \Phi_2, \Psi_2 \rangle_{1 \frac{3}{2} \frac{1}{2}} + \frac{1}{2} \langle \Phi_3, \Psi_3 \rangle_{\frac{3}{2} \frac{3}{2} 1}, \]

which we found by explicit calculation, cf. Braun, Manashov, Rohrwild, NPB826 (2010) 235

- Thanks to hermiticity, it is sufficient to solve an inverse problem:

Need: \( \rightarrow Q(z_1, z_2, z_3) = \sum_{Nk} \vec{b}_{Nk}(z_1, z_2, z_3) \partial_+^k (\partial O)_N + \text{geniune twist-4 operators} \)

Do: \( 2(\partial O)_N = \frac{ig \rho_N}{(N + 1)^2} \left[ \langle \rightarrow \Psi_N, \rightarrow Q \rangle - \langle \rightarrow \Psi_N, \rightarrow Q \rangle \right] + \text{quasipartonic operators} \)
**Kinematic projection operators**

- If $\vec{\Psi}_N = \{\Psi^{(1)}, \Psi^{(2)}, \Psi^{(3)}\}$ is known, adding “plus” derivatives is easy:

$$2\partial_+^k (\partial\mathcal{O})_N = \frac{ig\rho_N}{(N+1)^2} \left[ \langle (S_+)^k \vec{\Psi}_N, \vec{Q} \rangle - \langle (S_+)^k \vec{\Psi}_N, \vec{\bar{Q}} \rangle \right] + \ldots$$

$$S^+ \vec{\Psi} = \begin{pmatrix} S_+^{(1,1,1)} \Psi^{(1,1,1)} \\ S_+^{(1,\frac{3}{2},\frac{1}{2})} \Psi^{(1,\frac{3}{2},\frac{1}{2})} \\ S_+^{(\frac{3}{2},\frac{3}{2},1)} \Psi^{(\frac{3}{2},\frac{3}{2},1)} \end{pmatrix} \quad ||\vec{\Psi}_{Nk}||^2 = p_{Nk}^{-1} ||\vec{\Psi}_N||^2$$

- Functions $\vec{\Psi}_{Nk} = (S_+)^k \vec{\Psi}_N$ are mutually orthogonal and also orthogonal to the coefficient functions of the other existing multiplicatively renormalizable operators, hence

$$ig \vec{Q}(z_1, z_2, z_3) = \sum_{N=0}^{\infty} \sum_{k=0}^{\infty} \frac{p_{Nk}(N+1)^2}{\rho_N ||\Psi_N||^2} \vec{\Psi}_{Nk}(z_1, z_2, z_3) \partial_+^k (\partial\mathcal{O})_N + \ldots$$

- Ellipses stand for genuine (dynamical) higher-twist operators.
Divergence of a conformal operator

The conformal operator $\mathcal{O}_N$ is obtained by the projection

$$\mathcal{O}_N = n^\mu n^{\mu_1} \ldots n^{\mu_N} (\mathcal{O}_N)_{\mu_1 \ldots \mu_N}$$

where $n$ is an auxiliary light-like vector. Define a divergence of the conformal operator as

$$(\partial \mathcal{O})_N = n^{\mu_1} \ldots n^{\mu_N} \partial^\mu (\mathcal{O}_N)_{\mu_1 \ldots \mu_N} = n^{\mu_1} \ldots n^{\mu_N} \left[ i P^\mu, (\mathcal{O}_N)_{\mu_1 \mu_2 \ldots \mu_N} \right]$$

where $P_\mu$ is the usual four-momentum operator

$$P_\mu |p\rangle = p_\mu |p\rangle, \quad i [P_\mu, \Phi(x)] = \frac{\partial}{\partial x^\mu} \Phi(x)$$

Taking into account that $n^\mu = \frac{1}{2} (\lambda \sigma^\mu \bar{\lambda})$ the same definition can be rewritten as

$$(\partial \mathcal{O})_N = \frac{1}{N+1} \left[ i P^\mu, \frac{\partial}{\partial n^\mu} \mathcal{O}_N(n) \right] = \frac{1}{(N+1)^2} \left[ i \bar{P}^{\dot{\alpha} \alpha}, \frac{\partial}{\partial \lambda^\alpha} \frac{\partial}{\partial \bar{\lambda}^{\dot{\alpha}}} \mathcal{O}_N(\lambda, \bar{\lambda}) \right],$$
Divergence of a conformal operator (2)

It is convenient to use the integral representation for $O_N$ in terms of the light-ray operator

$$O_{++}(z_1, z_2) = \bar{\psi}(z_1 n)n[z_1 n, z_2 n]\psi(z_2 n)$$  \hspace{1cm} (2)

Let

$$(\partial O)_{++}(z_1, z_2) = \left[ i\bar{P}^{\dot{\alpha}\alpha}, \frac{\partial}{\partial \lambda^\alpha} \frac{\partial}{\partial \bar{\lambda}^{\dot{\alpha}}} O_{++}(z_1, z_2) \right]$$

then

$$(\partial O)_N = \frac{\rho_N}{(N+1)^2} \left< z_{12}^N, (\partial O)_{++}(z_1, z_2) \right> = \frac{\rho_N}{(N+1)^2} \int \int Dz_1 Dz_2 \bar{z}_{12}^N (\partial O)_{++}(z_1, z_2)$$

By a direct calculation obtain

$$2(\partial O)_{++}(z_1, z_2) = ig \left[ A(z_1, z_2) - \bar{A}(z_1, z_2) \right] + \ldots ,$$

where

$$A(z_1, z_2) = \partial_{z_2} z_{12}^2 \left\{ Q_1(z_1, z_1, z_2) + \int_0^1 du \left[ Q_2(z_1, z_{21}^u, z_2) + z_{12} Q_3(z_1, z_{21}^u, z_2) \right] \right\}$$

$$+ \partial_{z_1} \partial_{z_2} z_{12}^3 \int_0^1 du \left[ - Q_1(z_1, z_{21}^u, z_2) + \bar{u} Q_2(z_1, z_{21}^u, z_2) \right]$$

The ellipses stand for EOM, contributions of quasipartonic operators and terms $\propto S_{12}^+$ which do not contribute to the projection that defines the conformal operator.
Coefficient functions

Obtain:

\[ 2(\partial \mathcal{O})_N = \frac{ig \rho N}{(N + 1)^2} \left[ \langle z_{12}^N, A(z_1, z_2) \rangle_{11} - \langle z_{12}^N, \bar{A}(z_1, z_2) \rangle_{11} \right] \]

This involves the sum of terms of the type, schematically

\[ (\partial \mathcal{O})_N \sim \langle z_{12}^N, [\mathbb{B}_i Q_i](z_1, z_2) \rangle_{11}, \]

where \( \mathbb{B}_i \) are some integral operators, and the scalar product is calculated for conformal spins \( j_1 = j_2 = 1 \). The idea is to rewrite this answer as a sum of terms

\[ (\partial \mathcal{O})_N \sim \langle \Psi_i^N(z_1, z_2, z_3), Q_i(z_1, z_2, z_3) \rangle_{j_1 j_2 j_3}, \]

where \( j_1, j_2, j_3 \) are the conformal spins of the \( Q_i \), so that the functions \( \Psi_i^N(z_1, z_2, z_3) \) can be identified with the coefficient functions of the \( Q \)-operators

Consider the first two terms as an example:

\[ A = \partial z_2 z_{12}^2 \left[ \phi_1(z_1, z_2) + \phi_2(z_1, z_2) + \ldots \right] + \ldots \]

where

\[ \phi_1(z_1, z_2) = Q_1(z_1, z_1, z_2), \quad \phi_2(z_1, z_2) = \int_0^1 du \ u \ Q_2(z_1, z_{21}^u, z_2) \]
Coefficient functions (2)

Step One: Notice that any operator $Q_i(z_1, z_2, z_3)$ can be written as

$$Q_i(z_1, z_2, z_3) = \left\langle \prod_{k=1}^{3} K_{j_k} (z_k, \bar{w}_k), Q_i(w_1, w_2, w_3) \right\rangle_{j_1 j_2 j_3},$$

where $K_{j_k}$ are the reproducing kernels. Thus

$$\varphi_1(z_1, z_2) = \left\langle K_1(z_1, \bar{w}_1) K_1(z_1, \bar{w}_2) K_1(z_2, \bar{w}_3) Q_1(w_1, w_2, w_3) \right\rangle_{111},$$

$$\varphi_2(z_1, z_2) = \int_0^1 du \left\langle K_1(z_1, \bar{w}_1) K_{3/2}(z_{21}^u, \bar{w}_2) K_{1/2}(z_2, \bar{w}_3) Q_1(w_1, w_2, w_3) \right\rangle_{13/2 1/2},$$

$$= \frac{1}{2} \left\langle K_1(z_1, \bar{w}_1) K_1(z_1, \bar{w}_2) K_{1/2}(z_2, \bar{w}_2) K_{1/2}(z_2, \bar{w}_3) Q_1(w_1, w_2, w_3) \right\rangle_{13/2 1/2},$$

where we used that

$$\int_0^1 du \, K_{3/2}(z_{21}^u, \bar{w}_2) = \int_0^1 du \, \frac{1}{(1 - (z_2 \bar{u} + z_1 u) \bar{w}_2)^3} = \int_0^1 du \, \frac{u}{(u(1 - z_1 \bar{w}_2) + \bar{u}(1 + z_2 \bar{w}_2))^3}$$

$$= \frac{1}{2} \frac{1}{(1 - z_1 \bar{w}_2)^2} \frac{1}{(1 - z_2 \bar{w}_2)} = \frac{1}{2} K_1(z_1, \bar{w}_2) K_{1/2}(z_2, \bar{w}_2).$$

- Note analogy between reproducing kernels and (Feynman) propagators
Coefficient functions (3)

• Making the conformal transformation \( z \to z', w \to w' \) and taking into account

\[
\mathcal{K}_j(z', \bar{w}') = (\bar{b}z + \bar{a})^{2j} \mathcal{K}_j(z, \bar{w})(b\bar{w} + a)^{2j},
\]

it is straightforward to check that both \( \varphi_1(z_1, z_2) \) and \( \varphi_2(z_1, z_2) \) transform according to the representation \( T^{j_1=2} \otimes T^{j_2=1} \)

• Using this result, it is easy to show that the both \( \partial_2 \bar{z}_1^2 \varphi_1(z_1, z_2) \) and \( \partial_2 \bar{z}_2^2 \varphi_2(z_1, z_2) \), transform according to \( T^{j_1=1} \otimes T^{j_2=1} \), as they should.

The transformation properties follow immediately from the following statements which can be checked by a direct calculation:

- if a function \( \varphi(z_1, z_2) \) transforms according to \( T^{j_1} \otimes T^{j_2} \), then the function \( \psi(z_1, z_2) = z_{12} \varphi(z_1, z_2) \) transforms as \( T^{j_1-1/2} \otimes T^{j_2-1/2} \), i.e. multiplication by \( z_{12} = z_1 - z_2 \) intertwines the representations
  \[
  z_{12} T^{j_1} \otimes T^{j_2} = T^{j_1-1/2} \otimes T^{j_2-1/2} z_{12}
  \]

- if a function \( \varphi(z) \) transforms according to \( T^{j=0} \) then its derivative \( \partial_z \varphi(z) \) transforms according to \( T^{j=1} \), i.e.
  \[
  \partial_z T^{j=0} = T^{j=1} \partial_z
  \]

• We have verified that each contribution to \( A(z_1, z_2) \) (and \( \bar{A}(z_1, z_2) \)) transforms according to the same representation, \( T^{j_1=1} \otimes T^{j_2=1} \), which provides a strong check of the calculation.
Step Two: The contribution of $\varphi_{1,2}(z_1, z_2)$ to $(\partial \mathcal{O})_N$ has the form

$$(\partial \mathcal{O})_N \sim \langle z_{12}^N, \mathbb{B} \varphi(z_1, z_2) \rangle_{11}, \quad \mathbb{B} = \partial_2 z_{12}^2$$

where $\mathbb{B}$ is an intertwining operator:

$$\mathbb{B} \mathcal{S}_{0, \pm}^{(j_1=2, j_2=1)} = \mathcal{S}_{0, \pm}^{(j_1=1, j_2=1)} \mathbb{B}$$

We can rewrite this as

$$\langle z_{12}^N, \mathbb{B} \varphi_1(z_1, z_2) \rangle_{11} = \langle \mathbb{B}^\dagger z_{12}^N, \varphi_1(z_1, z_2) \rangle_{2,1} = b_N \langle z_{12}^{N-1}, \varphi_1(z_1, z_2) \rangle_{2,1}$$

Thus we obtain

$$\langle z_{12}^N, \partial_2 z_{12}^2 \varphi_1(z_1, z_2) \rangle_{11} = b_N \langle z_{12}^{N-1}, \varphi_1(z_1, z_2) \rangle_{2,1} = \langle \Psi_1(a(w_1, w_2, w_3), Q_1(w_1, w_2, w_3)) \rangle_{111}$$

with

$$\Psi_1(a(w_1, w_2, w_3) = b_N \int \int \int_{|z_k|<1} \mathcal{D}_2 z_1 \mathcal{D}_1 z_2 \mathcal{S}_{12}^{N-1} \mathcal{K}_1(w_1, \bar{z}_1) \mathcal{K}_1(w_2, \bar{z}_1) \mathcal{K}_1(w_3, \bar{z}_2).$$

The numerical coefficient $b_N$ can be fixed as follows:

$$\langle z_{12}^N, \mathbb{B} z_{12}^{N-1} \rangle_{11} = -(N+1)\|z_{12}^N\|_{11}^2 = \langle \mathbb{B}^\dagger z_{12}^N, z_{12}^{N-1} \rangle_{21} = b_N \|z_{12}^{N-1}\|_{21}^2$$

where from

$$b_N = -(1/6)N(N+2)(N+3)$$
Step Three:

Combining the two reproducing kernels with the help of Feynman parametrization

\[ K_1(w_1, \bar{z}_1)K_1(w_2, \bar{z}_1) = 6 \int_0^1 d\alpha \alpha \bar{\alpha} K_2(w_1^\alpha, \bar{z}_1) \]

and using that

\[ \int \int \int_{|z_k|<1} D_2 z_1 D_1 z_2 z_{12}^{N-1} K_2(w_1^\alpha, \bar{z}_1)K_1(w_3, \bar{z}_2) = (w_{12}^\alpha - w_3)^{N-1} \]

one obtains

\[ \Psi_{1a}(w_1, w_2, w_3) = -N(N+2)(N+3) \int_0^1 d\alpha \alpha \bar{\alpha} (w_{12}^\alpha - w_3)^{N-1} \]
**Coefficient functions (6)**

**Final results:**

| Equation | Expression |
|----------|------------|
| $\Psi^{(1)}_N (w)$ | $4a_N \left[ \int_0^1 d\alpha \bar{\alpha} \int_0^1 d\beta \bar{\beta} (w_{12}^\alpha - w_{32}^\beta)^{N-1} - \frac{1}{N+1} \int_0^1 d\alpha \alpha \bar{\alpha} (w_{12}^\alpha - w_{32}^3)^{N-1} \right] $ |
| $\Psi^{(2)}_N (w)$ | $-4a_N \int_0^1 d\alpha \bar{\alpha} \int_0^1 d\beta \left( \beta + \frac{1}{N+1} \alpha \right) (w_{12}^\alpha - w_{32}^\beta)^{N-1}$ |
| $\Psi^{(3)}_N (w)$ | $-24c_N \int_0^1 d\alpha \bar{\alpha}^2 \alpha \int_0^1 d\beta \bar{\beta} (w_{12}^\alpha - w_{32}^\beta)^{N-2}$ |

where

- $a_N = \frac{1}{8} (N+3)(N+2)(N+1)N$,
- $b_N = -\frac{1}{6} (N+3)(N+2)N$,
- $c_N = \frac{1}{48} \frac{(N+4)!}{(N+1)(N-2)!}$
Coefficient functions (7)

Calculation of the norm is straightforward

\[ \| \vec{\Psi}_N \|^2 = 2\| \Psi_N^{(1)} \|^2_{111} + \| \Psi_N^{(2)} \|^2_{1\frac{3}{2}1\frac{3}{2}} + \frac{1}{2} \| \Psi_N^{(3)} \|^2_{\frac{3}{2}3\frac{3}{2}1} . \]

\[ \| \vec{\Psi}_N \|^2 = \frac{1}{2} \| z_{12}^N \|^2_{11} (N+2)^2 (N+1)^2 \left[ \psi(N+3) + \psi(N+1) - \psi(3) - \psi(1) \right] \]

Compare: leading twist anomalous dimension

\[ \gamma_N = C_F \left( 1 - \frac{2}{(N+1)(N+2)} + 4 \sum_{m=2}^{N+1} \frac{1}{m} \right) = 2C_F \left[ \psi(N+3) + \psi(N+1) - \psi(3) - \psi(1) \right] \]
Summary: Kinematic projection operators

\[ 2(\partial O)_N = \frac{ig\rho_N}{(N+1)^2} \left[ \langle \vec{\Psi}_N, \vec{Q} \rangle - \langle \vec{\Psi}_N, \vec{\bar{Q}} \rangle \right] + \ldots \]

\[ \langle \vec{\Phi}, \vec{\Psi} \rangle = 2\langle \Phi_1, \Psi_1 \rangle_{111} + \langle \Phi_2, \Psi_2 \rangle_{1\frac{3}{2}\frac{1}{2}} + \frac{1}{2} \langle \Phi_3, \Psi_3 \rangle_{3\frac{3}{2}\frac{1}{2}} \]

\[ ig \vec{Q}(z_1, z_2, z_3) = \sum_{N=0}^{\infty} \sum_{k=0}^{\infty} \frac{p_{Nk}(N+1)^2}{\rho_N \| \Psi_N \|^2} \frac{\Psi_{Nk}(z_1, z_2, z_3)}{\partial^k_+ (\partial O)_N} + \ldots \]

- All entries known explicitly
- The ellipses stand for “dynamic” operators
T-product of two electromagnetic currents

\[ T_{\mu\nu}(z_1, z_2) = iT\{j_{\mu}(z_1 x)j_{\nu}(z_2 x)\} \]

\[ j_{\mu}(x) = \bar{q}(x)\gamma_{\mu}Qq(x) \]

Need:

\[ T_{\mu\nu}(z_1, z_2) = T_{\mu\nu}^{t=2}(z_1, z_2) + T_{\mu\nu}^{t=3}(z_1, z_2) + T_{\mu\nu}^{t=4}(z_1, z_2) + \ldots \]

Ward identities:

\[ \partial^\mu T_{\mu\nu}(z_1, z_2) = z_2 \left[ i\mathbf{P}_\mu, T_{\mu\nu}(z_1, z_2) \right] \]

\[ \partial^\nu T_{\mu\nu}(z_1, z_2) = z_1 \left[ i\mathbf{P}_\nu, T_{\mu\nu}(z_1, z_2) \right] \]

Translation invariance:

\[ e^{iz\mathbf{P}\cdot x} T_{\mu\nu}(z_1, z_2) e^{-iz\mathbf{P}\cdot x} = T_{\mu\nu}(z_1 + z, z_2 + z) \]

Both relations are only valid in the sum of all twists but not for each twist separately.
**T-product of two electromagnetic currents (2)**

- **Major simplification after the integration over gluon light-cone coordinate**

  \[
  \int_0^1 du \ u \left[(S^1_{+}, \frac{3}{2}, \frac{1}{2})^k \Psi_N^{(2)}\right](z_1, z_{21}^u, z_2) = (S^1_{+}, \frac{3}{2}, \frac{1}{2})^k \int_0^1 du \ u \Psi_N^{(2)}(z_1, z_{21}^u, z_2),
  \]

  \[
  \int_0^1 du \ \bar{u} \left[(S^1_{+}, \frac{3}{2}, \frac{1}{2})^k \Psi_N^{(2)}\right](z_1, z_{21}^u, z_2) = (S^1_{+}, \frac{3}{2}, \frac{1}{2})^k \int_0^1 du \ \bar{u} \Psi_N^{(2)}(z_1, z_{21}^u, z_2),
  \]

  \[
  \int_0^1 du \ \left[(S^1_{+}, \frac{1}{2}, \frac{1}{2})^k \Psi_N^{(1)}\right](z_1, z_{21}^u, z_2) = (S^1_{+}, \frac{3}{2}, \frac{3}{2})^k \int_0^1 du \ \Psi_N^{(1)}(z_1, z_{21}^u, z_2)
  \]

  \[
  \int_0^1 du \ u \left[(S^1_{+}, \frac{1}{2}, \frac{1}{2})^k \Psi_N^{(1)}\right](z_1, z_{21}^u, z_2) = cannot\ be\ simplified
  \]

- **Another major simplification:** \(1/\gamma_N\) gets cancelled
T-product of two electromagnetic currents: Results

\[ T_{\alpha\beta \dot{\alpha} \dot{\beta}}(z_1, z_2) = -\frac{2}{\pi^2 x^4 z_{12}^3} \left\{ x_{\alpha\beta} B_{\beta \dot{\alpha}}(z_1, z_2) - x_{\beta \dot{\alpha}} B_{\alpha \dot{\beta}}(z_2, z_1) + x_{\alpha \dot{\beta}} x_{\beta \dot{\alpha}} A(z_1, z_2) \right. \]
\[ \left. + x^2 \left[ x_{\beta \dot{\alpha}} \partial_{\alpha \dot{\beta}} C(z_1, z_2) - x_{\alpha \dot{\beta}} \partial_{\beta \dot{\alpha}} C(z_2, z_1) \right] + \ldots \right\} \]

\[ B_{\alpha \dot{\alpha}}(z_1, z_2) = B_{\alpha \dot{\alpha}}^{t=2}(z_1, z_2) + B_{\alpha \dot{\alpha}}^{t=3}(z_1, z_2) + B_{\alpha \dot{\alpha}}^{t=4}(z_1, z_2) + \ldots \]
\[ A(z_1, z_2) = A^{t=2}(z_1, z_2) \]
\[ C(z_1, z_2) = C^{t=2}(z_1, z_2) \]
T-product of two electromagnetic currents: twist 2+3

\[ \mathcal{B}_{\alpha \dot{\alpha}}^{t=2}(z_1, z_2) = \frac{1}{2} \partial_{\alpha \dot{\alpha}} \int_{0}^{1} du \mathcal{O}_{++}^{t=2}(uz_1 x, uz_2 x), \]

\[ \mathcal{B}_{\alpha \dot{\alpha}}^{t=3}(z_1, z_2) = \frac{1}{4} \int_{0}^{1} udv \int_{z_2}^{z_1} dv \left\{ \left[ i \mathbf{P}_\mu, (x \bar{\sigma}^\mu \partial)_{\alpha \dot{\alpha}} z_1 \mathcal{O}_{++}^{t=2}(z_1 u, vu) + (\bar{x} \sigma^\mu \bar{\partial})_{\dot{\alpha} \alpha} z_2 \mathcal{O}_{++}^{t=2}(vu, z_2 u) \right] 
\right. 
\left. + \frac{1}{2} \ln u \partial_{\alpha \dot{\alpha}} x^2 \partial_{\beta \dot{\beta}} \left[ i \mathbf{P}_{\beta \dot{\beta}} \right. \left. , z_1 \mathcal{O}_{++}^{t=2}(z_1 u, vu) + z_2 \mathcal{O}_{++}^{t=2}(vu, z_2 u) \right] \right\}, \]

- The last term in \( \mathcal{B}_{\alpha \dot{\alpha}}^{t=3} \) is new
T-product of two electromagnetic currents: twist 4

Notation:

\[ \mathcal{O}_1(z_1, z_2) = [i P^\mu, [i P_\mu, \mathcal{O}^{t=2}_{++}(z_1 x, z_2 x)]] , \]

\[ \mathcal{O}_2(z_1, z_2) = [i P^\mu, \frac{\partial}{\partial x^\mu} \mathcal{O}^{t=2}_{++}(z_1 x, z_2 x) ] , \]

and

\[ \Delta \mathcal{O}_1(z_1, z_2) = \mathcal{O}_1(z_1, z_2) - \mathcal{O}_1(z_2, z_1) , \]

\[ \Delta \mathcal{O}_2(z_1, z_2) = \mathcal{O}_2(z_1, z_2) - \mathcal{O}_2(z_2, z_1) . \]

\[ A = \frac{1}{4} \int_0^1 du \left\{ u^2 \ln u z_1 z_2 \Delta \mathcal{O}_1(u z_1, u z_2) - u^2 \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta z_{12}^2 \Delta \mathcal{O}_1(u z_{12}, u z_{21}) \right. \]

\[ - u^2 \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta \left[ z_{21}^\beta S_{+}^{(3 1 2 3 1 2)} \Delta \mathcal{O}_1(u z_{12}, u z_{21}) + z_{12}^\beta S_{+}^{(3 1 2 3 1 2)} \Delta \mathcal{O}_1(u z_{12}, u z_{21}) \right] \]

\[ - u^2 \ln u \int_0^1 d\alpha \left[ z_{2} S_{+}^{(1 3 2 3 1 2)} \Delta \mathcal{O}_1(u z_{12}, u z_{21}) + z_{1} S_{+}^{(3 1 2 3 1 2)} \Delta \mathcal{O}_1(u z_{12}, u z_{21}) \right] \]

\[ - u[1 - \delta(\bar{u})] \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta \left[ z_{21}^\beta \Delta \mathcal{O}_2(u z_{12}, u z_{21}) + z_{12}^\beta \Delta \mathcal{O}_2(u z_{12}, u z_{21}) \right] \]

\[ - u(1 + \ln u) \int_0^1 d\alpha \left[ z_{2} \Delta \mathcal{O}_2(u z_{12}, u z_{21}) + z_{1} \Delta \mathcal{O}_2(u z_{12}, u z_{21}) \right] \} \]
T-product of two electromagnetic currents: twist 4 (2)

\[ \mathcal{B}_{\alpha\dot{\alpha}}^{t=4}(z_1, z_2) = x^2 \partial_{\alpha\dot{\alpha}} \mathcal{B}(z_1, z_2) \]

\[ \mathcal{B} = \frac{1}{8} \int_0^1 du \left\{ (1-u^2 + u^2 \ln u) z_1 z_2 \mathcal{O}_1(uz_1, uz_2) + (1-u^2) \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta \ z_{12}^2 \mathcal{O}_1(uz_{12}^\alpha, uz_{21}^\beta) \right. \\
+ (1-u^2) \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta \ \frac{\bar{\alpha}}{\beta} \left[ z_{21}^\beta S_+^{(3/2, 1/2)} \mathcal{O}_1(uz_{12}^\alpha, uz_{21}^\beta) + z_{12}^\beta S_+^{(1/2, 3/2)} \mathcal{O}_1(uz_{12}^\beta, uz_{21}^\alpha) \right] \\
- (1-u^2 + u^2 \ln u) \int_0^1 d\alpha \left[ z_2 S_+^{(3/2, 1/2)} \mathcal{O}_1(uz_{12}^\alpha, uz_2) + z_1 S_+^{(3/2, 1/2)} \mathcal{O}_1(uz_1, uz_{21}^\alpha) \right] \\
- \frac{1}{u} (1+u^2) \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta \ \frac{\bar{\alpha}}{\beta} \left[ z_{21}^\beta \mathcal{O}_2(uz_{12}^\alpha, uz_{21}^\beta) + z_{12}^\beta \mathcal{O}_2(uz_{12}^\beta, uz_{21}^\alpha) \right] \\
+ \frac{1}{u} (1-u^2 \ln u) \int_0^1 d\alpha \left[ z_2 \mathcal{O}_2(uz_{12}^\alpha, uz_2) + z_1 \mathcal{O}_2(uz_1, uz_{21}^\alpha) \right] \} \]
$$C = -\frac{1}{8} z_{12} \int_0^1 du \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta \frac{\beta}{\bar{\beta}} \left\{ S^{(\frac{3}{2}, \frac{1}{2})} \Delta O_1(u z_{12}^\alpha, u z_{21}^\beta) - \left( \delta(\bar{u}) + \frac{1}{u} \right) \Delta O_2(u z_{12}^\alpha, u z_{21}^\beta) \right\}$$
Outlook

• Done:

A theoretical framework for the calculation of finite $t$ and target mass corrections in hard off-forward processes

• To do:

Factorization of kinematic contributions to DVCS to twist-4 accuracy
Concrete predictions and applications to data analysis in DVCS, $\gamma^* \rightarrow \eta \gamma$
Meson distribution amplitudes, applications to B-decays
An alternative derivation