Embedding theorems in constructive approximation

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Abstract

In this paper necessary and sufficient conditions for the accuracy of embedding theorems of different function classes are obtained. The main result of the paper is the criterion for embedding between the generalized Weyl-Nikol’skii and Lipschitz classes. To define the Weyl-Nikol’skii classes, we use the concept of a $(\lambda, \beta)$-derivative, which is a generalization of the derivative in the sense of Weyl. As corollaries, we obtain estimates of norms and moduli of smoothness of transformed Fourier series.

Keywords: Embedding theorems, Lipschitz classes, Weyl-Nikol’skii classes, $(\lambda, \beta)$-derivative, Moduli of smoothness.

1 Introduction

History of the question. One of the main problems of constructive theory of functions is in finding a relationship between differential properties of functions and its structural or constructive characteristics. This topic started to develop more than a century ago and in many cases the research was conducted as follows: authors considered a given functional class and by investigating the properties of its elements obtained embedding theorems with other functional classes. We recommend the recent articles [29], [47], [57] for a historical survey.

Below we write the three results, which influenced substantially further research and actually gave rise to the development of new areas within the approximation theory.

(A) \[ f^{(r)} \in \text{Lip } \alpha \iff E_n(f) = O \left( \frac{1}{n^{r+\alpha}} \right) \quad (0 < \alpha < 1, r \in \mathbb{Z}_+) \]

(B) \[ f^{(r)} \in \text{Lip } \alpha \iff \omega_{r+1} \left( f, \frac{1}{n} \right) = O \left( \frac{1}{n^{r+\alpha}} \right) \quad (0 < \alpha < 1, r \in \mathbb{Z}_+) \]

(C) \[ f \in \text{Lip } \alpha \Rightarrow \tilde{f} \in \text{Lip } \alpha \quad (0 < \alpha < 1) \]

Criterion (A) was proved by D. Jackson (1911, [22]) and S.B. Stechkin ([42]) in the necessity part, and by S. N. Bernstein (1912, [8], [6]) and Ch. de la Vallée-Poussin (1919, [56]) in the sufficiency part.

The theorems of this type are called direct and inverse theorems of approximation theory. Direct theorems for $L_p, 1 \leq p \leq \infty$ (see e.g. [15], [47], [57]) are written as follows:

\[ E_n(f)_p \leq C(k) \omega_k \left( f, \frac{1}{n} \right)_p, \quad k, n \in \mathbb{N}, \]  \hspace{1cm} (1)

\[ E_n(f)_p \leq \frac{C(k)}{n^r} \omega_k \left( f^{(r)}, \frac{1}{n} \right)_p, \quad k, n, r \in \mathbb{N}. \]  \hspace{1cm} (2)

Inverse theorems for $L_p, 1 \leq p \leq \infty$ (see e.g. [15], [47], [57]):

\[ \omega_k \left( f, \frac{1}{n} \right)_p \leq \frac{C(k)}{n^k} \sum_{\nu=0}^{n} (\nu + 1)^{k-1} E_{\nu}(f)_p, \quad k, n \in \mathbb{N}, \]  \hspace{1cm} (3)

\footnote{The concept became well known through S.N. Bersntein’s paper [9] V. 2, p.295-300; p. 349-360].}

\footnote{See [7] for a detailed review of the question before the 30ths of the 20th century.}
\[ \omega_k \left( f^{(r)}, \frac{1}{n} \right)_p \leq C(k) \left( \frac{1}{n^k} \sum_{\nu=0}^{n} (\nu + 1)^{k+r-1} E_{\nu}(f)_p + \sum_{\nu=n+1}^{\infty} \nu^{r-1} E_{\nu}(f)_p \right), \quad k, n \in \mathbb{N}. \] (4)

Here and further, the best trigonometric approximation \( E_n(f)_p \) and the modulus of smoothness \( \omega_k(f, \delta)_p \) are defined as follows:

\[ E_n(f)_p = \min \left( \| f - T \|_p : T \in \mathbf{T}_n \right), \quad \mathbf{T}_n = \text{span} \{ \cos mx, \sin mx : |m| \leq n \} \]

and

\[ \omega_k(f, \delta)_p = \sup_{|h| \leq \delta} \left\| \Delta_h^k f(x) \right\|_p, \]

\[ \Delta_h^k f(x) = \Delta_h^{k-1} (\Delta_h f(x)) \quad \Delta_h f(x) = f(x + h) - f(x), \]

respectively.

In the case of \( 1 < p < \infty \) one can write ([15, p. 210], [47, 53]) the following improvement of estimates (11) and (3):

\[ \frac{C(k)}{n^k} \left( \sum_{\nu=0}^{n} (\nu + 1)^{k+r-1} E_{\nu}(f)_p^r \right)^{\frac{1}{r}} \leq \omega_k \left( f, \frac{1}{n} \right)_p \leq \frac{C(k)}{n^k} \left( \sum_{\nu=0}^{n} (\nu + 1)^{k\theta - 1} E_{\nu}(f)^{\theta}_p \right)^{\frac{1}{\theta}} \]

where \( k, n \in \mathbb{N}, \theta = \min(2, p), \) and \( \tau = \max(2, p) \).

In view of estimates (2) and (11), we note that investigation of the question on existence of the \( r \)-th derivative of \( f \) from a given function space has been initiated by Bernstein [8]. He proved that the condition \( \sum_{\nu=1}^{\infty} \nu^{r-1} E_{\nu}(f)_\infty < \infty \) implies \( f^{(r)} \in C \). Later on, for \( L_p \) \( (1 \leq p \leq \infty) \), the following results were obtained (see the review [57] and the paper by O.V. Besov [10]). For convenience, we write these embeddings in terms of the Besov space \( B^r \) and the Sobolev space \( W^r \):

\[ B^r_{p,1} \subset W^r_p \subset B^r_{p,\infty} \quad p = 1, \infty, \]
\[ B^r_{p,p} \subset W^r_p \subset B^r_{p,2} \quad 1 < p \leq 2, \]
\[ B^r_{p,2} \subset W^r_p \subset B^r_{p,p} \quad 2 \leq p < \infty. \]

**Criterion (B)** was proved by A. Zygmund (1945, [58]). He was one of the first to use the modulus of smoothness concept of an integer order introduced by Bernstein in 1912 ([8]). At present, the moduli of smoothness properties are well-studied ([24, 57]) and the result (B) follows from the following inequalities (see [15, Chapters 2 and 6], [25]): \( 1 \leq p \leq \infty \)

\[ \omega_{k+r} \left( f, \frac{1}{n} \right)_p \leq C(k,r) \omega_k \left( f^{(r)}, \frac{1}{n} \right)_p, \quad k, r, n \in \mathbb{N} \]

(7)

\[ \omega_k \left( f^{(r)}, \frac{1}{n} \right)_p \leq C(k,r) \sum_{\nu=n+1}^{\infty} \nu^{r-1} \omega_{k+r} \left( f, \frac{1}{\nu} \right)_p, \quad k, r, n \in \mathbb{N}. \]

(8)

Comparing the last two inequalities and inequalities (2) and (11) we see that from (7) and (8), using (11) and (3), it is easy to get (2) and (4).

We also mention the paper by J. Marcinkiewicz (1938, [26]), where the following two inequalities were proved:

\[ ||f'||_p \leq C(p) \left( \int_0^{2\pi} \frac{\omega_2(f,u)^p}{u^{p+1}} du \right)^{\frac{1}{p}}, \quad 1 < p \leq 2 \]

(9)
and
\[
\left( \frac{2\pi}{\omega_2(f, u)^p}{u^{p+1}}du \right)^{\frac{1}{p}} \leq C(p)\|f'\|_p, \quad 2 \leq p < \infty. \tag{10}
\]
It is easy to show \cite[Rem 3.5]{17} that \eqref{8} and \eqref{9} are corollaries of the estimate \cite[Th 3.1]{17}

\[
\omega_k\left((r), \frac{1}{n}\right)_p \leq C(k, r) \left( \sum_{\nu=n+1}^\infty \nu^r \omega_k^{\theta}(f, \frac{1}{\nu})_p \right)^{\frac{1}{p}}, \tag{11}
\]
where \( k, r, n \in \mathbb{N}, \theta = \min(2, p), \) and \( 1 \leq p < \infty. \)

**Criterion (C)** was proved by I.I. Privalov (1919, \cite{34}). The following inequality, which implies embedding (C), was obtained by A. Zygmund \cite{58} and N.K. Bary and S.B. Stechkin \cite{3} \((p = 1, \infty):\)

\[
\omega_k\left((r), \frac{1}{n}\right)_p \leq C(k, r) \left( n^{-k} \sum_{\nu=1}^n \nu^{k+r-1} \omega_k^{\theta}(f, \frac{1}{\nu})_p + \sum_{\nu=1}^\infty \nu^{r-1} \omega_k^{\theta}(f, \frac{1}{\nu})_p \right), \tag{12}
\]
Here and further, \( \tilde{f} \) denotes the conjugate function to \( f \) \cite[V. 1, Ch. 2]{59}.

Finally, we mention some improvements of above written inequalities for the case of the generalized derivatives and moduli of smoothness, as these estimates are of a particular interest for us: \eqref{11} and \eqref{12} are proved for any \( k > 0 \) in \cite{13} and \cite{45}; analogues of inequality \eqref{2} for the \( \beta_0 \)-derivatives are shown in \cite[6.3]{39}, and analogues of \eqref{11} and \eqref{12} are proved in \cite{25, 40, 41, 46}.

**Embedding theorems for functional classes.** The results \((A) - (C)\) as well as their generalizations mentioned above can be written as the embedding theorems of the following functional classes:

\[
W_p^r = \left\{ f \in L_p : f^{(r)} \in L_p \right\},
\]
\[
\tilde{W}_p^r = \left\{ f \in L_p : \tilde{f}^{(r)} \in L_p \right\},
\]
\[
W_p^rH_\alpha[\varphi] = \left\{ f \in W_p^r : \omega_\alpha\left(f^{(r)}, \delta\right)_p = O[\varphi(\delta)] \right\},
\]
\[
\tilde{W}_p^rH_\alpha[\varphi] = \left\{ f \in \tilde{W}_p^r : \omega_\alpha\left(\tilde{f}^{(r)}, \delta\right)_p = O[\varphi(\delta)] \right\},
\]
\[
W_p^rE[\xi] = \left\{ f \in W_p^r : E_n\left(f^{(r)}\right)_p = O[\xi(\delta)] \right\}.
\]

We will study more general classes such that \( W_p^r, \tilde{W}_p^r, W_p^rH_\alpha[\varphi], \tilde{W}_p^rH_\alpha[\varphi], W_p^rE[\xi] \) are their particular cases.

**Transformed Fourier series.** Let \( L_p = L_p[0, 2\pi] \) \((1 \leq p < \infty)\) be a space of \(2\pi\)-periodic measurable functions such that \( |f|^p \) is integrable, and \( L_\infty \equiv C[0, 2\pi] \) be the space of \(2\pi\)-periodic continuous functions with the uniform norm, that is, \( \|f\|_\infty = \max \{|f(x)|, 0 \leq x \leq 2\pi\}. \)

Let the Fourier series of a summable function \( f(x) \) be written as

\[
f(x) \sim \sigma(f) := \frac{a_0(f)}{2} + \sum_{\nu=1}^\infty (a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x) \equiv \sum_{\nu=0}^\infty A_\nu(f, x). \tag{13}
\]

*The transformed Fourier series* for series \eqref{13} is defined as follows:

\[
\sigma(f, \lambda, \beta) := \sum_{\nu=1}^\infty \lambda_\nu \left[ a_\nu \cos\left(\nu x + \frac{\pi \beta}{2}\right) + b_\nu \sin\left(\nu x + \frac{\pi \beta}{2}\right) \right],
\]
where \( \beta \in \mathbb{R} \) and \( \lambda = \{ \lambda_n \} \) is a given sequence of positive numbers.

This definition is well-known in the literature (see, for example, \([59]\) ch. 12 §8-9, \([28]\), \([30]\), \([37]\), \([39]\), \([14]\), \([35]\), \([44]\) and remark 3 in this paper). We also note that \( \sigma(\lambda, \beta, \nu) \) coincides up to notations with the Fourier series of so-called \( f_\beta^\nu \)-derivatives, using the terminology of \([39]\) p. 132.

Studies of the transformed Fourier series are naturally related to the problems of Fourier multipliers theory (see \([11]\), \([59]\) V. 1, Ch. III, \([5]\), \([54]\) Chapter 7), summability methods (see \([59]\) V. 1, Ch. III, \([12]\) Chapter 1.2, \([54]\) Chapter 8)) and the so-called fractional Sobolev classes or the Weyl classes \([39]\) V. 1, Ch. III.

The function class

\[
W_{p,\lambda,\beta} = \left\{ f \in L_p : \exists g \in L_p, \ \sigma(g) = \sigma(f, \lambda, \beta) \right\}
\]

is called the \textit{Weyl class} (see for example \([30]\), \([39]\), \([31]\)). It is named so, because for \( \lambda_n = n^r, r > 0 \) and \( \beta = r \) the class \( W_{p,\lambda,\beta} \) coincides with the class \( W_r^p \), which is defined in terms of fractional derivatives \( f^{(r)} \) in the Weyl sense \(([59] V. 2, \text{Ch. XII})\). In the case of \( \lambda_n = n^r, r > 0 \) and \( \beta = r + 1 \) the class \( W_{p,\lambda,\beta} \) coincides with the class \( W_r^p \). A function \( g(x) \sim \sigma(f, \lambda, \beta) \) is called the \( (\lambda, \beta) \)-derivative of a function \( f(x) \) and is denoted by \( f^{(\lambda,\beta)}(x) \). Using the terminology of \([39]\), we have \( f_\beta^\nu = f^{(\lambda,\beta)} \) for \( \psi^{-1}(k) = \lambda_k \).

The \textit{generalized Weyl-Nikolskii class}. In the definition of this functional class we use the \textit{modulus of smoothness} concept \( \omega_\alpha(f, \delta) \) of fractional order\(^3\) of a function \( f(x) \in L_p \), i.e.,

\[
\omega_\alpha(f, \delta) = \sup_{|h| \leq \delta} \| \Delta_\delta^\alpha f(x) \|_p,
\]

where

\[
\Delta_\delta^\alpha f(x) = \sum_{\nu=0}^{\infty} (-1)^\nu \binom{\alpha}{\nu} f(x + (\alpha - \nu)h), \quad \alpha > 0
\]

is the \( \alpha \)-th difference\(^4\) of a function \( f \) with step \( h \) at the point \( x \). It is clear that for \( \alpha \in \mathbb{N} \) this definition is the same as \([53]\).

Let \( \Phi_\alpha (\alpha > 0) \) be the class of functions \( \varphi(\delta) \), defined and non-negative on \((0, \pi]\) such that

1. \( \varphi(\delta) \to 0 \) \( \delta \to 0 \),
2. \( \varphi(\delta) \) is non-decreasing,
3. \( \delta^{-\alpha} \varphi(\delta) \) is non-increasing.

For functions \( \varphi \in \Phi_\alpha, \alpha > 0 \) and for \( \lambda = \{ \lambda_n \} \) we define the \textit{generalized Weyl-Nikolskii class} similarly to the classes \( W_{p,\lambda,\beta} \) \( H^\alpha \) and \( W_r^p \) \( H^p \) (see, for example, \([31]\)):

\[
W_{p,\lambda,\beta} \varphi = \left\{ f \in W_{p,\lambda,\beta} : \omega_\alpha \left( f^{(\lambda,\beta)}, \delta \right) = O(\varphi(\delta)), \delta \to +0 \right\}.
\]

It is clear that if \( \lambda_n = n^r, r > 0 \) and \( \beta = r \), then \( W_{p,\lambda,\beta} \varphi = W_{p}^{r} \varphi \); and if \( \lambda_n = n^r, r > 0 \) and \( \beta = r + 1 \), then \( W_{p,\lambda,\beta} \varphi = W_{p}^{r+1} \varphi \).

In case \( \lambda_n \equiv 1 \) and \( \beta = 0 \) the class \( W_{p,\lambda,\beta} \varphi \) coincides with the \textit{generalized Lipschitz class} \( H^{\alpha}_\varphi \), i.e.,

\[
H^{\alpha}_\varphi = \left\{ f \in L_p : \omega_\alpha(f, \delta) = O(\varphi(\delta)) \right\} \delta \to +0
\]

In particular, for \( 0 < \gamma \leq 1 \),

\[
\text{Lip}(\gamma, L_p) = H^{\gamma}_\varphi = \left\{ f \in L_p : \omega_1(f, \delta) = O(\varphi(\delta)) \right\} \delta \to +0
\]

The problem setting and the structure of the paper. In this paper, we obtain embedding theorems for the Weyl classes \( W_{p,\lambda,\beta} \), for the generalized Weyl-Nikolskii classes \( W_{p,\lambda,\beta} \varphi \) and for the generalized

\(^3\)See also references to \([3]\) §13, Ch. II.

\(^4\) The term "fractional" can be found in earlier papers \(([13]\) and \([45]\) which used this definition. As in the case of fractional derivatives, the positive number \( \alpha \) that defines the modulus order is not necessarily rational.

\(^5\)As usual, \( \binom{\alpha}{\nu} = \frac{\beta(\beta-1)\cdots(\beta-\nu+1)}{\nu!} \) for \( \nu > 1 \), \( \binom{\alpha}{1} = \beta \) for \( \nu = 1 \), and \( \binom{\alpha}{0} = 1 \) for \( \nu = 0 \).
Lipschitz classes $H^p_1[\omega]$. We show how the parameters $\alpha$ and $\gamma$ are related to each other depending on the behavior of the sequence $\{\lambda_n\}$ and on the choice of the metric $L_p$.

This paper is organized as follows. In section 2 we formulate the main theorem. Sections 3 and 4 contain the proofs of the sufficiency and necessity parts of the main theorem respectively. In section 5 we prove several corollaries. In particular, we describe the difference in results for metrics $L_p, 1 < p < \infty$ and $L_p, p = 1, \infty$. Also, the estimates of $\omega_\gamma(f(r), \delta)_p$ and $\omega_\gamma(\tilde{f}(r), \delta)_p$ are written in terms of $\omega_\beta(f, \delta)_p$ for different values of $r, \gamma,$ and $\beta$. The concluding remarks are given in section 6.

2 Embedding theorems for the generalized Lipschitz and Weyl-Nikolskii classes

For $\lambda = \{\lambda_n\}_{n \in \mathbb{N}}$ we define $\Delta \lambda_n := \lambda_n - \lambda_{n+1}; \Delta^2 \lambda_n := \Delta (\Delta \lambda_n)$.

**Theorem 1** Let $\theta = \min(2, p), \alpha \in \mathbb{R}_+, \beta \in \mathbb{R}, \rho \in \mathbb{R}_+ \cup \{0\}$ and $\lambda = \{\lambda_n\}$ be a non-decreasing sequence of positive numbers such that $\{n^{-\rho} \lambda_n\}$ is non-increasing.

I. If $1 < p < \infty$, then

$$H^p_{\alpha+\rho}[\omega] \subset W^{\lambda, \beta}_p \iff \sum_{n=1}^{\infty} \left( \lambda_{n+1}^\theta - \lambda_n^\theta \right) \omega^\theta \left( \frac{1}{n} \right) < \infty,$$

$$W^{\lambda, \beta}_p \subset H^p_{\alpha+\rho}[\omega] \iff \frac{1}{\lambda_n} = O \left[ \omega \left( \frac{1}{n} \right) \right],$$

$$W^{\lambda, \beta}_p \subset H^p_{\alpha+\rho}[\omega] \iff \frac{\varphi \left( \frac{1}{\lambda_n} \right)}{\lambda_n} = O \left[ \omega \left( \frac{1}{n} \right) \right].$$

II. Let $p = 1$ or $p = \infty$.

(a) If $\Delta^2 \lambda_n \geq 0$ or $\Delta^2 \lambda_n \leq 0$, then

$$H^p_{\alpha+\rho}[\omega] \subset W^{\lambda, \beta}_p \iff \left| \cos \frac{\beta \pi}{2} \right| \sum_{n=1}^{\infty} (\lambda_{n+1} - \lambda_n) \omega \left( \frac{1}{n} \right)$$

$$+ \left| \sin \frac{\beta \pi}{2} \right| \sum_{n=1}^{\infty} \lambda_n \omega \left( \frac{1}{n} \right) < \infty;$$

and if, additionally, for some $\tau > 0$ the following inequality holds,

$$\Delta^2 \left( \frac{\lambda_n}{n^{\tau r}} \right) \geq 0 \text{ for } r = \rho + \tau \text{ sign } \sin \left( \frac{\beta - \rho \pi}{2} \right),$$

then

$$H^p_{\alpha+r}[\omega] \subset W^{\lambda, \beta}_p \iff n^{-\alpha} \sum_{\nu=1}^{n} (\nu^{-\tau} \lambda_{\nu} - (\nu + 1)^{-\tau} \lambda_{\nu+1}) \omega \left( \frac{1}{\nu} \right)$$

$$+ \left| \cos \frac{\beta \pi}{2} \right| \sum_{\nu=n+2}^{\infty} (\lambda_{\nu+1} - \lambda_{\nu}) \omega \left( \frac{1}{\nu} \right)$$
\[ \begin{align*}
\sum_{\nu=n+2}^{\infty} \frac{\lambda_{\nu} \omega \left( \frac{1}{\nu} \right)}{\nu} + \lambda_{n+1} \omega \left( \frac{1}{n+1} \right) &= O \left[ \varphi \left( \frac{1}{n+1} \right) \right].
\end{align*} \]

(b) If for \( \beta = 2k, k \in \mathbb{Z} \), the condition \( \Delta^2 (1/\lambda_n) \geq 0 \) holds, and for \( \beta \neq 2k, k \in \mathbb{Z} \) conditions \( \Delta^2 (1/\lambda_n) \geq 0 \) and \( \sum_{\nu=n+1}^{\infty} \frac{1}{\nu^2} \leq \frac{C}{\lambda_n} \) are fulfilled, then
\[ W_{\lambda,\beta}^p \subset H_{\alpha+p}^p[\omega] \iff \frac{1}{\lambda_n} = O \left[ \omega \left( \frac{1}{n} \right) \right]; \]
and if, additionally, for some \( \tau > 0 \) the following inequality holds,
\[ \Delta^2 \left( \frac{n^r}{\lambda_n} \right) \geq 0 \quad \text{or} \quad \Delta^2 \left( \frac{n^r}{\lambda_n} \right) \leq 0 \quad \text{for} \quad r = \rho + \tau \text{ sign } \left| \sin \left( \frac{\beta - \rho}{2} \pi \right) \right|,
\]
then
\[ W_{\lambda,\beta}^p H_{\alpha}[\varphi] \subset H_{\alpha+r}^p[\omega] \iff \frac{\varphi \left( \frac{1}{\lambda_n} \right)}{\lambda_n} = O \left[ \omega \left( \frac{1}{n} \right) \right]. \]

3 Proof of sufficiency in Theorem 1.

We will use the following notations.
Let series \([13]\) be the Fourier series of a function \( f(x) \in L \). Then \( S_n(f) \) denotes the \( n \)-th partial sum of series \([13]\), \( V_n(f) \) denotes the de la Vallée-Poussin sum and \( K_n(x) \) is the Fejér kernel, i.e.,
\[ S_n(f) = \sum_{\nu=0}^{n} A_\nu(x), \quad V_n(f) = \frac{1}{n} \sum_{\nu=0}^{n-1} S_\nu(f), \quad K_n(x) = \frac{1}{n+1} \sum_{\nu=0}^{n} \left( \frac{1}{2 + \sum_{m=1}^{\nu} \cos \nu} \right) . \]
The following lemmas will play an important role in the proof of the main theorem.

Lemma 3.1 If \( f(x) \in L_p, 1 \leq p \leq \infty \) and \( \alpha > 0 \), then
\[ C_1(p, \alpha) \omega( f, \frac{1}{n} ) \leq \left( n^{-\alpha} \left\| V_n^{(\alpha)}(f, x) \right\|_p + \left\| f(x) - V_n(f, x) \right\|_p \right) \leq C_2(p, \alpha) \omega( f, \frac{1}{n} ) . \]

If \( f(x) \in L_p, 1 < p < \infty \), then
\[ C_1(p, \alpha) \omega( f, \frac{1}{n} ) \leq \left( n^{-\alpha} \left\| S_n^{(\alpha)}(f, x) \right\|_p + \left\| f(x) - S_n(f, x) \right\|_p \right) \leq C_2(p, \alpha) \omega( f, \frac{1}{n} ) . \]

Proof of Lemma 3.1 The estimate of \( \omega( f, \frac{1}{n^\alpha} ) \) from above follows from the inequality (see \([13]\))
\[ \omega( T_n, \frac{1}{n^\alpha} ) \leq C(p, \alpha) n^{-\alpha} \left\| T_n^{(\alpha)} \right\|_p \] ,
where \( T_n \) is a trigonometric polynomial of order \( n \). Indeed,
\[ \omega( f, \frac{1}{n^\alpha} ) \leq C(p, \alpha) \left( \omega( T_n, \frac{1}{n^\alpha} ) \right) \left\| f - T_n \right\|_p \leq C(p, \alpha) \left( n^{-\alpha} \left\| T_n^{(\alpha)} \right\|_p + \left\| f - T_n \right\|_p \right). \]

To estimate \( \omega( f, \frac{1}{n^\alpha} ) \) from below, we will use the generalized Nikol’skii-Stechkin inequality (see \([15]\))
\[ n^{-\alpha} \left\| T_n^{(\alpha)} \right\|_p \leq C(p, \alpha) \omega( T_n, \frac{1}{n^\alpha} ) \] and the generalized Jackson inequality (see \([13]\) for \( \alpha > 0 \))
\[ E_n(f)_p \leq C(\alpha) \omega( f, \frac{1}{n+1} ) . \]
Also, it is well known that the Vallée-Poussin mean is the near best approximant, i.e.,

$$\|f - V_n(f)\|_p \leq CE_n(f)_p.$$  \hfill (25)

Then

$$n^{-\alpha} \left\| v^{(\alpha)}_n(f, x) \right\|_p + \|f(x) - V_n(f, x)\|_p \leq C(p, \alpha) \left( \omega_\alpha \left( V_n, \frac{1}{n} \right)_p + E_n(f)_p \right) \leq C(p, \alpha) \left( \omega_\alpha \left( f, \frac{1}{n} \right)_p + \omega_\alpha \left( f - V_n, \frac{1}{n} \right)_p \right) \leq C(p, \alpha) \omega_\alpha \left( f, \frac{1}{n} \right)_p,$$

i.e., (22) is proved. Using

$$\|f - S_n(f)\|_p \leq C(p) E_n(f)_p$$  \hfill (26)

for $1 < p < \infty$, we obtain (23) analogously. Lemma 3.1 is proved.

We note that (22) and (23) are the realization results for modulus of smoothness (see the original paper [16] by Z. Ditzian, V. Hristov, K. Ivanov).

**Lemma 3.2** ([13]). Let $f(x) \in L_p$, $p = 1, \infty$ and let the condition $\sum n^{-1} E_n(f)_p < \infty$ hold. Then $\tilde{f}(x) \in L_p$ and

$$E_n(\tilde{f})_p \leq C \left( E_n(f)_p + \sum_{k=n+1}^{\infty} k^{-1} E_k(f)_p \right), \quad n \in \mathbb{N}.$$

**Lemma 3.3** Let $p = 1, \infty$ and let $\{\lambda_n\}$ be monotone concave (or convex) sequence. Let

$$T_n(x) = \sum_{\nu=0}^{n} a_\nu \cos \nu x + b_\nu \sin \nu x,$$

$$T_n(\lambda, x) = \sum_{\nu=0}^{n} \lambda_\nu \left( a_\nu \cos \nu x + b_\nu \sin \nu x \right).$$

Then for any integer $M > N + 2$ we have

$$\|T_M(\lambda, x) - T_N(\lambda, x)\|_p \leq \mu(M, N) \|T_M(x) - T_N(x)\|_p,$$

where

$$\mu(M, N) = \begin{cases} 2M(\lambda_M - \lambda_{M-1}) + \lambda_{N+1} - (N + 1)(\lambda_{N+2} - \lambda_{N+1}), & \text{if } \lambda_n \uparrow (n \uparrow), \triangle^2 \lambda_n \geq 0; \\ 2\lambda_M + (N + 1)(\lambda_{N+2} - \lambda_{N+1}) - \lambda_{N+1}, & \text{if } \lambda_n \uparrow (n \uparrow), \triangle^2 \lambda_n \leq 0; \\ (N + 1)(\lambda_{N+1} - \lambda_{N+2}) + \lambda_{N+1}, & \text{if } \lambda_n \downarrow (n \uparrow), \triangle^2 \lambda_n \geq 0, \end{cases}$$

for $M = N + 1$, $\mu(M, N) = \lambda_M$, and for $M = N + 2$, $\mu(M, N) = 2|\lambda_{N+1} - \lambda_{N+2}| + \lambda_{N+2}$.

**Proof of Lemma 3.3** First we consider the case when $M > N + 2$. Applying twice Abel’s transformation, we write

$$\|T_M(\lambda, x) - T_N(\lambda, x)\|_p = \left\| \frac{1}{\pi} \int_{-\pi}^{\pi} \left( T_M - T_N \right)(x + u) \sum_{\nu=N+1}^{M} \lambda_\nu \cos \nu u \, du \right\|_p \leq$$

$$\leq \| \frac{1}{\pi} \int_{-\pi}^{\pi} \left( T_M - T_N \right)(x + u) \left\{ \sum_{\nu=N+1}^{M-2} (\lambda_\nu - 2\lambda_{\nu+1} + \lambda_{\nu+2}) (\nu + 1) K_\nu(u) \right\} \|_p.$$
Finally, if \( \lambda \) Applying Lemma 3.3 and the Bernstein inequality, we have
\[
\frac{\text{of lemma 3.4 is now complete.}}{\lambda}
\]
The proof of Lemma 3.3 is complete.

If \( \lambda \uparrow (n \uparrow) \), \( \triangle^2 \lambda_n \geq 0 \). Then
\[
I(M, N) = \sum_{\nu=n+1}^{M-2} (\lambda_{\nu} - 2\lambda_{\nu+1} + \lambda_{\nu+2}) (\nu + 1) + (\lambda_M - \lambda_{M-1}) M + \lambda_M
\]
\[
= -(N+1)(\lambda_{N+2} - \lambda_{N+1}) + (\lambda_{N+1} - \lambda_{M+1}) + \lambda_M + (2M-1)(\lambda_M - \lambda_{M-1})
\]
\[
= -(N+1)(\lambda_{N+2} - \lambda_{N+1}) + \lambda_{N+1} + 2M(\lambda_M - \lambda_{M-1})
\]
If \( \lambda \uparrow (n \uparrow) \), \( \triangle^2 \lambda_n \leq 0 \), then
\[
I(M, N) = -(N+1)(\lambda_{N+2} - \lambda_{N+1}) + \lambda_{N+1}
\]
Finally, if \( \lambda \downarrow (n \downarrow) \), \( \triangle^2 \lambda_n \geq 0 \), then
\[
I(M, N) = -(N+1)(\lambda_{N+2} - \lambda_{N+1}) + \lambda_{N+1}
\]
The estimate for the case of \( M = N+1 \) is trivial and for the case of \( M = N+2 \) immediately follows from the equation
\[
T_M(\lambda, x) - T_N(\lambda, x) = \frac{\lambda_M}{\pi} \int_{-\pi}^{\pi} (T_M - T_N)(x + u) \left( \mu = \sum_{\nu=0}^{N+2} \cos \nu u \right) du
\]
\[
+ \frac{\lambda_{N+1} - \lambda_{N+2}}{\pi} \int_{-\pi}^{\pi} (T_M - T_N)(x + u) \cos (N + 1) du.
\]
The proof of Lemma 3.3 is complete.

**Lemma 3.4** Let \( p = 1, \infty \). If \( T_{2^n, 2^{n+1}}(x) = \sum_{\nu=2^n}^{2^{n+1}} \left( c_{\nu} \cos \nu x + d_{\nu} \sin \nu x \right) \), then
\[
C_1 \| \tilde{T}_{2^n, 2^{n+1}}(x) \|_p \leq \| T_{2^n, 2^{n+1}}(x) \|_p \leq C_2 \| \tilde{T}_{2^n, 2^{n+1}}(x) \|_p.
\]

**Proof of Lemma 3.4** We rewrite \( T_{2^n, 2^{n+1}}(x) \) in the following way
\[
T_{2^n, 2^{n+1}}(x) = \sum_{\nu=2^n}^{2^{n+1}} \frac{1}{\nu} \left( \nu c_{\nu} \cos \nu x + \nu d_{\nu} \sin \nu x \right).
\]
Applying Lemma 3.3 and the Bernstein inequality, we have
\[
\| T_{2^n, 2^{n+1}}(x) \|_p \leq C \left( \sum_{\nu=2^n}^{2^{n+1}} \left| \nu \cos \nu x + \nu d_{\nu} \sin \nu x \right| \right)_p
\]
\[
= C \left( \sum_{\nu=2^n}^{2^{n+1}} \left| \nu \cos \nu x + c_{\nu} \sin \nu x \right| \right)_p
\]
\[
\leq C \| \tilde{T}_{2^n, 2^{n+1}}(x) \|_p.
\]
Similar reasoning for \( \tilde{T}_{2^n, 2^{n+1}}(x) \) allows us to obtain the left-hand side inequality in (27). The proof of lemma 3.4 is now complete.
Sufficiency in (14) - (21).

I. $1 < p < \infty$. In this case, for $\lambda_n \equiv 1$, the Riesz inequality ([59], V. 1, p. 253) \( \|f\|_p \leq C(p)\|f\|_p \) implies

\[
\|f(\lambda, \beta)\|_p \leq C(p, \beta)\|f\|_p
\]  

(28)

(Here and henceforth, by $C(s, t, \cdots)$ we understand positive constants that depend only on $s, t, \cdots$ and in general, may be different in different inequalities).

Let the series in the right part of (14) be convergent and $f \in H^p_{\alpha+\rho}[\omega]$. We will use the following representation

\[
\lambda^\theta_{2n} = \begin{cases} 
\lambda^\theta_1 + \sum_{\nu=2}^{n+1} (\lambda^\theta_{2\nu-1} - \lambda^\theta_{2\nu-2}), & \text{if } n \geq 1; \\
\lambda^\theta_1, & \text{if } n = 0.
\end{cases}
\]

Applying Minkowski’s inequality, we get (here and further $\Delta_1 := A_1(f, x), \Delta_{n+2} := \sum_{\nu=2^{n+1}}^{2^{n+1}} A_\nu(f, x)$, where $A_\nu(f, x)$ is from (13))

\[
I_1 := \left\{ 2\pi \int_0^{2\pi} \left[ \sum_{n=1}^{\infty} \lambda^\theta_{2n-1} \Delta^2_n \right] \frac{dx}{p} \right\}^{\frac{\theta}{p}} \leq C(p) \left( \lambda^\theta_1 \left\{ 2\pi \int_0^{2\pi} \left[ \sum_{n=1}^{\infty} \Delta^2_n \right] \frac{dx}{p} \right\}^{\frac{\theta}{p}} + \sum_{s=2}^{\infty} \left( \lambda^\theta_{2s-1} - \lambda^\theta_{2s-2} \right) \left\{ 2\pi \int_0^{2\pi} \left[ \sum_{n=s}^{\infty} \Delta^2_n \right] \frac{dx}{p} \right\}^{\frac{\theta}{p}} \right)^{\frac{1}{p}}. 
\]

(29)

By the Littlewood-Paley theorem (see, for example, [59], p. 349) and inequality (20), we obtain

\[
I_1 \leq C(p) \left\{ \lambda^\theta_1 \|f\|_p^\theta + \sum_{s=1}^{\infty} \left( \lambda^\theta_{2s-1} - \lambda^\theta_{2s-2} \right) E_{2s-1}(f) \right\}^{\frac{1}{p}}. 
\]

(30)

Then, both the generalized Jackson inequality (24) and the condition $f \in H^p_{\alpha+\rho}[\omega]$ imply $I_1 < \infty$. Thus, there exists a function $g \in L_p$ with the Fourier series

\[
\sum_{n=1}^{\infty} \lambda_{2n-1} \Delta_n,
\]

(31)

and $\|g\|_p \leq C(p)I_1$. We write series (31) in the form of $\sum_{n=1}^{\infty} \gamma_n A_n(f, x)$, where $\gamma_i := \lambda_i, i = 1, 2$ and $\gamma_{2^\nu} := \lambda_{2^n}$ for $2^{n-1} + 1 \leq \nu \leq 2^n (n = 2, 3, \cdots)$. Further, we consider the series

\[
\sum_{n=1}^{\infty} \lambda_n A_n(f, x) = \sum_{n=1}^{\infty} \gamma_n A_n(f, x), 
\]

(32)

where $A_1 = A_2 = 1$, $A_\nu := \lambda_\nu/\gamma_\nu = \lambda_\nu/\lambda_{2^n}$ for $2^{n-1} + 1 \leq \nu \leq 2^n (n = 2, 3, \cdots)$.

Since the sequence $\{A_n\}$ satisfies the conditions of the Marcinkiewicz multiplier theorem ([59], p.346), series (32) is the Fourier series of a function $f^{(\lambda, 0)} \in L_p$ and $\|f^{(\lambda, 0)}\|_p \leq C(p)\|g\|_p$. Then from inequalities (24), (28) and (30) we get

\[
\|f^{(\lambda, \beta)}\|_p \leq C(p, \beta) \left\{ \lambda^\theta_1 \|f\|_p^\theta + \sum_{s=1}^{\infty} E_{2s-1}(f) \sum_{n=2^{s-1}+1}^{2^s-1} \left( \lambda^\theta_{n+1} - \lambda^\theta_n \right) \right\}^{\frac{1}{p}}. 
\]

(33)
Collecting estimates (35), (36) and the inequality in the right-hand side of (15), we get

\[
C(p, \beta, \alpha, \rho) \left\{ \lambda^\theta_n \| f \|_p + \sum_{n=1}^{\infty} \left( \lambda^\theta_{n+1} - \lambda^\theta_n \right) \omega_{\alpha+\rho} \left( f, \frac{1}{n} \right)_p \right\}^{\frac{1}{\theta}},
\]

i.e., the sufficiency in (14) is proved.

Let the relation in the right-hand side in (15) hold, and \( f \in H^p_{\alpha+\rho}[\varphi] \). Let us prove \( f \in W^\lambda_{p, \alpha+\rho} H_{\alpha}[\varphi] \). First, we estimate \( \omega_{\alpha}(f(\lambda, \beta), \frac{1}{n})_p \). By Lemma 3.1,

\[
\omega_{\alpha} \left( f(\lambda, \beta), \frac{1}{n} \right)_p \leq C(p, \alpha) \left( \| f(\lambda, \beta) - S_n(f(\lambda, \beta)) \|_p + n^{-\alpha} \| S_n^{(\alpha)}(f(\lambda, \beta)) \|_p \right).
\]

Using (33) for the function \( f \), we have \((a)\) is the integer part of \( a \)

\[
\| f(\lambda, \beta) - S_n(f(\lambda, \beta)) \|_p \leq C(p, \beta, \alpha, \rho) \left\{ \lambda^\theta_{n+1} \| f - S_n \|_p + E^\theta_\infty(f)_p \sum_{s=1}^{2n} \left( \lambda^\theta_{s+1} - \lambda^\theta_s \right) \right\}^{\frac{1}{\theta}}
\]

\[
+ \sum_{s=n+1}^{\infty} \left( \lambda^\theta_{s+1} - \lambda^\theta_s \right) E^\theta_\infty(f)_p \leq C(p, \beta, \alpha, \rho) \left\{ \lambda^\theta_{n+1} \omega^\theta_{\alpha+\rho} \left( f, \frac{1}{n} \right)_p + \sum_{n+1}^{\infty} \left( \lambda^\theta_{n+1} - \lambda^\theta_n \right) \omega^\theta_{\alpha+\rho} \left( f, \frac{1}{n} \right)_p \right\}^{\frac{1}{\theta}}.
\]

Further, we estimate the second term of (34). Let \( m \) be an integer such that \( 2^m \leq n + 1 < 2^{m+1} \).

We will use the identity

\[
2^{-\nu \rho^\theta} \lambda^\theta_2 = 2^{-(m+1) \rho^\theta} \lambda^\theta_{2^m+1} + \sum_{\nu=m}^{m} \left( 2^{-\nu \rho^\theta} \lambda^\theta_{2^\nu} - 2^{-(\nu+1) \rho^\theta} \lambda^\theta_{2^{\nu+1}} \right).
\]

Then using Lemmas 3.1 and 3.3 we follow the proof of typical estimates (29)-(33). Then we get

\[
n^{-\alpha} \| S_n^{(\alpha)}(f(\lambda, \beta)) \|_p \leq C(p, \beta, \alpha, \rho) \left\{ \lambda^\theta_{n+1} \omega^\theta_{\alpha+\rho} \left( f, \frac{1}{n} \right)_p + n^{-\alpha} \sum_{\nu=1}^{\infty} \left( \nu^\theta - \lambda^\theta_\nu \right) \omega^\theta_{\alpha+\rho} \left( f, \frac{1}{\nu} \right)_p \right\}^{\frac{1}{\theta}}.
\]

Collecting estimates (35), (36) and the inequality in the right-hand side of (15), we get \( f \in W^\lambda_{p, \alpha+\rho} H_{\alpha}[\varphi] \).

Now we prove that conditions \( \frac{1}{\lambda^\alpha_n} = O \left[ \omega \left( \frac{1}{n} \right) \right] \) and \( \frac{\rho_0(1)}{\lambda^\alpha_n} = O \left[ \omega \left( \frac{1}{n} \right) \right] \) are sufficient for embeddings \( W^\lambda_{p, \alpha+\rho} \subset H^p_{\alpha+\rho}[\varphi] \) and \( W^\lambda_{p, \alpha+\rho} H_{\alpha}[\varphi] \subset H^p_{\alpha+\rho}[\varphi] \), respectively.

Using the Littlewood-Paley and the Marcinkiewicz multiplier theorems and the properties of the sequence \( \{ \lambda_n \} \) (following the proof of (29)-(33)), we get

\[
\omega_{\alpha+\rho} \left( f, \frac{1}{n} \right)_p \leq C(p, \beta, \alpha, \rho) \left( \| f - S_n(f) \|_p + n^{-(\alpha+\rho)} \| S_n^{(\alpha+\rho)}(f) \|_p \right)
\]

\[
\leq C(p, \beta, \alpha, \rho) \left( \lambda^{-1}_n \| f(\lambda, \beta) - S_n(f(\lambda, \beta)) \|_p + \lambda^{-1}_n n^{-\alpha} \| S_n^{(\alpha)}(f(\lambda, \beta)) \|_p \right).
\]

Then, by Lemma 3.1 we have the following inequalities

\[
\omega_{\alpha+\rho} \left( f, \frac{1}{n} \right)_p \leq C(p, \beta, \alpha, \rho) \lambda^{-1}_n \omega_{\alpha} \left( f(\lambda, \beta), \frac{1}{n} \right)_p \leq C(p, \beta, \alpha, \rho) \lambda^{-1}_n \| f(\lambda, \beta) \|_p,
\]

where the first one implies sufficiency in (17) and the second one shows sufficiency in (16).
\[ (18) \]

Let \( M > N > 0 \). Let the series in the right-hand side of (18) converge, and let \( f \in H^p_{\alpha+p}[\omega] \).

Consider the series
\[ \sum_{n=1}^{\infty} \left\{ \cos \frac{\pi \beta}{2} (V_{2n} (\lambda, f) - V_{2n-1} (\lambda, f)) - \sin \frac{\pi \beta}{2} (\widetilde{V}_{2n} (\lambda, f) - \widetilde{V}_{2n-1} (\lambda, f)) \right\}, \]

where \( V_1 (\lambda, f) := \lambda_1 A_1 (f, x) \),

\[ V_n (\lambda, f) := \sigma (\lambda, V_n (f)) = \sum_{m=1}^{n} \lambda_m A_m (f, x) + \sum_{m=n+1}^{2n-1} \lambda_m \left( 1 - \frac{m-n}{n} \right) A_m (f, x) \quad (n \geq 2). \]

Let \( M > N > 0 \).

Using the inequality \( \| f - V_n (f) \|_p \leq C E_n (f)_p \), the Jackson theorem (24), and the properties of \( \{ \lambda_n \} \), and following the proof of Lemma 3.3 we get

\[ A := \left\| \sum_{n=N+1}^{M} \left\{ \cos \frac{\pi \beta}{2} (V_{2n+1} (\lambda, f) - V_{2n} (\lambda, f)) - \sin \frac{\pi \beta}{2} (\widetilde{V}_{2n+1} (\lambda, f) - \widetilde{V}_{2n} (\lambda, f)) \right\} \right\|_p \]

\[ \leq \sum_{n=N+1}^{M} \left\{ \cos \frac{\pi \beta}{2} \| V_{2n+1} (f) - V_{2n} (f) \|_p \left( \sum_{m=2n}^{2n+2-1} |\Delta^2 \lambda_m | (m+1) + 2^{n+2} |\Delta \lambda_{2n+2} | \right) \right\} \]

\[ + \left\{ \sin \frac{\pi \beta}{2} \| V_{2n+1} (f) - V_{2n} (f) \|_p \left( \sum_{m=2n}^{2n+2-1} |\Delta^2 \lambda_m | (m+1) + 2^{n+2} |\Delta \lambda_{2n+2-1} | \right) \right\} \]

\[ + \left\{ \cos \frac{\pi \beta}{2} \left( \sum_{n=N+1}^{M} \lambda_{2n+2} (V_{2n+2} (f) - V_{2n} (f)) \right) + \sin \frac{\pi \beta}{2} \left( \sum_{n=N+1}^{M} \lambda_{2n+2} (\widetilde{V}_{2n+2} (f) - \widetilde{V}_{2n} (f)) \right) \right\} \]

\[ \leq C \left( \sum_{n=2N+1}^{\infty} (\lambda_{n+1} - \lambda_n) \left( \cos \frac{\pi \beta}{2} \left| \omega_{\alpha+p} \left( f, \frac{1}{n} \right) \right| + \sin \frac{\pi \beta}{2} \left| \omega_{\alpha+p} \left( f, \frac{1}{n} \right) \right| \right) \right). \]

To complete the proof of the sufficiency part in (18), we apply Lemma 3.2 inequality (3) (see 15 for the case \( k > 0 \)), and inequality (24).

Then the convergence of the series in the right-hand side of (18) and the condition \( f \in H^p_{\alpha+p}[\omega] \) imply the fact that the sequence
\[ \left\{ V_{2n} (\lambda, \beta, f) := \cos \frac{\pi \beta}{2} V_{2n} (\lambda, f) - \sin \frac{\pi \beta}{2} \widetilde{V}_{2n} (\lambda, f) \right\} \]

is fundamental in \( L_p \). If \( p = 1 \), since \( L_1 \) is complete, there exists a subsequence \( \{ n_k \} \) such that \( V_{2n_k} (\lambda, \beta, f) \) converges almost everywhere to a function \( \varphi \in L_1 \). Then from the mean convergence we obtain that, say for cosine coefficients,

\[ a_n (\varphi) = \frac{1}{\pi} \int_{-\pi}^{\pi} \varphi (x) \cos nx \, dx = \lim_{k \to \infty} \frac{1}{\pi} \int_{-\pi}^{\pi} V_{2n_k} (\lambda, \beta, f) \cos nx \, dx = a_n (f(\lambda, \beta)). \]

Therefore, \( \sigma (\varphi) = \sigma (f(\lambda, \beta)) \).

For \( p = \infty \) the proof is similar. This completes the proof of the sufficiency part of (18).
Let now the condition in the right-hand side of \((19)\) hold and \(f \in H^{p}_{\alpha+r}[\omega]\). Let us estimate \(\omega_{\alpha} \left( f(\lambda,\beta), \frac{1}{n} \right)_p \) from above. By Lemma 3.4
\[
\omega_{\alpha} \left( f(\lambda,\beta), \frac{1}{n} \right)_p \leq C(\alpha) \left( \left\| f(\lambda,\beta) - V_n(f(\lambda,\beta)) \right\|_p + n^{-\alpha} \left\| V^{(\alpha)}_n(f(\lambda,\beta)) \right\|_p \right).
\]

Let us show that
\[
\left\| f(\lambda,\beta) - V_n(f(\lambda,\beta)) \right\|_p \leq C(\beta, \alpha, r) \left( \lambda_n \omega_{\alpha+r} \left( f, \frac{1}{n} \right)_p + \left| \cos \frac{\pi \beta}{2} \right| \sum_{\nu=\alpha+1}^{\infty} (\lambda_{\nu+1} - \lambda_{\nu}) \omega_{\alpha+r} \left( f, \frac{1}{\nu} \right)_p \right.
\]
\[
\left. + \left| \sin \frac{\pi \beta}{2} \right| \sum_{\nu=\alpha+1}^{\infty} \lambda_{\nu} \omega_{\alpha+r} \left( f, \frac{1}{\nu} \right)_p \right)
\]
\[
(39)
\]

It has been proved above that
\[
A = \left\| \sum_{n=N}^{M} \left( V_{2n+1}(f(\lambda,\beta)) - V_{2n}(f(\lambda,\beta)) \right) \right\|_p.
\]

Then for \(2^m \leq n < 2^{m+1}\)
\[
\left\| f(\lambda,\beta) - V_n(f(\lambda,\beta)) \right\|_p \leq \left\| V_n(f(\lambda,\beta)) - V_{2m+1}(f(\lambda,\beta)) \right\|_p + \left\| V_{2m+1}(f(\lambda,\beta)) - V_{2m+1}(f(\lambda,\beta)) \right\|_p
\]
\[
=: I_1 + I_2.
\]

By Lemma 3.4, we have
\[
I_1 = \left\| \cos \frac{\pi \beta}{2} (V_n(\lambda, f) - V_{2m+1}(\lambda, f)) - \sin \frac{\pi \beta}{2} (V_n(\lambda, f) - V_{2m+1}(\lambda, f)) \right\|_p
\]
\[
\leq \left\| V_n(\lambda, f) - V_{2m+1}(\lambda, f) \right\|_p + \left\| V_n(\lambda, f) - V_{2m+1}(\lambda, f) \right\|_p
\]
\[
\leq C \left\| V_n(\lambda, f) - V_{2m+1}(\lambda, f) \right\|_p
\]

and, by Lemma 3.3, we write \(I_1 \leq \lambda_n \omega_{\alpha+r} \left( f, \frac{1}{n} \right)_p\).

Further, we estimate
\[
I_2 \leq \left| \cos \frac{\pi \beta}{2} \right| \left\| \sum_{\nu=m+1}^{\infty} (V_{2\nu}(\lambda, f) - V_{2\nu+1}(\lambda, f)) \right\|_p + \left| \sin \frac{\pi \beta}{2} \right| \left\| \sum_{\nu=m+1}^{\infty} \left( V_{2\nu}(\lambda, f) - V_{2\nu+1}(\lambda, f) \right) \right\|_p.
\]

As in \((38)\), we write
\[
\left\| \sum_{\nu=N}^{M} \left( V_{2\nu+1}(\lambda, f) - V_{2\nu}(\lambda, f) \right) \right\|_p \leq C \left( \lambda_{2N} \omega_{\alpha+r} \left( f, \frac{1}{2N} \right)_p + \sum_{\nu=2N+1}^{\infty} (\lambda_{\nu+1} - \lambda_{\nu}) \omega_{\alpha+r} \left( f, \frac{1}{\nu} \right)_p \right)
\]
\[
+ \sum_{\nu=N}^{M} \left\| V_{2\nu}(\lambda, f) - V_{2\nu+1}(\lambda, f) \right\|_p \left( \sum_{m=2^{\nu}}^{2^{\nu+2}-1} |\lambda^2|\lambda_{m}(m+1) + 2^{\nu+2} |\Delta \lambda_{2\nu+2}| \right) =: I_{21} + I_{22}.
\]

Applying Lemma 3.2
\[
I_{21} \leq \lambda_{2N+1} \left\| V_{2M+1}(f) - V_{2N}(f) \right\|_p + \sum_{\nu=N}^{M} (\lambda_{2\nu+2} - \lambda_{2\nu+1}) \left\| V_{2\nu+1}(f) - V_{2\nu}(f) \right\|_p
\]
\[
\leq C \left( \frac{1}{2^{N+1}} \sum_{\nu=N}^{\infty} E_{2\nu}^p(f) + \sum_{\nu=N}^{\infty} \frac{(\lambda_{2\nu+2} - \lambda_{2\nu+1})}{\lambda_{2\nu+2}} \sum_{\nu=\nu}^{\infty} E_{2\nu}^p(f) \right)
\]
\leq C \sum_{\nu=2N}^{\infty} \frac{\lambda_{2\nu}}{\nu^r} \left( f, \frac{1}{\nu} \right)_p,
\]
\[
I_{22} \leq C \sum_{\nu=N}^{M} (\lambda_{2\nu+3} - \lambda_{2\nu-1}) E_{2\nu}^p(f) \leq C \sum_{\nu=2N}^{\infty} \frac{\lambda_{2\nu}}{\nu^r} \left( f, \frac{1}{\nu} \right)_p,
\]
and (39) follows.

Repeating the arguments which were used in (38), we estimate \( n^{-\alpha} \left\| V_n^{(\alpha)}(f,\lambda,\beta) \right\|_p \). Using Lemma 3.3 and inequalities \( 24 \) and \( 25 \), we write
\[
\left\| V_n^{(\alpha)}(f,\lambda,\beta) \right\|_p \leq C(\beta, \alpha, r) \left( n^\alpha \lambda_n \omega_{\alpha+r} \left( f, \frac{1}{n} \right)_p + \sum_{\nu=1}^{n} \frac{\nu^r}{(\nu + 1)^r} \right) n^{-r}. \]
Collecting (39) and (40) and using the condition in the right-hand side of (19), we get \( f \in W^p_{\lambda,\beta} H_{\alpha}[\varphi] \).

Let us prove (20). Let \( f \in W^p_{\alpha+p}(\omega) \). To establish \( f \in H^p_{\alpha+p}(\omega) \), we will first estimate \( E_n(f)_p \). Let \( \beta = \rho + 2m \), and therefore, \( r = \rho \), then
\[
E_n(f)_p \leq C \sum_{\nu=m}^{\infty} \frac{1}{\lambda_{2\nu}} \left\| (V_{2\nu-1} - V_{2\nu})\left( f, \lambda_n^{\alpha+r} \right) \right\|_p \leq C E_{[2m-1]}(f,\lambda,\beta)_p \sum_{\nu=m}^{\infty} \frac{1}{\lambda_{2\nu}} \leq C(\rho, p) \left\| f(\lambda,\beta) \right\|_p. \]
If \( \sin \frac{\beta}{2} = 0 \), then it is easy to see that
\[
E_n(f)_p \leq C \frac{C}{\lambda_n} E_n(f,\lambda,\beta)_p \leq C \frac{C}{\lambda_n} \left\| f(\lambda,\beta) \right\|_p.
\]
Hence, substituting the obtained bound for \( E_n(f)_p \) into inequality (3) and using the fact that \( n^\rho \lambda_n^{-1} \uparrow (n \uparrow) \), we obtain
\[
\omega_{\alpha+p} \left( f, \frac{1}{n} \right)_p \leq C(\alpha, \rho) \frac{1}{n^\rho} \sum_{\nu=0}^{n} \nu^{\alpha+p-1} E_{\nu-1}(f)_p \leq C(\alpha, \rho) \frac{C}{\lambda_n} = O \left[ \omega \left( \frac{1}{n} \right) \right],
\]
i.e., \( f \in H^p_{\alpha+p}(\omega) \). This completes the proof of the sufficiency part in (21).

Let the right-hand side part of (21) hold true and \( f \in H^p_{\alpha+p}[\omega] \). First let us prove that
\[
\left\| V_n^{(\alpha+r)}(f) \right\|_p \leq C(\alpha, r) \left\| V_n^{(\alpha)}(f,\lambda,\beta) \right\|_p. \quad (41)
\]
If \( \beta = \rho + 2m \), and therefore, \( r = \rho \), then
\[
V_n^{(\alpha+r)}(f) = V_n^{(\alpha)} \left( \frac{\nu^r}{\lambda_n} \right) f(\lambda,\beta)
\]
and, by Lemma 3.3
\[
\left\| V_n^{(\alpha+r)}(f) \right\|_p \leq C(\alpha, r) \frac{n^r}{\lambda_n} \left\| V_n^{(\alpha)}(f,\lambda,\beta) \right\|_p \leq C(\alpha, r) \frac{n^r}{\lambda_n} \left\| V_{2n}^{(\alpha)}(f,\lambda,\beta) \right\|_p.
\]
Thus, by (41) and the estimate (48), we can write

\[ \|V_n^{(\alpha+r)}(f)\|_p \leq \sum_{\nu=1}^{n} \left\| V_n^{(\alpha+r)}(f) - V_{2^{\nu-1}}^{(\alpha+r)}(f) \right\|_p + \left\| V_1^{(\alpha+r)}(f) \right\|_p \]

\[ \leq C \sum_{\nu=1}^{n} \frac{2^{\nu r}}{\lambda_{2^{\nu}}} \left\| V_2^{(\alpha)}(f(\lambda,\beta)) - V_{2^{\nu-1}}^{(\alpha)}(f(\lambda,\beta)) \right\|_p + \frac{1}{\lambda_1} \left\| V_1^{(\alpha)}(f(\lambda,\beta)) \right\|_p \]

\[ \leq C \left\| V_{2^{n+1}}^{(\alpha)}(f(\lambda,\beta)) \right\|_p \left( \sum_{\nu=1}^{n} \frac{2^{\nu r}}{\lambda_{2^{\nu}}} + \frac{1}{\lambda_1} \right) \]

\[ \leq C(\alpha,r) \frac{2^{nr}}{\lambda_2} \left\| V_{2^{n+1}}^{(\alpha)}(f(\lambda,\beta)) \right\|_p. \]

Thus, by (11) and the estimate

\[ E_n(f)_p \leq \frac{C}{\lambda_n} E_{[\frac{\alpha}{2}]}(f(\lambda,\beta))_p \]

we can write

\[ \omega_{\alpha+r}\left( f, \frac{1}{n} \right)_p \leq C(\alpha,r) \left( n^{-(\alpha+r)} \|V_n^{(\alpha+r)}(f)\|_p + E_n(f)_p \right) \]

\[ \leq C(\alpha,r) \frac{1}{\lambda_n} \left( n^{-\alpha} \|V_n^{(\alpha)}(f(\lambda,\beta))\|_p + E_{[\frac{\alpha}{2}]}(f(\lambda,\beta))_p \right) \]

\[ \leq C(\alpha,r) \frac{\omega_\alpha \left( f(\lambda,\beta), \frac{1}{n} \right)_p = O \left[ \varphi \left( \frac{1}{n} \right) \right] = O \left[ \omega \left( \frac{1}{n} \right) \right]. \]

Thus, the condition \( \frac{\varphi(\frac{1}{n})}{\lambda_n} = O \left[ \omega \left( \frac{1}{n} \right) \right] \) is sufficient for the embedding \( W^\lambda_{p} H_\alpha[\varphi] \subset H^p_{\alpha+r}[\omega]. \)

4 Proof of necessity in Theorem 1.

We define the trigonometric polynomials \( \tau_{n+1}(x) \):

\[ \tau_{n+1}(x) = \sum_{j=1}^{n+1} \alpha_j \sin jx, \quad \text{where} \quad \alpha_j = \begin{cases} \frac{j}{n+2}, & 1 \leq j \leq \frac{n+2}{2}, \\ 1 - \frac{j}{n+2}, & \frac{n+2}{2} \leq j \leq n+1. \end{cases} \]

We will use the following lemmas, as well as Lemmas 3.1, 3.2.

Lemma 4.1 ([14]). Let series (13) be the Fourier series of a function \( f(x) \in L_1 \). Then

\[ E_n(f)_1 \geq C \left| \sum_{\nu=n+1}^{\infty} \frac{b_\nu}{\nu} \right|. \]

Lemma 4.2 ([59], V.1, p. 345; V.2, p. 198]). Let \( 1 \leq p < \infty \).

(a) If the series \( \sum_{\nu=1}^{\infty} (a_\nu \cos 2^\nu x + b_\nu \sin 2^\nu x) \) is the Fourier series of a function \( f(x) \in L_p \), then

\[ \left\{ \sum_{\nu=1}^{\infty} (a_\nu^2 + b_\nu^2) \right\}^{\frac{1}{2}} \leq C \|f\|_p. \]

(b) Let \( a_n, b_n (n \in \mathbf{N}) \) be real numbers such that \( \sum_{\nu=1}^{\infty} (a_\nu^2 + b_\nu^2) < \infty \). Then the series \( \sum_{\nu=1}^{\infty} (a_\nu \cos 2^\nu x + b_\nu \sin 2^\nu x) \) is the Fourier series of a function \( f(x) \in L_p \), and at the same time

\[ \|f\|_p \leq C \left\{ \sum_{\nu=1}^{\infty} (a_\nu^2 + b_\nu^2) \right\}^{\frac{1}{2}}. \]
Lemma 4.3 ([3] Ch.11,§12). If the series \( \sum_{\nu=1}^{\infty} (a_{\nu} \cos 2^\nu x + b_{\nu} \sin 2^\nu x) \), \( a_{\nu}, b_{\nu} \geq 0 \) is the Fourier series of a function \( f(x) \) \( \in L_\infty \), then

\[
C_1 \sum_{\xi=n}^{\infty} (a_\xi + b_\xi) \leq E_{2n-1}(f)_{\infty} \leq C_2 \sum_{\xi=n}^{\infty} (a_\xi + b_\xi).
\]

We will use the following definitions. Let \( \omega(\cdot) \in \Phi_\alpha \).

A sequence \( \psi \) is called \( Q_{\alpha, \theta}(\omega) \)-sequence if

\[
0 < \psi_n \leq n^\alpha \omega \left( \frac{1}{n} \right), \quad \psi_n \uparrow (n \uparrow)
\]

\[
C_1 \omega \left( \frac{1}{n} \right) \leq \left\{ \sum_{\nu=n}^{\infty} \nu^{-\alpha\theta-1} \psi_\nu^\theta \right\} \frac{1}{\theta} \leq C_2 \omega \left( \frac{1}{n} \right).
\]

A sequence \( \varepsilon \) is called \( q_{\alpha, \theta}(\omega) \)-sequence if

\[
0 < \varepsilon_n \leq \omega \left( \frac{1}{n+1} \right), \quad \varepsilon_n \downarrow (n \downarrow)
\]

\[
C_1 \omega \left( \frac{1}{n+1} \right) \leq \left\{ (n+1)^{-\alpha\theta} \sum_{\nu=1}^{n+1} \nu^{\alpha\theta-1} \varepsilon_\nu^\theta \right\} \frac{1}{\theta} \leq C_2 \omega \left( \frac{1}{n+1} \right).
\]

Necessity in (14) - (21).

We prove the necessity part by constructing corresponding examples. The proof consists of eight steps.

1. \( 1 < p < \infty \). Step 1. Let us show the necessity part in (14).

Let \( \omega(\cdot) \in \Phi_{\alpha+\rho} \) and \( \theta = \min(2, p) \). We will construct a \( Q_{\alpha+\rho, \theta}(\omega) \)-sequence \( \psi \).

Assume that integers \( 1 = n_1 < n_2 < \cdots < n_s \) are chosen. Then as \( n_{s+1} \) we take the minimum number \( N > n_s \) such that

\[
\omega \left( \frac{1}{N} \right) < \frac{1}{2} \omega \left( \frac{1}{n_s} \right) \leq \omega \left( \frac{1}{N-1} \right).
\]

We set

\[
\psi_n = \begin{cases} 
\frac{n_s^{\rho+\alpha}}{n_s^\alpha} \omega \left( \frac{1}{n_s} \right), & \text{if } n_s \leq n < n_{s+1}, \quad s = 1, 2, \cdots; \\
0, & \text{if } n = 0.
\end{cases}
\]

It is easy to see that this sequence is what we need.

Let \( H^p_{\alpha+\rho}[\omega] \subset W^\lambda_{p} \) and let the series in (14) be divergent. By means of properties of sequence \( \{\psi_n\} \), we have

\[
\infty = \sum_{\nu=1}^{\infty} \left( \lambda_{\nu+1}^\theta - \lambda_\nu^\theta \right) \omega^\theta \left( \frac{1}{\nu} \right) \leq C(\alpha, \rho, \theta) \sum_{\nu=1}^{\infty} \left( \lambda_{\nu+1}^\theta - \lambda_\nu^\theta \right) \sum_{m=\nu}^{\infty} m^{-(\alpha+\rho)\theta-1} \psi_m^\theta
\]

\[
\leq C(\alpha, \rho, \theta) \sum_{\nu=1}^{\infty} \lambda_\nu^\theta \nu^{-(\alpha+\rho)\theta-1} \psi_\nu^\theta.
\]

Step 1(a): \( 2 \leq p < \infty \). We consider the series

\[
\sum_{\nu=1}^{\infty} 2^{-\nu(\alpha+\rho)} \left( \psi^2_{2\nu} - \psi^2_{2\nu-1} \right) \frac{1}{\theta} \cos 2^\nu x.
\]

Since

\[
\sum_{\nu=1}^{\infty} 2^{-\nu(\alpha+\rho)} \left( \psi^2_{2\nu} - \psi^2_{2\nu-1} \right) \leq \sum_{\nu=1}^{\infty} \left( \psi^2_{2\nu} - \psi^2_{2\nu-1} \right) \sum_{\xi=\nu}^{\infty} 2^{-2\xi(\alpha+\rho)}
\]

\[
\sum_{\nu=1}^{\infty} 2^{-\nu(\alpha+\rho)} \left( \psi^2_{2\nu} - \psi^2_{2\nu-1} \right) \leq \sum_{\nu=1}^{\infty} \left( \psi^2_{2\nu} - \psi^2_{2\nu-1} \right) \sum_{\xi=\nu}^{\infty} 2^{-2\xi(\alpha+\rho)}
\]
then, by Zygmund’s Lemma \(4.2\) series \(46\) is the Fourier series of a function \(f_1(x) \in L_p\). Applying Lemmas \(3.1\) and \(4.2\) we get

\[
C(\alpha, \rho)\omega_{\alpha+\rho} (f_1, \frac{1}{2^n})_p \leq 2^{-n(\alpha+\rho)} \left( \sum_{\nu=1}^{n} a_\nu^2 2^{(\alpha+\rho)\nu} \right)^{\frac{1}{2}} + \left( \sum_{\nu=n+1}^{\infty} a_\nu^2 \right)^{\frac{1}{2}} =: I_1 + I_2,
\]

where \(a_\nu = 2^{-(\alpha+\rho)} \left( \psi_{2^\nu}^2 - \psi_{2^{\nu-1}}^2 \right)^{\frac{1}{2}}\). Similarly to estimate \(47\), by means of \(4.2\), we get

\[
I_1 \leq 2^{-n(\alpha+\rho)} \psi_{2^n} \leq \omega \left( \frac{1}{2^n} \right),
\]

and by means of \(43\),

\[
I_2 \leq C(\alpha, \rho) \left( \sum_{\nu=n+1}^{\infty} 2^{-2\nu(\alpha+\rho)} \psi_{2^\nu}^2 \right)^{\frac{1}{2}} \leq C(\alpha, \rho)\omega \left( \frac{1}{2^n} \right).
\]

Thus, \(f_1(x) \in H^p_{\alpha+\rho}[\omega]\). Then from our assumption, \(f_1(x) \in W^\lambda_{p,\beta}\). On the other hand,

\[
\left\| f_1^{(\lambda,\beta)} \right\|_p \geq C(\alpha, \rho, \theta) \left( \sum_{\nu=1}^{\infty} \lambda_\nu^2 2^{-(\alpha+\rho)-1} \psi_\nu^2 \right)^{\frac{1}{2}} = \infty.
\]

This contradiction proves the convergence of series in \(44\).

Step 1(b): \(1 < p \leq 2\). Consider series \(48\)

\[
\psi_1 \cos x + \sum_{\nu=1}^{\infty} 2^{-\nu(\alpha+\rho)} 2^{\nu(\frac{1}{p}-1)} \left( \psi_{2^\nu}^p - \psi_{2^{\nu-1}}^p \right)^{\frac{1}{p}} \sum_{\mu=2^{\nu-1}+1}^{2^\nu} \cos \mu x.
\]

Using the Jensen inequality \(\left( \sum_{n=1}^{\infty} a_n^\alpha \right)^{\frac{1}{\alpha}} \leq \left( \sum_{n=1}^{\infty} a_n^\beta \right)^{\frac{1}{\beta}}\) \((a_n \geq 0 \text{ and } 0 < \beta \leq \alpha < \infty)\), we write

\[
\int_0^{2\pi} \left[ \sum_{\nu=1}^{\infty} 2^{-\nu(\alpha+\rho)} 2^{\nu(\frac{1}{p}-1)} \left( \psi_{2^\nu}^p - \psi_{2^{\nu-1}}^p \right)^{\frac{1}{p}} \sum_{\mu=2^{\nu-1}+1}^{2^\nu} \cos \mu x \right]^p dx \leq \int_0^{2\pi} \left[ \sum_{\nu=1}^{\infty} 2^{-\nu(p-1)} \left( \psi_{2^\nu}^p - \psi_{2^{\nu-1}}^p \right) \sum_{\mu=2^{\nu-1}+1}^{2^\nu} \cos \mu x \right]^p dx \leq C(p) \sum_{\nu=1}^{\infty} \left( \psi_{2^\nu}^p - \psi_{2^{\nu-1}}^p \right) 2^{-\nu(p-1)} \leq C(p)\omega^p(1),
\]

because of \(C_1(p)2^{\nu(p-1)} \leq \left\| \sum_{\mu=2^{\nu-1}+1}^{2^\nu} \cos \mu x \right\|_p^p \leq C_2(p)2^{\nu(p-1)}\).

By the Littlewood-Paley theorem (see [59] Vol. 2, p. 349), there exists a function \(f_2 \in L_p\) with the Fourier series \(48\). One can easily check that \(f_2 \in H^p_{\alpha+\rho}[\omega]\). By our assumption, \(f_2(x) \in W^\lambda_{p,\beta}\). On the other hand, Paley’s theorem on Fourier coefficients [59 V.2, p. 182] implies that for \(f_2 \in L_p\)

\[
\left\| f_2^{(\lambda,\beta)} \right\|_p \geq C(p) \sum_{\nu=1}^{\infty} 2^{-\nu(p-1)} \left( \psi_{2^\nu}^p - \psi_{2^{\nu-1}}^p \right) \sum_{\mu=2^{\nu-1}+1}^{2^\nu} \lambda_\mu^p \mu^{p-2}.
\]

Series of this type was considered in [92].
\[ C(\alpha, \rho, p) \sum_{\nu=1}^{\infty} \left( \psi_{2\nu}^p - \psi_{2\nu-1}^p \right) \sum_{\xi=\nu}^{\infty} \left( 2^{-2(\alpha+\rho)p} \lambda_{2\xi}^p - 2^{-2(\xi+1)(\alpha+\rho)p} \lambda_{2\xi+1}^p \right) \geq C_1(\alpha, \rho, p) \sum_{\nu=1}^{\infty} \psi_{2\nu}^p \lambda_{2\nu}^p \nu^{-(\alpha+\rho)-1} - C_2(\alpha, \rho, p) \psi_{2\nu}^p \lambda_{2\nu}^p 2^{-(\alpha+\rho)-1} = \infty. \]

This contradiction shows that the series in the right-hand side of (14) converges. This completes the proof of the necessity part of (14).

**Step 2.** Let us prove the necessity in (15) for the case \( 2 \leq p < \infty \).

We notice that using Lemmas 3.1 and 4.2 we get for the function \( f \)

\[ f(x) \sim \sum_{\nu=1}^{\infty} (a_{2\nu} \cos 2\nu x + b_{2\nu} \sin 2\nu x) \]

the following relation

\[ \omega_\alpha \left( f, \frac{1}{2m} \right)_p \geq \left( 2^{-2m\alpha} \sum_{\nu=1}^{m} (a_{2\nu}^2 + b_{2\nu}^2) 2^{2\nu} \right)^{\frac{1}{2}} + \left( \sum_{\nu=m+1}^{\infty} (a_{2\nu}^2 + b_{2\nu}^2) \right)^{\frac{1}{2}}. \] (49)

Let \( \omega(\cdot) \in \Phi_{\alpha+p} \). Then one can construct a sequence \( \varepsilon \) such that it is a \( q_{\alpha+p, \rho}(\omega) \)-sequence. In this case, we consider

\[ \varepsilon_0 + \left( \varepsilon_1^2 - \varepsilon_2^2 \right)^{\frac{1}{2}} \cos x + \sum_{\nu=1}^{\infty} \left( \varepsilon_{2\nu}^2 - \varepsilon_{2\nu+1}^2 \right)^{\frac{1}{2}} \cos 2\nu x. \] (50)

Repeating the argument used for series (46), we obtain that series (50) is the Fourier series of a function \( f_3 \in L_p \). Since \( E_{2n-1}(f_3) \leq C(p) \varepsilon_{2n} \), then by (13) and (45) we have \( f_3 \in H_{\alpha+p}^{p}[\omega] \). We define \( f_{13} := f_1 + f_3 \). Then \( f_{13} \in H_{\alpha+p}^{p}[\omega] \subset W_p^\lambda \lambda H_{\alpha}[\varphi] \). It is easy to see from (49) that

\[ C(\alpha, \beta) \omega_\alpha \left( f_{13}, \frac{1}{n+1} \right)_p \geq \omega_\alpha \left( f_1, \frac{1}{n+1} \right)_p + \omega_\alpha \left( f_3, \frac{1}{n+1} \right)_p. \] (51)

Let us estimate \( \omega_\alpha \left( f_1, \frac{1}{n+1} \right)_p \). Applying (49) and using the properties of the sequence \( \{\psi_{2\nu}\} \), we write \((2m \leq n+1 < 2^{m+1})\)

\[ \omega_\alpha^2 \left( f_1, \frac{1}{n+1} \right)_p \geq C(\alpha, \beta) \sum_{\nu=m}^{\infty} \psi_{2\nu}^2 2^{-2\nu(\alpha+\beta)} \left( \sum_{k=m}^{\nu} (\lambda_{2k}^2 - \lambda_{2k-1}^2) + \lambda_{2m-1}^2 \right) \]

\[ \geq C(\alpha, \beta) \left( \lambda_{2m}^2 \omega^2 \left( \frac{1}{2m} \right) + \sum_{k=m}^{\infty} (\lambda_{2k}^2 - \lambda_{2k-1}^2) \omega^2 \left( \frac{1}{2k} \right) \right) \]

\[ \geq C(\alpha, \beta) \left( \lambda_{n+1}^2 \omega^2 \left( \frac{1}{n+1} \right) + \sum_{\nu=n+2}^{\infty} (\lambda_{\nu+1}^2 - \lambda_{\nu}^2) \omega^2 \left( \frac{1}{\nu} \right) \right). \] (52)

Let us proceed to the estimation of \( \omega_\alpha \left( f_3, \frac{1}{n+1} \right)_p \). Using (49), we have \((2m \leq n+1 < 2^{m+1})\):

\[ \omega_\alpha^2 \left( f_3, \frac{1}{n+1} \right)_p \geq C(\alpha, \beta) 2^{-2m\alpha} \sum_{\nu=0}^{m} 2^{2\nu\alpha} \lambda_{2\nu}^2 \left( \varepsilon_{2\nu}^2 - \varepsilon_{2\nu+1}^2 \right) \]

\[ \geq C_1(\alpha, \beta) 2^{-2m\alpha} \sum_{\nu=0}^{m} 2^{2\nu\alpha} \lambda_{2\nu}^2 \varepsilon_{2\nu}^2 - C_2(\alpha, \rho) \lambda_{2m+1}^2 \varepsilon_{2m+1}^2. \] (53)

\footnote{See, for example, [11, 19].}
Then Jackson’s inequality implies
\[
\omega_\alpha^2 \left( f_{13}(\lambda, \beta), \frac{1}{n+1} \right)_p & \geq C(\alpha) E_{2m-1} (f_{13}(\lambda, \beta))_p \\
& \geq C(\alpha, \beta) \sum_{\nu=m}^{\infty} \lambda_{2\nu}^2 (\varepsilon_{2\nu}^2 - \varepsilon_{2\nu+1}^2) \\
& \geq C(\alpha, \beta) \lambda_{2m}^2 \varepsilon_{2m}^2.
\]

Applying estimates (53) and (54), we get
\[
\omega_{\alpha} \left( f_{13}(\lambda, \beta), \frac{1}{n+1} \right)_p \geq C(\alpha, \beta) \left( (n+1)^{-2\alpha} \sum_{\nu=1}^{n+1} \lambda_{\nu}^2 \nu^{2\alpha-1} \varepsilon_{\nu}^2 \right)^{\frac{1}{2}}.
\]

Further, (45) and \( \nu^{-\alpha} \lambda_{\nu} \downarrow \) allow us to write the estimates
\[
\lambda_{n+1}^2 \omega^2 \left( \frac{1}{n+1} \right) + (n+1)^{-2\alpha} \sum_{\nu=1}^{n+1} \nu^{2\alpha-1} \omega^2 \left( \frac{1}{\nu} \right) (\lambda_{\nu}^2 \nu^{-2\rho} - \lambda_{\nu+1}^2 (\nu+1)^{-2\rho}) \\
\leq C(\alpha, \rho) (n+1)^{-2\alpha} \sum_{\nu=1}^{n+1} \nu^{2\alpha-1} \lambda_{\nu}^2 \varepsilon_{\nu}^2.
\]

Combining estimates (51), (52), (55), (56), and \( \omega_{\alpha} \left( f_{13}(\lambda, \beta), \frac{1}{n+1} \right)_p = O \left( \varphi \left( \frac{1}{n} \right) \right) \), we obtain the condition in the right-hand side part of (13).

**Step 3.** Here we show the necessity in (15) for the case of \( 1 < p < 2 \). In this case the proof is similar to the proof of the case \( 2 \leq p < \infty \). The only difference is that we use Paley’s theorem on Fourier coefficients instead of Zygmund’s Lemma [42]. In this case we consider the sum of \( f_2(x) \) and the following function
\[
\varepsilon_0 + (\varepsilon_1^p - \varepsilon_2^p)^{\frac{1}{p}} \cos x + \sum_{\nu=0}^{\infty} 2^{\nu} (\frac{1}{p} - 1) (\varepsilon_{2\nu+1}^p - \varepsilon_{2\nu+2}^p)^{\frac{1}{p}} \sum_{\mu=2\nu+1}^{2\nu+1} \cos \mu x.
\]

**Step 4.** To prove the necessity in (15) and (20), we consider the general case of \( 1 \leq p \leq \infty \). Let \( \Phi \) be the class of all decreasing null-sequences. It is clear that
\[
\frac{1}{\lambda_n} = O \left[ \omega \left( \frac{1}{n} \right) \right] \iff \forall \gamma = \{\gamma_n\} \in \Phi \quad \frac{\gamma_n}{\lambda_n} = O \left[ \omega \left( \frac{1}{n} \right) \right].
\]

Let us assume that \( \frac{\gamma_n}{\lambda_n} = O \left[ \omega \left( \frac{1}{n} \right) \right] \) does not hold for all \( \gamma \in \Phi \) and \( W_p^{\lambda, \beta} \subset H_p^{(\alpha, \rho)}[\omega] \). Then there exist \( \gamma = \{\gamma_n\} \in \Phi \) and \( \{C_n \uparrow \infty\} \) such that \( \frac{\gamma_n}{\lambda_n} \geq C_n \omega \left( \frac{1}{m_n} \right) \). Further, we choose a subsequence \( \{m_n\} \) such that \( \frac{m_{n+k+1}}{m_{n+k}} \geq 2 \) and \( \gamma_{m_n} \leq 2^{-k} \). Consider the series
\[
\sum_{k=0}^{\infty} \frac{\gamma_{m_n}}{\lambda_{m_n}} \cos(m_n + 1)x.
\]

Since \( \sum_{k=0}^{\infty} \frac{\gamma_{m_n}}{\lambda_{m_n}} \leq \frac{1}{\lambda_{m_n}} \sum_{k=0}^{\infty} \frac{1}{2^k} < \infty \), there exists a function \( f_4 \in L_p \) with the Fourier series (59).

Because of \( \sum_{k=0}^{\infty} \frac{\gamma_{m_n}}{\lambda_{m_n}} \leq \sum_{k=0}^{\infty} \frac{1}{2^k} < \infty \), we have \( f_4(\lambda, \beta) \in L_p \), i.e., \( f_4 \in W_p^{\lambda, \beta} \).

On the other hand, using (24) and \( E_{n-1}(f)_p \geq C \left( |a_n| + |b_n| \right) \),
\[
\omega_{\alpha+p} \left( f_4, \frac{1}{m_{n+k}} \right)_p \geq C(\alpha, \rho) E_{m_{n+k}} (f_4)_p \geq C(\alpha, \rho) \frac{\gamma_{m_n}}{\lambda_{m_n}} \geq C(\alpha, \rho) C_n \omega \left( \frac{1}{m_n} \right).
\]
i.e., \( f_4 \notin H^p_{\alpha+\rho} [\omega] \). This contradiction proves that the condition \( \frac{1}{\lambda_n} = O \left[ \omega \left( \frac{1}{n} \right) \right] \) is necessary for \( W^\lambda_{\alpha+\rho} \subset H^p_{\alpha+\rho} [\omega] \).

**Step 5.** To prove the necessity in (17) and (21), we verify that for any \( \rho > 0 \) and \( 1 \leq p \leq \infty \),

\[
W^\lambda_{\alpha} H_\alpha [\varphi] \subset H^p_{\alpha+\rho} [\omega] \implies \frac{\varphi_n}{\lambda_n} = O \left[ \omega \left( \frac{1}{n} \right) \right].
\]  
(59)

First, we remark that

\[
\frac{\varphi_n}{\lambda_n} = O \left[ \omega \left( \frac{1}{n} \right) \right] \iff \forall \gamma = \{ \gamma_n \} \in \Phi \quad \frac{\gamma_n \varphi_n}{\lambda_n} = O \left[ \omega \left( \frac{1}{n} \right) \right].
\]

We assume that the relation in the right-hand side of (59) does not hold. Then there exist \( \gamma = \{ \gamma_n \} \in \Phi \) and \( \{ C_n \uparrow \infty \} \) such that \( \frac{\gamma_n \varphi_n}{\lambda_n} \geq C_n \omega \left( \frac{1}{m_n} \right) \). We choose a subsequence \( \{ m_{nk} \} \) such that \( \frac{m_{nk+1}}{m_{nk}} \geq 2 \gamma_{m_{nk}} \leq 2^{-k} \). Because of \( \sum_{k=0}^{\infty} \gamma_{m_{nk}} \varphi \left( \frac{1}{m_n} \right) \leq \varphi \left( \frac{1}{m_0} \right) \sum_{k=0}^{\infty} \frac{1}{2^k} < \infty \), there exists a function \( f_5 \in L_p \) with the Fourier series

\[
\sum_{k=0}^{\infty} \gamma_{m_{nk}} \varphi \left( \frac{1}{m_{nk}} \right) \cos (m_{nk} + 1)x.
\]  
(60)

For \( m_{nk} \leq n < m_{nk+1} \) using Lemmas 3.3 and 4.3 we have

\[
\omega_\alpha \left( f_5, \frac{1}{n} \right)_p \leq C \omega_\alpha \left( f_5, \frac{1}{n} \right)_\infty \leq C \left( n^{-\alpha} \sum_{s=0}^{k} \gamma_{m_{ns}} \varphi \left( \frac{1}{m_{ns}} \right) m_{\alpha s} + \sum_{s=k+1}^{\infty} \gamma_{m_{ns}} \varphi \left( \frac{1}{m_{ns}} \right) \right)
\]

\[
\leq C \varphi \left( \frac{1}{n} \right) \sum_{s=0}^{k} \gamma_{m_{ns}} + \varphi \left( \frac{1}{n} \right) \sum_{s=k+1}^{\infty} \gamma_{m_{ns}} \leq C \varphi \left( \frac{1}{n} \right).
\]

Then \( f_5 \in H^p_{\alpha} [\varphi] \), i.e., setting \( \frac{1}{x} := \{ \frac{1}{\lambda_n}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \cdots \} \), we have \( f_5(\frac{1}{x}, \beta) \in W^\lambda_{p} H_\alpha [\varphi] \).

On the other hand,

\[
\omega_{\alpha+p} \left( f_5(\frac{1}{x}, \beta), \frac{1}{m_{nk}} \right)_p \geq C E_{m_{nk}} \left( f_5(\frac{1}{x}, \beta) \right)_p \geq C \frac{\gamma_{m_{nk}} \varphi \left( \frac{1}{m_{nk}} \right)}{\lambda_{m_{nk}}} \geq CC_{nk} \omega \left( \frac{1}{m_{nk}} \right),
\]

i.e., \( f_5(\frac{1}{x}, \beta) \notin H^p_{\alpha+p} [\omega] \). This contradicts our assumption. The proof of the necessity part in (16)-(17) and (20)-(21) is now complete.

**II.** \( p = 1 \) or \( p = \infty \). **Step 6.** Let us prove the necessity in (18). Let \( H^p_{\alpha+p} [\omega] \subset W^\lambda_{p} \) and the series in (18) be divergent.

**Step 6(a):** \( \sin \frac{\beta \pi x}{2} \neq 0 \). In this case a divergence of the series in (18) is equivalent to the divergence of the series \( \sum_{n=1}^{\infty} \frac{\lambda_n}{n} \omega \left( \frac{1}{n} \right) \).

Let \( p = 1 \). We take a \( q_{\alpha+p, 1} (\omega) \)-sequence \( \varepsilon \) and consider the series

\[
\sum_{\nu=1}^{\infty} (\varepsilon_{\nu} - \varepsilon_{\nu+1}) K_\nu(x).
\]  
(61)

This series converges in \( L_1 \) (see [18]) to a function \( f_6(x) \) and \( E_n(f_6)_1 = O \left( \varepsilon_n \right) \). Applying (3) and (45), we get \( f_6 \in H^p_{\alpha+p} [\omega] \subset W^\lambda_{p} \). One can also rewrite (61) in the following form

\[
\sum_{\nu=1}^{\infty} a_\nu \cos \nu x, \quad \text{where} \quad a_\nu = \varepsilon_\nu - \nu \sum_{j=\nu}^{\infty} \frac{\varepsilon_j - \varepsilon_{j+1}}{j+1}.
\]
By Lemma 4.1,

\[
\left\| f_6(\lambda, \beta) \right\|_1 \geq C(\beta) \sum_{\nu=1}^{\infty} \frac{\lambda_\nu}{\nu} a_\nu = C(\beta) \left( \sum_{\nu=1}^{\infty} \frac{\lambda_\nu}{\nu} \varepsilon_\nu - \sum_{\nu=1}^{\infty} \frac{\lambda_\nu}{\nu} \sum_{j=\nu}^{\infty} \frac{\varepsilon_j - \varepsilon_{j+1}}{j+1} \right)
\]

\[
= C(\beta) \left( \sum_{\nu=1}^{\infty} \frac{\lambda_\nu}{\nu} \varepsilon_\nu - \sum_{\nu=1}^{\infty} (a_\nu - a_{\nu+1}) \lambda_\nu \right)
\]

\[
\geq C_1(\beta) \sum_{\nu=1}^{\infty} \frac{\lambda_\nu}{\nu} \varepsilon_\nu - C_2(\beta) \left( \lambda_1 a_1 + \sum_{n=1}^{\infty} (\lambda_{n+1} - \lambda_n) a_n \right).
\]

Further, using (44) and (45), we get

\[
\sum_{\nu=1}^{\infty} \frac{\lambda_\nu}{\nu} \varepsilon_\nu \leq C(p) \sum_{\nu=1}^{\infty} \frac{\lambda_\nu}{\nu} \omega \left( \frac{1}{\nu} \right) \leq C(p) \sum_{\nu=1}^{\infty} \lambda_\nu \nu^{-(\alpha+\rho)-1} \sum_{m=1}^{\infty} m^{\alpha+\rho-1} \varepsilon_m
\]

\[
= C(p) \sum_{m=1}^{\infty} m^{\alpha+\rho-1} \varepsilon_m \sum_{\nu=m}^{\infty} \frac{\lambda_\nu}{\nu^{\alpha}} \nu^{\alpha-1} \leq C(p, \alpha) \sum_{m=1}^{\infty} \lambda_m \varepsilon_m.
\]

Then

\[
\left\| f_6(\lambda, \beta) \right\|_1 \geq C_1(\alpha, \rho, \beta) \sum_{\nu=1}^{\infty} \lambda_\nu \nu^{-1} \omega \left( \frac{1}{\nu} \right) - C_2(\alpha, \rho, \beta) \left( \lambda_1 a_1 + \sum_{\nu=1}^{\infty} (\lambda_{\nu+1} - \lambda_\nu) a_\nu \right). \tag{62}
\]

On the other hand, using monotonicity of \(\{a_\nu\}\) and Lemma 4.1, we have

\[
C(\beta, \rho) \left\| f_6(\lambda, \beta) \right\|_1 \geq C(p) \left( \lambda_1 a_1 + \sum_{\nu=0}^{\infty} \frac{\lambda_\nu a_\nu}{\nu} \right) \geq C(p) \left( \lambda_1 a_1 + \sum_{\nu=0}^{\infty} \lambda_2^\nu a_2^\nu \right)
\]

\[
\geq \lambda_1 a_1 + \sum_{\nu=0}^{\infty} a_2^\nu (\lambda_{\nu+1} - \lambda_\nu) \geq \lambda_1 a_1 + \sum_{\nu=1}^{\infty} (\lambda_{\nu+1} - \lambda_\nu) a_\nu. \tag{63}
\]

Using (62) and (63), we get

\[
\left\| f_6(\lambda, \beta) \right\|_1 \geq C(\alpha, \rho, \beta) \sum_{\nu=1}^{\infty} \lambda_\nu \nu^{-1} \omega \left( \frac{1}{\nu} \right) = \infty.
\]

The obtained contradiction implies the convergence of series in (18).

Let now \(p = \infty\). Define the function (see [2]) \(f_7(x) = \sum_{\nu=1}^{\infty} \varepsilon_\nu \nu^{-1} \sin \nu x\), where \(\varepsilon\) is a \(q_{\alpha+\rho, 1}(\omega)\)-sequence. Then \(E_n(f_7)_{\infty} \leq C \varepsilon_{n+1}\). Using (3) and (15), we get \(f_7 \in H^p_{\alpha+\rho}[\omega] \subset W^\lambda_\beta_p\). On the other hand,

\[
\infty = \left\| f_7(\lambda, \beta) \right\|_\infty \geq C(\beta) \sum_{\nu=0}^{\infty} 2^{\nu(\alpha+\rho)} \varepsilon_2^\nu \frac{\lambda_\nu}{2^{\nu(\alpha+\rho)}}
\]

\[
\geq C(\alpha, \rho, \beta) \sum_{\nu=0}^{\infty} 2^{\nu(\alpha+\rho)} \varepsilon_2^\nu \sum_{m=\nu}^{\infty} \frac{\lambda_\nu^m}{2^{m(\alpha+\rho)}}
\]

\[
\geq C(\alpha, \rho, \beta) \sum_{\nu=1}^{\infty} \lambda_\nu \nu^{-1} \omega \left( \frac{1}{\nu} \right).
\]

The obtained contradiction proves the convergence of the series in (18).
Applying Lemma 4.3, we get which contradicts \( f \). Similarly, as for the function \( f \), converges \((18)\) to \( f_8 \in L_1 \) and \( E_n(f_8) \leq C \varepsilon_{n+1} \). Hence, we have \( f_8 \in H^p_{\alpha+\rho}[\omega] \subset W^\lambda_\beta \).

We write series \((65)\) in the following way
\[
\sum_{\nu=1}^{\infty} b_\nu \sin \nu x, \quad \text{where} \quad b_\nu = \sum_{j=\nu}^{2\nu-2} \left( 1 - \frac{\nu}{j+1} \right) (\varepsilon_\nu - \varepsilon_{\nu+1}) + \sum_{j=2\nu-1}^{\infty} \frac{\nu}{j+1} (\varepsilon_\nu - \varepsilon_{\nu+1}).
\]

By Lemma 4.1, we write
\[
\left\| f_{\lambda,\beta} \right\|_1 \geq C(\beta) \sum_{\nu=1}^{\infty} \lambda_\nu b_\nu \geq C(\beta) \sum_{\nu=0}^{\infty} \lambda_\nu (\varepsilon_\nu - \varepsilon_{\nu+1}).
\]

This contradicts \((64)\). Thus, the series in \((18)\) converges.

Let now \( p = \infty \). Let \( \psi \) be a \( Q_{\alpha+\rho,1} \)-sequence. Then, we define
\[
f_9(x) = \psi_1 \cos x + \sum_{\nu=1}^{\infty} 2^{-\nu(\rho+\alpha)} (\psi_{2\nu} - \psi_{2\nu-1}) \cos 2^{\nu} x.
\]

Similarly, as for the function \( f_1 \) in the case of \( 2 \leq p < \infty \), it is easy to check that \( f_9 \in H^p_{\alpha+\rho}[\omega] \subset W^\lambda_\beta \).

Applying Lemma 4.3, we get
\[
\left\| f_{\lambda,\beta} \right\|_\infty \geq C(\beta) \left( \lambda_1 \psi_1 + \sum_{\nu=1}^{\infty} \lambda_2 \cdot 2^{-\nu(\rho+\alpha)} (\psi_{2\nu} - \psi_{2\nu-1}) \right)
\geq C(\beta) \sum_{\nu=1}^{\infty} \lambda_\nu \nu^{-\theta} \psi_\nu
\geq C(\beta) \sum_{\nu=1}^{\infty} (\lambda_{\nu+1} - \lambda_\nu) \omega(1/\nu) = \infty,
\]

which contradicts \( f_9 \in W^\lambda_\beta \). This completes the proof of the necessity in \((18)\).

Step 7. We will show the necessity in \((19)\) for the case of \( \sin \frac{\pi \beta}{2} \neq 0 \). Let \( H^p_{\alpha+\rho}[\omega] \subset W^\lambda_\beta H_{\alpha}[\varphi] \). We take a \( q_{\alpha+r,1}(\omega) \)-sequence \( \varepsilon \). Then \((15)\) holds for \( \alpha + r \) instead of \( \alpha \) and for 1 instead of \( \theta \). Since \( \sin \frac{\pi \beta}{2} \neq 0 \),
\[
J := \lambda_{\nu+1} \omega \left( \frac{1}{n+1} \right) + n^{-\alpha} \sum_{\nu=1}^{n} \nu^{r+\alpha} (\nu^{-\rho} \lambda_\nu - (\nu+1)^{-\rho} \lambda_{\nu+1}) \omega \left( \frac{1}{\nu} \right)
\]
We will use several times the following evident relations:

\[ J \quad E \]

It is clear that applying Step 7(b):

\[ \pi \beta \]

At the same time,

\[ p = \infty \]

In this case by Lemma 3.1, we have

\[ f_7(\lambda, \beta) := (f_7(\lambda, \beta))_+ \]

Using (24) and \( \sum_{k=2n}^{\infty} a_k \leq 4E_n(f)_\infty \) (see [4]), we write

\[ \omega_\alpha\left( f_7(\lambda, \beta), \frac{1}{n} \right)_p \geq C(\alpha) E_\infty(f) \geq C(\alpha, \beta)J_1. \]  

It is proved above that \( f_7 \in H^p_{2\alpha+\varphi}[\omega] \subset W^\lambda_\varphi H^\alpha_\varphi[\varphi]. \) Collecting inequalities (68), (69), and (66), we obtain the estimate in the right-hand side of (19).

Step 7(b): \( p = \infty \) and \( \cos \frac{\pi \beta}{2} \neq 0. \) If \( \cos \frac{\pi \beta}{2} \neq 0, \) then we use (69) and

\[ \omega_\alpha\left( f_7(\lambda, \beta), \frac{1}{n} \right)_p \geq C(\alpha) n^{-\alpha} \left\| V_n^{(\alpha)}(f_7(\lambda, \beta))(\cdot) \right\|_p \geq C(\alpha, \beta)J_2. \]

If \( \cos \frac{\pi \beta}{2} = 0, \) then \( f_7 = \pm f_7^\pm \) and

\[ \omega_\alpha\left( f_7(\lambda, \beta), \frac{1}{n} \right)_p \geq C(\alpha) E_\infty(f_7(\lambda, \beta))_p \geq C(\alpha, \beta)J_1. \]

To obtain the estimate of \( J_2, \) we define

\[ f_{10}(x) = \frac{\varepsilon_0}{2} + (\varepsilon_1 - \varepsilon_2) \cos x + \sum_{\nu=1}^{\infty} (\varepsilon_2^\nu - \varepsilon_2^{\nu+1}) \cos 2^\nu x. \]

It is clear that \( E_n(f_{10})_p \leq \varepsilon_{n+1}. \) Then using (45), we get \( f_{10} \in H^p_{\alpha+\varphi}[\omega] \subset W^\lambda_\varphi H^\alpha_\varphi[\varphi]. \) Further, applying \( \omega_\alpha\left( f_n, \frac{1}{n} \right)_p \geq C(\alpha) n^{-\alpha} \left\| V_n^{(\alpha)}(f) \right\|_p \) and Lemma 143, we write (2m \leq n + 1 < 2^{m+1})

\[ \omega_\alpha\left( f_{10}(\lambda, \beta), \frac{1}{2m} \right)_p \geq C(\alpha, \beta) 2^{-\alpha m} \sum_{\nu=0}^{m} \lambda_{2\nu} 2^{\nu \alpha} (\varepsilon_2^\nu - \varepsilon_2^{\nu+1}) \]

\[ \geq C_1(\alpha, \beta, r) 2^{-\alpha m} \sum_{\nu=0}^{m} \lambda_{2\nu} 2^{\nu \alpha} \varepsilon_2^\nu - C_2(\alpha, \beta, r) \lambda_{2m+1} \varepsilon_2^{m+1}. \]

At the same time, \( \omega_\alpha\left( f_{10}(\lambda, \beta), \frac{1}{2m} \right)_p \geq C(\alpha, \beta, \lambda_{2m+1} \varepsilon_2^{m+1}. \) Then we have

\[ \omega_\alpha\left( f_{10}(\lambda, \beta), \frac{1}{n} \right)_p \geq C(\alpha, \beta, r) J_2 \]
and applying (67),
\[ C(\alpha, \beta, r) \left[ J_1 + J_2 \right] \leq \omega_\alpha \left( f_{10}^{(\lambda, \beta)}, \frac{1}{2m} \right)_p + \omega_\alpha \left( f_{12}^{(\lambda, \beta)}, \frac{1}{2m} \right)_p \]
\[ \times \omega_\alpha \left( (f_7 + f_{10})^{(\lambda, \beta)}, \frac{1}{n} \right)_p = O \left[ \varphi \left( \frac{1}{n} \right) \right]. \]

The necessity in (19) follows.

Step 7(c): \( p = 1 \) and \( \cos \frac{\pi \alpha}{2} \neq 0 \). In this case we use the function \( f_6 \).

It is known that \( f_6 \in \mathcal{H}_{p+1}^\beta [\omega] \) and by Lemma 4.1 we have
\[ \omega_\alpha \left( f_6^{(\lambda, \beta)}, \frac{1}{n} \right)_1 \geq C(\alpha, \beta, r) \left( a_{n+1} \lambda_n + \sum_{\nu=n+1}^{\infty} (\lambda_\nu - \lambda_{\nu-1}) a_\nu \right). \]

On the other hand,
\[ \omega_\alpha \left( f_6^{(\lambda, \beta)}, \frac{1}{n} \right)_1 \geq C(\alpha, \beta, r) \left( a_{n+1} \lambda_n + \sum_{\nu=n+1}^{\infty} (\lambda_\nu - \lambda_{\nu-1}) a_\nu \right). \]

Therefore, the last two inequalities imply \( \omega_\alpha \left( f_6^{(\lambda, \beta)}, \frac{1}{n} \right)_1 \geq C(\alpha, \beta, r) J_1 \). Moreover,
\[ \omega_\alpha \left( f_6^{(\lambda, \beta)}, \frac{1}{n} \right)_p \geq C(\alpha, \beta, \gamma) \left\| V_n^{(\alpha)}(f_6^{(\lambda, \beta)})(\cdot) \right\|_p \geq C(\alpha, \beta, J_2). \]

Step 7(d): \( p = 1 \) and \( \cos \frac{\pi \alpha}{2} = 0 \). If \( \cos \frac{\pi \alpha}{2} \neq 0 \), then we use \( \omega_\alpha \left( f_6^{(\lambda, \beta)}, \frac{1}{n} \right)_1 \geq C(\alpha, \beta, r) J_1 \) and
\[ \omega_\alpha \left( f_6^{(\lambda, \beta)}, \frac{1}{n} \right)_p \geq C(\alpha, \beta, \gamma) \left\| V_n^{(\alpha)}(f_6^{(\lambda, \beta)})(\cdot) \right\|_p \geq C(\alpha, \beta, J_2). \]

If \( \cos \frac{\pi \alpha}{2} = 0 \), we consider \( f_6 + f_8 \). Using Lemmas 3.1 and 4.1 we get
\[ \omega_\alpha \left( f_8^{(\lambda, \beta)}, \frac{1}{n} \right)_1 \geq C(\alpha, \beta, \gamma) \left\| V_n^{(\alpha)}(f_8^{(\lambda, \beta)})(\cdot) \right\|_p \geq C(\alpha, \beta, J_2). \]

Step 8. We prove the necessity in (19) in the case of \( \sin \frac{\pi \beta}{2} = 0 \). Let \( \mathcal{H}_{p+1}^\beta [\omega] \subset W_p^{\lambda, \beta} \mathcal{H}_\alpha [\varphi] \) and let \( \varepsilon \) be a \( q_{a+r, 1}(\omega) \)-sequence. Since \( \sin \frac{\pi \beta}{2} = 0 \), from (15) it follows that
\[ \lambda_{n+1} \omega \left( \frac{1}{n+1} \right) + n^{-\alpha} \sum_{\nu=1}^{n} \nu^{\alpha+1} (\nu^{-r} \lambda_\nu - (\nu + 1)^{-r} \lambda_{\nu+1}) \omega \left( \frac{1}{\nu} \right) \]
\[ + \mid \cos \frac{\beta \pi}{2} \mid \sum_{\nu=n+2}^{\infty} (\lambda_{\nu+1} - \lambda_\nu) \omega \left( \frac{1}{\nu} \right) \]
Then

\[ f \leq C(\alpha, \beta, r) \left( \sum_{\nu=n+1}^{\infty} \lambda_{\nu} (\varepsilon_{\nu} - \varepsilon_{\nu+1}) + n^{-\alpha} \sum_{\nu=1}^{n} \lambda_{\nu} \varepsilon_{\nu} \nu^{\alpha-1} \right) \]

\[ =: C(\alpha, \beta, r) (J_3 + J_4). \]  

(73)

Step 8(a): \( p = \infty \) and \( \cos \frac{\pi \alpha}{2} \neq 0 \). Applying the Jackson inequality and Lemma 4.3, we get

\[ \omega_{\alpha} \left( f^{(\lambda, \beta)}_{10}, \frac{1}{2m} \right) \geq C(\alpha, \beta) \sum_{\nu=m}^{\infty} \lambda_{2\nu} (\varepsilon_{2\nu} - \varepsilon_{2\nu+1}). \]  

(74)

We also note that by Lemma (4.3), (71) holds for all \( \alpha > 0 \). This and (74) allow us to write

\[ \omega_{\alpha} \left( f^{(\lambda, \beta)}_{10}, \frac{1}{n+1} \right) \geq C(\alpha, \beta) (J_3 + J_4). \]

Using condition (73) and \( \lambda, \beta \in W_{p}^\lambda \beta H_{\alpha}[\varphi] \), we obtain the relation in the right-hand side of (19).

Step 8(b): \( p = \infty \) and \( \cos \frac{\pi \alpha}{2} = 0 \). Then we consider \( f_{10} \) and \( f_{11} := \tilde{f}_{10} \). It is clear that \( f_{11} \in L_p \) and \( f_{10} + f_{11} \in H_{\alpha+r}[\omega] \). Besides,

\[ \omega_{\alpha} \left( f^{(\lambda, \beta)}_{10}, \frac{1}{n+1} \right) \geq C(\alpha, \beta) J_3, \]

\[ \omega_{\alpha} \left( f^{(\lambda, \beta)}_{11}, \frac{1}{n+1} \right) \geq C(\alpha, \beta) J_4 \]

and

\[ \omega_{\alpha} \left( f^{(\lambda, \beta)}_{10}, \frac{1}{n+1} \right) + \omega_{\alpha} \left( f^{(\lambda, \beta)}_{11}, \frac{1}{n+1} \right) \sim \omega_{\alpha} \left( f^{(\lambda, \beta)}_{10}, \frac{1}{n+1} \right). \]

Step 8(c): \( p = 1 \) and \( \cos \frac{\pi \alpha}{2} \neq 0 \). Since \( f^{(\lambda, \beta)}_{8}(x) \sim \pm \sum_{\nu=1}^{\infty} \lambda_{\nu} b_{\nu} \sin \nu x \), then by Lemmas 3.1 and 4.1 we write (see Step 7(d))

\[ \omega_{\alpha} \left( f^{(\lambda, \beta)}_{8}, \frac{1}{n} \right) \geq C(\alpha) n^{-\alpha} \left\| V_{n}^{(\alpha)} f^{(\lambda, \beta)}_{8} \right\| \geq C(\alpha, \beta) J_4. \]  

(75)

Further, using Lemma 4.1 and the Jackson inequality (24), we have

\[ \omega_{\alpha} \left( f^{(\lambda, \beta)}_{8}, \frac{1}{n} \right) \geq C(\alpha, \beta) \sum_{\nu=n+1}^{\infty} \frac{\lambda_{\nu} b_{\nu}}{\nu} \]

\[ \geq C(\alpha, \beta) \sum_{\nu=n+1}^{\infty} \lambda_{\nu} \sum_{j=2\nu-1}^{\infty} \frac{\varepsilon_{j} - \varepsilon_{j+1}}{j+1} \]

\[ \geq C(\alpha, \beta, r) \sum_{j=4n-1}^{\infty} \frac{(\varepsilon_{j} - \varepsilon_{j+1}) \lambda_{\nu}}{\nu}. \]

Using the properties of the modulus of smoothness, we get (19).

Step 8(d): \( p = 1 \) and \( \cos \frac{\pi \alpha}{2} = 0 \). We use the fact that \( f_{12} := \tilde{f}_{8} \in L_{1} \) and \( E_{n}(f_{12}) \leq C \varepsilon_{n+1} \) (see [18]). Then \( f_{8} + f_{12} \in H_{\alpha+r}[\omega] \) and

\[ \omega_{\alpha} \left( f^{(\lambda, \beta)}_{8}, \frac{1}{n} \right) \geq C(\alpha, \beta) J_3, \]

\[ \omega_{\alpha} \left( f^{(\lambda, \beta)}_{12}, \frac{1}{n} \right) \geq C(\alpha, \beta) J_4. \]

Theorem 1 is fully proved.
5 Corollaries. Estimates of transformed Fourier series.

Theorem 1 actually provides estimates of the norms and moduli of smoothness of the transformed Fourier series, i.e., the estimates of \( \| \varphi \|_p \) and \( \omega_\alpha(\varphi, \delta)_p \), where \( \varphi \sim \sigma(f, \lambda) \) in terms of \( \omega_\gamma(f, \delta)_p \). Analyzing the obtained results, one can see that the following two conditions play a crucial role for these estimates. The first is the behavior of the transforming sequence \( \{\lambda_n\} \) and the second is the choice between the considered space (as the Riesz inequality \([28]\) holds for \( L_p, 1 < p < \infty \) and no such inequality exists for \( L_p, p = 1, \infty \)).

We will investigate in detail some important examples for \( L_p, 1 < p < \infty \) and for \( L_p, p = 1, \infty \), separately.

1. The case of \( 1 < p < \infty \).

**Theorem 2** Let \( 1 < p < \infty, \theta = \min(2, p) \tau = \max(2, p) \alpha, \rho \in \mathbb{R}_+ \), and \( \lambda = \{\lambda_n\} \) be a non-decreasing sequence of positive numbers such that \( \{n^{-\rho} \lambda_n\} \) is non-increasing.

**I.** If for \( f \in L_p^0 \) the series

\[
\sum_{n=1}^{\infty} \frac{(\lambda_{n+1}^\theta - \lambda_n^\theta)}{\omega_{\alpha+\rho}^\theta(f, 1/n)} = C(p, \lambda, \alpha, \rho) \left\{ \lambda_1^\theta \left\| f \right\|_p^\theta + \sum_{n=1}^{\infty} \frac{\left(\lambda_{n+1}^\theta - \lambda_n^\theta\right)}{\omega_{\alpha+\rho}^\theta(f, 1/n)} \right\}^\frac{1}{\theta}, \tag{76}
\]

converges, then there exists a function \( \varphi \in L_p^0 \) with the Fourier series \( \sigma(f, \lambda) \), and

\[
\omega_\alpha^\theta(\varphi, 1/(n+1)) \leq C(p, \lambda, \alpha, \rho) \left\{ n^{-\alpha\theta} \sum_{n=1}^{\infty} \lambda_n^\theta \omega_{\alpha+\rho}^\theta(f, 1/n) + \lambda_{n+1}^\theta \omega_{\alpha+\rho}^\theta(f, 1/n) \right\}^\frac{1}{\theta}, \tag{77}
\]

**II.** If for \( f \in L_p^0 \) there exists a function \( \varphi \in L_p \) with the Fourier series \( \sigma(f, \lambda) \), then

\[
\left\{ \lambda_1^\tau \left\| f \right\|_p^\tau + \sum_{n=1}^{\infty} \frac{(\lambda_{n+1}^\tau - \lambda_n^\tau)}{\omega_{\alpha+\rho}^\tau(f, 1/n)} \right\}^\frac{1}{\tau} \leq C(p, \lambda, \alpha, \rho) \left\| \varphi \right\|_p, \tag{78}
\]

\[
\sum_{n=1}^{\infty} n^{-\alpha\tau} \sum_{n=1}^{\infty} \lambda_{n+1} - \lambda_n \omega_{\alpha+\rho}^\tau(f, 1/n) + \omega_{\alpha+\rho}^\tau(f, 1/n) \right\} \leq C(p, \lambda, \alpha, \rho) \omega_\alpha^\tau(\varphi, 1/(n+1)), \tag{79}
\]

\[
\omega_{\alpha+\rho}^\tau(f, 1/n) \leq C(p, \lambda, \alpha, \rho) \frac{\left\| \varphi \right\|_p}{\lambda_n}, \tag{80}
\]

\[
\omega_{\alpha+\rho}^\tau(f, 1/n) \leq C(p, \lambda, \alpha, \rho) \frac{\omega_\alpha(\varphi, 1/n)}{\lambda_n}. \tag{81}
\]

Inequalities \([76]-[77]\) and \([80]-[81]\) were actually proved in Theorem 1 (see the proof of sufficiency in the part **I**). The estimates \([78]-[79]\) are proved analogously, using Lemma 3.1 the theorems by Littlewood-Paley, Marcinkiewicz, and the Minkowski’s inequality (see also [37] and [31]).

An important corollary of Theorem 2 is the following.
Corollary 1  Let $1 < p < \infty$, $\theta = \min(2, p)$, and $\tau = \max(2, p)$. Then for any $k, r > 0$ we have

\[
C_1 \left\{ \sum_{\nu=n+1}^\infty \nu^{r\tau-1} \omega_{k+r}^{\tau} \left( f, \frac{1}{\nu} \right)_p \right\}^{\frac{1}{\tau}} \leq \omega_k \left( f^{(r)}, \frac{1}{n} \right)_p \leq C_2 \left\{ \sum_{\nu=n+1}^\infty \nu^{r\theta-1} \omega_{k+r}^{\theta} \left( f, \frac{1}{\nu} \right)_p \right\}^{\frac{1}{\theta}}, \tag{82}
\]

where $C_1 = C_1(p, k, r), C_2 = C_2(p, k, r), n \in \mathbb{N}$.

See also [17], where the right-hand side estimate in (82) was shown for integers $k$ and $r$. The last two inequalities provide sharper bounds in the sense of order than (7) and (8). Indeed, using properties of the modulus of smoothness and the Jensen inequality, we have

\[
n^r \omega_{k+r} \left( f, \frac{1}{n} \right)_p \leq C(k, r) \left\{ \sum_{\nu=n+1}^\infty \nu^{r\tau-1} \omega_{k+r}^{\tau} \left( f, \frac{1}{\nu} \right)_p \right\}^{\frac{1}{\tau}};
\]

\[
\left\{ \sum_{\nu=n+1}^\infty \nu^{r\theta-1} \omega_{k+r}^{\theta} \left( f, \frac{1}{\nu} \right)_p \right\}^{\frac{1}{\theta}} \leq C(k, r) \sum_{\nu=n+1}^\infty \nu^{-r} \omega_{k+r} \left( f, \frac{1}{\nu} \right)_p.
\]

Example.  Let $\psi(t) = t^\tau \ln^{-A}(1/t)$ and $2 \leq p < \infty, \frac{1}{2} < A < 1$. If $\omega_{k+r} \left( f, t \right)_p \asymp \psi(t)$, then inequalities (7) and (8) give only $C \ln^{-A}(1/t) \leq \omega_k \left( f^{(r)}, t \right)_p$. At the same time, (82) implies $C_1 \ln^{-A+1/p}(1/t) \leq \omega_k \left( f^{(r)}, t \right)_p \leq C_2 \ln^{-A+1/2}(1/t)$, which is sharper.

Proof of Corollary 1 follows from (77) and (79) with $r = \rho$, because if $f \in L_p$, $1 < p < \infty$, then one can assume that $f^{(r)} \sim \sigma(f, \lambda)$ for \( \lambda_n = n^r \).

The estimates (7), (8) and (82) show that it is natural to estimate $\omega_\alpha \left( f^{(\gamma)}, \frac{1}{n} \right)_p$ in terms of $\omega_{\alpha+r} \left( f, \frac{1}{n} \right)_p$. Further analysis allowed us to distinguish three different types of such estimates. It will be convenient for us to write inequalities in the integral form:

1. $\gamma = r$ (see Corollary 1):

\[
\left\{ \int_0^\delta t^{-(r-1)} \omega_{r+\alpha}^{\tau}(f, t)_p dt \right\}^{\frac{1}{\tau}} \ll \omega_\alpha \left( f^{(r)}, \delta \right)_p \ll \left\{ \int_0^\delta t^{-(r-1)} \omega_{r+\alpha}^{\theta}(f, t)_p dt \right\}^{\frac{1}{\theta}}, \tag{83}
\]

2. $\gamma = r - \varepsilon$, $0 < \varepsilon < r$ (see Theorem 2 for $\rho = r$ and $\lambda_n = n^{r-\varepsilon}$):

\[
\left\{ \int_0^\delta t^{-(r-\varepsilon-1)} \omega_{r+\alpha}^{\tau}(f, t)_p dt + \delta^{a\tau} \int_0^1 t^{-(r-\varepsilon-\alpha)\tau-1} \omega_{r+\alpha}^{\tau}(f, t)_p dt \right\}^{\frac{1}{\tau}} \ll \omega_\alpha \left( f^{(r-\varepsilon)}, \delta \right)_p, \tag{84}
\]

3. $\gamma = r + \varepsilon$, $0 < \varepsilon < \alpha$, (see [32]):

\[
\left\{ \int_0^\delta t^{-(r+\varepsilon-1)} \omega_{r+\alpha}^{\tau}(f, t)_p dt \right\}^{\frac{1}{\tau}} \ll \delta^{a-\varepsilon} \left\{ \int_0^1 t^{-(a-\varepsilon)\theta-1} \omega_\alpha^{\theta} \left( f^{(r+\varepsilon)}, t \right)_p dt \right\}^{\frac{1}{\theta}}, \tag{86}
\]

\[
\delta^{a-\varepsilon} \left\{ \int_0^\delta t^{-(a-\varepsilon)\tau-1} \omega_\alpha^{\tau} \left( f^{(r+\varepsilon)}, t \right)_p dt \right\}^{\frac{1}{\tau}} \ll \left\{ \int_0^\delta t^{-(r+\varepsilon)\theta-1} \omega_{r+\alpha}^{\theta}(f, t)_p dt \right\}^{\frac{1}{\theta}}. \tag{87}
\]

\(^{4}\)Here and further, $\tau = \max(2, p), \theta = \min(2, p)$. If $A_1 \leq CA_2$, $C \geq 1$, we write $A_1 \ll A_2$. Also, if $A_1 \ll A_2$ and $A_2 \ll A_1$, then $A_1 \asymp A_2$. 
Some more general estimates of the type \((54)-(57)\) for moduli of smoothness of the transformed Fourier series can be obtained \((32, 33, 31)\) using the sequences of the type (see, for example, \([35], [31]\)) \(\{\Lambda_n(s) := \Lambda(s, \frac{1}{n})\}\), where

\[
\Lambda(s, t) = \Lambda(s, r, t) = \left(\int_1^t \xi(u)du + t^{-rs} \int_0^t u^{rs} \xi(u)du\right)^{\frac{1}{r}} \tag{88}
\]

and a non-negative function \(\xi(u)\) on \([0,1]\) is such that \(u^{rs} \xi(u)\) is summable.

2. The case of \(p = 1, \infty\).

Estimates of \(\omega_\alpha(\varphi, t)_p\) in terms of \(\omega_{r+\alpha}(f, t)_p\) for this case follow from Theorem 1 (see item II). We will write only the commonly used estimates of \(\omega(\varphi, t)_p\) and \(\omega_{r}(f, t)_p\) in terms of \(\omega_{r+\alpha}(f, t)_p\) (see also \([35], [34-36]\)).

**Corollary 2** If \(p = 1, \infty\), then inequalities \((7), (8)\) hold true for any \(k, r > 0\).

If \(\{\lambda_n = n^\rho\}, \rho \geq 0\) and \(\beta = r + 1\), Theorem 1 implies the following

**Corollary 3** Let \(p = 1, \infty\). Then

\[
H_{\alpha+\rho}[\omega] \subset \widetilde{W}_p^\rho \iff \sum_{n=1}^{\infty} n^{\rho-1} \omega \left(\frac{1}{n}\right) < \infty.
\]

Note that in the case of \(p = 1, \rho = 0\) and \(\alpha = 1\), Corollary 3 gives the answer for the question by F. Móricz \((27), 1995\) on necessary conditions for the embedding \(H_{\alpha+\rho}[\omega] \subset \widetilde{W}_p^\rho\). We also mention the papers \([8], [33], [31]\), where the embedding theorems were proved in the necessity part.

**Corollary 4** Let \(p = 1, \infty\) and \(r, \alpha, \varepsilon > 0\).

I. If for \(f \in L_p\) the series \(\sum_{\nu=1}^{\infty} \nu^{r-1} \omega_{r+\alpha+\varepsilon} \left(f, \frac{1}{\nu}\right)_p\) converges, then there exists \(\tilde{f}(r) \in L_p\) and

\[
\omega_\alpha \left(\tilde{f}(r), \frac{1}{n}\right)_p \leq C(r, \alpha, \varepsilon) \left(n^{-\alpha} \sum_{\nu=1}^{n} \nu^{r+\alpha-1} \omega_{r+\alpha+\varepsilon} \left(f, \frac{1}{\nu}\right)_p + \sum_{\nu=n+1}^{\infty} \nu^{r-1} \omega_{r+\alpha+\varepsilon} \left(f, \frac{1}{\nu}\right)_p\right), \quad n \in \mathbb{N}.
\]

II. If for \(f \in L_p\) there exists \(\tilde{f}(r) \in L_p\), then

\[
\omega_{r+\alpha+\varepsilon} \left(f, \frac{1}{n}\right)_p \leq \frac{C(r, \alpha, \varepsilon)}{n^r} \omega_\alpha \left(\tilde{f}(r), \frac{1}{n}\right)_p, \quad n \in \mathbb{N}. \tag{89}
\]

Using the direct and inverse approximation theorems, we can write inequality from the item I of the previous corollary in the following equivalent form (see also \([3], [35], [39]\) Vol. 2, Ch. 6 and 7-41):

**Corollary 5** Let \(p = 1, \infty\) and \(r \geq 0, \alpha > 0\). If for \(f \in L_p\) the series \(\sum_{\nu=1}^{\infty} \nu^{r-1} E_\nu \left(f, \frac{1}{\nu}\right)_p\) converges, then there exists \(\tilde{f}(r) \in L_p\) and

\[
\omega_\alpha \left(\tilde{f}(r), \frac{1}{n}\right)_p \leq C(r, \alpha, \varepsilon) \left(n^{-\alpha} \sum_{\nu=1}^{n} \nu^{r+\alpha-1} E_\nu \left(f, \frac{1}{\nu}\right)_p + \sum_{\nu=n+1}^{\infty} \nu^{r-1} E_\nu \left(f, \frac{1}{\nu}\right)_p\right), \quad n \in \mathbb{N}.
\]
6 Remarks

1. The embedding theorems for the classes $H_p^r[\omega], W_p^r,$ and $W_p^r H_k[\varphi]$ in the necessity part were investigated, for example, in the papers by N.K. Bary and S.B. Stechkin (see [3], $H_p^r[\varphi] \subset H_p^r[\omega]), V. E Geit (see [19], H_p^r[\varphi] \subset H_p^r[\omega], H_p^r[\varphi] \subset H_p^r[\omega], p = 1, \infty), N. A. Il'yasov (see [23], H_p^r[\varphi] \subset W_p^r, H_p^r[\varphi] \subset W_p^r H_k[\varphi], r, k \in \mathbb{N}).$ Note also that all estimates in Theorem 1 are "correct" (in the terminology of Stechkin), that is, the sharpness from the point of view of order, is realized with the help of individual functions. We thank N. A. Il'yasov for this remark.

Theorem 1 specifies the previous results both in the necessity and sufficiency parts. Sufficient conditions for the embeddings $H_p^r[\omega] \subset W_p^{\lambda,\beta} H_k[\varphi]$ and $H_p^r[\omega] \subset W_p^{\lambda,\beta}$ were studied in the papers [39]-[41]. From these articles, particularly, for an important model example $f(\lambda,\beta) \equiv f(r), r > 0,$ we have the following estimates

$$E_n(f^{(\lambda)}) = \sum_{k=n+1}^{\infty} k^{r-1} E_k(f)^{\frac{1}{2}},$$

$$\omega_k(f^{(\lambda)}) \leq C \left( \frac{k}{n} \right)^{\frac{1}{2}},$$

where $\theta = 1.$ At the same time, for the case of $1 < p < \infty,$ Theorem 3 and Theorem 11 respectively, imply that these estimates hold for $\theta = \min(2, p)$ and this exponent is sharp the best possible.

2. A generalization of the class $W_p^r E[\xi]$ is the class

$$W_p^{\lambda,\beta} E[\omega] = \left\{ f \in W_p^{\lambda,\beta} : E_n \left( f^{(\lambda,\beta)} \right) = O [\omega_n] \right\}.$$

In this paper we do not consider in detail the embedding theorems between the classes $W_p^{\lambda,\beta} E[\omega], W_p^{\lambda,\beta} H_n[\varphi], E_p[\xi] \equiv W_p^{(1,0)} E[\xi].$ We only notice that some results of such types easily follow from direct and inverse theorems (11-14), and some are given in [28], [39], Vol. 2, Ch. 6 and 7, [11] and [20]. For the case when $\lambda_n$ satisfies the $\Delta_2$-condition, a complete solution of the problem on embedding between the classes $W_p^{\lambda,\beta}, W_p^{\lambda,\beta} E[\xi],$ and $E_p[\xi]$ is described in the following result.

**Theorem 3** [39] Let $1 < p < \infty, \theta = \min(2, p), \beta \in \mathbb{R},$ and $\lambda = \{\lambda_n\}$ be a non-decreasing sequence of positive numbers satisfying the $\Delta_2$-condition, i.e., $\lambda_{2n} \leq C \lambda_n.$ Let also $\varepsilon = \{\varepsilon_n\}$ and $\omega = \{\omega_n\}$ be non-increasing null-sequences.

**I.** If $1 < p < \infty,$ then

$$E_p[\xi] \subset W_p^{\lambda,\beta} \iff \sum_{n=1}^{\infty} \left( \lambda_{n+1}^{\theta} - \lambda_n^{\theta} \right) \varepsilon_n^{\theta} < \infty,$$

$$E_p[\xi] \subset W_p^{\lambda,\beta} E[\omega] \iff \left\{ \sum_{n=1}^{\infty} \left( \lambda_{n+1}^{\theta} - \lambda_n^{\theta} \right) \varepsilon_n^{\theta} \right\}^{\frac{1}{\theta}} + \lambda_n \varepsilon_n = O [\omega_n],$$

$$W_p^{\lambda,\beta} \subset E_p[\xi] \iff \frac{1}{\lambda_n} = O [\varepsilon_n],$$

$$W_p^{\lambda,\beta} E[\omega] \subset E_p[\xi] \iff \frac{\omega_n}{\lambda_n} = O [\varepsilon_n].$$

**II.** Let $p = 1$ or $p = \infty.$

(a) If $\Delta \lambda_n \leq C \Delta \lambda_{2n}$ and $\Delta_x \lambda_n \geq 0$ (or $\leq 0$), then

$$E_p[\xi] \subset W_p^{\lambda,\beta} \iff \mid \cos \frac{\beta \pi}{2} \sum_{n=1}^{\infty} \left( \lambda_{n+1} - \lambda_n \right) \varepsilon_n$$

Theorem 1 implies inequality [39], which is equivalent (using the direct and inverse theorems) to inequality [41].
Theorem 4 (see also [32]) Let $f(x_1, x_2) \in L_p$, $1 < p < \infty$, $\theta = \min(2, p)$, $\tau = \max(2, p)$ and let $\alpha_1, \alpha_2, \tau_1, \tau_2 > 0$.

I. If

$$J_1(\theta) := \left( \int_0^1 \int_0^1 t_1^{-\tau_1 \theta - 1} t_2^{-\tau_2 \theta - 1} \omega_{\alpha_1, \alpha_2, \tau_1, \tau_2}^{\theta} (f, t_1, t_2) dt_1 dt_2 \right)^{\frac{1}{\theta}} < \infty,$$

(b) If for $\beta = 2k$, $k \in \mathbb{Z}$ the condition $\Delta^2 (1/\lambda_n) \geq 0$ holds, and for $\beta \neq 2k$, $k \in \mathbb{Z}$ the conditions $\Delta^2 (1/\lambda_n) \geq 0$ and $\sum_{\nu=\lambda+1}^{\infty} \frac{1}{\nu^2} \leq \frac{C}{\lambda_n}$ hold, then

$$W_p^{\lambda, \beta} \subset E_p[\varepsilon] \iff \frac{1}{\lambda_n} = O[\varepsilon_n],$$

$$W_p^{\lambda, \beta} E[\omega] \subset E_p[\varepsilon] \iff \frac{\omega_n}{\lambda_n} = O[\varepsilon_n].$$

3. The Weyl class $W_p^{\lambda, \beta}$ coincides with the class of functions from $L(0, 2\pi)$ such that their Fourier series can be presented in the following form

$$a_0(f) + \sum_{\nu=1}^{\infty} \frac{1}{\pi \lambda_\nu} \int_0^{2\pi} \psi(x-t) \cos \left( \frac{\nu \beta}{2} \right) dt, \quad \psi(x) \in L^0.$$

Further, consider the case when

$$\sum_{\nu=1}^{\infty} \frac{1}{\lambda_\nu} \cos \left( \frac{\nu \beta}{2} \right)$$

is the Fourier series of a summable function $D_{\lambda, \beta}(t)$.

For example, it is certainly so if $\{\lambda_\nu \uparrow \infty\} (n \uparrow)$ and $\sum_{\nu=1}^{\infty} \frac{1}{\lambda_\nu} < \infty$ (see [59] Ch. 5). Then elements of $W_p^{\lambda, \beta}$ can differ only by the mean value from functions $f$, which have the following representation by convolution,

$$f(x) = \frac{1}{\pi} \int_0^{2\pi} \psi(x-t) D_{\lambda, \beta}(t) dt, \quad \psi(x) \in L^0.$$

Here, $\psi$ coincides almost everywhere with $f^{(\lambda, \beta)}$. See, for example, [12] Ch. 11. Also, similar questions are discussed in detail in the book [39] Vol. 1, Ch. 3. Note that a representation by convolution was first considered in [44].

4. The results of sections 2 and 5 can be extended to a multidimensional case. We only write the following estimates for the mixed modulus of smoothness $\omega_{\alpha_1, \alpha_2}(f, \delta_1, \delta_2)$ of orders $\alpha_1$ and $\alpha_2$ $(\alpha_1, \alpha_2 > 0)$ of a function $f$ (in the $L_p$ metric) with respect to the variables $x_1$ and $x_2$, respectively.\[10\]

10 The definition of the mixed modulus of smoothness and the mixed derivative in the Weyl sense can be found in, for example, [32].
then \( f \) has the mixed derivative in the Weyl sense \( f^{(r_1,r_2)} \in L^p_\theta \). Moreover,

\[
\| f^{(r_1,r_2)} \|_p \leq C(p, r_1, r_2)J_1(\theta)
\]

and

\[
\omega_{\alpha_1, \alpha_2}(f^{(r_1,r_2)}, \delta_1, \delta_2)_p \leq C(p, \alpha_1, \alpha_2, r_1, r_2) \left( \int_0^\delta \int_0^{\delta_2} t_1^{-r_1-1}t_2^{-r_2-1} \omega_{r_1, \alpha_1, r_2, \alpha_2}(f, t_1, t_2)_p dt_1 dt_2 \right) \leq C(\delta)J_2(\theta).
\]

II. If \( f \) has the mixed derivative in the Weyl sense \( f^{(r_1,r_2)} \in L^p_\theta \), then

\[
J_1(\tau) \leq C(p, r_1, r_2)\| f^{(r_1,r_2)} \|_p
\]

and

\[
J_2(\tau) \leq C(p, \alpha_1, \alpha_2, r_1, r_2)\omega_{\alpha_1, \alpha_2}(f^{(r_1,r_2)}, \delta_1, \delta_2)_p.
\]

Theorem 5 Let \( f(x_1, x_2) \in L^p_\theta, p = 1, \infty \) and let \( \alpha_1, \alpha_2, r_1, r_2 > 0 \).

I. If

\[
J_3 := \int_0^1 \int_0^1 t_1^{-r_1-1}t_2^{-r_2-1} \omega_{r_1, \alpha_1, r_2, \alpha_2}(f, t_1, t_2)_p dt_1 dt_2 < \infty,
\]

then \( f \) has the mixed derivative in the Weyl sense \( f^{(r_1,r_2)} \in L^p_\theta \). Moreover,

\[
\| f^{(r_1,r_2)} \|_p \leq C(r_1, r_2)J_3
\]

and

\[
\omega_{\alpha_1, \alpha_2}(f^{(r_1,r_2)}, \delta_1, \delta_2)_p \leq C(\alpha_1, \alpha_2, r_1, r_2) \left( \int_0^\delta \int_0^{\delta_2} t_1^{-r_1-1}t_2^{-r_2-1} \omega_{r_1, \alpha_1, r_2, \alpha_2}(f, t_1, t_2)_p dt_1 dt_2 \right).
\]

II. If \( f \) has the mixed derivative in the Weyl sense \( f^{(r_1,r_2)} \in L^p_\theta \), then

\[
\omega_{r_1, \alpha_1, r_2, \alpha_2}(f, \delta_1, \delta_2)_p \leq C(\alpha_1, \alpha_2, r_1, r_2)\delta_1^{r_1} \delta_2^{r_2} \| f^{(r_1,r_2)} \|_p
\]

and

\[
\omega_{r_1, \alpha_1, r_2, \alpha_2}(f, \delta_1, \delta_2)_p \leq C(\alpha_1, \alpha_2, r_1, r_2)\delta_1^{r_1} \delta_2^{r_2} \omega_{\alpha_1, \alpha_2}(f^{(r_1,r_2)}, \delta_1, \delta_2)_p.
\]

For more details on the estimates of transformed series in a multidimensional case, see the articles 32-33.

5. In view of inequalities (83) and (84) - (87), the problem of finding the estimates of \( \omega_{\alpha}(\varphi, t)_p \) in terms of \( \omega_{\alpha+r}(f, t)_p \) arises, e.g., in the case \( \varphi \sim \sigma(f, \lambda) \), where \( \lambda_n = n^r \ln^4 n \). If \( A < 0 \) (which is an analogue of the case \( \lambda_n = n^{r-\epsilon} \)), then estimates \( \omega_{\alpha}(\varphi, t)_p \) follow from [49].

For example, if \( p = 2 \) and \( \varphi \sim \sigma(f, n^r \ln^4 n) \), \( A < 0 \), then

\[
\omega_{\alpha}^{2}(\varphi, \delta) \sim \int_0^\delta \int_{\ln^{2|A|}(\frac{\delta}{k})} t^{-2r-1} \omega_{r+\beta}^{2}(f, t)_2 dt + \delta \int_\delta^1 \int_{\ln^{1+2|A|}(\frac{\delta}{k})} t^{-2(r+\beta)-1} \omega_{r+\beta}^{2}(f, t)_2 dt. \tag{92}
\]

Note that the differences between (92) and (84)-(85) are related only to the replacement of \( n^{-\epsilon} \) by \( \ln^4 n \).
The case $A > 0$ (which is an analogue of the case $\lambda n = n^r+\varepsilon$) is interesting. For $p = 2$ and $\varphi \sim \sigma(f, n^r \ln^A n), A > 0$ we have

$$
\int_0^\delta t \left( \frac{2}{t} \right) \omega_2^2(\varphi, t) dt \leq \int_0^\delta t \left( \frac{2}{t} \right) \omega_2^2(\varphi, \delta) dt. \quad (93)
$$

Comparing these relations with estimates (86)-(87), one can remark that the new term $\omega_2^2(\varphi, \delta)$ appears in (93). Thus, this case has essential distinctions. See for detail the papers [33], [49].

6. Defining the class $W^\lambda_\beta H_\alpha[\varphi]$ we assume that $\varphi \in \Phi_\alpha$. This restriction is natural for a majorant of the modulus of smoothness of order $\alpha$ (see [51]).

7. In Theorem 1 (item II) we used the inequality $\sum_{\nu=n+1}^{\infty} \frac{1}{\nu^{\lambda n}} \leq \frac{C}{\lambda n}$. It is equivalent to the following condition: there exists $\varepsilon > 0$ such that the sequence $\{n^{-\varepsilon} \lambda n\}$ is almost increasing, i.e., $n^{-\varepsilon} \lambda n \leq C n^{-\varepsilon} \lambda m, C \geq 1, n \leq m$. This and other conditions can be found in [3] and [51].

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