LINEARITY DEFECTS OF FACE RINGS

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Dedicated to Professor Jürgen Herzog on his 65th birthday

Abstract. Let \( S = K[x_1, \ldots, x_n] \) be a polynomial ring over a field \( K \), and \( E = \wedge \langle y_1, \ldots, y_n \rangle \) an exterior algebra. The linearity defect \( \text{ld}_E(N) \) of a finitely generated graded \( E \)-module \( N \) measures how far \( N \) departs from “componentwise linear”. It is known that \( \text{ld}_E(N) < \infty \) for all \( N \). But the value can be arbitrary large, while the similar invariant \( \text{ld}_S(M) \) for an \( S \)-module \( M \) is always at most \( n \). We will show that if \( I \) \( \Delta \) (resp. \( J \) \( \Delta \)) is the squarefree monomial ideal of \( S \) (resp. \( E \)) corresponding to a simplicial complex \( \Delta \subset \{1, \ldots, n\} \), then \( \text{ld}_E(E/J_\Delta) = \text{ld}_S(S/I_\Delta) \). Moreover, except some extremal cases, \( \text{ld}_E(E/J_\Delta) \) is a topological invariant of the geometric realization \( |\Delta^\vee| \) of the Alexander dual \( \Delta^\vee \) of \( \Delta \). We also show that, when \( n \geq 4 \), \( \text{ld}_E(E/J_\Delta) = n - 2 \) (this is the largest possible value) if and only if \( \Delta \) is an \( n \)-gon.

1. Introduction

Let \( A = \bigoplus_{i \in \mathbb{N}} A_i \) be a graded (not necessarily commutative) noetherian algebra over a field \( K \). Let \( M \) be a finitely generated graded left \( A \)-module, and \( P_* \) its minimal free resolution. Eisenbud et al. [4] defined the linear part \( \text{lin}(P_*) \) of \( P_* \), which is the complex obtained by erasing all terms of degree \( \geq 2 \) from the matrices representing the differential maps of \( P_* \) (hence \( \text{lin}(P_*)_i = P_i \) for all \( i \)). Following Herzog and Iyengar [7], we call \( \text{ld}_A(M) = \sup \{ i \mid H_i(\text{lin}(P_*)) \neq 0 \} \) the linearity defect of \( M \). This invariant and related concepts have been studied by several authors (e.g., [4, 7, 10, 13, 20]). We say a finitely generated graded \( A \)-module \( M \) is componentwise linear (or, (weakly) Koszul in some literature) if \( M_{(i)} \) has a linear free resolution for all \( i \). Here \( M_{(i)} \) is the submodule of \( M \) generated by its degree \( i \) part \( M_i \). Then we have

\[
\text{ld}_A(M) = \min \{ i \mid \text{the } i^{th} \text{ syzygy of } M \text{ is componentwise linear} \}.
\]

For this invariant, a remarkable result holds over an exterior algebra \( E = \wedge \langle y_1, \ldots, y_n \rangle \). In [4 Theorem 3.1], Eisenbud et al. showed that any finitely generated graded \( E \)-module \( N \) satisfies \( \text{ld}_E(N) < \infty \) while \( \text{proj. dim}_E(N) = \infty \) in most cases. (We also remark that Martinez-Villa and Zacharia [10] proved the same result for many selfinjective Koszul algebras). If \( n \geq 2 \), then we have \( \sup \{ \text{ld}_E(N) \mid N \text{ a finitely generated graded } E \text{-module} \} = \infty \). But Herzog and Römer proved that if \( J \subset E \) is a monomial ideal then \( \text{ld}_E(E/J) \leq n - 1 \) (c.f. [13]).

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A monomial ideal of $E = \bigwedge \langle y_1, \ldots, y_n \rangle$ is always of the form $J_\Delta := (\prod_{i \in F} y_i \mid F \not\in \Delta)$ for a simplicial complex $\Delta \subset 2^{\{1, \ldots, n\}}$. Similarly, we have the Stanley-Reisner ideal $I_\Delta := (\prod_{i \in F} x_i \mid F \not\in \Delta)$ of a polynomial ring $S = K[x_1, \ldots, x_n]$. In this paper, we will show the following.

**Theorem 1.1.** With the above notation, we have $\text{ld}_E(E/J_\Delta) = \text{ld}_S(S/I_\Delta)$. Moreover, if $\text{ld}_E(E/J_\Delta) > 0$ (equivalently, $\Delta \neq 2^T$ for any $T \subset [n]$), then $\text{ld}_E(E/J_\Delta)$ is a topological invariant of the geometric realization $|\Delta^\vee|$ of the Alexander dual $\Delta^\vee$.

(But $\text{ld}(E/J_\Delta)$ may depend on char$(K)$.)

By virtue of the above theorem, we can put $\text{ld}(\Delta) := \text{ld}_E(E/J_\Delta) = \text{ld}_S(S/I_\Delta)$. If we set $d := \min\{ i \mid [J_i] \neq 0 \} = \min\{ i \mid [J_i] \neq 0 \}$, then $\text{ld}(\Delta) \leq \max\{1, n - d\}$. But, if $d = 1$ (i.e., $\{i\} \not\in \Delta$ for some $1 \leq i \leq n$), then $\text{ld}(\Delta) \leq \max\{1, n - 3\}$. Hence, if $n \geq 3$, we have $\text{ld}(\Delta) \leq n - 2$ for all $\Delta$.

**Theorem 1.2.** Assume that $n \geq 4$. Then $\text{ld}(\Delta) = n - 2$ if and only if $\Delta$ is an $n$-gon.

While we treat $S$ and $E$ in most part of the paper, some results on $S$ can be generalized to a normal semigroup ring, and this generalization makes the topological meaning of $\text{ld}(\Delta)$ clear. So §2 concerns a normal semigroup ring. But, in this case, we use an irreducible resolution (something analogous to an injective resolution), not a projective resolution.

2. **LINEARITY DEFECTS FOR IRREDUCIBLE RESOLUTIONS**

Let $C \subset \mathbb{Z}^n \subset \mathbb{R}^n$ be an affine semigroup (i.e., $C$ is a finitely generated additive submonoid of $\mathbb{Z}^n$), and $R := K[\mathbf{x}^c \mid c \in C] \subset K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ the semigroup ring of $C$ over the field $K$. Here $\mathbf{x}^c$ for $c = (c_1, \ldots, c_n) \in C$ denotes the monomial $\prod_{i=1}^n x_i^{c_i}$. Let $P := \mathbb{R}_{\geq 0}C \subset \mathbb{R}^n$ be the polyhedral cone spanned by $C$. We always assume that $ZC = \mathbb{Z}^n$, $\mathbb{Z}^n \cap P = C$ and $C \cap (-C) = \{0\}$. Thus $R$ is a normal Cohen-Macaulay integral domain of dimension $n$ with a maximal ideal $m := (\mathbf{x}^c \mid 0 \neq c \in C)$.

Clearly, $R = \bigoplus_{c \in C} K\mathbf{x}^c$ is a $\mathbb{Z}^n$-graded ring. We say a $\mathbb{Z}^n$-graded ideal of $R$ is a monomial ideal. Let $\text{mod}^* R$ be the category of finitely generated $\mathbb{Z}^n$-graded $R$-modules and degree preserving $R$-homomorphisms. As usual, for $M \in \text{mod}^* R$ and $a \in \mathbb{Z}^n$, $M_a$ denotes the degree $a$ component of $M$, and $M(a)$ denotes the shifted module of $M$ with $M(a)_b = M_{a+b}$.

Let $\textbf{L}$ be the set of non-empty faces of the polyhedral cone $P$. Note that $\{0\}$ and $P$ itself belong to $\textbf{L}$. For $F \in \textbf{L}$, $P_F := (\mathbf{x}^c \mid c \in C \setminus F)$ is a prime ideal of $R$. Conversely, any monomial prime ideal is of the form $P_F$ for some $F \in \textbf{L}$. Note that $P_{\{0\}} = m$ and $P_{\emptyset} = (0)$. Set $K[F] := R/P_F \cong K[\mathbf{x}^c \mid c \in C \cap F]$ for $F \in \textbf{L}$. The Krull dimension of $K[F]$ equals the dimension $\dim F$ of the polyhedral cone $F$.

For a point $u \in P$, we always have a unique face $F \in \textbf{L}$ whose relative interior contains $u$. Here we denote $s(u) = F$.

**Definition 2.1** ([17]). We say a module $M \in \text{mod}^* R$ is squarefree, if it is $C$-graded (i.e., $M_a = 0$ for all $a \not\in C$), and the multiplication map $M_a \ni y \mapsto \mathbf{x}^b y \in M_{a+b}$ is bijective for all $a, b \in C$ with $s(a + b) = s(a)$. 

For a monomial ideal \( I \), \( R/I \) is a squarefree \( R \)-module if and only if \( I \) is a radical ideal (i.e., \( \sqrt{I} = I \)). Regarding \( L \) as a partially ordered set by inclusion, we say \( \Delta \subset L \) is an order ideal, if \( \Delta \supseteq F \supseteq F' \in L \) implies \( F' \in \Delta \). If \( \Delta \) is an order ideal, then \( I_\Delta := (x^c | c \in C, s(c) \notin \Delta) \subset R \) is a radical ideal. Conversely, any radical monomial ideal is of the form \( I_\Delta \) for some \( \Delta \). Set \( K[\Delta] := R/I_\Delta \). Clearly,

\[
K[\Delta]_a \cong \begin{cases} 
K & \text{if } a \in C \text{ and } s(a) \in \Delta, \\
0 & \text{otherwise}.
\end{cases}
\]

In particular, if \( \Delta = L \) (resp. \( \Delta = \{ \{0\} \} \)), then \( I_\Delta = 0 \) (resp. \( I_\Delta = m \)) and \( K[\Delta] = R \) (resp. \( K[\Delta] = K \)). When \( R \) is a polynomial ring, \( K[\Delta] \) is nothing else than the Stanley-Reisner ring of a simplicial complex \( \Delta \). (If \( R \) is a polynomial ring, then the partially ordered set \( L \) is isomorphic to the power set \( 2^{\{1,...,n\}} \), and \( \Delta \) can be seen as a simplicial complex.)

For each \( F \in L \), take some \( c(F) \in C \cap \text{rel-int}(F) \) (i.e., \( s(c(F)) = F \)). For a squarefree \( R \)-module \( M \) and \( F,G \in L \) with \( G \supseteq F \), [17] Theorem 3.3 gives a \( K \)-linear map \( \varphi^M_{G,F} : M_{c(F)} \to M_{c(G)} \). They satisfy \( \varphi^M_{F,F} = \text{Id} \) and \( \varphi^M_{H,G} \circ \varphi^M_{G,F} = \varphi^M_{H,F} \) for all \( H \supseteq G \supseteq F \). We have \( M_c \cong M_{c'} \) for \( c,c' \in C \) with \( s(c) = s(c') \). Under these isomorphisms, the maps \( \varphi^M_{G,F} \) do not depend on the particular choice of \( c(F) \)'s.

Let \( \text{Sq}(R) \) be the full subcategory of \( \text{^mod} R \) consisting of squarefree modules. As shown in [17], \( \text{Sq}(R) \) is an abelian category with enough injectives. For an indecomposable squarefree module \( M \), it is injective in \( \text{Sq}(R) \) if and only if \( M \cong K[F] \) for some \( F \in L \). Each \( M \in \text{Sq}(R) \) has a minimal injective resolution in \( \text{Sq}(R) \), and we call it a minimal irreducible resolution (see [21] for further information). A minimal irreducible resolution is unique up to isomorphism, and its length is at most \( n \).

Let \( \omega_R \) be the \( \mathbb{Z}^n \)-graded canonical module of \( R \). It is well-known that \( \omega_R \) is isomorphic to the radical monomial ideal (\( x^c | c \in C, s(c) = \mathbb{P} \)). Since we have \( \text{Ext}_R^i(M^\bullet, \omega_R) \in \text{Sq}(R) \) for all \( M^\bullet \in \text{Sq}(R) \), \( D(-) := \text{RHom}_R(-, \omega_R) \) gives a duality functor from the derived category \( D^b(\text{Sq}(R)) \) \( \cong D^b_{\text{sq}(R)}(\text{^mod} R) \) to itself.

In the sequel, for a \( K \)-vector space \( V \), \( V^* \) denotes its dual space. But, even if \( V = M_a \) for some \( M \in \text{^mod} R \) and \( a \in \mathbb{Z}^n \), we set the degree of \( V^* \) to be 0.

**Lemma 2.2** ([21] Lemma 3.8). If \( M \in \text{Sq}(R) \), then \( D(M) \) is quasi-isomorphic to the complex \( D^\bullet : 0 \to D^0 \to D^1 \to \cdots \to D^n \to 0 \) with

\[
D^i = \bigoplus_{F \in L, \dim F = n-i} (M_{c(F)})^* \otimes_K K[F].
\]

Here the differential is the sum of the maps

\[
(\pm \varphi^M_{F,F'})^* \otimes \text{nat} : (M_{c(F)})^* \otimes_K K[F] \to (M_{c(F')})^* \otimes_K K[F']
\]

for \( F,F' \in L \) with \( F \supseteq F' \) and \( \dim F = \dim F' + 1 \), and \( \text{nat} \) denotes the natural surjection \( K[F] \to K[F'] \). We can also describe \( D(M^\bullet) \) for a complex \( M^\bullet \in D^b(\text{Sq}(R)) \) in a similar way.

**Convention.** In the sequel, as an explicit complex, \( D(M^\bullet) \) for \( M^\bullet \in D^b(\text{Sq}(R)) \) means the complex described in Lemma 2.2.
Since $D \circ D \cong \text{Id}_{D^n(Sq(R))}$, $D \circ D(M)$ is an irreducible resolution of $M$, but it is far from being minimal. Let $(I^\bullet, \partial^\bullet)$ be a minimal irreducible resolution of $M$. For each $i \in \mathbb{N}$ and $F \in L$, we have a natural number $\nu_i(F, M)$ such that

$$I^i \cong \bigoplus_{F \in L} K[F]^{
u_i(F, M)}.$$

Since $I^\bullet$ is minimal, $z \in K[F] \subset I^i$ with $\dim F = d$ is sent to

$$\partial^i(z) \in \bigoplus_{\dim G < d} K[G]^{
u_{i+1}(G, M)} \subset I^{i+1}.$$

The above observation on $D \circ D(M)$ gives the formula ([17, Theorem 4.15])

$$\nu_i(F, M) = \dim_{\mathbb{R}}[-n-i-\dim F](M, \omega_R)_{c(F)}.$$

For each $l \in \mathbb{N}$ with $0 \leq l \leq n$, we define the $l$-linear strand $\text{lin}_l(I^\bullet)$ of $I^\bullet$ as follows: The term $\text{lin}_l(I^\bullet)^i$ of cohomological degree $i$ is

$$\bigoplus_{\dim F = i-l} K[F]^{
u_i(F, M)},$$

which is a direct summand of $I^i$, and the differential $\text{lin}_l(I^\bullet)^i \to \text{lin}_l(I^\bullet)^{i+1}$ is the corresponding component of the differential $\partial^i : I^i \to I^{i+1}$ of $I^\bullet$. By the minimality of $I^\bullet$, we can see that $\text{lin}_l(I^\bullet)$ are cochain complexes. Set $\text{lin}(I^\bullet) := \bigoplus_{0 \leq i \leq n} \text{lin}_l(I^\bullet)$. Then we have the following. For a complex $M^\bullet$ and an integer $p$, let $M^\bullet[p]$ be the $p^{th}$ translation of $M^\bullet$. That is, $M^\bullet[p]$ is a complex with $M^i[p] = M^{i+p}$.

**Theorem 2.3** ([21, Theorem 3.9]). With the above notation, we have

$$\text{lin}_l(I^\bullet) \cong D(\text{Ext}_{\mathbb{R}}^{n-l}(M, \omega_R))[n-l].$$

Hence

$$\text{lin}(I^\bullet) \cong \bigoplus_{i \in \mathbb{Z}} D(\text{Ext}_{\mathbb{R}}^i(M, \omega_R))[i].$$

**Definition 2.4.** Let $I^\bullet$ be a minimal irreducible resolution of $M \in \text{Sq}(R)$. We call $\max\{i \mid H^i(\text{lin}(I^\bullet)) \neq 0\}$ the linearity defect of the minimal irreducible resolution of $M$, and denote it by $\text{ld. irr}_R(M)$.

**Corollary 2.5.** With the above notation, we have

$$\max\{i \mid H^i(\text{lin}(I^\bullet)) \neq 0\} = l - \text{depth}_{\mathbb{R}}(\text{Ext}_{\mathbb{R}}^{n-l}(M, \omega_R)),$$

and hence

$$\text{ld. irr}_R(M) = \max\{i - \text{depth}_{\mathbb{R}}(\text{Ext}_{\mathbb{R}}^{n-i}(M, \omega_R)) \mid 0 \leq i \leq n\}.$$

Here we set the depth of the 0 module to be $+\infty$.

**Proof.** By Theorem 2.3, we have $H^i(\text{lin}(I^\bullet)) = \text{Ext}_{\mathbb{R}}^{i+l}(\text{Ext}_{\mathbb{R}}^l(M, \omega_R), \omega_R)$. Since $\text{depth}_{\mathbb{R}} N = \min\{i \mid \text{Ext}_{\mathbb{R}}^{n-i}(N, \omega_R) \neq 0\}$ for a finitely generated graded $R$-module $N$, the assertion follows. \qed
Definition 2.6 (Stanley [15]). Let $M \in \text{mod} \ R$. We say $M$ is \textit{sequentially Cohen-Macaulay} if there is a finite filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

of $M$ by graded submodules $M_i$ satisfying the following conditions.

(a) Each quotient $M_i/M_{i-1}$ is Cohen-Macaulay.

(b) dim$(M_i/M_{i-1}) < \text{dim}(M_{i+1}/M_i)$ for all $i$.

Remark that the notion of sequentially Cohen-Macaulay module is also studied under the name of a “Cohen-Macaulay filtered module” ([14]).

Sequentially Cohen-Macaulay property is getting important in the theory of Stanley-Reisner rings. It is known that $M \in \text{mod} \ R$ is sequentially Cohen-Macaulay if and only if $\text{Ext}^n_R(M, \omega_R)$ is a zero module or a Cohen-Macaulay module of dimension $i$ for all $i$ (c.f. [15, III. Theorem 2.11]). Let us go back to Corollary 2.5. If $N := \text{Ext}^{n-i}_R(M, \omega_R) \neq 0$, then $\text{depth}_R N \leq \text{dim}_R N \leq i$. Hence $\text{depth}_R N = i$ if and only if $N$ is a Cohen-Macaulay module of dimension $i$. Thus, as stated in [21, Corollary 3.11], $\text{ld} \ irr_R(M) = 0$ if and only if $M$ is sequentially Cohen-Macaulay.

Let $I^* : 0 \rightarrow I^0 \xrightarrow{\partial^0} I^1 \xrightarrow{\partial^1} I^2 \rightarrow \cdots$ be an irreducible resolution of $M \in \text{Sq}(R)$. Then it is easy to see that $\ker(\partial^i)$ is sequentially Cohen-Macaulay if and only if $i \geq \text{ld} \ irr_R(M)$. In particular,

$$\text{ld} \ irr_R(M) = \min \{ i \mid \ker(\partial^i) \text{ is sequentially Cohen-Macaulay}\}.$$

We have a hyperplane $H \subset \mathbb{R}^n$ such that $B := H \cap L$ is an $(n-1)$-dimensional polytope. Clearly, $B$ is homeomorphic to a closed ball of dimension $n-1$. For a face $F \in L$, set $|F|$ to be the relative interior of $F \cap H$. If $\Delta \subset L$ is an order ideal, then $|\Delta| := \bigcup_{F \in \Delta} |F|$ is a closed subset of $B$, and $\bigcup_{F \in \Delta} |F|$ is a \textit{regular cell decomposition} (c.f. [2, §6.2]) of $|\Delta|$. Up to homeomorphism, (the regular cell decomposition of) $|\Delta|$ does not depend on the particular choice of the hyperplane $H$. The dimension $\text{dim} |\Delta|$ of $|\Delta|$ is given by $\max \{ \text{dim} |F| \mid F \in \Delta \}$. Here $\text{dim} |F|$ denotes the dimension of $|F|$ as a cell (we set $\text{dim} \emptyset = -1$), that is, $\text{dim} |F| = \text{dim} F - 1 = \text{dim} K[F] - 1$. Hence we have $\text{dim} K[\Delta] = \text{dim} |\Delta| + 1$.

If $F \in \Delta$, then $U_F := \bigcup_{F' \supset F} |F'|$ is an open set of $B$. Note that $\{ U_F \mid \{0\} \neq F \in L \}$ is an open covering of $B$. In [18], from $M \in \text{Sq}(R)$, we constructed a sheaf $M^+$ on $B$. More precisely, the assignment

$$\Gamma(U_F, M^+) = M_{c(F)}$$

for each $F \neq \{0\}$ and the map

$$\varphi^M_{F,F'} : \Gamma(U_{F'}, M^+) = M_{c(F')} \rightarrow M_{c(F)} = \Gamma(U_F, M^+)$$

for $F, F' \neq \{0\}$ with $F \supset F'$ (equivalently, $U_{F'} \supset U_F$) defines a sheaf. Note that $M^+$ is a \textit{constructible sheaf} with respect to the cell decomposition $B = \bigcup_{F \in L} |F|$. In fact, for all $\{0\} \neq F \in L$, the restriction $M^+|_{|F|}$ of $M^+$ to $|F| \subset B$ is a constant sheaf with coefficients in $M_{c(F)}$. Note that $M_0$ is “irrelevant” to $M^+$, where $0$ denotes $(0,0,\ldots,0) \in \mathbb{Z}^n$. 
It is easy to see that $K[\Delta]^+ \cong j_*K_{|\Delta|}$, where $K_{|\Delta|}$ is the constant sheaf on $|\Delta|$ with coefficients in $K$, and $j$ denotes the embedding map $|\Delta| \hookrightarrow B$. Similarly, we have that $(\omega_R)^+ \cong h_*K_{B^0}$, where $K_{B^0}$ is the constant sheaf on the relative interior $B^0$ of $B$, and $h$ denotes the embedding map $B^0 \hookrightarrow B$. Note that $(\omega_R)^+$ is the orientation sheaf of $B$ over $K$.

**Theorem 2.7** ([18, Theorem 3.3]). For $M \in \text{Sq}(R)$, we have an isomorphism

$$H^i(B; M^+) \cong [H_{m}^{i+1}(M)]_0 \quad \text{for all } i \geq 1,$$

and an exact sequence

$$0 \to [H_{m}^{0}(M)]_0 \to M_0 \to H^0(B; M^+) \to [H_{m}^{1}(M)]_0 \to 0.$$  

In particular, we have $[H_{m}^{i+1}(K[\Delta])]_0 \cong \tilde{H}^i(|\Delta|; K)$ for all $i \geq 0$, where $\tilde{H}^i(|\Delta|; K)$ denotes the $i^{\text{th}}$ reduced cohomology of $|\Delta|$ with coefficients in $K$.

Let $\Delta \subset L$ be an order ideal and $X := |\Delta|$. Then $X$ admits Verdier’s dualizing complex $D_X^*$, which is a complex of sheaves of $K$-vector spaces. For example, $D_B^*$ is quasi-isomorphic to $(\omega_R)^+[n-1]$.

**Theorem 2.8** ([18, Theorem 4.2]). With the above notation, if $\text{ann}(M) \supset I_\Delta$ (equivalently, $\text{supp}(M^+) := \{ x \in B \mid (M^*)_x \neq 0 \} \subset X$), then we have

$$\text{supp}(\text{Ext}^i_R(M, \omega_R)^+) \subset X \quad \text{and} \quad \text{Ext}^i_R(M, \omega_R)^+|_X \cong \text{Ext}^{i-n+1}(M^+)|_X; D_X^*).$$

**Theorem 2.9.** Let $M$ be a squarefree $R$-module with $M \neq 0$ and $[H_{m}^{0}(M)]_0 = 0$, and $X$ the closure of $\text{supp}(M^+)$. Then $\text{ld. irr}_R(M)$ only depends on the sheaf $M^+|_X$ (also independent from $R$).

**Proof.** We use Corollary [2.5] In the notation there, the case when $i = 0$ is always unnecessary to check. Moreover, by the present assumption, we have $\text{depth}_R(\text{Ext}^{n-1}_R(M, \omega_R)) \geq 1$ (in fact, $\text{Ext}^{n-1}_R(M, \omega_R)$ is either the 0 module, or a 1-dimensional Cohen-Macaulay module). So we may assume that $i > 1$.

Recall that

$$\text{depth}_R(\text{Ext}^{n-i}_R(M, \omega_R)) = \min\{ j \mid \text{Ext}^{n-j}_R(\text{Ext}^{n-i}_R(M, \omega_R), \omega_R) \neq 0 \}.$$

By Theorem [2.8] $[\text{Ext}^{n-j}_R(\text{Ext}^{n-i}_R(M, \omega_R), \omega_R)]_a$ can be determined by $M^+|_X$ for all $i$, $j$ and all $a \neq 0$. If $j > 1$, then $[\text{Ext}^{n-j}_R(\text{Ext}^{n-i}_R(M, \omega_R), \omega_R)]_0$ is isomorphic to

$$[H^i_{m}(\text{Ext}^{n-i}_R(M, \omega_R))]_0^* \cong H^{j-1}(B; \text{Ext}^{n-i}_R(M, \omega_R)^*) \cong H^{j-1}(X; \text{Ext}^{i-n+1}(M^+|_X; D_X^*))^*$$

(the first and the second isomorphisms follow from Theorem [2.7] and Theorem [2.8] respectively), and determined by $M^+|_X$. So only $[\text{Ext}^{n-j}_R(\text{Ext}^{n-i}_R(M, \omega_R), \omega_R)]_0$ for $j = 0, 1$ remain. As above, they are isomorphic to $[H^j_{m}(\text{Ext}^{n-j}_R(M, \omega_R))]_0$. But, by [21, Lemma 5.11], we can compute $[H^j_{m}(\text{Ext}^{n-j}_R(M, \omega_R))]_0$ for $i > 1$ and $j = 0, 1$ from the sheaf $M^+|_X$. So we are done.

**Theorem 2.10.** For an order ideal $\Delta \subset L$ with $\Delta \neq \emptyset$, $\text{ld. irr}_R(K[\Delta])$ depends only on the topological space $|\Delta|$.
Note that \( \text{ld. irr}_R(K[\Delta]) \) may depend on \( \text{char}(K) \). For example, if \( |\Delta| \) is homeomorphic to a real projective plane, then \( \text{ld. irr}_R(K[\Delta]) = 0 \) if \( \text{char}(K) \neq 2 \), but \( \text{ld. irr}_R(K[\Delta]) = 2 \) if \( \text{char}(K) = 2 \).

Similarly, some other invariants and conditions (e.g., the Cohen-Macaulay property of \( K[\Delta] \)) studied in this paper depend on \( \text{char}(K) \). But, since we fix the base field \( K \), we always omit the phrase “over \( K \).

**Proof.** If \( |\Delta| \) is not connected, then \( [H^1_m(K[\Delta])]_0 \neq 0 \) by Theorem 2.7 and we cannot use Theorem 2.9 directly. But even in this case, \( \text{depth}_R(\text{Ext}^{n-1}_R(K[\Delta], \omega_R)) \) can be computed for all \( i \neq 1 \) by the same way as in Theorem 2.9. In particular, they only depend on \( |\Delta| \). So the assertion follows from the next lemma. □

**Lemma 2.11.** We have \( \text{depth}_R(\text{Ext}^{n-1}_R(K[\Delta], \omega_R)) \in \{0, 1, +\infty\} \), and

\[
\text{depth}_R(\text{Ext}^{n-1}_R(K[\Delta], \omega_R)) = 0 \quad \text{if and only if} \quad |\Delta'| \text{ is not connected.}
\]

Here \( \Delta' := \Delta \setminus \{ F \mid F \text{ is a maximal element of } \Delta \text{ and } |F| = 0 \} \).

**Proof.** Since \( \dim \text{dim} R \text{Ext}^{n-1}_R(K[\Delta], \omega_R) \leq 1 \), the first statement is clear. If \( \dim |\Delta| \leq 0 \), then \( |\Delta'| = \emptyset \) and \( \text{depth}_R(\text{Ext}^{n-1}_R(K[\Delta], \omega_R)) \geq 1 \). So, to see the second statement, we may assume that \( \dim |\Delta| > 1 \). Set \( J := I_{\Delta'}/I_{\Delta} \) to be an ideal of \( K[\Delta] \).

Note that either \( J \) is a 1-dimensional Cohen-Macaulay module or \( J = 0 \). From the short exact sequence \( 0 \to J \to K[\Delta] \to K[\Delta'] \to 0 \), we have an exact sequence

\[
0 \to \text{Ext}^{n-1}_R(K[\Delta], \omega_R) \to \text{Ext}^{n-1}_R(K[\Delta], \omega_R) \to \text{Ext}^{n-1}_R(J, \omega_R) \to 0.
\]

Since \( \text{Ext}^{n-1}_R(J, \omega_R) \) has positive depth, \( \text{depth}_R(\text{Ext}^{n-1}_R(K[\Delta]', \omega_R)) = 0 \) if and only if \( \text{depth}_R(\text{Ext}^{n-1}_R(K[\Delta], \omega_R)) = 0 \). But, since \( K[\Delta'] \) does not have 1-dimensional associated primes, \( \text{Ext}^{n-1}_R(K[\Delta], \omega_R) \) is an artinian module. Hence we have the following.

\[
\text{depth}_R(\text{Ext}^{n-1}_R(K[\Delta], \omega_R)) = 0 \iff [\text{Ext}^{n-1}_R(K[\Delta], \omega_R)]_0 \neq 0 \\
\iff [H^1_m(K[\Delta])]_0 \neq 0 \\
\iff |\Delta'| \text{ is not connected.}
\]

□

3. Linearity Defects of Symmetric and Exterior Face Rings

Let \( S := K[x_1, \ldots, x_n] \) be a polynomial ring, and consider its natural \( \mathbb{Z}^n \)-grading. Since \( S = K[\mathbb{N}^n] \) is a normal semigroup ring, we can use the notation and the results in the previous section.

Now we introduce some conventions which are compatible with the previous notation. Let \( e_i := (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^n \) be the \( i \)-th unit vector, and \( \mathbf{P} \) the cone spanned by \( e_1, \ldots, e_n \). We identify a face \( F \) of \( \mathbf{P} \) with the subset \( \{ i \mid e_i \in F \} \) of \( [n] := \{1, 2, \ldots, n\} \). Hence the set \( \mathbf{L} \) of nonempty faces of \( \mathbf{P} \) can be identified with the power set \( 2^{[n]} \) of \( [n] \). We say \( \mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n \) is squarefree, if \( a_i = 0, 1 \) for all \( i \). A squarefree vector \( \mathbf{a} \in \mathbb{N}^n \) will be identified with the subset \( \{ i \mid a_i = 1 \} \) of \( [n] \). Recall that we took a vector \( \mathbf{c}(F) \in C \) for each \( F \in \mathbf{L} \) in the previous section. Here we assume that \( \mathbf{c}(F) \) is the squarefree vector corresponding
to \( F \in L \cong 2^n \). So, for a \( \mathbb{Z}^n \)-graded \( S \)-module \( M \), we simply denote \( M_{c(F)} \) by \( M_F \). In the first principle, we regard \( F \) as a subset of \([n]\), or a squarefree vector in \( \mathbb{N}^n \), rather than the corresponding face of \( P \). For example, we write \( P_F = (x_i \mid i \notin F) \), \( K[F] \cong K[x_i \mid i \in F] \). And \( S(-F) \) denotes the rank 1 free \( S \)-module \( S(-a) \), where \( a \in \mathbb{N}^n \) is the squarefree vector corresponding to \( F \).

Squarefree \( S \)-modules are defined by the same way as Definition 2.1. Note that the free module \( S(-a) \), \( a \in \mathbb{Z}^n \), is squarefree if and only if \( a \) is squarefree. Let \( *\text{mod} S \) (resp. \( \text{Sq}(S) \)) be the category of finitely generated \( \mathbb{Z}^n \)-graded \( S \)-modules (resp. squarefree \( S \)-modules). Let \( P_* \) be a \( \mathbb{Z}^n \)-graded minimal free resolution of \( M \in *\text{mod} S \). Then \( M \) is squarefree if and only if each \( P_i \) is a direct sum of copies of \( S(-F) \) for various \( F \subseteq [n] \). In the present case, an order ideal \( \Delta \) of \( L \) (\( \cong 2^n \)) is essentially a simplicial complex, and the ring \( K[\Delta] \) defined in the previous section is nothing other than the Stanley-Reisner ring (c.f. [2, 13]) of \( \Delta \).

Let \( E = \bigwedge \langle y_1, \ldots, y_n \rangle \) be the exterior algebra over \( K \). Under the Bernstein-Gel’fand-Gel’fand correspondence (c.f. [3]), \( E \) is the counter part of \( S \). We regard \( E \) as a \( \mathbb{Z}^n \)-graded ring by \( \deg y_i = e_i = \deg x_i \) for each \( i \). Clearly, any monomial ideal of \( E \) is “squarefree”, and of the form \( J_{\Delta} := (\prod_{i \in F} y_i \mid F \subseteq [n], F \notin \Delta) \) for a simplicial complex \( \Delta \subseteq 2^n \). We say \( K(\Delta) := E/J_{\Delta} \) is the exterior face ring of \( \Delta \).

Let \( *\text{mod} E \) be the category of finitely generated \( \mathbb{Z}^n \)-graded \( E \)-modules and degree preserving \( E \)-homomorphisms. Note that, for graded \( E \)-modules, we do not have to distinguish left modules from right ones. Hence

\[
\mathbf{D}_E(-) := \bigoplus_{a \in \mathbb{Z}^n} \text{Hom}_{*\text{mod} E}(-, E(a))
\]
gives an exact contravariant functor from \( *\text{mod} E \) to itself satisfying \( \mathbf{D}_E \circ \mathbf{D}_E = \text{Id} \).

**Definition 3.1** (Römer [12]). We say \( N \in *\text{mod} E \) is squarefree, if \( N = \bigoplus_{F \subseteq [n]} N_F \) (i.e., if \( a \in \mathbb{Z}^n \) is not squarefree, then \( N_a = 0 \)).

An exterior face ring \( K(\Delta) \) is a squarefree \( E \)-module. But, since a free module \( E(a) \) is not squarefree for \( a \neq 0 \), the syzygies of a squarefree \( E \)-module are not squarefree. Let \( \text{Sq}(E) \) be the full subcategory of \( *\text{mod} E \) consisting of squarefree modules. If \( N \) is a squarefree \( E \)-module, then so is \( \mathbf{D}_E(N) \). That is, \( \mathbf{D}_E \) gives a contravariant functor from \( \text{Sq}(E) \) to itself.

We have functors \( \mathcal{S} : \text{Sq}(E) \to \text{Sq}(S) \) and \( \mathcal{E} : \text{Sq}(S) \to \text{Sq}(E) \) giving an equivalence \( \text{Sq}(S) \cong \text{Sq}(E) \). Here \( \mathcal{S}(N)_F = N_F \) for \( N \in \text{Sq}(E) \) and \( F \subseteq [n] \), and the multiplication map \( \mathcal{S}(N)_F \ni z \mapsto x_i z \in \mathcal{S}(N)_{F \cup \{i\}} \) for \( i \notin F \) is given by \( \mathcal{S}(N)_F = N_F \ni z \mapsto (-1)^{\alpha(i,F)} y_i z \in N_{F \cup \{i\}} = \mathcal{S}(N)_{F \cup \{i\}} \), where \( \alpha(i,F) = \# \{ j \in F \mid j < i \} \). For example. \( \mathcal{S}(K(\Delta)) \cong K[\Delta] \). See [12] for detail.

Note that \( \mathbf{A} := \mathcal{S} \circ \mathbf{D}_E \circ \mathcal{E} \) is an exact contravariant functor from \( \text{Sq}(S) \) to itself satisfying \( \mathbf{A} \circ \mathbf{A} = \text{Id} \). It is easy to see that \( \mathbf{A}(K[F]) \cong S(-F^c) \), where \( F^c := [n] \setminus F \). We also have \( \mathbf{A}(K[\Delta]) \cong I_{\Delta^\vee} \), where

\[
\Delta^\vee := \{ F \subseteq [n] \mid F^c \notin \Delta \}
\]
is the Alexander dual complex of \( \Delta \). Since \( \mathbf{A} \) is exact, it exchanges a (minimal) free resolution with a (minimal) irreducible resolution.
Eisenbud et al. ([3, 11]) introduced the notion of the linear strands and the linear part of a minimal free resolution of a graded \( S \)-module. Let \( P_\bullet : \cdots \to P_1 \to P_0 \to 0 \) be a \( \mathbb{Z}^n \)-graded minimal \( S \)-free resolution of \( M \in \ast \text{mod} \ S \). We have natural numbers \( \beta_{i,a}(M) \) for \( i \in \mathbb{N} \) and \( a \in \mathbb{Z}^n \) such that \( P_i = \bigoplus_{a \in \mathbb{Z}^n} S(-a)_{\beta_i,a}(M) \). We call \( \beta_{i,a}(M) \) the graded Betti numbers of \( M \). Set \( |a| = \sum_{i=1}^{n} a_i \) for \( a = (a_1, \ldots, a_n) \in \mathbb{N}^n \). For each \( l \in \mathbb{Z} \), we define the \( l \)-linear strand \( \text{lin}_l(P_\bullet) \) of \( P_\bullet \) as follows: The term \( \text{lin}_l(P_\bullet)_i \) of homological degree \( i \) is
\[
\bigoplus_{|a| = l+i} S(-a)_{\beta_i,a}(M),
\]
which is a direct summand of \( P_i \), and the differential \( \text{lin}_l(P_\bullet)_i \to \text{lin}_l(P_\bullet)_{i-1} \) is the corresponding component of the differential \( P_i \to P_{i-1} \) of \( P_\bullet \). By the minimality of \( P_\bullet \), we can easily verify that \( \text{lin}_l(P_\bullet) \) are chain complexes (see also [3, §7A]). We call \( \text{lin}_l(P_\bullet) := \bigoplus_{l \in \mathbb{Z}} \text{lin}_l(P_\bullet) \) the linear part of \( P_\bullet \). Note that the differential maps of \( \text{lin}(P_\bullet) \) are represented by matrices of linear forms. We call
\[
\text{ld}_S(M) := \max \{ i \mid H_i(\text{lin}_l(P_\bullet)) \neq 0 \}
\]
the linearity defect of \( M \).

Sometimes, we regard \( M \in \ast \text{mod} \ S \) as a \( \mathbb{Z} \)-graded module by \( M_j = \bigoplus_{|a| = j} M_a \). In this case, we set \( \beta_{i,j}(M) := \bigoplus_{|a| = j} \beta_{i,a}(M). \) Then \( \text{lin}_l(P_\bullet)_i = S(-l-i)_{\beta_{i,l+i}(M)}. \)

**Remark 3.2.** For \( M \in \ast \text{mod} \ S \), it is clear that \( \text{ld}_S(M) \leq \text{proj. dim}_S(M) \leq n \), and there are many examples attaining the equalities. In fact, \( \text{ld}_S(S/(x_1^2, \ldots, x_n^2)) = n \). But if \( M \in \text{Sq}(S) \), then we always have \( \text{ld}_S(M) \leq n - 1 \). In fact, for a squarefree module \( M \), \( \text{proj. dim}_S(M) = n \), if and only if \( \text{depth}_S M = 0 \), if and only if \( M \cong K \oplus M' \) for some \( M' \in \text{Sq}(S) \). But \( \text{ld}_S(K) = 0 \) and \( \text{ld}_S(M' \oplus K) = \text{ld}_S(M') \). So we may assume that \( \text{proj. dim}_S M' \leq n - 1 \).

**Proposition 3.3.** Let \( M \in \text{Sq}(S) \), and \( P_\bullet \) its minimal graded free resolution. We have
\[
\max \{ i \mid H_i(\text{lin}_l(P_\bullet)) \neq 0 \} = n - l - \text{depth}_S(\text{Ext}^*_S(A(M),S)),
\]
and hence
\[
\text{ld}_S(M) = \max \{ i - \text{depth}_S(\text{Ext}^{n-i}_S(A(M),S)) \mid 0 \leq i \leq n \}.\]

**Proof.** Note that \( I^\bullet := A(P_\bullet) \) is a minimal irreducible resolution of \( A(M) \). Moreover, we have \( A(\text{lin}_l(P_\bullet)) \cong \text{lin}_{n-l}(I^\bullet) \). Since \( A \) is exact,
\[
\max \{ i \mid H_i(\text{lin}_l(P_\bullet)) \neq 0 \} = \max \{ i \mid H_i(\text{lin}_{n-l}(I^\bullet)) \neq 0 \},
\]
and hence
\[
(3.1) \quad \text{ld}_S(M) = \text{ld. irr}_S(A(M)).
\]
Hence the assertions follow from Corollary 2.3 (note that \( S \cong \omega_S \) as underlying modules).

For \( N \in \ast \text{mod} \ E \), we have a \( \mathbb{Z}^n \)-graded minimal \( E \)-free resolution \( P_\bullet \) of \( N \). By the similar way to the \( S \)-module case, we can define the linear part \( \text{lin}_l(P_\bullet) \) of \( P_\bullet \), and set \( \text{ld}_E(N) := \max \{ i \mid H_i(\text{lin}_l(P_\bullet)) \neq 0 \}. \) (In [13, 20], \( \text{ld}_E(N) \) is
denoted by \( \text{lpd}(N) \). “\( \text{lpd} \)” is an abbreviation for “linear part dominate”.) In [4, Theorem 3.1], Eisenbud et al. showed that \( \text{ld}_E(N) < \infty \) for all \( N \in \text{mod} \ E \). Since \( \text{proj} \dim_E(N) = \infty \) in most cases, this is a strong result. If \( n \geq 2 \), then we have \( \sup \{ \text{ld}_E(N) \mid N \in \text{mod} \ E \} = \infty \). In fact, since \( E \) is selfinjective, we can take “cosyzygies”. But, if \( N \in \text{Sq}(E) \), then \( \text{ld}_E(N) \) behaves quite nicely.

**Theorem 3.4.** For \( N \in \text{Sq}(E) \), we have \( \text{ld}_E(N) = \text{ld}_S(S(N)) \leq n - 1 \). In particular, for a simplicial complex \( \Delta \subset 2^{[n]} \), we have \( \text{ld}_E(K(\Delta)) = \text{ld}_S(K[\Delta]) \).

*Proof.* Using the Bernstein-Gel’fand-Gel’fand correspondence, the second author described \( \text{ld}_E(N) \) in [20, Lemma 4.12]. This description is the first equality of the following computation, which proves the assertion.

\[
\text{ld}_E(N) = \max \{ i - \text{depth}_S(\text{Ext}^{n-i}_S(\mathcal{D}_E(N), S)) \mid 0 \leq i \leq n \} \quad \text{(by [20])}
= \max \{ i - \text{depth}_S(\text{Ext}^{n-i}_S(A \circ \mathcal{S}(N), S)) \mid 0 \leq i \leq n \} \quad \text{(see below)}
= \text{ld}_S(S(N)) \quad \text{(by Proposition 3.3)}.
\]

Here the second equality follows from the isomorphisms \( A \circ \mathcal{S}(N) \cong \mathcal{S} \circ \mathcal{D}_E(N) \cong \mathcal{S} \circ \mathcal{D}_E \circ \mathcal{E} \circ S(N) \cong A \circ S(N) \).

**Remark 3.5.** Herzog and Römer showed that \( \text{ld}_E(N) \leq \text{proj} \dim_S(S(N)) \) for \( N \in \text{Sq}(E) \) ([13, Corollary 3.3.5]). Since \( \text{ld}_S(S(N)) \leq \text{proj} \dim_S(S(N)) \) (the inequality is strict quite often), Theorem 3.4 refines their result. Our equality might follow from the argument in [13], which constructs a minimal \( E \)-free resolution of \( N \) from a minimal \( S \)-free resolution of \( S(N) \). But it seems that certain amount of computation will be required.

Theorem 3.4 suggests that we may set

\[
\text{ld}(\Delta) := \text{ld}_S(K[\Delta]) = \text{ld}_E(K(\Delta)).
\]

**Theorem 3.6.** If \( I_\Delta \neq (0) \) (equivalently, \( \Delta \neq 2^{[n]} \)), then \( \text{ld}_S(I_\Delta) \) is a topological invariant of the geometric realization \( |\Delta^\vee| \) of the Alexander dual \( \Delta^\vee \) of \( \Delta \). If \( \Delta \neq 2^T \) for any \( T \subset [n] \), then \( \text{ld}(\Delta) \) is also a topological invariant of \( |\Delta^\vee| \) (also independent from the number \( n = \text{dim} S \)).

*Proof.* Since \( A(I_\Delta) = K[\Delta^\vee] \) and \( \Delta^\vee \neq \emptyset \), the first assertion follows from Theorem 2.10 and the equality (3.1) in the proof of Proposition 3.3.

It is easy to see that \( \Delta \neq 2^T \) for any \( T \) if and only if \( \text{ld}(\Delta) \geq 1 \). If this is the case, \( \text{ld}(\Delta) = \text{ld}_S(I_\Delta) + 1 \), and the second assertion follows from the first. \( \square \)

**Remark 3.7.** (1) For the first statement of Theorem 3.6, the assumption that \( I_\Delta \neq (0) \) is necessary. In fact, if \( I_\Delta = (0) \), then \( \Delta = 2^{[n]} \) and \( \Delta^\vee = \emptyset \). On the other hand, if we set \( \Gamma := 2^{[n]} \setminus [n] \), then \( \Gamma^\vee = \{\emptyset\} \) and \( \text{dim} \Gamma^\vee = \emptyset = |\Delta^\vee| \). In view of Proposition 3.3, it might be natural to set \( \text{ld}_S(I_\Delta) = \text{ld}_S(\emptyset) = -\infty \). But, \( I_\Gamma = \omega_S \) and hence \( \text{ld}_S(I_\Gamma) = 0 \). One might think it is better to set \( \text{ld}_S(\emptyset) = 0 \) to avoid the problem. But this convention does not help so much, if we consider \( K[\Delta] \) and \( K[\Gamma] \). In fact, \( \text{ld}_S(K[\Delta]) = \text{ld}_S(S) = 0 \) and \( \text{ld}_S(K[\Gamma]) = \text{ld}_S(S/\omega_S) = 1 \).

(2) Let us think about the second statement of the theorem. Even if we forget the assumption that \( \Delta \neq 2^T \), \( \text{ld}(\Delta) \) is almost a topological invariant. Under the assumption that \( I_\Delta \neq 0 \), we have the following.
• \( \text{ld}(\Delta) \leq 1 \) if and only if \( K[\Delta^\vee] \) is sequentially Cohen-Macaulay. Hence we can determine whether \( \text{ld}(\Delta) \leq 1 \) from the topological space \( |\Delta^\vee| \).
• \( \text{ld}(\Delta) = 0 \), if and only if all facets of \( \Delta^\vee \) have dimension \( n - 2 \), if and only if \( |\Delta^\vee| \) is Cohen-Macaulay and has dimension \( n - 2 \).

Hence, if we forget the number “\( n \)”, we can not determine whether \( \text{ld}(\Delta) = 0 \) from \( |\Delta^\vee| \).

4. AN UPPER BOUND OF LINEARITY DEFECTS.

In the previous section, we have seen that \( \text{ld}_E(N) = \text{ld}_S(S(N)) \) for \( N \in \text{Sq}(E) \), in particular \( \text{ld}_E(K(\Delta)) = \text{ld}_S(K[\Delta]) \) for a simplicial complex \( \Delta \). In this section, we will give an upper bound of them, and see that the bound is sharp.

For \( 0 \neq N \in \text{mod } E \), regarding \( N \) as a \( \mathbb{Z} \)-graded module, we set \( \text{indeg}_E(N) := \min\{ i \mid N_i \neq 0 \} \), which is called the initial degree of \( N \), and \( \text{indeg}_S(M) \) is similarly defined as \( \text{indeg}_S(M) := \min\{ i \mid M_i \neq 0 \} \) for \( 0 \neq M \in \text{mod } S \). If \( \Delta \neq 2^{[n]} \) (equivalently \( I_\Delta \neq 0 \) or \( J_\Delta \neq 0 \)), then we have \( \text{indeg}_S(I_\Delta) = \text{indeg}_E(J_\Delta) = \min\{ \sharp F \mid F \subset [n], F \not\subset \Delta \} \), where \( \sharp F \) denotes the cardinal number of \( F \). So we set

\[
\text{indeg}(\Delta) := \text{indeg}_S(I_\Delta) = \text{indeg}_E(J_\Delta).
\]

Since \( \text{ld}(2^{[n]}) = \text{ld}_S(S) = \text{ld}_E(E) = 0 \) holds, we henceforth exclude this trivial case; we assume that \( \Delta \neq 2^{[n]} \).

We often make use of the following facts:

**Lemma 4.1.** Let \( 0 \neq M \in \text{mod } S \) and let \( P_* \) be a minimal graded free resolution of \( M \). Then

1. \( \text{lin}_i(P_*) = 0 \) for all \( i < \text{indeg}_S(M) \), i.e., there are only \( l \)-linear strands with \( l \geq \text{indeg}_S(M) \) in \( P_* \);
2. \( \text{lin}_{\text{indeg}_S(M)}(P_*) \) is a subcomplex of \( P_* \);
3. if \( M \in \text{Sq}(S) \), then \( \text{lin}(P_*) = \bigoplus_{0 \leq l \leq n} \text{lin}_l(P_*) \), and \( \text{lin}(P_*)_i = 0 \) for all \( i > n - l \) and all \( 0 \leq l \leq n \), where the subscript \( i \) is a homological degree.

**Proof.** (1) and (2) are clear. (3) holds from the fact that \( P_i \cong \bigoplus_{F \subset [n]} S(-F)^{\beta_i,F} \).

**Theorem 4.2.** For \( 0 \neq N \in \text{Sq}(E) \), it follows that

\[
\text{ld}_E(N) \leq \max\{0, n - \text{indeg}_E(N) - 1\}.
\]

By Theorem 3.4, this is equivalent to say that for \( M \in \text{Sq}(S) \),

\[
\text{ld}_S(M) \leq \max\{0, n - \text{indeg}_S(M) - 1\}.
\]

**Proof.** It suffices to show the assertion for \( M \in \text{Sq}(S) \). Set \( \text{indeg}_S(M) = d \) and let \( P_* \) be a minimal graded free resolution of \( M \). The case \( d = n \) is trivial by Lemma 4.1 (1), (3). Assume that \( d \leq n - 1 \). Observing that \( \text{lin}(P_*)_i = S(-l - i)^{\beta_{i+l},l} \)

where \( \beta_{i+l} \) are \( \mathbb{Z} \)-graded Betti numbers of \( M \), Lemma 4.1 (1), (3) implies that the last few steps of \( P_* \) are of the form

\[
0 \longrightarrow S(-n)^{\beta_{n-d,n}} \longrightarrow S(-n)^{\beta_{n-d-1,n}} \oplus S(-n+1)^{\beta_{n-d-1,n-1}} \longrightarrow \ldots
\]
Hence \( \operatorname{lin}_d(P_\bullet)_{n-d} = S(-n)^{\beta_{n-d,n}} = P_{n-d} \). Since \( \operatorname{lin}_d(P_\bullet) \) is a subcomplex of the acyclic complex \( P_\bullet \) by Lemma 4.1 (2), we have \( H_{n-d}(\operatorname{lin}_d(P_\bullet)) = 0 \), so that \( \operatorname{ld}_S(M) \leq n - d - 1 \). \( \square \)

Note that \( J_\Delta \in \text{Sq}(E) \) (resp. \( I_\Delta \in \text{Sq}(S) \)). Since \( \operatorname{ld}(\Delta) \leq \operatorname{ld}_E(J_\Delta) + 1 \) (resp. \( \operatorname{ld}(\Delta) \leq \operatorname{ld}_S(I_\Delta) + 1 \)) holds, we have a bound for \( \operatorname{ld}(\Delta) \), applying Theorem 4.2 to \( J_\Delta \) (resp. \( I_\Delta \)).

**Corollary 4.3.** For a simplicial complex \( \Delta \) on \([n]\), we have

\[
\operatorname{ld}(\Delta) \leq \max\{1, n - \operatorname{indeg}(\Delta)\}.
\]

Let \( \Delta, \Gamma \) be simplicial complexes on \([n]\). We denote \( \Delta * \Gamma \) for the join

\[
\{ F \cup G \mid F \in \Delta, G \in \Gamma \}
\]

of \( \Delta \) and \( \Gamma \), and for our convenience, set

\[
\operatorname{ver}(\Delta) := \{ v \in [n] \mid \{v\} \in \Delta \}.
\]

**Lemma 4.4.** Let \( \Delta \) be a simplicial complex on \([n]\). Assume that \( \operatorname{indeg}(\Delta) = 1 \), or equivalently \( \operatorname{ver}(\Delta) \neq [n] \). Then we have

\[
\operatorname{ld}(\Delta) = \operatorname{ld}(\Delta * \{v\})
\]

for \( v \in [n] \setminus \operatorname{ver}(\Delta) \).

**Proof.** We may assume that \( v = 1 \). Let \( P_\bullet \) be a minimal graded free resolution of \( K[\Delta * \{1\}] \) and \( \mathcal{K}(x_1) \) the Koszul complex

\[
0 \longrightarrow S(-1) \xrightarrow{x_1} S \longrightarrow 0
\]

with respect to \( x_1 \). Consider the mapping cone \( P_\bullet \otimes_S \mathcal{K}(x_1) \) of the map \( P_\bullet(-1) \xrightarrow{x_1} P_\bullet \). There is the short exact sequence

\[
0 \longrightarrow P_\bullet \longrightarrow P_\bullet \otimes_S \mathcal{K}(x_1) \longrightarrow P_\bullet(-1)[-1] \longrightarrow 0,
\]

whence we have \( H_i(P_\bullet \otimes_S \mathcal{K}(x_1)) = 0 \) for all \( i \geq 2 \) and the exact sequence

\[
0 \longrightarrow H_1(P_\bullet \otimes_S \mathcal{K}(x_1)) \longrightarrow H_0(P_\bullet(-1)) \xrightarrow{x_1} H_0(P_\bullet).
\]

But since \( H_0(P_\bullet) = K[\Delta * \{1\}] \) and \( x_1 \) is regular on it, we have \( H_1(P_\bullet \otimes_S \mathcal{K}(x_1)) = 0 \). Thus \( P_\bullet \otimes_S \mathcal{K}(x_1) \) is acyclic and hence a minimal graded free resolution of \( K[\Delta] \).

Note that \( \operatorname{lin}(P_\bullet \otimes_S \mathcal{K}(x_1)) = \operatorname{lin}(P_\bullet) \otimes_S \mathcal{K}(x_1) \): in fact, we have

\[
\operatorname{lin}(P_\bullet \otimes_S \mathcal{K}(x_1))_i = \operatorname{lin}(P_\bullet \otimes_S S)_i \oplus \operatorname{lin}(P_\bullet[-1] \otimes_S S(-1))_i
\]

\[
= (\operatorname{lin}(P_\bullet)_i \otimes_S S) \oplus (\operatorname{lin}(P_\bullet)_{i-1} \otimes_S S(-1))
\]

\[
= (\operatorname{lin}(P_\bullet) \otimes_S \mathcal{K}(x_1))_i,
\]

where the subscripts \( i \) denote homological degrees, and the differential map

\[
\operatorname{lin}(P_\bullet \otimes_S \mathcal{K}(x_1))_i \longrightarrow \operatorname{lin}(P_\bullet \otimes_S \mathcal{K}(x_1))_{i-1}
\]

is composed by \( \partial_i^{(l)} \), \( -\partial_i^{(l)}_{i-1} \), and the multiplication map by \( x_1 \), where \( \partial_i^{(l)} \) (resp. \( \partial_i^{(l)}_{i-1} \)) is the \( i \)th (resp. \( (i-1) \)th) differential map of the \( l \)-linear strand of \( P_\bullet \). Hence there is the short exact sequence
0 \longrightarrow \text{lin}(P_\bullet) \longrightarrow \text{lin}(P_\bullet \otimes_S K(x_1)) \longrightarrow \text{lin}(P_\bullet)(-1)[-1] \longrightarrow 0,

which yields that \( H_i(\text{lin}(P_\bullet \otimes_S K(x_1))) = 0 \) for all \( i \geq \text{ld}(\Delta_* \{1\}) + 2 \), and the exact sequence

\[
0 \longrightarrow H_{\text{ld}(\Delta_* \{1\}) + 1}(\text{lin}(P_\bullet \otimes_S K(x_1))) \longrightarrow H_{\text{ld}(\Delta_* \{1\})}(\text{lin}(P_\bullet(-1)))
\]

\[
\xrightarrow{x_1} H_{\text{ld}(\Delta_* \{1\})}(\text{lin}(P_\bullet)) \longrightarrow H_{\text{ld}(\Delta_* \{1\})}(\text{lin}(P_\bullet \otimes_S K(x_1)))).
\]

Since \( x_1 \) does not appear in any entry of the matrices representing the differentials of \( \text{lin}(P_\bullet) \), it is regular on \( H_*(\text{lin}(P_\bullet)) \), and hence we have

\[
H_{\text{ld}(\Delta_* \{1\}) + 1}(\text{lin}(P_\bullet \otimes_S K(x_1))) = 0
\]

and

\[
H_{\text{ld}(\Delta_* \{1\})}(\text{lin}(P_\bullet \otimes_S K(x_1)))) \neq 0,
\]

since \( H_{\text{ld}(\Delta_* \{1\})}(\text{lin}(P_\bullet)) \neq 0 \). Therefore \( \text{ld}(\Delta) = \text{ld}(\Delta_* \{1\}) \). \( \Box \)

Let \( \Delta \) be a simplicial complex on \([n]\). For \( F \subset [n] \), we set

\[
\Delta_F := \{ G \in \Delta \mid G \subset F \}.
\]

The following fact, due to Hochster, is well known, but because of our frequent use, we mention it.

**Proposition 4.5** (c.f. [2, 15]). For a simplicial complex \( \Delta \) on \([n]\), we have

\[
\beta_{i,j}(K[\Delta]) = \sum_{F \subset [n], \sharp F = j} \dim K \tilde{H}_{j-i-1}(\Delta_F; K),
\]

where \( \beta_{i,j}(K[\Delta]) \) are the \( \mathbb{Z} \)-graded Betti numbers of \( K[\Delta] \).

Now we can give a new proof of [20, Proposition 4.15], which is the latter part of the next result.

**Proposition 4.6** (cf. [20, Proposition 4.15]). Let \( \Delta \) be a simplicial complex on \([n]\). If \( \text{ind deg} \Delta = 1 \), then we have

\[
\text{ld}(\Delta) \leq \max\{1, n - 3\}.
\]

Hence, for any \( \Delta \), we have

\[
\text{ld}(\Delta) \leq \max\{1, n - 2\}.
\]

**Proof.** The second inequality follows from the first one and Corollary 4.3. So it suffices to show the first. We set \( V := [n] \setminus \text{ver}(\Delta) \). Our hypothesis \( \text{ind deg} \Delta = 1 \) implies that \( V \neq \emptyset \). By Lemma 4.4, the proof can be reduced to the case \( \# V = 1 \).

We may then assume that \( V = \{1\} \). Thus we have only to show that \( \text{ld}(\Delta_* \{1\}) \leq \max\{1, n - 3\} \). Since we have \( \text{ind deg}(\Delta_* \{1\}) \geq 2 \), we may assume \( n \geq 4 \) by Corollary 4.3. The length of the 0-linear strand of \( K[\Delta_* \{1\}] \) is 0, and hence we concentrate on the \( l \)-linear strands with \( l \geq 1 \). Let \( P_\bullet \) be a minimal graded free resolution of \( K[\Delta_* \{1\}] \). Since, as is well known, the cone of a simplicial complex, i.e., the join with a point, is acyclic, we have

\[
\beta_{i,n}(K[\Delta_* \{1\}]) = \dim_K \tilde{H}_{n-i-1}(\Delta_* \{1\}; K) = 0
\]
by Proposition 4.5. Thus \( \dim(P_\bullet)_{n-l} = 0 \) for all \( l \geq 1 \). Now applying the same argument as the last part of the proof of Theorem 4.2 (but we need to replace \( n \) by \( n - 1 \)), we have

\[
H_{n-2}(\lin(P_\bullet)) = 0,
\]
and so \( \ld(\Delta \ast \{1\}) \leq n - 3 \).

According to [20 Proposition 4.14], we can construct a squarefree module \( N \in \Sq(E) \) with \( \ld_E(N) = \proj \dim_S(S(N)) = n - 1 \). By Theorems 3.4 and 4.2, \( M := S(N) \) satisfies that \( \indeg_S(M) = 0 \) and \( \ld_S(M) = n - 1 \). For \( 0 \leq i \leq n - 1 \), let \( \Omega_i(M) \) be the \( i \)th syzygy of \( M \). Then \( \Omega_i(M) \) is squarefree, and we have that \( \ld_S(\Omega_i(M)) = \ld_S(M) - i = n - i - 1 \) and \( \indeg_S(\Omega_i(M)) \geq \indeg_S(M) + i = i \). Thus by Theorem 4.2, we know that \( \indeg_S(\Omega_i(M)) = i \) and \( \ld_S(\Omega_i(M)) = n - \indeg_S(\Omega_i(M)) - 1 \). So the bound in Theorem 4.2 is optimal.

In the following, we will give an example of a simplicial complex \( \Delta \) with \( \ld(\Delta) = n - \indeg(\Delta) \) for \( 2 \leq \indeg(\Delta) \leq n - 2 \), and so we know the bound in Proposition 4.3 is optimal if \( \indeg(\Delta) \geq 2 \), that is, \( \text{ver}(\Delta) = [n] \).

Given a simplicial complex \( \Delta \) on \([n]\), we denote \( \Delta^{(i)} \) for the \( i \)th skeleton of \( \Delta \), which is defined as

\[
\Delta^{(i)} := \{ F \in \Delta \mid \#F \leq i + 1 \}.
\]

**Example 4.7.** Set \( \Sigma := 2^{[n]} \), and let \( \Gamma \) be a simplicial complex on \([n]\) whose geometric realization \( |\Gamma| \) is homeomorphic to the \((d-1)\)dimensional sphere with \( 2 \leq d < n - 1 \), which we denote by \( S^{d-1} \). (For \( m > d \) there exists a triangulation of \( S^{d-1} \) with \( m \) vertices. See, for example, [2 Proposition 5.2.10]). Consider the simplicial complex \( \Delta := \Gamma \cup \Sigma^{(d-2)} \). We will verify that \( \Delta \) is a desired complex, that is, \( \ld(\Delta) = n - \indeg(\Delta) \). For brief notation, we put \( t := \indeg \Delta \) and \( l := \ld(\Delta) \). First, from our definition, it is clear that \( t \geq d \). Thus it is enough to show that \( n - d \leq l \): in fact we have that \( l \leq n - t \leq n - d \leq l \) by Corollary 4.3 and hence that \( t = d \) and \( l = n - d \). Our aim is to prove that

\[
\beta_{n-d,n}(K[\Delta]) \neq 0 \quad \text{and} \quad \beta_{n-d-1,n-1}(K[\Delta]) = 0,
\]

since, in this case, we have \( H_{n-d}(\lin_d(P_\bullet)) \neq 0 \), and hence \( n - d \leq l \).

Now, let \( F \subset [n] \), and \( \tilde{C}_\bullet(\Delta_F; K) \), \( \tilde{C}_\bullet(\Gamma_F; K) \) be the augmented chain complexes of \( \Delta_F \) and \( \Gamma_F \), respectively. Since \( \Sigma^{(d-2)} \) have no faces of dimension \( \geq d - 1 \), we have \( \tilde{C}_{d-1}(\Delta_F; K) = \tilde{C}_{d-1}(\Gamma_F; K) \) and hence \( \tilde{H}_{d-1}(\Delta_F; K) = \tilde{H}_{d-1}(\Gamma_F; K) \). On the other hand, our assumption that \( |\Gamma| \approx S^{d-1} \) implies that \( \Gamma \) is Gorenstein, and hence that

\[
\tilde{H}_{d-1}(\Gamma_F; K) = \begin{cases} K & \text{if } F = [n]; \\ 0 & \text{otherwise.} \end{cases}
\]

Therefore, by Proposition 4.5, we have that

\[
\beta_{n-d,n}(K[\Delta]) = \dim_K \tilde{H}_{d-1}(\Gamma; K) = 1 \neq 0;
\]

\[
\beta_{n-d-1,n-1}(K[\Delta]) = \sum_{F \subset [n], |F| = n-1} \dim_K \tilde{H}_{d-1}(\Gamma_F; K) = 0.
\]
5. A simplicial complex $\Delta$ with $\text{ld}(\Delta) = n-2$ is an $n$-gon

Following the previous section, we assume that $\Delta \neq [n]$, throughout this section. We say a simplicial complex on $[n]$ is an $n$-gon if its facets are $\{1, 2\}, \{2, 3\}, \cdots, \{n-1, n\}$, and $\{n, 1\}$ after a suitable permutation of vertices. Consider the simplicial complex $\Delta$ on $[n]$ given in Example 4.7. If we set $d = 2$, then $\Delta$ is an $n$-gon. Thus if a simplicial complex $\Delta$ on $[n]$ is an $n$-gon, we have $\text{ld}(\Delta) = n-2$. Actually, the inverse holds, that is, if $\text{ld}(\Delta) = n-2$ with $n \geq 4$, $\Delta$ is nothing but an $n$-gon.

**Theorem 5.1.** Let $\Delta$ be a simplicial complex on $[n]$ with $n \geq 4$. Then $\text{ld}(\Delta) = n-2$ if and only if $\Delta$ is an $n$-gon.

In the previous section, we introduced Hochster’s formula (Proposition 4.5), but in this section, we need explicit correspondence between $\text{Tor}^S(K[\Delta], K)_F$ and reduced cohomologies of $\Delta$.

**Lemma 5.2.** Let $\Delta$ be a simplicial complex on $[n]$ with $\text{indeg}(\Delta) \geq 2$, and $P_\bullet$ a minimal graded free resolution of $K[\Delta]$. We denote $Q_\bullet$ for the subcomplex of $P_\bullet$ such that $Q_i := \bigoplus_{j \leq i+1} S(-j)^{\delta_{i,j}} \subset \bigoplus_{j \in \mathbb{Z}} S(-j)^{\delta_{i,j}} = P_i$. Assume $n \geq 4$. Then the following are equivalent.

1. $\text{ld}(\Delta) = n-2$;
2. $H_{n-2} (\text{lin}_2(P_\bullet)) \neq 0$;
3. $H_{n-3}(Q_\bullet) \neq 0$.

In the case $n \geq 5$, the condition (3) is equivalent to $H_{n-3}(\text{lin}_1(P_\bullet)) \neq 0$.

**Proof.** Since $\text{indeg}(\Delta) \geq 2$, $\text{lin}_0(P_\bullet)_i = 0$ holds for $i \geq 1$. Clearly, $H_i(Q_\bullet) = H_i(\text{lin}_1(P_\bullet))$ for $i \geq 2$. Since $\text{lin}_1(P_\bullet)_i = 0$ for $i \geq n-2$ and $l \geq 3$ by Lemma 4.1 and that $\text{ld}(\Delta) \leq n-2$ by Proposition 4.6, it suffices to show the following.

(5.2) $H_{n-2} (\text{lin}_2(P_\bullet)) \cong H_{n-3}(Q_\bullet)$ and $H_i(Q_\bullet) = 0$ for $i \geq n-2$. 
Since $Q_\bullet$ is a subcomplex of $P_\bullet$, there exists the following short exact sequence of complexes.

$$0 \to Q_\bullet \to P_\bullet \to \tilde{P}_\bullet := P_\bullet / Q_\bullet \to 0,$$

which induces the exact sequence of homology groups

$$H_i(P_\bullet) \to H_i(\tilde{P}_\bullet) \to H_{i-1}(Q_\bullet) \to H_{i-1}(P_\bullet).$$

Hence the acyclicity of $P_\bullet$ implies that $H_i(\tilde{P}_\bullet) \cong H_{i-1}(Q_\bullet)$ for all $i \geq 2$. Now $H_i(\tilde{P}_\bullet) = 0$ for $i \geq n - 1$ by Lemma 4.1 and the fact that $\tilde{P}_i = \oplus_{l \geq 2} \text{lin}_l(P_\bullet)$. So the latter assertion of (5.2) holds, since $n - 2 \geq 2$. The former follows from the equality $H_{n-2}(\tilde{P}_\bullet) = H_{n-2}(\text{lin}_2(P_\bullet))$, which is a direct consequence of the fact that $\text{lin}_2(P_\bullet)$ is a subcomplex of $\tilde{P}_\bullet$, that $\tilde{P}_{n-2} = \text{lin}_2(P_{n-2})$, and that $\tilde{P}_{n-1} = 0$. 

Let $\Delta$ be a 1-dimensional simplicial complex on $[n]$ (i.e., $\Delta$ is essentially a simple graph). A cycle $C$ in $\Delta$ of length $t \geq 3$ is a sequence of edges of $\Delta$ of the form $(v_1, v_2), (v_2, v_3), \ldots, (v_t, v_1)$ joining distinct vertices $v_1, \ldots, v_t$.

Now we are ready for the proof of Theorem 5.1.

**Proof of Theorem 5.1.** The implication “$\Leftarrow$” has been already done in the beginning of this section. So we shall show the inverse. By Proposition 4.6 we may assume that $\text{indeg}(\Delta) \geq 2$. Let $P_\Delta$ be a minimal graded free resolution of $K[\Delta]$ and $Q_\bullet$ as in Lemma 5.2. Note that $Q_\bullet$ is determined only by $[I_\Delta]_2$ and that it follows $[I_\Delta]_2 = [I_{\Delta(2)}]_2$. If the 1-skeleton $\Delta^{(1)}$ of $\Delta$ is an $n$-gon, then so is $\Delta$ itself. Thus by Lemma 5.2 we may assume that $\dim \Delta = 1$. Since $\text{ld}(\Delta) = n - 2$, by Lemma 5.2 we have

$$\tilde{H}_1(\Delta; K) \cong \tilde{H}^1(\Delta; K) \cong [\text{Tor}_{n-2}^S(K[\Delta], K)][n] \neq 0,$$

and hence $\Delta$ contains at least one cycle as a subcomplex. So it suffices to show that $\Delta$ has no cycles of length $\leq n - 1$. Suppose not, i.e., $\Delta$ has some cycles of length $\leq n - 1$. To give a contradiction, we shall show

$$0 \to \text{lin}_2(P_\bullet)_{n-2} \to \text{lin}_2(P_\bullet)_{n-3}$$

is exact; in fact it follows $H_{n-2}(\text{lin}_2(P_\bullet)) = 0$, which contradicts to Lemma 5.2. For that, we need some observations (this is a similar argument to that done in Theorem 4.1 of [16]). Consider the chain complex $K[\Delta] \otimes_K \bigwedge V \otimes_K S$ where $V$ is the $K$-vector space with the basis $x_1, \ldots, x_n$. We can define two differential map $\vartheta, \partial$ on it as follows:

$$\vartheta(f \otimes \bigwedge^G x \otimes g) = \sum_{i \in G} (-1)^{\alpha(i,G)} (x_i f \otimes \bigwedge^{G\setminus\{i\}} x \otimes g);$$

$$\partial(f \otimes \bigwedge^G x \otimes g) = \sum_{i \in G} (-1)^{\alpha(i,G)} (f \otimes \bigwedge^{G\setminus\{i\}} x \otimes x_i g).$$

By a routine, we have that $\partial \vartheta + \vartheta \partial = 0$, and easily we can check that the $i$th homology group of the chain complex $(K[\Delta] \otimes_K \bigwedge V \otimes_K S, \vartheta)$ is isomorphic to the $i$th graded free module of a minimal free resolution $P_\bullet$ of $K[\Delta]$. Since, moreover, the
differential maps of \( \text{lin}(P_*) \) is induced by \( \partial \) due to Eisenbud-Goto \([5]\) and Herzog-Simis-Vasconcelos \([9]\), \( \text{lin}_i(P_*) \to \text{lin}_i(P_*)_{-1} \) can be identified with

\[
\bigoplus_{F \subset [n], 1 \leq F = i + l} \left[ \text{Tor}^S_i(K[\Delta], K) \right]_F \otimes_K S \xrightarrow{\delta} \bigoplus_{F \subset [n], 1 \leq F = i - 1 - l} \left[ \text{Tor}^S_{i-1}(K[\Delta], K) \right]_F \otimes_K S,
\]

where \( \delta \) is induced by \( \partial \). In the sequel, \(-\{i\}\) denotes the subset \([n] \setminus \{i\}\) of \([n]\). Then we may identify the sequence (5.3) with

\[
0 \to [\text{Tor}^S_{n-2}(K[\Delta], K)]_{\{i\}} \otimes_K S \xrightarrow{\tilde{\eta}} \bigoplus_{i \in [n]} [\text{Tor}^S_{n-3}(K[\Delta], K)]_{-\{i\}} \otimes_K S
\]

and hence, by the isomorphism (5.1), with

\[
(5.4) \quad 0 \to \tilde{H}^1(\Delta; K) \otimes_K S \xrightarrow{\tilde{\varepsilon}} \bigoplus_{i \in [n]} \tilde{H}^1(\Delta_{-\{i\}}; K) \otimes_K S.
\]

Here \( \tilde{\varepsilon} \) is composed by \( \tilde{\varepsilon}_i : \tilde{H}^1(\Delta; K) \otimes_K S \to \tilde{H}^1(\Delta_{-\{i\}}; K) \otimes_K S \) which is induced by the chain map

\[
\varepsilon_i : \tilde{C}^*(\Delta; K) \otimes_K S \to \tilde{C}^*(\Delta_{-\{i\}}; K) \otimes_K S,
\]

\[
\varepsilon_i(e_G^* \otimes 1) = \begin{cases} (-1)^{\alpha(i,G)} e_G^* \otimes x_i & \text{if } i \notin G; \\ 0 & \text{otherwise}. \end{cases}
\]

Well, let \( C \) be a cycle in \( \Delta \) of the form \((v_1, v_2), (v_2, v_3), \ldots, (v_i, v_1)\) with distinct vertices \( v_1, \ldots, v_i \). We say \( C \) has a chord if there exists an edge \((v_i, v_j)\) of \( G \) such that \( j \neq i + 1 \) (mod \( t \)), and \( C \) is said to be minimal if it has no chord. It is easy to see that the 1\textsuperscript{st} homology of \( \Delta \) is generated by those of minimal cycles contained in \( \Delta \), that is, we have the surjective map:

\[
\bigoplus_{C \subset \Delta, \text{minimal cycle}} \tilde{H}_1(C; K) \to \tilde{H}_1(\Delta; K).
\]

Now by our assumption that \( \Delta \) contains a cycle of length \( \leq n - 1 \) (that is, \( \Delta \) itself is not a minimal cycle), we have the surjective map

\[
(5.5) \quad \bigoplus_{i \in [n]} \tilde{H}_1(\Delta_{-\{i\}}; K) \xrightarrow{\tilde{\eta}} \tilde{H}_1(\Delta; K)
\]

where \( \tilde{\eta} \) is induced by the chain map \( \eta : \bigoplus \tilde{C}_*(\Delta_{-\{i\}}; K) \to \tilde{C}_*(\Delta; K) \), and \( \eta \) is the sum of

\[
\eta_h : \tilde{C}_*(\Delta_{-\{i\}}; K) \ni e_G \mapsto (-1)^{\alpha(i,G)} e_G \in \tilde{C}_*(\Delta; K).
\]

Taking the \( K \)-dual of (5.5), we have the injective map

\[
\tilde{H}_1(\Delta; K) \xrightarrow{\eta^*} \bigoplus_{i \in [n]} \tilde{H}_1(\Delta_{-\{i\}}; K),
\]

where \( \eta^* \) is the \( K \)-dual map of \( \tilde{\eta} \), and composed by the \( K \)-dual

\[
\tilde{\eta}_i^* : \tilde{H}_1(\Delta; K) \to \tilde{H}_1(\Delta_{-\{i\}}; K)
\]
of \( \bar{\eta}_i \). Then for all \( 0 \neq z \in \tilde{H}^1(\Delta; K) \), we have \( \bar{\eta}_i^*(z) \neq 0 \) for some \( i \). Recalling the map \( \bar{\varepsilon} : \tilde{H}^1(\Delta; K) \otimes_K S \to \bigoplus \tilde{H}^1(\Delta_{-\{i\}}; K) \otimes_K S \) in (5.4) and its construction, we know for \( z \in \tilde{H}^1(\Delta; K) \),

\[
\bar{\varepsilon}(z \otimes y) = \sum_{i=1}^{n} \bar{\eta}_i^*(z) \otimes x_i y,
\]

and hence \( \bar{\varepsilon} \) is injective. \( \square \)

Remark 5.3. (1) If \( \Delta \) is an \( n \)-gon, then \( \Delta^\vee \) is an \((n-3)\)-dimensional Buchsbaum complex with \( \tilde{H}_{n-4}(\Delta^\vee; K) = K \). If \( n = 5 \), then \( \Delta^\vee \) is a triangulation of the Möbius band. But, for \( n \geq 6 \), \( \Delta^\vee \) is not a homology manifold. In fact, let \( \{1,2\}, \{2,3\}, \ldots, \{n-1,n\}, \{n,1\} \) be the facets of \( \Delta \), then if \( F = [n] \setminus \{1,3,5\} \), easy computation shows that \( \text{lk}_{\Delta^\vee} F \) is a 0-dimensional complex with 3 vertices, and hence \( \tilde{H}_0(\text{lk}_{\Delta^\vee} F; K) = K^2 \).

(2) If \( \text{indeg} \Delta \geq 3 \), then the simplicial complexes given in Example 4.7 are not the only examples which attain the equality \( \text{ld}(\Delta) = n - \text{indeg}(\Delta) \). We shall give two examples of such complexes.

Let \( \Delta \) be the triangulation of the real projective plane \( \mathbb{P}^2 \mathbb{R} \) with 6 vertices which is given in [2] figure 5.8, p.236. Since \( \mathbb{P}^2 \mathbb{R} \) is a manifold, \( K[\Delta] \) is Buchsbaum. Hence we have \( H^2_m(K[\Delta]) = [H^2_m(K[\Delta])]_0 \cong H_1(\Delta; K) \). So, if \( \text{char}(K) = 2 \), then we have \( \text{depth}_S(\text{Ext}^4_S(K[\Delta], \omega_S)) = 0 \). Note that we have \( \Delta = \Delta^\vee \) in this case. Therefore, easy computation shows that

\[
\text{ld}(\Delta^\vee) = \text{ld}(\Delta) = 3 = 6 - 3 = 6 - \text{indeg}(\Delta).
\]

Next, as is well known, there is a triangulation of the torus with 7 vertices. Let \( \Delta \) be the triangulation. Since \( \dim \Delta = 2 \), we have \( \text{indeg}(\Delta^\vee) = 7 - \dim \Delta - 1 = 4 \). Observing that \( K[\Delta] \) is Buchsbaum, we have, by easy computation, that

\[
\text{ld}(\Delta^\vee) = 3 = 7 - 4 = 7 - \text{indeg}(\Delta^\vee).
\]

Thus \( \Delta^\vee \) attains the equality, but is not a simplicial complex given in Example 4.7 since it follows, from Alexander’s duality, that

\[
\dim_K \tilde{H}_i(\Delta^\vee; K) = \dim_K \tilde{H}_{4-i}(\Delta; K) = \begin{cases} 2 \neq 1 & \text{for } i = 3; \\ 0 & \text{for } i \geq 4. \end{cases}
\]

More generally, the dual complexes of \( d \)-dimensional Buchsbaum complexes \( \Delta \) with \( \tilde{H}_{d-1}(\Delta; K) \neq 0 \) satisfy the equality

\[
\text{ld}(\Delta^\vee) = n - \text{indeg}(\Delta^\vee),
\]

but many of them differ from the examples in Example 4.7 and we can construct such complexes more easily as \( \text{indeg}(\Delta^\vee) \) is larger.

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