Black Hole with Non-Commutative Hair

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Abstract

The specific nonlinear vector $\sigma$-model coupled to Einstein gravity is investigated. The model arises in the studies of the gravitating matter in non-commutative geometry. The static spherically symmetric spacetimes are identified by direct solving of the field equations. The asymptotically flat black hole with the “non-commutative” vector hair appears for the special choice of the integration constants, giving thus another counterexample to the famous “no-hair” theorem.

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1. Introduction

The black-hole solutions of the Einstein equations have constituted one of the most fascinating features, with which Einstein’s general relativity has enriched the theoretical physics. The curvature singularities and the event horizons have been qualitatively novel structures, the role of which in our world is still to be clarified, in particular in quantum context. Moreover, the black holes are in a sense exclusive, as the famous “no-hair” theorem says. The developments in eighties led to considerable interest in the various “exotic” gravitating matter theories. For example, the important role of the classical solutions of the pure Yang-Mills theory had to be elucidated from the point of view of the coupling to gravity and the string theory effective action also incorporates the matter described by the higher spin fields. As the product of those efforts, the possibility of the higher spin hair in the Einstein-Yang-Mills, Einstein-Skyrme and Einstein-axion equations was discovered \[1, 2, 3\].

In this contribution, we give another “counterexample” to the no-hair theorem, showing explicitly the black hole supporting the “non-commutative” hair in 3+1 dimensions. The action of the system is that of the nonlinear vector $\sigma$-model coupled to the Einstein gravity. The explicit form of the action is inspired, in fact, by the recent studies of the gravitating matter in the non-commutative geometry. The non-commutative pure Einstein-Hilbert action for the Connes’ double-sheeted manifolds gives the specific nonlinear vector $\sigma$-model coupled to the Einstein gravity. This theory then possesses the black hole solution with the $\sigma$-model vector field hair.

In what follows, we present the detailed way of solving of the field equations for the static spherically symmetric ansatz. We pick up the black hole solution and provide a detailed analysis of the curvature tensor and regularity condition for the hair at the asymptotic region, near the horizon and below the horizon.
2. The Model and the Solutions of the Field Equations

We wish to find static spherically symmetric solutions of the gravitating non-linear vector $\sigma$-model with the (non-commutative Einstein-Hilbert) action \[ I = \int_Y d^4x \sqrt{-g} \left[ 2R + Q^{\alpha\beta\gamma\delta}(V) D_\alpha V_\beta D_\gamma V_\delta \right], \] (2.1)

where \[ Q^{\alpha\beta\gamma\delta}(V) = \frac{4}{(V^2)^3} (V^\alpha V^\beta V^\gamma V^\delta - g^{\alpha\beta} V^\gamma V^\delta V^2 - g^{\gamma\beta} V^\alpha V^\delta V^2), \] (2.2)
and $D_\alpha$ is the covariant derivative. This vector $\sigma$-model action can be rewritten as \[ I = \int_Y d^4x \sqrt{-g} \left[ 2R + 4\kappa D_\beta \left( \frac{V^\alpha V^\beta}{\sqrt{V^2}} \right) D_\alpha \left( \frac{1}{\sqrt{V^2}} \right) \right], \] (2.3)

with $\kappa = 1$. In what follows, we shall consider a general gravitational constant $\kappa > 0$. Let us introduce the Schwarzschild-like coordinates, in which the metric has a form \[ ds^2 = -e^\nu(r) dt^2 + e^\lambda(r) dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \]

We shall look for the solutions with $V^0$ and $V^1$ as the only non-zero components (0 and 1 correspond to $t$ and $r$ respectively). Assume also that $V^2$ is positive. Then $V^\alpha$ can be renormalized, introducing new functions $f^\alpha(r)$, $\sigma(r)$ \[ f^\alpha = \frac{V^\alpha}{\sqrt{V^2}}, \quad \sigma = \frac{1}{\sqrt{V^2}}. \]

Now, the action can be rewritten, using a Lagrange multiplier $\Lambda$ \[ I = \int_Y d^4x \sqrt{-g} \left[ 2R - 4\kappa f^\alpha f^\beta \frac{D_\alpha D_\beta \sigma}{\sigma} - 4\kappa \Lambda (f^\alpha f_{\alpha} - 1) \right]. \]

Variation of this action with respect to all variables yields the equations of motion for this model \[ f^\alpha f^\beta g_{\alpha\beta} = 1, \]
\[ f^\alpha \left( \frac{D_\mu D_\alpha \sigma}{\sigma} + \Lambda g_{\mu\alpha} \right) = 0, \]
\[ D_\beta \left( 2 f^\alpha f^\beta \frac{D_\alpha \sigma}{\sigma} - D_\alpha (f^\alpha f^\beta) \right) = 0, \] (2.4)
\[ R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = \kappa \left[ - g_{\mu \nu} \left( f^\alpha f^\beta \left( \frac{D_\alpha D_\beta \sigma}{\sigma} + \Lambda g_{\alpha \beta} \right) - \Lambda \right) + \right.
\]
\[ + 2 f^\alpha f_{[\mu} \left( \frac{D_\alpha D_{\nu]} \sigma}{\sigma} + \frac{D_\alpha \sigma D_{\nu]} \sigma}{\sigma^2} \right) + 2 f_{\mu \nu} \Lambda + \]
\[ + D^\alpha \left( f_{\mu \nu} \frac{D_\alpha \sigma}{\sigma} \right) - 2 D_\alpha (f^\alpha f_{[\nu} \frac{D_\mu \sigma}{\sigma}) \right]. \]

where \([\alpha \beta]\) means the symmetrization in the indices. In our ansatz these general equations acquire the following form (the equation (2.8) is in fact the first integral of (2.4))

\[- e^\nu f^0 f^0 + e^\lambda f^1 f^1 = 1, \tag{2.5} \]
\[
\frac{\sigma' \nu'}{\sigma} e^{-\lambda} + \Lambda = 0, \tag{2.6} \]
\[
\frac{1}{\sigma} \left( \sigma'' - \frac{\lambda'}{2 \sigma} \right) + e^\lambda \Lambda = 0, \tag{2.7} \]
\[
\frac{1}{r^2} - e^{-\lambda} \left( \frac{1}{r^2} - \frac{\lambda'}{r} \right) = \kappa \left[ f^0 f^0 \left( - \frac{\nu'}{2} e^{\nu-\lambda} \frac{\sigma'}{\sigma} - \Lambda e^\nu \right) + \right.
\]
\[ + f^1 f^1 \left( \frac{\sigma''}{\sigma} - \frac{\lambda'}{2 \sigma} + \Lambda e^\lambda \right) - \Lambda \right] + \]
\[ + \kappa \left( f^0 f^0 \right)' \frac{\sigma'}{\sigma} e^{\nu-\lambda} + \]
\[ + \kappa f^0 f^0 e^{\nu-\lambda} \left[ \frac{\sigma'}{\sigma} \left( \frac{5}{2} \nu' - \frac{\sigma'}{\sigma} - \frac{\lambda'}{2} + \frac{2}{r} \right) + \]
\[ + \frac{\sigma''}{\sigma} + 2 \Lambda e^\lambda \right], \tag{2.9} \]
\[
\frac{1}{r^2} - e^{-\lambda} \left( \frac{1}{r^2} + \frac{\nu'}{r} \right) = \kappa \left[ f^0 f^0 \left( - \frac{\nu'}{2} e^{\nu-\lambda} \frac{\sigma'}{\sigma} - \Lambda e^\nu \right) + \right.
\]
\[ + f^1 f^1 \left( \frac{\sigma''}{\sigma} - \frac{\lambda'}{2 \sigma} + \Lambda e^\lambda \right) - \Lambda \right] + \]
\[ +\kappa f_1^0 f_1^1 \frac{\sigma'}{\sigma} + \kappa f_1^0 f_1^0 \nu' e^{\nu - \lambda} \frac{\sigma'}{\sigma} + \]
\[ +\kappa f_1 f_1^1 \left[ \frac{\sigma'}{\sigma} \left( \frac{\sigma'}{\sigma} + \frac{5}{2} \nu' + \frac{2}{r} \right) - \right.\]
\[ \left. - 3 \frac{\sigma''}{\sigma} - 2 e^\lambda \right], \quad (2.10)\]
\[ e^{-\lambda} \left( \nu'' + \frac{(\nu')^2}{2} + \frac{\nu' - \lambda'}{r} - \frac{\nu' \lambda'}{2} \right) = 2\kappa \Lambda, \quad (2.11)\]

where \( A \) in Eq. (2.8) is an integration constant.

These equations look formidable, nevertheless, they can be patiently disentangled as follows. From Eq. (2.6) and Eq. (2.7) we can write
\[ 2\sigma'' = \sigma' (\nu' + \lambda'), \]
thus (if \( \sigma' \neq 0 \))
\[ e^\nu = \sigma^2 e^{-\lambda} e^{-2K}, \quad (2.12)\]

where \( K \) is a constant. Substituting Eq. (2.12) and Eq. (2.5) into Eq. (2.8) we obtain
\[ (f_1 f_1^1)' + 2 f_1 f_1^1 \left( \frac{1}{r} + \frac{\sigma''}{\sigma'} - \frac{\sigma'}{\sigma} \right) = \frac{A e^K}{r^2 \sigma'} + e^{-\lambda} \left( \frac{\sigma''}{\sigma'} - \frac{\lambda'}{2} \right). \quad (2.13)\]

The difference of Eq. (2.9) and Eq. (2.10) results in
\[ \frac{\sigma''}{\sigma} + \kappa \left( \frac{\sigma^2}{\sigma^2} - \frac{r \sigma^3}{2 \sigma^3} \right) = 0, \quad (2.14)\]
and Eq. (2.10), combined with Eq. (2.13), yields
\[ f_1 f_1^1 = \frac{\sigma^2}{\kappa r^2 \sigma^2} - \frac{e^{-\lambda}}{\kappa} \left( \frac{\sigma^2}{r^2 \sigma^2} + \frac{2\sigma^2 \sigma''}{r \sigma^3} - \frac{\lambda \sigma^2}{r \sigma^2} \right) - \frac{A e^K}{\sigma^2 r^2}. \quad (2.15)\]

Using Eq. (2.5) and inserting \( f_1 f_1^1 \) from Eq. (2.15) into Eq. (2.4), we obtain
\[ \frac{\sigma^2}{\sigma^2} Q'' + \left( \frac{\sigma}{\sigma'} - \frac{\sigma'' \sigma^2}{\sigma^3} \right) Q' - \frac{\kappa Q}{2} = 0, \quad (2.16)\]

\(^{1}\)In the case \( \sigma' = 0 \) we end up with the standard Schwarzschild metric.
where we have introduced $Q(r)$, defined by

$$Q(r) \equiv e^{\nu(r)}r.\]$$

Both equations (2.14) and (2.16) can be easily solved after changing variable $r \rightarrow \sigma$, as then they become

$$\frac{d^2r}{d\sigma^2} - \frac{\kappa}{\sigma d\sigma} + \frac{\kappa r}{2\sigma^2} = 0, \quad (2.17)$$

$$\frac{d^2Q}{d\sigma^2} + \frac{\kappa Q}{\sigma d\sigma} - \frac{\kappa Q}{2\sigma^2} = 0. \quad (2.18)$$

The general solution of Eq. (2.17) and Eq. (2.18) is

$$r = c_1 \sigma^{\alpha_1} + c_2 \sigma^{\alpha_2}, \quad (2.19)$$

$$Q = k_1 \sigma^{\beta_1} + k_2 \sigma^{\beta_2}, \quad (2.20)$$

where $\alpha_{1,2} = (1 + \kappa \pm \sqrt{\kappa^2 + 1})/2$, $\beta_{1,2} = (1 - \kappa \pm \sqrt{\kappa^2 + 1})/2$, and $c_{1,2}$ and $k_{1,2}$ are (real) constants. Formulae (2.12), (2.19), and (2.20) fully determine the metric, and the hair $f^1f^1$ is given by Eq. (2.15) and $f^0f^0$ by Eq. (2.3). It is easy to check that this solution solves also Eq. (2.11), hence, it is the most general solution for our ansatz. Let us study the behavior of the scalar curvature of the solutions. We have

$$R = \frac{2}{r^2} - e^{-\lambda} \left( \frac{2}{r^2} + \frac{\nu''}{2} + \nu' + 2\left(\frac{\nu' - \lambda'}{r} - \frac{\nu'\lambda'}{2}\right) \right).$$

This expression diverges for $r \rightarrow 0$ and tends to zero for $r \rightarrow \infty$. Is there any special choice of parameters giving the black-hole solutions? The answer is affirmative. Indeed, set $c_1 = 0$, $c_2 > 0$, $k_1 > 0$ and $k_2 < 0$, then from Eq. (2.19) it can be explicitly extracted $\sigma(r)$ and Eq. (2.20) and Eq. (2.14) yield

$$e^{\nu} = \frac{k_1}{r} \left( \frac{r}{c_2} \right)^{1+\sqrt{\kappa^2+1}} + \frac{k_2}{r} \left( \frac{r}{c_2} \right)^{-\kappa-\sqrt{\kappa^2+1}}, \quad (2.21)$$

$$e^{\lambda} = \frac{e^{-2\kappa}}{\alpha^2 r} \left( \frac{r}{c_2} \right)^{2\kappa+2+\sqrt{\kappa^2+1}} + \frac{1}{k_1 \left( \frac{r}{c_2} \right)^{2\alpha^2+2+\sqrt{\kappa^2+1}}} \quad \alpha^2 \frac{1}{c_2} \left( \frac{r}{c_2} \right)^{-\kappa^2+2\alpha^2+2+2\kappa+\sqrt{\kappa^2+1}}. \quad (2.22)$$
There is obvious horizon for
\[ r = c_2 \left( -\frac{k_2}{k_1} \right)^{\frac{\kappa}{\sqrt{\kappa^2 + 1}}}. \]

From the formulae (2.21) and (2.22) it turns out that the scalar curvature is bounded at the horizon. Let us analyze the behavior of the curvature tensor in detail. Following orthonormal vierbein is parallelly propagated along the geodesics respecting spherical symmetry

\[ e^{(0)} = \left( e^{\mu/2}, 0, 0, 0 \right), \]
\[ e^{(1)} = \left( 0, e^{\lambda/2}, 0, 0 \right), \]
\[ e^{(2)} = \left( 0, 0, r, 0 \right), \]
\[ e^{(3)} = \left( 0, 0, 0, r \sin \theta \right). \]

The explicit evaluation of all (nontrivial) vierbein components of Riemann tensor gives the following result. All of them vanish for \( r \to \infty \), are finite at horizon, and diverge for \( r \to 0 \). We conclude that the metric (2.21), (2.22) is black hole metric for \( c_2 > 0, k_1 > 0 \) and \( k_2 < 0 \). It is asymptotically flat, singular at \( r = 0 \) and regular at the horizon. It remains to analyze the behavior of the remaining component \( V^\alpha \) of the non-commutative metric.

First of all, we discuss the behavior of \( V^2 \), which is the only independent scalar quantity which can be built up from the field \( V^\alpha \). It is very simple

\[ V^2 = \left( \frac{r}{c_2} \right)^{-2e^2+2+2\sqrt{\kappa^2+1} \kappa}. \quad (2.23) \]

Hence \( V^2 \) (the distance of the Connes’ sheets) tends to zero for \( r \to \infty \), it is bounded near the horizon and diverges for \( r \to 0 \). Thus the behavior of \( V^2 \) at infinity and near the horizon is satisfactory. As far as the behavior of the components is concerned, from Eq. (2.23) it is easy to identify the explicit form of the \( (V^1)^2 \) for the black-hole metric Eq. (2.21), (2.22). It reads

\[
(V^1)^2 = \kappa^2 + \kappa + 1 - \frac{(1 + \kappa)\sqrt{\kappa^2 + 1}}{2} \left( \frac{r}{c_2} \right)^{-2e^2+2+2\sqrt{\kappa^2+1} \kappa} \times \\
\times \left[ 1 + c_2 k_2 \sqrt{\kappa^2 + 1} - 1 \right] e^{2\kappa} \left( \frac{r}{c_2} \right)^{-2e^2+2+2\sqrt{\kappa^2+1} \kappa}.
\]
\[-\left(A e^K + c_2 k_1 \frac{\sqrt{K^2 + 1} - 1}{2} e^{2K}\right) \left(\frac{r}{c_2}\right)^{-\frac{\kappa+1+\sqrt{\kappa^2+1}}{\kappa}}\].

Then \((V^0)^2\) is given by

\[
(V^0)^2 = e^{-2K} \left[k_1 \left(\frac{r}{c_2}\right)^{\frac{1+\sqrt{\kappa^2+1}}{\kappa}} + k_2 \left(\frac{r}{c_2}\right)^{-\frac{\kappa-\sqrt{\kappa^2+1}}{\kappa}}\right] - \\
\times \left[1 + c_2 k_2 e^{2K} \frac{\sqrt{K^2 + 1} - 3}{2} \left(\frac{r}{c_2}\right)^{-\frac{\kappa^2+(\kappa+2)\sqrt{K^2+1}}{\kappa}}\right] - \\
- \left(c_2 k_1 e^{2K} \frac{\sqrt{K^2 + 1} + 1}{2} + A e^K\right) \left(\frac{r}{c_2}\right)^{-\frac{\kappa+1+\sqrt{\kappa^2+1}}{\kappa}}. \tag{2.24}\]

Which choice of the integration constants ensures the regularity of the hair? The non-commutative geometry requires that the components of the non-commutative metric \(g_{\alpha\beta}, V_\alpha\) must be real [4, 5]. But \((V^1)^2\) is positive only for \(r > r_{cr}\) where \(r_{cr}\) is given implicitly by the formula

\[
A = c_2 k_2 \frac{\sqrt{K^2 + 1} - 1}{2} e^K \left(\frac{r_{cr}}{c_2}\right)^{-\frac{\kappa^2+(\kappa+1)\sqrt{K^2+1}}{\kappa}} - \\
- c_2 k_1 \frac{\sqrt{K^2 + 1} + 1}{2} e^K + e^{-K} \left(\frac{r_{cr}}{c_2}\right)^{\frac{\kappa+1+\sqrt{\kappa^2+1}}{\kappa}}. \tag{2.25}\]

Therefore, from the geometrical point of view, the space-time becomes singular at \(r = r_{cr} > 0\) and not at \(r = 0\) as it could seem at the first sight. There is no curvature singularity at \(r_{cr}\), however, the (non-commutative) metric ceases to be real [4, 5]. The parameters of the general solution have to be chosen in such way that \(r_{cr}\) is below the horizon, otherwise we would loose the black hole. Moreover, we have to ensure that \((V^0)^2\) is positive for \(r > r_{cr}\). The following choice of the integration constants meets these criteria

\[
\left(-\frac{k_2}{k_1}\right)^{\frac{1}{\sqrt{\kappa^2+1}}} \geq e^{2K} c_2 k_1 \frac{3\sqrt{K^2 + 1} - \kappa^2 - 1}{2}, \\
r_{cr}(A) < c_2 \left(-\frac{k_2}{k_1}\right)^{\frac{\kappa^2+(\kappa+1)\sqrt{K^2+1}}{2\kappa^2+1}}.
\]
where the dependence $r_{\sigma^*}(A)$ is given by Eq. (2.23).

The next subtle point consists in a behavior of the $(V^0)^2$ on the horizon. This quantity is obviously divergent, as it can be seen from Eq. (2.24). Is this divergence pathological? It need not to be necessarily so, because the invariant quantity $V^2$ is smooth and bounded. Moreover, as the experience from the Bekenstein (conformal scalar) black-hole teaches [6], even the diverging (invariant) hair on the horizon need not mean a pathology. Certainly, the careful and independent analysis of the issue is required as in the Bekenstein case [7]. However, apart from the regular behavior of $V^2$ on the horizon, we may give another invariant quantity which is regular on the horizon, namely, the non-commutative Ricci scalar or—what is the same thing—the Lagrangian of our model.

3. Conclusion

We have solved the field equations of the nonlinear vector $\sigma$-model (2.3) for the static spherically symmetric ansatz with $V^\alpha$ spacelike$^2$. We picked up the particular black-hole solution which is asymptotically flat, it has the hair asymptotically falling off and it has the smooth horizon covering the singularity inside.

The singularity is not the standard curvature singularity, but it corresponds to the point where the components of the non-commutative vielbein (see [4]) become complex. The situation is somewhat analogous to that reported in [8]. In that case, there occurred a specific singular behavior between the horizon and the curvature singularity, where the metric has lost its proper signature.

We hope to understand better the origin of the peculiar singularities inside the black-hole in our future works.

$^2$ In the time-like case the hair singularities cannot be hidden beyond the horizon.
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