A special case of a conjecture of Hellerstein, Shen and Williamson

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Dedicated to the memory of I.V. Ostrovskii

Abstract

The paper proves a special case of a conjecture of Hellerstein, Shen and Williamson concerning non-real zeros of derivatives of real meromorphic functions. Keywords: meromorphic function, non-real zeros. MSC 2010: 30D20, 30D35.

1 Introduction

This paper concerns the problem of classifying real meromorphic functions in the plane which, together with some of their derivatives, have only real zeros and poles [10, 13, 14]. Here a meromorphic function $f$ is called real if $f(\mathbb{R}) \subseteq \mathbb{R} \cup \{\infty\}$, and it is known that if $f$ is real entire and $f$ and $f''$ have only real zeros then $f$ belongs to the Laguerre-Polya class $LP$, as conjectured by Wiman and proved in [2, 27, 32]. For the meromorphic case, the following conjecture was advanced in [10].

Conjecture 1.1 ([10]) Let $f$ be a real transcendental meromorphic function in the plane with at least one pole, and assume that all zeros and poles of $f$, $f'$ and $f''$ are real, and that all poles of $f$ are simple. Then $f$ satisfies

$$f(z) = C \tan(az + b) + Dz + E, \quad a, b, C, D, E \in \mathbb{R}. \quad (1)$$

In the absence of the assumption that $f$ has only simple poles, further examples arise for which $f$, $f'$ and $f''$ have only real zeros and poles [12]. Conjecture 1.1 is known to be true if any of the following additional hypotheses holds:

(a) $f'$ omits some finite value [10, 15, 19, 21, 23, 29, 31];
(b) $f$ has infinitely many poles and $f''/f'$ has finitely many zeros [24, Theorem 1.5];
(c) $f$ has infinitely many zeros and poles, all real, simple and interlaced – that is, between any two consecutive poles of $f$ there is a zero, and between consecutive zeros of $f$ lies a pole [25].

The following theorem implies a further special case of Conjecture 1.1.
Theorem 1.1 Let \( f \) be a real meromorphic function in the plane, such that \( f \) and \( f' \) have no zeros or poles in \( \mathbb{C} \setminus \mathbb{R} \), while \( f''/f \) has no zeros in \( \mathbb{C} \). Then there exist \( \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \), with \( \alpha_1 \alpha_2 \neq 0 \), such that \( g(z) = \alpha_1 f(\alpha_2 z + \alpha_3) \) satisfies one of the following:

(i) \( g(z) = f_1(z) = \sin z \);
(ii) \( g(z) = f_2(z) = e^z \);
(iii) \( g(z) = f_3(z) = \tan z \);
(iv) \( g(z) = f_4(z) \), where

\[
f_4(z) = \sum_{k=0}^{\infty} \frac{z^{k+1}}{k!(k+1)!} = z + \frac{z^2}{2} + \frac{z^3}{12} + \ldots \tag{2}
\]

solves

\[
zy''(z) = g(z); \tag{3}
\]

(v) \( g(z) = F_1(z) = z^Q \) for some \( Q \in \mathbb{Z} \setminus \{0, 1\} \);
(vi) \( g(z) = F_2(z) \), where

\[
F_2(z) = \frac{d^{n-2}}{dz^{n-2}} \left( z^{n-1}(z-1)^{n-1} \right) \tag{4}
\]

for some integer \( n \geq 2 \), and \( F_2 \) solves

\[
z(z-1)y''(z) = n(n-1)y(z); \tag{5}
\]

(vii) \( g(z) = F_3(z) \), where \( F_3 \) is given by

\[
F_3(z) = (z-K)H_n \left( \frac{K+1}{K-1} - \frac{2K}{(K-1)z} \right) \tag{6}
\]

for some integer \( n \geq 1 \) and \( K \in \mathbb{R} \setminus \{0, 1\} \), in which

\[
H_n(w) = \frac{d^n}{dw^n} \left( (w-1)^{n-1}(w+1)^{n+1} \right), \tag{7}
\]

while \( F_3 \) solves

\[
z^2(z-1)(z-K)y''(z) = Kn(n+1)y(z); \tag{8}
\]

(viii) \( g(z) = F_4(z) \), where

\[
F_4(z) = H_n \left( 1 - \frac{2}{z} \right), \tag{9}
\]

in which \( 1 \leq n \in \mathbb{N} \) and \( H_n \) is given by (7), while \( F_4 \) solves

\[
z^2(z-1)y''(z) = -n(n+1)y(z). \tag{10}
\]

Conversely, the equations (3), (5), (8) (for \( K > 1 \)) and (10) all supply examples satisfying the hypotheses of the theorem. The function \( f_4 \) in (2) and its connection to Bessel functions will be discussed in Section 2.1, while the rational functions \( F_2, F_3, F_4 \) in (vi), (vii) and (viii), which are linked to hypergeometric functions, will be treated in detail in Sections 2.2, 2.3 and 2.4.

Of course, the condition that \( f''/f \) has no zeros makes Theorem 1.1 a very special case of Conjecture 1.1, albeit without the assumption that \( f \) is transcendental and all poles of \( f \) are
simple, but the fact that the proofs of all the resolved special cases are lengthy tends to suggest that the full conjecture is difficult. The result may also be viewed as a special case of the problem of determining all meromorphic functions \( f \) such that \( f''/f \) has no zeros: in this direction, it was proved in [17] that if \( f \) is entire of order less than 1, or meromorphic of order less than \( 1/2 \), and \( f''/f \) is transcendental, then \( f''/f \) has infinitely many zeros.

Note that the corresponding problem for the case where \( f \) is strictly non-real, that is, \( f \) is not a constant multiple of a real meromorphic function, was completely settled in [9], the main result of which classified all strictly non-real meromorphic functions \( f \) in the plane for which \( f, f' \) and \( f'' \) have only real zeros and poles.

In common with much of the work on non-real zeros of derivatives, this paper relies heavily on key results and methods developed by B. Ja. Levin and I.V. Ostrovskii, as set out in the paper [27] and the textbook [5] - in particular, the factorisation of the logarithmic derivative (Section 3.1) and an integral inequality linking the Tsuji and Nevanlinna proximity functions (Lemma 3.4).

2 Preliminaries and examples

First, let \( D \) be a real entire function, whose zeros \( x_k \) are all real and simple. Then a standard application of the Mittag-Leffler theorem gives a real entire function \( C \) such that

\[
\frac{e^{C(z)}}{D(z)} = \frac{-2}{z - x_k} + O(|z - x_k|)
\]

as \( z \to x_k \), for each \( k \). The formula \( g'/g = e^C/D \) then defines a real meromorphic function \( g \), such that \( g \) and \( g' \) have no zeros at all, while for each \( k \) there exists \( c_k \in \mathbb{R} \setminus \{0\} \) with \( g(z) = c_k(z - x_k)^{-2} + O(1) \) as \( z \to x_k \). Hence there exists a real meromorphic function \( f \) with \( f' = g \) and \( f' f'' \) zero-free. However, this construction of course gives no control over the location of the zeros of \( f \) itself.

The remainder of this section will make use of the following standard lemma.

**Lemma 2.1** Let \( P \) be a polynomial with a simple zero at \( a \in \mathbb{C} \). If the equation

\[
P(z) y''(z) = y(z)
\]

has a solution \( f \) which is meromorphic in the plane and has \( f(a) \in \mathbb{C} \), then every solution which is meromorphic in the plane is a constant multiple of \( f \).

**Proof.** It may be assumed that \( a = 0 \). The assumptions force \( f(0) = 0 \), and the zero of \( f \) at 0 must be simple, because otherwise \( P = f/f'' \) has a double zero at \( a \). Hence \( c_1 = f'(0) \neq 0 \), and it follows from (11) that \( 2c_2 = f''(0) \neq 0 \). A second solution \( g \) may then be obtained by writing

\[
\left( \frac{g}{f} \right)'(z) = \frac{1}{f(z)^2} = \frac{1}{(c_1z + c_2z^2 + \ldots)^2} = \frac{1}{c_1^2 z^2}(1 - 2(c_2/c_1)z + \ldots),
\]

and integration clearly gives rise to a logarithm. \( \square \)
2.1 The equation (3)

Let \( f_4 \) be as in (2). Then differentiating \( f_4 \) twice leads to

\[
z f_4''(z) = \sum_{k=1}^{\infty} \frac{z^k}{(k-1)!k!} = \sum_{k=0}^{\infty} \frac{z^{k+1}}{k!(k+1)!} = f_4(z),
\]

after replacing \( k \) by \( k + 1 \), and so \( f_4 \) is a solution of (3). Lemma 2.1, with \( a = 0 \), shows that any solution of (3) which is meromorphic in \( \mathbb{C} \) is a constant multiple of \( f_4 \).

It turns out that \( f_4 \) has a representation in terms of Bessel functions: write \( z = w^2 \) and

\[
f_4(z) = \sum_{k=0}^{\infty} \frac{w^{2k+2}}{k!(k+1)!} = \frac{w^2}{i} \sum_{k=0}^{\infty} \frac{(-1)^k(2iw)^{2k+1}}{2^{2k+1}k!(k+1)!} = \frac{w^2}{i} J_1(2iw),
\]

where \( J_1 \) is the Bessel function of the first kind of order 1 [11]. This relation can be used to prove that all zeros of \( f_4 \) are real and non-positive, but the following approach applies Green’s transform [11, pp.286-8] directly to \( f_4 \) and (3).

Suppose then that \( R > 0 \) and \( s \in \mathbb{R} \) and \( Re^{is} \) is a zero of \( f_4 \). Set

\[
F(r) = f_4(re^{is}), \quad H(r) = \overline{F(r)}F'(r).
\]

This yields, for \( r > 0 \), by (3),

\[
H'(r) = |F'(r)|^2 + \overline{F(r)}F''(r) = |F'(r)|^2 + e^{2is}\overline{F(r)}f_4''(re^{is}) = |F'(r)|^2 + \frac{e^{is}|F(r)|^2}{r}.
\]

Since \( H(R) = H(0) = 0 \), integration from 0 to \( R \) results in

\[
\int_0^R |F'(r)|^2 \, dr = -e^{is} \int_0^R \frac{|F(r)|^2}{r} \, dr,
\]

which forces \( e^{is} = -1 \), so that \( Re^{is} \) lies on the negative real axis.

Next, a straightforward application of the Wiman-Valiron theory [8] in (3) shows that the order of \( f_4 \) is \( 1/2 \), and so a standard generalisation of the Gauss-Lucas theorem [34] implies that all zeros of \( f_4' \) are also real and non-positive. This completes the proof of the following.

**Lemma 2.2** The real entire function \( f_4 \) given by (2) is a solution of (3), and all zeros of \( f_4 \) and \( f_4' \) are real and non-positive. Moreover, any solution of (3) which is meromorphic in \( \mathbb{C} \) is a constant multiple of \( f_4 \).

\[\square\]

2.2 The equation (5)

Let \( n \geq 2 \) be an integer, and consider the equation (5). MAPLE gives solutions in terms of hypergeometric functions, but an explicit solution will be derived as follows. Let \( F = F_2 \) be given
by (4) and write

\[
F(z) = \frac{d^{n-2}}{dz^{n-2}} \left( z^{n-1}(z - 1)^{n-1} \right)
\]

\[
= \frac{d^{n-2}}{dz^{n-2}} \left( \sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-1-k)!} z^{k+n-1}(-1)^{n-1-k} \right)
\]

\[
= \sum_{k=0}^{n-1} \frac{(n-1)!(n+k-1)!}{k!(n-1-k)!(k+1)!} z^{k+1}(-1)^{n-1-k}
\]

\[
= \sum_{k=0}^{n-1} a_k z^{k+1}, \quad a_k \in \mathbb{R}, \quad a_0 \neq 0.
\]

It is then clear that \( F \) is a polynomial of degree \( n \), with a simple zero at 0, and that all zeros of \( F \) and \( F' \) lie in \([0, 1]\), by repeated application of the Gauss-Lucas theorem. Moreover, \( a_k \) satisfies

\[
\frac{a_{k+1}}{a_k} = -\frac{(k+n)(n-1-k)}{(k+1)(k+2)} \quad \text{for } k = 0, \ldots, n-2.
\]

Hence differentiating \( F \) twice yields

\[
z(z - 1)F''(z) = (z^2 - z) \sum_{k=0}^{n-1} (k+1)ka_kz^{k-1}
\]

\[
= \sum_{k=0}^{n-1} (k+1)ka_kz^{k+1} - \sum_{k=1}^{n-1} (k+1)ka_kz^k
\]

\[
= \sum_{k=0}^{n-1} (k+1)ka_kz^{k+1} - \sum_{k=0}^{n-2} (k+2)(k+1)a_{k+1}z^{k+1}
\]

\[
= \sum_{k=0}^{n-1} (k+1)ka_kz^{k+1} + \sum_{k=0}^{n-2} (k+n)(n-1-k)a_kz^{k+1}
\]

\[
= \sum_{k=0}^{n-1} (k+1)ka_kz^{k+1} + \sum_{k=0}^{n-1} (k+n)(n-1-k)a_kz^{k+1}
\]

\[
= \sum_{k=0}^{n-1} (n^2 - n)a_kz^{k+1} = n(n-1)F(z).
\]

Thus \( F \) solves (5). Applying Lemma 2.1, with \( a = 0 \), completes the proof of the following.

**Lemma 2.3** For \( 2 \leq n \in \mathbb{Z} \), the real polynomial \( F_2 \) given by (4) has degree \( n \) and solves (5). Moreover, all zeros of \( F_2 \) and \( F'_2 \) are real, and any solution of (5) which is meromorphic in the plane must be a constant multiple of \( F_2 \).

\( \square \)
2.3 The equation (8)

Let \( n \geq 1 \) be an integer and let \( K \in \mathbb{R} \setminus \{0, 1\} \). MAPLE gives solutions of (8) in terms of hypergeometric functions, and the following direct determination of a rational solution was found via properties of the related Jacobi polynomials [30, p. 254]. Using the substitution

\[
 w = \frac{K + 1}{K - 1} - \frac{2K}{(K - 1)z}, \quad z = \phi(w) = \frac{2K}{K + 1 - (K - 1)w}, \quad y(z) = (z - K)h(w), \quad (12)
\]

write

\[
 y'(z) = h(w) + \frac{2K(z - K)}{(K - 1)z^2} h'(w), \quad y''(z) = \frac{2K}{(K - 1)z^2} h'(w) + \frac{2K}{(K - 1)z^2} h''(w) - \frac{4K(z - K)}{(K - 1)z^3} h'(w) + \frac{4K^2(z - K)}{(K - 1)^2 z^4} h''(w)
\]

\[
 = \frac{4K^2}{(K - 1)z^3} h'(w) + \frac{4K^2(z - K)}{(K - 1)^2 z^4} h''(w).
\]

Observe next that, by (12),

\[
 \frac{z - 1}{z} = \frac{(K - 1)(w + 1)}{2K}, \quad \frac{z - K}{z} = \frac{(K - 1)(w - 1)}{2}.
\]

Thus substituting for \( y \) and \( y'' \) delivers

\[
 R(z) = Kn(n + 1)y(z) - z^2(z - 1)(z - K)y''(z)
\]

\[
 = Kn(n + 1)(z - K)h(w) - z^2(z - 1)(z - K) \left( \frac{4K^2}{(K - 1)z^3} h'(w) + \frac{4K^2(z - K)}{(K - 1)^2 z^4} h''(w) \right),
\]

\[
 = K(z - K) \left[ n(n + 1)h(w) - \left( \frac{4K}{K - 1} \right) \left( \frac{z - 1}{z} \right) h'(w) \right]
\]

\[
 - K(z - K) \left[ \frac{4K}{(K - 1)^2} \left( \frac{(z - 1)(z - K)}{z^2} \right) h''(w) \right]
\]

\[
 = K(z - K) \left[ n(n + 1)h(w) - 2(w + 1)h'(w) + (1 - w^2)h''(w) \right].
\]

Thus \( y \) solves (8) if and only if \( h \) solves

\[
 (1 - w^2)h''(w) - 2(w + 1)h'(w) + n(n + 1)h(w) = 0. \quad (13)
\]

**Lemma 2.4** Let \( H(w) = H_n(w) \), with \( H_n \) as in (7). Then \( H \) is a polynomial of degree \( n \) and solves (13). Moreover, \( H(-1) = 0 \), all \( n \) zeros of \( H \) are simple, and they all lie in \([-1, 1]\).

**Proof.** Write (7) in the form

\[
 H(w) = \frac{d^n}{d w^n} \left( (w + 1 - 2)^{n-1}(w + 1)^{n+1} \right)
\]

\[
 = \frac{d^n}{d w^n} \left( \sum_{k=0}^{n-1} \binom{n-1}{k} (-2)^{n-1-k}(w + 1)^{n+1+k} \right)
\]

\[
 = \sum_{k=0}^{n-1} \binom{n-1}{k} (-2)^{n-1-k} \frac{(n + 1 + k)!}{(k + 1)!} (w + 1)^{k+1},
\]
which leads to
\[
H(w) = \sum_{k=0}^{n-1} b_k (w + 1)^{k+1}, \quad b_k = \frac{(n - 1)! (n + 1 + k)!}{k! (n - k - 1)! (k + 1)! (-2)^{n-k}},
\]  
(14)
in which \(b_0 \neq 0\) and
\[
\frac{b_{k+1}}{b_k} = \frac{(n - k - 1) (n + 2 + k)}{(k + 1) (k + 2) (-2)} = \frac{(k + 1 - n) (k + 2 + n)}{2 (k + 1) (k + 2)} \quad \text{for } k = 0, \ldots, n - 2.
\]  
(15)
Substitution of (14) into the right-hand side of (13), followed by application of (15), delivers
\[
Q(w) = (1 - w^2) H''(w) - 2 (w + 1) H'(w) + n (n + 1) H(w)
\]
\[
\quad = (2 - (w + 1)) (w + 1) \sum_{k=0}^{n-1} (k + 1) b_k (w + 1)^{k-1}
\]
\[
\quad - 2 (w + 1) \sum_{k=0}^{n-1} (k + 1) b_k (w + 1)^k + n (n + 1) \sum_{k=0}^{n-1} b_k (w + 1)^{k+1}
\]
\[
\quad = 2 \sum_{k=1}^{n-1} (k + 1) b_k (w + 1)^k + \sum_{k=0}^{n-1} (n(n + 1) - (k+1) - 2 (k+1)) b_k (w + 1)^{k+1}
\]
\[
\quad = 2 \sum_{k=0}^{n-2} (k + 2) (k + 1) b_{k+1} (w + 1)^{k+1} + \sum_{k=0}^{n-1} (n(n + 1) - (k+2) (k+1)) b_k (w + 1)^{k+1}
\]
\[
\quad = \sum_{k=0}^{n-2} (k + 1 - n) (k + 2 + n) b_k (w + 1)^{k+1} + \sum_{k=0}^{n-1} (n(n + 1) - (k+2) (k+1)) b_k (w + 1)^{k+1}
\]
\[
\quad = \sum_{k=0}^{n-1} (k + 1 - n) (k + 2 + n) b_k (w + 1)^{k+1} + \sum_{k=0}^{n-1} (n(n + 1) - (k+2) (k+1)) b_k (w + 1)^{k+1}
\]
\[
\quad = 0.
\]
Thus \(H(w)\) is a polynomial solution of (13), of degree \(n\), with a simple zero at \(-1\), since \(b_0 \neq 0\) in (14). Repeated application of the Gauss-Lucas theorem to \(G(w) = (w - 1)^{n-1} (w + 1)^{n+1}\) shows that all zeros of \(H(w)\) lie in \([-1, 1]\). Moreover, since \(G\) has a zero of multiplicity \(n - 1\) at \(1\), all zeros of \(E = G^{(n-1)}\) lie in \([-1, 1]\) and therefore so do all zeros of \(H = E'.\) Finally, all zeros of \(H\) in \((-1, 1)\) are simple, by the existence-uniqueness theorem and (13).

\[\square\]

\textbf{Lemma 2.5} With \(H_0\) as in (7) and \(n \geq 1\), and with \(w\) defined by (12), the function \(F_3(z)\) in (6) is a rational solution of (3) with \(F_3(1) = F_3(K) = 0\) and all its zeros real and simple, and every solution of (8) which is meromorphic in the plane is a constant multiple of \(F_3\). Moreover, \(F_3(z) = P(z) z^{-n}\), where \(P\) is a real polynomial with \(P(0) \neq 0\), and \(P\) has degree \(n\) or \(n + 1\).

Furthermore, if \(P\) has degree \(n + 1\) and all zeros of \(P\) lie in \((0, +\infty)\), then \(F_3, F'_3\) and \(F''_3\) have no zeros or poles in \(\mathbb{C} \setminus \mathbb{R}\). In particular, this holds if \(K > 1\).

\textbf{Proof.} First, \(F_3\) solves (3) and has a pole at \(z = 0\) of order \(n\), since \(z = 0\) corresponds to \(w = \infty\) and \(H\) has degree \(n\). Clearly, \(F_3\) has no other poles in \(\mathbb{C}\).
Next, \( H(-1) = 0 \) and all \( n \) zeros of \( H \) are simple and lie in \([-1, 1)\). Since one of them may be mapped to \( \infty \) by \( z = \phi(w) \), it follows from (12) that \( F_3(\infty) \neq 0 \) and \( F_3 \) has \( n \) or \( n + 1 \) zeros in \( \mathbb{C} \). In particular, \( F_3 \) has zeros at \( z = 1 \) and \( z = K \), which correspond to \( w = -1 \) and \( w = 1 \) respectively, and any solution of (8) which is meromorphic in the plane is a constant multiple of \( F_3 \), by Lemma 2.1 with \( a = K \) or \( a = 1 \).

Now suppose that \( P \) has degree \( n + 1 \) and all zeros of \( P \) lie in \((0, +\infty)\). This will certainly hold if \( K > 1 \), because in this case the function \( z = \phi(w) \) is finite and increasing for \(-1 \leq w < 1\), and maps \([-1, 1) \) to \([1, K)\), so that \( F_3(z) = (z - K)H(w) \) inherits all \( n \) zeros of \( H(w) \), as well as having a zero at \( z = K \). Under these assumptions, a consideration of leading terms shows that \( zP'(z) - nP(z) \) has degree \( n + 1 \), and so

\[
F_3'(z) = \frac{zP'(z) - nP(z)}{z^{n+1}}
\]

has \( n + 1 \) zeros in \( \mathbb{C} \). Of these, \( n \) arise from Rolle’s theorem and lie in \((0, +\infty)\), while one lies in \((-\infty, 0)\) because, with \( x \in \mathbb{R} \),

\[
\frac{F_3'(x)}{F_3(x)} \sim \frac{1}{x} \quad \text{as} \quad x \to -\infty, \quad \frac{F_3'(x)}{F_3(x)} \sim -\frac{n}{x} \quad \text{as} \quad x \to 0-.\]

Thus \( F_3 \) and \( F_3' \) have no zeros in \( \mathbb{C} \setminus \mathbb{R} \), and nor has \( F_3'' \), because of (8).

Taking \( K = 2 \) delivers \( F_3(z) = (z - K)H_n(w) = (z - 2)H_n(3 - 4/z) \) and (with help from MAPLE) the following:

\[
\begin{align*}
n = 1, & \quad g_1(z) = \frac{8(z - 1)(z - 2)}{z}; \\
n = 2, & \quad g_2(z) = \frac{144(z - 1)(z - 4/3)(z - 2)}{z^2}; \\
n = 3, & \quad g_3(z) = \frac{384(z - 1)(z - 2)(11z^2 - 30z + 20)}{z^3}.
\end{align*}
\]

In these examples, \( g_j \), \( g_j' \) and \( g_j'' \) have no zeros in \( \mathbb{C} \setminus \mathbb{R} \), by Lemma 2.5. On the other hand, choosing \( K = -1 \) leads to \( F_3(z) = (z - K)H_n(w) = (z + 1)H_n(-1/z) \), as well as:

\[
\begin{align*}
n = 1, & \quad h_1(z) = \frac{2(z^2 - 1)}{z}; \\
n = 2, & \quad h_2(z) = \frac{-12(z^2 - 1)}{z^2}; \\
n = 3, & \quad h_3(z) = \frac{-24(z^2 - 1)(z^2 - 5)}{z^3}; \\
n = 4, & \quad h_4(z) = \frac{720(z^2 - 1)(z^2 - 7/3)}{z^4}.
\end{align*}
\]

Here \( h_j \), \( h_j' \) and \( h_j'' \) have no zeros in \( \mathbb{C} \setminus \mathbb{R} \) for \( j = 2, 4 \), but \( h_j' \) has non-real zeros for \( j = 1, 3 \).
2.4 The equation (10)

A solution of (10) is obtained by the following limiting process with $K$ real: let $n \geq 1$ and let $F_3, H_n$ be as in (6) and (7), and set

$$F_4(z) = \lim_{K \to +\infty} \frac{F_3(z)}{-K} = \lim_{K \to +\infty} \left( \frac{z - K}{-K} \right) H_n \left( \frac{K + 1}{K - 1} - \frac{2K}{(K - 1)z} \right) = H_n \left( 1 - \frac{2}{z} \right).$$

Since all zeros of $H_n$ and $H'_n$ lie in $[-1, 1]$, by Lemma 2.4, $F_4$ and $F'_4$ have no zeros in $\mathbb{C} \setminus [1, +\infty)$, and $F_4$ has a pole of order $n$ at $0$. Applying Weierstrass’ theorem yields, since $F_3$ solves (8),

$$\frac{F''_4(z)}{F_4(z)} = \lim_{K \to +\infty} \frac{F''_3(z)}{F_3(z)} = -\frac{n(n + 1)}{z^2(z - 1)}$$

as required. Furthermore, $F_4$ has a simple zero at $z = 1$, inherited from the simple zero of $H_n$ at $w = -1$, which completes the proof of the following.

Lemma 2.6 With $H_n$ as in (7) and $n \geq 1$, the function $F_4(z) = H_n(1 - 2/z)$ is a rational solution of (10) with a simple zero at 1, and every solution of (10) which is meromorphic in the plane is a constant multiple of $F_4$. Furthermore, $F_4, F'_4$ and $F''_4$ have no zeros or poles in $\mathbb{C} \setminus \mathbb{R}$.

Calculating $F_4(z) = H_n(1 - 2/z)$ using MAPLE delivers:

- $n = 1$, $p_1(z) = \frac{4(z - 1)}{z}$;
- $n = 2$, $p_2(z) = \frac{24(z - 1)(z - 2)}{z^2}$;
- $n = 3$, $p_3(z) = \frac{192(z - 1)(z^2 - 5z + 5)}{z^3}$.

3 Lemmas needed for the proof of Theorem 1.1

Lemma 3.1 Let $g$ be a real meromorphic function in the plane, of order at most 1, and with infinitely many zeros, all but finitely many of them real, and assume that $g$ has finitely many poles. Then

$$\lim_{y \to +\infty, y \in \mathbb{R}} \frac{\log |g(iy)|}{\log y} = +\infty.$$

Proof. It is enough to prove this when $g$ is real entire, with only real zeros, and with $g(0) \neq 0$. The hypotheses then imply that

$$g(z) = e^{\alpha z + \beta} \prod_{n=1}^{\infty} \left( 1 - \frac{z}{a_n} \right) e^{z/a_n},$$

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with \( \alpha, \beta, a, n \) real. As \( y \to +\infty \) with \( y \in \mathbb{R} \) this gives

\[
2 \log |g(iy)| \geq 2 \sum_{n=1}^{\infty} \log \left| 1 - \frac{i y}{a_n} \right| - O(1) = \sum_{n=1}^{\infty} \log \left(1 + \frac{y^2}{a_n^2}\right) - O(1)
\]

\[
\geq \sum_{|a_n| \leq y} \log (1 + y) - O(1) \geq n(\sqrt{y}, 1/g) \log y - O(1).
\]

\[\square\]

**Lemma 3.2 ([4])** Let \( D \subseteq \mathbb{C} \) be a domain and let \( \mathcal{F} \) be a family of meromorphic functions \( f \) on \( D \) such that \( f \) and \( f'' \) have no zeros in \( D \). Then the family \( \{f''/f : f \in \mathcal{F}\} \) is normal on \( D \).

\[\square\]

Next, suppose that \( G \) is a transcendental meromorphic function in the plane, and that \( G(z) \to a \in \mathbb{C} \cup \{\infty\} \) as \( z \to \infty \) along a path \( \gamma \); then the inverse \( G^{-1} \) is said to have a transcendental singularity over the asymptotic value \( a \) \([1, 28]\). If \( a \in \mathbb{C} \) then for each \( \varepsilon > 0 \) there exists a component \( \Omega = \Omega(a, \varepsilon, G) \) of the set \( \{z \in \mathbb{C} : |G(z) - a| < \varepsilon\} \) such that \( \gamma \setminus \Omega \) is bounded: these components are referred to as neighbourhoods of the singularity \([1]\). Two such paths \( \gamma, \gamma' \) on which \( G(z) \to a \) determine distinct singularities if the corresponding components \( \Omega(a, \varepsilon, G), \Omega'(a, \varepsilon, G) \) are disjoint for some \( \varepsilon > 0 \). The singularity is called direct \([1]\) if \( \Omega(a, \varepsilon, G), \) for some \( \varepsilon > 0, \) contains finitely many zeros of \( G - a, \) and indirect otherwise. A direct singularity is called logarithmic if there exists \( \varepsilon > 0 \) such that \( w = \log 1/(G(z) - a) \) is a conformal bijection from \( \Omega(a, \varepsilon, G) \) to the half-plane \( \mathbb{R} w > \log 1/\varepsilon \). Finally, transcendental singularities over \( \infty \) may be classified using \( 1/G, \) and a transcendental singularity will be referred to as lying in an open set \( D \) if \( \Omega(a, \varepsilon, G) \subseteq D \) for some \( \varepsilon > 0. \)

The next lemma combines \([20], \text{Lemma 2.4}\) and \([23], \text{Lemma 2.2}\), and links asymptotic values approached on paths in the upper half-plane \( H^+ \) with the growth of the Tsuji characteristic \( \Omega(r, L) = \Omega(r, G) + \mathcal{M}(r, G) \) for functions \( g \) that are meromorphic on the closed upper half-plane \([2, 5, 35]\).

**Lemma 3.3 ([20, 23])** Let \( L \neq 0 \) be a real meromorphic function in the plane such that \( \Omega(r, L) = O(\log r) \) as \( r \to \infty \), and let \( F(z) = z - 1/L(z) \). Assume that at least one of \( L \) and \( 1/L \) has finitely many non-real poles. Then there exist finitely many \( \alpha \in \mathbb{C} \) such that \( F(z) \) or \( L(z) \) tends to \( \alpha \) as \( z \) tends to infinity along a path in \( \mathbb{C} \setminus \mathbb{R} \).

Moreover, there exists at most one direct transcendental singularity of \( F^{-1} \) lying in \( H^+ \).

\[\square\]

The following result of Levin and Ostrovskii \([27]\) (see also \([5, \text{Ch. 6, Lemma 5.2}]\) and \([20, \text{Lemma 2.4}]\)) will be required.

**Lemma 3.4 ([27])** Let \( G \) be a meromorphic function in the plane: then, for each \( R \geq 1, \)

\[
\frac{1}{2\pi} \int_{R}^{+\infty} \frac{1}{r^3} \int_{0}^{\pi} \log^+ |G(re^{i\theta})| d\theta dr \leq \int_{R}^{+\infty} \frac{\Omega(r, G)}{r^2} dr.
\]

If, in addition, \( G \) is real meromorphic with finitely many poles, and satisfies \( \Omega(r, G) = O(\log r) \) as \( r \to \infty \), then \( T(R, G) = O(R \log R) \) as \( R \to +\infty \).
Lemma 3.5 There exists a positive constant $c_0$ such that if the function $\psi$ maps the upper half-plane $H^+$ analytically into itself then, for $r \geq 1$ and $\theta \in (0, \pi)$,
\begin{equation}
\frac{|\psi(i)| \sin \theta}{5r} < |\psi(re^{i\theta})| < \frac{5r|\psi(i)|}{\sin \theta} \quad \text{and} \quad \left| \frac{\psi'(re^{i\theta})}{\psi(re^{i\theta})} \right| \leq \frac{c_0}{r \sin \theta}.
\end{equation}
Both of these estimates are standard: the first is essentially just Schwarz’ lemma \[26, \text{Ch. I.6, Thm } 8\], while the second follows from Bloch’s theorem applied to $\log \psi$.

3.1 The Levin-Ostrovskii factorisation

The following constructions are standard \[2, 27\]. Suppose that $(u_k), (v_k)$ are sequences satisfying $u_k < v_k < u_{k+1}$ for $-\infty \leq M < k < N \leq +\infty$. Then there exists $k_0 \in \mathbb{N}$ such that $u_k$ and $v_k$ have the same sign for $|k| \geq k_0$, and
\[ \psi(z) = \prod_{|k| \geq k_0} \frac{1 - z/v_k}{1 - z/u_k} \]
converges on $\mathbb{C}$ by the alternating series test. Furthermore, $\psi$ satisfies, for $z$ in the upper half-plane $H^+$,
\[ \arg \psi(z) = \sum_{|k| \geq k_0} \arg \frac{1 - z/v_k}{1 - z/u_k} = \sum_{|k| \geq k_0} \arg \frac{v_k - z}{u_k - z} \in (0, \pi). \]
This leads to the Levin-Ostrovskii factorisation \[2, 27\] of the logarithmic derivative of a real entire function $f$ with real zeros. If $f$ has finitely many zeros, set $\psi(z) = 1$, while if $f$ has infinitely many zeros $u_k$ then zeros of $f'$ given by Rolle’s theorem can be labelled $v_k$ so that $u_k < v_k < u_{k+1}$, whereupon $\psi$ may be constructed as above. It follows that $f' / f = P \psi$, where $P$ is real meromorphic with finitely many poles and either $\psi \equiv 1$ or $\psi(H^+) \subseteq H^+$.

4 Proof of Theorem 1.1: first steps

Let $f$ be as in the hypotheses and write
\begin{equation}
L = \frac{f'}{f}, \quad F(z) = z - \frac{f(z)}{f'(z)}, \quad F' = \frac{ff''}{(f')^2}.
\end{equation}

Lemma 4.1 Let $0 < \delta < \pi/2$ and $\delta < \sigma < \pi - \delta$.
(i) If $rL(re^{i\sigma})$ is bounded as $r \to +\infty$ then $zL(z)$ is bounded as $z \to \infty$ with $\delta < \arg z < \pi - \delta$.
(ii) If $\lim_{r \to +\infty} rL(re^{i\sigma}) = 0$, then $zL(z) \to 0$ uniformly as $z \to \infty$ with $\delta < \arg z < \pi - \delta$.

Proof. The functions $u_R(z) = RL(Rz)$, $R \geq 1$, form a normal family on the domain $D_1 = \{z \in \mathbb{C} : 1/2 < |z| < 2, \delta/2 < \arg z < \pi - \delta/2\}$: this follows from Lemma 3.2 applied to the functions $f(Rz)$. Take a sequence $R_n \to +\infty$ such that $(u_{R_n})$ converges locally spherically uniformly on $D_1$. In case (i), $(u_{R_n})$ cannot have $\infty$ as limit, while in case (ii) the limit function must vanish identically, by the identity theorem. \[ \square \]
Lemma 4.2 Poles of $F$ in $\mathbb{C}$ coincide with zeros of $L = f'/f$, all of which are real and simple. All zeros of $F'$ in $\mathbb{C}$ are real zeros of $f'$ and super-attracting fixpoints of $F'$; furthermore, simple zeros of $F'$ in $\mathbb{C}$ are zeros of $f$ which are not zeros of $f''$, while multiple zeros of $F'$ in $\mathbb{C}$ have multiplicity 2 and are common simple zeros of $f$ and $f''$.

Proof. This is standard, and all assertions follow from (17). First, any multiple zero of $L = f'/f$ would be a zero of $f''$, and hence of $f$, and thus a pole of $f'/f$, an obvious contradiction. Next, zeros of $F'$ are zeros of $f$ or $f''$, and hence of $f$. But multiple zeros of $f$ are not zeros of $F'$, and so all zeros of $F'$ must be simple zeros of $f$, and since $f''/f$ has no zeros they cannot be zeros of $f''/f$ of multiplicity greater than 1. □

Define the sets $W^+$ and $W^-$ using

$$H^+ = \{ z \in \mathbb{C} : \text{Im } z > 0 \}, \quad W^\pm = \{ z \in H^+ : \pm F(z) \in H^+ \}. \quad (18)$$

The next lemma is fairly standard and goes back to Sheil-Small [32].

Lemma 4.3 Let $x_0 \in \mathbb{R}$ be a zero of $f'/f$. If $(f'/f)'(x_0) < 0$ then $x_0 \in \partial W^-$, while if $(f'/f)'(x_0) > 0$ then $x_0 \in \partial W^+$. Poles of $f$ are repelling fixpoints of $F$ and lie in $\partial W^+ \setminus \partial W^-$. Proof. The first two assertions hold since as $z \to x_0$ from within $H^+$ the sign of $\text{Im } (-f(z)/f'(z))$ is the same as that of $\text{Im } (f'(z)/f(z))$. Furthermore, if $x_1$ is a pole of $f$ of multiplicity $m_1$ then $F(x_1) = x_1$ and $F'(x_1) = 1 + 1/m_1 > 1$. □

Lemma 4.4 The following statements hold.

(i) If $F$ is transcendental and has finitely many asymptotic values then all but finitely many zeros of $f$ are simple.

(ii) If $F$ is rational and either $F(\infty) = \infty$ or $\infty$ is not a multiple point of $F$, then all zeros of $f$ in $\mathbb{C}$ are simple.

Proof. This uses standard facts involving iteration [33]. To prove (i) observe that a multiple zero of $f$ is an attracting, but not super-attracting, fixpoint of $F$, and so under iteration of $F$ attracts a critical or asymptotic value of $F$, while zeros of $F'$ in $\mathbb{C}$ are fixpoints of $F$. Now (ii) follows since the only singular values of $F^{-1}$ are the values taken by $F$ at multiple points in $\mathbb{C} \cup \{ \infty \}$, all of which are fixpoints of $F$ by Lemma 4.2 and the assumptions of (ii). □

Denote by $\partial D$ the boundary of a domain $D$ with respect to $\mathbb{C}$.

Lemma 4.5 Let $C, D$ be domains with $C \subseteq D \subseteq H^+$ and $\mathbb{R} \subseteq \partial D$, such that $F$ maps $C$ univalently onto $D$. Then $\partial C$ contains at most one point which is a pole of $f$.

Proof. Suppose that $y_1, y_2 \in \partial C$ are distinct poles of $f$. Each $y_j$ is a real repelling fixpoint of $F$ and the branch of $F^{-1}$ mapping $D$ to $C$ extends by reflection to a small neighbourhood $U_j$ of $y_j$, with an attracting fixpoint at $y_j$. The iterates $(F^{-1})^n$ of $F^{-1} : D \to C \subseteq D \subseteq H^+$ form a normal family on $D$, but as $n \to \infty$ they tend to the constant $y_j$ on $D \cap U_j$, a contradiction. □
Lemma 4.6 Let $A$ be a component of $W^+$, and suppose that a closed interval $[a, b]$ lies in $\partial A \cap \mathbb{R}$, with $a < b$ and $f(a), f(b) \in \{0, \infty\}$, and with $f(x) \neq 0, \infty$ on $(a, b)$. Then one of the following holds: 
(A) $f(a) \neq f(b)$ and $L = f'/f$ has no zeros in $(a, b)$; 
(B) $f(a) = f(b) = \infty$, the function $L$ has exactly one zero $c$ in $(a, b)$, and $c$ satisfies $L'(c) > 0$, while $F$ does not map $A$ univalently onto $H^+$. 

Proof. Observe first that all zeros of $L$ in $\mathbb{C}$ are simple, by Lemma 4.2, and that if $f(a) = f(b) = \infty$ then $F$ cannot map $A$ univalently onto $H^+$, by Lemma 4.5. Moreover, if $f(a) \neq f(b)$ then $L = f'/f$ has an even number of zeros in $(a, b)$. It follows that if neither (A) nor (B) holds then there exists at least one zero $d$ of $L$ in $(a, b)$ with $L'(d) < 0$, contradicting Lemma 4.3. \hfill \blacksquare

Lemma 4.7 Let $A$ be a component of $W^+$ which is mapped univalently onto $H^+$ by $F$, and assume that $x_1 \in \partial A \cap \mathbb{R}$ is a zero of $L = f'/f$. Then at least one of $(-\infty, x_1]$ and $[x_1, +\infty)$ lies in $\partial A$. 

Proof. Assume the contrary. Since all multiple points of $F$ in $\mathbb{C}$ are zeros of $f$, by Lemma 4.2, it is possible to start at $x_1$ and follow $\mathbb{R}$ in each direction until the first encounter with a zero or pole of $f$, giving a closed interval $[a, b] \subseteq \partial A \cap \mathbb{R}$, with $a < x_1 < b$, satisfying the hypotheses of Lemma 4.6; this is impossible, since alternative (A) is incompatible with the existence of $x_1$ and (B) with $F$ mapping $A$ univalently onto $H^+$. \hfill \blacksquare

Lemma 4.8 Let $A$ be a component of $W^+$. Then $A$ is unbounded. 

Proof. Assume the contrary: since $F$ has no critical values in $\mathbb{C} \setminus \mathbb{R}$, the mapping $F : A \to H^+$ is univalent and onto. Thus $F$ must have a pole on $\partial A \cap \mathbb{R}$, which contradicts Lemma 4.7. \hfill \blacksquare

Lemma 4.9 Let $A$ be a bounded component of $W^-$. Then $-F$ maps $A$ univalently onto $H^+$, and $\partial A$ consists of a closed interval $[a, b]$, where $-\infty < a < b < +\infty$ and $f(a) = f(b) = 0$, together with a Jordan curve $\lambda$ which joins $a$ to $b$ via $H^+$. Moreover, $\partial A$ contains precisely one zero $x_0 \in (a, b)$ of $L = f'/f$. 

Proof. First, $F$ must have a pole on $\partial A$, and so on $\partial A \cap \mathbb{R}$. Second, the mapping is univalent since $F$ has no critical values in $\mathbb{C} \setminus \mathbb{R}$. Finally, the nature of the boundary follows from the absence of bounded components of $W^+$. \hfill \blacksquare

Definition 4.1 A finite chain $D$ of bounded components of $W^-$ will mean the following: 
(a) $D$ is the union of $N \in \mathbb{N}$ bounded components $C_1, \ldots, C_N$ of $W^-$, each as in Lemma 4.9, 
(b) the boundary of each $C_j$ consists of a closed interval $[a_j, b_j]$, where $-\infty < a_j < b_j < +\infty$, together with a Jordan curve $\lambda_j$ which joins $a_j$ to $b_j$ via $H^+$; 
(c) the boundaries of the $C_j$ are disjoint except that $b_{j-1} = a_j$.

Such a finite chain $D$ will be called maximal if $D' = D$ whenever $D'$ is a finite chain of bounded components of $W^-$ with $D \subseteq D'$. 

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Lemma 4.10 Let $D$ be a maximal finite chain of bounded components of $W^-$ as in Definition 4.4. Then $a_2 = b_1, \ldots, a_N = b_{N-1}$ are common simple zeros of $f$ and $f''$, and double zeros of $F'$, and there exists a component $A$ of $W^+$ such that

$$\lambda_1 \cup \ldots \cup \lambda_N \subseteq \partial A.$$ 

Moreover, if $x^* = a_1$ or $x^* = b_N$ then $x^*$ satisfies exactly one of the following: (i) $x^*$ is a simple zero of $F'$ and a simple zero of $f$, but not a zero of $f''$; (ii) $x^*$ is a double zero of $F'$, and a common zero of $f$ and $f''$, lying on the boundary of an unbounded component $B$ of $W^-$. 

Proof. This follows from the maximality of $D$ and Lemmas 4.2 and 4.9.

5 The case where $f$ is a rational function

Proposition 5.1 Assume that $f$ is a rational function which satisfies the hypotheses of Theorem 1.1. Then there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ with $\alpha_1\alpha_2 \neq 0$ such that $g(z) = \alpha_1 f(\alpha_2 z + \alpha_3)$ is one of the functions $F_j$ in (v), (vi), (vii) and (viii) of Theorem 1.1.

The whole of this section will be occupied with the proof of Proposition 5.1. First, $f/f''$ is a polynomial and

$$L(z) = \frac{f'(z)}{f(z)} = \frac{m}{z} + O(|z|^{-2}) \quad \text{as } z \to \infty,$$

where $m$ is the number of zeros minus the number of poles, counting multiplicities, of $f$ in the finite plane. Further, $F(\infty)$ exists and is real or infinite, and all components of $W^\pm$ are mapped univalently onto $H^+$ by $\pm F$.

5.1 The case where $m \neq 0, 1$

Assume that $m \neq 0, 1$ in (19): then

$$\frac{f''(z)}{f(z)} = L'(z) + L(z)^2 = \frac{m(m-1)}{z^2} + O(|z|^{-3}) \quad \text{as } z \to \infty.$$ 

(20)

Thus $f/f''$ has degree 2 and so has either one real double zero, or two simple real zeros.

Suppose that $f/f''$ has a double zero. Then applying a real translation in the $z$ plane leads to $f(z)/f''(z) = cz^2$ for some real constant $c$, and comparison with (20) forces $f$ to satisfy

$$z^2 y''(z) = m(m-1)y(z),$$

which has linearly independent solutions $z^{d_j}$, where $d_1 = m \neq 0, 1$ and $d_2 = 1 - m \neq 0, 1$. If $f$ is a constant multiple of $z^{d_j}$, for some $j$, then clearly $f$ satisfies conclusion (v) of Theorem 1.1.

The only remaining possibility in this subcase is that there exist $a_1, a_2 \in \mathbb{C} \setminus \{0\}$ with

$$f(z) = a_1 z^m + a_2 z^{1-m} = z^m(a_1 + a_2 z^{1-2m}).$$

Since $f$ has only real zeros, the odd integer $1 - 2m$ must be $\pm 1$, and either possibility gives $m = 0$ or $m = 1$, a contradiction.
Assume next that $f/f''$ has two simple zeros, which implies that $f$ has no poles in $\mathbb{C}$ and is a polynomial of degree $n \geq 2$. A real linear change of variables then leads to
\[ z(z-1)f''(z) = df(z), \quad d \in \mathbb{R} \setminus \{0\}. \]
A comparison of leading terms shows that $d = n(n-1)$, giving equation (5), so that $f$ satisfies conclusion (vi) of Theorem 1.1 by Lemma 2.3.

5.2 The case $m = 1$

Suppose that $m = 1$ in (19): if $f$ has no poles in $\mathbb{C}$ then evidently $f$ is a linear function, contradicting the assumption that $f''/f$ has no zeros. Assume for the remainder of this section that $f$ has at least one pole in $\mathbb{C}$. Then a real linear re-scaling delivers $c \in \mathbb{R} \setminus \{0\}$ and $q \geq 1$ such that, as $z \to \infty$ and $\zeta = 1/z \to 0$,
\[
 f(z) = z \left(1 + \frac{c}{z^q + 1} + \ldots\right), \quad J(\zeta) = \frac{1}{f(1/\zeta)} = \zeta(1 - c\zeta^q + 1 + \ldots) = \zeta - c\zeta^{q+2} + \ldots.
\]
The flower theorem from complex dynamics [33] (see [3, Lemma 10] for a convenient statement of the theorem as applied here) gives $q + 1$ components $U_j$ of the Fatou set of $J$, each with $0 \in \partial U_j$ and containing a critical value $\zeta_j$ of $J$, such that the iterates $J^n$ tend to $0$ on $U_j$. Moreover the $U_j$ can be labelled so that, as $n \to +\infty$,
\[
 \arg J^n(\zeta_j) \to \frac{2\pi j - \arg c}{q + 1}.
\]
Since all critical values of $J$ belong to $\mathbb{R} \cup \{\infty\}$, because those of $f$ do, while $J(\mathbb{R} \cup \{\infty\}) \subseteq \mathbb{R} \cup \{\infty\}$, this gives a contradiction unless $q = 1$.

It follows that, again as $z \to \infty$,
\[
 f(z) = z + c/z + \ldots, \quad f'(z) = 1 - c/z^2 + \ldots, \quad f''(z) = 2c/z^3 + \ldots,
\]
\[
 \frac{f(z)}{f'(z)} = z(1 + c/z^2 + \ldots) = z(1 + 2c/z^2 + \ldots) = z + 2c/z + \ldots,
\]
\[
 F(z) = -2c/z + \ldots, \quad \frac{f''(z)}{f(z)} = 2c/z^4 + \ldots
\]
Hence $W^+$ and $W^-$ have one unbounded component $A$ between them. Moreover, all zeros of $f$ are simple by Lemma 4.4.

Since $f$ has at least one pole in $\mathbb{C}$, Lemmas 4.3, 4.5 and 4.8 imply that $A$ is a component of $W^+$ and $f$ has exactly one pole $x_0$ in $\mathbb{C}$, of order $n$ say, and hence $n + 1$ zeros in $\mathbb{C}$, all simple. Furthermore, each of these simple zeros $u$ of $f$ is a multiple point of $F$ and so lies in $\partial W^+ \cap \partial W^-$. Because $W^-$ has only bounded components, each $u$ belongs to the boundary of a maximal finite chain $D$ of bounded components of $W^-$ as in Definition 4.1.

Take such a maximal finite chain $D$: then the unique pole $x_0$ of $f$ does not lie on $\partial D$, by Lemma 4.3. Moreover, with the notation of Definition 4.1 and Lemma 4.10 $a_1$ and $b_N$ are simple
poles of $f''/f$, but the intermediate points $a_j = b_{j-1}$, $j = 2, \ldots, N$, are neither zeros nor poles of $f''/f$. Hence the closure of each such $D$ contributes exactly 2 to the number of poles of $f''/f$ in $\mathbb{C}$. Since $f''/f$ has a double pole at $x_0$, and no zeros in $\mathbb{C}$, (21) implies that there exists precisely one maximal finite chain $D$.

Hence among the $n + 1$ zeros of $f$, precisely $n - 1$ are also zeros of $f''$, and $f/f''$ has degree 4 and two simple zeros at the ends $a_1, b_N$ of $D$, plus one double zero at $x_0$. Since $x_0 \in \partial A \setminus \partial D$, a real linear change of independent variable makes it possible to assume that $x_0 = 0$ and $\partial D \cap \mathbb{R} = [1, K]$ for some $K > 1$. Hence $f$ satisfies

$$z^2(z - 1)(z - K)f''(z) = df(z), \quad d \in \mathbb{R},$$

and expanding about $z = 0$ shows that $d = Kn(n + 1)$, giving equation (8). Lemma 2.5 then implies that $f$ satisfies conclusion (vii) of Theorem 3.1. This completes the discussion of the case $m = 1$.

5.3 The case $m = 0$

Suppose that $m = 0$ in (19): then $f$ has as many zeros as poles in $\mathbb{C}$, counting multiplicities. Moreover, $f(\infty)$ is finite and real but non-zero, and it may be assumed that $f(\infty) = 1$. Further, there exist $c \in \mathbb{R} \setminus \{0\}$ and $s \geq 1$ such that, as $z \to \infty$,

$$f(z) - 1 \sim cz^{-s}, \quad L(z) = \frac{f''(z)}{f(z)} \sim f'(z) \sim -csz^{-1-s}, \quad F(z) \sim \frac{z^{1+s}}{cs}, \quad (22)$$

as well as

$$\frac{f''(z)}{f(z)} = L'(z) + L(z)^2 \sim cs(s + 1)z^{-2-s}. \quad (23)$$

Thus $F$ has a pole of multiplicity $1 + s \geq 2$ at infinity and so a super-attracting fixpoint there. Moreover, Lemma 4.4 again implies that all zeros of $f$ are simple. Assume that $W^+$ has $p$ components, all necessarily unbounded by Lemma 4.6 and $W^-$ has $q$ unbounded components, while the polynomial $f/f''$ has $r$ zeros in $\mathbb{C}$ arising from zeros of $f$, all of which must be simple zeros of $f$ which are not zeros of $f''$.

Each component $C$ of $W^+$ has at most one pole of $f$ on its boundary, by Lemma 4.5 and so precisely one, by Lemma 4.3 and the Denjoy-Wolff theorem [33] applied to the inverse function $F^{-1} : H^+ \to C$, coupled with the fact that $\infty$ is a super-attracting fixpoint of $F$ (which implies in particular that $F$ is not a Möbius transformation and $C \neq H^+$). Thus $f$ has poles at precisely $p$ points, and each is a double zero of $f/f''$. It now follows, in light of (22) and (23), that

$$|p-q| \leq 1, \quad p+q = 1 + s \geq 2, \quad 2p+r = 2 + s = p+q+1, \quad r = q-p+1, \quad 0 \leq r \leq 2. \quad (24)$$

Lemma 5.1 There do not exist $x_1, x_2, x_3 \in \mathbb{R}$ such that $x_1 < x_2 < x_3$ and $x_1, x_3$ are poles of $f$ while $x_2$ is a zero of $f'/f$.

Proof. Assume that such a triple $x_1, x_2, x_3$ does exist, and without loss of generality that $f$ has no poles in $(x_1, x_3)$. No zero of $f'/f$ can lie on the boundary of an unbounded component $B$ of $W^\pm$, by the univalence of $F$ on $B$ and the fact that $F(\infty) = \infty$. In particular, by Lemma 4.8
Suppose that \( x_1 \neq x_0 \). Then \( f \) has no poles in \((x_1, a_1)\), by the choice of \( x_1 \) and \( x_3 \), and \( x_1 \) lies on the boundary of some component \( A' \neq A \) of \( W^+ \). Since \( a_1 \), which is a zero of \( f \), lies on \( \partial A \), there must exist at least one zero of \( f' \), and so of \( f \), in \((x_1, a_1)\): let \( c_1 \) be the nearest such zero to \( a_1 \). Then there must exist a zero \( d_1 \) of \( L = f'/f \) with \( c_1 < d_1 < a_1 \) and \( L'(d_1) < 0 \), which forces \( d_1 \in \partial W^- \), so that \( d_1 \) lies on the boundary of a bounded component of \( W^- \). Because \( f \) has no zeros or poles in \((c_1, a_1)\), this contradicts the maximality of the finite chain \( D \).

Similar reasoning if \( x_3 \neq x_0 \) completes the proof of the lemma. \( \square \)

**Lemma 5.2** The integer \( s \) in (22) satisfies \( s \leq 2 \), and if \( s = 2 \) then \( c < 0 \).

**Proof.** Suppose that \( s > 2 \), or \( s = 2 \) and \( c > 0 \). Then there exist a large positive \( R \) and \( \theta_j \) satisfying \( 0 < \theta_1 < \theta_2 < \theta_3 < \pi \), with the property that \((-1)^{j+1}(f(Re^{i\theta_j}) - 1)\) is small, real and positive.

Thus \( Re^{i\theta_2} \) lies on a level curve \( \lambda_2 \) in the closed upper half-plane on which \( f(z) \) is real and \( 0 < f(z) < 1 \), and following \( \lambda_2 \) in the direction of decreasing \( f \) leads to a real zero \( y_2 \) of \( f \), possibly via one or more real zeros of \( f' \). Similarly, for \( j = 1, 3 \), the point \( Re^{i\theta_j} \) lies on a level curve \( \gamma_j \) in the closed upper half-plane on which \( f(z) \) is real and \( 1 < f(z) < +\infty \). Follow each \( \gamma_j \) in the direction of increasing \( f \): then \( \gamma_j \) must approach a real pole \( x_j \) of \( f \).

Furthermore, \( \gamma_1 \) and \( \gamma_3 \) do not meet \( \lambda_2 \) at all, and do not meet each other in the open half-plane \( H^+ \). Hence it must be the case that \( x_1 > y_2 > x_3 \). Thus \( y_2 \) lies in an unbounded component \( U \) of the set \( \{ z \in \mathbb{C} : |f(z)| < 1 \} \), which cannot contain a zero of \( f' \), by Lemma 5.1. By the Riemann-Hurwitz formula [33], or by analytic continuation of \( f^{-1} \), the function \( f \) is univalent on \( U \). But this contradicts the fact that \( y_2, \lambda_2 \) and the reflection of \( \lambda_2 \) across \( \mathbb{R} \) must all lie in \( U \). \( \square \)

If \( s = 1 \) then \( r = 1 \) and \( p = 1 \) by (24), and \( f/f'' \) has degree 3, and after a real linear re-scaling it may be assumed that \( f \) has a pole at 0, of order \( n \) say, while the remaining zero of \( f'/f'' \) lies at 1. Thus \( f \) satisfies (10) and is, by Lemma 2.6 a constant multiple of the function \( F_4 \) in conclusion (viii) of the theorem.

Now suppose \( s = 2 \) and \( c < 0 \). Then \( q = 2, p = 1 \) and \( r = 2 \) by (22) and (24). Hence \( f/f'' \) has degree 4, with one double zero at the unique pole of \( f \) and two simple zeros. After a linear re-scaling it may be assumed that 0 is a pole of \( f \) of order \( n \), and that the simple zeros of \( f/f'' \) are 1 and \( K \neq 0, 1 \). This leads to (8) and, in view of Lemma 2.5 to conclusion (viii), which completes the proof of Proposition 5.1. \( \square \)

### 6 Continuation of the proof in the transcendental case

Assume henceforth that \( f \) is transcendental and satisfies the hypotheses of Theorem 1.1. Since all zeros and poles of \( f \) and \( f'' \) are real, the Tsuji characteristic of \( L = f'/f \) satisfies [2]

\[
\Im(r, L) = O(\log r) \quad \text{as } r \to +\infty.
\] (25)
**Lemma 6.1** The function \( f/f'' \) is real entire and has order of growth at most 1. If \( f/f'' \) is a polynomial then \( f \) satisfies conclusion (i), (ii) or (iv) of Theorem 1.1.

**Proof.** The growth estimate follows from (25) and Lemma 3.4. If \( f/f'' \) is a polynomial then \( f \) has finitely many poles and a standard application of the Wiman-Valiron theory [8] implies that \( f/f'' \) has degree at most 1. If \( f/f'' \) is constant then evidently \( f \) satisfies conclusion (i) or (ii), whereas if \( f/f'' \) has degree 1 a real linear change of variables leads to \( f \) satisfying equation (3), in which case Lemma 2.2 delivers conclusion (iv). \( \blacksquare \)

Assume henceforth that \( f \) and \( f/f'' \) are both transcendental. Then Lemmas 3.1 and 6.1 together imply that

\[
\lim_{y \to +\infty, y \in \mathbb{R}} \frac{\log |f(iy)/f''(iy)|}{\log y} = +\infty.
\]

(26)

**Lemma 6.2** \( f \) has infinitely many zeros.

**Proof.** Suppose that \( f \) has finitely many zeros. Then so has \( f'' \), and hence \( f'/f \) is a rational function, by the main result of [16], and so is \( f''/f \), contrary to the assumption just made. \( \blacksquare \)

**Lemma 6.3** \( f \) has infinitely many poles.

**Proof.** Suppose that \( f \) has finitely many poles. Then Section 3.1 shows that \( L = f'/f \) has a representation \( L = P\psi \), where \( P \) and \( \psi \) are real meromorphic functions such that \( \psi(H^+) \subseteq H^+ \) and \( P \) has finitely many poles. Combining (16) with (25) delivers \( \Sigma(r, P) = O(\log r) \) as \( r \to +\infty \), and so \( P \) has order at most 1, by Lemma 3.4.

Suppose first that \( P \) is transcendental with infinitely many zeros. Then Lemma 3.1 implies that \( L(z) \) tends to infinity as \( z \to \infty \) on \( i\mathbb{R}^+ \), faster than any power of \( |z| \), and so does \( L(z) \), by (16). The fact that \( P \) has real zeros implies that, again as \( z \to \infty \) on \( i\mathbb{R}^+ \),

\[
\frac{L'(z)}{L(z)} = \frac{P'(z)}{P(z)} + \frac{\psi'(z)}{\psi(z)} = O(|z|), \quad \frac{f''(z)}{f(z)} = L(z)^2 + L'(z) = L(z)^2 + O(|z|)L(z) \to \infty,
\]

which contradicts (26).

Next, if \( P \) is transcendental with finitely many zeros then \( zL(z) \) tends to 0 on one of the rays \( \arg z = \pi/4, 3\pi/4 \) and to \( \infty \) on the other, by (16), contradicting Lemma 4.1.

Hence \( P \) must be a rational function, and \( f \) has finite order [2]. Moreover, it follows from (16) and (26) that, as \( z \to \infty \) on \( i\mathbb{R}^+ \),

\[
L(z)^2 + L'(z) = L(z)^2 + O \left( \frac{1}{|z|} \right) L(z) = \frac{f''(z)}{f(z)} = O \left( \frac{1}{|z|^2} \right), \quad zL(z) = O(1).
\]

Let \( \delta \) be small and positive. Then Lemma 4.1 implies that, as \( z \to \infty \) with \( \delta < \arg z < \pi - \delta \), \( zL(z) \) is bounded and \( \log |f(z)| = O(\log |z|) \). An application of the Phragmén-Lindelöf principle now shows that \( f \) is a rational function, contrary to assumption. \( \blacksquare \)
Lemma 6.4 The following statements hold for asymptotic values $\beta \in \mathbb{C} \cup \{\infty\}$ of $F$, that is, values $\beta$ such that $F(z) \to \beta$ as $z \to \infty$ on a path $\Gamma_\beta$.

(i) There exist at most two $\beta \in \mathbb{C} \cup \{\infty\}$ such that $\Gamma_\beta \cap \mathbb{R}$ is unbounded.

(ii) There exist finitely many $\beta \in \mathbb{C}$ for which $\Gamma_\beta \cap \mathbb{R}$ is bounded, and $F$ has finitely many asymptotic values.

(iii) All transcendental singularities of $F^{-1}$ over finite values are logarithmic.

(iv) $F$ has at most one asymptotic value $\beta \in \mathbb{C} \setminus \mathbb{R}$ with $\Gamma_\beta \setminus H^+$ bounded.

Proof. To prove (i) just note that if $\Gamma_\beta \cap \mathbb{R}$ is unbounded then $\beta \in \mathbb{R} \cup \{\infty\}$ and it may be assumed that $\Gamma_\beta$ lies in the closed upper half-plane; hence there is at most one $\beta$ such that $\Gamma_\beta \cap \mathbb{R}^+$ is unbounded, and at most one for which $\Gamma_\beta \cap \mathbb{R}^-$ is unbounded. Next, the first assertion of (ii) follows from Lemma 3.3 and on combination with (i) shows that $F$ has finitely many asymptotic values. Since all critical points of $F$ are fixpoints of $F$, all finite singular values of $F^{-1}$ are isolated, so that (iii) is a consequence of the argument from [28, p.287]. The fact that $F^{-1}$ has at most one direct singularity lying in $H^+$, by Lemma 3.3 then delivers (iv). □

Lemma 6.5 Let $D$ be a neighbourhood of a logarithmic singularity of $F^{-1}$ over $\beta \in \mathbb{R}$, such that $D \cap \mathbb{R}^+$ is unbounded. Then there exists $a \in \mathbb{R}$ with $[a, +\infty) \subseteq D$, and $f$ has finitely many zeros and poles on $\mathbb{R}^+$. Moreover, there cannot exist a neighbourhood $E \subseteq \mathbb{C} \setminus \mathbb{R}$ of a transcendental singularity of $F^{-1}$ over a finite value $\gamma \neq \beta$.

Proof. The first two assertions hold since $D$ is simply connected and symmetric with respect to $\mathbb{R}$, while all zeros and poles of $f$ are fixpoints of $F$.

Next, assume that $E$ and $\gamma$ do exist, without loss of generality with $E \subseteq H^+$. There must exist a path tending to infinity in $D \cap H^+$ on which $F(z) \to \beta$, and so $F^{-1}$ has a direct singularity over $\gamma$, lying in $H^+$, by Lemma 6.4 plus one over $\infty$, which contradicts Lemma 3.3. □

Lemma 6.6 The finite asymptotic values of $F$ comprise either a pair $\beta, \overline{\beta}$, where $\beta \in \mathbb{C} \setminus \mathbb{R}$, or one value $\beta \in \mathbb{R}$. Furthermore, all but finitely many zeros of $f$ are simple.

Proof. Suppose that $\beta, \gamma \in \mathbb{C}$ are distinct asymptotic values of $F$: then there exist simply connected neighbourhoods $D, E$ of logarithmic singularities of $F^{-1}$ over $\beta, \gamma$ respectively, by Lemma 6.4. If $D \cap \mathbb{R}^+$ and $E \cap \mathbb{R}^-$ are both unbounded then $\beta, \gamma$ are real and Lemma 6.5 applied to $f(z)$ and $f(-z)$, implies that $f$ has finitely many poles, contrary to assumption.

It may therefore be assumed that either $D$ or $E$ lies in $\mathbb{C} \setminus \mathbb{R}$, and hence that both do, by Lemma 6.5 again. But then it must be the case that one of $D, E$ lies in $H^+$ and the other in the lower half-plane $H^-$, by Lemma 3.3 and moreover that $\gamma = \overline{\beta}$.

The last assertion then follows from Lemma 4.4. □

Lemma 6.7 Suppose that $f'/f$ has finitely many zeros. Then $f$ satisfies conclusion (c) of Theorem 1.1

Proof. This can be deduced from [19] but the following proof is included in order to keep the account self-contained. The function $f/f'$ has finitely many poles, and so has order at most 1 by (25) and Lemma 3.4. On the other hand, $f/f'$ is transcendental, by Lemma 6.2.
Since all but finitely many zeros of $f$ are simple, by Lemma 6.6, the function $f$ can be written in the form $f = f_1/f_2$, in which $f_1, f_2$ are real entire functions with real zeros and no common zeros, and $f_1$ has order at most 1. Here each $f_j$ has infinitely many zeros, by Lemmas 6.2 and 6.3. Use the Levin-Ostrovskii factorisation of $f_j/f_j$ to write

$$\frac{f_j'}{f_j} = \frac{f_j'}{f_1} - \frac{f_j'}{f_2} = \phi_j \psi_1 - \phi_j \psi_2,$$

in which $\phi_j$ and $\psi_j$ are real meromorphic, while $\psi_j(H^+) \subseteq H^+$ and $\phi_j$ has finitely many poles. Since $f_1$ has finite order, (16) leads to $m(r, \phi_1) = O(\log r)$ as $r \to \infty$, and so $\phi_1$ must be a rational function. It then follows from (16), (25) and standard properties of the Tsuji characteristic that $T(r, \phi_2) = O(\log r)$ as $r \to +\infty$ and so $\phi_2$ has order at most 1 by Lemma 3.4.

Let $\delta$ be small and positive and apply Lemma 3.1 to $f/f'$. On combination with Lemma 4.1 and a standard estimate for $f_j'/f_j$ [6], this yields, as $z \to \infty$ with $\delta < \arg z < \pi - \delta$,

$$z L(z) \to 0, \quad \frac{f_j'(z)}{f_j(z)} = O(\log |z|) \quad \text{and} \quad \frac{f_j''(z)}{f_j(z)} = O(\log |z|). \quad (27)$$

It then follows in view of (16) that $\log^+ |\phi_j(z)| \leq 3 \log |z|$ as $z \to \infty$ in the same sector. Since $\delta$ may be chosen arbitrarily small, an application of the Phragmén-Lindelöf principle now shows that $\phi_2$ is a rational function, so that $f_2$ has finite order [2] and so has $f$.

The next step is to show that $f$ and $f''$ have, with finitely many exceptions, the same zeros. Since $f''/f$ has no zeros, and all but finitely many zeros of $f$ are simple, it suffices to show that all but finitely many zeros of $f$ are zeros of $f''$. Suppose then that $x_1, x_2, x_3 \in \mathbb{R}$ are zeros of $f$ but not of $f''$, such that $x_1 < x_2 < x_3$, while $|x_1|$ and $|x_3|$ are large and $x_1 x_3 > 0$. Thus $x_2$ is a simple zero of $f''$ and lies on the boundary of a component of $A$ of $W^{-}$. Hence it is possible to move along the real axis, away from $x_2$, while remaining on $\partial A$. Since $f$ cannot have a pole on $\partial A$, by Lemma 4.3 it follows that continuing along $\mathbb{R}$ in the same direction until the first encounter with a pole or zero of $f$ gives rise to a closed interval $I \subseteq \partial A$, its endpoints being zeros of $f$. This interval $I$ must then contain a zero of $f'/f$, a contradiction.

Since $f'/f$ has finitely many zeros and all but finitely many zeros of $f$ are simple, it now follows that the function

$$R = \frac{f''}{ff'}$$

has finite order and finitely many poles. As $z \to \infty$ in $\delta < \arg z < \pi - \delta$, integration of $f'/f$ using (27), coupled with a standard estimate for $f''/f'$ from [6], yields

$$\log^+ |R(z)| \leq \log^+ \frac{1}{|f(z)|} + \log^+ \left| \frac{f''(z)}{f'(z)} \right| = O(\log |z|).$$

The Phragmén-Lindelöf principle forces $R$ to be a real rational function, and all but finitely many poles of $f$ are simple. Now write $R = 2/S$ and

$$2ff' = Sf'', \quad (f^2 - Sf' + S'S) = 2ff' - Sf'' - Sf' + S'Sf + f' = Sf.$$

Hence $S''f$ is the derivative of a meromorphic function and, since $f$ has infinitely many simple poles, by Lemma 6.3, the rational function $S''$ must vanish identically and $f^2 - Sf' + Sf' = Sf$ must be a constant $c$. This yields a Riccati equation

$$Sf' = f^2 + Sf - c = P_2(f) = (f - A_1)(f - A_2), \quad A_1, A_2 \in \mathbb{C}, \quad -c = A_1 A_2. \quad (28)$$
If \( A_1 = A_2 \), then \( 1/S \) is the derivative of the transcendental meromorphic function \( -(f - A_1)^{-1} \), which is obviously impossible. Assume that \( A_1 \neq A_2 \): then \( A_1, A_2 \) are distinct Picard values, and hence asymptotic values, of \( f \) and so \( A_2 = \overline{A_1} \) by Lemma 6.6. Moreover, partial fractions yields

\[
\frac{f'}{f - A_1} - \frac{f'}{f - A_2} = \frac{A_1 - A_2}{S}.
\]

Thus \( S \) must be constant, since otherwise \( f \) is rational, a contradiction. It now follows from (28) that \( Sf' = f^2 - c \) and so \( A_1 + A_2 = 0 \). Hence \( A_1 \) and \( A_2 = \overline{A_1} \) are purely imaginary, while \( -c = A_1A_2 > 0 \), and conclusion (iii) of Theorem 1.1 follows easily.

\[\square\]

It may be assumed henceforth that \( f \) and \( f'/f \) each have infinitely many zeros and infinitely many poles.

### 7 Non-real asymptotic values

**Proposition 7.1** \( F \) has no finite non-real asymptotic values.

To prove Proposition 7.1, assume for the remainder of this section that \( F \) has an asymptotic value \( \beta \in \mathbb{C} \setminus \mathbb{R} \). Then by Lemma 6.6 and the argument of [28, p. 287], it may be assumed that the only finite asymptotic values of \( F \) are \( \beta \) and \( \overline{\beta} \) and that there exists an unbounded component \( A \) of \( W^\pm \) which contains no \( \beta \)-points of \( F \), and which is mapped “infinite to one” by \( F \) onto \( H^\pm \setminus \{\beta\} \), where \( H^- \) denotes the open lower half-plane. Moreover, the corresponding transcendental singularity of \( F^{-1} \) over \( \beta \) is logarithmic, and \( A \) is simply connected. The first main step in the proof of Proposition 7.1 will be accomplished via the following lemma.

**Lemma 7.1** There does not exist a component \( B \) of \( W^\pm \) such that \( \pm F \) maps \( B \) univalently onto \( H^+ \) and \( F(z) \to \infty \) as \( z \to \infty \) on a path in \( B \).

**Proof.** Assume the contrary, let \( \delta \) be small and positive, and set

\[
u(z) = \log^{+} \frac{\delta}{|F(z) - \beta|}, \quad (z \in A), \quad \nu(z) = 0 \quad (z \notin A),
\]

as well as \( B(t, u) = \max \{u(z) : |z| = t\} \). Then \( \nu \) is subharmonic and non-constant in the plane and Lemma 3.4 yields, for \( R \geq 1 \),

\[
\int_{R}^{+\infty} \frac{B(r/2, u)}{r^3} dr \leq \frac{3}{2\pi} \int_{R}^{+\infty} \left( \int_{0}^{\pi} u(re^{i\theta}) \frac{d\theta}{r^2} \right) \frac{dr}{r^3} \leq \frac{3}{2\pi} \int_{R}^{+\infty} \left( \int_{0}^{\pi} \log^{+} 1/|F(re^{i\theta}) - \beta| \frac{d\theta}{r^3} \right) \frac{dr}{r^3} \leq 3 \int_{R}^{+\infty} m(r, 1/(F - \beta)) \frac{dr}{r^2} \leq 3 \int_{R}^{+\infty} \Sigma(r, F) + O(\log r) \frac{dr}{r^2} \leq O \left( \frac{\log R}{R} \right).
\]
Since $B(r/2, u)$ is non-decreasing this yields $B(R, u) = O(R \log R)$ as $R \to +\infty$.

Let $\delta$ and $1/r_0$ be small and positive and denote by $\theta_A(r), \theta_B(r)$ the angular measure of the intersection with the circle $|z| = r \geq r_0$ of $A, B$ respectively. Suppose first that $\theta_A(r) < \pi(1-\delta)$ on a set $F_1$ of upper logarithmic density at least $\delta$. Then, since $A \subseteq H^+$, all sufficiently large $r \in F_1$ satisfy [2, Lemma 2.1]

\[
(1 + o(1)) \log r \geq \log B(2r, u) \geq \int_{r_0}^r \frac{\pi dt}{\theta_A(t)} - O(1)
\]

\[
\geq \int_{[r_0, r] \cap F_1} \frac{dt}{(1-\delta)t} + \int_{[r_0, r] \setminus F_1} \frac{dt}{t} - O(1)
\]

\[
\geq \int_{[r_0, r] \cap F_1} \frac{\delta dt}{(1-\delta)t} + \log r - O(1) \geq \left( \frac{\delta^2}{2(1-\delta)} + 1 \right) \log r,
\]

an evident contradiction.

Hence there exists a set $E_1$ of lower logarithmic density at least $1-\delta$ on which $\theta_A(r) \geq \pi(1-\delta)$ and so $\theta_B(r) \leq \pi \delta$, since $A, B$ are evidently not the same component of $W^\pm$. The function $w = \pm iF(z)$ maps $B$ conformally onto the right half-plane: let $z = G(w)$ be the inverse mapping, and let $\gamma_0$ be the image in $B$ under $G$ of the real interval $[1, +\infty)$, starting from $z_0 = G(1)$. Then $\gamma_0$ tends either to infinity or to a pole of $F$, and so to infinity since $F$ is univalent on $B$. Let $r^* = |z_0|$ and let $z = G(x) \in \gamma_0$, with $x \geq 1$ and $r = |z|$ large. Then applying Koebe’s quarter theorem to $G$ on the disc of centre $w \in [1, X]$ and radius $w$ leads to

\[
\log |F(z)| = \log X = \int_{[1, X]} \frac{|dw|}{|w|} = \int_{z_0}^z \frac{|dz|}{|w||G'(w)|}
\]

\[
\geq \int_{z_0}^z \frac{|dz|}{4|z|\theta_B(|z|)} \geq \int_{r^*}^r \frac{dt}{4t \theta_B(t)}
\]

\[
\geq \int_{[r^*, r] \cap E_1} \frac{dt}{4\pi \delta t} \geq \left( \frac{1 - 2\delta}{4\pi \delta} \right) \log r \geq 2 \log r,
\]

in which the second integral is from $z_0$ to $z$ along $\gamma_0$. This delivers, as $z \to \infty$ on $\gamma_0$,

\[
|F(z)| = |z(1 - 1/z L(z))| \geq |z|^2, \quad zL(z) \to 0.
\]

On the other hand, there evidently exists a path $\gamma_1$, tending to infinity in $A$, on which $F(z) \to \beta$ and hence $zL(z) \to 1$. Since $L$ has only real poles, the inverse of $zL(z)$ must have a direct singularity over $\infty$, lying in $H^+$ and separating $\gamma_0$ from $\gamma_1$. But $L$ has only real zeros, and so the inverse of $zL(z)$ must have a direct singularity over $0$, lying in $H^+$ and separating the singularity over $\infty$ from $\gamma_1$. This contradicts Lemma [3,3].

Because the component $A$ of $W^\pm$ is unbounded and simply connected and $F$ has no finite real asymptotic values, the boundary of $A$ consists of countably many pairwise disjoint piecewise analytic simple curves $\gamma_j$, each going to infinity in both directions and mapped by $F$ onto $\mathbb{R}$ or $\mathbb{R} \cup \{\infty\}$, and if $F(\gamma_j) = \mathbb{R} \cup \{\infty\}$ then $\gamma_j$ must meet $\mathbb{R}$, since $F$ has only real poles. Suppose that one of these curves, $\gamma$ say, lies wholly in $H^+$; then $\gamma$ is mapped by $F$ onto $\mathbb{R}$ and forms part of the boundary of a component $A' \neq A$ of $W^\pm$. Let $z^* \in A'$: since $F^{-1}$ cannot have two logarithmic singularities lying in $H^+$, by Lemma [3,3] analytic continuation of a local branch of the
inverse of $\mp F$ shows that $z^*$ lies in a component of $W^+$ which is mapped univalently onto $H^+$, and which must be $A'$, so that $\gamma = \partial A'$ and $F(z)$ tends to $\infty$ along a path in $A'$, contradicting Lemma 6.6. Hence each $\gamma_j$ meets $\mathbb{R}$, and there is only one, because if $\gamma_j$ meets $\mathbb{R}$ then it must separate any other $\gamma_j'$ from $\mathbb{R}$.

Thus $\partial A$ consists of a single curve, which is mapped "infinite to one" onto $\mathbb{R} \cup \{\infty\}$, and passes in each direction through infinitely many poles of $F$, all of which are real. In particular, $F^{-1}(\{\infty\})$ and $\partial A \cap \mathbb{R}$ are neither bounded above nor bounded below, and neither $W^+ \cap W^-$ has any unbounded component other than $A$. Since $W^+$ has no bounded components, by Lemma 4.7 while $f$ has by (29) infinitely many poles, all of which lie in $\partial W^+$ by Lemma 4.3, it must be the case that $A = W^+$ and $\beta \in H^+$, and all components of $W^-$ must be bounded.

Let $K_0 = \{\beta + it : 0 \leq t < +\infty\}$. Then each pole of $F$ on $\partial A$ is the starting point of a simple curve $\Lambda$ which tends to infinity in $A$ and is mapped injectively onto $\{\beta + it : 0 < t \leq +\infty\}$ by $F$. There are infinitely many of these $\Lambda$ and they are pairwise disjoint. Moreover, at most finitely many such $\Lambda$ meet the vertical line segment $I_0 = [\Re \beta, \beta]$, because otherwise $\Re F$ would be constant on $I_0$ and on the curves $\Lambda$, contradicting the absence of non-real critical points of $F$. Choose a component $I_1$ of $\mathbb{R} \setminus \{\Re \beta\}$ which contains infinitely many zeros of $f$: this is possible by Lemma 6.2. Because $\partial A$ passes in each direction through infinitely many real poles of $F$, one of the curves $\Lambda$ can be chosen to start at a pole $y_1 \in \partial A \cap I_1$ of $F$ and not meet $I_0$. If this curve is labelled $K_1$ then the set $H^+ \setminus K_1$ is the union of two disjoint domains $U_1, U_2$, with $\beta \in U_1$ and with infinitely many zeros of $f$ lying on the boundary of $U_2$.

**Lemma 7.2** Choose a simple path $\Gamma = K_2$ which starts at $\beta$, tends to infinity and lies in $U_1$, such that $K_2$ does not meet $K_0$ except at $\beta$ itself. If $y_3 \in \partial A \cap \partial U_2$ is a pole of $F$ with $y_3 \neq y_1$, then there exists a path $K_3 \subset A \cup \{y_3\}$, starting at $y_3$ and tending to infinity, which is mapped injectively by $F$ onto $K_2 \cup \{\infty\} \setminus \{\beta\}$. Moreover, $K_3$ lies in $U_2 \cup \{y_3\}$.

*Proof.* Here $K_2$ can be constructed using the fact that $K_1$ does not meet the line segment $I_0$: just follow $I_0$ vertically downwards from $\beta$ and then go to infinity within $U_1$, keeping sufficiently close to the real axis to avoid $K_1$.

The existence of $K_3$ follows from analytic continuation along $K_2$ of the branch of $F^{-1}$ which maps $\infty$ to $y_3$. The path $K_3$ lies in $A \cup \{y_3\}$ and meets $U_2$, but cannot meet $K_1$ because $K_3$ and $K_1$ start at different poles of $F$ and

\[
F(K_3 \cap K_1) \subset F(K_3) \cap F(K_1) = (K_2 \cap K_0) \cup \{\infty\} \setminus \{\beta\} = \{\infty\}.
\]

It follows that $K_3 \subset U_2 \cup \{y_3\}$.

Take a zero $x_1 \in \partial U_2$ of $f$ with $|x_1|$ large, which is possible by the choice of $U_2$. Then $x_1$ is a simple zero of $f$ and a multiple point of $F$, and $x^*$ lies on $\partial W^+ = \partial A$. Moreover, $F(z)$ describes $\mathbb{R} \cup \{\infty\}$ monotonically and "infinite to one" as the curve $\partial A$ is followed in each direction: let $y_3, y_4$ be the first poles of $F$ which are thereby reached. Since $|x_1|$ is large it may be assumed that $|y_3|$ and $|y_4|$ are large and $y_3 < x_1 < y_4$, and that $y_3, y_4 \in \partial U_2$.

Let $\Omega = H^+ \setminus K_2$. Then, since $F(x_1) = x_1 \in \mathbb{R}$, the point $x_1$ lies on the boundary of a component $C \subset A$ of $F^{-1}(\Omega)$, and $F$ maps $C$ univalently onto $\Omega$, by analytic continuation of $F^{-1}$ and the fact that $\beta \in K_2$. Furthermore, the parts of $\partial A$ described in reaching $y_3, y_4$ from $x_1$ belong also to $\partial C$. In particular, $y_3, y_4$ both lie in $\partial U_2 \cap \partial C \cap \partial A$.  

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Lemma \[7.2\] gives paths \(K_3, K_4\) with \(K_j \subseteq (A \cap U_2) \cup \{y_j\}\), each starting at \(y_j\) and tending to infinity, mapped by \(F\) onto \(K_j \cup \{\infty\} \setminus \{\beta\}\). Since \(K_2\) lies in \(U_1\), and no path in \(C\) can cross \(K_3\) or \(K_4\), it follows that \(C\) lies in \(U_2\) and thus in \(\Omega\).

Now start at \(y_3\), which is a simple pole of \(F\) in \(\partial A \cap \mathbb{R}\), and let \(z\) follow \(\partial A\) in each direction until the first encounter with a pole of \(F\) or a zero or a pole of \(f\): then \(z\) does not leave \(\mathbb{R}\) as this is done, since all critical points of \(F\) of \(f\) until the first encounter with a pole of \(F\) a pole of \(K\) that both these points must be poles of \(f\) between \(y_3\) and \(x_1\), which also lies in \(\partial C\). Doing the same for \(y_4\) shows that \(\partial C\) contains at least two distinct poles \(y_3', y_4'\) of \(f\). But this conclusion is incompatible with the choice \(D = \Omega\) in Lemma \[4.5\] giving a contradiction and hence completing the proof of Proposition \[7.1\].

\[\Box\]

8 Completion of the proof in the transcendental case

**Lemma 8.1** All components of \(W^\pm\) are mapped univalently onto \(H^\pm\) by \(\pm F\), and if \(x_1\) is a zero of \(f'/f\) with \(|x_1|\) large then \(L'(x_1) < 0\) and \(x_1\) does not lie on the boundary of a component of \(W^+\).

**Proof.** This follows from Lemmas \[4.3\] and \[4.7\] in conjunction with Proposition \[7.1\].

**Proposition 8.1** There do not exist sequences \(x_j\) of zeros of \(f'/f\) and \(y_j\) of poles of \(f\) both tending to \(+\infty\).

**Proof.** Assume the contrary: then it is possible to choose a large positive \(X_0\) and enumerate all the zeros \(x_j\) of \(f'/f\) and distinct poles \(y_j\) of \(f\) in \((X_0, +\infty)\) as

\[X_0 < x_0 < x_1 < x_2 < \ldots, \quad X_0 < y_0 < y_1 < y_2 < \ldots\]

Let \(A_j\) be the component of \(W^+\) with \(y_j \in \partial A_j\). By Lemmas \[4.5\] and \[8.1\] it may be assumed that the \(A_j\) are distinct and their boundaries contain no zeros of \(f'/f\). It then follows that each \(A_j\) contains a path tending to infinity on which \(F(z) \to \infty\). Hence at most finitely many of these \(A_j\) also contain a path tending to infinity on which \(F(z)\) tends to a finite real asymptotic value, because otherwise \(F^{-1}\) would have at least two direct singularities over \(\infty\) lying in \(H^+\), contradicting Lemma \[3.3\]. Thus it may be assumed further, for each \(j\), that \(\infty\) is the one and only asymptotic value approached by \(F\) along a path tending to infinity in \(A_j\).

Similar reasoning shows that it may now also be assumed that each \(x_j\) lies on the boundary of a component \(B_j\) of \(W^-\), these \(B_j\) being distinct and mapped univalently onto \(H^+\) by \(-F\). Hence no \(B_j\) contains a path tending to infinity on which \(F(z) \to \infty\), and again Lemma \[3.3\] implies that at most finitely many \(B_j\) contain a path on which \(F(z)\) tends to a finite real asymptotic value. Thus each of these \(B_j\) may be assumed to be bounded.

After re-labelling if necessary, poles \(y_1, y_2\) of \(f\) and a zero \(x_m\) of \(f'/f\) may be chosen with \(y_1\) large and positive and \(y_1 < x_m < y_2\). Then \(x_m\) lies on the boundary of a bounded component \(B_m\).
of $W^-$, and hence on the boundary of a maximal finite chain $D$ of bounded components of $W^-$ as in Definition 4.1 and its notation, and Lemma 4.10 applies to $D$, with $[a_1, b_N] \subseteq \partial D \cap (y_1, y_2)$. Let $A$ be the component of $W^+$ given by Lemma 4.10, then $F$ maps $A$ univalently onto $H^+$.

Suppose first that $a_1, b_N$ are both simple zeros of $F'$. Then there exist $c_1 < a_1$ and $d_N > b_N$ with $[c_1, a_1] \cup [b_N, d_N] \subseteq \partial A$. Continue along $\mathbb{R}$ leftwards from $a_1$ and rightwards from $b_N$ until the first encounter with a zero or pole of $f$: this is possible since $y_1 < a_1 < b_N < y_2$. But then conclusion (A) of Lemma 4.6 must hold, which gives at least two poles of $f$ on $\partial A$, contradicting Lemma 4.5.

Hence Lemma 4.10 forces some $x^* \in \{a_1, b_N\}$ to be a zero of $F'$ of multiplicity 2, lying on the boundary of an unbounded component $B$ of $W^-$. Starting from $x^*$, follow the real axis, in the direction away from $[a_1, b_N]$, until the first encounter with a zero or pole of $f$, again possible since $y_1 < a_1 < b_N < y_2$. Then the point so reached lies on $\partial B$ and must be a zero of $f$, by Lemma 4.3. But this gives a zero of $L = f'/f$ and hence a pole of $F$ lying on $\partial B$, so that $B$ is one of the $B_j$ and hence bounded, a contradiction. □

**Lemma 8.2** It may be assumed that:

(I) $f$ has finitely many positive poles but infinitely many negative zeros;

(II) $f'/f$ has infinitely many positive zeros but finitely many negative zeros;

(III) there exists a large $X_1 \in (0, +\infty)$ such that the zeros of $f$ and $f'/f$ in $(X_1, +\infty)$ are simple and interlaced in the sense that if $X_1 < a < b$ and $a, b$ are zeros of $f$ then $f'/f$ has a zero in $(a, b)$, while if if $X_1 < a < b$ and $a, b$ are zeros of $f'/f$ then $f$ has a zero in $(a, b)$.

**Proof.** It can certainly be assumed, by (29) and an application of Proposition 8.1 to $f(z)$ and $f(-z)$, that $f$ has finitely many positive poles but infinitely many negative zeros, while $f'/f$ has finitely many negative zeros and infinitely many positive zeros. It then follows that $f$ must also have infinitely many negative zeros, which proves (I) and (II). Together (I) and (II) imply (III), on combination with Lemmas 4.2, 4.3, 6.6 and 3.1 and the fact that if $X_1 > 0$ is large and two zeros of $f'/f$ in $(X_1, +\infty)$ are not separated by a pole or zero of $f$ or $f'/f$ then one of them has $(f'/f)' > 0$. □

It is now possible to write

$$f' = P\psi, \quad \text{(30)}$$

in which: $\psi$ is formed as Section 3.1 using zeros $0 < u_1 < u_2 < \ldots$ of $f$ and zeros $v_j \in (u_j, u_{j+1})$ of $f'/f$, and $\psi$ satisfies $\psi(H^+) \subseteq H^+$; the function $P$ is real meromorphic, with finitely many zeros in $\mathbb{C}$, and finitely many positive poles, but infinitely many negative poles.

**Lemma 8.3** The function $P$ has order of growth at most 1 and satisfies

$$\lim_{x \to +\infty, x \in \mathbb{R}} \frac{\log |P(x)|}{\log x} = -\infty.$$

**Proof.** The first assertion follows from (25), (30) and Lemma 3.4, applied to $1/P$. Next, since $P$ is transcendental, applying Lemma 3.1 to $1/P$ leads to

$$\lim_{y \to +\infty, y \in \mathbb{R}} \frac{\log |P(iy)|}{\log y} = -\infty, \quad \lim_{y \to +\infty, y \in \mathbb{R}} yL(iy) = 0. \quad \text{(31)}$$
Now let $\delta$ be small and positive. Then (16), (30) and Lemma 4.1 imply that $zL(z) \to 0$ and $P(z) \to 0$ as $z \to \infty$ with $\delta < |\arg z| < \pi - \delta$. Because $P$ has finite order and finitely many poles on $\mathbb{R}^+$, it follows from the Phragmén-Lindelöf principle that $P(z) \to 0$ as $z \to \infty$ with $|\arg z| \leq \delta$. If $N_1 \in \mathbb{N}$ is large then (31) and the Phragmén-Lindelöf principle now show that $z^{N_1}P(z)$ tends to 0 on the sector $|\arg z| \leq \pi/2$, which completes the proof. 

Since $\psi$ maps $H^+$ into itself there exists a series representation

$$
\psi(z) = az + b + \sum_{k=1}^{\infty} A_k \left( \frac{1}{u_k - z} - \frac{1}{u_k} \right),
$$

in which the $u_k$ are the poles of $\psi$, all of which are positive and zeros of $f$, while $a, b, A_k$ are real and $A_k > 0$, $\sum_{k=1}^{\infty} A_k u_k^{-2} < \infty$. On combination with Lemma 8.3 this implies that if $k$ is large then the residue of $f'/f = P\psi$ at $u_k$ is $-P(u_k)A_k = o(u_k^{-2})o(u_k^2) = o(1)$, an obvious contradiction. This completes the proof of Theorem 1.1.

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