Two-Loop Scalar Kinks

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Abstract

At one loop, quantum kinks are described by a sum of quantum harmonic oscillator Hamiltonians, and the ground state is just the product of the oscillator ground states. Two-loop kink masses are only known in integrable and supersymmetric cases and two-loop states have never been found. We find the two-loop kink mass and explicitly construct the two-loop kink ground state in a scalar field theory with an arbitrary nonderivative potential. We use a coherent state operator which maps the vacuum sector to the kink sector, allowing all states to be treated with a single Hamiltonian which needs to be renormalized only once, eliminating the need for regulator matching conditions. Our calculation is greatly simplified by a recently introduced alternative to collective coordinates, in which the kink momentum is fixed perturbatively.

1 Introduction

Quantum solitons at strong coupling are poorly understood, and yet are widely believed to be somehow responsible for confinement in Yang-Mills and QCD. Understanding them is therefore of critical importance. However we would like to suggest that this is premature as solitons at weak coupling are also not understood.

Early papers on quantum solitons produced consistent results. Beginning with the pioneering paper \cite{1}, one-loop corrections to kink masses were calculated by introducing a vacuum sector and a kink sector Hamiltonian, regularizing them both, identifying the regulators and renormalizing. In the 1970s, the regulator was a cutoff in the number of modes. In the 1980s, authors instead calculated one-loop corrections to the masses of supersymmetric kinks, regularizing with an energy cutoff. It was only in the following decade that Ref. \cite{2} reported that, when applied to the same kink, these two methods yielded different masses.

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The basic problem is as follows. A theory is defined by its Hamiltonian together with a regulator and renormalization scheme. One thus expects masses to depend on these three choices. However, once these are fixed, the theory is fixed as are all observables. In particular, nothing may depend on an arbitrary choice of matching conditions for regulators. At most one such inequivalent choice may be correct, but which?

Many responses to this question have since appeared in the literature. The most common interpretation is that some regulator matching conditions give answers which are “bad” \cite{3}. Another response is that the problem is caused by linear divergences, but these may be made logarithmic by taking a derivative with respect to a mass scale and then integrating using a physical principle to fix the constant of integration \cite{11}. This strategy has been successfully employed to reproduce the two-loop mass of the Sine-Gordon soliton. However, as noted by an overlapping collection of authors in \cite{5}, this strategy fails with some choices of boundary conditions and, more importantly, it does not shed light on which matching conditions should be allowed. Perhaps the most interesting suggestion, proposed in Ref. \cite{5}, is that an ultraviolet cutoff may only be imposed if the nontrivial background itself has no effect above that cutoff. It is an appealing physical principle, however in practice it does not entirely determine how the density of states is to be corrected. Ultimately the authors chose this correction to reproduce the known answer, leading one to wonder just what prescription works when the answer is not already known. Later it was proposed \cite{6} that instead the matching condition should keep the same mode density in every sector. However the authors note that this proposal is only expected to work at one loop.

This state of affairs has motivated our program to systematically study perturbation theory about quantum solitons in a formalism with no matching conditions. Instead, following \cite{7}, we introduce a nonlocal operator which maps the vacuum sector to the one soliton sector. This allows all computations involving both sectors to be performed using the original Hamiltonian, with no need to introduce another Hamiltonian for the soliton sector. We thus need to renormalize only once, obviating the need for regulator matching. In Refs. \cite{9,10} this was carried out at one loop in the 1+1d $\phi^4$ and Sine-Gordon models. At one loop these results were known as the theory is free. The first correction to the states was reported in Ref. \cite{11}. The present paper continues to two loops, for a general scalar kink in 1+1 dimensions. The kink ground state and mass are found.

We begin in Sec. 2 with a review of our formalism. Then we calculate the two-loop quantum ground states in two steps. Our states are decomposed in a power series in the zero mode $\phi_0$ of the scalar field. We refer to the constant terms in this decomposition as $\phi_0$-primaries and others as $\phi_0$-descendants. In Sec. 3 we use translation invariance to fix all

\footnote{For a computationally similar approach without the nonlocal operator, see Ref. \cite{8}.}
\(\phi_0\)-descendants in terms of \(\phi_0\)-primaries. Next in Sec. 4 we use Schrodinger’s equation to find the \(\phi_0\)-primaries. As an application, in Sec. 5 we present a formula for the two-loop mass correction to kinks in 1+1 dimensional scalar theories with an arbitrary potential. In Appendix A we show that the states that we have constructed indeed solve Schrodinger’s equation.

2 Review

In this section we will review the formalism for treating quantum kinks presented in Refs. [12, 9, 13]. Table 1 summarizes some of our notation.

Let \(\phi(x)\) and \(\pi(x)\) be a Schrodinger picture real scalar field and its conjugate in 1+1 dimensions, whose dynamics are described by the Hamiltonian

\[
H = \int dx \mathcal{H}(x), \quad \mathcal{H}(x) = \frac{1}{2} :\pi(x)\pi(x):_a + \frac{1}{2} :\partial_x \phi(x)\partial_x \phi(x):_a + \frac{M^2}{\hbar^2} :V[g\phi(x)]:_a .
\]

(2.1)

Here \(M\) and \(g\) have dimensions of mass and action\(^{-1/2}\) respectively. We expand in \(g^2\hbar\) and set \(\hbar = 1\). Also we will define the dimensionful potential

\[
V = M^2 \mathcal{V}.
\]

(2.2)

The normal ordering \(:a_\rangle\) is defined below.

If \(V\) has degenerate minima then there will be a classical kink solution

\[
\phi(x, t) = f(x).
\]

(2.3)

In the Schrodinger picture, where we will always work, the translation operator

\[
\mathcal{D}_f = \exp \left( -i \int dx f(x) \pi(x) \right)
\]

(2.4)

satisfies [9]

\[
:F[\pi(x), \phi(x)]:_a \mathcal{D}_f = \mathcal{D}_f :F[\pi(x), \phi(x) + f(x)]:_a
\]

(2.5)

where \(F\) is an arbitrary functional. This operator takes the vacuum sector to the kink sector. In particular one may relate the ground states \(|\Omega\rangle\) and \(|K\rangle\) of the two respective sectors

\[
|K\rangle = \mathcal{D}_f |\Omega\rangle
\]

(2.6)

using the perturbative operator \(\mathcal{O}\). The kink ground state \(|K\rangle\) is an eigenstate of the Hamiltonian \(H\) and so \(\mathcal{O}|\Omega\rangle\) must be an eigenstate of the Hamiltonian

\[
H' = \mathcal{D}_f^{-1}H\mathcal{D}_f = Q_0 + H_2 + H_1
\]

(2.7)

\[
H_2 = \frac{1}{2} \int dx \left[ :\pi^2(x):_a + :\partial_x \phi(x)^2:_a + V''[g\phi(x)]:\phi^2(x):_a \right]
\]
### Table 1: Summary of Notation

| Operator | Description |
|----------|-------------|
| $\phi(x), \pi(x)$ | The real scalar field and its conjugate momentum |
| $a_p^\dagger, a_p, A_p^\dagger, A_p$ | Creation and annihilation operators in plane wave basis |
| $b_k^\dagger, b_k, B_k^\dagger, B_k$ | Creation and annihilation operators in normal mode basis |
| $\phi_0, \pi_0$ | Zero mode of $\phi(x)$ and $\pi(x)$ in normal mode basis |
| ::a, ::b | Normal ordering with respect to a or b operators respectively |

| Hamiltonian | Description |
|------------|-------------|
| $H$ | The original Hamiltonian |
| $H'$ | $H$ with $\phi(x)$ shifted by kink solution $f(x)$ |
| $H_n$ | The $\phi^n$ term in $H'$ |

| Symbol | Description |
|--------|-------------|
| $f(x)$ | The classical kink solution |
| $D_f$ | Operator that translates $\phi(x)$ by the classical kink solution |
| $g_B(x)$ | The kink linearized translation mode |
| $g_k(x)$ | Continuum normal mode or breather |
| $\gamma_{imn}$ | Coefficient of $\phi_0^m B^n |0\rangle$ in order $i$ ground state |
| $\Gamma_i^{mn}$ | Coefficient of $\phi_0^m B^n |0\rangle$ in order $i$ Schrodinger Equation |
| $V_{ijk}$ | Derivative of the potential contracted with various functions |
| $Y_{ijk}$ | $V_{ijk}$ divided by a sum of frequencies |
| $I(x)$ | Contraction factor from Wick’s theorem |
| $p$ | Momentum |
| $k_i$ | The analog of momentum for normal modes |
| $\omega_k, \omega_p$ | The frequency corresponding to $k$ or $p$ |
| $\Omega_i$ | Sum of frequencies $\omega_k$ |
| $\tilde{g}$ | Inverse Fourier transform of $g$ |
| $Q_n$ | n-loop correction to kink energy |

| State | Description |
|-------|-------------|
| $|K\rangle, |\Omega\rangle$ | Kink and vacuum sector ground states |
| $O|\Omega\rangle$ | Translation of $|K\rangle$ by $D_f^{-1}$ |
| $O_n|\Omega\rangle$ | Translation of $|K\rangle$ by $D_f^{-1}$ at order $n$ |
Here \( Q_0 \) is the classical mass of the solution \( f(x) \) and \( H_I \) contains all higher order terms in \( g \).

The free Hamiltonian \( H_2 \) leads to classical linear equations of motion whose constant frequency solutions are the normal modes \( g(x) \) of the kink
\[
\phi(x, t) = e^{-i\omega t} g(x), \quad V''[g f(x)] g(x) = \omega^2 g(x) + g''(x).
\] (2.8)

There will be continuum solutions \( g_k(x) \) labeled by an index \( k \) such that
\[
2 \omega_k = \sqrt{M^2 + k^2}, \quad \omega_k = 0.
\]

For brevity of notation, we will not distinguish between continuum solutions and breathers, and so it will be implicit that integrals over the continuous variable \( k \) include a sum over the breathers.

We adopt the normalization conditions
\[
\int dx g_k(x) g^*_k(x) = 2\pi \delta(k_1 - k_2), \quad \int dx |g_B(x)|^2 = 1
\] (2.10)
and we choose the phases such that
\[
g_k(-x) = g_k^*(x) = g_{-k}(x).
\] (2.11)

We also define inverse Fourier transforms
\[
\tilde{g}(p) = \int dx g(x) e^{ipx}
\] (2.12)
satisfying the completeness relations
\[
\tilde{g}_B(p)\tilde{g}_B(q) + \int \frac{dk}{2\pi} \tilde{g}_k(p)\tilde{g}_{-k}(q) = 2\pi \delta(p + q).
\] (2.13)

The same quantum field and its conjugate may be expanded in terms of plane waves
\[
\phi(x) = \int \frac{dp}{2\pi} \frac{1}{\sqrt{2\omega_p}} (a^+_p + a_{-p}) e^{-ipx}, \quad \omega_p = \sqrt{M^2 + p^2}
\] (2.14)
\[
\pi(x) = i \int \frac{dp}{2\pi} \frac{\omega_p}{2} (a^+_p - a_{-p}) e^{-ipx}
\]

\(^2\)The sign of \( k \) is chosen to agree with the momentum of the corresponding plane wave at \( |x| >> 0. \)
or normal modes

\[ \phi(x) = \phi_B(x) + \phi_C(x), \quad \pi(x) = \pi_B(x) + \pi_C(x) \quad (2.15) \]

\[ \phi_B(x) = \phi_0 g_B(x), \quad \phi_C(x) = \int \frac{dk}{2\pi} \frac{1}{\sqrt{2\omega_k}} \left( b_k^+ + b_{-k} \right) g_k(x) \]

\[ \pi_B(x) = \pi_0 g_B(x), \quad \pi_C(x) = i \int \frac{dk}{2\pi} \sqrt{\frac{\omega_k}{2}} \left( b_k^+ - b_{-k} \right) g_k(x). \]

We define the plane wave normal ordering :\( a \_a \) by placing the \( a \_a \) to the left of the \( a \) and normal mode normal ordering :\( b \_b \) by placing \( b^\dagger \) and \( \phi_0 \) to the left of \( b \) and \( \pi_0 \).

Using the canonical algebra satisfied by \( \phi(x) \) and \( \pi(x) \) together with the completeness of the solutions [11]

\[ g_B(x)g_B(y) + \int \frac{dk}{2\pi} g_k(x)g_{-k}(y) = \delta(x - y) \quad (2.16) \]

one finds

\[ [a_p, a_q^\dagger] = 2\pi \delta(p - q), \quad [\phi_0, \pi_0] = i, \quad [b_{k_1}, b_{k_2}^\dagger] = 2\pi \delta(k_1 - k_2). \]

These allow the plane wave normal ordered \( H_2 \) to be rewritten in terms of a normal mode normal ordered free Hamiltonian plus a constant \( Q_1 \), which is the one-loop correction to the kink mass. This can be achieved one term a time

\[ :\pi_B^2(x) :_a = :\pi_B^2(x) :_b + g_B(x)\hat{g}_B(x), \quad \hat{g}_B(x) = -\int \frac{dp}{2\pi} e^{-ip \omega_p} \frac{\omega_p}{2} g_B(p) \quad (2.17) \]

\[ :\pi_C^2(x) :_a = :\pi_C^2(x) :_b + \int \frac{dk}{2\pi} g_k(x)\hat{g}_{-k}(x), \quad \hat{g}_k(x) = \int \frac{dp}{2\pi} e^{-ip \omega_p} \frac{\omega_p}{2} g_k(p) \]

\[ :\phi_B^2(x) :_a = :\phi_B^2(x) :_b + g_B(x)\hat{g}_B(x), \quad \hat{g}_B(x) = -\int \frac{dp}{2\pi} e^{-ip \omega_p} \frac{1}{2\omega_p} g_B(p) \]

\[ :\phi_C^2(x) :_a = :\phi_C^2(x) :_b + \int \frac{dk}{2\pi} g_k(x)\hat{g}_{-k}(x), \quad \hat{g}_k(x) = \int \frac{dp}{2\pi} e^{-ip \omega_p} \left( \frac{1}{2\omega_k} - \frac{1}{2\omega_p} \right) g_k(p). \]

Applying the classical equations of motion [2.8] one finds

\[ V''[gf(x)] :\phi_B^2(x) :_a = V''[gf(x)] :\phi_B^2(x) :_b + g_B''(x)\hat{g}_B(x) \quad (2.18) \]

\[ V''[gf(x)] :\phi_C^2(x) :_a = V''[gf(x)] :\phi_C^2(x) :_b + \int \frac{dk}{2\pi} \left( \omega_k^2 g_k(x) + g_k''(x) \right) \hat{g}_{-k}(x). \]

The \( g'' \) terms cancel : \( \partial\phi(x)\partial\phi(x) :_a - \partial\phi(x)\partial\phi(x) :_b \) after an integration by parts, leaving

\[ H_2 = Q_1 + \frac{\pi_0^2}{2} + \int \frac{dk}{2\pi} \omega_k b_k^+ b_k \quad (2.19) \]

\[ Q_1 = -\frac{1}{4} \int \frac{dk}{2\pi} \int \frac{dp}{2\pi} \frac{(\omega_p - \omega_k)^2}{\omega_p} \hat{g}_k^2(p) - \frac{1}{4} \int \frac{dp}{2\pi} \omega_p \hat{g}_B(p) \hat{g}_B(p). \]
We perform a semiclassical expansion of the kink ground state in powers of $g$

$$O|\Omega\rangle = \sum_{i=0}^{\infty} |0\rangle_i.$$  \hfill (2.20)

The one-loop kink ground state $|0\rangle_0$ is a product of free vacua

$$\pi_0|0\rangle_0 = b_k|0\rangle_0 = 0.$$ \hfill (2.21)

In Ref. [14] we found a general Wick’s formula for the conversion of plane wave to normal mode normal ordering. For powers of $\phi(x)$ it reads

$$\hat{\phi}^n(x) :_a = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{2^m m!(n-2m)!} \mathcal{I}^m(x) : \hat{\phi}^{n-2m}(x) :_b$$ \hfill (2.22)

where

$$\mathcal{I}(x) = g_B(x)\hat{g}_B(x) + \int \frac{dk}{2\pi} g_{-k}(x)\hat{g}_k(x)$$ \hfill (2.23)

$$\hat{g}_B(x) = -\int \frac{dp}{2\pi} e^{-ipx} \tilde{g}_B(p), \quad \hat{g}_k(x) = \int \frac{dp}{2\pi} e^{-ipx} \tilde{g}_k(p) \left( \frac{1}{2\omega_k} - \frac{1}{2\omega_p} \right).$$

Using the completeness relations (2.13) one can show [11] [14] that $\mathcal{I}(x)$ is determined by

$$\partial_x \mathcal{I}(x) = \int \frac{dk}{2\pi} 1 \frac{1}{2\omega_k} \partial_x |g_k(x)|^2$$ \hfill (2.24)

together with the condition that it vanish at spatial infinity.

3 Translation Invariance

In this section we will calculate the translation operator that acts on our states $O|\Omega\rangle$ and will use it to fix all $\phi_0$-descendants (components of states that include operators $\phi_0$).

3.1 The Translation Operator

Let us define the shorthand

$$\Delta_{ij} = \int dx g_i(x) g'_j(x) = i \int \frac{dp}{2\pi} p\tilde{g}_i(p)\tilde{g}'_j(-p)$$ \hfill (3.1)
where $i$ and $j$ may be a bound state or a momentum $k$. Note that $\Delta$ is antisymmetric. We will use reweighted creation and annihilation operators

\[ B_k^\dagger = \frac{b^\dagger_k}{\sqrt{2\omega_k}}, \quad B_k = \sqrt{2\omega_k}b_k \]

which satisfy the same Heisenberg commutation relations as $b^\dagger$ and $b$.

The identity

\[ PD_f = D_f \left( P - \sqrt{Q_0}\pi_0 \right) \]

implies that translation invariance

\[ P|K\rangle = PD_f \sum_i |0\rangle_i = 0 \]

is equivalent to

\[ P|0\rangle_i = \sqrt{Q_0}\pi_0|0\rangle_{i+1}. \]

Our strategy will be to solve this equation by inverting $\pi_0$. Thus translation invariance fixes our states entirely up to an element of the kernel of $\pi_0$. We then only use the Schrödinger equation to fix the element of the kernel of $\pi_0$, thus greatly simplifying the problem. Note that the kernel of $\pi_0$ consists precisely of the $\phi_0$-primary states.

Let us write the translation operator as

\[ P = -\int dx \pi(x) \partial_x \phi(x) \]

\[ = -\int dx \left[ \pi_0 g_B(x) \int \frac{dk}{2\pi} \phi_k g_k'(x) + \left( \int \frac{dk}{2\pi} \pi_k g_k(x) \right) \phi_0 g_B'(x) \right] 
\]

\[ + \int \frac{d^2k}{(2\pi)^2} \pi_{k_1} \pi_{k_2} g_{k_1} g_{k_2} \phi_0 g_B'(x) \]

\[ = \int \frac{dk}{2\pi} \Delta_{kB} \left[ i\phi_0 \left( -\omega_k B_k^\dagger + \frac{B_{-k}}{2} \right) + \pi_0 \left( B_k^\dagger + \frac{B_{-k}}{2\omega_k} \right) \right] 
\]

\[ + i \int \frac{d^2k}{(2\pi)^2} \Delta_{k_1k_2} \left[ -\omega_{k_1} B_{k_1}^\dagger B_{k_2}^\dagger + \frac{B_{-k_1} B_{-k_2}}{4\omega_{k_2}} - \frac{1}{2} \left( 1 + \frac{\omega_{k_1}}{\omega_{k_2}} \right) B_{k_1}^\dagger B_{-k_2} \right] \]

and expand the $i$th order kink ground state as

\[ |0\rangle_i = \sum_{m,n=0}^\infty |0\rangle^{mn}_i, \quad |0\rangle^{mn}_i = Q_0^{-i/2} \int \frac{d^nk}{(2\pi)^n} \phi_0 B_{k_1}^\dagger \cdots B_{k_n}^\dagger |0\rangle_0. \]
We will refer to \( m = 0 \) states or matrix elements \( \gamma_{0n} \) as \( \phi_0 \)-primary and \( m > 0 \) states as \( \phi_0 \)-descendants. Then translation invariance \( (3.5) \) yields the recursion relation

\[
\gamma_{i+1}(k_1 \cdots k_n) = \Delta_{kn} \left( \gamma_{i,n-1}(k_1 \cdots k_{n-1}) + \frac{\omega_{kn}}{m} \gamma_{i-1,n-1}(k_1 \cdots k_{n-1}) \right) 
\]

\( (3.8) \)

\[
+ \int \frac{dk'}{2\pi} \Delta_{k'n} \sum_{j=0}^{n} \left( \frac{\gamma_{i+1,n+1}(k_1 \cdots k_j, k', k_{j+1} \cdots k_n)}{2\omega_{k'}} - \frac{\gamma_{i-1,n+1}(k_1 \cdots k_j, k', k_{j+1} \cdots k_n)}{2m} \right) 
\]

\[
+ \frac{1}{2m} \sum_{j=1}^{n} \int \frac{dk'}{2\pi} \Delta_{kn-j} \left( 1 + \frac{\omega_{kn}}{\omega_{k'}} \right) \gamma_{i-1,n-1}(k_i \cdots k_{j-1}, k', k_{j} \cdots k_{n-1}) 
\]

\[
+ \frac{\omega_{kn-1}}{m} \Delta_{kn-j} \gamma_{i-1,n-2}(k_1 \cdots k_{n-2}) 
\]

\[
- \int \frac{d^2k'}{(2\pi)^2} \frac{\Delta_{k'n}}{2m\omega_{k'}} \sum_{j=1}^{n} \sum_{j'=j+1}^{n+2} \gamma_{i-1,n+2}(k_1 \cdots k_{j-1}, k', k_j \cdots k_{j-2}, k_{j+1}', k_{j+2} \cdots k_n) 
\]

This recursion relation determines all \( \phi_0 \)-descendants in terms of \( \phi_0 \)-primary states plus the free state corresponding to the one-loop initial condition \( \gamma_0 \). It does not determine the \( \phi_0 \)-primaries, as it corresponds to a particular solution of Eq. \( (3.5) \) and the addition of any element of the kernel of \( \pi_0 \), in other words any \( \phi_0 \)-primary state, is another solution.

### 3.2 Constructing Translation-Invariant States

At one loop, the quantum kink is described by a series of harmonic oscillators and so its spectrum is known precisely \[1\]. To find a Hamiltonian eigenstate at higher but finite order, one need only start the recursion \( (3.8) \) at \( i = 0 \) with the one-loop avatar of the state of interest.

In this note we will apply this strategy to the ground state, corresponding to the initial condition

\[
\gamma_{0n}^{mn} = \delta_{m0}\delta_{n0} \gamma_{00}^{00} \ \ (3.9) 
\]

One recursion yields

\[
\gamma_{12}^{12}(k_1, k_2) = \omega_{k_1}\Delta_{k_1,k_2} \gamma_{00}^{00}, \ \ \gamma_{11}^{21}(k_1) = \frac{\omega_{k_1}}{2} \Delta_{k_1B} \gamma_{00}^{00} \ \ (3.10) 
\]

We are not interested in calculating the \( \phi_0 \)-primaries \( (m = 0 \text{ terms}) \) because these are in the kernel of \( \pi_0 \) and so they are not determined by translation invariance. These will be calculated using Schrödinger’s equation in Sec. 4.

We may continue by simply plugging in to our recursion relation \( (3.8) \). But we can simplify things somewhat by noticing that \( (3.7) \) does not completely determine the functions
\( \gamma_{i+1}^{mn}(k_1 \cdots k_n) \). For example, one may add any function which is antisymmetric under the exchange of any \( k_i \) and \( k_j \) without affecting \( |0\rangle \). Therefore we are free to symmetrize each function. As this will simplify our answer, that will be our convention: It will be understood that after calculating each \( \gamma \) using (3.8) it should be symmetrized before the next recursion. This convention allows one to perform all of the sums in our recursion relation (3.8), leaving

\[
\gamma_{i+1}^{mn}(k_1 \cdots k_n) = \Delta_{k_n}B \left( \gamma_i^{m,n-1}(k_1 \cdots k_{n-1}) + \frac{\omega_{k_n}}{m} \gamma_i^{m-2,n-1}(k_1 \cdots k_{n-1}) \right) \\
+ (n+1) \int \frac{dk'}{2\pi} \Delta_{-k'}B \left( \frac{\gamma_i^{m,n+1}(k_1 \cdots k_n, k')}{2\omega_{k'}} - \frac{\gamma_i^{m-2,n+1}(k_1 \cdots k_n, k')}{2m} \right) \\
+ \frac{\omega_{k_{n-1}}}{m} \Delta_{k_{n-1}k_n} \gamma_i^{m-1,n-2}(k_1 \cdots k_{n-2}) \\
+ \frac{n}{2m} \int \frac{dk'}{2\pi} \Delta_{k_{n-1}k'} \left( 1 + \frac{\omega_{k_n}}{\omega_{k'}} \right) \gamma_i^{m-1,n}(k_1 \cdots k_{n-1}, k') \\
- \frac{(n+2)(n+1)}{2m} \int \frac{d^2k'}{(2\pi)^2} \Delta_{-k_1'-k_2'} \gamma_i^{m-1,n+2}(k_1 \cdots k_n, k'_1, k'_2). 
\]

In summary, the recursion relation (3.8) always yields a correct \( \gamma_{i+1} \) whereas the simpler (3.11) is also correct if one first symmetrizes each \( \gamma_i^{mn}(k_1 \cdots k_n) \) with respect to its arguments \( k_1 \cdots k_n \). Thus to apply (3.11) to derive \( \gamma_2 \) we must first symmetrize all \( \gamma_1^{mn} \) with \( n \geq 2 \). We only found one such element, which after symmetrizing using the antisymmetry of \( \Delta \) becomes

\[
\gamma_1^{12}(k_1, k_2) = \frac{(\omega_{k_1} - \omega_{k_2})}{2} \Delta_{k_1 k_2} |0\rangle \langle 0| .
\]

What about the \( \phi_0 \)-primaries \( \gamma_1^{0n} \)? These are not fixed by translation invariance as they are in the kernel of \( \pi_0 \). Rather they are determined using the Schrodinger equation. In a scalar theory with a canonical kinetic term, \( \phi \) will have dimensions of \([\text{action}]^{1/2}\). As each \( |0\rangle_i \) is suppressed by \( h^{1/2} \) with respect to \( |0\rangle_{i-1} \), it may only depend on terms in the potential up to \( \phi^{2+i} \). Therefore \( |0\rangle_1 \) and so \( \gamma_1 \) only depend on \( \phi^3 \) terms. As a result the only nonzero entries resulting from the Schrodinger equation can be \( \gamma_1^{01} \) and \( \gamma_1^{03} \).

Finally we are ready to apply (3.11) to calculate \( \gamma_2 \). Remember that the recursion relations only determine \( \phi_0 \)-descendants \( (m > 0) \), so over all we expect 3, 4, 5 and 6 contributions.
Figure 1: The $\gamma_{m}^n$ generated by the recursion relation at $i = 1$ (left) and $i = 2$ (right). Green stars, blue squares and red circles represent elements at $i = 0$, $i = 1$ and $i = 2$ respectively. As $\phi_0$-primaries ($m = 0$ elements) are in the kernel of $\pi_0$, they are not fixed by (3.5) and so arrows to such elements are not shown.

From $\gamma_1^{01}$, $\gamma_1^{03}$, $\gamma_1^{21}$ and $\gamma_1^{12}$ respectively. At $m = 1$ we find

\[
\gamma_{11}^{11}(k_1) = \int \frac{dk'}{2\pi} \frac{\Delta_{-k'B} \gamma_1^{12}(k_1, k')}{\omega_{k'}} - \frac{3}{4} \int \frac{d^2k'}{(2\pi)^2} \frac{\Delta_{-k_{1}'-k_{2}'} \gamma_1^{03}(k_1, k_1', k_2')}{\omega_{k_{2}'}},
\]

\[
+ \frac{1}{2} \int \frac{dk'}{2\pi} \Delta_{k_1, -k'} \left(1 + \frac{\omega_{k_1}}{\omega_{k'}}\right) \gamma_1^{01}(k')
\]

\[
= \frac{1}{2} \int \frac{dk'}{2\pi} \left(\frac{\omega_{k_1}}{\omega_{k'}} - 1\right) \Delta_{k_{1}k'} \Delta_{-k'B} \gamma_0^{00} - \frac{3}{2} \int \frac{d^2k'}{(2\pi)^2} \frac{\Delta_{-k_{1}'-k_{2}'} \gamma_1^{03}(k_1, k_1', k_2')}{\omega_{k_{2}'}},
\]

\[
+ \frac{1}{2} \int \frac{dk'}{2\pi} \Delta_{k_1, -k'} \left(1 + \frac{\omega_{k_1}}{\omega_{k'}}\right) \gamma_1^{01}(k')
\]

and

\[
\gamma_{12}^{13}(k_1, k_2, k_3) = \omega_{k_2} \Delta_{k_2k_3} \gamma_1^{01}(k_1) + \Delta_{k_3B} \gamma_1^{12}(k_1, k_2)
\]

\[
+ \frac{3}{2} \int \frac{dk'}{2\pi} \Delta_{k_3, -k'} \left(1 + \frac{\omega_{k_3}}{\omega_{k'}}\right) \gamma_1^{03}(k_1, k_2, k')
\]

\[
= \omega_{k_2} \Delta_{k_2k_3} \gamma_1^{01}(k_1) + \frac{1}{2} \Delta_{k_3B} (\omega_{k_1} - \omega_{k_2}) \Delta_{k_1k_2} \gamma_0^{00}
\]

\[
+ \frac{3}{2} \int \frac{dk'}{2\pi} \Delta_{k_3, -k'} \left(1 + \frac{\omega_{k_3}}{\omega_{k'}}\right) \gamma_1^{03}(k_1, k_2, k')
\]

\[
\gamma_{15}^{15}(k_1 \cdots k_5) = \omega_{k_4} \Delta_{k_4k_5} \gamma_1^{03}(k_1, k_2, k_3).
\]
Next at $m = 2$

\[
\gamma_{20} = \int \frac{dk'}{2\pi} \Delta_{-k'B} \left( \frac{\gamma_{1}^{21}(k') - \gamma_{1}^{01}(k')}{2\omega_{k'}} \right) - \frac{1}{4} \int \frac{d^{2}k'}{(2\pi)^{2}} \frac{\Delta_{-k_{1}',-k_{2}'} \gamma_{12}^{0}(k_{1}',k_{2}')}{\omega_{k_{2}'}} \tag{3.15}
\]

\[
\gamma_{20} = \frac{1}{4} \int \frac{dk'}{2\pi} \Delta_{-k'B} \left( \Delta_{k'B} \gamma_{00}^{[0]} - \gamma_{1}^{01}(k') \right) + \frac{1}{8} \int \frac{d^{2}k'}{(2\pi)^{2}} \left( 1 - \frac{\omega_{k_{1}'} \gamma_{12}}{\omega_{k_{2}'}} \right) \Delta_{k_{1}'k_{2}'} \Delta_{-k_{1}',-k_{2}'} \gamma_{00}^{[0]}
\]

and

\[
\gamma_{22}^{(k_{1}, k_{2})} = \Delta_{k_{2}B} \left( \gamma_{1}^{21}(k_{1}) + \frac{\omega_{k_{2}}}{2} \gamma_{1}^{01}(k_{1}) \right) - \frac{3}{4} \int \frac{dk'}{2\pi} \Delta_{-k'B} \gamma_{03}^{[0]}(k_{1}, k_{2}, k') \tag{3.16}
\]

\[
+ \frac{1}{2} \int \frac{dk'}{2\pi} \Delta_{k_{2},-k'} \left( 1 + \frac{\omega_{k_{2}}}{\omega_{k'}} \right) \gamma_{12}^{1}(k_{1}, k')
\]

\[
= \frac{\Delta_{k_{2}B}}{2} \left( \omega_{k_{1}} \Delta_{k_{1}B} \gamma_{00}^{[0]} + \omega_{k_{2}} \gamma_{1}^{01}(k_{1}) \right) - \frac{3}{4} \int \frac{dk'}{2\pi} \Delta_{-k'B} \gamma_{03}^{[0]}(k_{1}, k_{2}, k')
\]

\[
+ \frac{1}{4} \int \frac{dk'}{2\pi} \Delta_{k_{2},-k'} \left( 1 + \frac{\omega_{k_{2}}}{\omega_{k'}} \right) \left( \omega_{k_{1}} - \omega_{k'} \right) \Delta_{k_{1}k_{2}} \gamma_{00}^{[0]}
\]

\[
\gamma_{24}^{(k_{1} \cdots k_{4})} = \frac{\omega_{k_{3}} \Delta_{k_{3}k_{4}}}{2} \gamma_{1}^{12}(k_{1}, k_{2}) + \Delta_{k_{4}B} \frac{\omega_{k_{4}}}{2} \gamma_{03}^{[0]}(k_{1} \cdots k_{3})
\]

\[
= \frac{\omega_{k_{1}} \omega_{k_{3}} \Delta_{k_{1}k_{2}} \Delta_{k_{3}k_{4}}}{2} \gamma_{00}^{[0]} + \frac{\omega_{k_{4}} \Delta_{k_{4}B}}{2} \gamma_{03}^{[0]}(k_{1} \cdots k_{3}).
\]

Continuing to $m = 3$ we find

\[
\gamma_{31}(k_{1}) = -\frac{1}{3} \int \frac{dk'}{2\pi} \Delta_{-k'B} \gamma_{12}^{1}(k_{1}, k') + \frac{1}{6} \int \frac{dk'}{2\pi} \Delta_{k_{1},-k'} \left( 1 + \frac{\omega_{k_{1}}}{\omega_{k'}} \right) \gamma_{1}^{21}(k')
\]

\[
= \gamma_{00}^{[0]} \int \frac{dk'}{2\pi} \left[ \left( \omega_{k'} - \omega_{k_{1}} \right) \Delta_{k_{1}k'} \Delta_{-k'B} + \frac{1}{2} \Delta_{k_{1},-k'} \left( \omega_{k_{1}} + \omega_{k'} \right) \omega_{k'} \Delta_{k'B} \right]
\]

\[
= \gamma_{00}^{[0]} \int \frac{dk'}{2\pi} \left( \frac{\omega_{k'}}{4} - \frac{\omega_{k_{1}}}{12} \right) \Delta_{k_{1}k'} \Delta_{-k'B}
\]

\[
\gamma_{23}^{(k_{1}, k_{2}, k_{3})} = \frac{\omega_{k_{3}} \Delta_{k_{3}B}}{3} \gamma_{1}^{12}(k_{1}, k_{2}) + \frac{\omega_{k_{2}} \Delta_{k_{2}k_{3}}}{3} \gamma_{1}^{21}(k_{1})
\]

\[
= \left( \omega_{k_{3}} \Delta_{k_{3}B} \left( \omega_{k_{1}} - \omega_{k_{2}} \right) \Delta_{k_{1}k_{2}} + \omega_{k_{2}} \Delta_{k_{2}k_{3}} \omega_{k_{1}} \Delta_{k_{1}B} \right) \gamma_{00}^{[0]} / 6.
\tag{3.17}
\]

Note that, since $\gamma_{23}^{33}$ is defined by its symmetric contraction with $B_{k_{1}}^{+} B_{k_{2}}^{+} B_{k_{3}}^{+}$, one is free to add any term which is annihilated by symmetrization of $k_{1}$, $k_{2}$ and $k_{3}$. Thus one may freely redefine

\[
\gamma_{33}^{(k_{1}, k_{2}, k_{3})} = \gamma_{00}^{[0]} \frac{\omega_{k_{2}} \Delta_{k_{1}B} \gamma_{00}^{[0]} \Delta_{k_{2}k_{3}}}{2}.
\tag{3.18}
\]

In other words, different paths from $\gamma_{00}^{[0]}$ to $\gamma_{33}$ lead to contributions which are proportional. This suggests that to some extent it may be possible to explicitly solve our recursion formula.
Finally the $m = 4$ terms are

$$
\gamma_2^{40} = -\frac{1}{2\pi} \int dk' \Delta_{-k'B} \frac{\gamma_1^{21}(k')}{8} = \frac{\gamma_0^{00}}{16} \int \frac{dk'}{2\pi} \omega_{k'} \Delta B k' \Delta_{-k'B} = \frac{\gamma_0^{00}}{16}.
$$

$$
\gamma_2^{42}(k_1, k_2) = \frac{\Delta_{k_2B}}{4} \frac{\omega_{k_2}}{8} \gamma_1^{21}(k_1) = \frac{\omega_{k_1} \Delta_{k_1B} \omega_{k_2} \Delta_{k_2B}}{8} \gamma_0^{00}.
$$

4 Schrodinger Equation

Let us define the symbol $\Gamma$ by any solution of

$$
\sum_{j=0}^{i} \left( H_{i+2-j} - Q_{\frac{j}{2}+1} \right) |0\rangle_j = Q_{0}^{-i/2} \sum_{mn} \frac{d^n k}{(2\pi)^n} \Gamma_{i}^{mn}(k_1 \cdots k_n) \phi_0^{m} B^\dagger_{k_1} \cdots B^\dagger_{k_n} |0\rangle_0.
$$

Then the Schrodinger Equation

$$
(H - Q)|0\rangle = 0
$$

is solved if

$$
\Gamma_{i}^{mn} = 0.
$$

Note that $\Gamma$ is not uniquely defined by (4.1). A necessary and sufficient condition for a solution to Schrodinger’s equations is that $\Gamma_{i}^{mn}$ vanishes when summed over all permutations of the $k_j$. The number of loops can be defined by counting powers of $\hbar$ and is equal to $i/2 + 1$. Note that only integral numbers of loops correct the energy, and so $Q$ vanishes if its subscript is a half-integer. Here $Q$ is defined to be the energy of the ground state. For applications to other states, $Q$ should be replaced with their respective energies.

Let us begin with the one-loop approximation, $i = 0$. Using

$$
H_2 - Q_1 = \frac{\pi_0^2}{2} + \int \frac{dk}{2\pi} \omega_k B^\dagger_k B_k
$$

one finds that the Schrodinger equation is satisfied if

$$
\pi_0 |0\rangle_0 = B_k |0\rangle_0 = 0.
$$

These are both satisfied by the initial condition $\gamma_{0}^{mn} = \delta_{m0}\delta_{n0}$ of our recursion.

4.1 Leading Corrections

At $i = 1$ the Schrodinger equation is

$$
H_3 |0\rangle_0 + (H_2 - Q_1) |0\rangle_1 = 0.
$$
Using
\[ H_3 = \frac{1}{6} \int dx V^{(3)}[gf(x)] : \phi^3(x) : = \frac{1}{6} \int dx V^{(3)}[gf(x)] : \phi^3(x) : + \frac{1}{2} \int dx V^{(3)}[gf(x)] \phi(x) I(x) \] (4.7)
where we have defined \( V^{(n)}[gf(x)] \) to be the \( n \)th derivative of \( g^{-2}V[g\phi(x)] \) with respect to \( \phi(x) \), evaluated at \( \phi(x) = f(x) \), one finds that the leading correction to the states \((3.10)\) yields
\[
\Gamma_{1}^{12} = \sqrt{\epsilon_{0}} B_{k_{1}k_{2}} \frac{(\omega_{k_{1}} - \omega_{k_{2}})(\omega_{k_{1}} + \omega_{k_{2}})}{2} \Delta_{k_{1}k_{2}}
\] (4.8)

where we have introduced the notation
\[
V_{\lambda_{m},\alpha_{1}...\alpha_{n}} = \int dx V^{(2m+n)}[gf(x)] I^{m}(x) g_{\alpha_{1}}(x) ... g_{\alpha_{n}}(x)
\] (4.9)

where \( \alpha_{j} \) can be \( B \) or \( k_{j} \).

Substituting the identities [11]
\[
V_{Bk} = \int dx V^{(3)}[gf(x)] g_{B}(x) \frac{f'(x)}{\sqrt{\epsilon_{0}}} g_{k}(x) = \frac{1}{\sqrt{\epsilon_{0}}} \int dx \partial_{x} \left( V^{(2)}[gf(x)] \right) g_{B}(x) g_{k}(x)
\]
\[
= -\frac{1}{\sqrt{\epsilon_{0}}} \int dx V^{(2)}[gf(x)] \left( g_{B}'(x) g_{k}(x) + g_{B}(x) g_{k}'(x) \right)
\]
\[
= -\frac{1}{\sqrt{\epsilon_{0}}} \int dx \left( g_{B}'(x) \omega_{k}^{2} g_{k}(x) + g_{B}(x) g_{k}''(x) + g_{B}'(x) g_{k}'(x) \right)
\]
\[
= -\frac{\omega_{k}^{2}}{\sqrt{\epsilon_{0}}} \Delta_{kB}
\]
\[
V_{Bk_{1}k_{2}} = -\frac{1}{\sqrt{\epsilon_{0}}} \int dx V^{(2)}[gf(x)] \left( g_{k_{1}}'(x) g_{k_{2}}(x) + g_{k_{1}}(x) g_{k_{2}}'(x) \right)
\]
\[
= -\frac{1}{\sqrt{\epsilon_{0}}} \int dx \left( g_{k_{1}}'(x) \omega_{k_{2}}^{2} g_{k_{2}}(x) + g_{k_{1}}(x) g_{k_{2}}''(x) + \omega_{k_{2}}^{2} g_{k_{1}}(x) g_{k_{2}}'(x) + g_{k_{1}}'(x) g_{k_{2}}'(x) \right)
\]
\[
= \frac{\omega_{k_{2}}^{2} - \omega_{k_{1}}^{2}}{\sqrt{\epsilon_{0}}} \Delta_{k_{1}k_{2}}
\] (4.10)

into (4.8) one finds \( \Gamma = 0 \), and so these matrix elements of Schrödinger’s equation are satisfied by the states \((3.10)\), which were derived from translation invariance alone. This is consistent with our claim that all \( \phi_{0} \)-descendants \((m > 0) \) components of states) are determined in terms of \( \phi_{0} \)-primaries by imposing the eigenvalue of the momentum, in this case zero.
The other components of the Schrödinger equation at $i = 1$ are

$$\Gamma_{01}^i = \sqrt{Q_0} V_{1k_1} - \frac{\omega_{k_1} \Delta_{k_1} B}{2} + \omega_{k_1} \gamma_{01}^i, \quad \Gamma_{03}^i = \sqrt{Q_0} V_{k_1 k_2 k_3} + (\omega_{k_1} + \omega_{k_2} + \omega_{k_3}) \gamma_{01}^i$$  \hspace{1cm} (4.11)

and so the state at order $i = 1$ is given by the $\phi_0$-descendants in Eqs. (3.10) and (3.12) together with the $\phi_0$-primaries

$$\gamma_{01}^i = \Delta_{k_1} B - \frac{\sqrt{Q_0}}{2} Y_{1k_1}, \quad \gamma_{03}^i = -\frac{\sqrt{Q_0}}{6} Y_{k_1 k_2 k_3}$$  \hspace{1cm} (4.12)

where we have defined the reduced potential

$$Y_{k_1 \ldots k_j} = \frac{V_{k_1 \ldots k_j}}{\omega_{k_1} + \cdots + \omega_{k_j}}, \quad Y_{\bar{I}, k_1 \ldots k_j} = \frac{V_{\bar{I}, k_1 \ldots k_j}}{\omega_{k_1} + \cdots + \omega_{k_j}}.$$  \hspace{1cm} (4.13)

Note that in models like the $\phi^4$ double well, in which the third derivative of the potential is nonzero at the minima, $V_{k_1 k_2 k_3}$ will have a divergence of the form $\delta (\sum_i k_i)$. When integrated over $k$ to determine the state, this divergence leads to finite coefficients. However at two loops it leads to an infrared divergence in the energy of the kink state. As we will see in Subsec. 5.2, this infrared divergence also appears in the vacuum energy and so the kink mass, which is the difference between the energies of the two states, is finite.

### 4.2 The Kink Ground State at Two Loops

Translation invariance fixes all $\phi_0$-descendant components $\gamma_{imn}$ in any Hamiltonian eigenstate. The $\phi_0$-primary terms $\gamma_{i0n}^m$, at each order $i$ are fixed by the Schrödinger equation. Interaction terms relate these coefficients to those at lower orders. Thus the other $\gamma_{imn}$ are only related to $\gamma_{i0n}^m$ by the free Hamiltonian \[4.4\]. More specifically, $\gamma_{i0n}^m$ is related to $\gamma_{i2n}^m$ by the $\pi_0^2/2$ term and to $\gamma_{i0n}^m$ by the oscillator term. This allows each $\phi_0$-primary $\gamma_{i0n}^m$ to be determined from $\gamma_{i2n}^m$ and the state at orders less than $i$. In theories, like those considered here, with nonderivative interactions the situation is even simpler because interactions never decrease $m$. Thus Schrödinger’s equation determines $\phi_0$-primaries $\gamma_{i0n}^m$ in terms of $\gamma_{i2n}^m$ and $\phi_0$-primaries $\gamma_{j0n}^m$ with $j < i$. In other words, only the $\phi_0$-descendants at $m = 2$ are needed. Similarly the energy at each order $i$ is determined by $\gamma_{i20}^m$ together with the $\phi_0$-primaries $\gamma_{j0n}^m$ at lower orders $j < i$.

This observation in practice leads to a dramatic reduction in the complexity of calculations of states and energies. For example, to compute the two-loop energy of the kink ground state, one only needs to know $\gamma_{220}^0$, $\gamma_{01}^0$ and $\gamma_{03}^0$, which themselves are determined from $\gamma_{12}^0$ and $\gamma_{13}^0$. In this subsection we will complete the calculation the kink ground state at two loops, corresponding to $i = 2(2 - 1) = 2$, by finding the $\phi_0$-primaries.
Figure 2: Terms in the Schrodinger equation: (top left) $H_3|0\rangle_0$ in black and $(H_2 - Q_1)|0\rangle_1$ in red, (top right) $(H_4 - Q_2)|0\rangle_0$ in blue and $(H_2 - Q_1)|0\rangle_2$ in green, (bottom) $H_3|0\rangle_1$ in black.
1 \ (m = 0, n = 6)

At \( i = 2 \) the Schrödinger equation is

\[
(H_4 - Q_2)|0\rangle_0 + H_3|0\rangle_1 + (H_2 - Q_1)|0\rangle_2 = 0. \tag{4.14}
\]

Let us begin with the simplest element, \( \Gamma_{06}^2 \). The previous argument agrees with Fig. 2 showing that there are two contributions, arising from \( \gamma_{03}^1 \), which was found at the previous order, and from \( \gamma_{06}^2 \) which is to be found now. Defining the total energy

\[
\Omega_n = \sum_{j=1}^{n} \omega_{k_j} \tag{4.15}
\]

these contributions are

\[
H_3|0\rangle_1^{03} \supset -\frac{1}{36} \int \frac{d^4k}{(2\pi)^4} Y_{k_1 k_2 k_3} V_{k_1 k_2 k_3} B_{k_1}^\dagger \cdots B_{k_6}^\dagger |0\rangle_0 \tag{4.16}
\]

\[
H_2|0\rangle_2^{06} = \frac{1}{Q_1} \int \frac{d^4k}{(2\pi)^4} \Omega_6 \gamma_{06}^2 B_{k_1}^\dagger \cdots B_{k_6}^\dagger |0\rangle_0
\]

and so one finds the matrix element

\[
\gamma_{06}^2 = \frac{Q_0}{36} Y_{k_1 k_2 k_3} V_{k_1 k_2 k_3} \Omega_6. \tag{4.17}
\]

2 \ (m = 0, n = 4)

To organize the calculations of the other matrix elements, we note that \( \Gamma \) may be decomposed into contributions which do not mix with one another. In particular contributions with different numbers of dummy momenta \( k' \) and with different numbers of powers of the undifferentiated contraction factor \( \mathcal{I}(x) \) together with \( V^{(3)} \) do not mix. We will include this decomposition in the subscript of \( \Gamma \). Of course each \( \Gamma_i^{0n} \) determines \( \gamma_i^{0n} \) whose form is not known before \( \Gamma_i^{0n} \) is calculated, so terms resulting from \( \gamma_i^{0n} \) will not be included in this decomposition.

Let us begin with all contributions \( \Gamma_{04}^{06} \) containing a single power of the contraction factor \( \mathcal{I}(x) \). These contributions arise from two terms

\[
H_3|0\rangle_1^{03} \supset -\frac{1}{12} \int \frac{d^4k}{(2\pi)^4} V_{k_1} Y_{k_1 k_2 k_3} B_{k_1}^\dagger \cdots B_{k_4}^\dagger |0\rangle_0 \tag{4.18}
\]

\[
H_3|0\rangle_1^{01} \supset -\frac{1}{12} \int \frac{d^4k}{(2\pi)^4} V_{k_1} Y_{k_1 k_2 k_3} B_{k_1}^\dagger \cdots B_{k_4}^\dagger |0\rangle_0
\]

\footnote{If multiplied by \( V^{(4)}_{BB}(x) \) then an integration by parts leads to a differentiated \( \mathcal{I}(x) \) which can be evaluated using (2.24) and this argument does not apply. This situation does not arise in the calculation of \( \gamma_2^{0n} \) but does arise when verifying that the Schrödinger equation is satisfied in Appendix A.}
whose sum yields
\[ \Gamma_{22}^{04} = -\frac{Q_0}{12} Y_{k_1 k_2 k_3} \Omega_4. \] (4.19)

Next let us consider the contributions with one contracted momentum \( k' \). There is only one
\[ H_3 |0\rangle_1^{03} \supset -\frac{1}{8} \int \frac{d^4k}{(2\pi)^4} \int \frac{dk'}{2\pi} Y_{k_1 k_2 - k'} \frac{V_{k_3 k_4 - k'}}{\omega_{k'}} B_{k_1}^\dagger \cdots B_{k_4}^\dagger |0\rangle_0 \] (4.20)
yielding
\[ \Gamma_{2k'}^{04} = -\frac{Q_0}{8} \int \frac{dk'}{2\pi} Y_{k_1 k_2 - k'} \frac{V_{k_3 k_4 - k'}}{\omega_{k'}}. \] (4.21)

Finally there are three contributions with no \( k' \)
\[ H_3 |0\rangle_1^{01} \supset \frac{1}{12\sqrt{Q_0}} \int \frac{d^4k}{(2\pi)^4} \Delta_{k_1 B} V_{k_2 k_3 k_4} B_{k_1}^\dagger \cdots B_{k_4}^\dagger |0\rangle_0 \] (4.22)
\[ \frac{\pi Q_0}{2} |0\rangle_2^{24} = \frac{1}{12Q_0} \int \frac{d^4k}{(2\pi)^4} \left[ -6\omega_{k_1} \omega_{k_3} \Delta_{k_1 k_2} \Delta_{k_3 k_4} + \sqrt{Q_0} Y_{k_1 k_2 k_3} \omega_{k_4} \Delta_{k_4 B} \right] B_{k_1}^\dagger \cdots B_{k_4}^\dagger |0\rangle_0 \]
\[ H_4 |0\rangle_0 = \frac{1}{24} \int \frac{d^4k}{(2\pi)^4} V_{k_1 k_2 k_3 k_4} B_{k_1}^\dagger \cdots B_{k_4}^\dagger |0\rangle_0 \]
which sum to
\[ \Gamma_{2k^0}^{04} = \frac{\sqrt{Q_0}}{12} Y_{k_1 k_2 k_3} \Delta_{k_4 B} \Omega_4 - \frac{\omega_{k_1} \omega_{k_3}}{2} \Delta_{k_1 k_2} \Delta_{k_3 k_4} + \frac{Q_0}{24} V_{k_1 k_2 k_3 k_4}. \] (4.23)

The final contribution to \( \Gamma_{22}^{04} \) arises from
\[ \int \frac{dk}{2\pi} \omega_{k} B_{k}^\dagger B_{-k} |0\rangle_2^{04} = \frac{1}{Q_0} \int \frac{d^4k}{(2\pi)^4} \Omega_4 \gamma_2^{04} B_{k_1}^\dagger \cdots B_{k_4}^\dagger |0\rangle_0 \] (4.24)
and is
\[ \Gamma_{2f}^{04} = \Omega_4 \gamma_2^{04}. \] (4.25)

The Schrodinger equation
\[ 0 = \Gamma_{22}^{04} = \Gamma_{2f}^{04} + \Gamma_{2k^0}^{04} + \Gamma_{2k'}^{04} \] (4.26)
then yields the matrix element
\[ \gamma_2^{04} = -\frac{\Gamma_{22}^{04} + \Gamma_{2k^0}^{04} + \Gamma_{2k'}^{04}}{\Omega_4} \] (4.27)
\[ = \frac{Q_0}{12} Y_{k_1 k_2 k_3} - \frac{\sqrt{Q_0}}{12} Y_{k_1 k_2 k_3} \Delta_{k_4 B} + \frac{\omega_{k_1} \omega_{k_3}}{2\Omega_4} \Delta_{k_1 k_2} \Delta_{k_3 k_4} - \frac{Q_0}{24} Y_{k_1 k_2 k_3 k_4} \]
\[ + \frac{Q_0}{8\Omega_4} \int \frac{dk'}{2\pi} \frac{Y_{k_1 k_2 k_4 - k'}}{\omega_{k'}}. \]
Note that in models like the Sine-Gordon model, in which the fourth derivative of the potential is nonzero at the minima, $Y_{k_1 k_2 k_3 k_4}$ will have a divergence of the form $\delta(\sum_i k_i)$. When integrated over $k$ to determine the state, this divergence leads to finite coefficients. However at three loops it leads to an infrared divergence in the energy of the kink state. As in the two-loop divergence in the $\phi^4$ kink energy, this divergence also appears in the vacuum energy and so the kink mass remains finite. We expect such cancellations at all loops, as the infrared divergences arise from a regime in $x$ where $f(x)$ is equal to a vacuum value, and so the energy contribution from the kink and vacuum sector should agree.

3 $\ (m = 0, n = 2)$

The last matrix element needed to fix the ground state at two loops is $(m = 0, n = 2)$. There is one contribution with two powers of the contraction factor $I$

$$H_3|0\rangle_{01}^{01} \supset -\frac{1}{4} \int \frac{d^2k}{(2\pi)^2} Y_{k_1 k_2} V_{k_1, k_2} B_{k_1}^\dagger B_{k_2}^\dagger |0\rangle_0 \quad (4.28)$$

which, after adding an antisymmetric term which does not affect the sum, yields

$$\Gamma_{02}^{02} = -\frac{Q_0}{8} Y_{k_1} Y_{k_2} \Omega_2. \quad (4.29)$$

There are four contributions with a single power of $I$

$$\frac{\pi_0^2}{2} |0\rangle_{22}^{02} \supset \frac{1}{4\sqrt{Q_0}} \int \frac{d^2k}{(2\pi)^2} Y_{k_1, k_2} \Delta_{k_2 B} B_{k_1}^\dagger B_{k_2}^\dagger |0\rangle_0 \quad (4.30)$$

$$H_3|0\rangle_{01}^{01} \supset \int \frac{d^2k}{(2\pi)^2} \left( \frac{1}{4\sqrt{Q_0}} V_{k_1, k_2} \Delta_{k_1 B} - \frac{1}{8} \int \frac{dk'}{2\pi} Y_{k, k'} \frac{V_{k_1, k_2 - k'}}{\omega_{k'}} \right) B_{k_1}^\dagger B_{k_2}^\dagger |0\rangle_0$$

$$H_4|0\rangle_0 \supset \frac{1}{4} \int \frac{d^2k}{(2\pi)^2} V_{k_1, k_2} B_{k_1}^\dagger B_{k_2}^\dagger |0\rangle_0$$

$$H_3|0\rangle_{03}^{03} \supset -\frac{1}{8} \int \frac{d^2k}{(2\pi)^2} \int \frac{dk'}{2\pi} Y_{k, k'} Y_{k_1, k_2 - k'} B_{k_1}^\dagger B_{k_2}^\dagger |0\rangle_0$$

which together contribute

$$\Gamma_{02}^{02} = Q_0^2 Y_{k_1, k_2} \Omega_2 + \frac{Q_0 V_{k_1, k_2}}{4} - \frac{Q_0}{8} \int \frac{dk'}{2\pi} Y_{k, k'} Y_{k_1, k_2 - k'} \left( 2 + \frac{\Omega_2}{\omega_{k'}} \right). \quad (4.31)$$

Now we will organize the terms with no powers of $I$ by the number of contracted momenta $k'$. There is one term with two contracted momenta $k'$. This is $H_3|0\rangle_{01}^{03}$

$$H_3|0\rangle_{01}^{03} \supset -\frac{1}{8} \int \frac{d^2k}{(2\pi)^2} \int \frac{d^2k'}{(2\pi)^2} Y_{k_1, k_1', k_2, k_2'} \frac{V_{k_2 - k_1', k_2'} B_{k_1}^\dagger B_{k_2}^\dagger}{\omega_{k_1'} \omega_{k_2'}} |0\rangle_0 \quad (4.32)$$
yields
\[ \Gamma_{2k^2}^{02} = -\frac{Q_0}{8} \int \frac{d^2k'}{(2\pi)^2} Y_{k_1k_2} V_{k_2-k'_1-k'_2} \omega_{k_1} \omega_{k_2}. \]  
(4.33)

There are two sources of terms with no \( I \) and a single \( k' \)

\[ H_3|0\rangle^0_1 \supset \frac{1}{8\sqrt{Q_0}} \int \frac{d^2k}{(2\pi)^2} \frac{dk'}{2\pi} Y_{k_1k_2k'} \Delta_{-k'B} B_{k_1}^{\dagger} B_{k_2}^{\dagger} |0\rangle_0 \]  
(4.34)

\[ \frac{\pi^2}{2} |0\rangle^2_2 \supset \int \frac{d^2k}{(2\pi)^2} \frac{dk'}{2\pi} \left[ -\frac{1}{8\sqrt{Q_0}} Y_{k_1k_2k'} \Delta_{-k'B} \right. 
\left. + \frac{1}{4Q_0} \Delta_{k_1k'} \Delta_{-k'k_2} \left( 1 + \frac{\omega_{k_1}}{\omega_{k'}} \right) \left( \omega_{k_1} - \omega_{k'} \right) \left( \omega_{k_1} - \omega_{k'} \right) \right] B_{k_1}^{\dagger} B_{k_2}^{\dagger} |0\rangle_0 \]

which contribute

\[ \Gamma_{2k^2}^{02} = \int \frac{dk'}{2\pi} \left[ \frac{\sqrt{Q_0}}{8} \frac{\Omega_2}{\omega_{k'}} Y_{k_1k_2k'} \Delta_{-k'B} \right. 
\left. + \frac{1}{4} \Delta_{k_1k'} \Delta_{-k'k_2} \left( \frac{\omega_{k_1} \omega_{k_2}}{\omega_{k'} - \omega_{k'}} \right) \left( \omega_{k_1} - \omega_{k'} \right) \right]. \]  
(4.35)

Finally the terms with neither \( I \) nor \( k' \) are

\[ \frac{\pi^2}{2} |0\rangle^2_2 \supset -\frac{3}{8Q_0} \int \frac{d^2k}{(2\pi)^2} \Omega_2 \Delta_{k_1B} \Delta_{k_2B} B_{k_1}^{\dagger} B_{k_2}^{\dagger} |0\rangle_0 \]  
(4.36)

and so

\[ \Gamma_{2k^0}^{02} = -\frac{3}{8} \Omega_2 \Delta_{k_1B} \Delta_{k_2B}. \]  
(4.37)

As in the previous cases,

\[ \Gamma_{2f}^{02} = \Omega_2 \gamma_{2}^{02} \]  
(4.38)

and so the Schrodinger equation

\[ 0 = \Gamma_{2}^{02} = \Gamma_{2f}^{02} + \Gamma_{2\gamma}^{02} + \Gamma_{2k^2}^{02} + \Gamma_{2k^0}^{02} + \Gamma_{2k^2}^{02} + \Gamma_{2k^0}^{02} \]  
(4.39)

fixes the last matrix element

\[ \gamma_{2}^{02} = -\frac{\Gamma_{2\gamma}^{02} + \Gamma_{2k^2}^{02} + \Gamma_{2k^0}^{02} + \Gamma_{2k^2}^{02}}{\Omega_2} \]  
(4.40)
5 The Kink Mass

5.1 The Energy of the Kink Ground State

The last Schrödinger equation is \( \Gamma_{00}^{00} = 0 \). This does not fix \( \gamma_{00}^0 \) because \( \Gamma_{00}^{00} \) does not depend on \( \gamma_{00}^0 \). This is reasonable because any value of \( \gamma_{00}^0 \) can be absorbed into the normalization of the state. Thus one may normalize the ground state so that

\[
\gamma_{00}^0 = \delta_{00}.
\]

Let us now solve this last Schrödinger equation. There are two terms with two powers of the contraction factor \( I \)

\[
H_1|0\rangle_0 \supset \frac{V_{II}}{8}|0\rangle_0
\]

\[
H_1|0\rangle^0_1 \supset -\frac{1}{8} \int \frac{dk'}{2\pi} Y_{I'k'} Y_{I-k'}|0\rangle_0
\]

yielding

\[
\Gamma_{22}^{00} \supset \frac{Q_0}{8} \left( V_{II} - \int \frac{dk'}{2\pi} Y_{I'k'} Y_{I-k'} \right).
\]

There are also two terms with a single factor of \( I \)

\[
H_1|0\rangle^0_1 \supset \frac{1}{8\sqrt{Q_0}} \int \frac{dk'}{2\pi} Y_{I'k'} \Delta_{-k'B}|0\rangle_0
\]

\[
\frac{\pi_0^2}{2}|0\rangle^2_2 \supset -\frac{1}{8\sqrt{Q_0}} \int \frac{dk'}{2\pi} Y_{I'k'} \Delta_{-k'B}|0\rangle_0
\]

which precisely cancel. The terms with no factors of \( I \) can be organized by the number of contracted momenta \( k' \). There is one term with 3, 2 and 1 momenta respectively, which for brevity we summarize together

\[
H_3|0\rangle^0_1 \supset -\frac{1}{48} \int \frac{d^3k'}{(2\pi)^3} Y_{k'k''k'''} \frac{V_{-k'_1-k'_2-k'_3}}{\omega_{k'_1} \omega_{k'_2} \omega_{k'_3}}
\]

\[
\frac{\pi_0^2}{2}|0\rangle^2_2 \supset \frac{1}{16Q_0} \int \frac{d^2k'}{(2\pi)^2} \frac{\left(\omega_{k'_1} - \omega_{k'_2}\right)^2}{\omega_{k'_1} \omega_{k'_2}} \Delta_{k'_1k'_2} \Delta_{-k'_1-k'_2}
\]

\[
\frac{\pi_0^2}{2}|0\rangle^2_2 \supset -\frac{1}{8Q_0} \int \frac{dk'}{2\pi} \Delta_{Bk'} \Delta_{B-k'}.
\]

As

\[
g^*_k(x) = g_{-k}(x)
\]
the symbols $\Delta$, $V$ and $Y$ are all complex conjugated when all of their $k$ arguments are negated. Therefore these contributions can each be rewritten as norms squared and so are real. The corresponding $\Gamma$ can therefore be written

$$
\Gamma_{2k'^3}^{00} = -\frac{Q_0}{48} \int \frac{d^3k'}{(2\pi)^3} \frac{|V_{k'k'k}|^2}{\omega_{k'} \omega_{k'} \omega_{k'} \left(\omega_{k'} + \omega_{k'} + \omega_{k'}\right)} \tag{5.7}
$$

$$
\Gamma_{2k'^2}^{00} = \frac{1}{16} \int \frac{d^2k'}{(2\pi)^2} \frac{\left|\left(\omega_{k'} - \omega_{k'}\right) \Delta_{k'k'}\right|^2}{\omega_{k'} \omega_{k'}}
$$

$$
\Gamma_{2k'}^{00} = -\frac{1}{8} \int \frac{dk'}{2\pi} |\Delta_{k'B}|^2.
$$

The last term may be written in a more convenient form using the completeness relation

$$
\int \frac{dk}{2\pi} \Delta_{kB} \Delta_{-kB} = \frac{1}{Q_0} \int dx \int dy \int \frac{dk}{2\pi} g_k(x) g_{-k}(y) f''(x) f''(y)
$$

$$
= \frac{1}{Q_0} \int dx \int dy (\delta(x-y) - g_B(x) g_B(y)) f''(x) f''(y) = \frac{1}{Q_0} \int dx |f''(x)|^2
$$

where the $g_B(x) f''(x)$ integrals vanish because they are proportional to the total derivative of $g_B^2(x)$.

The Schrodinger equation then gives the two-loop energy

$$
Q_2 = \frac{V_{\Xi \Xi}}{8} - \frac{1}{8} \int \frac{dk'}{2\pi} |Y_{\Xi k'}|^2 - \frac{1}{48} \int \frac{d^3k'}{(2\pi)^3} \frac{|V_{k'k'k'}|^2}{\omega_{k'} \omega_{k'} \omega_{k'} \left(\omega_{k'} + \omega_{k'} + \omega_{k'}\right)}
$$

$$
+ \frac{1}{16Q_0} \int \frac{d^2k'}{(2\pi)^2} \frac{\left|\left(\omega_{k'} - \omega_{k'}\right) \Delta_{k'k'}\right|^2}{\omega_{k'} \omega_{k'}} - \frac{1}{8Q_0} \int \frac{dk'}{2\pi} |f''(x)|^2.
$$

To our knowledge, this is the first time that the two-loop energy has been calculated for kinks that need be neither integrable nor supersymmetric. The explicit calculation of Refs. [15, 16], in the case of the Sine-Gordon model, did not require integrability and so could be repeated in this general setting. However in that case we stress that the energy was found by summing 13 divergent Feynman diagrams, and carefully regulating and subtracting the divergences. Here instead we find 5 terms, each of which is already UV finite. Let us identify each of these terms.

When changing from plane wave to normal mode normal ordering, so that $H_2$ annihilates the one-loop kink ground state, the interaction Hamiltonian acquired constant and tadpole
terms. The first two terms in \([5.9]\) are just the corresponding leading shifts to the energy, equal to the constant \(V_{II}\) plus the first perturbative contribution from the tadpole \(V_{k}\). The next term is the usual one-loop perturbation theory correction to an energy arising from a cubic interaction, and is given by the same expression as in the vacuum sector \([5.16]\) with plane waves replaced by normal modes. The fourth term is a correction to the third term arising from the fact that derivative operators mix the normal modes, which is not the case for plane waves. The last term was found long ago \([17, 15]\) using the collective coordinate approach, where it appeared as the leading term in an expansion of the denominator of an effective Hamiltonian, which came from a canonical transformation that separated a nonlinear extension of \(\phi_B(x)\). The manipulations which led to its appearance here are very different from those in the collective coordinate approach, but it is reassuring to see this agreement in the result.

The only trace of renormalization can be found in the first two terms, in the function \(I(x)\) which is the expected difference between two divergent sums weighted by \(1/\omega_k\) and \(1/\omega_p\) respectively. In models such as the \(\phi^4\) double well, in which the potential has a nonvanishing third derivative at the minima, the third term will be IR divergent. This divergence arises from the region far from the kink, and so its contribution to the kink mass will be canceled by the same IR divergence in the vacuum energy, which we will now calculate.

### 5.2 Vacuum Sector Energy

The kink mass is generally not \(Q_2\). It is \(Q_2 - E_1\) where \(E_1\) is the 1-loop correction to the vacuum sector energy, as this contributes at the same order. It is easily computed in perturbation theory. Decompose the field in terms of plane waves as

\[
\phi(x) = \int \frac{dp}{2\pi} \left( A_p^\dagger + \frac{A_{-p}}{2\omega_p} \right) e^{-ipx}
\]

and the free and interaction Hamiltonians can be written

\[
H_2 = \int \frac{dp}{2\pi} \omega_p A_p^\dagger A_p, \quad H_{n>2} = \frac{1}{n!} \int dx V^{(n)}(\phi_0) \phi^n(x) :a
\]

where \(\phi_0\) is the minimum of \(V\) corresponding to the vacuum. Then the first order of perturbation theory

\[
H_3|\Omega\rangle_0 + H_2|\Omega\rangle_1 = 0
\]

yields the first order correction \(|\Omega\rangle_1\) to the vacuum state \(|\Omega\rangle\)

\[
|\Omega\rangle_1 = -\frac{V^{(3)}(\phi_0)}{6} \int \frac{d^3p}{(2\pi)^3} \frac{2\pi \delta(p_1 + p_2 + p_3)}{\omega_{p_1} + \omega_{p_2} + \omega_{p_3}} A_{p_1}^\dagger A_{p_2}^\dagger A_{p_3}^\dagger |\Omega\rangle_0.
\]
Acting again with $H_3$, the $|\Omega\rangle_0$ term yields the one loop correction to the energy

$$H_3|\Omega\rangle_1 \supset -\left(\frac{(V^{(3)}[\phi_0])^2}{48}\right) \int dx \int \frac{d^3p'}{(2\pi)^3} e^{-ix(p_1^0+p_2^0+p_3^0)} \frac{2\pi\delta(p_1+p_2+p_3)}{\omega_{p_1'}\omega_{p_2'}\omega_{p_3'}(\omega_{p_1'}+\omega_{p_2'}+\omega_{p_3'})} |\Omega\rangle_0$$

$$= -\left(\frac{(V^{(3)}[\phi_0])^2}{48}\right) L \int \frac{d^3p'}{(2\pi)^3} \frac{2\pi\delta(p_1+p_2+p_3)}{\omega_{p_1'}\omega_{p_2'}\omega_{p_3'}(\omega_{p_1'}+\omega_{p_2'}+\omega_{p_3'})} |\Omega\rangle_0$$

(5.14)

where $L$ is the length of the spatial direction which serves as an infrared cut off.

The subleading correction to the Schrodinger equation is

$$(H_4 - E_1)|\Omega\rangle_0 + H_3|\Omega\rangle_1 + H_2|\Omega\rangle_2 = 0.$$  

(5.15)

As $H_4$ is normal ordered, $H_4|\Omega\rangle_0$ is orthogonal to $|\Omega\rangle_0$ and so does not contribute to $E_1$. We will choose $|\Omega\rangle_2$ to be orthogonal to $|\Omega\rangle_0$ so that the last term does not contribute to $E_1$. Then $E_1$ can be read off of (5.14). Evaluating the delta function, this is

$$E_1 = -\left(\frac{(V^{(3)}[\phi_0])^2}{48}\right) L \int \frac{d^2p'}{(2\pi)^2} \frac{1}{\omega_{p_1'}\omega_{p_2'}\omega_{p_3'}(\omega_{p_1'}+\omega_{p_2'}+\omega_{p_3'})}$$

(5.16)

The dependence on the infrared cutoff $L$ implies that we have calculated an energy density, and not an energy. When this energy density is nonvanishing, it must be subtracted from the kink ground state energy to obtain the kink mass. The kink mass will be finite only if these divergences cancel. This procedure depends on the matching of the infrared divergences, which can be achieved for example if the energy densities are subtracted before they are integrated. If the potential is symmetric about the minimum $\phi_0$, as it is in the case of the Sine-Gordon model but not the $\phi^4$ double well, $V^{(3)}[\phi_0]$ vanishes and so $E_1 = 0$ and this complication is avoided.

### 5.3 The Sine-Gordon Model

In the case of the Sine-Gordon model, the two-loop mass has been conjectured in [18] and calculated in [15, 16, 4, 19]. It is of course dependent upon the renormalization scheme [18] although in some schemes there is a renormalization group flow invariant coupling which provides a universal relation between the kink and meson mass. No such relation may be

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5Here we are cavalier with boundary conditions, as the theory contains only scalar fields. In practice, we simply subtract the kink and vacuum energy densities before performing the $x$ integration, in which case the integral converges. In a theory with fermions a more careful approach may be warranted, for example adding a distant antikink to each kink to allow identical boundary conditions in each sector.
expected to hold in general as there are other schemes in which the coupling may be shifted by any finite amount at any scale.

Using the well-known Sine-Gordon normal modes \[11\]
\[ g_k(x) = \frac{e^{-ikx \text{sign}(k)}}{\sqrt{1 + m^2/k^2}} (k - im \tanh(mx)) \], \[ g_B(x) = \sqrt{\frac{m}{2}} \text{sech}(mx) \]. (5.17)
a contour integration yields
\[ \Delta_{kB} = \frac{i\pi\omega_k}{\sqrt{8M}} \text{sech} \left( \frac{k\pi}{2M} \right) \] (5.18)
\[ \Delta_{k_1k_2} = -i(k_1 - k_2)\pi\delta(k_1 + k_2) + \frac{i\pi}{2} \frac{(k_2^2 - k_1^2)}{\omega_{k_1}\omega_{k_2}} \text{csch} \left( \frac{\pi(k_1 + k_2)}{2m} \right) \text{sign}(k_1k_2). \]
We have evaluated the energy (5.9) term by term.

In the first two terms, \( I(x) \) appears. It was calculated in Ref. \[11\] by integrating the general identity (2.24). Using the present conventions
\[ I(x) = -\frac{\text{sech}^2(Mx)}{2\pi}. \] (5.19)
This leads to \( Mg^2/(40\pi^2) \) and \(-Mg^2/(120\pi^2) \) for the first and second terms of (5.9). In the fourth term, the delta function in (5.18) is multiplied by a zero which leads to a vanishing contribution, as can be checked directly by considering the case \( k_1 = k_2 \) separately from the beginning. Fixing the mass \( M \) and coupling \( g \) to unity, the third, fourth and fifth terms are equal to terms which may be found in Ref. \[15\] and they were evaluated analytically by Verwaest who found that the sum of the third and fourth is \(-Mg^2/(60\pi^2) \) while the fifth is \(-Mg^2/192 \). Altogether we find that the two-loop correction to the kink mass is
\[ Q_2 = -\frac{Mg^2}{192} \] (5.20)
in agreement with the literature. As shown in Appendix B our normal ordering prescription yields the same meson mass as Ref. \[15\] and so the soliton to meson mass ratio agrees, in accordance with Refs. \[18, 19\].

6 Remarks

Calculations of masses of quantum kinks have been an industry from Ref. \[1\] to Refs. \[20, 21\]. So far these calculations have been largely at one loop, where they are described by a free theory, with the exception of integrable and supersymmetric models. In this paper, we have
calculated the two-loop masses of scalar kinks in theories with arbitrary potentials. We have also explicitly constructed their states, with the \( \phi_0 \)-descendants calculated in Sec. 3 using translation invariance and the \( \phi_0 \)-primary components in Sec. 4 using the Schrödinger equation. These constructions we feel are even more interesting than the masses, as they allow one to compute matrix elements and so open the door to understanding the phenomenology [22], such as scattering [23, 24, 25] and acceleration [26, 27] of quantum kinks beyond the harmonic oscillator approximation. For example, one may calculate form factors [28, 29].

While we only calculated the ground state, starting our recursion with a superposition of normal modes would have allowed us to apply the same strategy to an arbitrary state in the one-kink sector.

The key step in our calculation was perturbatively imposing the translation invariance conditions, which fixed most matrix elements of the state, the \( \phi_0 \)-descendants, in terms of a few coefficients, the \( \phi_0 \)-primaries. The \( \phi_0 \)-primary components needed to be fixed using ordinary perturbation theory. More generally, in the case of any translation-invariant Hamiltonian, as the Hamiltonian and momentum operators commute, a basis of all Hamiltonian eigenstates may be obtained by first fixing the momentum to obtain the \( \phi_0 \)-descendant matrix elements in terms of \( \phi_0 \)-primary matrix elements \( \gamma_{in} \) and then using the Schrödinger equation to fix the \( \phi_0 \)-primary matrix elements.

In the case of a BPS state in a supersymmetric model one may first impose both translation invariance and also that the state be invariant under the preserved supersymmetries. Presumably this will strongly constrain the state. The big question is whether in a sufficiently supersymmetric model, this may constrain the state sufficiently that perturbation theory is no longer required. In this case, one would have finally opened the door to a truly quantum understanding of nonperturbative solitons. More precisely, one could understand the physical mechanisms at work behind the nonrenormalization theorems. This of course is a prerequisite for applying lessons from supersymmetric theories to Yang-Mills.

**Appendix A  Checking Schrödinger’s Equation**

We have derived the two-loop ground state using translation invariance together with Schrödinger’s equation. We restricted Schrödinger’s equation to the \( \phi_0 \)-primaries, the subspace of the Fock space with no \( \phi_0 \) acting on \( |0\rangle_0 \), but we argued that, since the Hamiltonian and momentum operators commute, we expect our solutions to solve the Schrödinger equation in the full Fock space. By imposing a condition on the momentum it is not possible that we lose the ground state solution, since it indeed must have zero momentum. Furthermore,
since the solution that we find, given the one-loop contribution, is unique, it must be the
ground state.

In this Appendix we explicitly check this claim by inserting our two-loop state into the
Schrodinger equation and showing that it vanishes on the full Fock space. More precisely,
we compute the various $\phi^0_0$-descendant components $\Gamma_{m}^{0n}$ at $m > 0$ and show that they
each vanish as claimed. Recall that in Subsec. 4.2 we found the $\phi^0_0$-primaries $\gamma^0_0$ by imposing
that $\Gamma^0_0$ vanishes, and so we already know that the $m = 0$ Schrodinger equation is satisfied.

1 $m = 5, n = 1$

The only contribution

$$H_3|0\rangle^{21}_1 \supset \frac{1}{6} \int dx V^{(3)}[g f(x)] g^3_B(x) \phi^0_0|0\rangle^{21}_1 = 0$$

(A.1)

vanishes because

$$V_{BBB} = \int dx V^{(3)}[g f(x)] g^3_B(x) = \frac{1}{\sqrt{Q_0}} \int dx \left( \partial_x V^{(2)}[g f(x)] \right) g^3_B(x)$$

(A.2)

$$= -\frac{2}{\sqrt{Q_0}} \int dx V^{(2)}[g f(x)] g_B(x) g'_B(x) = -\frac{2}{\sqrt{Q_0}} \int dx g''_B(x) g'_B(x) = 0$$

is a total derivative.

2 $m = 4, n = 2$

$$H_3|0\rangle^{21}_1 \supset \frac{1}{4\sqrt{Q_0}} \int \frac{d^2 k}{(2\pi)^2} \omega_{k_1} \Delta_{k_1 B} V_{k_2 B B} \phi^4_{k_1} \phi^4_{k_2} |0\rangle^{21}_0$$

(A.3)

exactly cancels

$$\int \frac{d^2 k'}{2\pi} \omega_{k'} B_{k'}^1 B_{k'} |0\rangle^2_{2} = \frac{1}{4Q_0} \int \frac{d^2 k}{(2\pi)^2} \omega^2_{k_1} \omega^2_{k_2} \Delta_{k_1 B} \Delta_{k_2 B} \phi^4_{k_1} \phi^4_{k_2} B^1_{k_1} B^1_{k_2} |0\rangle^2_0$$

(A.4)

as a result of (4.10).

3 $m = 4, n = 0$

$$H_3|0\rangle^{21}_1 \supset -\frac{1}{8} \int \frac{d^3 k}{(2\pi)^3} Y_{k B B} Y_{-k B B} \phi^4_0 |0\rangle^2_0$$

(A.5)

exactly cancels

$$H_4|0\rangle_0 \supset \frac{V_{BBB}}{24} \phi^4_0 |0\rangle_0$$

(A.6)
as

\[ V_{BBBB} = \int dx V^{(4)}[g f(x)] g^4_B(x) = \int dx V^{(4)}[g f(x)] g^2_B(x) \frac{f'(x)}{\sqrt{Q_0}} \]  

\[ = \frac{1}{\sqrt{Q_0}} \int dx \partial_x (V^{(3)}[g f(x)]) g^3_B(x) = -\frac{3}{\sqrt{Q_0}} \int dx V^{(3)}[g f(x)] g^2_B(x) g_B'(x) \]

\[ = -\frac{3}{\sqrt{Q_0}} \int dx V^{(3)}[g f(x)] g^2_B(x) \int dy \delta(x - y) g_B'(y) \]

\[ = -\frac{3}{\sqrt{Q_0}} \int \frac{dk}{2\pi} V_{kBB} \Delta_{-kB} = 3 \int \frac{dk}{2\pi} Y_{kBB} Y_{-kBB}. \]  

4 \hspace{1cm} m = 3, \hspace{0.5cm} n = 3

The two terms in

\[ \int \frac{dk'}{2\pi} \omega_{k'} B^\dagger_{k'} B_{k'} |0\rangle \right] \]

\[ = -\frac{1}{4\sqrt{Q_0}} \int \frac{dk}{(2\pi)^3} [V_{k_1 k_2 B} \omega_{k_3} \Delta_{k_3 B} + V_{k_3 B B} (\omega_{k_1} - \omega_{k_2}) \Delta_{k_1 k_2}] \phi^3_{k_1} B^\dagger_{k_1} B^\dagger_{k_2} B^\dagger_{k_3} |0\rangle \]

are respectively canceled by \( H_3 |0\rangle \) and \( H_3 |1\rangle \).

5 \hspace{1cm} m = 3, \hspace{0.5cm} n = 1

We will need the identity

\[ V_{IB} = \int dx V^{(3)}[g f(x)] g_B(x) I(x) = \frac{1}{\sqrt{Q_0}} \int dx \left( \partial_x V^{(2)}[g f(x)] \right) I(x) \]

\[ = -\frac{1}{\sqrt{Q_0}} \int dx V^{(2)}[g f(x)] I'(x) = -\frac{1}{\sqrt{Q_0}} \int dx V^{(2)}[g f(x)] \int \frac{dk}{2\pi} g_k(x) g'_{-k}(x) \frac{\omega_k}{\omega_k} \]

\[ = -\frac{1}{\sqrt{Q_0}} \int \frac{dk}{2\pi} \int dx \left( \omega_k g_k(x) + \frac{g''_k(x)}{\omega_k} \right) g'_{-k}(x) \]

\[ = -\frac{1}{2\sqrt{Q_0}} \int \frac{dk}{\omega_k} \int dx \partial_x \left( \omega_k |g_k(x)|^2 + \frac{|g'_k(x)|^2}{\omega_k} \right) = 0. \]  

(A.10)

Note that although this is the integral of a total derivative, the differentiated function does not vanish at infinity. The integral vanishes because the differentiated function is even in \( x \).
This is true at each $k$ if the potential is symmetric under an inversion that exchanges the two minima responsible for the kink, as it is in the Sine-Gordon model and the $\phi^4$ model. More generally, in the large $x$ region which fixes this integral by the fundamental theorem of calculus, the functions $g_k(x)$ are plane waves and their norm is constant and independent of the potential. In the case of a reflectionless potential, the norm is equal in both asymptotic regions. This is true at each $k$ when summed over $k$ and $-k$ as the summed norms squared are equal in the two asymptotic regions.

Using this identity, one evaluates the contribution of $|0\rangle_1^{21}$ to be

$$H_3|0\rangle_1^{21} \supset \frac{1}{4\sqrt{Q_0}} \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} V_{Bk-k'} \Delta_{k'} \phi_0^3 B_k^1 |0\rangle_0.$$  \hspace{1cm} (A.11)

Similarly

$$H_3|0\rangle_1^{12} \supset \frac{1}{4\sqrt{Q_0}} \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} \left( \frac{\omega_k}{\omega_{k'}} - 1 \right) V_{Bk-k'} \Delta_{k'} \phi_0^3 B_k^1 |0\rangle_0 \hspace{1cm} (A.12)$$

$$\int \frac{dk'}{2\pi} \omega_{k'} B_{k'}^1 B_{k'}^1 |0\rangle_2^{31} = \frac{1}{12Q_0} \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} \left( 3\omega_{k'} - \omega_k \right) \omega_k \Delta_{k'} \Delta_{k'} \phi_0^3 B_k^1 |0\rangle_0.$$  \hspace{1cm}

The final contribution is

$$H_4|0\rangle_0 = \frac{1}{6} \int \frac{dk}{2\pi} V_{BBBk} \phi_0^3 B_k^1 |0\rangle_0.$$  \hspace{1cm} (A.13)

The $V$’s may all be traded for $\Delta$’s using (4.10) and, as may be derived similarly to (A.7),

$$V_{BBBk} = \frac{1}{\sqrt{Q_0}} \int dx \partial_x \left( V^{(3)}[g f(x)] \right) g_B(x)^2 g_k(x)$$

$$= - \frac{1}{\sqrt{Q_0}} \int dx V^{(3)}[g f(x)]$$

$$\times \left[ 2g_B(x) g_k(x) \int dy \left( g_B(x) g_B(y) + \int \frac{dk'}{2\pi} g_{k'}(x) g_{-k'}(y) \right) g'_B(y) \right. \right.$$  \hspace{1cm}

$$+ g_B^2(x) \int dy \left( g_B(x) g_B(y) + \int \frac{dk'}{2\pi} g_{k'}(x) g_{-k'}(y) \right) g'_B(y) \right]$$

$$= - \frac{1}{\sqrt{Q_0}} \int \frac{dk'}{2\pi} \left( 2V_{BBk} \Delta_{k'} + V_{BBk'} \Delta_{-k'} \right) \Delta_{k'} \Delta_{k'}.$$  \hspace{1cm}

Combining these contributions

$$\Gamma_2^{31}(k) = \int \frac{dk'}{2\pi} \Delta_{k'} \Delta_{-k'} \left( \frac{\omega_{k'}^2 - \omega_{k}^2}{4} - \frac{\omega_{k'}^2}{4} \left( \frac{\omega_k}{\omega_{k'}} - 1 \right) + \frac{3\omega_{k'} \omega_k - \omega_k^2}{12} + \frac{2\omega_{k'}^2 - 3\omega_k^2}{6} \right)$$

$$= 0.$$  \hspace{1cm} (A.15)
The contributions are

\[ H_3|0\>_{11} \supset \frac{1}{12\sqrt{Q_0}} \int \frac{d^4k}{(2\pi)^4} \omega_{k_1} \Delta_{k_1 B} V_{k_2 k_3 k_4} \phi_0^2 B_{k_1}^\dagger B_{k_2}^\dagger B_{k_3}^\dagger B_{k_4}^\dagger |0\>_0 \]  

(A.16)

\[ H_3|0\>_{12} \supset \frac{1}{4\sqrt{Q_0}} \int \frac{d^4k}{(2\pi)^4} \left( \omega_{k_1} - \omega_{k_2} \right) \Delta_{k_1 k_2} V_{k_3 k_4} \phi_0^2 B_{k_1}^\dagger B_{k_2}^\dagger B_{k_3}^\dagger B_{k_4}^\dagger |0\>_0 \]

(A.17)

\[ H_3|0\>_{13} \supset - \frac{1}{12} \int \frac{d^4k}{(2\pi)^4} V_{Bk_1} Y_{k_2 k_3 k_4} \phi_0^2 B_{k_1}^\dagger B_{k_2}^\dagger B_{k_3}^\dagger B_{k_4}^\dagger |0\>_0 \]

(A.18)

Therefore

\[ \Gamma^{24}_2 (k_1 \cdots k_4) = \frac{\sqrt{Q_0} \Delta_{k_1 B} V_{k_2 k_3 k_4}}{12} \left[ \frac{\omega_{k_1}}{\omega_{k_2} + \omega_{k_3} + \omega_{k_4}} - \frac{\omega_{k_1} \Omega_4}{\omega_{k_2} + \omega_{k_3} + \omega_{k_4}} \right] 
\]

\[ + \frac{\sqrt{Q_0} \Delta_{k_1 k_2} V_{B k_3 k_4}}{4} \left[ \frac{\omega_{k_1} - \omega_{k_2}}{\omega_{k_1} + \omega_{k_2}} \right] = 0 \]  

(A.19)

as the terms in each square bracket vanish.

7  \( m = 2, \ n = 2 \)

From here on there will be many more contributions to each \( \Gamma^{mn}_2 \), and so we will decompose them into pieces that are not expected to mix as was done in Subsec. [4.2]. First let us consider contributions that depend on \( \mathcal{I}(x) \) and so on our renormalization scheme. As we have seen that \( V_{IB} \) vanishes, there are three contributions

\[ H_3|0\>_{21} \supset \frac{1}{4\sqrt{Q_0}} \int \frac{d^2k}{(2\pi)^2} V_{Ik_1} \omega_{k_2} \Delta_{k_2 B} \phi_0^2 B_{k_1}^\dagger B_{k_2}^\dagger |0\>_0 \]  

(A.20)

\[ H_3|0\>_{01} \supset \frac{1}{4\sqrt{Q_0}} \int \frac{d^2k}{(2\pi)^2} Y_{Ik_1} \omega_{k_2} \Delta_{k_2 B} \phi_0^2 B_{k_1}^\dagger B_{k_2}^\dagger |0\>_0 \]

\[ \int \frac{d^2k}{2\pi} \omega_{k'} B_{k'}^\dagger B_{k'}^\dagger |0\> \supset - \frac{1}{4\sqrt{Q_0}} \int \frac{d^2k}{(2\pi)^2} \left( \omega_{k_1} + \omega_{k_2} \right) Y_{Ik_1} \omega_{k_2} \Delta_{k_2 B} \phi_0^2 B_{k_1}^\dagger B_{k_2}^\dagger |0\>_0 \]

whose sum is easily seen to vanish.
Similarly to (A.14) one may derive
\[
V_{BBk_1k_2} = \frac{1}{Q_0} \left[ - (\omega_{k_1}^2 + \omega_{k_2}^2) \Delta_{k_1B} \Delta_{k_2B} \right. \\
+ \left. \int \frac{dk'}{2\pi} \left[ - \sqrt{Q_0} V_{k_1k_2k'} \Delta_{-k'B} + (\omega_{k_1}^2 + \omega_{k_2}^2 - 2\omega_{k'}^2) \Delta_{k_2k'} \Delta_{-k_1'k_1} \right] \right].
\] (A.21)

There are four terms that contain \(V_{k_1k_2k_3}\)
\[
H_4|0\rangle_0 \supset - \frac{1}{4\sqrt{Q_0}} \int \frac{d^2k}{(2\pi)^2} \int \frac{dk'}{2\pi} V_{k_1k_2k'} \Delta_{-k'B} \phi_0^2 B_{k_1}^\dagger B_{k_2} |0\rangle_0
\]
(A.22)
\[
H_3|0\rangle_{10} \supset \frac{1}{8\sqrt{Q_0}} \int \frac{d^2k}{(2\pi)^2} \int \frac{dk'}{2\pi} Y_{k_1k_2k'} \omega_{k'} \Delta_{-k'B} \phi_0 \phi_0 B_{k_1}^\dagger B_{k_2}^\dagger |0\rangle_0
\]
\[
H_3|0\rangle_{21} \supset \frac{1}{8\sqrt{Q_0}} \int \frac{d^2k}{(2\pi)^2} \int \frac{dk'}{2\pi} V_{k_1k_2k'} \Delta_{-k'B} \phi_0^2 B_{k_1} B_{k_2}^\dagger |0\rangle_0
\]
\[
\int \frac{dk'}{2\pi} \omega_{k'} B_{k'}^\dagger B_{k'} |0\rangle_{22} \supset \frac{1}{8\sqrt{Q_0}} \int \frac{d^2k}{(2\pi)^2} \int \frac{dk'}{2\pi} (\omega_{k_1} + \omega_{k_2}) Y_{k_1k_2k'} \Delta_{-k'B} \phi_0 \phi_0 B_{k_1}^\dagger B_{k_2}^\dagger |0\rangle_0
\]
which again sum to zero, as the second plus the fourth and also the third are equal to minus one half of the first. The four terms with no \(k'\) integral are
\[
H_4|0\rangle_0 \supset - \frac{1}{4Q_0} \int \frac{d^2k}{(2\pi)^2} \left( \omega_{k_1}^2 + \omega_{k_2}^2 \right) \Delta_{k_1B} \Delta_{k_2B} \phi_0^2 B_{k_1}^\dagger B_{k_2}^\dagger |0\rangle_0
\]
(A.23)
\[
H_3|0\rangle_{10} \supset - \frac{1}{8Q_0} \int \frac{d^2k}{(2\pi)^2} \left( \omega_{k_1}^2 + \omega_{k_2}^2 \right) \Delta_{k_1B} \Delta_{k_2B} \phi_0 \phi_0 B_{k_1}^\dagger B_{k_2}^\dagger |0\rangle_0
\]
\[
\frac{n_0^2}{2} |0\rangle_{22} = - \frac{3}{4Q_0} \int \frac{d^2k}{(2\pi)^2} \omega_{k_1} \omega_{k_2} \Delta_{k_1B} \Delta_{k_2B} \phi_0 \phi_0 B_{k_1}^\dagger B_{k_2}^\dagger |0\rangle_0
\]
\[
\int \frac{dk'}{2\pi} \omega_{k'} B_{k'}^\dagger B_{k'} |0\rangle_{22} \supset \frac{3}{8Q_0} \int \frac{d^2k}{(2\pi)^2} \Omega_2 \Delta_{k_1B} \Delta_{k_2B} \phi_0 \phi_0 B_{k_1}^\dagger B_{k_2}^\dagger |0\rangle_0
\]
which sum easily to zero as well. Finally the three terms with \(k'\) but no \(V_{k_1k_2k_3}\) are
\[
H_4|0\rangle_0 \supset \frac{1}{4Q_0} \int \frac{d^2k}{(2\pi)^2} \int \frac{dk'}{2\pi} \left( \omega_{k_1}^2 + \omega_{k_2}^2 - 2\omega_{k'}^2 \right) \Delta_{k_1k'} \Delta_{-k'k_2} \phi_0 \phi_0^2 B_{k_1}^\dagger B_{k_2}^\dagger |0\rangle_0
\]
(A.24)
\[
H_3|0\rangle_{12} \supset \frac{1}{2Q_0} \int \frac{d^2k}{(2\pi)^2} \int \frac{dk'}{2\pi} \left( \omega_{k_2}^2 + \omega_{k'}^2 \right) \Delta_{k_1k'} \Delta_{-k'k_2} \phi_0 \phi_0^2 B_{k_1}^\dagger B_{k_2}^\dagger |0\rangle_0
\]
\[
\int \frac{dk'}{2\pi} \omega_{k'} B_{k'}^\dagger B_{k'} |0\rangle_{22} \supset - \frac{1}{4Q_0} \int \frac{d^2k}{(2\pi)^2} \int \frac{dk'}{2\pi} \Omega_2 \left( \frac{\omega_{k_1} \omega_{k_2}}{\omega_{k'}} - \omega_{k'} \right) \Delta_{k_1k'} \Delta_{-k'k_2} \phi_0 \phi_0^2 B_{k_1}^\dagger B_{k_2}^\dagger |0\rangle_0.
\]
Notice that in the second term, all factors are symmetric with respect to \(k_1 \leftrightarrow k_2\) except for the factors of \(\omega\). Therefore these may be symmetrized to
\[
\frac{1}{2} \left( \frac{\omega_{k_1} \omega_{k_2} \Omega_2}{\omega_{k'}} - \omega_{k_1}^2 - \omega_{k_2}^2 - \omega_{k'} \Omega_2 \right) + \omega_{k'}^2
\] (A.25)
which exactly cancels the corresponding contributions from the first and third terms. We thus conclude that $\Gamma_{22}^{32} = 0$.

8 $m = 2, n = 0$

We will see that this is the most interesting case so far, because it is the first that strongly depends on the form of $\mathcal{I}(x)$. To see this, let us try to proceed as above. The terms that depend on $\mathcal{I}(x)$ are

$$H_3|0\rangle_0 \supset \frac{1}{4} V_{BB} \phi_0^2|0\rangle_0$$

$$H_3|0\rangle_1 \supset \frac{1}{8 \sqrt{Q_0}} \int \frac{dk'}{2\pi} V_{kk'} \Delta_{-k'k} \phi_0^2|0\rangle_0$$

$$H_3|0\rangle_1 \supset \frac{1}{8 \sqrt{Q_0}} \int \frac{dk'}{2\pi} V_{kk'} \Delta_{-k'k} \phi_0^2|0\rangle_0.$$ 

As above, we may eliminate $V^{(4)}$ using integration by parts and then inserting the completeness relation (2.16)

$$\partial_x \mathcal{I}(x) = \int \frac{dx}{2\pi} \left( g_k(x) g_k'(x) + g_k'(x) g_k(x) \right).$$

The second term cancels the contributions from $H_3|0\rangle_1^{21}$ and $H_3|0\rangle_1^{01}$, leaving

$$\Gamma_{21}^{20} = -\frac{\sqrt{Q_0}}{4} V_{TB}.$$ 

Let us rewrite $V_{TB}$ in terms of quantities that we expect to find in other contributions.

Writing the identity (2.24) as

$$\partial_x \mathcal{I}(x) = \int \frac{dk}{2\pi} \frac{1}{2\omega_k} \left( g_k(x) g_{-k}(x) + g_k'(x) g_{-k}(x) \right),$$

one finds

$$V_{TB} = \int \frac{dk}{2\pi} \frac{1}{2\omega_k} \int dx V^{(3)}[g(x)] \mathcal{I}(x) g_{k'}(x) \left( g_k(x) g_{-k}(x) + g_k'(x) g_{-k}(x) \right)$$

$$= \int \frac{dk}{2\pi} \frac{1}{\omega_k} \left( V_{BB} \Delta_{Bk} + \int \frac{dk'}{2\pi} V_{kk'} \Delta_{-k'B} \right)$$

$$= -\frac{1}{\sqrt{Q_0}} \int \frac{dk'}{(2\pi)^2} \frac{\omega_{k_1} - \omega_{k_2}}{\omega_{k_1}} \Delta_{k_1'k_1} \Delta_{-k_1'k_2} - \frac{1}{\sqrt{Q_0}} \int \frac{dk'}{2\pi} \omega_{k'} \Delta_{Bk'} \Delta_{-k'B}.$$
Inserting this into (A.28) and symmetrizing dummy indices one finds

\[ \Gamma_{20}^{20} = \frac{1}{8} \int \frac{d^2 k'}{(2\pi)^2} \left( -\omega_{k_1'} - \omega_{k_2'} + \frac{\omega_{k_1}^2}{\omega_{k_2'}} + \frac{\omega_{k_2}^2}{\omega_{k_1'}} \right) \Delta_{k_1'k_2'} \Delta_{-k_1'-k_2'} + \frac{1}{4} \int \frac{d k'}{2\pi} \omega_{k'} \Delta_{Bk'} \Delta_{-k'B}. \]  

(A.31)

Schrodinger’s equation will only be satisfied if these terms are canceled by contributions with no \( i(x) \).

There is only one contribution with no \( i(x) \) that has two contracted momenta \( k' \)

\[ H_3|0\rangle_{11} \supset \frac{1}{8Q_0} \int \frac{d^2 k'}{(2\pi)^2} \left( \omega_{k_1'} + \omega_{k_2'} - \frac{\omega_{k_1}^2}{\omega_{k_2'}} - \frac{\omega_{k_2}^2}{\omega_{k_1'}} \right) \Delta_{k_1'k_2'} \Delta_{-k_1'-k_2'} \phi_0^2|0\rangle_0 \]  

(A.32)

which indeed cancels the first term in (A.31). There are two contributions with no \( i(x) \) and a single \( k' \)

\[ H_3|0\rangle_{01} \supset \frac{1}{8Q_0} \int \frac{d k'}{2\pi} \omega_{k'} \Delta_{Bk'} \Delta_{-k'B} \phi_0^2|0\rangle_0 \]  

(A.33)

\[ \frac{\pi^2}{2} |0\rangle_{40} = - \frac{3}{8Q_0} \int \frac{d k'}{2\pi} \omega_{k'} \Delta_{Bk'} \Delta_{-k'B} \phi_0^2|0\rangle_0 \]

which are equal to 1/2 and \(-3/2\) of the second term in (A.31), and so altogether they cancel, leaving \( \Gamma_{20}^{20} = 0 \).

9 \( m = 1, \ n = 5 \)

There are only three contributions to this term

\[ H_3|0\rangle_{03} \supset \frac{1}{12\sqrt{Q_0}} \int \frac{d^5 k}{(2\pi)^5} \left( \omega_{k_4}^2 - \omega_{k_5}^2 \right) Y_{k_1k_2k_3} \Delta_{k_4k_5} \phi_0 B_{k_1}^\dagger \cdots B_{k_5}^\dagger |0\rangle_0 \]

\[ H_3|0\rangle_{12} \supset \frac{1}{12\sqrt{Q_0}} \int \frac{d^5 k}{(2\pi)^5} \left( \omega_{k_4} - \omega_{k_5} \right) V_{k_1k_2k_3} \Delta_{k_4k_5} \phi_0 B_{k_1}^\dagger \cdots B_{k_5}^\dagger |0\rangle_0 \]

\[ \int \frac{d k'}{2\pi} \omega_{k'} B_{k'}^\dagger B_{k'} |0\rangle_2 \supset = - \frac{1}{12\sqrt{Q_0}} \int \frac{d^5 k}{(2\pi)^5} \left( \omega_{k_4} - \omega_{k_5} + \frac{\omega_{k_4}^2 - \omega_{k_5}^2}{\Omega_3} \right) \times V_{k_1k_2k_3} \Delta_{k_4k_5} \phi_0 B_{k_1}^\dagger \cdots B_{k_5}^\dagger |0\rangle_0 \]  

(A.34)

whose sum is readily seen to vanish.
Again let us divide the 11 contributions to $\Gamma_{12}^{13}$ into 3 subsets that are expected to cancel separately. First, terms involving $I(x)$ are

$$H_3|0\rangle_{12}^3 \supset \frac{1}{4\sqrt{Q_0}} \int \frac{d^3 k}{(2\pi)^3} (\omega_{k_2} - \omega_{k_3}) V_{I_k} \Delta_{k_2 k_3} \phi_0 B_{k_1}^\dagger B_{k_2}^\dagger B_{k_3}^\dagger |0\rangle_0 \quad \text{(A.35)}$$

$$H_3|0\rangle_{11}^3 \supset \frac{1}{4\sqrt{Q_0}} \int \frac{d^3 k}{(2\pi)^3} (\omega_{k_2} - \omega_{k_3}) Y_{I_k} \Delta_{k_2 k_3} \phi_0 B_{k_1}^\dagger B_{k_2}^\dagger B_{k_3}^\dagger |0\rangle_0$$

$$\int \frac{d k'}{2\pi} \omega_{k'} B_{k'}^\dagger B_{k'} |0\rangle_2^3 \supset -\frac{1}{4\sqrt{Q_0}} \int \frac{d^3 k}{(2\pi)^3} (\omega_{k_2} - \omega_{k_3}) \Omega_3 Y_{I_k} \Delta_{k_2 k_3} \phi_0 B_{k_1}^\dagger B_{k_2}^\dagger B_{k_3}^\dagger |0\rangle_0$$

which sum to zero.

We will now need

$$V_{B_{k_1 k_2 k_3}} = \frac{1}{Q_0} \left[ (\omega_{k_2} - \omega_{k_3}) \Delta_{B_{k_1 k_2}} \Delta_{k_2 k_3} + (\omega_{k_1} - \omega_{k_3}) \Delta_{B_{k_1 k_3}} \Delta_{k_1 k_3} + (\omega_{k_1} - \omega_{k_2}) \Delta_{B_{k_1 k_2}} \Delta_{k_1 k_2} \right]$$

$$- \frac{1}{\sqrt{Q_0}} \int \frac{d k'}{2\pi} \left[V_{B_{k_1 k_2 k'}} \Delta_{-k'_{k_1}} + V_{B_{k_1 k_2 k'}} \Delta_{-k'_{k_2}} + V_{B_{k_1 k_2 k'}} \Delta_{-k'_{k_3}} \right]. \quad \text{(A.36)}$$

The terms which have a contracted index $k'$ are

$$H_4|0\rangle_0 \supset -\frac{1}{2\sqrt{Q_0}} \int \frac{d^3 k}{(2\pi)^3} \int \frac{d k'}{2\pi} V_{B_{k_1 k_2 k'}} \Delta_{-k'_{k_1}} \phi_0 B_{k_1}^\dagger B_{k_2}^\dagger B_{k_3}^\dagger |0\rangle_0 \quad \text{(A.37)}$$

$$H_3|0\rangle_{03}^3 \supset \frac{1}{4\sqrt{Q_0}} \int \frac{d^3 k}{(2\pi)^3} \int \frac{d k'}{2\pi} \omega_{k'}^2 - \omega_{k_3}^2 \omega_{k_2}^2 \Omega_3 Y_{I_k} \Delta_{-k'_{k_3}} \phi_0 B_{k_1}^\dagger B_{k_2}^\dagger B_{k_3}^\dagger |0\rangle_0$$

$$H_3|0\rangle_{12}^3 \supset \frac{1}{4\sqrt{Q_0}} \int \frac{d^3 k}{(2\pi)^3} \int \frac{d k'}{2\pi} \omega_{k'}^2 - \omega_{k_3}^2 \omega_{k_2}^2 V_{B_{k_1 k_2 k'}} \Delta_{-k'_{k_3}} \phi_0 B_{k_1}^\dagger B_{k_2}^\dagger B_{k_3}^\dagger |0\rangle_0$$

$$\int \frac{d k'}{2\pi} \omega_{k'} B_{k'}^\dagger B_{k'} |0\rangle_2^3 \supset \frac{1}{4\sqrt{Q_0}} \int \frac{d^3 k}{(2\pi)^3} \int \frac{d k'}{2\pi} \Omega_3 \omega_{k'}^2 + \omega_{k_3}^2 Y_{I_k} \Delta_{-k'_{k_3}} \phi_0 B_{k_1}^\dagger B_{k_2}^\dagger B_{k_3}^\dagger |0\rangle_0$$

whose sum also vanishes. Finally the terms with neither $I(x)$ nor $k'$ are

$$H_4|0\rangle_0 \supset \frac{1}{2Q_0} \int \frac{d^3 k}{(2\pi)^3} (\omega_{k_2} - \omega_{k_3}) \Delta_{B_{k_1 k_2}} \Delta_{k_2 k_3} \phi_0 B_{k_1}^\dagger B_{k_2}^\dagger B_{k_3}^\dagger |0\rangle_0 \quad \text{(A.38)}$$

$$H_3|0\rangle_{01}^3 \supset \frac{1}{4Q_0} \int \frac{d^3 k}{(2\pi)^3} (\omega_{k_2} - \omega_{k_3}) \Delta_{B_{k_1 k_2}} \Delta_{k_2 k_3} \phi_0 B_{k_1}^\dagger B_{k_2}^\dagger B_{k_3}^\dagger |0\rangle_0$$

$$\frac{3}{2Q_0} |0\rangle_2^{33} \supset \frac{3}{4Q_0} \int \frac{d^3 k}{(2\pi)^3} \omega_{k_2}^2 - \omega_{k_3}^2 \Delta_{B_{k_1 k_2}} \Delta_{k_2 k_3} \phi_0 B_{k_1}^\dagger B_{k_2}^\dagger B_{k_3}^\dagger |0\rangle_0$$

$$\int \frac{d k'}{2\pi} \omega_{k'} B_{k'}^\dagger B_{k'} |0\rangle_2^3 \supset \frac{3}{4Q_0} \int \frac{d^3 k}{(2\pi)^3} \left[ \omega_{k_2}^2 - \omega_{k_3}^2 \right] \Delta_{B_{k_1 k_2}} \Delta_{k_2 k_3} \phi_0 B_{k_1}^\dagger B_{k_2}^\dagger B_{k_3}^\dagger |0\rangle_0$$

$$\times \Delta_{B_{k_1 k_2}} \Delta_{k_2 k_3} \phi_0 B_{k_1}^\dagger B_{k_2}^\dagger B_{k_3}^\dagger |0\rangle_0$$
which again trivially cancel, leaving $\Gamma_{2}^{13} = 0$.

11 $m = 1$, $n = 1$

Finally we turn our attention to $\Gamma_{2}^{11}$. Like $\Gamma_{2}^{20}$ we will see that it only vanishes if $\mathcal{I}(x)$ satisfies (2.24). The terms involving $\mathcal{I}(x)$ are

$$ H_{4}[0]_{0} = \frac{1}{2} \int \frac{dk}{2\pi} V_{IBk} \Phi_{0} B_{k}^{1}[0]_{0} \quad (A.39) $$

$$ H_{3}[0]_{1}^{12} = \frac{1}{4\sqrt{Q_{0}}} \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} \left( \omega_{k'} - \omega_{k} \right) Y_{k'} \Delta_{-k'} \phi_{0} B_{k}^{1}[0]_{0} $$

$$ H_{3}[0]_{1}^{11} = \frac{1}{4\sqrt{Q_{0}}} \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} \left( \omega_{k'} - \omega_{k} \right) Y_{k} \Delta_{-k} \phi_{0} B_{k}^{1}[0]_{0} $$

$$ \int \frac{dk'}{2\pi} \omega_{k'} B_{k'}^{1} B_{k'}^{1}[0]_{11} = \frac{1}{4\sqrt{Q_{0}}} \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} \omega_{k} \left( \omega_{k'} + \omega_{k} \right) Y_{k} \Delta_{-k} \phi_{0} B_{k}^{1}[0]_{0}. $$

Again, as in the case of $\Gamma_{2}^{20}$ integration by parts allows us to remove the fourth derivative

$$ V_{IBk} = -\frac{1}{\sqrt{Q_{0}}} \int dx V^{3}[gf(x)] (\mathcal{I}(x) g_k(x) + \mathcal{I}(x) g'_k(x)) \quad (A.40) $$

and the two terms can be simplified using completeness (2.16) and the formula (2.24) for $\mathcal{I}(x)$

$$ -\frac{1}{\sqrt{Q_{0}}} \int dx V^{3}[gf(x)] \mathcal{I}(x) g_k(x) = \frac{1}{Q_{0}} \int \frac{dk}{2\pi} \left( \frac{\omega_{k}^{2} - \omega_{k'}^{2}}{\omega_{k'}} \right) \Delta_{kk'} \Delta_{-k'B} \quad (A.41) $$

$$ -\frac{1}{2\sqrt{Q_{0}}} \int \frac{d^{2}k}{(2\pi)^{2}} \left( \frac{\omega_{k'} - \omega_{k''}}{\omega_{k'} \omega_{k''}} \right) V_{kk'B} \Delta_{-k'B-k''} $$

$$ -\frac{1}{\sqrt{Q_{0}}} \int dx V^{3}[gf(x)] \mathcal{I}(x) g_k(x) = -\frac{1}{\sqrt{Q_{0}}} V_{IBk} \Delta_{-k'B}. $$

The second equation in (A.41) substituted into the first term in (A.39) cancels the second, third and fourth terms. This leaves only the first equation in (A.41), which when substituted into (A.39) yields

$$ \Gamma_{2,\mathcal{I}}^{11} = \frac{1}{2} \int \frac{dk'}{2\pi} \left( \frac{\omega_{k}^{2} - \omega_{k'}^{2}}{\omega_{k'}} \right) \Delta_{kk'} \Delta_{-k'B} - \frac{\sqrt{Q_{0}}}{4} \int \frac{d^{2}k'}{(2\pi)^{2}} \left( \frac{\omega_{k'} - \omega_{k''}}{\omega_{k'} \omega_{k''}} \right) V_{kk'B} \Delta_{-k'B-k''}. \quad (A.42) $$
Summing the three contributions with integrals over $k'_1$ and $k'_2$

$$H_3|0\rangle_{1}^{03} \supset \frac{1}{8\sqrt{Q_0}} \int \frac{dk'}{2\pi} \int \frac{d^2k'}{(2\pi)^2} \left( \frac{\omega_{k'_1}^2 - \omega_{k'_2}^2}{\omega_{k'_1} \omega_{k'_2}} \right) Y_{kk'_1k'_2} \Delta_{-k'_1-k'_2} \phi_0 B_{k'_10}|0\rangle_0$$

$$H_3|0\rangle_{1}^{12} \supset \frac{1}{8\sqrt{Q_0}} \int \frac{dk'}{2\pi} \int \frac{d^2k'}{(2\pi)^2} \left( \frac{\omega_{k'_1} - \omega_{k'_2}}{\omega_{k'_1} \omega_{k'_2}} \right) V_{kk'_1k'_2} \Delta_{-k'_1-k'_2} \phi_0 B_{k'_1}^\dagger |0\rangle_0$$

$$\int \frac{dk'}{2\pi} \omega_{k'} B_{k'}^\dagger B_{k'} |0\rangle_{11} \supset \frac{1}{8\sqrt{Q_0}} \int \frac{dk'}{2\pi} \int \frac{d^2k'}{(2\pi)^2} \omega_{k'_1} \left( \frac{\omega_{k'_1} - \omega_{k'_2}}{\omega_{k'_1} \omega_{k'_2}} \right) Y_{kk'_1k'_2} \Delta_{-k'_1-k'_2} \phi_0 B_{k'}^\dagger |0\rangle_0$$

one obtains

$$\Gamma_{2,2k'}^{11} = \frac{\sqrt{Q_0}}{4} \int \frac{d^2k'}{(2\pi)^2} \left( \frac{\omega_{k'_1} - \omega_{k'_2}}{\omega_{k'_1} \omega_{k'_2}} \right) V_{kk'_1k'_2} \Delta_{-k'_1-k'_2} \quad (A.43)$$

which cancels the second term in $\Gamma_{2,\mathcal{I}}^{11}$.

Finally the terms with no $\mathcal{I}(x)$ and one $k'$ are

$$H_3|0\rangle_{1}^{01} \supset \frac{1}{4Q_0} \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} \left( \omega_{k'} - \frac{\omega_k^2}{\omega_{k'}} \right) \Delta_{kk'} \Delta_{-k'B} \phi_0 B_{k}^\dagger |0\rangle_0 \quad (A.44)$$

$$\pi_0^2 \int \frac{d^2k'}{(2\pi)^2} \left( \frac{\omega_k - 3\omega_{k'}}{\omega_{k'}} \right) \Delta_{kk'} \Delta_{-k'B} \phi_0 B_{k}^\dagger |0\rangle_0$$

$$\int \frac{dk'}{2\pi} \omega_{k'} B_{k'}^\dagger B_{k'} |0\rangle_{11} \supset \frac{1}{4Q_0} \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} \left( -\omega_k + \frac{3\omega_k^2}{\omega_{k'}} \right) \Delta_{kk'} \Delta_{-k'B} \phi_0 B_{k}^\dagger |0\rangle_0$$

which sum to

$$\Gamma_{2,1k'}^{11} = \frac{1}{2} \int \frac{dk'}{2\pi} \left( \frac{\omega_k^2 - \omega_{k'}^2}{\omega_k \omega_{k'}} \right) \Delta_{kk'} \Delta_{-k'B} \quad (A.45)$$

canceling the first term in $\Gamma_{2,\mathcal{I}}^{11}$. Summarizing, we have verified that

$$\Gamma_2^{11} = \Gamma_{2,\mathcal{I}}^{11} + \Gamma_{2,1k'}^{11} + \Gamma_{2,2k'}^{11} = 0 \quad (A.46)$$

and so the state $|0\rangle_2$ that we have found indeed solves Schrodinger’s equation at two loops.

**Appendix B  The Meson Mass in the Sine-Gordon Model**

In this Appendix we briefly review the Schrodinger picture derivation of the two-loop meson mass in the normal-ordered Sine-Gordon model. First one expands the scalar field in terms of Heisenberg operators

$$\phi(x) = \int \frac{dp}{2\pi} \left( A_p^\dagger + A_{-p} \frac{A_{-p}}{2\omega_p} \right) e^{-ipx}.$$  \quad (B.1)
The Sine-Gordon potential

\[ V(x) = \frac{M^2}{g^2} (1 - \cos (g\phi(x))) \]  

(B.2)

at fourth order is the interaction

\[ H_4 = -\frac{M^2 g^2}{24} \int dx : \phi^4(x) : \]  

(B.3)

while the free Hamiltonian is

\[ H_2 = \int \frac{dp}{2\pi} \omega_p A_p^\dagger A_p. \]  

(B.4)

Let the meson state \(|p\rangle\) be an eigenstate of the full Hamiltonian. Expand it in powers of \(g\)

\[ |p\rangle = \sum_{n=0}^{\infty} |p\rangle_n \]  

(B.5)

where

\[ |p\rangle_0 = A_p^\dagger |\Omega\rangle. \]  

(B.6)

The tree level Schrodinger Equation

\[ H_2 |p\rangle_0 = E_0 |p\rangle_0 \]  

(B.7)

is solved by \(E_0 = \omega_p\). At one loop

\[ 0 = (H_4 - E_1) |p\rangle_0 + (H_2 - E_0) |p\rangle_1 \]  

(B.8)

together with the convention\(^6\)

\[ \delta(p|p)_i = \delta_{0i} \]  

(B.9)

are solved by \(E_1 = 0\) and

\[ |p\rangle_1 = \frac{M^2 g^2}{24} \int dx \int \frac{d^4 q}{(2\pi)^4} \frac{e^{-ix \sum_j q_j}}{\sum_j \omega_{q_j}} A_{q_1}^\dagger \cdots A_{q_4}^\dagger A_p^\dagger |\Omega\rangle \]

\[ + \frac{M^2 g^2}{12} \int dx \int \frac{d^3 q}{(2\pi)^3} \frac{e^{-ix(-p + \sum_j q_j)}}{\omega_p \left(-\omega_p + \sum_j \omega_{q_j}\right)} A_{q_1}^\dagger \cdots A_{q_3}^\dagger |\Omega\rangle. \]  

(B.10)

At two loops, the Schrodinger equation is

\[ 0 = (H_6 - E_2) |p\rangle_0 + (H_4 - E_1) |p\rangle_1 + (H_2 - E_0) |p\rangle_2. \]  

(B.11)

\(^6\)For simplicity we have fixed \(p\) and applied a \(p\)-dependent normalization condition on \(|\Omega\rangle\).
Let us left multiply $0 \langle p \mid H_4 \mid p \rangle$ and use the orthogonality condition (B.9). As $H_6$ is normal ordered, its matrix element vanishes and one finds

$$E_2 = 0 \langle p \mid H_4 \mid p \rangle_1 = A + B$$

(B.12)

where

$$A = -\frac{M^4 g^4}{48\omega_p} \int \frac{d^2 q}{(2\pi)^2} \frac{\omega_{q_1} + \omega_{q_2} + \omega_{p-q_1-q_2}}{\omega_{q_1} \omega_{q_2} (\omega_{q_1} + \omega_{q_2} + \omega_{p-q_1-q_2})^2 - \omega_p^2}$$

(B.13)

$$B = -\frac{M^4 g^4}{384} \int \frac{d^3 q}{(2\pi)^3} \frac{1}{\omega_{q_1} \omega_{q_2} \omega_{q_3} (\omega_{q_1} + \omega_{q_2} + \omega_{q_3})^2}.$$  

The infrared divergent term $B$ is equal to the two-loop energy of the vacuum state $\langle \Omega \rangle$ and so it does not contribute to the meson mass. Therefore the two-loop meson mass correction $M_2$ is equal to $A$ evaluated at $p = 0$

$$M_2 = -\frac{M^2 g^4}{48} \int \frac{d^2 q}{(2\pi)^2} \frac{\omega_{q_1} + \omega_{q_2} + \omega_{q_1+q_2}}{\omega_{q_1} \omega_{q_2} (\omega_{q_1} + \omega_{q_2} + \omega_{q_1+q_2})^2 - M^2} = -\frac{M g^4}{768}$$

(B.14)

in agreement with the pole mass in Ref. [15].

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