From constructive field theory to fractional stochastic calculus. (I) The Lévy area of fractional Brownian motion with Hurst index $\alpha \in \left(\frac{1}{8}, \frac{1}{4}\right)$

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Let $B = (B_1(t), \ldots, B_d(t))$ be a $d$-dimensional fractional Brownian motion with Hurst index $\alpha < 1/4$. Defining properly iterated integrals of $B$ is a difficult task because of the low Hölder regularity index of its paths. Yet rough path theory shows it is the key to the construction of a stochastic calculus with respect to $B$, or to solving differential equations driven by $B$.

We show in this paper how to obtain second-order iterated integrals as the limit when the ultra-violet cut-off goes to infinity of iterated integrals of weakly interacting fields defined using the tools of constructive field theory, in particular, cluster expansion and renormalization. The construction extends to a large class of Gaussian fields with the same short-distance behaviour, called multi-scale Gaussian fields. Previous constructions \cite{36, 35} were of algebraic nature and did not provide such a limiting procedure.

**Keywords:** fractional Brownian motion, stochastic integrals, rough paths, constructive field theory, Feynman diagrams, renormalization, cluster expansion.

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## 0 Introduction

A major achievement of the probabilistic school since the middle of the 20th century is the study of diffusion equations, in connection with Brownian motion or more generally Markov processes – and also with partial differential equations, through the Feynman-Kac formula – with many applications in physics and chemistry [38]. One of the main tools is stochastic calculus with respect to semi-martingales $M$. An adapted integral such as $\int_s^t X(u) dM(u)$ may be understood as a limit in some sense to be defined. Classically one uses piecewise linear interpolations, $\sum_{s \leq t_1 < \ldots < t_N \leq t} X(t_i)(M(t_{i+1}) - M(t_i))$ or $\sum_{s \leq t_1 < \ldots < t_N \leq t} \frac{X(t_i) + X(t_{i+1})}{2}(M(t_{i+1}) - M(t_i))$; these approximations define in the limit $N \to \infty$ the Itô, resp. Stratonovich integral. The latter one is actually obtained e.g. if $M = W$ is Brownian motion and $X(t) = f(W_t)$ with $f$ smooth as the limit $\lim_{\varepsilon \to 0} \int_s^t f(W_\varepsilon(u)) dW_\varepsilon(u)$ for any smooth approximation $(W_\varepsilon)_\varepsilon > 0$ of $W$ converging a.s. to $W$ (see [10], or [17] p. 169).
The Stratonovich integral \( \int_s^t X(u) d^{\text{Strato}} M(u) \) has an advantage over the Itô integral in that it agrees with the fundamental theorem of calculus, namely, \( F(M(t)) = F(M(s)) + \int_s^t F'(M(u)) d^{\text{Strato}} M(u) \).

The semi-martingale approach fails altogether when considering stochastic processes with lower regularity. Brownian motion, and more generally semi-martingales (up to time reparametrization), are \((1/2)^-\) Hölder, i.e. \( \alpha \)-Hölder for any \( \alpha < 1/2 \). Processes with \( \alpha \)-Hölder paths, where \( \alpha \ll 1/2 \), are maybe less common in nature but still deserve interest. Among these, the family of multifractional Gaussian processes is perhaps the most widely studied \[28\], but one may also cite diffusions on fractals \[15\], sub- or superdiffusions in porous media \[11, 18\] and the fascinating multi-fractal random measures/walks in connection with turbulence and two-dimensional quantum gravity \[4, 6\]. Although some of these processes appear in applications rather as a random landscape in \( D \geq 1 \) space dimensions (which raises related but different problems), we view them here – restricting ourselves to \( D = 1 \) – as some colored noise integrated in time, and wish to define stochastic integration with respect to them or to solve differential equations driven by them.

We concentrate in this article on \textit{multiscale Gaussian processes} (the terminology is ours) with \textit{scaling dimension} or more or less equivalently \textit{Hölder regularity} \( \alpha \in (0, 1/2) \), the best-known example of which being \textit{fractional Brownian motion} (fBm for short) with Hurst index \( \alpha \), \( B^\alpha(t) \) or simply \( B(t) \). \footnote{Recall that a continuous path \( X : [0,T] \to \mathbb{R} \) is \( \alpha \)-Hölder, \( \alpha \in (0,1) \), if \( \sup_{s,t \in [0,T]} \frac{|X_t - X_s|}{|t-s|^{\alpha}} < \infty \).} We consider more precisely a \textit{two-dimensional fBm}, \( B(t) = (B_1(t), B_2(t)) \), with independent, identically distributed components \footnote{It is (up to a constant) the unique self-similar Gaussian process with stationary increments. The last property implies that its derivative is a \textit{stationary field}.}. The simplest non-trivial stochastic integral is then \( A(s,t) := \int_s^t dB_1(t_1) \int_s^{t_1} dB_2(t_2) = \int_s^t (B_2(u) - B_2(s)) dB_1(u) \), a twice iterated integral, where \( B = (B_1(t), B_2(t)) \) is a two-component fBm. Since \( \int_s^t dB_1(t_1) \int_s^{t_1} dB_2(t_2) + \int_s^t dB_2(t_2) \int_s^{t_2} dB_1(t_1) = (B_1(t) - B_1(s))(B_2(t) - B_2(s)) \), one is mainly interested in the antisymmetrized quantity (measuring an area, as follows from \footnote{The one-dimensional case is very different and much simpler, and has been treated in \[12\].}}
the Green-Riemann formula),

\[ \mathcal{L}A(s,t) := \int_s^t dB_1(t_1) \int_s^{t_1} dB_2(t_2) = \int_s^t dB_1(t_1) \int_s^{t_2} dB_2(t_2) - \int_s^t dB_2(t_2) \int_s^{t_1} dB_1(t_1) \]

\[ = \int_s^t (B_2(u) - B_2(s)) dB_1(u) - (B_1(u) - B_1(s)) dB_2(u), \]

(0.1)
called Lévy area. The corresponding Stratonovich integral, obtained as a limit either by linear interpolation or by more refined Gaussian approximations \[5, 27, 32, 33\], has been shown to diverge as soon as \( \alpha \leq 1/4 \).

This seemingly no-go theorem, although clear and derived by straightforward computations that we reproduce in short in section 1, appears to be a puzzle when put in front of the results of rough path theory \(23, 24, 13, 20, 21, 10\). We shall not enter into the details of this fascinating theory, with its deep connections with nilpotent groups, Hopf algebras and sub-Riemannian geometry, leaving this to a forthcoming paper \(25\) where these structures will be instrumental to define higher-order integrals. Let us only state that rough path theory allows one to define integration with respect to an \( \alpha \)-Hölder path \( \Gamma \) in terms of substitutes of iterated integrals of \( \Gamma \) up to order \( \lfloor 1/\alpha \rfloor \), \( \lfloor \cdot \rfloor \) = entire part, called rough path over \( \Gamma \), satisfying some Hölder regularity property and also two algebraic properties, called respectively Chen and shuffle property. The former one is of geometric nature and may be seen as an additivity property for the areas or volumes generated by double or multiple iterated integrals. In particular, it implies the following compatibility property between the twice iterated integral \( A \) and the original process \( B \),

\[ A(s,t) = A(s,u) + A(u,t) + (B_2(u) - B_2(s))(B_1(t) - B_1(u)). \]

(0.2)
The shuffle property, on the other hand, ensures once again that the fundamental theorem of calculus holds. Whatever the exact formulation of these two properties, they hold true trivially for the usual iterated integrals of \( \Gamma \) if \( \Gamma \) is smooth, and (being of algebraic nature), also hold true trivially when the rough path over \( \Gamma \) is constructed out of true iterated integrals of smooth paths by some limiting procedure as explained above.

The main point is now the following. General theorems show (i) that rough paths always exist (existence theorem); (ii) that any rough path over \( \Gamma \) may be constructed by some rather abstract limiting procedure\( ^4\) (approximation theorem). Unfortunately, these theorems are not constructive; also,\n
\[^4\text{using pieces of horizontal sub-Riemannian geodesics.}\]
they are of pathwise nature, and thus not necessarily appropriate in the probabilistic setting which is our starting point. Recent results using the underlying Hopf algebra structures \cite{36, 35, 37, 8, 34} give an explicit, general solution to the existence problem, well-suited in particular for Gaussian processes, but no one knows how to obtain these rough paths by an explicit limiting procedure.

Our project in this series of papers is to define an explicit rough path over fBm with arbitrary Hurst index, or more generally multiscale Gaussian fields (to be defined below) by an explicit, probabilistically meaningful limiting procedure, thus solving at last the problem of constructing a full-fledged, Stratonovich-like integration with respect to fBm.

Roughly speaking, it is obtained by making

$$B = (B(1), B(2))$$

interact through a weak but singular quartic, non-local interaction, which plays the role of a squared kinetic momentum, associated to the rotation of the path, and makes its Lévy area finite. Following the common use of quantum field theory, this is implemented by multiplying the Gaussian measure by the exponential weight

$$e^{-\frac{1}{2}c_\alpha \int \int L_{\text{int}}(\phi_1, \phi_2)(t_1, t_2) \left| t_1 - t_2 \right|^{-4\alpha} dt_1 dt_2},$$

with

$$L_{\text{int}}(\phi_1, \phi_2)(t_1, t_2) = \lambda^2 \left\{ (\partial A^+)(t_1)(\partial A^+)(t_2) + (\partial A^-)(t_1)(\partial A^-)(t_2) \right\},$$

(0.3)

where: \(\lambda\) (the coupling parameter) is a small, positive constant; \(\phi_1, \phi_2\) are the (infra-red divergent) stationary fields associated to \(B_1, B_2\), with covariance kernel as in eq. (2.21), and similarly, \(A^\pm\) are stationary left- and right-turning fields, built out of \(\phi_1, \phi_2\), and representing the singular part of the Lévy area (see section 1 for details). By assumption \(\alpha < 1/4\), so that the kernel \(\left| t_1 - t_2 \right|^{-4\alpha}\) is locally integrable. The statistical weight is maximal when \(\partial A^+ = \partial A^- = 0\), i.e. for sample paths which are “essentially” straight lines. Another way to motivate this interaction is to understand that the divergence of the Lévy area is due to the accumulation in a small region of space \cite{20}; the statistical weight is unfavorable to such an accumulation. On the other hand, the quantities in the first-order Gaussian chaos (in other words, the n-point functions \(\langle B_{i_1}(x_1) \ldots B_{i_n}(x_n) \rangle \lambda = \frac{1}{Z} E \left[ B_{i_1}(x_1) \ldots B_{i_n}(x_n) e^{-\frac{1}{2}c_\alpha \int \int L_{\text{int}}(\phi_1, \phi_2)(t_1, t_2) \left| t_1 - t_2 \right|^{-4\alpha} dt_1 dt_2} \right]\), \(i_1, \ldots, i_n = 1, 2\),

where \(Z := E \left[ e^{-\frac{1}{2}c_\alpha \int \int L_{\text{int}}(\phi_1, \phi_2)(t_1, t_2) \left| t_1 - t_2 \right|^{-4\alpha} dt_1 dt_2} \right]\) is a normalization constant called partition function) – associated to pure translation – are insensitive to the interaction. Thus the compatibility property (0.2) is satisfied,

\footnote{The unessential constant \(c_\alpha\) is fixed e.g. by demanding that the Fourier transform of the kernel \(c_\alpha \left| t_1 - t_2 \right|^{-4\alpha}\) is the function \(|\xi|^{4\alpha - 1}\).}
and one has constructed a Lévy area for fBm.

It would be very interesting to find an equivalent pathwise description of this interaction in terms of a constrained, long-memory self-interacting walk, see section 1 for more comments. The interaction \( \mathcal{L}_{\text{int}} \) would thus play the rôle of a Ginzburg-Landau functional, whose usual, local version is conjectured to give e.g. a field-theoretic description of the Ising model. Proving such an equivalence should be much easier here (despite the unusual non-local features) since it is a one-dimensional model.

Starting from the above field-theoretic description, the proof of finiteness and Hölder regularity of the Lévy area for \( \lambda > 0 \) small enough follows the well-established scheme of constructive field theory. The main theorem may be stated as follows. As a rule, we denote in this article by \( \mathbb{E}[\ldots] \) the Gaussian expectation and by \( \langle \ldots \rangle_{\lambda, \rho} \) the expectation with respect to the \( \lambda \)-weighted interaction measure with scale \( \rho \) ultraviolet cut-off, so that in particular \( \mathbb{E}[\ldots] = \langle \ldots \rangle_{0, \infty} \).

**Theorem 0.1** Assume \( \lambda \in \left( \frac{1}{8}, \frac{1}{4} \right) \). Consider for \( \lambda > 0 \) small enough the family of probability measures (also called: \( (\phi, \partial \phi, \sigma) \)-model)

\[
\mathbb{P}^\lambda \to \rho(\phi_1, \phi_2) = e^{\frac{1}{2}c} \int dt_1 dt_2 \int_{|t_1 - t_2|^{-\alpha}} dt_1 dt_2 |t_1 - t_2|^{-4\alpha} \mathcal{L}_{\text{int}}(\phi_1, \phi_2)(t_1, t_2) d\mu \rightarrow \rho(\phi_1) d\mu \rightarrow \rho(\phi_2),
\]

(0.4)

where \( d\mu \rightarrow \rho(\phi_i) = d\mu(\phi_i \rightarrow \rho) \) is a Gaussian measure obtained by an ultraviolet cut-off at Fourier momentum \( |\xi| \approx M \rho \) \( (M > 1) \), see Definition 2.10 and section 5. Then \( (\mathbb{P}^\lambda \to \rho) \) converges in law when \( \rho \rightarrow \infty \) to some measure \( \mathbb{P}_\lambda \), and the associated twice iterated integral \( \int_s^t d\phi_1 \rightarrow \rho(t_1) \int_s^{t_1} d\phi_2 \rightarrow \rho(t_2) \) converges to a well-defined quantity \( A(s, t) \) satisfying the Chen and shuffle properties, with variance

\[
\langle A(s, t)^2 \rangle_\lambda := \lim_{\rho \rightarrow \infty} \langle A(s, t)^2 \rangle_{\lambda, \rho} = \left( \frac{4}{\lambda^2} K_1 + K_2 \right) |t - s|^{4\alpha}. \tag{0.5}
\]

Furthermore, the connected moments of \( A(s, t) \) in the interacting theory – except its variance – coincide with those of the free, Gaussian theory, allowing for the computation of moments of \( A(s, t) \) in terms of explicit Gaussian integrals. In particular, one finds:

\[
\langle A(s, t)^{2n} \rangle_\lambda \leq C_n |t - s|^{4n\alpha}, \quad n \geq 0. \tag{0.6}
\]

As an immediate corollary following from the Kolmogorov-Centsov lemma \[20\], \( A(s, t) \) has a.s. \((2\alpha)^-\)-Hölder paths.

\[^6\]This lemma states that a process \( X \) such that \( \mathbb{E}|X(t) - X(s)|^p \leq C |t - s|^{1+p\beta} \) has almost surely \( \beta^- \)-Hölder paths. Classical hypercontractivity arguments implies the same conclusion for a Gaussian process provided simply \( \mathbb{E}|X(t) - X(s)|^2 \leq C |t - s|^{2\beta} \).
The only difficulty is to prove the convergence of the moments of finite distributions of the Lévy area when $\rho \to \infty$; let us see how to deduce the assertions of the theorem from this.

The weak convergence of (some subsequence of) the family of laws $(\mathbb{P}^{\rho,\lambda})_\rho$ is actually a consequence – using standard arguments including Prohorov’s theorem, [2], chap. 6, 8, 12 – of the following estimates which imply that $(\mathbb{P}^{\rho,\lambda})_\rho$ is tight,

$$\langle |\phi(t) - \phi(s)|^2 \rangle_{\lambda,\rho} \leq K |t - s|^{2\alpha}, \quad (0.7)$$

$K$ being a constant uniform in $\rho$. This last estimate is proved in a stronger Fourier form

$$\langle |\phi(\xi)|^2 \rangle_{\lambda,\rho} \leq \frac{K}{|\xi|^{1+2\alpha}}. \quad (0.8)$$

Finally, standard multi-scale estimates, sketched in section 1, show that the generating function of connected moments of finite distributions of the Lévy area, $(u_1,\ldots,u_n) \mapsto \log \langle \exp i (u_1\mathcal{A}(s_1,t_1) + \ldots + u_n\mathcal{A}(s_n,t_n)) \rangle_{\lambda,\infty}$, has a non-zero radius of convergence, hence (see [7], Chap. XV, §4) the moments of the finite distributions of the Lévy area determine its law, and every subsequence of $(\mathbb{P}^{\rho,\lambda})_\rho$ converges to the same limit. It would also be possible (by a simple extension of the results of section 6) to prove a speed of convergence for these moments, e.g. $\langle \phi^j(s)\phi^j(t) \rangle_{\lambda,\rho} - \langle \phi^j(s)\phi^j(t) \rangle_{\lambda,\rho'} = K_r\frac{M^{-2\alpha j}}{(1+M|t-s|)^r}O(M^{-(8\alpha-1)(\rho'-j)})$ (for some constant $K_r$ depending on $r = 1, 2, \ldots$) if $j < \rho' < \rho$.

The result is not difficult to understand using the non-rigorous perturbation theory. First, by a trick explained in section 1, one replaces the non-local interaction $L(\phi_1,\phi_2)(t_1,t_2)|t_1 - t_2|^{-4\alpha}$ with a local interaction $L(\phi_1,\phi_2,\sigma)(t)$ depending on a two-component exchange particle field $\sigma = (\sigma_+(t), \sigma_-(t))$. Then a Schwinger-Dyson identity (a functional integration by parts) relates the moments of $\mathcal{A}$ to those of $\sigma$. Simple power-counting arguments show that a connected $2n$-point function of $\sigma$ alone is superficially divergent if and only if $1 - 4n\alpha \geq 0$. Thus, restricting to $\alpha > 1/8$, one only needs to renormalize the two-point function. Since the renormalized propagator of $\sigma$ is screened by an infinite mass term (as we shall see in section 1), the theory is free once one has integrated out the $\sigma$-field, hence one retrieves the underlying Gaussian theory $(\phi_1, \phi_2)$. The Schwinger-Dyson identity then shows that the two-point functions of $\mathcal{A}$ have been made finite.

\footnote{Namely, $|\phi(t) - \phi(s)|^2 \leq \int_{|\xi|>\frac{K}{|t-s|}} K |\xi|^2 d\xi + \int_{|\xi|<\frac{K}{|t-s|}} K \frac{1-M^2\xi^2}{|\xi|^2} d\xi \leq K^*|t-s|^{2\alpha}$.}
Since constructive field theory \cite{26,30} is not a standard tool in probability, we shall spend some time explaining in full details and generality its objects, in particular the *cluster expansions* which lie at its heart. Cluster expansions are based on a simplified wavelet decomposition \((\psi^j_\Delta)\) of the fields \(\psi = \phi_1, \phi_2, \sigma_\pm\) where \(j\) is a *vertical* (Fourier) scale index, and \(\Delta\) a *horizontal* (i.e. in direct space) interval of size \(M^{-j}\) around the center of the wavelet component. Each \(\psi^j_\Delta\) is to be seen as a *degree of freedom* of the theory, relatively independent from the others, so that the interaction may be expressed as a divergent doubly-infinite vertical and horizontal sum.

The horizontal (H) and vertical (V) cluster expansions allow to rewrite the partition function \(Z_{V^\rightarrow}^\rho\) over a finite volume, with ultraviolet truncation at scale \(\rho\), as a sum,

\[
Z_{V^\rightarrow}^\rho = \sum_n \frac{1}{n!} \sum_{P_1, \ldots, P_n \text{ non-overlapping}} F_{HV}(P_1) \cdots F_{HV}(P_n), \tag{0.9}
\]

where:
- \(P_1, \ldots, P_n\) are disjoint *polymers*, i.e. sets of intervals \(\Delta\) connected by vertical and horizontal links; during the course of the expansion, the Gaussian measure has been modified so that the field components belonging to different polymers have become independent;
- \(F_{HV}(P)\), \(P = P_1, \ldots, P_n\) is the \(\lambda\)-weighted expectation value, \(F_{HV}(P) = \langle f_{HV}(P) \rangle_\lambda\), of some function \(f_{HV}\) depending only on the field components located in the support of \(P\).

The fundamental idea is that (i) the *polymer evaluation function* \(F(P)\) is all the smaller as the polymer \(P\) is large, due to the polynomial decrease of correlation at large distances (for the horizontal direction), and to power-counting arguments developed in section 4 for the vertical direction, leading to the image of horizontal islands maintained together by vertical *springs*; (ii) the horizontal and vertical links in \(P\) (once one interval belonging to \(P\) has been fixed) suppress the invariance by translation, which normally leads to a divergence when \(|V| \to \infty\). A classical combinatorial trick, called *Mayer expansion*, allows one to rewrite eq. (0.9) as a similar sum over *trees of polymers*, also called *Mayer-extended polymers* and denoted by the same letter \(P\), but *without non-overlap conditions*, \(Z_{V^\rightarrow}^\rho = \sum_n \frac{1}{n!} \sum_{P_1, \ldots, P_n} F(P_1) \cdots F(P_n)\), where \(F = F_{HV,M}\) is the *Mayer-extended polymer evaluation function*, so that \(\ln Z_{V^\rightarrow}^\rho = \sum_P F(P)\). In the process, *local parts of diverging graphs* have been resummed into an exponential, leading to a *counterterm in the interaction*; this is the essence of *renormalization*. On the whole, one finds that in the limit \(|V|, \rho \to \infty\), the free energy \(\ln Z_{V^\rightarrow}^\rho\) is a sum over each scale
of scale-dependent extensive quantities, i.e. \( \ln Z_{V}^{\rho} = |V| \sum_{j=0}^{\infty} M_j f_{V}^{j-\rho} \), where \( f_{V}^{j-\rho} \) converges when \( |V| \to \infty \) to a finite quantity of order \( O(\lambda) \). One retrieves the idea that each interval of scale \( j \) contains one degree of freedom. Finally, \( n \)-point functions are computed as derivatives of an external-field dependent version of the free energy.

Here is the outline of the article. Section 1 recalls the classical results on the divergence of the Lévy area for \( \alpha \leq 1/4 \) and recasts them into notations which are more appropriate for cluster expansions. It also contains a heuristic section using perturbative field theory and giving some intuition on why adding the interaction term \( L_{\text{int}} \) briefly described above should make the Lévy area convergent. Section 2 contains the definition of multiscale Gaussian fields, including fBm, and many useful notations and general results concerning the scale (wavelet) decompositions. Section 3 is on cluster expansions, section 4 on renormalization. Sections 2 to 4 are extremely general, valid also for fields living on \( \mathbb{R}^D \), in the hope that they may serve as a basis for future work, possibly also for \( D > 1 \)-models. Section 5, on the contrary, concentrates on the definition of our specific \( (\phi, \partial \phi, \sigma) \)-model. The proof of finiteness of \( n \)-point functions of the Lévy area and freeness of the \( \phi \)-field is given in section 6. It depends on Gaussian bounds which hold in great generality, and on domination arguments which are very specific of our model.

Notations. Cluster expansions imply the use of many indices and letters. Let us summarize here some of our most important conventions, in the hope that this will help the reader not to get lost.

1. Quite generally, \( \psi = (\psi_1(x), \ldots, \psi_d(x)) \) is a \( d \)-dimensional field living on \( \mathbb{R}^D \), \( D \geq 1 \), with scaling dimensions \( \beta_1, \ldots, \beta_d \). Roughly speaking, for probabilists, \( \beta \) is the Hölder continuity index, at least when \( \beta > 0 \); physicists usually call scaling dimension \( -\beta \). Most of the model-independent results here are valid for arbitrary \( D \). The Fourier transform is denoted by \( \mathcal{F} \).

2. If \( \psi = \psi(x) \) is a Gaussian field, then its scale decomposition (see section 1) reads \( \psi = \sum_{j \geq 0} \psi^j \). The low-momentum, resp. high-momentum field with respect to scale \( j \) is denoted by \( \psi^{-j} = \sum_{h \leq j} \psi^h \), resp. \( \psi^{j-} = \sum_{k \geq j} \psi^k \). All through the text, we observe the following convention: if \( h, j, k \) are scale indices, then \( h \leq j \leq k \); any primed scale index (for instance \( j', \rho', \ldots \)) is less than the original scale index.
Secondary fields (i.e. low-momentum fields ψ minus their average) are denoted by δψ. We also introduce restricted high-momentum fields, see section 1, denoted by Res ψ.

3. (products of fields) If ψ = (ψ₁(x), ..., ψ₅(x)) is a d-dimensional field, and I = (i₁, ..., iₙ) ∈ {1, ..., d}ⁿ (n ≥ 1) is a multi-index, we denote by ψ_I the product of fields ψ_I(x) := ψ_{i₁}(x) ... ψ_{iₙ}(x). The interaction Lagrangian L_{int} is written in general as \( \sum_{q=1}^{p} K_q \lambda_{\psi}^q \psi_{I_q} \) for some constants K_q and exponents κ_q. Cluster expansions produce propagators and products of fields which are linear combinations of monomials (also called: G-monomials), generically denoted by G.

4. (constants) K is a constant depending only (possibly) on the details of the model, such as the degree of the interaction, the scaling dimension of the fields... It may vary from line to line. M > 1 is the basis of the scale decomposition; it is an absolute constant, whose value is unimportant. γ is some constant > 1, or sometimes simply ≥ 1. N_{ext,max} is such that every Feynman diagram with ≥ N_{ext,max} external legs is superficially convergent; this perturbative notion also plays a central role in constructive field theory. Our model (φ, ∂φ, σ) has N_{ext,max} = 4.

5. (variables) The maximum scale index is ρ. Other scale indices are denoted by h, j, k, or any of those with primes. Indices i are summation indices, with finite range, used in various contexts, for instance for the field components. The parameters of the horizontal, resp. Mayer, resp. vertical (also called momentum-decoupling) cluster expansion, are denoted by s, resp. S, resp. t. The scale of an interval Δ is denoted by j(Δ). If Δ is an interval, then n(Δ) is the coordination number of Δ inside the tree defined by the horizontal cluster expansion, while N(Δ), resp. N_i(Δ) is the total number of fields, resp. the number of fields ψ_i located in the interval Δ. τ stands for a number of derivations, usually with respect to the t-parameters, in which case it is assumed to be ≤ N_{ext,max} + O(n(Δ)).

1 Statement of the problem and heuristics

The quantity we want to define in the case of fractional Brownian motion is the following.

**Definition 1.1 (Lévy area)** The Lévy area of a two-dimensional path Γ : \( \mathbb{R} \to \mathbb{R}^2 \) between s and t is the area between the straight line connecting
(\Gamma_1(s), \Gamma_2(s)) to (\Gamma_1(t), \Gamma_2(t)) and the graph \{ (\Gamma_1(u), \Gamma_2(u)) ; s \leq u \leq t \}. It is given by the following antisymmetric quantity,

\[ \mathcal{L}A_\Gamma(s, t) := \int_s^t d\Gamma_1(t_1) \int_{t_1}^t d\Gamma_2(t_2) - \int_s^t d\Gamma_2(t_2) \int_{t_2}^t d\Gamma_1(t_1). \] (1.1)

1.1 A Fourier analysis of the Lévy area

In order to understand the analytic properties of the Lévy area of fBm, we shall resort to a Fourier transform. One obtains, using the harmonizable homogeneous in \( \xi \) is given by the following antisymmetric quantity, \( A_{L_\zeta} \) is homogeneous of degree 2\( \alpha \) in \( |t - s| \) since \( B(ct) - B(cs), c > 0 \) has same law as \( e^{it}(B(t) - B(s)) \) by self-similarity.

Expanding the right-hand side yields an expression which is not homogeneous in \( \xi \). Hence it is preferable to define instead the skeleton integral, depending only on one variable,

\[ \mathcal{A}(s, t) := \int_s^t dB_1(t_1) \int_{t_1}^t dB_2(t_2) = \frac{1}{2\pi c_\alpha} \int \frac{dW_1(\xi_1) dW_2(\xi_2)}{|\xi_1|^{\alpha-1/2} |\xi_2|^{\alpha-1/2}} \int_s^t dt_1 \int_{t_1}^t dt_2 \cdot e^{i(t_1 \xi_1 + t_2 \xi_2)}. \] (1.2)

The \( \mathcal{L}\mathcal{A}(s, t) := \mathcal{L}A_B(s, t) \) is obtained from this twice iterated integral by antisymmetrization. Note that \( \mathcal{L}\mathcal{A}(s, t) \) is homogeneous of degree 2\( \alpha \) in \( |t - s| \) since \( B(ct) - B(cs), c > 0 \) has same law as \( e^{it}(B(t) - B(s)) \) by self-similarity.

Expanding the right-hand side yields an expression which is not homogeneous in \( \xi \). Hence it is preferable to define instead the skeleton integral, depending only on one variable,

\[ \mathcal{A}(t) := \int_t^s dB_1(t_1) \int_{t_1}^s dB_2(t_2) = \frac{1}{2\pi c_\alpha} \int \frac{dW_1(\xi_1) dW_2(\xi_2)}{|\xi_1|^{\alpha-1/2} |\xi_2|^{\alpha-1/2}} \int_t^s dt_1 \int_{t_1}^s dt_2 \cdot e^{i(t_1 \xi_1 + t_2 \xi_2)} = \frac{1}{2\pi c_\alpha} \int \frac{dW_1(\xi_1) dW_2(\xi_2)}{|\xi_1|^{\alpha-1/2} |\xi_2|^{\alpha-1/2}} \cdot \frac{e^{it(\xi_1 + \xi_2)}}{|i(\xi_1 + \xi_2)||i\xi_2|}. \] (1.3)

where by definition \( \int t e^{iu\xi} du = \frac{e^{it\xi}}{i\xi} \). From \( \mathcal{A}(t) \) and the one-dimensional skeleton integral

\[ \phi_1(t) = (2\pi c_\alpha)^{-\frac{1}{2}} \int_0^t dB_1(u) = \int \frac{dW_i(\xi)}{|\xi|^{\alpha-1/2}} \cdot \frac{e^{i\xi}}{i\xi}, \] (1.4)

which is the infra-red divergent stationary process associated to \( B \), see section 2, one easily retrieves \( \mathcal{A}(s, t) \) since

\[ \mathcal{A}(s, t) = \int_s^t dB_1(t_1) \left( \int_{t_1}^t dB_2(t_2) - \int_s^t dB_2(t_2) \right) = \mathcal{A}(t) - \mathcal{A}(s) + \mathcal{A}_\theta(s, t), \] (1.5)

where \( (2\pi c_\alpha)^{\frac{1}{2}} \mathcal{A}_\theta(s, t) := (B_1(t) - B_1(s)) \phi_2(s) \) (called boundary term) is a product of first-order integrals.
One may easily estimate these quantities in each sector \(|\xi_1| \gtrsim |\xi_2|\). In practice, it turns out that estimates are easiest to get after a permutation of the integrals (applying Fubini’s theorem) such that (for twice or multiple iterated integrals equally well) innermost (or rightmost) integrals bear highest Fourier frequencies; this is the essence of Fourier normal ordering \cite{[36][38][37]}. This gives a somewhat different decomposition with respect to \((1.5)\) since \(\int_0^t dB_1(t_1) \int_0^{t_1} dB_2(t_2)\) is rewritten as \(-\int_0^t dB_2(t_2) \int_0^{t_2} dB_1(t_1)\) in the "negative" sector \(|\xi_1| > |\xi_2|\). After some elementary computations, one gets the following.

**Lemma 1.2** Let

\[
A^+(t) := 2\pi c_\alpha \int_0^t dt_1 \int_0^{t_1} dt_2 F_1^{-1} ((\xi_1, \xi_2) \mapsto 1_{|\xi_1| < |\xi_2|} (FB_1^1(\xi_1)(FB_2^1(\xi_2))) (t_1, t_2)
\]

\[
= \int_{|\xi_1| < |\xi_2|} \frac{dW_1(\xi_1) dW_2(\xi_2)}{|\xi_1|^{\alpha-1/2} |\xi_2|^{\alpha-1/2}} \cdot \frac{e^{i\xi_1 + \xi_2}}{[i(\xi_1 + \xi_2)][i\xi_2]} \quad (1.6)
\]

and

\[
A^-(t) := 2\pi c_\alpha \int_0^t dt_2 \int_0^{t_2} dt_1 F_1^{-1} ((\xi_1, \xi_2) \mapsto 1_{|\xi_2| < |\xi_1|} (FB_1^1(\xi_1)(FB_2^1(\xi_2))) (t_1, t_2)
\]

\[
= \int_{|\xi_2| < |\xi_1|} \frac{dW_1(\xi_1) dW_2(\xi_2)}{|\xi_1|^{\alpha-1/2} |\xi_2|^{\alpha-1/2}} \cdot \frac{e^{i\xi_1 + \xi_2}}{[i(\xi_1 + \xi_2)][i\xi_1]} . \quad (1.7)
\]

Then

\[
A(s, t) = \frac{1}{2\pi c_\alpha} \left\{ (A^+(t) - A^+(s)) - (A^-(t) - A^-(s)) + (A^+_\delta(s, t) - A^-_\delta(s, t)) \right\},
\]

where

\[
A^+_\delta(s, t) - A^-_\delta(s, t) = \left\{ -\int_{|\xi_1| < |\xi_2|} \frac{(e^{i\xi_1} - e^{i\xi_2}) e^{i\xi_2}}{[i\xi_1][i\xi_2]} + \int_{|\xi_2| < |\xi_1|} \frac{(e^{i\xi_2} - e^{i\xi_1}) e^{i\xi_1}}{[i\xi_1][i\xi_2]} \right\} \cdot \frac{dW_1(\xi_1) dW_2(\xi_2)}{|\xi_1|^{\alpha-1/2} |\xi_2|^{\alpha-1/2}} .\quad (1.9)
\]

Two lines of computations show immediately that the variance of \(A^+_\delta\) is always finite (essentially, considering e.g. \(A^+_\delta\), because \(\int_1^{+\infty} \frac{d\xi_1}{\xi_1} \int_1^{+\infty} \frac{d\xi_2}{\xi_2} < \infty\), while the artificial infrared divergence at \(\xi_1 = 0\) disappears when Taylor expanding \(e^{i\xi_1} - e^{i\xi_2}\)). On the other hand, letting \(\xi := \xi_1 + \xi_2\) and
introducing an ultra-violet cut-off at \( |\xi_2| = \Lambda \gg 1 \), one may see for instance \( \mathcal{A}^+ (t) \) as an inverse random Fourier transform of the integral \( \xi \mapsto \int_{|\xi - \xi_2| < |\xi_2|}^\Lambda \frac{dW_2 (\xi_2)}{\xi_2} |\xi - \xi_2|^{\alpha - 1/2} |\xi_2|^{\alpha - 1/2} \), whose variance diverges like \( \int_{|\xi|}^\Lambda d\xi \propto \Lambda^{1 - 4\alpha} \) or \( O(\ln \Lambda) \) in the limit \( \Lambda \to \infty \) as soon as \( \alpha \leq 1/4 \). Note that the ultraviolet divergence is in the region \( |\xi_1|, |\xi_2| \gg |\xi| \). Later on, we shall introduce the field \( \sigma \), and this region will correspond to a splitting of the vertex \( \int \partial \phi_1 (x) \phi_2 (x) \sigma (x) dx = \int d\xi_1 d\xi_2 d\xi \delta (\xi_1 + \xi_2 + \xi = 0) (\mathcal{F} \partial \phi_1) (\xi_1) \mathcal{F} \phi_2 (\xi_2) \mathcal{F} \sigma (\xi) \) such that the momentum of the \( \sigma \)-field, \( \xi \), is much lower than that of the two \( \phi \)-fields. Note also that, had we not operated the Fourier normal ordering, the above integral would have been infrared divergent near \( \xi_2 = 0 \).

It is apparent that the central rôle in this decomposition is played by the Fourier projection operator \( D (1_{|\xi_1| < |\xi_2|}) = \mathcal{F}^{-1} (1_{|\xi_1| < |\xi_2|} \cdot \mathcal{F} (\cdot)) \). Since \( \mathcal{A}^\pm \) are obtained by Fourier projecting \((B_1 (t) - B_1 (s)) \phi_2 (s)\), or \((B_2 (t) - B_2 (s)) \phi_1 (t)\), which are perfectly well-defined products of continuous fields \( \phi \), it was clear from the onset that these would be regular terms. Hence singularities come only from the one-time quantity \( \mathcal{A}^\pm (t) \), which does not split into a product of first-order integrals.

In order to go one step further towards our final formulation, let us consider instead of the brute-force projection \( D (1_{|\xi_1| < |\xi_2|}) \) the following projection operators, which are more illuminating, and also analytically more robust, allowing for the extension of our theory to more general multiscale Gaussian fields as defined in section 2.

**Definition 1.3 (Fourier projections)** Let \( \psi_i = \psi_i (x), i = 1, 2 \) be Gaussian fields, and \( \psi_i = \sum_j^{\infty} \psi_i^j \) their scale decomposition along the Fourier partition of unity \( (\chi^j) \) as in Definition 2.10. Then one lets

\[
D^+ (\psi_1 \otimes \psi_2) := \sum_{0 \leq j < k < \infty} \psi_1^j \otimes \psi_2^k + \frac{1}{2} \sum_{j=0}^{+\infty} \psi_1^j \otimes \psi_2^j \tag{1.10}
\]

and similarly

\[
D^- (\psi_1 \otimes \psi_2) := \sum_{0 \leq j < k < \infty} \psi_1^k \otimes \psi_2^j + \frac{1}{2} \sum_{j=0}^{+\infty} \psi_1^j \otimes \psi_2^j \tag{1.11}
\]

Clearly \( (D^+ + D^-) (\psi_1 \otimes \psi_2) = \psi_1 \otimes \psi_2 \).
Thus one abandons – to the great benefit of notations – the above complicated expression for $A^\pm(t)$ in favour of the analytically equivalent

$$A^+(t) := D^+\left(\int^t \partial\phi_1(t_1)\phi_2(t_1)dt_1\right) = \int^t \left[ \sum_{0 \leq j < k < \infty} \partial\phi_1^j \phi_2^k + \frac{1}{2} \sum_{j=0}^{+\infty} \partial\phi_1^j \phi_2^j \right] (t_1)dt_1, \quad (1.12)$$

$$A^-(t) := D^-\left(\int^t \partial\phi_2(t_1)\phi_1(t_1)dt_1\right) = \int^t \left[ \sum_{0 \leq j < k < \infty} \partial\phi_2^j \phi_1^k + \frac{1}{2} \sum_{j=0}^{+\infty} \partial\phi_2^j \phi_1^j \right] (t_1)dt_1. \quad (1.13)$$

### 1.2 Definition of the interaction

We recall that $\int_s^t dB_1(t_1) \int_s^t dB_2(t_2)$ represents the area between the straight line connecting $(B_1(s), B_2(s))$ to $(B_1(t), B_2(t))$ and the graph. If the curve turns right, resp. left, then the Lévy area increases, resp. decreases. We have seen that $A^\pm$ represents in some sense the singular part of the Lévy area. So $A^+$, resp. $A^-$ plays the rôle of a right-, resp. left-turning (singular) field.

It is conceivable that $B_1, B_2$ or $\phi_1, \phi_2$ represent the idealized, strongly self-correlated motion in $\mathbb{R}^2$ of a particle, which – although rotation-invariant – may not (probably as a consequence of a mechanical or electromagnetic rigidity due to the macroscopic dimension of the particle, or any other similar phenomenon) turn absolutely freely. A natural quantum field theoretic description of this rigidity phenomenon is to add an interaction Lagrangian of the form $L_{\text{int}} = (\partial A^\pm)^2$. The fundamental intuition here is that the field $B$ is in some sense a mesoscopic field, while $A^\pm$ depends on microscopic details of the theory. This is explained in great accuracy in [21], in a mathematical language. A. Lejay explains how the paths of $B$ may be modified by inserting microscopic bubbles along the paths of $B$, resulting in the limit in paths which are indistinguishable from the original ones, while the Lévy area has been corrected by an arbitrary amount. Hence one must search for an interaction which cures the ultra-violet divergences of the microscopic scale, without modifying the theory at mesoscopic scale. Understanding the mesoscopic scale as low-frequency (which is not really appropriate, since there are two reference scales here, instead of one), a natural candidate would be an interaction theory which is asymptotically free at large (mesoscopic scale) distances. The best-known example of that is probably the infra-red $\phi^4$-theory in 4 dimensions [9]; in that case, the coupling constant increases
indefinitely at short distances, and forces an ultra-violet cut-off. In our case, the coupling constant \( \lambda \) will not flow, and the theory will be well-defined at all scales, which suggests a just renormalizable theory (or, in other terms, an integrated interaction which is homogeneous of degree 0). Since \((\partial A^\pm)^2\) is homogeneous of degree \((4\alpha - 2)\) in time, one shall use in fact a non-local interaction lagrangian, \(\frac{1}{2} c^2 \int |t_1 - t_2|^{-4\alpha} \mathcal{L}_{\text{int}}(\phi_1, \phi_2)(t_1, t_2)\), where

\[
\mathcal{L}_{\text{int}}(\phi_1, \phi_2)(t_1, t_2) = \lambda^2 \left\{ \partial A^+(t_1) \partial A^+(t_2) + \partial A^-(t_1) \partial A^-(t_2) \right\},
\]

(1.14)

which is positive for \(\alpha < 1/4\) since the kernel \(|t_1 - t_2|^{-4\alpha}\) is locally integrable and positive definite. Thus the Gaussian measure is penalized by the singular exponential weight \(e^{-\frac{c^2}{2} \int \mathcal{L}_{\text{int}}(\phi_1, \phi_2)(t_1, t_2) |t_1 - t_2|^{-4\alpha} dt_1 dt_2}\). Equivalently, using the so-called Hubbard-Stratonovich transformation we introduce two independent exchange particle fields \(\sigma_\pm = \sigma_\pm(t)\) with covariance kernel \(\mathbb{E}[\sigma_\pm(s)\sigma_\pm(t)] = c\alpha|s - t|^{-4\alpha}\) and rewrite (letting \(d\mu(\sigma)\), resp. \(d\mu(\sigma)\) be the Gaussian measure associated to \(\phi\), resp. \(\sigma = (\sigma_+, \sigma_-)\)) the partition function \(Z := Z(\lambda)\),

\[
Z := \int e^{-\frac{c^2}{2} \int \mathcal{L}_{\text{int}}(\phi_1, \phi_2)(t_1, t_2) dt_1 dt_2} d\mu(\phi)
\]

(1.15)

as

\[
Z := \int e^{-\int \mathcal{L}_{\text{int}}(\phi_1, \phi_2, \sigma)(t) dt} d\mu(\phi) d\mu(\sigma),
\]

(1.16)

where

\[
\mathcal{L}_{\text{int}}(\phi_1, \phi_2, \sigma)(t) = i\lambda \left( \partial A^+(t)\sigma_+(t) - \partial A^-(t)\sigma_-(t) \right).
\]

(1.17)

All of this is ill-defined mathematically since (1) \(\sigma\) is a distribution-valued process and \(\partial A^\pm\) is not defined at all when \(\alpha \leq 1/4\); (2) one integrates over \(\mathbb{R}\) a translation-invariant quantity (note that \(\phi_1, \phi_2, \sigma\) are all stationary fields).

### 1.3 Towards a heuristic expression for the Lévy area

Assume the coupling parameter \(\lambda\) is small enough. Perturbative quantum field theory suggests then to expand formally the exponential of the Lagrangian and compute polynomial moments,

\[
\frac{1}{n!} \mathbb{E} \left[ \psi_1(x_1) \ldots \psi_n(x_n) e^{-\int \mathcal{L}_{\text{int}}(\phi_1, \phi_2, \sigma)(t) dt} \right],
\]

also called \(n\)-point functions and denoted by \(\langle \psi_1(x_1) \ldots \psi_n(x_n) \rangle_\lambda\), \(\psi_i = \ldots \psi_n = \mathcal{L}_{\text{int}}(\phi_1, \phi_2, \sigma)(t) dt\)

\[\mathbb{E}[e^{i\lambda X}] = e^{-\sigma^2\lambda^2/2}\] for a random variable \(X \sim \mathcal{N}(0, \sigma^2)\)
\[ \phi_1, \phi_2, \sigma_+ \text{ or } \sigma_-, \text{ as } \frac{1}{Z} \sum_{n \geq 0} \frac{(-1)^n}{n!} E \left[ \psi_1(x_1) \ldots \psi_n(x_n) \left( \int L_{int}(\cdot ; t) dt \right)^n \right]. \]

Using Wick's formula (recalled in §6.1), one may represent \( \langle \psi_1(x_1) \ldots \psi_n(x_n) \rangle_\lambda \) as a sum over Feynman diagrams, \( \sum \Gamma A(\Gamma) \), where \( \Gamma \) ranges over a set of diagrams with \( n \) external legs, and \( A(\Gamma) \in \mathbb{C} \) is the evaluation of the corresponding diagram (see section 4). The Gaussian integration by parts formula yields a so-called Schwinger-Dyson identity,

\[
\langle \partial A^\pm(x) \partial A^\pm(y) \rangle_\lambda = -\frac{1}{\lambda^2 Z(\lambda)} E \left[ \frac{\delta}{\delta \sigma^+(y)} \frac{\delta}{\delta \sigma^+(x)} e^{-\int L_{int}(\phi_1, \phi_2, \sigma_+)(t) dt} \right]
\]

\[
= -\frac{1}{\lambda^2 Z(\lambda)} E \left[ (C^{-1}_{\sigma^+})(y) \frac{\delta}{\delta \sigma^+(x)} e^{-\int L_{int}(\phi_1, \phi_2, \sigma_+)(t) dt} \right]
\]

\[
= -\frac{1}{\lambda^2} \left[ -C^{-1}_{\sigma^+}(x, y) + \langle (C^{-1}_{\sigma^+})(x) (C^{-1}_{\sigma^+})(y) \rangle_\lambda \right],
\]

(1.18)

with Fourier transform

\[
\langle |F(\partial A^\pm)(\xi)|^2 \rangle_\lambda = \frac{1}{\lambda^2} |\xi|^{-4\alpha} \left[ 1 - |\xi|^{-4\alpha} \langle |(F\sigma^+)(\xi)|^2 \rangle_\lambda \right].
\]

(1.19)

\[ \xi \xi_1 \xi \]

Figure 1: Bubble diagram. Bold lines are \( \phi \)-fields, plain lines are \( \sigma \)-fields.

\[ \xi \xi_1 \xi \]

\[ \xi \xi_1 \xi \]

\[ \xi \xi_1 \xi \] + \ldots

Figure 2: First two terms of the bubble series.

Consider (using a brute-force ultraviolet cut-off at momentum \( \Lambda \)) the term of lowest degree in \( \lambda \) in the term between square brackets: it is essentially equal, up to a sign, to the evaluation of the half-amputated bubble
\(-\xi^{1-4\alpha} \cdot (-i\lambda)^2 \int_{|\xi_1|<|\xi-\xi_1|} d\xi_1 \left\{ (\mathbb{E}|\mathcal{F}\sigma_\pm(\xi)|^2)^2 \mathbb{E}|\mathcal{F}(\partial\phi_1)(\xi_1)|^2 \mathbb{E}|\mathcal{F}\phi_2(\xi-\xi_1)|^2 \right\} \right.

\begin{align*}
\lambda^2 |\xi|^{4\alpha-1} \int_{|\xi_1|<|\xi-\xi_1|} d\xi_1 |\xi_1|^{-2\alpha} |\xi-\xi_1|^{-1-2\alpha} \sim_{\Lambda \to \infty} K \lambda^2 (\Lambda/|\xi|)^{1-4\alpha}, \quad (1.20)
\end{align*}

hence a diverging positive quantity. (Using the Fourier truncation of Definition 1.3 only changes the constant \(K\).) However, resumming formally the bubble series as in Fig. 2 yields – taking into account the possible insertion of \(\sigma_-\)-propagators between \(\sigma_+\)-propagators –

\begin{align*}
\frac{1}{\lambda^2} |\xi|^{1-4\alpha} \left[ 1 - \frac{1}{1 + K' \lambda^2 (\Lambda/|\xi|)^{1-4\alpha}} \right] = \frac{1}{\lambda^2} |\xi|^{1-4\alpha} \cdot \frac{K' \lambda^2 (\Lambda/|\xi|)^{1-4\alpha}}{1 + K' \lambda^2 (\Lambda/|\xi|)^{1-4\alpha}} \to_{\Lambda \to \infty} \frac{1}{\lambda^2} |\xi|^{1-4\alpha}.
\end{align*}

Thus the bare \(\sigma\)-propagator \(\frac{1}{|\xi|^{1-4\alpha}}\) has been replaced with the renormalized propagator \(\frac{1}{|\xi|^{1-4\alpha} + K\lambda^2 (\Lambda/|\xi|)^{1-4\alpha}}\), which vanishes in the limit \(\Lambda \to \infty\). In physical terms, the interaction in \(\frac{1}{|\xi|^{1-4\alpha}}\) has been totally screened by an infinite mass term \(K' \lambda^2 \Lambda^{1-4\alpha}\). More complicated diagrams contributing to \(\langle |\mathcal{F}\sigma_\pm(\xi)|^2 \rangle_\lambda\), and involving internal \(\sigma\)-links, also vanish when \(\Lambda \to \infty\). There remains simply:

\begin{align*}
\langle |\mathcal{F}\mathcal{A}^\pm(\xi)|^2 \rangle_\lambda = \frac{1}{\lambda^2} |\xi|^{-1-4\alpha}.
\end{align*}

As for the mixed term \(\langle \partial \mathcal{A}^\pm(x) \partial \mathcal{A}^\mp(y) \rangle_\lambda\), its Fourier transform is given by

\begin{align*}
\frac{1}{\lambda^2} |\xi|^{1-4\alpha} \left[ \frac{1}{1 + K'' \lambda^2 (\Lambda/|\xi|)^{1-4\alpha}} \right],
\end{align*}

where \(K'' < K'\) due to the constraints on the scales for bubbles of mixed type with one \(\sigma_+\)- and one \(\sigma_-\)-leg \({}^{11}\), which vanishes in the limit \(\Lambda \to \infty\) (note the disappearance of the factor 1 compared to eq. (1.21)), due to the fact that \(\mathbb{E}\sigma^+(x)\sigma^-(y) = 0\). Thus the covariance of the two-component \(\sigma\)-field has been renormalized to \(\frac{1}{|\xi|^{1-4\alpha} \mathbb{I}_d + b'},\)

where \(b'\) is a two-by-two positive field with eigenvalues \(\lambda^2 M_{\rho(1-4\alpha)}\).

\({}^{11}\)Namely, coupling \((\phi_1, \phi_2)\) simultaneously to \(\sigma_+\) and \(\sigma_-\) leaves only the scale-diagonal projected field \(D^+ D^- (\phi_1 \otimes \phi_2) = \frac{1}{4} \sum_j \phi'_1 \otimes \phi'_2\).
Using eq. (1.8), one obtains:

\[
(2\pi c_\alpha)^2 \langle A(s,t)^2 \rangle_\lambda = \langle |A^+(t) - A^+(s)|^2 \rangle_\lambda + \langle |A^-(t) - A^-(s)|^2 \rangle_\lambda + E |A^+(s,t) - A^-(s,t)|^2
\]

\[
= \frac{4}{\lambda^2} \int (1 - \cos(t-s)\xi)|\xi|^{-1-4\alpha} d\xi + E |A^+(s,t) - A^-(s,t)|^2
\]

\[
= \left( \frac{4}{\lambda^2} K_1 + K_2 \right) |t-s|^{4\alpha}
\]

for some constants \( K_1, K_2 \).

Figure 3: Higher connected moments of the Lévy area.

A simple power-counting argument (see subsection 4.1) shows that the overall degree of divergence of a connected diagram with \( 2n \) external \( \sigma \)-legs is \( 1 - 4n\alpha \). For \( n \geq 2 \), this is \( \leq 1 - 8\alpha < 0 \) since \( \alpha > \frac{1}{8} \) by hypothesis, so such diagrams are convergent. By the above arguments, there remain only the connected diagrams in the limit \( \Lambda \to \infty \), see Fig. 3 whose evaluation is independent of \( \lambda \). Then these may be shown by standard multi-scale arguments for single Feynman diagrams (see e.g. [39]) to be of order \( K^n \), which implies that the generating series for connected moments has a non-zero radius of convergence.

On the other hand, the law of the field \( \phi \) is left unchanged by the interaction. Namely, all non-trivial diagrams contributing e.g. to \( \langle \phi_1(x)\phi_2(x) \rangle_\lambda \) involve internal \( \sigma \)-links which (as previously "shown") vanish in the limit \( \Lambda \to \infty \).

On the whole, this is the content of Theorem 0.1.

The art of constructive field theory is to make the previous speculations rigorous.
2 Multiscale Gaussian fields

2.1 Scale decompositions

We fix some constant $M > 1$. The next definition is borrowed from [31].

Definition 2.1 (Fourier partition of unity) Let $\chi^0 : \mathbb{R} \to [0, 1]$ be an even, $C^\infty$ function such that $\chi^0_{[-1,1]} \equiv 1$ and $\text{supp} \chi^0 \subset [-M, M]$, and

$$\chi^j(\xi) := \chi^0(M^{-j}\xi) - \chi^0(M^{1-j}\xi), \quad j \geq 1$$

so that $\text{supp} \chi^j \subset [M^{-j}, M^{j+1}] \cup [-M^{j+1}, -M^{j-1}]$.

The $(\chi_j)_{j \geq 0}$ define a $C^\infty$ partition of unity, namely, $\sum_{j=0}^{\infty} \chi_j \equiv 1$.

Note that $\text{supp} \chi_j \cap \text{supp} \chi_{j'}$ has empty interior if $|j - j'| = 2$, and is empty if $|j - j'| \geq 3$.

Definition 2.2 ($M$-adic intervals) Let $\mathbb{D}^j := \{[kM^{-j}, (k + 1)M^{-j}), k \in \mathbb{Z}\}, j \geq 0$ be the set of $M$-adic intervals of scale $j$, and $\mathbb{D} := \cup_{j \geq 0} \mathbb{D}^j$ the disjoint sum of these sets over all scales. The set $\mathbb{D}$ is a tree with links (called: inclusion links) connecting each interval $\Delta \in \mathbb{D}^j$ to the unique interval $\Delta' \in \mathbb{D}^{j-1}$ such that $\Delta \subset \Delta'$ (see Definition 3.3 below, and left part of Fig. 4 in subsection 3.3). An element of $\mathbb{D}^j$ is usually denoted by $\Delta^j$, or simply $\Delta$ if no confusion may arise. The volume $|\Delta^j|$ is simply $M^{-j}$. If $\Delta \in \mathbb{D}^j$, then one denotes by $j(\Delta) = j$ the scale of $\Delta$.

If $x \in \mathbb{R}$, then $x$ belongs to a single $M$-adic interval of scale $j$, denoted by $\Delta^j_x$.

If $\Delta^j \in \mathbb{D}^j$, then the set of intervals $\Delta \in \cup_{h < j} \mathbb{D}^h$ such that $\Delta$ lies below $\Delta^j$, i.e. $\Delta \supset \Delta^j$, is denoted by $\{\Delta^j\}^\parallel$.

We denote by $d^j(\Delta, \Delta')$, $\Delta, \Delta' \in \mathbb{D}^j$, the distance in terms of number of $M$-adic intervals of scale $j$ between $\Delta$ and $\Delta'$, namely,

$$d^j([kM^{-j}, (k + 1)M^{-j}), [k'M^{-j}, (k' + 1)M^{-j})]) = |k' - k|.$$  \hfill (2.2)

By extension, one may also define the $d^j$-distance of two points or two $M$-adic intervals of scale $j' > j$, namely, $d^j(x, y) = M^j|x - y|$ and

$$d^j(\Delta, \Delta') := M^{j-j'}d^{j'}(\Delta, \Delta'), \quad \Delta, \Delta' \in \mathbb{D}^{j'}.$$

Remark. It is preferable not to define $d^j(\Delta, \Delta')$ for $\Delta, \Delta' \in \mathbb{D}^{j'}$ with $j' < j$, since $d^j(x, x')$, $x \in \Delta, x' \in \Delta'$ depends strongly on the choice of the points $x, x'$ then.

The definition extends in a natural way to a $D$-dimensional setting by decomposing $\mathbb{R}^D$ into a disjoint union of hypercubes of size side $M^{-j}$.
Definition 2.3 (multiscale decomposition) (i) Let $\psi = \psi(x)$ be some integrable function or distribution in $S'$. Then the multiscale decomposition of $\psi$ is $\psi = \sum_{j \geq 0} \psi^j$, where

$$\psi^j(x) := \mathcal{F}^{-1} \left( \xi \mapsto \chi^j(\xi)(\mathcal{F}\psi)(\xi) \right)(x) = \frac{1}{\sqrt{2\pi}} \int e^{ix\xi} \chi^j(\xi)(\mathcal{F}\psi)(\xi) \, d\xi.$$  

(2.4)

(ii) (low-momentum field) Let $\psi^\rightarrow := \sum_{j=0}^{k} \psi^j$.

(iii) (high-momentum field) Let $\psi^\leftarrow := \sum_{j=k}^{\infty} \psi^j$.

Now comes a general remark. Let $\hat{f} = \mathcal{F} f(\xi)$ be some function with support in $|\xi| \leq M^j$ such that $|\mathcal{F}(f')(\xi)| = |\hat{f}'(\xi)|$ is bounded. Then

$$|\mathcal{F}^{-1}\hat{f}(x) - \mathcal{F}^{-1}\hat{f}(y)| \leq |x-y| \int_{0}^{M^j} |\xi||\hat{f}(\xi)| \, d\xi \leq K \cdot M^j |x-y|, \quad (2.5)$$

so $\psi^j(x)$ or $\psi^\rightarrow_j(x)$ varies slowly inside intervals of scale $k$ if $k > j$. Hence it makes sense in first approximation to consider $\psi^j(x)$ or $\psi^\rightarrow_j(x)$ to be approximately equal to the averaged, locally constant function

$$\psi_{av}^j(x) := \sum_{\Delta^j \in D^j} 1_{x \in \Delta^j} \frac{1}{|\Delta^j|} \int_{\Delta^j} \psi^j(y) dy = \frac{1}{|\Delta^j_x|} \int_{\Delta^j_x} \psi^j(y) dy, \quad (2.6)$$

or similarly for the low-momentum field $\psi^\rightarrow_j$

$$\psi_{av}^\rightarrow_j(x) := \sum_{\Delta^j \in D^j} 1_{x \in \Delta^j} \frac{1}{|\Delta^j|} \int_{\Delta^j} \psi^\rightarrow_j(y) dy = \frac{1}{|\Delta^j_x|} \int_{\Delta^j_x} \psi^\rightarrow_j(y) dy. \quad (2.7)$$

Summing the $\psi_{av}^j$ over $j$ would give a new function $\sum_{j \geq 0} \psi_{av}^j(x)$ which is a sort of “naive” wavelet expansion of the original function $\psi$; with a little extra care, one could arrange that the two functions be equal, but we shall not need to do so.

Conversely, if $\psi$ is a ‘reasonable’ random Gaussian field, then the covariance $\langle \psi^j(x)\psi^j(y) \rangle$ or, more generally, $\langle \psi^\rightarrow_j(x)\psi^\rightarrow_j(y) \rangle$ is usually small if the corresponding $M$-adic intervals are far apart, i.e. if $d^j(\Delta^j_x, \Delta^j_y) \gg 1$.

These two remarks may be made precise in the case when $\psi$ is a multi-scale Gaussian field, see Definition 2.4 below. Then, for any scale $j,$
the field \( \psi \rightarrow j \) may be decomposed into the sum of the \textit{locally averaged field} at scale \( j \), namely, \( \psi_{av}^j(x) \), and a secondary field, denoted by \( \delta^j \psi \rightarrow j \), whose low momentum components of scale \( h < j \) decrease like \( M^{-\gamma(j-h)} \) for some \( \gamma > 0 \), see Lemma 2.6 and Corollary 2.7. For reasons explained below, it is customary to use the nickname of \textit{spring factor} for a decrease factor of the type \( M^{-\gamma(j-h)} \), \( \gamma > 0 \).

since the covariance decreases with the distance in terms of number of \( M \)-adic intervals of scale \( j \), it makes sense to try and find some expansion of the functional \( L(\psi) \) in which the field values over far enough \( M \)-adic intervals have been made independent. This is called a \textit{cluster expansion} (see section 2).

**Definition 2.4 (multiscale Gaussian field)** A multiscale Gaussian field \( \psi \) with scaling dimension \( \beta < 1 \) is a field \( \psi = \psi(x) \) such that, for every \( r \geq 0 \) and \( \tau, \tau' = 0, 1, 2, \ldots \),

\[
|\langle \partial^\tau \psi^j(x) \partial^{\tau'} \psi^j(y) \rangle| \leq K_{\tau, \tau', r} M^{(\tau + \tau' - 2\beta)j} (1 + M^j|x - y|^r) \tag{2.8}
\]

with some constant \( K_{\tau, \tau', r} \) depending only on \( \tau, \tau' \) and \( r \).

**Remark.** A multiscale Gaussian field \( \psi \) with scaling dimension \( \beta \in (0, 1) \) has almost surely \( \beta^- \)-Hölder paths. Namely, \( \mathbb{E}|\psi^j(x) - \psi^j(y)|^2 = \int_x^y \int_x^y dz' dz'' \langle \partial \psi^j(z) \partial \psi^j(z') \rangle \) is bounded by \( K(x - y)^2 M^{(2-2\beta)j} \) if \(|x - y| < M^{-j}\), and by

\[
2 \int_{|x - y|}^{x - y} dz' \langle \psi^j(0) \partial \psi^j(z') \rangle \leq K \int_0^{x - y} dz' M^{(1-2\beta)j} (1 + M^j z')^2 \leq K' M^{-2\beta j} \tag{2.9}
\]

otherwise. Summing over \( j \) yields

\[
\mathbb{E}|\psi(x) - \psi(y)|^2 \leq K \left( (x - y)^2 \sum_{j \leq -\log|x - y|} M^{(2-2\beta)j} + \sum_{j \geq -\log|x - y|} M^{-2\beta j} \right) \\
\leq K' |x - y|^{2\beta}, \tag{2.10}
\]

and one concludes by using the Kolmogorov-Centsov lemma.

As we shall see in the next paragraph, fractional Brownian motion with Hurst index \( \alpha \) is the paramount example of multiscale Gaussian field with scaling dimension \( \beta = \alpha \in (0, 1) \).
Definition 2.5 (averaged and secondary fields) Choose some scale \( k \geq 0 \). Let \( f: \mathbb{R} \to \mathbb{R} \) be a function, typically, \( f(x) = \psi^{-k}(x) \) or \( \psi^{j}(x) \) for some \( j < k \). Then:

(i) the averaged field at scale \( k \) is the locally constant field
\[
 f_{av}(x) := \sum_{\Delta^k \in \mathbb{D}^k} 1_{x \in \Delta^k} \frac{1}{|\Delta^k|} \int_{\Delta^k} f(y)dy. \tag{2.11}
\]

It is more convenient to consider \( f_{av} \) as some function on \( \mathbb{D}^k \), still denoted by \( f \),
\[
 f(\Delta^k) := \frac{1}{|\Delta^k|} \int_{\Delta^k} f(y)dy, \quad \Delta^k \in \mathbb{D}^k, \tag{2.12}
\]
so that \( f_{av}(x) = f(\Delta^k_x) \). This notation fixes unambiguously the scale of the average.

(ii) the secondary field at scale \( k \) is the difference between the original field and the averaged field at scale \( k \), namely,
\[
 \delta^k f(x) := f(x) - f(\Delta^k_x). \tag{2.13}
\]

These definitions imply the following easy Lemma:

**Lemma 2.6**
\[
 \delta^k f(x) = \frac{1}{|\Delta^k|} \int_{\Delta^k_x} \left( \int_{u}^{x} f'(v)dv \right) du = \int_{\Delta^k_x} dv f'(v) \delta^k(x;v) \tag{2.14}
\]

where
\[
 \delta^k(x;v) := \frac{1}{|\Delta^k|} \left\{ (v - \inf \Delta^k_x)1_{v<x} + (v - \sup \Delta^k_x)1_{v>x} \right\} \in [-1,1] \tag{2.15}
\]
is a signed distance from \( v \) to the boundary of the interval \( \Delta^k_x = [\inf \Delta^k_x, \sup \Delta^k_x] \) measured in terms of the rescaled \( d^k \)-distance.

**Proof.** Straightforward. \( \Box \)

**Corollary 2.7** Let \( k, k' > j \). Assume \( \psi = \psi(x) \) is a multiscale Gaussian field with scaling dimension \( \beta < 1 \). Then
\[
 |\langle \delta^k \psi^j(x) \delta^{k'} \psi^j(y) \rangle| \leq K_r \frac{M^{-2\beta j}}{(1 + M|y-x|)^r} M^{-(k-j)}M^{-(k'-j)}(2.16)
\]
\[
 = K_r M^{-\beta(k+k')} \frac{M^{-(1-\beta)(k-j)}M^{-(1-\beta)(k'-j)}}{(1 + M|y-x|)^r}. \tag{2.17}
\]
**Proof.** Straightforward.

Eq. (2.16), resp. (2.17) emphasizes the “spring-factor” $M^{-(k-j)}$, resp. $M^{-(1-\beta)(k-j)}$ gained for each secondary field with respect to the covariance of a multiscale Gaussian field with scaling dimension $\beta$ at scale $j$, resp. at scale $k$. Had one considered directly \[\langle \psi^j(x)\psi^j(y) \rangle,\] the spring factor – called rescaling spring factor in (2.17) would have been simply $M^\beta(k-j)$, see introduction to §6.1.2.

**Remarks.**

1. The same spring factors appear in $D$ dimensions. It is sometimes useful to take a cleverer definition of the secondary field by replacing the simple average $f(\Delta^h_j)$ with a wavelet component of $f$, where the wavelets have vanishing first moments up to order $\tau \geq 1$. This allows one to enhance the spring-factor from $M^{-(k-j)}$ to $M^{-(\tau+1)(k-j)}$, resp. from $M^{-(1-\beta)(k-j)}$ to $M^{-(\tau+1-\beta)(k-j)}$.

2. The separation of low-momentum fields into a sum (field average)+(secondary field) is required only for fields with scaling dimension $\beta > -D/2$, and is not performed otherwise (see explanation after Definition 3.9 and in subsection 6.1).

Consider conversely a high-momentum field $\psi^j(x)$, $j > h$ with $x \in \mathbb{D}^h$. Typically (see section 3), $\psi^j(x) = \psi^j(x_{\Delta^h_j})$, $\psi^j(x'_{\Delta^h_j})$, or $\psi^j(x_{\Delta^h_{j'}})$, $x \in \Delta^h$ coming from the scale $h$ horizontal cluster expansion or a scale $h$-derivation in an interval $\Delta^h \in \mathbb{D}^h$, and one must integrate over $x \in \Delta^h$. Then $\psi^j(x')$ and $\psi^j(x'')$ are almost decorrelated if $x', x'' \in \Delta^h$ but $d^j(x, x') \gg 1$. Hence it makes sense to restrict $\psi^j$ over each sub-interval $\Delta_j \subset \Delta^h$ of scale $j$:

**Definition 2.8 (restriction of high-momentum fields)** Let $\Delta^h \in \mathbb{D}^h$ and $j > h$. Then the high-momentum field $\psi^j(x)$, $x \in \Delta^h$, splits into

$$\psi^j(x) = \sum_{\Delta^h \in \mathbb{D}^h, \Delta_j \subset \Delta^h} Res^h_{\Delta_j} \psi^j(x), \quad x \in \mathbb{D}^h \tag{2.18}$$

where $Res^h_{\Delta_j} \psi^j(x) := 1_{x \in \Delta_j} \psi^j(x)$.

**2.2 Multiscale Gaussian fields in one dimension**

We introduce here more specifically the infra-red divergent stationary field $\phi$ associated to fBm $B$, and the fields $\sigma = \sigma_\pm$ conjugate to the turning fields $\mathcal{A}_\pm$. In this paragraph $D = 1$. All fields come implicitly with an ultra-violet
cut-off at scale $\rho$, so that $\psi = \phi, \sigma$ should be understood as $\psi \to \phi$, $\psi^j \to$ as $\psi^j \to \rho$, and so on.

**Definition 2.9 (Harmonizable representation of fBm)** Let $W(\xi), \xi \in \mathbb{R}$ be a complex Brownian motion such that $W(-\xi) = W(\xi)$, and

$$B_t := (2\pi c_\alpha)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} \frac{e^{i\xi} - 1}{i\xi} |\xi|^{\frac{1}{2} - \alpha} dW(\xi), \quad t \in \mathbb{R}.$$  \hspace{1cm} (2.19)

The field $B_t, t \in \mathbb{R}$ is called *fractional Brownian motion* \(^{12}\). Its paths are almost surely $\alpha$-Hölder, i.e. $(\alpha - \varepsilon)$-Hölder for every $\varepsilon > 0$. It has dependent but identically distributed (or in other words, stationary) increments $B_t - B_s$.

In order to gain translation invariance, we shall rather use the closely related stationary process

$$\phi(t) := \int_{-\infty}^{+\infty} e^{i\xi} \frac{1}{|\xi|^{\frac{1}{2} - \alpha}} dW(\xi), \quad t \in \mathbb{R}$$  \hspace{1cm} (2.20)

– with covariance

$$\langle \phi(x)\phi(y) \rangle = \int e^{i\xi(x-y)} \frac{1}{|\xi|^{1+2\alpha}} d\xi$$  \hspace{1cm} (2.21)

– which is infrared divergent. However, as already discussed in section 1, the increments $\phi(t) - \phi(s)$ are well-defined for any pair of variables $(s,t)$.

**Definition 2.10**

1. Let

$$C^j_{\phi}(x,y) := \int e^{i\xi(x-y)} \frac{\chi^j(\xi)}{|\xi|^{1+2\alpha}} d\xi, \quad j \in \mathbb{N}.$$  \hspace{1cm} (2.22)

Then $C_\phi := \sum_{j \in \mathbb{N}} C^j_{\phi}$ is the covariance of the field $\phi$. We denote by $\phi := \sum_{j \in \mathbb{N}} \phi^j$ the corresponding multiscale decomposition of the field $\phi$ into independent components $\phi^j, j \in \mathbb{N}$.

2. Let $\phi \to j = \sum_{h=0}^{j} \phi^h$ and $\phi^j \to = \phi^j \to \rho = \sum_{k=j}^{\rho} \phi^k$. The covariance of $\phi$, resp. $\phi \to j$, resp. $\phi^j \to$ is $D(\chi^j)C_\phi$, resp. $D(\chi^\to j)C_\phi$, resp. $D(\chi^j)C_\phi$, where $\chi^\to j := \sum_{h=0}^{j} \chi^h, \chi^j \to = \sum_{k=j}^{\rho} \chi^k$.

**Remark.** Note that this multiscale decomposition does not coincide exactly with that of Definition \(^{23}\) With this slightly modified definition, the $\phi^j$ have been made independent.

\(^{12}\)The constant $c_\alpha$ is conventionally chosen so that $\mathbb{E}(B_t - B_s)^2 = |t - s|^{2\alpha}$.

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Lemma 2.11  The stationary field \( \phi \) associated to fBm is a multiscale Gaussian field with scaling dimension \( \alpha \).

Proof. A simple scaling of the variable of integration yields

\[
C^j_\phi(x, y) = M^{-2\alpha(j-1)} \int e^{iM^{-1}(x-y)\xi'} \frac{1}{|\xi'|^{1+2\alpha}} \chi^1(\xi') d\xi' \tag{2.23}
\]

with \( \text{supp} \chi^1 \subset [1, M^2] \cup [-M^2, -1] \) bounded away from 0, hence \( |C^j_\phi(x, y)| \lesssim M^{-2\alpha j} \), and more precisely (by integrating by parts \( n \) times) \( |C^j_\phi(x, y)| = O(M^{-2\alpha j}(M^j|x-y|)^{-n}) \) when \( M^j|x-y| \to \infty \). The bound for \( \langle \partial^\tau \phi^j(x) \partial^{\tau'} \phi^j(y) \rangle \) is obtained in the same way (simply multiply by \( |\xi'|^{\tau+\tau'} \) in Fourier coordinates).

We may now define the field \( \sigma \). As indicated in the Introduction and in section 1, the Fourier transform of its bare covariance \( \langle \sigma_i \sigma_{i'} \rangle \), \( i, i' = \pm \) is \( \frac{\delta_{i, i'}}{|\xi|^{1-4\alpha}} \). On the other hand, the renormalized covariance (up to the overlap between the supports of the Fourier multipliers \( \chi^j \)) is essentially \( \frac{1}{|\xi|^{1-4\alpha} \text{Id} + \sum_{j=0}^\infty b^j \chi^{(j-1)}(\xi)} \), where the scale \( j \) mass counterterm \( b^j \) for diagrams of scale \( j \) with two low-momentum external legs \( \sigma^{-}(j-1) \) – a two-by-two, positive matrix – is defined inductively (see subsection 3.5, section 4 and subsection 6.3) and shown to be of order \( \lambda^2 M^j(1-4\alpha) \). For technical reasons one chooses to retain in the covariance of \( \sigma \) only a simplified version of the counterterm (essentially the term of highest scale \( \rho \)), which is of the same order as the sum of all mass counterterms.

Definition 2.12  (i) Let \( \sigma \) be the stationary two-component massive Gaussian field with covariance kernel

\[
C^\sigma^{-}(x - y) := \int_{-\infty}^{+\infty} \frac{e^{i\xi(x-y)}}{|\xi|^{1-4\alpha} \text{Id} + b^0} \chi^0(\xi) \ d\xi. \tag{2.24}
\]

(ii) Decompose \( C^\sigma^{-} \) into \( \sum_{j \geq 0} C^j_\sigma \) and \( \sigma \) into a sum of independent fields, \( \sum_{j \geq 0} \sigma^j \) as in Definition 2.10 by setting

\[
C^j_\sigma(x, y) = \int e^{i\xi(x-y)} \frac{\chi^j(\xi)}{|\xi|^{1-4\alpha} \text{Id} + b^0} \ d\xi, \tag{2.25}
\]
Lemma 2.13 \( \sigma \) is a multiscale Gaussian field with scaling dimension \(-2\alpha\). More precisely,

\[
\left| \langle \partial^\tau \sigma^j(x) \partial^\tau' \sigma^j(y) \rangle \right| \leq K_{\tau,\tau'} M^{(\tau+\tau'-4\alpha)j} \cdot \inf(1, \frac{M^j(1-4\alpha)}{b^\rho}). \tag{2.26}
\]

For \( \rho \) large enough, \( \inf(1, \frac{M^j(1-4\alpha)}{b^\rho}) \approx \lambda^{-2} M^{-(\rho-j)(1-4\alpha)} \). Thus \( \sigma \) vanishes in the limit \( \rho \to \infty \) because of the infinite mass counterterm.

**Proof.** Same as for Lemma 2.11. Note that the rescaled denominator \( \chi^j(\xi) \left| \xi \right|^{1-4\alpha + b^\rho M^{-\Delta}(1-4\alpha)} \) is bounded by \( \inf(1, \frac{M^j(1-4\alpha)}{b^\rho}) \). \( \square \)

Note the following to conclude:

– multiscale Gaussian fields with *positive scaling dimension* (as \( \phi \) for instance) are ultra-violet convergent and define continuous fields, but may be infra-red divergent, necessitating some infra-red cut-off (however, if \( \beta < 1 \), the increments \( \phi_t - \phi_s \) are well-defined);

– multiscale Gaussian fields with *negative scaling dimension* (as in the case of the field \( \sigma \), or of most fields in quantum field theory) are ultra-violet divergent and define distribution-valued fields whose covariance kernel is singular on the diagonal. On the other hand, they are infra-red convergent.

## 3 Cluster expansions: an outline

The general aim of horizontal and vertical cluster expansions has already been explained in the Introduction and in §2.1. The (non restricted) *horizontal cluster expansion* has been given by D. Brydges and T. Kennedy a beautiful combinatorial structure in terms of forests of intervals (see §3.1 and 3.2). The vertical or *momentum-decoupling expansion*, in terms of \( t \)-parameters, is somewhat looser, relying on a Taylor expansion to some order \( \tau_\Delta \) in each interval \( \Delta \). Putting together these two expansions, one obtains so-called *polymers* (see §3.3). Polymers with too few external legs must still be renormalized by resumming the local part of diverging graphs into an exponential making up an interaction counterterm (see section 4 for detailed explanations); a Mayer expansion makes it possible to get rid of their non-overlap constraints with the other polymers, and finally to divide out the polymers with no external legs (called: *vacuum polymers*) when computing \( n \)-point functions (see §3.4). All these provide series depending on the small parameter \( \lambda \) which will be shown to converge in section 6.
Summarizing, the general expansion scheme which one must have in mind is made up of a sequence of transformations for each scale, starting from highest scale $\rho$, namely

\[
(\text{Horizontal cluster expansion at scale } \rho \rightarrow \text{Vertical cluster expansion at scale } \rho \rightarrow \text{separation of local part of diverging graphs of lowest scale } \rho \rightarrow \text{Mayer expansion at scale } \rho \rightarrow \text{resummation of local parts of diverging graphs of lowest scale } \rho) \rightarrow \ldots
\]

down to lowest scale $\rho = 0$.

3.1 The general Brydges-Kennedy formula

Let us start with the following general definition.

**Definition 3.1** Let:

(i) $\mathcal{O}$ be an arbitrary set, whose elements are called objects;

(ii) $L(\mathcal{O})$ be the set of links of the total graph associated to $\mathcal{O}$, or in other words, the set of pairs of objects, so that $\ell \in L(\mathcal{O})$ is represented as a pair, $\ell \sim \{o_\ell, o'_\ell\} \subset \mathcal{O}$, $o_\ell \neq o'_\ell$;

(iii) $[0,1]^{L(\mathcal{O})} := \{z = (z_\ell)_{\ell \in L(\mathcal{O})}, 0 \leq z_\ell \leq 1\}$ be the convex set of link weakenings of $\mathcal{O}$;

(iv) $\mathcal{F}(\mathcal{O})$ be the set of forests connecting (some, not necessarily all) vertices of $\mathcal{O}$. A typical element of $\mathcal{F}(\mathcal{O})$ is denoted by $\mathcal{F}$, and its set of links by $L(\mathcal{F}) \subset L(\mathcal{O})$.

Assume finally that link weakenings have been made to act in some smooth way on $Z$ (a functional depending on some extra parameters), thus defining a $C^\infty$ link-weakened functional $Z : L(\mathcal{O}) \rightarrow \mathbb{R}, z = (z_\ell)_{\ell \in L(\mathcal{O})} \mapsto Z(z)$ on the set of pairs of objects, still denoted by $Z$ by a slight abuse of notation, such that $Z$ (the original functional) is equal to $Z(1, \ldots, 1) = Z(1_{L(\mathcal{O})})$.

**Definition 3.2** (one step of BK expansion) The Brydges-Kennedy decoupling expansion consists in the following steps:
(i) Taylor-expand $Z(1,\ldots,1)$ with respect to the parameters $(z_\ell)$ simultaneously, namely,

$$Z(1,\ldots,1) = Z((z_\ell)_{\ell \in L(O)} = 0) \quad (3.1)$$

$$+ \sum_{\ell_1 \in L(O)} \int_0^1 dw_1 \partial_{z_{\ell_1}} Z((z_\ell)_{\ell \in L(O)} = w_1).$$

(ii) Choose some link $\ell_1$ in the above sum. Draw a link of strength $w_1$ between $o_{\ell_1}$ and $o'_{\ell_1}$ and consider the new set $O_1$ made up of the simple objects $o \in O \setminus \{o_{\ell_1}, o'_{\ell_1}\}$ and of the composite object $\{(o_{\ell_1}, o'_{\ell_1})\}$, with set of links $L(O_1) = L(O) \setminus \{\ell_1\}$. Then $\partial_{z_{\ell_1}} Z(z_{\ell_1} = w_1, (z_\ell)_{\ell \in L(O_1)})$ must now be considered as a functional of $(z_\ell)_{\ell \in L(O_1)}$.

When iterating this procedure, composite objects grow up to be trees. This leads to the following result:

**Proposition 3.3 (Brydges-Kennedy or BK1 cluster formula)** $Z(1,\ldots,1)$ may be computed as an integral over weakening parameters $w_\ell \in [0,1]$, where $\ell$ does not range over the links of the total graph, but, more restrictively, over the links of a forest on $O$:

$$Z(1,\ldots,1) = \sum_{F \in F(O)} \left( \prod_{\ell \in L(F)} \int_0^1 dw_\ell \right) \left( \prod_{\ell \in L(F)} \frac{\partial}{\partial z_\ell} \right) \left( z(w) \right), \quad (3.2)$$

where $z_\ell(w), \ell \in L(O)$ is the infimum of the $w_{\ell'}$ for $\ell'$ running over the unique path from $o_\ell$ to $o'_{\ell}$ if $o_\ell$ and $o'_{\ell}$ are connected by $F$, and $z_\ell(w) = 0$ otherwise.

If it is not desirable to make a cluster expansion with respect to the links between certain objects (of type 2 in Proposition 3.4 below), then it is sufficient to consider all these objects as belonging to the same composite object. This yields:

**Proposition 3.4 (restricted 2-type cluster or BK2 formula)** Assume $O = O_1 \sqcup O_2$. Choose as initial object an object $o_1 \in O_1$ of type 1, and stop the Brydges-Kennedy expansion as soon as a link to an object of type 2 has appeared. Then choose a new object of type 1, and so on. This leads to a restricted expansion, for which only the link variables $z_\ell$, with $\ell \notin O_2 \times O_2$, have been weakened. The following closed formula holds. Let $F_{\text{res}}(O)$ be the
set of forests \( \mathbb{F} \) on \( \mathbb{O} \), each component of which is (i) either a tree of objects of type 1, called unrooted tree; (ii) or a rooted tree such that only the root is of type 2. Then

\[
Z(1, \ldots, 1) = \sum_{F \in \mathbb{F}_{\text{res}}(\mathbb{O})} \left( \prod_{\ell \in L(F)} \int_0^1 dw_\ell \right) \left( \prod_{\ell \in L(F)} \frac{\partial}{\partial S_\ell} \right) Z(S_\ell(w)) ,
\]

where \( S_\ell(w) \) is either 0 or the minimum of the \( w \)-variables running along the unique path in \( \bar{F} \) from \( o_\ell \) to \( o_\ell' \), and \( \bar{F} \) is the forest obtained from \( F \) by merging all roots of \( F \) into a single vertex.

This restricted cluster expansion will be useful for the Mayer expansion (see section 3.4).

### 3.2 Single scale cluster expansion

**Definition 3.5**

(i) A horizontal cluster forest of level \( \rho' \leq \rho \), associated to a \( d \)-dimensional vector Gaussian field \( \psi = (\psi_1(x), \ldots, \psi_d(x)) \), is a finite number of \( M \)-adic intervals in \( D^\rho' \), seen as vertices, connected by links, without loops. Any link \( \ell \) connects \( \Delta_\ell \) to \( \Delta'_\ell \) \((\Delta_\ell, \Delta'_\ell \in D^\rho')\), bears a double index, \((i_\ell, i'_\ell) \in \{1, \ldots, d\} \times \{1, \ldots, d\}\), and may be represented as two half-propagators or simply half-segments put end to end, one starting from \( \Delta_\ell \) with index \( i_\ell \), and the other starting from \( \Delta'_\ell \) with index \( i'_\ell \).

The set of all horizontal cluster forests of level \( \rho' \) is denoted by \( \mathbb{F}^\rho' \). If \( \mathbb{F}' \in \mathbb{F}^\rho' \), then \( L(\mathbb{F}') \) is its set of links.

(ii) A horizontal cluster tree is a connected horizontal cluster forest. Any horizontal cluster forest decomposes into a product of cluster trees which are its connected components.

(iii) If there exists a link between \( \Delta \) and \( \Delta' \) \((\Delta, \Delta' \in D^\rho')\) then we shall write \( \Delta \sim_{\mathbb{F}^\rho'} \Delta' \), or simply (if no ambiguity may arise) \( \Delta \sim \Delta' \).

The following result is an easy consequence of Proposition 3.3, see [1].

**Proposition 3.6 (single-scale cluster expansion)** Let

\[
Z_V(\lambda) := \int e^{-\int_V L(\psi)(x) dx} d\mu(\psi),
\]

(3.4)
where \( d\mu \) is the Gaussian measure associated to a Gaussian field with \( d \geq 1 \) components, \( \psi(x) = (\psi_1(x), \ldots, \psi_d(x)) \), with covariance matrix \( C = C(i, x; j, y) \), and \( L \) is a local functional.

Choose \( \rho' \leq \rho \).

Let \( d\mu_s(\psi) \), \( s = (s_{\Delta, \Delta'})_{(\Delta, \Delta') \in \mathbb{D}^{\rho'} \times \mathbb{D}^{\rho'} \in [0, 1]} \) such that \( s_{\Delta, \Delta'} = s_{\Delta', \Delta} \) and \( s_{\Delta, \Delta} = 1 \), be the Gaussian measure with covariance kernel (if positive definite)

\[
C_s(i, x; i', x') = s_{\Delta_x, \Delta_{x'}}C(i, x; i', x')
\]

if \( \Delta_x \ni x \), resp. \( \Delta_x' \ni x' \), are the intervals of size \( M^{-\rho'} \) containing \( x \), resp. \( y \).

Then

\[
Z_V(\lambda) := \sum_{\rho' \in \mathbb{F}^{\rho'}} \left[ \prod_{\ell \in L(\mathbb{F}^{\rho'})} \int_0^1 dw_{\ell} \int_{\Delta_\ell} dx_{\ell} \int_{\Delta'_{\ell}} dx'_{\ell} C_s(w)(i_{\ell}, x_{\ell}; i'_{\ell}, x'_{\ell}) \right] \int d\mu_s(\psi) \text{Hor}^{\rho'}(e^{-\int_V L(\psi)(x)dx})
\]

where:

\[
\text{Hor}^{\rho'} = \text{Hor}^{\rho'}(\mathbb{R}^{\rho'}; (x_{\ell})_{\ell \in L(\mathbb{F}^{\rho'})}, (x'_{\ell})_{\ell \in L(\mathbb{F}^{\rho'})}) := \prod_{\ell \in L(\mathbb{F}^{\rho'})} \left( \frac{\delta}{\delta \psi_{\ell}^{\rho'}(x_{\ell})} \frac{\delta}{\delta \psi_{\ell}^{\rho'}(x'_{\ell})} \right);
\]

\[
s(w) = (s_{\Delta, \Delta'}(w))_{\Delta, \Delta' \in \mathbb{D}^{\rho'}}, \ s_{\Delta, \Delta'}(w), \ \Delta \neq \Delta' \text{ being the infimum of the } w_{\ell}
\]

for \( \ell \) running over the unique path from \( \Delta \) to \( \Delta' \) in \( \mathbb{F}^{\rho'} \) if \( \Delta \sim_{\mathbb{F}^{\rho'}} \Delta' \), and \( s_{\Delta, \Delta'}(w) = 0 \) else.

Note that \( C_s(w) \) is positive-definite with this definition \( \prod \), as a convex sum of evidently positive-definite kernels.

### 3.3 Multi-scale cluster expansion

As explained in the Introduction, cluster expansion has two main objectives. The first one is to express the partition function as a sum over quantities depending essentially on a finite number of degrees of freedom. For a given scale \( j \), the horizontal cluster expansion perfectly meets this aim. The second one is to get rid of the invariance by translation, which necessarily produces divergent quantities.

Multi-scale cluster expansion fulfills this program by building inductively, starting from the highest scale \( \rho \) and going down to scale 0, connected
multi-scale clusters called polymers. These are finite, connected subsets of $D$, extended over several scales, and containing at least one fixed interval $\Delta^j$ at the bottom which breaks invariance by translation. Like the horizontal cluster expansion, they are obtained by a Taylor expansion with respect to some $t$-parameters $(t_\Delta)_\Delta$ depending on the interval.

Let us give a precise definition of such an object (see left part of Fig. 4).

Figure 4: Polymer. A horizontal bold line below an interval means that it is not connected to the intervals below it.

**Definition 3.7**

(i) A polymer of scale $j$ (typically denoted by $P^j$) is a tree of $M$-adic intervals of scale $j$, i.e. a connected element of $F^j$. The set of all polymers of scale $j$ is denoted by $P^j$.

(ii) A polymer down to scale $j$ (typically denoted by $P^j \rightarrow$) is a connected graph connecting $M$-adic intervals of scale $k = j, j+1, \ldots, \rho$, whose links are of two types:

- horizontal links connecting $M$-adic intervals of the same scale; the restriction of $P^j \rightarrow$ to any given scale, $P^j \rightarrow \cap D^k$, $k \geq j$, is required to be a disjoint union of polymers of scale $k$, or simply in other words, an element of $F^k$; furthermore, $P^j \rightarrow \cap D^j$ is assumed to be non-empty;

- vertical links or more explicitly inclusion links connecting an $M$-adic interval $\Delta \in D^k$, $k > j$ to the unique interval $\Delta' \in D^{k-1}$ below $\Delta$,
i.e. such that $\Delta \subset \Delta'$. These may be multiple links, with degree of multiplicity $\tau_\Delta$.

A polymer $P^{j\rightarrow}$ is also allowed to have an external structure, characterized by a subset $\Delta \subset P^{j\rightarrow} \cap \Delta'$ of intervals of scale $j$, such that each $\Delta \in \Delta'$ is connected below by an external inclusion link to $\Delta' - 1$.

(iii) The horizontal skeleton of a polynomial $P^{j\rightarrow}$ is the disjoint union of the single-scale cluster forests $P^{j\rightarrow} \cap \Delta'$, $k \geq j$ (all vertical links removed).

One denotes by $P^{j\rightarrow}_{\text{ext}}$ the set of polymers down to scale $j$, and $P^{j\rightarrow}_{\text{ext}} \subset P^{j\rightarrow}$ the subset of polymers with $N_{\text{ext}} := \sum_{\Delta \in \Delta'} \tau_\Delta$ external links.

In particular, polymers in $P^{j\rightarrow}_{0\rightarrow}$, without external links, are called vacuum polymers.

The easiest example is that of so-called full-inclusion polymers $P^{j\rightarrow} \in P^{j\rightarrow}$ containing all inclusion links. Such polymers may be obtained from arbitrary multiscale horizontal cluster forests $F^{j\rightarrow} = (F^{j\rightarrow}, F^{j+1\rightarrow}, \ldots, F^{\rho\rightarrow})$ by linking all pairs $(\Delta, \Delta')$, $\Delta, \Delta' \in P^{j\rightarrow}$ such that $\Delta' \subset \Delta$.

In general, if $\Delta \subset \Delta' \subset \Delta \subset P^{j\rightarrow}$, then there is an inclusion link from $\Delta$ to $\Delta'$ inside $P^{j\rightarrow}$ (by definition) if and only if $\tau_\Delta \neq 0$.

The integer $\tau_\Delta$ corresponds in the momentum-decoupling expansion to the number of derivatives $\partial \partial_\Delta$.

Whereas horizontal links are built in by independent horizontal cluster expansions at each scale, the construction of vertical links depends on a procedure called momentum decoupling (or sometimes vertical cluster) expansion, which we now set about to describe.

Let us first give some definitions.

Assume positive numbers $t_x^j$, $0 \leq j \leq \rho$, depending on $x$ but locally constant in each $M$-adic interval $\Delta^j \subset \Delta'$, have been defined. Let us introduce operators $T_x^{j\rightarrow}$ ($0 \leq k \leq \rho, x \in \mathbb{R}$) acting on the low-momentum components of the fields at the point $x$, i.e. on $\phi^j(x)$ or $\sigma^j(x)$, $j < k$, by

$$T_x^{j\rightarrow} \phi^j(x) = T_x^{j\rightarrow} \phi^j(x) = t_x^j t_x^j \ldots \psi^j(x), \quad k > j, \quad (3.8)$$

$$T_x^{j\rightarrow} \sigma^j(x) = T_x^{j\rightarrow} \sigma^j(x) = \sigma^j(x), \quad (3.9)$$

where $\psi = \phi, \partial \phi$ or $\sigma$.

The shorthand $(T_x^{j\rightarrow} \psi)^j$ emphasizes the idea that $T_x^{j\rightarrow}$ is not simply a multiplication by some product of $t$-variables, but an operator acting diagonally on the whole field $\psi$, or rather on $\psi^{j\rightarrow}$ (seen as a vector).
In other words, writing $t^j_x = t_{Δj}$ if $x \in Δ^j$, so that $t$ is a real-valued function on $D^{→ρ} = ψ_{0≤j≤ρ}D^j$, one has for $k ≥ j$: $(T^{→k}ψ)^j(x) = \prod_{k'=j+1}^{k} t_{Δk'} \cdot ψ^j(x)$.

We shall also use the following notation:

$$(Tψ)^{→k} := \sum_{j≤k} (T^{→k}ψ)^j.$$  \hspace{1cm} (3.10)

This weakened field, depending on the reference scale $k$, will be called the \textit{dressed low-momentum field at scale $k$}. Separating the $ψ^k$-component and the $t^k$-variables from the others yields equivalently:

$$(Tψ)^{→k}(x) = ψ^k(x) + t^k_x \sum_{j<k} (T^{→(k-1)}ψ)^j(x).$$  \hspace{1cm} (3.11)

Dressed interactions may be built quite generally out of dressed low-momentum fields. Let us give a general definition.

\textbf{Definition 3.8 (dressed interaction)} \hspace{1cm} (i) Let $L^{→ρ}(ψ)(x) := λ^q ψ_I(x) = λ^q \prod_{i \in I} ψ_i(x)$ be some arbitrary local functional built out of a product of fields. Then the momentum decoupling of $L_q$, or simply dressed interaction, is the following quantity,

$$L^{→ρ}(.; t)(x) := λ^q \left\{ \prod_{i \in I_q} (Tψ_i)^{→ρ}(x) + \sum_{ρ'≤ρ} (1 - (t^1_x)|I_q|) \prod_{i \in I_q} (Tψ_i)^{→(ρ'-1)}(x) \right\}.$$  \hspace{1cm} (3.12)

(ii) \hspace{1cm} (generalization to scale-dependent case)

Let $L^{→ρ}(ψ)(x) := (λ^ρ)^q \sum_{(j_1) \in I_q} K^{(j_1)}_q \prod_{i \in I_q} ψ_i^{j_1}(x)$ be a local functional with coupling constant $λ^ρ$ and coefficients $K^{(j_1)}_q$ depending on the scales. Then one defines

$$L^{→ρ}(.; t)(x) := (λ^ρ)^q \sum_{0≤(j_1)\in I_q≤ρ} K^{(j_1)}_q \prod_{i \in I_q} (T^{→ρ}ψ_i)^{j_1}(x) + \sum_{ρ'≤ρ} (λ^{ρ'-1})^q (1 - (t^1_x)|I_q|) \sum_{0≤(j_1)\in I_q≤ρ'-1} K^{(j_1)}_q \prod_{i \in I_q} (T^{→(ρ'-1)}ψ_i)^{j_1}(x).$$  \hspace{1cm} (3.13)

where $λ^{ρ'}$, $ρ' ≤ ρ$ are the renormalized coupling constants.
Summing up in general the contribution of the different interaction terms, \( \mathcal{L}_{int} := \sum_{q=1}^{p} K_q \mathcal{L}_q \), this defines a dressed interaction \( \mathcal{L}_{int}^{\rho}(\mathcal{L};t) \) and a dressed partition function

\[
Z_{\mathcal{V}}^{\rho}(\lambda; t) = \int e^{-\int_{\mathcal{V}} \mathcal{L}_{int}^{\rho}(\mathcal{L}; t)(x) dx} d\mu(\psi).
\]  

(3.14)

For the moment, vertical links have not been constructed, and nothing prevents a priori the different horizontal clusters to move freely in space one with respect to the other, nor on the contrary to generate via Wick pairings infinite multi-scale clusters. As we shall now see, the momentum-decoupling expansion provides the mechanism responsible both for the translation-invariance breaking and the separation of polymers.

Since this is an inductive procedure, let us first consider the result of horizontal cluster expansion at highest scale \( \rho \). It may be expressed as a sum of monomials split over each connected component \( T^{\rho}_{c} \), with generic term (called \( G \)-monomial or simply monomial)

\[
G_{c}^{\rho} := \left[ \prod_{\ell \in L(T_{c}^{\rho})} \left( \psi_{I_{\ell}}^{\rho}(x_{\ell}) \psi_{I'_{\ell}}^{\rho}(x'_{\ell}) \right) \right] \cdot \left[ \prod_{\ell \in L(T_{c}^{\rho})} \left( t_{x_{\ell}}^{\rho}(T_{\psi_{J_{\ell}}}^{\rightarrow (\rho-1)}(x_{\ell})) \cdot (t_{x'_{\ell}}^{\rho}(T_{\psi_{J'_{\ell}}}^{\rightarrow (\rho-1)}(x'_{\ell})) \right) \right]
\]  

(3.15)

for some (possibly empty) index subsets \( I_{\ell}, I'_{\ell}, J_{\ell}, J'_{\ell} \subset \{1, \ldots, d\} \), multiplied by a product of propagators as in eq. (3.6); the \( t^{\rho} \)-variables dressing low-momentum components have been written explicitly as in eq. (3.11).

Let us draw an oriented, downward dashed line from \( \Delta^{\rho} \in T_{c}^{\rho} \) to some \( \Delta'^{\rho} \in (\Delta^{\rho})^{ll}, \rho' < \rho \) below \( \Delta^{\rho} \) if \( G_{c}^{\rho} \) contains some low-momentum field \( \psi_{I_{\ell}}^{\rho}(x_{\ell}) \) with \( x_{\ell} \in \Delta^{\rho} \). Either \( \Delta'^{\rho} \not\in F^{\rho'} \), or \( \Delta'^{\rho} \) belongs to some connected component \( T_{c'}^{\rho'} \) of \( F^{\rho'} \). In the latter case, one has attached \( T_{c}^{\rho} \) to some cluster tree below, \( T_{c'}^{\rho'} \), by the inclusion constraint \( \Delta^{\rho} \subset \Delta'^{\rho} \), which prevents \( T_{c}^{\rho} \) from moving freely in space with respect to \( T_{c'}^{\rho'} \). It may also happen that \( G_{c}^{\rho} \) contains no low momentum field component \( \psi_{I_{\ell}}^{\rho} \), \( \rho' < \rho \), so that \( T_{c}^{\rho} \) looks isolated; unfortunately, nothing prevents some horizontal cluster expansion at a lower scale \( \rho' \) from generating some high-momentum field component
\(\psi^\rho_i(x), \ x \in \Delta^\rho \in (\Delta^\rho)^\parallel,\) which may be represented as a reversed \textit{upward} dashed line from \(\Delta^\rho\) to \(\Delta^\rho\).

In order to have an effective mechanism of separation of scales, we shall make a Taylor expansion to order 1 with respect to the \((t^\rho_\Delta)_{\Delta \in T^\rho_c}\)-variables of the product \((G\text{-monomial}) \times (t^\rho_\Delta)_{\Delta \in T^\rho_c}\)-dependent part of the dressed interaction, \(\tilde{G}^\rho_c := G^\rho_c e^{-\int_{\Delta^\rho} L_{\text{int}}(x)dx},\) namely (splitting \(\tilde{G}^\rho_c\) into a product \(\prod_{\Delta} \tilde{G}^\rho_\Delta = \prod_{\Delta} G^\rho_\Delta e^{-\int_{\Delta^\rho} L_{\text{int}}(x)dx}\) over the fields located in each interval \(\Delta\) of scale \(\rho\) in \(T^\rho_c)\)

\[
\tilde{G}^\rho_\Delta(t^\rho_\Delta = 1) = \tilde{G}^\rho_\Delta((t^\rho_\Delta)_{\Delta \in T^\rho_c} = 0) + \int_0^1 dt^\rho_\Delta \partial^\rho_\Delta \tilde{G}^\rho_\Delta(t^\rho_\Delta), \quad (3.16)
\]

thus producing a new set of monomials multiplied by the interaction.

Setting all \((t^\rho_\Delta)_{\Delta \in T^\rho_c}\) to zero has the effect, see eq. (3.12) and (3.15), of killing in the interaction – and hence in \(G^\rho_c\) which is a derivative of the interaction – \textit{all mixed terms} containing both \(\psi^\rho_i(x)\) and \(\psi^\rho_j(x')\), with \(x \in \Delta^\rho \subset \mathbb{D}^\rho, \ x' \in \Delta^\rho' \subset \mathbb{D}^\rho', \ \Delta^\rho' \supset \Delta^\rho\) as above. Hence, in all the terms \(\tilde{G}^\rho_c((t^\rho_\Delta)_{\Delta \in T^\rho_c} = 0)\), \(T^\rho_c\) has been effectively isolated from all other intervals in \(\mathbb{D}\); it constitutes a (single-scale) isolated polymer. Thus, by letting \(t^\rho_\Delta = 0\), one \textit{cuts} all dashed lines crossing from \(\Delta\) (or above \(\Delta\) in general) to \(\Delta^\parallel\), and sets up a wall between \(\Delta\) and \(\Delta^\parallel\).

On the contrary, derivating with respect to \(t^\rho_\Delta, \ \Delta \in T^\rho_c\), produces necessarily low-momentum components, and hence vertical links materialized by dashed lines as above. In this case, one links \(\Delta^\rho\) to \(\Delta^\rho^{-1}\) by a \textit{full line}, signifying that all corresponding monomials will contain some low-momentum field; this implies in turn the existence of a \textit{downward dashed line} or \textit{wire} connecting some \(\Delta^\rho \in T^\rho_c\) to some \(\Delta^\rho' \in (\Delta^\rho)^\parallel\) as above (note however that there isn’t necessarily a dashed line from \(\Delta^\rho\) to \(\Delta^\rho^{-1}\)).

Due to the necessity of \textit{renormalization} (see section 4) and to the \textit{domination problem} (see §6.2), one may need to Taylor expand to higher order, up to order \(N_{\text{ext,max}} + O(n(\Delta))\). One obtains thus a polymer with a certain number of external legs per interval. Choosing the Taylor integral rest term for some \(\Delta\) leads to a polymer with \(\geq N_{\text{ext,max}}\) external legs, which does not need to be renormalized. On the other hand, a polymer whose \textit{total} number of external legs is \(< N_{\text{ext,max}}\) requires renormalization (see section 4).

Let us emphasize at this point that high-momentum fields have \textit{two scales} attached to them: \(j\) and \(k\) for a field \(\psi^h_k(x), \ x \in \Delta^j\) \((k > j)\), produced by the horizontal/vertical cluster expansion at scale \(j\); but low-momentum fields \(\psi^h_i\) have \textit{three scales}. There are in fact two cases:
(i) If \( \beta_i < -D/2 \) then \( \psi_i \) is not separated into a sum (low-momentum field average)+(secondary field). The genesis of \( \psi_i^h \) (contrary to the high-momentum case) is actually a process which may not be understood apart from the multi-scale cluster expansion. At their production scale \( k \), low-momentum fields are of the form \((T\psi_i)^{(k-1)}(x) = \sum_{j=0}^{k-1} \prod_{k'=j+1}^{k-1} t_{\Delta'}^j(x)\). Successive \( t \)-derivations of scale \( k-1, k-2, \ldots \) "push" \((T\psi_i)^{(k-1)}(x)\) downward like a down-going elevator, in the sense that \( \partial_{t_{\Delta}^{k-1}} (T\psi_i)^{(k-1)}(x) \) is by construction of scale \( \leq k-2 \), \( \partial_{t_{\Delta}^{k-2}} (T\psi_i)^{(k-1)}(x) \) of scale \( \leq k-3 \) and so on. But of course, \( t \)-derivations may act on other fields instead. The last \( t \)-derivation acting on \((T\psi_i)^{(k-1)}(x)\) drops \((T\psi_i)^{(k-1)}(x)\) at a certain scale \( j < k \). Then \((T\psi_i)^{(j-1)}(x)\) leaves the elevator and is torn apart into its scale components \((T\rightarrow(T\psi_i)^{(h)})_{h<j}\) through free falling. Thus \( \psi_i^h \) has a production scale \( k \) and a dropping scale \( j \), while \( h \) itself may be called its free falling scale or simply its scale.

(ii) If \( \beta_i \geq -D/2 \), then, at the dropping scale, \((T\psi_i)^{(j-1)}(x)\) is separated from its average \((T\psi_i)^{(j-1)}(\Delta_{\beta,i})\) which must be dominated apart, while \((T\psi_i)^{(j-1)}(x) \rightarrow (T\psi_i)^{(j-1)}(\Delta_{\beta,i})\) splits into its scale components \((T\rightarrow(T\psi_i)^{(j-1)}\Delta_{\beta,i})_{h<j}\) through free falling as in case (i).

The extension of the above procedure to lower scales is straightforward and leads to the following result.

**Definition 3.9 (multi-scale cluster expansion)**

1. Fix a multi-scale horizontal cluster expansion \( \mathbb{F}^{j\rightarrow} = (\mathbb{F}^{j_1}, \ldots, \mathbb{F}^{j_\rho}) \) and consider a polymer down to scale \( j \), \( \mathbb{F}^{j\rightarrow} \), with horizontal skeleton \( \mathbb{F}^{j\rightarrow} \). To such a polymer is associated a sum of products \((G\text{-monomial}) \times (dressed interaction } L_{\text{int}}(\cdot; t)\), where all \( t \)-variables such that \( \Delta \in \mathbb{F}^{j\rightarrow} \) and \( \tau_\Delta < N_{\text{ext,max}} + O(n(\Delta)) \) have been set to 0, and \( G \) is one of the monomials obtained by expanding \( \prod_{k \geq j} \text{Vert}^k \cdot \text{Hor}^k \cdot e^{-\int_t \text{L}_{\text{int}}(\cdot; t)(x) dx} \), where \( \text{Hor}^k \) is as in Proposition 5.5 and \( \text{Vert}^k \) is the following operator, with \( N'_{\text{ext,max}} = N_{\text{ext,max}} + O(n(\Delta)) \),

\[
\text{Vert}^k = \prod_{\Delta \in \mathbb{F}^k} \left( \sum_{\tau_\Delta = 0}^{N'_{\text{ext,max}} - 1} \partial_{t_{\Delta}}^{i_{\Delta}}|_{t_\Delta = 0} + \int_0^1 dt_\Delta \frac{(1 - t_\Delta)(N'_{\text{ext,max}} - 1)!}{N_{\text{ext,max}}(N'_{\text{ext,max}} - 1)!} \partial_{t_{\Delta}}^{i_{\Delta}} \right).
\]

(3.17)
2. (definition of \(n(\Delta)\)) Fix \(F_j \to \) and let \(\Delta \in D^j \to \). Then \(n(\Delta)\) is the number of intervals of scale \(j(\Delta)\) connected to \(\Delta\) by the forest \(F_j(\Delta)\).

3. (definition of \(N(\Delta)\)) Fix \(F_j \to \) and some \(G\)-monomial, and let \(\Delta \in D^j \to \). Then \(N(\Delta)\) is the number of fields \(\psi_j(\Delta)(x), x \in \Delta\) of scale \(j(\Delta)\) lying in the interval \(\Delta\).

Classically, one sets apart polymers made up of one isolated interval \(\Delta_j^j\) where no vertex has been produced; this means that \(t_{\Delta_j+1} = 0\) for all intervals \(\Delta_j \subset \Delta\); \(s_{\Delta_j, \Delta'} = 0\) if \(\Delta' \in D^j \setminus \{\Delta_j\}\). Write as usual \(L_{\text{int}} := \sum_{q=1}^p K_q \lambda^q L_q\). Their contribution to the partition functions reads simply

\[
\int \mu(\psi) e^{-\int L \to (\ : t=0)}(x) dx \\
= 1 - \sum_{q=1}^p K_q \lambda^q \int \mu(\psi) \cdot \int_0^1 dv \int L_q \to (\ : t = 0)(x) dx \cdot e^{-v \int L_q \to (\ : t=0)(x) dx} \\
=: 1 - A. \tag{3.18}
\]

by using a Taylor expansion to order 1. In the above expression, all fields have same scale \(j\) since all \(t\)-coefficients connecting \(\Delta_j^j\) have been set to zero.

In all cases where the exponentiated interaction is bounded by 1 – which is the case in our \((\phi, \partial \phi, \sigma)\)-model – the Cauchy-Schwarz inequality implies that \(|A|^2 \leq K\lambda^{2\kappa}\), where \(\kappa = \min\{\kappa_q; q = 1, \ldots, p\}\) since the integrated vertex is homogeneous of degree 0. Now, one eliminates the factor 1 by releasing the constraint that the disjoint union of all polymers must span the whole volume \(V\), and ends up with a polymer whose evaluation is of order \(O(\lambda^\kappa)\). This makes it possible to treat such polymers on an equal footing with the other polymers where some vertex has been produced (see introduction to section 6.1 for the general principles of the bounds for cluster expansions).

**Remark.** Note that \(N(\Delta)\) is at most of order \(O(n(\Delta))\) for a single-scale cluster expansion. This is not true for a multi-scale cluster expansion. Namely, fix \(\Delta^h \in D^h\). Assume a low-momentum field \(\psi^h(x) (\beta_i < -D/2)\) or \(\delta^i \psi^h(x) (\beta_i \geq -D/2)\) has been produced at scale \(k\) and dropped inside \(\Delta^j\) at scale \(j\), with \(k > j > h\). Although the number of fields produced at scale \(k\) in an interval \(\Delta^k \subset \Delta^j\) increases exponentially with \(k - j\), there are \(\leq N_{\text{ext,max}} + O(n(\Delta^j))\) low-momentum fields dropped inside \(\Delta^j\) for a given \(G\)-monomial, since \(\partial^i_{\Delta^j}\) occurs at a power \(\leq N_{\text{ext,max}} + O(n(\Delta^j))\). On the other hand, the number of low-momentum fields \(\psi^h(x), x \in \Delta^j\) with \(\Delta^j \subset\)
\(\Delta^h\) originating from vertices of scale \(j\) may be of order \(#\{\Delta^j \in \mathbb{D}^j; \Delta^j \subset \Delta^h\} = M^{D(j-h)}\). This is a well-known phenomenon, called \textit{accumulation of low-momentum fields}. This "negative" spring-factor must be combined with the rescaling spring factor \(M^{2\beta(j-h)}\), see Corollary 2.7, resulting in \(M^{(D+2\beta)(j-h)}\), a \textit{positive} spring-factor if \(\beta < -D/2\). This accounts for the (already mentioned) fact that secondary fields need not be produced for fields with scaling dimension \(< -D/2\), see subsection 6.1 for details.

At a given scale \(j\), composing the horizontal cluster and momentum-decoupling expansions at all scales \(\ge j\) yields the following result, easy to show by induction:

**Lemma 3.10 (result of the expansion above scale \(j\))**

1. 

\[
Z_{V}^{j}\left(\lambda; t^{\rightarrow(j-1)}\right) = \int d\mu(\psi^{\rightarrow(j-1)})Z_{V}^{j\rightarrow r}(\lambda; (\psi^{h})_{h \le j-1}; t^{\rightarrow(j-1)}), \tag{3.19}
\]

with

\[
z_{V}^{j\rightarrow r}(\left.\cdot\right) = \left(\sum_{\psi^{\rightarrow j}(\psi^{j})}^{\psi^{j}} \prod_{\ell \in L(\psi^{j})} \int_{0}^{1} \partial x_{\ell} \int_{\Delta_{\ell}^{j}} \partial x_{\ell} \int_{\Delta_{\ell}^{j}} \partial C_{\psi^{\rightarrow j}(\psi^{j}, \xi_{\ell}^{j}, \xi_{j}^{j})} \right)
\]

\[
\ldots \left(\sum_{\psi^{\rightarrow j}(\psi^{j})}^{\psi^{j}} \prod_{\ell \in L(\psi^{j})} \int_{0}^{1} \partial w_{k} \int_{\Delta_{\ell}^{j}} \partial x_{\ell} \int_{\Delta_{\ell}^{j}} \partial C_{\psi^{\rightarrow j}(\psi^{j}, \xi_{\ell}^{j}, \xi_{j}^{j})} \right)
\]

\[
\int d\mu_{s(u)}(\psi^{j}) \left(\prod_{k \ge j} \text{Vert}^{k} \text{Hor}^{k}\right) e^{-f_{V}(\psi^{j})} dx
\]

(3.20)

where \(d\mu_{s(u)}(\psi^{j})\) is a short-hand for \(\prod_{k=j}^{\infty} d\mu_{s(k)}(\psi^{k})\).

2. The right-hand side (3.20) depends on the low-momentum fields \((\psi^{h})_{h < j}\) only through the dressed fields \((T\psi^{h})_{h \le j-1}\), since the \(t\)-variables of scale \(\le j - 1\) have not been touched. Hence it makes sense to consider the quantity \(Z_{V}^{j\rightarrow r}(\lambda) := Z_{V}^{j\rightarrow r}(\lambda; (\psi^{h})_{h \le j-1} = 0; t^{\rightarrow(j-1)} = 1)\).

3. The partition function \(Z_{V}^{j\rightarrow r}(\lambda)\) writes

\[
Z_{V}^{j\rightarrow r}(\lambda) = \sum_{N=1}^{\infty} \frac{1}{N!} \sum_{\text{non--overlapping } P_{1}, \ldots, P_{N}}^{N} \prod_{n=1}^{N} F_{HV}(P_{n})
\]

\[
= \sum_{N=1}^{\infty} \frac{1}{N!} \sum_{\text{non--overlapping } P_{1}, \ldots, P_{N}}^{N} \prod_{n=1}^{N} F_{HV}(P_{n}) \cdot \prod_{\ell \in \{P, P'\}} 1_{P, P'}
\]

(3.21)
where $F_{HV}$, called polymer functional associated to horizontal (H) and vertical (V) cluster expansion, is the contribution of each polymer $P^j \rightarrow$ to the right-hand side of (3.20).

Later on, the polymer functional $F_{HV}$ will be replaced with the renormalized (R) polymer functional $F_{HVR}$, or simply $F$, hence eq. (3.21) shall not be used in this form. The general discussion in the next subsection, which does not depend on the precise form of $F$, shows how to get rid of the non-overlapping conditions.

### 3.4 Mayer expansion

Recall the polynomial evaluation function $F_{HV}$ introduced in the expansion of $Z_{\rho \rightarrow \rho}(\lambda; t)$, see eq. (3.21). It is unsatisfactory in this form because of the non-overlapping conditions which make it impossible to compute directly a finite quantity out of it, see Introduction. Were this the only source of trouble, it would suffice to make a global Mayer expansion for $Z^{0 \rightarrow \rho}(\lambda)$ after the multi-scale cluster expansion has been completed. However, it is also unsatisfactory when renormalization is required; local parts of diverging graphs (for reasons accounted for in section 4) must be discarded and resummed into an exponential, thus leading to a counterterm in the interaction. This forces upon us a sequence of three moves at each scale $j$, starting from highest scale $\rho$: (1) a separation of the local part of diverging graphs; (2) the Mayer expansion proper, at scale $j$; (3) the construction of the interaction counterterm (also called renormalization phase).

Let us formalize this into the following:

**Induction hypothesis at scale $j$.** After completing all expansions of scale $\geq j + 1$ and the horizontal/vertical cluster expansion at scale $j$, $Z_{V^{\rightarrow \rho}}(\lambda; t)$ has been rewritten as

$$Z_{V^{\rightarrow \rho}}(\lambda; t^{(j-1)}) = \prod_{k=j+1}^{\rho} e^{|V|M^{k \rightarrow \rho}(\lambda)} \cdot \int d\mu(\psi^{(j-1)}) Z_{V^{\rightarrow \rho}}(\lambda; t^{(j-1)}; \psi^{(j-1)}),$$

(3.22)

where $f^{k \rightarrow \rho}(\lambda)$ may be reinterpreted as a scale $k$ free energy density per
\[ Z^{j\rightarrow \rho}(\lambda; t \rightarrow (j-1)); \psi \rightarrow (j-1)) = \]
\[ \sum_{N=1}^{\infty} \frac{1}{N!} \sum_{\text{non-}j\text{-overlapping}} \int dw^{j\rightarrow} \int d\mu_{s(w)}(\psi^{j\rightarrow}) \prod_{n=1}^{N} F^{j}_{HV}(P_{n}; \psi), \]
\[ (3.23) \]

see Lemma 3.10 for notations, where "non-\(j\)-overlapping" means that \(P_{1} \cap \mathbb{D}^{j}, \ldots, P_{N} \cap \mathbb{D}^{j}\) are non-overlapping single-scale polymers, and \(F^{j}_{HV}(P_{n}; \psi)\) depends only on the values of \(\psi\) on the support of \(P_{n}\).

For \(j = \rho\), this formula is equivalent to the single-scale expansion of \(Z^{\rho\rightarrow \rho}(\lambda)\) of eq. (3.21).

According to the general expansion scheme (see introduction to section 3), we must now perform three tasks.

1. Separation of local part of diverging graphs

Consider the polymer evaluation function \(F^{j}_{HV}(P_{n}; \psi)\). If \(P_{n}\) has \(N_{ext} = \sum_{\Delta^{i} \in P_{n} \cap \mathbb{D}^{j}} \tau_{\Delta^{i}} < N_{ext,max}\) external legs, situated in (possibly coinciding) intervals \(\Delta^{i}_{j}\), it may be superficially divergent, in which case one separates its local part according to the rule explained in section 4: letting \((\psi^{(j-1)}_{i}(x_{i}))_{i \in I_{q}}\), with \(x_{i} \in \Delta^{i}_{j}\), be its external structure, \(F^{j}_{HV}(P_{n}; \psi)\) writes

\[ \prod_{i \in I_{q}} \int_{\Delta^{i}_{j}} dx_{i} F^{j}_{HV, \text{amputated}}(P_{n}; \psi; (x_{i})) \quad \prod_{i \in I_{q}} \psi^{(j-1)}_{i}(x_{i}) \]
\[ = F^{j}_{HV, \text{local}}(P_{n}; \psi) + \delta F^{j}_{HV}(\delta N_{ext,max} - N_{ext}P_{n}; \psi), \]
\[ (3.24) \]

where

\[ F^{j}_{HV, \text{local}}(P_{n}; \psi) = \prod_{i \in I_{q}} \int_{\Delta^{i}_{j}} dx_{i} F^{j}_{HV, \text{amputated}}(P_{n}; \psi; (x_{i})); \quad \left( \frac{1}{|I_{q}|} \sum_{i \in I_{q}} \psi^{(j-1)}_{i}(x_{i}) \right) \]
\[ (3.25) \]

and \(F^{j}_{HV}(P_{n}; \psi)\) minus its local part is now thought as if it had \(N_{ext,max} - N_{ext}\) supplementary external legs shared in an arbitrary way among the intervals \(\Delta^{i}_{j}\), which produces a polymer denoted by \(\delta N_{ext,max} - N_{ext}P_{n}\) belonging to \(P^{j}_{N_{ext,max}}\).
Equivalently (see subsection 4.1), considering only local parts, according to our new convention,

\[ F^j_{HV}(P_n; \psi) = \prod_{i \in I_q} \int_{\Delta^j_i} dx_i F^j_{HV, \text{amputated}}(P_n; \psi; (x_i)) \cdot \prod_{i \in I_q} \psi^{(j-1)}(x_i), \]

where now

\[ F^j_{HV, \text{amputated}}(P_n; \psi; (x_i)) = \frac{1}{|I_q|} \sum_{i \in I_q} \int \prod_{i' \in I_q, i' \neq i} dx_{i'} F^j_{HV}(P_n; \psi; (x_i)). \]

(3.26)

2. Mayer expansion at scale \( j \)

We shall now apply the restricted cluster expansion, see Proposition 3.4, to the functional \( Z^{j \rightarrow \rho}_V(\lambda; t; \psi^{(j-1)}) \). The ”objects” are multi-scale polymers \( P \in O = \{P_1, \ldots, P_N\} \) (see induction hypothesis above); a link \( \ell \in L(O) \) is a pair of polymers \( \{P_n, P_{n'}\}, n \neq n' \). Objects of type 2 are polymers with \( \geq N_{\text{ext, max}} \) external legs whose non-overlap conditions may not be removed properly at this stage, because they are in any case dependent of each other in an as yet unforeseeable way through the low-momentum fields. All other objects are of type 1, they belong to \( P^{j \rightarrow N_{\text{ext, max}} - 1}_N \); they are either vacuum polymers, i.e. polymers with no external legs, or superficially divergent polymers whose contribution to the interaction counterterm one would like to compute.

Link weakenings \( S = (S_{[P, P']}_{[P, P']})_{[P, P'] \in L(O)} \in [0, 1]^L(O) \) act on \( Z^{j \rightarrow \rho}_V(\lambda; t^{(j-1)}; \psi^{(j-1)}) \) by replacing the non-overlapping condition

\[ \text{NonOverlap}(P_1, \ldots, P_N) := \prod_{(P_n, P_{n'})} 1_{P_n, P_{n'} \text{ non-} j \text{-overlapping}} \]

\[ = \prod_{(P_n, P_{n'})} \prod_{\Delta \in P_n \cap \Delta', \Delta' \in P_{n'} \cap \Delta} (1 + (1_{\Delta \neq \Delta'} - 1)) \]

(3.28)

with a weakened non-overlapping condition
\[
\prod_{(P_n, P_{n'})} \prod_{\Delta \in \Delta_{\text{ext}}(P_n), \Delta' \in \Delta_{\text{ext}}(P_{n'})} 1_{\Delta \neq \Delta'} \cdot \\
\left(1 + S_{\{P_n, P_{n'}\}} \left( \prod_{\Delta \in P_n \cap D_j, \Delta' \in P_{n'} \cap D_j, \Delta, \Delta' \in \Delta_{\text{ext}}(P_n) \times \Delta_{\text{ext}}(P_{n'})} 1_{\Delta \neq \Delta'} - 1 \right) \right),
\]

(3.29)

where \(\Delta_{\text{ext}}(P) \subset P \cap D_j\) is the subset of intervals \(\Delta^j\) with external legs, i.e. such that \(\tau_{\Delta^j} \neq 0\).

Note that each factor in (3.29) ranges in \([0, 1]\). We ask the reader to accept this definition as it is and wait till the remark after Proposition 3.11 for explanations.

Let us now give some necessary precisions. The Mayer expansion is really applied to the non-overlap function \(\text{NonOverlap}^n\) and not to \(Z^j_{\psi^j \rightarrow \psi}(\lambda; t \rightarrow (j-1); \psi \rightarrow (j-1))\). Hence one must still extend the function \(\int dw^j \rightarrow \int d\tilde{\mu}_j^j(s) \prod_{i=1}^N F^j_{HV}(P_n; \psi)\) to the case when the \(P_n, n = 1, \ldots, N\) have some overlap. The natural way to do this is to assume that the random variables \((\psi^j|_{P_n})_{n=1, \ldots, N}\) remain independent even when they overlap. This may be understood in the following way. Choose a different color for each polymer \(P_n = P_1, \ldots, P_N\), and paint with that color all intervals \(\Delta^j \in P_n \cap D_j\). If \(\Delta^j \in \Delta_{\text{ext}}(P_n)\), then its external links to the interval \(\Delta^{j-1}\) below it are left in black. The rules (3.29) imply that intervals with different colors may superpose; on the other hand, external inclusion links may not, so that: (i) low-momentum fields \(\psi^j \rightarrow (j-1)(x)\), \(x \in \Delta^j\) with \(\Delta^j \in \Delta_{\text{ext}}(P_n)\), do not superpose and may be left in black (till the next expansion stage at scale \(j - 1\) at least); (ii) the color of high-momentum fields of scale \(j\) created at a later stage may be determined without ambiguity.

Hence one must see \(\psi^j\) as living on a two-dimensional set, \(\mathbb{D}^j \times \{\text{colors}\}\), so that copies of \(\psi^j\) with different colors are independent of each other. This defines a new, extended Gaussian measure \(d\tilde{\mu}_j^j(s^j)\) associated to an extended field \(\tilde{\psi}^j : \mathbb{R}^D \times \{\text{colors}\} \rightarrow \mathbb{R}\), and Mayer-extended polymers. By abuse of notation, we shall skip the tilde in the sequel, and always implicitly extend the fields and the measures of each scale. Mayer-extended polymers shall be considered as (colored) polymers in section 6.
This gives the following expansion for $Z_{Vj}^{j\rightarrow\rho}(\lambda; t^{(j-1)}; \psi^{(j-1)})$.

**Proposition 3.11 (Mayer expansion)** Let $\mathcal{F}(\mathcal{P}^{j\rightarrow})$ be the set of all forests of polymers whose each component $T$ is (i) either a tree of polymers of type 1 (called: unrooted tree); (ii) or a rooted tree of polymers such that only the root is of type 2. Then

$$Z_{Vj}^{j\rightarrow\rho}(\lambda; t^{(j-1)}; \psi^{(j-1)}) = \sum_{F \in \mathcal{F}(\mathcal{P}^{j\rightarrow})} \text{Mayer}(Z_{Vj}^{j\rightarrow\rho}(\lambda; t^{(j-1)}; \psi^{(j-1)}); F),$$

(3.30)

with

$$\text{Mayer}(Z_{Vj}^{j\rightarrow\rho}(\lambda; t^{(j-1)}; \psi^{(j-1)}); F) = \left( \prod_{\ell \in L(F)} \int_0^1 dW_{\ell} \right) \left( \prod_{\ell \in L(F)} \frac{\partial}{\partial S_{\ell}} \right) Z_{Vj}^{j\rightarrow\rho}(\lambda; t^{(j-1)}; \psi^{(j-1)}) \right) (S(W))$$

(3.31)

where $S_{\ell}(W)$ is either 0 or the minimum of the $W$-variables running along the unique path in $F$ from $o_\ell$ to $o'_\ell$, and $\overline{F}$ is the forest obtained from $F$ by merging all roots in $\mathcal{P}^{j\rightarrow}_{N_{\text{ext,max}}}$ into a single vertex.

As a result, (i) polymers $\mathcal{P}_\ell, \mathcal{P}'_\ell$ linked by a Mayer link are $j$-overlapping (otherwise the derivative $\partial S_{\ell}$ would produce a zero factor); (ii) pairs of vacuum polymers $\mathcal{P}, \mathcal{P}'$ belonging to different Mayer trees come with the factor 1: they have lost their non-overlap conditions and may superpose each other freely (in other words, they have become transparent to each other). Hence $Z_{Vj}^{j\rightarrow\rho}(\lambda; t^{(j-1)}; \psi)$ factorizes as a product

$$Z_{Vj}^{j\rightarrow\rho}(\lambda; t^{(j-1)}; \psi) = e^{F_{j\rightarrow\rho}^{\text{HVM}}} \int d\mu(\psi^{(j-1)}).$$

$$\cdot \sum_{N=1}^{\infty} \frac{1}{N!} \sum_{\text{non-}j\text{-overlapping } \mathcal{P}'_1, \ldots, \mathcal{P}'_n \in \mathcal{P}^{j\rightarrow}} \prod_{n=1}^N F_{j\rightarrow\rho}^{\text{HVM}}(\mathcal{P}'_n; \psi)$$

(3.32)

where $F_{j\rightarrow\rho}^{\text{HVM}}$ is the contribution of all (unrooted) trees of vacuum polymers. Denote by $f_{j\rightarrow\rho}(\lambda)$ the quantity obtained by fixing one interval $\Delta^j$ of scale $j$ belonging to one of the polymers of the tree. Summing over all $\Delta^j$, one obtains an overall factor $e^{V|M^j f_{j\rightarrow\rho}(\lambda)}$. 

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3. Renormalization phase

For simplicity we shall assume that only 2-point functions $\langle \psi_i \psi_{i'} \rangle$, $1 \leq i, i' \leq d$ need to be renormalized (which is the case for the $(\phi, \partial \phi, \sigma)$-model). Denote by $-\frac{1}{2} \int_{\Delta^j} dx b^j_{i,i'} \psi_i \rightarrow (j-1)(x) \psi_{i'} \rightarrow (j-1)(x)$ the contribution to $F^j_{HVM}$ of all unrooted trees of polymers containing exactly one polymer with 2 external legs in $\Delta^j$, plus a cloud of vacuum polymers, attached to it, directly or indirectly, by Mayer links. The intervals without external legs of the different unrooted trees of polymers have become transparent to each other, and to the rooted trees too, hence $b^j_{i,i'}$ may be computed by considering one unrooted tree of polymers with 2 external legs, irrespectively of the position of the other trees of polymers. By translation invariance, $b^j_{i,i'}$ is a constant, which is fixed by induction on $j$ by demanding that

$$0 \equiv \int dy \frac{\partial^2}{\partial \psi_i \rightarrow (j-1)(x) \partial \psi_{i'} \rightarrow (j-1)(y)} Z^{j \rightarrow \rho} \lambda; \psi \rightarrow (j-1) \bigg|_{\psi \rightarrow (j-1)=0}.$$  

(3.33)

or better (so as to eliminate higher scale polymers which depend only on $b^{j+1}_{i,i'}$)

$$0 \equiv \int dy \frac{\partial^2}{\partial \psi_i \rightarrow (j-1)(x) \partial \psi_{i'} \rightarrow (j-1)(y)} \left( Z^{j \rightarrow \rho} - Z^{(j+1) \rightarrow \rho} \right) \lambda; \psi \rightarrow (j-1) \bigg|_{\psi \rightarrow (j-1)=0}.$$  

(3.34)

Hence eq. (3.34) yields $b^j_{i,i'}$ as a sum over polymers (containing at least one interval of scale $j$) of an expression depending itself on $b^j_{i,i'}$, – an implicit equation.

Local parts of 2-point functions are generated by the exponential over the whole volume, $e^{-\frac{1}{2} \int_V dx b^j_{i,i'} \psi_i \rightarrow (j-1)(x) \psi_{i'} \rightarrow (j-1)(x)}$, which may be seen as a counterterm in the interaction. This counterterm disappears by renormalizing the Gaussian covariance kernel $C_\psi = K^{-1}_0$ – previously renormalized down to scale $j + 1$ – to $K^{-1}$, where

$$K(\psi, \psi) = K^0_0(\psi, \psi) + \delta K^j(\psi, \psi) = K^0_0(\psi, \psi) + \int_V dx b^j_{i,i'} \psi_i \rightarrow (j-1)(x) \psi_{i'} \rightarrow (j-1)(x).$$  

(3.35)

\[\text{Note that } K - \text{ contrary to the original bare kernel } - \text{ is only almost-diagonal, namely, } \int \psi'_i(x) \psi''_{i'}(x) dx = 0 \text{ by momentum conservation if } |\text{supp}(\chi'_i) \cap \text{supp}(\chi''_{i'})| = 0, \text{ i.e. if } |j - j'| \geq 2.\]
Note that the renormalization of $C_\sigma$ in the case of the $(\phi, \partial \phi, \sigma)$ shall only be performed at scale $\rho$, while the other counterterms shall be left out as a counterterm in the interaction (see section 5).

After applying a horizontal/vertical expansion at scale $(j-1)$ to $Z^{(j-1)\to \rho}$, we are finally back to the induction hypothesis, at scale $(j - 1)$ this time.

Let us now show the following single scale Mayer bounds.

**Proposition 3.12 (Mayer bounds)**

1. (vacuum polymers) Let $\mathcal{P}_0^{j\to \Delta j}$ be the set of vacuum polymers down to scale $j$ containing some fixed interval $\Delta j \in \mathbb{D}^j$. Assume

$$\sum_{\mathcal{P} \in \mathcal{P}_0^{j\to \Delta j}} e^{\vert \mathcal{P} \vert - 1} \vert F_{HV}^j(\mathcal{P}) \vert \leq K' \lambda, \quad (3.36)$$

where $F_{HV}^j(\mathcal{P}) = \int d\mu(\psi^{j\to}) F_{HV,amputated}^j(\mathcal{P};\psi^{j\to})$. Then

$$\vert f^{j\to \rho}(\lambda) \vert \leq K' \lambda (1 + O(\lambda)) \quad (3.37)$$

with the same constant $K'$.

2. (counterterm) Let $\mathcal{P}_{i,i'}^{j\to \Delta j}$ be the set of polymers down to scale $j$ with exactly two external legs, $\psi_i^{j\to(j-1)}$ and $\psi_{i'}^{j\to(j-1)}$, displaced into the same point in a fixed interval $\Delta j$. Rewrite $F_{HV,amputated}^j(\mathcal{P}) := \int d\mu(\psi^{j\to}) F_{HV,amputated}^j(\mathcal{P};\psi^{j\to})$, $\mathcal{P} \in \mathcal{P}_{i,i'}^{j\to \Delta j}$ as

$$M^j(1+\beta_i+\beta_{i'}) F_{HV,amputated}^{j,\text{rescaled}}(\lambda, \mathcal{P};\psi^{j\to(j-1)}), \quad (3.38)$$

with:

- $\sum_{\mathcal{P} \in \mathcal{P}_{i,i'}^{j\to \Delta j}, \# \{\text{vertices of } \mathcal{P}\} = 2} F_{HV,amputated}^{j,\text{rescaled}}(\lambda;\mathcal{P}) = -K \lambda^2 (1 + O(\lambda));$

- $\sum_{\mathcal{P} \in \mathcal{P}_{i,i'}^{j\to \Delta j}, \# \{\text{vertices of } \mathcal{P}\} \geq 3} e^{\vert \mathcal{P} \vert} F_{HV,amputated}^{j,\text{rescaled}}(\lambda, \mathcal{P})$

$$= -K' \lambda^3 (1 + O(\lambda) + O(M^{-j(1+\beta_i+\beta_{i'})} b_{i,i'}^j)), \quad (3.39)$$

$^{14}$Since $\mathcal{P}$ is a vacuum polymer, no high-momentum fields of scale $j$ may be produced at a later stage, hence one may integrate out the field components $\psi^{j\to}$. 45
$$b_{i,i'}^j = K\lambda^2 M^{j(1+\beta_i+\beta_{i'})}(1 + O(\lambda)).$$  \hspace{1cm} (3.40)

We shall see in section 6.1 how to obtain estimates which are valid for a succession of Mayer expansions at each scale.

**Proof.**

1. Fix some interval $\Delta^j \in \mathbb{D}^j$ and compute $f^{j\rightarrow\rho}(\lambda)$ using Proposition 3.11 as

$$\sum_{N \geq 1} \sum_{\mathbb{P}_1,\ldots,\mathbb{P}_N \in \mathbb{P}_0^j \rightarrow (\Delta^j)} \sum_{\mathbb{T}} \text{Mayer}(Z^{j\rightarrow\rho}(\lambda); \mathbb{T}),$$  \hspace{1cm} (3.41)

where

$$|\text{Mayer}(Z^{j\rightarrow\rho}(\lambda); \mathbb{T})| \leq \frac{1}{N!} \prod_{n=1}^{N} |F^j_{\mathbb{P}}(\mathbb{P}_n)|.$$  \hspace{1cm} (3.42)

The $\frac{1}{N!}$ factor is matched by Cayley’s theorem, which states that the number of trees over $\mathbb{P}_1,\ldots,\mathbb{P}_N$ with fixed coordination number $(n(\mathbb{P}_i))_{i=1,\ldots,N}$ equals $\frac{N!}{(n(\mathbb{P}_i)-1)!}$. Recall that $\mathbb{P}_\ell$ and $\mathbb{P}'_\ell$ are necessarily overlapping if $\ell \in L(\mathbb{T})$. Start from the leaves and go down the branches of the tree inductively. Let $\mathbb{P}_1,\ldots,\mathbb{P}_{n(\mathbb{P})}-1$ be the leaves attached onto one and the same vertex $\mathbb{P}'$ of $\mathbb{T}$. Choose $n(\mathbb{P}') - 1$ (possibly non-distinct) vertices of $\mathbb{P}' \cap \mathbb{D}^j$ (there are $|\mathbb{P}' \cap \mathbb{D}^j|^{|n(\mathbb{P}')|-1}$ possibilities), fix their spatial location, $\Delta^j_1,\ldots,\Delta^j_{n(\mathbb{P})}-1$, and assume that $\Delta^j_i \in \mathbb{P}_i \cap \mathbb{D}^j$. For each choice of polymer $\mathbb{P}'$, this gives a supplementary factor $\leq (K'\lambda|\mathbb{P}' \cap \Delta^j|)^{n(\mathbb{P}')-1}$, to be multiplied by $\frac{1}{(n(\mathbb{P}')-1)!}$ coming from Cayley’s theorem. Summing over $n(\mathbb{P}') = 2,3,\ldots$, yields $e^{K'\lambda|\mathbb{P}' \cap \Delta^j|}-1$, which is $\leq 2|\mathbb{P}'|(K'\lambda)$ for $\lambda$ small enough. By induction, one gets $|f^{j\rightarrow\rho}(\lambda)| \leq \sum_{h \geq 1} (K'\lambda)^h = K'\lambda(1 + O(\lambda))$, where $h$ is the height of the tree.

2. The definition of $b_{i,i'}^j$ and arguments analogous to those used in (1) yield (letting $\tilde{b}_{i,i'}^j := \lambda^{-2}M^{-j(1+\beta_i+\beta_{i'})}b_{i,i'}^j$)

$$\tilde{b}_{i,i'}^j = -K(1 + O(\lambda) + O(\lambda^2 \lambda_{i,i'}^j)),$$  \hspace{1cm} (3.43)

hence the result by the implicit function theorem.

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Remark. Consider the case when the Mayer expansion has surrounded \( P \in \mathcal{P}^{j} \rightarrow \), a polymer with \( \geq N_{\text{ext,\,max}} \) external legs, by a cloud of vacuum polymers. Then the contribution of the cloud to \( F_{\mathcal{H}V}^{j}(P;\psi) \) may be computed as in Proposition 3.12 (1), see eq. (3.37), leading at most to an overall multiplicative coefficient \( \leq 1 + K\lambda (1 + O(\lambda)) + \ldots + n! \left( \frac{(K\lambda(1 + O(\lambda)))^n}{n!} \right) + \ldots = 1 + O(\lambda) \) per interval \( \Delta \in \mathcal{P} \) instead of \( e \) as in eq. (3.36). This is important if \( \Delta \in \Delta_{\text{ext}}(P) \) is crossed by downward dashed lines as in §3.3 but contains no field, for an arbitrary product of factors \( e \) down to the scale of the low-momentum fields, possibly produced by successive Mayer expansions, may lead to a disaster. On the other hand (see end of §6.1.2) a factor \( 1 + O(\lambda) \) per interval of \( \mathcal{P} \) is not a problem.

4 Power-counting and renormalization

This section is divided into two parts. The first one gives a quick overview of how divergences in quantum field theory are discarded by extracting the local part of diverging diagrams; these ideas come from classical power-counting arguments which are recalled here. The whole idea of renormalization is then to transfer the sum of these local parts to the interaction as a counterterm (see §3.4). The second one is an informal discussion of the domination problem of low-momentum field averages, in particular in the case of the \((\phi, \partial \phi, \sigma)\)-model). The full treatment of this problem is postponed to §6.2.

4.1 Power-counting and diverging graphs

Definition 4.1 (Feynman diagrams) Let \( \psi = (\psi_1(x), \ldots, \psi_d(x)) \) be a multiscale Gaussian field with covariance kernel \( C_\psi \), and \( \mathcal{L}_{\text{int}} = \sum_{q=1}^{p} K_q \lambda^q \psi I_q \) be an interaction. Then:

1. a Feynman diagram for this theory is a connected graph \( \Gamma \) (whose lines, resp. vertices are generically denoted by \( \ell \), resp. \( z \)) with
   (i) external vertices \( y = (y_1, \ldots, y_n) \) of type \( 1, \ldots, p \);
   (ii) internal vertices \( x \) of type \( 1, \ldots, p \);
   (iii) internal lines \( \ell \) connecting \( z_\ell \) to \( z'_\ell \) with double index \( (i_\ell, i'_\ell) \).

Since the lines do not have a preferred orientation, and in order to avoid confusion, one writes \( \ell \simeq (i_\ell, z_\ell; i'_\ell, z'_\ell) \) or indifferently \( (i'_\ell, z'_\ell; i_\ell, z_\ell) \). For every vertex \( z \) of type \( q \), one may order the
lines leaving or ending in $z$ as $\ell_{z,1} \simeq (i_1, z; i'_1, z'_1), \ldots, \ell_{z,n(z)} \simeq (i_{n(z)}, z; i'_{n(z)}, z'_{n(z)})$ so that $n(z) = q$ and $(i_1, \ldots, i_q) = I_q$ if $z$ is an internal vertex, and $n(z) < q, (i_1, \ldots, i_q) \subset I_q$ if $z$ is external.

(iv) external lines $\ell \simeq (i_\ell, y_\ell; i'_\ell, y'_\ell)$, where $y'_\ell \in \mathbb{R}^D$ is an external point not belonging to $\Gamma$; $|I_q| - n(y)$ of them per external vertex $y$ of type $q$. The total number of external lines is $N_{\text{ext}}(\Gamma) := \sum_q \sum_y$ of type $q (|I_q| - n(y))$.

The evaluation $A(\Gamma)$ of an amputated Feynman diagram is given by

$$A(\Gamma)(y) = \int \prod_x dx \left( \prod_{\ell \in L_{\text{int}}(\Gamma)} C_{\psi}(i_\ell, z_\ell; i'_\ell, z'_\ell) \right), \quad (4.1)$$

where $x$ ranges over all internal vertices, and $L_{\text{int}}(\Gamma)$ is the set of all internal lines.

The evaluation $A(\Gamma)$ of a full Feynman diagram is given by

$$\bar{A}(\Gamma)(y') = \int \prod_y dy \left( \prod_{\ell \in L_{\text{ext}}(\Gamma)} C_{\psi}(i_\ell, y_\ell; i'_\ell, y'_\ell) \right) A(\Gamma)(y), \quad (4.2)$$

where $L_{\text{ext}}(\Gamma)$ is the set of all external lines.

2. A multi-scale Feynman diagram $(\Gamma; (j(\ell)))$ is obtained from $\Gamma$ by choosing a scale $j(\ell)$ for each (internal or external) line and splitting each vertex into these different scales, with the following constraint,

$$\text{height}(\Gamma) := \min\{j(\ell); \ell \in L_{\text{int}}(\Gamma)\} - \max\{j(\ell); \ell \in L_{\text{ext}}(\Gamma)\} \geq 0. \quad (4.3)$$

Note that $\text{height}(\Gamma)$ measures the height of internal lines of $\Gamma$ with respect to the external lines it is attached to.

The evaluation $A(\Gamma)$ of a multi-scale Feynman diagram is given by

$$A(\Gamma; (j(\ell))_{\ell \in L_{\text{int}}(\Gamma)})(y) = \int \prod_x dx \left( \prod_{\ell \in L_{\text{int}}(\Gamma)} C_{\psi}^{j(\ell)}(i_\ell, z_\ell; i'_\ell, z'_\ell) \right), \quad (4.4)$$

while
Cluster expansions produce single-scale (horizontal) Feynman trees, multiplied by some $G$-polynomial and by the exponential of the interaction, up to the modification of the measure by the weakening coefficients $s$. In the final bounds, see section 6, one bounds the exponential by a constant and applies the Cauchy-Schwarz inequality to the rest, yielding a multi-scale Feynman diagram. The scale $j(z)$ of a vertex $z$ is then the scale at which the vertex has been produced.

Let us from now on assume that the interaction is just renormalizable. This means that, for every $q = 1, \ldots, p$, $\sum_{i \in I_q} \beta_i = -D$, or in other words, $\int \psi_{I_q}(x) dx$ is homogeneous of degree 0. In that case, the following very simple power-counting rules hold.

**Proposition 4.2 (power-counting)**

1. The power-counting of an amputated Feynman diagram $\Gamma$ is the product $\Lambda^{\omega(\Gamma)} := \Lambda^D \prod_{z} \Lambda^{-D} \prod_{\ell \in L_{int}(\Gamma)} \Lambda^{-j(\ell) + \beta_\ell}$, see Definition 2.4 for notations, where $\Lambda$ is some large, indefinite constant representing an ultra-violet cut-off.

Then the degree of divergence $\omega(\Gamma)$ of $\Gamma$ is equal to $D + \sum_{\ell \in L_{ext}(\Gamma)} \beta_\ell$.

2. The power-counting of a full multi-scale Feynman diagram $\Gamma$ is the product

$$M^{\omega_m.s.}(\Gamma) := M^{D \cdot \text{height}(\Gamma)} \prod_{z} M^{-D j(z)} \prod_{\ell \in L(\Gamma)} M^{-j(\ell) (\beta_\ell + \beta_\ell')}. \quad (4.6)$$

The multi-scale degree of divergence $\omega_{m.s.}(\Gamma)$ is equal to

$$\omega_{m.s.}(\Gamma) = D \cdot \text{height}(\Gamma) + \sum_{z} \sum_{\ell \in L_z} \beta_\ell (j(z) - j(\ell)), \quad (4.7)$$

where $L_z$ is the set of internal or external lines leaving or ending in $z$.

Let us give brief explanations. The power-counting in 1. may be obtained from Definition 2.4 by assuming all internal lines of $\Gamma$ to be of scale $\ln \Lambda / \ln M$ and summing over all vertices except one, due to overall (approximate or
exact) translation invariance, see [39]. The multi-scale power counting is a refined version of the previous one, which takes into account the scale of the external legs; the definition of the height of $\Gamma$ is somewhat \textit{ad-hoc} and measures the horizontal freedom of movement of $\Gamma$ with respect to its external legs if these are supposed to be fixed. More precise computations in the spirit of cluster expansion appear in the final bounds in section 6.

The principle of the power-counting rule explained in section §6.1.2 is to rescale all fields produced at scale $j$ of a multiscale Feynman diagram as if they were of scale $j$. The production scale of a given vertex $z$ being $j(z)$, this leads to a rescaled degree of divergence,

$$
\omega_{\text{rescaled}} \; (\Gamma) := \omega \; (\Gamma) - \sum_{z} \sum_{\ell \in L_{z,\text{int}}} \beta_{i_{\ell}} (j(z) - j(\ell)),
$$

(4.8)

where the sum on $\ell$ ranges over all internal lines leaving or ending in $x$. Hence

$$
\omega_{\text{rescaled}} \; (\Gamma) = D \cdot \text{height}(\Gamma) + \sum_{y} \sum_{\ell \in L_{y,\text{ext}}} \beta_{i_{\ell}} (j(y) - j(\ell)),
$$

(4.9)

where the sum on $\ell$ ranges over all external lines leaving or ending in $y$.

If $\text{height}(\Gamma)$ and all $j(y) - j(\ell)$ in (4.9) are equal to the same positive constant, then $\omega_{\text{rescaled}} \; (\Gamma)$ is proportional to the naive degree of divergence defined in Proposition 4.2 (1).

\textbf{Definition 4.3 (renormalization)} If the degree of divergence $\omega \; (\Gamma)$ of a Feynman diagram, resp. multiscale Feynman diagram, is $\geq 0$, then it needs to be renormalized.

The local part of an amputated Feynman diagram or multiscale Feynman diagram is obtained by integrating over all external vertices except one (in order to take into account the global invariance by translation),

$$
A(\Gamma; (j(\ell)))(y = (y_{1}, \ldots, y_{n})) \sim \int dy_{2} \ldots dy_{n} A(\Gamma; (j(\ell)))(y) \; =: \text{Local}(A(\Gamma; (j(\ell)))).
$$

(4.10)

By invariance by translation again, it does not depend on $y_{1}$.

Integrating over external points, one obtains the local part of a full Feyn-
man diagram,

\[ \tilde{A}(\Gamma; (j(\ell)))(y') \sim \text{Local}(\tilde{A}(\Gamma; (j(\ell)))(y') : = \]

\[ \int \prod_y \left( \prod_{\ell \in L_{\text{ext}}(\Gamma)} C^{j(\ell)}_\psi (i_\ell, y_\ell; i'_\ell, y'_\ell) \right), \]

(4.11)

which may be rewritten as

\[ \text{Local}(\tilde{A}(\Gamma; (j(\ell)))(y') = \int \prod_y \left( \prod_{\ell \in L_{\text{ext}}(\Gamma)} C^{j(\ell)}_\psi (i_\ell, y_1; i'_\ell, y'_\ell) \right) A(\Gamma; (j(\ell)))(y). \]

(4.12)

In other terms, all external legs of \( \Gamma \) have been displaced at the same arbitrary external vertex, here \( y_1 \).

The renormalized amplitude \( \mathcal{R}A \) or \( \tilde{\mathcal{R}} \tilde{A} \) of a Feynman diagram or multiscale Feynman diagram is the difference between its amplitude and its local part.

**Remark.** Using a Fourier transform, the local part is equivalent to the classical evaluation at zero external momenta.

A simple Taylor expansion of order 1 yields

\[ \mathcal{R} \tilde{A}(\Gamma; (j(\ell)))(y') = \int \prod_y \left( \prod_{\ell \in L_{\text{ext}}(\Gamma)} C^{j(\ell)}_\psi (i_\ell, y_1; i'_\ell, y'_\ell) \right) A(\Gamma; (j(\ell)))(y). \]

(4.13)

In principle, this formula shows that the renormalized diagram amplitude \( \mathcal{R}A(\Gamma; (j(\ell))) \) comes with a supplementary spring factor \( M^{-\text{height}(\Gamma)} \), leading to an equivalent degree of divergence \( \omega^*(\Gamma) := \omega(\Gamma) - 1 \). Namely, \( y_\ell - y_1 \) should be at most of order \( M^{-\min(j(\ell), \ell \in L_{\text{int}}(\Gamma))} \), while by definition (see section 2) \( E \left[ \partial \psi^{j(\ell)}_{i_\ell}(y_1 + t(y_\ell - y_1)) \psi_{i'_\ell}^{j(\ell)}(y'_\ell) \right] \) is of order \( M^{j(\ell)} M^{-(\beta_{i_\ell} + \beta_{i'_\ell})} \).

If \( \omega(\Gamma) < 1 - D \), then \( \omega^*(\Gamma) < -D \) and the diagram has become convergent.

15 Otherwise one should renormalize to a higher order by removing the beginning of the Taylor expansion of the diagram in the external momenta. We shall not describe this straightforward extension of the procedure since we shall not use it in our context.
Let us give a formal proof of this general statement. This requires a little care because the covariance is not exponentially decreasing at large distances in our setting, so using systematically the Taylor expansion (4.13) does not work. Consider a given multi-scale diagram \((\Gamma; (j_\ell))\). We consider only divergent two-point subdiagrams for the sake of notations. Start from the divergent diagram of lowest scale, \(j_{\text{min}}\), say, with external legs \(y, y'\), and use the Taylor expansion (4.13). Choose a tree of propagators of scale \(\geq j_{\text{min}}\) connecting \(y\) to \(y'\) through intermediary points \(x_2, \ldots, x_n\), and set \(x_1 = y, x_{n+1} = y'\), so that \(|y - y'| \leq \sum_{m=1}^{n} |x_m - x_{m+1}|\). The divergent diagram with external legs \(y, y'\) has been renormalized as explained above. Letting \(j_m\) be the scale of the link \((x_m, x_{m+1})\), one has lost a rescaled factor \(\sum_{m=1}^{n} d^{j_{\text{min}}}(x_m, x_{m+1}) = \sum_{m=1}^{n} M^{-j_m-j_{\text{min}}} d^{j_m}(x_m, x_{m+1})\). Each term in this sum has obtained a spring factor \(M^{-j_m-j_{\text{min}}}\), which shows that the corresponding diagram with external legs \(x_m, x_{m+1}\) is already \textit{de facto} superficially renormalized, although subdiagrams of higher scale may still need renormalization. In the process, it is clear that every propagator of the diagram may appear only in \textit{one} tree of propagators at most. In the bounds of section 6, this yields an overall factor per polymer \(P\) which is bounded by \(\prod_{\ell \in L(\rho)} (1 + d^{j_{\ell}}(x_{\ell}, x'_{\ell}))\), which is easily controlled by the polynomial decrease of the covariance at large distances.

Let us summarize our brief discussion.

\textbf{Definition 4.4 (diverging graphs)} A Feynman graph or multi-scale Feynman graph \(\Gamma\) is divergent if and only if

\[ \omega(\Gamma) := D + \sum_{\ell \in L_{\text{ext}}(\Gamma)} \beta_i \geq 0. \]  

We shall call \(N_{\text{ext},\text{max}}\) the minimum value of \(N_{\text{ext}}\) such that every diagram with \(\geq N_{\text{ext},\text{max}}\) external legs is convergent.

Renormalizing the evaluation of a Feynman graph or multi-scale Feynman graph \(\Gamma\) yields a quantity \(\mathcal{R}A(\Gamma)\) for which the displaced external legs have obtained a supplementary spring factor, which is equivalent to replacing one of the scaling dimensions \(\beta_i\), \(\ell \in L_{\text{ext}}(\Gamma)\) with \(\beta^*_i := \beta_i - 1\), or globally \(\omega(\Gamma)\) by \(\omega^*(\Gamma) := \omega(\Gamma) - 1\).

Let \(\omega^*_{\text{max}} < 0\) the maximal value of the set \(\{\omega^*(\Gamma)\}\), where \(\Gamma\) ranges over the set of all Feynman graphs.

\textbf{Example} \(((\phi, \partial \phi, \sigma)\)-model)

In our case \(\beta_{\phi} = \alpha, \beta_{\partial \phi} = \alpha - 1, \beta_{\sigma} = -2\alpha\). Note however that the interaction (see section 5) is such that split vertices are either of type
\[ \partial \phi^j \phi^k \sigma^{k'} \text{ or } \sigma^j \partial \phi^k \phi^{k'} , \text{ with } k' = k \pm 1, \text{ hence (assuming } k' = k \text{ which does not change anything) } \phi \text{ may not be an external leg of a multi-scale Feynman diagram (in particular, the vertex } \phi \partial \phi \sigma \text{ is not renormalized). Let } N_{\partial \phi}, \text{ resp. } N_{\sigma} \text{ be the number of external } \partial \phi- \text{ or } \sigma- \text{legs of such a diagram } \Gamma. \text{ The power-counting rule yields } \sum_{\ell \in \text{LExt}(\Gamma)} \beta_i^\ell = (\alpha - 1)N_{\partial \phi} - 2\alpha N_{\sigma} < -1 \text{ as soon as } N_{\partial \phi} \geq 2 \text{ or } N_{\sigma} \geq 4, \text{ so } N_{\text{ext,max}} = 4. \text{ However, by symmetry arguments, local parts of diagrams with } N_{\partial \phi} \text{ or } N_{\sigma} \text{ odd vanish, so there remains only the } \sigma- \text{propagator } \langle \sigma_\pm(x)\sigma_\pm(y) \rangle \text{ to renormalize.}

Let us write down precisely these power-counting rules in the framework of multi-scale cluster expansion, where Feynman diagrams are replaced by polymers. Recall from §3.3 that a low momentum field \((T^{-j-1})\psi_i^h(x)\) – or \((T^{-j-1})\delta^j \psi_i^h(x)\) – has three scales attached to it. It is the difference between the two highest ones – the production scale \(k\) and the dropping scale \(j\), with \(k > j\) – that counts here. Namely, considering \(h < j' < j\) there are two cases:

(i) either \(t^j \Delta_x^j\) (the number of \(t\)-derivatives produced inside \(\Delta_x^j\)) is \(\geq N_{\text{ext,max}}\). Then rescaling the fields to which the \(t\)-derivatives are applied is enough to ensure the convergence of the polymer \(\mathbb{P}^{j' \to \text{ containing } \Delta_x^j}\);

(ii) or \(t^j \Delta_x^j < N_{\text{ext,max}}\). Then (by definition) \(t^j \Delta_x^j\) is set to 0, which kills \((T^{-j-1})\psi_i^h(x)\) or \((T^{-j-1})\delta^j \psi_i^h(x)\). Hence one must count only on the derived fields for the power-counting. As explained above, this may give a non-negative degree of divergence, in which case one must renormalize by removing the local part of the polymer.

This means that the rescaling spring factor \(M^{\beta_i(k-h)}\) of these low-momentum fields must be split into \(M^{\beta_i(k-j)}\) – ensuring the horizontal fixing of the polymer, and counting in the right-hand side of eq. \((4.9)\) – and \(M^{\beta_i(j-h)}\) – which helps control the accumulation of low-momentum fields as explained briefly in §3.3.

4.2 Domination problem and boundary term in the interaction

Assume \(\psi_i\) is a multi-scale Gaussian field with scaling dimension \(\beta_i \in (-D/2,0)\). Then \(2\beta_i > -D\) so, by Definition 4.4, the associated two-point functions must be renormalized. If \(\psi_i\) occurs in some interaction term
\( \lambda^{\kappa} \psi_{I_{q}} \), i.e. \( i \in I_{q} \), then renormalization produces a scale-dependent counterterm of the form

\[
\delta \mathcal{L}(\psi_{i}; x) = \sum_{j \geq 0} \lambda_{j}^{2 \kappa} C_{j} M^{(D + 2 \beta_{j})/2} \left( (T \psi_{i})^{-2} \right)(x),
\]

where \( \lambda_{j} \) is the running coupling constant, and \( C_{j} \) is a scale-dependent constant. In this paper we generally assume that \( \lambda_{j} = \lambda \) is not renormalized. (In the case of our \((\phi, \partial \phi, \sigma)\)-model, \( \delta \mathcal{L} \) is of the form \( \sum_{j \geq 0} b^{j}((T \sigma)^{-2})^{2} \), with \( b^{j} \approx \lambda^{2} M^{(1 - 4 \alpha)j} \) and \( C_{j} \approx 1 \). See the counterterm \( \delta \mathcal{L}_{4} \) in section 5 below for details.) Separating the low-momentum field \( \psi_{i}^{-2}(j - 1)(x), x \in \Delta^{j} \) into \( \psi_{i}^{-2}(j - 1)(\Delta^{j}) + \delta \psi_{i}^{-2}(j - 1)(x) \) as in Definition 2.3 produces a secondary field \( \delta \psi_{i}^{-2}(j - 1) \) whose contributions to the partition function are easily bounded thanks to the "spring factor" (see subsection 6.1). Unfortunately, the averaged fields \( \psi_{i}^{-2}(j - 1)(\Delta^{j}) \) do not come with such a spring factor and (unless \( \beta_{i} < -D/2 \), see subsection 6.1 again) must be dominated apart, using the positivity of the interaction. We assume that \( C_{j} > 0 \). Then Lemma \( \text{6.9} \) (1) shows that \( |\lambda^{\kappa} (T \psi_{i})^{-2}(j - 1)(\Delta^{j})|^{n(\Delta)} e^{-\int \delta \mathcal{L}(\psi_{i}; x)dx} \) is bounded by \( O \left( K n(\Delta)^{\frac{1}{2}} C_{j}^{-\frac{1}{2}} M^{-j \beta_{i}} \right)^{n(\Delta)} \), which agrees – up to unessential local factorials \( n(\Delta)^{n(\Delta)/2} \), see §6.1 – with the correct power-counting, but the "petit facteur" \( \lambda_{j}^{\kappa} \) has been entirely used, in contradiction with the general guideline of cluster expansions. So – unless \( C_{j}^{-\frac{1}{2}} \) is small – one must use a different strategy.

Several strategies have been used, depending on the model. Here things are particularly simple because the mass counterterm of highest scale, \( b^{\rho} \approx \lambda^{2} M^{\rho(1 - 4 \alpha)} \), couples with all scales of the field \( \sigma \) and is much better than the term of order \( \lambda^{2} M^{\rho(1 - 4 \alpha)} \) appearing in the right-hand side of \( \text{4.15} \). This is the reason why this counterterm has been set apart from the counterterm and put into the covariance of \( \sigma \) right from the beginning. The supplementary spring factor \( M^{-\frac{1}{2}(1 - 4 \alpha)(\rho - j)} \) per field plays the role of a "petit facteur", except in a small scale interval \( \rho - q, \ldots, \rho \), where \( q \approx \ln(1/\lambda) \) is \( \rho \)-independent. Add now e.g. to the interaction a term of the form \( M^{-(4 \alpha - 1) \rho} \lambda^{\kappa} \sum_{\rho' \leq \rho} \| (T \sigma)^{-\rho'}(x) \|^{2n} \) with \( 2n \geq 4 \), which is homogeneous of degree 0, and totally negligible away from the highest scales because of the evanescent coupling coefficient \( M^{-(4 \alpha - 1) \rho} \leq M^{-(8 \alpha - 1) \rho} < 1 \). If \( \kappa < 2n \), then each \( \sigma \)-field in the interaction is coupled to \( \lambda^{\kappa/2n} \gg \lambda \), which makes it possible to dominate the low-momentum fields for the highest scales.
$\rho - q + 1, \ldots, \rho$. In this sense this term is a (Fourier) boundary term. The term $\delta L_{12}$ in section 5 below plays this rôle. The choice of $4n = 12$ is arbitrary.

5 Definition of the model

As a general rule, we shall denote by $\mu_K$ the Gaussian measure with covariance $K^{-1}$ if $K$ is a positive-definite kernel.

Recall from the heuristics in section 1.2 that one wants to define as interaction $\int \mathcal{L}_{\text{int}}(\phi, \sigma)(x)dx = i\lambda \int (\partial A^+(x)\sigma_+(x) - \partial A^-(x)\sigma_-(x))dx$. Integrating over $\mathbb{R}$ and splitting the fields $\phi_1, \phi_2, \sigma_\pm$ into their different momentum scales yields a momentum conservation rule; as a result, products $\partial \phi_1^j(x)\phi_2^j(x)\sigma_\pm^k(x)$ ($j_1 \leq j_2$) or $\partial \phi_1^j(x)\phi_1^j(x)\sigma_\pm^k(x)$ ($j_2 \leq j_1$) contribute to the action integral only if the two highest scales differ by at most 1 (provided $M \geq 2$, say). The notation $\sum_{j_1,j_2,k}^{\text{adm}}$ means that the sum is restricted to this admissible subset; explicitly, $\sum_{(j_1,j_2,k)\in I(\cdot)}^{\text{adm}} = \sum_{(j_1,j_2,k)\in I \cap I^{\text{adm}},}$, with $I^{\text{adm}} = \{(j_1,j_2,k); j_1 \simeq j_2 \geq k \text{ or } j_2 \simeq j_1 \geq k \text{ or } j \simeq j_1 \geq j_2\}$, where $j \simeq k$ means $j = k$ or $k \pm 1$. This scale restriction is added explicitly by hand because the dressing procedure (which is not translation-invariant) spoils momentum conservation.

Definition 5.1 (propagators)  
(i) Let $d\mu_{\rightarrow \rho}(\phi) = d\mu_{[\xi]^{1+2\alpha}}(\phi^{\rightarrow \rho})$ be the Gaussian measure associated to the field $\phi^{\rightarrow \rho}$ defined in section 2.2.

(ii) Let $d\mu_{\rightarrow \rho}(\sigma_\pm) = d\mu_{[\xi]^{1-4\alpha+4\alpha}}(\sigma_\pm^{\rightarrow \rho})$ be the Gaussian measure associated to the fields $\sigma_\pm^{\rightarrow \rho}$ defined in section 2.2.

The two-by-two matrix coefficient $b^{\rho}$ is called the renormalized mass coefficient of the $\sigma$-field at scale $\rho$. It is equal to the local part at scale $\rho$ of the two-point function of the $\sigma$-field (see precise definition below). Note that $d\mu_{[\xi]^{1-4\alpha+4\alpha}}(\sigma^{\rightarrow \rho}) = Z e^{-\frac{1}{2}b^{\rho}\int (\sigma^{\rightarrow \rho})^2(x)dx}d\mu_{[\xi]^{1-4\alpha}}$, where $Z'$ is a normalization constant. This quadratic term appears with the opposite sign in the interaction Lagrangian $\mathcal{L}_{\text{int}}^{\rightarrow \rho}(\phi, \sigma)(x)$ below, so the interacting measure has really been constructed initially by using a $\sigma$-field with bare covariance $d\mu_{[\xi]^{1-4\alpha}}$, as explained in section 1.

Note that all summations over scales in the interaction start from the scale $j = 1$. Namely, it is really meant to cure ultra-violet divergence and absolutely useless in the infra-red region. However, the $\sigma$-field is not coupled to the infra-red part of the Lévy area, which yields a regularized area which

55
Definition 5.2 (interaction) (i) Let 

\[ b_{\sigma,\sigma} = | \prod (\text{leave details to the reader and concentrate on the region } | \xi | \lessim 1, \text{with } \supp \chi^j \subset [M^{j-1}, M^{j+1}] \text{ for all } j \in \mathbb{Z}. \text{ One may then sum over all scales from } j = \rho \text{ to } -\rho, \text{ without bothering to renormalize at scales } j < 0, \text{ and let } \rho \to \infty. \text{ We leave details to the reader and concentrate on the region } | \xi | \gtrsim 1. \]

In the sequel, expressions such as \( b_{\sigma,\sigma} = b_{+,+}(\sigma_+)^2 + 2b_{+,-}\sigma_+\sigma_- + b_{-,-}(\sigma_-)^2. \)

\( \text{Definition 5.2 (interaction)} \)

(i) Let 

\[ \mathcal{L}_{\text{int}}^{-\rho}(\phi, \sigma)(x) := \mathcal{L}_4^{-\rho}(\phi, \sigma)(x) - \frac{b^\rho}{2} (\sigma^{-\rho}(x))^2, \]

where

\[ \mathcal{L}_4^{-\rho}(\phi, \sigma)(x) := \lambda \left( D_+ \left( \sum_{1 \leq j, j_2, 0 < k < \rho} \partial \phi_1^2(x) \phi_1^2(x) \sigma_+^k(x) \right) - D_- \left( \sum_{1 \leq j, j_2, 0 < k < \rho} \partial \phi_2^2(x) \phi_1^2(x) \sigma_+^k(x) \right) \right) \]

\[ = \lambda \left\{ \sum_{1 \leq j, j_2, 0 < k < \rho} \partial \phi_1^2(x) \phi_2^2(x) \sigma_+^k(x) - \sum_{1 \leq j < j_2, 0 < k < \rho} \partial \phi_2^2(x) \phi_2^2(x) \sigma_+^k(x) \right\} ; \]

(ii) (dressed interaction)

Let

\[ \mathcal{L}^{-\rho}_{\text{int}}(\phi, \sigma; t)(x) := \mathcal{L}_4^{-\rho}(\phi, \sigma)(x) - \frac{b^\rho}{2} (T^t \sigma^{-\rho}(x))^2 + \delta \mathcal{L}_4^{-\rho}(\phi, \sigma; t)(x) + \delta \mathcal{L}_{12}^{-\rho}(\phi, \sigma; t)(x), \]

where:

\[ \mathcal{L}_4^{-\rho}(\phi, \sigma; t)(x) = \mathcal{L}_{4,+}^{-\rho}(\phi, \sigma; t)(x) - \mathcal{L}_{4,-}^{-\rho}(\phi, \sigma; t)(x), \]

\[ \mathcal{L}_{12}^{-\rho} := \lambda D^+(\sum_{1 \leq j, j_2, 0 < k < \rho} \partial (T^{-\rho} \phi_1)^j_1(x)(T^{-\rho} \phi_2)^j_2(x)(T^{-\rho} \sigma_+)^k(x) \]

\[ + \sum_{2 \leq \rho' < \rho} (1 - (t^\rho)^2) \sum_{1 \leq j, j_2, 0 < k < \rho' - 1} \partial (T^{-\rho'} \phi_1)^j_1(x)(T^{-\rho'} \phi_2)^j_2(x)(T^{-\rho'} \sigma_+)^k(x) \]
and similarly for $\mathcal{L}_{4}^{\rho\to\rho'}$ (the Fourier projections $D_\pm$ have been defined in section 1);

$$\delta\mathcal{L}_{4}^{\rho\to\rho'}(\cdot; t)(x) := \frac{1}{2} \sum_{2 \leq \rho' \leq \rho} b^{\rho'-1} \sum_{2 \leq \rho'' \leq \rho'} (1 - (t^\rho_{x})^2) \left((T\sigma)^{\to(\rho'-1)}(x)\right)^2;$$

(5.7)

$$\delta\mathcal{L}_{12}^{\rho\to\rho'}(\cdot; t)(x) := M^{-((2\alpha-1)\lambda)^3} \left\{ \|(T\sigma)^{\to\rho}(x)\|_6 + \sum_{2 \leq \rho' \leq \rho} (1 - (t^\rho_{x})^6) \int \Delta \|(T\sigma)^{\to(\rho'-1)}(x)\|_6 dx \right\}$$

(5.8)

for the Euclidean norm $\|\sigma(x)\| = \sqrt{|\sigma_+(x)|^2 + |\sigma_-(x)|^2}$.

By construction $\mathcal{L}_{\text{int}}^{\rightarrow\rho}(\phi, \sigma) = \mathcal{L}_{\text{int}}^{\rightarrow\rho}(\phi, \sigma; t = 1)$.

(iii) Let

$$Z_{V}^{\rho\to\rho'}(\lambda) := \int e^{-\int_{V} \mathcal{L}_{\text{int}}^{\rightarrow\rho}(\phi, \sigma; t=1)(x) dx} d\mu^{\to\rho}(\phi) d\mu^{\to\rho}(\sigma),$$

(5.9)

and, more generally,

$$Z_{V}^{\rho\to\rho'}(\lambda; (\phi^{h})_{h<\rho'}, (\sigma^{h})_{h<\rho'}) = \int e^{-\int_{V} \mathcal{L}_{\text{int}}^{\rightarrow\rho}(\phi, \sigma; t=1)(x) dx} d\mu^{\to\rho}(\phi) d\mu^{\to\rho}(\sigma)$$

(5.10)

which is a function of the low-momentum components of the fields, considered as external fields.

(iv) (definition of the renormalized mass coefficient $b^{\rho'}$) Fix $b^{\rho'}$ by requiring that

$$0 = \int dy \frac{\partial^2}{\partial\sigma^{\rightarrow(\rho'-1)}(y) \partial\sigma^{\rightarrow(\rho'-1)}(y)} \left. Z_{V}^{\rho\to\rho'}(\lambda; (\phi^{h})_{h<\rho'}, (\sigma^{h})_{h<\rho'}) \right|_{\sigma^{\rightarrow(\rho'-1)} = \phi^{\rightarrow(\rho'-1)} = 0},$$

(5.11)

Note that $\delta\mathcal{L}_{4}^{\rho\to\rho'}(\cdot; t)(x)$ may be seen as a dressed interaction as in Definition 3.8 if one sets $b^{\rho'} := 0$ (the counterterm of scale $\rho'$ has been treated separately). The interaction before dressing therefore vanishes, which is
coherent with the fact that it has not been put into the model from the beginning, but built inductively to compensate local parts of diverging graphs.

Although the actual form of $L^\lambda_4 \rightarrow \rho$ looks complicated, it is very close to the interaction term $i\lambda (\partial A^+ (x)\sigma_+ (x) - \partial A^- (x)\sigma_- (x))$ of section 1, see eq. (1.12), (1.13), (1.17).

6 Bounds

6.1 Gaussian bounds

This paragraph is the backbone of the section, since it provides a means (i) to bound the sum all possible Wick pairings of all possible $G$-monomial associated to a given polymer; (ii) to bound the sum over all possible polymers containing some fixed interval $\Delta$ at its lowest scale. The general idea (as explained in the introduction) is that a polymer is connected either by horizontal cluster links which are polynomially decreasing at large distances, or by vertical inclusion links which create spring factors. The computations below for a polymer $P$ (see §6.1.2 and 6.1.3) give in the end a bound which is of order $\prod_v \lambda^\kappa \prod_\Delta M^{-\varepsilon}$ for some positive exponents $\kappa, \varepsilon$, where the product ranges over all vertices $v$ and intervals $\Delta$ of $P$. This is the general principle of the bounds for cluster expansions. Both terms $\lambda^\kappa$ and $M^{-\varepsilon}$ are called a “petit facteur par carré” (small factor per interval, in French). It does not include combinatorial factors (see §6.1.4), dominated averaged low-momentum fields (see §6.2), and possibly some other terms (see first step in the Proof of Theorem 6.2 in §6.3), but all these are proved to give for each choice of cluster expansion and $G$-polynomial a supplementary factor of the type $\prod_\Delta (1 + O(\lambda^\kappa'))$ for some exponent $\kappa' > 0$, which is eaten up by the factor $\prod_\Delta M^{-\varepsilon}$.

Since all computations are Gaussian in this paragraph, we shall take the liberty to write $\langle \cdots \rangle$ instead of $E[\cdots]$, without any risk of confusion.

6.1.1 Wick’s formula and applications

We first recall the classical Wick formula.

**Proposition 6.1 (Wick’s formula)** Let $(X_1, \ldots, X_{2N})$ be a (centered) Gaussian vector. Pair the indices $1, \ldots, 2N$; the result may be represented as a graph $F$ with $n$ connected components, linking the vertices $1, \ldots, 2N$ two by two.
two. As in Definition 3.5 we use the pair notation \( \ell = \{i_\ell,i'_\ell\} \) for links. Then

\[
\langle X_1 \ldots X_{2N} \rangle = \sum_{\text{pairing of } \{1,\ldots,2N\}} \prod_{\ell \in F} \langle X_{i_\ell} X_{i'_\ell} \rangle. \tag{6.1}
\]

**Proof.** see e.g. [10], §5.1.2. \( \Box \)

**Corollary 6.2 (simple Wick bound)** Let \((X_1,\ldots,X_{2N})\) be a Gaussian vector. Then, for every \(K > 0\),

\[
|\langle X_1 \ldots X_{2N} \rangle| \leq K^{-N} \prod_{i=1}^{2N-1} \left[ 1 + K \sum_{j > i} |\langle X_i X_j \rangle| \right]. \tag{6.2}
\]

In particular,

\[
|\langle X_1 \ldots X_{2N} \rangle| \leq \prod_{i=1}^{2N} \left[ 1 + \sum_{j \neq i} |\langle X_i X_j \rangle| \right]. \tag{6.3}
\]

**Proof.** Expand the right-hand side and use Wick’s formula to get eq. (6.2) with \(K = 1\). The bound with \(K \neq 1\) may be obtained from the previous one by a simple rescaling \(X_i \rightarrow \sqrt{K}X_i, i = 1,\ldots,2N\). \( \Box \)

The above bound eq. (6.2) depends on the ordering of the variables \(X_1,\ldots,X_{2N}\), although \(\langle X_1 \ldots X_{2N} \rangle\) doesn’t, of course. The idea conveyed by this bound is that it may be important to choose the right order. Similarly, eq. (6.2) is clearly optimal when the factors \(K \sum_{j > i} |\langle X_i X_j \rangle|, 1 \leq i \leq 2N - 1\), are of order 1.

However this bound is too simple to apply in most cases, and we shall need refined versions of it using the spatial structure of the Gaussian variables. The following lemmas, for the reasons we have just explained, are to be used after a suitable rescaling.

**Corollary 6.3 (Wick bound with spatial structure)** 1. (single-scale bound) Let \((X_\Delta,n)\) be a Gaussian vector indexed by \(M\)-adic intervals of scale \(j\). Denote by \(I = \{(\Delta,n) ; \Delta \in \mathbb{D}^j, 1 \leq n \leq N(\Delta)\}\) the total set of indices. Call connecting pairing a partial pairing \(F\) of the indices \((\Delta,n)\) such that its spatial projection \(\bar{F}\) with vertices \(\{\Delta \in \mathbb{D}^j ; \exists n \leq N_{\text{max}}(\Delta) \mid (\Delta,n) \in \bar{F}\}\) and links \(\{(\Delta,\Delta') \in \)
\(\mathbb{D}^j \times \mathbb{D}^j; \exists n \leq N_{\text{max}}(\Delta), n' \leq N_{\text{max}}(\Delta') \mid (\Delta, n) \sim_F (\Delta', n')\) is connected. Fix some \(M\)-adic interval \(\Delta_1 \in \mathbb{D}^j\). Then

\[
\sum_{\mathcal{F}\text{ connecting pairing of } I, |\mathcal{F}| = 2N, \Delta_1 \in \mathcal{F}} \prod_{\ell \in \mathcal{F}} |\langle X(\Delta_\ell, n_\ell)X(\Delta'_\ell, n'_\ell)\rangle| \\
\leq \left(1 + \sup_{\Delta \in \mathbb{D}^j} \left(\sum_{n=1}^{N_{\text{max}}(\Delta)} \sum_{(\Delta', n') \in I, (\Delta', n') \neq (\Delta, n)} |\langle X(\Delta, n)X(\Delta', n')\rangle|\right)\right)^{3N}.
\]

(6.4)

2. \text{(single-scale bound, improved version)}

More generally, assume \(N_{\text{max}}(\Delta) = \infty\) and \(K : (\mathbb{D}^j \times \{1, \ldots, d\}) \times (\mathbb{D}^j \times \{1, \ldots, d\}) \rightarrow \mathbb{R}_+\) is some kernel, copied an infinite number of times, so that \(K((\Delta, pd + i), (\Delta', p'd + i')) = K((\Delta, i), (\Delta', i'))\), with \(p, p' = 1, 2, \ldots\) (see Remark below). Let \(I = \mathbb{D}^j \times \{1, 2, \ldots\}\). Connecting pairings of \(I\) must be understood modulo the pair identifications \(((\Delta, pd + i), (\Delta', p'd + i')) \sim ((\Delta, i), (\Delta', i'))\).

Then, for any \(\gamma \geq 1\), letting \(N(\Delta) := \#\{(\Delta, n); (\Delta, n) \in \mathcal{F}\} - \) to be interpreted as the number of fields lying in a given interval \(\Delta\),

\[
\sum_{\mathcal{F}\text{ connecting pairing of } I, |\mathcal{F}| = 2N, \Delta_1 \in \mathcal{F}} \prod_{\ell \in \mathcal{F}} (N(\Delta_\ell)N(\Delta'_\ell))^{-\gamma} K((\Delta_\ell, i_\ell), (\Delta'_\ell, i'_\ell)) \\
\leq \left(1 + \sup_{\Delta \in \mathbb{D}^j} \sum_{\Delta' \in \mathbb{D}^j} \sum_{i, i' = 1}^d K((\Delta, i), (\Delta', i'))\right)^{3N}.
\]

(6.5)

3. \text{(multi-scale bound)}

Let \(K : (\mathbb{D}^{j\rightarrow} \times \{1, \ldots, d\}) \times (\mathbb{D}^{j\rightarrow} \times \{1, \ldots, d\}) \rightarrow \mathbb{R}_+\) be some kernel indexed by \(M\)-adic intervals of scale \(\geq j\), copied an infinite number of times as in 2. We denote once again by \(I\) the total set of indices. Let \(I^k := \{(\Delta, n) \in I; \Delta \in \mathbb{D}^k\}, k \geq j\) and \(I^{k\rightarrow} := \cup_{j'=j}^{k} I^{j\rightarrow}\), \(k^j \geq j\). Fix a certain number of (non necessarily distinct) \(M\)-adic intervals for each scale \(k = j, j + 1, \ldots, \rho, \) say, \(\Delta^k_1, \ldots, \Delta^k_c\) \((k \geq j)\), with \(c_k \geq 0\) \((k > j)\) and \(c_j = 1\); write for short \(\Delta^{j\rightarrow} = \{(\Delta^k)_{k\geq j, 1 \leq c \leq c_k} \subset \mathbb{D}^{j\rightarrow}\). Let \(\mathcal{F}^{j\rightarrow}(\Delta^{j\rightarrow})\) be the set of multi-scale cluster forests \(\mathbb{F}^{j\rightarrow}\) (called: \(\Delta^{j\rightarrow}\)-connected multiscale cluster forests) such that, for each \(j^j \geq j\), each vertex of \(\mathbb{F}^{j\rightarrow}\) is connected by horizontal cluster links or inclusion links.
to some (possibly many) of the selected intervals \((\Delta_k')_{k' \geq 1, 1 \leq c \leq c_{k'}}\), and the intervals \(\Delta_1', \ldots, \Delta_{c_{j'}}'\) are not connected within \(\mathbb{F}_j \rightarrow\) by horizontal cluster nor inclusion links. (In other words, each selected interval \(\Delta_j', \ldots, \Delta_{c_{j'}}'\) lies within a different horizontal cluster and inclusion connected component of \(\mathbb{F}_j \rightarrow\), and these \(c_{j'}\) connected components exhaust the set of horizontal cluster and inclusion connected components of \(\mathbb{F}_j \rightarrow\) which contain at least one \(M\)-adic interval of scale \(j'\)). Call \(\Delta_j \rightarrow\)-connecting pairing a partial pairing \(\mathbb{F}\) of the indices \((\Delta, n)\) such that its spatial projection \(\overline{\mathbb{F}}\) has the same set of vertices and links as some \(\Delta_j \rightarrow\)-connected forest, plus possibly some supplementary links, possibly reducing the number of connected components. Then:

\[
\sum_{\mathbb{F}\Delta_j \rightarrow\text{-connecting pairing of } I, |\mathbb{F}| = 2^n} \prod_{\ell \in \mathbb{F}} (N(\Delta_\ell) N(\Delta_\ell'))^{\gamma} K((\Delta_\ell, i_\ell), (\Delta_\ell', i'_\ell))
\]

\[
\leq \left( 1 + \max_{k' \geq j} \sup_{\Delta \in \mathbb{D}_j} \sum_{\Delta' \in \mathbb{D}_j \rightarrow k'} \sum_{i, i' = 1}^d K((\Delta_k', i'), (\Delta_{k'}', i''_{k'})) \right)^{3N}
\]

\[
\text{where } \Delta_{k'}' \text{ is the unique interval of scale } k' \text{ such that } \Delta_{k'}' \supset \Delta_k'.
\]

**Remark.** When \(N_{\max}(\Delta) = \infty\), which is due to the fact that the total number of fields in a given \(M\)-adic interval, \(N(\Delta)\), may be of order \(n(\Delta)\), hence unbounded, the bound in 1. is infinite. In practice, a cluster expansion generates – thanks to the polynomial decrease in the distance of the covariance of multi-scale Gaussian fields – extremely small factors per interval when \(n(\Delta)\) is large. The idea is then to bound \(|\langle \psi_j(x) \psi_j'(x') \rangle|, x \in \Delta, x' \in \Delta'\) by \(\frac{1}{(1 + d (\Delta, \Delta'))^r} K((\Delta, i), (\Delta', i'))\), where some of the polynomial decrease in the distance has been retained in the kernel \(K\) (see §6.1.2).

**Proof.**

1. Consider first the left-hand side of (6.4). Consider a connecting pairing \(\mathbb{F}\) such that \(|\mathbb{F}| = 2^n\) and containing the \(M\)-adic interval \(\Delta_1\), and a spanning tree \(\overline{T}\) of \(\overline{\mathbb{F}}\) containing \(\Delta_1\). Associate to \(\mathbb{F}\) the following sequence of links and of factors 1:

- consider all the pairings of the indices \((\Delta_1, n)\), \(1 \leq n \leq N_{\max}(\Delta_1)\) among themselves and with indices \((\Delta_1', n')\), \(\Delta_1' \neq \Delta_1\); say (in some
arbitrary order), \((\Delta_1, n_1)\) pairs with \((\Delta_1', n_1')\), \ldots, \((\Delta_1, n_{N_1-1})\) pairs with \((\Delta_{N_1-1}', n_{N_1-1}')\). Insert after these \(N_1-1\) links a factor 1, signifying that all paired Gaussian variables lying in the interval \(\Delta_1\) have been exhausted;

– continue to explore new vertices of \(\bar{F}\) by going along the branches of \(\bar{T}\). Always insert a factor 1 after all the pairings of the Gaussian variables lying in a given interval have been exhausted.

Since \(\bar{T}\) is connected, all \(M\)-adic intervals in \(\bar{F}\) and all indices in \(F\) will eventually have been explored. The number of factors is \(\ell(\bar{F}) + |\bar{F}| \leq N + |F| = 3N\), to the completed by the required number of factors 1 so that there are exactly 3\(N\) factors.

Consider now the right-hand side. Let

\[
K_\Delta := \sum_{1 \leq n \leq N_{\text{max}}(\Delta)} \sum_{\Delta' \neq \Delta, n} |\langle X_{(\Delta, n)} X_{(\Delta', n')} \rangle| \tag{6.7}
\]

and \(K_\emptyset = 1\), and expand \(K^{3N} := (K_\emptyset + \sup_{\Delta \in \mathbb{D}, \Delta} K_\Delta)^{3N}\). One gets

\[
K^{3N} = \sum_{p} \sum_{N_1 < \ldots < N_p} (\sup_{\Delta} K_\Delta)^{N_{1}-1} \cdot 1 \cdot (\sup_{\Delta} K_\Delta)^{N_{2}-N_{1}-1} \cdot 1 \ldots 1 \cdot (\sup_{\Delta} K_\Delta)^{N_{p-1}-N_{p-1}-1} \cdot 1 \ldots 1. \tag{6.8}
\]

Replace the first sequence of \(N_1-1\) factors by \(K_\Delta^{N_{1}-1} \leq (\sup_{\Delta} K_\Delta)^{N_{1}-1}\) and expand them, which encodes in particular all possible pairings of the Gaussian variables lying in \(\Delta_1\). Consider now an interval \(\Delta_2 \neq \Delta_1\) linked by \(\bar{T}\) to \(\Delta_1\), and replace the second sequence of factors by \(K_\Delta^{N_{2}-N_{1}-1}\), and so on. Thus one has encoded all possible connecting pairings containing the interval \(\Delta_1\).

2. Considering as in the previous case all the pairings of the indices \((\Delta_1, n), n = 1, 2, \ldots\) for a given pairing \(\bar{F}\), one gets the factor

\[
K_{\Delta_1} := \sum \left( (N(\Delta_1)N(\Delta'_1))^{-\gamma} K((\Delta_1, i_1), (\Delta'_1, i'_1)) \right) \ldots \left( (N(\Delta_1)N(\Delta'_{N_1-1}))^{-\gamma} K((\Delta_1, i_{N_1-1}), (\Delta'_{N_1-1}, i'_{N_1-1})) \right), \tag{6.9}
\]

where the sum ranges over all possible \((N_1-1)-\text{uple couplings}\) \(((\Delta_1, i), (\Delta'_1, i'))\) with multiplicities (note that \(\frac{N(\Delta_1)}{2} \leq N_1 - 1 \leq N(\Delta_1)\), depending on the number of couplings of fields inside \(\Delta_1\)). Then
Apart from this slight difference, the exploration procedure is the same.

3. Choose a spanning tree of $\overline{F} \cap D^\rho$, complete it into a spanning tree of $\overline{F} \cap D^{(\rho-1)}$, and so on. As in 1., explore the horizontal cluster connected components of scale $\rho$ starting from the selected intervals $\Delta^\rho$, $1 \leq c \leq c^\rho$, then the connected components of scale $\rho - 1$, and so on, down to scale $j$. The only difference is that two different horizontal cluster connected components of $\overline{F}$ of the same scale $j'$ may be connected from above by inclusion links and horizontal cluster links of higher scale; in this case, this procedure may not explore all vertices of $\overline{F}$. Fortunately, the bound in eq. (6.6) gives the possibility, starting from some interval $\Delta^k \in \mathbb{D}^k$, to go on to explore all the Gaussian variables located below $\Delta^k$, i.e. in some interval $\Delta^{k'} \supset \Delta^k$ with $k' < k$.

6.1.2 Gaussian bounds for cluster expansions

We assume here that $\mathcal{L}_{int}$ is just renormalizable, so that (assuming just for simplicity of notations that its coefficients are scale-independent) $\mathcal{L}_{int} = K_1 \lambda^{\kappa_1} \psi_{I_1} + \ldots + K_p \lambda^{\kappa_p} \psi_{I_p}$, where $\kappa_1, \ldots, \kappa_p > 0$ and $\sum_{i \in I_1} \beta_i = \ldots = \sum_{i \in I_p} \beta_i = -D$. Each term $\psi_{I_1}, \ldots, \psi_{I_p}$ is called a vertex by reference to the Feynman diagram representation (see section 4). Let us recall briefly that the $G$-monomials are produced:

- either by horizontal cluster expansions; if $i_\ell \in I_q$, $\ell$ being a link at scale $j$, then $\frac{\delta}{\delta \psi^q_{I_q}(x_\ell)} e^{-\lambda^\kappa_q \int \psi_{I_q}(x)dx}$ produces $\lambda^{\kappa_q} \psi_{I_q \setminus \{i_\ell\}}(x_\ell)$. On the other hand, $\frac{\delta}{\delta \psi^q_{I_q}(x_\ell)}$ may derivate the low-momentum components of monomials produced at scales $\geq j + 1$, which lowers the degree of $G$;

- or by $t$-derivations acting on $e^{-\int \mathcal{L}_{int}(x)dx}$, yielding (up to $t$-coefficients) some (scale components) of the $\lambda^{\kappa_q} \psi_{I_q}(x_\Delta)$, $1 \leq q \leq p$, integrated over

\[
K_{\Delta_1} \leq \left( \sum_{\Delta' \in D, i, i' \in 1 \leq d} K((\Delta_1, i), (\Delta', i')) \sum_{n=1}^{N(\Delta_1)} \sum_{n'=1}^{N(\Delta')} (N(\Delta_1)N(\Delta'))^{-\gamma} \right)^{N_1-1} \leq \left( \sum_{\Delta' \in D} \sum_{i, i' = 1}^d K((\Delta_1, i), (\Delta', i')) \right)^{N_1-1}. \tag{6.10}
\]
some $M$-adic interval $\Delta$. Again, $t$-derivations may derivgate the monomials produced at scales $\geq j$, which does not change the degree of $G$.

The above products of fields must now be split according to their scale decomposition. Thus one obtains a certain number $v$ of vertices split into different scales.

We suggest the following notations in order to avoid the proliferation of indices. Make a list $(\psi_1, \ldots, \psi_d)$, $d = |I_1| + \ldots + |I_p|$, of all the fields involved in the interaction, possibly with repetitions. Thus the cluster expansion at scale $j$ generates at the same scale $\lambda^{q_i} \psi_{I_k}(x_{i_k})$, where $I_k \subset I_q \setminus \{i_k\} \subset \{1, \ldots, d\} \setminus \{i_{\ell}\}$ for some $q \leq p$; on the other hand, each $t$-derivation in an interval $\Delta \subset \mathbb{D}^j$ generates (up to $t$-coefficients) some $\lambda^{q_p} \psi_{I_\Delta}(x_{\Delta})$, $I_\Delta \subset I_q$, or (with an extra index $\tau_\Delta$ for the order of derivation) $(\lambda^{q_p} \psi_{I_{\Delta, \tau}}(x_{\Delta, \tau}))_{\tau_1, \ldots, \tau_\Delta}$.

But other field components of scale $j$, lying in some fixed interval $\Delta^j \subset \mathbb{D}^j$, are produced, either at an earlier stage $k > j$, in the form of a low-momentum field, $\psi^{(i)}_{\Delta}(x)$ or $\delta_i^{(j)} \psi^{(j)}(x)$ (secondary field) with $x \in \Delta^k$, $\Delta^k \subset \mathbb{D}^j$, $\Delta^k \subset \Delta^j$, or at a later stage $h < j$, in the form of a high-momentum field, $\text{Res}^{h}_{\Delta^j} \psi^{(j)}(x)$, $x \in \Delta^h$, where $\Delta^h \subset \mathbb{D}^h$, $\Delta^h \subset \Delta^j$.

The general principle of bounds for cluster expansions in quantum field theory (as explained at the beginning of §6.1) is to (1) use the polynomial decrease in the distance of the covariance of the field components; (2) find out a “petit facteur par carré” (small factor per cube, or rather per interval in one dimension). This means essentially the following: chose some possibly derivated interaction term $\lambda^{q_p} \psi_{I_k}(x_{i_k}) \in \Delta^j$ or $\lambda^{q_p} \psi_{I_q}(x_{\Delta^j})$ coming from a vertex at scale $j$; the fields $\psi_{i_k}^{(j)}$, $i_k \in I_q$ scale like $M^{-\beta_i j}$, and the integration over the interval $\Delta^j \subset \mathbb{D}^j$ produces a factor $M^{-j}$ (or $M^{-D_j}$ in general). As for the cluster expansion at scale $j$, it has produced a factor $C^j_{\psi}(i_{i_{\ell}}, x_{i_{\ell}})$ which scales like $M^{-\beta_i j}$, times the same quantity with a prime. Supposing one chooses a splitting of the vertex such that all fields are of scale $j$, then the product of these factors is $\lambda^{q_i} M^{-D_j} M^{-\sum_{i_k \in I_q} \beta_i} = \lambda^{q_i} \ll 1$, which is the “petit facteur”. Unfortunately the splittings of the vertex produce much more complicated situations; however, the guideline is to compare the scalings of the high-momentum fields (rewritten as a sum of restricted fields) and of the low-momentum fields (possibly rewritten as secondary fields, modulo averaged fields) with the scaling they would produce if they were of scale $j$. The high-, resp. low-momentum rescaled fields, write

\[
\psi^{(k)}_i(x) = M^{-\beta_i(k-j)} \psi^{(k, \text{rescaled})}_i(x) \quad (k > j), \quad \text{resp. } \psi^{(h)}_i(x) = M^{\beta_i(h-j)} \psi^{(h, \text{rescaled})}_i(x) \quad (h < j),
\]

(6.11)
yielding positive or negative rescaling spring-factors (depending also on the sign of $\beta$), see after Corollary 2.7. The factor $\lambda^{\kappa_0}$ is split into the different scales of the vertex, so that each field $\psi_i(x)$, $i = i_1, \ldots, i_q$ is accompanied by a small factor $\leq \lambda^{\kappa_0/|I_0|}$ for some $\kappa_0 > 0$.

Averaged fields must be treated apart and account for the so-called domination problem; bounding them may require part of the small factor $\lambda^{\kappa_0}$, so that, generally speaking, the “petit facteur” is of order $\lambda$, for some $\kappa > 0$ but small.

Let us first consider the following single scale situation, throwing away all low- or high-momentum fields for the time being.

**Lemma 6.4** Let $\psi = (\psi_1(x), \ldots, \psi_d(x))$ be a Gaussian field with $d$ components such that

$$|C^{\delta}_{\psi}(i, x; i', x')| = |\langle \psi_i^\delta(x) \psi_i^{\delta'}(x') \rangle| \leq K_r \frac{M^{-j(\beta_i + \beta_i')}}{(1 + M|x - x'|)^r} \quad (6.12)$$

for every $r \geq 0$, with some constant $K_r$ depending only on $r$; these bounds hold in particular if $\psi_1, \ldots, \psi_d \subset \{(\partial^n \tilde{\psi}_1)_{n \geq 0}, \ldots, (\partial^n \tilde{\psi}_d)_{n \geq 0}\}$ are derivatives of some independent multiscale Gaussian fields $\tilde{\psi}_1, \ldots, \tilde{\psi}_d$.

Consider a horizontal cluster forest $F^j \in \mathcal{F}^j$ of scale $j$, and associated cluster points $x_\ell, x'_\ell$, $\ell \in L(F^j)$, $x_\ell \in \Delta_\ell$, $x'_\ell \in \Delta'_\ell$. Choose:

- for each link $\ell \in L(F^j)$, a subset $I_\ell$ of $\{1, \ldots, d\} \setminus \{i_\ell\}$ and a subset $I'_\ell$ of $\{1, \ldots, d\} \setminus \{i'_\ell\}$;
- for each $M$-adic interval $\Delta \in \mathcal{F}^j$, $\tau_\Delta \leq N_{\text{ext}, \text{max}} + O(n(\Delta))$ subsets $(I_{\Delta, \tau})_{\tau = 1, \ldots, \tau_\Delta}$ of $\{1, \ldots, d\}$, and additional integration points $(x_{\Delta, \tau})_{\tau = 1, \ldots, \tau_\Delta}$ in $\Delta$.

Such a choice defines uniquely a monomial $G^{i,j} = G^{i,j}(F^j; (I_\ell), (I'_\ell); (I_{\Delta, \tau}), (x_{\Delta, \tau}))$ in the fields $\psi^\ell_i$, $i = 1, \ldots, d$ taken at the cluster points $(x_\ell)$, $(x'_\ell)$ and the $t$-derivation points $(x_{\Delta, \tau})$, namely,

$$G^{i,j} := \lambda^{\text{cov}(F^j; G^{i,j})} \left[ \prod_{\ell \in L(F^j)} \psi^\ell_{I_\ell}(x_\ell) \psi^\ell_{I'_\ell}(x'_\ell) \right] \cdot \left[ \prod_{\Delta \in \mathcal{F}^j} \prod_{\tau = 1}^{\tau_\Delta} \psi^\ell_{I_{\Delta, \tau}}(x_{\Delta, \tau}) \right] \quad (6.13)$$

where $v(F^j; G^{i,j}) := 2L(F^j) + \sum_{\Delta \in \mathcal{F}^j} \tau_\Delta$ is the total number of vertices obtained from the horizontal cluster and the $t$-derivations is (two per horizontal cluster link, one per $t$-derivation acting on the exponential of the interaction).

Denote by $N^i_j(G^{i,j}; \Delta)$, $i = 1, \ldots, d$ the number of fields $\psi^i_j(x)$, $x \in \Delta$ occurring in $G^{i,j}$ if $\Delta \in \mathcal{F}^j$, summing up to $N^j(G^{i,j}; \Delta) := \sum_{i=1}^d N^i_j(G^{i,j}; \Delta)$,
and by $N^j_i(G^{j,j}) := \sum_{\Delta \in F^j_i} N^j_i(G^{j,j}; \Delta)$ the total number of fields $\psi^j_i$ occurring in $G^{j,j}$. Similarly, we denote by $N^j_i(F^j_i)$ the number of half-propagators of type $i$ in $F^j_i$. Note that $N^j_i(G^{j,j}; \Delta) = O(n(\Delta))$, see (6.13).

1. Let

$$I^j_{\text{Gaussian}}([F^j_i; G^{j,j}]) := \int d\mu(\psi^j) \prod_{\ell \in L(F^j)} C^j_{\psi}(i_\ell; x_\ell; i'_\ell; x'_\ell) \cdot G^{j,j}. \quad (6.14)$$

Then

$$|I^j_{\text{Gaussian}}([F^j_i; G^{j,j}])| \leq K |F^j_i| \lambda \kappa_0 v(F^j_i) \prod_{i=1}^d M^{-j\beta_i(N^j_i(G^{j,j})+N^j_i(F^j))}. \quad (6.15)$$

2. Fix the total number of vertices, $v = v(F^j_i; G^{j,j})$, and fix one $M$-adic interval $\Delta^j_1 \in D^j$. Let $F^j_{\Delta^j_1} \subset F^j$ be the subset of connected horizontal cluster forests of scale $j$ containing $\Delta^j_1$. Consider the rescaled quantity

$$I^j_{\text{Gaussian}, \text{rescaled}}(v; G^{j,j}; \Delta^j_1) := \frac{1}{d} \prod_{i=1}^d \left[ M^{j\beta_i(N^j_i(G^{j,j})+N^j_i(F^j))} I^j_{\text{Gaussian}}([F^j_i; G^{j,j}]) \right]$$

and:

- sum over all $F^j \in F^j_{\Delta^j_1}$, $(I_\ell) \subset \{1, \ldots, d\} \setminus \{i_\ell\}$, $(I'_\ell) \subset \{1, \ldots, d\} \setminus \{i'_\ell\}$, $(I_{\Delta, \tau}) \subset \{1, \ldots, d\}$;
- maximize for $(x_\ell), (x'_\ell), (x_{\Delta, \tau})$, each one ranging over its associated interval in $D^j$.

Call $I^j_{\text{Gaussian}, \text{rescaled}}(v; \Delta^j_1)$ the result. Then

$$I^j_{\text{Gaussian}, \text{rescaled}}(v; \Delta^j_1) \leq (K \lambda \kappa_0)^v. \quad (6.17)$$

**Proof.**

1. The integral $I^j_{\text{Gaussian}}([F^j_i; G^{j,j}])$ may be evaluated by using Wick’s lemma. Each choice of contractions leads, using the numerator in the right-hand side of eq. (6.12), to some term with the correct homogeneity factor, $\lambda^{\kappa_0 v(F^j_i; G^{j,j})} \prod_{i=1}^d M^{-j\beta_i(N^j_i(G^{j,j})+N^j_i(F^j))}$. Consider the rescaled fields $\psi^j_{i, \text{rescaled}} := M^{j\beta_i} \psi^j_i$. For reasons to be discussed presently, we
shall apply Corollary 6.2 to the rescaled fields \( n(\Delta x) - \gamma \psi_j^i : \) for some power \( \gamma \geq 1. \)

The possibility to introduce this supplementary scaling factor \( n(\Delta)^{-\gamma} \) comes from the following argument. Split \( \frac{1}{(1 + M'(x-x'))^r} \) into \( \frac{1}{(1 + M'(x-x'))^r} \cdot \), with \( r = r' + r'', \) \( r', r'' > 0. \) The product of propagators \( \prod_{t \in L(F)} C_p(i_t, x_i; i_t', x_i') \) contributes, see denominator in the right-hand side of eq. (6.12), a convergence factor \( K_{r''}^{L(F)} \).

\[
\prod_{\Delta \in F} \prod_{\Delta' \sim \Delta} \left( d^j(\Delta, \Delta') \right)^{-r''/2}, \text{ for some constant } K_{r''} \text{ depending only on } r''.
\]

Since the number of intervals \( \Delta' \in F \) such that \( d^j(\Delta, \Delta') \leq n(\Delta)/4 \) is \( \leq n(\Delta)/2 \), this means that at least half of the intervals \( \Delta' \sim \Delta \) are at a \( d^j \)-distance \( > n(\Delta)/4 \) from \( \Delta \), so that \( 16 \)

\[
\prod_{\Delta' \sim \Delta} \left( d^j(\Delta, \Delta') \right)^{-r''/2} \leq \left( (n(\Delta)/4)^{-r''/2} \right)^{n(\Delta)/2} = K^{n(\Delta)} n(\Delta) - K' r'' n(\Delta).
\]

(6.18)

On the other hand, taking into account the \( n(\Delta)^{\gamma} \), resp. \( n(\Delta')^{\gamma} \) factors separated from the rescaled fields in cluster intervals contributes \( \prod_{\Delta \in F} n(\Delta)^{\gamma N_j(G^{j;j}; \Delta)} \leq \prod_{\Delta \in F} n(\Delta)^{O(n(\Delta))}, \) a product of so-called \textit{local factorials}, which is compensated by the above convergence factor as soon as \( r'' \) is chosen large enough.

We may now apply Corollary 6.2, eq. (6.3) to the rescaled fields, which yields, using once again \( N_j(G^{j;j}; \Delta) = O(n(\Delta)) \),

\[
I_{G_j}^{j, \text{rescaled}}(\mathbb{R}^j; G^{j;j}) \leq K \sum_{\Delta \in F} n(\Delta)^{\lambda_{n(\Delta); G^{j;j}}} \prod_{\Delta' \in F} \left[ 1 + (n(\Delta))^{-\gamma} \sum_{\Delta' \in F} (n(\Delta'))^{-\gamma} \frac{O(n(\Delta'))}{(1 + d^j(\Delta, \Delta'))^{r''}} \right]^{O(n(\Delta))}.
\]

(6.19)

Now the sum \( \sum_{\Delta' \in F} \frac{1}{(1 + d^j(\Delta, \Delta'))^{r''}} \) converges as soon as \( r' > D. \) Hence each term between square brackets is bounded by a constant. Since \( \sum_{\Delta \in F} n(\Delta) = 2|F| - 2 = O(|F|) \), one gets:

\[
I_{G_j}^{j, \text{rescaled}} \leq K |F|^{\lambda_{n(\Delta); G^{j;j}}},
\]

(6.20)

\[\text{In } D \text{ dimensions, } \frac{n(\Delta)}{4} \text{ becomes } Kn(\Delta)^{1/D} \text{ and eq. (6.15) holds, with different constants.}\]
2. Associate to a connected forest $F^j \in \mathcal{F}^j_{\Delta_i}$ and a monomial $G^{j,j}$ as in (6.13) its Wick expansion, represented as a sum over a set of connecting pairings of $D^j$ as in Corollary 6.3 (1), except that $N(\Delta) \leq d(n(\Delta) + \tau_\Delta) + n(\Delta) - \text{the number of fields and half-propagators in the } M\text{-adic interval } \Delta - \text{depends on } F^j \text{ and } G^{j,j}$, and is unbounded since $n(\Delta)$ may be arbitrarily large. Hence (to get a finite bound for our sum of Gaussian integrals) we shall use the $F^j$-dependent rescaling by $n(\Delta)^{-\gamma} = O(N(\Delta)^{-\gamma})$ of the fields defined in 1., at the price of the extension of the exploration procedure described in the proof of Corollary 6.3 (2). Note however that the mapping $(F^j, G^{j,j}) \mapsto \text{connecting pairing is not one-to-one, since a link of the resulting pairing } F \text{ may come either from the links of } F^j \text{ or from the pairings of } G^{j,j}$; this contributes at most a factor 2 per pairing, hence at most $2^{dv/2}$.

Now, the factor $\sum_{\Delta' \in D^j} \sum_{i,i'=1}^d K((\Delta, i), (\Delta', i'))$ of Corollary 6.3 (2) associated to the rescaled fields defined in 1. is bounded up to a constant by $\sum_{\Delta' \in D^j} (1+d(\Delta, \Delta'))^{-r} < \infty$, hence the result.

\[\Box\]

The above arguments extend easily to single scale Mayer trees of polymers of scale $j$. The new rules are:

(i) there may be some undetermined number of copies of each interval $\Delta^j_i$, each with a different color;

(ii) fields in intervals with different colors are uncorrelated;

(iii) each cluster forest of a given color is connected; one of them (the red one, say) contains a fixed interval, $\Delta^j_1$;

(iv) the different cluster forests are connected by Mayer links. These define a tree structure on the set of colors, and imply for each link between 2 colors, say, red and blue, an overlap between one red interval and one blue interval (chosen at random if they have several overlaps).

The proof of Lemma 6.4 (2) is the same as before, except that the exploration procedure must now take into account Mayer links. Let $n_{\text{Mayer}}(P')$ be the coordination number of a (red, say) polymer $P'$ in the Mayer tree. The overlap constraint between $P'$ and its neighbours $P_1, \ldots, P_{n_{\text{Mayer}}(P')-1}$ in the tree splits into multiple overlaps of order $n_i = n_1, \ldots, n_c \geq 1$ between an interval $\Delta_i$ in $P'$ and $n_i$ intervals in $n_i$ neighbouring trees, $P_{i,1}, \ldots, P_{i,n_i}$.  

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with \( n_1 + \ldots + n_c = n_{\text{Mayer}}(\mathbb{P}') - 1 \). The exploration procedure at the red interval \( \Delta_i \) adds \( n_i \) to \( K_{\Delta_i} \), see eq. (6.7), corresponding to the number of possible choices of neighbouring trees, but Cayley’s theorem, see proof of Proposition 3.12 yields a factor \( \frac{1}{(n_{\text{Mayer}}(\mathbb{P}') - 1)!} \leq \frac{1}{n_1! \ldots n_c!} \). Summing over all possible values of \( n_i \) leads to replacing \( K_{\Delta_i} \) by \( \sum_{n_i \geq 0} K_{\Delta_i} n_i^{n_i!} = O(1) \).

The whole procedure must be slightly amended to take into account rooted Mayer trees (see §3.4) connecting possibly an interval \( \Delta \in \Delta_{\text{ext}}(\mathbb{P}) \), where \( \mathbb{P} \in \mathcal{P} \) is a polymer with \( \geq N_{\text{ext,max}} \) external legs, to intervals without external legs of polymers of type 1. Then one should associate some small power of \( \lambda \), say \( \lambda^\kappa \) with \( \kappa \ll \kappa_0 \) (at the price of reducing slightly \( \kappa_0 \) in eq. (6.17)) to each interval with a field lying in it, while the intervals \( \Delta \) of the above type and containing moreover no field define intervals of a new type (of type 2, say), with no small factor attached to them. The whole discussion is very similar to the Remark after Proposition 3.12. For such intervals, \( K_{\Delta} \equiv 1 \) must be replaced with \( 1 + \sum_{n_i \geq 1} \lambda^\kappa \frac{\Delta \cdot \Delta_{\text{ext}}}{\Delta} = 1 + O(\lambda^\kappa) \). As explained at the end of this paragraph, such a factor is not a problem.

We may now give a more general, multiscale bound which takes into account secondary fields and high-momentum fields. We rescale the low-momentum fields by reference to their dropping scale \( j \), and not to their production scale \( k \), see §3.3, which leaves outside a supplementary spring factor that will be used to fix the horizontal motion of the polymers as in §4.1. As mentioned in the Remarks following Corollary 2.7 and Definition 3.9 we shall not split low-momentum fields \( \psi_i^- \) into a sum (field average)+(secondary field) if \( \beta_i < -D/2 \). The following definition is valid in all cases:

**Definition 6.5 (spring factors)** Let \( \tilde{\delta}^j \psi_i^h := \delta^j \psi_i^h \) if \( \beta_i \geq -D/2 \), with \( \delta^j \) defined by means of a wavelet admitting \( \lfloor \beta_i + \frac{D}{2} \rfloor \) vanishing moments, and \( \tilde{\psi}_i^h := \psi_i^h \) if \( \beta_i < -D/2 \), so that, by Corollary 2.7,

\[
|\langle \tilde{\delta}_{\text{rescaled}}^j \tilde{\psi}_i^h(x) \delta_{\text{rescaled}}^j \tilde{\psi}_j^h(x') \rangle| \leq K_r \frac{M \tilde{\beta}^{(j-h)} M \tilde{\beta}'^{(j'-h)}}{(1 + d^h(\Delta_j, \Delta_j'))^r},
\]

where \( \tilde{\delta}_{\text{rescaled}}^j \tilde{\psi}_i^h := M^j \beta_i \delta^j \psi_i^h \) is the rescaled field, and

\[
\tilde{\beta}_i = \beta_i \ (\beta_i < -D/2), \quad \tilde{\beta}_i = \beta_i - \lfloor \beta_i + \frac{D}{2} \rfloor + 1 \in [-1-D/2, -D/2) \quad (\beta_i \geq -D/2).
\]
Example. The $\sigma$-field in the $(\phi, \partial \phi, \sigma)$-model has $\beta_\sigma = -2\alpha > -1/2$. Thus low-momentum fields are severed from their averages, and $\bar{\beta}_\sigma = \beta_\sigma - 1 = -1 - 2\alpha$.

**Hypothesis 6.6 (high-momentum fields)** Assume

(i) either that all $\beta_i$, $i = 1, \ldots, d$ are $< 0$\(^{17}\).

(ii) or, more generally, that there is a scale constraint on $L_{\text{int}}$ of the following form: rewriting $L_{\text{int}}$ as

$$L_{\text{int}}(x) = \sum_{q \geq 2} \sum_{1 \leq i_1, \ldots, i_q \leq d} \sum_{j_1 \leq \ldots \leq j_q} K_{i_1, \ldots, i_q}^{j_1, \ldots, j_q} \phi_{i_1}^{j_1}(x) \cdots \phi_{i_q}^{j_q}(x), \quad (6.23)$$

then

$$\left(K_{i_1, \ldots, i_q}^{j_1, \ldots, j_q} \neq 0\right) \implies (\beta_{i_1} < 0, \beta_{i_1} + \beta_{i_2} < 0, \ldots, \beta_{i_1} + \ldots + \beta_{i_q} < 0). \quad (6.24)$$

This condition on the scales of the low-momentum fields is of course equivalent to a condition on the scales of the high-momentum fields due to the homogeneity of the vertices.

Note that Hypothesis (6.24) holds true for our $(\phi, \partial \phi, \sigma)$-model, since splitting a vertex leads to one low-momentum field, either $\partial \phi$ or $\sigma$, with respective scaling exponents $\alpha - 1, -2\alpha < 0$. In general, it has the following obvious consequence.

**Lemma 6.7** Assume Hypothesis (6.24) holds, and fix $I_q = (i_1, \ldots, i_q)$. Choose $\varepsilon > 0$ such that

$$\varepsilon < \min \left( |\beta_{i_1}|, |\beta_{i_1} + \beta_{i_2}|, \ldots, |\beta_{i_1} + \ldots + \beta_{i_q}| \right) \quad (6.25)$$

whenever there exists $j_1 \leq \ldots \leq j_q$ such that $K_{i_1, \ldots, i_q}^{j_1, \ldots, j_q} \neq 0$, and let

$$\gamma_i := \frac{\varepsilon}{q} - \beta_i, \quad i = i_1, \ldots, i_q; \quad \gamma_I := \sum_{i \in I_q} \gamma_i. \quad (6.26)$$

Then $\beta_i + \gamma_i > 0$ for all $i$, and $\gamma_{i_{q'}} + \ldots + \gamma_{i_q} < D$ for every $q' = 1, \ldots, q$.

\(^{17}\) which is the case of $\phi^4$-theory for $D > 2$ for instance
Lemma 6.8 (multiscale generalization) Assume $\psi_1, \ldots, \psi_d \subset \{(\partial^n \tilde{\psi}_1)_{n \geq 0}, \ldots, (\partial^n \tilde{\psi}_d)_{n \geq 0}\}$ are derivatives of some independent multiscale Gaussian fields $\tilde{\psi}_1, \ldots, \tilde{\psi}_d$. Fix some constant $\kappa_0 \in (0,1)$ as in the previous lemma, as well as some reference scale $j_{\text{min}} \leq j$.

1. For each $k \geq j$, consider a horizontal cluster forest $\mathbb{F}^k \in \mathcal{F}^k$ and associated cluster points $x_{ek}, x'_{ek}$, and choose subsets $(I_{ek}), (I'_{ek}), (x_{ek}), (x_{ek})$ as in the previous lemma. Do the same for each $h < j$, and choose for each point $x = x_{eh}, x'_{eh}$ or $x_{eh}, x_{eh}$ a restriction interval $\Delta_j = \Delta_{eh}, \Delta_{eh}, \Delta_{eh}$ such that $\Delta_j \subset \Delta_{eh}, \Delta_{eh}^j$ or $\Delta_j$ respectively.

Such as choice defines uniquely a monomial $G^{i,j}$ as before, and one more monomial per different scale,

\[
G^{i,k} := \lambda_{\text{cov}}(\mathbb{F}^k;k, G^{i,k}) \left[ \prod_{\ell^k \in L(\mathbb{F}^k)} \tilde{\delta}^k \psi_i^{j_{\ell^k}} \left( x_{\ell^k} \right) \tilde{\delta}^k \psi_i^{j_{\ell^k}} \left( x'_{\ell^k} \right) \right] \prod_{\Delta_k \in \mathbb{F}^k} \prod_{\tau=1}^{\tau_{\Delta_k}} \tilde{\psi}_i^{\Delta_k, \tau} \left( x_{\Delta_k} \right). \tag{6.28}
\]

for $k > j$, and, see eq. (6.26) for notations,

\[
G^{i,h} := \lambda_{\text{cov}}(\mathbb{F}^h; G^{i,h}) \left[ \prod_{\ell^h \in L(\mathbb{F}^h)} \sum_{\Delta \subset \Delta_{eh}} \sum_{\Delta_j \subset \Delta_{eh}^j} M^{-\gamma_i, \ell^h (j-h)} \text{Res}_{\Delta_j} \psi_i^{j_{\ell^h}} \left( x_{\ell^h} \right) M^{-\gamma_i, \ell^h (j-h)} \text{Res}_{\Delta_j} \psi_i^{j_{\ell^h}} \left( x'_{\ell^h} \right) \right]. \tag{6.29}
\]
for \( h < j \), where the \( \Delta^j \), \( \Delta'^j \) are restriction intervals.

Let \( v(\mathcal{F}, \mathcal{G}^j) := v(\mathcal{F}^j; \mathcal{G}^j) + \sum_{k > j} v(\mathcal{F}^k; \mathcal{G}^{j,k}) + \sum_{h < j} v(\mathcal{F}^j; \mathcal{G}^{j,h}) \) be the total number of vertices. Let finally \( \mathcal{G}^j := \prod_{k > j} \mathcal{G}^{j,k} \prod_{h < j} \mathcal{G}^{j,h} \)

and

\[
P_{\text{Gaussian}}(\mathcal{F}; \mathcal{G}^j) := \int d\mu(\psi^j) \prod_{\ell \in L(\mathcal{F}^j)} C^j_\psi(i_{\ell,j}, x_{\ell,j}, i'_{\ell,j}, x'_{\ell,j}) \mathcal{G}^j. \tag{6.30}
\]

Then

\[
P_{\text{Gaussian}}(\mathcal{F}; \mathcal{G}^j) \leq K |\mathcal{F}| M^{-(D+|\omega^\ast_{\max}|)\#\Delta^\min_{\to}} \prod_{a \geq \min} \prod_{i=1}^d M^{-a\beta_i(N_i^a(\mathcal{G}^{j,a}) + N_i^a(\mathcal{F}^a))}, \tag{6.31}
\]

where \( a \) stands either for a low-momentum scale \( h \leq j \) or a high-momentum scale \( k > j \), and \( \omega_{\max} < 0 \) is an in Definition 4.4.

2. Fix the total number of vertices, \( v := v(\mathcal{F}, \mathcal{G}^j) \). Define \( \mathcal{F}^j_{\min \to}(\Delta^j_{\min \to}) \) as in Corollary 6.3 (3). Consider, similarly to the previous lemma, the rescaled quantity

\[
P^{\min \to}_{\text{rescaled Gaussian}}(v; \mathcal{G}^j) := \prod_{i=1}^d M^{j\beta_i(N_i^j(\mathcal{G}^{j,j}) + N_i^j(\mathcal{F}^j))} \prod_{a \geq \min, a \neq j} M^{a\beta_iN_i^j(\mathcal{G}^{j,a})} I^j_{\text{Gaussian}}(\mathcal{F}; \mathcal{G}^j). \tag{6.32}
\]

and:

- sum over all \( \mathcal{F} \in \mathcal{F}^j_{\min \to}(\Delta^j_{\min \to}) \), \( (I_{\mathcal{F}^k}) \subset \{1, \ldots, d\} \setminus \{i_{\mathcal{F}^k}\} \), \( (I'_{\mathcal{F}^k}) \subset \{1, \ldots, d\} \setminus \{i'_{\mathcal{F}^k}\} \), \( (I_{\Delta^k,\tau}) \subset \{1, \ldots, d\} \) \( k \geq j \), and similarly for \( h < j \);
- sum over all possibly choices of the restriction intervals \( \Delta^j \);
- maximize for \( (x_{\mathcal{F}^k}), (x'_{\mathcal{F}^k}), (x_{\Delta^k,\tau}), (x_{\Delta^h,\tau}) \), each one over its associated interval in \( \mathbb{D}^k \), \( k \geq j \), and for \( (x_{\mathcal{F}^h}), (x'_{\mathcal{F}^h}), (x_{\Delta^h,\tau}), h < j \), each one ranging over its associated restriction interval in \( \mathbb{D}^j \) (and not \( \mathbb{D}^h \)).

Call \( P^{\min \to}_{\text{rescaled Gaussian}}(v; \Delta^j_{\min \to}) \) the result. Then

\[
P^{\min \to}_{\text{rescaled Gaussian}}(v; \Delta^j_{\min \to}) \leq M^{-(D+|\omega^\ast_{\max}|)\#\Delta^\min_{\to}} (K\lambda^{\ast_0})^v. \tag{6.33}
\]

Proof.
Consider first the factor \( M^{-(D+|\omega_{\text{max}}|)} \Delta_{j_{\text{min}}} \rightarrow \). It comes from the part of the rescaling spring factors used for fixing horizontally the polymers (see §4.1). Let \( \mathbb{P}_c^j \rightarrow \) be one of the connected components of the multi-scale forest at scale \( j \), and \( \Delta^j_i \subset \Delta_{j_{\text{min}}} \rightarrow \cap \Delta^j \) be its intervals of scale \( j \) with external legs. Then the rescaling spring factors of the corresponding low-momentum fields \((T\psi_{i_{\text{next}}})^{(j-1)}(x_{n_{\text{next}}})\) \( n_{\text{ext}} = 1, \ldots, N_{\text{ext}}(\mathbb{P}_c^j \rightarrow) \), yield a factor \( \prod_{n_{\text{ext}}=1}^{N_{\text{ext}}}(\mathbb{P}_c^j \rightarrow) \beta^*_{n_{\text{ext}}} \) when going down from scale \( j \) to scale \( j - 1 \), where \( \beta^*_{n_{\text{ext}}} = \beta_{n_{\text{ext}}} \) or \( \beta_{n_{\text{ext}}} - 1 \), see §4.1, are such that \( \sum_{n_{\text{ext}}=1}^{N_{\text{ext}}}(\mathbb{P}_c^j \rightarrow) \beta^*_{n_{\text{ext}}} \leq -D - |\omega^*_{\text{max}}| \).

Let us now prove 2. directly, since 1. is a weaker form of 2. Note that by the Cauchy-Schwarz type inequality \( \int |fgh| \leq (\int |f|^3 \int |g|^3 \int |h|^3)^{1/3} \), one may separate low-momentum fields from high-momentum fields and from the fields produced at scale \( j \), to which the previous lemma applies.

Let us first consider low-momentum fields. We use the same rescaling as in the proof of the previous lemma, namely, we consider the rescaled fields \( n(\Delta^j_k)^{-\gamma} \delta^k, \text{rescaled} \psi^{j_i}_j \), with \( k > j \) in the specific case of low-momentum fields.

Cluster links of scale \( k > j \) (or \( h < j \)) contribute a factor \( \frac{1}{(1 + \Delta^k(\Delta^j)^k)^r} \), which is required both for the bound on \( I^j \) and for that on \( I^k \). Since \( r \) is arbitrary, one chooses it large enough and splits the above factor among the different scales of the vertices. On the other hand, the scaling of the propagators \( C^k_\psi \) or \( C^h_\psi \) is left for the computation of \( I^k \) or \( I^h \). Note the possible existence of chains of propagators of scale \( k \) connecting two vertices with low-momentum fields of scale \( j \); summing over all possible chains yields the same factor of order \( \frac{1}{(1 + \Delta^k(\Delta^j)^k)^r} \) as soon as \( r > D \).

By Definition 6.5, the term between square brackets in eq. (6.6) is bounded up to a constant (see proof of Corollary 6.3 (3)) by \( A_{\text{cluster}}(\Delta^j) + \)

\(^{18}\) Of course, fields produced at scale \( j \) may also be treated on an equal foot with high-momentum fields for instance.
$$A^{\text{low}}(\Delta^k),$$ where

$$A_{\text{cluster}}(\Delta^k) := \sum_{i,i'=1}^{d} \sum_{k' = j}^{k} M^{\tilde{\beta}(k'-j)} M^{\tilde{\beta}_i(k'-j)} \frac{1}{(1 + d^k(\Delta^k, \Delta'))^r}$$  

(6.34)

where $\Delta^{k'}$ is the unique interval of scale $k'$ such that $\Delta^{k'} \supset \Delta^k$, and

$$A^{\text{low}}(\Delta^k) := \sum_{i,i'=1}^{d} \sum_{k' = j}^{k} M^{\tilde{\beta}(k'-j)} \sum_{k'' = j}^{k'} M^{\tilde{\beta}_i(k''-j)} \frac{1}{(1 + d^k(\Delta^k, \Delta'))^r} \frac{1}{\Delta'' \in \mathbb{D}^{k''}}.$$  

(6.35)

The above sum, $\sum_{\Delta'' \in \mathbb{D}^{k''}} \frac{1}{(1 + d^k(\Delta^k, \Delta'))^r}$, is of order $M^{D(k''-j)}$, which yields

$$A^{\text{low}}(\Delta^k) \leq K \sum_{i,i'=1}^{d} A^{\text{low}}_{\tilde{\beta}_i, \tilde{\beta}_i, \tilde{\beta}_i} \quad A^{\text{low}}_{\tilde{\beta}_i, \tilde{\beta}_i, \tilde{\beta}_i} := \sum_{k' = j}^{k'} M^{\tilde{\beta}(k'-j)} \sum_{k'' = j}^{k'} M^{(\tilde{\beta}_i + \tilde{\beta}_i)(k''-j)},$$  

(6.36)

while clearly $A_{\text{cluster}}(\Delta^k)$ is finite since $\tilde{\beta}_i, \tilde{\beta}_i < 0$.

Let us finish the proof with the assumption that $D = 1$. There are 3 different cases:

- either $\beta_i' < -1$; then $\tilde{\beta}_i' = \beta_i'$ and $\sum_{k'' = j}^{k'} M^{(\beta_i + 1)(k''-j)} = O(1), \sum_{k'' = j}^{k'} M^{\tilde{\beta}_i'(k''-j)} = O(1)$ since $\tilde{\beta}_i < 0$;

- or $-1 \leq \beta_i' < -\frac{1}{2}$, resp. $0 \leq \beta_i' < \frac{1}{2}$; then $\tilde{\beta}_i' = \beta_i'$, resp. $\beta_i' - 1$, and $\sum_{k'' = j}^{k'} M^{(\tilde{\beta}_i' + 1)(k''-j)} = O((k' - j) M^{(\tilde{\beta}_i' + 1)(k' - j)}), \sum_{k'' = j}^{k'} (k' - j) M^{(\tilde{\beta}_i' + \tilde{\beta}_i')(k''-j)} = O(1)$ since $\tilde{\beta}_i, \tilde{\beta}_i' < -\frac{1}{2}$;

- or $-\frac{1}{2} \leq \beta_i' < 0$, resp. $\frac{1}{2} \leq \beta_i' < 1$; then $\tilde{\beta}_i' = \beta_i' - 1$, resp. $\beta_i' - 2$, and $\sum_{k'' = j}^{k'} M^{(\tilde{\beta}_i' + 1)(k''-j)} = O(1), \sum_{k'' = j}^{k'} (k' - j) M^{(\tilde{\beta}_i' + \tilde{\beta}_i')(k''-j)} = O(1)$, while the sum over $k'$ converges as in the first case.

The simpler case $D \geq 2$ is left to the reader (there are only 2 subcases: $\beta < -D/2$, or $\beta \geq -D/2$, the latter subcase to be split according to the value of $|\beta + \frac{D}{2}|$).

Consider now high-momentum fields, produced at a scale $h < j$ (or $h \leq j$). By the same method, one ends up with the following quantity to bound instead of eq. (6.35):

$$A^{\text{high}}(\Delta^j) := \sum_{i,i'=1}^{d} \sum_{h' = j_{\min}}^{h} M^{-(\beta_i + \gamma_i)(j-h')} \sum_{h'' = j_{\min}}^{h'} M^{-(\beta_i' + \gamma_i')(j-h'')} \sum_{\Delta'' \in \mathbb{D}^{j}} \frac{1}{(1 + d^j(\Delta^j, \Delta''))^r},$$  

(6.37)
where $\Delta^j, \Delta'^j$ range over restriction intervals, plus the finite term $A_{\text{cluster}}(\Delta^h)$ due to cluster links of scale $h$ as before. Now $\sum_{\Delta^j \in D^j} \frac{1}{(1 + d_{\Delta}(\Delta^j))^v} < K$ and $\beta_i + \gamma_i, \beta'_i + \gamma' > 0$ by Lemma 6.4 hence $A_{\text{high}}(\Delta^j)$ is bounded by a constant.

Note that a small part of the rescaling spring factor $M_\beta_i(k - j)$ for low-momentum fields may be used to obtain a factor $M^{\varepsilon} < 1$ for $\varepsilon > 0$ small enough per interval $\Delta$ belonging to a fixed polymer $P$ – in particular in empty intervals where there is no field. This simple remark is essential in the sequel since various estimates yield a factor $1 + \mathcal{O}(\kappa)$ per interval, which may be compensated by this (not so) small factor $M^{\varepsilon}$. On the other hand, to each vertex or equivalently to each non-empty interval – or to each field – is associated a factor of order $\lambda^{-\kappa}$, which may be arbitrarily small. This is a general principle for the bounds to come now.

### 6.1.3 Gaussian bounds for polymers

It remains to be seen how these Gaussian estimates, valid for each scale, combine to give estimates for the (Mayer-extended) polymer evaluation functions defined in subsection 3.4. Note that (rescaled) low-momentum field averages have been left out; the domination estimates in subsection 6.2 prove that it is possible in the case of the $(\phi, \partial \phi, \sigma)$-model to bound them while leaving a small factor per field, $\lambda^{\kappa_0'}$, $\kappa_0, \kappa_0' > 0$, negligible with respect to $\lambda^{\kappa_0}$.

We take this into account for our next Theorem by choosing a small enough exponent $\kappa_0$.

**Theorem 6.1** Fix some reference scale $j_{\text{min}} \geq 0$ and some exponent $\kappa > 0$.

Let $\mathcal{P}^j_{0 \rightarrow \text{min}}(\Delta^j_{\text{min}})$ be the set of vacuum (Mayer-extended) polymers down to scale $j_{\text{min}}$ containing some fixed interval $\Delta^j_{\text{min}}$ of scale $j_{\text{min}}$. Let $\text{Eval}^{\kappa_0}(\mathbb{P})$, $\mathbb{P} \in \mathcal{P}^j_{0 \rightarrow \text{min}}(\Delta^j_{\text{min}})$ be the sum over all multi-scale splitting of vertices into cluster forests $F^j$ extending over $\mathbb{P}$ and monomials $G^j$ of the product integrals $\prod_{j=\text{min}}^j \int d\mu(\psi^j) \prod_{\ell \in U(F^j)} C^j_{\psi}(i_{\ell j}, x_{\ell j}; i'_{\ell j}, x'_{\ell j}) G^j$, all fields in $G^j$ being accompanied as before with the factor $\lambda^{\kappa_0}$. Then

$$\sum_{\mathbb{P} \in \mathcal{P}^j_{0 \rightarrow \text{min}}(\Delta^j_{\text{min}}); \# \text{vertices of } \mathbb{P} = \nu} \left| \text{Eval}^{\kappa_0 + \kappa_0'}(\mathbb{P}) \right| \leq (K \lambda^{\kappa_0})^\nu. \quad (6.38)$$

In particular, for $\lambda$ small enough,

$$\sum_{\mathbb{P} \in \mathcal{P}^j_{0 \rightarrow \text{min}}(\Delta^j_{\text{min}})} \left| \text{Eval}^{\kappa_0 + \kappa_0'}(\mathbb{P}) \right| < \infty. \quad (6.39)$$
Proof.

Let \( \mathcal{P}^{\jmath_{\text{min}} \rightarrow} (\Delta^{\jmath_{\text{min}} \rightarrow}) \) be the set of vacuum (Mayer-extended) polymers down to scale \( \jmath_{\text{min}} \) with fixed set of intervals \( \Delta^{\jmath_{\text{min}} \rightarrow} \) as in the previous Lemma. Note that the horizontal fixing scaling factor \( M^{-(D+|\omega_{\text{max}}|)} \# \Delta^{\jmath_{\text{min}} \rightarrow} \) makes it possible to sum over all inclusion links of the polymers. Namely, each inclusion link \( \Delta^j \subset \Delta^{-1} \) – implying necessarily some \( \ell \)-derivative in \( \Delta^j \) – produces a factor \( M^D \) due to the choice of \( \Delta^j \) among the \( M^D \) intervals \( \Delta \in \mathbb{D}^j \) such that \( \Delta \subset \Delta^{-1} \), which is compensated by the factor \( M^{-(D+|\omega_{\text{max}}|)} \) attached to \( \Delta^j \).

Hence it suffices to prove eq. (6.38) for \( \mathbb{P} \) ranging in the set \( \mathcal{P}^{\jmath_{\text{min}} \rightarrow} (\Delta^{\jmath_{\text{min}} \rightarrow}) \).

Let \( \mathcal{A}^{\jmath_{\text{min}} \rightarrow} (\Delta^{\jmath_{\text{min}} \rightarrow}) := \sum_{\mathbb{P} \in \mathcal{P}^{\jmath_{\text{min}} \rightarrow} (\Delta^{\jmath_{\text{min}} \rightarrow})} \left| \left| \text{Eval}_{\kappa_0 + \kappa_0} (\mathbb{P}) \right| \right| \).

Split the small factor per vertex into \( \lambda^{\kappa_0} \lambda^{\kappa_0} \), and set apart \( \lambda^{\kappa_0} \) to get a global homogeneity factor \( \lambda^{\kappa_0} \). Then

\[
\mathcal{A}^{\jmath_{\text{min}} \rightarrow} (\Delta^{\jmath_{\text{min}} \rightarrow}) \leq \lambda^{\kappa_0} \sum_{\mathbb{P} \in \mathcal{P}^{\jmath_{\text{min}} \rightarrow} (\Delta^{\jmath_{\text{min}} \rightarrow})} \left| \left| \text{Eval}_{\kappa_0} (\mathbb{P}) \right| \right|, \tag{6.40}
\]

where the total number of vertices is now unrestricted.

Let \( \mathbb{P} \in \mathcal{P}^{\jmath_{\text{min}} \rightarrow} (\Delta^{\jmath_{\text{min}} \rightarrow}) \). Consider some product of fields of type \( q \) produced in some interval \( \Delta^j \) of scale \( j \), \( \psi_{I_q} (x_{\Delta^j, \tau}) \) or \( \psi_{I_q \setminus (i_i)} (x_{\ell}) \), interpreted as some pairing of \( \psi_{I_q} (x_{\ell}) \) with \( \psi_{I_q'} (x_{\ell'}) \), and:

- choose some non-empty subset \( I_{\text{high}} = [i_q', \ldots, i_q] \subset I_q \);
- choose some high-momentum scales \( (k_i)_{i \in I_{\text{high}}} \), with \( j \leq k_i \leq \ldots \leq k_{i_q} \), as in Hypothesis 6.6, and restriction intervals \( (\Delta^j_{k_i})_{i \in I_{\text{high}}}, \Delta^j_{k_i} \subset \mathbb{D}^{k_i}, \Delta^j_{k_i} \subset \Delta^j \);
- letting \( I_{\text{low}} := I_q \setminus I_{\text{high}} \), choose some low-momentum scales \( (h_i)_{i \in I_{\text{low}}} \), \( h_i < j \).

Then

\[
\psi_{I_q} := \sum_{I_{\text{high}} \subset I} \sum_{(k_i)} \sum_{\Delta^j_{k_i}} \left( \prod_{i \in I_{\text{high}}} \text{Res}^j_{k_i} \psi^k_{I_q} \right) \cdot \left( \prod_{i \in I_{\text{low}}} \psi^{h_i}_{I_q} \right) \tag{6.41}
\]

is the decomposition of \( \psi_{I_q} \) into all possible splittings. Any given splitting of \( \psi_{I_q} \) is supported on an \( M \)-adic interval \( \cap_{i \in I_{\text{high}}} \Delta^j_{k_i} \) of size bounded by

\[
M^{-k_{i_q} D} = M^{-j D} \cdot M^{-(k_{i_q} - j) D} \ldots M^{-(k_{i_q} - k_{i_{q-1}}) D} \leq M^{-j D} M^{-(\gamma + \ldots + \gamma_q)(k_{i_q} - j)} \ldots M^{-\gamma_q (k_{i_q} - k_{i_{q-1}})} = M^{-j D} M^{-\sum_{i \in I_{\text{high}}} \gamma_i (k_i - j)} \tag{6.42}
\]
by Lemma 6.7.

Fixing $k_i$, letting $j$ range over all scales $\leq k_i$ and changing notations $(k_i, j) \to (j, h)$ yields the spring factors $M^{-\gamma_i} I_{ij} (j-h), M^{-\gamma_i} I_{ij} (j-h), M^{-\gamma_i} I_{ij} (j-h)$ of eq. (6.29). The remaining factor $M^{-jD} = |\Delta j|$ may be rewritten as $M^j \sum_{k \in I_k} \beta_j$, which is distributed between the different fields, $\psi_i^j \to \psi_i^{j, \text{rescaled}} = M^j \beta_i \psi_i^j$, as in §6.1.2. Recall from the end of the preceding paragraph that each interval of $P$ comes with a factor $M^{-\varepsilon} < 1$. Hence, letting $j_{\text{max}}$ be the maximal scale of a given polymer $P$ one has by Lemma 6.8,

$$\sum_{P \in P_0^j \cap (\Delta j_{\text{min}} \to)} \left| \text{Eval}_{\kappa_i} (P) \right| \leq \sum_{j_{\text{max}} = j_{\text{min}}}^{j_{\text{max}}} \prod_{j = j_{\text{min}}}^{j_{\text{max}}} \left( M^{-\varepsilon} + \sum_{v \geq 1} I_{\text{Gaussian}} (v; \Delta j_{\text{min}} \to) \right)$$

$$\leq \sum_{j_{\text{max}} = j_{\text{min}}}^{j_{\text{max}}} (M^{-\varepsilon} + K' \lambda^{n_0})^{j_{\text{max}} - j_{\text{min}} + 1} < \infty \quad (6.43)$$

if $\lambda$ is chosen small enough so that in particular (with the constant $K$ of eq. (6.33)) $\sum_{v \geq 1} (K' \lambda^{n_0})^v \leq K' \lambda^{n_0} < 1 - M^{-\varepsilon}$.

### 6.1.4 Combinatorial factors

The last point – for the Gaussian part of the final bounds – is to control the combinatorial factors due to the horizontal and vertical cluster expansions. Let us show briefly how to do this.

As a general rule, differentiating a product of $n$ fields yields $n$ terms (by the Leibniz formula). This implies supplementary combinatorial factors when estimating the polymer evaluation functions $F(P)$, compared to the estimates of Theorem 6.1. Consider the $O(n(\Delta^j))$ derivations in a given interval $\Delta^j$ due to horizontal/vertical cluster expansion at scale $j$. A field produced by one such derivation may be acted upon by another one in the same interval, yielding a local factorial $O(n(\Delta^j))!$. Otherwise, derivations in $\Delta^j$ act on fields produced at an earlier stage in some interval $\Delta^k \subset \Delta^j$ belonging to the same polymer $P$ as $\Delta^j$. Integrating over these, and borrowing some small power of $\lambda$ from one of the differentiated fields, yields at most an averaged factor in $\Delta^j$ of order $K := \frac{1}{|\Delta^j|} \lambda^n \sum_{k > j} \sum_{\Delta^k \subset \Delta^j} \int_{\Delta^k} dx$. Once again, there are $O(n(\Delta^j))$ such derivations. One may always gain a local factorial $\prod_{\Delta^j \subset \Delta^k}$ to some arbitrary power (see proof of Lemma 6.4); using $\frac{\lambda^{n(\Delta^j)}}{n(\Delta^j)!!} \leq e^K$, and
multiplying over all scales \( j \) and all intervals \( \Delta^j \in \mathbb{P} \cap \mathbb{D}^j \), one obtains

\[
\exp \lambda^\kappa \sum_k \sum_{\Delta^k \in \mathbb{P} \cap \mathbb{D}^j} \sum_{j < k} M^{-(k-j)},
\]

hence a factor of order \( 1 + O(\lambda^\kappa) \) per interval \( \Delta \in \mathbb{P} \), compensated by some factor \( M^{-\varepsilon} \) as explained at the end of §6.1.2.

### 6.2 Domination bounds

Unlike Gaussian bounds, which are rather sophisticated, these are essentially based on the simple fact that \( |x|e^{-A|x|} = A^{-1}(A|x|)e^{-A|x|} \leq KA^{-1} \) if \( A > 0 \).

**Lemma 6.9 (domination)** Let \( \psi \) be a multiscale Gaussian field with scaling dimension \( \beta \). Then

\[
|((T\psi)^{(k-1)}(\Delta^k)|^n \exp -\lambda^\kappa M^{\kappa n \beta^k} \cdot \frac{1}{|\Delta^k|} \int_{\Delta^k} ((T\psi)^{(k-1)}(x)^m dx) \leq K^{-n n/m} \lambda^{-\kappa n/m} M^{-n \beta^k}.
\]

**Proof.** Let \( u := (T\psi)^{(k-1)}(\Delta^k) \) and \( v := \frac{1}{|\Delta^k|} \int_{\Delta^k} ((T\psi)^{(k-1)}(x)^m dx; \) by Hölder’s inequality, \( |u| \leq v^{1/m} \), so that

\[
|u|^n e^{-\lambda^\kappa M^{\kappa n \beta^k} v} \leq \left( \frac{\lambda^\kappa}{n} M^{\kappa n \beta^k} \right)^{-n/m} \left( \frac{\lambda^\kappa}{n} M^{\kappa n \beta^k} v \right)^m \exp -\frac{\lambda^\kappa}{n} M^{\kappa n \beta^k} v \right)^n \leq K^{-n n/m} \lambda^{-\kappa n/m} M^{-n \beta^k}.
\]

**Example ((\( \phi, \partial \phi, \sigma \))-model).** Lemma 6.9 implies in particular the following four kinds of low-momentum field domination:

(i) \( \mathcal{A}_\mathcal{L}_4 < \delta \mathcal{L}_4 \) terms

Consider low-momentum fields \( \sigma \) produced at a scale \( k \) by letting some derivation \( \frac{\delta}{\delta \sigma} \) or \( \partial_t \) (due resp. to horizontal and vertical cluster expansions of scale \( k \)) act on \( \mathcal{L}_4 \). When \( \rho - k \) is large enough, they will be dominated by the part of the counterterm \( \delta \mathcal{L}_4 \) which is coupled to \( b^{\rho-1} \). Eq. (6.45) yields (using \( 1 - t^2 \geq (1 - t)^2 \) for \( t \in [0,1] \))

\[
\left( \frac{1 - t k_{\Delta^k}}{n!} \right)^n |\lambda(T\sigma)^{(k-1)}(\Delta^k)|^n \exp -\lambda^2(1 - (t k_{\Delta^k})^2)M^{(1 - 4\alpha)k} \int_{\Delta^k} |(T\sigma)^{(k-1)}(x)|^2 dx \leq \left( KM^{2k \alpha n - \frac{1}{2}} \right)^n.
\]
Note the factor \((1 - t_k^\Delta)^n\) coming from the rest term in the Taylor-Lagrange expansion as in §3.3, with \(n = N_{ext,max} + O(n(\Delta))\). This is precisely the reason why we chose to Taylor expand to order \(N_{ext,max} + O(n(\Delta))\) and not simply to order \(N_{ext,max}\). The other terms in the Taylor-Lagrange expansion have \(t_k^\Delta = 0\).

Replacing in the exponential \(\lambda^2 M^{(1-4\alpha)k}\) by the term \(b^\rho - 1 \approx \lambda^2 M^{(1-4\alpha)(\rho-1)}\), one gains a supplementary "petit facteur" \(\left(M^{-\frac{1}{2}(1-4\alpha)(\rho-1-k)}\right)^n\).

(ii) \(\text{Av}_{\mathcal{L}_4 < \delta \mathcal{L}_{12}}\)-terms

When \(\rho - k\) is too small, the "petit facteur" is not small any more. One dominates by some fraction (say, one tenth) of the boundary term, \(\frac{1}{10} \delta \mathcal{L}_{12}\) instead. Again, eq. (6.45) yields (using \(1 - (t_k^\Delta)^6 \geq (1 - t_k^\Delta)^6\))

\[
\frac{(1 - t_k^\Delta)^n}{n!} \left|\lambda(T\sigma)^{(k-1)}(\Delta^k)\right|^n e^{-\frac{1}{10} \lambda^3 (1 - (t_k^\Delta)^6) M^{(1-12\alpha)k} \int_{\Delta^k} |(T\sigma)^{(k-1)}|^6(x)dx} \leq \frac{n^{n/6}}{n!} \left(K' \lambda^\frac{1}{2} M^{2\alpha k}\right)^n \leq \left(K' \lambda^\frac{1}{2} M^{2\alpha k}\right)^n. \tag{6.48}
\]

Replacing in the exponential \(M^{(1-12\alpha)k}\) by the term \(M^{(1-12\alpha)\rho}\) present in \(\delta \mathcal{L}_{12}\) (note that \(1 - 12\alpha < 0\) !), one loses this time a supplementary large factor \(\left(M^{12\alpha-1}_{\rho} \left(\rho-k\right)\right)^n\). However, it is accompanied by a small factor \(\lambda^\frac{1}{2}\) per field.

Assume \(\rho - k = 0, 1, \ldots, q\). We fix \(q\) such that \(\lambda^\frac{1}{2} M^q_{\rho} = \lambda^{1/4}\), i.e. \(q = \frac{3}{2(12\alpha - 1)} \ln(1/\lambda) \ln M\). Thus the "petit facteur" in the preceding \(\text{Av}_{\mathcal{L}_4 < \delta \mathcal{L}_4}\) case – where \(\rho - k > q\) – is at most \(\lambda^\kappa\), where \(\kappa = \frac{3(1-4\alpha)}{4(12\alpha - 1)}\).

(iii) \(\text{Av}_{\delta \mathcal{L}_{12} < \delta \mathcal{L}_{12}}\)-terms

Consider low-momentum fields \(\sigma\) produced from some vertex \(\Delta^\rho\) by letting some derivation act on \(\delta \mathcal{L}_{12}\) this time. It produces \(i \leq 5\) low-momentum \(\sigma\)-fields, accompanied by \(\lambda^3\). Again, eq. (6.45) yields, by dominating by one tenth of the boundary term, \(\frac{1}{10} \delta \mathcal{L}_{12}\), and leaving out once and for all the \(t\)-coefficients,
Consider low-momentum fields $\sigma$ the counterterm. Again, eq. (6.45) yields, by dominating as in (i) by the part of $M$ per high-momentum field, and by $\Delta \rho \leq \text{factor}$ all intervals $\Delta k \in \mathbb{P}$. Such terms may be produced only in intervals $\Delta^k \in \mathbb{P}$ which is coupled to $b^{\rho-1}$, a factor $O(\lambda^{6\alpha} \cdot M^{-\frac{1}{2}(1-4\alpha)(\rho-k)})$ per field, alas multiplied by $b^{\rho}/\lambda \approx \lambda M^{\rho(1-4\alpha)}$. The rest of the argument goes as in subsection 6.1.4 – a general argument called ”aplatissement du fortement connexe” in (colloquial) French. The factor $M^{-\frac{1}{2}(1-4\alpha)(\rho-k)}$ may be simply bounded by 1.

Such terms may be produced only in intervals $\Delta^{\rho'} \subset \Delta^k$ such that $\Delta^{\rho'} \in \mathbb{P}$. Integrating over all such intervals – and taking into account the $M^{-2\alpha}$ scaling of the high-momentum field $\sigma^k$ left behind – yields at most $\lambda K := \lambda \sum_{\rho' \geq k} M^{-4\alpha(\rho'-k)} \# \{(\mathbb{P} \cap D^{\rho'}) \text{ per low-momentum field produced, all together } (\lambda^{1/2})^n \cdot (\lambda^{\frac{1}{2}} K)^n \text{. One may always gain a local factorial } \frac{1}{n} \text{; using } K^n \leq e^{K} \text{ and multiplying over all scales } k \text{ and all intervals } \Delta^k \in \mathbb{P} \cap D^k \text{, one gets}

$$\lambda^{v/2} \cdot \exp \lambda^{\frac{1}{2}} \sum_{\rho'} \sum_{\Delta^{\rho'} \in \mathbb{P}} \sum_{k \leq \rho'} M^{-4\alpha(\rho'-k)}$$

(6.50)

where $v$ is the number of vertices where such low-momentum fields have been produced, hence a factor $\lambda^{v/2}$ per vertex and a factor of order $1 + O(\lambda^{\frac{1}{2}})$ per interval $\Delta \in \mathbb{P}$.
6.3 Final bounds

The main Theorem is the following.

**Theorem 6.2**

1. There exists a constant, positive two-by-two matrix $K$ such that
   
   $$b^j = K \lambda^2 M^{1-4\alpha}(1 + O(\lambda)).$$  \hspace{1cm} (6.51)

2. There exists a constant $K'$ such that the Mayer bound of Proposition 3.12 for the scale $j$ free energy,
   
   $$|f^{j\rightarrow \rho}(\lambda)| \leq K' \lambda(1 + O(\lambda))$$ \hspace{1cm} (6.52)

holds uniformly in $j$.

**Proof.**

Let us first prove eq. (6.51) for $b^j$.

Consider a product (G-monomial) × (product of propagators) as in subsection 3.3, written generically as $G_{C}$. Multiply it by a product of averaged low-momentum fields of one of the four types $A_{\theta_{4},\delta L_{4}}$, $A_{\theta_{4},\delta L_{12}}$, or $A_{\theta_{4},\delta L_{12}}$, see subsection 6.2, generically written as $A_{\theta_{low}}$.

We make the following induction hypothesis:

**Induction hypothesis.** $\tilde{b}^k = K(1 + O(\lambda))$ for all $k > j$ for some scale-independent constant $K$, where $\tilde{b}^k := \lambda^{-2} M^{-(1-4\alpha)k}b^k$ is the rescaled mass counterterm of scale $k$.

We shall soon see how to compute the constant $K$. For the time being, we must bound

$$\int d\mu_\theta(\phi) d\mu_\alpha(\sigma) \sum_{P \in \mathcal{P}_{j,\theta}(\Delta)} \sum \text{Av}_{\theta_{low}} \text{G}\text{C} e^{-\int_{\partial} (L_{4} + \delta L_{4} + \frac{1}{2} \delta L_{12})(; t)(x) dx} \cdot e^{-\int_{\partial} \left[ \frac{1}{2} \delta L_{12}^{\theta_{low}}(; t)(x) - \frac{1}{2} (\frac{2}{3})^2 ((T\sigma)^{-\rho}(x))^2 \right] dx},$$ \hspace{1cm} (6.53)

where $|P|$ is the support of the polymer $P$ with two external $\sigma$-legs, in the notation of Proposition 3.11

**First step (domination of the mass counterterm).**

Note first that the term between square brackets in eq. (6.53) is negative when $X := \lambda M^{-2\alpha} \|(T\sigma)^{-\rho}\|$ is small. Up to unessential coefficients, it is
equal to \( M^\rho (\lambda^{-3} X^6 - (t_\rho^0)^2 X^2) \), which is minimal, of order \( M^\rho \lambda^{9/2} \) – for \( t_\rho^0 \approx 1 \), which is the worst case – when \( X \) is of order \( \lambda^{3/4} \). The factor \( t_\rho^0 \) in front of \( (T \sigma)^{-\epsilon(\rho-1)} \) selects the intervals in \( \mathbb{D}^\rho \) belonging to the polymer. Hence the exponential in eq. (6.53) is bounded by \( e^{K \int_{|P| \cap \mathbb{D}^\rho} M^\rho \lambda^{9/2} dx} \), all together a factor of order \( 1 + O(\lambda^{9/2}) \) per interval \( \Delta \in P \cap \mathbb{D}^\rho \) of the polymer with \( t_\Delta^0 \neq 0 \).

Second step (domination of low-momentum fields). Split \( A_{\text{low}} \) from the expression in eq. (6.53). As shown in subsection 6.2, these produce a small factor of order \( \lambda^\kappa \), \( \kappa > 0 \) per field. More precisely, any \( \kappa < \inf(\frac{3(1-4\alpha)}{4(12\alpha-1)}, \frac{3(1-4\alpha)}{4(12\alpha-1)}) \) is suitable. There remains (by using the Cauchy-Schwarz inequality) to bound \( \sum_{P \in \mathcal{P}_{j,+}} \sum_{C \in \mathcal{G},C} (\int d\mu_s(\sigma) d\mu_s(\sigma) |GC|^2)^{1/2} \).

Third step (computation of \( b_j \)).

Let us now estimate \( b_j \) by means of our induction hypothesis and of the Gaussian bounds of subsection 6.1. Consider for instance the diagonal term \( b_{+,+} \). The terms with the fewest number of vertices are:

- the term with 0 vertex obtained by applying twice \( \frac{\partial}{\partial \sigma} \) to the counterterm \( \delta \mathcal{L}_4 \), namely, \( b_{+,+}^0 \) (by definition);

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{diagram5.png}
\end{array}
\]

Figure 5: Main part of the mass counterterm of scale \( j \).

- the polymers with two vertices, see Fig. 5 which sum up to

\[
\lambda^2 \sum_{k \geq j} \int_V C_{\phi_2}^k (x,y) (-\partial^2) C_{\phi_1}^j (x,y) dy =: \lambda^2 M^j (1-4\alpha) K_V,
\]

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where
\[
K_V := M^{-j(1-4\alpha)} \int_V C^j\phi^\rho(x,y)(-\partial^2)C^j\phi(x,y)dy
\rightarrow_{|V|\to\infty} K := M^{-j(1-4\alpha)} \int |\xi|^{-4\alpha} \chi^j(\xi)\chi^{j\to}(\xi) d\xi = \int |\xi|^{-4\alpha} \chi^1(\xi)\chi^{1\to2}(\xi) d\xi,
\]
(6.54)
a scale-independent quantity. As in section 1, the non-diagonal counterterm \(b^j_{-,-} \) or \(b^j_{-,+} \) may be computed in the same way, yielding in the end a scale-independent positive two-by-two matrix.

More complicated polymers are of order \(M^j(1-4\alpha)\frac{\lambda^2}{(\tilde{b}^j,\tilde{b}^k)}_{k>j},\lambda) \), where \(\tilde{b}^j := \lambda^{-2}M^{-j(1-4\alpha)}b^j \) is the rescaled mass counterterm as in the induction hypothesis. The function \(g \) is \(C^\infty \) in a neighbourhood of 0 and vanishes at 0. Hence, by the implicit function theorem, \(b^j = KM^j(1-4\alpha)(1+O(\lambda)) \) as in Proposition 3.11.

The bound for the scale \(j \) free energy \(f^{j\to \rho} \) is now straightforward. \( \square \)

Bounds for \(n\)-point functions are easy generalizations of the preceding Theorem. Consider for instance the 2-point function \(\langle |F\phi^1(\xi)|^2 \rangle_{\lambda} \), with \(M^j \leq |\xi| \leq M^{j+1} \). By momentum conservation, and by definition of the Fourier partition of unity, see subsection 2.1, this is equal to the sum over \(j_1, j_2 = j, j \pm 1 \) of \(\langle F\phi^{j_1}(\xi)F\phi^{j_2}(-\xi) \rangle_{\lambda} \). The term of order 0 in \(\lambda \) is given by the Gaussian evaluation \(\mathbb{E}\left[F\phi^{j_1}(\xi)F\phi^{j_2}(-\xi)\right] \). Further terms involve at least one \(\sigma\)-propagator with a small factor (see Lemma 2.13) of order \(\inf(1, \frac{M^j(1-4\alpha)}{\rho^{j\to \rho}}) \leq K \inf(1, \lambda^{-2}M^{-j(1-4\alpha)}) \) which goes to 0 when \(\rho \to \infty \).

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