Translation-invariance of two-dimensional Gibbsian point processes

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Abstract

The conservation of translation as a symmetry in two-dimensional systems with interaction is a classical subject of statistical mechanics. Here we establish such a result for Gibbsian particle systems with two-body interaction, where the interesting cases of singular, hard-core and discontinuous interaction are included. We start with the special case of pure hard core repulsion in order to show how to treat hard cores in general.

Key words: Gibbsian point processes, Mermin-Wagner theorem, translation, hard-core potential, singular potential, pure hard core repulsion, percolation, superstability.

1 Introduction

Gibbsian processes were introduced by R. L. Dobrushin (see [D1] and [D2]), O. E. Lanford and D. Ruelle (see [LR]) as a model for equilibrium states in statistical physics. (For general results on Gibbs measures on a d-dimensional lattice we refer to the books of H.-O. Georgii [G], B. Simon [Sim] and Y. G. Sinai [Sin], which cover a wide range of phenomena.) The first results concerned existence and uniqueness of Gibbs measures and the structure of the set of Gibbs measures related to a given potential. The question of uniqueness is of special importance, as the non-uniqueness of Gibbs measures can be interpreted as a certain type of phase transition occurring within the particle system. A phase transition occurs whenever a symmetry of the potential is broken, so it is natural to ask, under which conditions symmetries are broken or conserved. The answer to this question depends on the type of the symmetry (discrete or continuous), the number of spatial dimensions and smoothness and decay conditions on the potential (see [G], chapters 6.2, 8, 9 and 20). It turns out that the case of continuous symmetries in two dimensions is especially interesting. The first progress in this case was achieved by M. D. Mermin and H. Wagner, who showed for special two-dimensional lattice models that continuous internal symmetries are conserved ([MW] and [M]). In [DS] R. L. Dobrushin and S. B. Shlosman established conservation of symmetries for more general potentials which satisfy smoothness and decay conditions, and C.-E. Pfister improved this in [P]. Later also continuum
systems were considered: S. Shlosman obtained results for continuous internal symmetries (Shl), while J. Fröhlich and C.-E. Pfister treated the case of translation of point particles (FP1 and FP2). All these results rely on the smoothness of the interaction, but in [ISV] D. Ioffe, S. Shlosman and Y. Velenik were able to relax this condition. Considering a lattice model they showed that continuous internal symmetries are conserved, whenever the interaction can be decomposed into a smooth part and a part which is small with respect to $L^1$-norm, using a perturbation expansion and percolation theory. We generalised this to a point particle setting (RII).

Here we will investigate the conservation of translational symmetry for non-smooth, singular or hard-core potentials in a point particle setting. While we treat non-smoothness by generalising ideas used in [RII], we will give an approach to singular potentials which is different from the one given in [FP1] and [FP2]. The advantage of our approach is that integrability condition (2.13) of [FP2] is simplified and relaxed and the case of hard-core potentials can easily be included. Thus we are able to show the conservation of translational symmetry for the pure hard core model, for example.

In Section 2 we will first confine ourselves to this special case of pure hard core repulsion. The corresponding result (Theorem 1) is of interest on its own and its proof shows how to deal with hard cores in the general case. For this general case we then define a suitable class of potentials (Definition 1), give some sufficient conditions for potentials to belong to that class (Lemmas 1 and 2) and state the general result obtained (Theorem 2). The precise setting is then given in Section 3. The proofs of the lemmas from Sections 2 and 3 are relegated to Section 4. In Sections 5 and 7 we will give the proofs of Theorems 1 and 2 respectively. The proofs of the corresponding lemmas are relegated to Sections 6 and 8 respectively. In the proof of the general case arguments of the special case have to be modified and refined by new concepts and ideas at several instances. So for sake of clarity we will repeat arguments from the proof of Theorem 1 in the proof of Theorem 2 whenever necessary.

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## 2 Result

We consider particles in the plane $\mathbb{R}^2$ without internal degrees of freedom. The chemical potential $- \log z$ of the system is given via an activity parameter $z > 0$. The interaction between particles is modelled by a translation-invariant pair potential $U$, i.e. a measurable function

\[ U : \mathbb{R}^2 \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}, \]

which is assumed to be symmetric in that $U(x) = U(-x)$ for all $x \in \mathbb{R}^2$. The potential of two particles $x_1, x_2 \in \mathbb{R}^2$ is then given by $U(x_1 - x_2)$.
We first consider the particular case of pure hard core repulsion, where the size and the shape of the hard core are given by a norm $|.|_h$ on $\mathbb{R}^2$. The corresponding pure hard-core potential $U_{hc}$ is defined by

$$U_{hc}(x) := \begin{cases} \infty & \text{for } |x|_h \leq 1 \\ 0 & \text{for } |x|_h > 1. \end{cases}$$

**Theorem 1** Let $z > 0$ be an activity parameter, $|.|_h$ be a norm on $\mathbb{R}^2$ and $U_{hc}$ be the corresponding pure hard-core potential. Then every Gibbs measure corresponding to $U_{hc}$ and $z$ is translation-invariant.

The proof of Theorem 1, which is given in Section 5, will show how to deal with hard cores in the general case presented below.

In order to describe a class of potentials for which translational symmetry is conserved we will define important properties of sets, functions and potentials. A set $A \subset \mathbb{R}^2$ is called symmetric if $A = -A$. We call $U$ a standard potential if $U$ is a measurable, symmetric pair potential and its hard core $K_U := \{U = +\infty\}$ is bounded. Usually the hard core will be empty, $\{0\}$ or a disc, but in our setup we are able to treat fairly general hard cores. For a given function $\psi : \mathbb{R}^2 \to \mathbb{R}_+$ we say that a standard potential $U$ has $\psi$-dominated derivatives on the set $A$ if

$$\partial_i^2 U(x + te_i) \leq \psi(x) \quad \text{for all } x \in A, t \in [-1,1] \text{ s.t. } x + te_i \in A$$

for $i = 1,2$. Here $e_1 = (1,0)$, $e_2 = (0,1)$ and $\partial_i$ is the partial derivative in direction $e_i$. The above definition is meant to imply that these derivatives exist. In the context of $\psi$-domination we will use the notion of a decay function, which is defined to satisfy

$$\|\psi\| < \infty \quad \text{and} \quad \int \psi(x)|x|^2 dx < \infty.$$

This definition of course does not depend on the choice of norm $|.|$, but for sake of definiteness let $|.|$ be the maximum norm on $\mathbb{R}^2$.

If $U$ is a potential, $z$ is an activity parameter and $\mathcal{X}_0$ is a set of boundary conditions, we say that the triple $(U, z, \mathcal{X}_0)$ is admissible if all conditional Gibbs distributions corresponding to $U$ and $z$ with boundary condition taken from $\mathcal{X}_0$ are well defined, see Definition 2 in Section 3.3. Important examples are the cases of superstable potentials with tempered boundary configurations and nonnegative potentials with arbitrary boundary conditions, see Section 3.4. For admissible $(U, z, \mathcal{X}_0)$ the set of Gibbs measures $\mathcal{G}_{\mathcal{X}_0}(U, z)$ corresponding to $U$ and $z$ with full weight on configurations in $\mathcal{X}_0$ is a well defined object. Finally we need bounded correlations: For admissible $(U, z, \mathcal{X}_0)$ we call $\xi \in \mathbb{R}$ a Ruelle bound if the correlation function of every Gibbs measure $\mu \in \mathcal{G}_{\mathcal{X}_0}(U, z)$ is bounded by powers of $\xi$ in the sense of (3.3) in Section 3.3.
Definition 1 Let \((U, z, X_0)\) be an admissible triple with Ruelle bound \(\xi\), where \(U : \mathbb{R}^2 \to \mathbb{R}\) is a translation-invariant standard potential. We say that \(U\) is smoothly approximable if there is a decomposition of \(U\) into a smooth part \(\bar{U}\) and a small part \(u\) in the following sense: We have a symmetric, compact set \(K \supset K^U\), a decay function \(\psi\) and measurable symmetric functions \(\bar{U}, u : K^c \to \mathbb{R}\) such that

\[
U = \bar{U} - u \quad \text{and} \quad u \geq 0 \quad \text{on} \quad K^c,
\]

\[
\bar{U} \quad \text{has} \quad \psi\text{-dominated derivatives on} \quad K^c,
\]

\[
\int_{K^c} \bar{u}(x)|x|^2 \, dx < \infty \quad \text{and} \quad \lambda^2(K \setminus K^U) + \int_{K^c} \bar{u}(x) \, dx < \frac{1}{z \xi},
\]

where \(\bar{u} := 1 - e^{-u} \leq u \land 1\).

The class of smoothly approximable standard potentials is a rich class of potentials. A smoothly approximable standard potential \(U\) may have a singularity or a hard core at the origin, and the type of convergence into the singularity or the hard core is fairly arbitrary, as we have not imposed any condition on \(U\) in the set \(K \setminus K^U\). For small activity \(z\) the last condition of (2.1) holds for large sets \(K\), which relaxes the conditions on \(U\). The small part \(u\) of \(U\) is not assumed to satisfy any regularity conditions, so that \(U\) doesn’t have to be smooth or continuous. We note that Definition 1 does not depend on the choice of the norm \(|.|\). If we know a potential to be smooth outside of its hard core the above conditions simplify:

Lemma 1 Let \((U, z, X_0)\) be an admissible triple with Ruelle bound \(\xi\), where \(U : \mathbb{R}^2 \to \mathbb{R}\) is a translation-invariant standard potential. Suppose we have a symmetric compact set \(K \supset K^U\) and a decay function \(\psi\) such that \(U\) has \(\psi\)-dominated derivatives on \(K^c\) and \(\lambda^2(K \setminus K^U) < 1/(z \xi)\). Then \(U\) is smoothly approximable.

This is an immediate consequence of Definition 1. In the non-smooth case, the following lemma gives important examples of smoothly approximable potentials:

Lemma 2 Let \((U, z, X_0)\) be an admissible triple with Ruelle bound \(\xi\), where \(U : \mathbb{R}^2 \to \mathbb{R}\) is a translation-invariant standard potential such that \(K^U\) is compact and \(U\) is continuous in \((K^U)^c\). Suppose we have a decay function \(\psi\) and a compact set \(\tilde{K} \subset \mathbb{R}^2\) such that one of the following properties holds:

(a) \(U\) has \(\psi\)-dominated derivatives in \(\tilde{K}^c\).

(b) There is a standard potential \(\tilde{U} \geq 0\) such that \(|U| \leq \tilde{U}\) in \(\tilde{K}^c\), \(\tilde{U}\) has \(\psi\)-dominated derivatives in \(\tilde{K}^c\) and \(\int_{\tilde{K}^c} \tilde{U}(x)|x|^2 \, dx < \infty\).

Then \(U\) is smoothly approximable.

For example, \(a\) holds trivially when \(U\) has finite range, and \(b\) includes the case that there are \(\epsilon > 0\) and \(k \geq 0\) such that \(|U(x)| \leq k/|x|^{4+\epsilon}\) for large \(|x|\). Our main result is now the following:
**Theorem 2** Let \((U, z, X_0)\) be admissible with Ruelle bound, where \(U : \mathbb{R}^2 \to \mathbb{R}\) is a translation-invariant standard potential. If \(U\) is smoothly approximable then every Gibbs measure \(\mu \in \mathcal{G}_{X_0}(U, z)\) is translation-invariant.

For a generalisation of the above result to the case of particles with inner degrees of freedom, i.e. Gibbsian systems of marked particles, we refer to [Ri2].

### 3 Setting

#### 3.1 State space

We will use the notations \(\mathbb{N} := \{0, 1, \ldots\}\), \(\mathbb{R}_+ := [0, \infty[\), \(\mathbb{R} := \mathbb{R} \cup \{+\infty\}\),

\[r_1 \vee r_2 := \max\{r_1, r_2\}\] and \(r_1 \wedge r_2 := \min\{r_1, r_2\}\) for \(r_1, r_2 \in \mathbb{R}\).

On \(\mathbb{R}^2\) we consider the maximum norm \(|.|\) and the Euclidean norm \(|.|_2\). For \(\epsilon > 0\) the \(\epsilon\)-enlargement of a set \(A \subset \mathbb{R}^2\) is defined by

\[A_\epsilon := \{x + x' : x \in A, |x'|_2 < \epsilon\}.

The state space of a particle is the plane \(\mathbb{R}^2\). The Borel-\(\sigma\)-algebra \(\mathcal{B}^2\) on \(\mathbb{R}^2\) is induced by any norm on \(\mathbb{R}^2\). Let \(\mathcal{B}^2\) be the set of all bounded Borel sets and \(\lambda^2\) be the Lebesgue measure on \((\mathbb{R}^2, \mathcal{B}^2)\). Integration with respect to this measure will be abbreviated by \(dx := d\lambda^2(x)\). Often we consider the centred squares

\[\Lambda_r := [-r, r]^2 \subset \mathbb{R}^2 \quad (r \in \mathbb{R}_+).

We also want to consider bonds between particles. For a set \(X\) we denote the set of all bonds in \(X\) by

\[E(X) := \{A \subset X : \#A = 2\}.

A bond will be denoted by \(xx' := \{x, x'\}\), where \(x, x' \in X\) such that \(x \neq x'\). For a bond set \(B \subset E(X)\) \((X, B)\) is an (undirected) graph, and we set

\[x \xrightarrow{X, B} x' : \iff \exists m \in \mathbb{N}, x_0, \ldots, x_m \in X : x = x_0, x' = x_m,
\]

\[x_{i-1} x_i \in B \text{ for all } 1 \leq i \leq m.

This connectedness relation is an equivalence relation on \(X\) whose equivalence classes are called the \(B\)-clusters of \(X\). Let

\[C_{X, B}(x) := \{x' \in X : x \xrightarrow{X, B} x'\}\] and \(C_{X, B}(\Lambda) := \bigcup_{x' \in X \cap \Lambda} C_{X, B}(x')\)

denote the \(B\)-clusters of a point \(x\) and a set \(\Lambda\) respectively. Primarily we are interested in the case \(X = \mathbb{R}^2\). On the corresponding bond set \(E(\mathbb{R}^2)\) we consider the \(\sigma\)-algebra

\[\mathcal{F}_{E(\mathbb{R}^2)} := \left\{\{x_1 x_2 \in E(\mathbb{R}^2) : (x_1, x_2) \in M\} : M \in (\mathcal{B}^2)^2\right\}.

Every symmetric function \(u\) on \(\mathbb{R}^2\) can be considered a function on \(E(\mathbb{R}^2)\) via \(u(xx') := u(x - x')\).
3.2 Configuration space

A set of particles \( X \subset \mathbb{R}^2 \) is called finite if \( \#X < \infty \), and locally finite if \( \#(X \cap \Lambda) < \infty \) for all \( \Lambda \in \mathcal{B}^2 \), where \( \# \) denotes the cardinality of a set. The configuration space \( \mathcal{X} \) of particles is defined as the set of all locally finite subsets of \( \mathbb{R}^2 \), and its elements are called configurations of particles. For \( X, \bar{X} \in \mathcal{X} \) let \( XX := X \cup \bar{X} \). For \( X \in \mathcal{X} \) and \( \Lambda \in \mathcal{B}^2 \) let

\[
\begin{align*}
X_\Lambda := X \cap \Lambda & \quad \text{(restriction of } X \text{ to } \Lambda), \\
\mathcal{X}_\Lambda := \{ X \in \mathcal{X} : X \subset \Lambda \} & \quad \text{(set of all configurations in } \Lambda) \text{ and} \\
N_\Lambda(X) := \# X_\Lambda & \quad \text{(number of particles of } X \text{ in } \Lambda).
\end{align*}
\]

The counting variables \( (N_\Lambda)_{\Lambda \in \mathcal{B}^2} \) generate a \( \sigma \)-algebra on \( \mathcal{X} \), which will be denoted by \( \mathcal{F}_X \). For \( \Lambda \in \mathcal{B}^2 \) let \( \mathcal{F}'_{\mathcal{X},\Lambda} \) be the \( \sigma \)-algebra on \( \mathcal{X}_\Lambda \) obtained by restricting \( \mathcal{F}_X \) to \( \mathcal{X}_\Lambda \), and let \( \mathcal{F}_\mathcal{X},\Lambda := e_{\Lambda}^{-1} \mathcal{F}'_{\mathcal{X},\Lambda} \) be the \( \sigma \)-algebra on \( \mathcal{X} \) obtained from \( \mathcal{F}'_{\mathcal{X},\Lambda} \) by the restriction mapping \( e_\Lambda : \mathcal{X} \to \mathcal{X}_\Lambda, X \mapsto X_\Lambda \). The tail \( \sigma \)-algebra or \( \sigma \)-algebra of the events far from the origin is defined by

\[
\mathcal{F}_{\mathcal{X},\infty} := \bigcap_{n \geq 1} \mathcal{F}_{\mathcal{X},\Lambda_n},
\]

Let \( \nu \) be the distribution of the Poisson point process on \( (\mathcal{X}, \mathcal{F}_X) \), i.e.

\[
\int \nu(dX)f(X) = e^{-\lambda^2(\Lambda)} \sum_{k \geq 0} \frac{1}{k!} \int_{\Lambda^k} dx_1 \ldots dx_k f(\{x_i : 1 \leq i \leq k\}),
\]

for any \( \mathcal{F}_{\mathcal{X},\Lambda} \)-measurable function \( f : \mathcal{X} \to \mathbb{R}_+ \), where \( \Lambda \in \mathcal{B}^2_+ \). For \( \Lambda \in \mathcal{B}^2_+ \) and \( \bar{X} \in \mathcal{X} \) let \( \nu_\Lambda(\cdot | \bar{X}) \) be the distribution of the Poisson point process in \( \Lambda \) with boundary condition \( \bar{X} \), i.e.

\[
\int \nu_\Lambda(dX | \bar{X})f(X) = \int \nu(dX)f(X_\Lambda \bar{X}_\Lambda^c)
\]

for any \( \mathcal{F}_\mathcal{X} \)-measurable function \( f : \mathcal{X} \to \mathbb{R}_+ \). It is easy to see that \( \nu_\Lambda \) is a stochastic kernel from \( (\mathcal{X}, \mathcal{F}_{\mathcal{X},\Lambda^c}) \) to \( (\mathcal{X}, \mathcal{F}_X) \).

The configuration space of bonds \( \mathcal{E} \) is defined to be the set of all locally finite bond sets, i.e.

\[
\mathcal{E} := \{ B \subset E(\mathbb{R}^2) : \#\{xx' \in B : xx' \subset \Lambda \} < \infty \text{ for all } \Lambda \in \mathcal{B}^2_+ \}.
\]

On \( \mathcal{E} \) the \( \sigma \)-algebra \( \mathcal{F}_\mathcal{E} \) is defined to be generated by the counting variables \( N_\mathcal{E} : \mathcal{E} \to \mathbb{N}, B \mapsto \#(E \cap B) \) \( (E \in \mathcal{F}_{E(\mathbb{R}^2)}) \). For a countable set \( E \in \mathcal{E} \) one can also consider the Bernoulli-\( \sigma \)-algebra \( \mathcal{B}_E \) on \( \mathcal{E}_E := \mathcal{P}(E) \subset \mathcal{E} \), which is defined to be generated by the family of sets \( \{B \subset E : e \in B\}_{e \in E} \). Given a family \( (p_e)_{e \in E} \) of reals in \( [0,1] \) the Bernoulli measure on \( (\mathcal{E}_E, \mathcal{B}_E) \) is defined as the unique probability measure for which the events \( \{B \subset E : e \in B\}_{e \in E} \) are independent with probabilities \( (p_e)_{e \in E} \). It is easy to check that the inclusion \( (\mathcal{E}_E, \mathcal{B}_E) \to (\mathcal{E}, \mathcal{F}_\mathcal{E}) \) is measurable. Thus any probability measure on \( (\mathcal{E}_E, \mathcal{B}_E) \) can trivially be extended to \( (\mathcal{E}, \mathcal{F}_\mathcal{E}) \).
3.3 Gibbs measures

Let \( U : \mathbb{R}^2 \to \mathbb{R} \) be a potential and \( z > 0 \) an activity parameter. For finite configurations \( X, X' \in \mathcal{X} \) we consider the energy terms

\[
H^U(X) := \sum_{x_1x_2 \in E(X)} U(x_1 - x_2) \quad \text{and} \quad W^U(X, X') := \sum_{x_1 \in X} \sum_{x_2 \in X'} U(x_1 - x_2).
\]

The last definition can be extended to infinite configurations \( X' \) whenever \( W^U(X, X'_\Lambda) \) converges as \( \Lambda \uparrow \mathbb{R}^2 \) through the net \( \mathcal{B}_b^2 \). The Hamiltonian of a configuration \( X \in \mathcal{X} \) in \( \Lambda \in \mathcal{B}_b^2 \) is given by

\[
H^U_\Lambda(X) := H^U(X_\Lambda) + W^U(X_\Lambda, X_{\Lambda^c}) = \sum_{x_1x_2 \in E_\Lambda(X)} U(x_1 - x_2),
\]

where

\[
E_\Lambda(X) := \{x_1x_2 \in E(X) : x_1x_2 \cap \Lambda \neq \emptyset\}.
\]

The integral

\[
Z^{U,z}_\Lambda(\bar{X}) := \int \nu_\Lambda(dX|\bar{X}) e^{-H^U_\Lambda(X)z\#X_\Lambda}
\]

is called the partition function in \( \Lambda \in \mathcal{B}_b^2 \) for the boundary condition \( \bar{X}_{\Lambda^c} \in \mathcal{X} \). In order to ensure that the above objects are well defined and the partition function is finite and positive we need the following definition:

**Definition 2** A triple \((U, z, X_0)\) consisting of a potential \( U : \mathbb{R}^2 \to \mathbb{R}\), an activity parameter \( z > 0 \) and a set of boundary conditions \( X_0 \in \mathcal{F}_{X,\infty} \) is called admissible if for all \( \bar{X} \in X_0 \) and \( \Lambda \in \mathcal{B}_b^2 \) the following holds: \( W^U(\bar{X}_\Lambda, \bar{X}_{\Lambda^c}) \) has a well defined value in \( \mathbb{R} \) and \( Z^{U,z}_\Lambda(\bar{X}) \) is finite.

If \((U, z, X_0)\) is admissible, \( \Lambda \in \mathcal{B}_b^2 \) and \( \bar{X} \in X_0 \) then \( W^U(\bar{X}_\Lambda, \bar{X}_{\Lambda^c}) \subseteq \mathbb{R} \) is well defined for every \( X \in \mathcal{X} \), because \( X_0 \in \mathcal{F}_{X,\infty} \) implies \( X_\Lambda \bar{X}_{\Lambda^c} \subseteq X_0 \). As a consequence the partition function \( Z^{U,z}_\Lambda(\bar{X}) \) is well defined. Furthermore by definition it is finite and by considering the empty configuration one can show that it is positive. The conditional Gibbs distribution \( \gamma^{U,z}_\Lambda(\cdot|\bar{X}) \) in \( \Lambda \in \mathcal{B}_b^2 \) with boundary condition \( \bar{X} \in X_0 \) is thus well defined by

\[
\gamma^{U,z}_\Lambda(A|\bar{X}) := \frac{1}{Z^{U,z}_\Lambda(\bar{X})} \int \nu_\Lambda(dX|\bar{X}) e^{-H^U_\Lambda(X)z\#X_\Lambda} 1_A(X) \quad \text{for} \quad A \in \mathcal{F}_X.
\]

\( \gamma^{U,z}_\Lambda \) is a probability kernel from \((X_0, \mathcal{F}_{X_0,\Lambda^c})\) to \((\mathcal{X}, \mathcal{F}_X)\). Let

\[
\mathcal{S}_{X_0}(U, z) := \{ \mu \in \mathcal{P}_1(\mathcal{X}, \mathcal{F}_X) : \mu(X_0) = 1 \} \quad \text{and} \quad \mu(A|\mathcal{F}_{X,\Lambda^c}) = \gamma^{U,z}_\Lambda(A|.) \quad \text{\( \mu \)-a.s.} \quad \forall A \in \mathcal{F}_X, \Lambda \in \mathcal{B}_b^2
\]

be the set of all Gibbs measures corresponding to \( U \) and \( z \) with whole weight on boundary conditions in \( X_0 \). It is easy to see that for any probability measure \( \mu \in \mathcal{P}_1(\mathcal{X}, \mathcal{F}_X) \) such that \( \mu(X_0) = 1 \) we have the equivalence

\[
\mu \in \mathcal{S}_{X_0}(U, z) \iff (\mu \otimes \gamma^{U,z}_\Lambda = \mu \forall \Lambda \in \mathcal{B}_b^2).
\]
So for every \( \mu \in \mathcal{G}_{X_0}(U, z) \), \( f : X \to \mathbb{R}_+ \) measurable and \( \Lambda \in \mathcal{B}_2^2 \) we have

\[
\int \mu(dX) f(X) = \int \mu(d\hat{X}) \int \gamma^U_{\Lambda}(dX|\hat{X}) f(X). \tag{3.1}
\]

If we consider a fixed potential and a fixed activity we will omit the dependence on \( U \) and \( z \) in the notations \( \gamma^U_{\Lambda} \) and \( Z^U_{\Lambda} \). As a consequence of (3.1) the hard core \( K^U \) of a potential \( U \) implies that particles are not allowed to get too close to each other, i.e. for admissible \( (U, z, X_0) \) and \( \mu \in \mathcal{G}_{X_0}(U, z) \) we have

\[
\mu(\{X \in X : \exists x, x' \in X : x \neq x', x - x' \in K^U\}) = 0. \tag{3.2}
\]

For admissible \( (U, z, X_0) \) and a Gibbs measure \( \mu \in \mathcal{G}_{X_0}(U, z) \) we define the correlation function \( \rho^U_{\mu} \) by

\[
\rho^U_{\mu}(X) = e^{-\mathcal{H}^U(X)} \int \mu(d\hat{X}) e^{-\mathcal{W}^U(X, \hat{X})}
\]

for any finite configuration \( X \in \mathcal{X} \). If there is a \( \xi = \xi(U, z, X_0) \geq 0 \) such that

\[
\rho^U_{\mu}(X) \leq \xi^\#X \quad \text{for all finite } X \in \mathcal{X} \text{ and all } \mu \in \mathcal{G}_{X_0}(U, z), \tag{3.3}
\]

then we call \( \xi \) a Ruelle bound for \( (U, z, X_0) \). Actually we need this bound on the correlation function in the following way:

**Lemma 3** Let \( (U, z, X_0) \) be admissible with Ruelle bound \( \xi \). For every Gibbs measure \( \mu \in \mathcal{G}_{X_0}(U, z) \) and every measurable \( f : \mathbb{R}^m \to \mathbb{R}_+, m \in \mathbb{N} \) we have

\[
\int \mu(dX) \sum_{x_1, \ldots, x_m \in X} f(x_1, \ldots, x_m) \leq (z\xi)^m \int dx_1 \ldots dx_m f(x_1, \ldots, x_m). \tag{3.4}
\]

We use \( \Sigma^\# \) as a shorthand notation for a multiple sum such that the summation indices are assumed to be pairwise distinct.

### 3.4 Superstability and admissibility

Now we will discuss some conditions on potentials which imply that \( (U, z, X_0) \) is admissible and has a Ruelle bound whenever the set of boundary conditions \( X_0 \) is suitably chosen. Apart from purely repulsive potentials such as the pure hard-core potential considered in Theorem 1 we also want to consider superstables in the sense of Ruelle, see [R]. Therefore let

\[
\Gamma_r := r + [-1/2, 1/2]^2 \subset \mathbb{R}^2 \quad (r \in \mathbb{Z}^2)
\]

be the unit square centred at \( r \) and let

\[
\mathbb{Z}^2(X) := \{r \in \mathbb{Z}^2 : N_{\Gamma_r}(X) > 0\}
\]

be the minimal set of lattice points such that the corresponding squares cover the configuration. A potential \( U : \mathbb{R}^2 \to \overline{\mathbb{R}} \) is called superstables if there are real constants \( a > 0 \) and \( b \geq 0 \) such that for all finite configurations \( X \in \mathcal{X} \)

\[
H^U(X) \geq \sum_{r \in \mathbb{Z}^2(X)} [aN_{\Gamma_r}(X)^2 - bN_{\Gamma_r}(X)].
\]
$U$ is called lower regular if there is a decreasing function $\Psi : \mathbb{N} \to \mathbb{R}_+$ with $
abla \Psi(|r|) < \infty$ such that

$$W_U(X, X') \geq - \sum_{r \in \mathbb{Z}^2} \sum_{s \in \mathbb{Z}^2} \Psi(|r - s|) \left[ \frac{1}{2} N_{r_0}(X)^2 + \frac{1}{2} N_{r_0}(X')^2 \right]$$

for all finite configurations $X, X' \in \mathcal{X}$. So superstability and lower regularity give lower bounds on energies in terms of particle densities. In order to be able to control these densities, a configuration $X \in \mathcal{X}$ is defined to be tempered if

$$\bar{s}(X) := \sup_{n \in \mathbb{N}} s_n(X) < \infty,$$

where $s_n(X) := \frac{1}{(2n + 1)^2} \sum_{r \in \mathbb{Z}^2 \cap \Lambda_{n+1/2}} N_{r_0}^2(X)$.

By $\mathcal{X}_t$ we denote the set of all tempered configurations. We note that $\mathcal{X}_t \in \mathcal{F}_{\mathcal{X}, \infty}$.

\begin{lemma}
Let $z > 0$ and $U : \mathbb{R}^2 \to \mathbb{R}$ be a translation-invariant pair potential.

(a) If $U$ is purely repulsive, i.e. $U \geq 0$, then $(U, z, \mathcal{X})$ is admissible with Ruelle bound $\xi := 1$.

(b) If $U$ is superstable and lower regular then $(U, z, \mathcal{X}_t)$ is admissible and admits a Ruelle bound.

The first assertion is a straightforward consequence of the fact that all energy terms are nonnegative. For the second assertion see [R].

3.5 Conservation of translational symmetry

Every $\vec{\tau} \in \mathbb{R}^2$ gives a translation on the configuration space $\mathcal{X}$ via

$$g_{\vec{\tau}}(X) := X + \vec{\tau} := \{x + \vec{\tau} : x \in X\}.$$

We say that a measure $\mu$ on $(\mathcal{X}, \mathcal{F}_\mathcal{X})$ is $\vec{\tau}$-invariant if $\mu \circ g_{\vec{\tau}^{-1}} = \mu$, and $\mu$ is translation-invariant if it is $\vec{\tau}$-invariant for every $\vec{\tau} \in \mathbb{R}^2$. The following lemma gives a sufficient condition for the conservation of $\vec{\tau}$-symmetry.

\begin{lemma}
Let $(U, z, \mathcal{X}_0)$ be admissible, where $U : \mathbb{R}^2 \to \mathbb{R}$ is a translation-invariant potential. If for all cylinder events $D \in \mathcal{F}_{\mathcal{X}, \Lambda_m}$ ($m \in \mathbb{N}$) and all Gibbs measures $\mu \in \mathcal{G}_{\mathcal{X}_0}(U, z)$ we have

$$\mu(D + \vec{\tau}) + \mu(D - \vec{\tau}) \geq \mu(D),$$

then every Gibbs measure $\mu \in \mathcal{G}_{\mathcal{X}_0}(U, z)$ is $\vec{\tau}$-invariant.

We further note that $\mathbb{R}^2$ is generated by the set $\{\tau_i e_i : 0 \leq \tau_i < 1/2, i \in \{1, 2\}\}$, so we only have to consider translations of this special form in order to establish translation-invariance of a set of Gibbs measures.
3.6 Concerning measurability

We will consider various types of random objects, all of which have to be shown to be measurable with respect to the considered \( \sigma \)-algebras. However we will not prove measurability of every such object in detail. Instead we will now give a list of operations that preserve measurability.

**Lemma 6** Let \( X, X' \in \mathcal{X}, B, B' \in \mathcal{E}, x \in \mathbb{R}^2 \) and \( p \in \Omega \) be variables, where \( (\Omega, \mathcal{F}) \) is a measurable space. Let \( f : \Omega \times \mathbb{R}^2 \to \mathbb{R} \) and \( g : \Omega \times E(\mathbb{R}^2) \to \mathbb{R} \) be measurable. Then the following functions of the given variables are measurable with respect to the considered \( \sigma \)-algebras:

\[
\begin{align*}
\sum_{x' \in X} f(p, x'), & \quad X \cap X', \quad X \cup X', \quad X \setminus X', \quad X + x, \\
\sum_{b' \in B} g(p, b'), & \quad B \cap B', \quad B \cup B', \quad B \setminus B', \quad B + x, \\
\inf_{x' \in X} f(p, x'), & \quad \{x' \in X : f(p, x') = 0\}, \quad C_{X,B}(x), \quad E(X), \\
\text{the number of different clusters of } (X, B).
\end{align*}
\]

Using this lemma and well known theorems, such as the measurability part of Fubini’s theorem, we can check the measurability of all objects considered.

4 Proof of the lemmas from Sections 2 and 3

4.1 Smoothly approximable potentials: Lemma 2

Let \((U, z, X_0), \xi, \psi, \tilde{K}\) and \(\tilde{U}\) (in case (b)) be as in the formulation of Lemma 2. By compactness of \(K^U\) we can choose an \(\epsilon > 0\) such that the \(\epsilon\)-enlargement \(K := (K^U)_\epsilon\) of the hard core \(K^U\) has the property

\[
c := 1/(z\xi) - \lambda^2(K \setminus K^U) > 0.
\]

In case (a) let \(U_1 := U\) and in case (b) let \(U_1 := \tilde{U}\). Let \(R \geq 1\) such that

\[
K \cup \tilde{K} \subset \Lambda_R \quad \text{and furthermore} \quad \int_{\Lambda_R} 2\tilde{U}(x)|x|^2 dx < \frac{c}{2} \quad \text{in case (b)}.
\]

In both cases \(U_1\) serves as an approximation of \(U\) on \(\Lambda_R\). Let \(C := \Lambda_{R+1} \setminus K\), \(\delta > 0\) and \(f_\delta : \mathbb{R} \to \mathbb{R}_+\) be a symmetric smooth probability density with support in the \(|,|^-2\)-disc \(B_2(\delta)\), e.g.

\[
f_\delta(x) := \frac{1}{c_\delta}1_{B_2(\delta)}(x)e^{-(1-|x|^2/\delta^2)^{-1}}, \quad \text{where } c_\delta := \int_{B_2(\delta)} e^{-(1-|x|^2/\delta^2)^{-1}} dx.
\]

Then

\[
U_2(x) := U * f_\delta(x) := \int dx' f_\delta(x')U(x - x')
\]
is a smooth approximation of $U$ on $C$. By continuity of $U$ and compactness of $C$ a sufficiently small $\delta$ guarantees

$$|U_2(x) - U(x)| < c' := \frac{c}{4\lambda^2(C)}$$

for $x \in C$.

Let $g : \mathbb{R}^2 \to [0, 1]$ be a smooth symmetric function such that $g = 0$ on $\Lambda_R$ and $g = 1$ on $\Lambda_R^{c+1}$. Now we can define $\bar{U}, u : K^c \to \mathbb{R}$ by

$$\bar{U} := (1 - g)(U_2 + c') + gU_1 \quad \text{and} \quad u := \bar{U} - U.$$  

It is easy to verify that the constructed objects have all the properties described in Definition 1 in both cases (a) and (b).

### 4.2 Property of the Ruelle bound: Lemma 3

For every $n \in \mathbb{N}$, every measurable $g : X_{\Lambda_n} \to \mathbb{R}_+$ and every $\bar{X} \in X_0$ we have

$$\int \nu_{\Lambda_n}(dX|\bar{X}) \sum_{x_1,\ldots,x_m \in X_{\Lambda_n} \neq \emptyset} f(x_1,\ldots,x_m) g(X)$$

$$= \int_{\Lambda_n^m} dx_1 \ldots dx_m f(x_1,\ldots,x_m) \int \nu_{\Lambda_n}(dX|\bar{X}) g(\{x_1,\ldots,x_m\}X').$$

Combining this with (3.1), the definition of the conditional Gibbs distribution and the definition of the correlation function we get

$$\int \mu(dX) \sum_{x_1,\ldots,x_m \in X_{\Lambda_n} \neq \emptyset} f(x_1,\ldots,x_m)$$

$$= \int \mu(d\bar{X}) \frac{1}{Z_{\Lambda_n^m}(\bar{X})} \int \nu_{\Lambda_n}(dX|\bar{X}) \sum_{x_1,\ldots,x_m \in X_{\Lambda_n} \neq \emptyset} f(x_1,\ldots,x_m) e^{-H_{\Lambda_n}^U(X)\#X_{\Lambda_n}}$$

$$= \int_{\Lambda_n^m} dx_1 \ldots dx_m f(x_1,\ldots,x_m) z^m \rho^{U,\mu}(\{x_1,\ldots,x_m\}).$$

Now we use (3.3) to estimate the correlation function by the Ruelle bound $\xi$. Letting $n \to \infty$ the assertion follows from the monotone limit theorem.

### 4.3 Sufficient condition: Lemma 5

The lemma can be shown exactly as Proposition (9.1) in [G] and we will only outline the proof: We first note that $(X, F_X)$ is a standard Borel space, which follows from [DV], Theorem A2.6.III. Hence the point particle version of Theorem (7.26) in [G] implies that every Gibbs measure can be decomposed into extremal Gibbs measures. Thus without loss of generality we may assume $\mu$ to be extremal. Suppose now that $\mu$ is not $\bar{\tau}$-invariant, i.e. $\mu \circ g_{\bar{\tau}}^{-1} \neq \mu$, which also implies $\mu \circ g_{\bar{\tau}} \neq \mu$. As the extremality of $\mu$ implies the extremality of $\mu \circ g_{\bar{\tau}}^{-1}$ and $\mu \circ g_{\bar{\tau}}$, the point particle version of Theorem (7.7) guarantees the existence of sets $A_-, A_+ \in F_{X,\infty}$ such that $\mu \circ g_{\bar{\tau}}^{-1}(A_-) = 0$, $\mu \circ g_{\bar{\tau}}(A_+) = 0$ and $\mu(A_-) = \mu(A_+) = 1$. Hence for $A := A_- \cap A_+$ we have

$$\mu \circ g_{\bar{\tau}}(A) + \mu \circ g_{\bar{\tau}}^{-1}(A) = 0 < 1 = \mu(A).$$
On the other hand by assumption (3.5) we know that \( \mu \circ g_{x} + \mu \circ g_{-1} \geq \mu \) on the algebra of all cylinder events. By the monotone class theorem this inequality even holds on all of \( \mathcal{F}_X \), which contradicts the above inequality.

### 4.4 Measurability: Lemma \([6]\)

Details concerning measurability of functions of point processes can be found in [DV], [K] or [MKM], for example. The first part of (3.6) is the measurability part of Campbell’s theorem. For the rest of (3.6) it suffices to observe that for \( \Lambda \in \mathcal{B}_2^b \) we have

\[
N_{\Lambda}(X \cap X') = \sum_{x \in X} \sum_{x' \in \Lambda} 1_{\{x-x' \in \Lambda\}}, \quad N_{\Lambda}(X \setminus X') = N_{\Lambda}(X) - N_{\Lambda}(X \cap X'),
\]

\[
N_{\Lambda}(X + x) = \sum_{x' \in \Lambda} 1_{\Lambda}(x' + x) \quad \text{and} \quad N_{\Lambda}(X \cup X') = N_{\Lambda}(X) + N_{\Lambda}(X' \setminus X).
\]

(3.7) can be proved similarly. For \( c \in \mathbb{R}, \Lambda \in \mathcal{B}_2^b, x' \in \mathbb{R}^2 \) and \( L \in \mathcal{F}_{E(\mathbb{R}^2)} \)

\[
\inf_{x' \in X} f(p, x') < c \iff \sum_{x' \in X} 1_{\{f(p, x') < c\}} \geq 1,
\]

\[
N_{\Lambda}\{x' \in X : f(p, x') = 0\} = \sum_{x' \in X} 1_{\{f(p, x') = 0, x' \in \Lambda\}},
\]

\[
N_{\Lambda}(C_{X,B}(x)) = \sum_{x' \in X} 1_{\{x' \in C_{X,B}(x), x' \in \Lambda\}},
\]

\[
x' \in C_{X,B}(x) \iff \sum_{m \geq 0} \sum_{x_0, \ldots, x_m \in X} 1_{\{x=x_0, x'=x_m\}} \prod_{i=1}^{m} 1_{\{x, x_i+1 \in B\}} \geq 1 \quad \text{and}
\]

\[
N_{L}(E(X)) = \frac{1}{2} \sum_{x_1 \in X} \sum_{x_2 \in X \setminus \{x_1\}} 1_{L}(x_1x_2).
\]

Using these relations, the measurability of the terms in (3.8) follows easily. For (3.9) it suffices to observe that there are at most \( k \) different clusters of \( (X, B) \) iff

\[
\sum_{x_1, \ldots, x_k \in X} 1_{\{X \cap (C_{X,B}(x_1) \cup \ldots \cup C_{X,B}(x_k)) = \emptyset\}} \geq 1.
\]

### 5 Proof of Theorem \([1]\): Main steps

#### 5.1 Basic constants

Let \( z > 0 \). Let \( |.|_h \) be a norm on \( \mathbb{R}^2 \) and \( U := U_{hc} \) the corresponding pure hard-core potential. As \( U \) is purely repulsive we know that \( (U, z, X) \) is admissible with Ruelle bound \( \xi := 1 \) by Lemma \([4]\) part (a). Let \( K := K^U \) and \( \epsilon > 0 \). If we choose \( \epsilon \) sufficiently small we have

\[
c_{\xi} := \lambda^2(K^U \setminus K^\epsilon) < \frac{1}{z\xi}, \quad (5.1)
\]
where \( K_\epsilon \) is the \( \epsilon \)-enlargement of \( K \). Let \( f_K: \mathbb{R}^2 \to \mathbb{R} \) be a function such that

\[
f_K \text{ is smooth, } f_K = 0 \text{ on } K \quad \text{and} \quad f_K = 1 \text{ on } (K_\epsilon)^c.
\]

Furthermore we need the following finite constants:

\[
c_K := \sup \{|x| : x \in K_\epsilon\} \quad \text{and} \quad c_f := \sup \{|f_K'(x)| : x \in \mathbb{R}^2\}.
\] (5.2)

On \( \mathbb{R}^2 \) let \( \leq \) be the lexicographic order and let the partial order \( \leq_{e_1} \) be defined by

\[
(r_1, r_2) \leq_{e_1} (r'_1, r'_2) \iff r_1 \leq r'_1, r_2 = r'_2.
\]

In order to show the conservation of translational symmetry we fix a Gibbs measure \( \mu \in \mathcal{G}_{X_0(U, z)} \) and a cylinder event \( D \in \mathcal{F}_{X, \Lambda_{n'}} \) where \( n' \in \mathbb{N} \), see Subsection 3.5. As mentioned there it suffices to consider translations \( \tau e \), where \( \tau \in [0, 1/2] \) and \( e = e_1 \) or \( e_2 \). Hence we fix \( \tau \in [0, 1/2] \), and by symmetry we may assume that \( e = e_1 \). We also fix an arbitrarily small real \( \delta > 0 \) in order to control probabilities close to 0. As all the above objects are fixed for the whole proof we will ignore dependence on them in our notations.

### 5.2 Generalised translation

Let \( n > n' \) and \( X \in \mathcal{X} \). We consider the bond set

\[
K^n_\epsilon := \{x_1x_2 \in E(X) : x_1x_2 \cap \Lambda_n \neq \emptyset, x_1 - x_2 \in K_\epsilon\}.
\]

Every time we use this notation it will be clear from the context which configuration \( X \) it refers to. Note that \( K^n_\epsilon \) is finite as \( X \) is locally finite and \( K_\epsilon \) is bounded. For a bounded set \( \Lambda \in \mathcal{B}_2 \) let

\[
r_{n,X}(\Lambda) = \sup\{|y'| : y' \in C_{X,K^n_\epsilon}(\Lambda)\}
\]

denote the range of the corresponding \( K^n_\epsilon \)-cluster. In the following lemma we consider the case \( \Lambda = \Lambda_{n'} \), where \( n' \in \mathbb{N} \) is the number fixed in Section 5.1.

**Lemma 7** We have

\[
\sup_{n > n'} \int \mu(dX) r_{n,X}(\Lambda_{n'}) < \infty.
\]

By the Chebyshev inequality we therefore can choose an integer \( R > n' \), such that for every \( n > n' \) we have

\[
\mu(G'_n) \geq 1 - \frac{\delta}{2} \quad \text{for} \quad G'_n := \{X \in \mathcal{X} : r_{n,X}(\Lambda_{n'}) < R\} \in \mathcal{F}_X.
\]

For \( n > R \) we define the functions

\[
q: \mathbb{R}_+ \to \mathbb{R}, \quad Q: \mathbb{R}_+ \to \mathbb{R}, \quad r: \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R} \quad \text{and} \quad \tau_n: \mathbb{R} \to \mathbb{R} \quad \text{by}
\]

\[
q(s) := \frac{1}{1 \vee (s \log(s))}, \quad Q(k) := \int_0^k q(s)ds, \quad r(s, k) := \int_{(s \vee 0) \land k} \frac{q(s')}{Q(k)}ds', \quad \tau_n(s) := \tau(r(s - R, n - R)).
\]

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Some important properties of $\tau_n$ are the following:

\[ \tau_n(s) = \tau \text{ for } s \leq R, \quad \tau_n(s) = 0 \text{ for } s \geq n \]
and $\tau_n$ is decreasing. (5.3)

For $X \in \mathcal{X}$ and $x \in X$ we define $a_{n,X}(x)$ to be the point of $C_{X,K_n^0}(x)$ with maximal $|.|$-distance to the origin. (If there is more than one such point we choose the maximal one with respect to the lexicographic order for the sake of definiteness.) Then (5.3) implies

\[ |a_{n,X}(x)| \geq |x| \quad \text{and} \quad \tau_n(|a_{n,X}(x)|) = \min\{\tau_n(|x'|) : x' \in C_{X,K_n^0}(x)\}. \]

The transformation $T^0_n : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $T^0_n(x) := x + \tau_n(|x|)e_1$ can also be viewed as a transformation on $\mathcal{X}$, such that every point $x$ of a configuration $X$ is translated the distance $\tau_n(|x|)$ in direction $e_1$. We would like to use this generalised translation $T^0_n$ as a tool for our proof just as in [FP1] and [FP2].

### 5.3 Good configurations

In order to deal with the hard core we will replace the above translation $T^0_n$ by a transformation

\[ \tau_n : \mathcal{X} \rightarrow \mathcal{X} \]

which is required to have the following properties:

1. For $X \in \mathcal{X}$ the transformed configuration $\tilde{X} = \tau_n(X)$ is constructed by translating every $x \in X$ a certain distance $t_{n,X}(x)$ in direction $e_1$. We note that we do not require the particles to be translated independently.

2. Particles in the inner region $\Lambda_{n-1}$ are translated by $\tau e_1$, and particles in the outer region $\Lambda_n^c$ are not translated at all.

3. $\tau_n$ is bijective, the density of the transformed process with respect to the untransformed process under the measure $\nu$ can be calculated explicitly and we have a suitable estimate on this density.

4. The Hamiltonian $H_{\Lambda_n}^{\tilde{X}}(X)$ is invariant under $\tau_n$, i.e. particles within hard core distance remain within hard core distance and particles at larger distance remain at larger distance.
Property (2) implies that the translation of the chosen cylinder event $D$ is the same as the transformation of $D$ by $\mathcal{T}_n$. Properties (3) and (4) imply that the density of the transformed process with respect to the untransformed process under the measure $\mu$ can be estimated. Therefore a transformation with these properties seems to be a good tool for proving (3.5). However, in general it is difficult to construct a transformation with all the given properties. For example properties (2) and (4) cannot both be satisfied if $X$ is a configuration of densely packed hard-core particles. If $n > R$ and $X \in G'_n$ then such a situation can not occur, and by Lemma 7 this is the case with high probability. Similar problems arise for the other properties, so we will content ourselves with a transformation satisfying the above properties only for configurations $X$ from a set of good configurations

$$G_n := \{ X \in G'_n : \sum_{i=1}^{3} \Sigma_i(n, X) < 1 \} \in \mathcal{F}_X.$$ (5.4)

The functions $\Sigma_i(n, X)$ will be defined whenever we want good configurations to have a certain property. In Lemma 13 we then will prove that the set of good configurations $G_n$ has probability close to 1 when $n$ is big enough. Up to that point we consider a fixed $n \geq R + 1$.

### 5.4 Modifying the generalised translation

With a view to properties (1) and the second part of (2) we define the transformation $\mathcal{T}_n : X \rightarrow X$ by

$$\mathcal{T}_n(X) := X_{\Lambda_n} \cup \{ P_{n,X}^k + \tau_{n,X}^k e_1 : 1 \leq k \leq m(X) \} = \{ x + t_{n,X}(x)e_1 : x \in X \}$$

for every $X \in \mathcal{X}$, where $m(X) := \# X_{\Lambda_n}$, $\{ P_{n,X}^k : 1 \leq k \leq m(X) \} = X_{\Lambda_n}$, $\tau_{n,X}^k$ is the translation distance of $P_{n,X}^k$ and the translation distance function $t_{n,X} : X \rightarrow \mathbb{R}$ is defined by $t_{n,X}(x) := 0$ for $x \in X_{\Lambda_n}$ and $t_{n,X}(P_{n,X}^k) := \tau_{n,X}^k$ for $1 \leq k \leq m(X)$. We are left to identify the points $P_{n,X}^k$ of $X$ and their translation

$$\begin{align*}
\Lambda_n &\sim \\
\Lambda_R &\sim \\
\tau_1^{n} &\sim \\
\tau_2^{n} &\sim \\
\tau_3^{n} &\sim \\
\tau_4^{n} & = \tau
\end{align*}$$

Figure 2: Every point $P_n^k$ is translated by $\tau_n^k e_1$

distances $\tau_{n,X}^k$. In order to simplify notation we will omit the dependence on
X in m(X), \( t_n, P_n^k, \) and \( \tau_n^k \) if it is clear which configuration is considered. In our construction we would like to ensure that the points \( P_n^k \) are ordered in a way such that

\[
0 =: \tau_n^0 \leq \tau_n^1 \leq \ldots \leq \tau_n^m.
\]  

(5.5)

This relation will be an important tool for showing the bijectivity of the transformation as required in property (3) of the last subsection. As required in (4) we also would like to have

\[
x_1, x_2 \in X, x_1 - x_2 \in K \Rightarrow t_n(x_1) = t_n(x_2),
\]  

(5.6)

\[
x_1, x_2 \in X, x_1 - x_2 \notin K \Rightarrow (x_1 + t_n(x_1)e_1) - (x_2 + t_n(x_2)e_1) \notin K.
\]  

(5.7)

With these properties in mind we will now give a recursive definition of \( P_n^k \) and \( \tau_n^k \) for a fixed configuration \( X \in \mathcal{X} \) using a translation distance function \( t_n^k := t_n^k : \mathbb{R}^2 \to \mathbb{R} \) in each step. In the k-th construction step (1 \( \leq k \leq m \)) let

\[
t_k^0 := t_0^0 \land \bigwedge_{0 \leq i < k} m_{p_i^k, \tau_i^k} = t_n^{k-1} \land \bigwedge_{0 \leq i < k} m_{p_i^{k-1}, \tau_i^{k-1}},
\]  

where \( t_0^0 := \tau_n(|\cdot|) \) and \( m_{p_0^0, \tau_0^0} := \bigwedge_{x \in X_n^0} m_{x,0} \).

The auxiliary functions \( m_{x', t} \) will be defined later.

\[\text{Figure 3: Construction of } t_n^k\]

Let \( P_n^k \) be the point of \( X_{\Lambda_n} \setminus \{P_n^1, \ldots, P_n^{k-1}\} \) at which the minimum of \( t_n^k \) is attained. If there is more than one such point then take the smallest point with respect to the lexicographic order for the sake of definiteness. Let \( \tau_n^k := t_n^k(P_n^k) \) be the corresponding minimal value of \( t_n^k \) and \( T_n^k := T_n^k := id + t_n^k e_1 \).

\( t_n^k \) is defined to be \( \tau_n^0 \) modified by local distortions \( m_{x', t} \). On the one hand we have thus ensured that \( t_n^k - \tau_n^0 \) is small, i.e. \( \tau_n^k \approx \tau_n(|P_n^k|) \), which will give us hold on the density in property (3). On the other hand the auxiliary functions of the form \( m_{x', t} \) slow down the translation locally near every point \( x' \) with known translation distance \( t \), see Figure 3. This will ensure properties (5.6) and (5.7).
Figure 4: One-dimensional sketch of the graph of $m_{x',t}$

For $x' \in \mathbb{R}^2$ and $t \in \mathbb{R}$ let the auxiliary function $m_{x',t} : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$m_{x',t}(x) := \begin{cases} t & \text{if } h_{x',t} c_f > \frac{1}{2} \\ t + h_{x',t} f_K(x - x') + \infty 1_{\{f_K(x - x') = 1\}} & \text{else,} \end{cases}$$

where $h_{x',t} := |\tau_n (|x'| - c_K) - t|$.

Note that the first case in the definition of $m_{x',t}$ has been introduced in order to bound the slope of $m_{x',t}$. In Section 6.2 we will show important properties of this auxiliary function, but for the moment we will content ourselves with the intuition given by Figure 4. Using Lemma 6 one can show that all above objects are measurable with respect to the considered $\sigma$-algebras. In the rest of this section we will convince ourselves that the above construction has indeed all the required properties.

**Lemma 8** The construction satisfies (5.5), (5.6) and (5.7).

**Lemma 9** For good configurations $X \in G_n$ we have

$$(\Xi_n X - \tau e_1)_{\Lambda_n^{-1}} = X_{\Lambda_n^{-1}} \quad \text{and} \quad (\Xi_n X)_{\Lambda_n} = X_{\Lambda_n}.$$

(5.8)

**Lemma 10** The transformation $\Xi_n : X \to \bar{X}$ is bijective.

Actually in the proof of Lemma 11 we construct the inverse of $\Xi_n$. This is needed in the proof Lemma 11 where we will show for every $\bar{X} \in \bar{X}$ that $\nu_{\Lambda_n} (|\bar{X}|)$ is absolutely continuous with respect to $\nu_{\Lambda_n} (|\bar{X}|) \circ \Xi_n^{-1}$ with density $\varphi_n \circ \Xi_n^{-1}$, where

$$\varphi_n(X) := \prod_{k=1}^{m(X)} |1 + \partial_1 t^k_{n,X}(P_{n,X})|.$$

(5.9)

The proof will also show that definition (5.9) makes sense $\nu_{\Lambda_n} (|\bar{X}|)$-a.s., in that the considered derivatives exist.

**Lemma 11** For every $\bar{X} \in \bar{X}$ and every $\mathcal{F}_X$-measurable function $f \geq 0$

$$\int d\nu_{\Lambda_n} (|\bar{X}|) (f \circ \Xi_n \cdot \varphi_n) = \int d\nu_{\Lambda_n} (|\bar{X}|) f.$$

(5.10)
Considering (3.5) we also need the backwards translation. So let $\tilde{T}_n$ and $\tilde{\phi}_n$ be defined analogously to the above objects, where now $e_1$ is replaced by $-e_1$. The previous lemmas apply analogously to this deformed backwards translation. We note that $\tilde{T}_n$ is not the inverse of $T_n$.

5.5 Final steps of the proof

From (3.1) and Lemma 11 we deduce

$$\mu(\tilde{T}_n(D \cap G_n)) = \int \mu(d\tilde{X}) \frac{1}{Z_{\Lambda_n}(X)} \nu_{\Lambda_n}(dX|\tilde{X}) 1_{\tilde{T}_n(D \cap G_n)}(X) z^{#X_{\Lambda_n}} e^{-H_{\Lambda_n}^U(X)}$$

$$= \int \mu(d\tilde{X}) \frac{1}{Z_{\Lambda_n}(X)} \nu_{\Lambda_n}(dX|\tilde{X}) 1_{\tilde{T}_n(D \cap G_n)}(X) \tilde{\phi}_n(X) \phi_n(X)$$

By Lemma 10 $\tilde{T}_n$ is bijective, by (5.8) $\#(\tilde{T}_nX)_{\Lambda_n} = #X_{\Lambda_n}$ and by (5.6) and (5.7) we have $H_{\Lambda_n}^U(\tilde{T}_nX) = H_{\Lambda_n}^U(X)$. Hence the above integrand simplifies to

$$1_{D \cap G_n}(X) z^{#X_{\Lambda_n}} e^{-H_{\Lambda_n}^U(X)} \tilde{\phi}_n(X),$$

and we have an analogous expression for the backwards transformation $\tilde{T}_n$. So

$$\mu(\tilde{T}_n(D \cap G_n)) + \mu(\tilde{T}_n(D \cap G_n)) - \mu(D \cap G_n) = \int \mu(d\tilde{X}) \frac{1}{Z_{\Lambda_n}(X)} \nu_{\Lambda_n}(dX|\tilde{X}) 1_{D \cap G_n}(X) z^{#X_{\Lambda_n}} e^{-H_{\Lambda_n}^U(X)}$$

$$\times [\tilde{\phi}_n(X) + \phi_n(X) - 1].$$

We note that for $X \in G_n$ we have

$$\tilde{\phi}_n(X) + \phi_n(X) \geq 2 (\tilde{\phi}_n(X) \phi_n(X))^{\frac{1}{2}} \geq 2 e^{-\frac{1}{2}} \geq 1,$$

where we have used the arithmetic-geometric-mean inequality in the first step and the following estimate in the second step:

**Lemma 12** For $X \in G_n$ we have

$$\log \tilde{\phi}_n(X) + \log \phi_n(X) \geq -1. \quad (5.11)$$

Hence we have shown that

$$\mu(\tilde{T}_n(D \cap G_n)) + \mu(\tilde{T}_n(D \cap G_n)) \geq \mu(D \cap G_n). \quad (5.12)$$

In (5.12) we would like to replace $D \cap G_n$ by $D$, and for this we need $G_n$ to have high probability:

**Lemma 13** If $n \geq R + 1$ is chosen big enough, then $\mu(G_n^c) \leq \delta$. 

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For the proof of Theorem 1 we choose such an \( n \geq R + 1 \). Because of \( D \in F_{X,\Lambda_{n'}} \) and (5.8) we have
\[
\forall X \in D \cap G_n : \ (\mathfrak{T}_n X - \tau e_1)_{\Lambda_{n'}} \in D, \quad \text{i.e.} \ \mathfrak{T}_n X \in D + \tau e_1,
\]
and an analogous result for the backwards transformation. Hence
\[
\mathfrak{T}_n (D \cap G_n) \subset D + \tau e_1 \quad \text{and} \quad \mathfrak{T}_n (D \cap G_n) \subset D - \tau e_1.
\]
Using these inclusions and Lemma 13 we deduce from (5.12)
\[
\mu(D - \tau e_1) + \mu(D + \tau e_1) \geq \mu(D) - \delta.
\]
\( \delta > 0 \) was chosen to be an arbitrary positive real, so we get the estimate (3.5) by taking the limit \( \delta \to 0 \). Now the claim of the theorem follows from Lemma 5.

6 Proof of the lemmas from Section 5

6.1 Cluster bounds: Lemma 7

For \( n > n' \) and \( X \in X \) we want to estimate \( r_{n,X}(\Lambda_{n'}) \). For any path \( x_0, ..., x_m \) in the graph \( (X,K^n) \) such that \( x_0 \in \Lambda_{n'} \) we have
\[
|x_m| \leq |x_0| + \sum_{i=1}^{m} |x_i - x_{i-1}| \leq n' + mc_K.
\]
By considering all possibilities for such paths we obtain
\[
r_{n,X}(\Lambda_{n'}) \leq n' + \sum_{m \geq 1} \sum_{x_0, ..., x_m \in X} \# 1_{\{x_0 \in \Lambda_{n'}\}} mc_K \prod_{i=1}^{m} 1_{\{x_i, x_{i-1} \in K^n\}}.
\]
Using the hard core property (3.2) and Lemma 3 we get
\[
R_n := \int \mu(dx) r_{n,X}(\Lambda_{n'}) - n' \leq \sum_{m \geq 1} \int \mu(dx) \sum_{x_0, ..., x_m \in X} \# 1_{\{x_0 \in \Lambda_{n'}\}} mc_K \prod_{i=1}^{m} 1_{K_i \setminus K^U} (x_i - x_{i-1}) \leq \sum_{m \geq 1} (z\xi)^{m+1} \int dx_0 ... dx_m 1_{\{x_0 \in \Lambda_{n'}\}} mc_K \prod_{i=1}^{m} 1_{K_i \setminus K^U} (x_i - x_{i-1}).
\]
By (5.1) we can estimate the integrals over \( dx_i \) in the above expression beginning with \( i = m \). This gives \( m \) times a factor \( c_\xi \) and the integration over \( dx_0 \) gives an additional factor \( \lambda^2(\Lambda_{n'}) = (2n')^2 \). Thus
\[
R_n \leq (2n')^2 z\xi c_K \sum_{m \geq 1} m (c_\xi z\xi)^m < \infty,
\]
where the last sum is finite because \( c_\xi z\xi < 1 \).
6.2 Properties of the auxiliary function

Let \( f : I \rightarrow \mathbb{R} \) be a function on an interval. \( f \) is called 1/2-Lipschitz-continuous if

\[
|f(r) - f(r')| \leq \frac{1}{2}|r - r'| \quad \text{for all } r, r' \in I.
\]

\( f \) is called piecewise continuously differentiable if it is continuous and if

\[ \exists \text{ countable and closed } M \subset I : f \text{ is continuously differentiable on } I \setminus M. \]

As \( M \) is closed, the connected components of \( I \setminus M \) are countably many intervals. For a strictly monotone piecewise continuously differentiable transformation \( f \) on \( \mathbb{R} \) we can apply the Lebesgue transformation theorem: The derivative \( f' \) is well defined \( \lambda^1 \)-a.s. and for every \( \mathcal{B}^1 \)-measurable function \( g \geq 0 \) we have

\[
\int g(f(x))|f'(x)|dx = \int g(x')dx'.
\]

(6.1)

The above properties are inherited as follows:

**Lemma 14** Let \( f_1, f_2 : I \rightarrow \mathbb{R} \) be functions on an interval \( I \).

(a) If \( f_1 \) and \( f_2 \) are 1/2-Lipschitz-continuous, then so is \( f_1 \wedge f_2 \).

(b) If \( f_1 \) and \( f_2 \) are piecewise continuously differentiable, then so is \( f_1 \wedge f_2 \).

For the proof of these easy facts we refer to [Ri2]. A function \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) is called 1/2-\(e_1\)-Lipschitz-continuous or piecewise \(e_1\)-differentiable if for all \( r_2 \in \mathbb{R} \) the function \( f(., r_2) \) is 1/2-Lipschitz-continuous or piecewise continuously differentiable respectively.

**Lemma 15** For \( x' \in \mathbb{R}^2 \) and \( t \in \mathbb{R} \) the function \( \tau_n(|.|) \wedge m_{x',t} \) is 1/2-\(e_1\)-Lipschitz-continuous and piecewise \(e_1\)-differentiable.

For details of the proof we again refer to [Ri2]. Basically Lemma 15 follows from Lemma 14. The only difficulty is to show the continuity of \( \tau_n(|.|) \wedge m_{x',t} \), which might be a problem because of the jump to infinity of \( m_{x',t} \) in case of \( h_{x',t} \leq 1/2 \). But if \( x \in \partial\{m_{x',t} < \infty\} = \partial\{f_K(., x') < 1\} \) then \( x - x' \) is contained in the closure of \( K_x \). Hence \( |x - x'| \leq c_K \), which implies \( |x'| - c_K \leq |x| \).

As \( \tau_n \) is decreasing we obtain

\[
\tau_n(|x|) \leq \tau_n(|x'|-c_K) \leq t + h_{x',t} \leq m_{x',t}(x)
\]

by definition of \( h_{x',t} \), which implies the claimed continuity.

6.3 Properties of the construction: Lemma 8

We will first investigate monotonicity and regularity properties of \( t^k_n \) and \( T^k_n \):

**Lemma 16** For \( X \in \mathcal{X} \) and \( k \geq 0 \)

\( t^k_n \) is 1/2-\(e_1\)-Lipschitz-continuous and piecewise \( e_1\)-differentiable, \( T^k_n \) is \( \leq e_1\)-increasing and bijective.

(6.2)  

(6.3)
In order to show this we fix $r$ that $X$.

Proof: $t^k_n$ is the minimum of finitely many functions of the form $\tau_n(|.|) \land m_{x',t}$, where $x' \in \mathbb{R}^2$ and $t \in \mathbb{R}$. Hence (6.2) is an immediate consequence of Lemmas 13 and 14. (6.2) implies that $T^k_n$ is $e_1$-continuous and $\leq e_1$-increasing, and hence bijective. This shows (6.3).

For (5.5) it suffices to observe that for every $2 \leq k \leq m$ we have

$$\tau^k_n = t^k_n(P^k_n) = t^{k-1}_n(P^k_n) \land m_{P^{k-1}_n, \tau^{k-1}_n}(P^k_n) \geq \tau^{k-1}_n.$$  

This follows from the definition of $\tau^k_n$ and $t^k_n$, from $t^{k-1}_n(P^k_n) \geq \tau^{k-1}_n$ by the definition of $P^{k-1}_n$ and from $m_{x',t} \geq t$.

For (5.6) and (5.7) let $x_1, x_2 \in X$. Without loss of generality we may suppose that $x_1 = P^i_n$ and $x_2 = P^j_n$, where $0 \leq i \leq j$. Here $P^0_n$ is interpreted to be any point of $X_{\Lambda_n}$. We first observe that $P^i_n \in \Lambda^i := \{x \in \mathbb{R}^2 : t^i_n(x) \geq \tau^i_n\}$ and

$$\forall x \in (P^i_n + K) \cap \Lambda^i : \ t^i_n(x) = t^i_n(x) \land \bigwedge_{i \leq k \leq j} m_{P^k_n, \tau^k_n}(x) = \tau^i_n.$$  

This holds as $t^i_n(x) \geq \tau^i_n$ by definition of $\Lambda^i$, $m_{P^k_n, \tau^k_n} \geq \tau^i_n$ by (5.5) and $m_{P^k_n, \tau^k_n}(x) = \tau^i_n$ by $x \in P^i_n + K$. If $P^i_n - P^j_n \in K$, then $P^i_n \in (P^i_n + K) \cap \Lambda^i$, so (6.4) implies $\tau^i_n = t^i_n(P^i_n) = \tau^i_n$, which shows (6.6). For (6.7) suppose $P^i_n - P^j_n \notin K$. We have $P^j_n \in \Lambda^i \setminus (P^i_n + K)$ and $\tau^j_n = t^j_n(P^j_n)$ by definition, so it suffices to show

$$T^j_n(\Lambda^i \setminus (P^i_n + K)) = \Lambda^i \setminus (P^i_n + K) + \tau^j_n e_1.$$  

In order to show this we fix $r \in \mathbb{R}$. Continuity of $t^i_n(\cdot, r)$ implies $t^i_n = \tau^i_n$ on $\partial \Lambda^i(\cdot, r)$. Just as in the proof of (6.4) it follows that $t^i_n = \tau^i_n$ on $\partial \Lambda^i(\cdot, r)$. But $T^j_n(\cdot, r)$ is increasing, continuous and bijective by (6.3), so

$$T^j_n(\Lambda^i) = \Lambda^i + \tau^j_n e_1,$$

and combining this with (6.4) we are done.

### 6.4 Properties of the deformed translation: Lemma 9

The following lemma shows how to estimate the translation distances $\tau^k_n$.

**Lemma 17** For $X \in \mathcal{X}$ and $k \geq 0$ we have

$$\tau^k_n \leq t^k_n(P^k_n) \quad \text{for all } k' \geq k, \quad (6.6)$$

$$\tau^k_n \geq t^0_n(a_{n,X}(P^0_n)) \quad \text{if } X \in G_n. \quad (6.7)$$

**Proof:** (6.6) follows from the definition of $P^k_n$ and from $t^k_n \leq t^0_n$. For the proof of (6.7) let $X \in G_n$. We first would like to show that

$$\forall x, x' \in X : |x| \leq |x'|, \ x \xrightarrow{X,K^0_n} x' \Rightarrow |\tau_n(|x| - c_K) - \tau_n(|x'|)| c_f \leq 1/2. \quad (6.8)$$

Defining

$$\Sigma_1(n, X) := \sum_{x, x' \in X} 1_{|x| \leq |x'|} 1_{x \xrightarrow{X,K^0_n} x'} 4(\tau_n(|x| - c_K) - \tau_n(|x'|))^2 c_f^2 \quad (6.9)$$

and recalling that $\tau_n(|x| - c_K) \leq \tau_n(|x'|)$ implies

$$\tau_n(|x| - c_K) \leq \tau_n(|x'|)^2 c_f^2 \leq \tau_n(|x'|) c_f^2,$$

we have

$$\forall x, x' \in X : |x| \leq |x'|, \ x \xrightarrow{X,K^0_n} x' \Rightarrow |\tau_n(|x| - c_K) - \tau_n(|x'|)| c_f \leq 1/2. \quad (6.8)$$

Defining

$$\Sigma_1(n, X) := \sum_{x, x' \in X} 1_{|x| \leq |x'|} 1_{x \xrightarrow{X,K^0_n} x'} 4(\tau_n(|x| - c_K) - \tau_n(|x'|))^2 c_f^2 \quad (6.9)$$

and recalling that $\tau_n(|x| - c_K) \leq \tau_n(|x'|)$ implies

$$\tau_n(|x| - c_K) \leq \tau_n(|x'|)^2 c_f^2 \leq \tau_n(|x'|) c_f^2,$$
we have $\Sigma_1(n, X) < 1$ by definition of the set $G_n$ of good configurations in (5.4) and by $X \in G_n$. Hence every summand of $\Sigma_1$ is less than 1, which implies (6.8). We now can prove (6.7) by induction on $k$. For $k = 0$ we have equality if the right hand side is defined to be 0. For the inductive step $k - 1 \rightarrow k$ let $i \leq k - 1$. By (6.6), the inductive hypothesis and (6.8) we have

$$0 \leq (\tau_n(\lvert P_n^i \rvert - c_K) - \tau_n^i) f \leq (\tau_n(\lvert P_n^i \rvert - c_K) - \tau_n(a_{n,X}(P_n^i))) f \leq 1/2,$$

so $h_{P_n^i, \tau_n} f \leq 1/2$. Therefore $m_{P_n^i, \tau_n}^k(P_n^k) = \infty$ whenever $P_n^k - P_n^i \notin K_\epsilon$. Thus

$$\tau_n^k = t_n^k(P_n^k) = t_n^0(P_n^k) \wedge \bigwedge_{i < k} m_{P_n^i, \tau_n}^k(P_n^k) \geq t_n^0(a_{n,X}(P_n^k)),$$

where the last step follows from $m_{P_n^i, \tau_n}^k(P_n^k) \geq \tau_n^i \geq t_n^0(a_{n,X}(P_n^0))$, which holds by induction hypothesis, and $a_{n,X}(P_n^k) = a_{n,X}(P_n^k)$ for $P_n^k - P_n^i \in K_\epsilon$. \hfill \Box

In the proof of (6.7) we have also shown that good configurations $X \in G_n$ have the following property: In the construction of $\tilde{\Sigma}_n(X)$ we have $h_{P_n^i, \tau_n} f \leq 1/2$ for every $k$, i.e. in the definition of $m_{P_n^i, \tau_n}^k$ we always have the second case. Now we will prove Lemma 9. It suffices to show for all $X \in G_n$ and $x \in X$ that

$$x \in \Lambda_{n'} \Rightarrow t_{n,X}(x) = \tau, \quad x \in \Lambda_{n'}^c \Rightarrow x + t_{n,X}(x)e_1 - \tau e_1 \notin \Lambda_{n'-1} \quad x \in \Lambda_n \Rightarrow x + t_{n,X}(x)e_1 \in \Lambda_n.$$ (6.10)

So let $X \in G_n$ and $x \in X$. We first note that

$$0 \leq \tau_n(\lvert a_{n,X}(x) \rvert) \leq t_{n,X}(x) \leq \tau_n(\lvert x \rvert) \leq \tau,$$ (6.11)

which is an immediate consequence of (6.7) and (6.6). We observe

$$x \in \Lambda_{n'} \Rightarrow a_{n,X}(x) \in \Lambda_R \Rightarrow \tau_n(\lvert a_{n,X}(x) \rvert) = \tau \Rightarrow t_{n,X}(x) = \tau,$$

where we have used the definition of $R$, $X \in G_n'$, (5.3) and (6.11). This gives the first assertion of (6.10). The second assertion is an immediate consequence of $0 \leq \tau - t_{n,X}(x) \leq 1$, which follows from (6.11) and $\tau \leq 1$. The third assertion follows from (6.11) and (5.3), and for the fourth assertion let $x \in \Lambda_n$. As

$$x \leq e_1 \Rightarrow x + t_{n,X}(x)e_1 \leq e_1 \Rightarrow T_n^0(x)$$

by (6.11), it suffices to show that also $T_n^0(x) \in \Lambda_n$. This however follows from $T_n^0 = id$ on $\Lambda_n^c$ and the bijectivity of $T_n^0$ from (6.3).

### 6.5 Bijectivity of the transformation: Lemma 10

We will construct the inverse transformation $\tilde{\tilde{\Sigma}}_n$ recursively just as in the construction of $\tilde{\Sigma}_n$, i.e. from a given configuration $\tilde{X}$ we will choose points $\tilde{P}_n^k$ and translate them by $\tilde{\tau}_n^k$ in direction $-e_1$.

To get an idea how to define the inverse transformation we start with $X \in \mathcal{X}$ and set $\tilde{X} := \tilde{\Sigma}_n(X)$. In the construction of $\tilde{X}$ we defined points $\tilde{P}_n^k$ and translation distances $\tilde{\tau}_n^k$. We denote the corresponding image points by $\tilde{P}_n^k := P_n^k + \tilde{\tau}_n^k e_1,$
For the construction of the inverse transformation we have to find a method to identify the points $\tilde{P}_n^k$ among the points of $\tilde{X}$ without knowing $X$. Suppose now that we have already found $\tilde{\tau}_n$ to be equalities, so for the given $\tilde{\tau}_n$ a method to identify the points $\tilde{\tau}_n$ is defined in terms of $P^0_n$. Then inductively we are able to construct the translation distances $\tau^i_n$ for all $1 \leq i < k$, because $t^i_n$ is defined in terms of $P^j_n$ and $\tau^j_n$ where $j < i$, $T^i_n = \text{id} + t^i_n e_1$, $P^i_n = (T^i_n)^{-1}(\tilde{P}_n^i)$ and $\tau^i_n = t^i_n(P^n_i)$. So in particular we know the transformation functions $\tilde{\tau}_n^k$ and $T^k_n$. Thus the following lemma gives a characterisation of $\tilde{P}_n^k$ just as needed:

**Lemma 18** Let $1 \leq k \leq m$. For every $\tilde{x} \in \tilde{X}_\Lambda \setminus \{\tilde{P}_n^1, \ldots, \tilde{P}_n^{k-1}\}$ we have

$$ t^k_n \circ (T^k_n)^{-1}(\tilde{P}_n^k) \leq t^k_n \circ (T^k_n)^{-1}(\tilde{x}). $$

For all $\tilde{x}$ for which equality occurs we have $(T^k_n)^{-1}(\tilde{P}_n^k) \leq (T^k_n)^{-1}(\tilde{x})$.

**Proof:** We first observe that for all $k$ by definition of $T^k_n$ we have

$$ (T^k_n)^{-1} + t^k_n \circ (T^k_n)^{-1} e_1 = \text{id}. \tag{6.12} $$

Since $t^{k+1}_n \leq t^k_n$, we also have $T^{k+1}_n \leq e_1 T^k_n$, and therefore $(T^k_n)^{-1} \leq e_1 (T^{k+1}_n)^{-1}$ by the $e_1$-monotonicity of $(T^{k+1}_n)^{-1}$ from (6.3). Together with (6.12) this implies

$$ t^{k+1}_n \circ (T^{k+1}_n)^{-1} \leq t^k_n \circ (T^k_n)^{-1}. \tag{6.13} $$

Now let $1 \leq k \leq m$ and $\tilde{x} \in \tilde{X}_\Lambda \setminus \{\tilde{P}_n^1, \ldots, \tilde{P}_n^{k-1}\}$, i.e. $\tilde{x} = \tilde{P}_n^l$ for some $l \geq k$. By definition we have $t^l_n(P^n_l) = \tau^l_n$, $T^l_n(P^n_l) = \tilde{P}_n^l$ and $\tilde{P}_n^k = T^k_n(P^n_k)$. Using (5.5) and (6.13) we deduce

$$ t^k_n(T^k_n)^{-1}(\tilde{P}_n^k) = \tau^k_n \leq \tau^l_n = t^l_n(P^n_l) = t^l_n(T^l_n)^{-1}(\tilde{x}) \leq t^k_n(T^k_n)^{-1}(\tilde{x}). $$

If for the given $\tilde{x}$ we have equality, all inequalities in the previous line have to be equalities, so $\tau^k_n = \tau^l_n$ and $t^l_n(T^l_n)^{-1}(\tilde{x}) = t^k_n(T^k_n)^{-1}(\tilde{x})$. Combining this with (6.12) we get $\tilde{P}_n^l = (T^l_n)^{-1}(\tilde{x}) = (T^k_n)^{-1}(\tilde{x})$, so $T^k_n(P^n_l) = \tilde{x}$, and thus $t^l_n(P^n_l) = \tau^l_n = \tau^k_n$. By definition of $P^n_k$ we conclude $(T^k_n)^{-1}(\tilde{P}_n^k) = P^n_k \leq P^n_l$ and we are done. \hfill $\Box$
Lemma 18 tells us exactly how to construct the inverse of $\Sigma_n$ recursively. So let $\tilde{X} \in \mathcal{X}$. Let $\tilde{m} = \tilde{m}(\tilde{X}) := \#\tilde{X}_{\Lambda_n}$, $\tilde{t}_n^0 = \tau_n(|.|)$ and $\tilde{z}_n^0 := 0$. In the $k$-th construction step ($1 \leq k \leq \tilde{m}$) let

$$\tilde{t}_n^k := \tilde{t}_n^{k-1} \wedge m \tilde{t}_n^{k-1} - \tilde{z}_n^{k-1}, \quad \text{where} \quad m \tilde{t}_n^0 - \tilde{z}_n^0 := \bigwedge_{\tilde{x} \in \tilde{X}_{\Lambda_n}} m\tilde{x},0.$$ 

Let $\tilde{T}_{n,\tilde{X}} = id + \tilde{t}_n^k e_1$ and let $\tilde{P}_{n,\tilde{X}}^k$ be the point of $\tilde{X}_{\Lambda_n} \setminus \{\tilde{P}_{n,\tilde{X}}^1, \ldots, \tilde{P}_{n,\tilde{X}}^{k-1}\}$ at which the minimum of $\tilde{t}_n^k \circ (\tilde{T}_{n,\tilde{X}})^{-1}$ is attained. If there is more than one such point then take the point $y$ such that $(\tilde{T}_{n,\tilde{X}})^{-1}(y)$ is minimal with respect to the lexicographic order $\leq$. Let $\tilde{\tau}_n^k := \tilde{t}_n^k \circ (\tilde{T}_{n,\tilde{X}})^{-1}(\tilde{P}_{n,\tilde{X}}^k)$ be the corresponding minimal value. In the above notations we will omit dependencies on $\tilde{X}$ if it is clear which configuration is considered. We need to show that the above construction is well defined, i.e. that $\tilde{T}_{n,\tilde{X}}^k$ is invertible in every step. Furthermore we need some more properties of the construction:

**Lemma 19** Let $\tilde{X} \in \mathcal{X}$ and $k \geq 0$. Then

$$\tilde{t}_n^k$$ is $1/2$-Lipschitz-continuous, $\tilde{T}_{n,\tilde{X}}^k$ is bijective and $\leq e_1$-increasing, \hspace{0.5cm} (6.14)

$$(\tilde{T}_{n,\tilde{X}})^{-1} + \tilde{t}_n^k \circ (\tilde{T}_{n,\tilde{X}})^{-1} e_1 = id,$$ \hspace{0.5cm} (6.15)

$$\forall c \in \mathbb{R}, x \in \mathbb{R}^2 : \tilde{t}_n^k \circ (\tilde{T}_{n,\tilde{X}})^{-1}(x) \geq c \iff \tilde{t}_n^k(x - ce_1) \geq c,$$ \hspace{0.5cm} (6.16)

$$\tilde{t}_n^k \leq \tilde{t}_n^{k-1} \quad \text{and} \quad \tilde{z}_n^{k-1} \leq \tilde{z}_n^k.$$ \hspace{0.5cm} (6.17)

**Proof:** The definitions of $\tilde{t}_n^k$ and $\tilde{T}_{n,\tilde{X}}^k$ are similar to those of $t_n^k$ and $T_n^k$ so we can show (6.14) and (6.15) just as the corresponding properties in (6.2), (6.3) and (6.12). For (6.16) we note that for $c \in \mathbb{R}$ and $x \in \mathbb{R}^2$ the equivalence

$$\tilde{t}_n^k \circ (\tilde{T}_{n,\tilde{X}})^{-1}(x) \geq c \iff (\tilde{T}_{n,\tilde{X}})^{-1}(x) \leq e_1 x - ce_1$$

$$\iff x \leq e_1 \tilde{T}_{n,\tilde{X}}(x - ce_1) = x - ce_1 + \tilde{t}_n^k(x - ce_1)e_1$$

follows from (6.15) and (6.14). The first part of (6.17) is obvious and for the second part we observe that

$$\tilde{t}_n^{k-1} \circ (\tilde{T}_{n,\tilde{X}})^{-1}(\tilde{P}_{n,\tilde{X}}^k) \geq \tilde{z}_n^{k-1} \quad \Rightarrow \quad \tilde{t}_n^{k-1}(\tilde{P}_{n,\tilde{X}}^k - \tilde{z}_n^{k-1}e_1) \geq \tilde{z}_n^{k-1} \quad \Rightarrow \quad \tilde{z}_n^k = \tilde{t}_n^k \circ (\tilde{T}_{n,\tilde{X}})^{-1}(\tilde{P}_{n,\tilde{X}}^k) \geq \tilde{z}_n^{k-1},$$

where the first statement holds by definition of $\tilde{P}_{n,\tilde{X}}^k$, the first and the third implication hold by (6.16) and the second holds by definition of $\tilde{t}_n^k$. \hfill $\square$

Let $\tilde{t}_{n,\tilde{X},\tilde{X}}^k := \tilde{t}_{n,\tilde{X}}^k$ and $\tilde{t}_{n,\tilde{X}}(x) = 0$ for $x \in \tilde{X}_{\Lambda_n}$. This defines a translation distance function $\tilde{t}_{n,\tilde{X}} : \tilde{X} \to \mathbb{R}$. Let $\tilde{\Sigma}_n : \mathcal{X} \to \mathcal{X}$ be defined by

$$\tilde{\Sigma}_n(\tilde{X}) := \tilde{X}_{\Lambda_n} \cup \{\tilde{P}_{n,\tilde{X}}^k - \tilde{z}_n^k e_1 : 1 \leq k \leq m\} = \{x - \tilde{t}_{n,\tilde{X}}(x)e_1 : x \in \tilde{X}\}.$$ 

By Lemma 8 we again see that all above objects are measurable with respect to the considered $\sigma$-algebras. The only difficulty is to show that the functions
For the second part let \((6.18)\) by induction on

\[
\tilde{p}_n^k(P_n^k) \leq \tilde{p}_n^k(x).
\]

For all \(x\) for which equality occurs we have \(P_n^k \leq x\).

**Proof:** Let \(1 \leq k \leq \tilde{m}\) and \(x \in X_{\Lambda_n} \setminus \{P_n^1, \ldots, P_n^{k-1}\}\), i.e. \(x = P_n^l\) for some \(l \geq k\). By definition of \(\tilde{p}_n^k\) and \(\tilde{p}_n^l\) and \((6.15)\) we have \((\tilde{T}_n^k)^{-1}(\tilde{P}_n^k) = P_n^k\) and \((\tilde{T}_n^l)^{-1}(\tilde{P}_n^l) = x\). Using \((6.17)\) we obtain

\[
\tilde{p}_n^k(P_n^k) = \tilde{p}_n^k = \tilde{p}_n^l(\tilde{T}_n^l)^{-1}(\tilde{P}_n^l) = \tilde{p}_n^l(x) \leq \tilde{p}_n^k(x).
\]

If for the given \(x\) we have equality, all inequalities in the previous line have to be equalities, so \(\tilde{p}_n^k = \tilde{p}_n^l\) and \(\tilde{p}_n^l(x) = \tilde{p}_n^l\), i.e. \(\tilde{T}_n^k(x) = x + \tilde{T}_n^l e_1 = \tilde{P}_n^l\). This gives \(\tilde{p}_n^k = \tilde{p}_n^l = \tilde{p}_n^k(x) = \tilde{p}_n^l \circ (\tilde{T}_n^k)^{-1}(\tilde{P}_n^k)\). So \(P_n^k = (\tilde{T}_n^k)^{-1}(\tilde{P}_n^k) \leq (\tilde{T}_n^k)^{-1}(\tilde{P}_n^k) = x\) by definition of \(\tilde{P}_n^k\) and we are done. \(\square\)

**Lemma 21** On \(\mathcal{X}\) we have \(\tilde{\varsigma}_n \circ \varsigma_n = id\) and \(\varsigma_n \circ \tilde{\varsigma}_n = id\).

**Proof:** For the first part let \(X \in \mathcal{X}\) and \(\tilde{X} := \tilde{\varsigma}_n(X)\). We have \(\tilde{m}(\tilde{X}) = m(X)\) by construction and we have \(X_{\Lambda_n^c} = \tilde{X}_{\Lambda_n^c}\) by \((5.8)\). Now it suffices to prove

\[
\tilde{p}_{n,X}^k = p_{n,X}^k, \quad \tilde{T}_{n,X}^k = T_{n,X}^k, \quad \tilde{\tau}_{n,X}^k = \tau_{n,X}^k \quad \text{and} \quad \tilde{P}_{n,X}^k = P_{n,X}^k + \tau_{n,X}^k \quad (6.18)
\]

for every \(k \geq 0\) by induction on \(k\). Here \(\tilde{P}_{n,X}^k = P_{n,X}^k + \tau_{n,X}^k\) is interpreted as \(X_{\Lambda_n^c} = \tilde{X}_{\Lambda_n^c}\). The case \(k = 0\) is trivial. For the inductive step \(k \rightarrow k + 1\) we observe that \(\tilde{p}_{n,X}^k = p_{n,X}^k\) by induction hypothesis, and \(\tilde{T}_{n,X}^k = T_{n,X}^k\) is an immediate consequence. Combining this with Lemma \((6.18)\) and the definition of \(\tilde{P}_{n,X}^k\) we get \(\tilde{P}_{n,X}^k = P_{n,X}^k + \tau_{n,X}^k\) and \(\tilde{p}_{n,X}^k = \tilde{p}_{n,X}^k\).

For the second part let \(\tilde{X} \in \mathcal{X}\) and \(X := \tilde{\varsigma}_n(\tilde{X})\). As above it suffices to show \((6.18)\) by induction on \(k\). Here \(X_{\Lambda_n^c} = X_{\Lambda_n^c}\) follows from an analogue of \((5.8)\) and the inductive step follows from Lemma \((20)\) \(\square\)

### 6.6 Density of the transformed process: Lemma \((11)\)

By definition the left hand side of \((5.10)\) equals

\[
e^{-4n^2} \sum_{k \geq 0} \frac{1}{k!} I(k), \quad \text{where} \quad I(k) = \int_{\Lambda_n^c} dx (f \circ \varsigma_n \cdot \varphi_n)(\tilde{X}_x),
\]

using the shorthand notation \(\tilde{X}_x := \{x_i : i \in J\} \cup \tilde{X}_{\Lambda_n^c}\) for \(x \in \Lambda_n^c\). To compute \(I(k)\) we need to calculate \(\varsigma_n(\tilde{X}_x)\), and for this we must identify the
points \( P^i_{n,\bar{X}_x} \) among the particles \( x_j \). So let \( \Pi \) be the set of all permutations \( \eta : \{1, \ldots, k\} \to \{1, \ldots, k\} \). For \( \eta \in \Pi \) let

\[
A_{k,\eta} := \{ x \in \Lambda_n^k : \forall 1 \leq j \leq k : x_{\eta(j)} = P^j_{n,\bar{X}_x} \} \quad \text{and} \quad \bar{A}_{k,\eta} := \{ x \in \Lambda_n^k : \forall 1 \leq j \leq k : x_{\eta(j)} = \bar{P}^j_{n,\bar{X}_x} \},
\]

where \( \bar{P}^j_{n,\bar{X}_x} \) are the points from the construction of the inverse transformation in Subsection 6.5. Now we can write

\[
I(k) = \sum_{\eta \in \Pi} I(k, \eta), \quad \text{where} \quad I(k, \eta) = \int_{\Lambda_n^k} dx 1_{A_{k,\eta}}(x)(f \circ \tau_n \cdot \varphi_n)(\bar{X}_x).
\]

If \( x \in A_{k,\eta} \) we can derive a simple expression for \( \tau_n(\bar{X}_x) \): For \( x \in \Lambda_n^k \) we define a formal transformation \( T_\eta(x) := (T^\eta_{\eta,x}(x_i))_{1 \leq i \leq k} \), where

\[
T^\eta_{\eta,x}(j) := id + t^\eta_{\eta,x}j e_1, \quad t^\eta_{\eta,x}(j) := \tilde{t}^j_{n,\bar{X}_x \eta, j-1} \quad \text{and} \quad x_{\eta,j-1} := (x_{\eta(i)})_{1 \leq i \leq j-1}.
\]

Clearly, \( T^\eta_{\eta,x}(j) \) doesn’t depend on all components of \( x \), but only on those \( x_{\eta(l)} \) such that \( l \leq j-1 \). By definition we now have

\[
\forall x \in A_{k,\eta} : \tau_n(\bar{X}_x) = \bar{X}_{T_\eta(x)} \quad \text{and} \quad T^j_{n,\bar{X}_x} = T^\eta_{\eta,x}(j) \quad \text{for all} \quad j \leq k. \quad (6.19)
\]

Furthermore we observe that for all \( x \in (\mathbb{R}^2)^k \) we have

\[
x \in A_{k,\eta} \iff T^j_{n,\bar{X}_x} \in \bar{A}_{k,\eta}. \quad (6.20)
\]

Here “\( \Rightarrow \)” holds by (6.19) and (6.18) from the proof of Lemma 21. For “\( \Leftarrow \)” let \( x \in (\mathbb{R}^2)^k \) such that \( T^j_{n,\bar{X}_x} \in \bar{A}_{k,\eta} \) and let \( X' := \tilde{\tau}_n(\bar{X}_{T^j_{n,\bar{X}_x}(x)}) \), where \( \tilde{\tau}_n \) is the inverse of \( \tau_n \) as defined in the last subsection. By induction on \( j \) we can show

\[
\forall 1 \leq j \leq k : T^j_{n,X'} = T^\eta_{\eta,x}(j) \quad \text{and} \quad x_{\eta(j)} = P^j_{n,X'}.
\]

In the inductive step \( j - 1 \to j \) the first assertion follows from the induction hypothesis and the second follows from the bijectivity of \( T^j_{n,X'} \), and

\[
T^j_{n,X'}(x_{\eta(j)}) = T^\eta_{\eta,x}(j)(x_{\eta(j)}) = \tilde{P}^j_{n,\bar{X}_{T^j_{n,\bar{X}_x}(x)}} = P^j_{n,X'} + \tau^j_{n,X'} = T^j_{n,X'}(P^j_{n,X'}),
\]

which follows from \( T^j_{n,X'} = T^\eta_{\eta,x} \), the definition of \( \bar{A}_{k,\eta} \) and (6.18) from the proof of Lemma 21. This completes the proof of the above assertion and we conclude \( \bar{X}_x = X' \), which implies \( x_{\eta(j)} = P^j_{n,X'} = P^j_{n,\bar{X}_x} \). Thus (6.20) holds.

Now let \( g : (\mathbb{R}^2)^k \to \mathbb{R}, g(x) := 1_{\tilde{A}_{k,\eta}}(x)f(\bar{X}_x) \). Then (6.19) and (6.20) imply

\[
I(k, \eta) = \left[ \prod_{j=1}^{k} \int dx_{\eta(j)} \left| 1 + \partial_t t^\eta_{\eta,x}(j)(x_{\eta(j)}) \right| \right] g(T^j_{n,\bar{X}_x}(x)),
\]

where we have also inserted the definition of \( \varphi_n \) (5.49). Now we transform the integrals. For \( j = k \to 1 \) we substitute \( x_i' := T^i_{n,\bar{X}_x}x_i \), where \( i := \eta(j) \). The
transformation only concerns the first component of \( x_i = (r_i, \bar{r}_i) \). For fixed \( \bar{r}_i \), \( r_i \) is transformed by \( id + \bar{r}_i t_{\eta,x}(., \bar{r}_i) \). From (6.2) we know that \( t_{\eta,x}(., \bar{r}_i) \) is 1/2-Lipschitz-continuous and piecewise continuously differentiable, so \( id + \bar{r}_i t_{\eta,x}(., \bar{r}_i) \) is strictly increasing and piecewise continuously differentiable. Therefore the Lebesgue transformation theorem (6.1) gives

\[
d x'_i = dx_i |1 + \partial_1 t_{\eta,x}^i(x_i)|.
\]

Thus

\[
I(k, \eta) = \left[ \prod_{j=1}^{k} \int dx'_{\eta(j)} \right] g(x') = \int_{\Lambda_n^k} dx 1_{\Lambda_{k,\eta}}(x) f(\bar{X}_x),
\]

and we are done as the same arguments show that the right hand side of (5.10) equals

\[
e^{-4n^2} \sum_{k \geq 0} \frac{1}{k!} \sum_{\eta \in \Pi} \int_{\Lambda_n^k} dx 1_{\Lambda_{k,\eta}}(x) f(\bar{X}_x).
\]

An analogous argument shows that the density function is well defined:

\[
\forall \bar{X} \in \mathcal{X}: \quad \nu_{\Lambda_n}(\text{"\varphi_n is well defined"}|\bar{X}) = e^{-4n^2} \sum_{k \geq 0} \frac{1}{k!} \sum_{\eta \in \Pi} \int_{\Lambda_n^k} dx 1_{\Lambda_{k,\eta}}(x) \prod_{j=0}^{k} 1_{\{\partial_1 t_{\eta,j}^0(x_{\eta(j)}) \exists \}}.
\]

As \( t_{\eta,x}^0 \) is piecewise continuously \( e_1 \)-differentiable, we have for arbitrary \( r \in \mathbb{R} \), \( k, \eta \) and \( x \) as above that \( \partial_1 t_{\eta,j}^0(., r) \) exists \( \lambda^1 \)-a.s.. So we may replace all indicator functions in the above product by 1 using Fubini’s theorem. Hence the above probability equals 1.

### 6.7 Estimation of the densities: Lemma 12

Let \( X \in G_n \). By the 1/2-\( e_1 \)-Lipschitz-continuity from (6.2) we have

\[
|\partial_1 t_{n,X}^k(P_{n,X}^k)| \leq 1/2.
\]

Using \( -\log(1-a) \leq 2a \) for \( 0 \leq a \leq 1/2 \) we obtain

\[
f_n(X) := -\log \varphi_n(X) - \log \varphi_n(X) = -\sum_{1 \leq k \leq m} \log \left(1 - (\partial_1 t_{n,X}^k(P_{n,X}^k))^2\right) \leq \sum_{1 \leq k \leq m} 2(\partial_1 t_{n,X}^k(P_{n,X}^k))^2.
\]

If \( \partial_1 t_{n}^k(P_{n}^k) \) exists it equals either \( \partial_1 t_{n}^0(P_{n}^k) \) or \( \partial_1 m_{x,t_n}(x)(P_{n}^k) \) for some \( x \in X \) such that \( x \neq P_{n}^k \) and \( P_{n}^k \in x + K_e \). By using (6.7) we see that

\[
|\partial_1 m_{x,t_n}(x)(P_{n}^k)| \leq (\tau_n(|x| - c_K) - t_n(x))c_f \leq (\tau_n(|x| - c_K) - \tau_n(|a_{n,X}(x)|))c_f.
\]
Furthermore $|\partial t_0^0(P_n^k)| \leq \tau q(|P_n^k| - R)/Q(n - R)$ by definition of $t_0^0 = \tau_n(|.|)$, so we can estimate $f_n(X)$ by the sum of the two following terms:

$$
\Sigma_2(n, X) := 2\tau^2 \sum_{x \in X} 1_{\{x \in \Lambda_n\}} \frac{q(|x| - R)^2}{Q(n - R)^2},
$$

$$
\Sigma_3(n, X) := 2c_f^2 \sum_{x \in X} 1_{\{x \in \Lambda_0\}} \frac{q(|x| - R)^2}{Q(n - R)^2} - 2 \tau^2 \sum_{x \in X} 1_{\{x \in \Lambda_n\}} \frac{q(|x| - R)^2}{Q(n - R)^2} \times (\tau_n(|x| - c_K) - \tau_n(|x''|))^2.
$$

Using these terms in the definition (5.4) of $G_n$ we are done.

### 6.8 Set of good configurations: Lemma 13

The functions $\Sigma_i(n, X)$ from the definition of the set of good configurations $G_n$ in (5.4) have been specified in (6.9) and (6.21). Using the shorthand

$$
\tau_n^q(x, x'') := 1_{\{|x| \leq |x''|\}} \tau_n(|x| - c_K) - \tau_n(|x''|)^2
$$

we have

$$
\Sigma_1 = 4c_f^2 \sum_{x \in X} 1_{\{x \in \Lambda_p\}} \tau_n^q(x, x''), \quad \Sigma_2 = 2\tau^2 \sum_{x \in X} 1_{\{x \in \Lambda_n\}} \frac{q(|x| - R)^2}{Q(n - R)^2} \quad \text{and}
$$

$$
\Sigma_3 = 2c_f^2 \sum_{x \in X} 1_{\{x \in \Lambda_0\}} \frac{q(|x| - R)^2}{Q(n - R)^2} - 2 \tau^2 \sum_{x \in X} 1_{\{x \in \Lambda_n\}} \frac{q(|x| - R)^2}{Q(n - R)^2} \times (\tau_n(|x| - c_K) - \tau_n(|x''|))^2.
$$

We will show that the expectation of every $\Sigma_i$ can be made arbitrarily small when $n$ is chosen big enough. But first we will give some relations used later.

Let $n \geq R + 1$. For $s' > s$ such that $s' > R$ and $s < n$ we have

$$
0 \leq r(s - R, n - R) - r(s' - R, n - R) = \int_{R/s}^{s'/n} \frac{q(t - R)}{Q(n - R)} dt \leq (s' - s) \frac{q(s - R)}{Q(n - R)}
$$

by the monotonicity of $q$. Defining $\bar{n} := n + c_K$ and $\bar{R} := R + c_K$ we thus have

$$
\tau_n^q(x, x') \leq 1_{\{x \in \Lambda_n\}} \tau^2(|x'| - |x| + c_K)^2 \frac{q(|x| - \bar{R})^2}{Q(\bar{n} - \bar{R})^2} \quad \text{for } x, x' \in \mathbb{R}^2,
$$

using the substitution $s' := |x'|$ and $s := |x| - c_K$. (If $s' \leq R$ or $s \geq n$ then $\tau_n^q(x, x') = 0$.) The following relations will give us control over the relevant terms of the right hand side of (6.22). We first observe that

$$
\int_{\Lambda_n} dx q(|x| - \bar{R})^2 \leq 16\bar{R}^2 + 32Q(\bar{n} - \bar{R}) \quad \text{for } \bar{n} \geq 2\bar{R}.
$$

Indeed, writing $s := |x|$ we obtain

$$
\int_{\Lambda_n} dx q(|x| - \bar{R})^2 \leq \int_0^{2\bar{R}} ds 8s + \int_{\bar{R}}^{\bar{n} - \bar{R}} ds 8(s + \bar{R})q(s)^2 \leq 16\bar{R}^2 + 32 \int_0^{\bar{n} - \bar{R}} q(s)ds \leq 16\bar{R}^2 + 32Q(\bar{n} - \bar{R}).
$$
In the first step we used $q \leq 1$, and in the second step $\bar{R} \leq s$ and $sq(s) \leq 2$. We observe $\lim_{n \to \infty} Q(n) = \infty$, which is a consequence of $\log \log n \leq Q(n)$ for $n > 1$. Therefore by \eqref{6.23}

$$
\lim_{n \to \infty} c(n) = 0 \quad \text{for} \quad c(n) := \int_{\Lambda_n} dx q(|x| - \bar{R})^2 \frac{Q(n - R)^2}{\bar{R}},
$$

Finally, for $x_0, \ldots, x_m \in \mathbb{R}^2$ such that $x_i - x_{i-1} \in K_\epsilon$ we have $|x_i - x_{i-1}| \leq c_K$, so

\[(|x_m| - |x_0| + c_K)^2 \leq (m + 1)^2 c_K^2.\]

Now we will use the ideas of the proof of Lemma \[7\]. For $X \in \mathcal{X}$ we can estimate the summands of $\Sigma_1(n, X)$ by considering all paths $x_0, \ldots, x_m$ in the graph $(X, \mathcal{K}^n)$ connecting $x = x_0$ and $x'' = x_m$. By \eqref{6.22} and \eqref{6.25} we can estimate $\Sigma_1(n, X)$ by a constant $c$ times

$$
\sum_{m \geq 0} (m + 1)^2 \sum_{x_0, \ldots, x_m \in \mathcal{X}} 1\{x_0 \in \Lambda_n\} q(|x_0| - \bar{R})^2 \frac{Q(n - R)^2}{\bar{R}} \prod_{i=1}^{m} 1\{x_i, x_{i-1} \in \mathcal{K}^n\}.
$$

Using Lemma \[8\] we can thus proceed as in the proof of Lemma \[7\]

$$
\int \mu(dX) \Sigma_1(n, X) \leq z\xi c \sum_{m \geq 0} (m + 1)^2 (c\xi z\xi)^m c(n).
$$

Likewise,

$$
\int \mu(dX) \Sigma_2(n, X) \leq 2z\xi^2 c(n).
$$

Finally, we can estimate $\Sigma_3(n, X)$ by a constant $c$ times

$$
\sum_{m \geq 0} (m + 1)^2 \sum_{x_0, \ldots, x_m \in \mathcal{X}} 1\{x_0 \in \Lambda_n\} q(|x_0| - \bar{R})^2 \frac{Q(n - R)^2}{\bar{R}} \times \prod_{i=1}^{m} 1\{x_i, x_{i-1} \in \mathcal{K}^n\} \left[ \sum_{x' \in \mathcal{X}, x' \neq x_0} 1\{x_i - x_0\} + \sum_{j=1}^{m} 1\{x_j - x_0\} \right].
$$

The second sum in the brackets can be estimated by $m$. As above

$$
\int \mu(dX) \Sigma_3(n, X) \leq z\xi c \sum_{m \geq 0} (m + 1)^2 (c\xi z\xi)^m c(n)(z\xi c\xi + m).
$$

In the bounds on the expectations of $\Sigma_1$ and $\Sigma_3$ the sums over $m$ are finite by \eqref{5.11}. Collecting all estimates and using \eqref{6.24} we thus find that

$$
\int \mu(dX) \sum_{i=1}^{3} \Sigma_i(n, X) \leq \frac{\delta}{2}
$$

for sufficiently large $n$, and $\mu(G'_{n}) \leq \delta$ follows from the high probability of $G'_{n}$, the Chebyshev inequality and the definition of $G_{n}$ in \eqref{5.4}.
7 Proof of Theorem 2: Main steps

7.1 Basic constants

Let \((U, z, \mathcal{X}_0)\) be admissible with Ruelle bound \(\xi\), where \(U : \mathbb{R}^2 \to \mathbb{R}\) is a translation-invariant, smoothly approximable standard potential. We choose \(K, \psi, \bar{U}\) and \(u\) according to Definition 1. W.l.o.g. we may assume \(0 \in K\), \(\bar{U} = U\) and \(u = 0\) on \(K\). We then let \(\epsilon > 0\) so small that

\[
c_\xi := \lambda^2(K \setminus K^{U}) + \int_{K}\tilde{u}(x)dx < \frac{1}{z_\xi}. \tag{7.1}
\]

In addition to the function \(f_K\) and the constants \(c_K\) and \(c_f\) introduced in Section 5.1 we also define

\[
c_u := \int_{K} \tilde{u}(x)|x|^2dx \quad \text{and} \quad c_\psi := \|\psi\| \vee \int dx \psi(x)(|x|^2 \vee 1). \tag{7.2}
\]

These constants are finite by our assumptions. Finally, we fix a Gibbs measure \(\mu \in \mathcal{G}_{\mathcal{X}_0}(U, z)\), a cylinder event \(D \in \mathcal{F}_{\mathcal{X}_0, n'}\) where \(n' \in \mathbb{N}\), a translation distance \(\tau \in [0, 1/2]\), the translation direction \(e_1\) and a real \(\delta > 0\).

7.2 Decomposition of \(\mu\) and the bond process

For \(n \in \mathbb{N}\) and \(X \in \mathcal{X}\) we consider the bond set

\[
E_n(X) := E_{\Lambda_n}(X) = \{x_1 x_2 \in E(X) : x_1 x_2 \cap \Lambda_n \neq \emptyset\}.
\]

On \((\mathcal{E}_{E_n(X)}, \mathcal{B}_{E_n(X)})\) we introduce the Bernoulli measure \(\pi_n(\cdot|X)\) with bond probabilities

\[
(\bar{u}(b))_{b \in E_n(X)} \quad \text{where} \quad \bar{u}(b) := 1 - e^{-u(b)},
\]

using the shorthand notation \(u(x_1 x_2) := u(x_1 - x_2)\) for \(x_1, x_2 \in \mathbb{R}^2\). We note that \(0 \leq \bar{u}(b) < 1\) for all \(b \in E_n(X)\) as \(0 \leq u < \infty\). As remarked earlier \(\pi_n(\cdot|X)\) can be extended to a probability measure on \((\mathcal{E}, \mathcal{F}_\mathcal{E})\). For all \(D \in \mathcal{F}_\mathcal{E}\) \(\pi_n(D|.)\) is \(\mathcal{F}_\mathcal{X}\)-measurable, so \(\pi_n\) is a probability kernel from \((\mathcal{X}, \mathcal{F}_\mathcal{X})\) to \((\mathcal{E}, \mathcal{F}_\mathcal{E})\).

Lemma 22 Let \(n \in \mathbb{N}\). We have

\[
\mu \otimes \nu_{\Lambda_n}(G''_n) = 1 \quad \text{and} \quad \mu(G''_n) = 1 \quad \text{for} \quad G''_n := \{X \in \mathcal{X}_0 : \sum_{b \in E_n(X)} \bar{u}(b) < \infty\}.
\]

For \(X \in G''_n\) the Borel-Cantelli lemma implies that every bond set is finite \(\pi_n(\cdot|X)\)-a.s., so

\[
\sum'_{B \subset E_n(X)} \pi_n(\{B\}|X) = 1,
\]

where the summation symbol \(\sum'\) indicates that the sum extends over finite subsets only. We have

\[
\pi_n(\{B\}|X) = \prod_{b \in B} \bar{u}(b) \prod_{b \in E_n(X) \setminus B} (1 - \bar{u}(b)) = e^{-H^X_n(X)} \prod_{b \in B} (e^{u(b)} - 1),
\]

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so for every $X \in G_n^\ast$ the Hamiltonian $H_{\Lambda_n}^u(X)$ is finite, and thus the decomposition of the potential gives a corresponding decomposition of the Hamiltonian
\[ H_{\Lambda_n}^U(X) = H_{\Lambda_n}^{U}(X) - H_{\Lambda_n}^u(X). \]

Using (3.1) we conclude that for every $F \in \mathcal{F}_X \otimes \mathcal{F}_E$-measurable function $f \geq 0$
\[
\int d\mu \otimes \pi_n f = \int \mu(d\bar{X}) \frac{1}{Z_{\Lambda_n}(X)} \int \nu_{\Lambda_n}(dX|x\bar{X}) \sum_{B \subset E_n(X)} f(B, x) \times z^{|X_{\Lambda_n}|} e^{-H_{\Lambda_n}^{U}(X)} \prod_{b \in B} (e^{u(b)} - 1).
\]

7.3 Generalised translation

First of all, we need to augment each bond set $B$ by additional bonds between all particles that are close to each other. That is, for $n > n'$, $X \in \mathcal{X}$ and $B \subset E_n(X)$ we introduce the $K_\varepsilon$-enlargement of $B$ by
\[ B_+ := B \cup \{x_1x_2 \in E_n(X) : x_1 - x_2 \in K_\varepsilon\}. \]

We then consider the range of the $B_+$-cluster of $\Lambda \in \mathcal{B}_B^X$
\[ r_{n,X,B_+}(\Lambda) = \sup\{|x'| : x' \in C_{X,B_+}(\Lambda)\}. \]

Lemma 23 We have
\[ \sup_{n > n'} \int \mu \otimes \pi_n(dX, dB) r_{n,X,B_+}(\Lambda_{n'}) < \infty. \]

By the Chebyshev inequality we therefore can choose an integer $R > n'$, such that for every $n > n'$ the event
\[ G_n' := \{(X, B) \in \mathcal{X} \times \mathcal{E} : r_{n,X,B_+}(\Lambda_{n'}) < R, B \subset E_n(X) \text{ finite} \} \in \mathcal{F}_X \otimes \mathcal{F}_E \]
has probability
\[ \mu \otimes \pi_n(G_n') \geq 1 - \delta/2. \]

For $n > R$ we define the functions $q$, $Q$, $r$ and $\tau_n$ exactly as in Section 5.2. For $X \in \mathcal{X}$, $B \in E_n(X)$ and $x \in X$ we define $a_{n,X,B_+}(x)$ to be a point of $C_{X,B_+}(y)$ such that
\[ |a_{n,X,B_+}(x)| \geq |x|, \quad \tau_n(|a_{n,X,B_+}(x)|) = \min\{\tau_n(|x'|) : x' \in C_{X,B_+}(y)\} \]
and $a_{n,X,B_+}(x)$ is a measurable function of $x$, $X$ and $B$. 31
7.4 Good configurations

In order to deal with the hard core and the perturbation encoded in the bond process, we will introduce a transformation

$$\Upsilon_n : \mathcal{X} \times \mathcal{E} \to \mathcal{X} \times \mathcal{E}$$

which is required to have the following properties:

1. Whenever $B$ is a set of bonds between particles in $X$, the transformed configuration $(\tilde{X}, \tilde{B}) = \Upsilon_n(X, B)$ is constructed by translating every particle $x \in X$ by a certain distance $t_{n,X,B}(x)$ in direction $e_1$, and by translating bonds along with the corresponding particles.

2. Particles in the inner region $\Lambda_{n'}^{-1}$ are translated by $\tau e_1$, and particles in the outer region $\Lambda_{nc}$ are not translated at all.

3. Particles connected by a bond in $B$ are translated the same distance.

4. $\Upsilon_n$ is bijective, and the density of the transformed process with respect to the untransformed process under the measure $\nu \otimes \pi_n'$ can be calculated explicitly.

5. We have suitable estimates on this density and on $H_{\Lambda_n}(\tilde{X}) - H_{\Lambda_n}(X)$. For the last assumption we need particles within hard core distance to remain within hard core distance and particles at larger distance to remain at larger distance.

Property (2) implies that the translation of the chosen cylinder event $D$ is the same as the transformation of $D$ by $\Upsilon_n$. Properties (3)-(5) are chosen with a view to the right hand side of (7.3). If $\Upsilon_n$ has these properties then the density of the transformed process with respect to the untransformed process under the measure $\mu \otimes \pi_n$ can be estimated. We will content ourselves with a transformation satisfying the above properties only for $(X, B)$ from a set of good configurations

$$G_n := \{(X, B) \in G'_n : \sum_{i=1}^{5} \Sigma_i(n, X, B) < 1/2\} \in \mathcal{F}_X \otimes \mathcal{F}_E. \quad (7.4)$$

The functions $\Sigma_i(n, X, B)$ will be defined whenever we want good configurations to have a certain property. In Lemma 28 we then will prove that the set of good configurations $G_n$ has probability close to 1 when $n$ is big enough. Up to that point we consider a fixed $n \geq R + 1$.

7.5 Modifying the generalised translation

The construction of the deformed translation $\Upsilon_n$ will go along the same lines as the corresponding construction in section 5.4. However, here we also have to consider bonds between particles, and by property (3) from the last section we know that we have to translate not just particles, but whole $B$-clusters.
For a rigorous recursive definition of \( \Xi_n(X, B) \) we first consider the case that \( B \) is a finite subset of \( E_n(X) \). Let \( t_0^n, X, B := \tau_n(\cdot, |) \), \( C_0^n, X, B \) the \( B \)-cluster of the outer region \( \Lambda'_n \), \( m = m(X, B) \) the number of different \( B \)-clusters of \( X \setminus C_0^n, X, B \) and \( \tau_0^n, X, B := 0 \). In the \( k \)-th construction step (\( 1 \leq k \leq m \)) let

\[
  t_k^n, X, B := t_{k-1}^{n, X, B} \land \bigwedge_{x \in C_{k-1}^{n, X, B}} m_{x, t_{k-1}^{n, X, B}} = t_0^n, X, B \land \bigwedge_{0 \leq i \leq k} \bigwedge_{x \in C_i^{n, X, B}} m_{x, t_k^{n, X, B}},
\]

where the auxiliary function \( m_{x, t} : \mathbb{R}^2 \to \mathbb{R} \) is defined as in Section 5.4. Let the pivotal point \( P_k^n, X, B \) be the point of \( X \setminus (C_0^n, X, B \cup \ldots \cup C_{k-1}^{n, X, B}) \) at which the minimum of \( t_k^n, X, B \) is attained. If there is more than one such point then take the smallest point with respect to the lexicographic order for the sake of definiteness. Let \( \tau_k^n, X, B := t_k^n, X, B(P_k^n, X, B) \) be the corresponding minimal value of \( t_k^n, X, B \), \( C_k^{n, X, B} \) the \( B \)-cluster of the point \( P_k^n, X, B \) and \( T_k^n, X, B := id + t_k^n, X, B e_1 \).

For \( k = m + 1 \) we can still define \( t_{m+1}^n, X, B \), but then the recursions stops as \( X \setminus (C_0^n, X, B \cup \ldots \cup C_m^{n, X, B}) = \emptyset \). In the above notations we will omit dependence on \( X \) and \( B \) if it is clear which configuration is considered. Now for \( x \in C_k^{n, X, B} \) let \( t^n, X, B(x) := \tau_k^n, X, B \) be the deformed translation distance function and let

\[
  \Xi_n, B(X) := \bigcup_{k=0}^{m(X, B)} (C_k^{n, X, B} + \tau_k^n, X, B e_1) = \{ x + t^n, X, B(x) e_1 : x \in X \} \quad \text{and} \quad \Xi_n, X(B) := \{ (x + t^n, X, B(x) e_1)(x' + t^n, X, B(x') e_1) : x' \in B \}.
\]

If \( B \) is not a finite subset of \( E_n(X) \) we define \( \Xi_n, B = id \) and \( \Xi_n, X = id \). The deformed transformation can now be defined to be

\[
  \Xi_n : X \times \mathcal{E} \to X \times \mathcal{E}, \quad \Xi_n(X, B) := (\Xi_n, B(X), \Xi_n, X(B)).
\]

Using Lemma 6 one can show that all above objects are measurable with respect to the considered \( \sigma \)-algebras. In the rest of this section we will convince ourselves that the above construction has indeed the required properties.

**Lemma 24** For good configurations \( (X, B) \in G_n \) we have

\[
  (\Xi_n, B X - \tau e_1)_{A_{n'}^{-1}} = X_{A_{n'}^{-1}} \quad \text{and} \quad (\Xi_n, B X)_{A_n e} = X_{A_n e}.
\]

**Lemma 25** The transformation \( \Xi_n : X \times \mathcal{E} \to X \times \mathcal{E} \) is bijective.

In the proof of Lemma 25 we again construct the inverse of \( \Xi_n \), which is needed in the proof of the following lemma. There we will also show that

\[
  \varphi_n(X, B) := \prod_{k=1}^{m(X, B)} \left| 1 + \partial_1 t_k^n, X, B(P_k^n, X, B) \right|
\]

is well defined \( \nu_{A_n} \otimes \pi'_n(\cdot, X)\text{-a.s.} \), in that the considered derivatives exist.
Lemma 26 For every $\bar{X} \in \mathcal{X}$ and every $\mathcal{F}_X \otimes \mathcal{F}_\mathcal{E}$-measurable function $f \geq 0$

$$\int d\nu_n \otimes \pi_n'(\cdot, \bar{X}) (f \circ \Sigma_n \cdot \varphi_n) = \int d\nu_n \otimes \pi_n'(\cdot, \bar{X}) f. \quad (7.7)$$

We also need the backwards translation. So let $\check{\Sigma}_n, \check{\Sigma}_{n,b}, \check{\Sigma}_{n,x}$ and $\bar{\varphi}_n$ be defined analogously to the above objects, where now $e_1$ is replaced by $-e_1$. The previous lemmas apply analogously to this deformed backwards translation. We note that $\check{\Sigma}_n$ is not the inverse of $\Sigma_n$.

7.6 Final steps of the proof

From (7.3) and Lemma 26 we deduce

$$\mu \otimes \pi_n(\bar{\Sigma}_n(D \cap G_n))$$

$$= \int \mu(d\bar{X}) \frac{1}{Z_{\Lambda_n}(X)} \int \nu_n(dX, dB) = \mu \otimes \pi_n(D \cap G_n)$$

$$1_{\bar{\Sigma}_n(D \cap G_n)} \circ \bar{\Sigma}_n(X, B) z^\#(\bar{\Sigma}_{n,b}X) \phi_n(X, B) e^{-H_{\Lambda_n}^U(\bar{\Sigma}_{n,b}X)} \prod_{b \in \bar{\Sigma}_{n,x}B} (e^{u(b)} - 1).$$

Here we have identified $D$ and $D \times \mathcal{E}$. By the bijectivity of $\bar{\Sigma}_n$ from Lemma 25 by (7.5) and by construction of $\bar{\Sigma}_{n,x}$ the above integrand simplifies to

$$1_{D \cap G_n}(X, B) z^\#X \phi_n(X, B) e^{-H_{\Lambda_n}^U} \prod_{b \in B} (e^{u(b)} - 1).$$

The backwards transformation $\check{\Sigma}_n$ can be treated analogously, hence

$$\mu \otimes \pi_n(\check{\Sigma}_n(D \cap G_n)) + \mu \otimes \pi_n(\check{\Sigma}_n(D \cap G_n)) - \mu \otimes \pi_n(D \cap G_n)$$

$$= \int \mu(d\bar{X}) \frac{1}{Z_{\Lambda_n}(X)} \int \nu_n \otimes \pi_n'(dX, dB) 1_{D \cap G_n}(X, B) z^\#X \phi_n(X, B) \prod_{b \in B} (e^{u(b)} - 1)$$

$$\times \left[ e^{\log \varphi_n(X, B) - H_{\Lambda_n}^U(T_n, b X)} + e^{\log \varphi_n(X, B) - H_{\Lambda_n}^U(T_n, b X)} - e^{-H_{\Lambda_n}^U(X)} \right].$$

We note that for $(X, B) \in G_n$ we have

$$e^{\log \varphi_n(X, B) - H_{\Lambda_n}^U(T_n, b X)} + e^{\log \varphi_n(X, B) - H_{\Lambda_n}^U(T_n, b X)}$$

$$\geq 2 e^{\frac{1}{2} \log \varphi_n(X, B) + \log \varphi_n(X, B) - H_{\Lambda_n}^U(T_n, b X) - H_{\Lambda_n}^U(T_n, b X)}$$

$$\geq 2 e^{-\frac{1}{2} H_{\Lambda_n}^U(X)} \geq e^{-H_{\Lambda_n}^U(X)},$$

where we have used the convexity of the exponential function in the first step and the following estimates in the second step:

Lemma 27 For $(X, B) \in G_n$ we have

$$H_{\Lambda_n}^U(\check{\Sigma}_{n,b}X) + H_{\Lambda_n}^U(\check{\Sigma}_{n,b}X) \leq 2H_{\Lambda_n}^U(X) + 1/2 \quad \text{and} \quad (7.8)$$

$$\log \bar{\varphi}_n(X, B) + \log \varphi_n(X, B) \geq -1/2. \quad (7.9)$$

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Hence we have shown that
\[
\mu \otimes \pi_n(\mathcal{F}_n(D \cap G_n)) + \mu \otimes \pi_n(\mathcal{F}_n(D \cap G_n)) \geq \mu \otimes \pi_n(D \cap G_n). \quad (7.10)
\]
In (7.10) we would like to replace \( D \cap G_n \) by \( D \), and for this we need \( G_n \) to have high probability:

**Lemma 28** If \( n \geq R + 1 \) is chosen big enough, then \( \mu \otimes \pi_n(G_n^c) \leq \delta \).

For the proof of Theorem 2 we choose such an \( n \geq R + 1 \). The rest of the argument is then the same as that in Section 5.5.

### 8 Proof of the lemmas from Section 7

#### 8.1 Convergence of energy sums: Lemma 22

Let \( n \in \mathbb{N} \). For every \( X \in \mathcal{X} \) we have
\[
H^\mu_{\Lambda_n}(X) = \sum_{b \in E_n(X)} \tilde{u}(b) \leq \sum_{x_1, x_2 \in X} 1_{\{x_1 \in \Lambda_n\}} \tilde{u}(x_1 - x_2), \quad \text{and so}
\]
\[
\int \nu_{\Lambda_n}(dX|\bar{X})H^\mu_{\Lambda_n}(X) \leq \int_{\Lambda_n} dx_1 \left( \int_{\Lambda_n} dx_2 \tilde{u}(x_1 - x_2) + \sum_{x_2 \in X_{\bar{X}}^c} \tilde{u}(x_1 - x_2) \right)
\]
for all \( \bar{X} \in \mathcal{X} \). By Lemma 3 we get
\[
\int \mu \otimes \nu_{\Lambda_n}(dX|\bar{X})H^\mu_{\Lambda_n}(X) \leq \int_{\Lambda_n} dx_1 \left( \int_{\Lambda_n} dx_2 \tilde{u}(x_1 - x_2) + z\xi \int_{\Lambda_n^c} dx_2 \tilde{u}(x_1 - x_2) \right)
\]
\[
\leq \int_{\Lambda_n} dx_1 (1 + z\xi) c_\xi \leq 4n^2 (1 + z\xi) c_\xi < \infty,
\]
where we have estimated the integrals over \( x_2 \) by \( c_\xi \) using (7.1). Thus we have proved the first assertion. However, \( \mu \) is absolutely continuous with respect to \( \mu \otimes \nu_{\Lambda_n} \), which follows from (3.1) and the definition of the conditional Gibbs distribution. So the first assertion implies the second one.

#### 8.2 Cluster bounds: Lemma 23

Let us refine the argument of Section 6.1 as follows. For \( n > n' \), \( X \in \mathcal{X} \) and \( B \subset E_n(X) \) we consider a path \( x_0, \ldots, x_m \) in the graph \((X, B_+)\) such that \( x_0 \in \Lambda_n \), and we consider an integer \( 1 \leq k \leq m \) and the bond \( x_{k-1}x_k \) has maximal \(|.|\)-length among all bonds on the path. We have
\[
|x_m| \leq |x_0| + \sum_{i=1}^m |x_i - x_{i-1}| \leq n' + m|x_k - x_{k-1}|.
\]
By considering all paths and bonds of maximal length we obtain
\[
r_{n,X,B_+}(\Lambda_{n'}) \leq n' + \sum_{m \geq 1} \sum_{k=1}^m \sum_{x_0, \ldots, x_m \in X} 1_{\{x_0 \in \Lambda_{n'}\}} m|x_k - x_{k-1}| \prod_{i=1}^m 1_{\{x_{i-1} \in B_+\}}.
\]

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Under the Bernoulli measure $\pi_n(dB|X)$, the events \( \{x_i x_{i-1} \in B_+\} \) are independent, and for $g := 1_{K_0 K'} + u$ we have

$$
\int \pi_n(dB|X)1_{\{x_i x_{i-1} \in B_+\}} \leq 1_{K'}(x_i - x_{i-1}) + g(x_i - x_{i-1}). \quad (8.1)
$$

Using the hard core property (3.2) and Lemma 3 we thus find

$$
R_n := \int \mu(dX) \int \pi_n(dB|X)r_{n,X,B}(\Lambda_{n'}) - n' \leq \sum_{m \geq 1} \sum_{k=1}^m \int \mu(dX) \sum_{x_0 \in \Lambda_{n'}} m|\{x_k - x_{k-1}\}| \prod_{i=1}^m g(x_i - x_{i-1}) \leq \sum_{m \geq 1} \sum_{k=1}^m (z\xi)^{m+1} \int dx_0 \ldots dx_m 1_{\{x_0 \in \Lambda_{n'}\}} m|\{x_k - x_{k-1}\}| \prod_{i=1}^m g(x_i - x_{i-1}).
$$

Setting $c_g := (1 + c_K^2) c_\xi + c_u$ we conclude from (7.1) and (7.2) that

$$
\int g(x)|x| \, dx \leq \int g(x)(1 + |x|^2) \, dx \leq c_g \quad \text{and} \quad \int g(x) \, dx \leq c_\xi, \quad (8.2)
$$

hence we can estimate the integrals over $dx_i$ in the above expression beginning with $i = m$. This gives $m - 1$ times a factor $c_\xi$ and once a factor $c_g$. Finally the integration over $dx_0$ gives an additional factor $\lambda^2(\Lambda_{n'}) = (2n')^2$. Thus

$$
R_n \leq (2n'z\xi)^2 c_g \sum_{m \geq 1} m^2(c_g z\xi)^{m-1}.
$$

The last sum is finite because $c_\xi z\xi < 1$.

8.3 Properties of the deformed translation: Lemma 24

We will show properties of the construction which are analogous to properties of the corresponding objects from the proof of the special case in Sections 6.3 and 6.4. Additionally we need a way to calculate the translation distance of an arbitrary particle $x \in C_n^k$ without knowing $P_n^k$. This can be done using the first relation of the following lemma.

**Lemma 29** For $X \in \mathcal{X}$, finite $B \subset E(X)$, $k \geq 0$, $x, x' \in X$ and $s \in [-1, 1]$

- $\tau_n^k = \ell_n^{k+1}(x)$ if $x \in C_n^k$, \quad (8.3)
- $\tau_n^k \leq \tau_n^{k+1}$, \quad (8.4)
- $\ell_n^k$ is $1/2$-Lipschitz-continuous and piecewise cont. $e_1$-differentiable, \quad (8.5)
- $T_n^k$ is $\leq_{e_1}$-increasing and bijective, \quad (8.6)
- $x - x' \in K \Rightarrow t_{n,X,B}(x) = t_{n,X,B}(x')$, \quad (8.7)
- $x - x' \notin K \Rightarrow x - x' + s(t_{n,X,B}(x) - t_{n,X,B}(x')) e_1 \notin K$, \quad (8.8)
- $\tau_n^k \leq \ell_n^0(x)$ for all $x \in C_n^{k'}$ such that $k' \geq k$, \quad (8.9)
- $\tau_n^k \geq \ell_n^0(a_{n,X,B}(P_n^k))$ if $(X, B) \in G_n$, \quad (8.10)
Proof: For (8.3) let \( x \in C_n^k \). By definition of \( P_n^k \) we have \( t_n^k(x) \geq \tau_n^k \), so
\[
i_n^{k+1}(x) = t_n^k(x) \wedge \bigwedge_{x' \in C_n^k} m_{x', \tau_n^k}(x) = \tau_n^k,
\]
where we have also used \( m_{x', \tau_n^k}(x) \geq \frac{\tau_n^k}{2} \) and \( m_{x, \tau_n^k}(x) = \tau_n^k \). The other assertions can be shown as in Sections 6.3 and 6.4. Here for the proof of (8.7) and (8.8) we have to use (8.3), and the key observations are the following: For \( i \leq j \), \( x_i \in C_n^i \) and \( T_{n,s}^{i+1} := id + s \cdot t_n^i e_1 \) we have
\[
\forall x \in K(x_i) \cap \Lambda^i : \ t_n^{i+1}(x) = t_n^i(x) \wedge \bigwedge_{i \leq k \leq j} \bigwedge_{x' \in C_n^k} m_{x', \tau_n^k}(x) = \tau_n^i,
\]
\[
T_{n,s}^{i+1}(\Lambda^i) = \Lambda^i + s \tau_n^i e_1 \quad \text{and} \quad T_{n,s}^{i+1}(\Lambda^i \setminus K(x_i)) = \Lambda^i \setminus K(x_i) + s \tau_n^i e_1.
\]
To obtain (8.10) here we specify the function
\[
\Sigma_1(n, X, B) := \sum_{x, x' \in X} 1_{\{|x| \leq |x'|\}} 1_{\{x \rightarrow x'\}} 4 \left( \tau_n(|x| - c_K) - \tau_n(|x'|) \right)^2 c_f^2 \quad (8.11)
\]
used in the definition of \( G_n \).

\[\square\]

Lemma 24 follows from (8.9) and (8.10), just as in the proof of Lemma 22.

8.4 Bijectivity of the transformation: Lemma 25

The construction of the inverse transformation is analogous to the one in Section 6.3. Let \( \tilde{X} \subset X \) and \( \tilde{B} \subset E_n(\tilde{X}) \) be finite. Let \( \tilde{C}_0^0_{n,\tilde{X},\tilde{B}} \) be the \( \tilde{B} \)-cluster of \( \Lambda_n^0 \), \( \tilde{m} = \tilde{m}(\tilde{X}, \tilde{B}) \) the number of different \( \tilde{B} \)-clusters of \( \tilde{X} \setminus \tilde{C}_0^0_{n,\tilde{X},\tilde{B}} \), \( \tilde{p}_{n,\tilde{X},\tilde{B}}^0 = \tau_n(|.|) \) and \( \tilde{p}_{n,\tilde{X},\tilde{B}}^0 := 0 \). In the \( k \)-th construction step \((k \geq 1)\) let
\[
\hat{\tau}_{n,\tilde{X},\tilde{B}}^k := \hat{\tau}_{n,\tilde{X},\tilde{B}}^{k-1} \wedge \bigwedge_{x \in \tilde{C}_n^{k-1}_{n,\tilde{X},\tilde{B}}} m_{x, \hat{\tau}_{n,\tilde{X},\tilde{B}}^{k-1}}.
\]
Let \( \hat{\tau}_{n,\tilde{X},\tilde{B}}^k := id + \hat{\tau}_{n,\tilde{X},\tilde{B}}^k e_1 \) and \( \hat{\tau}_{n,\tilde{X},\tilde{B}}^k \) be the point of \( \tilde{X} \setminus (\tilde{C}_n^{0}_{n,\tilde{X},\tilde{B}} \cup \ldots \cup \tilde{C}_n^{k-1}_{n,\tilde{X},\tilde{B}}) \) at which the minimum of \( \hat{\tau}_{n,\tilde{X},\tilde{B}}^k \circ (\hat{\tau}_{n,\tilde{X},\tilde{B}}^k)^{-1} \) is attained. If there is more than one such point then take the point \( x \) such that \((\hat{\tau}_{n,\tilde{X},\tilde{B}}^k)^{-1}(x)\) is minimal with respect to the lexicographic order \( \leq \). Let \( \hat{\tau}_{n,\tilde{X},\tilde{B}}^k := \hat{\tau}_{n,\tilde{X},\tilde{B}}^k \circ (\hat{\tau}_{n,\tilde{X},\tilde{B}}^k)^{-1}(\hat{\tau}_{n,\tilde{X},\tilde{B}}^k) \) be the corresponding minimal value and \( \tilde{C}_n^{0}_{n,\tilde{X},\tilde{B}} \) be the \( \tilde{B} \)-cluster of the pivotal point \( \hat{\tau}_{n,\tilde{X},\tilde{B}}^k \). The recursion stops for \( k = \tilde{m} + 1 \). In the above notations we will omit dependence on \( \tilde{X} \) and \( \tilde{B} \) if it is clear which configuration is considered. We need to show that the above construction is well defined, i.e. that \( \hat{\tau}_{n,\tilde{X},\tilde{B}}^k \) is invertible in every step. Furthermore we need some more properties of the construction. All this is done in the following lemma:
Lemma 30  Let $\bar{X} \in \mathcal{X}$, $\bar{B} \subset E_n(\bar{X})$ finite and $k \geq 0$. Then

\begin{align}
\hat{t}_n^k \text{ is } 1/2-c_1\text{-Lipschitz-continuous, } \hat{T}_n^k \text{ is bijective and } \leq_{e_1}\text{-increasing,} & \quad \text{(8.12)} \\
(\hat{T}_n^k)^{-1} + \hat{t}_n^k \circ (\hat{T}_n^k)^{-1}e_1 = \text{id}, & \quad \text{(8.13)} \\
\forall c \in \mathbb{R}, x \in \mathbb{R}^2 : \hat{t}_n^k \circ (\hat{T}_n^k)^{-1}(x) \geq c \Leftrightarrow \hat{t}_n^k(x - ce_1) \geq c, & \quad \text{(8.14)} \\
\hat{t}_n^k \leq \hat{t}_n^{k-1} \text{ and } \hat{t}_n^{k-1} \leq \hat{t}_n^k, & \quad \text{(8.15)} \\
\forall x \in \bar{C}_n^k : \hat{t}_n^{k+1} \circ (\hat{T}_n^{k+1})^{-1}(x) = \hat{z}_n^k & \quad \text{(8.16)}
\end{align}

Proof: Assertions (8.12) - (8.15) can be shown exactly as the corresponding assertions from Lemma 19. For (8.16) let $x \in \bar{C}_n^k$. We have

\[
\hat{t}_n^k \circ (\hat{T}_n^k)^{-1}(x) \geq \hat{z}_n^k \Rightarrow \hat{t}_n^k(x - \hat{t}_n^ke_1) \geq \hat{z}_n^k \\
\Rightarrow \hat{t}_n^{k+1}(x - \hat{t}_n^ke_1) = \hat{z}_n^k \Rightarrow \hat{t}_n^{k+1} \circ (\hat{T}_n^{k+1})^{-1}(x) = \hat{z}_n^k,
\]

where the first statement holds by definition, and the implications follow from (8.14), $x - \hat{t}_n^ke_1 \in \bar{C}_n^k - \hat{t}_n^ke_1$ and (8.13) respectively.

For $x \in \bar{C}_{n,X,B}^k$ let $\hat{t}_{n,X,B}(x) := \hat{t}_{n,X,B}^k$ be the distance the particle $x$ is translated. We define

\[
\bar{\xi}_{n,B}(\bar{X}) := \bigcup_{k=0}^m (\bar{C}_{n,X,B}^k - \hat{t}_{n,X,B}^ke_1) = \{x - \hat{t}_{n,X,B}^k(x)e_1 : x \in \bar{X}\} \quad \text{and}
\]

\[
\bar{\xi}_{n,X}(\bar{B}) := \{(x - \hat{t}_{n,X,B}^k(x)e_1)(x' - \hat{t}_{n,X,B}^k(x')e_1) : xx' \in \bar{B}\}.
\]

Now if $\bar{B}$ is a not a finite subset of $E_n(\bar{X})$ we define $\bar{\xi}_{n,B} = \text{id}$ and $\bar{\xi}_{n,X} = \text{id}$. $\bar{\xi}_n$ is then defined by

\[
\bar{\xi}_n : \bar{X} \times \bar{E} \to \bar{X} \times \bar{E}, \quad \bar{\xi}_n(X, B) := (\bar{\xi}_{n,B}(X), \bar{\xi}_{n,X}(B)).
\]

By Lemma 6 we see again that all above objects are measurable with respect to the considered $\sigma$-algebras. The following two lemmas are the key to show that $\bar{\xi}_n$ is indeed the inverse of $\xi_n$. The proofs differ from the proofs of Lemmas 18 and 20 only, in that we have to use (8.13) and (8.16), whenever we want to calculate the translation distance of a point explicitly. In the first lemma we consider $X \in \mathcal{X}$, finite $B \subset E_n(X)$, $t_n^k, T_n^k, \bar{C}_n^k, \bar{P}_n^k$ and $\bar{t}_n^k (0 \leq k \leq m)$ as in the construction of $\bar{\xi}_n(X, B)$, and we define $(X, \bar{B}) := \bar{\xi}_n(X, B), \bar{P}_n^k := P_n^k + \bar{t}_n^ke_1$ and $\bar{C}_n^k := C_n^k + \bar{t}_n^ke_1$, see Figure 6.

Lemma 31  Let $1 \leq k \leq m$. For every $\hat{x} \in \bar{X} \setminus (\bar{C}_n^0 \cup \ldots \cup \bar{C}_n^{k-1})$ we have

\[
\hat{t}_n^k \circ (T_n^k)^{-1}(\bar{P}_n^k) \leq t_n^k \circ (T_n^k)^{-1}(\hat{x}).
\]

For all $\hat{x}$ for which equality occurs we have $(T_n^k)^{-1}(\bar{P}_n^k) \leq (T_n^k)^{-1}(\hat{x})$.

For the second lemma we consider $\bar{X} \in \mathcal{X}$, finite $\bar{B} \subset E_n(\bar{X})$, $\hat{t}_n^k, \hat{T}_n^k, \bar{C}_n^k, \bar{P}_n^k$ and $\bar{t}_n^k (0 \leq k \leq m)$ as in the construction of $\bar{\xi}_n(X, \bar{B})$, and we define $(X, B) := \bar{\xi}_n(X, \bar{B}), P_n^k := \bar{P}_n^k - \bar{t}_n^ke_1$ and $C_n^k := \bar{C}_n^k - \bar{t}_n^ke_1$, see Figure 6.
Figure 6: Construction of the inverse $\tilde{\Sigma}_n$ of $\Sigma_n$.

**Lemma 32** Let $1 \leq k \leq m$. For every $x \in X \setminus (C_0^k \cup \ldots \cup C_{n-1}^k)$ we have

$$\tilde{t}_n^k(P_n^k) \leq \tilde{t}_n^k(x).$$

For all $x$ for which equality occurs we have $P_n^k \leq x$.

Now the following lemma follows exactly as in the proof of Lemma 21.

**Lemma 33** On $X \times E$ we have $\tilde{T}_n \circ T_n = id$ and $T_n \circ \tilde{T}_n = id$.

### 8.5 Density of the transformed process: Lemma 26

By definition the left hand side of (7.7) equals

$$e^{-4n^2 \sum_{k \geq 0} \frac{1}{k!} I(k)},$$

where $I(k) = \int_{\Lambda^n_k} dx \sum_{B \subset E_n(X_k)} (f \circ \Sigma_n \cdot \varphi_n)(\tilde{X}_x, B)$, using the shorthand notation $\tilde{X}_x = \{x_1, \ldots, x_k\} \cup \tilde{X}_{\Lambda^n_k}$. We would like to fix the bond set $B$ before we choose the positions $x_i$ of the particles. Thus we introduce bonds between indices of particles instead of bonds between particles. Let $\mathbb{N}_k := \{1, \ldots, k\}$,

$$\tilde{X}^k := \mathbb{N}_k \cup \tilde{X}_{\Lambda^n_k} \text{ and } E_n(\tilde{X}^k) := \{x_1x_2 \in E(\tilde{X}^k) : x_1x_2 \cap \mathbb{N}_k \neq \emptyset\}.$$ 

For $B \subset E_n(\tilde{X}^k)$ and $x \in \Lambda^n_k$ ($I \subset \mathbb{N}_k$) we define $B_x$ to be the bond set constructed from $B$ by replacing the point $i \in I$ by $x_i$ in every bond of $B$ and by deleting every bond $B$ that contains a point $i \in \mathbb{N}_k \setminus I$. Analogously let $\tilde{X}_x := \{x_i : i \in I\} \cup \tilde{X}_{\Lambda^n_k}$ be the configuration corresponding to the sequence and let $\Sigma_n(\tilde{X}, B) := (\tilde{X}_x, B_x)$. Using this notation we obtain

$$I(k) = \sum_{B \subset E_n(\tilde{X}^k)}' I(k, B), \text{ where } I(k, B) := \int_{\Lambda^n_k} dx (f \circ \Sigma_n \cdot \varphi_n)(\tilde{X}, B)x.$$ 

To compute $I(k, B)$ we need to calculate $\Sigma_n(\tilde{X}, B)_x$, and for this we must identify the points $P_n^k, \tilde{X}_x, B_x$ among the particles $x_j$. So let $m_B$ be the number
of different $B$-clusters of $\bar{X}^k \setminus C_{\bar{X}^k,B}(\Lambda_n^\epsilon)$, $C_{\bar{X}^k,B}(\eta(0)) := C_{\bar{X}^k,B}(\Lambda_n^\epsilon) \cap \bar{X}$ and $\Pi(B)$ be the set of all mappings $\eta : \{1, \ldots, m_B\} \to (\bar{X}^k \setminus C_{\bar{X}^k,B}(\Lambda_n^\epsilon))$ such that every $\eta(i)$ is in a different $B$-cluster. For $\eta \in \Pi(B)$ let

$$A_{k,B,\eta} := \{x \in \Lambda_n^k : \forall 1 \leq j \leq m_B : x_{\eta(j)} = P_{n,X_i,B_i}^j\} \quad \text{and}$$

$$\tilde{A}_{k,B,\eta} := \{x \in \Lambda_n^k : \forall 1 \leq j \leq m_B : x_{\eta(j)} = \tilde{P}_{n,X_i,B_i}^j\},$$

where $\tilde{P}_{n,X_i,B_i}^j$ are the pivotal points from the construction of the inverse transformation in Subsection 8.4. Now we can write

$$I(k, B) = \sum_{\eta \in \Pi(B)} \int_{\Lambda_n^k} dx 1_{A_{k,B,\eta}}(x)(f \circ \iota_n \cdot \varphi_n)(\bar{X}, B)_x$$

and we denote the summands in the last term by $I(k, B, \eta)$. If $x \in A_{k,B,\eta}$ we can derive a simple expression for $\iota_n(\bar{X}, B)_x$. For $x \in \Lambda_n^k$ and $\eta \in \Pi$ we define a formal transformation $T_{B,\eta}(x) := (T_{B,\eta,x}(i))_{1 \leq i \leq k}$, where

$$t^{\eta(j)}_{B,\eta,x} := t^{\eta(j)}_{n,\bar{X},B_{x}}(x_{\eta(j)}) e_1$$

for $0 \leq j \leq m_B$ and $i \in C_{\bar{X}^k,B}(\eta(j)), i \neq \eta(j)$. Here $x^{\eta,j}$ is defined to be the subsequence of $x$ corresponding to the index set $C_{\bar{X}^k,B}(\eta(i))$. Clearly, for $i \in C_{\bar{X}^k,B}(\eta(j))$, $T_{B,\eta,x}^i$ doesn’t depend on all components of $x$, but only on those $x_i$ such that $l \in C_{\bar{X}^k,B}(\eta(j))$ and additionally on $x_{\eta(j)}$ if $i \neq \eta(j)$. By definition we now have

$$x \in A_{k,B,\eta} \Rightarrow \begin{cases} \iota_n(\bar{X}, B)_x = (\bar{X}, B)_{T_{B,\eta}(x)} \text{ and} \\ T_{n,\bar{X},B_{x}}^j = T_{B,\eta,x}^j \text{ for all } j \leq m_B. \end{cases} \tag{8.17}$$

Furthermore we observe that for all $x \in (\mathbb{R}^2)^k$ we have

$$x \in A_{k,B,\eta} \iff T_{B,\eta}(x) \in \tilde{A}_{k,B,\eta}. \quad \tag{8.18}$$

This can be shown exactly as (6.20) in Section 6.6. Let $g : (\mathbb{R}^2)^k \to \mathbb{R}$, $g(x) := 1_{\tilde{A}_{k,B,\eta}}(x)f(\bar{X}, B_x)$. Then (8.17) and (8.18) imply

$$I(k, B, \eta) = \left[ \prod_{j=0}^{m_B} \prod_{i \in C_{\bar{X}^k,B}(\eta(j))} \int dx_i \right] \left[ 1 + \partial_1 t^{\eta(j)}_{B,\eta,x}(x_{\eta(j)}) \right] g(T_{B,\eta}(x)),$$

where we have also inserted the definition of $\varphi_n$ (7.6). Now we transform the integrals. For $j = m_B$ to 1 and $i \in C_{\bar{X}^k,B}(\eta(j))$ we substitute $x'_i := T_{B,\eta,x}^i x_i$. For $i \neq \eta(j)$ $T_{B,\eta,x}^i$ is a translation by a constant vector, so $dx'_i = dx_i$. For $i = \eta(j)$ the Lebesgue transformation theorem (5.1) gives

$$dx_{\eta(j)}' = \left| 1 + \partial_1 t^{\eta(j)}_{B,\eta,x}(x_{\eta(j)}) \right| dx_{\eta(j)}.$$
as in Section 6.6. Thus
\[ I(k, B, \eta) = \left( \prod_{j=0}^{m_B} \prod_{i \in C_{x_k, B}^{(j)}} \int dx'_i \right) g(x') = \int_{\Lambda_n} dx \bar{\Lambda}_{k,B,\eta}(x) f(\bar{X}_x, B_x) \]
and we are done as the same arguments show that the right hand side of (7.7) equals
\[ e^{-4n^2} \sum_{k \geq 1} \frac{1}{k!} \sum'_{B \in E_n(\bar{X}^k)} \sum_{\eta \in \Pi(B)} \int_{\Lambda_n} dx \bar{\Lambda}_{k,B,\eta}(x) f(\bar{X}_x, B_x). \]
Combining the above ideas with the reasoning in Section 6.6 also shows that the density function is well defined.

### 8.6 Key estimates: Lemma [27]
For all \( x \in \mathbb{R}^2 \) and \( \vartheta \in [-1, 1] \) such that \( x + se \notin K \) for all \( s \in [-\vartheta, \vartheta] \) we have
\[ \bar{U}(x + \vartheta e_1) + \bar{U}(x - \vartheta e_1) - 2\bar{U}(x) \leq \sup_{s \in [-\vartheta, \vartheta]} \partial^2 \bar{U}(x + se_1) \vartheta^2 \leq \psi(x) \vartheta^2 \]
by Taylor expansion of \( \bar{U} \) at \( x \) using the \( e_1 \)-smoothness of \( \bar{U} \) and by the \( \psi \)-domination of the derivatives. Let \((X, B) \in G_n\). W.l.o.g. we may assume that the right hand side of (8.8) is finite. Introducing
\[ \eta_{x,x'} := x - x', \quad \vartheta_{x,x'} := t_{n,X,B}(x') - t_{n,X,B}(x) \quad \text{for} \quad x, x' \in E_n(X) \]
and \( E_{n,K}(X) := \{ xx' \in E_n(X) : x - x' \notin K \} \) for \( X \in \mathcal{X} \) we have
\[ H^U_{\Lambda_n}(\mathcal{X}_{n,B}X) + H^U_{\Lambda_n}(\mathcal{X}_{n,B}X) - 2H^U_{\Lambda_n}(X) = \sum_{xx' \in E_{n,K}(X)} \left[ \bar{U}(\eta_{x,x'} + \vartheta_{x,x'}e_1) + \bar{U}(\eta_{x,x'} - \vartheta_{x,x'}e_1) - 2\bar{U}(\eta_{x,x'}) \right] \leq \sum_{xx' \in E_{n,K}(X)} \psi(x - x')(t_{n,X,B}(x) - t_{n,X,B}(x'))^2 =: f_n(X, B). \]
In the first step we have used that for \( x - x' \in K \) we have \( \vartheta_{x,x'} = 0. \) In the second step we are allowed to apply the above Taylor estimate as for \( x - x' \notin K \) we have \( x - x' + se \notin K \) for all \( s \in [-\vartheta_{x,x'}, \vartheta_{x,x'}] \) by [8.8]. The arithmetic-quadratic mean inequality gives
\[ \frac{1}{3} \left( (t_{n,X,B}(x) - \tau_n(|x|)) + (\tau_n(|x|) - \tau_n(|x'|)) + (\tau_n(|x'|) - t_{n,X,B}(x')) \right)^2 \leq (t_{n,X,B}(x) - \tau_n(|x|))^2 + (\tau_n(|x|) - \tau_n(|x'|))^2 + (\tau_n(|x'|) - t_{n,X,B}(x'))^2, \]
and thus
\[ f_n(X, B) \leq 6 \sum_{x,x' \in X} \psi(x - x')(\tau_n(|x|) - t_{n,X,B}(x))^2 + 3 \sum_{x,x' \in X} 1_{\{|x| \leq |x'|\}} \psi(x - x')(\tau_n(|x|) - \tau_n(|x'|))^2. \]
In the first sum on the right hand side we estimate
\[
(\tau_n(|x|) - t_{n,X,B}(x))^2 \leq (\tau_n(|x|) - \tau_n(|a_{n,X,B_+}(x)|))^2
\leq \sum_{x'' \in X} 1_{\{|x| \leq |x''|\}} 1_{\{x \xrightarrow{B_+} x''\}} (\tau_n(|x|) - \tau_n(|x''|))^2
\]
using (8.10). By distinguishing the cases $x'' \neq x, x'$ and $x'' = x'$ we thus can estimate $f_n(X, B)$ by the sum of the two following expressions:
\[
\Sigma_2(n, X) := 9 \sum_{x, x'' \in X} 1 \neq \psi(x - x'') 1_{\{|x| \leq |x''|\}} |\tau_n(|x|) - c_K - \tau_n(|x''|)|^2,
\]
\[
\Sigma_3(n, X, B) := 6 \sum_{x, x', x'' \in X} 1 \neq \psi(x - x') 1_{\{|x| \leq |x''|\}} \psi(x - x'') 1_{\{|x| \leq |x''|\}}
\times |\tau_n(|x|) - c_K - \tau_n(|x''|)|^2.
\]
Inserting these sums into the definition of $G_n$ in (7.4), we obtain assertion (7.8).

Assertion (7.9) can be proved as in Section 6.7 using
\[
\Sigma_4(n, X) := 2\tau^2 \sum_{x \in X} 1_{\{x \in \Lambda_n\}} q(|x| - R)^2 \frac{Q(n - R)^2}{Q(n - R)^2},
\]
\[
\Sigma_5(n, X, B) := 2\tau^2 \sum_{x, x' \in X} \sum_{x'' \in X} 1_{K_0}(x - x') 1 \neq \psi(x - x'') 1_{\{|x| \leq |x''|\}} \psi(x - x')
\times |\tau_n(|x|) - c_K - \tau_n(|x''|)|^2.
\]
in the definition (7.4) of $G_n$.

### 8.7 Set of good configurations: Lemma 28

The functions $\Sigma_i(n, X, B)$ from the definition of the set of good configurations $G_n$ in (7.4) have been specified in (8.11), (8.19) and (8.20). Using the shorthand
\[
\tau_n^q(x, x'') := 1_{\{|x| \leq |x''|\}} |\tau_n(|x|) - c_K - \tau_n(|x''|)|^2
\]
we have
\[
\Sigma_1 = 4c^2 \sum_{x, x'' \in X} 1 \neq \psi(x - x'') \tau_n^q(x, x''), \quad \Sigma_2 = 9 \sum_{x, x'' \in X} \psi(x - x'') \tau_n^q(x, x''),
\]
\[
\Sigma_3 = 6 \sum_{x, x', x'' \in X} 1 \neq \psi(x - x') \tau_n^q(x, x''), \quad \Sigma_4 = 2\tau^2 \sum_{x \in X} 1_{\{x \in \Lambda_n\}} q(|x| - R)^2 \frac{Q(n - R)^2}{Q(n - R)^2},
\]
\[
\Sigma_5 = 2c^2 \sum_{x, x' \in X} \sum_{x'' \in X} 1_{K_0}(x - x') 1 \neq \psi(x - x'') \tau_n^q(x, x'').
\]

To estimate these sums we set $\bar{n} := n + c_K$ and $\bar{R} := R + c_K$ and use the assertions (6.22) and (6.24) of Section 6.8. As a refinement of (6.25), we note that for $x_0, \ldots, x_m \in \mathbb{R}^2$
\[
|x_m| - |x_0| + c_K \leq m \bigvee_{i=1}^m |x_i - x_{i-1}| + c_K \leq (m + 1)(1 \lor c_K) \left(1 \lor \bigvee_{i=1}^m |x_i - x_{i-1}|\right),
\]

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\[
so \quad (|x_m| - |x_0| + c_K)^2 \leq (m + 1)^2(1 \lor c_K^2) \sqrt[m]{(1 \lor |x_i - x_{i-1}|^2)^m}.
\] (8.21)

For the estimation of the expectations of \(\Sigma_i\) we combine the ideas from Section 6.8 and from the proof of Lemma 23. Using (6.22), (6.25), (7.1), (7.2), (8.1) and (8.2) we obtain

\[
\int \mu(dX) \int \pi_n(dB|X) \Sigma_i(n, X, B) \leq c_i c(n),
\]

where \(c_i\) are finite constants. By (6.21) we find that

\[
\int \mu \otimes \pi_n(d(X, B)) \sum_{i=1}^5 \Sigma_i(n, X, B) \leq \frac{\delta}{4}
\]

for sufficiently large \(n\). Now \(\mu \otimes \pi_n(G_n^c) \leq \delta\) follows from the high probability of \(G_n^c\), the Chebyshev inequality and the definition of \(G_n\) in (7.4).

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