Higher genus FJRW invariants of a Fermat cubic

Jun Li, Yefeng Shen and Jie Zhou

Abstract. We reconstruct all-genus Fan-Jarvis-Ruan-Witten invariants of a Fermat cubic Landau-Ginzburg space \((x_1^3 + x_2^3 + x_3^3 : [C^3/\mu_3] \to C)\) from genus-one primary invariants, using tautological relations and axioms of Cohomological Field Theories. The genus-one primary invariants satisfy a Chazy equation by the Belorousski-Pandharipande relation. They are completely determined by a single genus-one invariant, which can be obtained from cosection localization and intersection theory on moduli of three spin curves.

We solve an all-genus Landau-Ginzburg/Calabi-Yau Correspondence Conjecture for the Fermat cubic Landau-Ginzburg space using Cayley transformation on quasi-modular forms. This transformation relates two non-semisimple CohFT theories: the Fan-Jarvis-Ruan-Witten theory of the Fermat cubic polynomial and the Gromov-Witten theory of the Fermat cubic curve. As a consequence, Fan-Jarvis-Ruan-Witten invariants at any genus can be computed using Gromov-Witten invariants of the elliptic curve. They also satisfy nice structures including holomorphic anomaly equations and Virasoro constraints.

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2010 Mathematics Subject Classification. 14N35, 11Fxx.
1. Introduction

Let \((d; \delta)\) be a weight system such that \(\delta = (\delta_1, \ldots, \delta_N) \in \mathbb{Z}_+^N\) is a primitive \(N\)-tuple with \(w_i := d / \delta_i \in \mathbb{Z}_+\). We say the system is of Calabi-Yau (CY) type if

\[
d = \delta_1 + \cdots + \delta_N, \quad \text{i.e.,} \quad \sum_{i=1}^N \frac{1}{w_i} = 1.
\]

The dimension of the CY type weight system \((d; \delta)\) is defined to be

\[
\tilde{e} = \sum_{i=1}^N \left(1 - \frac{2\delta_i}{d}\right) = N - 2.
\]

Let \(\mu_d\) be the multiplicative group consisting of \(d\)-th roots of unity and

\[
J_\delta = (\xi_1^{\delta_1}, \ldots, \xi_N^{\delta_N}) \in \mu_d, \quad \xi_d := \exp(2\pi \sqrt{-1}/d).
\]

We call the data \(((\mathbb{C}^N/\langle J_\delta \rangle), W)\) a Landau-Ginzburg (LG) space, where \(W\) is a non-degenerate quasi-homogeneous polynomial on \(\mathbb{C}^N\) satisfying

\[
W(\lambda_{\delta_1} x_1, \ldots, \lambda_{\delta_N} x_N) = \lambda^d W(x_1, \ldots, x_N), \quad \forall \lambda \in \mathbb{C}^*.
\]

The polynomial \(W\) is assumed to have only an isolated critical point at the origin and not involve quadratic terms \(x_i x_j, i \neq j\). In general, we can consider Landau-Ginzburg spaces \(((\mathbb{C}^N/G), W)\) for a group \(G\) which is a subgroup of the group of diagonal symmetries with \(J_\delta \in G\) (see [FJR13, CLL15]). Two enumerative theories can be associated to such a LG space:

- The Gromov-Witten (GW) theory of the \(G/\langle J_\delta \rangle\)-quotient of the hypersurface defined by the vanishing of \(W\) in the corresponding weighted projective space \(\mathbb{P}^{N-1}(\delta_1, \ldots, \delta_N)\).
- The Fan-Jarvis-Ruan-Witten (FJRW) theory of the pair \((W, G)\) as introduced in [FJR07, FJR13].

Both the GW theory and the FJRW theory associated to a CY type weight system are Cohomological Field Theories (CohFT, for short) in the sense of [KM94].

In this work we shall focus on the theories arising from one-dimensional CY type weight systems. These systems are classified by

\[
(d; \delta) = (3; 1, 1, 1), \quad (4; 1, 1, 2), \quad (6; 1, 2, 3).
\]

The LG space we consider are \(((\mathbb{C}^3/\langle J_\delta \rangle), W)\), with \(W\) the Fermat polynomials

\[
W = x_1^{d/\delta_1} + x_2^{d/\delta_2} + x_3^{d/\delta_3}.
\]

On the CY-side, the hypersurface \(W = 0\) in the weighted projective space \(\mathbb{P}^2(\delta_1, \delta_2, \delta_3)\) is an elliptic curve, denoted by \(E_d\) or \(E\) (when the degree \(d\) is implicit or unimportant in the discussion) for simplicity. We focus on the GW theory of \(E\). The GW state space is then defined to be \(\mathcal{H}_E := H^* (E, \mathbb{C})\).

Let \(\overline{M}_{g,n}(E, \beta)\) be the moduli stack of degree-\(\beta\) stable maps from a connected genus \(g\) curve with \(n\) markings to the target \(E\). Let \(ev_k, k = 1, 2, \ldots, n\) be the evaluation morphisms, \(\pi\) be the forgetful morphism, and \([\overline{M}_{g,n}(E, \beta)]^\text{vir}\) be the virtual fundamental cycle of \(\overline{M}_{g,n}(E, \beta)\). The ancestor GW invariants are given by

\[
\langle \alpha_1 \psi_1^f, \ldots, \alpha_n \psi_n^f \rangle_{g,n,\beta}^E = \int_{[\overline{M}_{g,n}(E, \beta)]^\text{vir}} \prod_{k=1}^n ev_k^* (\alpha_k) \pi^* \psi_k^f.
\]
The ancestor GW correlation function is the formal $q$-series
\begin{equation}
\langle \alpha_1 \psi_1^{e_1}, \ldots, \alpha_n \psi_n^{e_n} \rangle^\mathcal{E}_{g,n}(q) = \sum_{d \geq 0} q^d \langle \alpha_1 \psi_1^{e_1}, \ldots, \alpha_n \psi_n^{e_n} \rangle^\mathcal{E}_{g,n,d}.
\end{equation}

By the virtual degree counting of $[\overline{\mathcal{M}}_{g,n}(\mathcal{E}, \beta)]^\text{vir}$, if the series
\[ \langle \alpha_1 \psi_1^{e_1}, \ldots, \alpha_n \psi_n^{e_n} \rangle^\mathcal{E}_{g,n}(q) \]

in (1.4) is nontrivial, then
\begin{equation}
\sum_{k=1}^n \left( \frac{\deg \alpha_k}{2} + \ell_k \right) = (3 - \dim C \mathcal{E})(g - 1) + n = 2g - 2 + n.
\end{equation}

On the LG-side, we consider the FJRW theory of the pair $(W, \langle J \rangle)$ as originally constructed in [FJR07, FJR13]. The main ingredients consist of a CohFT
\[ \mathcal{H}(W, \langle J \rangle), \langle \cdot, \cdot \rangle, 1, \Lambda^{(W, \langle J \rangle)} \]
and FJRW invariants (see Section 2.1 for details)
\[ \langle \alpha_1 \psi_1^{e_1}, \ldots, \alpha_n \psi_n^{e_n} \rangle^{(W, \langle J \rangle)}_{g,n}, \]
with $\alpha_i$ elements in the vector space $\mathcal{H}(W, \langle J \rangle)$. The space $\mathcal{H}(W, \langle J \rangle)$ contains a canonical degree-2 element, denoted by $\phi$ below. We assemble the FJRW invariants into an ancestor FJRW correlation function (as a formal series in $s$)
\begin{equation}
\langle \alpha_1 \psi_1^{e_1}, \ldots, \alpha_n \psi_n^{e_n} \rangle^{(W, \langle J \rangle)}_{g,n}(s) := \sum_{m=0}^{\infty} \frac{1}{m!} \langle \alpha_1 \psi_1^{e_1}, \ldots, \alpha_n \psi_n^{e_n}, s\phi, \ldots, s\phi \rangle^{(W, \langle J \rangle)}_{g,n+m}.
\end{equation}

### 1.1. LG/CY correspondence via modularity

One of the motivation in constructing the FJRW invariants [FJR07, FJR13] is to understand mathematically the so-called Landau-Ginzburg/Calabi-Yau correspondence proposed by physicists [VW89, GVW89, Mar90, Wit93]. The Landau-Ginzburg/Calabi-Yau Correspondence Conjecture [FJR13, CR11b, Rua12] predicts that for a CY type weight system the corresponding GW theory and the FJRW theory are related. In the past decade, a lot of effort has been made to formulate and solve this conjecture:

- An LG/CY correspondence between the vector spaces is solved in [CR11a].
- Genus-zero LG/CY correspondence for various pairs $(W, G)$ have been studied using Givental’s I-functions, see [CR10, LS14, CIR14, LPS16, Cla17, BaP18].
- For the quintic 3-fold, the correspondence has been pushed to genus one [GR19b].
- For higher genus, the only known examples in the work [KS11, MR11, MS16, SZ18, IMRS16] are all generically semisimple and therefore the correspondence at higher genus is a consequence of the genus-zero correspondence, based on Givental-Teleman’s classification of semisimple CohFTs [Giv01a, Tel12].

One of the main results of the present work is to solve this conjecture for the Fermat cubic pair $(W = x_1^3 + x_2^3 + x_3^3, \langle J \rangle)$ at all genus, using the properties of moduli spaces and quasi-modular forms. We remark that the GW CohFT and the FJRW CohFT for such a pair are not generically semisimple and therefore this case is beyond the scope of Givental-Teleman’s results.
1.1.1. Quasi-modular forms and Chazy equation. Specializing to the cases of one-dimensional CY type weight systems, it is known [BO00, OP06a] that the GW correlation functions for an elliptic curve are quasi-modular forms [KZ95]. The key of this work is to relate the generating series in (1.4) and (1.6) using transformations on quasi-modular forms.

Consider the Eisenstein series

\[
E_{2k}(\tau) := \frac{1}{2\zeta(2k)} \sum_{(c,d) \in \mathbb{Z}} \frac{1}{(c\tau + d)^{2k}}, \quad \tau \in \mathbb{H},
\]

where \(\zeta(2k)\) are the zeta-values. These are holomorphic functions on the upper-half plane \(\mathbb{H}\), of which \(E_{2k}, k \geq 2\), are modular under the group \(\Gamma := \text{SL}(2, \mathbb{Z})/\{\pm 1\}\); while \(E_2\) is quasi-modular [KZ95]. To be more precise, \(E_2\) is not modular, but its non-holomorphic modification \(\hat{E}_2(\tau, \bar{\tau})\) is modular where

\[
\hat{E}_2(\tau, \bar{\tau}) := E_2(\tau) - \frac{3}{\pi \text{Im}(\tau)}.
\]

The set of quasi-modular forms (we regard modular forms as special cases of quasi-modular forms) for \(\Gamma\) form a ring [KZ95].

\[
\tilde{M}_*(\Gamma) := \mathbb{C}[E_2(\tau), E_4(\tau), E_6(\tau)].
\]

The set of almost-holomorphic modular forms as introduced [KZ95] also gives rise to a ring that is isomorphic to \(\tilde{M}_*(\Gamma)\)

\[
\hat{M}_*(\Gamma) := \mathbb{C}[\hat{E}_2(\tau, \bar{\tau}), E_4(\tau), E_6(\tau)].
\]

Let \(q = \exp(2\pi \sqrt{-1}\tau)\). The GW invariants of elliptic curves are [OP06a] Fourier coefficients expanded around the infinity cusp \(\tau = \sqrt{-1}\infty\) of certain quasi-modular forms. For example\(^1\), let \(\omega \in H^2(\mathcal{E})\) be the Poincaré dual of the point class, then

\[
-24 \langle \langle \omega \rangle \rangle_{1,1}^{\mathcal{E}}(q) = E_2(q) = 1 - \frac{1}{24} \sum_{n=1}^{\infty} n \frac{q^n}{1 - q^n}.
\]

For any \(f \in \hat{M}_*(\Gamma)\), we define

\[
f'(\tau) := \frac{1}{2\pi \sqrt{-1}} \frac{df}{d\tau}.
\]

The Eisenstein series \(E_2, E_4,\) and \(E_6\) satisfy the so-called Ramanujan identities

\[
E_2' = \frac{E_2^2 - E_4}{12}, \quad E_4' = \frac{E_2 E_4 - E_6}{3}, \quad E_6' = \frac{E_2 E_6 - E_4^2}{2}.
\]

Eliminating \(E_4, E_6\), we see that \(E_2\) is a solution to the so-called Chazy equation,

\[
2f''' - 2f \cdot f'' + 3(f')^2 = 0.
\]

Our key observation is that the Chazy equation (1.12) appears in both GW/FJRW theory for one-dimensional CY weight systems, thanks to the Belorousski-Pandharipande relation discovered in [BP00].

\(^1\)We are sometimes sloppy about the argument for a quasi-modular form when no confusion should arise. For instance we shall occasionally write \(E_k(q)\) for \(E_k(\tau)\).
Proposition 1. Consider the LG space \( ([\mathbb{C}^3 / \langle \mathcal{I}_5 \rangle], W) \) given by (1.2) and (1.3). Then both the genus-one GW correlation function \(-24\langle \omega \rangle^{E}_{1,1}(q)\) and the genus-one FJRW correlation function \(-24\langle \phi \rangle^{(W, \langle \mathcal{I}_5 \rangle)}_{1,1}(s)\) are solutions to the Chazy equation (1.12).

Here for a function \( f(q) \) in \( q \), we use the convention \( f'(q) = q \partial_q f \); for a function \( f(s) \) in \( s \), \( f'(s) = \partial_s f \).

Furthermore, using more tautological relations discovered in [Ion02, FP05], we can show that both the GW and FJRW correlation functions in (1.4) and (1.6) are determined by the genus-one correlation functions in Proposition 1.

Proposition 2. Consider the LG space \( ([\mathbb{C}^3 / \langle \mathcal{I}_5 \rangle], W) \) given by (1.2) and (1.3). Let
\[
f = -24\langle \omega \rangle^{E}_{1,1} \quad \text{or} \quad -24\langle \phi \rangle^{(W, \langle \mathcal{I}_5 \rangle)}_{1,1}.
\]
then the GW correlation functions in (1.4) (or the FJRW correlation functions in (1.6)) are determined from \( f \) by tautological relations and are elements in the ring \( \mathbb{C}[f, f', f''] \).

1.1.2. LG/CY correspondence via Cayley transformation. By direct calculation, we can show \( \langle \omega \rangle^{E}_{1,1}(q) \) and \( \langle \phi \rangle^{(W, \langle \mathcal{I}_5 \rangle)}_{1,1}(s) \) are expansions of the same quasi-modular form \(-\left(1/24\right) \cdot E_2(\tau)\) at two different points on the upper-half plane. In particular, the GW functions are Fourier expansion around the cusp \( \tau = \sqrt{-1} \infty \). This viewpoint allows us to relate the GW functions in (1.4) and the FJRW functions in (1.6) by a variant of the Cayley transformation which we now briefly review following [SZ18].

For any point \( \tau_s \in \mathbb{H} \), there exists a Cayley transform that maps a point \( \tau \) on the upper half-plane \( \mathbb{H} \) to a point \( s(\tau) \) in the unit disk \( \mathbb{D} \), that is,
\[
s(\tau) = (\tau_s - \tau_s) \frac{\tau - \tau_s}{\tau - \tau_s}.
\]

This transform is biholomorphic and we denote its inverse by \( \tau(s) \). Following [Zag08] and [SZ18], there exists a Cayley transformation that maps a weight-\( k \) almost-holomorphic modular form
\[
\hat{f} \in \tilde{M}_s(\Gamma) = \mathbb{C}[\tilde{E}_2(\tau, \bar{\tau}), E_4(\tau), E_6(\tau)]
\]
to
\[
\left( \frac{\tau(s) - \tau_s}{\tau - \tau_s} \right)^k \cdot \hat{f} \left( \tau(s), \bar{\tau(s)} \right).
\]
The Taylor expansion of the image gives a natural way to expand the almost-holomorphic modular form \( \hat{f} \) near \( \tau = \tau_s \), where the local complex coordinate is \( s(\tau) \).

Using the fact that the two rings \( \tilde{M}_s(\Gamma) \) and \( \tilde{M}_s(\Gamma) \) are isomorphic differential ring, a holomorphic Cayley transformation \( \mathcal{C}^{\text{hol}}_{\tau_s} \) (see Section 4) can then be defined [SZ18]. This turns out to be the correct transformation that relates the GW correlation functions in (1.4) and the FJRW correlation functions in (1.6), both of which are holomorphic. It allows us to solve the LG/CY Correspondence Conjecture for the Fermat cubic pair.

Theorem 1. Consider the Fermat cubic polynomial \( W = x_1^3 + x_2^3 + x_3^3 \) and the LG space \( ([\mathbb{C}^3 / \mu_3], W) \). There exists a degree- and grading-preserving vector space isomorphism
\[
\Psi : \mathcal{H}^E = H^*(E) \longrightarrow \mathcal{H}^{(W, \mu_3)}
\]
and a holomorphic Cayley transformation $\mathcal{C}^\text{hol}_{\tau^*}$ with

$$\tau^* = -\frac{\sqrt{-1}}{\sqrt{3}} \exp\left(\frac{2\pi \sqrt{-1}}{3}\right) \in \mathbb{H},$$

such that $\mathcal{C}^\text{hol}_{\tau^*} \left(\langle\langle \alpha_1 \psi_1^{\ell_1}, \cdots, \alpha_n \psi_n^{\ell_n} \rangle\rangle_g \right) = \langle\langle \Psi(\alpha_1) \psi_1^{\ell_1}, \cdots, \Psi(\alpha_n) \psi_n^{\ell_n} \rangle\rangle_g^{(W, \mu_3)}(s)$.

The explicit construction of $\Psi$ and $\mathcal{C}^\text{hol}_{\tau^*}$ will be given in Section 4.

Theorem 1 can be generalized to the rest of the one-dimensional CY type weight systems in (1.2) straightforwardly: the only difference lies in the technical computations on the initial genus-one FJRW invariants. This approach of using modular forms was previously introduced in [SZ18] for elliptic orbifold curves.

It is worthwhile to mention that for one-dimensional CY type weight systems, our approach of the LG/CY correspondence is compatible with the I-function approach introduced in [CR10, MR11]. In fact, the automorphy factor in the Cayley transformation (1.13) provides the equivalent information as the symplectic transformation that appears in [CR10, Corollary 4.2.4].

1.2. Applications: higher-genus FJRW invariants and their structures. The higher-genus FJRW invariants are very difficult to compute in general. In our example, with the identification of the correlation functions with quasi-modular forms, various results from the GW-side can be transformed into the LG-side, by the virtue of the holomorphic Cayley transformation which respects the differential ring structure of quasi-modular forms. In particular, higher-genus FJRW invariants can be computed easily and nice structures of the FJRW correlation functions can be obtained for free.

Indeed, higher-genus FJRW invariants are determined from the results on descendent GW invariants of elliptic curves given by Bloch-Okounkov [BO00], whose generating series admit very concrete and beautiful formulae. The following gives a sample of the computations.

Corollary 1. For the $d = 3$ case, the following holds for the ancestor FJRW correlation functions

$$\langle\langle \phi_1^{2g-2} \rangle\rangle_g^{(W, \mu_3)} = \sum_{\ell, m, n \geq 0 \atop \ell + 2m + 3n = g} b_{m,n} \left(\mathcal{C}^\text{hol}_{\tau^*}(E_2)\right)^{\ell} \left(\mathcal{C}^\text{hol}_{\tau^*}(E_4)\right)^{m} \left(\mathcal{C}^\text{hol}_{\tau^*}(E_6)\right)^{n},$$

where $\mathcal{C}^\text{hol}_{\tau^*}(E_i), i = 1, 2, 3$ are holomorphic Cayley transformations of the Eisenstein series $E_2, E_4, E_6$ whose expansions can be computed explicitly, while $\{b_{m,n}\}_{m,n}$ are rational numbers that can be obtained recursively.

The holomorphic anomaly equations (HAE) discovered in [OP18] and the Virasoro constraints discovered in [OP06b] for the GW theory of elliptic curves also carry over to the corresponding FJRW theory. See Corollary 3 and Corollary 4 for the explicit statements.

Plan of the paper. In Section 2 we review the basic construction of CohFTs, and use tautological relations in particular the Belorousski-Pandharipande relation to prove Proposition 1 and Proposition 2. In Section 3 we calculate a genus-one FJRW invariant for the $d = 3$ case using cosection localization. In Section 4 we prove Theorem 1 using properties of quasi-modular forms. In Section 5 we review some results on GW invariants for the elliptic curve and discuss the ancestor/descendent correspondence. In Section 6 we give some applications of the quasi-modularity of the GW and FJRW theory for the $d = 3$ case, such
as the explicit computations of higher-genus FJRW invariants basing on the results on the
GW invariants of the elliptic curve, the derivation of holomorphic anomaly equations and
Virasoro constraints they satisfy.

Acknowledgement. Y. Shen would like to thank Qizheng Yin, Aaron Pixton, and Felix Janda
for inspiring discussions on tautological relations. J. Zhou thanks Baosen Wu and Zijun
Zhou for useful discussions.

J. Li is partially supported by National Natural Science Foundation of China no. 12071079.
Y. Shen is partially supported by Simons Collaboration Grant 587119. J. Zhou is supported
by a start-up grant at Tsinghua University, the Young overseas high-level talents introd-
uction plan of China, and the national key research and development program of China
(No. 2020YFA0713000). Part of J. Zhou’s work was done while he was a postdoc at the
Mathematical Institute of University of Cologne and was partially supported by German
Research Foundation Grant CRC/TRR 191.

2. Belorousski-Pandharipande relation and Chazy equation

We study the two Cohomological Field Theories (both GW theory and FJRW theory)
for the one-dimensional CY type weight systems using tautological relations and axioms
of CohFTs. The key is the identification between Belorousski-Pandharipande relation and
Chazy equation.

2.1. Cohomological field theories. Both the GW theory and FJRW theory of the LG space
([CN/G], W) satisfy axioms of Cohomological Field Theories (CohFT) in the sense of [KM94],
which we briefly recall now.

Let \( M_{g,n} \) be the Deligne-Mumford moduli stack of genus \( g \) stable (i.e., \( 2g - 2 + n > 0 \))
curves with \( n \) markings. A Cohomological Field Theory with a flat identity is a quadruple
\((\mathcal{H}, \eta, 1, \Lambda)\),

where the state space
\[ \mathcal{H} := \mathcal{H}^{\text{even}} \oplus \mathcal{H}^{\text{odd}} \]
is a \( \mathbb{Z}_2 \)-graded finite dimensional \( \mathbb{C} \)-vector space (called superspace in [KM94]), \( \eta \) is a
non-degenerate pairing on \( \mathcal{H} \), \( 1 \in \mathcal{H} \) is the flat identity, and
\[ \Lambda := \{ \Lambda_{g,n} \in \text{Hom}(\mathcal{H}^{\otimes n}, H^*(\mathcal{M}_{g,n}, \mathbb{C})) \} \]
is a set of multi-linear maps satisfying the CohFT axioms below:

(i) Let \( | \cdot | \) be the grading. The maps \( \Lambda_{g,n} \) satisfy
\[ \Lambda_{g,n}(\cdot, \cdot, \cdot) = (-1)^{|a_1||a_2|} \Lambda_{g,n}(\cdot, \cdot, \cdot). \]

(ii) The maps in \( \Lambda \) are compatible with the gluing and the forgetful morphisms
\begin{itemize}
  \item \( \mathcal{M}_{g_1,n_1+1} \times \mathcal{M}_{g_2,n_2+1} \to \mathcal{M}_{g,n} \) and \( \mathcal{M}_{g-1,n+2} \to \mathcal{M}_{g,n} \),
  \item \( \pi : \mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n} \) forgetting one of the markings.
\end{itemize}
For example, the compatibility with the forgetting morphism is
\[ \Lambda_{g,n+1}(a_1, \cdot, a_n, 1) = \pi^* \Lambda_{g,n}(a_1, \cdot, a_n). \]

(iii) The pairing \( \eta \) is compatible with \( \Lambda_{0,3} \):
\[ \int_{\mathcal{M}_{0,3}} \Lambda_{0,3}(a_1, a_2, 1) = \eta(a_1, a_2). \]
Let $\psi_k \in H^2(\overline{M}_{g,n})$ be the cotangent line class at the $k$-th marking. For each CohFT $(\mathcal{H}, \eta, 1, \Lambda)$, one defines the quantum invariants from $\Lambda$ by

$$\langle \alpha_1 \psi_1^{\ell_1}, \ldots, \alpha_n \psi_n^{\ell_n} \rangle_{g,n} := \int_{\overline{M}_{g,n}} \Lambda_{g,n}(\alpha_1, \ldots, \alpha_n) \prod_{k=1}^n \psi_k^{\ell_k}, \quad \alpha_k \in \mathcal{H}.$$  

Such invariants are called the ancestor GW invariants for the GW CohFT and FJRW invariants for the LG CohFT. Our focus is the relation between these two types of invariants arising from the same CY type LG space $([\mathbb{C}^N/G], W)$.

Fix a basis $\mathcal{B}$ for $\mathcal{H}$. It is convenient to choose the elements $\alpha_k$ from $\mathcal{B}$ and parametrize $\alpha_k$ by $s_k$. We introduce the genus-zero primary potential of the CohFT as a formal power series

$$F^0_0 := \sum_{n \geq 0} \sum_{\alpha_k \in \mathcal{B}} \frac{1}{n!} \langle \alpha_1, \ldots, \alpha_n \rangle_{0,n} \prod_{k=1}^n s_k.$$  

Here primary means all $\ell_k = 0$ in (2.3).

2.1.1. FJRW invariants. The CohFTs arising from GW theories have become a familiar topic since [KM94]. Here we only recall some basics on the LG CohFT constructed from the FJRW invariants defined in [FJR07, FJR13]. See also [CLL15, PV16, KL18, CKL18] for various CohFT constructions for LG models.

As $G$ acts on $\mathbb{C}^N$, for any $\gamma \in G$, the fixed-point set $\text{Fix}(\gamma)$ is an $N_\gamma$-dimensional subspace of $\mathbb{C}^N$. Let $W_\gamma$ be the restriction of $W$ on $\text{Fix}(\gamma)$. Following [FJR13], one considers the graded vector space (called the FJR state space)

$$\mathcal{H}_{(W,G)} = \bigoplus_{\gamma \in G} \mathcal{H}_\gamma,$$

where each $\mathcal{H}_\gamma$ is the space of $G$-invariants of the middle-dimensional relative cohomology in $\text{Fix}(\gamma)$. There is a natural pairing $\langle , \rangle$ and an isomorphism (see [FJR13, Section 5.1])

$$\left( \mathcal{H}_{(W,G)}, \langle , \rangle \right) \cong \left( \bigoplus_{\gamma \in G} \left( \text{Jac}(W_\gamma) \Omega_{\text{Fix}(\gamma)} \right)^G, \text{Res} \right).$$

Here $\text{Jac}(W_\gamma)$ is the Jacobi algebra of $W_\gamma$, $\Omega_{\text{Fix}(\gamma)}$ is the standard holomorphic volume form on $\text{Fix}(\gamma)$ and $\text{Res}$ is the residue pairing.

In [FJR07, FJR13], Fan-Jarvis-Ruan constructed the virtual fundamental cycle over the moduli space of $W$-spin structures, and a corresponding CohFT

$$\left( \mathcal{H}_{(W,G)}, \langle , \rangle, 1, \Lambda^{(W,G)} \right).$$

This CohFT defines the so-called FJRW invariants $\langle \alpha_1 \psi_1^{\ell_1}, \ldots, \alpha_n \psi_n^{\ell_n} \rangle_{(W,G)}$ through (2.3).

We now specialize to a pair $(W, G)$ given in (1.3) with $G = \langle J_\delta \rangle$. For a set of homogeneous elements $\alpha_k \in \mathcal{H}_\gamma, k = 1, 2, \ldots, n$, the dimension formula in [FJR13, Theorem 4.1.8] shows if $\langle \alpha_1 \psi_1^{\ell_1}, \ldots, \alpha_n \psi_n^{\ell_n} \rangle_{(W,\{J_\delta\})}$ is non-trivial, then

$$2g - 2 + n = \sum_{k=1}^n \frac{\deg \alpha_k}{2} + \sum_{k=1}^n \ell_k.$$
We define a linear map $\Psi$. The constraint (2.7) allows us to define the following ancestor FJRW correlation function (as a formal series in $s$)

$$
(2.8) \quad \langle \alpha_1 \psi_1^{l_1}, \ldots, \alpha_n \psi_n^{l_n} \rangle^{(W_s(t_b))}_{g,n} (s) := \sum_{m=0}^{\infty} \frac{1}{m!} \langle \alpha_1 \psi_1^{l_1}, \ldots, \alpha_n \psi_n^{l_n}, s_\phi, \ldots, s_\phi \rangle^{(W_s(t_b))}_{g,n+m} .
$$

In the following, we will use the subscript "d" to label the CY type weight systems in (1.2). Let $\Omega = dx_1 \wedge dx_2 \wedge dx_3$. For each polynomial $W_d$, when $d = 3$ (resp. 4; resp. 6), we consider the following element

$$
(2.11) \quad h(W_d) = x_1 x_2 x_3 / 27 \text{ (resp. } x_1^2 x_2^2 / 32; \text{ resp. } x_1^3 x_2 / 36) .
$$

According to (2.6), the FJRW state space is

$$
(2.10) \quad H_{(W_s, G_d)} = H_{I_s} \bigoplus H_{I_s}^{-1} \bigoplus H_{G_d} = C \{ 1, \phi, b_1, b_2 \} .
$$

Here the even part is spanned by $1 \in H_{I_s}$ and $\phi \in H_{I_s}^{-1}$; while the odd part is spanned by $b_1 = h(W_d) \Omega, \quad b_2 = \Omega \in (Jac(W_d) \Omega)^G \subseteq H_{G_d}$.

The degrees are

$$
(2.11) \quad \deg 1 = 0 , \quad \deg b_1 = \deg b_2 = 1 , \quad \deg \phi = 2 .
$$

2.1.2. Genus-zero comparison. We begin with a comparison between the genus-zero parts of the two theories. On the GW-side, recall the state space for the elliptic curve $E_d$ is $H^* (E_d, \mathbb{C})$.

Let $1 \in H^0$ be the identity of the cup product, and $\omega \in H^2$ be the Poincaré dual of the point class. We choose a symplectic basis $\{ e_1, e_2 \}$ of $H^1$ such that

$$
e_1 \cup e_2 = -e_2 \cup e_1 = \omega .
$$

We define a linear map $\Psi : H^* (E_d) \to H_{(W_s, (I_s))}$ by

$$
(2.12) \quad \Psi (1) = 1 , \quad \Psi (\omega) = \phi , \quad \Psi (e_i) = b_i , \quad i = 1, 2 .
$$

Let $(t_0, t_1, t_2, t)$ be the coordinates with respect to the basis $\{ 1, e_1, e_2, \omega \}$. Similarly we let $(u_0, u_1, u_2, u)$ be the coordinates with respect to the basis $\{ 1, b_1, b_2, \phi \}$.

The moduli stack $\tilde{M}_{g,n} (E_d, \beta)$ is empty when $g = 0$ and $\beta > 0$. Then according to (2.4), the genus-zero primary GW potential is

$$
F_{0}^{E_d} = \frac{1}{2} \cdot t_0^2 t + t_0 t_1 t_2 .
$$

A calculation on residue shows that

$$
(2.13) \quad \langle 1, 1, \phi \rangle_{0,3}^{W_d} = \langle 1, b_1, b_2 \rangle_{0,3}^{W_d} = 1 , \quad \langle 1, b_2, b_1 \rangle_{0,3}^{W_d} = -1 .
$$

Thus the genus-zero primary FJRW potential is

$$
F_{0}^{W_d} = \frac{1}{2} \cdot u_0^2 u + u_0 u_1 u_2 + \text{quantum corrections} .
$$

These quantum corrections vanish as shown below. This was firstly observed by Francis [Fra15, Section 4.2] using WDVV equations.

**Proposition 3.** The map $\Psi$ in (2.12) is a degree- and grading-preserving ring isomorphism, and

$$
(2.14) \quad F_{0}^{W_d} = \frac{1}{2} \cdot u_0^2 u + u_0 u_1 u_2 .
$$
Proof. It is easy to see $\Psi$ preserves the degree and grading. To show $\Psi$ is a ring isomorphism, it is enough to prove (2.14). The compatibility condition (2.2) implies the String Equation in FJRW theory. Combining the degree constraints (2.11) and (2.7), we find that the quantum corrections are encoded in $C_i(s)$, where $C_i(s)$ is the correlation function with $i$ copies of $b_1$-insertions and $(4 - i)$ copies of $b_2$-insertions. For example,

$C_0(s) = \langle\langle b_1, b_1, b_1, b_1 \rangle\rangle_{0,4}^{W_d}$, $C_3(s) = \langle\langle b_1, b_2, b_2, b_2 \rangle\rangle_{0,4}^{W_d}$.

The $\mathbb{Z}_2$-grading (2.1) shows $C_i(s) = 0$ because for $\alpha = b_1$ or $b_2$

$\langle\langle \alpha, \alpha, \cdots \rangle\rangle_{g,n}^{W_d} = (-1)^{|\alpha| - |\alpha|} \langle\langle \alpha, \alpha, \cdots \rangle\rangle_{g,n}^{W_d} = -\langle\langle \alpha, \alpha, \cdots \rangle\rangle_{g,n}^{W_d}$.

This proves the claim. □

2.2. Belorousski-Pandharipande’s relation and $g$-reduction. The tautological rings $RH(\overline{\mathcal{M}}_{g,n})$ of $\overline{\mathcal{M}}_{g,n}$ are defined (see [FP05] for example) as the smallest system of subrings of $H^*(\overline{\mathcal{M}}_{g,n})$ stable under push-forward and pull-back by the gluing and forgetful morphisms. Thus pulling back the tautological relations in $RH(\overline{\mathcal{M}}_{g,n})$ via the CohFT maps $\Lambda_{g,n}$ gives relations among quantum invariants. We use this technique to prove Proposition 1 and Proposition 2.

2.2.1. Belorousski-Pandharipande’s relation for a genus-one correlation function. The degree constraints (2.11) and (2.7) show that the non-vanishing genus-one primary FJRW invariants could only come from the coefficients in $\langle\langle \psi \rangle\rangle_{1,1}^{W_d}(s)$. We determine this series and the GW correlation function $\langle\langle \omega \rangle\rangle_{1,1}^{C_1}(q)$ up to some initial values, using the tautological relation found by Belorousski and Pandharipande [BP00, Theorem 1]. The relation is a nontrivial rational equivalence among codimension-2 descendent stratum classes in $\overline{\mathcal{M}}_{2,3}$ shown in Figure 1 below.

![Figure 1. Belorousski-Pandharipande relation. Taken from [BP00, Theorem 1, formula (4)].](image)

Each stratum in the relation is represented by the topological type of the stable curve corresponding to the generic moduli point in the stratum. The markings on the stratum are
unassigned. The geometric genera of the components are underlined. The cotangent line class $\psi$ always appears on the genus-2 component.

Proof of Proposition 1. On the FJRW-side, we integrate
$$\Lambda_{2,3}^{W_d}(\phi,\phi,\phi) \in H^4(\overline{M}_{2,3})$$
over the Belorousski-Pandharipande relation. We read off one term from each stratum.

Strata in the 1st row of Figure 1. Let us consider the 1st stratum in the 1st row. The integration over this stratum gives the term
$$-2 \sum_{a,a',\beta,\beta' \in \mathcal{H}_{W_d}} \langle\langle a \rangle\rangle_{2,1}^{W_d}(s) \eta^{a,a'} \langle\langle \alpha',\phi,\beta \rangle\rangle_{0,3}^{W_d}(s) \eta^{\beta,\beta'} \langle\langle \beta',\phi,\phi \rangle\rangle_{0,3}^{W_d}(s).$$
Here the notations $\eta^{a,a'}$ stands for the $(a,a')$ component of the inverse of the paring $\eta$, etc.
For any homogeneous element $a \in \mathcal{H}_{W_d}$, the degree constraint (2.7) implies that if $\langle\langle a \rangle\rangle_{2,1}^{W_d}(s)$ is nonzero, then we must have
$$2(2-1) + 1 = \deg a = 2.$$
This contradicts (2.11), where we have $\deg a = 0,1,2$. Thus we have that $\langle\langle a \rangle\rangle_{2,1}^{W_d}(s) = 0$ and hence the contribution from this stratum is 0. Similar arguments imply that the contribution from all the strata in the 1st row of Figure 1 vanish, since the contribution from each stratum must contain one of the following terms as a factor
$$\langle\langle a \rangle\rangle_{2,1}^{W_d}(s) = \langle\langle a \psi_1 \rangle\rangle_{2,1}^{W_d}(s) = \langle\langle \phi \psi_1, a \rangle\rangle_{2,2}^{W_d}(s) = \langle\langle \phi, a \psi_2 \rangle\rangle_{2,2}^{W_d}(s) = 0.$$

Other vanishing strata. Now we look at the 1st, 2nd, and 5th stratum in the 2nd row, the 3rd, 4th and 5th stratum in the 3rd row, and the 2nd, 3rd, 5th, 6th stratum in the last row. Each stratum has a genus-zero component with at least 4 markings (including the nodes). According to Proposition 3, one has for the primary invariants
$$\langle\langle \ldots \rangle\rangle_{0,n}^{W_d} = 0, \quad \forall n \geq 4.$$
Thus the integration of $\Lambda_{2,3}^{W_d}(\phi,\phi,\phi) \in H^4(\overline{M}_{2,3})$ over each of these strata vanishes.

For the 1st and 2nd stratum in the 3rd row, the genus-zero component only contains 3 markings, but at least 2 of the markings are labeled with the class $\phi$. Again by Proposition 3, we have
$$\langle\langle \phi, \phi, a \rangle\rangle_{0,3}^{W_d} = 0, \quad \forall a \in \mathcal{H}_{W_d}.$$
So the contribution from these two strata also vanish.

Finally, the integration on the 1st stratum in the 4th row also vanishes. This is a consequence of the $\mathbb{Z}_2$-grading. In fact, we apply the degree constraint (2.7) to the genus-one component and find that the non-vanishing contribution from this stratum, if exists, should be of the form
$$-\frac{1}{60} \sum_{a,a'} \langle\langle (\phi, \phi, \phi) \rangle\rangle_{1,4}^{W_d}(s) \eta^{a,a'} \langle\langle (1, a) \rangle\rangle_{0,3}^{W_d} \eta^{a,a'}.$$
The vanishing of this term is a direct consequence of the formula (2.13), where
\[ \eta^{b_1} = \eta^{b_2} = 1, \quad \eta^{b_2} = -1. \]

**Non-vanishing terms.** Now we see that all the possibly non-vanishing terms are from the 3rd and 4th stratum in the 2nd row, and the 4th stratum in the last row. Let us calculate them term by term. The 3rd stratum of the 2nd row gives a possibly non-vanishing term
\[
\frac{12}{5} \langle \langle \phi \rangle \rangle_{1,1}(s) \eta^{b_1,1} \langle \langle 1, \phi, 1 \rangle \rangle_{0,3}(s) \eta^{1,1} \langle \langle \phi, \phi, \phi \rangle \rangle_{1,3}(s) = \frac{12}{5} g \cdot g''.
\]
The 4th stratum of the 2nd row gives a possibly non-vanishing term
\[
-\frac{18}{5} \langle \langle \phi, \phi \rangle \rangle_{1,2}(s) \eta^{b_1,1} \langle \langle 1, \phi, 1 \rangle \rangle_{0,3}(s) \eta^{1,1} \langle \langle \phi, \phi, \phi \rangle \rangle_{1,2}(s) = -\frac{18}{5} g' \cdot g'.
\]
The 4th stratum of the last row gives a possibly non-vanishing term
\[
\frac{1}{5} \cdot \frac{1}{2} \cdot \langle \langle 1, \phi, 1 \rangle \rangle_{0,3}(s) \eta^{1,1} \langle \langle \phi, \phi, \phi \rangle \rangle_{1,4}(s) \eta^{b_1,1} = \frac{1}{5} \cdot \frac{8}{2}.
\]
Here the denominator 2 in the term above comes from the automorphism of the graph.

Putting all these together, we see the Belorousski-Pandharipande relation in Figure 1 allows us to verify by brute-force computation that the correlation function \( g := \langle \langle \phi \rangle \rangle_{1,1}(s) \) is a solution to
(2.15)
\[
\frac{12}{5} g \cdot g'' - \frac{18}{5} g' \cdot g' + \frac{1}{5} \cdot \frac{8}{2} = 0.
\]
Thus \(-24 \langle \langle \phi \rangle \rangle_{1,1}(s)\) is a solution of the Chazy equation (1.12).

Similarly, by integrating the GW cycle \( \Lambda_{\mathcal{E}_{1,1}}(\omega, \omega, \omega) \) over the Belorousski-Pandharipande relation in Figure 1, we see that \(-24 \langle \langle \phi \rangle \rangle_{1,1}(q)\) is a solution of the Chazy equation (1.12). This completes the proof of Proposition 1. \(\square\)

The identity (2.15) is independent of the specific form \( \mathcal{E}_{d} \), as should be the case since the GW invariants are independent of the choice of complex structures put on the elliptic curve.

**Remark 1.** For the elliptic orbifold curve \( \chi_{N} := \mathcal{E}^{(N)} / \mu_{N} \) for some particular elliptic curve \( \mathcal{E}^{(N)} \) that admits \( \mu_{N} \) as its automorphism group, the first stratum in the fourth line does not vanish. Let \( \mu \) be the rank of the Chen-Ruan cohomology \( H^{*}_{CR}(\chi_{N}) \) which satisfies
\[
1 - \frac{\mu}{12} = \frac{1}{N}.
\]
Define similarly \( g := \langle \langle \mathcal{P} \rangle \rangle_{1,1}(s) \) where \( \mathcal{P} \) is the point class on \( \chi_{N} \). The Belorousski-Pandharipande relation now gives
\[
\frac{12}{5} g \cdot g'' - \frac{18}{5} (g')^2 + \left( -\frac{\mu}{60} + \frac{1}{5} \right) \frac{8}{2} = 0,
\]
where \( \mu' = Q \partial_{Q} \) is now the derivative with respect to the parameter for the point class \( \mathcal{P} \). Then \( f = -24g \) satisfies
\[
2f \cdot f'' - 3(f')^2 - 2 \left( 1 - \frac{\mu}{12} \right) f''' = 0.
\]
Its solutions coincide with the ones to (2.15) via the relation \( Q = q^{N} \), see [SZ17] for more details.
2.2.3. \(g\)-reduction for higher-genus correlation functions. Now we prove Proposition 2 using the \(g\)-reduction technique introduced in [FSZ10]. We recall the following result.

**Lemma 1.** [Ion02, FP05] Let \(M(\psi, \kappa)\) be a monomial of \(\psi\)-classes and \(\kappa\)-classes \(\mathcal{M}_{g,n}\). Assume \(\deg M \geq g\) when \(g \geq 1\), and \(\deg M \geq 1\) when \(g = 0\), then \(M(\psi, \kappa)\) is equal to a linear combination of dual graphs on the boundary of \(\mathcal{M}_{g,n}\).

**Proof of Proposition 2.** Consider the GW or FJRW correlation function of the form

\[
\langle\langle \alpha_1 \psi_1^{\ell_1}, \cdots, \alpha_n \psi_n^{\ell_n} \rangle\rangle_{g,n}^\bullet = E_d \text{ or } W_d.
\]

Using that the cohomology classes have \(0 \leq \deg \alpha_k \leq 2\), and using (1.5) and (2.7), we deduce that the correlation function is trivial if

\[
\sum_{k=1}^n \ell_k < 2g - 2.
\]

Now we assume it is nontrivial and \(\sum_{k=1}^n \ell_k \geq 1\), then we must have

\[
\deg \left( \prod_{k=1}^n \psi_k^\ell_k \right) = \sum_{k=1}^n \ell_k \geq \begin{cases} 2g - 2 & g \geq 2, \\ 1 & g = 0, 1. \end{cases}
\]

Then \(\prod_{k=1}^n \psi_k^\ell_k\) is a monomial satisfying the condition in Lemma 1, thus we can apply this technique and use the Splitting Axiom in GW/FJRW theory to rewrite the function as a linear combination of products of other correlation functions, with smaller genera.

We then repeat the process for nontrivial correlation functions with smaller genera and eventually rewrite the correlation function as a linear combination of products of primary (all \(\ell_k = 0\)) correlation functions in genus-zero (which are just constants) and in genus-one, which must be \(f_d^{(n-1)} = \langle\langle \omega, \cdots, \omega \rangle\rangle_{1,n}^E_d\) or \(\langle\langle \phi, \cdots, \phi \rangle\rangle_{1,n}^W_d\). Thus we have

\[
\langle\langle \alpha_1 \psi_1^{\ell_1}, \cdots, \alpha_n \psi_n^{\ell_n} \rangle\rangle_{g,n}^\bullet \in C \left[ f_d, f'_d, f''_d, \cdots \right] = C \left[ f_d, f'_d, f''_d \right].
\]

The last equality follows from (2.15). \(\square\)

### 3. A genus-one FJRW invariant

Throughout this section, we consider the \(d = 3\) case, with \(W_3 = x_1^3 + x_2^3 + x_3^3\) and \(G = \mu_3\).

We focus on the following genus-one FJRW invariant (see (1.6)) with \(n = 3\)

\[
\Theta_{1,3} := \langle\langle \phi, \cdots, \phi \rangle\rangle_{1,3}^{(W_3, \mu_3)}.
\]

Combining the computations in [LLSZ20], we will prove

**Proposition 4.** [LLSZ20, Theorem 1.1] For the \((W_3, \mu_3)\) case, one has the following FJRW invariant

\[
\Theta_{1,3} = \langle\langle \phi, \phi, \phi \rangle\rangle_{1,3}^{(W_3, \mu_3)} = \frac{1}{108}.
\]

We first obtain a formula that express the Witten’s top Chern class for \(\Theta_{1,3}\) in terms of a Witten’s top Chern class of three spin curves in Lemma 2. Then in Proposition 5 and Corollary 2, we analyze the later virtual class explicitly by cosection localization. Finally, Proposition 4 will be deduced from these results and explicit computations in [LLSZ20].
#### 3.1. Witten’s top Chern class

We begin with a formula for a Witten’s top Chern class of the moduli of three-spin curves. The relevant moduli $\overline{M}_{g=1,23}(W_3, \mu_3)$ (defined in [CLL15]) is the moduli of families

$$\xi = \{ \Sigma \subset C, (\mathcal{L}_i, \rho_i)_{i=1}^3 \}$$

such that $\Sigma \subset C$ is a family of genus-one 3-pointed twisted nodal curves, each marking is a stacky point of automorphism group $\mu_3, \rho_i : \mathcal{L}_i^{\otimes 3} \cong \omega_\xi^{log}$ are isomorphisms together with isomorphisms $\mathcal{L}_i \cong \mathcal{L}_1$ for $i = 2$ and 3 understood, the monodromy of $\mathcal{L}_1$ along $\Sigma_i \subset \Sigma$ is $\frac{3-1}{3}$.

Because of the isomorphisms $\mathcal{L}_i \cong \mathcal{L}_1$, we have canonical isomorphism

$$\mathcal{W}_3 := \overline{M}_{1,23}^{1/3} \cong \overline{M}_{1,23} (W_3, \mu_3),$$

where recall that $\mathcal{W}_3$ parameterizes families of $\xi = \{ \Sigma \subset C, \mathcal{L}, \rho \}$ with objects $\Sigma, C, \mathcal{L}$ and $\rho$ as before.

Let

$$[\overline{M}_{1,23} (W_3, \mu_3)^p]^\text{vir} \in A_* \overline{M}_{1,23} (W_3, \mu_3)$$

be the FJR invariant of the pair $(W_3, \mu_3)$, which is defined in [CLL15] as the cosection localized virtual cycles of the moduli stack $\overline{M}_{1,23} (W_3, \mu_3)^p$, parameterizing

$$\xi = \{ (C, \Sigma, \mathcal{L}_1, \cdots, \varphi_1, \varphi_2, \varphi_3) : (C, \Sigma, \mathcal{L}_1, \cdots) \in \overline{M}_{1,23} (W_3, \mu_3); \varphi_i \in \Gamma (\mathcal{L}_i) \}.$$ 

As shown in [CLL15], it has a cosection localized virtual cycle, denoted by $[\overline{M}_{1,23} (W_3, \mu_3)^p]^\text{vir}$. We let

$$[\overline{M}_{1,23}^{1/3}^p]^\text{vir} \in A_* \overline{M}_{1,23}^{1/3}$$

be the similarly defined its cosection localized virtual cycle.

**Lemma 2.** We have identity

$$[\overline{M}_{1,23} (W_3, \mu_3)^p]^\text{vir} = ([\overline{M}_{1,23}^{1/3}^p]^\text{vir})^3 \in A^3 \mathcal{W}_3 \equiv A_0 \mathcal{W}_3.$$

**Proof.** First we have the following Cartesian product

$$\begin{array}{ccc}
\overline{M}_{1,23} (x^3, \mu_3)^p \times \overline{M}_{1,23} (x^3, \mu_3)^p & \xrightarrow{f} & \overline{M}_{1,23} (x^3 + y^3, (\mu_3)^2)^p \\
\downarrow & & \downarrow \\
\overline{M}_{1,23} (x^3, \mu_3) \times \overline{M}_{1,23} (x^3, \mu_3) & \xrightarrow{\ell} & \overline{M}_{1,23} (x^3 + y^3, (\mu_3)^2),
\end{array}$$

where the morphism $f$ is defined via sending $(C, \Sigma, \mathcal{L}_1, \mathcal{L}_2)$ to

$$((C, \Sigma, \mathcal{L}_1), (C, \Sigma, \mathcal{L}_2)).$$

Applying [CLL15, Thm 4.11], we get that

$$[\overline{M}_{1,23} (x^3 + y^3, (\mu_3)^2)^p]^\text{vir} = f^* ([\overline{M}_{1,23} (x^3, \mu_3)^p]^\text{vir} \times [\overline{M}_{1,23} (x^3, \mu_3)^p]^\text{vir}).$$

Now let

$$g : \overline{M}_{1,23} (x^3 + y^3, \mu_3) = \overline{M}_{1,23} (x^3, \mu_3) \longrightarrow \overline{M}_{1,23} (x^3 + y^3, (\mu_3)^2)$$

be the diagonal morphism, then

$$f \circ g : \overline{M}_{1,23} (x^3, \mu_3) \longrightarrow \overline{M}_{1,23} (x^3, \mu_3) \times \overline{M}_{1,23} (x^3, \mu_3)$$

Our convention is that for $C = [A^1/\mu_3]$ and an invertible sheaf of $O_C$-modules having monodromy $\xi \in [0, 1]$ at $[0]$, then locally the sheaf takes the form $O_{A^1} (a[0]) / \mu_3$. 


is the diagonal morphism. As $g$ is étale and proper, we conclude
\begin{equation}
[M_{1,2^3}(x^3 + y^3, \mu_3)^p]_{\text{vir}} = g^*[M_{1,2^3}(x^3 + y^3, (\mu_3)^2)^p]_{\text{vir}}.
\end{equation}

Combined with (3.4) and (3.5), we obtain
\begin{equation}
[M_{1,2^3}(x^3 + y^3, \mu_3)^p]_{\text{vir}} = (f \circ g)^*([M_{1,2^3}(x^3, \mu_3)^p]_{\text{vir}} \times [M_{1,2^3}(x^3, \mu_3)^p]_{\text{vir}}),
\end{equation}

which is \( ([M_{1,2^3}(x^3, \mu_3)^p]_{\text{vir}})^2 \). Here we have used that \( M_{1,2^3}(x^3, \mu_3) \) is smooth. Repeating the same argument, go from \( x^3 + y^3 \) to \( W_3 \), we prove the lemma.

3.1.1. Cosection localized virtual cycles. Let \( W \) be a smooth DM stack, with a complex of locally free sheaves of \( \mathcal{O}_W \)-modules
\begin{equation}
\mathcal{E}^\bullet := [\mathcal{O}_W(E_0) \overset{s}{\rightarrow} \mathcal{O}_W(E_1)],
\end{equation}
of rank \( a_0 \) and \( a_1 = a_0 + 1 \), respectively. Let \( \pi : E_0 \rightarrow W \) be the projection; the section \( s \) induces a section \( \tilde{s} \in \Gamma(\hat{E}_1) \) of the pullback bundle \( \hat{E}_1 := \pi^*E_1 \). We define
\begin{equation}
M := (\tilde{s} = 0) \subset E_0.
\end{equation}

**Assumption-I.** We assume \( D = (\ker s \neq 0) \subset W \) is a smooth Cartier divisor; \( \text{Im}(s|_D) \) is a rank \( a_0 - 1 \) subbundle of \( E_1|_D \).

Because \( D \) is a smooth Cartier divisor, we can find a vector bundle \( F \) on \( W \) fitting into
\begin{equation}
\mathcal{O}_W(E_0) \overset{\eta_1}{\rightarrow} \mathcal{O}_W(F) \overset{\eta_2}{\rightarrow} \mathcal{O}_W(E_1)
\end{equation}
so that \( \eta_1|_{W-D} = \tilde{s}|_{W-D} \) is an isomorphism, \( F \rightarrow E_1 \) is a subvector bundle, and \( s = \eta_2 \circ \eta_1 \).

We let \( A = H^1(\mathcal{E}^\bullet) \). By Assumption-I, it fits into the exact sequence
\begin{equation}
0 \rightarrow \mathcal{O}_W(E_0) \overset{\phi}{\rightarrow} \mathcal{O}_W(F) \rightarrow A \rightarrow 0.
\end{equation}

Further, there is a line bundle \( A \) on \( D \) so that \( A = \mathcal{O}_D(A) \). In the following, we will view \( c_1(A) \in A^1\mathcal{D} \). Then for the inclusion \( \iota : D \rightarrow W \), \( \iota_*c_1(A) \in A^2W \). Since \( A \) is a line bundle on \( D \), we have \( c_1(A) = [\mathcal{D}] \), thus

**Lemma 3.** We have identity \( c_1(E_1 - F) = c_1(E_1 - E_0) - [\mathcal{D}] \).

We let \( J \subset E_0|_D \) be the kernel of \( s|_D \); by our assumption it is a line bundle on \( D \). We relate \( A \) to \( J \).

**Lemma 4.** Let the situation be as stated, and assume Assumption-I, then \( A \cong J(D) \).

**Proof.** Let \( J = \mathcal{O}_D(J) \) and let \( \eta = \ker\{\mathcal{O}_D(F) \rightarrow A\} \). Then \( \eta \) fits into the exact sequences
\begin{equation}
0 \rightarrow \mathcal{O}_D(J) \rightarrow \mathcal{O}_D(E_0) \rightarrow \eta \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \eta \rightarrow \mathcal{O}_D(F) \rightarrow \mathcal{O}_D(A) \rightarrow 0.
\end{equation}

Let \( \xi \in \mathcal{O}_D(J) \) be any (local) section. Let \( \tilde{\xi} \in \mathcal{O}_W(E_0) \) be a lift of the image of \( \xi \) in \( \mathcal{O}_D(E_0) \). Then \( \phi(\tilde{\xi}) \in \mathcal{O}_W(F) \), where \( \phi \) is as in (3.9). Clearly, \( \phi(\tilde{\xi})|_D = 0 \). Let \( t \in \mathcal{O}_W(D) \) be the defining equation of \( D \). Then \( t^{-1}\phi(\tilde{\xi}) \in \mathcal{O}_W(F(-D)) \). We define \( \phi(\xi) \) be the image of \( t^{-1}\phi(\tilde{\xi}) \) in \( \mathcal{O}_D(A(-D)) \), under the composition
\begin{equation}
\mathcal{O}_W(F)(-D) \rightarrow \mathcal{O}_D(F(-D)) \rightarrow \mathcal{O}_D(A(-D)).
\end{equation}

It is direct to check that \( \phi \) is a well-defined homomorphism of sheaves \( \mathcal{O}_D(J) \rightarrow \mathcal{O}_D(A(-D)) \), and is an isomorphism. This proves the Lemma.
This way, $\mathcal{M}$ (cf. (3.7)) is a union of $\mathcal{W} \subset E_0$ (the 0-section) and the subbundle $J \subset E_0|_D \subset E_0$. As $\mathcal{M} \subset E_0$ is defined by the vanishing of $\tilde{s}$, it comes with a normal cone (3.10)
\[ C := \lim_{t \to 0} \Gamma_{t^{-1}s} \subset \bar{E}_1|_\mathcal{M}. \]

**Lemma 5.** With Assumption-I, the cone $C \subset \bar{E}_1|_\mathcal{M}$ is a union of two subvector bundles $\eta_2(F) \subset E_1$ and $\pi^*\eta_2(F)|_J \subset \bar{E}_1|_J$.

**Proof.** This is local, thus without loss of generality we can assume $a_0 = 1$. Since $D = (s = 0)$ is a smooth divisor in $\mathcal{W}$, near a point in $D$ we can give $\mathcal{W}$ an analytic neighborhood $U$ with chart $(u, x)$, where $u$ is a multi-variable, so that $D = (x = 0)$ and $s|_U : E_0|_U \to E_1|_U$ takes the form
\[ s|_U = (x, 0) : \mathcal{O}_U \to \mathcal{O}_U \oplus \mathcal{O}_U^{(a_1-1)} \cong \mathcal{O}_U(E_1). \]
We let $y$ be the fiber-direction coordinate of $E_0|_U$. Then $\pi^{-1}(U) \subset E_0$ has the chart $(u, x, y)$, with $s|_{\pi^{-1}(U)} = (xy, 0)$. Therefore, the cone $C \subset E_0$ over $\pi^{-1}(U)$ is the line bundle
\[ \mathcal{O}_{\pi^{-1}(U) \cap \mathcal{M}} \subset \mathcal{O}_{\pi^{-1}(U) \cap \mathcal{M}} \oplus \mathcal{O}_{\pi^{-1}(U) \cap \mathcal{M}} \cong \mathcal{O}_{\pi^{-1}(U) \cap \mathcal{M}}(\bar{E}_1). \]
This proves the Lemma. \[ \square \]

**Assumption-II.** We assume that there is a homomorphism (cosection)
\[ \sigma : \bar{E}_1|_\mathcal{M} \to \mathcal{O}_\mathcal{M} \]
so that $\sigma|_\mathcal{W} = 0$, and $\pi^*\eta_2(F)|_J$ lies in the kernel of $\sigma$.

Let
\[ [\mathcal{M}]^\text{vir}_\sigma := 0^1_\sigma[C] \in A^{a_1-a_0}\mathcal{W} \]
be the image of $[C]$ under the cosection localized Gysin map.

**Proposition 5.** Let the situation be as mentioned, and the cosection $\sigma$ is fiberwise homogeneous of degree $e$. Then
\[ [\mathcal{M}]^\text{vir}_\sigma = -c_1(E_0 - E_1) - (e + 1)[D] \in A^1\mathcal{W}, \quad \text{when } a_1 - a_0 = 1. \]

**Proof.** Following the discussion leading to [CL15, Lemma 6.4], we compactify $\mathcal{M}$ by compactifying $J$ by $\mathcal{P} := \mathbb{P}_D(J \oplus 1)$. Let $\mathcal{D}_\infty = \mathbb{P}_D(J \oplus 0) \subset \mathbb{P}_D(J \oplus 1)$. Then $\mathcal{P} = J \cup \mathcal{D}_\infty$, and $\mathcal{M} = \mathcal{P} \cup \mathcal{W}$. Let $\pi : \mathcal{P} \to \mathcal{D}$ be the tautological projection. Then $\pi^*F|_J \subset \bar{E}_1|_J$ extends to $\pi^*F \subset \pi^*E_1$, a subbundle. Because $\sigma$ is fiberwise homogeneous of degree $e$, we see that $\sigma|_J : \bar{E}_1|_J = \pi^*E_1|_J \to \mathcal{O}_J$ extends to a homomorphism
\[ \sigma : \pi^*E_1(-eD_\infty) \to \mathcal{O}_\mathcal{P}, \]
surjective along $\mathcal{D}_\infty = \mathcal{M} - \mathcal{M}$.

We let $\pi^*F(-eD_\infty) \subset \pi^*E_1(-eD_\infty)$ be the associated twisting of the subbundle $\pi^*F \subset \pi^*E_1$. Applying [CL15, Lemma 6.4], we conclude that
\[ 0^1_\sigma[C] = 0^1_{E_1}[F] + \pi_*\left(0^1_{\pi^*E_1(-eD_\infty)}[\pi^*F(-eD_\infty)]\right). \]

When $a_1 - a_0 = 1$,
\[ 0^1_{\pi^*E_1(-eD_\infty)}[\pi^*F(-eD_\infty)] = c_1(\pi^*(E_1/F)(-eD_\infty)) = \pi^*c_1(E_1/F) - e[D_\infty]. \]
Thus $\pi_*\left(0^1_{\pi^*E_1(-2D_\infty)}[\pi^*F(-eD_\infty)]\right) = -e[D]$. Combined with Lemma 3, we prove the lemma. \[ \square \]
3.2. Applying to FJRWh invariant. We let \( \mathcal{M} = \overline{\mathcal{M}}_{1,2^3}^{1/3,p} \). We claim that there is a complex of vector bundle as in (3.6) so that \( \mathcal{M} \) is defined as in (3.7), and there is a cosection \( \sigma \) as in Assumption-II satisfying the condition stated.

Indeed, let \( \overline{\mathcal{M}}_{1,2^3} \) be the moduli of 3-pointed genus one twisted curves with all markings are \( \mu_3 \) stacky. Then the forgetful morphism \( q : \overline{\mathcal{M}}_{1,2^3}^{1/3} \to \overline{\mathcal{M}}_{1,2^3} \) is finite and smooth. Further, let \( (\Sigma \subset \mathcal{C}, \mathcal{L}) \) be the universal family of \( \overline{\mathcal{M}}_{1,2^3}^{1/3} \), then \( (\Sigma \subset \mathcal{C}) \) is the pull back of the universal family of \( \mathcal{M}_{1,2^3} \). Then a standard method shows that we can find a complex \( \mathcal{E}^* = [s : \mathcal{O}_\mathcal{C}(E_0) \to \mathcal{O}_\mathcal{C}(E_1)] \) of locally free sheaves so that \( \mathcal{E}^* = R^* \pi_* \mathcal{L} \), in the derived category. Here \( \pi : \mathcal{C} \to \overline{\mathcal{M}}_{1,2^3}^{1/3} \) is the projection. Then a standard argument shows that this complex \( \mathcal{E}^* \) is the desired one, giving a canonical embedding of \( \mathcal{M} = \overline{\mathcal{M}}_{1,2^3}^{1/3,p} \) into the total space of \( E_0 \), as the vanishing locus of \( \tilde{s} \).

The choice of cosection \( \sigma \) is induced by \( \mathcal{O}_\mathcal{W}(E_1) \to H^1(\mathcal{E}^*) \), following that in [CLL15], and satisfies Assumption-II. Finally, following the construction of \( [\overline{\mathcal{M}}_{1,2^3}^{1/3,p}]_{\text{vir}} \), we see that

\[
[\mathcal{M}]_{\text{vir}} = [\overline{\mathcal{M}}_{1,2^3}^{1/3,p}]_{\text{vir}}.
\]

We skip the details here.

We next check that the Assumption-I holds in this case.

**Lemma 6.** Let \( D \subset \mathcal{W} (= \overline{\mathcal{M}}_{1,2^3}^{1/3}) \) be the locus where \( R^0 \pi_* \mathcal{L} \) is non-trivial, then it is a smooth divisor of \( \mathcal{W} \).

**Proof.** Let \( (\mathcal{C}, \Sigma, \mathcal{L}) \in \mathcal{W} \) be a closed point so that \( H^0(\mathcal{L}) \neq 0 \). Then a direct calculation shows that \( \mathcal{C} \) has a node \( q \in \mathcal{C} \) that separates \( \mathcal{C} \) into two irreducible components \( \mathcal{E} \) and \( \mathcal{R} \), so that \( q \subset \mathcal{E} \) is a 1-pointed (twisted) elliptic curve with \( h^0(\mathcal{L}|_\mathcal{E}) = 1 \), and \( q \cup \Sigma \subset \mathcal{R} \) is a 4-pointed (twisted) rational curve. The same argument shows that the converse is also true. Therefore, letting \( D \subset \overline{\mathcal{M}}_{1,2^3}^{1/3} \) be the closed locus (see Fig. 2 below) where \( R^0 \pi_* \mathcal{L} \) is non-trivial, \( R^0 \pi_* \mathcal{L} \) is a locally free sheaf of \( \mathcal{O}_D \)-modules. Equivalently, this says that, letting

\[
\pi_D : \mathcal{C}_D = \mathcal{C} \times_{\overline{\mathcal{M}}_{1,2^3}^{1/3}} D \longrightarrow D
\]

be the projection, then \( \pi_D, (\mathcal{L}|_{\mathcal{C}_D}) \) is a rank one locally free sheaf of \( \mathcal{O}_D \)-modules. Let \( t \) be a local section of this sheaf, then \( (t = 0) \subset \mathcal{C}_D \) becomes a family of rational curves, the family that contains all those \( q \cup \Sigma \subset \mathcal{R} \) mentioned. This shows that \( \mathcal{C}_D \to D \) is exactly the subfamily in \( \overline{\mathcal{M}}_{1,2^3}^{1/3} \) that can be decomposed into 1-pointed twisted elliptic curves \( q \subset \mathcal{E} \) with \( h^0(\mathcal{L}|_\mathcal{E}) = 1 \), and 4-pointed twisted rational curves \( q \cup \Sigma \subset \mathcal{R} \). This implies that \( D \) is a smooth divisor of \( \mathcal{W} = \overline{\mathcal{M}}_{1,2^3}^{1/3} \). \( \square \)

We illustrate the divisor \( D \) by a decorated graph in Figure 2 below. A generic point in \( D \) consists a nodal curve with a genus one component (in blue) and a genus zero component (in green). The monodromy along the node is \( \frac{1}{3} \) on the genus-one component and \( \frac{2}{3} \) on the genus-zero component. Here \( h^0 = 1 \) is the rank of \( R^0 \pi_* \mathcal{L} \) restricted on the genus-one component.
Finally, to apply Proposition 5, we need to show that the cosection is fiberwise homogeneous of degree \( e = 2 \). This follows from the definition of the cosection in [CLL15], and the degree \( e \) is \( 3 - 1 \), where 3 is the denominator of \( 1/3 \). Applying Proposition 5, we obtain

**Corollary 2.** The Witten’s top Chern class of the moduli of three-spin curves \( \overline{M}_{1,2^3}^{1/3} \) is

\[
[\overline{M}_{1,2^3}^{1/3},p]_{\text{vir}} = -c_1(R^*\pi_*\mathcal{L}) - 3[D].
\]

Applying Lemma 2, we get

\[
\Theta_{1,3} = \deg([\overline{M}_{1,2^3}(W_3, \mu_3)]_{\text{vir}}) = \deg([\overline{M}_{1,2^3}^{1/3}]_{\text{vir}}) \cdot 3.
\]

Thus the FJRW invariant \( \Theta_{1,3} \) in Proposition 4 can be calculated explicitly from the triple self-intersection of the cycle (3.12). Note that the first term in (3.12) can be calculated by Chiodo’s formula [Chi08]. The calculation is subtle and lengthy. The details are given in [LLSZ20]. An alternative approach in computing this invariant using the Mixed-Spin-P fields method developed in [CLLL19, CLLL16] is also presented in [LLSZ20].

4. LG/CY correspondence for the Fermat cubic

This section is devoted to proving Theorem 1. We shall show that the GW/FJRW correlation functions as Fourier/Taylor expansions of the same quasi-modular form around different points (the infinity cusp and an interior point on the upper-half plane) which are related by the so-called holomorphic Cayley transformation that we shall introduce.

4.1. Cayley transformation and elliptic expansions of quasi-modular forms. It is well known that the Eisenstein series \( E_2(\tau) \) is not modular, however its non-holomorphic modification

\[
\hat{E}_2(\tau, \bar{\tau}) := E_2(\tau) - \frac{3}{\pi \text{Im}(\tau)}
\]

is modular. The map (called modular completion) sending \( E_2 \) to \( \hat{E}_2 \), and \( E_4, E_6 \) to themselves is an isomorphism from \( \hat{M}_*(\Gamma) \) to the ring of almost-holomorphic modular forms

\[
\hat{M}_*(\Gamma) := \mathbb{C}[\hat{E}_2, E_4, E_6].
\]

\(^3\)This formula is a special case of a sequence of formulas for moduli of \( r \)-spin curves, conjectured by Janda [Jan].
More precisely, for any quasi-modular form \( f(\tau) \in \tilde{M}_s(\Gamma) \) of weight \( k \), we denote by \( \hat{f}(\tau, \bar{\tau}) \in \tilde{M}_s(\Gamma) \) its modular completion. The function \( \hat{f} \) can be regarded as a polynomial in the formal variable \( 1/\text{Im}\tau \)

\[
(4.3) \quad \hat{f} = f + \sum_{j=1}^{k} f_j \cdot \left( \frac{1}{\text{Im}\tau} \right)^j,
\]

with coefficients some holomorphic functions \( f_j, j = 1, 2, \cdots, k \), in \( \tau \). We call the inverse of the modular completion the holomorphic limit: it maps the almost-holomorphic modular form \( \hat{f} \) in (4.3) to its degree zero term \( f \) in the formal variable \( 1/\text{Im}\tau \).

For any point \( \tau_\ast \in \mathcal{H} \), we form the Cayley transform from \( \mathcal{H} \) to a disk \( D \) (of appropriate radius determined by \( \tau_\ast \) and \( c \neq 0 \))

\[
(4.4) \quad \tau \mapsto s(\tau) := c \cdot 2\pi \sqrt{-1} \left( \tau_\ast - \bar{\tau}_\ast \right) \frac{\tau - \tau_\ast}{\tau - \tau_\ast} \in D.
\]

It is biholomorphic and we denote its inverse by \( \tau(s) \).

Following [Zag08], [SZ18] defined a Cayley transformation \( \mathcal{C}_{\tau_\ast} \), based on the action (4.4) on the space of almost-holomorphic modular forms: it maps the almost-holomorphic modular form \( \hat{f} \in \tilde{M}_s(\Gamma) \) to

\[
(4.5) \quad \mathcal{C}_{\tau_\ast} (\hat{f})(s, \bar{s}) := (2\pi \sqrt{-1} c)^{-\frac{k}{2}} \cdot \left( \frac{\tau(s) - \tau_\ast}{\tau(s) - \bar{\tau}_\ast} \right)^k \cdot \hat{f}(\tau(s), \bar{\tau}(s)).
\]

This gives a natural way to expand an almost-holomorphic modular form near \( \tau = \tau_\ast \).

A similar notion of holomorphic limit can be defined near the interior point \( \tau_\ast \). Computationally, this amounts to taking the degree zero term in the \( s \)-expansion of (4.5) (now regarded as a real-analytic function in \( s, \bar{s} \)) using the structure (4.3). This procedure induces a transformation \( \mathcal{C}_{\tau_\ast}^{\text{hol}} \) on quasi-modular forms. The transformation \( \mathcal{C}_{\tau_\ast}^{\text{hol}} \) will be called the holomorphic Cayley transformation in the present work. This transformation can be shown to respect the differential ring isomorphism between the differential ring of quasi-modular forms and the differential ring of almost-holomorphic modular forms. We illustrate the construction by the commutative diagram Figure 3 below. Interested readers are referred to [SZ18] for details.

\[\begin{array}{c}
\tilde{M}_s(\Gamma) \\
\downarrow \text{constant term map} \\
\tilde{M}_s(\Gamma)
\end{array}\]

\[\begin{array}{c}
\mathcal{C}_{\tau_\ast} \\
\downarrow \text{modular completion} \\
\mathcal{C}_{\tau_\ast}^{\text{hol}} (\tilde{M}_s(\Gamma)) \\
\downarrow \text{holomorphic limit} \\
\mathcal{C}_{\tau_\ast}^{\text{hol}} (\tilde{M}_s(\Gamma))
\end{array}\]

Figure 3. Cayley transformation on quasi-modular and almost-holomorphic modular forms.
In this work we are mainly concerned with the expansions of the quasi-modular form $E_2$ around the infinity cusp $\sqrt{-1}\infty$ and the elliptic points

$$\tau_\ast = -\frac{1}{2\pi\sqrt{-1}} \cdot \frac{1}{d} \cdot \frac{\Gamma \left( \frac{1}{d} \right)}{\Gamma \left( 1 - \frac{1}{d} \right)} e^{-\frac{\pi\sqrt{-1}}{d}}, \quad d \in \{3, 4, 6\}.$$  

For the Fermat cubic polynomial case $d = 3$, in (4.4) we take

$$c = \frac{1}{2\pi\sqrt{-1}} \cdot \frac{\Gamma \left( \frac{1}{7} \right)}{\Gamma \left( 1 - \frac{1}{7} \right)^2} e^{-\frac{\pi\sqrt{-1}}{d}}.$$  

The choices in (4.6) and (4.7) then lead to the following rational expansion of $E_2$ around $\tau_\ast$

$$\mathcal{C}_{\tau_\ast}(E_2) = -\frac{s^2}{9} - \frac{s^5}{1215} - \frac{s^8}{459270} + \cdots$$

The other cases $d = 4, 6$ are similar. All of these computations are easy following those in [SZ18].

4.2. LG/CY correspondence. We consider the elliptic points (4.6) and the value (4.7) for $c$ in (4.4). Theorem 1 then follows from Theorem 2 below.

**Theorem 2.** Consider the LG space $([C^3/\langle J_0 \rangle], W)$ given by (1.2) and (1.3), with $d = 3$.

(i) The genus-one GW correlation function is

$$-24 \langle \langle \omega \rangle \rangle_{1,1}^{E_d}(q) = E_2(q).$$

(ii) The GW correlation functions $\langle \langle \cdots \rangle \rangle_{g,n}^{E_d}$ are quasi-modular forms in the ring $\mathbb{C}[E_2, E_1', E_2'']$.

(iii) The genus-one FJRW correlation function $\langle \langle \phi \rangle \rangle_{1,1}^{W_d}(s)$ is the Taylor expansion of $-\frac{1}{24} \cdot E_2$ around the elliptic point

$$\tau_\ast = -\sqrt{-1} \exp\left(\frac{2\pi\sqrt{-1}}{3}\right) \in \mathbb{H}.$$  

That is,

$$\langle \langle \phi \rangle \rangle_{1,1}^{W_d}(s) = \mathcal{C}_{\tau_\ast}^{\text{hol}} \left( \langle \langle \omega \rangle \rangle_{1,1}^{E_d}(q) \right).$$  

(iv) The FJRW correlation functions $\langle \langle \cdots \rangle \rangle_{g,n}^{W_d}$ are holomorphic Cayley transformations of quasi-modular forms in the ring $\mathbb{C}[\mathcal{C}_{\tau_\ast}^{\text{hol}}(E_2), \mathcal{C}_{\tau_\ast}^{\text{hol}}(E_1'), \mathcal{C}_{\tau_\ast}^{\text{hol}}(E_2'')]$,

such that

$$\mathcal{C}_{\tau_\ast}^{\text{hol}} \left( \langle \langle \alpha_1 \psi_1^{\ell_1}, \cdots, \alpha_n \psi_n^{\ell_n} \rangle \rangle_{g,n}^{E_d}(q) \right) = \langle \langle \Psi(\alpha_1) \psi_1^{\ell_1}, \cdots, \Psi(\alpha_n) \psi_n^{\ell_n} \rangle \rangle_{g,n}^{W_d}(s).$$

**Proof.** Part (i) is a well-known result in the literature, see e.g. [OP06a]. We give a new proof based on the Chazy equation. In order to get (4.9), it suffices to check\footnote{Note that only two initial conditions are needed to determine a solution from the space of formal power series in $q = e^{2\pi i \tau}$.}

$$\langle \omega \rangle_{1,1,0}^{E_d} = -\frac{1}{24}, \quad \langle \omega \rangle_{1,1,1}^{E_d} = 1.$$  

Both invariants can be obtained by analyzing the virtual fundamental classes explicitly.
Part (ii) is a consequence of Part (i), the Ramanujan identities (1.11), and Proposition 2.

For part (iii), the Selection rule [FJR13, Proposition 2.2.8] implies
\[ \Theta_{1,1} = \Theta_{1,2} = 0 \]
as the corresponding moduli spaces are empty. On the other hand, according to Proposition 4,
\[ \Theta_{1,3} = \frac{1}{108}. \]
Now we see that as a formal power series in s, the first three terms of \( \langle \langle \phi \rangle \rangle_W^{1,1}(s) \) matches with those obtained from \( \tau^\text{hol}_r(E_2) \) in (4.8). Since both \( \langle \langle \phi \rangle \rangle_W^{1,1}(s) \) and \( \tau^\text{hol}_r(E_2) \) satisfies the Chazy equation (1.12), we conclude that
\[ \langle \langle \phi \rangle \rangle_W^{1,1}(s) = -\frac{1}{24} \tau^\text{hol}_r(E_2). \]

For part (iv), we recall that by g-reduction, in either theory all non-trivial correlation functions are differential polynomials in the building block \( \langle \langle \omega \rangle \rangle_E \) or \( \langle \langle \phi \rangle \rangle_W^{1,1}(s) \). Since the holomorphic Cayley transformation respects the differential ring structure and the g-reduction is independent of the CohFT in consideration, part (iv) is a consequence of part (iii), the Ramanujan identities (1.11), and Proposition 2. □

**Remark 2.** Proposition 1 and Proposition 2 hold for all of the one-dimensional CY weight systems in (1.2) and (1.3). Provided the analogue of Proposition 4 for the \( d = 4 \) or 6 case is obtained, the same argument in the proof of Theorem 2 generalizes straightforwardly.

5. Ancestor GW invariants for elliptic curves

The tautological relations used in establishing Proposition 2 are not constructive and hence not so useful for actual calculation of higher-genus invariants. For this reason, we make use of the beautiful formulae for the descendent GW invariants of elliptic curves given by Bloch-Okounkov [BO00] reviewed below. For later use we also discuss the ancestor/descendent correspondence.

5.1. Higher-genus descendent GW invariants of elliptic curves. In [OP06a], Okounkov and Pandharipande proved a correspondence between the stationary GW invariants and Hurwitz covers, called Gromov-Witten/Hurwitz correspondence. To be more precise, let
\[ \langle \prod_{i=1}^N \omega \psi_i^{\ell_i} \rangle^{\epsilon}_g, \]
be the disconnected, stationary, descendent GW invariant of genus \( g \) and degree \( d \) (the number \( N \) of markings is self-explanatory in the notation). Here \( \psi_i \) is the descendant cotangent line class attached to the \( i \)th marking, and the symbol \( \bullet \) stands for disconnected counting. The invariant is called stationary as the insertions only involve the descendents of \( \omega \).

Following [OP06a], we define the \( N \)-point generating function
\[ F_N(z_1, \ldots, z_N, q) := \sum_{\ell_1, \ldots, \ell_N \geq -2} \langle \prod_{i=1}^N \omega \bar{\psi}_i^{\ell_i} \rangle^{\epsilon}_g \prod_{i=1}^N z_i^{\ell_i + 1}, \]
with the convention
\[ \langle \omega \bar{\psi}^{-2} \rangle_0^{\epsilon}(q) = 1. \]
The GW/Hurwitz correspondence [OP06a, Theorem 5] allows one to rewrite the $N$-point generating function $F_N(z_1, \cdots, z_N, q)$ by a beautiful character formula from [BO00]

$$\text{(5.2)} \quad F_N(z_1, z_2, \cdots, z_N, q) = (q)^{-1} \sum_{\text{all permutations of } z_1, \cdots, z_N} \frac{\det M_N(z_1, z_2, \cdots, z_N)}{\Theta(z_1 + z_2 + \cdots + z_N)},$$

where $M_N(z_1, z_2, \cdots, z_N)$ is the matrix whose $(i, j), j \neq N$ entries are zero for $i > j + 1$ and otherwise are given by

$$\frac{\Theta^{(j-i+1)}(z_1 + \cdots + z_{N-j})}{(j-i+1)!\Theta(z_1 + \cdots + z_{N-j})}, \quad j \neq N, \quad \frac{\Theta^{(N-i+1)}(0)}{(N-i+1)!}, \quad j = N.$$

Recall that $\Theta$ is defined to be the prime form

$$\text{(5.3)} \quad \Theta(z) = \left. \frac{\theta_{(\frac{1}{2}, \frac{1}{2})}(z, q)}{\theta_{(\frac{1}{2}, \frac{1}{2})}(z, q)} \right|_{z=0} = 2\pi \sqrt{-1} \frac{-1}{2\pi i^3} = 2\pi \sqrt{-1} e^{\frac{\pi}{2} E_2 z^2} \sigma(z).$$

with

(i) the Euler function

$$(q)^\infty := \prod_{n=1}^{\infty} (1 - q^n)$$

is related to the Dedekind eta function by $\eta = q^{\frac{1}{2}}(q)^\infty$;

(ii) the Jacobi $\theta$-function

$$\theta_{(\frac{1}{2}, \frac{1}{2})}(z, q) := \sum_{n \in \mathbb{Z}} q^{n^2} e^{n \frac{1}{2} z}$$

has characteristic $(\frac{1}{2}, \frac{1}{2})$;

(iii) the Weierstrass $\sigma$-function $\sigma(z)$ satisfies the following well-known formula\(^5\) (see [Sil09]),

$$\text{(5.4)} \quad \sigma(z) = \frac{z}{2\pi \sqrt{-1}} \exp \left( \sum_{k=2}^{\infty} \frac{B_{2k}}{2k(2k)!} z^{2k} E_{2k} \right),$$

where $B_{2k}, k \geq 1$ are Bernoulli numbers determined from

$$\frac{x}{e^x - 1} = 1 + \frac{x}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} x^{2k}.$$

Note that we often omit the subscript $g$ in the correlation function

$$\langle \prod_{i=1}^{N} \omega_{\bar{\psi}_i}^{(g)} \rangle^g$$

which can be read off from the degree of the insertion according to the dimension axiom, we shall also omit the argument $q$ in the functions for ease of notation.

\(^5\)Note that the $z$-variable here differs from the usual one by a $2\pi \sqrt{-1}$ factor.
The formula (5.2) provides an effective algorithm in computing the stationary descendent GW invariants. For example, as already computed in [BO00], one has

\begin{align}
F_1(z_1) &= \frac{1}{(q)_\infty \Theta(z_1)}, \\
F_2(z_1, z_2) &= \frac{1}{(q)_\infty \Theta(z_1 + z_2)} \left( \partial_{z_1} \ln \Theta(z_1) + \partial_{z_2} \log \Theta(z_2) \right), \\
\cdots
\end{align}

Remark 3. Let $\langle \langle \omega \rangle \rangle \circ E$ be the generating series of stable maps with connected domains with no descendent nor ancestor classes. Then one has the well known formula

\begin{equation}
\langle \langle \omega \rangle \rangle \circ E = -\frac{1}{24} E_2. \tag{5.6}
\end{equation}

It is easy to see that

\begin{equation}
\langle \langle \omega \rangle \rangle^\bullet = \langle \langle \omega \rangle \rangle \circ E \cdot \exp(G(q)), \quad G = \sum_{d \geq 1} \langle \langle \rangle \rangle_{g=1,d} q^d. \tag{5.7}
\end{equation}

One can show in this case by enumerating stable maps with connected domains that

\begin{equation}
q \frac{d}{dq} G = \sum_{d \geq 1} \langle \langle \rangle \rangle_{g=1,d} q^d = -q \frac{d}{dq} \log(q)_\infty. \tag{5.8}
\end{equation}

Solving this equation and using the initial terms of $G$ which can be easily computed, one obtains

\begin{equation}
G = -\log(q)_\infty. \tag{5.9}
\end{equation}

This then gives

\begin{equation}
\langle \langle \omega \rangle \rangle^\bullet = (q)_\infty^{-1} \cdot \langle \langle \omega \rangle \rangle \circ E = (q)_\infty^{-1} \cdot -\frac{1}{24} E_2. \tag{5.10}
\end{equation}

More generally, for the one-point GW correlation function, the same reasoning implies that

$$
\langle \langle \omega \varphi^k \rangle \rangle^\bullet = (q)_\infty^{-1} \cdot \langle \langle \omega \varphi^k \rangle \rangle \circ E.
$$

The result (5.9), indicates that one can add an extra contribution from the degree zero part to $G$, whose corresponding moduli is an Artin stack. This contribution can be defined to be $\log q^{-\frac{1}{2}}$. In this way, after applying the divisor equation, it yields the contribution $-1/24$ for the degree zero part in $\langle \langle \omega \rangle \rangle \circ E$. This definition of the extra contribution for the Artin stack changes $(q)_\infty$ to $\eta$. What one gains from the inclusion of this is the quasi-modularity of the GW generating functions. The discrepancy will be further discussed from the viewpoint of ancestor/descendent correspondence below.

It is shown in [BO00] by manipulating the series expansions that the descendent GW correlation functions are essentially (modulo the issue discussed in Remark 3) quasi-modular forms. By induction, the weight of $(q)_\infty \cdot \langle \langle \prod_{i=1}^N \omega \varphi_i^k \rangle \rangle^\bullet$ is $\sum (k_i + 2)$. This can also be seen easily by using (5.3), (5.4).
5.2. **Ancestor/descendent correspondence.** Since explicit formulae in [BO00] are available only for descendent GW invariants while we are mainly concerned with ancestor GW
invariants, we shall first exhibit the relation between these two types of GW invariants. The
relation between the descendent GW invariants and the ancestor GW invariants are described for general targets in [KM98, Theorem 1.1]. This is the so-called ancestor/descendent correspondence. This correspondence is written down elegantly using a quantization formula of quadratic Hamiltonians in [Giv01b, Theorem 5.1].

We summarize some basics of quantization of quadratic Hamiltonians from [Giv01b]. Let 
\[ H \] be a vector space of finite rank, equipped with a non-degenerating pairing \( \langle -,- \rangle \). Let 
\( H(\langle z \rangle) \) be the loop space of the vector space \( H \), equipped with a symplectic form \( \Omega \)
\( \Omega(f(z),g(z)) := \text{Res}_{z=0} \langle f(-z), g(z) \rangle \).

Let \( t_k \) be the collection of variables \( t_k = \{ t_{k,i}^a \} \) where \( a \) runs over a basis of \( H \), and \( t \) be the
\begin{align*}
\{ t_0, t_1, \cdots \}.
\end{align*}
We organize the collection \( t_k \) into a formal series \( t_k \)
\begin{align*}
t_k(z) = \sum_i t_{k,i}^a \cdot a_i \cdot z^k.
\end{align*}
Similar notations are used for \( s_k, s \) below. Introduce the dilaton shift
\begin{equation}
(5.11) \quad q(z) = t(z) - z \cdot 1.
\end{equation}

We consider an upper-triangular symplectic operator on \( H(\langle z \rangle) \), defined by
\begin{align*}
S(z^{-1}) := 1 + \sum_{i=1}^{\infty} z^{-i} S_i, \quad S_i \in \text{End}(H).
\end{align*}

Given an element \( G(q) \) in certain Fock space, the quantization operator \( \hat{S} \) of a symplectic
operators \( S \) gives another Fock space element
\begin{equation}
(5.12) \quad \hat{S}^{-1}G(q) = e^{W(q,q)/2h} G([S q]_+),
\end{equation}
where \( [S q]_+ \) is the power series truncation of the function \( S(z^{-1})q(z) \), and the quadratic
form \( W = \sum (W_{k,\ell} q_k q_\ell) \) is defined by
\begin{align*}
\sum_{k,\ell \geq 0} W_{k,\ell} \frac{w^k z^\ell}{w^{-1} z^{-1}} := \frac{S^*(w^{-1}) S(z^{-1}) - \text{Id}}{w^{-1} z^{-1}}.
\end{align*}
Here \( \text{Id} \) is the identity operator on \( H(\langle z \rangle) \) and \( S^* \) is the adjoint operator of \( S \).

Following Givental [Giv01b, Section 5], for the descendent theory we define a particular
symplectic operator \( S_t \) by
\begin{equation}
(5.13) \quad (a, S_t b) := \langle a, \frac{b}{z-\psi} \rangle =: (a, b) + \sum_{k=0}^{\infty} \langle \langle a, b \psi^k \rangle \rangle_0 \psi^{k-1} z^{1-k}.
\end{equation}

Now we specialize to the elliptic curve case and write down the quantization formula for
the ancestor/descendent correspondence explicitly. Henceforward, we use the following convention.

- Recall \( \{ 1, b_1, b_2, \phi \} \) is a basis of the FJRW state space \( \mathcal{H}_{(W_{ij}, G_{j})} \) given in (2.10). We parametrize the ancestor classes \( 1 \psi^t, b_1 \psi^t, b_2 \psi^t, \phi \psi^t \) by
\begin{equation}
(5.14) \quad s_0^t, s_1^t, s_2^t, s_3^t.
\end{equation}
Recall \( \{1, e_1, e_2, \omega\} \) is a basis of the cohomology space \( H^*(\mathcal{E}) \). We parametrize the ancestor classes \( 1\psi^\ell, e_1\psi^\ell, e_2\psi^\ell, \omega\psi^\ell \) and descendent classes \( 1\bar{\psi}^\ell, e_1\bar{\psi}^\ell, e_2\bar{\psi}^\ell, \omega\bar{\psi}^\ell \) by

\[
\tilde{t}^0, t_1, t_2, t_3 ; \quad \tilde{t}^0, t_1, t_2, t_3
\]

respectively.

The total descendent potential of the GW theory of \( \mathcal{E} \) is defined by

\[
\mathcal{D}^\mathcal{E}(\tilde{t}) := \exp \left( \sum_{g \geq 0} h^{g-1} \mathcal{F}^\mathcal{E}_g(\tilde{t}) \right) := \exp \left( \sum_{g \geq 0} h^{g-1} \sum_{n \geq 0} \langle \tilde{t}, \cdots, \tilde{t} \rangle^\mathcal{E}_{g,n} \right).
\]

The total ancestor potential of the GW theory of \( \mathcal{E} \) is defined by

\[
\mathcal{A}^\mathcal{E}(t) := \exp \left( \sum_{g \geq 0} h^{g-1} \mathcal{F}^\mathcal{E}_g(t) \right) := \exp \left( \sum_{g \geq 0} h^{g-1} \sum_{n \geq 0, \sum 2g - 2n > 0} \langle t, \cdots, t \rangle^\mathcal{E}_{g,n} \right).
\]

The total ancestor FJRW potential is defined similarly.

The quantity \( \mathcal{F}^\mathcal{E}_1(t) \) is the genus-one primary potential of the GW theory of \( \mathcal{E} \) appearing in \( \mathcal{A}^\mathcal{E} \), with \( q = e^t \) the parameter keeping track of the degree. According to [Giv01b, Theorem 5.1], the Ancestor/descendent correspondence of the elliptic curve is given by

\[
\mathcal{D}^\mathcal{E} = e^{\mathcal{F}^\mathcal{E}_1(t)} \mathcal{S}_t^{-1} \mathcal{A}^\mathcal{E},
\]

under the identification \( \tilde{t}^i = t^i \).

According to (5.9), the genus-one potential is

\[
\mathcal{F}^\mathcal{E}_1(t) = G(q) = \sum_{d \geq 1} \langle \rangle_1^\mathcal{E} q^d = -\log(q)_\infty, \quad q = e^t.
\]

Thus we obtain

\[
\mathcal{S}_t^{-1} \mathcal{A}^\mathcal{E} = e^{-\mathcal{F}^\mathcal{E}_1(t)} \mathcal{D}^\mathcal{E} = (q)_\infty \cdot \mathcal{D}^\mathcal{E} = (q)_\infty \cdot \sum_{g,n \in \mathbb{Z}} h^{g-1} \langle \tilde{t}, \cdots, \tilde{t} \rangle_{g,n}^\mathcal{E}.
\]

A direct calculation of (5.13) shows the restriction of \( \mathcal{S}_t \) on the odd cohomology is the identity operator, and the restriction to even cohomology is given by

\[
\mathcal{S}_t \begin{pmatrix} 1 \\ \omega \end{pmatrix} = \begin{pmatrix} 1 & 1/t \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \omega \end{pmatrix}.
\]

Now we write down an explicit formula for the quantization operator (5.12). The symplectic operator \( \mathcal{S}_t \) is given in terms of infinitesimal symplectic operator \( h(t)/z \),

\[
\mathcal{S}_t = \exp \left( \frac{h(t)}{z} \right),
\]

where \( h(t) \in \text{End}(H) \) such that \( h(t)(1) = t\omega, h(t)(\omega) = 0, \) and \( h(t)(e_j) = 0 \) otherwise. In terms of the Darboux coordinates \( \tilde{q}^i_k, \tilde{p}^i_k \), the corresponding quadratic Hamiltonian has the form (see [Lee09, Section 3] for example)

\[
P \left( \frac{h(t)}{z} \right) = -t \frac{(\tilde{q}^0)^2}{2} - t \sum_{k \geq 0} \tilde{q}^0_k \tilde{p}^0_k.
\]
Applying the quantization formula, we get

\[ \hat{S}_t = \exp \left( P \left( \frac{h(t)}{z} \right) \right) = \exp \left( -t \frac{q_0^2}{2} - t \sum_{k \geq 0} \frac{q_k}{\partial q_k} \right). \]

As a consequence, we observe that this operator has no influence on the parameter \( q_k^3 \) for the descendent \( \omega q^k \). Thus we obtain

**Proposition 6.** The relation between the stationary descendent invariants and the corresponding ancestor invariants is given by

\[ (q)_\infty \cdot \langle \prod_{i=1}^N \omega q^{\ell_i} \rangle_g^\bullet = \langle \prod_{i=1}^N \omega q^{\ell_i} \rangle_g^\bullet. \]

Quasi-modularity for the correlation functions in the disconnected theory is equivalent to those for the connected theory, as one can see by examining the generating series. Hence our Theorem 2(ii) is consistent with the results in [BO00, OP06a] about the quasi-modularity via the above proposition.

6. **Higher-genus FJRW invariants for the Fermat cubic**

In this section, we give several applications of Theorem 1. With the help of the Bloch-Okounkov formula [BO00], Cayley transformation allows us to compute the FJRW invariants of the Fermat elliptic polynomials at all genera. It also transforms various structures for the GW theory of elliptic curves, such as the holomorphic anomaly equations [OP06a, OP18] and Virasoro constraints [OP06b], to those in the corresponding FJRW theory.

**6.1. Higher-genus ancestor FJRW invariants for the cubic.** Consider the Laurent expansion of the \( N \)-point generating function \( F_N(z_1, z_2, \ldots, z_N, q) \). The Laurent expansion of \( \partial^m \ln \Theta \) is clear from (5.4), while that of \( 1/\Theta \) or \( 1/\sigma \) can be obtained by applying the Faà de Bruno formula to the exponential term in \( 1/\sigma \) which in the current case is determined by the Bell polynomials in \( -B_{2k}E_{2k}/2k, k \geq 2 \). However, this only gives the Laurent coefficients in terms of the generators \( E_{2k}, k \geq 2 \) for the ring of modular forms. The expansions obtained are not particularly useful for our later purpose which prefers a finite set of generators only.

We proceed as follows. First the Taylor expansion of the Weierstrass \( \sigma \)-function is given by the classical result [Wer94]

\[ \sigma = \sum_{m,n \geq 0} \frac{a_{m,n}}{(4m + 6n + 1)!} \left( \frac{2\pi^4}{3} E_4 \right)^m \left( \frac{16\pi^6}{27} E_6 \right)^n \left( \frac{z}{2\pi \sqrt{-1}} \right)^{4m + 6n + 1}, \]

where the coefficients \( a_{m,n} \) are complex numbers determined from the Weierstrass recursion

\[ a_{m,n} = 3(m + 1)a_{m+1,n-1} + \frac{16}{3}(n + 1)a_{m-2,n+1} \]

\[ - \frac{1}{6} (4m + 6n - 1)(4m + 6n - 2)a_{m-1,n}, \]

with the initial values \( a_{0,0} = 1 \) and \( a_{m,n} = 0 \) if either of \( m, n \) is strictly negative. The Laurent expansion of \( 1/\sigma \) is then obtained from the above. It takes the form

\[ \frac{1}{\sigma} = \sum_{m,n \geq 0} b_{m,n} \left( \frac{2\pi^4}{3} E_4 \right)^m \left( \frac{16\pi^6}{27} E_6 \right)^n \left( \frac{z}{2\pi \sqrt{-1}} \right)^{4m + 6n - 1}, \]
for some $b_{mn}$ that can also be obtained recursively. The formula in (6.1) also gives rise to the Laurent expansion of $\partial \ln \sigma$ and hence of $\partial \ln \Theta$ in terms of the generators $E_2, E_4, E_6$. Together with that of $\partial \ln \Theta$ it can be used to compute the Laurent expansion of $F_N(z_1, z_2, \cdots, z_N, q)$.

Consider the $N = 1$ case first. According to (5.5) the Laurent expansion of $F_1$ is given by

$$F_1(z, q) = \frac{1}{2\pi \sqrt{-1}} \cdot (q)_{\infty} e^{-\frac{1}{2} E_2 z^2} z^{-1}.$$  

$$= \frac{1}{z \cdot (q)_{\infty}} \sum_{\ell,m,n \geq 0} \frac{b_{m,n}}{\ell!} \left( -\frac{E_2}{24} \right)^\ell \left( \frac{E_4}{24} \right)^m \left( -\frac{E_6}{108} \right)^n z^{2\ell + 4m + 6n}.$$  

We therefore arrive at the following relation for the descendent GW correlation functions

$$\langle \langle \omega \psi^k \rangle \rangle \text{•} = \sum_{\ell,m,n \geq 0} \frac{b_{m,n}}{\ell!} \left( -\frac{E_2}{24} \right)^\ell \left( \frac{E_4}{24} \right)^m \left( -\frac{E_6}{108} \right)^n, \ k \geq -2. \quad (6.3)$$

As explained in Proposition 6, this is the corresponding ancestor GW correlation function and is indeed a quasi-modular form of weight $k + 2$. The first few Laurent coefficients are

$$1, -\frac{1}{24} E_2, -\frac{1}{2632} \left( \frac{1}{5} E_4 + \frac{1}{2} E_2^2 \right), \cdots \quad (6.4)$$

The other cases are similar. For example, for the $N = 2$ case from (5.5) we write

$$\langle \langle \omega \psi^k \rangle \rangle \text{•} F_2(z_1, z_2) = \frac{z_1 + z_2}{\Theta(z_1 + z_2)} \cdot \frac{\partial z_1 \ln \Theta(z_1) + \partial z_2 \log \Theta(z_2)}{z_1 + z_2}.$$  

The first term on the right hand side is expanded as in the $N = 1$ case, while the second term using (5.3) and (5.4).

Recall that the derivative on the level of generating series corresponds to the divisor equation in GW theory, and that taking derivatives commute with Cayley transformations as shown in [SZ18]. The generators of the differential ring of quasi-modular forms are $E_2, E_4, E_6$. To deal with the differential structure, it is in fact more convenient to use the generators $E_2, E_2', E_2''$ for the ring of quasi-modular forms as opposed to $E_2, E_4, E_6$. By Theorem 2, the ancestor GW correlation functions satisfy

$$\langle \langle \prod_{i=1}^N \omega \psi_i^{k_i} \rangle \rangle \text{•} = \mathcal{C}[E_2, E_2', E_2''] \in \mathbb{C}[E_2, E_2', E_2''].$$

Theorem 1 applies to the disconnected invariants (by examining the relation between the generating series) and we have

$$\langle \langle \phi \psi_1^{k_1}, \cdots, \phi \psi_N^{k_N} \rangle \rangle_{\text{•}}^{\text{W}} = \mathcal{C}_{\tau_e} \left( \langle \langle \omega \psi_1^{k_1}, \cdots, \omega \psi_N^{k_N} \rangle \rangle_{\text{•}}^{\text{E}} \right).$$

Now we can apply Cayley the transformation directly to the disconnected, ancestor GW correlation functions and obtain the disconnected, ancestor FJRW correlation functions. As computed in (4.8), for the $d = 3$ case we have

$$\mathcal{C}_{\tau_e}(E_2) = -\frac{s^2}{9} - \frac{s^3}{1215} - \frac{s^8}{459270} + \cdots \quad (6.7)$$
Since \( \varphi_{\text{hol}} \) respects the product and the differential structure \([SZ18]\), the differential equations (1.11) imply

\[
\begin{align*}
\varphi_{\text{hol}}(E_4) &= \varphi_{\text{hol}}(E_2^2 - 12E_2') = \frac{8s^3}{3} + \frac{5s^4}{81} + \frac{2s^7}{5103} + \cdots \\
\varphi_{\text{hol}}(E_6) &= \varphi_{\text{hol}}(E_2E_4 - 3E_4') = -8 - \frac{28s^3}{27} - \frac{7s^6}{405} + \cdots
\end{align*}
\]

From (6.3), Proposition 6, Theorem 1 and the degree formula (1.5), we immediately obtain

\[
\langle \langle \phi \psi^{2s-2} \rangle \rangle_{g,1} = \sum_{\ell,m,n \geq 0} b_{m,n} \left( -\frac{\varphi_{\text{hol}}(E_2)}{24} \right)^{\ell} \left( -\frac{\varphi_{\text{hol}}(E_4)}{24} \right)^{m} \left( -\frac{\varphi_{\text{hol}}(E_6)}{108} \right)^{n}.
\]

Now Corollary 1 follows from the fact that the disconnect and connected one-point ancestor functions are the same.

6.2. Holomorphic anomaly equations. We now describe holomorphic anomaly equations for the FJRW correlation functions. In the rest of the paper we shall only discuss connected invariants and hence omit the superscript “\( \text{c} \)” from the notations.

6.2.1. HAE for ancestor GW correlation functions. In [OP18], Oberdieck and Pixton use the polynomiality of double ramification cycles to prove that the GW cycles \( \Lambda_{g,n}(\alpha_1, \ldots, \alpha_n) \) of the elliptic curves are cycle-valued quasi-modular forms. Take the derivative of those cycles with respect to the second Eisenstein series \( E_2(q) \), they obtain a holomorphic anomaly equation [OP18, Theorem 3]. As a consequence, intersecting the corresponding GW cycles with \( \prod_k \psi_k^{\ell_k} \) on \( \overline{\mathcal{M}}_{g,n} \) leads to a holomorphic anomaly equation for the ancestor GW functions

\[
\langle \langle \alpha_1 \psi_1^{\ell_1}, \ldots, \alpha_n \psi_n^{\ell_n} \rangle \rangle_{g,n}(q) \in \mathbb{C}[E_2, E_4, E_6].
\]

For each subset \( I \subseteq \{1, \ldots, n\} \), we use the following convention

\[
\alpha_I := \{ \alpha_i \psi_i^{\ell_i}, \forall i \in I \}.
\]

For convenience we introduce the normalized Eisenstein series

\[
C_2(q) = -\frac{1}{24} E_2(q).
\]

It is a classical fact that the Eisenstein series \( E_2, E_4, E_6 \) are algebraically independent. One has [OP18] for the ancestor GW correlation functions

\[
\frac{\partial}{\partial C_2} \langle \langle \alpha_1 \psi_1^{\ell_1}, \ldots, \alpha_n \psi_n^{\ell_n} \rangle \rangle_{g,n}(q) = \langle \langle \alpha_1 \psi_1^{\ell_1}, \ldots, \alpha_n \psi_n^{\ell_n}, 1 \rangle \rangle_{g-1,n+2}(q) + \sum_{g_1 + g_2 = g, \{1, \ldots, n\} = I_1 \cup I_2} \langle \langle \alpha_{I_1}, 1 \rangle \rangle_{g_1}(q) \langle \langle 1, \alpha_{I_2} \rangle \rangle_{g_2}(q) - 2 \sum_{i=1}^{n} \left( \int_{\mathcal{E}} \alpha_i \right) \langle \langle \alpha_1 \psi_1^{\ell_1}, \ldots, \psi_i^{\ell_i+1}, \ldots, \alpha_n \psi_n^{\ell_n} \rangle \rangle_{g,n}(q).
\]

Remark 4. This equation can also be proved using only the combinatorial results reviewed in Section 5.1, see Pixton [Pix08].
6.2.2. HAE for ancestor FJRW correlation functions. Recall that the holomorphic Cayley transformation $\varphi^{\text{hol}}_{\tau_c}$ respect the differential ring structure of the set of quasi-modular forms. Applying the holomorphic Cayley transformation to (6.9), using Theorem 2 we immediately obtain the following HAE for the ancestor FJRW correlation functions.

**Corollary 3.** Let the notations be as in Theorem 1. For the $d = 3$ case, the ancestor FJRW correlation function

$$
\langle \langle \alpha_1 \psi_1^{\ell_1}, \ldots, \alpha_n \psi_n^{\ell_n} \rangle \rangle_{g,n}^{W_d} \in \mathbb{C} [c^{\text{hol}}_{\tau_c}(C_2), c^{\text{hol}}_{\tau_c}(E_4), c^{\text{hol}}_{\tau_c}(E_6)], \quad C_2 = -\frac{1}{24} E_2
$$

satisfies

$$
\frac{\partial}{\partial \varphi^{\text{hol}}_{\tau_c}(C_2)} \langle \langle \alpha_1 \psi_1^{\ell_1}, \ldots, \alpha_n \psi_n^{\ell_n} \rangle \rangle_{g,n}^{W_d} = \langle \langle \alpha_1 \psi_1^{\ell_1}, \ldots, \alpha_n \psi_n^{\ell_n}, 1, 1 \rangle \rangle_{g-1,n+2}^{W_d} + \sum_{g_1 + g_2 = g, \{1, \ldots, n\} = h \cup I_2} \langle \langle \alpha_{I_1}, 1 \rangle \rangle_{g_1}^{W_d} \langle \langle 1, \alpha_{I_2} \rangle \rangle_{g_2}^{W_d} - 2 \sum_{i=1}^n \langle \langle \alpha_1 \psi_1^{\ell_1}, \ldots, \delta_{\phi}^{\mu} 1 \psi_i^{\ell_i+1}, \ldots, \alpha_n \psi_n^{\ell_n} \rangle \rangle_{g,n}^{W_d},
$$

where $\delta_{\phi}^{\mu}$ is the Kronecker symbol.

6.3. Virasoro constraints. Virasoro operators in Gromov-Witten theory were proposed by Eguchi, Hori, and Xiong [EHX97] for Fano manifolds and later generalized to more general targets [DZ99, Giv01b]. The famous Virasoro Conjecture predicts that the total descendent potentials in GW theory are annihilated by the Virasoro operators. It is one of the most fascinating conjectures in GW theory. Despite significant developments in the literature, it remains open for a large category of targets.

The Virasoro conjecture for nonsingular target curves is solved by Okounkov and Pandharipande [OP06b]. In particular, when the target is an elliptic curve, the formulas are particularly simple. To be more explicit, using the coordinates induced by (5.15) and let

$$(\ell)_n := \ell \cdot (\ell + 1) \cdots (\ell + n - 1)$$

be the Pochhammer symbol with the convention $(\ell)_0 := 1$, then the Virasoro operators \{\lambda_k^\ell \mid k \in \mathbb{Z}; k \geq -1\} are given by

$$
L_k^\ell = -(k + 1) + \sum_{\ell \geq 0} \left( (\ell)_{k+1} \frac{\partial}{\partial F_{k+\ell}^0} + (\ell + 1)_{k+1} \frac{\partial}{\partial F_{k+\ell}^3} \right) + \sum_{\ell \geq 0} \left( (\ell + 1)_{k+1} \frac{\partial}{\partial F_{k+\ell}^1} + (\ell)_k \frac{\partial}{\partial F_{k+\ell}^2} \right).
$$

According to [OP06b, Theorem 1], the total descendent GW potential defined in (5.16) is annihilated by these Virasoro operators

$$
L_k^\ell \mathcal{D}^{\ell} (\mathbb{t}) = 0.
$$

Recently in [HS21], using Givental’s quantization formula of quadratic Hamiltonians [Giv01b], the second author and his collaborator study Virasoro operators in FJRW theory and conjecture that the total ancestor FJRW potential of any admissible LG pair $(W, G)$ is annihilated by the defining Virasoro operators. Besides various generically semisimple cases, they also verified the conjecture for the non-semisimple Fermat cubic pair $(W_3, \mu_3)$, using
Theorem 1. More explicitly, using the coordinates induced by (5.14), the Virasoro operators \( \{ L_k^{W_3, \mu_3} | k \in \mathbb{Z}; k \geq -1 \} \) for the Fermat cubic pair \((W_3, \mu_3)\) are

\[
L_k^{W_3, \mu_3} := -(k+1)! \frac{\partial}{\partial t^0_k} + \sum_{\ell \geq 0} \left( (\ell)_{k+1} s^0_\ell \frac{\partial}{\partial s^0_{k+\ell}} + (\ell+1)_{k+1} s^3_\ell \frac{\partial}{\partial s^3_{k+\ell}} \right) + \sum_{\ell \geq 0} \left( (\ell+1)_{k+1} s^3_\ell \frac{\partial}{\partial s^3_{k+\ell}} + (\ell)_{k+1} s^3_\ell \frac{\partial}{\partial s^3_{k+\ell}} \right).
\]

It is not hard to see that these operators commute with the quantization operator \( \hat{S}_t^{-1} \) in the ancestor/descendent correspondence formula (5.17) and the holomorphic Cayley transformation \( \chi^{\text{hol}}_t \) in Theorem 1. Therefore, Virasoro constraints for the FJRW theory is a consequence of Theorem 1.

Corollary 4. [HS21] The total ancestor FJRW potential of the pair \((W_3, \mu_3)\) is annihilated by the Virasoro operators \( \{ L_k^{W_3, \mu_3} \} \),

\[
L_k^{W_3, \mu_3} A^{W_3, \mu_3}(s) = 0.
\]

Appendix A.

A.1. A genus-one formula for Fermat cubic polynomial. For the examples studied in this paper, the connection between modular forms and periods of families of elliptic curves give rise to nice formulae for the holomorphic Cayley transformation of quasi-modular forms in terms of hypergeometric series and Givental’s I-functions. In the following, we shall only consider the \( d = 3 \) case as an example, the other cases are similar.

Let us first recall some facts of quasi-modular forms following the exposition in [SZ17]. Let \( \Gamma(3) \) be the level-3 principal congruence subgroup of \( \Gamma = \text{SL}(2, \mathbb{Z})/\{ \pm 1 \} \). It is well known that the ring of quasi-modular forms (with a certain Dirichlet character) for \( \Gamma(3) \) is generated by

\[
A = \theta_{A_2}(2\tau)
\]

and

\[
E = \frac{3E_2(3\tau) + E_2(\tau)}{4},
\]

where \( \theta_{A_2} \) is the theta function for the \( A_2 \)-lattice. Define further the quantities (where \( \eta \) is the Dedekind eta function)

\[
C = \frac{3\eta(3\tau)^3}{\eta(\tau)}, \quad \alpha = \frac{C^3}{A^3}.
\]

These quantities satisfy

\[
A = 2 F_1\left(\frac{1}{3}; \frac{2}{3}; 1; \alpha\right),
\]

and furthermore

\[
\begin{cases}
A^2 = \frac{1}{2} (3E_2(3\tau) - E_2(\tau)) = \frac{1}{2\pi \sqrt{-1}} \frac{1}{\alpha(1-\alpha)} \frac{\partial}{\partial \tau} \alpha, \\
E = \frac{6}{2\pi \sqrt{-1}} \frac{\partial}{\partial \tau} \log A - \frac{2C^3 - A^3}{A}.
\end{cases}
\]
Using (A.1), (A.2) and (A.4), we can rewrite the quasi-modular form $E_2$ as

\begin{equation}
E_2(\tau) = \frac{12}{2\pi \sqrt{-1}} \frac{\partial}{\partial \tau} \log A \left( 4\alpha - 1 \right) A^2
\end{equation}

\begin{equation}
= \frac{1}{2\pi \sqrt{-1}} \frac{\partial}{\partial \tau} \left( 12 \log A + \log(1 - A) \right).
\end{equation}

In [SZ18] the following was obtained from period calculation. Taking $\tau_0 = 1/(1 - \zeta_3)$ as given in (4.6) and $c$ as in (4.7), then one has

$$s(\tau) = 2\pi \sqrt{-1} c(\tau_0 - \tau) \frac{\tau - \tau_0}{\tau - \tau_0}$$

$$= -2\pi \sqrt{-1} c(\tau_0 - \tau) \Gamma(-\frac{1}{3}) \Gamma(\frac{2}{3})^2 (-\alpha)^{-\frac{1}{3}} \frac{2 F_1(\frac{2}{3}, \frac{2}{3}; \frac{1}{3}; \alpha^{-1})}{2 F_1(\frac{1}{3}, \frac{1}{3}; \frac{2}{3}; \alpha^{-1})}.$$

Combining the properties of the holomorphic Cayley transformation, Theorem 2, and (A.5), we immediately get

$$\langle \langle \phi \rangle \rangle_{1,1}^{\omega_3} = \langle \langle \omega \rangle \rangle_{1,1}^{E_2}$$

$$= c^{-1} \frac{\partial}{\partial s} \left( -\frac{1}{2} \log 2 F_1 \left( \frac{1}{3}, \frac{2}{3}; \frac{1}{3}; \alpha^{-1} \right) - \frac{1}{8} \log(1 - \alpha) \right) .$$

In the above GW generating series, the divisor class $\omega$ which corresponds to the first Chern class of a degree one line bundle on $E$ is used as the insertion. According to the divisor axiom, it follows that

$$\langle \langle \rangle \rangle_{1,0}^{E_3} = -\log \eta(\tau),$$

up to an additive constant. Results derived for a plane cubic curve $E_3$, such as those in Givental’s formalism, use the pull-back of the hyperplane class on the ambient space $P^2$ as the insertion. The corresponding class $H$ is related to the one $\omega$ above by $H = 3\omega$. Hence we have up to an additive constant

$$\langle \langle \rangle \rangle_{1,0}^{E_3} = -\log \eta(3\tau),$$

and thus

$$\langle \langle H \rangle \rangle_{1,0}^{E_3} = -\frac{1}{24} \cdot 3 \cdot E_2(3\tau).$$

Using (A.1), (A.2) and (A.4), one can rewrite it as

$$\langle \langle H \rangle \rangle_{1,0}^{E_3} = \frac{1}{2\pi \sqrt{-1}} \frac{\partial}{\partial \tau} \left( -\frac{1}{2} \log 2 F_1 \left( \frac{1}{3}, \frac{2}{3}; \frac{1}{3}; \alpha \right) - \frac{1}{24} \log(\alpha^3(1 - \alpha)) \right).$$

This matches the results in [Zin09, Pop13] obtained using virtual localization. Its holomorphic Cayley transformation is

$$\langle \langle H \rangle \rangle_{1,0}^{E_3} = c^{-1} \frac{\partial}{\partial s} \left( -\frac{1}{2} \log 2 F_1 \left( \frac{1}{3}, \frac{2}{3}; \frac{1}{3}; \alpha^{-1} \right) - \frac{1}{24} \log(1 - \alpha^{-1}) \right).$$
This agrees with the result derived using the wall-crossing method in Guo-Ross [GR19a].

A.2. Cayley transformation and $I$-functions. Now we discuss the connection between our formulation of LG/CY correspondence and the original formulation in [CR10, Conjecture 3.2.1] using $I$-functions.

A.2.1. $I$-functions and analytic continuation. Following [CR10, Section 4.2], the cohomology-valued Givental $I$-function for the GW theory of the cubic hypersurface

$$\{W_3 = x_1^3 + x_2^3 + x_3^3 = 0 \} \subset \mathbb{P}^2$$

is given by\(^{6}\)

$$I_{GW}(q, z) := \sum_{d \geq 0} z q^{H/z+d} \prod_{k=1}^{3d} (3H + kz) \prod_{k=1}^{d} (H + kz)^3$$

$$= I_0^{GW}(q) z 1 + I_1^{GW}(q) H,$$

where $H$ is the hyperplane class of $\mathbb{P}^2$. While the $I$-function for the FJRW theory of the pair $(W_3, \mu_3)$ is given by

$$I_{FJRW}(t, z) := \sum_{k=1}^{2} \frac{1}{\Gamma(k)} \cdot \sum_{\ell \geq 0} \frac{\left( \frac{k}{3} \right)^{3} t^{k+3\ell}}{\left( k \right)_{3}z^{k-1}} \phi_{k-1}$$

$$= I_0^{FJRW}(t) z 1 + I_1^{FJRW}(t) \phi,$$

where $\phi_0 = 1$ and $\phi_1 = \phi$ are nontrivial degree-zero and two elements in the state space. The genus-zero LG/CY correspondence [CR10] relates these two $I$-functions by analytic continuation via $q = t^{-3}$. To be more explicit, one has the following analytic continuation

$$\begin{pmatrix}
\frac{I_1^{FJRW}(t)}{3} \\
\frac{I_0^{FJRW}(t)}{3}
\end{pmatrix} = \begin{pmatrix}
\frac{\left( -1 \right)}{\Gamma(5/3)} \frac{2 \pi \sqrt{-1} \Gamma_3}{\Gamma(1)} & \frac{\left( -1 \right)}{\Gamma(1)} \frac{2 \pi \sqrt{-1} \Gamma_3}{\Gamma(-1)} \\
\frac{\left( -1 \right)}{\Gamma(1)} \frac{2 \pi \sqrt{-1} \Gamma_3}{\Gamma(1)} & \frac{\left( -1 \right)}{\Gamma(1)} \frac{2 \pi \sqrt{-1} \Gamma_3}{\Gamma(-1)}
\end{pmatrix} \begin{pmatrix}
I_1^{GW}(t(q)) \\
I_0^{GW}(t(q))
\end{pmatrix},$$

where the normalization factor $1/3$ on the basis \{ $t_0^{FJRW}$, $t_1^{FJRW}$ \} is introduced such that the connection matrix lies in $\text{SL}_2(\mathbb{C})$. In particular, define

$$t_{GW} := \frac{I_1^{GW}(q)}{I_0^{GW}(q)}, \quad t_{FJRW} := \frac{I_1^{FJRW}(t)}{I_0^{FJRW}(t)}.$$

Then one has

$$t_{FJRW} = -e^{\pi i} \cdot \frac{\Gamma(3)}{\Gamma(-\frac{1}{3}) \Gamma(\frac{1}{2})^2} \cdot \frac{t_{GW} - 2 \pi i \tau_s}{t_{GW} - 2 \pi i \tau_s}.$$\(^{6}\)Here the variable $q$ should not be confused with the variable $q = e^{2 \pi i \tau}$ in modular forms.
A.2.2. Cayley transformation. Following the computations in [SZ18] as in Appendix A.1, we can relate the above $I$-functions to modular forms. In particular, we see that

\[(A.11) \quad t_{GW} := \frac{I_{GW}^1(q)}{I_{GW}^0(q)} = 2\pi i \tau, \quad t_{FJRW} := \frac{I_{FJRW}^1(t)}{I_{FJRW}^0(t)} = e^{2\pi i \tau} \cdot \frac{3}{i} \cdot \frac{\Gamma(\frac{1}{3})^2}{\Gamma(-\frac{1}{3})} s.\]

Here $s$ is the coordinate given in (4.4), with again $\tau_s = 1/(1 - \zeta_3)$ as given in (4.6) and $c$ as in (4.7). Analytical continuations on the $I$-functions, induced by (A.10), coincide with Cayley transformations on them induced by (4.4) by construction [SZ18].

Through the connection to modular forms, LG/CY correspondence on $I$-functions can be restated as follows. Let $\mathcal{M} = \Gamma(3) \backslash \mathbb{H}^*$ be the modular curve as the global moduli space, where $\mathbb{H}^* = \mathbb{H} \cup \mathbb{P}_1(\mathbb{Q})$. Denote its canonical bundle by $K_{\mathcal{M}}$. Then $I_{GW}$ and $I_{FJRW}$ correspond to descriptions of the same holomorphic section of the line bundle that is isomorphic to $K_{\mathcal{M}}^\otimes n$, but on different patches of the moduli space. Their coordinate expressions $I_{GW}^0, I_{FJRW}^0$, with respect to the trivializations $(d\tau)^{\frac{1}{2}}, (ds)^{\frac{1}{2}}$ respectively, are modular forms related by Cayley transformation.

A.2.3. Stationary correlation functions. At higher genus, consider the stationary correlation function

\[\mathcal{I} = \langle \langle a_1 \psi_1^{\ell_1}, \ldots, a_n \psi_n^{\ell_n} \rangle \rangle_{g,n},\]

with $a_i = \omega$ when $\bullet = E_3$ and $a_i = \phi$ when $\bullet = W_3$. By applying the $g$-reduction technique in Lemma 1 inductively, we see that under the map (A.11) this correlation function on the GW side is the Fourier expansion of a quasi-modular form of weight $2g - 2 + 2n$ near the cusp, and on the FJRW side is the Taylor expansion (in terms of the parameter $s$) of the same quasi-modular form near the point $\tau_s$.

According to standard facts in the theory of modular forms (see e.g., [Urb14, Zag08]) on the transition between quasi-modular forms and almost-holomorphic modular forms, we see that on the level of GW correlation functions the modular completion is induced by the transformation mapping the frame of $H^{even}(E_3, \mathbb{C})$ from \{1 + 2\pi i \tau H, 2\pi i H\} to \{1 + 2\pi i \tau H, \frac{1}{\tau} \cdot (1 - 2\pi i \tau H)\}. This transformation also induces the modular completion on the FJRW correlation functions by compositing with the aforementioned transformation that relates $I_{GW}$ with $I_{FJRW}$.

One succinct way to reformulate our higher-genus LG/CY correspondence result on $\langle \langle a_1 \psi_1^{\ell_1}, \ldots, a_n \psi_n^{\ell_n} \rangle \rangle_{g,n}$ is then as follows. Denote its modular completion by

\[\mathcal{I} = \langle \langle a_1 \psi_1^{\ell_1}, \ldots, a_n \psi_n^{\ell_n} \rangle \rangle_{g,n}.\]

Let $I_0^\bullet = I_{GW}^0, d\tau = ds$ for $\bullet = E_3$, and $I_0^\bullet = I_{FJRW}^0, d\tau = ds$ for $\bullet = W_3$. Then the quantity

\[\langle I_0^\bullet \rangle_{g,n}^{2-2g} \langle \langle a_1 \psi_1^{\ell_1}, \ldots, a_n \psi_n^{\ell_n} \rangle \rangle_{g,n} (d\tau)^{\otimes n}\]

is a global (smooth with holomorphic pole) section of the holomorphic line bundle $K_{\mathcal{M}}^{\otimes n}$ on the modular curve $\mathcal{M} = \Gamma(3) \backslash \mathbb{H}^*$.

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