Greybody factors for rotating black holes in four dimensions

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Abstract

We present the wave equation for a minimally coupled scalar field in the background of a rotating four-dimensional black hole that is parametrized by its mass, angular momentum, and four independent U(1) charges. The near horizon structure is identical to the five-dimensional case, and suggestive of an underlying description in string theory that is valid in the general non-extremal case. We calculate the greybody factors for the Hawking radiation. For sufficiently large partial wave number the emission spectrum can be calculated for general non-extremal black holes and any particle energy. We interpret this spectrum in terms of a multi-body process in an effective string theory.
1 Introduction

It is now widely accepted that a large class of extremal and near-extremal black holes can be described in string theory. The evidence for this correspondence falls in two broad categories: first, the counting of string states agrees with the degeneracies inferred from the Bekenstein-Hawking area law [1, 2] (also [3, 4, 5]); and second, the rate and spectrum of Hawking radiation agrees in the two descriptions [2, 4, 7]. The string theory side of the counting involves two distinct contributions to the entropy, from the right and left moving excitations of the string; and similarly the emission spectrum of the string is characterized by two independent temperatures, because the initial state has quanta in both sectors. The agreement requires that the classical geometry reproduces these duplications of thermodynamic variables; and this is indeed the case, at least in the so-called dilute gas regime.

The appearance of two independent sets of thermodynamic variables is related to the presence of two event horizons. For example the two contributions to the entropy are proportional to the sum and difference of the inner and outer horizon area, respectively. This geometric interpretation of the thermodynamic variables gives access to properties of the underlying microscopic theory that seems to be valid for all black holes. In a previous paper we explained these results for the most general black holes in five dimensions [8] (see also [9, 10]). The purpose of the present paper is to present the corresponding calculations for a large class of rotating black holes in four dimensions. Our focus will be on the differences between the two cases; so a number of interpretive remarks will not be repeated.

The description of black holes in string theory is more involved in four dimensions than in five dimensions. In particular, the effective string models for the entropy are less rigorous [4, 11, 12, 13, 14]; and the microscopic foundation of the proposed exact BPS-degeneracy [15] is less clear than the analogous one in five dimensions [1]. The regime of agreement between the emission spectrum of the string model and the Hawking radiation of semiclassical description is also more restrictive in four dimensions [16, 17, 18].

The interpretation of greybody factors is simplest in the regime where the radiation is the result of a two-body process in the underlying string theory. In [8] we
found that for five dimensional black holes the rotational parameters can be tuned so that greybody factors retain their characteristic two-body form, without restrictions on the charges. Here we will find no comparable simplifications in four dimensions. Despite this limitation many cases remain where the radiation can be described as the result of a two-body process. An important example is the low energy limit of the radiation from the most general black holes.

New possibilities arise if we also consider multi-body processes in the effective string theory. It is particularly interesting to consider processes with higher partial wave number. In both four and five dimensions the partial wave number can be chosen so that these processes can be described analytically in the classical theory for general non-extremal black holes, and even at high energy. The structure of the resulting emission rates has a characteristic multi-body form that can be interpreted in an effective string theory.

The paper is organized as follows. In sec. 2, we present the wave equation for a minimally coupled scalar field in the background of a rotating black hole in four dimensions. We interpret the various terms. In sec. 3, we solve the equation in various regions, and match these exact solutions to find approximate wave functions that are valid throughout. This leads to a general expression for the absorption cross-section. In sec. 4 the region of applicability of this result is determined, and we present examples. In sec. 5 we discuss the microscopic interpretation of our classical calculations in terms of an effective string theory with emphasis on applications to general non-extremal black holes.

## 2 The Wave Equation

We consider a class of four-dimensional rotating black hole in toroidally compactified string theory. It is parametrized by the mass $M$, angular momentum $J$, and four independent $U(1)$ charges $Q_i$. This black hole solution was given explicitly in [20]; and its non-rotating form appeared in [21, 22]. In special cases the black hole reduces to a solution of Einstein-Maxwell gravity. For example the Kerr-Newman black hole

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1The most general black hole in four dimensions is parametrized by five charges [3, 14]. It has not yet been constructed.
corresponds to all four charges equal.

It is often convenient to parametrize the physical variables \(M, Q_i,\) and \(J\) in terms of auxiliary quantities \(\mu, \delta_i,\) and \(l\) defined as:

\[
M = \frac{1}{2} \mu \sum_{i=1}^{4} \cosh 2\delta_i ,
\]

\[
Q_i = \frac{1}{2} \mu \sinh 2\delta_i \quad i = 1, 2, 3, 4 \quad ,
\]

\[
J = \frac{1}{2} \mu l \left( \prod_{i=1}^{4} \cosh \delta_i - \prod_{i=1}^{4} \sinh \delta_i \right) .
\]

The gravitational coupling constant in four dimensions is \(G_4 = \frac{1}{8} .\) In string units this corresponds to \((2\pi)^6 (\alpha')^4 / V_6\) With these conventions the areas \(A_\pm\) of the inner and outer event horizons imply the entropies:

\[
S_\pm = \frac{A_\pm}{4G_N} = 2\pi \left[ \frac{1}{2} \mu^2 \left( \prod_i \cosh \delta_i + \prod_i \sinh \delta_i \right) \pm \sqrt{\frac{1}{2} \mu^4 \left( \prod_i \cosh \delta_i - \prod_i \sinh \delta_i \right)^2 - J^2} \right]
\]

\[
= 2\pi \left[ \frac{1}{2} \mu^2 \left( \prod_i \cosh \delta_i + \prod_i \sinh \delta_i \right) \pm \frac{1}{2} \mu \sqrt{\mu^2 - l^2 \left( \prod_i \cosh \delta_i - \prod_i \sinh \delta_i \right)} \right] .
\]

In the string theory interpretation the two terms:

\[
S_L = \pi \mu^2 \left( \prod_i \cosh \delta_i + \prod_i \sinh \delta_i \right) ,
\]

\[
S_R = \frac{1}{2} \mu \sqrt{\mu^2 - l^2 \left( \prod_i \cosh \delta_i - \prod_i \sinh \delta_i \right)} ,
\]

are the entropies of the left (L) and right (R) moving excitations.

We consider a minimally coupled scalar field propagating in the background of the black hole. The wave equation is the Klein-Gordon equation:

\[
\frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} g^{\mu\nu} \partial_{\nu} \Phi) = 0 .
\]

It is convenient to introduce the dimensionless radial coordinate:

\[
x \equiv \frac{r - \frac{1}{2} (r_+ + r_-)}{r_+ - r_-} ,
\]

so that the outer and inner event horizons are located at \(x = \pm \frac{1}{2},\) respectively; and the asymptotic space is at large \(x.\) The wave function is written in spherical coordinates as:

\[
\Phi \equiv \Phi_r (r) \chi(\theta) e^{-i\omega t + im\phi} .
\]

\(^2\)The \(m\) of [20] is \(m = \frac{1}{4} \mu\) and also \(l_{\text{here}} = \frac{1}{4} l_{\text{here}}.\) The \(r_0\) of [22] is \(r_0 = \frac{1}{2} \mu.\)
Then the radial part of the wave equation becomes:

\[
\frac{\partial}{\partial x} \left( x^2 - \frac{1}{4} \right) \frac{\partial}{\partial x} \Phi_r + \frac{1}{4} \left[ x^2 \Delta^2 \omega^2 + x M \Delta \omega^2 - 4 \tilde{\Lambda} \right] \Phi_r = 0 .
\]

(10)

This equation is our main technical result\(^3\). It is not more complicated than special cases that have been considered previously \[23, 17, 18\], but the present context is more general. In the following we explain the notation and interpret the various terms.

The variable \(\Delta\) is defined as:

\[
\Delta = 2(r_+ - r_-) = \sqrt{\mu^2 - l^2} = \beta_H^{-1} S .
\]

(11)

\(T_H = \beta_H^{-1}\) is the Hawking temperature.) The first equation, and the definition of the radial variable \(x\), ensures that the wave equation reduces at large \(x\) to the Klein-Gordon equation in flat space, as it should. Accordingly the term \(\frac{1}{4} x^2 \Delta^2 \omega^2\), dominant at large \(x\), can be interpreted as the energy of the scalar field at infinity.

At large \(x\) the mass-term \(\frac{1}{4} x M \Delta \omega^2\) is suppressed relative to the energy of the scalar field at infinity by one power of \(x \propto r\), as expected for a long range Coulomb type interaction in four dimensions.

The effective angular momentum barrier \(\tilde{\Lambda}\) is the term that is suppressed by \(x^2 \propto r^2\) for large \(x\). It is:

\[
\tilde{\Lambda} = \Lambda - \frac{1}{16} \mu^2 \omega^2 \left( 1 + \sum_{i<j} \cosh 2 \delta_i \cosh 2 \delta_j \right) ,
\]

(12)

where \(\Lambda\) is the eigenvalue of the operator:

\[
\hat{\Lambda} = -\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} - \frac{1}{\sin^2 \theta} \frac{\partial}{\partial \phi^2} - \frac{1}{16} l^2 \omega^2 \cos^2 \theta ,
\]

(13)

which is simply the angular Laplacian in four flat dimensions, except for the last term which reflects the rotation of the black hole background.

The terms that are most important for the microscopic interpretation are the horizon terms at \(x = \pm \frac{1}{2}\). Their form is such that \(\kappa_\pm\) are the physical surface

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\(^3\)The main intermediate step is to calculate the determinant of the metric. It turns out that the result is simple: \(-g = \Delta \sin^2 \theta\), in the notation of \[21\]
accelerations at the inner and outer horizons. They are given by:

\[
\frac{1}{\kappa_{\pm}} = \frac{1}{2} \mu^3 \left( \prod_i \cosh^2 \delta_i - \prod_i \sinh^2 \delta_i \right) \pm \frac{1}{2} \mu \left( \prod_i \cosh \delta_i - \prod_i \sinh \delta_i \right) \nonumber \\
\pm \frac{1}{2} \mu^2 \left( \prod_i \cosh \delta_i \prod_i \sinh \delta_i \right)^2 - J^2 \pm \frac{1}{2} \mu \left( \prod_i \cosh \delta_i - \prod_i \sinh \delta_i \right) .
\] (14)

Similarly, \( \Omega \) is the angular velocity at the outer event horizon, given by:

\[
\frac{1}{\kappa_+} \Omega = \frac{J}{\sqrt{\frac{1}{4} \mu^4 \left( \prod_i \cosh \delta_i - \prod_i \sinh \delta_i \right)^2 - J^2}} = \frac{l}{\sqrt{\mu^2 - l^2}} .
\] (15)

These expressions for \( \kappa_{\pm} \) and \( \Omega \) agree with those that follow from the entropy (eq. 4), by use of thermodynamic relations.

### 3 Calculation of Absorption Cross-section

The wave equation cannot in general be solved exactly. However we can find solutions valid in the asymptotic region and match them with solutions valid in the horizon region to find approximate wave functions that apply throughout spacetime. The calculations follow previous work closely [7, 17, 18, 24, 25].

**The angular equation:** Consider first the angular Laplacian \( \hat{\Lambda} \) (eq. 13). The eigenfunctions are the spheroidal functions (see eg. [26]):

\[
\chi(\theta) = S_{mn}(\frac{l\omega}{4}, \cos \theta) .
\] (16)

The corresponding eigenvalues \( \Lambda \) are labelled by the orbital angular momentum \( n \) and the azimuthal quantum number \( m \). They can be represented by a power series in \( (\frac{l\omega}{4})^2 \) as:

\[
\Lambda(n, m) = n(n + 1) + \frac{1}{2} \left[ 1 - \frac{(2m - 1)(2m + 1)}{(2n - 1)(2n + 3)} \right] \left( \frac{l\omega}{4} \right)^2 + \cdots
\] (17)

\(^4\)The angular velocity at the inner event horizon is given through \( \frac{1}{\kappa_-} \Omega_- = \frac{1}{\kappa_+} \Omega \). The appearance of \( \kappa_+ \) in the inner horizon term is due to the convention that \( \Omega \) is measured at the outer horizon.
The asymptotic region: Next consider the radial equation. At large \( x \gg 1 \) we omit the horizon terms so that the radial wave equation becomes:

\[
\frac{\partial}{\partial x} x^2 \frac{\partial}{\partial x} \Phi_\infty + \frac{1}{4} x^2 \Delta^2 \varpi^2 + x M \Delta \omega^2 - 4 \tilde{\Lambda} \Phi_\infty = 0. \tag{18}
\]

Denoting the solutions in this region \( \Phi_\infty^\pm \) we find:

\[
\Phi_\infty^\pm = x^{-\frac{1}{2} \pm \xi} e^{-\frac{i}{2} \Delta \omega x} C_\pm M_K \left( \frac{1}{2} \pm \zeta - \frac{i}{4} M \omega, 1 \pm 2 \zeta, i \Delta \omega x \right), \tag{19}
\]

where the function \( M_K \) is Kummer's function (see e.g. [26]), the parameter \( \zeta \) is\(^5\)

\[
\zeta = \sqrt{\frac{1}{4} + \tilde{\Lambda}}, \tag{20}
\]

and the normalization constants \( C_\pm \) are:

\[
C_\pm = \frac{1}{2} (\Delta \omega)^{\frac{1}{2} \pm \xi} e^{-\frac{i}{8} \pi M \omega} \frac{\left| \Gamma \left( \frac{1}{2} \pm \zeta + \frac{i}{4} M \omega \right) \right|}{\Gamma (1 \pm 2 \zeta)}. \tag{21}
\]

The wave functions eq. 19 are generalizations of the Coulomb functions to non-integer angular momentum. They are normalized so that:

\[
\Phi_\infty^\pm \sim x^{-\frac{1}{2} \pm \xi} C_\pm, \tag{22}
\]

for \( \Delta \omega x \ll 1 \) and:

\[
\Phi_\infty^\pm \sim \frac{1}{x} \sin \left( \frac{1}{2} \Delta \omega x - \frac{1}{4} M \omega \log x + \text{const} \right), \tag{23}
\]

for \( \Delta \omega x \gg 1 \).

The horizon region: In the horizon region we ignore the energy of the scalar field at infinity and also the Coulomb type screening, due to the mass of the black hole. Denoting by \( \Phi_0 \) the radial wave function in this regime the radial equation becomes:

\[
\frac{\partial}{\partial x} (x^2 - \frac{1}{4}) \frac{\partial}{\partial x} \Phi_0 - \tilde{\Lambda} + \frac{1}{x - \frac{1}{2}} \left( \frac{\omega}{\kappa_+} - m \frac{\Omega}{\kappa_+} \right)^2 - \frac{1}{x + \frac{1}{2}} \left( \frac{\omega}{\kappa_-} - m \frac{\Omega}{\kappa_-} \right)^2 \Phi_0 = 0 \tag{24}
\]

The form of this equation is identical to the corresponding one for the most general black holes in five dimensions [8]. This is in harmony with the intuition that the

\(^5\)The parameter \( \xi \) of [8] can be introduced as \( \xi = \frac{1}{2} + \zeta \). The present notation gives more symmetric formulae.
near horizon terms express universal physics related to the underlying microscopic structure. In particular the $SL(2, R)_L \times SL(2, R)_R$ symmetry of the horizon region, exhibited for the five-dimensional case in [8], carries over to four dimensions.

The solution relevant in the following has only an infalling component at the outer horizon. Taking the azimuthal quantum number $m = 0$ for typographical simplicity it is:

$$\Phi_{\text{in}}^0 = \left(1 - \frac{1}{x} \right)^{-\frac{\beta_H \omega}{4\pi}} \left(x + \frac{1}{2} \right)^{-\frac{1}{2} - \zeta} F \left(\frac{1}{2} + \zeta - i \frac{\beta_R \omega}{4\pi}, \frac{1}{2} + \zeta - i \frac{\beta_L \omega}{4\pi}, 1 - i \frac{\beta_H \omega}{2\pi}, x - \frac{1}{2} \right). \quad (25)$$

We introduced the inverse Hawking temperature $T_H^{-1} = \beta_H = \frac{2\pi}{\kappa}$, and the corresponding right (R) and left (L) components:

$$\beta_R = \frac{2\pi}{\kappa_+} + \frac{2\pi}{\kappa_-} = \frac{2\pi \mu^2}{\sqrt{\mu^2 - l^2}} (\prod_i \cosh \delta_i + \prod_i \sinh \delta_i), \quad (26)$$

$$\beta_L = \frac{2\pi}{\kappa_+} - \frac{2\pi}{\kappa_-} = 2\pi \mu (\prod_i \cosh \delta_i - \prod_i \sinh \delta_i). \quad (27)$$

A linearly independent solution can be chosen as the purely outgoing wave, given by time reversal $\omega \rightarrow -\omega$. The asymptotic behavior of $\Phi_{\text{in}}^0$ at large $x$ can be extracted, by using the modular properties of the hypergeometric function $F$. It is:

$$\Phi_{\text{in}}^0 \sim x^{-\frac{1}{2} - \zeta} \frac{\Gamma(1 - i \frac{\beta_H \omega}{2\pi}) \Gamma(-2\zeta)}{\Gamma(\frac{1}{2} - \zeta - i \frac{\beta_H \omega}{4\pi}) \Gamma(\frac{1}{2} - \zeta - i \frac{\beta_L \omega}{4\pi})} + x^{-\frac{1}{2} - \zeta} \frac{\Gamma(1 - i \frac{\beta_H \omega}{2\pi}) \Gamma(2\zeta)}{\Gamma(\frac{1}{2} + \zeta - i \frac{\beta_H \omega}{4\pi}) \Gamma(\frac{1}{2} + \zeta - i \frac{\beta_L \omega}{4\pi})}. \quad (28)$$

It is now straightforward to calculate the absorption cross-section. We take $A_{\infty} \Phi_\infty^+$ as the wave function in the asymptotic region, and in some intermediate "matching region" we identify this with $A_0 \Phi_0^+$, using eqs. 22 and 28. Postponing the justification of this procedure to the subsequent section, the ratio of amplitudes becomes:

$$\left| \frac{A_0}{A_{\infty}} \right| = \frac{1}{2} (\Delta \omega)^{\frac{1}{2} + \zeta} e^{\pi M \omega} |\Gamma(\frac{1}{2} + \zeta + i \frac{\beta_L \omega}{4\pi})| \left| \frac{1}{\Gamma(2\zeta)} \Gamma(2\zeta + 1) \Gamma(1 - i \frac{\beta_H \omega}{2\pi}) \right|. \quad (29)$$

From eq. 23 we find the flux factor:

$$\text{flux} = \frac{1}{2i} (\Phi^2 \partial_t \Phi - \text{c.c.}) = \frac{\Delta^2 \omega}{16} |A_{\infty}|^2, \quad (30)$$

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6We take $m = 0$ in the remainder of the paper. It can be restored by the replacements $\beta_{R,L} \omega / 2 \rightarrow \beta_{R,L} \omega / 2 - m \beta_H \Omega$ and $\beta_H \omega \rightarrow \beta_H \omega - m \beta_H \Omega$. 

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at large distances; and from eq. 25 we find the flux factor:

\[ \text{flux} = \frac{1}{2i} \langle \Phi r^2 \sqrt{g^{rr}} \partial_r \Phi - c.c. \rangle = \frac{\beta_H \omega \Delta}{8\pi} |A_0|^2, \tag{31} \]

at the horizon. The effective two dimensional transmission coefficient \( |T_n|^2 \) is the ratio of these two fluxes. Using a standard relation from scattering theory we present the result as the absorption cross-section of the \( n \)th partial wave:

\[
\sigma_{\text{abs}}^{(n)}(\omega) = \frac{\pi (2n + 1)}{\omega^2} |T_n|^2 = A (2n + 1)(\Delta \omega)^{2\zeta - 1} \frac{\Gamma(\frac{1}{2} + \zeta - i\frac{\beta_H \omega}{4\pi}) \Gamma(\frac{1}{2} + \zeta - i\frac{\beta_H \omega}{4\pi})}{\Gamma(2\zeta) \Gamma(2\zeta + 1) \Gamma(1 - i\frac{\beta_H \omega}{4\pi})}^2 \times e^{\frac{\pi}{4}M\omega |\Gamma(\frac{1}{2} + \zeta + i\frac{\beta_H \omega}{4\pi})|^2}. \tag{32} \]

In the intermediate step we used \( \frac{1}{2} \beta_H \Delta = \frac{1}{2} S = A \) (which follows from \( G_N = \frac{1}{8} \) and the last definition of \( \Delta \) in eq. 11.)

The absorption cross-section is almost identical to the corresponding expression in five dimensions \[8\]; except for the presence of the last term. In the limit where the effective angular momentum barrier \( \Lambda \) vanishes we have \( \zeta \sim \frac{1}{2} \); so this factor becomes:

\[
e^{\frac{\pi}{4}M\omega |\Gamma(1 + i\frac{\beta_H \omega}{4\pi})|^2} = \frac{\frac{1}{2} \pi M\omega}{1 - e^{-\frac{\pi}{4}M\omega}}. \tag{33} \]

The so-called Coulomb enhancement of the absorption is due to long range attractive interactions of Coulomb type. It is well-known from \textit{e.g.} nuclear physics. Note that it is unique to four dimensions where the gravitational potential falls off as \( r^{-1} \).

## 4 Estimates and Examples

### 4.1 Matching on constant potential

We now determine the regime of validity of the absorption cross-section eq. 32. The most generous condition arises when the matching region can be chosen so that the effective angular momentum barrier \( \Lambda \) dominates over all other terms. This is matching on a \textit{constant} potential, in the parlance of \[8\]. The logic is that the asymptotic wave function can be used at large \( x \), the horizon wave function can be used at small \( x \), and both are valid in the matching region.
The precise condition is that there is a range of large $x$ so that:

$$1 \ll x ; x^2 \Delta^2 \omega^2 \ll |\tilde{\Lambda}| ; xM\Delta\omega^2 \ll |\tilde{\Lambda}| ; \frac{\beta_R\beta_L\omega^2}{x} \ll |\tilde{\Lambda}|$$  \hspace{1cm} (34)

The necessary and sufficient conditions are that either:

$$M\omega \gg |\tilde{\Lambda}|^{1/2} ; \beta_R\beta_L M\omega^4 \Delta \ll |\tilde{\Lambda}|^2 ; M\omega^2 \Delta \ll |\tilde{\Lambda}| ,$$  \hspace{1cm} (35)

or:

$$M\omega \ll |\tilde{\Lambda}|^{1/2} ; \beta_R\beta_L \omega^3 \Delta \ll |\tilde{\Lambda}|^{3/2} ; \Delta\omega \ll |\tilde{\Lambda}|^{1/2} .$$  \hspace{1cm} (36)

In the S-wave $\tilde{\Lambda} \propto \omega^2$; so in this case all the conditions are independent of frequency. This is reasonable because in this case the only frequency dependence of the potential is an overall factor that cannot influence the relative size of the terms.

In the calculation of the absorption cross-section we also assumed that the $x^{-1/2+\zeta}$ term dominates over the $x^{-1/2-\zeta}$ term in the matching region. The large value of the matching $x$ is sufficiently to ensure this, provided $\zeta > 0$. In the regime where $\zeta$ is imaginary a closely related calculation is valid. (It is given in the appendix of [17].)

**Higher partial waves:** The constant term in the potential is the effective angular momentum barrier. Thus an important example is that of higher partial waves. In particular, for any given background and perturbation frequency, the partial wave number can be chosen so that $\tilde{\Lambda} \sim n^2$; and moreover so that all the conditions eq. 36 are satisfied. For natural frequencies $\omega \sim \beta_H^{-1}$ and generic black holes ($\delta_i \sim 1$ and $l \sim \mu$) it is sufficient that $n \gg 1$. This is interesting because the higher partial waves can be understood microscopically from multi-particle processes [18, 27, 28]. We will consider this example further in the discussion in sec. 5.

**Two large boosts:** This case is the S-wave, with two large boosts $\delta \sim \delta_i \gg 1$, and the other two boosts of order unity. A background angular momentum of order $l \sim \mu$ may be included. The inverse temperatures are comparable: $\beta_R \sim \beta_L \sim \mu e^{2\delta}$; and the other parameters are $M \sim \mu e^{2\delta}$, $|\tilde{\Lambda}| \sim \mu^2 \omega^2 e^{4\delta}$, and $\Delta \sim \mu$. The conditions eq. 36 are satisfied for all frequencies.
We note that the case of S-wave and one large boost violates the conditions, even if the freedom to tune the angular momentum is employed. In this case the absorption cross-section cannot be found in closed form.

4.2 Matching on vanishing potential

The calculation of Maldacena and Strominger [7] employed matching on a vanishing potential. This is in general more restrictive than the matching on a constant potential, but it also provides a clearer distinction between the horizon terms, of presumed microscopic importance, and the long range fields. In this case there must be a range of large $x$ so that all the potential terms can be neglected:

$$1 \ll x ; \ x^2 \Delta^2 \omega^2 \ll 1 ; \ xM \Delta \omega^2 \ll 1 ; \ |\tilde{A}| \ll 1 ; \ \frac{\beta_R \beta_L \omega^2}{x} \ll 1 .$$

The necessary and sufficient conditions are that either:

$$M \omega \gg 1 ; \ |\tilde{A}| \ll 1 ; \ \beta_R \beta_L M \Delta \omega^4 \ll 1 ; \ M \Delta \omega^2 \ll 1 ,$$

or:

$$M \omega < 1 ; \ |\tilde{A}| \ll 1 ; \ \beta_R \beta_L \Delta \omega^3 \ll 1 ; \ \Delta \omega \ll 1 .$$

We have not found any significant example that takes advantage of the first conditions, and a similar comment applies to scattering that is not in the S-wave; so these possibilities will be disregarded. Then the conditions eq. 39 are low energy conditions.

Matching on a vanishing potential automatically implies $\zeta \sim \frac{1}{2}$; so the general absorption cross-section eq. 32 simplifies dramatically:

$$\sigma_{\text{abs}}^{(0)}(\omega) = A \frac{\Gamma(1 - i \frac{\beta_L \omega}{2\pi}) \Gamma(1 - i \frac{\beta_R \omega}{2\pi})}{\Gamma(1 - i \frac{\beta \omega}{2\pi})} \left| e^{\frac{i}{4} M \omega} \right|^2 \left| \Gamma(1 + i \frac{\beta \omega}{4}) \right|^2$$

$$= A \frac{\beta_L \omega \beta_R \omega}{\beta_H \omega} \frac{e^{\beta_H \omega} - 1}{(e^{\beta_L \frac{\omega}{2}} - 1)(e^{\beta_R \frac{\omega}{2}} - 1)} \frac{1}{1 - e^{-\frac{\pi}{2} M \omega}}$$

The Coulomb enhancement factor will play no role in our discussion. Next we consider some examples:

**Low energy:** All conditions are automatically satisfied when the wave length $\lambda \sim \omega^{-1}$ is larger than all other scales in the problem. In this case:

$$\sigma_{\text{abs}}(\omega \rightarrow 0) = A .$$
This is the universal low energy absorption cross-section (see [29] and references therein.)

**Dilute gas regime:** In this case three boosts are large $\delta \sim \delta_i \gg 1$ but the last one is of order unity [23]. An angular momentum of order $l \sim \mu$ may be included. The inverse temperatures are comparable $\beta_R \sim \beta_L \sim \mu e^{3\delta}$; and the other parameters are $M \sim \mu e^{2\delta}$, $|\tilde{A}| \sim \mu^2 \omega^2 e^{4\delta}$, and $\Delta \sim \mu$. The effective angular momentum barrier and the horizons terms give the strongest condition on the frequency, namely $\mu \omega e^{2\delta} \ll 1$. However this is still consistent with the interesting regime $\beta_R \omega \sim \beta_L \omega \sim 1$.

**Near BPS black hole:** Here all boosts are large $\delta \sim \delta_i \gg 1$. An angular momentum of order $l \sim \mu$ may be included. There is a hierarchy between the temperatures: $\beta_L \sim \mu e^{2\delta}$ and $\beta_R \sim \mu e^{4\delta}$; and the other parameters are $M \sim \mu e^{2\delta}$, $|\tilde{A}| \sim \mu^2 \omega^2 e^{4\delta}$, and $\Delta \sim \mu$. The horizon terms and the effective angular momentum barrier both give the condition $\mu \omega e^{2\delta} \ll 1$ on the frequency. This is consistent with the structure in the range $\beta_R \omega \sim 1$ but not the other interesting range $\beta_H \omega \sim \beta_L \omega \sim 1$.

**Near extremal Kerr-Newman:** The angular momentum is tuned so that $\mu^2 - l^2 = \mu^2 e^2 \ll \mu^2$, but the charges are left general. In this limit there is a hierarchy between the temperatures: $\beta_L \sim \mu$ and $\beta_R \sim \mu e^{-1}$; and $|\tilde{A}| \sim \mu^2 \omega^2$, $M \sim \mu$, and $\Delta \sim \mu e$. The strongest condition on the frequency, from the horizon terms and the effective angular momentum barrier, is $\mu \omega \ll 1$. This is analogous to the near BPS case: the region $\beta_R \omega \sim 1$ can be probed, but $\beta_H \omega \sim \beta_L \omega \sim 1$ cannot.

In five dimensions there are two angular momenta and it is possible to tune them so that the inverse temperatures are large and comparable [8]. In this limit the calculation is sensitive to all boost parameters. However in four dimensions the angular momentum appears only in the $R$ sector. It plays no role in the estimates unless $l$ is tuned to be near $\mu$, and this implies $\beta_R \gg \beta_L$; so there is necessarily a hierarchy between the inverse temperatures. There is therefore no “rapidly spinning black hole” regime in four dimensions.
5 Discussion

In the region where matching on a vanishing potential can be justified, the Hawking emission rate is\[7\]

\[
\Gamma^{(0)}_{\text{em}}(\omega) = \sigma_{\text{abs}}(\omega) \frac{1}{e^{\beta H \omega} - 1} \frac{d^4k}{(2\pi)^4},
\]

\[
= 2\pi G_N \omega \frac{1}{(e^{\beta L \omega} - 1)(e^{\beta R \omega} - 1)} \frac{d^4k}{(2\pi)^4},
\]

where:

\[
\mathcal{L} = 2\pi \mu^3 \left( \prod_{i=1}^{4} \cosh^2 \delta_i - \prod_{i=1}^{4} \sinh^2 \delta_i \right).
\]

The emission rate eq. 44 is identical to the two-body annihilation rate for quanta propagating on an effective string of length $L$ \[6\]. From our classical calculation we expect that this microscopic model of the emission process can be applied in the full range of validity (eq. 39) of the two-body form of the emission rate (eq. 44). In particular, this includes the emission of very low energy quanta from all black holes; so in this sense we can model all black holes as effective strings, and their Hawking radiation as two-body processes.

However, it is only for special black holes that the two-body description can be used at the energies of typical Hawking particles; and only for these black holes is the emission spectrum eq. 44 applicable in the range where it has characteristic features. In four dimensions there is less flexibility than in five dimensions; so the characteristic two-body features are only significant in regimes that involve restrictions on the charges.

Eventually we would like to have a microscopic description of all processes, at least in principle. However, at this point it seems more rewarding to consider specific processes that go beyond the S-wave of a minimally coupled scalar field \[39, 17, 18, 27, 28\]. Higher partial waves are particularly interesting because, for sufficiently high partial wave number (eq. 36), their absorption cross-section can be calculated in the classical theory for all black holes and arbitrary frequencies\[8\]; and the result has a

\[7\]We ignore the Coulomb-type factor. Note that in regimes where it gives an enhancement of the absorption cross-section it gives a reduction of the emission rate. The naive application of detailed balance that leads to eq. 43 is modified by the presence of long range interactions.

\[8\]We ignore backreaction on the geometry and ultimately this sets on both frequency and partial
form that is characteristic of string theory\textsuperscript{9}. The corresponding emission rate can be written\textsuperscript{10}:

\[
\Gamma_{\text{em}}^{(n)}(\omega) = \left(\frac{8G_N\mathcal{L}\omega}{2\pi}\right)^{2n+1}(2n+1)\left(\frac{(2\pi)^2(n!)^2}{8(2n)!^2(2n+1)!}\right) \frac{\prod_{j=1}^{n}[(\frac{\omega}{2})^2 + (\frac{2\pi j}{\beta})^2]}{(e^{\beta\frac{\omega}{2}} - 1)} \times
\]

\[
\times \frac{\prod_{j=1}^{n}[(\frac{\omega}{2})^2 + (\frac{2\pi j}{\beta_R})^2]}{(e^{\beta_R \frac{\omega}{2}} - 1)} \times \frac{\prod_{j=1}^{n}[(\frac{\omega}{2})^2 + (\frac{2\pi j}{\beta_L})^2]}{(e^{\beta_L \frac{\omega}{2}} - 1)} \times \frac{d^3k}{(2\pi)^3}.
\]

(46)

This expression depends only on quantities that have a microscopic interpretation: \(\beta_{R,L}\) are the inverse temperatures of the right and left moving string excitations, \(\mathcal{L}\) is the length of the effective string, and Newton’s coupling constant \(G_N\) is the \(U\)-duality invariant form of the string coupling. The angular momentum of the black hole background enters only through \(\beta_R\). This is expected from a microscopic point of view, because here the introduction of angular momentum is implemented as a projection acting on the Hilbert space of the right movers. (In five dimensions there are two angular momenta and they enter through \(\beta_R\) and \(\beta_L\), respectively.)

In an effective string theory description the emission of a partial wave with an angular momentum \(n\) is dominated by an operator that has dimension \(n + 1\), both in the right and left moving sectors. It can be realized as a composite operator of \(n + 1\) free boson fields (each boson can be traded for 2 fermions). It is simplest to calculate the thermal phase space factors of the initial string state in the bosonic realization:

\[
I^{(n+1)}(\beta) = \int_{-\infty}^{\infty} \frac{\omega}{2} - \sum_{i=0}^{n} p_i^2 dp^i \prod_{i=0}^{n} e^{p_i^2/\beta} - 1 = \frac{1}{(2n+1)!} \frac{\omega}{2} \prod_{i=1}^{n}[(\frac{\omega}{2})^2 + (\frac{2\pi i}{\beta})^2] (e^{\beta\frac{\omega}{2}} - 1) \times \frac{d^3k}{(2\pi)^3}.
\]

(47)

(Useful relations are given in \textsuperscript{28}. ) The final state is the \(n\)th partial wave of a minimally coupled scalar field. Microscopically this corresponds a vertex operator of a scalar field with \(n\) spacetime derivatives; so this gives a further frequency dependence \(\omega^n\) in the amplitude, and therefore \(\omega^{2n}\) in the rate. Finally, the normalization of the outgoing state gives a factor \(\omega^{-1}\). Collecting the factors, we arrive at a microscopic interpretation of the frequency dependence of the emission rate eq. 46.

The dependence on coupling constant can be understood as follows: in the bosonic representation there are \(n + 1\) operators on both the right and left sides, and there

\textsuperscript{9}These statements are also true in the five dimensional case.

\textsuperscript{10}As before we suppress the Coulomb-type factor due to long range fields.
is one outgoing state. The sphere amplitude with these \(2n + 3\) vertices has a factor of \(g^{-2}g^{2n+3} = g^{2n+1}\); so the rate has a factor of \(g^{4n+2} \propto G_{N}^{2n+1}\). Moreover, there is a suppressed factor of \(\mathcal{L}\) in the measure of each of the phase space integrals above, and the outgoing particle accounts for a factor of \(\mathcal{L}^{-1}\); so the dependence on the effective string length becomes \(\mathcal{L}^{2n+1}\). Dimensional analysis serves as a check on these arguments. Combining these results we account for eq. [16], up to the numerical prefactor.

It would clearly be desirable to understand these arguments in the framework of a detailed microscopic model. Specifically the numerical prefactor of the emission rate should be calculable. However, already in its present form the analysis indicates a striking connection between the classical result and an effective string theory.

It is apparent from the preceding discussion that the emission rate eq. [16] is closely related to similar ones that appear in the context of near-extremal black holes. An effort to understand these cases microscopically, including prefactor, is far advanced and our discussion is adapted from this context [18, 27, 28]. However, our result generalizes these limiting cases: (i) it treats all four charges independently, and no hierarchy between them is assumed, (ii) it is not limited to near-extremal black holes, (iii) it includes a background angular momentum, and (iv) there is no low energy requirement. The expression eq. [16] is nevertheless of the form that has been considered previously. It is reasonable to assume that an effective string model can be devised that accounts for the details of the limiting cases; and then our classical result seems to indicate that the model will automatically apply also in general.

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References

\footnote{An important issue in these calculations is the dependence on the volume of the compact space. Our approach is manifestly duality invariant; so it supports the resolution proposed in [27].}
[1] A. Strominger and C. Vafa. Microscopic origin of the Bekenstein-Hawking entropy. *Phys. Lett. B*, 379:99–104, 1996. [hep-th/9601029](https://arxiv.org/abs/hep-th/9601029).

[2] C. Callan and J. Maldacena. D-brane approach to black hole quantum mechanics. *Nucl. Phys. B*, 472:591–610, 1996. [hep-th/9602043](https://arxiv.org/abs/hep-th/9602043).

[3] A. Sen. Extremal black holes and elementary string states. *Mod. Phys. Lett. A*, 10:2081, 1995. [hep-th/9504147](https://arxiv.org/abs/hep-th/9504147).

[4] F. Larsen and F. Wilczek. Internal structure of black holes. *Phys. Lett. B*, 375:37–42, 1996. [hep-th/9511064](https://arxiv.org/abs/hep-th/9511064).

[5] M. Cvetič and A. Tseytlin. Solitonic strings and BPS saturated dyonic black holes. *Phys. Rev. D*, 53:5619–5633, 1996. [hep-th/9512031](https://arxiv.org/abs/hep-th/9512031).

[6] S. Das and S. Mathur. Comparing decay rates for black holes and D-branes. *Nucl. Phys. B*, 478:561–576, 1996. [hep-th/9606185](https://arxiv.org/abs/hep-th/9606185).

[7] J. Maldacena and A. Strominger. Black hole grey body factors and D-brane spectroscopy. *Phys. Rev. D*, 55:861–870, 1996. [hep-th/9609026](https://arxiv.org/abs/hep-th/9609026).

[8] M. Cvetič and F. Larsen. General rotating black holes in string theory: Greybody factors and event horizons. [hep-th/9705192](https://arxiv.org/abs/hep-th/9705192).

[9] M. Cvetič. Properties of black holes in toroidally compactified string theory. [hep-th/9701152](https://arxiv.org/abs/hep-th/9701152).

[10] F. Larsen. A string model of black hole microstates. [hep-th/9702158](https://arxiv.org/abs/hep-th/9702158), to appear in *Phys. Rev. D*.

[11] G. T. Horowitz, J. M. Maldacena, and A. Strominger. Nonextremal black hole microstates and U-duality. *Phys. Lett. B*, 383:151–159, 1996. [hep-th/9603109](https://arxiv.org/abs/hep-th/9603109).

[12] C.V. Johnson, R.R. Khuri, and R.C. Myers. Entropy of 4D extremal black holes. *Phys. Lett. B*, 378:78–86, 1996. [hep-th/9603061](https://arxiv.org/abs/hep-th/9603061).

[13] I. Klebanov and A. Tseytlin. Intersecting M-branes as four-dimensional black holes. *Nucl. Phys.B*, 475:179–192, 1996. [hep-th/9604166](https://arxiv.org/abs/hep-th/9604166).
[14] V. Balasubramanian and F. Larsen. On D-branes and black holes in four dimensions. *Nucl. Phys.B*, 478:199, 1996. [hep-th/9604189](http://arxiv.org/abs/hep-th/9604189).

[15] R. Dijkgraaf, E. Verlinde, and H. Verlinde. Counting dyons in N=4 string theory. *Nucl. Phys. B*, 484:543–561, 1997. [hep-th/9607026](http://arxiv.org/abs/hep-th/9607026).

[16] S. S. Gubser and I. R. Klebanov. Four-dimensional greybody factors and the effective string. *Phys. Rev. Lett.*, 77:4491–4494, 1996. [hep-th/9609076](http://arxiv.org/abs/hep-th/9609076).

[17] I. R. Klebanov and S. Mathur. Black hole grey body factors and absorption of scalars by effective strings. [hep-th/9701187](http://arxiv.org/abs/hep-th/9701187).

[18] J. Maldacena and A. Strominger. Universal low energy dynamics for rotating black holes. [hep-th/9702015](http://arxiv.org/abs/hep-th/9702015).

[19] M. Cvetič and C. Hull. Black holes and U-duality. *Nucl. Phys. B*, 480:296–316, 1996. [hep-th/9606193](http://arxiv.org/abs/hep-th/9606193).

[20] M. Cvetič and D. Youm. Entropy of non-extreme charged rotating black holes in string theory. *Phys. Rev. D*, 54:2612–2620, 1996. [hep-th/9603147](http://arxiv.org/abs/hep-th/9603147).

[21] M. Cvetič and D. Youm. BPS saturated and nonextreme states in abelian Kaluza-Klein theory and effective N=4 supersymmetric vacua. [hep-th/9508058](http://arxiv.org/abs/hep-th/9508058); (talk presented at STRINGS ’95).

[22] G. T. Horowitz, D. A. Lowe, and J. M. Maldacena. Nonextremal black hole microstates and U-duality. *Phys. Rev. Lett.*, 77:430–433, 1996. [hep-th/9603193](http://arxiv.org/abs/hep-th/9603193).

[23] S. S. Gubser and I. R. Klebanov. Emission of charged particles from four-dimensional and five-dimensional black holes. *Nucl. Phys. B*, 482:173–186, 1996. [hep-th/9608108](http://arxiv.org/abs/hep-th/9608108).

[24] F. Dowker, D. Kastor, and J. Traschen. U-duality, D-branes and black hole emission rates: agreements and disagreements. [hep-th/9702109](http://arxiv.org/abs/hep-th/9702109).

[25] S. W. Hawking and M. M. Taylor-Robinson. Evolution of near extremal black holes. [hep-th/9702043](http://arxiv.org/abs/hep-th/9702043), to appear in Phys. Rev. D.
[26] M. Abramowitz and A. Stegun. *Handbook of Mathematical Functions*. Dover, 1965.

[27] S. Mathur. Absorption of angular momentum by black holes and D-branes. hep-th/9704156.

[28] S. Gubser. Can the effective string see higher partial waves? hep-th/9704195.

[29] S. Das, G. Gibbons, and S. Mathur. Universality of low-energy absorption cross-sections for black holes. *Phys. Rev. Lett.*, 78:417–419, 1997. hep-th/9609052.

[30] C. G. Callan, S. S. Gubser, I. R. Klebanov, and A. A. Tseytlin. Absorption of fixed scalars and the D-brane approach to black holes. *Nucl. Phys. B*, 489:65–94, 1997. hep-th/9610172.