THE REAL \textsc{3x + 1} PROBLEM

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1. Introduction

First of all, let’s fix some notations.

As usual, \(\mathbb{R}\), \(\mathbb{Q}\) and \(\mathbb{Z}\) will denote the sets of all real, rational and integer numbers, respectively.

Put \(\mathbb{R}_1 = \{x \in \mathbb{R} : x \geq 1\}\), \(\mathbb{Q}_1 = \mathbb{Q} \cap \mathbb{R}_1\) and \(\mathbb{N}_1 = \mathbb{Z} \cap \mathbb{R}_1 = \{1, 2, 3, \ldots\}\).

Put \(\mathbb{R}_0 = \{x \in \mathbb{R} : x \geq 0\}\), \(\mathbb{Q}_0 = \mathbb{Q} \cap \mathbb{R}_0\) and \(\mathbb{N}_0 = \mathbb{Z} \cap \mathbb{R}_0 = \{0, 1, 2, \ldots\}\).

For \(x \in \mathbb{R}\), \(\lfloor x \rfloor\) will denote the floor or integer part of \(x\), that is to say, \(\lfloor x \rfloor = \max \{k \in \mathbb{Z} : k \leq x\}\).

The well-known \(3n + 1\) function (see, e.g., [8] and [10]) is the function \(T : \mathbb{N}_1 \rightarrow \mathbb{N}_1\) given by

\[
T(n) = \begin{cases} 
T_0(n) = \frac{n}{2} & \text{if } n \text{ is even}, \\
T_1(n) = \frac{3n+1}{2} & \text{if } n \text{ is odd}.
\end{cases}
\]

In this work, we introduce another extension of \(T\), namely the function \(U : \mathbb{R}_1 \rightarrow \mathbb{R}_1\) defined by

\[
U(x) = \begin{cases} 
U_0(x) = \frac{x}{2} & \text{if } \lfloor x \rfloor \text{ is even}, \\
U_1(x) = \frac{3x+1}{2} & \text{if } \lfloor x \rfloor \text{ is odd}.
\end{cases}
\]

Note that \(U|_{\mathbb{N}_1}\) (the restriction of \(U\) to \(\mathbb{N}_1\)) is indeed \(T\). We shall call \(U\) the \textit{real} \(3x + 1\) function (in contrast to the \textit{integer} \(3n + 1\) function \(T\)). In Section 2, we’ll propose a conjecture about the iterates of \(U\) that generalizes the famous \(3n + 1\) conjecture. We’ll then prove our main result about the iterates of \(U\) (Theorem 2.1), which is directly related to both of these conjectures. We’ll also introduce the \textit{flipped} \(3x + 1\) function \(\tilde{U}\) and prove an analogous result for its iterates. In Section 3, we’ll show a couple of simple propositions about the iterates of \(U\) and \(\tilde{U}\), introduce other related functions and propose some questions and conjectures about their iterates.

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2. The Conjecture and the Main Results

Given a (nonempty) set $X$ and a function $f : X \to X$, the iterates of $f$ will be denoted by $f^i$ ($i \in \mathbb{N}_0$). They are defined by $f^0 = \text{id}_X$ (the identity function on $X$) and by $f^i = f \circ f^{i-1}$, for $i > 0$. For any given $x \in X$, the $f$-trajectory of $x$ or starting at $x$ is the sequence $T_f(x) = (f^i(x))_{i=0}^\infty$. An $f$-periodic trajectory or, simply, an $f$-cycle is the $f$-trajectory of some $z \in X$ such that $f^n(z) = z$, for some $n \in \mathbb{N}_1$ (in this case, the $f$-cycles starting at $f^k(z)$, $k \in \mathbb{N}_0$, will sometimes be considered as being one and the same $f$-cycle). By an $f$-cycle of length $l \in \mathbb{N}_1$ we mean any sequence in the set \{$(x, f(x), \ldots, f^l(x)) : x \in X$, $f^l(x) = x$\}.

Now, let $\mathbb{Q}[(2)]$ denote the set of all rational numbers having an odd denominator when written in lowest terms (see [3]). A number $a/b \in \mathbb{Q}[(2)]$ (with an odd $b$) is even (odd) if its numerator $a$ is even (odd). The rational Collatz sequence generated by $r_0 \in \mathbb{Q}[(2)]$ is the $g$-trajectory of $r_0$, where $g : \mathbb{Q}[(2)] \to \mathbb{Q}[(2)]$ is given by $g(r) = g_0(r) = r/2$, if $r$ is even, and $g(r) = g_1(r) = (3r+1)/2$, if $r$ is odd. A rational Collatz cycle (of length $l$) is simply a $g$-cycle (of length $l$). Given $l \in \mathbb{N}_1$ and $n \in \mathbb{N}_0$, let $S_{l,n}$ be the set of all 0-1 sequences of length $l$ containing exactly $n$ 1’s, and put $S_l = \bigcup_{n=0}^l S_{l,n}$ and $S = \bigcup_{l=1}^\infty S_l$. If $s \in S$, we’ll denote the number of 1’s in $s$ by $n(s)$ and the length of $s$ by $l(s)$. Given $s = (s_1, s_2, \ldots, s_l) \in S$, define $\phi_s : \mathbb{R} \to \mathbb{R}$ by $\phi_s = g_{s_1} \circ \cdots \circ g_{s_{l-1}} \circ g_{s_l}$. A sequence $(x_0, x_1, \ldots, x_l)$ of numbers $x_i \in \mathbb{R}$ is called a pseudo-cycle of length $l$ if there exists $s = (s_1, s_2, \ldots, s_l) \in S$ such that $x_i = x_0$ and $x_{i} = g_{s_i}(x_{i-1})$, for $i = 1, 2, \ldots, l$ (note that $x_l = \phi_s(x_0)$).

Finally, define $\varphi : S \to \mathbb{N}_0$ by $\varphi(s) = \sum_{j=1}^{l(s)} s_j 2^{j-1}3^{s_{j+1}+s_{j+2}+\cdots+s_{l(s)}}$.

Let $x_0 \in \mathbb{R}_1$ be given. If $\{\lim_{k \to \infty} U^{2k}(x_0), \lim_{k \to \infty} U^{2k+1}(x_0)\} = \{1, 2\}$, then we’ll say that its $U$-trajectory $T_U(x_0)$ tends to $\{1, 2\}$ and this will be denoted by $T_U(x_0) \to \{1, 2\}$. Our real $3x + 1$ conjecture is

RU: For all $x \in \mathbb{R}_1$, $T_U(x) \to \{1, 2\}$.

Note that, for all $n \in \mathbb{N}_1$, $T_U(n) = T_U(n)$. The famous (integer) $3n + 1$ conjecture may then be stated as
NU: For all \( n \in \mathbb{N}_1 \), \( T_U(n) \to \{1, 2\} \).

One could also state both of these conjectures in terms of the \( U \)-parity sequence associated with \( x \in \mathbb{R}_1 \), which is simply the infinite 0-1 sequence \( P_U(x) = (\lfloor U^i(x) \rfloor \mod 2)_{i=0}^\infty \). Note that this sequence encodes which branch of \( U \) (\( U_0 \) or \( U_1 \)) is used in each step of \( T_U(x) \). Now, an infinite 0-1 sequence \( (p_i)_{i=0}^\infty \) will be called eventually periodic with period \((0, 1)\) if there exists \( j \in \mathbb{N}_0 \) such that \( (p_i, p_{i+1}) = (0, 1) \), for all \( i = j + 2m \), \( m \in \mathbb{N}_0 \). It’s a simple matter (see Proposition 3.1) to show that, for each \( x \in \mathbb{R}_1 \), \( P_U(x) \) is eventually periodic with period \((0, 1)\), if, and only if, \( T_U(x) \to \{1, 2\} \).

In other words, the conjectures RU and NU above can be stated in the following alternative, equivalent forms.

RU: For all \( x \in \mathbb{R}_1 \), \( P_U(x) \) is eventually periodic with period \((0, 1)\).

NU: For all \( n \in \mathbb{N}_1 \), \( P_U(n) \) is eventually periodic with period \((0, 1)\).

Now, we observe that our RU conjecture clearly implies both of the following two conjectures.

OU: The only \( U \)-cycle is the trivial \( T \)-cycle \((1, 2, 1, 2, 1, \ldots)\).

BU: Every \( U \)-trajectory is bounded.

Of course, all \( T \)-cycles are \( U \)-cycles, and one would naturally expect to find (many) more \( U \)-cycles than \( T \)-cycles. However, our main result, which is directly related to the conjectures RU and OU above, tells us that in fact quite the opposite happens.

**Theorem 2.1.** The only \( U \)-cycles are the \( T \)-cycles.

**Proof.** Let’s first state next two lemmas that will be used in this and subsequent proofs. The reader may find their proofs in [5] and [7] (the basic idea of most of Lemma 2.2 below is due originally to Böhm and Sontacchi [1]).

**Lemma 2.2.** (Böhm and Sontacchi, Lagarias, Halbeisen and Hungerbühler). A sequence \( (x_0, x_1, \ldots, x_l) \) is a rational Collatz cycle of length \( l \) if, and only if, it is a pseudo-cycle of length \( l \). Moreover, if a rational Collatz cycle is not the cycle \((0, 0, \ldots)\), then it’s elements are either all strictly positive or all strictly negative.

**Lemma 2.3.** (Lagarias). For any \( s \in S \) and any \( x \in \mathbb{R} \), we have that

\[
\phi_s(x) = \frac{3^{n(s)}x + \varphi(s)}{2^{l(s)}}.
\]
Therefore, given \( s \in S \),
\[
x_0(s) = \frac{\varphi(s)}{2^{l(s)} - 3^{n(s)}} \in \mathbb{Q} \[(2)\]
\]

is the unique number that generates the rational Collatz cycle of length \( l(s) \) that is also the pseudo-cycle of length \( l(s) \) determined by \( s \). \( \square \)

To begin with, we note that all \( U \)-cycles start at numbers in \( \mathbb{Q}_1 \), since, for each \( k \in \mathbb{N}_1 \), we have that every solution of \( x = U^k(x) \) is rational. Let’s suppose then that there exist \( x_0 \in \mathbb{Q}_1 \setminus \mathbb{N}_1 \) and \( l \in \mathbb{N}_1 \) such that there’s a \( U \)-cycle of length \( l \) starting at \( x_0 \), namely \( \Omega(x_0) = (x_0, U(x_0), \ldots, U^l(x_0) = x_0) \). If we derive a contradiction from this hypothesis, then we’ll be done. Note that it’s immediate (by inspection) that the only \( U \)-cycle of length less than 4 is the \( T \)-cycle (1, 2, 1). Hence, without loss of generality, we may assume that \( l \geq 4 \), which avoids our having to treat some trivial cases separately in what follows. Now, since \( U_i \equiv g_i \) (\( i = 0, 1 \)), \( \Omega(x_0) \) is a pseudo-cycle of length \( l \). Thus, by Lemma 2.2, \( \Omega(x_0) \) is a rational Collatz cycle of length \( l \) as well. Therefore, by using Lemma 2.3 and the fact that \( U|\mathbb{N}_1 = T \), one obtains both that all \( U^i(x_0) \in \mathbb{Q} \[(2)\] \cap \mathbb{Q}_1 \setminus \mathbb{N}_1 \) and that
\[
x_0 = x_0(s) = \frac{\varphi(s)}{2^{l(s)} - 3^{n(s)}},
\]
where \( s = (s_1, s_2, \ldots, s_l) \in S \) is the 0-1 sequence associated with (the pseudo-cycle) \( \Omega(x_0) \), i.e., \( s \) consists of the first \( l = l(s) \) terms in \( \mathcal{P}_U(x_0) \). For convenience, put \( n = n(s) \) and \( d = 2^l - 3^n \). Now, given any \( a/b \) in \( \mathbb{Q} \[(2)\] \) (with an odd \( b \)), it’s clear that every term in the rational Collatz sequence generated by \( a/b \) may be written with denominator \( b \). As \( d \) happens to be odd, one may, for \( i = 0, 1, \ldots, l \), write that
\[
x_i = U^i(x_0) = \frac{c_i}{d} = \frac{q_i d + r_i}{d} = q_i + \frac{r_i}{d},
\]
where \( q_i \) is the quotient and \( r_i \) the remainder in the Euclidean division of \( c_i \) by \( d \). Note that all \( c_i, q_i \) and \( r_i \) lie in \( \mathbb{N}_1 \) and that \( d \geq 5 \) (for all \( x_i = c_i/d \) are in \( \mathbb{Q}_1 \setminus \mathbb{N}_1 \), \( \varphi(s) > 0 \) and 3 doesn’t divide \( d \) ). In particular, no \( r_i \) is 0, and so all \( r_i \) satisfy \( 0 < r_i < d \). Moreover, because \( d = 2^l - 3^n > 0 \), one has
\[
n < l \log_3 2.
\]
Now, since \( \Omega(x_0) = (x_0, x_1, \ldots, x_l = x_0) \) is both a \( U \)-cycle and a rational Collatz cycle (of length \( l \)), we have, for \( i = 0, 1, \ldots, l \), that \( q_i = [x_i] \) is even (odd) if, and only if, \( c_i \) is even (odd). Thus, all \( r_i = c_i - dq_i \) are even. Write \( r_i = 2^{e_i} a_i \), where \( e_i \geq 1 \) and \( a_i \) is odd, think of \( r_0, r_1, \ldots, r_l = r_0 \) as being arranged (in this order) in a circular manner and observe that, for
\[ i = 0, 1, \ldots, l, \]

\[
\begin{cases}
\frac{1}{2} r_{i-1} & \text{if } q_{i-1} \text{ is even}, \\
\frac{3}{2} r_{i-1} & \text{if } q_{i-1} \text{ is odd and } r_{i-1} < \frac{2}{3} d, \\
\frac{3}{2} r_{i-1} - d & \text{if } q_{i-1} \text{ is odd and } r_{i-1} > \frac{2}{3} d.
\end{cases}
\]  

(8)

Note that, since 3 doesn’t divide \( d \), it’s never the case that \( r_{i-1} = 2d/3 \) in \( (8) \). As usual, indices are to be considered modulo \( l \) whenever it’s the case to do so. Now, if \( r_i \) is such that \( r_i = 3r_{i-1}/2 - d \), then we’ll say that this \( r_i \) is new. Note that, if \( r_{j+1} \) is not new, we have that either \( r_{j+1} = 2e_{j-1}o_j \) or \( r_{j+1} = 2^{e_{j-1}}(3o_j) \). This clearly means that at least one of \( r_0, r_1, \ldots, r_{l-1} \) is new. By renaming the \( x_i \)’s if necessary, we can assume that \( r_0 (= r_l) \) is new. Now, let \( 0 \leq p < q \leq l \) be such that \( r_p \) and \( r_q \) are consecutive new, that is, both \( r_p \) and \( r_q \) are new and, for all \( p < k < q \), \( r_k \) is not new (if \( r_0 \) is the only new one, then put \( p = 0 \) and \( q = l \)). Because there’re no new \( r_k \)’s strictly between \( r_p \) and \( r_q \), one has

\[
r_{q-1} = 2 \left( 3^{n(p, q-1)}o_p \right) \quad \text{and} \quad e_p = q - p,
\]

where, for any \( 0 \leq i \leq j \leq l \), \( n(i, j) \) is the number of times \( U_i \) is used from \( x_i \) to \( x_j \), i.e., \( n(i, j) \) is the number of 1’s in \( \{s_{i+1}, s_{i+2}, \ldots, s_j\} \). Since \( r_q \) is new, we have both that \( n(p, q) = n(p, q-1) + 1 \) and that \( d < 3r_{q-1}/2 \). From this and (9), it follows that

\[
d < 3^{n(p, q-1)+1}o_p = \frac{3^{n(p, q-1)+1}}{2e_p}r_p = \frac{3^{n(p, q)}}{2q-p}r_p.
\]

(10)

Now, because \( r_p \) is new, \( r_p = 3r_{p-1}/2 - d \), and so, since \( 0 < r_{p-1} < d \), we obtain \( r_p < d/2 < 2d/3 \). From this and (10), one gets

\[
d < \frac{3^{n(p, q)-1}}{2q-p-1} \quad \Rightarrow \quad 3^{n(p, q)-1} > 2^{q-p-1} \quad \Rightarrow \quad n(p, q) > \log_3 2^{q-p-1} + 1.
\]

(11)

Therefore, \( n(p, q) > \log_3 2^{q-p-1} + \log_3 2 = \log_3 2^{q-p} \), and so one has that

\[
n(p, q) > (q - p) \log_3 2.
\]

(12)

Now, let \( 0 = i_0 < i_1 < \cdots < i_m = l \), \( m \geq 1 \), be such that \( r_{i_0}, r_{i_1}, \ldots, r_{i_m} \) are all the new \( r_i \)’s in \( \{r_0, r_1, \ldots, r_l\} \). We have that \( n = \sum_{k=1}^m n(i_{k-1}, i_k) \), \( l = \sum_{k=1}^m (i_k - i_{k-1}) \) and that \( r_{i_{k-1}} \) and \( r_{i_k} \) are consecutive new for all \( k = 1, 2, \ldots, m \). Consequently, inequality (12) gives us \( n > l \log_3 2 \), but this last inequality contradicts inequality (7).

We note that some authors have already investigated a variety of interesting smooth extensions of \( T \) to the real (and even complex) numbers (see,
e.g., [2], [3], [4], [6] and [9]). Unlike the conjectured case of \( U \), however, the dynamics of these extensions outside the integers are always extraneous to the \( 3x + 1 \) conjecture (i.e., there exist periodic and divergent trajectories).

Now, the previous theorem illustrated the relative ease one has in obtaining some results if he is allowed the freedom to work in \( \mathbb{R} \) (instead of his having to concentrate solely on \( \mathbb{N} \)). For another example along these lines, consider the flipped \( 3x + 1 \) function \( \tilde{U} : \mathbb{R}_0 \to \mathbb{R}_0 \) defined by

\[
\tilde{U}(x) = \begin{cases} 
\tilde{U}_0(x) = U_1(x) & \text{if } \lfloor x \rfloor \text{ is even}, \\
\tilde{U}_1(x) = U_0(x) & \text{if } \lfloor x \rfloor \text{ is odd}.
\end{cases}
\]

Clearly, \( \tilde{U}|_{\mathbb{N}_0} \) is not a function from \( \mathbb{N}_0 \) to \( \mathbb{N}_0 \). Naturally, one would like to know what happens to the \( \tilde{U} \)-trajectories. In particular, one would try to obtain all \( \tilde{U} \)-cycles. This is in fact done in our next theorem, which is a bonus result we’ve gotten from the method we’ve used to prove Theorem 2.1.

**Theorem 2.4.** There are no \( \tilde{U} \)-cycles.

*Proof.* The proof is almost entirely analogous to the proof of Theorem 2.1 above, and so we’ll be brief and point out only the required modifications. Clearly, no \( \tilde{U} \)-cycles start at numbers in \( \mathbb{N}_0 \). Let’s suppose then that there exist \( x_0 \in \mathbb{Q}_0 \setminus \mathbb{N}_0 \) and \( l \in \mathbb{N}_1 \) such that there’s a \( \tilde{U} \)-cycle of length \( l \) starting at \( x_0 \), namely \( \tilde{\Omega}(x_0) = (x_0, \tilde{U}(x_0), \ldots, \tilde{U}^l(x_0) = x_0) \). If this assumption leads us to a contradiction, then we’ll be done. By inspection, there’re no \( \tilde{U} \)-cycles of length less than 4, and so we may, without loss of generality, assume that \( l \geq 4 \) (again, this assumption is made so as to avoid trivialities in what follows). Now, with similar notations and the same arguments from the proof of Theorem 2.1, one obtains, for \( i = 0, 1, \ldots, l \), that

\[
x_i = \tilde{U}^i(x_0) = \frac{c_i}{d} = \frac{qd_i + r_i}{d} = q_i + \frac{r_i}{d} = (q_i + 1) - \frac{d - r_i}{d},
\]

where \( q_i \) is the quotient and \( r_i \) the remainder in the Euclidean division of \( c_i \) by \( d \). Since no \( x_i \)'s belong to \( \mathbb{N}_0 \), we have that all \( r_i \) satisfy \( 0 < d - r_i < d \). Moreover, because \( d = 2^l - 3^n > 0 \), we have, as before, that

\[
n < l \log_2 3.
\]

Since \( \tilde{\Omega}(x_0) = (x_0, x_1, \ldots, x_l = x_0) \) is both a \( \tilde{U} \)-cycle and a rational Collatz cycle (of length \( l \)), it follows, for \( i = 0, 1, \ldots, l \), that \( q_i = \lfloor x_i \rfloor \) is even (odd) if, and only if, \( c_i \) is odd (even). Thus, all \( r_i = c_i - dq_i \) are odd, i.e., all \( d - r_i \) are even. Now, think of \( d - r_0, d - r_1, \ldots, d - r_l = d - r_0 \) as being arranged...
(in this order) in a circular fashion and note that, for \( i = 0, 1, \ldots, l, \)
\[
(16) \quad d - r_i = \begin{cases} 
\frac{1}{2} (d - r_{i-1}) & \text{if } q_{i-1} \text{ is odd,} \\
\frac{3}{2} (d - r_{i-1}) & \text{if } q_{i-1} \text{ is even and } d - r_{i-1} < \frac{2}{3} d, \\
\frac{3}{2} (d - r_{i-1}) - d & \text{if } q_{i-1} \text{ is even and } d - r_{i-1} > \frac{2}{3} d.
\end{cases}
\]

Now, arguing exactly in the same way as we’ve done in the proof of Theorem 2.1, we conclude that \( n > l \log_3 2, \) which contradicts (15). \( \square \)

Note that yet another equivalent way of phrasing the conjecture \( RU \) is to say that, for every \( x \in \mathbb{R}_1, \) there exists \( k \in \mathbb{N}_0 \) such that \( U^k(x) \in [1, 3). \)

Our corresponding conjecture for the iterates of \( \tilde{U} \) is

\( R\tilde{U}: \) For every \( x \in \mathbb{R}_0 \) there exists \( k \in \mathbb{N}_0 \) such that \( \tilde{U}^k(x) \in [0, 2). \)

Of course, Theorem 2.4 is directly related to the conjecture \( R\tilde{U} \) above.

Let’s conclude this section by observing that our \( R\tilde{U} \) conjecture clearly implies the following conjecture.

\( B\tilde{U}: \) Every \( \tilde{U} \)-trajectory is bounded.

3. Other Results, Conjectures and Questions

One way to find out if studying what happens to the iterates of \( U \) can shed some new light on the \( 3n + 1 \) conjecture or not would be to try and answer our first question.

\( Q1: \) Does the \( 3n + 1 \) conjecture imply our real \( 3x + 1 \) conjecture \( RU? \)

On one hand, if the answer to this question is \textit{yes}, then this would show that looking at the iterates of \( \tilde{U} \) amounts to essentially the same thing as looking at those of \( T \) (as far as the \( 3n + 1 \) conjecture is concerned). On the other hand, we note that, if the \( 3n + 1 \) conjecture is true, then the answer to the question \( Q1 \) above could very well be \textit{no}. To see how this might be so, suppose that, instead of \( T, \) one considered the original Collatz function, i.e., the function \( f : \mathbb{N}_1 \to \mathbb{N}_1 \) given by

\[
(17) \quad f(n) = \begin{cases} 
f_0(n) = \frac{n}{2} & \text{if } n \text{ is even,} \\
f_1(n) = 3n + 1 & \text{if } n \text{ is odd.}
\end{cases}
\]
Its extension to \( \mathbb{R}_1 \) (in our sense) is the function \( F : \mathbb{R}_1 \to \mathbb{R}_1 \) given by

\[
F(x) = \begin{cases} 
    \frac{x}{2} & \text{if } \lfloor x \rfloor \text{ is even}, \\
    3x + 1 & \text{if } \lfloor x \rfloor \text{ is odd.}
\end{cases}
\]

(18)

The statement for the \( F \)-trajectories which corresponds to the conjecture \( RU \) would be the claim that, for all \( x \in \mathbb{R}_1 \), \( \mathcal{T}_F(x) \to \{1, 2\} \). However, this is readily seen to be false, since one has, for example, that all \( F \)-trajectories starting at \( 2m + 3/2, m \in \mathbb{N}_0 \), diverge (monotonically) to \( +\infty \). Now, the \( 3n + 1 \) conjecture for the iterates of \( T \) is equivalent to the (same) one for the iterates of \( f \). Thus, if the \( 3n + 1 \) conjecture turns out to be true, then the question for the \( F \)-trajectories that is the counterpart to question \( Q1 \) will have a negative answer. Moreover, if our real \( 3x + 1 \) conjecture \( RU \) is true, then the \( U \)-trajectories and the \( F \)-trajectories will be seen to have quite different behaviors in \( \mathbb{R}_1 \) (as opposed to what happens in \( \mathbb{N}_1 \)). In our view, comparisons between the \( U \)-trajectories and the \( F \)-trajectories may play an important rôle in some future \( 3x + 1 \)-type investigations. Let our next question emphasize this point.

\[Q2: \text{Are the } F \text{-trajectories starting at } 2m + 3/2, m = 0, 1, 2, \ldots, \text{ the only } F \text{-trajectories that do not tend to } \{1, 2\}?\]

Of course, analogous questions on similar notions regarding the iterates of \( \tilde{U} \) could be posed as well. We'll now show a simple result about the iterates of \( U \). Its proof will suggest a new approach one might consider in trying to prove the conjecture \( OU \) (see Remark 3.3). A corresponding result for the iterates of \( \tilde{U} \) will be then obtained as a corollary. Before we can state these results, a couple of definitions are needed.

Given \( x_0 \in \mathbb{R}_1 \), we’ll say that \( \mathcal{P}_U(x_0) = (p_i)_{i=0}^{\infty} \) is eventually periodic with period \( s = (s_0, s_1, \ldots, s_{l(s)-1}) \in S \) if there exists \( j \in \mathbb{N}_0 \) such that \( (p_i, p_{i+1}, \ldots, p_{i+l(s)-1}) = (s_0, s_1, \ldots, s_{l(s)-1}) \) for all \( i = j + ml(s), m \in \mathbb{N}_0 \). Moreover, if \( a \in \mathbb{N}_1 \) is such that there’s a \( U \)-cycle of length \( l \) starting at \( a \), then we’ll say that \( \mathcal{T}_U(x_0) \) tends to \( \{U^t(a)\} \) from above (in symbols, \( \mathcal{T}_U(x_0) \uparrow \{U^t(a)\} \)) if there is \( j_0 \in \mathbb{N}_0 \) such that, for all \( j \in \{0, 1, \ldots, l - 1\} \), \( U^{kl}(U^{j_0+j}(x_0)) \to U^j(a)^+, \) as \( k \to +\infty \).

**Proposition 3.1.** If \( a \in \mathbb{N}_1 \) is such that there’s a \( U \)-cycle of length \( l \) starting at \( a \), then, for all \( x \in \mathbb{R}_1 \), we have that \( \mathcal{P}_U(x) \) is eventually periodic with period \( (a \mod 2, U(a) \mod 2, \ldots, U^{l-1}(a) \mod 2) \), if, and only if, \( \mathcal{T}_U(x) \) tends to \( \{U^t(a)\} \) from above.
Proof. Suppose at first that \( x \in \mathbb{R}_1 \) is such that \( T_U(x) \not\rightarrow \{U^l(a)\} \). Now, from the fact that there’s a \( U \)-cycle of length \( l \) starting at \( a \), it clearly follows that there’s some \( 0 < \theta \in \mathbb{R} \) such that, for all \( y \in [a, a + \theta) \) and all \( m \in \mathbb{N}_0 \),

\[
(19) \quad ([U^{ml}(y)], [U^{ml+1}(y)], \ldots, [U^{ml+l-1}(y)]) = (a, U(a), \ldots, U^{l-1}(a))
\]

For instance, any \( 0 < \theta < (2/3)^l \) will do. Since \( T_U(x) \not\rightarrow \{U^l(a)\} \), there’s some \( k_0 \in \mathbb{N}_0 \) such that \( U^{k_0}(x) \in [a, a + \theta) \). Hence, \( P_U(x) \) is eventually periodic with period \( (a \mod 2, U(a) \mod 2, \ldots, U^{l-1}(a) \mod 2) \). For the other direction, suppose now that \( x \in \mathbb{R}_1 \) is such that \( P_U(x) \) is eventually periodic with period \( s = (a \mod 2, U(a) \mod 2, \ldots, U^{l-1}(a) \mod 2) \in S \). By using Lemma 2.3, one sees that there’s some \( j \in \mathbb{N}_0 \) such that

\[
U^{l(s)}(U^{j_0}(x)) = \frac{3^{n(s)}U^{j_0}(x) + \varphi(s)}{2^{l(s)}} = \frac{3^{n(s)}(a + U^{j_0}(x) - a) + \varphi(s)}{2^{l(s)}} = \frac{3^{n(s)}a + \varphi(s)}{2^{l(s)}} + \frac{3^{n(s)}(U^{j_0}(x) - a)}{2^{l(s)}} = a + \frac{3^{n(s)}(U^{j_0}(x) - a)}{2^{l(s)}}.
\]

Analogously, we have, for \( j = 0, 1, \ldots, l - 1 \), that

\[
U^{l(s)}(U^{j+j_0}(x)) = U^j(a) + \frac{3^{n(s)}(U^{j+j_0}(x) - U^j(a))}{2^{l(s)}}.
\]

Therefore, for all \( m \in \mathbb{N}_0 \) and all \( j \in \{0, 1, \ldots, l - 1\} \),

\[
(20) \quad U^{ml(s)}(U^{j+j_0}(x)) = U^j(a) + \left(\frac{3^{n(s)}}{2^{l(s)}}\right)^m(U^{j+j_0}(x) - U^j(a)).
\]

Since \( 3^{n(s)} < 2^{l(s)} \), it’s not hard to conclude now that \( T_U(x) \not\rightarrow \{U^l(a)\} \). \( \square \)

Now, with the appropriate analogous definitions for the iterates of \( \tilde{U} \), the same argument presented in the proof of Proposition 3.1 above gives us the following result as well.

**Proposition 3.2.** If \( a \in \mathbb{N}_1 \) is such that there’s a \( U \)-cycle of length \( l \) starting at \( a \), then, for all \( x \in \mathbb{R}_1 \), we have that \( P_{\tilde{U}}(x) \) is eventually periodic with period \( (1 - a \mod 2, 1 - (U(a) \mod 2), \ldots, 1 - (U^{l-1}(a) \mod 2)) \) if, and only if, \( T_{\tilde{U}}(x) \) tends to \( \{U^l(a)\} \) from below. \( \square \)

Note that if the \( 3n + 1 \) conjecture is true and \( x_0 \in \mathbb{R}_1 \) is such that \( T_U(x_0) \rightarrow a_0^+ \) for some \( a_0 \in \mathbb{N}_1 \), then \( T_U(x_0) \rightarrow \{1, 2\} \). This indicates one way in which one may try and give a positive answer to question Q1.

Now, consider \( \mathcal{I}_U(\mathbb{N}_1) = \{x \in \mathbb{R}_1 : \exists k \in \mathbb{N}_0 \text{ with } U^k(x) \in \mathbb{N}_1\} \) and \( \mathcal{N}_U(\mathbb{N}_1) = \mathbb{R}_1 \setminus \mathcal{I}_U(\mathbb{N}_1) \). Of course, our RU conjecture implies the following conjecture.
\(\mathcal{N}U\): For all \(x \in \mathcal{N}_U(N_1), T_U(x) \to \{1, 2\}\).

We may pose now our next question, which can also be thought of as being one of the possible (non-trivial) ways of turning question \(Q_1\) around.

**Q3:** Does the \(\mathcal{N}U\) conjecture above imply the \(3n + 1\) conjecture?

**Remark 3.3.** Let’s just note here an interesting corollary of the proof of Proposition 3.1 if one proves that, for all \(n \in N_1\) and all \(0 < \rho \in \mathbb{R}\), there exists some \(z \in (n, n + \rho) \cap \mathcal{N}_U(N_1)\) such that \(T_U(z) \to \{1, 2\}\), then it will follow that the \(\mathcal{O}U\) conjecture (which, in light of Theorem 2.1, is in fact the “there are no non-trivial \(T\)-cycles” conjecture) is true.

To try and answer the question \(Q_3\) above might be an even better way of seeing whether there’re some real advantages in shifting one’s attention from \(T\) to \(U\). Let’s end this line of inquiries now by our registering the following very broad (but also potentially very productive) question.

**Q4:** What kind of results for the iterates of \(U\) does one get by attempting to translate known results for the iterates of \(T\)?

In conclusion, let’s just remark that the apparent general project would be for one to study the dynamical system in \(\mathbb{R}\) generated by the iterates of the (discontinuous) piecewise linear functions of the following “simple” kind.

Let \(\alpha, \beta, \gamma, \delta, \tau \in \mathbb{R}\) be fixed, with \(\tau \in [0, 2]\), and consider the function \(\Phi = \Phi(\alpha, \beta, \gamma, \delta, \tau) : \mathbb{R} \to \mathbb{R}\) defined by

\[
\Phi(x) = \begin{cases} 
\Phi_0(x) = \alpha x + \beta & \text{if } \lfloor x + \tau \rfloor \text{ is even}, \\
\Phi_1(x) = \gamma x + \delta & \text{if } \lfloor x + \tau \rfloor \text{ is odd}.
\end{cases}
\]

(21)

Naturally, the crux of the matter here is to find out how the parameters \(\alpha, \beta, \gamma, \delta\) and \(\tau\) affect the behavior of the \(\Phi = \Phi(\alpha, \beta, \gamma, \delta, \tau)\)-trajectories. This brings us to our final (albeit seemingly intractable as of yet!) question.

**Q5:** How do the general properties of the dynamical system in \(\mathbb{R}\) generated by the iterates of the function \(\Phi = \Phi(\alpha, \beta, \gamma, \delta, \tau)\) defined as in (21) depend on the values of the real parameters \(\alpha, \beta, \gamma, \delta\) and \(\tau\)?

E.g., \(U = \Phi(1/2, 0, 3/2, 1/2, 0)|_{\mathbb{R}_1}\) and \(\tilde{U} = \Phi(1/2, 0, 3/2, 1/2, 1)|_{\mathbb{R}_0}\). Note also that the functions \(\Phi(1/2, 0, 3/2, 1/2, \tau_0)|_{\mathbb{R}_1}\), with \(0 \leq \tau_0 < 1\), are all extensions of \(T\). Finally, we bring into attention \(V = \Phi(1/2, 0, 3/2, 0, 0)|_{\mathbb{R}_1}\).
i.e., the function \( V : \mathbb{R}_1 \to \mathbb{R}_1 \) given by

\[
V(x) = \begin{cases} 
0.5x & \text{if } \lfloor x \rfloor \text{ is even,} \\
0.75x & \text{if } \lfloor x \rfloor \text{ is odd.}
\end{cases}
\]

Of course, there’re no \( V \)-cycles. It might be worthwhile for one to try and find out the status of our following final two conjectures, as well as their possible connections to the 3n + 1 and RU conjectures, if any.

**RV:** For every \( x \in \mathbb{R}_1 \) there exists \( k \in \mathbb{N}_0 \) such that \( V^k(x) \in [1, 3) \).

**BV:** Every \( V \)-trajjectory is bounded.

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