COUNTING INTERSECTIONS OF NORMAL CURVES

IVAN DYNNIKOV

Abstract. A fast algorithm for counting intersections of two normal curves on a triangulated surface is proposed. It yields a convenient way for treating mapping class groups of punctured surfaces by presenting mapping classes by matrices, and the composition by an exotic matrix multiplication. An efficient solution of the word problem for mapping class groups of punctured surfaces is proposed, with efficiency understood in a more restrictive way than the most common one.

1. Introduction

Among all finitely generated groups there are those that come with a naturally defined geometry due to their geometric or topological origin. By a geometry here we mean a quasi-isometry class of a left-invariant (or right-invariant) metric on the group. When the efficiency of an algorithm solving some decision problem for such a group is discussed, it is natural to evaluate it in terms of the accompanying geometry, which may be quite different from the word-length geometry used as the default option for abstract finitely generated groups.

Basic examples to look at are the groups $GL(n, \mathbb{Z})$ of integral invertible matrices, in which the ‘natural’ geometry is defined by the distance $d(x, y) = \log \|x^{-1}y\|$, where $\|\|$ stands for an operator norm. The complexity $c(x)$ of an element $x \in GL(n, \mathbb{Z})$ defined by $c(x) = d(1, x)$ is asymptotically comparable to the amount of space needed for recording $x$ in the conventional way. If $x$ is conjugate to a Jordan block, then $c(x^n)$ grows with $n$ as $\log(n)$ whereas the word length of $x^n$ grows as $n$. This means that an algorithm operating with elements of $GL(n, \mathbb{Z})$ which is polynomial-time with respect to the word-length geometry, may appear to be exponential-time in worst cases with respect to the ‘natural’ geometry. Such a divergence is unavoidable if the algorithm uses a presentation of the group elements as decompositions into a product of generators and reads such presentations on input letter by letter.

Mapping class groups $\text{MCG}(M, \mathcal{P})$ of punctured surfaces are similar to $GL(n, \mathbb{Z})$ in this respect. There are several equivalent natural ways to define a geometry on them, and the ‘natural’ complexity of the $n$th power of a Dehn twist grows logarithmically with $n$. A polynomial-time solution to the word problem for these groups with respect to the word-length geometry is given by Lee Mosher in [19], but it is not equally efficient in the worst case with respect to the ‘natural’ geometry, since the algorithm is based on finite state automata.

The present paper proposes a viewpoint on the groups $\text{MCG}(M, \mathcal{P})$ from which these groups appear very much like integral matrix groups, and mimic, in a certain sense, orthogonal groups. The group elements are presented by specific integral $N \times N$ matrices with $N$ depending on the surface and the number of punctures. The geometry on $\text{MCG}(M, \mathcal{P})$ arising from this presentation coincides with the one coming from the action on the thick part of the respective Teichmüller space.

To define the matrix presentation, a triangulation $T$ is fixed on $M$, and the elements of a certain subset $L \subset \mathbb{Z}^N$ are interpreted as multiple curves on the surface encoded by their
normal coordinates with respect to $T$. For any two multiple curves $\gamma_1, \gamma_2$, one defines their geometric intersection index denoted by $\langle \gamma_1, \gamma_2 \rangle$, which becomes a function on $L \times L$ once the triangulation $T$ has been fixed.

The matrices representing elements of $\text{MCG}(M, \mathcal{P})$ are ‘orthogonal’ with respect to $-\langle \cdot, \cdot \rangle$ in the sense that the rows (equivalently, columns) of those matrices form an orthonormal family provided that $-\langle \cdot, \cdot \rangle$ is used instead of the standard scalar product. The $ij$th element of the matrix representing the product $xy$ is equal to $\langle r_i, c_j \rangle$, where $r_i$ is the $i$th row of the matrix representing $x$, and $c_j$ is the $j$th column of the matrix representing $y$. (However, not all integral matrices satisfying the above mentioned ‘orthogonality’ condition represent elements of $\text{MCG}(M, \mathcal{P})$. What they do represent is the set of isotopy classes of all triangulations of $M$ with vertices at $\mathcal{P}$.)

So, one can operate with elements of $\text{MCG}(M, \mathcal{P})$ as efficiently as with those of $\text{GL}(n, \mathbb{Z})$ provided that the geometric intersection index can be computed as efficiently as the standard scalar product. And this is exactly what the technical part of the paper is devoted to—an algorithm for computing the geometric intersection index $\langle \gamma_1, \gamma_2 \rangle$ of two multiple curves $\gamma_1, \gamma_2$ represented by their normal coordinates. The asymptotic running time of the algorithm, for fixed $M$ and $\mathcal{P}$, is $O(|\gamma_1| \cdot |\gamma_2|)$ where $|\gamma|$ stands for a complexity measure comparable to the amount of space needed for writing the normal coordinates of $\gamma$ (see Proposition 10.1).

This is used to construct an algorithm solving the word problem for $\text{MCG}(M, \mathcal{P})$ that accepts as the input zipped words, and whose running time is quadratic in the size of the input provided that the generating set satisfies certain conditions (see Theorem 2.1). By saying ‘zipped’ here we mean that powers of generators $a^k$ are encoded as pairs $(a, k)$ with $k$ written in a positional numeral system, so the contribution of such a term to the size of the input is $O(\log k)$.

However, from the practical point of view, the matrix presentation of $\text{MCG}(M, \mathcal{P})$ has an advantage over the zipped-word presentation in the following respects:

- the asymptotic time for computing the matrix presentations $g_1 g_2$ can be made linear in the size of each of the matrix presentations of $g_1$ and $g_2$ (see Corollary 10.5);
- inverting a group element requires constant time (provided that the result is returned by reference), since it amounts just to transposing the respective matrix;
- the dependence of the running time of these algorithms on the complexity of the punctured surface $(M, \mathcal{P})$ is polynomial (see Theorem 10.6).

This is important because, for computations involving a non-abelian group, one might prefer not only to have a solution of the word problem for this group, but also an algorithm for computing a normal form of any element, which would allow to avoid operating with unreasonably long presentations of group elements without knowing that they can be simplified. When a normal form is defined for all group elements, the key question is how efficiently one can compute the normal form of $g_1 g_2$ from the normal forms of $g_1$ and $g_2$, and the normal form of $g^{-1}$ from the normal form of $g$.

The author is unaware of any approach based on decompositions of group elements into products of generators that yields a solution of these problems with the above mentioned properties.

Many ideas we use are pretty well known to date (such as representing curves by measured train tracks and simplifying them by a procedure similar in nature to the accelerated Euclidean algorithm). The feature of the method proposed here is that to compute $\langle \gamma_1, \gamma_2 \rangle$ we simplify the presentation of both $\gamma_1$ and $\gamma_2$ simultaneously to the extent in which their simplifications go in parallel, which makes all intersections detectable without fully untangling any of $\gamma_1$ and $\gamma_2$. In cases when only few simplification steps are needed (which seem to be typical in a sense),
this allows to benefit from fast multiplication algorithms for integers, which are known since the work of A. Karatsuba [11, 12].

Another possible way to compute $\langle \gamma_1, \gamma_2 \rangle$ could be by simplifying the presentation of $\gamma_1$ as much as possible at the expense of possibly getting the presentation of $\gamma_2$ more complicated. This idea is realized in [4, 5] where a method is suggested to change the triangulation of the surface so that normal coordinates of $\gamma_1$ become small. Another way of simplifying the presentation of a normal curve on a surface is given in [10] by means of constructing a special cell decomposition of the surface called the street complex. The computational efficiency of these approaches, for a fixed surface, would be comparable, in worst cases, to the one proposed here, but in good cases, no acceleration due to Karatsuba type algorithms can be achieved.

It is worth noting that in order to have a polynomial bound for the running time of the algorithms simplifying the presentation of a normal curve it is important to devlop an analogue of the accelerated version of the Euclidean algorithm (the one that uses Euclidean division instead of subtraction). This means that the algorithm must somehow detect ‘large spirals’ in the given normal curve and untwist each of them in a single step. Without such a feature the running time of the algorithm would be, in worst cases, exponential in the size of the presentation of a normal curve by normal coordinates (as in [26], where intersections of closed multiple curves in a punctured disc are counted using the ‘relaxation’ algorithm from [8]).

Saul Schleimer pointed out to the author that a similar computational efficiency (in worst cases) to the one of the method proposed here can also be achieved by means of straight-line programs [2], which provide for another way to efficiently treat normal curves on a surface (see [22]).

The paper is organized as follows. In Section 2 we introduce the ‘natural’ geometry on $\text{MCG}(M, P)$ in purely algebraic terms and formulate in these terms the claim about the efficiency of our approach. Sections 3 and 4 are devoted to preliminaries. The matrix presentation for $\text{MCG}(M, P)$ is constructed in Sections 5 and 6. In Sections 7–10 the algorithmic issues are discussed. Section 11 proposes a direction of further research motivated by the matrix presentation of the mapping class groups.

2. GROUP PRESENTATIONS AND COMPLEXITY

For two non-negative functions $c_1$ and $c_2$ on a group $G$, we write $c_1 \preceq c_2$, if there exists a constant $C$ such that $c_1(g) < C \cdot c_2(g)$ for all $g \in G$, $g \neq 1$. If both $c_1 \preceq c_2$ and $c_2 \preceq c_1$ hold we say that $c_1$ and $c_2$ are comparable. If only $c_1 \preceq c_2$ holds but not $c_2 \preceq c_1$, we write $c_1 \prec c_2$.

We say that $c : G \to \mathbb{R}_{\geq 0}$ is a complexity function if there exists a finite alphabet $\mathcal{A}$ and a language $\mathcal{L}$ (= a subset of the set of all words) in $\mathcal{A}$ with an onto mapping $\pi : \mathcal{L} \to G$ such that

(i) if $w_1, w_2 \in \mathcal{L}$, then $w_1w_2 \in \mathcal{L}$ and $\pi(w_1w_2) = \pi(w_1)\pi(w_2)$;
(ii) $c$ is comparable to the following function $f$:

$$f(g) = \inf_{\pi(w)=g} |w|,$$

where by $|w|$ we denote the word length.

A couple $(\mathcal{L}, \pi)$ satisfying (i) will be referred to as a $G$-presentation, and if (ii) also holds then it will be said to be appropriate for $c$.

**Example 2.1.** An ordinary word length complexity function $\text{wl}_{\mathcal{A}}$, where $\mathcal{A} \subset G$ is a finite generating set for $G$, is a typical example of a complexity function. An appropriate $G$-presentation is obtained by letting $\mathcal{L}$ be the set of all words in $\mathcal{A}$. One can see that $\text{wl}_{\mathcal{A}}$ is always a maximal complexity function with respect to $\preceq$. 

Example 2.2. The conventional way for encoding integral matrices, by listing their entries written in a positional numeral system, yields a \(\text{GL}(n, \mathbb{Z})\)-presentation appropriate for the complexity function \(c(x) = \log \|x\|\) (any element can also be represented as a product of other elements encoded in this way).

A complexity function on \(\text{GL}(n, \mathbb{Z})\) (viewed as an abstract finitely presented group) comparable to \(c\) in Example 2.2 can also be defined without an explicit reference to the matrix presentation using the following general construction.

For a finite generating set \(\mathcal{A}\) of a group \(G\) we define the zipped word length function \(\zwl_{\mathcal{A}}\) as follows:

\[
\zwl_{\mathcal{A}}(g) = \min_{g = a_{k_1} \ldots a_{k_m}} \sum_{i=1}^{m} \log_2(|k_i| + 1),
\]

where \(a_1, \ldots, a_m \in \mathcal{A}\), \(k_1, \ldots, k_m \in \mathbb{Z}\).

Obviously, this is a complexity function, for which an appropriate \(G\)-presentation is obtained by choosing a reasonable encoding for sequences of the form \(((a_1, k_1), \ldots, (a_m, k_m))\), where \(a_i \in \mathcal{A}\), \(k_i \in \mathbb{Z}\), and interpreting such a sequence as the product \(a_{k_1} \ldots a_{k_m} \in G\). We call it the zipped word presentation.

One can show that the complexity function \(c(x) = \log \|x\|\) on \(\text{GL}(n, \mathbb{Z})\) is comparable to \(\zwl_{\mathcal{A}}\) if the generating set \(\mathcal{A}\) is chosen appropriately. Namely, it suffices that, for each \(\ell = 2, 3, \ldots, n\), the subset \(\mathcal{A}\) contains an element whose Jordan normal form has a single Jordan block of size \(\ell\) and \(n - \ell\) blocks of size 1 with all eigenvalues equal to \(\pm 1\), and there are no elements in \(\mathcal{A}\) having eigenvalues other than \(\pm 1\).

There is a direct analogy of this statement for the mapping class groups. The ‘natural’ geometry on \(\text{MCG}(M, \mathcal{P})\) can be defined in terms of the matrix presentation introduced below in Section 6, and this geometry coincides with the one defined by the zipped word length function provided that the generating set is chosen appropriately (see Proposition 9.2). As shown in [23] this geometry also coincides with the one coming from the action of the group on the thick part of the corresponding Teichmüller space.

Definition 2.1. For a complexity function \(c\) on a group \(G\), we call an efficient solution of the word problem for \(G\) with respect to \(c\) an appropriate \(G\)-presentation \((\mathcal{L}, \pi)\) together with

(i) a mapping \(\text{nf} : G \to \mathcal{L}\) (the word \(\text{nf}(g)\) is thought of as the normal form of \(g\)) such that we have \(\pi \circ \text{nf} = \text{id}_G\) and the function \(g \mapsto |\text{nf}(g)|\) is comparable to \(c\), and

(ii) polynomial-time algorithms to decide whether \(w \in \mathcal{L}\) or not and to compute \(\text{nf}(\pi(w))\) from \(w\) if \(w \in \mathcal{L}\).

Definition 2.2. Let \(a, b\) be elements of a group \(G\). We say that \(a\) is a fractional power of \(b\) if \(a^k = b^l\) for some \(k, l \in \mathbb{Z}\), \(k > 0\).

In particular, any torsion element is a fractional power any other group element.

Theorem 2.1. Let \(M\) be a compact surface, \(P_1, \ldots, P_n \in M\) a non-empty collection of pairwise distinct points such that the mapping class group \(G = \text{MCG}(M, \{P_1, \ldots, P_n\})\) is infinite. Let \(\mathcal{A}\) be a finite generating set for \(G\) such that

(i) every element in \(\mathcal{A}\) is a fractional power of a Dehn twist;

(ii) every Dehn twist in \(G\) is conjugate to a fractional power of an element from \(\mathcal{A}\).

Then the word problem in \(G\) is efficiently solvable with respect to \(\zwl_{\mathcal{A}}\). Moreover, the algorithms for this solution can be made quadratic-time.
There are various generating sets known satisfying Condition (i) for the mapping class groups, see \[15, 16, 17, 14, 6, 24\]. Condition (ii) can always be met by adding a few Dehn twists to the generating set, since up to a homeomorphism there are only finitely many distinct simple closed curves in \(M \setminus \{P_1, \ldots, P_n\}\).

Our settings here are slightly more general than those that one typically considers (we allow multiple \(P_i\)'s on a single boundary component and include orientation reversing homeotopies into the mapping class group of an orientable surface), but extending the existing results so as to obtain a generating set satisfying (i) and (ii) is easy. So, Theorem 2.1 applies to any infinite mapping class group of a compact surface with \(n \geq 1\) punctures.

In the particular case when \(G\) is the braid group \(B_n\) and the generating set \(\mathcal{A}\) consists of all Garside-like elements \(\Delta_{ij}\) (half-twists of strands \(i\) through \(j\)), Theorem 2.1 was established by the present author and Bert Wiest in [9].

Theorem 2.1 will be proved in Section 10.

3. Notation, Terminology, and Conventions

Once and for all until Subsection 10.8 we fix a connected compact surface \(M\), orientable or not, which will be referred to simply as the surface, and a non-empty set of punctures \(\mathcal{P} = \{P_1, \ldots, P_n\} \subset M\). If \(M\) is a sphere we require \(n \geq 4\); if \(M\) is a projective plane, a disk, an annulus, or a Möbius band we require \(n \geq 3\); and if \(M\) is a torus or a Klein bottle we require \(n \geq 2\). The excluded cases will be referred to as sporadic and the remaining ones nonsporadic.

By \(G\) we will denote the mapping class group \(\text{MCG}(M, \mathcal{P})\), that is, the quotient of the group \(\text{Homeo}(M, \mathcal{P})\) of self-homeomorphisms of \(M\) preserving the subset \(\mathcal{P}\) by the connected component \(\text{Homeo}_0(M, \mathcal{P})\) containing the identity homeomorphism.

We assume that every boundary component \(\gamma\) of \(M\) contains at least one of \(P_i\)'s. This is not a loss of generality because otherwise one can contract \(\gamma\) to a point and treat it as a puncture, which does not affect the mapping class group.

The punctures located at \(\partial M\) will be called boundary punctures and all the others internal punctures.

By a proper arc on \(M\) we mean an open simple arc \(\alpha\) in \(M \setminus \mathcal{P}\) approaching some punctures \(P_i, P_j\) at the ends such that the closure \(\overline{\alpha}\) of \(\alpha\) does not bound an empty disk, i.e. a disk with no puncture inside. It is allowed, however, that \(\overline{\alpha}\) forms a loop.

By a simple curve on \(M\) we mean a smooth simple closed curve in \(M \setminus \mathcal{P}\) that does not bound an empty disk.

By a multiple curve on \(M\) we mean a possibly empty union of pairwise disjoint simple curves and proper arcs on \(M\).

Two proper arcs are parallel if they coincide or enclose an empty disk. Two simple curves are parallel if they enclose an empty annulus.

Two curves \(\gamma_1, \gamma_2\) are said to be tight (with respect to each other) if they either do not meet or meet transversely, and there is no empty disk \(D \subset M\) bounded by two subarcs \(\alpha_1 \subset \overline{\gamma_1}\) and \(\alpha_2 \subset \overline{\gamma_2}\) such that at least one of the common endpoints of \(\alpha_1\) and \(\alpha_2\) is not a puncture.

Let \(\gamma\) and \(\gamma'\) be two multiple curves. We write \(\gamma \sim \gamma'\) if they are isotopic relative to \(\mathcal{P}\).

**Definition 3.1.** If two multiple curves \(\gamma_1\) and \(\gamma_2\) are tight we define their geometric intersection index \(\langle \gamma_1, \gamma_2 \rangle\) to be the number of intersections \(\|\gamma_1 \cap \gamma_2\|\) less the number of pairs \((\alpha_1, \alpha_2)\) of parallel proper arcs such that \(\alpha_i \subset \gamma_i\), and the number of pairs \((\beta_1, \beta_2)\) of isotopic one-sided simple curves such that \(\beta_i \subset \gamma_i\). For arbitrary multiple curves \(\gamma_1, \gamma_2\), the geometric intersection index \(\langle \gamma_1, \gamma_2 \rangle\) is defined as \(\langle \gamma'_1, \gamma'_2 \rangle\) with any tight pair \((\gamma'_1, \gamma'_2)\) of multiple curves such that \(\gamma'_i \sim \gamma_i, i = 1, 2\). As we will see below (Proposition 4.1) this number is well defined.
Note that, according to this definition, a pair $\beta_1, \beta_2$ of isotopic simple curves such that $\beta_i \subset \gamma_i$ does not contribute anything to $\langle \gamma_1, \gamma_2 \rangle$. Indeed, if these curves are two-sided, then they are disjoint in tight position. If they are one-sided, then they have a single intersection in tight position, but this contribution is cancelled by subtracting the total number of such pairs.

Note also that, for any proper arc $\alpha$, we have $\langle \alpha, \alpha \rangle = -1$. This will be justified by Proposition 5.1 (see also Section 11).

By a triangulation of $M$ with vertices at $\mathcal{P}$ we mean a maximal collection of proper arcs $(e_1, \ldots, e_N)$ such that they are pairwise disjoint and nonparallel. The arcs $e_i$ are called edges of the triangulations. We assume additionally that the boundary $\partial M$ is covered by $\bigcup_{i=1}^{N} e_i$.

In nonsporadic cases, the edges of a triangulation cut the surface $M$ into triangles, which are homeomorphic images of the interior of a 2-simplex under a continuous map that sends the interior of each side of the simplex into an edge of the triangulation.

It is standard to check that the number of edges of any triangulation of $M$ with vertices at $\mathcal{P}$ is equal to

$$N = -3\chi + 3n - m,$$

where $\chi$ is the Euler characteristics of $M$ and $m$ is the number of punctures at $\partial M$, and the number of triangles equals

$$F = -2\chi + 2n - m.$$

In order not to overload the exposition by technical details, we postpone the discussion on the dependence of the asymptotic complexity of the proposed algorithms on the complexity of the surface till Subsection 10.8 and discuss it there very briefly.

‘Isotopic’ in this paper always means ‘isotopic relative to $\mathcal{P}$’.

4. The Pulling Tight Procedure

Here we recall some standard facts about curves on a surface, adapted to our settings. An experienced reader may safely skip this section.

**Proposition 4.1.** Let $\gamma_1$ and $\gamma_2$ be two multiple curves in $M$. Then there exist multiple curves $\gamma'_1$ and $\gamma'_2$ such that $\gamma'_i \sim \gamma_i$, $i = 1, 2$, and $\gamma'_1$ and $\gamma'_2$ are tight. If $\gamma''_1$ and $\gamma''_2$ are another such pair of multiple curves then $\langle \gamma'_1, \gamma'_2 \rangle = \langle \gamma''_1, \gamma''_2 \rangle$.

**Proof.** The standard method to produce the desired $\gamma'_1, \gamma'_2$ is known as the *pulling tight procedure*. We start from $\gamma'_i = \gamma_i$, $i = 1, 2$, and then modify them. First, we disturb $\gamma'_1$ and $\gamma'_2$ slightly to make them transverse to each other.

Assume there is a bigon, i.e. a 2-disk $D \subset M$ bounded by arcs $\overline{\alpha_1}$ and $\overline{\alpha_2}$ with $\alpha_i \subset \gamma'_i$, $i = 1, 2$, being non-proper open arcs such that the interior of $D$ is disjoint from $\gamma'_1$ and $\gamma'_2$. We replace $\gamma'_1$ and $\gamma'_2$ by $(\gamma'_1 \setminus \alpha_1) \cup \alpha_2$ and $(\gamma'_2 \setminus \alpha_2) \cup \alpha_1$, respectively, and then smooth out the obtained curves at the breaking point(s) (see Figure 1). This reduces the number of intersections of $\gamma'_1$ and $\gamma'_2$, so the process terminates after finitely many steps. Obviously the isotopy class of each curve stays unchanged.

Figure 2 illustrates the fact that the order in which we reduce bigons does not matter. More precisely, the isotopy class of the union $\gamma'_1 \cup \gamma'_2$ does not depend on that order.

![Figure 1. Reducing bigons](image-url)
Remark 4.1. However, the isotopy class of the pair \((\gamma_1', \gamma_2')\) can depend on the arbitrariness in the pulling tight process if there are connected components \(\beta_1 \subset \gamma_1, \beta_2 \subset \gamma_2\) that are isotopic to each other. The issue is illustrated in Figure 3. Whichever bigon we reduce we get the same unordered pair of curves but which of them will be \(\beta_1'\) and which \(\beta_2'\) depends on the choice of the bigon(s) being reduced.

Let \(\beta_1\) and \(\beta_2\) be isotopic to \(\gamma_1\) and \(\gamma_2\), respectively, and the multiple curves in both pairs are transverse to each other. Let \((\beta_1', \beta_2')\) and \((\gamma_1', \gamma_2')\) be obtained from the corresponding pairs by pulling them tight. Then we claim that \(\beta_1' \cup \beta_2'\) is isotopic to \(\gamma_1' \cup \gamma_2'\).

Indeed by choosing a generic isotopy from \(\beta_1\) to \(\gamma_1\) and from \(\beta_2\) to \(\gamma_2\) we get a finite sequence of bigon reduction and inverse operations, that produces \(\gamma_1 \cup \gamma_2\) from \(\beta_1 \cup \beta_2\). So, it suffices to prove the claim for a single bigon reduction. If \(\beta_1 \cup \beta_2 \mapsto \gamma_1 \cup \gamma_2\) is a bigon reduction, then it may be taken for the first step of pulling \(\beta_1\) and \(\beta_2\) tight, so the result of pulling tight procedure for \((\beta_1, \beta_2)\) and \((\gamma_1, \gamma_2)\) will be exactly the same.

Applying this to \(\beta_i = \gamma_i'', i = 1, 2\), we get \(\gamma_1' \cup \gamma_2' \sim \gamma_1'' \cup \gamma_2''\). This implies the second claim of the proposition.

The first claim in Proposition 4.1 can be strengthened as follows.
Proposition 4.2. Let $\gamma_1$ and $\gamma_2$ be two multiple curves in $M$. Then there exists a multiple curve $\gamma'_2$ such that we have $\gamma'_2 \sim \gamma_2$ and $\gamma_1, \gamma'_2$ are tight.

Proof. One only needs to apply an isotopy that carries $\gamma'_1$ to $\gamma_1$ at the end of the pulling tight procedure described in the proof of Proposition 4.1. One can restore $\gamma_1$ by an isotopy not only at the very end but also at every step of the procedure. $\square$

Proposition 4.3. For any three multiple curves $\gamma_1$, $\gamma_2$, and $\gamma_3$, there are multiple curves $\gamma'_2$, $\gamma'_3$ such that $\gamma'_i \sim \gamma_i$, $i = 2, 3$, and the curves $\gamma_1$, $\gamma'_2$, and $\gamma'_3$ are pairwise tight.

Proof. Due to Proposition 4.2 we may assume without loss of generality that the pairs $(\gamma_1, \gamma_2)$ and $(\gamma_1, \gamma_3)$ are already tight. We may also assume that there are no triple intersections, i.e. $\gamma_1 \cap \gamma_2 \cap \gamma_3 = \emptyset$ as we can achieve this by a small deformation of $\gamma_3$.

Now we apply the pulling tight procedure to $(\gamma_2, \gamma_3)$. It produces $\gamma'_2$, $\gamma'_3$ that are still tight with respect to $\gamma_1$. Indeed, $\gamma'_2 \cup \gamma'_3$ is obtained from $\gamma_2 \cup \gamma_3$ by resolution of intersections, which occur far from $\gamma_1$. So, the number of intersection points in $\gamma_1 \cap (\gamma'_2 \cup \gamma'_3)$ is equal to that in $\gamma_1 \cap (\gamma_2 \cup \gamma_3)$. If $(\gamma_1, \gamma'_2)$ or $(\gamma_1, \gamma'_3)$ were not tight, we could have applied the pulling tight process again and get $\gamma''_2$ and $\gamma''_3$ such that $\gamma''_2 \cup \gamma''_3$ has a smaller number of intersections with $\gamma_1$ than $\gamma_2 \cup \gamma_3$ has, which contradicts Proposition 4.1. $\square$

5. Normal coordinates

The idea of a normal curve and normal coordinates goes back to H. Kneser [13] who introduced the concept of a normal surface in a 3-manifold, which has had a big impact on low-dimensional topology. We use a modification the classical notion of a normal curve which allows arcs emanating from the punctures.

Let $T = (e_1, \ldots, e_N)$ be a triangulation of $M$. A multiple curve $\gamma$ is said to be a normal curve with respect to $T$ if $\gamma$ and $\bigcup_{i=1}^N e_i$ are tight. This is equivalent to saying that the intersection of $\gamma$ with any triangle $\tau$ of $T$ consists of arcs each of which either connects points on different sides of $\tau$ or a vertex to a point on the opposite side or two vertices (in the latter case such an arc is parallel to a side of $\tau$); see Figure 4. Such arcs will be called normal. By saying that an arc is normal we will also assume that its endpoints do not lie in the interior of boundary edges of $M$ as this never happens to intersections of multiple curves with triangles. We will call a normal arc side-to-side, vertex-to-side, or vertex-to-vertex according the location of its endpoints.

![Normal arcs](image)

**Figure 4. Normal arcs**

If $\gamma$ is not normal with respect to $T$, then the pulling tight process for $\gamma$ and $\bigcup_i e_i$ with the latter staying fixed (see Proposition 4.2) will be referred to as normalization of $\gamma$ with respect to $T$.

Definition 5.1. For a multiple curve $\gamma$, the numbers $\langle \gamma, e_i \rangle$, $i = 1, \ldots, N$, are called normal coordinates of $\gamma$ with respect to $T$. 
Proposition 5.1. Let $\gamma_1$ and $\gamma_2$ be multiple curves such that $\langle \gamma_1, e_i \rangle = \langle \gamma_2, e_i \rangle$ for all $i = 1, \ldots, N$. Then $\gamma_1$ and $\gamma_2$ are isotopic.

Proof. Due to Proposition 4.2 we may restrict ourselves to the case when $\gamma_1$ and $\gamma_2$ are normal with respect to $T$.

Let $\gamma'_1$ and $\gamma'_2$ be multiple curves obtained from $\gamma_1$ and $\gamma_2$, respectively, by removing all proper arcs parallel to edges of $T$. Then we still have $\langle \gamma'_1, e_i \rangle = \langle \gamma'_2, e_i \rangle$ for all $i = 1, \ldots, N$, and, moreover, all these geometric intersection indexes are non-negative.

Since the number of intersections of $\gamma'_1$ with $e_i$ coincides with that of $\gamma'_2$ for any $i = 1, \ldots, N$, we can apply an isotopy that preserves all $e_i$s and carries $\gamma'_1 \cap e_i$ to $\gamma'_2 \cap e_i$ for all $i$. So, we may assume that $\gamma'_1 \cap e_i = \gamma'_2 \cap e_i$ for $i = 1, \ldots, N$.

Now we focus on a single triangle $\tau$ of $T$.

Lemma 5.2. The intersection of a normal curve $\gamma$ with $\tau$ can be recovered from $\gamma \cap \partial \tau$ uniquely up to isotopy relative to $\partial \tau$ provided that $\gamma$ has no components parallel to the sides of $\tau$.

Proof. Indeed, let $\tau$ be bounded by the edges $e_1$, $e_2$, $e_3$. Denote $x_i = \langle \gamma, e_i \rangle$. Each normal arc in $\gamma \cap \tau$ connects either a point at $e_i$, $i \in \{1, 2, 3\}$, to the opposite vertex, in which case we say that it has type $(0i)$, or two points on different sides $e_i$ and $e_j$, $i, j \in \{1, 2, 3\}$, $i < j$, in which case we attribute it type $(ij)$; see Figure 5.

![Figure 5. Six types of normal arcs in a single triangle](image)

One can see that some of these types are incompatible meaning that normal arcs of those types can not occur in $\gamma \cap \tau$ simultaneously. For example, type $(01)$ is incompatible with types $(02)$, $(03)$, and $(23)$. Therefore, if a normal arc of type $(01)$ is present in $\gamma \cap \tau$, then the other arcs may have only types $(12)$ and $(13)$. One can readily see that, in this case, we have $x_1 > x_2 + x_3$, and the number of normal arcs of type $(01)$, $(12)$, and $(13)$ is equal to $(x_1 - x_2 - x_3)$, $x_2$, and $x_3$, respectively.

Similarly, if arcs of type $(02)$ or $(03)$ are present, we will have $x_2 > x_1 + x_3$ or $x_3 > x_1 + x_2$, respectively; and, in each case, recover the number of arcs of each type from $x_1, x_2, x_3$.

If only arcs of types $(12)$, $(13)$ are present, then the triangle inequalities hold for $x_1$, $x_2$, and $x_3$, and the number of arcs of type $(12)$, $(23)$, and $(13)$ is equal to $(x_1 + x_2 - x_3)/2$, $(x_2 + x_3 - x_1)/2$, $(x_1 + x_3 - x_2)/2$, respectively.

The sets of all possible triples $(x_1, x_2, x_3)$ obtained in these four cases do not overlap; hence, from the knowledge of $x_1, x_2, x_3$ we can always decide which case occurs.

Clearly, the number of normal arcs of each type defines $\gamma \cap \tau$ up to isotopy relative to $\partial \tau$. □

We resume the proof of Proposition 5.1. It follows from Lemma 5.2 that $\gamma'_1$ and $\gamma'_2$ are isotopic relative to $\bigcup_{i=1}^{N} e_i$. Thus, we may assume $\gamma'_1 = \gamma'_2$ from the beginning.
Now let $k_i = \max(0, -\langle \gamma_1, e_i \rangle) = \max(0, -\langle \gamma_2, e_i \rangle)$. Each of $\gamma_1$ and $\gamma_2$ is obtained from $\gamma_1' = \gamma_2'$ by adding $k_i$ proper arcs parallel to $e_i$ for all $i = 1, \ldots, N$. Obviously, the result is unique up to isotopy.

**Remark 5.1.** One can see from the proof of Proposition 5.1 that the collections of normal coordinates of normal curves form a subset $L$ of $\mathbb{Z}^N$ that can be characterized as follows: $(x_1, x_2, \ldots, x_N) \in \mathbb{Z}^N \setminus L$ if and only if either there is a boundary edge $e_i$ such that $x_i > 0$ or there is a triangle of $T$ with sides $e_i$, $e_j$, and $e_k$ such that the respective coordinates $x_i$, $x_j$, and $x_k$ are all positive, satisfy the triangle inequalities, and sum up to an odd number.

Let $T' = (e'_1, \ldots, e'_N)$ be another triangulation of $M$ with vertices at $\mathcal{P}$. We denote by $\langle T, T' \rangle$ the $N \times N$ matrix whose $(ij)$th entry is 

$$
\langle T, T' \rangle_{ij} = \langle e_i, e'_j \rangle.
$$

**Remark 5.2.** The determinant of the matrix $\langle T, T' \rangle$ is always an integral power of two. This fact plays no role here, but the reader might enjoy trying to prove this.

Recall from Section 3 that we deal only with nonsporadic cases.

**Proposition 5.3.** Let $\varphi$ be a diffeomorphism of $M$. Then the following two statements are equivalent:

(i) $\langle T, T' \rangle = \langle T, \varphi(T') \rangle$, where $\varphi(T') = (\varphi(e'_1), \ldots, \varphi(e'_N))$;

(ii) $\varphi$ is isotopic to the identity.

**Proof.** Implication (ii)⇒(i) follows from Proposition 4.1.

Suppose now that (i) holds. Then it follows from Proposition 5.1 that $\varphi(T')$ is isotopic to $T'$. So, without loss of generality we may assume $\varphi(T') = T'$.

Let us choose an orientation for each triangle of $T'$. Clearly, if $\varphi$ carries each triangle to itself and preserves its orientation, then $\varphi$ is isotopic to identity. Suppose, to the contrary, that this is not the case.

Then either there are two different triangles $\tau, \tau'$ of $T'$ such that $\varphi(\tau) = \tau'$, or $\varphi$ preserves each triangle but flips the orientation.

In the former case, the triangles $\tau$ and $\tau'$ have the same sides, hence $\overline{\tau \cup \tau'}$ is a closed surface, which is the whole of $M$. By gluing up two triangles along all three sides one can obtain only the following surfaces: a sphere $\mathbb{S}^2$ with 3 punctures, a torus $\mathbb{T}^2$ with a single puncture, a projective plane $\mathbb{R}P^2$ with two punctures, and a Klein bottle $\mathbb{K}^2$ with a single puncture. All these are sporadic cases.

In the latter case, two sides of every triangle $\tau$ of $T'$ are glued together, hence $\overline{\tau}$ is either a disk $\mathbb{D}^2$ with a single puncture inside and a single puncture at the boundary, or a Möbius band $\mathbb{M}^2$ with a single puncture at the boundary. Both are sporadic cases, so there must be more than one triangle of $T'$. Two such surfaces glued along the boundary form a closed surface, so the number of triangles cannot be greater than two. From two triangles we get either $(M, n) = (\mathbb{S}^2, 3)$ or $(M, n) = (\mathbb{R}P^2, 2)$ or $(M, n) = (\mathbb{K}^2, 1)$, which are also sporadic cases.

So, in all nonsporadic cases we have (i)⇒(ii).

6. A MATRIX PRESENTATION OF THE MAPPING CLASS GROUPS

Whenever $\gamma$ is a multiple curve and $g$ is an element of $G = \text{MCG}(M, \mathcal{P})$ we will use the notation $g(\gamma)$ for $\varphi(\gamma)$, where $\varphi$ is any diffeomorphism representing $g$. We will do it when only the isotopy class of $\varphi(\gamma)$ matters. This will apply also to triangulations in place of curves.

Let us fix a triangulation $T = (e_1, \ldots, e_N)$ of $M$ with vertices at $\mathcal{P}$. It follows from Proposition 5.3 that an element $g \in G$ can be recovered uniquely from the matrix $\langle T, g(T) \rangle$. Thus, by
choosing a proper encoding for $N \times N$-matrices we get a $G$-presentation in which an element $g \in G$ can be presented by any sequence of matrices $(m_1, \ldots, m_k)$ such that $m_i = \langle T, g_i(T) \rangle$ with $g_1, \ldots, g_k \in G$, $g_1 \cdot \ldots \cdot g_k = g$.

For any $g \in G$, we let the intersection matrix $\langle T, g(T) \rangle$ (encoded in a reasonable way using a finite alphabet) be the normal from $n_f(g)$ of $g$ and define the complexity of $g$ as

$$c_T(g) = \sum_{i,j=1}^{N} \log_2(|\langle T, g(T) \rangle|_{ij} + \delta_{ij}| + 1),$$

where $\delta_{ij}$ is the Kronecker delta (which is added just to set the complexity of the identity element to zero and plays no role otherwise). One can see that $c_T(g)$ is comparable to the amount of space needed to encode $\langle T, g(T) \rangle$. The rest of this section is devoted to showing that this setup satisfies Condition (i) of Definition 2.1.

The key question about the efficiency of this approach is how to compute $\langle T, g_1(g_2(T)) \rangle$ from $\langle T, g_1(T) \rangle$ and $\langle T, g_2(T) \rangle$ for arbitrary $g_1, g_2 \in G$. We start by observing that this computation has much in common with the ordinary matrix multiplication.

**Proposition 6.1.** The matrix element $\langle T, g_1(g_2(T)) \rangle_{ij}$ equals $\langle \gamma, \gamma' \rangle$, where $\gamma$ and $\gamma'$ are the normal curves whose normal coordinates with respect to $T$ form the $i$th row of $\langle T, g_1(T) \rangle$ and the $j$th column of $\langle T, g_2(T) \rangle$, respectively.

**Proof.** Let $\gamma = g_1^{-1}(e_i)$ and $\gamma' = g_2(e_j)$. Then we have

$$\langle T, g_1(g_2(T)) \rangle_{ij} = \langle g_1^{-1}(T), g_2(T) \rangle_{ij} = \langle \gamma, \gamma' \rangle.$$

The $k$th coordinate of $\gamma$ is

$$\langle \gamma, e_k \rangle = \langle g_1^{-1}(e_i), e_k \rangle = \langle e_i, g_1(e_k) \rangle = \langle T, g_1(T) \rangle_{ik}.$$

The $k$th coordinate of $\gamma'$ is

$$\langle \gamma', e_k \rangle = \langle e_k, g_2(e_j) \rangle = \langle T, g_2(T) \rangle_{kj}.$$

The claim follows. \hfill \Box

For $1 \leq i, j \leq N$ let $\mu_{ij}$ be equal to the number of triangles of $T$ adjacent to both $e_i$ and $e_j$ if $i \neq j$, and 1 otherwise.

**Proposition 6.2.** For any two curves $\gamma$, $\gamma'$ we have

$$|\langle \gamma, \gamma' \rangle| \leq \sum_{i,j=1}^{N} |\langle \gamma, e_i \rangle| \cdot \mu_{ij} \cdot |\langle \gamma', e_j \rangle|.$$

**Proof.** Due to Proposition 4.3 we may assume that $\gamma$ and $\gamma'$ are tight and each of them is normal with respect to $T$. We may also assume that $\gamma \cap \gamma'$ is disjoint from the edges of $T$.

Denote by $X_i$ the intersection set $\gamma \cap e_i$, if $\langle \gamma, e_i \rangle \geq 0$, and the set of proper arcs in $\gamma$ parallel to $e_i$ otherwise. In both cases we have $|X_i| = |\langle \gamma, e_i \rangle|$. We define $X'_i$ similarly, with $\gamma'$ in place of $\gamma$.

Denote by $Y$ the set of transverse intersections in $\gamma \cap \gamma'$ joined with the set of all pairs $(\alpha, \alpha')$ of parallel proper arcs with $\alpha \subset \gamma$, $\alpha' \subset \gamma'$. We clearly have $|\langle \gamma, \gamma' \rangle| \leq |Y|.$

Now define maps $f, f'$ from $Y$ to $(\bigcup_i X_i)$ and $(\bigcup_i X'_i)$, respectively, as follows. Let $P \in Y$ be an intersection point of $\gamma$ and $\gamma'$. Let $\tau$ be the triangle of $T$ in which this intersection occurs, and $\alpha \subset \gamma$, $\alpha' \subset \gamma'$ be the normal arcs that contain $P$. If $\alpha$ is a proper arc parallel to an edge $e_i$ we put $f(P) = \alpha$. Otherwise, $\alpha$ must have an endpoint $Q$ at some edge of $T$. In this case we put $f(P) = Q$. We define the map $f'$ similarly, by replacing $\alpha$ with $\alpha'$.\hfill \Box
Now let $P = (\alpha, \alpha') \in Y$ be a pair of parallel proper arcs. If they are parallel to some $e_i$ we put $f(P) = \alpha$ and $f'(P) = \alpha'$. Otherwise, $\alpha$ and $\alpha'$ must intersect some edge $e_i$. Then we choose $Q \in \alpha \cap e_i$ and $Q' \in \alpha' \cap e_i$ and put $f(P) = Q$, $f'(P) = Q'$.

It is now easy to check that due to normality and tightness of $\gamma, \gamma'$ the number of preimages of any $(Q, Q') \in (\cup_i X_i) \times (\cup_i X'_i)$ under the map $f \times f'$ does not exceed $\mu_{ij}$ if $Q \in X_i$ and $Q' \in X'_j$. Therefore, we have

$$|\langle \gamma, \gamma' \rangle| \leq |Y| \leq \sum_{i,j=1}^{N} |X_i| \cdot \mu_{ij} \cdot |X'_j| = \sum_{i,j=1}^{N} |\langle \gamma, e_i \rangle| \cdot \mu_{ij} \cdot |\langle \gamma', e_j \rangle|. \quad \Box$$

**Proposition 6.3.** There exists a constant $C$ depending on $M$ and $\mathcal{P}$ such that

$$c_T(g_1 g_2 \ldots g_k) \leq C(c_T(g_1) + c_T(g_2) + \ldots + c_T(g_k)).$$

for any $k \in \mathbb{N}$, $g_1, g_2, \ldots, g_k \in G$.

**Proof.** For a matrix $A$, we denote by $\|A\|_E$ the standard Euclidean norm of $A$:

$$\|A\|_E = \sqrt{\sum_{i,j} A_{ij}^2}. \quad (3)$$

Since the numbers of summands in (1) and (3) are fixed, the function $E : G \to \mathbb{R}$ defined by

$$E(g) = \log_2 \|\langle T, g(T) \rangle\|_E$$

is comparable to $c_T$. Therefore, it suffices to prove (2) for $E$ in place of $c_T$. This is done by using Propositions 6.1 and 6.2 which imply

$$E(g_1 g_2) \leq E(g_1) + E(g_2) + \mu,$$

where $\mu = \log_2 \|\langle \mu_{ij} \rangle\|_E$. The rest of the proof is easy. \quad \Box

Thus, we are done with showing that the matrix presentation introduced in this section satisfies Condition (i) of Definition 2.1. The key question now is how to compute $\langle \gamma, \gamma' \rangle$ efficiently for two normal curves given by their normal coordinates.

### 7. Train tracks

Train tracks, which have been introduced by W. Thurston, are widely used for studying homeomorphisms of surfaces and related problems [3] [13] [20] [21] [25]. Here by a train track track we mean what is known as a train track with terminals [20].

Whenever we deal with a finite graph (i.e. a 1-dimensional CW-complex) $\theta$ embedded in $M$ we assume that all edges of $\theta$ are smooth images of a closed interval, and that an open contractible neighborhood $U_v$ is chosen around each vertex $v$ of $\theta$ so that $U_v$ and $U_{v'}$ do not overlap for any two different vertices $v$ and $v'$, and the intersection $U_v \cap \theta$ is contractible for all $v$. The closure $t$ of any connected component of $(U_v \cap \theta) \setminus \{v\}$ is called a tail of the edge whose closure contains $t$.

Loops and multiple edges with the same endpoints are allowed for graphs.

The edges of graphs that we consider are not allowed to pass through a puncture, but a puncture may be a vertex of a graph.

A connected component $A$ of $U_v \setminus \theta$ is called a cusp in two cases:

(a) the boundary $\partial A$ contains two tails whose tangent rays at $v$ coincide;

(b) the vertex $v$ coincides with a puncture, i.e. $v \in \mathcal{P}$.

In the latter case the cusp is called special, and otherwise ordinary.
**Definition 7.1.** By a *train track* we mean an embedded 1-dimensional CW-complex \( \theta \subset M \) consisting of two disjoint parts \( \theta_1, \theta_2 \) such that

(i) \( \theta_1 \) is a union of pairwise disjoint smooth simple closed curves disjoint from \( \mathcal{P} \);

(ii) \( \theta_2 \) is a graph whose edges have interiors disjoint from \( \mathcal{P} \);

(iii) every vertex \( v \) of \( \theta \) such that \( v \not\in \mathcal{P} \) is a 3-valent *switch*, which means the following.

There are exactly three tails attached to \( v \), and they can be numbered \( t_1, t_2, t_3 \) so that \( t_1 \) forms a smooth arc together with any of \( t_2 \) and \( t_3 \) (thus, \( t_2 \) and \( t_3 \) give rise to a cusp). The tail \( t_1 \) will be referred to as *outgoing*, and \( t_2 \), \( t_3 \) *ingoing*;

(iv) no connected component of \( M \setminus \theta \) is an empty disk with exactly two ordinary cusps and no special cusp;

(v) no connected component of \( M \setminus \theta \) is an empty disk with less than two cusps.

Connected components of \( \theta_1 \) and edges of \( \theta_2 \) will be referred to as *branches* of \( \theta \). Branches that are not attached to at least one switch are called *free*. In particular, all branches contained in \( \theta_1 \) are such.

If both tails of an edge of \( \theta_2 \) are outgoing or one is outgoing and the other is attached to a puncture, then the edge is called a *wide branch* of \( \theta \).

So, our train tracks may have vertices at punctures, and those vertices are *not* switches. For instance, the closure \( \gamma \) of any multiple curve \( \gamma \) is a train track in our sense.

Let \( (\theta, w) \) be a pair in which \( \theta \) is a train track, and \( w \) is an assignment to every branch a non-negative integer, which is referred to as *the width* of the branch, such that, for every switch, the sum of the widths of the ingoing tails equals to the width of the outgoing one. We will call such a pair a *measured train track*.

The *complexity* \( |(\theta, w)| \) of a measured train track \( (\theta, w) \) is defined as

\[
|(\theta, w)| = \sum_{\alpha} \left( 1 + \log_2(w(\alpha) + 1) \right),
\]

where the sum is taken over all branches of \( \theta \). One can see from this formula that the complexity decreases whenever the width of a branch decreases or a branch with zero width is removed.

Every measured train track \( (\theta, w) \) encodes a multiple curve as follows. Each branch \( \alpha \) of \( \theta \) is replaced by as many as \( w(\alpha) \) ‘parallel’ copies \( \alpha_1, \ldots, \alpha_{w(\alpha)} \) of \( \alpha \). If \( \alpha \) is attached to a puncture \( P_i \), then the corresponding arcs \( \alpha_j \) approach \( P_i \) at the corresponding end. At every switch, the parallel copies of ingoing tails are attached to that of the outgoing one so as to get a non-selfintersecting curve; see Figure 6. (If \( \alpha \) is a branch of \( \theta \) having the form of a one-sided closed simple curve, then ‘\( w(\alpha) \) parallel copies of \( \alpha \)’ should be understood ‘locally’. Precisely this means ‘\( [w(\alpha)/2] \) parallel copies of the boundary of a small tubular neighborhood of \( \alpha \) and, if \( w(\alpha) \) is odd, \( \alpha \) itself’. Here \([x]\) stands for the integral part of \( x \).)

A curve \( \gamma \) obtained in this way from \( \theta \) for some choice of branch widths is said to be *carried* by the train track \( \theta \).

![Figure 6. Turning a measured train track into a multiple curve](image)
More formally, the correspondence between measured train tracks and curves can be described as follows. For every train track \( \theta \), we fix a singular foliation \( \mathcal{F}_\theta \) on \( M \) such that:

(i) every branch of \( \theta \) is transverse to \( \mathcal{F}_\theta \) everywhere except at the punctures;
(ii) \( \mathcal{F}_\theta \) has only isolated singularities;
(iii) \( \mathcal{F}_\theta \) has a center-like singularity at every puncture. All other singularities are outside of \( \theta \) (see Figure 7);
(iv) every connected component of \( M \setminus \theta \) contains a singularity of \( \mathcal{F}_\theta \).

In order to construct such a foliation one first defines it in a small neighborhood of \( \theta \) so as to enforce (i) and (iii), then in a small disk in every connected component of \( M \setminus \theta \) so as to enforce (iv), and then continue to the whole surface generically.

**Figure 7.** Foliation \( \mathcal{F}_\theta \) near punctures and switches

**Definition 7.2.** A union \( \gamma \) of pairwise disjoint proper arcs and simple curves is said to be carried by a train track \( \theta \) if \( \gamma \) is transverse to \( \mathcal{F}_\theta \) and there exists a homotopy \( f : \gamma \times [0, 1] \rightarrow M \) such that:

(i) there are no singularities of \( \mathcal{F}_\theta \) in \( f(\gamma \times [0, 1]) \);
(ii) for all \( x \in \gamma \) we have \( f(x, 0) = x, f(x, 1) \in \theta \);
(iii) all the leaves of the foliation on \( \gamma \times [0, 1] \) induced by \( f \) from \( \mathcal{F}_\theta \) have the form \( x \times [0, 1], x \in \gamma \).

The map \( \pi_\gamma : \gamma \rightarrow \theta \) defined by \( \pi_\gamma(x) = f(x, 1) \) is called the projection of \( \gamma \) to \( \theta \). Due to Condition (i) in this definition and Condition (iv) in the definition of \( \mathcal{F}_\theta \) one can see that \( \pi_\gamma \) does not depend on a particular choice of the homotopy \( f \).

The measured train track \( (\theta, w) \) that encodes \( \gamma \) is defined by letting \( w(\alpha) \), where \( \alpha \) is a branch of \( \theta \), be the number of points in \( \pi_\gamma^{-1}(y) \) with \( y \) a point from the interior of \( \alpha \).

We will use the well known relation between the Euler characteristics of a compact surface \( D \) and singularities of a generic foliation \( \mathcal{F} \) on \( D \). Namely, the Euler characteristics \( \chi(D) \) is equal to the sum of topological indexes of all singularities of \( \mathcal{F} \) provided that the points of \( \partial D \) in which the leaves of \( \mathcal{F} \) are not transverse to \( \partial D \) are also regarded as singularities. The simplest singularities and their topological indexes are shown in Figure 8. Thus, Conditions (iv) and

**Figure 8.** Generic singularities of a foliation and their topological indexes

\[ \text{[v]} \] in Definition [vi] simply mean that the sum of indexes of all singularities of \( \mathcal{F}_\theta \) inside any
connected component $D$ of $M \setminus \theta$ is non-positive unless there is a puncture inside $D$, and the sum is strictly negative unless there is a puncture inside $D$ or at the boundary $\partial D$.

This implies, in particular, the following.

**Proposition 7.1.** Any curve $\gamma$ encoded by a measured train track satisfies the conventions that we introduced in Section 3. Namely, if $D \subset M$ is a disk that is bounded by the closure of a connected component of $\gamma$, then $D$ is not empty (i.e. contains a puncture).

**Proof.** Indeed, if the connected component in question is a closed curve, then it is transverse to $\mathcal{F}_\theta$. If it is an arc whose closure forms a loop, then (after an appropriate smoothing) it will contribute just $1/2$ to the sum of the singularity indexes whereas we have $\chi(D) = 1$. So, in both cases the total contribution of singularities from the interior of $D$ must be positive.

Since the boundary $\partial D$ can be homotoped to its projection $\pi_\gamma(\partial D) \subset \theta$ through a family of curves that remain transverse to $\mathcal{F}_\theta$ (except at one point in the case when $\partial D$ is the closure of a proper arc) there is a family of connected components $D_1, \ldots, D_k$ of $M \setminus \theta$ such that the set of singularities inside $D$ coincides with that inside $D_1 \cup \ldots \cup D_k$. Since the sum of the topological indexes of singularities inside $D$ is positive, some $D_i$ contains a puncture, and so does $D$. □

## 8. Universal train tracks

With every triangulation $T = (e_1, \ldots, e_N)$ we associate a train track $\theta_T$ having the following property: for any multiple curve $\gamma$, there is another multiple curve $\gamma'$ isotopic to $\gamma$ such that $\theta_T$ carries $\gamma'$. For this reason we call this train track *universal*. It is not uniquely defined but the arbitrariness in its definition will not matter.

We construct $\theta_T$ in three steps.

**Step 1.** Put three switches in each triangle of $T$ and mark a single point in each edge of $T$. Outgoing tails are connected by arcs to the marked points, and ingoing ones are paired so as to make three-cusped disk in each triangle (see Figure 9 on the left).

**Step 2.** Orient connected components of $\partial M$ arbitrarily. Then we detach the edges of the graph under construction from the marked points at $\partial M$ and pull them in the direction defined by the orientation of the corresponding edge of $T$ toward the nearest puncture at $\partial M$ (see Figure 9 in the center).

**Step 3.** Let $\theta$ be the graph constructed so far. For every internal puncture $P$, the connected component of $M \setminus \theta$ containing $P$ is a disk with smooth boundary, and this disk contains no other puncture. We put an additional switch at its boundary and connect it by a new branch with $P$. We put the new switches at wide branches of $\theta$ and position the new branches as shown in Figure 9 on the right. Namely, each new branch must be contained entirely in a single triangle, and its smooth extension through the new switch should point to the nearest edge of the triangle, i.e. away of the 3-cusped disk located inside the triangle.

The result may look as shown in Figure 10.
**Proposition 8.1.**

(i) For any multiple curve $\gamma$ there is an isotopic multiple curve $\gamma'$ such that $\theta_T$ carries $\gamma'$.

(ii) Among multiple curves isotopic to $\gamma$ and carried by $\theta_T$ there is a multiple curve $\gamma_{\min}$ that is minimal in the following sense: if $(\theta_T, w_{\min})$ encodes $\gamma_{\min}$ and $(\theta_T, w)$ encodes any other multiple curve isotopic to $\gamma$, then $w_{\min}(\alpha) \leq w(\alpha)$ for any branch $\alpha \subset \theta_T$. Clearly, such width assignment $w_{\min}$ is unique.

(iii) There is a linear time algorithm to produce $w_{\min}$ from normal coordinates of $\gamma$, and we have $|(\theta_T, w_{\min})| \leq C|\gamma|_T$, with $C$ not depending on $\gamma$, where by $|\gamma|_T$ we denote the following complexity measure:

$$|\gamma|_T = \sum_i \log_2(|\langle \gamma, e_i \rangle| + 1).$$

**Proof.** By construction, for every puncture $P$, we have a single branch of $\theta_T$ approaching $P$. Denote this branch by $\alpha_P$, and the triangle of $T$ containing $\alpha_P$ by $\tau_P$. If $P$ is an internal puncture, then the other end of $\alpha_P$ approaches a switch from an ingoing side, and the latter will be used to choose an orientation of $M$ at $P$, by which we mean a sign designation to either rotation direction. Namely, if the cusp at the switch occurs on the left when one travels along $\alpha_P$ from $P$ to the switch, then the counterclockwise direction will be positive and clockwise negative, and vice versa if the cusp occurs on the right; see Figure 11.

![Figure 11. The positive rotation direction at an internal puncture P](image)

We also choose an orientation of the surface at every boundary vertex $P$ so that a tangent vector to the boundary $\partial M$ having positive direction will point inward $M$ after a small rotation in the positive direction around $P$; see Figure 12.

An arc in a triangle $\tau$ of $T$ will be called *almost normal* if it connects a vertex of $\tau$ with an interior point of an adjacent side; see Figure 13.
Two arcs in a triangle $\tau$ of $T$ are called *similar* if they are ambient isotopic in $\tau$ relative to the vertices of $\tau$. If an arc is similar to a smooth arc contained in $\theta_T$ we say that it is *supported* by $\theta_T$.

Our train track $\theta_T$ is designed so that any side-to-side normal arc is supported by $\theta_T$ (and in a unique way). For a vertex-to-side or vertex-to-vertex normal arc this is typically not true. Exceptions occur in triangles having one or two edges at the boundary, see Figure 12 and Figure 14. The intersection of $\theta_T$ with such a triangle supports one vertex-to-side arc and, in the case of two boundary edges, one vertex-to-vertex arc.

The idea behind the construction of $\gamma'$ is to normalize the original curve with respect to $T$ and then deform all normal arcs that are not supported by $\theta_T$ so as to obtain a composition of normal and almost normal arcs that are supported. After that we can push the obtained curve toward $\theta_T$ so that all normal and almost normal arcs become close to the corresponding arcs in $\theta_T$.

In order to see how it works we start from the opposite side, i.e. from a multiple curve $\gamma'$ that is carried by $\theta_T$. Let $w$ be the corresponding width assignment to branches of $\theta_T$.

If $P$ is not a boundary puncture with just one triangle adjacent to it (consult Figure 14), then there is a unique, up to similarity, almost normal arc attached to $P$ that is supported by $\theta_T$. It is obtained by a smooth extension of $\alpha_P$ along $\theta_T$ up to the boundary of the triangle. Denote this almost normal arc by $\tilde{\alpha}_P$.

If $w(\alpha_P) > 0$, then $\gamma'$ contains an arc similar to $\tilde{\alpha}_P$, hence, it is not normal with respect to $T$ as $\tilde{\alpha}_P$ cuts a bigon off $\tau_P$. Now see what happens if we run the normalization procedure for $\gamma'$.

The following assertions remain true during the normalization process:
(i) at every normalization step the multiple curve $\gamma'$ is composed of normal and almost normal arcs;
(ii) every bigon reduction results in rotating the tail of an almost normal arc around the corresponding puncture in the negative direction.

Indeed, it is easy to see that a bigon whose boundary is disjoint from punctures cannot appear in the pulling tight process unless it was present at the beginning. Figure 15 demonstrates a single bigon reduction for all possible types of arcs extending the almost normal arc being reduces. In the first three cases, a normal arc is produced. In the last case, a new almost normal arc appears, and it is ‘oriented’ in the same way as the original one meaning that a small rotation in the positive direction around the puncture pushes it off the corresponding bigon.

Figure 15 shows all possible ways in which a normal arc that is not supported by $\theta_T$ may appear. So, it is clear how to invert this procedure.

Namely, we start from a normal curve $\gamma'$ isotopic to $\gamma$. Then we keep repeating the following step until $\gamma'$ is carried by $\theta_T$: if $\gamma'$ contains an unsupported almost normal arc, we apply an isotopy to $\gamma'$ that modifies such an arc by a transformation inverse to one of those shown in Fig 15. It is not hard to see that the procedure will eventually stop.

There is an arbitrariness in the process affecting the result in the following two ways. First, at some stages of the process there may be more than one unsupported arc to modify and more than one way to modify the chosen unsupported arc (the latter case may occur for a normal vertex-to-vertex arc). Second, if $\gamma'$ contains proper arcs isotopic to edges of $T$, then their initial position is not unique. Both issues are illustrated in Figure 16, where the universal train track is shown in grey dashed line.

However, the possible results are not very much different from each other. Namely, each one can be obtained from any other by creating and/or removing additional spiral turns like the one shown in Figure 17.

Figure 15. Reduction of a bigon cut off by an almost normal arc

Clearly, each spiral turn contributes positively into widths of the branches of $\theta_T$ that make a full turn around a puncture, so, in order to minimize the widths we must avoid the spiral turns. The branches $\alpha_P$ were constructed so that the spiral turns around different punctures do not overlap. So, there is always a unique way (up to isotopy preserving the triangulation) to remove them, which gives the sought-for $\gamma_{\min}$.

Computing the width assignment $w_{\min}$ corresponding to $\gamma_{\min}$ is now very simple. There are only finitely many different types of normal arcs. For each of them we implement the procedure described above and find an isotopic arc decomposed in the optimal way into normal and almost normal arcs supported by $\theta_T$. In this way the contribution of each normal arc type into the width assignment is computed and recorded. This is done only once, before any multiple curve is given. Note that a single normal arc of the normalized form of the original curve contributes at most two to the width of any branch of $\theta_T$. 
Then, given the normal coordinates of a multiple curve $\gamma$ one computes the number of normal arcs of each type (as described in the proof of Lemma 5.2) and sums up their contributions. The running time estimation and that for the complexity of the result are straightforward.

**Example 8.1.** Figure 18 illustrates how the curve $\gamma_{\text{min}}$ and the corresponding width assignment (the non-zero widths) look like for $\gamma$ the union of the five edges connecting the four punctures in Figure 10 where the choice of $\theta_T$ is shown in grey dashed line.

9. **Simplifying train tracks**

Simplification procedure introduced in this section is one of the many similar ones that mimic the accelerated Euclidean algorithm. The general principle for constructing such algorithms in low-dimensional topology settings was learnt by the author from the work of I. Agol, J. Hass, and W. Thurston.
Here we describe transformations \((\theta, w) \mapsto (\theta', w')\) of measured train tracks such that the multiple curves encoded by \((\theta, w)\) and \((\theta', w')\) are isotopic. To every such transformation we assign two numbers that are called the gain and the cost of the transformation. Vaguely speaking, the former indicates how much \((\theta', w')\) is simpler than \((\theta, w)\), and the latter measures ‘the algorithmic complexity’ of the operation.

Recall that by complexity \(|(\theta, w)|\) of a measured train track \((\theta, w)\) we mean the sum
\[
|(\theta, w)| = \sum_{\alpha} \left(1 + \log_2(w(\alpha) + 1)\right),
\]

taken over all branches of \(\theta\). It is comparable to the amount of space needed to encode \((\theta, w)\). However, for technical reasons, we will need a slightly more subtle measure of complexity.

Denote by \(A(\theta)\) the set of non-free branches of \(\theta\). Define
\[
|(\theta, w)|_0 = |A(\theta)| + \sum_{\alpha \in A(\theta)} \log_2(w(\alpha) + 1).
\]

This is obtained from (5) by dropping the contribution of free branches.

Whatever a transformation \((\theta, w) \mapsto (\theta', w')\) is the gain of this transformation is defined as the difference \(|(\theta, w)|_0 - |(\theta', w')|_0\). If we have a sequence
\[
(\theta_0, w_0) \mapsto (\theta_1, w_1) \mapsto \ldots \mapsto (\theta_k, w_k)
\]
of transformations, then the total gain of the sequence is set to \(|(\theta_0, w_0)|_0 - |(\theta_k, w_k)|_0\).

Now we introduce transformations \((\theta, w) \mapsto (\theta', w')\) of our interest. They will be referred to as simplification moves and include removing trivial branches, splittings (ordinary and multiple), and slidings defined below.

Removing trivial branches. The train track \(\theta'\) is obtained from \(\theta\) by removing all non-free branches \(\alpha\) such that \(w(\alpha) = 0\). If \(\theta\) contains a switch to which exactly one free branch of \(\theta\) is attached, then the other two branches approaching this switch become parts of a single branch of \(\theta'\). The width \(w'(\alpha')\) of any branch \(\alpha'\) of \(\theta'\) is set to \(w(\alpha)\) with any branch \(\alpha\) of \(\theta\) such that \(\alpha \subset \alpha'\) (clearly the choice of \(\alpha\) does not matter).

We set the cost of this operation to be equal to the number of non-free branches \(\alpha \subset \theta\) such that \(w(\alpha) = 0\).
Ordinary splitting. Recall that a branch $\alpha$ of $\theta$ is called wide in the following two cases:

1. both tails of $\alpha$ are outgoing for some switches;
2. one tail of $\alpha$ is outgoing, and the other approaches a puncture.

An ordinary splitting $\theta, w \mapsto (\theta', w')$ on a wide branch $\alpha$ is a modification of the measured train track $\theta, w$ that occurs in a small neighborhood of $\alpha$ and has the form shown in Figure 19, where widths of the involved branches are also indicated. Widths of the other branches are preserved.

To every ordinary splitting we assign cost 1.

Multiple splitting. Suppose that the train track $\theta$ contains two branches $\beta$ and $\gamma$, say, whose union is a two-sided simple closed curve. There must be two tails outside of $\beta \cup \gamma$ that approach switches at $\beta \cup \gamma$. We additionally suppose that they do it from different sides of $\beta \cup \gamma$. Finally, we suppose $w(\beta) < w(\gamma) \leq 2w(\beta)$.

Then $\gamma$ must be a wide branch, and we have a situation shown in Figure 20 on the left, where the widths $b = w(\beta)$, $c = w(\gamma)$, $a = c - b$ are indicated near the respective branches. By assumption, we have $c \leq 2b$, hence $b \geq a$. After a splitting on the branch $\gamma$ we get a measured train track $(\theta', w')$ that is obtained from $(\theta, w)$ by a Dehn twist along $\beta \cup \gamma$ and making the branches $\beta$, $\gamma$ narrower by $a$, see Figure 20. So, if $b \geq ka$, $k \in \mathbb{N}$, we can apply $k$ successive splittings to this portion of $\theta$, which result in the application of the $k$th power of a Dehn twist along $\alpha \cup \beta$ to $\theta$ and making the branches $\alpha$ and $\beta$ narrower by $ka$.

Such application of $k$ successive splittings will be treated as a single operation called a $k$-times multiple splitting on the circle $\beta \cup \gamma$. Its cost is set to $\log_2(k + 1)$.

Sliding. Let $\alpha$ be a branch of $\theta$ having one ingoing and one outgoing tail. A sliding along $\alpha$ is a modification of $(\theta, w)$ that occurs in a small neighborhood of $\alpha$ as shown in Figure 21. The widths of all branches outside of this neighborhood are preserved. The cost is set to 1. Note that unlike other simplification moves a sliding may have a negative gain, so, sometimes it does not justify the name ‘simplification move’.

**Figure 19.** Ordinary splittings
Figure 20. If $b/a \geq k \in \mathbb{N}$, then we can apply $k$ splittings at once, which will be a multiple splitting

\[ \alpha \mapsto - \rightarrow \]

Figure 21. A sliding

If $(\theta_0, w_0), (\theta_1, w_1), \ldots, (\theta_k, w_k)$ is a sequence of measured train tracks in which every transition $(\theta_i, w_i) \mapsto (\theta_{i+1}, w_{i+1})$ is a simplification move, then the sum of their costs is called the total cost of the sequence.

**Proposition 9.1.** Let $(\theta, w)$ be a measured train track. Then there exists a sequence of simplification moves starting from $(\theta, w)$ and ending with a measured train track without switches, such that the total cost of the sequence does not exceed $3 \cdot |(\theta, w)|$.

There is an algorithm that produces such a sequence in $O(|(\theta, w)|^2)$ operations on a RAM machine.

**Proof.** We prove the first statement with $|(\theta, w)|$ replaced by $|(\theta, w)|_0$, which is stronger as we always have $|(\theta, w)| \geq |(\theta, w)|_0$. We proceed by induction in $|(\theta, w)|_0$, where $\lfloor \cdot \rfloor$ stands for the integral part. The equality $|(\theta, w)|_0 = 0$ means that $\theta$ has no switches, and we are done.

For the induction step we just need to find a sequence $(\theta, w) = (\theta_0, w_0) \mapsto (\theta_1, w_1) \mapsto \ldots \mapsto (\theta_l, w_l)$ of simplification moves such that its total gain $g$ and total cost $p$ satisfy the following inequalities:

\[ g \geq 1, \quad g \geq p/3. \]

If $(\theta, w)$ has trivial branches, we remove them, which gives $g \geq p \geq 1$. In the sequel we assume that all non-free branches have positive widths.

If there is a wide branch that is attached to a puncture (see the lower part of Figure 19) we apply an ordinary splitting on it, which gives $g > p = 1$. In the sequel we assume that there is no such branch.

Among all wide branches of $(\theta, w)$ choose a widest one $\alpha$, say, i.e. having the largest width. By the assumption we have just made, both ends of $\alpha$ are switches. An ordinary splitting on $\alpha$ may then have arbitrarily small gain, so, our strategy will depend on the structure of $(\theta, w)$ around $\alpha$. We consider below a bunch of cases that are summarized in Figure 22, where $\alpha$ is the branch that has width $a + b + c$. The sign ‘$\otimes$’ in the pictures denotes an orientation flip.
Figure 22. The chart of simplification cases

**Case A.** We have \( a + b \geq c + 1 \), hence, for an ordinary splitting on \( \alpha \), we have

\[
2^g = \frac{a + b + c + 1}{c + 1} \geq 2.
\]

Thus, \( g \geq 1 = p \).
**Case B.** Let \( e = d - a > 0, \ f = c - e > 0; \) consult Figure 23. After one sliding and one ordinary splitting the two branches of width \( a + b + e + f \) and \( a + e + f \) are replaced by those of width \( b + f \) and \( e \). We have

\[
2^g = \frac{(a + b + e + f + 1)(a + e + f + 1)}{(b + f + 1)(e + 1)} \geq \frac{4(a + b + e + f + 1)(a + e + f + 1)}{(a + b + e + f + 1)^2} = \frac{4(a + e + f + 1)}{(a + b + e + f + 1)} > \frac{2(a + e + f + 1)}{e + f + 1} > 2
\]
as \( e + f = c \geq a + b \) in this case. Thus, \( g > 1, \ p = 2 \).

**Case C.** Let \( e = a - d \); see Figure 24. We apply two ordinary splittings, which give

\[
2^g = \frac{(b + c + d + e + 1)(c + d + e + 1)}{(c + 1)(e + 1)} > 2
\]
as \( c \geq a + b = b + d + e > e + 1 \). Thus, we have \( g > 1, \ p = 2 \).

**Case D.** We apply an ordinary splitting and a sliding as shown in Figure 25. Since the branch \( \alpha \) is the widest one, we have \( d \leq b \). Together with \( c \geq a + b \) this gives:

\[
2^g = \frac{(a + b + c + 1)(a + c + 1)}{(a + d + 1)(c + 1)} \geq \frac{4(a + c)}{a + b + c + 2} \geq \frac{2(a + c)}{c + 1} \geq 2.
\]
Hence, in this case, \( g > 1, \ p = 2 \).

**Case E.** Consult Figure 26 for notation. We have \( c \geq a + b \). Since \( \alpha \) is a widest branch we also have \( e + f \leq b \). Thus, we have \( e < b < c < a + c + f \), which implies that splitting on the
upper wide branch in Figure 26 will do the job (the situation is identical to Case A). We will have $g \geq 1 = p$.

**Case F.** After one splitting we come to the situation of Case B or Case C; see Figure 27. Thus, after applying two more simplification moves we have $g > 1$, $p = 3$.

**Case G.** Consult Figure 28 for notation. Since there are no branches wider than $a + b + c$, we must have $d + e \leq b$. An ordinary splitting followed by two slidings gives:

$$2^g = \frac{(a + b + c + 1)(a + c + 1)(a + c + d + 1)}{(c + 1)(a + e + 1)(c + d + 1)} \geq \frac{(a + b + c + 1)(a + c + 1)(a + c + d + 1)}{(c + 1)(a + b)(c + d + 1)} \geq \frac{4(a + c + 1)(a + c + d + 1)}{(a + b + c + 1)(c + d + 1)} > \frac{2(a + c + 1)(a + c + d + 1)}{(c + 1)(c + d + 1)} > 2.$$  

Thus, we have $g > 1$, $p = 3$.  

**Figure 25.** Simplification in Case D

**Figure 26.** Simplification in Case E

**Figure 27.** Simplification in Case F
Case H. Since there are no branches wider than \( a + b + c \), we have \( d + e \leq b \) (see Figure 29). After an ordinary splitting and a sliding shown in Figure 29 we have

\[
\begin{align*}
2^g &= \frac{(a + b + c + 1)(a + c + d + 1)}{(c + 1)(d + e + 1)} \geq \frac{(a + b + c + 1)(a + c + d + 1)}{(b + 1)(c + 1)} > 2
\end{align*}
\]

as \( c \geq a + b \). So, \( g > 1, p = 2 \).

Case I. The fragment of \((\theta, w)\) contains the configuration symmetric to that covered by cases B and C; see Figure 30.

Case J. After two splittings we can remove a trivial branch, which gives \( g > 3, p = 3 \) (see Figure 31).

Case K. Let \( k = \lfloor c/a \rfloor + 1 \), \( d = c - (k - 1)a \). We have \( 0 \leq d < a, k \geq 3 \). We apply a \( k \)-times multiple splitting (see Figure 32), which gives
Thus, \( g > \log_2(k+1) = p. \)

**Case L.** We apply two splittings (see Figure 33), which gives

\[
2^g = \frac{(ka + d + 1)((k+1)a + d + 1)}{(d+1)(a + d + 1)} > \frac{(ka + d + 1)((k+1)a + d + 1)}{2(d+1)a} > \frac{(k+1)^2}{2} > k+1.
\]

This completes the proof of the first claim of the proposition.

The procedure above gives explicitly an algorithm to find the desired simplification sequence. The only thing we need is to estimate the number of operations. Throughout the procedure we operate with integers whose absolute value is bounded by \(2^{\max(|\theta, w|)}\).
In every case except Case K we need to perform a bounded number of additions (subtractions) of such numbers. Thus, the amount of work at every step where we don’t have Case K is $O(|(\theta, w)|)$. In Case K we perform additionally a single division, which consumes time $O(|(\theta, w)| \cdot \log_2(k + 1))$, where $k$ is the multiplicity of the splitting.

Thus, in all cases the time consumed at each step of the algorithm is bounded by the total cost of the step multiplied by $|(\theta, w)|$, which implies the second claim of the Proposition. 

Proposition 9.2. Let $\mathcal{A} \subset G$ be a generating set as in Theorem 2.4 and let $T$ be a triangulation of $M$. Then the zipped word length function $\text{zw}l_{\mathcal{A}}$ is comparable to the matrix complexity function $c_T$.

Proof. We start from proving that $c_T \preceq \text{zw}l_{\mathcal{A}}$.

Due to Proposition 6.3 it suffices to show that $c_T(g^k)$ grows with $k$ not faster than $\log |k|$ when $g$ is a Dehn twist.

Let $g$ be a Dehn twist along a simple closed curve $\gamma$. For any multiple curve $\alpha$ such that $\gamma$ and $\alpha$ are tight, the image $g^k(\alpha)$ can be obtained from the union of $\alpha$ and $|k| \cdot \langle \gamma, \alpha \rangle$ parallel copies of $\gamma$ by resolving intersections of those copies with $\alpha$. Therefore, for any edge $e_i$ of the triangulation $T$ we have

$$\langle g^k(\alpha), e_i \rangle \leq |k| \cdot \langle \gamma, \alpha \rangle \cdot \langle \gamma, e_i \rangle + |\langle e_i, \alpha \rangle|,$$

which implies

$$c_T(g^k) \leq C \cdot \log_2(|k| + 1)$$

for some constant $C$.

Now we will show that $\text{zw}l_{\mathcal{A}} \preceq c_T$.

Denote by $K$ the number of branches in a universal train track (which is clearly independent on the choice of the latter). There are only finitely many, up to a self-homeomorphism of $(M, \mathcal{P})$, train tracks in $M$ with at most $K$ branches. So, we can fix a finite subset $X$ of train tracks such that:

(i) every train track in $X$ carries a triangulation;
(ii) for any train track $\theta$ that carries a triangulation and has not more than $K$ branches, there is an element $g \in G$ such that $g(\theta) \in X$.

If a train track $\theta$ carries a triangulation, then the set of $g \in G$ such that $g(\theta) \sim \theta$ is also finite. Therefore, there is a finite subset $H$ of $G$ such that for any simplification move $(\theta_1, w_1) \mapsto (\theta_2, w_2)$ with $\theta_1, \theta_2$ carrying a triangulation and having at most $K$ branches, and any $g_1, g_2 \in G$ such that $g_1(\theta_1), g_2(\theta_2) \in X$, the following holds:

(i) if $(\theta_1, w_1) \mapsto (\theta_2, w_2)$ is not a multiple splitting, then $g_1g_2^{-1} \in H$;
(ii) if $(\theta_1, w_1) \mapsto (\theta_2, w_2)$ is a $k$-times multiple splitting, then there is a Dehn twist $d \in H$ such that $g_1g_2^{-1} = ad^k$ for some $a \in H$.

Thus, in both cases $\text{zw}l_{\mathcal{A}}(g_1g_2^{-1})$ is bounded from above by $C \cdot p$, where $C$ is a constant and $p$ is the cost of the move $(\theta_1, w_1) \mapsto (\theta_2, w_2)$. In the multiple splitting case this is due to the hypothesis that every Dehn twist is conjugate to a fractional power of an element from $\mathcal{A}$.

We may assume without loss of generality that $\cup_{i=1}^N e_i \in X$ and $\theta_T \in X$, where $\{e_i\}_{i=1, \ldots, N}$ is the set of all edges of $T$.

Now let $g \in G$ be any element different from 1, and let $w_0$ be a width assignment to the branches of $\theta_T$ such that $(\theta_T, w_0)$ encodes $g(\cup_{i=1}^N e_i)$ in the minimal way. Pick a sequence of simplification moves

$$(\theta_T = \theta_0, w_0) \mapsto (\theta_1, w_1) \mapsto \ldots \mapsto (\theta_r = g(\cup_{i=1}^N e_i), w_r)$$
with total cost not larger than $3|\langle \theta_T, w_0 \rangle|$. Such a sequence exists according to Proposition \ref{prop:counting_intersections}. Since the number of branches never grows under a simplification move, all train tracks in this sequence have at most $K$ branches.

Now for every $i = 0, \ldots, r$ chose $g_i \in G$ so that $g_i(\theta_i) \in X$. Specifically for $i = 0$ and $r$ we put $g_0 = 1$ and $g_r = g^{-1}$. We will have

$$zwln(g) = zwln\left(\left(g_0 g_1^{-1}\right) \left(g_1 g_2^{-1}\right) \cdots \left(g_{r-1} g_r^{-1}\right)\right) \leq \sum_{i=1}^{r} zwln(g_{i-1} g_i^{-1}) \leq 3C|\langle \theta_T, w_0 \rangle|.$$ 

An application of Proposition \ref{prop:counting_intersections} completes the proof. \hfill \Box

### 10. Counting intersections

Here we present the main technical result of the paper and prove Theorem \ref{thm:counting_intersections}, which asserts the existence of an efficient solution of the word problem with respect to $zwln$. Before starting the actual proof we mention briefly a strategy that we are not going to follow, but which yields another proof of the theorem.

For any fractional power $a$ of a fixed Dehn twist, one can construct an algorithm that produces the normal coordinates of $a^k(\gamma)$ from the normal coordinates of a multiple curve $\gamma$, and an integer $k$ in time $O(|\gamma_T| \cdot \log_2 k)$, where $|\gamma_T|$ is defined by (1).

Doing so for all generators from $\mathcal{A}$ yields a translation algorithm from the zipped word presentation to the matrix presentation. Given a zipped word representing an element $g \in G$ it computes $\langle T, g(T) \rangle$ in time $O(zwln(g)^2)$ (if implemented properly).

The procedure from the proof of Proposition \ref{prop:counting_intersections} used to establish $zwln \leq c_T$ can be turned into an actual algorithm that performs the inverse translation, from the matrix presentation to the zipped word presentation, and also consumes $O(zwln(g)^2)$ amount of time. The output of the algorithm is a zipped word representing $g$ and depending only on $g$ but not on the original presentation. Thus, this output can be taken for the normal form of $g$.

The strategy that we do follow is not to translate back and forth, and use only the matrix presentation. The key ingredient missing so far is the following statement.

**Proposition 10.1.** There exists an algorithm that, given the normal coordinates of two multiple curves $\gamma_1$ and $\gamma_2$, computes $\langle \gamma_1, \gamma_2 \rangle$ in time $O(|\gamma_1|T \cdot |\gamma_2|T)$ on a RAM machine.

**Proof.** We will use a modification of the procedure from the proof of Proposition \ref{prop:counting_intersections} This time we are going to simplify two train tracks simultaneously to an extent that allows to detect all intersections between $\gamma_1$ and $\gamma_2$. We subdivide the proof into several subsections.

#### 10.1 General strategy and notation

At every step of the algorithm, the multiple curves $\gamma_1$ and $\gamma_2$ are encoded by measured train tracks denoted $(\theta_1, w_1)$ and $(\theta_2, w_2)$, respectively, which are being modified during the process. What data representing $\theta_1$ and $\theta_2$ is actually kept in computer’s memory is described in Subsection \ref{subsec:memory}.

Here is the skeleton of the algorithm.

- The algorithm receives as input the vectors of normal coordinates of $\gamma_1$ and $\gamma_2$.
- We start from $\theta_1 = \theta_2 = \theta_T$ and compute $w_1$ and $w_2$ so as to obtain the minimized representation of the isotopy classes of $\gamma_1$ and $\gamma_2$ by $\theta_T$ as described in the proof of Proposition \ref{prop:counting_intersections}.
- We run the simplification process for $(\theta_1, w_1)$ and $(\theta_2, w_2)$ as described in Subsections \ref{subsec:general_strategy} and \ref{subsec:notation}. For branches $\alpha$ and $\beta$ of $\theta_1$ and $\theta_2$, respectively, the number of their transverse intersections is counted during the simplification process. By abusing notation slightly we denote this number by $\langle \alpha, \beta \rangle$. If $\alpha = \beta$ is the closure of a proper arc we set $\langle \alpha, \beta \rangle = -1$. 

Initially we set $\langle \alpha, \beta \rangle = 0$ for all branches $\alpha$ and $\beta$ of $\theta_1$ and $\theta_2$, respectively, as there are no transverse intersections of $\theta_1$ and $\theta_2$ and no proper arcs in any of them. These numbers are updated during the simplification process whenever a new intersection (or coincidence of proper arcs) is detected or any of $\theta_1$ and $\theta_2$ is modified.

- When the simplification finishes, we compute

$$\langle \gamma_1, \gamma_2 \rangle = \sum_{\alpha, \beta} \langle \alpha, \beta \rangle w_1(\alpha)w_2(\beta), \tag{7}$$

where the sum is taken over all branches $\alpha$ of $\theta_1$ and $\beta$ of $\theta_2$, and this is the output.

At every step of the simplification process, the train tracks $\theta_1$ and $\theta_2$ partially coincide and have a finite number of transverse intersection points. The latter may not occur at switches of $\theta_1$ and $\theta_2$. We denote by $\theta_\cap$ the set of all non-isolated points of $\theta_1 \cap \theta_2$, and by $\theta_\partial$ the set of all isolated ones. At every step, the intersection $\theta_1 \cap \theta_2$ is homeomorphic to a simplicial complex of dimension $\leq 1$, with $\theta_\cap$ being the 1-dimensional part of $\theta_1 \cap \theta_2$ and $\theta_\partial$ the 0-dimensional one.

We think of $\theta_1 \cup \theta_2$ as ‘a train track with self-intersections’ and use notation $\theta_\cup$ for the ‘abstract train track’ of which $\theta_1 \cup \theta_2$ is the image under an immersion $\theta_\cup \to M$. The formal meaning of $\theta_\cup$ will not be needed, but it will be handy to define branches and switches of $\theta_\cup$.

**Definition 10.1.** By a branch of $\theta_\cup$ we mean any of the following:

(i) a connected component of $\alpha \cap \beta$ different from a single point, where $\alpha$ and $\beta$ are branches of $\theta_1$ and $\theta_2$, respectively;

(ii) the closure of a connected component of $\alpha \setminus \theta_\cap$, where $\alpha$ is a branch of $\theta_1$ or $\theta_2$.

By a switch of $\theta_\cup$ we mean a point $p \in (\theta_1 \cup \theta_2) \setminus (\theta_\partial \cup \mathcal{P})$ such that the intersection of $\theta_1 \cup \theta_2$ with any small neighborhood of $p$ is not an arc.

We allow only 3-valent switches of $\theta_\cup$, which means that exactly three branches of $\theta_\cup$ (counted with multiplicity) join at every switch. There are, however, four ways how a switch of $\theta_\cup$ can arise.

A switch of $\theta_\cup$ can be a switch of both $\theta_1$ and $\theta_2$. For a small enough neighborhood $U$ of such a switch we have $U \cap \theta_1 = U \cap \theta_2$.

A switch of $\theta_\cup$ can be a switch of $\theta_1$ but not of $\theta_2$. For a small enough neighborhood $U$ of such a switch we have $\theta_2 \cap U \subset \theta_1 \cap U$. In particular, the intersection $\theta_2 \cap U$ may be empty.

Similarly, a switch of $\theta_\cup$ can be a switch of $\theta_2$ but not of $\theta_1$.

Finally, a switch of $\theta_\cup$ can be neither a switch of $\theta_1$ nor a switch of $\theta_2$. The intersection of a small enough neighborhood $U$ of such a point with either $\theta_1$ or $\theta_2$ is an arc. We call such a switch a divergence point. Among three branches of $\theta_\cup$ joining at a divergence point, exactly one is contained in $\theta_\cap$, one in $\theta_1$ but not in $\theta_2$, and one in $\theta_2$ but not in $\theta_1$ (see Figure 34).

![Figure 34. A divergence point](image)

By the width $w(\alpha)$ of a branch $\alpha$ of $\theta_\cup$ we call the pair $(u_1, u_2)$ in which $u_i$ is equal to $w_i(\beta)$ if a branch $\beta$ of $\theta_i$ contains $\alpha$, and 0 if $\alpha$ is not contained in $\theta_i$.

The terms ‘ingoing’, ‘outgoing’, ‘free’, and ‘wide’ have the same meaning for branches of $\theta_\cup$ as for branches of an ordinary train track.
10.2. **Simplification moves.** Here we introduce certain transformations of the pair \(((\theta_1, w_1), (\theta_2, w_2))\) under which each of the two measured train tracks is modified either by an isotopy or by a simplification move introduced in Section 9.

**Removing trivial branches.** For each of the measured train tracks \((\theta_1, w_1)\) and \((\theta_2, w_2)\) this move consists, as before, in removing trivial branches. On the level of \(\theta_n\) this means that all non-free branches of \(\theta_n\) width \((0,0)\) are removed, and those whose width has the form \((0,u)\) (respectively, \((u,0)\)) with \(u > 0\) are thought of as being contained in \(\theta_2\) but not in \(\theta_1\) (respectively, in \(\theta_1\) but not in \(\theta_2\)).

This move has the highest priority. So, in the sequel, whenever the width of a branch has the form \((0,u)\) of \((u,0)\) we assume that it is no longer contained in \(\theta_n\) and any non-free branch of \(\theta_n\) of width \((0,0)\) has to be erased.

**Ordinary splitting of \(\theta_n\).** If \(\alpha\) is a wide branch of \(\theta_n\) we can perform a splitting on \(\alpha\). This will be done only if \(w_1(\alpha) > 0\) and \(w_2(\alpha) > 0\). If one of the endpoints of \(\alpha\) is a puncture, then it is done exactly as in the case of a single train track, see the bottom of Figure 19.

For each of the measured train tracks \((\theta_i, w_i), i = 1, 2\), this will result in an ordinary splitting or just an isotopy depending on whether or not \(\alpha\) is a wide branch of \(\theta_i\).

If both endpoints of \(\alpha\) are switches of \(\theta_n\), a splitting on \(\alpha\) will mean the modification of \(\theta_n\), \(w_1, w_2\) shown in Figure 35 where the indicated widths of the branches are related as follows:

\[
c_1 = \max(0, a_1 + b_1 - a'_1 - b'_1), \quad c_2 = \max(0, a_2 + b_2 - a'_2 - b'_2),
\]
\[
d_1 = \max(0, a'_1 + b'_1 - a_1 - b_1), \quad d_2 = \max(0, a'_2 + b'_2 - a_2 - b_2).
\]

and it is understood that branches of width \((0,0)\), if any, must be erased.

**Figure 35.** Splitting two train tracks simultaneously

For each of the measured train tracks \((\theta_i, w_i), i = 1, 2\), this operation may be an isotopy, an ordinary splitting, or an ordinary splitting followed by removing a trivial branch. All combinations can occur. A new point of \(\theta_n\) may or may not be introduced depending of the widths of the branches adjacent to the endpoints of \(\alpha\). If it is introduced we say that \(\theta_1\) and \(\theta_2\) disagree on the branch being splitted.

**Multiple splitting of \(\theta_n\).** We refer again to Figure 20. Now all widths are not just integers but integral vectors from \(\mathbb{Z}_2^2\): \(a = (a_1, a_2), \) etc. Multiple splitting of \(\theta_n\) will be used only if \(b_i = a_i > 0, i = 1, 2\). Again, it is equivalent to applying \(k\) times an ordinary splitting on the wide branch in the fragment, where for \(k\) we take the largest integer satisfying \(b_1 \geq ka_1, b_2 \geq ka_2\). For each of the measured train tracks \((\theta_i, w_i), i = 1, 2\), this will result in a \(k\)-multiple splitting in the previously defined sense.

Separation of circles. Suppose \(\partial_\gamma\) has a connected component \(\sigma\) that is a two-sided simple curve containing exactly four switches of \(\partial_\gamma\) on \(\sigma\) two of which are switches of \(\partial_\gamma\) and the other two of \(\partial_\gamma\). Suppose also that branches of \(\partial_\gamma\) approach \(\sigma\) from both sides, \(i = 1, 2\). Thus, the parts of \(\gamma_1\) and \(\gamma_2\) located in a small neighborhood of \(\sigma\) have the form of ‘spirals’.

Finally, suppose that the ‘spirals’ of \(\gamma_1\) and \(\gamma_2\) are twisted in opposite ways. Formally this means the following. Let \(\alpha\) be a smooth arc in a small neighborhood of \(\sigma\) such that:

(i) \(\alpha\) is contained in \(\partial_\gamma\);
(ii) one of the endpoints of $\alpha$ is in $\theta_1 \setminus \theta_2$ and the other in $\theta_2 \setminus \theta_1$;
(iii) the intersection $\alpha \cap \theta_\gamma$ is an arc contained in $\sigma$.

Then the endpoints of $\alpha$ are on the same side of $\sigma$, see the left picture on Figure 36.

Then we can deform $\theta_1$ and $\theta_2$ so as to obtain disjoint simple closed curves $\sigma_1 \subset \theta_1$, $\sigma_2 \subset \theta_2$ close to $\sigma$ and such that $\sigma_1$ has a single intersection point with $\theta_2$ and so does $\sigma_2$ with $\theta_1$; see Figure 36.

Such modification of $\theta_1$, $\theta_2$ will be referred to as a separation of circles.

We also will use this name in the situation when a connected component $\sigma$ of $\theta_\gamma$ has the form of a two-sided simple curve that is a free branch of $\theta_1$ and contains two switches of $\theta_2$ such that there are branches of $\theta_2$ approaching $\sigma$ from both sides. Separation of circles in this case works as before, Figure 36 will illustrate this if the branches of width $(a_1, 0)$ are erased.

10.3. Simplification rules. The simplification procedure starts with two measured train tracks $(\theta_1, w_1)$, $(\theta_2, w_2)$ such that $\theta_1 = \theta_2 = \theta_\gamma = \theta_T$ and modifies them so as to end up with the situation in which $\theta_\gamma$ consists of common free branches of $\theta_1$ and $\theta_2$.

The simplification rules are not symmetric with respect to $\theta_1$ and $\theta_2$. The process starts from checking which of the multiple curves $\gamma_1$ and $\gamma_2$ is simpler. If $|\gamma_1|_T > |\gamma_2|_T$, then their roles are exchanged. So, we suppose in the sequel that $|\gamma_1|_T \leq |\gamma_2|_T$. (More honestly, we compute an approximate value of $|\gamma_i|_T$ by using the $[\log_2] \gamma$ function instead of $\log_2$, which is much quicker. This produces a bounded error, which can be ignored in this context.)

The simplification runs as follows.

**Step 1:** Remove all trivial branches of $\theta_1$, $\theta_2$.

**Step 2:** If $\theta_\gamma \setminus \mathcal{P}$ has a contractible connected component with at least one switch of $\theta_\cup$ in it then:
(i) do a splitting on a wide branch of $\theta_\cup$ contained in (the closure of) this component;
(ii) repeat this step.

**Step 3:** If there is a switch of $\theta_1$ contained in $\theta_\gamma$ such that its outgoing branch $\alpha$ of $\theta_1$ contains a divergence point (equivalently, is not covered by $\theta_\gamma$) then:
(i) run the cleanup process described below, on $\alpha$;
(ii) return to Step 2;

**Step 4:** If there is a branch of $\theta_1$ that is also a wide branch of $\theta_\cup$ on which $\theta_1$ and $\theta_2$ disagree then:
(i) split this branch;
(ii) return to Step 2;

**Step 5:** Run the simplification procedure from the proof of Proposition 9.1 for $(\theta_1, w_1)$ with the following modifications:
(i) at every step, the widest branch $\alpha$ should be chosen only among wide branches of $\theta_1$ contained in $\theta_\cap$;
(ii) do not perform any splitting or sliding on a branch of $\theta_1$ not contained in $\theta_\cap$. In any of the Cases B–L shown in Figure 22, if the branch of width $a+c$ (in $(\theta_1, w_1)$) is not covered by $\theta_2$ just perform a splitting of the branch of width $a+b+c$;
(iii) should an ordinary splitting or sliding be performed on a branch $\alpha$ of $\theta_1$, first remove all switches of $\theta_\cup$ from $\alpha$ by running the cleanup process on $\alpha$ and then perform a splitting or sliding on $\alpha$ with $\theta_\cup$ so as to have the desired modification of $\theta_1$;
(iv) should a multiple splitting be performed on a circle $\sigma$ of $\theta_1$, first run the cleanup process for $\sigma$ and then do either a multiple splitting on $\sigma$ with $\theta_\cup$ or a separation of circles, whichever is applicable. If none of these can be applied perform an ordinary splitting of $\theta_\cup$ on the wide branch contained in $\sigma$;
(v) whenever during this process the number of branches of $\theta_1$ contained in $\theta_\cap$ decreases interrupt the process and return to Step 2;
(vi) after each round of simplification return to Step 3.

**Step 6:** If $\theta_\cap$ has a connected component $\sigma$ that is a free branch of $\theta_1$ having the form of a simple curve and containing a switch of $\theta_2$, then:
(i) run the cleanup procedure for this component;
(ii) if two switches of $\theta_2$ remain on $\sigma$ do a separation of circles on $\sigma$;
(iii) repeat this step.

Now we describe the cleanup procedure. The general principles are as follows:

(i) we apply simplification moves to $\theta_\cup$ so that $\theta_1$ does not change (or changes by isotopy);
(ii) we remove switches of $\theta_\cup$ from a branch or a circle consisting of two branches of $\theta_1$ so that either the desired simplification move of $\theta_1$ becomes extendable to a simplification move of $\theta_\cup$ or it becomes possible to apply a separation of circles.

The cleanup procedure appears in three different versions.

*Cleanup of a single branch of $\theta_1$ having the form of an arc.* Let $\alpha$ be a branch of $\theta_1$ not forming a simple closed curve. We suppose that there are some switches of $\theta_\cup$ in the interior of $\alpha$ and we want to get rid of them. A tail of a branch of $\theta_\cup$ not contained in $\theta_1$ is attached to every such switch. We call these tails *shavings.*

The branch $\alpha$ is locally two-sided, so we can choose one side to be top and the other to be bottom. We can also orient $\alpha$ and think of this orientation as being from left to right. Having fixed this orientations we can sort shavings and the corresponding switches of $\theta_\cup$ contained in $\alpha$ into four types: bottom-left, bottom-right, top-left, and top-right according to the direction from which the corresponding shaving approaches the switch.

At the first stage of the cleanup we move left (top and bottom) shavings to the right and right shavings to the left of $\alpha$ as well as reduce the number of shavings of each type to at most one. This is done by performing splittings on wide branches of $\theta_\cup$ contained in the interior of $\alpha$ (see Figure 37a,b), slidings on branches of $\theta_\cup$ connecting switches of the same type (Figure 37c), and, if neither of these is possible but still there are two shavings of the same type, slidings on branches of $\theta_\cup$ contained in the interior of $\alpha$ followed by another sliding reducing the number of shavings (Figure 37d).

At the second stage of the cleanup we do splittings (Figure 38a, b) and/or slidings (Figure 38c, d), whichever are applicable, on branches of $\theta_\cup$ contained in $\alpha$ and sharing an endpoint with $\alpha$. Each operation removes one switch of $\theta_\cup$ from $\alpha$.

*Cleanup of a circle consisting of two branches of $\theta_1$.* Let $\sigma$ be a circle consisting of two branches of $\theta_1$ such that a multiple splitting of $\theta_1$ can be performed on $\sigma$. We denote by $\alpha$ the wide
branch of $\theta_1$ contained in $\sigma$, and the other branch by $\beta$. We do the cleanup for $\alpha$ as described above, then for $\beta$, and again for $\alpha$. One can see that after the second cleanup (for $\beta$) at most two shavings may remain on $\sigma$, and after the third one they either escape from $\sigma$ or shift to $\beta$, in which case we get the situation shown in Figure 36 on the left.

**Cleanup of a free branch of $\theta_1$ having the form of a simple closed curve.** Let $\sigma$ be a circular free branch of $\theta_1$ with some switches of $\theta_\cup$ on it. Choose a point $p \in \sigma$ disjoint from those switches. Run the first stage of cleanup for $\sigma \setminus \{p\}$ as if it is an ordinary branch of $\theta_1$. At most four switches of $\theta_\cup$ will remain on $\sigma$. Then, if necessary, do splittings on wide branches of $\theta_\cup$ contained in $\sigma$ until at most one shaving remains on each side of $\sigma$. At most three splittings are needed for that.

In Subsection 10.6 we will show that the simplification process defined above eventually stops, (and estimate the asymptotic complexity of the algorithm). For the moment, we take it for granted.

**Lemma 10.2.** When the simplification is completed, $\theta_1$ and $\theta_2$ satisfy the following conditions:
(i) each branch of $\theta_1$ is transverse to all branches of $\theta_2$ with an exception that some free branches of $\theta_1$ may coincide with free branches of $\theta_2$;
(ii) the switches of both $\theta_1$ and $\theta_2$ are disjoint from $\theta_1 \cap \theta_2$.

Proof. The simplification rules are designed so that after any elementary operation that can result in creating a new contractible connected component of $\theta_1 \setminus \mathcal{P}$ (these are operations at Step 3, Step 4, and those at Step 5 that satisfy Condition (v)) we return to Step 2 and destroy such components or make them free of switches of $\theta_1 \cup \theta_2$.

At each of Steps 3, 4, and 5 we either change something in $\theta_1 \cup \theta_2$ and return to one of the previous steps or do nothing. So, proceeding with Step 6 means that, just before that, we passed Steps 3, 4, and 5 with no action.

Let $\Omega$ be a connected component of $\theta_1 \cap \theta_2$ at the moment when we proceed to Step 6, and let $\alpha$ be the widest branch of $\theta_1$ having a non-empty intersection with $\Omega$. Then $\alpha$ cannot be a wide branch of $\theta_1$ contained in $\Omega$, since otherwise we would do something non-trivial at Step 4 or 5. It cannot be an ingoing branch of $\theta_1$ for a switch contained in $\theta_1 \cap \theta_2$, since otherwise a wider branch of $\theta_1$ than $\alpha$ would have a non-empty intersection with $\Omega$. It cannot also be an outgoing branch of $\theta_1$ for a switch contained in $\theta_1 \cap \theta_2$, since otherwise we would either do something non-trivial at Step 3 or one of the previously ruled out cases occurs. Finally, it cannot happen that $\Omega$ is a portion of $\alpha$ not containing the endpoints, since whenever such connected components of $\theta_1 \setminus \mathcal{P}$ appear they are destroyed by running Step 2.

We are left with the following two options.

(i) $\Omega$ is a proper arc and $\alpha = \Omega$ is a free branch of $\theta_1$. In this case, $\alpha$ is also a free branch of $\theta_2$. Indeed, due to rule (v) at Step 5, whenever a free branch of $\theta_1$ having the form of the closure of a proper arc emerges as a result of a splitting, it is either disjoint from $\theta_1 \setminus \mathcal{P}$ or we return to Step 2 and make this branch free of switches of $\theta_1 \cup \theta_2$.

(ii) $\Omega = \alpha$ is a free branch of $\theta_1$ having the form of a simple closed curve. If some switches of $\theta_2$ remain on $\Omega$ they are removed at Step 6.

This completes the proof of the first claim of the proposition. The second claim follows from the definition of the simplification moves: transverse intersections of $\theta_1$ and $\theta_2$ are never created at switches.

10.4. Data representation. The efficiency of implementation of the simplification procedure described above depends heavily on the way in which the combinatorial data is represented. The topological description of the algorithm might suggest that at every step we should somehow keep track of how $\theta_1$ and $\theta_2$ are embedded in $M$, but this is not the case. We only need ‘a local description’ of $\theta_1 \cup \theta_2$ to implement the algorithm, which means the following.

We create a family of data objects, one per each branch and each switch of $\theta_1 \cup \theta_2$, and each puncture. For every switch and puncture we fix (arbitrarily) a surface orientation in its small neighborhood. Each object keeps references to related objects (e.g., each puncture keeps references to branches of $\theta_1$ adjacent to it) together with orientation information: for each puncture and each switch it is a cyclic order of the attached tails, and for each branch there is a boolean saying weather the orientations at the endpoints agree along the branch.

Additionally, for every branch, we keep information about its width, and, for every pair of branches $\alpha$ and $\beta$ of $\theta_1$ and $\theta_2$, respectively, the number $\langle \alpha, \beta \rangle$ of their transverse intersections which is set to $-1$ if $\alpha = \beta$ is the closure of a proper arc). All these data are updated at every simplification step in the way reflecting the change of $(\theta_1, w_1)$, $(\theta_2, w_2)$, and $\theta_1 \cup \theta_2$.

One can see that the result of each simplification move can be computed in terms of this data without any reference to the actual immersion $\theta_1 \cup \theta_2 : M \to M$, and the number of operations needed
for implementing a single move is bounded by a constant. By an operation here we mean a creation or removal of an object, an assignment, or an arithmetic operation.

10.5. Why the output is $\langle \gamma_1, \gamma_2 \rangle$? The aim of this subsection is to show that equality (7) holds when the simplification described in the Subsections 10.2 and 10.3 halts. The proof is based on the following fact.

Lemma 10.3. The multiple curves $\gamma_1$ and $\gamma_2$ can be deformed by isotopies, and the simplification of the train tracks $\theta_1$, $\theta_2$ can be implemented so as to satisfy the following two conditions:

(i) $\gamma_1$ and $\gamma_2$ are in tight position;
(ii) $\gamma_1$ and $\gamma_2$ are carried by $\theta_1$ and $\theta_2$, respectively, at any stage of the simplification process in a consistent way. The latter means that the foliations $F_{\theta_1}$ and $F_{\theta_2}$ (see Section 7 for the notation), to which $\gamma_1$ and $\gamma_2$ are transverse, coincide and remain unchanged during the simplification. Initially, $\gamma_1$ and $\gamma_2$ are carried by $\theta_T$ in the minimal way (see Proposition 8.7).

Proof. We use the construction of the foliation $F_\theta$ from Section 7. Let $\mathcal{F} = F_{\theta_T}$. All modifications of the train tracks $\theta_1$ and $\theta_2$ made during the simplification process can be performed so that all branches of $\theta_1$ and $\theta_2$ remain transverse to $\mathcal{F}$ and close to $\theta_T$, which means that $\mathcal{F}$ can be taken for $F_{\theta_1}$ and $F_{\theta_2}$ at any stage of the process. Moreover, if the curves $\gamma_1, \gamma_2$ have been chosen initially to be carried by $\theta_T$, then they need not be changed during the simplification to be carried by $\theta_1$ and $\theta_2$, respectively. We assume that they are carried by $\theta_T$ in the minimal way.

Let $U$ be an open neighborhood of $\theta_T \setminus \mathcal{P}$ such that $\mathcal{F}$ has no singularities in $U$ and every leaf of $\mathcal{F}|_U$ is an arc intersecting $\theta_T \setminus \mathcal{P}$. Clearly, we may assume that $\gamma_1, \gamma_2$ are contained in $U$, and so are $\theta_1 \setminus \mathcal{P}$ and $\theta_2 \setminus \mathcal{P}$ during the whole simplification process.

Now consider the pulling tight process for $\gamma_1, \gamma_2$. Let $D$ be a bigon of $\gamma_1$ and $\gamma_2$ which is going to be reduced, and let $p$ be one of its corners. Smoothing out the boundary $\partial D$ near $p$ by a small perturbation (see Figure 39) either makes it transverse to $\mathcal{F}$ or creates an isolated tangency point of topological index $1/2$ (like the one on the last picture of Figure 8). We claim that the former case is impossible. Indeed, the sum of the indexes of the singularities of $\mathcal{F}$ located in the interior of $D$ is non-positive, since every connected component of $M \setminus U$ having a non-empty intersection with $D$ is contained entirely in $D$. So, the only way how we can have $\chi(D) = 1$ is that both corners contribute $1/2$.

This means that, after the reduction of $D$, the new curves $\gamma_1, \gamma_2$ remain transverse to $\mathcal{F}$, and hence, are still carried by $\theta_T$. Clearly, the sum $w_1 + w_2$ of the width functions corresponding to $\gamma_1, \gamma_2$ does not change. Since both have been chosen initially in the minimal way, this implies that neither of them changes under a bigon reduction. □

In what follows we assume that $\gamma_1$ and $\gamma_2$ satisfy the conditions from Lemma 10.3 and keep the notation $U$ from its proof.
A kind of tightness holds also for $\theta_1$ and $\theta_2$. Namely, we have the following.

**Lemma 10.4.** At no stage of the simplification procedure, there is a 2-disc $D \subset \overline{U}$ such that $D \cap (\theta_1 \cup \theta_2) = \partial D$ and $\partial D$ consists of two smooth arcs.

**Proof.** Indeed, this is true initially, since $\theta_1$ and $\theta_2$ coincide with $\theta_T$, and for no connected component $V$ of $U \setminus \theta_T$ we have $\partial V \subset \theta_T$. During the subsequent simplification, new connected components of $U \setminus (\theta_1 \cup \theta_2)$ are created (consult Figures 33 and 36), but the boundary of each of them has three breaking points. Also, some of these connected components may join during the simplification forming either an open 2-disc with at least four breaking points at the boundary or a non-simply-connected domain. \hfill $\square$

Denote by $\pi_1$ and $\pi_2$ the projections $\pi_1 : \gamma_1 \to \theta_1$ and $\pi_2 : \gamma_2 \to \theta_2$, respectively, defined for the state of $\theta_1, \theta_2$ after the simplification has finished. We extend $\pi_1$ and $\pi_2$ to the closures $\overline{\gamma}_1$, $\overline{\gamma}_2$ by continuity, that is, put $\pi_i(P_j) = P_j$ whenever $P_j \in \overline{\gamma}_i$, $i = 1, 2$.

For any $p \in \gamma_i$, $i = 1, 2$, define $\delta_i(p)$ to be a (unique) closed subarc, possibly degenerate to a point, of the leaf of $\mathcal{F}_U$ passing through $p$ such that $\partial \delta_i(p) = \{p, \pi_i(p)\}$. Denote by $\Gamma$ be the following subset of $\gamma_1 \times \gamma_2$:

$$\Gamma = \{(p_1, p_2) \in \gamma_1 \times \gamma_2 : \delta_1(p_1) \cap \delta_2(p_2) \neq \emptyset\},$$

and by $\mathcal{X}$ the set of connected components of $\Gamma$. Define a map $s : \mathcal{X} \to \{-1, 0, 1\}$ as follows:

$$s(\beta) = \begin{cases} -1, & \text{if } \beta \text{ is homeomorphic to an open interval;} \\ 1, & \text{if } \beta \text{ is homeomorphic to a closed interval and } (p, p) \in \beta \text{ for some } p \in \gamma_1 \cap \gamma_2; \\ 0, & \text{otherwise.} \end{cases}$$

First, we claim that the following equality holds:

$$\langle \gamma_1, \gamma_2 \rangle = \sum_{\beta \in \mathcal{X}} s(\beta). \quad (8)$$

Indeed, let $\beta$ be a connected component of $\Gamma$. Denote the projections of $\beta$ to $\gamma_1$ and $\gamma_2$ by $\beta_1$ and $\beta_2$, respectively.

Suppose that $\beta$ is an open arc. Since $\Gamma$ is a closed subset of $\gamma_1 \times \gamma_2$, this means that $\beta_1$ and $\beta_2$ are proper arcs. By construction, their projections $\pi_1(\beta_1)$ and $\pi_2(\beta_2)$ are isotopic, which implies, by Lemma 10.4, that $\pi_1(\beta_1) = \pi_2(\beta_2) \subset \theta_n$. We also have $\pi_1(\beta_1') = \pi_2(\beta_2') \subset \theta_n$ for any parallel proper arcs $\beta_1' \subset \gamma_1$ and $\beta_2' \subset \gamma_2$. Therefore, the number of connected components $\beta$ of $\Gamma$ with $s(\beta) = -1$ is exactly the number of such pairs $(\beta_1', \beta_2')$.

Now suppose that $\beta$ is a connected component of $\Gamma$ such that $(p, p) \in \beta$ for some $p \in \gamma_1 \cap \gamma_2$. Due to the tightness of $\gamma_1$ and $\gamma_2$ there is no $p' \in \gamma_1 \cap \gamma_2$ distinct from $p$ with $(p', p') \in \beta$, and $\beta$ is homeomorphic either to a closed interval or to a circle. In the latter case, the closed curves $\beta_1$, $\beta_2$ are isotopic to one another and intersect once, which implies that they are one-sided. In the former case, the connected components of $\gamma_1$, $\gamma_2$ containing the arcs $\beta_1$, $\beta_2$, respectively, are not isotopic. Thus, the number of connected components $\beta$ of $\Gamma$ with $s(\beta) = 1$ is equal to the number of points in $\gamma_1 \cap \gamma_2$ less the number of pairs of isotopic one-sided curves $\beta_1' \subset \gamma_1$, $\beta_2' \subset \gamma_2$.

Thus, equality (8) is settled.

Now we show that

$$\sum_{\beta \in \mathcal{X}} s(\beta) = \sum_{\alpha_1, \alpha_2} \langle \alpha_1, \alpha_2 \rangle w_1(\alpha_1)w_2(\alpha_2), \quad (9)$$

Indeed, let $\beta$ be a connected component of $\Gamma$. Denote the projections of $\beta$ to $\gamma_1$ and $\gamma_2$ by $\beta_1$ and $\beta_2$, respectively.
where the sum is taken over all branches \( \alpha_1 \) of \( \theta_1 \) and \( \alpha_2 \) of \( \theta_2 \). For a connected component \( \beta \) of \( \Gamma \), we again denote by \( \beta_1 \) and \( \beta_2 \) the projections of \( \beta \) to \( \gamma_1, \gamma_2 \), respectively.

As we have seen above, the equality \( s(\beta) = -1 \) means that \( \pi_1(\beta_1) = \pi_2(\beta_2) \) is a common proper arc contained in \( \theta_\gamma \). The number of such \( \beta \) is thus equal to

\[
\sum_\alpha w_1(\alpha)w_2(\alpha),
\]

where the sum is taken over all common free branches of \( \theta_1 \) and \( \theta_2 \).

If \( s(\beta) = 1 \), then \( \beta_1 \) and \( \beta_2 \) intersect once and do not belong to isotopic one-sided closed components of \( \gamma_1, \gamma_2 \), respectively. This implies that \( \pi_1(\beta_1) \) and \( \pi_2(\beta_2) \) also intersect once (more intersections would contradict Lemma 10.2 or Lemma 10.4). Therefore, the number of connected components \( \beta \subset \Gamma \) with \( s(\beta) = 1 \) is equal to the number of triples \((p, p', q) \in \gamma_1 \times \gamma_2 \times \theta_\gamma \) such that \( \pi_1(p) = \pi_2(p') = q \), which, in turn, is equal to

\[
\sum_{\alpha_1, \alpha_2} \langle \alpha_1, \alpha_2 \rangle w_1(\alpha_1)w_2(\alpha_2),
\]

where the sum is taken over all pairs of branches \( \alpha_1 \subset \theta_1, \alpha_2 \subset \theta_2 \) intersecting transversely.

Thus, equality (9) is also settled. Together with (5), it implies (7).

### 10.6. The asymptotic complexity of the algorithm.

Recall that we assume the surface \( M \) and the set of punctures \( \mathcal{P} \) to be fixed once and for all. So, in what follows, ‘bounded’ means ‘bounded from above by a constant depending on \( M \) and \( \mathcal{P} \) but not on anything else’.

We need to show that the number of elementary arithmetic operations needed to compute \( \langle \gamma_1, \gamma_2 \rangle \) has growth \( O(|\gamma_1|_T \cdot |\gamma_2|_T) \). The way to prove this is essentially the same as that of Proposition 9.1 so, we stop only on the differences.

We may assume that the number of switches of \( \theta_\gamma \) remains bounded during the whole simplification process. Indeed, the number of switches of \( \theta_1 \) and \( \theta_2 \) never increases, so, we should worry only about the number of divergence points.

When no branch of \( \theta_\gamma \) connecting two divergence points is present, the number of divergence points is clearly not larger than \( n + 3q \), where \( q \) is the total number of switches of \( \theta_1 \) and \( \theta_2 \).

Suppose that, at some stage of the simplification process, there is a branch \( \alpha \) of \( \theta_\gamma \) having the form of an arc whose both ends are divergence points. Such a branch is untouched during the simplification until a moment when a splitting is performed on it. As a result of this splitting, the branch \( \alpha \) and the two divergence points disappear. We may reorder the simplification moves so that such splittings are performed immediately as they become possible. Any other simplification move may create at most two such branches of \( \theta_\gamma \), which contribute at most four divergence points in excess of the previous estimate. Thus, we may assume that the number of divergence points never exceeds \( n + 4 + 3q \).

This implies, in particular, that every cleanup procedure takes a bounded number of elementary operations. Also bounded is the number of simplification moves performed in a row under Step 2.

We think of the procedure under Steps 1 and 5 as the regular simplification, which is interrupted several times to perform simplification moves under Steps 2–4. Let \( C_1, C_2 \) be the number of branches and the number of switches of \( \theta_1 \), respectively, contained in \( \theta_\gamma \). These numbers may only decrease during the simplification.

Interruptions of the regular simplification occur either immediately after one of \( C_1, C_2 \) decreases or just before such an event. Therefore, the total number of interruptions is bounded, and so is the total cost of all simplification moves performed at the interruptions of the regular simplification.
Now we reconsider the regular simplification. To make the gain/cost ratio of every simplification step bounded away from zero we put \((\theta, w) = (\theta_1, w_1)\) and redefine the gain by substituting \(A(\theta)\) in (1) with the set of non-free branches of \(\theta_1\) contained in \(\theta_2\).

In any of the Cases B–L shown in Figure 22 if the branch of width \(a + c\) (in \((\theta_1, w_1)\)) is not covered by \(\theta_2\), a splitting of the branch of width \(a + b + c\) will reduce \(C_1\), and thus will have gain at least one.

Another feature of the simultaneous simplification of two train tracks is that at every step we recompute \(w_2\). So, every simplification step that does not involve a multiple splitting consumes time \(O(\gamma_2|T|)\).

A special care is needed only in Case K if a multiple splitting is involved. This occurs when the two branches forming a circle are contained in \(\theta_2\). Let their widths in \((\theta_2, w_2)\) be \(a' + c'\) and \(2a' + c'\).

First, we compute \(k = [c/a] + 1\), as before. Then we check whether or not \(ka' \leq c'\). If the inequality holds true, we make a \(k\)-times multiple splitting, which has the same gain as before, that is at least \(\log(k + 1)\), and consumes time \(O(\log(k + 1) \cdot |\gamma_2|)\).

If we find out that \(ka' > c'\) we compute \(k' = [c'/a'] + 1\) and perform a \(k'\)-times multiple splitting, which produces a wide branch \(\alpha\) at which \(w_1, w_2\) are equal to \((c - (k' - 2)a)\) and \((c' - (k' - 2)a')\), respectively, such that \(\theta_1, \theta_2\) disagree on \(\alpha\). Subsequent ordinary splitting on \(\alpha\) (which is done under Step 4) will produce a branch of \(\theta_1\) of width \((c - k'a)\) which is no longer covered by \(\theta_2\). The total gain will be at least

\[
\log_2 \frac{(2a + c + 1)(a + c + 1)}{(c - (k' - 1)a + 1)} \geq \log_2(2a + c + 1) > \log_2(k + 1),
\]

whereas the computational time is again \(O(\log_2(k + 1) \cdot |\gamma_2|)\).

When the regular simplification is finished, it is only bounded number of circle separations performed at Step 6, which consumes bounded computational time.

Thus, the whole simplification procedure consumes time \(O(|\gamma_1|T \cdot |\gamma_2|)\). The same estimate works for the final computation of \(\langle \gamma_1, \gamma_2 \rangle\) using formula (17). This completes the proof of Proposition 10.1.

Proposition 10.1 gives an estimate for worst cases, but for ‘typical’ cases the running time of the algorithm might be better than \(O(|\gamma_1|T \cdot |\gamma_2|)\). This is due to the fact that, for random \(\gamma_1, \gamma_2\) having large complexity, it is likely that the simplification procedures for \((\theta_1, w_1)\) and \((\theta_2, w_2)\) diverge well before \((\theta_1, w_1)\) and \((\theta_2, w_2)\) get much simpler, which means that the simplification process described in the proof of Proposition 10.1 will actually have much smaller cost than \(|\gamma_1|T\). This means, in turn, that the main contribution to the running time will come from the computation of the expression in the right hand side of (17), that is, from multiplying large numbers \(w_1(\alpha)\) and \(w_2(\beta)\) for all appropriate pairs of branches \((\alpha, \beta)\).

There exist faster methods for multiplication of natural numbers than the grade-school algorithm. The first such method was proposed by A. Karatsuba [11, 12]. Thus, it is plausible that the average running time for computing \(\langle \gamma_1, \gamma_2 \rangle\) can also be improved.

Propositions 6.1 and 10.1 immediately imply the following.

**Corollary 10.5.** There is an algorithm for computing \(\langle T, (g_1, g_2)(T) \rangle\) from \(\langle T, g_1(T) \rangle\) and \(\langle T, g_2(T) \rangle\) in time \(O(c_T(g_1) \cdot c_T(g_2))\) on a RAM machine.

10.7. **Proof of Theorem 2.1.** Due to Proposition 9.2 we can choose a triangulation \(T\) of \(M\) with vertices at \(\mathcal{P}\) and substitute \(c_T\) for \(zw_{1,\mathcal{A}}\) in the formulation of the theorem. So, we use the matrix presentation for elements of \(G\).
Computing the normal form of an element amounts to computing \( \langle T, (g_1 \circ g_2 \circ \ldots \circ g_k)(T) \rangle \) from \( \langle T, g_i(T) \rangle, i = 1, \ldots, k \), which can be done in quadratic time due to Propositions 6.1 and 10.1.

It remains to describe a procedure for checking whether a given word represents a sequence of matrices of the form \( \langle T, g(T) \rangle, g \in G \). First, it is a simple syntax check whether the given word represents a sequence of integral matrices of the appropriate size. Then each of the matrices should be tested for having the form \( \langle T, g(T) \rangle \).

Proposition 10.1 allows to check in quadratic time whether any two distinct columns of an \( N \times N \)-matrix represent normal curves having zero geometric intersection index. Further, if this is true, take the linear combination of the columns with coefficients 1, 2, \ldots, \( N \) and simplify the obtained multiple curve by using Propositions 8.1 and 9.1. After the simplification, check whether the result is, up to a homeomorphism, the triangulation \( T \) with the \( i \)th edge repeated \( i \) times, \( i = 1, \ldots, N \).

10.8. Asymptotic dependence on \( N \). Throughout the paper, we have been assuming the punctured surface \( (M, \mathcal{P}) \) to be fixed, and have been ignoring the question on the dependence of the complexity of the proposed algorithms on the complexity of the surface. Now we will have a quick look on this dependence. For the measure of complexity of \( (M, \mathcal{P}) \) we use the number \( N \) of edges of any triangulation of \( M \) with vertices at \( \mathcal{P} \) (which is clearly comparable to the number of faces).

We make the following, quite realistic, assumption on the computational model: we use a RAM machine with standard memory unit size large enough to store integers comparable to the size of the input. So, numbers like \( N \), \( |\gamma|_T \), or \( c_T(g) \) are thought of as fitting the standard memory unit size, whereas the numbers like \( \langle \gamma, e_i \rangle \) may occupy many standard memory units.

With these settings, Proposition 10.1 and Corollary 10.5 can be strengthened as follows.

**Theorem 10.6.** There is an algorithm that accepts as input a triangulation \( T \) of a punctured surface \( (M, \mathcal{P}) \) and two vectors of the normal coordinates of multiple curves \( \gamma_1, \gamma_2 \) on \( M \) (or two matrices \( \langle T, g_1(T) \rangle, \langle T, g_2(T) \rangle \), where \( g_1, g_2 \in \text{MCG}(M, \mathcal{P}) \)) and computes \( \langle \gamma_1, \gamma_2 \rangle \) (respectively, \( \langle T, g_1 g_2(T) \rangle \)) in time \( O(N^3 \cdot |\gamma_1|_T \cdot |\gamma_2|_T) \) (respectively, \( O(N^5 \cdot c_T(g_1) \cdot c_T(g_2)) \)), where \( N \) is the number of edges of \( T \).

**Sketch of proof.** We revisit the proofs of all previously made statements about proposed algorithms. First, consider the problem of computing \( \langle \gamma_1, \gamma_2 \rangle \).

With a reasonable way to encode \( T \) at the input, the amount of work to produce a universal train track \( \theta_T \) and to encode \( \gamma_1 \) and \( \gamma_2 \) by assigning weights to the branches of \( \theta_T \) as described in Section 8 takes time \( O(N \cdot (|\gamma_1|_T + |\gamma_2|_T)) \), which is considerably faster than the required \( O(N^3 \cdot |\gamma_1|_T \cdot |\gamma_2|_T) \). Deciding whether we need to exchange \( \gamma_1 \) and \( \gamma_2 \) to have \( \gamma_1 \) not (much) more complicated than \( \gamma_2 \) is also pretty quick.

In the transition from a normal multiple curve \( \gamma \) to an isotopic multiple curve carried by \( \theta_T \), as described in the proof of Proposition 8.1 a normal arc in \( \gamma \) converts into \( O(N) \) almost normal arcs supported by \( \theta_T \), since there are only \( O(N) \) branches of \( \theta_T \). This means, that, for the obtained with assignment \( w \), we have in worst case \( |(\theta_T, w)| = O(N \cdot |\gamma|_T) \).

So, the total gain that we should collect during the simplification process described in the proof of Proposition 9.1 with modifications described in Subsections 10.2 and 10.3 does not exceed \( O(N \cdot |\gamma_1|_T) \).

Every round of regular simplification of \( \theta_U \) may include clean up procedures, each of which requires \( O(N) \) simplification moves. For every round, we also need to find the appropriate place in \( \theta_U \) where the simplification should occur, and this also consumes \( O(N) \) arithmetic operations. Therefore, the estimates for the cost/gain ratio from the proof of Proposition 9.1...
should be multiplied by $O(N)$. In total, we get the estimate $O(N^2 \cdot |\gamma_1|_T \cdot |\gamma_2|_T)$ for the time consumed by the regular part of the simplification.

The regular simplification is interrupted $O(N)$ times to destroy contractible connected components of $\theta \setminus \mathcal{P}$ containing switches of $\theta \cup$. To discover that the given connected component is contractible it suffices to perform $O(N)$ arithmetic operations on ‘small’ numbers. One can show that a contractible connected components of $\theta \setminus \mathcal{P}$ having $k$ switches in it is destroyed in $O(k^2)$ splittings. So, the total amount of splittings involved at Step 2 is bounded by $O(N^3)$, and the time lost on the interruptions is estimated as $O(N^3 \cdot |\gamma_2|_T)$.

At Step 6 we need to perform only $O(N)$ splittings and slidings, which consume time $O(N \cdot |\gamma_2|_T)$.

Finally, we use formula (7) to compute the output, which requires time $O(N^2 \cdot |\gamma_1|_T \cdot |\gamma_2|_T)$. We see that none of the asymptotic terms we have encountered exceeds $O(N^3 \cdot |\gamma_1|_T \cdot |\gamma_2|_T)$.

The estimate for the running time to compute $\langle T, g_1 g_2(T) \rangle$ from $\langle T, g_1(T) \rangle$ and $\langle T, g_2(T) \rangle$ now follows from that for computing $\langle \gamma_1, \gamma_2 \rangle$, since we simply need to compute $N^2$ geometric intersection indexes.

Our estimations in the proof of Theorem 10.6 are very rough, so it is likely that the power of $N$ can be lowered.

To conclude this section, we note that the question on the dependence of the running time on the complexity of the surface makes sense for algorithms whose existence is established in Theorem 2.1 only when a concrete family of generating sets for all mapping class groups has been fixed. The author is unaware if such a family can be chosen so as to make this dependence polynomial.

11. A CONCLUDING REMARK

Suppose for time being that a triangulation $T$ with vertices at $\mathcal{P}$ is fixed on $M$ such that the first $N_0$ edges of $T$ are non-boundary edges, and the last $N - N_0$ are boundary ones. Using Proposition 5.1 let us identify the set of all multiple curves on $(M, \mathcal{P})$ not containing a proper arc parallel to a boundary edge with a subset $L \subset \mathbb{Z}^{N_0}$, and treat the geometric intersection index (see Definition 3.1) as a function $\langle \cdot, \cdot \rangle : L \times L \to \mathbb{Z}$. It is not hard to show that this function can be extended, in a unique way, to a continuous function $\mathbb{R}^{N_0} \times \mathbb{R}^{N_0} \to \mathbb{R}$ satisfying the following homogeneity condition: $\langle \lambda x, y \rangle = \langle x, \lambda y \rangle = \lambda \langle x, y \rangle$ for all $x, y \in \mathbb{R}^{N_0}$, $\lambda \in \mathbb{R}_{>0}$.

The whole space $\mathbb{R}^{N_0}$ in this construction admits a natural topological interpretation as the space of measured laminations in the sense of W. Thurston [25], which can be defined without a reference to a concrete triangulation $T$. We denote this space by $\mathcal{M}(M, \mathcal{P})$. Topologically it is just an open $N_0$-dimensional ball. Different triangulations just give rise to different global coordinate systems on it.

Now every triangulation with vertices at $\mathcal{P}$ (and with non-boundary edges numbered first) gives rise also to a point of the following space:

$$\mathcal{Z}(M, \mathcal{P}) = \{(e_1, e_2, \ldots, e_{N_0}) \in \mathcal{M}(M, \mathcal{P})^{N_0} : \langle e_i, e_j \rangle = -\delta_{ij} \ \forall i, j = 1, \ldots, N_0\}.$$  

The following question sounds intriguing to the author: is $\mathcal{Z}(M, \mathcal{P})$ a topological manifold of dimension $N_0(N_0 - 1)/2$? If the answer is positive, what kind of manifold is it?

This is interesting because the natural action of $\text{MCG}(M, \mathcal{P})$ on $\mathcal{Z}(M, \mathcal{P})$ is likely to be free and properly discontinuous, so knowing the structure of this space may be useful for studying the group $\text{MCG}(M, \mathcal{P})$.

The author was able to answer the questions above only in the following three simplest cases.

(i) $M = \mathbb{D}^2$, $|\mathcal{P}| = 5$, $\mathcal{P} \subset \partial \mathbb{D}^2$. In this case, $\mathcal{Z}(M, \mathcal{P}) \cong S^1 \times \mathbb{Z}_2$.

(ii) $M = \mathbb{D}^2$, $|\mathcal{P}| = 6$, $\mathcal{P} \subset \partial \mathbb{D}^2$. In this case, $\mathcal{Z}(M, \mathcal{P}) \cong \mathbb{R}P^3 \times \mathbb{Z}_2$. 

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(iii) $M = \mathbb{T}^2$, $|\mathcal{P}| = 1$. In this case, $\mathcal{Z}(M, \mathcal{P}) \cong \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{Z}_2$. (This case was classified as sporadic and excluded from the main consideration in this paper, but this was only because the matrix presentation discussed throughout the paper gives rise to the quotient group $\operatorname{MCG}(\mathbb{T}^2, \{P_1\})/(\pm 1) \cong \operatorname{PSL}(2, \mathbb{Z})$ instead of the mapping class group itself.)

In cases (ii) and (iii), establishing this was far from being straightforward. However, these cases may still be too simple to reflect the general picture.

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