Optimal Control of Dissipative Nonlinear Dynamical Systems with Triggers of Coupled Singularities

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Abstract. This paper analyses the controllability of motion of nonconservative nonlinear dynamical systems in which triggers of coupled singularities exist or appear. It is shown that the phase plane method is useful for the analysis of nonlinear dynamics of nonconservative systems with one degree of freedom of control strategies and also shows the way it can be used for controlling the relative motion in rheonomic systems having equivalent scleronomic conservative or nonconservative system. For the system with one generalized coordinate described by nonlinear differential equation of nonlinear dynamics with trigger of coupled singularities, the functions of system potential energy and conservative force must satisfy some conditions defined by a Theorem on the existence of a trigger of coupled singularities and the separatrix in the form of "an open spiral form" of number eight. Task of the defined dynamical nonconservative system optimal control is: by using controlling force acting to the system, transfer initial state of the nonlinear dynamics of the system into the final state of the nonlinear dynamics in the minimal time for that optimal control task

1. Introduction

In systems with the change of some kinetic parameters of the system, the process of losing stability of one static equilibrium position is followed with appearance of two close stable dynamical equilibrium positions (see Refs. [1-11]). Also, these two newly appeared singularities with the previous stable position, which lost its stability, make a trigger of coupled singularities [11]. In an engineering system (see Refs. [9-10] and [12]) with mass deviation properties to the axes of rotation, in the gravitation field, where hybrid, coupled rotation motion exists, in the case of some kinetic parameter relations, the phenomenon of bifurcation equilibrium position (see Refs. [1] and [8]) or appearance of two new relative dynamic stable equilibrium positions in relation to precession rotation motion are nonlinear properties of such system. Trigger of coupled singularities, with corresponding choice of kinetic parameters of system dynamics, which degenerates into one threefold (triple) singular point, corresponds to stable equilibrium position. Coupled singularities existence is coupled with existence of two types of homoclinic orbits–separatrix trajectories: newly appeared, second homoclinic orbit in the form of "an open spiral form" of the number eight, which appear with bifurcation of equilibrium position, inside of previous homoclinic orbit, whose shape is deformed in the form of one pair of central symmetric "waves". Examples of engineering systems with coupled rotational motions in the
nonlinear dynamics of which there are triggers of couples singularities which are the cause of chaotic
dynamics and the system response to periodic excitation are given.

2. Theorem on the existence of a trigger of the coupled singularities and the separatrix in the
"an open a spiral form" of number eight in the system with turbulent damping

This paper analyses the controllability of non-linear dynamics of non-conservative system in which a
triggers of coupled singularities exists. The differential equation of non-linear dynamics, of a
non-conservative system with trigger of the coupled singularities, is in the form of:

$$\ddot{x} + 2\delta \dot{x} + g[k, F(x)]f(x) = 0,$$  \hspace{1cm} (1)

where $x$ is generalized coordinate and in the special cases in same time relative coordinate,
$\delta$. damping coefficient of the turbulent damping (see Refs. [14] and [10]) generalized force
proportional to the square of the velocity, and potential energy is in the form:

$$E = \int_0^1 g[k, F(x)]f(x) dx = G[k, F(x)]$$

in which the functions $f(x)$ and $g(x)$ are: $F(x) = \int_0^x f(x') dx'$ and

$$G(k, x) = \int_0^1 g(k, x') dx'$$

and satisfy the conditions of the trigger of coupled singularities existence. For that
case when in the system existed a trigger of the coupled singularities, than the functions $f(x)$, $F(x)$
and $g[k, F(x)]$ must be satisfy some conditions defined by a Theorem on the existence of a trigger of
the coupled singularities (see Refs. [5] and [11]) and the separatrix in the form of “an open a spiral form” of a number eight:

$$a^* \hspace{0.2cm} x_r = sT_0, \hspace{0.2cm} s = 0, 1, 2, 3, 4, ... \hspace{0.2cm} \text{for} \hspace{0.2cm} f(0) = 0 \hspace{0.2cm} \text{and} \hspace{0.2cm} f(x) = 0$$

$$b^* \hspace{0.2cm} x_r = sT_0, \hspace{0.2cm} s = 0, 1, 2, 3, 4, ... \hspace{0.2cm} \text{for} \hspace{0.2cm} f(0) = 0 \hspace{0.2cm} \text{and} \hspace{0.2cm} f(x) = 0$$

$$x_r = \pm x_0 \pm rT_0, \hspace{0.2cm} r = 0, 1, 2, 3, 4, ... \hspace{0.2cm} \text{for} \hspace{0.2cm} g[k, F(x_r)] = 0, \hspace{0.2cm} \text{for} \hspace{0.2cm} k \in (k_1, k_2) \cup (k_2, k_1) \hspace{0.2cm} \text{and} \hspace{0.2cm} |x| < \frac{T_0}{2}$$  \hspace{1cm} (3)

It is possible to define the following Theorem (see Refs. [5] and [11]): In the system with turbulent
damping, whose dynamics can be described with the use of non-linear differential equation in the form
(1) and whose potential energy $G(k, x)$ is in the form (2) in which the functions $f(x)$ and $g(x)$ are
in the form (2) and (3) and both functions $f(x)$ and $g(x)$ have one maximum or minimum in the
interval between two zero roots: $a^*$ for parameters values $k \notin (k_1, k_2) \cup (k_2, k_1)$ , outside of the
intervals $(k_1, k_2) \cup (k_2, k_1)$..., the trigger of coupled singularities in the local area does not exist; $b^*$
for parameters values $k \in (k_1, k_2) \cup (k_2, k_1)$... , inside of the intervals $(k_1, k_2) \cup (k_2, k_1)$..., the
series of triggers of coupled singularities in the local domains exist, as well as corresponding
homoclinic orbit - the separatrix in the form of "open spiral number eight".

Equation of the homoclinic orbit in the form of "an open a spiral form of number eight". For the
system with turbulent damping, whose dynamics can be described with the use of non-linear
differential equation in the form (1) we can obtain phase trajectories by integrating previous equation
by use the following notation: $\dot{x} = v$ for system equation transformation into:

$$\frac{dv^2}{dx} \pm 4\delta v^2 = -2g[k, F(x)]f(x)$$

In the following form:
\[ v^2 + 2G[k, F(x)] - 8e^{\frac{\gamma}{4\delta}} \int_0^x G[k, F(x)] e^{\frac{\gamma}{4\delta} dx} + C_{1,2} e^{\frac{\gamma}{4\delta} x} = 0 \]  

(5)

where integral constant: \( C_{1,2} = -e^{\frac{\gamma}{4\delta} x_0} \left\{ \frac{x_0}{8e^{\frac{\gamma}{4\delta} x_0}} \int_0^x G[k, F(x)] e^{\frac{\gamma}{4\delta} dx} \right\} \) depend on initial condition of motions. Equation of homoclinic orbit in the form "an open a spiral form of number eight" through homoklinic point \((0,0)\) is in the form:

\[ v^2 + 2G[k, F(x)] - 8e^{\frac{\gamma}{4\delta}} \int_0^x G[k, F(x)] e^{\frac{\gamma}{4\delta} dx} - 2G[k, F(0)] e^{\frac{\gamma}{4\delta} x} = 0 \]  

(6)

where constant of integration is: \( C_{1,2} = -\frac{2G[k, F(0)]}{g[k, F(x)] = 0, \text{ for } k \in (k_i, k_j) \cup (k_j, k_i) ...} \)

3. Optimal Control of Nonlinear Dynamics with trigger of coupled singularities in the nonconservative system

Task of the defined dynamical system optimal control is: By using controlling force \( \tilde{u}(t) \) acting to the system, transfer initial state of the nonlinear dynamics of the nonconservative system defined by \( x_1(0) = \alpha \) and \( x_2(0) = \beta \) into the final state of the nonlinear dynamics defined by \( x_1(T) = \gamma \) and \( x_2(T) = \chi \), where \( T \) is minimal time for that optimal control task. Than we can write two new form of nonlinear differential equations first order for optimal control task in the following form:

\[ \dot{x}_1 = x_2 \quad \dot{x}_2 = -2\alpha x_2 |x_2| - g[k, F(x)] f(x_1) \pm \tilde{u}(t) \]  

(7)

with previously defined initial conditions state, in the form \( x_1(0) = \alpha \) and \( x_2(0) = \beta \) and with final conditions in the form \( x_1(T) = \gamma \) and \( x_2(T) = \chi \), where \( T \) is time necessary for this motion.

Pontrijagin’s maximum principle (see Refs. [7] and [4]) is used. For minimization of the time \( T \) to the previous system dynamics defined by nonlinear differential equations (7), we add to these equations the following functional \( I = \int_0^T 1 \cdot dt \), as a criterion of the optimality – time minimization, or “criterion of quality” of the motion control, with addition in the form of the controlling force \( \tilde{u}(t) \) constrained in the form: \(-\tilde{u}_0 \leq \tilde{u}(t) \leq +\tilde{u}_0\). The concept of controllability of motion (or system dynamics) implies the possibility that the mechanical system motion (or dynamics) is realized according to a given program under the excitation of special generalized forces. Controlling force \( \tilde{u}(t) \) is, here, considered as generalized force of controllability corresponding generalized coordinate of the system. The phrase “system motion or system dynamics control” implies the process of realizing a given or programming motion or dynamics. Programs can be of a great variety.

Formulation of the Theorem of the motion controllability (see Ref. [15]) gives the following explanation: The mechanical system motion or dynamics is controllable according to the program given in advance if there are such controlling forces of such magnitude, depending upon the program parameters, which are by their absolute value greater than other respective active forces if the controlling force direct the motion opposite to the motion direction under the influence of the other forces.

Concept of optimal motion of system dynamics implies here motion or dynamics of the mechanical systems whose particular attributes have extreme values with respect to some dynamics parameters. Now, let’s determine the controlling force \( \tilde{u}(t) \) that can control the dynamics of the system of
accordance with defined control task. It is possible criterion of the optimal control of dynamics to write in the following way:

$$\tilde{I} = \int_0^T \left\{ \mathbf{H}[x_1, x_2, p_1, p_2, \tilde{u}] - p_1(t)x - p_2(t)\left[ - 2\delta \dot{x}_2 x_2 - g[k, F(x)]/f(x) \right] \right\} dt$$

(8)

This previous functional is for the case with unspecified interval of time. As one of the boundaries time interval which is a functional, then it is necessary to take into account nonisochronous variations of the functional. On this basis we can write:

$$\Delta \tilde{I} = \left[ p_1 \delta \dot{x}_1 - p_2 \delta x_2 \right] t_0 + \left[ \mathbf{H} - p_1 \dot{x}_1 - p_2 \dot{x}_2 \right] M(t) + \int_0^T \left\{ \sum_{i=1}^2 \left( \frac{\partial \mathbf{H}}{\partial x_i} - \dot{p}_i \right) \delta x_i + \sum_{i=1}^2 \left( \frac{\partial \mathbf{H}}{\partial p_i} - \dot{\delta} \right) \right\} dt = 0$$

(9)

Optimal dynamics is defined by solving the following system of differential equations with corresponding additional conditions of the optimal control of motion:

$$\dot{x}_i = \frac{\partial \mathbf{H}}{\partial p_i}; \quad \dot{p}_i = -\frac{\partial \mathbf{H}}{\partial x_i}; \quad \frac{\partial \mathbf{H}}{\partial u} = 0; \quad -u_0 \leq u(t) \leq u_0; \quad \mathbf{H}|_{u=T} = 0 \quad i = 1, 2.$$  

(10)

By using theory from Ref. [15], as well as previously obtained conditions of the optimal control of motions, we can write the following Hamilton functional in the form:

$$\mathbf{H}[x_1, x_2, p_1, p_2, \tilde{u}] = 1 + p_1(t)x - p_2(t)\left[ - 2\delta \dot{x}_2 x_2 - g[k, F(x)]/f(x) \right] \tilde{u}(t)$$

(11)

For the system with turbulent damping, whose dynamics can be described with the use of nonlinear differential equation in the form (7) we can solve phase trajectories by integrating previous equation by introducing following notation:

$$\dot{x} = v$$

$$v^2 + 2G[k, F(x)] - 8\delta \dot{x}_2 \int_0^T G[k, F(x)] dx + C_{1,2} e^{4\delta \dot{x}} + \frac{1}{2\delta} \tilde{u}_0 = 0$$

(12)

where:

$$C_{1,2} = -e^{4\delta \dot{x}} \left\{ \delta \dot{x}_2 + 2G[k, F(x)]\right\} - 8\delta \dot{x}_2 \int_0^T G[k, F(x)] e^{4\delta \dot{x}} dx + \frac{1}{2\delta} \tilde{u}_0$$

(13)

Previously defined trajectories must be with at least one common point - cross section - representative point $N(x_c, \dot{x}_c)$ in the phase plane of dynamical state as a cross section of the previous trajectories. first through the initial condition point $x_1(0) = \alpha \quad x_2(0) = \beta \quad$ and second through final condition point $x_1(T) = \gamma \quad x_2(T) = \chi$ corresponds to the final dynamic state of the system and with force with alternative directions: If this cross section is real. Control of the motion is possible and system is controllable, if this cross section exists. If it does not exist control of this motion is not possible.

**1st First case**  Phase coordinates of the dynamic state $C(x_c, \dot{x}_c)$ in which it is optimal to change direction of the optimal control force are in the form:

$$\begin{align*}
\dot{x}_c &= \pm \frac{1}{4\delta} \ln \left\{ \frac{\delta}{\tilde{u}_0} e^{4\delta \tilde{x}} \left\{ \delta \dot{x}_2 + 2G[k, F(x)]\right\} - 8\delta \dot{x}_2 \int_0^T G[k, F(x)] e^{4\delta \dot{x}} dx + \frac{1}{2\delta} \tilde{u}_0 \right\} - \\
\dot{\tilde{u}}_0 &= \frac{\delta}{\tilde{u}_0} e^{4\delta \tilde{x}} \left\{ \delta \dot{x}_2 + 2G[k, F(x)]\right\} - 8\delta \dot{x}_2 \int_0^T G[k, F(x)] e^{4\delta \dot{x}} dx + \frac{1}{2\delta} \tilde{u}_0 \right\}
\end{align*}$$

(13)
\begin{equation}
\nu(x_c) = \dot{x}_c = \pm \sqrt{-2G[k,F(x_c)] + 8\delta_k^{2\delta} \int_0^{x_c} G[k,F(x)] e^{2\delta x} dx - C_{12}(x_0,\dot{x}_0)e^{2\delta x_c} - \frac{1}{2\delta} \tilde{u}_0}.
\end{equation}

Optimal time period $T$ for transfer of nonlinear system from one state dynamics $C(x_0,\dot{x}_0)$ along one phase trajectory to the other, $C(x_1,\dot{x}_1)$, along other phase trajectory successive passing through common state $C(x_c,\dot{x}_c)$ on both phase trajectories which correspond to cases with different control motion force direction, we can obtain as sum of times $T_{OC}$ and $T_{CT}$ as times of the motion of the system phase representative point along the first and the second trajectory. Optimal time period $T_{opt}$ for transfer of nonlinear system from one, initial, state dynamics $(0,0,x_0,\dot{x}_0)$ along one phase trajectory to the other final dynamic state $(T,T,x_T,\dot{x}_T)$ along other phase trajectory successive passing through common state $C(x_c,\dot{x}_c)$ is obtained in the form:

\begin{align}
T_{opt} &= T_{OC} + T_{CT} = \int_{x_c}^{x_0} dx + \int_{x_c}^{x_T} dx \\
&= \int_{x_c}^{x_0} \left[ \frac{8\delta_k^{2\delta}}{2G[k,F(x)]^{2\delta}} e^{2\delta x} dx - \frac{1}{2\delta} \tilde{u}_0 \right] \pm \int_{x_c}^{x_T} \left[ \frac{8\delta_k^{2\delta}}{2G[k,F(x)]^{2\delta}} e^{2\delta x} dx - \frac{1}{2\delta} \tilde{u}_0 \right] \\
&= \frac{1}{2\delta} \left[ \tilde{u}_0 \delta e^{2\delta x_c} - \tilde{u}_0 \delta e^{2\delta x_T} \right] \pm \frac{1}{2\delta} \tilde{u}_0 \delta e^{2\delta x_c} - \frac{1}{2\delta} \tilde{u}_0 \delta e^{2\delta x_T} \\
&= \frac{1}{2\delta} \left[ \tilde{u}_0 \delta e^{2\delta x_c} - \tilde{u}_0 \delta e^{2\delta x_T} \right] + \frac{1}{2\delta} \tilde{u}_0 \delta e^{2\delta x_c} - \frac{1}{2\delta} \tilde{u}_0 \delta e^{2\delta x_T} \quad (14)
\end{align}

For solution of the problem it is necessary to find cross section $(t_c = T_{OC},x_c,\dot{x}_c)$ (mutual phase state) between previous phase trajectories in which the control force changes direction. Initial branch of the phase trajectory contains the representative point $(t_0 = 0,x_0,\dot{x}_0)$, and final branch of the phase trajectory contains the point $(t_f = T,T,x_T,\dot{x}_T)$. By using phase trajectories of the nonlinear dynamics we can find the time moment $T_c$ in which we must change control force direction and final minimal time for optimal control motion:

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