Acceleration of the universe, vacuum metamorphosis, and the large-time asymptotic form of the heat kernel

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Abstract

We investigate the possibility that the late acceleration observed in the rate of expansion of the universe is due to vacuum quantum effects arising in curved spacetime. The theoretical basis of the vacuum cold dark matter (VCDM), or vacuum metamorphosis, cosmological model of Parker and Raval is revisited and improved. We show, by means of a manifestly nonperturbative approach, how the infrared behavior of the propagator (related to the large-time asymptotic form of the heat kernel) of a free scalar field in curved spacetime leads to nonperturbative terms in the effective action similar to those appearing in the earlier version of the VCDM model. The asymptotic form that we adopt for the propagator or heat kernel at large proper time $s$ is motivated by, and consistent with, particular cases where the heat kernel has been calculated exactly, namely in de Sitter spacetime, in the Einstein static universe, and in the linearly-expanding spatially-flat FRW universe. This large-$s$ asymptotic form generalizes somewhat the one suggested by the Gaussian approximation and the $R$-summed form of the propagator that earlier served as a theoretical basis for the VCDM model. The vacuum expectation value for the energy-momentum tensor of the free scalar field, obtained through variation of the effective action, exhibits a resonance effect when the scalar curvature $R$ of the spacetime reaches a particular value related to the mass of the field. Modeling our universe by an FRW spacetime filled with classical matter and radiation, we show that the back reaction caused by this resonance drives the universe through a transition to an accelerating expansion phase, very much in the same way as originally proposed by Parker and Raval. Our analysis includes higher derivatives that were neglected in the earlier analysis, and takes into account the possible runaway solutions that can follow from these higher-derivative terms. We find that the runaway solutions do not occur if the universe was described by the usual classical FRW solution prior to the growth of vacuum energy-density and negative pressure (i.e., vacuum metamorphosis) that causes the transition to an accelerating expansion of the universe in this theory.

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I. INTRODUCTION

Recent observations of type Ia supernovae (SNe-Ia) appear to support the view that the expansion of the universe started speeding up about 5 Gyr before the present time, and that it was slowing down prior to that time [1]-[4]. Together with the evidence from the power spectrum of the cosmic microwave background radiation (CMBR) [5]-[12] establishing that the universe is nearly spatially flat, these observations imply that a significant fraction of the energy density of the universe is in the form of a dark energy that exerts negative pressure. The focus of this paper will be on one possible candidate for this dark energy.

It has been suggested [13, 14] that the recent acceleration of the universe may be caused by the vacuum energy and pressure of a quantized scalar field in curved spacetime. This scalar field is assumed to be free, meaning that it has no non-gravitational interactions. Furthermore, the mass $m$ of the field is assumed to have a very small non-zero value, so the Compton wavelength of the particle is a significant fraction of the present Hubble radius of the universe. The origin of the acceleration in this theory is not the zero-point energy that would give rise to a cosmological constant $\Lambda$. The latter would be affected by renormalization and would be expected to be very large, if it is not forced to be zero or nearly zero by some symmetry, dynamical process, or other principle that is not well understood. The curved spacetime effect considered here has a magnitude that depends on the ratio of the mass of the free scalar field to a dimensionless constant of order unity. For a free scalar field in curved spacetime, the mass of the field is not renormalized [15, 16]. Hence, the mass of this free field may be taken to be very small and will remain small within the context of quantum field theory in curved spacetime. Thus, the theory studied here, with a very small value for $m$, is internally consistent.

The gravitational field is treated as a classical field here. One may ask if quantizing the gravitational field, by regarding gravitons as quantized fluctuations of the metric on the curved spacetime background, would necessarily force the effective mass $m$ of the scalar field to be large? The self-energy contribution to the scalar field propagator caused by virtual gravitons is evidently not renormalizable, so a cut-off must be introduced. The magnitude of the effective mass of the scalar particle would then depend on the cut-off. If the cut-off is somewhat smaller than the Planck mass, then the graviton contribution to the effective mass would appear to be small. This would not affect our cosmological predictions because
they depend on a quantity $\bar{m}^2$, of order $m^2$, that is determined by the cosmological data. We make no attempt here to obtain the value of $m$ from first principles.

The VCDM model gives a satisfactory fit to the observed CMBR power spectrum and SNe-Ia data \[17\]. It may also lead to the observed suppression of the CMBR power spectrum at very low values of $l$ \[18\].

The question of how the very small mass scale fits in with fundamental theories of elementary particles and strings is not one that has a clear answer. It has been suggested \[19\] that a scalar particle of the required small mass may arise via symmetry breaking as a pseudo-Nambu-Goldstone boson (pngb). It seems possible that such a pngb, when quantized in curved spacetime, could lead to similar effects in curved spacetime through an effective action \[20\] or other methods \[21\]. In the present paper, only the free field will be considered.

In the previous work of Parker and Raval \[13, 14\], the approximation was made that the propagator (the analytic continuation of the heat kernel) was proportional to \[
\exp \left\{ -i \left( m^2 + (\xi - 1/6) R \right) s \right\} \left( 1 + (is)^2 \bar{f}_2 \right),
\]
where $m$ is the mass of the free scalar field, $R$ is the scalar curvature, $\xi$ is a dimensionless coupling constant that appears in the equation governing the free scalar field in curved spacetime, and $\bar{f}_2$ is a quantity constructed from covariant derivatives of $R$ and contractions of products of two Riemann tensors [see Eq. (3.4)].

This approximation was obtained from the expansion of the heat kernel in powers of the proper-time parameter $s$, with the exponential factor involving $R$ coming from summing all terms that involve one or more factors of $R$ to all orders in $s$ \[22, 23\]. Alternatively, the exponential involving $R$ can be obtained by means of a Feynman path integral solution of the Schrödinger equation for the propagator \[24\].

The presence of the exponential in $R$ was shown by Parker and Raval to lead to a growth in $\langle T_{\mu\nu} \rangle$, the expectation value of the energy-momentum tensor of the scalar field, when $R$ falls to a value of the order of $m^2$. The reaction back of this growth of $\langle T_{\mu\nu} \rangle$ in the semiclassical Einstein equations was shown to cause the expansion of the universe to accelerate in such a way as to keep the scalar curvature, $R$, of the spacetime nearly constant.

There are several natural questions that one can raise about the previous approach. For example, one may question the validity of the above approximation to the heat kernel, and in particular the use of that form of the heat kernel for large values of $s$. It is the large-$s$ form of the propagator that gives rise to the terms in $\langle T_{\mu\nu} \rangle$ that grow large as $R$ approaches a critical
value of order \( m^2 \). A partial justification is that the factor \( \exp \left[ -i (m^2 + (\xi - 1/6) R) s \right] \) in the propagator comes from all powers of \( s \), and is thus nonperturbative. However, the series in powers of \( s \) is in general asymptotic, not convergent, so the convergent subset of terms involving \( R \) that are summed may not fully reflect the form of the propagator at large \( s \). It should be mentioned that the Gaussian approximation to the path integral of Bekenstein and Parker \[24\] is independent of the power series in \( s \) and gives reason to expect an exponential factor at large \( s \). However, it is also an approximation, and thus leaves room for some modification.

In the present paper, we directly examine the large-\( s \) asymptotic form of the heat kernel for several cases in which the heat kernel is exactly known. We postulate a general expression for the asymptotic form that is consistent with the cases studied. We show that the transition to constant \( R \) occurs as a result of the large-\( s \) asymptotic form of the heat kernel. In our numerical integration of the Einstein equations in the FRW universe, we use the postulated asymptotic form with a particular choice of a factor that is quadratic in the Riemann tensor. This choice (namely, \( \bar{f}_2 \)) is consistent with the exact asymptotic forms considered, but there are also other consistent possibilities, as we discuss. We expect that the transition is fairly generic, as the choice we make of the quadratic factor is sufficiently complicated to be representative. As noted below, the possible runaway solutions of the field equations obtained from this higher derivative effective action do not occur with physically acceptable initial conditions. This approach based on the asymptotic form of the heat kernel is general enough that it may be useful in other fundamental theories, such as string theory, that have higher derivative terms in their low-energy effective actions.

One may also question the validity, in the earlier work of Parker and Raval, of neglecting derivatives of the Riemann tensor in arriving at the expression for \( \langle T_{\mu\nu} \rangle \). The reason for neglecting such derivatives was that the present expansion of the universe is slow enough that terms involving derivatives of invariants were expected to be small with respect to other terms in the effective action that do not involve such derivatives. However, the solution for the expansion of the universe that they found went through a relatively rapid transition from a matter-dominated expansion to a constant-\( R \) expansion. During that transition it may be necessary to include the derivative terms that were neglected. Furthermore, by including the higher derivative terms one may generate unstable runaway solutions of the Einstein equations.
In the present work, we keep the derivative terms that were neglected in the earlier work. Then the covariant conservation of $\langle T_{\mu\nu} \rangle$ is satisfied at all times without neglecting derivatives. We find that the transition to constant $R$ still occurs. We also carefully consider runaway solutions. We find that if the early evolution, prior to the transition, is described by the usual classical solution to the Einstein equations (as one expects because $\langle T_{\mu\nu} \rangle$ is negligible before the transition), then there is a transition to the constant $R$ solution with no runaway solutions. There are runaway solutions only if one takes the initial evolution to be significantly different from that of the classical solution prior to the transition.

We also generalize the previous work by including the neglected effect of dissipative processes. These could come from very small interactions of the scalar field $\phi$ with other fields, which could involve dissipative processes such as very slow decay of the scalar particles into less massive fields, or weak radiative processes, such as the production of gravitational waves. In the present paper, we model such dissipation phenomenologically by introducing a small, but nonzero, imaginary part in the mass term.

In their previous work, Parker and Raval mentioned the possibility that the mechanism they considered may be relevant to early inflation. However, because the series in powers of $s$ is more difficult to justify as an approximation when the curvature terms are large, they applied their work only to the recent universe. However, now that the mechanism has been further justified by considering known exact asymptotic forms of the heat kernel, the time may be ripe to apply this mechanism to large-mass scalar fields that are present in elementary particle theories. The acceleration effect will still come from the large-$s$ behavior of the heat kernel using the same postulated asymptotic form. The fact that the mass is large would not seem to prevent the growth of $\langle T_{\mu\nu} \rangle$. Because of the large mass, these bosons could give rise to early inflation. At the very least, our mechanism could modify the considerations of early inflation that come from possible self-interaction potentials of these fields. It is also possible that our mechanism could give rise to a satisfactory new model of early inflation in the absence of self-interaction potentials. Thus, this type of field might serve as a candidate for an inflaton. The dissipative term would then be larger because of the interactions and decay channels of the massive scalar boson. One would of course have to consider the questions of reheating and the perturbation spectrum produced in such an inflationary epoch. This mechanism for early inflation is worthy of investigation, but it is not our main focus here, and will not be considered further in the present paper.
The very low-mass, free, scalar particle that we consider in this paper as a possible cause of the recent acceleration of the universe, may appropriately be called an \textit{acceletron}, to distinguish it from an inflaton. Even in the absence of other mechanisms for early inflation, the acceletron itself would produce an early inflation at the Planck scale, similar to Starobinsky inflation \cite{25}. Assuming that a successful exit from early inflation occurs, then at a much later time, the same acceletron would produce the presently observed acceleration of the universe.

In Section \textbf{II}, we outline the basic theoretical structure, including the regularized expression for the effective action. In Section \textbf{III}, we discuss the asymptotic form of the heat kernel for large-$s$, and make a postulate for this asymptotic form that is consistent with the asymptotic forms of several exact solutions for the heat kernel. In Section \textbf{IV}, we show that the large-$s$ asymptotic form of the heat kernel is what gives rise to the terms that make large contributions to the vacuum energy-density and corresponding negative pressure when the scalar curvature falls to a magnitude comparable to the square of the mass of the acceletron field, $\phi$. In Section \textbf{V}, we write the renormalized effective action, and give the corresponding energy-momentum tensor. The relevant variations, including terms with derivatives of the Riemann tensor, are given in Appendix A. In Section \textbf{VI}, we briefly summarize results of numerical integration of the Einstein equations in an FRW cosmological spacetime. More detailed results of the numerical integration will be given in a later paper that is now in preparation. In Section \textbf{VII}, we present our conclusions.

\section{Effective-Action Formalism}

In this section we outline the method of relating the one-loop effective action of a quantum field in curved spacetime to the generalized $\zeta$-function and the propagator in the proper-time formalism.

The action $S$ of an uncharged scalar field $\phi$ in curved spacetime can be written in the form

$$S = S[g_{\mu\nu}, \phi] \equiv -\frac{1}{2} \int d^4x \sqrt{-g} \phi(x)H(x)\phi(x),$$

(2.1)

with $H(x)$ being the differential operator

$$H(x) \equiv -\Box + m^2 + \xi R,$$

(2.2)
\[ \Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu, \text{ and } \xi \text{ being a coupling constant between the field } \phi \text{ and the scalar curvature } R = R(x) \text{ of the spacetime.} \]

The corrections to the classical action of the gravitational field that are caused by quantum fluctuations of \( \phi \) can be found by evaluating the effective action \( W_q = W_q[g_{\mu\nu}] \). The latter is defined for a given spacetime with metric \( g_{\mu\nu} \) through the functional integral

\[ e^{iW_q} = \int d[\phi] e^{iS}. \quad (2.3) \]

In analogy with the Feynman path integral, the functional integral is proportional to a probability amplitude to go from an “in” state to an “out” state determined by the configurations of \( \phi \) that one sums over. Rather than explicitly specifying the initial and final configurations of \( \phi \), the “in” and “out” states are specified implicitly by boundary conditions used later in evaluating the functional integral. [To insure convergence of the functional integral, the operator \( H(x) \) is understood to be \( H(x) - i\epsilon \), where \( \epsilon \) is a positive infinitesimal.]

A nonzero imaginary part of \( W_q \) implies a nonzero rate of particle production of this scalar field \[26\]. If the spacetime is such that we can neglect the imaginary part of \( W_q \), then the “in” and “out” states of the \( \phi \) field are equivalent, differing only by a phase factor. The expectation value of the energy-momentum tensor of the scalar field in this state is then given by

\[ \langle T^{\mu\nu} \rangle = \frac{2}{\sqrt{-g}} \frac{\delta W_q}{\delta g_{\mu\nu}}. \quad (2.4) \]

In the semi-classical theory, this expectation value appears on the right-hand-side of the Einstein gravitational field equations, along with the other sources of the gravitational field. Later, we will apply this formulation to a spacetime for which it appears quite suitable; namely, the Friedmann-Robertson-Walker (FRW) universe describing the averaged behavior of the recent expansion of the universe.

Because the action \( S \) of the free scalar field is quadratic in \( \phi \), the functional integral reduces to a Gaussian integral which gives (to within a constant normalization factor that does not affect the variation of \( W_q \)):

\[ e^{iW_q} = \frac{1}{\sqrt{\text{Det} \left( \hat{H}/\mu^2 \right) }}, \quad (2.5) \]

where \( \hat{H} \) is the abstract operator defined through

\[ \langle x | \hat{H} | x' \rangle = H(x)\delta(x,x'), \quad (2.6) \]
with \(|x⟩\) being position eigenstates normalized through
\[
⟨x| x'⟩ = δ(x, x') \equiv \frac{δ^4(x - x')}{\sqrt{-g}}.
\] (2.7)

Then
\[
W_q = (i/2) \ln \text{Det} \left( \hat{H}/\mu^2 \right) = (i/2) \text{Tr} \ln \left( \hat{H}/\mu^2 \right),
\] (2.8)
with \(\mu\) an arbitrary constant having the units of mass. The renormalized effective action will not depend on \(\mu\).

The expression for \(W_q\) can be regularized by writing it in terms of the generalized \(ζ\)-function [27, 28, 29]. The latter is defined as
\[
ζ(ν) = \text{Tr} \hat{H}^{-ν} = \text{Tr} e^{-ν \ln \hat{H}}.
\] (2.9)

We can rewrite Eq. (2.8) as
\[
W_q = -\frac{i}{2} \left[ ζ'(0) + \ln(\mu^2) ζ(0) \right],
\] (2.10)
where \(ζ'(ν) = dζ(ν)/dν\). This expression can be regularized by analytic continuation of \(ζ(ν)\) in the parameter \(ν\) to make it and its first derivative well-defined at \(ν = 0\). The dependence on the arbitrary constant \(μ\) can then be absorbed into the definition of the renormalized constants such as \(G_N\) and \(Λ\) that appear in the Einstein action. As is well known, additional terms quadratic in the Riemann tensor must be added to the Einstein action to absorb all the dependence on \(μ\) through renormalization.

It is convenient to introduce an integral representation of the generalized \(ζ\)-function:
\[
ζ(ν) = \text{Tr} \hat{H}^{-ν} = \text{Tr} \left\{ \Gamma(ν)^{-1} \int_0^∞ ids (is)^{ν-1} e^{-is(\hat{H}-iν)} \right\}.
\] (2.11)

The operator \(H(x)\) of Eq. (2.2) may be regarded as the Hamiltonian of a fictitious nonrelativistic particle moving on a curved 4-dimensional hypersurface having coordinates \(x^μ\). The operator \(\exp(-is\hat{H})\) is the quantum mechanical evolution operator of this fictitious particle, with \(s\) being the “proper time.” This idea was introduced by Schwinger in the context of quantum electrodynamics and applied by DeWitt in curved spacetime. The trace can be represented as an integral over a complete set of position eigenstates \(|x⟩\) of the fictitious particle:
\[
ζ(ν) = \frac{1}{Γ(ν)} \int d^4x \sqrt{-g} \int_0^∞ ids (is)^{ν-1} ⟨x| e^{-is(\hat{H}-iν)} |x⟩.
\] (2.12)
The propagation amplitude for the particle to go from position \( x' \) at time \( s = 0 \) to position \( x \) at time \( s \) is
\[
K(x, x'; is) \equiv \langle x | e^{-is(H - is)} | x' \rangle.
\] (2.13)

This propagator satisfies the Schrödinger equation
\[
i \frac{\partial}{\partial s} K(x, x'; is) = H(x) K(x, x'; is),
\] (2.14)
and satisfies the “initial” condition
\[
\lim_{s \to 0} K(x, x'; is) = \delta(x, x').
\] (2.15)

Note that what appears in Eq. (2.12) is the coincidence limit \( x' \to x \) of Eq. (2.13). If one were to replace \( is \) by \( s \) in the Schrödinger equation, then it would become the equation governing heat flow on the 4-dimensional hypersurface, in which case \( K(x, x', s) \) is the heat kernel. Thus, the heat kernel and propagator are related by analytic continuation in the proper time \( s \).

The asymptotic expansion of the propagator in powers of \( s \) can be obtained from the Schrödinger equation by iteration. The proper time \( s \) has the dimensions of length squared, so the coefficients of successive powers of \( s \) in the series contain contracted products of increasing numbers of curvature tensors and/or covariant derivatives to balance the dimensions. The leading terms in this power series covariantly characterize the short wavelength behavior of the quantum field. In four dimensions, the terms of order \( s^{-2}, s^{-1}, \) and \( s^0 \) in the series are the ones that would give rise to ultraviolet (UV) divergences in the unregularized expression for the expectation value of the energy-momentum tensor. Those are the terms that are absorbed through renormalization of the coupling constants of the curvature terms in the classical Einstein action (with counterterms quadratic in the Riemann curvature tensor included).

Writing the coincidence limit, \( K(x, x; is) \), of the propagator (in four dimensions) in the form
\[
K(x, x; is) = \frac{i}{16\pi^2(is)^2} e^{-i(m^2 - is)s} F(x, x; is),
\] (2.16)
the proper-time series for \( F(x, x; is) \) has the form
\[
F(x, x; is) = \sum_{j=0}^{\infty} (is)^j f_j,
\] (2.17)
where the first three terms are given by

\[ f_0 \equiv f_0(x, x) = 1, \quad (2.18) \]

\[ f_1 \equiv f_1(x, x) = -\bar{\xi} R, \quad (2.19) \]

\[ f_2 \equiv f_2(x, x) = \frac{1}{2} \bar{\xi}^2 R^2 + \frac{1}{180} \left[ (1 - 30\bar{\xi}) \square R + R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - R_{\alpha\beta} R^{\alpha\beta} \right], \quad (2.20) \]

with \( \bar{\xi} \equiv \xi - 1/6 \). In general, this proper time series for \( F(x, x; is) \) is an asymptotic series that does not converge to the exact solution for \( F(x, x; is) \). Thus, it can only supply limited information about the large \( s \) behavior of \( F \).

It turns out that the Schwinger-deWitt proper-time series given in Eq. (2.17) is sufficient to calculate one of the terms appearing in Eq. (2.10), namely the term involving \( \zeta(0) \). In order to do so, we first have to analytic continue Eq. (2.12) to the value \( \nu = 0 \). Substituting Eq. (2.16) into Eq. (2.12) and performing three integration by parts, one has:

\[ \zeta(\nu) = -\frac{i}{16\pi^2(\nu - 2)(\nu - 1)\Gamma(\nu + 1)} \int d^4x \sqrt{-g} \int_0^\infty ids (is)^\nu \frac{\partial^3}{\partial (is)^3} \left[ e^{-i(m^2 - i\epsilon)s} F(x, x; is) \right], \quad (2.21) \]

which is regular at \( \nu = 0 \). In fact, one easily obtains:

\[ \zeta(0) = \frac{i}{32\pi^2} \int d^4x \sqrt{-g} \left\{ \frac{\partial^2}{\partial (is)} \left[ e^{-i(m^2 - i\epsilon)s} F(x, x; is) \right]\right\}_{s=0} = \frac{i}{32\pi^2} \int d^4x \sqrt{-g} \left( m^4 - 2m^2 f_1 + 2 f_2 \right). \quad (2.22) \]

The remaining term in Eq. (2.10) involving \( \zeta'(0) \), on the other hand, cannot be calculated only based on the small-\( s \) behavior of \( F(x, x; is) \). Taking the derivative of Eq. (2.21) with respect to \( \nu \), and applying the result to \( \nu = 0 \), we have

\[ \zeta'(0) = \left( \frac{3}{2} + \gamma \right) \zeta(0) - \frac{i}{32\pi^2} \int d^4x \sqrt{-g} \int_0^\infty ids \ln(is) \frac{\partial^3}{\partial (is)^3} \left[ e^{-i(m^2 - i\epsilon)s} F(x, x; is) \right], \quad (2.23) \]

where \( \gamma \equiv -(d/dz) \ln \Gamma(z)|_{z=1} \) is the Euler’s constant. As we shall show in Sec. \[V \] a most important contribution to \( \zeta'(0) \) comes from the large-\( s \) regime of the integrand of Eq. (2.23).

Combining Eqs. (2.22) and (2.23) into Eq. (2.10), we have the following expression for the one-loop effective action:

\[ W_q = -\frac{1}{64\pi^2} \int d^4x \sqrt{-g} \left[ I(0, +\infty) - (m^4 - 2m^2 f_1 + 2 f_2) \ln \bar{\mu}^2 \right], \quad (2.24) \]
where, for given \( 0 \leq \alpha < \beta \leq +\infty \),

\[
\mathcal{I}(\alpha, \beta) \equiv \int_{\alpha}^{\beta} ids \ \ln(is) \frac{\partial^3}{\partial(is)^3} \left[ e^{-i(m^2-\xi)s} F(x, x; is) \right],
\]

(2.25)

and \( \tilde{\mu} \) is such that \( \ln \tilde{\mu}^2 = 3/2 + \gamma + \ln \mu^2 \).

Before analyzing in detail the properties of \( \mathcal{I}(0, +\infty) \), we turn to some known results about the coincidence limit of the propagator, \( K(x, x; is) \). In the next section, we consider several cases in which the exact solution for \( K(x, x; is) \) is known. Based on these cases, we postulate a general form for the large-\( s \) asymptotic form of \( K(x, x; is) \). A similar asymptotic form is also suggested by the summation of the subset of terms in the proper-time series for \( F(x, x; is) \) that involve one or more factors of the scalar curvature \( R \) \[22, 23\]. This summation involves all powers of \( s \), and thus may give some information about the asymptotic form of \( F(x, x; is) \) for large \( s \). In addition, the Feynman path integral solution of the Schrödinger equation \[24\] to Gaussian order suggests a similar large-\( s \) asymptotic behavior. It is the asymptotic form of \( F(x, x; is) \) for large \( s \) that determines the contributions of long wavelength infrared (IR) quantum fluctuations of the field \( \phi \) to the functional integral for the effective action. If the mass of the scalar particle is very small, it has a very large Compton wavelength which may significantly influence the long wavelength quantum fluctuations of the \( \phi \) field. As we shall see, there are spacetimes in which this influence can cause the expectation value of the energy-momentum tensor of the free scalar field to become large.

III. IN SEARCH OF THE LARGE-S BEHAVIOR OF THE PROPAGATOR

It is well known that the series for \( F(x, x; is) \) given in Eq. (2.17) is not convergent in general. However, as conjectured by Parker and Toms \[22\], and later proved by Jack and Parker \[23\], Eq. (2.17) contains a convergent sub-series involving powers of the scalar curvature \( R \), with the property that when it is summed, the (asymptotic) series that is left does not possess any power of the scalar curvature (without derivatives applied to it). More precisely, the series for \( F(x, x; is) \) in powers of \( s \) can be written (with \( \bar{\xi} = \xi - 1/6 \)) as

\[
F(x, x; is) = e^{-i\bar{\xi}Ra} \sum_{j=0}^{\infty} (is)^j \bar{f}_j,
\]

(3.1)
with the first three terms being
\[\bar{f}_0 = \bar{f}_0(x, x) = 1,\] (3.2)
\[\bar{f}_1 = \bar{f}_1(x, x) = 0,\] (3.3)
\[\bar{f}_2 = \bar{f}_2(x, x) = \frac{1}{180} \left[ (1 - 30\bar{\xi}) \Box R + R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - R_{\alpha\beta} R^\alpha_{\beta} \right],\] (3.4)
and the subsequent terms being free of any terms containing undifferentiated factors of \(R\). Obviously, fixing a particular background might allow one to write the contractions of the Riemann and Ricci tensors appearing in the \(\bar{f}_j\) in terms of the scalar curvature \(R\). For example, the symmetries of de Sitter spacetime (a case that will be analyzed later) make it possible to express any geometrical scalar quantity in terms of \(R\) alone. However, the dependence of \(\bar{f}_j\) on \(R\) would be different for different backgrounds. Similarly, there are identities that depend on the dimension of spacetime that would allow one to reexpress some of the coefficients \(\bar{f}_j\) in such a way that terms involving \(R\) would appear. The expressions for the \(\bar{f}_j\) that are valid for general metrics and in spacetimes of arbitrary dimension do not include factors of \(R\). The form of the coincidence limit of the propagator given in Eq. (3.1) is called the \(R\)-summed, or the partially-summed, form of the propagator or heat kernel. The factor of \(\exp(-i\bar{\xi}Rs)\) in Eq. (3.1) sums a covariant and dimensionally-invariant set of terms to all orders in \(s\). Therefore, it may contain information about the large-\(s\) behavior of \(K(x, x; is)\). Earlier, Bekenstein and Parker [24], using Fermi coordinates in curved spacetime, were able to obtain the Gaussian approximation to the Feynman path integral solution of the Schrödinger equation (2.14). This approximation did not restrict \(s\) to small values. The result they obtained for \(K(x, x'; is)\) reduces in the coincidence limit to the result obtained from the first term of the series in Eq. (3.1). This gives further support to the impression that the exponential factor has relevance to the nonperturbative large-\(s\) behavior of \(K(x, x; is)\).

In the work of Parker and Raval [13, 14], they considered the effects of this nonperturbative term in \(K(x, x; is)\) on the effective action. Here we clarify and extend their work in a number of respects. First, by considering several exactly known heat kernels we generalize the expression for the large-\(s\) behavior of \(K(x, x; is)\) that was suggested by the \(R\)-summed form and by the Gaussian approximation. Second, we prove that the growth in the vacuum expectation value of \(T_{\mu\nu}\) that occurs in certain spacetimes comes directly from the large-\(s\) asymptotic form of the propagator. Third, we incorporate the possibility of dissipative
processes into our expression for the effective action. We also summarize numerical solutions of the resulting semi-classical Einstein equations that we obtained in the FRW universe. These numerical solutions include the contributions of higher derivatives of the Riemann tensor.

In order to arrive at a sufficiently general conjecture for the large-$s$ asymptotic form of $K(x, x; is)$, let us analyze three particular cases where the propagator $K(x, x'; is)$ is known exactly, namely, de Sitter spacetime, the Einstein static universe, and the linearly-expanding spatially-flat FRW universe (the latter one assuming conformal coupling $\xi = 1/6$). The exact Euclidean heat kernel $K_E(x, x'; s)$ (which is the Euclideanized form of the propagator $K$) in de Sitter spacetime for a massive scalar field was calculated by Dowker and Critchley [35] and given by:

$$K_E(p, s) = \frac{1}{4\pi^2 a^4} \frac{d}{dp} \sum_{j=0}^{\infty} (j + 1/2) \exp \left\{ \frac{i s}{a^2} \left[ \frac{9}{4} - (j + 1/2)^2 \right] \right\} P_j(p), \quad (3.5)$$

where $p \equiv \cos(\sqrt{2/3 \sigma}/a)$, $\sigma = \sigma(x, x')$ is half the square of the geodesic distance between $x$ and $x'$, $a$ is a constant related to the scalar curvature by $R = 12/a^2$, and $P_j$ are the Legendre polynomials. [Here we use signature $(-, +, +, +)$, so that we have performed the substitution $\sigma \mapsto -\sigma$ in the results of Ref. [35], which uses signature $(+, -, -, -).$]

Using $dP_j(p)/dp|_{p=1} = j(j+1)/2$, factoring out the $j$-independent exponential factor, and multiplying Eq. (3.5) by $i$ to go from the Euclidean to the Lorentzian metric, we obtain the exact de Sitter propagator in the coincidence limit:

$$K(x, x; is) = e^{-i(M^2 + \xi R - i\epsilon)s} iK_E(1, s)$$

$$= \frac{i}{8\pi^2 144} e^{-i(M^2 - i\epsilon)s}$$

$$\times \sum_{j=0}^{\infty} j(j + 1)(j + 1/2) \exp \left[ \frac{-iRs}{12} j(j + 1) \right], \quad (3.6)$$

where $M^2 \equiv m^2 + \xi R$. (The exponential factor multiplying $iK_E(1, s)$ sets the propagator to the notation used here.) Notice that the exponential for the term $j = 0$ would be exactly the same as the exponential obtained from the $R$-summed form of the propagator, $e^{-iM^2s}$. However, this exponential in Eq. (3.6) appears multiplying a factor proportional to $j$, and therefore gives no contribution to the summation (possibly due to the high degree of symmetry of de Sitter spacetime). Since the effective action $W_q$ depends linearly on
the propagator \( K \) [see Eqs. (2.10), (2.12), and (2.13)], we can investigate separately the contributions to \( W_q \) coming from each term in the summation of Eq. (3.6):

\[
K_j(x, x; is) \equiv -\frac{i}{16\pi^2 s^2} e^{-i(M_j^2 - i\epsilon)s} \mathcal{R}_{2(j)}(is)^2, \quad j \geq 0,
\]

(3.7)

with \( M_j^2 \equiv M^2 + j(j + 1)R/12 \) and \( \mathcal{R}_{2(j)} \equiv j(j + 1)(j + 1/2)R^2/72 \). Note that

\[
K(x, x; is) = \sum_{j=0}^{\infty} K_j(x, x; is).
\]

(3.8)

As will become clear in the next section, the dominant term in Eq. (3.8) for large \( s \) is the one with \( j > 0 \) for which \( |M_j^2| \) is the smallest. This value of \( j \), which we denote by \( k \), will of course depend on the value of \( \xi \). For example, if \( \xi > -1/6 \), then \( k = 1 \) and \( M_k^2 = m^2 + \xi R \).

More generally, we can say that (in the distributional sense) the dominant behavior of the propagator in de Sitter spacetime, for large \( s \), is given by

\[
K(x, x; is) \sim -\frac{i}{16\pi^2 s^2} e^{-i(M_k^2 - i\epsilon)s} \mathcal{R}_{2(k)}(is)^2,
\]

(3.9)

with \( k \geq 1 \).

Our second example, is the Einstein static universe. In this case, the exact form of the coincidence limit of the propagator is given by

\[
K(x, x; is) = -\frac{i}{16\pi^2 s^2} e^{-i(M^2 - i\epsilon)s} \left\{ 1 + 2 \sum_{j=1}^{\infty} e^{ij^2\pi^2 a^2/s} \left( 1 + 2ij^2\pi^2 a^2/s \right) \right\},
\]

(3.10)

where \( a \) is related to the scalar curvature through \( R = 6/a^2 \). The summation in \( j \) appearing in Eq. (3.10) is related to the fact that in the Einstein static universe there are infinitely many geodesics connecting any point \( x \) to itself. The contribution to the propagator given by the direct path connecting \( x \) to itself (i.e., the trivial path), is encompassed by the factor 1 inside the curly brackets of Eq. (3.10). Note then that the Gaussian approximation for the propagator gives the exact direct-path contribution in the Einstein static universe. If we restrict our attention to the direct-path contribution, we have the large-\( s \) regime of Eq. (3.10) given by

\[
K(x, x; is) \sim -\frac{i}{16\pi^2 s^2} e^{-i(M^2 - i\epsilon)s}.
\]

(3.11)

We also find that approximation of the infinite sum in Eq. (3.10) by an integral strongly suggests that the contributions of the indirect paths sum to a quantity that grows much
more slowly than \( s^2 \). Therefore, we conclude that the rate of growth of the de Sitter heat kernel for large \( s \) is faster than that of the heat kernel in the Einstein static universe.

As our last example, let us consider the spatially-flat FRW universe that is expanding linearly in the FRW proper time coordinate. With conformal coupling, \( \xi = 1/6 \), between the field and the scalar curvature, the propagator was calculated by Chitre and Hartle \[32\] and Charach and Parker \[33\]. In the coincidence limit, it can be put into the form

\[
K(x, x; is) = -\frac{i}{16\pi^2 s^2} e^{-i(m^2-\imath)is} \left\{ 1 + \beta \frac{Rs}{6\pi} \int_{-\infty}^{+\infty} dv e^{-6i(cosh v)^2/(Rs)} \right\}, \tag{3.12}
\]

where \( \beta \) is an arbitrary constant. Note again that the overall exponential factor appearing in Eq. (3.12) is consistent with the \( R \)-summed form of the propagator, recalling that in the case of conformal coupling one has \( \bar{\xi} = 0 \), i.e., \( M^2 = m^2 \). Another point worth mentioning about Eq. (3.12) is that for large values of \( s \), the integral inside curly brackets gives no contribution to order \( s^j \) for \( j \geq 0 \). [This conclusion can be drawn by taking the limit \( s \to \infty \) in the integrand in Eq. (3.12) and noting that the integral vanishes in this limit.] Therefore, we have that in the large-\( s \) regime the dominant behavior of Eq. (3.12) can be written as

\[
K(x, x; is) \sim -\frac{i}{16\pi^2 s^2} e^{-i(m^2-\imath)is} R^\lambda(is)^\lambda, \tag{3.13}
\]

where \( \lambda \) is some number smaller than 1 and \( R^\lambda \) is a scalar quantity with the same dimension as \( R^\lambda \). For \( \xi \neq 1/6 \), we would expect \( m^2 \) in (3.13) to be replaced by \( M^2 \).

Of the FRW metrics considered so far, the large \( s \) asymptotic form of the heat kernel of de Sitter spacetime, Eq. (3.9), has the highest power of \( s \) multiplying the exponentials that appear in all the examples. Therefore, in arriving at an ansatz for the large \( s \) asymptotic form of the heat kernel in a general (but not pathological) FRW universe, we must include an exponential multiplied by a power of \( s \) that is at least as large as the power that appears in (3.9). Therefore, based on the results and discussion presented so far in this section, about the form of the coincidence limit of the propagator, \( K(x, x; is) \), in de Sitter spacetime, the Einstein static universe, and the linearly-expanding spatially-flat FRW universe, it seems reasonable to make the ansatz that, at least in the four-dimensional FRW universes that will be considered in a later section, the general form of the dominant term in the large-\( s \) behavior of \( K(x, x; is) \) is given by Eq. (2.16) with

\[
F(x, x; is) \sim R_n e^{-i\chi_n Rs(is)^n} \tag{3.14}
\]
for some integer $n$ (to be discussed in the next paragraph), with $\chi_n$ being a dimensionless number and $\mathcal{R}_n = \mathcal{R}_n(x)$ a scalar quantity constructed from the metric $g_{\mu\nu}$ and having the same dimension as the $n$-th power of the scalar curvature $R$. It seems clear that the value of $\chi_n$ will depend on the constant $\xi$ that appears in the field equation for $\phi$. It may also be related to the topological properties of the spacetime because the large-s asymptotic form of $F(x, x; is)$ could conceivably sense the large-scale structure of the spacetime. We will assume that the value of $\xi$ is chosen such that $\chi_n$ is negative, as negative values of $\chi_n$ are of interest for the cosmological effect that we consider in a later section. The parameter that determines the cosmological effect is the ratio $-m^2/\chi_n \equiv \bar{m}^2$.

The particular cases analyzed here may give some hints on the values of $n$ and $\mathcal{R}_n$. Since we are looking for the general dominant large-$s$ behavior of $K(x, x; is)$, the de Sitter case analyzed above seems to suggest that $n$ is 2. (It could be larger than 2, but our examples give no evidence of that.) Then the Einstein static universe and the linearly expanding universe would be special cases in which subdominant asymptotic terms become dominant as a result of the vanishing of the coefficient of the dominant asymptotic term. Should this be true, then $\mathcal{R}_2$ must be a scalar that vanishes when calculated in the Einstein static universe and in the linearly-expanding spatially-flat FRW universe, while being non-zero when calculated in de Sitter spacetime (recalling also that its dimension is the same as the dimension of $R^2$).

It is not difficult to verify that some of the candidates for the general expression of $\mathcal{R}_2$ in four dimensions are given by the integrand of the *Gauss-Bonnet invariant*,

$$ G \equiv R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2, $$

the second-order term in the $R$-summed form of the Schwinger-deWitt proper-time series [see Eq. (3.1)],

$$ \bar{f}_2 \equiv \frac{1}{180} \left[ (1 - 30\tilde{\xi})\Box R + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - R_{\alpha\beta}R^{\alpha\beta} \right], $$

and the scalar quantity

$$ S \equiv R_{\alpha\beta}R^{\alpha\beta} - \frac{R^2}{3}. $$

Actually, linear combinations of $G$, $\bar{f}_2$, and $S$ are also candidates for the general expression of $\mathcal{R}_2$.

It is already clear from the work of Parker and Raval, that it is the exponential that appears in the asymptotic form of the propagator that is the key to causing an acceleration
of the expansion of the universe. In our numerical work, to be discussed in a later section, we will take \( n = 2 \) and a particular choice for \( \mathcal{R}_2 \). We will also mention numerical results we obtained by taking \( n = 0 \), which confirm that the exponential is responsible for bringing about acceleration of the expansion. Although the case \( n = 0 \) was worth considering because it involves fewer time derivatives of the metric, the case \( n = 2 \) appears to behave better, as will be discussed.

It is important to stress that the results we shall derive in the next section are independent of the specific form of the term \( \mathcal{R}_n \) in Eq. (3.14) and of the assumption that \( n = 2 \) (as long as \( n \geq 0 \) and \( \mathcal{R}_n \) is not identically zero for a general metric, which one immediately sees to be true from the cases we have considered). For this reason, in the next section we will use the generic ansatz for the large-\( s \) behavior of \( F(x, x; is) \) given in Eq. (3.14) and only in later sections will we take \( n = 2 \) and assume a particular form for \( \mathcal{R}_2 \).

### IV. LARGE-\( s \) ASYMPTOTIC BEHAVIOR OF THE PROPAGATOR AND NON-PERTURBATIVE INFRARED QUANTUM EFFECTS

Returning to the form of the effective action given in Eq. (2.24), we will split the quantity \( \mathcal{I}(0, +\infty) \) in two terms [see Eq. (2.25)]:

\[
\mathcal{I}(0, +\infty) = \mathcal{I}_{\text{reg}} + \mathcal{I}_{\text{IR}},
\]

where

\[
\mathcal{I}_{\text{reg}} = \mathcal{I}_{\text{reg}}(\lambda_{IR}) \equiv \mathcal{I}(0, \lambda_{IR}), \tag{4.2}
\]

\[
\mathcal{I}_{\text{IR}} = \mathcal{I}_{\text{IR}}(\lambda_{IR}) \equiv \mathcal{I}(\lambda_{IR}, +\infty), \tag{4.3}
\]

with \( \lambda_{IR} \) some “large” but fixed parameter with dimension \((\text{length})^2\). The contribution of long wavelength (IR) fluctuations of the field \( \phi \) to the effective action is given by the integration over large \( s \).

We will be most interested in the quantity \( \mathcal{I}_{\text{IR}} \) and will assume that \( \mathcal{I}_{\text{reg}} \) is a well-behaved function of the metric if \( x \) is a non-singular point of the spacetime. This assumption seems quite reasonable because we do not expect the function \( F(x, x; is) \) appearing in the definition of \( \mathcal{I}_{\text{reg}} \) to give any problem in the limited interval of integration \((0, \lambda_{IR})\). Moreover, from the Schwinger-deWitt proper-time series we even know that \( F(x, x; is) \to 1 \) as \( s \to 0 \). The situation is different, however, for the large-\( s \) contribution \( \mathcal{I}_{\text{IR}} \), as we will analyze next.
Combining Eqs. (4.3), (2.25), and (3.14), we have, for sufficiently large $\lambda_{IR}$,

\[ I_{IR} \approx R_n \int_{\lambda_{IR}}^{\infty} ids \ln(is) \frac{\partial^3}{\partial(is)^3} e^{-i(M_n^2-\epsilon)s} (is)^n \]

= \((-1)^{n+1}R_n\frac{\partial^n}{\partial(M_n^2)^n} \left[(M_n^2 - i\epsilon)^3 J(\lambda_{IR}, M_n^2)\right], \tag{4.4} \]

where

\[ M_n^2 \equiv m^2 + \chi_n R \tag{4.5} \]

and the quantity $J$ is defined by

\[ J(\lambda_{IR}, M_n^2) \equiv \int_{\lambda_{IR}}^{\infty} ids \ln(is) e^{-i(M_n^2-\epsilon)s} \]

= $i\lambda_{IR} \int_1^{\infty} d\tilde{s} \left[\ln \tilde{s} + \ln(i\lambda_{IR})\right] e^{-i\lambda_{IR}(M_n^2-\epsilon)\tilde{s}}$

= $\frac{\ln(i\lambda_{IR}) e^{-i\lambda_{IR}(M_n^2-\epsilon)}}{(M_n^2 - i\epsilon)}$

\[ + i\lambda_{IR} \int_1^{\infty} d\tilde{s} \left(\ln \tilde{s}\right) e^{-i\lambda_{IR}(M_n^2-\epsilon)\tilde{s}}, \tag{4.6} \]

with $\tilde{s} \equiv s/\lambda_{IR}$. Using Eq. (4.358.1) of Ref. [34], the integral appearing in Eq. (4.6) can be evaluated:

\[ \int_1^{\infty} d\tilde{s} \left(\ln \tilde{s}\right) e^{-i\lambda_{IR}(M_n^2-\epsilon)\tilde{s}} = \frac{\partial}{\partial\beta} \left(\Gamma(\beta, i\lambda_{IR}M_n^2 + \epsilon)\right) \bigg|_{\beta=1} \]

= $\left[-[i\lambda_{IR}(M_n^2 - i\epsilon)]^{-1} \left(\gamma + \ln[i\lambda_{IR}(M_n^2 - i\epsilon)]\right)\right]$

$+ \sum_{j=0}^{\infty} \frac{(-1)^j(i\lambda_{IR}M_n^2)^j}{j!(j+1)^2}, \tag{4.7} \]

where $\Gamma(\beta, \alpha)$ is the incomplete gamma function and in passing from the first to the second line of Eq. (4.7) we have used the expansion of $\Gamma(\beta, \alpha)$ in powers of $\alpha$ (see Eq. (8.354.2) of Ref. [34]). Then, Eq. (4.6) becomes

\[ J(\lambda_{IR}, M_n^2) = -(M_n^2 - i\epsilon)^{-1} \left[\gamma + \ln(M_n^2 - i\epsilon)\right] \]

\[ + i\lambda_{IR} \sum_{j=0}^{\infty} \frac{(-1)^j(i\lambda_{IR}M_n^2)^j}{j!(j+1)^2} \left[\frac{1}{j+1} - \ln(i\lambda_{IR})\right]. \tag{4.8} \]

Finally, using Eq. (4.8) to evaluate Eq. (4.4), we obtain for the dominant infrared contribu-
tion to the effective action

\[ I_{IR} \approx (-1)^n \mathcal{R}_n \frac{\partial^n}{\partial (M^2_n)^n} \left[ (M^2_n - i\epsilon)^2 \ln(M^2_n - i\epsilon) \right] \]

\[ + (-1)^n \mathcal{R}_n \frac{\partial^n}{\partial (M^2_n)^n} \left\{ \gamma M^4_n \right\} \]

\[ + (i\lambda_{IR})^{-2} \sum_{j=3}^{\infty} \left( -1 \right)^j \left( i\lambda_{IR} M^2_n \right)^j \left( \frac{1}{j - 2} - \ln(i\lambda_{IR}) \right) \]  

(4.9)

Notice that the summation now starts at \( j = 3 \).

There are a few points worth mentioning about Eq. (4.9). First, note that the series appearing above is (absolutely) convergent for any (finite) value of \( \lambda_{IR} M^2_n \). Moreover, the result of any number of derivatives with respect to \( M^2_n \) applied to this series is still (absolutely) convergent. These facts imply that the first term on the right-hand-side of Eq. (4.9) is the dominant one when \( M^2_n \) is sufficiently small. (In particular, for \( n \geq 2 \), the value of this dominant term is unbounded when \( M^2_n \to 0 \), i.e., when \( R \to -m^2/\chi_n \).) Notice that this dominant term in Eq. (4.9) is independent of \( \lambda_{IR} \). Thus, the effective action given in Eq. (2.24) can be approximated by

\[ W_q \approx W_q^{reg} + \frac{(-1)^{n+1}}{64\pi^2} \int d^4x \sqrt{-g} \mathcal{R}_n \]

\[ \times \frac{\partial^n}{\partial (M^2_n)^n} \left[ (M^2_n - i\epsilon)^2 \ln \left( \frac{M^2_n - i\epsilon}{m^2} \right) \right], \]  

(4.10)

where

\[ W_q^{reg} \equiv - \frac{1}{64\pi^2} \int d^4x \sqrt{-g} \left\{ \mathcal{I}_{reg} - (m^4 - 2m^2 f_1 + 2f_2) \ln \tilde{\mu}^2 \right. \]

\[ + (-1)^n \mathcal{R}_n \frac{\partial^n}{\partial (M^2_n)^n} \left[ (\gamma + \ln m^2) M^4_n \right] \]

\[ + (i\lambda_{IR})^{-2} \sum_{j=3}^{\infty} \left( -1 \right)^j \left( i\lambda_{IR} M^2_n \right)^j \left( \frac{1}{j - 2} - \ln(i\lambda_{IR}) \right) \]  

(4.11)

denotes the part of the effective action whose value and variation with respect to the metric \( g_{\mu\nu} \) are well-behaved for any value of \( M^2_n \).

The positive infinitesimal \( \epsilon \) in Eq. (4.10) was originally introduced to ensure convergence of the functional integral for \( \exp(iW_q) \). Dissipation through small interactions with external systems can be modeled by allowing \( \epsilon \) to be a finite quantity, small with respect to \( m^2 \). For example, a long but finite lifetime of the particle associated with the field \( \phi \) may be modeled in this way.
V. RENORMALIZATION AND VARIATION OF THE EFFECTIVE ACTION

Now, let us carry out the renormalization of the effective action given in Eq. (4.10). Notice that because we have used the $\zeta$-function formalism, the effective action $W_q$ already at this stage is free from “unphysical” divergences. However, $W_q$ given in Eq. (4.10) depends on the unobservable parameter $\tilde{\mu}$ through terms up to second order in the curvature of the spacetime [see Eq. (4.11)]. Adding a bare gravitational action containing terms up to second order in curvature,

$$W_g \equiv \int d^4x \sqrt{-g} \left( -2\kappa \Lambda + \kappa R + \alpha_1 R^2 + \alpha_2 R_{\mu\nu} R^{\mu\nu} + \alpha_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right)$$

with $\kappa, \Lambda, \alpha_1, \alpha_2, \alpha_3$ being bare gravitational constants, we can absorb $\tilde{\mu}$ into the definition of observable low-curvature gravitational constants $\kappa_o, \Lambda_o, \alpha_{1o}, \alpha_{2o}, \alpha_{3o}$ with the result that the low-curvature limit of the total action has the form

$$W \equiv W_g + W_1 \sim \int d^4x \sqrt{-g} \left( -2\kappa_o \Lambda_o + \kappa_o R + \alpha_{1o} R^2 \right.$$

$$+ \alpha_{2o} R_{\mu\nu} R^{\mu\nu} + \alpha_{3o} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + ...) \, .$$

Since this procedure was performed by Parker and Raval in Ref. [13], using the Schwinger-deWitt proper-time series, we will skip this detailed calculation here. What we will compute now is the form of the total effective action after renormalization. In order to do so we note that the renormalization procedure described above has the net effect of replacing the bare gravitational constants in the bare gravitational action by the observable ones (thus giving the renormalized gravitational action), while adding to $W_q$ terms up to second order in the curvature in such a way to cancel the terms up to second order originally in the low-curvature expansion of $W_q$ (in this way giving the renormalized effective action). Therefore, we have:

$$W = W_g + W_q = (W_g)_{ren} + (W_q)_{ren} \, .$$

with

$$(W_g)_{ren} = \int d^4x \sqrt{-g} \left( -2\kappa_o \Lambda_o + \kappa_o R + \alpha_{1o} R^2 \right.$$

$$+ \alpha_{2o} R_{\mu\nu} R^{\mu\nu} + \alpha_{3o} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right)$$

(5.4)
and assuming, for example, $n = 2$ in Eq. (4.10) motivated by the discussion of Sec. III,

$$(W_q)_{\text{ren}} \approx (W_{q}^{\text{reg}})_{\text{ren}} \approx \frac{1}{32\pi^2} \int d^4 x \sqrt{-g} \mathcal{R}_2 \ln \left( \frac{M_2^2 - i\epsilon}{m^2} \right)$$

$$\approx (W_{q}^{\text{reg}})_{\text{ren}} - \frac{i}{32\pi} \int d^4 x \sqrt{-g} \mathcal{R}_2 \Theta(-M_2^2)$$

$$- \frac{1}{64\pi^2} \int d^4 x \sqrt{-g} \mathcal{R}_2 \ln \left( \frac{M_4^2 + \epsilon^2}{m^4} \right). \tag{5.5}$$

Recall that $M_2^2 = m^2 + \chi_2 R$ [see Eq. (4.5)] and that $\mathcal{R}_2$ is quadratic in the curvature (see Sec. III). In passing from the first to the second line of Eq. (5.5) we have used $\ln(M_2^2 - i\epsilon) \approx \ln |M_2^2 + i\epsilon| + i\pi \Theta(-M_2^2)$, where

$$\Theta(x) \equiv \begin{cases} 
0 & , x < 0, \\
1/2 & , x = 0, \\
1 & , x > 0 
\end{cases} \tag{5.6}$$

is the Heaviside step function. The quantity $(W_{q}^{\text{reg}})_{\text{ren}}$ in Eq. (5.5) is obtained by applying the renormalization procedure to $W_{q}^{\text{reg}}$ alone.

The imaginary part of Eq. (5.5) is related to particle production [14]. However, since the order of magnitude of the imaginary term written explicitly in Eq. (5.5) could be comparable to the order of magnitude of $(W_{q}^{\text{reg}})_{\text{ren}}$, which may also have an imaginary part that we are not obtaining explicitly, we will not analyze this phenomenon here.

Setting the observable gravitational constants $\alpha_1, \alpha_2, \text{and } \alpha_3$ to zero (to reproduce the classical vacuum Einstein equations, with a cosmological constant, in the low-curvature limit), the condition that the variation of the total effective action vanishes for arbitrary variations of the metric gives us the **semi-classical** vacuum Einstein equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda_0 g_{\mu\nu} = 8\pi G_N \langle T_{\mu\nu} \rangle, \tag{5.7}$$

where $G_N \equiv 1/(16\pi\kappa_o)$ is Newton’s constant, and the vacuum expectation value of the energy-momentum tensor of the scalar field is defined through

$$\langle T^{\mu\nu} \rangle = \frac{2}{\sqrt{-g} \delta g_{\mu\nu}} (W_q)_{\text{ren}}. \tag{5.8}$$

As noted earlier, we will set $\Lambda_0 = 0$ to see if the expression for $\langle T^{\mu\nu} \rangle$ could explain the observed acceleration of the expansion of the universe.

In order to numerically integrate the semi-classical Einstein equations in an FRW universe, we must adopt an explicit form of $\langle T^{\mu\nu} \rangle$. For this purpose, we assume again $n = 2$
and adopt an expression for $\mathcal{R}_2$ that satisfies all the properties discussed in Sec. III. Namely, we let

$$\mathcal{R}_2 = \alpha \bar{f}_2 = \frac{\alpha}{180} \left[ (1 - 30\xi)\Box R + R_{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} - R_{\alpha\beta} R^{\alpha\beta} \right],$$

(5.9)

where $\alpha$ is some dimensionless constant. This is the term that appeared in the $R$-summed form of the heat kernel. It was used by Parker and Raval in their effective action. Therefore, it allows us to directly compare our results to the ones that were obtained from their analytic solution that neglected derivatives of curvature invariants during the transition to an expansion of the universe in which $R$ is constant. This choice for $\mathcal{R}_2$ is sufficiently complicated to be representative of other possibilities that contain higher derivatives of the metric. In our numerical integration, we do not neglect higher derivatives of the metric in the relevant terms, and we carefully analyze the possibility of runaway solutions. As it would be too lengthy to give the details here, we will present the full numerical analysis in a paper now in preparation [37]; but here we give our main conclusions and plot a representative numerical solution.

As we noted earlier, there are other possible choices of $\mathcal{R}_2$. Our numerical analysis using the expression in Eq. (5.9) suggests that the transition from a classical expanding universe to an expansion with $R$ constant depends only on certain general features of $\mathcal{R}_2$, and thus may be a fairly robust generic feature of a wide class of asymptotic forms of the heat kernel. Therefore, it seems reasonable to also look at effective actions suggested by other underlying theories that have similar higher derivative terms. For example, string theory leads to low-energy effective actions that contain higher derivative terms analogous to those that appear in our low-energy effective action. Thus, our methods and analysis may be of interest to string theorists.

Combining Eqs. (5.8), (5.5), and (5.9), and using the variations presented in Appendix A [see Eqs. (A1)-(A6)], we have (after dropping the subscripts of $\chi_2$ and $M_2$ for simplicity):

$$\langle T^{\mu\nu} \rangle = \langle T^{\mu\nu}_{\text{reg}} \rangle - \frac{\alpha}{16\pi^2} \left\{ A^{\mu\nu}_{(0)} \ln \left( \frac{M_\xi^4}{m^4} \right) + \frac{\chi}{M_\xi^2} A^{\mu\nu}_{(1)} + \frac{\chi^2}{M_\xi^4} A^{\mu\nu}_{(2)} + \frac{\chi^3}{M_\xi^6} A^{\mu\nu}_{(3)} + \frac{\chi^4}{M_\xi^8} A^{\mu\nu}_{(4)} \right\},$$

(5.10)

where we have defined

$$M_\xi^2 \equiv \sqrt{M^4 + \bar{\epsilon}^2} = \sqrt{(m^2 + \chi R)^2 + \bar{\epsilon}^2},$$

(5.11)
and the regular tensors \( A_{\mu j}^{\nu}, j = 0, 1, 2, 3, 4 \), are given by

\[
A_{(0)}^{\mu \nu} = \frac{1}{720} \left\{ 2\nabla^\nu R - 6\Box R^{\mu \nu} + g^{\mu \nu} \left( \Box R + R^\alpha_{\beta \gamma \delta} R_{\alpha \beta \gamma \delta} - R_{\alpha \beta} R_{\alpha \beta} \right) \\
-4R^{\mu \alpha \beta \gamma} R_{\alpha \beta \gamma} - 4R^{\mu \alpha \beta \gamma} R^\rho_{\alpha \beta \gamma \rho} + 8R^\mu_{\alpha \rho} R^\nu_{\alpha \rho} \right\},
\]

\( M \)

\[
A_{(1)}^{\mu \nu} = \frac{1}{360} \left( \frac{M^2}{\sqrt{M^4 + \epsilon^2}} \right) \left\{ (1 - 30\bar{\xi}) g^{\mu \nu} \nabla^\alpha R \nabla_\alpha R + 12\nabla^\alpha R \nabla^\nu (R^\mu_\alpha) - 12\nabla^\alpha R \nabla_\alpha R^{\mu \nu} \\
+ 2(\nabla^\mu \nabla^\nu - g^{\mu \nu} \Box - R^{\mu \nu}) \left[ 2(1 - 30\bar{\xi}) \Box R + R^\alpha_{\beta \gamma \delta} R_{\alpha \beta \gamma \delta} - R_{\alpha \beta} R_{\alpha \beta} \right] + 2R^{\mu \nu} \Box R \\
-60\bar{\xi} \nabla^\mu R \nabla^\nu R + 2g^{\mu \nu} R^\alpha \nabla_\alpha \nabla_\beta R - 4R^{\alpha (\mu} \nabla^\nu) \nabla_\alpha R - 8R^{\mu \alpha \nu \beta} \nabla_\alpha \nabla_\beta R \right\},
\]

\( \alpha, \beta, \gamma, \delta, \rho \)

\[
A_{(2)}^{\mu \nu} = -\frac{1}{180} \left( \frac{M^4 - \epsilon^2}{M^4 + \epsilon^2} \right) \left\{ (1 - 30\bar{\xi}) (\nabla^\mu \nabla^\nu - g^{\mu \nu} \Box - R^{\mu \nu}) (\nabla^\alpha R \nabla_\alpha R) \\
+ [\nabla^\mu \nabla^\nu R - g^{\mu \nu} \Box R + 2\nabla^{(\mu} R \nabla^{\nu)} - 2g^{\mu \nu} \nabla^\lambda R \nabla_\lambda] \left[ 2(1 - 30\bar{\xi}) \Box R + R^\alpha_{\beta \gamma \delta} R_{\alpha \beta \gamma \delta} - R_{\alpha \beta} R_{\alpha \beta} \right] \\
+ R^{\mu \nu} g^{\alpha \beta} + R^{\mu \nu} g^{\alpha \beta} - 2R^{\alpha (\mu} g^{\nu) \beta} - 4R^{\mu \alpha \nu \beta} \right\} (\nabla^\alpha R \nabla_\beta R),
\]

\( \alpha, \beta, \gamma, \delta, \rho \)

\[
A_{(3)}^{\mu \nu} = \frac{1}{90} \left( \frac{M^6 - 3\epsilon^2 M^2}{(M^4 + \epsilon^2)^{3/2}} \right) \left\{ (1 - 30\bar{\xi}) (\nabla^\mu \nabla^\nu R - g^{\mu \nu} \Box R) (\nabla^\lambda R \nabla_\lambda R) \\
+ (\nabla^\mu R \nabla^\nu R - g^{\mu \nu} \nabla^\lambda R \nabla_\lambda R) \left[ 2(1 - 30\bar{\xi}) \Box R + R^\alpha_{\beta \gamma \delta} R_{\alpha \beta \gamma \delta} - R_{\alpha \beta} R_{\alpha \beta} \right] \\
+ 2(1 - 30\bar{\xi}) \nabla^\alpha R \nabla_\alpha \left( \nabla^\mu R \nabla^\nu R - g^{\mu \nu} \nabla^\lambda R \nabla_\lambda R \right) \right\},
\]

\( \alpha, \beta, \gamma, \delta, \rho \)

\[
A_{(4)}^{\mu \nu} = -\frac{1}{30} \left( \frac{M^8 - 6\epsilon^2 M^4 + \epsilon^4}{(M^4 + \epsilon^2)^2} \right) \left( 1 - 30\bar{\xi} \right) (\nabla^\lambda R \nabla_\lambda R) (\nabla^\mu R \nabla^\nu R - g^{\mu \nu} \nabla^\alpha R \nabla_\alpha R) .
\]

\( \alpha, \beta, \gamma, \delta, \rho \)

The term \( \langle T^{\mu \nu}_{\text{reg}} \rangle \) in Eq. (5.10) stands for the part of the expectation value of the energy-momentum tensor coming from the regular part of the effective action.

Notice that the ratios involving \( M \) and \( \epsilon \) appearing in parenthesis in each one of the Eqs. (5.13)-(5.16) are bounded functions of \( M \). In fact, for \( \epsilon \) very small, these ratios are of order 1 except for very particular values of \( M \) for which the ratios become close to zero.

Notice also that in the limit \( \epsilon \to 0 \) all these ratios become equal to 1. In the following section we will apply the expectation value given in Eq. (5.10) to an FRW spacetime and analyze its cosmological consequences.

VI. QUANTUM SCALAR FIELD IN AN FRW UNIVERSE: THE VCDM COSMOLOGICAL MODEL

Now we shall apply the results obtained in the previous sections to a cosmological spacetime. The goal is to show that the present accelerating expansion of the universe may be
explained as due to the nonperturbative-infrared form of the effective action calculated in Sec. V. Such a model for the “dark energy” was first proposed by Parker and Raval and it is known as the VCDM or vacuum metamorphosis model [13]-[17].

As an idealization, we will consider our universe as being described by a spatially-flat FRW spacetime, with line element

\[ ds^2 = -dt^2 + a(t)^2 \left( dx_1^2 + dx_2^2 + dx_3^2 \right), \]  

(6.1)

filled with non-interacting matter, radiation, and a scalar field with zero expectation value, \( \langle \phi \rangle = 0 \), and a expectation value for its energy-momentum tensor \( \langle T^{\mu \nu} \rangle \) given by Eqs. (5.10)-(5.16). The semi-classical Einstein equations (with zero cosmological constant) then read

\[ R^{\mu \nu} - \frac{1}{2} g^{\mu \nu} R = 8 \pi G_N \left( T^{\mu \nu}_r + T^{\mu \nu}_m + \langle T^{\mu \nu} \rangle \right), \]  

(6.2)

where \( T^{\mu \nu}_r = \rho_r (g^{\mu \nu} + 4u^{\mu}u^{\nu})/3 \) and \( T^{\mu \nu}_m = \rho_m u^{\mu}u^{\nu} \), with \( u^\mu \) being the future-pointing, normalized vector field orthogonal to the homogeneous and isotropic hypersurfaces, and \( \rho_r \) and \( \rho_m \) being the radiation and matter energy densities, respectively, as measured by the family of geodesic observers with four-velocity \( u^\mu \) (comoving observers). Moreover, \( \rho_r \) and \( \rho_m \) are constant over each homogeneous and isotropic hypersurface.

The symmetries of the FRW spacetime greatly simplify the task of solving Eq. (6.2). It is not difficult to see that the most general form of a rank-two tensor field \( T^{\mu \nu} \) that is consistent with the FRW symmetries is

\[ T^{\mu \nu} = Au^\mu u^\nu + B \left( g^{\mu \nu} + u^\mu u^\nu \right), \]  

(6.3)

with \( A \equiv T^{\mu \nu} u_\mu u_\nu \) and \( B \equiv (T^{\mu \mu}_\mu + A) / 3 \) being scalar functions that are constant over each homogeneous and isotropic hypersurface, i.e.,

\[ \nabla_\mu A = -u_\mu u^\nu \nabla_\nu A \quad \text{and} \quad \nabla_\mu B = -u_\mu u^\nu \nabla_\nu B. \]  

(6.4)

Take, then, \( T^{\mu \nu} \) to be the difference between the left-hand-side (l.h.s.) and the right-hand-side (r.h.s.) of Eq. (6.2), which in an FRW spacetime clearly exhibits the form presented in Eq. (6.3). This shows that the usually ten (semi-classical) Einstein equations, \( T^{\mu \nu} = 0 \), are reduced to two equations, \( A = B = 0 \), in an FRW spacetime. Moreover, considering that \( \nabla_\mu T^{\mu \nu} = 0 \) is also satisfied [for both sides of Eq. (6.2) satisfy this condition], we have, after
using Eq. (6.4) and the fact that $u^\mu$ is a geodesic field,

$$0 = \nabla_\mu T^{\mu\nu} = u^\nu [\nabla_\mu (\mathcal{A} u^\mu) + \mathcal{B} \nabla_\mu u^\mu] ,$$

(6.5)

which tells us that equation $\nabla_\mu T^{\mu\nu} = 0$ is equivalent to $\mathcal{B} = (\nabla_\alpha u^\alpha)^{-1} \nabla_\mu (\mathcal{A} u^\mu)$, provided $\nabla_\alpha u^\alpha \neq 0$. In this case, $\mathcal{A} = 0$ and $\nabla_\mu T^{\mu\nu} = 0$ imply that $\mathcal{B} = 0$. Then, summarizing what we have shown so far, solving Eq. (5.2) in an FRW spacetime is equivalent to solving $\mathcal{A} = \mathcal{B} = 0$, which, in turn, is equivalent to solving $\mathcal{A} = 0$ and $\nabla_\mu T^{\mu\nu} = 0$ (assuming $\nabla_\alpha u^\alpha \neq 0$). In other words, we are left with the problem of solving

$$R^{\mu\nu} u_\mu u_\nu + \frac{1}{2} R = 8\pi G_N (\rho_m + \rho_m + \langle T^{\mu\nu} \rangle u_\mu u_\nu) ,$$

(6.6)

$$\nabla_\mu (T^{\mu\nu}_r + T^{\mu\nu}_m) = 0 .$$

(6.7)

[To obtain Eq. (6.7) we have used the fact that $\nabla_\mu (R^{\mu\nu} - g^{\mu\nu} R/2) = \nabla_\mu \langle T^{\mu\nu} \rangle = 0$ are identities since both $\sqrt{-g} (R^{\mu\nu} - g^{\mu\nu} R/2)$ and $\sqrt{-g} \langle T^{\mu\nu} \rangle$ are given by the functional derivative of an action with respect to the metric.] Eq. (6.7) is simple to solve analytically, but the Eq. (6.6) can only be solved numerically.

Space does not permit us to go into the details of solving numerically the ordinary differential equation for $a(t)$ obtained from Eq. (6.6). Here, we show representative plots, and summarize the main conclusions of the numerical calculations. In Fig. 1 we show the result for the scale parameter $a(t)$ for a universe with present value of matter and radiation energy densities such that $\Omega_{m0} \equiv \rho_m(t_0)/\rho_c = 0.34$ and $\Omega_{r0} \equiv \rho_r(t_0)/\rho_c = 8.33 \times 10^{-5}$, with $\rho_c \equiv 3H_0^2/(8\pi G_N)$, $H_0$ being the present value of the Hubble constant. We compare this result (solid line) with the approximation for $a(t)$ given by the earlier version of the VCDM model, where a constant-scalar curvature stage follows the usual matter dominated phase of the universe (dashed line). We can see that the “constant-$R$ approximation” in fact provides a very good (analytical) approximation to the numerical solution $a(t)$. The mass of the scalar field was chosen to be such that $\bar{m}/H_0 = 3.26$, and in order to facilitate the numerical analysis a non-zero value for $\epsilon$ was assumed (see discussion at the end of Sec. IV for the physical meaning of $\epsilon$). Also for numerical reasons, the value of the dimensionless parameter $\alpha_0 \equiv \alpha G_N H_0^2$, on which $a(t)$ depends through the quantum energy-momentum tensor $\langle T^{\mu\nu} \rangle$, was taken to be much larger than its physical value $\alpha_0 \approx 10^{-122}$. This, however, should not be a problem since we find that the smaller the value of $\alpha_0 > 0$, the closer the numerical solution for $a(t)$ becomes to the constant-$R$ approximation.
FIG. 1: Plot of the numerical solution (solid line) and constant-$R$ approximation (dashed line) for the scale parameter $a(t)$, as a function of time $t$, of a universe with matter and radiation content given by $\Omega_{m0} = 0.34$ and $\Omega_{r0} = 8.33 \times 10^{-5}$. In order to facilitate the numerical analysis, we have used $\epsilon_{H0} \equiv \epsilon/H_0^2 = 10^{-3}$ and $\alpha_0 \equiv \alpha G_N H_0^2 = 10^{-6}$. Also, we used $\bar{m}/H_0 = 3.26$.

In Fig. 2, we plot the Hubble parameter $H(z)$ (normalized by its present value $H_0$) as a function of red-shift $z$. Note again that the numerical solution for $H(z)$ (solid line) is very well approximated by the constant-$R$ approximation (dashed line).

Finally, in Fig. 3, we present the numerical evolution of the universe (solid line) in a diagram showing its scalar curvature $R$ and the square of the Hubble parameter $H^2$. Notice that after spending some time in the matter-dominated stage (represented by the dotted line), the universe makes a transition (not as sharp as in the constant-$R$ approximation) to an era dominated by the expectation value of the energy-momentum tensor of the quantized scalar field. During this latter era, the universe enters a period of accelerating expansion that lasts forever (in the spatially-flat FRW case) and that approaches, asymptotically in the future, an exponentially fast expansion with Hubble parameter $H \to \bar{m}/(2\sqrt{3})$.

It is important to mention that in numerically solving the semi-classical Einstein equations presented here, we have taken into account the higher derivative terms. Note that Eq. (6.6) leads to a fifth-order differential equation for $a(t)$. For this reason, it is natural to expect that there exist solutions to Eq. (6.6) that are not physically acceptable (e.g., runaway solutions). Notwithstanding, we find that taking initial conditions such that $a(t)$ behaves...
FIG. 2: Plot of the numerical solution (solid line) and constant-$R$ approximation (dashed line) for the Hubble parameter $H(z)$, as a function of redshift $z$, of a universe with matter and radiation content given by $\Omega_{m0} = 0.34$ and $\Omega_{r0} = 8.33 \times 10^{-5}$. In order to facilitate the numerical analysis, we have used $\epsilon_{H0} \equiv \epsilon/H_0^2 = 10^{-3}$ and $\alpha_0 \equiv \alpha G_N H_0^2 = 10^{-6}$. Also, we used $\bar{m}/H_0 = 3.26$.

classically (e.g., describing a matter-dominated universe) during some period of time in the past is enough to select only physically acceptable solutions, all of them evolving, eventually, to a phase of accelerating expansion. Since the vacuum energy density and pressure are negligible at early times, the classical initial conditions are the natural ones to impose. Initial conditions sufficiently far away from the classical ones in the past do appear to give rise to unphysical solutions, but it is remarkable that the class of natural initial conditions evidently gives only physically reasonable solutions.

As noted earlier, we also numerically integrated the semiclassical Einstein equations that result from using the ansatz of Eq. (3.14) with $n = 0$. These equations are simpler to solve numerically because they contain fewer time derivatives of the metric. We found that, as expected, the rate of change of the scalar curvature $R$ decreases as $R$ approaches the value $\bar{m}^2$ from above. However, instead of making a transition to an expansion in which $R$ remains close to $\bar{m}^2$, the scalar curvature effectively bounces off the value $\bar{m}^2$ and evolves toward increasing values of $R$. The figures that we show are all for the case of $n = 2$, which seems to be of more physical interest.
FIG. 3: Plot of the evolution of the square of the Hubble parameter $H^2$ and scalar curvature $R$ of a universe with matter and radiation content given by $\Omega_m = 0.34$ and $\Omega_r = 8.33 \times 10^{-5}$. Note that after a classical period of expansion (represented by the dotted line), the universe enters an era dominated by the energy-momentum tensor of the quantum scalar field that prevents the scalar curvature $R$ from dropping below the value $\bar{m}^2$. Eventually, the universe enters a stage of accelerating expansion that leads to an asymptotic exponentially-fast expansion. In order to facilitate the numerical analysis, we have used $\epsilon_{H_0} \equiv \epsilon/H_0^2 = 10^{-3}$ and $\alpha_0 \equiv \alpha G_N H_0^2 = 10^{-6}$. Also, we used $\bar{m}/H_0 = 3.26$.

VII. DISCUSSION

We have reconsidered the theory of the vacuum metamorphosis transition that occurs in the vacuum cold dark matter cosmological model from a manifestly nonperturbative point of view. We showed first that the terms in the vacuum energy-momentum tensor that become large when $R$ is close to the value $\bar{m}^2 = -m^2/\chi$ derive from the large-$s$ asymptotic behavior of the heat kernel. Then by examining the large-$s$ asymptotic form of the exact heat kernel in the de Sitter, Einstein static, and linearly expanding FRW universes, we arrived at a reasonable ansatz for the dominant asymptotic behavior of the heat kernel in a general FRW universe. The key feature of the heat kernel that ultimately causes the vacuum energy density and pressure to become large is the presence of a term of the form $\exp(-i\chi Rs)$ in the large-$s$ regime of the heat kernel. Our approach is manifestly non-perturbative in $s$ because
it involves the large-$s$ asymptotic form of the heat kernel, and does not involve summation of a series in powers of $s$. [There is also a factor $\exp(-im^2s)$ in the heat kernel that is exact, and is thus valid for any nonzero mass $m$ and for all values of $s$.]

We then obtained the explicit renormalized expression for the terms in the vacuum expectation value of $T_{\mu\nu}$ that become large when the scalar curvature $R$ is close to $\bar{m}^2$. These terms involve up to fifth-order time derivatives of the FRW scale factor $a(t)$. We also introduced a small parameter $\epsilon$ (of dimension $m^2$) that phenomenologically describes possible weakly dissipative effects, and softens the growth of $T_{\mu\nu}$ as $R$ approaches $\bar{m}^2$.

Finally, we adopted a specific form for the invariant factor, quadratic in the Riemann tensor, that multiplies the exponentials in the heat kernel, and we numerically integrated the Einstein equations in a spatially flat FRW universe, with the source consisting of the energy-momentum tensor of classical matter and radiation and the expectation value of $T_{\mu\nu}$ of the quantized scalar field $\phi$. We numerically evolved the terms having the higher time derivatives of $a(t)$. We found that if the universe in the past evolved classically (when the vacuum expectation values of $T_{\mu\nu}$ are negligible), then there are no runaway or unphysical solutions. All solutions with such classical initial conditions undergo a transition to an accelerating expansion that approaches the de Sitter expansion at late times.

We also noted (see the Introduction) that the theory considered here may provide a mechanism for early inflation, which could be caused by heavy bosons, or by the very same ultralight acceletron that may be responsible for the presently observed acceleration of the universe.

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APPENDIX A: VARIATIONS

We present here the list of the variations that were used in Sec. VI (recall that $M^2 \equiv m^2 + \chi R$ and $M^2_\epsilon \equiv \sqrt{M^4 + \epsilon^2}$):

\[
\delta \left( \int d^n x \sqrt{|g|} \ln \left( \frac{M^4}{m^4} \right) \right) = \int d^n x \sqrt{|g|} \left\{ \frac{1}{2} g^{\mu \nu} \ln \left( \frac{M^4}{m^4} \right) + 2 \chi (\nabla^\mu \nabla^\nu - g^{\mu \nu} \Box - R^{\mu \nu}) \left( \frac{M^2}{M^4_\epsilon} \right) \right\} \delta g_{\mu \nu}, \tag{A1}
\]

\[
\delta \left( \int d^n x \sqrt{|g|} R \ln \left( \frac{M^4}{m^4} \right) \right) = \int d^n x \sqrt{|g|} \left\{ \frac{R}{2} g^{\mu \nu} \ln \left( \frac{M^4}{m^4} \right) + (\nabla^\mu \nabla^\nu - g^{\mu \nu} \Box - R^{\mu \nu}) \left[ \ln \left( \frac{M^4}{m^4} \right) + \frac{2 \chi RM^2}{M^4_\epsilon} \right] \right\} \delta g_{\mu \nu}, \tag{A2}
\]

\[
\delta \left( \int d^n x \sqrt{|g|} R^2 \ln \left( \frac{M^4}{m^4} \right) \right) = \int d^n x \sqrt{|g|} \left\{ \frac{R^2}{2} g^{\mu \nu} \ln \left( \frac{M^4}{m^4} \right) + 2 (\nabla^\mu \nabla^\nu - g^{\mu \nu} \Box - R^{\mu \nu}) \left[ R \ln \left( \frac{M^4}{m^4} \right) + \frac{\chi R^2 M^2}{M^4_\epsilon} \right] \right\} \delta g_{\mu \nu}, \tag{A3}
\]

\[
\delta \left( \int d^n x \sqrt{|g|} \Box R \ln \left( \frac{M^4}{m^4} \right) \right) = \int d^n x \sqrt{|g|} \left\{ \frac{\chi M^2}{M^4_\epsilon} g^{\mu \nu} \nabla^\alpha R \nabla_\alpha R + (\nabla^\mu \nabla^\nu - g^{\mu \nu} \Box - R^{\mu \nu}) \left[ \frac{2 \chi M^2}{M^4_\epsilon} \Box R + \Box \ln \left( \frac{M^4}{m^4} \right) \right] \right\} \delta g_{\mu \nu}, \tag{A4}
\]

\[
\delta \left( \int d^n x \sqrt{|g|} R^{\mu \nu \sigma \rho} R_{\mu \nu \rho \sigma} \ln \left( \frac{M^4}{m^4} \right) \right) = \int d^n x \sqrt{|g|} \left\{ \frac{1}{2} g^{\mu \nu} R^\alpha_\beta R^\gamma_\delta \ln \left( \frac{M^4}{m^4} \right) + 2 \chi (\nabla^\mu \nabla^\nu - g^{\mu \nu} \Box - R^{\mu \nu}) \left( \frac{M^2}{M^4_\epsilon} R^\alpha_\beta R^\gamma_\delta \right) - 2 R^\mu_\nu R^\rho_\sigma \ln \left( \frac{M^4}{m^4} \right) \right\} \delta g_{\mu \nu}, \tag{A5}
\]

\[
\delta \left( \int d^n x \sqrt{|g|} R^{\mu \nu} R_{\mu \nu} \ln \left( \frac{M^4}{m^4} \right) \right) = \int d^n x \sqrt{|g|} \left\{ \frac{1}{2} g^{\mu \nu} R^{\alpha_\beta} \ln \left( \frac{M^4}{m^4} \right) + 2 \chi (\nabla^\mu \nabla^\nu - g^{\mu \nu} \Box - R^{\mu \nu}) \left( \frac{M^2}{M^4_\epsilon} R^{\alpha_\beta} \ln \left( \frac{M^4}{m^4} \right) \right) - 2 R^\alpha_\mu R^\nu_\sigma \ln \left( \frac{M^4}{m^4} \right) \right\} \delta g_{\mu \nu}, \tag{A6}
\]

\[
+2 \nabla_\alpha \nabla^{(\mu} \ln \left( \frac{M^4}{m^4} \right) \nabla^{\nu)} \ln \left( \frac{M^4}{m^4} \right) - \Box \left[ R^{\mu \nu} \ln \left( \frac{M^4}{m^4} \right) - g^{\mu \nu} \nabla_\alpha \nabla_\beta \ln \left( \frac{M^4}{m^4} \right) \right] \delta g_{\mu \nu}.
\]
Notice that one can obtain the variation of the factors multiplying the logarithmic term in the l.h.s. of the Eqs. (A1)-(A6) as the particular case where $\chi = 0$.

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