OVERGROUPS OF LEVI SUBGROUPS I. THE CASE OF ABELIAN UNIPOTENT RADICAL

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ABSTRACT. In the present paper we prove sandwich classification for the overgroups of the subsystem subgroup $E(\Delta, R)$ of the Chevalley group $G(\Phi, R)$ for the three types of pairs $(\Phi, \Delta)$ (the root system and its subsystem) such that the group $G(\Delta, R)$ is (up to torus) a Levi subgroup of the parabolic subgroup with abelian unipotent radical. Namely, we show that for any such an overgroup $H$ there exists a unique pair of ideals $\sigma$ of the ring $R$ such that $E(\Phi, \Delta, R, \sigma) \leq H \leq N_{G(\Phi, R)}(E(\Phi, \Delta, R, \sigma))$.

1. INTRODUCTION

In the paper [20], dedicated to the maximal subgroups project, Michael Aschbacher introduced eight classes $C_1$-$C_8$ of subgroups of finite simple classical groups. The groups from these classes are "obvious" maximal subgroups of a finite classical groups. To be precise, each subgroup from an Aschbacher class either is maximal itself or is contained in a maximal subgroup that in its turn either also belongs to an Aschbacher class or can be constructed by a certain explicit procedure.

Nikolai Vavilov defined five classes of "large" subgroups of the Chevalley groups (including exceptional ones) over arbitrary rings (see [17] for details). Although these subgroups are not maximal, he conjectured that they are sufficiently large for the corresponding overgroup lattice to admit a description. One of these classes is the class of subsystem subgroups (the definition will be given in Subsection 2.1).

Overgroups of (elementary) subsystem subgroups of the general linear group were studied in the papers [4], [1], [2], [3], [5]. In that case, subsystem subgroups are precisely block-diagonal subgroups. This results were generalized to orthogonal and symplectic groups under the assumption $2 \in R^*$ in the thesis of Nikolai Vavilov (see also [7], [8] and [10]). After that, this assumption was lifted in the thesis of Alexander Shchegolev [28]. In that thesis, Shchegolev also solved the problem for unitary groups (see also [18] and [19]).

The problem we discuss in the present paper is to describe the overgroups of subsystem subgroups of exceptional groups (over a commutative ring). This problem was posed in [13] (Problem 7). The first step of solving this problem was done in [14]. The table from that paper contains the list of pairs $(\Phi, \Delta)$ for which such a description may be possible in

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principle, along with the number of ideals determining the level and along with certain links between these ideals.

By a standard description we mean "sandwich classification". Let $G$ be an abstract group, and let $\mathcal{L}$ be a certain lattice of its subgroups. The lattice $\mathcal{L}$ admits sandwich classification if it is a disjoint union of "sandwiches":

$$\mathcal{L} = \bigsqcup_i L(F_i, N_i),$$

where $i$ runs through some index set, and $F_i$ is a normal subgroup of $N_i$ for all $i$. To study such a lattice, it suffices to study the quotients $N_i/F_i$. In [14] it was conjectured that the lattice of subgroups of a Chevalley group that contain a sufficiently large subsystem subgroup admits sandwich classification.

In the present paper, we prove a sandwich classification theorem for the embeddings $A_{l-1} \leq D_l$ (where $l \geq 5$), $D_5 \leq E_6$, and $E_6 \leq E_7$. Our approach allows us to consider all three cases simultaneously. In all these cases, the corresponding subsystem subgroup is (up to a torus) a Levi subgroup $L_\alpha$, where $\alpha$ is a fundamental root such that its coefficient in the decomposition of a maximal root is equal to one. In that case, the level is determined by two ideals. One can add to this list the embedding $A_{l-1} \leq A_l$, for which the main result follows, of course, from the paper [3]. The fact that in all these cases the corresponding unipotent radical is Abelian simplifies our problem from a technical point of view.

Overgroup description for the embedding $A_{l-1} \leq D_l$ is a special case of the results obtained in [6] and [28]. Another two cases considered in the present paper are new.

We mention two previous results that are closely related to the present paper.

- Over a field (distinct from $F_3$ and of characteristic not equal to 2), overgroup description in the cases considered in the present paper was obtained by Wang Dengyin in [35]. His proof involves Bruhat decomposition.
- Description of subgroups of a maximal parabolic subgroup that contain the elementary Levi subgroup was obtained in the paper of Anastasia Stavrova [29]. Note that over a field such a description had been obtained before in the papers of Wang Dengyin and Li ShangZhi [36] and Victoria Kozakevich and Anastasia Stavrova [15].

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2. Basic notation

2.1. Root systems and Chevalley groups. Let $\Phi$ be an irreducible root system, $\mathcal{P}$ a lattice that is intermediate between the root lattice $\mathcal{Q}(\Phi)$ and the weight lattice $\mathcal{P}(\Phi)$, $R$ a commutative associative ring with unity, $G(\Phi, R) = G_\mathcal{P}(\Phi, R)$ a Chevalley group of type $\Phi$ over $R$, $T(\Phi, R) = T_\mathcal{P}(\Phi, R)$ a split maximal torus of $G_\mathcal{P}(\Phi, R)$. For every root $\alpha \in \Phi$ we
denote by \( X_\alpha = \{ x_\alpha(\xi) : \xi \in R \} \) the corresponding unipotent root subgroup with respect to \( T \). We denote by \( E(\Phi, R) = E_P(\Phi, R) \) the elementary subgroup generated by all \( X_\alpha \).

Let \( \Delta \) be a subsystem of \( \Phi \). We denote by \( E(\Delta, R) \) the subgroup of \( G(\Phi, R) \) generated by all \( X_\alpha \), where \( \alpha \in \Delta \). It is called an (elementary) subsystem subgroup. It can be shown that it is an elementary subgroup of a Chevalley group \( G(\Delta, R) \) embedded into the group \( G(\Phi, R) \). Here the lattice between \( \mathcal{Q}(\Delta) \) and \( \mathcal{P}(\Delta) \) is an orthogonal projection of \( \mathcal{P} \) to the corresponding subspace.

We are going to describe intermediate subgroups between \( E(\Delta, R) \) and \( G(\Phi, R) \). For each case considered in the present paper, we fix a basic representation \( V \) of the group \( G(\Phi, R) \) (i.e., a representation that has a basis consisting of weight vectors such that the Weyl group acts transitively on the set of nonzero weights, see [27]) and we assume that the lattice \( \mathcal{P} \) is such that this representation is faithful (therefore, \( \mathcal{P} \) is uniquely determined by the representation).

Here is the list of the cases we consider:

(a) \( \Phi = D_l, \Delta = A_{l-1} \) (\( l \geq 5 \)), \( V = V_{\omega_l} \) (half-spin representation).
(b) \( \Phi = E_6, \Delta = D_5 \), \( V = V_{\omega_l} \).
(c) \( \Phi = E_7, \Delta = E_6 \), \( V = V_{\omega_l} \).

The embedding of \( \Delta \) into \( \Phi \) is obtained by crossing out a vertex of the Dynkin diagram as shown in Figure 1.

Let us fix an order on the system \( \Phi \) and denote by \( \Pi = \{ \alpha_1, \ldots, \alpha_l \} \) the corresponding set of fundamental roots; its enumeration is also shown in Figure 1. Let \( \alpha^{(1)} \) be the root that corresponds to the vertex crossed out, and \( \alpha^{(2)} \) the root that corresponds to the vertex that is adjacent to the crossed out one.

Let \( \Delta' \) be the subsystem of \( \Delta \) obtained by crossing out both \( \alpha^{(1)} \) and \( \alpha^{(2)} \). Furthermore, let \( \Delta'' \) be equal to \( \Delta' \) in cases (b) and (c) and to its irreducible component distinct from \( A_1 \) (it is important that \( l \geq 5 \)) in case (a).

The symbols \( \Omega^\pm, \Sigma^\pm, \lambda^1, \Delta^1, \) and \( (\Delta \cap \Delta^1)' \) will be introduced in Lemmas 6 and 11.

2.2. Affine schemes. The functor \( G(\Phi, -) \) from the category of rings to the category of groups is an affine group scheme (a Chevalley–Demazure scheme). This means that its composition with the forgetful functor to the category of sets is representable, i.e.,

\[ G(\Phi, R) = \text{Hom}(\mathbb{Z}[G], R). \]

The ring \( \mathbb{Z}[G] \) here is called the ring of regular functions on the scheme \( G(\Phi, -) \).

An element \( g_{gen} \in G(\Phi, \mathbb{Z}[G]) \), that corresponds to the identity ring homomorphism is called the generic element of the scheme \( G(\Phi, -) \). This element has a universal property: for any ring \( R \) and for any \( g \in G(\Phi, R) \), there exists a unique ring homomorphism

\[ f : \mathbb{Z}[G] \to R \]

such that \( f(g_{gen}) = g \). For details about application of the method of generic element to the problems similar to our problem see the paper of Alexei Stepanov [32]. We will use this method in the second part of Lemma 32 to avoid problems with zero divisors of the ring \( R \).
Later, we will define the scheme $\mathfrak{R}(-)$ of the root type elements and introduce a similar notation for it.

The identity element of the group $G(\Phi, \mathbb{Z})$ corresponds to the augmentation homomorphism

$$\varepsilon: \mathbb{Z}[G] \to \mathbb{Z}. $$

Its kernel is called the augmentation ideal. We denote it by $I_{\text{aug}}$.

2.3. **Representation $V$.** Let $\Lambda$ be the set of weights of the representation $V$. A basis of $V$ is indexed by the weights from $\Lambda$. We denote this basis by $\{v^\lambda: \lambda \in \Lambda\}$.

The order we fixed on the system $\Phi$ induces a partial order on the set $\Lambda$: a weight $\lambda$ is bigger than a weight $\mu$ if $\lambda - \mu$ is the sum of positive roots.

Root elements of the group $G(\Phi, R)$ act on the elements of this basis as follows:

$$x_\alpha(\xi)v^\lambda = \begin{cases} v^\lambda + v^{\lambda+\alpha}c_{\lambda\alpha}\xi & \text{if } \lambda + \alpha \in \Lambda, \\ v^\lambda & \text{if } \lambda + \alpha \notin \Lambda, \end{cases}$$

where the structure constants $c_{\lambda\alpha}$ of the action are equal to $\pm 1$ (Matsumoto’s lemma [26], [30]). Moreover, we choose $\{v^\lambda\}$ to be a crystal basis, i.e., $c_{\lambda\alpha} = 1$ if $\alpha \in \pm \Pi$. The existence of such a basis was proved in [34], [22], [9].
Therefore, the weight diagram describes the action of $E(\Phi, R)$ on the module $V$ completely. This action has a unique extension that is an algebraic action of the group $G(\Phi, R)$.

Weights from $\Lambda$ correspond to the vertices of the weight diagram (see [27]). To describe the action of the group $G(\Delta, R)$ on the module $V$, one should remove the edges of the weight diagram labeled by $\alpha^{(1)}$. The new diagram will have a number of connected components, which can be enumerated $\Lambda_0, \Lambda_1, \ldots$ in such a way that whenever $i > j$ any weight from $\Lambda_i$ is less that any weight from $\Lambda_j$ with respect to the order we fix. The set $\Lambda_0$ contains only one weight — the highest one, we denote it by $\lambda_0$.

For any pair of weights $\lambda, \mu \in \Lambda$ we denote by $d(\lambda, \mu)$ the distance between them (i.e., the length of the shortest path) in the weight graph (two weights are adjacent in the weight graph if their difference belongs to $\Phi$).

2.4. Group theoretic notation.

- Recall that for abstract groups $A, B \leq G$, the transporter from $A$ to $B$ is the set $\text{Tran}_G(A, B) = \{g \in G: gAg^{-1} \subseteq B\}$.
- We denote by $N_G(\Gamma)$ the normaliser of the group $\Gamma$ in the group $G$.
- If the group $G$ acts on the set $X$ and $x \in X$, we denote by $\text{Stab}_G(x)$ the stabiliser of the element $x$.
- Commutators are left normalised:
  $$[x, y] = xyx^{-1}y^{-1}.$$  
- If $X$ is a subset of the group $G$, we denote by $\langle X \rangle$ the subgroup generated by $X$.
- We denote by $D_i^G$ the $i$th member of the derived series, i.e., $D^0G = G$ and $D^{i+1}G = [D^iG, D^iG]$.

2.5. Matrices. We identify any element $g$ of the group $G(\Phi, R)$ with the matrix of its action on the module $V$ in the crystal basis. Rows and columns of such a matrix are indexed by weights from $\Lambda$. We denote by $g_{\lambda, \mu}$ the entry of this matrix in the $\lambda$th row and the $\mu$th column. We denote by $g_{*, \mu}$ the $\mu$th column of the matrix $g$, which we identify with a vector from $V$. Similarly, we denote by $g_{\lambda, *}$ the $\lambda$th row of the matrix $g$, which we identify with a covector from $V^*$.

2.6. Relative subgroups. For a root system $\Phi$, a ring $R$ and an ideal $I \subseteq R$ we denote by $\rho_I$ the reduction homomorphism

$$\rho_I: G(\Phi, R) \to G(\Phi, R/I),$$

induced by the projection of $R$ to the quotient ring $R/I$. We denote by $G(\Phi, R, I)$ the principal congruence subgroup of the group $G(\Phi, R)$

$$G(\Phi, R, I) = \text{Ker } \rho_I.$$  

We denote by $CG(\Phi, R, I)$ the full congruence subgroup, i.e., the inverse image of the center of the group $G(\Phi, R/I)$ under the reduction homomorphism $\rho_I$.

We denote by $E(\Phi, I)$ the subgroup generated by the root elements of the level $I$:

$$E(\Phi, I) = \langle \{x_\alpha(\xi): \alpha \in \Phi, \xi \in I\} \rangle.$$
We denote by $E(\Phi, R, I)$ the relative elementary subgroup of the level $I$, i.e., the normal closure of the group $E(\Phi, I)$ in the group $E(\Phi, R)$. As a subgroup, $E(\Phi, R, I)$ is generated by the elements $z_\alpha(\xi, \zeta) = x_\alpha(\zeta)x_{-\alpha}(\xi)x_\alpha(-\zeta)$, where $\alpha \in \Phi$, $\xi \in I$ and $\zeta \in R$ (see [33]).

The following facts about this group are well known.

**Lemma 1.** (Standard commutator formula) If $\text{rk} \Phi \geq 2$, then $E(\Phi, R, I)$ is a normal subgroup of $G(\Phi, R)$ and

$$[E(\Phi, R), CG(\Phi, R, I)] \subseteq E(\Phi, R, I).$$

**Lemma 2.** (see [33]) Let $I \leq R$. Then

$$E(\Phi, R, I^2) \subseteq E(\Phi, I).$$

2.7. Nilpotent structure of $K_1$. We will use the following fact proved in [21].

**Lemma 3.** For an arbitrary commutative ring $R$ of finite Bass–Serre dimension (in particular, for any finitely generated ring) and an arbitrary idea $I \leq R$, the quotient group $G(\Phi, R, I)/E(\Phi, R, I)$ has a nilpotent normal subgroup with Abelian quotient group (in particular, it is solvable).

2.8. Parabolic subgroups. Assume that we consider one of cases (a)–(c).

We denote by $P = P_{\alpha(1)}$ the parabolic subgroup of the group $G(\Phi, R)$, that corresponds to the root $\alpha^{(1)}$.

This subgroup $P$ coincides with the stabiliser of the line generated by the highest weight vector. The stabiliser of the line generated by the vector of the weight $\lambda \in \Lambda$ is also a parabolic subgroup (it is conjugate to $P$ by an element of the Weyl group $W$ that maps the highest weight to the weight $\lambda$), we will denote it by $P_\lambda$.

We denote the corresponding opposite parabolic subgroups by $P^{-}$ and $P^-_{\lambda}$, respectively. The subgroup $P^-_{\lambda}$ can be described as the stabiliser of the line generated by the covector that corresponds to the weight $\lambda$. In other words, the subgroup $P_\lambda$ consist of the matrices such that their $\lambda$th column is a multiple of the corresponding column of the identity matrix. Similarly, the subgroup $P^-_{\lambda}$ consists of the matrices such that their $\lambda$th row is a multiple of the corresponding row of the identity matrix.

The unipotent radicals of these parabolic subgroups will be denoted by $U_\lambda$, $U^{-}_\lambda$, and $U^-_{\lambda}$, respectively. The correspondent Levi subgroups will be denoted by $L$ and $L_{\lambda}$ (opposite parabolic subgroups have the same Levi subgroups).

The following statements are well known.

**Lemma 4.** In each of cases (a)–(c)

(1)

$L_{\lambda} = P_\lambda \cap P^-_{\lambda}$.

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1We are considering basic representation without zero weight. This means exactly that the Weyl group acts transitively on the set of weights.
overgroups of Levi subgroups I.

\[ P_\lambda = U_\lambda \ltimes L_\lambda \quad \text{and} \quad P^-_\lambda = U^-_\lambda \ltimes L_\lambda. \]

\[ U_\lambda = \langle \{ x_\alpha(\xi) : \xi \in R, \ \lambda - \alpha \in \Lambda \} \rangle, \]
\[ U^-_\lambda = \langle \{ x_\alpha(\xi) : \xi \in R, \ \lambda + \alpha \in \Lambda \} \rangle. \]

(4) The groups \( U_\lambda \) and \( U^-_\lambda \) are Abelian.

3. Combinatorial lemmas

In this section, we assume that we are considering one of cases (a)–(c), which were listed at the beginning of the paper.

Let \( W(\Phi) \) and \( W(\Delta) \) be the Weyl groups of the corresponding root systems.

Lemma 5. The coefficient of \( \alpha^{(1)} \) in the decomposition of the maximal root is equal to one. The coefficient of \( \alpha^{(2)} \) is equal to two.

Proof. Direct calculation in each case.

Lemma 6. The set \( \Phi \setminus \Delta \) has exactly two \( W(\Delta) \)-orbits: the roots such that in their expansion in fundamental roots the coefficient of \( \alpha^{(1)} \) is equal to 1, and the roots such that it is equal to \(-1\). We will denote these orbits by \( \Omega^+ \) and \( \Omega^- \) respectively.

Proof. We prove that if a root \( \alpha \in \Phi \), has coefficient of \( \alpha^{(1)} \) equal to 1, then it lies in one \( W(\Delta) \)-orbit with the maximal root (the proof for the opposite roots is the same).

One can obtain the maximal root from the root \( \alpha \) by adding fundamental roots step by step. All these fundamental roots are distinct from \( \alpha^{(1)} \), (hence they belong to \( \Delta \)) because the coefficient of \( \alpha^{(1)} \) in the expansion of \( \alpha \) is already maximal possible. If one root is obtained from another by adding a root from \( \Delta \), then it is also obtained by reflection with respect to this root (because all roots have the same length). Therefore, all roots in the chain from \( \alpha \) to the maximal root lie in the same \( W(\Delta) \)-orbit.

Lemma 7. The set \( \Delta \setminus \Delta' \) has exactly two \( W(\Delta') \)-orbits: the roots such that in their expansion in fundamental roots the coefficient of \( \alpha^{(2)} \) is equal to 1, and the roots such that it is equal to \(-1\).

Proof. The arguments are similar to the proof of the previous lemma.

Lemma 8. Suppose \( \beta \in \Phi \setminus \Delta \); then there exists a root \( \alpha \in \Delta \) such that \( \alpha + \beta \in \Phi \).

Proof. Using the action of the Weyl group \( W(\Delta) \) we may assume that \( \beta = \pm \alpha^{(1)} \). In this case, we can take \( \alpha = \pm \alpha^{(2)} \).

Lemma 9. Suppose \( \alpha \in \Omega^+ \); then \( \lambda_0 - \alpha \in \Lambda \).

Proof. Note that the weight \( \lambda_0 \) is fixed by the group \( W(\Delta) \). Hence using the action of the Weyl group \( W(\Delta) \) we may assume that \( \alpha = \alpha^{(1)} \). In each of cases (a)–(c), one can see on the weight diagram that the highest weight is, indeed, adjacent to another one by the edge labeled with \( \alpha^{(1)} \).
\textbf{Lemma 10.} Suppose \( \alpha \in \Phi \) and \( \lambda, \rho, \lambda - \alpha, \rho + \alpha \in \Lambda \) are such that \( \lambda - \alpha \neq \rho \); then \( d(\lambda, \rho) \geq 2 \).

\textit{Proof.} First, using the action of the group \( W(\Phi) \), we may assume that \( \lambda = \lambda_0 \) is the highest weight. Second, using the action of the group \( W(\Delta) \), we may also assume that \( \alpha = \alpha^{(1)} \).

Consider the difference \( \lambda - (\rho + \alpha) \). To calculate this difference, one should consider the path on the weight diagram between the vertices \( \lambda \) and \( \rho + \alpha \) (since \( \lambda \) is the highest weight, this path can always be chosen going from the left to the right), and take the sum of the fundamental roots corresponding to the edges of this path. Note that \( \alpha^{(1)} \) appears in this sum at least once because this is the only edge incident to the vertex \( \lambda_0 \), and \( \lambda \neq \rho + \alpha \) by assumption. Therefore, in the expansion of \( \lambda - \rho \) in fundamental roots, the coefficient of \( \alpha^{(1)} \) is at least 2; hence by Lemma 5, \( \lambda - \rho \) is not a root. \( \Box \)

\textbf{Lemma 11.} Suppose \( \lambda_1 \in \Lambda_1 \). We introduce the following notation: \( \Sigma_{\lambda_1} = \{ \alpha \in \Phi: \lambda_1 - \alpha \in \Lambda \} \). Next, we split this set in three disjoint subsets:

\[ \Sigma_{\lambda_1} = \Sigma_{\lambda_1}^- \cup \Sigma_{\lambda_1}^0 \cup \Sigma_{\lambda_1}^+, \]

where

\[ \Sigma_{\lambda_1}^- = \Sigma_{\lambda_1} \cap \Omega^-, \quad \Sigma_{\lambda_1}^0 = \Sigma_{\lambda_1} \cap \Delta, \quad \Sigma_{\lambda_1}^+ = \Sigma_{\lambda_1} \cap \Omega^+ . \]

Let \( \Delta_{\lambda_1} \) be the image of \( \Delta \) under reflection with respect to \( \lambda_0 - \lambda_1 \). Then the following holds.

1. \( \Sigma_{\lambda_1}^- = \{ \lambda_1 - \lambda_0 \} \).
2. \( \Sigma_{\lambda_1}^0 \neq \emptyset \) and for any \( \alpha \in \Sigma_{\lambda_1}^0 \) there exists a root \( \beta \in \Sigma_{\lambda_1}^0 \) such that \( \alpha - \beta \in \Phi \).
3. For any \( \beta \in \Sigma_{\lambda_1}^+ \) there exists a root \( \gamma \in \Delta \cap \Delta_{\lambda_1} \) such that \( \beta + \gamma \in \Phi \), but \( \lambda_1 - \lambda_0 + \gamma \notin \Phi \).
4. The subsystem \( \Delta \cap \Delta_{\lambda_1} \) has an irreducible component \( (\Delta \cap \Delta_{\lambda_1})' \) distinct from \( \Lambda_1 \). Next, for any weight \( \mu \in \Lambda_1 \setminus \{ \lambda_1 \} \) there exists a weight \( \nu \in \Lambda_1 \setminus \{ \lambda_1 \} \) such that \( \mu - \nu \in (\Delta \cap \Delta_{\lambda_1})' \).
5. For any \( \alpha \in \Omega^+ \) such that \( \langle \alpha, \lambda_0 - \lambda_1 \rangle = 1 \), there exists a root \( \gamma \in \Sigma_{\lambda_1}^0 \) such that \( \alpha + \gamma \in \Phi \).

\textit{Proof.} Using the action of the group \( W(\Delta) \), we may assume that \( \lambda_0 - \lambda_1 = \alpha^{(1)} \). Therefore, we have \( \Delta \cap \Delta_{\lambda_1} = \Delta' \), and in Item (4) we have \( (\Delta \cap \Delta_{\lambda_1})' = \Delta'' \). Let \( w_{\alpha^{(1)}} \in W(\Phi) \) be reflection with respect to the root \( \alpha^{(1)} \). This reflection maps \( \lambda_0 \) to \( \lambda_1 \); hence it maps \( \Omega^+ \) to \( \Sigma_{\lambda_1} \).

We prove items (1) and (2). Since \( w_{\alpha^{(1)}} \alpha^{(1)} = -\alpha^{(1)} \), we have \( -\alpha^{(1)} \in \Sigma_{\lambda_1}^- \). Suppose that \( \alpha \in \Omega^+ \) and \( \alpha \neq \alpha^{(1)} \), then \( \langle \alpha, \alpha^{(1)} \rangle \) is equal either to 0 or to 1 (because \( \alpha + \alpha^{(1)} \notin \Phi \)). In the first case, we have

\[ w_{\alpha^{(1)}} \alpha = \alpha \in \Sigma_{\lambda_1}^+ . \]

In the second case, we have

\[ w_{\alpha^{(1)}} \alpha = \alpha - \alpha^{(1)} \in \Sigma_{\lambda_1}^0 . \]

Item (1) is, therefore, proved.
Lemma 13. In the paper \( [14] \) (Lemmas 1 and 3), the following was proved.

\[
\mathcal{E} \quad \text{i.e.} \quad \mathcal{F} \quad \text{correspondence is bijective. Therefore, this lemma is merely a restatement of Item 2 of Lemma 12.}
\]

Proof. To each weight \( \alpha \in \Lambda \), let \( \nu = \gamma \), equal to 1. Also the root \( \gamma \) is adjacent to the additional vertex in the affine diagram.

Looking at the weight diagram, one can easily check the second part.

We prove item (3). From the above, it follows that \( \langle \beta, \alpha \rangle = 0 \). Hence the coefficient of \( \alpha \) in the expansion of \( \beta \) is equal to 2. Therefore, the maximal root can be obtained from \( \beta \) by adding (= reflecting with respect to) simple roots from \( \Delta \) step by step. Using the action of the group \( W(\Delta') \) we may assume that \( \beta \) is a maximal root. Now we can take \( \gamma = -\alpha_k \), where \( \alpha_k \) is a simple root that corresponds to the vertex of the Dynkin diagram that is adjacent to the additional vertex in the affine diagram.\(^2\)

The first part of Item (4) has already been proved above. Moreover, \( (\Delta \cap \Delta_{\lambda_i})' = \Delta'' \). Looking at the weight diagram, one can easily check the second part.

Let us prove Item (5). By hypothesis, \( \alpha - \alpha^{(1)} \in \Delta \), and coefficient of \( \alpha \) in its decomposition is equal to one. Using the action of the group \( W(\Delta') \), we can assume that the sum of coefficients in the decomposition of \( \alpha \) in simple roots is maximal among all such sums for the roots from the \( W(\Delta') \)-orbit of \( \alpha \). There is a simple root \( \alpha_k \), such that \( \alpha + \alpha_k \in \Phi \). The root \( \alpha_k \) cannot be equal to \( \alpha^{(1)} \) because the coefficient of \( \alpha^{(1)} \) is already equal to 1. Also the root \( \alpha_k \) cannot be in \( \Delta' \); otherwise, \( \alpha + \alpha_k \) would be a root from the \( W(\Delta') \)-orbit of \( \alpha \) with bigger sum of coefficients. Hence \( \alpha + \alpha^{(2)} \in \Phi \) and we can take \( \gamma = \alpha^{(2)} \).

\[ \square \]

Lemma 12. Let \( \lambda_1 \in \Lambda_1 \); then:

\begin{enumerate}
  \item there exists \( \mu \in \Lambda \) such that \( d(\lambda_1, \mu) = 1 \);
  \item let \( \nu \in \Lambda_1 \) be a weight such that \( d(\lambda_1, \nu) = 1 \); then there exists \( \mu \in \Lambda_1 \) such that \( d(\mu, \nu) = d(\lambda_1, \mu) = 1 \).
\end{enumerate}

Proof. To each weight \( \mu \in \Lambda_1 \) such that \( d(\lambda_1, \mu) = 1 \), we assign a root \( \lambda_1 - \mu \in \Sigma_{\lambda_1} \). This correspondence is bijective. Therefore, this lemma is merely a restatement of Item 2 of Lemma 11.\[ \square \]

4. Level computation

For now, let \( \Delta, \Phi, \) and \( \mathcal{P} \) be arbitrary. Let \( H \) be an overgroup of the group \( E(\Delta, R) \), i.e. \( E(\Delta, R) \leq H \leq G(\Phi, R) \). For each root \( \alpha \in \Phi \setminus \Delta \) we set \( I_\alpha = \{ \xi \in R : x_\alpha(\xi) \in H \} \).

In the paper \[ \square \] (Lemmas 1 and 3), the following was proved.

Lemma 13. \( \quad \) (1) The set \( I_\alpha \) is an ideal of the ring \( R \).

\[ \square \]

\(^2\)It is important that \( l \geq 5 \), and that we do not consider the case \( D_{l-1} \leq D_l \). Theorem \[ \square \] is not true for these subsystems because we can take \( H = G(B_{l-1}, R) \).
The ideal $I_\alpha$ depends only on the $W(\Delta)$-orbit of the root $\alpha$.

Now we assume that we are considering one of cases (a)–(c) that were listed in the beginning of the paper. Recall that by Lemma 6, the set $\Phi \setminus \Delta$ has exactly two $W(\Delta)$-orbits: $\Omega^+$ and $\Omega^-$. We denote the corresponding ideals $I_\alpha$ by $I_\alpha^+$ and $I_\alpha^-$ respectively.

The pair $\sigma = (I_\alpha^+, I_\alpha^-)$ is called the level of the overgroup $H$. We will write $\sigma = \text{lev}(H)$.

Now let $\sigma = (I_\alpha^+, I_\alpha^-)$ be an arbitrary pair of ideals of the ring $R$, We introduce the following notation:

$$E(\Phi, \Delta, R, \sigma) = \langle x_\alpha(\xi) : \alpha \in \Delta, \xi \in R \text{ or } \alpha \in \Omega^+, \xi \in I^+ \text{ or } \alpha \in \Omega^-, \xi \in I^- \rangle \leq G(\Phi, R),$$
$$G(\Phi, \Delta, R, (R, I^-)) = (\rho_{I^-})^{-1}(P) \leq G(\Phi, R),$$
$$G(\Phi, \Delta, R, (I^+, R)) = (\rho_{I^+})^{-1}(P^-) \leq G(\Phi, R),$$
$$G(\Phi, \Delta, R, \sigma) = G(\Phi, \Delta, R, (R, I^-)) \cap G(\Phi, \Delta, R, (I^+, R)) \leq G(\Phi, R).$$

It is easily seen that

$$E(\Phi, \Delta, R, \sigma) \leq G(\Phi, \Delta, R, \sigma),$$

and also that

$$\sigma \leq \text{lev}(E(\Phi, \Delta, R, \sigma)) \leq \text{lev}(G(\Phi, \Delta, R, \sigma)) \leq \sigma$$

(we write $\sigma_1 \leq \sigma_2$ if $I_1^+ \subseteq I_2^+$ and $I_1^- \subseteq I_2^-$. Hence in fact, we have the following lemma.

**Lemma 14.** We have

$$\text{lev}(E(\Phi, \Delta, R, \sigma)) = \text{lev}(G(\Phi, \Delta, R, \sigma)) = \sigma.$$

Therefore, any pair of ideals is a level of some overgroup of the group $E(\Delta, R)$. Let us also compute the level of the group $E(\Phi, \Delta, R, \sigma)$.

**Lemma 15.** We have

$$\text{lev}(N_{G(\Phi, R)}(E(\Phi, \Delta, R, \sigma))) = \sigma.$$

**Proof.** First,

$$\text{lev}(N_{G(\Phi, R)}(E(\Phi, \Delta, R, \sigma))) \geq \text{lev}(E(\Phi, \Delta, R, \sigma)) = \sigma.$$  

We prove the inverse inclusion. Assume that $x_\beta(\xi) \in N_{G(\Phi, R)}(E(\Phi, \Delta, R, \sigma))$, where $\beta \in \Omega^+$, but $\xi \notin I^+$ (the proof for $\Omega^-$ is similar). Take $\alpha$ from Lemma 8, then $x_\alpha(1) \in E(\Delta, R) \leq E(\Phi, \Delta, R, \sigma)$. Hence we have $[x_\alpha(1), x_\beta(\xi)] \in E(\Phi, \Delta, R, \sigma)$. On the other hand, we have $[x_\alpha(1), x_\beta(\xi)] = x_{\alpha + \beta}(\pm \xi)$, where $\alpha + \beta \in \Omega^+$, which contradicts the fact that $\text{lev}(E(\Phi, \Delta, R, \sigma)) = \sigma.$

$\square$
overgroups of Levi subgroups I.

5. The statement of the main result

In the present paper, we prove the following theorem

**Theorem 1.** Assume that we are considering one of cases (a)–(c) listed at the beginning of the paper, i.e.

(a) \( \Phi = D_l, \Delta = A_{l-1} \) (where \( l \geq 5 \), \( V = V_{\varpi_l} \) (half-spin representation).
(b) \( \Phi = E_6, \Delta = D_5, V = V_{\varpi_6} \).
(c) \( \Phi = E_7, \Delta = E_6, V = V_{\varpi_7} \).

Let \( R \) be a commutative ring (associative with unit). Then for any group \( H \) such that

\[
E(\Delta, R) \leq H \leq G(\Phi, R),
\]

there exists a unique pair of ideals \( \sigma = (I^+, I^-) \) of the ring \( R \), such that

\[
E(\Phi, \Delta, R, \sigma) \leq H \leq N_{G(\Phi, R)}(E(\Phi, \Delta, R, \sigma)).
\]

Note that uniqueness has already been proved: indeed, by Lemmas 14 and 15, the pair \( \sigma \) must be equal to the pair \( \text{lev}(H) \). If we set \( \sigma = \text{lev}(H) \), then the left inclusion in the theorem holds true by the definition of a level. Thus, it remains to prove the right inclusion.

It is natural to call the subgroup \( E(\Phi, \Delta, R, (R, 0)) \) the elementary parabolic subgroup. Note that Theorem 1 gives us an overgroup description for such a subgroup.

**Corollary 1.** Assume that we are considering one of cases (a)–(c) that are listed at the beginning of the paper. Let \( R \) be a commutative ring. Then for any group \( H \) such that

\[
E(\Phi, \Delta, R, (R, 0)) \leq H \leq G(\Phi, R),
\]

there exists a unique ideal \( I^- \) of \( R \) such that

\[
E(\Phi, \Delta, R, (R, I^-)) \leq H \leq N_{G(\Phi, R)}(E(\Phi, \Delta, R, (R, I^-))).
\]

Similarly, we obtain the same result for the subgroup \( E(\Phi, \Delta, R, (0, R)). \)

To formulate the second main result, we divide cases (a)–(c) into two groups. Let \( n+1 \) be the number of connected components \( \Lambda_0, \ldots, \Lambda_n \) of the weight diagram after removing the edges labeled with \( \alpha^{(1)} \). Note that for \( i \neq 0, n \) the component \( \Lambda_i \) has at least two weights. Indeed, given a weight from such a component we can always add a simple root to it, and we can always subtract a simple root from it (so that the result is still in \( \Lambda \)). At least one of these roots is distinct from \( \alpha^{(1)} \) because the representation is minuscule.

Therefore, two cases are possible. We say that we are considering the case of the first type if \( \Lambda_0 \) is the only component containing only one weight (that includes case (a) with \( l \) odd, and also case (b)). We say that we are considering the case of the second type if there are two such components, \( \Lambda_0 \) and \( \Lambda_n \) (that includes case (a) with \( l \) even, and also case (c)). In the second case it is easily seen that \( \Lambda_n = \{-\lambda_0\} \).

**Theorem 2.** Let \( \sigma = (I^+, I^-) \) be a pair of ideals of the ring \( R \); then:

(1) for the cases of the first type, we have
\[ N_{G(\Phi, R)}(E(\Phi, \Delta, R, \sigma)) = G(\Phi, \Delta, R, \sigma). \]

(2) for the cases of the second type, the group \( N_{G(\Phi, R)}(E(\Phi, \Delta, R, \sigma)) \) consists of exactly the elements \( g \) of the group \( G(\Phi, R) \) that satisfy the following conditions:
\[ g_{\lambda_0, \lambda} \in I^+ \quad \forall \lambda \in \Lambda \setminus \{\lambda_0, -\lambda_0\}, \]
\[ g_{\lambda_0, -\lambda_0} I^- \subseteq I^+, \]
\[ (g^{-1})_{\lambda, \lambda_0} \in I^- \quad \forall \lambda \in \Lambda \setminus \{\lambda_0, -\lambda_0\}, \]
\[ (g^{-1})_{-\lambda_0, \lambda_0} I^+ \subseteq I^- \]

**Agreement**

At this point, we begin the proof of Theorem 1, i.e., below we always assume that we are considering one of cases (a)–(c) that were listed at the begining of the paper

6. **Normalizer of the subgroup** \( E(\Phi, \Delta, R, \sigma) \)

**Lemma 16.** Let \( I \trianglelefteq R \); then
\[ U \cap G(\Phi, R, I) = \langle \{x_\alpha(\xi) : \xi \in I, \ \alpha \in \Omega^+\} \rangle, \]
and the same is true for \( U^- \).

**Proof.** Obviously, the right-hand side is contained in the left-hand side, let us prove the inverse inclusion. Let \( g \in U \cap G(\Phi, R, I) \). Since \( g \in U \), by Items 3 and 4 of Lemma 4, the element \( g \) can be written as the product
\[ g = \prod_{\alpha \in \Omega^+} x_\alpha(\xi_\alpha), \quad \text{where} \ \xi_\alpha \in R. \]

Fixing \( \overline{\alpha} \in \Omega^+ \), we prove that \( \xi_{\overline{\alpha}} \in I \). Let \( \rho = \lambda_0 - \overline{\alpha} \); then by Lemma 9 we have \( \rho \in \Lambda \). Note that for an arbitrary matrix \( h \) and for \( \alpha \in \Omega^+ \setminus \{\overline{\alpha}\} \) we have
\[ (x_\alpha(\xi_\alpha)h)_{\lambda_0, \rho} = h_{\lambda_0, \rho}. \]

Indeed, otherwise we have \( \rho + \alpha \in \Lambda \), and by Lemma 9 we also have \( \lambda_0 - \alpha \in \Lambda \), which contradicts Lemma 10.

Therefore,
\[ \xi_{\overline{\alpha}} = \pm(x_{\overline{\alpha}}(\xi_{\overline{\alpha}}))_{\lambda_0, \rho} = \pm g_{\lambda_0, \rho} \in I. \]

In the lemmas below, let \( \sigma = (I^+, I^-) \), \( I^+ \trianglelefteq R \), and let \( J = (I^+ \cap I^-)^2 \).

**Lemma 17.** Assume that \( J = 0 \); then
\begin{enumerate}
  \item \( G(\Phi, \Delta, R, \sigma) = E(\Phi, \Delta, R, \sigma)T(\Phi, R)G(\Delta, R) \).
  \item the subgroup \( E(\Phi, \Delta, R, \sigma) \) is normal in \( G(\Phi, \Delta, R, \sigma) \).
\end{enumerate}
Proof. First, note that the subgroup \( T(\Phi, R)G(\Delta, R) \) normalises the subgroup \( E(\Phi, \Delta, R, \sigma) \); hence Item 2 follows from Item 1. Indeed, \( T(\Phi, R) \) normalises each of the subgroups \( \{ x_\alpha(\xi) : \xi \in I \} \), where \( \alpha \in \Phi \) and \( I \) equals \( I^\pm \) or \( R \). Furthermore, \( G(\Delta, R) \) normalises \( E(\Delta, R) \) (this is a special case of Lemma 1). Finally, if \( \alpha \in \Omega^+ \), \( \xi \in I^+ \) and \( g \in G(\Delta, R) \), then by Lemma 16 we have
\[
[x_\alpha(\xi), g] \in U \cap G(\Phi, R, I^+) = \langle \{ x_\alpha(\xi) : \xi \in I^+, \ \alpha \in \Omega^+ \} \rangle \subseteq E(\Phi, \Delta, R, \sigma).
\]
The case of \( \Omega^- \) and \( I^- \) is similar.

Therefore, it remains to prove Item 1. Obviously, the right-hand side is contained in the left-hand side, let us prove the inverse inclusion.

Let \( g \in G(\Phi, \Delta, R, \sigma) \). Since all the rows and columns of an invertible matrix are unimodular, and all the entries in the first column except \( g \) entry are invertible modulo \( I \), the subgroup \( \langle g \rangle \) is invertible modulo \( I^- \). Similarly, we deduce that it is invertible modulo \( I^+ \). Hence it is invertible modulo \( J \), i.e., it is simply an invertible element of the ring \( R \).

Therefore, we can apply to the matrix \( g \) the special case of the Chevalley–Matsumoto decomposition (see [23, 26, 30]) asserting that if \( g \in G(\Phi, R) \) and \( g_{\lambda_0, \lambda_0} \in R^* \), then
\[
g = v g_1 u,
\]
where \( u \in U \), \( v \in U^- \) and \( g_1 \in T(\Phi, R)G(\Delta, R) \). Arguing as in the proof of Lemma 16 and using the fact that \( g \in G(\Phi, \Delta, R, \sigma) \), we see that actually
\[
u \in \langle \{ x_\alpha(\xi) : \xi \in I^+, \ \alpha \in \Omega^+ \} \rangle \subseteq E(\Phi, \Delta, R, \sigma),
\]
and
\[
u \in \langle \{ x_\alpha(\xi) : \xi \in I^-, \ \alpha \in \Omega^- \} \rangle \subseteq E(\Phi, \Delta, R, \sigma).
\]
To finish the proof, it remains to note that the subgroup \( T(\Phi, R)G(\Delta, R) \) normalises the subgroup \( E(\Phi, \Delta, R, \sigma) \).

Lemma 18. If the ring \( R \) is finitely generated, then there exists a natural number \( N \) such that
\[
D^N G(\Phi, \Delta, R, \sigma) = E(\Phi, \Delta, R, \sigma).
\]

Proof. First, the right-hand side is contained in the left-hand side for every \( N \) because the group \( E(\Phi, \Delta, R, \sigma) \) is obviously perfect. We prove the inverse inclusion.

We start with the case where \( J = 0 \) and prove that
\[
D^i(G(\Phi, \Delta, R, \sigma)) \leq E(\Phi, \Delta, R, \sigma) D^i(T(\Phi, R)G(\Delta, R))
\]
The proof is by induction on \( i \). The base of induction, for \( i = 0 \), follows from Lemma 17.

Now, we pass to the induction step. By the inductive hypothesis, we have
\[
D^{i+1}(G(\Phi, \Delta, R, \sigma)) \leq [E(\Phi, \Delta, R, \sigma) D^i(T(\Phi, R)G(\Delta, R)), E(\Phi, \Delta, R, \sigma) D^i(T(\Phi, R)G(\Delta, R))]
\]
As a normal subgroup of \( G(\Phi, \Delta, R, \sigma) \), the last group is generated by the commutators \([x, y], \) where \( x, y \in E(\Phi, \Delta, R, \sigma) \cup D^i(T(\Phi, R)G(\Delta, R)) \). By item 2 of Lemma 17 all these
commutators belong to $E(\Phi, \Delta, R, \sigma)D^{i+1}(T(\Phi, R)G(\Delta, R))$; hence it remains to prove that this subgroup is normal in $G(\Phi, \Delta, R, \sigma)$, but this follows immediately from the fact that the subgroup $E(\Phi, \Delta, R, \sigma)$ is normal in the group $G(\Phi, \Delta, R, \sigma)$, and $D^{i+1}(T(\Phi, R)G(\Delta, R))$ is normal in $T(\Phi, R)G(\Delta, R)$.

Further, note that
\[ E(\Phi, \Delta, R, \sigma)D^i(T(\Phi, R)G(\Delta, R)) \leq E(\Phi, \Delta, R, \sigma)D^{i-1}(G(\Delta, R)), \]
and by Lemma 3, if $i$ is sufficiently large, then the right-hand side is contained in
\[ E(\Phi, \Delta, R, \sigma)E(\Delta, R) = E(\Phi, \Delta, R, \sigma). \]

Now we consider the general case. From what has been proved previously, it follows that for $N_1$ sufficiently large, we have:
\[ \rho_J(D^{N_1}G(\Phi, \Delta, R, \sigma)) \leq E(\Phi, \Delta, R/J, \sigma/J). \]
Since $E(\Phi, \Delta, R, \sigma)$ maps surjectively onto $E(\Phi, \Delta, R/J, \sigma/J)$, we obtain
\[ D^{N_1}G(\Phi, \Delta, R, \sigma) \leq E(\Phi, \Delta, R, \sigma)G(\Phi, R, J). \]

Now, we prove that
\[ D^i(E(\Phi, \Delta, R, \sigma)G(\Phi, R, J)) \leq E(\Phi, \Delta, R, \sigma)D^i(G(\Phi, R, J)). \]
As before, it suffices to verify the fact that $E(\Phi, \Delta, R, \sigma)D^{i+1}(G(\Phi, R, J))$ is normal in
\[ E(\Phi, \Delta, R, \sigma)G(\Phi, R, J) \]
and contains the commutators of elements from $E(\Phi, \Delta, R, \sigma) \cup D^i(G(\Phi, R, J))$. Both claims follow from Lemma 1 and the fact that by Lemma 2 we have
\[ E(\Phi, R, J) \leq E(\Phi, I^+ \cap I^-) \leq E(\Phi, \Delta, R, \sigma). \]
Using Lemmas 3 and 2 once again, we deduce that for a large $N_2$ we have
\[ D^{N_2}(E(\Phi, \Delta, R, \sigma)G(\Phi, R, J)) \leq E(\Phi, \Delta, R, \sigma)E(\Phi, R, J) \leq E(\Phi, \Delta, R, \sigma)E(\Phi, I^+ \cap I^-) = E(\Phi, \Delta, R, \sigma). \]

It remains to set $N = N_1 + N_2$. □

**Lemma 19.** The subgroup $E(\Phi, \Delta, R, \sigma)$ is normal in $G(\Phi, \Delta, R, \sigma)$.

**Proof.** For a finitely generated ring, the statement follows from Lemma 18, and every ring is an inductive limit of its finitely generated subrings. □

**Proposition 1.** We have the identity
\[ N_{G(\Phi, R)}(E(\Phi, \Delta, R, \sigma)) = \text{Tran}(E(\Phi, \Delta, R, \sigma), G(\Phi, \Delta, R, \sigma)). \]

**Proof.** Obviously, the right-hand side is contained in the left-hand side. We prove the inverse inclusion. As in the previous lemma, without loss of generality we may assume that the ring $R$ is finitely generated.
Then by Lemma 18 we have:
\[ \text{Tran}(E(\Phi, \Delta, R, \sigma), G(\Phi, \Delta, R, \sigma)) \leq \text{Tran}(D^N E(\Phi, \Delta, R, \sigma), D^N G(\Phi, \Delta, R, \sigma)) = \text{Tran}(E(\Phi, \Delta, R, \sigma), E(\Phi, \Delta, R, \sigma)). \]

It remains to prove that if \( g \in \text{Tran}(E(\Phi, \Delta, R, \sigma), E(\Phi, \Delta, R, \sigma)) \), then the same is true for \( g^{-1} \).

First, if \( g \in \text{Tran}(E(\Phi, \Delta, R, \sigma), E(\Phi, \Delta, R, \sigma)) \), then for every \( k \in \mathbb{N} \) we have \( g^k \in \text{Tran}(E(\Phi, \Delta, R, \sigma), E(\Phi, \Delta, R, \sigma)) \).

Let \( \tilde{g} \) be the operator on the space of matrices \( M(\Lambda) \) given by:
\[
\tilde{g}: M(\Lambda) \to M(\Lambda) \quad m \mapsto gm^{-1}.
\]

From the Cayley–Hamilton theorem, it follows that the operator \( \tilde{g}^{-1} \) is a polynomial of \( \tilde{g} \).

Set
\[
L = \{ m \in M(\Lambda): \forall \lambda \neq \lambda_0 \ m_{\lambda_0, \lambda} \in I^+ \& m_{\lambda, \lambda_0} \in I^- \}.
\]

Since all operators \( \tilde{g}^k \) preserve \( E(\Phi, \Delta, R, \sigma) \), all polynomials of \( \tilde{g} \) map \( E(\Phi, \Delta, R, \sigma) \) to \( L \).

Since
\[
L \cap G(\Phi, R) = G(\Phi, \Delta, R, \sigma),
\]
we obtain
\[
g^{-1} \in \text{Tran}(E(\Phi, \Delta, R, \sigma), G(\Phi, \Delta, R, \sigma)) \leq \text{Tran}(E(\Phi, \Delta, R, \sigma), E(\Phi, \Delta, R, \sigma)).
\]

The idea to use the nilpotent structure \( K_1 \) for the computation of normalisers was suggested by A. Stepanov in [31].

7. THE PROOF OF THEOREM 2 AND ITS COROLLARY

Proof. From Proposition 1 it follows that \( N_{G(\Phi, R)}(E(\Phi, \Delta, R, \sigma)) \) consist exactly of elements \( g \in G(\Phi, R) \) such that the group \( E(\Phi, \Delta, R, \sigma) \) preserves the line spanned by the covector \( g_{\lambda_0, \lambda} \), modulo \( I^+ \), and preserves the line spanned by the vector \( (g^{-1})_{* \lambda_0, \lambda} \) modulo \( I^- \). Since the group \( E(\Phi, \Delta, R, \sigma) \) is perfect, the preservation of lines implies the preservation of the vector and covector themselves. By writing out the meaning of this preservation for the generators of the group \( E(\Phi, \Delta, R, \sigma) \) in terms of matrices, we obtain Theorem 2.

To obtain the corollary we are interested in, we need the following two lemmas.

Lemma 20. In the cases of the second type, for any \( \lambda \in \Lambda \) the weight \( -\lambda \) also belongs to \( \Lambda \), and there exists \( G(\Phi, R) \)-invariant bilinear form on the module \( V \) given by
\[
h\left( \sum_{\lambda \in \Lambda} x_\lambda v^\lambda, \sum_{\lambda \in \Lambda} y_\lambda v^\lambda \right) = \sum_{\lambda \in \Lambda} \pm x_\lambda y_{-\lambda}.
\]
Proof. The fact that $-\lambda_0 \in \Lambda$ has already been mentioned. Thus, the first statement follows from the fact that $W(\Phi)$ acts transitively on weights. Next, consider the dual representation $V^*$ and its basis $\{(v^\lambda)^*\}$ that is dual to the basis $\{v^\lambda\}$. Then the vector $(v^\lambda)^*$ is a weight vector with the weight $-\lambda$. Therefore, the dual representation has the same set of weights; hence there is an isomorphism between $V$ and $V^*$ given by
\[ \tilde{h}: V \to V^*, \quad v^\lambda \mapsto \pm(v^{-\lambda})^*, \]

which is the same as an invariant bilinear form. □

Note that for the case of $E_7$ such a form was written out explicitly in \[16\].

Lemma 21. In the cases of the second type, there exists a quadratic form on the module $V$:
\[ q \left( \sum_{\lambda \in \Lambda} x_{\lambda} v^\lambda \right) = \sum q_{\mu,\nu} x_{\mu,\nu}, \]

(\text{where } (\mu, \nu) \text{ runs over the set of nonordered pairs of weights}) that satisfies the following conditions:

(1) $q(v) = 0$ for any vector $v$ from the orbit of the highest weight vector (i.e., for $v = g_\ast, \lambda_0$ for any $g \in G(\Phi, R)$; since the Weyl group acts transitively, the vectors $v = g_\ast, \lambda$ for the other weights $\lambda$ also belong to that orbit);

(2) $q_{\lambda_0, -\lambda_0} = \pm 1$.

In \[25\], such equations on the orbit of the highest weight vector were called $\pi$-equations.

Proof. Let $\mu_1, \mu_2, \ldots, \mu_k$ be the shortest path in the weight graph such that $\mu_1 \in \Lambda_2$ and $\mu_k = -\lambda_0$. Then $\gamma_i = \mu_i - \mu_{i+1} \in \Phi$ and $d(\mu_i, -\lambda_0) = k - i$.

We will prove by induction on $i$ that there exists a quadratic form
\[ q^{(i)} \left( \sum_{\lambda \in \Lambda} x_{\lambda} v^\lambda \right) = \sum q^{(i)}_{\mu,\nu} x_{\mu,\nu}, \]

where $(\mu, \nu)$ runs over the set of nonordered pairs of weights such that $d(\mu, -\lambda_0) \geq k - i$ and $d(\nu, -\lambda_0) \geq k - i$, that satisfies the following conditions:

(1) $q^{(i)}(v) = 0$ for any vector $v$ from the orbit of the highest weight vector;

(2) $q^{(i)}_{\lambda_0, \mu_i} = \pm 1$.

The base of induction, $i = 1$ follows from \[11\]. Theorem 2 of that paper implies that, since $d(\lambda_0, \mu_1) = 2$, the set
\[ \Omega(\lambda_0, \mu_1) = \{ \nu \in \Lambda: d(\nu, \lambda_0) = d(\nu, \mu_1) = 1 \} \cup \{ \lambda_0, \mu_1 \} \subseteq \Lambda_1 \cup \{ \lambda_0, \mu_1 \} \]

is a square\[3\]. Hence the vectors from the orbit of the highest weight vector satisfy the corresponding square equation, which has the required shape. It only remains to observe that the weights from $\Lambda_1$ and the weight $\lambda_0$ are at a distance of at least $k$ from the weight $-\lambda_0$ in the weight graph. Indeed, otherwise the corresponding path goes through $\Lambda_2$ (by

\[3\] subset $\Omega \subseteq \Lambda$ is called a square, if $|\Omega| \geq 4$, and for any $\lambda \in \Omega$ exactly one of the differences $\{\lambda - \mu: \mu \in \Omega \setminus \{\lambda\}\}$ is NOT a root.\]
Lemma 5. Every root has coefficient of \( \alpha^{(1)} \) not greater than one in absolute value; hence every vertex in the path is either in the same component as the previous one, or in the adjacent component; in other words, the path cannot jump through \( \Lambda_2 \), and we get a shorter path to \( \Lambda_2 \).

Now we do the induction step from \( i \) to \( i + 1 \). Set

\[ q^{(i+1)}(v) = q^{(i)}(x_{\gamma_i}(1)v). \]

After this transformation, the minimal distance from a weight participating in the formula to the weight \(-\lambda_0\) decreases at most by 1. Obviously, the new form still annihilates the vectors from the orbit of the highest weight vector. Since \( d(\mu_{i+1}, -\lambda_0) = k - i - 1 \), the weight \( \mu_{i+1} \) does not participate in the formula for \( q^{(i)} \); hence the monomial \( x_{\lambda_0}x_{\mu_{i+1}} \) of the form \( q^{(i)} \)'s the only one that contributes to the coefficient of \( x_{\lambda_0}x_{\mu_{i+1}} \) in the form \( q^{(i+1)} \).

Therefore, this coefficient is equal to \( \pm 1 \). □

Corollary 2. (to Theorem 2) Let \( R \) be a finitely generated ring, and let \( \sigma = (I^+, I^-) \) be a pair of ideals of \( R \). Then there exists a natural number \( N \) such that

\[ D_N N_{G(\Phi, R)} (E(\Phi, \Delta, R, \sigma)) = E(\Phi, \Delta, R, \sigma). \]

Proof. In the cases of the first type, the statement follows from Theorem 2 and Lemma 18. We consider the case of the second type.

By Lemma 18, it suffices to prove the inclusion

\[ [N_{G(\Phi, R)} (E(\Phi, \Delta, R, \sigma)), N_{G(\Phi, R)} (E(\Phi, \Delta, R, \sigma))] \leq G(\Phi, \Delta, R, \sigma). \]

We prove the inclusion

\[ [N_{G(\Phi, R)} (E(\Phi, \Delta, R, \sigma)), N_{G(\Phi, R)} (E(\Phi, \Delta, R, \sigma))] \leq G(\Phi, \Delta, R, (R, I^-)). \]

The proof of the inclusion into \( G(\Phi, \Delta, R, (I^+, R)) \) is similar (the form \( q \) from Lemma 21 should be transferred to the dual representation via the isomorphism described in Lemma 20).

Replacing the ring by its quotient \( R/I^- \), we may assume that \( I^- = (0) \). Therefore, we need to prove the inclusion

\[ [N_{G(\Phi, R)} (E(\Phi, \Delta, R, \sigma)), N_{G(\Phi, R)} (E(\Phi, \Delta, R, \sigma))] \leq P. \]

Since the ring \( R \) is finitely generated, it is Noetherian; hence it is a direct product of rings with connected spectra. Thus, without loss of generality, we may assume that the spectrum of \( R \) is connected.

Case 1. \( I^+ \neq (0) \).

We claim that in this case, even a stronger inclusion holds:

\[ N_{G(\Phi, R)} (E(\Phi, \Delta, R, \sigma)) \leq P. \]

Indeed, let \( g \in N_{G(\Phi, R)} (E(\Phi, \Delta, R, \sigma)) \). Then \( g^{-1} \in N_{G(\Phi, R)} (E(\Phi, \Delta, R, \sigma)) \), and by Theorem 2 we have

\[ g_{\lambda, \lambda_0} \in I^- = (0) \quad \forall \lambda \in \Lambda \setminus \{\lambda_0, -\lambda_0\}. \]
Next, consider the form \( q \) from Lemma 21. Since it annihilates the vectors from the orbit of the highest weight vector, the coefficients \( g_{\lambda,\lambda}(= q(v^\lambda)) \) vanish; hence, using that \( g_{e,-\lambda_0} = g_{\lambda,\lambda_0} v^{\lambda_0} + g_{-\lambda,-\lambda_0} v^{-\lambda_0} \), we obtain
\[
g_{\lambda,\lambda_0} g_{\lambda_0,\lambda} = \pm q(g_{e,\lambda_0}) = 0.
\]
The pair \( g_{\lambda_0,\lambda_0}, g_{-\lambda_0,\lambda_0} \) is unimodular. Hence every prime ideal of the ring \( R \) contains exactly one of the elements \( g_{\lambda_0,\lambda_0} \) and \( g_{-\lambda_0,\lambda_0} \). Since the spectrum of the ring \( R \) is connected, one of those elements is invertible, and the other (since \( g_{\lambda_0,\lambda_0} g_{-\lambda_0,\lambda_0} = 0 \)) is equal to zero. By Theorem 2 we have the inclusion
\[
g_{-\lambda_0,\lambda_0} I^+ \subseteq I^- = 0.
\]
Hence the element \( g_{-\lambda_0,\lambda_0} \) is not invertible, i.e., it is equal to zero. Therefore, \( g \in P \).

**Case 2.** \( I^+ = (0) \).

First we assume that \( g \in N_G(\Phi, R) (E(\Phi, \Delta, R, \sigma)) \). Then \( g^{-1} \in N_G(\Phi, R) (E(\Phi, \Delta, R, \sigma)) \), and by Theorem 2 we have:
\[
g_{\lambda_0,\lambda} \in I^+ = (0) \quad \forall \lambda \in \Lambda \setminus \{\lambda_0, -\lambda_0\},
\]
\[
g_{\lambda_0,\lambda} \in I^+ = (0) \quad \forall \lambda \in \Lambda \setminus \{\lambda_0, -\lambda_0\}.
\]
The same is true for \( g^{-1} \); hence for \( \lambda \in \Lambda \setminus \{\lambda_0, -\lambda_0\} \), we have
\[
g_{-\lambda_0,\lambda} = h(g_{e,\lambda}, v^{\lambda_0}) = h(g_{e,\lambda}, v^{\lambda_0}) = h(v^\lambda, v^{-1} v^{\lambda_0}) = (g^{-1})_{-\lambda,\lambda_0} = 0,
\]
where \( h \) is a form occurring in Lemma 20. Similarly, we deduce that \( g_{\lambda,-\lambda_0} = 0 \). Therefore, the submodule generated by the vectors \( v^{\lambda_0} \) and \( v^{-\lambda_0} \), is invariant with respect to the action of the group \( N_G(\Phi, R) (E(\Phi, \Delta, R, \sigma)) \), and the element \( g \) acts on it by the following matrix:
\[
\begin{pmatrix}
g_{\lambda_0,\lambda_0} & g_{\lambda_0,-\lambda_0} \\
g_{-\lambda_0,\lambda_0} & g_{-\lambda_0,-\lambda_0}
\end{pmatrix}.
\]
Arguing as in the first case, we see that each column and each row of this matrix contain an invertible entry and a zero entry. Therefore, this matrix can be of one of the following shapes:
\[
\begin{pmatrix}
g_{\lambda_0,\lambda_0} & 0 \\
0 & g_{-\lambda_0,-\lambda_0}
\end{pmatrix}
or
\begin{pmatrix}
0 & g_{\lambda_0,-\lambda_0} \\
g_{-\lambda_0,\lambda_0} & 0
\end{pmatrix}.
\]
If
\[
g \in \left[ N_G(\Phi, R) (E(\Phi, \Delta, R, \sigma)), N_G(\Phi, R) (E(\Phi, \Delta, R, \sigma)) \right],
\]
then the matrix has the first shape, i.e. \( g_{-\lambda_0,\lambda_0} = 0 \) and \( g \in P \). \( \square \)

## 8. Root type elements

Let \( \mathfrak{R}(\mathbb{R}) \) be the smallest closed subscheme (over \( \mathbb{Z} \)) in \( G(\Phi, -) \), such that for any \( R \) and any \( h \in G(\Phi, R) \) we have
\[
x_{\alpha_1}(1)^h \in \mathfrak{R}(R).
\]
Obviously, one can replace \( \alpha_1 \) with any other root because all elements \( x_{\alpha}(1) \) are conjugate by elements of the Weyl group.
The elements of \( \mathcal{R}(R) \) are what we call the root type elements.

Let \( \mathbb{Z}[G] \) be the ring of regular functions on the scheme \( G(\Phi, -) \), and let \( \mathbb{Z}[\mathcal{R}] \) be the ring of regular functions on the scheme \( \mathcal{R}(-) \). By definition, the second ring is a quotient of the first one. However, using the following lemma, we will also view the ring \( \mathbb{Z}[\mathcal{R}] \) as a subring of \( \mathbb{Z}[G] \).

We denote by \( g_{\text{gen}} \in G(\Phi, \mathbb{Z}[G]) \) and \( r_{\text{gen}} \in \mathcal{R}(\mathbb{Z}[\mathcal{R}]) \) the generic elements of the corresponding schemes.

**Lemma 22.** There exists an injective map \( i: \mathcal{R}[\mathcal{R}] \to \mathbb{Z}[G] \) such that \( i(r_{\text{gen}}) = x_{\alpha_1}(1)^{g_{\text{gen}}} \).

**Proof.** The collection of maps

\[
G(\Phi, R) \to \mathcal{R}(R) \\
g \mapsto x_{\alpha_1}(1)^g
\]

gives a morphism of schemes \( G(\Phi, -) \to \mathcal{R}(-) \). Take \( i \) to be the corresponding homomorphism of rings of regular functions. The relation \( i(r_{\text{gen}}) = x_{\alpha_1}(1)^{g_{\text{gen}}} \) is true in this case by definition. It remains to prove that the homomorphism \( i \) is injective.

Let \( I = \text{Ker} \ i \), and let \( \mathcal{R}(-) \) be the subscheme of \( \mathcal{R}(-) \) determined by the ideal \( I \). Then by the definition of \( I \) we have \( x_{\alpha_1}(1)^{g_{\text{gen}}} \in \mathcal{R}(\mathbb{Z}[G]) \). Hence by the universal property of \( g_{\text{gen}} \), we have \( x_{\alpha_1}(1)^{h} \in \mathcal{R}(R) \), for any ring \( R \) and any \( h \in G(\Phi, R) \). However, \( \mathcal{R}(-) \) is the smallest subscheme with this property. Hence \( \mathcal{R}(-) \) coincides with \( \mathcal{R}(-) \), i.e. \( I = 0 \). □

We provide several important examples of root type elements.

**Lemma 23.** Let \( R \) be a commutative ring; then the following holds

1. \( x_{\alpha}(t) \in \mathcal{R}(R) \) for any \( \alpha \in \Phi, \ t \in R \). In particular, the identity element is a root type element.
2. An element conjugate to a root type element is a root type element.
3. Let \( \alpha, \beta \in \Phi \) be such that \( \angle(\alpha, \beta) = \frac{\pi}{3} \). Then \( x_{\alpha}(\xi)x_{\beta}(\zeta) \in \mathcal{R}(R) \).

**Proof.**

1. It suffices to consider the case where \( t \) is a free variable \( (R = \mathbb{Z}[t]) \). We need to check that the element \( x_{\alpha}(t) \) satisfies certain equations. Since the ring \( \mathbb{Z}[t] \) is embedded into the ring \( \mathbb{Z}[t, t^{-1}] \), we can replace one with another. In the group \( G(\Phi, \mathbb{Z}[t, t^{-1}]) \) the element \( x_{\alpha}(t) \) is conjugate to the element \( x_{\alpha_1}(1) \); hence it satisfies those equations.

2. It suffices to consider the element \( (r_{\text{gen}})^{g_{\text{gen}}} \in G(\Phi, \mathbb{Z}[\mathcal{R}] \otimes_{\mathbb{Z}} \mathbb{Z}[G]) \). As before, it suffices to map our ring to another ring injectively with the image of the element \( r_{\text{gen}} \) (and hence of the element \( (r_{\text{gen}})^{g_{\text{gen}}} \) being conjugate to \( x_{\alpha_1}(1) \)). Those requirements are satisfied if we take the map

\[
i \otimes \mathbb{Z}[G]: \mathbb{Z}[\mathcal{R}] \otimes_{\mathbb{Z}} \mathbb{Z}[G] \to \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G],
\]

where \( i \) is from Lemma 22. This map is injective because \( i \) is injective, and the scheme \( G(\Phi, -) \) is flat, i.e \( \mathbb{Z}[G] \) is a flat \( \mathbb{Z} \)-module.
(3) As before, it suffices to consider the ring $R = \mathbb{Z}[\xi, \zeta, \zeta^{-1}]$. Note that in the group $\text{SL}(3, R)$ we have

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & \zeta^{-1} & 1
\end{pmatrix}
\begin{pmatrix}
1 & \xi & \zeta \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -\zeta^{-1} & 1
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & \zeta \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
$$

Applying the corresponding map from $\text{SL}(3, R)$ to $G(\Phi, R)$, we see that the element $x_\alpha(\xi)x_\beta(\zeta)$ is conjugate to the element $x_\beta(\zeta)$; hence it is conjugate to $x_\alpha(1)$.

Lemma 24. Every root type element is an exponential of a certain element of the Lie algebra of the group $G(\Phi, R)$. Moreover, in terms of matrices in the representation $V$ taking the exponential is addition of the identity matrix.

Proof. It suffices to prove the statement for the element $r_{\text{gen}} \in \mathfrak{N}(\mathbb{R}[\mathfrak{N}])$. We view $\mathbb{Z}[\mathfrak{N}]$ as a subring in $\mathbb{Z}[G]$ (Lemma 22). Over $\mathbb{Z}[G]$, the element $r_{\text{gen}}$ becomes conjugate to the element $x_\alpha(1)$; hence the statement is true for it (the second part holds true because the representation is minuscule). It remains to note that after subtraction of the identity matrix, all the entries remain in the ring $\mathbb{Z}[\mathfrak{N}]$; hence the corresponding element of the Lie algebra is defined over $\mathbb{Z}[\mathfrak{N}]$.

Lemma 25. Let $g \in \mathfrak{N}(R)$. Then the following holds true:

1. $g_{\lambda, \mu} = 0$ for any pair of weights $\lambda, \mu$ such that $d(\lambda, \mu) \geq 2$.
2. $g_{\lambda, \mu} = \pm g_{\rho, \sigma}$ for any $\lambda, \mu, \rho, \sigma \in \Lambda$ such that $d(\lambda, \mu) = d(\rho, \sigma) = 1$ and $\lambda - \mu = \rho - \sigma$.

Proof. The two statements follow from Lemma 24 and the Cartan decomposition.

9. Extraction of elementary root elements

We need the following obvious lemma.

Lemma 26. Let $\lambda \in \Lambda$ and $\alpha, \beta_1, \ldots, \beta_k \in \Phi$ be such that $\lambda + \alpha \notin \Lambda$, and $\lambda - \beta_i \in \Lambda$ for all $i$. Then for any $\zeta, \xi_1, \xi_k \in R$ we have

$$
\prod_{i=1}^{k} x_{\beta_i}(\xi_i), x_\alpha(\zeta) = \prod_{i=1}^{k} [x_{\beta_i}(\xi_i), x_\alpha(\zeta)].
$$

9.1. Extraction from the subgroups $P$ and $P^-$. The proposition below follows from the paper [29]. However, in our case the proof is quite simple, so we give it here.

Proposition 2. Let $H \supseteq H$ be an overgroup for $E(\Delta, R)$.

1. Let $I^+ \subseteq R$ be such that $(H \cap P) \setminus G(\Phi, \Delta, R, (I^+, 0)) \neq \emptyset$. Then $H$ contains an element $x_\alpha(\xi)$, where $\alpha \in \Omega^+$ and $\xi \in R \setminus I^+$.
2. The same is true for $P^-$ and $\Omega^-$. 
Proof. We prove Item 1, the proof of the second item is similar.

Suppose the contrary; then the first component of \( \text{lev}(H) \) is contained in \( I^+ \). Without loss of generality, we may assume that it is equal to \( I^+ \) (if we replace \( I^+ \) with this component, then the assumption remains true).

Let \( g \in (H \cap P) \setminus G(\Phi, \Delta, R, (I^+, 0)) \), then \( g = g_1g_2 \), where \( g_1 \in U \) and \( g_2 \in L \). Let

\[
g_1 = \prod_{i=1}^{l} x_{\beta'_i}^{\gamma}(\xi_i), \quad \text{where } \beta'_i \in \Omega^+.
\]

Since \( g / \notin G(\Phi, \Delta, R, (I^+, 0)) \), without loss of generality we may assume that \( \xi_1 / \notin I^+ \). Take \( \alpha \in \Delta \) such that \( \beta'_1 + \alpha \in \Phi \) (Lemma 8). Then we have:

\[
[g_1, x_\alpha(1)] = [gg_2^{-1}, x_\alpha(1)] = g[g_2^{-1}, x_\alpha(1)] \cdot [g, x_\alpha(1)] \in H,
\]

because \( g, x_\alpha(1) \in H \), and

\[
[g_2^{-1}, x_\alpha(1)] \in [L, E(\Delta, R)] = [G(\Phi, \Delta, R, (0, 0)), E(\Phi, \Delta, R, (0, 0))] = E(\Phi, \Delta, R, (0, 0)) \leq H.
\]

On the other hand, by Lemma 26 we get:

\[
[g_1, x_\alpha(1)] = \prod_{i=1}^{l} [x_{\beta'_i}(\xi_i), x_\alpha(1)].
\]

Without loss of generality, we may assume that the first \( k \) commutators and only they are nontrivial, i.e.,

\[
[g_1, x_\alpha(1)] = \prod_{i=1}^{k} x_{\beta_i}(\pm \xi_i), \quad \text{where } \beta_i = \beta'_i + \alpha.
\]

Therefore, \( H \) contains an element of the form

\[
\prod_{i=1}^{k} x_{\beta_i}(\xi_i),
\]

where \( \beta_i \in \Omega^+ \) and \( \xi_i / \notin I^+ \). Among all such elements of \( H \) consider those that have the smallest \( k \), and among them, take the one with the greatest sum of the \( \beta_i \). Then none of the \( \xi_i \) belongs to \( I^+ \) (otherwise, the corresponding root elements belong to \( H \), and we can remove them to reduce \( k \)), and all the \( \beta_i \) are distinct (otherwise we reduce \( k \), by using additivity). If at least one of the \( \beta_i \) is not the maximal root, then we can take a simple root \( \alpha_m \) such that \( \beta_i + \alpha_m \in \Phi \) (\( \alpha_m \neq \alpha^{(1)} \) because \( \beta_i \in \Omega^+ \); hence \( \alpha_m \in \Delta \)). In this case, we see that the element

\[
\prod_{i=1}^{k} x_{\beta_i}(\xi_1), x_{\alpha_m}(1) \in H
\]

has the same form by Lemma 26. Furthermore, we either reduce \( k \), or retain \( k \) the same, but increase the sum of \( \beta_i \), which contradicts our choice.

Therefore, our element is \( x_{\delta}(\xi_1) \), where \( \delta \) is the maximal root, and \( \xi_1 / \notin I^+ \). \( \Box \)
9.2. Extraction from the subgroups $P_{\lambda}$ and $P_{\lambda}^-$. 

**Lemma 27.** Let $H$ be an overgroup of $E(\Delta, R)$, of level $\sigma = (I^+, I^-)$, and let $\lambda_1 \in \Lambda_1$. Then the following inclusion holds:

$$H \cap U_{\lambda_1} \subseteq G(\Phi, \Delta, R, \sigma).$$

**Proof.** Let $g \in H \cap U_{\lambda_1} \setminus G(\Phi, \Delta, R, \sigma)$. 

Case 1. $g \notin G(\Phi, \Delta, R, (I^+, R))$.

In this case, we have

$$g = \prod_{i=0}^{l} x_{\beta_i}(\xi_i), \quad \text{where } \beta_i' \in \Sigma_{\lambda_1}.$$ 

We may assume that $\beta_0' = \lambda_1 - \lambda_0$, and $x_{\beta'_1}(\xi_1) \notin G(\Phi, \Delta, R, (I^+, R))$. Then $\beta'_1 \in \Sigma^+_{\lambda_1}$ and $\xi_1 \notin I^+$. 

By Item 3 of Lemma 11 there exists $\gamma \in \Delta \cap \Delta_{\lambda_1}$ such that $\beta_1 + \gamma \in \Phi$ (hence, in fact, $\beta_1 + \gamma \in \Sigma^+_{\lambda_1}$), but $\gamma + \beta_0 \notin \Phi$. Then $[g, x_\gamma(1)] \in H$. On the other hand, by Lemma 26 we have:

$$[g, x_\gamma(1)] = \prod_{i=0}^{l} [x_{\beta'_i}(\xi_i), x_\gamma(1)] = \prod_{i=1}^{l} [x_{\beta'_i}(\xi_i), x_\gamma(1)].$$

Without loss of generality, we may assume that the first $k$ commutators and only they are nontrivial, i.e.,

$$[g, x_\gamma(1)] = \prod_{i=1}^{k} x_{\beta_i}(\pm \xi_i),$$

where all the $\beta_i$ belong to $\Sigma^0_{\lambda_1} \cup \Sigma^+_{\lambda_1}$ (because by Item 1 of Lemma 11 $\beta'_0$ is the only element of $\Sigma^-_{\lambda_1}$); next, $\beta_1 \in \Sigma^+_{\lambda_1}$ and $\xi_1 \notin I^+$. Arguing as in the end of the proof of Proposition 2, we get a contradiction.

Case 2. $g \in G(\Phi, \Delta, R, (I^+, R))$.

Similarly,

$$g = \prod_{i=0}^{l} x_{\beta'_i}(\xi_i).$$

However, now we have $\xi_i \in I^+$ ($1 \leq i \leq l$), but $\xi_0 \notin I^-$. Thus, all the factors except the first one belong to $H$; hence we deduce that $x_{\beta_0'}(\xi_0) \in H$, which contradicts the fact that $\xi \notin I^-$. \hfill \Box

**Lemma 28.** Let $I^+ \subseteq R$, $\lambda_1 \in \Lambda_1$, and let

$$g \in (P_{\lambda_1} \cap R(R)) \setminus G(\Phi, \Delta, R, (I^+, R)),$$

Then there exists $\gamma \in (\Delta \cap \Delta_{\lambda_1})'$ (see Item (4) of Lemma 11) such that

$$gx_\gamma(1)g^{-1} \notin G(\Phi, \Delta, R, (I^+, R)).$$
Proof. First, we prove the following statement:

\[ gE((\Delta \cap \Delta_{\lambda_1})', R)g^{-1} \not\subseteq G(\Phi, \Delta, R, (I^+, R)). \]

Suppose the contrary; then our assumption means exactly that the group \( E((\Delta \cap \Delta_{\lambda_1})', R) \) stabilizes the line spanned by the covector \( g_{\lambda_0} \), modulo \( I^+ \). Since this group is perfect (because \((\Delta \cap \Delta_{\lambda_1})'\) is an irreducible system distinct from \( A_1 \)), it must stabilize the covector itself.

Further, since \( g \notin G(\Phi, \Delta, R, (I^+, R)) \), there exists \( \mu \in \Lambda \setminus \{\lambda_0\} \) such that \( g_{\lambda_0, \mu} \notin I^+ \). By Lemma 25 \( \mu \in \Lambda_1 \), and since \( g \in P_{\lambda_1} \), we see that \( \mu \neq \lambda_1 \). Take \( \nu \) from Item 4 of Lemma 25 and set \( \tilde{\gamma} = \mu - \nu \in (\Delta \cap \Delta_{\lambda_1})' \).

Since the element \( g_{\lambda_0} \) stabilizes the covector \( x_\gamma(1) \) we obtain the following congruence

\[ (gx_\gamma(1))_{\lambda_0, \nu} \equiv g_{\lambda_0, \nu} \mod I^+. \]

However, we also have the identity

\[ (gx_\gamma(1))_{\lambda_0, \nu} = g_{\lambda_0, \nu} \pm g_{\lambda_0, \mu}. \]

Hence \( g_{\lambda_0, \mu} \in I^+ \), which contradict the choice of \( \mu \).

Therefore, for some \( \gamma \in (\Delta \cap \Delta_{\lambda_1})' \) and some \( t \in R \), we have

\[ gx_\gamma(t)g^{-1} \notin G(\Phi, \Delta, R, (I^+, R)). \]

We also know that

\[ gx_\gamma(t)g^{-1} = e + t(gx_\gamma(1)g^{-1} - e), \]

Hence we obtain

\[ gx_\gamma(1)g^{-1} \notin G(\Phi, \Delta, R, (I^+, R)). \]

\[ \square \]

Lemma 29. Let \( \sigma = (I^+, I^-) \) be the pair of ideals of the ring \( R \), let \( \lambda_1 \in \Lambda_1 \), and let \( g \in (L_{\lambda_1} \cap \mathfrak{R}(R)) \setminus G(\Phi, \Delta, R, \sigma) \).

Then there exists \( \gamma \in \Sigma_{\lambda_1}^0 \) such that \( g^{-1}x_\gamma(1)g \notin G(\Phi, \Delta, R, \sigma) \).

Proof. Case 1. \( g \notin G(\Phi, \Delta, R, (R, I^-)) \).

In this case, there exists a weight \( \mu \) such that \( g_{\mu, \lambda_0} \notin I^- \). By Item 1 of Lemma 25 \( \mu \in \Lambda_1 \), and, since \( g \in L_{\lambda_1} \), we see that \( \mu \neq \lambda_1 \). Next, let \( w \in W(\Phi) \) be the reflection with respect to \( \lambda_0 - \lambda_1 \). Note that the roots \( \lambda_0 - \mu \) and \( \lambda_0 - \lambda_1 \) are not orthogonal. Indeed, otherwise we obtain:

\[ \lambda_0 - \mu = w(\lambda_0 - \mu) = \lambda_1 - w(\mu). \]

Then, applying Item 2 of Lemma 25 and the fact that \( g \in L_{\lambda_1} \), we deduce that \( g_{\mu, \lambda_0} = g_{w(\mu), \lambda_1} = 0 \).

Therefore, their inner product should be equal to one (it cannot be equal to minus one because they both belong to \( \Omega^+ \)), and we can set

\[ \gamma = \lambda_1 - \mu = (\lambda_0 - \mu) - (\lambda_0 - \lambda_1) \in \Phi. \]

By definition, \( \gamma \in \Sigma_{\lambda_1} \), and since \( \mu \in \Lambda_1 \), we have \( \gamma \in \Sigma_{\lambda_1}^0 \). We show that it satisfies the requirement.
Indeed, the column $(x_\gamma(1)g)_{\delta,\lambda_0}$ is not a multiple of the column $g_{\lambda_1,\lambda_0}$ modulo $I^-$, because $g_{\lambda_1,\lambda_0} = 0$, but $(x_\gamma(1)g)_{\lambda_1,\lambda_0} \notin I^-$. This means exactly that

$$g^{-1}x_\gamma(1)g \notin G(\Phi, \Delta, R, (R, I^-)).$$

**Case 2.** $g \notin G(\Phi, \Delta, R, (I^+, R))$.

In this case $g^{-1} \notin G(\Phi, \Delta, R, (I^+, R))$, i.e., the set

$$M = \{ \mu \in \Lambda_1 : g^{-1}_{\lambda_0, \mu} \notin I^+ \}$$

is nonempty.

Consider the strict partial order $\rightarrow$ on the set $M$ given by

$$\mu_1 \rightarrow \mu_2 \iff \exists \varepsilon \in R/I^+ : \left( g^{-1}_{\lambda_0, \mu_1} = \varepsilon g^{-1}_{\lambda_0, \mu_2} \right. \text{ and } \left. \varepsilon g^{-1}_{\lambda_0, \mu_1} = 0 \right),$$

where the bar stand for the image in $R/I^+$. Multiplying the corresponding $\varepsilon$, we get transitivity; antireflexivity follows by the definition of $M$.

Let $\mu \in M$ be a maximal element (i.e., such that there are no arrows from it). As in the first case, we prove that the root $\alpha = \lambda_0 - \mu$ has the inner product with $\lambda_0 - \lambda_1$ equal to one. Take $\gamma$ from item (5) of Lemma 11. Then

$$\nu = \mu - \gamma = \lambda_0 - (\alpha + \gamma) \in \Lambda.$$

Clearly, $\nu \in \Lambda_1$.

Assume that $g^{-1}x_\gamma(1)g \in G(\Phi, \Delta, R, \sigma)$, i.e., the element $x_\gamma(1)$ stabilises the line spanned by the covector $g^{-1}_{\lambda_0, \sigma}$ modulo $I^+$. Thus, it multiplies this covector by a scalar, which we denote by $(1 + \varepsilon) \in R/I^+$. We show that $\nu \in M$ and $\mu \rightarrow \nu$. If we do this, then we get a contradiction.

First, we have:

$$g^{-1}_{\lambda_0, \nu} \pm g^{-1}_{\lambda_0, \mu} = (g^{-1}_{\lambda_0, \nu}x_\gamma(1))_{\nu} \equiv (1 + \varepsilon)g^{-1}_{\lambda_0, \nu} \mod I^+,\n$$

i.e., $\overline{g^{-1}_{\lambda_0, \nu} = (\pm \varepsilon)g^{-1}_{\lambda_0, \mu}}$. In particular, this implies that $\nu \in M$.

Second, we have:

$$g^{-1}_{\lambda_0, \mu} = (g^{-1}_{\lambda_0, \nu}x_\gamma(1))_{\mu} \equiv (1 + \varepsilon)g^{-1}_{\lambda_0, \nu},$$

i.e., $\overline{\varepsilon g^{-1}_{\lambda_0, \mu}} = 0$. The lemma is proved. \qed

**Proposition 3.** Let $H$ be an overgroup of $E(\Delta, R)$.

1. Let $I^+ \trianglelefteq R$, and $\lambda_1 \in \Lambda_1$ be such that there exists an element

$$g \in (H \cap P_{\lambda_1} \cap \mathcal{R}(R)) \setminus G(\Phi, \Delta, R, (I^+, R)).$$

Then $H$ contains an element of the form $x_\alpha(\xi)$, where $\alpha \in \Omega^+$ and $\xi \in R \setminus I^+.$

2. The same is true for $P_{\lambda_1}^-$ and $\Omega^-$. 24
Proof. We prove Item (1); the proof of Item (2) is similar.
Suppose the contrary; then the first component of \( \text{lev}(H) \) is contained in \( I^+ \). Without loss of generality, we may assume that it is equal to \( I^+ \) if we replace \( I^+ \) with this component, then the assumption remains true. Let \( I^- \) be the second component of \( \text{lev}(H) \) and let \( \sigma = (I^+, I^-) \).

By Lemma 28, the element \( g_1 = gx_{\gamma_1}(1)g^{-1} \) satisfies the same assumptions as the element \( g \) for a suitable \( \gamma_1 \in (\Delta \cap \Delta_{\lambda_1})' \).

Let \( g = ul \) and \( g_1 = u_1l_1 \), where \( l, l_1 \in L_{\lambda_1} \) and \( u, u_1 \in U_{\lambda_1} \). Then, taking the projection to \( L_{\lambda_1} \) in the definition of \( g_1 \), we obtain

\[
l_1 = l^{-1}x_{\gamma_1}(1)l \in \mathcal{R}(R).
\]

**Case 1.** \( l_1 \notin G(\Phi, \Delta, R, \sigma) \).

In this case, applying Lemma 29, we obtain the element \( h = l_1^{-1}x_{\gamma_2}(1)l_1 \). Applying Lemma 27, we get a contradiction.

**Case 2.** \( l_1 \in G(\Phi, \Delta, R, \sigma) \).

Applying Lemma 28 once again, we obtain the element \( g_2 = g_1x_{\gamma_2}(1)g_1^{-1} \) that satisfies the same assumptions as the initial element \( g \). By assumption, we have \( l_1 \in G(\Phi, \Delta, R, \sigma) \); hence by Lemma 19, we have

\[
l_2 \in E(\Phi, \Delta, R, \sigma) \subseteq H \cap G(\Phi, \Delta, R, \sigma).
\]

Hence \( u_2 \in H \setminus G(\Phi, \Delta, R, \sigma) \), and we can apply Lemma 27 again. \( \square \)

9.3. **Extraction from a congruence subgroup of a nilpotent level.**

**Lemma 30.** Let \( S \) be a commutative ring, and let \( I \subseteq S \) be an ideal such that the following holds true.

1. \( I^2 = 0 \)
2. The ideal \( I \) is finitely generated.
3. As an Abelian group, \( S = \mathbb{Z} \oplus I \), where \((1, 0)\) is the identity element of \( S \).
4. The ideal \( I \) is torsion free as an Abelian group.

Further, let \( \xi \in I \setminus \{0\} \). Then there exists a ring homomorphism

\[
\varphi: S \to \mathbb{C}[\varepsilon]/(\varepsilon^2)
\]

such that \( \varphi(\xi) \neq 0 \).

**Proof.** Since the ideal \( I \) acts on itself by zero and is finitely generated as an ideal, it is finitely generated as an Abelian group. Hence, as an Abelian group,

\[
I \simeq \bigoplus_{i=1}^{N} \mathbb{Z}.
\]
Clearly, any subgroup of $I$ is an ideal. Choosing a direct summand with the projection of $\xi$ to it being nonzero, and taking the quotient of $S$ by the sum of all other summands, we may assume that $N = 1$. In this case

$$S \simeq \mathbb{Z}[e]/(e^2),$$

which can naturally be embedded into $\mathbb{C}[e]/(e^2)$.

\[\square\]

**Lemma 31.** Let $R$ be an arbitrary commutative ring, and let $\mathfrak{B} \subseteq R$ be an ideal such that $\mathfrak{B}^2 = 0$. Next, let

$$g \in G(\Phi, R, \mathfrak{B}).$$

Then $g_{\lambda_0, \mu_0} = 0$ for any pair of weights $\lambda_0, \mu_0$ such that $d(\lambda_0, \mu_0) \geq 2$.

**Proof.** Consider the ring

$$\widetilde{S} = \mathbb{Z}\{a_{\lambda, \mu} : \lambda, \mu \in \Lambda\}/\langle\{a_{\lambda, \mu}a_{\lambda', \mu'} : \lambda, \mu, \lambda', \mu' \in \Lambda\}\rangle,$$

and the ideal $\widetilde{I} \subseteq \widetilde{S}$ generated by all the $a_{\lambda, \mu}$. Clearly, they satisfy the assumptions in the previous lemma.

Set $S = \mathbb{Z}[G]/I^2_{\text{aug}}$, and $I = I_{\text{aug}}/I^2_{\text{aug}}$, where $I_{\text{aug}}$ is the augmentation ideal. Further, set

$$\pi : \widetilde{S} \to S$$

$$a_{\lambda, \mu} \mapsto \overline{(g_{\text{gen}})_{\lambda, \mu} - \delta_{\lambda, \mu}}.$$

We prove that the pair $S, I$ satisfies the assumptions in the previous lemma. The first two conditions are obvious, the third one is fulfilled because the augmentation homomorphism splits; it remains to show that $I$ is torsion free as an Abelian group. The scheme $G(\Phi, -)$ is smooth, the homomorphism $\pi$ is surjective, and $(\text{Ker } \pi)^2 = 0$; hence the reduction homomorphism

$$\pi_* : G(\Phi, \widetilde{S}) \to G(\Phi, S)$$

is surjective (see [24, §4, Item 4.6], set $k = \mathbb{Z}$). Hence there exists a matrix

$$e + (b_{\lambda, \mu}) \in G(\Phi, \widetilde{S})$$

such that $\pi(\delta_{\lambda, \mu} + b_{\lambda, \mu}) = \overline{(g_{\text{gen}})_{\lambda, \mu}}$. In particular, $b_{\lambda, \mu} \in \widetilde{I}$; hence the ring homomorphism

$$i : S \to \widetilde{S}$$

$$(g_{\text{gen}})_{\lambda, \mu} \mapsto b_{\lambda, \mu} + \delta_{\lambda, \mu}.$$ 

is well defined. Then the homomorphism of Abelian groups

$$i|_{I} : I \to \widetilde{I}$$

is a right inverse to $\pi|_{\widetilde{I}}$; hence $I$ is isomorphic to a direct summand of $\widetilde{I}$ and, therefore, is torsion free.

The assumption implies that the ring homomorphism

$$\psi : S \to R$$

$$\overline{(g_{\text{gen}})_{\lambda, \mu}} \mapsto g_{\lambda, \mu}$$
Proof. We prove Item (1); the proof of Item (2) is similar.

Let \( \lambda_0, \mu_0 \) be such that \( d(\lambda_0, \mu_0) \geq 2 \). We prove that \( g_{\lambda_0, \mu_0} = 0 \). To do this, it suffices to prove that \( (\overline{g_{\text{gen}}})_{\lambda_0, \mu_0} = 0 \). Suppose the contrary; then by the previous lemma, there exists a ring homomorphism

\[ \varphi : S \to \mathbb{C}[\varepsilon]/(\varepsilon^2) \]

such that \( \varphi((\overline{g_{\text{gen}}})_{\lambda, \mu}) \neq 0 \). The matrix \( \overline{g_{\text{gen}}} \) belongs to \( G(\Phi, S, I) \). Since \( I^2 = 0 \), we see that \( \varphi(I) \) belongs to the ideal generated by \( \varepsilon \); hence

\[ \varphi_*((\overline{g_{\text{gen}}})_{\lambda, \mu}) \in G(\Phi, \mathbb{C}[\varepsilon]/(\varepsilon^2), (\varepsilon)) = \text{Lie}(G(\Phi, -)_{\mathbb{C}}). \]

This Lie algebra admits a Cartan decomposition, i.e., the matrix \( (\varphi(a_{\lambda, \mu})) \) is a linear combination of elementary root elements and diagonal elements of this algebra. Thus, since \( d(\lambda_0, \mu_0) \geq 2 \), its entry \( \varphi((\overline{g_{\text{gen}}})_{\lambda, \mu}) \) is equal to 0, which contradicts the choice of \( \varphi \). \( \square \)

**Proposition 4.** Let \( H \) be an overgroup of \( E(\Delta, R) \).

1. Let \( (H \cap G(\Phi, R, \mathfrak{B})) \backslash P^\perp \neq \emptyset \), where \( \mathfrak{B} \subseteq R \) is such that \( \mathfrak{B}^2 = 0 \). Then \( H \) contains an element of the form \( x_\alpha(\xi) \), where \( \alpha \in \Omega^+ \) and \( \xi \neq 0 \).

2. The same holds for \( P \) and \( \Omega^- \).

**Proof.** We prove Item (1); the proof of Item (2) is similar.

Let \( g \in (H \cap G(\Phi, R, \mathfrak{B})) \backslash P^- \). Since \( g \notin P^- \), there exists \( \lambda_1 \in \Lambda \setminus \{\lambda_0\} \) such that \( g_{\lambda_0, \lambda_1} \neq 0 \). By the previous lemma \( \lambda_1 \in \Lambda_1 \). Take \( \alpha \in \Delta \) such that \( \lambda_1 + \alpha = \nu \in \Lambda_1 \)(Lemma 12). If we prove that

\[ h = gx_\alpha(1)g^{-1} \in (H \cap P_{\lambda_1}) \setminus P^- , \]

then by Proposition 3, we are done.

We prove that the element \( x_\alpha(1) \) stabilizes the line spanned by the vector \( (g^{-1})_{*, \lambda_1} \).

Indeed, we have:

\[ (x_\alpha(1)(g^{-1})_{*, \lambda_1})_{\mu} = \begin{cases} (g^{-1})_{\mu, \lambda_1} & \mu + \alpha \notin \Phi, \\ (g^{-1})_{\mu, \lambda_1} \pm (g^{-1})_{\mu + \alpha, \lambda_1} & \mu + \alpha \in \Phi. \end{cases} \]

Note that since \( \lambda_1 + \alpha \in \Lambda \), in the second case, by Lemma 10 we have either \( \mu = \lambda_1 \) or \( d(\mu + \alpha, \lambda_1) \geq 2 \); hence \( (g^{-1})_{\mu + \alpha, \lambda_1} = 0 \) by the previous lemma. Therefore,

\[ (x_\alpha(1)(g^{-1})_{*, \lambda_1})_{\mu} = \begin{cases} (g^{-1})_{\mu, \lambda_1} & \mu \neq \lambda_1, \\ (g^{-1})_{\lambda_1, \lambda_1} \pm (g^{-1})_{\nu, \lambda_1} & \mu = \lambda_1. \end{cases} \]

Hence since \( \mathfrak{B}^2 = 0 \), we obtain

\[ x_\alpha(1)(g^{-1})_{*, \lambda_1} = (g^{-1})_{*, \lambda_1} \pm (g^{-1})_{\nu, \lambda_1} v_{\lambda_1} = (g^{-1})_{*, \lambda_1} (1 \pm (g^{-1})_{\nu, \lambda_1}). \]

Thus \( h \in P_{\lambda_1} \).

Finally, let \( w \) be a row such that

\[ g_{\lambda_0, x}(1) = g_{\lambda_0, x} + w. \]
Thus $w_\mu \in \mathfrak{B}$ for all $\mu \in \Lambda$ (note that $\lambda_0 - \alpha \notin \Lambda$ because $\alpha \in \Delta$). Furthermore, $w_\nu = \pm g_{\lambda_0, \lambda_1} \neq 0$. Then since $g^{-1} \in G(\Phi, R, \mathfrak{B})$, we obtain:

$$h_{\lambda_0, \nu} = (g_{\lambda_0, \nu} g^{-1} + w g^{-1})_\nu = 0 + w_\nu \neq 0.$$ 

Thus $h \notin P^-$.

10. An $A_2$-proof

The idea of the argument below is borrowed from [12].

**Lemma 32.** Let $g \in \mathfrak{R}(R)$ and $\lambda_1 \in \Lambda$. For every pair of weights $\mu, \nu \in \Lambda$ such that $d(\lambda_1, \mu) = d(\lambda_1, \nu) = d(\mu, \nu) = 1$, set

$$x(\mu, \nu) = x_\alpha(c_{\mu, \alpha} \cdot g_{\nu, \lambda_1})x_\beta(-c_{\nu, \beta} \cdot g_{\mu, \lambda_1}),$$

where $\alpha = \lambda_1 - \mu$ and $\beta = \lambda_1 - \nu$.

Then the following holds.

1. The element $x(\mu, \nu)$ stabilises the column $g_{*, \lambda_1}$.
2. We have

$$g^{-1}x(\mu, \nu)g = x(\mu, \nu) + [x(\mu, \nu), g]_{\text{ring}},$$

where $[\cdot, \cdot]_{\text{ring}}$ is a ring commutator of matrices.

**Proof.**

1. We act by the element $x_\beta(-c_{\nu, \beta} \cdot g_{\mu, \lambda_1})$ on the column $g_{*, \lambda_1}$

$$(x_\beta(-c_{\nu, \beta} \cdot g_{\mu, \lambda_1})g_{*, \lambda_1})_\rho = \begin{cases} g_{\rho, \lambda_1} & \rho - \beta \notin \Lambda, \\ g_{\rho, \lambda_1} - c_{\rho - \beta, \beta} c_{\nu, \beta} \cdot g_{\mu, \lambda_1}g_{\rho - \beta, \lambda_1} & \rho - \beta \in \Lambda. \end{cases}$$

Note that if the second case occurs, Lemma 10 shows that we have either $\rho = \lambda_1$ or $d(\rho - \beta, \lambda_1) \geq 2$; hence by Lemma 25 $g_{\rho - \beta, \lambda_1} = 0$. Therefore, we have:

$$x_\beta(-c_{\nu, \beta} \cdot g_{\mu, \lambda_1})g_{*, \lambda_1} = g_{*, \lambda_1} - g_{\mu, \lambda_1} g_{\nu, \lambda_1}.$$

Similarly, applying $x_\alpha(c_{\mu, \alpha} \cdot g_{\nu, \lambda_1})$ to the resulting column, we obtain the initial column $g_{*, \lambda_1}$.

2. It suffices to consider the case where $R = \mathbb{Z}[\mathfrak{R}]$ and $g = r_{\text{gen}}$. Since what we need to prove is a polynomial identity, we may extend the ring up to $\mathbb{Z}[G]$, using Lemma 22 and then extend it up to $\mathbb{Z}[G]([\frac{1}{3}])$.

Let $\bar{x} = x(\mu, \nu) - e$ and $\bar{r} = r_{\text{gen}} - e$ be the corresponding elements of the Lie algebra (see Lemma 24). Here it should be noted that $\mathcal{L}(\alpha, \beta) = \frac{\pi}{3}$ because $\alpha - \beta = \nu - \mu \in \Phi$. Hence $x(\mu, \nu) \in \mathfrak{R}(R)$ by Lemma 23. Then by Lemma 24 we have the following identity:

$$r_{\text{gen}}^{-1} x(\mu, \nu) r_{\text{gen}} = e + \exp(-\text{ad} \bar{r})\bar{x} = x(\mu, \nu) + [\bar{x}, \bar{r}]_{\text{ring}} + \frac{1}{2}(\text{ad} \bar{r})^2 \bar{x} = x(\mu, \nu) + [\bar{x}, \bar{r}]_{\text{ring}} + \frac{1}{2}(\text{ad} \bar{r})^2 \bar{x}.$$

Next, note that for any element $y$ of our Lie algebra, we have $(\text{ad} e_{\alpha_1})^2 y = \xi_y e_{\alpha_1}$ for some $\xi_y \in R$, where $e_{\alpha_1} = x_{\alpha_1}(1) - e$ is a root element. This relation is obtained
from the Cartan decomposition of the element $y$ by expanding (the only nonzero term comes from the coefficient of $e_{-\alpha_1}$). Conjugating this relation by the element $g_{\text{gen}}$, we obtain a similar statement for $r_{\text{gen}}$. Therefore, we have the formula

$$r_{\text{gen}}^{-1}x(\mu, \nu)r_{\text{gen}} = x(\mu, \nu) + [x(\mu, \nu), r_{\text{gen}}]\text{ring} + \xi \tilde{r} \tag{*}$$

for some $\xi \in \mathbb{Z}[G][\frac{1}{2}]$. We claim that in fact, we have $\xi = 0$. Indeed, let us compare the $\lambda_1$th columns of the left-hand and right-hand sides of (*). The first item implies the formula

$$(r_{\text{gen}}^{-1}x(\mu, \nu)r_{\text{gen}})_{*, \lambda_1} = v_{\lambda_1},$$

as well as the relation

$$(x(\mu, \nu)r_{\text{gen}})_{*, \lambda_1} = (r_{\text{gen}})_{*, \lambda_1}.$$

Further, it is easy to see that the element $x(\mu, \nu)$ also stabilises the column $v_{\lambda_1}$. Hence $x(\mu, \nu)_{*, \lambda_1} = v_{\lambda_1}$. Finally, calculating the product $r_{\text{gen}}x(\mu, \nu)$ we see that the matrix $x(\mu, \nu)$ acts on each row of the matrix $r_{\text{gen}}$, without changing the entry in the $\lambda_1$th column (indeed, $\lambda_1 - \alpha$ and $\lambda_1 - \beta$ belong to $\Lambda$; hence $\lambda_1 + \alpha$ and $\lambda_1 + \beta$ do not belong to $\Lambda$, because the representation is minuscule), i.e., we have

$$(r_{\text{gen}}x(\mu, \nu))_{*, \lambda_1} = (r_{\text{gen}})_{*, \lambda_1}.$$

Therefore, we obtain the identity:

$$v_{\lambda_1} = v_{\lambda_1} + (r_{\text{gen}})_{*, \lambda_1} - (r_{\text{gen}})_{*, \lambda_1} + \xi \tilde{r}_{*, \lambda_1},$$

i.e., $\xi \tilde{r}_{*, \lambda_1} = 0$. The ring $\mathbb{Z}[G][\frac{1}{2}]$ has no zero divisors, and the column $\tilde{r}_{*, \lambda_1}$ is nonzero. Hence $\xi = 0$, and (*) gives us what we need.

□

Lemma 33. Let $g \in \mathfrak{g}(R)$ and $\lambda_1 \in \Lambda$. Let $\mu \in \Lambda$ be such that $d(\lambda_1, \mu) = 1$, and let $\xi \in R$ be such that $\xi g_{\rho, \lambda_1} = 0$. Then the element $x_{\alpha}(\xi)$, where $\alpha = \lambda_1 - \mu$, stabilises the column $g_{*, \lambda_1}$.

Proof. We have

$$(x_{\alpha}(\xi)g_{*, \lambda_1})_{\rho} = \begin{cases} g_{\rho, \lambda_1} & \rho - \alpha \notin \Lambda, \\ g_{\rho, \lambda_1} \pm \xi g_{\rho - \alpha, \lambda_1} & \rho - \alpha \in \Lambda, \end{cases}$$

moreover, if the second case occurs, Lemma 10 shows that either $\rho = \lambda_1$, or $d(\rho - \alpha, \lambda_1) \geq 2$; hence $g_{\rho - \alpha, \lambda_1} = 0$ by Lemma 25. Therefore, we obtain

$$x_{\alpha}(\xi)g_{*, \lambda_1} = g_{*, \lambda_1} \pm \xi g_{\mu, \lambda_1}v_{\lambda_1} = g_{*, \lambda_1}.$$

□
Proof. (1) Let \( R \) the following four ideals of the ring \( R \) and let \( \lambda \)

\[
\mathfrak{A}(g, \lambda_1) = \langle \{g_{\mu, \lambda_1} : \mu \in \Lambda_1 \setminus \{\lambda_1\}\} \rangle, \\
\mathfrak{B}(g, \lambda_1) = \langle g_{\lambda_0, \lambda_1} \rangle, \\
\mathfrak{A}'(g, \lambda_1) = \langle \{g_{\lambda_{1, \mu}} : \mu \in \Lambda_1 \setminus \{\lambda_1\}\} \rangle, \\
\mathfrak{B}'(g, \lambda_1) = \langle g_{\lambda_{1, \mu_0}} \rangle.
\]

Proposition 5. Let \( H \) be an overgroup of \( E(\Delta, R) \) of level \( \sigma = (I^+, I^-) \), let \( g \in \mathfrak{A}(R) \cap H \), and let \( \lambda_1 \in \Lambda_1 \). Then we have:

1. \( \mathfrak{A}(g, \lambda_1)\mathfrak{B}(g, \lambda_1) \subseteq I^+ \);
2. if \( I^+ = 0 \), then \( (\mathfrak{B}(g, \lambda_1))^3 = (0) \);
3. \( \mathfrak{A}'(g, \lambda_1)\mathfrak{B}'(g, \lambda_1) \subseteq I^- \);
4. if \( I^- = 0 \), then \( (\mathfrak{B}'(g, \lambda_1))^3 = (0) \).

Proof. (1) Let \( \nu \in \Lambda_1 \setminus \{\lambda_1\} \). We prove that \( g_{\nu, \lambda_1}g_{\lambda_0, \lambda_1} \in I^+ \). It suffices to consider the case where \( d(\nu, \lambda_1) = 1 \) because otherwise \( g_{\nu, \lambda_1} = 0 \) by Lemma 25.

Take a weight \( \mu \) from Item 2 of Lemma 12. Let \( \alpha = \lambda_1 - \mu, \beta = \lambda_1 - \nu \) and let \( x(\mu, \nu) \) be as in Lemma 32. Then this lemma implies that the element \( g^{-1}x(\mu, \nu)g \) stabilises \( v^{\lambda_1} \) (i.e., belongs to \( P_{\lambda_1} \)).

By Lemma 23 this element belongs to \( \mathfrak{A}(R) \). It also belongs to \( H \) (indeed, since \( \lambda_1, \mu, \nu \in \Lambda_1 \), we have \( \alpha, \beta \in \Delta \); hence \( x(\mu, \nu) \in H \)). If it does not belong to \( G(\Phi, \Delta, R, (I^+, R)) \), then by Proposition 3 the group \( H \) cannot be of level \( \sigma \). Hence it belongs to \( G(\Phi, \Delta, R, (I^+, R)) \), and by Item 2 of Lemma 32 we obtain the congruence

\[(gx(\mu, \nu))_{\lambda_0, \mu} \equiv (x(\mu, \nu)g)_{\lambda_0, \mu} \mod I^+. \tag{1}\]

We compute the left-hand side of the congruence (1). Note that \( \mu - \beta \notin \Lambda \), because otherwise, by Lemma 10, the relation \( \lambda_1 + \beta \in \Lambda \), implies that either \( d(\lambda_1, \mu) \geq 2 \), or \( \mu = \nu \), which contradicts the choice of \( \mu \). Hence,

\[gx(\mu, \nu)v^\mu = g\alpha(\pm g_{\nu, \lambda_1})x_\beta(\pm g_{\mu, \lambda_1})v^\mu = g\alpha(\pm g_{\nu, \lambda_1})v^\mu = g(v^\mu \pm g_{\nu, \lambda_1}v^{\lambda_1}), \]

which implies that

\[(gx(\mu, \nu))_{\lambda_0, \mu} = g_{\lambda_0, \mu} \pm g_{\nu, \lambda_1}g_{\lambda_0, \lambda_1}. \]

Now we compute the right-hand side of the congruence (1). Note that \( \lambda_0 - \alpha \notin \Lambda \) because \( \alpha \in \Delta \). Hence,

\[(x_\alpha(\pm g_{\nu, \lambda_1})x_\beta(\pm g_{\mu, \lambda_1})g)_{\lambda_0, \mu} = (x_\beta(\pm g_{\mu, \lambda_1})g)_{\lambda_0, \mu}. \]

Similarly, \( \lambda_0 - \beta \) does not belong to \( \Lambda \); hence we obtain

\[(x_\beta(\pm g_{\mu, \lambda_1})g)_{\lambda_0, \mu} = g_{\lambda_0, \mu}. \]

Therefore, the congruence (1) tells us exactly that \( g_{\nu, \lambda_1}g_{\lambda_0, \lambda_1} \in I^+ \).
(2) We prove that \((g_{\lambda_0,\lambda_1})^3 = 0\).

Take a weight \(\mu\) from Item 1 of Lemma 12. Put \(\alpha = \lambda_1 - \mu\), and \(x = x_\alpha(g_{\lambda_0,\lambda_1})\). By what we proved before, we have \(g_{\lambda_0,\lambda_1}g_{\mu,\lambda_1} = 0\). Then Lemma 33 implies that the element \(g^{-1}xg\) stabilizes \(v^{\lambda_1}\). It also belongs to \(H \cap R(R)\). If it does not belong to \(P^-\) (i.e., to \(G(\Phi, \Delta, R, (0, R))\)), then by Proposition 3 the group \(H\) cannot be of level \(\sigma\) (because \(I^+ = 0\)). Hence it belongs to \(P^-\), i.e., the matrix \(x\) stabilizes the line spanned by the covector \(g_{\lambda_0,\lambda_1}^{-1}\). Thus it multiplies this covector by a scalar, which we denote by \((1 + \varepsilon)\). Then, first, we have:

\[
(1 + \varepsilon)(g^{-1})_{\lambda_0,\lambda_1} = (g^{-1})_{\lambda_0,\lambda_1},
\]

i.e., \(\varepsilon(g^{-1})_{\lambda_0,\lambda_1} = 0\). Second, we have:

\[
(1 + \varepsilon)(g^{-1})_{\lambda_0,\mu} = (g^{-1}x)_{\lambda_0,\mu} = (g^{-1})_{\lambda_0,\mu} \pm (g^{-1})^2_{\lambda_0,\lambda_1},
\]

i.e., \(\varepsilon(g^{-1})_{\lambda_0,\mu} = \pm (g^{-1})^2_{\lambda_0,\lambda_1}\). Therefore, we obtain the identity

\[
(g^{-1})^3_{\lambda_0,\lambda_1} = \pm \varepsilon(g^{-1})_{\lambda_0,\mu}(g^{-1})_{\lambda_0,\lambda_1} = 0.
\]

It remains to note that by Lemma 24 we have: \((g^{-1})_{\lambda_0,\lambda_1} = -g_{\lambda_0,\lambda_1}\).

The proofs of Items 3 and 4 are similar. \(\square\)

**Corollary 3.** Let \(I\) be an ideal of \(R\), and let \(H\) be an overgroup of \(E(\Delta, R)\) of level \(\sigma = (I^+, I^-)\). Then for the level of \(\rho_I(H)\) as an overgroup of \(E(\Delta, R/I)\) we have:

\[\text{lev}(\rho_I(H)) = (\rho_I(I^+), \rho_I(I^-)).\]

**Proof.** Obviously, the right-hand side is contained in the left-hand side, let us prove the inverse inclusion.

Suppose the contrary. Without loss of generality, we may assume that the inclusion fails for the first component, i.e., for some \(g \in H\), \(\xi \in R/I\) such that \(\xi \notin \rho_I(I^+)\), and \(\alpha \in \Omega^+\) we have \(\rho_I(g) = x_{\alpha}(\xi)\).

Let \(\nu = \lambda_0 - \alpha \in \Lambda_1\) (Lemma 9). Take \(\beta \in \Delta\) such that \(\alpha + \beta \in \Phi\) (Lemma 8). Then \(\alpha + \beta \in \Omega^+,\) and \(\lambda_1 = \lambda_0 - \alpha - \beta \in \Lambda_1\).

Set \(h = (x_{\beta}(1))^\beta\). Clearly, we have \(h \in H \cap R(R)\). Next, we have

\[
\rho_\beta(h) = (x_{\beta}(1))^{x_{\alpha}(\xi)} = x_{\beta}(-1) \cdot [x_{\beta}(1), x_{\alpha}(-\xi)] = x_{\beta}(-1)x_{\alpha + \beta}(\pm \xi)
\]

By the previous proposition, we have \(h_{\nu, \lambda_1}h_{\lambda_0, \lambda_1} \in I^+\). Hence we have

\[
\rho_\beta(h)_{\nu, \lambda_1} \rho_\beta(h)_{\lambda_0, \lambda_1} \in \rho_I(I^+).
\]

On the other hand, the following identity holds:

\[
\rho_\beta(h)^\lambda = x_{\beta}(-1)x_{\alpha + \beta}(\pm \xi)^\lambda = x_{\beta}(-1)(v^\lambda \pm \xi v^{\lambda + \alpha + \beta}) =
\]
In any case, we obtain $\rho_B(h)_{\lambda+\beta+\alpha,\lambda} = \pm \xi$, and $\rho_B(h)_{\lambda+\beta,\lambda} = \pm 1$. Hence $\xi \in \rho_I(I^+)$, which contradicts the assumption. \qed

11. Root type elements in an overgroup

Lemma 34. Let $\sigma = (I^+, I^-)$ be a pair of ideals, and let $g \in \mathcal{R}(R) \setminus G(\Phi, \Delta, R, \sigma)$. Then there exists $\gamma \in \Delta$ such that

$$gx_\gamma(1)g^{-1} \notin G(\Phi, \Delta, R, \sigma).$$

Proof. Clearly, without loss of generality we may assume that $g \notin G(\Phi, \Delta, R, (I^+, R))$. First we prove the following statement:

$$gE(\Delta, R)g^{-1} \not\subseteq G(\Phi, \Delta, R, (I^+, R)).$$

Suppose the contrary; then our assumption means exactly that the group $E(\Delta, R)$ stabilizes the line spanned by the covector $g_{\lambda_0}$ modulo $I^+$. Since this group is perfect, it must stabilize the covector itself.

Further since $g \notin G(\Phi, \Delta, R, (I^+, R))$, there exists $\mu \in \Lambda \setminus \{\lambda_0\}$ such that $g_{\lambda_0,\mu} \notin I^+$. By Lemma 25, we have $\mu \in \Lambda_1$. Take $\tilde{\gamma} \in \Delta$ such that

$$\nu = \mu + \tilde{\gamma} \in \Lambda_1$$

(apply Lemma 8 for $\beta = \lambda_0 - \nu$).

Since the element $x_{\tilde{\gamma}}(1)$ stabilizes the covector $g_{\lambda_0}$, we obtain the following congruence:

$$(gx_{\tilde{\gamma}}(1))_{\lambda_0,\nu} \equiv g_{\lambda_0,\nu} \mod I^+.$$  

However, we also have

$$(gx_{\tilde{\gamma}}(1))_{\lambda_0,\nu} = g_{\lambda_0,\nu} \pm g_{\lambda_0,\mu}.$$  

This implies that $g_{\lambda_0,\mu} \in I^+$, which contradicts the choice of $\mu$.

Therefore, we have proved that for some $\gamma \in \Delta$ and some $t \in R$, we have

$$gx_\gamma(t)g^{-1} \notin G(\Phi, \Delta, R, (I^+, R)).$$

We also know that

$$gx_\gamma(t)g^{-1} = e + t(gx_\gamma(1)g^{-1} - e),$$  

which implies

$$gx_\gamma(1)g^{-1} \notin G(\Phi, \Delta, R, (I^+, R)).$$  

\qed

Proposition 6. Let $H$ be an overgroup of $E(\Delta, R)$ of level $\sigma = (I^+, I^-)$, and let $g \in H \cap \mathcal{R}(R)$. Then $g \in G(\Phi, \Delta, R, \sigma)$.

Proof. First, we reduce the proof to the case where one of the ideals in $\sigma$ is equal to zero. Assume that we can prove the proposition for such $\sigma$, and let us prove it in general case. Note that $\rho_{I^+}(g) \in \rho_{I^+}(H) \cap \mathcal{R}(R/I^+)$. By Corollary 3 we have:

$$\text{lev}(\rho_{I^+}(H)) = ((0), \rho_{I^+}(I^-)).$$
Hence by assumption
\[ \rho_{I^+}(g) \in G(\Phi, \Delta, R/I^+, \ldots) \leq P^- . \]
Similarly, we obtain \( \rho_{I^-}(g) \in P^- \), q.e.d.

Therefore, without loss of generality, we may assume that \( I^+ = 0 \). Now we prove the proposition in this case.

Suppose the contrary. Take \( \alpha \) from Lemma 34, applied to the element \( g^{-1} \), i.e.,
\[ g^{-1}(x_\alpha(1))g \notin G(\Phi, \Delta, R, \sigma). \]

Set
\[ M = \{ \lambda_1 \in \Lambda_1 : g_{\lambda_0, \lambda_1} \neq 0 \} . \]

For any \( \lambda_1 \in M \), set
\[ k(\lambda_1) = \min \{ k : (g_{\lambda_0, \lambda_1})^k = 0 \} \]
(by Proposition 3, we have \( k(\lambda_1) \leq 3 \)). Finally, set
\[ K = \sum_{\lambda_1 \in M} k(\lambda_1) . \]

We prove by induction on \( K \) that our assumption leads to a contradiction.

The base of induction: \( K = 0 \); this means that \( M = \emptyset \). Together with Item 1 of Lemma 25 this means exactly that \( g \in P^- \). Then \( g \in G(\Phi, \Delta, R, \sigma) \) because otherwise, by Item 2 of Proposition 2 the group \( H \) cannot be of level \( \sigma \).

Now we pass to the induction step. Assume that \( M \neq \emptyset \). Fix \( \lambda_1 \in M \). Set
\[ \mathfrak{B} = \begin{cases} \mathfrak{B}(g, \lambda_1) & k(\lambda_1) = 2 \\ \mathfrak{B}(g, \lambda_1)^2 & k(\lambda_1) = 3 \end{cases} \]

Then \( \mathfrak{B}^2 = (0) \).

**Case 1.** \( \rho_{\mathfrak{B}}(g) \notin G(\Phi, \Delta, R/\mathfrak{B}, \rho_{\mathfrak{B}}(\sigma)) \).

Note that \( \rho_{\mathfrak{B}}(g) \in \rho_{\mathfrak{B}}(H) \cap \mathfrak{B}(R/\mathfrak{B}) \). By Corollary 3 \( \text{lev}(\rho_{\mathfrak{B}}(H)) = \rho_{\mathfrak{B}}(\sigma) \), next, the number \( K \) for \( \rho_{\mathfrak{B}}(g) \) is less than that for \( g \). Applying the inductive hypothesis, we obtain a contradiction.

**Case 2.** \( \rho_{\mathfrak{B}}(g) \in G(\Phi, \Delta, R/\mathfrak{B}, \rho_{\mathfrak{B}}(\sigma)) \).

By Lemma 19
\[ (x_\alpha(1))^{\rho_{\mathfrak{B}}(g)} \in E(\Phi, \Delta, R/\mathfrak{B}, \rho_{\mathfrak{B}}(\sigma)). \]

The group \( E(\Phi, \Delta, R, \sigma) \) maps surjectively onto the group \( E(\Phi, \Delta, R/\mathfrak{B}, \rho_{\mathfrak{B}}(\sigma)) \). Hence there exists \( h \in E(\Phi, \Delta, R, \sigma) \) such that
\[ \rho_{\mathfrak{B}}(h) = (x_\alpha(1))^{\rho_{\mathfrak{B}}(g)} = \rho_{\mathfrak{B}}((x_\alpha(1))^g) . \]

Then we have
\[ g_1 = h^{-1}(x_\alpha(1))^g \in (H \cap G(\Phi, R/\mathfrak{B})) \setminus G(\Phi, \Delta, R, \sigma). \]

If \( g_1 \in P^- \), then the group \( H \) cannot be of level \( \sigma \) by Proposition 2, and otherwise it cannot by Proposition 4. This is a contradiction. \( \square \)
12. Finishing the proof of Theorem 1

Let \( H \) be an overgroup of \( E(\Delta, R) \) of level \( \sigma = (I^+, I^-) \). We want to prove that
\[
H \leq N_{G(\Phi, R)}(E(\Phi, \Delta, R, \sigma)).
\]

By Proposition 1, it suffices to prove that for any \( g \in H \) and any generator \( x_\alpha(\xi) \) of the group \( E(\Phi, \Delta, R, \sigma) \) we have \( g^{-1}x_\alpha(\xi)g \in G(\Phi, \Delta, R, \sigma) \). which is true indeed by Proposition 6.

13. Changing the lattice \( \mathcal{P} \)

In this section we generalise Theorem 1 to other weight lattices. First note that Theorem 1 and Corollary 2 imply the following fact.

**Corollary 4.** Let \( R \) be a finitely generated ring, and let \( H \) be an overgroup of \( E(\Delta, R) \) of level \( \sigma \). Then there exists a natural number \( N \) such that \( D^N H = E(\Phi, \Delta, R, \sigma) \).

Note also that Corollary 4 is a stronger statement than Theorem 1. Indeed, for finitely generated rings, Theorem 1 follows from Corollary 2 directly, and the general case can be derived by using the fact that any ring is an inductive limit of its finitely generated subrings. Let us transfer the result of Theorem 1 to other weight lattices in this stronger form.

Let \( \mathcal{P}' \) be another weight lattice between \( \Omega(\Phi) \) and \( \mathcal{P}(\Phi) \). We denote by \( G_{\mathcal{P}'}(\Phi, R) \) the corresponding Chevalley group. The subgroups \( E_{\mathcal{P}'}(\Delta, R) \) and \( E_{\mathcal{P}'}(\Phi, \Delta, R, \sigma) \) for it are defined similarly. In the case of \( \mathcal{P}' = \mathcal{P}(\Phi) \) we will use the subscript sc. Let \( C_{\text{sc}}(\Phi, R) \) be the center of the group \( G_{\text{sc}}(\Phi, R) \).

For any intermediate subgroup \( E_{\mathcal{P}'}(\Delta, R) \leq H \leq G_{\mathcal{P}'}(\Phi, R) \), one can similarly define its level.

**Proposition 7.** Corollary 4, and hence Theorem 1, can be transferred to the simply connected group.

**Proof.** Let \( H \) be an overgroup of \( E_{\text{sc}}(\Phi, R) \) of level \( \sigma = (I^+, I^-) \).

Consider the natural homomorphism
\[
\pi: G_{\text{sc}}(\Phi, R) \to G(\Phi, R).
\]

Obviously, we have \( \text{lev } \pi(H) \geq \sigma \). Let us prove the inverse inclusion. Assume that \( x_\beta(\xi) \in \pi(H) \), for some \( \beta \in \Omega^+ \) and \( \xi \in R \setminus I^+ \) (the proof for \( \Omega^- \) is similar). Then for some \( g \in C_{\text{sc}}(\Phi, R) \) we have \( gx_\beta(\xi) \in H \). Take \( \alpha \) from Lemma 8 then we have
\[
x_{\alpha + \beta}(\pm \xi) = [x_\alpha(1), x_\beta(\xi)] = [x_\alpha(1), gx_\beta(\xi)] \in H,
\]
which contradicts the assumption.
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Thus \( \lev \pi(H) = \sigma \). Hence for a large \( N \) the following inclusion holds
\[
D^N \pi(H) \leq E(\Phi, \Delta, R, \sigma).
\]

Therefore, we have the inclusion
\[
D^N H \leq C(\Phi, R) E(\Phi, \Delta, R, \sigma),
\]
which, in its turn, implies that
\[
D^{N+1} H \leq E(\Phi, \Delta, R, \sigma).
\]

\( \square \)

**Proposition 8.** Corollary \([4]\) and hence Theorem \([1]\) can be transferred to an arbitrary weight lattice \( \mathcal{P}' \).

**Proof.** Let
\[
\pi : G_{sc} \to G_{\mathcal{P}'}(\Phi, R)
\]
be the natural homomorphism.

The sequence
\[
1 \longrightarrow \text{Ker } \pi \longrightarrow G_{sc}(\Phi, -) \overset{\pi}{\longrightarrow} G_{\mathcal{P}'}(\Phi, -) \longrightarrow 1.
\]
is exact in the category of fpqc-sheaves. Hence for groups we have the exact sequence
\[
1 \longrightarrow \text{Ker } \pi \longrightarrow G_{sc}(\Phi, R) \overset{\pi}{\longrightarrow} G_{\mathcal{P}'}(\Phi, R) \longrightarrow H^1_{\text{fpqc}}(R, \text{Ker } \pi).
\]

\( \square \)

Therefore, \( \text{Im } \pi \) is a normal subgroup of \( G_{\mathcal{P}'}(\Phi, R) \) with an Abelian quotient (which can be embedded into the Abelian group \( H^1_{\text{fpqc}}(R, \text{Ker } \pi) \)).

Now let \( H \) be an overgroup of \( E_{\mathcal{P}'}(\Delta, R) \) of level \( \sigma \). Replacing \( H \) with its commutant, we may assume that \( H \subseteq \text{Im } \pi \). Clearly, we have \( \lev(\pi^{-1}(H)) = \sigma \). Then for a large \( N \) we have
\[
D^N \pi^{-1}(H) = E_{sc}(\Phi, \Delta, R, \sigma).
\]

Since \( H \leq \text{Im } \pi \), we have \( H = \pi(\pi^{-1}(H)) \), which implies the formula
\[
D^N H = \pi(D^N \pi^{-1}(H)) = \pi(E_{sc}(\Phi, \Delta, R, \sigma)) = E_{\mathcal{P}'}(\Phi, \Delta, R, \sigma).
\]

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