Revisiting canonical gravity with fermions

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Abstract

Fermions constitute an important component of matter and their quantization in presence of dynamical gravity is essential for any theory of quantum gravity. We revisit the classical formulation adapted for a background free quantization. The analysis is carried out with the Hilbert-Palatini form for gravity together with the Nieh-Yan topological term which keeps the nature of Barbero-Immirzi parameter independent of inclusion of arbitrary matter with arbitrary couplings. With dynamical gravity, a priori, there are two distinct notions of ‘parity’—orientation reversing diffeomorphisms and improper Lorentz rotations. The invariance properties of the action and the canonical framework are different with respect to these and gravitational origin of parity violation seems ambiguous.

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I. INTRODUCTION

Incorporation of fermions in the background free quantum theory gravity has been discussed in the literature [1]. In the formulation of general relativity in terms of real SU(2) connection, Thiemann discussed loop quantization of standard model fields [2]. The fermions were treated in the second order form i.e. fermions couple to gravity through the spin connection (torsion free Lorentz connection). Perez and Rovelli returned to fermions in presence of the Holst term and found that its coefficient playing the role of the inverse of the Barbero-Immirzi parameter, $\gamma$, becomes classically observable [3]. Mercuri [4] discovered that with a further addition of suitable non-minimal fermionic couplings, $\gamma$ can be made classically unobservable. He also noted that these added terms (Holst plus non-minimal) can be expressed as the Nieh-Yan topological term once the connection equations of motion are used. The strategy of adding non-minimal couplings to keep $\gamma$ classically unobservable was followed for $N = 1, 2$ and $4$ supergravities also [5]. Canonical analysis and loop quantization of fermions with non-minimal couplings was discussed by Bojowald and Das [6]. It was subsequently realised that $\gamma$ will automatically be classically unobservable provided it is the (inverse) of the coefficient of the Nieh-Yan term (a total divergence) in the Lagrangian density. Thus, instead of the Holst terms alone, if the Nieh-Yan term (Holst + torsion$^2$ piece) is used in conjunction with the Hilbert-Palatini, then for arbitrary matter and their couplings, the $\gamma$ will drop out of the classical equations of motion. Furthermore, it is possible to systematically derive the real SU(2) Hamiltonian formulation from such an action [7]. Fermions were also included in the canonical analysis. A necessary condition for a topological origin of $\gamma$ is thus satisfied. The canonical analysis leading to real SU(2) formulation has since been extended to supergravities [8] as well as further inclusion of the other two topological terms namely the Pontryagin and the Euler classes [9, 10].

Fermions are also tied with possible parity violations [6, 11]. There are two distinct notions of ‘parity’: one related to orientation of the space-time manifold and one related to the improper Lorentz transformation. Depending upon the definitions of the basic canonical variables (with or without the sgn($\epsilon$) factors in this work (section [V]), the canonical framework and the action are (non-)invariant under one of the notions of parity. These possibilities are not distinguished in the previous works. The work in [7] is a little incomplete in the constraint analysis although the final results are correct. The constraint expressions
are also not in a form which is suitable for loop quantization. This work seeks to fill these gaps.

We re-derive the real SU(2) formulation including a Dirac fermion. The analysis is done using restricted fields corresponding to ‘time gauge’. Elimination of the second class constraints, leads to the usual formulation. When a Dirac fermion is included, the solution of the second class constraint leads to non-trivial Dirac brackets between the SU(2) connection and the fermions. One can however make natural shifts in the definition of the connection to recover the canonical brackets. This also simplifies the constraints. Four Fermi interaction terms however survive in the Hamiltonian and are signatures of first order formulation. In the second order formulation where fermions couple to the torsion free connection, there are no terms quartic in the fermions.

The straightforward derivation also introduces factors of \( \text{sgn}(e) := \text{sign} \left( \det(e^i_\mu) \right) \) (which equals \( N \text{sign} \left( \det(V^i_a) \right) \) in the time gauge parametrization) in appropriate places. For instance, we are naturally lead to the definitions: \( E^a_i := \text{sgn}(e) \sqrt{q} V^a_i \), \( A^i_a := \text{sgn}(e) K^i_a + \eta \Gamma^i_a \) (eqn. (14)). Under an improper orthogonal transformation \((O(3))\) acting on the index \(i\), the triad changes sign and so does the \( \text{sgn}(e) \) factor leaving \( E^a_i \) invariant. This is as it should be since the index \(i\) on \(E\) represents adjoint representation of SO(3) while on \(V\) it represents the defining representation. For SO(3), both are equivalent but not for \(O(3)\). Under an inversion, \( \Lambda_{ij} = -\delta_{ij} \), quantities in the defining representation change sign while those in the adjoint don’t. The same reasoning applies to the definition of the connection. Now the connection is also even under inversion. The \( \text{sgn}(e) \) factor also change the behaviour of \( E^a_i \) and \( A^i_a \) under the action of orientation reversing diffeomorphisms. These factors of \( \text{sgn}(e) \) however occur only in the intermediate derivations, the final form of constraints and basic variables are independent of these factors.

The paper is organized as follows. Section II deals with the Hamiltonian formulation for purely gravitational sector starting with the Hilbert-Palatini plus the Nieh-Yan action. Here steps in the analysis are given and a gap in the constraint analysis in [7] is filled in. The constraints are presented in the more standard form. The sign factors which appear in the action itself, are followed through in all expressions. Section III deals with inclusion of a single Dirac fermion. Constraint analysis as well as simplification of constraints is given here. The section ends with the final form of the canonical formulation in the standard notation. Section IV discusses the two distinct notions of parities and the invariance properties of the
action and the canonical framework. Section VII contains brief concluding remarks.

II. THE HAMILTONIAN FORMULATION

The starting point is a choice of (tensor/spinor) fields and a corresponding generally covariant, local action on 4 dimensional space-time $M \simeq \mathbb{R} \times \Sigma_3$. The next step is to carry out a $3 + 1$ decomposition to identify the Lagrangian which is a function of (tensor) fields on $\Sigma_3$ together with their velocities with respect to the chosen time coordinate. The fields whose velocities appear in the Lagrangian are potentially the configuration space variables while those without velocities appearing in the Lagrangian are Lagrange multipliers whose coefficients will be primary constraints. This Lagrangian leads to the kinematical phase space. Now a constraint analysis a la Dirac is performed. If there are second class constraints, one may hope to simplify the analysis by solving the second class constraints. However, now one must use the Dirac brackets. These may not have the canonical form for the remaining variables (i.e. may not be Darboux coordinates) and a new choice of variables may be necessary. This is particularly relevant for Lagrangians which are linear in the velocities such as the Hilbert-Palatini-Nieh-Yan and the Dirac Lagrangians which typically do have primary, second class constraints. The classical Hamiltonian formulation is completed when the action is expressed in the Hamiltonian form together with first class constraints. We also have fields coordinatizing the kinematical phase space (after the second class constraints are eliminated) with the configuration space coordinates identified.

A. Pure gravity

We begin with the Lagrangian 4-forms built from the basic fields the co-tetrad $e^I_\mu dx^\mu$ and the Lorentz connection $\omega^{IJ}_\mu dx^\mu$.

\[
\mathcal{L}_{\text{HP}}(e, \omega) = \frac{1}{2\kappa} \left[ \text{sgn}(e) \frac{1}{2} \varepsilon^{ijkl} R^{ij}(\omega) \wedge e^k \wedge e^l \right] \tag{1}
\]

\[
\mathcal{L}_{\text{NY}}(e, \omega) = \left[ T^I(e, \omega) \wedge T_I(e, \omega) - R_{IJ}(\omega) \wedge e^I \wedge e^J \right] \tag{2}
\]

\[
\mathcal{L}_{\text{grav}} := \mathcal{L}_{\text{HP}} + \frac{\eta}{2\kappa} \mathcal{L}_{\text{NY}} \tag{3}
\]
Here $R := d\omega + \omega \wedge \omega$ and $T := de + \omega \wedge e$ are the usual curvature and torsion 2-forms\(^1\) and $\kappa = 8\pi G$. The factor of $\text{sgn}(e)$ is present because only then the Hilbert-Palatini Lagrangian matches with the $\sqrt{|g|} R(g)$. This arises from noting that determinant of the co-tetrad is given by, $e = \text{sgn}(e) \sqrt{|g|}$.

A $3+1$ decomposition can be expressed using,

\[
d := d_\perp + d_\parallel := dt \partial_t + dx^a \partial_a \\
e^I = e^I_\perp + e^I_\parallel, \quad \omega^{IJ} = \omega^{IJ}_\perp + \omega^{IJ}_\parallel \quad \text{etc} \quad \text{(4)}
\]

which leads to,

\[
\mathcal{L}_{\text{grav}} = \frac{1}{2\kappa} \left[ \Sigma^{IJ}_\parallel \wedge d_\perp \left( \text{sgn}(e)\tilde{\omega}^{IJ}_\parallel - \eta\omega^{IJ}_\parallel \right) + 2\eta T^{IJ}_\parallel \wedge d_\parallel e^{IJ} + 2\eta e^{I}_\parallel \wedge D_\parallel T^{IJ}_\parallel \right. \\
+ \left. \omega^{IJ}_\perp \wedge \left\{ D_\parallel \left( \text{sgn}(e)\Sigma^{IJ}_\parallel - \eta\Sigma^{IJ}_\parallel \right) - 2\eta e^{I}_\parallel \wedge T^{IJ}_\parallel \right\} \\
+ e^{I}_\parallel \wedge \left\{ e^{I}_\parallel \wedge \left( \text{sgn}(e)\tilde{\Sigma}^{IJ}_\parallel - \Sigma^{IJ}_\parallel \right) \right\} \right] + \text{surface terms} \quad \text{where,} \quad \text{(5)}
\]

$\Sigma^{IJ} := e^I \wedge e^J$, $D := d + \omega \wedge$ is the Lorentz covariant derivative and $e$ is the determinant of the co-tetrad, assumed to be non-zero. The $\tilde{}$ denotes the Lorentz dual defined for any quantity antisymmetric in a pair of Lorentz indices, $X^{IJ}$, as,

\[
\tilde{X}^{IJ} := \frac{1}{2} \varepsilon^{IJK} X_{KL}.
\]

A $3+1$ decomposition is carried out as usual by choosing a foliation defined by a time function $T : M \rightarrow \mathbb{R}$ and a vector field $t^\mu \partial_\mu$, transversal to its leaves. The vector field is normalised by $t \cdot \partial T = 1$ so that the parameters of its integral curves, serve as the time coordinate. Given such a decomposition, we choose a parametrization of the tetrad and the co-tetrad as,

\[
e^I_t = N n^I + N^a V^I_a, \quad e^I_a = V^I_a, \quad n^I n_I = -1, \quad n^I V^J_a \eta_{IJ} = 0; \quad \text{(6)}
\]

\[
e^I_t = -N^{-1} n^I, \quad e^I_a = N^{-1} n_I V^a + V^a_I, \quad n^I V^a_I = 0,
\]

with

\[
V^a_I V^J_a = \delta^J_I + n^J n^I, \quad V^a_I V^b_a = \delta^b_I. \quad \text{(7)}
\]

\(^1\) Our conventions are such that (i) $\eta = \gamma^{-1}$, our action matches with Ashtekar-Lewandowski (AL) action including signs and factors. Our identification of $K^i_a$ is same is that of AL while our $\Gamma^i_a$ is minus that of AL. To match with the standard notation we first change $A \rightarrow \eta A$ and then set $\eta = \gamma^{-1}$. This results in standard definition of $F^i_{ab}$, our $\mathcal{G}^i \rightarrow \gamma^{-1} \mathcal{G}^i_{std}$, the diffeomorphism and the Hamiltonian constraints remain unchanged. The symplectic form becomes $\gamma^{-1} E^i_a \partial_i A^a_b$. Therefore the Poisson brackets also become the standard ones.
Expressing the wedge products in terms of the components, using $dx^i \wedge dx^a \wedge dx^b \wedge dx^c := \mathcal{E}^{abc} dx^i \wedge dx^j \wedge dx^k$ and separating the $IJ$ sums into $(0i), (jk)$ sums, we write the gravitational Lagrangian density as,

$$L_{\text{grav}} = \frac{\mathcal{E}^{abc}}{2\kappa} \left[ 2V^0_b V^i_c \partial_i (\text{sgn}(e)\omega_{a0i} - \eta\omega_{a0i}) + V^i_b V^j_c \partial_i (\text{sgn}(e)\omega_{aij} - \eta\omega_{aij}) ight]$$

$$+ \omega_{0i} \left\{ D_a \left( \text{sgn}(e)\Sigma_{bc}^0 - \eta\Sigma_{bc}^0 \right) - D_a \left( \text{sgn}(e)\Sigma_{bc}^0 - \eta\Sigma_{bc}^0 \right) \right\}$$

$$+ \omega_{ij} \left\{ D_a \left( \text{sgn}(e)\Sigma_{bc}^j - \eta\Sigma_{bc}^j \right) \right\}$$

$$+ N \left\{ n^0 \left( V^i_a \left( \text{sgn}(e)\tilde{R}_{qibc} - \eta\tilde{R}_{qibc} \right) \right) + n^i \left( -V^0_a \left( \text{sgn}(e)\tilde{R}_{qibc} - \eta\tilde{R}_{qibc} \right) + V^j_i \left( \text{sgn}(e)\tilde{R}_{ijbc} - \eta\tilde{R}_{ijbc} \right) \right) \right\}$$

$$+ N' \left\{ -V^0_i \left( \text{sgn}(e)\tilde{R}_{qibc} - \eta\tilde{R}_{qibc} \right) \right\} + V^j_i \left( \text{sgn}(e)\tilde{R}_{ijbc} - \eta\tilde{R}_{ijbc} \right) \right\}$$

$$+ \frac{1}{\kappa} \left[ T^{ai} \partial_i V_{aI} + \omega_{0i} \left\{ -V^0_i T^{ai} + V^j_i T^{aj} \right\} \right]$$

$$+ N \left\{ n^0 \left( (D_a T^0) + n^i \left( D_a T^i \right) \right) + N' \left\{ V^0_i \left( D_a T^0 \right) + V^j_i \left( D_a T^j \right) \right\} \right\}$$

(8)

The terms in the last two lines, come from the torsion piece of the Nieh-Yan term. We have also defined,

$$T^{ai} := \frac{\eta}{2} \mathcal{E}^{abc} T^{bc}_a = \frac{\eta}{2} \mathcal{E}^{abc} \left( \partial_b V^i_c - \partial_c V^i_b + \omega_{b}^{ij} V^j_c - \omega_{c}^{ij} V^j_b \right)$$

(9)

We will now restrict to configurations such that $n_i = 0, n_0 = -1$. This also implies that $V^0_a = 0 = V^a_0$ and that $V^i_a$ are invertible with $V^i_a$ as the inverse. We also define the 3-metric $q_{ab} := V^i_a V^j_b \delta_{ij}$ (which is positive definite in classical theory) and denote $q := \text{det}(q_{ab})$. Many terms in the above equation drop out. In particular, there isn’t any time derivative of $V_{a0}$ in the last square bracket. We also promote the torsion components $T^{ai}$ to new independent variables $\tilde{T}^{ai}$ and $V^i_a$ to $\hat{V}^i_a$, for the terms in the last square bracket. The number of variables is restored back by introducing two primary constraints: $\nu^i_a := \hat{V}^i_a - V^i_a \approx 0$ and $\tau^a_i := \tilde{T}^a_i - T^a_i \approx 0$ with $\xi^a_i, \phi^i_a$ as the corresponding Lagrange multipliers. Note that $T^{a0}$ is not promoted to a new variable. This simplifies the Lagrangian to,

$$L_{\text{grav}} = \frac{\mathcal{E}^{abc}}{2\kappa} \left[ V^i_b V^j_c \partial_i (\text{sgn}(e)\omega_{aij} - \eta\omega_{aij}) \right]$$

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2 This would correspond to the choice of the so-called time gauge if we started without restricting the configurations a priori.
\[ + \omega_{0i} \left\{ D_a \left( \text{sgn}(e) \tilde{\Sigma}^0_{bc} - \eta \Sigma^0_{bc} \right) - D_a \left( \text{sgn}(e) \tilde{\Sigma}^0_{bc} - \eta \Sigma^0_{bc} \right) \right\} \\
+ \omega_{ij} \left\{ D_a \left( \text{sgn}(e) \tilde{\Sigma}^ij_{bc} - \eta \Sigma^ij_{bc} \right) \right\} \\
+ N \left\{ V^i_a \left( \text{sgn}(e) \tilde{\Sigma}^0_{0ibc} - \eta \Sigma^0_{0ibc} \right) \right\} \\
+ N^a \left\{ V^i_a V^j_a \left( \text{sgn}(e) \tilde{\Sigma}^ij_{0ibc} - \eta \Sigma^ij_{0ibc} \right) \right\} \\
+ \frac{1}{\kappa} \left[ \tilde{T}^m_i \partial_i \hat{V}^m - \xi^m_i (\hat{V}^m_i - V^m_i) - \delta^m_i (\tilde{T}^a_i - T^a_i) \right] \\
+ \omega_{0i} \left\{ \hat{V}^i_a T^a_0 \right\} + \omega_{ij} \left\{ -\tilde{V}^i_a T^a_0 \right\} \\
+ N \left\{ (D_a T^0) \right\} + N^a \left\{ \tilde{V}^i_a (D_a T^a_i) \right\} \right\} \] (10)

At this stage it is convenient to introduce new notations for certain combinations of the components of the Lorentz connection as well as those of the (co)tetrad. These are,

\[ E^a_i := e^i_0 e^a_0 - e^i_0 e^a_0 \Rightarrow E^a_i = \text{sgn}(e) \sqrt{q} V^a_i \] (11)

\[ K^i_a := \omega^a_0 i , \quad \Gamma^i_a := \frac{1}{2} \varepsilon_{ijk} \omega^j_k \Rightarrow \] (12)

\[ \omega_{aij} = \varepsilon_{ijk} \Gamma^k_a \ , \quad \omega_{a0i} = \Gamma^i_a \ , \quad \omega_{aij} = \varepsilon_{ijkl} K^l_a \] (13)

and we have used \( e = N \det(V^i_a) = N \text{sgn}(e) \sqrt{q} \). Note that \( N > 0 \) implies that \( \text{sgn}(e) = \text{sgn}(\det(V^i_a)) \) and therefore we continue to use \( \text{sgn}(e) \) every where.

With these, the first velocity term becomes, \( 2E^a_i \partial_i (\text{sgn}(e) K^i_a - \eta \Gamma^i_a) \) suggesting the identification of a canonical pair, \( (A^i_a, E^a_i) \) with

\[ A^i_a := \text{sgn}(e) K^i_a - \eta \Gamma^i_a \] (14)

The velocity terms become,

\[ (\mathcal{L}_{\text{grav}})_{\text{velocity}} = \frac{1}{\kappa} \left\{ E^a_i \partial_i A^i_a + \tilde{T}^a_i \partial_i \hat{V}^i_a \right\} \] (15)

At this stage we have 10 Lagrange multiplier fields, \( N, N^a, \omega_{IJ} \) and 18 canonical variables, \( A^i_a, E^a_i \) while the co-tetrad and the Lorentz connection constitute \( 13 + 24 = 37 \) fields. Nine more fields are yet ‘unaccounted’. If we define

\[ \tilde{\omega}^a_{IJ} := \omega^a_{IJ} + \eta \tilde{\omega}^a_{IJ}, \]

we can see that, \( A^i_a = \tilde{\omega}^{0i}_a \).
The nine remaining variables, $M_{kl}, \zeta^k$, are identified through the definition\(^3\),
\[
\omega^a_{ij} := \frac{\text{sgn}(e)}{2} (E^i_j \zeta^j - E^j_i \zeta^i) + \frac{1}{2} \epsilon^{ijk} M_{kl} E^i_a, \quad M_{kl} = M_{lk}, \quad E^a_i E^b_j = \delta^b_a, \quad E^a_i E^a_j = \delta^j_i.
\]
(16)
The 18 components of $\omega^a_{IJ}$ are organized as the 9 $A^i_a$, the 3 $\zeta^i$ and the 6 $M_{kl}$. The components $K, \Gamma$ of the Lorentz connection $\omega$ can be expressed in terms of the $A, \zeta$ and $M$ using the inverse formula,
\[
\omega^a_{IJ} = \frac{1}{1 + \eta^2} \left( \text{sgn}(e) \omega^a_{IJ} - \frac{\eta}{2} \epsilon^{IJK} \omega^a_{KL} \right).
\]
The $K, \Gamma$ are given in terms of the independent canonical variables $A, \zeta, M$ as,
\[
K^i_a = \frac{1}{1 + \eta^2} \left\{ \text{sgn}(e) A^i_a + \frac{\eta}{2} \left( M^j_a E^j_i + \text{sgn}(e) \epsilon^i_{jk} E^j_a \zeta^k \right) \right\}, \quad \Gamma^i_a = \frac{1}{\eta} (\text{sgn}(e) K^i_a - A^i_a).
\]
(17)
For later convenience, we eliminate $\Gamma$ in favour of $K, A$. We get,
\[
\mathcal{L}_{\text{grav}} = \frac{1}{\kappa} \left[ E^a_i \partial_t A^i_a + \hat{T}^a_i \partial_i \hat{V}^a_i \right. \\
- \xi^i_a v^a_i - \phi^i_a \tau^a_i - \Lambda_i G^0i - \lambda^k G^k - N \mathcal{H} - N^a \mathcal{H}_a \bigg], \quad (18)
\]
\[
k^i_a := \hat{V}^i_a - V^i_a, \quad \kappa^i_a := \hat{T}^a_i - T^a_i, \quad (19)
\]
\[
\kappa G^{0i} := - \frac{1}{2} \epsilon^{abc} \left[ D_a \left( \text{sgn}(e) \hat{\Sigma}^{0i} - \eta \Sigma^{0i} \right) - D_a \left( \text{sgn}(e) \hat{\Sigma}_{bc}^{0i} - \eta \Sigma_{bc}^{0i} \right) \right] - \hat{V}^i_a T^{a0}, \quad (21)
\]
\[
\kappa G^k := \frac{1}{2} \epsilon^{kij} G^{ij} := - \epsilon^{kij} \left[ \frac{\epsilon^{abc}}{2} D_a \left( \text{sgn}(e) \hat{\Sigma}_{bc}^{ij} - \eta \Sigma_{bc}^{ij} \right) - \hat{V}^i_a \hat{T}^{aj} \right], \quad (22)
\]
\[
\kappa H^i_a := - V^a_i \left[ \frac{\epsilon^{abc}}{2} V^j_a \left( \text{sgn}(e) \hat{R}_{ijbc} - \eta R_{ijbc} \right) + D_a \hat{T}^a_i \right] \quad , \quad (23)
\]
\[
\kappa H := - \left[ \frac{\epsilon^{abc}}{2} V^a_i \left( \text{sgn}(e) \hat{R}_{0abc} - \eta R_{0abc} \right) + D_a T^{a0} \right], \quad (24)
\]
In anticipation, we will refer to the $G^{0i}$ as the boost constraints, $G^k$ as the rotation constraints, $H^i_a$ as the diffeomorphism constraints and $H$ as the Hamiltonian constraint although at this stage the interpretation of the transformations generated by these is ambiguous. The other two terms in the second line are the two additional primary constraints which identify the canonical coordinates $\hat{T}, \hat{V}$ with the torsion and the co-tetrad components respectively.

\(^3\) It is possible to bypass this decomposition and work directly with $K^i_a$ and $A^i_a$ as independent canonical variables. The conjugate momenta of $M_{ij}, \zeta^i$ will be subsumed by the conjugate momenta of $K^i_a$ \cite{10}. We will however continue to use these variables.
We have also used $\omega_{ti} := \Lambda_i$, $\omega_{ij} := \mathcal{E}_{ijk} \lambda^k$. Straightforward algebra then leads to\footnote{We set $\kappa = 1$ for notational convenience. It will be restored back later.},

\[
G^{0i} = \text{sgn}(e) \partial_a E^{ai} - \text{sgn}(e) \zeta^i - \hat{V}_a^i T_{a0}^0
\]

\[
G_k = \eta \partial_a E^a_k + \mathcal{E}_{kij} A^i_a E^{aj} + \mathcal{E}_{kij} \hat{V}^i_a T_{aj}^0
\]

\[
\mathcal{H}_{d'} := \eta A^i_b E^b_i + \left( \frac{1 + \eta^2}{\eta} \right) V^i_a K^b_b K^c_e V^{i}j \mathcal{E}^{bec}
\]

\[
- V^i_a \left\{ \partial_a \hat{T}^a_i - K_a^i T_{00}^a + \eta^{-1} \mathcal{E}_{ijk} \hat{T}^{aj} (\text{sgn}(e) K^k_a - A^k_a) \right\} \quad \text{where,}
\]

\[
F_{ab} := \frac{\partial A^i_b - \partial_b A^i_a}{\eta} + \frac{\mathcal{E}^{i,jk} A^j_a A^k_b}{\eta^2},
\]

\[
\mathcal{H} := -(\partial_a T^a_0 - K_a^i T^a_i) + \frac{E^b_k E^c_l}{\sqrt{\eta}} \left\{ \mathcal{E}^{i,jk} \eta (\partial_b K^j_a - \partial_a K^i_b) \right\}
\]

As there are no velocities of $M_{ij}, \zeta^k$ variables, we have their conjugate momenta $\pi^{ij} \approx 0 \approx \pi_k$ as additional primary constraints. In Dirac’s terminology, the total Hamiltonian will be the minus terms in the second line of eqn (18) plus linear combinations of $\pi_i \approx \pi_{ij} \approx 0$. All these are to be preserved during an evolution and this leads to secondary constraints and determination of some Lagrange multipliers.

Consider the preservation of the constraints $\pi_i, \pi_{ij}, v^i_a, \tau^a_i \approx 0$. This leads to the equations:

\[
\{ \pi_k, H_{tot} \} \approx 0 \approx \int \left[ \Lambda_i \frac{\delta G^{0i}}{\delta \zeta^k} + \cdots \right]
\]

\[
\{ \pi_{kl}, H_{tot} \} \approx 0 \approx \int \left[ \Lambda_i \frac{\delta G^{0i}}{\delta M^{kl}} + \frac{\lambda^i}{\delta M^{kl}} + N^a \frac{\delta H_a}{\delta M^{kl}} + N \frac{\delta H}{\delta M^{kl}} - \phi^i_a \frac{\delta T^a_i}{\delta M^{kl}} + \zeta^a \times \right]
\]

\[
\{ v^i_a, H_{tot} \} \approx 0 \approx \int \left\{ v^i_a, \tau^a_i \right\} \phi^i_a + \left\{ v^i_a, \int \Lambda_j G^{0j} + \lambda^i \mathcal{G}_j + N^b \mathcal{H}_b + N H \right\}
\]

\[
\{ \tau^a_i, H_{tot} \} \approx 0 \approx \int \left\{ \tau^a_i, v^i_a \right\} \xi^b + \cdots
\]
remain true also in the presence of fermionic matter. Let us use the abbreviation \( \hat{H} := \int N^a \mathcal{H}_a + N \mathcal{H} \). Then, using equation \((32)\), we obtain,

\[
\frac{\phi_i(x)}{1 + \eta^2} = - \left\{ v^i_a(x), \int \Lambda_i \mathcal{G}^{bi} + \lambda^i \mathcal{G}_b \right\} - \left\{ v^i_a(x), \hat{H} \right\}
\]

(34)

Now,

\[
\{ v^i_a, X \} = \{ \dot{V}^i_a - V^i_a, X \} = \frac{\delta X}{\delta T^a_i} + \frac{\text{sgn}(e)}{\sqrt{q}} \left( \frac{V^i_a V^k}{} - 2 \right) \frac{\delta X}{\delta A^k} \quad \forall X.
\]

It is easy to see that for \( X = \int \Lambda \cdot \mathcal{G} + \lambda \cdot \mathcal{G} \), the Poisson bracket with \( v^i_a \) is \textit{weakly zero}, with or without matter (fermionic). Using these, the equation \((31)\) becomes,

\[
0 \approx \frac{\delta \hat{H}}{\delta M^{kl}} + (1 + \eta^2) \left( \frac{\delta T^a_i}{\delta M^{kl}} \right) \left\{ \frac{\delta}{\delta T^a_i} + \frac{\text{sgn}(e)}{\sqrt{q}} \left( \frac{V^i_a V^j}{} - 2 \right) \frac{\delta}{\delta A^j} \right\} \hat{H},
\]

(35)

\[
\frac{\delta T^a_i}{\delta M^{kl}} = \frac{\eta \text{sgn}(e)}{2(1 + \eta^2)} \left( \delta_{ik} V^a_k + \delta_{il} V^a_l - 2 \delta_{kl} V^a_i \right)
\]

and

\[
\frac{\delta \hat{H}}{\delta T^a_i} = \partial_a (N^b V^i_b) + \mathcal{E}^{jk}_a V^j_a N^b \Gamma^k_a + NK^i_a
\]

(36)

(37)

\[
0 \approx \left[ \frac{\delta}{\delta M^{kl}} - \frac{\eta \text{sgn}(e)}{2} \left( E_{kl} \delta^i_j + E_{il} \delta^j_k \right) \frac{\delta}{\delta A^j} \right] \hat{H} + \frac{\eta \text{sgn}(e)}{2} \left( \delta_{ik} V^a_k + \delta_{il} V^a_l - 2 \delta_{kl} V^a_i \right) \left\{ \partial_a (N^b V^i_b) + \mathcal{E}^{jk}_a V^j_a N^b \Gamma^k_a + NK^i_a \right\}
\]

(38)

The second line of the last equation is independent of matter contribution. These appear only in the first line of that equation. Noting that the \( M^{kl} \) dependence in \( \hat{H} \) appears \textit{only through} \( K^i_a \), we can trade derivative w.r.t. \( M^{kl} \) to that with respect to \( K^i_a \). Furthermore, the \( A^i_a \) dependence is both \textit{explicit} as well as \textit{implicit through} \( K^i_a \). This simplifies equation \((38)\) to,

\[
(E_{ak} \delta^i_j + E_{al} \delta^i_j) \frac{\delta \hat{H}}{\delta A^j_a} \approx \left( \delta_{ik} V^a_k + \delta_{il} V^a_l - 2 \delta_{kl} V^a_i \right) \left\{ \partial_a (N^b V^i_b) + \mathcal{E}^{jk}_a V^j_a N^b \Gamma^k_a + NK^i_a \right\}
\]

(39)

On the left hand side, the \( \hat{\delta} \) signifies that only the \textit{explicit} dependence on \( A^i_a \) is to be picked up. The right hand side is independent of matter contributions. The sgn factors cancel out.

A somewhat lengthy but straight forward computation yields,

\[
0 \approx \frac{N}{\eta^2} \mathcal{S}_{kl} + \frac{N_a}{\eta} \left( E_{ak} \mathcal{G}_l + E_{al} \mathcal{G}_k \right)
\]

where,

\[
\mathcal{S}_{kl} := \frac{1}{1 + \eta^2} \left\{ \text{sgn}(e) (M^{kl} - \delta_{kl} M^i_i) - \eta (A_{ak} E^a_l + A_{al} E^a_k - 2 \delta_{kl} A^i_a E^a_i) \right\}
\]

\[
+ \mathcal{E}^{abc} (V_{ak} \partial_b V_{cl} + V_{al} \partial_b V_{ck})
\]

(40)

(41)
All the dependence on the derivatives of the lapse and shift variables disappears. The secondary constraint is just \( S_{kl} \approx 0 \):

\[
0 = \frac{\eta}{1 + \eta^2} \{ \text{sgn}(\epsilon) M^{ij} - \eta \left( A^i_a E^{aj} + A^j_a E^{ai} \right) - \delta^{ij} \left( \text{sgn}(\epsilon) M^k_k - 2\eta A^k_a E^k_a \right) \} \\
+ \eta E^{abc} \left( V^i_a \partial_b V^j_c + V^j_a \partial_b V^i_c - \delta^{ij} V^k_a \partial_b V^k_c \right) \\
\Rightarrow \quad \text{sgn}(\epsilon) M^{ij} = -(1 + \eta^2) E^{abc} \left\{ V^i_a \partial_b V^j_c + V^j_a \partial_b V^i_c - \delta^{ij} V^k_a \partial_b V^k_c \right\} + \eta \left( A^i_a E^{aj} + A^j_a E^{ai} \right) \quad (42)
\]

When fermionic matter is added, this constraint gets modified and is discussed in the next section.

There are no further tertiary constraints.

From eqn. (30), we notice that \( (\pi_k, G^{0i}) \) forms a second class pair of constraints, one of which is a canonical variable. Defining a Dirac bracket relative to this pair will allow us to set these constraints strongly equal to zero. Because \( \pi_k \) is a canonical variable, the Dirac brackets of variables other than \( \pi_k, \zeta^i \), among themselves, coincide with the corresponding Poisson brackets. This follows by noting (schematically),

\[
\Delta := \{ \pi, G(q^i, p_j, \pi, \zeta) \} \\
\{ f(q^i, p_j, \pi, \zeta), g(q^i, p_j, \pi, \zeta) \}_* := \{ f, g \} + \{ f, \pi \} \Delta^{-1} \{ G, g \} - \{ f, G \} \Delta^{-1} \{ \pi, g \} \\
\Rightarrow \quad \{ q^i, p_j \}_* = \{ q^i, p_j \} \quad , \quad (44)
\]

Since there is no explicit \( \pi_k \) dependence in any of the remaining constraints, these expressions remain the same. We impose \( G^{0i} = 0 \) strongly and eliminate \( \zeta^i = \partial_a E^{ai} - \dot{V}_a T^{0a} \).

From eqn. (A21) of the appendix, this sets \( T^{ai} = \frac{1}{2} S^{ij} V_j^a \).

Similarly, \( (\pi_{kl}, S^{ij}) \) form a second class pair hence we can define Dirac brackets relative to these and impose these strongly. The constraints \( \pi_{kl} \) being canonical variables, the Dirac brackets among the remaining canonical variables remain the same as their Poisson brackets. Once again, setting \( \pi_{ij} = 0 \) changes no expressions but \( S^{ij} = 0 \) gives \( M^{ij} \) in terms of \( A^i_a \) and \( E^a_i \). Also \( S^{ij} = 0 \) implies that \( T^{ia} = 0 \) and sets the \( \tau^a_i = \dot{T}^a_i \approx 0 \).

But now we can define Dirac brackets relative to the \( (v^i_a, \tau^b_j) \) second class pair. In this \( \tau^a_i \) is now a canonical variable and therefore the Dirac brackets among the remaining variables, \( A^i_a, E^b_j \), remain equal to their Poisson brackets.

When fermionic matter is included, the first two steps of elimination of second class constraints will remain the same but with non-zero torsion and this will lead to non-trivial Dirac brackets among the final set of variables in the third step.
Note: Solving $S_{ij} = 0$ and $\mathcal{G}^{0i} = 0$ strongly, determines $M_{ij}$ and $\zeta^k$ in terms of $A, E$. Substitution in the expression for $\Gamma_a^i$ (eqn. 17), leads to,

$$
\Gamma_a^i = -\frac{\varepsilon_{bde}}{2} E_{aj} (V_b^i \partial_a V_d^j + V_b^j \partial_a V_d^i - \delta_{ij} V_b^k \partial_a V_d^k) + \frac{\varepsilon_{ijk}}{2} E_a^j \partial_b E^{bk}.
$$

Using the identities, $\varepsilon^{ijk} E_k^a = \delta^{abc} V_i^a V_j^b$ and $\varepsilon^{cab}(V_i^a V_j^b - V_i^b V_j^a) = \varepsilon^{cab}\varepsilon^{ij} E_k^m V_m^b V_n^a$, one can see that this is precisely the the usual expression for $\Gamma_a^i$ in terms of the triad alone,

$$
\Gamma_a^i = -\frac{\varepsilon^{ijk}}{2} V_k^b \{ \partial_b V_{aj} - \partial_a V_{bj} + V_{c}^j V_{l}^a \partial_b V_{cl} \}. \quad (45)
$$

Thus, in the pure gravity case, we get the Hamiltonian formulation in terms of the $A_a^i, E_j^b$ variables with the original Poisson brackets, we are left with only the rotation, the diffeomorphism and the Hamiltonian constraints and the total Hamiltonian is made up of these alone. It remains to simplify the expressions for these constraints.

B. Simplification of Constraints

After the second class constraints are imposed, we get $T^{0a} = -\text{sgn}(e) \eta V_a^i \mathcal{G}^i \approx 0$, $\bar{T}_a^i = T_a^i \approx 0$ and $\dot{V}_a^i = V_a^i$. The boost constraint, eqn. (25), is strongly zero and the rotation constraint, eqn (26), takes the usual form:

$$
\mathcal{G}^i = \eta \partial_a E^{ai} + \varepsilon^{ijk} A_a^j E^{ak}. \quad (46)
$$

Consider the diffeomorphism constraint. The second term in equation (27) simplifies as,

$$
\frac{1 + \eta^2}{\eta} V_a^i K_{ib} K_{jc} V_a^j \varepsilon^{bca} = \left( 1 + \frac{\eta^2}{\eta} \right) V_a^i K_{ib} K_{jc} \text{sgn}(e) \frac{E_k^b E_i^c}{\sqrt{q}} \varepsilon^{jkl} = \left( 1 + \frac{\eta^2}{\eta} \right) V_a^i \text{sgn}(e) K_{ib} E_k^b \left( -\varepsilon^{jkl} K_{jb} E_l^b \right)
$$

$$
= -\left( 1 + \frac{\eta^2}{\eta} \right) V_a^i K_{ib} V_k^b \left[ \text{sgn}(e) \left( \eta \xi^k + \varepsilon^{kjl} A_{bj} E_l^b \right) \right]
$$

$$
= -\left( 1 + \frac{\eta^2}{\eta} \right) V_a^i K_{ib} V_k^b \varepsilon^{k} \text{sgn}(e) \left( \eta F_{aj}^b E_i^b \right)
$$

$$
\therefore \mathcal{H}_a = \eta F_{aj}^b E_i^b - \frac{1}{\eta} V_a^i K_{ib} V_k^b \varepsilon^{k} \text{sgn}(e) \approx \eta F_{aj}^b E_i^b \quad (48)
$$

In the last but one line we have used the equation (A23).
Next consider the Hamiltonian constraint (eqn. (29)), after using the expressions for torsion.

\[ \mathcal{H} := \frac{1}{2} \frac{E^b_j E^c_k}{\sqrt{q}} \left[ \mathcal{E}^j_k F^l_{bc} \right] = \partial_d \left( \eta V^d_i \mathcal{G}^i \right) + \left( 1 + \eta^2 \right) \left( \frac{1}{2} E^b_j E^c_k \right) \left[ (K^j_b K^k_c - K^j_c K^k_b) \right. \\
- \eta \left. \left( \mathcal{E}^j_k \{ \eta (\partial_b K^l_c - \partial_c K^l_b) + \mathcal{E}^l_m (A^m_b K^l_c - A^m_c K^l_b) \} \right) \right] \] (49)

The \( K \)-dependent terms can be written as (after taking out the \((1 + \eta^2)/(2\eta^2)\) factor) and using \( E^j_i E^i_k \eta^{jkl} = \eta \mathcal{E}^j_i \mathcal{E}^j_k \mathcal{E}^j_l \mathcal{E}^j_m \mathcal{E}^j_n \mathcal{E}^j_p \mathcal{E}^j_q \mathcal{E}^j_r \mathcal{E}^j_s \mathcal{E}^j_t \mathcal{E}^j_u \mathcal{E}^j_v \mathcal{E}^j_w \mathcal{E}^j_x \mathcal{E}^j_y \mathcal{E}^j_z \) factor and \( \delta \mathcal{E}^j_i \mathcal{E}^j_k \mathcal{E}^j_l \mathcal{E}^j_m \mathcal{E}^j_n \mathcal{E}^j_p \mathcal{E}^j_q \mathcal{E}^j_r \mathcal{E}^j_s \mathcal{E}^j_t \mathcal{E}^j_u \mathcal{E}^j_v \mathcal{E}^j_w \mathcal{E}^j_x \mathcal{E}^j_y \mathcal{E}^j_z \) factor, we get,

\[ \mathcal{E}^j_k K^j_c E^c_k = \frac{\eta}{\eta^2} \left[ \mathcal{E}^j_k A^j_c E^c_k + \frac{\eta}{2} \mathcal{E}^j_k \mathcal{E}^l_m \mathcal{E}^l_n \mathcal{E}^l_p \mathcal{E}^l_q \mathcal{E}^l_r \mathcal{E}^l_s \mathcal{E}^l_t \mathcal{E}^l_u \mathcal{E}^l_v \mathcal{E}^l_w \mathcal{E}^l_x \mathcal{E}^l_y \mathcal{E}^l_z \right] \\
= -\frac{\eta}{\eta^2} \left[ \mathcal{E}^j_k A^j_c E^c_k + \eta \mathcal{E}^c_j \mathcal{E}^j_c \mathcal{E}^j_l \mathcal{E}^j_m \mathcal{E}^j_n \mathcal{E}^j_p \mathcal{E}^j_q \mathcal{E}^j_r \mathcal{E}^j_s \mathcal{E}^j_t \mathcal{E}^j_u \mathcal{E}^j_v \mathcal{E}^j_w \mathcal{E}^j_x \mathcal{E}^j_y \mathcal{E}^j_z \right] = -\eta \mathcal{E}^j_k \mathcal{E}^j_c \mathcal{E}^j_l \mathcal{E}^j_m \mathcal{E}^j_n \mathcal{E}^j_p \mathcal{E}^j_q \mathcal{E}^j_r \mathcal{E}^j_s \mathcal{E}^j_t \mathcal{E}^j_u \mathcal{E}^j_v \mathcal{E}^j_w \mathcal{E}^j_x \mathcal{E}^j_y \mathcal{E}^j_z \] (50)

Using the \( (12) \) equation for \( M_{ij} \) and the boost constraint for eliminating \( \zeta^i \), we get,

\[ \text{sgn}(e) K^i_a E^{aj} - A^i_a E^{aj} = -\frac{\eta}{2} \mathcal{E}^{abc} \left\{ 2V^i_a \partial_b V^j_c - \delta^i_j V^k_c \partial_b V^k_c \right\}. \]

Writing \( A^i_a E^{am} = (A^i_a E^{am} \text{sgn}(e) K^i_a E^{am}) + \text{sgn}(e) K^i_a E^{am} \) in the 3rd term, the algebraic \( K \)-dependence simplifies to,

\[ -E^b_j E^c_k \left( K^j_b K^k_c - K^j_c K^k_b \right). \]

Combining all the terms, the Hamiltonian constraint takes the simplified form,

\[ \mathcal{H} := \frac{1}{2} \frac{E^b_j E^c_k}{\sqrt{q}} \left[ \mathcal{E}^j_k F^l_{bc} - \frac{1 + \eta^2}{\eta^2} \left( K^j_b K^k_c - K^j_c K^k_b \right) \right] + \frac{1}{\eta} \partial_d \left( \text{sgn}(e) V^d_i \mathcal{G}^i \right). \] (51)

The total derivative term differs from eqn. 2.24 of [12], because that expression is derived from the Holst action while ours is derived from the Hilbert-Palatini-Nieh-Yan action.

With this we recover the usual form of the constraints (in the time gauge) starting from the Hilbert-Palatini action with Nieh-Yan terms added. In the next sub-section we add a Dirac fermion minimally coupled to gravity.
III. ADDITION OF A DIRAC FERMION

The Lagrangian we begin with is,

\[ \mathcal{L}_{\text{Dirac}} = -\frac{i}{2} \bar{\lambda} \gamma^I D_\mu(\omega, A, \ldots) \lambda - D_\mu(\omega, A, \ldots) \bar{\lambda} \gamma^I \lambda \]  

(52)

\[ D_\mu(\omega) \lambda := \partial_\mu \lambda + \frac{1}{2} \omega_\mu \lambda + i e' A_\mu \lambda + \ldots \lambda \]

(53)

The \ldots refer to possible couplings of the Dirac fermion to other gauge fields e.g. the Maxwell field. These are suppressed in the following. The conventions for the \( \gamma^I \) matrices (space-time independent) are given in the appendix A 1. The factor in front of the Dirac Lagrangian is \(-i\) because our metric signature is \((- + + +)\). This is crucial for the Dirac brackets and subsequent passage to quantization via ‘Quantum brackets = \( i \hbar \) Dirac brackets’ rule, with the quantum brackets being realised on a Hilbert space.

For future convenience we introduce \( \Psi := q^{1/4} \lambda, \quad \Psi^\dagger := q^{1/4} \lambda^\dagger \). This absorbs away the \( \sqrt{q} \) factors in the Lagrangian as well as in the constraints. Note that the terms involving the derivatives of \( \sqrt{q} \) cancel out. The \( \lambda \) fermionic variables being of density weight zero, the \( \Psi \) fermionic variables are of density weight 1/2. From now on we will use the half density variables.

Substituting the 3 + 1 parametrization of the tetrad and using the time-gauge, the Lagrangian can be written as,

\[ \mathcal{L}_{\text{Dirac}} = \frac{i}{2} (\bar{\Psi} \gamma^0 \partial_t \Psi - \partial_t (\Psi^\dagger) \Psi - \frac{1}{2} \omega_{IJ} \mathcal{G}^{IJ}_F - N^{a'} \mathcal{H}^F_{a'} - N \mathcal{H}_F \]  

where,

\[ \mathcal{G}^{0i}_F = 0 \quad , \quad \mathcal{G}^F_i = -\frac{i}{2} \epsilon_{ijk} \Psi^\dagger \sigma^{jk} \Psi \]

(54)

\[ \mathcal{H}^F_{a'} = -\frac{i}{2} (\bar{\Psi} \gamma^0 D_{a'} \Psi - D_{a'} \bar{\Psi} \gamma^0 \Psi) \quad , \quad \mathcal{H}_F = \frac{i}{2} V^a_i \left( \bar{\Psi} \gamma^i D_a \Psi - D_a \bar{\Psi} \gamma^i \Psi \right) \]

(55)

As in the vacuum case, the fermionic contribution to the boost and the rotation constraints is independent of \( K \) and hence of \( M \) and \( \zeta \). Hence, the \( \zeta, \mathcal{G}^{0i} \) pair of second class constraint can be eliminated in exactly the same manner as before leading to the same determination of \( \zeta \) and of course without affecting the Poisson brackets among the remaining variables. Furthermore, as in the vacuum case, the \( \Lambda_i, \lambda^i \), terms drop out from the equation (31). The secondary constraint is thus determined from equation (39) with \( \bar{H} \) now including the fermionic contribution. We only need to pick out the explicit \( A \) dependence.
This is easily done and leads to,

$$\left( E_{ak}^i \delta_i^k + k \leftrightarrow l \right) \frac{\delta}{\delta A_a^i} H_F |_A = \frac{i}{2\eta} \left[ N^a E_{ak} \mathcal{E}^{mn}_{l} \Psi^\dagger \sigma_{mn} \Psi + k \leftrightarrow l \right. $$

$$- \frac{\text{sgn}(e) N}{2\sqrt{q}} \left\{ \mathcal{E}^{mn}_{l} \Psi (\gamma_k \sigma_{mn} + \sigma_{mn} \gamma_k) \Psi + k \leftrightarrow l \right\} \right]$$

$$= \frac{N^a}{\eta} \left( E_{ak} G^F_l + E_{al} G^F_k \right) - \frac{\text{sgn}(e) N}{\eta \sqrt{q}} \left( \delta_{kl} \Psi^\dagger \gamma_5 \Psi \right)$$

Adding these to the eqn. (39) gives,

$$\frac{N}{\eta^2 \sqrt{q}} \left[ S_{kl} - \text{sgn}(e) \eta \delta_{kl} \Psi^\dagger \gamma_5 \Psi \right] + \frac{N^a}{\eta} \left[ E_{ak} G^F_l + E_{al} G^F_k \right] \approx 0 \tag{58}$$

Hence the secondary constraint becomes,

$$S_{ij} \approx \text{sgn}(e) \eta \delta_{ij} \Psi^\dagger \gamma_5 \Psi =: S^F_{ij} \tag{59}$$

and the solution for $M_{ij}$ becomes,

$$\text{sgn}(e) M_{ij} = -(1 + \eta^2) \mathcal{E}^{abc}_{ij} \left\{ V_{a}^i \partial_b V_{c}^j + V_{a}^j \partial_b V_{c}^i - \delta_{ij} V_{a}^i \partial_b V_{c}^i \right\} + \eta \left( A^i_a E^a_j + A^j_a E^a_i \right)$$

$$- \frac{(1 + \eta^2)}{2} \text{sgn}(e) \delta_{ij} \Psi^\dagger \gamma_5 \Psi \tag{60}$$

After imposing the second class constraints strongly, we get,

$$\hat{T}_i^a = T_i^a = \frac{1}{2} S_{ij} V^{aj} = \text{sgn}(e) \eta \frac{V_{i}^a}{2} \Psi^\dagger \gamma_5 \Psi$$

and this leads to non-trivial Dirac brackets. While we could simplify the remaining constraints as before, it turns out to be better to check the Dirac brackets relative to the $(v^a_i, \tau^a_i)$ second class pair which suggests a new definition of $A$ and simplifies the constraints considerably.

For the fermions, the action being linear in velocities, we have primary constraints, $\pi_\lambda \sim \lambda^i, \pi_{\lambda^i} \sim \lambda$, which are second class. Also these variables fail to be Darboux coordinates - do not have vanishing Poisson brackets\(^5\) with the gravitational variables due to the $\sqrt{q}$ factor. The shift to $\Psi, \Psi^\dagger$ variables makes the matter and gravitational variables Poisson-commute. Defining Dirac brackets relative to these primary, second class constraints allows us to use

\(^5\) strictly Generalised Poisson brackets\(^{13}\), due to the Grassmann nature of the fermions.
\[ \{ \Psi^\alpha(x), \Psi^\dagger_\beta(y) \}_+ = -i\delta^\alpha_\beta \delta^3(x,y) \]  

(61)

Now we define Dirac brackets relative to the \((\nu, \tau)\) constraints:

\[ \nu^i_a = \hat{V}_a^i - V^i_a, \quad \tau^a_i = \hat{T}^a_i - \frac{1}{2} S_{ij} V^{aj} = \hat{\eta}_a^i \frac{\text{sgn}(e)}{2} V^a \Psi^\dagger \gamma_5 \Psi . \]

The Dirac bracket is defined as,

\[ \{ f(x), g(y) \}^* := \{ f(x), g(y) \} - \int dz \{ f(x), \nu^i_a(z) \} \{ \tau^a_i(z), g(y) \} + \int dz \{ f(x), \tau^a_i(z) \} \{ \nu^i_a(z), g(y) \} . \]  

(62)

Due to the presence of the fermionic term in \(\tau^a_i\), the Dirac bracket between \(A\) and the fermions is non-trivial while all other basic Dirac brackets remain the same as the corresponding Poisson brackets. Specifically,

\[ \{ A^i_a(x), \Psi^\alpha(y) \}^* = -\text{sgn}(e) \frac{i}{4} E^i_a(\gamma_5 \Psi)^\alpha \delta^3(x,y) , \]

\[ \{ A^i_a(x), \Psi^\dagger_\alpha(y) \}^* = +\text{sgn}(e) \frac{i}{4} E^i_a(\Psi^\dagger \gamma_5)\alpha \delta^3(x,y) . \]  

(63)

This suggests that if we use,

\[ A^i_a := A^i_a - \text{sgn}(e) \frac{\eta}{4} E^i_a \Psi^\dagger \gamma_5 , \]

all Dirac brackets become standard Poisson brackets. Thus, while simplifying the constraints we will first effect the shift \(A = A + \text{sgn}(e) \frac{\eta}{4} E \Psi^\dagger \gamma_5\). This results in very natural expressions for the constraints.

The solution (60) for \(M^{ij}\) simplifies as,

\[ M^{ij}(A, E, \Psi) = \overline{M}^{ij}(A, E) - \frac{1}{2} \delta^{ij} \Psi^\dagger \gamma_5 \Psi \quad \text{where,} \quad \overline{M}(A, E) \text{ is defined in (43)} , \]  

(64)

while the \(K, \Gamma\) simplify as,

\[ K^i_a(A, E, \zeta) = K^i_a(A, E, \zeta) , \quad \Gamma^i_a(A, E, \zeta) = \Gamma^i_a(A, E, \zeta) - \text{sgn}(e) \frac{\eta}{4} E^i_a \Psi^\dagger \gamma_5 \Psi \]  

(65)

\[ \text{The minus sign in the basic Dirac bracket, is correlated with the sign in the Lagrangian. Upon quantization, we will have anti-commutator, } [\Psi, \Psi^\dagger]_+ \sim +\hbar \text{ which is consistent with the Hilbert space inner product.} \]
The underlined functions are the functions obtained in the pure gravity case (‘torsion-free’). In particular, \( \Gamma_i^{ai}(A, E, \zeta) = \Gamma_i^{ai}(V) \) as given in equation (45), when the second class constraints are strongly solved. This is relevant for regularization of the Hamiltonian constraint later on.

The shift of \( A \) does not affect the expression for the boost constraint (A12) since the extra term is killed by the \( E \). For the same reason the rotation constraint (A13) is also unaffected.

The diffeomorphism constraint simplifies as follows. Recall eqn.(27),

\[
\begin{align*}
H^a = & \eta F_{ai} E^b_i + \frac{1}{\eta} V^i_i A^{k}_{ai} \mathcal{E}_{ijk} \hat{T}^{aoj} - V^i_i \partial_a \hat{T}^aoi \\
& + \frac{1 + \eta^2}{\eta} V^i_i K_{i} j c V^j_e \mathcal{E}^{bca} - V^i_i K_{i0} T^{ao0} - \frac{\text{sgn}(e)}{\eta} V^i_i K^k_{ia} \mathcal{E}_{ijk} \hat{T}^{aoj} \\
& - \frac{i}{2} \left\{ \bar{\Psi} \gamma^0 D_a \Psi - \overline{D_a \Psi} \gamma^0 \Psi \right\}
\end{align*}
\]

where,

\[
\begin{align*}
\hat{T}^a_i = & \text{sgn}(e) \eta V^a_i \Psi^* \gamma_5 \Psi, \quad T^{ao0} = -\eta \text{sgn}(e) V^a_i G_{i}^{vac} \approx \text{sgn}(e) \eta V^a_i G_{i}^{vac} = \Psi^* \gamma_5 \sigma_0 \Psi \\
D_a \Psi = & \left( \partial_a \Psi - \frac{i}{\eta} A^i_a \gamma_5 \sigma_{0i} \Psi \right) + K^i_a \left( 1 + \text{sgn}(e) \frac{i \gamma_5}{\eta} \right) \sigma_{0i} \Psi \\
\overline{D_a \Psi} = & \left( \partial_a \Psi^* + \frac{i}{\eta} A^i_{a^*} \Psi^* \gamma_5 \sigma_{0i} \right) - K^i_a \Psi \left( 1 + \text{sgn}(e) \frac{i \gamma_5}{\eta} \right) \sigma_{0i}
\end{align*}
\]

In the above expression, we have separated the \( K \)-dependent terms and also solved the second class constraints.

The first step in simplification is to put \( A^i_a = A^i_a + \text{sgn}(e) \frac{\eta}{4} E^i_a \Psi^* \gamma_5 \Psi \). The \( K \) independent pieces just have the \( A \) dependence being replaced by \( A \) dependence with one additional term,

\[
[K-\text{indep. terms}] \ (A) = \ [K-\text{indep. terms}] \ (A) - \text{sgn}(e) \frac{\eta}{2} E^i_a (\partial_a E^a_i)(\Psi^* \gamma_5 \Psi)
\]

The \( K \)-dependent terms combine to give,

\[
[K-\text{dependent terms}] = -\text{sgn}(e) \frac{1}{2} E^i_a G^F_i (\Psi^* \gamma_5 \Psi)
\]

In getting this simplification, we have used the boost constraint, the rotation constraint as well as \( K = K(A, M, \zeta) \) expressions.

The middle term of the first line of equation (66) and the last term of equation (68), combine to give \(-\text{sgn}(e) \frac{1}{2} E^i_a \Psi^* \gamma_5 \Psi G^{vac}_i\). This in turn combines with the simplified \( K \)-dependent
term to give a term proportional to the total rotation constraint which is weakly zero. Thus, finally, we get the simplified diffeomorphism constraint as,

\[ \mathcal{H}_a(A, E, \Psi) = \eta F_a^i \gamma_b \partial_i F^b_{\text{be}}(A) + \theta_a T^a \theta^b + \left( \frac{1 + \eta^2}{\eta^2} \right) \left( \frac{E^b_i E^c_j}{2\sqrt{q}} \right) \left\{ K^j_b K^j_c - K^j_c K^j_b \right\} - \text{sgn}(e) \left( \eta(\partial_b K^i_c - \partial_i K^i_b) + \mathcal{E}^l_{mn} (A^m_b K^n_c - A^m_c K^n_b) \right) + K^i_a \right\} \]

where,

\[ K^i_a := \left( \partial_a \Psi - i \frac{A^i_a}{\eta} \sqrt{\gamma} \gamma_5 \sigma_{0i} \right) \]  

Notice that this is a simple additive form of the vacuum diffeomorphism and the Dirac diffeomorphism constraint (coupled to the SU(2) gauge connection \( A \)). Four fermion terms generated by the second class constraint have been neatly absorbed in the shifted \( A \).

Lastly, we simplify the Hamiltonian constraint. Recall eqn. (29),

\[ \mathcal{H} = \frac{1}{2} \left( \frac{E^b_i E^c_j}{\sqrt{q}} \mathcal{E}^{jk}_i F^b_{\text{be}}(A) \right) + \partial_a T^a + \left( \frac{1 + \eta^2}{\eta^2} \right) \left( \frac{E^b_i E^c_j}{2\sqrt{q}} \right) \left\{ K^j_b K^j_c - K^j_c K^j_b \right\} - \text{sgn}(e) \left( \eta(\partial_b K^i_c - \partial_i K^i_b) + \mathcal{E}^l_{mn} (A^m_b K^n_c - A^m_c K^n_b) \right) + K^i_a \]

As before, second class constraints have been solved and the \( K \)-dependent terms are separated. Substitute \( A^i_a = A^i_a + \text{sgn}(e) \frac{\eta^2}{4} E^a_i \Psi^\dagger \gamma_5 \Psi \). Thanks to eqn. (65), we replace everywhere \( K \rightarrow K \). The \( A \)-dependent terms generate additional terms.

\[ \frac{1}{2} \left( \frac{E^b_i E^c_j}{\sqrt{q}} \mathcal{E}^{jk}_i F^b_{\text{be}}(A) \right) \rightarrow \]

\[ \frac{1}{2} \left( \frac{E^b_i E^c_j}{\sqrt{q}} \mathcal{E}^{jk}_i F^b_{\text{be}}(A) \right) + \left\{ \frac{\mathcal{E}^{abc}}{4} V_a^l \partial_b E^c_l + \frac{A^i_a V_i^a}{2\eta} \right\} \text{sgn}(e) \Psi^\dagger \gamma_5 \Psi + \frac{3}{16\sqrt{q}} \left( \Psi^\dagger \gamma_5 \Psi \right)^2 , \]

\[ - \text{sgn}(e) \left( \frac{1 + \eta^2}{\eta^2} \right) \left( \frac{E^b_i E^c_j}{2\sqrt{q}} \mathcal{E}^{jk}_i F^b_{\text{be}}(A) \right) \rightarrow \]

\[ - \text{sgn}(e) \left( \frac{1 + \eta^2}{\eta^2} \right) \left( \frac{E^b_i E^c_j}{2\sqrt{q}} \mathcal{E}^{jk}_i F^b_{\text{be}}(A) \right) - \left( \frac{1 + \eta^2}{2\eta^2} \right) K^i_a E^a_i \Psi^\dagger \gamma_5 \Psi , \]

and,

\[ \frac{i}{2} V_i^a \left\{ \Psi^\dagger D_a \Psi - \overline{D_a \Psi} \gamma^i \Psi \right\} (A, K) \rightarrow \]

\[ \frac{i}{2} V_i^a \left\{ \Psi^\dagger D_a \Psi - \overline{D_a \Psi} \gamma^i \Psi \right\} (A, K) - \frac{3}{8\sqrt{q}} \left( \Psi^\dagger \gamma_5 \Psi \right)^2 \]
The $\mathbf{K}$–dependent terms in the fermionic Hamiltonian and the torsion pieces combine to give,
\[
-\text{sgn}(e) \mathcal{E}^{k}_{ij} K^{a}_{i} E^{aj}_{F} \frac{G^{k}_{F}}{\sqrt{q}} + \text{sgn}(e) \partial_{a} \left( \eta \psi^{a}_{i} G^{i}_{F} \right) + \frac{1 + \eta^{2}}{2\eta \sqrt{q}} K^{a}_{i} E^{i}_{F} \left( \chi^{\gamma}_{5} \psi \right)
\]

The terms generated by the shift in $A$, simplify to,
\[
\frac{K^{a}_{i} E^{i}_{F}}{2\eta \sqrt{q}} \left( \chi^{\gamma}_{5} \psi \right) - \frac{1 + \eta^{2}}{2\eta \sqrt{q}} K^{a}_{i} E^{i}_{F} \left( \chi^{\gamma}_{5} \psi \right) - \frac{3}{16} \left( \chi^{\gamma}_{5} \psi \right)^{2}
\]

Using $G^{k}_{F} = G^{k}_{tot} - G^{k}_{\text{vac}}$ and dropping the term containing $G^{k}_{tot}$, leads to
\[
\mathcal{H} := \frac{1}{2} \frac{E^{b}_{j} E^{c}_{k}}{\sqrt{q}} \left\{ \mathcal{E}^{jk} F_{bc}(A) + \frac{1 + \eta^{2}}{\eta^{2}} \left( K^{i}_{j} K^{k}_{i} - K^{j}_{i} K^{k}_{i} \right) \right\} - \partial_{a} \left( \eta \text{sgn}(e) V^{a}_{i} G^{i}_{\text{vac}} \right)
\]
\[
- \frac{1 + \eta^{2}}{\eta^{2}} \frac{E^{b}_{j} E^{c}_{k}}{2\sqrt{q}} \mathcal{E}^{jk} \left\{ \eta \left( \partial_{b} K^{k}_{i} - \partial_{i} K^{k}_{j} \right) + \mathcal{E}^{lm}_{mn} (A^{m}_{b} K^{n}_{k} - A^{m}_{b} K^{n}_{k}) \right\} \text{sgn}(e)
\]
\[
+ \frac{i}{2} V^{a}_{i} \left\{ \chi^{\gamma}_{5} \mathcal{D}_{a} \psi - \mathcal{D}_{a} \chi^{\gamma}_{5} \psi \right\} (A) + \frac{\mathcal{E}^{a}_{bc} V^{i}_{a} \partial_{b} V^{a} + 1}{2\sqrt{q}} \frac{A^{i}_{a} E^{a}_{i}}{\eta} \left( \chi^{\gamma}_{5} \psi \right) \text{sgn}(e) + \frac{G^{k}_{F} G^{k}_{F}}{\sqrt{q}}
\]

The first two lines of the above equation are exactly the same as the Hamiltonian for vacuum case and therefore are simplified in exactly the same way as before. In particular, the derivatives of $K$ terms after partial integration, generates another total derivative term, namely, $(\eta^{-1} + \eta) \partial_{a} (\text{sgn}(e) V^{a}_{i} G^{i}_{\text{vac}})$, which combines with the last term in the first line leaving us with $\eta^{-1} \partial_{a} (\text{sgn}(e) V^{a}_{i} G^{i}_{\text{vac}})$. The $k$–dependent terms just produce minus the second term in the first line as before.

The third line is the contribution from the Dirac Hamiltonian coupled to the $A$ field while the terms in the fourth line are the extra term including 4-fermions terms.

Using the relation,
\[
\left( \frac{1}{4\sqrt{q}} \mathcal{E}^{abc} V^{i}_{a} \partial_{b} V^{a} + \frac{1}{2\sqrt{q}} \frac{A^{i}_{a} E^{a}_{i}}{\eta} \right) = \frac{\text{sgn}(e)}{2\eta \sqrt{q}} K^{a}_{i} E^{i}_{F}
\]

The final form of the Hamiltonian constraint is,
\[
\mathcal{H} := \frac{1}{2} \frac{E^{b}_{j} E^{c}_{k}}{\sqrt{q}} \left\{ \mathcal{E}^{jk} F_{bc}(A) - \frac{1 + \eta^{2}}{\eta^{2}} \left( K^{i}_{j} K^{k}_{i} - K^{j}_{i} K^{k}_{i} \right) \right\} + \frac{1}{\eta} \partial_{a} \left( \text{sgn}(e) V^{a}_{i} G^{i}_{\text{vac}} \right)
\]
\[
+ \frac{i}{2} V^{a}_{i} \left\{ \chi^{\gamma}_{5} \mathcal{D}_{a} \psi - \mathcal{D}_{a} \chi^{\gamma}_{5} \psi \right\} (A) + \left[ \left( \frac{1}{2\eta \sqrt{q}} K^{a}_{i} E^{i}_{F} \right) \left( \chi^{\gamma}_{5} \psi \right) \right] - \left[ \frac{3}{16} \frac{\left( \chi^{\gamma}_{5} \psi \right)^{2}}{\sqrt{q}} - \frac{G^{k}_{F} G^{k}_{F}}{\sqrt{q}} \right], \text{ where } G^{i}_{F} = \chi^{\gamma}_{5} \sigma_{0i} \psi
\]
The Hamiltonian constraint thus consists of additive combination of the vacuum and the Dirac Hamiltonian (coupled to $A_i^a$). However, unlike the diffeomorphism constraint (70), there are the additional terms in the square bracket. These extra terms contain contributions quartic in the fermions as well as quadratic in fermions. Notice that there are no explicit factors of $\text{sgn}(e)$ in the final expressions. The $K_i$ appears which is the same as in the vacuum case and therefore the properties needed in using the Thiemann identities in the quantization of the gravitational Hamiltonian constraint continue to hold. We have also checked that if the connection equations of motion for the spatial components are substituted back in the action, the above Hamiltonian is recovered.

We close this section by reverting to the standard notation e.g. (12).

For this, we first substitute $A_i^a \rightarrow \eta A_i^a$ (and $K_i \rightarrow K_i^\eta$). This removes the factors of $\eta^{-1}$ from the definition of $F_{ab}^i$ given in eqn. (28). It also extracts a common factor of $\eta$ from the rotation constraint, $G^i$. The diffeomorphism and the Hamiltonian constraints remains the same except for removing the factor of $\eta^{-1}$ from the total derivative term in the Hamiltonian constraint. The symplectic term, $E_i^a \partial_t A_i^a \rightarrow (\eta E_i^a) \partial_t A_i^a$. And finally we put $\eta = \gamma^{-1}$ and restore $\kappa$ (Recall the overall $\kappa^{-1}$ factor in the Lagrangian density of eqn (18)).

Here are the final expressions:

\[
P_a = (\kappa \gamma)^{-1} E_i^a = (\kappa \gamma)^{-1} \text{sgn}(e) V_i^a \sqrt{q} ; \quad A_i^a = \gamma \text{sgn}(e) K_i^a - \Gamma_i^a (V). \tag{76}
\]

\[
\{ A_i^a(x), P_j^b(y) \} = \delta_i^b \delta_j^a \delta^3(x,y) , \quad \{ \Psi(x), \Psi^\dagger(y) \}_+ = -i \delta_i^a \delta^3(x,y). \tag{77}
\]

\[
G_i = \partial_a P_a^i + \mathcal{E}_{ij} K_a^i P_k^a - \frac{i}{2} \mathcal{E}_{ijk} \Psi^\dagger \sigma_{jk} \Psi , \quad \sigma_{jk} := \frac{1}{4} [\gamma_j, \gamma_k]; \tag{78}
\]

\[
\mathcal{H}_a = F_{ab}^i P_i^b - \frac{i}{2} (\Psi \gamma^0 D_a \Psi - \overline{D_a \Psi} \gamma^0 \Psi) , \quad D_a \Psi := (\partial_a - i A_i^a \gamma_5 \sigma_0 i) \Psi; \tag{79}
\]

\[
\mathcal{H} = \kappa \gamma^2 \frac{1}{2} \frac{P_b P_c}{\sqrt{q}} \left\{ \mathcal{E}_{jk} F_{ic}^a (A) - (1 + \gamma^2) (K^j_b K_c^k - K^j_c K_b^k) \right\} + \gamma \partial_a (\text{sgn}(e) \quad V_i^a G^i_{\text{vac}})
\]

\[
+ \frac{i}{2} \kappa \gamma \text{sgn}(e) P_a^i \left\{ \overline{\Psi} \gamma^i D_a \Psi - \overline{D_a \Psi} \gamma^i \Psi \right\} (A) + \left[ \left( \frac{\kappa \gamma^2}{2 \sqrt{q}} K_i^a P_a \right) \left( \Psi^\dagger \gamma_5 \Psi \right) \right]
\]

\[
- \left[ \frac{3}{16} \kappa \gamma \left( \Psi^\dagger \gamma_5 \Psi \right)^2 \right] - \kappa \left( \Psi^\dagger \gamma_5 \sigma_{0i} \Psi \right) \left( \Psi^\dagger \gamma_5 \sigma_{0i} \Psi \right) \right] \right] \right] \tag{80}
\]

Note: Dimensionally, $\kappa \sim L^2 , \quad (A, K, \partial) \sim L^{-1} , \quad E \sim L^0 , \quad P \sim L^{-2} , \quad (\Psi, \overline{\Psi}) \sim L^{-3/2} , \quad \mathcal{G} \sim L^{-3} , \quad (\mathcal{H}_a, \mathcal{H}) \sim L^{-4}$.
The inverse square root of $q$ and $K^i_a$ appearing above are manipulated exactly as in the vacuum case. As remarked earlier, the $A$ and $K$ above correspond to the vacuum case for which the Thiemann identities hold. Explicitly, the identities we would use are:

$$ \text{sgn}(e) \mathcal{E}^{bca} V^i_a = \mathcal{E}^{ijk} \frac{E^b_j E^c_k}{\sqrt{\det(E^a_i)}} , \quad q := (\det(V^i_a))^2 = \det(E^a_i) \quad (81) $$

$$ \kappa \gamma \frac{\text{sgn}(e)}{2} V^i_a(x) = \left\{ A^i_a(x), \int d^3 y \sqrt{q} \right\} \Rightarrow $$

$$ \mathcal{E}^{ijk} \frac{E^b_j E^c_k}{\sqrt{\det(E^a_i)}} = \frac{2}{\kappa \gamma} \mathcal{E}^{bca} \left\{ A^i_a(x), \int d^3 y \sqrt{q} \right\} . \quad (82) $$

$$ H_E(1) := \frac{\kappa \gamma^2}{2} \int \frac{P^b_j P^c_k}{\sqrt{q}} \mathcal{E}^{ijk} F_{bc}^l , \quad \mathcal{K} := \int d^3 y \text{sgn}(e) K^i_a P^a_i \Rightarrow $$

$$ H_E(1) , \int d^3 y \sqrt{q} \right\} , \quad \text{sgn}(e) K^i_a(x) = \left\{ A^i_a(x) , \mathcal{K} \right\} \quad (84) $$

These identities suffice to derive a quantization the Hamiltonian constraint from that of the ‘Euclidean Hamiltonian constraint’ (the first term in the Hamiltonian constraint) and of the volume operator.

**IV. ACTION OF CONSTRAINTS AND THEIR ALGEBRA**

It is easy to see that the gauge constraint generates correct gauge transformation of the basic fields. Specifically, with $G(\Lambda) := \int_{\Sigma_3} d^3 x \Lambda^i G_i$,

$$ \left\{ A^i_a(x), G(\Lambda) \right\} = -D_a \Lambda^i = -\partial_a \Lambda^i - \mathcal{E}^{ijk} A^j_a \Lambda^k \quad (85) $$

$$ \left\{ P^a_i(x), G(\Lambda) \right\} = \mathcal{E}^{jik} \Lambda^j P^a_k \quad (86) $$

$$ \left\{ \Psi^\alpha(x), G(\Lambda) \right\} = -i \Lambda^i (\gamma_5 \sigma_{0i} \Psi)^\alpha \quad (87) $$

$$ \left\{ \Psi^\dagger_\alpha(x), G(\Lambda) \right\} = +i \Lambda^i (\Psi^\dagger_\alpha \gamma_5 \sigma_{0i}) \quad (88) $$

If we compute the infinitesimal action of the $H_a$ constraint on the basic variables, we see that it equals the Lie derivatives of the basic variables only up to an SU(2) gauge transformation. We are however free to modify the constraints by adding suitable combinations of themselves. So we define the diffeomorphism constraint as:

$$ C(\tilde{N}) := \int_{\Sigma_3} d^3 x N^a C_a \quad \text{with,} $$

$$ C_a := H_a - A^i_a G_i = P^b_i \partial_a A^i_b - \partial_b (A^i_a P^b_i) + \frac{i}{2} \left( \Psi^\dagger_\alpha \partial_a \Psi - \partial_a \Psi^\dagger \Psi \right) \quad (89) $$
which leads to the infinitesimal transformations,

\[
\{ A^i_a(x), C(\vec{N}) \} = L_N A^i_a = \partial_a (N^b A^i_b) + N^b (\partial_b A^i_a - \partial_a A^i_b) \tag{90}
\]

\[
\{ P^a_i(x), C(\vec{N}) \} = L_N P^a_i = N^b \partial_b P^a_i - P^b_i \partial_b N^a + 1 \cdot (\partial_b N^b) P^a_i \tag{91}
\]

\[
\{ \Psi^a(x), C(\vec{N}) \} = L_N \Psi^a = N^b \partial_b \Psi^a + \frac{1}{2} \cdot (\partial_b N^b) \Psi^a \tag{92}
\]

\[
\{ \Psi^\dagger_\alpha(x), C(\vec{N}) \} = L_N \Psi^\dagger_\alpha = N^b \partial_b \Psi^\dagger_\alpha + \frac{1}{2} \cdot (\partial_b N^b) \Psi^\dagger_\alpha \tag{93}
\]

This implies that \( \{ \text{var}, \int N^a C_a \} = L_N (\text{var}) \) for all variables. The Gauge constraint already generates the correct gauge transformation of the basic variables. By inspection, it follows that the gauge constraints (weakly) commute with the diffeomorphism and the Hamiltonian constraint, the gauge constraint and the diffeomorphism constraints form sub-algebras and the diffeomorphism constraint transforms the Hamiltonian constraint by the Lie derivative. The non-trivial bracket is the bracket of two Hamiltonian constraints.

V. PARITY AND INTERNAL PARITY

Recall that we begin with the (co-)tetrad field \( e^I_{\mu} \), the Lorentz connection \( \omega^{IJ}_{\mu} \) and the fermion fields \( \lambda, \bar{\lambda} \) (or \( \Psi, \bar{\Psi} \)) defined over a manifold \( M \sim \mathbb{R} \times \Sigma_3 \) which is assumed to be orientable. With the topology specified, \( M \) can be taken to be time-orientable with respect to all the metric tensors constructed by the parametrization (6,7). Obviously, \( \Sigma_3 \) is orientable as well.

There are two distinct sets of discrete transformations: orientation reversing diffeomorphism of \( M \) and a \( O(1,3) \) transformation with determinant = -1. We will keep the time orientation fixed. Orientation reversing diffeomorphism of \( M \) will then be reversing the orientation of \( \Sigma_3 \). We will refer to these as parity transformations. The improper Lorentz transformations \( \Lambda^I_J \), will also be taken so that \( \det \Lambda = -1 \) and \( \Lambda^0_0 = 1 \) and will be referred to as an Lorentz parity transformation.

After going to the canonical framework in the ‘time gauge’, we have the fields \( A^i_a, P^a_i, \Psi, \bar{\Psi} \) defined on \( \Sigma_3 \). The parity transformations are the orientation reversing diffeomorphism of \( \Sigma_3 \) and the improper \( O(3) \) transformation, inversion (say) will be the Lorentz parity transformation.

In the Lagrangian framework, the Hilbert-Palatini action (also the Euler and the cosmological terms) are invariant under both sets of transformations while the Nieh-Yan (as
well as the Pontryagin and the Holst) actions change signs under parity but are invariant under Lorentz parity. Hence the combined action is invariant under Lorentz parity and non-invariant under parity.

The variables of the canonical framework are defined in terms of those of the Lagrangian framework. These definitions of the $SU(2)$ connection in terms of $K$ and $\Gamma$ and the conjugate momentum in terms of the triad are consistent with the $SO(3)$ gauge transformation extended to include the Lorentz parity. Thus the triad which transforms by the defining representation changes sign under Lorentz parity. The ‘densitized triad’ (or the conjugate momentum) transforms by the adjoint representation and should be invariant under Lorentz parity. The $\text{sgn}(e)$ factor in their definitions precisely takes care of this. The same can be seen in the definition of the connection. It is easy to see that the symplectic structure and the constraints (vacuum) are all invariant under Lorentz parity.

When fermions are included, these are scalars under diffeomorphism and transform as $\Psi \rightarrow \gamma^0 \Psi$ under Lorentz parity. All the constraints including fermions are invariant under Lorentz parity. This is true in both the Lagrangian and the canonical frameworks.

With regards to parity the situation is different. The action is not invariant under parity, due to the Nieh-Yan term. In the canonical framework, the connection is not simply even/odd under parity since the $K$ term changes sign while the $\Gamma$ does not. The ‘densitized triad’ also acquires an extra minus sign under parity (behaves as a ‘pseudo-vector of weight 1’). The symplectic structure thus is not invariant. The constraints also are not invariant under parity. This is consistent with the non-invariance of the action.

The action is invariant under parity combined with $\gamma \rightarrow -\gamma$. Our definitions have the appropriate factors of $\gamma$ to restore the simple (even) behaviour of the basic canonical variable resulting also in the invariance of the Poisson brackets and constraints.

In short, Lorentz parity is an invariance of the action as well as the canonical framework and parity is not. However, parity combined with $\gamma \rightarrow -\gamma$ is an invariance of both action and the canonical framework. It is not our intention to suggest that the combined operation be a physical symmetry (which depends on the quantum theory), but it is useful in checking the algebra.

One could try to change the definitions of the basic variables by dropping the $\text{sgn}(e)$ factors. This will result in expressions which can be obtained from the above by putting $\text{sgn}(e) = 1$. This will restore the ‘densitized triad’ to its usual density weight 1 vector density
status and the connection to its 1-form status. The canonical framework is then invariant under parity (without changing sign of $\gamma$). The action however is still not invariant.

Under Lorentz parity, the connection does not have simple behaviour and the conjugate momentum will be odd. This results in non-invariance of the canonical framework. The action however is still invariant under Lorentz parity. If the sign of $\gamma$ is changed along with the Lorentz parity transformation, then the basic variables are even, the symplectic structure is invariant and so are the constraints and the action. Thus, the definitions without the $\text{sgn}(e)$ factors, interchanges the role of Lorentz parity and parity appropriately combined with $\gamma \rightarrow -\gamma$.

With two different definitions (with and without the $\text{sgn}(e)$ factors), we can ensure either invariance of the canonical framework under parity or under Lorentz parity and this is independent of minimally coupled fermions. The action however unambiguously remains invariant under Lorentz parity and non-invariant under parity. Which of these is more appropriate?

Observe that if we were to consider formulation in terms of the metric tensor, then the notion of Lorentz parity is not even definable as there is no internal Lorentz transformation. On the other hand, existence of fermions (spinorial fields) requires an orientable manifold and using the tetrad formulation making the Lorentz parity notion available. If the orientation of the manifold is regarded as a fixed background structure, then parity transformation is excluded by definition and Lorentz parity alone is available. Which of these is relevant from an observational point of view is not very clear and so also the issue of ‘parity violation’ via gravitational interactions.

VI. CONCLUDING REMARKS

The next natural step is the loop quantization of the fermions with interacting gravity. The kinematic Hilbert space of this system has been already constructed [2]. The procedure given by Thiemann can be followed in toto. The extra feature, not available in Thiemann’s discussion are the quartic terms in the fermionic sector. Their regularization has been given in [6] and we don’t have any thing new to add to this.

In summary, we have presented a canonical form of a Dirac fermion, minimally coupled to the tetrad form of gravity including the Nieh-Yan term. The Canonical analysis shows that
the coefficient of the Nieh-Yan term is the inverse of the usual Barbero-Immirzi parameter while the topological nature of the Nieh-Yan term guarantees its non-appearance in the classical equations of motion. One could consider additional non-minimal couplings, not affecting the symplectic structure, but these will not change the status of the Barbero-Immirzi parameter. We saw the natural appearance of the sgn($e$) factors which also serve to give definitions consistent with the two distinct notions of parity and Lorentz parity. There is no Lorentz parity violation either in the Lagrangian framework or the Hamiltonian framework for this system. As noted earlier, among matter fields, fermions alone are sensitive to the SU(2) action and contribute to the rotation (Gauss) constraint. Its implications for the homogeneous, diagonal models has already been explored by Bojowald and Das [6]. For the same reasons, fermions are likely mediators in the black hole evaporation process. Due to the role of fermions in the chiral anomalies in the usual quantization, they are also probes to see how loop quantization does or does not accommodate chiral anomalies. For exploration and elaboration of these issues, it is necessary to have a sufficiently precise control over the fermion-gravity system and classical analysis is the first step in this direction.

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Appendix A

1. Notation and Conventions

The basic fields are the co-tetrad \( e^I_\mu \), the Lorentz connection \( \omega^IJ_\mu \) and a Dirac spinor \( \lambda^\alpha \). The co-tetrad is taken to be non-degenerate and defines a metric \( g_{\mu\nu} := e^I_\mu e^J_\nu \eta^{IJ} \), \( \eta^{IJ} = \text{diag}(-1, 1, 1, 1) \). The general coordinate indices, \( \mu, \nu \ldots \), are raised/lowered with \( g_{\mu\nu} \), \( g^{\mu\nu} \) while the Lorentz indices, \( I, J, \ldots \), are raised/lowered with \( \eta^{IJ} \), \( \eta_{IJ} \).

\[
\begin{align*}
e &:= \det(e^I_\mu) := \frac{1}{4!} \varepsilon^{\mu\nu\alpha\beta} \varepsilon_{IJKL} e^I_\mu e^J_\nu e^K_\alpha e^L_\beta, \quad \varepsilon^{abc} = 1 = \varepsilon_{0123}; \\
e \varepsilon^{IJKL} &= -\varepsilon^{\mu\nu\alpha\beta} e^I_\mu e^J_\nu e^K_\alpha e^L_\beta \\
2e\Sigma^{IJ}_{\mu} &:= e(e^{\mu I} e^\nu_j - e^{\mu J} e^\nu_i) = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} \varepsilon_{IJKL} e^K_\alpha e^L_\beta \\
\hat{X}^I_J &:= \frac{1}{2} \varepsilon^{IJKL} X^{KL}, \quad \hat{X} := -X \\
T^I_{\mu} &:= D_\mu e^I_\nu - D_\nu e^I_\mu, \quad D_\mu e^I_\nu := \partial_\mu e^I_\nu + \omega^I_\mu J e^J_\nu, \\
R^{IJ}_{\mu\nu} &:= \partial_\mu \omega^I_\nu J - \partial_\nu \omega^I_\mu J + \omega^I_\mu K \omega^K_J - \omega^I_K \omega^K_\mu,
\end{align*}
\]

\[
\begin{align*}2\eta^{IJ} \mathbb{1} &= \gamma^I \gamma^J + \gamma^J \gamma^I, \quad \sigma^{IJ} := \frac{1}{4} [\gamma^I, \gamma^J], \quad \gamma_5 := i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \\
\bar{\lambda} &= \lambda^\dagger \gamma^0, \quad \gamma_i^\dagger = \gamma^0 \gamma_i \gamma^0, \quad \varepsilon_{ij}^{jk} \sigma_{jk} = 2i \gamma_5 \sigma_{0i} \\
D_\mu(\omega) \lambda &:= \partial_\mu \lambda + \frac{1}{2} \omega^I_\mu J \sigma_{IJ} \lambda, \quad \overline{D_\mu(\omega) \lambda} := \left\{ \partial_\mu \lambda^\dagger + \frac{1}{2} \omega^I_\mu J \lambda^\dagger \sigma_{IJ}^\dagger \right\} \gamma^0.
\end{align*}
\]

An explicit representation for the Dirac matrices is chosen to be,

\[
\begin{align*}
\gamma^0 &:= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i &:= \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad \gamma_5 &:= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{align*}
\]

2. Torsion Components

The torsion is defined geometrically as the covariant derivative of the co-tetrad. These involve the components of the Lorentz connection which have been obtained as functions of the canonical variables \( A, E, M, \zeta \). Similarly the boost and the rotation constraint are also expressed as functions of \( A, E, \zeta \). Eliminating \( \zeta \) and some combinations of \( \varepsilon, A, E \), we can express the torsion in terms of the constraints and the \( \hat{V}, \hat{T} \) variables. Subsequently, we will set \( \hat{V} = V \) and \( \hat{T} = T \) to simplify the expressions for the torsion as well as the constraints.
The equations are,
\[ T^{a0} := \eta \epsilon^{abc} T^0_{bc} = \eta \epsilon^{abc} K_{ai} V^i_c \]  \hspace{1cm} (A10)
\[ T^{ai} := \frac{\eta}{2} \epsilon^{abc} T^{ri}_{bc} = \eta \epsilon^{abc} \left( \partial_b V^i_c + \epsilon_{ijk} T^{bk}_b V^j_c \right) \]  \hspace{1cm} (A11)
\[ G^{\text{bi}}_{\text{vac}} = \text{sgn}(e) (\partial_b E^{ai} - \zeta^i) - \dot{V}^i_a T^{a0} \]  \hspace{1cm} (A12)
\[ G^i_{\text{vac}} = \eta \partial_a E^a_i - \epsilon_{kij} A^i_a E^{aj} + \epsilon_{kij} \dot{V}_a^i \dot{V}^{aj} \]  \hspace{1cm} (A13)

Substitution of,
\[ \epsilon^{abc} V^k_c = \text{sgn}(e) \sqrt{q} \epsilon_{ijk} V^a_i V^b_j \]  \hspace{1cm} (A14)
\[ K_{ai} E^a_j = \frac{1}{1 + \eta^2} \left\{ \text{sgn}(e) A_{ai} E^a_j + \frac{\eta}{2} M_{ij} + \frac{\eta}{2} \text{sgn}(e) \epsilon_{ijk} \zeta^k \right\} \]  \hspace{1cm} (A15)
\[ \Gamma_{ai} E^a_j = \frac{1}{1 + \eta^2} \left\{ -\eta A_{ai} E^a_j + \frac{\text{sgn}(e)}{2} M_{ij} + \frac{1}{2} \epsilon_{ijk} \zeta^k \right\} \]  \hspace{1cm} (A16)
leads to, (using \( E^a_i = \text{sgn}(e) \sqrt{q} V^a_i \) and \( E^a_i = \text{sgn}(e) V^i_a / \sqrt{q} \))
\[ T^{a0} = -\frac{\eta}{1 + \eta^2} V^a_i \text{sgn}(e) \left\{ \frac{\eta}{2} + \epsilon^{ijk} A_{bj} E^b_k \right\} \]  \hspace{1cm} [See eqn. (11)]
\[ T^{ai} V^j_a + T^{aj} V^i_a = \frac{\eta}{1 + \eta^2} \left\{ \text{sgn}(e) \left( M_{ij} - \delta^i_j M^k_k \right) - \eta \left( A_{ai} E^{aj} + A_{aj} E^{ai} - 2 \delta^i_j A^k_a E^a_k \right) \right\} \]  \hspace{1cm} (A17)
\[ T^{ai} V^j_a - T^{aj} V^i_a = \eta \epsilon^{ij}_k \left[ \partial_b E^{bk} + \frac{\eta}{1 + \eta^2} \epsilon^{mn}_k A^m_b E^{bn} - \frac{1}{1 + \eta^2} \right] \]  \hspace{1cm} (A18)
\[ : T^{ai} V^j_a = \frac{1}{2} \text{sgn}(e) \epsilon^{ij}_k G^{0k}_{\text{vac}} \]  \hspace{1cm} (A20)

In the last line of the equation (A19) we have used equation (A17). Now if we further put \( \dot{T} = T, \dot{\dot{V}} = V, T^{a0} \) simplifies to give,
\[ T^{a0} = -\eta \text{sgn}(e) V^i_a G^{ai}_{\text{vac}} \]  \hspace{1cm} (A21)

Using these in the expressions of the boost (A12) and the Gauss constraints (A13), lead to,
\[ G^{\text{bi}}_{\text{vac}} = \text{sgn}(e) \left[ \partial_a E^{ai} - \frac{\zeta^i}{1 + \eta^2} + \frac{\eta}{1 + \eta^2} \epsilon^{ijk} A^j_a E^{ak} \right] \]  \hspace{1cm} (A22)
\[ G_{\text{vac}}^i = \frac{\eta}{1 + \eta^2} \xi^i + \frac{1}{1 + \eta^2} \xi^i_A j A^j_a E^{ak} \]  

\[(A23)\]

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