MARKOVIAN STRATEGIES FOR PIECEWISE DETERMINISTIC DIFFERENTIAL GAMES WITH CONTINUOUS AND IMPULSE CONTROLS

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Abstract. This paper is concerned with the Markovian feedback strategies of piecewise deterministic differential games and their applications to business and management decision-making problems that involve multiple agents and continuous and impulse controls. For a class of piecewise deterministic differential games in finite or infinite horizons we formulate conditions for the value functions in the form of quasi-variational inequalities, prove a verification theorem, and derive a criterion for the Markovian regime change in certain case. These results are applied to a technology adoption problem that involves multiple companies engaged in extraction of an exhaustible resource with different technologies. Using the model proposed by Long et al in [16], we show the existence of a pure Markovian strategy and develop an algorithm for computing the solutions.

1. Introduction. Business and management planning often involves two types of decisions, one leads to gradual and slow changes, and other leads to abrupt and sudden changes. The former includes setting the production rates; adjusting investment proportions; and raising or lowering tax. The latter includes adopting new technologies; enacting new policies; and entering into new business partnerships. Accordingly, business and management environment often has two types of features. One vary slowly, such as the available capitals and the market shares. The other vary abruptly, such as the technology adopted, and business regulations enforced. The tasks of business strategists are to make optimal gradual changes as well as choosing the optimal timing for taking the abrupt changes. For a single agent, this is an optimal control problem with continuous and impulse controls. For multiple agents, this is a piecewise in time multi-regime game.

Optimal timing for regime switch has been a subject of study in recent years, especially in the area of technology adoption and investment strategies. For single-agent optimal control problems, Tomiyama [21] studies the investment decision of a firm whose capital goods have a delivery lag. This work is continued by Tomiyama and Rossana in [22] who include the switch time as an argument of optimization. Amit [1] is interested in the petroleum recovery process that has two phases, the primary phase during which the resource recovers by itself and the secondary phase.
during which the resource recovers using artificial means. He derives necessary conditions for the optimal controls. Both Tomiyama and Amit deal with the optimal control problems in finite horizon. In contrast, Makris [17] derives necessary conditions for the optimal regime switch in infinite horizon, and use the results to study a model of capital controls. Boucekkine et al [5] study a two-stage optimal control problem for technology adoption that involves obsolescence and learning costs. Following this direction, Valente [23] studies the switching from exhaustible resources to renewable resources. Saglam [20] extends the results to a multi-stage problem. Boucekkine et al [3] consider a control problem with two types of regime switch, one concerns technological or institutional regimes, and the other features regimes on some given threshold values, such as pollution. Their method is applied to a problem of optimal management of natural resources under ecological irreversibility. For multi-agent differential game models, the results are much sparse. Reinganum [19] considers the optimal timing for adopting a new technology by finding a Nash equilibrium, Fudenberg and Tirole [9] consider the timing of adopting new technology for $n$ identical firms. Boucekkine et al [4] study political regime changes in resource-rich countries and examine the “oil impedes democracy” hypothesis. Finally, Long et al [16] study a two-company competition model of technology adoption for extraction of an exhaustible resource. It is interesting to note that although many of the above works provide closed-loop feedback strategies, none gives pure Markovian strategies.

In this paper we develop Markovian feedback strategies for autonomous piecewise deterministic differential games, and apply the results to a class of business and management problems. Differential game models have been used in economics and management, such as resources and environment management and marketing and investment strategies for a long time (cf. [7, 8, 11, 14, 15, 18]). However, ordinary differential games are mostly used to model gradual changing processes and thus not sufficient for processes that also involve regime changes. For processes that are intertwined with gradual and abrupt changes, strategies that synthesize both continuous and impulse controls are needed. Piecewise deterministic differential games are tools for determining such synthesized strategies. A piecewise deterministic differential game is a type of differential games that involve continuous evolution of the state variables and abrupt regimes changes at certain jump times. Between such instants the state variables evolve deterministically according to the system dynamics of the economic interrelationship. This theory has been developed in recent years and have been used in some models of economics ([2, 6, 7, 10, 12, 13, 16]). In this paper, we apply it to a class of business and management problems.

The problems under study are finding Markovian strategies for single or multiple agents in a system that consists of finite sets of state variables and regimes. The state variables are governed by a system of autonomous differential equations. The agents can use continuous controls to alter the rates of change of the state variables. At any time the system is in one of a finite number of regimes. The regimes are changed by the impulse controls of the agents. Finally, each agent has an instantaneous payoff as well as a lumpsum payoff when the regime switches. The objective of each agent is to choose the optimal continuous and impulse controls to obtain the maximum possible total payoff, assuming that other agents are doing the same. The precise mathematical form of the problems is given in Section 2. For such problems, we derive conditions for Markovian strategies as a set of quasi-variational inequalities. We show that the classical solutions to the quasi-variational
inequalities are optimal by proving a verification theorem. In addition, in the case where the system has only one state variable, we derive a necessary condition for regime change. To illustrate the application of our results, we perform a thorough analysis to the technology adoption problem in resource extraction process proposed in [16], showing the existence of the Markovian strategies and describe how the strategies can be numerically computed. These results are new because so far there exist no similar methods in the literature to find Markovian strategies that involve both continuous and impulse control for multi-company competition problems.

This paper is organized as follows. In Section 2 we derive a system of quasi-variational inequalities for Markovian strategies of the general piecewise deterministic differential game model and provide a verification theorem which guarantees the the optimality of the Markovian strategies. We also derive a necessary condition for the regime switch in the case where there is one state variable. In Section 3 we apply the results in Section 2 to the resource extraction model proposed in [16], deriving the Markovian strategies in the cases where there is one company with multiple technologies, and there are multiple companies with two technologies. Numerical examples are presented, including the one with the same parameter values as in [16] for comparison. In Section 4 we give brief concluding remarks. Finally in Appendix we give proofs of Theorems 2.1 and 3.3.

2. Markovian continuous and impulse controls. In this section we derive conditions for the Markovian strategies in the form of quasi-variational inequalities, provide a verification theorem, and derive a necessary condition for the regime change when the system involves only one state variable. In what follows, we use “player(s)” for “agent(s)” and “mode(s)” for “regime(s)” to be consistent with the terminologies used in the literature of differential games.

The mathematical form of the problems is formulated as follows. The general differential game involves \( n \) players denoted by \( i = 1, \ldots, n \), \( d \) state variables represented by an \( d \)-tuple \( y = (y_1, \ldots, y_d) \in \mathbb{R}^d \), and \( m \) modes \( \sigma \in \{\sigma_1, \ldots, \sigma_m\} \). We use \( \Omega \) and \( \Sigma \) to denote the set of possible values of \( y \) and \( \sigma \), respectively. The state variables evolve deterministically except at certain jump times \( t < t_1 < t_2 < \cdots < T \), at which times the mode and the values of the state variables change abruptly. Here, \( T \) is either a finite positive number or \( \infty \), and the jump times are determined endogenously as described in Subsection 2.1. Between two jumps the state variables are governed by a system of differential equations,

\[
\frac{dy(s)}{ds} = f_{\sigma_l}(y(s), u(s)) \quad \text{for} \quad t_l < s < t_{l+1},
\]

\[
y(t_l) = x_l, \quad l = 1, 2, \ldots \tag{2.1}
\]

where \( \sigma_l \) is the mode for \( t_l < s < t_{l+1} \), \( u = (u_1, \ldots, u_n) \) represents the continuous controls of the players, and \( x_l \in \Omega \) is the value of the state variables at \( t_l \). Switching of mode are caused by the impulse controls of the players given by a mapping

\[
\sigma' = F(x, \sigma, \xi), \tag{2.2}
\]

where \( F \) is continuous in \( x \) and \( \xi = (\xi_1, \ldots, \xi_n) \) represents the impulse controls of the players.

We assume that the payoffs are exponentially discounted. Let \( g^\sigma_i(x, u) \) be the instantaneous payoff for player \( i \) when the state is \( (x, \sigma) \). Also, let \( \gamma^\sigma_{\sigma'}(x) \) be the lumpsum payoff for player \( i \) when the mode changes from \( \sigma \) to \( \sigma' \). Then, the
exponentially discounted total payoff for player $i$ for $s > t$ when the initial values are $\theta(t) = \sigma$ and $y(t) = x$ is

$$J^x_\sigma(u(\cdot) ; \xi(\cdot)) = \mathbb{E}^i_\theta \left[ \int_t^T e^{-\rho_i(s-t)} g_i^\theta(s) (y(s), u(s)) \, ds + e^{-\rho_i(T-t)} V^\theta_i(y(T)) \, \chi(t,\infty)(T) \right. \right. \left. + \sum_{t < t_i \leq T} e^{-\rho_i(t_{i-1}-t)} \gamma_i^\sigma_{i-1,\sigma_i}(y(t_i)) \right],$$

(2.3)

where $\mathbb{E}^i_\theta$ is the expectation operator conditional on player $i$’s available information at $t$, $\rho_i > 0$ is the discount factor, $V^\theta_i$ is the lumpsum payoff for player $i$ in mode $\sigma$ at the time when the state variable $y(s)$ reaches the boundary $\partial \Omega_i$ and $\chi(t,\infty)(\cdot)$ is the characteristic function of the interval $(0, \infty)$. The sets of admissible continuous and impulse controls on any interval $[t, T]$ are

$$U_i[t, T) := \{u_i(\cdot) : [t, T) \rightarrow U_i \text{ is measurable}) \},$$

$$X_i[t, T) := \{\xi_i(\cdot) : [t, T) \rightarrow X_i \text{ is piecewise constant}) \},$$

respectively, for $i = 1, \ldots, n$, where $U_i$ and $X_i$ are the sets of possible values of $u_i$ and $\xi_i$, respectively. A set of optimal strategy profile is a pair $\{u^*, \xi^*\}$, where $u^* \equiv u^*(\cdot) = (u^*_1(\cdot), \ldots, u^*_n(\cdot))$ and $\xi^* \equiv \xi^*(\cdot) = (\xi^*_1(\cdot), \ldots, \xi^*_n(\cdot))$ satisfying

$$J^x_\sigma(u^*(\cdot) ; \xi^*(\cdot)) \geq J^x_\sigma(u_i(\cdot) ; \xi^*_i(\cdot), \xi^*_{-i})(\cdot),$$

(2.4)

for any $u_i(\cdot) \in U_i[t, T)$, $\xi_i(\cdot) \in X_i[t, T)$, $i = 1, \ldots, n$. (Throughout this paper, we use the subscript “$-i$” to denote the sub-vector with all the components $j \in \{1, \ldots, n\} \setminus \{i\}$.)

2.1. Quasi-variational inequalities. We derive conditions for the Markovian strategies. Let

$$V^x_i(\cdot) = J^x_\sigma(u^*(\cdot) ; \xi^*(\cdot)) \text{ for } i \in \{1, \ldots, n\}, \quad \sigma \in \Sigma$$

denote the value functions, where $\{u^*(\cdot), \xi^*(\cdot)\}$ satisfies (2.4). A Markovian strategy is a mapping $(x, \sigma) \rightarrow (u^*(\cdot), \xi^*(\cdot))$ where $u^* = (u^*_1, \ldots, u^*_n)$ are the continuous controls and $\xi^* = (\xi^*_1, \ldots, \xi^*_n)$ are the impulse controls depending only on $(x, \sigma)$. The impulse controls form a Nash equilibrium. For any $i \in \{1, \ldots, n\}$ and $\sigma \in \Sigma$ we define the function

$$G^x_i(x, \xi, q) = \gamma_i^\sigma \gamma_i' (x) + q^\sigma, \quad x \in \overline{\Omega}, \quad \xi \in X, \quad q = (q^\sigma)_{\sigma \in \Sigma} \in \mathbb{R}^m$$

where $\sigma' = F(x, \sigma, \xi)$ and $X = \prod_{i=1}^n X_i$. Then, the payoff of player $i$ at the state $(x, \sigma)$ if the players take the impulse controls $\xi = (\xi_1, \ldots, \xi_n)$ is

$$V^x_i(x) = V^x_i(x) + \gamma_i^\sigma \gamma_i' (x) \equiv G^x_i(x, \xi, \delta_i(x)),$$

where $\delta_i(x) = (V^x_i(x))_{\sigma \in \Sigma}$. A Nash equilibrium $\xi^*$ satisfies the inequalities

$$G^x_i(x, \xi^*, \delta_i(x)) \geq G^x_i (x, \xi^*_{i-1}, \delta_i(x)) \text{ for any } \xi^*_i \in X_i, \quad i \in \{1, \ldots, n\}.$$  

The result of the actions $\xi^*$ is the mode change $\sigma \rightarrow \sigma^*$ where

$$\sigma^* = F(x, \sigma, \xi^*).$$

(2.5)
Since $V_i^\sigma (x)$ is the value function, it is necessary that the inequality
\[ V_i^\sigma (x) \geq G_i^\sigma (x, \xi^*, V_i(x)) \]  
holds, and only when this relation is equal can the change $\sigma \rightarrow \sigma^*$ be made.

We derive the condition for $u^*$ as follows. If the mode does not immediately change at $t$, it must satisfy the inequality
\[ V_i^\sigma (x) \geq \int_{t}^{t'} e^{-\rho_i(s-t)} g_i^\sigma(y(s), u^*(s)) \, ds + e^{-\rho_i(t'-t)} V_i^\sigma (y(t')) \]  
for $t' > t$ and near $t$, where $u^* (\cdot) = (u_1^* (\cdot), \ldots, u_n^* (\cdot)) \in \mathcal{U}[t, t')$ is the part of the Nash equilibrium strategy profile $u^* (\cdot)$ on $[t, t')$, and $y(t)$ is the solution of the initial-value problem
\[ dy_k / ds = f_k^\sigma (y(s), u^* (s)) \quad \text{for } t < s < t', \quad k = 1, \ldots, d. \]

The result is a Markovian strategy for the differential game. Assuming that $V_i^\sigma (x)$ is differentiable at $x$, then
\[ V_i^\sigma (y(t')) = V_i^\sigma (x) + DV_i^\sigma (x) (y(t') - x) + o(|y(t') - x|) \]
\[ = V_i^\sigma (x) + DV_i^\sigma (x) \cdot f^\sigma (x, u^* (t)) (t' - t) + o(t' - t), \]
where $f^\sigma = (f_1^\sigma, \ldots, f_d^\sigma)$ and $D$ represents the gradient operator. Substituting the right-hand side into (2.7), dividing the result by $t' - t$, and then taking the limit $t' \rightarrow t$, we obtain
\[ \rho_i V_i^\sigma (x) \geq g_i^\sigma (x, u^* (t)) + DV_i^\sigma (x) \cdot f^\sigma (x, u^* (t)). \]

We define Hamiltonians $H_i^\sigma$ by
\[ H_i^\sigma (x, u, p) = g_i^\sigma (x, u) + p \cdot f^\sigma (x, u) \quad \text{for } (x, u, p) \in \Omega \times U \times \mathbb{R}^d \]
where $U = \prod_{i=1}^{n} U_i$. Then an equilibrium continuous controls $u^* = (u_1^*, \ldots, u_n^*)$ satisfy the inequalities
\[ H_i^\sigma (x, u^*, DV_i^\sigma (x)) \geq H_i^\sigma (x, u_i, u_{\text{i}^{-1}}, DV_i^\sigma (x)) \quad \text{for any } u_i \in U_i, \quad i = 1, \ldots, n. \]

It follows from the above derivations that if the value function $V_i^\sigma$ is differentiable in $\Omega$, both inequalities (2.6) and (2.9) must hold, and at least one of them holds equal. This leads to the quasi-variational inequalities
\[ \min \{ \rho_i V_i^\sigma (x) - H_i^\sigma (x, u^*, DV_i^\sigma (x)), V_i^\sigma (x) - G_i^\sigma (x, \xi^*, V_i(x)) \} = 0, \]
where $u^* \in U$, $\xi^* \in X$, and $\sigma^* \in \Sigma$ satisfy the conditions
\[ H_i^\sigma (x, u^*, DV_i^\sigma (x)) \geq H_i^\sigma (x, u_i, u_{\text{i}^{-1}}, DV_i^\sigma (x)) \quad \text{for any } u_i \in U_i, \]
\[ G_i^\sigma (x, \xi^*, V_i(x)) \geq G_i^\sigma (x, \xi_i, \xi_{\text{i}^{-1}}, V_i(x)) \quad \text{for any } \xi_i \in X_i, \quad i = 1, \ldots, n. \]

In addition, the inequalities are supplemented with the boundary conditions
\[ V_i^\sigma (x) = \hat{V}_i^\sigma (x) \quad \text{for } x \in \partial \Omega, \quad i = 1, \ldots, n, \quad \sigma \in \Sigma. \]

In what follows, we regard $V^\sigma (x) = (V_i^\sigma (x))$ as an $m \times n$ matrix-valued function defined in $\overline{\Omega}$ for $\sigma \in \Sigma \equiv \{ \sigma_1, \ldots, \sigma_m \}$ and $i \in \{1, \ldots, n\}$. Similarly, $u^*$ and $\xi^*$ are also $m \times n$ matrix-valued mappings from $x \in \overline{\Omega}$ to $U$ and $X$, respectively. We use $\phi_i^\sigma (x)$ and $\psi_i^\sigma (x)$ to denote the entries of $u^*$ and $\xi^*$, respectively. A triplet $(V, u^*, \xi^*)$ is called a classical solution of System (2.10)–(2.12) if for all $i \in \{1, \ldots, n\}$.
and \( \sigma \in \Sigma \), \( V_i^\sigma (x) \) is continuous in \( \overline{\Omega} \), continuously differentiable in \( \Omega \), and satisfy the inequalities and the boundary condition in the classical sense. For any classical solution \( \{ V, u^*, \xi^* \} \) the trajectory \( y(\cdot, x, \sigma) \) and the jump times \( \{ t_i^{\sigma} \}_{i=1,2,\ldots} \) are generated by (2.1) with \( u(s) = (\phi_i^\sigma (y(s)), \ldots, \phi_n^\sigma (s)) \) and having the initial state \( y(t) = x, \theta(t) = \sigma \). We provide a verification theorem for classical solutions under the following hypothesis.

\[ (H): \] Let \( \xi_i \) denote the impulse control of player \( i \) taking no action, and let \( \xi = (\xi_1, \ldots, \xi_n) \). Then \( F(x, \sigma, \xi) = \sigma \) for any \( (x, \sigma) \in \overline{\Omega} \times \Sigma \). Also, functions \( \gamma_i^{\sigma'}(x) \) satisfy the conditions

\[ \gamma_i^{\sigma'}(x) = 0 \quad \text{for any} \quad (x, \sigma) \in \overline{\Omega} \times \Sigma, \quad i \in \{1, \ldots, n\}. \]

The first part of Hypothesis (H) indicates that if all players take no action, the mode does not change, and the second part of (H) indicates that if the mode does not change, no player receives a lumpsum payoff. Under this hypothesis we assert

**Theorem 2.1.** Let Hypothesis (H) hold. Suppose functions \( q_i^\sigma, \gamma_i^{\sigma\sigma'}, \) and \( f^\sigma \) for \( i \in \{1, \ldots, n\} \) and \( \sigma, \sigma ' \in \Sigma \) are continuously differentiable with respect to \( x \) and \( u \).

Let \( \{ V, u^*, \xi^* \} \) be a classical solution of Problem (2.10)–(2.12), with the associated continuous and impulse controls \( \{ \phi_i^\sigma (x) \} \) and \( \{ \psi_i (x) \} \) and the trajectories \( y(\cdot, x, \sigma) \) and \( \{ t_i^{\sigma} \}_{i=1,2,\ldots} \). Then \( u^*(\cdot) = (u_1^*(\cdot), \ldots, u_n^*(\cdot)) \) and \( \xi^*(\cdot) = (\xi_1^*(\cdot), \ldots, \xi_n^*(\cdot)) \) defined by

\[ u_i^*(s) = \phi_i^\theta(s)(y(s)), \quad \xi_i^*(s) = \psi_i^\theta(s)(y(s)) \quad \text{for} \quad s > t, \quad i = 1, \ldots, n \]

satisfy

\[ V_i^\sigma (x) = J_i^\sigma (u^*(\cdot), \xi^*(\cdot)) \quad \text{for} \quad x \in \Omega, \quad \sigma \in \Sigma, \quad i = 1, \ldots, n \]  

(2.13)

and (2.4).

The long proof is deferred to Appendix.

2.2. **Mode change.** We next derive a criterion for mode changes when the state space is of one dimension. That is, when \( d = 1 \) and \( \Omega \) is an interval. In this case, \( V_i^\sigma (x) \) is a single variable function and \( DV_i^\sigma (x) \) is a scalar function for any \( i \) and \( \sigma \).

Substituting a solution \( u^* \) of the first inequality in (2.11) into \( H_i^\sigma (x, u^*, DV_i^\sigma (x)) \) and denote

\[ \tilde{H}_i^\sigma (x, DV_1^\sigma (x), \ldots, DV_n^\sigma (x)) = H_i^\sigma (x, u^*, DV_i^\sigma (x)) \]

for \( x \in \Omega, i \in \{1, \ldots, n\} \), \( \sigma \in \Sigma \). If in a mode \( \sigma \) the system of equations

\[ \rho_i y_i - \tilde{H}_i^\sigma (x, p_1, \ldots, p_n) = 0, \quad i = 1, \ldots, n \]  

(2.14)

can be solved for \( p \) for any \( x \in \Omega \), then the following condition holds for the mode change \( \sigma \to \sigma^* \).

**Theorem 2.2.** Let Hypothesis (H) hold and let \( d = 1 \). Suppose \( x^* \in \Omega \) and there is a neighborhood \( (a, b) \subset \Omega \) of \( x^* \) such that the mode is \( \sigma' \) for \( x \in (a, x^*) \) and is \( \sigma \) for \( x \in (x^*, b) \). Also suppose that the system (2.14) is uniquely solvable for \( (p_1, \ldots, p_n) \) for \( x \in (a, b) \). Let \( N' \) be the subset of \( \{1, \ldots, n\} \) such that \( V_i^\sigma (x^*) = V_i^{\sigma'} (x^*) + \gamma_i^{\sigma\sigma'} (x^*) \) if and only if \( i \in N' \). Then, there exist numbers \( p_1^*, \ldots, p_n^* \in \mathbb{R} \) such that \( p_i^* = D (V_i^\sigma + \gamma_i^{\sigma\sigma'}) (x^*) \) for \( i \in N' \) and the following relations hold:

\[ \rho_i \left( V_i^{\sigma'} (x^*) + \gamma_i^{\sigma\sigma'} (x^*) \right) - \tilde{H}_i^\sigma (x^*, p_1^*, \ldots, p_n^*) = 0 \quad \text{for} \quad i \in N', \]

\[ \rho_i V_i^{\sigma'} (x^*) - \tilde{H}_i^\sigma (x^*, p_1^*, \ldots, p_n^*) = 0 \quad \text{for} \quad i \notin N'. \]  

(2.15)
Proof. Since the mode is $\sigma'$ for $x \in (a, x^*)$ and is $\sigma$ for $x \in (x^*, b)$, by (2.10),
\[
p_i V_i^\sigma(x) - \tilde{H}_i^\sigma(x, DV_i^\sigma(x), \ldots, DV_n^\sigma(x)) = 0
\]
for $x \in (x^*, b)$, $i = 1, \ldots, n$. Let the solution $(p_1, \ldots, p_n)$ of (2.14) be written as
\[
p_i = Q_i(x, y_1, \ldots, y_n) \quad \text{for } i = 1, \ldots, n.
\]
Then, system (2.16) can be written as
\[
DV_i^\sigma = Q_i(x, V_i^\sigma, \ldots, V_n^\sigma), \quad \text{for } x \in (x^*, b), \quad i = 1, \ldots, n.
\]
Furthermore,
\[
V_i^\sigma(x^*) = V_i^\sigma(x^*) + \gamma_i^\sigma(x^*) \quad \text{for } i \in N',
\]
\[
V_i^\sigma(x^*) = V_i^\sigma(x^*) \quad \text{for } i \notin N'.
\]
Hence the system of differential equations can be written in the integral form
\[
V_i^\sigma(x) = \int_{x^*}^x Q_i(s, V_i^\sigma(s), \ldots, V_n^\sigma(s)) \, ds \quad \text{for } i \in N',
\]
\[
V_i^\sigma(x) = V_i^\sigma(x^*) + \int_{x^*}^x Q_i(s, V_i^\sigma(s), \ldots, V_n^\sigma(s)) \, ds \quad \text{for } i \notin N', \quad x \in (x^*, b).
\]
Since the value functions are optimal, $Q_{x^*} V_i^\sigma(x) = 0$ for any $x \in (x^*, b)$ and $i \in N'$. Thus, by differentiating the right-hand side of the above equation with respect to $x^*$, we obtain
\[
0 = D \left( V_i^\sigma(x^*) + \gamma_i^\sigma(x^*) \right) (x^*) - Q_i(x^*, V_i^\sigma(x^*), \ldots, V_n^\sigma(x^*)) \quad \text{for } i \in N'.
\]
Let $p_i^* = D \left( V_i^\sigma + \gamma_i^\sigma \right) (x^*)$ for $i \in N'$ and
\[
p_i^* = Q_i(x^*, V_i^\sigma(x^*), \ldots, V_n^\sigma(x^*)) \quad \text{for } i \notin N'.
\]
We obtain the equations
\[
p_i^* = Q_i(x^*, V_i^\sigma(x^*), \ldots, V_n^\sigma(x^*)) \quad \text{for } i = 1, \ldots, n.
\]
These equations are equivalent to
\[
p_i V_i^\sigma(x^*) - \tilde{H}_i^\sigma(x^*, p_i^*, \ldots, p_n^*) = 0 \quad \text{for } i = 1, \ldots, n.
\]
Using the initial conditions in (2.17), we obtain (2.15). This completes the proof.

3. Application: Technology adoption in resource extraction process. In this section, we solve a technology adoption problem for a resource extraction process as an illustration of application of the results in Section 2. In a recent paper [16] Long et al study a differential game model of two competing companies extracting an exhaustible resource with two available technologies. Each company has at its disposal a continuous control and an impulse control. The former is the day by day consumption rate of the resource, and the latter is the timing of adopting the new technology. The authors of [16] propose a kind of “closed-loop” strategies for the companies, with Markovian continuous controls and non-Markovian impulse controls. In this section we develop pure Markovian feedback strategies with both continuous and impulse controls, and prove their optimality.

To motivate our formulation, we first describe the model proposed in [16]. The differential game model consists of two companies, indicated by a subscript $i \in \{1, 2\}$, and two technologies, old and new. The new technology is more efficient, but there is a cost for adopting it. Depending on which technology each company
uses, there are four possible modes \((1,1), (1,2), (2,1),\) and \((2,2)\), where the first component represents the technology that Company 1 is using and the second component represents the one that Company 2 is using, with 1 representing the old technology and 2 the new technology. It is assumed that the companies extract the resource in proportion to their consumption rates. Let \(\gamma_i^j\) be the constant proportionality for Company \(i\) using Technology \(j\). Then, the reciprocal of \(\gamma_i^j\) represents the efficiency of Technology \(j\) for Company \(i\), and the rate of extraction of the resource by Company \(i\) is \(-\gamma_i^j u_i(t)\). Hence, the resource stock \(y(t)\) is governed by the differential equation
\[
\dot{y}(t) = -\gamma_1^{\mu_1} u_1(t) - \gamma_2^{\mu_2} u_2(t) \tag{3.1}
\]
in mode \(\sigma = (\mu_1, \mu_2)\). It is assumed in [16] that the instantaneous payoff for Company \(i\) is \(\ln u_i(t)\) and the lumpsum cost of Company \(i\) adopting the new technology at time \(t_i\)
\[
\omega_i(t_i) \equiv \alpha_i + \beta_i y(t_i),
\]
for some positive constants \(\alpha_i\) and \(\beta_i\). In addition, the payoff is exponentially discounted with a rate \(\rho \in (0,1)\). Thus the total payoff of Company \(i\) starting extraction at time \(t\) when the resource stock is \(x\) and the mode is \(\sigma\) is
\[
J_i^{y,\sigma}(u(\cdot), \xi(\cdot)) \equiv \int_t^T e^{-\rho(s-t)} \ln[u_i(s)] \, ds - e^{-\rho(t_i-t)} \omega_i(t_i) \chi_{(t,T)}(t_i)
\]
for \(i = 1, 2\), where \(T\), either finite or infinite, is the time when the extraction ends, and \(\chi_{(t,T)}(\cdot)\) is a function such that \(\chi_{(t,T)}(t_i) = 1\) if \(t_i \in (t,T)\) and \(\chi_{(t,T)}(t_i) = 0\) if \(t_i \notin (t,T)\). (It is shown below that \(T\) is actually a finite number.)

Each company has two types of controls, the continuous control, \(u_i(t)\), which is the instantaneous rate of consumption of the resource, and the impulse control, \(\xi_i\), which is switching to the new technology. A strategy of a company consists of both continuous and impulse controls at any time. A Markovian strategy is one such that both \(u_i\) and \(\xi_i\) depend directly on \((y, \sigma)\), but not directly on \(t\). This means there are functions \(\phi_i\) and \(\psi_i\) defined on \(\mathbb{R}_+ \times \Sigma\), where \(\Sigma\) is the set of possible modes, such that
\[
u_i(t) = \phi_i(y(t), \sigma(t)), \quad \xi_i = \psi_i(y(t), \sigma(t)).
\]
The closed-loop strategies proposed in [16] assume that the second company switching to the new technology chooses its switching time based on the state \((y, \sigma)\) at the time when the first company switches to the new technology. For example, if Company 1 first switches to the new technology at \(t_1\). Then Company 2 switches at \(t_2\) based on \(y(t_2)\) but on \(y(t_1)\). The authors term such a strategy a “piecewise closed-loop” strategy. Such a strategy discounts the effect of the observation that the company makes after the other company has switched to the new technology on the company’s switching time. It also causes the strategy rather complicated and difficult to generalize to multi-company and multi-technology cases.

We prove the existence of pure Markovian strategies of the continuous and impulse controls, and develop a computing algorithm of the Markovian strategies in two cases. One case is the optimal control problem with one company and any number of technologies. The other is the differential game model with any number of companies and two technologies. The more general case of \(n\) companies and \(m\) technologies quickly become extraordinary complicated as \(n\) and \(m\) increase. Therefore we content ourselves with these two special cases. It will be shown that in the
first case the value functions can be analytically solved, while in the second case the value functions cannot be analytically solved, but can be numerically computed.

3.1. One company with multiple technologies. In this subsection we consider the case where there is one company with multiple technologies. The company wants to choose the optimal rate of consumption and optimal times to change technology. This is a typical optimal control problem with continuous and impulse controls. We first give the form of the value function and derive the criterion for the value of the resource stock at the mode change, and then formulate a computing algorithm for determining the optimal continuous and impulse controls. An example is provided to illustrate the algorithm.

Suppose there are \( m \) technologies, denoted by \( \sigma = 1, \ldots, m \). The resource stock is governed by the differential equation

\[
y'(s) = -\gamma^\sigma u^\sigma(s) \quad \text{in mode } \sigma.
\]

The quasi-variational inequalities consists of

\[
\rho V^\sigma - \ln u^\sigma + \gamma^\sigma u^\sigma V_x^\sigma \geq 0, \quad \sigma = 1, 2,
\]

and

\[
V^\sigma(x) \geq V'^{\sigma'}(x) - \omega^{\sigma\sigma'}(x)
\]

at any \( x > 0 \), where

\[
\omega^{\sigma\sigma'}(x) = \alpha^{\sigma\sigma'} + \beta^{\sigma\sigma'}x
\]

is the cost of technology change from type \( \sigma \) to type \( \sigma' \). It is easy to see that

\[
u^\sigma = \frac{1}{(\gamma^\sigma V_x^\sigma)}.
\]

So the quasi-variational inequalities (2.10) have the form

\[
\min \left\{ \rho V^\sigma(x) + \ln (\gamma^\sigma V_x^\sigma(x)) + 1, V^\sigma(x) - V'^{\sigma'}(x) + \omega^{\sigma\sigma'}(x) \right\} = 0
\]

for \( \sigma = 1, \ldots, m \).

3.1.1. Value function in the terminal mode. The following theorem gives the value function in the terminal mode.

**Theorem 3.1.** Suppose mode \( \sigma \) continues until the end of extraction when the resource is exhausted. Also suppose that \( V^\sigma(x_0) = V_0^\sigma \) for some \( x_0 \geq 0 \). Then for any \( x > x_0 \) the value function \( V^\sigma \) is given by

\[
V^\sigma(x) = \frac{1}{\rho} \ln \left[ e^{\rho V_0^\sigma} + \frac{\rho}{\gamma^\sigma} e^{(x-x_0)} \right].
\]

**Proof.** In mode \( \sigma \), \( V^\sigma(x) \) satisfies the initial value problem

\[
\rho V^\sigma(x) + \ln (\gamma^\sigma V_x^\sigma(x)) + 1 = 0, \quad V^\sigma(x_0) = V_0^\sigma.
\]

By differentiating the two-sides of the differential equation, we find that the derivative \( P^\sigma = V_x^\sigma \) satisfies the initial value problem

\[
\rho P^\sigma(x) + P_x^\sigma(x)/P^\sigma(x) = 0, \quad P^\sigma(x_0) = e^{-1-\rho V_0^\sigma}/\gamma^\sigma.
\]

The separable equation has the solution

\[
P^\sigma(x) = 1/\left[ \gamma^\sigma e^{1-\rho V_0^\sigma} + \rho (x-x_0) \right].
\]
So
\[ V^\sigma(x) = -\frac{1}{\rho} \ln (\gamma^\sigma P^\sigma(x)) + 1 = \frac{1}{\rho} \ln \frac{\gamma^\sigma e^{1+\rho V_0^\sigma} + \rho(x-x_0)}{\gamma^\sigma e} = \frac{1}{\rho} \ln \left[ e^{\rho V_0^\sigma} + \frac{\rho}{\gamma^\sigma e} (x-x_0) \right]. \]

3.1.2. Criterion for change of technology. We show that if technology changes from type \(\sigma\) to type \(\sigma'\) as \(x\) decreases across \(x^*\), then the relation
\[ \rho \left[ V^{\sigma'}(x^*) - \omega^{\sigma\sigma'}(x^*) \right] + 1 + \ln \left[ \gamma^\sigma \left( V_x^{\sigma'}(x^*) - \beta^{\sigma\sigma'} \right) \right] = 0 \] (3.9)
holds. Furthermore, for \(x > x^*\) the value function \(V^\sigma(x)\) is given by
\[ V^\sigma(x) = \frac{1}{\rho} \ln \left[ e^{\rho \left[ V^{\sigma'}(x^*) - \omega^{\sigma\sigma'}(x^*) \right]} + \frac{\rho(x-x^*)}{\gamma^\sigma e} \right]. \] (3.10)

**Theorem 3.2.** Suppose the mode changes from \(\sigma\) to \(\sigma'\) as \(x\) decreases across \(x^*\) and there is no other mode change in a neighborhood of \(x^*\). Then \(x^*\) satisfies (3.9) and the value function \(V^\sigma(x)\) in mode \(\sigma\) is given by (3.10).

**Proof.** Note that (3.5) is a special case of (2.10) with \(n = 1\), \(\gamma^{\sigma\sigma'} = -\omega^{\sigma\sigma'}\), and \(H^\sigma(u,p) = \ln u - \gamma^\sigma up\). By (3.4),
\[ H^\sigma(x,V_x^\sigma) = -\ln [\gamma^\sigma V_x^\sigma] - 1. \]
Hence (2.14) has the form
\[ \rho y + \ln [\gamma^\sigma p] + 1 = 0. \]
This equation can be uniquely solved for \(p\). Hence, relation (3.9) follows from the first relation in (2.15) in Theorem 2.2. Furthermore, at \(x^*\) the inequality (3.3) holds equal. That is,
\[ V^\sigma(x^*) = V^{\sigma'}(x^*) - \omega^{\sigma\sigma'}(x^*). \] (3.11)
Hence (3.10) follows from (3.6). \(\square\)

3.1.3. Computing algorithm. Theorems 3.1 and 3.2 give the value function within each mode and the criterion of \(x^*\) at a mode change. Based on these results we can find the Markovian feedback controls as follows. Let \(\sigma'\) be the terminal mode in which the extraction of the resource ends. Tracing backward, the last technology change occurs at \(x^* = x_{\sigma'\sigma}^*\) that satisfies the equation (3.9), where \(\sigma\) is the mode before the technology change and \(V^{\sigma'}(x)\) is value function in mode \(\sigma'\) given by (3.6) with \(\sigma\) replaced by \(\sigma'\) and \(V_0^\sigma = 0\). Using (3.6) with \(V_0^\sigma = V^{\sigma'}(x^*) - \omega^{\sigma\sigma'}(x^*)\) we can find the value function \(V^\sigma(x)\) in mode \(\sigma\). We can then find the resource stock at previous technology change by (3.9), and the value function before the change by (3.6). This process can be repeated until \(x\) reaches the initial resource stock. In each mode, the optimal continuous control is given by (3.4), and at the change of mode, the impulse control \(\sigma \mapsto \sigma'\) is taken at \(x^*\) that is determined by (3.9).

Suppose the modes are ordered so that \(\gamma^{\sigma'} < \gamma^\sigma\) if \(\sigma' > \sigma\), and suppose that \(\alpha^{\sigma\sigma'} \geq 0, \beta^{\sigma\sigma'} \geq 0\) if \(\sigma' > \sigma\), and at least one of \(\alpha^{\sigma\sigma'}\) and \(\beta^{\sigma\sigma'}\) is positive. We propose the following algorithm for computing the optimal controls.

```markdown
Theorem 3.2. Suppose the mode changes from \(\sigma\) to \(\sigma'\) as \(x\) decreases across \(x^*\) and there is no other mode change in a neighborhood of \(x^*\). Then \(x^*\) satisfies (3.9) and the value function \(V^\sigma(x)\) in mode \(\sigma\) is given by (3.10).

Proof. Note that (3.5) is a special case of (2.10) with \(n = 1\), \(\gamma^{\sigma\sigma'} = -\omega^{\sigma\sigma'}\), and \(H^\sigma(u,p) = \ln u - \gamma^\sigma up\). By (3.4),
\[ H^\sigma(x,V_x^\sigma) = -\ln [\gamma^\sigma V_x^\sigma] - 1. \]
Hence (2.14) has the form
\[ \rho y + \ln [\gamma^\sigma p] + 1 = 0. \]
This equation can be uniquely solved for \(p\). Hence, relation (3.9) follows from the first relation in (2.15) in Theorem 2.2. Furthermore, at \(x^*\) the inequality (3.3) holds equal. That is,
\[ V^\sigma(x^*) = V^{\sigma'}(x^*) - \omega^{\sigma\sigma'}(x^*). \] (3.11)
Hence (3.10) follows from (3.6). \(\square\)

3.1.3. Computing algorithm. Theorems 3.1 and 3.2 give the value function within each mode and the criterion of \(x^*\) at a mode change. Based on these results we can find the Markovian feedback controls as follows. Let \(\sigma'\) be the terminal mode in which the extraction of the resource ends. Tracing backward, the last technology change occurs at \(x^* = x_{\sigma'\sigma}^*\) that satisfies the equation (3.9), where \(\sigma\) is the mode before the technology change and \(V^{\sigma'}(x)\) is value function in mode \(\sigma'\) given by (3.6) with \(\sigma\) replaced by \(\sigma'\) and \(V_0^\sigma = 0\). Using (3.6) with \(V_0^\sigma = V^{\sigma'}(x^*) - \omega^{\sigma\sigma'}(x^*)\) we can find the value function \(V^\sigma(x)\) in mode \(\sigma\). We can then find the resource stock at previous technology change by (3.9), and the value function before the change by (3.6). This process can be repeated until \(x\) reaches the initial resource stock. In each mode, the optimal continuous control is given by (3.4), and at the change of mode, the impulse control \(\sigma \mapsto \sigma'\) is taken at \(x^*\) that is determined by (3.9).

Suppose the modes are ordered so that \(\gamma^{\sigma'} < \gamma^\sigma\) if \(\sigma' > \sigma\), and suppose that \(\alpha^{\sigma\sigma'} \geq 0, \beta^{\sigma\sigma'} \geq 0\) if \(\sigma' > \sigma\), and at least one of \(\alpha^{\sigma\sigma'}\) and \(\beta^{\sigma\sigma'}\) is positive. We propose the following algorithm for computing the optimal controls.

```
Step 1.: Find the value functions \( V^m(x;0) \) by (3.6) with \( V^0_\sigma = 0 \) to get
\[
V^m(x;0) = \frac{1}{\rho} \ln \left[ 1 + \frac{\rho}{\gamma^m e} x \right].
\]
Then, for each \( \sigma < m \), find \( x^*_\sigma \) by solving
\[
\rho \left[ V^m(x^*_\sigma) - \omega^\sigma m (x^*_\sigma) \right] + 1 + \ln \left[ \gamma^\sigma \left( V^m(x^*_\sigma) - \beta^\sigma m \right) \right] = 0. \tag{3.12}
\]
These are the resource stocks when the company switches from mode \( \sigma \) to \( m \) directly. Draw lines connecting 0 and \( x^*_\sigma \) for all \( x^*_\sigma > 0 \) and label the line by \( m \). If there exists a \( \sigma < m \) that is not connected to \( m \), find the largest \( m' > \sigma \) such that \( x^*_m > 0 \). Draw a line connecting 0 and \( x^*_m \). If no such \( m' \) exists, connect 0 by \( \sigma \) to any \( x^*_m \) for all \( \sigma' < \sigma \) such that \( x^*_m > 0 \). Repeat this process until each mode is either labeled on a segment connecting 0 or there is a mode \( \sigma > \sigma \) such that \( x^*_\sigma > 0 \).

Step 2.: For any \( \sigma \) that is labeled on a line segment connecting 0 and any \( \sigma' < \sigma \) such that \( x^*_\sigma > 0 \), find \( V^{\sigma'}(x;x^*_\sigma) \) by
\[
V^{\sigma'}(x;x^*_\sigma) = \frac{1}{\rho} \ln \left[ \frac{\rho (x - x^*_\sigma)}{\gamma^\sigma e} + e^{\rho \left[ V^\sigma(x^*_\sigma) - \omega^\sigma \sigma' (x^*_\sigma) \right]} \right].
\]
Then, for each \( \sigma'' < \sigma' \), find \( x^*_{\sigma'';\sigma'} \) by solving the equation
\[
\rho \left[ V^{\sigma'}(x^*_{\sigma'';\sigma'};x^*_\sigma) - \omega^\sigma \sigma' (x^*_\sigma) \right] + 1 + \ln \left[ \gamma^{\sigma''} \left( V^{\sigma'}(x^*_{\sigma'';\sigma'}) - \beta^{\sigma''} \sigma' \right) \right] = 0. \tag{3.13}
\]
If \( x^*_{\sigma'';\sigma'} > x^*_\sigma \), draw a line connecting \( x^*_{\sigma'';\sigma'} \) and \( x^*_\sigma \) and label the line by \( \sigma' \). Repeat this step to all modes less than \( \sigma'' \), etc.

Step 3.: Let \( x_p \) and \( \sigma_1 \) denote the present resource stock and mode, respectively. Identify all paths that connect \( x_p \) and 0 by line segments in the form \( \sigma_1 \mapsto \sigma_2 \mapsto \cdots \mapsto \sigma_k \). For each such path, compute \( V^{\sigma_i}(x_p;x^*_\sigma) \) and choose the maximum of these values. The optimal impulse controls consist of switching modes labeled on the segments on the chosen path, at the resource stock which are the nodes of the path. The continuous controls in mode \( \sigma_i \) is
\[
u^{\sigma_i}(x) = 1/\left( \gamma^\sigma_p;P^\sigma_{\sigma_i}(x) \right) = e^{1+\rho V^{\sigma_i}(x^*_{\sigma_i+1})} + \frac{\rho}{\gamma^\sigma_{\sigma_i}} \left( x - x^*_{\sigma_\sigma_i+1} \right) \tag{3.14}
\]
for \( i = 1, \ldots, k - 1 \).

We can find the time \( t \) in terms of \( x \) as follows. From Eq. (3.2) we find
\[
dt = -\gamma^\sigma u^\sigma(y) \quad \text{dy} = -\frac{dy}{P^\sigma(y)} \quad \text{in mode} \; \sigma.
\]
Using (3.8), we derive
\[
t - t_0 = -\int_{x_0}^{x} \frac{dy}{\gamma^\sigma e^{1+\rho V^\sigma_{\sigma}(y)} + \rho (x - x_0)} = -\frac{1}{\rho} \ln \left[ 1 + \frac{\rho}{\gamma^\sigma e^{1+\rho V^\sigma_{\sigma}}(x - x_0)} \right]
\]
where \( x_0 = y(t_0) \) and \( V_0^\sigma = V^\sigma(y(t_0)) \). Let \( t = 0 \) be the present moment. From the above equation we find the time when the resource stock reaches \( x^*_{\sigma_i,\sigma_{i+1}} \),

\[
t_{\sigma_i,\sigma_{i+1}}^* = \frac{1}{\rho} \ln \left[ 1 + \frac{\rho}{\gamma_{\sigma_i}} e^{1+\rho V^{\sigma_i}(x^*_{\sigma_{i+1}})} (x^*_p - x^*_{\sigma_i,\sigma_{i+1}}) \right],
\]

\[
t_{\sigma_i,\sigma_{i+1}}^* = \frac{1}{\rho} \ln \left[ 1 + \frac{\rho}{\gamma_{\sigma_{i+1}} e^{1+\rho V^{\sigma_{i+1}}(x^*_{\sigma_{i+1}})}} (x^*_{\sigma_{i-1},\sigma_{i}} - x^*_{\sigma_i,\sigma_{i+1}}) \right],
\]

for \( i = 2, \ldots, k \). The extraction ends at

\[
t_{\sigma_k,m'} = t_{\sigma_{k-1},\sigma_k}^* + \frac{1}{\rho} \ln \left[ 1 + \frac{\rho}{\gamma_{m'} e^{x^*_{\sigma_{k-1},\sigma_k}}} \right].
\]

### 3.1.4. An example.

We present an example to illustrate the use of the algorithm. Suppose there are three technologies, with Technology 1 being the least efficient and Technology 3 the most efficient. The company is currently using Technology 1. Let \( \sigma = 1, 2, \) or 3 denote the mode when the company is using Technologies 1, 2, or 3, respectively.

Suppose the parameters are

\[
\rho = 0.04, \quad \gamma^1 = 2, \quad \gamma^2 = 1.2, \quad \gamma^3 = 1, \quad \alpha^{12} = 1, \quad \alpha^{13} = 2,
\]

\[
\alpha^{23} = 1.5, \quad \beta^{12} = 0.01, \quad \beta^{13} = 0.03, \quad \beta^{23} = 0.02,
\]

and at the beginning the resource stock is \( x_p = 1000 \), while the company is using Technology 1. Solving Eq. (3.12) in Step 1 for \( \sigma = 1, 2 \), which takes the form

\[
\ln \left[ 1 + \frac{\rho}{\gamma^\sigma e^{x^\sigma_{\sigma_3}}} \right] - \alpha^{\sigma 3} x^\sigma_{\sigma_1} + 1 + \ln \left[ \frac{\gamma^\sigma}{\gamma^\sigma e^{x^\sigma_{\sigma_1}}} \right] = 0,
\]

we find

\[
x^*_{13} = 301.87, \quad x^*_{23} = 50.28.
\]

Solving Eq. (3.13) in Step 2 for \( \sigma' = 1 \), we find

\[
x^*_{12} = 733.99.
\]

We draw the diagram as described in Steps 1 and 2 above. It can be seen that there are two possible paths: 1 \( \mapsto \) 3 and 1 \( \mapsto \) 2 \( \mapsto \) 3. This means that the company has two choices: (1) First switch to Technology 2 when \( x = x^*_{12} = 733.99 \), then switch to Technology 3 when \( x = x^*_{23} = 50.28 \); and (2) use Technology 1 until \( x = x^*_{13} = 301.87 \), then switch to Technology 3. Using strategy (1) the total payoff is

\[
V^1(x_p; x^*_{12}) = 55.18,
\]

and using strategy (2) the total payoff is

\[
V^1(x_p, x^*_{13}) = 53.89.
\]

This means strategy (1) is better. Therefore, the impulse controls are adopting Technology 2 when \( x = 733.99 \) and then adopting Technology 3 when \( x = 50.28 \).

The continuous controls are given by (3.14),

\[
u^1(x) = e^{1+\rho V^1(x^*_{12})} + \frac{\rho}{\gamma^1} (x - x^*_{12}) \quad \text{for} \quad x^*_{12} < x < x_p,
\]

\[
u^2(x) = e^{1+\rho V^2(x^*_{23})} + \frac{\rho}{\gamma^2} (x - x^*_{12}) \quad \text{for} \quad x^*_{23} < x < x^*_{12},
\]

\[
u^3(x) = e + \frac{\rho}{\gamma^3} x \quad \text{for} \quad 0 < x < x^*_{23}
\]
where

\[ V^1(x_{12}^{*}) = V^2(x_{12}^{*}; x_{23}^{*}) - \omega^{12}(x_{12}^{*}) = 49.12, \]
\[ V^2(x_{23}^{*}) = V^3(x_{23}^{*}; 0) - \omega^{23}(x_{23}^{*}) = 11.34. \]

Technology changes occur at

\[ t_{12}^{*} = \frac{1}{\rho} \ln \left[ 1 + \frac{\rho}{\gamma^1 e^{1 + \rho V^1(x_{12}^{*})}} (x_p - x_{12}^{*}) \right] = 6.06, \]
\[ t_{23}^{*} = t_{12}^{*} + \frac{1}{\rho} \ln \left[ 1 + \frac{\rho}{\gamma^2 e^{1 + \rho V^2(x_{23}^{*})}} (x_{12}^{*} - x_{23}^{*}) \right] = 52.18, \]

and the extraction ends at

\[ T = t_{23}^{*} + \frac{1}{\rho} \left[ 1 + \frac{\rho}{\gamma^3 e^{x_{23}^{*}}} \right] = 66.03. \]

Fig. 3.2 shows the extraction rate \( \gamma^\sigma u^\sigma \) and the consumption rate \( u^\sigma \) as time increases.

**Figure 3.1.** Possible strategies with parameter values given by (3.15).

**Figure 3.2.** Extraction and consumption rates using parameters in (3.15).
Suppose in the above example, $\gamma^2$ is changed to 1.6 and other parameters remain the same. Then $x^*_13, x^*_23$ and $x^*_12$ have the values

$$x^*_13 = 301.87, \quad x^*_23 = 276.35, \quad x^*_12 = 253.29.$$  

![Diagram](image)

**Figure 3.3.** Possible strategies with $\gamma^2 = 1.6$ and other parameter values given by (3.15).

As shown in Fig. 3.3, since $x^*_12 < x^*_23$, there is no connection between $x^*_23$ and $x_p$. The only path connecting 0 and $x_p$ is 1 $\rightarrow$ 3. This means Technology 2 will never be adopted, and the company will use Technology 1 until the resource stock reaches $x = 301.87$. At this time $t^*_13 = 22.59$. The continuous controls are

$$u^1(x) = e^{1+\rho V^1(x^*_13)} + \frac{\rho}{\gamma^1} (x - x^*_13) \quad \text{for} \quad x^*_13 < x < x_p,$$

$$u^3(x) = e + \frac{\rho}{\gamma^3} x \quad \text{for} \quad 0 < x < x^*_13,$$

where

$$V^1(x^*_13) = V^3(x^*_13, 0) - \omega^{13}(x^*_13) = 31.30.$$  

The extraction ends when $t = 64.95$. Fig. 3.4 shows how the extraction and the consumption rates change with time.

### 3.2. **Multiple companies with two technologies.**

In this subsection we consider the case where there are $n$ companies with two technologies. In this case the modes can be represented by the $n$-tuples

$$\sigma = (\mu_1, \ldots, \mu_n)$$

where $\mu_i \in \{1, 2\}$ meaning that Company $i$ is using Technology $\mu_i$. Let $\Sigma$ denote all $2^n$ possible modes. The resource stock equation has the form

$$y'(s) = -\sum_{i=1}^{n} \gamma^{1\sigma_i} u^{\mu_i}(s)$$

and the value functions $V_i^\sigma$ satisfy the inequalities

$$\rho V_i^\sigma - \ln u^{\mu_i} + DV_i^\sigma \sum_{j=1}^{n} \gamma^{1\mu_j} u^{\mu_j} \geq 0,$$

$$V_i^\sigma(x) \geq V_i^{\sigma'}(x) - \omega_i^{\mu_i\mu'_i}(x), \quad i = 1, \ldots, n$$

where

$$\sigma = (\mu_1, \ldots, \mu_n), \quad \sigma' = (\mu'_1, \ldots, \mu'_n)$$

and

$$\omega_i^{\mu_i\mu'_i}(x) = \alpha^{\mu_i\mu'_i} + x\beta_i^{\mu_i\mu'_i}$$
Figure 3.4. Extraction and consumption rates when $\gamma^2$ is changed to 1.6.

represents the cost of Company $i$ switching from Technology $\mu_i$ to Technology $\mu'_i$. The maximizing consumption rates $u^\mu_i$ satisfies

$$u^\mu_i = \arg \max_{v_i \geq 0} \left\{ \rho V^\sigma_i - \ln v_i + D V^\sigma_i \left[ \gamma^\mu_i v_i + \sum_{j \neq i} \gamma^\mu_j u^\mu_j \right] \right\}.$$

The solution is

$$u^\mu_i = 1 / (\gamma^\mu_i D V^\sigma_i), \quad i = 1, \ldots, n.$$

Therefore, the differential inequalities for the value functions become

$$\rho V^\sigma_i + \ln (\gamma^\mu_i D V^\sigma_i) + \sum_{j=1}^n D V^\sigma_j / D V^\sigma_j \geq 0, \quad \text{for } i = 1, \ldots, n. \quad (3.18)$$

3.2.1. Within one mode. Suppose the companies do not change the technologies on an interval of $x$ so that the mode $\sigma$ does not change until the resource stock reaches certain value $x_0$. Then (3.18) becomes the system of equations

$$\rho V^\sigma_i + \ln (\gamma^\mu_i D V^\sigma_i) + \sum_{j=1}^n D V^\sigma_j / D V^\sigma_j = 0 \quad \text{for } x > x_0, \quad V^\sigma_i (x_0) = V^\sigma_i. \quad (3.19)$$

This initial value problem cannot be solved explicitly. However, we show that it has a unique classical solution for any nonnegative initial values $V^\sigma_{i,0}, \ldots, V^\sigma_{n,0}$.

Theorem 3.3. For any $\sigma \in \Sigma$, Problem (3.19) has a unique classical solution $(V^\sigma_1, \ldots, V^\sigma_n)$ defined for any $x > 0$.

The lengthy proof of theorem is deferred to Appendix. Based on the proof of this theorem, the system of differential equations (3.19) can be solved in two steps:
Step 1.: Solve functions $P_i^\sigma (x) = DV_i^\sigma (x)$ for $x \geq 0$, $i = 1, \ldots, n$, from the differential equations
\[
\rho P_i^\sigma + \frac{DP_i^\sigma}{P_i^\sigma} + \sum_{j \neq i} \frac{P_j^\sigma DP_i^\sigma - P_i^\sigma DP_j^\sigma}{(P_j^\sigma)^2} = 0 \quad \text{for } i = 1, \ldots, n. \tag{3.20}
\]
(cf. (A.13)) with the initial values determined by (A.20). The solution exists and is unique. Even though the initial value problem cannot be solved analytically, it can be solved numerically by many popular numerical methods for differential equations, such as the Runge-Kutta method.

Step 2.: Find the solution $(V_1^\sigma (x), \ldots, V_n^\sigma (x))$ of Problem (3.19) by using the equations
\[
V_i^\sigma (x) = -\frac{1}{\rho} \left[ \ln (\gamma_i^\sigma P_i^\sigma (x)) + \sum_{j=1}^n \frac{P_j^\sigma (x)}{P_j^\sigma (x)} \right] \quad \text{for } x > 0. \tag{3.21}
\]

As shown in the proof of Theorem 3.3 in Appendix, $V_i^\sigma$ satisfies (3.19).

3.2.2. Change of mode. At certain values of $x$ one or more companies may change the technology, resulting in a mode change. Suppose that Company $i$ changes the technology, resulting in a mode change. Suppose that Company $i$ changes the technology from $\mu_i$ to $\mu_i^*$ as $x$ decreases across $x_*$. Let $\sigma$ and $\sigma'$ be the modes before and after the company changes the technology, respectively. We suppose that no other mode change occurs in a neighborhood of $x_*$. In this case Theorem 2.2 ensures that the relations
\[
\rho \left( V_i^{\sigma'} (x^*) - \omega_i^{\mu_i^*} (x^*) \right) + \ln \left[ \gamma_i^{\mu_i} (DV_i^{\sigma'} (x^*) - \beta_i^{\mu_i \mu_i^*}) \right] + \sum_{j \neq i} \left( DV_i^{\sigma'} (x^*) - \beta_i^{\mu_i \mu_i^*} \right) / p_j^* = -1,
\]
\[
\rho V_j^{\sigma'} (x^*) + \ln \left( \gamma_j^{\mu_j} p_j^* \right) + p_j^* / \left( DV_j^{\sigma'} (x^*) - \beta_j^{\mu_j \mu_j^*} \right) + \sum_{l \neq j, i} p_j^*/p_l^* = -1, \quad j \neq i \tag{3.22}
\]
hold for some positive constants $p_j^*, j \in \{1, \ldots, n\} \setminus \{i\}$, provided that the system
\[
\rho y_i + \ln (\gamma_i^{\mu_i} p_i) + \sum_{j=1}^n p_i/p_j = 0 \quad \text{for } i = 1, \ldots, n
\]
is uniquely solvable for $p_1, \ldots, p_n$ for any $y_1, \ldots, y_n \in \mathbb{R}$. The solvability of the system is given by Lemma A.1 in Appendix with $a_i = \rho y_i + \ln \gamma_i^{\mu_i}$. Therefore (3.22) holds. Furthermore, if a solution $x^*$ of (3.22) exists, the value functions $V_i^\sigma (x), \ldots, V_n^\sigma (x)$ satisfy the equations (3.19) with the initial conditions
\[
V_i^\sigma (x^*) = V_i^{\sigma'} (x^*) - \omega_i^{\mu_i^*} (x^*), \quad V_j^{\sigma'} (x^*) = V_j^{\sigma'} (x^*) \quad \text{for } j \neq i. \tag{3.23}
\]

3.2.3. Markovian strategies. Based on the algorithm for solving Problem (3.19) described below Theorem 3.3, and the conditions for the mode change in (3.22), we can propose the following algorithm for finding the Markovian feedback strategies for multi-company two-technology game.
Step 1.: Start with the mode \( \sigma_0 = (2, \ldots, 2) \). Solve the initial value problem (3.19) with \( \sigma = \sigma_0 \) and \( V^\sigma_{1,0} = 0 \). Then for any \( i = 1, \ldots, n \), solve (3.22) with \( \mu_i = 1 \) and \( \mu_i = 2 \). Let the solution be denoted as \( x^*_i \). Find \( \min \{x^*_1, \ldots, x^*_n\} \), say \( x^*_1 \). If either \( x^*_i < 0 \) or \( V^\sigma_{i,0} (x^*_i) - \omega_i (x^*_i) < 0 \), change \( \mu_i \) in \( \sigma_0 \) to 1 where \( i \) is the one such that \( x^*_i \) is the minimum. Repeat this process with the new mode. Continue this process until all \( x^*_i \) are positive. The resulting mode is the terminal mode, and the index \( i \) such that \( x^*_i \) is the minimum is the last company that adopt the new technology. Denote the terminal mode by \( \sigma_1 \), and denote the company that adopts the new technology by \( m_1 \).

Step 2.: Let \( \sigma_2 \) be the mode whose \( m_1 \)-th component is 1 and other component the same as that in \( \sigma_1 \). Solve the initial value problem (3.19) with the initial values

\[
V^\sigma_{m_1,0} = V^\sigma_{m_1} (x^*_m) - \omega_{m_1} (x^*_m), \quad V^\sigma_{j,0} = V^\sigma_j (x^*_m) \quad \text{for } j \neq m_1.
\]

Then for any \( i \in \{1, \ldots, n\} \) such that the \( i \)-th component in \( \sigma_2 \) is 2, solve (3.22) and denote the solution by \( x^*_i \). Find the minimum of \( \{x^*_i\} \) and denote the index by \( m_2 \). Repeat this process until the mode reaches the present mode, \( \sigma_p \).

Step 3.: The impulse controls are \( \sigma_p \mapsto \cdots \mapsto \sigma_2 \mapsto \sigma_1 \) at the resource stock \( x^*_{m_p}, \ldots, x^*_{m_1} \), respectively. The continuous control in mode \( \sigma_k \) is

\[
\mu^k_i (x) = 1/(\gamma^k_i DV^\sigma_k (x)) \quad \text{for } x^*_{m_{k-1}} < x < x^*_{m_k}, \quad i = 1, \ldots, n,
\]

where \( \mu^k_i \) is the \( i \)-th component of \( \sigma_k \), \( x^*_{m_0} = 0 \), and \( x^*_{m_p} = x_p \) is the present resource stock. The time at which a company adopts the new technology, \( t^*_{m_k} \), can be found by the differential equation (3.16), which can be written in the form

\[
dt = - \sum_{j=1}^{n} \frac{dy}{P^\sigma_j (y)}
\]

where \( P^\sigma_j = DV_j^\sigma \). Integrating the two sides on the interval \( (x^*_{m_{k-1}}, x^*_{m_k}) \) successively for \( k = p, \ldots, 2, 1 \), with \( t_p = 0 \), we find

\[
t^*_{m_{p-1}} = \sum_{j=1}^{n} \int_{x^*_{m_{p-1}}}^{x^*_{m_p}} \frac{dy}{P^\sigma_j (y)}, \quad t^*_{m_{k-1}} = t^*_{m_k} + \sum_{j=1}^{n} \int_{x^*_{m_{k-1}}}^{x^*_{m_k}} \frac{dy}{P^\sigma_j (y)}
\]

for \( k = p-1, \ldots, 1 \). The total extraction time is

\[
t^*_{m_0} = \sum_{k=1}^{p} \sum_{j=1}^{n} \int_{x^*_{m_{k-1}}}^{x^*_{m_k}} \frac{dy}{P^\sigma_j (y)}
\]

3.2.4. An example. We present the two-company two-technology example which is proposed in [16] with the same parameter values for comparison. The parameter values are

\[
\rho = 0.04, \quad \alpha_1 = 1, \quad \alpha_2 = 10, \quad \beta_1 = 0.001, \quad \beta_2 = 0.01, \quad \gamma^1_1 = \gamma^1_2 = 2, \quad \gamma^2_1 = 1.715, \quad \gamma^2_2 = 1.
\]

The present resource stock is \( x_p = 1500 \) and at that time both companies are using the old technology.
Step 1. We first solve (3.19) for \( \sigma = (2, 2) \) and \( V^\sigma_{0,0} = 0 \) numerically, using the methods described after Theorem 3.3. System (3.19) consists of equations

\[
\begin{align*}
\rho V_1^{2,2} + \ln \left( \gamma_1^2 D V_1^{2,2} \right) + 1 + DV_1^{2,2}/DV_2^{2,2} &= 0, \\
\rho V_2^{2,2} + \ln \left( \gamma_2^2 D V_2^{2,2} \right) + 1 + DV_2^{2,2}/DV_1^{2,2} &= 0
\end{align*}
\]

for \( x > 0 \),

and the initial conditions \( V_1^{2,2}(0) = V_2^{2,2}(0) = 0 \). We first solve System (3.20) which has the form

\[
\begin{align*}
\rho P_1 + \frac{P_1'}{P_1} + \frac{P_1' P_2 - P_1 P_2'}{P_2^2} &= 0, \\
\rho P_2 + \frac{P_2'}{P_2} + \frac{P_1 P_2 - P_2 P_1'}{P_1^2} &= 0
\end{align*}
\]

Where \( P_i = D V_i^{2,2} \). The equations can be solved for \( P_1' \) and \( P_2' \) as

\[
\begin{align*}
P_1' &= -\rho \frac{P_2^2 P_2 (2 P_1 + P_2)}{P_1^2 + P_1 P_2 + P_2^2}, \\
P_2' &= -\rho \frac{P_1 P_2^2 (P_1 + 2 P_2)}{P_1^2 + P_1 P_2 + P_2^2}.
\end{align*}
\]

(3.25)

The initial values \( P_1(0) \) and \( P_2(0) \) satisfy equations

\[
\begin{align*}
\ln \gamma_1^2 + \ln P_1(0) + 1 + \frac{P_1(0)}{P_2(0)} &= 0, \\
\ln \gamma_2^2 + \ln P_2(0) + 1 + \frac{P_2(0)}{P_1(0)} &= 0.
\end{align*}
\]

We find the numerical solution

\[
P_1(0) = 0.09, \quad P_2(0) = 0.11.
\]

(3.26)

After solving the initial value problem (3.25)-(3.26), we find \( V_1^{2,2}(x) \) and \( V_2^{2,2}(x) \) by (3.21), which has the form

\[
\begin{align*}
V_1^{2,2}(x) &= -\frac{1}{\rho} \left[ \ln \left( \gamma_1^2 P_1(x) \right) + 1 + P_1(x)/P_2(x) \right], \\
V_2^{2,2}(x) &= -\frac{1}{\rho} \left[ \ln \left( \gamma_2^2 P_2(x) \right) + 1 + P_2(x)/P_1(x) \right],
\end{align*}
\]

for \( x > 0 \).

We then solve system (3.22) for \( i = 1, 2 \). For \( i = 1 \) the system takes the form

\[
\begin{align*}
\rho \left( V_1^{2,2}(x_1^*) - \omega_1(x_1^*) \right) + \ln \left[ \gamma_1^1 (P_1(x_1^*) - \beta_1) \right] + 1 + (P_1(x_1^*) - \beta_1)/p_1^* &= 0, \\
\rho V_2^{2,2}(x_1^*) + \ln \left[ \gamma_2^2 p_2^* \right] + 1 + Y_1^*/(P_2(x_1^*) - \beta_1) &= 0,
\end{align*}
\]

and for \( i = 2 \) it takes the form

\[
\begin{align*}
\rho V_1^{2,2}(x_2^*) + \ln \left[ \gamma_1^1 p_1^* \right] + 1 + Y_1^*/(P_2(x_2^*) - \beta_2) &= 0, \\
\rho \left( V_2^{2,2}(x_2^*) - \omega_2(x_2^*) \right) + \ln \left[ \gamma_2^2 (P_2(x_2^*) - \beta_2) \right] + 1 + (P_2(x_2^*) - \beta_2)/p_1^* &= 0,
\end{align*}
\]

where \( p_1^* \) and \( p_2^* \) are positive constants. Solving the systems numerically, we find

\[
x_1^* = 975.26, \quad x_2^* = 132.80.
\]

Both \( x_1^* \) and \( x_2^* \) are positive. So \( \sigma = (2, 2) \) is the terminal mode. This completes Step 1.
Step 2. Since $x_2 < x_1^*$, the mode is $(2, 2)$ for $0 < x < x_2^*$ and $(2, 1)$ for $x_2^* < x < x_3^*$ where $x_3^*$ is the resource stock at which Company 1 changes the technology. We find $V_{1,2}^1(x)$ and $V_{2,2}^1(x)$ by solving the initial value problem

$$\begin{align*}
\rho V_{1,2}^1 + \ln \left( \frac{\gamma_1^2 D V_{1,2}^1}{\omega_1} \right) + 1 + \frac{D V_{1,2}^1}{D V_{2,2}^1} &= 0, \\
\rho V_{2,2}^1 + \ln \left( \frac{\gamma_2^2 D V_{2,2}^1}{\omega_2} \right) + 1 + \frac{D V_{2,2}^1}{D V_{1,2}^1} &= 0,
\end{align*}$$

for $x > x_2^*$,

and

$$V_{1,2}^2(x_2^*) = V_{1,2}^{2,2}(x_2^*) = 9.92, \quad V_{2,2}^2(x_2^*) = V_{2,2}^{2,2}(x_2^*) - \omega_2(x_2^*) = 0.39.$$  

We then solve the equations for $x_3^*$,

$$\begin{align*}
p (V_{1,2}^2(x_3^*) - \omega_1(x_3^*)) + \ln \left[ \frac{\gamma_1^1 (D V_{1,2}^2(x_3^*) - \beta_1)}{\omega_1(x_3^*)} \right] + 1 + \frac{(D V_{1,2}^2(x_3^*) - \beta_1)}{p^*} &= 0, \\
\rho V_{2,2}^2(x_3^*) + \ln \left[ \frac{\gamma_2^1 p^*}{\omega_2(x_3^*)} \right] + 1 + \frac{p^*}{(D V_{2,2}^2(x_3^*) - \beta_1)} &= 0,
\end{align*}$$

for some $p^*$. The solution is

$$x_3^* = 949.01.$$  

This completes Step 2.

Step 3. By the result of Steps 1 and 2, this is the resource stock when Company 1 adopts the new technology. So both companies use the old technology if $x > 949.01$. Company 1 switches the new technology when $x = 949.01$, and Company 2 switches when $x = 132.80$. These are the impulse controls of the companies. Using numerical integration, the times $t_2^*$ and $t_3^*$ when the resource stocks are $x_2^*$ and $x_3^*$ are

$$\begin{align*}
t_2^* &= \int_{x_2^*}^{x_3^*} \frac{1}{D V_{1,2}^1(y)} + \frac{1}{D V_{2,2}^1(y)} \, dy = 4.45, \\
t_3^* &= t_2^* + \int_{x_2^*}^{x_3^*} \frac{1}{D V_{1,2}^2(y)} + \frac{1}{D V_{2,2}^2(y)} \, dy = 17.91,
\end{align*}$$

and the resource is exhausted at

$$T = t_2^* + t_3^* + \int_{0}^{x_2^*} \frac{1}{D V_{1,2}^{2,2}(y)} + \frac{1}{D V_{2,2}^{2,2}(y)} \, dy = 23.27.$$  

The continuous controls of the companies are

$$u_i^\sigma(x) = 1 / (\gamma_i^\sigma D V_i^\sigma(x)) \quad \text{for} \quad i = 1, 2, \quad \sigma = (1, 1), (2, 1), (2, 2).$$

The graph of the total extraction rate and the consumption rates of the companies are shown in Fig. 3.5. This completes Step 3.

If $\gamma_2^2$ in (3.24) is changed to 1.2 while other parameters remain the same, by repeating the computation we find

$$x_3^* = 962.62, \quad x_2^* = -53.68.$$  

Since $x_2^* < 0$, the terminal mode is $(2, 1)$. In Steps 2 and 3, we find $x_3^* = 917.58$, which corresponds to $t_3^* = 4.60$. This is the time when Company 1 adopts the new technology. The resource is finally exhausted at $t = 20.89$. The graph of the resource stock and the consumption rates are shown in Fig. 3.6.

These results are substantially different from those in [16]. In particular, it is interesting to note that even when both companies are using the same old technology and having the same efficiency, their continuous controls may not be the same, and the different future timing of switching technologies have effect on the consumption rates of the companies early on.
Figure 3.5. Extraction and consumption rates. The blue and red curves on the right are the consumption rates of Companies 1 and 2, respectively.

Figure 3.6. Extraction and consumption rates when $\gamma_2$ is changed to 1.2. The blue and red curves on the right are the consumption rates of Companies 1 and 2, respectively.

4. Conclusions. For a class of piecewise deterministic differential games in finite or infinite horizon, we derive Markovian feedback strategies and formulate a set of
We fix an initial state \((x, \sigma)\) and show that the classical solution \(\{V, u^*, \xi^*\}\) satisfies (2.13). To simplify the notation, we suppress the superscripts \(x, \sigma\) and write \(y(s)\) for \(y(s, x, \sigma)\). Since \(V_i^\sigma\) is differentiable in \(\Omega\) for any \(\sigma\) and \(i\), and since \(y\) differentiable for \(s\) between \(t_l\) and \(t_{l+1}\), we derive, using the Chain Rule,

\[
e^{-\rho_i t_{l+1}} V_i^{\sigma_i} (y(t_{l+1})) - e^{-\rho_i t_l} V_i^{\sigma_i} (y(t_l)) = \int_{t_l}^{t_{l+1}} \frac{d}{ds} [e^{-\rho_i s} V_i^{\sigma_i} (y(s))] \, ds \]

where \(\sigma_l = \theta(s)\) for \(t_l < s < t_{l+1}\). Since \(V\) is a classical solution of (2.10)–(2.12), it follows that

\[
\rho_i V_i^{\sigma_i} (y(s)) = g_i^{\sigma_i} (y(s), u^*(s)) + D V_i^{\sigma_i} (y(s)) \cdot f^{\sigma_i} (y(s), u^*(s))
\]

for \(t_l < s < t_{l+1}\). Hence,

\[
e^{-\rho_i t_{l+1}} V_i^{\sigma_i} (y(t_{l+1})) - e^{-\rho_i t_l} V_i^{\sigma_i} (y(t_l)) = -\int_{t_l}^{t_{l+1}} e^{-\rho_i s} g_i^{\theta(s)} (y(s), u^*(s)) \, ds.
\]

Summing over \(l = 0, 1, \ldots\) and using the initial values \(y(t) = x, \theta(t) = \sigma\), the boundary condition (2.12), and the transition condition

\[
V_i^0 (y(t_{l+1})) = G_i^{\sigma_l} (x, \xi^*, V_i (x)) \equiv \gamma_i^{\sigma_l} (y(t_{l+1})) + V_i^{\sigma_{l+1}} (y(t_{l+1}))
\]
for \( l = 1, 2, \ldots \), we obtain

\[
V_i^\sigma(x) = \int_t^T e^{-\rho_i(s-t)} g_i^\theta(s) (y(s), u^*(s)) \, ds + e^{-\rho_i(T-t)} \tilde{V}_i^\theta(T) (y(T)) \chi_{(t,\infty)}(T) + \sum_l e^{-\rho_i(t_l-t)} \gamma_i \delta l l (y(t_l)) .
\]

Comparing with (2.3) we see that \( V_i^\sigma(x) = J_i^{x,\sigma} (u^*(\cdot), \xi^*(\cdot)) \).

It remains to show that \( u^*(\cdot) \) and \( \xi^*(\cdot) \) satisfy (2.4). Let \( \tilde{u}(\cdot) = (u_i(\cdot), u^*_{i-1}(\cdot)) \) and \( \tilde{\xi}(\cdot) = (\xi_i(\cdot), \xi^*_{i-1}(\cdot)) \) for some \( u_i(\cdot) \in \mathcal{U}_i[t, T] \) and \( \xi_i(\cdot) \in \mathcal{X}_i[t, T] \). Under the controls \( \tilde{u}(\cdot) \) and \( \tilde{\xi}(\cdot) \) the state \( (\tilde{y}(\cdot), \tilde{\theta}(s)) \) and the jumping times \( \tilde{t}_i, l = 1, 2, \ldots \) are determined by equations (2.1) and (2.2). Similar to above, we derive

\[
e^{-\rho_i \tilde{t}_{i+1}} V_i^{\sigma_l} (\tilde{y}(\tilde{t}_{i+1})) - e^{-\rho_i \tilde{t}_l} V_i^{\sigma_l} (y(\tilde{t}_l)) = \int_{\tilde{t}_l}^{\tilde{t}_{i+1}} \frac{d}{ds} \left[ e^{-\rho_i s} V_i^{\sigma_l} (\tilde{y}(s)) \right] \, ds
\]

By (2.10) and (2.11),

\[
\rho_i V_i^{\sigma_l} (\tilde{y}(s)) \geq H_i^{\sigma_l} (\tilde{y}(s), u^*(s), DV_i^{\sigma_l}) \geq H_i^{\sigma_l} (\tilde{y}(s), \tilde{u}(s), DV_i^{\sigma_l})
\]

Comparing with (2.3) we see that \( V_i^\sigma(x) = J_i^{x,\sigma} (u^*(\cdot), \xi^*(\cdot)) \).

Hence the inequality is true for any \( u_i(\cdot) \in \mathcal{U}_i[t, T] \) and \( \xi_i(\cdot) \in \mathcal{X}_i[t, T] \), it follows that \( u^*(\cdot) \) and \( \xi^*(\cdot) \) satisfy (2.4). This completes the proof.
Lemma A.1. For any \((a_1, \ldots, a_n) \in \mathbb{R}^n\) the system (A.1) has a unique solution \((p_1, \ldots, p_n)\).

Proof. Let \(z_i = 1/p_i\) for \(i = 1, \ldots, n\). The system can be written as

\[
\ln z_i = a_i z_i + \sum_{j=1}^n z_j, \quad i = 1, \ldots, n.
\]

Let \(b = \max_{i \in \{1, \ldots, n\}} |a_i|\). We add \(bz_i\) to the both sides of the equation to get

\[
z_i \ln z_i + bz_i = (b + a_i) z_i + \sum_{j=1}^n z_j.
\]

Let

\[
f(z_i) = z_i \ln z_i + bz_i, \quad l_i(z_1, \ldots, z_n) = (b + a_i) z_i + \sum_{j=1}^n z_j.
\]

The equations become

\[
f(z_i) = l_i(z_1, \ldots, z_n) \quad \text{for} \quad i = 1, \ldots, n. \tag{A.3}
\]

Note that each \(l_i\) is an increasing in all variables. Function \(f\) is defined for positive numbers, and is negative for \(z_i < e^{-b}\) and positive if \(z_i > e^{-b}\). From the derivatives

\[
f'(z_i) = \ln z_i + b + 1, \quad f''(z_i) = 1/z_i > 0.
\]

So \(f\) is increasing and convex for \(z_i > e^{-b-1}\).

We show that any solution \((z_1, \ldots, z_n)\) of (A.3) satisfies \(e^{-b} \leq z_i \leq e^{b+n}\) for \(i = 1, \ldots, n\). Indeed, if \(z_i < e^{-b}\), then

\[
f(z_i) = 0 < l_i(z_1, \ldots, z_n).
\]

Therefore \((z_1, \ldots, z_n)\) is not a solution. On the other hand, suppose there is a \(z_i > e^{b+n}\). We may assume that \(z_i = \max_{j \in \{1, \ldots, n\}} z_j\). Then by monotonicity of \(l_i\),

\[
l_i(z_1, \ldots, z_n) \leq (b + a_i) z_i + nz_i < (2b + n) z_i.
\]

On the other hand,

\[
f(z_i) = z_i \ln z_i + bz_i > z_i (b + n) + bz_i = (2b + n) z_i.
\]

So \((z_1, \ldots, z_n)\) is again not a solution. This proves that all components of a solution is bounded by \(e^{-b}\) and \(e^{b+n}\).

We next prove that (A.3) has at least one solution. Let \(\tilde{z}_i^{(1)} = e^{-b}\) and \(\bar{z}_i^{(1)} = e^{b+n}\) for \(i = 1, \ldots, n\), and define \(\tilde{z}_i^{(k)}\) and \(\bar{z}_i^{(k)}\) for \(k \geq 2\) by

\[
f(\tilde{z}_i^{(k)}) = l_i(z_1^{(k-1)}, \ldots, z_n^{(k-1)}), \quad f(\bar{z}_i^{(k)}) = l_i(z_1^{(k-1)}, \ldots, z_n^{(k-1)}).
\]

for \(i = 1, \ldots, n\). These equations are uniquely solvable from the intermediate value theorem and the invertibility of \(f\). We show that the inequalities

\[
\tilde{z}_i^{(1)} \leq \cdots \leq \tilde{z}_i^{(k)} \leq \cdots \leq \bar{z}_i^{(k)} \leq \cdots \leq \bar{z}_i^{(1)} \quad \text{for} \quad i = 1, \ldots, n, \quad k = 2, 3, \ldots \tag{A.4}
\]
For $k = 2$, since
\[ 0 = f \left( \bar{z}^{(1)} \right) < l_i \left( \bar{z}_1^{(1)}, \ldots, \bar{z}_n^{(1)} \right) < l \left( \bar{z}_1^{(1)}, \ldots, \bar{z}_n^{(1)} \right) = (b + a_i + n) e^{b+n} < f \left( z_i^{(1)} \right), \]
by the intermediate value theorem and invertibility of $f$, there are unique solutions $z_i^{(2)}$ and $\bar{z}_i^{(2)}$ for $i = 1, \ldots, n$ such that $\bar{z}_i^{(1)} \leq z_i^{(2)} \leq \bar{z}_i^{(2)}$, and
\[ f \left( z_i^{(2)} \right) = l_i \left( z_1^{(1)}, \ldots, z_n^{(1)} \right), \quad f \left( \bar{z}_i^{(2)} \right) = l_i \left( z_1^{(1)}, \ldots, \bar{z}_n^{(1)} \right) \quad \text{for } i = 1, \ldots, n. \]

By the monotonicity of $f$ and $l_i$, we also have $z_i^{(2)} \leq \bar{z}_i^{(2)}$ for $i = 1, \ldots, n$. Suppose
\[ z_i^{(k-1)} \leq z_i^{(k)} \leq \bar{z}_i^{(k-1)} \]
for some $k \geq 2$ and $i = 1, \ldots, n$. Then, by monotonicity
\[ f \left( z_i^{(k)} \right) = l_i \left( z_1^{(k-1)}, \ldots, z_n^{(k-1)} \right) \leq l_i \left( z_1^{(k)}, \ldots, z_n^{(k)} \right), \quad f \left( \bar{z}_i^{(k)} \right) = l_i \left( z_1^{(k-1)}, \ldots, \bar{z}_n^{(k-1)} \right) \geq l_i \left( z_1^{(k)}, \ldots, \bar{z}_n^{(k)} \right) \]
for each $i$. Thus by the intermediate value theorem and invertibility of $f$, there are unique $z_i^{(k+1)}$ and $\bar{z}_i^{(k+1)}$ that satisfy $z_i^{(k)} \leq z_i^{(k+1)} \leq \bar{z}_i^{(k+1)}$ and
\[ f \left( z_i^{(k+1)} \right) = l_i \left( z_1^{(k)}, \ldots, z_n^{(k)} \right), \quad l_i \left( z_1^{(k)}, \ldots, \bar{z}_n^{(k)} \right) = f \left( \bar{z}_i^{(k+1)} \right) \quad \text{for } i = 1, \ldots, n. \]

By the monotonicity of $f$ and $l_i$ we also have $z_i^{(k+1)} \leq \bar{z}_i^{(k+1)}$ for $i = 1, \ldots, n$. This completes the induction proof of (A.4). By the monotone convergence theorem, it follows that the sequences $\left( z_i^{(k)} \right)_{k=1}^{\infty}$ and $\left( \bar{z}_i^{(k)} \right)_{k=1}^{\infty}$ both converge, and by the continuity of $f$ and $l_i$ the limits $(z_1, \ldots, z_n)$ and $(\bar{z}_1, \ldots, \bar{z}_n)$ satisfy
\[ f \left( z_i \right) = l_i \left( z_1, \ldots, z_n \right), \quad f \left( \bar{z}_i \right) = l_i \left( \bar{z}_1, \ldots, \bar{z}_n \right) \quad \text{for } i = 1, \ldots, n. \]
So both $(z_1, \ldots, z_n)$ and $(\bar{z}_1, \ldots, \bar{z}_n)$ are solutions of (A.3). This proves the existence of the solution.

It remains to show the uniqueness of the solution. Suppose $(z_1, \ldots, z_n)$ is any solution of (A.3). Then
\[ e^{-b} = z_i^{(1)} \leq z_i \leq z_i^{(1)} = e^{b+n} \quad \text{for each } i. \]

By monotonicity,
\[ f \left( z_i^{(2)} \right) = l_i \left( z_1^{(1)}, \ldots, z_n^{(1)} \right) \leq f \left( z_i \right) = l_i \left( z_1, \ldots, z_n \right) \leq l_i \left( z_1^{(1)}, \ldots, z_n^{(1)} \right) = f \left( z_i^{(2)} \right). \]
Thus $z_i^{(2)} \leq z_i \leq z_i^{(2)}$ for each $i$. Repeating the process successively, we obtain $z_i^{(k)} \leq z_i \leq z_i^{(k)}$ for all $i$ and all $k$. Therefore $z_i \equiv \bar{z}_i$ for each $i$. We show that $z_i = \bar{z}_i$ for all $i$. This would imply that there can be no other solution of (A.3). Since $f \left( \bar{z}_i \right) = l_i \left( \bar{z}_1, \ldots, \bar{z}_n \right)$ and $f \left( \bar{z}_i \right) = l_i \left( \bar{z}_1, \ldots, \bar{z}_n \right)$, it follows that
\[ \hat{z}_i \ln \bar{z}_i + b \bar{z}_i = (b + a_i) \bar{z}_i + \sum_{j=1}^{n} \bar{z}_j, \quad \bar{z}_i \ln \bar{z}_i + b \bar{z}_i = (b + a_i) \bar{z}_i + \sum_{j=1}^{n} \bar{z}_j, \quad (A.5) \]
and therefore,
\[ \bar{z}_i \ln \bar{z}_i - \bar{z}_i \ln \bar{z}_i = a_i \left( \bar{z}_i - \bar{z}_j \right) + \sum_{j=1}^{n} \left( \bar{z}_j - \bar{z}_j \right), \quad (A.6) \]
Using the mean value theorem on the left-hand side, we have
\[(\ln \theta_i + 1)(\bar{z}_i - z_i) = a_i (\bar{z}_i - z_i) + \sum_{j=1}^{n} (\bar{z}_j - z_j) \quad \text{for } i = 1, \ldots, n,\]
where $\theta_i$ is between $z_i$ and $\bar{z}_i$. Thus $\ln \theta_i > \ln z_i$. The above system can be written as
\[b_i (\bar{z}_i - z_i) - \sum_{j \neq i} (\bar{z}_j - z_j) = 0 \quad \text{for } i = 1, \ldots, n,
\]
where $b_i = \ln \theta_i - a_i$. The coefficient matrix has the form
\[
\begin{pmatrix}
  b_1 & -1 & \cdots & -1 \\
  -1 & b_2 & \cdots & -1 \\
  \vdots & \vdots & \ddots & \vdots \\
  -1 & -1 & \cdots & b_n
\end{pmatrix}
\]
By multiplying $z_i$ to the $i$-th column of the coefficient matrix, we see that
\[
\det \begin{pmatrix}
  b_1 & -1 & \cdots & -1 \\
  -1 & b_2 & \cdots & -1 \\
  \vdots & \vdots & \ddots & \vdots \\
  -1 & -1 & \cdots & b_n
\end{pmatrix} = \prod_{j=1}^{n} z_j^{-1} \det \begin{pmatrix}
  b_1 z_1 & -z_2 & \cdots & -z_n \\
  -z_1 & b_2 z_2 & \cdots & -z_n \\
  \vdots & \vdots & \ddots & \vdots \\
  -z_1 & -z_2 & \cdots & b_n z_n
\end{pmatrix}
\]
Note that by the first relation in (A.5),
\[\ln \bar{z}_i = a_i + \frac{1}{\bar{z}_i} \sum_{j=1}^{n} \bar{z}_j,\]
it follows that
\[b_i > \ln \bar{z}_i - a_i > \frac{1}{\bar{z}_i} \sum_{j=1}^{n} \bar{z}_j.\]
Therefore, the matrix on the right-hand side of (A.7) is strictly diagonally dominant. By Levy-Desplanques Theorem, it is invertible. Therefore System (A.6) has only trivial solution. This proves the uniqueness of the solution of (A.3), which is equivalent to (A.1).

We denote the solution of (A.1) by
\[p_i = F_i(a_1, \ldots, a_n), \quad i = 1, \ldots, n.\]
Note that any solution of (A.1) is necessarily positive. So functions $F_1, \ldots, F_n$ are positive. Furthermore, by the Implicit Function Theorem, these functions are continuously differentiable with respect to $(a_1, \ldots, a_n)$.

The next lemma regards the determinant of the matrix
\[
B = \begin{pmatrix}
  \sum_{j=1}^{n} a_j & -a_2 & \cdots & -a_n \\
  -a_1 & \sum_{j=1}^{n} a_j & \cdots & -a_n \\
  \vdots & \vdots & \ddots & \vdots \\
  -a_1 & -a_2 & \cdots & \sum_{j=1}^{n} a_j
\end{pmatrix}
\]

**Lemma A.2.** Let $a_1, \ldots, a_n$ be nonnegative numbers. Then
\[
\det (B) \geq \frac{1}{n!} \left( \sum_{j=1}^{n} a_j \right)^n.
\]
Proof. The determinant det (B) has the expansion
\[
\det (B) = b^n - \sum_{k=1}^{n} (k - 1) b^{n-k} \left( \sum_{\{j_1, \ldots, j_k\} \subset \{1, \ldots, n\}} a_{j_1} \cdots a_{j_k} \right).
\]

We show that on the right-hand side, every product \(a_1^{\delta_1} \cdots a_n^{\delta_n}\) has a positive coefficient, where \(\delta_1, \ldots, \delta_n\) are nonnegative integers with the sum \(\delta_1 + \cdots + \delta_n = n\). The coefficient of this product in \(\det (B)\) is
\[
\frac{n!}{\delta_1! \cdots \delta_n!} - \sum_{k=1}^{n} (k - 1) \frac{(n - k)!}{\delta_1! \cdots \delta_n!} \sum_{\{j_1, \ldots, j_k\} \subset \{1, \ldots, n\}} \delta_{j_1} \cdots \delta_{j_k} \tag{A.11}
\]
It is positive if and only if
\[
n! > \sum_{k=1}^{n} (k - 1) (n - k)! \sum_{\{j_1, \ldots, j_k\} \subset \{1, \ldots, n\}} \delta_{j_1} \cdots \delta_{j_k}. \tag{A.12}
\]
The right-hand side is a function of \(\delta_1, \ldots, \delta_n\) in the set \(\{\delta_i \geq 0 : \delta_1 + \cdots + \delta_n = n\}\). Its maximum occurs at
\[
\delta_1 = \cdots = \delta_n = 1.
\]
At this point the right-hand side has the value
\[
\sum_{k=1}^{n} (k - 1) (n - k)! \left(\frac{n}{k}\right) = \sum_{k=1}^{n} \frac{k - 1}{k!} = n! - 1.
\]
So (A.12) is true for any \(\delta_1, \ldots, \delta_n\). This proves that the coefficient of \(a_1^{\delta_1} \cdots a_n^{\delta_n}\) in \(\det (B)\) is positive.

Since all coefficients of \((\sum_{k=1}^{n} a_k)^n\) are no more than \(n!\), inequality (A.10) follows. \(\square\)

Proof of Theorem 3.3. Let \(P_i^\sigma = DP_i^\sigma\) for \(i = 1, \ldots, n\). Differentiating the left-hand side of (3.19) with respect to \(x\), we obtain the equation for \(P_i\) in the form
\[
\rho P_i^\sigma + \frac{DP_i^\sigma}{P_i^\sigma} + \sum_{j \neq i} \frac{P_j^\sigma}{P_i^\sigma} \frac{DP_j^\sigma}{P_j^\sigma} - \frac{P_i^\sigma}{P_j^\sigma} \frac{DP_j^\sigma}{P_j^\sigma} = 0 \quad \text{for } i = 1, \ldots, n. \tag{A.13}
\]
We show that this system can be uniquely solved for \(DP_i^\sigma\) if \(P_i^\sigma > 0\) for \(i = 1, \ldots, n\). The linear system for \(DP_i^\sigma\), \(i = 1, \ldots, n\), can be written in the matrix form
\[
Aq = b \tag{A.14}
\]
where \(A\) is the coefficient matrix
\[
A = \begin{pmatrix}
\sum_{j=1}^{n} \frac{1}{P_j^\sigma} & -\frac{P_j^\sigma}{P_i^\sigma} & \cdots & -\frac{P_j^\sigma}{(P_i^\sigma)^2} \\
-\frac{P_j^\sigma}{P_i^\sigma} & \sum_{j=1}^{n} \frac{1}{P_j^\sigma} & \cdots & -\frac{P_j^\sigma}{(P_i^\sigma)^2} \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{P_n^\sigma}{P_i^\sigma} & -\frac{P_n^\sigma}{(P_i^\sigma)^2} & \cdots & \sum_{j=1}^{n} \frac{1}{P_j^\sigma}
\end{pmatrix}, \tag{A.15}
\]
\(q = (DP_1^\sigma, \ldots, DP_n^\sigma)^T\), and \(b = -\rho (P_1^\sigma, \ldots, P_n^\sigma)^T\). (Here “\(T\)” denotes transposition.) We first show that the determinant of \(A\) is positive. Dividing \(P_i^\sigma\) into the \(i\)-th
row of $A$ followed by multiplying $P^{\sigma}_i$ to the $i$-th column of $A$, the matrix becomes

$$B = \begin{pmatrix}
\sum_{j=1}^{n} \frac{1}{P^j} & -\frac{1}{P^2} & \cdots & -\frac{1}{P^n} \\
-\frac{1}{P^1} & \sum_{j=1}^{n} \frac{1}{P^j} & \cdots & -\frac{1}{P^n} \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{1}{P^1} & -\frac{1}{P^2} & \cdots & \sum_{j=1}^{n} \frac{1}{P^j}
\end{pmatrix}.$$ 

By the properties of determinant, $\det(A) = \det(B)$. By Lemma A.2 with $a_i = 1/P^{\sigma}_i$, $\det(B) > 0$. So, $\det(A) > 0$. Therefore system (A.14) is uniquely solvable.

We show that its solution is negative. By the symmetry of the form of the system, it suffices to show that $DP^\sigma_1 < 0$. Using Cramer’s rule,

$$DP^\sigma_1 = \frac{\det(C)}{\det(A)} \tag{A.16}$$

where

$$C = \begin{pmatrix}
-\rho P^\sigma_1 & -\frac{P^\sigma_1}{(P_2^\sigma)^2} & \cdots & -\frac{P^\sigma_1}{(P_n^\sigma)^2} \\
-\rho P^\sigma_2 & \sum_{j=1}^{n} \frac{1}{P^j} & \cdots & -\frac{P^\sigma_2}{(P_n^\sigma)^2} \\
\vdots & \vdots & \ddots & \vdots \\
-\rho P^\sigma_n & -\frac{P^\sigma_n}{(P_2^\sigma)^2} & \cdots & \sum_{j=1}^{n} \frac{1}{P^j}
\end{pmatrix} \tag{A.17}$$

By dividing the $i$-th row of $C$ by $P^\sigma_i$ followed by multiplying the $i$-th column of $C$ by $P^\sigma_i$, we see that $\det(C)$ is the same as the determinant of the matrix

$$\begin{pmatrix}
-\rho P^\sigma_1 & -\frac{1}{P^2} & \cdots & -\frac{1}{P^n} \\
-\rho P^\sigma_1 & \sum_{j=1}^{n} \frac{1}{P^j} & \cdots & -\frac{1}{P^n} \\
\vdots & \vdots & \ddots & \vdots \\
-\rho P^\sigma_n & -\frac{1}{P^2} & \cdots & \sum_{j=1}^{n} \frac{1}{P^j}
\end{pmatrix}$$

Factoring out $-\rho P^\sigma_1$ from the first column and subtracting rows 2 to $n$ by the first row in $D$, we see that

$$\det(C) = -\rho P^\sigma_1 \cdot \text{det} \left( \begin{array}{cccc}
1 & -\frac{1}{P^2} & \cdots & -\frac{1}{P^n} \\
0 & \sum_{j=1}^{n} \frac{1}{P^j} + \frac{1}{P^2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \sum_{j=1}^{n} \frac{1}{P^j} + \frac{1}{P^n}
\end{array} \right).$$

Note that the sub-matrix of the last $n - 1$ rows and columns of the matrix on the right-hand side is diagonal, we find

$$\det(C) = -\rho P^\sigma_1 \prod_{k=2}^{n} \left( \sum_{j=1}^{n} \frac{1}{P^j} + \frac{1}{P^k} \right) < 0. \tag{A.18}$$

This proves that system (A.13) can be uniquely solved for $DP^\sigma_i$ if $P^\sigma_i > 0$ for $i = 1, \ldots, n$, and each $DP^\sigma_i$ is negative. We write the differential equations in the form

$$DP^\sigma_i = f_i(P^\sigma_1, \ldots, P^\sigma_n), \quad i = 1, \ldots, n. \tag{A.19}$$

It can be seen that functions $f_i$ are differentiable with respect to $P^\sigma_j$ because they are quotients of determinants of matrices whose entries are differentiable functions.
of the variables $P_i^j$, $j = 1, \ldots, n$. The initial values $P_i^j(x_0)$, $i = 1, \ldots, n$, are derived by the initial values of $V_i^\sigma(x_0)$ in (3.19), which lead to the equations

$$pV_{i,0} + \ln \gamma_i^\sigma + \ln P_i^\sigma(x_0) + \sum_{j=1}^n \frac{P_i^\sigma(x_0)}{P_j^\sigma(x_0)} = 0, \quad i = 1, \ldots, n. \quad (A.20)$$

By Lemma A.1, this system has a unique solution $(P_1^\sigma(x_0), \ldots, P_n^\sigma(x_0))$.

Since functions $f_i$ in (A.19) are differentiable, the initial-value problem has a solution $(P_1(x), \ldots, P_n(x))$ on an interval $[x_0, x_0 + \delta)$ for some $\delta > 0$. We show that the solution $(P_1(x), \ldots, P_n(x))$ exists for all $x > x_0$. Suppose by contradiction that the solution exists on the maximum interval $[x_0, x_0 + \delta)$ for some $\delta < \infty$. If $P_i(\delta) > 0$ for all $i$, then by the differentiability of $f_i$, the solution can be extended beyond $\delta$. Therefore there is an $i \in \{1, \ldots, n\}$ such that $P_i(\delta) = 0$. We show that there is a constant $C > 0$ such that

$$P_i^\sigma(x) \geq \frac{P_i^\sigma(x_0)}{1 + CP_i^\sigma(x_0)(x - x_0)} \quad \text{for} \quad x \in [x_0, x_0 + \delta), \quad i = 1, \ldots, n. \quad (A.21)$$

This would lead to a contradiction.

Apply Lemma A.2 to matrix $B$ with $a_i = 1/P_i^\sigma$, we see that

$$\det (A) = \det (B) \geq \frac{1}{n!} \left( \sum_{j=1}^n \frac{1}{P_j^\sigma} \right)^n.$$ 

Hence, by (A.16) and (A.18),

$$DP_i^\sigma = \det (C) / \det (A) \geq -\rho n! P_i^\sigma \prod_{k=2}^n \left( \sum_{j=1}^n \frac{1}{P_j^\sigma} + \frac{1}{P_k^\sigma} \right) / \left( \sum_{j=1}^n \frac{1}{P_j^\sigma} \right)^n$$

$$= -\left( P_i^\sigma \right)^2 \left[ \rho n! \prod_{j \neq 1} P_j^\sigma \prod_{k=2}^n \left( 2 \prod_{l \neq k} P_l^\sigma + \sum_{j \neq k} \prod_{l \neq k \neq l} P_l^\sigma \right) / \left( \sum_{j=1}^n \prod_{l \neq k} P_l^\sigma \right)^n \right].$$

Observe that the quantity on the right-hand side in the brackets is positive and bounded. Let $C$ be its upper bound. Then

$$DP_i^\sigma \geq -C \left( P_i^\sigma \right)^2.$$ 

Hence

$$\frac{1}{P_i^\sigma(x_0)} - \frac{1}{P_i^\sigma(x)} = \int_{x_0}^x \frac{DP_i^\sigma(s)}{P_i^\sigma(s)^2} ds \geq -C (x - x_0).$$

This inequality leads to

$$P_i^\sigma(x) \geq \frac{P_i^\sigma(x_0)}{1 + C(x - x_0) P_i^\sigma(x_0)} \quad \text{for} \quad x \in [x_0, x_0 + \delta).$$

A similar derivation leads to the inequality for $i = 2, \ldots, n$. Hence the solution $(P_1^\sigma(x), \ldots, P_n^\sigma(x))$ exists for all $x > x_0$.

Finally, let

$$V_i^\sigma(x) = \frac{1}{\rho} \ln (\gamma_i^\mu_i P_i^\sigma(x)) - \frac{1}{\rho} \sum_{j=1}^n \frac{P_i^\sigma(x)}{P_j^\sigma(x)} \quad \text{for} \quad i = 1, \ldots, n. \quad (A.22)$$
Then, by differentiation,

\[ DV_i^\sigma(x) = \frac{DP_i^\sigma}{\rho P_i^\sigma(x)} - \frac{1}{\rho} \sum_{j \neq i} \frac{P_j^\sigma(x) DP_i^\sigma(x) - P_i^\sigma(x) DP_j^\sigma(x)}{(P_j^\sigma(x))^2}. \]

From (A.13) we find

\[ DV_i^\sigma(x) = P_i^\sigma(x). \]

Hence by (A.22) we see that the functions \( V_i^\sigma(x) \) satisfy (3.19). Furthermore, Eq. (A.20) implies that \( V_i^\sigma(x_0) = V_i^\sigma,0 \). This proves that \((V_1^\sigma(x),\ldots,V_n^\sigma(x))\) is a classical solution of Problem (3.19).

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