AN IMPROVED UPPER BOUND FOR THE WARING RANK OF THE DETERMINANT

GARRITT JOHNS AND ZACH TEITLER

Abstract. The Waring rank of the generic $d \times d$ determinant is bounded above by $d \cdot d!$. This improves previous upper bounds, which were of the form an exponential times the factorial. Our upper bound comes from an explicit power sum decomposition. We describe some of the symmetries of the decomposition and set-theoretic defining equations for the terms of the decomposition.

For a homogeneous polynomial $F$ of degree $d$, the Waring rank of $F$, denoted rank$(F)$, is the least integer $r$ such that $F$ can be expressed as a linear combination of $r$ terms which are each $d$th powers of linear forms. For example,

$$xy = \frac{1}{4}((x + y)^2 - (x - y)^2),$$

so rank$(xy) \leq 2$. Similarly

$$xyz = \frac{1}{24}((x + y + z)^3 - (x + y - z)^3 - (x - y + z)^3 + (x - y - z)^3),$$

so rank$(xyz) \leq 4$.

Let det$_d$ denote the generic $d \times d$ determinant, that is, the determinant of the $d \times d$ matrix $(x_{i,j})_{1 \leq i,j \leq d}$ whose entries are independent variables. Previous upper bounds for rank$(\text{det}_d)$ were $2^{d-1} \cdot d!$, later improved to $(\frac{5}{6})^{\lfloor d/3 \rfloor} 2^{d-1} \cdot d!$; these have the form, an exponential function times a factorial. We give an explicit expression to show a new upper bound for the Waring rank of the determinant, which is a linear function times the factorial, namely,

$$\text{rank}(\text{det}_d) \leq d \cdot d!. \tag{1}$$

This holds over the complex numbers, or more generally over any field or commutative ring where $d!$ is invertible and there is a primitive $d$th root of unity.$^1$

Specifically, we show that

$$d \cdot d! \cdot \text{det}_d = \sum_{\sigma \in S_d} (-1)^{\sigma} \sum_{j=1}^{d} (-1)^{(d+1)j} \left( \sum_{i=1}^{d} \omega^{ij} x_{i, \sigma_i} \right)^d, \tag{2}$$

where $\omega$ is a primitive $d$th root of unity and $S_d$ denotes the symmetric group on $d$ letters. This is a linear combination of $d \cdot d!$ terms which are $d$th powers of linear forms, with coefficients $\pm 1$.

In addition, we describe some of the symmetries of the decomposition, and set-theoretic defining equations for the terms of the decomposition.

\textit{Date:} April 18, 2020.

2020 Mathematics Subject Classification. 14N07.

Key words and phrases. Waring rank, symmetric rank, determinant.

$^1$J.M. Landsberg informed us of an unpublished result of Gurvits giving an upper bound of $(d+1) \cdot d!$, see below.
1. Background

Fix a degree \( d \). Let \( \mathbb{k} \) be a field or commutative ring in which \( d! \) is invertible; in particular, the characteristic is zero or greater than \( d \). We consider homogeneous polynomials \( F \in \mathbb{k}[x_1, \ldots, x_n] \) of degree \( d \).

We denote by \( \text{rank}_\mathbb{k}(F) \), or simply \( \text{rank}(F) \), the least number \( r \) of linear forms \( \ell_1, \ldots, \ell_r \in \mathbb{k}[x_1, \ldots, x_n] \) such that \( F = c_1 \ell_1^d + \cdots + c_r \ell_r^d \) for some \( c_1, \ldots, c_r \in \mathbb{k} \). (The term “Waring rank” is often reserved for the case that \( \mathbb{k} \) is a field, even an algebraically closed field.) If \( \mathbb{k} \subseteq \mathbb{K} \) is an extension field or ring, then evidently \( \text{rank}_\mathbb{k}(F) \geq \text{rank}_\mathbb{K}(F) \). Thus, upper bounds for \( \text{rank}_\mathbb{k}(F) \) are also upper bounds for \( \text{rank}_\mathbb{K}(F) \). For this reason we will describe our upper bound for \( \text{rank} \text{(det}_d \text{)} \) over the smallest \( \mathbb{k} \) possible. First, in this section, we describe previously known upper and lower bounds. All the results described in this section are valid at least for \( \mathbb{k} = \mathbb{C} \); we make some partial indications of more general \( \mathbb{k} \) where they hold.

We have

\[
2^{d-1} \cdot d! \ x_1 \cdots x_d = \sum_{\epsilon \in \{\pm 1\}^d} \left( \prod_{i=1}^d \epsilon_i \right) (\epsilon_1 x_1 + \cdots + \epsilon_d x_d)^d,
\]

which shows that \( \text{rank}(x_1 \cdots x_d) \leq 2^{d-1} < \infty \) whenever \( 2^{d-1} \cdot d! \) is a unit in \( \mathbb{k} \). By substitution, any monomial of degree \( d \) has finite rank, and then so does any homogeneous form, simply by decomposing each monomial into a sum of powers. This shows that every homogeneous form of degree \( d \) has finite rank.

One can show that in fact \( \text{rank}(xy) = 2 \) and \( \text{rank}(xyz) = 4 \), and more generally

\[
\text{rank}(x_1 \cdots x_d) = 2^{d-1},
\]

see [12].

The determinant \( \text{det}_d \) is a (signed) sum of \( d! \) monomials, each of which is of the form \( x_1 \cdots x_d \). For example, \( \text{det}_3 = x_{1,1}x_{2,2}x_{3,3} - \cdots \), a sum of 6 terms which each have the form \( xyz \). Therefore \( \text{det}_3 \) can be written as a sum of 6 \( \cdot \text{rank}(xyz) = 24 \) powers of linear forms.

In general,

\[
\text{det}_d = \sum_{\sigma \in S_d} (-1)^\sigma x_{1,\sigma_1} \cdots x_{d,\sigma_d}.
\]

Combining with (3), this yields the “classical” power sum decomposition

\[
2^{d-1} \cdot d! \ \text{det}_d = \sum_{\sigma \in S_d} (-1)^\sigma \sum_{\epsilon \in \{\pm 1\}^d} \left( \prod_{i=1}^d \epsilon_i \right) (\epsilon_1 x_{1,\sigma_1} + \cdots + \epsilon_d x_{d,\sigma_d})^d,
\]

with \( 2^{d-1} \cdot d! \) terms. That is,

\[
\text{rank}(\text{det}_d) \leq 2^{d-1} \cdot d!.
\]

So \( \text{rank}(\text{det}_d) \) for \( d = 3, 4, 5, 6, \ldots \) are bounded above by 24, 192, 1920, 23040, and so on.

Derksen [5] and, later, Krishna-Makam [10] found expressions for \( \text{det}_3 \) as a sum of 5 terms which are products of linear forms. The identity of Krishna-Makam, which is slightly simpler,
is as follows:

$$\det_3 = x_{1,1} (x_{2,2} + x_{2,3}) (x_{3,1} + x_{3,3})$$

$$+ (x_{1,2} + x_{1,3}) x_{2,1} x_{3,2}$$

$$- (x_{1,1} + x_{1,3}) x_{2,2} x_{3,1}$$

$$- x_{1,2} (x_{2,1} + x_{2,3}) (x_{3,2} + x_{3,3})$$

$$+ (x_{1,2} - x_{1,1}) x_{2,3} (x_{3,1} + x_{3,2} + x_{3,3}).$$

Notably, this holds over the integers (and even in characteristic 2). The Derksen and Krishna-Makam identities give \( \text{rank}(\det_3) \leq 5 \text{rank}(xyz) = 20. \) Derksen observed that by Laplace expansion,

$$\text{rank}(\det_d) \leq \left( \frac{5}{6} \right)^{\lfloor d/3 \rfloor} 2^{d-1} \cdot d!. \tag{6}$$

In particular \( \text{rank}(\det_d) \) for \( d = 3, 4, 5, 6, \ldots \) are bounded above by 20, 160, 1600, 16000, and so on.

Conner-Gesmundo-Landsberg-Ventura gave an explicit expression for \( \det_3 \) as a sum of 18 cubes of linear forms over \( \mathbb{C} \) \[3, Theorem 2.11\], showing

$$\text{rank}_{\mathbb{C}}(\det_3) \leq 18. \tag{7}$$

Our expression \( \tag{2} \) is a direct generalization of theirs; their expression is exactly the case \( d = 3 \) of ours.

J.M. Landsberg informed us of an unpublished result of Gurvits, that \( \text{rank}_k(\det_d) \leq (d + 1) \cdot d! \). This follows from the identity

$$d! \cdot \det_d = \sum_{\sigma \in S_d} (-1)^\sigma \left( \sum_{i=1}^d x_{i,\sigma i} \right)^d - \sum_{j=1}^d \left( \sum_{1 \leq i \leq d} x_{i,\sigma i} \right)^d. \tag{8}$$

This decomposition is valid over the integers, so the upper bound holds whenever \( d! \) is a unit in \( k \).

Lower bounds for \( \text{rank}(\det_d) \) have been studied by several authors, including \[11, 13, 6, 8, 1\]. Currently, the best lower bounds, for algebraically closed fields \( k \), are as follows. For all \( d \geq 3 \), \( \text{rank}(\det_d) \geq (2d^4) - (2d-2) \): thus for \( d = 3, 4, 5, 6, \ldots \), \( \text{rank}(\det_d) \) is bounded below by 14, 50, 182, 672, and so on, see \[3\]. Recently, \[1\] improved this to \( \text{rank}(\det_3) \geq 15 \). Very soon after that, \[4\] improved it further to \( \text{rank}_{\mathbb{C}}(\det_3) \geq 17 \).

We summarize the previous and new bounds in Table 1.

1.1. Notation. We use multi-index notation as follows. For a tuple \( \alpha = (\alpha_1, \ldots, \alpha_n) \), \( x^\alpha \) denotes \( x_1^{\alpha_1} \cdots x_n^{\alpha_n} \). We write \( |\alpha| = \sum \alpha_i \), \( \alpha! = \prod \alpha_i! \), and \( \binom{d}{\alpha} = \frac{d!}{\alpha!} \) when \( d = |\alpha| \).

For a pure (coefficient 1) monomial \( M \) and polynomial \( P \), \([M]P \) denotes the coefficient of \( M \) in \( P \). For example, \([xy](x+y)^2 = 2 \). More generally, \([x^\alpha](x_1 + \cdots + x_n)^{|\alpha|} = \binom{|\alpha|}{\alpha} \).

\( S_d \) denotes the symmetric group on \( |d| = \{1, \ldots, d\} \). For \( \sigma \in S_d \) and \( i \in |d| \) we write \( \sigma i \) for the result of the permutation \( \sigma \) applied to \( i \). We write \((-1)^{\sigma} \) for the sign of the permutation \( \sigma \).
Table 1. Summary of bounds for rank \( \text{rank}_C(\text{det}_d) \).

|          | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   |
|----------|-----|-----|-----|-----|-----|-----|-----|-----|
| classical upper bound | 4   | 24  | 192 | 1920| 23040| 322560| 5160960| 92897280|
| Derksen upper bound   | 6   | 20  | 160 | 1600| 224000| 3584000| 53760000|       |
| Gurvits upper bound   | 6   | 24  | 120 | 720 | 5040 | 40320 | 3628800 |       |
| CGLV upper bound      | 18  |     |     |     |     |     |     |     |
| new upper bound       | 4   | 18  | 96  | 600 | 4320 | 35280 | 322560 | 3265920|
| lower bound           | 4   | 17  | 50  | 182 | 672  | 2508  | 9438  | 35750 |

2. Proofs

**Theorem 1.** Let \( k \) be any commutative ring containing a primitive \( d \)th root of unity \( \omega \). Then over \( k \), (2) holds,

\[
d \cdot d! \det_d = \sum_{\sigma \in S_d} (-1)^\sigma \sum_{j=1}^d (-1)^{(d+1)j} \left( \sum_{i=1}^d \omega^{ij} x_{i,\sigma_i} \right)^d.
\]

In particular, if also \( d! \) is invertible in \( k \), then \( \text{rank}(\det_d) \leq d \cdot d! \).

We begin by proving the following lemma.

**Lemma 2.** Fix a degree \( d \) monomial \( x_{i_1} \cdots x_{i_d} \), where the indices \( 1 \leq i_1 \leq \cdots \leq i_d \leq d \) may repeat. Let \( \lambda = (\lambda_1, \ldots, \lambda_d) \) be the \( d \)-tuple of multiplicities of elements in the multiset \( I = \{i_1, \ldots, i_d\} \). That is, each \( \lambda_k \) is the number of \( j \) such that \( i_j = k \). Then

\[
[x_{i_1} \cdots x_{i_d}] \sum_{j=1}^d (-1)^{(d+1)j} \left( \sum_{i=1}^d \omega^{ij} x_i \right)^d = \begin{cases} 
(d) d & \text{if } \sum_{k=1}^d i_k \equiv \binom{d+1}{2} \pmod{d}, \\
0 & \text{otherwise}.
\end{cases}
\]

**Proof.** We write \( x_I = x_{i_1} x_{i_2} \cdots x_{i_d} = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_d^{\lambda_d} \). Let

\[
P = \sum_{j=1}^d (-1)^{(d+1)j} \left( \sum_{i=1}^d \omega^{ij} x_i \right)^d.
\]

We aim to find the coefficient of \( x_I \) in \( P \):

\[
[x_I] P = \sum_{j=1}^d (-1)^{(d+1)j} \left( \sum_{i=1}^d \omega^{ij} x_i \right)^d.
\]

After applying the multinomial theorem, this becomes

\[
[x_I] P = \sum_{j=1}^d (-1)^{(d+1)j} \left( \prod_{k=1}^d \omega^{k \cdot j} \right)^{d \choose \lambda} \sum_{j=1}^d ((-1)^{d+1} \omega \sum_{i=1}^d \omega x_i)^j.
\]

Now \(-1 = \omega^{d/2}\), so \((-1)^{d+1} = (-1)^{-(d+1)} = \omega^{-(d+1)/2}\). Thus

\[
[x_I] P = \left( \prod_{k=1}^d \left( \omega x_k - \frac{d+1}{2} \right)^{j} \right)^{d \choose \lambda}.
\]
The claim is proved when we recall that $\sum_{j=1}^{d} \omega^{pj}$ is $d$ if $p \equiv 0 \pmod{d}$, 0 otherwise.

Note that if $d$ is even, then $(d+1)_2 \equiv d/2 \pmod{d}$, while if $d$ is odd, then $(d+1)_2 \equiv 0 \pmod{d}$.

Either way, $(d+1)_2 \equiv -(d+1)_2 \pmod{d}$.

Now we proceed with the proof of Theorem 1.

Proof of Theorem 1. Let

$$R = \sum_{\sigma \in S_d} (-1)^{\sigma} \sum_{j=1}^{d} (-1)^{(d+1)j} \left( \sum_{i=1}^{d} \omega^{ij} x_{i,\sigma i} \right)^d,$$

The claim is that $R = d \cdot d! \det_d$. Let $I = (i_1, i_2, \ldots, i_d)$ and $J = (j_1, j_2, \ldots, j_d)$ be $d$-tuples of elements in $[d]$. We denote by $x_{I,J}$ the monomial $x_{i_1,j_1} \cdots x_{i_d,j_d}$.

The coefficient of $x_{I,J}$ in $\det_d$ is easy to find. Denote by $\text{Supp}(I)$ the support of $I$, that is, the subset of $[d]$ of values that appear in $I$; and similarly $\text{Supp}(J)$. If $\text{Supp}(I) = \text{Supp}(J) = [d]$, then there is a unique permutation $\sigma, I, J \in S_d$ such that $\sigma I_i = j_k$ for $k = 1, \ldots, d$. In this case $[x_{I,J}] \det_d = (-1)^{\sigma I, J}$. Otherwise, if $\text{Supp}(I) \neq [d]$ or $\text{Supp}(J) \neq [d]$, then $[x_{I,J}] \det_d = 0$.

Now we consider the coefficient of $x_{I,J}$ in $R$. Let $H = \{\sigma \in S_d \mid \sigma I_1 = j_1, \sigma I_2 = j_2, \ldots, \sigma I_d = j_d\}$. We have

$$[x_{I,J}] R = \sum_{\sigma \in H} (-1)^{\sigma} [x_{I,J}] \sum_{j=1}^{d} (-1)^{(d+1)j} \left( \sum_{i=1}^{d} \omega^{ij} x_{i,\sigma i} \right)^d,$$

since $x_{I,J}$ simply involves variables that do not appear in the $\sigma$ terms for $\sigma \notin H$. In particular, if $H = \emptyset$ then $[x_{I,J}] R = 0$. (We will not use this fact, but the case $H = \emptyset$ occurs when there are $k, k'$ such that $i_k = i_{k'}$ but $j_k \neq j_{k'}$, or $j_k = j_{k'}$ but $i_k \neq i_{k'}$.)

Now suppose $H \neq \emptyset$. For $\sigma \in H$ we have $x_{I,J} = x_{i_1,\sigma i_1} \cdots x_{i_d,\sigma i_d} = x_{i_1,\sigma i_1}^{\lambda_{i_1}} \cdots x_{i_d,\sigma i_d}^{\lambda_{i_d}}$. By Lemma 2 applied to the variables $x_k = x_{k,\sigma k}$, we have

$$[x_{I,J}] \sum_{j=1}^{d} (-1)^{(d+1)j} \left( \sum_{i=1}^{d} \omega^{ij} x_{i,\sigma i} \right)^d = \begin{cases} \binom{d}{2} d, & \text{if } \sum_{k=1}^{d} i_k \equiv \binom{d+1}{2} \pmod{d}, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore

$$[x_{I,J}] R = \begin{cases} \binom{d}{2} d \sigma H (-1)^{\sigma}, & \text{if } \sum_{k=1}^{d} i_k \equiv \binom{d+1}{2} \pmod{d}, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\sigma_0 \in H$ and let $H_0 = \sigma_0^{-1} H$. Observe $H_0$ is precisely the subgroup of $S_d$ consisting of elements that fix $\text{Supp}(I)$ pointwise, so $H_0$ is (isomorphic to) the symmetric group on $[d] \setminus \text{Supp}(I)$. Indeed, $\sigma \in H_0$ if and only if $\sigma_0 \sigma \in H$, if and only if $\sigma_0 \sigma i_k = j_k = \sigma_0 i_k$ for all $k$, if and only if $\sigma i_k = i_k$ for all $k$. That is, $H_0 = \{\sigma \in S_d \mid \sigma i_1 = i_1, \ldots, \sigma i_d = i_d\}$.

Now if $|H| \geq 2$ then the subgroup $H_0$ consists of an equal number of even and odd permutations, and so does its coset $H$. Therefore $\sum_{\sigma \in H} (-1)^{\sigma} = 0$, so $[x_{I,J}] R = 0$.

If $|H| = 1$ then $\text{Supp}(I)$ consists of either $d-1$ or $d$ elements of $[d]$.

Suppose $|\text{Supp}(I)| = d-1$, so (as multisets) $\{i_1, \ldots, i_d\} = ([d] \setminus \{m\}) \cup \{n\}$ for some $m, n \in [d], m \neq n$. In this case

$$\sum_{k=1}^{d} i_k = (n-m) + \sum_{k=1}^{d} k = (n-m) + \binom{d+1}{2}.$$
We evaluate the point whose coordinates are the coefficients of $\ell$ claimed. This is $d \cdot d!$ times the coefficient $[x_{I,J}] \det_d$, for all $I, J$. Therefore $R = d \cdot d! \det_d$, as claimed. \hfill \Box

3. Linear Independence

In this section $k$ is a field.

For $\sigma \in S_d$ and $j = 1, \ldots, d$, let $T_{\sigma,j} = (-1)^{\sigma}(-1)^{(d+1)j}(\omega^j x_{1,\sigma 1} + \cdots + \omega^j x_{d,\sigma d})^d$, so Theorem \[ is that

$$d \cdot d! \det_d = \sum_{\sigma, j} T_{\sigma,j}.$$

**Theorem 3.** The terms $T_{\sigma,j}$ appearing in \[ are linearly independent.

**Proof.** We show that there are linear functionals $\ell_{\sigma,j}$ on the space of homogeneous forms of degree $d$ such that $\ell_{\sigma,j}(T_{\psi,k})$ is nonzero if $\psi = \sigma$, $k = j$, and zero otherwise. Recall that linear functionals correspond to “dual” homogeneous forms of degree $d$ in such a way that evaluating such a functional on a $d$th power $\ell^d$ of a linear form $\ell$ corresponds to evaluating the dual form at the point whose coordinates are the coefficients of $\ell$. Let $P_{\sigma,j} \in \mathbb{K}^{d^2}$ be the point whose coordinates are the coefficients of $\sum_{i=1}^d \omega^{ij} x_{i,\sigma i}$, so the $x_{i,k}$ coefficient of $P_{\sigma,j}$ is $\omega^{ij}$ if $k = \sigma i$, 0 otherwise. Thus it is sufficient to show that, for each $\sigma, j$, there is a degree $d$ form which is nonvanishing at the point $P_{\sigma,j}$ and vanishing at all of the points $P_{\psi,k}$ for $\psi \neq \sigma$ or $k \neq j$.

In fact we explicitly produce such forms of degree $d - 1$. The appropriate forms of degree $d$ can be obtained by multiplying by appropriate linear forms (to increment the degree); any linear form nonvanishing at $P_{\sigma,j}$ will do.

Given $\sigma \in S_d$, for each $k$ let $e_{\sigma,k} = x_{1,\sigma 1} \cdots \widehat{x_{k,\sigma k}} \cdots x_{d,\sigma d}$. Now given $j$, let $L_{\sigma,j}$ be the degree $d - 1$ form

$$L_{\sigma,j} = \sum_{k=1}^d \omega^{kj} e_{\sigma,k}.$$ We evaluate $L_{\sigma,j}$ first at the point $P_{\sigma,j}$. Observe $e_{\sigma,k}(P_{\sigma,j}) = \omega^{ij} \cdots \widehat{\omega^{kj}} \cdots \omega^{dj} = \omega^{(d+1)j-kj}$.

Thus $L_{\sigma,j}(P_{\sigma,j}) = \sum_{k=1}^d \omega^{(d+1)j} = (-1)^{(d+1)j}d$,

which is indeed nonzero.
For \( m \neq j \),
\[
L_{\sigma,j}(P_{\sigma,m}) = \sum_{k=1}^{d} \omega^{kj} \left( \frac{d+1}{2} \right)^{m-km} = (-1)^{(d+1)m} \sum_{k=1}^{d} \omega^{(j-m)k}.
\]
This is zero when \( j \not\equiv m \pmod{d} \), which is equivalent to \( j \neq m \) since \( 1 \leq j, m \leq d \).

Finally for \( \psi \neq \sigma \), for every \( k \), there is some \( j \neq k \) such that \( \sigma j \neq \psi j \). (That is, the permutations \( \psi \) and \( \sigma \) have different values in at least two positions.) So every \( e_{\sigma,k}(P_{\psi,m}) \) is a product involving a factor given by the \( x_{j,\sigma j} \) coordinate of \( P_{\psi,m} \), but that coordinate is zero.

\[ \square \]

4. Symmetries

In this section \( k \) is a field.

Let \( V \) be the vector space over \( k \) spanned by the variables \( x_{i,j} \), so \( \det_d \in \text{Sym}^d(V) \). Linear automorphisms of \( V \) induce automorphisms of \( \text{Sym}^d(V) \). Following [2], it is natural to ask which of these transformations fix the decomposition [2], that is, which linear automorphisms of \( V \) leave the set \( \{ T_{\sigma,j} \} \) invariant. In other words, they should leave invariant the set of linear forms whose \( d \)th powers occur in [2]. However, note that said linear forms are only unique up to a \( d \)th root of unity, and the \( d \)th powers occur with coefficients of \( \pm 1 \), which must be preserved.

It is helpful to describe these linear forms in terms of their matrix of coefficients. We write \( E_{i,j} \) for the \( d \times d \) matrix with a 1 entry in the \((i,j)\) position and all other entries 0. We write \( P_\sigma \) for the permutation matrix \( P_\sigma = \sum_{i=1}^{d} E_{i,\sigma i} \). Note that for a \( d \times 1 \) column vector \((v_i)\), we have
\[
P_\sigma \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix} = \begin{pmatrix} v_{\sigma 1} \\ v_{\sigma 2} \\ \vdots \\ v_{\sigma d} \end{pmatrix}.
\]
Note also that for \( \sigma, \psi \in S_d \), \( P_\sigma P_\psi = P_{\sigma \psi} \). The permutation matrices are orthogonal: \( P_\sigma^{-1} = P_\sigma^t = P_{\sigma^{-1}} \). We have \( (-1)^{\sigma} = \det P_\sigma \). Let \( D = \text{diag}\{\omega, \omega^2, \ldots, \omega^d\} \). Now the linear form that appears in the term \( T_{\sigma,j} \) has its coefficients given by the matrix \( D^j P_\sigma \).

In conclusion, we are looking for linear automorphisms of \( V \) which preserve the set \( \{D^j P_\sigma\} \) up to factors of \( d \)th roots of unity, i.e., linear automorphisms which leave the set \( \{\omega^k D^j P_\sigma\} \) invariant. Beyond this, some terms occur in [2] with coefficient 1 and others with coefficient \(-1\); we want our linear automorphisms to respect those coefficients.

**Definition 4.** Let \( \tilde{G} \) be the group of linear automorphisms of \( V \) which leave the set of matrices \( M = \{\omega^k D^j P_\sigma\} \) invariant and let \( G \) be the subgroup of \( \tilde{G} \) which preserves the determinant. We call the elements of \( G \) the symmetries of the decomposition [2].

We’re looking for the group \( G \), which is a finite group. We are not able to give the full group, but we describe a subgroup of order \( d^3 \varphi(d) \cdot d!/2 \).

A theorem of Frobenius [9] (see also [14]) states that all linear automorphisms \( L \) of \( V \) that fix \( \det_d \) (in the sense that \( \det(L(X)) = \det(X) \) for all \( X \)) are of the form \( X \mapsto AXB \) or \( X \mapsto AX^tB \), where \( X = (x_{i,j}) \) and \( A, B \) are \( d \times d \) matrices with \( \det(AB) = 1 \). The transformations involving transposition are difficult to analyze, so we consider only the transformations \( AXB \). Our question, then, is which \( A \) and \( B \) satisfy \( A(D^m P_\rho)B \in M \).
The set $M$ is invariant under multiplication by powers of $\omega$, left multiplication by $D$, and right multiplication by arbitrary permutation matrices. Half of the permutations also preserve the determinant (the other half reverse the sign). Note that $\det(D) = (-1)^{d+1}$, so depending on the parity of $d$, multiplication by $D$ may also preserve or reverse the sign of the determinant.

As for left multiplication by permutation matrices, the decomposition is preserved by certain permutations corresponding to what one might call affine linear transformations of $\mathbb{Z}/d\mathbb{Z}$. Let $\text{Aff}_d$ be the set of permutations $\sigma$ in $S_d$ for which there exist $a, b \in [d]$ such that $\sigma i = ai + b \mod d$ for all $i \in [d]$. This is a subgroup of $S_d$ with order $d\varphi(d)$, where $\varphi$ is Euler’s totient function. In fact $\text{Aff}_d \cong \mathbb{Z}/d\mathbb{Z} \times (\mathbb{Z}/d\mathbb{Z})^*$. The permutation $\sigma i = i + b \mod d$ has sign $(-1)^{b(d+1)}$; for $a \in (\mathbb{Z}/d\mathbb{Z})^*$, the sign of the permutation $\sigma i = ai \mod d$ is the Jacobi symbol $\left(\frac{a}{d}\right)$ if $d$ is odd, or $(-1)^{(\frac{d-1}{2})(a^2-1)}$ if $d$ is even.

**Lemma 5.** Let $\pi \in S_d$. Then $P_\pi D \in M$ if and only if $\pi \in \text{Aff}_d$.

**Proof.** Suppose that $P_\pi D = \omega^b D^a P_\psi$. Then, by comparing the locations of nonzero entries on both sides we have that $\psi = \sigma$. Hence $P_\pi D P_{\pi^{-1}} = \omega^b D^a$. The two sides are equivalent to $\text{diag}\{\omega^{a_1}, \omega^{a_2}, \ldots, \omega^{a_d}\}$ and $\text{diag}\{\omega^{a+b}, \omega^{2a+b}, \ldots, \omega^{d(a+b)}\}$. Hence $\omega \in \text{Aff}_d$.

Conversely, if $\sigma \in \text{Aff}_d$ so that $\sigma i = ai + b \mod d$ then $P_\sigma D = \omega^b D^a P_\psi$, and hence $P_\sigma D \in M$.

This proves that the map $X \mapsto \omega^m D^n P_\pi X P_\sigma$ is an element of $\tilde{G}$. Let $\tilde{H} \subseteq \tilde{G}$ be the subgroup of elements of the form $X \mapsto \omega^m D^n P_\pi X P_\sigma$, $\pi \in \text{Aff}_d, \sigma \in S_d$. The proof of the above lemma shows that this is a subgroup, and in fact that it is a semidirect product:

$$\tilde{H} \cong ((\mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}) \rtimes \text{Aff}_d) \times S_d.$$ 

Here the $\mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$ factor corresponds to multiplication by $\omega$ and $D$.

An element of $\tilde{H}$ given by $X \mapsto \omega^m D^n P_\pi X P_\sigma$ preserves the determinant, meaning that $\det(\omega^m D^n P_\pi X P_\sigma) = \det(X)$, if and only if $\det(\omega^m D^n P_\pi P_\sigma) = 1$. For half the elements of $H$ this holds, and for the other half, this determinant is $-1$. Let $H \subset \tilde{H}$ be the index 2 subgroup that preserves the determinant. That is, $H = \tilde{H} \cap G$.

**Theorem 6.** Let $m, n \in [d], \pi \in \text{Aff}_d$, and $\sigma \in S_d$ such that $\det(D)^n (-1)^{\pi} (-1)^{\sigma} = 1$. Then $L : X \mapsto \omega^m D^n P_\pi X P_\sigma$ is a symmetry of the decomposition and is an element of $G$. □

This gives a subgroup of $G$ with order $d^3 \varphi(d)d!l/2$, as claimed. Table 2 shows the number of symmetries given by Theorem 6. However, this is not the full group of symmetries, as in the $2 \times 2$ and $3 \times 3$ cases one can check that transposition $X \mapsto X^t$ is another symmetry of the decomposition. But for $d > 3$ it is not: for example $DP_{(12)} \in M$ but $(DP_{(12)})^t = P_{(12)}D \notin M$.

| $d$  | 2   | 3   | 4   | 5   | 6   |
|------|-----|-----|-----|-----|-----|
| $d^3 \varphi(d)d!/2$ | 8   | 16  | 1536| 37500| 15552 |

**Table 2.** Number of symmetries in the subgroup $H$.

It would be interesting to fully characterize the whole group of symmetries $G$, where $A, B$ are not necessarily given by powers of $D$ or permutation matrices. Additionally, it would be interesting to characterize the symmetries of the decomposition of the form $X \mapsto AX^tB$. 

---

**Table 2.** Number of symmetries in the subgroup $H$.

| $d$  | 2   | 3   | 4   | 5   | 6   |
|------|-----|-----|-----|-----|-----|
| $d^3 \varphi(d)d!/2$ | 8   | 16  | 1536| 37500| 15552 |

It would be interesting to fully characterize the whole group of symmetries $G$, where $A, B$ are not necessarily given by powers of $D$ or permutation matrices. Additionally, it would be interesting to characterize the symmetries of the decomposition of the form $X \mapsto AX^tB$. 

---

**Table 2.** Number of symmetries in the subgroup $H$. 

| $d$  | 2   | 3   | 4   | 5   | 6   |
|------|-----|-----|-----|-----|-----|
| $d^3 \varphi(d)d!/2$ | 8   | 16  | 1536| 37500| 15552 |
Remark 7. The decomposition [2] is certainly not unique. For example, if $A, B$ are any two $d \times d$ matrices with $\det(AB) = 1$ then one obtains a decomposition of $\det(X) = \det(AXB)$ as a sum of $d$th powers of linear forms with coefficient matrices given by $AD^i P_{ij} B$. Transposition yields more decompositions.

5. Defining equations

We describe defining equations of the set of matrices $M$, up to scalar multiple. That is, we give set-theoretic defining equations for the set of projective points $\{[D^i P_{ij}]\}$. The equations we describe are quadrics of two types: first, monomials, products of two distinct matrix entries from the same row or column, and second, certain quadrics in row-sums.

In this theorem, $x_{i,j}$ are coordinates on the space of $d \times d$ matrices.

**Theorem 8.** For each $i$, let $\rho_i = x_{i,1} + \cdots + x_{i,d}$, the $i$th row-sum. Let $I$ be the ideal generated by the quadrics $x_{i,j} x_{i',j'}$ for all $i$ and $j$ and all $i_1 \neq i_2$, and $x_{i,j} x_{i',j'}$ for all $i$ and all $i_1 \neq i_2$, and $\rho_i^2 - \rho_{i-1} \rho_{i+1}$ for all $i$, with indices considered modulo $d$. The common zero locus of these equations is exactly the set of projective points $\{[D^i P_{ij}]\}$.

**Proof.** The monomial generators cut out the set of matrices with at most one nonzero entry in each row and column. Such matrices have the form $\Delta P_{ij}$ for some diagonal $\Delta$ and permutation $\sigma$, unique as long as $\Delta$ is nonsingular. On this set, the values of $\rho$ are the entries of $\Delta$. The equations $\rho_i^2 - \rho_{i-1} \rho_{i+1}$ for $2 \leq i \leq d-1$ ensure that the entries of $\Delta$ form a geometric series $(s^{d-1}, s^{d-2} t, \ldots, t^{d-1})$ for some $s, t$ (they are the familiar equations of the rational normal curve in $\mathbb{P}^d$ parametrized by $[s^{d-1} : s^{d-2} t : \ldots : t^{d-1}]$). The other equations at $i = 1, d$ ensure that $s^d = t^d$, i.e., $t/s$ is a $d$th root of unity. Then, up to a scalar multiple, the entries of $\Delta$ are $(w^j, w^{2j}, \ldots, w^{dj})$ for some $j$, i.e., $\Delta = D^j$. \qed

These are certainly not the generators of the full ideal of this set of points. For $d = 3$, additional generators are given as follows: $x_{i,j}^2 - P_{ij}$, where $P_{ij}$ is the permanent of the $2 \times 2$ submatrix complementary to the entry $x_{i,j}$. (Our equations $\rho_i^2 - \rho_{i-1} \rho_{i+1}$ are sums of those generators, up to some monomials.) For $d = 4$, some additional generators are given by $x_{i,j}^2 + x_{i,j}^2 - P_{i,i+2;j,j}$, where $P_{i,i+2;j,j}$ is the permanent of the $2 \times 2$ submatrix complementary to rows $i, i+2$ (i.e., having rows $i-1$ and $i+1$) and to columns $j, j$. More generators are given by $P_{i,i+2;j,j} - P_{i,i+2;j,j}$ where $\{i_1, \ldots, i_4\} = \{j_1, \ldots, j_4\} = \{1, \ldots, 4\}$ and $i_1 + i_2 \equiv i_3 + i_4 \pmod{4}$.

It would be interesting to describe these ideals in general.

ACKNOWLEDGEMENTS

This work was supported by a grant from the Simons Foundation (#354574, Zach Teitler). We thank Gianni Krakoff and Kayla Krakoff for contributions in early conversations about this project, the authors of [3] for sharing with us their work, Jaroslaw Buczyński and Fulvio Gesmundo for a number of very helpful suggestions, and J.M. Landsberg for sharing with us Gurvits’s result, as well as encouragement.
References

1. Mats Boij and Zach Teitler, A bound for the Waring rank of the determinant via syzygies, Linear Alg. Appl. 587 (2020), 195–214.

2. Austin Conner, A Rank 18 Waring Decomposition of $sM_{3}$ with 432 Symmetries, Experimental Mathematics (2019), 1–3.

3. Austin Conner, Fulvio Gesmundo, Joseph M. Landsberg, and Emanuele Ventura, Kronecker powers of tensors and Strassen’s laser method, arXiv:1909.04785 [cs.CC], 2019.

4. Austin Conner, Alicia Harper, and Joseph M. Landsberg, New lower bounds for matrix multiplication and the $3 \times 3$ determinant, arXiv:1911.07981 [math.AG], 2019.

5. Harm Derksen, On the nuclear norm and the singular value decomposition of tensors, Found. Comput. Math. 16 (2016), no. 3, 779–811.

6. Harm Derksen and Zach Teitler, Lower bound for ranks of invariant forms, J. Pure Appl. Algebra 219 (2015), no. 12, 5429–5441.

7. Georges Elencwajg (https://math.stackexchange.com/users/3217/georges-elencwajg), The parity of the permutation $a \mapsto ma$ mod $n$ (answer), URL (version: 2011-05-11): https://math.stackexchange.com/q/38563.

8. Cameron Farnsworth, Koszul-Young flattenings and symmetric border rank of the determinant, J. Algebra 447 (2016), 664–676.

9. Georg Frobenius, Über die Darstellung der endlichen Gruppen durch lineare Substitutionen, Sitzungsber., Preuss. Akad. Wiss., Berlin (1897), 994–1015.

10. Siddharth Krishna and Visu Makam, On the tensor rank of $3 \times 3$ permanent and determinant, arXiv:1801.00496 [math.CO], Jan 2018.

11. J. M. Landsberg and Zach Teitler, On the ranks and border ranks of symmetric tensors, Found. Comput. Math. 10 (2010), no. 3, 339–366.

12. Kristian Ranestad and Frank-Olaf Schreyer, On the rank of a symmetric form, J. Algebra 346 (2011), 340–342.

13. Sepideh Masoumeh Shafiei, Apolarity for determinants and permanents of generic matrices, J. Commut. Algebra 7 (2015), no. 1, 89–123.

14. Eric Wofsey (https://mathoverflow.net/users/75/eric-wofsey), Linear transformation that preserves the determinant (answer), URL (version: 2020-01-24): https://mathoverflow.net/q/5347.

Email address: garrittjohns@u.boisestate.edu

DEPARTMENT OF MATHEMATICS, BOISE STATE UNIVERSITY, 1910 UNIVERSITY DRIVE, BOISE, ID 83725-1555, USA

Email address: zteitler@boisestate.edu

DEPARTMENT OF MATHEMATICS, BOISE STATE UNIVERSITY, 1910 UNIVERSITY DRIVE, BOISE, ID 83725-1555, USA