On the stable set of an analytic gradient flow

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Abstract. Let \( f : \mathbb{R}^n \to \mathbb{R}, n \geq 2 \), be an analytic function. There are presented sufficient conditions for the stable set of the gradient flow \( \dot{x} = \nabla f(x) \) to have a non-empty interior.

1 Introduction.

Let \( f : \mathbb{R}^n, n \geq 2 \), be an analytic function. According to Łojasiewicz [8], the limit set of a trajectory of the dynamical system \( \dot{x} = \nabla f(x) \) is either empty or contains a single critical point of \( f \). So the family of integral curves which converge to a critical point is a natural object of study in the theory of gradient dynamical systems.

Let \( f : \mathbb{R}^n, 0 \to \mathbb{R}, 0 \) be an analytic function defined in a neighbourhood of the origin, having a critical point at \( 0 \). We shall write \( S(f) \) for the stable set of the origin, which is the union of all orbits of the solutions that converge to the origin. By [8], the stable set in closed near the origin. There is a natural problem: is the interior of \( S(f) \) non-empty? (In the planar case this is equivalent to the problem whether the set of integral curves converging to the origin is infinite?)

Of course, if the origin is a local maximum then \( \text{int} S(f) \neq \emptyset \). If the origin is a non-degenerate critical point then the opposite implication holds.

Let \( \omega : \mathbb{R}^n, 0 \to \mathbb{R}, 0 \) be the homogeneous initial form associated with \( f \). Put \( \Omega = S^{n-1} \cap \{ \omega < 0 \} \). Applying the Moussu results [10] one may show that \( \text{int} S(f) \neq \emptyset \) if there exists at least one non-degenerate critical point of \( \omega \mid \Omega \) which is a local maximum.

Let \( S_r = S^{n-1}_r \cap \{ f < 0 \} \), where \( S^{n-1}_r = \{ x \in \mathbb{R}^n \mid |x| = r \}, 0 < r \ll 1 \). The main result of this paper says that \( \text{int} S(f) \neq \emptyset \) if \( \text{rank} H^{n-2}(S_r) < \text{rank} H^{n-2}(\Omega) \), where \( H^{n-2}(\cdot) \) is the \((n-2)\)-th cohomology group with rational coefficients.

Let \( \Omega' = S^{n-1}\Omega \cap \{ \omega \geq 0 \} = S^{n-1}_r \setminus \Omega \), and \( S'_r = S^{n-1}_r \cap \{ f \geq 0 \} = S^{n-1}_r \setminus S_r, 0 < r \ll 1 \). Sets \( \Omega', S'_r \) are compact and semianalytic, hence they are triagulable. By the Alexander duality theorem, if \( S'_r \) and \( \Omega' \) are non-empty then \( \text{rank} H_0(S'_r) = 1 + \text{rank} H^{n-2}(S_r) \) and \( \text{rank} H_0(\Omega') = 1 + \text{rank} H^{n-2}(\Omega) \).

Thus, if \( S'_r \) has less connected components than \( \Omega' \) then the interior of \( S(f) \) is non-empty.
Let $f$ be as above. Assume that $g : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ analytic. We shall prove that $\text{int} S(g) \neq \emptyset$ if $g$ is right-equivalent to $f$.

In exposition and notation we follow closely [13], where there are presented sufficient conditions for existence of an infinite family of trajectories of the gradient flow converging to the origin.

The paper is organized as follows. In Section 2 we prove sufficient conditions for a compact subset of the sphere to have a non-empty interior. In Section 3 we investigate the stable set of an analytic gradient flow and we prove the main results (Theorems 3.6, 3.7). Section 4 is devoted to functions right-equivalent to the ones that satisfy assumptions of those theorems. References [1, 3, 4, 5, 6, 7, 12] present significant related results and applications.

2 Preliminaries.

**Lemma 2.1.** Suppose that $L \subset K$ are closed subsets of $S^{n-1}$, $n \geq 2$, and $\text{rank } \bar{H}^{n-2}(K) < \text{rank } \bar{H}^{n-2}(L) < \infty$, where $\bar{H}^{n-2}(\cdot)$ is the $(n-2)$-th Čech-Alexander cohomology group with rational coefficients. Then the interior of $K$ is non-empty.

*Proof.* As $\bar{H}^{n-2}(L) \neq 0$ then sets $L, K, S^{n-1} \setminus L$ are not void. If $K = S^{n-1}$ then the assertion holds. From now on we assume that $S^{n-1} \setminus K \neq \emptyset$ and $n \geq 3$.

By the Alexander duality theorem there are isomorphisms

$$
\bar{H}^{n-2}(L) \simeq \bar{H}_0(S^{n-1} \setminus L), \quad \bar{H}^{n-2}(K) \simeq \bar{H}_0(S^{n-1} \setminus K),
$$

where $\bar{H}_0(\cdot)$ is the 0-th reduced homology group.

Then $S^{n-1} \setminus L$ is a disjoint union of open connected components $U_1, \ldots, U_\ell$, where $\ell = 1 + \text{rank } \bar{H}_0(S^{n-1} \setminus L) = 1 + \text{rank } \bar{H}^{n-2}(L)$, and $S^{n-1} \setminus K$ is a disjoint union of open connected components $V_1, \ldots, V_k$, where $k = 1 + \text{rank } \bar{H}_0(S^{n-1} \setminus K) = 1 + \text{rank } \bar{H}^{n-2}(K)$.

Suppose that $U_i \setminus K \neq \emptyset$ for each $1 \leq i \leq \ell$, so that there are points $p_i \in U_i \setminus K$ and then $p_i \in V_{j(i)}$ for some $1 \leq j(i) \leq k$. As $V_{j(i)}$ is a connected subset of $U_1 \cup \ldots \cup U_\ell$, then $V_{j(i)} \subset U_i$.

Because $U_i$ are pairwise disjoint, then $V_{j(i)}$ are pairwise disjoint too. Hence $k \geq \ell$, contrary to our claim. Then at least one open connected component $U_i$ is a subset of $K$.

Similar arguments apply to the case where $n = 2$. \qed
Corollary 2.2. Suppose that $L \subset K \subset F$, where $L, K$ are compact, $n \geq 2$, $\text{rank} \, \check{H}^{n-2}(K) < \text{rank} \, \check{H}^{n-2}(L) < \infty$, and $F$ is an $(n-1)$-dimensional manifold homeomorphic to a subset of $S^{n-1}$. Then the interior of $K$ is non-empty.

3 Stable sets of gradient flows

Let $f : \mathbb{R}^n, 0 \to \mathbb{R}, 0$, $n \geq 2$, be an analytic function defined in an open neighbourhood of the origin. For $0 < -y \ll r \ll 1$ we shall write

$$B_r^n = \{ x \in \mathbb{R}^n \mid |x| \leq r \}, \quad S_r^{n-1} = \{ x \in \mathbb{R}^n \mid |x| = r \},$$

$$F_r(y) = B_r^n \cap f^{-1}(y), \quad S_r = \{ x \in S_r^{n-1} \mid f(x) < 0 \}.$$

We call the set $F_r(y)$ the real Milnor fibre. According to [9], it is either an $(n-1)$-dimensional compact manifold with boundary or an empty set. Moreover, the sets $F_r(y)$ and $S_r$ are homotopy equivalent.

Corollary 3.1. If $0 < -y \ll r \ll 1$ then the cohomology groups $H^*(S_r)$ and $H^*(F_r(y))$ are isomorphic.

According to [10], there are $0 < -y \ll r \ll 1$ such that each non-trivial trajectory of the gradient flow $\dot{x} = \nabla f(x)$ converging to the origin intersects $F_r(y)$ transversally at exactly one point. Let $\Gamma(f) \subset F_r(y)$ be the union of all those points. By [8], the set $\Gamma(f)$ is a closed subset of $F_r(y)$, so $\Gamma(f)$ is compact. Hence there is a natural one-to-one correspondence between trajectories converging to the origin and points in $\Gamma(f)$.

By [11, Theorem 12] we have

Theorem 3.2. If $0 < -y \ll r \ll 1$ then the inclusion $\Gamma(f) \subset F_r(y)$ induces an isomorphism

$$\check{H}^*(\Gamma(f)) \simeq H^*(F_r(y)),$$

where $\check{H}^*(\cdot)$ is the Čech-Alexander cohomology group.

Corollary 3.3. There is an isomorphism $\check{H}^*(\Gamma(f)) \simeq H^*(S_r)$.

Let $\omega$ be the initial form associated with $f$, and let $\Omega = S_r^{n-1} \cap \{ \omega < 0 \}$. In the same manner as in the proof of [13, Proposition 3.5] we can get

Proposition 3.4. There exists a compact set $\check{\Gamma}(f) \subset \Gamma(f)$ such that $\check{H}^*(\check{\Gamma}(f)) \simeq H^*(\Omega)$. As $\Omega$ is semi-algebraic, then $\text{rank} \, \check{H}^{n-2}(\check{\Gamma}(f)) = \text{rank} \, H^{n-2}(\Omega) < \infty$.
Corollary 3.5. If $\omega$ is a quadratic form which can be reduced to the diagonal form $-x_1^2 - \cdots - x_i^2 + x_{i+2}^2 + \cdots + x_j^2$, where $i \geq 1$, then

$$\bar{H}^*(\tilde{\Gamma}(f)) \approx H^*(\Omega) \approx H^*(S^i).$$

In that case $\text{rank } \bar{H}^{n-2}(\tilde{\Gamma}(f)) = \text{rank } H^{n-2}(\Omega) > 0$ if and only if $\omega$ can be reduced to the diagonal form $-x_1^2 - \cdots - x_{n-1}^2$.  

Theorem 3.6. Suppose that $f : \mathbb{R}^n, 0 \to \mathbb{R}, 0, n \geq 2$, is an analytic function defined in an open neighbourhood of the origin. Suppose that $\text{rank } H^{n-2}(S_r) < \text{rank } H^{n-2}(\Omega)$. Then the stable set of the origin of the gradient flow $\dot{x} = \nabla f(x)$ has a non-empty interior.

Proof. By [9, Lemma 5.10], if $0 < -y \ll r \ll 1$ then $F_r(y)$ is homeomorphic to an $(n-1)$-dimensional submanifold of $S^{n-1}$.

As $\tilde{\Gamma}(f) \subset \Gamma(f)$ are compact subsets of $F_r(y)$ with $\bar{H}^{n-2}(\tilde{\Gamma}(f)) = \text{rank } H^{n-2}(S_r) < \text{rank } H^{n-2}(\Omega) = \text{rank } \bar{H}^{n-2}(\tilde{\Gamma}(f)) < \infty$, then by Corollary 2.2 the set $\Gamma(f)$ has a non-empty interior in $F_r(y)$.

Trajectories of the flow $\dot{x} = \nabla f(x)$ converging to the origin cut transversally $F_r(y)$ at point of $\Gamma(f)$. Hence the stable set of the origin has a non-empty interior. \qed

Put $\Omega' = S^{n-1} \cap \{\omega \geq 0\} = S^{n-1} \setminus \Omega$, and $S'_r = S^{n-1} \cap \{f \geq 0\} = S^{n-1} \setminus S_r$, $0 < r \ll 1$. Sets $\Omega', S'_r$ are compact and semianalytic, hence they are triagulable. By the Alexander duality theorem, if $S'_r$ and $\Omega'$ are non-empty then $\text{rank } H_0(S'_r) = 1 + \text{rank } H^{n-2}(S_r)$ and $\text{rank } H_0(\Omega') = 1 + \text{rank } H^{n-2}(\Omega)$.

Theorem 3.7. Suppose that the set $S'_r$ has less connected components than $\Omega'$. Then the stable set of the origin of the gradient flow $\dot{x} = \nabla f(x)$ has a non-empty interior.

Proof. The set $\Omega'$ is obviously not empty. If $S'_r = \emptyset$ then the origin is a local maximum, and then $\text{int } S(f) \neq \emptyset$.

Suppose that $S'_r \neq \emptyset$. Sets $S'_r, \Omega'$ are compact, semianalytic. So they are triangulable, and the number of connected components of $S'_r$ (resp. $\Omega'$) equals the number of its path-components which is $\text{rank } H_0(S'_r)$ (resp. $\text{rank } H_0(\Omega')$).

By assumption, $\text{rank } H_0(S'_r) < \text{rank } H_0(\Omega')$ and then $\text{rank } H^{n-2}(S_r) < \text{rank } H^{n-2}(\Omega)$. By Theorem 3.6 the stable set $S(f)$ has a non-empty interior. \qed
Applying arguments presented by Moussu in [10, p.449] one can prove the next proposition. (As its proof would require to introduce other techniques, so we omit it here.)

**Proposition 3.8.** Suppose that there exists a non-degenerate critical point of \( \omega|\Omega \) which is a local maximum. Then the interior of \( S(f) \) is not-empty.

**Example 3.9.** Let \( f(x, y) = x^3 + 3xy^2 + x^2y^2 \), so that \( \omega = x^3 + 3xy^2 \). It is easy to see that \( \omega|S^1 \) has a non-degenerate local maximum at \( (-1, 0) \in \Omega \). Then the interior of \( S(f) \) is non-empty.

**Example 3.10.** Let \( f(x, y) = x^3 - y^2 \), so that \( \omega = -y^2 \). Then \( \Omega = \{(x, y) \in S^1 \mid -y^2 < 0\} = S^1 \setminus \{(1, 0)\} \), and \( \Omega' = \{(-1, 0), (1, 0)\} \). The function \( \omega|\Omega \) has exactly two critical (minimum) points at \( (0, \pm 1) \), so one cannot apply Proposition 3.8 in this case. As \( S_r \) is homeomorphic to a closed interval, then by Theorem 3.7 the interior of \( S(f) \) is non-empty.

**Example 3.11.** Let \( f(x, y, z) = -x^2y^2 - z^4 + x^5 \). Then \( \omega = -x^2y^2 - z^4 \) and \( \Omega' \) consists of four points. It is easy to see that \( S_r \) is homeomorphic to a disjoint union of a closed disc and two points. By Theorem 3.7 the interior of \( S(f) \) is non-empty.

## 4 Right-equivalent functions

Let \( g : \mathbb{R}^n, 0 \to \mathbb{R}, 0 \) be an analytic function which is right-equivalent to \( f \), i.e. there exists a \( C^\infty \)-diffeomorphism \( \phi : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0 \) defined in an open neighbourhood of the origin such that \( g = f \circ \phi \). Then in particular the derivative \( D\phi(0) : \mathbb{R}^n \to \mathbb{R}^n \) is a linear isomorphism.

Let \( \theta \) be the initial homogeneous form associated with \( g \), and let \( \Theta' = S^{n-1}_r \cap \{ \theta \geq 0 \} \). It is easy to see that \( \theta = \omega \circ D\phi(0) \). Hence sets \( \Omega' \) and \( \Theta' \) are diffeomorphic, and then \( H_0(\Omega') \simeq H_0(\Theta') \).

Both \( f \) and \( g \) are analytic, hence there exists small \( r_0 > 0 \) such that for each \( 0 < r \leq r_0 \) the number of connected components of \( S_r \) equals the number of connected components of \( B^n_r \setminus \{0\} \cap \{ f \geq 0 \} \), and the number of connected components of \( S^{n-1}_r \cap \{ g \geq 0 \} \) equals the number of connected components of \( B^n_r \setminus \{0\} \cap \{ g \geq 0 \} \). As \( g = f \circ \phi \) then \( (B^n_r \setminus \{0\}) \cap \{ g \geq 0 \} \) is homeomorphic to \( (\phi(B^n_r) \setminus \{0\}) \cap \{ f \geq 0 \} \).

There exist \( 0 < r_3 < r_2 < r_1 < r_0 \) such that \( \phi(B^n_{r_3}) \subset B^n_{r_2} \subset \phi(B^n_{r_1}) \subset B^n_{r_0} \).
The inclusion \((B^n \setminus \{0\}) \cap \{g \geq 0\} \subset (B^n \setminus \{0\}) \cap \{g \geq 0\}\) is a homotopy equivalence. Hence inclusions
\[
(\phi(B^n) \setminus \{0\}) \cap \{f \geq 0\} \subset (\phi(B^n) \setminus \{0\}) \cap \{f \geq 0\},
\]
\[
(B^n \setminus \{0\}) \cap \{f \geq 0\} \subset (B^n \setminus \{0\}) \cap \{f \geq 0\}
\]
are homotopy equivalencies, and then in particular sets \((B^n \setminus \{0\}) \cap \{g \geq 0\}, (\phi(B^n) \setminus \{0\}) \cap \{f \geq 0\}\) and \((B^n \setminus \{0\}) \cap \{f \geq 0\}\) have the same number of connected components.

Hence sets \(S^n_{r-1} \cap \{g \geq 0\}\) and \(S'_r\) have the same number of connected components too. By Theorem 3.7 we get

**Corollary 4.1.** Suppose that \(\text{rank } H_0(S'_r) < \text{rank } H_0(\Omega')\) and \(g\) is right-equivalent to \(f\). Then \(S(g)\) has a non-empty interior.

**Example 4.2.** Let \(g(x, y, z, w) = x^5 + z^5 + 2zw - x^2 - y^2 - z^2 - w^2 - 2xyz - y^2z^2\). Applying standard methods of the singularities theory (see [2]) one can show that \(g\) is right-equivalent to \(f(x, y, z, w) = x^5 - y^2 - z^2 - w^2\). Then \(\omega = -y^2 - z^2 - w^2\), and so \(\Omega'\) consists of two points. It is easy to see that \(S'_r\) is homeomorphic to a closed ball. By Corollary 4.1 the set \(S(g)\) has a non-empty interior.

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