On the accuracy of the Timoshenko beam theory above the critical frequency: best shear coefficient

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Abstract

We obtain values for the shear coefficient both below and above the critical frequency, by comparing the results of the Timoshenko beam theory with experimental results published very recently. The best results are obtained when different values of the shear coefficient are used below and above the critical frequency.

Keywords: Timoshenko beam theory, Timoshenko shear coefficient, critical frequency

1. Introduction

In the last years there has been a renewed interest in the flexural vibrations of beams. Even in the simplest case of a uniform beam their vibrations are of great interest since not only the validity of the theory is still under discussion, but also the physical parameters that should be used with it. This is the case of the (Timoshenko) shear coefficient $\kappa$. This adjustment parameter appears in the Timoshenko beam theory, or TBT, to estimate the shear force, at the cross section of a beam, in terms of the shear strain at the centroidal axis.

Up to now, on the one hand, there are several theoretical studies to get the best value for the shear coefficient \cite{1, 2, 3, 4, 5, 6, 7}. Some commonly used values are $\kappa = 5/6$, $\kappa = \pi^2/12$, etc., for a beam with rectangular cross-section but there is not a consensus about its value. Experimental studies, on the other hand, are scarce \cite{8, 9, 10}. This is due to the fact that measurements with frequencies...
above the Bernoulli-Euler regime are needed. Numerical simulations, using finite elements, have been also performed to calculate the shear coefficient [11].

To calculate the Timoshenko’s shear coefficient it is assumed that the cross-sectional area is plane [12]. This hypothesis is correct at low frequencies but it seems to fail at frequencies higher than the critical frequency \( f_c \). After a very long debate, it was shown that in this regime, i.e., for \( f > f_c \), being \( f \) the frequency, the TBT is still valid [13] and that two families of normal modes appear as doublets [13, 14, 15, 16, 17, 18, 19, 20]. One family of these doublets is called the “second TBT spectrum”. It was also shown, in Ref. [13], that the slope \( \Psi \), along the cross-sectional area, is not longer constant for the second TBT spectrum. Thus, it expected that the value of the shear coefficient \( \kappa \) changes for frequencies higher than \( f_c \). A qualitative picture of the classification of the first and second TBT spectra, in terms of the shape of the cross-sectional area, is given in Fig. 1.

Following the line of Ref. [10], in this paper we will give a new benchmark for the Timoshenko shear coefficient that is valid not only below but also above the critical frequency. In the next section we introduce the Timoshenko’s beam theory with the different values of the shear coefficient. In Section 3 we compare the different predictions of the TBT for different values of the shear coefficient \( \kappa \) with the experiment results published recently [13]. Some brief conclusions are then given.

2. Timoshenko beam theory

The vertical displacement \( \xi \) in the two-coefficient Timoshenko beam theory satisfy [12]

\[
\frac{EI}{\rho A} \frac{\partial^4 \xi}{\partial z^4} - \frac{I}{A} \left( 1 + \frac{E}{\kappa_3 G} \right) \left( \frac{\partial^4 \xi}{\partial z^2 \partial t^2} + \frac{\partial^2 \xi}{\partial t^2} + \frac{\rho I}{\kappa_1 GA} \frac{\partial^4 \xi}{\partial t^4} \right) = 0,
\]

where \( G \) and \( E \) are the shear and Young’s moduli, and \( A, \rho, \) and \( I \) are the cross-sectional area, the density and the second moment of area, respectively. This one-dimensional theory predicts correctly the doublets [13] but it assumes flat deformations of the cross-sectional area; any other deformation is absorbed in the shear coefficient \( \kappa \).

When the beam is vibrating in a normal-mode, the separation of variables can be used,

\[
\xi(z,t) = e^{i\omega t} \chi(z),
\]
where \( \chi(z) \) is the time-independent displacement amplitude. With the previous anzats the Timoshenko equation can be written as

\[
d^4 \chi \over dz^4 + \frac{\rho \omega^2}{M_t} d^2 \chi \over dz^2 + \frac{\rho^2 \omega^2}{\kappa_3 G E} (\omega^2 - \omega_c^2) \chi = 0,
\]

where \( M_t \) is the reduced modulus defined as

\[
\frac{1}{M_t} = \frac{1}{E} + \frac{1}{\kappa G},
\]

and

\[
\omega_c = 2\pi f_c = \sqrt{\frac{\kappa_3 G A}{\rho I}};
\]

\( f_c \) is known as the critical frequency.

For a free-free beam of length \( L \) the boundary conditions, given by the vanishing of moments and shear forces, can be written in terms of the time-independent displacement amplitude only as \([13, 16, 18]\)

\[
\begin{align*}
\frac{d^2 \chi}{dz^2} \bigg|_{z=0,L} &= -\rho \omega^2 \chi \bigg|_{z=0,L}, \\
\frac{d^3 \chi}{dz^3} \bigg|_{z=0,L} &= -\frac{\rho \omega^2}{M_t} \frac{d \chi}{dz} \bigg|_{z=0,L}.
\end{align*}
\]

Notice that Eqs. (6) and (7) depend on the frequency, and that they reduce to the Bernoulli-Euler boundary conditions, used in Ref. \([10]\), when the terms at the right are much smaller than the respective terms at the left, \( i.e., \) for low frequencies.

The solution to Eq. (3), obtained using standard methods for ordinary differential equations, is given by

\[
\chi(z) = a \exp(k_+ z) + b \exp(-k_+ z) + c \exp(ik_- z) + d \exp(-ik_- z),
\]

where \( \chi_+(z) = \exp(k_+ z) \), \( a, b, c, \) and \( d \) are constants determined by the boundary conditions and

\[
k^2_\pm(\omega) = -\frac{\rho \omega^2}{2M_t} \pm \frac{1}{2} \sqrt{\frac{\rho^2 \omega^4}{M^2_t} - \frac{4\rho^2 \omega^2 (\omega^2 - \omega_c^2)}{\kappa_3 G E}}.
\]

It can be seen from the previous equations that the critical frequency separates the behavior of the solutions in two regimes: when \( \omega < \omega_c \) there are two travelling
and two exponential terms in Eq. (8) since \( k_\pm \) are real; when \( \omega > \omega_c \) all the terms in Eq. (8) are travelling waves.

The frequency spectrum can be obtained inserting Eq. (8) in the boundary conditions (6) and (7); the normal-mode frequencies are then obtained when \( \det P = 0 \) with

\[
P = \begin{pmatrix}
(k_+^2 + \frac{\rho \omega^2}{\kappa G}) & (k_+^2 + \frac{\rho \omega^2}{\kappa G}) & (k_-^2 + \frac{\rho \omega^2}{\kappa G}) & (k_-^2 + \frac{\rho \omega^2}{\kappa G}) \\
(k_+^2 + \frac{\rho \omega^2 k}{M_r}) e^{k_+ L} & (k_+^2 + \frac{\rho \omega^2 k}{M_r}) e^{-k_+ L} & (k_-^2 + \frac{\rho \omega^2 k}{M_r}) e^{k_- L} & (k_-^2 + \frac{\rho \omega^2 k}{M_r}) e^{-k_- L} \\
(-k_+^2 + \frac{\rho \omega^2 k}{M_r}) & (-k_+^2 + \frac{\rho \omega^2 k}{M_r}) & (k_-^2 + \frac{\rho \omega^2 k}{M_r}) e^{k_- L} & (k_-^2 + \frac{\rho \omega^2 k}{M_r}) e^{-k_- L} \\
(-k_+^2 + \frac{\rho \omega^2 k}{M_r}) e^{k_+ L} & (-k_+^2 + \frac{\rho \omega^2 k}{M_r}) e^{-k_+ L} & (k_-^2 + \frac{\rho \omega^2 k}{M_r}) & (k_-^2 + \frac{\rho \omega^2 k}{M_r}) e^{-k_- L}
\end{pmatrix}.
\]

(10)

### 3. Results

The lower 24 normal-mode frequencies of a beam of rectangular cross-section, with length \( L = 0.500 \) m, height \( a = 0.0252 \) m and width \( b = 0.0504 \) m, were calculated finding the roots of the determinant of Eq. (10). The elastic constants of the beam, taken from Ref. [13], are \( G = 26.92 \) GPa, \( E = 67.42 \) GPa and \( \rho = 2699.04 \) kg/m³. Several values, found in the literature, for the Timoshenko shear coefficient \( \kappa = \kappa_1 = \kappa_3 \) were used (cases A, B, C, D, and E); these values are given in the Table 1. The results were compared with the experimental results of Ref. [13]. The error between the experimental results of Ref. [13] and the theory with the different values of \( \kappa_1 \) and \( \kappa_3 \) is plotted in Fig. 2. As it can be seen in this figure, an excellent agreement between the theory and the experiment is found at low frequencies; one can also notice that the error grows with the frequency. Close to \( f_c \), which varies between 196187 Hz and 202383 Hz for the different cases reported here, see Eq. (5), the error between the different theories and the experiment grows up to 3 %; the larger errors are found close to \( f_c \), around mode 13, where a peak in the error is found. In this figure one can also observe that the results above the critical frequency present larger errors than the results below \( f_c \).

We also calculated the best shear coefficients using least squares in three different ways: with one independent coefficient \( \kappa = \kappa_1 = \kappa_3 \) (case F); with \( \kappa_1 \) and \( \kappa_3 \) as independent coefficients (case G); and, due to the change of regime introduced by the critical frequency, with four coefficients, \( \kappa_1 \) and \( \kappa_3 \) for \( f < f_c \) and \( \kappa_1 \) and \( \kappa_3 \) for \( f > f_c \) (case H). As it can be seen from Fig. 2 in case H, the error is smaller than 0.5 % for all resonances below and above the critical frequency. The error is smaller than 0.62 % in case G.
Table 1: Different values of the Timoshenko coefficients used to test the TBT. The beam has length $L = 0.5$ m, height $a = 0.0252$ m, and width $b = 0.0504$ m. The elastic properties of the beam are $G = 26.92$ GPa, $E = 67.42$ GPa and $\rho = 2699.04$ kg/m$^3$ that correspond to aluminum. The values used in cases F, G and H were calculated using least squares (see text).

| case symbol | Timoshenko shear coefficient(s) |
|-------------|--------------------------------|
| A ▲         | $\kappa = 5/6$                |
| B ▲         | $\kappa = 0.83945$            |
| C ◦         | $\kappa = 10(1 + \nu)/(12 + 11\nu)$ |
| D □         | $\kappa = 5(1 + \nu)/(6 + 5\nu)$ |
| E ◊         | $\kappa_1 = 10(1 + \nu)/(12 + 11\nu), \kappa_3 = 5(1 + \nu)/(6 + 5\nu)$ |
| F •         | $\kappa = 0.8291$            |
| G ■         | $\kappa_1 = 0.80811, \kappa_3 = 0.83292$ |
| H ×         | below $f_c$ $\kappa_1 = 0.82003, \kappa_3 = 0.84651$ \above $f_c$ $\tilde{\kappa}_1 = 0.81687, \tilde{\kappa}_3 = 0.81923$ |

4. Conclusions

In this paper we have shown that the predictions of the Timoshenko’s beam theory can be very accurate not only below but also above the critical frequency $f_c$. This was done by comparing the theoretical results for different values of the Timoshenko shear coefficient with experimental results published very recently. When the two–coefficient Timoshenko beam theory is used, except close to the critical frequency, the difference between theory and experiment is smaller than 0.5%. This difference is smaller than 0.62% if one includes the results that are close to the critical frequency. The results strongly suggest that the value of the shear coefficient above the critical frequency is different from the value below it.

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Figure 1: Displacement of the cross-sectional area for modes of with frequency below the critical frequency $f_c$ (A), for modes belonging to the first TBT spectrum (A), and for the second TBT spectrum (B).
Figure 2: Percentage error in frequency between the experimental results of Ref. [13] and the theoretical predictions of the Timoshenko beam theory. The aluminum rod has a length $L = 0.500$ m, height $a = 0.0252$ m and width $b = 0.0504$ m; the elastic constants are $G = 26.92$ GPa, $E = 67.42$ GPa and $\rho = 2699.04$ kg/m$^3$. The filled diamonds correspond to $\kappa = 5/6$, the filled triangles to $\kappa = 0.83945$, the open circles to $\kappa = 10(1 + \nu)/(12 + 11\nu)$, the open squares to $\kappa = 5(1 + \nu)/(6 + 5\nu)$, the open diamonds to $\kappa_1 = 10(1 + \nu)/(12 + 11\nu)$ and $\kappa_3 = 5(1 + \nu)/(6 + 5\nu)$, the filled circles to $\kappa = 0.8291$, the filled squares to $\kappa_1 = 0.80811$ and $\kappa_3 = 0.83292$, the crosses below $f_c$ to $\kappa_1 = 0.82003$ and $\kappa_3 = 0.84651$, finally the crosses above $f_c$ to $\kappa_1 = 0.81687$ and $\kappa_3 = 0.81923$. 