Minimax Optimal Algorithms
for Unconstrained Linear Optimization

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Abstract

We design and analyze minimax-optimal algorithms for online linear optimization games where the player’s choice is unconstrained. The player strives to minimize regret, the difference between his loss and the loss of a post-hoc benchmark strategy. The standard benchmark is the loss of the best strategy chosen from a bounded comparator set. When the comparison set and the adversary’s gradients satisfy $L_\infty$ bounds, we give the value of the game in closed form and prove it approaches $\sqrt{2T}/\pi$ as $T \to \infty$.

Interesting algorithms result when we consider soft constraints on the comparator, rather than restricting it to a bounded set. As a warmup, we analyze the game with a quadratic penalty. The value of this game is exactly $T/2$, and this value is achieved by perhaps the simplest online algorithm of all: unprojected gradient descent with a constant learning rate. We then derive a minimax-optimal algorithm for a much softer penalty function. This algorithm achieves good bounds under the standard notion of regret for any comparator point, without needing to specify the comparator set in advance. The value of this game converges to $\sqrt{e}$ as $T \to \infty$; we give a closed-form for the exact value as a function of $T$. The resulting algorithm is natural in unconstrained investment or betting scenarios, since it guarantees at worst constant loss, while allowing for exponential reward against an “easy” adversary.

1 Introduction

Minimax analysis has recently been shown to be a powerful tool for the construction of online learning algorithms [Rakhlin et al., 2012]. Generally, these results use bounds on the value of the game (often based on the sequential Rademacher complexity) in order to construct efficient algorithms. In this work, we show that when the learner is unconstrained, it is often possible to efficiently compute an exact minimax strategy.

We consider a game where on each round $t = 1, \ldots, T$, first the learner selects $x_t \in \mathbb{R}^n$, and then an adversary chooses $g_t \in \mathcal{G} \subset \mathbb{R}^n$, and the learner suffers loss $g_t \cdot x_t$. The goal of the learner is to minimize regret, that is, loss in excess of that achieved by a benchmark strategy. We define

$$\text{Regret} = \text{Loss} - (\text{Benchmark Loss}) = \sum_{t=1}^{T} g_t \cdot x_t - L(g_1, \ldots, g_T)$$  (1)
as the regret with respect to benchmark performance $L$ (the $L$ intended will be clear from context). Letting $I(x \in \mathcal{X}) = 0$ for $x \in \mathcal{X}$ and $\infty$ otherwise, the standard definition of regret arises from the choice

$$L(g_1, \ldots, g_T) = \inf_{x \in \mathcal{X}} g_{1:T} \cdot x + I(x \in \mathcal{X}),$$

(2)

the loss of the best strategy in a bounded convex set $\mathcal{X}$ (we write $g_{1:T} = \sum_{s=1}^T g_s$ for a sum of scalars or vectors). When $L$ depends only on the sum $G \equiv g_{1:T}$ we write $L(G)$. We will be able to interpret the alternative benchmarks $L$ we consider as penalties $\Psi$ on comparator points, so $L(G) = \arg\min_x G \cdot x + \Psi(x)$, where $\Psi(x)$ has replaced $I(x \in \mathcal{X})$ in Eq. (2).

We view this interaction as a sequential zero-sum game played over $T$ rounds, where the player strives to minimize Eq. (1), and the adversary attempts to maximize it. We study the value of this game, $V^T$, and design minimax optimal algorithms for the player; formal definitions are given below. Some results are more naturally stated in terms of rewards rather than losses, and so we define $\text{Reward} = -\text{Loss} = -\sum_{t=1}^T g_t x_t$.

**Outline and Summary of Results** Section 2 provides motivation for the consideration of alternative benchmarks $L$. Section 3 then develops several theoretical tools for analyzing unconstrained games with concave benchmark functions $L$. Section 4 applies this theory to three particular instances; Figure 1 summarizes the results from this section. These games exhibit a strong combinatorial structure, which leads to interesting algorithms and perhaps surprising game values.

Section 4.1 serves as a warmup, where we show that constant step-size gradient descent is in fact minimax optimal for a natural choice of $L$, which can be though of as replacing the hard feasible set $\mathcal{X}$ in Eq. (2) with a quadratic penalty function on comparator points. Section 4.2 provides results analogous to those of Abernethy et al. 2008; we consider regret compared to the best $\hat{x}$ where $\|\hat{x}\|_\infty \leq 1$ against an adversary constrained to play $\|g_t\|_\infty \leq 1$, while Abernethy et al. considered $\|g_t\|_2 \leq 1$ and $\|\hat{x}\|_2 \leq 1$ for $n \geq 3$ dimensions. Interestingly, while we prove results for the unconstrained player, we show the optimal strategy in fact always plays points from $\mathcal{X} = \{x \mid \|x\|_\infty \leq 1\}$, and so applies to the constrained case as well. Our results hold for the $n = 1$ case (where $L_2$ and $L_\infty$ coincide), showing that the value of the game approaches $\sqrt{2T/\pi}$ as $T \to \infty$, as opposed to $\sqrt{T}$ as one might extrapolate from the results of Abernethy. This indicates an interesting change in the geometry of the $L_2$ game between $n = 1$ and $n = 3$. Finally, Section 4.3 gives a minimax optimal algorithm for the setting introduced by Streeter and McMahan [2012]. Following their work, our algorithm obtains standard regret at most $O(R\sqrt{T} \log ((1 + R)T))$ simultaneously for any comparator $\hat{x}$ with $|\hat{x}| = R$, without needing to choose $R$ in advance. However, we emphasize a slightly different interpretation of this setting, discussed in Section 2. It is worth noting that the regret (relative the the respective $L$) of these algorithms is $O(T)$, $O(\sqrt{T})$, and $O(1)$, respectively, though all three are minimax algorithms.

**The Minimax Value of the Game** Given a benchmark function $L$, the minimax value of the game is

$$V^T = \left(\inf_{x_t \in \mathcal{X}} \sup_{g_t \in G} \left(\sum_{t=1}^T g_t \cdot x_t - L(g_1, \ldots, g_T)\right)\right)$$

(3)
where \( \langle \inf_{x_t} \sup_{g_t} \rangle_{t=1}^T \) is a shorthand notation for \( \inf_{x_1} \sup_{g_1} \ldots \inf_{x_T} \sup_{g_T} \). Against a worst-case adversary, any algorithm must incur regret at least \( V^T \), and the minimax optimal algorithm will incur regret at most \( V^T \) against any adversary. Since in this work we study minimax algorithms, we will often use the value of the game \( V^T \) as an upper bound on Regret (as defined in Eq. (1)). Generally we will not assume our adversaries are minimax optimal.

We are also concerned with the conditional value of the game, \( V_t \), given \( x_1, \ldots, x_t \) and \( g_1, \ldots, g_t \) have already been played. That is, the Regret when we fix the plays on the first \( t \) rounds, and then assume minimax optimal play for rounds \( t + 1 \) through \( T \). However, following the approach of Rakhlin et al. [2012], we omit the terms \( \sum_{s=t+1}^T g_s \cdot x_s \) from Eq. (3).

We can view this as cost that the learner has already payed, and neither that cost nor the specific previous plays of the learner impact the value of the remaining terms in Eq. (1). Thus, we define

\[
V_t(g_1, \ldots, g_t) = \left\langle \inf_{x_t} \sup_{g_t} \right\rangle_{t=1}^T \left( \sum_{s=t+1}^T g_s \cdot x_s - L(g_1, \ldots, g_T) \right).
\]

Note the conditional value of the game before anything has been played, \( V_0() \), is exactly \( V^T \).

**Related Work**  Regret-based analysis has received extensive attention in recent years; see Shalev-Shwartz [2012] and Cesa-Bianchi and Lugosi [2006] for an introduction. The analysis of alternative notions of regret is also not new. In the expert setting, there has been much work on tracking a shifting sequence of experts rather than the single best expert; see Koolen et al. [2012] and references therein. Zinkevich [2003] considers drifting comparators in an online convex optimization framework. This notion can be expressed by an appropriate \( L(g_1, \ldots, g_T) \), but now the order of the gradients matters, unlike the benchmarks \( L \) considered in this work. Merhav et al. [2006] and Dekel et al. [2012] consider the stronger notion of policy regret in the online experts and bandit settings, respectively. For investing scenarios, Agarwal et al. [2006] and Hazan and Kale [2009] consider regret with respect to the best constant-rebalanced portfolio.
More recently, the field has seen minimax approaches to online learning. Abernethy and Warmuth [2010] give a minimax strategy for several zero-sum games against a budgeted adversary. Section 4.2 studies the online linear game of Abernethy et al. [2008] under different assumptions, and we adapt some techniques from Abernethy et al. [2009]. Rakhlin et al. [2012] takes powerful tools for non-constructive analysis of online learning problems and shows they can be used to design algorithms; our work differs in that we focus on cases where the exact minimax strategy can be computed.

2 Alternative Notions of Regret

One of our contributions is showing that interesting results can be obtained by choosing $L$ differently than in Eq. (2); in particular, we obtain minimax optimal algorithms for the problem considered by Streeter and McMahan [2012] by analyzing an appropriate choice of $L$.

One could choose $L(G) = 0$, but this leads to an uninteresting game: the adversary has no long-term constraints, and so can simply pick $g_t$ to maximize $g_t x_t$ for whatever $x_t$ the player selected. Thus, the player can do no better than always picking $x_t = 0$. This is exactly the reason for studying the standard notion of regret: we do not require that we do well in absolute terms, but rather relative to the best strategy from a fixed set.

That is, interesting games result when the player accepts the fact that it is impossible to do well in terms of the absolute loss $\sum_t g_t x_t$ for all sequences $g_1, \ldots, g_T$. However, the player can do better on some sequences at the expense of doing worse on others. The benchmark function $L$ makes this notion precise: sequences for which $L(g_1, \ldots, g_T)$ is large and negative are those on which the player desires good performance at the expense of allowing more loss (in absolute terms) on sequences where $L(g_1, \ldots, g_T)$ is large and positive. The value of the game $V_T$ tells us to what extent any online algorithm can hope to match the benchmark performance $L$. It follows by definition that if we add a constant $k$ to $L$ (making $L$ easier to achieve), we decrease the minimax value of the game by $k$, without changing the minimax optimal strategy.

We can use these ideas to derive algorithms for a setting that is quite different from typical online convex optimization. On each round $t$, the world (possibly adversarial, possibly not) offers the player a betting opportunity on a binary outcome; the player can take either side of the bet. The player begins with $\$1$, but on later rounds can wager up to whatever amount he currently has (based on previous wins and losses). The player selects an amount $x_t$ to bet, and then the world reveals whether the bet was won or lost; if the player won the bet, he receives $x_t$ dollars; otherwise, he loses $x_t$ dollars. The players net winnings are $-\sum_t g_t x_t$, where $g_t \in \{-1, 1\}$; the player wins the bet when $\text{sign}(x_t) \neq g_t$ (thus, the player strives to minimize $\sum_t g_t x_t$). How should the player bet in this game? Clearly if the world is adversarial, we cannot do better than always betting $x_t = 0$. But, we might have reason to believe the world is not fully adversarial; if we knew $g_t = 1$ with a fixed probability $p$, then following a Kelly betting scheme [Kelly Jr, 1956] might be appropriate, but knowing $p$ is often unrealistic in practice.

If the player is familiar with online linear optimization, he might try projected online

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1 It can be useful to think about $-L(G)$ as the benchmark reward for the sequence with gradient sum $G$. 

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gradient descent \cite{Zinkevich:2003} with a constant step size. If we restrict our bets to the feasible set $[-B, B]$, letting $G = g_{1:T}$, this algorithm guarantees $\text{Regret} = \text{Loss} + B|G| \leq 2B\sqrt{T}$. Then $\text{Winnings} = -\text{Loss} \geq B|G| - 2B\sqrt{T}$. Thus in the best case (when $|G| = T$) the player ends up with a little less than $BT$; but he can lose up to $B\sqrt{T}$ when $G = 0$. Thus, to ensure he loses no more than the \$1 he has on hand, he must choose $B = 1/\sqrt{T}$. With this restriction, in the best case the player wins less than $\sqrt{T}$ dollars. However, the post-hoc optimal strategy would have been to bet everything every round, netting winnings of $2^T$. Despite the theoretical guarantees, the player certainly might feel regret at having won only $\sqrt{T}$ in this situation!

One might also hope to use online algorithms for portfolio management, for example those of Hazan and Kale \cite{HazanKale:2009} and Agarwal et al. \cite{Agarwal:2006}. However, these algorithms require the assumption that you always retain at least an $\alpha > 0$ fraction of your bet, which is directly violated in our game.

By carefully crafting a suitable benchmark function $L$, we can provide the player with a more satisfying algorithm. Ideally, we would like an $L$ that satisfies three properties: 1) there exists an algorithm where regret is bounded by a constant $\epsilon$ (for any $T$) with respect to $L$, 2) $-L(G) \geq 0$, and 3) $-L(G)$ grows exponential in $|G|$. Properties 1) and 2) ensure the player never loses more than $\epsilon$ running this algorithm; by scaling the bets the algorithm suggests by $1/\epsilon$, he can ensure he never loses more than his starting \$1. Property 3 implies that for “easy” sequences, we get exponential reward; in fact, given 1) and 2) we would like $-L(G)$ to grow as quickly as possible.

Of course, if the adversary chooses $g_t$ uniformly at random from $\{-1, 1\}$ each round, we expect to frequently see $|G| \geq \sqrt{T}$, and so intuitively we will not be able to guarantee exponential winnings. This suggests the best we might hope for is a function like $L(G) = -\exp \left( \frac{|G|}{\sqrt{T}} \right)$. In fact, in Section 4.3 we show that constant regret against such a benchmark function is possible, and we derive a minimax algorithm.

A Comparator Set Interpretation The classic definition of regret defines $L$ indirectly as the loss of the best strategy from a fixed class $\mathcal{X}$ in hindsight, Eq. (2). As this work shows, it can be advantageous to state $L$ as an explicit function of $G$; however, useful intuition can be gained by interpreting $L$ as a penalty function on comparator points $\hat{x}$. That is, we wish to find a $\Psi$ such that

$$L(G) = \arg \min_{x} Gx + \Psi(x).$$

For the benchmark functions $L$ we consider, we also derive the corresponding penalty functions $\Psi$ using convex conjugates. These are summarized in our results in Figure 1.

The standard notion of regret correspond to a hard penalty $\Psi(x) = I(x \in \mathcal{X})$. Such a definition makes sense when the player by definition must select a strategy from some bounded set, for example a probability from the $n$-dimensional simplex, or a distribution on paths in a graph. For such problems, standard regret is really comparing the player’s performance to that of any fixed feasible strategy chosen with knowledge of $g_1, \ldots, g_T$; by putting an equal penalty on each of them, we do not indicate any prior belief that some strategies are more likely to be optimal than others.

\footnote{Any other algorithm that provides a bound on standard regret of $\mathcal{O}(B\sqrt{T})$ will behave similarly.}
However, in contexts such as machine learning where any $x \in \mathbb{R}^n$ corresponds to a valid model, such a hard constraint is difficult to justify; while any $x \in \mathbb{R}^n$ is technically feasible, in order to prove regret bounds we compare to a much more restrictive set. As an alternative, in Sections 4.1 and 4.3 we propose soft penalty functions that encode the belief that points near the origin are more likely to be optimal (we can always re-center the problem to match our beliefs in this regard), but do not rule out any $x \in \mathbb{R}^n$ a priori.

3 General Unconstrained Linear Optimization

In this section we prove a theorem that greatly simplifies the task of computing minimax values and deriving algorithms for the games we consider. We prove this result in the one-dimensional case; Corollary 2 then extends the result to $n$-dimensions.

**Theorem 1.** Consider the one-dimensional unconstrained game where the player selects $x_t \in \mathbb{R}$ and the adversary chooses $g_t \in \mathcal{G} = [-1, 1]$, and $L$ is concave in each of its arguments and bounded below on $\mathcal{G}^T$. Then,

$$V^T = \mathbb{E}_{g_t \sim \{-1, 1\}} [-L(g_1, \ldots, g_T)].$$

where the expectation is over each $g_t$ chosen independently and uniformly from $\{-1, 1\}$ (that is, the $g_t$ are Rademacher random variables). Further, the conditional value of the game is

$$V_t(g_1, \ldots, g_t) = \mathbb{E}_{g_{t+1}, \ldots, g_T \sim \{-1, 1\}} [-L(g_1, \ldots, g_T)].$$  \hspace{1cm} (5)

**Proof.** We argue by backwards induction (from $t = T$ to $t = 1$) on the conditional value of the game, with the induction hypothesis that

$$V_t(g_1, \ldots, g_t) = \mathbb{E}_{g_{t+1}, \ldots, g_T \sim \{-1, 1\}} [-L(g_1, \ldots, g_T)].$$  \hspace{1cm} (6)

and further that $V_t$ is convex in each of its arguments and bounded above on $\mathcal{G}^T$. The induction hypothesis holds trivially for $T = t$, using the assumption that $L$ is concave and bounded below for the second part. Now, suppose the induction hypothesis holds for $t$. We then have (by the definition of $V_t$)

$$V_{t-1}(g_1, \ldots, g_{t-1}) = \inf_{x_t} \sup_{g_t} g_t x_t + V_t(g_1, \ldots, g_{t-1}, g_t).$$

Note $V_{t-1}$ must be convex in each of its arguments, using the induction hypothesis on $V_t$. Let $M(g, x) = g x + V_t(g_1, \ldots, g_{t-1}, g)$. We would like to appeal to the minimax theorem to switch the inf and sup, but since $M$ is convex in $g$ (using the induction hypothesis) rather than concave, we cannot do so immediately. However, because we are choosing $g_t \in [-1, 1]$, it follows from the convexity of $M$ that the supremum is obtained at either $-1$ or $+1$. Thus, we can write

$$V_{t-1}(g_1, \ldots, g_{t-1}) = \inf_{x_t} \sup_{g_t \in [-1, 1]} M(g_t, x_t)$$

$$= \inf_{x_t} \sup_{g_t \in \{-1, 1\}} M(g_t, x_t)$$

$$= \inf_{x_t} \sup_{p_t \in \Delta([-1, 1])} \mathbb{E}_{g_t \sim p_t} [M(g_t, x_t)],$$

where the expectation is over each $g_t$ chosen independently and uniformly from $\{-1, 1\}$.
where \( p_t \in [0, 1] \) is the probability the adversary chooses \( g_t = +1 \) (otherwise, \( g_t = -1 \)). Now \( \mathbb{E}_{g_t \sim p_t}[M(g_t, x_t)] \) is linear in both \( p_t \) and \( x_t \), and so we can apply the minimax theorem (e.g., Theorem 7.1 from Cesa-Bianchi and Lugosi [2006]), which gives

\[
V_{t-1}(g_1, \ldots, g_{t-1}, g_t) = \sup_{p_t \in \Delta([-1,1])} \inf_{x_t} \mathbb{E}[g_t x_t + V_t(g_1, \ldots, g_{t-1}, g_t)]
\]

Now, the adversary (sup player) must choose \( p_t = 0.5 \) so \( \mathbb{E}[g_t] = 0 \), or otherwise the player can choose \( x_t \) to drive the value to \(-\infty\) (since \( V_t \) is bounded above). Thus, the first expectation term disappears, and the choice of the player becomes irrelevant, giving

\[
V_{t-1}(g_1, \ldots, g_{t-1}) = \mathbb{E}[V_t(g_1, \ldots, g_{t-1}, g_t)],
\]

where now the expectation is on \( g_t \) drawn i.i.d. from \([-1,1]\). Applying the induction hypothesis completes the proof, since then iterated expectation yields Eq. (6) for \( V_{t-1} \), and boundedness is immediate. \( \square \)

The use of randomization to allow the application of the minimax theorem is similar to the technique used by Abernethy et al. [2009].

A key insight from the proof is that an optimal adversary can always select from \([-1,1]\). With this knowledge, we can view the game as a binary tree of height \( T \). An algorithm for the player simply assigns a play \( x \in \mathbb{R} \) to each node, and the adversary chooses which outgoing edge to take: if the adversary chooses the left edge, the player suffers loss \( x \), otherwise the player wins \( x \) (suffers loss \(-x\)). Finally, when leaf \( \ell \) is reached, the adversary pays the player some amount \( L(\ell) \). Theorem 1 implies the value of the game is then simply the average value of \(-L(\ell)\).

Given Theorem 1 and the fact that the functions \( L \) of interest will generally depend only on \( g_{1:T} \), it will be useful to define \( B_T \) to be the distribution of \( g_{1:T} \) when each \( g_t \) is drawn independently and uniformly from \([-1,1] \) (that is, the sum of \( T \) Rademacher random variables).

Theorem 1 immediately yields bounds for games in \( n \)-dimensions where the adversary is constrained to play \( \|g_t\|_\infty \leq 1 \):

**Corollary 2.** Consider the game where the player chooses \( x_t \in \mathbb{R}^n \), and the adversary chooses \( g_t \in [-1,1]^n \), and the total payoff is

\[
\sum_{t=1}^{T} g_t \cdot x_t - \sum_{i=1}^{n} L(g_{i;1:T})
\]

for a concave function \( L \). Then, the value of the game is

\[
V_T = n \mathbb{E}_{G \sim B_T} \left[ -L(G) \right],
\]

Further, the conditional value of the game is

\[
V_t(g_1, \ldots, g_t) = \sum_{i=1}^{n} \mathbb{E}_{G_i \sim B_T-i} \left[ -L(g_{i;1:t} + G_i) \right].
\]
Proof sketch. The proof follows by noting the constraints on both players’ strategies and the value of the game fully decompose on a per-coordinate basis.

A recipe for minimax optimal algorithms in one dimension

For any function $L$,\[ \mathbb{E}_{G \sim B_T}[L(G)] = \frac{1}{2T} \sum_{i=0}^{T} \binom{T}{i} L(2i - T), \tag{7} \]

since $2^{-T} \binom{T}{i}$ is the binomial probability of getting exactly $i$ gradients of +1 over $T$ rounds, which implies $T - i$ gradients of −1, so $G = i - (T - i) = 2i - T$.

Since Eq. (5) gives the minimax value of the game if both players play optimally from round $t + 1$ forward, a minimax strategy for the learner on round $t + 1$ must be

\[ x_{t+1} = \arg\min_{x \in \mathbb{R}} \max_{g \in \{-1, 1\}} g \cdot x + V_{t+1}(g_1, \ldots, g_t, g) \]

\[ = \frac{1}{2}(V_{t+1}(g_1, \ldots, g_t, -1) - V_{t+1}(g_1, \ldots, g_t, +1)). \tag{8} \]

The second line follows because the argmin is simply over the max of two intersecting linear functions, which we can compute in closed form as the point of intersection. Thus, if we can derive a closed form for $V_t(g_1, \ldots, g_t)$, we will have an efficient minimax-optimal algorithm. In the next section, we explore cases where this is possible.

When $L$ depends only on $G = g_1 : T$, we may be able to run the minimax algorithm efficiently even if $V_t$ does not have a convenient closed form: if $\tau = T - t$, the number of rounds remaining, is small, then we can compute $V_t$ exactly by using the appropriate binomial probabilities (following Eq. (5) and Eq. (7)). On the other hand, if $\tau$ is large, then applying the Gaussian approximation to the binomial distribution may be sufficient.

4 Deriving Minimax Optimal Algorithms

In this sections, we explore three applications of the tools from the previous section. We begin with a relatively simple but interesting example which illustrates the technique.

4.1 Constant step-size gradient descent can be minimax optimal

Suppose we use a “soft” feasible set for the benchmark,

\[ L(G) = \min_x Gx + \frac{\sigma}{2} x^2 = \frac{1}{2\sigma} G^2, \tag{9} \]

for a constant $\sigma > 0$. Does a no-regret algorithm against this comparison class exist? Unfortunately, the general answer is no, as shown in the next theorem:

**Theorem 3.** The value of this game is $V_T = \mathbb{E}_{G \sim B_T} \left[ \frac{1}{2\sigma} G^2 \right] = \frac{T}{2\sigma}$. 

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Proof. Starting from Eq. (7),

$$E_{G \sim B_T} [G^2] = \frac{1}{2T} \sum_{i=0}^{T} \binom{T}{i} (2i - T)^2$$  \hspace{1cm} Eq. (7)

$$= \frac{1}{2T} \left( 4 \sum_{i=0}^{T} \binom{T}{i} i^2 - 4T \sum_{i=0}^{T} \binom{T}{i} i + T^2 \sum_{i=0}^{T} \binom{T}{i} \right)$$

and since $\sum_{t=0}^{T} \binom{T}{t} = 2T$, $\sum_{t=0}^{T} \binom{T}{t} t = T2^{T-1}$, $\sum_{t=0}^{T} \binom{T}{t} t^2 = (T + T^2)2^{T-2}$,

$$= \frac{1}{2T} \left( 4(T + T^2)2^{T-2} - 4T(T^2-1) + T^22^T \right)$$

$$= (T + T^2) - 2T^2 + T^2 = T.$$ 

The result then follows from linearity of expectation. \qed

This implies $\text{Reward} \geq -L(G) - \text{Regret} = \frac{1}{2\sigma}(G^2 - T)$, a fact noted by Streeter and McMahan [2012, Lemma 2].

Thus, for a fixed $\sigma$, we cannot have no a regret algorithm with respect to this $L$. However, if $T$ is known in advance, we could choose $\sigma = \sqrt{T}$ in order to claim no-regret. But this is a bit arbitrary: if the player could pick $\sigma$, and cares purely about Regret, obviously he would like to play the game where $\sigma \to \infty$, as that makes the value of the game (Regret) as small as possible. However, this choice also drives Reward to zero. If the lower-bound on reward is what matters, then the player should choose based on how he expects $G^2$ to relate to $T$.

To derive the minimax optimal algorithm, we can compute conditional values (using similar techniques to Theorem 3),

$$V_t(g_1, \ldots, g_t) = \mathbb{E}_{G \sim B_{T-t}} \left[ \frac{1}{2\sigma}(g_{1:t} + G)^2 \right] = \frac{1}{2\sigma}((g_{1:t})^2 + (T - t)),$$

and so following Eq. (8) the minimax-optimal algorithm must use

$$x_{t+1} = \frac{1}{4\sigma} ((g_{1:t} - 1)^2 + (T - t - 1)) - ((g_{1:t} + 1)^2 + (T - t - 1))$$

$$= \frac{1}{4\sigma}(-4g_{1:t}) = -\frac{1}{\sigma}g_{1:t}$$

Thus, a minimax-optimal algorithm is simply constant-learning-rate gradient descent with learning rate $\frac{1}{\sigma}$. Note that for a fixed $\sigma$, this is the optimal algorithm independent of $T$; this is atypical, as usually the minimax optimal algorithm depends on the horizon (as we will see in the next two cases).

### 4.2 Optimal regret against hypercube adversaries

Abernethy et al. [2008] gives a minimax optimal algorithm when the player’s $x_t$ and the comparator $\check{x}$ are constrained to an $L_2$ ball, and the adversary must also select $g_t$ from an $L_2$ ball, for $n \geq 3$ dimensions\footnote{Their results are actually more general than this, allowing the constraint on $\|g_t\|_2$ to vary on a per-round basis. Our work could also be extended in that manner.}. In contrast, we consider regret compared to the best $\check{x}$.
constrained to the unit $L_\infty$ ball, but allow the player to select any $x_t \in \mathbb{R}^n$; our adversary is constrained to select $g_t$ from the unit $L_\infty$ ball (the generalization to arbitrary hyper-rectangles is straightforward). Perhaps surprisingly, the optimal strategy for the player always plays from the unit $L_\infty$ ball as well, so our results immediately apply to the case of the constrained player.

Since we consider $L_\infty$ constraints on both the comparator and adversary, Corollary 2 implies it is sufficient to study the one-dimensional case. We consider the standard notion of regret, taking $L(G) = -|G|$ following Eq. (2). Our main result is the following:

**Theorem 4.** Consider the game between an adversary who chooses loss functions $g_t \in [-1,1]$, and a player who chooses $x_t \in \mathbb{R}$. For a given sequence of plays, $x_1, g_1, x_2, g_2, \ldots, x_T, g_T$, the value to the adversary is $\sum_{t=1}^T g_t x_t - |g_1:T|$. Then, when $T$ is even with $T = 2M$, the minimax value of this game is given by

$$V_T = 2^{-T} \frac{2M T!}{(T - M)! M!} \leq \sqrt{\frac{2T}{\pi}}.$$  

Further, as $T \to \infty$, $V_T \to \sqrt{\frac{2T}{\pi}}$.

**Proof.** Letting $T = 2M$ and working from Eq. (7),

$$V^T = -\mathbb{E}_{G \sim B_T} [L(G)] = \frac{2}{2^T} \sum_{i=0}^T \binom{T}{i} |i - M| = \frac{2M}{2^T} \binom{2M}{M} = 2^{-T} \frac{2M T!}{(T - M)! M!},$$  

where we have applied a classic formula of de Moivre [1718], for the mean absolute deviation of the binomial distribution (see also Diaconis and Zabell [1991]). Using a standard bound on the central binomial coefficient (based on Stirling's formula),

$$\binom{2M}{M} = \frac{A^M}{\sqrt{\pi M}} \left(1 - \frac{c_M}{M}\right),$$  

where $\frac{1}{9} < c_M < \frac{1}{8}$ for all $M \geq 1$, we have

$$V^T \leq 2M \frac{1}{\sqrt{\pi M}} = \sqrt{\frac{2T}{\pi}}.$$  

As implied by Eq. (11), this inequality quickly becomes tight as $T \to \infty$.\hfill \Box

**The minimax algorithm (for the constrained player, too)!** In order to compute the minimax algorithm, we would like a closed form for

$$V_t(G_t) = -\mathbb{E}_{G' \sim B_t} [L(G_t + G')] ,$$

where $G_t = g_{1:t}$ is the sum of the gradients so far, $\tau = T - t$ is the number of rounds to go, and and $G' = g_{t+1:T}$ is a random variable giving the sum of the remaining gradients. Unfortunately, the structure of the binomial coefficients exploited by de Moivre and used in Eq. (10) does not apply given an arbitrary offset $G'$. Nevertheless, we will be able to derive
a formula for the update that is readily computable. Writing $\Pr_T(b)$ for the probability a random draw from $B_T$ has value $b$, the update of Eq. (8) becomes

$$x_{t+1} = \frac{1}{2} \sum_{b=-\tau}^{\tau} \Pr_T(b) \left( |G_t + b - 1| - |G_t + b + 1| \right).$$

Whenever $G_t + b \geq 1$, the difference in absolute values is $-2$, and whenever $G_t + b \leq 1$, the difference is $2$. When $G_t + b = 0$, the difference is zero. Thus,

$$x_{t+1} = \frac{1}{2} \left( \Pr_T(b > -G)(-2) + \Pr_T(b < -G)(2) \right)$$

$$= \Pr_T(b < -G) - \Pr_T(b > -G). \quad (12)$$

While this update does not have a closed form, it can be efficiently computed numerically.

It follows from this expression that even though we allow the player to select $x_{t+1} \in \mathbb{R}$, the minimax optimal algorithm always selects points from $[-1,1]$. Thus, we have the following Corollary:

**Corollary 5.** Consider the game of Theorem 4, but suppose now we also constrain the player to choose $x_t \in [-1,1]$. This does not change the value of the game, as the minimax algorithm for the unconstrained case always plays from $[-1,1]$ regardless.

Abernethy et al. [2008] shows that for the linear game with $n \geq 3$ where both the learner and adversary select vectors from the unit sphere, the minimax value is exactly $\sqrt{T}$. Interestingly, in the $n = 1$ case (where $L_2$ and $L_\infty$ coincide), the value of the game is lower, about $0.8\sqrt{T}$ rather than $\sqrt{T}$. This indicates a fundamental difference in the geometry of the $n = 1$ space and $n \geq 3$. We conjecture the minimax value for the $L_2$ game with $n = 2$ lies somewhere in between.

4.3 Non-stochastic betting and No-regret for all feasible sets simultaneously

We derive a minimax optimal approach to the betting problem presented in Section 2 which also corresponds to the setting introduced by Streeter and McMahan [2012]. Again, it is sufficient to consider the one-dimensional case. In that work, the goal was to prove bounds like $\text{Regret} \leq \mathcal{O}(R\sqrt{T}\log((1 + R)T))$ simultaneously for any comparator $\hat{x}$ with $|\hat{x}| = R$. Stating their Theorem 1 in terms of losses, this bound is achieved by any algorithm that guarantees

$$\text{Loss} = \sum_{t=1}^{T} g_t x_t \leq -\exp \left( \frac{|g_{1:T}|}{\sqrt{T}} \right) + \mathcal{O}(1). \quad (13)$$

Note that whenever $|g_{1:T}|$ is large compared to $\sqrt{T}$ the algorithm must achieve significantly negative loss (positive reward).

\footnote{The CDF of the binomial distribution can be computed numerically using the regularized incomplete beta function, from which $\Pr_T(b \leq -G)$ can be derived. Then, $\Pr_T(b = -G)$ can be computed from the appropriate binomial coefficient, leading to both needed probabilities.}
We initially study the game where

\[ L(G) = -\exp\left(\frac{G}{\sqrt{T}}\right) \]  

(note \( G = g_{1:T} \in [-T, T] \) can be positive or negative). We prove the minimax algorithm achieves

\[ \sum_{t=1}^{T} g_t x_t - L(g_{1:T}) \leq \sqrt{e}, \]

implying Reward = \(-\sum_{t=1}^{T} g_t x_t \geq \exp\left(\frac{G}{\sqrt{T}}\right) - \sqrt{e}\). Thus, this algorithm guarantees large reward whenever the gradient sum \( G \) is large and positive. In order to satisfy Eq. (13), we must also achieve large reward when \( G \) is large and negative. Since \( L(g_{1:t}) + L(-g_{1:t}) \leq -\exp\left(\frac{|g_{1:T}|}{\sqrt{T}}\right) \), this can be accomplished by running two copies of the minimax algorithm simultaneously, switching the signs of the gradients and plays of the second copy. We formalize this in Appendix A.

**Interpretation as a soft feasible set**  
Before developing an algorithm it is worth noting an alternative characterization of this benchmark function. One can show, that for \( a \geq 0 \),

\[ \min_{x \in \mathbb{R}} (G x - ax \log(-ax) + ax) = -\exp\left(\frac{G}{a}\right) \]

Thus, if we take \( \Psi(x) = -ax \log(ax) + ax + I(x \leq 0) \), we have

\[ \min_{x \in \mathbb{R}} g_{1:T} x + \Psi(x) = -\exp\left(\frac{G}{a}\right). \]

Since this algorithm needs large Reward when \( G \) is large and positive, we might expect that the minimax optimal algorithm only plays \( x_t \leq 0 \). Another intuition for this is that the algorithm should not need to play any point \( \hat{x} \) to which \( \Psi \) assigns an infinite penalty. This intuition is confirmed by the analysis of this “one-sided” algorithm:

**Theorem 6.** Consider the game with benchmark \( L \) as defined in Eq. (14). The minimax value of this game is exactly

\[ V_T = \frac{\left(1 + \exp\left(\frac{2}{\sqrt{T}}\right)\right)^T}{2^T \exp\left(\sqrt{T}\right)} \leq \sqrt{e}, \]

and further \( \lim_{T \to \infty} V_T = \sqrt{e} \). Letting \( \tau = T - t \) be the number of rounds left to be played, and defining \( G_t = g_{1:t} \), the conditional value of the game is

\[ V_t(G_t) = 2^{-\tau} \exp\left(\frac{G_t - \tau}{\sqrt{T}}\right) \left(1 + \exp\left(2/\sqrt{T}\right)\right)^\tau, \]

which leads to the minimax optimal algorithm\(^5\) for the player

\[ x_{t+1} = -2^{-\tau} \exp\left(\frac{G_t - \tau - 1}{\sqrt{T}}\right) \left(\exp\left(\frac{2}{\sqrt{T}}\right) - 1\right) \left(\exp\left(\frac{2}{\sqrt{T}}\right) + 1\right)^\tau \leq 0. \]  

\(^5\)When computing the player’s strategy via Eq. (15), it is numerically preferable to do the calculation in log-space, and then exponentiate to get the final play.
Proof. First, we compute the value of the game:

\[ V_T = \mathbb{E}_{G \sim B_T} [ - L(G) ] = 2^{-T} \sum_{i=0}^{T} \binom{T}{i} \exp \left( \frac{2i - T}{\sqrt{T}} \right) \]

\[ = 2^{-T} \exp \left( - \sqrt{T} \right) \sum_{i=0}^{T} \binom{T}{i} \left( \exp \left( 2/\sqrt{T} \right) \right)^i \]

\[ = 2^{-T} \exp \left( - \sqrt{T} \right) \left( 1 + \exp \left( 2/\sqrt{T} \right) \right)^T, \]

where we have used the ordinary generating function, \( \sum_{i=0}^{T} \binom{T}{i} x^i = (1 + x)^T \). Manipulating the above expression for the value of the game, we arrive at

\[ V_T = \left( \cosh \left( 1/\sqrt{T} \right) \right)^T \leq \exp \left( \frac{1}{2T} \right)^T = e. \]

Using similar techniques, we can derive the conditional value of the game, letting \( \tau = T - t \) be the number of rounds left to be played:

\[ V_t(G_t) = 2^{-\tau} \sum_{i=0}^{\tau} \binom{\tau}{i} \exp \left( \frac{G_t + 2i - \tau}{\sqrt{T}} \right) = 2^{-\tau} \exp \left( \frac{G_t - \tau}{\sqrt{T}} \right) \left( 1 + \exp \left( 2/\sqrt{T} \right) \right)^\tau. \]

Following Eq. (8) and simplifying leads to the update of Eq. (14). It remains to show \( \lim_{T \to \infty} V_T = \sqrt{e} \). Using the change of variable \( x = 1/\sqrt{T} \), equivalently we have \( \lim_{x \to 0} \cosh(x) \frac{1}{x^2} = \exp \left( \lim_{x \to 0} \log \left( \cosh(x) \frac{1}{x^2} \right) \right) = \sqrt{e}. \)

Examine the log of this function,

\[ \lim_{x \to 0} \log \left( \cosh(x) \frac{1}{x^2} \right) = \lim_{x \to 0} \frac{1}{x^2} \log \cosh(x) = \lim_{x \to 0} \frac{1}{x^2} \left( \frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{45} - \frac{17x^8}{2520} + \ldots \right) = \frac{1}{2}, \]

where we have taken the Maclaurin series of \( \log \cosh(x) \). Using the continuity of \( \exp \), we have

\[ \lim_{x \to 0} \left( \cosh(x) \frac{1}{x^2} \right) = \exp \left( \lim_{x \to 0} \log \left( \cosh(x) \frac{1}{x^2} \right) \right) = e. \]

\[ \Box \]

A strong lower-bound. Recall from Section 2 that as long as \( -L(G) \geq 0 \) and we get constant regret with respect to \( L \), we can scale our bets so that we never risk losing more than a constant starting budget. This holds for any number of rounds \( T \) against any adversary. Given that constraint, we would like \( -L(G) \) to grow as fast as possible, so it is natural to consider the generalizing Eq. (14) as

\[ L_\alpha(G) = - \exp \left( \frac{G}{T^{\alpha}} \right) \]

for \( \alpha \in (0, \frac{1}{2}] \). Following the techniques used in the preceding proof, we can show for this game

\[ V_\alpha^T = \mathbb{E}[L(G)] = 2^{-T} \exp \left( -T^{1-\alpha} \right) \left( 1 + \exp \left( 2T^{-\alpha} \right) \right) = \cosh \left( T^{-\alpha} \right)^T. \]
By taking the first term in the series for log cosh \( x \), namely \( x^2/2 \), and plugging in \( x = 1/T^\alpha \leq 1 \), we get a good upper bound on the value of the game:

\[
V^T = \exp \left( T \log \cosh(T^{-\alpha}) \right) \leq \exp \left( T \frac{1}{2T^{2\alpha}} \right) = \exp \left( \frac{1}{2} T^{1-2\alpha} \right)
\]

This implies that, for any \( \alpha < 1/2 \), no algorithm can provide constant loss (that is, \( \sum_{t=1}^{T} g_t x_t \leq k \) for a constant \( k \geq 0 \)) for any sequence while also guaranteeing

\[
\text{Reward} = -\sum_{t=1}^{T} g_t x_t = \Omega \left( \exp \left( \frac{G}{T^\alpha} \right) \right)
\]

for any \( \alpha < 1/2 \). In fact, for \( \alpha < 1/2 \), no algorithm can guarantee even linear loss in the worst case while making the reward guarantee of Eq. (16).
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A A Symmetric Betting Algorithm

The one-sided algorithm of Theorem 6 has
\[
\text{Loss} = V^T + L(G) \leq -\exp\left(\frac{G}{\sqrt{T}}\right) + \sqrt{e}.
\]

In order to do well when \(g_1:T\) is large and negative, we can run a copy of the algorithm on \(-g_1, \ldots, -g_T\), switching the signs of each \(x_t\) it suggests. The combined algorithm then satisfies
\[
\text{Loss} \leq -\exp\left(\frac{G}{\sqrt{T}}\right) - \exp\left(-\frac{G}{\sqrt{T}}\right) + 2 \sqrt{e},
\]
and so following Eq. (13) and Theorem 1 of Streeter and McMahan [2012], we obtain the desired regret bounds. The following theorem implies the symmetric algorithm is in fact minimax optimal with respect to the combined benchmark
\[
L^C(G) = -\exp\left(\frac{G}{\sqrt{T}}\right) - \exp\left(-\frac{G}{\sqrt{T}}\right).
\]

**Theorem 7.** Consider two 1-D games where the adversary plays from \([-1, 1]\), defined by concave functions \(L_1\) and \(L_2\) respectively. Let \(x^1_t\) and \(x^2_t\) be minimax-optimal plays for \(L_1\) and \(L_2\) respectively, given that \(g_1, \ldots, g_{t-1}\) have been played so far in both games. Then \(x^1_t + x^2_t\) is also minimax optimal for the combined game that uses the benchmark \(L^C(G) = L_1(G) + L_2(G)\).

**Proof.** First, taking \(\tau = T - t\) and using Theorem 1 three times, we have
\[
V^C(g_1, \ldots, g_T) = -\mathbb{E}_{G^\tau \sim B} \left[ L_1(g_1, \ldots, g_T) + L_2(g_1, \ldots, g_T) \right]
= -\mathbb{E}_{G^\tau \sim B} \left[ L_1(g_1, \ldots, g_T) \right] - \mathbb{E}_{G^\tau \sim B} \left[ L_2(g_1, \ldots, g_T) \right]
= V^1(g_1, \ldots, g_T) + V^2(g_1, \ldots, g_T),
\]
using linearity of expectation. Then, using Eq. (8) for each of the three games, we have
\[
x^C_t = \arg\min_x \max_g gx + V^C(g_1, \ldots, g_{t-1}, g)
= \frac{1}{2}(V^1(g_1, \ldots, g_{t-1}, -1) - V^1(g_1, \ldots, g_{t-1}, 1))
= \frac{1}{2}(V^2(g_1, \ldots, g_{t-1}, -1) + V^2(g_1, \ldots, g_{t-1}, -1) - V_1(g_1, \ldots, g_{t-1}, -1) - V_2(g_1, \ldots, g_{t-1}, +1))
= x^1_t + x^2_t.
\]