On the detailed structure of quantum control landscape for fast single qubit phase-shift gate generation

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August 22, 2024

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Abstract

In this work, we study the detailed structure of quantum control landscape for the problem of single-qubit phase shift gate generation on the fast time scale. In previous works, the absence of traps for this problem was proved on various time scales. A special critical point which was known to exist in quantum control landscapes was shown to be either a saddle or a global extremum, depending on the parameters of the control system. However, in the case of saddle the numbers of negative and positive eigenvalues of Hessian at this point and their magnitudes have not been studied. At the same time, these numbers and magnitudes determine the relative ease or difficulty for practical optimization in a vicinity of the critical point. In this work, we compute the numbers of negative and positive eigenvalues of Hessian at this saddle point and moreover, give estimates on magnitude of these eigenvalues. We also significantly simplify our previous proof of the theorem about this saddle point of the Hessian [Theorem 3 in B.O. Volkov, O.V. Morzhin, A.N. Pechen, J. Phys. A: Math. Theor. \textbf{54}, 215303 (2021)].

1 Introduction

Optimal quantum control, which includes methods for manipulation of quantum systems, attracts now high attention due to various existing and prospective applications in quantum technologies \cite{1,2,3,4,5,6,7,8}. Among important topics, one problem which was posed in \cite{9} is the analysis of quantum control landscapes, that is, local and global extrema

\url{www.mathnet.ru/eng/person/94935} \url{www.mathnet.ru/eng/person/17991} \url{www.mi-ras.ru/eng/dep51}
of quantum control objective functionals. Various results have been obtained in this field e.g. in [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23] for closed and open quantum systems. For open quantum systems, a formulation of completely positive trace preserving dynamics as points of complex Stiefel manifold (strictly speaking, of some factors of complex Stiefel manifolds over some equivalence relation) was proposed and theory of open system’s quantum control as gradient flow optimization over complex Stiefel manifolds was developed in details for two-level [24] and general n–level quantum systems [25] and applied to the analysis of quantum control landscapes. Control landscapes for open-loop and closed-loop control were analyzed in a unified framework [26]. A unified analysis of classical and quantum kinematic control landscapes was performed [27]. Computation of numbers of positive and negative eigenvalues of the Hessian at saddles of the control landscape is an important problem [11].

For numerical optimization in quantum control, various local and global search methods are used including such as based on Pontryagin maximum principle [28, 29], GRadient Ascent Pulse Engineering (GRAPE) [30], gradient flows [31], Krotov type methods [32, 33], gradient free CRAB optimisation [34], Hessian based methods such as the Broyden–Fletcher–Goldfarb–Shanno (BFGS) algorithm [35], geometric methods [36], genetic algorithms [37], machine learning [38], dual annealing [39], etc. The importance of mathematical analysis of extrema of quantum control objective functionals is motivated by the fact that local but not global maxima (if they would exist) would impede the search for globally optimal controls using local search algorithms, which could be more efficient otherwise. For this reason, points of local but not global extrema of the objective functional are called traps.

In this work, we analytically study the detailed structure of quantum control landscape around special critical point for the problem of single-qubit phase shift gate generation on the fast time scale. The absence of traps for single-qubit gate generation was proven on long time scale in [13, 16] and on fast time scale [21, 22], where a single special control which is a critical point was studied and shown to be a saddle. However, the numbers of negative and positive eigenvalues of Hessian at this saddle point control were not studied. At the same time, these numbers are important as they determine the numbers of directions towards decreasing and increasing of the objective and hence determine the level of difficulty for practical optimization starting in a vicinity of the saddle point. The numbers of positive and negative eigenvalues of Hessian of the objective for some other examples of quantum systems were computed in [11]. In this work, we compute the numbers of negative and positive eigenvalues of Hessian at this saddle point, give estimates on magnitude of the eigenvalues and also significantly simplify our previous proof of the theorem about Hessian at this saddle point [22]. Numerical experiments for the problem of single-qubit phase shift gate generation on the fast time scale were performed in [22, 40].

In Sec. 2, we summarize results of previous works which are relevant for our study. In Sec. 3, the main theorem of this work is presented; its proof is provided in Sec. 4. Conclusions Sec. 5 summarizes this work.
2 Previous results

A single qubit driven by a coherent control $f \in L^2([0, T]; \mathbb{R})$, where $T > 0$ is the final time, in the absence of the environment evolves according to the Schrödinger equation for the unitary evolution operator $U^f_t$

$$\frac{dU^f_t}{dt} = -i(H_0 + f(t)V)U^f_t, \quad U^f_0 = I$$  \hfill (1)

where $H_0$ and $V$ are the free and interaction Hamiltonians. Common assumption is that $[H_0, V] \neq 0$. This assumption guarantees controllability of the two-level system for large time; otherwise the dynamics is trivial. In this work, we consider time scale smaller than the controllability time. A single qubit quantum gate is a unitary $2 \times 2$ operator $W$ defined up to a physically irrelevant phase, so that $W \in SU(2)$. The problem of single qubit gate generation can be formulated as

$$J_W[f] = \frac{1}{4} |\text{Tr}(U^f_TW^\dagger)|^2 \to \max$$  \hfill (2)

**Definition 1** Control $f^*$ is called trap for the problem (2) if $f^*$ is a point of local but not global maximum of $J_W$, i.e. $J(f^*) < \sup_f J_W[f]$.

In [16, 21, 22], results on the absence of traps for this problem were obtained. To explicitly formulate these results, consider the special constant control $f(t) = f_0$ and time $T_0$:

$$f_0 := -\frac{\text{Tr}H_0\text{Tr}V + 2\text{Tr}(H_0V)}{(\text{Tr}V^2)^2 - 2\text{Tr}(V^2)},$$

$$T_0 := \frac{\pi}{\|H_0 - \mathbb{I}\text{Tr}H_0/2 + f_0(V - \mathbb{I}\text{Tr}V/2)\|},$$  \hfill (3)

where $\| \cdot \|$ denotes the spectral norm.

The following theorem was proved in [16].

**Theorem 1** Let $W \in SU(2)$ be a single qubit quantum gate. If $[W, H_0 + f_0V] \neq 0$ then for any $T > 0$ traps do not exist. If $[W, H_0 + f_0V] = 0$ then any control, except possibly $f \equiv f_0$, is not trap for any $T > 0$ and the control $f_0$ is not trap for $T > T_0$.

The case of whether control $f_0$ can be trap for $T \leq T_0$ or not was partially studied in [21]. Without loss of generality it is sufficient to consider the case $H_0 = \sigma_z$ and $V = v_x\sigma_x + v_y\sigma_y$, where $v_x, v_y \in \mathbb{R}$ ($v_x^2 + v_y^2 > 0$) and $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  \hfill (5)

In this case, the special time is $T_0 = \frac{\pi}{2}$ and the special control is $f_0 = 0$.

By Theorem 1 if $[W, \sigma_z] \neq 0$, then for any $T > 0$ there are no traps for $J_W$. If $[W, \sigma_z] = 0$, then $W = e^{i\varphi_W\sigma_z + i\beta}$, where $\varphi_W \in (0, \pi)$ and $\beta \in [0, 2\pi)$. The phase can be neglected, so without loss of generality we set $\beta = 0$. Below we consider only such gates. The following result was proved in [21].
Theorem 2 Let $W = e^{i\varphi W_z}$. If $\varphi_W \in (0, \pi/2)$, then for any $T > 0$ there are no traps. If $\varphi_W \in [\pi/2, \pi]$, then for any $T > \pi - \varphi_W$ there are no traps.

For fixed $\varphi_W$ and $T$ the value of the objective evaluated at $f_0$ is

$$J_W[f_0] = \cos^2(\varphi_W + T).$$

If $\varphi_W + T = \pi$ then $J_W[f_0] = 1$ and $f_0$ is a point of global maximum. If $\varphi_W + T = 3\pi/2$ then $J_W[f_0] = 0$ and $f_0$ is a point of global minimum.

The Taylor expansion of the functional $J_W$ at $f$ up to the second order has the form (for the theory of calculus of variations in infinite dimensional spaces see [41]):

$$J_W[f + \delta f] = J_W[f] + J_W^{(1)}[f](\delta f) + \frac{1}{2} J_W^{(2)}[f](\delta f, \delta f) + o(\|\delta f\|^2)$$

as $\|\delta f\| \to 0$. (7)

The first Fréchet derivative is

$$J_W^{(1)}[f](\delta f) = \int_0^T \frac{\delta J_W}{\delta f(t)} \delta f(t) dt,$$

where the integral kernel, which determines the gradient of the objective, is

$$\frac{\delta J_W}{\delta f(t)} = \frac{1}{2} \mathfrak{H}(\text{Tr}Y^*\text{Tr}(YV_t))$$

Here as in [21] we use the notations $Y = W^\dagger U_T^f$ and $V_t = U_t^f U_t^V$. The second order term is

$$\frac{1}{2} J_W^{(2)}[f](\delta f, \delta f) = \frac{1}{2} (\text{Hess} \delta f, \delta f)_{L^2} = \frac{1}{2} \int_0^T \int_0^T \text{Hess}(t, s) f(t) f(s) dt ds,$$

where Hessian $\text{Hess}: L^2([0, T], \mathbb{R}) \to L^2([0, T], \mathbb{R})$ is an integral operator:

$$(\text{Hess} f)(t) = \int_0^T \text{Hess}(t, s) f(s) ds.$$ (9)

The integral kernel of the Hessian has the form

$$\text{Hess}(t, s) = \begin{cases} \frac{1}{2} \text{Re}(\text{Tr}(YV_t)\text{Tr}(Y^*V_s) - \text{Tr}(YV_s V_t)\text{Tr}Y^*), & \text{if } s \geq t \\ \frac{1}{2} \text{Re}(\text{Tr}(YV_s)\text{Tr}(Y^*V_t) - \text{Tr}(YV_t V_s)\text{Tr}Y^*), & \text{if } s < t. \end{cases}$$

The control $f_0 = 0$ is a critical point, i.e., gradient of the objective evaluated at this control is zero. The integral kernel of Hessian at $f_0 = 0$ has the form (see [21]):

$$\text{Hess}(s, t) = -2\nu^2 \cos \varphi \cos(2|t - s| + \varphi),$$

where $\varphi = -\varphi_W - T$ and $\nu = \sqrt{\nu_x^2 + \nu_y^2}$.

We consider for the values of the parameters $(\varphi_W, T)$ the following cases (see Fig. 4.1, where the set $D_2$ in addition is divided into three subsets described in Sec. 4):

- $(\varphi_W, T)$ belongs to the triangle domain

$$D_1 := \{ (\varphi_W, T) : 0 < T < \frac{\pi}{2}, \quad \frac{\pi}{2} \leq \varphi_W < \pi - T \};$$
• \((\varphi_W, T)\) belongs to the triangle domain
\[D_2 := \left\{(\varphi_W, T) : \ 0 < T \leq \frac{\pi}{2}, \ \pi - T < \varphi_W < \pi, \ (\varphi_W, T) \neq \left(\frac{\pi}{2}, \frac{\pi}{2}\right)\right\}.\]

• \((\varphi_W, T)\) belongs to the square domain without the diagonal
\[D_3 := \left\{(\varphi_W, T) : \ 0 < T \leq \frac{\pi}{2}, \ 0 < \varphi_W < \frac{\pi}{2}, \ \varphi_W + T \neq \frac{\pi}{2}\right\}.\]

• \((\varphi_W, T)\) belongs to the set
\[D_4 := \left\{(\varphi_W, T) : \ 0 < T \leq \frac{\pi}{2}, \ \varphi_W = \pi\right\}.\]

Remark 1 Note that these notations for the domains \(D_2, D_3, D_4\) are different from that used in [22]. The present notations seem to be more convenient.

The following theorem was obtained in [22].

**Theorem 3** If \((\varphi_W, T) \in D_1 \cup D_2 \cup D_3 \cup D_4\) then the Hessian of the objective functional \(J_W\) at \(f_0 = 0\) is an injective compact operator on \(L^2([0, T]; \mathbb{R})\). Moreover, the following holds.

1. If \((\varphi_W, T) \in D_1\), then Hessian at \(f_0\) has only negative eigenvalues.

2. If \((\varphi_W, T) \in D_2 \cup D_3 \cup D_4\) then Hessian at \(f_0\) has both negative and positive eigenvalues. In this case, the special control \(f_0 = 0\) is a saddle point for the objective functional.

Numerical experiments suggest that if \((\varphi_W, T) \in D_1\) then \(f_0\) is a point of global maximum [40]. Note that the numbers of positive and negative eigenvalues mentioned in item 2 of this theorem, as well as their magnitudes, were not computed in [22]. However, these numbers and magnitudes are important since they determine the numbers of directions towards increasing or decreasing of \(J_W\), and the magnitudes of the eigenvalues determine the speed of increasing and decreasing the objective along these directions. All of that affects the relative level of easy or difficulty of practical optimization in a vicinity of \(f_0\).

### 3 Main theorem

Our main result of this work is the following theorem.

**Theorem 4** One has the following.

1. If \((\varphi_W, T) \in D_1\), then Hessian at \(f_0\) has only negative eigenvalues.

2. If \((\varphi_W, T) \in D_2\), then Hessian at \(f_0\) has two positive and infinitely many negative eigenvalues.

3. If \((\varphi_W, T) \in D_3\) and \(\varphi_W + T < \pi/2\), then Hessian at \(f_0\) has one positive and infinitely many negative eigenvalues. If \((\varphi_W, T) \in D_3\) and \(\varphi_W + T > \pi/2\), then Hessian at \(f_0\) has one negative and infinitely many positive eigenvalues.

4. If \((\varphi_W, T) \in D_4\), then Hessian at \(f_0\) has one negative and infinitely many positive eigenvalues.
4 Proof of the main theorem

In this section, we will investigate the spectrum of Hessian $\mathbf{Hess}$ and prove Theorem 4.

If $(\varphi, W, T) \in D_1 \cup D_2 \cup D_3 \cup D_4$, then $\sin 2\varphi = -\sin 2(\varphi + T) \neq 0$. Instead of Hessian, we can consider the operator $\mathbf{K} = \frac{1}{\upsilon^2 \sin 2\varphi} \mathbf{Hess}$ which differs from $\mathbf{Hess}$ by a scalar factor that makes the calculations a bit simpler. Here $\mathbf{K}: L^2([0, T], \mathbb{R}) \rightarrow L^2([0, T], \mathbb{R})$ is an integral operator:

$$(Kf)(t) = \int_0^T K(t, s)f(s)ds$$

with the integral kernel

$$K(t, s) = -\frac{\cos (2|t - s| + \varphi)}{\sin \varphi},$$

Because operators $\mathbf{K}$ and $\mathbf{Hess}$ differ by a scalar factor, their spectra are related

$$\sigma(\mathbf{K}) = \frac{1}{\upsilon^2 \sin 2\varphi} \sigma(\mathbf{Hess}).$$

Let $g \in L^2([0, T], \mathbb{R})$ and $h = \mathbf{K}g$. Then

$$h(t) = -\frac{1}{\sin \varphi} \int_0^t \cos (2t - 2s + \varphi)g(s)ds - \frac{1}{\sin \varphi} \int_t^T \cos (2s - 2t + \varphi)g(s)ds. \quad (12)$$

Differentiating twice in a generalized sense expression (12), we obtain (see [22] for details)

$$h''(t) + 4h(t) = 4g(t). \quad (13)$$

Moreover, for any continuous $g$, we can find $h = \mathbf{K}g$ as a unique solution of ODE (13), which satisfies the initial conditions

$$h(0) = -\frac{1}{\sin \varphi} \int_0^T \cos (2s + \varphi)g(s)ds, \quad (14)$$

$$h'(0) = -\frac{2}{\sin \varphi} \int_0^T \sin (2s + \varphi)g(s)ds. \quad (15)$$

These initial conditions are obtained by substituting $t = 0$ into the right hand side of expression (12) and into its derivative.

Equality (13) implies that if $h \equiv 0$ then $g \equiv 0$. Hence $\mathbf{K}$ is an injective operator. Let $\mu \neq 0$ be an eigenvalue of the operator $\mathbf{K}$ and $g$ be a corresponding eigenfunction, then $h = \mathbf{K}g = \mu g$. Let $\lambda = 1/\mu$. Then (13), (14) and (15) together imply that the search for eigenvalues of the operator $\mathbf{K}$ reduces to the problem of finding such $h \in C^\infty([0, T], \mathbb{R})$ and nonzero $\lambda \in \mathbb{R}$ that

$$\begin{cases}
h''(t) = 4(\lambda - 1)h(t), \\
\lambda h(0) = -\frac{1}{\sin \varphi} \int_0^T \cos (2s + \varphi)h(s)ds, \\
\lambda h'(0) = -\frac{2}{\sin \varphi} \int_0^T \sin (2s + \varphi)h(s)ds.
\end{cases} \quad (16)$$

This problem is similar to the Sturm–Liouville problem (see [42]).
4.1 Case $\lambda < 1$

Consider the case of nonzero $\lambda < 1$. Let $a^2 = (1 - \lambda)$ and $a > 0$. If $h$ satisfies \(16\) then $h$ has the form

$$h(t) = b \cos 2at + c \sin 2at.$$ 

If we substitute $h$ in the boundary conditions of \(16\), then we get a system of two linear algebraic equations on $(b, c)$. It has a non-zero solution if the determinant of the coefficients of this system is not equal to zero. This determinant has the form (see [22])

$$F_{\varphi W, T}^1(a) = -2a - a^2 \sin (2aT) \sin (2\varphi_W) - \sin (2aT) \sin (2\varphi_W) + 2a \cos (2aT) \cos (2\varphi_W).$$

So the function $h$ and $\lambda < 1$ are a solution of problem \(16\) if and only if the function $F_{\varphi W, T}^1$ has a positive root.

It is easy to see that for $(\varphi_W, T) \in \mathcal{D}_4$ the roots of the function $F_{\varphi W, T}^1$ are $a_n = \frac{\pi n}{2T}$.

If $(\varphi_W, T) \in \mathcal{D}_1$ and $\varphi_W = \frac{\pi}{2}$ then the roots of the function $F_{\varphi W, T}^1$ are $a_n = \frac{(2n-1)\pi}{2T}$. Hence, in these cases, $\mu_n = \frac{\pi n}{1 - \frac{\pi}{2T}}$ belong to the spectrum of the operator $K$. We will show bellow that there is also only one positive eigenvalue for $(\varphi_W, T) \in \mathcal{D}_4$ and there are not negative eigenvalues for $(\varphi_W, T) \in \mathcal{D}_1$ such that $\varphi_W = \frac{\pi}{2}$.

**Lemma 1** Let $T \in (0, \frac{\pi}{2})$. The equation

$$\alpha x = \tan(Tx)$$

has only one root on $(0, 1)$ if $T < \alpha < \tan T$ and has not roots on $(0, 1)$ if $\alpha \in (-\infty, T]$ and $\alpha \in [\tan T, +\infty)$.

The proof of this lemma is illustrated on Fig. 4.1 (left subplot).

**Lemma 2** Let $T \in (0, \frac{\pi}{2})$. The equation

$$\alpha x = \cot(Tx)$$

has only one root on $(0, 1)$ if $\cot T < \alpha$ and has not roots on $(0, 1)$ if $\alpha \in (-\infty, \cot T]$.

The proof of this lemma is illustrated on Fig. 4.1 (right subplot).

In addition, we divide the domain $\mathcal{D}_2$ into the following three subdomains (see Fig. 4.1).

- $(\varphi_W, T)$ belongs to the set

  $$\mathcal{D}'_2 := \{(\varphi_W, T) : (\varphi_W, T) \in \mathcal{D}_2, \ T < -\tan \varphi_W\};$$

- $(\varphi_W, T)$ belongs to the set

  $$\mathcal{D}''_2 := \{(\varphi_W, T) : (\varphi_W, T) \in \mathcal{D}_2, \ T = -\tan \varphi_W\};$$

- $(\varphi_W, T)$ belongs to the set

  $$\mathcal{D}'''_2 := \{(\varphi_W, T) : (\varphi_W, T) \in \mathcal{D}_2, \ T > -\tan \varphi_W\}.$$

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of the operator \( K \) between these two straight lines. Hence on the interval \((0, 1)\) two inclined straight lines have angles of inclination \( T \) and \( y = \tan(T) \), respectively. The line \( y = \tan(Tx) \) on the interval \((0, 1)\) lies between these two straight lines. Hence on the interval \((0, 1)\) line \( y = \alpha x \) intersects line \( y = \tan(Tx) \) if and only if \( T < \alpha < \tan(T) \). Right: The inclined straight line \( y = \cot(T)x \) intersects with line \( y = \cot(Tx) \) at \( x = 1 \). Hence equation \( \alpha x = \cot(Tx) \) on \((0, 1)\) has one solution if \( \alpha > \cot(T) \) and has no solutions if \( \alpha \leq \cot(T) \).

**Figure 1:** Illustration for Lemma 1 (left) and for Lemma 2 (right). Left: Two inclined straight lines have angles of inclination \( y \) intersects line \( y = \tan(T) \), respectively. The line \( y = \tan(Tx) \) on the interval \((0, 1)\) lies between these two straight lines. Hence on the interval \((0, 1)\) line \( y = \alpha x \) intersects line \( y = \tan(Tx) \) if and only if \( T < \alpha < \tan(T) \). Right: The inclined straight line \( y = \cot(T)x \) intersects with line \( y = \cot(Tx) \) at \( x = 1 \). Hence equation \( \alpha x = \cot(Tx) \) on \((0, 1)\) has one solution if \( \alpha > \cot(T) \) and has no solutions if \( \alpha \leq \cot(T) \).

**Proposition 1** Positive roots of the function \( F_{\varphi_W}^1 \) are positive solutions \( \{a'_m\} \) and \( \{a_n\} \) of the equation

\[-x \tan \varphi_W = \tan (xT) \tag{17}\]

and the equation

\[-x \cot \varphi_W = \cot (xT) \tag{18}\]

respectively. Here \( \{a'_m\} \) and \( \{a_n\} \) are two countable sets whose elements are distributed in ascending order. Then the numbers \( \mu'_m = \frac{1}{1-a'_m} \) and \( \mu_n = \frac{1}{1-a_n} \), belong to the spectrum of the operator \( K \). Moreover, the following holds.

1. If \((\varphi_W, T) \in \mathcal{D}_1 \) and \( \varphi_W \neq \frac{\pi}{2} \), then \( a'_m \in \left( \frac{(m-1)\pi}{T}, \frac{(2m-1)\pi}{2T} \right) \) and \( a_n \in \left( \frac{(n-1)\pi}{T}, \frac{(2n-1)\pi}{2T} \right) \), where \( n, m \in \mathbb{N} \). In this case, the numbers \( \{\mu'_m\} \) and \( \{\mu_n\} \) are negative for all \( n, m \in \mathbb{N} \).

2. If \((\varphi_W, T) \in \mathcal{D}_2'' \cup \mathcal{D}_2''' \) or \( \varphi_W \neq \frac{\pi}{2} \), then \( a'_m \in \left( \frac{(m-1)\pi}{T}, \frac{(2m-1)\pi}{2T} \right) \) and \( a_n \in \left( \frac{(n-1)\pi}{T}, \frac{(2n-1)\pi}{2T} \right) \), where \( m \in \{2, 3, \ldots\} \) and \( n \in \mathbb{N} \). In this case, \( \mu_1 = \frac{1}{1-a_1^2} > 1 \) is positive and the numbers \( \{\mu'_m\} \) and \( \{\mu_n\} \) are negative for \( n > 1 \) and \( m > 1 \).

3. If \((\varphi_W, T) \in \mathcal{D}_2' \) or \( \varphi_W \neq \frac{\pi}{2} \), then \( a'_m \in \left( \frac{(m-1)\pi}{T}, \frac{(2m-1)\pi}{2T} \right) \) and \( a_n \in \left( \frac{(n-1)\pi}{T}, \frac{(2n-1)\pi}{2T} \right) \), where \( n, m \in \mathbb{N} \). In this case, \( \mu_1' = \frac{1}{1-a'_1^2} > 1 \) and \( \mu_1 = \frac{1}{1-a_1^2} > 1 \) are positive. The numbers \( \{\mu'_m\} \) and \( \{\mu_n\} \) are negative for \( n > 1 \) and \( m > 1 \).

4. If \((\varphi_W, T) \in \mathcal{D}_3 \) or \( \varphi_W \neq \frac{\pi}{2} \), then \( a'_m \in \left( \frac{(2m-1)\pi}{2T}, \frac{m\pi}{T} \right) \) and \( a_n \in \left( \frac{(2n-1)\pi}{2T}, \frac{n\pi}{T} \right) \), where \( n, m \in \mathbb{N} \). In this case, the numbers \( \{\mu'_m\} \) and \( \{\mu_n\} \) are negative for all \( n, m \in \mathbb{N} \).
Figure 2: The domains of the rectangle \((\varphi_W, T) \in [0, \pi] \times [0, \pi/2]\). \(D_3\) is the left square except of the dashed diagonal and borders. \(D_1\) is the bottom triangle in the right square with left vertical border and without the solid diagonal and bottom horizontal border. \(D_4\) is the vertical line marked with crosses. \(D'_2\) is the dash-dotted line. \(D''_2\) is the area between \(D_1\) and \(D'_2\). \(D''''_2\) is the area between \(D''_2\) and \(D_4\). Note that these notations for the domains \(D_2, D_3, D_4\) are different from that used in [22]. In the domain \(D_1\), the Hessian at critical point \(f_0 = 0\) has only negative eigenvalues. On the dashed diagonal (in the left square), the critical point \(f_0 = 0\) is a point of global minimum. On the solid diagonal (in the right square), the critical point \(f_0 = 0\) is a point of global maximum. On the domain \(D_2 = D'_2 \cup D''_2 \cup D''''_2\), the critical point \(f_0 = 0\) is a saddle point with two positive and infinitely many negative eigenvalues of Hessian. On the bottom triangle of \(D_3\), the critical point \(f_0 = 0\) is a saddle point with one positive and infinitely many negatives eigenvalues of the Hessian. On the top triangle of \(D_3\) and on the domain \(D_4\), the critical point \(f_0 = 0\) is a saddle point with one negative and infinitely many positive eigenvalues of the Hessian.

**Proof** Let us analyze positive roots of the function \(F_{\varphi_W,T}^1\). For this purpose we consider quadratic (with respect to \(x\)) equation:

\[
x^2 \sin(2aT)\sin(2\varphi_W) + 2x(1 - \cos(2aT)\cos(2\varphi_W)) + \sin(2aT)\sin(2\varphi_W) = 0
\]

The roots of this quadratic equation are

\[
x_1 = -\cot \varphi_W \tan(aT), \tag{19}
\]
\[
x_2 = -\tan \varphi_W \cot(aT) \tag{20}
\]

Hence \(a\) is a root of the function \(F_{\varphi_W,T}^1\) if and only if \(x = a\) is a solution of either equation (17) or equation (18). If \((\varphi_W, T) \in D_3\), then both equations have a single root on each interval \(\left[\frac{(2n-1)\pi}{2T}, \frac{n\pi}{2T}\right]\) for \(n \in \mathbb{N}\). If \((\varphi_W, T) \in D_1 \cup D_2\) and \(\varphi_W \neq \frac{\pi}{2}\), then equation (18) has a single root on each interval \(\left[\frac{(n-1)\pi}{T}, \frac{(2n-1)\pi}{2T}\right]\) for \(n \in \mathbb{N}\). If \((\varphi_W, T) \in D_1 \cup D'_2\) and \(\varphi_W \neq \frac{\pi}{2}\), then equation (17) has a single root on each interval \(\left[\frac{(n-1)\pi}{T}, \frac{(2n-1)\pi}{2T}\right]\) for \(n \in \mathbb{N}\).
Lemma 1 implies that, in the case \((\varphi_W, T) \in D''_2 \cup D'''_2\), equation (17) has not roots on \((0, \pi/2T)\) and has a single root on each interval \((n\pi/2T, (2n-1)\pi/2T)\) for \(n > 1\).

The number \(\mu = \frac{1}{1-a^2}\) is a positive eigenvalue of the operator \(K\) if and only if \(a \in (0,1)\). If \((\varphi_W, T) \in D_1\) and \(\varphi_W \neq \pi/2\) then
\[
\tan T < \tan (\pi - \varphi_W) = -\tan (\varphi_W).
\]

Due to Lemma 2 equation (18) has not roots on \((0,1)\). Due to Lemma 1 equation (17) has not roots on \((0,1)\).

If \((\varphi_W, T) \in D_2\), then
\[
\cot T < -\cot \varphi_W.
\]

Hence, due to Lemma 2 equation (18) has one root on \((0,1)\). If \(T \geq -\tan \varphi_W\) then Lemma 1 implies that (17) has not solution on \((0,1)\). So if \((\varphi_W, T) \in D''_2 \cup D'''_2\), then the function \(F^1_{\varphi_W, T}\) has only one root \(a_1\) on the interval \((0,1)\). If \((\varphi_W, T) \in D'_2\), then the function \(F^1_{\varphi_W, T}\) has two roots \(a_1\) and \(a_1'\) on the interval \((0,1)\).

If \((\varphi_W, T) \in D_3\), then \(-\tan \varphi_W < 0\) and Lemmas 1 and 2 imply that both equations (17) and (18) have not solutions on \((0, \pi/2T)\). Hence, the numbers \(\{\mu'_m\}\) and \(\{\mu_n\}\) are negative for all \(n, m \in \mathbb{N}\).

4.2 Case \(\lambda = 1\)

If \(\mu = 1\) is an eigenvalue of the operator \(K\) then the corresponding eigenfunctions should have the form
\[
h(t) = g(t) = ct + b.
\]

If we substitute \(g\) in (14) and (15), then we get a system of two linear algebraic equations on \((b,c)\). This system has a non-zero solution if and only if the determinant of the coefficients of this system is not equal to zero. This determinant has the form (see [22])
\[
\Delta = -2\sin \varphi_W (\sin \varphi_W + T \cos \varphi_W).
\]

Proposition 2 \(\mu'_1 = 1\) is an eigenvalue of \(K\) only in the following cases

1. \((\varphi_W, T) \in D_4\).
2. \((\varphi_W, T) \in D''_2\).

4.3 Case \(\lambda > 1\)

Consider the case \(\lambda > 1\). Let \(a^2 = (\lambda - 1)\) and \(a > 0\). If \(h\) satisfies (16) then \(h\) has the form
\[
h(t) = be^{2at} + ce^{-2at}.
\]

If we substitute \(h\) in boundary conditions of (16), then we get a system of two linear algebraic equations on \(b\) and \(c\). This system has a non-zero solution if and only if the determinant of the coefficients of this system is not equal to zero. This determinant has the form [22]
\[
F^2_{\varphi_W, T}(a) = -a^2 \sinh (2aT) \sin (2\varphi_W) + 2a(1 - \cosh (2aT) \cos (2\varphi_W)) + \sinh (2aT) \sin (2\varphi_W).
\]

 Proposition 2 \(\mu'_1 = 1\) is an eigenvalue of \(K\) only in the following cases

1. \((\varphi_W, T) \in D_4\).
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\]
So the function \( h \) and \( \lambda > 1 \) are a solution of problem \((16)\) if and only if the function \( F_{\varphi_{W,T}}^2 \) has a positive root.

It is easy to see that for \((\varphi_{W,T}) \in D_4\) and \((\varphi_{W,T}) \in D_1\) such that \(\varphi_W = \frac{\pi}{2}\) the function \( F_{\varphi_{W,T}}^2 \) has not positive roots.

**Proposition 3** One has the following.

1. If \((\varphi_{W,T}) \in D_3\), then \( F_{\varphi_{W,T}}^2 \) has only one positive root \( a'_{0} > 0 \), where \( a'_{0} \) is a solution of the equation
   \[
   x \cot \varphi_W = \coth Tx
   \]  
   Then \( \mu'_{0} = \frac{1}{1+a'_{0}} < 1 \) is a positive eigenvalue of the operator \( K \).

2. If \((\varphi_{W,T}) \in D_1 \cup D_2' \cup D_2''\) and \(\varphi_W \neq \frac{\pi}{2}\), then the function \( F_{\varphi_{W,T}}^2 \) has no positive roots.

3. If \((\varphi_{W,T}) \in D_2''\), then \( F_{\varphi_{W,T}}^2 \) has only one positive root \( a'_{1} > 0 \), where \( a'_{1} \) is a solution of the equation
   \[
   -x \tan \varphi_W = \tanh Tx.
   \]  
   Then \( \mu'_{1} = \frac{1}{1+a'_{1}} < 1 \) is a positive eigenvalue of the operator \( K \).

**Proof.** Let us analyze positive roots of the function \( F_{\varphi_{W,T}}^2 \). For this purpose we consider quadratic (with respect to \( x \)) equation:

\[
x^2 \sinh (aT) \sin (2\varphi_W) - 2x(1 - \cosh (aT) \cos (2\varphi_W)) - \sinh (aT) \sin (2\varphi_W) = 0
\]

The roots of this equation are

\[
x_1 = -\cot \varphi_W \tanh (aT)
\]

\[
x_2 = \tan \varphi_W \coth (aT)
\]

Then \( a \) is a root of the function \( F_{\varphi_{W,T}}^2 \) if and only if \( x = a \) is a solution of either equation \((23)\) or equation \((24)\). If \((\varphi_{W,T}) \in D_3\) then due to \(\tan \varphi_W > 0\) equation \((24)\) has not positive roots. Equation \((23)\) has only one positive root.

If \((\varphi_{W,T}) \in D_1 \cup D_2\) and \(\varphi_W \neq \frac{\pi}{2}\) then due to \(\cot \varphi_W < 0\) equation \((23)\) has not positive roots. If \((\varphi_{W,T}) \in D_1 \cup D_2' \cup D_2''\) and \(\varphi_W \neq \frac{\pi}{2}\) then \(T \leq -\tan \varphi_W\) and \(\tanh (Tx) < -\tan(\varphi_W)x\) for positive \(x\) and equation \((24)\) has not positive roots.

If \((\varphi_{W,T}) \in D_2''\) then equation \((24)\) has one positive root.

The statement of Theorem 4 follows directly from Propositions 1, 2, 3. Important is that these propositions in addition give estimates for the magnitudes of the eigenvalues.

**5 Conclusions**

Analysis of either existence or absence of traps (which are points of local, but not global, extrema of the objective quantum functional) is important for quantum control. It was known that in the problem of single-qubit phase shift quantum gate generation all controls, except maybe the special control \(f_0 = 0\) at small times, cannot be traps. In the previous work \([22]\), we studied the spectrum of the Hessian at this control \(f_0\) and investigated under what conditions this control is a saddle point of the quantum objective functional. In this work, we have calculated the numbers of negative and positive eigenvalues of the Hessian at this control point and obtained estimates for the magnitudes of these eigenvalues. At the same time, we significantly simplified the proof of Theorem 3 of the paper \([22]\).
Acknowledgements

Authors thank A. I. Mikhailov for helpful discussions and advice. This work was funded by Russian Federation represented by the Ministry of Science and Higher Education (grant number 075-15-2020-788).

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