Spectral triplets, statistical mechanics and emergent geometry in non-commutative quantum mechanics

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Abstract
We show that when non-commutative quantum mechanics is formulated on the Hilbert space of Hilbert–Schmidt operators acting on a classical configuration space, spectral triplets as introduced by Connes in the context of non-commutative geometry arise naturally. A distance function as defined by Connes can therefore also be introduced. We proceed to give a simple algorithm to compute this function in generic situations. Using this we compute the distance between pure and mixed states on the quantum Hilbert space and demonstrate a tantalizing link between statistics and geometry.

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1. Introduction

The idea that spacetime may be quantized in some sense and that a more abstract notion of geometry may be required has become firmly established in the physics literature in the past two decades. Connes was the first to introduce, in a rigorous fashion, the notion of non-commutative geometry [1], captured by a spectral triplet \((\mathcal{A}, \mathcal{H}, D)\) with \(\mathcal{A}\) an involutive algebra acting on a Hilbert space \(\mathcal{H}\) by a representation \(\pi\) and \(D\) is the so-called Dirac operator on \(\mathcal{H}\). The further technical conditions that this operator must satisfy can be found in [1].

In a separate, yet related development, it was argued by Doplicher et al [2] from the considerations of both general relativity and quantum mechanics that the localization of an event in spacetime with arbitrary accuracy is operationally impossible and this feature is captured by postulating a non-vanishing commutation relation between operator-valued coordinates. In its simplest form they are given as

\[
[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}, \tag{1}
\]

where \(\theta\) is anti-symmetric and its entries are viewed as new fundamental constants. This form of non-commutativity also follows from the low energy limit of string theory [3].

Despite the fact that the two-dimensional non-commutative (Moyal) plane also fits in the framework of non-commutative geometry for both compact [4–6] and non-compact cases...
[7], a detailed and explicit investigation into the link between these two versions of non-commutative spacetime and associated geometries seems to have been started only recently with explicit computations of the Connes’ spectral distance between pure state harmonic oscillator functions as well as translated states [8–10]. One possible reason could be that the explicit computation of the Connes’ distance function can be very challenging. Another reason may be that the Connes’ approach was primarily applied to ‘almost commutative spaces’ which are generically of the form \( M \times F \) with \( M \) representing four-dimensional ordinary commutative spacetime describing the gravitational part and \( F \) corresponding to a zero-dimensional discrete space described by matrices, which takes care of the gauge part of the standard model [11]. The spaces of the type \( M \times F \) have thus only a ‘mild’ non-commutativity stemming from this internal space \( F \). This is believed to be adequate to describe physics up to the GUT scale. However, at a still higher energy scale, like in the vicinity of the Planck scale, one may have to take the fuzziness of spacetime, captured by equations like (1), seriously.

Connes’ spectral distance function relies on the notion of states, which are positive linear functionals of norm 1, over the involutive algebra \( \mathcal{A} \). They are also closely related to the notion of density matrices, which describe the physical states of a quantum system. This naturally raises the question whether the formulation of non-commutative quantum mechanics as set out in [12, 13], which entails a representation of the quantum system on the Hilbert space of Hilbert–Schmidt operators, does not naturally encode geometric information in the context of Connes’ non-commutative geometry. This is the question we investigate here and we indeed find that this formulation of non-commutative quantum mechanics naturally encodes geometry in the form of spectral triplets. Moreover, we succeed in giving a simple algorithm to compute the Connes’ spectral distance between states. It should, however, be remarked that the notion of spectral triplets that we introduce here is stronger than the conventional one, which takes the involutive algebra to be the space of bounded operators on \( \mathcal{H} \), while here it is taken as the Hilbert–Schmidt operators and therefore is itself a Hilbert space. It is precisely this Hilbert space structure of the involutive algebra that facilitates the present analysis.

However, in contrast to the analysis in [8, 9] that used the Moyal star product, which, as shown in [14], stems from a particular choice of basis in the quantum Hilbert space, our analysis is carried out in a basis-independent operator approach that bypasses the use of any star product (as explained in [14]) and any ambiguities that may result therefrom. Indeed, the analysis carried out here becomes problematic for the Moyal star product as the necessary positivity condition is not satisfied [14]. On the other hand, the power of the operator approach was demonstrated by solving for the spectrum of a spherical well potential in a non-commutative plane [12] and the Coulomb problem in 3D non-commutative space [15] (with the coordinates satisfying a \( SU(2) \) algebra). The current approach and that of [8, 9] also differ on a more fundamental level in that the analysis in [8, 9] was essentially carried out in the classical configuration space \( \mathcal{H}_c \) (see section 2). Here, we generalize the notion of the Connes’ distance to the true physical states of a non-commutative system that are density matrices on the quantum Hilbert space, \( \mathcal{H}_q \), which has a natural tensorial structure \( \mathcal{H}_q \equiv \mathcal{H}_c \otimes \mathcal{H}_c^* \) (see section 2). This offers rather tantalizing possibilities in terms of the modification of the implied geometry and the possible statistical underpinnings thereof.

The paper is organized as follows: in section 2 we briefly review the formulation of non-commutative quantum mechanics on a two-dimensional non-commutative (Moyal) plane and in section 3 we provide a brief review of earlier notions of geometry in quantum mechanics, which used the inner product on the Hilbert space to induce a metric on the manifold of quantum states. In section 4 we show how the results of [8] can be recovered from a spectral triplet defined on the classical configuration space and establish a link with the earlier notions of geometry discussed in section 3. In section 5, we proceed to introduce a spectral triplet...
on the quantum Hilbert space and introduce distances between mixed states. We also explore
the possible statistical underpinnings of the emerging geometry. Section 6 summarizes and
concludes the paper.

2. Quantum mechanics on the non-commutative plane

In two dimensions the non-commutative plane is defined through the commutation relations

\[ \hat{x}_i, \hat{x}_j = i \epsilon_{ij} \theta. \]  

(2)

where \( \epsilon \) is the anti-symmetric tensor with \( \epsilon_{12} = -\epsilon_{21} = 1 \). We use a hat to emphasize
the operator nature of these coordinates. One constructs standard creation and annihilation
operators \( b^\dagger \) and \( b \),

\[ b = \frac{\hat{x}_1 + i \hat{x}_2}{\sqrt{2\theta}}, \quad b^\dagger = \frac{\hat{x}_1 - i \hat{x}_2}{\sqrt{2\theta}}, \]  

(3)

and view the non-commutative plane as a boson Fock space spanned by the eigenstate \(|n\rangle\) of
the radial operator \( b^\dagger b \). We refer to it as the classical configuration space \( \mathcal{H}_c \),

\[ \mathcal{H}_c = \text{span}\{|n\rangle = \frac{1}{\sqrt{n!}} (b^\dagger)^n |0\rangle\}. \]  

(4)

This space plays the same role as the classical configuration space \( \mathbb{R}^2 \) in commutative
quantum mechanics. Next we introduce the quantum Hilbert space in which the physical states
of the system and the non-commutative Heisenberg algebra are to be represented. This is taken
to be the set of Hilbert–Schmidt operators over \( \mathcal{H}_c \) and we refer to it as the quantum Hilbert
space \( \mathcal{H}_q \),

\[ \mathcal{H}_q = \{ \psi : \text{tr}_c(\psi^\dagger \psi) < \infty \}, \]  

(5)

where the subscript ‘c’ refers to tracing over \( \mathcal{H}_c \). Elements of \( \mathcal{H}_q \) are denoted by a round
bracket \(|\psi\rangle\) and the inner product is defined as

\[ (\phi | \psi) = \text{tr}_c (\phi^\dagger \psi). \]  

(6)

We reserve \( \dagger \) to denote Hermitian conjugation on the classical Hilbert space, while \( \hat{\ } \)
denotes Hermitian conjugation on the quantum Hilbert space. Note that \( \mathcal{H}_q \) has a natural
tensor product structure in that it can be viewed as the tensor product of \( \mathcal{H}_c \) and its dual, i.e.
\( \mathcal{H}_q = \mathcal{H}_c \otimes \mathcal{H}_c^* \). We can therefore also write the elements of \( \mathcal{H}_q \) in the form \(|\psi\rangle \equiv |\psi\rangle \langle \psi|\).
We shall use this notation quite often below and refer to \( \psi \) as the left-hand and \( \phi \) as the right-
hand sector.

In units of \( \hbar = 1 \), a unitary representation of the non-commutative Heisenberg algebra (we
use a hat to emphasize the operator nature of the coordinates and momenta and to distinguish
them from their classical counterparts)

\[ [\hat{x}_i, \hat{x}_j] = i \epsilon_{ij} \theta, \]  

(7)

\[ [\hat{x}_i, \hat{p}_j] = i \delta_{ij}, \]  

(8)

\[ [\hat{p}_i, \hat{p}_j] = 0 \]  

(9)

is obtained by the following action:

\[ X_i |\psi\rangle = |\hat{x}_i \psi\rangle, \]  

(10)

\[ P_i |\psi\rangle = \frac{1}{\theta} [\epsilon_{ij} |\hat{x}_j \psi\rangle]. \]  

(11)
Here we use capital letters to distinguish operators acting on the quantum Hilbert space from those acting on the classical Hilbert space. For notational simplicity, we also drop the hat notation as the capital letters distinguish these operators from their classical counterparts. It is also useful to introduce the following quantum operators:

\[ B = \frac{1}{\sqrt{2\theta}} (X_1 + iX_2), \]
\[ B^\dagger = \frac{1}{\sqrt{2\theta}} (X_1 - iX_2), \]
\[ P = P_1 + iP_2, \]
\[ P^\dagger = P_1 - iP_2. \] (12)

These operators act as follows:

\[ B |\psi\rangle = |b\psi\rangle, \]
\[ B^\dagger |\psi\rangle = |b^\dagger \psi\rangle, \]
\[ P |\psi\rangle = -i \sqrt{\frac{2}{\theta}} [b, |\psi\rangle], \]
\[ P^\dagger |\psi\rangle = i \sqrt{\frac{2}{\theta}} [b^\dagger, |\psi\rangle]. \] (13)

The interpretation of this quantum system now proceeds as for a standard one. The only modification required is that the position measurement must now be interpreted in the context of a weak measurement (positive operator valued measure, POVM) rather than a strong (projective valued measurement) one. The essence of the construction is based on the minimal uncertainty states on the non-commutative configuration space, which are the normalized coherent states

\[ |z\rangle = e^{-z^2/2} e^{i\theta z} |0\rangle, \] (14)

where \( z = \frac{1}{\sqrt{2\theta}} (x_1 + ix_2) \) is a dimensionless complex number. These states provide an overcomplete basis on the non-commutative configuration space. Corresponding to these states one constructs a state in the quantum Hilbert space as follows: introduce the non-orthogonal projection operators \(|z\rangle\langle z|\) on \( H_c \) and define

\[ |z, z\rangle \equiv |z\rangle\langle z|, \quad B|z, z\rangle = z |z, z\rangle, \] (15)

which leads to the natural interpretation of \((x_1, x_2)\) as the dimensionful position coordinates. These states provide an overcomplete set on the quantum Hilbert space in the form [13, 16]

\[ 1_q = \int \frac{dz d\bar{z}}{\pi} |z, z\rangle \tilde{e}^{\bar{\partial}z} (z, \bar{z}), \quad \tilde{e}^{\bar{\partial}z} = \sum_{n=0}^{\infty} \frac{1}{n!} \bar{z}^n \partial_z^n, \] (16)

where \( \tilde{\partial}_z \) denotes differentiation to the left and \( \bar{\partial}_\bar{z} \) differentiation to the right. The operators

\[ \pi_z = \frac{1}{2\pi \theta} |z, z\rangle \tilde{e}^{\bar{\partial}z} (z, \bar{z}) \] (17)

form a set of complete, positive, but non-orthogonal and un-normalized projection operators [16], i.e.

\[ \int d\bar{z} dz \pi_z = 1_q \quad (\psi |\pi_z|\psi) \geq 0, \forall \psi, \quad \pi_z \pi_w \neq \delta(z - w), \quad \pi_z^2 \propto \pi_z. \] (18)
They therefore provide a POVM that can be used to give a consistent probability interpretation by assigning the probability of finding the outcome of a position measurement to be \((x_1, x_2)\), given that the system is described by the density matrix \(\rho\), to be
\[
P(x_1, x_2) = \text{tr}_q(\pi_z \rho),
\]
where \(\text{tr}_q\) denotes the trace over the quantum Hilbert space. In particular, for a pure state density matrix \(\rho = |\psi\rangle \langle \psi|\), this probability is given by
\[
P(x_1, x_2) = \text{tr}_q(\pi_z \rho) = \frac{1}{2\pi \theta} \langle \psi | z \rangle e^{\frac{i\theta}{2\theta}} \langle z | \psi \rangle,
\]
which differs from the commutative probability only by the presence of the Voros product [13].

In [16, 17] it was shown that these operators can also be written as follows:
\[
\pi_z = \frac{1}{2\pi \theta} \sum_n |z, n\rangle (z, n),
\]
where \(|n\rangle\) is an arbitrary orthonormal basis in the classical configuration space, implying that this probability should be interpreted as the ‘total’ probability of finding \((x_1, x_2)\) as the outcome of a position measurement, which is insensitive to the value of the additional quantum number \(n\) labeling the right-hand sector.

3. Induced metric from the Hilbert space inner product

It was realized some time ago that the inner product on a generic Hilbert space \(\mathcal{H}\) quite naturally induces a metric on manifolds of quantum states (unit rays) [18, 19]. In this section we briefly review this; however, instead of working with the quantum states as unit rays, which invariably restricts the formalism to pure states, we use the density matrix as the primary object that represents the states of a quantum system. This has the advantage of generalizing this notion of geometry to mixed states and it also ensures that the results are manifestly gauge invariant. This will also establish a link to Connes’ notion of spectral distance in the following section.

Let us therefore consider a \(n\)-parameter family of density matrices \(\rho(\vec{s})\) with \(\vec{s} = (s_1, \ldots, s_n) \in \mathbb{R}^n\). The density matrices have the usual properties of hermiticity (\(\rho^\dagger = \rho\)), unit norm (\(\text{tr} \rho(\vec{s}) = 1\)), semi-positivity and we assume that the map \(s \mapsto \rho(s)\) is at least \(C^1\).

Realizing that the density matrices can be equipped with the natural inner product and trace norm on Hilbert–Schmidt operators,
\[
(A, B) = \text{tr}(A^\dagger B), \quad \|A\|^2 = \langle A, A \rangle,
\]
the distance between two neighboring states (density matrices) with coordinates \(\vec{s}\) and \((\vec{s} + d\vec{s})\) can be defined as
\[
d^2(\vec{s}, \vec{s} + d\vec{s}) = \frac{1}{2} \|d\rho\|^2 = \frac{1}{2} (\partial_i \rho, \partial_i \rho) \, ds^i \, ds^i.
\]

In the case of a family of pure state density matrices \(\rho(\vec{s}) = |\psi(\vec{s})\rangle \langle \psi(\vec{s})|\), \(|\psi(\vec{s})\rangle \in \mathcal{H}\) it is a matter of straightforward verification that the metric tensor
\[
g_{ij} = \frac{1}{2} (\partial_i \rho, \partial_j \rho),
\]
when re-written in terms of \(\psi(\vec{s})\) and its derivative, yields [18, 19]
\[
g_{ij} = \text{Re}(\partial_i \psi, \partial_j \psi) - A_i A_j.
\]
Here \(A_i = -i (\psi, \partial_i \psi)\) represents the pull-back of the connection 1-form of the \(U(1)\) principal bundle onto the parameter space, and the exterior derivative of this connection...
1-form $A = A_i\,ds^i$ gives the symplectic 2-form: $\sigma = dA = \text{Im}(\bar{\partial}\psi, \partial\psi)\,ds^i \wedge ds^j$. This can be further expressed in terms of the covariant derivative $D_i = (\partial_i - iA_i)$ as $g_{ij} = (D_i\psi, D_j\psi)$.

Finally, $d^2(\vec{s}, \vec{s} + d\vec{s})$ can indeed be shown to be the smallest distance in the space of rays in the projective Hilbert space [18]

$$d^2(\vec{s}, \vec{s} + d\vec{s}) = D^2(R_\phi, R_\phi') = \inf_{\omega_1, \omega_2} \| \psi_1 e^{i\omega_1} - \psi_2 e^{i\omega_2} \|^2,$$

where $R_\phi$ represents the ray associated with the state $\psi$.

### 4. Spectral triplets on classical configuration space

We start our analysis of the geometrical content of the construction in section 2 by identifying the spectral triplet $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A} = \mathcal{H}_q$, $\mathcal{H} = \mathcal{H}_c \otimes \mathbb{C}^2$, the action of elements of $a \in \mathcal{A}$ on $\mathcal{H}$ is defined through the representation $\pi(a)(|\psi\rangle, |\phi\rangle) = (a|\psi\rangle, a|\phi\rangle)$ and the Dirac operator is defined as

$$D = -i\sqrt{\frac{\hbar}{\theta}}\begin{pmatrix} 0 & b^i \\ b_i & 0 \end{pmatrix}.$$  

(27)

Here, we have chosen the Dirac operator as in [8] to enable direct comparison with their results.

States $\omega$ are positive linear functionals of norm 1 over $\mathcal{A}$. Pure states play a rather fundamental role and are defined as those functionals that cannot be written as a convex linear combination of two other functionals. The Connes’ spectral distance between two states is then defined by

$$d(\omega, \omega') = \sup_{a \in \mathcal{A}} |\omega(a) - \omega'(a)|,$$

$$B = \{ a \in \mathcal{A} : \|[D, \pi(a)]\|_{\text{op}} \leq 1 \},$$

$$\|A\|_{\text{op}} = \sup_{\phi \in \mathcal{H}} \frac{\|A\phi\|}{\|\phi\|}.$$  

(28)

This is a rather difficult quantity to compute and therefore we first cast it into a more tractable form for the purpose of explicit computation and interpretation. For this we shall restrict our analysis to states satisfying the following conditions.

- The states $\omega, \omega'$ are normal states (see [20] for the definition).
- The states $\omega$ and $\omega'$ are separately bounded on $B$, i.e. $\omega(a) < \infty$ and $\omega'(a) < \infty$, $\forall a \in B$.
- Let $V_0 = \{ a \in \mathcal{A} : \|[D, \pi(a)]\|_{\text{op}} = 0 \}$, then the states $\omega, \omega'$ are such that $\omega(a) - \omega'(a) = 0$, $\forall a \in V_0$.

The first two conditions are actually quite mild and essentially only imply that we restrict our analysis to states that can be represented by density matrices. Generally, the third condition places a restriction on the states for which our result for the Connes’ distance holds, but under some rather generic conditions on the Dirac operator, that implies a certain irreducibility condition, it turns out that the result holds for all states satisfying the first two conditions. To show this we note that there is a close link between normal states and density matrices. Indeed, any normal state over $\mathcal{A}$ can be uniquely written as [20]

$$\omega(a) = \text{tr}_c(\rho_{\omega}a),$$  

(29)

where $\rho_{\omega}$ is a Hermitian, semi-positive operator on $\mathcal{H}_c$ of trace 1. Note that since $\text{tr}_c(\rho_{\omega}^2) \leq \text{tr}_c(\rho_{\omega}) = 1$, $\rho_{\omega}$ is a Hilbert–Schmidt operator and therefore, $\rho_{\omega} \in \mathcal{A} = \mathcal{H}_q$. If $\omega$ is a pure state, so is $\rho_{\omega}$, i.e. it is of the form $\rho_{\omega} = |\psi\rangle\langle\psi|$ for some $|\psi\rangle \in \mathcal{H}_c$. This result can actually

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be understood quite easily from the Hilbert space structure of $A = \mathcal{H}_q$ as it follows simply from the Riesz–Frechet theorem, where the properties of $\rho_0$ derive from the properties of $\omega$.

Using this, and noting that our assumption $\omega(a) < \infty$ and $\omega'(a) < \infty, \forall a \in B$ implies $\text{tr}_c(\rho_0 a) < \infty$ and $\text{tr}_c(\rho_0^a) < \infty, \forall a \in B$ and setting $d\rho = \rho - \rho'$, we can write the third condition as

- the states are such that

$$\text{tr}_c( (d\rho) a ) = (d\rho, a) = 0, \quad \forall a \in V_0.$$  \hfill (30)

We use the notation $d\rho$ to indicate that we are interested in infinitesimal changes. Furthermore, note that since $\text{tr}_c( (d\rho^2) ) \leq 2(1 - \text{tr}_c(\rho\rho')) \leq 2$ from the semi-positiveness of $\rho$ and $\rho'$, $d\rho$ is also a Hilbert–Schmidt operator, as one would also have expected from the Hilbert space structure of these operators.

Let us consider a Dirac operator of the form (27)

$$D = -i \begin{pmatrix} 0 & c^\dagger \\ c & 0 \end{pmatrix},$$  \hfill (31)

with the requirement that $c, c^\dagger$ forms an irreducible pair on $\mathcal{H}_c$, i.e. $[c, a] = [c^\dagger, a] = 0 \Rightarrow a \propto 1_c$. It is easily checked that $||[D, \rho(a)]||_{op} = 0 \Rightarrow ||[c, a]||_{op} = ||[c^\dagger, a]||_{op} = 0 \Rightarrow [c, a] = [c^\dagger, a] = 0 \Rightarrow a \propto 1_c$. Since the identity element is not Hilbert–Schmidt, the set $V_0$ contains only the null element and the condition $\text{tr}_c( (d\rho) a ) = 0$ holds for all states satisfying the first two conditions. For the choice $c = b$ in (27), it immediately follows that the Dirac operator satisfies this condition since $\mathcal{H}_c$ carries an irreducible representation of the oscillator algebra.

Returning to the Connes’ distance function, we can write

$$d(\rho, \rho') = \sup_{a \in B} |\text{tr}_c((d\rho)a)| = \sup_{a \in B}|(d\rho, a)|,$$  \hfill (32)

We are now in a position to determine the supremum on the right of (32). Let us introduce the orthogonal complement, $V_0^\perp$, of $V_0$. Note from (30) that $V_0$ is a subspace of the space of elements orthogonal to $d\rho$. We can then decompose any $a \in A$ uniquely into its component $a_0 \in V_0$ and its component $a_1 \in V_0^\perp$ (note that $\{D, \rho(a)\} \neq 0, \forall a \in V_0^\perp$). From (30) we can then write the Connes’ distance function as

$$d(\rho, \rho') = \sup_{a \in B} |\text{tr}_c((d\rho)a)| = \sup_{a \in B}|(d\rho, a)|, \quad B' = \{a \in V_0^\perp : ||[D, \rho(a)]||_{op} \leq 1\}. \hfill (33)$$

The first step is to find a lower bound. To do this we note that the family of operators $\Lambda \equiv \{a \in A : a = \lambda d\rho, 0 < \lambda \leq \frac{1}{||[D, \rho(d\rho)]||_{op}}\} \subset B'$. Taking the extremal element $a = \frac{d\rho}{||[D, \rho(d\rho)]||_{op}}$ yields the lower bound

$$d(\rho, \rho') \geq \frac{|d\rho|^2}{||[D, \rho(d\rho)]||_{op}}. \hfill (34)$$

Next we show that this is also an upper bound. Consider any $a \in B'$ and write it as $a = ||a||_a \hat{a}$ where $||\hat{a}||_a = 1$. Then we have from (33)

$$||a||_a ||[D, \rho(\hat{a})]||_{op} \leq 1.$$  \hfill (35)

Since $||[D, \rho(\hat{a})]||_{op} > 0$, this places an upper bound $||a||_a \leq \frac{1}{s}$ with $s = \inf_{a \in B} ||[D, \rho(\hat{a})]||_{op}$. We decompose $\hat{a} = \cos(\theta) \hat{d}\rho + \sin(\theta) \hat{d}\rho_\perp(\hat{a})$ with $\hat{d}\rho$ and $\hat{d}\rho_\perp(\hat{a})$ unit vectors parallel and orthogonal to $d\rho$. Note that both components are still in $V_0^\perp$ and that the latter depends on $\hat{a}$. Since $||[D, \rho(\hat{a})]||_{op} \leq |\cos(\theta)|||[D, \rho(\hat{d}\rho)]||_{op} + |\sin(\theta)|||[D, \rho(\hat{d}\rho_\perp(\hat{a})]||_{op}$...
It can be easily verified that the overlap \( \langle \hat{\xi} | m, n \rangle \) of the basis \( | m, n \rangle \) with the Moyal basis \( | \hat{\xi} \rangle_M = \frac{1}{\sqrt{D}} \int d\hat{p} e^{-i\hat{p}\hat{\xi}} | \hat{\xi} \rangle \) as introduced in [14] indeed gives the Moyal representation \( h_{m,n}(\xi) \) used in [8, 9]. However, since the Moyal basis does not conform to the requirements of POVM [14] it is desirable to carry out this analysis using the Voros basis or, better still, to carry this out in an abstract framework, i.e. in a basis-independent approach—as we do here.
states are taken as $a_m(\alpha) = a_{m,m}$ with the corresponding pure state density matrix $\rho_m = |m\rangle\langle m|$. We then take $d\rho = \rho_{m+1} - \rho_m$. The calculation is now straightforward:

$$[D, \pi (d\rho)] = -i \frac{2}{\theta} \begin{pmatrix} 0 & [b^\dagger, d\rho] \\ [b, d\rho] & 0 \end{pmatrix},$$

$$[D, \pi (d\rho)]^\dagger [D, \pi (d\rho)] = \frac{2}{\theta} \begin{pmatrix} [b, d\rho] [b^\dagger, d\rho] & 0 \\ 0 & [b^\dagger, d\rho] [b^\dagger, d\rho] \end{pmatrix}.$$ (39)

A simple calculation yields

$$[b, d\rho] [b, d\rho] = 4(m + 1) |m + 1\rangle\langle m + 1| + |m\rangle\langle m| + (m + 2) |m + 2\rangle\langle m + 2|,$$

$$[b^\dagger, d\rho] [b^\dagger, d\rho] = (m + 2) |m + 1\rangle\langle m + 1| + 4(m + 1) |m\rangle\langle m| + |m - 1\rangle\langle m - 1|.$$ (40)

As both of these operators are already diagonal, the operator norm, which is nothing but the largest eigenvalue, can be read off exactly. This yields

$$||[D, \pi (d\rho)]||_\text{op} = 2 \sqrt{\frac{m + 1}{\theta}}$$ (41)

and

$$d(m + 1, m) = \sqrt{\frac{\theta}{2(m + 1)}}.$$ (42)

which agrees precisely with the result of [8].

Next we calculate the distance between two coherent states. We define the usual harmonic oscillator coherent states in $\mathcal{H}_c$ given by $|\zeta\rangle = e^{-\frac{\zeta^2}{2}} e^{\zeta b^\dagger} |0\rangle$ and consider the corresponding pure state density matrix $\rho = |\zeta\rangle\langle \zeta|$. Then

$$d\rho = |z + d\zeta\rangle\langle z + d\zeta| - |\zeta\rangle\langle \zeta|$$

$$= (b^\dagger - \bar{z}) |\zeta\rangle\langle \zeta| d\zeta + |\zeta\rangle (b - z) d\bar{z} + O(d^2)$$ \hspace{1cm} (43)

This can be written more simply as

$$d\rho = d\bar{z} (\bar{0}\langle 1 | + d\zeta | 1 \rangle \langle 0 |),$$ (44)

where we have introduced new bosons $\tilde{b} = b - z$, $\tilde{b}^\dagger = b^\dagger - \bar{z}$ and the vacuum $|\tilde{0}\rangle = |\zeta\rangle$ for which $\tilde{b}(\tilde{0}) = 0$, $\tilde{b}^\dagger (\tilde{0}) = \tilde{b}^\dagger |0\rangle$. From this

$$[b, d\rho] = d\bar{z} (\tilde{0}\langle 1 | - \sqrt{2} d\zeta \langle 0 | \tilde{2} - d\zeta | 1 \rangle \langle 1 |)$$ \hspace{1cm} (45)

and

$$[b^\dagger, d\rho] = d\bar{z} (\tilde{0}\langle 1 | - \sqrt{2} (d\zeta)^2 \langle 0 | \tilde{2} - \sqrt{2} (d\zeta)^2 \langle 2 | 0 \rangle - d\zeta^2 \langle 2 | 1 \rangle + d\zeta d\bar{z} | 2 \rangle \langle 1 |) + d\bar{z} d\zeta | 1 \rangle \langle 1 |)$$. (46)

This yields, on diagonalizing the first one,

$$||[D, \pi (d\rho)]||_\text{op} = \sqrt{\frac{6 d\zeta d\bar{z}}{\theta}}$$ (47)

and

$$d(z + d\zeta, z) = \sqrt{\frac{3\theta}{5}} \sqrt{d\zeta d\bar{z}},$$ (48)

yielding the Euclidean distance. One can quickly check from (40) and (46) that replacing the operator norm in (42) and (48) by the trace norm one recovers, up to numerical factors, the same results. In what follows we shall, for computational simplicity, therefore often
replace the operator norm with the trace norm and rather compute the closely related distance function
\[
\tilde{d} (\rho, \rho') = \frac{\text{tr}_c (d\rho)^2}{||[D, \pi (d\rho)]||_{op}}.
\]

In the examples above this distance function and the Connes’ distance function only differ by a numerical constant. Whether this holds generally is an open question that needs further investigation.

5. Spectral triplets on the quantum Hilbert space

In the construction of the previous section there is a one-to-one correspondence between states (points) \(|\psi\rangle\) in the classical configuration space and pure state density matrices \(\rho = |\psi\rangle\langle\psi|\) acting on the classical configuration space. These density matrices are also elements of the quantum Hilbert space corresponding to diagonal states \(|\psi, \psi\rangle\equiv |\psi\rangle\langle\psi|\). However, there are many more physical pure states in the quantum Hilbert space of the off-diagonal form \(|\psi, \phi\rangle\). These states are in a one-to-one relation with the pure state density matrices \(\rho_q = |\psi, \phi\rangle\langle\psi, \phi|\) acting on the quantum Hilbert space and in a many-to-one relation with states (points) in the classical configuration space, while only the diagonal density matrices \(\rho_d = |\psi, \psi\rangle\langle\psi, \psi|\) are in a one-to-one correspondence with states in the classical configuration space. Note that this is quite different to commutative quantum mechanics where there is a one-to-one relation between pure states density matrices and points in the configuration space. Specifically, in commutative quantum mechanics a particle localized at a point \(x\) is described by the pure state density matrix \(\rho = |x\rangle\langle x|\). On the other hand, as discussed in section 2 and further below, in non-commutative quantum mechanics a particle maximally localized at a point \((x_1, x_2)\), without any prior knowledge of the right-hand sector, is described by the mixed state density matrix \(\rho_q = \sum_n |z, n\rangle(z, n| (z = x_1 + ix_2). When the right-hand sector is also specified the particle is described by a pure state density matrix \(\rho = |z, n\rangle(z, n|, but note that there are infinitely many choices for the right-hand sector given a point \((x_1, x_2)\) in the classical configuration space.

We would therefore also like to calculate distances between off-diagonal pure states and mixed states on the quantum Hilbert space. To do this we introduce a further spectral triplet \((\mathcal{A}, \mathcal{H}, D)\) on the quantum Hilbert space where \(\mathcal{A}\) are the Hilbert–Schmidt operators on \(\mathcal{H}_q\), \(\mathcal{H}_c = \mathcal{H}_q \otimes \mathbb{C}^2; the action of elements of \(a \in \mathcal{A}\) on \(\mathcal{H}\) is defined through the representation \(\pi (a) (|\psi, \phi\rangle) = (a|\psi\rangle, a|\phi\rangle)\) and the Dirac operator is defined as
\[
D = -\frac{i}{\sqrt{2}} \begin{pmatrix} 0 & B^\dagger \\ B & 0 \end{pmatrix}.
\]

Note that the Dirac operator defined here acts only on the left sector as it does not involve any momentum operators.

Clearly, under the same conditions as in section 4 on the states, the analysis that led up to (38) holds here with the replacement \(\mathcal{H}_c \rightarrow \mathcal{H}_q\), i.e.
\[
d (\rho_q, \rho_q') = \frac{\text{tr}_q (d\rho_q)^2}{||[D, \pi (d\rho_q)]||_{op}}.
\]

However, since the Dirac operator (50) only acts on the left sector, the condition (30) becomes non-trivial. Indeed it is simple to verify that the general operators that commute with the Dirac operator are of the form
\[
\Gamma = \sum_p \sum_{k, \ell} a_{k, \ell} |p, k\rangle(p, \ell|.
\]
with $|p\rangle$ an orthonormal basis in the classical configuration space and $a_{q,j}$ arbitrary complex numbers independent of $\rho$. In this case these operators span $V_0$. Generically $(d\rho_q, \Gamma) \neq 0$, (30) is violated and we cannot expect (51) to yield the Connes’ distance between all states. At best it will yield a lower bound, while the true Connes’ distance may even be unbounded. However, (51) still applies for states such that $(d\rho_q, \Gamma) = 0, \quad \forall \Gamma$ and can be used to compute the Connes’ distance function between such states. Below we apply (51) in this context. Indeed, we shall compute (51) regardless of whether the condition $(d\rho_q, \Gamma) = 0$ is met or not and simply keep in mind that this will only yield the true Connes’ distance when this condition holds. This is particularly useful in the context of a variational calculation as performed below.

For an infinitesimal change $d\rho_q$ in the density matrix $\rho_q$, a calculation paralleling the calculation leading to equation (39) yields

$$
[D, \pi(d\rho_q)] = -i\frac{\sqrt{\theta}}{\theta} \left( [B, d\rho_q] \right). 
$$

To start, we compute the distance between the pure states corresponding to the density matrices $\rho_q(m, \phi) = |m, \phi\rangle\langle m, \phi| \: \text{with } |m\rangle$ a harmonic oscillator state and $|\phi\rangle$ an arbitrary state in the classical configuration space. Introducing $d\rho_q = |m + 1, \phi\rangle\langle m + 1, \phi| - |m, \phi\rangle\langle m, \phi|$, the calculation proceeds exactly as in section 4. Indeed, the state $|\phi\rangle$ is purely a spectator and the result obtained is exactly the same as in (42). The same is true for the pure states described by the density matrix $\rho_q(z, \phi) = |z, \phi\rangle\langle z, \phi| \: \text{where } |z\rangle$ is a coherent state in the classical configuration space. Computing the distance between these states yields again the result (48), independent of $|\phi\rangle$. The independence of these results from the choice of the right-hand sector specified by $|\phi\rangle$ is of course a consequence of the choice of the Dirac operator, which only acts on the left-hand sector. In these cases it is also simple to verify that $(d\rho_q, \Gamma) = 0, \quad \forall \Gamma$ and thus this will yield the true Connes’ distance function.

We can use the same Dirac operator to compute the more general spectral distance between the states $\rho_q(m, \phi) = |m, \phi\rangle\langle m, \phi|$ and $\rho_q(m + 1, \phi') = |m + 1, \phi'\rangle\langle m + 1, \phi'|$ where $\phi'$ is different and orthogonal to $\phi$, i.e. $\langle \phi | \phi' \rangle = \delta_{\phi, \phi'}$. Introducing $d\rho_q = |m + 1, \phi'\rangle\langle m + 1, \phi'| - |m, \phi\rangle\langle m, \phi|$, we obtain

$$
[B, d\rho_q]^2[B, d\rho_q] = (m + 1) | m + 1, \phi'(m + 1, \phi') + (m + 1) \delta_{\phi', \phi} | + (m + 1) \δ_{\phi', \phi} \big| + m, \phi\rangle\langle m, \phi | + (m + 2) | m + 2, \phi'(m + 2, \phi' | + (m + 1) \delta_{\phi', \phi} \big| + m, \phi\rangle\langle m, \phi | + (m + 1) | m + 1, \phi'(m + 1, \phi' | + (m + 1) | m + 1, \phi |.
$$

$$
[B^2, d\rho_q]^2[B^2, d\rho_q] = (m + 2) | m + 1, \phi'(m + 1, \phi') + (m + 1) | m, \phi\rangle\langle m, \phi | + (m + 1) \delta_{\phi', \phi} \big| + m, \phi\rangle\langle m, \phi | + (m + 1) | m, \phi'(m, \phi' | + m | m - 1, \phi\rangle\langle m - 1, \phi |.
$$

It is easily seen that for $\phi = \phi'$ we recover the earlier result (42), but for $\phi \neq \phi'$ we obtain

$$
\|D, \pi(d\rho_q)\|_\infty = \sqrt{\frac{2(m + 2)}{\theta}}.
$$

This eventually yields $d(\rho(m + 1, \phi'), \rho(m, \phi)) = \sqrt{\frac{20}{m^2 \tau^2}}$. In this case, however, $(d\rho_q, \Gamma) \neq 0, \quad \forall \Gamma$ so that this does not correspond to the true Connes’ distance. However, taking this quantity at face value, it does show that $d(\rho(m + 1, \phi), \rho(m, \phi)) < d(\rho(m + 1, \phi'), \rho(m, \phi))$ and demonstrates that despite the fact that the Dirac operator only acts on the
left-hand sector, the distance between two states depends on the right-hand sector and, indeed, that the distance increases when the right-hand sectors are taken differently. Here we have taken the right hand to be a pure state that depends on the left hand, i.e. it changes from point to point. One may contemplate an even more general situation by taking the right-hand sector to be a pure state that depends on the left hand, i.e. it changes from point to point. This implies that these states describe a system maximally localized at the point \((x_1, x_2)\). Note that the operators \(\pi_z\), introduced in (21) are density matrices of this type as they can be interpreted as the density matrix for a particle maximally localized at \(z\), but with complete ignorance of the right-hand sector, i.e. all probabilities are equal. Indeed, computing the probability of finding a particle at point \(z^0 = (\bar{x}_1 + i\bar{x}_2)/\sqrt{2}\theta\), given that it is prepared in the state described by the density matrix \(\pi_z\), with \(z = (x_1 + ix_2)/\sqrt{2}\theta\), yields a Gaussian \(P(x_1, x_2) \propto e^{-\frac{\theta}{4}((x_1-x)^2+(x_2-y)^2)}\). It is also for this reason that their use as a POVM leads to a measurement of position that disregards any information of the right-hand sector.

In these cases it is easily verified that if the probabilities \(p_n\) are position independent, i.e. they do not depend on \(m\) or \(z\), \((d\rho_q, \Gamma) = 0\). Thus, in this case (51) will yield the true Connes’ distance between these states.

The choice of probabilities in (57) is, of course, an open issue. In equilibrium statistical mechanics, the choice of probabilities is dictated by equilibrium considerations, i.e. one maximizes the entropy subject to constraints on averages, e.g., the average energy that leads to a Boltzmann distribution. Here it is not clear which criteria should be used to determine the probability distribution, but two obvious possibilities present themselves. The first is to choose the probabilities such that the path length between two points is minimized. The second is to fix the probabilities from a condition of local thermal equilibrium, i.e. the local average energy is fixed and the local entropy associated with the density matrices (57) is optimized. The rest of this section explores the interplay between geometry and statistics from these two perspectives. Ideally one would prefer that the probability distributions and geometries that emerge from these two perspectives coincide. However, it turns out below that this only happens in the \(T \to \infty\) limit.

To facilitate the computation below, we use the modified distance function (49), adapted to the quantum Hilbert space, rather than (51), i.e.

\[
\tilde{d}(\rho_q, \pi'_{\rho_q}) = \frac{\text{tr}_q(d\rho_q)^2}{||[D, \pi(d\rho_q)]||_W}. \tag{58}
\]

Setting \(d\rho_q = \rho_q(m + 1) - \rho_q(m)\) one obtains

\[
\tilde{d}(m + 1, m) = \frac{\sqrt{\theta}}{2} \frac{\sum_n \left(p_n^2(m + 1) + p_n^2(m)\right)}{\sqrt{\sum_n \left((2m + 3)p_n^2(m + 1) + (2m + 1)p_n^2(m) + 2(m + 1)p_n(m + 1)p_n(m)\right)}}. \tag{59}
\]
Let us first consider the first perspective and ask which choice of probabilities actually minimizes the distance function. Starting with (59), the distance between two points \(m_i\) and \(m_f\) is given by

\[
\tilde{d}(m_f, m_i) = \sum_{m=m_i}^{m_f-1} \tilde{d}(m + 1, m). \tag{60}
\]

One easily concludes that the probabilities that minimize the distance must satisfy the equation

\[
\Delta p_n = \lambda, \quad \forall n, \tag{61}
\]

where

\[
p_n = \begin{pmatrix}
p_n(m_i) \\
\vdots \\
p_n(m_i+1) \\
p_n(m_f)
\end{pmatrix}, \quad \lambda = \begin{pmatrix}
\lambda(m_i) \\
\vdots \\
\lambda(m_i+1) \\
\lambda(m_f)
\end{pmatrix}.
\]

\(\lambda(m)\) are the Lagrange multipliers imposing the constraints that the probabilities sum to 1 and \(\Delta\) is the tri-diagonal matrix

\[
\Delta = \begin{pmatrix}
a(m_i) & b(m_i) & b(m_{i+1}) \\
b(m_i) & a(m_{i+1}) & b(m_{i+1}) \\
\ddots & \ddots & \ddots \\
b(m_f) & a(m_f)
\end{pmatrix}.
\]

The matrix elements of the matrix \(\Delta\) are given by

\[
a(m) = \sqrt{\theta} \left( g(m) + g(m-1) + (2m+1)(f(m)+f(m-1)) \right),
\]

\[
b(m) = -\sqrt{\theta}(m+1)f(m+1),
\]

\[
g(m) = \frac{1}{\sqrt{\sum_n \left( (2m+3)p_n^2(m+1) + (2m+1)p_n^2(m) + 2(m+1)p_n(m+1)p_n(m) \right)}},
\]

\[
f(m) = \frac{1}{\sqrt{2}} \frac{\sum_n \left( p_n^2(m+1) + p_n^2(m) \right)}{\left( (2m+3)p_n^2(m+1) + (2m+1)p_n^2(m) + 2(m+1)p_n(m+1)p_n(m) \right)}^{\frac{1}{2}}.
\tag{62}
\]

It is important to note that although \(\Delta\) is a function of all the probabilities, it is independent of \(n\). Using this we can sum over \(n\) in (61), which shows that \(\lambda = \Delta p\), where \(p\) is the column vector with equal entries \(1/\Omega\), \(\Omega\) being a cut-off for the sum over \(n\). Substituting this back in (61) shows that the probabilities are independent of \(m\) and \(n\). In this case the distance function reduces to

\[
\tilde{d}(m+1, m) = \frac{\sqrt{\theta}}{\sqrt{6\Omega}} \frac{1}{\sqrt{m+1}}. \tag{63}
\]

which only differs from (42) by a probability-dependent global scale factor. In this case \((\delta p_q, \Gamma') = 0\), \(\forall \Gamma'\) and (63) should be closely related to the Connes’ distance, the only difference residing in the use of the trace norm in (58), rather than the operator norm. We expect this to only yield numerical factors.
Next we explore the geometry emerging from the second perspective. To do this we introduce a local entropy associated with the density matrices (57) in the usual manner

\[ S(m) = -\sum_n p_n(m) \log p_n(m). \]

\[ S(z, \bar{z}) = -\sum_n p_n(z, \bar{z}) \log p_n(z, \bar{z}). \]

(64)

A simple calculation shows that maximizing this local entropy leads, as above, to equally distributed probabilities. This, of course, corresponds to the limit where only entropy plays a role and all states are assigned equal probabilities.

If we associate an energy scale \( \epsilon_n \) with the right-hand states labeled by \( n \) and require the local average energy to be fixed on \( E(m) \), maximization of the local entropy yields a local Boltzmann distribution

\[ p_n(m) = \frac{e^{-\beta(m)\epsilon_n}}{Z(\beta(m))}, \quad Z(\beta(m)) = \sum_n e^{-\beta(m)\epsilon_n}, \]

(65)

where \( \beta(m) \) is the local inverse temperature. From this the distance function (59) can be computed. In particular, if we assume the local average energy and hence the temperature to be independent of \( m \), we obtain

\[ \tilde{d}(m + 1, m) = \frac{\sqrt{\beta(\beta)} \sqrt{Z(2\beta)}}{Z(\beta)} \frac{1}{\sqrt{m + 1}}, \]

(66)

which again only differs from (42) by a temperature-dependent global scale factor. The same remarks as for (63) regarding the relation to the Connes’ distance are applicable here.

This implied modified geometry for mixed states becomes even more explicit in the case of the continuous coherent state basis, which we consider next.

The distance between the coherent state density matrices simplifies considerably with a slight redefinition of the Dirac operator. We take the Dirac operator to be of the continuous coherent state basis, which we consider next.

The general operators that commute with it are of the form

\[ |\tilde{\theta} \rangle = |\tilde{\theta} \rangle \tilde{\theta} = |\theta \rangle \tilde{\theta} \]

(67)

Here, we have again introduced the bosons \( \tilde{b} = b - z, \tilde{b}^\dagger = b^\dagger - \bar{z} \) and the vacuum \( |\tilde{0} \rangle = |\theta \rangle \). Note that the Dirac operator is still insensitive to the right-hand sector. Apart from the simplification that occurs with this choice of the Dirac operator, it is also a more natural choice in that it is based on the local vacuum \( |\tilde{0} \rangle = |\theta \rangle \) and local excitations \( \tilde{b}^\dagger = b^\dagger - \bar{z} \) around this vacuum. The same remarks as made for the Dirac operator (50) apply to this Dirac operator. The general operators that commute with it are of the form

\[ \tilde{\Gamma} = \sum_p \sum_{k, \ell} a_{k, \ell} (\tilde{\rho}, k) (\tilde{\rho}, \ell). \]

(69)

As before, it is also straightforward to verify that for the density matrices (57), \( df_{\tilde{\rho}, \tilde{\Gamma}} = 0 \), \( \forall \tilde{\Gamma} \) if the probabilities are position independent.

The computation now follows the same route as before and one obtains

\[ \tilde{d}(z + dz, z) = \frac{\sqrt{\beta}}{2} \sqrt{\sum_n \left( p_n(z)^2 + \frac{\partial p_n(z)}{\partial z} \frac{\partial p_n(z)}{\partial z} \right) dzdz + \sum_n \left( \frac{\partial p_n(z)}{\partial z} \right)^2 dz^2 + \left( \frac{\partial p_n(z)}{\partial z} \right)^2 dz^2}. \]

(70)
This clearly exhibits a modified geometry. Assuming that $p_n$ is just a function of the dimensionful radial coordinate $r = \sqrt{2} \theta \bar{z} z$, one obtains
\[
\tilde{d}(z + dz, z) = \frac{1}{\sqrt{2}} \left( \sum_n (2 p_n(r)^2 + p'_n(r)^2) \right) \sqrt{dr^2 + r^2 d\phi^2},
\]
which is related to the Euclidean distance by a probability-dependent conformal transformation. Note that since the probabilities are position dependent, this will not correspond to the true Connes’ distance. We can of course again compute the probabilities from the condition of local equilibrium. In the $T \to \infty$ limit, the probabilities are obtained by maximizing (64), which yields equal, $r$-independent probabilities and subsequently, up to a global scale, an Euclidean geometry, which should be closely related to the Connes’ distance function. If we constrain the local average energy to $E(r)$, the temperature becomes position dependent and the probabilities again follow a Boltzmann distribution
\[
p_n(r) = \frac{e^{-\beta(r)\epsilon_n}}{Z(\beta(r))},
\]
yielding a conformal scale factor. If the local average energy is position independent, so is the temperature, and we obtain an Euclidean geometry with a global, temperature-dependent, scale factor
\[
\tilde{d}(z + dz, z) = \sqrt{\frac{Z(2\beta)}{Z(\beta)}} \sqrt{dr^2 + r^2 d\phi^2}.
\]

As the prefactor is bounded by $0 < \sqrt{\frac{Z(2\beta)}{Z(\beta)}} \leq \frac{1}{2}$, reaching its minimum at $T \to \infty$ and its maximum at $T \to 0$ this suggests, quite remarkably, that distances expand as the global temperature is lowered. We expect this distance function to be very closely related to the true Connes’ distance as the probabilities are position independent and therefore $(d_\rho, \Gamma_1) = 0$, $\forall \Gamma_1$. The only difference from the true Connes’ distance is therefore the use of the trace, rather than the operator norm, which we expect to yield only numerical prefactors.

6. Conclusions

The formulation of non-commutative quantum mechanics on the Hilbert space of Hilbert–Schmidt operators naturally encodes the notion of spectral triplets and thus geometry, albeit in a sense stronger than in the conventional sense. Conditions under which a more tractable form of the Connes’ distance formula apply have been derived. This form, and natural modifications of it, were used to make the geometry explicit on the level of the classical configuration space and the quantum Hilbert space. In the former case, as in commutative quantum mechanics, there is a one-to-one correspondence between pure state density matrices and points in the classical configuration space. In the latter case this no longer holds due to the tensor product structure of the quantum Hilbert space and the distance function depends on information encoded in the right-hand sector, even though the Dirac operator only acts on the left-hand sector. It then becomes natural to define a local entropy. The condition of local equilibrium then fully determines the distance function and, subsequently, the geometry. When the temperature is independent of the position, Euclidean geometry results with a scale determined by the global temperature. Remarkably the scale increases as the temperature is lowered, leading to expanding distances at lower temperature.

There are a number of open issues worthwhile exploring further. This includes the consequence of the stronger technical condition on the involutive algebra imposed here and the issue of the modified distance function that invokes the trace rather than the operator norm. Another interesting point to explore further would be to investigate the class of Dirac operators for which (51) holds for all states and the implied associated geometries.
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