Bayesian Inference for Non-Parametric Extreme Value Theory

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Abstract: Statistical inference for extreme values of random events is difficult in practice due to low sample sizes and inaccurate models for the studied rare events. If prior knowledge for extreme values is available, Bayesian statistics can be applied to reduce the sample complexity, but this requires a known probability distribution. By working with the quantiles for extremely low probabilities (in the order of 10^{-2} or lower) and relying on their asymptotic normality, inference can be carried out without assuming any distributions. Despite relying on asymptotic results, it is shown that a Bayesian framework that incorporates prior information can reduce the number of observations required to estimate a particular quantile to some level of accuracy.

1. INTRODUCTION

The study of extremely rare statistical events is important to many areas of science and engineering. In meteorology and hydrology, the probability of extreme natural hazards like storms or droughts must be estimated to properly design human-built infrastructure that can withstand these dangerous events (Bousquet and Bernardara, 2021). In finance, it is essential to quantify investments’ risks and assess the likelihood of negative events, e.g., stock market crashes (Rocco, 2014). In wireless communication, a new generation of ultra-reliable and low-latency networks are being deployed for critical communication applications such as automated vehicles, where characterizing the rare packet loss events is essential for safe operation (Popovski et al., 2019). The fundamental problem in extreme value theory is the need to statistically quantify values of a process that are rarely or previously never observed. As such, it is necessary to extrapolate from observed to unobserved values (Bousquet and Bernardara, 2021). Extreme value theory provides methods to estimate probabilities of extreme events based on limited observations, but they can suffer from model mismatch or being hyperparameter dependent. For example, the Pickands-Balkeman-de Haan theorem provides a distribution for values above some threshold, but selecting the threshold and distribution parameters can be difficult (Mehrinia and Coleri, 2021).

One approach to avoid the problems of fitting distributions to rare events is to work purely with summary statistics that are estimated directly from observations without assuming an underlying distribution. Examples of such statistics include the mean and variance, which are informative statistics that can be estimated efficiently for any distribution. For extreme value theory, the quantiles for p-values with p ≈ 0 or p ≈ 1 are interesting statistics as they describe the “out of distribution” values. Summary statistics are inherently less informative than fitted distributions, which may be an issue for statistical inference. However, by choosing sufficient statistic for the application, this problem can be mitigated. Another issue is the excessive number of observations required to estimate some statistics. For example, to non-parametrically estimate the quantile for a p-value of p = 10^{-m}, it is required to have at least 10^m observations (Angjelichinoski et al., 2019).

This paper seeks to lessen the required number of observations to estimate rare-event statistics by relying on prior information. In particular, if prior information about a quantile is available, it is possible to combine this information with new observations to increase the accuracy of the estimated quantile through Bayesian inference.

The paper is structured as follows. First, the concept of continuous quantiles and how to estimate them are introduced in section 2. The focus here will be low-value quantiles where p ≈ 0, but the framework could be extended to studying high-values. Section 3 then introduces Bayesian inference for quantiles based on prior information. Numerical results are presented in section 4, where quantile estimates from the Bayesian approach are compared to a sample-based approach that does not rely on prior information. Section 5 concludes on the general applications of paper, and section 6 gives a perspective on wireless communication as an example of where prior information can be obtained.

2. NON-PARAMETRIC EXTREME VALUE THEORY

Consider the continuous random variable X, e.g., the duration of a drought, the share price of a company, or the signal-to-noise ratio (SNR) of a wireless communication system. A continuous quantile with p-value p ∈ [0, 1], denoted x_p, is defined as the value such that the probability of X being below x_p is exactly p, i.e.,

\[ P(X \leq x_p) = p. \]  (1)

For example, if x_p = −30 dB with p = 10^{-5} in a wireless communication system, then the probability of the SNR being below −30 dB is 10^{-5}. If the distribution of X is known, its quantiles can be obtained through the inverse cumulative distribution function (CDF). Thus, given the CDF \( F(x) = P(X \leq x) \), the quantile is given as \( x_p = F^{-1}(p) \), assuming that the inverse function exists.
Several non-parametric quantile estimators exist, including methods based on order statistics, kernel density estimates, and bootstrapping (or some combination of these methods) (Cheung and Lee, 2005). An order statistic-based method will be used here due to its tractable statistical properties, which is convenient for Bayesian inference. Thus, assume that \( n \) independent observations \( x^n = \{x_i\}_{i=1}^n \) of \( X \) are given. The observations are sorted in descending order such that
\[
x(1) \leq x(2) \leq \cdots \leq x(n),
\]
where \( (\cdot) \) denotes the sorted index. The \( t \)th order statistic of \( x^n \) is then simply defined as \( x(t) \). The sample quantile estimate based on \( x^n \) is
\[
\hat{x}_p = x(r), \quad r = \lfloor np \rfloor,
\]
where \( \lfloor \cdot \rfloor \) denotes the floor function. Intuitively, say \( n = 100 \) and \( p = 0.2 \), then the average probability that \( X \) is below the 20th order statistic is 20%. When either \( p \) is low or \( n \) is high, the case when \( r = 0 \) may occur. In this context, \( x(0) \) is undefined since the observations do not have sufficient about the quantile. From a statistical point of view, the order statistic is unbiased and admits an asymptotic normal distribution as \( n \to \infty \). This result uses the central limit theorem, and it can be shown that (Cheung and Lee, 2005)
\[
\hat{x}_p - x_p \xrightarrow{d} \mathcal{N} \left( 0, \frac{p(1-p)}{nf(x_p)^2} \right),
\]
where \( f \) is the probability density function (PDF) of \( X \) and \( d \) denotes convergence in distribution.

### 3. BAYESIAN INFERENCE

The result in (4) reveals that the variance of the quantile scales inversely with the number of observations \( n \). To increase the accuracy of the sample quantile, a Bayesian framework can be applied when prior information is available (as mentioned of this is given in section 6). Specifically, it is assumed that the prior distribution for the quantile is normal with known parameters, i.e.,
\[
x_p \sim \mathcal{N}(\mu, \sigma^2)
\]
Assume (for now) also that the variance of the sample quantile \( \sigma^2_n = \text{Var}[\hat{x}_p] = p(1-p)/(nf(x_p)^2) \) is known. The idea of the Bayesian framework is to combine prior information about the quantile with the new observations \( x^n \). To do so, motivated by the asymptotic distribution in (4), it is assumed that the sample quantile \( \hat{x}_p \) follows a normal distribution with mean \( x_p \) and variance \( \sigma^2_n \), i.e.,
\[
\hat{x}_p | x_p \sim \mathcal{N}(x_p, \sigma^2_n),
\]
which is the likelihood distribution. Since both the prior and likelihood distributions are normal, the posterior distribution is also a normal distribution whose parameters are available on closed form. To obtain the distribution, it is shown that
\[
\text{Var}[\hat{x}_p] = \text{Var}[E[\hat{x}_p | x_p]] + E[\text{Var}[\hat{x}_p | x_p]]
= \text{Var}[x_p] + E[\sigma^2_n] = \sigma^2 + \sigma^2_n,
\]
where (7) and (8) use the law of total expectation and (9) the law of total variance. The normal posterior distribution of \( x_p \) given \( \hat{x}_p \) is then given by its mean
\[
E[x_p | \hat{x}_p] = E[x_p] + \frac{\text{Cov}(x_p, \hat{x}_p)}{\text{Var}[\hat{x}_p]} (\hat{x}_p - E[x_p])
= \mu + \frac{\sigma^2}{\sigma^2 + \sigma^2_n} (\hat{x}_p - \mu),
\]
and variance
\[
\text{Var}[x_p | \hat{x}_p] = \text{Var}[x_p] - \frac{\text{Cov}(x_p, \hat{x}_p)^2}{\text{Var}[\hat{x}_p]}
= \sigma^2 - \frac{\sigma^4}{\sigma^2 + \sigma^2_n}
= \left( \frac{1}{\sigma^2} + \frac{1}{\sigma^2_n} \right)^{-1}.
\]
Thus, the Bayesian quantile estimate is the expectation (10) with variance according to (11). The expectation (10) is interpreted as a weighted sum of the prior and estimated quantile with the extreme cases \( E[x_p | \hat{x}_p] \to \mu \) for \( \sigma^2_n \to \infty \) (rely on the prior knowledge when the sample variance is high) and \( E[x_p | \hat{x}_p] \to \hat{x}_p \) for \( \sigma^2_n \to 0 \) (rely on the observations when the sample variance is low). The variance (11) features the inverse variances (also known as precision) and is proportional to the harmonic mean of \( \sigma^2 \) and \( \sigma^2_n \). Importantly, the posterior variance is less than \( \sigma^2 \) and \( \sigma^2_n \), so the posterior estimate is generally more precise than the sample quantile.

If the variance of the sample quantile \( \sigma^2_n \) is unknown, it can be estimated from the observations \( x^n \). One method for this is bootstrapping, where multiple values of \( \hat{x}_p \) are estimated from random samples of \( x^n \) (with replacement), and the sample variance of these then approximates \( \sigma^2_n \). However, due to the simple method of estimating the sample quantile, it is possible to perform analytic bootstrapping. Specifically, it can be shown that with infinite bootstrapping samples of size \( n \), the variance estimate for the \( r \)th order statistic tends to
\[
\hat{\sigma}^2_n = \frac{1}{n} \sum_{i=1}^{n} (x(i) - x(r))^2 w_{n,i}, \quad \text{with}
\]
\[
w_{n,i} = r \left( \frac{n}{r} \right) \int_{(i-1)/n}^{i/n} y^{r-1}(1-y)^{n-r} \, dy,
\]
and has a relative error of the order \( O(n^{-1/4}) \) (Cheung and Lee, 2005) \(^1\).

### 4. NUMERICAL RESULTS

Since the Bayesian inference relies on asymptotic results, it is not necessarily guaranteed to improve the quantile estimate particularly when \( n \) is low. This section will compare the accuracy of the sample quantile in (3) and the Bayesian \(^1\) (Cheung and Lee, 2005) presents a similar estimate that has a smaller relative error, although it is not used here as it relies on a hyper-parameter that is computationally complex to estimate.
The quantile in (3), 2) The Bayesian estimate in (10) assuming the 10
between the estimates \( \hat{\sigma}_X \) estimated with (12). Note that the PDF of
\( x \) is then estimated from \( \lambda \) distributed with rate
for different choices of prior variance
seen that estimating \( \sigma \) is high and vise versa for lower variance. Finally, it is
quantile for \( \sigma \) is about one order of magnitude better than the sample
sample methods are similar for \( \sigma \) performance of the Bayesian approaches, where the Bayesian and
prior variance is seen to significantly affect the perfor-
become comparable to the sample quantile approach. The

Fig. 1. Quantile estimation error measured by the RMSE
for different p-values \( p \), number of observations \( n \), and
prior variance \( \sigma^2 \). Results from the three different
methods marked by solid/dashed/dotted lines and
results for different values of \( \sigma^2 \) are denoted on the
plot.

Simulation The quantile estimators are tested on the
random variable \( X = \log(Y) \), where \( Y \) is exponentially
distributed with rate \( \lambda \). The quantile \( x_p \) is simulated 1000
times from (5) for each setting to analyze the average
performance over the prior. For each simulated quantile \( x_p \),
\( x^n \) is drawn from \( X = \log(Y) \), where \( F(x_p) = p \) is achieved
by setting the rate of \( Y \) to \( \lambda = -\log(1-p)e^{-x_p} \). \( x_p \) is
then estimated from \( x^n \) with three methods: 1) The sample
quantile in (3), 2) The Bayesian estimate in (10) assuming \( \sigma_n^2 \) is known, and 3) The Bayesian estimate where \( \sigma_n^2 \) is estimated with (12). Note that the PDF of \( X \) is \( f(x) = \lambda e^{-x} e^{-x} \), which is used in the second method to compute
\( \sigma_n^2 \) according to (4). The root-mean-square error (RMSE)
between the estimates \( \hat{x}_p \) and \( x_p \) is then evaluated over the
10^4 simulated quantiles. This experiment is repeated for
different choices of prior variance \( \sigma^2 \), for different p-
values \( p \), and for different number of observations \( n \).
The prior mean is \( \mu = 0 \) for all settings.

Results from the simulations are shown in Fig. 1 and reveal
a few interesting observations. When \( n \) is low, the Bayesian
approaches have lower errors, and as \( n \) increases, the errors
become comparable to the sample quantile approach. The
prior variance is seen to significantly affect the perfor-
mance of the Bayesian approaches, where the Bayesian and
sample methods are similar for \( \sigma^2 = 1 \), and the Bayesian
is about one order of magnitude better than the sample
quantile for \( \sigma^2 = 0.01 \). Intuitively, this is because the prior
information is less informative when the prior variance
is high and vise versa for lower variance. Finally, it is
seen that estimating \( \sigma_n^2 \) does not significantly impact the

5. CONCLUSION

This paper discusses the problems of estimating rare-
event statistics in the field of extreme value theory. To
avoid the problems of parametric methods, it is proposed
to work with summary statistics, e.g., quantiles. It is
shown that Bayesian inference can be used to lessen the
number of required samples to estimate specific quantiles
for some level of accuracy. It was observed that the
performance gain of the Bayesian approach depends on
the prior variance. It was also observed that estimating
the sample quantile variance did not significantly worsen
the performance compared to known variance for the
Bayesian approaches. In future works, the performance of
quantile estimation for other types of distributions and
the performance with uncertain prior information could
be investigated.

6. PERSPECTIVES TO WIRELESS
COMMUNICATION

In practice, prior information about the quantiles is esti-
mated from knowledge about the event. An example is in
wireless communication with the problem of determining
the SNR. Say that a mobile device at a known location
wants to estimate the statistical properties of the SNR.
The device can measure the SNR over a period of time.
Still, it may be infeasible to acquire a sufficient number
of samples to estimate rare event statistics, such as the
quantiles for low p-values. In this case, the mobile device
can rely on measurements from other mobile devices in
the area, i.e., prior information, to estimate SNR statistics
using fewer measurements. The statistical knowledge of
the SNR can then be used to select, e.g., a communication
rate that ensures a reliable transmission (Angjelichinoski
et al., 2019).

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