Thermodynamic behavior and stability of Polytropic gas

H. Moradpour, A. Abri, H. Ebadi

Research Institute for Astronomy and Astrophysics of Maragha (RIAAM), P.O. Box 55134-441, Maragha, Iran.
Astrophysics Department, Physics Faculty, University of Tabriz, Tabriz, Iran.

We focus on the thermodynamic behavior of Polytropic gas as a candidate for dark energy. We use the general arguments of thermodynamics to investigate its properties and behavior. We find that a Polytropic gas may exhibit the dark energy like behavior in the large volume and low temperature limits. It may also be used to simulate a fluid with zero pressure at the small volume and high temperature limits. Briefly, our study shows that this gas may be used to describe the universe expansion history from the matter dominated era to the current accelerating era. By applying some initial condition to the system, we can establish a relation between the Polytropic gas parameters and initial conditions. Relationships with related works has also been addressed.

I. INTRODUCTION

In the Einstein General Relativity framework, Friedmann-Robertson-Walker metric (FRW) and its conformal form help us get a suitable model for describing the universe expansion and its inhomogeneities in scales smaller than 100–Mpc, respectively [1,2]. In standard cosmology, the universe is born from a singularity called big bang and is inflated in primary moments. Thereinafter, the rate of this expansion is decreased, since Radiation and Matter play the role of dominated fluid, which is the determinant of the universe expansion rate, in subsequent eras [1]. Finally, the universe deals with an accelerated expansion with positive rate, about 13700–Myr after big bang [1,3–4]. Although the latest and current phase of universe expansion satisfies the thermodynamics stability conditions [7–11] but, the nature of dominated fluid, which supports this phase of expansion and called dark energy (DE), is a mysterious puzzle [12–14].

In addition to the mentioned puzzle, standard cosmology suffers from a series of other weaknesses such as the coincidence and fine tuning problems [1]. Among these attempts, some models include a varying DE candidate with equation of state [15,16]. In astrophysics, a Polytropic gas with equation of state

\[ P = K \rho^{(1+\frac{1}{n})}, \]

where

\[ \rho = \frac{U}{V}, \]

is its energy density, while \( U \) and \( V \) are the total energy of gas and the volume of its container, respectively, has numerous applications [17]. In addition, \( n \) called the Polytropic index and \( K \) is a constant [17]. An adiabatic gas [18] and a degenerate electron gas [17] are two examples of such gases. Since the equation of state of this gas is in the \( P = f(\rho) \) form, it attracts the attention of cosmologist to itself as a probable way to describe DE and the coincidence as well as the fine tuning problems [19–54].

Karami et al. investigated the mutual interaction between Polytropic gas (as the DE candidate) and CDM, and find out that for some even and positive Polytropic index and some positive values of \( K \), the Polytropic gas behaves as the Phantom dark energy [19]. Moreover, the generalized second law of thermodynamics is always satisfied by a universe filled with a Polytropic gas model of DE together with a CDM which interact with each other [20]. Authors in [21] used the first law of thermodynamics, and qualitatively show that a Polytropic gas may exhibit a DE behavior. In addition, they show that this model leads to a suitable fitting with observational data about the current expanding era [21]. In fact, Asadzadeh et al. have argued that a Polytropic gas model of DE with Polytropic index \( n < -1 \) and \( K < 0 \) is in suitable agreement with the observational data [22]. It is also shown that the Polytropic gas model of DE may simulate the new agegraphic as well as the holographic dark energy models and thus the current accelerating phase of universe [23,24]. Interaction between the Polytropic gas model of DE and dark matter in Kaluza-Klein cosmology has also been studied [25].

It is useful to mention here that in all of Refs. [19–34], authors investigate the Polytropic gas properties and behaviors by considering the cosmological setup. As a common result, all of them find that, in the FRW background, the Polytropic gas may be a suitable candidate for describing DE. Therefore, one may ask what is the origin of the DE like behavior of Polytropic gas? Does this ability of the Polytropic gas to describe DE is due to the gravitational theory used to describe the universe expansion? Indeed, does thermodynamical properties of Polytropic gas is similar with those of DE without considering a cosmological background? Such confusions have been arisen in studying the Chaplygin gas as well as its...
generalized and modified forms \[33, 36\]. In fact, authors in \[33, 36\] by focusing on the thermodynamic arguments, have been shown that such gases have enough ability for exhibiting the DE like behavior. Although, the equation of state of Polytropic gas is similar to that of the generalized Chaplygin gas but the physics behind them are completely different. Whiles, the Chaplygin gas model comes from the Russian physicist attempts to study the lifting force which applies on the plane wings in aerodynamics \[37\], the Polytropic gas models are introduced to study the hydrostatic equilibrium equation and thus the star evolution \[17\]. Therefore, it is not straightforward to generalize the results of Refs. \[35, 36\] to Polytropic gas models.

Here, we want to study the thermodynamics of a Polytropic gas confined to volume \( V \). In fact, we are eager to know that under which thermodynamic conditions a Polytropic gas exhibits the DE like behavior. Does this model satisfies the thermodynamic stability conditions, and does it pleases the thermodynamic expectations? Our study shows that the Polytropic gas may naturally exhibit the DE like behavior. Since we only use thermodynamics to study the behavior of Polytropic gas, our results are independent of considering the gravitational theory used to describe the universe expansion.

The paper is organized as follows. In the next section, we consider an adiabatic Polytropic gas and evaluate its thermodynamical properties such as its energy as a function of volume. The thermodynamic stability conditions are also addressed. Section III includes more debates about the thermodynamic stability conditions and some examples. In section IV, by applying initial conditions to the system we get a relation between the Polytropic index, \( K \) and initial conditions. Section V includes summary and concluding remarks.

II. ADIABATIC POLYTROPIC GAS EOS

The pressure of a fluid with energy \( U \) confined into a cylinder with volume \( V \) is evaluated as \[18\]

\[
\left( \frac{\partial U}{\partial V} \right)_S = -P. \tag{3}
\]

Combining Eqs. (1) and (2), and inserting the result into (3) to get

\[
\left( \frac{\partial U}{\partial V} \right)_S = -K \left( \frac{U}{V} \right)^{1+ \frac{n}{b}}. \tag{4}
\]

By taking integral from this equation, one finds the energy of Polytropic gas as

\[
U = (-1)^{-n} K^{-n} V^{1 - \frac{n}{b}} + b^{-n}, \tag{5}
\]

where \( b = b(S) \). It is also apparent that Eqs. (4) and (5) are also available, if \( K = K(S) \). By defining \( \delta = \frac{b}{K} \) and simple calculations, we get

\[
U = (-1)^{-n} K^{-n} V^{1 - \frac{n}{\delta}} + \delta^{-n}, \tag{6}
\]

which leads to

\[
\rho = \frac{U}{V} = (-1)^{n} K^{-n} (1 + \frac{V}{\delta})^{-n}, \tag{7}
\]

for the energy density of Polytropic gas. This equation leads to

\[
\rho \sim (-1)^{n} K^{-n}, \tag{8}
\]

and

\[
\rho \sim (-1)^{n} K^{-n} \frac{\delta}{V}, \tag{9}
\]

for \( n > 0 \) and \( n < 0 \), respectively, in the small volume limits. Since it seems that density should be positive for even \( n \) \[17, 19, 24\], these equations state that \( \delta \) should meet the \( \delta > 0 \) condition. Since \( \delta > 0 \), one can use Eq. (7) to conclude that, independent of \( n \), density is positive for \( K < 0 \). Moreover, inserting this equation into (11) to reach

\[
P = (-1)^{n+1} K^{-n} (1 + \frac{V}{\delta})^{-\left(n+1\right)}, \tag{10}
\]

which yields

\[
P = \frac{-\rho}{\left(1 + \left(\frac{V}{\delta}\right)^{\frac{1}{n}}\right)}, \tag{11}
\]

and

\[
\omega = \frac{P}{\rho} = -\frac{1}{\left(1 + \left(\frac{V}{\delta}\right)^{\frac{1}{n}}\right)}, \tag{12}
\]

as the corresponding pressure and the state parameter of Polytropic gas, respectively. Therefore, the state parameter of Polytropic gas meets the \(-1 \leq \omega \leq 0\) condition. It is interesting to note that for \( b = 0 \), one obtains

\[
P = -\rho = (-1)^{n+1} K^{-n}, \tag{13}
\]

and

\[
\omega = \frac{P}{\rho} = -1, \tag{14}
\]

which are independent of the system volume. Therefore, the \( b = 0 \) case is capable to explain the primary inflationary era and the current expanding phase as well as the anti de-Sitter spacetime. For example, consider a situation in which \( K > 0 \). While \( \frac{\delta}{K} = 0 \), on one hand, for even \( n \) this equation is compatible with the equation of state of the cosmological constant (\( \rho > 0 \)) which may support the de-Sitter spacetime. On the other hand, an odd \( n \) points to a fluid with \( \rho < 0 \) which may support the anti de-Sitter spacetime.
Whenever \( n > 0 \) and \( \delta \) does not diverge, one gets
\[
P \approx (-1)^{n+1} K^{-n} \simeq -\rho, \tag{15}
\]
for small volumes, which is the same as the result of the \( b = 0 \) case \cite{13}. In order to derive this equation, we used Eqs. (7) and (11). This result is in accordance with the primary inflationary era. It is interesting to note that the generalized Chaplygin gas has similar behavior at large volumes \cite{35}. For large volumes, Eqs. (7) and (11) lead to
\[
\rho \approx \frac{(-1)^n K^{-n} \delta}{V}, \tag{16}
\]
and
\[
P \approx \frac{(-1)^n K^{-n} \delta^{\frac{n+1}{n}}}{V^{\frac{n}{n+1}}}, \tag{17}
\]
respectively. We should note that since \( V \gg 1 \) and \( n > 0 \), \( \omega \sim 0 \). In fact, while length is proportional to the cosmological scale factor \( a \), \( V \propto a^3 \) and we get
\[
\rho \approx \frac{(-1)^n K^{-n}}{a^3}, \quad P \sim 0, \tag{18}
\]
which indicates a pressureless matter. This behavior of Polytropic gas at large volume is the same as that of the generalized Chaplygin gas in small volumes \cite{35}. Therefore, a Polytropic gas with \( n > 0 \) and \( b \neq 0 \) cannot simulate a fluid with \( P = -\rho \neq 0 \) at large volumes. Briefly, a Polytropic gas with \( n > 0 \), while \( \delta \) does not diverge, behaves as a fluid with \( \omega \simeq -1 \) and a pressureless matter at the small and large volumes limits, respectively.

Meanwhile, while \( n < 0 \) and \( \delta \) does not diverge, by using Eqs. (7) and (11) one gets
\[
P \approx 0, \quad \rho \approx \frac{(-1)^n K^{-n}}{a^3}, \tag{19}
\]
and
\[
P \approx (-1)^{n+1} K^{-n} \simeq -\rho, \tag{20}
\]
as the pressure and density of Polytropic gas in small and large volumes, respectively. In order to get \cite{19}, we assumed again \( V \propto a^3 \). Therefore, a Polytropic gas with \( n < 0 \) behaves as the pressureless matter and a fluid with \( \omega \simeq -1 \) at small and large volumes limits, respectively. This behavior is fully consistent with that of the generalized Chaplygin gas \cite{35}. Let us summarize the above results. The \( b = 0 \) case indicates a fluid with \( \omega = -1 \) and therefore, may be used to simulate the primary and current inflationary eras. Moreover, since the state parameter of a Polytropic gas with \( n > 0 \) is increased from \( \omega \simeq -1 \) (for small volumes) to \( \omega \simeq 0 \) (for large volumes), it may be useful to describe an expanding universe which expands from an inflationary like primary era to a matter dominated like era. Finally, since the state parameter of a Polytropic gas with \( n < 0 \) is decreased from \( \omega \simeq 0 \) (for small volumes) to \( \omega \simeq -1 \) (for large volumes), it may be useful to describe the universe expansion from the matter dominated era to the current expanding phase.

The thermodynamic stability condition or the convexity of the energy surface requires that \cite{18}
\[
\frac{\partial P}{\partial V} \bigg|_S \leq 0, \tag{21}
\]
and
\[
C_P \geq C_V \geq 0, \tag{22}
\]
where \( C_V \) and \( C_P \) are the heat capacities at constant volume and pressure, respectively. Additionally, since for a fluid with \( N \) particle
\[
C_P - C_V = \frac{TV\alpha^2}{NK_T}, \tag{23}
\]
where \( \kappa_T = \frac{1}{TV\rho} \) and \( \alpha \) are the isothermal compressibility and the coefficient of thermal expansion, respectively \cite{18}, if \( \kappa_T > 0 \) and \( C_V \geq 0 \) are simultaneously satisfied condition (22) is also met. The latter means a thermodynamic system, which satisfies Eq. (21),
\[
C_V > 0, \tag{24}
\]
and
\[
\frac{\partial P}{\partial V} \bigg|_T \leq 0, \tag{25}
\]
also meets the thermodynamic stability condition. For the \( b = 0 \) case, from Eq. (13), simple calculations lead to \( (\frac{\partial P}{\partial V})_S = 0 \) and therefore, the stability condition (21) is marginally satisfied. Now, using Eq. (10) to reach
\[
\frac{\partial P}{\partial V} \bigg|_S = -\frac{\delta \frac{\partial (1 + \frac{n}{a^3} V)}{\partial V}}{1 + (\delta V)^{-\frac{n}{a^3}}}. \tag{26}
\]
Bearing the \( \delta > 0 \) condition together with Eqs. (7) and (11) in mind, a Polytropic gas with \( \rho > 0 \) satisfies condition (21), if \( -1 \leq n < 0 \). In addition, a Polytropic gas with negative density (\( \rho < 0 \)) satisfies condition (21) if its Polytropic index meets either the \( n > 0 \) or \( n < 0 \) conditions. It is also apparent that, just the same as the \( b = 0 \) case, the \( n = -1 \) case marginally satisfies condition (21). In the next section, we investigate the quality of validity of Eqs. (24) and (25).

### III. THERMAL POLYTROPIC GAS EOS

One should determine the thermal equation of state \( P(T,V) \) to investigate the quality of validity of Eq. (25). In addition, since
\[
C_V = T \frac{\partial S}{\partial T} \bigg|_V, \tag{27}
\]
we need also to determine the thermal equation of state $S = S(T, V)$ in order to study the behavior of $C_V$ \cite{Chaplygin}. Here, since in cosmological setups authors generally set $K$ to a constant value \cite{17, 19, 24}, we only consider the situation in which $\frac{dS}{dK} = 0$ which means that $K$ is constant. In order to evaluate the temperature of Polytropic gas, inserting Eq. (6) into

\[ T = \left(\frac{\partial U}{\partial S}\right)_V, \quad (28) \]

and get

\[ T = (-1)^{n+1}nV^{1+\frac{1}{n}}(K + bV^{\frac{1}{n}})^{-(n+1)} \frac{db}{dS}, \quad (29) \]

as the temperature of Polytropic gas. Bearing Eqs. (7) and (11) together with the definition of $\delta$ in mind, this equation can be written as

\[ T = -n\rho V^{1+\frac{1}{n}} \frac{db}{(1 + (\frac{V}{\rho})^{\frac{1}{n}})} dS, \quad (30) \]

In order to continue our analysis, we need an exact form for $b$. To achieve this aim, bearing Eq. (5) in mind, a dimensional analysis shows

\[ [b]^{-n} = [U]. \quad (31) \]

In addition, from Eq. (50) we have $db \propto dS$. Finally, since $[U] = [TS]$ in thermodynamics \cite{18}, a primary simple selection which satisfies Eq. (31) is

\[ b = (T, S)^{-\frac{1}{n}}, \quad (32) \]

where $T_s$ is a universal constant with temperature dimension, which should be evaluated from other parts of physics such as statistical mechanics or experimental data. Such analysis yields similar results for generalized Chaplygin gas and can be found in Ref. \cite{33}. Taking derivative with respect to $S$ from this equation to get

\[ \frac{db}{dS} = -\frac{1}{n}T_s^{-\frac{1}{n}} S^{-\frac{1}{n}-1}. \quad (33) \]

Inserting this result into (30) to obtain

\[ T = \frac{\rho V^{1+\frac{1}{n}}}{(1 + (\frac{V}{\rho})^{\frac{1}{n}})} T_s^{-\frac{1}{n}} S^{-\frac{1}{n}-1}. \quad (34) \]

Now, inserting Eqs. (7) and (32) into (33) to get

\[ T = (-1)^{n+1}nV^{1+\frac{1}{n}}(T_s^{-\frac{1}{n}} S^{-\frac{1}{n}-1})[K + T_s^{-\frac{1}{n}} S^{-\frac{1}{n}-1}]^{-(n+1)}, \quad (35) \]

which leads to

\[ S = (\frac{T}{T_s})^{\frac{1}{n+1}} (-1)^{\frac{n}{n+1}} - 1]^{n} \frac{V}{K^n T_s}. \quad (36) \]

for the entropy of Polytropic gas. By combining Eqs. (11), (34) and (36) one gets

\[ P = K^{-n}(-1)^{n+1}(1 + [(\frac{T}{T_s})^{\frac{1}{n+1}} (-1)^{\frac{n}{n+1}} - 1]^{n+1}), \quad (37) \]

as the last thermal equation of state ($P = P(T, V)$) needed to investigate the system. From this equation it is apparent that $(\frac{\partial P}{\partial T})_V = 0$ which indicates condition (26) is marginally satisfied by the Polytropic gas. Finally, since $P = P(T, \frac{\partial P}{\partial T})_V$ and $(\frac{\partial P}{\partial T})_V$ are also zero which means that there is no critical point in this situation \cite{18}. It is useful to note that this conclusion is the direct result of using Eq. (32) to obtain $b$. By inserting Eq. (37) into Eqs. (11) and (3), we get

\[ \rho = \frac{(-K)^{-n}}{[1 + ((\frac{V}{\rho})^{\frac{1}{n}})^{\frac{n}{n+1}} (-1)^{\frac{n}{n+1}} - 1]^{n}}, \quad (38) \]

and

\[ U = \frac{(-K)^{-n}V}{[1 + ((\frac{V}{\rho})^{\frac{1}{n}})^{\frac{n}{n+1}} (-1)^{\frac{n}{n+1}} - 1]^{n}}, \quad (39) \]

as the energy density and total energy of Polytropic gas, respectively. As we know, the entropy of a thermodynamical system should be positive \cite{18}. Moreover, Eqs. (37) and (38) can be used to get

\[ \omega = -\frac{1}{1 + ((\frac{V}{\rho})^{\frac{1}{n}})^{\frac{n}{n+1}} (-1)^{\frac{n}{n+1}} - 1} \quad (40) \]

as another relation for the state parameter of Polytropic gas. This equation indicates that for a Polytropic gas with Polytropic index satisfying the $(-1)^{\frac{n}{n+1}} = 1$ condition and temperature $T$, which meets the $0 \leq T \leq T_s$ range, the state parameter encounters the $-1 \leq \omega \leq 0$ range. By inserting this equation into Eq. (36), one reach

\[ S = \frac{V}{K^n T_s}(-\frac{\omega}{\omega + 1})^n. \quad (41) \]

for entropy. Hence, since from Eq. (12) $\omega$ meets the $-1 \leq \omega \leq 0$ condition, the $S > 0$ condition is met if $K^n T_s > 0$. Now, using Eq. (27) to get

\[ C_V = n\frac{V}{K^n T_s}(-1)^{\frac{n}{n+1}}[(\frac{T}{T_s})^{\frac{1}{n+1}} (-1)^{\frac{n}{n+1}} - 1]^{n-1}, \quad (42) \]

for the heat capacity at constant volume, which can be rewritten as

\[ C_V = n\frac{S[1 + ((\frac{V}{\rho})^{\frac{1}{n}})]^{\frac{n}{n+1}}}{K\rho^{\frac{1}{n+1}}}, \quad (43) \]

where we have used Eqs. (34) and (36) to obtain this equation. Now, inserting Eqs. (11) into this equation to obtain

\[ C_V = n\frac{V[1 + ((\frac{V}{\rho})^{\frac{1}{n}})]^{\frac{n}{n+1}}}{K\rho^{\frac{1}{n+1}}}(-\frac{\omega}{\omega + 1})^n. \quad (44) \]

Therefore, the $C_V > 0$ condition is satisfied if \( \frac{n}{K^n T_s \rho^{\frac{1}{n+1}}} > 0 \). By combining this result with the result obtained from the $S > 0$ condition, we get if the \( \frac{n}{K^n T_s \rho^{\frac{1}{n+1}}} > 0 \) condition is satisfied, then the $S > 0$ and $C_V > 0$ conditions are simultaneously satisfied.
Some examples

Consider a Polytropic gas with \( n = -\frac{2k}{2k+1} \), where \( k \) is an integer and positive number, which leads to \( \rho > 0 \). From Eq. (26), it is obvious that this gas satisfies condition (21). Eq. (41) indicates that condition (21) is fulfilled. Bearing Eq. (40) in mind, since Eq. (12) indicates \( K < 0 \), bearing the \( \delta \) definition in mind, in order to get positive values for \( \delta \), \( b \) should meet the \( b < 0 \) condition. Considering Eq. (32), the latter leads to \( T_s < 0 \). From Eqs. (11) and (14), it is apparent that the \( S > 0 \) and \( C_V > 0 \) conditions are simultaneously met in this situation, respectively. Bearing Eq. (10) in mind, in order to preserve the \( -1 \leq \omega \leq 0 \) condition, which comes from Eq. (12), one gets \( 0 \leq T \leq T_s \) as a permissible range for temperature. The latter is fully consistent with physical expectancy about positivity of \( T \). Finally, we may conclude that the investigated Polytropic gas may provide a suitable model for describing the sources of universe expansion in the matter and current accelerated eras.

As another example, focus on a Polytropic gas with positive density, which yields \( P < 0 \), and \( n = -\frac{2k+1}{2k+3} \), where \( k \) is again a positive integer number. Since we assumed \( \rho > 0 \), Eq. (17) implies \( K < 0 \). Eq. (29) shows that condition (21) is fulfilled. Bearing the \( \delta \) definition in mind, in order to get positive values for \( \delta \), \( b \) should meet the \( b < 0 \) condition. Considering Eq. (32), the latter leads to \( T_s < 0 \). From Eqs. (11) and (14), it is apparent that the \( S > 0 \) and \( C_V > 0 \) conditions are simultaneously met in this situation, respectively. Bearing Eq. (10) in mind, in order to preserve the \( -1 \leq \omega \leq 0 \) condition, which comes from Eq. (12), one gets \( 0 \leq T \leq T_s \), as a permissible range for temperature. The latter is fully consistent with physical expectancy about positivity of \( T \). Finally, we may conclude that the investigated Polytropic gas may provide a suitable model for describing the sources of universe expansion in the matter and current accelerated eras.

Finally, consider a Polytropic gas with \( n < -1 \) and \( K < 0 \). According to (1) and (10), it is obvious that \( \rho > 0 \) and \( P < 0 \), respectively. From Eq. (26), it is apparent that condition (21) is not satisfied. Also, it is easy to show that the \( S > 0 \) and \( C_V > 0 \) conditions may be met. For example, if \( T_s > 0 \) and \( K^n > 0 \) then the \( S > 0 \) and \( C_V > 0 \) conditions are satisfied. Bearing Eqs. (12) and (10) in mind, this gas behaves as a pressureless matter and a fluid with \( \omega \sim -1 \) for small volumes and at the \( T \rightarrow T_s \) limit and for large volumes and at the \( T \rightarrow 0 \) limit, respectively. It is shown that such a Polytropic gas may lead to a satisfactory fitting with the observational data [22].

IV. THE POLYTROPIC GAS PARAMETERS FROM THERMODYNAMIC ARGUMENTS

Consider a Polytropic gas with initial condition \( V = V_0, P = P_0, \rho = \rho_0 \) and \( T = T_0 \). From Eq. (30) we get

\[
b = -(\rho_0^{-\frac{1}{n}} + K)\frac{1}{V_0^n}.
\]

(45)

Now, inserting this result into Eqs. (7) and (11) to obtain

\[
\rho = \rho_0(-K\rho_0^{\frac{1}{n}} + (1 + K\rho_0^{\frac{1}{n}})(\frac{V}{V_0})^{\frac{1}{n}})^{-n},
\]

(46)

and

\[
P = K\rho^{1+\frac{1}{n}} = K\rho_0^{1+\frac{1}{n}}(-K\rho_0^{\frac{1}{n}} + (1 + K\rho_0^{\frac{1}{n}})(\frac{V}{V_0})^{\frac{1}{n}})^{-n+1},
\]

(47)

for the density and pressure of Polytropic gas, respectively. Here, we define new parameters \( \varepsilon, v, p, \gamma, t \) and \( t_s \) as

\[
\varepsilon = \frac{\rho}{\rho_0}, \quad v = \frac{V}{V_0}, \quad p = \frac{P}{K^{-n}},
\]

\[
\gamma = -K\rho_0^{\frac{1}{n}}, \quad t = \frac{T}{T_0}, \quad t_s = \frac{T_s}{T_0}.
\]

(48)

It is useful to mention that \( \gamma \) is nothing but \( \tilde{K} \) defined in [22]. Therefore, Eqs. (46) and (47) can be written as

\[
\varepsilon = (\gamma + (1 - \gamma)v^n)^{-n},
\]

(49)

and

\[
p = \left[(-1 + [\frac{1}{T} + \frac{1}{(1 - \gamma)v^n}] - 1)^{-1}\right]^{-n+1},
\]

(50)

respectively. Moreover, from Eq. (17) we obtain

\[
p = \frac{P}{K^{-n}} = K^{n+1}\rho^{\frac{1}{n+1}} = \frac{\left(-1\right)^{n+1}\lambda^{n+1}}{(\gamma + (1 - \gamma)v^n)^{n+1}}.
\]

(51)

Inserting initial conditions into Eqs. (50) and (51), and equating them to each other to get

\[
t_s = \left(-1\right)^n(1 - \gamma)^{n+1}.
\]

(52)

In fact, this equation helps us to relate \( K, n, T_s, T_0 \) and \( \rho_0 \) to each other. Now, consider a Polytropic gas with \( \rho > 0 \) and \( n = -\frac{2k}{2k+1} \), where \( k \) is an integer and positive number. Now, combining Eqs. (48) and (52) to obtain

\[
K = \left[\frac{1}{T_s}\right]^{\frac{1}{n+1}} - 1\rho_0^{-\frac{1}{n}}.
\]

(53)

Therefore, for \( 0 \leq T_0 < T_s \), since \( k > 0 \), this equation indicates \( K < 0 \) which is in line with the results obtained in the previous section. Indeed, for every Polytropic index \( n \) we get

\[
K = \left(\frac{T_0}{T_s}\right)^{\frac{1}{n+1}}(1 - \gamma)^{\frac{n+1}{n}} - 1, \quad \rho_0^{-\frac{1}{n}}.
\]

(54)
which indicates that in order to get real values for $K$, whiles $T_0 > 0$, $n$ should meet the $(-1)^{\frac{n}{K + 1}} = \pm 1$ condition. For example, if $0 \leq T_0 < T_*$ and $n$ meets the $(-1)^{\frac{n}{K + 1}} = 1$ condition, we get $K < 0$ and thus $\gamma > 0$ meaning that $\tilde{K} > 0$ which is fully consistent with the results obtained in Ref. [22]. Loosely speaking, by applying some initial conditions on the Polytropic gas we get a relation between the initial conditions and the Polytropic gas parameters including the Polytropic index ($n$) and constant $K$.

V. SUMMARY AND CONCLUDING REMARKS

Here, we have considered an adiabatic Polytropic gas and found out its energy density and pressure by using pressure definition in thermodynamics and the equation of state of Polytropic gas, respectively. Thereinafter, we got the corresponding relations for the state parameter as well as the total energy of Polytropic gas. In addition, by investigating the asymptotic behavior of energy density of Polytropic gas for small and large volumes, and using the positivity of its density for even Polytropic indexes condition $[17, 19, 24]$, we got the $\delta > 0$ condition for parameter $\delta$ appeared in relations. Our study shows that the state parameter of Polytropic gas meets the $-1 \leq \omega \leq 0$ range for the finite values of $\delta$. In continue, we reviewed the thermodynamic stability condition which comes from the energy surfaces convexity condition $[18]$. In order to investigate all of the stability conditions, we used dimensional analysis to get a primary relation for $b$ and thus temperature. Due to our special choice for temperature, no critical point observed for Polytropic gas. We should note that this result may be changed by choosing different functions for $b$. We also pointed to the entropy of Polytropic gas, and get to the $K^nT_* > 0$ condition to have positive entropy. Our study shows that whenever the $\frac{n}{K + 1} > 0$ condition is satisfied, the $C_V > 0$ condition is also met by Polytropic gas. We have pointed to the behavior of some Polytropic gases, and found out that a Polytropic gas with $n = -\frac{2k}{K + 1}$, where $k$ is a positive integer number, and $\rho > 0$ satisfies all of the stability conditions if $T_0 > 0$ and $K < 0$. Under such criterion, the state parameter and temperature of Polytropic gas meet the $-1 \leq \omega \leq 0$ and $0 \leq T \leq T_*$ ranges, respectively. Polytropic gas with $n = -\frac{2k}{K + 1}$, where $k$ is a positive integer number, and $K < 0$ is also addressed. We found out that the $S > 0$ and $C_V > 0$ conditions indicate $T_* < 0$. Finally, we saw that its temperature should meet the $0 \leq T \leq T_*$ range in order to preserve the $-1 \leq \omega \leq 0$ condition obtained from Eq. [12]. Our study also shows that a Polytropic gas with $n < -1$ and $K < 0$ may satisfy the $S > 0$ and $C_V > 0$ conditions whiles $T_* > 0$ and $\delta$ is considered as a finite and well-defined quantity. Such a Polytropic gas has positive density and behaves as a pressureless matter for small volumes and at the $T \to T_*$ limit and a fluid with $\omega \to -1$ for large volumes and at the $T \to 0$ limit, whiles $K^n > 0$. It is worthwhile to mention here that a Polytropic gas with $n < -1$ and $K < 0$ has an appropriate fitting with observational data $[22]$. Finally, by imposing some initial conditions to under investigation Polytropic gas, we could find a relation between the Polytropic gas parameters, including $n$ and $K$, and the initial conditions applied on system.

Acknowledgments

The work of H. M. has been supported financially by Research Institute for Astronomy & Astrophysics of Maragha (RIAAM) under research project No. 1/4165–5.

[1] M. Roos, *Introduction to Cosmology* (John Wiley and Sons, UK, 2003).
[2] N. Riazi, H. Moradpour and A. Amiri, Prog. Theor. Phys. 126, 6 (2011).
[3] A. G. Riess, et al., Astron. J. 116, 1009 (1998).
[4] S. Perlmutter, et al., Astrophys. J. 517, 565 (1999).
[5] P. de Bernardis, et al., Nature. 404, 955 (2000).
[6] S. Perlmutter, et al., Astrophys. J. 598, 102 (2003).
[7] S. Nojiri and S. D. Odintsov, Phys. Rev. D 70, (2004) 103522.
[8] I. H. Brevik, S. Nojiri and S. D. Odintsov, Phys. Rev. D 70, (2004) 043520.
[9] R. Recalce and D. Pavón, Gen. Relativ. Grav. 44, 685 (2012).
[10] H. Moradpour, A. Sheykhi, N. Riazi and B. Wang, AHEP. 2014, 718583 (2014).
[11] H. Moradpour and N. Riazi, Int. J. Theor. Phys. DOI 10.1007/s10773-015-2659-2 (2015).
[12] S. Capozziello and V. Faraoni, *Beyond Einstein Gravity* (Springer, NY, 2011).
[13] K. Bamba, S. Capozziello, S. Nojiri and S. D. Odintsov, Astrophys. Space. Sci. 342, (2012) 155.
[14] M. Li. X. D. Li, S. Wang and and Y. Wang, THE UNIVERSE. 1, (2013) 4.
[15] S. Nojiri, S. D. Odintsov, Phys. Rev. D 68, 123512 (2003).
[16] L. L. Jenkins, V. Zhdanov and E. J. Stukalo, Phys. Rev. D 90, 023529 (2014).
[17] J. Christensen-Dalsgard, *Lecture Notes on Stellar Structure and Evolution* (6th edn. Aarhus University Press, Aarhus 2004).
[18] H. B. Callen, *Thermodynamics and Introduction to Thermostatics* (New York: John Wiley and Sons, 1985).
[19] K. Karami, S. Ghaffari and J. Fehti, Eur. Phys. J. C 64, 85 (2009).
[20] K. Karami and S. Ghaffari, Phys. Lett. B 688, 125 (2010).
[21] K. Kleidis and N. K. Spyrou, Astron. Astrophys. 576, 23
[22] S. Asadzadeh, Z. Safari, K. Karami and A. Abdolmaleki, Int. J. Theor. Phys. 53, 1248 (2014).
[23] K. Karami and A. Abdolmaleki, Astrophys. Space Sci. 330, 133 (2010).
[24] M. Taji and M. Malekjani, Int. J. Theor. Phys. 52, 3405 (2013).
[25] K. S. Adhav, Eur. Phys. J. Plus, 126, 127 (2011).
[26] M. Malekjani, A. Khodam-Mohammadi and M. Taji, Int. J. Theor. Phys. 50, 312 (2011).
[27] K. Karami and A. Abdolmaleki, Journal of Physics: Conference Series. 375, 032009 (2012).
[28] K. Karami and M.S. Khaledian, Int. J. Mod. Phys. D 21, 1250083 (2012).
[29] M. Malekjani and A. Khodam-Mohammadi, Int. J. Theor. Phys. 51, 3141 (2012).
[30] M. Malekjani, Int. J. Theor. Phys. 52, 2674 (2013).
[31] M. Khurshudyan, J. Sadeghi, R. Myrzakulov, Antonio Pasqua and H. Farahani, AHEP. 2014, 878092 (2014).
[32] M. Azizur Rahman and M. Ansari, Astrophys. Space Sci. 354, 617 (2014).
[33] M. Salti, I. Acikgoz and H. Abedi, Chine. J. Phys. 52, 3 (2014).
[34] S. Sarkar, Int. J. Theor. Phys. DOI 10.1007/s10773-015-2682-3 (2015).
[35] F. C. Santos, M. L. Bedran and V. Soares, Phys. Lett. B 636, 86 (2006).
[36] F. C. Santos, M. L. Bedran and V. Soares, Phys. Lett. B 646, 215 (2007).
[37] S. Chaplygin, Sci. Mem. Moscow Univ. Math. 21, 1 (1904).