DEGREES OF MAPS BETWEEN ISOTROPIC GRASSMANN MANIFOLDS

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Abstract. Let \( \tilde{I}_{2n,k} \) denote the space of \( k \)-dimensional, oriented isotropic subspaces of \( \mathbb{R}^{2n} \), called the oriented isotropic Grassmannian. Let \( f: \tilde{I}_{2n,k} \to \tilde{I}_{2m,l} \) be a map between two oriented isotropic Grassmannians of the same dimension, where \( k, l \geq 2 \). We show that either \( (n, k) = (m, l) \) or \( \deg f = 0 \). Let \( \tilde{G}_{m,l} \) denote the oriented real Grassmann manifold. For \( k, l \geq 2 \) and \( \dim \tilde{I}_{2n,k} = \dim \tilde{G}_{m,l} \), we also show that the degree of maps \( g: \tilde{G}_{m,l} \to \tilde{I}_{2n,k} \) and \( h: \tilde{I}_{2n,k} \to \tilde{G}_{m,l} \) must be zero.

1. Introduction

It has been proved in [5] that maps between two different oriented real Grassmann manifolds of the same dimension cannot have non-zero degree, provided the target space is not a sphere. A similar result is obtained for complex Grassmann manifolds in [4], when the map is a morphism of projective varieties. For arbitrary maps, this result has been verified for the complex Grassmann manifolds for many cases in [5] and [6].

In this paper we consider the analogous question for the space \( \tilde{I}_{2n,k} \) of oriented \( k \)-dimensional isotropic subspaces of a symplectic vector space of dimension \( 2n \). The oriented isotropic Grassmannian was considered in [3] and its cohomology was computed with real coefficients. Their method involves identifying \( \tilde{I}_{2n,k} \) as a homogeneous space \( \tilde{I}_{2n,k} \cong U(n)/(SO(k) \times U(n-k)) \). One may similarly consider \( I_{2n,k} \), the isotropic Grassmannian of \( k \)-dimensional isotropic subspaces of a symplectic \( 2n \) dimensional vector space, which is \( \cong U(n)/(O(k) \times U(n-k)) \). It turns out that the isotropic Grassmannian is orientable if and only if \( k \) is odd ([4]). In this paper we consider maps between oriented isotropic Grassmannians of the same dimension and prove

Theorem 1.1. Let \( n, k, m, l \) be integers such that \( 2 \leq l \leq m \) and \( \dim \tilde{I}_{2n,k} = \dim \tilde{I}_{2m,l} \). Let \( f: \tilde{I}_{2n,k} \to \tilde{I}_{2m,l} \). Then either \( (n, k) = (m, l) \) or \( \deg f = 0 \).

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(see Theorem 4.1). Note that \( \widetilde{I}_{2n,1} \simeq S^{2n-1} \) and so it is possible to get maps of arbitrary, non-zero degree, \( \phi : \widetilde{I}_{2n,k} \rightarrow \widetilde{I}_{2m,1} \) whenever \( \dim(\widetilde{I}_{2n,k}) = 2m - 1 \). We also prove

**Theorem 1.2.** Consider maps \( h : \widetilde{I}_{2n,k} \rightarrow \mathbb{R}\widetilde{G}_{m,l} \) and \( g : \mathbb{R}\widetilde{G}_{m,l} \rightarrow \widetilde{I}_{2n,k} \), where \( 2 \leq l \leq m \), \( 2 \leq k \leq n \) and \( \dim \widetilde{I}_{2n,k} = \dim \mathbb{R}\widetilde{G}_{m,l} \). Then \( \deg g = \deg h = 0 \).

The main technique used to prove the statements above is the result that if \( f : X \rightarrow Y \) (with \( \dim X = \dim Y \)) is a map of non-zero degree, then \( f^* : H^*(Y; \mathbb{Q}) \rightarrow H^*(X; \mathbb{Q}) \) is a monomorphism. We obtain some results on the structure of the cohomology ring of \( \widetilde{I}_{2n,k} \) to deduce the above theorems.

The paper is organised as follows. In section 2 we recall the description of the spaces \( I_{2n,k} \) and \( \widetilde{I}_{2n,k} \) and express them as homogeneous spaces. In section 3 we compute the cohomology of \( \widetilde{I}_{2n,k} \). In section 4 we prove the main theorems.

### 2. ISOTROPIC GRASSMANNIAN

In this section we set up the relevant notation and describe the isotropic Grassmannian as a homogeneous space. For \( F = \mathbb{R} \) or \( \mathbb{C} \) and \( n \in \mathbb{N} \), let \( F^n \) denote the \( n \)-dimensional \( F \)-vector space (upto isomorphism). Further, let \( M(n, F) \) denote the group of linear maps \( F^n \rightarrow F^n \), and let \( GL(n, F) \) denote the group of automorphisms \( F^n \rightarrow F^n \). Let \( U(n) := U(n; \mathbb{C}) \) denote the group of unitary linear transformations \( \mathbb{C}^n \rightarrow \mathbb{C}^n \), and \( O(n) := O(n; \mathbb{R}) \) denote the group of orthogonal linear transformations \( \mathbb{R}^n \rightarrow \mathbb{R}^n \).

Choose a symplectic form \( \omega \) on \( \mathbb{R}^{2n} \). Let \( Sp(n) := Sp(n; \mathbb{R}) \) denote the set of linear transformations \( \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n} \) which preserve this symplectic form \( \omega \). Choose a basis \( \mathcal{B} = (e_1, \cdots, e_n; f_1, \cdots, f_n) \), of \( \mathbb{R}^{2n} \), such that with respect to this basis, \( \omega \) can be written as:

\[
\omega = de_1 \wedge df_1 + \cdots + de_n \wedge df_n
\]

We coordinatise \( \mathbb{R}^{2n} \) with respect to this basis and identify \( \mathbb{R}^{2n} \) with \( \mathbb{C}^n \) via the following map:

\[
r : \mathbb{R}^{2n} \rightarrow \mathbb{C}^n
\]

\[
(x_1, \cdots, x_n; y_1, \cdots, y_n) \mapsto (x_1 + iy_1, \cdots, x_n + iy_n)
\]

Then, \( r \) induces a map \( M(n; \mathbb{C}) \rightarrow M(2n; \mathbb{R}) \). By abuse of notation, we will also call this map \( r \). Let \( Sp(n) := Sp(n; \mathbb{R}) \in M(2n; \mathbb{R}) \) denote the set of isomorphisms which preserve the symplectic forms \( \omega \). Then the image of \( U(n) \) under \( r \) lies in \( Sp(n) \). Hence the usual action of \( U(n) \) on \( \mathbb{C}^n \) induces an action of \( U(n) \) on \( \mathbb{R}^{2n} \) via symplectic
morphisms.

Define the isotropic Grassmannian, $I_{2n,k}$, to be the space of $k$-dimensional isotropic vector subspaces of $\mathbb{R}^{2n}$. Then one has the following proposition.

**Proposition 2.1.** The isotropic Grassmannian, $I_{2n,k}$, is diffeomorphic to the quotient $U(n)/(O(k) \times U(n-k))$.

*Proof.* Note that $U(n)$ acts on $I_{2n,k}$. Consider any $k$-dimensional isotropic subspace, $V \subset \mathbb{R}^{2n}$. Via the identification of $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, the complement, denoted by $W$ say, of $V \oplus iV$ with respect to the form $\omega$, is a complex subspace of $\mathbb{C}^n$. Moreover, $(V \oplus iV) \oplus W \cong \mathbb{C}^n$ is a decomposition with respect to the standard Hermitian inner product on $\mathbb{C}^n$.

Now, given any two $k$-dimensional, isotropic subspaces $V$ and $V'$, one has identifications:

$$(V \oplus iV) \oplus W \cong \mathbb{C}^n$$

$$(V' \oplus iV') \oplus W' \cong \mathbb{C}^n$$

Therefore, one can choose an orthogonal transformation $\varphi: V \to V'$, which takes $V$ to $V'$ and an isometry $\psi$ which takes $W$ to $W'$. Since $\varphi$ is orthogonal, $i\varphi$ will take $iV$ to $iV'$. Thus one obtains an isometry $\mathbb{C}^n$ to $\mathbb{C}^n$ which takes $V$ to $V'$, and hence, the action of $U(n)$ is transitive.

Let $V_k$ denote the $k$-dimensional subspace of $\mathbb{R}^{2n}$ generated by the basis vectors $e_1, \cdots, e_k$. Then any isometry $A \in U(n)$ with $A(V_k) = V_k$, is orthogonal when restricted to $V_k$. Additionally, it gives isomorphisms $V_k \oplus iV_k \to V_k \oplus iV_k$ and $(V_k \oplus iV_k)^\perp \to (V_k \oplus iV_k)^\perp$. Therefore, the stabilizer of $V_k$ is $O(k) \times U(n-k)$.

It follows that $I_{2n,k}$ is a $2k(n-k) + [k(k+1)/2]$-dimensional manifold. In fact, for $k = 1$, $I_{2n,1}$ is the real projective space, $\mathbb{RP}^{2n-1}$. One notes that $I_{2n,k}$ is orientable if and only if $k$ is odd ([1]).

We consider $\tilde{I}_{2n,k}$, the space of $k$-dimensional, oriented, isotropic subspaces of $\mathbb{R}^{2n}$, called the oriented isotropic Grassmannian. The oriented isotropic Grassmannian, $\tilde{I}_{2n,k}$, is again a $2k(n-k) + [k(k+1)/2]$-dimensional manifold. As in Proposition 2.1, one has:

**Proposition 2.2.** The isotropic Grassmannian, $\tilde{I}_{2n,k}$, is diffeomorphic to the quotient $U(n)/(SO(k) \times U(n-k))$. 

3. Cohomology of the Oriented Isotropic Grassmannian

In this section we compute the cohomology of the oriented isotropic Grassmannian, \( \tilde{I}_{2n,k} \). The cohomology with \( \mathbb{R} \) coefficients was computed in [3] using formulas for the real cohomology of homogeneous spaces. We compute the same algebraically, fixing appropriate notation along the way. We use the Serre spectral sequence and the following fibrations:

\[
\begin{align*}
\tilde{I}_{2n,k} & \longrightarrow BSO(k) \times BU(n-k) \longrightarrow BU(n) \\
U(n) & \longrightarrow \tilde{I}_{2n,k} \longrightarrow BSO(k) \times BU(n-k)
\end{align*}
\]

The first fibration induces the Serre spectral sequence with \( E_2^{p,q} \) term given by

\[(3.1) \quad E_2^{p,q} = H^p(BU(n); \mathbb{Q}) \otimes H^q(\tilde{I}_{2n,k}; \mathbb{Q}) \]

which converges to \( H^{p+q}(BSO(k) \times BU(n-k); \mathbb{Q}) \).

The second fibration induces the Serre spectral sequence with \( E_2^{p,q} \) term given by

\[(3.2) \quad E_2^{p,q} = H^q(U(n); \mathbb{Q}) \otimes H^p(BSO(k) \otimes BU(n-k); \mathbb{Q}) \]

which converges to \( H^{p+q}(\tilde{I}_{2n,k}; \mathbb{Q}) \).

It is well-known ([2]) that

\[ H^*(BU(n-k); \mathbb{Q}) \cong \mathbb{Q}[c_1, c_2, \cdots, c_{n-k}] \]

where \( c_i \in H^{2i}(BU(n-k); \mathbb{Q}) \) is the \( i \text{th} \) Chern class of the universal complex \( (n-k) \)-plane bundle \( \gamma_{n-k} \); and,

\[ H^*(U(n); \mathbb{Q}) \cong \wedge_\mathbb{Q}[x_1, x_3, \cdots, x_{2n-1}] \]

where \( x_i \in H^i(U(n); \mathbb{Q}) \) and \( \wedge \) denotes the exterior algebra.

For odd \( k (= 2m+1) \)

\[ H^*(BSO(k); \mathbb{Q}) \cong \mathbb{Q}[p_1, p_2, \cdots, p_m] \]

where \( p_i \in H^{4i}(BSO(k); \mathbb{Q}) \) are the Pontrjagin classes of the universal oriented \( k \)-plane bundle \( \xi_k \). For the case \( k = 2m \)

\[ H^*(BSO(k); \mathbb{Q}) \cong \mathbb{Q}[p_1, p_2, \cdots, p_{m-1}, e_k] \]

where \( e_k \in H^k(BSO(k); \mathbb{Q}) \) is the Euler class of \( \xi_k \). In this case one has \( p_m(\xi_k) = e_k^2 \).

Note that the inclusion \( SO(k) \times U(n-k) \subset U(n) \) is induced by \( (\mathbb{R}^k \otimes \mathbb{C}) \oplus \mathbb{C}^{n-k} \cong \mathbb{C}^n \). It follows that on classifying spaces \( BSO(k) \times BU(n-k) \rightarrow BU(n) \) classifies the
complex bundle $\xi_k \otimes \mathbb{C} \oplus \gamma_{n-k}$. Hence we have a commutative diagram of fibrations

$$
\begin{array}{ccc}
U(n) & \longrightarrow & \tilde{I}_{2n,k} \\
\downarrow & & \downarrow \\
U(n) & \longrightarrow & EU(n)
\end{array}
\rightleftharpoons
\begin{array}{ccc}
BSO(k) \times BU(n-k) & \longrightarrow & (\xi_k \otimes \mathbb{C}) \oplus \gamma_{n-k} \\
\downarrow & & \downarrow \\
BU(n) & \longrightarrow & BU(n)
\end{array}
$$

This induces a diagram of spectral sequences which we may use to compute the differentials in $3.2$. Let $\lambda^*$ denote the homomorphism from the spectral sequence $\wedge \mathbb{Q}(x_1, x_3 \cdots) \otimes \mathbb{Q}[c_1, c_2, \cdots] \longrightarrow H^*(pt; \mathbb{Q})$ to $3.2$. As the classes $x_{2i-1}$ are transgressive with $d_{2i}(x_{2i-1}) = c_i$, so are the classes $\lambda^*(x_{2i-1})$. Therefore we obtain

$$
d(x_{2i-1}) = \lambda^*(c_i) = c_i((\xi_k \otimes \mathbb{C}) \oplus \gamma_{n-k}) = \sum_{j=0}^{\infty} c_j(\xi_k \otimes \mathbb{C})c_{i-j}(\gamma_{n-k}) = \sum_{j=0}^{\infty} i/2 P_j c_{i-2j} = c_i + \sum_{j=1}^{\infty} i/2 P_j c_{i-2j}
$$

(3.3)

**Proposition 3.1.** Let $2 < k \leq n$. If $k < n$, the cohomology groups $H^i(\tilde{I}_{2n,k})$ are 0 if $i \leq 3$ and $H^4(\tilde{I}_{2n,k})$ is generated by $p_1$. In the case $k = n$, the cohomology group $H^k(\tilde{I}_{2n,n})$ is isomorphic to $\mathbb{Z}$ and $H^4(\tilde{I}_{2n,n})$ is zero.

**Proof.** For $k = n$, the space $\tilde{I}_{2n,k} \cong U(n)/SO(n)$, thus the fundamental group and hence $H^1$ is $\cong \mathbb{Z}$. Otherwise in the spectral sequence $3.2$ one has a class $c_1$ in $E_2^{2,0}$. In degrees $\leq 3$ the spectral sequence $3.2$ is $\cong \wedge(x_1, x_3) \otimes \mathbb{Q}[c_1]$ and from $3.3$ we get that $d_2(x_1) = c_1$. Hence the only possible class in $H^{* \leq 3}$ is $x_3$.

Note that $3.3$ also gives $d_4(x_3) = c_2 + p_1$ if $k \geq 2$ and $d_4(x_3) = c_2$ if $k = 1$. Thus we conclude that $H^{* \leq 3}$ is 0 if $k < n$, and if in addition $n > k \geq 2$ then $H^4(\tilde{I}_{2n,k})(\cong \mathbb{Q})$ generated by $p_1$. In the case $k = n$, $d_4(x_3) = p_1$, and hence $H^4(\tilde{I}_{2n,n})$ becomes zero.

We may compute further in the spectral sequence $3.2$. Notice that the formula $3.3$ is of the form $d(x_{2i-1}) = c_i + \cdots$ and so the class $c_i$ is not zero if $i \leq n-k$. Thus the elements $d(x_1), d(x_3), \ldots d(x_{2(n-k)-1})$ form a regular sequence in $E_2^{*,0}$. It follows that no multiple of $x_{2j-1}$, for $j \leq n-k$, can be a permanent cycle. Therefore any positive degree classes surviving to the $E_\infty$-page must have degree $> 2(n-k)+1$. In fact we have the Proposition

**Proposition 3.2.** Suppose $2 \leq k < n$. The cohomology algebra $H^*(\tilde{I}_{2n,k}; \mathbb{Q})$ has algebra generators $p_1, \cdots, p_m$ in degrees $4, 8, \cdots$ when $k = 2m+1$ is odd. If $k = 2m$ there is an additional generator $e_m$ in degree $2m$ that satisfies $e_m^2 = p_m$. Other algebra generators are in degrees $\geq 2(n-k)+1$. 

Proof. In view of the discussion above it suffices to prove the first two statements for the horizontal 0-line \( E^*_\infty \). Note that for \( j \) odd, the equation \( 3.3 \) gives
\[
d(x_{2j-1}) = c_j + \sum_{l=1}^{[j/2]} p_l c_{j-2l}
\]

Inductively we conclude that \( d_{2j}(x_{2j-1}) = c_j \). We have \( d_2(x_1) = c_1 \) and thus \( c_1 \) is 0 in the \( E_3 \)-page. Inductively \( c_{j-1} \) is 0 in the group \( E^{2j-4}_2 \). Hence, the equation above implies \( d_2(x_{2j-1}) = c_j \) as the other \( c_{odd \leq j-1} \) are 0 in \( E_2 \). It follows that \( c_j \) is 0 in \( E_{2j+1} \).

The remaining classes in the horizontal 0-line are \( p_i \), \( c_{2j} \) and \( e_m \) if \( k \) is even. The remaining differentials are generated by
\[
d(x_{4j-1}) = \sum_{l=0}^{j} p_l c_{2j-2l}
\]

Hence the horizontal 0-line is the graded algebra \( A(n, k) \) below. The Proposition now follows from Lemma 3.3 and [3].

Let \( \mathbb{R}\tilde{G}_{n,k} \) denote the oriented real Grassmannian of all \( k \)-dimensional oriented subspaces of \( \mathbb{R}^n \). As a space this is \( \simeq SO(n)/SO(k) \times SO(n-k) \). Let \( CG_{n,k} \) denote the Grassmannian of \( k \)-planes in \( \mathbb{C}^n \). Define the graded algebras \( A(n, k) \) for \( n > k \) as (with notations as above)
\[
A(n, k) = \begin{cases} 
\mathbb{Q}[p_1, \ldots, p_n, c_2, c_4, \ldots, c_{n-2m-2}] & \text{if } n \text{ is even, } k = 2m + 1 \\
\mathbb{Q}[p_1, \ldots, p_n, c_2, c_4, \ldots, c_{n-2m-1}] & \text{if } n \text{ is odd, } k = 2m + 1 \\
\mathbb{Q}[p_1, \ldots, p_{n-1}, c_2, c_4, \ldots, c_{n-2m}] & \text{if } n \text{ is even, } k = 2m \\
\mathbb{Q}[p_1, \ldots, p_{n-1}, c_2, c_4, \ldots, c_{n-2m-2}] & \text{if } n \text{ is odd, } k = 2m 
\end{cases}
\]

**Lemma 3.3.** There are isomorphisms of graded algebras
\[\]
a) \( A(2s, 2m + 1) \cong H^{*+2}(CG_{s-1,m}; \mathbb{Q}) \)
b) \( A(2s + 1, 2m + 1) \cong H^{*+2}(CG_{s,m}; \mathbb{Q}) \)
c) \( A(2s, 2m) \cong H^*(\mathbb{R}\tilde{G}_{2s+1,2m}; \mathbb{Q}) \)
d) \( A(2s + 1, 2m) \cong H^*(\mathbb{R}\tilde{G}_{2s+1,2m}; \mathbb{Q}) \)

**Proof.** This follows from the computation of the cohomology algebras of the Grassmannians using (with \( \mathbb{Q} \) coefficients):
\[
H^*(CG_{n,k}) \cong H^*(BU(k)) \otimes H^*(BU(n-k))/H^+(BU(n))
\]
and if \( n \) is odd,
\[
H^*(\mathbb{R}\tilde{G}_{n,k}) \cong H^*(BSO(k)) \otimes H^*(BSO(n-k))/H^+(BSO(n))
\]
\( \square \)
Remark 3.4. One may compare this to the expression obtained in [3]. Observe that the ring of characteristic classes $A$ of the principal bundle $U(n) \to \tilde{I}_{2n,k}$ matches the graded algebra $A(n, k)$ above. Note that the cohomology of $\tilde{I}_{2n,k}$ is $A \otimes \Lambda$ where $\Lambda$ is an exterior algebra on classes in degrees $d \in S_{n,k}$ ([3], Theorem 1.7) with

$$S_{n,k} = \begin{cases} \{4\left[\frac{n-k+1}{2}\right] + 1, 4\left[\frac{n-k+1}{2}\right] + 3, \ldots, 2n - 3\} & \text{if both } n \text{ and } k \text{ are even} \\ \{4\left[\frac{n-k+1}{2}\right] + 1, 4\left[\frac{n-k+1}{2}\right] + 3, \ldots, 2n - 1\} & \text{otherwise} \end{cases}$$

It follows that the Poincaré polynomial of $\tilde{I}_{2n,k}$ is given by

$$p_{\tilde{I}_{2n,k}}(x) = p_{A(n,k)}(x)\prod_{d \in S_{n,k}}(1 + x^d)$$

which may be computed from the known formulas for complex and real Grassmannians.

Recall that height of a nilpotent element, $x$, in an algebra, is defined to be the least positive integer $n$, such that $x^n \neq 0$ but $x^{n+1} = 0$.

**Proposition 3.5.** The height of the element, $p_1$ in $H^*(\tilde{I}_{2n,k})$ is $t(s-t)$ for $(n, k) \in \{(2s + 2, 2t + 1), (2s + 1, 2t + 1), (2s, 2t), (2s + 1, 2t)\}$.

**Proof.** Follows from Lemma 3.3 and Lemma 4 of [3]. \hfill $\Box$

4. MAIN RESULTS

In this section we consider the question of possible Brouwer degrees of maps $f: \tilde{I}_{2n,k} \to \tilde{I}_{2m,l}$, where $\tilde{I}_{2n,k}$ and $\tilde{I}_{2m,l}$ are oriented isotropic Grassmannians, such that $\dim \tilde{I}_{2n,k} = \dim \tilde{I}_{2m,l}$.

Note that when $l = 1$, the space $\tilde{I}_{2m,l} \simeq U(m)/U(m-1) \simeq S^{2m-1}$. Also note that $\dim \tilde{I}_{2n,k} = 2k(n-k) + \frac{k(k+1)}{2}$ is odd if and only if $k \equiv 1, 2 \pmod{4}$. In these cases (dim $\tilde{I}_{2n,k} = 2m - 1$) given any $\lambda \in \mathbb{Z}$, there exists a map $f_\lambda: \tilde{I}_{2n,k} \to S^{2m-1}$ with deg $f_\lambda = \lambda$. We prove that these are the only possible cases of non zero degree.

**Theorem 4.1.** Let $n, k, m, l$ be integers such that $2 \leq k \leq n$ and $2 \leq l \leq m$ and $\dim \tilde{I}_{2n,k} = \dim \tilde{I}_{2m,l}$. Let $f: \tilde{I}_{2n,k} \to \tilde{I}_{2m,l}$. Then either $(n, k) = (m, l)$ or deg $f = 0$.

**Proof.** Suppose $n = k$ and $m = l$. Then $\dim \tilde{I}_{2n,k} = \dim \tilde{I}_{2m,l}$ implies $n(n+1)/2 = m(m+1)/2$ and it follows that $n = m$. Note that the space $\tilde{I}_{4,2} \simeq U(2)/SO(2)$ is an oriented manifold of dimension 3. Observe that $k \geq 2$ and $n \neq k$ implies $\dim \tilde{I}_{2n,k} \geq 4$. Hence $\dim \tilde{I}_{2n,k} = \dim \tilde{I}_{2m,l}$ implies $(n, k) = (m, l)$ if $n = k$, $m = l$ or one of $(n, k)$ or $(m, l)$ equals $(2, 2)$.

Consider the case where $n = k$, $n > 2$ and $m \neq l$. We use the fact that if deg $f \neq 0$ then $f^*$ is a monomorphism on cohomology with rational coefficients. By Proposition 3.1 $H^4(\tilde{I}_{2n,n}) = 0$ and $H^4(\tilde{I}_{2m,l}) \neq 0$. Hence deg $f = 0$. In the case $n \neq k$, $m = l$ and
$m > 2$, we have, again by Proposition 3.1, that $H^4(I_{2m,m}) \cong \mathbb{Z}$ and $H^4(I_{2n,k}) = 0$. Therefore $\deg f = 0$.

Now we proceed to the more general case $2 \leq k < n$ and $2 \leq l < m$. Consider $f^* : H^*(I_{2m,l}; \mathbb{Q}) \to H^*(I_{2n,k}; \mathbb{Q})$. Since $p_1$ is the generator of $H^4(I_{2m,l}; \mathbb{Q})$ we must have (denote by $p_1(m,l)$ the class $p_1 \in H^4(I_{2m,l}; \mathbb{Q})$)

$$f^*p_1(m,l) = \lambda p_1(n,k)$$

By Proposition 3.3 the height of $p_1(n,k)$ is $[k/2][(n-k)/2]$ and the height of $p_1(m,l)$ is $[l/2][(m-l)/2]$. (Here, $[t]$ denotes the integral part of $t$.)

If $\deg f \neq 0$ then $f^*$ is a monomorphism and so $\lambda \neq 0$. Moreover $p_1(m,l)^{[l/2][(m-l)/2]} \neq 0$ implies

$$f^*p_1(m,l)^{[l/2][(m-l)/2]} = \lambda^{[l/2][(m-l)/2]}p_1(n,k)^{[k/2][(n-k)/2]} \neq 0$$

$$\implies \left[ \frac{l}{2} \right] \left[ \frac{m-l}{2} \right] \leq \left[ \frac{k}{2} \right] \left[ \frac{n-k}{2} \right].$$

Since $f^*$ is a ring homomorphism, we have

$$0 = f^*p_1(m,l)^{[l/2][(m-l)/2]+1} = \lambda^{[l/2][(m-l)/2]+1}p_1(n,k)^{[k/2][(n-k)/2]+1}$$

$$\implies \left[ \frac{k}{2} \right] \left[ \frac{n-k}{2} \right] \leq \left[ \frac{l}{2} \right] \left[ \frac{m-l}{2} \right].$$

Therefore $[k/2][(n-k)/2] = [l/2][(m-l)/2]$. Together with the equation $2k(n-k) + \frac{k(k+1)}{2} = 2l(m-l) + \frac{l(l+1)}{2}$ we prove that it leads to a contradiction. Assume that $k \leq l$ (there is no loss of generality in doing this.) The above equality implies $l(m-l) - 4 \leq k(n-k) \leq l(m-l) + 4$. Rearranging terms we obtain $-16 \leq (k-l)(k+l+1) \leq 16$.

As both $k \geq 2$ and $l \geq 2$ we have $k + l + 1 \geq 5$ and so the above inequality can hold only when $k - l = 0, 1, 2$. Observe that $k = l$ implies $n = m$ so that $(n,k) = (m,l)$. Note also that $(k-l)(k+l+1)$ must also be divisible by 4 being equal to $k(n-k) - l(m-l)$. If $k = l + 2$, we have $(k-l)(k+l+1) = 2(2l + 3)$ is not divisible by 4. If $k = l + 1$ we have $(k-l)(k+l+1) = 2l + 2$ which is divisible by 4 only when $l$ is odd. Therefore the allowed values of $l$ are 3, 5, 7.

**Case $l = 3$** : We have $k = 4$ and the equation $8(n-4) + 10 = 6(m-3) + 6$ which implies $4n = 3m + 5$. This implies $m = 4s + 1, n = 3s + 2$ for some positive integer $s$. The equation $\left[ \frac{s}{2} \right] = \left[ \frac{s-1}{2} \right]$ implies $2s - 1 = 2\left[ \frac{s-1}{2} \right]$. But the LHS is bigger for $s = 1$ and the RHS is always bigger for $s > 1$.

**Case $l = 5$** : We have $k = 6$ and the equation $12(n-6) + 21 = 10(m-5) + 15$ which implies $6n = 5m + 8$ which has the only solution $m = 6s + 2, n = 5s + 3$. The equation $\left[ \frac{s}{2} \right] = \left[ \frac{s-2}{2} \right]$ implies $3\left[ \frac{s-2}{2} \right] = 2(3s - 2)$. The LHS is always bigger for $s > 0$. 

**Case** $l = 7$ : We have $k = 8$ and the equation $16(n - 8) + 36 = 14(m - 7) + 28$ which implies $8n = 7m + 11$ which has the only solution $m = 8s + 3, n = 7s + 4$. The equation $[\frac{l}{2}] [\frac{n - k}{2}] = [\frac{l}{2}] [\frac{m - l}{2}]$ implies $4[\frac{7s - 4}{2}] = 3(4s - 2)$. For $s = 1$, the LHS is 4 and the RHS is 6. For $s > 1$ the LHS is bigger.

The arguments in the above case can be extended to prove the following:

**Theorem 4.2.** Consider maps $h : \tilde{I}_{2n,k} \to \mathbb{R}\tilde{G}_{m,l}$ and $g : \mathbb{R}\tilde{G}_{m,l} \to \tilde{I}_{2n,k}$, where $2 \leq l < m$, $2 \leq k \leq n$ and $\dim \tilde{I}_{2n,k} = \dim \mathbb{R}\tilde{G}_{m,l}$. Then $\deg g = \deg h = 0$.

**Proof.** Note that when $l = 1$, $\mathbb{R}\tilde{G}_{m,l} \simeq S^{m-1}$. Hence there exists a map $h_\lambda : \tilde{I}_{2n,k} \to \mathbb{R}\tilde{G}_{m,1}$ of any degree $\lambda \in \mathbb{Z}$ whenever $\dim(\tilde{I}_{2n,k}) = m - 1$. Similarly, we have a map $g_\lambda : \mathbb{R}\tilde{G}_{m,l} \to \tilde{I}_{2n,1}$ of any specified degree $\lambda$ whenever $\dim(\mathbb{R}\tilde{G}_{m,l}) = 2n - 1$.

If $n = k = 2$, $\dim \tilde{I}_{2n,k} = \dim \mathbb{R}\tilde{G}_{m,l} = 3$ implies either $l = 1$ or $m - l = 1$. Since $\mathbb{R}\tilde{G}_{m,l}$ is diffeomorphic to $\mathbb{R}\tilde{G}_{m,m-1}$, both these cases reduce to the cases discussed in the previous paragraph.

Now consider the case where $n = k > 2$. Then, by Proposition 3.1, we have $H^4(\tilde{I}_{2n,n}) = 0$ and $\pi_1(\tilde{I}_{2n,n}) = 0$, which respectively imply $\deg h = 0$ and $\deg g = 0$.

Henceforth we restrict ourselves to the cases $2 \leq k < n$ and $2 \leq l < m$. Consider $h^* : H^*(\mathbb{R}\tilde{G}_{m,l}; \mathbb{Q}) \to H^*(\tilde{I}_{2n,k}; \mathbb{Q})$. Recall that $H^4(\mathbb{R}\tilde{G}_{m,l}; \mathbb{Q})$ is generated by $p_1$ which has order $[l/2][(m - l)/2]$. By Proposition 3.5, order of $p_1 \in H^4(\tilde{I}_{2n,k}; \mathbb{Q})$ is $[k/2][(n - k)/2]$.

And, $h^*$ takes $p_1 \in H^4(\mathbb{R}\tilde{G}_{m,l}; \mathbb{Q})$ to some multiple of $p_1$ in $H^4(\tilde{I}_{2n,k}; \mathbb{Q})$.

Therefore, as in the proof of Theorem 4.1, we have that if $\deg h \neq 0$, $l(m - l) - 4 \leq k(n - k) \leq l(m - l) + 4$. Observe that $\dim \tilde{I}_{2n,k} = \dim \mathbb{R}\tilde{G}_{m,l}$ implies $2k(n - k) + k(k + 1)/2 = l(m - l)$. Hence the bound gives us $k(n - k) + k(k + 1)/2 \leq 4$ which is not possible if $k \geq 2$ and $n > k$. Therefore we have $\deg h = 0$. The proof that $\deg g = 0$ is similar.

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**References**

[1] M. Mikosz, *Secondary characteristic classes for the isotropic Grassmannian*, Geometry and Topology of Caustics, Banach Center Publ. 50, Warsaw, 1999, 195–204.
[2] M. Mimura and H. Toda, *Topology of Lie groups. I, II*. Translated from the 1978 Japanese edition by the authors. Translations of Mathematical Monographs, 91. American Mathematical Society, Providence, RI, 1991.

[3] J. Morvan, L. Niglio, *Isotropic characteristic classes*, Compo. Math. 91 (1994) 67–89.

[4] K. H. Paranjpe, V. Srinivas, *Self maps of homogeneous spaces*, Invent. Math. 98 (1989) 425–444.

[5] V. Ramani and P. Sankaran, On degrees of maps between Grassmannians, Proc. Indian Acad. Sci. Math. Sci. 107 (1997), no.1, 13–19.

[6] P. Sankaran, S. Sarkar, *Degrees of maps between Grassmann manifolds*, Osaka J. Math. Volume 46, Number 4 (2009), 1143–1161.

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