Seiberg-Witten Theory, Symplectic Forms, and Hamiltonian Theory of Solitons

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Abstract

This is an expanded version of lectures given in Hangzhou and Beijing, on the symplectic forms common to Seiberg-Witten theory and the theory of solitons. Methods for evaluating the prepotential are discussed. The construction of new integrable models arising from supersymmetric gauge theories are reviewed, including twisted Calogero-Moser systems and spin chain models with twisted monodromy conditions. A practical framework is presented for evaluating the universal symplectic form in terms of Lax pairs. A subtle distinction between a Lie algebra and a Lie group version of this symplectic form is clarified, which is necessary in chain models.

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1 Introduction

Soliton equations and integrable models have usually a rich Hamiltonian structure, in terms of which they become Hamiltonian flows with a full set of integrals of motion in involution (see [1] and references therein). However, except for the $R$-matrix approach developed by Faddeev and Takhtajan, the Hamiltonian structure for each model had been found on a case by case basis. In particular, there was no general construction based directly on the characterizing feature of soliton equations and integrable models, namely that they can be expressed as a Lax or a zero curvature equation. Such a construction became available only recently [2, 3]. A key input came from supersymmetric gauge theories: the Seiberg-Witten Ansatz for their exact solution in the Coulomb phase can be expressed in terms of a symplectic form on a moduli space of spectral curves and divisors [4]. Many spectral curves arising in this way were recognized as identical with the spectral curves of some known integrable models [3, 4]. The problem became to identify the symplectic form for the integrable models which would correspond to the one from the Seiberg-Witten Ansatz. The symplectic form found in [2, 3] was the answer. It provided at the same time the direct and universal construction in terms of Lax pairs which had been lacking in the Hamiltonian theory of solitons.

The contents of this paper are as follows.

The key information provided by the spectral curves and their symplectic structure is the prepotential $F$. For integrable models, $F$ is the $\tau$-function of the Whitham hierarchy [8]. For supersymmetric gauge theories, it is the prepotential which determines entirely the Wilson effective action in the Coulomb phase (see [9] and references therein). In the perturbative regime, $F$ consists of the classical prepotential, together with one-loop and instanton corrections. The one-loop corrections characterize the field content of the corresponding supersymmetric gauge theory. They are a defining feature of the correspondence between gauge theories and integrable models, and several methods for determining them have been developed [10, 11, 12, 13]. Here we present another method which is efficient, and based on a $\delta$-regularization process different from the analytic continuation in [11, 12]. The instanton corrections can be recaptured from renormalization group equations [14]. We provide such equations in some models where they had not been available §2.

The correspondence between integrable models and supersymmetric gauge theories has been mutually beneficial. In one direction, Seiberg-Witten solutions of gauge theories have been found via integrable models. Such was the case for gauge theories with a matter hypermultiplet in the adjoint representation, where the solution came from twisted Calogero-Moser systems [15, 16, 17]. In the other direction, solutions of gauge theories which had been obtained by other methods such as M theory [18, 19] or geometric engineering [20], have led to the discovery of new integrable models [21, 22]. We review these developments in §4 and §5, with emphasis on the scaling limits of Calogero-Moser systems, and the general construction of spin chain models with twisted monodromy conditions.
The symplectic form of \[2,3\] is discussed in Section §5. Given the diversity of integrable models, it may be helpful to provide a framework which is at the same time broad enough for applications to supersymmetric gauge theories, and yet simple enough for the symplectic form to be easily worked out. We provide such a framework in §5.1. In practice, the general construction of \[2,3\] can lead to two slightly different symplectic forms, depending on whether the Lax operator is viewed as a Lie group element or as a Lie algebra element. This distinction is not relevant in most models, but it is important for chain models such as the Toda chain or the spin models of Section §4. We clarify it in Section §5.2. It is a remarkable fact that the formula for the finite-dimensional symplectic forms arising in Seiberg-Witten theory works extends to the partial differential equations of soliton theory. In particular, we obtain a symplectic structure even for zero curvature equations in \(2 + 1\) space-time variables, whose Hamiltonian formulation had formerly not been fully satisfactory. In section §5.4, we present a basic example of how this can be done, in the case of the Kadomtsev-Petviashvili hierarchy, following \[3\]. In the remaining sections §5.3 and §5.5, we describe some recent progress where the general construction of symplectic forms played a major role. This includes the construction of Lax equations with spectral parameter on a curve of higher genus, field analogues of Calogero-Moser equations, and isomonodromy problems \[23,24\].

2 Seiberg-Witten Solutions of \(\mathcal{N} = 2\) Super Yang-Mills Theories

2.1 The Wilson effective action

In four dimensions, \(\mathcal{N} = 2\) supersymmetric gauge theories can be classified by their gauge group \(G\) and the representation \(R\) of their matter hypermultiplets, subject to the requirement of asymptotic freedom or scale invariance. Let \(n\) denote the rank of the gauge group \(G\). The \(\mathcal{N} = 2\) multiplet of the gauge field \(A_\mu dx^\mu\) consists of \((A_\mu dx^\mu, \lambda_\pm, \phi)\), where \(\lambda_\pm\) are Weyl spinors, and \(\phi\) is a scalar, all valued in the adjoint representation. Classically, the equations of motion are

\[
F_{\mu\nu} = 0, \quad D_\mu \phi = 0, \quad [\phi, \phi^\dagger] = 0
\]

where \(F_{\mu\nu}\) is the curvature of the gauge field \(A_\mu dx^\mu\). Thus the theory admits an \(n\)-dimensional moduli space of classical vacua, corresponding to the diagonalizable elements in the Lie algebra of \(G\). Quantum mechanically, the gauge group is spontaneously broken to \(U(1)^n\), and the effective theory is a theory of \(n\) interacting \(\mathcal{N} = 2\) supersymmetric electromagnetic multiplets \((A_\mu^i dx^\mu, \lambda_\pm^i, \phi^i)\), \(1 \leq i \leq n\). In view of the \(\mathcal{N} = 2\) supersymmetry, the Wilson low-energy effective action \(\mathcal{L}_{eff}\) is characterized completely by a single function, the prepotential \(\mathcal{F}(\phi, \Lambda)\)

\[
\mathcal{L}_{eff} = \{\text{Im} \, \tau_{ij}(\phi)\} F^i_{\mu\nu} F^{i\mu\nu} + \{\text{Re} \, \tau_{ij}(\phi)\} \tilde{F}^i_{\mu\nu} \tilde{F}^{i\mu\nu} + \cdots
\]
\[ \tau_{ij} = \frac{\partial^2 F}{\partial \phi^i \partial \phi^j}(\phi, \Lambda) \]  

(2.2)

Here \( F^{\mu
u} = \epsilon^{\mu
u\lambda\rho} F_{\lambda\rho}^{ij} \), and \( \Lambda \) is a scale introduced by renormalization. In the perturbative regime where \( \Lambda \) is small compared to \( \phi \), the prepotential \( F \) admits an expansion of the form

\[ F(\phi, \Lambda) = \frac{i}{8\pi} \left( \sum_{\alpha \in \mathcal{R}(G)} (\alpha \cdot \phi)^2 \ln \frac{\alpha \cdot \phi}{\Lambda^2} - \sum_{\lambda \in \mathcal{W}(R)} (\lambda \cdot \phi + m)^2 \ln \frac{(\lambda \cdot \phi + m)^2}{\Lambda^2} \right) + \sum_{d=1}^{\infty} F_d \Lambda^d \]  

(2.3)

where \( \mathcal{R}(G) \) are the roots of \( G \), \( \mathcal{W}(R) \) are the weights of the representation \( R \), and \( m \) is the mass of the hypermultiplet. The expression \( I(R) \) is the Dynkin index of the representation \( R \). When \( R \) is the adjoint representation, it is also given by \( 2h^\vee_G \), where \( h^\vee_G \) is the dual Coxeter number of \( G \). In the above expansion, we have ignored the classical prepotential. The logarithmic singularities are due to one-loop effects, the higher loops do not contribute by non-renormalization theorems, and \( F_d \Lambda^d \) is the contribution of \( d \)-instanton processes.

So far we have been discussing asymptotically free theories. In scale invariant theories, the renormalization scale \( \Lambda \) is replaced by a coupling

\[ \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \]  

(2.4)

which is well-defined microscopically. Here \( \frac{1}{g^2} \) is the gauge coupling and \( \theta \) the \( \theta \)-angle. The expansion (2.3) for \( F \) is then replaced by

\[ F(\phi, \Lambda) = \frac{i}{8\pi} \left( \sum_{\alpha \in \mathcal{R}(G)} (\alpha \cdot \phi)^2 \ln \frac{\alpha \cdot \phi}{\Lambda^2} - \sum_{\lambda \in \mathcal{W}(R)} (\lambda \cdot \phi + m)^2 \ln \frac{(\lambda \cdot \phi + m)^2}{\Lambda^2} \right) + \sum_{d=1}^{\infty} F_d \tau^d \]  

(2.5)

### 2.2 The Seiberg-Witten Ansatz for the effective prepotential

An exact solution of the gauge theory is provided by the Seiberg-Witten Ansatz \[4\], which reduces the problem of finding \( F \) to finding a fibration of Riemann surfaces \( \Gamma(\Lambda) \) over the moduli space of vacua, equipped with a meromorphic 1-form \( d\lambda \) on each surface \( \Gamma(\Lambda) \). The prepotential \( F \) is then obtained from \( \Gamma(\Lambda) \) and \( d\lambda \) by the Ansatz

\[ a_i = \frac{1}{2\pi i} \oint_{A_i} d\lambda, \quad a_{Di} = \frac{1}{2\pi i} \oint_{B_i} d\lambda, \quad a_{Di} = \frac{\partial F}{\partial a_i}, \quad 1 \leq i \leq n \]  

(2.6)

where \( A_i, B_i \) are suitable cycles over the surface \( \Gamma(\Lambda) \). The fibration \( \Gamma(\Lambda) \) will usually include singular surfaces over some subvarieties of the moduli space of vacua, corresponding to when physical massless particles appear. There are also severe constraints on the fibrations and differentials can arise as the Seiberg-Witten solution of a gauge theory. The curves \( \Gamma(\Lambda) \) must be invariant under the Weyl group \( \text{Weyl}(G) \), which is the residual gauge invariance after the gauge group \( G \) has been broken down to its Cartan subalgebra, \( U(1)^n \).
The differential $\frac{\partial d\lambda}{\partial a_i}$ must be holomorphic, and the residues of $d\lambda$ must be independent of $a_i$ and linear in the masses $m$. Physically, this means that the hypermultiplet masses are not renormalized and are consistent with the BPS mass formula, in which the hypermultiplet mass parameters enter linearly.

As noted very early on in [4], the Seiberg-Witten solution of a gauge theory can already be derived from a natural symplectic form $\omega$ defined by the data $\Gamma(\Lambda), d\lambda$

$$\omega = \delta \left( \sum_{i=1}^{n} d\lambda(z_i) \right)$$  

(2.7)

where $\delta$ denotes exterior differential on the total space of the fibration of spectral curves and divisors $[z_1, \cdots, z_n]$. This symplectic form can often be more manageable than the data $(\Gamma(\Lambda), d\lambda)$ itself.

### 2.3 The logarithmic singularities of the effective prepotential

The Seiberg-Witten exact solution is now known for many gauge theories, although not for all. Methods for finding the solution include geometric engineering [20], M theory [18, 19], and integrable models [6, 25, 15, 16, 26], the latter being the main one considered here. Once certain models have been solved, the solution of others can also be derived by various decoupling limits. In all cases, it is a key requirement that the fibration $\Gamma(\Lambda)^n$ for a gauge group $G$ and a representation $R$ of the matter hypermultiplet must exhibit the corresponding logarithmic singularities (2.3) or (2.5) in the perturbative regime of large vacuum expectation values. The problem of identifying the logarithmic singularities (as well as the instanton corrections) from a given fibration $(\Gamma(\Lambda), d\lambda)$ has thus received considerable attention. Some of the many methods developed are Picard-Fuchs equations [10], the method of residues [11], non-hyperelliptic extensions of the method of residues [12], and others [13, 27, 28]. We take the opportunity in this paper to present a new method which may also be useful. To be specific, we shall consider the case of $G = SU(N)$, with either no hypermultiplet, or a hypermultiplet in the antisymmetric or the symmetric representation. The integrable models corresponding to these theories are also the focus of §4 below.

#### 2.3.1 The pure $SU(N)$ Yang-Mills theory

As a warm-up, we begin with the case of pure Yang-Mills. Suitably formulated, we shall see that the more difficult cases of a symmetric or an anti-symmetric hypermultiplet follow from the present method by simple modifications. For the pure $SU(N)$ Yang-Mills theory, the following Seiberg-Witten differentials and differential were proposed (see [29] and references therein)

$$\frac{1}{2} \left( k + \frac{\Lambda^N}{k} \right) = P(x), \quad d\lambda = x \, d\ln k$$  

(2.8)
where $P(x)$ is a monic polynomial of degree $N$ with no $u_{N-1}x^{N-1}$ term, whose $N-1$ coefficients can be viewed as parameters for the moduli of classical vacua. For the derivation of the effective prepotential, it is more convenient to introduce the variable $y = k - P(x)$, and to parametrize $P(x)$ as $P(x) = \prod_{k=1}^{N} (x - \bar{a}_k)$, so that the periods $a_k$ of $d\lambda$ will emerge naturally as renormalizations of the classical moduli parameters $\bar{a}_k$. With these variables, the curve and differential are now given by

$$y^2 = P(x)^2 - \Lambda^{2N}, \quad d\lambda = \frac{dP}{y}. \quad (2.9)$$

This is a hyperelliptic curve, made of two copies of the complex plane, glued along $N$ cuts going between pairs of zeroes of the right hand side. More specifically, let $x_k^\pm$ be the zeroes of the right hand side, with $x_k^- \to \bar{a}_k$ as $\Lambda \to 0$. For real values of $\bar{a}_k$ and $\bar{\Lambda}$, we let $x_k^+$ and $x_k^-$ be respectively the left and right edges of the cuts. We choose the cycles $A_k$ to be loops around the cuts from $x_k^- \to x_k^+$, and the cycles $B_k$ to be the cycles going from $x_k^+$ to $x_k^-$ on one sheet, and coming back on the other sheet. Our first task is to determine the logarithmic singularities of $a_{Dk}$

$$a_{Dk} = \frac{1}{2\pi i} \oint_{B_k} d\lambda = \frac{1}{\pi i} \int_{x_k^-}^{x_k^+} x \frac{dP}{\sqrt{P^2 - \Lambda^2}}, \quad 2 \leq k \leq N \quad (2.10)$$

Here we have set $\bar{\Lambda} = \Lambda^N$. Locally, $a_{Dk}$ is a holomorphic function of $\bar{\Lambda} \neq 0$. It is a multivalued function, due to the choice of branch points $x_k^+$ and $x_k^-$. If we analytically continue from $\bar{\Lambda}^2$ to $e^{2\pi i} \bar{\Lambda}^2$, the $B_k$ cycle transforms to $B_k + A_k - A_1$. Thus

$$2\pi i a_{Dk} = (\bar{a}_k - \bar{a}_1) \ln \bar{\Lambda}^2 + b(\bar{\Lambda}) \quad (2.11)$$

where $b(\bar{\Lambda})$ is a single valued holomorphic function on the punctured disk $\bar{\Lambda} \neq 0$. Our next goal is to show that $b(\bar{\Lambda})$ is bounded. This will imply that $b(\bar{\Lambda})$ is a holomorphic function of $\bar{\Lambda}$ on the whole disk. Fix $\delta$ small and rewrite (2.11) as the sum of three integrals

$$\pi i a_{Dk} = \int_{\bar{a}_1 + \delta}^{\bar{a}_k - \delta} x \frac{dP}{\sqrt{P^2 - \Lambda^2}} + \int_{\bar{a}_k - \delta}^{x_k^-} x \frac{dP}{\sqrt{P^2 - \Lambda^2}} - \int_{\bar{a}_1 + \delta}^{x_k^+} x \frac{dP}{\sqrt{P^2 - \Lambda^2}} \quad (2.12)$$

The first integral $I_1$ is a holomorphic function of $\bar{\Lambda}$ for $|\bar{\Lambda}| << \delta$. Therefore

$$I_1 = I_1^0 + \mathcal{O}(\bar{\Lambda}) \quad (2.13)$$

where

$$I_1^0 = \int_{\bar{a}_1 + \delta}^{\bar{a}_k - \delta} x \frac{dP}{P} = \int_{\bar{a}_1 + \delta}^{\bar{a}_k - \delta} \sum_{j=1}^{N} (1 + \frac{\bar{a}_j}{(x - \bar{a}_j)}) dx \quad (2.14)$$

$$= \bar{a}_k(1 + \ln(-\delta)) - \bar{a}_1(1 + \ln \delta) + \sum_{j \neq k} \bar{a}_j \ln(\bar{a}_k - \bar{a}_j) - \sum_{j \neq 1} \bar{a}_j \ln(\bar{a}_1 - \bar{a}_j) + \mathcal{O}(\delta)$$
The second integral in (2.12) is equal to

\[ I_2 = \bar{\alpha}_k \int_{x_k-\delta}^{x_k} \frac{dP}{\sqrt{P^2 - \bar{\Lambda}^2}} + \int_{x_k-\delta}^{x_k} \frac{(x - \bar{\alpha}_k) dP}{\sqrt{P^2 - \Lambda^2}} \quad (2.15) \]

In the range of integration, we have \((x - \bar{\alpha}_k) = \prod_{j \neq k}(\bar{\alpha}_k - \bar{\alpha}_j)^{-1}P(x)(1 + O(\delta))\). The term \(O(\delta)\) is readily seen to contribute only another \(O(\delta)\) term to \(I_2\). As for the other term, an explicit calculation gives

\[ \int_{x_k-\delta}^{x_k} \frac{(x - \bar{\alpha}_k) dP}{\sqrt{P^2 - \Lambda^2}} = \prod_{j \neq k}(\bar{\alpha}_k - \bar{\alpha}_j)^{-1} \left[ \sqrt{P^2 - \bar{\Lambda}^2} \right] \]

\[ \int_{x_k-\delta}^{x_k} \frac{dP}{\sqrt{P^2 - \bar{\Lambda}^2}} = \int_{u_k}^{0} du = -u_k \quad (2.16) \]

where the lower bound of integration is defined by

\[ e^{u_k} = \frac{P(a_k - \delta) + \sqrt{P^2(a_k - \delta) - \bar{\Lambda}^2}}{\Lambda} = \frac{2P(a_k - \delta)}{\Lambda} + O(\delta) \quad (2.18) \]

Thus

\[ I_2 = -\bar{\alpha}_k \ln \left( \frac{2(-\delta) \prod_{j \neq k}(\bar{\alpha}_k - \bar{\alpha}_j)}{\Lambda} \right) + O(\delta) \quad (2.19) \]

The third term in (2.12) admits a similar expression. Altogether, we find

\[ \pi i a_{Dk} = (\bar{\alpha}_k - \bar{\alpha}_1)(\ln \bar{\Lambda} + 1 - \ln 2) - \left( \sum_{j \neq k}(\bar{\alpha}_k - \bar{\alpha}_j) \ln(\bar{\alpha}_k - \bar{\alpha}_j) - (k \to 1) \right) + O(\delta) \quad (2.20) \]

This equation implies that \(b(\bar{\Lambda})\) is bounded, and hence holomorphic. Now \(b(\bar{\Lambda})\) does not depend on \(\delta\). Thus its leading term can be obtained by letting \(\delta \to 0\). Taking into account the fact that \(a_k = \bar{\alpha}_k + O(\Lambda)\), we conclude that

\[ \pi i a_{Dk} = (a_k - a_1)(\ln \bar{\Lambda} + 1 - \ln 2) - \left( \sum_{j \neq k}(a_k - a_j) \ln(a_k - a_j) - (k \to 1) \right) + O(\bar{\Lambda}) \quad (2.21) \]

These are indeed the logarithmic singularities of the effective prepotential characteristic of the pure \(SU(N)\) Yang-Mills theory.

The effective prepotential \(F\) of supersymmetric gauge theories has a lot of structure. Of particular interest to us is a renormalization group equation satisfied by \(F\). This renormalization group equation provides another important link with a Hamiltonian formulation of integrable models, and is also an efficient tool for the evaluation of instanton corrections.
It is a consequence of a closed formula for $F$. This formula goes back to a similar formula for the $\tau$-function of the Whitham hierarchy [8], and can be expressed as follows in the case of pure $SU(N)$ Yang-Mills

$$2F = \frac{1}{2\pi i} \left( \sum_{k=2}^{N} a_k \int_{B_k} d\lambda + \text{Res}_{P_+(x d\lambda)} \text{Res}_{P_+} (x^{-1} d\lambda) + \text{Res}_{P_-} (x d\lambda) \text{Res}_{P_-} (x^{-1} d\lambda) \right)$$

(2.22)

Here $P_\pm$ are the two points on $\Gamma$ above $x = \infty$. To establish this formula, it suffices to show that the derivative of the right hand side with respect to $a_m$ is equal to $2\partial \partial a_m$. Now the residues of $d\lambda$ as well as the residues of $x^{-1} d\lambda$ at $P_\pm$ are fixed. If we denote by $d\omega_k$ the basis of holomorphic differentials dual to $A_k$, it follows that

$$\frac{\partial}{\partial a_m} d\lambda = 2\pi i d\omega_m,$$

(2.23)

and hence the derivative of the above right hand side with respect to $a_m$ is given by

$$\frac{1}{2\pi i} \int_{B_m} d\lambda + \sum_{k=2}^{N} a_k \int_{B_k} d\omega_m + \frac{1}{2\pi i} (\text{Res}_{P_+(x^{-1} d\lambda)} \text{Res}_{P_+} (x d\omega_m) + \text{Res}_{P_-} (x^{-1} d\lambda) \text{Res}_{P_-} (x d\omega_m))$$

(2.24)

Let $d\Omega_\pm$ be the Abelian differentials of the second kind with a double pole at $P_\pm$ respectively, vanishing $A_k$-periods, and normalization $d\Omega_\pm = dx(1 + O(x))$. By the Riemann bilinear relations,

$$\int_{B_m} d\omega_m = \int_{B_m} d\omega_k, \quad \text{Res}_{P_\pm} (x d\omega_m) = \frac{1}{2\pi i} \int_{B_m} d\Omega_\pm$$

(2.25)

we can rewrite the last three terms on the above right hand side in terms of integrals over the cycle $B_m$. The resulting form is precisely $d\lambda$

$$d\lambda = \sum_{k=2}^{N} 2\pi i a_k d\omega_k + \text{Res}_{P_+} (x^{-1} d\lambda) d\Omega_+ + \text{Res}_{P_-} (x^{-1} d\lambda) d\Omega_-$$

(2.26)

as an inspection of the poles, residues, and periods of $d\lambda$ readily shows. The closed formula for $F$ can also be rewritten as

$$(\sum_{k=2}^{N} a_k \frac{\partial}{\partial a_k} - 2) F = -\frac{1}{2\pi i} (\text{Res}_{P_+} (x d\lambda) \text{Res}_{P_+} (x^{-1} d\lambda) + \text{Res}_{P_-} (x d\lambda) \text{Res}_{P_-} (x^{-1} d\lambda))$$

= $-\frac{N}{2\pi i} \sum_{k<m} \tilde{a}_k \tilde{a}_m.$

(2.27)

By dimensional analysis, we have $(\Lambda \frac{\partial}{\partial \Lambda} + \sum_{k=2}^{N} a_k \frac{\partial}{\partial a_k} - 2) F = 0$. Thus we obtain the following renormalization group equation, giving the dependence of the effective prepotential on the renormalization scale $\Lambda$ [7]

$$\Lambda \frac{\partial}{\partial \Lambda} F = \frac{N}{2\pi i} \sum_{k<m} \tilde{a}_k \tilde{a}_m.$$

(2.28)
The renormalization group equation provides no information on the logarithmic singularities of the effective prepotential, which have to be determined independently as we did above, in order to verify that the Seiberg-Witten curve proposed are indeed the solution of a given gauge theory. But it gives a relatively easy way of determining the instanton corrections to any order, since the terms $\bar{a}_k$, and hence the “beta function” $\frac{N}{2\pi i} \sum_{k<m} \bar{a}_k \bar{a}_m$, can be readily evaluated perturbatively in terms of the periods $a_k$.

We shall see later that the Seiberg-Witten curve (2.8) coincides with the spectral curve for the $SU(N)$ affine Toda system. Furthermore, the Toda system can be given a Hamiltonian formulation with Hamiltonian given precisely by the above right hand side. It had been pointed out by Donagi and Witten [25] that the Seiberg-Witten Ansatz provides a symplectic structure with respect to which the moduli vacua parameters become a maximal system of commuting Hamiltonians. The renormalization group equation suggests a more precise picture: the beta function is the Hamiltonian of a very specific integrable model, whose spectral curve coincides with the Seiberg-Witten curve of the gauge theory.

### 2.3.2 The $SU(N)$ Yang-Mills theory with antisymmetric matter

We come now to the derivation of the logarithmic singularities for the effective prepotential from the Landsteiner-Lopez curves for the gauge theory with a hypermultiplet in the antisymmetric representation. This had been carried out by Ennes, Naculich, Rhedin, and Schnitzer [12], by extending to the non-hyperelliptic case the method of residues introduced in [11]. Here we shall derive these singularities by the $\delta$ regularization method used in the previous section.

The Seiberg-Witten curve for the $SU(N)$ theory with a matter hypermultiplet in the antisymmetric representation had been proposed by Landsteiner-Lopez, using a brane construction [19]. It is of the form

$$
\Gamma : y^3 - (3\Lambda^{N+2} + x^2 P(x))y^2 + (3\Lambda^{N+2} + x^2 P(-x))\Lambda^{N+2}y - \Lambda^{3(N+2)} = 0,
$$

(2.29)

where $\Lambda$ is the renormalization scale, and $P(x)$ is again a monic polynomial of degree $N$, without $u_{N-1}x^{N-1}$ term. The Seiberg-Witten differential $d\lambda$ is given by

$$
d\lambda = x \frac{dy}{y}.
$$

(2.30)

We set $P(x) = \sum_{i=0}^{N} u_i x^i = \prod_{k=1}^{N} (x - \bar{a}_k)$, $u_N = 1$, $u_{N-1} = 0$. The curve $\Gamma$ is a three-sheeted cover of the complex plane. It is invariant with respect to the involution

$$
\sigma : y \rightarrow y^{-1} \Lambda^{2(N+2)}, \quad x \rightarrow -x.
$$

(2.31)

The quotient $\Gamma_0 = \Gamma/\sigma$ has genus $N-1$. We denote the three points on $\Gamma$ above $x = \infty$ by $P_1, P_2, P_3$, with $P_1$ characterized by $y \sim x^{N+2}$, $P_3 = \sigma(P_1)$, and $P_2 = \sigma(P_2)$. 

9
We consider now the \( \Lambda \to 0 \) limit. Set \( \bar{\Lambda}^2 = \Lambda^{(N+2)} \). Then the three branches \( y_i(x), i = 1, 2, 3 \) of the spectral curve (2.29) in the limit \( \Lambda \to 0 \) can be obtained in the form of series in \( \bar{\Lambda}^2 \):

\[
y_1 = x^2 P(x) + \frac{3P(x) - P(-x)}{P(x)} \bar{\Lambda}^2 + \sum_{s=2}^{\infty} \xi_{1,s}(x)\bar{\Lambda}^{2s} \quad (2.32)
\]

\[
y_2 = \bar{\Lambda}^2 \left( \frac{P(-x)}{P(x)} + \frac{(P(-x) - P(x))^3}{P^3(x)P(-x)} \bar{\Lambda}^2 + \sum_{s=2}^{\infty} \xi_{2,s}(x)\bar{\Lambda}^{2s} \right) \quad (2.33)
\]

\[
y_3 = \bar{\Lambda}^4 \frac{1}{y_1(-x)} = \bar{\Lambda}^4 \left( \frac{1}{x^2 P(-x)} + t(\bar{\Lambda}^2) \right) \quad (2.34)
\]

The first series is convergent outside the neighborhood of the zeroes of \( x^2P(x) \) where \( |\bar{\Lambda}^2/P(x)| < c_1 \) for some constant \( c_1 \). The second series is convergent outside neighborhoods of the zeroes of \( x^2P(x) \) and \( x^2P(-x) \).

The branch points of the spectral curve are defined by the equation of the curve together with the equation

\[
3y^2 - 2(3\bar{\Lambda}^2 + x^2P(x))y + \bar{\Lambda}^2(3\bar{\Lambda}^2 + x^2P(-x)) = 0. \quad (2.35)
\]

A set of \( 2N \) solutions of these equations can be found in the form of series in \( \bar{\Lambda} \) with the leading terms defined by

\[
y = \frac{1}{2} x^2 P(x) + \mathcal{O}(\bar{\Lambda}^2), \quad x^2 P^2(x) = \bar{\Lambda}^2 P(-x) + \mathcal{O}(\bar{\Lambda}^3). \quad (2.36)
\]

The corresponding branch points \( x^\pm_k \) have the form

\[
x^\pm_k = \bar{\alpha}_k \pm \bar{\Lambda} \left( \frac{(-1)^{N/2} \prod_j (\bar{\alpha}_k + \bar{\alpha}_j)^{1/2}}{\bar{\alpha}_k \prod_{j\neq k} (\bar{\alpha}_k - \bar{\alpha}_j)} \right) + \mathcal{O}(\bar{\Lambda}^2). \quad (2.37)
\]

They are branch points for the first and the second sheets. Another \( 2N \) branching points are \(-x^\pm_k\). Finally, there are two more branch points near \( x = 0 \).

Let the \( B_k \)-cycle on \( \Gamma_0 \) be covered by the cycle on \( \Gamma \) which goes from \( x_1^+ \) to \( x_k^- \) on the first sheet and returns back on the second sheet. Then

\[
2\pi i a_{Dk} = \oint_{B_k} d\lambda = \int_{x_1^+}^{x_k^-} x \frac{dy_1}{y_1} - \int_{x_1^+}^{x_k^-} x \frac{dy_2}{y_2}, \quad k = 2, \ldots N. \quad (2.38)
\]

We follow now closely the discussion of the case of pure Yang-Mills. The above integral is a multivalued holomorphic function of \( \bar{\Lambda} \neq 0 \), due to the choice of branch points \( x_1^+ \) and \( x_k^- \). For real values of \( \alpha_k \) and \( \bar{\Lambda} \) we choose \( x_k^\pm \) to be the left and the right edges of cuts. The path from \( \bar{\Lambda}^2 \) to \( e^{2\pi i} \bar{\Lambda}^2 \) generates a transformation of the \( B_k \)-period from \( B_k \) to \( B_k + A_k - A_1 \). Therefore,

\[
2\pi i a_{Dk} = (\bar{\alpha}_k - \bar{\alpha}_1) \ln \bar{\Lambda} + b(\bar{\Lambda}), \quad (2.39)
\]
where $b(\Lambda)$ is a single-valued holomorphic function in the punctured disc $\bar{\Lambda} \neq 0$. As before, our main goal is to show that $b(\Lambda)$ is bounded. That will imply that $b(\Lambda)$ is a holomorphic function of $\Lambda$. Let us again fix $\delta$ and rewrite (2.38) as a sum of three terms

$$2\pi i a_Dk = I_1 + I_2 + I_3,$$

(2.40)

where

$$I_1 = \int_{\bar{a}_1 + \delta}^{\bar{a}_k - \delta} x \frac{dy_1}{y_1} - \int_{\bar{a}_1 + \delta}^{\bar{a}_k - \delta} x \frac{dy_2}{y_2},$$

(2.41)

$$I_2 = \int_{\bar{a}_k - \delta}^{x_k^-} x \frac{dy_1}{y_1} - \int_{\bar{a}_k - \delta}^{x_k^-} x \frac{dy_2}{y_2},$$

(2.42)

$$I_3 = \int_{\bar{a}_1 + \delta}^{x_1^+} x \frac{dy_1}{y_1} - \int_{\bar{a}_1 + \delta}^{x_1^+} x \frac{dy_2}{y_2}.$$  

(2.43)

In the first integral $I_1$, for $|\bar{\Lambda}| < \delta$, we can replace $y_1, y_2$ by their leading terms in the expansions (2.32, 2.33). Therefore,

$$I_1 = 2 \int_{\bar{a}_1 + \delta}^{\bar{a}_k - \delta} x \ln (x^2 P(x)) - \int_{\bar{a}_1 + \delta}^{\bar{a}_k - \delta} x \ln (x^2 P(-x)) = (N + 2)(\bar{a}_k - \bar{a}_1)$$

$$+ 2 \left( \bar{a}_k \ln(\bar{a}_k - \bar{a}_1) \ln \delta + \sum_{j \neq k} \bar{a}_j \ln(\bar{a}_k - \bar{a}_j) - \sum_{j \neq k} \bar{a}_j \ln(\bar{a}_1 - \bar{a}_j) \right)$$

$$- \sum_j \bar{a}_j \ln \left( \frac{\bar{a}_k + \bar{a}_j}{\bar{a}_1 + \bar{a}_j} \right) + O(\delta)$$

(2.44)

Now let us consider the second term. The equations (2.36) imply that $|y(x_k^+) > c|\bar{\Lambda}|$. The constant $c$ can be chosen so that the inequality $|y| > c|\Lambda|$ holds for all points of the path of integration in (2.42). The equation (2.29) implies that

$$x^2 P(x)y^{-1} = 1 - 3\Lambda^{N+2} y^{-1} + (3\Lambda^{N+2} + x^2 P_N(-x))\Lambda^{N+2} y^{-2} - \Lambda^{3(N+2)} y^{-3}$$

(2.45)

Therefore $x^2 P(x)y^{-1}$ is uniformly bounded along the path of integration in (2.42). At the same time, we have $(x - \bar{a}_k) = \prod_{j \neq k}(\bar{a}_k - \bar{a}_j)^{-1} P(x)(1 + O(\delta))$. Hence,

$$I_2 = \bar{a}_k \left( \int_{\bar{a}_k - \delta}^{x_k^-} \frac{dy_1}{y_1} - \int_{\bar{a}_k - \delta}^{x_k^-} \frac{dy_2}{y_2} \right) + O(\delta)$$

$$= \bar{a}_k \left( \ln y_2(\bar{a}_k - \delta) - \ln y_1(\bar{a}_k - \delta) \right) + O(\delta).$$

(2.46)

The expansions (2.32 2.33) for $y_i$ imply

$$I_2 = \bar{a}_k \left[ 2 \ln \bar{\Lambda} - 2 \ln(\delta) - \ln \left( \frac{\bar{a}_k^2 \prod_{j \neq k}(\bar{a}_k - \bar{a}_j)^2}{(-1)^N \prod_j(\bar{a}_k + \bar{a}_j)} \right) \right] + O(\delta).$$

(2.47)
A similar expression can be derived for $I_3$. Altogether, we find

$$2\pi i a_{kD} = \bar{a}_k (2\ln 2\bar{\Lambda} + (N + 2) + N\ln(-1)) - \sum_{j \neq k} (\bar{a}_k - \bar{a}_j) \ln(\bar{a}_k - \bar{a}_j)^2 + \sum_{j < k} (\bar{a}_k + \bar{a}_j) \ln(\bar{a}_k + \bar{a}_j)^2 - (k \to 1) + O(\delta) \quad (2.48)$$

This equation implies that $b(\bar{\Lambda})$ in (2.39) is uniformly bounded. Therefore it is holomorphic. Since $b(\bar{\Lambda})$ does not depend on $\delta$, its leading term can be obtained by letting $\delta \to 0$.

Taking into account the fact that $a_k = \alpha_k + O(\Lambda)$ we conclude that

$$2\pi i a_{kD} = \alpha_k (2\ln 2\bar{\Lambda} + (N + 2) + N\ln(-1)) - \sum_{j \neq k} (\alpha_k - \alpha_j) \ln(\alpha_k - \alpha_j)^2 + \sum_{j < k} (\alpha_k + \alpha_j) \ln(\alpha_k + \alpha_j)^2 - (k \to 1) + O(\Lambda) \quad (2.49)$$

Since the weights of the antisymmetric representation are $e_j + e_k$, $j < k$, these are the logarithmic singularities of the $SU(N)$ theory with matter in the antisymmetric representation.

We derive now the renormalization group equation for this model. This is a consequence of the following closed formula for the prepotential

$$2\mathcal{F} = \frac{1}{2\pi i} \left[ \sum_{k=1}^{N} a_k \oint_{B_k^0} d\lambda + (N + 2)\text{Res}_{P_+}(xd\lambda) \right]. \quad (2.50)$$

Recall that $B_k^0$ is a $B$-cycle on $\Gamma_0$, which is covered twice by the $B_k$ cycle in $\Gamma$, and $P_+$ is the image of $P_1$ and $P_3$. This can be proved as in the previous section by differentiating the right hand side with respect to $a_k$, applying the Riemann bilinear relations, and recognizing $d\lambda$ as

$$d\lambda = (N + 2)d\Omega_2^+ + \sum_{k=1}^{N-1} 2\pi i a_k d\omega_k. \quad (2.51)$$

Here $d\Omega_2^+$ is an even normalized differential of the second kind with poles of the form

$$d\Omega_2^+ = \pm dx(1 + O(x^{-2})) \quad (2.52)$$

at the punctures $P_1$ and $P_3$. This differential can also be considered as a differential on $\Gamma_0 = \Gamma/\sigma$ with a pole at a point $P_+$, which is the projection of the points $P_1$ and $P_3$. Similarly, the differentials $d\omega_k$ are normalized even differentials on $\Gamma$ and therefore are preimages of the basic holomorphic differentials on $\Gamma_0$. The residue $\text{Res}_{P_+}(xd\lambda)$ can be easily calculated

$$\text{Res}_{P_1}(xd\lambda) = -\text{Res}_{P_3}(xd\lambda) = \text{Res}_{P_+}(xd\lambda) = 2u_{N-2} \quad (2.53)$$
In (2.50) we considered $d\lambda$ as a differential on $\Gamma_0$. If we lift it to $\Gamma$, $\mathcal{F}$ can be rewritten as

$$2\mathcal{F} = \frac{1}{4\pi i} \left[ \sum_{k=1}^N a_k \oint_{B_k^+} d\lambda + (N + 2)\text{Res}_{P_1}(xd\lambda) - (N + 2)\text{Res}_{P_3}(xd\lambda) \right]. \quad (2.54)$$

The closed formula for $\mathcal{F}$ can be converted as before into a renormalization group equation using the homogeneity relation $(\Lambda \partial_\Lambda + \sum_{k=2}^N a_k \partial a_k - 2)\mathcal{F} = 0$. The result is

$$\Lambda \frac{\partial}{\partial \Lambda} \mathcal{F} = \frac{N + 2}{\pi i} u_{N-2} \quad (2.55)$$

As noted before, the renormalization group equation can be used very effectively for the explicit evaluation of the contributions $\mathcal{F}_d$ of instanton processes.

### 2.3.3 The SU($N$) Yang-Mills theory with symmetric matter

For the $SU(N)$ theory with matter in the symmetric representation, Landsteiner and Lopez [19] have proposed the following spectral curve and differential

$$\Gamma : y^3 + P(x)y^2 + P(-x)x^2\Lambda^{N-2}y + \Lambda^{3(N-2)}x^6 = 0$$

$$d\lambda = \frac{dy}{y}, \quad (2.56)$$

where $P(x)$ is a polynomial as in the previous model, that is, it is monic, of degree $N$, with no $u_{N-1}x^{N-1}$ term. The weights for the symmetric representation are $e_i + e_j, i \leq j$, and the one-loop correction to the effective prepotential must then be of the form

$$\frac{1}{8\pi i} \left( \sum_{j \neq k} (a_j - a_k)^2 \ln \frac{(a_j - a_k)^2}{\Lambda^2} + \sum_{j \leq k} (a_j + a_k)^2 \ln \frac{(a_j + a_k)^2}{\Lambda^2} \right) \quad (2.57)$$

These logarithmic singularities can be derived from the proposed curve and differential in complete analogy with the previous models. We shall not give the details, but just the closed formula for the prepotential $\mathcal{F}$ and the corresponding renormalization group equation. As in the case of a hypermultiplet in the antisymmetric representation, the curve is a three-sheeted covering of the complex plane, with an involution $\sigma$ given this time by

$$\sigma : (x, y) \rightarrow (-x, \frac{\Lambda^{2(N-2)}x^4}{y}) \quad (2.58)$$

Let $P_1$ be the point above $x = \infty$ corresponding to $y \sim -x^N$, $P_3 = \sigma(P_1)$, and let the differentials $d\Omega_\pm$ be the even and odd Abelian differentials with double poles at $P_1$ and $P_3$, with the same normalizations as before (2.53). Then we have

$$d\lambda = (N - 2)d\Omega_+ + 2d\Omega_- + \sum_{k=2}^N 2\pi i a_k d\omega_k \quad (2.59)$$
from which the desired formula for $F$ follows

$$2F = \frac{1}{2\pi i} \left( \sum_{k=2}^{N} a_k \oint_{B_k} d\lambda + (4 - N) \text{Res}_{P_l}(x d\lambda) + N \text{Res}_{P_3}(x d\lambda) \right) \quad (2.60)$$

Evaluating the residues and expressing the equation in terms of $\Lambda$ derivatives, we obtain the following renormalization group equation

$$\Lambda \frac{\partial}{\partial \Lambda} F = \frac{N - 2}{\pi i} u_{N-2} \quad (2.61)$$

As could have been anticipated on general grounds, the coefficient of the beta function in each case is proportional to $2h_\vee - I(R)$ (the dual Coxeter number for $SU(N)$ is $h_\vee_{SU(N)} = N$, and the Dynkin index $I(R)$ is respectively $N - 2$ and $N + 2$ for the antisymmetric and the symmetric representations). A priori, there does not appear to be a reason for the remaining term $u_{N-2}$ in the beta function to be the Hamiltonian of an integrable dynamical system. This will however be shown in sections §3 and §4.

### 3 Hypermultiplets in the Adjoint Representation and Twisted Calogero-Moser Systems

#### 3.1 The $\mathcal{N} = 2$ SUSY Yang-Mills theory with adjoint hypermultiplet

In this section, we describe the solution of a class of supersymmetric gauge theories where the relation with integrable models plays a major role. These are the gauge theories with arbitrary simple gauge algebra $G$ and a matter hypermultiplet in the adjoint representation of $G$. Physically, they are of particular importance as scale invariant theories which admit a well-defined microscopic coupling $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$. They also admit scaling limits to many other theories of interest. These properties provide valuable clues in their eventual solution.

First, the parameter $\tau$ must figure in the Seiberg-Witten curves, and in view of Montonen-Olive duality, it is natural to expect that the Riemann surfaces $\Gamma(\tau)$ should be coverings of the torus $\mathbb{C}/\mathbb{Z} + \tau \mathbb{Z}$. Second, in the limit when the hypermultiplet becomes infinitely massive and decouples, the Seiberg-Witten solution of the theories should reduce to that of the pure Yang-Mills theory. Finally, as the hypermultiplet mass tends to 0, the theories acquire an $\mathcal{N} = 4$ symmetry, and the classical prepotential should receive no corrections.

We shall see how these considerations lead to the spectral curves of a new integrable model, namely twisted Calogero-Moser systems $[15, 16, 17]$.

#### 3.2 The $SU(N)$ theory and Hitchin Systems

The starting point is the solution first found by Donagi and Witten $[25]$ in the basic case where the gauge group is $SU(N)$. The key to their construction is a $SU(N)$ Hitchin
system, consisting of a $SU(N)$ connection coupled to a Higgs field $\phi$ on the torus $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ with a pole with given residue matrix at 0 \cite{1}. Let the matrix $\mu$ be defined by

$$
\mu = \begin{pmatrix}
1 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 1 & 0 \\
0 & \cdots & 0 & -(N-1)
\end{pmatrix}
$$

(3.1)

Then the moduli space

$$
X_\mu = \{(A, \phi); \bar{\partial}_A \phi = m \mu \delta(z,0), \; z \in \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}\}/\{\text{Gauge transformations}\},
$$

(3.2)

has the same dimension $N-1$ as the space of vacua of the four-dimensional gauge theory. The Seiberg-Witten exact solution for the $SU(N)$ theory with adjoint hypermultiplet is given by

$$
\Gamma = \{(k,z); \det(kI - \phi(z)) = 0\}, \; d\lambda = kdz
$$

(3.3)

By construction, it satisfies the Montonen-Olive $SL(2,\mathbb{Z})$ duality on $\tau$. It also satisfies the key scaling consistency checks described above. As $m \to 0$, the natural symplectic form on $X_\mu$ reduces to the uncoupled form $\omega = \sum_i dx_i \wedge da_i$ on $N-1$ copies of the torus. As $m \to \infty$, $\Gamma$ scales to $y^2 = \prod_{k=1}^{N} (x - \bar{a}_k)^2 - \Lambda^{2N}$, which is the Seiberg-Witten curve for the pure $SU(N)$ Yang-Mills theory that we had encountered before in §2.3.1.

### 3.3 The $SU(N)$ theory and Calogero-Moser systems

To see how twisted Calogero-Moser systems emerge in the solution of the general $\mathcal{N} = 2$ SUSY gauge theory with adjoint hypermultiplet and gauge group an arbitrary simple Lie group $G$, we need to reexamine the Donagi-Witten solution for $SU(N)$ in terms of $SU(N)$ Calogero-Moser systems. The $SU(N)$ Calogero-Moser system is the Hamiltonian system defined by

$$
H_{CM}^{SU(N)}(x, p) = \frac{1}{2} \sum_{i=1}^{N} p_i^2 - \frac{1}{2} m^2 \sum_{j \neq k} \wp(x_j - x_k)
$$

(3.4)

The basic property of the $SU(N)$ Calogero-Moser system is the existence of a Lax pair $L(z), M(z)$ with spectral parameter $z \in \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ discovered in \cite{30}

$$
\dot{x}_i = m^2 \sum_{j \neq i} \wp'(x_i - x_j) \leftrightarrow \dot{L}(z) = [M(z), L(z)]
$$

(3.5)

\footnote{Strictly speaking, $\wp(z)$ also has an essential singularity at 0, which can be gauged away by a singular gauge transformation. Alternatively, $\wp(z)$ is a meromorphic section, without essential singularity, of a suitable vector bundle on the torus}
Here $L(z)$, $M(z)$ are $N \times N$ matrices given explicitly by

\[
L_{ij} = \dot{x}_i \delta_{ij} - m(1 - \delta_{ij})\Phi(x_i - x_j, z) \\
M_{ij} = m \delta_{ij} \sum_{k \neq i} \varphi(x_k - x_i) + m(1 - \delta_{ij})\Phi'(x_i - x_j, z) 
\]

(3.6)

It is not difficult to see that $L(z)$, $M(z)$ is a Lax pair for the $SU(N)$ Calogero-Moser system if the function $\Phi(x, z)$ satisfies the following elliptic functional equation

\[
\Phi(x, z)\Phi'(y, z) - \Phi(y, z)\Phi'(x, z) = (\varphi(x) - \varphi(y))\Phi(x + y, z) 
\]

This functional equation is solved by

\[
\Phi(x, z) = \frac{\sigma(z - x)}{\sigma(z)\sigma(x)} e^{x\zeta(z)} 
\]

(3.8)

where $\sigma(z)$, $\zeta(z)$ are the usual elliptic Weierstrass functions. The fibration of spectral curves associated with the Lax pair $L(z), M(z)$ can now be defined by

\[
\Gamma = \{(k, z); \ det(kI - L(z)) = 0\}, \quad d\lambda = kdz 
\]

(3.9)

This fibration is the same as the fibration associated to the $SU(N)$ Hitchin system [7, 31]. In fact, the spectral curve $\Gamma(\tau)$ is clearly invariant under the transformation $L(z) \to L(z) = GL(z)G^{-1}$, for $N \times N$ matrices $G$. Choosing $G_{ij} = \delta_{ij} e^{\pm i\zeta(z)}$, we have $\Phi(x_i - x_j, z) \to \tilde{\Phi}(x_i - x_j, z) = \frac{\sigma(z-x_i+x_j)}{\sigma(z)\sigma(x_i-x_j)} = -\frac{1}{x_i-x_j} + \cdots$. This leads to $\tilde{L}_{ij}(z) = -m(1 - \delta_{ij})\frac{1}{x_i} + \cdots$, and

\[
\tilde{L}(z) - \frac{m}{z} = -\frac{m}{z} \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \sim -\frac{m}{z} \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & 0 & \vdots \\ 0 & 0 & N \end{pmatrix} \Rightarrow \tilde{L}(z) \sim \frac{m}{z} \mu + \cdots 
\]

(3.10)

This shows that $\tilde{L}(z)$ has exactly the same poles and residues as the Higgs field $\phi(z)$ of the $SU(N)$ Hitchin model. The equivalence of the two fibrations follows.

We check now the scaling limits. Clearly $m \to 0$ leads to a free system, so we concentrate on the limit $m \to \infty$. This limit was found in the late 1980’s by Inozemtsev [32], who showed that if $m$ tends to $\infty$ according to the following rule

\[
m = M q^{-\frac{i}{2N}}, \quad q = e^{2\pi i\tau} \to 0, \quad \omega_1 = -i\pi 
\]

(3.11)

and if new dynamical variables $(X_i, P_i)$ are defined by

\[
x_i = X_i - 2 i \omega_2 \frac{i}{N}, \quad p_i = P_i 
\]

(3.12)

then the Hamiltonian $H_{CM}^{SU(N)}$ tends to the Hamiltonian for the Toda system associated to the affine Lie algebra $SU(N)^{(1)}$}

\[
H_{CM}^{SU(N)} \to H_{Toda}^{SU(N)^{(1)}} = \frac{1}{2} \sum_{i=1}^{N} P_i^2 - \frac{1}{2} M^2 \sum_{i=1}^{N} e^{X_{i+1} - X_i}, \quad X_{N+1} = X_1. 
\]

(3.13)
The basic mechanism of this scaling limit is the following asymptotics for the Weierstrass \( \wp \) function

\[
\wp(u) = \frac{\eta_1}{i\pi} + \sum_{n=-\infty}^{\infty} \frac{1}{sh(u - 2n\omega_2)} \sim e^u + e^{-2\omega_2 - u}
\]

(3.14)

for \(-2\omega_2 < u < 0\). In the potential for the \( SU(N) \) Calogero-Moser system, we may assume without loss of generality that \( i > j \). Then \( x_i - x_j = (X_i - X_j) - \frac{2\omega_2}{N}(i - j) \) is in the range \((-2\omega_2, 0)\), and we have

\[
m^2 \wp(x_i - x_j) = M^2 \{ e^{X_i - X_j} e^{\frac{2\omega_2}{N}(1-i+j)} + e^{-(X_i - X_j)} e^{\frac{2\omega_2}{N}(1+i-j-N)} \}
\]

(3.15)

The only surviving terms arise from \( x_{i+1} - x_i \), \( 1 \leq i \leq N-1 \), and \( x_1 - x_N \). This establishes the scaling limit of the Hamiltonian asserted above. The Lax pair also admits a scaling limit

\[
L(z) \to L_{Toda}(Z), \quad e^z = Zq^{\frac{1}{2}}, \quad q \to 0
\]

(3.16)

since the function \( \Phi(u, z) \) scales as

\[
\Phi(u, z) \to \begin{cases} 
e^{-\frac{1}{2}u}(1 - Z^{-1}e^{u-\omega_2}), & \text{if } \text{Re } u \to +\infty; \\ -e^{\frac{1}{2}u}(1 - Z^{-1}e^{-u-\omega_2}), & \text{if } \text{Re } u \to -\infty. \end{cases}
\]

(3.17)

The existence of the scaling limit of the Lax pair insures the existence of the scaling limit of the fibration of spectral curves. In fact, it is easy to derive from the scaling limit of \( \Phi(u, z) \) an explicit formula for \( L_{Toda}(Z) \), and hence for the spectral curves of the Toda system

\[
\Gamma_{Toda} = \{(k, Z); \det(kI - L_{Toda}(Z)) = 0\}
\]

(3.18)

We find \( \det(kI - L_{Toda}(Z)) = Z + \frac{MN}{Z} - P(k) \), which is the Seiberg-Witten curve for the pure Yang-Mills theory. Thus the consistency of the scaling limit as \( m \to \infty \) has also been verified from the point of view of integrable models.

We have now recaptured the essential features of the Donagi-Witten solution of the \( SU(N) \) gauge theory in terms of Calogero-Moser systems. It turns out that the Calogero-Moser system formulation also provides another important check, namely that the effective prepotential \( F \) defined by its spectral curves does satisfy the logarithmic singularities required of the effective prepotential for the \( SU(N) \) gauge theory. This is because the \( SU(N) \) Calogero-Moser spectral curves turn out to admit a parametrization in closed form, from which the classical order parameters of the four-dimensional gauge theory can be read off, and the perturbative expansion of the periods \( a_{Dk} \) evaluated \[^{33}\].

### 3.4 G Calogero-Moser systems and \( G^{(1)} \) affine Toda systems

The \( SU(N) \) Calogero-Moser system admits a generalization to any simple Lie group \( G \), as introduced by Olshanetsky and Perelomov \[^{34}\] in the mid 1970’s

\[
H^G_{CM} = \frac{1}{2} \sum_{i=1}^{n} p_i^2 - \frac{1}{2} \sum_{\alpha \in \mathcal{R}(G)} m^2_{\alpha} \varphi(\alpha \cdot x)
\]

(3.19)
where $m_{[\alpha]}$ is a mass parameter which depends only on the length of the root $\alpha$. It is natural to look to these systems for the solution of the supersymmetric four-dimensional gauge theory with gauge group $G$, but for this, we need to examine their scaling limits. For $SU(N)$, we have seen that the root lattice of $SU(N)$ reduces to the set of simple roots for the affine Lie algebra $SU(N)_{(1)}$. This scaling limit can be generalized to all simple Lie algebras as follows [16]. We have

\[ m^2 \sum_{\alpha \in R(G)} \varphi(\alpha \cdot x) \to M^2 \sum_{\alpha \in R(G^{(1)}) \text{ simple}} e^{\alpha \cdot X} \]  

(3.20)

if $m^2$, $x$, and $p_i$ scale according to

\[ m^2 = M^2 q^{\frac{1}{h_G}} \]

\[ x = X - \frac{2\omega_2}{h_G} \rho^\vee, \quad p = P \]

(3.21)

where $\rho^\vee$ is the level vector and $h_G$ is the Coxeter number of $G$. The right hand side of (3.20) defines the Hamiltonian of the Toda system associated to the affine Lie algebra $G^{(1)}$, so that our result can be restated simply as

\[ H^G_{CM} \to H^{G^{(1)}}_{Toda} \]  

(3.22)

### 3.5 $G$ Twisted Calogero-Moser systems and $(G^{(1)})^\vee$ affine Toda systems

The scaling limit described above for the $G$ Calogero-Moser system is however not suitable for the Seiberg-Witten solution of the supersymmetric gauge theory with gauge group $G$. According to the renormalization group, the decoupling of the matter hypermultiplet should rather obey the following scaling law

\[ m^2 = M^2 q^{\frac{1}{h_G'}} \]

(3.23)

where $h_G'$ is the dual Coxeter number of $G$. When $G$ is not simply-laced, we have $h_G' < h_G$, and the $G$ Calogero-Moser system does not admit a finite limit under this scaling. Thus new generalizations of the $SU(N)$ elliptic Calogero-Moser system admitting finite limits under the scaling (3.23) are required. It turns out that these new systems are the twisted $G$ Calogero-Moser systems defined as follows [13]

\[ H^G_{twisted} = \frac{1}{2} P^2 - \frac{1}{2} \sum_{\alpha \in R(G)} m_{[\alpha]} \varphi(\nu(\alpha) \cdot x) \]

(3.24)

where $\nu(\alpha) = 1$ if $\alpha$ is a long root, $\nu(\alpha) = 2$ if $\alpha$ is a short root of $B_n, C_n, F_4$, and $\nu(\alpha) = 3$ if $\alpha$ is a short root of $G_2$. The key to the twisting is the improved asymptotics for the
twisted Weierstrass \( \wp \)-function

\[
\wp_\nu(u) = \sum_{k=0}^{\nu-1} \wp(u + 2\omega_2 k) = \frac{\nu^2}{2} \sum_{n=-\infty}^{\infty} \frac{1}{\text{ch} \nu(u - 2n\omega_2) - 1}, \quad \nu = 1, 2, 3.
\] (3.25)

Then under (3.23), with new dynamical variables \( X_i, P_i \) defined by

\[
x = X - \frac{2\omega_2}{h^\vee G} \rho, \quad p = P
\] (3.26)

where \( \rho \) is the Weyl vector, the improved asymptotics lead to the finite limit

\[
H_{\text{twisted}}^G \rightarrow H_{\text{Toda}}^{(G^{(1)})^\vee}
\] (3.27)

where \( H_{\text{Toda}}^{(G^{(1)})^\vee} \) is the Toda Hamiltonian associated to the dual of the affine Lie algebra \( G^{(1)} \). The emergence of \( H_{\text{Toda}}^{(G^{(1)})^\vee} \) as the limit of the twisted \( G \) Calogero-Moser system is another confirmation of the latter system as the solution of the \( G \) gauge theory with an adjoint hypermultiplet. Indeed, generalizing the case of \( SU(N) \), the system \( H_{\text{Toda}}^{(G^{(1)})^\vee} \) has been shown by Martinec and Warner [6] to be the solution of the supersymmetric pure Yang-Mills theory. Conversely, this result would follow from the solution of the theory with adjoint hypermultiplet by twisted Calogero-Moser systems.

### 3.6 Lax pairs with spectral parameter for Calogero-Moser systems

The twisted \( G \) Calogero-Moser systems have now been shown to be the correct models for the Seiberg-Witten solution of the \( G \) gauge theory with an adjoint hypermultiplet. However, for all Lie groups except \( SU(N) \), we still have to establish their integrability and the existence of a Lax pair with spectral parameter. Even the integrability of the untwisted \( G \) Calogero-Moser systems had not fully been established prior to [15]. The only known results at that time went back to the 1975 work of Olshanetsky and Perelomov [34], and were as follows

- \( G \) classical (\( B_n, C_n, \text{or} \ D_n \)): a Lax pair was known, but without spectral parameter.
- \( G \) exceptional (\( G_2, F_4, E_6, E_7, E_8 \)): no Lax pair was known.

The integrability of twisted Calogero-Moser systems had of course not even been an issue at this point. It turned out that all these systems admit Lax pairs with spectral parameter, and we now describe these Lax pairs.

#### 3.6.1 Construction of the Lax pairs

A major difficulty in the search for a Lax pair in the case of a general simple Lie algebra \( G \) is that, unlike in the case of \( SU(N) \), it cannot be found in the Lie algebra \( G \). Rather, we proceed as follows.
Let $\Lambda : G \rightarrow GL(N, \mathbb{C})$ be a representation of $G$ of dimension $N$, $\lambda_I (I = 1, \cdots, N)$ its weights, and
\[ \alpha_{IJ} = \lambda_I - \lambda_J \] (3.28)
the weights of $\Lambda \otimes \Lambda^*$. Let $h_i, i = 1, \cdots, n$ be generators of the Cartan subalgebra $H_G$ of $G$, and let $\tilde{h}_j, j = n + 1, \cdots, N$, satisfy $[h_i, \tilde{h}_j] = [\tilde{h}_i, \tilde{h}_j] = 0$. Let $H$ be generated by $h_i \oplus \tilde{h}_j$. Let $u_I$ be the weights of the fundamental representation of $GL(N, \mathbb{C})$. We can write
\[ su_I = \lambda_I + v_I \quad \text{with} \quad \lambda_I \perp v_J, \] (3.29)
where $s^2 = \frac{1}{n} \sum_{I=1}^{N} \lambda_I \cdot \lambda_I$ is the Dynkin index. Let $E_{IJ} = u_I u_J^T$ be the generators of $GL(N, \mathbb{C})$. We now look for a Lax pair $L(z), M(z)$ under the following form
\[ L = P + X, \quad M = D + Y \] (3.30)
where
\[
\begin{align*}
X &= \sum_{I \neq J} C_{IJ} \Phi_{IJ}(\alpha_{IJ} \cdot x, z) E_{IJ} \\
P &= p \cdot h \\
Y &= \sum_{I \neq J} C_{IJ} \Phi'_{IJ}(\alpha_{IJ} \cdot x, z) E_{IJ} \\
D &= d(h \oplus \tilde{h}) + \Delta
\end{align*}
\] (3.31)
with the coefficients $C_{IJ}$ and the functions $\Phi_{IJ}(x, z)$ yet to be determined. We observe that the case of $SU(N)$ corresponds to all functions $\Phi_{IJ}(u, z)$ equal to $\Phi(u, z)$, and that the coefficients $C_{IJ}$ are equivalent to the matrix of residues $\mu$. In the general case, they have to be solved for. To do so, we introduce the following notation
\[
\begin{align*}
\Phi_{IJ} &= \Phi_{IJ}(\alpha_{IJ} \cdot x, z) \\
\varphi'_{IJ} &= \Phi_{IJ}(\alpha \cdot x, z)\Phi'_{IJ}(-\alpha \cdot x, z) - \Phi_{IJ}(-\alpha \cdot x, z)\Phi'_{IJ}(\alpha \cdot x, z)
\end{align*}
\] (3.32)
Then the matrices $L(z), M(z)$ are a Lax pair for the (twisted or untwisted) Calogero-Moser system if and only if the following functional equations are satisfied
\[
\begin{align*}
\sum_{I \neq J} C_{IJ} C_{JK} \varphi'_{IJ}(v_I - v_J) &= 0 \\
\sum_{I \neq J} C_{IJ} C_{IK} \varphi'_{IJ}(v_I - v_J) &= 0 \\
\sum_{K \neq I, J} C_{IK} C_{KJ} (\Phi_{IK} \Phi'_{KJ} - \Phi'_{IK} \Phi_{KJ}) &= \sum_{K \neq I, J} \Delta_{IJ} C_{KJ} \Phi_{KJ} - \sum_{K \neq I, J} C_{IK} \Phi_{JK} \Delta_{KJ} + sC_{IJ} \Phi_{IJ} d \cdot (v_I - v_J)
\end{align*}
\] (3.33)
More specifically, it turns out that the condition $\dot{x} = p$ is equivalent to $\dot{X} = [P, Y]$. The second condition above combined with the Calogero-Moser equations of motion are
equivalent to $\hat{P} = [X,Y]_H$, where the subscript $H$ denotes projection onto that subspace. Finally, the third condition above is equivalent to $[X,Y]_{GL(N,C) \otimes H} + [X,d \cdot (h \oplus \tilde{h}) + \Delta] = 0$.

**Theorem 1.** [13]. Lax pairs $L(z), M(z)$ with spectral parameter $z \in \mathbb{C}/\mathbb{Z} + \tau \mathbb{Z}$ of the above form can be found for both twisted and untwisted Calogero-Moser systems, for all simple Lie algebras $G$, except possibly in the case of twisted $G_2$. In the case of $E_8$, we have to assume the existence of a sign assignment satisfying a cocycle-type condition.

### 3.6.2 The scaling limit of the Lax pair

The basic property of the Lax pairs for the twisted and untwisted Calogero-Moser systems which we just constructed is

**Theorem 2.** [16]. All Lax pairs constructed above admit a finite scaling limit. More precisely, the limit is taken with respect to the scaling law (3.21) for untwisted Calogero-Moser systems, and with respect to (3.23) for twisted Calogero-Moser systems.

We have seen that the scaling limit of the Calogero-Moser Hamiltonian was a consequence of some precise asymptotics for the Weierstrass $\wp$-function. For the Lax pairs, we need the precise asymptotics of the functions $\Phi_{IJ}(u,z)$. In the case of untwisted Calogero-Moser systems, set

$$C_{IJ} = M_{[\alpha]} e^{\delta \omega_2} c_{IJ}, \quad Z = e^{z \frac{q-1}{2}}$$

(3.34)

All the functions $\Phi_{IJ}(u,z)$ are given in this case by $\Phi(u,z)$, and the scaling limit of the Lax pair $L(z), M(z)$ followed from

$$L : C_{IJ} \Phi(\alpha \cdot x, z) \to \begin{cases} 
\pm c_{IJ} e^{\frac{i}{2} \alpha \cdot X}, & \text{if } l(\alpha) = \pm 1; \\
\mp c_{IJ} e^{\frac{i}{2} \alpha \cdot X}, & \text{if } l(\alpha) = \pm \ell_0; \\
0 & \text{if otherwise.}
\end{cases}$$

(3.35)

$$M : C_{IJ} \Phi'(\alpha \cdot x, z) \to \frac{1}{2} \epsilon_\alpha \lim C_{IJ} \Phi(\alpha \cdot x, z), \quad \epsilon_\alpha = \begin{cases} 
1, & \text{if } l(\alpha) = l_0 \text{ or } l(\alpha) = -1; \\
-1, & \text{if } l(\alpha) = -l_0 \text{ or } l(\alpha) = 1.
\end{cases}$$

In the case of twisted Calogero-Moser systems, set instead $C_{IJ} = M_{[\alpha]} e^{\delta \omega_2} c_{IJ}$. In this case, several distinct functions $\Phi_{IJ}(u,z)$ arise

$$\Phi_1(u,z) = \Phi(u,z) - \Phi(u + \omega_1, z) e^{\pi i \zeta(z) + z \zeta(\omega_1)}$$

$$\Phi_2(u,z) = \frac{\Phi(u,z) \Phi(u + \omega_1, z)}{\Phi(\omega_1, z)}$$

(3.36)

as well as $\Phi_2(u + \epsilon_{IJ} \omega_2, z)$ where $\epsilon_{IJ} = \pm 1$. The existence of the scaling limit of the Lax pairs for the twisted Calogero-Moser systems follows from the scaling limits of $\Phi_1(u,z), \Phi_2(u,z)$,

$$\Phi_1(u,z) \to \mp 2Z^{\mp 1} e^{\frac{i}{2} u - \omega_2}, \quad u \to \pm \infty$$

$$\Phi_2(u,z) \to \pm 2 e^{\mp u} (1 - Z^{\mp 1} e^{\pm 2u - 2\omega_2}), \quad u \to \pm \infty$$

(3.37)
3.7 Exact solution for super Yang-Mills with adjoint hypermultiplet

The Lax pair $L(z), M(z)$ of the twisted $G$ Calogero-Moser system provides now the Seiberg-Witten solution of the $\mathcal{N} = 2$ supersymmetric gauge theory with gauge group $G$ and a hypermultiplet in the adjoint representation [L7]. It is given by

$$\Gamma = \{(k, z); \ det(kI - L(z)) = 0\}, \ d\lambda = k dz$$

(3.38)

This can be checked explicitly for low $D_n$ and low rank $n$, by working out the logarithmic singularities of the above fibration in the trigonometric limit. In this limit,

$$\varphi(z) \rightarrow \frac{1}{Z^2} - \frac{1}{6}, \ \Phi(x, z) \rightarrow \frac{1}{2} \coth \frac{1}{2} x - \frac{1}{Z}, \ \frac{1}{Z} = \frac{1}{2} \coth \frac{1}{2} x$$

(3.39)

and the equation for the fibration simplifies considerably

$$\det (kI - L(z)) = \frac{m^2 + mA - 2kZ}{m^2 + 2mA} H(A) + \frac{mA + 2kZ}{m^2 + 2mA} H(A + m)$$

(3.40)

Here the moduli vacua are parametrized now by monic polynomials $H(A)$ of the form $H(A) = \prod_{j=1}^{n} (A^2 - p_j^2)$, and the variable $A$ is related to the variables $(k, z)$ by

$$A^2 + mA + 2kZ - k^2 = 0$$

(3.41)

The Seiberg-Witten form $d\lambda$ can then be re-expressed as

$$d\lambda = -Adu, \ e^u = \frac{(k + A + m)(k - A - m)}{k^2 - A^2}$$

(3.42)

The methods of [33] lead then to the desired prepotential

$$\mathcal{F} = -\frac{1}{8\pi i} \sum_{\alpha \in \mathbb{R}(D_n)} (\alpha \cdot \phi)^2 \ln (\alpha \cdot \phi)^2 - (\alpha \cdot \phi + m)^2 \ln (\alpha \cdot \phi + m)^2$$

(3.43)

3.8 Other developments

There has been many related developments since the works [15, 16, 17].

Seiberg-Witten solutions for the $\mathcal{N} = 2$ gauge theory with gauge group $G$ and several hypermultiplets in the adjoint representation have been proposed by Uranga and Yokono [35], under the assumption of total zero mass.

Lax pairs with spectral parameter for both twisted and untwisted Calogero-Moser systems, including the case of twisted $G_2$, have been constructed by Bordner, Sasaki, Corrigan, et al. [36]. These Lax pairs are different from the ones constructed in [15].
particular, they do not admit finite scaling limits under (3.23), and are not candidates for
the Seiberg-Witten solution of the $G$ gauge theory with adjoint hypermultiplet.

It is intriguing that, except for $SU(N)$, the Lax pairs obtained so far do not fit in
the framework of classical Hitchin systems. Recently, Hurtubise and Markman [37] have
proposed a new general for modified Hitchin systems, which can incorporate both Lax
pairs constructed in [15] and in [30], at least for the case of untwisted Calogero-Moser
systems.

The Calogero-Moser systems can be viewed as non-relativistic limits of Ruijsenaars-
Schneider systems [38]. Except for $SU(N)$, the integrability of these systems is still in-
completely understood. It may be hoped that advances in the theory of Calogero-Moser
systems may help advances on Ruijsenaars-Schneider systems. Recently, there has been
progress on the integrability of Ruijsenaars-Schneider systems for certain classical algebras,
thanks to the works of B.Y. Hou et al. [39].

Ruijsenaars-Schneider systems have been proposed by H. Braden, A. Marshakov, A.
Mironov, and A. Morozov [26] as the Seiberg-Witten solution of supersymmetric gauge
theories in dimensions 5 and 6, with the relativization coming from the contributions of the
infinite tower of Kaluza-Klein modes. This is an imrant direction of investigation, since
supersymmetric gauge theories in 5 and 6 dimensions are difficult to explore otherwise.

Relations with the E-string and the reduction on $T^2$ of the 6-dimensional (2,0) theory
may be found in [40]. Recently, Dijkgraaf and Vafa [11] have proposed a promising relation
between the effective prepotential for supersymmetric gauge theories and matrix models.
Some developments in this direction and related to the issues discussed here can be found
in [42].

4 New Spin Chain Models from M Theory

In the previous section, we have seen how integrable models can produce the Seiberg-
Witten exact solution of a supersymmetric gauge theory. The correspondence can also go
the other way: here we discuss how the Seiberg-Witten solution of a gauge theory, in this
case the $SU(N)$ theory with a hypermultiplet in either the symmetric or the antisymmetric
representation, can lead to new integrable models.

4.1 A periodic generalized spin chain model

We consider first the case of a hypermultiplet in the antisymmetric representation. Recall
that the fibration $\Gamma(\Lambda)$ and differential $d\lambda$ have been found by Landsteiner and Lopez, and
are given by (2.29). Postponing for the moment the choice of $d\lambda$, our problem is to find
$L(x), M(x)$ satisfying a Lax equation, with the spectrum of $L(x)$ determined by $\Gamma(\Lambda)$. We
shall look for such a Lax pair in a (generalized) spin chain model [21]. Henceforth, we set
$\Lambda = 1$ for notational simplicity.
In general, an integrable chain model \( \{ \psi_n \}_{n \in \mathbb{Z}} \) is defined as the compatibility condition for two linear equations of the form

\[
\begin{align*}
\dot{\psi}_{n+1} &= L_n(x)\psi_n \\
\psi_n &= M_n(x)\psi_n
\end{align*}
\]

\Rightarrow \dot{L}_n(x) = M_{n+1}(x)L_n(x) - L_n(x)M_n(x) \quad (4.1)

Under the periodicity condition \( L_{N+2}(x) = L_0(x) \), \( M_{N+2}(x) = M_0(x) \), the equation on the right hand side implies the Lax pair equation

\[
\dot{L}(x) = [M(x), L(x)] \quad (4.2)
\]

where \( L(x) \) is defined by \( L(x) = L_{N+1}(x)L_N(x) \cdots L_0(x) \equiv \prod_{j=0}^{N+1} L_j(x) \), and \( M(x) = M_0(x) \). The spectral curve \( \Gamma \) of such an integrable system can then be defined as usual by

\[
\Gamma = \{ (x, y) ; \det(yI - L(x)) = 0 \} \quad (4.3)
\]

Returning to the construction of the desired integrable model, the key property is the invariance of the Landsteiner-Lopez curve under the involution

\[
\sigma : (x, y) \leftrightarrow (-x, y^{-1}) \quad (4.4)
\]

We see then that the spectral curve \( \Gamma \) would reproduce the Landsteiner-Lopez curve if the matrix \( L(x) \) is \( 3 \times 3 \), and satisfies

\[
\det L(x) = 1, \quad L(x)^{-1} = -L(x), \quad Tr L(x) = 3 + \mathcal{O}(x^2) \quad (4.5)
\]

In analogy with the \( 2 \times 2 \) Lax matrix used in [44] for the integration of a quasi-classical approximation to a system of reggeons in QCD, we can achieve this by setting

\[
L_n(x) = 1 + x s_n s_n^T, \quad M_n(x) = x \frac{1}{s_{n+1}^T s_n} (s_{n-1} s_n^T + s_n s_{n-1}^T) \quad (4.6)
\]

where \( s_n \) is a periodic sequence of complex 3-vectors satisfying \( s_n^T s_n = 0 \), \( s_{n+N+2} = s_n \). We obtain in this way an integrable model, with \( s_n \) as dynamical variables satisfying the equation of motion

\[
\dot{s}_n = \frac{s_{n+1}}{s_{n+1}^T s_n} - \frac{s_{n-1}}{s_{n-1}^T s_n} \quad (4.7)
\]

This integrable model admits the same spectral curves as the \( SU(N) \) gauge theory with a hypermultiplet in the antisymmetric representation. However, it turns out that the associated differentials \( d\lambda \) in the two theories do not coincide. The reason is the parity of the differential \( d\lambda \) under the involution \( (x, y) \rightarrow (-x, y^{-1}) \), which requires instead the following model

\[
\begin{align*}
\dot{p}_n &= \frac{p_{n+1}}{p_{n+1}^T q_n} + \frac{p_{n-1}}{p_{n-1}^T q_n} + \mu_n p_n, \quad \dot{q}_n = -\frac{q_{n+1}}{p_{n+1}^T q_{n+1}} - \frac{q_{n-1}}{p_{n-1}^T q_{n-1}} - \mu_n q_n
\end{align*} \quad (4.8)
\]
Here \( \mu_n \) is an arbitrary multiplier, and the dynamical variables \( q_n, p_n \) are complex 3-vectors, satisfying the following conditions

\[
q_{n+N+2} = q_n, \quad p_{n+N+2} = p_n \tag{4.9}
\]

\[
p_T^T q_n = 0, \quad p_n = g_0 p_{n-1}, \quad q_n = g_0 q_{n-1} \tag{4.10}
\]

where \( g_0 \) is the \( 3 \times 3 \) diagonal matrix with \( g_{ii} = (-1)^{i+1} \). Then the system admits a Lax pair \( L(x), M(x) \) given by

\[
L(x) = \prod_{n=0}^{N+1} (1 + x q_n p_n^T), \quad M(x) = x \left( \frac{q_{n+1} p_n^T}{p_0 q_{n+1}} - \frac{q_0 p_{n+1}}{p_T^T p_{n+1} q_0} \right) \tag{4.11}
\]

The corresponding spectral curve \( \Gamma = \{(x,y); \det (yI - L(x)) = 0\} \) is the Landsteiner-Lopez curve. Furthermore, there is a correspondence between the variables \( (q_n, p_n) \) and pairs \( (\Gamma, [D]) \)

\[
(q_n, p_n) \leftrightarrow (\Gamma, [D]) \tag{4.12}
\]

where \( [D] = [z_1, \cdots, z_{2N+1}] \) is a divisor even under the involution \( \sigma \). For given \( (q_n, p_n) \), \( [D] \) is the divisor of poles of the Bloch eigenfunction \( \psi_n(x, y) \) defined by \( \psi_{n+1}(x, y) = L_n(x) \psi_n(x), \psi_{n+N+2}(x, y) = y \psi_n(x, y) \), for \( (x, y) \in \Gamma \). We shall denote also denote the points \( (x, y) \) on \( \Gamma \) by \( Q \). Let the action variables \( a_i \) and the angle variables \( \phi_i \) be defined on the \( 2(N-1) \)-dimensional space \( \mathcal{M}_0 \) by

\[
a_i = \oint_{A_i} d\lambda, \quad \phi_i = \sum_{i=1}^{2N+1} \int_{z_i} d\omega_i \tag{4.13}
\]

where \( \{A_i\}_{1 \leq i \leq N-1} \) and \( \{d\omega_i\}_{1 \leq i \leq N-1} \), are respectively a basis for the even cycles and a basis for the even holomorphic differentials on \( \Gamma \). Then the form \( \omega = \delta(\sum d\lambda(z_i)) \) defines a symplectic form on \( \mathcal{M}_0 \) which can also be expressed as

\[
\omega = \sum_{i=1}^{N-1} \delta a_i \wedge \delta \phi_i \tag{4.14}
\]

The dynamical system \( (4.8) \) is Hamiltonian with respect to this symplectic form, with Hamiltonian

\[
H = u_{N-2} = \sum_{n=0}^{N+1} \frac{(p_T^T q_{n-3})}{(p_T^T q_{n-1})(p_T^T q_{n-2})} - \frac{(p_T^T q_{n-2})^2}{2(p_T^T q_{n-1})^2(p_T^T q_{n-2})^2} \tag{4.15}
\]

Thus the system \( (4.8) \) is the integrable model that we were looking for. The correspondence between the Hamiltonian structure for the dynamical variables \( (q_n, p_n) \) and the geometric symplectic form \( \omega \) for \( (\Gamma, [D]) \) depends fundamentally on the fact that they both coincide with a symplectic form which can be defined in terms of the Lax pair

\[
\omega = \frac{1}{2} \sum_{\alpha=1}^{3} \text{Res}_{P_\alpha} \langle \Psi^*_{n+1}(Q) \delta L_n(x) \wedge \delta \Psi_n(Q) \rangle dx \tag{4.16}
\]

where \( P_\alpha \) are the points in \( \Gamma \) lying above \( x = \infty \).
4.2 Models with twisted monodromies

We turn now to the problem of finding an integrable model corresponding to the Seiberg-Witten solution of the $SU(N)$ gauge theory with matter in the symmetric representation. In this case, the spectral curves still admit an involution $\sigma$, but which is now of the form

$$\sigma : (x, y) \rightarrow (-x, x^4y^{-1})$$  \hspace{1cm} (4.17)

As suggested earlier \cite{2}, such shifts are indicative of twisted monodromy conditions. We consider then the following dynamical system \cite{21}

$$
\begin{align*}
\dot{p}_n &= \frac{p_{n+1}}{p_{n+1}q_n} + \frac{p_n}{p_nq_{n+1}} + \mu_n p_n, \\
\dot{q}_n &= -\frac{q_{n+1}}{p_nq_{n+1}} - \frac{q_n}{p_nq_{n+1}} - \mu_n p_n, \\
\dot{a} &= \left\{ \frac{q_m-1P_m}{p_mq_{m+1}}, b \right\}, \\
\dot{b} &= \left\{ \frac{q_m-1P_m}{p_mq_{m+1}}, c \right\}, \\
\dot{c} &= 0
\end{align*}
$$  \hspace{1cm} (4.18)

Here $\mu_n(t)$ is again an arbitrary scalar function, and we have set $m = -\frac{N}{2} + 1$ for $N$ even and $m = -\frac{N}{2} + \frac{1}{2}$ for $N$ odd. The above system appears uncoupled, but it will not be after imposing twisted monodromy conditions on $(q_n, p_n)$. More precisely, let $L_n(x) = 1 + xq_nP_nT$ as before. Then

- There are unique $3 \times 3$ matrices $g_n(x) = a_nx^2 + b_nx + c_n$ which satisfy the periodicity condition

$$g_{n+1}L_{n+N-2} = L_n g_n$$  \hspace{1cm} (4.21)

for any fixed data $a_r, b_r, c_r, (p_n, q_n)_{n=r-N+3}$ with the constraint $q_nT p_n = 0$.

- The above dynamical system with $a_m = ah, b_m = bh, c_m = ch$ is integrable, in the sense that it is equivalent to the following Lax equation

$$
\dot{L}_n = M_{n+1}L_n - L_n M_n, \quad M_n(x) \equiv x \left( \frac{q_{n+1}P_n}{p_{n+1}q_{n+1}} - \frac{q_nP_n}{p_nq_n} \right)
$$  \hspace{1cm} (4.22)

- The spectral curve $\Gamma = \{(x, y); \det(yI - g_n(x)L_{n+N-3}(x) \cdots L_n(x)) = 0\}$ is independent of $n$ and coincides with the Landsteiner-Lopez curve \cite{22}. The dynamical system \cite{4,18} is Hamiltonian with respect to the symplectic form $\omega = \sum_{i=1}^{N-2} \delta x(z_i) \wedge \delta y(z_i)$ on the reduced phase space $u_N = 1, u_{N-1} = 0$, through the usual correspondence between dynamical variables and curves and divisors. The Hamiltonian is $H = u_{N-2}$.

- The symplectic form $\omega$ can also be expressed in terms of the Lax operator $L_n(x)$ and the matrices $g_k(x)$ defining the twisted monodromy conditions

$$\omega = \frac{1}{2} \sum_{a=1}^{3} \text{Res}_{P_a} \left( \langle \psi_{n+1}^* (Q) \delta L_n(x) \wedge \delta \psi_n (Q) \rangle_k + \psi_{n}^* (\delta g_k g_k^{-1}) \wedge \delta \psi_k \right) dx$$  \hspace{1cm} (4.23)
Here $\langle f_n \rangle_k$ is defined to be $\sum_{n=k}^{N+k-3} f_n$, $\psi_n(Q)$ is the Baker-Akhiezer function, and $\psi_n^*(Q)$ the dual Baker-Akhiezer function characterized by

$$\psi_{n+1}^*(Q)L_n(x) = \psi_n^*(Q), \quad \psi_{k+N-2g_k}^{-1}(Q) = y^{-1}\psi_n^*(Q), \quad \psi_k^*(Q)\psi_k(Q) = 1. \quad (4.24)$$

In summary, we have

**Theorem 3.** \[21, 22\] The Seiberg-Witten solution for both $SU(N)$ gauge theories with a hypermultiplet in either the symmetric or the antisymmetric representation can be realized as the spectral curves and symplectic form of a Hamiltonian system admitting a Lax pair representation. These systems are given by spin chains, with twisted monodromy in the case of the symmetric representation. The Hamiltonian for each model turn is the beta function of the corresponding gauge theory.

## 5 Hamiltonian Formulation of Soliton Equations

The correspondence between $\mathcal{N} = 2$ supersymmetric gauge theories and integrable models begins with the identification of their spectral curves. However, as we have seen in the case of the $SU(N)$ with a hypermultiplet in the antisymmetric representation, the identification of spectral curves has to be supplemented by an identification of symplectic structures. More precisely, on the gauge theory side, the Seiberg-Witten meromorphic form $d\lambda$ equips a moduli space of spectral curves and divisors with a symplectic form. On the integrable model side, the phase space must be then equipped with a corresponding symplectic structure with respect to which the integrable model is Hamiltonian. What is this symplectic structure? Since the integrable model is given by its Lax pair, what is needed is a general construction of a Hamiltonian structure for an integrable model directly from its Lax representation. Such a construction was found in [3] [4], and developed further in [13]. The essential features of this symplectic form are summarized in the following general formula

$$\omega = \frac{1}{2} \sum_{\alpha} \text{Res}_\alpha \langle \psi^\dagger(x, k)\delta L(x) \wedge \delta \psi(x, k) \rangle dk \quad (5.1)$$

The set-up for this formula is broadly as follows. The phase space of the system is a space $\mathcal{L}$ of operators $L(x)$. The operators $L(x)$ can be finite-dimensional matrices, or differential operators in the variable $x$. The expression $\psi(x, k)$ is the Baker-Akhiezer (or Bloch) function, which is an eigenvector of $L(x)$. The variable $k$ is the spectral parameter, or an analogous quantity \[2\]. The expression $\psi^\dagger(x, k)$ is the analogous dual Baker-Akhiezer

\[2\] The variable $k$ here is the analogue of the variable $x$ of §4, and of the variable $z$ of §3. The notation of (5.1) is consistent with the case of the Korteweg-deVries equation, where $x$ corresponds to the variable in $L(x) = \partial_x^2 + u(x)$. Unfortunately, the diversity of integrable models and entrenched practices makes attempts at a uniform notation impractical.
function, which should be viewed as a row vector, if $\psi$ is a column vector. The functions $\psi(x, k)$, $\psi^\dagger(x, k)$ have compensating essential singularities in $k$, so that all combined expressions are meromorphic. The differential $\delta$ denotes exterior differentiation on the space $L$, and $\langle \cdot \rangle$ denotes averaging with respect to all variables $x$. The points $P_\alpha$ are given fixed poles. Thus (5.1) defines a 2-form on the phase space $L$.

The above formula provides a universal framework for the construction of symplectic forms in terms of the Lax operator $L(x)$. In practice, for each integrable model, a suitable ambient space $L$ for the operators $L(x)$ has to be specified, in order for all terms in (5.1) to have the desirable meromorphicity properties and the resulting formula to be independent of the normalization for the Baker-Akhiezer functions. The integrable model is then obtained by constructing an operator-valued function $M(x)$ on $L$ and considering the corresponding flow

$$\partial_t L(x) = [L(x), M(x)]$$

which will be Hamiltonian with respect to $\omega$. This was done in some generality in [2, 3], where $L$ was constructed from the leaves in a foliated moduli space of curves with two specified Abelian integrals, as well as from spaces of ordinary differential operators. It is also evident that the symplectic forms found in §4 in the context of the $SU(N)$ gauge theories with symmetric or antisymmetric hypermultiplets are examples of the same construction. It may be helpful to have an intermediate framework which is at the same time sufficiently specific for easy use, and broad enough to encompass all models encountered in supersymmetric gauge theories. We present such a framework below, together with a brief description of some further recent developments.

### 5.1 The universal symplectic form

For finite-dimensional integrable systems which admit a Lax pair representation with a rational spectral parameter, a practical framework is the following.

Let $L(D)$ be a space of meromorphic matrix functions

$$L(z) = u_0 + \sum_{m=1}^n \sum_{l=1}^{h_m} \frac{u_{ml}}{(z - z_m)^l}$$

with a fixed divisor of poles $D = \sum_{m=1}^n h_m z_m$. The integrable models we consider are flows on the space $L(D)$ which can be constructed as follows. For each $L \in L(D)$, the matrix functions $M_{n,p}(z, L)$ are defined by the formula

$$M_{n,p}(z, L) = \frac{L^n(p)}{z - p}, \quad p \neq z_m.$$  

Then the commutator $[M, L]$ has no pole at $z = p$. If we identify the vector space $L(D)$ with its own tangent space, $[M, L]$ can be regarded as a tangent vector field $\partial_{n,p}$ to $L(D)$. 

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The corresponding flow on $L(D)$

$$
\frac{\partial}{\partial t_{n,p}}L = [M_{n,p}, L]
$$

(5.5)

is a dynamical system which admits by construction a Lax representation. Here $t_{n,p}$ is the time of the flow defined by $M_{n,p}(z, L)$. Standard arguments from the theory of solitons show that all these flows commute with each other. We stress that the construction of the flows on $L(D)$ does not depend on a Hamiltonian structure.

Next, we define a two-form on $L(D)$ by the formula

$$
\omega = \frac{1}{2} \sum_a \text{Res}_{z_a} \text{Tr} \left( \Psi^{-1} \delta L(z) \wedge \delta \Psi(z) \right) dz
$$

(5.6)

The sum is taken over the set of all the poles of $L$ together with the pole of $dz$ at $z_0 = \infty$, i.e., $z_a = \{z_0, z_1, \ldots, z_n\}$. We shall assume for simplicity that the normalization point $z_0$ does not coincide with any of the other punctures $z_m$. The case when $z_0$ coincides with one of the punctures can be treated with only slight technical modifications. The various components of the above formula are as follows. The entries of matrices $u_0, u_{ml}$ can be viewed as coordinates on $L(D)$. If we denote the exterior differentiation on $L(D)$ by $\delta$, then $\delta L(z)$ can be regarded as a matrix valued one-form on $L(D)$

$$
\delta L(z) = \delta u_0 + \sum_m \sum_{l=1}^{h_m} \frac{\delta u_{ml}}{(z - z_m)^l}
$$

(5.7)

Let $\Psi(z)$ be the matrix whose columns are normalized eigenvectors of $L(z)$, i.e.

$$
L(z) \Psi(z) = \Psi(z) K(z), \quad e_0 \Psi = e_0
$$

(5.8)

where $K$ is a diagonal matrix $K^{ij} = k_i \delta^{ij}$, and $k_i$ are the eigenvalues of $L(z)$. The co-vector $e_0$ defining the normalization of the eigenvectors is $e_0 = (1, 1, \ldots, 1)$. The external differential $\delta \Psi$ of $\Psi$ can be viewed as a one-form on $L(D)$, and the formula (5.6) defines a two-form on $L(D)$.

A change of normalization vector $e_0$ leads to a transformation

$$
\Psi(z) \rightarrow \Psi(z)' = \Psi(z)h(z)
$$

(5.9)

where $h(z)$ is a diagonal matrix. Under such transformation $\omega$ gets changed to

$$
\omega' = \omega + \frac{1}{2} \sum_a \text{Res}_{z_a} \text{Tr} \left( \Psi^{-1} \delta L(z) \Psi(z) \wedge \delta hh^{-1} \right) dz
$$

(5.10)
We fix now a set of diagonal matrices \( C = (C_0, C_m) \)

\[
C_0(z) = C_{0,0} + C_{0,1}z^{-1}, \quad C_m(z) = \sum_{l=1}^{m} C_{m,l}(z - z_m)^{-l}, \quad m > 0
\]

(5.11)

and define a subspace \( \mathcal{M} = \mathcal{M}^C \) of \( L(D) \) by the constraints

\[
K(z) = C_0(z) + \mathcal{O}(z^{-2}), \quad z \to z_0
\]

(5.12)

\[
K(z) = C_m(z) + \mathcal{O}(1), \quad z \to z_m.
\]

(5.13)

The number of independent constraints is \((N + 2)r - 1\) because \( \text{Tr} \ K(z) = \text{Tr} \ L(z) \) is a meromorphic function of \( z \). Thus \( \dim \mathcal{M} = (\deg D) r(r - 1) - 2r + r^2 + 1 \). The restriction of \( \delta K \) to \( \mathcal{M} \) is regular at the poles of \( L \) and has a zero of order 2 at \( z_0 \). Therefore, the form \( \omega \) restricted to \( \mathcal{M} \) is independent on the choice of the normalization of the eigenvectors.

We can now define the phase space for our system, over which the form \( \omega \) will be intrinsic and non-degenerate. The space \( L(D) \) and its subspaces \( \mathcal{M}^C \) are invariant under the adjoint action \( L \to gLg^{-1} \) of \( SL_r \). Let

\[
\mathcal{P} = \mathcal{P}^C = \mathcal{M}^C / SL_r
\]

(5.14)

be the quotient space. Its dimension equals \( \dim \mathcal{P} = (\deg D) r(r - 1) - 2r + 2 \). Then we have

**Theorem 4.** (a) The two-form \( \omega \) defined by (5.6) restricted to \( \mathcal{M} \) is gauge invariant and descends to a symplectic form on \( \mathcal{P} \);
(b) The Lax equation (5.5) is Hamiltonian with respect to \( \omega \). The Hamiltonian is

\[
H_{n,p} = -\frac{1}{(n + 1) \text{Tr} \ L^{n+1}(p)}.
\]

(5.15)

(c) All the Hamiltonians \( H_{n,p} \) are in involution with respect to \( \omega \).

This provides a straightforward way of exhibiting Lax equations as Hamiltonian systems.

### 5.2 The universal symplectic form: logarithmic version

There are situations, such as chain models, where a modified version of the above symplectic form (5.1), defined on a slightly different phase space, can also be constructed. Which symplectic form is more appropriate for a given model can be subtle. It can be traced back to the fact that there are two basic algebraic structures on a space of operators. The first one is the Lie algebra structure defined by the commutator of operators. The second one is the Lie group structure. The basic symplectic form introduced in the previous section is
related to the Lie algebra structure. We present now a construction of another symplectic structure, related to the Lie group structure, defined on suitable leaves in \( \mathcal{L}(D) \).

Consider the open subspace of \( \mathcal{L}(D) \) consisting of meromorphic matrix functions which are invertible at a generic point \( z \), i.e., the subspace of matrices \( L(z) \in \mathcal{L}(D) \) such that \( L^{-1}(z) \) is also a meromorphic function. We define subspaces of \( \mathcal{L}(D) \) with fixed divisor for the poles of \( L^{-1}(z) \) as follows. Fix a set \( D^- \) of \( (\deg D^-) \) distinct points \( z^{-s} \) and define a subspace \( \mathcal{M}_1 \subset \mathcal{L}(D,D^-) \) by the constraints

\[
L(z) \in \mathcal{L}(D,D^-) : \det L(z) = c \prod_{a=1}^{Nr} (z - z^{-s}_a), \quad c = \text{const} \neq 0. \tag{5.16}
\]

If \( C_0(z) \) is the same as in (5.11), a subspace \( \mathcal{M}_1 = \mathcal{M}_1^{C_0} \subset \mathcal{L}(D,D^-) \) can be defined by the constraints (5.12). The following two-form on \( \mathcal{M}_1 \) is obviously a group version of (5.6)

\[
\omega^# = \frac{1}{2} \sum \text{Res}_{z_a} \text{Tr} \left( \Psi^{-1}L^{-1}(z)\delta L(z) \wedge \delta \Psi(z) \right) dz \tag{5.17}
\]

Here the sum is taken over all the punctures \( z_a = \{z_0, z_m, z^{-s}_s\} \). The subspace \( \mathcal{M}_1 \) is invariant under the flows defined by the same Lax equations (5.5), which are also gauge invariant and therefore define flows on the quotient space \( \mathcal{P}_1 = \mathcal{P}_1^{C_0} = \mathcal{M}_1^{C_0}/\text{SL}_r \).

**Theorem 5.** The two-form \( \omega^# \) restricted to \( \mathcal{M}_1 \) is independent on the normalization of the eigenvectors. It is gauge invariant and descents to a symplectic form on \( \mathcal{P}_1 \). The Lax equation (5.5) is Hamiltonian with respect to \( \omega^# \). The Hamiltonian is

\[
H_{n-1,p} = -\frac{1}{n} \text{Tr} \ L^n(p). \tag{5.18}
\]

All the Hamiltonians \( H_{n,p} \) are in involution with respect to \( \omega^# \).

Thus Theorems 4 and 5 provide a framework for the existence of so-called bi-Hamiltonian structures. It was first observed by Magri that the KdV hierarchy possesses a bi-Hamiltonian structure, in the sense that all the flows of the hierarchy are Hamiltonian with respect to two different symplectic structures. If \( H_n \) is the Hamiltonian generating the \( n \)-th flow of the KdV hierarchy with respect to the first Gardner-Zakharov-Faddeev symplectic form, then the same flow is generated by the Hamiltonian \( H_{n-1} \) with respect to the second Lenard-Magri symplectic form.

The two symplectic structures \( \omega \) and \( \omega^# \) are equally good in the case of a single matrix function \( L(z) \), but there is a marked difference between them when periodic chains of operators are considered. Let \( L_n(z) \in \mathcal{L}(D) \) be a periodic chain of matrix-valued functions with a pole divisor \( D, \ L_n = L_{n+N} \). The total space of such chains is \( \mathcal{L}(D)^{\otimes N} \). The monodromy matrix

\[
T_n(z) = L_{n+N-1}(z)L_{n+N-2}(z) \cdots L_n(z) \tag{5.19}
\]
is a meromorphic matrix function with poles of order $Nh_m$ at the puncture $z_m$, i.e. $T_n(z) \in \text{L}(ND)$. For different $n$ they are conjugated to each other. Thus the map

$$\text{L}(D)^{\otimes N} \longrightarrow \text{L}(ND)/\text{SL}_r$$ (5.20)

is well-defined. However, the natural attempt to obtain a symplect ic structure on the space $\text{L}(D)^{\otimes N}$ by pulling back the first symplectic form $\omega$ on $\text{L}(ND)$ runs immediately into obstacles. The main obstacle is that the form $\omega$ is only well-defined on the symplectic leaves of $\text{L}(ND)$ consisting of matrices with fixed singular parts for the eigenvalues at the punctures. These constraints are non-local, and cannot be described in terms of constraints for each matrix $L_n(z)$ separately. 

On the other hand, the second symplectic form $\omega^#$ has essentially the desired local property. Indeed, let $L_n$ be a chain of matrices such that $L_n \in \text{L}(D, D_-)$. Then the monodromy matrix defines a map

$$\hat{T} : \text{L}(D, D_-)^{\otimes N} \longrightarrow \text{L}(ND, ND_-)/\text{SL}_r$$ (5.21)

The group $\text{SL}_r^N$ of $z$-independent matrices $g_n \in \text{SL}_r, g_n = g_{n+N}$ acts on $\text{L}(D, D_-)^{\otimes N}$ by the gauge transformation

$$L_n \rightarrow g_{n+1}L_ng_n^{-1}$$ (5.22)

which is compatible with the monodromy matrix map (5.21). Let the space $\mathcal{P}_{\text{chain}}$ be defined as the corresponding quotient space of a preimage under $\hat{T}$ of a symplectic leaf $\Phi^C_0 \subset \text{L}(ND, ND_-)/\text{SL}_r$

$$\mathcal{P}_{\text{chain}} = (\hat{T}^{-1}(\Phi^C_0))/\text{SL}_r^N$$ (5.23)

The dimension of this space is equal to

$$\dim \mathcal{P}_{\text{chain}} = N(\deg D)r(r - 1) - 2r + 2.$$ (5.24)

**Theorem 6.** The pull-back by $\hat{T}$ of the second symplectic form $\omega_1$

$$\omega_{\text{chain}} = \hat{T}^*(\omega_2)$$ (5.25)

restricted to $\hat{T}^{-1}(\mathcal{P}^C_0)$ is gauge invariant and descends to a symplectic form on $\mathcal{P}_{\text{chain}}$. It can also be expressed by the local expression

$$\omega_{\text{chain}} = \frac{1}{2} \sum \text{Res}_{z_a} \sum_{n=1}^N \text{Tr} \left( \Psi_n^{-1} \delta L_n(z) \wedge \delta \Psi_n(z) \right) dz$$ (5.26)

where

$$\Psi_n = L_n \Psi_n, \quad \Psi_n^+ = \Psi_n K, \quad K^{ij} = \text{diag}(k_i) \delta^{ij},$$ (5.27)

All the coefficients of the characteristic polynomial of $T(z)$ are in involution with respect to this symplectic form. The number of independent integrals equals $\dim \mathcal{P}_{\text{chain}}/2$. 

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The symplectic forms (4.16) and (4.23) for the spin chain models corresponding to the SU(N) gauge theory with matter in the symmetric and the antisymmetric representations can be recognized as examples of this construction. So is of course the symplectic structure for the Toda chain.

5.3 Vector bundles and Lax equations on algebraic curves

As we have seen in the case of the elliptic Calogero-Moser systems, the spectral parameter of the Lax pair is sometimes defined on an elliptic curve. Below we present briefly results of [23], where it was shown that the scheme presented above can be extended to the case of the Lax equations on an arbitrary algebraic curve.

The Riemann-Roch theorem shows that the naive direct generalization of the zero curvature equation for matrix functions which are meromorphic on an algebraic curve of genus \( g > 0 \) leads to an overdetermined system of equations. Indeed, the dimension of \( r \times r \) matrix functions of fixed degree \( d \) divisor of poles in general position is \( r^2(d - g + 1) \). If the divisors of \( L \) and \( M \) have degrees \( n \) and \( m \), then the commutator \([L, M]\) is of degree \( n + m \). Thus the number of equations \( r^2(n + m - g + 1) \) exceeds the number \( r^2(n + m - 2g + 1) \) of unknown functions modulo gauge equivalence. There are two ways to overcome this difficulty in defining zero curvature equations on algebraic curves. The first way is based on a choice of \( L \) with essential singularity at some point or with entries as sections of some bundle over the curve. We have seen above that the standard Lax pair for the elliptic Calogero-Moser system falls into this category. The second way, based on a theory of high rank solutions of the Kadomtsev-Petviashvili equation, was discovered in [47]. There it was shown that if in addition to fixed poles the matrix functions \( L \) and \( M \) have \( rg \) moving poles of a special form, then the Lax equation is a well-defined system on the space of singular parts of \( L \) and \( M \) at fixed poles.

We begin by describing a suitable space of meromorphic matrix functions \( L(z) \) on an algebraic curve \( \Gamma \) of genus \( g \). As before, we fix a divisor \( D = \sum h_m z_m \) on \( \Gamma \), and introduce a space \( L(D) = L(D, \Gamma) \) of meromorphic matrix functions \( L \) on \( \Gamma \) with a pole of order \( h_m \) at \( z_m \), such that outside of \( D \) \( L \in L(D) \) has simple poles at a set of \( rg \) distinct points \( \gamma_s \). The Laurent expansion of \( L \) in the neighborhood of \( \gamma_s \)

\[
L = \frac{L_{s0}}{z - z_s} + L_{s1} + O(z - z_s), \quad z_s = z(\gamma_s),
\]

is assumed to satisfy the following constraints:

(i) the singular term \( L_0 \) is a traceless, rank 1 matrix, i.e. it can be represented in the form

\[
L_{s0} = \beta_s \alpha_s^T, \quad \alpha_s^T \beta_s = \text{tr} L_{s0} = 0;
\]

where \( \alpha_s, \beta_s \) are \( r \)-dimensional vectors;
(ii) \( \alpha_s^T \) is a left eigenvector of the matrix \( L_{s1} \)

\[
\alpha_s^T L_{s1} = \alpha_s^T \kappa_s. \tag{5.30}
\]

The following characterization of the constraints \((5.29, 5.30)\) is key: A meromorphic matrix-function \( L \) in the neighborhood \( U \) of \( \gamma_s \) with a pole at \( \gamma_s \) satisfies the constraints \((5.29)\) and \((5.30)\) if and only if it is of the form

\[
L = \Phi_s(z) \bar{L}_s(z) \Phi_s^{-1}(z), \tag{5.31}
\]

where \( \bar{L}_s \) and \( \Phi_s \) are holomorphic in \( U \), and \( \det \Phi_s \) has at most simple zero at \( \gamma_s \).

We would like to emphasize that the points \( \gamma_s \) are not fixed and are themselves dynamical variables. For a non-special degree \( N \geq g \) divisor \( D \), the space \( L(D) \) is of dimension

\[
\dim L(D) = r^2 (N + 1). \tag{5.32}
\]

As in the genus \( g = 0 \) (rational) case, we define a two-form \( \omega \) on \( L(D) \) by the formula

\[
\omega = \frac{1}{2} \left[ \sum_m \text{Res}_{z_m} \text{Tr} \left( \Psi^{-1} \delta L(z) \wedge \delta \Psi(z) \right) + \sum_s \text{Res}_{\gamma_s} \text{Tr} \left( \Psi^{-1} \delta L(z) \wedge \delta \Psi(z) \right) \right] dz \tag{5.33}
\]

Here \( dz \) is a holomorphic differential on \( \Gamma \). Here we follow again the rule of summing over all poles of \( L \).

Let \( k_i(z) \) be eigenvalues of \( L(z) \), i.e., the roots of the characteristic equation

\[
\det (kI - L(z)) = 0 \tag{5.34}
\]

Then we obtain as previously a foliation structure on \( L(D) \) by the constraint:

**The differentials** \( \delta k_i(z)dz \) **are holomorphic in the neighborhood of all the punctures** \( z_m \).

In other words, a leaf \( \mathcal{M} \) of the foliation is fixed by the singular parts of eigenvalues of \( L \) at the points \( z_m \) where \( dz(z_m) \neq 0 \).

**Theorem 7.** [23] If the divisor \( D \) contains the zero divisor \( \mathcal{K} \) of the holomorphic differential \( dz \), then the two-form \( \omega \) defined by \((5.33)\) is invariant under gauge transformations and descends to a symplectic form on the quotient space \( \mathcal{P} = \mathcal{M}/SL_r \). The functions \( H_{n,p} \) given by \((5.13)\) are in involution with respect to the symplectic form \( \omega \).

**Example 1.** Let \( D = \mathcal{K} \). Then the constraints defining \( \mathcal{M} \) are trivial and \( \mathcal{M} = L(K) \). As shown in [23], the corresponding quotient space \( L(K) \) is isomorphic to an open subspace of the cotangent bundle \( T^* (\text{Vect}) \) of a moduli space of stable, rank \( r \), and degree \( rg \) holomorphic vector bundles over \( \Gamma \). The symplectic form \( \omega \) coincides with a canonical symplectic structure on the cotangent bundle. The integrable structure of \( T^* (\text{Vect}) \) was established first by Hitchin [10] using a completely different approach.
Example 2. Consider the Lax matrices on an elliptic curve $G = C/(2\omega_1 Z + 2\omega_2 Z)$ with one puncture, which we can put at $z = 0$ without loss of generality. In this example we denote the parameters $\gamma_s$ and $\kappa_s$ by $q_s$ and $p_s$, respectively. In the gauge $\alpha_s = e_s, e_j^s = \delta_j^s$, the $j$-th column of the Lax matrix $L^{ij}$ has poles only at the points $q_j$ and $z = 0$. It follows that $L^{jj}$ is regular everywhere, i.e. it is a constant. Equation (5.30) implies that $L^{ji}(q_j) = 0, i \neq j$ and $L^{jj} = p_j$. An elliptic function with two poles and one zero fixed is uniquely defined up to a constant. It can be written in terms of the Weierstrass $\sigma$-function as follows

\[
L^{ij}(z) = f^{ij} \frac{\sigma(z + q_i - q_j) \sigma(z - q_i) \sigma(q_i)}{\sigma(z) \sigma(z - q_j) \sigma(q_i - q_j) \sigma(q_i)}, \quad i \neq j; \quad L^{ii} = p_i. \tag{5.35}
\]

Let $f^{ij}$ be a rank 1 matrix $f^{ij} = a^i b^j$. The equations $\alpha_i = e_i$ fix the gauge up to transformations by diagonal matrices. We can use these transformation to make $a^i = b^i$. The corresponding momentum is given then by the collection $(a^i)^2$ and we fix it to the values $(a^i)^2 = 1$. We compare now the different formalisms for the Lax operator of the elliptic Calogero-Moser system. In §3.2, the Lax operator $L(z)$ had entries with essential singularities. Its gauge-transform $\tilde{L}(z)$ by $G_{ij} = \delta_{ij} e^{q_i \zeta(z)}$ has meromorphic entries with poles only at the fixed puncture $z = 0$, but these entries are sections of a non-trivial bundle over the elliptic curve. The present Lax pair $L(z)$ is yet another gauge-transform of the Lax pair of §3.2, this time by the gauge-transformation

\[
G_{ij} = \delta_{ij} \Phi(q_i). \tag{5.36}
\]

As we have seen, its entries are now just meromorphic functions, but with poles at the points $q_j$ as well as at the puncture $z = 0$.

The new gauge in which the Lax matrix for the elliptic Calogero-Moser system is meromorphic gives a new geometric interpretation of the elliptic Calogero-Moser system. The dynamical variables $q_i$ can be identified with the so-called Tyurin parameters of the semi-stable vector bundle over the elliptic curve.

### 5.4 2 + 1 soliton equations of zero curvature form

The integrable models corresponding to the Seiberg-Witten solutions of gauge theories are mechanical systems with a finite number of degrees of freedom, and their symplectic forms are finite dimensional. Nevertheless, the formula (5.1) is quite general. It turns out that it provides also infinite-dimensional symplectic forms for general soliton equations admitting a Lax or zero curvature representation. These symplectic forms are defined on spaces of functions satisfying suitable constraints, which can be identified by a straightforward algorithm [3]. This is a welcome development, since except for the $R$-matrix approach pioneered by Faddeev and Takhtajan [1] (see also the recent proposal in [45]), there was no systematic Hamiltonian theory for soliton equations. Although a Hamiltonian structure was known for most of them, they did not appear to fit in any particular scheme.
To be specific, we shall construct a whole class of soliton equations in zero curvature form and their symplectic forms. Fix an integer \( n \geq 2 \), and consider the space \( \mathcal{M} = \{(u_0, \cdots, u_{n-2}); u_i(x+1, y) = u_i(x, y+1) = u_i(x, y)\} \) of smooth doubly periodic functions. The points of \( \mathcal{M} \) can be identified with the Lax operator

\[
L = \partial_x^n + \sum_{i=0}^{n-2} u_i \partial_x^i \tag{5.37}
\]

The integer \( n \) classifies different hierarchies of soliton equations. For example, the case \( n = 2 \) corresponds to the Kadomtsev-Petviashvili hierarchy, which reduces to the KdV hierarchy when \( L \) is independent of \( y \). The Baker-Akhiezer function \( \psi(x, y, k) \) is defined to be the unique function of the form

\[
\psi(x, y, k) = e^{kx+ky+\sum_{i=0}^{n-2} B_i(y)k^i} \left( 1 + \xi_1 k^{-1} + \xi_2 k^{-2} + \cdots \right) \tag{5.38}
\]

characterized by the normalization condition \( \psi(0, 0, k) = 1 \) and

\[
(\partial_y - L)\psi = 0, \quad \psi(x+1, y, k) = e^k \psi(x, y, k), \quad \psi(x, y+1, k) = e^{Kn} \psi(x, y, k) \tag{5.39}
\]

This key observation is that this determines uniquely both the functions \( B_i(y) \) and the coefficients \( \xi_s(x, y) \) as certain integro-differential expressions in \((u_0, \cdots, u_{n-2})\), which can be written down explicitly. The dual Baker-Akhiezer function \( \psi^\dagger(x, ym, k) \) is characterized by the condition that it be of the form

\[
\psi^\dagger(x, y, k) = e^{-(kx+ky+\sum_{i=0}^{n-2} B_i(y)k^i)} \left( 1 + \xi_1^\dagger k^{-1} + \xi_2^\dagger k^{-2} + \cdots \right) \tag{5.40}
\]

and satisfies

\[
\text{Res}_\infty (\psi^\dagger \partial_x^m \psi) \, dk = 0, \quad m = 0, 1, 2, \cdots \tag{5.41}
\]

We can now define the phase space of the flow to be the subspace \( \mathcal{M}(b) \) defined by the equations

\[
\frac{dB_i(y)}{dy} = b_i, \quad i = 0, \cdots, n-2 \tag{5.42}
\]

where \( b_0, \cdots, b_{n-2} \) are some fixed constants. For each \( m \), there is a unique operator \( A_m = \partial_x^m + \sum_{i=0}^{m-1} v_m \partial_x^i \) satisfying

\[
(A_m - K^m)\psi = \mathcal{O}(k^{-1})\psi \tag{5.43}
\]

The soliton equations we consider are the flows on \( \mathcal{L}(b) \) defined by

\[
\frac{\partial L}{\partial t_m} = \frac{\partial A_m}{\partial y} + [A_m, L] \tag{5.44}
\]

where \( t_m \) is the time variable of the \( m \)-th flow. By construction, these equations are in zero curvature form. Then
Theorem 8. The above flows form an infinite system of commuting flows on the space \( \mathcal{L}(b) \). The expression
\[
\omega = \frac{1}{2} \text{Res}_\infty \langle \psi^\dagger \delta L \wedge \delta \psi \rangle \, dk
\] (5.45)
defines a symplectic form on \( \mathcal{L}(b) \). With respect to this form, the flows are Hamiltonian, with Hamiltonian \( nH_{m+n} \), where \( H_s \) is defined by
\[
k = K + \sum_{s=1}^{\infty} H_s K^{-s}
\] (5.46)

It may be helpful to note that, in the context of KdV and the Kadomtsev-Petviashvili hierarchy, the parameter \( k \) above plays the role of spectral parameter. The role of the spectral curve is assumed here by the complex plane, with \( \infty \) as the distinguished point. Indeed, the Baker-Akhiezer function \( \psi(x, y, k) \) satisfies by definition the boundary condition \( \psi(x + 1, y, k) = e^{k\psi(x, y, k)} \), and the Lax operator \( L(x) \) acting on such a space should be viewed as dependent on \( k \). Thus the symplectic form in Theorem 8 is a close analogue of the symplectic forms constructed earlier for Lax operators with spectral parameter.

We illustrate this construction when \( n = 2 \) and when \( n = 3 \). The case \( n = 2 \) corresponds to the Kadomtsev-Petviashvili hierarchy, and the operator \( L \) is given by \( L = \partial_x^2 + u \). A straightforward calculation shows that
\[
\psi(x, y, k) = e^{kx + k^2y + B(y)}(1 + \xi_1 k^{-1} + \cdots)
\] (5.47)
with \( B(y) = \int_0^y \int_0^1 u(x, \alpha)dx \, d\alpha \), and \( \partial_x \xi_1 = -\frac{1}{2}u + \frac{\partial B}{\partial y} \). The space \( \mathcal{L}(b) \) is defined then by the constraint
\[
\int_0^1 u(x, y)dx = b
\] (5.48)
and we have \( \xi_1 = -\frac{1}{2} \int x u + bx \). It follows readily that \( \psi^\dagger = e^{-(kx + k^2y + B(y))}(1 - \xi_1 k^{-1} + \cdots) \), and the symplectic form becomes
\[
\omega = \text{Res}_\infty \langle (1-\xi_1 k^{-1} + \cdots) \delta u \wedge (\delta \xi_1 k^{-1} + \cdots) \rangle \, dk = \langle \delta u \wedge \delta \xi_1 \rangle = -\frac{1}{2} \langle \delta u(x) \wedge \int x \, \delta u \rangle
\] (5.49)
This is the Gardner-Faddeev-Zakharov symplectic form for the KdV equation.

The case \( n = 3 \) corresponds to the Boussinesq hierarchy. Here \( L = \partial_x^3 + u \partial_x + v \), and the space \( \mathcal{L}(b_0, b_1) \) is the space of doubly periodic functions \( u \) and \( v \) satisfying
\[
\int_0^1 u(x, y)dx = b_0, \quad \int_0^1 v(x, y) = b_1.
\] (5.50)
We find the following expression for the symplectic form
\[
\omega = -\frac{1}{3} \langle \delta u \wedge \int_{x_0}^x \delta v \, dx + \delta v \wedge \int_{x_0}^x \delta u \, dx \rangle
\] (5.51)
5.5 Calogero-Moser field equations and isomonodromy equations

Suitably interpreted, the universal formula \(5.1\) can produce a Hamiltonian structure for differential equations in diverse contexts, including field versions of Calogero-Moser equations and monodromy equations \([23],[24]\). We describe these symplectic structures briefly.

The field version of Calogero-Moser system arises as a zero curvature equation for a Lax pair \(L(x, q), M(x, q)\)

\[
L_t = M_x + [M, L] \tag{5.52}
\]

with elliptic spectral parameter \(q\). Here \(L(x, q)\) is a \(r \times r\) matrix, periodic of period 1 in \(x\) and meromorphic in \(q\). The set-up is analogous to the one considered in Example 2 in the previous section, with an additional space variable \(x\). The matrix \(L_{ij}(x, q)\) now has poles in \(q\) only at \(q = 0\) and \(q = q_j(x)\). On a suitable space of operators \(L(x, q)\) satisfying some specific constraints on their poles and residues, one can construct as before an infinite number of \(r \times r\) matrix valued operators \(M_n(x, q)\) whose coefficients are differential expressions in terms of the coefficients of \(L(x, q)\), which satisfy the condition that \(\partial_x M + [M, L]\) is tangent to \(L\). The equation \(5.52\) defines then an infinite number of commuting flows on \(L\). As in the construction of the symplectic form for the Kadomtsev-Petviashvili hierarchy, a uniquely suitably normalized solution \(\Psi(x, q)\) of the equation \((\partial_x - L(x, q))\Psi(x, q) = 0\) can be found, with monodromy of the form

\[
\Psi(x + 1, q) = e^{p(q)}\Psi(x, q) \tag{5.53}
\]

for some unique \(p(q)\). Then the following modification of the universal symplectic form \((5.1)\)

\[
\omega = -\frac{1}{2} \sum_{\alpha} \text{Res}_\alpha \frac{1}{2} \int_{x_0}^{x_{n+1}} \frac{\Psi^{-1} \delta L \wedge \delta \Psi}{\delta \Psi(x_0) \wedge \delta p} dx dz \tag{5.54}
\]

defines a symplectic structure on \(L\) with respect to which all the flows \((5.52)\) are Hamiltonian. For \(r = 2\), one finds the following expression for the corresponding Hamiltonian

\[
H = \int \left( p^2 (1 - q_x^2) - \frac{q_x^2}{2(1 - q_x)} - 2(1 - 3q_x^2) \varphi(2q) \right) dx \tag{5.55}
\]

giving the equations of motion for the poles \(q(x)\). In terms of \(q(x)\) and its conjugate variable \(p(x)\), the symplectic structure leads to the simple Poisson bracket

\[
\{p(x), q(y)\} = \delta(x - y). \tag{5.56}
\]

It is noteworthy that after a change of variables, this system coincides with the well-known Landau-Lifshitz equation.

Next, we discuss the isomonodromy problem. Let \(V\) be a stable rank \(r\) and degree \(rg\) holomorphic bundle on a genus \(g\) Riemann surface \(\Gamma\), and let \([V] = \{\gamma_s\}\) be the divisor of its determinant bundle. Let \(D = \sum_m(h_m + 1)P_m\) be a divisor which does not intersect \([V]\).
Consider meromorphic connections \( L = L(z)dz \) on \( V \) with poles at \( P_m \) of degree \( \leq h_m + 1 \). As shown in [24], they can be represented by meromorphic matrix-valued differentials \( L \) with a pole of order \( h_m + 1 \) at \( P_m \), with the property that outside \( D \), the differential \( L \) has simple poles at the points \( \gamma_s \). The Laurent expansion of \( L \) in a neighborhood of \( \gamma_s \)

\[
L = \left( \frac{L_{s0}}{z - z_s} + L_{s1} + \mathcal{O}(z - z_s) \right)dz, \quad z_s = z(\gamma_s)
\]

is assumed to satisfy the following constraints

(i) the singular term \( L_{s0} \) is a rank 1 and trace 1 matrix, i.e.,

\[
L_{s0} = \beta_s \alpha_s^T, \quad \alpha_s^T \beta_s = \text{Tr} L_{s0} = 1,
\]

where \( \alpha_s, \beta_s \) are \( r \)-dimensional vectors.

(ii) The vector \( \alpha_s^T \) is a left eigenvector of the matrix \( L_{s1} \)

\[
\alpha_s^T L_{s1} = \alpha_s^T \kappa_s.
\]

The following characterization of these constraints is essential: A meromorphic matrix-valued function \( L \) in a neighborhood \( U \) of \( \gamma_s \) with pole at \( \gamma_s \) satisfies them if and only if it is of the form

\[
L = d\Phi_s(z)\Phi_s(z)^{-1} + \Phi_s(z)\bar{L}_s(z)\Phi_s^{-1}(z),
\]

where \( \bar{L}_s \) and \( \Phi_s \) are holomorphic in \( U \), and \( \det \Phi_s \) has at most a simple zero at \( \gamma_s \). For any point \( P \in \Gamma \), it follows from (5.60) that the equation

\[
d\Psi = L\Psi
\]

admits a multi-valued solution in \( \Gamma \setminus D \). The analytic continuation of \( \Psi \) defines the monodromy representation of \( L \). The isomonodromy problem is the problem of describing the deformations of \( L \) which preserve the monodromy representation as well as certain local data at the singular points \( P_m \) called Stokes matrices and exponents. In order to define these data, it is necessary to fix a normalization point \( Q \in \Gamma \), and \( h_m \)-jets of local coordinates in the neighborhoods of the punctures \( P_m \).

Let \( h = \{ h_m, \sum_m (h_m + 1) = N \} \) be a set of non-negative integers. We denote by \( \mathcal{M}_{g,1}(h) \) the moduli space of smooth genus \( g \) algebraic curves with a puncture \( Q \in \Gamma \), and fixed \( h_m \)-jets \([w_m]\) of local coordinates \( w_m \) in the neighborhoods of the punctures \( P_m \). The space \( \mathcal{M}_{g,1}(h) \) has dimension

\[
\dim calM_{g,1}(h) = 3g - 2 + N
\]

The space \( \mathcal{A}(h) \) of admissible meromorphic differentials \( L \) on algebraic curves with fixed multiplicities \( h_m + 1 \) at the poles can be viewed as the total space of the bundle

\[
\mathcal{A}(h) \longrightarrow \mathcal{M}_{g,1}(h) = \{ \Gamma, P_m, [w_m], Q \}
\]
For a fixed point of $\mathcal{M}_{g,1}(h)$, the monodromy data, Stokes matrices, and exponents uniquely define $L$. In [24], special coordinates $t_a$ were introduced and it was shown that the isomonodromy deformations on the space $\mathcal{A}$ of meromorphic connections are described by the zero curvature equations

$$\partial_{t_a} \tilde{L} - d M^{(a)} + [\tilde{L}, M^{(a)}] = 0$$

(5.64)

for suitable $M^{(a)}$. The corresponding flows commute. Furthermore, as in all the models described previously, they turn out to be Hamiltonian with respect to the symplectic structure defined by

$$\omega = -\frac{1}{2}(\text{Res}_V Tr(\psi^{-1} \delta L \wedge \delta \psi) + \text{Res}_D Tr(\Psi^{-1}_m \delta L \wedge \delta \Psi_m))$$

(5.65)

Here $\psi$ and $\psi_m$ are the solutions of the equation (5.61) in the neighborhoods of the points in the divisors $[V]$ and $D$ respectively. An explicit expression of the symplectic form in terms of the monodromy data and Stokes matrices was also found in [24]. It should be stressed that, even in the case of genus 0, a symplectic structure on the space of Stokes matrices with one irregular singularity of order 2 and one regular singularity was found only recently [48]. In the case of elliptic curves, the monodromy data consists of just two matrices $A$ and $B$, modulo conjugation, and the monodromy matrix is $J = B^{-1}A^{-1}BA$. The symplectic form $\omega$ becomes in this case

$$\omega(A, B) = Tr(B^{-1} \delta B \wedge \delta AA^{-1} - A^{-1} \delta A \wedge \delta BB^{-1} + \delta JJ^{-1} \wedge B^{-1}A^{-1} \delta(AB))$$

(5.66)

It had also been found in [49] under a different form.
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