RIEMANN-HILBERT FACTORIZATION OF MATRICES INVARIANT UNDER INVERSION IN A CIRCLE

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Abstract. We consider matrix functions with certain invariance under inversion in the unit circle. If such a function satisfies a positivity assumption on the unit circle, then only zero partial indices appear in its Riemann-Hilbert (Wiener-Hopf) factorization. It implies the unique solvability of a certain class of Riemann-Hilbert boundary value problems. It includes the ones associated with the inverse scattering transform of the focusing/defocusing integrable discrete nonlinear Schrödinger equations.

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1. Introduction

Riemann-Hilbert problems (RHPs), formulated in various ways, are a powerful tool in the study of integrable systems. As is proved in [10], if a matrix function is invariant under Schwarz reflection, its Riemann-Hilbert (Wiener-Hopf) factorization involves only zero partial indices and it implies the unique solvability of the corresponding RHPs formulated in other ways (a singular integral equation and a boundary value problem). The zero partial indices property is a key in the argument in [11]. The main result there is the bijectivity of the scattering and inverse scattering maps. So bijectivity is known for NLS in sufficient detail (see also [4, 12]), but the integrable discrete nonlinear Schrödinger equation (IDNLS) still lacks a satisfactory theory. In order to construct such a theory, we need a detailed information about relevant RHPs. In the discrete case, the real axis must be replaced by the unit circle ([11] [7] [8] [9]). If a matrix function invariant under the inversion in $S^1$, namely $z \rightarrow 1/\bar{z}$, and satisfies a positivity condition on $S^1$, then it has only zero partial indices. It implies the unique solvability of a certain class of Riemann-Hilbert boundary value problems including the ones associated with IDNLS. This fact can be a basis of the bijectivity proof of the scattering/inverse scattering transforms for IDNLS. See also the approach in [5] based on a vanishing lemma.

Factorization of matrices given on $S^1$ is a topic that can be studied from other directions. See, e.g., [3] [6]. It is known that a positive Hermitian matrix function $v$ on $S^1$ has an expression $v = w^*w$, where

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$w$ is holomorphic inside $S^1$. We give a generalization of this fact to the case of an inversion invariant contour including $S^1$.

2. **Function spaces**

Let $\Sigma \subset \mathbb{C}$ be a finite disjoint union of smooth simple closed curves. More specifically, we assume $\Sigma = \bigcup_{j=1}^{J} \Sigma_j$, where each $\Sigma_j$ is a smooth simple closed curve and $\Sigma_j \cap \Sigma_k = \emptyset$ $(j \neq k)$. It is possible to assign an orientation on $\Sigma$ such that it is the positively oriented boundary of an open set $\Omega_+$. Set $\Omega_- = \mathbb{C} \setminus (\Sigma \cup \Omega_+)$. Then $\Sigma$ is the negatively oriented boundary of the open set $\Omega_-$.

We introduce function spaces following [10, 11]. The $L^2$ norm of a matrix function $f : \Sigma \to \mathbb{M}_n$ (where $\mathbb{M}_n$ is the complex $n \times n$ matrix algebra) is defined by

$$
\|f\|_2 = \left(\int_{\Sigma} |f|^2 |dz| \right)^{1/2},
$$

where $|\cdot|$ means the Hermitian conjugate. We write $L^2(\Sigma)$ for $L^2(\Sigma, \mathbb{M}_n)$. We denote by $H^k(\Sigma)$ $(k \geq 1)$ the space of all the matrix functions $f$ such that $f(j) \in L^2(\Sigma)$ for all $j = 0, \ldots, k$ in the distribution sense. Its norm is

$$
\|f\|_{2,k} = \left(\sum_{j=0}^{k} \|f(j)\|_2^2\right)^{1/2}
$$

and $H^k(\Sigma)$ is a Hilbert space with continuous pointwise multiplication. Sometimes we write $\|f\|_{2,k}$ as $\|f\|_k$ for brevity. A function $f \in H^k(\Sigma)$ is Hölder continuous.

In the present paper, we choose a formulation in which the contour is bounded. In [10] and [11], however, the author assumes that the contour is unbounded. At some places of the present paper, we reduce the proof to the unbounded case. Let $C_{\pm}$ be the Cauchy operators defined by

$$
C_{\pm}f(z) = \lim_{z' \to z} \frac{1}{2\pi} \int_{\Sigma} f(w) \frac{dw}{w - z'},
$$

where the nontangential limit $z' \to z$ is taken from $\Omega_{\pm}$ respectively. They are bounded from $L^2(\Sigma)$ to $L^2(\Sigma)$ and from $H^k(\Sigma)$ to $H^k(\Sigma)$. It is known that $C_+$ and $-C_-$ are complementary projections. Moreover, a function in $\text{Ker} C_{\pm} = \text{Range} C_{\mp}$ has a holomorphic extention to $\Omega_{\mp}$.

3. **Formulation of Riemann-Hilbert problems**

Assume that $v \in H^k(\Sigma)$ admits a factorization $v = (b^-)^{-1}b^+$ for invertible $b^\pm \in H^k(\Sigma)$. Since $\Sigma$ is bounded, $|\det b^-|$ is uniformly away from 0 and $(b^-)^{-1} \in H^k(\Sigma)$. A factorization as above always exists (we have only to choose $b^+ = I$ or $b^- = I$). Set $w^\pm = \pm (b^\pm - I)$, i.e. $b^\pm = I \pm w^\pm$. We call $w = (w^+ , w^-)$ a pair of factorization data of $v$.

The set of all such pairs is denoted by $FD_k$. We have

$$
FD_k = \left\{(w^+, w^-) \in \oplus^2 H^k(\Sigma); I \pm w^\pm \text{ is invertible}\right\}.
$$
**Definition 1.** An element \( \mu \in H^j(\Sigma)(j = 0, \ldots, k) \) is said to be a solution of the Riemann-Hilbert problem (RHP) of the pair of factorization data \( w \) if

\[
(3.1) \quad \mu b^\pm - h \in \text{Range} C_\pm
\]

for some constant matrix \( h \).

The definition in [10] has been modified here because \( \mu(\infty) \) is not defined in the present paper. Notice that \( m^\pm := \mu b^\pm \in H^j(\Sigma) \). Since they are in the ranges of the Cauchy operators modulo \( h \), they have a holomorphic extension to \( \mathbb{C} \setminus \Sigma \), which we denote by \( m \). We call it the solution of the Riemann-Hilbert problem of \( v \) or \( w \).

**Proposition 2.** If \( \mu \) is a solution of (3.1) for fixed \( h \), then the holomorphic extension \( m \) is a solution of a Riemann-Hilbert boundary value problem in the classical sense:

\[
m_+ = m_- v, \quad m(\infty) = \lim_{z \to \infty} m(z) = h.
\]

Conversely, if a holomorphic function \( m \) satisfies \( m(\infty) = h \) and \( m_+ = m_- v \), then \( \mu = m_+(b^+)^{-1} = m_-(b^-)^{-1} \) is a solution of (3.1).

**Proof.** We have \( m_+(b^+)^{-1} = \mu = m_-(b^-)^{-1} \) and \( m_+ = m_-(b^-)^{-1} b^+ = m_- v \). Next, \( m(\infty) = h \) follows from \( m = h + (\text{a Cauchy integral}) \). The converse is now easy. \( \square \)

For \( w = (w^+, w^-) \), set

\[
C_w\phi = C_+(\phi w^-) + C_-(\phi w^+).
\]

Then \( C_w \) is a bounded operator from \( H^j(\Sigma) \) to itself for every \( j = 0, 1, \ldots, k \).

**Proposition 3.** An element \( \mu \) of \( L^2(\Sigma) \) is a solution of (3.1) if and only if

\[
(3.2) \quad (I - C_w)\mu = h
\]

holds. If \( \text{Id} - C_w \) is a bijection, then a solution of (3.1) exists uniquely.

**Proof.** We follow the proof of [10] Prop 3.3. Recall that \( C_+ - C_- = \text{Id} \). If \( \mu \) satisfies (3.2), we have

\[
\mu b^+ - h = \mu (I + w^+) - (I - C_w)\mu = \mu w^+ + C_w\mu
\]

\[
= (C_+ - C_-)(\mu w^+) + C_+(\mu w^-) + C_- (\mu w^+)
\]

\[
= C_+(\mu w^+ + \mu w^-) \in \text{Range} C_+.
\]
and similarly $\mu b^- - h = C_-(\mu w^+ + \mu w^-) \in \text{Range } C_-$. 
Conversely, assume (3.1). Then $\mu b^\pm - h \in \text{Ker } C_\pm$. We have

$$
(I - C_w)\mu = (C_+ - C_-)\mu - [C_+(\mu w^-) + C_-(\mu w^+)] \\
= C_+(\mu b^-) - C_- (\mu b^+) \\
= C_+(\mu b^- - h) - C_- (\mu b^+ - h) + h = h.
$$

\[ \square \]

4. Factorization and partial indices

We introduce two classes of holomorphic matrix functions following [11]:

$$
\mathcal{H}^k(\mathbb{C} \setminus \Sigma) := \{ m; m_\pm - m(\infty) \in \text{Range } C_\pm \}, \\
G\mathcal{H}^k(\mathbb{C} \setminus \Sigma) := \{ m \in \mathcal{H}^k(\mathbb{C} \setminus \Sigma); \text{ det } m \text{ vanishes nowhere} \},
$$

where $C_\pm : H^k(\Sigma) \to H^k(\Sigma)$.

**Theorem 4.** Any $v \in H^k(\Sigma)$ with $\text{det } v \neq 0$ admits a Riemann-Hilbert (Wiener-Hopf) factorization $v = m^{-1}_- \theta m_+$ relative to $\Sigma$ in $H^k(\Sigma)$. Here $m_\pm$ are the boundary values of an element $m$ of $G\mathcal{H}^k(\mathbb{C} \setminus \Sigma)$. The matrix $\theta$ is

$$
(4.1) \quad \theta = \text{diag } \left[ \left( \frac{z - z_+}{z - z_-} \right)^{k_1}, \ldots, \left( \frac{z - z_+}{z - z_-} \right)^{k_n} \right],
$$

where $z_\pm \in \Omega^\pm$ and $k_1, \ldots, k_n$ are integers such that $k_1 \geq \cdots \geq k_n$. We call $k_1, \ldots, k_n$ the partial indices of $v$. They are uniquely determined.

**Proof.** Fix $z_\pm \in \Omega^\pm$. We embed our contour $\Sigma$ into $\hat{\Sigma} \ni \infty$ and reduce the proof to [10] Th 9.1 or [11] Th 2.1.3]. Let $\Sigma'$ be a line (a circle in the Riemann sphere) defined by $\text{Re } z = -p$, where $p$ is so large that $\Sigma'$ is far away from $\Sigma$ and $z_\pm$. First we assume that $\Omega_+$ is bounded and that $\Omega_-$ is unbounded. The orientation of $\Sigma'$ is from $-p - i\infty$ to $-p + i\infty$. If $p$ is sufficiently large, we have $z_+ \in \Omega_+ \subset \hat{\Omega}_+, z_- \in \hat{\Omega}_- \subset \Omega_-$. Set $\hat{\Sigma} = \Sigma \cup \Sigma'$. It has a compatible orientation in the sense that it is a positively oriented boundary of an open set $\hat{\Omega}_+$ and is a negatively oriented boundary of an open set $\hat{\Omega}_-$. Extend $v \in H^k(\Sigma)$ to $\hat{\Sigma}$ by setting $\hat{v}|_\Sigma = v, \hat{v}|_{\Sigma'} = I$. Then $\hat{v}$ is not an element of $H^k(\hat{\Sigma})$, but it belongs to $H^k(\hat{\Sigma}) = H^k(\Sigma) \oplus M_n$ introduced in [10] and [11]. It consists of matrix functions $f$ on $\hat{\Sigma}$ with the limit $f(\infty)$ such that $f - f(\infty) \in H^k(\hat{\Sigma})$. The norm is the square root of $|f(\infty)|^2 + \| f - f(\infty) \|^2_k$. Since there is no self-intersection, it is not necessary to introduce $H^k(\Sigma^\pm)$ and $H^k(\Sigma^\pm)$. 


By [10] Th 9.1 or [11] Th 2.1.3, \( \hat{v} \in H^k_I(\hat{\Sigma}) \) admits a Riemann-Hilbert factorization
\[
\hat{v} = \hat{m}_-^{-1}\theta\hat{m}_+, \\
\theta = \text{diag} \left[ \left( \frac{z - z_+}{z - z_-} \right)^{k_1}, \ldots, \left( \frac{z - z_+}{z - z_-} \right)^{k_n} \right].
\]

On \( \Sigma' \), we have \( \hat{v} = I = \hat{m}_-^{-1}\theta\hat{m}_+ \), which implies \( \hat{m}_+ = \theta^{-1}\hat{m}_- \). Set \( m = \theta^{-1}\hat{m} \) in \( \Re z < -p \) (the positive side of \( \Sigma' \)) and \( m = \hat{m} \) elsewhere. Then \( m \) is holomorphic for \( z \not\in \Sigma \) and we have a factorization \( v = m_-^{-1}\theta m_+ \) on \( \Sigma \). In particular, \( v \) and \( \hat{v} \) has the same partial indices.

Next, if \( \Omega_+ \) is unbounded and \( \Omega_- \) is bounded, we reverse the orientation of \( \Sigma' \). We get \( \hat{m}_- = \theta\hat{m}_+ \) on \( \Sigma' \) and set \( m = \theta\hat{m} \) in \( \Re z < -p \).

**Theorem 5.** The operator \( \text{Id} - C_w : H^k(\Sigma) \rightarrow H^k(\Sigma) \) is Fredholm. Let \( k_1, \ldots, k_n \) be the partial indices of \( v \). Then
\[
\dim \text{Ker}(\text{Id} - C_w) = \sum_{k_j > 0} k_j, \\
\dim \text{Coker}(\text{Id} - C_w) = -\sum_{k_j < 0} k_j.
\]

**Proof.** We employ the embedding argument in the proof of Theorem 4. We extend \( v \in H^k(\Sigma) \) and \( w \) to \( \hat{\Sigma} \) by setting \( \hat{v}|_{\Sigma} = v, \hat{v}|_{\Sigma'} = I \) and \( \hat{w}|_{\Sigma} = w, \hat{w}|_{\Sigma'} = (0, 0) \). Then \( \hat{w} \) is a pair of factorization data of \( \hat{v} \).

Recall that \( v \) and \( \hat{v} \) has the same partial indices.

On \( H^k_I(\hat{\Sigma}) = H^k(\Sigma) \oplus H^k_I(\Sigma') \), we have \( C_{\hat{w}} = C_w \oplus 0 \). By [10] Th 9.2 and [11] Th 2.1.6, we have \( \dim \text{Ker}(\text{Id} - C_w) = \dim \text{Ker}(\text{Id} - C_{\hat{w}}) = n \sum_{k_j > 0} k_j \). The assertion about the cokernel is proved in the same way.

**Corollary 6.** If the partial indices are all zero, the Riemann-Hilbert problem (3.1) has a unique solution.

**Proof.** Use Proposition 3 and Theorem 5. \( \square \)
5. INVERSION IN THE UNIT CIRCLE

For a subset $A$ of $\mathbb{C}$ and a matrix function $f$, we set $A^\pm = \{1/\bar{z}; z \in A\}$ and $f^\pm(z) = f(1/\bar{z})^\pm$. It is the inversion in the unit circle $S^1 = \{z; |z| = 1\}$. For example, if $\theta$ is as in (4.1) and $z \pm \neq 0$, then we have

$$\theta^\pm = \begin{pmatrix} \bar{z}_+ \cdot z - 1/\bar{z}_+ & \cdots & \bar{z}_- \cdot z - 1/\bar{z}_- \end{pmatrix} \begin{pmatrix} k_1 & \cdots & k_n \end{pmatrix}.$$  

**Theorem 7.** Let $\Sigma \supset S^1$ be a contour invariant under inversion in $S^1$. If $v \in H^k(\Sigma)$ with $\det v \neq 0$ satisfies

$$v = v^\pm \text{ on } \Sigma \setminus S^1 \text{ and } \Re v = (v + v^*)/2 > 0 \text{ on } S^1,$$

then it has only zero partial indices and a solution of (3.1) exists uniquely when $h$ is fixed.

**Proof.** Let $w = (w^+, w^-)$ be an arbitrary pair of factorization data of $v$ and set $b^\pm = I \pm w^\pm$. It is enough to prove the bijectivity of $\text{Id} - C^w$. Let $v = m^- \theta m^+$ be the factorization of $v$ on $\Sigma$ as in Theorem 4. By inversion, we have $v^\pm = m^\pm \theta^\pm (m^{-1})^\pm$ on $\Sigma^\pm = \Sigma$. Since $1/\bar{z}_+ \in \Omega_\pm^+ \subset \Omega_\pm^\pm \subset \Omega_\pm^\pm$, $m^\pm \in \text{Range } C_-$, $(m^{-1})^\pm \in \text{Range } C_+$, the expression (5.1) of $\theta^\pm$ implies that the partial indices of $v^\pm$ are $-k_n, \ldots, -k_1$. If $w^\pm$ is a pair of factorization data of $v^\pm$, we have by Theorem 5

$$\dim \text{Coker}(\text{Id} - C^w) = \dim \text{Ker}(\text{Id} - C^w).$$

Since $v^\pm$ also satisfies the assumptions of the theorem, it is enough to prove that $\text{Ker}(\text{Id} - C^w) = 0$.

Let $\Omega_\nu$ be a component of $\Sigma \setminus \Sigma$ outside $S^1$. In the figure, the orientation of $\Sigma$ is indicated by placing plus signs on the positive sides of the curves. We may assume that $S^1$ has the clockwise orientation following the convention of [1]. Assume $\mu \in \text{Ker}(\text{Id} - C^w)$. Then by Proposition 3, we have $m^\pm := \mu b^\pm \in \text{Range } C_\pm$ and they have a holomorphic extension, which we denote by $m$. Let $m_{\nu_1}, m_{\nu_2}$ be the boundary values of $m|_{\Omega_\nu}, m|_{\Omega_\nu^\mp}$ respectively. If $\Omega_\nu \subset \Omega_\pm$, then $\Omega_\nu^\mp \subset \Omega_\pm$. So if $m_{\nu_1}$ is the
boundary value from (a part of) $\Omega_\pm$, then $m_{\nu \pm}$ is the boundary value from (a part of) $\Omega_\mp$. We have

$$\int_{\partial \Omega_\nu} m_{\nu \pm} m_{\nu \pm}^\sharp = 0.$$ 

Here notice that the usual counterclockwise orientation of $\partial \Omega_\nu$ may or may not coincide with the orientation of $\Sigma$ depending on whether $\Omega_\nu \subset \Omega_+$ or $\Omega_\nu \subset \Omega_-$. We calculate the sum with respect to all the components $\Omega_\nu$ outside $S^1$, including the one whose boundary contains $S^1$ (like $\Omega_\lambda$ in the figure). We get

$$(5.2) \quad \sum_{\nu} \int_{\partial \Omega_\nu} m_{\nu \pm} m_{\nu \pm}^\sharp = 0.$$ 

It is possible that a single curve, not $S^1$, is included in both $\partial \Omega^+_{\nu}$ with $\Omega^+_{\nu} \subset \Omega^+$ and $\partial \Omega^-_{\nu}$ with $\Omega^-_{\nu} \subset \Omega^-$. In this case it has two orientations. So cancellation happens in the sum above. Now we show

$$(5.3) \quad \sum_{\nu} \int_{\partial \Omega_\nu} m_{\nu \pm} m_{\nu \pm}^\sharp = \int_{S^1} m_- v m_-^\sharp.$$ 

Let $\Sigma_\nu$ be a component of $\Sigma$ outside $S^1$. Then it is a part of the common boundary of some components $\Omega^+_{\nu} \subset \Omega^+$ and $\Omega^-_{\nu} \subset \Omega^-$. Let $m^+_{\nu \pm}, m^-_{\nu \pm}$ be the boundary values of $m$ on $\Sigma_\nu$ from $\Omega^+_{\nu}, \Omega^-_{\nu}$ respectively (hence $m^+_{\nu \pm} = m^-_{\nu \pm}$) and let $m^+_{\nu \pm}, m^-_{\nu \pm}$ be the boundary values of $m$ on $\Sigma^\pm_{\nu}$ from $\Omega^+_{\nu}, \Omega^-_{\nu}$ respectively (hence $m^+_{\nu \pm} = m^-_{\nu \pm}$). In the left-hand side of (5.3), the integral along $S^1$ appear only once as $\int_{S^1} m_+ m_-^\sharp = \int_{S^1} m_- v m_-^\sharp$. The integrals along $\Sigma_\nu$ appear twice, once as $\int_{\Sigma_\nu} m_{\nu \pm} m_{\nu \pm}^\sharp = \int_{\Sigma_\nu} m_- v m_+^\sharp$ and once again as $\int_{\Sigma_\nu} m_{\nu \pm} m_{\nu \pm}^\sharp = \int_{\Sigma_\nu} m_- v m_-^\sharp$. Since $v = v^\sharp$ on $\Sigma_\nu$, these integrals cancel each other and (5.3) has been proved. By (5.2) and (5.3), we have

$$\int_{S^1} m_- v m_-^\sharp = \int_{S^1} m_- v m_+^\sharp = 0.$$ 

Inversion (or Hermitian conjugation) gives $\int_{S^1} m_- v^* m_-^* = 0$. Adding these two equations, we get

$$\int_{S^1} m_- (v + v^*) m_-^* = 0.$$ 

By the positivity of $\text{Re} \, v$, we have $m_- = 0$ on $S^1$, which implies $m_+ = m_- v = 0$ there. We get $m = 0$ at least in the components of $\Omega_\pm$ whose boundaries include $S^1$ like $\Omega_\lambda$ and $\Omega^\sharp_\lambda$ in the figure. Then the boundary value $m_+$ or $m_-$ from such a component vanishes along any other part of the boundary. Since $v$ is invertible, the boundary value from the other side also vanishes and we have $m = 0$ in that side. We can repeat this process as many times as necessary (e.g. concentric circles) and finally we get $m_- = 0$ and $\mu = 0$ everywhere on $\Sigma$. \qed
Corollary 8. Let $\Sigma \supset S^1$ be a contour invariant under inversion in $S^1$. If $v \in H^k(\Sigma)$ with $\det v \neq 0$ satisfies
\[ v = v^\sharp \text{ on } \Sigma \quad \text{and} \quad v > 0 \text{ on } S^1, \]
then $v = (m_+)^4m_+$ for some $m \in G\mathcal{H}^k(C \setminus \Sigma)$.

Proof. By the preceding theorem, $v$ has only zero partial indices and we have $v = n_-n_+$ on $\Sigma$ for some $n = n(z) \in G\mathcal{H}^k(C \setminus \Sigma)$. Here we have replaced $n_-^{-1}$ by $n_-$. It is equivalent to replacing $n$ by its inverse in $\Omega_-$. We have $v^\sharp = n_+^*n_-^*$ Since $v = v^\sharp$, [3, p.11] implies that there exists a constant matrix $C$ such that $n_- = n_+^*C$ and $n_+ = C^{-1}n_-^*$. Therefore we have $v = n_+^*Cn_+$ on $\Sigma$. In particular, we have $v = n_+^*Cn_+$ on $S^1$. Since $v$ is Hermitian and positive on $S^1$, so is $C$. There exists a positive Hermitian matrix $R$ such that $R^2 = C$. We have $v = n_+^*R^2n_+ = (Rn)^2n_+$ everywhere on $\Sigma$. □

Example 9. Let $\Sigma$ be the unit circle $S^1$, and set
\[ v(z) := \begin{bmatrix} 1 - |r(z)|^2 & -z^{2n}\tilde{r}(z) \\ z^{-2n}r(z) & 1 \end{bmatrix} = \begin{bmatrix} 1 - |r'(z)|^2 & -\tilde{r}'(z) \\ r'(z) & 1 \end{bmatrix} \quad (z \in S^1). \]

Here $r(z)$ is a sufficiently smooth function on $S^1$, $r'(z) = z^{-2n}r(z)$ and $n$ is an integer. If $|r(z)| < 1$, then $\text{Re } v = \text{diag} [1 - |r(z)|^2, 1] > 0$ and Theorem [7] applies. The matrix $v(z)$ is modeled on the one corresponding to the defocusing integrable discrete nonlinear Schrödinger equation (IDNLS). See [1, 7, 8]. But in the present paper, it is not necessary to assume that $r(z)$ is obtained by the scattering transform. It can be prescribed without reference to a potential and we do not have to assume $r(-z) = -r(z)$ ([1, (3.2.76)]), a property of the reflection coefficient. The present author hopes this example and Theorem [10] below give a basis for establishing the bijectivity of the scattering/inverse scattering transforms (cf. [4, 11, 12]).

6. RHP modeled on the focusing IDNLS

In this section, we consider a problem modeled on the focusing IDNLS ([11]). Let $z_j (j = 1, 2, \ldots, J)$ be distinct points outside $S^1$. We consider
the Riemann-Hilbert boundary value problem

\begin{equation}
M_+(z) = M_-(z)V(z) \quad \text{on } S^1,
\end{equation}

\begin{equation}
V(z) = \begin{bmatrix}
1 + |r(z)|^2 & z^{2n}r(z) \\
z^{-2n}r(z) & 1
\end{bmatrix},
\end{equation}

\begin{equation}
\operatorname{Res}(M(z); z_j) = \lim_{z \to z_j} M(z) \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix},
\end{equation}

\begin{equation}
\operatorname{Res}(M(z); \bar{z}_j^{-1}) = \lim_{z \to \bar{z}_j^{-1}} M(z) \begin{bmatrix}
0 & \bar{z}_j^{-2n-2}c_j \\
0 & 0
\end{bmatrix},
\end{equation}

\begin{equation}
M(z) \to I \quad \text{as } z \to \infty.
\end{equation}

Here \( n \) is an integer, \( r(z) \) is a sufficiently smooth function on the unit circle and \( c_j \) is an arbitrary complex number. Moreover \( M_+ \) and \( M_- \) are the boundary values from the outside and inside of the unit circle respectively. The unit circle is oriented clockwise following the convention in [1]. In the study of the focusing IDNLS, one encounters quartets of the form \( \{ \pm z_j, \pm 1/\bar{z}_j \} \), but in the present paper we generalize the situation and consider pairs of the form \( \{ z_j, 1/\bar{z}_j \} \). Moreover we do not assume \( r(-z) = -r(z) \).

Following [9], we reduce this problem to one without poles.

Let \( C[z_j] \) be a sufficiently small circle centered at \( z_j \) for each \( j \). Assume that it is oriented clockwise. By inversion in \( S^1 \), we get \( C[z_j]^\# \), which is oriented counterclockwise. This simple closed curve encloses \( 1/\bar{z}_j \).

Set

\[ m(z) = \begin{cases}
M(z) \begin{bmatrix}
1 & 0 \\
-\bar{z}_j^{-2n}c_j & 1
\end{bmatrix} & \text{inside } C[z_j], \\
M(z) \begin{bmatrix}
1 & \bar{z}_j^{-2n-2}c_j \\
0 & 1
\end{bmatrix} & \text{inside } C[z_j]^\#.
\end{cases} \]

and \( m(z) = M(z) \) elsewhere. Then \( m(z) \) is holomorphic near \( z_j, \bar{z}_j^{-1} \).

Set \( \Sigma = S^1 \cup \cup_{j=1}^J C[z_j] \cup \cup_{j=1}^J C[z_j]^\# \). We introduce a matrix \( v(z) \) on \( \Sigma \) by

\[ v(z) = \begin{cases}
V(z) & \text{on } S^1, \\
M(z) \begin{bmatrix}
1 & 0 \\
z_j^{-2n}c_j & 1
\end{bmatrix} & \text{on } C[z_j], \\
M(z) \begin{bmatrix}
1 & \bar{z}_j^{-2n-2}c_j \\
0 & 1
\end{bmatrix} & \text{on } C[z_j]^\#.
\end{cases} \]
Then the RHP (6.1)-(6.5) is equivalent to the following RHP without poles:

\[
(6.6) \quad m_+(z) = m_-(z)v(z) \quad \text{on} \quad \Sigma \quad \text{and} \quad m(z) \to I \quad (z \to \infty).
\]

**Theorem 10.** The classical Riemann-Hilbert problem (6.6) has a unique solution and so does (6.1)-(6.5). Moreover \(v\) has only zero partial indices.

**Proof.** The jump matrix \(v(z)\) does not satisfy the assumption of Theorem 7 but can be converted to such a one by conjugation, i.e. by introducing a new unknown matrix \(m'(z)\).

Let \(C_R\) and \(C_{1/R}\) be the circles \(|z| = R\) and \(|z| = 1/R\) respectively, where \(R > 0\) is sufficiently large. We give them both counterclockwise orientation.

We define \(m' = m'(z)\) by the following set of rules: (1) \(m' = m\) inside \(C_{1/R}\) and outside \(C_R\). (2) \(m' = mA\) if \(z\) is between \(S^1\) and \(C_R\) and is outside \(C[z_j]\) for all \(j\). (3) \(m' = mB_j\) inside \(C[z_j]\). (4) \(m' = mC\) between \(C_{1/R}\) and \(S^1\) except on \(\bigcup_j C[z_j]\). Then the normalization condition at \(\infty\) remains the same. Now we calculate the jump matrix \(v' = v'(z)\) for \(m'\): \(m_+' = m'_-v'\) on \(\Sigma\).

We have \(v' = A\) on \(C_R\) and \(v' = C^{-1}\) on \(C_{1/R}\). Since \(A^* = C^{-1}\), we have \(v'^* (z) = v(z)\) for \(z \in C_R \cup C_{1/R}\).
On $C[z_j]$, we have $v' = B_j^{-1}vA$. We evaluate $v^{r-1} = A^{-1}v^{-1}B_j$ first, because $A^{-1}$ is easier than $B_j^{-1}$. Then we get

$$v' = (v^{r-1})^{-1} = \begin{bmatrix} 1 & 0 \\ \left(\prod_{k \neq j} z_k \right) z_j^{-2n} \bar{c}_j & 1 \end{bmatrix}$$

on $C[z_j]$. Next on $C[z_j]^\sharp$, we have

$$v' = C^{-1}vC = \begin{bmatrix} 1 & -\frac{z \left(\prod_{k \neq j} \bar{z}_k \right) \bar{z}_j^{-2n-1} \bar{c}_j}{z - \bar{z}_j^{-1}} \\ 0 & 1 \end{bmatrix}.$$

Therefore $v'^z(z) = v'(z)$ holds for $z \in C[z_j] \cup C[z_j]^\sharp$.

On $S^1$, we have $v' = C^{-1}vA = A^2vA = A^*vA$. Since $v$ is a positive Hermitian matrix, so are $v'$ and Re $v'$.

By Theorem 7, the matrix $v'$ has only zero partial indices. By Proposition 2 and Corollary 6, the classical RHP $m'_+ = m'_-v'$, $m' \to I(z \to \infty)$ has a unique solution and so does (6.6).

We have $v' = PvQ$, where $\{P, Q\} \subset \{A, B_j^{-1}, C, C^{-1}\}$. Let $v' = \hat{m}_-^{-1}\hat{m}_+$ be its factorization. We have $v = P^{-1}\hat{m}_-^{-1}\hat{m}_+Q^{-1}$. This factorization of $v$ means all the partial indices are zero. □

Remark 11. In Theorem 10 above, $\{(z_j, 1/\bar{z}_j), c_j; j = 1, \ldots, J\}$ and $r(z)$ are not true scattering data. True ones have two additional characteristics: poles appear in quartets of the form $(\pm z_j, \pm 1/\bar{z}_j)$ and the reflection coefficient satisfy $r(-z) = -r(z)$. According to Theorem 10, we can solve the associated RHP uniquely even for this kind of formal or generalized ‘scattering data’ and apply the potential reconstruction formula, but the ‘potential’ obtained this way is not necessarily a potential.

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