CANONICAL HEIGHTS, TRANSFINITE DIAMETERS, AND POLYNOMIAL DYNAMICS

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Abstract. Let \( \phi(z) \) be a polynomial of degree at least 2 with coefficients in a number field \( K \). Iterating \( \phi \) gives rise to a dynamical system and a corresponding canonical height function \( h_\phi \), as defined by Call and Silverman. We prove a simple product formula relating the transfinite diameters of the filled Julia sets of \( \phi \) over various completions of \( K \), and we apply this formula to give a generalization of Bilu’s equidistribution theorem for sequences of points whose canonical heights tend to zero.

1. Introduction

There are a number of interesting results in arithmetic geometry and in the theory of complex dynamical systems which deal with equidistribution properties of sequences of “small” points. Examples on the arithmetic side include Bilu’s equidistribution theorem (see [9]) and the equidistribution theorem of Szpiro, Ullmo, and Zhang (see [31]), which imply the “generalized Bogomolov conjecture” for subvarieties of algebraic tori and abelian varieties, respectively. (See [9], [32], and [34] for further details). On the complex dynamics side, there is for example an important equidistribution result for the backward iterates of a point under a polynomial due to Brolin (see [12] or [26, Theorem 6.5.8]). Brolin’s result is closely related to both the strengths and weaknesses of the so-called “inverse iteration method” for drawing Julia sets of polynomials on a computer. (See [26, Page 207] and [25, Chapter 2] for a further discussion).

One of our goals in this paper is to simultaneously generalize Bilu’s theorem and Brolin’s theorem, and to explore the interplay of both results with potential theory. The connection between Bilu’s theorem and potential theory was already pointed out by Rumely in [27] and by Bombieri in [10]. Because some of the sets we deal with in this paper are rather exotic (they can be thought of as “\( p \)-adic fractals”), we have found it convenient to base all of our proofs on the purely algebraic notion of transfinite diameter, rather than on the equivalent potential-theoretic notion of capacity.

Another goal of this paper is to promote the philosophy that interesting information about dynamical systems can be found by looking “adelically”, rather than just over \( \mathbb{C} \). A simple but instructive example is the observation that 0 is not a periodic point under iteration of the polynomial \( z^2 - 3/2 \). This is not obvious from

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plotting the iterates in the real line, but it is clear if we look at what is happening 2-adically. More generally, it seems useful to try to put the \(p\)-adic theory of polynomial iteration on an equal footing with the complex theory \([5, 6, 19, 20, 22]\). However, complications inevitably arise from the fact that \(\mathbb{C}_p\) is not locally compact. For example, in Section 6.2 we compute the transfinite diameter of the filled Julia set of a polynomial \(\phi\) over \(\mathbb{C}_p\), but the computation is not quite as simple as its complex analogue due to the failure of local compactness. This calculation leads to a simple global product formula, which in turn is closely related to the equidistribution results alluded to above.

We have aimed to make this paper as self-contained as possible. We hope that this makes the paper a useful introduction to certain aspects of the theory of canonical heights, potential theory, and dynamical systems over local and global fields.

2. Terminology and Conventions

Throughout this paper, \(K\) will denote a \textit{global field}, which for our purposes will mean a field equipped with a product formula and satisfying the Northcott finiteness property.\(^1\) The main examples are number fields and function fields of curves over finite fields.

To say that \(K\) is equipped with a product formula means that we are given a family \(M_K\) of absolute values \(x \mapsto |x|_v\) on \(K\) (where \(v\) denotes the corresponding additive valuation) and a family \(\{n_v\} (v \in M_K)\) of positive real numbers satisfying the following properties (compare with [30, Section 2.1]):

- All but finitely many \(v \in M_K\) are ultrametric\(^2\), meaning that \(|x|_v = c^{-v(x)}\) for some real additive valuation (which we also denote by \(v\)) on \(K\) and some \(c > 1\).
- The absolute values which are not ultrametric are induced by embeddings \(K \hookrightarrow \mathbb{C}\).
- For all \(x \in K^*\), we have \(|x|_v = 1\) for all but finitely many \(v \in M_K\).
- If we set \(\|\|_v := |\|_v^{n_v}\), then for all \(x \in K^*\), we have the product formula

\[
\prod_{v \in M_K} \|x\|_v = 1.
\]

Fix a place \(v \in M_K\), and let \(K_v\) denote the completion of \(K\) with respect to \(|\|_v\). If we fix an algebraic closure \(\bar{K}_v\) of \(K_v\), then it is well-known (see [21, Section II.1]) that \(|\|_v\) extends in a unique way to an absolute value on \(\bar{K}_v\). Moreover, if we denote by \(\mathbb{C}_v\) the completion of \(\bar{K}_v\), then \(|\|_v\) extends uniquely to \(\mathbb{C}_v\) by continuity. For convenience, we often fix without comment a distinguished embedding of \(K\) into \(\mathbb{C}_v\) for each \(v \in M_K\).

We say that \(K\) satisfies the \textit{Northcott finiteness property} if for each integer \(d \geq 1\) and each positive real number \(M\), there are only finitely many elements \(\alpha \in \bar{K}\) of degree at most \(d\) over \(K\) and absolute logarithmic height at most \(M\). (See [30, Section 2.2] or Section 4 below for a definition of the absolute logarithmic height). For example, it is well-known that number fields and function fields of curves over

\(^1\)This is not the standard definition of a global field.

\(^2\)We will use the terms ultrametric and non-archimedean interchangeably throughout this paper.
finite fields (equipped with the standard product formula structure as in [30]) have this property.

Throughout this paper, \( L \) will denote a valued field. For ease of exposition, we will say that a valued field \( L \) is a local field if \( L \) is either \( \mathbb{R} \) or \( \mathbb{C} \) (with the usual absolute value) or a field which is complete with respect to an ultrametric absolute value. If the absolute value on \( L \) arises from some \( v \in M_K \) then the absolute value will be denoted by \( | \cdot |_v \), as usual and the completion of \( L \) will be denoted by \( \mathbb{C}_v \). Otherwise we simply denote the absolute value on \( L \) by \( | \cdot | \) (dropping the subscript \( v \)). For a non-archimedean field \( L \), the set \( \{ z \in L : |z| \leq R \} \) will be called a closed disc (of radius \( R \)) and the set \( \{ z \in L : |z| < R \} \) will be called an open disc (of radius \( R \)).

3. Canonical heights associated to a polynomial map

3.1. Global canonical heights. In this subsection, we define the global canonical height function associated to a polynomial \( \phi \in K[z] \) of degree \( d \geq 2 \), and we summarize some of its main properties. Our formulation is essentially the same as that in [14], although our notation and terminology are slightly different.

If \( v \in M_K \), we define the standard local height function on \( \mathbb{C}_v \) to be the function \( h_v : \mathbb{C}_v \to \mathbb{R} \) given by

\[
\hat{h}_v(z) = \log^+ \|z\|_v,
\]

where for \( x \in \mathbb{R} \) we set \( \log^+ x = \log \max\{x, 1\} \).

If \( S \) is any finite, Galois-stable subset of \( \bar{K} \), we define the (absolute logarithmic) height of \( S \) to be the average of the (properly normalized) local heights of all elements in \( S \) (which makes sense because all but finitely many of these local heights will be zero). More precisely, we define

\[
h(S) := \frac{1}{\#S} \sum_{z \in S} \sum_{v \in M_K} \log^+ \|z\|_v.
\]

This definition makes sense, since we have fixed embeddings of \( \bar{K} \) into \( \mathbb{C}_v \) for each \( v \). However, since we are assuming that \( S \) is Galois-stable, the height of \( S \) is in fact independent of our choices of embeddings.

More generally, if \( T \) is any finite subset of \( \bar{K} \), we define \( h(T) \) to be the height of the smallest Galois-stable subset containing \( T \). In particular, note that this definition is compatible with the previous one in the special case where \( T \) is already Galois-stable. If \( T = \{ \alpha \} \) consists of a single algebraic number, the height \( h(\alpha) \) we have just defined coincides with the usual definition of the absolute logarithmic height (see e.g. [30]).

We now define the (global) canonical height associated to \( \phi \in K[z] \) by the formula

\[
\hat{h}_\phi(T) := \lim_{n \to \infty} \frac{1}{d^n} h(\phi^n(T)),
\]

where \( \phi^n \) denotes the \( n \)th iterate of \( \phi \). It follows from [14] that the limit exists, and in particular that \( \hat{h}_\phi \) is well-defined. (This of course uses the fact that \( d \geq 2 \)).

As a function from \( \bar{K} \) to \( \mathbb{R} \), we can characterize \( \hat{h}_\phi \) as the unique function such that

1. There exists a constant \( M \) (depending only on \( \phi \)) such that for all \( \alpha \in \bar{K} \), we have \( |\hat{h}_\phi(\alpha) - h(\alpha)| \leq M \).
(2) For all \( \alpha \in \bar{K} \), we have \( \hat{h}_\phi(\phi(\alpha)) = d \hat{h}_\phi(\alpha) \).

Since \( K \) satisfies the Northcott finiteness property, these two properties imply:

(3) If \( \alpha \in \bar{K} \), then \( \hat{h}_\phi(\alpha) \geq 0 \), and \( \hat{h}_\phi(\alpha) = 0 \) if and only if \( \alpha \) is a preperiodic point for \( \phi \), i.e., a point whose orbit \( \{\phi^n(\alpha) : n \geq 0\} \) under \( \phi \) is finite.

3.2. Canonical local heights. There is a decomposition of the global canonical height attached to \( \phi \) into a sum of canonical local heights. In our situation, this is quite straightforward.\(^3\)

We define \( \hat{h}_{\phi,v} : \mathbb{C}_v \to \mathbb{R} \) to be the function given by

\[
\hat{h}_{\phi,v}(z) = \lim_{n \to \infty} \frac{1}{d^n} \log^+ |\phi^n(z)|_v.
\]

It is shown in [14] that this limit exists, and that we have

\[
\hat{h}_\phi(\alpha) = \frac{1}{m} \sum_{v \in M_K} \sum_{\alpha_i} n_v \hat{h}_{\phi,v}(\alpha_i),
\]

for all \( \alpha \in \bar{K} \), where \( \alpha_1, \ldots, \alpha_m \) are the conjugates of \( \alpha \) over \( K \).

We can similarly define the local height \( \hat{h}_{\phi,v}(T) \) for any finite subset \( T \) of \( \mathbb{C}_v \).

If \( \phi(z) = a_0 + a_1 z + \cdots + a_d z^d \in K[z] \) with \( a_d \neq 0 \), and if \( v \) is non-archimedean, we let \( \alpha_v \) be the minimum of the \( d+1 \) numbers \( \frac{1}{d-1} v(\frac{a_i}{a_d}) \), \( 0 \leq i < d \) and \( \frac{1}{d-1} v(\frac{1}{a_d}) \).

(We consider \( v(0) \) to be \( +\infty \).)

We also define

\[
c_v(\phi) := |a_d|_v^{-1}.
\]

See Theorem 4.1 for the motivation behind this definition.

The following are some of the properties of the canonical local heights attached to \( \phi \) (see [14] and [13] for proofs):

(1). For each \( v \) and each \( z \in \mathbb{C}_v \), \( \hat{h}_{\phi,v}(\phi(z)) = d \hat{h}_{\phi,v}(z) \).

(2). For each \( v \), \( \hat{h}_{\phi,v} : \mathbb{C}_v \to \mathbb{R} \) is a continuous nonnegative function which is zero precisely on the \( v \)-adic filled Julia set \( F_v := \{ z \in \mathbb{C}_v : |\phi^n(z)|_v \not\to \infty \} \) of \( \phi \).

(3). For each \( v \), the difference \( |\hat{h}_{\phi,v} - \hat{h}_v| \) is a bounded function on \( \mathbb{C}_v \).

(4). For all but finitely many \( v \in M_K \), we have \( \hat{h}_{\phi,v} = h_v \).

(5). If \( v \) is archimedean, then \( \hat{h}_{\phi,v}(z) = \log |z|_v - \log c_v(\phi) + o(1) \) as \( |z|_v \to \infty \).

(6). If \( v \) is non-archimedean, then for \( |z|_v \), sufficiently large (depending only on \( \phi \)),

we have \( \hat{h}_{\phi,v}(z) = \log |z|_v - \log c_v(\phi) \).

Specifically, this formula is valid whenever \( v(z) < \alpha_v \).

The basic properties of local and global canonical heights provide a quantitative form of the following “local-global principle” for dynamical systems:

**Proposition 3.1.** An element \( \alpha \in \bar{K} \) is a preperiodic point for \( \phi \) if and only if every conjugate of \( \alpha \) is contained in the \( v \)-adic filled Julia set \( F_v \) of \( \phi \) for all \( v \in M_K \).

This principle can also be easily derived directly from the Northcott finiteness property.

\(^3\)Note however that one can define canonical heights attached to more general morphisms, such as a rational function \( \psi \) on \( \mathbb{P}^1 \), but the canonical local heights are in general not given by such a simple formula. This is due to the fact that as divisors on \( \mathbb{P}^1 \), we have \( \psi^*(\infty) = d(\infty) \), but \( \psi^*(\infty) \) and \( d(\infty) \) are merely linearly equivalent.
4. Statement of main results

Let $L$ be an algebraically closed local field, and let $\phi \in L[z]$ be a polynomial of degree $d \geq 2$. The filled Julia set $F = F(\phi)$ of $\phi$ is the set of points in $L$ which remain bounded under iteration of $\phi$, i.e.,

$$F := \{ z \in L : |\phi^n(z)| \not\to \infty \}.$$

Our first result concerns the transfinite diameter of the set $F$. The notion of transfinite diameter makes sense in a fairly general context. Let $(M,d)$ be a metric space, and let $A$ be a subset of $M$. We define the $n$th diameter of $A$, denoted $d_n(A)$, to be the supremum over all $n$-tuples of points in $A$ of the average pairwise distance between points of a given $n$-tuple. Here “average” means geometric mean; in other words:

$$d_n(A) = \sup_{x_1, \ldots, x_n \in A} \prod_{i \neq j} d(x_i, x_j)^{\frac{1}{n(n-1)}}.$$

The sequence of nonnegative real numbers $\{d_n(A)\}$ is decreasing, and therefore has a limit as $n$ tends to infinity. This is proved, for example, in [1]. By definition, the transfinite diameter of $A$ is

$$c(A) = \lim_{n \to \infty} d_n(A).$$

By convention, we say that $c(\emptyset) = 0$.

In the next section we will compute the transfinite diameter $c(F)$ of the filled Julia set of a polynomial $\phi \in L[z]$. The result is the following:

**Theorem 4.1.** Let $\phi(z) = \sum_{n=0}^{d} a_n z^n \in L[z]$ be a polynomial of degree $d \geq 2$, and let $F \subset L$ be its filled Julia set. Then\footnote{We use the letter $c$ for transfinite diameter because $d$ will be the degree of a polynomial $\phi$, and because over the complex numbers, the transfinite diameter of a compact set $F$ is the same thing as its capacity.}

$$c(F) = |a_d|^{-\frac{1}{d-1}}.$$

In particular, suppose $\phi \in K[z]$ for a global field $K$. Then for each $v \in M_K$, one has a filled Julia set $F_v = F_v(\phi) \subset C_v$ of $\phi$ considered as a polynomial in $C_v[z]$. Let $c(F_v) = c(F_v)^n_v$. Then as a corollary of Theorem 4.1, we obtain (using the product formula for $K$):

**Corollary 4.2.** If $\phi \in K[z]$ is a polynomial of degree $d \geq 2$, then

$$\prod_{v \in M_K} c(F_v) = 1.$$

For notational convenience, we will sometimes write $c_v(\phi) = |a_d|^{-1/(d-1)}$ instead of $c(F_v)$.

As in [27], the above product formula for the local transfinite diameters leads to an archimedean equidistribution theorem for points of small canonical height. We also obtain, under additional hypotheses, a family of non-archimedean equidistribution results.\footnote{We cannot directly apply the ideas in [27], however, because the sets $F_v$ are not in general compact.} In order to state these results, we need to introduce the notion of an equilibrium measure.

For notational convenience, we will sometimes write $c_v(\phi) = |a_d|^{-1/(d-1)}$ instead of $c(F_v)$.
Let \( v \in M_K \) be a fixed place of \( K \). For any compact subset \( F \subset \mathbb{C}_v \) with nonzero transfinite diameter, there exists a unique probability measure \( \mu_F \) which "minimizes energy" among all probability measures supported on \( F \). The energy of a (Borel) probability measure \( \mu \) on \( F \) is the double integral
\[
\int_F \int_F -\log |x - y|_v \, d\mu(x) d\mu(y).
\]
The measure \( \mu_F \) of minimal energy is called the equilibrium measure for \( F \), and the energy of \( \mu_F \) turns out (not surprisingly) to be \(-\log c(F)\). (For a more detailed discussion, as well as for proofs of the above assertions see [26, Chapters 3 and 5] for the complex case and [28, §3.1 and §4.1] for both archimedean and non-archimedean cases).

In particular, if \( F \) is compact in \( \mathbb{C}_v \) and is the filled Julia set for a polynomial \( \phi \in \mathbb{C}_v[z] \), then \( c(F) = c_v(\phi) > 0 \) by Theorem 4.1, so there exists a well-defined equilibrium measure \( \mu_F \).

When \( \mathbb{C}_v = \mathbb{C} \) and \( F \) is the filled Julia set of \( \phi \), the support of \( \mu_F \) is precisely the topological boundary \( \partial F \) of \( F \) (which coincides with the Julia set of \( \phi \) — see Section 5). Also, \( \mu_F \) is a \( \phi \)-invariant measure, in the sense that
\[
\mu_F(\phi^{-1}(B)) = \mu_F(B)
\]
for all Borel sets \( B \subset \partial F \). (See [26, Chapter 6] for these and other facts about \( \mu_F \) in the complex case).

Let \( M \) be a metric space. A sequence \( \{\mu_n\} \) of measures on \( M \) is said to converge weakly to a measure \( \mu \) if
\[
\lim_{n \to \infty} \int_M f \, d\mu_n = \int_M f \, d\mu
\]
for every bounded, continuous function \( f : M \to \mathbb{R} \). For a general theory of weak convergence of probability measures on metric spaces, see [24]. In this paper, the metric space \( M \) will always be a valued local field with the metric induced by the absolute value.

For any finite subset \( S \subset \mathbb{C}_v \), we denote by \( \delta_S \) the probability measure \( \delta_S := \frac{1}{|S|} \sum_{z \in S} \delta_z \), where \( \delta_z \) is the Dirac probability measure supported at the single point \( z \in \mathbb{C}_v \). Finally, we say a sequence \( \{S_n\} \) of finite subsets of \( \mathbb{C}_v \) is equidistributed with respect to the measure \( \mu \) if the sequence \( \{\delta_n\} := \{\delta_{S_n}\} \) of probability measures converges weakly to \( \mu \).

We now introduce a variant of the notion of equidistribution.

First, suppose \( F \) is a subset of the valued field \( L \), and define a logarithmic distance function\(^6\) for \( F \) to be a continuous function \( \lambda : L \to \mathbb{R} \) such that \( \lambda(z) = 0 \) for \( z \in F \), \( \lambda(z) > 0 \) for \( z \notin F \), and such that the difference \(|\lambda(z) - \log^+ |z||\) is bounded.

As our main example of such a function, if \( \phi \in K[z] \) has degree \( d \geq 2 \), then for each place \( v \in M_K \) the canonical local height function \( h_{\phi,v} : \mathbb{C}_v \to \mathbb{R} \) serves as a logarithmic distance function for the \( v \)-adic filled Julia set \( F_v \) (see Section 3.2). If we do not specify a particular set \( F \), we define a logarithmic distance function for \( L \) to be any nonnegative function \( \lambda : L \to \mathbb{R} \) such that the difference \(|\lambda(z) - \log^+ |z||\) is bounded.

\(^6\)Over \( \mathbb{C} \), this is similar to the notion of a Green’s function for \( F \), but we do not impose any harmonicity condition.
Now let \( S = \{ S_n \} \) be a sequence of finite subsets of \( L \). Given a logarithmic distance function \( \lambda \) for \( F \), we say that the sequence \( S \) is \( \lambda \)-taut if the average value of \( \lambda \) on \( S_n \) converges to zero, i.e., if
\[
\lim_{n \to \infty} \frac{1}{\#S_n} \sum_{z \in S_n} \lambda(z) = 0.
\]

We say that \( S \) is a generalized Fekete sequence for \( F \) if \( N_n := \#S_n \to \infty \) as \( n \to \infty \) and the limit
\[
\lim_{n \to \infty} \prod_{x,y \in S_n, x \neq y} |x - y|^{\frac{1}{N_n(N_n - 1)}}
\]
exists and equals the transfinite diameter of \( F \).

Finally, we say that \( S \) is pseudo-equidistributed with respect to the pair \((F, \lambda)\) if \( S \) is \( \lambda \)-taut and is a generalized Fekete sequence for \( F \).

With these definitions in mind, we have:

**Theorem 4.3.** Let \( K \) be a global field, let \( \phi \in K[z] \) be a polynomial of degree at least 2, and for each \( v \in M_K \) let \( F_v \subset \mathbb{C}_v \) be the \( v \)-adic filled Julia set of \( \phi \). Let \( \{ S_n \} \) be a sequence of distinct finite Galois-stable subsets of \( \mathbb{C} \) such that
\[
\lim_{n \to \infty} \hat{h}_\phi(S_n) = 0.
\]
Then for all \( v \in M_K \), the sequence \( \{ S_n \} \) is pseudo-equidistributed with respect to the pair \((F_v, \hat{h}_\phi,v)\).

Over the complex numbers, we will also prove the following result relating the notions of equidistribution and pseudo-equidistribution:

**Proposition 4.4.** Let \( F \subset \mathbb{C} \) be a compact set with nonzero transfinite diameter, and let \( \lambda \) be a logarithmic distance function for \( F \). Let \( S = \{ S_n \} \) be a sequence of finite subsets of \( \mathbb{C} \) which is pseudo-equidistributed with respect to the pair \((F, \lambda)\). Then \( \{ S_n \} \) is equidistributed with respect to the equilibrium measure on \( F \).

Combining this result with Theorem 4.3, we obtain a generalization of Bilu’s equidistribution theorem for sequences of small points with respect to the usual absolute logarithmic Weil height [9].

In the non-archimedean case, if the filled Julia set \( F_v \subset \mathbb{C}_v \) is compact, then we can also prove the following result:

**Proposition 4.5.** Let \( L \) be a non-archimedean algebraically closed local field and let \( \phi \in L[z] \) be a polynomial of degree \( d \geq 2 \). Assume that the filled Julia set \( F \) of \( \phi \) is compact, and let \( \lambda \) be the canonical local height as defined in Section 3.2. Let \( S = \{ S_n \} \) be a sequence of finite subsets of \( \mathbb{C}_v \) which is pseudo-equidistributed with respect to the pair \((F, \lambda)\). Then \( \{ S_n \} \) is equidistributed with respect to the equilibrium measure \( \mu_F \).

Combining Theorem 4.3 and Proposition 4.4, 4.5, we obtain:

**Corollary 4.6.** Let \( K \) and \( \phi \) be as in Theorem 4.3, and let \( F \subset \mathbb{C}_v \) be the \( v \)-adic filled Julia set of \( \phi \) with respect to some fixed place \( v \in M_K \). Assume that \( F \) is compact with respect to the \( v \)-adic topology. Then if \( \{ S_n \} \) is a sequence of distinct finite Galois-stable subsets of \( \hat{K} \) such that \( \lim_{n \to \infty} \hat{h}_\phi(S_n) = 0 \), the sequence \( \{ S_n \} \) is equidistributed with respect to \( \mu_F \).
One can apply Theorem 4.3 and Corollary 4.6 to prove interesting statements about the canonical height associated to $\phi$. As an example, we present in Section 8 a generalization to dynamical heights of a result of Schinzel concerning the usual (logarithmic Weil) height of totally real algebraic numbers. (See Proposition 8.4). We also present a $p$-adic analogue of this result (Theorem 8.1), which generalizes a theorem of Bombieri and Zannier for the usual height.

As another application, we show that there exists a positive constant $C > 0$ such that $\hat{h}_\phi(\alpha) + \hat{h}_\phi(\sigma(\alpha)) \geq C$ for all but finitely many $\alpha \in \mathbb{Q}$ whenever $\sigma$ is an affine linear transformation of the complex plane which does not belong to the symmetry group of the complex Julia set of $\phi$ (Theorem 8.8). This allows us to confirm an interesting special case of a conjecture of S. Zhang.

In Section 8.2, we extend our results to sets which are more general than the filled Julia sets of polynomial mappings, and as an application we establish an equidistribution result for the “adelic Mandelbrot set.”

5. Dynamical systems

In this section, we review some terminology and results from the theory of dynamical systems. We restrict our attention to polynomial dynamics, and for the reader’s benefit we give some illustrative examples.

Let $L$ be an algebraically closed local field, and let $\phi \in L[z]$ be a polynomial. In Section 4, we defined the filled Julia set $F$ of $\phi$. One can give another equivalent definition of $F$ as follows. First of all, it is easy to see that there exists a positive real number $R$ such that if $|z| > R$ then $|\phi(z)| > R$, so that the region $\{ z \in L : |z| > R \}$ is $\phi$-stable and contained in the complement of $F$. Let $F_0 := \{ z \in L : |z| \leq R \}$. We can then define a sequence $\{ F_n \}$ of closed subsets by setting $F_n = \phi^{-1}(F_{n-1}) = \phi^{-n}(F_0)$ for all $n \geq 1$. Then

$$F_0 \supseteq F_1 \supseteq \cdots \supseteq F_n \supseteq \cdots \supseteq F$$

and

$$F = \bigcap_{n=0}^{\infty} F_n.$$

Throughout this paper, we will implicitly choose a decreasing sequence $\{ F_n \}$ of closed subsets as above for the filled Julia set $F$ of $\phi$. Furthermore, if $L$ is a non-archimedean field then we will always assume that $R$ is chosen so that $|\phi(z)| = |a_d z^d| > R$ for $|z| > R$.

It is clear from the above construction that the filled Julia set is a closed and bounded subset of $L$. If $L = \mathbb{C}$ then $F$ is always compact, but if $L$ is non-archimedean, then $F$ might or might not be compact.

The Fatou set of $\phi$ is the set of all points in $L$ having a neighborhood on which the family of iterates $\{ \phi^n \}$ is equicontinuous (see [20] for a more detailed discussion of this definition). The Julia set of $\phi$ is defined to be the complement of the Fatou set. When $L = \mathbb{C}$, the Julia set of $\phi$ is never empty and is a compact subset of $\mathbb{C}$, but for non-archimedean $L$, the Julia set of $\phi$ can be empty or non-empty and non-compact (see Example 5.4 below). Intuitively, the Julia set is the locus of points which behave “chaotically” under iteration of $\phi$. Moreover, when the Julia set is non-empty, it is the topological boundary of the filled Julia set (and hence the name of the latter). When $L = \mathbb{C}$, this is well-known; when $L$ is non-archimedean, see [5, Theorem 5.1].
By definition, the Fatou set of $\phi$ is open. Moreover, one can show that the Fatou set, the Julia set, and the filled Julia set of $\phi$ are each completely invariant under $\phi$, i.e., that each is stable under both forward and backward iteration of $\phi$.

Now we turn to a discussion of periodic and preperiodic points.

Let $P \in \bar{\mathbb{L}}$ be a point of exact period $n$ for $\phi$, i.e., a point such that $\phi^n(P) = P$ but $\phi^m(P) \neq P$ for all positive integers $m < n$.

The multiplier of $P$ is defined to be $\lambda(P) = |(\phi^n)'(P)|$. It follows from the chain rule that $P$ and $\phi(P)$ have the same multiplier.

We say that $P$ is repelling (resp. neutral, attractive) if its multiplier satisfies $\lambda(P) > 1$ (resp. $\lambda(P) = 1$, $\lambda(P) < 1$).

**Lemma 5.1.** Every attracting periodic point is contained in the Fatou set of $\phi$, and every repelling periodic point is contained in the Julia set. If $\mathbb{L}$ is non-archimedean, then every neutral periodic point is also contained in the Fatou set.

**Proof.** See [6, Proposition 1.1].

For the next result, we introduce another definition. A point $P \in \bar{\mathbb{L}}$ is said to be grand orbit finite (or an exceptional point) if the set of all forward and backward iterates of $P$ under $\phi$ is finite. Such points are quite rare, as the following shows.

**Lemma 5.2.** If $\phi \in \mathbb{L}[z]$ is a polynomial of degree at least 2, then $\phi$ has at most one grand orbit finite point in $\bar{\mathbb{L}}$.

**Proof.** See [3] for the case $\mathbb{L} = \mathbb{C}$ and [20, Remark 2.7] for the non-archimedean case.

**Corollary 5.3.** $\phi$ has infinitely many distinct preperiodic points in $\bar{\mathbb{L}}$.

**Proof.** The set $P$ of all preperiodic points of $\phi$ is clearly completely invariant under $\phi$. If $P$ is a finite set, then all preperiodic points are grand orbit finite, and hence there can be only one preperiodic point. However, a simple calculation shows this is impossible: We may assume without loss of generality that the only preperiodic point of $\phi$ is 0, and then we must have $\phi(z) - z = z^d$, so that $\phi(z) = z^d + z$. But then the equation $\phi(z) = 0$ has a nonzero root in $\bar{\mathbb{L}}$ which is also preperiodic, a contradiction.

We now turn to some illustrative examples of Julia sets and filled Julia sets. Because numerous examples of such sets appear in most books on complex dynamics, we concentrate below on examples from non-archimedean dynamics.

**Example 5.4.** We give examples of polynomials $\phi$ for which the Julia set $J$ of $\phi$ is (1) empty, (2) nonempty and compact, and (3) non-compact.

(1) Let $\mathbb{L}$ be an algebraically closed non-archimedean local field. We say that a polynomial $\phi(z) \in \mathbb{L}[z]$ has good reduction if $\phi(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0 \in \mathbb{L}[z]$ with $|a_n| \leq 1$ for all $n$ and $|a_d| = 1$. (See [23] for a more general definition of good reduction for rational maps).

It is a simple exercise to verify that if $\phi(z)$ has good reduction, then the filled Julia set of $\phi$ is just the closed unit disc $\{|z| \leq 1\}$ in $\mathbb{L}$, and the Julia set of $\phi$ is empty (something which cannot happen over $\mathbb{C}$!).
(2) Let \( p \) be a prime number. Let \( k \) be either \( \mathbb{Q}_p \) or the field of Laurent series \( \mathbb{F}_p((t)) \). Let \( L \) be a finite extension of ramification index \( e \) and residue degree \( f \) over \( k \). Let \( | \cdot | \) be the absolute value on \( L \) so that \( |p| = 1/p \) or \( |t| = 1/p \). Let \( q \) be the cardinality of the residue field of \( L \), and fix a uniformizer \( \pi \) of \( L \), so that \( |\pi| = (1/p)^{1/e} \) and \( q = p^f \).

Let \( g(z) \in \mathcal{O}_L[z] \) be a monic polynomial of degree \( n \) such that \( \tilde{g}(z) := g(z) \mod \pi \) is a separable polynomial of degree \( n \) which splits over \( \mathbb{F}_q \). Take \( \phi(z) \) to be the polynomial \( g(z)/\pi \). Then the filled Julia set and Julia set of \( \phi \) are both equal to a compact subset \( J_\phi \) of \( \mathcal{O}_L \) (see [19] for more detailed description of \( J_\phi \)). In particular, if \( \tilde{g}(z) = z^q - z \) then \( J_\phi = \mathcal{O}_L \). For example, the Julia set of \( g(z) = (z^p - z)/p \) is the compact subset \( \mathbb{Z}_p \subseteq \mathbb{C}_p \).

(3) Let \( \phi(z) = (z^3 - z^2)/p \). If we denote by \( v(\cdot) \) the standard valuation on \( \mathbb{C}_p \), so that \( v(p) = 1 \) and \( |z| = p^{-v(z)} \), then it is easy to see that the filled Julia set \( F \) of \( \phi \) is contained in the unit disc \( B_0 := \{ v(z) \geq 0 \} \) and contains the disc \( B_\infty := \{ v(z) \geq 1 \} \). Note that since \( F \) contains a disc, it is not compact. The disc \( B_\infty \) is contained in the Fatou set of \( \phi \), but nonetheless (at least if \( p \neq 3 \)) we claim that the Julia set of \( \phi \) is not compact. To see this, note that by [20, Example 4.2], \( \phi \) has both infinitely many repelling and infinitely many non-repelling periodic points which are contained in \( B_0 \). (Note that over \( \mathbb{C} \), a rational map of degree at least 2 always has finitely many non-repelling periodic points.) The claim then follows from the following fact (see [8, Proposition 9] for a proof): If \( P \in \mathbb{C}_p[z] \) is a polynomial of degree at least 2 whose Julia set is non-empty and compact, then all periodic points of \( P \) are repelling.

If we set \( B_n := \{ v(z) \geq 1 - \frac{1}{p^n} \} \) for \( n = 0, 1, 2, \ldots \), then a simple computation shows that for \( n \geq 1 \) we have \( \phi(B_n) \subseteq B_{n-1} \), so that \( \phi^n(B_n) \subseteq B_0 \). Additionally, if we set \( C_n := \{ v(z) = 1 - \frac{1}{2^n} \} \), then for \( z \in \{ 1 - \frac{1}{2^n} < v(z) < 1 - \frac{1}{2^{n+1}} \} = B_n \setminus (B_{n+1} \cup C_n) \) \( (n \geq 0) \), we have \( \phi^{n+1}(z) \in B_0 \). In particular, \( F \) is contained in the union of \( B_{\infty} \) and \( \cup_{n=0}^\infty C_n \), and the Julia set is contained in \( \cup_{n=0}^{\infty} C_n \).

6. Transfinite diameters

In this section, we will discuss in more detail the general theory of transfinite diameters.

6.1. Fekete’s theorem and Chebyshev constants. The first result which we want to discuss is a classical theorem of Fekete (which is the easy half of the “Fekete-Szego theorem” – see [28] for further details). Since the proof is short and illustrative, we include a proof (c.f. [15, Theorem 5.1.2] and [28, Theorem 6.3.1]).

For the statement, we make the following definition. Let \( K \) be a global field. An \( \text{adelic set} \) with respect to \( K \) is a collection \( \{ F_v \} \) of subsets \( F_v \subseteq \mathbb{C}_v \) such that \( F_v \) is the closed unit disc in \( \mathbb{C}_v \) for almost all \( v \).

We define the transfinite diameter (or capacity) of an adelic set \( F = \{ F_v \} \) to be

\[
\text{c}(F) := \prod_v \text{c}(F_v).
\]

By Remark 6.4 below, almost all terms of this infinite product are 1.
Proposition 6.1 (Fekete). Let $K$ be a global field, and let $\mathbb{F} := \{ F_v \}$ be an adelic set with respect to $K$ such that $c(\mathbb{F}) < 1$. Then there are only finitely many elements $\alpha \in \hat{K}$ such that all Galois conjugates of $\alpha$ are contained in $F_v$ for all $v \in M_K$.

Proof. Assume to the contrary that there are infinitely many such elements $\alpha_1, \alpha_2, \ldots$, and let $S_n$ be the set of all Galois conjugates of $\alpha_n$. By the Northcott finiteness property, the cardinality $N_n$ of $S_n$ tends to infinity as $n \to \infty$, because the heights of the $\alpha_n$’s are uniformly bounded by assumption. Let $\Delta_n := \prod_{x, y \in S_n, x \neq y} (x - y)$ be the discriminant of $\alpha_n$, which is a nonzero element of $K$. The product formula for $K$ implies that $\prod_{v \in M_K} \| \Delta_n \|_v = 1$ for all $n$, so that by the definition of transfinite diameter we have

$$1 = \limsup_{n \to \infty} \prod_{v \in M_K} \| \Delta_n \|_v^{\frac{1}{n}} \leq \prod_{v \in M_K} c(F_v) < 1,$$

a contradiction. \hfill $\blacksquare$

We see that a certain amount of mileage can be gained from exploiting the interplay between the transfinite diameter and the discriminant. We will use this observation again in the proof of Theorem 4.3. In order to prove Theorem 4.1, however, we will also need some further properties of the transfinite diameter. The key result is that when the metric space $M$ in question is a valued field, the transfinite diameter of a bounded set $A \subset M$ coincides with what is sometimes called the “Chebyshev constant” of $A$. This is made precise in the following lemma, whose proof can be found in [1, Lemma 5.4.2] (see also [18, Theorem 16.2.1], which is stated for compact subsets of $C$, but whose proof can be easily adapted to the present situation).

Lemma 6.2. Let $B$ be a bounded subset of the valued field $L$, and denote by $P_n$ the set of monic polynomials of degree $n$ with coefficients in $L$. If $P \in P_n$, define $\| P \|_B = \sup_{x \in B} | P(x) |$, and let

$$S_n(B) = \inf_{P \in P_n} \| P \|_B, \quad s_n(B) = (S_n(B))^{1/n}.$$

Then $s_n(B) \to c(B)$ as $n \to \infty$.

The proof of the next lemma is adapted from the exercises for [18, Section 16.2].

Lemma 6.3. Let $L$ be an algebraically closed valued field and let $A$ be a bounded subset of $L$. Let $\phi \in L[z]$ be a polynomial of degree $d \geq 1$ with leading coefficient $a_d$, and set $A^\star := \{ z \in L : |\phi(z)| \in A \}$. Then $c(A^\star) = (c(A)/|a_d|)^{1/d}$.

Proof. It is an elementary exercise to show that the assertion is true in the case $d = 1$. For the general case, a simple change of variables shows that we only need to treat the case where $\phi$ is monic, in which case we want to show that $c(A^\star) = c(A)^{1/d}$.

Let $T_n \in P_n$ be arbitrary. Then $T_n \circ \phi \in P_{nd}$ and $\| T_n \circ \phi \|_{A^\star} \leq \| T_n \|_A$. It therefore follows from Lemma 6.2 that $c(A^\star) \leq c(A)^{1/d}$.

For the other direction, we again use Lemma 6.2. Let $n \geq 1$ be an integer, let $\epsilon > 0$ be arbitrary, and choose $T_n^\star \in P_n$ such that $\| T_n^\star \|_{A^\star} < S_n(A^\star) + \epsilon$. For each $w \in L$, let $z_1(w), \ldots, z_d(w)$ denote the roots of $\phi(z) = w$ (with multiplicities), and set $\phi_n(w) := \prod_{j=1}^d T_n^\star(z_j(w))$. Viewing $\phi_n(w)$ as the resultant of $T_n^\star(z)$ and
\[ \phi(z) - w, \text{ it is clear from Sylvester's determinant that } \phi_n \text{ is a polynomial of degree } n \text{ in } w \text{ with leading coefficient } \pm 1. \] Therefore

\[ S_n(A) \leq \|\phi_n\|_A \leq \prod_{j=1}^d \|T_n^j\|_{A^*} = \|T_n^d\|_{A^*} < (S_n(A^*) + \epsilon)^d. \]

Since \( \epsilon \) was arbitrary, it follows that \( s_n(A) \leq s_n(A^*)^d \) for all \( n \geq 1 \). Therefore

\[ c(A^*) \geq c(A)^d \] as desired. \[ \square \]

**Remark 6.4.** (1) Suppose \( L \) is an algebraically closed local field. If \( M \) is a positive real number such that \( |\alpha| = M \) for some \( \alpha \in L \) (in which case we say that \( M \) is in the value group of \( L \)), then the disc \( D_M := \{ z \in L : |z| \leq M \} \) has transfinite diameter \( M \). To see this, note that by making the substitution \( z \mapsto z/\alpha \), we may reduce to the statement that if \( D = D_1 \) is the unit disc in \( L \), then \( c(D) = 1 \). Since the polynomial \( z^n \) has sup-norm equal to 1 on \( D \) for all \( n \), it follows from Lemma 6.2 that \( c(D) \leq 1 \). On the other hand, for any \( n \) not divisible by the residue characteristic of \( L \), consider the \( n \) distinct roots \( \zeta_1, \ldots, \zeta_n \) of the separable polynomial \( f_n(z) := z^n - 1 \). We find that

\[ \prod_{i \neq j} |\zeta_i - \zeta_j| \geq \prod_i |f_n'(\zeta_i)| = |n^n| \geq 1, \]

so that \( c(D) \geq 1 \) by the definition of transfinite diameter. Therefore \( c(D) = 1 \) as claimed.

(2) Let \( \phi \in L[z] \) be a polynomial of degree \( d \geq 1 \) with leading coefficient \( a_d \). Applying Lemma 6.3 and (1), we see that the transfinite diameter of the region \( B := \{ z \in L : |\phi(z)| \leq M \} \) is equal to \( \left( \frac{M}{|a_d|} \right)^d \). We will refer to such regions as **lemniscates**.

### 6.2. Proof of Theorem 4.1.

The conclusion of Theorem 4.1 in the case \( L = \mathbb{C} \) is well-known (see e.g. [26]). However, the proof uses the following result (see [26, Theorem 5.1.3] or [18, Theorem 16.2.2(iii)] for a proof), which does not generalize to the more general fields \( L \) which we consider:

**Lemma 6.5.** Let \( F_1, F_2, \ldots \) be closed and bounded subsets of \( \mathbb{C} \) with \( F_1 \supseteq F_2 \supseteq \cdots \), and let \( F = \cap_{n=1}^\infty F_n \). Then

\[ c(F) = \lim_{n \to \infty} c(F_n). \]

**Remark 6.6.** The analogue of this lemma is false in general over a non-archimedean valued field \( L \) (such as \( \mathbb{C}_p \) for any prime \( p \)) which is not locally compact. (Of course, one always has an inequality \( c(F) \leq \lim_{n \to \infty} c(F_n) \).)

For example, if \( L = \mathbb{C}_p \), we can find an infinite sequence of disjoint closed discs \( B_1, B_2, \ldots \) of the form \( B_i = \{ z \in \mathbb{C}_p : |z - \alpha_i| \leq \frac{1}{p} \} \) for some sequence \( \alpha_i \in \mathbb{C}_p \) with each \( |\alpha_i| = 1 \). Then if we set \( F_n := \bigcup_{i=n}^\infty B_i \) for \( n \geq 1 \), then each \( F_n \) is closed and bounded and \( F_1 \supseteq F_2 \supseteq \cdots \). However, we have \( c(F_n) \geq c(B_n) = \frac{1}{p} \) for all \( n \) (see Remark 6.4), whereas the transfinite diameter of \( \cap_{n=1}^\infty F_n = \emptyset \) is zero.

**Remark 6.7.** The analogue of Lemma 6.5 over \( \mathbb{C}_p \) is still false even if we assume that each set \( F_n \) is a closed disc, since one can construct a sequence \( F_1 \supseteq F_2 \supseteq \cdots \) of closed discs of radius \( r_n \) in \( \mathbb{C}_p \) such that \( \lim_{n \to \infty} r_n > 0 \) but \( \cap_{n=1}^\infty F_n = \emptyset \).

We now prove Theorem 4.1.
Proof. Define $c(\phi) = |a_d|^{-1}$, where $a_d$ is the leading coefficient of $\phi \in L[z]$. The proof of the inequality $c(F) \leq c(\phi)$ is based on the argument found in [26, Theorem 6.5.1]. We give two proofs of the reverse inequality, one global and the other purely local. The global proof is in some sense simpler, but it only applies under additional hypotheses.

Recall that $\phi(z) \in L[z]$, where $L$ is either $\mathbb{C}$ or an algebraically closed ultrametric field, that the degree of $\phi$ is $d \geq 2$, and that $F$ is the filled Julia set of $\phi$. Fix a sequence $\{F_n\}$ of closed subsets as in Section 5 such that $F_n = \phi^{-1}(F_{n-1}) = \phi^{-n}(F_0)$ for all $n \geq 1$, so that

$$F_0 \supseteq F_1 \supseteq \cdots \supseteq F_n \supseteq \cdots \supseteq F$$

and

$$F = \cap_{n=0}^{\infty} F_n.$$

By Lemma 6.3, we have

$$c(F) \leq \lim_{n \to \infty} c(F_n) = \lim_{n \to \infty} c(F_0)^{1/|a_d|^{-(1/d + \cdots + 1/n)}} = |a_d|^{1/(d-1)},$$

which gives us the upper bound $c(F) \leq c(\phi)$.

We now present two proofs of the fact that $c(F) = c(\phi)$.

For the global proof, we make the additional assumption that $\phi \in K[z]$, where $K$ is a global field, and that $L = \mathbb{C}_v$ for some $v \in M_K$. We know already that $c(F) \leq c(\phi)$ for all $v \in M_K$, and the idea is to prove the reverse inequality for all $v \in M_K$ at once. To do this, we first recall that by Corollary 5.3, $\phi$ has infinitely many distinct preperiodic points in $\mathcal{K}$, and by definition, a preperiodic point of $\phi$ lies in $F_v$ for all places $v$.

Let $\mathbb{F}$ be the adelic set corresponding to the collection $\{F_v\}$. If $c(F_v) < c_v(\phi)$ for some $v \in M_K$, then $c(\mathbb{F}) \leq \prod_{v \in M_K} c_v(\phi) = 1$ and the preperiodic points furnish a contradiction to Fekete’s theorem (Theorem 6.1). Therefore $c(F_v) = c_v(\phi)$ for all $v$, as desired.

For the purely local proof, we return to the general setup where $\phi \in L[z]$ is a polynomial of degree at least 2, and we need to show that $c(F) \geq c(\phi)$. Note that if $L = \mathbb{C}$, then we can already conclude that $c(F) = c(\phi)$ by Lemma 6.5. So without loss of generality, we may assume that the field $L$ is non-archimedean.

Since $F$ is completely invariant under $\phi$, we have $F = \phi^{-1}(F) = \{z \in L : \phi(z) \in F\}$. By Lemma 6.3, we have $c(F)^d = c(\phi)^{d-1} c(F)$, so it suffices to show that $c(F) > 0$. If there is a non-repelling periodic point $Q$ in $F$, then $Q$ is in the Fatou set by Lemma 5.1. As the Fatou set is open and completely invariant under $\phi$, it follows that $F$ contains an entire disc around $Q$. It is then clear from Remark 6.4 (1) that $c(F) > 0$. Therefore we may reduce to the case where all periodic points of $F$ are repelling.

For each $n \geq 1$, let $\text{Per}_n$ denote the set of all points of period dividing $n$ (i.e., the set of all roots of $\phi^n(z) - z$) and let $N_n$ be its cardinality. By the assumption that all periodic points are repelling, together with the fact that the absolute value on $L$ is non-archimedean, we see that $|\phi^n(P)| - 1 = |\phi^n(P)| > 1$ for all $P \in \text{Per}_n$. In particular, the polynomial $\phi^n(z) - z$ is separable, so that $N_n = d^n$. Letting $d(\text{Per}_n) := \prod_{x,y \in \text{Per}_n, x \neq y} |x-y|^{1/N_n(N_n-1)}$, we find (using a standard algebraic
identity) that
\[ d(\text{Per}_n) = |a_d|^{1/n} \prod_{P \in \text{Per}_n} \vert (\phi^n)'(P) - 1 \vert^{1/N_n(N_n-1)} > c(\phi) \]
for all \( n \), so that \( c(F) \geq c(\phi) \), as desired. ■

**Example 6.8.** Let \( \phi(z) \) be as given in Example 5.4(2). Then, the formula in Theorem 4.1 says that the transfinite diameter of \( J_\phi \) is equal to \( |1/\pi|^{-1/(n-1)} = p^{-1/\epsilon(n-1)} \). In particular, if \( n = q = p^f \) then \( J_\phi = \mathcal{O}_L \) and the transfinite diameter of \( \mathcal{O}_L \) is equal to \( p^{-1/\epsilon(p^f-1)} \), in agreement with the computation found in [28, Example 4.1.24]. As a concrete special case, the transfinite diameter of \( Z_p \subseteq \mathbb{C}_p \) is \( p^{-1/\epsilon} \).

7. Equidistribution and pseudo-equidistribution

**7.1. Proof of Theorem 4.3.** We begin by proving a couple of important technical lemmas concerning the relationship between heights and transfinite diameters. First, we make some definitions. Suppose \( L \) is a valued field and that \( S = \{ S_n \} \) and \( S' = \{ S'_n \} \) are sequences of finite subsets of \( L \). Let \( N_n \) (resp. \( N'_n \)) denote the cardinality of \( S_n \) (resp. \( S'_n \)).

We say that \( S' \) is a full subsystem of \( S \) if:

- \( S'_n \subseteq S_n \) for all \( n \)
- \( \lim_{n \to \infty} N_n/N'_n = 1 \).

If \( T \) is any finite subset of \( L \), with cardinality \( N \), we define
\[ d(T) := \prod_{x,y \in T, x \neq y} |x - y|^{1/(N - 1)}. \]

With this terminology, we have:

**Lemma 7.1.** Let \( L \) be a valued field, let \( S = \{ S_n \} \) and \( S' = \{ S'_n \} \) be sequences of finite subsets of \( L \), and assume that \( S' \) a full subsystem of \( S \). Let \( N_n \) (resp. \( N'_n \)) denote the cardinality of \( S_n \) (resp. \( S'_n \)), and assume that \( \lim_{n \to \infty} N_n = \infty \). Furthermore, let \( \lambda \) be a logarithmic distance function for \( L \), and assume that \( S \) is \( \lambda \)-taut. Then
\[ \limsup d(S_n) \leq \limsup d(S'_n). \]

**Proof.** Set \( T_n = S_n \setminus S'_n \), then
\[ d(S_n) = \frac{1}{N_n(N_n - 1)} \sum_{x,y \in S'_n, x \neq y} \log |x - y| + \psi_n, \]
where
\[ \psi_n := \frac{2}{N_n(N_n - 1)} \sum_{x \in S'_n, y \in T_n} \log |x - y| + \frac{1}{N_n(N_n - 1)} \sum_{x \in S'_n, y \in T_n} \log |x - y|. \]

Since \( \lim_{n \to \infty} N_n/N'_n = 1 \), we find using Remark 7.3 below that
\[ \limsup d(S_n) \leq \limsup d(S'_n) + \limsup \psi_n. \]

Observe that \( \log |x - y| \leq \log^+ |x| + \log^+ |y| + \log 2 \) for all \( x, y \in L \), since by the triangle inequality we have
\[ |x - y| \leq 2 \max\{|x|, |y|\} \leq 2 \max\{1, |x|\} \max\{1, |y|\}. \]
By assumption, there exists a constant $C > 0$ such that $\log^+ |z| \leq \lambda(z) + C$. Therefore $\log |x - y| \leq \lambda(x) + \lambda(y) + C'$, where $C' = 2C + \log 2$.

If we set $R_n := \# T_n$, so that $\lim R_n / N_n = 0$ by assumption, then it follows that

$$
\psi_n \leq \frac{1}{N_n(N_n - 1)} \left\{ 2 \sum_{x \in S'_n, y \in T_n} (\lambda(x) + \lambda(y) + C') + \sum_{x, y \in T_n, x \neq y} (\lambda(x) + \lambda(y) + C') \right\}
= \frac{1}{N_n(N_n - 1)} \left\{ 2R_n \left( \sum_{z \in S'_n} \lambda(z) \right) + 2(N'_n + R_n - 1) \left( \sum_{z \in T_n} \lambda(z) \right) \right\}
+ \frac{1}{N_n(N_n - 1)} \{ R_n (2N'_n + R_n - 1)C' \}.
$$

Therefore $\limsup \psi_n \leq 0$ by our assumptions, which yields the desired inequality.

The following key lemma makes precise the intuition that if a set $T$ of points in $\mathbb{C}_v$ is close to the filled Julia set $F_v$ of $\phi$ (in the sense that the canonical local height $\hat{h}_{\phi, v}(T)$ is small$^7$), then the average pairwise distance between the points of $T$ can exceed the transfinite diameter of $F_v$ only by a small amount.

**Lemma 7.2.** Let $v \in M_K$, and suppose that $S_n$ is a sequence of finite subsets of $\mathbb{C}_v$ with the property that $N_n = \# S_n \to \infty$ and $\hat{h}_{\phi, v}(S_n) \to 0$. Then

$$
\limsup_{n \to \infty} d(S_n) \leq c_v(\phi).
$$

**Proof.** Set $\lambda(z) := \hat{h}_{\phi, v}(z)$ and put $L := \mathbb{C}_v$. By property (3) above of the local canonical height, $\lambda$ is a logarithmic distance function for $L$, so that there exists a constant $C > 0$ such that $|\lambda(z) - \log^+ |z|| \leq C$ for all $z \in L$. In particular (with the convention that $\log 0 = -\infty$), we have $\log |z| \leq \lambda(z) + C$ for all $z \in L$.

By property (1) of the local canonical height, we also have $\lambda(\phi(z)) = d\lambda(z)$ for all $z \in L$. Therefore, for all $m \geq 1$ and all $z \in L$ we have

$$
(1) \quad \frac{1}{d^m} \log |\phi^m(z)| \leq \lambda(z) + \frac{C}{d^m}.
$$

Fix a rational number $\beta > 0$, and define $S'_n := \{ z \in S_n : \lambda(z) \leq \beta/2 \}$. Then $S' := \{ S'_n \}$ is a full subsystem of $S$, since (setting $T_n = S_n \setminus S'_n$ and $R_n = \# T_n$) if $\limsup R_n / N_n = \delta > 0$, then we would have $\hat{h}_{\phi, v}(S_n) \geq \delta \beta > 0$ for infinitely many $n$, contradicting the fact that $\hat{h}_{\phi, v}(S_n) \to 0$.

By Lemma 7.1, it suffices to prove that $\limsup_{n \to \infty} d(S'_n) \leq c_v(\phi)$.

To see this, for each integer $m \geq 1$ we define

$$
U_{m, \beta} := \{ z \in L : \frac{1}{d^m} \log |\phi^m(z)| \leq \beta \}.
$$

$^7$We remind the reader that by definition, $\hat{h}_{\phi, v}(T) := \frac{1}{|T|} \sum_{z \in T} \hat{h}_{\phi, v}(z)$. Note that one should not take the above intuition too literally, however, since a sequence of points $z_n \in \mathbb{C}_v$ for $v$ non-archimedean can satisfy $\hat{h}_{\phi, v}(z_n) \to 0$ without $\text{dist}(z_n, F_v)$ converging to zero (see Example 7.6 below).
Then by (1) above, for any \( m \) such that \( \frac{C}{d^m} \leq \beta/2 \) we have \( S'_n \subset U_{m,\beta} \). By the definition of transfinite diameter, it follows that for \( m \) sufficiently large we have

\[
\limsup_{n \to \infty} d(S'_n) \leq c(U_{m,\beta}).
\]

On the other hand, note that \( \phi^m(z) \) has degree \( d^m \) and leading coefficient \( a(m) := a_d^{d+d^2+\ldots+d^{d-1}} \). Since

\[
|a(m)|^{-d^m} = \left( |a_d| \frac{d^m}{d^m-1} \right)^{-d^m} = \left( |a_d| \frac{1}{d-1} \right)^{-d^m} = \psi_v(\phi)^{1-d^m},
\]

it follows from Remark 6.4(2) that

\[
c(U_{m,\beta}) = \frac{e^\beta}{|a(m)|^{-d^m}} = e^\beta \psi_v(\phi)^{1-d^m}.
\]

Letting \( m \to \infty \) in (2), we find that \( \limsup_{n \to \infty} d(S'_n) \leq \psi_v(\phi) e^\beta \). Now let \( \beta \to 0 \), and we obtain \( \limsup_{n \to \infty} d(S'_n) \leq \psi_v(\phi) \) as desired.

We now turn to the proof of Theorem 4.3.

**Remark 7.3.** Before we give the proof of Theorem 4.3, note that if \( (a_n^{(j)})_{j=1,\ldots,k,n \geq 1} \) is a doubly indexed sequence of real numbers, then

\[
\limsup_{n \to \infty} \left( \sum_{j=1}^k a_n^{(j)} \right) \leq \sum_{j=1}^k \limsup_{n \to \infty} a_n^{(j)}.
\]

This is an easy consequence of Fatou’s lemma, applied to an appropriate step function. However, this inequality can fail if we replace the finite sum by an infinite sum.

**Proof.** Let \( N_n \) be the cardinality of \( S_n \). Since \( K \) has the Northcott finiteness property and the difference \( |\hat{h}_\phi(\alpha) - h(\alpha)| \) is bounded on \( \bar{K} \), it follows from the assumption \( \hat{h}_\phi(S_n) \to 0 \) that \( N_n \to \infty \) as \( n \to \infty \). By hypothesis, we have \( \sum_{v \in M_K} \hat{h}_\phi,v(S_n) \to 0 \). Since each of the functions \( \hat{h}_\phi,v \) takes nonnegative values, this implies that \( \frac{1}{N_n} \sum_{z \in S_n} \hat{h}_\phi,v(z) \to 0 \) for all \( v \in M_K \), i.e., that \( S \) is \( \hat{h}_\phi,v \)-taut for all \( v \).

We need to show that \( S \) is a generalized Fekete sequence for \( F_v \) for all \( v \in M_K \). First, we note that \( \log \psi_v(\phi) = 0 \) and \( \hat{h}_\phi,v(z) = h_v(z) \) for all but finitely many \( v \in M_K \). Let \( T \) be a finite subset of \( M_K \) containing the archimedean places and all places \( v \) where either \( \log \psi_v(\phi) \neq 0 \) or \( \hat{h}_\phi,v(z) \neq h_v(z) \).

Let \( \Delta_n := \prod_{x,y \in S_n,x \neq y} (x - y) \), a nonzero element of \( K \). For \( v \in M_K \setminus T \), we have

\[
\log |x - y|_v \leq \log^+ |x|_v + \log^+ |y|_v = h_v(x) + h_v(y)
\]

as \( v \) is a non-archimedean valuation. Therefore, we can bound \( \log \|\Delta_n\|_v \) in terms of local heights of \( S_n \) as follows:

\[
\frac{1}{N_n(N_n - 1)} \log \|\Delta_n\|_v \leq \frac{n_v}{N_n(N_n - 1)} \sum_{x,y \in S_n,x \neq y} \{h_v(x) + h_v(y)\} = \frac{2n_v}{N_n} \sum_{x \in S_n} h_v(x).
\]
Hence,
\[
\frac{1}{N_n(N_n-1)} \sum_{v \in T} \log \|\Delta_n\|_v \leq \frac{2}{N_n} \sum_{v \notin T} \sum_{x \in S_n} h_v(x) \\
\leq 2\hat{h}_\phi(S_n),
\]
where the last inequality comes from the non-negativity of local heights.

Using Lemma 7.2, we have
\[
\limsup_{n \to \infty} \frac{1}{N_n(N_n-1)} \log \|\Delta_n\|_v \leq \log c_v(\phi)
\]
for all \(v \in M_K\). By assumption, \(\lim_{n \to \infty} \hat{h}_\phi(S_n) = 0\), and since \(\Delta_n \neq 0\), the product formula gives \(\sum_{v \in T} \log c_v(\phi) = \sum_{v \in M_K} \log c_v(\phi) = 0\). Therefore,
\[
0 = \limsup_{n \to \infty} \frac{1}{N_n(N_n-1)} \sum_{v \in M_K} \log \|\Delta_n\|_v \\
\leq \sum_{v \in T} \limsup_{n \to \infty} \frac{1}{N_n(N_n-1)} \log \|\Delta_n\|_v + \limsup_{n \to \infty} \frac{1}{N_n(N_n-1)} \sum_{v \notin T} \log \|\Delta_n\|_v \\
\leq \sum_{v \in T} \log c_v(\phi) + 2 \lim_{n \to \infty} \hat{h}_\phi(S_n) = 0.
\]

It follows that \(\limsup_{n \to \infty} \frac{1}{N_n(N_n-1)} \log \|\Delta_n\|_v = \log c_v(\phi)\) for all \(v \in T\). As \(T\) can be chosen to contain any place \(v \in M_K\), we have shown that
\[
\limsup_{n \to \infty} \frac{1}{N_n(N_n-1)} \log \|\Delta_n\|_v = \log c_v(\phi)
\]
for all \(v \in M_K\).

To finish the proof, let \(w \in M_K\) be any fixed place. Then
\[
\log \|\Delta_n\|_w = -\sum_{v \in M_K, v \neq w} \log \|\Delta_n\|_v.
\]

Let \(T\) be a finite set of places containing \(w\). Then by the same argument as in (3), we have
\[
\limsup_{n \to \infty} \frac{1}{N_n(N_n-1)} \sum_{v \in M_K, v \neq w} \log \|\Delta_n\|_v \leq \sum_{v \in T, v \neq w} \log c_v(\phi) = -\log c_w(\phi).
\]

It follows that
\[
\liminf_{n \to \infty} \frac{1}{N_n(N_n-1)} \log \|\Delta_n\|_w = -\limsup_{n \to \infty} \frac{1}{N_n(N_n-1)} \sum_{v \in M_K, v \neq w} \log \|\Delta_n\|_v \\
\geq \log c_w(\phi).
\]

As \(w\) is arbitrary, we therefore have
\[
\lim_{n \to \infty} \frac{1}{N_n(N_n-1)} \log \|\Delta_n\|_v = \log c_v(\phi)
\]
for all \(v \in M_K\). Thus \(S\) is pseudo-equidistributed with respect to \((F_v, \hat{h}_\phi, v)\) as desired. ■
Example 7.4. Let $p$ be a prime number. Assume that $K$ is a number field and let $\phi(z) = z^2$. Fix a non-archimedean place $v \in M_K$ such that $F_v$ is the unit disc in $\mathbb{C}_v$, and let $\ell$ be the residue characteristic of $\mathbb{C}_v$. Let $S_n = \mu_{p^n}$ be the set of $p^n$-th roots of unity. Clearly $\{S_n\}$ satisfies the conditions required in Theorem 4.3. Hence, by Theorem 4.3 the sequence of sets $\{S_n\}$ is pseudo-equidistributed with respect to $(F_v, h_{\phi,v})$. We can see this concretely as follows.

If $p \neq \ell$ then distinct elements are in distinct residue classes, and it is easy to see for such a sequence of finite sets that $\limsup_{n \to \infty} d(S_n) = 1$, which is the transfinite diameter of the unit disc.

However, if $p = \ell$ then for all $\zeta \in \mu_{p^n}$ we have $\zeta \equiv 1 \pmod{v}$. That is, for all $n \geq 1$, $S_n$ is contained in the open unit disc centered at 1. A computation similar to that in Remark 6.4 (1) now shows that $\limsup_{n \to \infty} d(S_n) = 1$, which is not quite as obvious as in the case $p \neq \ell$.

7.2. Equidistribution: archimedean case. Our goal in this section is to prove Proposition 4.4. Before we do this, however, we need an easy lemma. Recall that a sequence of measures $\nu_n$ on a metric space $M$ is said to be tight if for every $\epsilon > 0$, there exists a compact set $K_\epsilon \subseteq M$ such that $\nu_n(M \setminus K_\epsilon) < \epsilon$ for all $n$ sufficiently large. Given $F \subseteq M$ and $\delta > 0$, we define

$$F_\delta := \{z \in M : \text{dist}(z, F) \leq \delta\}.$$ 

Suppose $S = \{S_n\}$ is a sequence of finite subsets of $M$. We say that $S$ is $F$-tight if for all $\delta, \epsilon > 0$, there exists $N$ such that for all $n \geq N$, at least $(1 - \epsilon)(\#S_n)$ of the elements of $S_n$ lie in $F_\delta$.

Lemma 7.5. Let $\phi \in \mathbb{C}[z]$ be a polynomial with filled Julia set $F \subseteq \mathbb{C}$. Let $\lambda$ be a logarithmic distance function for $F$, and let $S = \{S_n\}$ be a sequence of finite subsets of $\mathbb{C}$ which is $\lambda$-taut. Then $S$ is $F$-tight.

Proof. Let $\delta > 0$, and let $V_\delta$ be the complement of $F_\delta$. Recall that the function $\lambda$ is continuous, nonnegative, and zero exactly on $F$, and that $\lambda(z) \to \infty$ as $z \to \infty$. By the local compactness of $\mathbb{C}$, it therefore follows that $\lambda$ is bounded below by a positive constant $C_\delta$ on $V_\delta$. It follows immediately from this and the fact that

$$\frac{1}{\#S_n} \sum_{z \in S_n} \lambda(z) \to 0$$

that the sequence $\{S_n\}$ is $F$-tight.

We now give the proof of Proposition 4.4.

Proof. As usual, we let $N_n$ be the cardinality of $S_n$. Let $\mu_F$ denote the equilibrium measure for $F$, and let $\delta_n := \frac{1}{N_n} \sum_{z \in S_n} \delta_z$. Since the sequence $\{\delta_n\}$ is $F$-tight, it is in particular tight in the usual sense, and so by Prohorov’s theorem (see [24]) it has a weakly convergent subsequence. We may therefore assume that there is a probability measure $\mu$ such that $\delta_n$ converges weakly to $\mu$, and it suffices to show that $\mu = \mu_F$. As a first step, we note that it follows easily from the $F$-tightness of $\{\delta_n\}$ that $\mu$ is supported on $F$.

Recall that the equilibrium measure on $F$ is characterized among probability measures supported on $F$ as having the smallest possible energy, which is precisely $-\log \epsilon(F)$. Let $\Delta$ denote the diagonal in $\mathbb{C} \times \mathbb{C}$. Then it suffices to prove the inequality

$$\limsup_{n \to \infty} \int_{\mathbb{C} \times \mathbb{C} - \Delta} \log |x - y| d\delta_n(x) d\delta_n(y) \leq \int_{\mathbb{C} \times \mathbb{C}} \log |x - y| d\mu(x) d\mu(y),$$

for all $x, y \in \mathbb{C}$.
because we know the left-hand side is equal to
\[
\log \left( \limsup_{n \to \infty} \prod_{x, y \in S_n, x \neq y} |x - y|^{\frac{1}{n(n-1)}} \right),
\]
which equals \( \log c(F) \) since \( S \) is a generalized Fekete sequence.

Let \( \epsilon > 0 \) be given, and set \( S' := \{ S'_n \} \), where \( S'_n := \{ z \in S_n : z \in F \} \). Then Lemma 7.5 says that \( S' \) is a full subsystem of \( S \). By Lemma 7.1, it therefore suffices to show that

\[
\limsup_{n \to \infty} \int_{F_n \times F_n - \Delta} \log |x - y| d\delta_n(x) d\delta_n(y) \leq \int_{\mathbb{C} \times \mathbb{C}} \log |x - y| d\mu(x) d\mu(y),
\]

where \( \Delta \) is the diagonal in \( F_n \times F_n \).

This inequality follows in an essentially formal manner from our assumptions and some general measure theory. Specifically, for a fixed \( M \) we have

\[
\int \int \max\{-M, \log |x - y|\} d\delta_n(x) d\delta_n(y) = O\left( \frac{1}{N_n^\epsilon} \right)
\]

by the definition of \( \Delta \), and therefore

\[
\limsup_{n \to \infty} \int_{F_n \times F_n - \Delta} \log |x - y| d\delta_n(x) d\delta_n(y)
\]

\[
\leq \lim_{M \to \infty} \limsup_{n \to \infty} \int_{F_n \times F_n - \Delta} \max\{-M, \log |x - y|\} d\delta_n(x) d\delta_n(y)
\]

\[
= \lim_{M \to \infty} \limsup_{n \to \infty} \int_{F_n \times F_n} \max\{-M, \log |x - y|\} d\delta_n(x) d\delta_n(y)
\]

\[
= \lim_{M \to \infty} \int_{F_n \times F_n} \max\{-M, \log |x - y|\} d\mu(x) d\mu(y)
\]

\[
= \int \int \log |x - y| d\mu(x) d\mu(y).
\]

(\( \delta_n \to \mu \) weakly)

(monotone convergence theorem)

(\( \text{supp}(\mu) \subseteq F \))

\[
\]

\[
\]

\[
\]

7.3. Equidistribution: non-archimedean case. Our goal is now to prove a non-archimedean analogue of Proposition 4.4. Let \( L \) be an algebraically closed non-archimedean local field. If we try to transport arguments used in the archimedean case directly, we face some difficulties in the fact that the non-archimedean field \( L \) is not locally compact.

For example, the analogue of Lemma 7.5 is not in general true over a non-archimedean field. This is clear if \( F \subset L \) is the unit disc and \( \lambda(z) = \log^+ |z| \), since \( F_\delta = F \) for all \( \delta \leq 1 \). Another counterexample which may be illustrative is the following.

Example 7.6. Let \( \phi(z) = (z^3 - z^2) / p \) be as in Example 5.4(3). Then if \( |z|_p < \frac{1}{p} \), we have \( \hat{h}_{\phi, p}(z) = 0 \), and if \( |z|_p > 1 \), we have \( \hat{h}_{\phi, p}(z) = \log |z|_p + \frac{1}{p} \log p \). For \( \frac{1}{p} < |z|_p \leq 1 \), there is no simple formula for the canonical local height. One can merely say that \( \hat{h}_{\phi, p}(z) \) will be zero if \( z \in F \), and otherwise we will have \( |\phi^n(z)|_p > 1 \) for some \( n \), in which case \( \hat{h}_{\phi, p}(z) = \frac{1}{N^n}(\log |\phi^n(z)|_p + \frac{1}{p} \log p) \).
Recall from Example 5.4(3) that $F$ is contained in the union of the disc \{v(z) \geq 1\} and the sets $C_n := \{z \in \mathbb{C}_p : v(z) = 1 - \frac{1}{2^n}\}$ for $n = 0, 1, 2, \ldots$. Choose a rational number $0 < \epsilon_n < \frac{1}{2^n}$, and choose $z_n \in \mathbb{C}_p$ with $v(z_n) = 1 - \frac{1}{2^n} - \epsilon_n$. If we set $y_n := \phi^n(z_n)$, then $v(y_n) = -2^n \epsilon_n$ and therefore
\[
\hat{h}_{\phi,p}(z_n) = \frac{1}{3^n} \hat{h}_{\phi,p}(y_n) = \frac{1}{3^n}(2^n \epsilon_n \log p + \frac{1}{2} \log p).
\]
Now it’s clear that $\hat{h}_{\phi,p}(z_n) \rightarrow 0$ while the sequence $\{z_n\}$ keeps a positive distance away from $F$ since $\text{dist}(z_n, F) > \frac{1}{p}$.

On the other hand, if the filled Julia set $F \subset L$ is compact, then we will see that there is a non-archimedean analogue of Lemma 7.5. We begin with the following result:

**Lemma 7.7.** Let $\phi(z) \in L[z]$ be a polynomial of degree $d \geq 2$. Assume that the filled Julia set $F$ of $\phi$ is compact. Then for each $\delta \in \{|L^*|\}$, there exists a positive integer $N$ such that $F_n \subset F_\delta$ for all $n \geq N$.

**Proof.** Our assumption implies that the filled Julia set equals the Julia set of $\phi$, otherwise there would be a closed disc totally contained in $F$ by [5, Theorem 5.1 (3), (4)], and hence $F$ would not be compact.

Let $\frac{1}{|L^*|} \in \mathbb{C}_\infty$ be given. For $\omega \in L$, let $D_\delta(\omega) = \{z \in L : |z - \omega| < \delta\}$ denote the open disc centered at $\omega$ of radius $\delta$. Then the collection $\{D_\delta(\omega) : \omega \in F\}$ gives an open covering of $F$. Since $F$ is compact, there exist $\omega_1, \ldots, \omega_m(\delta) \in F$ such that $F \subset \bigcup_{i=1}^m D_\delta(\omega_i)$.

Since $F$ is the Julia set of $\phi$, for each $i$ there exists a positive integer $N_i$ such $\phi^{N_i}(D_\delta(\omega_i)) \supseteq F_0$. Take $N = \max\{N_1, \ldots, N_m(\delta)\}$. Then $\phi^{N}(D_\delta(\omega_i)) \supseteq F_0$ for all $i$. We claim that $F_N \subseteq U_\delta$.

Assume to the contrary that $F_N$ is not contained in $U_\delta$. We note that $F_N$ itself is a disjoint union of closed discs $F_N = \bigcup_j B_{N,j}$ where $\phi^N(B_{N,j}) = F_0$. As the valuation on $L$ is ultrametric, any two discs are either disjoint or one contains the other. Therefore, there must be some $B_{N,j}$ which is disjoint from $U_\delta$. This is impossible as $B_{N,j}$ contains the inverse image of some point of $F$ under $\phi^N$ and hence has non-empty intersection with $F$.

Our claim now shows that $F_N \subset F_\delta$. On the other hand, we have $F_N \supseteq F_{N+1} \supseteq \cdots$. Now the assertion of our proposition is clear. ■

The following analogue of Lemma 7.5 suffices in order to prove our non-archimedean equidistribution theorem.

**Proposition 7.8.** Let $\phi \in L[z]$ be a polynomial with compact filled Julia set $F \subset L$. Let $\lambda_\phi$ be the canonical local height associated to $\phi$ as defined in Section 3.2, and let $S = \{S_n\}$ be a sequence of finite subsets of $L$ which is $\lambda_\phi$-taut. Then $S$ is $F$-tight.

**Proof.** As in the proof of Lemma 7.5, it suffices to show that $\lambda_\phi$ is bounded below by a positive constant outside $F_\delta$ for any $\delta \in \{|L^*|\}$.

Recall that if $F_0 = \{|z| \leq R\}$ with $R$ sufficiently large, then $|\phi^m(z)| = |a(m)z^{d^m}| > \max(1, R)$ for all $z \notin F_0$ where $a(m) := a_d^{1+d+d^2+\cdots+d^{m-1}}$. It follows that $\lambda_\phi(z) = \log |z| - \log c(\phi)$ for all $z \notin F_0$, where $c(\phi) = |a_d|^{\frac{1}{d^m}}$ as usual. From this formula, it is clear that $\lambda_\phi$ is bounded below by a positive constant outside $F_0$. 

Let $\delta \in [L^\ast]$ be given. By Lemma 7.7 we know that there is a positive integer $N$ such that $\phi^n(z) \not\in F_0$ for $z \not\in F_3$ and $n \geq N$. As

$$\lambda_\phi(z) = (1/d^N)\lambda_\phi(\phi^N(z)),$$

for all $z \not\in F_3$, our assertion now follows in general. 

Recall from [27, Theorem 4.1.22] that for a compact subset $E \subseteq L$ of positive transfinite diameter, there exists a unique equilibrium probability measure $\mu_E$ supported on $E$. We now prove Proposition 4.5.

\textbf{Proof.} (of Proposition 4.5):

As in the proof of Proposition 4.4, we let $\delta_a := \frac{1}{h_{\phi}} \sum_{z \in S_a} \delta_z$.

In the first step, we need to show that the sequence of measures $\{\delta_a\}$ has a weakly convergent subsequence. Equivalently, we will show that for any given $\epsilon, \delta > 0$ there exists a set $F_{\epsilon, \delta}$ which is the union of a finite number of discs of radius $\delta$ such that $\delta_n(F_{\epsilon, \delta}) > 1 - \epsilon$ for all $n$ (see [24, p. 49]).

By Corollary 7.8 the sequence $S = \{S_n\}$ is $F$-tight, for any given $\epsilon, \delta$ it is immediate from the definition that there exists an integer $N$ such that $\delta_n(F_3) > 1 - \epsilon$ for all $n \geq N$. We note that $F_3$ is already a finite union of discs of radius $\delta$. Let $F_{\epsilon, \delta}$ be the union of $F_3$ and finitely many discs centered at points of the set $\cup_{i < N} S_i$. Then, clearly we have $\delta_n(F_{\epsilon, \delta}) > 1 - \epsilon$ for all $n$. Thus, we may assume that there is a probability measure $\mu$ such that $\delta_n$ converges weakly to $\mu$ and show that $\mu = \mu_F$.

The remaining arguments are the same as in the proof of Proposition 7.5 and we will not repeat them here. 

\section{Applications}

\subsection{Lower Bounds for Canonical Heights}

Let $v \in M_K$ be a non-archimedean place. We say an algebraic extension $E$ of $K$ is totally $v$-adic of type $(e, f)$ (over $K$) if for any embedding $E \hookrightarrow \mathbb{C}_v$, the image of $E$ is contained in a finite extension of $K_v$ whose ramification degree and residue degree are bounded by $e$ and $f$ respectively. Likewise, an $\alpha \in \bar{K}$ is said to be totally $v$-adic of type $(e, f)$ if $K(\alpha)$ is totally $v$-adic of type $(e, f)$.

\textbf{Theorem 8.1.} (c.f. [11, Theorem 2.]) Let $\phi(z) = a_0 + a_1 z + \cdots + a_d z^d \in K[z]$ and let $v \in M_K$ be a place of good reduction for $\phi$. Let $e, f$ be positive integers. Then there exists a positive constant $C = C(e, f, v)$ such that $h_\phi(\alpha) \geq C$ for all but finitely many $\alpha$ which are totally $v$-adic of type $(e, f)$. In particular, there are only finitely many preperiodic points of $\phi$ which are totally $v$-adic of type $(e, f)$.

\textbf{Remark 8.2.} If $K$ is a number field and $\phi(z) = z^d$ for some $d \geq 2$, then $h_\phi(\alpha) = h(\alpha)$ is just the absolute logarithmic height of $\alpha$. In this special case, Theorem 8.1 can be viewed as a qualitative version of [11, Theorem 2].

\textbf{Proof.} Let $E$ be the compositum of all finite extensions of $K_v$ whose ramification indices and residue degrees are bounded by $(e, f)$. Thus, $E$ contains all $\alpha \in \bar{K}$ which are totally $v$-adic of type $(e, f)$. Note that $E$ is a finite extension over $K_v$, since there are only finitely many extensions of $K_v$ with prescribed ramification indices and residue degrees. Let $e', f'$ be the ramification index and the residue degree, respectively, of $E$ over $K_v$.

It suffices to show that there does not exist a sequence $\{S_n\}$ in $E$ of distinct finite Galois-stable subsets of $\bar{K}$ such that $\lim_{n \to \infty} h_\phi(S_n) = 0$. Assume to the
contrary that there is such a sequence \( S = \{S_n\} \). Then \( S \) is \( \hat{h}_{\phi,v} \)-taut. Let \( S'_n = \{\alpha \in S_n : |\alpha|_v \leq 1\} \). We claim that \( S'_n \) is full subsystem of \( S_n \), i.e., that \( \lim_{n \to \infty} \frac{\#S'_n}{\#S_n} = 1 \). Equivalently, we need to show that \( \lim_{n \to \infty} \frac{\#T_n}{\#S_n} = 0 \), where \( T_n = S_n \setminus S'_n \).

By hypothesis, we have \( \hat{h}_{\phi,v}(S_n) = \frac{1}{\#S_n} \sum_{\alpha \in T_n} \log^+ |\alpha|_v \).

Let \( p \) be the residue characteristic of \( \mathbb{C}_v \). As the ramification degree of \( E \) over \( K_v \) is \( e' \), we see that \( \log^+ |\alpha|_v \geq (1/e') \log p \) for \( \alpha \in T_n \). It follows that \( \hat{h}_{\phi,v}(S_n) \geq \frac{\#T_n}{\#S_n} \log p \). If \( \#T_n/\#S_n \not\to 0 \) then \( S_n \) is not \( \hat{h}_{\phi,v} \)-taut which contradicts our assumption. Hence, \( S'_n \) is a full subsystem of \( S_n \) as desired.

By Lemma 7.1, \( \lim \sup d(S_n) \leq \lim \sup d(S'_n) \). However, we have \( \lim \sup d(S'_n) \leq c(\mathcal{O}_E) \), where \( \mathcal{O}_E \) is the ring of integers of \( E \). We have \( c(\mathcal{O}_E) = p^{-1/e'(p'^{\ell}-1)} < 1 \) as computed in Example 6.8(2). This shows that \( S = \{S_n\} \) is not pseudo-equidistributed and contradicts Theorem 4.3. \( \blacksquare \)

Let \( p \) be a rational prime. Following [11], we call an algebraic number \( \alpha \) totally \( p \)-adic if the prime \( p \) splits completely in \( \mathbb{Q}(\alpha) \). This is equivalent, in the above terminology, to saying that \( \alpha \) is totally \( p \)-adic of type \((1, 1)\) over \( \mathbb{Q} \). As a special case of Theorem 8.1, we have the following \( p \)-adic analogue of Schinzel’s theorem [29] on heights of totally real algebraic numbers:

**Corollary 8.3.** (c.f. [11, Example 1.]) Let \( \phi(z) = a_0 + a_1 z + \cdots + a_d z^d \in \mathbb{Q}[z] \), and let \( p \) be a prime where \( \phi \) has good reduction. Then there exists a constant \( C > 0 \), depending only on \( p \) and \( \phi \), such that \( \hat{h}_\phi(\alpha) \geq C \) for all but finitely many totally \( p \)-adic algebraic numbers \( \alpha \).

We can also give an archimedean generalization of Schinzel’s theorem. We use the fact (see [26, Proof of Theorem 6.5.8]) that if \( F \) is the filled Julia set of a polynomial \( \phi \in \mathbb{C}[z] \) of degree at least 2, then the support of \( \mu_F \) is precisely the Julia set of \( \phi \).

**Proposition 8.4.** Let \( K \) be a number field and let \( \phi \in K[z] \) of degree \( d \geq 2 \). Assume that the complex Julia set \( J_\phi \) of \( \phi \) is not contained in the real line. Then there exists a constant \( C > 0 \), depending only on \( \phi \), such that for all but finitely many totally real algebraic numbers \( \alpha \), we have \( \hat{h}_\phi(\alpha) \geq C \).

**Proof.** Assume to the contrary that there exists sequence of totally real algebraic numbers \( \{\alpha_n\} \) such that \( \hat{h}_\phi(\alpha_n) \to 0 \). By Corollary 4.6, \( \{\alpha_n\} \) is equidistributed with respect to \( \mu_F \), where \( F \) is the filled Julia set of \( \phi \). This implies that \( \mu_F \) is supported on a subset of the real line. On the other hand, the support of \( \mu_F \) is equal to the Julia set of \( \phi \), which by assumption is not totally contained in the real line. We thus have a contradiction. \( \blacksquare \)

For the rest of this section, we assume \( K \) is a number field, and we fix an embedding of \( K \) into the complex numbers. For the next application of our main result, we are concerned with an analogue of a result of Zhang [34, 33] for the usual height on \( \mathbb{Q} \) in the setting of dynamical canonical heights. We want to know how small the canonical height of algebraic points on a line in \( \mathbb{A}^2 \) can be.
Lemma 8.5. Let \( \{S_n\} \) be a sequence of distinct, finite, Galois-stable subsets of \( \bar{K} \) such that \( \lim_{n \to \infty} \text{h}_\phi(S_n) = 0 \). Let \( m(z) = \alpha(z + \beta) \) be an affine linear transformation defined over \( K \), with \( \alpha \neq 0 \), and suppose \( \lim_{n \to \infty} \text{h}_\phi(m(S_n)) = 0 \). Then

1. \( \alpha \) is a root of unity, and
2. \( m(J_\phi) = J_\phi \) where \( J_\phi \) denotes the complex Julia set of \( \phi \).

Proof. It follows from Theorem 4.3 that both sequences \( \{S_n\} \) and \( \{m(S_n)\} \) are pseudo-equidistributed with respect to the pair \( (F_v, \hat{h}_\phi, v) \) for each \( v \in M_K \). Therefore, for any \( v \in M_K \)

\[
\lim_{n \to \infty} d_v(S_n) = c_v(\phi) = \lim_{n \to \infty} d_v(m(S_n)).
\]

Clearly, \( d_v(m(S_n)) = |\alpha|_v d_v(S_n) \). Thus, since \( c_v(\phi) > 0 \), we have \( |\alpha|_v = 1 \) for all \( v \in M_K \), and it follows that \( \alpha \) is a root of unity. This proves (1).

For the proof of (2), note that by Corollary 4.6 both sequences \( \{S_n\} \) and \( \{m(S_n)\} \) are equidistributed with respect to \( \mu_F \), where \( \mu_F \) is the equilibrium measure on the filled Julia set \( F \). It follows easily from this that \( \mu_F = \mu_{m(F)} \). But the support of \( \mu_F \) is \( J_\phi \) and the support of \( \mu_{m(F)} \) is \( m(J_\phi) \), so \( J_\phi = m(J_\phi) \) as desired.

Remark 8.6. If \( \alpha = 1 \) then we must have \( \beta = 0 \) in Lemma 8.5. This follows immediately from the fact that \( J_\phi \) is a bounded subset of \( \mathbb{C} \), since if \( \beta \neq 0 \) then clearly \( J_\phi + \beta \neq J_\phi \), which contradicts (2) above.

We see that under the hypotheses of Lemma 8.5(2), the affine transformation \( m \) must be in the symmetry group of the Julia set. This group has been studied in [4] (see also the references cited in that paper). In order to state what we need from [4], we introduce some notation. We define the centroid of \( \phi(z) = a_dz^d + a_{d-1}z^{d-1} + \cdots + a_1z + a_0 \in \mathbb{C}[z] \) to be the complex number \( \zeta = -a_{d-1}/(da_d) \).

We also let \( \Sigma(\phi) \) denote the set of all affine linear transformations \( \sigma \) on the complex plane such that \( \sigma(J_\phi) = J_\phi \), and call \( \Sigma(\phi) \) the set of symmetries of \( J_\phi \).

Theorem 8.7.

1. The set \( \Sigma(\phi) \) of symmetries of \( J_\phi \) consists of the rotations about the centroid of \( \phi \).

2. If \( \sigma \) is any affine transformation, then \( \sigma \in \Sigma(\phi) \) if and only if \( \phi \circ \sigma = \sigma^d \circ \phi \).

Proof. See Theorem 5 and Lemma 7 of [4].

Note that part 2 of Theorem 8.7 implies that a point \( Q \) is preperiodic for \( \phi \) if and only if \( \sigma(Q) \) is. The only case where \( \Sigma(\phi) \) is infinite is when \( \phi \) is conjugate to a polynomial map of the form \( z \mapsto z^n \) for some integer \( n \geq 2 \) [4, Lemma 4].

We now consider a line \( L \) defined by an equation of the form \( ax + by = c \) \((a, b, c \in K)\), and we study the canonical height of algebraic points \((x, y) \in L(\bar{K})\). Here, we embed \( \mathbb{A}^2 \) into \( \mathbb{P}^1 \times \mathbb{P}^1 \) via \((x, y) \mapsto ([x : 1], [y : 1])\), we consider \( \Phi = (\phi(x), \phi(y)) \) as a self map on \( \mathbb{P}^1 \times \mathbb{P}^1 \), and we define the canonical height of an algebraic point \( Q = (x, y) \) to be \( \hat{h}_\phi((x, y)) = \hat{h}_\phi(x) + h_\phi(y) \).

If one of \( a, b \) is zero then the closure of \( L \) in \( \mathbb{P}^1 \times \mathbb{P}^1 \) is of the form \( \{\alpha\} \times \mathbb{P}^1 \) or \( \mathbb{P}^1 \times \{\beta\} \), and it is easy to see that \( L \) contains sequence \( \{S_n\} \) of Galois stable algebraic points with \( \hat{h}_\phi(S_n) \to 0 \) if and only if \( \alpha \) (respectively, \( \beta \)) is a preperiodic point of \( \phi \).
Proposition 8.8. Assume that $\Sigma(\phi)$ is finite. Let $a, b, c \in K$ with $ab \neq 0$, and assume either that $-a/b$ is not a root of unity or that $c/a \neq \zeta$, where $\zeta$ is the centroid of $\phi$. Let $L \subset \mathbb{A}^2$ be the line defined by the equation $ax + by = c$. Then there exists a positive constant $C = C(a, b, c)$ such that the set $\{(x, y) \in L(K) : \hat{h}_\phi((x, y)) := \hat{h}_\phi(x) + \hat{h}_\phi(y) < C\}$ is finite. In particular, $L$ contains only finitely many points $(s, t)$ such that both $s$ and $t$ are preperiodic points of $\phi$.

Proof. Assume to the contrary that there exist a sequence of points $\{(x_n, y_n) \in L(K)\}$ such that $\lim_{n \to \infty} \hat{h}_\phi((x_n, y_n)) = 0$. Then we have two sequences $\{x_n\}$ and $\{y_n\}$ such that $\hat{h}_\phi(x_n) \to 0$ and $\hat{h}_\phi(y_n) \to 0$.

As $y_n = \left(\frac{-b}{a} (x_n - \frac{c}{a})\right)$, we see by combining Lemma 8.5 and Theorem 8.7 that it is impossible for $\hat{h}_\phi(x_n) \to 0$ and $\hat{h}_\phi(y_n) \to 0$ to hold simultaneously. \hfill \blacksquare

Remark 8.9. One can replace the assumption that $-a/b$ is not a root of unity in the statement of Proposition 8.8 with the stronger hypothesis that $-a/b$ is not an $e$th root of unity, where $e$ is the exponent of $\Sigma(\phi)$. Moreover, whenever $\Sigma(\phi)$ is finite, one can show that the order of $\Sigma(\phi)$ is the largest integer $n$ such that $\phi$ can be written in the form $\phi(z) = z^n \psi(z^n)$ for some polynomial $\psi$ (see [4]).

This result motivates the following theorem, which is closely related to Conjecture 2.5 of [35].

Theorem 8.10. Let $L/K$ be any line in $\mathbb{A}^2$. Then the following are equivalent:

(a) $L$ is a preperiodic line under $\Phi$.
(b) $L$ is defined by an equation of the form $x = \sigma(x)$ or $y = \sigma(x)$ with $\sigma \in \Sigma(\phi)$.
(c) $L$ contains infinitely many preperiodic points of $\Phi$.
(d) $L$ contains an infinite sequence of algebraic points $(x_n, y_n)$ such that $\hat{h}_\phi(x_n, y_n) \to 0$.

Proof. We may assume without loss of generality that $\Sigma(\phi)$ is finite, since otherwise $\phi(z)$ is conjugate to $z^n$ for some $n \geq 2$ and the result follows from [34].

(a) implies (d): Since $L$ is preperiodic, there exist positive integers $k$ and $k'$ and a curve $C$ such that $\Phi^k L = C$ and $\Phi^{k'} C = C$. The maps $\Phi^k : L \to C$ and $\Phi^{k'} : C \to C$ of affine algebraic curves extend to nonconstant maps of projective algebraic curves, and therefore must be surjective. It follows that for any point $(x_0, y_0) \in C$ and any $n \geq 1$, we can find a point $(x_n, y_n) \in L$ such that $\phi^{nk+k'}(x_n, y_n) = (x_0, y_0)$. We then have $\hat{h}_\phi(x_n, y_n) = d^{-nk-k} \hat{h}_\phi(x_0, y_0) \to 0$ as $n \to \infty$. Moreover, if we choose $(x_0, y_0)$ so that it is not a preperiodic point for $\Phi^k$, then the points $(x_n, y_n)$ are distinct.

(d) implies (b): If the equation of $L$ is not of the form $x = \alpha$ or $y = \alpha$, with $\alpha$ a preperiodic point of $\phi$, then it follows by the same proof as that given for Proposition 8.8 that $L$ is given by an equation $y = \sigma(x)$ with $\sigma \in \Sigma(\phi)$.

(b) implies (a): If $L$ is defined by $y = \sigma(x)$ with $\sigma \in \Sigma(\phi)$, then $\Phi(L)$ is defined by $y = \sigma^d(x)$. As $\sigma$ has finite order, it follows that $L$ is preperiodic.

(b) implies (c): Let $x_0$ be any preperiodic point for $\phi$. If $L$ is defined by $y = \sigma(x)$ with $\sigma \in \Sigma(\phi)$, then since $\sigma$ preserves the set of $\phi$-preperiodic points, it follows that $L$ must contain the $\Phi$-preperiodic point $(x_0, \sigma(x_0))$.

(c) implies (d): Obvious. \hfill \blacksquare
8.2. Generalized height functions and the Mandelbrot set. In this section, we prove a more general version of Theorem 4.3. As an application, we attach a height function to the (adelic) Mandelbrot set and obtain a corresponding equidistribution theorem.

First, we define generalized Green’s functions. Let $L$ be an algebraically closed local field, let $F$ be a closed and bounded subset of $L$, and let $\lambda$ be a logarithmic distance function for $F$. If $F$ is a lemniscate of the form $\{z \in L : |P(z)| \leq R\}$ for some polynomial $P \in L[z]$ of degree $d$ and some $R$ in the value group of $L$, we call the function

$$g_F(z) = \frac{1}{d} \log^+ \frac{|P(z)|}{R}$$

the Green’s function attached to $F$. It follows from [28, Proposition 4.4.1] that $g_F(z)$ is well-defined, independent of the choice of a particular defining pair $(P(z), R)$.

More generally, we say that $\lambda$ is a generalized Green’s function for $F$ if there exists a descending sequence $F_1 \supseteq F_2 \supseteq \cdots$ of lemniscates of the form $F_n = \{z \in L : |P_n(z)| \leq R_n\}$ for some polynomial $P_n \in L[z]$ of degree $d_n$ and some positive number $R_n$ in the value group of $L$ such that:

- $\cap_{n=1}^{\infty} F_n = F$
- $\lim_{n \to \infty} c(F_n) = c(F)$ (this does not automatically follow from the previous condition – see Remark 6.7)
- $\lambda(z) = \lim_{n \to \infty} g_n(z)$, where $g_n = g_{F_n}$ is the Green’s function attached to $F_n$.

Remark 8.11. It follows from the results of [28, Section 4.4] (especially Theorem 4.4.4 and Lemma 4.4.7) that if $F$ is algebraically capacitable, then there exists a unique generalized Green’s function attached to $F$ called (naturally) the Green’s function of $F$ (with respect to the point $\infty$). As shown in [28], examples of algebraically capacitable sets include all finite unions of compact sets and lemniscates in $L$. However, we do not know, for example, whether or not the $v$-adic filled Julia set of a polynomial $\phi$ is always algebraically capacitable. This is why we have chosen to define generalized Green’s functions.

The following important fact is proved in [28, Proposition 4.4.1(C)]:

Lemma 8.12. If $F_1$ and $F_2$ are lemniscates with $F_1 \subseteq F_2$, then

$$g_{F_1}(z) \geq g_{F_2}(z)$$

for all $z \in L$.

We then have the following generalization of Lemma 7.2:

Lemma 8.13. Let $F$ be a closed and bounded subset of $L$, and suppose that $\lambda$ is a generalized Green’s function for $F$. Suppose that $S_n$ is a sequence of finite subsets of $L$ with the property that $N_n := \# S_n \to \infty$ and $\lambda(S_n) \to 0$. Then

$$\limsup_{n \to \infty} d(S_n) \leq c(F).$$

Proof. Fix $\beta > 0$ in the value group of $L$, and set $S'_n := \{z \in S_n : \lambda(z) \leq \beta\}$. Then as in the proof of Lemma 7.2, we see easily that $S' := \{S'_n\}$ is a full subsystem of $S$.

Moreover, it follows from Lemma 8.12 and the fact that $g_n \to \lambda$ that for all $m \geq 1$ we have

$$\{z \in L : \lambda(z) \leq \beta\} \subseteq F_{m, \beta} := \{z \in L : g_m(z) \leq \beta\}.$$
Therefore $S'_n \subset F_{m,\beta}$ for all $m$, so that

$$\limsup_{n \to \infty} d(S'_n) \leq c(\cap_{m=1}^\infty F_{m,\beta}) = c(F)e^\beta$$

by Remark 6.4 and the assumption that $c(F_n) \to c(F)$.

Letting $\beta \to 0$ now gives the desired result. \hfill \blacksquare

Now let $K$ be a global field, and let $F = \{ F_v \}$ be an adelic set with respect to $K$ (see Section 6.1). We assume that $F$ also comes equipped with an adelic generalized Green’s function, by which we mean a collection $g := \{ \lambda_v \}$ consisting of a generalized Green’s function $\lambda_v$ for $F_v$ for each $v \in M_K$.

The height function $h_F : \bar K \to \mathbb{R}$ attached to $F$ (or more properly to the pair $(F,g)$) is then defined by

$$h_F(\alpha) := \frac{1}{\#S} \sum_{v \in M_K} \sum_{\alpha_i \in S} n_v \lambda_v(\alpha_i),$$

where $S$ is the smallest Galois-stable subset of $\bar K$ containing $\alpha$.

In this context, we have:

**Theorem 8.14.** Let $K,F,g$ be as above, and assume that $c(F) = 1$. Let $\{ S_n \}$ be a sequence of distinct finite Galois-stable subsets of $\bar K$ such that $\lim_{n \to \infty} h_F(S_n) = 0$. Then for all $v \in M_K$, the sequence $\{ S_n \}$ is pseudo-equidistributed with respect to the pair $(F_v, \lambda_v)$.

The proof is very similar to the proof of Theorem 4.3, and we omit it.

As an application of Theorem 8.14, we consider the “adelic Mandelbrot set”.

The classical Mandelbrot set $M = M_\infty$ can be defined as the locus of points $c \in \mathbb{C}$ such that 0 stays bounded under iteration of the quadratic polynomial $z^2 + c$ (see [16, Ch. VIII] for more details). If we replace $\mathbb{C}$ by $\mathbb{C}_p$ for a prime number $p$ and take this as the definition of the $p$-adic Mandelbrot set $M_p$, then it is easy to see that $M_p$ is just the closed unit disc in $\mathbb{C}_p$.

It is well-known that the transfinite diameter of $M$ is 1. We can see this as follows. Let $\phi_0(z) = z^2 + c$, and for $n \geq 0$ define $F_n := \{ c \in \mathbb{C} : \phi_n^0(0) \leq 2 \}$. Then it is a straightforward exercise to show (see [16, Theorem VIII.1.2]) that $F_1 \supseteq F_2 \supseteq \cdots$ and $M = \cap_{n=0}^\infty F_n$. If we define $\psi_n(z) := \phi_n^0(0)$, then $\psi_n$ is a monic polynomial of degree $2^{n-1}$ in $z$, and $F_n := \{ z \in \mathbb{C} : \psi_n(z) \leq 2 \}$.

Therefore

$$c(M) = \lim_{n \to \infty} 2^{\frac{1}{2^{n-1}}} = 1.$$

In particular, since we clearly have $c(M_p) = 1$ for all primes $p$, we have the product formula $c(M) = 1$, where $M = \{ M_v \}_v$ is the adelic Mandelbrot set.

In view of the above comments, it is natural to attach to $M$ an adelic generalized Green’s function $g$ given by $\lambda_p(z) = \log^+ |z|_p$ at the finite places of $\mathbb{Q}$, and given at the archimedean place by

$$\lambda_\infty(z) := \lim_{n \to \infty} \frac{1}{2^{n-1}} \log^+ \left| \frac{\psi_n(z)}{2} \right|.$$

We then have a corresponding height function $h_M : \bar \mathbb{Q} \to \mathbb{R}$. It follows from the definitions that $h_M(\alpha) = 0$ if and only if the set $\{ \phi_n^0(0) \}$ of orbits of 0 under $\phi_0$ is finite.
Applying Theorem 8.14 and Proposition 4.4, we find:

**Theorem 8.15.** Let \( \{S_n\} \) be a sequence of distinct finite Galois-stable subsets of \( \mathbb{Q} \) such that \( \lim_{n \to \infty} h_{\mathbb{Q}}(S_n) = 0 \). Then for all \( v \in M_{\mathbb{Q}} \), the sequence \( \{S_n\} \) is pseudo-equidistributed with respect to the pair \((M_{\mathbb{Q}}, \lambda_v)\). In particular, \( \{S_n\} \) is equidistributed with respect to the equilibrium measure \( \mu_M \) for \( M \).

**9. Open questions**

We close the paper with a list of questions for further investigation.

1. If \( \phi \in \mathbb{C}_p[z] \) is a polynomial of degree \( d \geq 2 \), must the filled Julia set (resp. the Julia set) of \( \phi \) be algebraically capacitable? More generally, under what condition the set \( F \) determined by a chain of lemniscates as defined above is algebraically capacitable?
2. Here is a related question (see [8] for more details). Let \( \phi \) be as in the previous question, and suppose that all periodic points of \( \phi \) are repelling. Is the filled Julia set (resp. the Julia set) of \( \phi \) necessarily compact?
3. Can Theorem 4.3 be generalized to canonical heights with respect to more general dynamical systems on varieties?

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