A fast quantum algorithm for the affine Boolean function identification

Ahmed Younes

1 Department of Mathematics and Computer Science, Faculty of Science, Alexandria University, Alexandria, Egypt
2 School of Computer Science, University of Birmingham, Birmingham, B15 2TT, United Kingdom

Received: 25 October 2014 / Revised: 25 January 2015
Published online: 25 February 2015 – © Società Italiana di Fisica / Springer-Verlag 2015

Abstract. Bernstein-Vazirani algorithm (the one-query algorithm) can identify a completely specified linear Boolean function using a single query to the oracle with certainty. The first aim of the paper is to show that if the provided Boolean function is affine, then one more query to the oracle (the two-query algorithm) is required to identify the affinity of the function with certainty. The second aim of the paper is to show that if the provided Boolean function is incompletely defined, then the one-query and the two-query algorithms can be used as bounded-error quantum polynomial algorithms to identify certain classes of incompletely defined linear and affine Boolean functions respectively with probability of success at least 2/3.

1 Introduction

The oracle identification problem is to determine which oracle we have from a set of possible Boolean oracles [1]. Another related problem is the oracle property testing problem, where the task is to determine if a given oracle has a certain property. The complexity of both problems is usually measured by the minimum number of times it is required to query the oracle to accomplish that task.

The case when the oracle represents a linear (affine) Boolean function has a special importance. The linearity (and nonlinearity) of Boolean functions is used in cryptography, data encryption, error control codes, etc. [2–4]. Such analysis requires the fully defined form of a Boolean function. The incompletely defined Boolean functions, where the correct output for certain input vectors are missing, have many applications in synthesis and optimization of circuit design. When the function is provided in an incompletely defined form, it is important to design efficient methods to construct the completely specified form for the incompletely defined Boolean functions.

Classically, it is hard to predict if a given large incompletely defined Boolean function can be realized as affine. Many techniques such as spectral techniques have been used to analyze Boolean functions in many areas such as classification, testing and evaluation of logic complexity, checking if a partially defined Boolean function can be realized in an affine form [5–7].

Designing quantum algorithms for completely specified Boolean functions have gained much attention in the literature. The oracle identification problem is solved for linear Boolean functions by Bernstein-Vazirani algorithm using a single query to the oracle [8]. The oracle property testing problem to check if a given Boolean oracle is either constant or balanced is solved by Deutsch-Jozsa algorithm using a single query to the oracle [9]. In [10], quantum algorithms based on the Bernstein-Vazirani algorithm [8] for finding the variables used in a Boolean function are presented. In [11], a quantum algorithm is shown to test the linearity of a Boolean function using Bernstein-Vazirani’s algorithm and an amplitude amplification technique. In [12], an enhanced algorithm of [11] is proposed using Deutsch-Jozsa’s [9] and Grover’s algorithm [13]. In [14], a quantum algorithm for determining the linear structures of a Boolean function using Bernstein-Vazirani’s algorithm and Simon’s algorithm [15] is presented.

Bernstein-Vazirani’s algorithm (the one-query algorithm) is known to identify the linear Boolean function with certainty using a single query to the oracle. It has been noticed by [16,17] that if the provided function is for an affine Boolean function, then Bernstein-Vazirani’s algorithm will be blind to the shift experienced by the affinity of the function where the affinity of the function will be relegated to an unobservable global phase. To overcome this drawback,
an independent query to the oracle using \( f(0^n) \) would identify the affinity of the function. This two-independent-queries scenario is sufficient to indentify the affine function only if it is provided in a completely specified form.

The aim of the paper is to propose a single algorithm that requires two queries to the oracle similar to the above scenario. The proposed algorithm can identify the affine Boolean function if it is provided in either a completely specified form or in an incompletely defined form. The first aim of this paper is to show that if the given oracle represents an affine Boolean function, then one more query to the oracle (the two-query algorithm) is sufficient to identify the affine Boolean function with certainty. The second aim is to show that the one-query algorithm can identify the linear Boolean function even if the function is provided as an incompletely defined function for certain class of functions with probability of success at least 2/3, and the two-query algorithm can identify certain class of incompletely defined affine Boolean functions with probability of success at least 2/3.

The paper is organized as follows: Section 2 reviews the basic definitions. Section 3 proposes the one-query algorithm and the two-query algorithm for the completely specified Boolean functions and the incompletely defined Boolean functions. Section 4 gives a discussion about the performance of the one-query algorithm and the two-query algorithm, respectively. The paper ends up with a conclusion in sect. 5.

2 Basics

A Boolean function \( f \) with \( n \) inputs is a mapping \( f : X^n \rightarrow X \), where \( X = \{0, 1\} \), i.e. the domain of \( f \) is the set of \( 2^n \) binary vectors \((0, 0, \ldots, 0), (0, 0, \ldots, 1), \ldots, (1, 1, \ldots, 1)\), and \( f \) maps each of these vectors to the constant 0 or 1. If the domain \( B \) of Boolean function \( f \) is \( X^n \), then \( f \) is called completely specified Boolean function. If \( B \subset X^n \), i.e. some input vectors of the function \( f \) belong to the set \( X^n\setminus B \), then the function is called incompletely defined Boolean function.

Given an incompletely defined version \( g \) of a completely specified Boolean function \( f \), the input vectors that have a value 0 for \( f \) are called \( OFF_f \) cubes, the input vectors that have a value 1 for \( g \) are called \( ON_f \) cubes. Let \( n_0 \) and \( n_1 \) denote the number of input vectors in the sets \( OFF_f \) and \( ON_f \), respectively, then \( n_0 + n_1 = 2^n \). If the domain \( B \) of Boolean function \( f \) is \( X^n \), then \( f \) is called completely specified Boolean function. If \( B \subset X^n \), i.e. some input vectors of the function \( f \) belong to the set \( X^n\setminus B \), then the function is called incompletely defined Boolean function.

An affine Boolean function with \( n \) inputs is a Boolean function that can be represented as follows:

\[
f_A \left( x_0, x_1, \ldots, x_{n-1} \right) = c_0 x_0 \oplus c_1 x_1 \oplus \cdots \oplus c_{n-1} x_{n-1} \oplus c_n,
\]

where \( x_i, c_i \in X \), \( i = 0, 1, \ldots, n \) and \( \oplus \) denotes bitwise exclusive-or. The affine Boolean function is fully identified if the coefficients \( c_i \) are known.

If the coefficient \( c_n \) is strictly equal to 0, then the function is called a linear Boolean function and it can be represented as follows:

\[
f_L \left( x_0, x_1, \ldots, x_{n-1} \right) = c_0 x_0 \oplus c_1 x_1 \oplus \cdots \oplus c_{n-1} x_{n-1},
\]

where \( x_i, c_j \in X \), \( j = 0, 1, \ldots, n-1 \). The linear Boolean function is fully identified if the coefficients \( c_j \) are known, this will be denoted as the bit string \( C \), where \( C = (c_0 c_1 \ldots c_{n-1}) \).

There are \( 2^{n+1} \) possible \( f_A \) functions while there are \( 2^n \) possible \( f_L \) functions. Both types of functions could be balanced, i.e. truth table contains an equal number of 0’s and 1’s, and both types of functions could be constant in a different way, for example, if \( c_j = 0 \) for \( 0 \leq j \leq n-1 \), then \( f_A = 0 \) while \( f_A = 0 \) or 1 depends on the value of \( c_n \). The function \( f_A \) is constant \((f_A = 0)\) if \( c_i = 0 \) for \( 0 \leq i \leq n \). If at least one \( c_j \neq 0 \), then both \( f_L \) and \( f_A \) are balanced, i.e. \( n_0 = n_1 = N/2 \), where \( N = 2^n \). If \( g_L \) and \( g_A \) represents incompletely defined versions of \( f_L \) and \( f_A \), respectively, then \( 0 \leq d_0, d_1 \leq N/2 \), \( d_0 = N/2 - n_0 \), and \( d_1 = N/2 - n_1 \).

In the literature, a Boolean function is considered as an oracle that marks certain states in a superposition. There are two ways used to mark the states, one way is to conditionally apply certain phase shifts on the marked states [13] by using an oracle \( V_f \) that works as follows: \( V_f(x) = (-1)^{f(x)} |x \rangle \). The other way is to use an oracle \( U_f \) to entangle the required states with certain state of the extra qubit workspace [18] as follows: \( U_f |x, 0 \rangle = |x, f(x) \rangle \), where the state of the extra qubit workspace is required for further operations. The oracle \( U_f \) is used by initializing the \( n + 1 \) qubits quantum register to the state \( |0 \rangle^\otimes n + 1 \), then apply the operator \( H^\otimes n \otimes I \) to the register, where \( I \) is the \( 2 \times 2 \) identity matrix. The oracle \( U_f \) can perform as \( V_f \) by initializing the \( n + 1 \) qubits quantum register to the state \( |0 \rangle^\otimes n \otimes |1 \rangle \), then
apply the operator $H^\otimes_{n+1}$ to the register and ignore the extra qubit workspace afterward, where $H$ is the Hadamard gate defined as follows:

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \quad (3)$$

Applying the $H$ gate on a qubit in state $|0\rangle$ or $|1\rangle$ will produce a qubit in a perfect superposition. In general, the effect of applying the $H$ gate on an $n$-qubits quantum register is known as Walsh-Hadamard transform and can be represented as follows:

$$H^\otimes_n |x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} (-1)^{x \cdot y} |y\rangle, \quad (4)$$

where $x \cdot y = x_0 \cdot y_0 \oplus x_1 \cdot y_1 \oplus \ldots \oplus x_{n-1} \cdot y_{n-1}$, and $x_j, y_j$ is the bitwise-and between $x_j$ and $y_j$.

3 The proposed algorithm

3.1 Completely specified Boolean function

Given a quantum register of $n+1$ qubits in state $|0\rangle^\otimes_n \otimes |1\rangle$ and an oracle $U_f$ that represents an $n$ inputs completely specified affine Boolean function $f$, then the operations of the proposed algorithm $A_1$ (shown in fig. 1) can be written as follows:

$$A_1 = U_f H^{n+1} U_f H^{n+1}. \quad (5)$$

Tracing the algorithm

The operations of the proposed algorithm can be understood as follows where the first three steps are straightforward from Bernstein-Vazirani algorithm.

1) Prepare a quantum register of $n+1$ qubits, the first $n$ qubits in state $|0\rangle$ and an extra qubit is state $|1\rangle$ as follows:

$$|\Psi_0\rangle = |0\rangle^\otimes_n \otimes |1\rangle. \quad (6)$$

2) Apply $H^\otimes_{n+1}$,

$$|\Psi_1\rangle = (H^\otimes_{n+1}) |\Psi_0\rangle
= \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle \otimes \left( |0\rangle - |1\rangle \right) \sqrt{2}. \quad (7)$$

3) Apply $U_f$ on the $n+1$ qubits,

$$|\Psi_2\rangle = U_f |\Psi_1\rangle
= \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} (-1)^{f(x)} |x\rangle \otimes \left( |0\rangle - |1\rangle \right) \sqrt{2}. \quad (8)$$

4) Apply $H^\otimes_{n+1}$,

$$|\Psi_3\rangle = (H^\otimes_{n+1}) |\Psi_2\rangle
= \frac{1}{2^n} \sum_{x=0}^{2^n-1} \sum_{z=0}^{2^n-1} (-1)^{f(x+z)} |z\rangle \otimes |1\rangle, \quad (9)$$

Fig. 1. A quantum circuit for the proposed two-query algorithm.
An incompletely defined affine Boolean function $g$, where (a) the truth table of $g$, and (b) a quantum circuit to realize $U_g$, where $\circ$ and $\bullet$ in the CNOT gates and CHAD gates mean that the condition on the qubit will evaluate to true if and only if the state of that qubit is $|0\rangle$ and $|1\rangle$, respectively.

and since the vectors of the linear Boolean functions (ignoring $c_n$) form an orthonormal basis, i.e. the following identity holds:

$$\sum_{x=0}^{2^n-1} (-1)^{x \cdot z} = 2^n \delta_{z,0},$$

where $x$ and $z$ are $n$-bit strings, then, $|\Psi_3\rangle$ can be written as follows [11]:

$$|\Psi_3\rangle = (-1)^{c_n} |c_0c_1 \ldots c_{n-1}\rangle \otimes |1\rangle.$$  \hspace{1cm} (11)

It is important to notice that Bernstein-Vazirani algorithm is not sensitive to the affinity of the oracle, i.e. the value of $c_n$, where the affinity appears as a global phase shift of $(-1)^{c_n}$ which will not be detected when the quantum register is measured. So, one more query to the oracle is required to find the value of $c_n$.

5) To find the value of $c_n$, apply $U_f$ on the $n+1$ qubits [18],

$$|\Psi_4\rangle = U_f |\Psi_3\rangle = (-1)^{c_n} |c_0c_1 \ldots c_{n-1}\rangle \otimes |1 \oplus c_n \oplus p_c\rangle,$$

where $p_c = c_0 \oplus c_1 \oplus \ldots \oplus c_{n-1}$.

6) Measure the first $n$ qubits to get the bit string $|c_0c_1 \ldots c_{n-1}\rangle$.

7) Measure the extra qubit to read the value of $c_n$ as $|1 \oplus c_n \oplus p_c\rangle$ such that if the number of 1’s in the bit string $|c_0c_1 \ldots c_{n-1}\rangle$ is odd then the measured value in the extra qubit is $|1 \oplus c_n\rangle$, i.e. the negation of $c_n$, and if the number of 1’s in the bit string $|c_0c_1 \ldots c_{n-1}\rangle$ is even then the measured value in the extra qubit is $|c_n\rangle$.

### 3.2 Incompletely defined Boolean function

Given an $n$ inputs incompletely defined affine Boolean function $g$ as follows:

$$g(x) = \begin{cases} 
0 & \text{if } x \in OFF_g, \\
1 & \text{if } x \in ON_g, \\
2 & \text{if } x \in DC_g,
\end{cases}$$

where $g(x) = 2$ if $x \in DC_g$ represents a third choice for the don’t cares. To find the completely specified version of $g$, $g(x) = 2$ should be replaced with either $g(x) = 0$ or $g(x) = 1$, the correct replacement is not known in advance. Quantum parallelism can be exploited to examine both replacements simultaneously. This can be done by encoding the third choice, i.e. $g(x) = 2$, in a quantum version $U_g$ of the oracle as $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. This can be achieved by assuming that the don’t care input vector $x$ is in the set $ON_g$, and then replace the NOT gate with the Hadamard gate in the controlled gate representation for the minterms equivalent to the don’t care input vector $x$ [18]. One way to realize $U_g$ as a quantum circuit, let $CNOT|x\rangle|t\rangle$ be a gate where the target qubit $|t\rangle$ is controlled by a register of qubits $|x\rangle$ such that $t \notin x$; the state of the qubit $|t\rangle$ will be flipped if and only if all the qubits in $|x\rangle$ are satisfied. For example, the gate $CNOT|x_0x_1x_2\rangle|t\rangle$ will flip $|t\rangle$ if $x_0 = |0\rangle$, $|x_1 = |1\rangle$ and $|x_2 = |1\rangle$. Let $CHAD|x\rangle|t\rangle$ be a gate where the target qubit $|t\rangle$ is controlled by a set of qubits $|x\rangle$ such that $t \notin x$; the Hadamard gate is applied on $|t\rangle$ if and only if all the qubits in $|x\rangle$ are satisfied.

For example, assume the truth table of an incompletely defined affine Boolean function $g$ shown in Fig. 2(a), then the input vectors in $ON_g$ can be represented as $\{x_0x_1x_2, x_0x_1\overline{x}_2, x_0\overline{x}_1x_2\}$, the input vectors in $OFF_g$ can be represented
as \( \{ \tau_0 \tau_1 \tau_2, \tau_0 x_1 \tau_2, x_0 x_1 x_2 \} \), and the input vectors in \( D_C \) can be represented as \( \{ \tau_0 \tau_1 \tau_2, x_0 x_1 x_2 \} \). One way to realize a quantum circuit for \( U_g \), initialize an extra qubit to \( |0 \rangle \) and add \( CNOT \) gate for every input vector in \( ON_g \) and \( CHAD \) gate for every input vector in \( DC \), taking the extra qubit as the target qubit as shown in fig. 2(b). The size and the quantum cost of the synthesized circuit for \( U_g \) are subject to optimization [18, 19], where the cascading of the quantum gates in the circuit is equivalent to \( U_g \), and so, the evaluation of \( U_g \) is counted as a single operation in the query complexity of the algorithm [8].

The proposed algorithm to find the completely specified version of \( g \) is as follows: prepare a quantum register of \( n + 1 \) qubits, the first \( n \) qubits in state \( |0 \rangle \otimes |1 \rangle \) and the quantum oracle \( U_g \) that represents the \( n \) inputs incompletely defined affine Boolean function \( g \) defined as follows:

\[
U_g |x \rangle \otimes |t \rangle = \begin{cases} 
| x \rangle \otimes | t \rangle \otimes g(x) & \text{if } x \notin DC_g, \\
| x \rangle \otimes H | t \rangle & \text{if } x \in DC_g,
\end{cases}
\]

(14)

where the don’t cares for \( g \) are encoded as \( (|0 \rangle + |1 \rangle) / \sqrt{2} \). Then the operations of the proposed algorithm \( A_2 \) can be written as follows:

\[
A_2 = U_g H^{n+1} U_g H^{n+1}.
\]

(15)

Tracing the algorithm

The operations of the proposed algorithm can be understood in the following:

1) Prepare a quantum register of \( n + 1 \) qubits, the first \( n \) qubits in state \( |0 \rangle \otimes |1 \rangle \) and an extra qubit in state \( |1 \rangle \) as follows:

\[
| \Psi_0 \rangle = |0 \rangle \otimes |1 \rangle.
\]

(16)

2) Apply \( H^{\otimes n+1} \),

\[
| \Psi_1 \rangle = (H^{\otimes n+1}) | \Psi_0 \rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} | x \rangle \otimes (|0 \rangle - |1 \rangle / \sqrt{2}).
\]

(17)

3) Apply \( U_g \),

\[
| \psi_2 \rangle = U_g | \psi_1 \rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0, x \notin DC}^{2^n-1} (-1)^g(x) | x \rangle \otimes (|0 \rangle - |1 \rangle / \sqrt{2}) + \frac{1}{\sqrt{2^n}} \sum_{x=0, x \in DC}^{2^n-1} | x \rangle \otimes |1 \rangle.
\]

(18)

4) Apply \( H^{\otimes n} \otimes I \). To simplify calculations, first apply \( I^{\otimes n} \otimes H \),

\[
| \psi_3 \rangle = (I^{\otimes n} \otimes H) | \psi_2 \rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0, x \notin DC}^{2^n-1} (-1)^g(x) | x \rangle \otimes |1 \rangle + \frac{1}{\sqrt{2^n}} \sum_{x=0, x \in DC}^{2^n-1} | x \rangle \otimes (|0 \rangle - |1 \rangle / \sqrt{2})
\]

\[
= \frac{1}{\sqrt{2^n}} \sum_{x=0, x \notin DC}^{2^n-1} (-1)^g(x) | x \rangle \otimes |1 \rangle + \frac{1}{\sqrt{2^{n+1}}} \sum_{x=0, x \in DC}^{2^n-1} | x \rangle \otimes |0 \rangle - \frac{1}{\sqrt{2^{n+1}}} \sum_{x=0, x \in DC}^{2^n-1} | x \rangle \otimes |1 \rangle.
\]

(19)

then apply \( H^{\otimes n} \otimes I \),

\[
| \psi_4 \rangle = (H^{\otimes n} \otimes I) | \psi_3 \rangle = \frac{1}{2^n} \sum_{x=0}^{2^n-1} \sum_{z=0}^{2^n-1} (-1)^{g(x)+z} | z \rangle \otimes |1 \rangle + \frac{1}{2^n} \sum_{x=0}^{2^n-1} \sum_{z=0}^{2^n-1} (-1)^{z} | z \rangle \otimes (|0 \rangle - |1 \rangle / \sqrt{2})
\]

\[
= \sum_{z=0, z \notin C}^{2^n-1} \alpha_z | z \rangle \otimes |0 \rangle + \sum_{z=0, z \notin C}^{2^n-1} \beta_z | z \rangle \otimes |1 \rangle + \gamma_0 | C \rangle \otimes |0 \rangle + \gamma_1 | C \rangle \otimes |1 \rangle.
\]

(20)
where,
\[ \gamma_0 = \frac{(-1)^{c_n}}{\sqrt{2} 2^n} (d_0 - d_1), \quad \gamma_1 = (-1)^{c_n} \left( 1 - \frac{\sqrt{2d} + d_0 - d_1}{\sqrt{2} 2^n} \right). \]  
(21)

If it is sufficient to find the linear part of the function, then the probability of success to get the linear part correctly is \( P_L = \gamma_0^2 + \gamma_1^2 \). If it is required to find the affinity of the function, i.e., the value of \( c_n \), then apply \( U_g \) one more time. To simplify the calculations and since we are interested in the bit string \( C \), so the subsystem \( |\psi_C\rangle \) of interest is as follows:
\[ |\psi_C\rangle = \gamma_0 |C\rangle \otimes |0\rangle + \gamma_1 |C\rangle \otimes |1\rangle. \]
(22)

We have to consider if \( C \subseteq DC_g \) or not. If \( C \notin DC_g \) then applying \( U_g \) gives
\[ |\psi_{C(final)}\rangle = U_g |\psi_C\rangle = \gamma_0 |C\rangle \otimes |c_n + p_c\rangle + \gamma_1 |C\rangle \otimes |1 \oplus c_n \oplus p_c\rangle, \]
(23)
and if \( C \subseteq DC_g \) then applying \( U_g \) gives
\[ |\psi_{C(temp)}\rangle = U_g |\psi_C\rangle = \gamma_0 |C\rangle \otimes \left( \frac{|0\rangle + (-1)^{c_n} |1\rangle}{\sqrt{2}} \right) + \gamma_1 |C\rangle \otimes \left( \frac{|0\rangle - (-1)^{c_n} |1\rangle}{\sqrt{2}} \right), \]
\[ = \frac{1}{\sqrt{2}} (\gamma_0 + \gamma_1) |C\rangle \otimes |0\rangle + \frac{(-1)^{c_n}}{\sqrt{2}} (\gamma_0 - \gamma_1) |C\rangle \otimes |1\rangle. \]
(24)

To increase the probability of success of finding \( c_n \) in \( |\psi_{C(temp)}\rangle \) when \( C \subseteq DC_g \), then apply \((I^{\otimes n} \otimes H)\)
\[ (I^{\otimes n} \otimes H) |\psi_{C(temp)}\rangle = \frac{1}{\sqrt{2}} \left( \gamma_0 + \gamma_1 \right) |C\rangle \otimes \left( \frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) + \frac{(-1)^{c_n}}{\sqrt{2}} \left( \gamma_0 - \gamma_1 \right) |C\rangle \otimes \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right), \]
\[ = \frac{1}{2} \left( \gamma_0 + \gamma_1 + (-1)^{c_n} (\gamma_0 - \gamma_1) \right) |C\rangle \otimes |0\rangle + \frac{1}{2} \left( \gamma_0 + \gamma_1 - (-1)^{c_n} (\gamma_0 - \gamma_1) \right) |C\rangle \otimes |1\rangle, \]
\[ = \gamma_0 |C\rangle \otimes |c_n + p_c\rangle + \gamma_1 |C\rangle \otimes |1 \oplus c_n \oplus p_c\rangle, \]
(25)
which is equal to \( |\psi_{C(final)}\rangle \) for \( C \notin DC_g \) shown in eq. (23). So, \( |\psi_{C(final)}\rangle \) can be taken as the final state of the system in analyzing the probability of success of finding \( c_n \). We can assume that the probability of success to get the coefficients of the affine Boolean function is \( P_A = \gamma_1^2 \), since \( \gamma_0 \leq \gamma_1 \), where the value of \( c_n \) can be read as \(|1 \oplus c_n \oplus p_c\rangle\) similar to the case of completely specified functions.

4 Discussion

The probability of success to get the coefficients of the incompletely defined linear Boolean function is
\[ P_L = \gamma_0^2 + \gamma_1^2, \]
(26)
such that \( d = d_0 + d_1, 0 \leq d < N/2, \) and \( 0 \leq d_0, d_1 < N/2. \) Let \( D = d/N, D_0 = d_0/N, \) and \( D_1 = d_1/N, \) such that, \( 0 \leq D < 1/2, \) and \( 0 \leq D_0, D_1 < 1/2. \) So, \( P_L \) can be written as follows:
\[ P_L = \left( 1 - \left( 1 + \frac{1}{\sqrt{2}} \right) D + \sqrt{2} D_1 \right)^2 + \left( \frac{1}{\sqrt{2}} D - \sqrt{2} D_1 \right)^2. \]
(27)

The probability of success for the spectrum of incompletely defined linear Boolean functions depends on the number of don’t cares. If the set \( DC_g \) contains only members from \( DC1_g, \) i.e., \( D = D_1, \) then \( 0.85 \leq P_L \leq 1, \) and if the set \( DC_g \) contains only members from \( DC0_g, \) i.e., \( D = D_0, \) then \( 0.15 \leq P_L \leq 1 \) as shown in fig. 3(top).
The class of incompletely defined linear Boolean functions for which the one-query algorithm can succeed with probability at least 2/3 as shown in fig. 3(bottom), i.e. $P_L \geq 2/3$ must satisfy the following condition:

$$D_1 \geq \frac{\sqrt{K_1^2 + 4K_2} - K_1}{4} \geq 0,$$

$$D_0 \leq D - \frac{\sqrt{K_1^2 + 4K_2} - K_1}{4} \geq 0,$$  \hspace{1cm} (28)

where $K_1 = \sqrt{2} - (2 + \sqrt{2})D$, and $K_2 = (2 + \sqrt{2})D(1 - D) - 1/3$.

For $d \geq N/2$, the oracle $U_g$ might be equivalent to more than one completely specified linear Boolean function, since the Hamming distance between the truth table of any two completely specified linear functions is equal to $N/2$. For example, if $d = N/2$, then $U_g$ is equivalent to two completely specified linear Boolean functions $f_1(x)$ and $f_2(x)$, then $|\psi_4\rangle$ in eq. (20) can be re-written as follows (ignoring the affinity of $f_1(x)$ and $f_2(x)$):

$$|\psi_4\rangle = \frac{1}{2} (|C_1\rangle + |C_2\rangle) \otimes |1\rangle + \frac{1}{2} \left( |0\rangle \otimes |C_1 \oplus C_2\rangle \right) \otimes \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right),$$  \hspace{1cm} (29)

where $C_1$ and $C_2$ are the bit strings for $f_1(x)$ and $f_2(x)$ respectively. The probability of success to get $C_1$ or $C_2$ is 1/4 with a total probability of success of 1/2 which is outside the class of incompletely defined linear Boolean functions for which the one-query algorithm can succeed with probability at least 2/3.

The probability of success to get the coefficients of the incompletely defined affine Boolean function is

$$P_A = \gamma_1^2 = \left( 1 - \frac{(\sqrt{2}d + d_0 - d_1)}{2^n \sqrt{2}} \right)^2$$

$$= \left( 1 - \left( 1 + \frac{1}{\sqrt{2}} \right) D + \sqrt{2}D_1 \right)^2$$  \hspace{1cm} (30)
The probability of success for the spectrum of incompletely defined affine Boolean functions depends on the number of don’t cares. If the set $D_{g}$ contains only members from $DC_{1}g$, i.e. $D = D_{1}$, then $0.72 \leq P_{A} \leq 1$, and if the set $DC_{g}$ contains only members from $DC_{0}g$, i.e. $D = D_{0}$, then $0.02 \leq P_{A} \leq 1$ as shown in fig. 4(top).

The class of incompletely defined affine Boolean functions for which the two-query algorithm can succeed with probability at least 2/3 as shown in fig. 4(bottom), i.e. $P_{A} \geq 2/3$ must satisfy the following condition:

$$\begin{align*}
D_{1} & \geq \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{2}} + \left(\frac{1 + \sqrt{2}}{2}\right) D \geq 0, \\
D_{0} & \leq \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \left(\frac{1 - \sqrt{2}}{2}\right) D \geq 0.
\end{align*}$$

(31)

It can be seen that the probability of success shown in eqs. (27) and (30) favor the don’t cares that belong to the set $DC_{1}g$ over the don’t cares that belong to the set $DC_{0}g$, i.e. the probability of success is higher if the don’t cares in the provided incompletely defined Boolean function are supposed to be the value 1 in the corresponding completely specified Boolean function. The reason is that the oracle $U_{g}$ used, as shown in eq. (14), is mapping the don’t care vectors to the state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. If the oracle maps the don’t care vectors to the state $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ instead, then the algorithms will favor the don’t cares that belong to the set $DC_{0}g$. This can be done by encoding the third choice, $g(x) = 2$, in a quantum version $U_{g}'$ of the oracle as $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. This can be achieved by assuming that the don’t care input vector $x$ is in the set $OFF_{g}$, and then replace the NOT gate with the Hadamard gate in the controlled gate representation for the minterm equivalent to the don’t care input vector $x$ [18]. If the values of $n_{0}'$ and $n_{1}'$ are known, then the values of $d_{0}$ and $d_{1}$ can be calculated respectively, i.e. $d_{0} = N/2 - n_{0}'$, and $d_{1} = N/2 - n_{1}'$. If $d_{0} < d_{1}$ then the oracle shown in eq. (14) is used in the algorithms, otherwise the following oracle is used instead:

$$U_{g}'|x\rangle \otimes |t\rangle = \begin{cases} 
|x\rangle \otimes |t \oplus g(x)\rangle & \text{if } x \notin DC_{g}, \\
|x\rangle \otimes H(NOT|t\rangle) & \text{if } x \in DC_{g}.
\end{cases}$$

(32)
The ability to choose the correct oracle will double the number of the incompletely defined Boolean functions in the class of functions for which the algorithm can succeed with probability at least 2/3. If the values of \( n_0 \) and \( n_1 \) are not known, then the algorithms may run constant number of times using each of the oracles \( U_g \) and \( U'_g \) in turn, then

the winner with more votes in the majority vote from the two runs is taken as the correct output string.

5 Conclusion

Bernstein-Vazirani algorithm (the one-query algorithm) is known to identify a completely specified linear Boolean function using a single query to the oracle with certainty. It has been shown that Bernstein-Vazirani algorithm is not sensitive to the affinity of the oracle. So, one more query to the oracle is required after Bernstein-Vazirani algorithm (the two-query algorithm) to be able to identify a completely specified affine Boolean function with certainty.

The one-query algorithm and the two-query algorithm are also able to identify classes of incompletely defined Boolean functions with probability at least 2/3. The probability of success depends on the number of don’t cares and on the choice to encode the don’t care in the oracle as \( \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \) or \( \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \).

References

1. A. Ambainis, J. Comput. Syst. Sci. 64, 750 (2002).
2. S. Maitra, P. Sarkar, Theor. Comput. Sci. 276, 133 (2002).
3. S. Mister, C. Adams, in Workshop on Selected areas in Cryptography (1996).
4. J. Seberry, X. Zhang, Australas. J. Combin. 9, 21 (1994).
5. B.J. Falkowski, I. Schafer, M. Perkowski, IEEE TCAD 11, 1207 (1992).
6. P. Porwik, Fund. Inform. 77, 277 (2007).
7. P. Porwik, Comput. Inform. 29, 1073 (2010).
8. E. Bernstein, U. Vazirani, in Proceedings 25th Annual ACM Symposium on Theory of Computing (ACM, 1993) pp. 11-20.
9. D. Deutsch, R. Jozsa, Proc. R. Soc. London A 439, 553 (1992).
10. D.F. Floess, E. Andersson, M. Hillery, arXiv:1006.1423 [quant-ph] (2010).
11. M. Hillery, E. Andersson, Phys. Rev. A 84, 062326 (2011).
12. K. Chakraborty, S. Maitra, arXiv:1306.6195 [quant-ph] (2013).
13. L.K. Grover, Phys. Rev. Lett. 79, 325 (1997).
14. H. Li, L. Yang, arXiv:1404.0611 [quant-ph] (2013).
15. D. Simon, SIAM J. Comput. 26, 1474 (1994).
16. R. Cleve, A. Ekert, C. Macchiavello, M. Mosca, Proc. R. Soc. London A 454, 339 (1998).
17. A. Montanaro, Inform. Process Lett. 112, 438 (2012).
18. A. Younes, J. Miller, Int. J. Electron. 91, 431 (2004).
19. A. Barreno, C.H. Bennett, R. Cleve, D.P. DiVincenzo, N. Margolus, P. Shor, T. Sleator, J.A. Smolin, H. Weinfurter, Phys. Rev. A 52, 3457 (1995).