Tensor power spectrum and disformal transformations

Jacopo Fumagalli\textsuperscript{1}, Sander Mooij\textsuperscript{2} and Marieke Postma\textsuperscript{1}

\textsuperscript{1}Nikhef, Science Park 105, 1098 XG Amsterdam, The Netherlands

\textsuperscript{2}FCFM, Universidad de Chile, Blanco Encalada 2008, Santiago, Chile

ABSTRACT

In a general effective theory description of inflation a disformal transformation can be used to set the tensor sound speed to one. After the transformation, the tensor power spectrum then automatically only depends on the Hubble parameter. We show that this disformal transformation, however, is nothing else than a change of units. It is a very useful tool for simplifying and interpreting computations, but it cannot change any physics. While the apparent parametrical dependence of the tensor power spectrum does change under a disformal transformation, the physics described is frame invariant. We further illustrate the frame invariance of the tensor power spectrum by writing it exclusively in terms of separately invariant quantities.
1 Introduction

Gravitational waves, or tensor modes, are a generic prediction of inflation. In Einstein gravity, gravitational waves move at the speed of light. Therefore, the only unknown in the tensor power spectrum is the Hubble scale during inflation. Consequently, the detection of primordial gravitational waves from inflation would unambiguously pin down the Hubble scale during inflation.

However, the effective field theory (EFT) approach to inflation \[1\] does in general allow for a non-canonical (i.e. not equal to the speed of light) tensor speed \(c_T\). Physical motivations can be found in brane world models or models of modified gravity, which have been elegantly summarized in Horndeski’s most general scalar-tensor theory with second-order field equations \[2\]. It should be noted that although the tensor and light speed may differ during inflation, they should evolve to equal values today, in agreement with experiments.

Recently, there has been a lot of interest in disformal transformations of the spacetime metric, called “disformal” because the metric’s temporal component is treated in a different way than its spatial component. In general one writes

\[
g_{\mu\nu} \mapsto \tilde{g}_{\mu\nu} = \Omega^2(g_{\mu\nu} + (1 - B^2)n_\mu n_\nu),
\]

where \(B\) generates a pure disformal transformation and \(\Omega\) a subsequent conformal transformation, that can be used to normalize the Planck mass. In \[3\], and later in \[4\], it was proposed to use such a disformal transformation — with \(\Omega^{-2} = B\) and \(n^\mu\) the unit normal to surfaces of constant time \[1\] — to set any non-canonical tensor speed \(c_T\) equal to the light speed today (which is set to unity). In the new frame, the Einstein frame, gravity is of standard form. The tensor speed does not appear explicitly in the tensor power spectrum anymore. It thus follows immediately that all tensor observables, such as the power spectrum and the bispectrum, can only depend on the Hubble scale defined in this frame \[3\].

It is important to realize that this does not mean that in the Einstein frame all dependence on the tensor speed has dropped out of the model. For example, in the background Friedmann equation which relates the Hubble constant to the microscopic model of inflation now a factor of \(c_T\) appears. The disformal transformation does not only rescale the tensor speed, but the sound speeds of all other species in the universe as well. In fact the ratio of the tensor to light speed during inflation remains invariant \(c_T/c_A = \tilde{c}_T/\tilde{c}_A\). This is in agreement with \[5\] who showed that a pure disformal transformation \((\Omega = 1)\) cannot change the causal structure and the propagation speed of the fields, since it is equivalent to a redefinition of the time coordinate. See also the discussion in \[6\] \[7\], in which it is pointed out that only changing the time coordinate of the metric does not change the physical speed of light. Thus although the tensor propagator, and therefore tensor quantum loop corrections, are standard in the Einstein frame, the propagator equation and loop corrections of all other fields will generically be non-standard (and depend on \(c_T\)).

In this paper we point out that the disformal transformation can equally be seen as a change of units. In the set-up above, in the original frame the light speed is kept equal to the light speed today \(c_A = c\) throughout the history of the universe, with \(c \approx 3 \times 10^8\) m/s; in the final frame the tensor speed is kept constant and equal to the light speed today: \(\tilde{c}_T = c\). In both frames the Hubble parameter can be expressed in inverse seconds, but the definition of a second will not be the same; indeed, it is defined via a light clock in the original frame and via a tensor clock in the final frame. In practice, this is not a problem as observables are dimensionless and frame-independent. As stressed in \[3\], the scale of inflation
can be unambiguously defined in terms of the (frame-independent) tensor power spectrum. The disformal transformation, or equivalently, the change of units, is still very useful, as the tensor spectrum and bispectrum are most easily calculated in the Einstein frame [3].

The invariance of the power spectrum has been discussed in the literature [3, 4, 5, 9, 10, 11, 12, 13, 14, 15, 16, 17]. Even though the power spectrum when written in terms of initial or final frame variables may have a different functional form, its numerical value is invariant under a disformal transformation. A particular feature may in one frame show up in the tensor speed, and in another frame in the Hubble parameter, but the resulting power spectrum always comes out equally.

This article is organized as follows. To set notation, in the next section we introduce the effective field theory (EFT) action for single field inflation, augmented with the Maxwell-action to keep track of the behavior of light waves. In section 3 the effect of the disformal transformation on the perturbation modes is discussed. We first make explicit how ratios of speeds do not change in the transformed frame, e.g. \( c_T/c_A = \tilde{c}_T/\tilde{c}_A \), and subluminal modes remain subluminal and superluminal modes remain superluminal (modes can become superluminal with respect to the light speed today though). Then we show that the disformal transformation can equally be interpreted as a change of units. We first consider a constant tensor speed, and discuss the issue of units in this simplified setting. To tackle the time-dependent transformation, it is convenient to rewrite action and the power spectrum in terms of separately invariant quantities. We end with concluding remarks. We will keep the units explicit. The speed of sound of the tensor, scalar and gauge modes are denoted by \( c_\gamma \), \( c_\zeta \), \( c_A \). Quantities in the transformed frame are denoted by a tilde (for example \( \tilde{c}_i \)), and dimensionless quantities by an overbar (for example \( \bar{c}_i \)).

## 2 The set-up

In this section we will introduce the effective action for inflation supplemented with the Maxwell action describing light waves. From this the action for the inflationary scalar and tensor modes can be derived, as well as those for the U(1) vector modes. The action is formulated in the initial frame with the speed of light \( c_A = c \approx 3 \times 10^8 \text{m/s} \) kept constant throughout the history of the universe.

Since we are interested in the effect of a disformal transformation on the modes during inflation we consider a perturbed FRW metric, which in ADM decomposition reads [18, 19]

\[
\text{ds}^2 = -N^2 c^2 \text{dt}^2 + h_{ij}(N^i \text{cdt} + \text{dx}^i)(N^j \text{cdt} + \text{dx}^j),
\]

with \( c \) the speed of light today, and with scalar and tensor perturbations

\[
h_{ij} = a^2 e^{2\zeta} (e^\gamma)_{ij}, \quad \gamma_{ii} = 0 = \partial_i \gamma_{ij}.
\]

In today’s universe \( c_T = c_A = c \). In the original frame the light speed is kept constant and one can set \( c_A = c \) in all formulas; in the final frame the tensor speed is kept constant and one can set \( \tilde{c}_T = c \). This is why we use \( c \) in the expression for the metric and in the action. We can define the lightcone for the unperturbed metric by setting \( \text{ds}^2 = 0 \). Modes that propagate with speed \( c \) travel on the lightcone.

---

1 We use the U(1) boson as an example of a massless field moving at the speed \( c \) in the original frame, i.e. the speed at which all the massless fields move today when Lorentz invariance is recovered. We could have equivalently used any other massless field for our arguments.
It is useful to formulate the theory in terms of the dimensionless action $\bar{S} \equiv S/\hbar$. The effective field theory action for single field inflation in unitary gauge is \[^{[1]}\]

\[
\bar{S} = \frac{c^3 m_p^2}{2\hbar^2} \int dt d^3x \sqrt{-g} \left[ R - \rho(t) g^{00} - \Lambda(t) - (1 - \bar{c}_T^2)(\delta K_{\mu\nu} - \delta K^2) + M_A^4 (\delta g^{00})^2 \right] \\
+ \frac{c}{\hbar^2} \int dt d^3x \sqrt{-g} \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right]. \quad (4)
\]

The reduced Planck mass is defined via $m_p^2 = \hbar c/(8\pi G)$. The last two terms on the first line affect the speed of sound of tensor and scalar modes respectively; if non-zero the respective sound speeds will differ from unity. Here $\bar{c}_T = c_T/c_A$ is the dimensionless speed of the tensor modes. The dimensionless scalar speed of sound is related to $M_A$, the speed of light. The equations of motion are $\partial_\mu (\sqrt{-g}F^{\mu\nu}) = 0$. It is convenient to work in Coulomb gauge $A_0 = 0$. Then the constraint equation gives $\partial_i A_i = 0$, hence this is the same as transverse gauge. In this gauge the $\mu = i$-component of the Maxwell equations yields the equation for the transverse gauge fields:

\[
(g^{00}\partial_0^2 - 3H\partial_0 + g^{ii}\partial_i^2)A_j = 0. \quad (5)
\]

This can be used to find the action for the two physical polarizations. The quadratic action for tensor and scalar modes can be found in \[^{[1]}\&^{[3]}\]. The result is

\[
\begin{align*}
\bar{S}_\gamma &= \frac{c^3 m_p^2}{8\hbar^2} \int dt d^3x a^3 c_T^{-2} \left[ \dot{\gamma}_{ij}^2 - c_T^2 \left( \partial_k \gamma_{ij} \right)^2 \right], \\
\bar{S}_\zeta &= \frac{c^3 m_p^2}{\hbar^2} \int dt d^3x a^3 c_s^{-2} \left[ \dot{\zeta}^2 - c_s^2 \left( \partial_k \zeta \right)^2 \right], \\
\bar{S}_A &= \frac{c}{2\hbar^2} \int dt d^3x a^3 c_A^{-2} \left[ \dot{A}_i^2 - c_A^2 \left( \partial_k A_i \right)^2 \right],
\end{align*} \quad (6)
\]

with $\epsilon = -\dot{H}/H^2$ the first slow roll parameter. The difference in dimensional factors in front of the action between the scalar and tensor perturbations, and the vector perturbation arises because $\zeta, \gamma$ are dimensionless fields whereas the gauge field has the dimensions of a scalar field $[A^a] = L^{-1}$.

From the quadratic action we can derive the power spectrum for the tensor modes during inflation (up to first order in $\epsilon$ and $\epsilon_T \equiv \frac{\dot{c}_T}{c_T}$ \[^{[20]}\&^{[21]}\]):

\[
P_\gamma = \frac{\hbar^2}{m_p^2 c_T^3} \frac{2H^2}{\pi c_T} \left( 1 + 2\epsilon [1 - \gamma - \ln 2] + \epsilon_T [-\gamma - \ln 2] \right), \quad (7)
\]

with $\gamma \approx 0.577...$ the Euler-Mascheroni constant, and evaluated at the sound horizon exit $aH = c_T k$.

### 2.1 Disformal transformation

Consider the combined conformal (described by $\Omega$) and disformal (described by $B$) transformation

\[
g_{\mu\nu} \mapsto \tilde{g}_{\mu\nu} = \Omega^2 (g_{\mu\nu} + (1 - B^2)n_\mu n_\nu), \quad (8)
\]
with \( n^\mu \) the unit normal to surfaces of constant \( t \): \[ n_\mu = \frac{\partial_\mu t}{\sqrt{-g^\mu\nu \partial_\mu t \partial_\nu t}} = N \delta_{\mu0}. \] (9)

For \( \Omega^2 = 1 \) the transformation is purely disformal; it transforms only the lapse function \( N \mapsto BN \). As such it affects the dispersion relation of all fields in the theory. This can be used to change the tensor speed of sound — but at the same time the scalar and vector speeds are altered. Both the disformal and conformal transformation change the normalization of the Ricci tensor in the action, i.e. of the Planck mass, which can be chosen to transform as desired.

The disformal plus conformal transformation is equivalent to a redefinition of the coordinates and fields according to \[ d\tilde{t} = \Omega B dt, \quad \tilde{h}_{ij}(\tilde{t}) d\tilde{x}^i d\tilde{x}^j = \Omega^2 h_{ij}(t) dx^i dx^j, \quad \tilde{N}^i = (\Omega B)^{-1} N^i, \quad \tilde{N} = N. \] (10)

The transformation of the spatial metric can be rewritten in terms of a rescaling of the scale factor \( a(\tilde{t}) = \Omega a(t) \) (and \( d\tilde{x} = dx \)). Using (10) the transformed metric is brought back to the original form:

\[
\begin{align*}
\tilde{d}s^2 &= -\Omega^2 B^2 N^2 c^2 dt^2 + \Omega^2 h_{ij}(N^i c dt + dx^i)(N^j c dt + dx^j) \\
&= -\tilde{N}^2 c^2 dt^2 + \tilde{h}_{ij}(\tilde{N}^i c d\tilde{t} + d\tilde{x}^i)(\tilde{N}^j c d\tilde{t} + d\tilde{x}^j).
\end{align*}
\] (11)

The disformal transformation (3) followed by the field redefinitions (10) leaves the form of the action invariant. The effect of the disformal transformation can thus equivalently be implemented via the field/coordinate transformation [3][2]

### 3 Transformation of tensor, scalar and vector modes

In this section we will see how the quadratic action (5) transforms under the disformal transformation, and how this affects the sound speed of the perturbations. The transverse traceless tensor modes, the traceless vector modes and the scalar mode are all effective scalar degrees of freedom, with the same kinetic structure. They all transform in the same way under the field/coordinate redefinitions (10). The net result is

\[
\begin{align*}
\tilde{S}_\gamma &= \frac{c^2 m_p^2}{8\hbar^2} \int d\tilde{t} d\tilde{x}^3 \tilde{x}^a \tilde{x}^b \frac{1}{B\Omega^2 c_T^2} \left[ (\partial_t \tilde{\gamma}_{ij})^2 - \frac{c_T^2}{B^2} \frac{(\partial_k \tilde{\gamma}_{ij})^2}{\tilde{a}^2} \right], \\
\tilde{S}_\zeta &= \frac{c^2 m_p^2}{\hbar^2} \int d\tilde{t} d\tilde{x}^3 \tilde{x}^a \tilde{x}^b \frac{1}{B\Omega^2 c_s^2} \left[ (\partial_t \tilde{\zeta})^2 - \frac{c_s^2}{B^2} \frac{(\partial_k \tilde{\zeta})^2}{\tilde{a}^2} \right], \\
\tilde{S}_A &= \frac{c}{2\hbar^2} \int d\tilde{t} d\tilde{x}^3 \tilde{x}^a \tilde{x}^b \frac{1}{B\Omega^2 c_A^2} \left[ (\partial_t \tilde{A}_i)^2 - \frac{c_A^2}{B^2} \frac{(\partial_k \tilde{A}_i)^2}{\tilde{a}^2} \right].
\end{align*}
\] (12)
The propagation speed for all modes thus scales
\[ c_i^2 \mapsto \tilde{c}_i^2 = \frac{c_i^2}{B^2}, \]
where \( i = \gamma, \zeta, A \) denotes tensor, scalar and light waves. It seems that if one chooses \( B \) appropriately the tensor speed can take on any value. However, it should be noted that this is with respect to the light speed \( \text{today} \), i.e. \( \tilde{c}_T/c = \tilde{c}_T/c_A \) can be set to unity. The disformal transformation affects the speed of all propagation modes in exactly the same way. As a consequence, the tensor speed expressed in units of the speed of light \( \text{in the early universe} \) remains invariant under the disformal transformation
\[ \frac{c_T^2}{c_A^2} \mapsto \frac{\tilde{c}_T^2}{\tilde{c}_A^2} = \frac{B^2 c_T^2}{B^2 c_A^2} = \frac{c_T^2}{c_A^2}. \]
A corollary of this is that if \( c_T^2 > c_A^2 \) in the original frame and tensor waves are superluminal, they still travel faster than the speed of light in the new frame \( \tilde{c}_T^2 > \tilde{c}_A^2 \).

The power spectrum computed from the transformed action is given by (at leading order, i.e. up to slow-roll corrections)
\[ P_\gamma = \frac{\hbar^2}{c^4 m_p^2} \frac{2\tilde{H}^2}{\pi^2 \tilde{c}_T} \cdot B \Omega^2. \]
Equivalently, we can substitute in (1) the expressions of the “tilde” quantities in terms of the “untilde” ones to arrive at the same expression, as the value of the power spectrum (the physics) is frame independent. In [3] it was proposed to choose
\[ \Omega^2 B = 1 \quad \& \quad B = c_T/c. \]
It then follows from (13) that \( \tilde{c}_T = c \). Then choosing units
\[ \tilde{c}_T = c = 1 \quad \& \quad \hbar = 1 \quad \Rightarrow \quad P_\gamma = \frac{1}{m_p^2} \frac{2\tilde{H}^2}{\pi^2}, \]
all dependence on the sound speed drops out (even if we include corrections of first order in the slow-roll parameters, see Appendix A), and the power spectrum can be directly linked to the “scale of inflation” \( \tilde{H} \). This is not surprising, also in the original frame working in units of \( c_T = 1 \) the power spectrum does not depend on \( c_T \). It should also be noted that although one can express the Hubble scale in GeV units, these are defined with a tensor clock (rather than a light clock which defines the meter, and thus the GeV, in the original frame). Equivalently, to relate the Hubble scale \( H \) via the Friedmann equation to an energy density \( \rho \) we would need the frame invariant conversion factor \( c_T/c_A \).

In other words there is not much value in the actual numbers for dimensionful and frame dependent quantities like \( H, H/m_p \) and \( \rho \) as long as we haven’t specified the unit system in which we are expressing them. Physical observables are dimensionless and frame invariant. The dimensionless and frame invariant number that we (hope to) measure in the sky giving information about the scale of inflation is precisely the tensor power spectrum.

The great use of the disformal transformation, however, is that in units of \( c_T = 1 \) the tensor action, and thus gravity, is of standard form. In this Einstein frame all tensor observables therefore only depend on the ratio \( \tilde{H}/m_p \); this simplifies the calculation and the interpretation of the various tensor n-point functions enormously [3]. Moreover, as pointed out in [3], the disformal transformation maps a theory with a non-canonical tensor speed to one with a non-canonical scalar speed, which has been studied at great length in the literature already.
3.1 Time-independent transformation and units

In this subsection we will elaborate on the relation between a disformal transformation and the choice of unit system. We will first focus on a time-independent tensor speed, which can be set to unity by a time-independent disformal transformation; the time-dependent situation is postponed to the next section. We emphasize that the time-independent case can be valid for some time-interval during inflation, but not throughout the full history of the universe, as today the tensor and light speed are equal.

First note that the action of the disformal transformation on the quadratic action (10) is nothing but a redefinition of time and length:

\[ d\tilde{t} = \Omega B dt, \quad \tilde{a} d\tilde{x} = \Omega a dx \]

(18)

All dimensionful quantities scale accordingly. Indeed, we can immediately derive the sound speed in the new frame. The speed has dimensions length/time, hence it scales as \( \tilde{c}_s^2 = c_s^2 / B^2 \). Similarly, \( [h] = ML^2T^{-1} \) and \( [m_{pl}^2] = M^2 \), from which it follows

\[ \tilde{c}_s^2 = \frac{1}{B^2} c_s^2, \quad \tilde{h} = \frac{\Omega}{B} h, \quad \tilde{m}_{pl} = m_{pl}. \]

(19)

Writing the transformed action (12) in terms of the tilde variables gives for the tensor modes

\[ \tilde{S}_\gamma = \tilde{c}_s^3 \tilde{m}_{pl}^2 \int d\tilde{t} d^3 \tilde{x} \tilde{a}^3 \tilde{c}_T^{-2} \left[ (\partial_T \gamma_{ij})^2 - \tilde{c}_T^2 \left( \frac{\partial_k \gamma_{ij}}{\tilde{a}^2} \right)^2 \right]. \]

(20)

The power spectrum for the tensor modes transforms (for \( c_T \) time-independent, and at leading order in the slow-roll approximation)

\[ P_\gamma = \frac{\hbar^2}{c^3 m_{pl}^2 \pi^2 c_T} \rightarrow P_\gamma = \frac{\tilde{h}^2}{c_s^3 \tilde{m}_{pl}^2 \pi^2 \tilde{c}_T}. \]

(21)

The transformed power spectrum is of the same form. One way to interpret that result is to say that if in both frames the same units are used — for example \( \tilde{h} = m_{pl} = c_A = 1 \) in the initial frame and \( \tilde{h} = \tilde{m}_{pl} = \tilde{c}_A = 1 \) in the final frame — nothing changes. Indeed, the transformed action (20) and power spectrum (21) are of the same functional form as in the initial frame, and one can simply drop the tildes. This is independent of the particular disformal transformation, that is, of the choice of \( \Omega \) and \( B \).

The disformal transformation can be viewed as a change of units. Indeed if \( c_A = c = 1 \) and \( \hbar = 1 \) is fixed in the original frame, then \( \tilde{c}_A = c / B \neq 1 \) in the final frame, as follows from (19). The choice \( B = c / c_T \) as in (16) transforms from a frame with constant light speed to a frame with constant tensor speed. The choice \( \Omega^2 B = 1 \) is needed for the normalization [3], as only for this choice \( \tilde{h}^2 / c_s^3 = h^2 / c^3 = 1 \). Then the power spectrum (21) indeed reduces to the previous expression (13).

The change of units discussed above can be expressed in the speed of light/tensors today. However, there is no unique way to do this. One can keep the speed of light constant throughout the history of the universe \( c_A = c \simeq 3 \times 10^8 \text{m/s} \), or instead keep the tensor speed constant \( \tilde{c}_T = c \simeq 3 \times 10^8 \text{m/s} \). But since the two speeds are different during inflation, this means the meter and second are defined differently using these two options (which can be considered a
different choice of units), and one needs to know which choice is made to interpret the results. One can visualize the different definition of units as follows: if one defines the meter via a light clock or a tensor clock then today this gives the same result, but in the early universe it leads to a different definition of the meter.

If we set \( c_A = c \simeq 3 \times 10^8 \text{m/s} \) and \( \tilde{c}_A = \tilde{c} \simeq 3 \times 10^8 \text{m/s} \) in the final frame (that is, in both frames the meter is defined with a light clock) the change of units corresponds to

\[
c_A = \hbar = m_p = 1 \quad \mapsto \quad \tilde{c}_A = \frac{\hbar^2}{c_A} = \tilde{m}_p = 1,
\]

This indeed brings the right hand side of (21) in the form of (17). The other option is to set \( c_A = c \simeq 3 \times 10^8 \text{m/s} \) in the first frame and set \( \tilde{c}_T = c \simeq 3 \times 10^8 \text{m/s} \) in the second frame (that is, we switch from a light to a tensor clock) the change of units is

\[
c_A = c \quad \& \quad c_A = \hbar = m_p = 1 \quad \mapsto \quad \tilde{c}_T = c \quad \& \quad \tilde{c}_T = \hbar = m_p = 1.
\]

Although there is an ambiguity how meters, and thus GeVs, are defined in the early universe, since inflationary observables are dimensionless combinations constructed from scales and speeds at the inflationary time they do not depend on the unit choice (or equivalently, are frame independent). From this discussion it is also clear that the power spectrum (17) still implicitly depends on \( c_T \) via the choice of units.

### 3.2 Invariant action

In this section we rewrite the action and power spectrum in terms of dimensionless quantities, which are manifestly invariant under the time-dependent disformal transformation (8). To do so we have to be careful with quantities that involve time-derivatives such as the Hubble constant. The reason is that if we start with a frame/unit system in which the tensor mode is time-dependent, then in the unit system with the tensor speed set to one the light speed is time-dependent — it forces us to deal with measuring sticks that change with time. This approach is similar to the frame invariant constructions for a pure conformal transformation proposed in [22, 23]. In this formalism different units can be implemented by a different choice of measuring sticks.

We define \( L, T, M \) as our measuring sticks for length, time and mass respectively. For now we keep them general, but we will be more explicit later. Since the measuring sticks are dimensionful they transform under the disformal transformation (18)

\[
\tilde{L} = \Omega L, \quad \tilde{T} = \Omega BT, \quad \tilde{M} = M.
\]

The dimensionless length and time are

\[
\tilde{a} \tilde{d} \tilde{x} = \frac{a \, dx}{L} = \frac{\tilde{a} \, \tilde{d} \tilde{x}}{\tilde{L}}, \quad \tilde{t} = \frac{t}{T} = \frac{\tilde{t}}{\tilde{T}}.
\]
The dimensionless velocity and the Hubble parameter are defined via

\[ \bar{c}_i = \frac{c_i}{T} = \bar{c}_i T, \quad \bar{H} = \frac{\partial \bar{a}}{\bar{a}}, \tag{27} \]

and so on. Using (10, 24) it follows that the dimensionless line element is invariant \( ds^2 = L^{-2}d\bar{s}^2 = \tilde{L}^{-2}d\tilde{s}^2 \). The dimensionless tensor action is of the form

\[ \bar{S}_\gamma = \int d\bar{t} d^3\bar{x} c^2 m_p^2 \bar{a}^3 \bar{c}_T^{-2} \left[ (\partial_t \bar{\gamma}_{ij})^2 - \bar{c}_T^{-2} \frac{\partial_k \bar{\gamma}_{ij}}{\bar{a}^2} \right] \tag{28} \]

which leads to a tensor power spectrum, at leading order in the slow-roll approximation for example, the first order \( T \) dependence in the original frame

\[ P_\gamma = \frac{\bar{h}^2}{c^4 m_p^2 \pi^2 \bar{c}_T} \tag{29} \]

The action and power spectrum are now manifestly invariant under the disformal transformation. Restoring the dimensions, the result agrees with our previous expression for a constant transformation. The action and power spectrum can be written in either the untilde or tilde variables, just as is done in (27), which gives the same expression. It is thus clear that if the same unit system is used in both frames, the results are identical. In this respect, the disformal transformation can be seen as simply a change of units.

The change of units used in [3] can be implemented in two ways (22, 23), as discussed in the previous subsection for the time-independent transformation. Starting with \( c_A = c \) in the initial frame corresponds to choosing

\[ L = \frac{h}{m_p c_A}, \quad T = \frac{h}{m_p c_A^2}, \quad M = m_p. \tag{30} \]

Note that the quadratic tensor action only depends on the combinations \( LT \) and \( L/T \). The change of units (22) is obtained via

\[ LT = L/T = M = 1 \quad \rightarrow \quad \tilde{L}/\tilde{T} = \frac{\bar{c}_A}{\bar{c}_T} \quad \& \quad \tilde{L}\tilde{T} = \tilde{M} = 1. \tag{31} \]

with \( \frac{\bar{c}_A}{\bar{c}_T} \) the frame invariant ratio of light to tensor speed.

\(^3\)We take the coordinates dimensionless, and then the scale factor has dimension of length, and scales as given in (25). Another (equivalent) option, that avoids this choice, is to define the Hubble constant in terms of a physical length rather than the unphysical scale factor (which also how it is measured in practice), via

\[ \bar{H} = \frac{\partial (\bar{a}d\bar{x})}{(\bar{a}d\bar{x})}. \tag{26} \]

\(^4\)Writing the dimensionless Hubble parameter in terms of the initial and final frame quantities gives \( \bar{H} = \bar{T} (\bar{H} - (\partial \bar{t} \bar{L})/\bar{L}) \) and thus \( H = (\Omega_B)(\bar{H} - (\partial \bar{t} \bar{L})/(\Omega^2 \bar{B})) \).

\(^5\)The result can be straightforwardly generalized to include the slow roll corrections given in [4]. We note that what appears as a changing tensor speed in original frame shows up as a changing Hubble parameter in the final frame (as follows from footnote 3). Hence, to relate the results in both frames the calculation should be done at the same order in all slow roll parameters in both frames. This is the reason for the discrepancy at first order in slow-roll between eq. (12) in [3] and our eq. (7). This is shown explicitly in Appendix A.
Alternatively, one can set $\tilde{c}_T = c$ in the final frame (switching from a light to a tensor clock as discussed in section 3.1) and introduce a new set of measuring sticks

\[
\tilde{L}' = \frac{\hbar}{m_p \tilde{c}_T}, \quad \tilde{T}' = \frac{\hbar}{m_p \tilde{c}_T}, \quad \tilde{M}' = m_p. \tag{32}
\]

The generalization of (23) for a time-dependent transformation is then simply going from

\[
L = T = M = 1 \quad \mapsto \quad \tilde{L}' = \tilde{T}' = \tilde{M}' = 1. \tag{33}
\]

Both procedures are fully equivalent in that they give exactly the same form of the quadratic action in the tilde variables, and in both cases the power spectrum (29) reduces to (17). It also clear that a disformal transformation is nothing but a change of units; in fact one could simply drop the tildes on the righthand side of the equations above and do the change of measuring sticks and units all in terms of initial frame variables.

4 Conclusions

The disformal transformation (8) changes the box operator in the wave equation, and thus equally affects scalar, vector and tensor modes. The ratio of tensor speed to light speed and the tensor power spectrum are dimensionless and frame independent\(^6\). As a direct consequence, the power spectrum for the tensor modes will thus in every frame depend on both the Hubble parameter and the tensor speed. However, by choosing a unit system in which $c_T = 1$ one can apparently remove the tensor speed from the tensor power spectrum, and that is exactly what the disformal transformation (8) does. This may simplify computations considerably. Tensor n-point functions are now standard. Scalar n-point functions on the other hand now involve a non-canonical scalar speed $c_s$, but these have been studied at length in the literature.

The aim of this paper has been to show that the disformal transformation (8) can equally be viewed as a change of units. It is then immediately clear that its impact on the actual physics is zero. Physical information comes exclusively from dimensionless and frame invariant quantities. However, as already noted above, the difficulty of the computations needed to reach that physical information is for sure not frame invariant. The disformal transformation (8) is a very useful tool for simplifying (and interpreting) computations. Finally, to further stress the frame invariance of the tensor power spectrum, we have written it in terms of quantities which are all manifestly frame independent by themselves.

Acknowledgements

We wish to thank Paolo Creminelli, Jérôme Gleyzes, Jorge Noreña and Filippo Vernizzi for commenting on an early draft of this paper, and Sebastian Cespedes, Gonzalo Palma and Spyros Sypsas for useful further discussions. JF and MP are funded by the Netherlands Foundation for Fundamental Research of Matter (FOM) and the Netherlands Organisation for Scientific Research (NWO). SM is funded by the Fondecyt 2015 Postdoctoral Grant 3150126.

---

\(^6\)In fact, for the particular case $\Omega^2 B = 1$ that was proposed in [3], the two (dimensionful) components that make up the dimensionless product

\[
P_\gamma = \frac{\hbar^2}{c^3 m_p^2} \times \frac{2H^2}{\pi^3 c_T}. \tag{34}
\]

are even separately invariant under the disformal transformation (8).
A Frame invariance of slow-roll corrected tensor power spectrum

In this appendix we show explicitly the full (functional) invariance of the tensor power spectrum, including slow roll corrections to its amplitude, for a time-dependent disformal transformation. This generalizes the calculation in section 3.1. For definiteness we consider the transformation \((16)\), and set (following the notation in [3])

\[ B = \beta, \quad \Omega^2 = 1/\beta, \quad (35) \]

for general \(\beta\).

In the first frame, we begin from the dimensionless tensor quadratic action

\[ \tilde{S}_\gamma = \frac{m_p^2 c^3}{8} \frac{\hbar^2}{(\ell^* a^3 c_T)^2} \int d^3 x \ dt \ x \ a^3 c_T^{-2} \left[ (\partial_t \tilde{h}_{ij})^2 - c_T^2 \left( \frac{\partial_k \tilde{h}_{ij}}{a^2} \right)^2 \right], \quad (36) \]

which yields for the tensor power spectrum, up to first order in \(\dot{\epsilon}^2\) and \(\dot{\epsilon}_T \equiv \ddot{c}_T / aH\) \[P_\gamma = \frac{2}{\pi^2} \frac{\hbar^2}{m_p^2 c^3} \frac{H^2}{c_T} \left( 1 + 2\epsilon[1 - \gamma - \ln 2] + \epsilon_T[\gamma - \ln 2] \right) \left( \frac{c_T k}{aH} \right)^{-2\epsilon - \epsilon_T}, \quad (37) \]

with \(\gamma \approx 0.577...\) the Euler-Mascheroni constant.

Now, the disformal transformation in eq. (35) implies a rescaling for time and space intervals:

\[ dt \rightarrow \tilde{dt} = \frac{1}{\sqrt{\beta}} dt, \quad dx \rightarrow \tilde{dx} = dx, \quad a \rightarrow \tilde{a} = \sqrt{\beta} a \quad (38) \]

For the quantities appearing in the tensor quadratic action in eq. (36) that implies

\[ \tilde{h} = \beta^{-3/2} \tilde{h}, \quad G = \beta^{-5/2} G, \quad m_p = \tilde{m}_p, \quad c_i = \beta^{-1} \tilde{c}_i, \quad (39) \]

where \(i\) can denote scalars, vectors or tensors.

It is then straightforward to see that in the second frame we end up with the following quadratic action for tensors\[7\]

\[ \tilde{\tilde{S}}_\gamma = \frac{m_p^2 c^3}{8} \frac{\hbar^2}{\tilde{\ell}^* \tilde{a}^3 \tilde{c}_T^{-2}} \int d^3 \tilde{x} \ \tilde{dt} \ \tilde{x} \ \tilde{a}^3 \tilde{c}_T^{-2} \left[ \left( \tilde{\partial}_t \tilde{h}_{ij} \right)^2 - \tilde{c}_T^2 \left( \frac{\tilde{\partial}_k \tilde{h}_{ij}}{\tilde{a}^2} \right)^2 \right]. \quad (40) \]

All reference to \(\beta\) has vanished. The functional dependence of the quadratic action is identical in both frames. Therefore, computing the tensor power spectrum in the second frame will yield the same result as in eq. (37) but with a tilde on every quantity:

\[ \tilde{P}_\gamma = \frac{2}{\pi^2} \frac{\tilde{\ell}^2}{\tilde{m}_p^2 \tilde{c}^3} \frac{\tilde{H}^2}{c_T} \left( 1 + 2\tilde{\epsilon}[1 - \gamma - \ln 2] + \tilde{\epsilon}_T[\gamma - \ln 2] \right) \left( \frac{\tilde{\epsilon}_T k}{\tilde{a}H} \right)^{-2\tilde{\epsilon} - \tilde{\epsilon}_T}. \quad (41) \]

---

\[7\]Since the combination \(\frac{\hbar^2}{m_p^2 c^3}\) simply transforms to \(\frac{\tilde{h}^2}{\tilde{m}_p^2 \tilde{c}^3}\), i.e. again without any reference to \(\beta\), one can as well decide to keep \(h, c\) and \(m_p\) fixed under disformal transformations.
To check the consistency of the whole setup, we now write the original $P_\gamma$ in terms of quantities evaluated in the second frame. To that end we introduce

$$\beta_i = \frac{\partial_i \beta}{\beta \tilde{H}}, \quad \beta_{ii} = \frac{\partial_i \beta_i}{\tilde{H}}.$$  \hspace{1cm} (42)

In the setup of [3], $\beta$ is identified with (the inverse of) the tensor speed $c_T$, so it seems well motivated to work up to first order in $\epsilon$, $\epsilon_T$, $\beta_i$ and $\beta_{ii}$. Then we find for the transformation of the remaining quantities in the tensor power spectrum in eq. (37)

$$H = \beta^{-1/2} \tilde{H} \left( 1 - \frac{\beta_i}{2} \right), \quad \epsilon = \tilde{\epsilon} + \frac{\beta_i}{2}, \quad \epsilon_T = \tilde{\epsilon}_T - \beta_i.$$ \hspace{1cm} (43)

Here we have defined $\tilde{H} \equiv \frac{\partial \tilde{\alpha}}{\tilde{\alpha}}$, $\tilde{\epsilon} \equiv - \frac{\partial \tilde{H}}{\tilde{H} \tilde{c}_T}$, $\tilde{\epsilon}_T \equiv \frac{\partial \tilde{c}_T}{\tilde{c}_T \tilde{H}}$. We can now rewrite the tensor power spectrum in the first frame in eq. (37) as

$$P_\gamma = \frac{2}{\pi^2} \frac{\beta^{-3} \tilde{h}^2}{m_{pl}^2 \beta^{-3} \tilde{c}^3} \frac{\beta^{-1} \tilde{H}^2 (1 - \beta_i)}{\beta^{-1} \tilde{c}_T} \left( 1 + (2 \tilde{\epsilon} + \beta_i) [1 - \gamma - \ln 2] + (\tilde{\epsilon}_T - \beta_i) [-\gamma - \ln 2] \right) \times \left( \frac{\beta^{-1} \tilde{c}_T k}{\beta^{-1/2} \tilde{a} \beta^{-1/2} \tilde{H} (1 - \frac{\beta_i}{2})} \right)^{-2(\tilde{\epsilon} + \frac{\beta_i}{2}) - (\tilde{\epsilon}_T - \beta_i)}$$

$$= \frac{2}{\pi^2} \frac{\tilde{h}^2}{m_{pl}^2 \tilde{c}^3} \frac{\tilde{H}^2}{\tilde{c}_T} \left( 1 + 2 \tilde{\epsilon} [1 - \gamma - \ln 2] + \tilde{\epsilon}_T [-\gamma - \ln 2] \right) \left( \frac{\tilde{c}_T k}{\tilde{a} \tilde{H}} \right)^{-2\tilde{\epsilon} - \tilde{\epsilon}_T}.$$ \hspace{1cm} (44)

Indeed we find that in both frames the tensor power spectra come out equal: $P_\gamma = \tilde{P}_\gamma$.

In the scalar section, finally, the situation is slightly different. This is caused by the presence of $\epsilon$ in the scalar quadratic action, and therefore in the amplitude of the scalar power spectrum. Using the transformation of $\epsilon$ in eq. (43), one finds that the transformed scalar quadratic action, when written in terms of transformed quantities, still contains factors of $\beta_i$. There is no one-to-one correspondence as in the tensor case between eqs. (36) and (40). The same happens for the scalar power spectrum.

However, this does not mean that the scalar quadratic action and power spectrum are not frame invariant. The scalar action is invariant $\tilde{S}_\zeta = \tilde{S}_\zeta$ (this is by construction, as it follows directly from the implementation of the disformal transformation in (38)), and therefore we also have that $P_\zeta = \tilde{P}_\zeta$. See for example the appendix of [14]. Both the scalar and tensor sector are invariant under a disformal transformation. On top of that, for the tensor sector we have that $\tilde{S}_\gamma$ and $P_\gamma$ have the same functional dependence as $\tilde{S}_\gamma$ and $\tilde{P}_\gamma$.

References

[1] C. Cheung, P. Creminelli, A. L. Fitzpatrick, J. Kaplan, and L. Senatore, “The Effective Field Theory of Inflation,” JHEP 03 (2008) 014 [arXiv:0709.0293 [hep-th]].

[2] G. W. Horndeski, “Second-order scalar-tensor field equations in a four-dimensional space,” Int. J. Theor. Phys. 10 (1974) 363–384.
[3] P. Creminelli, J. Gleyzes, J. Noreña, and F. Vernizzi, “Resilience of the standard predictions for primordial tensor modes,” *Phys. Rev. Lett.* **113** no. 23, (2014) 231301 [arXiv:1407.8439 [astro-ph.CO]]

[4] A. De Felice and S. Tsujikawa, “Inflationary gravitational waves in the effective field theory of modified gravity,” *Phys. Rev. D* **91** no. 10, (2015) 103506 [arXiv:1411.0736 [hep-th]]

[5] G. Domènech, A. Naruko, and M. Sasaki, “Cosmological disformal invariance,” *JCAP* **1510** no. 10, (2015) 067 [arXiv:1505.00174 [gr-qc]]

[6] G. F. R. Ellis and J.-P. Uzan, “‘c’ is the speed of light, isn’t it?,” *Am. J. Phys.* **73** (2005) 240–247 [arXiv:gr-qc/0305099 [gr-qc]]

[7] G. F. R. Ellis, “Note on Varying Speed of Light Cosmologies,” *Gen. Rel. Grav.* **39** (2007) 511–520 [arXiv:astro-ph/0703751 [astro-ph]]

[8] M. Minamitsuji, “Disformal transformation of cosmological perturbations,” *Phys. Lett. B* **737** (2014) 139–150 [arXiv:1409.1566 [astro-ph.CO]]

[9] S. Tsujikawa, “Disformal invariance of cosmological perturbations in a generalized class of Horndeski theories,” *JCAP* **1504** no. 04, (2015) 043 [arXiv:1412.6210 [hep-th]]

[10] S. Tsujikawa, “Cosmological disformal transformations to the Einstein frame and gravitational couplings with matter perturbations,” *Phys. Rev. D* **92** no. 6, (2015) 064047 [arXiv:1506.08561 [gr-qc]]

[11] Y. Watanabe, A. Naruko, and M. Sasaki, “Multi-disformal invariance of non-linear primordial perturbations,” *Europhys. Lett.* **111** (2015) 39002 [arXiv:1504.00672 [gr-qc]]

[12] F. Arroja, N. Bartolo, P. Karmakar, and S. Matarrese, “The two faces of mimetic Horndeski gravity: disformal transformations and Lagrange multiplier,” *JCAP* **1509** (2015) 051 [arXiv:1506.08575 [gr-qc]]

[13] G. Domènech, S. Mukohyama, R. Namba, A. Naruko, R. Saitou, and Y. Watanabe, “Derivative-dependent metric transformation and physical degrees of freedom,” *Phys. Rev. D* **92** no. 8, (2015) 084027 [arXiv:1507.05390 [hep-th]]

[14] D. Baumann, H. Lee, and G. L. Pimentel, “High-Scale Inflation and the Tensor Tilt,” *JHEP* **01** (2016) 101 [arXiv:1507.07250 [hep-th]]

[15] Y. Cai, Y.-T. Wang, and Y.-S. Piao, “Oscillating modulation to B-mode polarization from varying propagating speed of primordial gravitational waves,” *Phys. Rev. D* **91** (2015) 103001 [arXiv:1501.06345 [astro-ph.CO]]

[16] Y. Cai, Y.-T. Wang, and Y.-S. Piao, “Is there an effect of a nontrivial $c_T$ during inflation?,” *Phys. Rev. D* **93** no. 6, (2016) 063005 [arXiv:1510.08716 [astro-ph.CO]]

[17] C. Burrage, S. Cespedes, and A.-C. Davis, “Disformal transformations on the CMB,” *JCAP* **1608** no. 08, (2016) 024 [arXiv:1604.08038 [gr-qc]]
[18] R. L. Arnowitt, S. Deser, and C. W. Misner, “Canonical variables for general relativity,” *Phys. Rev.* **117** (1960) 1595–1602.

[19] R. L. Arnowitt, S. Deser, and C. W. Misner, “The Dynamics of general relativity,” *Gen. Rel. Grav.* **40** (2008) 1997–2027, arXiv:gr-qc/0405109 [gr-qc].

[20] E. D. Stewart and D. H. Lyth, “A More accurate analytic calculation of the spectrum of cosmological perturbations produced during inflation,” *Phys. Lett.* **B302** (1993) 171–175, arXiv:gr-qc/9302019 [gr-qc].

[21] X. Chen, M.-x. Huang, S. Kachru, and G. Shiu, “Observational signatures and non-Gaussianities of general single field inflation,” *JCAP* **0701** (2007) 002, arXiv:hep-th/0605045 [hep-th].

[22] R. Catena, M. Pietroni, and L. Scarabello, “Einstein and Jordan reconciled: a frame-invariant approach to scalar-tensor cosmology,” *Phys. Rev.* **D76** (2007) 084039, arXiv:astro-ph/0604492 [astro-ph].

[23] M. Postma and M. Volponi, “Equivalence of the Einstein and Jordan frames,” *Phys. Rev.* **D90** no. 10, (2014) 103516, arXiv:1407.6874 [astro-ph.CO].