\textbf{T-Duality and Conformal Invariance at Two Loops}

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Abstract

We show that the conformal invariance conditions for a general $\sigma$-model with torsion are invariant under $T$-duality through two loops.

\section{I. INTRODUCTION}

Duality invariance is a tremendously powerful concept in string theory. One of the earliest forms of duality to be recognised was that now known as $T$-duality \cite{1}. This acts to transform the background fields of a $\sigma$-model so as to map one conformally-invariant background (or string vacuum) into another conformally-invariant background, at least at lowest order. In fact the $\sigma$-model and its dual should be equivalent, again at least at lowest order \cite{2} \cite{3}. The duality can be understood as a consequence of an isometry of the theory; upon gauging the isometry, by performing the path integral over the gauge field and the path integral over a lagrange multiplier in different orders, one obtains two equivalent descriptions of the theory with backgrounds related by the duality. This duality is straightforwardly checked at lowest order in $\alpha'$; conformal invariance requires the vanishing of “$B$”-functions, one for each background field (the metric $g$, the antisymmetric tensor $b$ and the dilaton $\phi$), which
are related to the $\beta$-functions for the corresponding background fields in a way which we will describe later. One can check that the duality transformations on backgrounds for which the $B$-functions vanish lead to dual backgrounds which also have vanishing $B$-functions. In fact, this property is equivalent to requiring that the $B$-functions are form-invariant under duality, and hence satisfy certain consistency conditions which were derived in Ref. [4] (see also Ref. [5]). However, it is far less clear whether, and if so how, the $T$-duality is maintained at higher orders in $\alpha'$. In Ref. [7] for a restricted background, and in Ref. [8] for the general case, it has been found that the two-loop string effective action can be made duality invariant by a redefinition of the background fields (see also Ref. [9]). Now the $B$-functions are related to the string effective action, and so this is grounds for hoping that the $B$-functions will also be invariant. However, this is not a fait accompli, as was explained in Refs. [4], [10]. Certainly the invariance of the one-loop effective action guarantees the invariance of the one-loop $B$-functions. On the other hand, the relation between the action and the $B$-functions is more complicated at higher orders. Moreover, once one has decided that the fields need to be redefined at higher orders to maintain duality invariance, then the $B$-functions will also be modified in a non-trivial way, because field redefinitions lead to changes in the $\beta$-functions. For these reasons it seems desirable to investigate explicitly whether the field redefinitions which make the action duality-invariant also lead to duality-invariant $B$-functions. This verification was carried out in Ref. [4] for the restricted case of Ref. [7]. Here we shall carry out the analysis for a different, complementary case which we believe displays most of the features of the full general situation.

II. DUALITY AT FIRST ORDER

The two-dimensional non-linear $\sigma$-model is defined by the action
\[ S = \frac{1}{4\pi\alpha'} \int d^2\sigma \left[ \sqrt{\gamma} g_{\mu\nu}(X) \partial_A X^\mu \partial_B X^\nu \gamma^{AB} + ib_{\mu\nu}(X) \partial_A X^\mu \partial_B X^\nu \epsilon^{AB} + \sqrt{\gamma} R^{(2)} \phi(X) \right], \] (2.1)

where \( g_{\mu\nu} \) is the metric, \( b_{\mu\nu} \) is the antisymmetric tensor field often referred to as torsion, and \( \phi \) is the dilaton. The indices \( \mu, \nu \) run over \( 1, \ldots D+1 \). \( \gamma_{AB} \) is the metric on the two-dimensional world sheet, with \( A, B = 1,2 \). \( \gamma = \det \gamma_{AB}, \epsilon_{AB} \) is the two-dimensional alternating symbol and \( R^{(2)} \) the worldsheet Ricci scalar. Note that \( b_{\mu\nu} \) is only defined up to a gauge transformation \( b_{\mu\nu} \mapsto b_{\mu\nu} + \nabla_{[\mu} \zeta_{\nu]} \). Conformal invariance of the \( \sigma \)-model requires the vanishing of the Weyl anomaly coefficients \( B^g \), \( B^b \) and \( B^\phi \) (we will refer to these as the \( B \)-functions), which are defined as follows [11]:

\[ B^g_{\mu\nu} = \beta^g_{\mu\nu} + 2\alpha' \nabla_{\mu} \nabla_{\nu} \phi + \nabla_{(\mu} S_{\nu)}, \] (2.2)

\[ B^b_{\mu\nu} = \beta^b_{\mu\nu} + \alpha' H^\rho_{\mu\nu} \nabla_{\rho} \phi + \frac{1}{2} H^\rho_{\mu\nu} S_{\rho}, \] (2.3)

\[ B^\phi = \beta^\phi + \alpha' \nabla^\rho \phi \nabla_{\rho} \phi + \frac{1}{2} \nabla^\rho \phi S_{\rho}. \] (2.4)

Here \( \beta^g \), \( \beta^b \) and \( \beta^\phi \) are the renormalisation group \( \beta \)-functions for the \( \sigma \)-model, and \( H_{\mu\nu\rho} \) is the field strength tensor for \( b_{\mu\nu} \), defined by \( H_{\mu\nu\rho} = 3 \nabla_{[\mu} b_{\nu\rho]} = \nabla_{\mu} b_{\nu\rho} + \text{cyclic} \). The vector \( S^\mu \) arises in the process of defining the trace of the energy-momentum tensor as a finite composite operator, and can be computed perturbatively. It will be sufficient to assume \( S^\mu = (S^0, 0) \). At one loop we have

\[ \beta^g_{\mu\nu}^{(1)} = R_{\mu\nu} - \frac{1}{4} H_{\mu\rho\sigma} H^{\rho\sigma}_{\nu}, \] (2.5)

\[ \beta^b_{\mu\nu}^{(1)} = -\frac{1}{2} \nabla_{\rho} H^\rho_{\mu\nu}, \] (2.6)

\[ \beta^\phi^{(1)} = -\frac{1}{2} \Box \phi - \frac{1}{24} H^2, \] (2.7)
where $H^2 = H_{\mu\nu\rho}H^{\mu\nu\rho}$. (Here and henceforth we set $\alpha' = 1$.)

The conformal invariance conditions are equivalent to the equations of motion of a string effective action

$$\Gamma = \int d^{D+1}X \sqrt{g}L(\lambda_M)$$

(2.8)

where $\lambda_M \equiv (g_{\mu\nu}, b_{\mu\nu}, \phi)$. More explicitly, we have to leading order

$$\Gamma^{(1)} = \int d^{D+1}X \sqrt{g}e^{-2\phi}\left[R(g) + 4(\nabla \phi)^2 - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right].$$

(2.9)

From this action it can be shown that

$$\frac{\partial \Gamma^{(1)}}{\partial \lambda_M} = 2\sqrt{g}e^{-2\phi}K^{(0)}_{MN}B^{(1)}_N$$

where

$$K^{(0)}_{MN} \equiv \begin{pmatrix} g^{\sigma}_{\mu}g^{\rho}_{\nu} & 0 & -g_{\mu\nu} \\ 0 & -g^{\rho}_{\mu}g^{\sigma}_{\nu} & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

(2.11)

and we have a basis for $B_M$ such that $B_M = (B^g_{\mu\nu}, B^b_{\mu\nu}, \tilde{B}^{\phi})$ and

$$\tilde{B}^{\phi} = B^{\phi} - \frac{1}{4}g^{\mu\nu}B^g_{\mu\nu}.$$  

(2.12)

We now consider the dual $\sigma$-model. This involves introducing an abelian isometry in the target space background of the model such that one can perform duality transformations. Background fields will now be locally independent of the coordinate $\theta \equiv X^0$ and locally dependent on $X^i$, $i = 1, \ldots, D$. The $\sigma$-model action now reads as

$$S = \int d^2\sigma \frac{1}{4\pi \alpha'} \left[\sqrt{-\gamma^{AB}}(\partial_0(X^k)\partial_A\theta\partial_B\theta + 2g_{0i}(X^k)\partial_A\theta\partial_BX^i + g_{ij}(X^k)\partial_AX^i\partial_BX^j) + \sqrt{-\gamma^R(2)\phi(X^k)} + i\epsilon^{AB}(2D_\alpha(X^k)\partial_A\theta\partial_BX^i + b_{ij}(X^k)\partial_AX^i\partial_BX^j)\right]$$

(2.13)

The classical $T$-duality transformations act on the background fields $\{g_{\mu\nu}, b_{\mu\nu}\}$ to give dual background fields $\{\tilde{g}_{\mu\nu}, \tilde{b}_{\mu\nu}\}$ given by [1]:

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\[ \tilde{g}_{00} = \frac{1}{g_{00}}, \]

\[ \tilde{g}_{0i} = \frac{b_{0i}}{g_{00}}, \quad \tilde{b}_{0i} = \frac{g_{0i}}{g_{00}}, \]

\[ \tilde{g}_{ij} = g_{ij} - \frac{g_{0i}g_{0j} - b_{0i}b_{0j}}{g_{00}}, \quad (2.14) \]

\[ \tilde{b}_{ij} = b_{ij} - \frac{g_{0i}b_{0j} - b_{0i}g_{0j}}{g_{00}}. \]

The origin of the two distinct models lies in the order in which one performs simple parts of the path integral. We shall deal with the transformation on the dilaton later.

We now parametrize the metric and torsion tensors in terms of reduced fields which appear in the Kaluza-Klein reduction to \( D \)-dimensions [12] [4] [8].

\[ g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} a & av_j \\ av_i & \tilde{g}_{ij} + av_iv_j \end{pmatrix}, \quad (2.15) \]

\[ b_{\mu\nu} = \begin{pmatrix} b_{00} & b_{0j} \\ b_{i0} & b_{ij} \end{pmatrix} = \begin{pmatrix} 0 & w_j \\ -w_i & b_{ij} \end{pmatrix}. \quad (2.16) \]

This choice simplifies the form of the classical transformations to

\[ a \mapsto \frac{1}{a}, \quad v_i \leftrightarrow w_i, \]

\[ b_{ij} \mapsto \tilde{b}_{ij} = b_{ij} + w_i v_j - w_j v_i. \quad (2.17) \]

Note that \( \tilde{g}_{ij} \) is unchanged under duality. It has been shown that a further simplification in the context of conformal invariance conditions can be employed, since the transformation properties of the one loop \( B \)-functions are manifest when mapped to tangent space [4]. That is, we construct a vielbein, \( e_\alpha^\mu \), such that
\[ \delta_{ab} = \epsilon_a^\mu \epsilon_b^\nu g_{\mu\nu}. \]  

(2.18)

In fact we can choose a block diagonal form

\[ e_a^\mu = \begin{pmatrix} e_0^0 & e_i^i \\ e_0^0 & e_i^i \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{a}} & 0 \\ -v_\alpha \bar{e}_\alpha^i \end{pmatrix}. \]  

(2.19)

with \( v_\alpha \equiv \bar{e}_\alpha^i v_i \). We can now define the tangent space anomaly coefficients as:

\[ B_{ab}^g = e_a^\mu e_b^\nu B_{\mu\nu}^g, \quad B_{ab}^b = e_a^\mu e_b^\nu B_{\mu\nu}^b, \]  

(2.20)

\[ B_{\hat{a}b}^\hat{g} = e_a^\mu e_b^\nu B_{\hat{\mu}\hat{\nu}}^\hat{g}, \quad B_{\hat{a}b}^\hat{b} = e_a^\mu e_b^\nu B_{\hat{\mu}\hat{\nu}}^\hat{b}. \]  

(2.21)

These coefficients transform as

\[ B_{\hat{a}00}^\hat{g} = -B_{\hat{0}00}^g, \]

\[ B_{\hat{a}\alpha}^\hat{g} = B_{\hat{0}\alpha}^b, \quad B_{\hat{0}\alpha}^\hat{g} = B_{\hat{0}\alpha}^b, \]  

(2.22)

\[ B_{\hat{a}\beta}^\hat{g} = B_{\hat{a}\beta}^g, \quad B_{\hat{0}\beta}^\hat{g} = B_{\hat{0}\beta}^b \]

With the exception of \( B_{\alpha\beta}^b \), we find that we can express each tangent space \( B \)-function in terms of a \( B \)-function for one reduced field, up to factors of \( g_{00} \).

\[ B_{\hat{0}00}^g = e_0^0 e_0^0 B_{\hat{0}00}^g = \frac{1}{a} B_a^a = \frac{1}{a} \beta^a + a^k \bar{\nabla}_k \phi, \]  

(2.23)

\[ B_{\hat{0}\alpha}^g = e_0^0 e_\alpha^i a B_i^\alpha = \bar{e}_\alpha^i \sqrt{a} B_i^\alpha \]

\[ = \bar{e}_\alpha^i \sqrt{a} \left[ \beta_i^\alpha - F_i^k \bar{\nabla}_k \phi + \frac{1}{2} \bar{\nabla}_i S_0 \right], \]  

(2.24)

\[ B_{\hat{0}\alpha}^b = e_0^0 e_\alpha^i B_i^\alpha = \bar{e}_\alpha^i \frac{1}{\sqrt{a}} B_i^\alpha \]

\[ = \bar{e}_\alpha^i \frac{1}{\sqrt{a}} \left[ \beta_i^\alpha - G_i^k \bar{\nabla}_k \phi \right], \]  

(2.25)
\[ B^a_{\alpha\beta} = \bar{e}_\alpha^i \bar{e}_\beta^j B^\eta_{ij} \]
\[ = \bar{e}_\alpha^i \bar{e}_\beta^j \left[ \beta^\eta_{ij} + 2\bar{\nabla}_i \bar{\nabla}_j \phi \right], \tag{2.26} \]

\[ B^b_{\alpha\beta} = \bar{e}_\alpha^i \bar{e}_\beta^j B^b_{ij} \]
\[ = \bar{e}_\alpha^i \bar{e}_\beta^j \left[ \beta^b_{ij} + \hat{H}^k_{ij} \bar{\nabla}_k \phi - \frac{1}{2} G_{ij} S^0 \right]. \tag{2.27} \]

We define

\[ B^b_{ij} = B^b_{ij} - v_i B^w_j + v_j B^w_i. \tag{2.28} \]

The one loop \(B\)-functions for the reduced tensors are

\[ B^{a(1)} = -\frac{a}{2} \bar{\nabla}_i a^i + a a^i \bar{\nabla}_i \Phi + \frac{1}{4} (a^2 F_{km} F^{km} - G_{km} G^{km}), \tag{2.29} \]

\[ B^{v(1)}_i = -F^k_i (\bar{\nabla}_k \Phi - \frac{1}{2} a_k) + \frac{1}{2} \bar{\nabla}^k F_{ik} + \frac{1}{4a} \hat{H}_{ikm} G^{km} + \frac{1}{2} \bar{\nabla}_i S^0, \tag{2.30} \]

\[ B^{w(1)}_i = -G^k_i (\bar{\nabla}_k \Phi + \frac{1}{2} a_k) + \frac{1}{2} \bar{\nabla}^k G_{ik} + \frac{a}{4} \hat{H}_{ikm} F^{km}, \tag{2.31} \]

\[ B^{\bar{b}(1)}_{ij} = \bar{R}_{ij} - \frac{1}{4} a_i a_j - \frac{1}{2} (a F_{ik} F^k_j + \frac{1}{a} G_{ik} G^k_j) - \frac{1}{4} \hat{H}_{ikm} \hat{H}^{km}_{ij} + 2\bar{\nabla}_i \bar{\nabla}_j \Phi, \tag{2.32} \]

\[ B^{b_1}_{ij} = -\frac{1}{2} \bar{\nabla}^k \hat{H}_{kij} + \hat{H}_{kij} \bar{\nabla}^k \phi - \frac{1}{2} G_{ij} S^0, \tag{2.33} \]

where \(a = \partial_i \ln a, F_{ij} = 2\bar{\nabla}_{[i} v_{j]}, G_{ij} = 2\bar{\nabla}_{[i} w_{j]}, = -H_{0ij}, \) and \( \hat{H}_{ijk} = H_{ijk} + 3v_{[i} G_{jk]} \). All barred tensors and covariant derivatives only have dependence on \( \bar{g}_{ij} \), i.e. that part of metric that is invariant under the classical \( T \)-duality transformations. Under the mapping (2.17) we have \( a_i \mapsto -a_i, F_{ij} \leftrightarrow G_{ij} \) and \( \hat{H}_{ijk} \) invariant. \( \Phi \) is the reduced dilaton defined by

\[ \Phi = \phi - \frac{1}{4} \ln a. \tag{2.34} \]
\(\Phi\) is another invariant under the duality transformation and this is seen to be case if one combines the shift in \(\phi\) needed to keep the one loop action invariant \([8]\) with the transformation on \(a\). The corresponding \(B\)-function can be calculated, though it need not be worked out in full detail. \((2.12)\) simplifies to

\[
\tilde{\mathcal{B}}^\phi = \mathcal{B}^\Phi - \frac{1}{4} B^\alpha_{\alpha}. \tag{2.35}
\]

This is manifestly invariant and indeed this fact is closely related to the invariance of the reduced one loop string effective action which can be similarly expressed \([8]\)

\[
\Gamma^{(1)}_R = \int d^dX \sqrt{\bar{g}} e^{-2\Phi} \left[ \bar{R}(\bar{g}) + 4 \nabla^i \Phi \nabla_i \Phi - \frac{1}{4} a^j a_i \right.
\]

\[
- \frac{1}{4} a F_{ij} F^{ij} - \frac{1}{4a} G_{ij} G^{ij} - \frac{1}{12} \bar{H}_{ijk} \bar{H}^{ijk}. \tag{2.36}
\]

From a reduced effective action one would expect the equations of motion for the reduced tensors to reproduce the appropriate conformal invariance conditions. In fact, we have the following (see (A4)—(A7)):

\[
ed_0^\mu e_0^\nu \left. \frac{\partial \Gamma_R}{\partial g^{\mu\nu}} \right|_{b_{\mu\nu}} = -2a \left. \frac{\partial \Gamma_R}{\partial a} \right|_{v^\mu}, \tag{2.37}
\]

\[
ed_0^\mu \bar{e}_0^\nu \left. \frac{\partial \Gamma_R}{\partial g^{\mu\nu}} \right|_{b_{\mu\nu}} = \bar{e}^i \bar{e}^j \left. \frac{\partial \Gamma_R}{\partial \bar{g}^{ij}} \right|_{v^\mu}, \tag{2.38}
\]

\[
ed_0^\mu e_0^\nu \left. \frac{\partial \Gamma_R}{\partial b^{\mu\nu}} \right|_{g_{\mu\nu}} = -\bar{e}^i \bar{e}^0 \left. \frac{\partial \Gamma_R}{\partial v^i} \right|_{w^\mu}, \tag{2.39}
\]

\[
ed_0^\mu \bar{e}_0^\nu \left. \frac{\partial \Gamma_R}{\partial b^{\mu\nu}} \right|_{g_{\mu\nu}} = \bar{e}^i \bar{e}^0 \left. \frac{\partial \Gamma_R}{\partial \bar{w}^i} \right|_{\bar{v}^\mu}, \tag{2.40}
\]

\[
ed_0^\mu e_0^\nu \left. \frac{\partial \Gamma_R}{\partial \bar{w}^{\mu\nu}} \right|_{g_{\mu\nu}} = \bar{e}^i \bar{e}^j \left. \frac{\partial \Gamma_R}{\partial \bar{w}^{ij}} \right|_{\bar{v}^\mu}. \tag{2.41}
\]

The validity of this set of equations at leading order is clear upon inspection of \((2.29)\)—\((2.33)\) and \((2.36)\), together with \((2.2)\)—\((2.7)\) and \((2.9)\). From now on we drop the \(R\) subscript, since all quantities will be assumed to be reduced.
III. DUALITY AT SECOND ORDER

We shall now proceed to illustrate the transformation properties of the next to leading order conformal invariance conditions. We would like to show that the two-loop conformal invariance conditions behave under duality as in (2.22), but we shall see that we have to redefine the fields in order to achieve this. There are two ways in which we might proceed. The more direct approach is to calculate the $B$-functions explicitly for the fields we are considering. These expressions are very complicated, so we abstain from this. The indirect method is to consider the corresponding result to (2.10) at second order

$$\frac{\partial \Gamma^{(2)}}{\partial \lambda_M} = 2\sqrt{g} e^{-2\phi} \left[ K^{(0)}_{MN} B^{(2)}_N + K^{(1)}_{MN} B^{(1)}_N \right],$$

where $\Gamma^{(2)}$ is given in Ref. [13]. $K^{(1)}_{MN}$ can be read off the following expressions [13],

$$\frac{1}{\sqrt{g} e^{-2\phi}} \frac{\partial \Gamma^{(2)}}{\partial \phi} = 8 \bar{B}^{(2)} + 2B^{(1)}_{\rho\sigma} B^{(1)\rho\sigma} + 2B^{(1)}_{\rho\sigma} B^{(1)\rho\sigma} - 2(\bar{B}^{(1)})^2,$$

$$\frac{1}{\sqrt{g} e^{-2\phi}} \frac{\partial \Gamma^{(2)}}{\partial g_{\mu\nu}} = B^{(2)}_{\mu\nu} - B^{(1)}_{\mu\rho} B^{(1)\rho\nu} + R_{\mu\rho\nu\sigma} B^{(1)\rho\sigma} + \bar{B}^{(1)}_{\mu\nu},$$

$$\frac{1}{\sqrt{g} e^{-2\phi}} \frac{\partial \Gamma^{(2)}}{\partial b_{\mu\nu}} = -B^{(2)}_{\mu\nu} + 3H^{(1)}_{\mu\rho\nu\sigma} B^{(1)\rho\sigma} + \frac{1}{2} \nabla_{(\mu} H_{\nu)\rho\sigma} B^{(1)\rho\sigma}$$

$$+ g_{\mu\nu} \left[ -4\bar{B}^{(2)} - B^{(1)}_{\rho\sigma} B^{(1)\rho\sigma} - B^{(1)}_{\rho\sigma} B^{(1)\rho\sigma} + (\bar{B}^{(1)})^2 \right],$$

$$\frac{1}{\sqrt{g} e^{-2\phi}} \frac{\partial \Gamma^{(2)}}{\partial g_{\mu\nu}} = -B^{(2)}_{\mu\nu} + R_{\rho\mu\nu\sigma} B^{(1)\rho\sigma} - \frac{1}{4} H_{\tau\mu\nu\rho\sigma} B^{(1)\rho\sigma} + \frac{1}{4} H_{\tau\rho\mu} H_{\nu\sigma} B^{(1)\rho\sigma}$$

$$- 2B^{(1)}_{\mu\rho} B^{(1)\rho\nu} - \frac{1}{2} \nabla^\sigma H_{\mu\nu}\rho B^{(1)\rho\sigma} - \bar{B}^{(1)}_{\mu\nu}. \quad (3.4)$$

With these relations, we could show the properties under duality of $B^{(2)}_M$ if we knew the properties of the second order action and those of $K^{(1)}_{MN}$. After all we are well informed to the behaviour of $B^{(1)}$. However we know already that $\Gamma^{(2)}$ is not the appropriate action. In fact in Ref. [8] it was proved that a shift in the reduced fields is required to obtain an invariant second order action. The shift in the one loop reduced effective action is then
\[ \delta \Gamma^{(1)} = \int d^d X \sqrt{g} e^{-2\Phi} \left[ \frac{1}{2} a^i \nabla_i \delta a - \frac{a}{2} \left( \frac{1}{2a} \delta a F^{ij} F_{ij} + F^{ij} \delta F_{ij} \right) + \frac{1}{2a} \left( \frac{1}{2a} \delta a G^{ij} G_{ij} - G^{ij} \delta G_{ij} \right) - \frac{1}{6} \dot{H}^{ijk} \delta \dot{H}_{ijk} \right]. \] (3.5)

The authors of Ref. [8] find the required leading order corrections to the reduced fields to be

\[ \delta a = a a_i a^i + \frac{1}{8} a^2 F^{ij} F_{ij} + \frac{1}{8} G^{ij} G_{ij}, \] (3.6)

\[ \delta v_i = - \frac{1}{4} F^k_i a_k - \frac{1}{8a} \dot{H}_{ikm} G^{km}, \] (3.7)

\[ \delta w_i = - \frac{1}{4} G^k_i a_k + \frac{1}{8a} \dot{H}_{ikm} F^{km}, \] (3.8)

\[ \delta \dot{H}_{ijk} = - \frac{3}{2} \nabla_i \left( G_j^{m} F_{k|m} \right) + 3 F_{[ij} \delta w_{k]} + 3 G_{[ij} \delta v_{k]}. \] (3.9)

We now have at our disposal an invariant action which we shall write as

\[ \Gamma'(2) = \Gamma^{(2)} + \delta \Gamma^{(1)}. \] (3.10)

However the consequence of (3.6)—(3.9) in (3.1) is not just that we have to replace \( \Gamma^{(2)} \) with \( \Gamma'(2) \) but also that the two-loop \( \beta \)-functions change, leading consequently to a modified \( K \)-matrix. The new \( \beta \)-functions are in general given by:

\[ \beta'_M = \beta_M - \delta \beta_M \] (3.11)

where

\[ \delta \beta_M(\lambda) = \delta \lambda \beta'_M(\lambda) - \mu \frac{d}{d\mu} \delta \lambda_M. \] (3.12)

The precise details of how we apply (3.12) will be given in the Appendix, and moreover we will leave for later the definition of the redefined \( K \)-matrix. We now seem to have all the required apparatus to achieve our task. In the expressions that follow, after all functional
derivatives have been taken such as in an equation of motion or $\delta_\beta M$, we will consider the simplified theory with $a = 1$ and $\Phi = 0$. (The complementary case with general $a$, but with $\Phi = b_{ij} = 0$, was considered in Ref. [3].)

We first turn our attention to the dilaton. This turns out to be the simplest calculation, mainly because there is no shift in the reduced dilaton, $\Phi$, and consequently the equations of motion are unaltered. By comparing (2.9) and (2.36) we see that the functional dependence on $\phi$ in the unreduced action is the same as on $\Phi$ in the reduced action. Therefore (3.2) becomes

$$\frac{1}{\sqrt{g}e^{-2\Phi}} \frac{\partial \Gamma^{(2)}}{\partial \Phi} = 8\tilde{B}^{\Phi(2)} + 2B^{b(1)}_{ab} B^{b(1)}_{ab} + 2B^{g(1)}_{ab} B^{g(1)}_{ab} - 2(\tilde{B}^{\Phi(1)})^2$$

$$= 8\tilde{B}^{\Phi(2)} + 4B^{b(1)}_{0\alpha} B^{b(1)}_{0\alpha} + 2B^{b(1)}_{\alpha\beta} B^{b(1)}_{\alpha\beta} + 4B^{g(1)}_{0\alpha} B^{g(1)}_{0\alpha}$$

$$+ 2B^{g(1)}_{00} B^{g(1)}_{00} + 2B^{g(1)}_{\alpha\beta} B^{g(1)}_{\alpha\beta} - 2(\tilde{B}^{\Phi(1)})^2,$$

(3.13)

where

$$\tilde{B}^{\Phi} = B^{\Phi} - \frac{1}{4} g^{ij} B_{ij}^g = \tilde{\phi}$$

(3.14)

and $\tilde{\phi}$ is given in (2.35). Given that the $B^{(1)}$-functions satisfy (2.22) and that $\tilde{B}^{\Phi(1)}$ is invariant, we conclude that that $\tilde{B}^{\Phi(2)}$ is invariant under duality.

We now turn our attention to $B^{g}_{\alpha\beta}$. The fundamental equation for us to deal with is

$$\frac{1}{2} e^\alpha_\mu e^\beta_\nu \frac{\partial \Gamma^{(2)}}{\partial g^{\mu\nu}} = e^\alpha_\mu e^\beta_\nu B^{(2)}_{\mu\nu} + e^\alpha_\mu e^\beta_\nu K^{g}_{\mu\nu}.$$ 

(3.15)

We can rewrite (3.15) using our analysis in the previous section of the equations of motion and conformal invariance conditions of the reduced fields, namely (2.26) and (2.38). This gives us

$$\frac{1}{2} \tilde{e}^i_\alpha \tilde{e}^j_\beta \frac{\partial \Gamma^{(2)}}{\partial \tilde{g}^{ij}} = \tilde{e}^i_\alpha \tilde{e}^j_\beta B^{(2)}_{ij} + e^\alpha_\mu e^\beta_\nu K^{g}_{\mu\nu}.$$ 

(3.16)

Correspondingly we can write
\[
\frac{1}{2} \varepsilon^{i \alpha} \varepsilon^{j \beta} \frac{\partial \delta \Gamma^{(1)}}{\partial \bar{g}^{ij}} = \varepsilon^{i \alpha} \varepsilon^{j \beta} \delta B^{(2)}_{ij} + \varepsilon^{\mu}_{\alpha} \varepsilon^{\nu}_{\beta} X^{g}_{\mu \nu}.
\]  

(3.17)

where \( \delta B^{(2)}_{ij} \) is given in the appendix. Finally we combine (3.16) and (3.17) to find

\[
\varepsilon^{i \alpha} \varepsilon^{j \beta} \left[ B^{g'}_{ij} \right] = \varepsilon^{i \alpha} \varepsilon^{j \beta} \frac{\partial}{\partial \bar{g}_{ij}} \left[ \Gamma^{(2)} \right] + K^{g'}_{\alpha \beta}
\]  

(3.18)

where

\[
K^{g'}_{\alpha \beta} = -\varepsilon^{\mu}_{\alpha} \varepsilon^{\nu}_{\beta} \left[ K^{g}_{\mu \nu} + X^{g}_{\mu \nu} \right]
\]

\[
= B^{g(1)}_{\alpha \gamma} B^{g(1)}_{\beta \gamma} + B^{g(1)}_{\alpha \delta} B^{g(1)}_{\beta \delta} - \bar{B}^{\Phi} B^{g(1)}_{\alpha \beta} + B^{b(1)}_{\alpha \delta} B^{b(1)}_{\beta \delta} + B^{b(1)}_{\alpha \gamma} B^{b(1)}_{\beta \gamma} \\
+ B^{g(1)}_{\alpha \delta} \varepsilon^{i \alpha} \varepsilon^{j \beta} \varepsilon^{k} \left[ \frac{1}{2} F^{i}_{kj} F^{k} - \frac{1}{2} G^{i}_{k} F^{k} \right] \\
+ B^{g(1)}_{\alpha \delta} \varepsilon^{i \alpha} \varepsilon^{j \beta} \varepsilon^{k} \left[ \nabla_{i} F_{kj} + \frac{1}{2} G_{ni} \hat{H}^{n}_{jk} + \frac{1}{2} G_{nj} \hat{H}^{n}_{ik} \right] \\
+ B^{b(1)}_{\alpha \delta} \varepsilon^{i \alpha} \varepsilon^{j \beta} \varepsilon^{k} \left[ \nabla_{i} G_{kj} + \frac{1}{2} F_{ni} \hat{H}^{n}_{jk} + \frac{1}{2} F_{nj} \hat{H}^{n}_{ik} \right] \\
+ B^{g(1)}_{\alpha \delta} \varepsilon^{i \alpha} \varepsilon^{j \beta} \varepsilon^{k} \varepsilon^{m} \left[ 3 \frac{1}{4} F^{i}_{kj} F^{m}_{jm} - 3 \frac{1}{4} G^{i}_{k} G^{m}_{jm} - 3 \frac{1}{4} \hat{H}^{m}_{ik} \hat{H}^{n}_{jm} \right] \\
+ B^{b(1)}_{\alpha \delta} \varepsilon^{i \alpha} \varepsilon^{j \beta} \varepsilon^{k} \varepsilon^{m} \left[ 3 \frac{1}{4} \nabla_{i} \hat{H}^{n}_{jkm} + 3 \frac{1}{4} \nabla_{j} \hat{H}^{n}_{ikm} \right] \\
+ \delta_{\alpha \beta} \left[ \bar{B}^{\Phi(2)} + B^{b(1)}_{\alpha \gamma} B^{b(1)}_{\beta \gamma} + B^{b(1)}_{\gamma \delta} B^{b(1)}_{\gamma \delta} + B^{g(1)}_{\alpha \delta \delta} B^{g(1)}_{\beta \delta \delta} \\
+ B^{g(1)}_{\alpha \gamma} B^{g(1)}_{\beta \gamma} + B^{g(1)}_{\gamma \delta} B^{g(1)}_{\gamma \delta} + (\bar{B}^{\Phi(1)})^2 \right]
\]  

(3.19)

So we have an invariant equation of motion and, from (2.22), an invariant \( K^{g'}_{\alpha \beta} \) and hence an invariant \( B^{g'(2)}_{\alpha \beta} \).

We now follow a similar argument to prove the transformation properties of \( B^{g(2)}_{\alpha \beta} \). Once again we combine the equation of motion of \( \Gamma^{(2)} \),

\[
- a \frac{\partial \Gamma^{(2)}}{\partial a} = \frac{1}{a} \left[ B^{a(2)} + K^{g}_{a} \right],
\]  

(3.20)

with its correction

\[
- a \frac{\partial \delta \Gamma^{(1)}}{\partial a} = \frac{1}{a} \left[ \delta B^{a(2)} + X^{a} \right]
\]  

(3.21)
We find
\[ \frac{1}{a} B^{a'}(2) = -a \frac{\partial \Gamma^{(2)}}{\partial a} + \frac{1}{a} K^{a'}, \] (3.22)
where
\[ K^{a'} = - [X^a + K^g_{00}] \]
\[ = B^{g(1)}_{\alpha\beta} \bar{e}_\alpha^i \bar{e}_\beta^j \left[ \frac{1}{2} F_{ik} F_{j}^k - \frac{1}{2} G_{ik} G_{j}^k \right] + B^{b(1)}_{\alpha\beta} \bar{e}_\alpha^i \bar{e}_\beta^j \left[ \frac{1}{4} F_{ik} G_{j}^k - \frac{1}{4} G_{ik} F_{j}^k \right] - B^{a(1)}_0 \tilde{B}^{(1)}. \] (3.23)

Since \( a \mapsto \frac{1}{a} \) and \( \frac{\partial \Gamma}{\partial a} = -\frac{1}{a} \frac{\partial \Gamma}{\partial (\frac{1}{a})} \) we have \( -a \frac{\partial \Gamma}{\partial a} \mapsto a \frac{\partial \Gamma}{\partial a} \). So both the equation of motion and \( K \)-matrix change sign under duality, and hence so does \( B^{g(2)}_{0\alpha} \).

The next check on duality concerns the mapping of \( B^{g}_{0\alpha} \leftrightarrow B^{b}_{0\alpha} \). For this case we will need to compare two equations of motion and two \( K \)-matrices. For the equations of motion, as can be seen in the appendix, we will deal with functional derivatives with respect to \( v_i \) and \( w_i \) of \( \Gamma \). In the case of the field equation for \( v_i \) we keep \( b_{ij} \) and \( w_i \) constant, while in the case of the field equation for \( w_i \) we keep \( \tilde{b}_{ij} \) and \( v_i \) constant (where \( \tilde{b}_{ij} \) was defined in (1.17)). Correspondingly, we can write \( \hat{H}_{ijk} \) either as
\[ \hat{H}_{ijk} = 3 \nabla [b_{jk}] + 3 v_i G_{jk} \] (3.24)
or as
\[ \hat{H}_{ijk} = 3 \nabla [\tilde{b}_{jk}] + 3 w_i F_{jk}. \] (3.25)
(Note that this displays the duality invariance of \( \hat{H}_{ijk} \).) We shall also use these two forms of \( \hat{H}_{ijk} \) when calculating \( \delta B^v_i \) and \( \delta B^w_i \) respectively. See the appendix for more details.

So for \( v_i \) we have
\[ e_0^0 \bar{e}_\alpha^i \left[ -\frac{1}{2} \frac{\partial \Gamma^{(2)}}{\partial v^i} \right] = e_0^0 \bar{e}_\alpha^i \left[ B^{v(2)}_i \right] + e_\alpha^\mu e_0^\nu K^g_{\mu\nu}, \] (3.26)
\( \delta B_i^{v(2)} = \delta \beta_i^{v(2)} + \frac{1}{2} \nabla_i \delta S^0, \)

with

\[ \delta S^0 = -\frac{1}{2} B_{km} G^{km}. \]

We sum these to give

\[ e_0^0 e_\alpha^i \left[ B_i^{v'(2)} \right] = -\frac{1}{2} e_0^0 e_\alpha^i \left[ \frac{\partial}{\partial v^i} \Gamma^{v(2)} \right] + K_{0\alpha}', \]

where

\[ K_{0\alpha}' = -e_\alpha^\mu e_0^\nu \left[ X_{\mu \nu} + K_{\mu \nu} \right] \]

\[ = B_{\alpha \gamma}^{(1)} B_{\theta \gamma}^{(1)} + B_{\gamma \delta}^{(1)} e_\alpha^i e_\gamma^k e_\delta^m \left[ \frac{1}{2} \nabla_k F_{im} + \frac{1}{2} G_{kn} \dot{H}_{imn} \right] \]

\[ + B_{\gamma \delta}^{(1)} e_\alpha^i e_\gamma^k e_\delta^m \left( \frac{1}{4} F_{in} \dot{H}_{km}^n + \frac{1}{4} F_{kn} \dot{H}_{mi}^n \right) \]

\[ + B_{\gamma \delta}^{(1)} e_\alpha^i e_\gamma^k \left[ \frac{1}{4} G_{in} G_{kn} - \frac{1}{4} F_{in} F_{kn} \right] \]

\[ + B_{\gamma \delta}^{(1)} e_\alpha^i e_\gamma^k \left[ \frac{1}{4} G_{in} F_{kn} - \frac{1}{4} F_{in} G_{kn} \right] - \frac{1}{8} e_\alpha^i \nabla_i \left( B_{km}^{h*} G^{km} \right). \]

For \( w_i \) we have

\[ e_0^0 e_\alpha^i \left[ \frac{1}{2} \partial \Gamma^{(2)} \right] = -e_0^0 e_\alpha^i \left[ B_i^{w(2)} \right] + e_\alpha^\mu e_0^\nu K_{\mu \nu}^b, \]

and a correction

\[ e_0^0 e_\alpha^i \left[ \frac{1}{2} \partial \delta \Gamma^{(1)} \right] = -e_0^0 e_\alpha^i \left[ \delta B_i^{w(2)} \right] + e_\alpha^\mu e_0^\nu X_{\mu \nu}^w. \]
\[ e_0^\alpha i e_0^\nu \left[ B_{i}^{\mu(2)} \right] = -\frac{1}{2} e_0^\alpha i \left[ \frac{\partial}{\partial \omega^i} \Gamma^{\mu(2)} \right] + K_{0\alpha}^{\nu}, \quad (3.34) \]

where
\[
K_{0\alpha}^{\nu} = e_\alpha^\mu e_0^\nu \left[ X_{\mu\nu} + K_{\mu\nu} \right] \\
= B_{0\gamma}^{\rho} B_{0\gamma}^{\sigma} B_{0\gamma}^{\rho} + e_\alpha^i e_\gamma^j e_\delta^m \left[ \frac{1}{2} \nabla_k G_{im} + \frac{1}{2} F_{kn} \hat{H}_m^{im} \right] \\
+ B_{\gamma\delta}^\mu e_\alpha^i e_\gamma^j e_\delta^m \left[ \frac{1}{4} G_{\gamma\delta} H_{km}^{n} + \frac{1}{4} G_{kn} \hat{H}_m^{n} \right] \\
+ B_{0\gamma}^\mu e_\alpha^i e_\gamma^j \left[ \frac{1}{4} F_{\mu\nu} F_{kn}^{n} - \frac{1}{4} G_{\gamma\delta} G_{\mu\nu} \right] \\
+ B_{0\gamma}^\mu e_\alpha^i e_\gamma^j \left[ \frac{1}{4} F_{\mu\nu} G_{\gamma\delta}^{n} - \frac{1}{4} G_{\gamma\delta} F_{\mu\nu}^{n} \right] - \frac{1}{8} e_\alpha^i \nabla_i \left( B_{0\gamma}^{\mu\nu} F^{\nu} \right). \quad (3.35) \]

It is readily apparent, given the dual nature of the equations of motion and \( K_{0\alpha}^{\nu} \leftrightarrow K_{0\alpha}^{\nu} \), that \( B_{0\alpha}^{\nu(2)} \leftrightarrow B_{0\alpha}^{\nu(2)} \).

We now consider \( B_{0\beta}^{\nu(2)} \). The equation of motion of \( \Gamma^{(2)} \) gives
\[ \frac{1}{2} \epsilon_\alpha^i \epsilon_\beta^j \frac{\partial \Gamma_{ij}^{(2)}}{\partial b_{ij}} = -\epsilon_\alpha^i \epsilon_\beta^j B_{ij}^{\nu(2)} + e_\alpha^\mu e_\beta^\nu K_{\mu\nu}. \quad (3.36) \]

Following from the previous calculations, one might expect our next step is to write an equation containing \( \delta B_{ij}^{\nu} \). However this is not well defined since as already stated in (2.28), there is no \( b_{ij}^{*} \) such that one can compute \( \mu \frac{\partial}{\partial \mu} b_{ij}^{*} \) to obtain \( \beta_{ij}^{\nu} \). Hence we cannot write down a \( \delta b_{ij}^{*} \). But we still have
\[ \frac{1}{2} \epsilon_\alpha^i \epsilon_\beta^j \frac{\partial \delta \Gamma_{ij}^{(1)}}{\partial b_{ij}} = -e_\alpha^\mu e_\beta^\nu \delta B_{\mu\nu}^{\nu(2)} + e_\alpha^\mu e_\beta^\nu X_{\mu\nu}^{\nu}. \quad (3.37) \]

The details concerning the computation of \( e_\alpha^\mu e_\beta^\nu X_{\mu\nu}^{\nu} \) are given in the appendix. So we now sum (3.36) and (3.37) to give
\[ B_{\alpha\beta}^{\nu(2)} = -\epsilon_\alpha^i \epsilon_\beta^j \frac{\partial \Gamma_{ij}^{(2)}}{\partial b_{ij}} + K_{\alpha\beta}^{\nu}, \quad (3.38) \]

where
\[ B_{\alpha\beta}^{\nu(2)} = \epsilon_\alpha^i \epsilon_\beta^j \left[ B_{ij}^{\nu(2)} - \frac{1}{2} G_{ij} \delta S^{0} + \Lambda_{ij} \right] + e_\alpha^\mu e_\beta^\nu \delta B_{\mu\nu}^{\nu(2)}. \quad (3.39) \]
and

\[ K^{b*} = e^\mu_\alpha e^\nu_\beta \left[ X^{b*}_{\mu\nu} + K^{b}_{\mu\nu} \right] \]

\[ = B^b_{\alpha\gamma} B^g_{\beta\gamma} - B^g_{\alpha\gamma} B^b_{\beta\gamma} + B^g_{\alpha\beta} e_\gamma^i e_\delta^j \left[ F_{inn} G^{n}_{jn} - G_{inn} F^{n}_{jn} \right] \]

\[ + \frac{1}{4} G_{ik} G_{jm} + \frac{1}{4} F_{ik} F_{jm} + \frac{1}{8} F_{ij} F_{km} + \frac{1}{8} G_{ij} G_{km} \]

where \( \delta S^0 \) is as given in (3.31). \( \Lambda_{ij} \) corresponds to the gauge freedom of the torsion tensor,

\[ \Lambda_{ij} = \frac{1}{4} \nabla_i \left[ G^{k}_j k^{\nu(1)}_k - F^{k}_j k^{w(1)}_k \right] - i \leftrightarrow j. \]

Although \( b_{ij} \mapsto \tilde{b}_{ij} \), the equations of motion for these two fields are the same. Given that the \( K \)-matrix (3.40) is also invariant we conclude that \( B^{b(2)}_{\alpha\beta} \) is invariant too.

**IV. CONCLUSIONS**

We have shown that the conformal invariance conditions are invariant for a model with non-vanishing torsion, but where we have set the reduced dilaton \( \Phi \) to zero, and taken \( g_{00} = 1 \). Since the fields require to be redefined, the conformal invariance conditions refer to a redefined renormalisation scheme. Alternatively, as was done in Ref. [8], one could leave the fields, and hence the renormalisation scheme, unchanged, but instead modify the duality transformations. The required consistency conditions for the conformal invariance conditions would then no longer be the simple ones we use, as given for instance in (2.22); but our results of course guarantee that these new consistency conditions will be satisfied through two loops.
Our results are complementary to those of Ref. [4], where the torsion was zero but a non-vanishing dilaton and non-constant $a$ were used. We believe our results display the main features of the general calculation and a non-vanishing dilaton and non-constant $a$ could be incorporated into our results without changing their basic form. As a consequence of the indirect method of calculation, we have not explicitly computed the various conformal invariance conditions; but if desired they could be obtained quite straightforwardly from our final results using the explicit expression for $\Gamma^{(2)}$ given in Ref. [8]—for instance, $B^g_{ij}^{(2)}$ could be obtained from Eqs. (3.18), (3.19).

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APPENDIX A: EQUATIONS OF MOTION

First we give $b^{\mu\nu}$

$$b^{0i} = \frac{1}{a} w^i - v^k \tilde{b}_k^i, \quad b^{ij} = \tilde{b}^{ij}.$$  \hspace{1cm} (A1)

We have the following relations, with $\Gamma$ as the reduced action

$$\left. \frac{\partial \Gamma}{\partial g^{ij}} \right|_{b^{\mu\nu}, v_i, a, \Phi} = \frac{\partial \Gamma}{\partial g^{kl}} \frac{\partial g^{kl}}{\partial g^{ij}} + \frac{\partial \Gamma}{\partial g^{k0}} \frac{\partial g^{k0}}{\partial g^{ij}} + \frac{\partial \Gamma}{\partial g^{00}} \frac{\partial g^{00}}{\partial g^{ij}}$$

$$= \frac{\partial \Gamma}{\partial g^{ij}} - \frac{\partial \Gamma}{\partial g^{i0}} v_j - \frac{\partial \Gamma}{\partial g^{0j}} v_i + v_i v_j \frac{\partial \Gamma}{\partial g^{00}} \hspace{1cm} (A2)$$

$$\left. \frac{\partial \Gamma}{\partial v^i} \right|_{b^{\mu\nu}, \tilde{g}^{ij}, a, \Phi} = \frac{\partial \Gamma}{\partial g^{kl}} \frac{\partial g^{kl}}{\partial v^i} + \frac{\partial \Gamma}{\partial g^{k0}} \frac{\partial g^{k0}}{\partial v^i} + \frac{\partial \Gamma}{\partial g^{00}} \frac{\partial g^{00}}{\partial v^i}$$

$$= - \frac{\partial \Gamma}{\partial g^{00}} + v_i \frac{\partial \Gamma}{\partial g^{00}} \hspace{1cm} (A3)$$
\[ \frac{\partial \Gamma}{\partial a} \bigg|_{g_{\mu\nu}, \bar{b}_{ij}, \Phi} = \frac{\partial \Gamma}{\partial g^{kl}} \frac{\partial g^{kl}}{\partial a} + \frac{\partial \Gamma}{\partial g^{k0}} \frac{\partial g^{k0}}{\partial a} + \frac{\partial \Gamma}{\partial g^{00}} \frac{\partial g^{00}}{\partial a} \\
= -\frac{1}{a^2} \frac{\partial \Gamma}{\partial g^{00}} \]  

(A4)

\[ \frac{\partial \Gamma}{\partial w^i} \bigg|_{g_{\mu\nu}, \bar{b}_{ij}, \Phi} = \frac{\partial \Gamma}{\partial b_{km}} \frac{\partial b_{km}}{\partial w^i} + \frac{\partial \Gamma}{\partial b_{0i}} \frac{\partial b_{0i}}{\partial w^i} \]

(A5)

In the calculation of \( \frac{\partial \delta \Gamma^{(1)}}{\partial \beta^i} \) the following identities were used

\[ \nabla_k \beta^{(1)} \bigg|_{a=1} = -\frac{1}{2} G^{km} \beta_{km}^{(1)}, \quad \nabla_k \beta^{(1)} \bigg|_{a=1} = -\frac{1}{2} F^{km} \beta_{km}^{(1)}. \]  

(A6)

**APPENDIX B: VARIATIONS OF ANOMALY COEFFICIENTS**

We now explain how to apply (3.12) in the context of this work. First for ease of reference we quote from Ref. \[8\] all the relevant shifts in the reduced fields

\[ \delta a = aa_i a^i + \frac{1}{8} a^2 F^{ij} F_{ij} + \frac{1}{8} G^{ij} G_{ij}, \]  

(B1)

\[ \delta v^i = -\frac{1}{4} F^k_i a_k - \frac{1}{8a} \hat{H}_{ikm} G^{km}, \]  

(B2)

\[ \delta w_i = -\frac{1}{4} G^k_i a_k + \frac{1}{8} a \hat{H}_{ikm} F^{km}, \]  

(B3)

\[ \delta \hat{H}_{ijk} = -\frac{3}{2} \nabla_i \left( G_j^m F_k^m \right) + 3 F_{ij} \delta w_k + 3 G_{ij} \delta v_k. \]  

(B4)

\[ \delta b_{ij} = \frac{1}{4} \left( G_{kj} F^k_i - G_{ki} F^k_j \right) - v_j \delta w_i + v_i \delta w_j \]  

(B5)

The variation for the metric B-function in tangent space is
\[ \delta B_{ab} = e_a^\mu e_b^\nu \delta B_{\mu\nu} \]
\[ = e_a^\mu e_b^\nu \left[ \delta \lambda, \frac{\partial}{\partial \lambda} \beta_{\mu\nu}^a - \mu \frac{d}{d\mu} \delta g_{\mu\nu} \right] \] (B6)

The corresponding equation for torsion follows directly. We now outline the manipulation of these equations required to express them in terms of variations of \( B \)-functions for reduced fields.

The results for \( B_{00}^g \) and \( B_{0a}^g \) are achieved with ease since \( e_0^i = 0 \). We have

\[ \delta B_{00}^g = e_0^0 e_0^0 \delta B_{00}^g \]
\[ = \frac{1}{a} \delta B^a = \frac{1}{a} \left( \delta \lambda, \frac{\partial}{\partial \lambda} \beta^a - \mu \frac{d}{d\mu} \delta a \right), \] (B7)

and

\[ \delta B_{0a}^g = e_0^0 \bar{e}_a^i \delta B_i^w \]
\[ = \bar{e}_a^i \frac{1}{\sqrt{a}} \left( \delta \lambda, \frac{\partial}{\partial \lambda} \beta_i^w - \frac{d}{d\mu} \delta w_i \right) \] (B8)

The manipulations for \( B_{\alpha\beta}^g \) and \( B_{0a}^g \) are similar in style to each other. We now illustrate the simpler latter case. It follows from (B6) that

\[ \delta B_{0a}^g = \frac{1}{\sqrt{a}} \bar{e}_a^i (\delta \beta_{\alpha}^w - v_i \delta \beta^a). \] (B9)

Since \( g_{oi} = av_i \) we can compute

\[ \delta g_{oi} = v_i \delta a + a \delta v_i, \] (B10)

and

\[ \beta_{0i}^g = \beta^a v_i + a \beta_i^w. \] (B11)

Hence we find

\[ \delta \beta_{0i}^g = v_i \delta \beta^a + a \delta \beta_i^w, \] (B12)
which upon substitution in (B.9) finally gives us

$$\delta B^g_{\alpha} = \bar{e}^i_{\alpha} \sqrt{a} \delta B^v_i.$$  

(B13)

Finally we illustrate the case for $B^b_{\alpha\beta}$. We need to manipulate the expression for the change of the B-function to calculate $e^{\mu}_\alpha e^{\nu}_\beta X^b_{\mu\nu}$ in (3.37). We have

$$\delta B^b_{\alpha\beta} = e^{\mu}_\alpha e^{\nu}_\beta \left( \delta \lambda \frac{\partial}{\partial \lambda} \beta^b_{\mu\nu} - \mu \frac{d}{d\mu} \delta b_{\mu\nu} \right)$$  

(B14)

We write out explicitly the first part of the right hand side of (B14)

$$e^{\mu}_\alpha e^{\nu}_\beta \delta \lambda \frac{\partial}{\partial \lambda} \beta^b_{\mu\nu} = \bar{e}^i_{\alpha} \bar{e}^j_{\beta} \left( \delta \lambda \frac{\partial}{\partial \lambda} \beta^b_{ij} - v_i \delta \lambda \frac{\partial}{\partial \lambda} \beta^w_{j} + v_j \delta \lambda \frac{\partial}{\partial \lambda} \beta^w_{i} \right).$$  

(B15)

Given the relation $B^b_{ij} = B^b_{ij} - v_i B^w_j + v_j B^w_i$ that we defined in (2.28), we can write (B15) in the following form

$$e^{\mu}_\alpha e^{\nu}_\beta \delta \lambda \frac{\partial}{\partial \lambda} \beta^b_{\mu\nu} = \bar{e}^i_{\alpha} \bar{e}^j_{\beta} \left( \delta \lambda \frac{\partial}{\partial \lambda} \beta^b_{ij} + \beta^w_{j} \delta v_i - \beta^w_{i} \delta v_j \right)$$  

(B16)

where $\beta^b_{ij}^{(1)}$ may be read off from (2.33). The approach for calculating the remaining part of (B14) is straightforward and hence one can compute $e^{\mu}_\alpha e^{\nu}_\beta X^b_{\mu\nu}$. 

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(B16)

where $\beta^b_{ij}^{(1)}$ may be read off from (2.33). The approach for calculating the remaining part of (B14) is straightforward and hence one can compute $e^{\mu}_\alpha e^{\nu}_\beta X^b_{\mu\nu}$.
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