CONSISTENCY OF SEMICLASSICAL GRAVITY

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Abstract

We discuss some subtleties which arise in the semiclassical approximation to quantum gravity. We show that integrability conditions prevent the existence of Tomonaga-Schwinger time functions on the space of three-metrics but admit them on superspace. The concept of semiclassical time is carefully examined. We point out that central charges in the matter sector spoil the consistency of the semiclassical approximation unless the full quantum theory of gravity and matter is anomaly-free. We finally discuss consequences of these considerations for quantum field theory in flat spacetime, but with arbitrary foliations.

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1 Introduction

Although a theory of quantum gravity is still elusive, formal schemes have been developed, which are thought to exhibit some of the important aspects of the full theory. One of these schemes is canonical quantisation of general relativity and its central constraint equation $H\Psi = 0$. Although the special form of the Hamiltonian might well differ from that of the fundamental theory, one would expect the constraint nature of this equation to remain true, since this is a general feature of quantum theories which are reparametrisation invariant at the classical level.

Given such a “model theory”, it is of central importance to recover from the timeless form of the full constraint equation the limit where spacetime can be considered as a classical object. There is general agreement on the fact that a kind of Born-Oppenheimer approximation with respect to the Planck scale plays a crucial rôle in this derivation (see [1] for a review). In particular, one recovers an approximate equation which is interpreted as a Tomonaga-Schwinger equation with respect to a “many-fingered time variable” $\tau(x)$ which controls the dynamical behaviour of non-gravitational fields [1, 2].

The main purpose of the present paper is a critical investigation into the meaning of this many-fingered time with regard to several aspects. First, its very existence crucially depends on some integrability conditions which, as it turns out, are not fulfilled on the space of three-metrics, $\text{Riem}\Sigma$ (where this time variable is usually written down), but only on the space of three-geometries (“superspace”). Second, the possible presence of anomalies in the non-gravitational sector prevents its existence even on superspace unless the full quantum theory of gravity and matter is anomaly-free. The consistency of semiclassical gravity is thus already tight to the consistency of full quantum gravity. Third, we shall show that semiclassical gravity can only be interpreted in a sensible way if one considers, for any given foliation of spacetime, a global Schrödinger equation with a global time parameter and that it is very likely that one obtains unitarily non-equivalent quantum theories for different foliations.

Our paper is organised as follows. Section 2 starts with a brief review of the semiclassical expansion. We then discuss in detail the integrability conditions for the existence of a time function. We show that these conditions close on the diffeomorphism constraints and thus prevent the existence of this time function on $\text{Riem}\Sigma$, but at the same time guarantee its existence on superspace. The equations are explicitly spelled out for the case of a scalar field.

In section 3 we point out that anomalies in the commutator between the matter Hamiltonian and the matter momentum might spoil the integrability conditions even on superspace unless full quantum gravity is anomaly-free. We illustrate this point by a two-dimensional example, the Virasoro algebra, which is well known from string theory.

In the last section we present a brief summary and discuss the general meaning
of our results for a diffeomorphism invariant quantum theory.

# The rôle of the diffeomorphism constraints in the semiclassical expansion

The basic idea in the semiclassical expansion is to start from the full Hamiltonian constraint equation in quantum general relativity and to perform an expansion in powers of the gravitational constant. In the geometrodynamical approach, in particular, the starting point is the Wheeler-DeWitt equation (we set $c = 1$)

$$\mathcal{H}_x \Psi[h_{ab}, \phi] \equiv \left( -16\pi G h^2 G_{abcd} \frac{\delta^2}{\delta h_{ab}\delta h_{cd}} - \frac{\sqrt{h}}{16\pi G} (R - 2\Lambda) + \mathcal{H}_m \right) \Psi = 0,$$

(1)

where $h_{ab}$ is the three-metric, $R$ the three-dimensional Ricci scalar, $\Lambda$ the cosmological constant, $G_{abcd}$ the contravariant DeWitt metric, and $\mathcal{H}_m$ is the Hamiltonian density for non-gravitational fields. We emphasise that (1) is actually an infinity of equations, one at each space point. More precisely, they should be understood to be distributional equations which have to be integrated against “lapse functions”, $N(x)$, from a suitably chosen space of test functions. This distinction is important when one considers open three-manifolds, since then surface terms appear and one can no longer write the equation in the local form (1). In this paper, however, we restrict attention to the closed case. The integrated form of (1) reads

$$\int d^3 x \ N \mathcal{H}_x \Psi \equiv \mathcal{H}_x^N \Psi \equiv (\mathcal{H}_x^N + \mathcal{H}_m^N) \Psi = 0,$$

(2)

where we have decomposed the full Hamiltonian into its gravitational and matter parts, respectively. The ansatz

$$\Psi = \exp \left( i(M S_0 + S_1 + M^{-1} S_2 + \ldots) / \hbar \right),$$

(3)

where $M \equiv (32\pi G)^{-1}$, then leads, when inserted into (1), to a set of equations at consecutive orders in $M$.

The highest order, $M^2$, yields that $S_0$ must only depend on the three-metric. The next order, $M$, gives the Hamilton-Jacobi equation for the gravitational field,

$$\mathcal{H}_x = \frac{1}{2} G_{abcd} \frac{\delta S_0}{\delta h_{ab}} \frac{\delta S_0}{\delta h_{cd}} - 2\sqrt{h} (R - 2\Lambda) = 0.$$

(4)

Again, these are infinitely many equations and should be interpreted in an integrated version as providing one equation for each lapse function $N(x)$ out of the specified class of test functions. The Hamilton-Jacobi equations (4) are, together with the momentum constraints in this order (see below), equivalent to all Einstein field equations, and a family of classical spacetimes can be constructed from a solution $S_0$ to the infinitely many equations (4).
At the next order, $M^0$, it turns out to be convenient to introduce a functional
\[ \psi \equiv D[h_{ab}] \exp(iS_1/\hbar) \] (5)
and demand that $D$ obeys
\[ G_{abcd} \frac{\delta S_0}{\delta h_{ab}} \frac{\delta D}{\delta h_{cd}} - \frac{1}{2} G_{abcd} \frac{\delta^2 S_0}{\delta h_{ab} \delta h_{cd}} D = 0. \] (6)
This corresponds to the usual equation for the WKB prefactor. The important observation at this order is that $\psi$ obeys the equation
\[ i\hbar G_{abcd} \frac{\delta S_0}{\delta h_{ab}} \frac{\delta \psi}{\delta h_{cd}} = H_m \psi. \] (7)
The left-hand side has the interpretation of being $i\hbar$ times the derivative along the vector fields (one vector field for each space point)
\[ \chi(x) = G_{abcd}(x) \frac{\delta S_0}{\delta h_{ab}(x)} \frac{\delta}{\delta h_{cd}(x)} = -2K_{ab}(x) \frac{\delta}{\delta h_{ab}(x)}. \] (8)
where
\[ K_{cd} = -\frac{1}{2} G_{abcd} \frac{\delta S_0}{\delta h_{ab}}, \]
which in the classical spacetime constructed from $S_0$ has the interpretation of the extrinsic curvature. Note, however, that $K_{ab}$ is here considered as a functional of the three-metric.

If we were allowed to write
\[ \chi(x) = \frac{\delta}{\delta \tau(x)}, \]
Eq. (8) would just be a Tomonaga-Schwinger equation with respect to the many-fingered time functionals $\tau(x)$. Note that $\tau$ is really a time functional on $\text{Riem}_\Sigma$ for each $x$. It would thus be more precise to write, as is often done, $\tau[h_{ab}, x]$, but we shall not use this notation explicitly.

However, if we used such time functionals in (8), this would lead us immediately to a contradiction: The commutator $[\delta/\delta \tau(x), \delta/\delta \tau(y)]$ necessarily vanishes but the commutator $[H_m(x), H_m(y)]$ arising from the right-hand side does not vanish – it closes on the momentum density of the matter field. We emphasise that this problem is different from the “global time problem” [3] which prevents the global existence of such a time variable in configuration space.

To simplify the discussion we write (7) in the form
\[ i\hbar \chi^N \psi = H^N_m \psi, \] (9)
where we have introduced the vector fields $\chi^N$ (giving one vector field for each choice of lapse) by integrating (8) with respect to $N$, and $H^N_m$ was defined above,
see Eq. (4). It is clear that $[\chi^N, \chi^M] = 0$ is a necessary condition for the local existence of time functionals $\tau^N$ such that $\delta/\delta \tau^N = \chi^N$.

It is straightforward to calculate the commutator

$$[\chi^N, \chi^M] = \delta S_0 / \delta h_{ab}(x) \delta h_{cd}(x),$$

$$\int_y M(y)G_{nmrs}(y) \delta S_0 / \delta h_{nm}(y) \delta h_{rs}(y),$$

$$\int_x \int_y (N(x)M(y) - M(x)N(y))G_{abcd}(x) \delta S_0 / \delta h_{cd}(x) \times$$

$$\frac{\delta^2 S_0}{\delta h_{rs}(y) \delta h_{ab}(x)} \int_z \delta(y, z)G_{rsnm}(z) \frac{\delta}{\delta h_{nm}(z)},$$

(10)

which vanishes only if

$$\int_x (N(x)M(y) - M(x)N(y))G_{abcd}(x) \delta S_0 / \delta h_{cd}(x) \times$$

$$\frac{\delta^2 S_0}{\delta h_{rs}(y) \delta h_{ab}(x)} \int_z \delta(y, z)G_{rsnm}(z) \frac{\delta}{\delta h_{nm}(z)},$$

vanishes for all $N$ and $M$. This is equivalent to the condition

$$\frac{\delta^2 S_0}{\delta h_{ab}(x) \delta h_{cd}(y)} \propto \delta(x, y).$$

(11)

We shall, however, show that this cannot occur. Differentiating the Hamilton-Jacobi equations (4) with respect to $h_{ab}(y)$, one obtains

$$\frac{\delta H_x}{\delta h_{ab}(y)} + \int_z \frac{\delta H_x}{\delta \pi^{cd}(z) \delta h_{cd}(z) \delta h_{ab}(y)} = 0,$$

(12)

where we have written $\pi^{cd}(z) = \delta S_0 / \delta h_{cd}(z)$ (this is $32\pi G$ times the classical geometrodynamical momentum). The first term in (12) captures the explicit dependence of $H_x$ on the three-metric, while the second term takes into account the implicit dependence through the replacement $\pi^{ab} \to \delta S_0 / \delta h_{ab}$.

From (4) and (12) one has

$$\frac{\delta H_x}{\delta h_{ab}(y)} = -G_{cdnm}(x) \delta S_0 / \delta h_{nm}(x) \delta h_{cd}(x) \delta h_{ab}(y).$$

(13)

The derivative on the left-hand side is given by the expression

$$\frac{\delta H_x}{\delta h_{ab}(y)} = 2G^{abcd}(y)\delta_{cd}(x, y) + F^{ab}(x)\delta(x, y),$$

(14)

where the explicit form of the second term will be irrelevant due to its ultralocal form (no derivatives of the delta function). We note that the second derivative
of the delta function occurs from differentiating the Ricci scalar. Inserting (13) and (14) into (10) then yields

\[
\left[ \chi^N, \chi^M \right] = -2 \int_x \int_y (N(x)M(y) - M(x)N(y)) \delta_{ab}(x, y) \frac{\delta}{\delta h_{ab}(y)},
\]

which after some partial integrations can be written as

\[
\left[ \chi^N, \chi^M \right] = -2 \int_x (N \partial_a M - M \partial_a N) \left( \frac{\delta}{\delta h_{ab}} \right) \mid_b = \int \mathcal{L}_K h_{ab} \frac{\delta}{\delta h_{ab}},
\]

where

\[ K^a = h_{ab} (N \partial_b M - M \partial_b N). \]

Thus, \( \left[ \chi^N, \chi^M \right] \neq 0 \), and time functions in the above sense can never be introduced. Formally, this happens because the Ricci scalar \( R \) is not ultralocal in \( h_{ab} \), which leads to the occurrence of the second derivative of the \( \delta \) distribution in (14). It is, however, not surprising that the commutator closes on the generator of a diffeomorphism – the vector fields \( \chi^N \) which appear in the semiclassical approximation are the generators of a hypersurface deformation normal to itself, whose commutator is known to generate deformations tangential (“stretchings”) to the hypersurface \( \mathbb{H} \).

In the De Sitter example discussed in [1] it was found that \( S_0 \propto \int d^3x \sqrt{h} \) which obeys relation (11). This is not in contradiction with the discussion above, where we assumed \( S_0 \) to be a solution to the full Hamilton-Jacobi equation. In the de Sitter example the expression for \( S_0 \) only solves the Hamilton-Jacobi equation on the submanifold of Riem\( \Sigma \) where the Ricci scalar vanishes. For De Sitter space this can always be achieved by a specific foliation.

A proper understanding of (16) and its compatibility with the semiclassical equations (7) is obtained if one expands, in addition to the Wheeler-DeWitt equation (1), the diffeomorphism constraints in powers of \( G \). These are given by

\[
2h_{bc} D_a \frac{\delta \Psi}{\delta h_{ab}} = \phi \frac{\delta \Psi}{\delta \phi},
\]

where we have, for simplicity, chosen a single scalar field for the non-gravitational sector. We emphasise that the gravitational constant is absent from this equation, which renders its expansion fairly trivial.

The highest order, \( M \), yields (since \( S_0 \) does not depend on \( \phi \))

\[
2h_{bc} D_a \frac{\delta S_0}{\delta h_{ab}} = 0.
\]

This expresses nothing but the diffeomorphism invariance of the solutions, \( S_0 \), to the Hamilton-Jacobi equations (10).
The next order, $M^0$, leads to a condition on the functional $\psi$, Eq. (19),

$$2h_{bc}D_a \left( \frac{\delta \psi}{\delta h_{ab}} - \frac{\psi}{D} \frac{\delta D}{\delta h_{ab}} \right) = \phi_{,c} \frac{\delta \psi}{\delta \phi}.$$  

(19)

Since the prefactor $D$ depends only on the three-metric, see (8), it is appropriate to demand that it be diffeomorphism invariant by itself, i.e.,

$$h_{bc}D_a \frac{\delta D}{\delta h_{ab}} = 0.$$  

(20)

From (19) one thus finds

$$2h_{bc}D_a \frac{\delta \psi}{\delta h_{ab}} = \phi_{,c} \frac{\delta \psi}{\delta \phi},$$  

(21)

which is of the same form as the general equation (17). Thus, it expresses the invariance of the wave functional $\psi[h_{ab}, \phi]$ with respect to simultaneous diffeomorphism of the metric and the matter field. The consistency condition for (9) reads

$$[\chi^N, \chi^M] \psi = [H^M_m, H^N_m] \psi.$$  

(22)

This, however, is nothing but the momentum constraint in this order of approximation, Eq. (21), since $[\chi^N, \chi^M]$ generates a diffeomorphism of the metric, Eq. (14), and $[H^M_m, H^N_m]$ closes on the momentum density of matter which generates a diffeomorphism of the matter field. Thus, as in the full theory (3), the momentum constraints provide the integrability conditions for the Tomonaga-Schwinger equations (3).

In the explicit case of a scalar field one has, for example,

$$[H^M_m, H^N_m] = -\int \left( N \partial_a M - M \partial_a N \right) h^{ab} \phi_{,b} \frac{\delta}{\delta \phi},$$  

(23)

which, together with (14), yields (21).

Although a family of time functions $\tau(x)$ on Riem$\Sigma$ does not exist, one can integrate (11) along the vector field $\chi^N$ for one particular choice of lapse $N$. This defines a global time parameter $t$ with respect to which one global Schrödinger equation can be written down. It is in this sense that quantum field theory with respect to a chosen foliation emerges from full quantum gravity. Such a picture has been implicitly used by many authors although (non-existing) time functions $\tau(x)$ have been used (11). One may then proceed to the next order of approximation and derive correction terms to the functional Schrödinger equation (11, 3). Again, these terms have to be interpreted with respect to a particular foliation.

We emphasise that the same criticism applies to the standard Tomonaga-Schwinger equations in flat spacetime since the matter Hamiltonians always close on the momentum density. One should thus be careful and consider these equations only in the context of a parametrised formalism where all possible embeddings of a spacelike hypersurface into spacetime are allowed. This leads to the
same interpretation as above. One would, however, not expect that the resulting quantum theories are foliation independent.

If there are no such time functions on Riem$\Sigma$, can there be such functions on superspace $S(\Sigma) \equiv \text{Riem}\Sigma/\text{Diff}\Sigma$? To answer this question we must project the vector fields $\chi^N$ down on superspace. This is possible since $\chi^N$ is invariant under diffeomorphisms – a property which it inherits from the diffeomorphism invariance of $S_0$ and $\int N(x)G_{abcd}(x,x')$. Moreover, $\chi^N$ is (with respect to DeWitt’s metric) orthogonal to the diffeomorphism orbits and thus can be projected onto non-vanishing vector fields $\bar{\chi}^N$ in specifiable regions of superspace $[7]$: $\chi^N \mapsto \bar{\chi}^N$.

One thus has

$$\pi_*[\chi^N, \chi^M] = [\pi_*\chi^N, \pi_*\chi^M] = [\bar{\chi}^N, \bar{\chi}^M] = 0.$$ 

Thus, there exist functions $\bar{\tau}^N$ on superspace such that

$$\bar{\chi}^N \equiv \delta \frac{\delta }{\delta \bar{\tau}^N}.$$ 

Since the Wheeler-DeWitt operator is, however, only defined on Riem$\Sigma$ and not on $S(\Sigma)$ (an object such as the second derivative with respect to three-geometry does not exist globally), the physical interpretation of these time functions is exhibited only implicitly by first performing the calculations on Riem$\Sigma$ and then projecting on superspace.

We finally note that the same discussion applies to the semiclassical approximation of connection dynamics $[1]$. One must, however, take care for the fact that the commutator between two Hamiltonians does not close on the diffeomorphism constraint, but contains an additional contribution from the gauge sector.

### 3 Anomalies

The “smeared out version” of the momentum constraints $[24]$ reads

$$\mathcal{H}_G^N \psi = \mathcal{H}_m^N \psi,$$

where $\mathcal{N}(x)$ denotes the shift vector field. This implies, together with $[3]$, the consistency condition

$$i\hbar [\chi^N, \mathcal{H}_G^N] \psi = [\mathcal{H}_m^N, \mathcal{H}_m^N] \psi.$$ 

Classically, the commutator on the right-hand side closes again on the matter Hamiltonian density $\mathcal{H}_m^M$, where $M = \mathcal{L}_\mathcal{N} N$. In the quantum theory, however, one knows that – under certain assumptions (see below) – the commutator on the right-hand side necessarily leads to central extensions (Schwinger terms, anomalies) $[8]$. It thus seems that the semiclassical expansion is inconsistent unless this anomaly is cancelled by a similar anomaly occurring on the left-hand side. It is, however, straightforward to show that this cannot happen. Basically, the reason is that $\chi^N$ is only the classical generator of hypersurface deformations and
thus contains, as well as $\mathcal{H}^N_m$, only single derivatives with respect to the metric. Therefore, the left-hand side of (25) yields the classical result and closes on a hypersurface deformation:

$$[\chi^N, \mathcal{H}^N_G] = \left[ \int_x N G_{abcd} \frac{\delta S_0}{\delta h_{ab}} \frac{\delta}{\delta h_{cd}}, -2 \int_y N^k h_{lk} D_m \frac{\delta}{\delta h_{ml}} \right] = \int_x M(x) G_{abcd} \frac{\delta S_0}{\delta h_{ab}} \frac{\delta}{\delta h_{cd}} = \chi^M,$$

(26)

where $M = \mathcal{L}_N N$. In the absence of an anomaly the condition (25) would thus again lead to (9).

At this point we recall that the necessary occurrence of anomalies was only shown under the following conditions [8]: (1) Locality, (2) the existence of a ground state for $\mathcal{H}_m$ (with $N = \text{constant}$), Lorentz invariance, and the existence of a positive definite Hilbert space product. But of these assumptions, Lorentz invariance is definitely an inappropriate assumption from the viewpoint of full quantum gravity, while the existence of a ground state is at least dubious. If the regularisation of the full theory can be performed in a diffeomorphism invariant manner (otherwise one would interpret the full theory as being inconsistent [9]), anomalies should be absent in the full theory and, consequently, also in the semiclassical consistency equation (25). If one nevertheless insists on a Lorentz covariant regularisation of the right-hand side of (25), anomalies appear, reflecting the inappropriateness of the regularisation procedure as viewed from the fundamental theory. Under the assumptions made in [8], the anomaly must be proportional to $\delta'''(x-y)$, where the constant of proportionality is given, in three space dimensions, by an inverse length squared, exhibiting an ultraviolet singularity as one approaches $L \to 0$. This anomaly would then have to be cancelled by a similar anomaly coming from the full gravitational commutator $[\mathcal{H}^N_G, \mathcal{H}^N_G]$. The latter must, by dimensional arguments, be proportional to $G^{-1} \delta'''(x-y)$ (recall that the gravitational constant, $G$, already appears, in contrast to the classical case, in pure quantum gravity). In other words, $G$ would then play the rôle of a “regulator” for ultraviolet divergencies. The total, regularised, commutator

$$[\mathcal{H}^N_G, \mathcal{H}^N_G]_{\text{reg}} \Psi = [\mathcal{H}^N_m, \mathcal{H}^N_m]_{\text{reg}} \Psi$$

would then exhibit no anomalies, and (25) would be consistent.

While little is known in four spacetime dimensions, the situation in two dimensions has been studied extensively, mostly in the context of string theories [10]. There, the central extension in $[\mathcal{H}^N_m, \mathcal{H}^N_m]$ is given by the central charge of the Virasoro algebra. Its appearance is usually found from a normal ordering prescription for the Hamiltonian. This, of course, does not mean that two-dimensional quantum gravity is inconsistent. The full theory can be regularised in a diffeomorphism invariant way but not in a way which respects both Weyl invariance and conformal invariance (this happens only in the critical string dimension). If
\[ \Gamma[g_{\mu\nu}] \] denotes the effective action which arises from integrating out matter in the full action \( S[g_{\mu\nu}, \phi] \), one has (see, e.g., [11])

\[ 0 = \delta\Gamma_{\text{Diff}} = \delta\Gamma_{\text{conf}} - \delta\Gamma_{\text{Weyl}}, \tag{27} \]

where the quantities on the right-hand side denote the change of the effective action with respect to conformal transformations and Weyl transformations, respectively. Both changes are, of course, related to the central charge, and the appearance of a Schwinger term in the Virasoro algebra is, in a sense, a consequence of the diffeomorphism invariance of the full theory. That the trace anomaly appears as a condition for the full Schwinger term to vanish has been explicitly studied in a two-dimensional model by Teitelboim [12].

Kuchař [13] has cast the theory of a massless free scalar field in two space-time dimensions into a parametrised form (this mimics the diffeomorphism invariance of quantum gravity). He could show that this enabled him to introduce an embedding-dependent anomaly potential which cancels the anomaly in the Dirac algebra. He argued that, provided certain topological conditions on the space of embeddings are met, the anomaly can be absorbed into a redefinition of momenta. These topological conditions are met in two dimensions, but not necessarily in four.

The two-dimensional case clearly demonstrates the dependence of anomalies on the regularisation prescription. One can, for example, consider a whole family of possible ground states with respect to which a normal ordering prescription can be performed [14]. Consequently, the obtained central charge depends on the special ground state chosen from this family. One can in this case make a choice such that the anomaly cancels against a similar anomaly from the ghost sector. This drastically illustrates how sensibly the concept of anomaly depends on the method of “quantisation”.

In summary, we have shown that the semiclassical expansion can be carried out consistently if the full theory does not possess central charges.

4 Conclusion

One of the important lessons to be drawn from the above considerations is in our opinion the fact that an approximate theory contains indications which point towards the more fundamental theory. In our case this happens through the occurrence of anomalies if a subsector of the theory is regularised with respect to approximate concepts: If the functional Schrödinger equation is approximately valid, and if the background gravitational field is very weak, it is very suggesting to take Lorentz invariance as a guiding principle for the approximate theory. Then

\footnote{The anomaly appears formally as a closed two-form on the manifold of surface embeddings. A sufficient topological condition for it to be exact is that the second cohomology vanishes (and not, as stated in [13], the first homotopy).}
anomalies necessarily occur which would lead to an inconsistency if one attempted to write down a Tomonaga-Schwinger equation. Full quantum gravity, however, is supposed to obey the more general diffeomorphism invariance which should allow a consistent regularisation of the theory and should thus be anomaly-free. Only if one imposed a restricted regularisation principle on subsectors of the theory (such as the matter fields alone) could one find anomalies which must then be cancelled by similar anomalies from the remaining part of the full theory. This is clearly illustrated in the two-dimensional theory.

Another important point concerns the interpretation of the Tomonaga-Schwinger equations in flat spacetime. As we have shown, even there one cannot introduce time functions $\tau(x)$ on the space of embeddings. One can only write down global Schrödinger equations for given foliations and a corresponding global time parameter. However, one would not expect that quantum field theories corresponding to different foliations would in general be unitarily equivalent. The reason for this expectation is the fact that particle number is not a diffeomorphism invariant quantity. If one produced, for example, a local “bump” in the hypersurfaces through a rapidly varying lapse in a local region, this would correspond to a strong gravitational field which would lead to strong particle creation and thus to a Fock space which is orthogonal to the standard Fock space referring to a flat foliation. Unitary equivalence would only be expected for restricted classes of foliations such as the ones corresponding to inertial observers.

Quite generally, symmetry groups which allow for central extensions and therefore anomalous commutators cease to do so if embedded in a larger group of symmetries. An effective reduction of the symmetries might therefore lead to spurious anomalies.

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