Convex unconditionality and summability of weakly null sequences

by

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Abstract. It is proved that every normalized weakly null sequence has a subsequence which is convexly unconditional. Further, an Hierarchy of summability methods is introduced and with this we give a complete classification of the complexity of weakly null sequences.

INTRODUCTION

In the present paper we investigate the behavior of the subsequences of a weakly null sequence \((x_n)_{n\in\mathbb{N}}\) of a Banach space \(X\) with respect to two fundamental properties. The first is the convex unconditionality which is investigated in the first section of the paper. This is defined as:

Definition:: A normalized sequence \((x_n)_{n\in\mathbb{N}}\) in a Banach space \(X\) is said to be convexly unconditional if for every \(\delta > 0\) there exists \(C(\delta) > 0\) such that if an absolutely convex combination \(x = \sum_{n=1}^{\infty} a_n x_n\) satisfies \(\|x\| > \delta\) then \(\|\sum_{n=1}^{\infty} \epsilon_n a_n x_n\| > C(\delta)\) for every choice of signs \((\epsilon_n)_{n\in\mathbb{N}}\).

The result we prove here is the following theorem.

Theorem A: If \((x_n)_{n\in\mathbb{N}}\) is a normalized weakly null sequence in a Banach space \(X\) then it has a convexly unconditional subsequence.

A fundamental example due to B. Maurey and H. Rosenthal [M–R] showed that we could not expect that every normalized weakly null sequence has an unconditional subsequence. The recent examples [G–M], [A–D] show that there are spaces without any unconditional basic sequence. On the other hand there are results where some weaker forms of unconditionality appear. One of them is due to J. Elton [E], [O]\(^1\) which is related to the unconditional behavior of the linear combinations with coefficients away from zero and the
other is due to E. Odell and it is related to the unconditionality of Schreier admissible linear combinations. Our theorem is in the same direction with Elton’s Theorem, more precisely it is the dual result, and the proof is based, as his proof, on infinite Ramsey Theorem. The result follows from a combinatorial principle (Lemma 1.2) which seems of independent interest and it is also used in the second part of the paper.

The existence of a convexly unconditional sequence is a strong evidence that the convex sets behave much better, with respect to the unconditionality than the subspaces of a Banach space.

In the second part we deal with summability methods. The starting point for our investigation is the following question.

As it follows from Mazur’s theorem, every weakly null sequence has convex combinations norm converging to zero. The general question is to describe “regular” convex combinations with this property. This problem was faced from the early days of the development of Banach space theory. Thus Banach and Saks proved that every \((x_n)_{n \in \mathbb{N}}\) bounded sequence in \(L^p(\mu), 1 < p < \infty\) has a norm Cesaro summable subsequence. This result was extended by W. Szlenk for weakly convergent sequences in \(L^1(\mu)\). Shortly after Banach-Saks Theorem, an example was given by J. Schreier [Sch] of a weakly null sequence with no norm Cesaro summable subsequence. Schreier’s example is defined as follows: First we define the following family.

\[
\mathcal{F} = \{F \in \mathbb{N} : \# F \leq \min F\}.
\]

Then on the vector space \(c_{00}(\mathbb{N})\) of eventually zero sequences we define the norm

\[
\|(a_n)_{n \in \mathbb{N}}\| = \sup \left\{ \sum_{n \in F} |a_n| : F \in \mathcal{F} \right\}.
\]

It is easy to see that \(\mathcal{F}\) is compact in the topology of pointwise convergence. Hence the standard basis \((e_n)_{n \in \mathbb{N}}\) is weakly null. Further from the definition of \(\mathcal{F}\) we get that for every

\[n_1 < n_2 < \cdots < n_k\] we have

\[
\left\| \frac{e_{n_1} + e_{n_2} + \cdots + e_{n_k}}{k} \right\| \geq \frac{1}{2}.
\]

So no subsequence of \((e_n)_{n \in \mathbb{N}}\) is norm Cesaro summable. Later it is proved by H. Rosenthal that if \((x_n)_{n \in \mathbb{N}}\) is weakly null and no subsequence is norm
Cesaro summable then there exists \((n_i)_{i \in \mathbb{N}}\) and \(\epsilon > 0\) such that
\[
\| \sum_{i \in F} a_i e_{n_i} \| > \epsilon \cdot \sum_{i \in F} |a_i|
\]
for all \(F \in \mathcal{F}\). Whenever this property appears, we say that the sequence \((x_{n_i})_{i \in \mathbb{N}}\) is an \(\ell^1\) spreading model. This result, in connection with a theorem proved by P. Erdös and M. Magidor [E-M] gives the following dichotomy.

**Theorem:** For every \((x_n)_{n \in \mathbb{N}}\) weakly null sequence exactly one of the following holds:

(a) For every \(M \in [\mathbb{N}]\) there exists \(L \in [M]\) such that for all \(P \in [L]\)
\(P = (n_i)_{i \in \mathbb{N}},\) the subsequence \((x_{n_i})_{i \in \mathbb{N}}\) is norm Cesaro summable

(b) There exists \(M \in [\mathbb{N}]\) such that \(M = (m_i)_{i \in \mathbb{N}}\) and the subsequence \((x_{m_i})_{i \in \mathbb{N}}\) is an \(\ell^1\) spreading model.

A proof of this theorem is also given in [M].

This theorem is sufficient when condition (a) appears. If (b) holds then there is no information on the structure of convex combinations that converge in norm to zero. Our aim is to give a full extension of the above theorem and through this to describe the complexity of weakly null sequences. For this we use two hierarchies, Schreier Hierarchy and the Repeated Averages Hierarchy.

**Schreier Hierarchy:** Schreier family \(\mathcal{F}\) is quite important in the theory of Banach space. Recall that it is one of the main ingredients in the definition of Tsirelson’s space [T]. D. Alspach and S. Argyros [A-A] defined a family \(\{ \mathcal{F}_\xi \}_{\xi < \omega_1}\) called generalized Schreier families. The definition of \(\mathcal{F}_\xi\) is given in the following way:

Set \(\mathcal{F}_0 = \{ \{n\} : n \in \mathbb{N}\}\) and \(\mathcal{F}_1 = \mathcal{F}\).

If \(\mathcal{F}_\xi\) has been defined then we set
\[
\mathcal{F}_{\xi+1} = \{ \bigcup_{i=1}^n F_i : n \leq F_1 < F_2 < \cdots < F_n, F_i \in \mathcal{F}_\xi \}.
\]

If \(\xi\) is a limit ordinal choose \((\xi_n)_{n \in \mathbb{N}}\) strictly increasing to \(\xi\) and we set
\[
\mathcal{F}_\xi = \{ F : F \in \mathcal{F}_{\xi_n}, n \leq \min F \}.
\]

We decided to call this family Schreier Hierarchy since it carries some strong universal properties some of which are described in the present paper. Roughly speaking, the complexity of every compact countable metric space is dominated by some member of \(\{ \mathcal{F}_\xi \}_{\xi < \omega_1}\). Further members of \(\{ \mathcal{F}_\xi \}_{\xi < \omega_1}\) appear naturally in several cases. For example, the \(n^{th}\) norm in the inductive definition of Tsirelson’s space is implicitly connected to the family \(\mathcal{F}_n\).
Explicitly the family \( (\mathcal{F}_n)_{n \in \mathbb{N}} \) appeared for the first time in an example constructed by E. Odell [A-O]. Recently \( \{ \mathcal{F}_\xi \}_{\xi < \omega_1} \) are used in the investigation of asymptotic \( \ell^p \) spaces. Connected to the family \( \{ \mathcal{F}_\xi \}_{\xi < \omega_1} \) is the following definition.

**Definition:** Let \( (x_n)_{n \in \mathbb{N}} \) be a bounded sequence in a Banach space \( X \). For \( M \in [\mathbb{N}] \), \( M = (m_i)_{i \in \mathbb{N}} \) we say that \( (x_{m_i})_{i \in \mathbb{N}} \) is an \( \ell^1 \) spreading model if there exists \( \epsilon > 0 \) such that for all choices, \( (a_i)_{i \in F} \), for \( F \in \mathcal{F}_\xi \) we have that:

\[
\| \sum_{i \in F} a_i x_{m_i} \| \geq \epsilon \cdot \sum_{i \in F} |a_i|.
\]

It is clear that \( \ell^1 \) spreading model is the usual \( \ell^1 \) spreading model. Since the families \( \{ \mathcal{F}_\xi \}_{\xi < \omega_1} \) are of increasing complexity, the existence of a subsequence which is an \( \ell^1 \) model, for large \( \xi \), describes strong \( \ell^1 \) behavior of the given sequences. As it is proved in [A-A], if a sequence contains \( \ell^1 \) spreading models for all \( \xi < \omega_1 \) then actually contains a subsequence equivalent to the usual basis of \( \ell^1 \).

The second hierarchy introduced here is that of Repeated Averages.

**Repeated Averages Hierarchy:** To introduce this we give some notations and definitions.

We denote by \( S^+_{\ell^1} \) the positive part of the unit sphere of \( \ell^1(\mathbb{N}) \) and if \( H = (x_n)_{n \in \mathbb{N}} \) is a bounded sequence in a Banach space, \( A = (a_n)_{n \in \mathbb{N}} \in S^+_{\ell^1} \) we set

\[
A \cdot H = \sum_{n=1}^{\infty} a_n x_n \in X.
\]

For \( M \in [\mathbb{N}] \) a sequence \( (A_n)_{n \in \mathbb{N}} \) of successive blocks in \( S^+_{\ell^1} \) defines an \( M \)-summability method, denoted by \( M - (A_n)_{n \in \mathbb{N}} \) if \( M = \bigcup_{n=1}^{\infty} \text{supp} A_n \).

**Definition:** A weakly null sequence \( H = (x_n)_{n \in \mathbb{N}} \) is \( M - (A_n)_{n \in \mathbb{N}} \) summable if the sequence \( (A_n \cdot H)_{n \in \mathbb{N}} \) is Cesaro summable.

The RA Hierarchy is defined, inductively, for every \( M \in [\mathbb{N}] \) and \( \xi < \omega_1 \) and it is an \( M \)-summability method denoted by \( (\xi^M_n)_{n \in \mathbb{N}} \). We also use the notation \( (M, \xi) \) for the same method. Thus the RA Hierarchy is the family

\[
\{(M, \xi) : M \in [\mathbb{N}], \xi < \omega_1\}.
\]

The precise definition is given at the beginning of the second section of the paper. A brief description of it goes as follows: For \( \xi = 0 \) and \( M = (m_n)_{n \in \mathbb{N}} \) we set \( \xi^M_n = e_{m_n} \). Thus the \( (M, \xi) \)-summability, for \( \xi = 0 \), of a weakly null
sequence \((x_n)_{n \in \mathbb{N}}\) is exactly the norm Cesaro summability of the subsequence \((x_{m_n})_{n \in \mathbb{N}}\) where \(M = (m_n)_{n \in \mathbb{N}}\).

If \((\xi^M_n)_{n \in \mathbb{N}}\) has been defined then for \(\zeta = \xi + 1\) we set \(\zeta^M_n\) to be the average of an appropriate number of successive elements of \((\xi^M_n)_{n \in \mathbb{N}}\). This justifies the term Repeated Averages. For \(\zeta\) limit ordinal \((\zeta^M_n)_{n \in \mathbb{N}}\) is constructed by a careful choice of terms of \(\{\zeta^M_n : \xi < \zeta, n \in \mathbb{N}\}\).

One property we would like to mention here is that \(\text{supp} \xi^M_n \in \mathcal{F}_\xi\) and moreover it is a maximal element of \(\mathcal{F}_\xi\). Thus \((\xi^M_n)_{n \in \mathbb{N}}\) exhausts the complexity of the family \(\mathcal{F}_\xi\). More important is that \((\xi^M_n)_{n \in \mathbb{N}}\) carries some nice stability properties (see P.3 – P.4 after the Definition) which allows us to handle them in the proofs of the theorems.

The difference between the RA Hierarchy and the summability methods described as an infinite matrix is that in RA Hierarchy the summability of a subsequence \((x_n)_{n \in \mathbb{N}}\) depends on the subset \(M\) while in the usual case, after reordering \((x_n)_{n \in \mathbb{N}}\) as \((x_n^k)_{k \in \mathbb{N}}\), we ignore the set \(M\) and apply the summability method with respect to the index \(k\). Thus in our case for fixed countable ordinal \(\xi\) we have \(2^\omega\) methods \(\{(M, \xi) : M \in [\mathbb{N}]\}\) which have uniformly bounded complexity. This is so, since for every \(M \in [\mathbb{N}], n \in \mathbb{N}\), the set \(\text{supp} \xi^M_n\) belongs to the compact family \(\mathcal{F}_\xi\).

For a given \(M \in [\mathbb{N}]\) the methods \(\{(M, \xi) : \xi < \omega_1\}\) are increasing very fast. It is worthwhile to remark that if for \(\xi < \omega_1\) and \(n \in \mathbb{N}\) we set \(k_\xi^N = \max \text{supp} \xi^N_n\) then the family \(\{(k_\xi^N) : n \in \mathbb{N}, \xi < \omega_1\}\) is the Ackerman Hierarchy, a well known hierarchy of Mathematical Logic.

**Theorem B:** For \((x_n)_{n \in \mathbb{N}}\) weakly null sequence in a Banach space \(X\) and \(\xi < \omega_1\) exactly one of the following holds.

(a) For every \(M \in [\mathbb{N}]\) there exists \(L \in [M]\) such that for every \(P \in [L]\) the sequence \((x_n)_{n \in \mathbb{N}}\) is \((P, \xi)\) summable

(b) There exists \(M \in [\mathbb{N}]\) \(M = (m_i)_{i \in \mathbb{N}}\) such that \((x_{m_i})_{i \in \mathbb{N}}\) is an \(\ell^1_{\xi+1}\) spreading model.

It is proved in [A-A] that every \((x_n)_{n \in \mathbb{N}}\) weakly null sequence there exists \(\xi < \omega_1\) such that for every \(\zeta \geq \xi\) no subsequence of \((x_n)_{n \in \mathbb{N}}\) is an \(\ell^1_{\xi}\) spreading model. So we introduce the Banach-Saks index of a weakly null sequence defined as

\[
\text{BS}[(x_n)_{n \in \mathbb{N}}] = \min\{\xi : \text{no subsequence of } (x_n)_{n \in \mathbb{N}} \text{ is an } \ell^1_{\xi} \text{ spreading model}\}
\]
and from Theorem B we get the following

**Theorem C:** Let $H = (x_n)_{n \in \mathbb{N}}$ be a weakly null sequence with $BS[(x_n)_{n \in \mathbb{N}}] = \xi$. Then: \(\xi\) is the unique ordinal satisfying the following

(a) For every $M \in [\mathbb{N}]$ there exists $L \in [M]$ such that for every $P \in [L]$
\[\lim_{n \in \mathbb{N}} \|\xi_P n \cdot H\| = 0.\]
(b) For every $\zeta < \xi$ there exists $L_\zeta \in [\mathbb{N}]$ such that $L_\zeta = (n_i)_{i \in \mathbb{N}}$ and $(x_{n_i})_{i \in \mathbb{N}}$ is an $\ell^1_\zeta$ spreading model.
(c) If $\xi = \zeta + 1$ there exists $\epsilon > 0$ and $L \in [\mathbb{N}]$ such that for all $P \in [L]$
\[\|\xi_P n \cdot H\| > \epsilon \text{ and } (\xi_P n \cdot H)_{n \in \mathbb{N}} \text{ is Cesaro summable.}\]

For $\xi = 0$ Theorem B implies exactly the dichotomy mentioned at the beginning of the introduction (Theorem). Theorem C gives the full description of the norm summability for a weakly null sequence in terms of the methods $\{(M, \xi) : M \in [\mathbb{N}], \xi < \omega_1\}$. This justifies the universal character of these summability methods as well as the universal character of Schreier Hierarchy since, as we mentioned above, the supp $\xi^M_n$ belongs to $\mathcal{F}_\xi$.

**Definition:** (a) A Banach space $X$ has the $\xi$-Banach Saks property ($\xi$-BS) if for every bounded sequence $(x_n)_{n \in \mathbb{N}}$ in $X$ there exists $L \in [\mathbb{N}]$ such that $(x_n)_{n \in \mathbb{N}}$ is $(L, \xi)$ summable.
(b) The space $X$ has the weak $\xi$ Banach Saks property (w $\xi$-BS) if the above property holds only for weakly convergent sequences.

From Theorem B and C follow the next corollaries.

**Corollary:** For every separable reflexive Banach space $X$ there exists a unique ordinal $\xi < \omega_1$ such that
(i) For all ordinals $\zeta \geq \xi$ the space $X$ has $\zeta$-BS.
(ii) For every $\zeta < \xi$ the space $X$ fails $\zeta$-BS.

**Corollary:** If $X$ is a separable Banach space not containing isomorphically $\ell^1$ then there exist a unique ordinal $\xi < \omega_1$ such that
(i) For all ordinals $\zeta \geq \xi$ the space $X$ has w $\zeta$-BS.
(ii) For every $\zeta < \xi$ the space $X$ fails w $\zeta$-BS.

The proofs of the above theorems use infinite Ramsey theorem and an index introduced here for compact families of infinite subsets of $\mathbb{N}$ that is called strong Cantor Bendixson index. This index helps us to develop a criterion for embedding the family $\mathcal{F}_\xi$ into a family $\mathcal{F}$ provided the index of
$\mathcal{F}$ is greater than $\omega^\xi$. Also, the proofs of these theorems make use of Lemma 1.2 and a variation of it.

**Notation:** For $N$ infinite subset of $\mathbb{N}$ we denote by $[N]$ the set of all infinite subsets of $N$. Further, we denote by $[N]^<\omega$ the set of all finite subsets of the set $N$. In the sequel for $F \in [N]^<\omega$ we will identify the set $F$ with its characteristic function. Thus for $A = (a_n)_{n \in \mathbb{N}}$ in $\ell^1(\mathbb{N})$ we will denote by $\langle A, F \rangle$ the quantity $\sum_{n \in F} a_n$. For $M \in [N]$ we will denote by $M = (m_i)_{i \in \mathbb{N}}$ the natural order of the set $M$.

As we mentioned above, our proofs use in an essential way the infinite Ramsey Theorem. This theorem, one of the most important principles in infinite combinatorics proved in several steps by Nash-Williams [N-W], Galvin and Prikry [G-P] and in the final form by Silver [Si]. Silver’s proof was model-theoretic. Later Ellenduck [Ell] gave a proof of Silver’s result using classical methods. We recall the statement of the theorem.

**0.1. Theorem:** Let $A$ be an analytic subset of $[N]$. Then for every $M \in [N]$ there exists $L \in [M]$ such that either $[L] \subset A$ or else $[L] \subset [M] \setminus A$.

In the sequel any set $A$ satisfying the above property will be called completely Ramsey. Here we consider the elements of $[M]$ as strictly increasing sequences and we topologize $[M]$ by the topology of the pointwise convergence.

### 1. Convex unconditionality

We start with two combinatorial lemmas.

**1.1. Lemma:** Let $F$ be a relatively weakly compact subset of $c_0(\mathbb{N})$. Then for every $N' \in [N]$ there exists $M \in [N']$ such that: if $l_1 < l_2 < \cdots < l_n$ are elements of $M$ and $f \in F$ such that for every $i = 2, \ldots, n \ f(l_i) \geq \delta$ then there exists $g \in F$ such that for every $i = 2, \ldots, n \ g(l_i) \geq \delta$ and $|g(l_1)| < \epsilon$.

**Proof.** For $n \in \mathbb{N}$ we set

$$S_n = \{ M \in [N'] : M = (m_i) \text{ and if there exists } f \in F \text{ such that }$$

$$\forall i = 2, \ldots, n \ f(m_i) \geq \delta \text{ then there exists } g \in F \text{ such that }$$

$$\forall i = 2, \ldots, n \ g(m_i) \geq \delta \text{ and } g(m_1) < \epsilon \}$$

It is clear that each $S_n$ is closed in the topology of pointwise convergence. Hence $S = \bigcap_{n=1}^{\infty} S_n$ is closed and therefore completely Ramsey. Choose $M \in$
\[ [N'] \text{ such that either } M \subset S \text{ or } [M] \subset [N'] \setminus S. \]

Suppose that \([M] \subset [N'] \setminus S\). Then \(M = (m_i)\) and consider any \(n \in \mathbb{N}\).

For \(1 \leq j \leq n\) set
\[
L_j = \{m_j, m_{n+1}, m_{n+2}, \ldots\},
\]
which does not belong to \(S\). Therefore, for any such \(j\) there are \(f_j \in F\) and \(l_j \in \mathbb{N}\) such that for all \(i = 1, \ldots, l_j\), \(f_j(m_{n+i}) \geq \delta\) and every \(g \in F\) with \(g(m_{n+i}) \geq \delta\) satisfies \(|g(m_j)| \geq \epsilon\). Set \(l_{j_0} = \max\{l_j : 1 \leq j \leq n\}\) and \(f^n = f_{j_0}\).

Observe that for \(1 \leq j \leq n\), \(f^n(m_{n+i}) \geq \delta\) for all \(i = 1, \ldots, l_j\) and hence \(f^n(m_j) \geq \epsilon\) for all \(j = 1, \ldots, n\). It is clear now that the sequence \(\{f^n\}\) does not have weakly convergent subsequence. Therefore the case \([M] \subset [N'] \setminus S\) is impossible and it is easy to check that if \([M] \subset S\) then \(M\) satisfies the conclusion of the lemma.

1.2. Lemma: Consider \(F\) relatively weakly compact subset of \(c_0[N], \delta > 0\) and \(0 < \epsilon < 1\). Then for every \(N' \in [N]\) there exists \(M = (m_i)\) such that:

For every \(f \in F, n \in \mathbb{N}, I \subset \{1, \ldots, n\}\) with \(\min_{i \in I} f(m_i) \geq \delta\) there exists \(g \in F\) satisfying the following two conditions
\[
\begin{align*}
(i) \quad \min_{i \in I} g(m_i) &> (1 - \epsilon) \delta \\
(ii) \quad \sum_{\{i \leq n : i \notin I\}} |g(m_i)| &< \epsilon \cdot \delta.
\end{align*}
\]

Proof. Choose \(a > 0\) with \(|f(m)| \leq a\) for \(m \in \mathbb{N}, f \in F\). Next we choose a strictly increasing sequence \((k_n)\) of natural numbers such that \(2^{k_1} > a\) and if \(\epsilon_n = \frac{1}{2^k}\) then \(\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \epsilon_k < \epsilon \cdot \delta\).

We divide the proof into two stages. In the first we will construct the set \(M\) and in the second we will show that it satisfies the conclusion of the lemma.

The set \(M = (m_i)\)

The set \(M = (m_i)\) is defined inductively so that the following conditions are fulfilled.

If \(I\) is a finite subset of \(\mathbb{N}\), and \(j < \min I\), then for every \(f \in F\) such that \(\min_I f(m_i) > \delta\) there exists \(g \in F\) with:
\[
\begin{align*}
\text{a) } \min_I g(m_i) &> \delta \\
\text{b) } |g(m_j)| &< \epsilon_j \\
\text{c) } |g(m_i) - f(m_i)| &\leq \epsilon_j \text{ for } i = 1, 2, \ldots, j - 1.
\end{align*}
\]
To find such an $M$ we choose inductively a decreasing sequence of infinite sets $N' \supset N_1 \supset \cdots \supset N_i \supset \cdots$ and we set $m_i = \min N_i$.

To choose $N_1$ we apply Lemma 1.1 to find $N_1$ subset of $N'$ such that the conclusion of Lemma 1 holds for the given $\delta$ and $\epsilon = \epsilon_1$. This finishes the choice of $N_1$.

Suppose that $N' \supset N_1 \supset \cdots \supset N_j$ have been chosen such that if $m_i = \min N_i$ then $m_1 < m_2 < \cdots m_j$ and if $1 < i \leq j$, $I$ is a finite subset of $N_i$ with $m_i < \min I$, $f \in F$ with $\min_{k \in I} f(k) > \delta$ then there exists $g \in F$ satisfying (a), (b), (c). To choose $N_{j+1}$ we consider the set $W$ of all closed dyadic intervals of length $\epsilon_{j+1} = \frac{1}{2^{j+1}}$ which are contained in the interval $[-2^{k_j}, 2^{k_j}]$. We denote by $W^j$ the $j$-times product of $W$ and for every $B \in W^j$, $B = (B_1, \ldots, B_j)$, we set

$$F_B = \{ f \in F : f(m_i) \in B_i \},$$

which clearly is relatively weakly compact.

Applying repeatedly Lemma 1.1, we find a set $N_{j+1}$ infinite subset of $N_j$ such that $m_j < \min N_{j+1}$ and the conclusion of Lemma 1.1 holds for the set $N_{j+1}$ and for every $F_B$, $B \in W^j$, the given $\delta$ and $\epsilon = \epsilon_{j+1}$. This completes the inductive construction of the sets $(N_i)$ and hence the set $M$ is defined.

It remains to show that $M$ satisfies the desired properties.

The set $M$ satisfies (I) and (II)

Given $n \in \mathbb{N}$, $I$ subset of $\{1, 2, \ldots, n\}$ and $f \in F$ such that $\min_{i \in I} f(m_i) > \delta$, we shall define the desired function $g$. For this, we inductively choose $g_0, g_1, \ldots, g_n$ elements of $F$ such that:

$\begin{align*}
  f &= g_0; \\
  &\text{if } k \in \mathbb{N}, 1 \leq k < n \text{ and } g_0, g_1, \ldots, g_k \text{ have been chosen satisfying the property:} \\
  &\text{for every } 1 \leq l \leq k \text{ and } i = 1, 2, \ldots, l, \\
  &|g_l(m_i) - g_{l-1}(m_i)| \leq \epsilon_l \text{ and} \\
  &\text{for } i \in \{l + 1, \ldots, n\} \cap I, \text{ we have that } g_l(m_i) > \delta \text{ and } g_l(m_i) > \delta \text{ if } l \in I \text{ or } |g_l(m_i)| < \epsilon_l \text{ otherwise.}
\end{align*}$

To choose $g_{k+1}$ we distinguish two cases.

Case 1: $k + 1 \in I$. Then we set $g_{k+1} = g_k$.

Case 2: $k + 1 \notin I$. Then we choose $g_{k+1}$ such that $|g_{k+1}(m_i) - g_k(m_i)| \leq \epsilon_{k+1}$, $g_{k+1}(m_i) > \delta$ for every $i \in \{k + 2, \ldots, n\} \cap I$, $|g_{k+1}(m_{k+1})| < \epsilon_{k+1}$. The existence of such a $g_{k+1}$ follows from the properties of the set $M$. 

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This completes the inductive definition of \( g_0, \ldots, g_n \). It is easy to see that the final function \( g_n \) is the desired \( g \). The proof of the lemma is complete.

1.3. **Theorem:** Every \( (x_n)_{n \in \mathbb{N}} \) normalized weakly null sequence in a Banach space \( X \) has a convexly unconditional subsequence.

**Proof.** Assume, by passing to a subsequence if it is needed, that \( (x_n)_{n \in \mathbb{N}} \) is Schauder basic with basic constant \( D \geq 1 \). We, inductively, apply Lemma 1.2 to choose a decreasing sequence \( (M_n)_{n \in \mathbb{N}} \) such that \( M_n \) satisfies the conclusion of the Lemma for \( F = \{(x^*(x_n))_{n \in \mathbb{N}}; \|x^*\| \leq 1\} \), \( \delta = \frac{1}{n^2} \), \( \epsilon = \frac{1}{n^3} \).

We select a strictly increasing sequence \( M = (m_n)_{n \in \mathbb{N}} \) such that \( m_n \in M_n \).

**Claim:** The sequence \( (x_n)_{n \in M} \) is convexly unconditional.

Indeed, given \( x = \sum_{n \in M} a_n x_n \) an absolutely convex combination with \( \|x\| > \frac{1}{k} \) and \((\varepsilon_n)_{n \in M}\) a sequence of signs we choose \( x^* \in \text{B}_X^* \) with \( x^*(x) > \frac{1}{k} \). There exists finite \( J \subset M \) such that \( x^*(\sum_{n \in J} a_n x_n) > \frac{1}{k} \). We set

\[ J_1 = \left\{ n \in J : |x^*(x_n)| > \frac{1}{2k} \right\}. \]

Then we have \( x^*(\sum_{n \not\in J \setminus J_1} a_n x_n) \leq \frac{1}{2k} \) and hence \( x^*(\sum_{n \in J_1} a_n x_n) > \frac{1}{2k} \). By splitting the set \( J_1 \) into four sets, in the obvious way, we find a subset \( I \subset J_1 \) such that:

\[
|\sum_{n \in I} \varepsilon_n a_n| > \frac{1}{2k}, \\
\{\varepsilon_n a_n : n \in I\} \text{ are either all non-negative or all negative and} \\
\{x^*(x_n) : n \in I\} \text{ are of the same sign.}
\]

We consider \( x^* \) if the sign of \( x^*(x_n) \) is positive and \( -x^* \) if the sign is negative and this we again denote by \( x^* \). Thus we also have \( x^*(x_n) > \frac{1}{2k} \) for \( n \in I \). For every \( r \in \mathbb{N} \) we denote by \( \text{B}(r) \) the unconditional constant of the \( \{x_{m_1}, \ldots, x_{m_r}\} \). This means that for \( G \subset \{1, \ldots, r\} \),

\[
\left\| \sum_{i \in I} b_i x_{m_i} \right\| \leq \text{B}(r) \cdot \left\| \sum_{i=1}^r b_i x_{m_i} \right\|.
\]

(This happens because the norm \( \| \cdot \| \) in the space of dimension \( r \) that is generated by \( x_{m_1}, \ldots, x_{m_r} \) is equivalent to the maximum norm with respect to to this basis).

We split \( I \) into two sets \( I_1 = I \cap \{m_1, \ldots, m_{2k-1}\} \) and \( I_2 = I \setminus I_1 \).

We have \( |\sum_{n \in I_1} \varepsilon_n a_n| > \frac{1}{16k} \) or \( |\sum_{n \in I_2} \varepsilon_n a_n| > \frac{1}{16k} \).
If the first condition holds, then
\[
\left\| \sum_{n \in M} \varepsilon_n a_n x_n \right\| \geq \frac{1}{D} \left\| \sum_{i=1}^{2k-1} \varepsilon_{m_i} a_{m_i} x_{m_i} \right\| \geq \frac{1}{D \cdot B(2k-1)} \left\| \sum_{n \in I_1} \varepsilon_n a_n x_n \right\| > \frac{1}{D \cdot B(2k-1)} \left\| \sum_{n \in I_2} \varepsilon_n a_n x_n \right\| \cdot \frac{1}{2^k} > \frac{1}{D \cdot B(2k-1)} \cdot \frac{1}{32k} \quad (1)
\]

In the second case, there exists \( y^* \in B_{X^*} \) such that
\begin{enumerate}
  \item \( \min I_2 \{ y^*(x_n) \} > \left( 1 - \frac{1}{(4k)^3} \right) \cdot \frac{1}{2k} \)
  \item \( \max \{ |y^*(x_n)| : n \in \{ m_{2k}, \ldots, m_l \} \setminus I_2 \} < \frac{1}{2k(4k)^3} \), where \( m_l = \max I \).
\end{enumerate}
Therefore
\[
\left| y^* \left( \sum_{i=2k}^l \varepsilon_i a_i x_i \right) \right| > \left| y^* \left( \sum_{n \in I_2} \varepsilon_n a_n x_n \right) \right| - \left| y^* \left( \sum_{n \in \{ m_{2k}, \ldots, m_l \} \setminus I_2} \varepsilon_n a_n x_n \right) \right|
\geq \frac{1}{16k} \left( 1 - \frac{1}{(4k)^3} \right) \cdot \frac{1}{2k} - \frac{1}{2k} \cdot \frac{1}{(4k)^3}
= \frac{1}{32k^2} \left( 1 - \frac{1}{(4k)^3} - \frac{1}{(2k)^2} \right) > \frac{1}{64k^2}
\]

Finally
\[
\left\| \sum_{n \in M} \varepsilon_n a_n x_n \right\| > \frac{1}{2D} \left\| \sum_{i=2k}^l \varepsilon_i a_i x_i \right\| > \frac{1}{D} \cdot \frac{1}{128k^2} \quad (2)
\]

From (1) and (2) we get that
\[
C \left( \frac{1}{k} \right) \geq \min \left\{ \frac{1}{D \cdot B(2k-1)} \cdot \frac{1}{32k} , \frac{1}{D \cdot 128k^2} \right\}.
\]

1.4. Corollary: Let \((x_n)_{n \in \mathbb{N}}\) be a normalized weakly null sequence. Then for every \( M \in [N] \) there exists \( L \in [M] \), \( L = (l_i)_{i \in \mathbb{N}} \) such that the following property is satisfied
For every \( k > 0 \) there exists \( C(k) > 0 \) such that for every \( x = \sum_{j=1}^{\infty} a_j x_{l_j} \) and \( \sum_{j=1}^{\infty} |a_j| < k \) then for every sequence \((\varepsilon_j)_{j \in \mathbb{N}}\) of signs we have \( C(k) < \left\| \sum_{j=1}^{\infty} \varepsilon_j a_j x_{l_j} \right\| \leq k. \)
Proof. Set \( d = \sum_{j=1}^{\infty} |a_j| \) and \( b_j = \frac{a_j}{d} \). Then \( \sum_{j=1}^{\infty} b_j x_{l_j} \) is absolutely convex combination and \( \| \sum b_j x_{l_j} \| > \frac{1}{d} > \frac{1}{k} \). Hence by Theorem 1.3 we get the left inequality with \( C(k) = C\left(\frac{1}{k}\right) \). The right is immediate from the triangle inequality.

The following consequence of Lemma 1.2 has been proved by H. Rosenthal with the use of transfinite induction.

1.5. Theorem: Let \( K \) be a compact space and \( (f_n)_{n \in \mathbb{N}} \) a sequence of continuous characteristic functions converging pointwise to zero. Then there exists \( L \in [\mathbb{N}], L = (l_j)_{j \in \mathbb{N}} \) such that \( (f_{l_j})_{j \in \mathbb{N}} \) is an unconditional basic sequence.

Proof. Define \( F : K \to c_0(\mathbb{N}) \) by the rule \( F(x) = (f_n(x))_{n \in \mathbb{N}} \). Then for each \( x \in K \), \( F(x) \) is a finite subset of \( \mathbb{N} \) and \( F[K] \) is weakly compact. By Lemma 1.2, there exists \( L \in [\mathbb{N}] \) such that for every \( x \in K \), \( G \subseteq F(x) \cap L \) there exists \( y \in K \) such that \( F(y) \cap L = G \). It is easy to check that the sequence \( (f_n)_{n \in L} \) is an unconditional sequence.

2. Summability methods

Schreier Hierarchy, The RA Hierarchy

Notation: We denote by \( S^+_\ell^1 \) the positive part of the unit sphere of \( \ell^1(\mathbb{N}) \). For \( A = (a_n)_{n \in \mathbb{N}} \) in \( S^+_\ell^1 \) and \( F = (x_n)_{n \in \mathbb{N}} \) bounded sequence in a Banach space \( X \) we denote by \( A \cdot F \) the usual matrices product, that is:

\[
A \cdot F = \sum_{n=1}^{\infty} a_n x_n.
\]

2.1.1. Definition: For an \( M \) infinite subset of \( \mathbb{N} \) an \( M \) summability method is a block sequence \( (A_n)_{n \in \mathbb{N}} \) with \( A_n \in S^+_\ell^1 \) and \( M = \bigcup_{n=1}^{\infty} \text{supp} A_n \) where \( \text{supp} A_n = \{ n \in \mathbb{N} : a_n \neq 0 \} \).

2.1.2. Definition: Suppose that \( (A_n)_{n \in \mathbb{N}} \) is an \( M \) summability method. A bounded sequence \( F = (x_n)_{n \in \mathbb{N}} \) is said to be \( M - (A_n)_{n \in \mathbb{N}} \) summable if the sequence \( (A_n \cdot F)_{n \in \mathbb{N}} \) is Cesaro summable. This means that the sequence \( z_n = \frac{1}{n} \sum_{k=1}^{n} A_k \cdot F \) is norm convergent.

2.1.3. Remark: To each \( M \in [\mathbb{N}], M = (m_n)_{n \in \mathbb{N}} \), we assign the \( M \)-summability method \( A_n = \{ e_{m_n} \} \). Then the \( M - (A_n)_{n \in \mathbb{N}} \) summability of \( (x_n)_{n \in \mathbb{N}} \) is exactly the usual Cesaro’s summability of the subsequence
Definition of Schreier Hierarchy

Next we recall the definition of the generalized Schreier families \((\mathcal{F}_\xi)_{\xi<\omega_1}\). These are defined inductively in the following manner.

**Notation:** For \(F_1, F_2 \in [\mathbb{N}]^{<\omega}\) we denote by \(F_1 < F_2\) the relation \(\max F_1 < \min F_2\).

We set \(\mathcal{F}_0 = \{\{n\} : n \in \mathbb{N}\} \cup \emptyset\).

Suppose that \(\xi = \zeta + 1\) and \(\mathcal{F}_\zeta\) has been defined. We set
\[
\mathcal{F}_\xi = \{F \in [\mathbb{N}]^{<\omega} : F = \bigcup_{i=1}^n F_i, F_i \in \mathcal{F}_\zeta, n \leq F_1 < \cdots < F_n\}.
\]

If \(\xi\) is a limit ordinal and \(\mathcal{F}_\zeta\) has been defined for all \(\zeta < \xi\) then we fix a strictly increasing family of non-limit ordinals \((\zeta_n)_{n \in \mathbb{N}}\) with \(\sup \zeta_n = \xi\) and we define
\[
\mathcal{F}_\xi = \{F \in [\mathbb{N}]^{<\omega} : n \leq \min F \text{ and } F \in \mathcal{F}_{\zeta_n}\}.
\]

**Remark:** The use of a sequence of non-limit ordinals \((\zeta_n)_{n \in \mathbb{N}}\) in the definition of \(\mathcal{F}_\xi\), \(\xi\) limit, is not important. We make this assumption in order to avoid some more complexity of the Approximation Lemma given below.

Definition of the RA Hierarchy

To each \(M \in [\mathbb{N}]\) and \(\xi < \omega_1\) we will assign inductively an \(M\) summability method \((\zeta_n^M)_{n \in \mathbb{N}}\) in the following manner.

(i) For \(\xi = 0\), \(M = (m_n)_{n \in \mathbb{N}}\) we set \(\zeta_n^M = \{e_{m_n}\}\).

(ii) If \(\xi = \zeta + 1\), \(M \in [\mathbb{N}]\) and \((\zeta_n^M)_{n \in \mathbb{N}}\) has been defined then we, inductively, define \((\zeta_n^M)_{n \in \mathbb{N}}\) as it follows. We set \(k_1 = 0, s_1 = \min \supp \zeta_1^M\), and
\[
\zeta_1^M = \frac{\zeta_1^M + \cdots + \zeta_{s_1}^M}{s_1}.
\]

Suppose that for \(j = 1, 2, \ldots, n-1 \) \(k_j, s_j\) have been defined and
\[
\zeta_j^M = \frac{\zeta_{k_j+1}^M + \cdots + \zeta_{k_j+s_j}^M}{s_j}.
\]

Then we set,
\[
k_n = k_{n-1} + s_{n-1}, \quad s_n = \min \supp \zeta_{k_n+1}^M \quad \text{and}
\]
\[ \xi_n^M = \xi_{k_n+1}^M + \cdots + \xi_{k_n+s_n}^M. \]

This completes the definition for successor ordinals.

(iii) If \( \xi \) is a limit ordinal and if we suppose that for every \( \zeta < \xi \), \( M \in [\mathbb{N}] \) the sequence \( (\xi_n^M)_{n \in \mathbb{N}} \) has been defined, then we define \( (\xi_n^M)_{n \in \mathbb{N}} \) as it follows:

We denote by \( (\xi_n)_{n \in \mathbb{N}} \) the strictly increasing sequence of ordinals with \( \sup \xi_n = \xi \) that defines the family \( \mathcal{F}_\xi \).

For \( M = (m_k)_{k \in \mathbb{N}} \) we inductively define \( M_1 = M \), \( n_1 = m_1 \),
\[
M_2 = \left\{ m_k : m_k \notin \text{supp} \left[ \xi_{n_1}^1 \right] \right\}, \quad n_2 = \min M_2,
\]
\[
M_3 = \left\{ m_k : m_k \notin \text{supp} \left[ \xi_{n_2}^2 \right] \right\}, \quad n_3 = \min M_3, \quad \text{and so on.}
\]

We set
\[
\xi_1^M = [\xi_{n_1}^1]^M_1, \quad \xi_2^M = [\xi_{n_2}^2]^M_2, \ldots, \quad \xi_k^M = [\xi_{n_k}^1]^M_k, \ldots.
\]

Hence \( (\xi_n^M)_{n \in \mathbb{N}} \) has been defined. This completes the definition of RA Hierarchy.

Properties of the two Hierarchies

The following properties can be established inductively.

**P.1**: For \( \xi < \omega_1 \), \( M \in [\mathbb{N}] \) \( (\xi_n^M)_{n \in \mathbb{N}} \) is an \( M \)-summability method, i.e., \( (\xi_n^M)_{n \in \mathbb{N}} \) is a block sequence of elements of \( S^*_\xi \) and \( M = \bigcup_{n=1}^{\infty} \text{supp} \xi_n^M \).

**P.2**: For every \( \xi < \omega_1 \), \( M \in [\mathbb{N}] \), \( n \in \mathbb{N} \) \( \xi_n^M \in \mathcal{F}_\xi \).

**P.3**: For every \( \xi < \omega_1 \) and every \( N, M \in [\mathbb{N}] \) such that \( \text{supp} \xi_i^M = \text{supp} \xi_i^N \) for \( i = 1, 2, \ldots, k \) we have \( \xi_i^M = \xi_i^N \) for \( i = 1, 2, \ldots, k \).

**P.4**: For every \( M \in [\mathbb{N}] \) and \( (n_k)_{k \in \mathbb{N}} \) subset of \( \mathbb{N} \) if \( M' = \bigcup_{k=1}^{\infty} \text{supp} \xi_{n_k}^M \) then \( \xi_{n_k}^M = \xi_{n_k}^{M'} \).

Remark: Properties P.3 and P.4 are important for our proofs and they indicate a strong stability of the methods \( (\xi_n^M)_{n \in \mathbb{N}} \).

2.1.4. Definition: A family \( \mathcal{F} \) of finite subsets of \( \mathbb{N} \) is said to be adequate if \( \mathcal{F} \) is compact and for every \( F \in \mathcal{F} \), if \( G \subset F \) then \( G \in \mathcal{F} \).

2.1.5. Notation: If \( \mathcal{F} \) is an adequate family and \( L \in [\mathbb{N}] \) we set \( \mathcal{F}[L] = \{ F \in \mathcal{F} : F \subset L \} \).

Clearly \( \mathcal{F}[L] \) is an adequate subfamily of \( \mathcal{F} \).
2.1.6. **Notation:** For an ordinal \( \xi < \omega_1 \) and \( M \in [N] \), \( M = (m_i)_{i \in \mathbb{N}} \) we define
\[
\mathcal{F}_\xi^M = \{ G : G = (m_i)_{i \in F}, F \in \mathcal{F}_\xi \}.
\]
It is easy to see that \( \mathcal{F}_\xi^M \) is an adequate family.

2.1.7. **Remark:** It is proved readily by induction that \( \mathcal{F}_\xi^M \) is a subfamily of \( \mathcal{F}_\xi[M] \); on the other hand, it is not true that \( \mathcal{F}_\xi[M] \) is contained in \( \mathcal{F}_\xi^M \). We will show that by going to a subset \( N \) of \( M \), \( \mathcal{F}_\xi^M[N] \) and \( \mathcal{F}_\xi[N] \) are in a sense comparable.

2.1.8. **Lemma:** For all ordinals \( \zeta < \xi < \omega_1 \) there exists \( n(\zeta, \xi) \in \mathbb{N} \) such that for every \( F \in \mathcal{F}_1 \), \( n(\zeta, \xi) \leq \min F, F \in \mathcal{F}_\xi \).
(b) The same holds for \( \mathcal{F}_M^\zeta \), \( \mathcal{F}_M^\xi \).

The proof of this lemma is obtained easily by induction.

2.1.9. **Lemma:** For \( M \in [N], \xi \) a limit ordinal, \( \epsilon > 0 \) and \( N \in [M] \) there exists \( L \in [N] \) satisfying the following property:
For every \( P \in [L], n \in \mathbb{N} \) there exists \( G \in \mathcal{F}_\xi^M \) such that \( \langle \zeta^P_n, G \rangle > 1 - \epsilon \).

**Proof:** We proceed by induction. We will establish the following.

**Inductive hypothesis:** For every limit ordinal \( \xi < \omega_1 \), \( N \in [M], \epsilon > 0 \) there exists \( L \in [N] \) such that:
For every ordinal \( \zeta, \zeta \leq \xi \) there exists \( l(\zeta, \xi) \) such that:
For every \( P \in [L], n \in \mathbb{N} \) with \( l(\zeta, \xi) \leq \min \supp \zeta^P_n \) there exists \( G \in \mathcal{F}_\xi^M \) such that \( \langle \zeta^P_n, G \rangle > 1 - \epsilon \).

**Note:** In the sequel for \( M \in [N], \zeta < \xi \) we shall denote by \( k(\zeta, \xi) \) the natural number appearing in part (b) of the previous lemma and satisfying the property: If \( F \in \mathcal{F}_\xi^M, k(\zeta, \xi) \leq \min F \) then \( F \in \mathcal{F}_\xi^M \).

We pass to prove the inductive hypothesis.

**Case 1:** \( \xi = 0 \). The proof follows immediately from the definitions.

**Case 2:** \( \xi \) is a limit ordinal and there exists an increasing sequence \( (\zeta_k)_{k \in \mathbb{N}} \) of smaller limit ordinals with \( \sup \zeta_k = \xi \). We also denote by \( (\zeta_n)_{n \in \mathbb{N}} \) the sequence of non limit ordinals, used in the definition of the family \( \mathcal{F}_\xi \).

Let \( N \) be a subset of \( M \) and \( \epsilon > 0 \) We, inductively, choose a decreasing sequence \( (L_k)_{k \in \mathbb{N}} \) of subsets of \( N \) such that each \( L_k \) satisfies the inductive
hypothesis for the ordinal $\zeta_k$ and the number $\epsilon' = \frac{\epsilon}{4}$. Further, for $n \in \mathbb{N}$ denote by $k_n$ the least natural number such that $\xi_n < \zeta_{k_n}$.

Inductively define a subset $L$ of $N$, $L = (l_i)_{i \in \mathbb{N}}$ in the following manner. Set $M = (m_s)_{s \in \mathbb{N}}$, choose $l_1 \in L_1$ such that $l_1 = m_{s_0}$ and $\frac{1}{2^0} < \frac{\epsilon}{4}$.

In general choose $l_{i+1} \in L_{i+1}$ such that

$$l_{i+1} > \max\{l_i, k(\zeta_{k_{i+2}}, \xi), l(\xi^{-1}_{l_i}, \zeta_{k_i})\},$$

where $\xi^{-1}_{l_i}$ denotes the predecessor of $\xi_{l_i}$.

The inductive definition of $L$ is complete and we prove the following:

Claim: The set $L$ satisfies the inductive hypothesis for the ordinal $\xi$ and the number $\epsilon$.

Suppose first that $\zeta < \xi$. Then there exists $\xi_k$ such that $\zeta < \xi_k$. By the inductive assumption there exists $l(\zeta, \xi_k) \in \mathbb{N}$ such that for $P \in [L_k]$, $n \in \mathbb{N}$ with $l(\zeta, \xi_k) < \min \operatorname{supp} \zeta^P_n$ there exists $G \in F^M_{\zeta_k}$ with $\langle \zeta^P_n, G \rangle > 1 - \epsilon'$.

We set

$$l(\zeta, \xi) = \max\{l(\zeta, \xi_k), k(\zeta_k, \xi)\}$$

and we show that it satisfies the inductive hypothesis for the pair $(\zeta, \xi)$ and the number $\epsilon$.

Indeed, if $P \in [L]$, $n \in \mathbb{N}$, with $l(\zeta, \xi) \leq \min \operatorname{supp} \zeta^P_n$ then there exists $G \in F^M_{\zeta_k}$ with $\langle \zeta^P_n, G \rangle > 1 - \epsilon'$.

Since $F^M_{\zeta_k}$ is adequate, we assume that $l(\zeta, \xi) \leq G$. Then $k(\zeta_k, \xi) \leq G$ hence $G \in F^M_{\zeta_k}$ and this completes the proof of this case.

We pass to the remaining case $\zeta = \xi$.

We shall show that $l(\zeta, \xi) = l_1$.

Indeed, suppose that $P \in [L]$, $n \in \mathbb{N}$. Then if $l_i = \min \operatorname{supp} \xi^P_n$ then there exists $P_1 \in [P]$ with $\xi^P_n = [\xi_{l_1}]^{P_1}_{l_1}$. Further $[\xi_{l_1}]^{P_1}_{l_1} = \frac{1}{l} \sum_{j=1}^{l_1} [\xi_{l_1}]^{P_1}_{l_1}$.

Set $F_j = \operatorname{supp} [\xi_{l_1}]^{P_1}_{l_1}$ and observe that:

(i) $F_1 < F_2 < \cdots < F_{l_1}$,

(ii) $l(\xi^{-1}_{l_1}, \zeta_{k_1}) \leq l_{i+1} \leq F_2$

Hence by the inductive assumption there exist $G_2, G_3, \ldots, G_{l_1}$ in $F^M_{\zeta_{k_1}}$ such that $G_j \subset F_j$ and $\langle [\xi_{l_1}]^{P_1}_{l_1}, G_j \rangle > 1 - \epsilon'$.

Notice also that $G_{l_1} < \cdots < G_{l}$.

We set $d_1 = \left\lfloor \frac{l}{2} \right\rfloor$ and observe that if $m_i = \min G_{d_{i+1}}$ then $i \geq d_1$ hence the set $W_1 = G_{d_{i+1}} \cup \cdots \cup G_{l_1}$ belongs to $F^M_{\zeta_{k_{l_1}}}$ and $\langle [\xi_{l_1}]^{P_1}_{l_1}, G_j \rangle > 1 - \epsilon'$. Repeat the same for $d_2 = \left\lfloor \frac{d_{i+1}}{2} \right\rfloor$ and for the set $W_2 = G_{d_{i+1}} \cup \cdots \cup G_{d_{i+1}}$ that also belongs to $F^M_{\zeta_{k_{l_1}}}$.
Therefore, following this procedure, we define $W_1, W_2, \ldots, W_s$ where $s \in \mathbb{N}$ such that $l_i = m_s$.

Since $l_1 = m_{s_0}$, \( \frac{1}{2^{s_0}} < \epsilon' = \frac{\epsilon}{4} \) we get that \( \frac{1}{2^{s_0}} < \epsilon' \).

Notice that $m_s \leq W_s < W_{s-1} < \cdots < W_1$, hence the set $G = \bigcup_{q=1}^s W_q$ belongs to $\mathcal{F}^M_{\zeta_{k_i}+2}$.

Further $\bigcup_{q=1}^s W_q = \bigcup_{j=j_0+1}^{P_i} G_j$ where $j_0$ satisfies the following:

\[
    j_0 < l_i - \sum_{p=1}^s \frac{l_p}{2^p} = \frac{l_i}{2^s} < l_i \cdot \epsilon'.
\]

Finally

\[
    \langle \xi^p_{n_i}, G \rangle = \frac{1}{l_i} \sum_{j=j_0+1}^{l_i} \langle [\xi \underline{n}_{j}]^p, G_j \rangle > (1 - \epsilon') - \frac{j_0}{l_i} > 1 - 2\epsilon' > 1 - \epsilon.
\]

Notice that $k(\zeta_{k_i}+2, \xi) \leq l_i + 1 \leq G$ and hence $G \in \mathcal{F}^M_{\xi}$.

This completes the proof for case 2.

**Case 3** $\xi = \zeta + \omega$ with $\zeta$ a limit ordinal.

Let $N$ be a subset of $M$ and $\epsilon > 0$. By the inductive assumption, there exists $L \in [N]$ satisfying the inductive hypothesis for the ordinal $\zeta$ and the number $\epsilon' = \frac{\epsilon}{4}$. Since every $L' \in [L]$ also satisfies the inductive hypothesis for the same $\zeta$ and $\epsilon'$, we may assume that $L$ satisfies the following.

If $M = (m_i)_{i \in N}$ and $L = (m_i)_{i \in D}$ then ordering $D$ in the natural manner, we have that $L = (m_{i_n})_{n \in \mathbb{N}}$. Under this notation we assume that

\[
    \frac{1}{m_{i_1}} < \frac{\epsilon}{4} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{2^{i_k}} < \frac{\epsilon}{4}.
\]

We first prove the following

**Claim 1:** For $n = 1, 2, \ldots$ there exists $k_n \geq n$ such that the following hold:

(i) If $P \in [L]$, $q \in \mathbb{N}$ with $m_{i_n} \leq \text{supp} [\zeta + n]^P_q$, there exists $G \in \mathcal{F}^M_{\zeta+k_n}$ such that:

\[
    \langle [\zeta + n]^P_q, G \rangle > 1 - \left( \epsilon' + \sum_{k=1}^{n} \frac{1}{2^k} \right) > 1 - 2\epsilon'.
\]

(ii) If $P \in [L]$, $q \in \mathbb{N}$, $m_i = \text{min supp} [\zeta + n]^P_q$ and

\[
    [\zeta + n]^P_q = \frac{1}{m_i} \sum_{j=1}^{m_i} [\zeta + n - 1]_j^P
\]

for some $P_i \in [P]$.

If $m_{i_n} \leq \text{supp} [\zeta + n - 1]_q^P$ then there exists $G \in \mathcal{F}^M_{\zeta+k_n}$ such that $m_{i_n} \leq \text{min} G$ and

\[
    \langle [\zeta + n - 1]^P_q, G \rangle > 1 - \left( 2\epsilon' + \sum_{k=1}^{n} \frac{1}{2^k} \right) > 1 - 3\epsilon'.
\]
Proof of Claim 1: We proceed by induction on $N$. The case $n = 1$ is proved by similar arguments as the general case. Suppose that the claim has been proved for $k = 1, 2, \ldots, n - 1$. We prove first part (i).

Suppose that $P \in [L]$ such that $m_i \leq \text{supp} \left[ \zeta + n \right]^P_1$.

The case $[ \zeta + n]^P_1$ follows from this since there always exists $P_1 \in [P]$ such that

$$[ \zeta + n]^P_1 = [ \zeta + n]_1^P.$$ 

We set $m_i = \min \text{supp} \left[ \zeta + n \right]^P_1$ and write

$$[ \zeta + n]^P_1 = \frac{1}{m_i} \sum_{j=1}^{m_i} [ \zeta + n - 1]^P_j.$$ 

Also set $F_j = \text{supp} \left[ \zeta + n - 1 \right]^P_j$ and notice that $F_1 < F_2 < \cdots < F_{m_i}$. (\ast) 

Since $m_{i_{n-1}} < F_j$, by the inductive assumption part (i), there exist $G_j \in \mathcal{F}_{\zeta + k_{n-1}}^M$, such that $G_j \subset F_j$ and

$$\langle [ \zeta + n - 1]^P_j, G \rangle > 1 - \left( \epsilon' + \sum_{k=1}^{n} \frac{1}{2^k} \right).$$ 

From (\ast) we also have, $m_i \leq G_1 < G_2 < \cdots < G_{m_i}$.

Set $d_1 = \left[ \frac{m_i}{2} \right]$ and notice that $m_{d_1+1} \leq \min G_{d_1}$ hence the set $W_1 = G_{d_1+1} \cup \cdots \cup G_{m_i}$ belongs to $\mathcal{F}_{\zeta + k_{n-1}}^M$. In the same manner we define $d_2 = \left[ \frac{d_1}{2} \right]$ and $W_2 = G_{d_2+1} \cup \cdots \cup G_{d_1}$ that also belongs to $\mathcal{F}_{\zeta + k_{n-1}}^M$. 

Thus, successively define $W_1, W_2, \ldots, W_i$ with $W_s \in \mathcal{F}_{\zeta + k_{n-1}}^M$ for $s = 1, 2, \ldots, i_n$ and

$m_i \leq W_1 < W_2 < \cdots < W_i$.

Hence the set $G = \bigcup_{s=1}^{i_n} W_s$ belongs to $\mathcal{F}_{\zeta + k_{n-1}+1}^M$.

Further, $G = \bigcup_{s=1}^{i_n} W_s = \bigcup_{j=j_0+1}^{m_i} G_j$ and $j_0$ is estimated by

$$j_0 \leq m_i - \sum_{s=1}^{i_n} m_i = m_i. $$

Finally we have,

$$\langle [ \zeta + n]^P_1, G \rangle = \frac{1}{m_i} \sum_{j=j_0+1}^{m_i} \langle [ \zeta + n - 1]^P_j, G \rangle > 1 - \left( \epsilon' + \sum_{k=1}^{n} \frac{1}{2^k} \right) - \frac{j_0}{m_i} > 1 - \left( \epsilon' + \sum_{k=1}^{n} \frac{1}{2^k} \right)$$

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and the proof of part (i) is complete.

The proof for part (ii) is similar to the above one.

Indeed, for \( P \in [L] \) we have as before
\[
[\zeta + \mathbf{n}]_j^P = \frac{1}{m_i} \sum_{j=1}^{m_i} [\zeta + \mathbf{n} - 1]_j^P
\]
and if \( F_j = \text{supp} [\zeta + \mathbf{n} - 1]_j^P \) then by our assumption we have that \( m_{i_n} \leq \min F_2 \).

Observe that \([\zeta + \mathbf{n} - 1]_j^P\) for \( j = 2, \ldots, m_i\) satisfy part (i) of the inductive assumption, hence there exist \( G_j \subset F_j \) such that \( G_j \in \mathcal{F}^M_{\zeta + n_{k-1}} \) and
\[
\langle [\zeta + \mathbf{n} - 1]_j^P, G_j \rangle > 1 - \left( \epsilon' + \sum_{k=1}^{n-1} \frac{1}{2^{n_k}} \right).
\]

As in the previous part of the proof, we build \( G \) in \( \mathcal{F}^M_{n_{k-1}+2} \) such that
\[
\langle \frac{1}{m_i} \sum_{j=2}^{m_i} [\zeta + \mathbf{n} - 1]_j^P, G \rangle > 1 - \left( \epsilon' + \sum_{k=1}^{n-1} \frac{1}{2^{n_k}} \right).
\]

Further, since \( \frac{1}{m_i} \leq \frac{1}{m_1} < \epsilon' \), we get that
\[
\langle [\zeta + \mathbf{n}]_1^P, G \rangle > 1 - \left( 2\epsilon' + \sum_{k=1}^{n-1} \frac{1}{2^{n_k}} \right).
\]

The proof of Claim 1 is complete.

To finish the proof of the lemma for case 3, it remains to define an appropriate \( L' \) subset of \( L \). The set \( L' = (l_j)_{j \in \mathbb{N}} \) is defined by induction. We denote by \( (\xi_k)_{k \in \mathbb{N}} \) the increasing sequence of smaller than \( \xi \) ordinals such that \( \lim \xi_k = \xi \). Notice that, since \( \xi = \zeta + \omega \) there exists \( k_0 \) such that for all \( k \geq k_0 \) \( \xi_k = \zeta + n_k \). Choose \( l_1 \in L \) with \( l_1 > k_0 \). Suppose that \( l_1 < \cdots < l_j \) have been defined. To define \( l_{j+1} \) we follow the next procedure. Since \( l_j > l_1 > k_0 \), there exists \( n_{k(j)} \) such that \( \xi_{l_j} = \zeta + n_{k(j)} \). From Claim 1 there exists \( k(n_{k(j)}) \) satisfying part (i) and (ii) of claim 1. Further there exists \( k'_{j} \) such that \( \zeta + n_{k(j)} < \xi_{k'_{j}} \).

Choose \( l_{j+1} \in L \) such that: \( l_{j+1} > \max\{l_j, m_{i_j}, k'_{j}\} \). This completes the inductive of \( L' \).

Claim 2: The set \( L' \) satisfies the inductive hypothesis for the ordinal \( \xi \) and the number \( \epsilon \).

Indeed, given \( \eta \leq \xi \)

Step 1: \( \eta \leq \zeta \). Then using the inductive hypothesis for \( \zeta \) and \( \eta \) we can easily establish the inductive hypothesis for \( \xi \) and \( \eta \).
Step 2: $\zeta < \eta < \xi$. Then $\eta = \zeta + n$ and we set $l(\eta, \xi) = l_{j_n}$ and by definition $l_{j_n} > \max\{m_{i_n}, k_\eta\}$ with $k_\eta \in \mathbb{N}$ such that $\xi_{k_\eta} > \zeta + k(n)$ and $k(n)$ is the number corresponding to $n$ in Claim 1.

If $P \in [L']$ and $n_0 \in \mathbb{N}$ and $l_{j_n} \leq \min \text{supp} \ \eta^P_{n_0}$ then by Claim 1 there exists $G \in \mathcal{F}^M_{\zeta + k_n}$ such that $\langle \eta^P_{n_0}, G \rangle > 1 - 2\epsilon'$.

Further, $G \in \mathcal{F}^M_{\xi_{k_n}}$ and $\min G > k_n$ which implies that $G \in \mathcal{F}^M_{\xi}$ and the proof of Step 2 is complete.

Step 3: $\eta = \xi$. In this case we shall show that $l(\eta, \xi) = l_1$.

Indeed, if $P \in [L']$ and $n_0 \in \mathbb{N}$ then, by definition, for $l_j = \min \text{supp} \ \xi^P_{n_0}$ choose $P_1 \in [P]$ such that: $\xi^P_{n_0} = [\xi_j]_1^{P_1}$.

Further,

$$[\xi_j]_1^{P_1} = \frac{1}{l_j} \sum_{i=1}^{l_j} [\xi_j]_i^{P_1}$$

where $\xi_j^{P_1}$ denotes the predecessor of $\xi_j$.

Also, $\xi_l = \zeta + n_{k(j)}$ and $l_{j+1} > m_{i_j}$. Hence $[\xi_j]_1^{P_1}$ satisfies part (ii) of Claim 1 and therefore there exists $G \in \mathcal{F}^M_{\xi_{k_j}}$ with $\min G \geq l_{j+1}$ and

$$\langle [\xi_j]_1^{P_1}, G \rangle > 1 - 3\epsilon' > 1 - \epsilon.$$

Since $l_{j+1} \geq k_j'$, we get that $G \in \mathcal{F}^M_{\xi}$ and the proof of step 3, as well as the proof of Case 3 and the proof of the lemma are complete.

2.1.10. Proposition: Let $\xi < \omega_1$. Then there exists $\delta_\xi > 0$ satisfying the following property

For every $M \in [\mathbb{N}]$ there exists $L \in [M]$ such that for every $P \in [L]$, $n \in \mathbb{N}$ there exists $G \in \mathcal{F}^M_{\xi}$ with $\langle \xi^P_n, G \rangle > \delta_\xi$.

Proof. If $\xi$ is a limit ordinal then we set $\delta_\xi = \frac{1}{2}$ and the previous lemma proves the desired result.

If $\xi$ is a successor ordinal then $\xi = \zeta + n$ with $\zeta$ a limit ordinal. Then we inductively prove that $\delta_\xi \geq \frac{1}{2^{n+1}}$.

Indeed, if $\xi = \zeta + 1$ choose $L \in [M]$ such that the conclusion of the previous lemma is satisfied for the ordinal $\zeta$ and the number $\epsilon = \frac{1}{2}$. We show that $L$ satisfies the conclusion of the proposition for $\xi = \zeta + 1$ and $\delta_\xi = \frac{1}{4}$. This is so, since for $P \in [L]$, $n \in \mathbb{N}$ there exists $P_1 \in [P]$ such that $\xi^P_n = \xi^P_1$ and further, if $l = \min \text{supp} \ \xi^P_1$ then $\xi^P_1 = \frac{1}{l} \sum_{j=1}^{l} \xi^P_j$. Choose
$G_j \subseteq \text{supp } \zeta_j^P$ such that $G_j \in \mathcal{F}_\zeta^M$ and $\langle \zeta_j^P, G_j \rangle > \frac{1}{2}$ and it is easy to see that if $d = \left\lceil \frac{1}{2} \right\rceil$ then the set $G = G_{d+1} \cup \cdots \cup G_l$ is in $\mathcal{F}_{\zeta+1}^M$ and that $\langle \xi_i^M, G \rangle > \frac{1}{4}$. This completes the proof for $\xi = \zeta + 1$. The general case is proved by a similar argument. The proof is complete.

2.1.12. Remark: A consequence of the above Lemma is that Schreier hierarchy is in a sense universal.
Indeed, consider $f : \mathbb{N} \to \mathbb{N}$ any strictly increasing function and define
$$\mathcal{F}_f^1 = \{ F \subseteq \mathbb{N} : \min F = n, \# F \leq f(n) \}.$$ $\mathcal{F}_f^1$ is an adequate family and the regular $\mathcal{F}_1$ is $\mathcal{F}_f^1$ for $f = \text{id}_{\mathbb{N}}$.
By iteration we produce $(\mathcal{F}_f^\xi)_{\xi < \omega_1}$ and the repeated averages hierarchy $[\xi]^M_\infty$ for $M \in [\mathbb{N}]$, $n \in \mathbb{N}$.
Next define the set $M = (m_i)_{i \in \mathbb{N}}$ by the rule $m_i = f(i)$. Then observe that if
$$\mathcal{F}_\xi^{f,M} = \{(m_i)_{i \in F}, F \in \mathcal{F}_f^I\},$$
we have that
$$\mathcal{F}_\xi^{f,M} = \mathcal{F}_\xi[M].$$
Therefore from the above lemma we get that there exists $L \in [M]$ such that for every $P \in [L]$, $n \in \mathbb{N}$ there exists $G \in \mathcal{F}_\xi^M$ with $\langle [\xi^M_p]_n, G \rangle > \delta_\xi$.
This shows that $\mathcal{F}_\xi^{f,M}$, $\mathcal{F}_\xi^M$ are comparable on the set $L$.

2.1.13. Lemma (Approximation Lemma) Let $\xi < \omega_1$, $M \in [\mathbb{N}]$, $\epsilon > 0$. We set $W = \text{co} (\{\xi_n^N : n \in \mathbb{N}, N \in [M]\})$.
Then for every ordinal $\zeta$ such that $\xi \leq \zeta < \omega_1$, $L \in [M]$ there exists $L_\zeta \in [L]$ satisfying the following property:
For every $L' \in [L_\zeta]$, $n \in \mathbb{N}$ we have that
$$d_{\mathcal{F}_\xi} (\zeta_n^{L'}, W) < \epsilon.$$

Proof. Fix $\xi < \omega_1$ $M \in [\mathbb{N}]$. We shall prove it by induction for $\zeta$ greater than $\xi$, every $L \in [M]$ and $\epsilon > 0$.
(i) $\zeta = \eta + 1$. Indeed, if $M \in [\mathbb{N}]$, $\epsilon > 0$ then there exist $L_\eta$ satisfying the conclusion for the ordinal $\eta$. Set $L_\zeta = L_\eta$ and it obvious that for every $L' \in [L_\zeta]$, $n \in \mathbb{N}$ we get the desired property $d_{\mathcal{F}_\xi} (\zeta_n^{L'}, W) < \epsilon$.
(ii) $\zeta$ is a limit ordinal. Then fix the strictly increasing sequence $(\zeta_n)_{n \in \mathbb{N}}$ of successor ordinals such that $\sup \zeta_n = \zeta$ and $(\zeta_n)_{n \in \mathbb{N}}$ defines the family $\mathcal{F}_\zeta$.
Since each $\zeta_n$ is a successor ordinal it has the form $\zeta_n = \xi_n + 1$. 21
Choose $L_0 \in [M]$ with $\min L_0 = m_1$ and $\frac{1}{m_1} < \frac{\epsilon}{4}$. We inductively choose $L_0 \supset L_1 \supset \cdots \supset L_k \supset \cdots$ such that if $n_k = \min L_k$ then $(n_k)_{k \in \mathbb{N}}$ is strictly increasing and $L_k = L_{n_k}$ for $M = L_{k-1}$, $\zeta = \xi_{n_{k-1}}$, $\epsilon = \frac{\epsilon}{4}$.

**Claim:** The set $N = (n_k)_{k \in \mathbb{N}}$ is strictly increasing and $L_k = L_{n_k} - 1$ for $M = L_k - 1$, $\zeta = \xi_{n_k} - 1$, $\epsilon = \frac{\epsilon}{4}$.

Indeed, let $L' \in [L_\xi]$, $n \in \mathbb{N}$. Then, by definition, $\zeta_{n_k}^{L'} = [\xi_{n_k}]_{L_k}^{L'}$, where $n_k = \min \text{supp} \zeta_{n_k}^{L'}$, $L_k' = \{m \in L' : n_k \leq m\}$. It is clear that $L_k' \setminus \{n_k\} \subset L_k$.

Since $\zeta_{n_k} = \xi_{n_k} + 1$, again by definition, $[\xi_{n_k}]_{L_k}^{L'}$ is an average of $n_k$ many successive elements of $[\xi_{n_k}]_{L_k}^{L'}$. Since all of them except the first one are $\frac{\epsilon}{2}$ approximated by convex combination of $W$ and $\frac{1}{n_k} < \frac{\epsilon}{4}$ we get that

$$d_\ell([\xi_{n_k}]_{L_k}^{L'}, W) < \epsilon$$

and hence $d_\ell([\xi_{n_k}]_{L_k}^{L'}, W) < \epsilon$. This completes the proof of the lemma.

### 2.2 Strong Cantor-Bendixson index

#### 2.2.1 Definition:
Let $\mathcal{F}$ be an adequate family. For $L \in [\mathbb{N}]$ we define the strong Cantor-Bendixson derivative of $\mathcal{F}[L]$ by the rule:

$$\mathcal{F}[L]^{(1)} = \{A \in \mathcal{F}[L] : \forall N \in [L] : A \text{ is a limit point of } \mathcal{F}[A \cup N]\}.$$

**Remark:** It is clear that $\mathcal{F}[L]^{(1)}$ is closed and nowhere dense subset of $\mathcal{F}[L]$ which is also adequate.

If $\xi = \zeta + 1$ then we, inductively, define $\mathcal{F}^{(\xi)}[L] = (\mathcal{F}^{(\zeta)}[L])^{(1)}$ if $\xi$ is a limit ordinal then we set

$$\mathcal{F}^{(\xi)}[L] = \bigcap_{\zeta < \xi} \mathcal{F}^{(\zeta)}[L].$$

We define the S.C.B. index of $\mathcal{F}[L]$ as the smallest ordinal $\xi_0$ such that $\mathcal{F}^{(\xi_0)}[L] = \emptyset$.

We denote this index by $s(\mathcal{F}[L])$.

#### 2.2.3 Proposition:
If $\xi$ is a limit ordinal and $L \in [\mathbb{N}]$ such that $s(\mathcal{F}[L]) > \xi$ then for every $N \in [L]$ we have that $s(\mathcal{F}[N]) > \xi$.

**Proof.** We will show that for every ordinal $\zeta$ satisfying $\zeta < \xi$ we have that $s(\mathcal{F}[N]) > \zeta$.

Indeed, if $\zeta < \xi$ then $\mathcal{F}^{(\zeta+1)}[L] \neq \emptyset$ hence there exists $A \in \mathcal{F}[L]$ with $A$ a limit point of $(\mathcal{F}^{(\xi)}[L])[A \cup N]$. Since $\mathcal{F}^{(\xi)}[L]$ is adequate, we get that
there exists at least one nonempty subset of \( N \) that belongs to \( \mathcal{F}^\xi[L] \). It is also easy to see that \( \mathcal{F}^\xi[L] \cap [N]^{<\omega} \subset \mathcal{F}^\xi[N] \) hence \( \mathcal{F}^\xi[N] \neq \emptyset \) and \( s(\mathcal{F}[N]) > \xi \). Since \( \xi \) is a limit ordinal and \( s(\mathcal{F}[N]) > \xi \) for all ordinals \( \zeta < \xi \) we get that \( s(\mathcal{F}[N]) > \xi \).

This completes the proof.

2.2.4 Proposition: If \( \mathcal{F} \) is an adequate family, \( \xi \) is a limit ordinal, \( L \in [N] \), such that \( L \) is almost contained in \( N \), then \( s(\mathcal{F}[N]) > \xi \) implies that \( s(\mathcal{F}[L]) > \xi \).

Proof. Similar to the previous one.

2.2.5 Notation: In the sequel we will denote by \( \mathcal{F}[L]^{\xi} \) the \( \xi \)-derivative of \( \mathcal{F}[L] \), while for \( N \in [L] \) we denote by \( \mathcal{F}^{\xi}[N] \) the restriction of \( \mathcal{F}^{\xi}[L] \) on the set \( N \).

2.2.6. Theorem: Let \( \mathcal{F} \) be an adequate family. If \( L \in [N] \) such that \( s(\mathcal{F}[L]) > \omega^\xi \) then there exists \( M \in [L] \), \( M = (m_i)_{i \in \mathbb{N}} \) satisfying the property: \( \mathcal{F}_M^\xi \) is a subfamily of \( \mathcal{F}[M] \).

Proof. We prove it by induction. For \( \xi = 0 \) the result is obvious. Suppose that we have proved it for all \( \zeta < \xi \). To prove it for \( \xi \) we will use a method created by Kiriakouli – Negrepontis [M-N]. This method consists in a double induction. We start with the next definition.

Definition: An \( n \)-tuple \( (\xi_1, \ldots, \xi_n) \) has property (A) if for every adequate family \( \mathcal{F} \) with
\[
s(\mathcal{F}) > \omega^{\xi_n} + \omega + \cdots + \omega^{\xi_1} + \omega
\]
and for every \( L \in [N] \) there exists \( N \in [L] \) such that \( N = (n_i)_{i \in \mathbb{N}} \), and for every \( F_1 \in \mathcal{F}_{\xi_1}, \ldots, F_n \in \mathcal{F}_{\xi_n} \) with \( F_1 < F_2 < \cdots < F_n \) the set \( \{n_i : i \in \bigcup_{k=1}^n F_k\} \) belongs to \( \mathcal{F}[N] \).

2.2.7 Lemma: Suppose that \( (\xi_1, \xi_2, \ldots, \xi_n) \) has property (A) and \( \zeta < \omega_1 \). Then \( (\zeta, \xi_1, \xi_2, \ldots, \xi_n) \) has also property (A).

Proof. We prove, by induction that for every \( \zeta < \omega_1, l \in \mathbb{N} \) the tuple
\[
(\zeta, \ldots, \zeta, \xi_1, \ldots, \xi_n) \text{ has property A.}
\]
Case 1. $\zeta = 0$. Given $(\eta_1, \ldots, \eta_k)$ with the property (A) we show that $(0, \eta_1, \ldots, \eta_k)$ has also property (A).

Indeed, start with $\mathcal{F}$ adequate such that

$$s(\mathcal{F}) > \omega^{n_k} + \omega + \cdots + \omega^{n_1} + \omega + 1 + \omega.$$ 

Since $s(\mathcal{F})$ is greater than a limit ordinal we get that for every $L \in \mathbb{N}$

$$s(\mathcal{F}[L]) > \omega^{n_k} + \omega + \cdots + \omega^{n_1} + \omega + \omega.$$ 

Therefore if we denote by $\zeta = \omega^{n_k} + \omega + \cdots + \omega^{n_1} + \omega$ we get that $\mathcal{F}[L] \zeta$ is an infinite set and since $\mathcal{F}[L] \zeta$ is an adequate family, there exists $M = (m_i)_{i \in \mathbb{N}}$ such that $\{m_i\} : i \in \mathbb{N}$ is a subfamily of $\mathcal{F}[L] \zeta$.

Observe that the set $G_{m_1} = \{ F \in \mathcal{F}[M \setminus \{m_1\}] : \{m_1\} \cup F \in \mathcal{F}[M] \}$ is an adequate family and $s(G_{m_1}) > \zeta$. This is so since $s(\mathcal{F}[M]) > \zeta$ and $\{m_1\} \in \mathcal{F}[M] \zeta$. From the inductive assumption there exists $M_1 \in [M]$ such that for every

$$F_1 \in \mathcal{F}_{n_1}, \ldots, F_k \in \mathcal{F}_{n_k} \text{ with } F_1 < F_2 < \cdots < F_k$$

implies that

$$D = \{m_i^1 : i \in \bigcup_{j=1}^k \mathcal{F}_{n_j}\} \in G_{m_1}[M_1],$$

where $M_1 = (m_i^1)_{i \in \mathbb{N}}$.

Then clearly for any such $D$ the set $\{m_1\} \cup D$ belongs to $\mathcal{F}[M]$.

Set $n_1 = m_1$, $n_2 = m_1^1$ and repeat the same procedure by defining $\mathcal{G}_{n_2}$ and finding $M_2 \in [M_1]$ such that $M_2 = (m_2^1)$, $n_2 < m_1^1$ and if

$$F_1 \in \mathcal{F}_{n_1}, \ldots, F_k \in \mathcal{F}_{n_k} \text{ satisfying } F_1 < F_2 < \cdots < F_k$$

then the set $\{n_2\} \cup \{m_i^2 : i \in \bigcup_{j=1}^k F_j\}$ belongs to $\mathcal{F}[M_2]$. Following the same procedure, we, inductively, choose $n_l$, $M_l$ with

$$M = M_0 \supset M_1 \supset M_2 \supset \cdots \supset M_l, \text{ and } n_l \in M_{l-1},$$

satisfying the above properties. It follows now immediately that the set $N = (n_l)_{l \in \mathbb{N}}$ satisfies the required properties and hence $(0, \eta_1, \eta_2, \ldots, \eta_k)$ has property (A).
Case 2. $\xi = \zeta + 1$. Then by the inductive hypothesis, for every $k$-tuple $(\eta_1, \eta_2, \ldots, \eta_k)$ with the property (A) and every $l \in \mathbb{N}$ the $l + k$-tuple $(\zeta, \zeta, \ldots, \zeta, \eta_1, \ldots, \eta_k)$ has the property (A).

For every $L \in [\mathbb{N}]$ such that

$$s(\mathcal{F}[L]) > \omega^n + \omega + \cdots + \omega^n + \omega + \omega^\xi + \omega$$

we have that for every $l \in \mathbb{N}$

$$s(\mathcal{F}[L]) > \omega^n + \omega + \cdots + \omega^n + \omega + \underbrace{\omega^\xi + \omega + \cdots + \omega^\xi + \omega}_{l\text{-times}}.$$

Hence we can find $L \supset L_1 \supset \cdots \supset L_l \supset \cdots$ with $L_l$ satisfying the property:

if $F_1 \in \mathcal{F}_\zeta, \ldots, F_l \in \mathcal{F}_\zeta, F_{l+1} \in \mathcal{F}_{\eta_1}, \ldots, F_{l+k} \in \mathcal{F}_{\eta_k}$

and $F_1 < \cdots < F_{l+k}$ then $\{m_l^i : i \in \bigcup_{j=1}^{l+k} F_j\} \in \mathcal{F}[L_l]$.

Then if $N = (n_l)_{l \in \mathbb{N}}$ with $n_l \in L_l$, it is easy to see that $N$ satisfies the required properties hence $(\xi, \eta_1, \ldots, \eta_k)$ has property (A).

Case 3. $\xi$ is a limit ordinal. The proof is similar to the previous case.

**Proof of the theorem: Case 1. $\xi = \zeta + 1$.**

Since $s(\mathcal{F}) > \omega^\xi > l \cdot (\omega^\zeta + \omega)$ and $(\zeta, \zeta, \ldots, \zeta)$ has property (A) then for any $L \in [\mathbb{N}]$ choose

$$L = L_0 \supset L_1 \supset L_2 \supset \cdots \supset L_l \supset \cdots$$

such that $L_l$ witnesses property (A) for the set $\mathcal{F}[L_{l-1}]$ and the $l$-tuple $(\zeta, \ldots, \zeta)$.

It is easy to show that for any $N \in [L]$ such that $N = (n_l)_{l \in \mathbb{N}}$ and $n_l \in L_l$ $\mathcal{F}[N]$ satisfies the inductive assumption for the ordinal $\xi$.

Case 2. $\xi$ is a limit ordinal. Let $(\xi_n)_{n \in \mathbb{N}}$ be the strictly increasing sequence with $\sup \xi_n = \xi$ that defines the family $\mathcal{F}_\xi$. For every $L \in [\mathbb{N}]$ we choose $L = L_0 \supset L_1 \supset \cdots \supset L_n \supset \cdots$ such that $L_n$ witnesses property (A) for the $(\xi_n)$ and the set $\mathcal{F}[L_{n-1}]$. If we set $N = (k_n)_{n \in \mathbb{N}}$ such that $k_n \in L_n$ then we easily check that $\mathcal{F}[N]$ satisfies the inductive assumption for the ordinal $\xi$.

The proof of the theorem is complete.
2.3 Large families

2.3.1 Definition: Let $\mathcal{F}$ be an adequate family, $M \in [N]$, $\xi < \omega_1$ and $\epsilon > 0$. We say that $\mathcal{F}$ is $(M, \xi, \epsilon)$ large if for every $N \in [M]$ and every $n \in \mathbb{N}$ we have that

$$\sup_{F \in \mathcal{F}} \langle \xi_n^N, F \rangle > \epsilon.$$ 

2.3.2 Proposition: If $\mathcal{F}$ is an adequate family which is $(M, \xi, \epsilon)$ large. Then for every $L \in [M]$ there exists $N \in [L]$ such that $s(\mathcal{F}[N]) > \omega^\xi$.

This proposition is one of the basic ingredients for the proof of the main Theorems of this section. The result of this in connection with Theorem 2.2.6 shows that every $(M, \xi, \epsilon)$ large family $\mathcal{F}$ contains the family $\mathcal{F}^{N'}_{\xi}$. Hence the summability methods \{$(\xi_n^N)_{n \in \mathbb{N}}, N \in [M]$\} are sufficiently many to describe the Schreier family $\mathcal{F}^{N'}_{\xi}$. The proof of the proposition depends strongly on infinite Ramsey theorem and the stability properties P.3 – P.4 of the RA hierarchy.

Proof of the proposition: We proceed by induction. The inductive hypothesis is the statement of the proposition.

Case 1. $\xi = 0$. This is the easiest case since the result immediately follows from the definitions.

Case 2. $\xi$ is a limit ordinal. In this case we prove first the following.

Claim: For every ordinal $\zeta$ with $\zeta < \xi$ and every $L \in [M]$ there exists $N \in [L]$ such that $s(\mathcal{F}[N]) > \omega^\zeta$.

Indeed, given $L \in [M]$ we define a partition of $[L]$ into $A_1, A_2$ by the rule:

$$A_1 = \left\{ N \in [L] : \sup_{F \in \mathcal{F}} \langle \xi^N_1, F \rangle \leq \frac{\epsilon}{2} \right\},$$

$$A_2 = [L] \setminus A_1.$$ Notice that if $N = (m_i)_{i \in \mathbb{N}}$ and $N' = (m'_i)_{i \in \mathbb{N}}$ are such that $m_i = m'_i$ for all $i \leq k = \max \sup \xi^N_1$ then by P.3 we get that $\xi^N_1 = \xi^{N'}_1$. Hence $A_1$ is an open set. Therefore from infinite Ramsey theorem we get that there exists $L_1 \in [L]$ such that either $[L_1] \subset A_1$ or $[L_1] \subset A_2$.

Assume that $[L_1] \subset A_1$. Then by P.4 we have that for every $N \in [L_1]$ and every $n \in \mathbb{N}$ we have that $\sup_{F \in \mathcal{F}} \langle \xi^N_n, F \rangle \leq \frac{\epsilon}{2}$. This is so since any such $\xi^N_n$ is equal to $\xi^{N'}_1$ for some $N' \in [N]$. But then, from Lemma 2.1.13 there exists $L_\xi \in [L_1]$ such that for every
\( n \in \mathbb{N} \ d_\ell (\xi_n^L, W) < \frac{\varepsilon}{2} \) where \( W = \text{co}(\{ \zeta_n^N : n \in \mathbb{N}, N \in [L_1] \}) \). Hence \( \sup_{F \in \mathcal{F}} \langle \xi_n^L, F \rangle \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \), a contradiction, therefore \( [L_1] \subset A_2 \).

This means that the family \( \mathcal{F}[L_1] \) is \( (L_1, \zeta, \frac{\varepsilon}{2}) \) large and by the inductive assumption we get that there exists \( N \in [L_1] \) such that \( s(\mathcal{F}[N]) > \omega^\xi \). Next choose a strictly increasing sequence of ordinals \( (\xi_n)_{n \in \mathbb{N}} \) with \( \sup \xi_n = \xi \).

Inductively we choose \( L = L_0 \supset L_1 \supset L_2 \supset \cdots \supset L_n \supset \cdots \) such that \( s(\mathcal{F}[L_n]) > \omega^\xi \).

It is easy to see that for every \( N, N \in [L] \) such that \( N \) almost contained in \( L_n \) we have that \( s(\mathcal{F}[N]) > \omega^\xi \) and therefore \( s(\mathcal{F}[N]) > \omega^\xi \). The proof for case 2 is complete.

**Case 3.** \( \xi = \zeta + 1 \). We start with the following Lemma, the proof of which uses again infinite Ramsey Theorem.

### Lemma

Let \( \xi = \zeta + 1, M \in [\mathbb{N}], \epsilon > 0 \) and \( \mathcal{F} \) an adequate family that is \( (M, \xi, \epsilon) \) large. Then for every \( L \in [M] \) and every \( n \in \mathbb{N} \) there exists \( L_n \in [L] \) such that for every \( N \in [L_n] \) and \( k \in \mathbb{N} \)

\[
\sup_{F \in \mathcal{F}} \min \{ \langle \zeta_k^N, F \rangle : 1 \leq k \leq n \} > \frac{\varepsilon}{2}.
\]

**Proof.** Consider \( L \in [M] \) and \( n \in \mathbb{N} \), and define a partition of \([L]\) into \( A_1, A_2 \) by the rule

\[
A_1 = \{ N \in [L] : \exists F \in \mathcal{F}, \{ \langle \zeta_k^N, F \rangle > \frac{\varepsilon}{2} \text{ for } k = 1, \ldots, n \} \}
\]

and \( A_2 = [L] \setminus A_1 \).

As in the previous lemma, \( A_1 \) is an open set hence by infinite Ramsey Theorem there exists \( L_n \in [L] \) such that \([L_n] \subset A_1 \) or \([L_n] \subset A_2 \).

We will show that the second case is not possible and this will prove the lemma.

Indeed, assuming that \([L_n] \subset A_2 \) we get that for every \( N \in [L_n] \) and every \( k_1 < k_2 < \cdots < k_n \) and every \( F \in \mathcal{F} \)

\[
\min \{ \langle \zeta_{k_i}^N, F \rangle, i = 1, 2, \ldots, n \} \leq \frac{\varepsilon}{2}.
\]

This follows from the fact that there exists \( N' \subset N \) such that \( \zeta_i^N = \zeta_i^{N'} \) for all \( i = 1, \ldots, n \).

Each \( \xi_n^L \) is an average of successive elements of \( (\zeta_k^L)_{k \in \mathbb{N}} \), that is \( \xi_n^L = \)
1 \sum_m \left( \zeta_{k_m}^{L_n} + \cdots + \zeta_{k_m+s_m}^{L_n} \right) \) and \((s_m)_{m \in \mathbb{N}}\) is strictly increasing. Choose \( F \in \mathcal{F} \) such that \( \langle \zeta_m^{L_n}, F \rangle > \epsilon \).

Then for large \( s_m \) we get that \# \( \{ i : \langle \zeta_{k_m+1}^{L_n}, F \rangle > \frac{\epsilon}{2} \} \) > \( n \).

But this contradicts (1) and the proof is complete.

2.3.4 Lemma: Assume that \( \xi, M, \epsilon, \mathcal{F} \) are as in the previous lemma. Then for every \( L \in [M] \) there exists \( N \in [L] \) such that for every \( N' \in [N] \) with \( \min N' \geq n \) we have that

\[
\sup_{F \in \mathcal{F}} \min \{ \langle \zeta_{k}^{N'}, F \rangle : k = 1, \ldots, n \} > \frac{\epsilon}{2}.
\]

Proof. We apply, inductively, the previous lemma and we choose \( L \supset L_1 \supset \cdots \supset L_n \supset \cdots \) such that the set \( L_n \) satisfy the conclusion of the previous lemma for the number \( n \). Then any set \( N = (m_n)_{n \in \mathbb{N}} \) with the property \( m_n \in L_n \) has the desired property.

2.3.5 Lemma: Let \( \zeta < \omega_1, M \in [\mathbb{N}], \epsilon > 0 \) and \( \mathcal{F} \) be an adequate family. Suppose that for some \( n \in \mathbb{N} \) we have that for every \( L \in [M] \)

\[
\sup_{F \in \mathcal{F}} \min \{ \langle \zeta_{k}^{L}, F \rangle : k = 1, \ldots, n \} > \epsilon.
\]

Suppose that \( \zeta \) satisfies the inductive assumption. Then for every \( L \in [M] \) there exists \( N \in [L] \) such that

\[
s(\mathcal{F}[N]) > n \cdot \omega^\zeta.
\]

Proof. We proceed by induction on \( \mathbb{N} \).

Case 1. \( k = 1 \). As we have shown in previous proofs, the fact that for \( N \in [M] \)

\[
\sup_{F \in \mathcal{F}} \langle \zeta_{k}^{N}, F \rangle > \epsilon \implies \mathcal{F} \text{ is } (M, \zeta, \epsilon) \text{ large hence by the inductive assumption every } L \in [M] \text{ contains infinite subset } N \text{ such that } \sup_{F \in \mathcal{F}} \langle \zeta_{k}^{N}, F \rangle > \omega^\zeta.
\]

Case 2. \( k = n \). Assume that the Lemma has been proved for all \( k = 1, 2, \ldots, n - 1 \).

Given \( L \in [M] \), since for every \( N \in [L] \) the vector \( \zeta_{k}^{N} \) has finite support and rational coefficients we get that the set \( \{ \zeta_{k}^{N} : N \in [L] \} \) is countable and we order it as \( (\zeta_{k})_{n \in \mathbb{N}} \). Consider \( \zeta_{1} \) and fix \( L_1 \in [L] \) with

\[
\max \text{ supp } \zeta_{1} < \min L_1.
\]

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Let \( \{F_i\}_{i=1}^d \) be an enumeration of all nonempty subsets of \( \text{supp } \zeta_1 \). We define a partition of \([L_1]\) into a family \((A_i)_{i=1}^d\) defined by the rule

\[ A_i = \{ N \in [L_1] : \text{if } N' = \text{supp } \zeta_1 \cup N \text{ and } \exists F \in \mathcal{F} \text{ satisfying} \]

\[ \min\{ \langle \zeta_k^{N'}, F \rangle : k = 1, \ldots, n \} > \epsilon \]

and \( F \cap \text{supp } \zeta_1 \cap \text{supp } \zeta_1 = F_i \} \).

Each \( A_i \) is an open set hence by infinite Ramsey theorem we get that there exist \( i_0 \) and \( S_1 \in [L_1] \) such that for every \( N \in [S_1] \) there exists \( F \in \mathcal{F} \) with:

\[ \min\{ \langle \zeta_k^{N'}, F \rangle : k = 1, \ldots, n \} > \epsilon \]

and

\[ \text{supp } \zeta_1^{N'} \cap F = F_{i_0}. \]

Set \( G_1 = F_{i_0} \) and consider the set

\[ \mathcal{F}_{G_1} = \{ F \in \mathcal{F} : F \cap \text{supp } \zeta_1 = G_1 \}. \]

Then it is easy to see that \( S_1, \zeta, \mathcal{F}_{G_1}, n-1 \) satisfy the inductive assumptions hence there exists \( N_1 \in [S_1] \) such that

\[ s(\mathcal{F}_{G_1}[N_1]) > (n - 1) \cdot \omega^\xi. \]

As a consequence of this we get that \( G_1 \in \mathcal{F}_{G_1}[N_1]^{(n-1) \cdot \omega^\xi}. \)

Inductively choose \( L \supset N_1 \supset \cdots \supset N_k \supset \cdots \) and \( (G_k)_{k \in \mathbb{N}} \) such that

(i) \( \max \text{supp } \zeta_k < \min N_k \)

(ii) \( G_k \subset \text{supp } \zeta_k, G_k \in \mathcal{F} \) and \( \langle \zeta_k, G_k \rangle > \epsilon \)

(iii) \( G_k \in \mathcal{F}_{G_k[N_k]}^{(n-1) \cdot \omega^\xi} \)

where \( \mathcal{F}_{G_k} = \{ F \in \mathcal{F} : F \cap \text{supp } \zeta_k = G_k \}. \)

The choice is done as in the case \( \zeta_1 \).

Choose a set \( N \) that is almost contained in \( N_k \) for all \( k \in \mathbb{N} \).

Claim: For every \( k \in \mathbb{N} \) the set \( G_k \) belongs to \( \mathcal{F}_{G_k[N]}^{(n-1) \cdot \omega^\xi} \), where \( \mathcal{F}_{G_k[N]} \) is defined as:

\[ \mathcal{F}_{G_k[N]} = \{ F \in \mathcal{F}[\text{supp } \zeta_k \cup N] : F \cap \text{supp } \zeta_k = G_k \}. \]

Indeed, set \( N^k = \{ m \in N : \max \text{supp } \zeta_k < m \}. \)

Then since \( N \setminus N^k \) is finite, we get that \( G_k \in \mathcal{F}_{G_k[N]} \) provided that \( G_k \in \mathcal{F}_{G_k[N^k]} \). Further, \( N^k \) is almost contained in \( N_k \) and from (iii) and the fact that \( (n - 1) \cdot \omega^\xi \) is a limit ordinal we get that \( G_k \in \mathcal{F}_{G_k[N]} \).
To finish the proof of the lemma we prove the following.

Claim 2: There exists $N' \in [N]$ such that $\mathcal{F}[N']^{n \omega^\xi} \neq \emptyset$.

Indeed, consider the family

$$\mathcal{G}[N] = \{ F : \exists k \in N \text{ with } G_k \subset N, F \subset G_k \}.$$ 

It is easy to see that $\mathcal{G}[N]$ is an adequate family. Further, for $G_k \subset N$ we have $\mathcal{F}_{G_k}[N] \subset \mathcal{F}[N]$ hence for every $F \in \mathcal{G}[N]$ we have that $F \in \mathcal{F}[N](n^{-1}) \omega^\xi$.

So we get that $\mathcal{G}[N] \subset \mathcal{F}[N](n^{-1}) \omega^\xi$.

Notice also that for every $S \in [N]$ there exists $G_k \in \mathcal{G}[N]$ such that $\langle \zeta^S, G_k \rangle > \epsilon$. Hence $\mathcal{G}[N]$ is $(N, \zeta, \epsilon)$ large and by the inductive assumption there exists $N' \in [N]$ such that $\mathcal{G}[N']^{n \omega^\xi} \neq \emptyset$. Hence

$$\mathcal{F}[N']^{n \omega^\xi} = \left[ \mathcal{F}[N']^{(n^{-1}) \omega^\xi} \right]^{n \omega^\xi} \supset G[N']^{\omega^\xi} \neq \emptyset.$$ 

This completes the inductive proof of the lemma.

**Completion of the proof of the proposition.** Using the previous lemmas, for a given $L \in [M]$, we choose $L \supset L_1 \supset L_2 \supset \cdots \supset L_n \supset \cdots$ such that $s(\mathcal{F}[L_n]) > n \cdot \omega^\xi$. Then it is easy to see that if $N$ is any set almost contained in $L_n$ for all $n \in \mathbb{N}$ then $s(\mathcal{F}[N]) > n \cdot \omega^\xi$ for all $n \in \mathbb{N}$ and hence $s(\mathcal{F}[N]) > \omega^{\xi+1} = \omega^\xi$. The proof of the proposition is complete.

We conclude this section with the following proposition.

**2.3.6 Proposition:** Let $\xi < \omega_1$, $M \in [N]$, $\epsilon > 0$ and $\mathcal{F}$ be an adequate family. Suppose that there exists $L \in [M]$, $L = (m_n)_{n \in \mathbb{N}}$ satisfying the property

for every $n \in \mathbb{N}$, $N \in [L]$, $n \leq \min N$

$$\sup_{F \in \mathcal{F}} \{ \langle \xi^N, F \rangle : k = 1, 2, \ldots, n \} > \epsilon.$$

Then there exists $N' \in [L]$ such that $s(\mathcal{F}[N]) > \omega^{\xi+1}$.

**Proof.** Notice that $\mathcal{F}$ satisfies the assumptions of the previous lemma hence there exists a decreasing sequence $(L_n)_{n \in \mathbb{N}}$ of subsets of $L$ such that $s(\mathcal{F}[L_n]) > n \cdot \omega^\xi$. Now, if $N$ is almost contained in $L_n$ for all $n \in \mathbb{N}$, it is easy to see that $s(\mathcal{F}[N]) > \omega^{\xi+1}$.

**2.4 The main results**
We pass now to give the statements and the proofs of the main results.

2.4.1 Theorem: For \((x_n)_{n \in \mathbb{N}}\) weakly null sequence in a Banach space \(X\) and \(\xi < \omega_1\) exactly one of the following holds.

(a) For every \(M \in [\mathbb{N}]\) there exists \(L \in [M]\) such that for every \(P \in [L]\) the sequence \((x_n)_{n \in \mathbb{N}}\) is \((P, \xi)\) summable.

(b) There exists \(M \in [\mathbb{N}]\) \(M = (m_i)_{i \in \mathbb{N}}\) such that \((x_{m_i})_{i \in \mathbb{N}}\) is an \(\ell_{\xi+1}\) spreading model.

To prove the theorem we begin with the following lemma.

2.4.2 Lemma: Assume that \(F = (x_n)_{n \in \mathbb{N}}\) is a weakly null sequence and \(\xi < \omega_1\) such that for every \(M \in [\mathbb{N}]\) there exists \(N \in [M]\) with \((x_n)_{n \in \mathbb{N}}\) not \((N, \xi)\) summable. Then there exists \(\epsilon > 0\) and \(L \in [\mathbb{N}]\) such that for every \(N \in [L]\)

\[
\lim \|z_n^L\| > \epsilon, \quad \text{where} \quad z_n^L = \frac{\sum_{k=1}^{n} \xi_k^F n}{n}.
\]

Proof. We prove it for \(M = \mathbb{N}\). The general case is similar. For given \(\epsilon > 0, n \in \mathbb{N}\), we consider the set

\[A_{\epsilon, n} = \{M \in [\mathbb{N}]: \|z_k^M\| \leq \epsilon \forall k \geq n\}.\]

Clearly each \(A_{\epsilon, n}\) is a closed set hence the set \(A_\epsilon = \bigcup_{n=1}^{\infty} A_{\epsilon, n}\) is a Ramsey set. Therefore there exists \(L_\epsilon \in [\mathbb{N}]\) such that \([L_\epsilon] \subset A_\epsilon\) or \([L_\epsilon] \subset [\mathbb{N}] \setminus A_\epsilon\).

If there exists some \(\epsilon > 0\) and \(L \in [\mathbb{N}]\) such that \([L] \subset [\mathbb{N}] \setminus A_\epsilon\) then the lemma has been proved. Assume that this does not occur. Then inductively choose \(\mathbb{N} \supset L_1 \supset L_2 \supset \cdots \supset L_n \supset \cdots\) such that \([L_n] \subset A_1/n\) and set \(L\) any infinite set almost contained in \(L_n\) for all \(n \in \mathbb{N}\). Set \(L\) any set that is almost contained in each \(L_n\).

Claim 1: If \(N \in [L]\) then \((x_n)_{n \in \mathbb{N}}\) is \((N, \xi)\) summable.

Indeed, for any such \(N\) and \(n \in \mathbb{N}\) there exists \(k_n \in \mathbb{N}\) such that for every \(k \in \mathbb{N}\) with \(k_n \leq k\) we have that supp \(\xi_k^N \subset L_n\). Therefore if \(N' = \bigcup_{k \geq k_n} \text{supp} \xi_k^N\), then by the property P.4 we get that \(\xi_s^{N'} = \xi_s^{N_{(s-1)+k_n}}\). Since \(N' \in [L_n]\), there exists \(s_0\) such that for all \(s \geq s_0\) we have that \(\|z_s^{N'}\| \leq \frac{1}{n}\). But then there exist large \(s_1\) such that for every \(s > s_1\) we have

\[
\|z_s^{N'}\| \leq \frac{\sum_{i=1}^{s} \xi_i^{N'} \cdot F}{s} = \frac{\sum_{i=1}^{k_{n-1}} \xi_i^{N'} \cdot F}{s} + \sum_{i=1}^{s} \xi_i^{N'} \cdot F - \frac{\sum_{i=s+1-k_n}^{s} \xi_i^{N'} \cdot F}{s}.
\]
\[
< \frac{k_n - 1}{s} + \frac{1}{n} + \frac{k_n - 1}{s} < \frac{2}{n}.
\]

This proves the Claim and it contradicts our assumptions. Hence there exists \( \epsilon > 0 \) and \( L \in [N] \) such that \([L] \) is a subset of \([N] \setminus A_\epsilon \) and this completes the proof of the lemma.

2.4.3 Lemma: Let \( F = (x_n)_{n \in \mathbb{N}} \) be a weakly null sequence. Suppose that for \( \xi < \omega_1, M \in [N] \) and \( \epsilon > 0 \) we have that for all \( N \in [M] \)

\[
\lim \|z^N_n\| > \epsilon \text{ for all } N \in [M], \text{ where } z^N_n = \frac{\sum_{i=1}^{n} \xi_i^N \cdot F}{n}.
\]

Then for every \( L \in [M] \) we have:

(a) For every \( n \in \mathbb{N} \) there exists \( L_n \in [L] \) such that for every \( N \in [L_n] \)

\[
a^N_n = \sup_{x^* \in B_{X^*}} \min \{x^*(\xi_k^N \cdot F) : k = 1, 2, \ldots, n \} > \frac{\epsilon}{2}.
\]

(b) There exists \( N \in [L], N = (m_n)_{n \in \mathbb{N}} \) such that:

for every \( N' \in [N] \) with \( m_n \leq \min N' \)

\[
a^{N'}_n > \frac{\epsilon}{2}.
\]

Proof. (a) For a given \( L \in [M] \) and \( n \in \mathbb{N} \) we define a partition of \([L] \) into \( A_1, A_2 \) such that:

\[
A_1 = \left\{ n \in [L] : a^N_n > \frac{\epsilon}{2} \right\}, \quad A_2 = [L] \setminus A_1.
\]

The set \( A_1 \) is a Ramsey set hence there exists \( L_n \) such that either \([L_n] \subset A_1 \) or \([L_n] \subset A_2 \). The first case proves part (a) of the Lemma. We show that the second case does not occur.

Indeed, if \([L_n] \subset A_2 \) then we get that for \( k_1 < k_2 < \cdots < k_n \) there exists \( N \in [L_n] \) such that \( \xi_1^{L_n} = \xi_1^N, \ldots, \xi_{k_n}^{L_n} = \xi_{k_n}^N \) and since \( N \in A_2, a^N_n \leq \frac{\epsilon}{2} \). Choose \( s \) large such that there exists \( x^* \in B_{X^*} \) with

\[
x^* \left( \frac{\sum_{i=1}^{s} \xi_i^{L_n} \cdot F}{s} \right) > \epsilon.
\]
Then from the choice of \( s \) we get that
\[
\# \left\{ i : i \leq s, \ x^* (\xi_i^L \cdot F) > \frac{\epsilon}{2} \right\} \geq n.
\]
But then there exists \( N \in [L_n] \) with \( a_{N}^{n} > \frac{x}{2} \), a contradiction and the proof of part (a) is complete.

(b) Choose, inductively, a decreasing sequence \( (L_n)_{n \in \mathbb{N}} \) such that \( L_n \in [L] \) and \( L_n \) satisfies the requirement for the number \( n \) of part (a). It is clear that any \( N \) almost contained in \( L_n \) for all \( n \in \mathbb{N} \) is the desired set.

Next we will prove two lemmas that will help us to reduce the proof of the theorem to the case of the sequence \( (\pi_n)_{n \in L} \) of the natural coordinate projections of \( \{0,1\}^N \) acting on an adequate family \( \mathcal{F} \) of finite subsets of \( \mathbb{N} \).

2.4.4 Definition: Let \( D \) be a weakly compact subset of \( c_0(\mathbb{N}) \) and \( \delta > 0 \). We set
\[
\mathcal{F}_{\delta} = \{ F \subset \mathbb{N} : \exists f \in D \text{ with } f(n) \geq \delta \ \forall \ n \in F \}.
\]

2.4.5 Remark: The weak compactness of \( D \) implies that \( \mathcal{F}_{\delta} \) is an adequate family of finite subsets of \( \mathbb{N} \).

2.4.6 Notation: In the sequel we denote by
\[
\mathcal{F}_{\delta}[\mathbb{N}] = \left\{ F : F \in [\mathbb{N}]^{<\omega}, \ \exists G \in \mathcal{F}_{\delta} : G \cap N = F \right\}.
\]
Notice that \( \mathcal{F}_{\delta}[\mathbb{N}] \) as a projection of \( \mathcal{F}_{\delta} \) is also a compact family.

The next Lemma is a consequence of Lemma 1.2.

2.4.7 Lemma: Let \( D \) be a weakly compact subset of \( c_0(\mathbb{N}) \). Then for every \( \delta > 0, \epsilon > 0 \) and \( M \in [\mathbb{N}] \) there exists \( N \in [M] \) such that for every \( F \in \mathcal{F}_{\delta}[\mathbb{N}] \) and \( G \subset F \) there exists \( f \in D \) such that
(i) \( \min \{ f(n) : n \in G \} \geq (1 - \epsilon)\delta \)
(ii) \( \sum_{n \notin G} |f(n)| < \epsilon \cdot \delta \).

Proof. From Lemma 1.2 for every \( M \in [\mathbb{N}] \) there exists \( N \in [M] \) such that for every \( k \in \mathbb{N} \), \( \max F \leq k \) there exists \( f_k \in D \)
(i) \( \min \{ f_k(n) : n \in G \} > (1 - \epsilon)\delta \)
(ii) \( \sum_{n \notin G} |f_k(n)| < \epsilon \cdot \delta \).

The desired \( f \) is weak limit of any weakly convergent subsequence of \( (f_k)_{k \in \mathbb{N}} \).
2.4.8 Lemma (Reduction lemma) Let $H = (x_n)_{n \in \mathbb{N}}$ be a weakly null sequence in a Banach space with $\|x_n\| \leq 1$. Then for every $\delta > 0$ and $\epsilon > 0$ there exists an adequate family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ and a function $f : B_{X^*} \rightarrow \mathcal{F}$ such that:

For every $M \in [\mathbb{N}]$ there exists $N \in [\mathbb{N}]$ satisfying the following properties

(a) If $A \in S^+_n$, supp $A \subseteq N$ then for every $x^* \in B_{X^*}$, $\langle x^*, A \cdot H \rangle > \delta$ we have $\langle A, f(x^*) \rangle > \frac{\delta^2}{4}$.

(b) If $A \in S^+_n$, supp $A \subseteq N$ and $F \in \mathcal{F}$ such that $\langle A, F \rangle \geq \frac{1}{2}$ then $\|A \cdot H\| \geq \frac{\epsilon \delta}{4}$.

Proof. We start by noticing that if $A \in S^+_n$ and $x^* \in B_{X^*}$ such that $A = (a_n)_{n \in \mathbb{N}}$ and $x^*(A \cdot H) > \delta$ then for $F = \{n \in \mathbb{N} : x^*(x_n) > \frac{\delta}{2}\}$ we get that $\sum_{n \in F} a_n > \frac{\delta}{2}$. Hence $\langle A, F \rangle > \frac{\delta^2}{4}$.

Since $(x_n)_{n \in \mathbb{N}}$ is a weakly null sequence, the set

$$D = \{(x^*(x_n))_{n \in \mathbb{N}} : x^* \in B_{X^*}\}$$

is a weakly compact subset of $c_0(\mathbb{N})$. Applying Lemma 2.4.7 for $\frac{\delta}{2}$, $\frac{\epsilon}{2}$ and $M \in [\mathbb{N}]$ we find $N \in [M]$ satisfying properties (i) and (ii) of that lemma. We let $\mathcal{F}$ be the adequate family defined as $\mathcal{F}_M$. We also define $f : B_{X^*} \rightarrow \mathcal{F}$ by the rule $f(x^*) = \{n \in \mathbb{N} : x^*(x_n) > \frac{\delta}{2}\}$. Using our note at the beginning of the proof, we get that property (a) holds for every $N \in [M]$. To see property (b), suppose that $A \in S^+_n$ with supp $A \subseteq N$ and $F \in \mathcal{F}$ such that $\langle A, F \rangle \geq \epsilon$. Then we may assume that $F \subseteq N \cap \{n \in \mathbb{N} : a_n > 0\}$ and by the definition of $\mathcal{F}$ there exists $G \in \mathcal{F}_M$ such that $F \subseteq G \cap N$. Then there exists $x^* \in B_{X^*}$ such that

(i) $\min\{x^*(x_n) : n \in F\} \geq \epsilon \left(1 - \frac{\epsilon}{2}\right) \frac{\delta}{2}$

(ii) $\sum_{n \notin F} |x^*(x_n)| < \frac{\epsilon}{2}$.

From (i), (ii) and the fact that $\langle A, F \rangle \geq \epsilon$ we get that $\|A \cdot H\| > \epsilon \cdot \frac{\delta}{4}$. The proof is complete.

Proof of the theorem: We prove first that the negation of (a) implies (b). Suppose that $\|x_n\| \leq 1$ and for a given $\xi < \omega_1$ the case (a) does not occur. Then from Lemma 2.4.2 there exists $M \in [\mathbb{N}]$ and $\delta > 0$ such that $\lim \|z_n^L\| > 2\delta$ for all $L \in [M]$. Going to a subset of $M$ if it necessary, we may assume that part (b) of Lemma 2.4.3 is also satisfied for $M$.

Consider the family $\mathcal{F}$ defined in Lemma 2.4.8 for the sequence $(x_n)_{n \in \mathbb{N}}$ and the number $\delta$. Let $N \in [M]$ such that (a) and (b) in Lemma 2.4.8
are satisfied. Property (a) in connection with the fact that $N$ satisfies the conclusion of Lemma 2.4.3 show that the assumptions of Proposition 2.3.6 are fulfilled. Hence there exists $N' \in [N]$ such that $s(\mathcal{F}[N']) > \omega^{\xi+1}$. From Theorem 2.2.6 there exists $N_1 \in [N']$ such that $N_1 = (m_i)_{i \in N}$ and for every $F \in \mathcal{F}_{\xi+1}$ the set $\{m_i : i \in F\} \in \mathcal{F}$.

Claim: For every $F \in \mathcal{F}_{\xi+1}$, $\|\sum_{i \in F} a_i x_{m_i}\| \geq \frac{\delta}{\xi} \sum_{i \in F} |a_i|$.

Indeed, by standard arguments, it is enough to show it for $(a_i)_{i \in F} \in S_{\ell^1}$. If $(a_i)_{i \in F} \in S_{\ell^1}$ then either $\sum\{a_i : i \in F, a_i > 0\} \geq \frac{1}{2}$ or $\sum\{a_i : i \in F, a_i < 0\} \leq -\frac{1}{2}$. We assume that the first case occurs. Otherwise we consider the $(b_i)_{i \in F}$ such that $b_i = -a_i$ for all $i \in F$. Set $F' = \{i \in F : a_i > 0\}$; then clearly $\langle A, F' \rangle \geq \frac{1}{2}$ and hence $\|A \cdot (x_n)_{n \in N}\| > \frac{\delta}{8}$, which proves the claim. The proof is complete.

We pass now to show that parts (a) and (b) of Theorem 2.4.1 are mutually exclusive.

2.4.9 Proposition: Let $(x_n)_{n \in N}$ be a weakly null sequence in a Banach space $X$. If $\xi < \omega_1$, $M \in [N]$ and $\delta > 0$ are such that $M = (m_i)_{i \in N}$ and

$$\|\sum_{i \in F} a_i x_{m_i}\| \geq \delta \sum_{i \in F} |a_i|$$

for every $F \in \mathcal{F}_{\xi+1}$

then there exists $L \in [M]$ such that for every $P \in [L]$ $(x_n)_{n \in N}$ is not $(P, \xi)$ summable.

Proof. Consider the adequate family $\mathcal{F}$ defined in Reduction lemma (Lemma 2.4.8) for the sequence $(x_n)_{n \in N}$ the number $\delta$ in our assumptions and $\epsilon = \delta_{\xi+1}$ (Proposition 2.1.10). Find $N \in [M]$ such that conditions (a), (b) of Reduction Lemma are fulfilled. Denote by $\left(\left[\xi^M\right]^P_n\right)_{n \in N}$, $P \in [N]$, the summabilidys methods defined by the rule $\left[\xi^M\right]^P_n = (a_{m_i})_{i \in N}$ where $\xi^P_n = (a_i)_{i \in N}$ and $a_{m_i} = a_i$. Then $\mathcal{F}$ is $(N, \xi + 1^M, \frac{\delta^2}{\xi})$ large hence there exists $L' \in [N]$ such that $\mathcal{F}_{\xi+1}$ is a subfamily of $\mathcal{F}$ and hence by Proposition 2.1.11 there exists $L'' \in [L']$ such that for every $P \in [L'']$, $n \in N$ there exists $G \in \mathcal{F}_{\xi+1}$ such that $\langle (\xi + 1)^P_n, G \rangle > \delta_{\xi+1} > 0$. Choose, as in Lemma 2.4.3 (part (b)), $L \in [L'']$ such that $L = (l_n)_{n \in N}$ and for every $P \in [L]$ $n \leq k_1 < k_2 < \cdots < k_n$ there exists $G \in \mathcal{F}$ such that $\langle \xi^P_{k_i}, G \rangle > \frac{\delta_{k_i+1}}{2}$.

Then by part (b) of Reduction Lemma there exists $x^* \in B_{X^*}$ such that
\[ \langle \xi_k^p \cdot H, X^* \rangle \geq \frac{\delta \cdot \delta \cdot k}{8} \] where \( H = \langle x_n \rangle_{n \in \mathbb{N}} \) and \( (k_i)_{i=1}^{n} \), \( P \) as above. It is clear now that for every \( P \in [L] \) the sequence \( (x_n)_{n \in \mathbb{N}} \) is not \((P, \xi)\) summable.

2.4.10 Remark: The above Proposition immediately shows that parts (a) and (b) in the Theorem 2.4.1 are mutually exclusive.

For the sequel we need the following result proved in [A-A].

2.4.11 Proposition: Let \( X \) be a Banach space and \( (x_n)_{n \in \mathbb{N}} \) a weakly null sequence in \( X \).

(a) There exists \( \xi < \omega \) such that for all \( \zeta < \omega \), \( \xi \leq \zeta \) \( (x_n)_{n \in \mathbb{N}} \) does not contain a subsequence which is \( \ell_1^\xi \) spreading model.

(b) If \( \ell_1 \) does not embed into \( X \) then there exists \( \xi < \omega \) such that for every \( \zeta < \omega \), \( \xi \leq \zeta \) and any bounded sequence \( (x_n)_{n \in \mathbb{N}} \) there is no subsequence of \( (x_n)_{n \in \mathbb{N}} \) which is \( \ell_1^\xi \) spreading model.

Sketch of proof: The proof of this Proposition part (a) follows from the fact that \( T_\varepsilon = \{ (x_{n_1}, \ldots, x_{n_k}) : \left\| \sum_{i=1}^{k} a_i x_{n_i} \right\| \geq \varepsilon \cdot \sum_{i=1}^{k} |a_i| \} \) ordered in the usual manner is a well-founded tree. If not, the sequence \( (x_n)_{n \in \mathbb{N}} \) should contain a subsequence equivalent to \( \ell_1 \) basis that contradicts the weak nullness of \( (x_n)_{n \in \mathbb{N}} \). Therefore the height of \( T_\varepsilon \), denoted by \( o(T_\varepsilon) \), is a countable ordinal \( \xi_\varepsilon \). Further, if \( (x_n)_{n \in \mathbb{N}} \) has a subsequence that is \( \ell_1^\xi \) spreading model with constant \( \delta_\xi > \varepsilon \) then \( \omega_\xi \leq \xi_\varepsilon \). So if \( \xi_0 = \sup\{\xi_\varepsilon, \varepsilon > 0\} \) then every \( \xi < \omega_1 \) such that \( (x_n)_{n \in \mathbb{N}} \) has a subsequence which is \( \ell_1^\xi \) spreading model should satisfy \( \omega_\xi \leq \xi_0 \) and this proves the result for part (a).

The proof of part (b) is the same and uses the technique developed by Bourgain [B].

The Banach - Saks index

2.4.12 Definition Let \( X \) be a Banach space and \( (x_n)_{n \in \mathbb{N}} \) a weakly null sequence in \( X \).

(a) The Banach-Saks index of \( (x_n)_{n \in \mathbb{N}} \) denoted by \( BS[(x_n)_{n \in \mathbb{N}}] \) is the least ordinal \( \xi \) such that there is no subsequence of \( (x_n)_{n \in \mathbb{N}} \) which is \( \ell_1^\xi \) spreading model.

(b) If \( X \) is a Banach space not containing \( \ell_1^1(\mathbb{N}) \) then we denote by \( BS[X] \) the least ordinal \( \xi \) such that no bounded sequence \( (x_n)_{n \in \mathbb{N}} \) in \( X \) is an \( \ell_1^\xi \) spreading model.
2.4.13 Theorem: Let $H = (x_n)_{n \in \mathbb{N}}$ be a weakly null sequence with $BS[(x_n)_{n \in \mathbb{N}}] = \xi$. Then: \( \xi \) is the unique ordinal satisfying the following

(a) For every $M \in [\mathbb{N}]$ there exists $L \in [M]$ such that for every $P \in [L]$ \( \lim_{n \in \mathbb{N}} \| \xi^P_H \| = 0. \)

(b) For every $\zeta < \xi$ there exists $L_\zeta \in [\mathbb{N}]$ such that $L_\zeta = (n_i)_{i \in \mathbb{N}}$ and $(x_{n_i})_{i \in \mathbb{N}}$ is an $\ell_1^\zeta$ spreading model.

(c) If $\xi = \zeta + 1$ there exists $\epsilon > 0$ and $L \in [\mathbb{N}]$ such that for all $P \in [L]$ \( \| \zeta^P_H \| > \epsilon \) and $(\zeta^P_H)_{n \in \mathbb{N}}$ is Cesaro summable.

Proof. (a) For $L \in [\mathbb{N}]$ and $n \in \mathbb{N}$ we define a partition of $[L]$ into sets $A, B$ by the rule $A = \{ P : \| \xi^P_H \| \leq \epsilon \}$ and $B = [L] \setminus A$. It is easy to see that $A$ is a closed subset of $[L]$ hence by infinite Ramsey Theorem there exists $N \in [L]$ such that either $[N] \subset A$ or $[N] \subset B$. If the second case holds then by Reduction lemma we get that $(x_n)_{n \in \mathbb{N}}$ has a subsequence which is $\ell_1^\zeta$ spreading model, a contradiction. Hence $[N] \subset A$. Choose, inductively, $M \supset L_1 \supset L_2 \supset \cdots \supset L_n \supset \cdots$ such that for every $P \in [L_n]$ $\| \xi^P_H \| < \frac{1}{n}$ and set $L = (l_n)_{n \in \mathbb{N}}$ such that $l_n \in L_n$. Then it is easy to see $L$ satisfies the conclusion of the first part of the theorem.

(b) It follows from the definition of $\xi$.

(c) Suppose now that $\xi = \zeta + 1$. Then, by the definition of $BS[(x_n)_{n \in \mathbb{N}}]$, there exists $M \in [\mathbb{N}]$ such that $(x_n)_{n \in M}$ is an $\ell_1^\zeta$ spreading model. Then by part (a) of Theorem 2.4.1 there exists $N \in [M]$ such that for every $P \in [N]$ $(x_n)_{n \in \mathbb{N}}$ is $(P, \xi)$ summable and finally from Proposition 2.4.9 there exists $L \in [N]$ and $\epsilon > 0$ such that for every $P \in [L], n \in \mathbb{N}$, $\| \zeta^P_H \| \geq \epsilon$. This proves part (c) and the proof is complete.

2.4.14 Remark: (i) The first part of the above Theorem is satisfied by any normalized sequence in Tsirelson’s space. Any such sequence has Banach-Saks index equal to $\omega$.

(ii) The third part gives a complete answer in the following question posed by the first named author: For what weakly null sequences there exists a sequence $(y_n)_{n \in \mathbb{N}}$ of block convex combinations such that $\| y_n \| > \epsilon$ and $(y_n)_{n \in \mathbb{N}}$ is Cesaro summable.

We conclude this Section with the following corollaries. Their proofs follow easily from the previous theorems.

2.4.15 Corollary: For every separable reflexive Banach space $X$ there
exists a unique ordinal $\xi < \omega_1$ such that

(i) For all ordinals $\zeta \geq \xi$ the space $X$ has $\zeta$-BS.

(ii) For every $\zeta < \xi$ the space $X$ fails $\zeta$-BS.

2.4.16 Corollary: If $X$ is a separable Banach space not containing isomorphically $\ell^1$ then there exist a unique ordinal $\xi < \omega_1$ such that

(i) For all ordinals $\zeta \geq \xi$ the space $X$ has $w\zeta$-BS.

(ii) For every $\zeta < \xi$ the space $X$ fails $w\zeta$-BS.

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