Bekenstein bound from the Pauli principle

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Assuming that the degrees of freedom of a black hole are finite in number and of fermionic nature, we naturally obtain, within a second-quantized toy model of the evaporation, that the Bekenstein bound is a consequence of the Pauli exclusion principle for these fundamental degrees of freedom. We show that entanglement, Bekenstein and thermodynamic entropies of the black hole all stem from one main proposal: i) the degrees of freedom (dof) responsible for the BH entropy have to take into account both matter and spacetime and hence must be of a new, more fundamental nature than the dof we know (with Feynman [3], here we call such dof “Xons”, see also [4] and [5]); ii) the Hilbert space $\mathcal{H}$ of the Xons of a given BH is necessarily finite dimensional

$$\dim \mathcal{H} = e^{S_{BH}},$$

with $S_{BH}$ the Bekenstein entropy. With these, in [1] it was shown that the (average) loss of information is an unavoidable consequence of the non-vanishing relic entanglement between the evaporated matter and spacetime.

In search of a unifying view of the various types of entropies involved in the BH evaporation (i.e., Bekenstein, thermodynamical, and entanglement entropies, see, e.g., [6]), we reverse here that logic. Namely, we start off by supposing that in a BH only free Xons exist (hence there can only be one kind of entropy at that level), and we suppose that they are finite in number and fermionic in nature. This amounts to have a finite dimensional $\mathcal{H}$. With these assumptions, we show here that the evaporation is a dynamical mechanism producing a maximal entanglement entropy, equal to the initial entropy of the BH.

This is an instance of the Bekenstein bound, obtained here with arguments that do not assume pre-existing geometrical (spatiotemporal) concepts. In fact, for a full identification with the standard formulae (see, e.g., [7]) one needs to associate a geometrical concept to the Xons, such as one dof per elementary Planck cell. Nonetheless, in our picture we do not need the exact expression of the bound. What is crucial is that the Xons are taken to be finite in number and fermionic, otherwise the entanglement entropy would just indefinitely grow without reaching a maximal value. It is suggestive, though, that taking on board the geometric picture of Xons as quanta of area (Planck cells), the horizon of the BH is of nonzero size as an effect of a Pauli exclusion principle. Before entering the details of what just discussed, let us now briefly put our work into the context of current literature.

Bekenstein entropy [8, 9] is traditionally regarded as a measure of our ignorance about the dof which formed the BH [9, 12] and as a consequence of the no-hair theorem [14]. However, other interpretations have been proposed in literature, as in Loop Quantum Gravity (LQG), where BH entropy is a counting of microstates corresponding to a given macroscopic horizon area $A$ [14, 15]. Along these lines, Bekenstein proposed a universal upper bound for the entropy of any physical system contained in a finite region [2], which is saturated by BHs. This implies [16] that the entropy of every system in a finite region is bounded from above by the Bekenstein entropy of a BH, whose horizon coincides with the area of the boundary of that region (see also [7]).

Using the approach of quantum field theory (QFT) in curved spacetimes, Hawking discovered the black body spectrum of BH radiation [17]. In the meantime, Umezawa and Takahashi developed their Thermofield dynamics (TFD) [18] (see also Ref. [19]), that immediately appeared to be a fruitful tool for the description of BH evaporation [20]. In [21], with the help of an entropy operator, whose structure is natural in TFD, the BH-radiation entropy is viewed as an entanglement entropy of radiation modes with their “TFD-double” (the modes beyond the horizon).

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*To our friend, colleague and inspirer Martin Scholtz (1984 - 2019).
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1 The exact proportionality factor, $1/4$, is beyond the reach of our analysis. Henceforth, we need not specify the exact number of Xons per the Planck cells, but only that this number is finite, as is the case for fermions.
Although the relation between QFT in curved spacetime and TFD was studied already in Refs. [22, 23], the renewed interest comes in connection with the AdS/CFT correspondence [24], where in a two-sided Anti-de Sitter (AdS) BH, the speculative asymptotic region is mapped into two copies of a conformal field theory (CFT). The thermal nature of the BH is then naturally seen through TFD. Extensions to incorporate dissipative effects are in the recent [25, 26].

Since a BH, initially described as a pure state, could end up in a mixed state (this is actually the view of [11]), questions arise on the unitary evolution, as first noticed by Hawking [27] and then extensively discussed, from different points of view, see e.g. Refs. [28, 29]. In particular, in Refs. [30, 31] Page studied the bipartite system BH-radiation, in a random (Haar distributed) pure state, computing the radiation entanglement entropy as function of the associated thermodynamical entropy. He found a symmetric curve (Page curve) which goes back to zero when the BH is completely evaporated. In Ref. [32] he postulated that entanglement, as function of time, follows the minimum between Bekenstein and radiation thermodynamic entropy (Conjectured Anorexic Triangular Hypothesis). Recently [33], Page curve was also derived from holographic computations [33].

As said, in this paper we reverse the line of reasoning of Ref. [11] and present a simple, purely quantum toy-model of the dynamics of BH evaporation, focusing on the fundamental dof. In Section IV the basic assumptions are the finiteness of slots (quantum levels) available for the system, and the fermionic nature of such dof. The finiteness of the Hilbert space of states follows from the Pauli exclusion principle. In Section III we compute the von Neumann entropy of the subsystems during their evolution. This is remarkably given by the expectation value of the TFD entropy operator [21] and it has the same qualitative behavior of Page curve: it starts from zero and ends in zero, while its maximum is reached at half of the evaporation process. That maximum is identified here with the Bekenstein entropy of the BH at the beginning of the evaporation. We can therefore argue that Bekenstein bound itself descends from the Pauli principle. In Section IV we explain the relation with TFD by mapping our model to an equivalent description as a dissipative-like system. The last Section is left to our conclusions, while in the Appendix we show the connection between TFD and von Neumann entropies in the present context.

II. BASIC ASSUMPTIONS AND MODEL OF BH EVAPORATION

We assume that the fundamental dof are fermionic (BH and models based on fermions are available in literature, see, e.g., the SYK model [32, 33]). As a consequence, each quantum level can be filled by no more than one fermion. This assures that the Hilbert space $\mathcal{H}$ of physical states with a finite number of levels is finite dimensional. In fact, if the fundamental modes were bosons, the requirement that the number of slots available were finite would not have been sufficient to guarantee the finiteness of dim $\mathcal{H}$. Let us recall now that, in the picture of [11], it is only at energy scales below those of quantum gravity (e.g., at the energy scales of ordinary matter) that the field modes are distinguishable from those “making” the spacetime, hence we can write

$$\mathcal{H}_F \otimes \mathcal{H}_G \subseteq \mathcal{H}.$$  

(2)

Here $F$ and $G$ stand for “fields” and “geometry”, respectively. In other words, at low energy, the $F$-modes will form quantum fields excitations, that is, the quasi-particles (from the Xons point of view) immersed into the spacetime formed by the $G$-modes.

Now, say $N$ is the total number of quantum levels (slots) available to the BH. The evaporation consists of the following, steady process: $N \rightarrow (N-1) \rightarrow (N-2) \rightarrow \cdots$. That is, the number of free Xons steadily decreases, in favor of the Xons that, having evaporated, are arranged into quasi-particles and the spacetime they live in. One might think of a counter that only sees free Xons, hence keeps clicking in one direction as the BH evaporates, till its complete stop.

In this picture: i) there is no pre-existing time, because the natural evolution parameter is the average number of free Xons; and ii) there is no pre-existing space to define the regions inside and outside the BH, because a distinction of the total system into two systems, say environment (I) and BH (II), naturally emerges in the way just depicted. With this in mind, in what follows we shall nonetheless refer to the system I as outside, and to system II as inside. It is a worthy remark that other authors do use the geometric notion of exterior and interior of BH, even at fundamental level [28]. Even though this can be justified, see, e.g., [41], and permits to produce meaningful models, see, e.g., [42], our approach does not require to do this. The Hilbert space of physical states is then built as a subspace of a larger tensor product (kinematical) Hilbert space

$$\mathcal{H} \subseteq \mathcal{H}_I \otimes \mathcal{H}_{\text{II}}.$$  

(3)

We now assume that such a Hilbert space can be constructed with the methods of second quantization. This provides a language contiguous to the language of QFT, which should be recovered in some limit. Therefore, BH and environment modes will be described by two sets of creation and annihilation operators, which satisfy the usual canonical anticommutation relations

$$\left\{ \chi_{\tau n}, \chi_{\tau' n'}^\dagger \right\} = \delta_{\tau, \tau'} \delta_{n n'},$$  

(4)

with $n, n' = 1, \ldots, N$, $\tau = I, \text{II}$, and all other anticommutators equal to zero. Then, we introduce the simplified notation

$$a_n \equiv \chi_{I \ n}, \quad b_n \equiv \chi_{\text{II} \ n}.$$

(5)
We initialize the system in the state
\[ |0, N\rangle \equiv |0, 0, \ldots, 0\rangle_t \otimes |1, 1, \ldots, 1\rangle_H, \] (6)
where both kets, I and II, have N entries and
\[ |1, 1, \ldots, 1\rangle_H = b_1^\dagger b_2^\dagger \ldots b_N^\dagger |0, 0, \ldots, 0\rangle_H. \] (7)
The state Eq. (6) represents the BH at the beginning of the evacuation process, with all the slots occupied by free Xons. Although the Xons, during the evacuation, are progressively arranged into less fundamental structures (and hence no longer are the dof to be used for the emergent description) we keep our focus on them. For us this “transmutation” only helps identifying what to call “inside” and what “outside”, so that evaporation is the process that moves the Xons from II to I. In this way, the final state (for which there are no free Xons left, as they all recombined to form fields and spacetime), has the form
\[ |N, 0\rangle \equiv |1, 1, \ldots, 1\rangle_t \otimes |0, 0, \ldots, 0\rangle_H, \] (8)
where
\[ |1, 1, \ldots, 1\rangle_t = a_1^\dagger a_2^\dagger \ldots a_N^\dagger |0, 0, \ldots, 0\rangle_t. \] (9)

In order to construct a state of the system, compatible with the previous assumptions, let us consider the evolved operators as
\[ c_n(\sigma) = e^{i\varphi_n} \left( b_n \cos \sigma + a_n e^{-i\varphi_n} \sin \sigma \right), \] (10)
\[ d_n(\sigma) = e^{i\varphi_n} \left( e^{-i\varphi_n} a_n \cos \sigma - b_n \sin \sigma \right), \] (11)
where on \( \sigma \) we shall soon comment. Eqs. (10) and (11) define a canonical transformation
\[ \{ c_n(\sigma), c_m^\dagger(\sigma) \} = \{ d_n(\sigma), d_m^\dagger(\sigma) \} = \delta_{nm}. \] (12)

Note that we are using the shorthand notation
\[ a_n \equiv a_n \otimes \mathbb{I}_H, \quad b_n \equiv \mathbb{I}_I \otimes b_n. \] (13)
We thus get the evolved of the initial state (6) as
\[ |\Psi(\sigma)\rangle \equiv \prod_{n=1}^{N} c_n^\dagger(\sigma) |0\rangle_t \otimes |0\rangle_H \]
\[ = \prod_{n=1}^{N} e^{-i\varphi_n} \left( b_n^\dagger \cos \sigma + a_n^\dagger e^{i\varphi_n} \sin \sigma \right) |0\rangle_t \otimes |0\rangle_H. \]
Strictly speaking, \( \sigma \) should be regarded as a discrete parameter, counting the free Xons that leave the BH, in the picture above described (see also the discussion in the next Section). Nonetheless, in order to simplify computations, and with no real loss of generality, we use a continuous approximation: given our initial (Eq. (6)) and final (Eq. (8)) states, \( \sigma \) can be seen as an interpolating parameter, which describes the evolution of the system, from \( \sigma = 0 \), corresponding to the start of the evacuation of the BH, till \( \sigma = \pi/2 \), corresponding to its complete evaporation.

Let us also notice that the linear canonical transformation defined in Eqs. (10), (11) is very general. In fact, we could think to extend it by mixing creation and annihilation operators, \( c_n(\sigma) \sim (a_n + b_n^\dagger) \). However, this choice does not permit to interpolate Eqs. (6) and (8). The choice of phases introduced in the canonical transformation defined by Eqs. (10) and (11) does not affect any of the results presented. This is a consequence of the fact that we are working with two types of modes (BH and environment). If we had more than two systems we had to deal with one or more physical phases, as is well known in quark and neutrino physics [44]. We can thus safely set \( \varphi_n = 0 = \psi_n \).

With our choice of parameters, the state (14), can also be written as
\[ |\Psi(\sigma)\rangle \equiv \prod_{n=1}^{N} \sum_{n_i=0,1} C_i(\sigma) (a_n^\dagger)^{n_i} (b_n^\dagger)^{1-n_i} |0\rangle_t \otimes |0\rangle_H, \] (15)
with \( C_i = (\sin \sigma)^{n_i} (\cos \sigma)^{1-n_i} \). This form would suggest the following generalization
\[ |\Phi(\sigma)\rangle \equiv \prod_{n=1}^{N} \sum_{n_i,m_i=0,1} D_i(\sigma) (a_n^\dagger)^{n_i} (b_n^\dagger)^{m_i} |0\rangle_t \otimes |0\rangle_H, \] (16)
with \( D_i = (\sin \sigma)^{n_i} (\cos \sigma)^{m_i} \). However, we easily compute
\[ |\Phi(0)\rangle = |0, 1, \ldots, 0\rangle_H \otimes |1, 1, \ldots, 1\rangle_H \]
\[ + |0, 1, \ldots, 0\rangle_H \otimes |0, 1, \ldots, 0\rangle_H. \] (17)
which is incompatible with our boundary condition (6). In order to enforce the latter, we need to impose the constraint \( m_i = 1 - n_i \).

### III. Entropy Operators, Page Curve and the Bekenstein Bound

The Hilbert space of physical states has dimension
\[ \Sigma \equiv \dim \mathcal{H} = 2^N. \] (18)
The state defined in Eq. (14) is an entangled state. This is due to the fact that \( c_n^\dagger(\sigma) \) cannot be factorized as \( a_n \) and \( b_n \) in Eq. (13), i.e. it cannot be written as \( c_n^\dagger = A_n \otimes B_n \), where \( A_n \) and \( B_n \) acts only on \( \mathcal{H}_I \) and \( \mathcal{H}_H \) respectively.

To quantify such entanglement we define the entropy operator for environment modes as in TFD [18, 19, 21]
\[ S_I(\sigma) = - \sum_{n=1}^{N} (a_n^\dagger a_n \ln \sin^2 \sigma + a_n^\dagger b_n \ln \cos^2 \sigma), \] (19)
We also define the entropy operator for BH modes

\[ S_{II}(\sigma) = - \sum_{n=1}^{N} \left( b_n^\dagger b_n \ln \cos^2\sigma + b_n^\dagger b_n \ln \sin^2\sigma \right). \]  

(20)

The reason for such an unconventional definition will be clear in the next Section. For the moment, notice that we have two different operators, for I and for II, but we see that, since

\[ \langle a_n^\dagger a_n \rangle_{\sigma} = \sin^2\sigma = 1 - \langle b_n^\dagger b_n \rangle_{\sigma}, \]  

(21)

then

\[ S_{I}(\sigma) = \langle S_{I}(\sigma) \rangle_{\sigma} = -N \left( \sin^2\sigma \ln \sin^2\sigma + \cos^2\sigma \ln \cos^2\sigma \right) = \langle S_{II}(\sigma) \rangle_{\sigma} = S_{II}(\sigma), \]

(22)

where \( \langle \ldots \rangle_{\sigma} \equiv \langle \Psi(\sigma) | \ldots | \Psi(\sigma) \rangle \). Therefore the averages of the operators coincide, as it must be for a bipartite system. This entropy is the entanglement entropy between environment and BH, when the system evolves. Remarkably, it has a behavior in many respect similar to that of the Page curve \[30\], as shown in Fig. 1.

![Graph of von Neumann entropy as a function of \( \sigma \), in the case \( N = 1000 \).](image)

The maximum value is

\[ S_{\text{max}} = N \ln 2 = \ln \Sigma, \]

(23)

so that

\[ \Sigma = e^{S_{\text{max}}}. \]  

(24)

As we see here through Eq. \[24\] (that is the analogue of Eq.\[1\]), in our model dim \( \mathcal{H} \) is related to the maximal entanglement (von Neumann) entropy of the environment with the BH (and, of course, vice versa). This happens exactly when the modes have half probability to be inside and half probability to be outside the BH\(^2\), and then a large amount of bits are necessary to describe the system. The system has thus an intrinsic way to know how big is the physical Hilbert space, hence how big is the BH at the beginning of the evaporation: when the maximal entanglement is reached, then that value of the entropy, \( S_{\text{max}} \), tells how big was the original BH. Hence \( S_{\text{max}} \) must be some function of \( M_0 \), with \( M_0 \) the original mass of the BH (we take here the simplest case of a neutral, \( Q_0 = 0 \), and static, \( J_0 = 0 \), BH). This is the Bekenstein bound in this picture, obtained as a consequence of the finiteness of the fermionic fundamental dof, hence of a Pauli principle.

For a full identification of \( S_{\text{max}} \) with \( S_{BH} \) we need more than what we have here. In particular, we need the concept of area, that somehow is what has been evoked in LQG \[14, 15\] when in \[23\] one identifies

\[ N \equiv \frac{\Lambda}{4\pi\gamma G \sqrt{3}}, \]  

(25)

where \( \gamma \) is the Immirzi parameter, which is fixed to \( \ln 2/(\sqrt{3}\pi) \).

We want now to bring into the picture the two missing pieces: how the entropy of the BH, that should always decrease, and the entropy of the environment, that should always increase (hence, can be related to a standard thermodynamical entropy), actually evolve in our model. To this end, let us introduce the following number operators

\[ \hat{N}_I = \sum_{n=1}^{N} a_n^\dagger a_n, \quad \hat{N}_{II} = \sum_{n=1}^{N} b_n^\dagger b_n, \]

(26)

that count the number of modes of the radiation and the number of modes of the BH, respectively. Although it should be clear from the above, it is nonetheless important to stress now again that, in our formalism, the full kinematical Hilbert spaces associated to both sides have fixed dimension (\( \text{dim} \mathcal{H}_I = \text{dim} \mathcal{H}_{II} = 2^N \)), while only a subspace \( \mathcal{H} \subseteq \mathcal{H}_I \otimes \mathcal{H}_{II} \) such that \( \text{dim} \mathcal{H} = 2^N \) is the one of physical states. Note that \( \mathcal{H} \) cannot be factorized and this is the origin of BH/environment entanglement.

Nonetheless, one could think that the physical Hilbert spaces of the two subsystems have to take into account only the number of modes truly occupied, at any given stage of the evaporation. Hence, the actual dimensions would be \( 2^{\hat{N}_I(\sigma)} \), and \( 2^{\hat{N}_{II}(\sigma)} \), where one easily finds that

\[ N_I(\sigma) \equiv \langle \hat{N}_I \rangle_{\sigma} = N \sin^2\sigma, \]  

(27)

and

\[ N_{II}(\sigma) \equiv \langle \hat{N}_{II} \rangle_{\sigma} = N - N_I(\sigma) = N \cos^2\sigma. \]  

(28)

Recall that \( \sigma \) is, in fact, a discrete parameter, essentially counting the diminishing number of free Xons (as said earlier, and shown in more details later).

\(^2\) Recall that we have an intrinsic, non-geometric notion of the partition into inside/outside.
In other words, when we take this view, the partition in I and II becomes in all respects similar to the one of Page [30], that is

\[ 2^N = 2^{N_{I\sigma}} \times 2^{N_{II}} \equiv n \times m, \]

with \( n = 1, \ldots, 2^{N-1}, 2^N, \) and \( m = 2^N, 2^{N-1}, \ldots, 1 \) while \( \sigma \) runs in discrete steps in the interval \([0, \pi/2]\). Number fluctuations, which make necessary to invoke the entire Hilbert space \( \mathcal{H} \) at each stage, represent a measure of entanglement of these modes, as we shall see below. It is then natural to define the Bekenstein entropy as

\[ S_{BH} \equiv \ln n = N \ln 2 \cos^2 \sigma, \]

and the environment entropy\(^3\) as

\[ S_{env} \equiv \ln m = N \ln 2 \sin^2 \sigma. \]

The plots of the three entropies, \( S_I, S_{BH}, S_{env} \) are shown in Fig. 2, and must be compared with similar results of Ref. [35]. There are, though, two main differences worth stressing. First, we have a common single origin behind all involved entropies, as explained. Second, since the overall system here is based on the most fundamental entities, the curve for \( S_I \) cannot be always below the other two, as happens in [35], but its maximum \( S_{\text{max}} \) must reach the starting point of \( S_{BH} \) (and the ending point of \( S_{env} \)). In our case, the inequality

\[ S_I \leq S_{BH} + S_{env} = S_{\text{max}}, \]

is always satisfied. Note also that the dynamics of our system is unitary, because we keep our focus on the evaporated Xons, and not on the emerging structures, as was done in [11]. Hence we are not in the position here to spot the relic entanglement between fields and spacetime, that would make the curves for \( S_{BH} \) and \( S_I \) end at a nonzero value, and that is the source of the information loss in the quasi-particle picture of [11]. Whether or not this formally unitary evolution is physically tenable at the emergent level, and the impact of this on the validity of von Neumann uniqueness theorem [19, 45, 46] in a quantum system with a finite-dimensional Hilbert space, is under scrutiny in ongoing research [47]. Recently relation between unitarity and the existence of a maximal entropy has been also investigated in Ref. [48].

It is maybe worthwhile to stress that Eq. (22) represents exactly a von Neumann entropy. We can write the density matrix \( \rho(\sigma) = |\Psi(\sigma)\rangle\langle\Psi(\sigma)| \) and evaluate the reduced density matrices \( \rho_I = \text{Tr}_{II}\rho \) and \( \rho_{II} = \text{Tr}_I\rho \) [49]. We can then easily check that (see Appendix A)

\[ S_I(\sigma) = -\text{Tr}_I (\rho_I(\sigma) \ln \rho_I(\sigma)) = -N \left( \cos^2 \sigma \ln \cos^2 \sigma + \sin^2 \sigma \ln \sin^2 \sigma \right) = -\text{Tr}_{II} (\rho_{II}(\sigma) \ln \rho_{II}(\sigma)) = S_{II}(\sigma). \]

Let us now consider some simple cases

- If \( N = 1 \) we have

\[ |\Psi(\sigma)\rangle = \cos \sigma |0\rangle_I \otimes |1\rangle_{II} + \sin \sigma |1\rangle_I \otimes |0\rangle_{II}. \]

This is generally an entangled state, whose maximal entanglement is reached for \( \sigma = \pi/4 \):

\[ |\Psi(\pi/4)\rangle = \frac{1}{\sqrt{2}} (|0\rangle_I \otimes |1\rangle_{II} + |1\rangle_I \otimes |0\rangle_{II}). \]

- For \( N = 2 \) we have

\[ |\Psi(\sigma)\rangle = \cos^2 \sigma |00\rangle + \sin^2 \sigma |11\rangle \otimes |00\rangle_{II} \]

\[ + \sin^2 \sigma |01\rangle_{II} \otimes |11\rangle + \cos \sigma \sin \sigma |01\rangle_{II} \otimes |10\rangle_{II} \]

\[ + \cos \sigma \sin \sigma |10\rangle_{II} \otimes |01\rangle_{II}. \]

It is then clear that the mean number [21] represents the probability of the \( n \)-th mode to “leave the BH phase” (to go from II to I).

As mentioned earlier, \( \sigma \) for us is a continuous approximation of a discrete parameter, that counts the Xons transmuting from being free (in the BH, II) to being arranged into fields and spacetime (that is what happens, eventually, in I). Now we can formalize that statement, by inverting Eq. (27) and getting

\[ \sigma(N_I) = \arcsin \sqrt{\frac{N_I}{N}}. \]

\[ \text{ FIG. 2: Here we plot: } S_{BH} \text{ in black, monotonically decreasing; } S_{env} \text{ in green, monotonically increasing; } S_I \text{ in red, with a Page-like behavior. Note that the maximal value of } S_I, S_{\text{max}}, \text{ coincides with the initial BH entropy, as well as with the final environment entropy, as inferred in the text. The plots are done for } N = 1000. \]

\[ \text{\(^3\) We could also call it thermodynamical entropy, in comparison with the nomenclature of Ref. [30].} \]
When \( N_I \) is constrained to be an integer \( N_I = m \), the
\( \sigma(N_I) = \sigma_m \) is discretized. Therefore, the evolution
parameter is just a way of counting how many modes
jumped out and cannot be regarded as time, which should
emerge, like space, at low energy from Xons dynamics.

The von Neumann entropy as a function of \( \sigma = \sigma_m \) is
reported in Fig. 1.

We could then expect that at each step the number of
\( \Delta N \) and \( \Delta N II \) on \( |\psi(\sigma)\rangle \)? A direct
computation shows that

\[
\Delta N_I(\sigma) = \Delta N_{II}(\sigma) = \frac{\sqrt{N} \sin(2\sigma)}{2},
\]

where \( \Delta N_j \equiv \sqrt{\langle \hat{N}_j^2 \rangle - \langle \hat{N}_j \rangle^2} \) is the standard deviation
of \( \hat{N}_j \) on \( |\Psi(\sigma)\rangle \). As evident, looking at Fig. 3, this is
a measurement of the entanglement, in accordance with
the general results of Ref. [50]. Moreover, for \( N = 1 \),
\( (\Delta N_j(\sigma))^2 \) is proportional to the linear entropy or impurity

\[
(\Delta N_j(\sigma))^2 = 2 S^L(\sigma), \quad j = I, II,
\]

defined as [49, 51]

\[
S^L = 1 - \text{Tr}\rho^2_j.
\]

Note that \( \Delta N_j \) can be easily discretized as explained
above. We finally turn our attention to the generator

\[
\Delta N_j(\sigma) = \Delta N_{II}(\sigma) = \frac{\sqrt{N} \sin(2\sigma)}{2},
\]

with the above recalled limitations, the existence of such
unitary generator is guaranteed by the Stone–von Neu-
man theorem and, in the general meaning of [15], it can be
seen to enter the Wheeler–DeWitt equation

\[
H |\Psi(\sigma)\rangle = 0,
\]

with \( H \equiv \imath \partial_\sigma - G \). This constrains the kinematical
Hilbert space \( \mathcal{H}_I \otimes \mathcal{H}_{II} \) to the physical Hilbert space \( \mathcal{H} \) as
previously extensively commented. Let us remark that
for \( \sigma = \sigma_m \), Eq. (44) becomes a linear difference (recursion)
equation.

IV. CONNECTION WITH DISSIPATIVE SYSTEMS

In the previous Section we have shown how our toy
model possesses a Page-like behavior for entanglement
entropy and this can be easily computed by means of the
TFD entropy operator. We now ask if this is a mere coincidence or if the connection with TFD can be made
more precise.

Let us perform the canonical transformation

\[
A_n = a_n, \quad B_n = b_n^\dagger.
\]

This is not a Special Bogoliubov transformation [52]. In
fact, this transformation can be obtained from

\[
A_n = a_n \cos \theta_n - b_n^\dagger \sin \theta_n,
\]

\[
B_n = b_n^\dagger \cos \theta_n + a_n \sin \theta_n,
\]

for \( \theta_n = 0 \). Then it is not connected with the identity.
However, we can still define vacua in the new representation

\[
A_n |0\rangle_A = B_n |0\rangle_B = 0.
\]

One can check that

\[
|0\rangle_A = |0\rangle_B,
\]

\[
|0\rangle_B = |1_1 1_2 \ldots 1_N\rangle_{II}.
\]

The second relation follows from the fact that \( (b_n^\dagger)^2 = 0 \).
Therefore, the generator [43] becomes

\[
G = -\imath \sum_{n=1}^{N} (A_n^\dagger B_n^\dagger - B_n A_n).
\]

This is nothing but the (fermionic version of the)
interaction Hamiltonian of quantum dissipative systems, as
introduced in Ref. [53] (see also [54]). This operator
noticeably coincides with the generator of a Special Bo-
goliubov transformation. Therefore \( |\Psi(\sigma)\rangle \) has a TFD-
vacuum-like structure [18, 19]

\[
|\Psi(\sigma)\rangle = \prod_{n=1}^{N} (\cos \sigma + \sin \sigma A_n^\dagger B_n^\dagger) |0\rangle_A \otimes |0\rangle_B
\]

\[
= e^{-\frac{1}{2} S_A(\sigma)} |I\rangle,
\]

where \( S_A(\sigma) \) is constrained to be an integer \( N_I = m \), the
\( \sigma(N_I) = \sigma_m \) is discretized. Therefore, the evolution
parameter is just a way of counting how many modes
jumped out and cannot be regarded as time, which should
emerge, like space, at low energy from Xons dynamics.
The von Neumann entropy as a function of \( \sigma = \sigma_m \) is
reported in Fig. 1.

We could then expect that at each step the number of
\( \Delta N \) and \( \Delta N II \) on \( |\psi(\sigma)\rangle \)? A direct
computation shows that

\[
\Delta N_I(\sigma) = \Delta N_{II}(\sigma) = \frac{\sqrt{N} \sin(2\sigma)}{2},
\]

where \( \Delta N_j \equiv \sqrt{\langle \hat{N}_j^2 \rangle - \langle \hat{N}_j \rangle^2} \) is the standard deviation
of \( \hat{N}_j \) on \( |\Psi(\sigma)\rangle \). As evident, looking at Fig. 3, this is
a measurement of the entanglement, in accordance with
the general results of Ref. [50]. Moreover, for \( N = 1 \),
\( (\Delta N_j(\sigma))^2 \) is proportional to the linear entropy or impurity

\[
(\Delta N_j(\sigma))^2 = 2 S^L(\sigma), \quad j = I, II,
\]

defined as [49, 51]

\[
S^L = 1 - \text{Tr}\rho^2_j.
\]

Note that \( \Delta N_j \) can be easily discretized as explained
above. We finally turn our attention to the generator

\[
\Delta N_j(\sigma) = \Delta N_{II}(\sigma) = \frac{\sqrt{N} \sin(2\sigma)}{2},
\]

with the above recalled limitations, the existence of such
unitary generator is guaranteed by the Stone–von Neu-
man theorem and, in the general meaning of [15], it can be
seen to enter the Wheeler–DeWitt equation

\[
H |\Psi(\sigma)\rangle = 0,
\]

with \( H \equiv \imath \partial_\sigma - G \). This constrains the kinematical
Hilbert space \( \mathcal{H}_I \otimes \mathcal{H}_{II} \) to the physical Hilbert space \( \mathcal{H} \) as
previously extensively commented. Let us remark that
for \( \sigma = \sigma_m \), Eq. (44) becomes a linear difference (recursion)
equation.

IV. CONNECTION WITH DISSIPATIVE SYSTEMS

In the previous Section we have shown how our toy
model possesses a Page-like behavior for entanglement
entropy and this can be easily computed by means of the
TFD entropy operator. We now ask if this is a mere coincidence or if the connection with TFD can be made
more precise.

Let us perform the canonical transformation

\[
A_n = a_n, \quad B_n = b_n^\dagger.
\]

This is not a Special Bogoliubov transformation [52]. In
fact, this transformation can be obtained from

\[
A_n = a_n \cos \theta_n - b_n^\dagger \sin \theta_n,
\]

\[
B_n = b_n^\dagger \cos \theta_n + a_n \sin \theta_n,
\]

for \( \theta_n = 0 \). Then it is not connected with the identity.
However, we can still define vacua in the new representation

\[
A_n |0\rangle_A = B_n |0\rangle_B = 0.
\]

One can check that

\[
|0\rangle_A = |0\rangle_B,
\]

\[
|0\rangle_B = |1_1 1_2 \ldots 1_N\rangle_{II}.
\]

The second relation follows from the fact that \( (b_n^\dagger)^2 = 0 \).
Therefore, the generator [43] becomes

\[
G = -\imath \sum_{n=1}^{N} (A_n^\dagger B_n^\dagger - B_n A_n).
\]

This is nothing but the (fermionic version of the)
interaction Hamiltonian of quantum dissipative systems, as
introduced in Ref. [53] (see also [54]). This operator
noticeably coincides with the generator of a Special Bo-
goliubov transformation. Therefore \( |\Psi(\sigma)\rangle \) has a TFD-
vacuum-like structure [18, 19]

\[
|\Psi(\sigma)\rangle = \prod_{n=1}^{N} (\cos \sigma + \sin \sigma A_n^\dagger B_n^\dagger) |0\rangle_A \otimes |0\rangle_B
\]

\[
= e^{-\frac{1}{2} S_A(\sigma)} |I\rangle,
\]
with \( |I \rangle = \exp \left( \sum_{n=1}^{N} A_n^I B_n^I \right) |0\rangle_A \otimes |0\rangle_B \) and the entropy operators already introduced

\[
S_A = - \sum_{n=1}^{N} \left( A_n^I A_n^I \ln \sin^2 \sigma + A_n^A A_n^A \ln \cos^2 \sigma \right) \tag{53}
\]

\[
S_B = - \sum_{n=1}^{N} \left( B_n^B B_n^B \ln \sin^2 \sigma + B_n^B B_n^B \ln \cos^2 \sigma \right) \tag{54}
\]

Therefore, through (45), we have now the usual entropy operators of TFD, to be compared with the unusual definitions of (20): \( S_1 = S_A \) and \( S_{II} = S_B \).

The physical picture here is that, when the system evolves, a pair of \( A \) and \( B \) particles is created. The \( A \)-modes enter into the BH, annihilating BH modes, while the \( A \)-modes form the environment. This mechanism is heuristically the same as the one proposed by Hawking \[17\], and lately formalized via the tunneling effect \[55\].

The \( A \)- and \( B \)-modes do not discern explicitly between fields and geometric dof. However, from the point of view of Ref. \[11\], we can think that some dof are indeed responsible for the reduction of the BH’s horizon area during the evaporation and that annihilators of geometric modes can be defined. In order to make this idea more precise we can decompose \( A_n \) in their geometric (\( G \)) and field (\( F \)) parts as follows

\[
A_n = \sum_k \left( g_{k,n} A_{G,n}^k \otimes \mathbb{I}_{F,n} + f_{k,n} \mathbb{I}_{G,n} \otimes A_{F,n}^k \right) \tag{55}
\]

where \( k \) labels the emergent modes. Eqs.(55) can be regarded as a dynamical (Haag) map at linear order \[19\].

The full dynamical map – available once the quantum theory of gravity is specified – should connect the fundamental dof to the emergent notions of geometry and fields. The coefficients of the map should then lead to the thermal behavior of the latter at emergent level. Note that the action of \( A_n^I \) on \( |0\rangle_A \), creates both a matter and a geometric mode outside the horizon: the region of spacetime surrounding the BH and available to an external observer increases, because the horizon area decreases.

V. CONCLUSIONS

We assumed here that the dof of a BH are finite in number and fermionic in nature, and hence obey a Pauli principle. Then, within the approach of second quantization, we naturally obtained that the BH evaporation is a dynamical mechanism producing a maximal entanglement entropy, equal to the initial entropy of the BH. This phenomenon is an instance of the Bekenstein bound, obtained here with arguments that do not assume pre-existing spatiotemporal concepts. Of course, for a full identification with the standard formulae (see, e.g., \[21\]), one needs to link geometrical concepts (such as elementary Planck cells) to such fundamental dof., something we have not attempted here.

We then showed that entanglement, Bekenstein and environment (thermodynamic) entropies here are all naturally obtained in the same approach, based on an entropy operator whose structure is the one typical of TFD. Through such operator, we have evaluated the von Neumann BH–environment entropy and noticeably obtained a Page-like evolution.

We finally have shown that the latter is a consequence of a duality between our model and a dissipative-like fermionic quantum system, and hence it has a natural TFD-like description.

Many directions for further research need be thoroughly explored, the most important being a reliable dynamical map from the fundamental modes to the emergent fields/spacetime structures. Nonetheless, we believe that our simple, although nontrivial, considerations are necessary to fully take into account the fascinating and far-reaching consequences of the Bekenstein bound.

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Appendix A: Equivalence of TFD entropy and von Neumann entropy

In this Appendix we explicitly show the equivalence of the TFD entropy of Eq.(22) with the von Neumann entropy of Eq.(33).

We first present the computation in the simplest cases of Eqs.(34) and (36). Let us note that the expectation value of TFD entropy operators (see Eq.(22)), immediately follows from Eq.(21), and then we shall focus on the computation of von Neumann entropy.

- In the case \( N = 1 \), the density matrix reads

\[
\rho(\sigma) = \cos^2 \sigma |0\rangle_{I,II} \langle 0| + \sin^2 \sigma |1\rangle_{I,II} \langle 0|, \quad (A1)
\]

where we omitted tensor product symbols. Reduced density matrices have the form

\[
\rho_i(\sigma) = \cos^2 \sigma |0\rangle_i \langle 0| + \sin^2 \sigma |1\rangle_i \langle 1|, \quad (A2)
\]

\[
\rho_{II}(\sigma) = \cos^2 \sigma |1\rangle_{II} \langle 1| + \sin^2 \sigma |0\rangle_{II} \langle 0|, \quad (A3)
\]

Eq.(33) follows immediately from these expressions.
In the case $N = 2$, we directly report reduced density matrices

$$\rho_i(\sigma) = \cos^4 \sigma |00\rangle_i |00\rangle + \sin^4 \sigma |11\rangle_i |11\rangle + \sin^2 2\sigma \left( |01\rangle_i |01\rangle + |10\rangle_i |10\rangle\right),$$

(A4)

$$\rho_{II}(\sigma) = \cos^4 \sigma |11\rangle_{II} |11\rangle + \sin^4 \sigma |00\rangle_{II} |00\rangle + \sin^2 2\sigma \left( |01\rangle_{II} |01\rangle + |10\rangle_{II} |10\rangle\right).$$

(A5)

It follows that

$$S_i(\sigma) = -\sin^4 \sigma \ln \sin^4 \sigma - \cos^4 \sigma \ln \cos^4 \sigma - 2 \sin^2 \sigma \cos^2 \sigma \ln (\sin^2 \sigma \cos^2 \sigma)
= S_{II}(\sigma).$$

By using that $\ln(ab) = \ln a + \ln b$, and $\cos^2 \sigma + \sin^2 \sigma = 1$, we get

$$S_i(\sigma) = -2 \left( \sin^2 \sigma \ln \sin^2 \sigma + \cos^2 \sigma \ln \cos^2 \sigma \right)
= S_{II}(\sigma),$$

(A7)

which is equal to Eq.(33) for $N = 2$.

One could repeat similar computations for all $N$. However it is simpler to use the correspondence of our model with a TFD/dissipative system via Eq.(43). As known in TFD, the “thermal vacuum” can be rewritten in the form

$$|\Psi(\sigma)\rangle = \sum_{n=0,1} \sqrt{w_n(\sigma)} |n\rangle_A |n\rangle_B,$$

(A8)

where $|n\rangle_A, |n\rangle_B$ are eigenstates of number operators

$$\hat{N}_A = \sum_{n=1}^{N} A_n^\dagger A_n, \quad \hat{N}_B = \sum_{n=1}^{N} B_n^\dagger B_n.$$  

Moreover, the coefficients $w_n$ are given by

$$w_{n}(\sigma) = \prod_{j=1}^{N} C_j^2(\sigma).$$

(A10)

and $C_j$ were firstly introduced in Eq.(15). Density matrix thus reads

$$\rho(\sigma) = \sum_{n=0,1} w_n(\sigma) |n\rangle_A |n\rangle_B \langle n|_A \langle n|.$$  

(A11)

Then, reduced density matrices are easily derived

$$\rho_A(\sigma) = \sum_{n=0,1} w_n(\sigma) |n\rangle_A \langle n|$$

(A12)

$$\rho_B(\sigma) = \sum_{n=0,1} w_n(\sigma) |n\rangle_B \langle n|.$$  

(A13)

Finally, the von Neumann entropy reads

$$S_A(\sigma) = S_B(\sigma) = -\sum_{n=0,1} w_n(\sigma) \ln w_n(\sigma).$$

(A14)

The expression Eq.(33) follows substituting the explicit form of $w_n$ (cf. Eq.(A10)).

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