A MULTIDIMENSIONAL LAW OF SINES

IGOR RIVIN

Abstract. We give a linear-algebraic proof of the law of sines, which also allows us to extend this theorem to simplices in $\mathbb{E}^n$, as Theorem 2.1.

Let $ABC$ be a triangle in the euclidean plane. The classical Law of Sines states that

\[
\frac{|AB|}{\sin \gamma} = \frac{|AC|}{\sin \beta} = \frac{|BC|}{\sin \alpha}.
\]

There are many proofs of this fact, some of them found in high school geometry textbook, but in this note we will derive this result as a special case of a theorem about simplices in $\mathbb{E}^n$ (Theorem 2.1), which, despite the fact that the classical Law of Sines has been known for at least two thousand years, seems to have not been noticed to date.

To begin, let $\Delta$ be a simplex in $\mathbb{E}^n$. This simplex will have $n + 1$ faces $f_1, \ldots, f_{n+1}$, and we will denote the $(n - 1$-dimensional) area of $f_i$ by $A_i$ and the outward unit normal to $f_i$ by $\mathbf{f}_i$. The following fact is fundamental:

Theorem 0.1.

\[
\sum_{i=1}^{n+1} A_i \mathbf{f}_i = 0.
\]

Proof. A vector $v$ in $\mathbb{E}^n$ is 0 if and only if its scalar product with any other vector $w$ is 0. Now let

\[
v = \sum_{i=1}^{n+1} A_i \mathbf{f}_i.
\]

Then,

\[
\langle v, w \rangle = \sum_{i=1}^{n+1} A_i \langle \mathbf{f}_i, w \rangle.
\]
Each summand $A_i(f_i, w)$ is simply the signed area of the projection of the face $f_i$ onto the plane $P_w$ through the origin normal to $w$. Since almost every point of $P_w$ either does not lie in the image of $\Delta$ under the orthogonal projection in the direction $w$ or is covered twice, with opposing signs, the result follows. \hfill \Box

Remark 0.2. The above is a special case of Stokes’ formula.

We now construct the matrix of columns

$$\mathcal{F}_\Delta = (f_1, \ldots, f_{n+1}),$$

and then the Gram matrix

$$G_\Delta = \mathcal{F}_\Delta^t \mathcal{F}_\Delta.$$  

(from here on we will drop the subscript $\Delta$.) The $ij$-th entry of $G$ is simply $\langle f_i, f_j \rangle$, which is the cosine of the exterior dihedral angle between $f_i$ and $f_j$.

If $\Delta$ is a non-degenerate simplex, then the rank of $\mathcal{F}$ is equal to $n$ (the one linear relation between the $f_i$ is given by the Theorem 0.1) and $G$ is a symmetric matrix with $n$ positive eigenvalues, and one 0 eigenvalue.

Theorem 0.3. The null-space of $G$ is spanned by the vector $\mathbf{a} = (A_1, \ldots, A_{n+1})$.

Proof.

$$Ga = \mathcal{F}^t \mathcal{F} \mathbf{a} = \mathcal{F}^t \left( \sum_{i=1}^{n+1} A_i f_i, e_1 \right), \ldots, \left( \sum_{i=1}^{n+1} A_i f_i, e_n \right) = 0.$$ \hfill \Box

At this point we need some linear algebra:

1. Some facts about matrices

Definition 1.1. Let $M$ be a matrix. The adjugate $\hat{M}$ of $M$ is the matrix of cofactors of $M$. That is, $\hat{M}_{ij} = (-1)^{i+j} \det M^{ij}$, where $M^{ij}$ is $M$ with the $i$-th row and $j$-th column removed.

The reason for this definition is

Theorem 1.2 (Cramer’s rule). For any $n \times n$ matrix $M$ (over any commutative ring)

$$M \hat{M} = \hat{M} M = (\det M) I(n),$$

where $I(n)$ is the $n \times n$ identity matrix.
We also need

**Definition 1.3.** The outer product of column vectors \( v = (v_1, \ldots, v_n) \) and \( w = (w_1, \ldots, w_n) \) is the matrix \( vw^t \).

Consider an arbitrary vector \( x = (x_1, \ldots, x_n) \). We see that

\[
(vw^t)x = \sum_{i=1}^{n} v_i w_i x_i = v_k \langle w, x \rangle,
\]

so that

\[
(vw^t)x = \langle w, x \rangle v.
\]

We see that \( vw^t \) is a multiple of the projection operator onto the subspace spanned by \( v \). In particular, in the case when \( \|v\| = 1 \), the operator \( vv^t \) is the orthogonal projection operator onto the subspace spanned by \( v \). Since \( vw^t \) is a rank 1 operator all but one of its eigenvalues are equal to 0. The one (potentially) nonzero eigenvalue equals \( \langle v, w \rangle \).

We now show:

**Theorem 1.4.** Suppose that \( M \) has nullity 1, and the null space of \( M \) is spanned by the vector \( v \), while the null space of \( M^t \) is spanned by the vector \( w \). Then

\[
\widehat{M} = cvw^t,
\]

*Proof.* Since \( M \) is singular, we know that \( \det M = 0 \), and so every column of \( \widehat{M} \) is in the null-space of \( M \). So, letting \( m_i \) denote the \( i \)th column of \( \widehat{M} \), we see that

\[
m_i = d_i v.
\]

However, \( \widehat{M}^t = (\widehat{M})^t \) so performing the computation on transposes we see that

\[
m_i^t = e_i w.
\]

We see that

\[
\widehat{M}_{ij} = d_j v_i = e_i w_j.
\]

Writing \( d_i = g_i w_i \), and \( e_j = h_j v_j \), we see that, for every pair \( i, j \),

\[
g_i w_i v_j = h_j w_i v_j.
\]

Hence \( g_i = h_j = c \), and the conclusion follows.  

**Theorem 1.5.** The constant \( c \) in the statement of the last theorem equals the product of the nonzero eigenvalues of \( M \) divided by the inner product of \( v \) and \( w \).
Proof. By considering the characteristic polynomial of $M$ we see that the product of the nonzero eigenvalues of $M$ equals the sum of the principal $n-1$ minors. On the other hand, the principal minors of $M$ equal the diagonal elements of $\hat{M}$, so

$$c \sum_{i=1}^{n} w_i v_i = \prod_{j=1}^{n-1} \lambda_j.$$ 

□

Remark 1.6. By the discussion following Eq. (3), the product of nonzero eigenvalues of $M$ equals

$$\frac{\det(M + w \otimes v)}{\langle v, w \rangle}.$$ 

2. Back to simplices

Let us now return to the Gram matrix $G$ of a simplex $\Delta$. The results in the preceding section, combined with Theorem 0.1 immediately imply:

Theorem 2.1 (Multidimensional theorem of sines). For any $1 \leq i, j, k, l \leq n + 1$

$$\frac{A_i A_j}{A_k A_l} = \frac{\hat{G}_{ij}}{\hat{G}_{kl}},$$

where $A_i, A_j, A_k, A_l$ refer to the areas of the corresponding faces of $\Delta$ and $\hat{G}_{ij}$ is the $ij$-th minor of the Gram matrix of $\Delta$.

Proof. Immediate from Theorem [1.4] together with Theorem 0.1. □

3. Examples

In two dimensions, the Gram matrix of a triangle $ABC$ is

$$G_{ABC} = \begin{pmatrix}
1 & -\cos \gamma & -\cos \beta \\
-\cos \gamma & 1 & -\cos \alpha \\
-\cos \beta & -\cos \alpha & 1
\end{pmatrix}.$$ 

Thus, $\hat{G}_{11} = \sin^2 \alpha$, while $\hat{G}_{22} = \sin^2 \beta$, so Theorem 2.1 implies that

$$\frac{|BC|^2}{|AC|^2} = \frac{\sin^2 \alpha}{\sin^2 \beta}.$$ 

The sign indeterminacy due to the squares is illusory, since all the quantities involved are a priori positive.
Note further that $\hat{G}_{12} = -\cos \alpha \cos \beta - \cos \gamma$, so Theorem 2.1 implies that

$$
\frac{|BC|}{|AC|} = \frac{\sin^2 \alpha}{-\cos \alpha \cos \beta - \cos \gamma},
$$

which, together with the Theorem of Sines proves either the addition formula for cosine (if we assume that the sum of the angles of a triangle is $\pi$) or that the sum of the angles of a triangle is $\pi$ (if we assume the addition formula for cosine).

In three dimensions, for an arbitrary (nondegenerate) tetrahedron $\Delta$,

$$
G_\Delta = \begin{pmatrix}
1 & -\cos \alpha_{12} & -\cos \alpha_{13} & -\cos \alpha_{14} \\
-\cos \alpha_{12} & 1 & -\cos \alpha_{23} & -\cos \alpha_{24} \\
-\cos \alpha_{13} & -\cos \alpha_{23} & 1 & -\cos \alpha_{34} \\
-\cos \alpha_{14} & -\cos \alpha_{24} & -\cos \alpha_{34} & 1
\end{pmatrix}.
$$

A quick computation shows that

$$
\frac{A_4^2}{A_3^2} = 1 - \cos^2 \alpha_{12} - \cos^2 \alpha_{13} - \cos^2 \alpha_{23} - 2 \cos \alpha_{12} \cos \alpha_{23} \cos \alpha_{13} - \cos^2 \alpha_{14} - \cos^2 \alpha_{24} - 2 \cos \alpha_{12} \cos \alpha_{24} \cos \alpha_{14},
$$

while using Theorem 2.1 with non-principal minors shows relationships analogous to the angle-sum relationship for a triangle.

Marina Pashkevich has pointed out that Eq. (4) can be considerably simplified (using the spherical Heron’s formula), as follows:

$$
\frac{A_4}{A_3} = \frac{\sin(S_4/2) \cos(\alpha_{13}/2) \cos(\alpha_{23}/2)}{\sin(S_3/2) \cos(\alpha_{14}/2) \cos(\alpha_{24}/2)},
$$

where $S_i$ is the (spherical) area of the link of the $i$-th vertex. M. Pashkevich has also succeeded in extending this formula (in 3 dimensions) to hyperbolic and spherical simplices.

**Mathematics Department, Temple University, 1805 N Broad St, Philadelphia, PA 19122**

**Mathematics Department, Princeton University, Fine Hall, Washington Rd, Princeton, NJ 08544**

E-mail address: rivin@math.temple.edu