THE $L^1$ EHRENPREIS CONJECTURE

T.M. GENDRON

ABSTRACT. Let $\tilde{\Sigma}$ be the algebraic universal cover of a closed surface of genus $g > 1$, $\mathcal{T}(\tilde{\Sigma})$ its Teichmüller space, $M(\Sigma, \ell)$ the group of mapping classes stabilizing a fixed leaf $\ell$. The $L^1$ Ehrenpreis conjecture asserts that $M(\Sigma, \ell)$ acts on $\mathcal{T}(\tilde{\Sigma})$ with dense orbits in the $L^1$ topology (the topology coming from the $L^1$ norm on quadratic differentials). We give a proof of this weaker version of the Ehrenpreis conjecture, announced first in [6].

1. INTRODUCTION

Let $\Sigma, \Sigma'$ be closed Riemann surfaces of genus greater than 1. The most succinct formulation of the Ehrenpreis conjecture (EC) uses the fact that $\Sigma, \Sigma'$ may be regarded as riemannian manifolds with metrics of curvature $-1$. While it is an elementary fact that the riemannian universal covers $\tilde{\Sigma}, \tilde{\Sigma}'$ are isometric to $\mathbb{H}^2$, the EC asserts a similar, asymptotic phenomenon for the family of finite riemannian covers:

Ehrenpreis Conjecture (Hyperbolic). For any $\varepsilon > 0$, there are finite degree isometric covers $Z \to \Sigma, Z' \to \Sigma'$ whose total spaces are $(1 + \varepsilon)$-quasiisometric.

In [6], we announced the solution of an $L^1$-version of this conjecture. In this paper, we provide the proof.

The traditional or conformal version of the EC [2] can be described in terms of Teichmüller theory. Let $\Sigma$ be a fixed compact surface of genus greater than 1, $\mathcal{T}(\Sigma)$ its Teichmüller space, $d_{\mathcal{T}(\Sigma)}$ the Teichmüller metric. Then given $\mu, \nu \in \mathcal{T}(\Sigma)$, the conformal version of the EC states

Ehrenpreis Conjecture (Conformal). For any $\varepsilon > 0$, there exists a surface $Z$ and finite covers $\rho, \sigma : Z \to \Sigma$ such that

$$d_{\mathcal{T}(Z)}(\rho^* \mu, \sigma^* \nu) < \varepsilon.$$ 

In the above, $\rho^*, \sigma^* : \mathcal{T}(\Sigma) \to \mathcal{T}(Z)$ are the isometric inclusions induced by $\rho, \sigma$. We remark that this version of the EC makes sense in genus 1, where it is not difficult to verify [6].

However, it is the genus independent or solenoidal version of the EC that will be most important for us. Let $\tilde{\Sigma}$ be the algebraic universal cover of the closed surface $\Sigma$, by definition the inverse limit of the total spaces of finite covers $\rho : Z \to \Sigma$ (one cover for each homotopy class of cover). The algebraic universal cover is a surface solenoid (a surface lamination with Cantor transversals), and as such has a Teichmüller space $\mathcal{T}(\tilde{\Sigma})$ of marked conformal structures [18]. The mapping class group $M(\tilde{\Sigma}, \ell)$ of homotopy classes of homeomorphisms of $\tilde{\Sigma}$ fixing a base leaf $\ell$ may be identified with the group of homotopy classes of lifts of correspondences $\sigma \circ \rho^{-1} : \Sigma \to \Sigma$.

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**Ehrenpreis Conjecture (Solenoidal).** \( M(\hat{\Sigma}, \ell) \) acts on \( \mathcal{T}(\hat{\Sigma}) \) with dense orbits.

The proof that all of these versions of the EC are equivalent can be found in [6]. The genus independent version of EC says that the "universal moduli space"

\[ \mathcal{M}(\hat{\Sigma}) = \mathcal{T}(\hat{\Sigma})/M(\hat{\Sigma}, \ell), \]

although an uncountable set, has the topology of a point (the discrete topology). It has the virtue of giving a certain explanation of the “moduli-rigidity gap” that separates the theory of compact hyperbolic surfaces from that of compact hyperbolic manifolds in dimension three or greater. From a practical point of view, working with \( \hat{\Sigma} \) allows us to regard Teichmüller theory of closed hyperbolic surfaces as concerning complex structures on a single topological type (as in the genus 1 case). In this way, we may isolate geometric properties of closed Riemann surfaces of hyperbolic type that do not depend on genera.

In this paper, we shall formulate and prove an \( L^1 \) version of the EC. On \( \mathcal{T}(\hat{\Sigma}) \) there are – in addition to the Teichmüller metric – three other metrics coming from the \( L^1 \), the \( L^\infty \), and the \( L^2 \) structures on the cotangent bundle of \( \mathcal{T}(\hat{\Sigma}) \). The \( L^1 \) version of the EC is obtained by asking that \( M(\hat{\Sigma}, \ell) \) act densely on \( \mathcal{T}(\hat{\Sigma}) \) with regard to the \( L^1 \) geometry.

The proof of the \( L^1 \) EC is as follows. A dense set of pairs \( \hat{\mu}, \hat{\nu} \in \mathcal{T}(\hat{\Sigma}) \) (dense in the Teichmüller geometry) lie along the axis \( A \) of a pseudo-Anosov homeomorphism \( \Phi : \hat{\Sigma} \to \hat{\Sigma} \). \( A \) is a Teichmüller geodesic and the action of \( \Phi \) on \( \mathcal{T}(\hat{\Sigma}) \) stabilizes \( A \), translating points along \( A \) a distance of \( \frac{1}{2} \log \lambda \), where \( \lambda \) is the entropy of \( \Phi \). Given \( n \in \mathbb{N} \), by an \( L^1 \) \( n \)th root of \( \Phi \) we mean a sequence of pseudo Anosov homeomorphisms \( \{\hat{\Psi}_m\} \) in which

- The entropies \( \lambda_m \) of \( \hat{\Psi}_m \) converge to \( \lambda^{1/n} \).
- The axes \( A_m \) of \( \hat{\Psi}_m \) converge to \( A \) in the \( L^1 \) Hausdorff topology.

We shall show that \( L^1 \) \( n \)th roots exist for every (lifted) pseudo Anosov homeomorphism of \( \hat{\Sigma} \). This will then imply the \( L^1 \) version of EC.

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2. **Topology of the Algebraic Universal Cover**

Let \( \Sigma \) be a fixed compact surface of genus at least two. We describe in this section the topology of the algebraic universal cover \( \hat{\Sigma} \). Unless otherwise noted, all proofs of statements in this section can be found in [6].

Let \( \pi = \pi_1 \Sigma \). For every finite index normal subgroup \( H < \pi \), choose a pointed cover \( \rho : (\tilde{Z}, x_\Sigma) \to (\Sigma, x) \) for which \( \rho_* \pi_1 \tilde{Z} = H \). By adding to this collection of covers all covers \( \tau : Z \to Z' \) between total spaces for which \( \rho' \circ \tau = \rho \), we obtain an inverse system of surfaces. The limit of this system \( \hat{\Sigma} \) is called the algebraic universal cover of \( \Sigma \), a compact topological space. Its topological type is independent of the choice of covers.

If we denote by \( \hat{\pi} \) the profinite completion of \( \pi \), then \( \hat{\Sigma} \) is homeomorphic to the quotient

\[ \left( \overline{\Sigma \times \hat{\pi}} \right)/\pi, \]

where \( \pi \) acts diagonally, and so has the structure of a surface solenoid: a surface lamination whose model transversals are Cantor sets. It also follows from (1) that \( \hat{\Sigma} \) is connected, its path components are its leaves, and each leaf is homeomorphic to \( \mathbb{R}^2 \) and dense in \( \hat{\Sigma} \). The point \( \ell = (x_\Sigma) \in \hat{\Sigma} \) – defined by the string of basepoints of the surfaces in the defining system – is contained in a leaf \( \ell \) which we call the base leaf. The Haar measure on \( \hat{\pi} \)
induces a transverse invariant measure \( \eta \) on \( \hat{\Sigma} \) which gives measure \( 1/\deg Z \) to the fibers of the natural projection \( \hat{\Sigma} \to Z \).

Any pointed finite cover \( \sigma : (Y,y) \to (\Sigma,x) \) lifts to a base leaf preserving homeomorphism \( \hat{\sigma} : \hat{Y} \to \hat{\Sigma} \), where \( \hat{Y} \) is the algebraic universal cover of \( Y \). If \( \rho : (Y,y) \to (\Sigma,x) \) is another such cover, the correspondence \( \sigma \circ \rho^{-1} \) lifts to the homeomorphism \( \hat{\sigma} \circ \hat{\rho}^{-1} \) of \( \hat{\Sigma} \) preserving \( \ell \). Let \( M(\hat{\Sigma}, \ell) \) denote the the group of homotopy classes of orientation preserving homeomorphisms of \( \hat{\Sigma} \) preserving \( \ell \). See \[7, 13\] for a proof of the following

**Theorem 1.** Every class \([h] \in M(\hat{\Sigma}, \ell)\) contains an element of the form \( \hat{\sigma} \circ \hat{\rho}^{-1} \).

### 3. Measured Laminations on \( \hat{\Sigma} \)

We begin by recalling a few facts about measured laminations on \( \Sigma \), a closed Riemann surface of genus \( g > 1 \). See \[11, 13, 3, 20\] for further discussion. A measured lamination \( \mathcal{f} \) on \( \Sigma \) is a closed 1-dimensional lamination smoothly embedded in \( \Sigma \) and possessing a transverse invariant measure \( m_{\mathcal{f}} \). Two measured laminations are equivalent if they are isotopic through an isotopy taking one measure to the other. The set of equivalence classes of measured laminations is denoted \( \mathcal{M}(\Sigma) \). Let \( \mathcal{C}(\Sigma) \) denote the set of isotopy classes of simple closed curves in \( \Sigma \). Given \( \mathcal{f} \in \mathcal{M}(\Sigma) \) and \( c \in \mathcal{C}(\Sigma) \), the intersection pairing is defined

\[
I(\mathcal{f}, c) = \inf_{c \in C} \int_{[c]} dm_{\mathcal{f}},
\]

where the infimum is taken over representatives of the classes of \( \mathcal{f} \) and \( c \). The intersection topology on \( \mathcal{M}(\Sigma) \) is the weak topology with respect to the intersection pairing. The space of projective classes of measured laminations is denoted \( \mathcal{P}(\Sigma) \) and is homeomorphic to a sphere of dimension \( 6g - 7 \). We have \( \mathcal{C}(\Sigma) \subset \mathcal{P}(\Sigma) \) with dense image. The intersection pairing extends to a map \( \mathcal{M}(\Sigma) \times \mathcal{M}(\Sigma) \to \mathbb{R} \) via the formula \[10\]

\[
I(\mathcal{f}, \mathcal{g}) = \inf_{\mathcal{g} \in \mathcal{G}} \int_{[\mathcal{g}]} dm_{\mathcal{f}} \otimes dm_{\mathcal{g}},
\]

A word is in order here regarding the allied concept of a measured foliation, a singular foliation \( \mathcal{F} \) of \( \Sigma \) equipped with a transverse invariant measure: these typically arise as trajectories of holomorphic quadratic differentials on \( \Sigma \) \[17\]. Two measured foliations are equivalent if after a finite number of Whitehead moves are applied to their singular leaves, they are isotopic through an isotopy taking one measure to the other. There is a bijective correspondence between classes of measured foliations and classes of measured laminations \[3\]. For example, to obtain a measured lamination starting with a measured foliation \( \mathcal{F} \), one chooses a nonsingular leaf from each minimal component of \( \mathcal{F} \), pulls each such leaf geodesic (with respect to the hyperbolic metric of \( \Sigma \)) and completes the resulting space. Our default will be to work with measured laminations, and – with the exception of the proof of Theorem 4 where we revert back to measured foliations – whenever a measured foliation happens to arise, we will assume it has been converted into its associated measured lamination.

A (homotopy class of) homeomorphism \( \Phi : \Sigma \to \Sigma \) induces a homeomorphism of \( \mathcal{M}(\Sigma) \) via pullback of measures, in particular inducing a homeomorphism of \( \mathcal{P}(\Sigma) \). According to the classification of surface diffeomorphisms, \[11, 3, 19\], \( \Phi \) is called pseudo Anosov if its induced action on \( \mathcal{P}(\Sigma) \) fixes precisely two classes \([p]\) and \([p']\). If \( \lambda \) is the entropy of \( \Phi \), then \( \lambda > 1 \); and if \( \mu \) \( \in [p] \) \((\mu' \in [p'])\) is a representative in \( \mathcal{M}(\Sigma) \), then there is a representative diffeomorphism in the class of \( \Phi \) (also denoted \( \Phi \)) such that \( \Phi(\mu) = \lambda \mu \) \((\Phi(\mu') = \lambda^{-1} \mu')\).
In this paper, we will be interested in the following class of pseudo Anosov homeomorphisms. Let \( \mathcal{C}, \mathcal{D} \) be families of pairwise nonisotopic simple closed curves, for which elements of \( \mathcal{C} \) intersect minimally in their isotopy classes with elements of \( \mathcal{D} \), and for which \( \Sigma \setminus (\mathcal{C} \cup \mathcal{D}) \) consists of a union of disks (the families are then said to be filling). For \( c \in \mathcal{C}, d \in \mathcal{D}, \) let \( F_c \) resp. \( G_d \) denote the right Dehn twist about \( c \) resp. \( d \). Then a homeomorphism of the form

\[
\Phi = G_{d_{k}}^{-N_{k}} \circ \cdots \circ G_{d_{1}}^{-N_{1}} \circ F_{c_{j}}^{M_{j}} \circ \cdots \circ F_{c_{1}}^{M_{1}},
\]

where the exponents \( M_{1}, \ldots, M_{j}, N_{1}, \ldots, N_{k} \) are positive and where all curves in \( \mathcal{C}, \mathcal{D} \) occur, is pseudo Anosov [15], [20]. We call these pseudo Anosovs of Thurston-Penner type.

We now extend the above considerations to \( \hat{\Sigma} \). A measured lamination \( \hat{\ell} \) on \( \hat{\Sigma} \) is a collection of measured laminations \( \{ \ell_{t} \} \), one on each leaf \( \ell \) of \( \hat{\Sigma} \), which have the same transversal model \( T \) and which vary in the following way with respect to the transversals of \( \hat{\Sigma} \). Let \( \mathcal{D} \approx D \times \hat{T} \) be a flowbox for \( \hat{\Sigma} \), such that \( \hat{\ell}_{t} := \hat{\ell}_{|_{D \times \{t\}}} \) is a flowbox for \( \ell_{t} \) if \( D \times \{t\} \subset \ell \). Then

1. The family of flowboxes \( \ell_{t} \) varies continuously in \( t \). (Thus \( \hat{\ell} \) gives rise to a smooth 1-dimensional sublamination of \( \hat{\Sigma} \) with transversal models \( \hat{T} \times T \).)
2. Given any continuous family of transversal models \( T_{i} \subset D \times \{t\} \), the function obtained by pairing with the measures of \( \hat{\ell} \) is continuous in \( t \).

The space of equivalence classes of measured laminations is denoted \( \mathcal{ML}(\hat{\Sigma}) \). If \( \hat{\ell} \) is a simple closed curve occurring in some surface \( Z \) in the defining system of \( \hat{\Sigma} \), its preimage \( \hat{\ell} \) in \( \hat{\Sigma} \) is a 1-dimensional solenoid which we call a simple closed solenoid (abusing, since such a \( \hat{\ell} \) always has infinitely many connected components). The set of such is denoted \( \mathcal{L}(\hat{\Sigma}) \).

The intersection pairing \( I(\hat{\ell}, \hat{\ell}') \) between a measured lamination and a simple closed solenoid is defined using the transverse invariant measure \( \eta \),

\[
I(\hat{\ell}, \hat{\ell}') = \int_{\ell \cap \ell'} dm_{\ell} \, d\eta.
\]

We equip \( \mathcal{ML}(\hat{\Sigma}) \) with the resulting weak topology. With its induced topology, \( \mathcal{ML}(\hat{\Sigma}) \) is precompact (being essentially a space of probability measures), but owing to its infinite-dimensionality, \( \mathcal{ML}(\hat{\Sigma}) \) is not compact. The simple closed solenoids \( \mathcal{L}(\hat{\Sigma}) \) are dense in \( \mathcal{ML}(\hat{\Sigma}) \).

If \( Z \) is a surface occurring in the defining limit of \( \hat{\Sigma} \), then we may pullback measured laminations on \( Z \) to measured laminations on \( \hat{\Sigma} \). The result is a direct system of inclusions \( \mathcal{ML}(Z) \hookrightarrow \mathcal{ML}(\hat{\Sigma}) \) whose limit \( \mathcal{ML}^{\infty} := \lim_{\rightarrow} \mathcal{ML}(Z) \) has dense image in \( \mathcal{ML}(\hat{\Sigma}) \). In this paper we will work exclusively with \( \mathcal{ML}^{\infty} \).

A pseudo Anosov diffeomorphism \( \Phi : Z \to Z \) lifts to a diffeomorphism of \( \hat{\Sigma} \) fixing precisely the lifted projective classes \( \hat{[P]} \) and \( \hat{[P']}. \) In what follows, the terminology “pseudo Anosov homeomorphism of \( \hat{\Sigma} \)” will always mean such a lift.

4. Train Tracks

We recall first some facts about train tracks on \( \Sigma \): details may be found in [14]. Let \( \tau \subset \Sigma \) be a smooth 1-dimensional branched manifold: thus \( \tau \) is a 1-dimensional CW-complex in which the interiors of edges are smooth curves, and the field of tangent lines \( T_{x}\tau, x \in \tau \setminus \{ \text{vertices} \}, \) extends to a continuous line field on \( \tau \). We say that \( \tau \) is a train track if it satisfies the following additional properties:
(1) The valency of any vertex is at least 3, except for simple closed curve components, which have a single vertex of valence 2.

(2) If $D(S)$ is the double of a component $S \subset \Sigma \setminus \tau$, then the Euler characteristic of $D(S)$ is negative.

We shall follow the custom of referring to the vertices of a train track as switches.

A bigon track is a smooth 1-dimensional branched manifold $\tau \subset \Sigma$ satisfying item (1) and which satisfies (2) after collapsing bigon complementary regions to curves. Bigon tracks arise naturally from a pair $\mathcal{C}, \mathcal{D}$ of transverse filling curves, by turning each intersection of a $\mathcal{C}$ curve with a $\mathcal{D}$ curve into a pair of 3-valent vertices as in Figure 1. Since such bigon tracks will be the only ones appearing in this article, we will assume from now on that all switches in bigon tracks have valency no more than three.

![Figure 1. Creating a bigon track from a filling pair of curves.](image)

Denote by $E$ the set of edges of the bigon track $\tau$. In a small disk neighborhood of a switch $v$, the ends of edges incident to $v$ may be divided into two classes, which for convenience we refer to as “incoming” and “outgoing”: each class consists of ends that are asymptotic to one another, and the decision of naming one class incoming, the other outgoing, is arbitrary. We write $e \in \text{in}(v)$ or $e \in \text{out}(v)$ if $e$ has an end belonging to the appropriate class. See Figure 2. (Note: it can happen that $e$ belongs to both $\text{in}(v)$ and $\text{out}(v)$.)

![Figure 2. Incoming and outgoing ends.](image)

A switch-additive measure on $\tau$ is a function $m : E \to \mathbb{R}_+$ for which

$$\sum_{e \in \text{in}(v)} m(e) = \sum_{e \in \text{out}(v)} m(e)$$

for all switches $v$. The set of all switch-additive measures forms a linear cone $C_\tau$ in $\mathbb{R}^E$.

Let $N(\tau)$ be a tubular neighborhood of $\tau$ equipped with a (singular) foliation by line segments transverse to $\tau$. See Figure 3.

A measured lamination $f \subset \Sigma$ is said to be carried by $\tau$ if it may by isotoped into $N(\tau)$ transverse to its foliation. We write in this case $f \prec \tau$. The subspace of isotopy classes of
measured laminations carried by $\tau$ is denoted $\mathcal{ML}_\tau(\Sigma)$. There is an open surjection

$$C_\tau \rightarrow \mathcal{ML}_\tau(\Sigma)$$

which is a homeomorphism if $\tau$ is a train track.

Let $\Phi : \Sigma \rightarrow \Sigma$ be a pseudo Anosov diffeomorphism. We say that $\Phi$ acts on $\tau$ if $\Phi(\tau)$ may be isotoped into $N(\tau)$ transverse to its foliation. We write then $\Phi(\tau) < \tau$. Fix a leaf $t_i \subset N(\tau)$ through each edge $e_i$ of $\tau$. The carrying matrix of $\Phi$ is by definition $M_\Phi = (a_{ij})$ where

$$a_{ij} = |\Phi(e_i) \cap t_j|.$$ 

$M_\Phi$ induces an inclusion $C_{\Phi(\tau)} \hookrightarrow C_\tau$ which when precomposed with the pushforward map $C_\tau \rightarrow C_{\Phi(\tau)}$ defines a linear map

$$M_\Phi : C_\tau \rightarrow C_\tau.$$ 

Note that the carrying matrix $M_\Phi$ is non-negative. Such a matrix has a unique eigenvalue of greatest modulus, which is positive-real and simple [4]. This eigenvalue is called the Perron root. A corresponding eigenvector may be taken non-negative, and is called a Perron vector. For $M_\Phi$, the Perron root coincides with the entropy $\lambda$ of $\Phi$, and the Perron vector parametrizes in track coordinates the unstable measured lamination $\mu$ of $\Phi$.

When $\Phi$ is a pseudo Anosov of Thurston-Penner type, one can recover the carrying matrix and all of its Perron data from a simpler matrix which records the action of $\Phi$ on the curves in the families $C$ and $D$. Indeed, let $\tau$ be the bigon track formed from $C \cup D$. If $e_i, e_j \in E_\tau$ are edges contained in say $c, d$ resp. then given $e_f \subset c$ another edge, there exists a unique $e_f \subset d$ with $a_{ij} = a_{ij}$. Conversely, for $e_f \subset d$, there exists $e_f \subset c$ with $a_{ij} = a_{ij}$. It follows that the carrying matrix of $\Phi$ can be subdivided into blocks indexed by pairs $(c, d)$, which are of the form $a_{c,d}$ where $I$ is a square matrix in which each column and row has exactly one non zero entry $= 1$. The matrix $(a_{c,d})$ whose columns and rows are indexed by $C \cup D$ has exactly the same Perron root as $M_\Phi$, and its Perron vector gives that of $M_\Phi$ in the obvious way. We shall call this matrix the curve matrix of the Thurston-Penner type pseudo Anosov $\Phi$, and we shall denote it $M_\Phi$ as well.

For example, if $C = \{c\}$, $D = \{d\}$ with $|c \cap d| = r$ and $\Phi = G_d^{-N} \circ F_c^N$, then

$$M_\Phi = \begin{pmatrix} 1 & rN \\ rN & (rN)^2 + 1 \end{pmatrix}.$$ 

We note that using the quadratic formula, it is easy to see that the eigenvalue not equal to the Perron root is $< 1$.

We now discuss tracks on the solenoid $\hat{\Sigma}$: in fact, we will only require tracks pulled back from surfaces appearing in its defining inverse system. Thus, if $\tau$ is a train track on such a surface $Z$, its preimage $\hat{\tau}$ is a smooth 1-dimensional branched solenoid with edge set
\( \hat{E} \approx E \times \tilde{T}_Z \), where \( E \) is the edge set of \( \tau \) and \( \tilde{T}_Z \) is the fiber over a point of \( Z \), homeomorphic to the Cantor group \( \hat{\pi}_1 Z \). With respect to this decomposition we define a measure on \( \hat{E} \) by the formula

\[
\mu_{\hat{E}} = \mu_E \times \eta_{\tilde{T}_Z}
\]

where \( \mu_E(e) = 1 \) for each edge of \( E \) and \( \eta_{\tilde{T}_Z} \) is the restriction to \( \tilde{T}_Z \) of the transverse invariant measure of \( \hat{\Sigma} \). In addition, if \( \tau \) is equipped with a switch additive measure \( \nu \), the pullback \( \tilde{\nu} \) is a transversally continuous switch-additive measure on \( \hat{\tau} \); the cone of such measures on \( \hat{\tau} \) is denoted \( C_\tau \). The relation \( \hat{\tau} < \hat{\tau} \) has exactly the same meaning as in the case of a surface.

5. Intersection Formulas

Let \( \tau, \kappa \) be bigon tracks in \( \Sigma \) that intersect transversally and minimally with edge sets \( E_\tau \) and \( E_\kappa \); let \( \hat{\tau}, \hat{\kappa} \) be measured laminations carried by them, parametrized by weights \( \nu, \omega \). The intersection pairing may be calculated by the following formula \([14]\):

\[
I(\hat{\tau}, \hat{\kappa}) = \sum_{e \in E_\tau, e' \in E_\kappa} \nu(e) \omega(e').
\]

It is useful to re-express \( I(\hat{\tau}, \hat{\kappa}) \) as a sum over edges in \( E_\tau \) only. Thus if we write

\[
\omega(e) = \sum_{e' \in E_\kappa} \omega(e'|e \cap e')
\]

then

\[
I(\hat{\tau}, \hat{\kappa}) = \sum_{e \in E_\tau} \nu(e) \omega(e).
\]

Suppose now that \( \hat{\tau}, \hat{\kappa} \) are measured laminations obtained as preimages of measured laminations \( \tau \subset Y \) and \( \kappa \subset Z \), surfaces occurring in the defining system of \( \hat{\Sigma} \). Let \( W \) be a surface finitely covering each of \( Y, Z \), and let \( \hat{\tau}, \hat{\kappa} \) be the preimages in \( W \) of \( \tau, \kappa \). Let \( \deg(W) \) be the degree of the covering \( W \to \Sigma \).

**Proposition 1.** \( I(\hat{\tau}, \hat{\kappa}) = \frac{I(\tau, \kappa)}{\deg(W)}. \)

**Proof.** The intersection locus of \( \hat{\tau} \) and \( \hat{\kappa} \) is of the form

\[
(\hat{\tau} \cap \hat{\kappa}) \times \tilde{T}_W
\]

where \( \tilde{T}_W \) is a fiber of \( \hat{\Sigma} \to W \). Since \( \tilde{T}_W \) has \( \eta \)-measure \( 1/\deg(W) \), the result follows. \[\square\]

Let \( \hat{\tau}, \hat{\kappa} \) be as in the previous paragraphs. Suppose now that \( \tau \) is parametrized by \( \nu : E \to \mathbb{R}_+ \) a weight on a bigon track \( \tau \subset Y \) and \( \kappa \) is parametrized by \( \omega : E' \to \mathbb{R}_+ \) a weight on a bigon track \( \tau' \subset Z \). The preimages \( \hat{\tau}, \hat{\kappa} \) are parametrized by the pullback weights \( \tilde{\nu}, \tilde{\omega} \) on the preimages \( \hat{\tau}, \hat{\kappa} \). Rewriting \( \omega \) as above as a function of the edge set \( E \), we have

\[
I(\hat{\tau}, \hat{\kappa}) = \sum_{\hat{e} \in \hat{E}} \tilde{\nu}(\hat{e}) \tilde{\omega}(\hat{e}).
\]

Let \( \hat{\tau} \) be the preimage of \( \tau \) in \( \hat{\Sigma} \), and let \( \tilde{\nu}, \tilde{\omega} \) be the pullbacks along the projection \( \hat{E} \to \tilde{E} \).

**Proposition 2.** \( I(\hat{\tau}, \hat{\kappa}) = \int_{\hat{E}} \tilde{\nu} \tilde{\omega} \ d\mu_{\hat{E}} \) where \( \mu_{\hat{E}} \) is the edge measure on \( \hat{E} \).
Proof. A calculation:

$$I(\hat{f}, \hat{g}) = \left( \sum_{\tilde{e} \in \tilde{E}} \hat{u}(\tilde{e}) \hat{w}(\tilde{e}) \right) \cdot \frac{1}{\deg(W)} = \left( \sum_{\tilde{e} \in \tilde{E}} \hat{u}(\tilde{e}) \hat{w}(\tilde{e}) \right) |\tilde{W}| = \int_{\tilde{E}} \hat{u} \hat{w} \, d\mu_{\tilde{E}}.$$  

\[\square\]

6. Teichmüller Theory of \(\hat{\Sigma}\)

References for material in this section are \[6\], \[12\], \[18\].

The definition of the Teichmüller space \(T(\hat{\Sigma})\) and of its metric \(d_{T(\hat{\Sigma})}\) copies that of a surface. In particular,

1. A conformal structure on \(\hat{\Sigma}\) is determined by a conformal structure on each leaf. These structures are required to vary continuously in the transverse direction.
2. Elements of \(\mathcal{T}(\hat{\Sigma})\) are represented by marked solenoids i.e. by homeomorphisms \(\mu : \hat{\Sigma} \to \hat{\Sigma}_{\mu}\), where \(\hat{\Sigma}_{\mu}\) is presumed to have a conformal structure.
3. The marked solenoid \(\mu' : \hat{\Sigma} \to \hat{\Sigma}_{\mu'}\) is equivalent to \(\mu\) if there exists an isomorphism \(\sigma : \hat{\Sigma}_{\mu} \to \hat{\Sigma}_{\mu'}\) such that \(\sigma \circ \mu \simeq \mu'\).

\(\mathcal{T}(\hat{\Sigma})\) has the structure of a separable Banach manifold. The canonical projection \(\hat{\mu} : \hat{\Sigma} \to Z\) onto any surface \(Z\) in the defining inverse system induces a direct system of isometric inclusions \(\hat{\mu}^* : \mathcal{T}(Z) \hookrightarrow \mathcal{T}(\hat{\Sigma})\).

Theorem 2 (\[12\]). The induced inclusion

\[i : \lim_{\to} \mathcal{T}(Z) \hookrightarrow \mathcal{T}(\hat{\Sigma})\]

is isometric with dense image.

For most of our purposes, it will be sufficient to work with the dense subspace \(\mathcal{T}^\omega := i(\lim_{\to} \mathcal{T}(Z))\), which is an incomplete metric space with respect to the direct limit of the Teichmüller metrics and a pre-Banach manifold. Unless otherwise said, all structures \(\hat{\mu}\) considered below will be assumed to be in \(\mathcal{T}^\omega\).

Let \(\hat{\Sigma}_{\hat{\mu}}\) be as above. By a holomorphic quadratic differential \(q\) on \(\hat{\Sigma}_{\hat{\mu}}\), we shall always mean the pull-back of a holomorphic quadratic differential \(q\) occurring on some surface \(Z_{\mu}\), where \(\hat{\mu}\) is the pull-back of \(\mu\). Thus, \(q\) is a choice of holomorphic quadratic differential on each leaf, constant along the fiber transversals \(\hat{T}_{Z}\) over \(Z\). The tangent space to \(\mathcal{T}^\omega\) at \(\hat{\mu}\) may be identified with the direct limit

\[Q^\omega_{\hat{\mu}} = \lim_{\to} Q_{\mu}(Z)\]

The tangent bundle of \(\mathcal{T}^\omega\) is then identified with \(\mathcal{Q}^\omega = \lim_{\to} \mathcal{Q}(Z)\) where \(\mathcal{Q}(Z)\) is the space of holomorphic quadratic differentials on \(Z\) (with respect to all possible complex structures).

The \(L^1\) norm on \(Q^\omega_{\hat{\mu}}\) is defined

\[\|q\| = \int_{\hat{\Sigma}_{\hat{\mu}}} |q| \, d\eta.\]

From the pre-Finsler norm \(\| \cdot \|\) we induce a path metric \(d_{L^1}\) on \(\mathcal{T}^\omega\) that defines 1) the \(L^1\) topology on \(\mathcal{T}^\omega\) and 2) along with \(\| \cdot \|\), the \(L^1\) topology on \(\mathcal{Q}^\omega\).
To any quadratic differential \( \hat{q} \) one associates two transverse, measured laminations \( \hat{f}^h \) and \( \hat{f}^v \) (those that correspond to the horizontal and vertical trajectories of \( \hat{q} \)). We have the following generalization to \( \hat{\Sigma} \) of a well-known formula for surfaces:

**Lemma 1.** \( I(\hat{f}^h, \hat{f}^v) = \|\hat{q}\| \).

**Proof.** \( \hat{q} \) is the lift of a holomorphic quadratic differential on some surface \( Z_\mu \), where \( \hat{\mu} \) is the lift of \( \mu \). The result now follows from the classical formula and the fact that the lift of an area measure on \( Z_\mu \) scales by \( 1/(\deg Z) \) in \( \hat{\Sigma} \).

In the same way, we may also avail ourselves of a direct limit version of the theorem of Hubbard and Masur [9]:

**Theorem 3.** Any pair of measured laminations \( \hat{\gamma}, \hat{\alpha} \in \mathcal{ML}^m \) determines a unique quadratic differential \( \hat{q} \).

**Theorem 4.** Let \( \hat{q}, \hat{q}_i \in \mathcal{ML}^m, \ i = 1, 2, \ldots , \) be quadratic differentials. If \( \hat{f}^h_i \rightarrow \hat{f}^h \) and \( \hat{f}^v_i \rightarrow \hat{f}^v \) in the intersection topology, then \( \hat{q}_i \rightarrow \hat{q} \) in the \( L^1 \) topology.

**Proof.** We assume first that there exists \( \hat{\mu} \in \mathcal{ML}^m \) with \( \hat{q}_i, \hat{q} \in \mathcal{Q}^\omega_\hat{\mu} \). Let \( \hat{f}^h_i, \hat{f}^v_i \) be the pairs of measured foliations which are the horizontal and vertical line fields of \( \hat{q} \) resp. \( \hat{q}_i \). By the comments of §3, the hypothesis on the convergence of measured laminations is equivalent to the corresponding statement for measured foliations. In particular we have,

\[
\lim I(\hat{f}^h_i, \hat{f}^v_i) = 0 \quad \text{and} \quad \lim I(\hat{f}^v_i, \hat{f}^h_i) = 0.
\]

Consider smooth measured foliations \( \hat{g}^h_i, \hat{g}^v_i \) equivalent to \( \hat{f}^h_i, \hat{f}^v_i \) whose heights with respect to \( \hat{f}^h, \hat{f}^v \) nearly give the intersections \( I(\hat{f}^h_i, \hat{f}^h), I(\hat{f}^v_i, \hat{f}^v) \). More precisely, for \( \varepsilon_i \rightarrow 0 \),

\[
\int_{\hat{g}^h_i} |\Re \sqrt{\hat{q}}| \ d\eta - I(\hat{f}^h_i, \hat{f}^h) < \varepsilon_i \quad \text{and} \quad \int_{\hat{g}^v_i} |\Im \sqrt{\hat{q}}| \ d\eta - I(\hat{f}^v_i, \hat{f}^v) < \varepsilon_i.
\]

Let \( \hat{q}'_i \) denote the smooth quatratic differential whose horizontal and vertical foliations are \( \hat{g}^h_i, \hat{g}^v_i \). Given \( \delta > 0 \), let \( \hat{A}_i \) be the set of points for which \( |\hat{q} - \hat{q}'_i| \) is uniformly \( \delta \) small. Let \( \hat{B}_i = \hat{\Sigma} \setminus \hat{A}_i \).

We begin by showing that

\[
\int_{\hat{B}_i} |\hat{q} - \hat{q}'_i| \ d\eta \rightarrow 0
\]
as \( i \rightarrow 0 \). By \( 4 \) and \( 5 \), \( \int_{\hat{B}} |\hat{q}| \ d\eta \rightarrow 0 \). If for \( i \) large, there is a \( m > 0 \) with

\[
0 < m \leq \int_{\hat{B}_i} |\hat{q} - \hat{q}'_i| \ d\eta
\]
we must also have

\[
0 < m_0 \leq \int_{\hat{B}_i} |\hat{q}'_i| \ d\eta
\]
for some \( m_0 > 0 \). The fact that \( \int_{\hat{B}} |\hat{q}| \ d\eta \rightarrow 0 \) whereas \( \int_{\hat{B}_i} |\hat{q}'_i| \ d\eta \) does not would violate \( 4 \) and \( 5 \) as well. Thus \( \int_{\hat{B}_i} |\hat{q}'_i| \ d\eta \rightarrow 0 \), proving \( 6 \). In particular, we have

\[
\lim \int |\hat{q} - \hat{q}'_i| \ d\eta = \lim \int |\hat{q} - \hat{q}'_i|.
\]

Now \( \hat{q}'_i \) is measure equivalent to \( \hat{q}_i \), hence \( \hat{q} - \hat{q}_i \) is measure equivalent to \( \hat{q} - \hat{q}'_i \). By the second minimal norm property \( 5 \), it follows that

\[
\|\hat{q} - \hat{q}_i\| \leq \|\hat{q} - \hat{q}'_i\|.
\]
Letting $\varepsilon \to 0$, we obtain $\|\hat{q} - \tilde{q}_i\| \to 0$. This proves the theorem in the special case where $\hat{\mu} = \tilde{\mu}_i$ for $i$ large.

Now we suppose that $\hat{q} \in Q^\infty_{\tilde{\mu}_i}, \tilde{q}_i \in Q^\infty_{\hat{\mu}_i}$ with $\hat{\mu}_i \neq \tilde{\mu}_i$. Let $\hat{C}_i$ be the set of points where the foliations $\hat{f}^h_i, \hat{f}^v_i$ are uniformly $\varepsilon$-close to $\tilde{f}^h, \tilde{f}^v$ in the $\hat{q}$-metric. Then for $i$ large, in $\hat{C}_i$ the complex structures defined by $\hat{q}_i, \hat{q}$ are uniformly nearly conformal. On the other hand, in $\tilde{D}_i = \tilde{\Sigma} \setminus \hat{C}_i$ they are not, but the $|\hat{q}|$-volume of this set limits to zero. Therefore, if $\hat{p}_i \in Q^\infty_{\hat{\mu}_i}$ generates the Teichmüller geodesic connecting $\hat{\mu}$ to $\tilde{\mu}_i$ in time 1, it follows that $\|\hat{p}_i\| \to 0$ so that $\hat{\mu}_i$ converges to $\hat{\mu}$ in the $L^1$ path metric. Moreover, the induced flow of quadratic differentials takes $\hat{q} L^1$ close to $\tilde{q}_i$ so that $\tilde{q}_i$ converges to $\hat{q}$ in the $L^1$ topology on $Q^\infty$.

\[\Box\]

7. The $L^1$ Ehrenpreis Conjecture

The $L^1$ EC is the following statement:

**$L^1$ Ehrenpreis Conjecture.** The mapping class group $M(\tilde{\Sigma}, \ell)$ acts with $L^1$ dense orbits on $T(\tilde{\Sigma})$.

Since $T^\infty$ is dense in $T(\tilde{\Sigma})$ and $M(\tilde{\Sigma}, \ell)$ stabilizes $T^\infty$ [6], it will be enough to demonstrate that $M(\tilde{\Sigma}, \ell)$ acts with $L^1$ dense orbits on $T^\infty$.

In [11] it is shown that for a closed surface $Z$, a Teichmüller dense subset of pairs $\mu, \nu \in T(Z)$ lie on the axes of pseudo Anosov homeomorphisms. By its definition as an isometric direct limit, $T^\infty$ enjoys the same property. Fix a pair $\hat{\mu}, \hat{\nu} \in T^\infty$; without loss of generality, we may then assume that $\hat{\mu}, \hat{\nu}$ lie on the axis $A$ of a pseudo Anosov diffeomorphism $\Phi$ which is the lift of a pseudo Anosov $\Phi : Z \to Z$, for some surface $Z$ occurring in the defining system of $\tilde{\Sigma}$.

By an $L^1$ nth root of $\Phi$ we mean a sequence $\{\Psi_m\}$ of pseudo Anosov homeomorphisms for which

1. If $\lambda_m$ is the entropy of $\Psi_m$, then $\lim \lambda_m = \lambda^{1/n}$.
2. If $A_m$ is the axis of $\Psi_m$ then $A_m \to A$ converges in the Hausdorff topology induced from the $L^1$ metric.

**Theorem 5.** If for every pseudo Anosov $\Phi$ and every $n > 0$, $\Phi$ has an $L^1$ nth root, then the $L^1$ EC is true.

**Proof.** Suppose that $\hat{\mu}, \hat{\nu}$ lie on the axis $A$ of $\Phi$ and let $\{\Psi_m\}$ be an $L^1$ nth root, $n$ large. Since the $A_m \to A$ in the $L^1$ Hausdorff topology, there exists $\mu', \nu'$ lying on some axis $A_m$ with $d_{L^1}(\hat{\mu}, \tilde{\mu}) < \varepsilon, d_{L^1}(\tilde{\nu}, \tilde{\nu}) < \varepsilon$. On the axis $A_m$, we may move via a power of $\Psi_m$ to $\tilde{\nu}'$ close to $\nu'$, which implies that $\tilde{\mu}$ is moved close to $\tilde{\nu}$ by the same power of $\Psi_m$ as well. \[\Box\]

**Note 1.** The existence of Teichmüller roots for all $\Phi$ and all $n$ implies the classical EC.

8. Directional Density

A family $\mathcal{P}$ of pseudo Anosov homeomorphisms is said to be *directionally dense* (in $Q^\infty$) if the set of quadratic differentials tangent to axes of elements of $\mathcal{P}$ is Teichmüller dense in $Q^\infty$. By [11], the family of all pseudo Anosov maps is directionally dense. In fact, it follows easily from the arguments in [11] that the family of lifts of pseudo Anosovs $\Phi$ of the type $\Phi = G^{-2N} \circ F^{2N}$, where $F, G$ are right Dehn twists about simple closed curves $c, d$ that fill $Z$, where $Z$ ranges over all surfaces in the defining system of $\tilde{\Sigma}$, is
directionally dense. By Corollary 2.6 in [11], the subfamily obtained by demanding that $c$ is nonseparating is also directionally dense.

Now given a nonseparating simple closed curve $\gamma \subset Z$, let $\rho_\gamma : Z_\gamma \to Z$ be the degree 2 cover obtained by cutting two copies of $Z$ along $\gamma$ and gluing ends. We say that a pair $(c, d)$ of filling, simple closed curves is interlacing if there exists a pair of nonseparating simple closed curves $\alpha, \beta$ such that $\rho^{\alpha^{-1}}(c)$ is connected whereas $\rho^{\alpha^{-1}}(d)$ is not and $\rho^{\beta^{-1}}(d)$ is connected whereas $\rho^{\beta^{-1}}(c)$ is not. See Figure 4.

![Figure 4](image.png)

**Figure 4.** A filling pair $c, d$ and a pair $\alpha, \beta$ interlacing them.

Let $\mathcal{P}$ be the family of pseudo Anosov homeomorphisms of $\hat{\Sigma}$ which are lifts of pseudo Anosovs of the form

$$\Phi = G^{-2N} \circ F^{2N} : Z \to Z,$$

where

1. $F, G$ are right Dehn twists about $c, d \subset Z$.
2. $c$ is nonseparating and $(c, d)$ is an interlacing pair.
3. $Z$ ranges over all surfaces in the defining system of $\hat{\Sigma}$.

**Lemma 2.** $\mathcal{P}$ is directionally dense.

**Proof.** It is enough to show that for a fixed surface $Z$, the family of maps satisfying (1) and (2) is directionally dense in $\mathcal{D}(Z)$. Assume that $c$ and $d$ are filling, generating the pseudo Anosov homeomorphism $\Phi = G^{-2N} \circ F^{2N}$. If $(c, d)$ is not an interlacing pair, there exists a simple closed curve $\delta$ for which the pair $(c, \delta)$ is interlacing, though not necessarily filling. Indeed, one may assume after a homeomorphism that $c$ is the curve appearing in Figure 5; then taking $\delta, \alpha, \beta$ as indicated there, $(c, \delta)$ is interlacing with respect to the pair $(\alpha, \beta)$. Now for $j$ large, $\delta_j = G^j(\delta)$ is close to $d$, hence $(c, \delta_j)$ is eventually filling. If $j$ is in addition even, $G^j$ lifts to the total space of any degree 2 cover of $\Sigma$, thus the pair $(c, \delta_j)$
is interlacable with respect to the same curves interlacing \((c, \delta)\). For \(j\) large, the pseudo Anosov \(\Phi_j = G_{\delta_j}^{2N} \circ F^{2N}\) has axis close to that of \(\Phi\), and since the maps of the form \(\Phi\) are already directionally dense, we are done. \(\square\)

**Figure 5.** Every nonseparating \(c\) is a member of a (not necessarily filling) interlacing pair.

### 9. Necklace Roots

Let \(n \in \mathbb{N}\) and let \(\hat{\Phi} \in \mathcal{P}\), the family appearing in Lemma 2, so that in particular \(\hat{\Phi}\) is the lift of a pseudo Anosov of the form \(\Phi = G^{-2N} \circ F^{2N} : Z \to Z\). Let \(\rho_{mn} : Z_{mn} \to Z\) be the cover obtained by cutting \(2mn\) copies of \(Z\) along a pair \(\alpha\) and \(\beta\) interlacing \(c, d\) and gluing in a circular fashion. We call \(\rho_{mn}\) the necklace cover associated to \((c, d)\). In Figure 6, we illustrate the construction of the necklace \(Z_{mn}\) and the formation of the lifts of the curve \(c\). In Figure 7 we display the finished necklace.

There are \(mn\) lifts \(c_1, \ldots, c_{mn}\) and \(d_1, \ldots, d_{mn}\) of each of \(c\) and \(d\), each mapping with degree two onto their ancestor. On \(Z_{mn}\), \(\Phi\) lifts to

\[
\tilde{\Phi} = G_{mn}^{-N} \circ \cdots \circ G_{1, m}^{-N} \circ F_{nm}^{N} \circ \cdots \circ F_{1, m}^{N}
\]

where \(F_i, G_i\) is the right Dehn twist about \(c_i, d_i\). Let \(\chi\) denote the clockwise rotation of \(Z_{mn}\) by an angle of \(2\pi/n\), so that the pair \(c_i, d_i\) is taken to \(c_{j+m}, d_{j+m}\) (indices taken mod \(mn\)).

We define the necklace \(n\)th root to be the sequence of lifts of pseudo Anosovs \(\{\sqrt[n]{\Phi_m}\}\) to \(\hat{\Sigma}\) where

\[
\sqrt[n]{\Phi_m} = \chi \circ G_{m}^{-N} \circ \cdots \circ G_{1, m}^{-N} \circ F_{m}^{N} \circ \cdots \circ F_{1, m}^{N},
\]

\(m = 2, 3, \ldots\). The necklace \(n\)th root is the basic construction used in the formation of \(L^1\) roots. The construction of \(\sqrt[n]{\Phi_m}\) is a generalization of one that first appeared in [16], where branched covers were used.

**Lemma 3.** \(\sqrt[n]{\Phi_m}\) is pseudo Anosov for all \(m\).

**Proof.** For \(i = 1, \ldots, n\), let

\[
T_i = G_{im}^{-N} \circ \cdots \circ G_{(i-1)m+1}^{-N} \circ F_{im}^{N} \circ \cdots \circ F_{(i-1)m+1}^{N}.
\]

Then it is easy to see that

\[
(\sqrt[n]{\Phi_m})^n = T_2 \circ \cdots \circ T_n \circ T_1,
\]

which is of Thurston-Penner type, hence [14] \(\sqrt[n]{\Phi_m}\) is pseudo Anosov, implying \(\sqrt[n]{\Phi_m}\) is pseudo Anosov as well. \(\square\)
10. Existence of $L^1$ Roots

Denote by $\mathcal{C}_Z(\hat{\Sigma})$ the family of simple closed solenoids which are lifts of simple closed curves on $Z$. We begin by constructing a family $\{\hat{\Psi}_m\}$ whose stable and unstable laminations intersection converge to those of $\hat{\Phi}$ with respect to test solenoids in $\mathcal{C}_Z(\hat{\Sigma})$.

For each $m = 2, 3, \ldots$, let $\sqrt[2]{\Phi}_m$ denote the $m$th element in the sequence of pseudo Anosovs whose lifts define the $m$th necklace root of $\hat{\Phi}$. Define the sequence $\{\hat{\Psi}_m\}$ as the lifts to $\hat{\Sigma}$ of the pseudo Anosov homeomorphisms

$$\Psi_m = \left(\sqrt[2]{\Phi}_m\right)^m.$$ 

Observe that the stable and unstable foliations of $\Psi_m$ and $\sqrt[2]{\Phi}_m$ are equal.

**Note 2.** $\Psi_m$ is not the same as $\sqrt[2]{\Phi}_m$. In fact, if we lift $\sqrt[2]{\Phi}_m$ to $Z_{m^n}$, where $\Psi_m$ is defined -- we see that this lift twists along $m$ disjoint “blocks” of curves, each block consisting of a succession of $m$ lifts of $c$ and $d$. On the other hand, $\Psi_m$ consists of twists along one block consisting of a succession of $m^2$ lifts of $c$ and $d$. As we shall see, the stable and unstable laminations of the family $\{\Psi_m\}$ have better convergence properties than those of $\{\sqrt[2]{\Phi}_m\}$.

Denote by $\hat{f}_{\mu}^{\mu}$, $\hat{f}_{\mu}^{s}$ and by $\hat{f}^{\mu}, \hat{f}^{s}$ the unstable and stable laminations of $\hat{\Psi}_m$ and $\hat{\Phi}$.

**Lemma 4.** For all $\hat{c} \in \mathcal{C}_Z(\hat{\Sigma})$,

$$I(\hat{f}_{\mu}^{\mu}, \hat{c}) \to I(\hat{f}^{\mu}, \hat{c}) \quad \text{and} \quad I(\hat{f}_{\mu}^{s}, \hat{c}) \to I(\hat{f}^{s}, \hat{c}).$$
**Proof.** The proof will be through examination of curve matrices. We begin with $\Phi$. Let $r = I(c, d)$. The action of $\Phi$ along the curves $c, d$ is given by the matrix

$$M_\Phi = \begin{pmatrix} 1 & 2rN \\ 2rN & (2rN)^2 + 1 \end{pmatrix}.$$ 

Let $Z_{mn}$ be the surface where $\Psi_m$ is defined. The curve families $\mathcal{C} = \{c_1, \ldots, c_{m^2n}\}$, $\mathcal{D} = \{d_1, \ldots, d_{m^2n}\}$ are filling, and the action of $\tilde{\Phi}$ on $\mathcal{C} \cup \mathcal{D}$ is prescribed schematically by the matrix in Figure 8, where all entries not contained in the boxed vectors are zero, and where the “broken” vectors indicate that only that portion of the corresponding vector is used. For example, in the upper right hand corner we have the entry $rN$, which is the bottom half of the $B$-vector; in the lower right hand corner, we have the vector entry $((rN)^2 - 2(rN)^2 + 1)^T$, which is the top two thirds of the vector $D$, and so on. The curve matrix of $(\Psi_m)^n = (\sqrt[nn]{\Phi_m})^{mn}$ is displayed in Figure 9. The black vectors indicate regions...
where $M_{\Psi_m^n} = (M\Psi_m)^n = (M\sqrt{\nabla_m})^{mn}$ differs from $\tilde{M}_\Phi$. Figure 10 contains the curve matrix of $m$. 

\[
\tilde{M}_\Phi = \begin{pmatrix}
\epsilon_{i,j} & \epsilon_{m} & d_i \\
\epsilon_{m} & d_i & b_i \\
d_i & b_i & \epsilon_j
\end{pmatrix}
\]

**Figure 8.** The curve matrix for $\Phi$.

Denote by $\lambda_m$ the Perron root of $M\Psi_m$ and by 

$$v_m = (a_1, \ldots, a_m, b_1, \ldots, b_m)$$

the corresponding Perron vector, normalized to have $L^1$ norm 1, i.e., so that $v_m$ is a probability vector. Let $v_m^{\text{avg}}$ be the vector 

$$v_m^{\text{avg}} = (a_1 + \cdots + a_m, b_1 + \cdots + b_m),$$

which is also a probability vector. In the case of $\Phi$ and $\tilde{\Phi}$, the Perron roots are identical and will be denoted $\tilde{\lambda}$; if $v = (a, b)^T$ is the $L^1$ norm 1 Perron vector of $M\Phi$, $\tilde{v} = (1/2m^2n(a_1, \ldots, a, b, \ldots, b)^T$ is the $L^1$ norm 1 Perron vector for $M\Phi$. We recall that this spectral data has the following interpretation:

1. The Perron roots $\lambda_m, \tilde{\lambda}$ are equal to the entropies of $\Psi_m, \Phi$.
2. Let $\tau_m$ be the bigon track formed from the curve families $\mathcal{C}, \mathcal{D}$. The measures $\mu_m$, $\tilde{\mu}$ formed from the Perron vectors $v_m, \tilde{v}$, parametrize the unstable laminations $f_m^\mu$, $\tilde{f}_m^{\tilde{\mu}}$ in the cone $C(\tau_m)$.

Note that the column sums of $M_i\Psi_m^n$ have uniform upper and lower bounds $B$ and $b > 1$. We thus obtain the bound $1 < b < (\lambda_m)^n < B$.

We may then assume, after passing to a subsequence if necessary, that the $\lambda_m$ converge to some value $\lambda^* > 1$. We shall need to control the following entries of $v_m$:

**Claim 1.** $a_m, a_{m+1}, b_m, b_{m+1} \to 0$ as $m \to \infty$. 

\[ M_{(\Psi_m)^n} = \]

\[
\begin{array}{cccc}
\text{\(A\)} & \text{\(B\)} & \text{\(C\)} & \text{\(D\)} \\
\text{\(A\)} & \text{\(B\)} & \text{\(C\)} & \text{\(D\)} \\
\text{\(A\)} & \text{\(B\)} & \text{\(C\)} & \text{\(D\)} \\
\text{\(A\)} & \text{\(B\)} & \text{\(C\)} & \text{\(D\)} \\
\end{array}
\]

\[ a = (\alpha N)^i \]
\[ A = (\alpha N)^i + 1 \]
\[ B = \left( \frac{\alpha N}{(\alpha N)^i + 1} \right) \]
\[ C = (\alpha N)^i \]
\[ D = \left( \frac{\alpha N}{(\alpha N)^i + 1} \right) \]
\[ D = \left( \frac{9}{2(\alpha N)^i + 1} \right) \]

**Figure 9.** The curve matrix for \(\Psi_m^n\).

\[ M_{\sqrt{\Phi_m}} = \]

\[
\begin{array}{cccc}
\text{\(A\)} & \text{\(B\)} & \text{\(C\)} & \text{\(D\)} \\
\text{\(A\)} & \text{\(B\)} & \text{\(C\)} & \text{\(D\)} \\
\text{\(A\)} & \text{\(B\)} & \text{\(C\)} & \text{\(D\)} \\
\text{\(A\)} & \text{\(B\)} & \text{\(C\)} & \text{\(D\)} \\
\end{array}
\]

**Figure 10.** The curve matrix for \(\sqrt{\Phi_m}\).
Proof of claim. Let \( \xi_m \) be the Perron root of \( M_{\Psi_m} \); thus \( (\xi_m)^m = \lambda_m \). If one of the four entries listed in the statement does not converge to 0, then consideration of the matrix \( M_{\Psi_m} \) shows that none of them do. Thus, let us suppose that \( a_m \neq 0 \). It follows that eventually \( a_{2m} \geq \delta \) for some positive \( \delta \). Examination of the action of \( M_{\Psi_m} \) on \( \upsilon_m \) shows that \( a_{m+im} = \xi_{2m}^{-1}a_{2m} \) for \( i = 1, \ldots, mn \). However \( \xi_{m}^{-1} > 1 \) for all \( i \), and since \( \upsilon_m \) is a probability vector, this would imply that \( m\delta < mna_{2m} < 1 \), impossible since \( m \to \infty \). This proves the claim.

Let us shorten notation, writing \( \hat{\xi} = \hat{\xi}^\mu \) and \( \hat{\upsilon}_m = \hat{\upsilon}_m^\upsilon \) for the unstable foliations of \( \hat{\Phi} \) and \( \hat{\Psi}_m \). Let \( \hat{c} \) be any simple closed solenoid in \( \mathcal{H}(\Sigma) \).

Then
\[
I((\lambda_m)^m, \hat{c}) = I((\Psi_m)^m, \hat{c}) = \int E M_{\Psi_m}^m \upsilon_m | \hat{\xi} \cap \hat{c}|,
\]
where \( M_{\Psi_m}^m \upsilon_m \) is the lift of the vector \( M_{\Psi_m}^m \upsilon_m \) to \( \hat{E} \). Now since \( \hat{c} \) is the lift of a simple closed curve \( c \) in \( Z \), we have that
\[
I(\hat{\xi}_m, \hat{c}) = \frac{1}{\text{deg}(Z)} I(\hat{\upsilon}_m^\upsilon, c),
\]
where \( \hat{\upsilon}_m^\upsilon \) is the measured lamination in \( Z \) corresponding to the weight \( \upsilon_m^\upsilon \). However an examination of the matrices \( M_{\Phi} \) and \( M_{\Psi_m}^m \) yields
\[
(\lambda_m)^m \upsilon_m^\upsilon = M_{\Phi} \upsilon_m^\upsilon + \epsilon_m
\]
where
\[
\epsilon_m = ((rN)^2 (a_m + a_m + 1) - (rN)^2 b_m + (rN)^2 a_m + (rN)^2 a_m + (rN)^2 b_m)^T.
\]
By Claim \( \mathbb{1} \) it follows that \( \epsilon_m \to (0,0)^T \) or \( \upsilon_m^\upsilon \) converges to an eigenvector of \( M_{\Phi} \) of eigenvalue \( \lambda_\infty \). But since \( \lambda_\infty > 1 \) and the second eigenvalue of \( M_{\Phi} \) is strictly less than 1, we must have that \( \upsilon_m^\upsilon \to \upsilon \) and \( \lambda_\infty = \lambda_1^{1/n} \). In particular,
\[
\lim_{m \to \infty} |\upsilon_m^\upsilon, c| = I(\hat{\xi}, \hat{c}).
\]
This takes care of the unstable part of the theorem; the stable part is handled by repeating the above argument for \( \Phi^{-1} \) and \( \Psi_m^{-1} \).

**Theorem 6.** Every pseudo Anosov \( \Phi \) has an \( L^1 \) nth root for all \( n \).

**Proof.** Since \( \mathcal{P} \) is directionally dense, there exists a sequence \( \{ \Phi^{(g)} \} \subset \mathcal{P} \), where \( \Phi^{(g)} \) is the lift of a pseudo Anosov \( \Phi^{(g)} : X_g \to X_g \) in which \( X_g \) is a surface of genus \( g \to \infty \), and the axes \( A_g \to A = \text{axis of } \Phi \). For each \( g \), let \( \{ \Psi_m^{(g)} \} \) be the sequence of pseudo Anosovs constructed above. Then by Lemma \( \mathbb{4} \) we may obtain an \( L^1 \) nth root \( \{ \Psi_m \} \) of \( \Phi \) by extracting a suitable diagonal subsequence of \( \{ \Psi_m^{(g)} \} \). Indeed, a suitable diagonal subsequence \( \{ \Psi_m \} \) yields a sequence of pseudo Anosov homeomorphisms whose stable and unstable laminations intersection converge to those of \( \Phi \). By Theorems \( \mathbb{3} \) and \( \mathbb{4} \) this gives rise to a sequence of quadratic differentials \( \hat{q}_i \) along the associated axes \( A_i \) which \( L^1 \)-converge to the quadratic differential \( \hat{q} \) determined by \( \hat{\xi}^\mu \) and \( \hat{\upsilon} \). \( \square \)

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**Instituto de Matemáticas – Unidad Cuernavaca, Universidad Nacional Autónoma de México, Av. Universidad S/N, C.P. 62210 Cuernavaca, Morelos, México**

E-mail address: tim@matcuer.unam.mx