ON HAWKING’S LOCAL RIGIDITY THEOREMS FOR CHARGED BLACK HOLES

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Abstract. We show the existence of a Hawking vector field in a full neighborhood of a local, regular, bifurcate, non-expanding horizon embedded in a smooth Einstein-Maxwell space-time without assuming the underlying space-time is analytic. It extends one result of Friedrich, Rácz and Wald, see [3], which was limited to the interior of the black hole region. Moreover, we also show, in the presence of an additional Killing vector field $T$ which tangent to the horizon and not vanishing on the bifurcate sphere, then space-time must be locally axially symmetric without the analyticity assumption. This axial symmetry plays a fundamental role in the classification theory of stationary black holes.

1. Introduction

Let $(M, g, F)$ be a smooth and time oriented Einstein-Maxwell space-time of dimension $3 + 1$ with electromagnetic field $F$. Let $S$ be an smoothly embedded space-like 2-sphere in $M$ and $N^+, N^-$ be the corresponding null boundaries of the causal future and the causal past of $S$. We also assume that both $N^+$ and $N^-$ are regular, achronal, null hypersurfaces in a neighborhood $O$ of $S$. The triplet $(S, N^+, N^-)$ is called a local, regular bifurcate horizon in $O$. The main result of the paper asserts if $(S, N^+, N^-)$ is non-expanding (see Definition 2.1), then it must be a Killing bifurcate horizon. More precisely, we have the following theorem:

Theorem 1.1. Given a local, regular, bifurcate, non-expanding horizon $(S, N^+, N^-)$ in a smooth and time oriented Einstein-Maxwell space-time $(O, g, F)$, there exists an neighborhood $O' \subset O$ of $S$ and a non-trivial Killing vector field $K$ in $O'$, which is tangent to the null generators of $N^+$ and $N^-$. Moreover, the Lie derivative $\mathcal{L}_KF = 0$.

The vector field $K$ is called the Hawking vector field in the literature. Its existence is already known under the assumption that the space-time is real analytic. In the work of [3], the authors showed, by solving wave equations, the existence of Hawking vector field $K$ without the analyticity assumption, but $K$ could only be constructed inside the domain of dependence of $N^+ \cup N^-$ due to the fact that the corresponding wave equations are ill-posed outside this region. So the new ingredient of our theorem is to extend the Hawking vector field $K$ to a full neighborhood of the bifurcate sphere $S$, without making any additional regularity assumptions on the underlying space-time $(M, g)$. We use the idea of S. Alexakis, A. Ionescu and S. Klainerman, who proved a similar theorem for Einstein vacuum space-time, see [2] for details.
We also prove the following theorem:

**Theorem 1.2.** Given a local, regular, bifurcate horizon \((S, N^+, N^-)\) in a smooth and time oriented Einstein-Maxwell space-time \((\mathcal{O}, g, F)\). If there is a Killing vector field \(T\) tangent to \(N^+ \cup N^-\) and non-vanishing on \(S\). Then there is a neighborhood \(\mathcal{O}' \subset \mathcal{O}\) of \(S\), such that we can find a rotational Killing vector \(Z\) in \(\mathcal{O}'\), i.e. \(Z\) has closed orbits. Moreover, \([Z, T] = 0\). If in addition \(L_T F = 0\), then \(L_Z F = 0\).

Although, we don’t make the non-expanding assumption on the horizon, it’s a well known fact that the non-expansion is a consequence of the fact that the Killing vector field \(T\) is tangent to \(N^+ \cup N^-\). So the first theorem will produce a Hawking vector field \(K\) in a full neighborhood of \(S\). The rotational vector field can be written as a linear combination of \(T\) and \(K\), i.e. we show that the existence a constant \(\lambda\) such that

\[
Z = T + \lambda K
\]

is a rotation with period \(t_0\). So the part \(L_Z F = 0\) in the theorem follows immediately.

In the proof, we will focus on other parts of the theorem. The period \(t_0\) is determined on the bifurcate sphere \(S\), while to determine \(\lambda\), we need the information on \(S\) and the information of one particular null geodesic on \(N^+ \cup N^-\), see the proof for more details.

Once more, under the restrictive additional assumption of analyticity of the space-time \((M, g)\), this second theorem is also known for Einstein vacuum space-times. It’s usually called Hawking’s rigidity theorem, see [4], which asserts that under some global causality, asymptotic flatness and connectivity assumptions, a stationary, non-degenerate analytic space-time must be axially symmetric. In the smooth category, one can find a proof in [2] based on the idea that, under a suitable conformal rescaling of null generators on the bifurcate sphere, the level sets of the affine parameters of the null generators on the horizon should represent the integrable surface ruled out by the closed rotational orbits. We will give a more geometric construction.

These two theorems play an important role in the classification theory of stationary black holes, since they reduce the classifications to the cases which are covered by the well-known uniqueness theorems for electrovac black holes in general relativity, see [7], [4].

We now describe the main ideas of the proofs. The first step is to construct the Hawking vector field \(K\). Since \(K\) is a Killing vector field, it must satisfy the following covariant linear wave equations:

\[
\square_g K_\alpha = -R_\alpha \beta K_\beta
\]  

(1.1)

where \(R_\alpha \beta\) is the Ricci curvature tensor for the Lorentzian metric \(g\). We hope to reconstruct \(K\) by solving this wave equation. This is precisely the strategy used in [3]. The equation can be solved in the domain of dependence if initial data is prescribed on the characteristic hypersurfaces, see [3] for a proof. The choice of initial data can be rediscovered by the following heuristic argument: because \(K\) is Killing, its restriction on a geodesic should be a Jacobi field, so it’s reasonable to guess the initial data on \(N^+\) should be the non-trivial parallel Jacobi field \(u L\) where \(L\) is one null geodesic generator on \(N^+\).
and $u$ is the corresponding affine parameter, i.e. $L(u) = 1$; another way to guess the initial data is to check the explicit formula for the exact Kerr-Newman solutions. While the Cauchy problem for (1.1) is ill-posed on complement of the domain of dependence, solving (1.1) can not construct the Hawking vector field in bad region. We have to rely on the new techniques used in [2]. A careful calculation shows $K$ also solves an ordinary differential equation which is well-posed in the ill-posed region for (1.1). So one can extend $K$ into the bad region by solving this ordinary equation. That’s how we construct $K$ in a full neighborhood of $S$. Notice that although $K$ is constructed, it’s not automatically a Killing vector field. One turns to prove the one parameter group $\phi_t$ generated by $K$ acts isometrically. We need to show that, for each small $t$, the pull-back metric $\phi^*_t g$ must coincide with $g$, in view of the fact that they are both solutions of Einstein-Maxwell equations and coincide on $N^+ \cup N^-$. Now the uniqueness for metric type problems come into play. The results of Ionescu-Klainerman [5], [6], Alexakis [1] and Alexakis-Ionescu-Klainerman [2] provide hints to the answer.

The paper is organized as follows. In section 2, we construct a canonical null frame associated to the bifurcate horizon $(S, N^+, N^-)$ and derive a set of partial differential equations for various geometric quantities, as consequences of non-expansion condition and the Einstein-Maxwell equations; in section 3, we give a self-contained proof of Theorem 1.1 in the domain of dependence of $N^+ \cup N^-$, which is the Proposition B.1 in [3]; in section 4, based on the Carleman estimates proved in [5] and [6], we extend the Hawking vector field to a full neighborhood of $S$ which completes the proof of Theorem 1.1; the last section is devoted to a geometric proof of Theorem 1.2.

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2. Preliminaries

In this paper, the indices $\alpha, \beta, \gamma, \delta, \rho$ are from 1 to 4, $a, b, c$ are from 1 to 2; the curvature convention is $R_{\alpha\beta\gamma\delta} = g(D_\alpha D_\beta e_\gamma - D_\beta D_\alpha e_\gamma, e_\delta)$, where $D_\alpha D_\beta X = D_\beta (D_\alpha X) - D_{[\alpha\beta]}X$; repeat indices are always understood as Einstein summation convention; since during the proof of our main theorems, we will keep shrinking the open neighborhood $\mathcal{O}$ of $S$ mentioned in the introduction, we keep denoting such neighborhoods by $\mathcal{O}$ for simplicity.

One can choose a smooth future-directed null pair $(L, L)$ along $S$ with normalization

$$g(L, L) = g(L, L) = 0, \quad g(L, L) = -1$$

such that $L$ is tangent to $N^+$ and $L$ is tangent to $N^-$. In a small neighborhood of $S$, we extend $L$ along the null geodesic generators of $N^+$ via parallel transport; we also extend $L$ along the null geodesic generators of $N^-$ via parallel transport. So $D_L L = 0$ and $D_L L = 0$. We now define two optical functions $u$ and $u$ near $S$. The function $u$ (resp. $u$) is defined along $N^+$ (resp. $N^-$) by setting initial value $u = 0$ (resp. $u = 0$) on $S$ and solving $L(u) = 1$ (resp. $L(u) = 1$). Let $S_u$ (resp. $S_u$) be the level surfaces of $u$(resp.
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We define \( L \) (resp. \( L \)) on each point of the hypersurface \( N^+ \) (resp. \( N^- \)) to be unique, future directed null vector orthogonal to the surface \( S_u \) (resp. \( S_u \)) passing though that point and such that \( g(L, L) = -1 \). The null hypersurface \( N^- \) (resp. \( N^+_u \)) is defined to be the congruence of null geodesics initiating on \( S_u \subset N^+ \) (resp. \( S_u \subset N^- \)) in the direction of \( L \) (resp. \( L \)). We require the null hypersurfaces \( N^- \) (resp. \( N^+_u \)) are the level sets of the function \( u \) (resp. \( u \)), by this condition, \( u \) and \( u \) are extended into a neighborhood of \( S \) from the null hypersurface \( N^+ \cup N^- \). The we can extend both \( L \) and \( L \) into a neighborhood of \( S \) as gradients of the optical functions

\[
L = -g^{\mu\nu} \partial_\mu u \partial_\nu, \quad L = -g^{\mu\nu} \partial_\mu u \partial_\nu.
\]

Since \( u \) and \( u \) are null optical functions, we know \( g(L, L) = g(L, L) = 0 \) while \( g(L, L) = -1 \) only holds on the null surface \( N^+ \cup N^- \). Moreover, we have

\[
L(u) = 1 \quad \text{on} \quad N^+, \quad L(u) = 1 \quad \text{on} \quad N^-.
\]

We define \( S_{uu} = N^+_u \cap N^-_u \). Using the null pair \( (L, L) \) one can choose a null frame \( \{e_1, e_2, e_3 = L, e_4 = L\} \) such that

\[
g(e_a, e_b) = \delta_{ab}, \quad g(e_a, e_3) = g(e_a, e_4) = 0, \quad a, b = 1, 2.
\]

At each point \( p \in S_{uu} \subset \emptyset \), \( e_1, e_2 \) form an orthonormal frame along the 2-surface \( S_{uu} \).

We will modify the frame by Fermi transport later. Recall the null second fundamental forms \( \chi \), \( \chi \) and torsion \( \zeta \) are defined on \( N^+ \cup N^- \) via the given null pair \( (L, L) \):

\[
\chi_{ab} = g(D_{e_a} L, e_b), \quad \chi_{ab} = g(D_{e_a} L, e_b), \quad \zeta_a = g(D_{e_a} L, L).
\]

The traces of \( \chi \) is defined by \( tr\chi = \chi^a_a \), similarly for \( tr\chi \).

**Definition 2.1.** We say that \( N^+ \) is non-expanding if \( tr\chi = 0 \) on \( N^+ \); similarly \( N^- \) is non-expanding if \( tr\chi = 0 \) on \( N^- \). The bifurcate horizon \( (S, N^+, N^-) \) is called non-expanding if both \( N^+, N^- \) are non-expanding.

The non-expansion condition has a very strong restriction on the geometry of the Einstein-Maxwell space-time. We recall the Einstein-Maxwell equations:

\[
\begin{cases}
R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = T_{\alpha\beta} \\
D_{[\alpha} F_{\beta\gamma]} = 0 \\
D^\alpha F_{\alpha\beta} = 0
\end{cases}
\]

where \( T_{\alpha\beta} = F_{\alpha\mu} F_{\beta\mu} - \frac{1}{4} g_{\alpha\beta} F_{\mu\nu} F_{\mu\nu} \) is the energy-momentum tensor for the corresponding electromagnetic field. Since the dimension of the underlying manifold is 4, the field theory is conformal, i.e. \( trT = 0 \). So by tracing the first equation in the system, we know the scalar curvature \( R = 0 \). We can rewrite the system as

\[
\begin{cases}
R_{\alpha\beta} = F_{\alpha\mu} F_{\beta\mu} - \frac{1}{4} g_{\alpha\beta} F_{\mu\nu} F_{\mu\nu} \\
D_{[\alpha} F_{\beta\gamma]} = 0 \\
D^\alpha F_{\alpha\beta} = 0
\end{cases}
\]  
(2.1)
We recall that the positive energy condition is valid for Einstein-Maxwell energy-momentum tensor, i.e.

\[ T(X, Y) \geq 0 \]

where \((X, Y)\) are an arbitrary pair of future-directed causal vectors. Let \(\hat{\chi}\) be the traceless part of \(\chi\), so on \(\mathbb{N}^+\), according to Raychaudhuri equation:

\[ L(tr\chi) = -R_{LL} - |\hat{\chi}|^2 - \frac{1}{2}(tr\chi)^2 \]

So non-expansion condition on the black hole boundary implies

\[ R_{LL} + |\hat{\chi}|^2 = 0 \]

One can take advantage of the positive energy condition to conclude

\[ R_{LL} = 0, \quad \hat{\chi} = 0 \quad \text{on} \quad \mathbb{N}^+. \]

So \(\chi = 0\) on \(\mathbb{N}^+\). According to untraced formulation of Raychaudhuri equation:

\[ L(\chi) + \chi^2 + R(\cdot, L)L = 0 \]

we know for all \(X \in T\mathbb{N}^+\),

\[ R(X, L)L = 0 \]

In view of the first equation in (2.1), \(R_{LL} = 0\) implies \(F_{4a} = 0\), and this last vanishing quantities imply \(\hat{R}_{4a} = 0\), combined with \(R(X, L)L = 0\), we know \(R_{4aba} = 0\). To summarize, the non-expansion condition implies, on the null hypersurface \(\mathbb{N}^+\)

\[
\begin{cases}
\chi = 0 \\
R_{4a} = 0 \\
R_{4aba} = 0 \\
R_{344a} = 0 \\
F_{4a} = 0
\end{cases}
\] (2.2)

Similar identities hold on \(\mathbb{N}^-\) by replacing the index 4 by 3. It’s precisely this set of geometric information that we use in the proof of our main theorems. Recall also our choice of the frame \(e_1, e_2\) is arbitrary on \(\mathbb{N}^+\). Since we know \(\chi = 0\), we can make this choice more rigid by using Fermi transport along \(L\), i.e. we first pick up an local orthonormal basis on \(S\), the use the Lie transport relation \(\mathcal{L}_Le_a = 0\) to get a basis on \(S_u\) (which needs not to be orthonormal), the vanishing of \(\chi\) on \(\mathbb{N}^+\) guarantees \(\{e_1, e_2\}\) is still an orthonormal basis. We summarize the computation formulas in the null frame \(\{e_1, e_2, e_3 = L, e_4 = L\}\) on \(\mathbb{N}^+\):

\[
\begin{cases}
D_LL = 0, \\
D_LL = -\zeta_a e_a \\
D_L e_a = -\zeta_a L \\
D_{e_a} e_b = -\zeta_a e_b + \chi_{ab} e_b + \chi_{ab} L
\end{cases}
\] (2.3)

where \(\nabla_{e_a} e_b\) is the projection of \(D_{e_a} e_b\) onto the surface \(S_u\). A similar set of identities hold on \(\mathbb{N}^-\).
Lemma 2.2. On $\mathbf{N}^+$, we have
\[ R(-, L, L, -) = -D\zeta - \nabla_{LX} + \zeta \otimes \zeta \]
i.e. for all $X, Y \in TS_{\mathbf{N}}$,
\[ R(X, L, L, Y) = -(D\zeta)(X, Y) - (\nabla_{LX})(X, Y) + \zeta(X)\zeta(Y). \]
where $\nabla$ denotes the restriction of $D$ on $S_{\mathbf{N}}$; similar result holds on $\mathbf{N}^-$.

Proof. For $X, Y \in TS_{\mathbf{N}}$, we have
\[
R(X, L, L, Y) = g(D_X D_L L, Y) - g(D_L D_X L, Y) - g(D_{D_{L}X} L, Y) + g(D_{D_{L}X} L, Y) \\
= g(D_X \zeta^2, Y) - g(D_L \zeta(X) + \zeta(X) L, Y) \\
+ \zeta(X)g(D_L L, Y) + g(D_{\nabla_{LX}} \zeta(X) L, Y) \\
= -(D\zeta)(X, Y) - g(D_L \zeta(X)), Y) + g(D_{\nabla_{LX}} L, Y) - \zeta(X)g(D_L L, Y) \\
= -(D\zeta)(X, Y) - (\nabla_{LX})(X, Y) + \zeta(X)\zeta(Y). \\
\]

\[
\Box
\]

3. Hawking vector field inside black hole

We define the following four regions $\mathbf{I}^{++}, \mathbf{I}^{--}, \mathbf{I}^{-+}$ and $\mathbf{I}^{+-}$:
\[
\mathbf{I}^{++} = \{ p \in \mathcal{O} | u(p) \geq 0 \text{ and } \bar{u}(p) \geq 0 \}, \quad \mathbf{I}^{--} = \{ p \in \mathcal{O} | u(p) \leq 0 \text{ and } \bar{u}(p) \leq 0 \}, \\
\mathbf{I}^{-+} = \{ p \in \mathcal{O} | u(p) \geq 0 \text{ and } \bar{u}(p) \leq 0 \}, \quad \mathbf{I}^{+-} = \{ p \in \mathcal{O} | u(p) \leq 0 \text{ and } \bar{u}(p) \geq 0 \}. \quad (3.1)
\]
In this section, we will prove the following proposition

Proposition 3.1. Under the assumptions of Theorem [14], in a small neighborhood $\mathcal{O}$ of $S$, there exists a smooth Killing vector field $K$ in $\mathcal{O} \cap (\mathbf{I}^{++} \cup \mathbf{I}^{--})$ such that
\[ K = uL - u L \] on $(\mathbf{N}^+ \cup \mathbf{N}^-) \cap \mathcal{O}.$
Moreover, $L_K F = 0$ and $[L, K] = -L.$

The region $\mathcal{O} \cap (\mathbf{I}^{++} \cup \mathbf{I}^{--})$ is the domain of dependence of $\mathbf{N}^+ \cup \mathbf{N}^-$. As we mentioned in the introduction, by using the Newman-Penrose formalism, the first part of the proposition is shown by H. Friedrich, I. Rácz and R. Wald, see [3]. For the sake of completeness, we provide a direct proof without Newman-Penrose formalism. As mentioned in the introduction, we consider the following characteristic initial value problem
\[
\begin{align*}
\Box g K_{\alpha} &= -R_{\alpha \beta} K_{\beta} \\
K &= uL - u L \quad \text{on } (\mathbf{N}^+ \cup \mathbf{N}^-) \cap \mathcal{O}
\end{align*} \quad (3.2)
\]
According to [8], it’s well-posed in $\mathcal{O} \cap (\mathbf{I}^{++} \cup \mathbf{I}^{--})$. So a smooth vector field $K$ is now constructed in the domain of dependence of $\mathbf{N}^+ \cup \mathbf{N}^-$. To show $K$ is indeed a Killing vector field, one has to show the deformation tensor of $K$
\[ \pi_{\alpha \beta} = L_K g = D_{\alpha} K_{\beta} + D_{\beta} K_{\alpha} \]
is zero in $\emptyset \cap (I^+ \cup I^-)$.

Since $K$ solves (3.2), by commuting derivatives, we know the deformation tensor $\pi_{\alpha\beta}$ solves the following covariant wave equation:

$$\Box g \pi_{\alpha\beta} = -2R^\rho_{\alpha\beta} \pi_{\rho\delta} + R_{\alpha\rho} \pi^\rho_{\beta} + R_{\beta\rho} \pi^\rho_{\alpha} - 2\mathcal{L}_K R_{\alpha\beta}$$

The geometric part of Einstein-Maxwell equations (2.1) provides

$$\mathcal{L}_K R_{\alpha\beta} = \mathcal{L}_K T_{\alpha\beta}$$

$$= F^\rho_{\alpha} \mathcal{L}_K F_{\beta\rho} + F^\rho_{\beta} \mathcal{L}_K F_{\alpha\rho} - \pi_{\rho\delta} F^\rho_{\alpha} F^\delta_{\beta}$$

$$- \frac{1}{4} \pi_{\alpha\beta} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} g_{\alpha\beta} F_{\mu\nu} L_{\mathcal{K}F_{\mu\nu}} + \frac{1}{2} g_{\alpha\beta} \pi_{\rho\delta} F^\delta_{\gamma} F^\gamma_{\rho}$$

This formula requires one to consider the partial differential equations satisfied by $\mathcal{L}_K F_{\alpha\beta}$, which follows directly from the electromagnetic part of the Einstein-Maxwell equations (2.1):

$$\begin{cases}
D_{[\alpha} \mathcal{L}_K F_{\beta\gamma]} = 0 \\
D^\alpha \mathcal{L}_K F_{\alpha\beta} = \pi_{\alpha\gamma} D^\gamma F^\alpha_{\beta} + \frac{1}{2} (D_{\alpha} \pi_{\beta\gamma} + D_{\beta} \pi_{\alpha\gamma} - D_{\gamma} \pi_{\alpha\beta})
\end{cases}$$

Put all the equations together, we know $\pi_{\alpha\beta}$ and $\mathcal{L}_K F_{\alpha\beta}$ solve the characteristic initial value problem for the following closed symmetric hyperbolic system:

$$\begin{cases}
\Box g \pi_{\alpha\beta} = -2R^\rho_{\alpha\beta} \pi_{\rho\delta} + R_{\alpha\rho} \pi^\rho_{\beta} + R_{\beta\rho} \pi^\rho_{\alpha} - 2\mathcal{L}_K R_{\alpha\beta} \\
D_{[\alpha} \mathcal{L}_K F_{\beta\gamma]} = 0 \\
D^\alpha \mathcal{L}_K F_{\alpha\beta} = \pi_{\alpha\gamma} D^\gamma F^\alpha_{\beta} + \frac{1}{2} (D_{\alpha} \pi_{\beta\gamma} + D_{\beta} \pi_{\alpha\gamma} - D_{\gamma} \pi_{\alpha\beta})
\end{cases}$$

(3.3)

So to show $\pi_{\alpha\beta} = 0$ and $\mathcal{L}_K F = 0$ in $\emptyset$, it suffices to show

$$\pi_{\alpha\beta} = 0 \quad \mathcal{L}_K F = 0 \quad \text{on} \quad N^+ \cup N^-.$$  

(3.4)

We only check (3.4) on $N^+$; on $N^-$, the argument is exactly the same. In view of the expression of $K = uL$ on $N^+$ (since $u = 0$ on it) and (2.3), it’s easy to see

$$\begin{cases}
D_a K_b = D_a K_a = D_a K_4 = D_4 K_a = 0, \quad D_4 K_3 = -1 \\
D_c D_a K_b = D D_a K_b = D_b D_a K_a = D_4 D_4 K_a = D_a D_b K_4 = 0 \\
D_4 D_a K_4 = D_a D_4 K_4 = D_4 D_a K_4 = D_4 D_4 K_a = D_a D_4 K_3 = 0.
\end{cases}$$

(3.5)

So one knows each component of $\pi_{\alpha\beta}$, which does not have the bad direction $L$, is zero, i.e.

$$\pi_{ab} = \pi_{4a} = \pi_{44} = 0 \quad \text{on} \quad N^+.$$  

(3.6)

To prove the remaining components of $\pi$ vanish, we need to make a serious use of (3.2) to get derivatives in $L$ direction. The equation (3.2) gives

$$D_3 D_4 K_3 + D_4 D_3 K_3 = \sum_{a=1}^{2} D_a D_a K_3 + R_{\beta\rho} K_{\rho}$$
Combine this with curvature identity $D_3 D_4 K_\beta - D_4 D_3 K_\beta = -R_{34\beta\rho} K_\rho$, then we have

$$2D_4 D_3 K_\beta = \sum_{\alpha=1}^{2} D_\alpha D_\alpha K_\beta + R_{\beta\rho} K_\rho + R_{34\beta\rho} K_\rho$$  \hspace{1cm} (3.7)

**Claim 3.2.** We have $D_3 K_4 = 1$, $D_\alpha D_3 K_4 = D_4 D_3 K_4 = 0$.

**Proof.** We set $\beta = 4$ in (3.7), it’s easy to check the left hand side of (3.7) is

$$2D_4 D_3 K_4 = 2L(D_3 K_4)$$

while the right hand side is 0 by (2.2). So $L(D_3 K_4) = 0$ on $N^+$. It implies the value of $D_3 K_4$ on $N^+$ is determined by its value on $S$ which is 1. The other identities are also easy to check, this completes the proof of the claim. \hfill $\square$

Apparently, Claim 3.2 implies $\pi_{34} = 0$.

**Claim 3.3.** We have $D_\alpha K_3 = u\zeta_\alpha$, $D_3 K_\alpha = -u\zeta_\alpha$.

**Proof.** The first identity in the claim is easy to verify by direct computations; we now prove the second one. We first prove that

$$L(\zeta_\alpha) = 0.$$  \hspace{1cm} (3.8)

We use (2.2):

$$L(\zeta_\alpha) = L(g(D_\alpha L, L)) = g(D_\alpha L, D_L L) + g(D_L D_\alpha L, L)$$
$$= g(D_L D_\alpha L, L) = R_{L\alpha L} L = 0$$  \hspace{1cm} (3.9)

We now set $\beta = b$ in (3.7), it implies $D_4 D_3 K_b = 0$, then by using the fact that $D_3 K_4 = 1$, we can show

$$L(D_3 K_\alpha) = -\zeta_\alpha$$

Combined with (3.8), it shows $D_3 K_\alpha = -u\zeta_\alpha$. \hfill $\square$

Apparently, Claim 3.3 implies $\pi_{3\alpha} = 0$.

**Claim 3.4.** We have $\pi_{33} = 2D_3 K_3 = 0$.

**Proof.** Before proving the claim, one needs more support from the Einstein-Maxwell equations (2.1). Since $F_{4\alpha} = 0$, we have

$$L(R_{44}) = L(F_{4\alpha}^2) = 2F_{4\alpha} L(F_{4\alpha}) = 0$$

which implies

$$L(R_{4\alpha 4\alpha}) = 0$$  \hspace{1cm} (3.11)

Recall one of the second Bianchi identities:

$$D_4 R_{3\alpha 4\alpha} + D_3 R_{a 4\alpha} + D_a R_{43\alpha 4} = 0$$  \hspace{1cm} (3.12)

A simple computation with the help (2.2) and (3.11) shows the last two terms in (3.12) are zeroes. So we have

$$L(R_{3\alpha a}) = D_4 R_{3\alpha a} = 0.$$  \hspace{1cm} (3.13)
We compute $L(tr\chi)$ along $N^+$:
\[
L(tr\chi) = L(g(D_\alpha L_\alpha, e_\alpha)) = g(D_L D_\alpha L_\alpha, e_\alpha) + g(D_\alpha L_\alpha, D_L e_\alpha)
\]
\[
= R_{\alpha\beta} L_\alpha + g(D_\alpha D_L L_\alpha, e_\alpha) + |\zeta|^2
\]
\[
= R_{4a3a} + |\zeta|^2 - g(D_\alpha \zeta^a, e_\alpha)
\]

In view of (3.13) and (3.8), we have
\[
LL(tr\chi) = -L(g(D_\alpha \zeta^a, e_\alpha)) = -g(D_L D_\alpha \zeta^a, e_\alpha)
\]
\[
= -R_{4a3a} - g(D_\alpha D_L \zeta^a, e_\alpha)
\]
\[
= -g(D_\alpha D_L (\zeta^a e_\alpha), e_\alpha)
\]
\[
= -\zeta_a g(D_\alpha (\zeta^a e_\alpha), e_\alpha)
\]

This shows
\[
LL(tr\chi) = 0. \tag{3.14}
\]

Now we are ready to prove the claim. We set $\beta = 3$ in (3.7), so
\[
2D_4D_3K_3 = D_\alpha D_\alpha K_3 + R_{3\rho} K^\rho + R_{3434} K^4
\]
\[
= D_\alpha D_\alpha K_3 + u R_{34} + u R_{3434}
\]
\[
= D_\alpha D_\alpha K_3 + u R_{3a4a}
\]

By Lemma 2.2, we have
\[
R_{3a4a} = -(D_\alpha \zeta^a)(e_\alpha, e_\alpha) - (\nabla_L \chi)(e_\alpha, e_\alpha) + \zeta_a^2
\]
\[
= -(D_\alpha \zeta^a)(e_\alpha) - L(tr\chi) + |\zeta|^2
\]
\[
= -\text{div}\zeta + \zeta(\nabla_\alpha e_\alpha) - L(tr\chi) + |\zeta|^2
\]

We also can compute
\[
D_\alpha D_\alpha K_3 = u(\text{div}\zeta - \zeta(\nabla_\alpha e_\alpha) - |\zeta|^2) + tr\chi
\]

The previous computations showed
\[
2D_4D_3K_3 = tr\chi - uL(tr\chi)
\]

So in view of (3.14)
\[
L(D_4D_3K_3) = -uLL(tr\chi) = 0 \tag{3.15}
\]

Since on $S$, on check easily that $D_4D_3K_3 = 0$, so $D_4D_3K_3 = 0$ on $N^+$, which once again implies $D_3K_3 = 0$ by solving transport equations along $L$. \qed

So we proved $\pi_{\alpha\beta} = 0$ on $N^+$. One still needs to show $L_K F_{\alpha\beta} = 0$.

Claim 3.5. We have the following identities:
\[
D_\alpha F_{Lb} = D_L F_{Lb} = D_L F_{ab} = D_L F_{L\alpha L} = 0 \tag{3.16}
\]
Proof. We will use (2.2) repeatedly:

\[ D_a F_{Lb} = (D_a F)(L \otimes e_b) \]
\[ = e_a (F_{Lb}) - F(D_a L \otimes e_b) - F(L \otimes D_a e_b) \]
\[ = 0. \]

Same argument shows \( D_L F_{Lb} = 0. \) We use Bianchi identity:

\[ D_L F_{ab} = -D_a F_{bL} - D_b F_{La} = 0. \]

We now use the last equality in Einstein-Maxwell equations (2.1):

\[ D^\alpha F_{\alpha L} = 0 \rightarrow D_a F_{aL} - D_L F_{LL} = 0 \]

so \( D_L F_{LL} = D_a F_{aL} = 0. \)

\[ \square \]

Claim 3.6. On \( N^+ \), we have

\[ \mathcal{L}_K F = 0 \tag{3.17} \]

Proof. Recall that

\[ \mathcal{L}_K F_{\alpha \beta} = D_K F_{\alpha \beta} + g^{\rho \delta} D_\alpha K_\delta F_{\rho \beta} + g^{\rho \delta} D_\beta K_\delta F_{\alpha \rho}. \]

We show each component of \( \mathcal{L}_K F \) vanishes on \( N^+ \):

\[ \mathcal{L}_K F_{ab} = D_K F_{ab} + g^{\rho \delta} D_a K_\delta F_{\rho b} + g^{\rho \delta} D_b K_\delta F_{a \rho} \]
\[ = u D_L F_{ab} = 0. \]

\[ \mathcal{L}_K F_{aL} = D_K F_{aL} + g^{\rho \delta} D_a K_\delta F_{\rho L} + g^{\rho \delta} D_L K_\delta F_{a \rho} \]
\[ = u D_L F_{aL} = 0. \]

\[ \mathcal{L}_K F_{LL} = D_K F_{LL} + g^{\rho \delta} D_L K_\delta F_{\rho L} + g^{\rho \delta} D_L K_\delta F_{L \rho} \]
\[ = D_K F_{LL} - D_L K_L F_{LL} - D_L K_L F_{LL} \]
\[ = u D_L F_{LL} - \pi L F_{LL} = 0. \]

We need some preparations to show the most difficult term \( \mathcal{L}_K F_{Lb} \) vanishes. From the electromagnetic part of the Einstein-Maxwell equations (2.1), we have

\[ D_L F_{Lb} - D_L F_{Lb} + D_b F_{LL} = 0 \]
\[ - (D_L F_{Lb} + D_L F_{Lb}) + D_a F_{ab} = 0 \]

So one derives

\[ 2D_L F_{Lb} = D_a F_{ab} - D_b F_{LL} \tag{3.18} \]
Apply \( L \) on (3.18), we have

\[
2L(D_L F_{Lb}) = L(D_a F_{ab}) - L(D_b F_{LL})
\]

\[
= (D_L D_a F_{ab} + D_a F_{D(a)b} + D_a F_{(D_l)b}) - (D_L D_b F_{LL} + D_b F_{(D_L)L} + D_b F_{(D_L)b})
\]

\[
= D_L D_a F_{ab} - D_1 D_b F_{LL}
\]

\[
= (D_a D_L F_{ab} - R_{Lab} \rho F_{\rho b} - R_{Lab} \rho F_{\rho a}) - (D_b D_L F_{LL} - R_{Lab} \rho F_{\rho b} - R_{Lab} \rho F_{\rho a})
\]

\[
= - [c_a(D_L F_{ab} - D_L F_{(D_a)b}) - D_L F_{(D_b)L} - D_L F_{(D_L)b}] = 0.
\]

So we have

\[
L(D_L F_{Lb}) = 0 \tag{3.19}
\]

Now we are ready to show \( \mathcal{L}_K F_{Lb} = 0 \).

\[
\mathcal{L}_K F_{Lb} = D_K F_{Lb} + g^{\rho \delta} D_L K_{\rho} F_{\delta b} + g^{\rho \delta} D_b K_{\rho} F_{L L}
\]

\[
= u D_L F_{Lb} - D_L K_{a} F_{ab} - D_L K_{L} F_{Lb} - D_b K_{L} F_{L L}
\]

In particular, this shows \( \mathcal{L}_K F_{Lb} = 0 \) on \( S \). Notice that \( L(F_{Lb}) = L(F_{ab}) = L(F_{LL}) = 0 \), now apply \( L \) on \( \mathcal{L}_K F_{Lb} \), so we have

\[
L(\mathcal{L}_K F_{Lb}) = L(uD_L F_{Lb}) - L(u\zeta_a F_{ab}) - L(F_{Lb}) - L(u\zeta_b F_{L L})
\]

\[
= D_L F_{Lb} - \zeta_a F_{ab} - [D_L F_{Lb} + F_{(D_L)Lb} + F_{L(D_L)b}] - \zeta_b F_{L L}
\]

\[
= 0 \tag{3.19}
\]

Now solving this ordinary differential equation on \( N^+ \) completes the proof. \( \square \)

**Remark 3.7.** It follows from the previous computation that \( D_3 \pi_{\alpha \beta} = 0 \) on \( N^+ \). That’s the nature of hyperbolic equations with initial on a characteristic surface. In fact, \( D_3 \pi_{\alpha \beta} = 0 \) and \( D_L \pi_{\alpha \beta} = 0 \) trivially comes from the fact that \( \pi_{\alpha \beta} = 0 \) on \( N^+ \); to see \( D_3 \pi_{\alpha \beta} = 0 \), we need to investigate the first equation in (3.3), utilizing \( \pi_{\alpha \beta} = 0 \) and \( \mathcal{L}_K F = 0 \), it gives

\[
D_4 D_3 \pi_{\alpha \beta} + D_3 D_4 \pi_{\alpha \beta} = 0.
\]

Combined the curvature identity

\[
D_4 D_3 \pi_{\alpha \beta} - D_3 D_4 \pi_{\alpha \beta} = -R_{34a}^{\rho} \pi_{\rho \beta} - -R_{34b}^{\rho} \pi_{\alpha \rho} = 0,
\]

it gives \( L(D_3 \pi_{\alpha \beta}) = 0 \). So \( D_3 \pi_{\alpha \beta} = 0 \) follows from the fact that it vanishes on \( S \).
The last statement of Proposition 3.1, $[L, K] = -L$ in the domain of dependence, follows from the fact that

$$
\begin{align*}
\left\{ \begin{array}{l}
D_L W &= -D_W L \\
W &= 0
\end{array} \right. \\
\text{where} & \\
W &= [L, K] + L.
\end{align*}
$$

(3.20)

We first prove this ordinary differential equation holds. Since $K$ is Killing vector field, we know that for arbitrary vector fields $X$ and $Y$, we have

$$
\mathcal{L}_K(D_X Y) = D_X(\mathcal{L}_K Y) + D_X(\mathcal{L}_K Y).
$$

Therefore,

$$
D_L W = D_L(-\mathcal{L}_K L + L) = -D_L(\mathcal{L}_K L) = -(\mathcal{L}_K(D_L L) - D_L(\mathcal{L}_K L))
$$

$$
= D_{\mathcal{L}_K L} L = -D_{[L, K] + L} L = -D_W L.
$$

It remains to show $W = 0$ on $N^+$.

$$
W = D_L K - D_K L + L
$$

$$
= D_L K - uD_L L + L
$$

Since we have already computed the components $D_3 K_\alpha$, it’s almost trivial to check $W = 0$ on $N^+$. This completes the proof of Proposition 3.1.

4. HAWKING VECTOR FIELD OUTSIDE THE BLACK HOLE

In the previous section we have constructed the Hawking vector field $K$ inside the black hole region. To be able to extend it outside the black hole, because the characteristic initial value problem is ill-posed in this region, as we explained in the introduction, we need to rely on a completely different strategy. The idea is, instead of solving a hyperbolic system, we now can solve $[L, K] = -L$ for $K$. This ordinary differential equation is well-posed in the complement of the domain of dependence. That’s how $K$ is constructed. Let $\phi_t$ be the one parameter diffeomorphisms generated by $K$. When $t$ is small, we show that $(g, F)$ and $(\phi_t^* g, \phi_t^* F)$ both verify Einstein-Maxwell equations and they coincide on $N^+ \cup N^-$. We show that the must be coincide in a full neighborhood of $S$. In particular, it shows $K$ is Killing. So it’s the Hawking vector field. In the vacuum case, this is due to Alexakis, Ionescu and Klainerman, see [2].

To realize this strategy, we first define a vector field $K'$ by setting $K' = uL$ on $N^+ \cap \emptyset$ and solving the ordinary differential equation $[L, K'] = -L$. The vector field $K'$ is well-defined and smooth in a small neighborhood of $S$ (since $L \neq 0$ on $S$) and coincides with $K$ in $I^{++} \cup I^{--}$ in $\emptyset$. Thus $K := K'$ defines the desired extension. This proves the following:

**Lemma 4.1.** There exists a smooth extension of the vector field $K$ to a full neighborhood $\emptyset$ of $S$ such that

$$
[L, K] = -L \quad \text{in } \emptyset.
$$

(4.1)
Let \( g_t = \phi^*_t g \) and \( L = (\phi_{-t})_* L \). In view of the definition (4.1) of \( K \), we know
\[
\frac{d}{dt} L = - L.
\]
It implies that
\[
L = e^{-t} L. \tag{4.2}
\]
Let \( D^t \) be the Levi-Civita connection of \( g_t \), by the tensorial nature, we know that
\[
D^t L, L = D L, L = 0, \tag{4.2}
\]
infers that \( 0 = D^t L, L = e^{-2t} D^t L. \) This proves the following

**Lemma 4.2.** Assume \( K \) is a smooth vector field constructed in (4.1) and \( D^t \) the covariant derivative induced by the metric \( \phi^*_t g \). Then,
\[
D^t L, L = 0 \quad \text{in a full neighborhood of } S. \tag{4.3}
\]

To summarize, let \( F_t = \phi^*_t F \), we have a family of metrics and 2 forms \((g_t, F_t)\) which verify the Einstein-Maxwell equations (2.1) in the domain of dependence of \( N^+ \cup N^- \) and such that \( D^t L, L = 0 \). So the Theorem 1.1 is an immediate consequence of the following uniqueness statement:

**Proposition 4.3.** Assume in a full neighborhood \( \mathcal{O} \) of \( S \), \( g' \) is a smooth Lorentzian metric and \( F' \) is a smooth 2 form, such that \((g', F')\) solves Einstein-Maxwell equations (2.1). If
\[
g' = g \quad \text{in } (I^{++} \cup I^{--}) \cap \mathcal{O} \quad \text{and} \quad D' L, L = 0 \quad \text{in } \mathcal{O},
\]
where \( D' \) denotes the Levi-Civita connection of the metric \( g' \). Then \( g' = g \) and \( F' = F \) in a full neighborhood \( \mathcal{O}' \subset \mathcal{O} \) of \( S \).

The similar proposition for Einstein vacuum space-times was first proved in [1]. A simplified version can be found in [2]. In [5], the authors proved uniqueness results for covariant semi-linear wave equations of a fixed metric. But for the uniqueness at the level of metrics, since the corresponding partial differential equations are quasi-linear, one has to couple the system with a system of ordinary differential equations to recover the semi-linearity. In this section, we use this idea to prove uniqueness for the full curvature tensor and the electromagnetic field. Since the metric is uniquely determined by the curvature, that will prove Proposition 4.3.

**Proof.** We first derive a system of covariant wave equations for the full curvature tensor \( R_{\alpha\beta\gamma\delta} \) of the metric \( g \) and \( F_{\alpha\beta} \). Recall the second Bianchi identities and once contracted Bianchi identities:
\[
D_\alpha R_{\beta\gamma\delta} + D_\beta R_{\gamma\alpha\delta} + D_\gamma R_{\alpha\beta\delta} = 0 \tag{4.4}
\]
\[
D^\alpha R_{\alpha\beta\gamma\delta} = D_\gamma R_{\beta\delta} - D_\beta R_{\gamma\delta} \tag{4.5}
\]
We apply $D^\alpha$ on (4.4) and commute derivatives, we have

$$D^\alpha D_\alpha R_{\beta\gamma\delta} = -[D^\alpha, D_\beta]R_{\gamma\alpha\delta} - [D^\alpha, D_\gamma]R_{\alpha\beta\delta} - D_\beta D^\alpha R_{\gamma\alpha\delta} - D_\gamma D^\alpha R_{\alpha\beta\delta}$$

$$= R^{\alpha}_{\beta\gamma\delta} R^\mu_{\alpha\delta} + R^\alpha_{\beta\mu} R^\mu_{\gamma\delta} + R^\alpha_{\beta\mu} R^\mu_{\gamma\alpha\delta} + R^\alpha_{\beta\mu} R^\mu_{\gamma\alpha\delta}$$

$$+ R^\alpha_{\gamma\mu} R^\mu_{\beta\delta} + R^\alpha_{\gamma\mu} R^\mu_{\beta\gamma\delta} + R^\alpha_{\gamma\mu} R^\mu_{\beta\gamma\delta} + R^\alpha_{\gamma\mu} R^\mu_{\beta\gamma\delta} + D_\beta D_\gamma R_{\gamma\delta} - D_\delta D_\gamma R_{\gamma\delta}$$

To simplify the formulae, without losing information, we will use notation. The expression $A * B$ is a linear combination of tensors, each formed by starting with $A \otimes B$, using the metric to take any number of contractions. So the algorithm to get $A * B$ is independent of the choices of tensors $A$ and $B$ of respective types.

Schematically, we write it as

$$\Box_g R_{\alpha\beta\gamma\delta} = (R * R)_{\alpha\beta\gamma\delta} + D_\gamma D_\delta R_{\alpha\beta\gamma\delta}$$

We need to compute the Hessian of Ricci tensor. By the gravitational part of (2.1), we have the following schematically expression:

$$D_\gamma D_\delta R_{\alpha\beta} = F^\mu_{\beta\mu} D_\gamma D_\delta F^\mu_{\alpha\mu} + F^\mu_{\alpha\mu} D_\gamma D_\delta F^\mu_{\beta\mu} + D_\delta F^\mu_{\alpha\mu} D_\gamma F^\mu_{\beta\mu} + D_\gamma F^\mu_{\alpha\mu} D_\delta F^\mu_{\beta\mu}$$

$$- \frac{1}{2} g_{\alpha\beta}(D^\mu F^\nu_{\mu\delta} D_\delta D_\mu F^\mu_{\nu\delta})$$

Plug this in (4.6), we have

$$\Box_g R_{\alpha\beta\gamma\delta} = (R * R)_{\alpha\beta\gamma\delta} + (F * D^2 F)_{\alpha\beta\gamma\delta} + (D F * DF)_{\alpha\beta\gamma\delta}$$

(Apparently, this equation involves two derivatives of $F$. In principle, the electromagnetic part of the Einstein-Maxwell equations (2.1) controls only one derivative of $F$ through the second order system:

$$D^\alpha D_\alpha F_{\beta\gamma} = -D^\alpha D_\beta F_{\gamma\alpha} - D^\alpha D_\alpha F_{\alpha\beta}$$

$$= -[D^\alpha, D_\beta]F_{\gamma\alpha} - [D^\alpha, D_\gamma]F_{\alpha\beta} - D_\beta D^\alpha F_{\gamma\alpha} - D_\gamma D^\alpha F_{\alpha\beta}$$

$$= R^\alpha_{\beta\gamma\mu} F^\mu_{\alpha\beta} + R^\alpha_{\beta\mu\alpha} F^\mu_{\gamma\beta}$$

Schematically, it’s expressed as

$$\Box_g F_{\alpha\beta} = (R * F)_{\alpha\beta}$$

Since for the Einstein-Maxwell equations, the electromagnetic part of is almost decoupled from the gravitational part, we can actually control second derivative of $F$ by a cost of one derivative on the curvature tensor $R_{\alpha\beta\gamma\delta}$. Let’s apply $D_\rho$ on the second equation of (2.1) and commute derivatives:

$$d(D_\delta F)_{\alpha\beta} = D_\alpha D_\delta F_{\beta\gamma} + D_\beta D_\delta F_{\gamma\alpha} + D_\gamma D_\delta F_{\alpha\beta}$$

$$= [D_\alpha, D_\delta]F_{\beta\gamma} + [D_\beta, D_\delta]F_{\gamma\alpha} + [D_\gamma, D_\delta]F_{\alpha\beta} + D_\rho (D_{[\alpha} F_{\beta\gamma])$$

$$= -R_{\alpha\beta\gamma\mu} F^\mu_{\gamma\beta} - R_{\alpha\gamma\delta\mu} F^\mu_{\beta\gamma} - R_{\beta\gamma\delta\mu} F^\mu_{\alpha\gamma} - R_{\gamma\delta\alpha\mu} F^\mu_{\beta\gamma} - R_{\gamma\delta\beta\mu} F^\mu_{\alpha\gamma}$$

$$- R_{\delta\beta\gamma\mu} F^\mu_{\alpha\gamma} - R_{\delta\gamma\alpha\mu} F^\mu_{\beta\gamma} - R_{\gamma\delta\beta\mu} F^\mu_{\alpha\gamma} - R_{\gamma\delta\alpha\mu} F^\mu_{\beta\gamma}$$
where $d$ stands for the exterior derivative on 2 forms. Schematically, it gives

$$D_{[\alpha}(DF)_{\beta\gamma]} = (R * F)_{\alpha\beta\gamma}$$

Similarly, we have

$$D^\alpha(DF)_{\alpha\beta} = (R * F)^\beta$$

Apply covariant derivative on these last two equations, it implies

$$\Box_g(DF)_{\alpha\beta} = (R * DF)_{\alpha\beta} + (DR * F)_{\alpha\beta}$$

(4.9)

We summarize (4.7), (4.8) and (4.9) in following system of equations

\begin{align*}
\Box_g R_{\alpha\beta\gamma\delta} &= (R * R)_{\alpha\beta\gamma\delta} + (F * D^2F)_{\alpha\beta\gamma\delta} + (DF * DF)_{\alpha\beta\gamma\delta} \\
\Box_g F_{\alpha\beta} &= (R * F)_{\alpha\beta} \\
\Box_g (DF)_{\alpha\beta} &= (R * DF)_{\alpha\beta} + (DR * F)_{\alpha\beta}
\end{align*}

(4.10)

We have a similar covariant system of equations for $R'_{\alpha\beta\gamma\delta}$ and $F'_{\alpha\beta}$.

We'll prove Proposition 4.3 in a neighborhood $O(p)$ of a point $p \in S$ where introduce a fixed coordinate system $x_k$ for $k = 1, 2, 3, 4$ such that it is fixed for both metrics $g$ and $g'$. In the proof we shall keep shrinking the neighborhoods of $p$; to simplify notations we keep denoting such neighborhoods by $O(p)$.

We now fix null frame $\{e_1, e_2, e_3 = L, e_4 = L\}$ on the null hypersurface $N^+ \cap O(p)$. Since $g$ and $g'$ agree to infinity order on $N^+$, this null frame is the same for both metrics. Recall also that the vector field $L$ is also the same for both metrics. We use two different Levi-Civita connections to parallel transport the given null frame along $\dot{L}$:

\begin{align*}
\begin{cases}
D_L v_\alpha = 0 & \text{with } v_\alpha = e_\alpha \text{ on } N^+ \cap O(p) \\
D_L' v'_\alpha = 0 & \text{with } v'_\alpha = e_\alpha \text{ on } N^+ \cap O(p)
\end{cases}
\end{align*}

(4.11)

The frames $\{v_\alpha\}$ and $\{v'_\alpha\}$ are smoothly defined in $O(p)$. We will express all the geometric quantities in these frames. Let $g_{\alpha\beta} = g(v_\alpha, v_\beta)$, $g'_{\alpha\beta} = g'(v'_\alpha, v'_\beta)$. Since $D_L v_\alpha = D_L' v'_\alpha = 0$, we know $L(g_{\alpha\beta}) = L(g'_{\alpha\beta}) = 0$, so $g_{\alpha\beta} = g'_{\alpha\beta}$. It follows that

$$h_{\alpha\beta} \triangleq g_{\alpha\beta} = g'_{\alpha\beta} \quad L(h_{\alpha\beta}) = 0 \quad \text{in } O(p).$$

(4.12)

Now define the Christoffel symbols, curvature tensors and their differences,

$$\Gamma^\gamma_{\alpha\beta} \triangleq g(D_{v_\alpha} v_\beta, v_\gamma), \quad \Gamma'^\gamma_{\alpha\beta} \triangleq g'(D'_{v'_\alpha} v'_\beta, v'_\gamma), \quad \delta \Gamma^\gamma_{\alpha\beta} \triangleq \Gamma'^\gamma_{\alpha\beta} - \Gamma^\gamma_{\alpha\beta}$$

$$R_{\alpha\beta\gamma\delta} \triangleq g(R(v_\alpha, v_\beta) v_\gamma, v_\delta), R'_{\alpha\beta\gamma\delta} \triangleq g'(R'(v'_\alpha, v'_\beta) v'_\gamma, v'_\delta), \delta R_{\alpha\beta\gamma\delta} \triangleq R'_{\alpha\beta\gamma\delta} - R_{\alpha\beta\gamma\delta}$$

Clearly, we have $\Gamma^\gamma_{3\beta} = \Gamma'^\gamma_{3\beta} = \delta \Gamma^\gamma_{3\beta} = 0$. The fact that $D_L v_\alpha = 0$ allow us to drive a system of ordinary differential equations for $\Gamma^\gamma_{\alpha\beta}$ and $\Gamma'^\gamma_{\alpha\beta}$:

$$L(\Gamma^\gamma_{\alpha\beta}) = L(g(D_{v_\alpha} v_\beta, v_\gamma)) = g(D_{v_\alpha} D_{v_\beta} v_\gamma, v_\gamma) + g(D_{v_\alpha} v_\beta, D_{v_\gamma} v_\gamma)$$

$$= R_{3\alpha\beta\gamma} + g(D_{v_3} D_{v_\alpha} v_\beta, v_\gamma) + g(D_{v_\alpha} v_\beta, D_{v_3} v_\gamma)$$

$$= R_{3\alpha\beta\gamma} + \Gamma^\rho_{3\alpha} \Gamma^\gamma_{\rho\beta} - \Gamma^\rho_{\alpha\beta} \Gamma^\gamma_{\rho3} + g_{\rho\delta} \Gamma^\delta_{\alpha\beta} \Gamma^\gamma_{\rho3}$$

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We take the difference of (4.13) and (4.14), so we have
\[ L(\Gamma^\gamma_{\alpha\beta}) = R_{3\alpha\beta\gamma} + (\Gamma * \Gamma)^\gamma_{\alpha\beta} \]  
(4.13)
\[ L(\Gamma'^\gamma_{\alpha\beta}) = R'_{3\alpha\beta\gamma} + (\Gamma' * \Gamma')^\gamma_{\alpha\beta} \]  
(4.14)
We take the difference of (4.13) and (4.14), so we have
\[ L(\delta \Gamma^\gamma_{\alpha\beta}) = \delta R_{3\alpha\beta\gamma} + (\Gamma' * \Gamma' - \Gamma * \Gamma)^\gamma_{\alpha\beta} \]
\[ = \delta R_{3\alpha\beta\gamma} + (\Gamma' * \delta \Gamma)^\gamma_{\alpha\beta} + (\Gamma * \delta \Gamma)^\gamma_{\alpha\beta} \]
Schematically, we have the following expression:
\[ L(\delta \Gamma) = M_\infty(\delta \Gamma) + M_\infty(\delta R). \]  
(4.15)

**Remark 4.4.** In general, given \( B = (B_1, \ldots, B_L) : \mathcal{O}(p) \to \mathbb{R}^L \) we let \( M_\infty(B) : \mathcal{O}(p) \to \mathbb{R}^{L'} \) denote vector-valued functions of the form \( M_\infty(B)v = \sum_{l=1}^L A_l^i B_l \), where the coefficients \( A^i_l \) are smooth on \( \mathcal{O}(p) \). So (4.15) holds because \( g, g' \) are fixed smooth metrics.

Now we also need to express the frames \( \{v^\alpha_3\} \) and \( \{v'^\alpha_3\} \) in terms of the fixed coordinate vector fields \( \partial_k \) relative to our local coordinates \( x_k \). We define
\[ v^\alpha_3 = v^k_\alpha \partial_k, \quad v'^\alpha_3 = v'^k_\alpha \partial_k, \quad (\delta v)^\alpha_k = v'^k_\alpha - v^k_\alpha \]
Consider \([v_3, v^\alpha_3] = -D_{v^\alpha_3} v_3 = -\Gamma^\beta_{\alpha 3} v_\beta = -\Gamma^\beta_{\alpha 3} v'^k_\beta \partial_k \), it implies
\[ v'^3_\alpha j(v'^k_\alpha) - v^3_\alpha j(v^k_\alpha) = -\Gamma^\beta_{\alpha 3} v'^k_\beta \]
i.e.
\[ L(v'^k_\alpha) = \partial_j(v'^k_\alpha) v^j_\alpha - \Gamma^\beta_{\alpha 3} v'^k_\beta \]
Now a similar relation holds for \( v^k_\alpha \), we take the difference, noticing that \( \partial_j(v'^k_\alpha) \) are fixed functions (since \( v_3 = v'_3 = L \)), so
\[ L(\delta v) = M_\infty(\delta \Gamma) + M_\infty(\delta v). \]  
(4.16)
We can also apply coordinate derivatives \( \partial_k \) on (4.13) and (4.14), so we have
\[ L(\delta \Gamma) = M_\infty(\delta \Gamma) + M_\infty(\delta \Gamma) + M_\infty(\delta R) + M_\infty(\delta R). \]  
(4.17)
\[ L(\delta \delta v) = M_\infty(\delta \Gamma) + M_\infty(\delta \Gamma) + M_\infty(\delta v) + M_\infty(\delta v). \]  
(4.18)
Finally, we derive a set covariant of wave equations for \( \delta R \) and \( \delta F \), \( \delta DF \) which are similarly defined for the difference of the corresponding quantities. In view of (4.10), the most difficult terms come from the following differences
\[ (\square_g - \square_{g'})R, \quad (\square_g - \square_{g'})F \quad \text{and} \quad (\square_g - \square_{g'})DF \]
For the first one, since \( g_{\alpha\beta} = g'_{\alpha\beta} \), it’s easy to see it has the following form
\[ (\square_g - \square_{g'})R = M_\infty(\delta \Gamma) + M_\infty(\delta \Gamma). \]
Similar relations hold for the other terms. Together with (4.13), (4.16), (4.17) and (4.18), we have the following system of ordinary-partial differential equations:
Proposition 4.5. Assume $G_i, H_j : \mathcal{O}(p) \to \mathbb{R}$ are smooth functions, $i = 1, \ldots, I$, $j = 1, \ldots, J$. Let $G = (G_1, \ldots, G_I)$, $H = (H_1, \ldots, H_J)$, $\partial G = (\partial_1 G_1, \partial_2 G_1, \partial_3 G_1, \partial_4 G_1, \ldots, \partial_3 G_I)$ and assume that in $\mathcal{O}(p)$,

\[
\begin{align*}
\square_g G &= \mathcal{M}_\infty(G) + \mathcal{M}_\infty(\partial G) + \mathcal{M}_\infty(H); \\
\mathcal{L}(H) &= \mathcal{M}_\infty(G) + \mathcal{M}_\infty(\partial G) + \mathcal{M}_\infty(H).
\end{align*}
\]

Assume that $G = 0$ and $H = 0$ on $(\mathbb{N}^+ \cup \mathbb{N}^-) \cap \mathcal{O}(p)$. Then, there exists a neighborhood $\mathcal{O}'(p) \subset \mathcal{O}(p)$ of $x_0$ such that $G = 0$ and $H = 0$ in $(I^+ \cup I^-) \cap \mathcal{O}'(p)$.

Apparently, this proposition finishes the proof of Proposition 4.3 which implies that the vector field $K$ is Killing in a full neighborhood of $S$. \hfill \Box

Remark 4.6. The vector field $K$ is time-like outside the black hole, i.e. $g(K, K) \leq 0$ in $I^+ \cup I^-$, which follows directly from the fact that $\mathcal{L}(g(K, K)) \geq 0$.

5. Rotational Killing vector field

The purpose of this section is to prove Theorem 1.2. In addition to the Hawking vector field $K$ we just constructed, we assume $(\mathcal{O}, g, F)$ has another Killing vector field $T$ such that it’s tangent to $\mathbb{N}^+ \cup \mathbb{N}^-$, non-vanishing on $S$ and $\mathcal{L}_K F = 0$. We need to find a constant $\lambda$, such that $Z = T + \lambda K$ is a rotational Killing vector fields, i.e. all the orbits are closed.

One needs to study the action of $T$ on the bifurcate sphere $S$. Since $T$ is a smooth vector field tangent the bifurcate horizon $\mathbb{N}^+ \cup \mathbb{N}^-$, it must be tangent to $S$. We can conclude that the existence of such a non-vanishing Killing vector field $T$ on $S$ forces the restriction of the metric $g$ on $S$ to be rotational symmetric thanks to Lemma A.1 in the appendix. In our case $X = T|_S$ on $S$ with induced metric from $g$. It has a period $t_0$. It has two zeroes and we choose one of them, denoting it by $p \in S$. To get a space-time
rotational vector field, we need to study $T$ on the black hold boundary $N^+ \cup N^-$. On $N^+$, we define $\lambda(T) = \frac{g(T, L)}{g(K, L)}$ which is essentially the $K$ direction of $T$. We prove the following lemma

**Claim 5.1.** On $N^+$, $L(T)$ is constant along each null geodesic, i.e. $L(\lambda(T)) = 0$.

*Proof.* We first show that $[T, L]$ is parallel to $L$, i.e. there is a function $f : N^+ \rightarrow \mathbb{R}$, such that

$$[T, L] = fL.$$

Since both vectors are tangent to $N^+$, so is $[T, L]$. It suffices to show $g([T, L], e_a) = 0$.

$$g([T, L], e_a) = g(D_T L, e_a) - g(D_L T, e_a) = g(D_T L, e_a) + g(D_a T, L) = g(D_T L, e_a) - g(T, D_a L) = \chi(T, e_a) - \chi(e_a, T) = 0$$

We then show that $L(f) = 0$. Since $D_L L = 0$ and $T$ is Killing, we have

$$0 = \mathcal{L}_T(D_L L) = D_{\mathcal{L}_T L} L + D_L (\mathcal{L}_T L) = D_{fL} L + D_L (fL) = L(f)L$$

It implies that $f$ is determined on $S$. We can assume $f : S \rightarrow \mathbb{R}$.

$$f = fL(u) = [T, L](u) = -L(T(u))$$

So

$$T(u) = -fu.$$ 

Now we compute $L(\lambda(T))$ by recalling that $L$ is the gradient of $u$ under the metric $g$:

$$L(\lambda(T)) = L\left(\frac{g(T, L)}{g(K, L)}\right) = -L\left(\frac{T(u)}{u}\right) = L(f) = 0$$

□

Now we can find the rotational vector field $Z$:

**Claim 5.2.** Let $\lambda = f(p)$, then $Z = T - \lambda K$ is a rotational vector field with period $t_0$.

*Proof.* Since $K = 0$ on $S$, $Z|_S = T|_S$ has the same period $t_0$. We denote $\psi_t$ the one parameter isometry group generated by $Z$ on space-time. We are going to prove that $\psi_{t_0} = id$ which concludes the proof of the claim.

We study the action of $\psi_t$ on the null geodesic $\gamma$ starting at $p$ and pointing at the $L$ direction. For each $t$, since $p$ is a fixed point of $\psi_t$ and $\psi_t$ is an isometry, we know that $\psi_t(\gamma) \subset \gamma$ is a reparametrization of $\gamma$ with a possible stretch. In particular, it implies $Z|_{\gamma}$ is proportional to $K|_{\gamma}$. In view of the definition of $\lambda$, we know that $Z|_{\gamma} = 0$ since we have subtracted the corresponding portion of $K$ from $T$. So $\psi_t|_{\gamma} = id$. In particular, $\psi_{t_0}|_{\gamma} = id$.

Now we look at the action of $\psi_{t_0}$ on the full tangent space of $p$. The previous argument shows $(\psi_{t_0})_* L = L$. Since it fixes the whole space slice $S$, then $(\psi_{t_0})_* e_a = e_a$. Now
using the fact that $\psi_{t_0}$ is an isometry, we know $L$ is also fixed. So $(\psi_{t_0})_*$ is the identity map on the tangent space of $p$, now we can use Lemma [A.2] in the appendix to conclude that $\psi_{t_0}$ is identity in a small neighborhood of $p$. Now on can use the compactness of $S$ and the standard open-closed argument on $S$ to conclude $\psi_{t_0}$ is identity map in a small neighborhood of $S$. □

We need one more claim to finish the proof of Theorem 1.2:

Claim 5.3. $Z$ is the vector field we constructed, then $[Z, K] = 0$.

Proof. It suffices to show $[T, K] = 0$. Since both $K$ and $T$ are Killing, in view of the fact that all the Killing vector fields on a manifold form a Lie algebra under $[−, −]$, we know that $W = [T, K]$ is Killing, so it solves the following equation:

$$\square_g W_\alpha = -W_\alpha^\beta W_\beta$$  \hspace{1cm} (5.1)

Once again, due to the well-posedness of the characteristic initial-value problem, $W = 0$ in the domain of dependence follows from the fact that $W = 0$ on $N^+ \cup N^-$. It is immediate from the calculations in the proof of Claim 5.1:

$$W = [T, K] = [T, uL] = u[T, L] + T(u)L$$

$$= uL - uL = 0$$

For ill-posed region $I^+ \cup I^-$, once again the vanishing of $W$ follows easily from setting $H = 0$ in Proposition 4.5. □

Appendix A. Two lemmas on geometry

Lemma A.1. Assume $h$ is a Riemannian metric on the topological sphere $S^2$ which admits a non-trivial Killing vector field $X$, then $(S^2, h)$ is a Riemannian wrapped product $([0, 1], dr^2) \times_{\phi(r)} (S^1, d\sigma^2)$. In particular, each orbit of $X$ is closed and has a common period $t_0$.

Proof. First, we observe that, if $X$ is non-trivial, then the set $Z(X)$, which consists all zeroes of $X$, is discrete. It follows from the fact that, the zero locus of a Killing vector field is a disjoint union of totally geodesic sub-manifolds each of even dimension. Since we are on a surface, the zeroes must be discrete. In particular, since the $S^2$ is compact, $X$ has only finite many zeroes.

The second observations is that, for each zero $p$ of $X$, $ind_X(p)$ the index of $X$ at $p$ is either 1 or $-1$. It following from the fact that, $X$ induces an isometry on $T_pS^2$, which is a 2-dimensional rotation. So its index must be 1 or $-1$.

Now we can apply the Poincaré-Hopf index Theorem:

$$\sum_{p \in Z(X)} ind_X(p) = \chi(S^2) = 2.$$
The previous observation imply that the cardinal number $|Z(X)| \geq 2$. We can pick up two points $p, q \in Z(X)$. Now let us fix a minimal geodesic $\gamma(t)$ between $p$ and $q$. Let $\phi_t$ be the flow generated by $X$. Since on $T_pM$, $(\phi_t)_*$ is a rotation, it has a period $t_0$. Let $x \neq p, q$ be a point on $\gamma$. We show that the orbit of $x$ under $\phi_t$ is a closed non-degenerate circle, more precisely, it is exactly the image $\{\phi_t(x)| t \in [0, t_0]\}$. It trivially holds when $x$ is close to either $p$ or $q$, i.e. in the normal coordinate of $p$ or $q$, since it will stay on the geodesic sphere which is a circle around either $p$ or $q$. Since $\gamma$ is minimal and $X(q) = 0$, so $\phi_t(\gamma)$ is also a minimal geodesic between $p$ and $q$. When $t$ varies, $\phi_t(\gamma)$ sweeps the whole $S^2$, we know that all points except $q$ is in the normal coordinate of $p$, so the orbit $x$ is closed. Apparently, this finishes the proof of the lemma.

Lemma A.2. Assume $(M, g)$ is a Lorentzian manifold, $\phi : M \to M$ is an isometry and $p \in M$ is one fixed point of $\phi$. If $\phi_{sp} = id$, the $\phi = id$ locally around $p$.

Proof. In Riemannian geometry, it’s easy since we have the concept of length; in our case, the difficulty comes from the fact that on the light-cone, we don’t have the concept of length. But the proposition holds inside light-cone since we can consider the maximal time-like geodesics. Since locally light-cone is the boundary of the future of the point $p$, the identity map can be continued to the boundary.

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