Beyond a question of Markus Linckelmann

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Abstract: In the 2002 Durham Symposium, Markus Linckelmann conjectured the existence of a regular central $k^*$-extension of the full subcategory over the selfcentralizing Brauer pairs of the Frobenius $P$-category $\mathcal{F}_{(b,G)}$ associated with a block $b$ of defect group $P$ of a finite group $G$, which would include, as $k^*$-automorphism groups of the objects, the $k^*$-groups associated with the automizers of the corresponding selfcentralizing Brauer pairs, introduced in [4, 6.6]. As a matter of fact, in this question the selfcentralizing Brauer pairs can be replaced by the nilcentralized Brauer pairs, still getting a positive answer. But the condition on the $k^*$-automorphism groups of the objects is not precise enough to guarantee the uniqueness of a solution, as showed in [3, Theorem 1.3]. This uniqueness depends on the folder structure [6, Section 2] associated with $\mathcal{F}_{(b,G)}$ in [5, Theorem 11.32], and here we prove the existence and the uniqueness for any folded Frobenius $P$-category.

1. Introduction

1.1. Let $p$ be a prime number and $\mathcal{O}$ a complete discrete valuation ring with a field of quotients $K$ of characteristic zero and a residue field $k$ of characteristic $p$; we assume that $k$ is algebraically closed. Let $G$ be a finite group, $b$ a block of $G$ — namely a primitive idempotent in the center $Z(\mathcal{O}G)$ of the group $\mathcal{O}$-algebra $\mathcal{O}G$ — and $(P,e)$ a maximal Brauer $(b,G)$-pair [5, 1.16]; recall that the Frobenius $P$-category $\mathcal{F}_{(b,G)}$ associated with $b$ is the subcategory of the category of finite groups where the objects are all the subgroups of $P$ and, for any pair of subgroups $Q$ and $R$ of $P$, the morphisms $\varphi$ from $R$ to $Q$ are the group homomorphisms $\varphi: R \to Q$ induced by the conjugation of some element $x \in G$ fulfilling

$$(R, g) \subset (Q, f)^x$$

where $(Q, f)$ and $(R, g)$ are the corresponding Brauer $(b,G)$-pairs contained in $(P,e)$ [5, Ch. 3].

1.2. Moreover, we say that a Brauer $(b,G)$-pair $(Q,f)$ is nilcentralized if $f$ is a nilpotent block of $C_G(Q)$ [5, 7.4], and that $(Q,f)$ is selfcentralizing if the image $\bar{f}$ of $f$ is a block of defect zero of $\bar{C}_G(Q) = C_G(Q)/Z(Q)$ [5, 7.4]; thus, a selfcentralizing Brauer $(b,G)$-pair is still nilcentralized. We respectively denote by $\mathcal{F}^{nc}_{(b,G)}$ or $\mathcal{F}^{sc}_{(b,G)}$ the full subcategories of $\mathcal{F}_{(b,G)}$ over the set of subgroups $Q$ of $P$ such that the Brauer $(b,G)$-pair $(Q,f)$ contained in $(P,e)$ is respectively nilcentralized or selfcentralizing.

1.3. Recall that a $k^*$-group $\hat{G}$ is a group endowed with an injective group homomorphism $\theta: k^* \to Z(\hat{G})$ [4, §5], that $G = \hat{G}/\theta(k^*)$ is called the
k*-quotient of $\hat{G}$ and that a k*-group homomorphism is a group homomorphism which preserves the “multiplication” by $k^*$; let us denote by $k^*$-$\mathcal{G}$ the category of k*-groups with finite k*-quotient. In the case of the Frobenius $P$-category $\mathcal{F}_{(b,G)}$ above, for any nilcentralized Brauer $(b,G)$-pair $(Q,f)$ contained in $(P,e)$ it is well-known that the action of $N_G(Q,f)$ on the simple algebra $OC_G(Q)f/J(OC_G(Q)f)$ supplies a k*-group $\hat{N}_G(Q,f)/C_G(Q)$ of k*-quotient $\mathcal{F}_{(b,G)}(Q) \cong N_G(Q,f)/C_G(Q)$ [5, 7.4].

1.4. On the other hand, for any category $\mathcal{C}$ and any Abelian group $Z$ let us call regular central $Z$-extension of $\mathcal{C}$ any category $\mathcal{E}$ over the same objects endowed with a full functor $\epsilon : \mathcal{E} \to \mathcal{C}$, which is the identity over the objects, and, for any pair of $\mathcal{C}$-objects $A$ and $B$, with a regular action of $Z$ over the fibers of the map

$$\hat{\epsilon}(B,A) \longrightarrow \epsilon(B,A)$$

1.4.1

induced by $\epsilon$ — where $\mathcal{C}(B,A)$ and $\mathcal{E}(B,A)$ denote the corresponding sets of $\mathcal{C}$- and $\mathcal{E}$-morphisms from $A$ to $B$ — in such a way that these $Z$-actions are compatible with the composition of $\mathcal{E}$-morphisms. Note that, if $\mathcal{C}'$ is a second category and $\epsilon : \mathcal{C} \to \mathcal{C}'$ an equivalence of categories, we easily can obtain a regular central $Z$-extension $\hat{\mathcal{E}}$ of $\mathcal{C}'$ and a $Z$-compatible equivalence of categories $\hat{\epsilon} : \hat{\mathcal{E}} \to \hat{\mathcal{C}}$. In short, we call k*-category any regular central k*-extension of a category.

1.5. In the 2002 Durham Symposium, Markus Linckelmann conjectured the existence of a regular central k*-extension $\hat{\mathcal{F}}_{(b,G)}^{sc}$ of $\mathcal{F}_{(b,G)}$ admitting a k*-group isomorphism

$$\hat{\mathcal{F}}_{(b,G)}^{sc}(Q) \cong \hat{N}_G(Q,f)/C_G(Q)$$

1.5.1

for any selfcentralizing Brauer $(b,G)$-pair $(Q,f)$ contained in $(P,e)$. Here we show the existence of a regular central k*-extension $\hat{\mathcal{F}}_{(b,G)}^{sc}$ of $\mathcal{F}_{(b,G)}^{sc}$ admitting a k*-group isomorphism

$$\hat{\mathcal{F}}_{(b,G)}^{sc}(Q) \cong \hat{N}_G(Q,f)/C_G(Q)$$

1.5.2

for any nilcentralized Brauer $(b,G)$-pair $(Q,f)$ contained in $(P,e)$, proving Linckelmann’s conjecture.

1.6. In both cases, these k*-group isomorphisms are not precise enough to guarantee the uniqueness either of $\hat{\mathcal{F}}_{(b,G)}^{sc}$, or of $\hat{\mathcal{F}}_{(b,G)}^{sc}$ as showed in [3, Theorem 1.3]. More explicitly, if $(Q,f)$ and $(R,g)$ are nilcentralized Brauer $(b,G)$-pairs contained in $(P,e)$ such that $(R,g)$ is contained and normal in $(Q,f)$ then, denoting by $\hat{N}_G(Q,f)_R$ the stabilizer of $R$ in $\hat{N}_G(Q,f)$, Proposition 11.23 in [5] supplies a particular k*-group homomorphism

$$\hat{N}_G(Q,f)_R/C_G(Q) \longrightarrow \hat{N}_G(R,g)/C_G(R)$$

1.6.1.
But, a regular central $k^*$-extension $\hat{F}^{nc}_{(b,G)}$ of $\mathcal{F}^{nc}_{(b,G)}$ also supplies a $k^*$-group homomorphism

$$\hat{F}^{nc}_{(b,G)}(Q)_R \rightarrow \hat{F}^{nc}_{(b,G)}(R)$$

where $\hat{F}^{nc}_{(b,G)}(Q)_R$ denotes the stabilizer of $R$ in $\hat{F}^{nc}_{(b,G)}(Q)$, sending any $\hat{s}$ in $\hat{F}^{nc}_{(b,G)}(Q)_R$ on the unique element $\hat{t} \in \hat{F}^{nc}_{(b,G)}(R)$ fulfilling $i^Q_R \circ \hat{t} = \hat{s} \circ i^Q_R$, where $i^Q_R$ is a lifting to $\hat{F}^{nc}_{(b,G)}(Q,R)$ of the inclusion map $R \subset Q$. The uniqueness of a suitable regular central $k^*$-extension $\hat{F}^{nc}_{(b,G)}$ depends on the compatibility of all the $k^*$-group homomorphisms 1.6.1 and 1.6.2 with the corresponding $k^*$-group isomorphisms 1.5.2 or, more generally, it depends on the folded structure of $\mathcal{F}^{nc}_{(b,G)}$ determined by [5, Theorem 11.32].

2. Folded Frobenius $P$-categories

2.1. Denoting by $P$ a finite $p$-group, by $i\mathfrak{G}$ the category formed by the finite groups and by the injective group homomorphisms, and by $\mathcal{F}_P$ the subcategory of $i\mathfrak{G}$ where the objects are all the subgroups of $P$ and the morphisms are the group homomorphisms induced by the conjugation by elements of $P$, recall that a Frobenius $P$-category $\mathcal{F}$ is a subcategory of $i\mathfrak{G}$ containing $\mathcal{F}_P$ where the objects are all the subgroups of $P$ and the morphisms fulfill the following three conditions [5, 2.8 and Proposition 2.11]

2.1.1 If $Q$, $R$ and $T$ are subgroups of $P$, for any $\varphi \in \mathcal{F}(Q,R)$ and any group homomorphism $\psi : T \rightarrow R$ the composition $\varphi \circ \psi$ belongs to $\mathcal{F}(Q,T)$ (if and) only if $\psi \in \mathcal{F}(R,T)$.

2.1.2 $\mathcal{F}_P(P)$ is a Sylow $p$-subgroup of $\mathcal{F}(P)$.

Let us say that a subgroup $Q$ of $P$ is fully centralized in $\mathcal{F}$ if for any $\mathcal{F}$-morphism $\xi : Q \rightarrow CP(Q) \rightarrow P$ we have $\xi(C_P(Q)) = C_P(\xi(Q))$.

2.1.3 For any subgroup $Q$ of $P$ fully centralized in $\mathcal{F}$, any $\mathcal{F}$-morphism $\varphi : Q \rightarrow P$ and any subgroup $R$ of $NP(\varphi(Q))$ containing $\varphi(Q)$ such that $\mathcal{F}_P(Q)$ contains the action of $\mathcal{F}_R(\varphi(Q))$ over $Q$ via $\varphi$, there is an $\mathcal{F}$-morphism $\zeta : R \rightarrow P$ fulfilling $\zeta(\varphi(u)) = u$ for any $u \in Q$.

2.2. With the notation in 1.1 above, it follows from [5, Theorem 3.7] that $\mathcal{F}_{(b,G)}$ is a Frobenius $P$-category. Moreover, we say that a subgroup $Q$ of $P$ is $\mathcal{F}$-nilcentralized if, for any $\varphi \in \mathcal{F}(P,Q)$ such that $Q' = \varphi(Q)$ is fully centralized in $\mathcal{F}$, the $C_P(Q')$-categories $\mathcal{F}_{C_P(Q')}$ [5, 2.14] and $\mathcal{F}_{C_P(Q')}$ coincide; note that, according to [5, Proposition 7.2], in $\mathcal{F}_{(b,G)}$ this definition agree with 1.2 above. Similarly, we say that $Q$ is $\mathcal{F}$-selfcentralizing if we have

$$C_P(\varphi(Q)) \subset \varphi(Q)$$

for any $\varphi \in \mathcal{F}(P,Q)$; once again, according to [5, Corollary 7.3], in $\mathcal{F}_{(b,G)}$ this definition agree with 1.2 above. Finally, we say that a subgroup $R$ of $P$
is $\mathcal{F}$-radical if it is $\mathcal{F}$-selfcentralizing and we have

$$O_p(\tilde{\mathcal{F}}(R)) = \{1\}$$

where $\tilde{\mathcal{F}}(R) = \mathcal{F}(R)/\mathcal{F}_R(R)$ [5, 1.3]. We respectively denote by $\mathcal{F}^{nc}, \mathcal{F}^{sc}$ and $\mathcal{F}^{rd}$ the full subcategories of $\mathcal{F}$ over the respective sets of $\mathcal{F}$-nilcentralized, $\mathcal{F}$-selfcentralizing and $\mathcal{F}$-radical subgroups of $P$.

2.3. We call $\mathcal{F}^{sc}$-chain any functor $q: \Delta_n \to \mathcal{F}^{sc}$ where the $n$-simplex $\Delta_n$ is considered as a category with the morphisms — denoted by $i \circ i'$ — defined by the order [5, A2.2]; for any $\mathcal{F}$-nilcentralized subgroup $Q$ of $P$, let us denote by $\tilde{\mathcal{F}}(R)$ the obvious $\mathcal{F}^{sc}$-chain sending $0$ to $Q$. Following [5, A2.8], we denote by $\chi^*(\mathcal{F}^{sc})$ the category where the objects are all the $\mathcal{F}^{sc}$-chains $(q, \Delta_n)$ and the morphisms from $q: \Delta_n \to \mathcal{F}^{sc}$ to another $\mathcal{F}^{sc}$-chain $r: \Delta_m \to \mathcal{F}^{sc}$ are the pairs $(\nu, \delta)$ formed by an order preserving map $\delta: \Delta_m \to \Delta_n$ and by a natural isomorphism $\nu: q \circ \delta \cong r$, the composition being defined by the formula

$$((\mu, \varepsilon) \circ (\nu, \delta)) = (\mu \circ (\nu \ast \varepsilon), \delta \circ \varepsilon)$$

Recall that we have a canonical functor [5, Proposition A2.10]

$$\text{aut}_{\mathcal{F}^{nc}} : \chi^*(\mathcal{F}^{sc}) \to \mathfrak{Gr}$$

mapping any $\mathcal{F}^{sc}$-chain $q: \Delta_n \to \mathcal{F}^{sc}$ to the group of natural automorphisms of $q$.

2.4. In [6, §2] we introduce a folded Frobenius $P$-category $(\mathcal{F}, \hat{\text{aut}}_{\mathcal{F}^{sc}})$ as a pair formed by a Frobenius $P$-category $\mathcal{F}$ and a functor

$$\hat{\text{aut}}_{\mathcal{F}^{sc}} : \chi^*(\mathcal{F}^{sc}) \to k^\ast \cdot \mathfrak{Gr}$$

lifting the canonical functor $\text{aut}_{\mathcal{F}^{sc}}$; here, we replace selfcentraling by nilcentralized: we call folded Frobenius $P$-category $(\mathcal{F}, \hat{\text{aut}}_{\mathcal{F}^{nc}})$ a pair formed by $\mathcal{F}$ and a functor

$$\hat{\text{aut}}_{\mathcal{F}^{nc}} : \chi^*(\mathcal{F}^{nc}) \to k^\ast \cdot \mathfrak{Gr}$$

lifting the canonical functor $\text{aut}_{\mathcal{F}^{nc}}$; we also say that $\hat{\text{aut}}_{\mathcal{F}^{nc}}$ is a folder structure of $\mathcal{F}$. With the notation of 1.1 above, Theorem 11.32 in [5] exhibits a folder structure of $\mathcal{F}_{(b,G)}$, namely a functor $\hat{\text{aut}}_{\mathcal{F}_{(b,G)}}$ lifting $\text{aut}_{\mathcal{F}_{(b,G)}}$, that we call Brauer folder structure of $\mathcal{F}_{(b,G)}$. Actually, both definitions coincide since any functor $\hat{\text{aut}}_{\mathcal{F}^{nc}}$ lifting $\text{aut}_{\mathcal{F}^{nc}}$ can be extended to a unique functor $\hat{\text{aut}}_{\mathcal{F}^{nc}}$, as it shows our next result.
Theorem 2.5. Any functor \( \hat{\text{aut}}_{F^\infty} \) lifting \( \text{aut}_{F^\infty} \) to the category \( k^*\cdot\mathfrak{G} \) can be extended to a unique functor lifting \( \text{aut}_{F^\infty} \):

\[
\hat{\text{aut}}_{F^\infty} : \text{ch}^*(F^\infty) \longrightarrow k^*\cdot\mathfrak{G} \tag{2.5.1}
\]

Proof: Let \( \mathfrak{X} \) be a set of \( F \)-nilcentralized subgroups of \( P \) which contains all the \( F \)-selfcentralizing subgroups of \( P \) and is stable by \( F \)-isomorphisms; denoting by \( F^\infty \) the full subcategory of \( F \) over \( X \), assume that \( \hat{\text{aut}}_{F^\infty} \) can be extended to a unique functor

\[
\hat{\text{aut}}_{F^\infty} : \text{ch}^*(F^\infty) \longrightarrow k^*\cdot\mathfrak{G} \tag{2.5.2}
\]

Assuming that \( \mathfrak{X} \) does not coincide with the set of all the \( F \)-nilcentralized subgroups of \( P \), let \( V \) be a maximal \( F \)-nilcentralized subgroup which is not in \( \mathfrak{X} \); denoting by \( \mathcal{Y} \) the union of \( \mathfrak{X} \) with all the subgroups of \( P \) \( F \)-isomorphic to \( V \), it is clear that it suffices to prove that \( \hat{\text{aut}}_{F^\infty} \) admits a unique extension to \( \text{ch}^*(F^\infty) \).

For any chain \( q : \Delta \rightarrow F^\infty \), we choose an \( F \)-morphism \( \alpha : q(n) \rightarrow P \) such that \( \alpha(q(n)) \) is fully centralized in \( F \) [5, Proposition 2.7] and denote by \( \alpha^n : \Delta_{n+1} \rightarrow F^\infty \) the chain which extends \( \alpha \) and which maps \( n + 1 \) on \( \alpha(q(n))\cdot C_P(\alpha(q(n))) \) and \( (n \cdot n + 1) \) on the \( F \)-morphism from \( q(n) \) to \( \alpha(q(n))\cdot C_P(\alpha(q(n))) \) induced by \( \alpha \); we have an obvious \( \text{ch}^*(F^\infty) \)-morphism [5, A3.1]

\[
(\text{id}_q, \delta_{n+1}^n) : (q^n, \Delta_{n+1}) \longrightarrow (q, \Delta_n) \tag{2.5.3}
\]

and the functor \( \text{aut}_{F^\infty} \) maps \( (\text{id}_q, \delta_{n+1}^n) \) on a group homomorphism

\[
\mathcal{F}(q^n) \longrightarrow \mathcal{F}(q) \tag{2.5.4}
\]

which is surjective since any \( \sigma \in \mathcal{F}(q) \subset \mathcal{F}(q(n)) \) can be “extended” to an \( F \)-automorphism of \( q(n + 1) \) [5, statement 2.10.1].

Then, since \( \alpha(q(n)) \) is is \( F \)-nilcentralized and fully centralized in \( F \), the kernel of homomorphism 2.5.4 is a \( p \)-group [5, Corollary 4.7]; moreover, since \( q^n(n + 1) \) belongs to \( \mathfrak{X} \), the functor \( \hat{\text{aut}}_{F^\infty} \) and the structural inclusion \( \mathcal{F}(q^n) \subset \mathcal{F}(q^n(n + 1)) \) determine a \( k^* \)-subgroup

\[
\hat{\mathcal{F}}(q^n) \subset \hat{\mathcal{F}}(q^n(n + 1)) = \hat{\text{aut}}_{F^\infty} (q^n(n + 1)) \tag{2.5.5}
\]

and, since the kernel of homomorphism 2.5.4 is a \( p \)-group, this \( k^* \)-subgroup induces a central \( k^* \)-extension \( \hat{\mathcal{F}}(q) \) of \( \mathcal{F}(q) \) such that we have a \( k^* \)-group homomorphism

\[
\hat{\mathcal{F}}(q^n) \longrightarrow \hat{\mathcal{F}}(q) \tag{2.5.6}
\]

lifting homomorphism 2.5.4.
Note that, for a different choice \( \alpha’ : \mathfrak{q}(n) \to P \) of \( \alpha \), we have an \( \mathcal{F} \)-isomorphism \( \alpha'(\mathfrak{q}(n)) \cong \alpha'(\mathfrak{q}(n)) \) which can be extended to an \( \mathcal{F} \)-isomorphism \( \mathfrak{q}^\alpha(n + 1) \cong \mathfrak{q}^\alpha(n + 1) \) [5, statement 2.10.1] and then \( \hat{\text{aut}}_{\mathcal{F},x} \) determines a \( k^* \)-isomorphism
\[
\hat{\text{aut}}_{\mathcal{F},x}(\mathfrak{q}^\alpha(n + 1)) \cong \hat{\text{aut}}_{\mathcal{F},x}(\mathfrak{q}^\alpha(n + 1)) \tag{2.5.7}
\]
mapping \( \hat{\mathcal{F}}(\mathfrak{q}^\alpha) \) onto \( \hat{\mathcal{F}}(\mathfrak{q}^\alpha) \); moreover, it follows from [5, Proposition 4.6] that two such \( \mathcal{F} \)-isomorphisms are \( C_P(\alpha’(\mathfrak{q}(n))) \)-conjugate and therefore our definition of \( \hat{\mathcal{F}}(\mathfrak{q}) \) does not depend on our choice of \( \alpha \). Similarly, if \( \mathfrak{q}(n) \) belongs to \( \mathfrak{X} \) then the functor \( \hat{\text{aut}}_{\mathcal{F},x} \) already defines a \( k^* \)-group \( \hat{\text{aut}}_{\mathcal{F}}(\mathfrak{q}(n)) \) and, denoting by \( \mathfrak{q}^\alpha(n+1) : \Delta_1 \to \mathcal{F}^X \) the chain mapping 0 on \( \mathfrak{q}(n) \), 1 on \( \mathfrak{q}^\alpha(n+1) \) and \((0\cdot 1) \) on \( \mathfrak{q}^\alpha(n\cdot n+1) \), also defines a \( k^* \)-group homomorphism
\[
\hat{\text{aut}}_{\mathcal{F},x}(\mathfrak{q}_n^{\alpha,n+1}) \longrightarrow \hat{\text{aut}}_{\mathcal{F},x}(\mathfrak{q}(n)) \tag{2.5.8}
\]
inducing a canonical \( k^* \)-group isomorphism from \( \hat{\mathcal{F}}(\mathfrak{q}) \) in 2.5.6 above onto the inverse image of \( \text{aut}_{\mathcal{F},x}(\mathfrak{q}) \subset \text{aut}_{\mathcal{F},x}(\mathfrak{q}(n)) \) in \( \hat{\text{aut}}_{\mathcal{F},x}(\mathfrak{q}(n)) \); in particular, if the image of \( \mathfrak{q} \) is contained in \( \mathfrak{X} \), we get a canonical \( k^* \)-group isomorphism \( \hat{\mathcal{F}}(\mathfrak{q}) \cong \text{aut}_{\mathcal{F},x}(\mathfrak{q}) \).

Now, for any \( \text{ch}(\mathcal{F}^\delta) \)-morphism \( (\nu, \delta) : (\tau, \Delta_m) \to (\mathfrak{q}, \Delta_n) \), choosing suitable \( \mathcal{F} \)-morphisms \( \alpha : \mathfrak{q}(n) \to P \) and \( \beta : \tau(m) \to P \) as above, we have to exhibit a \( k^* \)-group homomorphism \( \hat{\mathcal{F}}(\tau) \to \hat{\mathcal{F}}(\mathfrak{q}) \) lifting \( \text{aut}_{\mathcal{F},x}(\nu, \delta) \). Firstly, we assume that the image of \( \tau(\delta(n)) \) via \( \tau(\delta(n) \cdot m) \) is normal in \( \tau(m) \); in this case, \( \beta(\tau(\delta(n) \cdot m)(\tau(\delta(n)))) \) is normal in \( \tau^\theta(m + 1) \) and, according to [5, statement 2.10.1], there is an \( \mathcal{F} \)-morphism
\[
\hat{\nu} : \tau^\theta(m + 1) \longrightarrow N_P(\alpha(\mathfrak{q}(n))) \tag{2.5.9}
\]
extending the \( \mathcal{F} \)-morphism
\[
\beta(\tau(\delta(n) \cdot m)(\tau(\delta(n)))) \cong \tau(\delta(n)) \overset{\nu_0}{\cong} \mathfrak{q}(n) \cong \alpha(\mathfrak{q}(n)) \subset P \tag{2.5.10},
\]
and we set \( U = \hat{\nu}(\tau^\theta(m + 1))C_P(\alpha(\mathfrak{q}(n))) \). Then, we consider the chains
\[
\mathfrak{q}^\alpha,\nu : \Delta_{n+2} \longrightarrow \mathcal{F}^\gamma \quad \text{and} \quad \tau^\theta,\nu : \Delta_{m+2} \longrightarrow \mathcal{F}^\gamma \tag{2.5.11}
\]
respectively extending the chains \( \mathfrak{q}^\alpha \) and \( \tau^\theta \) defined above, fulfilling
\[
\mathfrak{q}^\alpha,\nu(n + 2) = U = \tau^\theta,\nu(m + 2) \tag{2.5.12}
\]
and, since \( \alpha(\mathfrak{q}(n)) \subset \hat{\nu}(\beta(\tau(m))) \), respectively mapping \((n + 1 \cdot n + 2) \) and \((m + 1 \cdot m + 2) \) on the inclusion \( \mathfrak{q}^\alpha(n + 1) \subset U \) and on the \( \mathcal{F} \)-morphism from
\( r^\beta (m + 1) \) to \( U \) induced by \( \hat{\nu} \). Note that, since the centralizer of \( \alpha(q(n)) \) contains \( C_P \left( \hat{\nu} (\beta(\tau(m))) \right) \) and \( \beta(\tau(m)) \) is fully centralized in \( \mathcal{F} \), we still have \( U = \hat{\nu} (\beta(\tau(m))) \cdot C_P (\alpha(q(n))) \). Moreover, it follows from [5, Proposition 4.6] that another choice \( \hat{\nu}' \) of the \( \mathcal{F} \)-morphism 2.5.9 is \( C_P (\alpha(q(n))) \)-conjugate of \( \hat{\nu} \) and, in particular, the group \( U \) does not change.

With all this notation, we have obvious \( \mathfrak{ch}_J (\mathcal{F}^\beta) \)-morphisms

\[
(id_{q^n}, \delta_{n+1}) : (q^{\alpha, \nu}, \Delta_n) \longrightarrow (q^{\alpha}, \Delta_n) \tag{2.5.13}
\]

and, considering the maps

\[
\Delta_n + 1 \xrightarrow{\infty} \Delta_1 \xrightarrow{\tau} \Delta_{n+1} \quad \text{and} \quad \Delta_n + 1 \xrightarrow{\infty} \Delta_0 \xrightarrow{\tau} \Delta_{n+1} \tag{2.5.14}
\]

respectively mapping \( i \) on \( i + n + 1 \) and \( i + m + 1 \), the \( \mathfrak{ch}_J (\mathcal{F}^\beta) \)-morphisms above determine the following \( \mathfrak{ch}_J (\mathcal{F}^\beta) \)-morphisms

\[
(q^{\alpha, \nu} \circ \sigma_n, \Delta_1) \longrightarrow (q^{\alpha} \circ \tau_n, \Delta_0) \quad \text{and} \quad (q^{\beta, \nu} \circ \sigma_m, \Delta_1) \longrightarrow (q^{\beta} \circ \tau_m, \Delta_0) \tag{2.5.15}
\]

Then, the functor \( \mathfrak{ut}_J \) maps these morphisms on \( k^* \)-group homomorphisms

\[
\mathcal{F}(q^{\alpha, \nu} \circ \sigma_n) \longrightarrow \mathcal{F}(q^{\alpha} \circ \tau_n) \quad \text{and} \quad \mathcal{F}(q^{\beta, \nu} \circ \sigma_m) \longrightarrow \mathcal{F}(q^{\beta} \circ \tau_m) \tag{2.5.16}
\]

But note that \( \mathcal{F}(q^{\alpha, \nu}) \), \( \mathcal{F}(q^{\alpha}) \), \( \mathcal{F}(q^{\beta, \nu}) \) and \( \mathcal{F}(q^{\beta}) \) are respectively contained in \( \mathcal{F}(q^{\alpha, \nu} \circ \sigma_n) \), \( \mathcal{F}(q^{\alpha} \circ \tau_n) \), \( \mathcal{F}(q^{\beta, \nu} \circ \sigma_m) \) and \( \mathcal{F}(q^{\beta} \circ \tau_m) \), and therefore, considering the corresponding inverse images in \( \mathcal{F}(q^{\alpha, \nu} \circ \sigma_n) \), \( \mathcal{F}(q^{\alpha} \circ \tau_n) \), \( \mathcal{F}(q^{\beta, \nu} \circ \sigma_m) \) and \( \mathcal{F}(q^{\beta} \circ \tau_m) \), the \( k^* \)-group homomorphisms 2.5.16 induce \( k^* \)-group homomorphisms (cf. 2.5.8)

\[
\mathcal{F}(q^{\alpha, \nu}) \longrightarrow \mathcal{F}(q^{\alpha}) \quad \text{and} \quad \mathcal{F}(q^{\beta, \nu}) \longrightarrow \mathcal{F}(q^{\beta}) \tag{2.5.17}
\]

More explicitly, we actually have

\[
\mathcal{F}(q^{\alpha, \nu} \circ \sigma_n) = \mathcal{F}(U) = \mathcal{F}(q^{\beta, \nu} \circ \sigma_m) \tag{2.5.18}
\]

and the structural inclusions \( \mathcal{F}(q^{\alpha, \nu}) \subset \mathcal{F}(U) \) and \( \mathcal{F}(q^{\beta, \nu}) \subset \mathcal{F}(U) \) induce an inclusion \( \mathcal{F}(q^{\alpha, \nu}) \subset \mathcal{F}(q^{\alpha, \nu}) \); indeed, an element \( \theta \) in \( \mathcal{F}(q^{\beta, \nu}) \) stabilizes the subgroups \( \hat{\nu} (\beta(\tau(i \cdot m)(\tau(i)))) \) of \( U \) for any \( i \in \Delta_m \), so that it stabilizes

\[
\alpha(q(n)) = \hat{\nu} (\beta(\tau(i \cdot m)(\tau(n)))) \tag{2.5.19}
\]

and therefore \( \theta \) also stabilizes \( C_P (\alpha(q(n))) = C_U (\alpha(q(n))) \); thus, it stabilizes the subgroup \( q^n(n + 1) \) of \( U \) and therefore \( \theta \) belongs to \( \mathcal{F}(q^{\alpha, \nu}) \).
Moreover, we claim that
\[ \left( \text{aut}_{\mathcal{F}}(\text{id}_S, \delta_{m+2}^{m+1}) \right) (\mathcal{F}(\tau^{\beta, \nu})) = \mathcal{F}(\tau^\beta) \] 2.5.20.

Indeed, an element \( \theta \) in \( \mathcal{F}(\tau^\beta) \) acts on \( \beta(\tau(m)) \) determining an automorphism \( \hat{\theta} \) of \( \hat{\nu}(\beta(\tau(m))) \) and, as above, this automorphism stabilizes \( \alpha(q(n)) \) inducing an \( \mathcal{F} \)-morphism
\[ \eta : \alpha(q(n)) \cong \alpha(q(n)) \subset P \] 2.5.21;
but, we are assuming that \( \alpha(q(n)) \) is normal in \( \hat{\nu}(\beta(\tau(m))) \), so that this group is normal in \( \tau^{\beta, \nu}(m+2) \) (cf. 2.5.12). Hence, it follows from [5, statement 2.10.1] that \( \eta \) can be extended to an \( \mathcal{F} \)-morphism \( \hat{\eta} : \tau^{\beta, \nu}(m+2) \to P \); then, the restriction of \( \hat{\eta} \) to \( \hat{\nu}(\beta(\tau(m))) \) and the \( \mathcal{F} \)-morphism
\[ \hat{\nu}(\beta(\tau(m))) \cong \hat{\nu}(\beta(\tau(m))) \subset P \] 2.5.22
coincide over the subgroup \( \alpha(q(n)) \) and therefore, according to [5, Proposition 4.6], these homomorphisms are \( C_P(\alpha(q(n))) \)-conjugate. In conclusion, up to a modification in our choice of \( \hat{\eta} \), we may assume that the restriction of \( \hat{\eta} \) to \( \hat{\nu}(\beta(\tau(m))) \) coincides with \( \hat{\theta} \) and therefore that \( \hat{\eta} \) stabilizes \( \hat{\nu}(\tau^{\beta, \nu}(m+1)) \) and \( \hat{\nu}(\tau^{\beta, \nu}(m+2)) \), so that \( \hat{\eta} \) induces an element of \( \mathcal{F}(\tau^{\beta, \nu}) \) lifting \( \theta \).

Consequently, we have the following commutative diagram
\[
\begin{array}{c}
\mathcal{F}(U) \supset \mathcal{F}(q^{\alpha, \nu}) \supset \mathcal{F}(q^\alpha) \supset \mathcal{F}(q) \\
\mathcal{F}(U) \supset \mathcal{F}(\tau^{\beta, \nu}) \supset \mathcal{F}(\tau^\beta) \supset \mathcal{F}(\tau)
\end{array}
\]
2.5.23;

Moreover, since \( q^\alpha(n+1) \) and \( \tau^\beta(m+1) \) are \( \mathcal{F} \)-selfcentralizing, the kernels of the compositions of the horizontal arrows are \( \mathcal{F}_{C_U(\alpha(q(n)))}(U) \) for the top and \( \mathcal{F}_{C_U(\hat{\nu}(\beta(\tau(m))))}(U) \) for the bottom, and the bottom composition is surjective; hence, since \( \mathcal{F}_{C_U(\hat{\nu}(\beta(\tau(m))))}(U) \) is contained in \( \mathcal{F}_{C_U(\alpha(q(n)))}(U) \) and they respectively lift canonically to \( \hat{\mathcal{F}}(\tau^{\beta, \nu}) \) and to \( \hat{\mathcal{F}}(q^{\alpha, \nu}) \) [5, Corollaire 4.7], we get a unique \( k^* \)-group homomorphism
\[ \hat{\text{aut}}_{\mathcal{F}}(\nu, \delta) : \hat{\mathcal{F}}(\nu) \longrightarrow \hat{\mathcal{F}}(q) \] 2.5.24
lifting \( \text{aut}_{\mathcal{F}}(\nu, \delta) \) and such that the corresponding diagram of \( k^* \)-group homomorphisms
\[
\begin{array}{c}
\hat{\mathcal{F}}(U) \supset \hat{\mathcal{F}}(q^{\alpha, \nu}) \supset \hat{\mathcal{F}}(q^\alpha) \supset \hat{\mathcal{F}}(q) \\
\hat{\mathcal{F}}(U) \supset \hat{\mathcal{F}}(\tau^{\beta, \nu}) \supset \hat{\mathcal{F}}(\tau^\beta) \supset \hat{\mathcal{F}}(\tau)
\end{array}
\]
2.5.25
is commutative.
Consider another $\mathfrak{ch}^{*}(\mathcal{F}_{\vartheta})$-morphism $(\mu, \varepsilon): (\ell, \Delta_{\ell}) \to (\tau, \Delta_{\tau})$, so that
\[(\nu, \delta) \circ (\mu, \varepsilon) = (\nu \circ (\mu \ast \delta), \varepsilon \circ \delta)\] 2.5.26
and set $\lambda = \nu \circ (\mu \ast \delta)$ and $\phi = \varepsilon \circ \delta$; then, choosing a suitable $\mathcal{F}$-morphism $\gamma: t(\ell) \to P$ as above, we still assume that the images of $t(\varphi(n))$ via $t(\varphi(n) \bullet \ell)$ and of $t(\varepsilon(m))$ via $t(\varepsilon(m) \bullet \ell)$ are normal in $t(\ell)$. In particular, this implies that the image of $t(\delta(n))$ via $t(\delta(n) \bullet m)$ is normal in $t(m)$; that is to say, we have already defined the $k^{*}$-group homomorphisms $\hat{\text{aut}}_{\mathcal{F}_{\vartheta}}(\nu, \delta)$, $\hat{\text{aut}}_{\mathcal{F}_{\vartheta}}(\mu, \varepsilon)$ and $\hat{\text{aut}}_{\mathcal{F}_{\vartheta}}(\lambda, \phi)$ respectively lifting $\text{aut}_{\mathcal{F}_{\vartheta}}(\nu, \delta)$, $\text{aut}_{\mathcal{F}_{\vartheta}}(\mu, \varepsilon)$ and $\text{aut}_{\mathcal{F}_{\vartheta}}(\lambda, \phi)$ and we want to prove that
\[\hat{\text{aut}}_{\mathcal{F}_{\vartheta}}(\lambda, \phi) = \hat{\text{aut}}_{\mathcal{F}_{\vartheta}}(\nu, \delta) \circ \hat{\text{aut}}_{\mathcal{F}_{\vartheta}}(\mu, \varepsilon)\] 2.5.27.

More explicitly, applying the construction in 2.5.9 above to the $\mathfrak{ch}^{*}(\mathcal{F}_{\vartheta})$-morphisms $(\nu, \delta)$, $(\mu, \varepsilon)$ and $(\phi, \lambda)$, we get $\mathcal{F}$-morphisms
\[\hat{\nu}: r^{\beta}(m + 1) \longrightarrow N_{P}\left(\alpha(q(n))\right)\]
\[\hat{\mu}: r^{\gamma}(\ell + 1) \longrightarrow N_{P}\left(\beta(\tau(m))\right)\]
\[\hat{\lambda}: r^{\gamma}(\ell + 1) \longrightarrow N_{P}\left(\alpha(q(n))\right)\] 2.5.28;

actually, it is clear that the respective images of $\hat{\nu}$, $\hat{\mu}$ and $\hat{\lambda}$ are respectively contained in $q^{\alpha}(n + 1)$, $r^{\beta}(m + 1)$ and $q^{\alpha}(n + 1)$ and, with evident notation, our construction can be explicit in the following commutative diagram
\[
\begin{array}{ccc}
t(\ell) & \cong & t(\ell) \\
\downarrow & & \downarrow \lambda \\
t(\varepsilon(m)) & \cong & t(\varepsilon(m)) \\
\downarrow & & \downarrow \mu \\
t(\varphi(n)) & \cong & t(\varphi(n)) \\
\end{array}
\]
\[
\begin{array}{ccc}
\gamma(t(\ell)) & \subset & t^{\gamma}(\ell + 1) \\
\| & & \| \\
\beta(t(m)) & \subset & r^{\beta}(m + 1) \\
\| & & \| \\
\alpha(q(n)) & \cong & q^{\alpha}(n + 1) \\
\end{array}
\]

That is to say, according to 2.5.10 above, $\hat{\lambda}$, $\hat{\mu}$ and $\hat{\nu}$ respectively extend the $\mathcal{F}$-morphisms
\[\gamma\left(t(\varphi(n) \bullet \ell)\left(t(\varphi(n))\right)\right) \cong t(\varphi(n)) \cong q(n) \cong \alpha(q(n)) \subset P\]
\[\gamma\left(t(\varepsilon(m) \bullet \ell)\left(t(\varepsilon(m))\right)\right) \cong t(\varepsilon(m)) \cong \tau(m) \cong \beta(\tau(m)) \subset P\] 2.5.30
\[\beta\left(t(\delta(n) \bullet m)\left(t(\delta(n))\right)\right) \cong \tau(\delta(n)) \cong q(n) \cong \alpha(q(n)) \subset P\]
and, since \( \beta \left( \tau(\delta(n) \cdot m) \right) \) is contained in \( \beta(\tau(m)) \), it is easily checked that the composition \( \hat{\nu} \circ \hat{\mu} \) also extends the top \( \mathcal{F} \)-morphism in 2.5.30; then, as above, it follows from [5, Proposition 4.6] that \( \hat{\lambda} \) and \( \hat{\nu} \circ \hat{\mu} \) are \( C_P \left( \alpha(q(n)) \right) \)-conjugate; actually, up to a modification of our choice of \( \hat{\lambda} \), we may assume that they coincide.

Moreover, we have to consider chains

\[
\begin{align*}
q^{\alpha,\nu,\lambda} & : \Delta_{n+3} \to \mathcal{F}^g \\
\tau^{\beta,\mu,\nu} & : \Delta_{m+3} \to \mathcal{F}^g \\
t^{\gamma,\mu,\nu} & : \Delta_{\ell+3} \to \mathcal{F}^g
\end{align*}
\]

respectively extending the chains \( q^{\alpha,\nu} \), \( \tau^{\beta,\mu} \) and \( t^{\gamma,\mu} \); recall that (cf. 2.5.12)

\[
\begin{align*}
q^{\alpha,\nu} & (n + 2) = \hat{\nu} \left( \beta(\tau(m)) \right) \cdot C_P \left( \alpha(q(n)) \right) \\
\tau^{\beta,\mu} & (m + 2) = \hat{\mu} \left( \gamma(t(\ell)) \right) \cdot C_P \left( \beta(\tau(m)) \right) = t^{\gamma,\mu}(\ell + 2)
\end{align*}
\]

and that, according to our remark above and since we assume that \( \hat{\lambda} = \hat{\nu} \circ \hat{\mu} \), we still have

\[
q^{\alpha,\lambda} (n + 2) = \hat{\nu} \left( \hat{\mu} \left( \gamma(t(\ell)) \right) \right) \cdot C_P \left( \alpha(q(n)) \right)
\]

thus, since \( \beta(\tau(m)) \subset \hat{\mu} \left( \gamma(t(\ell)) \right) \), we get \( q^{\alpha,\nu}(n + 2) \subset q^{\alpha,\lambda}(n + 2) \) and, since the centralizer of \( \alpha(q(n)) \) contains the centralizer of \( \hat{\nu} \left( \beta(\tau(m)) \right) \), \( \hat{\nu} \) induces an \( \mathcal{F} \)-morphism

\[
\tau^{\beta,\mu}(m + 2) = t^{\gamma,\mu}(\ell + 2) \to q^{\alpha,\lambda}(n + 2)
\]

then, we complete our definition of \( q^{\alpha,\nu,\lambda} \), \( \tau^{\beta,\mu,\nu} \) and \( t^{\gamma,\mu,\nu} \) by setting

\[
q^{\alpha,\nu,\lambda}(n + 3) = \tau^{\beta,\mu,\nu}(m + 3) = t^{\gamma,\mu,\nu}(\ell + 3) = q^{\alpha,\lambda}(n + 2)
\]

and respectively mapping \( (n + 2 \cdot n + 3) \), \( (m + 2 \cdot m + 3) \) and \( (\ell + 2 \cdot \ell + 3) \) on the inclusion \( q^{\alpha,\nu}(n + 2) \subset q^{\alpha,\lambda}(n + 2) \) and on the \( \mathcal{F} \)-morphism 2.5.34 induced by \( \hat{\nu} \).

Now, it is clear that the functor \( \text{aut}_{\mathcal{F}^g} \) applied to the obvious \( \mathfrak{g}^*(\mathcal{F}^g) \)-morphisms

\[
\begin{align*}
(id_{q^{\alpha,\nu}}, \delta_{n+3}^{\alpha,\nu}) : (q^{\alpha,\nu,\lambda}, \Delta_{n+3}) & \to (q^{\alpha,\nu}, \Delta_{n+2}) \\
(id_{\tau^{\beta,\mu}}, \delta_{m+3}^{\beta,\mu}) : (\tau^{\beta,\mu,\nu}, \Delta_{m+3}) & \to (\tau^{\beta,\mu}, \Delta_{m+2}) \\
(id_{t^{\gamma,\mu}}, \delta_{m+3}^{\gamma,\mu}) : (t^{\gamma,\mu,\nu}, \Delta_{m+3}) & \to (t^{\gamma,\mu}, \Delta_{m+2})
\end{align*}
\]

2.5.35.
yields group homomorphisms
\[ F(\alpha, \nu, \lambda) \rightarrow F(\alpha, \nu), \quad F(\beta, \mu, \nu) \rightarrow F(\beta, \mu), \quad F(\gamma, \mu, \nu) \rightarrow F(\gamma, \mu) \]
\[ 2.5.37; \]
as in 2.5.16 above, considering the maps
\[ \hat{\sigma}_n : \Delta_1 \rightarrow \Delta_{n+3} \quad \text{and} \quad \hat{\tau}_n : \Delta_0 \rightarrow \Delta_{n+2} \]
\[ \hat{\sigma}_m : \Delta_1 \rightarrow \Delta_{m+3} \quad \text{and} \quad \hat{\tau}_m : \Delta_0 \rightarrow \Delta_{m+2} \]
\[ \hat{\sigma}_\ell : \Delta_1 \rightarrow \Delta_{\ell+3} \quad \text{and} \quad \hat{\tau}_\ell : \Delta_0 \rightarrow \Delta_{\ell+2} \]
respectively sending \( i \) to \( i + n + 2 \), to \( i + m + 2 \) and to \( i + \ell + 2 \), the functor \( \hat{\text{aut}}_{\mathcal{F}^*} \) still induces \( k^* \)-group homomorphisms
\[ \hat{F}(\alpha, \nu, \lambda) \rightarrow \hat{F}(\alpha, \nu), \quad \hat{F}(\beta, \mu, \nu) \rightarrow \hat{F}(\beta, \mu), \quad \hat{F}(\gamma, \mu, \nu) \rightarrow \hat{F}(\gamma, \mu) \]
\[ 2.5.39; \]
moreover it is quite clear that \( \hat{F}(\gamma, \mu, \nu) = \hat{F}(\gamma, \nu) \). Consequently, the functoriality of \( \hat{\text{aut}}_{\mathcal{F}^*} \) guarantees the commutativity of the following diagram
\[ \begin{array}{c}
\hat{F}(\alpha, \mu, \lambda) \rightarrow \hat{F}(\alpha, \mu, \nu) \\
\hat{F}(\beta, \mu, \nu) \rightarrow \hat{F}(\beta, \mu, \nu) \\
\hat{F}(\gamma, \mu, \nu) \rightarrow \hat{F}(\gamma, \mu, \nu)
\end{array} \]
\[ 2.5.40; \]
thus, by uniqueness, in this case we obtain
\[ \hat{\text{aut}}_{\mathcal{F}^*}(\nu, \delta) \circ \hat{\text{aut}}_{\mathcal{F}^*}(\mu, \varepsilon) = \hat{\text{aut}}_{\mathcal{F}^*}((\nu, \delta) \circ (\mu, \varepsilon)) \]
\[ 2.5.41. \]

Secondly, assume that the image of \( r(\delta(n)) \) by \( r(\delta(n) \bullet m) \) is not normal in \( r(m) \); let \( m' \) be the maximal element in \( \Delta_m - \Delta_{\delta(n)-1} \) such that the image of \( r(\delta(n)) \) by \( r(\delta(n) \bullet m') \) is normal in \( r(m') \) and denote by \( R_{(\nu, \delta)} \) the
normalizer of the image of $\tau(\delta(n))$ in $\tau(m' + 1)$, by $\tau(\nu, \delta) \colon \Delta_{m+1} \to \mathcal{F}^\circ$ the functor fulfilling
\[\tau(\nu, \delta) \circ \delta^m_{m+1} = \tau \quad \text{and} \quad \tau(\nu, \delta)(m' + 1) = R(\nu, \delta)\]
and mapping $(m' + 1 \cdot m' + 2)$ on the inclusion map $R(\nu, \delta) \to \tau(m' + 1)$, and by $\tau'(\nu, \delta)$ the restriction of $\tau(\nu, \delta)$ to $\Delta_{m'+1}$; then, it is quite clear that $\mathcal{F}(\tau(\nu, \delta)) = \mathcal{F}(\tau)$ and it is easily checked that $\hat{\mathcal{F}}(\tau(\nu, \delta)) = \hat{\mathcal{F}}(\tau)$; moreover, we have an evident $\text{ch}^*(\mathcal{F}^\circ)$-morphism
\[(\nu', \delta') : (\tau'(\nu, \delta), \Delta_{m'+1}) \to (q, \Delta_n)\]
such that
\[(\nu', \delta') \circ (\text{id}_{\nu'(\nu, \delta)}, \pi^m_{m'}) = (\nu, \delta) \circ (\text{id}_{\nu}, \delta^m_{m'+1})\]
where $\pi^m_{m'} : \Delta_{m'+1} \to \Delta_{m+1}$ denotes the natural inclusion, we clearly have $\text{aut}_{\mathcal{F}^\circ} (\text{id}_{\nu}, \delta^m_{m'+1}) = \hat{\mathcal{F}}(\tau)$ and in 2.5.4 above we have already defined $\text{aut}_{\mathcal{F}^\circ} (\nu', \delta')$; on the other hand, arguing by induction on $|\tau(m)||q(n)|$, we may assume that $\text{aut}_{\mathcal{F}^\circ} (\text{id}_{\nu'}, \delta')$ is already defined and then we set
\[\hat{\text{aut}}_{\mathcal{F}^\circ} (\nu, \delta) = \hat{\text{aut}}_{\mathcal{F}^\circ} (\nu', \delta') \circ \hat{\text{aut}}_{\mathcal{F}^\circ} (\text{id}_{\nu'}, \delta')\]

For another $\text{ch}^*(\mathcal{F}^\circ)$-morphism $(\mu, \varepsilon) : (t, \Delta_\ell) \to (\tau, \Delta_m)$, we claim that
\[\hat{\text{aut}}_{\mathcal{F}^\circ} (\mu, \varepsilon) = \hat{\text{aut}}_{\mathcal{F}^\circ} (\nu, \delta) \circ (\mu, \varepsilon)\]

we argue by induction firstly on $|t(\ell)|/|q(n)|$ and after on $|t(\ell)|/|\tau(m)|$. First of all, we assume that the image of $\tau(\delta(n))$ in $\tau(m)$ by $\tau(\delta(n) \cdot m)$ is not normal; with the notation above, denote by $\ell'$ the maximal element in $\Delta_{m - \Delta(\tau(\delta))} \cdot 1 \cdot \varepsilon(n)$ such that the image of $\varepsilon((\tau(\delta(n)))$ in $t(\ell')$ is normal in $t(\ell')$; then, it is clear that $\varepsilon(m') \leq \ell' < \varepsilon(m)$ and easily checked that we have a $\text{ch}^*(\mathcal{F}^\circ)$-morphism
\[(\mu(\nu, \delta), \varepsilon(\nu, \delta)) : (t(\nu, \delta) \circ (\mu, \varepsilon), \Delta_{\ell+1}) \to (\tau(\nu, \delta), \Delta_{m+1})\]
such that
\[(\text{id}_{\nu}, \delta^m_{m'+1}) \circ (\mu(\nu, \delta), \varepsilon(\nu, \delta)) = (\mu, \varepsilon) \circ (\text{id}_{\nu}, \delta^m_{m'+1})\]
that $\varepsilon(\nu, \delta)(m' + 1) = \ell' + 1$ and that $(\mu(\nu, \delta))_{m'+1}$ from $t(\nu, \delta) \circ (\mu, \varepsilon)(\ell' + 1)$ to $\tau(\nu, \delta)(m'+1)$ is determined by $\mu_{m'+1}$ and $t(\ell' + 1 \cdot \varepsilon(m' + 1))$; moreover, we consider the corresponding restriction
\[(\mu'(\nu, \delta), \varepsilon'(\nu, \delta)) : (t'(\nu, \delta) \circ (\mu, \varepsilon), \Delta_{\ell'+1}) \to (\tau'(\nu, \delta), \Delta_{m'+1})\]
which obviously fulfills
\[(\text{id}_{\nu,\delta}, \ell_{\nu,\delta}) \circ (\mu_{\nu,\delta}, \varepsilon_{\nu,\delta}) = (\mu'_{\nu,\delta}, \varepsilon'_{\nu,\delta}) \circ (\text{id}_{\nu,\delta}, \ell_{\nu,\delta}) \] 2.5.50.

Now, it is easily checked that the composition \((\nu', \delta') \circ (\mu'_{\nu,\delta}, \varepsilon'_{\nu,\delta})\) coincides with the corresponding morphism 2.5.43 for the \(\text{ch}^*(\mathcal{F})\)-morphism \((\nu, \delta) \circ (\mu, \varepsilon)\) and therefore, by the very definition 2.5.45, we have
\[\widehat{\text{aut}}_{\mathcal{F}}((\nu, \delta) \circ (\mu, \varepsilon)) = \widehat{\text{aut}}_{\mathcal{F}}((\nu', \delta') \circ (\mu'_{\nu,\delta}, \varepsilon'_{\nu,\delta})) \circ \widehat{\text{aut}}_{\mathcal{F}}((\text{id}_{\nu,\delta}, \ell_{\nu,\delta})) \] 2.5.51;

but, since \(|R_{\nu,\delta}|/|q(n)| < |t(\ell)|/|q(n)|\), it follows from the induction hypothesis that
\[\widehat{\text{aut}}_{\mathcal{F}}((\nu', \delta') \circ (\mu'_{\nu,\delta}, \varepsilon'_{\nu,\delta})) = \widehat{\text{aut}}_{\mathcal{F}}((\nu', \delta') \circ \widehat{\text{aut}}_{\mathcal{F}}((\mu'_{\nu,\delta}, \varepsilon'_{\nu,\delta})) \] 2.5.52;

similarly, since we have \(|t(\ell)|/|R_{\nu,\delta}| < |t(\ell)|/|q(n)|\) and
\[\widehat{\text{aut}}_{\mathcal{F}}((\mu_{\nu,\delta}, \varepsilon_{\nu,\delta})) = \widehat{\text{aut}}_{\mathcal{F}}((\mu, \varepsilon)) \] 2.5.53,
we still get
\[\widehat{\text{aut}}_{\mathcal{F}}((\nu, \delta) \circ (\mu, \varepsilon)) = \widehat{\text{aut}}_{\mathcal{F}}((\nu', \delta') \circ (\mu'_{\nu,\delta}, \varepsilon'_{\nu,\delta}) \circ \widehat{\text{aut}}_{\mathcal{F}}((\text{id}_{\nu,\delta}, \ell_{\nu,\delta})) \] \[= \widehat{\text{aut}}_{\mathcal{F}}((\nu', \delta') \circ \widehat{\text{aut}}_{\mathcal{F}}((\text{id}_{\nu,\delta}, \ell_{\nu,\delta}) \circ (\mu_{\nu,\delta}, \varepsilon_{\nu,\delta}))) \] \[= \widehat{\text{aut}}_{\mathcal{F}}((\nu, \delta) \circ (\mu, \varepsilon)) \] 2.5.54.

Finally, we may assume that the image of \(r(\delta(n))\) by \(r(\delta(n) \bullet m)\) is normal in \(r(m)\), so that the image of \(t(\varepsilon \circ \delta(n))\) by \(t((\varepsilon \circ \delta(n) \bullet \varepsilon(m))\) is normal in \(t(\varepsilon(m))\); in particular, denoting by \(\ell'\) the maximal element in \(\Delta_l - \Delta_{(\varepsilon(n)) \bullet 1}\) such that the image of \(t(\varepsilon \circ \delta(n))\) by \(t((\varepsilon \circ \delta(n) \bullet \ell')\) is normal in \(t(\ell')\), we have \(\varepsilon(m) \leq \ell'\). If \(\ell' = \ell\) then, by 2.5.41, we may assume that the image of \(t(\varepsilon(m))\) by \(t(\varepsilon(m) \bullet \ell'')\) is normal in \(t(\ell'')\), by our very definition (cf. 2.5.45) we have
\[\widehat{\text{aut}}_{\mathcal{F}}((\mu, \varepsilon)) = \widehat{\text{aut}}_{\mathcal{F}}((\mu', \varepsilon') \circ \widehat{\text{aut}}_{\mathcal{F}}((\text{id}_{\nu,\delta}, \ell_{\nu,\delta})) \] 2.5.55;

but, according to equality 2.5.41, we have
\[\widehat{\text{aut}}_{\mathcal{F}}((\nu, \delta) \circ \widehat{\text{aut}}_{\mathcal{F}}((\mu', \varepsilon')) = \widehat{\text{aut}}_{\mathcal{F}}((\nu, \delta) \circ (\mu', \varepsilon')) \] 2.5.56.
Consequently, since in the compositions of \((\nu, \delta)\) with \((\mu, \varepsilon)\) and of \(((\nu, \delta) \circ (\mu', \varepsilon'))\) with \((\text{id}_{\ell_{(\nu, \delta)}}, \ell_{(\mu', \varepsilon')})\) the first induction indices coincide with each other and the second ones strictly decrease, it follows from the induction hypothesis that

\[
\widetilde{\text{aut}}_{F^\Psi} (\nu, \delta) \circ \widetilde{\text{aut}}_{F^\Psi} (\mu, \varepsilon)
= \widetilde{\text{aut}}_{F^\Psi} (\nu, \delta) \circ \widetilde{\text{aut}}_{F^\Psi} (\mu', \varepsilon') \circ \widetilde{\text{aut}}_{F^\Psi} (\text{id}_{\ell_{(\mu', \varepsilon')}}, \ell_{(\mu', \varepsilon')}^\ell) \quad 2.5.57.
\]

In any case, we have a \(\text{ch}^*(F^\Psi)\)-morphism

\[
(\mu_{(\nu, \delta)}, \varepsilon_{(\nu, \delta)}') : (\ell_{(\nu, \delta)\circ (\mu, \varepsilon)}, \Delta_{\nu} + 1) \longrightarrow (\tau, \Delta_m) \quad 2.5.58
\]

fulfilling

\[
(\mu_{(\nu, \delta)}, \varepsilon_{(\nu, \delta)}') \circ (\text{id}_{\ell_{(\nu, \delta)\circ (\mu, \varepsilon)}}, \ell_{\nu}^\ell) = (\mu, \varepsilon) \circ (\text{id}_{\ell_{\nu}^\ell}, \delta_{\nu} + 1) \quad 2.5.59;
\]

as above, it is easily checked that the composition \((\nu, \delta) \circ (\mu_{(\nu, \delta)}, \varepsilon_{(\nu, \delta)}')\) coincides with the corresponding morphism 2.5.43 for the \(\text{ch}^*(F^\Psi)\)-morphism \((\nu, \delta) \circ (\mu, \varepsilon)\) and therefore, by the very definition 2.5.45, we have

\[
\widetilde{\text{aut}}_{F^\Psi} (\nu, \delta) \circ (\mu, \varepsilon)
= \widetilde{\text{aut}}_{F^\Psi} (\nu, \delta) \circ (\mu_{(\nu, \delta)}, \varepsilon_{(\nu, \delta)}') \circ \widetilde{\text{aut}}_{F^\Psi} (\text{id}_{\ell_{(\nu, \delta)\circ (\mu, \varepsilon)}}, \ell_{\nu}^\ell) \quad 2.5.60;
\]

since \(\widetilde{\text{aut}}_{F^\Psi} (\text{id}_{\ell_{\nu}^\ell}, \delta_{\nu} + 1) = \text{id}_{F^\Psi} (\ell_{\nu}^\ell)\) and we may assume that \(\ell' \neq \ell\), it follows from the induction hypothesis applied to the composition of \((\nu, \delta)\) with \((\mu_{(\nu, \delta)}, \varepsilon_{(\nu, \delta)}')\) that

\[
\widetilde{\text{aut}}_{F^\Psi} (\nu, \delta) \circ (\mu_{(\nu, \delta)}, \varepsilon_{(\nu, \delta)}') = \widetilde{\text{aut}}_{F^\Psi} (\nu, \delta) \circ \widetilde{\text{aut}}_{F^\Psi} (\mu_{(\nu, \delta)}, \varepsilon_{(\nu, \delta)}) \quad 2.5.61;
\]

moreover, if \(|q(n)| < |r(m)|\), we can apply the induction hypothesis to both members of equality 2.5.59 and then we get

\[
\widetilde{\text{aut}}_{F^\Psi} (\mu_{(\nu, \delta)}, \varepsilon_{(\nu, \delta)}') \circ \widetilde{\text{aut}}_{F^\Psi} (\text{id}_{\ell_{(\nu, \delta)\circ (\mu, \varepsilon)}}, \ell_{\nu}^\ell) = \widetilde{\text{aut}}_{F^\Psi} (\mu, \varepsilon) \quad 2.5.62.
\]

Consequently, once again we have

\[
\widetilde{\text{aut}}_{F^\Psi} ((\nu, \delta) \circ (\mu, \varepsilon)) = \widetilde{\text{aut}}_{F^\Psi} ((\nu, \delta) \circ \text{id}_{(\mu, \varepsilon)}) \quad 2.5.63.
\]
If \(|q(n)| = |r(m)|\) then it follows from the definitions of \(\hat{\text{aut}}_{\mathcal{F}^\nu}(\mu, \varepsilon)\) and of \(\hat{\text{aut}}_{\mathcal{F}^\nu}((\nu, \delta) \circ (\mu, \varepsilon))\) (cf. 2.5.45) that \(\ell'\) coincides with both induction indices, that we get \(t'_\nu = t'_\nu \circ (\mu, \varepsilon)\) and that the homomorphism 2.5.43

\[
(t'_\nu \circ (\mu, \varepsilon), \Delta e + 1) \rightarrow (q, \Delta n)
\]

2.5.64

corresponding to the composition \((\nu, \delta) \circ (\mu, \varepsilon)\) coincides with \((\nu, \delta) \circ (\mu', \varepsilon')\); at this point, we can apply equality 2.5.41 to obtain

\[
\hat{\text{aut}}_{\mathcal{F}^\nu}((\nu, \delta) \circ (\mu, \varepsilon)) = \hat{\text{aut}}_{\mathcal{F}^\nu}((\nu, \delta) \circ (\mu', \varepsilon'))
\]

2.5.65;

then, composing this equality with \(\hat{\text{aut}}_{\mathcal{F}^\nu}(id_{(\mu, \varepsilon)}, \ell'_\nu)\), from definition 2.5.45 we get

\[
\hat{\text{aut}}_{\mathcal{F}^\nu}((\nu, \delta) \circ (\mu, \varepsilon)) = \hat{\text{aut}}_{\mathcal{F}^\nu}((\nu, \delta) \circ (\mu, \varepsilon))
\]

2.5.66.

We are done.

Theorem 2.6.[6, Theorem 2.5] Any functor \(\hat{\text{aut}}_{\mathcal{F}^\nu}\) lifting \(\text{aut}_{\mathcal{F}^\nu}\) to the category \(k^*-\mathfrak{Gc}\) can be extended to a unique folder structure of \(\mathcal{F}\).

Theorem 2.7.[5, Theorem 11.32] The Frobenius \(P\)-category \(\mathcal{F}_{(b, G)}\) associated with a block \(b\) of a finite group \(G\) has a unique isomorphism class of folded structures admitting a \(k^*-\)group isomorphism

\[
\hat{\text{aut}}_{\mathcal{F}^\nu}(q_Q) \cong \tilde{N}_G(Q, f)/C_G(Q)
\]

2.7.1

for any \(\mathcal{F}_{(b, G)}\)-selfcentralizing subgroup \(Q\) of \(P\).

2.8. An obvious way for getting a folded structure of \(\mathcal{F}\) is to start with a regular central \(k^*-\)extension \(\bar{\mathcal{F}}^\nu\) of \(\mathcal{F}\); indeed, in this case it follows again from [5, Proposition A2.10] that we have a canonical functor

\[
\text{aut}_{\mathcal{F}^\nu} : \text{ch}^*(\bar{\mathcal{F}}^\nu) \rightarrow k^*-\mathfrak{Gc}
\]

2.8.1

mapping any \(\bar{\mathcal{F}}^\nu\)-chain \(q : \Delta_n \rightarrow \bar{\mathcal{F}}^\nu\) to the stabilizer \(\bar{\mathcal{F}}^\nu(q)\) in \(\bar{\mathcal{F}}^\nu(q(n))\) of all the subgroups \(\text{Im}(q(i \bullet n))\) for \(i \in \Delta_n\), where \(q : \Delta_n \rightarrow \bar{\mathcal{F}}^\nu\) denotes the corresponding \(\mathcal{F}\)-chain; then, this functor factorizes throughout a folder structure of \(\mathcal{F}\)

\[
\hat{\text{aut}}_{\mathcal{F}^\nu} : \text{ch}^*(\bar{\mathcal{F}}^\nu) \rightarrow k^*-\mathfrak{Gc}
\]

2.8.2.

Conversely, our main purpose here is to prove that any folder structure of \(\mathcal{F}\) comes from a regular central \(k^*-\)extension \(\bar{\mathcal{F}}^\nu\) of \(\mathcal{F}\); consequently, once this result was obtained, to consider a folded Frobenius \(P\)-category is equivalent to consider a pair \((\mathcal{F}, \bar{\mathcal{F}}^\nu)\) formed by a Frobenius \(P\)-category \(\mathcal{F}\) and by a regular central \(k^*-\)extension \(\bar{\mathcal{F}}^\nu\) of \(\bar{\mathcal{F}}^\nu\).
2.9. On the other hand, in [1], [2], [7] and [8] it has been recently proved that there exists a unique perfect $\mathcal{F}^{ac}$-locality $\mathcal{P}^{ac}$ [5, 17.4 and 17.13]. More explicitly, denote by $\mathcal{T}_P$ the category where the objects are all the $\mathcal{F}$-self-centralizing subgroups of $P$ and, for a pair of $\mathcal{F}$-selfcentralizing subgroups $Q$ and $R$ of $P$, the set of morphisms from $R$ to $Q$ is the $P$-transporter $\mathcal{T}_P(R,Q)$, the composition being induced by the product in $P$; then [8, §4]

2.9.1 there is a unique Abelian extension $\pi^*: \mathcal{P}^{ac} \to \mathcal{F}^{ac}$ of $\mathcal{F}^{ac}$ endowed with a functor $\tau^*: \mathcal{T}_P \to \mathcal{P}^{ac}$ in such a way that the composition $\pi^* \circ \tau^*$ is the canonical functor defined by the conjugation in $P$, that $\mathcal{P}^{ac}(Q)$ is an $\mathcal{F}$-localizer of $Q$ [5, Theorem 18.6] and that $\mathcal{P}^{ac}$ acts regularly over the fibers of the map $\mathcal{P}^{ac}(Q,R) \to \mathcal{F}^{ac}(Q,R)$ induced by $\pi^*$ [5, 17.7], for any pair of $\mathcal{F}$-selfcentralizing subgroups $Q$ and $R$ of $P$.

2.10. Presently, the so-called $\mathcal{F}$-localizing functor considered in [6, 3.2.1]

\[ \mathfrak{lo}_{\mathcal{F}^{ac}} : \mathfrak{ch}^*(\mathcal{F}^{ac}) \to \widehat{\mathfrak{Lo}}_{\mathcal{F}^{ac}} \] 2.10.1

is just a quotient of the canonical functor [5, Proposition A2.10]

\[ \mathfrak{aut}_{\mathcal{P}^{ac}} : \mathfrak{ch}^*(\mathcal{P}^{ac}) \to \mathfrak{Gr} \] 2.10.2.

Moreover, any regular central $k^*$-extension $\hat{\mathcal{F}}^{ac}$ of $\mathcal{F}^{ac}$ determines via $\pi^*$ a regular central $k^*$-extension $\hat{\mathcal{P}}^{ac}$ of $\mathcal{P}^{ac}$; then, the corresponding functor

\[ \hat{\mathfrak{lo}}_{\mathcal{F}^{ac}} : \mathfrak{ch}^*(\hat{\mathcal{F}}^{ac}) \to k^* \cdot \hat{\Lo}_{\mathcal{F}^{ac}} \] 2.10.3

considered in [6, 3.3.1] is just a quotient of the obvious canonical functor [5, Proposition A2.10]

\[ \mathfrak{aut}_{\mathcal{P}^{ac}} : \mathfrak{ch}^*(\hat{\mathcal{P}}^{ac}) \to k^* \cdot \mathfrak{Gr} \] 2.10.4.

Actually, it is clear that $\pi^*$ induces an equivalence between the so-called exterior quotients $\hat{\mathcal{F}}^{ac}$ of $\mathcal{F}^{ac}$ and $\hat{\mathcal{P}}^{ac}$ of $\mathcal{P}^{ac}$ [5, 1.3]; that is to say, the quotients of $\mathcal{F}^{ac}$ and $\mathcal{P}^{ac}$ by the inner automorphisms of the objects are just isomorphic and, in particular, the regular central $k^*$-extensions of $\hat{\mathcal{F}}^{ac}$, $\mathcal{F}^{ac}$ and $\mathcal{P}^{ac}$ are clearly in bijective correspondence. In particular, a folder structure in $\mathcal{F}$ is equivalent to a functor

\[ \hat{\mathfrak{aut}}_{\mathcal{P}^{ac}} : \mathfrak{ch}^*(\mathcal{P}^{ac}) \to k^* \cdot \mathfrak{Gr} \] 2.10.5

lifting the canonical functor $\mathfrak{aut}_{\mathcal{P}^{ac}}$.

3. Regular central $k^*$-extensions of $\mathcal{F}^{ac}$

3.1. Let $(\mathcal{F}, \hat{\mathfrak{ut}}_{\mathcal{F}^{ac}})$ be a folded Frobenius $P$-category (cf. 2.4) and denote by $\mathcal{P}$ and $\mathcal{P}^{ac}$ the respective perfect $\mathcal{F}$- and $\mathcal{F}^{ac}$-localities [7, §6 and §7] and by $\pi: \mathcal{P} \to \mathcal{F}$ and $\tau: \mathcal{T}_P \to \mathcal{P}$ the structural functors [5, 17.3]. Our main purpose is to show that $(\mathcal{F}, \hat{\mathfrak{ut}}_{\mathcal{F}^{ac}})$ or, equivalently, $(\mathcal{P}, \hat{\mathfrak{ut}}_{\mathcal{P}^{ac}})$ (cf. 2.10.5) is determined by a regular central $k^*$-extension $\hat{\mathcal{P}}^{ac}$ of $\mathcal{P}^{ac}$; we choose to work on $\mathcal{P}^{ac}$ rather than on $\mathcal{F}^{ac}$, which is equivalent as mentioned above, since in $\mathcal{P}^{ac}$ all the morphisms are monomorphisms and epimorphisms [5, Proposition 24.2].
3.2. In particular, if $Q$ and $Q'$ are $\mathcal{F}$-isomorphic $\mathcal{F}$-selfcentralizing subgroups of $P$, for any pair of $\mathcal{F}$-selfcentralizing subgroups $R$ of $Q$ and $R'$ of $Q'$ condition 2.1.1 in $\mathcal{F}$ induces an injective restriction map

$$r_{R,R'}^{Q,Q'} : \mathcal{P}(Q',Q)_{R',R} \rightarrow \mathcal{P}(R',R)$$

where $\mathcal{P}(Q',Q)_{R',R}$ denotes the set of $x \in \mathcal{P}(Q',Q)$ such that $\pi_{Q',Q}(x)$ maps $R$ on $R'$; in particular, we may identify the stabilizer $\mathcal{P}(Q)_R$ of $R$ in $\mathcal{P}(Q)$ with a subgroup of $\mathcal{P}(R)$. First of all, note the following consequence of condition 2.1.3.

**Lemma 3.3.** With the notation above, assume that $R$ and $R'$ are $\mathcal{F}$-isomorphic and fully normalized in $\mathcal{F}$; set $N = N_P(R)$ and $N' = N_P(R')$. Then the restriction map and the composition induce a bijection

$$\mathcal{P}(N',N)_{R',R} \times \mathcal{P}(N)_R \mathcal{P}(R) \cong \mathcal{P}(R',R)$$

**Proof:** It is clear that, for any $x \in \mathcal{P}(N',N)_{R',R}$ and any $s \in \mathcal{P}(R)$, the composition $r_{R,R'}^{N',N}(x) \cdot s$ belongs to $\mathcal{P}(R',R)$; moreover, for any $y \in \mathcal{P}(N',N)_{R',R}$ and any $t \in \mathcal{P}(R)$ such that $r_{R',R}^{N',N}(y) \cdot t = r_{R',R}^{N',N}(x) \cdot s$, we clearly have that $r_{R,R}^{N,N}(x^{-1}y) = s \cdot t^{-1}$ which implies that $x^{-1}y$ belongs to $\mathcal{P}(N)_R$; consequently, the pairs $(x, s)$ and $(y, t)$ have the same image in the quotient set

$$\mathcal{P}(N',N)_{R',R} \times \mathcal{P}(N)_R \mathcal{P}(R) = (\mathcal{P}(N',N)_{R',R} \times \mathcal{P}(R))/\mathcal{P}(N)_R$$

Conversely, any $x \in \mathcal{P}(R',R)$ induces by conjugation a group isomorphism $\mathcal{P}(R) \cong \mathcal{P}(R')$; then, since $\tau_R(N)$ and $\tau_{R'}(N')$ are respective Sylow $p$-subgroups of $\mathcal{P}(N)$ and $\mathcal{P}(N')$ [5, 2.11.4], there is $s \in \mathcal{P}(R)$ such that the isomorphism $\mathcal{P}(R) \cong \mathcal{P}(R')$ induced by $s \cdot s$ sends $\tau_R(N)$ onto $\tau_{R'}(N')$; at this point, it follows from condition 2.1.3 that there is $y \in \mathcal{P}(N',N)$ such that $r_{R,R'}^{N',N}(y) = x \cdot s$, so that $y$ belongs to $\mathcal{P}(N',N)_{R',R}$ and $x$ is the image of the pair $(y, s^{-1})$.

3.4. In order to discuss the uniqueness of the announced $k^*$-category $\hat{\mathcal{P}}^x$, note that the coherent $\mathcal{F}^x$-locality structure of $\mathcal{P}^x$ [5, 17.9] can be lifted to a coherent $\mathcal{F}^x$-locality structure of $\hat{\mathcal{P}}^x$. More precisely, let us consider a nonempty set $\mathcal{X}$ of $\mathcal{F}$-selfcentralizing subgroups of $P$ which contains any subgroup of $P$ admitting an $\mathcal{F}$-morphism from some subgroup in $\mathcal{X}$, and respectively denote by $\mathcal{T}_P$, $\mathcal{F}^x$ and $\mathcal{P}^x$ the full subcategories of $\mathcal{T}_P$, $\mathcal{F}^x$ and $\mathcal{P}^x$ over $\mathcal{X}$ as the set of objects; we actually will prove that there exists an essentially unique regular central $k^*$-extension $\hat{\mathcal{P}}^x$ of $\mathcal{P}^x$ inducing the obvious restricted functor (cf. 3.1)

$$\widehat{\text{aut}}_{P^x} : \text{ch}^x(\mathcal{P}^x) \rightarrow k^*\text{-Gr}$$
first of all, we claim that the coherent $\mathcal{F}^x$-locality structure of $\mathcal{P}^x$ [5, 17.9] can be lifted to a coherent $\mathcal{F}^x$-locality structure of $\hat{\mathcal{P}}^x$.

**Proposition 3.5.** With the notation above, the first structural functor $\tau^x : T^x_P \rightarrow \mathcal{P}^x$ can be lifted to a functor $\hat{\tau}^x : T^x_P \rightarrow \hat{\mathcal{P}}^x$ and such a lifting fulfills

$$\hat{x} \cdot \hat{\tau}^x_R(v) = \hat{\tau}^x_Q\left((\pi_{Q,R}(x))(v)\right) \hat{x}$$  \hspace{1cm} (3.5.1)

for any pair of subgroups $Q$ and $R$ in $\mathfrak{X}$, any $x \in \mathcal{P}(Q,R)$, any $\hat{x} \in \hat{\mathcal{P}}^x(Q,R)$ lifting $x$ and any $v \in R$.

**Proof:** We already know that $\tau^x : \mathcal{P} \rightarrow \mathcal{P}(\mathcal{P})$ is injective and thus, it can be uniquely lifted to an injective group homomorphism $\hat{\tau}^x : \mathcal{P} \rightarrow \hat{\mathcal{P}}^x(\mathcal{P})$; then, choosing $\hat{\tau}^x_{P,Q}(1)$ lifting $\tau^x_{P,Q}(1)$ in $\hat{\mathcal{P}}^x(P,Q)$ for any subgroup $Q \neq P$ in $\mathcal{A}$, the functor $\hat{\tau}^x$ maps any $T^x_P$-morphism $u : R \rightarrow Q$ on the unique element $\hat{\tau}^x_{Q,R}(u)$ in $\hat{\mathcal{P}}^x(Q,R)$ fulfilling

$$\hat{\tau}^x_{P,Q}(1) \cdot \hat{\tau}^x_{Q,R}(u) = \hat{\tau}^x_{P,R}(1)$$  \hspace{1cm} (3.5.2)

which makes sense since $u$ belongs to the transporter $T^x_P(R,Q)$.

With such a choice, $\hat{\mathcal{P}}^x$ becomes a divisible $\mathcal{F}^x$-locality [5, 17.7], the divisibility being an easy consequence of the divisibility of $\mathcal{P}$ and of the regularity of the $k^*$-extension $\hat{\mathcal{P}}^x$; thus, our argument in [5, Proposition 17.10] applies to $\hat{\mathcal{P}}^x$ and therefore it suffices to prove condition [5, 17.10.1]; but, note that for any $\hat{x} \in \hat{\mathcal{P}}^x(Q)$ the homomorphisms sending $v \in Q$ to $\hat{x} \cdot \hat{\tau}^x_Q(v) \hat{x}^{-1}$ and to $\hat{\tau}^x_Q(\pi_Q(x)(v))$ lift the same group homomorphism from $Q$ to $\mathcal{P}(Q)$ and therefore they coincide with each other.

3.6. Note that, since a regular central $k^*$-extension $\hat{\mathcal{P}}^x$ of $\mathcal{P}^x$ endowed with a functor $\hat{\tau}^x : T^x_P \rightarrow \hat{\mathcal{P}}^x$ lifting the first structural functor $\tau^x : T^x_P \rightarrow \mathcal{P}^x$ and fulfilling condition 3.5.1 is actually a coherent $\mathcal{F}^x$-locality [5, 17.7], with the notation in 3.2 above we also have an injective $k^*$-restriction map

$$\hat{\tau}^x_{Q',R} : \hat{\mathcal{P}}^x(Q',Q)_{R',R} \rightarrow \hat{\mathcal{P}}^x(R',R)$$  \hspace{1cm} (3.6.1)

where $\hat{\mathcal{P}}^x(Q',Q)_{R',R}$ is the converse image of $\mathcal{P}(Q',Q)_{R',R}$ in $\hat{\mathcal{P}}^x(Q',Q)$.

**Theorem 3.7.** With the notation above, there exists a regular central $k^*$-extension $\hat{\mathcal{P}}^x$ of $\mathcal{P}^x$, unique up to $k^*$-equivalences, inducing the folded Frobenius $\mathcal{P}$-category $(\mathcal{F}, \text{aut}_{\mathcal{P}})$.
Proof: We choose a set \( \mathcal{X} \) as above and, arguing by induction on \(|\mathcal{X}|\), we will prove that there exists a regular central \( k^* \)-extension \( \hat{\mathcal{P}}^x \) of \( \mathcal{P}^x \) inducing the obvious restricted functor (cf. 3.1)

\[
\hat{\text{aut}}_{x^*} : \text{ch}^*(\mathcal{F}^x) \longrightarrow k^*:\mathfrak{Gr}
\]

and that such a \( \hat{\mathcal{P}}^x \) endowed with a lifting \( \hat{\tau}^x : \mathcal{T}_{P}^x \rightarrow \hat{\mathcal{P}}^x \) of \( \tau^x \), which fulfills condition 3.5.1, is unique up to \( k^* \)-equivalences.

If \( \mathcal{X} = \{P\} \) then \( \mathcal{P}^x \) has just one object \( P \) and its automorphism group is \( \mathcal{P}(P) \); then, the folder structure maps the trivial \( \mathcal{F}^x \)-chain \( \Delta_0 \rightarrow \mathcal{F}^x \) sending \( 0 \) to \( P \) on a \( k^* \)-group \( \hat{\mathcal{F}}(P) \) which, by restriction, determines a \( k^* \)-group \( \hat{\mathcal{P}}(P) \); that is to say, we get a \( k^* \)-category \( \hat{\mathcal{P}}^x \) with one object \( P \) and with the \( k^* \)-group automorphism \( \hat{\mathcal{P}}(P) \), which clearly induces the corresponding functor 3.7.1 again; the uniqueness is clear.

Otherwise, choose a minimal element \( U \) in \( \mathcal{X} \) fully normalized in \( \mathcal{F} \) and set

\[\mathfrak{Y} = \mathcal{X} - \{\theta(U) \mid \theta \in \mathcal{F}(P,U)\}\]

that is to say, according to our induction hypothesis, there exists a regular central \( k^* \)-extension \( \hat{\mathcal{P}}^\mathfrak{Y} \) of \( \mathcal{P}^\mathfrak{Y} \) inducing the obvious restricted functor (cf. 3.1)

\[
\hat{\text{aut}}_{x^*} : \text{ch}^*(\mathcal{F}^\mathfrak{Y}) \longrightarrow k^*:\mathfrak{Gr}
\]

and such a \( k^* \)-category \( \hat{\mathcal{P}}^\mathfrak{Y} \) endowed with a lifting \( \hat{\tau}^\mathfrak{Y} : \mathcal{T}_{P}^\mathfrak{Y} \rightarrow \hat{\mathcal{P}}^\mathfrak{Y} \) of \( \tau^\mathfrak{Y} \) which fulfills condition 3.5.1 (cf. Proposition 3.5) is unique up to \( k^* \)-isomorphisms.

If \( N_{\mathcal{F}}(U) = \mathcal{F} \) [5, Proposition 2.16], we also have \( N_{\mathcal{P}}(U) = \mathcal{P} \) [5, 17.5] and then it is easily checked from 3.2.1 that \( \mathcal{P}^x \) actually coincides with the category \( \mathcal{T}_{\mathcal{P}(U)} \) where \( \mathcal{X} \) is the set of objects and where, for a pair of subgroups \( Q \) and \( R \) in \( \mathcal{X} \), the set of morphisms from \( R \) to \( Q \) is the \( \mathcal{P}(U) \)-transporter

\[
\mathcal{T}_{\mathcal{P}(U)}(Q,R) = \{x \in \mathcal{P}(U) \mid x \cdot \tau_U(R) \cdot x^{-1} \subset \tau_U(Q)\}
\]

the composition being defined by the product in \( \mathcal{P}(U) \); but, once again, the folder structure maps the trivial \( \mathcal{F}^x \)-chain \( \Delta_0 \rightarrow \mathcal{F}^x \) sending \( 0 \) to \( U \) on a \( k^* \)-group \( \hat{\mathcal{F}}(U) \) which, by restriction, determines a \( k^* \)-group \( \hat{\mathcal{P}}(U) \); hence, denoting by \( \hat{\tau}_U(Q) \) and \( \hat{\tau}_U(R) \) the finite \( p \)-subgroups of \( \hat{\mathcal{P}}(U) \) respectively lifting \( \tau_U(Q) \) and \( \tau_U(R) \), we can consider the corresponding transporter in the \( k^* \)-group \( \hat{\mathcal{P}}(U) \)

\[
\mathcal{T}_{\hat{\mathcal{P}}(U)}(Q,R) = \{\hat{x} \in \hat{\mathcal{P}}(U) \mid \hat{x} \cdot \hat{\tau}_U(R) \cdot \hat{x}^{-1} \subset \hat{\tau}_U(Q)\}
\]
Now, it is clear that the $k^*$-category $\mathcal{T}_\mathcal{P}(U)_k$ where $\mathcal{X}$ is the set of objects, where the obvious $k^*$-set $\mathcal{T}_\mathcal{P}(U)_k(Q, R)$ is the $k^*$-set of morphisms from $R$ to $Q$ for any pair of subgroups $Q$ and $R$ in $\mathcal{X}$, and where the composition is defined by the product in $\mathcal{P}(U)$ determines a regular central $k^*$-extension of $\mathcal{T}_\mathcal{P}(U)_k = \mathcal{P}_k^X$ together with an obvious lifting of $\tau^X$, which fulfills condition 3.5.1.

On the other hand, it is easily checked that such a regular central $k^*$-extension $\hat{\mathcal{P}}_k^X$ is also divisible [5, 17.7] and therefore that, for any pair of subgroups $Q$ and $R$ in $\mathcal{X}$, we may assume that there exists a $k^*$-set homomorphism

$$\hat{\mathcal{P}}_k^X(Q, R)_k \cong \hat{\mathcal{P}}_k^X(Q, R)$$

which is always injective; moreover, since we have $\mathcal{N}_\mathcal{P}(U) = \mathcal{P}_k^X$, always by the divisibility of $\hat{\mathcal{P}}_k^X$ we get a $k^*$-set isomorphism

$$\hat{\mathcal{P}}_k^X(Q, R)_k \cong \hat{\mathcal{P}}_k^X(Q, R)$$

From these remarks, it is easily checked the uniqueness of $\hat{\mathcal{P}}_k^X$ and the fact that this $k^*$-category determines the restricted functor $\hat{\mathcal{F}}_k^X$.

Otherwise recall that, according to [6, 3.1], for any subgroup $Q$ of $P$ fully normalized in $\mathcal{F}$, our folded Frobenius $P$-category induces a folded Frobenius $N_\mathcal{F}(Q)$-category $(N_\mathcal{F}(Q), \hat{\mathcal{F}}_k^X)$ where

$$\hat{\mathcal{F}}_k^X : \mathcal{X} \rightarrow \mathcal{Y}_k$$

is the unique functor lifting $\mathcal{F}_k^X$ and extending the restriction of $\mathcal{F}_k^X$ to $N_\mathcal{F}(Q)^{\text{rd}}$ (cf. Theorem 2.6 and [6, Lemma 2.5]).

Thus, if we have $N_\mathcal{F}(U) \neq \mathcal{F}$, arguing by induction on the size of $\mathcal{F}$, for any $V \in \mathcal{X} - \mathcal{F}$ fully normalized in $\mathcal{F}$ we may assume that there exists a regular central $k^*$-extension $\hat{N}_\mathcal{F}(V)^{\text{rd}}$ of $N_\mathcal{F}(V)^{\text{rd}}$ determining $\hat{\mathcal{F}}_k^X$, and that such a $k^*$-category $\hat{N}_\mathcal{F}(V)^{\text{rd}}$, endowed with a lifting $\hat{\mathcal{F}}_k^X : \mathcal{N}_\mathcal{F}(V)^{\text{rd}} \rightarrow \hat{N}_\mathcal{F}(V)^{\text{rd}}$ of the first structural functor of $N_\mathcal{F}(V)^{\text{rd}}$ which fulfills condition 3.5.1 (cf. Proposition 3.5), is unique up to $k^*$-isomorphisms. Actually, we are only interested in the full $k^*$-subcategory of $\hat{N}_\mathcal{F}(V)^{\text{rd}}$ over the set $N_\mathcal{F}(V)$ of subgroups in $\mathcal{X}$ contained in $N_\mathcal{P}(V)$ and may assume that the lifting

$$\hat{\mathcal{F}}_k^X : \mathcal{N}_\mathcal{F}(V)^{\text{rd}} \rightarrow \hat{N}_\mathcal{F}(V)^{\text{rd}}$$

coincides with the restriction of $\hat{\mathcal{F}}_k^X$; then, it follows from Proposition 3.5 that we can identify $\hat{N}_\mathcal{F}(V)^{\text{rd}}$ with the full $k^*$-subcategory of $\hat{\mathcal{F}}_k^X$ over the set $N_\mathcal{F}(V)$. 
Moreover, setting \( N = N_P(V) \) and considering the \( \mathcal{N}_F(V)^* \)-chains \( q_V : \Delta_0 \to \mathcal{N}_F(V)^* \), \( q_N : \Delta_0 \to \mathcal{N}_F(V)^* \) (cf. 2.2) and \( n : \Delta_1 \to \mathcal{N}_F(V)^* \) which map 0 on \( V \), 1 on \( N \) and 0 \( \bullet 1 \) on the inclusion of \( V \) in \( N \), noted \( \mathcal{N}_N \), and the obvious \( \mathbb{C}^*(\mathcal{N}_F(V)^*) \)-morphisms (cf. 2.2)

\[
(\mathbb{I}_V, \delta^V_0) : (n, \Delta_1) \to (q_V, \Delta_0) \quad \text{and} \quad (\mathbb{I}_N, \delta^N_0) : (n, \Delta_1) \to (q_N, \Delta_0)
\]

3.7.10, the functors \( \mathbb{N}_{\mathcal{N}_F(V)^*} \) and \( \mathbb{N}_{\mathcal{N}_F(V)^*} \) send \( n \), \( q_V \) and \( q_N \) to the same respective \( k^* \)-groups \( \hat{\mathcal{F}}(N)_V \), \( \hat{\mathcal{F}}(V) \) and \( \hat{\mathcal{F}}(N) \), and they send the \( \mathbb{C}^*(\mathcal{N}_F(Q)^*) \)-morphisms \( (\mathbb{I}_V, \delta^V_0) \) and \( (\mathbb{I}_N, \delta^N_0) \) to the same respective \( k^* \)-group homomorphisms

\[
\text{\hat{\mathcal{F}}}(N)_V \to \hat{\mathcal{F}}(V) \quad \text{and} \quad \text{\hat{\mathcal{F}}}(N)_V \to \hat{\mathcal{F}}(N)
\]

3.7.11; note that the images of \( \hat{\mathcal{F}}(N)_V \) are respectively \( N_{\mathcal{F}(V)}(\mathcal{F}_N(V)) \) and the stabilizer \( \hat{\mathcal{F}}(N)_V \) of \( V \) in \( \hat{\mathcal{F}}(N) \).

Since \( N \) belongs to \( \mathcal{Y} \), the restriction of \( \hat{\mathcal{F}}(N) \) from \( \mathcal{F}(N) \) to \( \mathcal{P}(N) \) necessarily coincides with \( \hat{\mathcal{P}}^\mathcal{Y}(N) \) and therefore the restriction of \( \hat{\mathcal{F}}(N)_V \) from \( \mathcal{F}(N)_V \) to \( \mathcal{P}(N)_V \) also coincides with the stabilizer \( \hat{\mathcal{P}}^\mathcal{Y}(N)_V \) of \( V \) in \( \hat{\mathcal{P}}^\mathcal{Y}(N) \).

Then, for any \( V' \in \mathcal{X} - \mathcal{Y} \) fully normalized in \( \mathcal{F} \), setting \( N' = N_P(V') \) and denoting by \( \hat{\mathcal{P}}^\mathcal{Y}(N', \mathcal{N}_V) \) the converse image of \( \mathcal{P}(N', \mathcal{N}_V) \) in \( \hat{\mathcal{P}}^\mathcal{Y}(N', \mathcal{N}_V) \) and by \( \hat{\mathcal{P}}^\mathcal{Y}(V) \) the restriction of \( \hat{\mathcal{F}}(V) \) from \( \mathcal{F}(V) \) to \( \mathcal{P}(V) \), it is clear that \( \hat{\mathcal{P}}^\mathcal{Y}(N)_V \) acts on the \( k^* \)-set \( \hat{\mathcal{P}}^\mathcal{Y}(N', \mathcal{N}_V \times \hat{\mathcal{P}}^\mathcal{Y}(V) \)

3.7.12

and then, from isomorphism 3.3.1, we get a canonical map

\[
\hat{\mathcal{P}}^\mathcal{Y}(V', V) \to \mathcal{P}(V, V)
\]

3.7.13.

Note that, in the case where \( V' = V \), our notation is coherent. Moreover, for another \( V'' \in \mathcal{X} - \mathcal{Y} \) fully normalized in \( \mathcal{F} \), setting \( N'' = N_P(V'') \) and considering \( \hat{\mathcal{P}}^\mathcal{Y}(N'', \mathcal{N}_V) \), \( \hat{\mathcal{P}}^\mathcal{Y}(N', \mathcal{N}_V) \) and \( \hat{\mathcal{P}}^\mathcal{Y}(V') \) as above, we also have the \( k^* \)-sets

\[
\hat{\mathcal{P}}^\mathcal{Y}(V', V) = \hat{\mathcal{P}}^\mathcal{Y}(N', \mathcal{N}_V) \times \hat{\mathcal{P}}^\mathcal{Y}(V)
\]

3.7.14

and we claim that the composition in \( \hat{\mathcal{P}}^\mathcal{Y} \) and in the corresponding \( k^* \)-groups induces a \( k^* \)-composition

\[
e_{V', V''} : \hat{\mathcal{P}}^\mathcal{Y}(V'', V') \times \hat{\mathcal{P}}^\mathcal{Y}(V', V) \to \hat{\mathcal{P}}^\mathcal{Y}(V'', V)
\]

3.7.15

lifting the composition in \( \mathcal{P} \) via the canonical maps 3.7.13.
First of all, *mutatis mutandis* denote by $q_{V'}$, $q_{N'}$ and $n'$, the analogous $N_\mathcal{F}(V')$-chains and by $(i_0, \delta_0)$ and $(i_0', \delta_0')$ the analogous $\mathcal{F}(N_\mathcal{F}(V'))$-morphisms, as in 3.7.10 above; it is clear that any $\mathcal{F}$-morphism $\varphi: N \rightarrow N'$ fulfilling $\varphi(V) = V'$ determines natural isomorphisms $q_V \cong q_{V'}$, $q_N \cong q_{N'}$ and $n \cong n'$ which induce commutative $\mathcal{F}_*^{\mathcal{F}}$-diagrams (cf. 3.7.10)

$$
\begin{align*}
(n', \Delta_1) & \rightarrow (q_{V'}, \Delta_0) & (n', \Delta_1) & \rightarrow (q_{N'}, \Delta_0) \\
(n, \Delta_1) & \rightarrow (q_V, \Delta_0) & (n, \Delta_1) & \rightarrow (q_N, \Delta_0)
\end{align*}
$$

3.7.16;

at this point, the functor $\widehat{\text{aut}}_{\mathcal{F}}$ sends these commutative $\mathcal{F}_*^{\mathcal{F}}$-diagrams to the commutative diagrams of $k$-groups

$$
\begin{align*}
\tilde{\mathcal{F}}(N')_{V'} & \rightarrow \tilde{\mathcal{F}}(V') & \tilde{\mathcal{F}}(N')_{V'} & \rightarrow \tilde{\mathcal{F}}(N') \\
\tilde{\mathcal{F}}(N)_V & \rightarrow \tilde{\mathcal{F}}(V) & \tilde{\mathcal{F}}(N)_V & \rightarrow \tilde{\mathcal{F}}(N)
\end{align*}
$$

3.7.17.

Consequently, for any $x \in \mathcal{P}(N', N)_{V', V}$ lifting $\varphi$ we get the commutative diagrams of $k$-groups

$$
\begin{align*}
\hat{\mathcal{P}}'(N')_{V'} & \rightarrow \hat{\mathcal{P}}'(V') & \hat{\mathcal{P}}'(N')_{V'} & \rightarrow \hat{\mathcal{P}}'(N') \\
\hat{\mathcal{P}}'(N)_V & \rightarrow \hat{\mathcal{P}}'(V) & \hat{\mathcal{P}}'(N)_V & \rightarrow \hat{\mathcal{P}}'(N)
\end{align*}
$$

3.7.18

and note that the $k$-group isomorphism $\hat{g}_x$ has to be induced by the composition in $\hat{\mathcal{P}}'(N)$ (cf. 3.7.3); that is to say, for any $\hat{x} \in \hat{\mathcal{P}}'(N', N)_{V', V}$ lifting $x$ and any $\hat{s} \in \hat{\mathcal{P}}'(N)$, we actually have $\hat{g}_x(\hat{s}) = \hat{x} \cdot \hat{s} \cdot \hat{x}^{-1}$.

We are ready to define the $k$-composition $c_{\hat{V}}^{\hat{x}, \hat{V}, \hat{V}, \hat{V}}$ in 3.7.15; any element in $\hat{\mathcal{P}}'(V', V)$ is the class $\overline{(\hat{x}, \hat{s})}$ of some pair $\hat{x}, \hat{s}$ where $\hat{x}$ and $\hat{s}$ respectively belong to $\hat{\mathcal{P}}'(N', N)_{V', V'}$ and to $\hat{\mathcal{P}}'(V)$; similarly, if $\overline{(\hat{x'}, \hat{s'})}$ is an element of $\hat{\mathcal{P}}'(V'\prime, V')$, it is clear that, in the $k$-category $\hat{\mathcal{P}}'$, the composition $\hat{x'}: \hat{x}$ makes sense and belongs to $\hat{\mathcal{P}}'(N', N)_{V', V}$; moreover, denoting by $x$ the image of $\hat{x}$ in $\mathcal{P}(N', N)$, we have the $k$-group isomorphism $\hat{h}_x$ from $\hat{\mathcal{P}}'(V)$ to $\hat{\mathcal{P}}'(V')$ and therefore $(\hat{h}_x)^{-1}(\hat{s'})$ belongs to $\hat{\mathcal{P}}'(V)$; then, we set

$$
c_{\hat{V}}^{\hat{x}, \hat{V}, \hat{V}, \hat{V}}((\hat{x'}, \hat{s'}), (\hat{x}, \hat{s})) = \overline{(\hat{x'}: \hat{x}, (\hat{h}_x)^{-1}(\hat{s'}) \cdot \hat{s})}
$$

3.7.19;

the compatibility with the action of $k^*$ is clear.

This makes sense since, for any $\hat{\ell} \in \hat{\mathcal{P}}'(N)_V$ and any $\hat{\ell}' \in \hat{\mathcal{P}}'(N')_{V'}$,
denoting by $t$ the image of $\hat{t}$ in $\mathcal{P}(N)$ we get (cf. 3.7.18)
$$
(\hat{x} \cdot \hat{y}) \cdot (\hat{x} \cdot \hat{z}) = \hat{x} \cdot (\hat{y} \cdot (\hat{z} \cdot (\hat{x} \cdot \hat{y})))
$$
$$
(\hat{x} \cdot \hat{y}) = \hat{x} \cdot (\hat{y} \cdot (\hat{x} \cdot \hat{y}))
$$
3.7.20.

The $k^*$-composition is associative since, for any $V'' \in \mathcal{X} - \mathcal{Q}$ fully normalized in $\mathcal{F}$ and any element $(\hat{x}'' \cdot \hat{y}'' \cdot \hat{z}'')$ in $\hat{\mathcal{P}}^k(V'', V')$, denoting by $x'$ the image of $\hat{x}'$ in $\mathcal{P}(N', N)$ we obtain
$$
c_{x'' \cdot y'', V, V'}(\hat{x}'' \cdot \hat{y}'', (\hat{x}' \cdot \hat{y}'', \hat{x})) = c_{x'' \cdot y'', V, V'}(\hat{x}'' \cdot \hat{y}'', (\hat{x} \cdot (\hat{y} \cdot (\hat{x} \cdot \hat{y})))
$$
$$
3.7.21.

According to our definition of $\hat{\mathcal{P}}^k(V', V)$ in 3.7.12, the unity element of $\hat{\mathcal{P}}^k(V)$ defines a canonical $k^*$-set homomorphism
$$
\hat{r}_{V', V}^{N', N} : \hat{\mathcal{P}}^k(N', N)_{V', V} \rightarrow \hat{\mathcal{P}}^k(V', V)
$$
3.7.22.

lifting $r_{V', V}^{N', N}$. More generally, let $Q$ and $Q'$ be a pair of subgroups of $\mathcal{P}$ respectively contained in $N$ and $N'$, and strictly containing $V$ and $V'$; we define as follows an injective $k^*$-set homomorphism
$$
\hat{r}_{V', V}^{Q', Q} : \hat{\mathcal{P}}^k(Q', Q)_{V', V} \rightarrow \hat{\mathcal{P}}^k(V', V)
$$
3.7.23.

lifting the restriction map (cf. 3.2.1)
$$
\hat{r}_{V', V}^{Q', Q} : \mathcal{P}(Q', Q)_{V', V} \rightarrow \mathcal{P}(V', V)
$$
3.7.24.

If $\hat{z} \in \hat{\mathcal{P}}^k(Q', Q)_{V', V}$ and $x$ denotes its image in $\mathcal{P}(Q', Q)_{V', V}$, it follows from Lemma 3.3 that $r_{V', V}^{Q', Q}(x) = r_{V', V}^{N', N}(y) \cdot z$ for suitable $y \in \mathcal{P}(N', N)_{V', V}$ and $z \in \mathcal{P}(V)$; thus, setting $Q'' = (\pi_{N, N}(y^{-1}))(Q') \subset N$, we get
$$
z = r_{V', V}^{Q'', Q}(r_{V', V}^{N', N}(y^{-1}) \cdot x)
$$
3.7.25.

and therefore, setting $s = r_{V', V}^{N', N}(y^{-1}) \cdot x$, by injectivity of $r_{V', V}^{Q', Q}$ (cf. 3.2) we still get $x = r_{V', V}^{N', N}(y) \cdot s$. 
Hence, choosing a lifting \( \hat{y} \) of \( y \) in \( \hat{\mathcal{P}}^\mathbb{R}(N', N)_{V', V} \), in the \( k^- \)-category \( \hat{\mathcal{P}}^\mathbb{R} \) we have the restriction \( \hat{r}_N^N;_{Q', Q''} (\hat{y}) \) (cf. 3.6) as an element of \( \hat{\mathcal{P}}^\mathbb{R}(Q', Q'')_{V', V} \); then, there is a unique lifting \( \hat{s} \) of \( s \) in \( \hat{\mathcal{P}}^\mathbb{R}(Q'', Q)_{V, V} \) fulfilling \( \hat{x} = \hat{r}_N^N;_{Q', Q''} (\hat{y}) \hat{s} \). Moreover, since \( \hat{N}_P(V)^{N_\mathbb{R}(V)} \) can be identified with the full \( k^- \)-subcategory of \( \mathcal{P}^\mathbb{R} \) over the set \( N_\mathbb{R}(V) \), actually \( \hat{s} \) can be identified with an element of \( \hat{N}_P(V)^{N_\mathbb{R}(V)}(Q'', Q) \) stabilizing \( V \) and therefore in the \( k^- \)-category \( \hat{N}_P(V)^{N_\mathbb{R}(V)} \) we have the restriction \( \hat{r}_V^Q;_V (\hat{s}) \) (cf. 3.6) lifting \( z \) to \( \hat{N}_P(V)^{N_\mathbb{R}(V)}(V) \) which coincides with \( \hat{F}(V) \) since we have

\[
\hat{N}_F(V)^{\mathbb{R}}(V) = \hat{\text{aut}}_{N_F(V)^{\mathbb{R}}}(q_V) = \hat{\text{aut}}_{\mathcal{F}}(q_V) = \hat{\mathcal{F}}(V) \quad 3.7.26.
\]

Then, we define (cf. 3.7.12)

\[
\hat{r}_V^Q;_V (\hat{x}) = (\hat{y}, \hat{r}_V^Q;_V (\hat{s})) \quad 3.7.27;
\]

it is independent of our choice of \( y \in \mathcal{P}(N', N)_{V', V} \) since, for another decomposition \( \hat{r}_V^Q;_V (x) = \hat{r}_N^N;_{Q', Q''}(y') \cdot z' \), we actually have \( y' = y \cdot t \) and \( z' = \hat{r}_V^Q(t^{-1}) \cdot z \) for some \( t \in \mathcal{P}(N)_V \); thus, setting \( Q''' = (\pi_N(t^{-1}))(Q'') \), once again an element \( \hat{t} \) of \( \hat{\mathcal{P}}^\mathbb{R}(N)_V \) lifting \( t \) can be identified with an element of \( \hat{N}_P(V)^{N_\mathbb{R}(V)}(N) \) stabilizing \( V \) and we also obtain

\[
\hat{x} = \hat{r}_N^N;_{Q', Q''} (\hat{y}) \hat{s} = (\hat{r}_N^N;_{Q', Q''} (\hat{y}) \hat{t}) \cdot (\hat{r}_N^N;_{Q', Q''} (\hat{t}^{-1}) \hat{s}) \quad 3.7.28;
\]

but, the pairs \( (\hat{y}, \hat{r}_V^Q;_V (\hat{s})) \) and \( (\hat{y} \hat{t}, \hat{r}_V^Q;_V (\hat{s} \hat{t}^{-1} \hat{s})) \) have the same class in \( \hat{\mathcal{P}}^\mathbb{R}(V', V) \).

At present, if \( R \) and \( R' \) are a pair of subgroups of \( P \) respectively contained in \( Q \) and \( Q' \), and strictly containing \( V \) and \( V' \), we claim that the corresponding restriction \( \hat{r}_V^Q;_V (x) \) agree with \( \hat{r}_V^Q;_V (x) \); if \( \hat{x} \in \hat{\mathcal{P}}^\mathbb{R}(Q', Q)_{V', V} \) has an image in \( \mathcal{F}(Q', Q) \) mapping \( R \) on \( R' \), it follows from 3.6 above that we have the restriction \( \hat{r}_V^Q;_V (\hat{x}) \) in \( \hat{\mathcal{P}}^\mathbb{R}(R', R)_{V', V} \) and we claim that

\[
\hat{r}_V^Q;_V (\hat{x}) = \hat{r}_V^Q;_V (\hat{x}) \quad 3.7.29;
\]

indeed, with the notation above we may assume that \( \hat{x} = \hat{r}_N^N;_{Q', Q''} (\hat{y}) \hat{s} \); then, setting \( R'' = (\pi_N, N_N(y^{-1}))(R') \subset N \), we clearly have

\[
\hat{r}_V^Q;_V (\hat{x}) = \hat{r}_V^Q;_V (\hat{y}) \hat{s} \hat{r}_V^Q;_V (\hat{y}) \hat{s} \quad 3.7.30;
\]
consequently, considering the set \( N_\mathcal{X}(V) \) defined above, since the restriction in the \( k^* \)-category \( \tilde{N}_P(V) \) is transitive (cf. 3.6), we clearly obtain

\[
\mathcal{N}_{R'}^V(y, \tilde{r}_{R'}^Q(\tilde{x})) = \mathcal{N}_{R'}^V(y, \tilde{r}_{R'}^Q(\tilde{s})) = \mathcal{N}_{R'}^V(y, \tilde{r}_{R'}^Q(\tilde{\delta})) = \mathcal{N}_{R'}^V(y, \tilde{r}_{R'}^Q(\tilde{\delta})) = \mathcal{N}_{R'}^V(y, \tilde{r}_{R'}^Q(\tilde{\delta})) \quad 3.7.31.
\]

As above, consider a third \( V'' \in \mathcal{X} - \mathcal{G} \) fully normalized in \( \mathcal{F} \), and a subgroup \( Q'' \) of \( P \) contained in \( N'' = N_P(V'') \) and strictly containing \( V'' \); thus, we have the three \( k^* \)-set homomorphisms \( \tilde{r}_{V', V}^Q, \tilde{r}_{V', V}^Q, \tilde{r}_{V', V}^Q \) and we claim that they are compatible with the \( k^* \)-compositions, namely that we have the following commutative diagram

\[
\begin{array}{ccc}
\hat{\mathcal{P}}^\mathcal{N}_{V'', V'}(Q'' \times Q') & \longrightarrow & \hat{\mathcal{P}}^\mathcal{N}_V(Q', Q) \\
\downarrow & & \downarrow \\
\hat{\mathcal{P}}^\mathcal{N}_V(V'' \times V') & \longrightarrow & \hat{\mathcal{P}}^\mathcal{N}_V(V', V)
\end{array}
\quad 3.7.32.
\]

Indeed, let \( \hat{x} \) and \( \hat{x}' \) be respective elements in \( \hat{\mathcal{P}}^\mathcal{N}_V(Q', Q) \) and in \( \hat{\mathcal{P}}^\mathcal{N}_V(Q'', Q') \); we actually may assume that

\[
\hat{x} = \hat{r}_{Q', Q}^N(\hat{y}) \cdot \hat{s} \quad \text{and} \quad \hat{x}' = \hat{r}_{Q', Q}^N(\hat{y}') \cdot \hat{s}'
\quad 3.7.33.
\]

where \( \hat{y} \) and \( \hat{y}' \) are suitable elements respectively belonging to \( \hat{\mathcal{P}}^\mathcal{N}_V(N', V) \) and \( \hat{\mathcal{P}}^\mathcal{N}_V(N'', N') \), and where, denoting by \( y \) and \( y' \) their images in \( \mathcal{P} \) and setting

\[
R = (\pi_{N', N}(y^{-1}))(Q') \quad \text{and} \quad R' = (\pi_{N', N''}(y'^{-1}))(Q'')
\quad 3.7.34.
\]

\( \hat{s} \) and \( \hat{s}' \) are suitable elements respectively belonging to \( \hat{\mathcal{P}}^\mathcal{N}_V(R, Q) \) and to \( \hat{\mathcal{P}}^\mathcal{N}_V(R', Q') \). Then, setting

\[
R'' = (\pi_{N', N}(y^{-1}))(R') = (\pi_{N', N''}(y'^{-1}))(Q'')
\quad 3.7.35.
\]

we clearly have

\[
\hat{x}' \cdot \hat{x} = (\hat{r}_{Q', R}^N(\hat{y}') \cdot \hat{s}') \cdot (\hat{r}_{Q', R}^N(\hat{y}) \cdot \hat{s})
= \hat{r}_{Q', R}^N(\hat{y}') \cdot (\hat{r}_{R', R}^N(\hat{y}^{-1}) \cdot \hat{s}' \cdot \hat{r}_{Q', R}^N(\hat{y})) \cdot \hat{s}
\quad 3.7.36.
\]

Hence, setting \( \hat{s}'' = \hat{r}_{R', R}^N(\hat{y}^{-1}) \cdot \hat{s}' \cdot \hat{r}_{Q', R}^N(\hat{y}) \), we get (cf. 3.7.27)

\[
\hat{r}_{V', V}^Q(\hat{x}' \cdot \hat{x}) = (\hat{y}' \cdot \hat{y} \cdot \hat{r}_{V', V}^Q(\hat{s}' \cdot \hat{s}))
\quad 3.7.37.
\]
On the other hand, from equalities 3.7.33 we obtain (cf. 3.7.27)

\[ \hat{r}_{V',V}^Q(\hat{x}) = (\hat{y}, \hat{r}_{V',V}^Q(\hat{s})) \quad \text{and} \quad \hat{r}_{V,V'}^Q(\hat{x}') = (\hat{y}', \hat{r}_{V,V'}^Q(\hat{s}')) \]

3.7.38;

but, according to our definition in 3.7.19, we get

\[ c_{V'',V',V}^Q \left( (\hat{y}', \hat{r}_{V',V'}^R(\hat{s}')), (\hat{y}, \hat{r}_{V,V}^R(\hat{s})) \right) = (\hat{y}' \cdot \hat{y}, (\hat{h}_y)^{-1} (\hat{r}_{V',V'}^R(\hat{s}')) \cdot \hat{r}_{V,V}^R(\hat{s})) \]

3.7.39

and we claim that we have \((\hat{h}_y)^{-1} (\hat{r}_{V',V'}^R(\hat{s}')) = \hat{r}_{V,V}^R(\hat{s}'')\) which will force (cf. 3.7.37)

\[ c_{V'',V',V}^Q \left( (\hat{y}', \hat{r}_{V',V'}^R(\hat{s}')), (\hat{y}, \hat{r}_{V,V}^R(\hat{s})) \right) = (\hat{y}' \cdot \hat{y}, \hat{r}_{V',V'}^R(\hat{z}', \hat{x})) \]

3.7.40

completing the proof of the commutativity of diagram 3.7.32.

Denoting by \(\varphi'\) the image of \(\hat{r}_{N,S}^N(1) \cdot s'\) in \((N_S(V'))(N', Q')\) (cf. 3.7.9) and employing the terminology in [5, 5.15], we argue by induction on the length \(\ell(\varphi')\) of \(\varphi';\) if \(\ell(\varphi') = 0\) we have \(\varphi' = \sigma' \circ \iota_{N'}^N\) for \(\sigma' \in (N_S(V'))(N')\) [5, Corollary 5.14] and therefore we get \(\hat{r}_{N,S}^N(1) \cdot s' = \hat{t}' \cdot \hat{r}_{N,S}^N(1)\) for a suitable \(\hat{t}' \in \hat{\mathcal{P}}_{\mathcal{Q}}(N')_{V'}\), so that we obtain (cf. 3.7.18)

\[ (\hat{h}_y)^{-1} (\hat{r}_{V',V'}^R(\hat{s}')) = \hat{r}_{V}^N (\hat{y} \hat{t}') = \hat{r}_{V}^N (\hat{y}^{-1} \hat{t}' \cdot \hat{y}) = \hat{r}_{V,V}^R (\hat{s}'') \]

3.7.41

Otherwise, we have [5, 5.15.1]

\[ \varphi' = \iota_{N'}^N \circ \tau' \circ \eta' \quad \text{and} \quad \ell(\iota_{N'}^N \circ \eta') = \ell(\varphi') - 1 \]

3.7.42

for some \(T' \in N_{\mathcal{P}}(V')\), some \(\eta' \in (N_S(V'))(T', Q')\) and some \(\tau' \in (N_S(V'))(T')\), and therefore we get \(s' = \hat{r}_{N,S,T'}^N(1) \cdot \hat{t}' \cdot \hat{u}'\) for suitable elements \(\hat{t}' \in \hat{\mathcal{P}}_{\mathcal{Q}}(T')_{V'}\) and \(\hat{u}' \in \hat{\mathcal{P}}_{\mathcal{Q}}(T', Q')_{V', V'}\) respectively lifting \(\tau'\) and \(\eta'\); hence,

we obtain

\[ \hat{r}_{V',V'}^R(\hat{s}') = \hat{r}_{V'}^T(\hat{t}') \cdot \hat{r}_{V',V'}^{T',Q'}(\hat{u}') \]

3.7.43

and therefore we still obtain

\[ (\hat{h}_y)^{-1} (\hat{r}_{V',V'}^R(\hat{s}')) = (\hat{h}_y)^{-1} (\hat{r}_{V'}^T(\hat{t}')) \cdot (\hat{h}_y)^{-1} (\hat{r}_{V',V'}^{T',Q'}(\hat{u}')) \]

3.7.44.

Then, by the induction hypothesis, setting \(T = (\pi_{N',N}(y^{-1}))(T')\) and \(\hat{u}'' = \hat{r}_{S,N,S}(\hat{y}^{-1}) \cdot \hat{u}' \cdot \hat{r}_{N',N}(\hat{y})\), we have \((\hat{h}_y)^{-1} (\hat{r}_{V',V'}^{T',Q'}(\hat{u}')) = \hat{r}_{V,V}^{T,R}(\hat{u}'')\); moreover, it is quite clear that in 3.7.18 replacing \(N\) by \(T\) and \(N'\) by \(T'\) we still
get the commutative diagrams of $k^*$-groups

\[
\hat{\mathcal{P}}^\pi(T')_V \rightarrow \hat{\mathcal{P}}^\pi(V') \quad \text{and} \quad \hat{\mathcal{P}}^\pi(T')_V \rightarrow \hat{\mathcal{P}}^\pi(T')
\]

and thus, since $\hat{\nu}$ belongs to $\hat{\mathcal{P}}^\pi(T')_V$, setting $\hat{\nu}' = \hat{\tau}^{N',N'}_T(y^{-1})\hat{\tau}^{N',N}_T(y)$ we still have $(\hat{h}_y)^{-1}(\hat{r}_V^n,\hat{\nu}) = \hat{r}_V^n(\hat{\nu}')$. Finally, it is easy to check that

\[
\hat{r}^T_{V,V}(\hat{\nu}') \hat{r}^T_{V,V}(\hat{\nu}) = \hat{r}_V^n(\hat{\nu}') \hat{r}_V^n(\hat{\nu}) \quad \text{which completes the proof of our claim.}
\]

We are ready to define the $k^*$-set $\hat{\mathcal{P}}^\pi(V',V)$ for any pair of subgroups $V$ and $V'$ in $\mathcal{K} \setminus \mathcal{G}$; we clearly have $N = N_P(V) \neq V$ and it follows from [5, Proposition 2.7] that there is a $\mathcal{F}$-morphism $\nu : N \rightarrow P$ such that $\nu(V)$ is fully normalized in $\mathcal{F}$; moreover, we choose $\hat{n} \in \hat{\mathcal{P}}^\pi(\nu(N),N)$ lifting $\nu$, for a $\mathcal{F}$-morphism $\nu : N \rightarrow P$ such that $\nu(V)$ is fully normalized in $\mathcal{F}$.

We denote by $\hat{\mathcal{R}}(V)$ the set of such pairs and often we write $\hat{n}$ instead of $(N,\hat{n})$, setting $\hat{n}N = \nu(N)$, $\hat{n}V = \nu(V)$, and $\pi_n = \nu_n$ where $n$ is the image of $\hat{n}$ in $\mathcal{R}(\nu(N),N)$.

For another pair $(\hat{N},\hat{n})$ in $\hat{\mathcal{R}}(V)$, denoting by $\hat{\nu} : \hat{N} \rightarrow P$ the $\mathcal{F}$-morphism determined by $\hat{n}$, setting $M = (N,\hat{N})$ and considering a new $\mathcal{F}$-morphism $\mu : M \rightarrow P$ such that $\mu(V)$ is fully normalized in $\mathcal{F}$, we can obtain a third pair $(M,\hat{m})$ in $\hat{\mathcal{R}}(V)$; then, $\hat{r}_{m,N,N}^\nu(\hat{m})^{-1}$ and $\hat{r}_{m,N,N}^\nu(\hat{m})^{-1}$ respectively belong to $\hat{\mathcal{P}}^\pi(mN,\hat{N})$ and to $\hat{\mathcal{P}}^\pi(m\hat{N},\hat{N})$; in particular, since $\hat{n}V$, $\hat{m}V$ and $mV$ are fully normalized in $\mathcal{F}$, the $k^*$-sets $\hat{\mathcal{P}}^\pi(\hat{m}V,V)$, $\hat{\mathcal{P}}^\pi(\hat{m}V,V)$ and $\hat{\mathcal{P}}^\pi(\hat{m}V,V)$ have been already defined above, and we consider the element (cf. 3.7.19)

\[
\hat{g}_{\hat{n},\hat{n}} = \hat{r}_{m,V,V}^{\hat{m}N,N}(\hat{m})^{-1} \hat{r}_{m,N,N}^{\hat{m}}(\hat{m})^{-1} \hat{r}_{m,N,N}^{\hat{m}}(\hat{m})^{-1}
\]

in $\hat{\mathcal{P}}^\pi(V',\hat{V})$, which actually does not depend on the choice of $m$.

Indeed, for another pair $(M,\hat{m}')$ in $\hat{\mathcal{R}}(V)$ we have

\[
\hat{r}_{m',\hat{N},\hat{N}}(\hat{m}') = \hat{r}_{m',\hat{N},\hat{N}}(\hat{m}') \quad \text{and} \quad \hat{r}_{m',\hat{N},\hat{N}}(\hat{m}) = \hat{r}_{m',\hat{N},\hat{N}}(\hat{m})
\]
and therefore it follows from equality 3.7.29 that we get

\[ \hat{r}^{m,N,N} \circ \hat{r}^{m,M,M} (\hat{m}' \cdot \hat{n}^{-1}) \]

\[ = \hat{r}^{m,N,N} (\hat{r}^{m,M,M} (\hat{m}' \cdot \hat{n}^{-1}) \circ \hat{r}^{m,M,M} (\hat{m} \cdot \hat{n}^{-1})) \]

\[ = \hat{r}^{m,N,N} (\hat{r}^{m,M,M} (\hat{m}' \cdot \hat{n}^{-1}) \circ \hat{r}^{m,M,M} (\hat{m} \cdot \hat{n}^{-1})) \]

which proves our claim. Similarly, for any triple of pairs \((N, \hat{n}), (N', \hat{n}' \cdot \hat{n}^{-1})\) and \((\hat{N}, \hat{n})\) in \(\hat{\mathcal{N}}(V)\), considering a pair \(((N, \hat{n}), \hat{N}, \hat{n})\) in \(\hat{\mathcal{N}}(V)\), it follows from equality 3.7.29 and from the commutativity of diagram 3.7.32 that

\[ \check{g}_{\hat{n}, \hat{n}'} \circ \hat{g}_{\hat{n}, \hat{n}} = \hat{g}_{\hat{n}, \hat{n}'} \]

Note that if \(V\) is fully normalized in \(\mathcal{F}\) then the pair formed by \(N = N_{\mathcal{P}}(V)\) and by the identity element \(\mathcal{N}\) in \(\hat{\mathcal{P}}(N)\) belongs to \(\hat{\mathcal{R}}(V)\).

Then, for any pair of subgroups \(V'\) and \(V''\) in \(\mathcal{X} - \mathcal{Y}\), since for any \((N, \hat{n}) \in \mathcal{N}(V)\) and any \((N', \hat{n}') \in \mathcal{N}(V)\) the \(k^*\)-set \(\hat{\mathcal{P}}^X(\hat{n}' V', \hat{n} V)\) is already defined, we denote by \(\hat{\mathcal{P}}^X(V', V)\) the \(k^*\)-subset of the product

\[ \prod_{\hat{n} \in \hat{\mathcal{R}}(V)} \prod_{\hat{n}' \in \hat{\mathcal{R}}(V')} \hat{\mathcal{P}}^X(\hat{n}' V', \hat{n} V) \]

formed by the families \(\check{x}_{\hat{n}, \hat{n}'} \circ \hat{x}_{\hat{n}', \hat{n}'} = \hat{x}_{\hat{n}', \hat{n}'} \circ \hat{g}_{\hat{n}, \hat{n}'} \circ \hat{g}_{\hat{n}, \hat{n}'} \)

In other words, the set \(\hat{\mathcal{P}}^X(V', V)\) is the inverse limit of the family formed by the \(k^*\)-sets \(\hat{\mathcal{P}}^X(\hat{n}' V', \hat{n} V)\) and by the bijections between them induced by the \(\hat{\mathcal{P}}^X\)-morphisms \(\hat{g}_{\hat{n}, \hat{n}'}\) and \(\hat{g}_{\hat{n}', \hat{n}'}\).

Note that, according to equalities 3.7.50, the projection map onto the factor labeled by the pair \(((N, \hat{n}), (N', \hat{n}'))\) induces a \(k^*\)-set isomorphism

\[ n_{\hat{n}', \hat{n}} : \hat{\mathcal{P}}^X(V', V) \cong \hat{\mathcal{P}}^X(\hat{n}' V', \hat{n} V) \]
in particular, if \( V \) and \( V' \) are fully normalized in \( \mathcal{F} \), setting \( N = N_P(V) \) and \( N' = N_P(V') \), the pairs \((N, i_N)\) and \((N', i_{N'})\) respectively belong to \( \mathcal{R}(V) \) and to \( \mathcal{R}(V') \), and therefore we have a canonical bijection

\[
n_{i_{N'}, i_N} : \hat{\mathcal{P}}^\times(V', V) \cong \hat{\mathcal{P}}^\times(i_N V', i_N V) \quad 3.7.54,
\]

so that our notation is coherent. Moreover, we have an obvious map

\[
\hat{\mathcal{P}}^\times(V', V) \longrightarrow \mathcal{P}(V', V) \quad 3.7.55
\]

and, for any \( u \in \mathcal{T}_P(V', V) \) and a suitable pair \(((N, \hat{n}), (N', \hat{n}'))\), we may assume that \( u \) belongs to \( \mathcal{T}_P(N', N) \) too; then, we consider the map

\[
\tilde{r}_{V', V}^\times : \mathcal{T}_P(V', V) \longrightarrow \hat{\mathcal{P}}^\times(V', V) \quad 3.7.56
\]

determined by

\[
n_{\hat{n}', \hat{n}}(\tilde{r}_{V', V}^\times(u)) = \tilde{r}_{\hat{n}', \hat{n}}^\times((\hat{n}', \hat{\mathcal{P}}^\times(u)) \cdot \hat{n}^{-1}) \quad 3.7.57,
\]

which does not depend on our choice.

Analogously, for any pair of subgroups \( Q \) and \( Q' \) of \( P \) respectively normalizing and strictly containing \( V \) and \( V' \), we can define an injective \( k^*\)-set homomorphism

\[
\hat{r}_{V', V}^Q : \hat{\mathcal{P}}^\times(Q', Q)_{V', V} \longrightarrow \hat{\mathcal{P}}^\times(V', V) \quad 3.7.58
\]

which lifts the restriction map (cf. 3.2.1)

\[
r_{V', V}^Q : \mathcal{P}(Q', Q)_{V', V} \longrightarrow \mathcal{P}(V', V) \quad 3.7.59
\]

and coincides with the \( k^*\)-set homomorphism 3.7.23 whenever \( V \) and \( V' \) are fully normalized in \( \mathcal{F} \); indeed, it is clear that we have pairs \((Q, \hat{n})\) in \( \mathcal{R}(V) \) and \((Q', \hat{n}')\) in \( \mathcal{R}(V') \), and then, for any \( \hat{x} \in \hat{\mathcal{P}}^\times(Q', Q)_{V', V} \), we set

\[
n_{\hat{n}', \hat{n}}(\hat{r}_{V', V}^Q(\hat{x})) = \hat{r}_{\hat{n}', \hat{n}}^\times((\hat{n}', \hat{x}) \cdot \hat{n}^{-1}) \quad 3.7.60,
\]

which does not depend on our choices. Moreover, it is easily checked that equality 3.7.29 still holds in this general situation.

On the other hand, for any \( V'' \in \mathfrak{X} - \mathfrak{Q} \), the \( k^*\)-composition map defined in 3.7.19 — and just noted \( \cdot \) from now on — can be extended to a new \( k^*\)-composition map

\[
\hat{\mathcal{P}}^\times(V'', V') \times \hat{\mathcal{P}}^\times(V', V) \longrightarrow \hat{\mathcal{P}}^\times(V'', V) \quad 3.7.61
\]

sending \((\hat{x}'', \hat{x}) \in \hat{\mathcal{P}}^\times(V'', V') \times \hat{\mathcal{P}}^\times(V', V)\) to

\[
\hat{x}' \cdot \hat{x} = (n_{\hat{n}'', \hat{n}})^{-1}(n_{\hat{n}'', \hat{n}}(\hat{x}') \cdot n_{\hat{n}'', \hat{n}}(\hat{x})) \quad 3.7.62
\]
for a choice of \((N, \hat{n})\) in \(\hat{\mathcal{H}}(V)\), of \((N', \hat{n}')\) in \(\hat{\mathcal{H}}(V')\) and of \((N'', \hat{n}'')\) in \(\hat{\mathcal{H}}(V'')\).

This \(k^*\)-composition map does not depend on our choice; indeed, for another choice of pairs \((\hat{N}, \hat{n})\) in \(\hat{\mathcal{H}}(V)\), \((\hat{N}', \hat{n}')\) in \(\hat{\mathcal{H}}(V')\) and \((\hat{N}'', \hat{n}'')\) in \(\hat{\mathcal{H}}(V'')\), we get (cf. 3.7.52)

\[
\hat{g}_{\hat{n}'', \hat{n}'} \cdot n_{\hat{n}'', \hat{n}'}(\hat{x}') : n_{\hat{n}'', \hat{n}'}(\hat{x}) = n_{\hat{n}', \hat{n}'}(\hat{x}') \cdot \hat{g}_{\hat{n}'', \hat{n}'}(\hat{x})
\]

In particular, for any triple of subgroups \(Q, Q'\) and \(Q''\) of \(P\) respectively normalizing and strictly containing \(V, V'\) and \(V''\), choosing pairs \((Q, \hat{n})\) in \(\hat{\mathcal{H}}(V)\), \((Q', \hat{n}')\) in \(\hat{\mathcal{H}}(V')\) and \((Q'', \hat{n}'')\) in \(\hat{\mathcal{H}}(V'')\). The commutativity of the corresponding diagram 3.7.32 forces the commutativity of the analogous diagram in the general situation

\[
\hat{P}^\partial(Q'', Q')_{V''V'} \times \hat{P}^\partial(Q', Q)_V \longrightarrow \hat{P}^\partial(Q'', Q)_{V''V}
\]

Finally, for any \(V''' \in \mathcal{X} - \mathcal{Y}\) and any \(\hat{x}'' \in \hat{P}(V'', V''')\), it follows from 3.7.21 that

\[
(\hat{x}'' \cdot \hat{x}) \cdot \hat{x} = \hat{x}'' \cdot (\hat{x} \cdot \hat{x})
\]

We are ready to complete our construction of the announced regular central \(k^*\)-extension \(\hat{P}^x\) of \(P^x\), endowed with a lifting \(\hat{\tau}^x : \mathcal{T}_P^x \rightarrow \hat{P}^x\) of \(\tau^x\) fulfilling condition 3.5.1; we are already assuming that \(\hat{P}\) contains \(P^\partial\) as a full \(k^*\)-subcategory over \(\mathcal{Y}\) and that \(\hat{\tau}\) extends \(\tau^\partial\). For any subgroups \(V\) in \(\mathcal{X} - \mathcal{Y}\) and \(Q\) in \(\mathcal{Y}\) we define

\[
\hat{P}^x(V, Q) = \emptyset \quad \text{and} \quad \hat{P}^x(Q, V) = \bigcup_{V'} \hat{P}^x(V', V)
\]

where \(V\) runs over the set of subgroups \(V' \in \mathcal{X} - \mathcal{Y}\) contained in \(Q\) and the \(k^*\)-subset \(\hat{P}^x(V', V)\) of \(\hat{P}^x(Q, V)\) coincides with the converse image of the subset \(\tau_{Q, V}(1) \cdot \hat{P}(V', V)\) in \(P(Q, V)\); moreover, any \(u \in \mathcal{T}_P(Q, V)\) also belongs to \(\mathcal{T}_{P}(uV)\) we define \(\hat{\tau}_{Q, V}(u)\) as the element \(\hat{\tau}_{uV}^x(u)\) in \(\hat{\mathcal{X}}\) in the union above.

In order to define the composition of two \(\hat{P}^x\)-morphisms \(\hat{x} : R \rightarrow Q\) and \(\hat{y} : T \rightarrow R\) we already may assume that \(\hat{T}\) does not belong to \(\mathcal{Y}\); if \(Q\) does not belong to \(\mathcal{Y}\) then the composition \(\hat{x} \cdot \hat{y}\) is given by the map 3.7.61; if \(Q\) belongs to \(\mathcal{Y}\) but \(R\) does not then, setting \(R' = \varphi(R)\) where \(\varphi\) is the image of \(\hat{x}\) in \(\mathcal{F}(Q, R)\), it follows from definition 3.7.66 that \(\hat{x}\) is actually an element of \(\hat{P}^x(R', T)\), that \(\hat{y}\) is an element of \(\hat{P}^x(R, T)\) and that the element \(\hat{x} \cdot \hat{y}\)
defined by the map 3.7.61 belongs to $\hat{\mathcal{P}}^x(R', T) \subset \hat{\mathcal{P}}^x(Q, T)$, so that we can define the composition of $\hat{x}$ and $\hat{y}$ by this element $\hat{x} \cdot \hat{y}$. Finally, assume that $R$ belongs to $\mathcal{Q}$ and, denoting by $\psi$ the image of $\hat{y}$ in $\mathcal{F}(R, T)$, consider the subgroups $T' = \psi(T)$ of $R$ and $T'' = \varphi(T')$ of $Q$; then, it follows again from definition 3.7.66 that $\hat{y}$ is actually an element of $\hat{\mathcal{P}}^x(T', T)$; moreover, setting $\tilde{R} = N_R(T')$ and $\tilde{Q} = N_Q(T'')$, it is clear that $\hat{r}_{\tilde{Q}, \tilde{R}}^{\tilde{Q}, \tilde{R}}(\hat{x})$ belongs to $\hat{\mathcal{P}}^x(\tilde{Q}, \tilde{R})$ (cf. 3.6) and we can define (cf. 3.7.58 and 3.7.61)

$$\hat{x} \cdot \hat{y} = r_{\tilde{T}', \tilde{T}}^{\tilde{Q}, \tilde{R}}(r_{\tilde{Q}, \tilde{R}}^{\tilde{Q}, \tilde{R}}(\hat{x})) \cdot \hat{y} \tag{3.7.67}.$$ 

This composition is clearly compatible with the action of $k^*$. Moreover, for a third $\hat{\mathcal{P}}^x$-morphism $\hat{z} : V \to T$ we claim that

$$(\hat{x} \cdot \hat{y}) \cdot \hat{z} = \hat{x} \cdot (\hat{y} \cdot \hat{z}) \tag{3.7.68}.$$ 

Once again, we may assume that $V$ does not belong to $\mathcal{Q}$; if $Q$ does not belong to $\mathcal{Q}$ then this equality follows from equality 3.7.65; if $Q$ belongs to $\mathcal{Q}$ but $R$ does not then $\hat{x}$ is actually an element of $\hat{\mathcal{P}}^x(R', R)$ and this equality follows again from equality 3.7.65. From now on, assume that $R$ belongs to $\mathcal{Q}$; then, if $T \in \mathcal{Q}$, denoting by $\eta$ the image of $\hat{z}$ in $\mathcal{F}(T, V)$, considering the subgroups $V' = \eta(V)$ of $T$, $V'' = \psi(V')$ and $V''' = \varphi(V'')$ and setting $\tilde{T} = N_T(V')$, $\tilde{R} = N_R(V'')$ and $\tilde{Q} = N_Q(V''')$, then we have (cf. 3.7.67)

$$(\hat{x} \cdot \hat{y}) \cdot \hat{z} = \left(\hat{r}_{\tilde{V}'', \tilde{V}}^{\tilde{Q}, \tilde{T}}(\hat{r}_{\tilde{Q}, \tilde{R}}^{\tilde{Q}, \tilde{R}}(\hat{x})) \cdot \hat{y}\right) \cdot \hat{z} \tag{3.7.69}.$$ 

but, it follows from 3.6 and from the commutativity of diagram 3.64 that

$$\hat{r}_{\tilde{V}'', \tilde{V}}^{\tilde{Q}, \tilde{T}}(\hat{r}_{\tilde{Q}, \tilde{R}}^{\tilde{Q}, \tilde{R}}(\hat{x})) = \hat{r}_{\tilde{V}'', \tilde{V}}^{\tilde{Q}, \tilde{T}}(\hat{r}_{\tilde{Q}, \tilde{R}}^{\tilde{Q}, \tilde{R}}(\hat{x})) \cdot \hat{r}_{\tilde{V}'', \tilde{V}}^{\tilde{Q}, \tilde{T}}(\hat{r}_{\tilde{R}, \tilde{T}}^{\tilde{R}, \tilde{T}}(\hat{y})) \tag{3.7.70};$$

consequently, since $\hat{y} \cdot \hat{z}$ is actually an element of $\hat{\mathcal{P}}^x(V'', V)$, it follows from equality 3.6.75 that

$$(\hat{x} \cdot \hat{y}) \cdot \hat{z} = \hat{r}_{\tilde{V}'', \tilde{V}}^{\tilde{Q}, \tilde{T}}(\hat{r}_{\tilde{Q}, \tilde{R}}^{\tilde{Q}, \tilde{R}}(\hat{x})) \cdot \hat{r}_{\tilde{V}'', \tilde{V}}^{\tilde{Q}, \tilde{T}}(\hat{r}_{\tilde{R}, \tilde{T}}^{\tilde{R}, \tilde{T}}(\hat{y})) \cdot \hat{z} = \hat{r}_{\tilde{V}'', \tilde{V}}^{\tilde{Q}, \tilde{T}}(\hat{r}_{\tilde{Q}, \tilde{R}}^{\tilde{Q}, \tilde{R}}(\hat{x})) \cdot (\hat{y} \cdot \hat{z}) \tag{3.7.71}.$$ 

Finally, assume that $T$ does not belong to $\mathcal{Q}$; then, we actually have $V' = T$, $V'' = T'$ and $V''' = T''$, and it follows from 3.7.65 and 3.7.67 that

$$(\hat{x} \cdot \hat{y}) \cdot \hat{z} = \left(\hat{r}_{\tilde{T}', \tilde{T}}^{\tilde{Q}, \tilde{R}}(\hat{r}_{\tilde{Q}, \tilde{R}}^{\tilde{Q}, \tilde{R}}(\hat{x})) \cdot \hat{y}\right) \cdot \hat{z} = \hat{r}_{\tilde{V}'', \tilde{V}}^{\tilde{Q}, \tilde{T}}(\hat{r}_{\tilde{Q}, \tilde{R}}^{\tilde{Q}, \tilde{R}}(\hat{x})) \cdot (\hat{y} \cdot \hat{z}) = \hat{x} \cdot (\hat{y} \cdot \hat{z}) \tag{3.7.72}.$$
It remains to prove the functoriality of $\hat{\tau}^x$; that is to say, for any pair of $T^x_p$-morphisms $u: R \to Q$ and $v: T \to R$ we claim that

$$\hat{\tau}^x_{Q,T}(uv) = \hat{\tau}^x_{Q,R}(u) \cdot \hat{\tau}^x_{R,T}(v)$$

3.7.73;

once again, we may assume that $T$ does not belong to $\mathfrak{Y}$; setting $T' = vTv^{-1}$ and $T'' = uT'u^{-1}$, it follows easily from our definition and from 3.7.57 that we have

$$\hat{\tau}^x_{Q,T}(uv) = \hat{\tau}^x_{T''_Q,T}(uv) = \hat{\tau}^x_{T''_Q,T'}(u) \cdot \hat{\tau}^x_{T',T}(v)$$

3.7.74;

if $R$ does not belong to $\mathfrak{Y}$ then we have $R = T'$ and, according to our definition, we still have $\hat{\tau}^x_{T'',T'}(u) = \hat{\tau}^x_{Q,R}(u)$; otherwise, setting $\bar{R} = N_R(T')$ and $\bar{Q} = N_Q(T'')$, it follows from 3.7.67 and 3.7.57 that

$$\hat{\tau}^x_{\bar{Q},R}(u) \cdot \hat{\tau}^x_{R,T}(v) = \bar{T} \circ \hat{\tau}^x_{Q,R}(u) \cdot \hat{\tau}^x_{T',T}(v)$$

3.7.75.

In order to prove the uniqueness of $\hat{\tau}^x$, let $\hat{\mathcal{P}}^x$ be another regular central $k^*$-extension of $\mathcal{P}^x$, endowed with a functor $\hat{\tau}: T^x_p \to \hat{\mathcal{P}}^x$ fulfilling condition 3.5.1, inducing the folded Frobenius $P$-category $(\mathcal{F}, \hat{\mathcal{A}}\mathcal{T}_{p,x})$ or, equivalently, $(\mathcal{P}, \hat{\mathcal{A}}\mathcal{T}_{p,x})$. We may assume that $\mathcal{X} \neq \{P\}$ and then, choosing a minimal element $U$ in $\mathcal{X}$ fully normalized in $\mathcal{F}$ and setting

$$\mathfrak{Y} = \mathcal{X} - \{\theta(U) \mid \theta \in \mathcal{F}(P,U)\}$$

3.7.76,

we may also assume that $N_U(U) \neq \mathcal{F}$.

In particular, for any group $Q$ in $\mathcal{X}$, denoting by $q_Q: \Delta_0 \to \mathcal{P}^x$ the functor sending $0$ to $Q$, we have

$$\hat{\mathcal{P}}^x(Q) = \hat{\mathcal{A}}\mathcal{T}_{p,x}(q_Q) = \hat{\mathcal{P}}^x(Q)$$

3.7.77;

similarly, for any group $V$ in $\mathcal{X} - \mathfrak{Y}$ fully normalized in $\mathcal{F}$, setting $N = N_P(V)$ and denoting by $n_V: \Delta_1 \to P^x$ the functor sending $0$ to $V$, $1$ to $N$ and $0 \cdot 1$ to $\hat{\tau}^x_{N,V}(1)$, and by $\hat{\mathcal{P}}^x(N)_V$ and $\hat{\mathcal{P}}^x(N)_V$ the corresponding stabilizers of $V$ in $\hat{\mathcal{P}}^x(N)$ and $\hat{\mathcal{P}}^x(N)$, we have

$$\hat{\mathcal{P}}^x(N)_V = \hat{\mathcal{A}}\mathcal{T}_{p,x}(n_V) = \hat{\mathcal{P}}^x(N)_V$$

3.7.78;

moreover, $\hat{\mathcal{A}}\mathcal{T}_{p,x}$ sends the obvious $ch^*(\mathcal{P}^x)$-morphism $(n_V, \Delta_1) \to (q_V, \Delta_0)$ to the injective restriction from $\hat{\mathcal{P}}^x(N)_V = \hat{\mathcal{P}}^x(N)_V$ to $\hat{\mathcal{P}}^x(V) = \hat{\mathcal{P}}^x(V)$. 
Arguing by induction on \(|X|\) we may assume that we have an equivalence of categories \(f^\circ : \hat{P}^0 \to \hat{P}^0\) inducing the identity on \(\hat{P}^0(Q) = \hat{P}^0(Q)\) for any group \(Q\) in \(\mathcal{Q}\) and fulfilling \(f^\circ \circ \tau^\circ = \tau^\circ\). We will extend \(f^\circ\) to a functor \(\hat{f}^\circ : \hat{P}^x \to \hat{P}^x\) inducing the identity on \(\hat{P}^0(Q) = \hat{P}^0(Q)\) for any group \(Q\) in \(\mathcal{X}\) and fulfilling \(\hat{f}^\circ \circ \tau^\circ \hat{f}^\circ = \tau^\circ\); for any pair of groups \(V\) and \(V'\) in \(\mathcal{X} - \mathcal{Q}\) fully normalized in \(\mathcal{F}\), any \(\hat{y} \in \hat{P}^0(N', N)_{V', V}\) where \(N' = N_P(V')\) and \(N = N_P(V)\), and any \(\hat{s} \in \hat{P}^x(V)\), we define

\[
\hat{f}^\circ (\hat{r}^x_{V', V}(\hat{y}), \hat{s}) = \hat{r}^x_{V', V}(f^\circ(\hat{y})), \hat{s} \quad 3.7.79;
\]

the definition is correct since for any \(\hat{t} \in \hat{P}^0(N)_V\) we have

\[
\hat{f}^\circ (\hat{r}^x_{V', V}(\hat{y}, \hat{t})).(\hat{r}^x_{V', V}(\hat{t}^{-1}), \hat{s}) = \hat{r}^x_{V', V}(f^\circ(\hat{y}, \hat{t})).(\hat{r}^x_{V', V}(\hat{t}^{-1}), \hat{s})
\]

\[
= \hat{r}^x_{V', V}(f^\circ(\hat{y})).(\hat{r}^x_{V', V}(\hat{t}^{-1}), \hat{s}) \quad 3.7.80.
\]

It follows from Lemma 3.3 that \(\hat{f}^\circ\) induces a bijection from \(\hat{P}^x(V', V)\) onto \(\hat{P}^x(V', V)\); moreover, if \(V''\) is a third group in \(\mathcal{X} - \mathcal{Q}\) fully normalized in \(\mathcal{F}\), setting \(N'' = N_P(V'')\) and considering \(\hat{y}' \in \hat{P}^0(N'', N')_{V'', V'}\) and \(\hat{s}' \in \hat{P}^x(V')\), it follows from [5, Condition 2.8.2] that \(\hat{s}' = \hat{r}^x_{V'', V'}(\hat{z}')\) for some \(\hat{z}' \in \hat{P}^x(N')\) and therefore we get

\[
\hat{f}^\circ (\hat{r}^x_{V'', V'}(\hat{y}'), \hat{s}').\hat{r}^x_{V', V}(\hat{y}).\hat{s} = \hat{f}^\circ (\hat{r}^x_{V'', V'}(\hat{y}'), \hat{s}' \hat{r}^x_{V', V}(\hat{y})).\hat{s}
\]

\[
= \hat{r}^x_{V'', V'}(f^\circ(\hat{y}')).\hat{s}' \hat{r}^x_{V', V}(f^\circ(\hat{y})).\hat{s} \quad 3.7.81.
\]

In particular, for any group \(V\) in \(\mathcal{X} - \mathcal{Q}\), as in 3.7.46 we can define an analogous set \(\tilde{H}(V)\) of pairs \((N, \tilde{n})\) formed by a subgroup \(N\) of \(P\) which strictly contains and normalizes \(V\), and by an \(\hat{P}^0\)-isomorphism \(\tilde{n}\) from \(N\) such that \(^nV\), where \(n\) is the image of \(\tilde{n}\) in \(N\), is fully normalized in \(\mathcal{F}\); similarly, for any pair of elements \((N, \tilde{n})\) and \((\tilde{N}, \tilde{n})\) in \(\tilde{H}(V)\), we can define an element \(\tilde{g}_{n, \tilde{n}}\) in \(\hat{P}^x(\tilde{n}V, nV)\) analogous to the element \(\hat{g}_{n, \tilde{n}} \in \hat{P}^x(\tilde{n}V, nV)\).
defined in 3.7.47 above and clearly get \( \hat{f}^X(\hat{g}_{\hat{n},\hat{n}}) = \hat{g}_{\hat{n},\hat{n}} \). Then, for any group \( V' \in X \setminus \mathcal{F} \), we have an obvious bijection from \( \mathcal{P}^X(V',V) \) onto the \( k^* \)-subset of the product

\[
\prod_{\hat{n} \in \mathcal{R}(V)} \prod_{\hat{n}' \in \mathcal{R}(V')} \mathcal{P}^X(\hat{n}'V',\hat{n}V)
\]

formed by the families \( \{\hat{x}_{\hat{n},\hat{n}'}\}_{\hat{n} \in \mathcal{R}(V), \hat{n}' \in \mathcal{R}(V')} \) fulfilling

\[
\hat{g}_{\hat{n}',\hat{n}} \cdot \hat{x}_{\hat{n},\hat{n}'} = \hat{x}_{\hat{n}',\hat{n}} \cdot \hat{g}_{\hat{n},\hat{n}}
\]

hence, \( \hat{f}^X \) can be extended to a bijection from \( \mathcal{P}^X(V',V) \) onto \( \mathcal{P}^X(V',V) \). At present, it is quite clear that \( \hat{f}^X \) can be extended to an equivalence of categories from \( \mathcal{P}^X \) onto \( \mathcal{P}^X \). We are done.

**Corollary 3.8.** Let \( G \) be a finite group, \( b \) a block of \( G \) and \( P \) a defect group of \( b \). There is a regular central \( k^* \)-extension \( \check{F}_{(b,G)} \) of \( F_{(b,G)} \) admitting a \( k^* \)-group isomorphism

\[
\check{F}_{(b,G)}(Q) \cong \hat{N}_G(Q,f)/C_G(Q)
\]

for any \( F_{(b,G)} \)-selfcentralizing subgroup \( Q \) of \( P \).

**Proof:** It is an easy consequence of [5, Theorem 11.32] and Theorem 3.7.

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