Categorification of Negative Information using Enrichment

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In many engineering applications it is useful to reason about “negative information”. For example, in planning problems, providing an optimal solution is the same as giving a feasible solution (the “positive” information) together with a proof of the fact that there cannot be feasible solutions better than the one given (the “negative” information). We model negative information by introducing the concept of “norphisms”, as opposed to the positive information of morphisms. A “nategory” is a category that has “nom”-sets in addition to hom-sets, and specifies the interaction between norphisms and morphisms. In particular, we have composition rules of the form morphism + norphism → norphism. Norphisms do not compose by themselves; rather, they use morphisms as catalysts. After providing several applied examples, we connect nategories to enriched category theory. Specifically, we prove that categories enriched in de Paiva’s dialectica categories GC, in the case C = Set and equipped with a modified monoidal product, define nategories which satisfy additional regularity properties. This formalizes negative information categorically in a way that makes negative and positive morphisms equal citizens.

1 Introduction

1.1 Manipulation of negative information is important in applications of category theory

Our research group’s background is in robotics and systems theory. In these fields, we have found that category theory can describe well many of the structures in our problems, but something is often missing: we find ourselves in the position of reasoning and writing algorithms that manipulate “negative information”, but we do not know what is an appropriate categorical concept for it. We give some examples.

Robot motion planning can be formalized as the problem of finding a trajectory through an environment, respecting some constraint (e.g., avoiding obstacles). One can think of the robot configuration manifold \( M \) as a category where the objects are elements of the tangent bundle and the morphisms are the feasible paths according to the problem constraints. The output of planning problems has an intuitive representation in category theory, if the problem is feasible. A path planning algorithm is given two objects and must compute a morphism as a solution. A motion planning algorithm would compute a trajectory, which could be seen as a functor \( F \) from the manifold \([0, T]\) to \( M \) with \( F(0) = A \) and \( F(T) = B \). However, if the problem is infeasible—if no morphisms between two points can be found—if the algorithm must present a certificate of infeasibility—what is the equivalent concept in category theory?

In many cases, the problems are not binary (either a solution exists or not, either a proposition is true or not) but we care about the performance of solutions. For example, consider the case of the weighted shortest path problem in dynamic programming. The problem is to find a path through a graph that minimizes the sum of the weights of the edges on the path. In robotics, this can be used for planning problems, where the weights could represent the time, the distance, or the energy required by a robot to traverse an edge, and the nodes are either regions of space or, more generally, joint states of the world and environment. Proving that a path is optimal means producing the path together with a proof that there are no shorter paths. This is called a “certificate of optimality” and, like certificates of infeasibility, is negative information as it consists in negating the existence of a certain class of paths. Interestingly, one
can see algorithms such as Dijkstra’s algorithm as constructing both positive and negative information at the same time, such that when a path is finally found, we are sure that there are no shorter ones. In some cases, the negative information is a first-class citizen which is critical to the efficiency. Algorithms such as A* require the definition of heuristic functions, which is negative information: they provide a lower bound on the cost of a path between two points. And better heuristics make the algorithm faster. Again, we ask, what could be the categorical counterpart of heuristics?

In co-design, a morphism $F \to R$ describes what functionality can be achieved with which resources. They are characterized as boolean profunctors, that is, monotone functions $F^{op} \times R \to \text{Bool}$.

The negative information would be a “nesign” problem that characterizes an impossibility. For example, if $F = R = \text{Energy}$, we expect that in this universe we cannot find a realizable morphism $d$ that satisfies $d(2J, 1J)$ (obtaining 2 Joules from 1 Joule). Can this be expressed as some sort of morphism? In which category does it live?

1.2 Our approach: “Categorification” of negative information

We briefly describe our thought process in finding a formalization for dealing with negative information. One approach could have been to build structure on top of a category, at a higher level, using logic. We eschew this approach because of the belief that we should find a duality between positive and negative information that puts them “at the same level”.

Our approach has been one in the spirit of “categorification”: representing the negative information with a concrete structure for which to find axioms and inference rules.

An early influence in our thinking was the paper of Shulman about “proofs and refutations”. What follows is a simplified explanation of one of the concepts of the paper. Consider a category where objects are propositions and morphisms $X \to Y$ are propositions $X \Rightarrow Y$ (with the particular case of $X \cong (\top \to X)$). We can then consider the type $P(X \to Y)$ of proofs and the type $R(X \to Y)$ of refutations, which correspond to positive and negative information. According to intuitionist logic, $P(X \to Y) = (P(X) \to P(Y)) \times (R(Y) \to R(X))$: a proof of $X \Rightarrow Y$ is a way to convert a proof of $X$ into a proof of $Y$ together with a way to convert a refutation of $Y$ into a refutation of $X$.

In that paper, proofs and refutations, positive and negative information, are treated at the same level but not symmetrically—proof and refutations have different semantics, and $P$ and $R$ map products and coproducts ($\lor$, $\land$) to different linear logic operators. This led to the idea that negative information should be at the same level of positive information: if positive information is represented by morphisms, then also the negative information should be described as “negative arrows” between objects, which we called norphisms (for negative morphisms).

We also realized that the positive/negative information duality we are looking for is richer than the structure of proofs/refutations in logic. In (classical/intuitionistic) logic, one expects the existence of either a proof of a proposition $A$, a refutation of $A$, or neither, but not both. Instead, in our formalization, norphisms are a more general notion, which can coexist with morphisms and give complementary information, as in the planning examples in the introduction.

An initial idea was to consider for each category a “twin” category, whose morphisms would be the norphisms we were looking for to represent the negative information; however, this idea failed. In the course of the paper, it will be clear that positive/negative information cannot be decoupled, because negative information cannot be composed independently of positive information. In the end, we unite them by viewing them as part of a single enrichment structure.

1.3 Plan of the paper

This paper follows an inductive exposition and is divided in two parts.

In the first part we provide the motivation and several examples of representing negative information with “norphism” structure. In Section 2 we consider the case of a thin category. In this simple setting we can already see that norphisms compose differently than morphisms, and that we need two
composition rules for them. In Section 3 we state our main definition, that of a “nategory”, and in Section 4 we show some canonical ways to build a nategory out of a category. In Sections 5 and 6 we discuss two examples, Berg and DP, which have norphism structures in which norphisms and morphisms are not mutually exclusive.

In the second part our goal is to provide an elegant way to think of norphisms and their composition by using enriched category theory. By doing so, we show that the additional structure of norphisms and their composition, rules which might initially appear “funky”, is not an arbitrary structure, but rather it is as “natural” as the positive information of morphisms. In Section 7 we introduce the dialectica category GSet and define a monoidal product for it which is slightly different than the ones usually used as linear logic connectives. Then, in Section 8, we prove that GSet-enriched categories encode nategories which satisfy some additional compatibilities between morphisms and norphisms. These additional compatibilities are not satisfied in certain examples of interest to us, therefore we have refrained from including them directly in our definition of nategory.

2 Building intuition: the case of thin categories

To build an intuition about norphisms, we look at the case of “thin” categories, in which each hom-set contains at most one morphism. Thin categories are essentially pre-orders. To aid the interpretation, one can think of a pre-order as defining a reachability relation, in which a morphism $X \rightarrow Y$ represents “I can reach $Y$ from $X$”. Or, we can think of morphisms as (proof-irrelevant) implications: $X \rightarrow Y$ represents “I can prove $Y$ from $X$”. In a thin category, negative information is limited to indicate the refutation of positive information. Therefore, a norphism $n: X \nrightarrow Y$ is equivalent to “There are no morphisms from $X$ to $Y$”. Particularly, this means “I cannot reach $Y$ from $X$” or “I cannot prove $Y$ from $X$”.

We will later see that, in general, norphisms need not necessarily be mutually exclusive with morphisms. Still, this example is sufficient to get us started in appreciating how morphisms and norphisms compose differently. The composition rule for morphisms reads:

$$
\frac{f: X \rightarrow Y \quad g: Y \rightarrow Z}{(f \circ g): X \rightarrow Z} \quad (1)
$$

Mimicking this, one could start with two norphisms $n: X \nrightarrow Y$ and $m: Y \nrightarrow Z$ and expect to be able to say something about a norphism $X \nrightarrow Z$, with a composition rule of the form:

$$
\frac{n: X \nrightarrow Y \quad m: Y \nrightarrow Z}{???: X \nrightarrow Z} \quad (2)
$$

However, norphisms do not compose this way. In fact, one can derive the following rule:

$$
\frac{o: X \nrightarrow Z \quad Y: \text{Ob}_C}{(n: X \nrightarrow Y) \vee (m: Y \nrightarrow Z)} \quad (3)
$$

This rule is “the dual” of (1) in the same sense as these two axioms are dual:

$$
\frac{\top}{X \rightarrow X} \quad \frac{X \rightarrow X}{\bot} \quad (4)
$$

that is, in the sense of flipping vertically and negating the propositions.

We read (3) as saying that if there is no morphism $X \rightarrow Z$, it is because, for every possible intermediate $Y$, there cannot be a morphism $X \rightarrow Y$ or $Y \rightarrow Z$. Note that composition goes in the “opposite” direction meaning that from one norphism, we get some information about the existence of one or two in a pair. The composition (3) is not constructive: from the “$\vee$”, we do not know which side we can create. Indeed, this composition highlights the asymmetry between morphisms and norphisms: morphisms
compose constructively by themselves (i.e., without taking into account norphisms); norphisms, instead, do not “compose”, but rather “decompose” by themselves. To construct norphisms, we need to start from a norphism and a morphism that acts as a “catalyst”.

When interpreting a thin category as a graph, if there is a norphism $n : X \rightarrow Y$, it means that for any $Y$, the path $X \rightarrow Y \rightarrow Z$ must be interrupted in one part or the other, because otherwise we would have a contradiction. Indeed, if we know that morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ exist, then their composition $f \circ g : X \rightarrow Z$ must exist, and therefore no norphism $n : X \rightarrow Z$ can exist. This observation can be turned around in a constructive way. Starting from a morphism $f : X \rightarrow Y$ and a norphism $n : X \rightarrow Z$ (i.e., morphisms and norphisms with the same source), we can infer a norphism $f \circ n : Y \rightarrow Z$ (i.e., there cannot be a morphism $Y \rightarrow Z$):

$$
\begin{array}{c}
\begin{array}{c}
X \\
\uparrow{n}
\end{array} & f & \rightarrow & Y \\
\downarrow{n}
\end{array} \quad \Rightarrow \quad \begin{array}{c}
\begin{array}{c}
Z \\
\uparrow{n}
\end{array} & f \circ n & \leftarrow & Y \\
\downarrow{n}
\end{array} \quad \frac{Y \leftarrow X \rightarrow Z}{Y \rightarrow Z}.
\end{array}
$$

Symmetrically, starting from a morphism $g : Y \rightarrow Z$ and a norphism $n : X \rightarrow Z$ (i.e., morphisms and norphisms with the same target), we can infer a norphism $n \circ f : X \rightarrow Y$:

$$
\begin{array}{c}
\begin{array}{c}
X \\
\uparrow{n}
\end{array} & g & \leftarrow & Y \\
\downarrow{n}
\end{array} \quad \Rightarrow \quad \begin{array}{c}
\begin{array}{c}
Z \\
\uparrow{n}
\end{array} & g \circ n & \rightarrow & Y \\
\downarrow{n}
\end{array} \quad \frac{X \rightarrow Z \rightarrow Y}{X \rightarrow Y}.
\end{array}
$$

Note that the new norphism is pointing in the “same direction” as the starting one, meaning that either source or target are preserved.

3 Describing negative information: natices

In this section we make the notion of norphisms more precise, by defining the additional structure which a category must have in order to encode negative information.

**Definition 1** (Nategorie). A locally small nategorie $C$ is a locally small category with the following additional structure. For each pair of objects $X, Y \in \text{Ob}_C$, in addition to the set of morphisms $\text{Hom}_C(X; Y)$, we also specify:

- A set of norphisms $\text{Nom}_C(X; Y)$.
- An incompatibility relation, which we write as a binary function

$$
i_{XY} : \text{Nom}_C(X; Y) \times \text{Hom}_C(X; Y) \rightarrow \{\bot, \top\}.
$$

For all triples $X, Y, Z$, in addition to the morphism composition function

$$
i_{XYZ} : \text{Hom}_C(X; Y) \times \text{Hom}_C(Y; Z) \rightarrow \text{Hom}_C(X; Z),
$$

we require the existence of two norphism composition functions

$$
i_{XYZ} : \text{Hom}_C(X; Y) \times \text{Nom}_C(X; Z) \rightarrow \text{Nom}_C(Y; Z),$$

$$
i_{XYZ} : \text{Nom}_C(X; Z) \times \text{Hom}_C(Y; Z) \rightarrow \text{Nom}_C(X; Y),$$

and we ask that they satisfy two “equivariance” conditions:

$$
i_{YZ}(f \circ n, g) \Rightarrow i_{XZ}(n, f \circ g), \quad \text{(equiv-1)}$$

$$
i_{XY}(n \circ g, f) \Rightarrow i_{XZ}(n, f \circ g). \quad \text{(equiv-2)}$$
The morphism $f : X \to Y$ must be pulled back incompatible morphisms.

We find incompatibility with $n$.

Catalyst morphism need to find a morphism here.

Condition (equiv-1) says that the morphism $f \leftarrow n$ can exclude the morphism $g$ only if $f \circ g$ is excluded by $n$. The idea is that such a $g$ should not be excluded for any "additional reasons", but only on the grounds that $f \circ g$ is excluded by $n$.

We draw some figures to develop further intuition (Fig. 1). Let $J_{XY}$ denote the function which maps a morphism to the set of morphisms with which it is incompatible:

$$J_{XY} : \text{Nom}_C(X; Y) \to \text{Pow}(\text{Hom}_C(X; Y)),$$

$$n \mapsto \{ f \in \text{Hom}_C(X; Y) : i_{XY}(n, f) \}. \quad (10)$$

We start in Fig. 1a with a morphism $n : X \to Z$ and a morphism $f : X \to Y$. In Fig. 1b we apply $J_{XZ}$ to find the set of incompatible morphisms $J_{XZ}(n)$. In Fig. 1c we use the precomposition map

$$\text{pre}_f : \text{Hom}_C(Y; Z) \to \text{Hom}_C(X; Z),$$

$$g \mapsto f \circ g, \quad (11)$$

to obtain the set of morphisms

$$\text{pre}_f^{-1}(J_{XZ}(n)). \quad (12)$$

These are to be prohibited because when pre-composed with $f$ they give a morphism that is forbidden by $n$. Now, in principle, it could be that our morphism inference is so powerful that $f \leftarrow n$ manages to exclude all of these:

$$J_{YZ}(f \leftarrow n) = \text{pre}_f^{-1}(J_{XZ}(n)). \quad (13)$$

In general, we are happy with the composition operation if it excludes part of those (but not more):

$$J_{YZ}(f \leftarrow n) \subseteq \text{pre}_f^{-1}(J_{XZ}(n)). \quad (14)$$
It can readily be checked that (14) is equivalent to (equiv-1). Similarly, (equiv-2) is equivalent to requiring
\[ JYZ(n \rightarrow g) \subseteq \text{post}^{-1}_g(J_{XZ}(n)), \]
where \( \text{post}_g \) is the map “post-composition with \( g \”).

4 Canonical nategory constructions

Here are three canonical constructions that allow us to get a nategory out of a category in a more or less straightforward way:

1. Setting the norphism sets to be empty (Example 3);
2. Setting the norphism sets to be singletons that negate the entire respective hom-sets (Example 4);
3. Setting the norphism sets to be the powerset of the respective hom-sets (Example 5).

Example 3 (A nategory with no norphisms). For any category \( \mathcal{C} \), let
\[ \text{Nom}_{\mathcal{C}}(X;Y) := \emptyset. \]
For all pairs \( X, Y \) the function \( i_{XY} \) is uniquely defined as it has an empty domain. The functions \( \leftrightarrow, \dashv \) also have empty domains. The conditions (equiv-1) and (equiv-2) are trivially verified. A nategory with no norphisms is just a category.

Example 4 (Singleton norphism sets negating all norphisms). In this construction, we turn a category into a nategory by making the choice that a norphism is a witness for the fact that the corresponding hom-set is empty. For any category \( \mathcal{C} \), let
\[ \text{Nom}_{\mathcal{C}}(X;Y) := \{ \bullet \}, \]
and for any pair \( X, Y \) and any \( f : X \rightarrow Y \) let
\[ i_{XY}(\bullet, f) = \top. \]
In this case, the element \( \bullet \) is a witness for “\( \text{Hom}_{\mathcal{C}}(X;Y) \) is empty”. Next, we need to define the two maps:

\[ \leftarrow : \text{Hom}_{\mathcal{C}}(X;Y) \times \text{Nom}_{\mathcal{C}}(X;Z) \rightarrow \text{Nom}_{\mathcal{C}}(Y;Z), \quad (19) \]
\[ \rightarrow : \text{Nom}_{\mathcal{C}}(X;Z) \times \text{Hom}_{\mathcal{C}}(Y;Z) \rightarrow \text{Nom}_{\mathcal{C}}(X;Y). \quad (20) \]
The choice is forced, as there is only one norphism in the codomains. We obtain:
\[ f \leftrightarrow \bullet = \bullet, \quad (21) \]
\[ \bullet \rightarrow g = \bullet. \quad (22) \]
The conditions (equiv-1) and (equiv-2) are easily verified because \( i_{XY} \) always evaluates to \( \top \).

Example 5 (Norphism sets are subsets of hom-sets). For any category \( \mathcal{C} \), let
\[ \text{Nom}_{\mathcal{C}}(X;Y) = \text{Pow}(\text{Hom}_{\mathcal{C}}(X;Y)). \]
Set the incompatibility relation as
\[ i_{XY}(n,f) = f \in n. \]
Define the composition operations as
\[ f \leftrightarrow n = \text{pre}^{-1}_f(n), \quad (25) \]
\[ n \rightarrow g = \text{post}_g^{-1}(n), \quad (26) \]
where \( \text{pre}_f \) and \( \text{post}_g \) are the pre- and post-composition maps
\[
\text{pre}_f : \text{Hom}_C(Y; Z) \rightarrow \text{Hom}_C(X; Z), \\
g \mapsto f \circ g,
\]
\[
\text{post}_g : \text{Hom}_C(X; Y) \rightarrow \text{Hom}_C(X; Z), \\
f \mapsto f \circ g.
\]
Let’s check the condition (equiv-1): \( i_{YZ}(f \leftrightarrow n, g) \Rightarrow i_{XZ}(n, f \circ g) \). Using our definitions, we get
\[
g \in f \leftrightarrow n \Rightarrow f \circ g \in n.
\]
Expanding the left-hand side we find
\[
g \in \text{pre}^{-1}(n) \Rightarrow f \circ g \in n.
\]
Another expansion shows that both sides are the same:
\[
f \circ g \in n \Rightarrow f \circ g \in n.
\]
Checking condition (equiv-2) is analogous. Note that this category is exact.

Finally, we provide an example of a category that we will use later as a counter-example.

**Example 6** (Very weak composition operations). For any category \( C \), as in the previous example, use subsets of morphisms as the morphisms
\[
\text{Nom}_C(X; Y) = \text{Pow}(\text{Hom}_C(X; Y)),
\]
and set the incompatibility relation as
\[
i_{XY}(n, f) = f \in n.
\]
However, define the composition operations as
\[
f \leftrightarrow n = \emptyset,
\]
\[
n \rightarrow g = \emptyset.
\]
The equivariance conditions are still satisfied. For example condition (equiv-1),
\[
i_{YZ}(f \leftrightarrow n, g) \Rightarrow i_{XZ}(n, f \circ g),
\]
becomes
\[
g \in \emptyset \Rightarrow f \circ g \in \emptyset,
\]
which is vacuously satisfied, because the premise is always false.

5 **Example: hiking on the Swiss mountains**

In this section we present an example of planning, giving a more concrete description of the path planning problems mentioned in the introduction. We describe \( \text{Berg} \), a category whose morphisms are hiking paths of various difficulty on a mountain. We then consider the problem of finding paths of minimum length.

**Definition 7 (Berg).** Let \( h: \mathbb{R}^2 \rightarrow \mathbb{R} \) be a \( C^1 \) function, describing the elevation of a mountain. The set with elements \( \langle a, b, h(a, b) \rangle \) is a manifold \( M \) that is embedded in \( \mathbb{R}^3 \). Let \( \sigma = [a_L, a_U] \subset \mathbb{R} \) be a closed interval of real numbers. The category \( \text{Berg}_{h, \sigma} \) is specified as follows:

1. An object \( X \) is a pair \( \langle p, v \rangle \in \mathcal{TM} \), where \( p = \langle p_x, p_y, p_z \rangle \) is the position, \( v \) is the velocity, and \( \mathcal{TM} \) is the tangent bundle of the manifold.
2. Morphisms are $C^1$ paths $f : [0, \tau] \to M$ on the manifold satisfying evident boundary conditions (here $\tau \in \mathbb{R}$ may vary). We also define, formally by decree, that for each $\langle p, v \rangle$ there is a trivial path $"[0, 0] \to M"$ which is $C^1$, has trace $p$, and velocity $v$. At each point $p = f(t)$ of a path we define the steepness via the formula

$$s(\langle p, v \rangle) := \frac{v_z}{\sqrt{v_x^2 + v_y^2}},$$

(38)

where $v = \frac{d}{dt} f(t)$. We choose as morphisms only the paths that have the steepness values contained in the interval $\sigma$:

$$\text{Hom}_{\text{Berg}_\sigma}(X; Y) = \{ f \text{ is a } C^1 \text{ path from } X \text{ to } Y \text{ and } s(f) \subseteq \sigma \},$$

(39)

3. Morphism composition is given by concatenation of paths.
4. Identity morphisms are given by trivial paths.

\[ \begin{array}{c}
\mathcal{F}M \\
\mathcal{Y} \\
\mathcal{X} \\
\mathcal{M} \\
\end{array} \]

The steepness interval $\sigma$ allows considering different categories on the same mountain, with possible hikes varying in difficulty, measured via minimum/maximum steepness. For example, a good hiker can handle $\sigma = [-0.57, 0.57]$ (positive/negative 30° slope). If $\sigma = [-0.57, 0]$, we are only allowed to climb down. If $\sigma = [0, 0]$, we can only walk along isoclines.

**Interpretation of norphisms in Berg**

What might a norphism be in this case?

One possibility is to let a norphism $n : X \to Y$ mean “there exists no path from $X$ to $Y$”. This is a simple choice that is similar to Example 4 and that makes morphisms and norphisms mutually exclusive.

We can obtain a more useful theory by letting norphisms carry information that is complementary to morphisms by interpreting them as lower bounds on distances. To see how this can work, let the set of norphisms be the real numbers completed by positive infinity:

$$\text{Nom}_{\text{Berg}}(X; Y) := \mathbb{R}_{\geq 0} \cup \{+\infty\},$$

(40)

Let $\text{length}(f)$ be the length of the path (according to the manifold metric). Then we interpret a norphism $n : X \to Y$ as a witness of “for all paths $f : X \to Y$, we have $\text{length}(f) \geq n$”. The case $n = +\infty$ negates any path from $X$ to $Y$. The incompatibility relation $i_{XY}$ can be written as follows:

$$i_{XY}(n, f) = \text{length}(f) < n.$$  

(41)

To say that a path $f$ is optimal means saying that $f$ is feasible and that $\text{length}(f)$ is a norphism:

$$f : X \to Y \quad \text{length}(f) : X \to Y \quad \text{f is optimal}.$$  

(42)
Next, we define the following two composition rules
\[ \begin{align*}
    f \leftrightarrow n &= \max\{n - \text{length}(f), 0\}, \\
    n \rightarrow g &= \max\{n - \text{length}(g), 0\},
\end{align*} \]
which are the equivalent of (5) and (6). See Fig. 2. Our reasoning is as follows. If for example \( f \) is a path from \( X \) to \( Y \), and we know that going from \( X \) to \( Z \) has a distance of at least \( n \), then any path from \( Y \) to \( Z \) must be at least \( n - \text{length}(f) \) long. In this case,
\[ J_{XZ}(f \leftrightarrow n) = \{ g : \text{length}(g) < \max\{n - \text{length}(f), 0\} \}. \]
If \( n < \text{length}(f) \), then \( J_{XZ}(f \leftrightarrow n) \) is empty, which differs from
\[ \text{pre}^{-1}(J_{XY}(n)) = \{ g : \text{length}(g) + \text{length}(f) < n \}. \]
The category is not exact. However, since
\[ \{ g : \text{length}(g) < \max\{n - \text{length}(f), 0\} \} \subseteq \{ g : \text{length}(g) + \text{length}(f) < n \}, \]
the category satisfies (equiv-1). The check for (equiv-2) is analogous.

**Example 8.** As a variant of the above, if we set
\[ \text{Nom}_{\text{Berg}}(X;Y) := \mathbb{R} \cup \{+\infty\}, \]
and define the composition operations as
\[ \begin{align*}
    f \leftrightarrow n &= n - \text{length}(f), \\
    n \rightarrow g &= n - \text{length}(g),
\end{align*} \]
then the category is exact. Indeed, for this case one has
\[ J_{XZ}(f \leftrightarrow n) = \{ g : \text{length}(g) < n - \text{length}(f) \}, \]
\[ = \{ g : \text{length}(g) + \text{length}(f) < n \} \]
\[ = \text{pre}^{-1}(J_{XY}(n)). \]

**Example 9.** We may also think of a variation in which the morphisms are integers:
\[ \text{Nom}_{\text{Berg}}(X;Y) := \mathbb{Z} \cup \{+\infty\}. \]
In this case we are limited to express constraints of the type
\[ \text{length}(f) \geq 0, \text{length}(f) \geq 1, \text{length}(f) \geq 2, \ldots \]
We then define the composition rules as
\[ \begin{align*}
    f \leftrightarrow n &= \text{floor}(n - \text{length}(f)), \\
    n \rightarrow g &= \text{floor}(n - \text{length}(g)).
\end{align*} \]
In this case, (equiv-1) is satisfied, however our category is not exact, because in general
\[ J_{XZ}(f \leftrightarrow n) = \{ g : \text{length}(g) < \text{floor}(n - \text{length}(f)) \} \]
\[ \subset \{ g : \text{length}(g) < n - \text{length}(f) \} \]
\[ = \text{pre}^{-1}(J_{XY}(n)), \]
since \( \text{floor}(n - \text{length}(f)) + \text{length}(f) \leq n \). An analogous reasoning applies to (equiv-2). We note that using \( \text{round}(\cdot) \) or \( \text{ceil}(\cdot) \) in (52) would violate (equiv-1) and (equiv-2).
Norphism schemas So far, we have not discussed heuristics for actually choosing a set $\text{Nom}$ for each pair of objects in $\text{Berg}$. Here are some different ways.

1. Non-negativity of lengths. Since path lengths cannot be negative, for all pair of objects $X, Y$ we can say that we have a norphism

$$0 : X \rightarrow Y.$$ (54)

If these are our only norphisms, we are providing no new information about paths.

2. Bound based on distance in $\mathbb{R}^3$. Any path along the mountain cannot be shorter than the distance of a straight line (“as the crow flies”). Therefore, for two objects $\langle p^1, v^1 \rangle, \langle p^2, v^2 \rangle$, we might choose the distance $\|p^1 - p^2\| : \langle p^1, v^1 \rangle \rightarrow \langle p^2, v^2 \rangle$. (55)

3. Bound based on geodesic distance. More accurate bounds are given by taking geodesic distance as our norphisms. This is defined using the metric $d_M$ of the manifold:

$$d_M(p^1, p^2) : \langle p^1, v^1 \rangle \rightarrow \langle p^2, v^2 \rangle.$$ (56)

4. Bound based on steepness interval. A different kind of norphism is to encode steepness information, and relate it to the steepness of paths, instead of their length. Given two objects $\langle p^1, v^1 \rangle, \langle p^2, v^2 \rangle$, we can use one of the following bounds

$$\frac{p^1_x - p^2_x}{p^1_y - p^2_y} < 0$$

$$\frac{p^1_z - p^2_z}{p^1_z - p^2_z} : \langle p^1, v^1 \rangle \rightarrow \langle p^2, v^2 \rangle.$$ (57)

$$\frac{p^1_z - p^2_z}{p^1_z - p^2_z} : \langle p^1, v^1 \rangle \rightarrow \langle p^2, v^2 \rangle.$$ (58)

6 Example: co-design

The next example revolves around the construction of norphisms for the category of design problems $\text{DP}$ [2, 6]; this is called $\text{FeasBool}$ in [6]. The objects of $\text{DP}$ are posets. The norphisms are design problems (also referred to as feasibility relations or boolean profunctors). A design problem $d : \text{P} \rightarrow \text{Q}$ is a monotone map of the form $d : \text{P} \rightarrow \text{Q} \rightarrow \text{Pos Bool}$, where $\text{P}, \text{Q}$ are arbitrary posets and $\text{Bool}$ denotes the poset with elements $\{\bot, \top\}$, with $\bot \leq \top$.

The semantics for a DP is that it describes a process which provides a certain functionality, by requiring certain resources. A design problem $d$ is a monotone map, since lowering the requested functionalities will not require more resources, and increasing the available functionalities will not provide less functionalities.

Morphism composition is defined as follows. Given DPs $d : \text{P} \rightarrow \text{Q}$ and $e : \text{Q} \rightarrow \text{R}$, their composite is

$$(d \circ e) : \text{P} \rightarrow \text{R}, \quad \langle p, r \rangle \mapsto \bigvee_{q \in \text{Q}} d(p, q) \land e(q, r).$$ (59)

For any poset $\text{P}$, the identity DP $id_{\text{P}} : \text{P} \rightarrow \text{P}$ is the monotone map

$$id_{\text{P}} : \text{P} \rightarrow \text{P}, \quad \langle p_1, p_2 \rangle \mapsto p_1 \leq p_2.$$ (60)
**Interpretation of norphisms in DP** Given that the morphisms of DP are feasibility relations, we expect that the norphisms of DP (“nesign problems” NP), should be infeasibility relations. We define a nesign problem \( n : F \to R \) to be a monotonous map \( n : F \times R^\text{op} \to \text{Bool} \), and we interpret \( n(f, r) = \top \) to mean that it is not possible to produce \( f \) from \( r \). The idea is that if \( \langle f_1, r_1 \rangle \) is infeasible, then \( f_1 \preceq f_2 \) implies that \( \langle f_2, r_1 \rangle \) is also infeasible and \( r_2 \preceq r_1 \) implies that \( \langle f_1, r_2 \rangle \) is also infeasible. Note that the source poset of a nesign problem is the \( \text{op} \) of the source poset for a design problem.

**Compatibility of morphisms and norphisms** Consider a DP \( d : F \to R \) and a NP \( n : F \to R \). The compatibility relation between DP and NP should ensure that there are no contradictions. We ask that, for any pair of functionality/resources \( \langle f, r \rangle \), it cannot happen that they are declared feasible by the DP \( d(f, r) \) and declared infeasible by the NP \( n(f, r) \):

\[
i_{FR}(n, d) = \exists f \in F, r \in R : d(f, r) \land n(f, r).
\]  

(61)

**Composition rules for norphisms** Given a NP \( n : P \to Q \) and a DP \( d : R \to Q \), one can compose them to get a NP \( n \circ d : P \to Q \):

\[
(n \circ d)(p, r) = \bigvee_{q \in Q} n(p, q) \land d(r, q).
\]  

(62)

And given a DP \( d : Q \to P \) and a NP \( n : Q \to R \), one can compose them to get a NP \( d \circ n : P \to R \):

\[
(d \circ n)(p, r) = \bigvee_{q \in Q} d(q, p) \land n(q, r).
\]  

(63)

The composition rules satisfy (equiv-1) and (equiv-2) and are exact, as may easily be checked.

**Example 10.** Consider the posets \( P = \langle N_{\text{kg pears}}, \preceq \rangle \), \( Q = \langle R_{\geq 0, \text{CHF}}, \preceq \rangle \), and \( R = \langle N_{\text{kg raisins}}, \preceq \rangle \). Consider the design problem \( d : R \to Q \) and the nesign problem \( n : P \to Q \) given, respectively, by the (in)feasibility relations

\[
\begin{align*}
    d(r, q) & : r \cdot 10 \leq q, \\
    n(p, q) & : p \cdot 5 > q.
\end{align*}
\]

These say that it is possible to buy raisins at 10 CHF/kg or more, and never possible to buy pears at less than 5 CHF/kg. We can evaluate the composition \( (n \circ d) : P \to R \) in a particular point to understand its meaning. For instance:

\[
(n \circ d)(10, 4) = \bigvee_{q \in Q} n(10, q) \land d(4, q) = \bigvee_{q \in Q} (40 \leq q < 50) = \top.
\]

This equation is saying that we cannot get 10 kilos of pears from 4 kilos of raisins. The rationale is that, if I could, then I would be able to start with 40 CHF and use \( d \) to get 4 kilos of raisins, which I could then use to obtain 10 kilos of pears. But this would contradict the norphism \( n \), because \( n(10, 40) = \top \) holds and this means that it is infeasible to exchange 40 CHF for 10 kilos of pears.

**Norphism schemas** Considerations about how to define norphisms might follow from specific knowledge about particular designs that we know are (in)feasible, as well as from more general principles of physics or information theory. One very general rule that is arguably valid across all fields: in this universe, physically realizable designs can never produce strictly more of the same resource than one started with. This rule can be encoded as a norphism. For each object \( P \), we postulate a NP

\[
n_P : P \to P,
\]  

(64)
such that
\[ n_{F}(q, p) = p \preceq_{F} q, \]  
where \( p \preceq_{F} q = (p \preceq_{p} q) \land (p \neq q). \)

Interestingly, starting from any morphism
\[ d : F \rightarrow \mathbb{R}, \]
one can directly obtain two NPs that go in the opposite direction, \( R \rightarrow F. \) These are
\[ (n_{R} \rightarrow d)(r, f) = \bigvee_{r' \in R} n_{R}(r, r') \land d(f, r'), \]
\[ (d \leftarrow n_{F})(r, f) = \bigvee_{f' \in F} d(f', r) \land n_{F}(r, f'). \]

which gives two impossibility results. The first states infeasibility because, while it is possible to get \( f \) from \( r' \) via \( d \) for a certain \( r' \), it is not possible to obtain \( r \) from \( r' \). The second states infeasibility because, while it is possible to get \( f' \) from \( r \) via \( d \) for a certain \( f' \), it is not possible to obtain \( f' \) from \( f \). In this category, we see that positive information induces negative information in the other direction.

7 The category GSet

The dialectica construction \( GC \) is due to De Paiva [3, 4], and its instantiation in the case \( C = \text{Set} \) has been studied from a “questions and answers” perspective, for example in [1]. We will focus on \( GSet \), however our discussion is also interesting for other cases of the GC construction.

Definition 11 (GSet). An object of \( GSet \) is a tuple
\[ \langle Q, A, C \rangle, \]
where \( Q \) and \( A \) are sets, and \( C : Q \rightarrow \text{Rel} A \) is a relation.

A morphism \( r : \langle Q_1, A_1, C_1 \rangle \rightarrow_{GC} \langle Q_2, A_2, C_2 \rangle \) is a pair of maps
\[ r = \langle r_1, r_2 \rangle, \]
\[ r_1 : Q_1 \rightarrow \text{set} Q_2, \]
\[ r_2 : A_1 \rightarrow \text{set} A_2, \]
that satisfy the property
\[ \forall q_2 : Q_2 \\forall a_1 : A_1 \\quad r_1(q_2) C_1 a_1 \Rightarrow q_2 C_2 r_2(a_1). \]  
(73)

Morphism composition is defined component-wise
\[ (r; s)_a = s_a \circ r_a, \]
\[ (r; s)^2 = r^2 \circ s^2, \]
and satisfies (73) via composition of implications.

The identity at \( \langle Q, A, C \rangle \) is \( \text{id} \langle Q, A, C \rangle = \langle \text{id}_{Q}, \text{id}_{A} \rangle. \)

Remark 12. Our notation was chosen to facilitate a “questions and answers” interpretation [1]. In this perspective, an object of \( GSet \) is a “problem”: a relation \( C \) between a set of questions \( Q \) and a set of answers \( A \). For a particular question \( q \in Q \) and answer \( a \in A \), \( q C a \) means that the answer is correct for the question. A morphism \( r : \langle Q_1, A_1, C_1 \rangle \rightarrow_{GC} \langle Q_2, A_2, C_2 \rangle \) is a reduction of problem 2 to problem 1, in the sense that we can use a solution to problem 1 to solve problem 2. The idea is that we start from a question \( q_2 \) and transform it to a question \( q_1 = r_1(q_2) \) of the first problem. Assuming we can find an answer \( a_1 \) to \( q_1 \), we can then transform it in an answer of the second problem \( a_2 = r_2(a_1) \). The condition (73) ensures that the answer so produced is correct for the second problem.
We now rewrite the objects of $\text{GSet}$ in a slightly different way. Instead of

$$C : Q \rightarrow_{\text{Rel}} A,$$

we can write this relation as a boolean function

$$\kappa : Q \times A \rightarrow \{\bot, \top\}.\tag{77}$$

Letting $\text{Bool}$ denote the category with two objects “$\bot$” and “$\top$” and a single non-identity morphism $\Rightarrow : \bot \rightarrow_{\text{Bool}} \top$, we can rewrite (77) again as

$$\kappa : Q \times A \rightarrow \text{Ob}_{\text{Bool}}.\tag{78}$$

And then condition (73) can be rewritten as a dependent function

$$r^s : \{q_2 : Q_2, a_1 : A_1\} \rightarrow \kappa_1(r_3(q_2), a_1) \rightarrow_{\text{Bool}} \kappa_2(q_2, r^s(a_1)).\tag{79}$$

The value $r^s(q_2, a_1)$ is a morphism in $\text{Bool}$ that witnesses an implication.

**Remark 13.** One idea that we find interesting is to replace $\text{Bool}$ with some other category $\text{B}$ (on which we may wish to place suitable assumptions) and can consider maps of the form

$$\kappa : Q \times A \rightarrow \text{Ob}_B.\tag{80}$$

If $\text{B}$ is some category whose morphisms are “proofs”, the analogue of (79) chooses a proof which is just one among many possible proofs (and such proofs might themselves be ordered by relevance or other criteria).

### 7.1 A monoidal product for GSet

The categories $\text{GC}$ have a very rich structure. In particular they provide models of linear logic with four distinct monoidal products $\otimes$, $\triangleright\triangleright$, $\oplus$, and $\&$. We define here a monoidal product $\sqcup$ for $\text{GSet}$ which one might say is “in between” the the multiplicative connectives $\otimes$ and $\triangleright\triangleright$ (which are denoted $\otimes$ and $\sqcup$, respectively, in $\mathbb{B}$).

**Definition 14 (Monoidal product $\sqcup$).** On objects,

$$\langle Q_1, A_1, \kappa_1 \rangle \sqcup \langle Q_2, A_2, \kappa_2 \rangle = \langle Q_1^A \times Q_2^A, A_1 \times A_2, \kappa_1 \sqcup \kappa_2 \rangle,$$

where

$$\kappa_1 \sqcup \kappa_2 : \langle q_1, q_2, \langle a_1, a_2 \rangle \rangle \mapsto \kappa_1(q_1(a_2), a_1) \triangleright \kappa_2(q_2(a_1), a_2).\tag{82}$$

The product of morphisms $r : \langle Q_1, A_1, \kappa_1 \rangle \rightarrow \langle Q_3, A_3, \kappa_3 \rangle$ and $s : \langle Q_2, A_2, \kappa_2 \rangle \rightarrow \langle Q_4, A_4, \kappa_4 \rangle$ is

$$r \sqcup s : \langle Q_1^A \times Q_2^A, A_1 \times A_2, \kappa_1 \sqcup \kappa_2 \rangle \rightarrow \langle Q_3^A \times Q_4^A, A_3 \times A_4, \kappa_3 \sqcup \kappa_4 \rangle,$$

with

$$(r \sqcup s)^r = \langle s^r - r^r, r^r - s^r \rangle,$$

$$(r \sqcup s)^s = r^s \times s^r,$$

$$(r \sqcup s)^t : \langle q_3, q_4, \langle a_1, a_2 \rangle \rangle \mapsto r^t(s^t \triangleright q_3(a_2), a_1) \triangleright s^t(r^t \triangleright q_3(a_1), a_2).\tag{86}$$

The monoidal unit is

$$1_{\sqcup} = \langle \{\ast\}, \{\ast\}, \bot \rangle, \quad \bot : \langle \ast, \ast \rangle \mapsto \bot.\tag{87}$$

For the associator and unitors we make the canonical choices, which are easily inferred from their signatures. We refrain from writing them out explicitly here. For reasons of space we also omit the the proof that $(\text{GSet}, \sqcup)$ is indeed a monoidal category.

**Remark 15.** In the generalization where we replace $\text{Bool}$ with some category $\text{B}$, the operation $\triangleright\triangleright$ and the object $\bot$ in $\text{Bool}$ which are used in the above definition would be replaced by suitable substitutes in $\text{B}$. 
8 Describing categories using enrichment

We recall the following standard definition of enriched category [7], for easy reference and to fix notation.

**Definition 16 (Enriched category).** Let \((V, \otimes, 1, as, lu, ru)\) be a monoidal category.

A \(V\)-enriched category \(E\) is a tuple \(\langle \text{Ob}_E, \alpha, \beta, \gamma \rangle\), where

1. \(\text{Ob}_E\) is a collection of objects.
2. \(\alpha\) is a function such that, for all pairs of objects \(X, Y \in \text{Ob}_E\), its value \(\alpha_{XY}\) is an object of \(V\), called a hom-object.
3. \(\beta\) is a function such that, for all \(X, Y, Z \in \text{Ob}_E\), there exists a morphism \(\beta_{XYZ}\) of \(V\)

\[\beta_{XYZ} : \alpha_{XY} \otimes \alpha_{YZ} \rightarrow \alpha_{XZ},\]  

called a composition morphism.
4. \(\gamma\) is a function such that, for each \(X \in \text{Ob}_E\), there exists a morphism of \(V\)

\[\gamma_X : 1 \rightarrow \alpha_{XX},\]  

called an identity-choosing morphism.

Moreover, for any \(X, Y, Z, U \in \text{Ob}_E\), the following diagrams must commute.

\[
\begin{align*}
\alpha_{XY} \otimes \alpha_{YZ} & \xrightarrow{\alpha_{XY} \otimes \alpha_{YZ}} \alpha_{XZ} \otimes \alpha_{U} \\
\beta_{XYZ} \otimes \text{id}_{\alpha_{ZY}} & \xrightarrow{\alpha_{XY} \otimes \alpha_{YZ}} \alpha_{XZ} \otimes \alpha_{U} \\
\alpha_{XZ} & \xrightarrow{\text{id}_{\alpha_{XZ}}} \alpha_{XZ} \\
\beta_{XZU} & \xrightarrow{\alpha_{XY} \otimes \alpha_{YZ}} \alpha_{XZ} \otimes \alpha_{U} \\
\alpha_{XY} & \xrightarrow{\gamma_X} \alpha_{XY} \\
\end{align*}
\]

Recall that specifying the data of an ordinary (locally small) category is equivalent to specifying a category enriched in the monoidal category \(P = (\text{Set}, \times, 1)\). In this case, both the enriched category and the ordinary category have the same objects, the hom-objects of the enriched category correspond to the hom-sets of the ordinary category, the composition morphisms encode the composition operations, and the identity-choosing morphisms select an element of each of the hom-sets of the type \(\alpha_{XX}\), corresponding to identity morphisms. The diagrams (90) and (91) encode the associativity of the composition operations, and that the identity morphisms act neutrally for composition.

The proof of our main result below follows a similar pattern – we show that to specify a nategory which satisfies some additional conditions it is sufficient to specify a category enriched in the monoidal category \((\text{GSet}, \sqcup)\). We will denote \((\text{GSet}, \sqcup)\) by \(\text{PN}\), which stands for “positive” and “negative”.

**Proposition 17.** A \(\text{PN}\)-enriched category provides the data necessary to specify a nategory. However, not all nategories can be specified by the data of a \(\text{PN}\)-enriched category, because the nategory produced has the following additional properties, which encode a covariant and a contravariant “action” of morphisms on norphisms.

**Identities act neutrally:**

\[\text{id} \mapsto n = n,\]  

\[n \mapsto \text{id} = n,\]  

(neut-1)  

(neut-2)
Compatibility with composition:

\[(f \circ g) \rightarrow n = g \rightarrow (f \rightarrow n), \quad (\text{covar})\]

\[n \rightarrow (g \circ h) = (n \rightarrow h) \rightarrow g. \quad (\text{contravar})\]

The actions commute:

\[f \rightarrow (n \rightarrow h) = (f \rightarrow n) \rightarrow h. \quad (\text{comm})\]

These conditions are not satisfied by all nategories.

**Proof.** Suppose somebody has provided us with a PN-enriched category \(E = (\text{Ob}_E, \alpha, \beta, \gamma)\). Using this data we will describe a nategory \(C\) with the above-stated properties.

For the objects of \(C\), we set \(\text{Ob}_C := \text{Ob}_E\).

For every pair of objects \(X, Y \in \text{Ob}_C\), we have an object \(\alpha_{XY}\) of \(\text{PN}\). This is a tuple

\[\alpha_{XY} = \langle Q, A, \kappa \rangle, \quad (92)\]

which we interpret as

\[\alpha_{XY} = \langle \text{Nom}_C(X; Y), \text{Hom}_C(X; Y), i_{XY} \rangle, \quad (93)\]

thereby setting \(\text{Nom}_C(X; Y) := Q, \text{Hom}_C(X; Y) := A, \text{and } i_{XY} := \kappa\).

Next, for each \(X \in \text{Ob}\) we have an identity-choosing morphism

\[\gamma_X : \text{I}_{\text{PN}} \rightarrow_{\text{PN}} \alpha_{XX}. \quad (94)\]

Because \(\text{I}_{\text{PN}} = \langle \{\bullet\}, \{\bullet\}, \bot \rangle\), this is a morphism

\[\gamma_X : \langle \{\bullet\}, \{\bullet\}, \bot \rangle \rightarrow_{\text{PN}} \langle \text{Nom}_C(X; X), \text{Hom}_C(X; X), i_{XX} \rangle, \quad (95)\]

which consists of three functions \(\langle r_3, r^2, r^* \rangle\). The forward map \(r^2 : \{\bullet\} \rightarrow \text{Hom}_C(X; X)\) chooses our (candidate) identity morphism, so we set \(\text{id}_X := r^2(\bullet)\). The backward map \(r_3 : \text{Nom}_C(X; X) \rightarrow \{\bullet\}\) is uniquely determined and does not carry any information. As for \(r^*\), it is a dependent function of the type

\[r^* : \{g_2 : \text{Nom}_C(X; X), a_1 : \{\bullet\}\} \rightarrow \bot (r_3(g_2), a_1) \rightarrow_{\text{Bool}} i_{XX}(q_2, r^2(a_1)). \quad (96)\]

Evaluated at \(q_2 = n\) and \(a_1 = \bullet\), we have

\[r^*(n, \bullet) : \bot \rightarrow_{\text{Bool}} i_{XX}(n, \text{id}_X). \quad (97)\]

Because \(\bot\) is an initial object in \(\text{Bool}\), such a morphism always exists, no matter what the right-hand side is. Therefore, this condition does not carry any additional information.

Now let us fix three objects \(X, Y, Z\) and consider the composition morphism

\[\beta_{XYZ} : \alpha_{XY} \otimes_{\text{PN}} \alpha_{YZ} \rightarrow_{\text{PN}} \alpha_{XZ}. \quad (98)\]

Rewriting the hom-objects as tuples and using abbreviated notation we have

\[\beta_{XYZ} : \langle N_{XY}, H_{XY}, i_{XY} \rangle \otimes_{\text{PN}} \langle N_{YZ}, H_{YZ}, i_{YZ} \rangle \rightarrow_{\text{PN}} \langle N_{XZ}, H_{XZ}, i_{XZ} \rangle. \quad (99)\]

Expanding using the definition of \(\otimes_{\text{PN}}\) we find

\[\beta_{XYZ} : \langle N_{XY} \times N_{YZ}, H_{XY} \times H_{YZ}, i_{XY} \sqcup i_{YZ} \rangle \rightarrow_{\text{PN}} \langle N_{XZ}, H_{XZ}, i_{XZ} \rangle. \quad (100)\]

Such a morphism corresponds to three maps \(\langle s_3, s^2, s^* \rangle\). The forward map \(s^2\) has type

\[s^2 : \text{Hom}_C(X; Y) \times \text{Hom}_C(Y; Z) \rightarrow \text{Hom}_C(X; Z), \quad (101)\]
and we use it to define morphism composition “\( \cdot_C \)” in our category. The backward map \( s_\beta \) has type

\[
s_\beta : N_{XY} \rightarrow N_{YZ}^{H_Y} \times N_{YZ}^{H_Y},
\]

which, after splitting into two maps (using the universal property of the product) and currying, specifies two maps

\[
\begin{align*}
&\rightarrow : N_{XZ} \times H_{YZ} \rightarrow N_{XY}, \\
&\leftarrow : H_{XY} \times N_{XZ} \rightarrow N_{YZ}.
\end{align*}
\]

As for the dependent function \( s^* \), given \( n : N_{XZ}, f : H_{XY}, g : H_{YZ} \) we have

\[
s^*(n, (f, g)) : (i_{XY} \sqcup i_{YZ})(((n \rightarrow -), (- \leftrightarrow n)), (f, g)) \rightarrow \text{Bool}_XZ(n, f \circ g).
\]

Expanding more,

\[
s^*(n, (f, g)) : i_{XY}(n \rightarrow g, f) \lor i_{YZ}(f \leftrightarrow n, g) \rightarrow \text{Bool}_XZ(n, f \circ g),
\]

which is equivalent to having two maps

\[
\begin{align*}
s_1^*(n, (f, g)) & : i_{XY}(n \rightarrow g, f) \rightarrow \text{Bool}_XZ(n, f \circ g), \\
s_2^*(n, (f, g)) & : i_{YZ}(f \leftrightarrow n, g) \rightarrow \text{Bool}_XZ(n, f \circ g).
\end{align*}
\]

These witness the implications which give us (equiv-1) and (equiv-2).

Next we move to the commutative diagrams in the definition of enriched category. As a general observation, we note that for diagrams in \( \text{GSet} \) we only need to consider commutativity on the level of “forward maps” and “backward maps” respectively. We do not need to worry about the conditions (73), because for any two parallel morphisms this condition is the same, and hence “commutativity” is trivially satisfied.

**Conditions from the associativity diagram for enriched categories** We now consider the diagram (90), in the case of \( \text{PN} \). On the level of “forward” maps, this commutative diagram encodes that morphism composition must be associative. One the level of “backward” maps, it implies that the following diagram must commute:

\[
\begin{array}{c}
\left( Q_{AY}^{A_X} \times Q_{AY}^{A_Y} \right)^{A_Z} \times Q_{AZ}^{A_Y} \times A_X \\
\downarrow \left( \beta_{XYZ} \sqcup \text{id}_{ZU} \right) \leftarrow \downarrow \left( \text{id}_{XY} \sqcup \beta_{YZU} \right) \\
Q_{AX}^{A_Z} \times Q_{AX}^{A_Y} \leftarrow Q_{AX}^{A_Y} \times A_X \leftarrow Q_{AX}^{A_Y} \times A_X
\end{array}
\]

Let us look at the two different routes through this diagram. For the left-hand route, note that

\[
\beta_{XYZ} \sqcup \text{id}_{ZU} = \left( \text{id}_U \sqcup (\beta_{XYZ}) \left( (-) \right)^\beta \right) \left( \text{id}_{ZU} \right) = \left( (-) \right)^\beta \beta_{XYZ} \left( (-) \right)^\beta \left( \text{id}_{ZU} \right)
\]

and so

\[
\beta_{YZU} \left( \beta_{XYZ} \sqcup \text{id}_{ZU} \right) : Q_{AX} \rightarrow Q_{AY} \times Q_{AZ} \left( \left( Q_{AY} \times Q_{AZ} \right)^{A_Z} \times Q_{AZ}^{A_Y} \times A_Z \right)
\]

\[
q \mapsto \left( q \rightarrow (-), (-) \leftrightarrow q \right) \mapsto \left( q \rightarrow (-) \right)^\beta \beta_{XYZ} \left( (-) \right)^\beta \left( q \right).
\]
For the right-hand route, note that
\[
\begin{align*}
(\text{id}_{XY} \sqcup \beta_{YZU})_\gamma &= \langle \beta_{YZU} \hat{\gamma}, \gamma \rangle \cdot (\text{id}_{XY} \sqcup \beta_{YZU})_\gamma \\
&= \langle \beta_{YZU} \hat{\gamma}, \gamma \rangle \\
\end{align*}
\]
and so
\[
\begin{align*}
\beta_{XYU} \hat{\gamma} (\text{id}_{XY} \sqcup \beta_{YZU})_\gamma : Q_{XYU} &\rightarrow Q_{XYU}^A \times Q_{XYU}^B \\
&\rightarrow Q_{XYU}^{A \times A} \times (Q_{YZU}^A \times Q_{ZU}^B)^{A \times B} \\
q &\mapsto \langle q \mapsto (-), (-) \mapsto q \rangle \mapsto \langle \beta_{YZU} \hat{\gamma}, \gamma \rangle \\
\end{align*}
\]
Instead of now applying \(\alpha_S\) directly, which is an obvious map but messy to write down, we evaluate the functions we obtained from our calculations for the left- and right-hand routes. Given \(\langle f, g, h \rangle \in A_{XY} \times A_{YZ} \times A_{ZU}\), evaluating the two components of (110) we find
\[
\begin{align*}
\langle q \mapsto (-) ; \beta_{XYZ} \rangle (h) &= \beta_{XYZ} (q \mapsto h) : \langle g, f \rangle \mapsto \langle (q \mapsto h) \mapsto g, f \mapsto (q \mapsto h) \rangle, \\
\beta_{XYZ} (f) &= \beta_{XYZ} (\langle f \mapsto g \rangle) = \langle f \mapsto g \rangle, \\
\end{align*}
\]
respectively.

For the right-hand route, evaluating (112) gives
\[
\begin{align*}
\beta_{YZU} \hat{\gamma} q \mapsto (-) \beta_{XYU} \hat{\gamma} (\langle g, h \rangle) &= q \mapsto (g \mapsto h), \\
\end{align*}
\]
and
\[
\begin{align*}
\beta_{YZU} \hat{\gamma} f (\langle g \mapsto h \rangle) &= \beta_{XYU} \hat{\gamma} \langle g \mapsto h \rangle \\
&\mapsto \langle (g \mapsto h) \mapsto f, g \mapsto (f \mapsto h) \rangle.
\end{align*}
\]
By comparing the two routes, we obtain the conditions
\[
\begin{align*}
f \mapsto g \mapsto q &= g \mapsto (f \mapsto q), \\
q \mapsto (g \mapsto h) &= (q \mapsto h) \mapsto g, \\
f \mapsto (q \mapsto h) &= (f \mapsto q) \mapsto h.
\end{align*}
\]

**Conditions from the unitality diagrams for enriched categories** Consider the right-hand portion of the diagram (91), now for the case of PN. On the level of forward maps, this diagram encodes the condition that \(\text{id}_X \leftrightarrow a = a\) for any morphism \(a : X \rightarrow Y\). On the level of backward maps, it amounts to the commutative diagram

\[
\begin{align*}
\array{Q_{XY} \ar[r]^<<<<{\beta_{XYU} \hat{\gamma}} \ar[dr]_{\text{id}_{XY}} & Q_{XY}^{A \times B} \times Q_{XY}^{A \times B} \ar[d]_{(\gamma \sqcup \text{id}_{AXY})_\gamma} & \langle \gamma \sqcup \text{id}_{AXY} \rangle_\gamma \ar[l]_{1^{A \times B} \times Q_{XY}^{A \times B}} \\
& & 1^{A \times B} \times Q_{XY}^{A \times B}
}
\end{align*}
\]
Computing the right-hand route, we have
\[
\begin{align*}
(\gamma \sqcup \text{id}_{AXY})_\gamma : Q_{XX}^{A \times B} \times Q_{XX}^{B \times C} &\rightarrow 1^{A \times B} \times Q_{XY}^{A \times B} \\
\langle \varphi_{XX}, \varphi_{YX} \rangle &\mapsto \langle !, \gamma \rangle \cdot (\varphi_{XX} \cdot \text{id}_{AXY})_\gamma = \langle !, \bullet \mapsto \varphi_{XX}(\text{id}_X) \rangle,
\end{align*}
\]
and
\[
\begin{align*}
(\gamma \sqcup \text{id}_{AXY})_\gamma \beta_{XYU} \hat{\gamma} : Q_{XY} &\rightarrow Q_{XY}^{A \times B} \times Q_{XY}^{B \times C} \\
q &\mapsto \langle \beta_{XYU} \hat{\gamma}, \gamma \rangle (q \mapsto (-), (-) \mapsto q) \\
&\mapsto \langle \text{id}_{AXY} \cdot \gamma \rangle q \mapsto (-), \gamma \mapsto (\gamma \mapsto g \mapsto \text{id}_{AXY})_\gamma \\
&= \langle !, \bullet \mapsto (\text{id}_X \mapsto q) \rangle,
\end{align*}
\]
which, when compared with $\mu_0 : q \mapsto (\lnot, \bullet \mapsto q)$, gives the condition

$$\text{id} \mapsto q = q.$$  

(120)

The left-hand portion of the diagram (91) may be treated analogously and gives rise to the conditions

$$a \mapsto \text{id} = a \quad \text{and} \quad q \mapsto \text{id} = q.$$  

(121)

Remark 18. Exact categories are those in which the implications in (equiv-1) and (equiv-2) are in fact equivalences. In the above proof, this corresponds to the morphisms (107) and (108) in $\text{Bool}$ being identities.

(17) begs the question as to why we don’t include the additional properties stated there as part of our definition of what a category is. Our reason is that these properties fail for examples of interest to us.

For example, in applications it is normal that physical measurements and numerical representations on a computer are given only to a certain accuracy. This motivates (9), where we only allow integer values for norphisms. However, in that example, the properties (contravar) and (covar) are not satisfied. To see this, consider (contravar) and consider morphisms $g$ and $h$ in $\text{Berg}$ with length$(g) = \text{length}(h) = 1.5$. For a norphism $q$ of the appropriate signature we have

$$q \mapsto (g \circ h) = \text{floor}(q - \text{length}(g \circ f)) = \text{floor}(q - 3)$$  

(122)

on the one hand, and

$$(q \circ h) \mapsto g = \text{floor}(\text{floor}(q - 1.5) - 1.5)$$  

(123)

on the other. If we choose $q = 10$, for instance, the previous expressions evaluate to 7 and 6, respectively.

As a different example, the properties (neut-1) and (neut-2) fail in Example 6. Indeed, there $\text{id}_X \mapsto n = \emptyset$ and $n \mapsto \text{id}_Y = \emptyset$, even though, in general, we would have $n \neq \emptyset$.

9 Conclusions

This work showed that we can encode negative information in a categorical manner such that norphisms (negative arrows) and morphisms (positive arrows) are equal citizens in the theory. Norphisms and morphisms are, in general, not mutually exclusive; they give complementary information.

We have seen how, in the category $\text{Berg}$, norphisms can represent negative results such as lower bounds on distances between two locations. A path planning algorithm must construct a morphism (a path) and construct a norphism (a bound) to prove that the path is optimal. We have also seen how, in the category $\text{DP}$, norphisms can represent design impossibility results.

After defining nategories as categories with extra structure, we showed a way to encode this new concept using categories enriched in the dialecta category $G\text{Set}$. This approach, however, introduces some compatibility properties for norphism composition that we do not wish to include in our general notion of nategory. Future work includes exploring if nategories might be recovered, on the nose, via a different enrichment, as well as studying various typical categorical concepts in the context of nategories. We would also be very happy to discover further interesting examples.

Acknowledgments

The authors would like to thank David I. Spivak for fruitful discussions. We also thank Valeria de Paiva, Brendan Fong, and David Yetter for helpful comments. Gioele Zardini was supported by the Swiss National Science Foundation, under NCCR Automation, grant agreement 51NF40_180545.
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