Energy distribution of Kerr spacetime using Möller energy momentum complex

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Using the energy momentum complex given by Möller in 1978 based on the absolute parallelism, the energy distribution in Kerr spacetime is evaluated. The energy with this spacetime is found to be the same as it was earlier evaluated using different definitions mainly based on the metric tensor.
1. Introduction

After Einstein’s original pseudotensor, a large number of expressions for the energy distribution in a general relativistic system have been proposed by many authors [1]∼[10]. To get a meaningful results for the energy in the prescription of Einstein, Tolman or Landau and Lifshitz one is compelled to use a quasi-Galilean*. Møller revived [5] the issue of energy and momentum in general relativity, and required [11] that any energy-momentum complex $\tau_{\mu\nu}$ must satisfy the following properties: (A) It must be an affine tensor density which satisfies conservation law, (B) for an isolated system the quantities $P_\mu$ are constant in time and transform as the covariant components of a 4-vector under linear coordinate transformations, and (C) the superpotential $U_{\mu}^{\nu\lambda} = -U_{\mu}^{\lambda\nu}$ transforms as a tensor density of rank 3 under the group of spacetime transformations.

It is not possible to satisfy all the above requirements if the gravitational field is described by the metric tensor alone [12]. In a series of papers [12, 13, 14] therefore, Møller was led to the tetrad description of gravitation, and constructed a formal form of energy-momentum complex that satisfies all the requirements. The metric tensor is uniquely fixed by the tetrad field, but the reverse is not true, since the tetrad has six extra degrees of freedom. In the tetrad formulation of general relativity, the tetrad field is allowed to undergo local Lorentz transformations with six arbitrary functions. The energy-momentum complex is not a tensor and changes its form under such transformations. Therefore, unless one can find a good physical argument for fixing the tetrad throughout the system, one cannot speak about the energy distribution inside the system. The total energy-momentum obtained by the complex, however, is invariant under local Lorentz transformations with appropriate boundary conditions [14, 15].

Dymnikova [16] derived a static spherically symmetric nonsingular black hole solution in orthodox general relativity assuming a specific form of the stress-energy momentum tensor. This solution practically coincides with the Schwarzschild solution for large $r$, for small $r$ it behaves like the de Sitter solution and describes a spherically symmetric black hole singularity free everywhere [16]. The calculations of the energy associated with this metric was done by Radinschi [17].

Nashed [18] has obtained two general spherically symmetric non singular black hole solutions in Møller’s tetrad theory of gravitation. One of those solutions is characterized by an arbitrary function while the other by a constant. The associated metric of these solutions are the same and gave the metric obtained before by Dymnikova [16]. The energy content of those two solutions are calculated using the energy momentum complex given by Møller. It is shown that the energy of the two solutions depends on the arbitrary function and on the constant respectively and they are different from each other and also different from the energy given before by Radinschi [17]. Nashed [19] also show that the calculations of energy using the prescription of Møller in the framework of absolute parallelism spacetime are more accurate than that given in the framework of the Riemannian spacetime. Toma [20] gives an exact solution to the vacuum field equation of the new general relativity.

It is the aim of the present work to calculate the energy distribution of the Kerr spacetime given by Toma [20] using the energy momentum complex given by Møller. In section 2 we briefly review the new general relativity theory of gravitation. The Kerr solution obtained

*By a quasi-Galilean coordinate we mean a coordinate in which the metric tensor $g_{\mu\nu}$ approaches the Minkowski spacetime metric $\eta_{\mu\nu} = diag(1, -1, -1, -1)$ at spatial infinity.
by Toma [20] and its energy distribution are given in section 3. Section 4 is devoted to discussion.

2. The new general relativity theory of gravitation

The fundamental fields of gravitation are the parallel vector fields $b^k_\mu$. The component of the metric tensor $g_{\mu\nu}$ are related to the dual components $b^k_\mu$ of the parallel vector fields by the relation

$$ g_{\mu\nu} = b^i_\mu b^i_\nu. \quad (1) $$

The nonsymmetric connection $\Gamma^\lambda_{\mu\nu}$ are defined by

$$ \Gamma^\lambda_{\mu\nu} = b^k_\lambda b^k_{\mu,\nu}, \quad (2) $$
as a result of the absolute parallelism [21].

The gravitational Lagrangian $L$ of this theory is an invariant constructed from the quadratic terms of the torsion tensor

$$ T^\lambda_{\mu\nu} \overset{\text{def}}{=} \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}. \quad (3) $$

The following Lagrangian

$$ L \overset{\text{def}}{=} -\frac{1}{3\kappa} (t^{\mu\nu\lambda} t_{\mu\nu\lambda} - v^\mu v_\mu) + \zeta a^\mu a_\mu, \quad (4) $$
is quite favorable experimentally [21]. Here $\zeta$ is a constant parameter, $\kappa$ is the Einstein gravitational constant and $t_{\mu\nu\lambda}, v_\mu$ and $a_\mu$ are the irreducible components of the torsion tensor:

$$ t_{\lambda\mu\nu} = \frac{1}{2} (T_{\lambda\mu\nu} + T_{\mu\lambda\nu}) + \frac{1}{6} (g_{\nu\lambda} V_\mu + g_{\mu\nu} V_\lambda) - \frac{1}{3} g_{\lambda\mu} V_\nu, $$

$$ V_\mu = T^\lambda_{\lambda\mu}, $$

$$ a_\mu = \frac{1}{6} \epsilon_{\mu\nu\rho\sigma} T^{\nu\rho\sigma}, \quad (5) $$

where $\epsilon_{\mu\nu\rho\sigma}$ is defined by

$$ \epsilon_{\mu\nu\rho\sigma} \overset{\text{def}}{=} \sqrt{-g} \delta_{\mu\nu\rho\sigma} \quad (6) $$

with $\delta_{\mu\nu\rho\sigma}$ being completely antisymmetric and normalized as $\delta_{0123} = -1$.

*Latin indices $(i, j, k, \cdots)$ designate the vector number, which runs from (0) to (3), while Greek indices $(\mu, \nu, \rho, \cdots)$ designate the world-vector components running from 0 to 3. The spatial part of Latin indices is denoted by $(a, b, c, \cdots)$, while that of Greek indices by $(\alpha, \beta, \gamma, \cdots)$.*
By applying the variational principle to the Lagrangian (4), the gravitational field equation are given by [21]†:

\[ G_{\mu\nu} + K_{\mu\nu} = -\kappa T_{(\mu\nu)}, \]  
\[ b^\mu b^\nu \partial_\lambda (\sqrt{-g} J_{ij}^\lambda) = \lambda \sqrt{-g} T_{[\mu\nu]}, \]

where the Einstein tensor \( G_{\mu\nu} \) is defined by

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \]

\[ R^\rho_{\sigma\mu\nu} = \partial_\mu \{ \rho_{\sigma\nu} \} - \partial_\nu \{ \rho_{\sigma\mu} \} + \{ \rho_{\tau\mu} \} \{ \tau_{\sigma\nu} \} - \{ \tau_{\sigma\mu} \} \{ \tau_{\tau\nu} \}, \]

\[ R_{\mu\nu} = R^\rho_{\mu\rho\nu}, \]

\[ R = g^{\mu\nu} R_{\mu\nu}, \]

and \( T_{\mu\nu} \) is the energy-momentum tensor of a source field of the Lagrangian \( L_m \)

\[ T^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta L_M}{\delta b^k_{\mu}} \]

with \( L_M = L_M/\sqrt{-g} \). The tensors \( K_{\mu\nu} \) and \( J_{ij\mu} \) are defined as

\[ K_{\mu\nu} = \frac{\kappa}{\lambda} \left( \frac{1}{2} \left[ \epsilon^\rho_{\rho\sigma\lambda}(T_{\nu\rho\sigma} - T_{\rho\sigma\nu}) + \epsilon^\rho_{\rho\sigma\lambda}(T_{\mu\rho\sigma} - T_{\rho\sigma\mu}) \right] a_\lambda + \frac{3}{2} a_\mu a_\nu - \frac{3}{4} g_{\mu\nu} a^\lambda a_\lambda \), \]

\[ J_{ij\mu} = -\frac{3}{2} b_j^\rho b_i^\sigma \epsilon_{\rho\sigma\mu\nu}, \]

respectively. The dimensionless parameter \( \lambda \) is defined by

\[ \frac{1}{\lambda} = \frac{4}{9} \zeta + \frac{1}{3\kappa}. \]

In this paper we are going to consider the vacuum gravitational field:

\[ T_{(\mu\nu)} = T_{[\mu\nu]} = 0. \]

†We will denote the symmetric part by ( ), for example, \( A_{(\mu\nu)} = (1/2)(A_{\mu\nu} + A_{\nu\mu}) \) and the antisymmetric part by the square bracket [ ], \( A_{[\mu\nu]} = (1/2)(A_{\mu\nu} - A_{\nu\mu}) \).
3. Kerr Solution and its energy contents

Toma [20] has obtained a solution in the new general relativity which gave the Kerr metric. The covariant form of the parallel vector field of this solution is given in the Boyer-Lindquist coordinate \((t, \rho, \theta, \phi)\) by

\[
\begin{align*}
    b_{00} &= (1 - \frac{a \rho}{2 \Sigma}), & b_{01} &= \frac{a \rho}{2 \Delta}, \\
    b_{02} &= 0, & b_{03} &= -\frac{a \rho h \sin^2 \theta}{2 \Sigma}, \\
    b_{10} &= i \frac{a \rho \sin \theta \cos \Phi}{2 \Sigma}, & b_{11} &= i \frac{\rho \sin \theta (X - a \cos \Phi)}{\Delta}, \\
    b_{12} &= i X \cos \theta, & b_{13} &= i \frac{(a \rho h \sin^3 \theta \cos \Phi - Y \sin \theta)}{2 \Sigma}, \\
    b_{20} &= i \frac{a \rho \sin \theta \sin \Phi}{2 \Sigma}, & b_{21} &= i \frac{\rho \sin \theta (Y - a \sin \Phi)}{\Delta}, \\
    b_{22} &= i Y \cos \theta, & b_{23} &= i (X \sin \theta + \frac{a \rho h \sin^3 \theta \sin \Phi}{2 \Sigma}), \\
    b_{30} &= i \frac{a \rho \cos \theta}{2 \Sigma}, & b_{31} &= i (1 + \frac{a \rho}{2 \Delta}) \cos \theta, \\
    b_{32} &= -i \rho \sin \theta, & b_{33} &= i \frac{a \rho h \sin^2 \theta \cos \theta}{2 \Sigma}, \\
\end{align*}
\]  

where

\[
\begin{align*}
    \Sigma &= \rho^2 + h^2 \cos^2 \theta, & \Delta &= \rho^2 + h^2 - a \rho, \\
    X &= \rho \cos \Phi + h \sin \Phi, & Y &= \rho \sin \Phi - h \cos \Phi, \\
    \Phi &= \phi - h B, & B &= \int^\rho \frac{d\rho}{\Delta}. \\
\end{align*}
\]  

The parallel vector field (18) is axially symmetric, i.e., it is invariant under the transformation [20]

\[
\begin{align*}
    \bar{\phi} &= \phi + \delta \phi, & \bar{b}_{00} &= b_{00}, & \bar{b}_{11} &= b_{11} \cos \delta \phi - b_{22} \sin \delta \phi, \\
    \bar{b}_{22} &= b_{22} \cos \delta \phi + b_{11} \sin \delta \phi, & \bar{b}_{33} &= b_{33}. \\
\end{align*}
\]  

The associated metric of the parallel vector field (18) is given by

\[
ds^2 = (1 - \frac{a \rho}{\Sigma}) dt^2 - \frac{\Sigma}{\Delta} d\rho^2 - \Sigma d\theta^2 - \sin^2 \theta \left( (\rho^2 + h^2) + \frac{a \rho h^2 \sin^2 \theta}{\Sigma} \right) d\phi^2 - 2 \frac{a \rho h \sin^2 \theta}{\Sigma} dt d\phi, \tag{23}\]

which is the Kerr metric written in the Boyer-Lindquist coordinates and \(a\) and \(h\) are respectively the mass and rotation parameter. This spacetime has null hypersurface which is given by

\[
\rho_\pm = \frac{a \pm \sqrt{a^2 - 4 h^2}}{2}. \tag{24}\]

There is a ring curvature singularity \(\Sigma = 0\) in the Kerr spacetime. This spacetime has an event horizon at \(\rho = \rho_+\), it describes a black hole if and only if \(a^2 \geq 4 h^2\). The coordinates
are singular at \( \rho = \rho_{\pm} \). Therefore, \( t \) is replaced by a null coordinate

\[
dt = dv - \frac{\rho^2 + h^2}{\Delta} d\rho,
\]
\[
d\phi = d\varphi - \frac{h}{\Delta} d\rho,
\]
and the Kerr metric is expressed in the advanced Eddington-Finkelstein coordinate \((v, \rho, \theta, \varphi)\) [22]

\[
ds^2 = (1 - \frac{a \rho}{\Sigma^2}) dv^2 - 2dv d\rho + \frac{2a h \rho \sin^2 \theta}{\Sigma^2} dv d\varphi - \Sigma^2 d\theta^2 + 2h \sin^2 \theta dp d\varphi
\]
\[
- \sin^2 \theta \left[ (\rho^2 + h^2) + \frac{a h^2 \rho \sin^2 \theta}{\Sigma^2} \right] d\varphi^2.
\]

Transforming the above to Kerr-Schild Cartesian coordinate \((T, x, y, z)\) according to

\[
T = v - \rho, \quad x = \sin \theta (\rho \cos \varphi - h \sin \varphi),
\]
\[
y = \sin \theta (\rho \sin \varphi + h \cos \varphi), \quad z = \rho \cos \theta,
\]
the line-element (26) takes the form

\[
ds^2 = dT^2 - dx^2 - dy^2 - dz^2 - \frac{a \rho^3}{\rho^4 + h^2 z^2} \left( dT + \frac{1}{\rho^2 + h^2} \left[ (xp + yh) dx + (yp - xh) dy \right] + \frac{z}{\rho} dz \right)^2.
\]

The superpotential of Møller’s theory is given by Mikhail et al. [23] as

\[
\mathcal{U}_{\mu}^{\nu \lambda} = \frac{(-\gamma)^{1/2}}{2\kappa} P_{\chi \rho \sigma}^{\tau \nu \lambda} \left[ \Phi^\mu g^\sigma_{\chi \tau} g_{\mu \tau} - \lambda g_{\tau \mu} \gamma_{\chi \rho \sigma} - (1 - 2\lambda) g_{\tau \mu} \gamma_{\rho \chi} \right],
\]

where \( P_{\chi \rho \sigma}^{\tau \nu \lambda} \) is

\[
P_{\chi \rho \sigma}^{\tau \nu \lambda} \overset{\text{def.}}{=} \delta^\tau_{\rho \nu \lambda} + \delta^\tau_{\sigma \rho \lambda} - \delta^\tau_{\rho \sigma \nu \lambda} - \delta^\tau_{\rho \nu \sigma \lambda} + \delta^\tau_{\rho \sigma \nu \lambda} - \delta^\tau_{\rho \nu \sigma \lambda} - \delta^\tau_{\rho \nu \sigma \lambda} + \delta^\tau_{\rho \nu \sigma \lambda}.
\]

The energy is expressed by the surface integral [5]

\[
E = \lim_{r \to \infty} \int_{r = \text{constant}} \mathcal{U}_0^{0 \alpha} n_\alpha dS,
\]

where \( n_\alpha \) is the unit 3-vector normal to the surface element \( dS \).

Now we are in a position to calculate the energy associated with the solution (18) using the superpotential (29). Thus substituting from (18) into (29) we obtain the following nonvanishing values

\[
\mathcal{U}_0^{01} = \frac{a \sin \theta}{16 \pi ( (\rho^2 + h^2 \cos^2 \theta)(a \rho - h^2 - \rho) )} \left( h^6 \cos^2 \theta - \rho^2 h^4 - \sigma \rho h^2 a \cos^2 \theta + \rho^3 h^2 a \cos^2 \theta - \rho h^4 \cos^2 \theta - 3 \rho^3 h^2 + \rho a h^4 \cos \theta - \rho^4 h^2 \cos^2 \theta - \rho^2 h^4 \cos^2 \theta \right)
\]
\[
- 2 \rho^6 + \sigma \rho h^2 a + \rho^3 ah^2 - h^6 \cos^4 \theta + 2 a \rho^5
\]
\[
\mathcal{U}_0^{02} = \frac{a\rho h^2 \sin^2 \theta \cos \theta}{16\pi((\rho^2 + h^2 \cos^2 \theta)^2(a\rho - h^2 - \rho^2))}(h^2 \cos^2 \theta + 2h^2 - 2a\rho - 3\rho^2) \quad (34)
\]

\[
\mathcal{U}_0^{03} = \frac{h \sin \theta}{16\pi((\rho^2 + h^2 \cos^2 \theta)^2(a\rho - h^2 - \rho^2))}(2\rho h^2 \cos^2 \theta - ah^2 \cos^2 \theta - 2a\rho^2 + 2\rho^3). \quad (35)
\]

Keeping terms of order \( a \) we get

\[
\mathcal{U}_0^{01} = \frac{a \sin \theta}{16\pi(\rho^2 + h^2 \cos^2 \theta)^2}(\rho^2 h^2 \cos^2 \theta + h^4 \cos^4 \theta - h^4 \cos^2 \theta + 2\rho^4 + \rho^2 h^2), \quad (36)
\]

\[
\mathcal{U}_0^{02} = \frac{-a\rho h^2 \sin^2 \theta \cos \theta}{16\pi((\rho^2 + h^2 \cos^2 \theta)^2(\rho^2 + h^2))}(h^2 \cos^2 \theta + 2h^2 + 3\rho^2), \quad (37)
\]

\[
\mathcal{U}_0^{03} = -\frac{\rho h \sin \theta}{8\pi(\rho^2 + h^2)} + \frac{a h^3 \sin \theta}{16\pi((\rho^2 + h^2 \cos^2 \theta)(\rho^2 + h^2)^2)}(h^2 \cos^2 \theta + \rho^2 \cos^2 \theta + 2\rho^2). \quad (38)
\]

Substituting (36), (37) and (38) into (32) we get

\[
E = \frac{a}{2}. \quad (39)
\]

This result is very satisfactory since if \( a = 2m \) then (39) becomes

\[
E = m, \quad (40)
\]

as it should be.

4. Summary

The results of the preceding sections can be summarized as follows

1) The exact form of the solutions (18) is a solution to the field equations (7) and (8) [20] and it is also a solution to the field equation of Einstein.

2) The parallel vector field resulting from the transformation (27) is an exact solution to the field equations (7) and (8).

3) The energy distribution of the solution (18) is calculated using the energy momentum complex given by Møller [15]. Using the superpotential (29) we obtained the necessary components (36,37,38) required for the calculations of energy. Substituting these values of the superpotential in (32) we get the satisfactory results \( E = m \) as it should be.
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