Autoequivalences of derived categories of elliptic surfaces with non-zero Kodaira dimensions

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Abstract

We study the group of autoequivalences of the derived categories of coherent sheaves on smooth projective elliptic surfaces with non-zero Kodaira dimensions. We find a description of it when each reducible fiber is a cycle of $(-2)$-curves.

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1 Introduction

1.1 Motivations and results

Let $X$ be a smooth projective variety over $\mathbb{C}$ and $D(X)$ the bounded derived category of coherent sheaves on $X$. For smooth projective varieties $X$ and $Y$, if they have mutually equivalent derived categories, then we call $X$ and $Y$ are Fourier–Mukai partners. We define the set of isomorphism classes of Fourier–Mukai partner of $X$ as

$$\text{FM}(X) := \{Y \text{ smooth projective varieties } | D(X) \cong D(Y)\} / \cong.$$ 

It is an interesting problem to determines the set $\text{FM}(X)$ for a given $X$. There are several known results in this direction. For example, Bondal and Orlov show that if $K_X$ or $-K_X$ is ample, then $X$ can be entirely reconstructed from $D(X)$, namely $\text{FM}(X) = \{X\}$ [BO95]. To the contrary, there are examples of mutually non-isomorphic varieties $X$ and $Y$ having mutually equivalent derived categories. For example, in dimension 2, if $\text{FM}(X) \neq \{X\}$, then $X$ is K3 surfaces, abelian surface or relatively minimal elliptic surfaces with non-zero Kodaira dimension ([BM01], [Ka02]). In dimension 3, some results are shown by Toda [To03]. Moreover, Orlov gives a complete answer in [Or02] to this problem for abelian varieties.

It is also natural to study the isomorphism classes of autoequivalences of $D(X)$. The group consisting of all exact $\mathbb{C}$-linear autoequivalences of $D(X)$ is denoted by

$$\text{Auteq} D(X).$$
We note that \( \text{Auteq} D(X) \) always contains the group

\[ A(X) := \text{Pic} X \rtimes \text{Aut} X \rtimes \mathbb{Z}[1], \]

generated by \textit{standard autoequivalences}, namely the functors of tensoring with invertible sheaves, automorphisms of \( X \) and the shift functor \([1]\).

When \( K_X \) or \(-K_X\) is ample, Bondal and Orlov show that \( \text{Auteq} D(X) \cong A(X) \).

When \( X \) is an abelian variety, Orlov determines the structure of \( \text{Auteq} D(X) \) \([\text{Or02}]\). When \( X \) is an elliptic curve as a special case, the autoequivalence group is described as

\[ 1 \to \hat{X} \rtimes \text{Aut} X \rtimes \mathbb{Z}[2] \to \text{Auteq} D(X) \xrightarrow{\theta} \text{SL}(2,\mathbb{Z}) \to 1. \]

Here \( \theta \) is given by the action of \( \text{Auteq} D(X) \) on the even integral cohomology group \( H^0 \oplus H^2(X,\mathbb{Z}) \), which is isomorphic to \( \mathbb{Z}^2 \). In this case the group \( \text{Auteq} D(X) \) contains the Fourier–Mukai transform \( \Phi_{J_X(a,b)\to X} \) determined by a universal sheaf \( U \) of the fine moduli space \( J_X(a,b) \) of stable vector bundles of the rank \( a \) and the degree \( b \) with \((a,b) = 1\). By the work of Atiyah \([\text{At57, Theorem 7}]\), \( J_X(a,b) \) is isomorphic to \( X \), and hence \( \Phi_{J_X(a,b)\to X} \) can be regarded as an autoequivalence of \( D(X) \).

For the minimal resolution \( X \) of \( A_n \)-singularities on a surface, Ishii and Uehara determine the structure of \( \text{Auteq} D(X) \) \([\text{IU05}]\). It is generated by the group \( A(X) \) and twist functors of the form \( T_{O_G(a)} \) \([\text{ST01}]\) associated with the line bundle \( O_G(a) \) on a \((-2)\)-curve \( G(\cong \mathbb{P}^1) \) on \( X \).

We can see that both autoequivalences, \( \Phi_{J_X(a,b)\to X} \) and \( T_{O_G(a)} \), do not belong to the group \( A(X) \).

The case of smooth projective elliptic surfaces \( \pi: S \to C \) with non-zero Kodaira dimension is a mixture of the last two cases. If \( S \) has a reducible fiber, each component of it is a \((-2)\)-curve. Hence \( \text{Auteq} D(S) \) contains twist functors as in the case \([\text{IU05}]\). On the other hand, let us consider the fine moduli space \( J_S(a,b) \) of pure 1-dimensional stable sheaves on \( S \), the general point of which represents a rank \( a \), degree \( b \) stable vector bundle supported on a smooth fiber of \( \pi \). It often occurs that there is an isomorphism \( S \cong J_S(a,b) \), and then the Fourier–Mukai transform \( \Phi_{J_S(a,b)\to S} \) determined by a universal sheaf \( U \) on \( J_S(a,b) \times S \) gives a non-standard autoequivalence.

For an object \( E \) of \( D(S) \), we define the fiber degree of \( E \)

\[ d(E) = c_1(E) \cdot F, \]

where \( F \) is a general fiber of \( \pi \). Let us denote by \( \lambda_S \) the highest common factor of the fiber degrees of objects of \( D(S) \). It is known that if \( a\lambda_S \) and \( b \) are coprime, the above mentioned fine moduli space \( J_S(a,b) \) exists. We denote \( J_S(b) := J_S(1,b) \).

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Define 
\[ B := \langle T_{O_C(a)} \mid G \text{ is a } (-2)\text{-curve} \rangle \]
and the congruence subgroup of SL(2, \Z) by
\[ \Gamma_0(m) := \left\{ \begin{pmatrix} c & a \\ d & b \end{pmatrix} \in \SL(2, \Z) \mid d \in m\Z \right\} \]
for \( m \in \Z \).

**Conjecture 1.1.** For a smooth projective elliptic surface \( S \) with \( \kappa(S) \neq 0 \), we have
\[ 1 \to \langle B, \otimes \mathcal{O}_S(D) \mid D \cdot F = 0, \ F \text{ is a fiber} \rangle \rtimes \mathrm{Aut} S \times \Z[2] \to \mathrm{Auteq} D(S) \]
\[ \Theta \to \left\{ \begin{pmatrix} c & a \\ d & b \end{pmatrix} \in \Gamma_0(\lambda_S) \mid J_S(b) \cong S \right\} \to 1. \]

Here \( \Theta \) is induced by the action of \( \mathrm{Auteq} D(S) \) on the integral cohomology groups of the even degree part \((H^0 \oplus H^2)(\Z) \cong \Z^2\) on smooth fibers.

**Remark 1.2.**
(i) The quotient group \( \Gamma_0(\lambda_S)/\mathrm{Im} \Theta \) is naturally identified with the set of Fourier–Mukai partners \( \FM(S) \) of \( S \). See Remark \[L.1\]
(ii) If \( \pi \) has a section, we know that \( \mathrm{Im} \Theta \cong \SL(2, \Z) \). See Remark \[L.1\]
(iii) By \[L05\] Proposition 4.18], we have
\[ B \cap (\otimes \mathcal{O}_S(D) \mid D \cdot F = 0, \ F \text{ is a fiber}) = \langle \mathcal{O}_S(G) \mid \ G \text{ is a } (-2)\text{-curve} \rangle. \]

The following is the main result in this article.

**Theorem 1.3** (=Theorem \[L.1\].) Suppose that each reducible fiber on the elliptic surface \( S \) is a cycle of \((-2)\text{-curves, i.e. of type } L_n \text{ for some } n > 1. \) Then Conjecture \[L.1\] is true.

In the case that \( \pi \) has only irreducible fibers, and \( \pi \) has a section, Conjecture \[L.1\] is essentially shown in \[LST13\].

### 1.2 Outline of the proof of Theorem \[1.3\]

Let \( S \) be a projective elliptic surface with \( \kappa(S) \neq 0 \), and \( Z \) be the set of union of all reducible fibers of the elliptic fibration \( \pi : S \to C \), and \( U \) be the complement of \( Z \) in \( S \). We introduce a group homomorphism
\[ \iota_U : \mathrm{Auteq} D(S) \to \mathrm{Auteq} D(U) \]

\[ ^1 \text{They only consider autoequivalences } \Phi \text{ with } \Phi \in \mathrm{Ker} \delta. \text{ See the definition of } \delta \text{ in Remark } \[L.9\]. \text{ Instead they do not put any restriction on the dimension of } S. \]
in Proposition 4.8 and denote $\text{Im} \, \iota_U$ by $\text{Auteq}^1 D(U)$. We can classify all elements of $\text{Auteq}^1 D(U)$ by Proposition 4.5 and determine the structure of $\text{Auteq}^1 D(U)$ in Theorem 4.10.

Assume furthermore that all reducible fibers of $\pi$ are of type $I_n$ ($n > 1$) as in Theorem 1.3. Then we can show that

$$B = \text{Ker} \, \iota_U$$

(1)

by using Proposition 1.4 (= Proposition 5.2). We use the notation in Theorem 1.3 and take a connected component $Z_0$ of $Z$. Let us consider the irreducible decomposition $Z_0 = C_1 \cup \cdots \cup C_n$, where each $C_i$ is a $(-2)$-curve. Suppose that we are given an autoequivalence $\Phi$ of $DZ_0(S)$ preserving the cohomology class $[\mathcal{O}_x] \in H^4(S, \mathbb{Q})$ for some point $x \in Z_0$. Then, there are integers $a$, $b$ ($1 \leq b \leq n$) and $i$, and there is an autoequivalence

$$\Psi \in \langle T_{\mathcal{O}_G(a)} \mid G \text{ is a } (-2)\text{-curves contained in } Z_0 \rangle$$

such that

$$\Psi \circ \Phi(\mathcal{O}_{C_1}) \cong \mathcal{O}_{C_b}(a)[i]$$

and

$$\Psi \circ \Phi(\mathcal{O}_{C_1}(-1)) \cong \mathcal{O}_{C_b}(a-1)[i].$$

In particular, for any point $x \in C_1$, we can find a point $y \in C_b$ with $\Psi \circ \Phi(\mathcal{O}_x) \cong \mathcal{O}_y[i]$.

Proposition 1.4 is proved in §7 by a method developed in [IU05]. Then we can deduce Theorem 1.3 from the equation (1) and the description of $\text{Auteq}^1 D(U)$ obtained in Theorem 4.10.

The construction of this article is as follows. In §2 we show several preliminaries results and give definitions needed afterwards. In §3 we consider the set of Fourier–Mukai partners $\text{FM}(S)$ of elliptic ruled surfaces. The results in §3 are used to describe the autoequivalence group $\text{Auteq} D(S)$ of certain elliptic ruled surfaces $S$ in §8. In §4 we consider the structure of $\text{Auteq}^1 D(U)$ intensively. In §5 we reduce the proof of Theorem 1.3 to showing the equation (1), and furthermore we reduce the proof of (1) to showing Proposition 1.4. In §6 we also show several lemmas used in §4 and §7. In §8 we show Proposition 5.4 which is the first step to prove Proposition 1.4. In §9 we prove Proposition 1.4. Finally in §8 we treat examples of elliptic surfaces satisfying the assumption in Theorem 1.3. In the examples, we can determine the set $\text{FM}(S)$, and also know when $J_S(b) \cong S$ holds. By the description of $\text{Im} \, \Theta$ in Conjecture 1.1, this information gives us better description of $\text{Im} \, \Theta$ in these examples.
1.3 Notation and conventions

All varieties $X$ will be defined over $\mathbb{C}$. A point on $X$ will always mean a closed point. For a closed subscheme $Z$ with a compact support, $[Z]$ denotes its cohomology class. By an elliptic surface, we will always mean a smooth surface $S$ together with a smooth curve $C$ and a relatively minimal projective morphism $\pi: S \to C$ whose general fiber is an elliptic curve.

Let $\pi: S \to C$ and $\pi': S' \to C'$ be projective elliptic surfaces, and suppose that each of $S$ and $S'$ has a unique elliptic fibration. Then every isomorphism $\varphi: S \to S'$ induces an isomorphism $C \to C'$. We always consider the cases (namely the cases $S' = J_S(a,b)$ in §2.4) that there is a natural identification between $C$ and $C'$, hence the induced isomorphism is naturally regarded as an automorphism of $C$. We denote it by $\varphi_C$. In other words, $\varphi_C$ satisfies $\pi' \circ \varphi = \varphi_C \circ \pi$ by the identification between $C$ and $C'$. We define

$$\text{Aut}_S C := \{ \varphi_C \in \text{Aut} C \mid \varphi \in \text{Aut} S \}$$
and

$$\text{Aut}_C S := \{ \varphi \in \text{Aut} S \mid \varphi_C = \text{id}_C \}.$$  

As its consequence we have a short exact sequence

$$1 \to \text{Aut}_C S \to \text{Aut} S \to \text{Aut}_S C \to 1.$$  

For two elliptic surfaces $\pi: S \to C$ and $\pi': S' \to C$, the isomorphism $\varphi: S \to S'$ satisfying $\pi = \pi' \circ \varphi$ is called an isomorphism over $C$.

For an elliptic curve $E$ and some positive integer $m$, $mE$ denotes the set of points on the group scheme $E$ whose order is $m$. $\hat{E}$ denotes the dual elliptic curve of $E$, namely $\hat{E}$ is just the group scheme $\text{Pic}^0 E$ of line bundles on $E$ of the degree 0.

$D(X)$ denotes the bounded derived category of coherent sheaves on $X$. For a closed subset $Z$ of $X$, $D_Z(X)$ denotes the full subcategory of $D(X)$ consisting of objects supported on $Z$. Here, the support of an object of $D(X)$ is, by definition, the union of the set-theoretically supports of its cohomology sheaves. For objects $\alpha, \beta \in D(X)$ with compact supports, we define the Euler form as

$$\chi(\alpha, \beta) := \sum_i (-1)^i \dim \text{Hom}^i_{D(X)}(\alpha, \beta).$$

An object $\alpha$ in $D(X)$ is said to be simple (respectively rigid) if

$$\text{Hom}_{D(X)}(\alpha, \alpha) \cong \mathbb{C} \ (\text{respectively } \text{Hom}^1_{D(X)}(\alpha, \alpha) \cong 0).$$

Suppose that the closed subset $Z$ of $X$ has a scheme structure, and denote the closed embedding by $i: Z \hookrightarrow X$, then we often denote the derived pull back $L_i^* \alpha$ simply by $\alpha|_Z$.

$\text{Auteq} \ T$ denotes the group of $\mathbb{C}$-linear exact autoequivalences of a $\mathbb{C}$-linear triangulated category $T$. 

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2 Preliminaries

2.1 General results for Fourier–Mukai transforms

Let $X$ and $Y$ be smooth projective varieties. We call $Y$ a Fourier–Mukai partner of $X$ if $D(X)$ is $\mathbb{C}$-linear triangulated equivalent to $D(Y)$. We denote by

$$\text{FM}(X)$$

the set of isomorphism classes of Fourier–Mukai partners of $X$.

For an object $\mathcal{P} \in D(X \times Y)$, we define an exact functor $\Phi^\mathcal{P}$, called an integral functor, to be

$$\Phi^\mathcal{P} := \mathbb{R}p_Y^*(\mathcal{P} \otimes p_X^*(-)) : D(X) \to D(Y),$$

where we denote the projections by $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$. We also sometimes write $\Phi^\mathcal{P}$ as $\Phi_{X \to Y}^\mathcal{P}$ to emphasize that it is a functor from $D(X)$ to $D(Y)$.

Next suppose that $X$ and $Y$ are not necessarily projective. Then in general $\mathbb{R}p_Y^*$ is not well-defined as the functor $D(X \times Y) \to D(Y)$, since $p_Y$ is not projective. Instead, suppose that there are projective morphisms $X \to C$ and $Y \to C$ over a smooth variety $C$, and $\mathcal{P}$ is a perfect complex in $D(X \times_C Y)$. Then we can also define the integral functor in this case, by replacing the projections $p_Y$ and $p_X$ with $p_Y : X \times_C Y \to Y$ and $p_X : X \times_C Y \to X$ respectively (here we use the same notation, by abuse of notation).

If we want to emphasize this situation, $\Phi$ is called an integral functor over $C$. We use relative integral transforms in the case of elliptic surfaces over a non-projective base $C$ afterwards.

By the result of Orlov ([Or97]), for smooth projective varieties $X$ and $Y$, and for a fully faithful functor $\Phi : D(X) \to D(Y)$, there is a unique object $\mathcal{P} \in D(X \times Y)$, up to isomorphisms, such that

$$\Phi \cong \Phi^\mathcal{P}.$$

If an integral functor (over $C$) is an equivalence, it is called a Fourier–Mukai transform (over $C$).
Since the left adjoint to an integral functor $\Phi^P$ over $C$ is given by an integral functor $\Phi^Q$ over $C$, where

$$Q := \mathbb{R}\text{Hom}_{X \times C Y}(P, \mathcal{O}_{X \times C Y}) \overset{\sim}{\rightarrow} p_X^*\omega_{X/C}[\dim X - \dim C]$$

(see the proof of [Hu06 Proposition 5.9]). Hence if $\Phi^P$ is an equivalence, its quasi-inverse is also given by $\Phi^Q$.

We can also see that the composition of integral functors over $C$ is again an integral functor over $C$ (cf. [Hu06 Proposition 5.10]).

**Lemma 2.1.** Let $\Phi: D(X) \to D(Y)$ be a Fourier–Mukai transform over a smooth variety $C$ between smooth varieties $X,Y$, projective over $C$. Then the set of points $x \in X$ for which the object $\Phi(\mathcal{O}_x)$ is a sheaf forms an (possibly empty) open subset of $X$.

**Proof.** This is a special case of [BM01, Proposition 2.4]. See also the proof of [BM01, Lemma 2.5].

The following is well-known.

**Lemma 2.2.** Let $\Phi: D(X) \to D(Y)$ be a Fourier–Mukai transform over a smooth variety $C$ between smooth varieties $X,Y$, projective over $C$. Assume that it satisfies that $\Phi(\mathcal{O}_x)$ is a shift of a sheaf supported on a finite set on $Y$ for all points $x \in X$. Then we have

$$\Phi \cong \phi_* \circ (-) \otimes \mathcal{L}[n]$$

for a line bundle $\mathcal{L}$ on $X$, an isomorphism $\phi: X \to Y$ over $C$ and some integer $n$.

**Proof.** We know that $\text{Hom}_{D(Y)}(\Phi(\mathcal{O}_x), \Phi(\mathcal{O}_x)) = \mathbb{C}$, and hence $\Phi(\mathcal{O}_x)$ satisfies the condition in [Hu06 Lemma 4.5]. In particular, $\Phi(\mathcal{O}_x) \cong \mathcal{O}_y[n]$ for some $y \in Y$ and $n \in \mathbb{Z}$. Note that the integer $n$ does not depend on the choice of a point $x$ by Lemma 2.1. Then apply the result in [BM98 §3.3] to get the conclusion.

**Lemma 2.3.** Let $X$ and $Y$ be smooth projective varieties, $Z$ and $W$ be closed subsets of $X$ and $Y$ respectively. Suppose that a Fourier–Mukai transform $\Phi = \Phi_{X \to Y}$ and its quasi-inverse $\Psi$ satisfy that

$$\text{Supp} \Phi(\mathcal{O}_x) \subset W \text{ and } \text{Supp} \Psi(\mathcal{O}_y) \subset Z.$$
Proof. We repeatedly use [BM02, Lemma 4.2] which states that for an object $E \in D(X)$, a point $x$ is contained in $\text{Supp } E$ if and only if $R\text{Hom}_{D(X)}(E, O_x) \neq 0$. Note that the Grothendieck–Serre duality implies that the latter condition is equivalent to the condition $R\text{Hom}_{D(X)}(O_x, E) \neq 0$.

Take a point $x \in Z$ and $y \in Y \setminus W$. Then

$$R\text{Hom}_{D(X)}(\Psi(O_y), O_x) = R\text{Hom}_{D(Y)}(O_y, \Phi(O_x))$$

vanishes by the assumption. This means that $\text{Supp } \Psi(O_y) \cap Z = \emptyset$, and thus for an object $E \in D_Z(X)$, we have

$$R\text{Hom}_{D(Y)}(\Phi(E), O_y) = R\text{Hom}_{D(X)}(E, \Psi(O_y)) = 0,$$

which implies that $\Phi(E) \in D_W(Y)$. Here the last equality follows from e.g. [Hu06, Lemma 3.9]. By a similar way, we can prove that $\Psi(F) \in D_Z(S)$ for any objects $F \in D_W(S)$. Therefore we obtain the conclusion. 

The following is also well-known.

Lemma 2.4 (cf. Proposition 2.15 in [HLS09]). Let $\pi: X \to C$ and $\pi': Y \to C$ be flat projective morphisms between smooth varieties, and take a point $c$ of $C$. Suppose that $X_c$ and $Y_c$ are the fibers of $\pi$ and $\pi'$ respectively over the point $c$ and that we are given an integral functor $\Phi = \Phi^P_{X \to Y}$ over $C$.

Let us consider the integral functor $\Psi = \Phi^P|_{X_c \times Y_c}$, and denote the inclusions by $k: X_c \hookrightarrow X$ and $k': Y_c \hookrightarrow Y$. Then we have the following.

(i) $\Phi$ and $\Psi$ satisfies $k'_* \circ \Psi \cong \Phi \circ k_*$.

(ii) Assume furthermore that $\Phi$ is an Fourier–Mukai transform. Then so is $\Psi$.

Proof. (i) directly follows from the projection formula and the flat base change formula (cf. [Hu06, Pages 83, 85]). For (ii), suppose that $X = Y$ and $\Phi \cong \text{id}_X$. Then obviously $\Psi \cong \text{id}_{X_c}$ also holds. Apply this argument for the functor $\Phi \circ \Phi^{-1}$, and then we get the result.

2.2 Twist functors
We give an important example of autoequivalences.

Definition 2.5 ([ST01]). Let $X$ be a smooth variety.

(i) We say that an object $\alpha \in D(X)$ is spherical if we have $\alpha \otimes \omega_X \cong \alpha$ and

$$\text{Hom}^k_{D(X)}(\alpha, \alpha) \cong \begin{cases} 0 & k \neq 0, \dim X \\ C & k = 0, \dim X. \end{cases}$$

Although we do not assume that $X_c$ and $Y_c$ are smooth, the perfectness of $P|_{X_c \times Y_c}$ assures that $\Psi$ defines a functor from $D(X_c)$ to $D(Y_c)$.
Let $\alpha \in D(X)$ be a spherical object. We consider the mapping cone

$$C = \text{Cone}(\pi_1^* \alpha \otimes L \otimes \pi_2^* \alpha \to O_\Delta)$$

of the natural evaluation $\pi_1^* \alpha \otimes L \otimes \pi_2^* \alpha \to O_\Delta$, where $\Delta \subset X \times X$ is the diagonal, and $\pi_i$ is the projection of $X \times X$ to the $i$-th factor. Then the integral functor $T_\alpha := \Phi^C_{X \times X}$ defines an autoequivalence of $D(X)$, called the twist functor along the spherical object $\alpha$.

Remark 2.6. By the definition, for any $\beta \in D(X)$ and a spherical object $\alpha \in D(X)$, we have an exact triangle

$$R \text{Hom}_{D(X)}(\alpha, \beta) \otimes C_\alpha \to \beta \to T_\alpha(\beta).$$

Suppose that $\text{Supp} \beta \cap \text{Supp} \alpha = \emptyset$. Then since $R \text{Hom}(\alpha, \beta) = 0$, we have $T_\alpha(\beta) \cong \beta$. We use this remark in §5.1. Furthermore in the Grothendieck group $K(X)$, we have

$$[T_\alpha(\beta)] = [\beta] - \chi(\alpha, \beta)[\alpha].$$

Example 2.7. (i) Let $S$ be a smooth surface, and $G$ be a $(-2)$-curve. Then a line bundle $O_G(a)$ is a spherical object. It follows from the equality (3) that $[T_{O_G(a)}(O_x)] = [O_x]$ for any point $x \in S$.

By using (2), we can see that

$$H^{-1}(T_{O_G}(O_G(2))) = O_G(1)^{\oplus 2} \text{ and } H^0(T_{O_G}(O_G(2))) = O_G,$$

and hence $T_{O_G} \notin A(S)$.

(ii) Let $E$ be an elliptic curve, and $\alpha$ be a simple coherent sheaf, e.g. line bundles and the structure sheaf $O_x$ on a point $x \in E$. Then $\alpha$ is spherical. Usually twist functors are not standard autoequivalences, but we can see in this case that $T_{O_x} \cong \otimes O_E(x) \in A(E)$ (cf. [Hu06, Example 8.10]).

Theorem 2.8 (Theorem 1.3 in [IU05], Appendix A in [IU010]). Let $X$ be a minimal resolution of $A_n$-singularity $\text{Spec} \mathbb{C}[x,y,z]/(x^2 + y^2 + z^{n+1})$. Define

$$B := \langle T_{O_G(a)} \mid G \text{ is a } (-2)\text{-curve} \rangle$$

and denote by $Z$ the exceptional set of the resolution. Then we have

$$\text{Auteq } D_Z(X) = \langle B, \text{Pic } X \rangle \rtimes \text{Aut } X \times \mathbb{Z}.$$
2.3 Autoequivalences of elliptic curves

Let $E$ be an elliptic curve. For a given $\Phi = \Phi^P \in \text{Auteq} D(E)$, we associate

to it a group automorphism $\rho(\Phi)$ of $H^*(E, \mathbb{Z}) \cong \mathbb{Z}^4$ such that

$$\rho(\Phi^P)(-):= p_{2*}(\text{ch}(\mathcal{P}) \cdot p_1^*(-))$$

(cf. [Hu06, Corollary 9.43]). Then we obtain a group homomorphism

$$\rho: \text{Auteq} D(E) \to \text{GL}(H^*(E, \mathbb{Z})),$$

and it is known that $\rho$ restricts the parity, i.e. it is decomposed as $\rho = \eta \oplus \theta$, where $\eta(\Phi) \in \text{GL}(H^{\text{odd}}(E, \mathbb{Z}))$ and $\theta(\Phi) \in \text{GL}(H^{\text{ev}}(E, \mathbb{Z}))$.

Because $\theta(\Phi) \in \text{GL}(2, \mathbb{Z})$ preserves the Euler form $\chi(-, -)$ we can see that $\theta(\Phi)$ actually gives an element of $\text{SL}(2, \mathbb{Z})$. Take the classes $[\mathcal{O}_E]$ and $[\mathcal{O}_x]$ for some point $x$ as a basis of $H^{\text{ev}}(E, \mathbb{Z}) \cong \mathbb{Z}^2$. Then we can check that $T_{\mathcal{O}_E}, T_{\mathcal{O}_x}(\cong \otimes \mathcal{O}_E(x))$ and $\Phi^U$ give elements

$$(1 \hspace{1cm} -1), \hspace{1cm} (1 \hspace{1cm} 0), \hspace{1cm} (0 \hspace{1cm} 1)$$

respectively, where $U$ is the normalized Poincare bundle on $E \times E$. Here note that every elliptic curve is principally polarized, and hence we can identify $E$ with $\hat{E}$. Two of these elements actually generate the group $\text{SL}(2, \mathbb{Z})$, and therefore the map

$$\theta: \text{Auteq} D(E) \to \text{SL}(2, \mathbb{Z})$$

is surjective. Compute the kernel of this map and then we have a short exact sequence of groups

$$1 \to \hat{E} \times \text{Aut} E \times \mathbb{Z}[2] \to \text{Auteq} D(E) \to \text{SL}(2, \mathbb{Z}) \to 1.$$

2.4 Bridgeland theory on Fourier–Mukai transforms on elliptic surfaces

Bridgeland, Maciocia and Kawamata show ([BM01], [Ka02]) that for a smooth projective surface $S$, if $S$ has a non-trivial Fourier–Mukai partner $T$ of $S$, that is $|\text{FM}(S)| \neq 1$, then both of $S$ and $T$ are abelian varieties, K3 surfaces or minimal elliptic surfaces with non-zero Kodaira dimensions.

We consider the last case more intensively. Let $\pi : S \to C$ be an elliptic surface. The results referred in §2.4 originally stated under the assumption that $S$ is projective, but some of them still hold true without the projectivity of $S$. Moreover for our purpose, it is sometimes important to consider non-projective elliptic surfaces, hence we do not assume that $S$ is projective without specified otherwise.

For an object $E$ of $D(S)$, we define the fiber degree of $E$ as

$$d(E) = c_1(E) \cdot F,$$
where $F$ is a general fiber of $\pi$. Let us denote by $r(E)$ the rank of $E$, and by $\lambda_S$ the highest common factor of the fiber degrees of objects of $D(S)$. Equivalently, $\lambda_S$ is the smallest number $d$ such that there is a holomorphic $d$-section of $\pi$. For integers $a > 0$ and $b$ with $b$ coprime to $a\lambda_S$, by [Br98] there exists a smooth, 2-dimensional component $J_S(a,b)$ of the moduli space of pure dimension one stable sheaves on $S$, the general point of which represents a rank $a$, degree $b$ stable vector bundle supported on a smooth fiber of $\pi$. There is a natural morphism $J_S(a,b) \to C$, taking a point representing a sheaf supported on the fiber $\pi^{-1}(x)$ of $S$ to the point $x$. This morphism is a minimal elliptic fibration ([Br98]). Put $J_S(b) := J_S(1,b)$. Obviously, $J^0(S) \cong J(S)$, the Jacobian surface associated to $S$, and $J^1(S) \cong S$. It can be also shown in [BM01, Lemma 4.2] that there is an isomorphism

$$J_S(a,b) \cong J_S(b) \quad (4)$$

over $C$.

**Theorem 2.9** (Theorem 5.3 in [Br98]). $\pi: S \to C$ be an elliptic surface and take an element

$$M = \left( \begin{array}{cc} c & a \\ d & b \end{array} \right) \in SL(2, \mathbb{Z})$$

such that $\lambda_S$ divides $d$ and $a > 0$. Then there exists universal sheaves $U$ on $J_S(a,b) \times S$, flat over both factors, such that for any point $(x,y) \in J_S(a,b) \times S$, $U|_{x \times S}$ has Chern class $(0,af,-b)$ on $S$ and $U|_{J_S(a,b) \times y}$ has Chern class $(0,af,-c)$ on $J_S(a,b)$. The resulting functor $\Phi^U_{J_S(a,b) \to S}$ is an equivalence and satisfies

$$\left( \frac{r(\Phi(E))}{d(\Phi(E))} \right) = M \left( \frac{r(E)}{d(E)} \right) \quad (5)$$

for all objects $E \in D(J_S(a,b))$

**Remark 2.10.** For integers $a > 0$ and $b$ with $b$ coprime to $a\lambda_S$, let us consider the Fourier–Mukai transform $\Phi = \Phi^U_{J_S(a,b) \to S}$. Take a smooth fiber $F$ of $\pi$ over a point $c \in C$, and denote by $F'$ the smooth fiber of the morphism $J_S(a,b) \to C$ over the point $c$. It turns out that $F'$ is isomorphic to $J_F(a,b)$, and hence $F'$ is isomorphic to $F$ by [Al57, Theorem 7]. The integral transform defined by the kernel $U|_{F \times F'} \in D(F \times F')$ induces an equivalence between $D(F)$ and $D(F')$ by Lemma 2.3. By a fixed isomorphism $F \cong F'$, regard the equivalence $\Phi^U_{F \times F'}$ as an autoequivalence of $D(F)$.

Note that $M = \theta(\Phi^U_{F \times F'})$ satisfies (5). (See the definition of $\theta$ in [2.3])

Put

$$\left( \begin{array}{cc} c & a \\ d & b \end{array} \right) := \theta(\Phi^U_{F \times F'})$$

for some $c,d \in \mathbb{Z}$. Then we see that $\lambda_S$ divides $d$, since the integer $d(\Phi(\mathcal{O}_S))$ divides $\lambda_S$. We will use this fact in [4.4]

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Theorem 2.9 implies that \( J_S(b)(\cong J_S(a, b)) \) is a Fourier–Mukai partner of \( S \) when \( (b, \lambda_S) = 1 \). Actually the converse is also true for projective elliptic surfaces \( S \) with non-zero Kodaira dimensions:

**Theorem 2.11** (Proposition 4.4 in [BM01]). Let \( \pi : S \to C \) be a projective elliptic surface and \( S' \) a smooth projective variety. Assume that the Kodaira dimension \( \kappa(S) \) is non-zero. Then the following are equivalent.

(i) \( S' \) is a Fourier–Mukai partner of \( S \).

(ii) \( S' \) is isomorphic to \( J_S(b) \) for some integer \( b \) with \( (b, \lambda_S) = 1 \).

**Remark 2.12.** Theorem 2.11 tells us that the Fourier–Mukai partner \( S' \) of \( S \) has an elliptic fibration \( \pi' : S' \to C \). Moreover we can see \( \kappa(S') = \kappa(S) \neq 0 \) (cf. [BM01, Lemma 4.3]), and hence it is a unique elliptic fibration structure on \( S' \). In particular, if two elliptic surfaces are mutually Fourier–Mukai partners, the base curves of elliptic fibrations can be identified. We use this fact implicitly afterwards.

There is a natural isomorphism

\[ J_S(b) \cong J_S(b + \lambda_S) \cong J_S(-b) \]

over \( C \) (see [BM01] Remark 4.5). Therefore we can define the subset

\[ H_S := \{ b \in (\mathbb{Z}/\lambda_S\mathbb{Z})^* \mid J_S(b) \cong S \} \]

of the multiplicative group \( (\mathbb{Z}/\lambda_S\mathbb{Z})^* \). Since we have

\[ J_{J_S(b)}(b) \cong J_S(bc) \text{ and } J_S(1) \cong S \]

(see, e.g. [Ue11] §2.2 and [BM01] Remark 4.5)), we can see that the condition \( J_S(b) \cong S \) implies that \( J_S(c) \cong S \) for \( c \in \mathbb{Z} \) with \( bc \equiv 1 \pmod{m} \). Therefore it turns out that \( H_S \) is a subgroup of \( (\mathbb{Z}/\lambda_S\mathbb{Z})^* \). In particular, there is a natural one-to-one correspondence between the set \( \text{FM}(S) \) and the quotient group \( (\mathbb{Z}/\lambda_S\mathbb{Z})^*/H_S \).

It is not easy to describe the group \( H_S \) concretely in general, which is equivalent to determine the set \( \text{FM}(S) \). See [Ue04] and [Ue11]. However when \( \lambda_S \leq 2 \), \( (\mathbb{Z}/\lambda_S\mathbb{Z})^* \) should be trivial, and hence \( \text{FM}(S) \) contains a unique element \( S \) itself.

As far as \( \lambda_S > 2 \), the group \( H_S \) contains at least two elements \( 1, \lambda_S - 1 \in (\mathbb{Z}/\lambda_S\mathbb{Z})^* \). Hence we have

\[ |\text{FM}(S)| \leq \varphi(\lambda_S)/2, \]

where \( \varphi \) is the Euler function. There are several examples in which we can compute the set \( \text{FM}(S) \) in [Ue11] Example 2.6]. In [K30] we also give examples in which we can compute the set \( \text{FM}(S) \).
Remark 2.13. In the above notation, we know that there is an isomorphism $J(J_S(a,b)) \cong J(S)$ over the curve $C$ (cf. [Ue11 §2.2]). Since it is known that the reduced induced structures of the fibers on $S$ and $J(S)$ over the same point in $C$ are isomorphic to each others, the same thing holds for $S$ and $J_S(a,b)$. Furthermore if $(b, \lambda_S) = 1$, the multiplicities of the fibers on $S$ and $J_S(a,b)$ are equal ([BM01 Lemma 4.3] or [Fr95 page 38]). Therefore, the fibers on $S$ and $J_S(a,b)$ over the same point are isomorphic to each other.

3 Fourier–Mukai partners of elliptic ruled surfaces

3.1 Fourier–Mukai transforms of quotient varieties

We review a method to construct Fourier–Mukai transforms between quotient varieties, due to [BM98]. Let $X$ and $Y$ be a smooth projective varieties on which a finite group $G$ acts. Let

\[ p_X: X \to X/G, \quad p_Y: Y \to Y/G \]

be the quotient morphisms.

Definition 3.1. Let $\Phi = \Phi^U: D(X) \to D(Y)$ be a Fourier–Mukai transform. $\Phi$ is called $G$-equivalent if there is an automorphism $\sigma \in \text{Aut} G$, and an isomorphism of functors

\[ g^* \circ \Phi \cong \Phi \circ \sigma(g)^* \]

for each $g \in G$.

The $G$-equivalence of $\Phi$ is equivalent to the existence of $\sigma$ satisfying

\[ (\text{id}_Y \times g)^* \mathcal{U} \cong (\sigma(g) \times \text{id}_X)^* \mathcal{U} \]

for any $g \in G$. It is also equivalent to that the existence of $\sigma$ satisfying

\[ (\sigma(g) \times g)^* \mathcal{U} \cong \mathcal{U} \]

for any $g$.

Proposition 3.2 (cf. [BM98]). Suppose that a given Fourier–Mukai transform $\Phi = \Phi^U: D(X) \to D(Y)$ is $G$-equivalent. Then there is a Fourier–Mukai transform $\Phi^{U'}: D(X/G) \to D(Y/G)$ for some $U' \in D((X/G) \times (Y/G))$.

Proof. The proof is scattered in [BM98], so we give only a sketch of it. The $G$-equivalence implies that $(\text{id}_Y \times p_X)_* \mathcal{U}$ is $G$-invariant, and hence

\[ (p_Y \times \text{id}_{X/G})^* \mathcal{U}' \cong (\text{id}_Y \times p_X)_* \mathcal{U} \]

(6)
holds for some $U'$ of $D((X/G) \times (Y/G))$. We define an integral functor

$$\Phi' := \Phi^U : D(X/G) \to D(Y/G).$$

Then by [BM98, Lemma 4.4], we conclude that there are isomorphisms

$$p_{X*} \circ \Phi \cong \Phi' \circ p_{Y*}, \quad p_X^* \circ \Phi' \cong \Phi \circ p_Y^*.$$

(When this happens, $\Phi$ is called a lift of $\Phi'$, or $\Phi'$ is a descent of $\Phi$.) Recall that the quasi-inverse of $\Psi$ is also a $G$-equivariant, hence is the lift of some integral functor $\Psi'$. Then $\Psi \circ \Phi \cong \text{id}_{D(X)}$ is the lift of $\Psi' \circ \Phi'$. Now [BM98, Lemma 4.3] implies the result. \qed

Remark 3.3. Take points $y \in Y$. Then for the object $U' \in D((X/G\times Y/G))$ in the proof of Proposition 3.2, the isomorphism (6) yields the isomorphism

$$U'|\text{p}_{X}(y) \times (X/G) \cong p_{X*}(U|y \times X).$$

3.2 Some technical lemmas on elliptic curves

Let us fix an elliptic curve $F$ and an element $a \in mF$ with a positive integer $m$. Let us denote by $E$ the quotient variety $F/\langle a \rangle$, by

$$q : F \to E$$

the quotient morphism, and by

$$\hat{q} : \hat{E} \to \hat{F}$$

the dual isogeny of $q$. Define a subgroup of $(\mathbb{Z}/m\mathbb{Z})^*$ as

$$H^n_F := \{k \in (\mathbb{Z}/m\mathbb{Z})^* \mid \exists \phi \in \text{Aut}_0 F \text{ such that } \phi(a) = k \cdot a\}.$$ 

We use the following technical lemmas in the proof of Theorem 3.7.

Lemma 3.4. Suppose that $m > 2$. Then exactly one of the following three cases for $F$ and $a \in mF$ occurs.

(i) The equality $H^n_F = \{\pm 1\}$ holds.

(ii) The $j$-invariant $j(F)$ is 1728, and there is an integer $n$ such that $m$ divides $n^2 + 1$. Moreover the point $a \in F$ is an element in the subgroup

$$\left\langle \frac{n}{m} + \frac{1}{m\sqrt{-1}} \right\rangle$$

of $F \cong \mathbb{C}/(\mathbb{Z} + \sqrt{-1}\mathbb{Z})$, and the equality $H^n_F = \{\pm 1, \pm n\}$ holds.
(iii) The \( j \)-invariant \( j(F) \) is 0, and there is an integer \( n \) such that \( m \) divides \( n^2 + n + 1 \). Moreover the point \( a \) is an element in the subgroup

\[
\left\langle \frac{n + 1}{m} + \frac{1}{m} \omega \right\rangle
\]

of \( F \cong \mathbb{C}/(\mathbb{Z} + \omega \mathbb{Z}) \), and the equality \( H_F^a = \{ \pm 1, \pm n, \pm n^2 \} \) holds. Here we put \( \omega = \frac{-1 + \sqrt{-3}}{2} \).

Proof. Recall that \( \text{Aut}_0 F \) consists of two elements \( \{ \pm 1 \} \) when \( j(F) \neq 0, 1728 \). In this case, obviously the case (i) occurs.

When \( j(F) = 1728 \), let us put \( a = \frac{x}{m} + \frac{y}{m} \sqrt{-1} \) \((x, y \in \mathbb{Z})\), and suppose first that an equality \( \sqrt{-1} a = k \cdot a \) (7) holds for some \( k \in \mathbb{Z} \).

Hence we know that \( a = \frac{ky}{m} + \frac{y}{m} \sqrt{-1} \), and since the order of \( a \) in \( F \) is \( m \), \( m \) and \( y \) are coprime. This and the equality (7) yield that \( m \) divides \( k^2 + 1 \). The coprimality also implies that the subgroups \( \langle a \rangle \) and \( \langle \frac{k}{m} + \frac{1}{m} \sqrt{-1} \rangle \) coincide. In particular, every element in this group also satisfies (7). It is easy to see that \( H_F^a = \{ \pm 1, \pm k \} \) holds.

Next suppose that the equality \( -\sqrt{-1} a = k \cdot a \) holds for some \( k \in \mathbb{Z} \), which is equivalent to \( \sqrt{-1} a = -k \cdot a \). Hence \( a \in \langle \frac{k}{m} + \frac{1}{m} \sqrt{-1} \rangle \), and a similar result to the previous case holds.

In the case (ii), the proof is similar.

It follows from the condition \( m > 2 \) that \( |H_F^a| = 2, 4 \) and 6 in the case (i), (ii) and (iii) respectively, hence the two cases do not occur at the same time.

As the definition of \( H_F^a \), we define

\[
H_E^L := \{ k \in (\mathbb{Z}/m\mathbb{Z})^* \mid \exists \psi \in \text{Aut}_0 E \text{ such that } \psi_1^* \mathcal{L} = \mathcal{L}^k \}.
\]

for a line bundle \( \mathcal{L} \in _m \hat{E} \). By the natural identification between \( E \) and \( \hat{E} \), we denote by \( b \in E \) the point corresponding to \( \mathcal{L} \in \hat{E} \). Then of course, the equality \( H_E^L = H_E^b \) holds.

Lemma 3.5. In any cases in Lemma 3.4 the equality \( H_F^a = H_E^L \) holds for any \( \mathcal{L} \in _m \hat{E} \) with \( \ker \hat{q} = \langle \mathcal{L} \rangle \). (This particularly implies that there is an isomorphism \( F \cong E \) in the cases (ii) and (iii).)

Proof. Let us consider the case (ii) first. Let \( L \) be the lattice generated by 1 and \( \sqrt{-1} \) in \( \mathbb{C} \) so that \( F \) with \( j(F) = 1728 \) is isomorphic to \( \mathbb{C}/L \). Moreover the elliptic curve \( E = F/\langle a \rangle \) is isomorphic to \( \mathbb{C}/(L + \langle a \rangle) \). We can see
that the lattice $L + \langle a \rangle$ is preserved by the complex multiplication by $\sqrt{-1}$.

(Hence $j(E) = 1728$, which implies the isomorphism $F \cong E$.) It turns out that the quotient morphism

$$q: F \cong \mathbb{C}/L \to E \cong \mathbb{C}/(L + \langle a \rangle)$$

induced by the inclusion $L \hookrightarrow L + \langle a \rangle$ is commutative with the complex multiplication by $\sqrt{-1}$.

Take an element $\frac{1}{m} \in \mathbb{C}/L(\cong F)$, and put

$$a := \frac{ny}{m} + \frac{y}{m}\sqrt{-1}$$

for the integer $n$ in (ii) and some $y \in (\mathbb{Z}/m\mathbb{Z})^*$. We define $L'$ to be the element in $E$ corresponding to $q(\frac{1}{m}) \in E$. Then we have

$$\sqrt{-1}q(\frac{1}{m}) = q(\frac{1}{m}\sqrt{-1}) = q(y^{-1}a - \frac{n}{m}) = -nq(\frac{1}{m}),$$

and this implies the equality $H^p_E = H^p_{E'}$. We can also see that $a$ and $\frac{1}{m}$
generated by the subgroup of $m$-torsion points on $F$, that is, the kernel of the multiplication map $[m](= \hat{q}q)$ by $m$. Consequently we have ker $\hat{q} = \langle L' \rangle$.

For any $L \in m\hat{E}$ with ker $\hat{q} = \langle L \rangle$, the equality $H^E_L = H^E_{L'}$ holds, which gives the assertion.

The proof in the case (iii) is similar.

Next let us take an element $L \in \ker \hat{q}$, and suppose that $|H^E_L| = 4$ or 6, namely the case (ii) or (iii) occurs for $E$ and $L \in m\hat{E}$. Then we have already shown above that $H^E_L = H^E_{L'}$ (just by replacing the roles of $\hat{E}$ and $F$). Consequently in the case (i), we again obtain the assertion.

3.3 Fourier–Mukai partners of elliptic ruled surfaces

Let $\mathcal{E}$ be a normalized, in the sense of [Ha77, Ch. 5. §2], rank 2 vector bundle on an elliptic curve $E$ and

$$f: S = \mathbb{P}(\mathcal{E}) \to E$$

be a $\mathbb{P}^1$-bundle on $E$ defined by $\mathcal{E}$. Let us put $e = -\deg \mathcal{E}$.

We use the following result to compute the set $FM(S)$ for elliptic ruled surfaces $S$.

Theorem 3.6 ([TU14]). In the above notation, $S$ has an elliptic fibration $\pi$ if and only if either

(i) $\mathcal{E} \cong \mathcal{O}_E \oplus L$ for $L \in m\hat{E}$ ($m \geq 1$), in particular $e = 0$, or

(ii) $e = -1$ and $\mathcal{E}$ is indecomposable.
Furthermore in the case $e = 0$ and $m > 0$, the set of singular fibers of $\pi$ consists of exactly two multiple fibers of type $mI_0$. In the case $e = 0$ and $m = 0$, then $S \cong E \times \mathbb{P}^1$, and hence $\pi$ has no multiple fibers. In the case $e = -1$, the set of singular fibers of $\pi$ consists of exactly three multiple fibers of type $2I_0$.

Our main result in this section is as follows.

**Theorem 3.7.** Let $f: S = \mathbb{P}(\mathcal{E}) \to E$ be a $\mathbb{P}^1$-bundle over an elliptic curve $E$ and $\mathcal{E}$ is a normalized rank 2 vector bundle. If $|FM(S)| \neq 1$, then we can take $E = \mathcal{O}_E \oplus L$ for some element $L \in \hat{m}E$ with $m > 2$. Furthermore in this case, the set $FM(S)$ consists of exactly $\varphi(m)/|H^L_E|$ elements of the form $\mathbb{P}((\mathcal{O}_E \oplus L_i))$ for some integer $i$ with $(i, m) = 1$ and $1 \leq i < m$. Here the group $H^L_E$ is defined in §3.2, and $\varphi$ is the Euler function.

We give the proof of this theorem in the last of §3.3. Before giving the proof, we need to show several claims.

Let $F$ be an elliptic curve. For points $x_1, x_2 \in F$, to distinguish the summations as divisors and as elements in the group scheme $F$, we denote by $x_1 \oplus x_2$ (respectively, $x_1 \ominus x_2$) the sum (respectively, the difference) of them by the operation of $F$, and

$$i \cdot x_1 := x_1 \oplus \cdots \oplus x_1 \quad (i \text{ times}).$$

We also denote by

$$ix_1 := x_1 + \cdots + x_1 \quad (i \text{ times})$$

the divisor on $F$ of degree $i$. As is well-known, there is a group isomorphism

$$F \to \hat{F} \quad x \mapsto \mathcal{O}_F(x - O),$$

where $O$ is the origin of $F$.

Let us take a cyclic group $G = \mathbb{Z}/m\mathbb{Z}$ with $m > 2$ and a generator $g$ of $G$. For integers $i \in (\mathbb{Z}/m\mathbb{Z})^*$, define representations

$$\rho: G \to \text{Aut}(\mathbb{P}^1) \quad \text{as} \quad \rho(g)(y) = \zeta y,$$

and

$$\rho_i: G \to \text{Aut}(F \times \mathbb{P}^1) \quad \text{as} \quad \rho_i(g)(x, y) = (T_{ia}x, \rho(g)(y)), \quad (9)$$

where $a$ is an element of $mF$, $T_a$ is the translation by $a$ and $\zeta$ is a primitive $m$-th root of unity in $\mathbb{C}$. Let us define

$$S_i := (F \times \mathbb{P}^1)/iG$$
the quotient of $F \times \mathbb{P}^1$ by the action $\rho_i$. Then we have the following commutative diagram:

\[
\begin{array}{ccc}
F & \overset{P_1}{\longrightarrow} & F \times \mathbb{P}^1 \\
\downarrow{q} & & \downarrow{q_i} \\
E & \overset{f_i}{\longrightarrow} & S_i \\
\downarrow{\pi_i} & & \downarrow{q_{\mathbb{P}^1}} \\
\mathbb{P}^1/G & \cong & \mathbb{P}^1
\end{array}
\]

Here every vertical arrow is the quotient morphism. Then $f_i$ is a $\mathbb{P}^1$-bundle and $\pi_i$ is an elliptic fibration.

Note that the quotient morphism $q$ does not depend on the choice of $i$, and that the left square in (10) is a fiber product diagram.

We can see that $\pi_i$ has exactly two multiple fibers of type $mI_0$ over the branch points of the quotient morphism $q_{\mathbb{P}^1}$, and it fits into the case in Theorem 3.6 (i). Consequently there is a line bundle $L_i \in m\hat{E}$ such that

\[ S_i \cong \mathbb{P}(O_E \oplus L_i) \]

for each $i$. Furthermore because the left square in (10) is a fiber product, we have $q_i^*L_i = O_F$, which implies that $\langle L_i \rangle = \text{Ker} \hat{q}$ for the dual isogeny $\hat{q}: \hat{E} \rightarrow \hat{F}$ of $q$. Therefore the subgroup $\langle L_i \rangle$ of $\hat{E}$ does not depend on the choice of $i$. In particular, we have an inclusion

\[ \{ S_i \mid i \in (\mathbb{Z}/m\mathbb{Z})^* \} / \cong \{ \mathbb{P}(O_E \oplus L_i) \mid i \in (\mathbb{Z}/m\mathbb{Z})^* \} / \cong . \]

We will see below that these sets actually coincide by checking their cardinality.

Let us start the following claim.

**Claim 3.8.** Take a line bundle $L \in m\hat{E}$. For $i, j \in (\mathbb{Z}/m\mathbb{Z})^*$, $\mathbb{P}(O_E \oplus L_i) \cong \mathbb{P}(O_E \oplus L_j)$ if and only if there is a group automorphism $\psi_1 \in \text{Aut}_0 E$ such that $\psi_1^*L \cong L^{\pm i-j}$ holds.

**Proof.** Since each of $\mathbb{P}(O_E \oplus L_i)$ and $\mathbb{P}(O_E \oplus L_i)$ has a unique $\mathbb{P}^1$-bundle structure, any isomorphism $\psi: \mathbb{P}(O_E \oplus L_i) \rightarrow \mathbb{P}(O_E \oplus L_j)$ induces an automorphism $\psi_1$ of $E$, which is compatible with $\psi$. We can see that $\psi_1$ satisfies the desired property. The opposite direction is obvious. \(\square\)

Now we have the following.

**Claim 3.9.** For $i, j \in (\mathbb{Z}/m\mathbb{Z})^*$, $S_i \cong S_j$ if and only if there is a group automorphism $\phi_1 \in \text{Aut}_0 F$ such that $\phi_1(a) = (\pm i-j) \cdot a$ holds.

**Proof.** Suppose that there is an isomorphism $\psi: S_i \rightarrow S_j$. As the proof of Claim 3.8, $\psi$ induces an automorphism $\psi_1$ of $E$ which is compatible with $\psi$. It is also satisfied that the dual isogeny $\psi_1$ preserves the subgroup $\langle L_i \rangle$ of $\hat{E}$, and hence $\psi_1$ lifts an automorphism $\phi_1$ of $F$. Since the left square
is a fiber product diagram, \( \psi \) lifts to an automorphism \( \phi \) of \( F \times \mathbb{P}^1 \). We can see that \( \phi \) is of the form \( \phi_1 \times \phi_2 \) for some \( \phi_2 \in \text{Aut} \mathbb{P}^1 \). We can see that any translation on \( F \) descends to a translation on \( E \), and hence replacing \( \phi_1 \) if necessary, we may assume that \( \phi_1 \in \text{Aut}_0 F \). Since \( \phi \) descends to \( \psi \), it should satisfy

\[
\phi \circ \rho_i(g) = \rho_j(g^k) \circ \phi
\]

for any \( g \in G \) and some \( k \in \mathbb{Z} \). By observing the action on \( \mathbb{P}^1 \), we know that \( k = 1 \) or \( m - 1 \), and moreover

\[
\phi_2(x) = \lambda x \quad \text{(in the case } k = 1) \quad \text{or} \quad \phi_2(x) = \lambda/x \quad \text{(in the case } k = m - 1)\]

for some \( \lambda \in \mathbb{C}^* \). In the former case, we obtain that \( \phi_1(a) = (i-1j) \cdot a \) holds, and in the latter case, \( \phi_1(a) = (-i-1j) \cdot a \) holds.

It follows from Claims 3.8 and 3.9 that we need to know the cardinality of the sets \( H^a_F \) and \( \hat{H}^\xi_E \) defined in §3.2, in order to know the cardinality in the sets in the both sides in (11). But by Lemma 3.5, we always have \( H^a_F = \hat{H}^\xi_E \), and by Claims 3.8 and 3.9, we also have

\[
\left\{ S_i \mid i \in (\mathbb{Z}/m\mathbb{Z})^* \right\}/\sim = \left\{ \mathbb{P}(O_E \oplus L_1^i) \mid i \in (\mathbb{Z}/m\mathbb{Z})^* \right\}/\sim.
\]

(12)

The cardinality of this set is

\[
\varphi(m)/|\hat{H}^\xi_E|
\]

and the cardinality \( |\hat{H}^\xi_E| \) is one of 2, 4 or 6, depending on the choice of \( \hat{E} \) and \( L \in mE \) as in Lemma 3.4, equivalently, the choice of \( F \) and \( a \in mF \).

**Claim 3.10.** In the above notation, \( S_i \cong J_{S_1}(i) \) for all \( i \) with \( i \in (\mathbb{Z}/m\mathbb{Z})^* \).

**Proof.** First we prove that there is a Fourier–Mukai transform between \( D(S_1) \) and \( D(S_i) \) by applying Proposition 3.2. It suffices to show that there is an object \( U \in D((F \times \mathbb{P}^1) \times (F \times \mathbb{P}^1)) \) satisfying

\[
(\rho_1(g) \times \rho_i(g))^* U \cong U
\]

(13)

and \( \Phi^U \) gives an autoequivalence on \( D(F \times \mathbb{P}^1) \).

Take an element \( j \in (\mathbb{Z}/m\mathbb{Z})^* \) such that \( ij = 1 \). Henceforth we identify \( F \) and \( \hat{F} \) as group schemes. Take the normalized Poincare line bundle \( \mathcal{P}_0 \), which means a Poincare line bundle \( P_0 \) on \( F \times F \) satisfies \( \mathcal{P}_0|_{O \times F} \cong O_F \) and \( \mathcal{P}_0|_{F \times O} \cong O_F \). Let us define

\[
\mathcal{P} := \mathcal{P}_0 \otimes p_1^*O_F(iO) \otimes p_2^*O_F(jO).
\]
Here we regard elements \( i, j \in (\mathbb{Z}/m\mathbb{Z})^* \) as integers satisfying \( 1 \leq i, j \leq m - 1 \). Then the line bundle \( \mathcal{P} \) satisfies
\[
\mathcal{P}|_{x \times F} \cong \mathcal{O}_F(x + (j - 1)O) \quad \text{and} \quad \mathcal{P}|_{F \times y} \cong \mathcal{O}_F(y + (i - 1)O)
\]
for any \( x, y \in F \). By the diagram
\[
\begin{array}{c}
x \times F \quad \xrightarrow{T_{i,a}} \quad F \times F \\
\quad \xrightarrow{T_{a,T_{i,a}}} \quad F \times y \quad \xleftarrow{T_{a,T_{i,a}}} \\
(x \oplus a) \times F \quad \xrightarrow{T_{i,a}} \quad F \times F \\
\quad \xleftarrow{T_{a,T_{i,a}}} \quad F \times (y \oplus i \cdot a)
\end{array}
\]
we have
\[
((T_{a} \times T_{i,a})^*\mathcal{P})|_{F \times y} \cong T_{a}^*(\mathcal{P}|_{(y \oplus i \cdot a) \times F})
\]
\[
\cong \mathcal{P}|_{F \times (y \oplus i \cdot a)} \otimes \mathcal{O}_F(a - O)^{-i}
\]
\[
\cong \mathcal{O}_F(y + ia - O) \otimes \mathcal{O}_F(a - O)^{-i}
\]
\[
\cong \mathcal{O}_F(y + (i - 1)O)
\]
\[
\cong \mathcal{P}|_{F \times y}.
\]

Using \( \text{ord} a = m \), we also have
\[
((T_{a} \times T_{i,a})^*\mathcal{P})|_{x \times F} \cong T_{i,a}^*(\mathcal{P}|_{(x \oplus a) \times F})
\]
\[
\cong \mathcal{P}|_{(x \oplus a) \times F} \otimes \mathcal{O}_F(ia - iO)^{-j}
\]
\[
\cong \mathcal{O}_F(x + a + (n - 2)O) \otimes \mathcal{O}_F(ia - iO)^{-j}
\]
\[
\cong \mathcal{O}_F(x + (j - 1)O)
\]
\[
\cong \mathcal{P}|_{x \times F}.
\]

Hence we obtain \((T_{a} \times T_{i,a})^*\mathcal{P} \cong \mathcal{P} \).

Let us define \( \Delta_{\mathbb{P}^1} (\cong \mathbb{P}^1) \) to be the diagonal of \( \mathbb{P}^1 \) in \( \mathbb{P}^1 \times \mathbb{P}^1 \). For the projection
\[
p_1 : (F \times F) \times \Delta_{\mathbb{P}^1} \to F \times F,
\]
define an object
\[
U := p_1^*\mathcal{P}.
\]
We restrict the action \( \rho_1 \times \rho_i \) on \( F \times \mathbb{P}^1 \times F \times \mathbb{P}^1 \) to the action on \( (F \times F) \times \Delta_{\mathbb{P}^1} \), which we also call by \( \rho_1 \times \rho_i \). Then we can see that
\[
((\rho_1(g) \times \rho_i(g))^*U)|_{x \times y \times \mathbb{P}^1} \cong \rho(g)^*U|(x \oplus a) \times (y \oplus i \cdot a) \times \mathbb{P}^1
\]
\[
\cong \rho(g)^*\mathcal{O}_{\mathbb{P}^1}
\]
\[
\cong \mathcal{O}_{\mathbb{P}^1}
\]
\[
\cong U|_{x \times y \times \mathbb{P}^1}
\]
and
\[
((\rho_1(g) \times \rho_i(g))^*\mathcal{U})|_{F \times F \times z} \cong (T_a \times T_i)^*(\mathcal{U}|_{F \times F \times \zeta z}) \\
\cong (T_a \times T_i)^*\mathcal{P} \\
\cong \mathcal{P} \\
\cong \mathcal{U}|_{F \times F \times z}
\]
for any \( z \in \mathbb{P}^1 \). Then by the See-saw theorem [Mu08 II Corollary 6] we can check that \( \mathcal{U} \) satisfies (13). Here we regard \( \mathcal{U} \) as a sheaf on \((F \times \mathbb{P}^1) \times (F \times \mathbb{P}^1)\).

Let us call by \( \mathcal{U}' \) the kernel object in \( D(S_1 \times S_i) \) of the decent of \( \Phi \). For \( y \times z \in F \times \mathbb{P}^1 \), we have \( \mathcal{U}|_{F \times z \times y \times z} = \mathcal{P}|_{F \times y} \), which is a degree \( i \) line bundle by (14). Since the restriction \( q_i|_{F \times z} \) for \( z \in \mathbb{P}^1 \), not a ramified point \( \{0, \infty\} \) of \( q_{P1} \), is isomorphic, it follows from Remark 3.3 that \( \mathcal{U}'|_{S_1 \times q_i(x \times z)} \) is also a degree \( i \) line bundle for such \( z \). Then by the universal property of \( J_i(S_1) \), there is a birational morphism between \( S_i \) and \( J_i(S_1) \) over \( \mathbb{P}^1 \setminus \{q_{P1}(0), q_{P1}(\infty)\} \).

Now Claim 4.1 implies the result.

Now we obtain the following.

**Proposition 3.11.** Let \( E \) be an elliptic surface, and define \( S \) to be an elliptic ruled surface \( P(\mathcal{O}_E \oplus L) \) for a line bundle \( L \in m \hat{E} \) with \( m > 2 \). Then we have

\[
\text{FM}(P(\mathcal{O}_E \oplus L)) = \{P(\mathcal{O}_E \oplus L^i) \mid i \in (\mathbb{Z}/m\mathbb{Z})^*\}/ \cong .
\]

This set consists of \( \varphi(m)/|H_E^*| \) elements.

**Proof.** Claim 3.8 yields the last statement for the cardinality. The first statement is a direct consequence of Theorem 2.11, the equation (12) and Claim 3.10. Only what we have to remark here is that for a given elliptic ruled surface \( P(\mathcal{O}_E \oplus L) \) is always obtained as a quotient of \( F \times \mathbb{P}^1 \) for some elliptic curve \( F \) by the action considered in (9).

Let us define \( F \) to be the dual of \( \hat{E}/\langle L \rangle \). Then we obtain an isogeny \( q: F \to E \) as the dual of the isogeny \( \hat{E} \to \hat{E}/\langle L \rangle \). Take an element \( a \in F \) generating the cyclic group \( \ker q(\cong \mathbb{Z}/m\mathbb{Z}) \), and consider the action on \( F \times \mathbb{P}^1 \) in (9). Then we obtain from (12) that \( S \cong S_i \) for some \( i \in (\mathbb{Z}/m\mathbb{Z})^* \). This is what we required.

Now we are in a position to show Theorem 3.7.

**Proof of Theorem 3.7.** First of all, the condition \( |\text{FM}(S)| \neq 1 \) implies that \( S \) has an elliptic fibration structure \( \pi: S \to \mathbb{P}^1 \) with \( \lambda_S > 2 \) (see [Ka02 Theorem 3.2] and [2.1]). In each cases in Theorem 3.6 we can see that \( \lambda_S = m \), hence \( S \) actually fits into the case in Theorem 3.6 (i) and \( m > 2 \). Now we can write \( S = P(\mathcal{O}_E \oplus L) \) for some \( L \in m\hat{E} \) for some \( m > 2 \). Then the assertion follows from Proposition 3.11. \( \square \)
4 Autoequivalences of elliptic surfaces with non-zero Kodaira dimensions

4.1 Notation and the setting

Let \( \pi: S \to C \) and \( \pi': S' \to C \)
be projective elliptic surfaces with non-zero Kodaira dimensions, and we denote the projections by

\[
p: S \times S' \to S \quad \text{and} \quad p': S \times S' \to S'.
\]

Let \( \Phi = \Phi^P: D(S) \to D(S') \) be a Fourier-Mukai transform. In this case, the cohomology class of \( K_{S'} \) is a non-zero rational multiple of the cohomology class of a fiber of \( \pi' \). On the other hand, since the Serre functor commutes with the equivalence \( \Phi \), there is an isomorphism

\[
\Phi(O_y) \cong \Phi(O_y) \otimes \omega_{S'}
\]

for each \( y \in S \). Furthermore since \( \Phi(O_y) \) is simple, we know that \( \text{Supp} \Phi(O_y) \) is connected. These facts imply that \( \text{Supp} \Phi(O_y) \) is contained in a single fiber of \( \pi' \). Denote the inclusion \( S' \cong y \times S' \hookrightarrow S \times S' \) by \( i \). Notice that

\[
P|_{y \times S'} \cong \Phi(O_y)
\]

and

\[
\text{Supp} P|_{y \times S'} = i^{-1}(\text{Supp} P)
\]

(see [Hu06, Lemma 3.29]). This gives

\[
\dim \text{Supp} P = 2 \text{ or } 3.
\]

Let us denote by \( Z \) the set of union of all \((-2)\)-curves on \( S \). Note that the set \( Z \) coincides with the set of union of all reducible fibers. We also denote by \( U \) the complement of \( Z \) in \( S \), by \( V \) the image of \( U \) by \( \pi \), and by \( F \) a smooth fiber of \( \pi \).

We define \( Z', U', V' \) and \( F' \) on \( S' \) similarly.

Claim 4.1. Let \( V_1 \) and \( V'_1 \) be non-empty open subsets of \( C \). Let us define \( U_1 := \pi^{-1}(V_1) \) and \( U'_1 := \pi'^{-1}(V'_1) \). Suppose that there is an isomorphism \( \varphi_{U_1}: U_1 \to U'_1 \). Then there is an automorphism \( \varphi_C \in \text{Aut } C \) and an isomorphism \( \varphi: S \to S' \) extending \( \varphi_{U_1} \) such that \( \varphi_C \circ \pi = \pi' \circ \varphi \) holds.

Proof. We can see that the support of \( \varphi^*_{U_1}(H') \) is contained in a finite union of fibers of \( \pi \) for a base point free divisor \( H' \) defining the morphism \( \pi'|_{U'_1} \). Hence it defines the morphism \( \pi|_{U_1} \), and hence there is an isomorphism \( \varphi_{V_1}: V_1 \to V'_1 \) such that \( \varphi_{V_1} \circ \pi|_{V_1} = \pi'|_{V'_1} \circ \varphi_{V_1} \). Since \( C \) is a smooth projective curve, \( \varphi_{V_1} \) extends to an isomorphism \( \varphi_C \). [BHPV, Proposition III.8.4] assures the existence of \( \varphi \) in the statement. \( \Box \)
Suppose that there is an isomorphism $\varphi: S \to S'$. Then since $S$ and $S'$ have a unique elliptic fibration, it induces an automorphism $\varphi_C$ of $C$ which satisfies $\varphi_C \circ \pi = \pi' \circ \varphi$. Moreover an isomorphism $\varphi_U: U \to U'$ extends to an isomorphism $\varphi: S \to S'$ by Claim 4.1. In particular we have

$$\text{Aut}_S \cong \text{Aut}_U, \quad \text{Aut}_C S \cong \text{Aut}_V U.$$ 

4.2 Autoequivalences associated with reducible fibers

In this subsection, we show that any autoequivalences of $D(S)$ induce autoequivalences of $D_Z(S)$.

The following is crucial to show Proposition 4.3, the main result in §4.2.

**Lemma 4.2.** In the notation in §4.1, take a point $x \in Z$. Then we have $\text{Supp } \Phi(O_x)$ is contained in $Z'$.

**Proof.** First let us consider the case $\dim \text{Supp } P = 2$. Then there is an irreducible component $W$ of $\text{Supp } P$ such that the restrictions $p|_W: W \to S$, $p'|_W: W \to S$ of projections $p, p'$ are birational morphism (see the proof of [Ka02, Theorem 2.3]). We put

$$q := p'|_W \circ p|^{-1}_W: S \dashrightarrow S'.$$

But as Kawamata pointed out in [Ka02, Lemma 4.2], $q$ is isomorphic in codimension 1, and hence in the surface case, it is an isomorphism. Hence if a $(-2)$-curve $G$ contains the point $x$, the $(-2)$-curve $q(G)$ contains the point $q(x) \in \text{Supp } \Phi(O_x)$. Because each $(-2)$-curve on $S'$ and $\text{Supp } \Phi(O_x)$ are always contained in a single fiber of $\pi'$, $\text{Supp } \Phi(O_x)$ is contained in the set $Z'$.

Next let us consider the case $\dim \text{Supp } P = 3$. Suppose that $\Phi(O_x)$ is a sheaf on $S'$, after replacing $\Phi$ with $\Phi \circ [n]$ for some $n \in \mathbb{Z}$. Take a point $y$, which is sufficiently near the point $x$, but not in $Z$. Then $\Phi(O_y)$ is also a sheaf by Lemma 2.2. Recall that $\Phi(O_y)$ is a simple, locally free sheaf on a smooth elliptic curve $F'$, hence it is known to be stable. Denote the Chern class of the stable sheaf $\Phi(O_y)$ on the fiber $F'$ by $(0, a_{F'}, -b)$ for some integer $a, b$. Then we know that $(a, b) = 1$ (see the proof of [BM01, Proposition 4.4]), and hence we can define an elliptic surface $J_{S'}(a, b) \to C$ and a universal sheaf $U$ on $J_{S'}(a, b) \times S'$. For the point $w \in J_{S'}(a, b)$ representing a stable sheaf $\Phi(O_y)$, it is satisfied that

$$O_w \cong (\Phi^{d_{J_{S'}(a, b)} \to S'})^{-1} \circ \Phi(O_y).$$

It follows that the kernel of the Fourier–Mukai transform $(\Phi^{d_{J_{S'}(a, b)} \to S'})^{-1} \circ \Phi$ has a 2-dimensional support. Apply the above argument in the case $\dim \text{Supp } P = 2$, and then we can find a point $z$, contained in a $(-2)$-curve on $J_{S'}(a, b)$, such that

$$O_z \cong (\Phi^{d_{J_{S'}(a, b)} \to S'})^{-1} \circ \Phi(O_x).$$
Then Remark 2.13 implies that $\text{Supp } \Phi(O_x)$ is contained in the set $Z'$.

Finally suppose that $\dim \text{Supp } P = 3$ and $\Phi(O_x)$ is not a shift of a sheaf. Take an integer $i$ such that $H^i(\Phi(O_x))$ is non-zero. Assume by contradiction that $c_1(H^i(\Phi(O_x)))$ is some multiple of $F'$ for some $i \in \mathbb{Z}$. Then we have

$$\chi(H^i(\Phi(O_x)), H^i(\Phi(O_x))) = -c_1(H^i(\Phi(O_x)))c_1(H^i(\Phi(O_x))) = 0,$$

and hence

$$\dim \text{Ext}^1_S(H^i(\Phi(O_x)), H^i(\Phi(O_x))) \geq 2.$$ 

But then [BM01, Proposition 2.9] implies that $i$ is a unique integer such that $H^i(\Phi(O_x)) \neq 0$. Namely $\Phi(O_x)$ is a shift of sheaf, which gives a contradiction. Therefore $c_1(H^i(\Phi(O_x)))$ is never a multiple of $F'$. Anyhow, because $H^i(\Phi(O_x))$ is contained in a single fiber, this means that $\text{Supp } H^i(\Phi(O_x))$ is contained in a reducible fiber, and hence in $Z'$.

Applying Lemmas 2.3 and 4.2, we obtain the following.

**Proposition 4.3.** In the notation in §4.1, there is a natural group homomorphism

$$\iota_Z : \text{Auteq } D(S) \to \text{Auteq } D_Z(S).$$

Let us define

$$\text{Auteq}^\dag D_Z(S) := \text{Im } \iota_Z.$$

The following is used in the proof of Lemma 4.6.

**Corollary 4.4.** In the notation in §4.1, take a point $x \in U$. Then $\text{Supp } \Phi(O_x) \subset U'$.

**Proof.** Let us denote by $\Psi$ the quasi-inverse of $\Phi$. Take any point $y \in Z'$. Then we have $\mathbb{R}\text{Hom}(\Phi(O_x), O_y) \cong \mathbb{R}\text{Hom}(O_x, \Psi(O_y)) = 0$ by Lemma 4.2 (See the proof of Lemma 2.3) This implies the assertion.

4.3 Autoequivalences associated with irreducible fibers: The classification

We begin §4.3 by classifying Fourier–Mukai transforms between elliptic surfaces without reducible fibers.

**Proposition 4.5.** Let $\pi : S \to C$ and $\pi' : S' \to C$ be elliptic surfaces without reducible fibers. Here we do NOT assume that $C$ is projective. Let $\Phi = \Phi_{S \to S'} : D(S \times_C S')$ be a Fourier–Mukai transform over $C$ such that $\dim \text{Supp } P = 2$ or 3.

(i) The object $P \in D(S \times_C S')$ is a shift of a sheaf, flat over $S$ by the first projection.

(ii) The following are equivalent.
(1) $\mathcal{P}$ is a sheaf on $S \times_C S'$ with $\dim \text{Supp} \mathcal{P} = 2$.

(2) There are points $x \in S, y \in S'$ such that $\Phi(\mathcal{O}_x) \cong \mathcal{O}_y$.

(3) There is a line bundle $\mathcal{L}$ on $S$ and an isomorphism $\varphi: S \to S'$ over $C$ such that $\Phi \cong \varphi_*(\mathcal{L})$.

(iii) Suppose that $\mathcal{P}$ is a sheaf with $\dim \text{Supp} \mathcal{P} = 3$. Then there are integers $a, b$ with $(a \lambda_{S'}, b) = 1$, a universal sheaf $\mathcal{U}$ on $S' \times J_{S}(a, b)$ and isomorphism $\phi: J_{S}(a, b) \to S$ over $C$ such that

$$\Phi \cong \Phi_{J_{S}(a, b) \to S'} \circ \phi^*.$$

Proof. (i) Take any point $x \in S$. By the irreducibility of fibers of $\pi'$, we have

$$\chi(H^i(\Phi(\mathcal{O}_x)), H^i(\Phi(\mathcal{O}_x))) = -c_1(H^i(\Phi(\mathcal{O}_x)))c_1(H^i(\Phi(\mathcal{O}_x))) = 0$$

for all $i \in \mathbb{Z}$, which implies that $\Phi(\mathcal{O}_x)$ is a shift of a sheaf by $[BM01$, Proposition 2.9$]$. Hence $[BM01$, Lemma 2.1$]$ imply that $\mathcal{P}$ is a shift of a sheaf, flat over $S$ by the first projection.

(ii) As a consequence of (i), the dimension of support of $\mathcal{P}|_{x \times S'} \cong \Phi(\mathcal{O}_x)$ does not depends on the choice of point $x \in S$, and in the situation (1) or (2), every $\Phi(\mathcal{O}_x)$ has a finite support. Then we get (3) by Lemma 2.2. Here recall that $\mathcal{P}$ is a sheaf on $S \times_C S'$, and hence $\varphi$ is defined over $C$. Obviously (3) implies (1) and (2), and (2) implies (1).

(iii) The proof goes parallel to that of $[BM01$, Proposition 4.4$]$ (see also the proof of Lemma 4.2). Let us take a general point $x \in S$, and denote the Chern class of the stable sheaf $\Phi(\mathcal{O}_x)$ on a smooth fiber $F'$ by $(a[F'], -b)$ for some integers $a, b$. Then we know that $(a \lambda_{S'}, b) = 1$, and hence we can define an elliptic surface $J_{S'}(a, b) \to C$ and a universal sheaf $\mathcal{U}$ on $S' \times J_{S'}(a, b)$. For the point $y \in J_{S'}(a, b)$ representing a stable sheaf $\Phi(\mathcal{O}_x)$, it is satisfied that

$$\Phi^{-1}(\Phi_{J_{S'}(a, b) \to S'}^{\mathcal{O}_y}) \cong \mathcal{O}_x.$$

Apply Lemma 2.2 and replace a universal bundle $\mathcal{U}$ with $\mathcal{U} \otimes p_{J_{S'}(a, b)}^* \mathcal{L}$ for a line bundle $\mathcal{L}$ on $J_{S'}(a, b)$ if necessary, then we obtain the assertion. $\square$

If $\pi$ has a reducible fiber, the implication from (2) to (3) in Proposition 4.5 (ii) fails because of the existence of twist functors associated to $(-2)$-curves.

Now we can prove the following important observation.

**Lemma 4.6.** In the notation in §4.7, there is an automorphism $\delta(\Phi) \in \text{Aut } C$ satisfying $\delta(\Phi)(V) = V'$ such that $\mathcal{P}|_{U \times U'}$ is a shift of a coherent sheaf on $U_{\delta(\Phi)} \times V' U'$. Here $U_{\delta(\Phi)} \times V' U'$ is the fiber product of $U$ and $U'$ over $V'$ via the morphisms $(\delta(\Phi) \circ \pi)|_U$ and $\pi'|_{U'}$. 

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Proof. Below we freely compose \( \Phi = \Phi^P \) with a shift functor if necessary. First suppose that \( \dim \text{Supp} \mathcal{P} = 2 \). Then [Hu06, Lemma 4.5] and Corollary 4.4 imply that for any point \( x \in U \), there is a point \( y \in U' \) such that \( \Phi(O_x) \cong O_y \), that is,

\[
\mathcal{P}|_{x \times U'} \cong O_y. \tag{17}
\]

Let us regard \( U' \) as Hilb\(^1\)(\( U' \)), and \( O_\Delta \) as the universal sheaf on Hilb\(^1\)(\( U' \)) \( \times U' \), where \( \Delta \) is the diagonal in \( U' \times U' \). Then there is a morphism \( \varphi_U : U \to U' \) such that \( (\varphi \times \text{id}_{U'})^*O_\Delta = \mathcal{P}|_{U \times U'} \). Since \( \Phi \) is an equivalence, we know that \( x \in U \) in (17) uniquely exists for a given \( y \in U' \), and in particular that \( \varphi_U \) is an isomorphism. Hence we can apply Claim 4.1 for \( \varphi_U \) to obtain the automorphism \( \varphi_C \) of \( C \). This \( \varphi_C \) plays the role of \( \delta(\Phi) \) in the assertion.

Next we consider the case \( \dim \text{Supp} \mathcal{P} = 3 \). By a similar way to the proof of Proposition 4.5, for any point \( x \in U \), there is a point \( y \in J_{U'}(a,b) \) satisfying

\[
\Phi^{-1} \circ \Phi^J_{J_{U'}(a,b) \to S'}(O_y) \cong O_x. \]

Here \( \mathcal{U} \) is a universal sheaf on \( J_{S'}(a,b) \times S' \) of a fine moduli space \( J_{S'}(a,b) \) for some \( a, b \in \mathbb{Z} \) with \( (a \lambda, b) = 1 \). And note that we can regard \( J_{U'}(a,b) \) as the inverse image of \( V' \) by the elliptic fibration \( J_{S'}(a,b) \to C \). Let us denote by \( \mathcal{Q} \in D(J_{S'}(a,b) \times S) \) the kernel of \( \Phi^{-1} \circ \Phi^J_{J_{U'}(a,b) \to S'} \). Since \( \dim \text{Supp} \mathcal{Q} = 2 \), we can apply the above argument to \( \mathcal{Q} \), and then we obtain the assertion for \( \mathcal{Q}|_{J_{U'}(a,b) \times U'} \). Since \( \mathcal{U}|_{J_{U'}(a,b) \times U'} \) is a coherent sheaf on \( J_{U'}(a,b) \times V' \times U' \), we obtain the assertion for \( \mathcal{P}|_{U \times U'} \).

Remark 4.7. For a point \( x \in S \), we put \( c := \delta(\Phi)(\pi(x)) \in C \). Furthermore if the point \( x \) belongs to \( U \), we know from the definition of \( \delta \) that \( \text{Supp} \Phi(O_x) \) is contained in the fiber \( \pi^{-1}(c) \). Recall the facts that \( \text{Supp} \Phi(O_x) = \text{Supp} \mathcal{P} \cap (x \times S') \) by (15) and (16), and that \( \text{Supp} \Phi(O_x) \) is contained in a single fiber. These facts imply that \( \text{Supp} \Phi(O_x) \) is contained in \( \pi^{-1}(c) \) for any \( x \in S \). Therefore we conclude that \( \mathcal{P} \) is an object of \( D_{S(\delta(\Phi) \times C S')}^\delta(S \times S') \).

On the other hand, \( \mathcal{P} \) is not necessarily an object of \( D(S(\delta(\Phi) \times C S')) \). For instance, consider a twist functor \( T_{O_G} \) for a \((-2)\)-curve \( G \) on \( S \). Because the spherical object \( T_{O_G}(O_G(2)) \) is simple, the computation in Example 2.7 (i) implies that it is not of a form \( k_{x,\alpha} \alpha \in D(F_c) \) for a fiber \( F_c \) and the inclusion \( k : F_c \to S \). Consequently, Lemma 2.3 (ii) tells us that the kernel of \( T_{O_G} \) is not an object of \( D(S \times C S) \). Here note that \( \delta(T_{O_G}) = \text{id}_C \).

Proposition 4.8. In the notation in 4.4, assume furthermore that \( S = S' \). Then the Fourier–Mukai autoequivalence \( \Phi = \Phi^P \) of \( D(S) \) induces a Fourier–Mukai autoequivalence \( \iota_U(\Phi) \) of \( D(U) \) over \( V \), by restricting the kernel \( \mathcal{P} \) to \( U \times U \). This \( \iota_U \) defines a group homomorphism

\[
\iota_U : \text{Auteq} D(S) \to \text{Auteq} D(U)
\]

satisfying that

\[
\mathcal{L}^* \circ \Phi \cong \iota_U(\Phi) \circ \mathcal{L}^*, \tag{18}
\]

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where $i$ is the inclusion $U \hookrightarrow S$.

**Proof.** The statement is a direct consequence of Lemma 4.6 and the isomorphisms

$$i_U \cong i_U (\Phi \circ \Phi^{-1}) \cong i_U (\Phi) \circ i_U (\Phi^{-1}).$$

The isomorphism follows from a direct computation.

Let us define

$$\text{Auteq}^\dagger D(U) := \text{Im} \ i_U.$$

Note that elements in $\text{Auteq}^\dagger D(U)$ are classified in Proposition 4.5.

**Remark 4.9.** Lemma 4.6 tells us that there is a group homomorphism

$$\delta: \text{Auteq} D(S) \rightarrow \text{Aut} C.$$

The map $\delta$ factors through the map $i_U$, and hence it induces a map

$$\delta_U: \text{Auteq}^\dagger D(U) \rightarrow \text{Aut} V (\cong \text{Aut} C).$$

Then it follows from Proposition 4.5 that

$$\text{Im} \delta = \{ \varphi_C \mid \varphi \in \text{Aut} S, \text{ or } \varphi: S \rightarrow J_S(a, b) \text{ isomorphism with } (a \lambda_S, b) = 1 \}\,$$

$$= \{ \varphi_C \mid \varphi \in \text{Aut} S, \text{ or } \varphi: S \rightarrow J_S(b) \text{ isomorphism with } (\lambda_S, b) = 1 \}.$$

The second equality follows from 4.5.

### 4.4 Autoequivalences associated with irreducible fibers: The structure of the group

We use the notation in §4.1 in this subsection. The aim of §4.4 is to study the structure of the group $\text{Auteq}^\dagger D(U)$.

Let us define the congruence subgroup of $\text{SL}(2, \mathbb{Z})$ by

$$\Gamma_0(m) := \{ \begin{pmatrix} c & a \\ d & b \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \mid d \in m\mathbb{Z} \}$$

for $m \in \mathbb{Z}$. We show the following.

**Theorem 4.10.** There is a short exact sequence

$$1 \rightarrow \langle \otimes \mathcal{O}_U(D) \mid D \cdot F = 0 \rangle \times \text{Aut} S \times \mathbb{Z}[2] \rightarrow \text{Auteq}^\dagger D(U) \rightarrow \{ \begin{pmatrix} c & a \\ d & b \end{pmatrix} \in \Gamma_0(\lambda_S) \mid J_S(b) \cong S \} \rightarrow 1.$$
Proof. For $\Phi^P \in \text{Auteq}^\dagger D(U)$, note that $\mathcal{P}$ is a shift of a sheaf on $U_{\delta(\Phi)} \times_U U$ by Lemma 4.6.

Take a smooth fiber $F = F_c$ over a point $c \in V$, and denote by $F' = F_{c'}$ a smooth fiber over the point $c' = \delta(\Phi)(c)$. Let $k: F \hookrightarrow U$, and $k': F' \hookrightarrow U$ be the natural inclusions.

It follows from Lemma 2.3 that we obtain a Fourier–Mukai transform $\Phi^P|_{F \times F'} : F \to F'$ such that there is an isomorphism of functors

$$\Phi^P|_{F \times F'} \circ \mathbb{L}k^* \cong \mathbb{L}k'^* \circ \Phi^P.$$  

There is a natural identification between $H^{ev}(F, \mathbb{Z})$ and $H^{ev}(F', \mathbb{Z})$, hence by choosing a basis as in §2.3 we obtain a group homomorphism

$$\Theta_U : \text{Auteq}^\dagger D(U) \to \text{SL}(2, \mathbb{Z}).$$

Notice that this morphism does not depend on the choice of a smooth fiber $F$ by the classification of elements in $\text{Auteq}^\dagger D(U)$ in Proposition 4.5.

Since $F$ and $F'$ are isomorphic, we fix an isomorphism. Then $\Phi^P|_{F \times F'}$ can be regarded as an autoequivalence of $D(F)$. Then we have

$$\Theta_U(\Phi^P) = \theta(\Phi^P|_{F \times F'}).$$

This is important for the latter computation. (Recall the definition of $\theta$ in §2.3)

Suppose that $\Phi = \Phi^P \in \text{Ker } \Theta_U$. Proposition 4.5 forces that $\dim \text{Supp } \mathcal{P} = 2$, and hence

$$\Phi \in \langle \otimes \mathcal{O}_U(D) | D \cdot F = 0 \rangle \rtimes \text{Aut } S \times \mathbb{Z}[2],$$

that is,

$$\text{Ker } \Theta_U \cong \langle \otimes \mathcal{O}_U(D) | D \cdot F = 0 \rangle \rtimes \text{Aut } S \times \mathbb{Z}[2].$$

Next let us consider the image of $\Theta_U$. Take integers $a, b$ with $a > 0$ and $(a \lambda_S, b) = 1$. Assume that there is an isomorphism $\phi: J_S(a, b) \to S$. Then

$$\Phi^P|_{J_S(a, b) \to S} \circ \phi^*$$

(19)

gives an autoequivalence of $D(S)$. In this case, it follows from Remark 2.10 that

$$\Theta_U \circ i_U(\Phi^P|_{J_S(a, b) \to S} \circ \phi^*) = \left( \begin{array}{cc} c & a \\ d & b \end{array} \right) \in \text{SL}(2, \mathbb{Z})$$

holds for some $c, d \in \mathbb{Z}$ such that $\lambda_S$ divides $d$. We also have

$$\Theta_U \circ i_U(\text{Pic } S) = \left\{ \left( \begin{array}{cc} 1 \\ \lambda_S \\ 1 \end{array} \right) \right\}$$

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and

$$\Theta_U([1]) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

Note that

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & a \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

for any $a \in \mathbb{Z}$. Then it follows from Proposition 4.5 that

$$\text{Im } \Theta_U = \langle \begin{pmatrix} 1 & 0 \\ \lambda S & 1 \end{pmatrix}, \begin{pmatrix} c & a \\ d & b \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \mid a \neq 0, bc - ad = 1, d \in \lambda S\mathbb{Z}, J_S(b) \cong S \rangle.$$ 

Furthermore we can see that

$$\begin{pmatrix} -1 & 0 \\ \lambda S & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\lambda S & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \text{Im } \Theta_U.$$ 

Therefore we conclude that

$$\text{Im } \Theta_U = \{ \begin{pmatrix} c & a \\ d & b \end{pmatrix} \in \Gamma_0(\lambda_S) \mid J_S(b) \cong S \},$$

which completes the proof of Theorem 4.10. \(\square\)

We define

$$\Theta : \text{Auteq } D(S) \to \text{SL}(2, \mathbb{Z})$$

to be the composition $\Theta_U \circ \uparrow_U$, where we regard $\uparrow_U$ as a surjective homomorphism from $\text{Auteq } D(S)$ to $\text{Auteq}^\dagger D(U)$. Thus by definition, $\text{Im } \Theta = \text{Im } \Theta_U$ holds.

**Remark 4.11.** (i) Let us consider the surjective map

$$\Gamma_0(\lambda_S) \to (\mathbb{Z}/\lambda S\mathbb{Z})^*/H_S \quad \begin{pmatrix} c & a \\ d & b \end{pmatrix} \mapsto b.$$ 

This is actually a group homomorphism and its kernel coincides with the group $\text{Im } \Theta$. That is, we have an group isomorphism

$$\Gamma_0(\lambda_S)/\text{Im } \Theta \cong (\mathbb{Z}/\lambda S\mathbb{Z})^*/H_S.$$ 

The latter group is naturally identified with the set $\text{FM}(S)$ as in §2.3.

(ii) When $|\mathbb{Z}/\lambda S\mathbb{Z}| \leq 2$ (e.g. $\lambda_S \leq 4$), we have $(\mathbb{Z}/\lambda S\mathbb{Z})^* = H_S$. Hence we can see that

$$\text{Im } \Theta = \Gamma_0(\lambda_S).$$ 

In particular, when $\lambda_S = 1$, we see that

$$\text{Im } \Theta = \text{SL}(2, \mathbb{Z}).$$
4.5 Kernels of $\iota_Z$ and $\iota_U$

Now let us study the kernel of the homomorphisms $\iota_Z$ given in Proposition 4.3. First of all, we may assume that $Z \neq \emptyset$ because otherwise, we have

$$\text{Ker } \iota_Z = \text{Auteq } D(S) = \text{Auteq}^\dagger D(U),$$

and the last group is studied intensively in §4.4. Take $\Phi \in \text{Ker } \iota_Z$. Then for any $x \in Z$, we have

$$\Phi(O_x) \cong O_x.$$

Hence by Lemma 2.2 there is a point $y \in U$ such that

$$\Phi(O_y) \cong O_w$$

for some $w \in U$. We can apply Proposition 4.5 (ii) for $\iota_U(\Phi)$, and then obtain that for all points $y \in U$, there is a point $w$ satisfying (20). Therefore Lemma 2.2 implies that Ker $\iota_Z$ is contained in the group $A(S)$ of the standard autoequivalences (see §1.1), and hence

$$\text{Ker } \iota_Z = \langle \otimes O_S(F_c) \mid F_c \text{ a fiber of } \pi \rangle \rtimes \text{Aut } S,$$

where we define a normal subgroup of Aut $S$ by

$$\text{Aut } S := \{ \varphi \in \text{Aut } S \mid \varphi(z) = z \text{ for all } z \in Z \}.$$

Let us denote

$$B := \langle T_{O_G(a)} \mid G \text{ is a } (-2)\text{-curve} \rangle.$$

For the homomorphism $\iota_U$ given in Proposition 4.8 we believe that the equality

$$B = \text{Ker } \iota_U$$

holds. Actually we have the following.

Lemma 4.12. Suppose that equality (22) holds. Then Conjecture 1.1 is true.

Proof. The result follows from the description of $\text{Auteq}^\dagger D(U)$ in Theorem 4.10. \hfill \Box

In §5, 6 and 7 we shall check the equality (22) in the case that any reducible fibers on a given projective elliptic surface are of type $I_n$ ($n > 1$), and consequently show Conjecture 1.1 in this case.
5 Autoequivalences associated with singular fibers of type $I_n$ for $n > 1$

Throughout this section, $\pi : S \to C$ is a projective elliptic surface whose reducible fibers are cycles of $(-2)$-curves, that is, of type $I_n$ for $n > 1$. In this case the set $Z$ is a disjoint union of cycles of projective lines. Below we regard $Z$ as a closed subscheme of $S$ with respect to the reduced induced structure. In our setting, line bundles on $(-2)$-curves are spherical in the sense of Definition 2.5. Therefore $\text{Auteq } D_Z(S)$ contains twist functors, and hence it is highly involved. The following is the main result in this article.

**Theorem 5.1.** Let $S$ be a smooth projective elliptic surface with $\kappa(S) \neq 0$. Suppose that any reducible fibers of the elliptic fibration are cycles of $(-2)$-curves. Then Conjecture 1.1 is true. Namely we have

$$1 \to \langle B, \otimes O_S(D) \mid D \cdot F = 0 \rangle \times \text{Aut } S \times \mathbb{Z}[2] \to \text{Auteq } D(S)$$

$$\Theta \to \{ \left( \begin{array}{cc} c & a \\ d & b \end{array} \right) \in \Gamma_0(\lambda_S) \mid J_S(a, b) \cong S \} \to 1.$$  

Theorem 5.1 concerns the autoequivalences of the derived categories of surfaces containing $(-2)$-curves in $\tilde{A}_n$-configurations. Many ideas of the proof come from [IU05], in which we study the autoequivalences of the derived categories of surfaces containing $(-2)$-curves in $A_n$-configurations.

5.1 Reduction of the proof of Theorem 5.1

Recall that Lemma 4.12 tells us that if we can show the equation (22), we obtain Theorem 5.1.

Let $\{ Z_j \}_{j=1}^M$ be the set of connected components of $Z$, that is, each $Z_j$ is a singular fiber of type $I_{n_j}$ for some $n_j > 1$. We define

$$B_j := \langle T_{\lambda_G(a)} \mid G \text{ is a } (-2)\text{-curves contained in } Z_j \rangle.$$

Take a connected component of $Z$, and denote it by

$$Z_0.$$

We put

$$Z_0 = C_1 \cup \cdots \cup C_n$$

such that each $C_i$ is a $(-2)$-curve on $S$, and they satisfy

$$C_i \cdot C_m = \begin{cases} 1 & \text{if } |l - m| = 1, \text{ or } |l - m| = n - 1 \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} (23)
in the case \( n > 2 \). In the case \( n = 2 \), \( C_1 \) and \( C_2 \) intersect each other transversely at two points. For any \( m \in \mathbb{Z} \) satisfying \( m - i \in n\mathbb{Z} \) with some \( i, 1 \leq i \leq n \), we also define 
\[
C_m := C_i.
\]

We can reduce the proof of (22), and hence, that of Theorem 5.1 to showing the following.

**Proposition 5.2** (cf. Proposition 1.7 in [IU05]). We use the notation in Theorem 5.1. Suppose that we are given an autoequivalence \( \Phi \) of \( D_{Z_0}(S) \) preserving the cohomology class \([O_x] \in H^4(S, \mathbb{Q})\) for some point \( x \in Z_0 \). Then, there are integers \( a, b (1 \leq b \leq n) \) and \( i \), and there is an autoequivalence \( \Psi \in B_0 \) such that
\[
\Psi \circ \Phi(O_{C_1}) \cong O_{C_b}(a)[i]
\]
and
\[
\Psi \circ \Phi(O_{C_1}(-1)) \cong O_{C_b}(a-1)[i].
\]
In particular, for any point \( x \in C_1 \), we can find a point \( y \in C_b \) with \( \Psi \circ \Phi(O_x) \cong O_y[i] \).

Suppose that we have shown Proposition 5.2. Take \( \Phi \in \text{Ker } \iota_U \). Since \( \delta(\Phi) = \text{id}_{C} \) for \( \delta \) given in §4.3, the induced autoequivalence \( \iota_Z(\Phi) \) of \( D_{Z_j}(S) \) also induces autoequivalences of each \( D_{Z_j}(S) \). We fix some \( j \) and put \( n = n_j \) for simplicity. Denote the induced autoequivalence by
\[
\Phi_j \in \text{Auteq } D_{Z_j}(S).
\]
We take the irreducible decomposition of \( Z_j \) as \( Z_j = C_1 \cup \cdots \cup C_n \). Now we can apply Proposition 5.2 for \( \Phi_j \), and then we find
\[
\Psi_j \in B_j
\]
as in Proposition 5.2. Since \( \Psi_j \circ \Phi_j \) also belongs to \( \text{Ker } \iota_U \), we know that \( b = 1 \) and \( i = 0 \) in Proposition 5.2. We also know that \( x = y \), and hence \( \Psi_j \circ \Phi_j \) gives an autoequivalence of \( D_{Z_j}(S) \), where \( Z'_j = C_2 \cup \cdots \cup C_n \). Since \( Z'_j \) is a chain of \((-2)\)-curves in \( A_{n-1} \)-configuration, [IU05 Proposition 1.7] implies that
\[
\Psi_j \circ \Phi_j \in ((B_j', \text{Pic } S) \rtimes \text{Aut } S) \cap \text{Ker } \iota_U
\]
\[
\cong B'_j \times \bigoplus \mathcal{O}_S(C_i) \mid i = 1, \ldots, n
\]
\[
\subset B_j,
\]
where we put
\[
B'_j := \left\langle T_{\mathcal{O}_{C_i}(a)} \mid i = 2, \ldots, n \right\rangle.
\]
and the last inclusion is a consequence of [IU05, Proposition 4.18 (i)]. Hence we know that $\Phi_j \in B_j$.

We apply this argument for each $Z_j$, and then we can see that

$$\Psi_1 \circ \cdots \circ \Psi_M \circ \Phi \in B$$

and hence $\Phi \in B$. Therefore we have

$$\iota_Z(\ker \iota_U) \subset B.$$ 

The other inclusion follows from Remark 2.6. Because $\ker \iota_U \cap \ker \iota_Z = \{1\}$ by (21), we obtain (22) as desired.

**Remark 5.3.** In the $A_n$-case in [IU05, Proposition 1.7], we do not need the assumption that $\Phi$ preserves the cohomology class $[\mathcal{O}_x]$, since it is always true. To the contrary, in the $\widetilde{A}_n$-case in Proposition 5.2 the image of $\Phi_{J_0}^{M(a,b) \rightarrow S} \circ \phi^*$ in (19) by $\iota_Z$ does not preserve the cohomology class $[\mathcal{O}_x]$. The existence of such elements forces us to put this assumption in Proposition 5.2.

The subgroup

$$\langle B, (\text{Pic } S/ \oplus \mathcal{O}_S(F_c) \mid c \in V) \rangle \rtimes (\text{Aut } S/ \text{Aut}_Z S) \times \mathbb{Z}[1]$$

of $\text{Auteq} \ D_Z(S)$ preserves $H^4(S, \mathbb{Q})$, and therefore it is strictly smaller than $\text{Auteq} \ D_Z(S)$. This is a contrast to the $A_n$-case in Theorem 2.8.

We shall prove Proposition 5.2 in §7. For the proof of Proposition 5.2 we first show the following.

**Proposition 5.4** (cf. Proposition 1.6 in [IU05]). We use the notation in Theorem 5.1. Let $\alpha$ be a spherical object in $D_{Z_0}(S)$. Then there are integers $a, b$ ($1 \leq b \leq n$) and $i$, and there is an autoequivalence $\Psi \in B_0$ such that

$$\Psi(\alpha) \cong \mathcal{O}_{C_b}(a)[i].$$

Proposition 5.4 is proved in §6.

**5.2 The cohomology sheaves of spherical objects**

**Torsion free sheaves on a chain of projective lines**

We denote by $\mathbb{I}_n$ a cycle of $n$ projective lines:

$$\mathbb{I}_n = C_1 \cup \cdots \cup C_n.$$ 

The curves $C_i$‘s are labelled as in (23) when $n > 2$.

We denote by $\mathbb{I}_n$ a chain of $n$ projective lines. We put $\mathbb{I}_n = C_1' \cup \cdots \cup C_n'$ such that each $C_i'$ is a projective line, and they satisfy

$$C_i' \cdot C_m' = \begin{cases} 1 & |l - m| = 1 \\ 0 & \text{otherwise}. \end{cases}$$

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For a coherent sheaf $R$ on $\mathbb{I}_n$ or $\tilde{\mathbb{I}}_n$, we denote by $\text{deg}_C R$ the degree of the restriction $R|_C$ on the component $C$ of $\mathbb{I}_n$ or $\tilde{\mathbb{I}}_n$. It is known that line bundles $L$ on $\mathbb{I}_n$ is determined by the degree $L|_C$ on each component $C$, that is,

$$\text{Pic} \mathbb{I}_n \cong \mathbb{Z}^n.$$ 

The line bundle corresponding to the vector $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ is denoted by $\mathcal{O}_{\mathbb{I}_n}(a_1, \ldots, a_n)$.

When we write $*$ instead of $a_i$, we do not specify the degree at $C'_l$. For instance, when we put $R_1 = \mathcal{O}_{\mathbb{I}_3}(a, b, *)$, this means that $R_1$ is a line bundle on $\mathbb{I}_3$ such that $\text{deg}_{C'_1} R_1 = a$, $\text{deg}_{C'_2} R_1 = b$ and $\text{deg}_{C'_3} R_1$ arbitrary. The expression

$$R_2 = \mathcal{O}_{C'_1 \cup \cdots}(a, *)$$

means that there exists $k \geq 2$ with $R_2 = \mathcal{O}_{\mathbb{I}_k}(a, *, \ldots, *)$. Note that the support of $R_2$ is strictly larger than $C'_1$. We often use figures

$$R_1 : \quad \begin{array}{ccc}
\otimes & C'_2 & C'_3 \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\end{array}$$

$$R_2 : \quad \begin{array}{c}
\otimes \\
\otimes \\
\otimes \\
\end{array}$$

R_1, R_2 above. We use a dotted line

$$R_3 : \quad \begin{array}{c}
\otimes \\
\circ \quad - \quad - \\
\circ \quad - \quad - \\
\end{array}$$

to indicate that $R_3$ is either $\mathcal{O}_{C'_1}(a)$ or $\mathcal{O}_{C'_1 \cup \cdots}(a, *)$.

**Torsion free, but not locally free sheaves on $\tilde{\mathbb{I}}_n$** The following is useful.

**Proposition 5.5** (Theorem 19 in [BBDG]). If an indecomposable torsion free $\mathcal{O}_{\mathbb{I}_n}$-module $S$ is not locally free, then there is a finite surjective morphism

$$p_k : \mathbb{I}_k \rightarrow \tilde{\mathbb{I}}_n,$$

some integer $s$, and a line bundle $\mathcal{L}$ on $\mathbb{I}_k$ such that $p_k(C'_l) = C_{l+s-1}$ with $l = 1, \ldots, k$, and

$$S \cong p_\ast \mathcal{L}.$$
In the situation in Proposition \[5.5\] assume that
\[
\mathcal{L} \cong \mathcal{O}_{1_k}(a_s, \ldots, a_{s+k-1}).
\]
In this case, we denote as
\[
S_s(a_s, \ldots, a_{s+k-1}) := \mathcal{S}, \quad \text{or} \quad S_{C_1 \cup \cdots \cup C_{s+k-1}}(a_s, \ldots, a_{s+k-1}) := \mathcal{S}.
\]
We can see that
\[
S_s(a_s, \ldots, a_{s+k-1}) \mid_{\mathcal{C}_m} \cong \bigoplus_{l \in \mathbb{Z}, s \leq l \leq s+k-1} \mathcal{O}_{C_m}(a_l).
\]
(26)
Notice that, for instance,
\[
S_{C_1 \cup \cdots \cup C_n}(0, \ldots, 0) = S_1(0, \ldots, 0) \not\cong \mathcal{O}_{\mathbb{I}_n},
\]
but
\[
S_{C_1 \cup \cdots \cup C_{n-1}}(0, \ldots, 0) = S_1(0, \ldots, 0) \cong \mathcal{O}_{C_1 \cup \cdots \cup C_{n-1}}.
\]

**The cohomology sheaves of spherical objects** Recall that \(Z_0\) is a connected component of \(Z\), and it is a cycle of \(n\) projective lines labelled as in \[23\], that is, \(Z_0 \cong \mathbb{I}_n\).

Henceforth we freely use the notations and results on \(\mathbb{I}_n\) mentioned above. For a spherical object \(\alpha \in D_{Z_0}(S)\), the direct sum \(\bigoplus_p H^p(\alpha)\) of its cohomology sheaves is rigid as an \(\mathcal{O}_S\)-module, and a torsion free \(\mathcal{O}_{Z_0}\)-module by \[IU05, Proposition 4.5, Lemmas 4.8, 4.9\].

For a non-empty open subset \(U\) of \(S\) and a spherical object \(\alpha \in D(S)\), let
\[
\Sigma(\alpha)_{U}
\]
be the set of all indecomposable summands of \(\bigoplus_p H^p(\alpha)_{|U}\). If \(U = S\), we just denote it by
\[
\Sigma(\alpha)
\]
For a connected union of \((-2)\)-curves
\[
Z' := C_s \cup C_{s+1} \cup \cdots \cup C_t
\]
contained in \(Z_0\), take a sufficiently small open neighbourhood of \(Z'\), and we often denote it by
\[
U_{s, \ldots, t}.
\]
For a torsion free sheaf \(\mathcal{E}\) on \(Z_0\) such that \(c_1(\mathcal{E})\) is some multiple of \([Z_0]\), we can see that \(\chi_S(\mathcal{E}, \mathcal{E}) = -c_1(\mathcal{E})^2 = 0\), which implies that \(\mathcal{E}\) is not rigid. In particular, any \(\mathcal{R} \in \Sigma(\alpha)\) cannot be a locally free \(\mathcal{O}_{Z_0}\)-module, and it is of the form \(S_i(a_i, a_{i+1}, \ldots, a_j)\) for some integers \(i, j\) with \(C_{i-1} \neq C_j\).
Define
\[ l_i(\alpha) := \sum_p \text{length}_{\mathcal{O}_{S, \eta_i}} H^p(\alpha)_{\eta_i} \]
for each curve \( C_i \), where \( \mathcal{O}_{S, \eta_i} \) is the local ring of \( S \) at the generic point \( \eta_i \) of \( C_i \), \( H^p(\alpha)_{\eta_i} \) is the stalk over \( \eta_i \) and \( \text{length}_{\mathcal{O}_{S, \eta_i}} \) measures the length over \( \mathcal{O}_{S, \eta_i} \). Furthermore we define
\[ l(\alpha) := \sum_{i=1}^n l_i(\alpha). \]

This invariant plays an important role in the induction step in the proofs of Propositions 5.2 and 5.4.

The rigidity of \( \bigoplus_p H^p(\alpha) \) gives strong conditions on elements of \( \Sigma(\alpha) \) as Lemma 5.7 below. A similar statement in the \( A_n \)-case to Lemma 5.7 is fully used in \cite{IU05}.

Before giving a general statement in Lemma 5.7, we first give a special one for the case \( n=2 \), since the proof is slightly different from one in the case \( n>2 \).

**Lemma 5.6.** Let us consider the case \( n=2 \), i.e. \( Z_0 \cong \mathbb{T}_2 \), and let \( \alpha \) be a spherical object in \( D_{\mathcal{Z}_0}(S) \). Assume that there is an element \( S := S_{(a_s, a_{s+1}, \ldots, a_t)} \in \Sigma(\alpha) \) with \( s \neq t \), equivalently to say that \( l(S) > 1 \). Then there are integers \( a \) and \( i = 1 \) or 2 satisfying the following conditions:

(i) We have \( C_s = C_t \) and \( a_s = a_t = a \). The integers \( a \) and \( i \) do not depend on the choice of \( S \in \Sigma(\alpha) \) with \( l(S) > 1 \).

(ii) Assume that there is an element \( S' \in \Sigma(\alpha) \) with \( l(S') = 1 \). Then \( S' \) is isomorphic to \( \mathcal{O}_{C_i}(a) \) or \( \mathcal{O}_{C_i}(a-1) \).

(iii) Assume furthermore that the above \( S \) satisfies \( l(S) > 3 \). Then we have

\[ a_{s+2} = a_{s+4} = \cdots = a_{t-2}. \]

Moreover if we put \( a' := a_{s+2} \), then \( a' = a \) or \( a+1 \), and \( S' \) is isomorphic to \( \mathcal{O}_{C_i}(a'-1) \).

**Proof.** Take elements
\[ S_1 := S_{(a_{s1}, \ldots, a_{t1})}, \quad S_2 := S_{(b_{s2}, \ldots, b_{t2})} \in \Sigma(\alpha). \]

First we show \( C_{s1} = C_{t2} \). To the contrary, suppose that \( C_{s1} \neq C_{t2} \), namely \( C_{s1} = C_{t2-1} \) holds. Then there is a non-split exact sequence
\[ 0 \to S_1 \to S_{(a_{s1}, \ldots, a_{t1-1}, a_{t1} - 1, b_{s1}, \ldots, b_{t2})} \to S_2 \to 0, \]

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which gives a contradiction with the rigidity of $S_1 \oplus S_2$. Thus we obtain $C_{s_1} = C_{t_2}$.

If we replace the role of $S_1$ and $S_2$ in the above argument, then we obtain $C_{s_2} = C_{t_1}$. Furthermore consider the special case $S_1 = S_2$. Then this particularly implies the equalities

$$C_{s_1} = C_{t_1} = C_{s_2} = C_{t_2},$$

and hence we obtain the assertion $C_t = C_s = C_i$ in (i), and the assertion $\text{Supp} S' = C_i$ in (ii) and (iii). The equalities (27) and the rigidity of $S_1 \oplus S_2$ also imply that

$$2 = -c_1(S_1)c_1(S_2) = \chi_S(S_1, S_2) = \dim \text{Hom}_S(S_1, S_2) + \dim \text{Ext}^2_S(S_1, S_2).$$

(i) Now suppose that $l(S_1) > 1$ and $l(S_2) > 1$. Applying (28) in the case $S_1 = S_2$, we conclude that $a_{s_1} = a_{t_1}$ and $b_{s_1} = b_{t_1}$. Furthermore if $a_{s_1} \neq b_{s_1}$, we see that

$$\dim \text{Hom}_S(S_1, S_2) \geq 4 \text{ or } \dim \text{Ext}^2_S(S_1, S_2) \geq 4,$$

which gives a contradiction with (28).

(ii), (iii) The rigidity of $S \oplus S'$ implies that $S' = O_{C_i}(a+1)$, $O_{C_i}(a)$ or $O_{C_i}(a-1)$, and that $|a_j - a| \leq 1$ for all $j - i \in 2\mathbb{Z}$ (see Lemma 5.7 (iii-1)). Then the equality (28) implies the assertions (ii) and (iii).

Let us proceed a general statement for any $n$.

**Lemma 5.7.** Let $\alpha$ be a spherical object in $D_{Z_0}(S)$. Take elements

$$S_1 := S_{s_1}(a_1, \ldots, a_{t_1}), \quad S_2 := S_{s_2}(b_1, \ldots, b_{t_2}) \in \Sigma(\alpha).$$

(i) Let us take a reduced closed subscheme $Z' = C_{l-1} \cup C_{l+1} \cup \cdots \cup C_j$ of $Z_0$ for some $i, j \in \mathbb{Z}$ with $1 \leq j - i \leq n - 1$. Then $(S_1 \oplus S_2)|_{Z'}$ is a rigid $O_S$-module.

(ii) We have $C_{l_1} \neq C_{s_2-1}$.

(iii) For integer $l, m$ satisfying $s_1 \leq l \leq t_1$ and $s_2 \leq m \leq t_2$ such that $C_l = C_m$, we have the following.

(1) $|a_l - b_m| \leq 1$.

(2) Suppose that $s_1 < l \leq t_1$ and $s_2 = m \leq t_2$:

$$\begin{array}{cccc}
A \text{ part of } S_1 : & & C_{l-1} & C_l & C_{l+1} \\
& & \Theta & - & - \\
The \text{ beginning of } S_2 : & & - & - & -
\end{array}$$

Then we have $a_l \geq b_m$.

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(3) Suppose that $s_1 < l < t_1$ and $s_2 = m = t_2$:

\[
\begin{array}{c}
\text{A part of } S_1 : \\
\text{S}_2 :
\end{array}
\]

Then we have $a_l = b_m + 1$.

(4) Suppose that $s_1 < l = t_1$ and $s_2 = m < t_2$.

\[
\begin{array}{c}
\text{The end of } S_1 : \\
\text{The beginning of } S_2 :
\end{array}
\]

Then we have $a_l = b_m$.

Proof. (i) We may assume that $Z' \neq Z$. Let us consider the restriction map

\[ S_1 \oplus S_2 \to (S_1 \oplus S_2)|_{Z'} \]

Note that there are no homomorphism from its kernel to $(S_1 \oplus S_2)|_{Z'}$, since their supports intersect at finite points. Then we can apply the Mukai’s lemma (see [KO95, Lemma 2.2 (2)]), and hence the rigidity of $S_1 \oplus S_2$ implies the rigidity of $(S_1 \oplus S_2)|_{Z'}$ as an $S$-module.

(ii) To the contrary, assume that $C_{t_1} = C_{s_2-1}$. Then we can actually deduce a contradiction in a completely similar way to one in the proof of $C_s = C_l$ in Lemma 5.6 (i). But for $n > 2$, we can also deduce a contradiction as follows:

Apply (i) for $Z' = C_{t_1} \cup C_{s_2}$, and then the rigidity of $(S_1 \oplus S_2)|_{Z'}$ follows. This contradicts with the fact that $S_1|_{Z'}$ and $S_2|_{Z'}$ contain $O_{C_{t_1}}(a_{t_1})$ and $O_{C_{s_2}}(a_{s_2})$ respectively as direct summands.

(iii) Take $Z' = C_l$ in (i) for (1), and $Z' = C_{l-1} \cup C_l$ in (i) for (2) in the case $n > 2$. Then (1) follows immediately from the rigidity of

\[ O_{C_l}(a_l) \oplus O_{C_m}(b_m) = O_{C_l}(a_l) \oplus O_{C_l}(b_m), \]

which is a direct summand of the rigid sheaf $(S_1 \oplus S_2)|_{C_l}$. In the situation in (2) in the case $n > 2$, we know the rigidity of

\[ O_{C_{l-1}} \cup O_{C_l}(a_{l-1}, a_l) \oplus O_{C_m}(b_m) = O_{C_{l-1}} \cup O_{C_l}(a_{l-1}, a_l) \oplus O_{C_l}(b_m), \]

which forces the result. The statement in (2) in the case $n = 2$ is proved in Lemma 5.6.

Taking $Z' = C_{l-1} \cup C_l \cup C_{l+1}$ in (i) for (3) in the case $n > 3$, then we can prove (3) in the case $n > 3$ similarly. The statement (3) in the case $n = 2$ is shown in Lemma 5.6 (iii). We leave the reader to show (3) in the case $n = 3$, since all ideas have already appeared in the other cases. The statement (4) is a direct consequence of (2).
6 The proof of Proposition 5.4

We prove Proposition 5.4 in this section.

6.1 Auxiliary results

Let us begin the following lemma.

**Lemma 6.1.** Let $S$ be a coherent sheaf on $S$, supported by $Z_0$. Take a small open neighbourhood $U = U_s$ of $C = C_s$ for some $s \in Z$. Then we have

$$(T_{O_C(a)}(S)|_{U}) = T_{O_C(a)}(S|_{U}).$$

**Proof.** Let us consider the exact triangle

$$\mathbb{R}\Gamma_{S\setminus U}(S, O_C(a)^{\vee} \otimes S) \rightarrow \mathbb{R}\Gamma(S, O_C(a)^{\vee} \otimes S) \rightarrow \mathbb{R}\Gamma(U, O_C(a)^{\vee} \otimes S),$$

where $\mathbb{R}\Gamma_{S\setminus U}$ is the derived functor of the left exact functor $\Gamma_{S\setminus U}$ taking the global sections whose support lies in $S\setminus U$. The first term vanishes, since $\text{Supp} O_C(a)^{\vee} \otimes S \subset C$. Hence the second morphism is isomorphic. Consider the morphism between exact triangles in $D(U)$:

$$\mathbb{R}\text{Hom}_{D(S)}(O_C(a), S) \otimes O_C(a) \rightarrow S|_{U} \rightarrow T_{O_C(a)}(S)|_{U}$$

$$\mathbb{R}\text{Hom}_{D(U)}(O_C(a), S|_{U}) \otimes O_C(a) \rightarrow S|_{U} \rightarrow T_{O_C(a)}(S|_{U})$$

The left vertical arrow is isomorphic as above, and thus so is the right one.

Suppose that we are given a spherical object $\alpha \in D_{Z_0}(S)$ with $l(\alpha) = 1$. Then we get $\alpha \cong O_{C_b}(a)[i]$ for some $a, b, i \in Z$. (Or, assume that $l(\alpha) < n$. Then $\text{Supp} \alpha \not\subset Z_0$, and then [IU05 Proposition 1.6] implies Proposition 5.4.) If we prove that for a spherical $\alpha$ with $l(\alpha) > 1$, there is an autoequivalence $\Psi \in B_0$ such that

$$l(\Psi(\alpha)) < l(\alpha),$$

then, since $\Psi(\alpha)$ is again spherical, the induction on $l(\alpha)$ yields Proposition 5.4. On the other hand, we can show by a similar way to [IU05 Lemma 4.11] that

$$l(\Psi(\alpha)) \leq \sum q l(\Psi(H^q(\alpha)))$$

for any $\Psi \in \text{Auteq } D_{Z_0}(S)$. Thus to get (29), it is enough to show that

$$l(\sum q \Psi(H^q(\alpha))) < \sum q l(H^q(\alpha))(= l(\alpha)).$$
Lemma 6.1 is useful to compute the left-hand side in (30) for \( \Psi = T_{\mathcal{O}_C(a)} \), and allows us to apply many results in [IU05] for the \( A_n \)-case to our \( \tilde{A}_n \)-case.

**Lemma A** (cf. Lemma A in [IU05]). Let \( \alpha \in D_{Z_0}(S) \) be a spherical object. Take a small connected open neighbourhood \( U = U_s \) of \( C = C_s \) for some \( s \in \mathbb{Z} \). Assume that we can write

\[
\bigoplus_p H^p(\alpha)|_U = \bigoplus_{j} \mathcal{R}_{1,j} \oplus \bigoplus_{j} \mathcal{R}_{2,j} \oplus \bigoplus_{j} \mathcal{R}_{3,j} \oplus \bigoplus_{j} \mathcal{R}_{4,j},
\]

such that \( \mathcal{R}_{k,j} \in \Sigma(\alpha)_U \), and they are of the form

\[
\begin{array}{ccc}
C_{s-1} & C_s & C_{s+1} \\
\mathcal{R}_{1,j} : & \bigcirc & \bigcirc \\
\mathcal{R}_{2,j} : & \bigcirc & \bigcirc \\
\mathcal{R}_{3,j} : & \bigcirc & \bigcirc \\
\mathcal{R}_{4,j} : & \bigcirc & \bigcirc \\
\end{array}
\]

Suppose that either \( r_3 \neq 0 \) or \( r_2 \cdot r_4 \neq 0 \) holds, and suppose furthermore that \( \text{Supp} \alpha \neq C \). Then, there is an integer \( a \) such that \( l(T_{\mathcal{O}_C(a)}(\alpha)) < l(\alpha) \).

**Proof.** The proof goes parallel to that of [IU05, Lemma A]. In the proof, we need several Lemmas in [IU05, §6]. We give comments why we can apply them in our situation.

On [IU05, Lemma 6.1], just use Proposition 5.5 instead of it. [IU05, Lemma 6.2] is true without any changes in our situation (except replacing the notation \( X \) with \( S \)). In [IU05, Lemma 6.3], we take a small neighbourhood \( U \) of the curve \( C \), to assure the existence of the decomposition of \( H^p(\alpha) \). The indecomposable sheaf on \( S \) (= \( X \) in [IU05]) may be decomposed if we restrict it to \( U \). For instance, one indecomposable sheaf on \( S \) can be decomposed into the factors \( \mathcal{R}_{1,j} \) and \( \mathcal{R}_{2,j} \) on \( U \), and thus contributes to the number \( r_1^p \) more than once and the number \( r_2^p \) at the same time, but \( r_3^p \) does not cause any problem in the proof of [IU05, Lemma 6.3]. [IU05, Lemmas 6.4, 6.5] hold without any changes in our situation. [IU05, Lemmas 6.6] holds after taking a small neighbourhood \( U \) of the curve \( C \).

**Lemma B** (cf. Lemma B in [IU05]). Let \( \alpha \in D_{Z_0}(S) \) be a spherical object and fix positive integers \( s, t \) with \( 1 \leq t - s \leq n - 1 \). Take a small connected open neighbourhood \( U = U_{s,\ldots,t} \) of \( C_s \cup \cdots \cup C_t \) and assume that we can write

\[
\bigoplus_p H^p(\alpha)|_U = \bigoplus_{j} \mathcal{R}_{1,j} \oplus \bigoplus_{j} \mathcal{R}_{2,j} \oplus \bigoplus_{j} \mathcal{R}_{3,j} \oplus \bigoplus_{j} \mathcal{R}_{4,j},
\]
where $R_{k,j} \in \Sigma(\alpha)_U$, and they are of the forms

| $R_{1,j}$ | $C_{s-1}$ | $C_s$ | $C_{s+1}$ | $C_{t-1}$ | $C_t$ | $C_{t+1}$ |
|-----------|-----------|-------|-----------|-----------|-------|-----------|
| $R_{2,j}$ |          |       |           |           |       |           |
| $R_{3,j}$ |          |       |           |           |       |           |
| $R_{4,j}$ |          |       |           |           |       |           |

Suppose that either $r_3 \neq 0$ or $r_2 \cdot r_4 \neq 0$ holds. Then there is

$$\Phi \in \left\{ T_{O_{C_j}(a)} \mid a \in \mathbb{Z}, s \leq t \leq t \right\}$$

such that $l(\Phi(\alpha)) < l(\alpha)$.

**Proof.** The proof goes parallel to that of [IU05, Lemma B]. We again give comments why we can apply Lemmas after [IU05, Lemma A] in [IU05, §6]. [IU05, Lemmas 6.7, 6.8] hold for $H^p(\alpha)_U$, and assuming that integers $s, t$ satisfy $1 \leq t - s \leq n - 1$.

### 6.2 The proof of Proposition 5.4

Notice that if we show the existence of an autoequivalence $\Phi \in B_0$ such that $l(\alpha) > l(\Phi(\alpha))$, under the assumption of $l(\alpha) > 1$, then we can prove the statement by induction on $l(\alpha)$. We may assume $\text{Supp} \alpha = Z_0 = C_1 \cup \cdots \cup C_n$. Recall that the proof is already done in [IU05, Proposition 1.7] in the case $\text{Supp} \alpha \neq Z_0$, since in this case $\alpha$ is supported by a chain of projective lines, contained in $Z_0$.

For $S = S(a_s, \ldots, a_t) \in \Sigma(\alpha)$, define an integer $s(S)$ as $s(S) := s$, and an integer $t(S)$ as $C_{t(S)} = C_t$ and $s(S) \leq t(S) \leq s(S) + n - 2$. Here note that Lemma 5.7 (ii) guarantees that for $R \in \Sigma(\alpha)$, there are no elements $S \in \Sigma(\alpha)$ such that $C_{t(S)} = C_{s(R) - 1}$ or $C_s(S) = C_{t(R) + 1}$. Thus we have

$$l_{s(R)-1}(\alpha) < l_s(R)(\alpha) \quad \text{and} \quad l_{t(R)}(\alpha) > l_{t(R)+1}(\alpha).$$

Let $s$ and $t$ be integers such that

$$l_{s-1}(\alpha) < l_s(\alpha) = l_{s+1}(\alpha) = \cdots = l_t(\alpha) > l_{t+1}(\alpha).$$

Then we are in the situation of Lemma A (if $s = t$) or Lemma B (if $s < t$). So the proof is done.

**Remark 6.2.** Take an arbitrary element $R \in \Sigma(\alpha)$. Then, in the proof above, we can find $s, t$ such that $s(R) \leq s \leq t \leq t(R)$. Thus Lemma A or B provides

$$\Phi \in \left\{ T_{O_{C_j}(a)} \mid a \in \mathbb{Z}, s(R) \leq l \leq t(R) \right\}$$

such that $l(\alpha) > l(\Phi(\alpha))$. We shall use this remark in [7].

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7 The proof of Proposition 5.2

In this section we prove Proposition 5.2.

7.1 Conditions needed in the proof of Proposition 5.2

Put $\alpha = \Phi(O_{C_1})$ and $\beta = \Phi(O_{C_1}(-1))$. By Proposition 5.4 and applying the shift functor $[1]$, we may assume that $l(\alpha) = 1$, and in particular,

$$\alpha = O_{C_b}(a)$$

for some $a, b \in \mathbb{Z}$ with $1 \leq b \leq n$. To prove Proposition 5.2, it suffices to show the following:

**Claim 7.1.** Suppose that $l(\beta) > 1$. There is an autoequivalence $\Psi \in B_0$ such that $l(\Psi(\alpha)) = 1$ and $l(\beta) > l(\Psi(\beta))$.

In fact, Proposition 5.2 easily follows from this:

**Proof of Proposition 5.2.** By Claim 7.1, we can reduce the problem to the case $l(\alpha) = l(\beta) = 1$. In this case, the supports of $\alpha$ and $\beta$ must be the same, since $\chi(\alpha, \beta) = 2$. Therefore, we get the conclusion from the $A_1$-case [IU05 Proposition 1.6]. Therefore we can complete the proof of Proposition 5.2 by induction on $l(\beta)$. \qed

The most of the proof of Claim 7.1 goes parallel to that of [IU05, Claim 7.1]. First we may assume

**Condition 7.2** (cf. Condition 7.2 in [IU05]).

$$\max\{\deg_{C_b} R \mid R \in \Sigma(\beta)_{U_b}, \text{Supp } R \supset C_b\} = 0.$$  

Especially, $\deg_{C_b} R = 0$ or $-1$ for all $R \in \Sigma(\beta)_{U_b}$ with $\text{Supp } R \supset C_b$ by Lemma 5.7.

Relations between $O_{C_1}$ and $O_{C_1}(-1)$ impose conditions on $a$ and $\beta$. From the spectral sequence

$$E_2^{p,q} = \text{Ext}_S^p(H^{-q}(\beta), O_{C_b}(a)) \implies \text{Hom}_{D(S)}^{p+q}(\beta, \alpha) = \begin{cases} \mathbb{C}^2 & p + q = 0 \\ 0 & p + q \neq 0, \end{cases}$$

we obtain

**Condition 7.3** (cf. Condition 7.3 in [IU05]). $E_2^{1,q} = 0$ for $q \neq -1$

and

**Condition 7.4** (cf. Condition 7.4 in [IU05]). $d_2^{0,1} : E_2^{0,-1} \to E_2^{2,-2}$ is injective, $d_2^{0,0} : E_2^{0,0} \to E_2^{2,-1}$ is surjective, and $d_2^{0,q} : E_2^{0,q} \to E_2^{2,q+1}$ are isomorphic for all $q \neq 0, -1$. 

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In addition to Conditions 7.3 and 7.4, (31) implies
\[
\dim \operatorname{Coker} d_2^{0,-1} + \dim \operatorname{Ker} d_2^{0,0} + \dim E_2^{1,-1} = 2. \tag{32}
\]

To show the following, we need the assumption that \( \Phi \) preserves the cohomology class \([\mathcal{O}_x]\) in \( H^4(S, \mathbb{Q}) \).

**Condition 7.5** (cf. Condition 7.5 in [IU05]). The equality \( c_1(\beta) = [C_b] \) holds in the cohomology group \( H^2(S, \mathbb{Q}) \).

**Proof.** By the choice of \( \Phi \), we have
\[
0 = c_1(\mathcal{O}_x) = c_1(\Phi(\mathcal{O}_x)) = c_1(\alpha) - c_1(\beta),
\]
which gives the assertion. \( \square \)

**Claim 7.6** (cf. Claim 7.6 in [IU05]). We have \( a \geq -1 \).

**Proof.** The proof is similar to that of [IU05] Claim 7.6. \( \square \)

**Claim 7.7** (cf. Claim 7.7 in [IU05]). Fix \( q \neq 0 \). If \( E_2^{2-q-1} = 0 \) in (31), then we have \( \deg_{\mathcal{C}_b} \mathcal{R} > a \) for all direct summands \( \mathcal{R} \in \Sigma(\beta)_{\mathcal{U}_b} \) of \( H^4(\beta)_{\mathcal{U}_b} \) with \( \operatorname{Supp} \mathcal{R} \supset C_b \). If, in addition, we suppose that \( a \geq 0 \), then we get \( C_b \not\subset \operatorname{Supp} H^4(\beta) \).

**Proof.** The proof of Claim 7.7 is similar to that of [IU05] Claim 7.7. \( \square \)

The following remark is useful.

**Remark 7.8.** If there is an element \( \mathcal{R} \in \Sigma(\beta) \) with either
\[
t(\mathcal{R}) - n < b < s(\mathcal{R}) \text{ or } s(\mathcal{R}) \leq t(\mathcal{R}) < b,
\]
then we can find \( \Psi \in \left\langle T_{\mathcal{O}_x}(a) \mid a \in \mathbb{Z}, C_l \subset \operatorname{Supp} \mathcal{R} \right\rangle \) such that \( \Psi(\alpha) \cong \alpha \) and \( l(\beta) > l(\Psi(\beta)) \) by Remark 6.2. Therefore we may assume that \( s(\mathcal{R}) \leq b \leq t(\mathcal{R}) \) or \( s(\mathcal{R}) \leq b + n \leq t(\mathcal{R}) \) for all \( \mathcal{R} \in \Sigma(\beta) \).

Now we divide the proof into cases as in [IU05] Division into cases in page 426. We have only to consider the three cases:

**Division into Cases.**

(i) \( C_b \subset \operatorname{Supp} \mathcal{R} \) for all \( \mathcal{R} \in \Sigma(\beta)_{\mathcal{U}_b} \),

(ii) there is \( \mathcal{R} \in \Sigma(\beta)_{\mathcal{U}_b} \) with \( \operatorname{Supp} \mathcal{R} \cap C_b = C_{b+1} \cap C_b \), but there is not \( \mathcal{R}' \in \Sigma(\beta)_{\mathcal{U}_b} \) with \( \operatorname{Supp} \mathcal{R}' \cap C_b = C_{b-1} \cap C_b \),

(iii) there are \( \mathcal{R}, \mathcal{R}' \in \Sigma(\beta)_{\mathcal{U}_b} \) with \( \operatorname{Supp} \mathcal{R} \cap C_b = C_{b+1} \cap C_b \) and \( \operatorname{Supp} \mathcal{R}' \cap C_b = C_{b-1} \cap C_b \).

\( ^3 \)If we drop this assumption, we can just conclude \( c_1(\beta) = [C_b] + k[Z_0] \) for some \( k \in \mathbb{Z} \) by a similar proof to that of Condition 7.5 in [IU05].
7.2 Case (i)

In this case, we can find \( \Psi \) in Claim 7.1 by a similar way to that of [IU05 §7.3. Case (i)], after some obvious changes; for instance [IU05 Claim 7.8] should be replaced to

Claim 7.9.

\[
O\cup C_b^{(\ast, -1)}|_{U_{b-1, b, b+1}}, \quad O\cup C_{b+1}^{(-1, \ast)}|_{U_{b-1, b, b+1}},
\]

\[
O\cup C_b^{(\ast, -1)}|_{U_{b-1, b, b+1}} \notin \Sigma(U_{b-1, b, b+1}).
\]

7.3 Case (ii)

The existence of \( R \in \Sigma(\beta) \) with \( \text{Supp} R \cap C_b = C_b \cap C_{b+1} \) and Lemma 5.7 imply the non-existence of \( S \in \Sigma(\beta) \) with \( \text{Supp} S \cap C_b = C_b \cap C_{b+1} \).

Thus we have

\[
\Sigma(\beta)_{U_{b-1, b, b+1}} \subset \{ O\cup C_b^{(\ast, a'), \ast)}|_{U_{b-1, b, b+1}}, O\cup C_{b+1}^{(\ast, a')}|_{U_{b-1, b, b+1}} \mid a' = -1, 0 \}.
\]

Then we can find \( \Psi \) in Claim 7.1 by a similar way to that of [IU05 Case (ii)].

7.4 Case (iii)

Condition 7.4 implies that \( R \) and \( R' \) above must be in \( H^1(\beta) \). Moreover, they are unique in a decomposition of \( H^1(\beta) \), by virtue of the inequality \( \dim E_{12}^{1,-1} \leq 2 \) from (32). Thus Lemma 5.7 allows us to write

\[
\bigoplus_{p} H_p^p(\beta)_{U_{b-2,..., b+2}} = \bigoplus_{j} R_{1,j} \oplus \bigoplus_{j} R_{2,j} \oplus R_3 \oplus R_4,
\]

where \( R_{k,j} \)'s, \( R_3 \) and \( R_4 \) are sheaves in \( \Sigma(\beta)_{U_{b-2,..., b+2}} \) of the following forms:

\[
R_{1,j} : \begin{array}{cccc}
C_{b-1} & C_b & C_{b+1} \\
\circ & \circ & \circ \oplus \bullet
\end{array}
\]

\[
R_{2,j} : \begin{array}{cccc}
C_{b-1} & C_b & C_{b+1} \\
\circ & \circ & \circ \oplus \bullet
\end{array}
\]

\[
R_3 : \begin{array}{ccc}
\circ & \circ & \circ \oplus \bullet
\end{array}
\]

\[
\alpha : \begin{array}{ccc}
\circ & \circ & \circ \oplus \bullet
\end{array}
\]

Here we assume that \( \deg_{C_{b-1}} R_3 = -1 \) for simplicity.

Claim 7.10. We have \( a = -1 \).

Proof. A similar proof to that of [IU05 Claim 7.11] works. \( \square \)
The inequality \( \dim E_2^{1,-1} \leq 2 \) from (32) also implies that

\[
\text{Ext}_S^1(\mathcal{R}_{k,j}, \mathcal{O} \mathcal{C}_b(-1)) = 0
\]

for \( k = 1, 2 \) and for all \( j \). In particular we get

\[
\deg \mathcal{R}_{1,j} = \deg \mathcal{R}_{2,j} = 0.
\]

Let \( \mathcal{R} \in \Sigma(\beta) \) be a sheaf satisfying \( \mathcal{C}_t(\mathcal{R}) = \mathcal{C}_b^{-1} \), that is \( \mathcal{R}_3 \subset \mathcal{R}|_{U_{b-2},...,b+2} \).

Now we give a proof for Case (iii) by induction on \( l(\mathcal{R}) \). First suppose \( l(\mathcal{R}) = 1 \). In this case \( \mathcal{R} = \mathcal{R}_3 \), and we write

\[
\bigoplus_j \mathcal{R}_{2,j} = \bigoplus_j \mathcal{S}_{1,j} \oplus \bigoplus_j \mathcal{S}_{2,j},
\]

where \( \mathcal{S}_{k,j} \)'s are sheaves in \( \Sigma(\beta)|_{U_{b-2},...,b+2} \) of the following forms.

\[
\begin{align*}
\mathcal{R}_{1,j} : & \quad C_{b-1} \quad C_b \quad C_{b+1} \\
\mathcal{S}_{1,j} : & \quad \circ \quad \circ \quad \circ - - - \\
\mathcal{S}_{2,j} : & \quad \circ \quad \circ \quad \circ - - - \\
\mathcal{R}_3 : & \quad \bigcirc \quad \circ - - - : \quad \mathcal{R}_4 \\
\alpha : & \quad \bigcirc 
\end{align*}
\]

Because of the existence of \( \mathcal{R}_3 \), we have \( s_1 \neq s_2 \) by [IU05] Lemma 6.6. Define

\[
\Psi_0 = \begin{cases} 
T_{\mathcal{O} \mathcal{C}_b^{-1} \cup \mathcal{C}_b(-1,-1)} & \text{if } s_1 < s_2, \\
T_{\mathcal{O} \mathcal{C}_b^{-1} \cup \mathcal{C}_b} & \text{if } s_2 < s_1.
\end{cases}
\]

Then \((\Psi_0(\alpha), \Psi_0(\beta))\) fits in Case (ii) and \( \Psi_0(\beta) \) satisfies \( l(\Psi_0(\beta)) \leq l(\beta) \). Since we have proved Case (ii), we finish the case \( l(\mathcal{R}) = 1 \).

Next suppose \( l(\mathcal{R}) > 1 \). In this case, Lemma [14.1] (iii-4) implies

\[
\deg \mathcal{R}_{b-1} \mathcal{R}_{2,j} = -1.
\]

Define

\[
\Psi' = T_{\mathcal{O} \mathcal{C}_b(-1)} \circ T_{\mathcal{O} \mathcal{C}_{b-1}(-2)}.
\]

Then we have \( \Psi'(\alpha) \equiv \mathcal{O} \mathcal{C}_{b-1}(-2) \) and \( l(\Psi'(\beta)) \leq l(\beta) \). First let us consider the case \( \mathcal{C}_s(\mathcal{R}) = \mathcal{C}_{b+1} \). In this case there is an element \( \mathcal{R}' \in \Sigma(\Psi'(\beta)) \) such that

\[
t(\mathcal{R}') - n < b < s(\mathcal{R}') \text{ or } s(\mathcal{R}') \leq t(\mathcal{R}') < b.
\]

\[\footnote{Note that \( \Psi'^{-1}(\mathcal{R}') \) is a sheaf satisfying \( \Psi'^{-1}(\mathcal{R}')|_{U_{b-2},...,b+2} \cong \mathcal{R}_{2,j} \) for some \( j \).} \]

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and hence we can find $\Psi$ in Claim 7.1 by Remark 7.8. Therefore we may assume that $C_s(R) \neq C_{b+1}$, and in this case we can see that $\Psi'(\beta)$ satisfies the induction hypothesis (on $l(R)$). This finishes the proof of Case (iii) and we get the assertion of Proposition 5.2.

8 Examples

As we mentioned before, it is not easy to describe the group $H_S$ concretely for a given elliptic surface $S$.

In this section, we give two examples of elliptic surfaces for which $\text{Im } \Theta$ in Theorem 5.1 is not equal to $\text{SL}(2, \mathbb{Z})$, but we are able to describe $H_S$, and hence $\text{Im } \Theta$ more concretely.

8.1 Autoequivalences of elliptic ruled surfaces

Let $S := \mathbb{P}(O_E \oplus L)$ be an elliptic ruled surface with non-trivial Fourier–Mukai partners, where $E$ is an elliptic curve, and $L$ is in $mE$ for some $m > 2$, as in Theorem 3.7. Then the group $H_S$ coincides with the group $H^E_L$ by the result in [3.7]. Note also that $\text{Auteq } D(S) = \text{Auteq } \Gamma(D(U))$, since $\pi$ has no reducible fibers by Theorem 3.6.

Therefore by Theorem 4.10, we have the following short exact sequence:

$$1 \rightarrow \langle \otimes O_S(D) \mid D \cdot F = 0 \rangle \rtimes \text{Aut } S \times \mathbb{Z}[2] \rightarrow \text{Auteq } D(S) \Theta \rightarrow \{ \begin{pmatrix} c & a \\ d & b \end{pmatrix} \in \Gamma_0(m) \mid b \in H^E_L \} \rightarrow 1.$$ 

Here for an integer $b$, coprime with $m$, we again denote by $b$ the corresponding element in $H^E_L(\subset (\mathbb{Z}/m\mathbb{Z})^*)$.

8.2 Autoequivalences of certain rational elliptic surfaces

Let us consider a rational elliptic surface $\pi: J \rightarrow \mathbb{P}^1$ with a section, and assume that $\pi$ has four singular fibers of types $I_7$, $I_2$, $II$ and $I_1$. Such a surface exists by Persson’s list [Pe90]. Take a point $s \in \mathbb{P}^1$ over which the fiber of $\pi$ is not of type $II$. Apply a logarithmic transformation along the point $s$, and then we obtain a rational elliptic surface $S$ whose Jacobian surface is $J$, and $S$ has a multiple fiber of the multiplicity $m$ over the point $s$. Suppose that $m > 2$. Then as in [Uc11 Example 2.6], we can show that $H_S = \{ \pm1 \}$. Therefore Theorem 1.3 assures that there is a short exact sequence:

$$1 \rightarrow \langle B, \otimes O_S(D) \mid D \cdot F = 0 \rangle \rtimes \text{Aut } S \times \mathbb{Z}[2] \rightarrow \text{Auteq } D(S) \Theta \rightarrow \{ \begin{pmatrix} c & a \\ d & b \end{pmatrix} \in \Gamma_0(m) \mid b \equiv \pm1 \pmod{m} \} \rightarrow 1.$$
In this case, \( \text{Aut} S = \text{Aut}_{\mathbb{P}^1} S \) is just the semi-direct product of the Mordell-Weil group of \( S \) and the subgroup of automorphism preserving the zero section (cf. [FM94 Theorem 1.3.14]). Hence it can be calculated by using [OS90].

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