RATIONAL CURVES ON CUBIC HYPERSURFACES OVER FINITE FIELDS

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Abstract. Given a smooth cubic hypersurface $X$ over a finite field of characteristic greater than 3 and two generic points on $X$, we use a function field analogue of the Hardy–Littlewood circle method to obtain an asymptotic formula for the number of degree $d$ rational curves on $X$ passing through those two points. We use this to deduce the dimension and irreducibility of the moduli space parametrising such curves, for large enough $d$.

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1. Introduction

Let $k = \mathbb{F}_q$ be a finite field and let $F \in k[x_1, \ldots, x_n]$ denote a non-singular homogeneous polynomial of degree 3. Moreover, let $X \subset \mathbb{P}^{n-1}_k$ be the smooth cubic hypersurface defined over $k$ by $F = 0$. Let $C$ be a smooth projective curve over $k$. Then $k(C)$ has transcendence degree 1 over a $C_1$-field and, by the Lang–Tsen theorem [6, Theorem 3.6], the set $X(k(C))$ of $k(C)$-rational points on $X$ is non-empty for $n \geq 10$. This still holds for $X$ singular. In this paper we are interested in the case $C = \mathbb{P}^1$, writing $K = k(t)$ denote the function field of $C$ over $k$. A degree $d$ rational curve on $X$ is a non-constant morphism $f : \mathbb{P}^1_k \to X$ given by

$$f = (f_1(u, v), \ldots, f_n(u, v)),$$

(1.1)
where \( f_i \in \bar{k}[u,v] \) are homogeneous polynomials of degree \( d \geq 1 \), with no non-constant common factor in \( \bar{k}[u,v] \), such that
\[
F(f_1(u,v), \ldots, f_n(u,v)) \equiv 0.
\]
Such a curve is said to be \( m \)-pointed if it is equipped with a choice of \( m \) distinct points \( P_1, \ldots, P_m \in X(k) \) called the \textit{marks} through which the curve passes. Up to isomorphism, these curves are parametrised by the moduli space \( M_{0,m}(\mathbb{P}^1_k, X, d) \). The compactification of this space, \( \overline{M}_{0,m}(\mathbb{P}^1_k, X, d) \), is the Kontsevich moduli space of stable maps.

Suppose from now on that \( \#k = q \) and \( \text{char}(k) > 3 \). In [13, Example 7.6], Kollár proves that there exists a constant \( c_n \) depending only on \( n \) such that for any \( q > c_n \) and any point \( x \in X(k) \), there exists a \( k \)-rational curve of degree at most 216 on \( X \) passing through \( x \). In our investigation we focus on the case \( m = 2 \) of 2-pointed rational curves on \( X \).

Associate to \( F \) the \textit{Hessian matrix}
\[
H(x) = \left( \frac{\partial^2 F}{\partial x_i \partial x_j} \right)_{1 \leq i,j \leq n}
\]
and the \textit{Hessian hypersurface} \( H = 0 \), where \( H(x) = \det H(x) \). Now let \( a, b \in \mathbb{F}_q^n \setminus \{0\} \) be such that \( F(a) = F(b) = 0 \) and \( H(b) \neq 0 \). Write \( a = [a], b = [b] \) for the corresponding points in \( X(k) \). As is well-known (see [9, Lemma 1], for example), the Hessian \( H(x) \) does not vanish identically on \( X \), since \( \text{char}(k) > 3 \). The main goal of this paper is to obtain an asymptotic formula for the number of rational curves of degree \( d \) on \( X \) passing through \( a \) and \( b \).

Denote the space of such curves by \( \text{Mor}_{d,a,b}(\mathbb{P}^1_k, X) \). We can write the \( f_i \) in (1.1) explicitly as
\[
\alpha_d^{(i)} u^d + \alpha_{d-1}^{(i)} u^{d-1} v + \ldots + \alpha_0^{(i)} v^d,
\]
where \( \alpha_j^{(i)} \in k \) for \( 0 \leq j \leq d \) and \( 1 \leq i \leq n \). Then, we capture the condition that the rational curve \( f \) passes through the points \( a \) and \( b \) by selecting
\[
\begin{align*}
\left( \alpha_0^{(1)}, \ldots, \alpha_0^{(n)} \right) &= a, \\
\left( \alpha_d^{(1)}, \ldots, \alpha_d^{(n)} \right) &= b.
\end{align*}
\] (1.2)

There exists a correspondence between the rational curves on \( X \) of bounded degree and the \( K \)-points on \( X \) of bounded height. Define \( N_{a,b}(d) \) to be the number of polynomials \( f_1, \ldots, f_n \in \mathbb{F}_q[t] \) of degree at most \( d \) whose constant coefficients are given by \( a \) and whose leading coefficients are given by \( b \), such that \( F(f_1, \ldots, f_n) = 0 \). Thus, \( N_{a,b}(d) \) counts the \( \mathbb{F}_q \)-points \( (f_1, \ldots, f_n) \) on the affine cone of \( \text{Mor}_{d,a,b}(\mathbb{P}^1_k, X) \), where the condition that \( f_1, \ldots, f_n \) have no common factor is dropped. Using a version of the Hardy–Littlewood circle
method for the function field $K$ developed by Lee [18, 19], and further by Browning–Vishe [1], we shall obtain the following result.

**Theorem 1.1.** Fix $k = \mathbb{F}_q$ with $\text{char}(k) > 3$. Fix a smooth cubic hypersurface $X \subset \mathbb{P}^{n-1}_k$, where $n \geq 10$. Let $a, b \in X(k)$, not both on the Hessian. Then, we have

$$N_{a,b}(d) = q^{(d-1)n-(3d-1)} + O \left( q^{\frac{5(d+2)n}{6} - \frac{5d+16}{3}} + q^{\frac{(5d+8)n}{6} - \frac{3d}{2} + \frac{14}{3}} + q^{\frac{3(d+5)n}{4} - \frac{3(d+5)}{4}} \right),$$

where the implied constant in the estimate depends only on $d$ and $X$.

The condition that one of the two fixed points is not on the Hessian comes from our analysis of certain oscillatory integrals (see Lemma 3.5).

Although it would be possible to generalise Theorem 1.1 to handle rational curves passing through any generic finite set of points in $X(k)$, the main motivation for considering rational curves through two fixed points comes from the notion of rational connectedness. In [21], Manin defined $R$-equivalence on the set of rational points of a variety in order to study the parametrisation of rational points on cubic surfaces. We say that two points $a, b \in X(k)$ are directly $R$-equivalent if there is a morphism $f : \mathbb{P}^1 \to X$ (defined over $k$) with $f(0,1) = a$ and $f(1,0) = b$; the generated equivalence relation is called $R$-equivalence. In [22], Swinnerton-Dyer proved that $R$-equivalence is trivial on smooth cubic surfaces over finite fields; that is, all $k$-points are $R$-equivalent. Next, the result was generalised for smooth cubic hypersurfaces $X \subset \mathbb{P}^{n-1}_k$, if $n \geq 6$ by Madore in [20], and if $n \geq 4$ and $q \geq 11$ by Kollár in [13]. Moreover, Madore’s result holds for $X$ defined over any $C_1$ field. The study of $R$-equivalence is closely related to understanding the geometry of the moduli space of rational curves. In particular, it is interesting to study $R$-equivalence in the case of varieties with many rational curves. Such varieties are called rationally connected and were first studied by Kollár, Miyaoka and Mori in [14], and independently by Campana in [3]. Roughly speaking, $Y$ is rationally connected if for two general points of $Y$ there is a rational curve on $Y$ passing through them. Thus, rationally connected varieties are varieties for which $R$-equivalence becomes trivial when one extends the ground field to an arbitrary algebraically closed field. Note that in the case of fields of positive characteristic one should consider separably rationally connected varieties. For precise definitions and a thorough introduction to the theory see Kollár [11, 12], and Kollár–Szabó [15].

**Corollary 1.2.** Fix $k = \mathbb{F}_q$ with $\text{char}(k) > 3$. Fix a smooth cubic hypersurface $X \subset \mathbb{P}^{n-1}_k$, where $n \geq 10$. Then there exists a constant $c_X > 0$ such that for any points $a, b \in X(k)$, not both on the Hessian, and any $d \geq \frac{10(n-1)}{n-9}$, if $q \geq c_X$, then there exists a $\mathbb{F}_q$-rational curve $C \subset X$ of degree $d$ that passes through $a$ and $b$. 
This can also be seen as a corollary of Pirutka [22, Proposition 4.3] which states that any two points \(a, b \in X(k)\) can be joined by two lines on \(X\) defined over \(k\).

Keeping track of the dependence on \(q\) allows us to deduce further results regarding the geometry of the moduli space \(\text{Mor}_{d,a,b}(\mathbb{P}^1_k, X)\), in the spirit of those obtained by Browning–Vishe [2]. We can regard \(f\) in (1.1) under the conditions given by (1.2) as a point in \(\mathbb{P}^{n(d-1)-1}\). Then the space \(\text{Mor}_{d,a,b}(\mathbb{P}^1_k, X)\) is an open subvariety of \(\mathbb{P}^{n(d-1)-1}\) cut out by \(3d - 1\) equations and so has expected naive dimension \(\mu = (n - 3)d - n\).

**Corollary 1.3.** Fix \(k = \mathbb{F}_q\) of \(\text{char}(k) > 3\). Fix a smooth cubic hypersurface \(X \subset \mathbb{P}^{n-1}_k\), where \(n \geq 10\). Pick any points \(a, b \in X(k)\), not both on the Hessian. Then for \(d \geq \frac{19(n-1)}{n-9}\) we have

\[
\lim_{q \to \infty} q^{-\hat{\mu}} N_{a,b}(d) \leq 1,
\]

where \(\hat{\mu} = \mu + 1\).

A result similar to [2, Theorem 2.1] concerning \(\text{Mor}_{d,a,b}(\mathbb{P}^1_k, X)\) follows from Corollary 1.3. Now, by [11, Theorem II.1.2], all irreducible components of \(\text{Mor}_{d,a,b}(\mathbb{P}^1_k, X)\) have dimension at least \(\mu\). Then, comparing this with the Lang–Weil estimate [17], we obtain that the space \(\text{Mor}_{d,a,b}(\mathbb{P}^1_k, X)\) is irreducible and of expected dimension \(\mu\). Following the same “spreading out” argument (see [5, §10.4.11] and [24]) as in [2, §2], the problem over \(\mathbb{C}\) can be related to the problem over \(\mathbb{F}_q\), and this leads to the following corollary.

**Corollary 1.4.** Fix a smooth cubic hypersurface \(X \subset \mathbb{P}^{n-1}_\mathbb{C}\) defined over \(\mathbb{C}\), where \(n \geq 10\). Pick any points two points in \(X(\mathbb{C})\), not both on the Hessian. Then for each \(d \geq \frac{19(n-1)}{n-9}\), the space \(\mathcal{M}_{0,2}(\mathbb{P}^1_\mathbb{C}, X, d)\) is irreducible and of expected dimension \(\bar{\mu} = \mu - 3\).

In the case of stable maps, Harris–Roth–Starr [7] prove that for a general hypersurface \(X \subset \mathbb{P}^{n-1}_\mathbb{C}\) of degree at most \(n - 2\), the Kontsevich moduli space \(\overline{\mathcal{M}}_{0,m}(\mathbb{P}^1_\mathbb{C}, X, d)\) is a generically smooth, irreducible local complete intersection stack of the expected dimension.

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2. Preliminaries

In this section we establish notation and record some basic definitions and facts. Throughout this paper \(S \ll T\) denotes an estimate of the form \(S \leq CT\), where \(C\) is some constant that does not depend on \(q\). Similarly, the implied
constants in the notation $S = O(T)$ are independent of $q$. Let $k = \mathbb{F}_q$, $K = k(t)$, and $\mathcal{O} = k[t]$. Finite primes $\mathfrak{p}$ in $\mathcal{O}$ are monic irreducible polynomials and we let $s = t^{-1}$ be the prime at infinity. These have associated absolute values which extend to give absolute values $| \cdot |_\infty$ and $| \cdot | = | \cdot |_\infty$ on $K$. We let $K_\mathfrak{p}$ and $K_\infty$ be the completions. We have

$$K_\infty = \mathbb{F}_q \left( \left( t^{-1} \right) \right) = \left\{ \sum_{i \leq N} a_i t^i : a_i \in \mathbb{F}_q, N \in \mathbb{Z} \right\}.$$ 

Set $\mathbb{T} = \left\{ \sum_{1 \leq i \leq N} a_i t^i | a_i \in \mathbb{F}_q \right\}$. Locally compact topological spaces have Haar measures, hence there is a (Haar) measure on $K_\infty$, and so on $\mathbb{T}$. This is normalised such that $\int_\mathbb{T} d\alpha = 1$ and is extended to $K_\infty$ in such a way that

$$\int_{\{\alpha \in K_\infty : |\alpha| < \hat{N}\}} d\alpha = q^N,$$

for any positive integer $N$. Moreover, this can be extended to $\mathbb{T}^n$ and $K_\infty^n$ for any $n \in \mathbb{Z}_{>0}$. Denote by $\psi : K_\infty \to \mathbb{C}^*$ the non-trivial additive character on $K_\infty$, given by

$$\sum_{i \leq N} a_i t^i \mapsto \exp \left( \frac{2\pi i \text{Tr}_{\mathbb{F}_q / \mathbb{F}_p}(a_{i-1})}{p} \right),$$

where $q$ is a power of $p$. Throughout this paper, for any real number $R$, let $\hat{R} = q^R$. The following orthogonality property in [16, Lemma 7] holds.

**Lemma 2.1.** For any $N \in \mathbb{Z}_{\geq 0}$ and any $\gamma \in K_\infty$, we have

$$\sum_{b \in \mathcal{O} \atop |b| < \hat{N}} \psi(\gamma b) = \begin{cases} \hat{N}, & \text{if } |\gamma| < \hat{N}^{-1}, \\ 0, & \text{else.} \end{cases}$$

The following lemma corresponds to [1, Lemma 2.2] and a proof can also be found in [16, Lemma 1(f)].

**Lemma 2.2.** Let $Y \in \mathbb{Z}$ and $\gamma \in K_\infty$. Then

$$\int_{|\alpha| < \hat{Y}} \psi(\alpha \gamma) \, d\alpha = \begin{cases} \hat{Y}, & \text{if } |\gamma| < \hat{Y}^{-1}, \\ 0, & \text{otherwise.} \end{cases}$$

Taking $Y = 0$, it follows that

$$\int_{\alpha \in \mathbb{T}} \psi(\alpha \gamma) \, d\alpha = \begin{cases} 1, & \text{if } \gamma = 0, \\ 0, & \text{if } \gamma \in \mathcal{O} \setminus \{0\}. \end{cases}$$

The next three results are standard, but are proved here since we require versions in which the implied constant is independent of $q$. 
Lemma 2.3. Let $\tau(f)$ be the number of monic divisors of a polynomial $f \in \mathbb{F}_q[t]$. Then for any $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$, depending only on $\varepsilon$, such that $\tau(f) \leq C(\varepsilon)|f|^{\varepsilon}$.

Proof. First note that

$$\frac{\tau(f)}{|f|^{\varepsilon}} = \prod_{\omega^{|f|\omega^{\alpha}|f| |\omega| < 2^{1/\varepsilon}}} \frac{\alpha + 1}{|\omega^{\alpha}|^{\varepsilon}} \prod_{\omega^{|f|\omega^{\alpha}|f| |\omega| \geq 2^{1/\varepsilon}}} \frac{\alpha + 1}{|\omega^{\alpha}|^{\varepsilon}},$$

where $\omega$ denotes a prime in $\mathcal{O}$. The second factor is less than or equal to 1. In the first factor $|\omega| < 2^{1/\varepsilon}$, which is equivalent to $d := \deg(\omega) < \frac{1}{\varepsilon} \log q =: D$. Then,

$$\prod_{\omega^{\alpha}|f} \frac{\alpha + 1}{|\omega^{\alpha}|^{\varepsilon}} \leq \prod_{d < D} \prod_{\omega^{\alpha}|f} \frac{\alpha + 1}{q^{\alpha \varepsilon}} \leq \prod_{d < D} \left(1 + \frac{\alpha}{q^{\alpha \varepsilon}}\right).$$

Now, if $g(\alpha) = \frac{\alpha}{y}$, then $g'(\alpha) = \frac{1}{y} y^{-\alpha}(1 - \alpha \log y)$. Thus, $g$ is maximised at $\alpha = \frac{1}{\log y}$ when $g\left(\frac{1}{\log y}\right) = \frac{1}{e \log y}$. Thus,

$$\prod_{\omega^{\alpha}|f} \frac{\alpha + 1}{|\omega^{\alpha}|^{\varepsilon}} \leq \prod_{d < D} \prod_{\omega^{\alpha}|f} \left(1 + \frac{1}{e \log q^{\alpha \varepsilon}}\right) \leq \prod_{d < D} \left(1 + \frac{1}{e \log q^{\alpha \varepsilon}}\right)^{2 q^d/d},$$

since by [23, Chapter 2], the number $a_d$ of primes of degree $d$ satisfies

$$\left|a_d - \frac{q^d}{d}\right| \leq \frac{q^d}{d} + q^d. \quad (2.1)$$

Then, using $1 + x \leq e^x$, we obtain

$$\prod_{\omega^{\alpha}|f} \frac{\alpha + 1}{|\omega^{\alpha}|^{\varepsilon}} \leq \prod_{d < D} \left(\exp\left(\frac{1}{e \log q^{\alpha \varepsilon}}\right)\right)^{2 q^d/d} = \exp\left(2 \sum_{d < D} \frac{q^d}{d} \cdot \frac{1}{e \varepsilon \log q}\right).$$

Now, $q^d/d^2$ is increasing with $d$ for $q \geq 4$ and thus, in this case we have $\sum_{d < D} q^d/d^2 < q^D/D$. In fact, we have $\sum_{d < D} q^d/d^2 < 2 q^D/D$, for any $q \geq 2$. Thus,

$$\prod_{\omega^{\alpha}|f} \frac{\alpha + 1}{|\omega^{\alpha}|^{\varepsilon}} \leq \exp\left(\frac{4 q^D}{D} \cdot \frac{1}{e \varepsilon \log q}\right) = \exp\left(\frac{2^{2+1/\varepsilon}}{e \log 2}\right),$$

which concludes the proof. \qed
Lemma 2.4. Let $\omega(f)$ denote the number of prime divisors of a polynomial $f \in \mathbb{F}_q[t]$. Then for any $\varepsilon > 0$ and any integer $k \geq 2$, we have $k^{\omega(f)} \ll_{\varepsilon,k} |f|^{\varepsilon}$.

Proof. Let $\tau_k(f)$ denote the number of factorisations of a polynomial $f \in \mathbb{F}_q[t]$ into $k$ factors. Write $f = \varpi_1^{a_1} \cdots \varpi_m^{a_m}$, where $\varpi_i$ are distinct primes in $\mathbb{F}_q[t]$. Then,

$$\tau_k(f) = \prod_{j=1}^m \left( \frac{a_j + k - 1}{a_j} \right) = \prod_{j=1}^m \frac{(a_j + k - 1) \cdots (a_j + 1)}{(k - 1)!} \geq \prod_{j=1}^m \frac{k!}{(k - 1)!} = k^m.$$ 

Thus, $\tau_k(f) \geq k^{\omega(f)}$. We will prove $\tau_k(f) \ll_{\varepsilon} |f|^{\varepsilon}$, by induction. For $k = 2$, the result follows from Lemma 2.3. For $k > 2$, use the fact that $\tau_k(f) = \sum_{d|f} \tau_{k-1}(f/d_k)$. \hfill $\Box$

Lemma 2.5. Let $Y \in \mathbb{N}$. Then

$$\sum_{m \in \mathcal{O}} \frac{1}{|m|} = Y + 1.$$ 

Proof. We have

$$\sum_{m \in \mathcal{O}} \frac{1}{|m|} = \sum_{n=0}^{Y} \frac{1}{q^n} \# \{ m \in \mathcal{O} : |m| = q^n, m \text{ monic} \} = Y + 1,$$

as claimed. \hfill $\Box$

3. The circle method over function fields

Recall that $k = \mathbb{F}_q$ has characteristic $> 3$ and $F \in k[x_1, \ldots, x_n]$ denotes a non-singular homogeneous polynomial of degree 3. Moreover, let $X \subset \mathbb{P}_k^n$ be the smooth cubic hypersurface defined by $F = 0$, and let $a, b \in \mathbb{F}_q^n \setminus \{0\}$ such that $F(a) = F(b) = 0$, $H(b) \neq 0$. We want $x_i \in \mathbb{F}_q[t]$ such that $F(x) = 0$ and

$$x(0) = a,$$

$$x(\infty) = b.$$ 

(3.1) 

(3.2)

Now, $t$ is a prime in $\mathbb{F}_q[t]$, and $s = t^{-1}$ is the prime at infinity. Moreover,

$$x_i = \sum_{0 \leq j \leq d} x_{ij} t^j = t^d \left( \sum_{0 \leq j \leq d} x_{ij} s^{d-j} \right) = t^d y_i,$$

say, for $x_{ij} \in \mathbb{F}_q$. Then $y_i = t^{-d} x_i$ and (3.1) is equivalent to $x \equiv a \mod t$, while (3.2) is equivalent to $y \equiv b \mod s$. 
Define a weight function \( \omega : K_\infty^n \to \mathbb{R}_{>0} \) such that
\[
\omega(x) = \begin{cases} 
1, & \text{if } |tx - b| < 1, \\
0, & \text{otherwise}.
\end{cases}
\]
This is the weight function \( w(t^L(x - x_0)) \) defined in [1, (7.2)], with \( x_0 = t^{-1}b \) and \( L = 1 \). We now check that [1, (7.1)] and [1, (7.3)] hold. Recall that \( H(x) = \det H(x) \) and note that \( F(t^{-1}b) = t^{-3}F(b) = 0, H(t^{-1}b) = H(t^{-1}b) = t^{-n}H(b) \neq 0 \), and \( |t^{-1}b| = 1/q < 1 \). Furthermore, \( |x| < 1 \) for any \( x \in K_\infty^n \) such that \( \omega(x) \neq 0 \). Moreover, any \( x \in K_\infty^n \) such that \( \omega(x) \neq 0 \) can be written in the form \( x = t^{-1}(b + z) \), where \( z \in \mathbb{T}^n \). Then, for any \( x \in K_\infty^n \) such that \( \omega(x) \neq 0 \),
\[
|\det H(x)| = |H(t^{-1}(b + z))| = q^{-n}|H(b) + z.\nabla H(b) + \ldots| = q^{-n},
\]
since \( H(b) \in \mathbb{F}_q^* \) and \( z \in \mathbb{T}^n \). But \( |H(t^{-1}b)| = |t^{-n}H(b)| = q^{-n} \), and thus \( |H(x)| = |H(t^{-1}b)| \), for any \( x \in K_\infty^n \) such that \( \omega(x) \neq 0 \). This confirms that [1, (7.1) and (7.3)] hold.

We have
\[
N_{a,b}(d) = \sum_{\substack{x \in \mathbb{O}^n \\ F(x) = 0 \\ x \equiv a \mod t}} \omega \left( \frac{x}{td+1} \right),
\]
where \( a \) and \( b \) are the corresponding points of \( a \) and \( b \) in \( X(k) \). We remark that any \( x \) in the sum has \( |x| = q^d \). To simplify notation, we write \( N_{a,b}(d) = N(d) \) and \( P = td+1 \). Then, \( \omega(x/P) \neq 0 \) implies that \( t^{-d}x \equiv b \mod s \).

Define
\[
S(\alpha) = \sum_{\substack{x \in \mathbb{O}^n \\ x \equiv a \mod t}} \psi(\alpha F(x))\omega \left( \frac{x}{P} \right).
\]
Then, by Lemma 2.2, we have
\[
N(d) = \int_{\alpha \in \mathbb{T}} S(\alpha) \, d\alpha.
\]
By [1, Lemma 4.1], \( \mathbb{T} \) can be partitioned into a union of intervals centred at rationals and since \( K \) is non-archimedean, the intervals do not overlap. Thus, for any \( Q \geq 1 \), we have
\[
N(d) = \sum_{r \in \mathbb{O}} \sum_{|a| < |r|} \int_{|\theta| < \frac{1}{|r|Q}} S \left( \frac{a}{r} + \theta \right) \, d\theta, \quad (3.3)
\]
where \( \sum^* \) denotes a restriction to \( (a, r) = 1 \). We shall take \( Q = \frac{3(d+1)}{2} \) in our work. We now note that \( S(a/r + \theta) \) is the same as the exponential
sum $S(a/r + \theta)$ appearing in \cite{1} pg. 690, where $b$ is $a$ and $M = t$. Define $r_M = rM/(r, M)$, for any $M$,

$$S_{r,M,a} (c) = \sum_{|a| < |r|} \sum_{\substack{y \in \mathcal{O}_n^m \cap |y| < |r_M| \ \ y \equiv a \mod M}} \psi \left( \frac{aF(y)}{r} \right) \psi \left( \frac{-c.y}{r_M} \right), \quad (3.4)$$

$$I_r (\theta; c) = \int_{K^n_{\infty}} \omega (u) \psi \left( \theta P^3 F(u) + \frac{Pc.u}{r} \right) \, du. \quad (3.5)$$

\textbf{Lemma 3.1.} Let $P = t^{d+1}$. We have

$$N(d) = |P|^n \sum_{r \in \mathcal{O}} \sum_{|r| < \hat{Q}} |\tilde{r}|^{-n} \int_{|\theta| < |r|} \sum_{c \in \mathcal{O}_n^m} S_{r,t,a} (c) I_{\tilde{r}} (\theta; c) \, d\theta, \quad (3.6)$$

where

$$\tilde{r} = \frac{rt}{(r,t)} = \begin{cases} rt, & \text{if } t \nmid r, \\ r, & \text{otherwise.} \end{cases}$$

and $\hat{C} = q|\tilde{r}| |P|^{-1} \max \{1, |\theta||P|^3\}$. \hfill \Box

\textit{Proof.} Applying \cite{1} (7.7) with $x_0 = t^{-b}b$ and $L = 1$, we have

$$I_{\tilde{r}} (\theta; c) = \frac{1}{q^n} \psi \left( \frac{Pc.b}{\tilde{r}t} \right) J_G \left( \theta P^3; \frac{Pt^{-1}c}{\tilde{r}} \right),$$

where $G(v) = F(t^{-1}b + t^{-1}v)$ and

$$J_G \left( \theta P^3; \frac{Pt^{-1}c}{\tilde{r}} \right) = \int_{T^n} \psi (\theta P^3 G(x) + \frac{Pt^{-1}c.x}{\tilde{r}}) \, dx,$$

using the notation in \cite{1} (2.4). According to \cite{1} Lemma 2.6 we have

$$J_G \left( \theta P^3; \frac{Pt^{-1}c}{\tilde{r}} \right) = 0$$

if $|c| > q|\tilde{r}| |P|^{-1} \max \{1, |\theta||P|^3\}$. Now apply \cite{1} Lemma 4.4. \hfill \Box

We note that $C \in \mathbb{Z}$. Our strategy is now to go through the remaining arguments in \cite{1} Sections 4 – 9] for our particular exponential sums and integrals, paying special attention to the uniformity in the $q$-aspect. Furthermore, we keep the same notation as in \cite{1} Definition 4.6] for the factorisation of any $r \in \mathcal{O}$. Thus, for any $j \in \mathbb{Z}_{>0}$ we have $r = r_{j+1} \prod_{i=1}^{j} b_i = r_{j+1} \prod_{i=1}^{j} k_i^{i}$, with $(j+1)$-full $r_{j+1}$, where for any $i \in \mathbb{Z}_{>0}$ we have

$$b_i = \prod_{\omega_i | r} \omega_i, \quad k_i = \prod_{\omega_i | r} \omega_i, \quad r_i = \prod_{\omega_i | r} \omega_i^{i}.$$
3.1. Exponential Sums. We continue to assume that char \((\mathbb{F}_q) > 3\). Moreover, we note that \(S_{r,M,a}(c)\) satisfies the multiplicativity property recorded in [1, Lemma 4.5]. We are interested in the case when \(M \mid t\).

**Lemma 3.2.** Let \(r = uv\) for coprime \(u, v \in \mathcal{O}\) and \(t \nmid u\). Then there exist non-zero \(a', a'' \in k^n\), depending on \(a\) and the residues of \(u, v\) modulo \(t\), such that

\[
S_{r,M,a}(c) = \begin{cases} 
S_{u,1,0}(c)S_{v,1,0}(c), & \text{if } M = 1, \\
S_{u,1,0}(c)S_{v,t,a'}(c), & \text{if } M = t \text{ and } t \mid r, \\
S_{u,1,0}(c)S_{v,1,0}(c)\psi\left(\frac{c.a''}{t}\right), & \text{if } M = t \text{ and } t \nmid r.
\end{cases}
\]

Furthermore, the estimates in [1, Lemma 5.1], [1, (5.2)] and [1, (5.3)] all hold and are independent of \(q\). Next we record the following result.

**Lemma 3.3.** Let \(r \in K^\infty_n, C \in \mathbb{N}, M \in \mathcal{O}\) and \(\varepsilon > 0\). Then there exists a constant \(c_{n,\varepsilon} > 0\), depending only on \(n\) and \(\varepsilon\), such that

\[
\sum_{c \in \mathcal{O}^n \mid |c| < \hat{C}} |S_{r,M,a}(c)| \leq c_{n,\varepsilon}|M|^{n/2} |r_3|^{n/2+1+\varepsilon} \left(|r_3|^{n/3} + \hat{C}^n\right).
\]

**Proof.** This follows directly from [1, Lemma 6.4] on noting that \(H_F = |\Delta_F| = 1\) in our situation.

Let \(F^* \in k[x_1, \ldots, x_n]\) be the dual form of \(F\). Its zero locus parametrises the set of hyperplanes whose intersection with the cubic hypersurface \(F = 0\) produces a singular variety. Moreover, \(F^*\) is absolutely irreducible and has degree \(3 \cdot 2^{n-2}\). We shall need the following variation of [1, Lemma 6.4] in which the sum is restricted to zeros of \(F^*\).

**Lemma 3.4.** Let \(C \in \mathbb{N}, M \in \mathcal{O}\) and \(\varepsilon > 0\). Then there exists a constant \(c_{n,\varepsilon} > 0\), depending only on \(n\) and \(\varepsilon\), such that

\[
\sum_{c \in \mathcal{O}^n \mid |c| < \hat{C} \atop F^*(c)=0} |S_{r,M,a}(c)| \leq c_{n,\varepsilon}|M|^{2n-5+\varepsilon} |r_3|^{2n+7+\varepsilon} \left(|r_3|^{n/3} + \hat{C}^n\right)^{n-\frac{3}{2}+\varepsilon}.
\]

**Proof.** This proof uses the same methods as in Section 7 of [8]. By (3.4), we have

\[
S_{r,M,a}(c) = \sum_{|a| < |r_3|} \sum_{y \in \mathcal{O}^n \mid |y| < |r_3|} \psi\left(\frac{aF(y) - c.y}{r_3}\right),
\]
since \( r_{3_M} = r_3 M / (r_3, M) = r_3 \). Setting \( y = a + Mz \), we have

\[
S_{r_3,M,a}(c) = \psi \left( \frac{-c.a}{r_3} \right) \sum_{|a|<|r_3|}^* S_a(c),
\]

where

\[
S_a(c) = \sum_{\substack{z \in \mathbb{O}^n \\text{mod } c | z|<|a|}} \psi \left( \frac{aM^{-1}F(a + Mz) - c.z}{l} \right)
\]

and \( l = r_3/M \). Denote

\[
g(z) = M^{-1}F(a + Mz),
\]

write \( l = c^2d \), where \( d \) is square-free, \( d \mid c \), and put \( z = z_1 + cdc_2z \), with \( |z_1| < |cd| \). Then,

\[
S_a(c) = |c|^n \sum_{\substack{z_1 \in \mathbb{O}^n \\text{mod } c \mid z_1|<|cd|}} \sum_{a \in \mathbb{O}^n} \psi \left( \frac{ag(z_1) - c.z_1}{c^2d} \right).
\]

Write \( a = a_1 + Mc_2 \) with \( |a_1| < |Mc| \). Then \( (a, r_3) = 1 \) if and only if \( (a_1, Mc) = 1 \) and thus,

\[
\sum_{|a|<|r_3|}^* S_a(c) = |c|^n |cd| \sum_{|a_1|<|Mc|} \sum_{\substack{z_1 \in \mathbb{O}^n \\text{mod } c \mid z_1|<|cd|}} \psi \left( \frac{a_1g(z_1) - c.z_1}{c^2d} \right) .
\]

Writing \( z_1 = h + cj \) with \( |h| < |c| \), we have \( g(z_1) \equiv g(h) + cjv(g(h)) \mod cd \) and \( a_1v(g(z_1)) \equiv a_1v(g(h)) \mod c \), since \( cd \mid c^2 \). Now, \( g(z_1) \equiv 0 \mod cd \) is equivalent to \( g(h) + cjv(g(h)) \equiv 0 \mod cd \). Thus, \( g(h) \equiv 0 \mod c \) and we can write \( g(h) = mc \). Thus, \( m + jv(g(h)) \equiv 0 \mod d \). Moreover, if \( a_1v(g(h)) = c + ck \), then \( a_1g(z_1) - c.z_1 \equiv a_1g(h) - c.h + c^2(k.j + a_1h\nabla g(j)) \mod c^2d \), and thus, the sum over \( z_1 \) becomes

\[
\sum_{\substack{h \in \mathbb{O}^n \\text{mod } c \mid |h|<|c|}} \psi \left( \frac{a_1g(h) - c.h}{c^2d} \right) \sum_{\substack{j \in \mathbb{O}^n \\text{mod } c \mid |j|<|d|}} \psi \left( \frac{k.j + a_1h\nabla g(j)}{d} \right) .
\]

Denote the sum over \( j \) by \( S_{k,h} \) and estimate it by writing

\[
|S_{k,h}|^2 = \sum_{\substack{j_1,j_2 \in \mathbb{O}^n \\text{mod } d \mid |j_1|,|j_2|<|d|}} \psi \left( \frac{k.(j_1 - j_2) + a_1h(\nabla g(j_1) - \nabla g(j_2))}{d} \right) .
\]
Writing \( j_1 = j_2 + j_3 \) and recalling (3.7), we note that

\[
h (\nabla g(j_1) - \nabla g(j_2)) = \frac{1}{2} j_3^T \nabla^2 g(h) j_3 + j_2^T \nabla^2 g(h) j_3
\]

and therefore,

\[
|S_{k,h}|^2 \leq \sum_{j_3 \in \mathcal{O}^n \atop \|j_3\| < |d|} \left| \sum_{j_2 \in \mathcal{O}^n \atop \|j_2\| < |d|} \psi^j \left( \frac{j_2 \cdot (a_1 \nabla^2 g(h) j_3)}{d} \right) \right| \leq |d|^n M_d(h),
\]

where \( M_d(h) = \# \{ j_3 \in \mathcal{O}^n : \|j_3\| < |d|, \nabla^2 g(h) j_3 \equiv 0 \mod d \} \). Thus,

\[
\sum_{c \in \mathcal{O}^n \atop |c| < \hat{C}} |S_{r_3,M,a}(c)| \leq |c|^{n+2} |d|^{n/2+1} |M| \sum_{c \in \mathcal{O}^n \atop |c| < \hat{C}} \sum_{g(h) \equiv 0 \mod c} M_d(x)^{1/2}.
\]

Now note that there exist elements \( c' = cM \) and \( d' = \frac{d}{(d,M)} \) with \( d' \mid c' \), such that

\[
\sum_{h \in \mathcal{O}^n \atop |h| < |c|} M_d(h)^{1/2} \leq \|(d, M)|^{n/2} \sum_{x \in \mathcal{O}^n \atop |x| < |c'|} N_{d'}(x)^{1/2},
\]

where \( x = a + Mh \) and \( N_{d'}(x) = \# \{ y \in \mathcal{O}^n : |y| < |d'|, H(x)y \equiv 0 \mod d' \} \).

Thus,

\[
\sum_{c \in \mathcal{O}^n \atop |c| < \hat{C}} |S_{r_3,M,a}(c)| \leq |c|^{n+2} |d|^{n/2+1} |M| |(d, M)|^{n/2} \mathcal{N} \sum_{x \in \mathcal{O}^n \atop |x| < |c'|} N_{d'}(x)^{1/2},
\]

where \( \mathcal{N} := \# \{ c \in \mathcal{O}^n : |c| < \hat{C}, F^*(c) = 0 \} \ll (1 + \hat{C})^{-3/2+\epsilon} \) for any \( \epsilon > 0 \), by [11, Lemma 2.10].

It remains to bound the inner sum. As is [11], let

\[
S(c, d) = \sum_{x \in \mathcal{O}^n \atop |x| < |c|} N_{d}(x)^{1/2},
\]

for given \( c, d \) in \( \mathcal{O} \), where \( d \mid c \) and \( d \) is square-free. This sum satisfies a multiplicitivity property, i.e. for any \( c_i, d_i \) in \( \mathcal{O} \) such that \((c_1d_1, c_2d_2) = 1 \) and \( d_i \mid c_i \) we have \( S(c_1c_2, d_1d_2) = S(c_1, d_1)S(c_2, d_2) \). Thus, we only need to look at the cases when \( c = \varpi^e \) and \( d = 1 \), and \( c = \varpi^e \) and \( d = \varpi \), for any \( e \in \mathbb{Z}_{>0} \) and any prime \( \varpi \). Note that \( F \) is non-singular modulo any prime \( \varpi \).
The arguments that follow are similar to [8, p. 244]. Define
\[ S_0(\varpi^e) = \# \{ x \in \mathcal{O}^n : |x| < |\varpi|^e, F(x) \equiv 0 \mod \varpi^e \}, \]
\[ S_1(\varpi^e) = \# \{ x \in \mathcal{O}^n : |x| < |\varpi|^e, \varpi \nmid x, F(x) \equiv 0 \mod \varpi^e \}, \]
for \( e \geq 1 \). Then, as in [8, (7.4), (7.5)], we have
\[ S_0(\varpi^e) = S_1(\varpi^e) + |\varpi|^{2n} S_0(\varpi^{e-3}), \quad \text{for } e \geq 4, \tag{3.8} \]
\[ S_0(\varpi^e) = S_1(\varpi^e) + |\varpi|^{(e-1)n}, \quad \text{for } 1 \leq e \leq 3, \tag{3.9} \]
\[ S_1(\varpi^{e+1}) = |\varpi|^{n-1} S_1(\varpi^e), \quad \text{for } e \geq 1. \tag{3.10} \]
Since, \( N_1(x) = 1 \), we have \( S(\varpi^e, 1) = S_0(\varpi^e) \). Moreover, \( S_1(\varpi) \ll |\varpi|^{n-1} \), and thus,
\[ S_1(\varpi^e) \ll |\varpi|^{e(n-1)}, \tag{3.11} \]
for \( e \geq 1 \). Thus, for \( 1 \leq e \leq 3 \) and \( n \geq 4 \), we have \( S_0(\varpi^e) \ll |\varpi|^{e(n-1)} \).
Similarly, for \( e \geq 4 \) and \( n \geq 4 \), we can use an induction argument to get
\( S_0(\varpi^e) \ll |\varpi|^{e(n-1)} \). Thus, for \( c = \varpi^e \) and \( d = 1 \), \( S(c, d) \leq A_1^{(c)}|c|^{n-1} \).
Consider now the case when \( c = \varpi^e \) and \( d = \varpi \). After a change of variables, \( S(\varpi^e, \varpi) \) is equal to
\[ \sum_{\overset{z \in \mathcal{O}^n}{|z| < |\varpi|}} N_{\varpi}(z)^{1/2} \# \{ x \in \mathcal{O}^n : |x| < |\varpi|^e, x \equiv z \mod \varpi, F(x) \equiv 0 \mod \varpi^e \}. \]
First analyse the contribution to \( S(\varpi^e, \varpi) \) coming from \( z \) such that \( \varpi \nmid z \).
Then, as in [8], by Cauchy’s inequality, it follows that this contribution is
\[ \ll \varpi^{(e-1)(n-1)} \sum_{\overset{z \in \mathcal{O}^n}{|z| < |\varpi|}} N_{\varpi}(z)^{1/2} \leq \varpi^{(e-1)(n-1)} S_{N_{\varpi}}(z)^{1/2} S_0(\varpi)^{1/2}, \]
where
\[ S_{N_{\varpi}}(z) = \# \{ z, y \in \mathcal{O}^n : |z| < |\varpi|, |y| < |\varpi|, \varpi \nmid z, H(z), y \equiv 0 \mod \varpi \}. \]
Then, by [8, 3.10], 3.11 and [8, Lemma 4], there exists some constant \( A \) such that the contribution to \( S(\varpi^e, \varpi) \) coming from \( z \) such that \( \varpi \nmid z \) is
\[ \leq A_{\varpi(\varpi)} \varpi^{(e-1)(n-1)} |\varpi|^{\frac{n}{2}} |\varpi|^{\frac{e}{2}} = A \varpi^{(n-1)+\frac{e}{2}}. \]
The remaining contribution to \( S(\varpi^e, \varpi) \) comes from \( z = 0 \). In this case, \( N_{\varpi}(0) = |\varpi|^n \), and thus this contribution is
\[ |\varpi|^{\frac{e}{2}} \# \{ y \in \mathcal{O}^n : |y| < |\varpi|^{e-1}, \varpi F(y) \equiv 0 \mod \varpi^e \}. \tag{3.12} \]
Then, as in [8], if \( 1 \leq e \leq 3 \), 3.12 becomes
\[ |\varpi|^{\frac{e}{2}} \# \{ y \in \mathcal{O}^n : |y| < |\varpi|^{e-1} \} = |\varpi|^{n(e-1)/2}, \tag{3.13} \]
and if $4 \leq e$, there exists a constant $A$ such that (3.12) is equal to
\[
|\omega|^3 S_0(\omega^{-3}) \leq A^\omega(\omega)|\omega|^3 = A|\omega|^{e(n-1) - \frac{n}{2} + 3}.
\]
Note that if $n \geq 5$, then the contributions in (3.13) and (3.14) are both
\[
\ll |\omega|^{e(n-1) + \frac{1}{2}},
\]
\[
\text{and thus } S(c, d) \ll A^\omega(c)|c|^{n-1}|d|^{1/2}.
\]
Putting everything together, we have
\[
\sum_{c \in \mathcal{O}^n, \quad |c| \leq \tilde{c}, \quad F^*(c) = 0} |S_{r,3,M,a}(c)| \leq |c|^{n+2}|d|^2 + |M||H(d, M)|^2 A^\omega(c')|c'|^{n-1}|d'\frac{1}{2} \left(1 + \tilde{C}\right)^{n - \frac{1}{2} + e}.
\]
Then, an application of Lemma 2.4 concludes the proof.

3.2. Exponential Integral. The following result is similar to [1] Lemma 7.3. It gives a good upper bound for $I_\tilde{F}(\theta; c)$, for $r, \theta, c$ appearing in the expression for $N(d)$ in Lemma 3.1.

Lemma 3.5. We have
\[
|I_{\tilde{F}}(\theta; c)| \ll \min \left\{ q^{-n}, q^n|\theta P^3|^{-n/2} \right\},
\]
where the implicit constant is independent of $q$.

Proof. As in [1] Lemma 7.3, we have $|I_{\tilde{F}}(\theta; c)| \leq \text{meas}(\mathcal{R})$, where
\[
\mathcal{R} = \left\{ x \in \mathbb{T}^n : |tx - b| < 1, |\theta P^3 \nabla F(x) + Pt^{-1}c/\tilde{r}| \leq \max \left\{ 1, |\theta P^3|^{1/2} \right\} \right\}.
\]
If $|\theta P^3| \leq 1$, then we have the trivial bound $\text{meas}(\mathcal{R}) \leq q^{-n}$. Otherwise, given $x \in \mathcal{R}$, we can write it as $x = bt^{-1} + d$, where $d \in \mathbb{T}^n, |d| \leq q^{-2}$. Then
\[
|H(x)| = |t^n H(b) + d \nabla H(bt^{-1}) + \ldots | = q^{-n}(1 + O(q^{-1})).
\]
Since the entries in the adjugate of $H(x)$ have norms equal to $q^{-n+1}(1 + O(q^{-1}))$, the inverse of $H(x)$ has entries with absolute value $q + O(1)$. Thus, if $x$ and $x + x'$ are in $\mathcal{R}$, we have $|x'| \ll q|\theta P^3|^{-1/2}$ and thus, $\text{meas}(\mathcal{R}) \ll q^n|\theta P^3|^{-n/2}$. \hfill \square

Note that this result uses crucially the condition that one of the two fixed points in Theorem 3.1 does not lie on the Hessian of $X$.

4. The main term

In this section we investigate the contribution to $N(d)$ in Lemma 3.1 coming from $c = 0$. Preserving the notation in [1], denote this term by $M(d)$. We will always assume $n \geq 10$. Thus,
\[
M(d) = |P|^n \sum_{r \in \mathcal{O}, \quad |r| \leq Q} \sum_{r \text{ monic}} |\tilde{r}|^{-n} S_{r,t,a}(0) K_r,
\]
where

\[ K_r = \int_{|\theta|<\frac{1}{|r|Q}} I_\hat{r}(\theta;0)\,d\theta. \]

Recall that \( \nabla F(t^{-1}b) \neq 0 \) and, in particular, \( q^{-2} = |\nabla F(t^{-1}b)| \). This corresponds to taking \( \xi = -2 \) in [I, Section 7.3]. The following result gives a similar bound to that in [I, Lemma 7.4].

Lemma 4.1. For any \( Y \in \mathbb{N} \) and any \( \varepsilon > 0 \) we have

\[ \sum_{r \in \mathcal{O}, |r|=Y, r \text{ monic}} |\hat{r}|^{-n} |S_{r,t,a}(0)| \ll q^{2n}\hat{Y}^{-\frac{2}{3}+\frac{4}{3}+\varepsilon}(q^{-n} + \hat{Y}^{\frac{2}{3}+\varepsilon}), \]

where the implicit constant is independent of \( q \).

Proof. Write \( r = b_1b_2r_3 \). Then, by the multiplicativity property in Lemma 5.6, we have

\[ |S_{r,t,a}(0)| = |S_{b_1b_2,M,a'}(0)||S_{r_3,M_3,a''}(0)|, \]

where \( M, M_3 \in \{1,t\} \) and \( a', a'' \in k^n \) depend on \( \text{ord}(r) \). By [I, Lemma 5.1],

\[ |S_{b_1b_2,M,a}(0)| \ll |b_1b_2|^{\frac{2}{3}+\varepsilon}, \]

where the implicit constant is independent of \( q \). Moreover,

\[ |S_{r_3,M_3,a}(0)| \leq \sum_{c \in \mathcal{O}^n, |c|<\hat{Y}} |S_{r_3,M_3,a}(c)|, \]

for any \( C > 0 \). Taking \( C = 1 \) and using [I, Lemma 6.4], we get

\[ |S_{r_3,M_3,a}(0)| \leq \sum_{c \in \mathcal{O}^n, |c|<q} |S_{r_3,M_3,a}(c)| \ll |M_3|^n|r_3|^{n/2+1+\varepsilon}(|r_3|^{n/3} + q^n), \]

where the the implicit constant depends only on \( n \) and \( \varepsilon \). On noting that for \( |r| = \hat{Y} \) we have \( |\hat{r}|^{-n} = q^{-n}\hat{Y}^{-n} \) if \( t \nmid r \) and \( |\hat{r}|^{-n} = \hat{Y}^{-n} \), otherwise, we obtain

\[ \sum_{r \in \mathcal{O}, |r|=\hat{Y}, r \text{ monic}} |\hat{r}|^{-n} |S_{r,t,a}(0)| \ll |M_3|^n\hat{Y}^{-\frac{2}{3}+\frac{4}{3}+\varepsilon} \sum_{r_3 \in \mathcal{O}, |r_3|\leq \hat{Y}, r_3 \text{ monic}} (|r_3|^{n/3} + q^n) \frac{\hat{Y}}{|r_3|}. \]

Then, since \# \( \{ r_3 \in \mathcal{O} : |r_3| \leq \hat{Y} \} = O(\hat{Y}^{1/3}) \) and \( M_3 \in \{1,t\} \), we can bound the above by

\[ \ll q^n\hat{Y}^{-\frac{2}{3}+\frac{4}{3}+\varepsilon}\left(\hat{Y}^{\frac{n}{3}-1} + q^n\right), \]
which concludes the proof.

Put $C = \tilde{L} - \xi = q^3$. Then, if $C^{-1} \tilde{Q} \leq |r| \leq \tilde{Q}$, we have $|\theta| < |r|^{-1} \tilde{Q}^{-1} \leq q^3 |P|^{-3}$, and thus, $|\theta P^3| \leq q^2$. Then, by Lemma 3.3 we have $K_r = O(q^{-n} |P|^{-3})$ in this case. On noting that the exponents of $\tilde{Y}$ in the bound given by Lemma 4.1 are negative for $n > 8$, we obtain that

$$\sum_{r \in \mathcal{O}, \ |r| \leq \tilde{Q}, \ \text{monic}} |P|^n |\tilde{r}|^{-n} S_{r,t,a} (0) K_r \ll |P|^{n-3} q^{3n+q^3 (\frac{2n}{3} + 4 + \varepsilon)} (q^{-n} + q^{Q-3})^{\frac{1}{3}}.$$ 

Thus, recalling that $\tilde{Q}^2 = |P|^3$, the contribution to $M(d)$ coming from such $r$ is $\ll q^{11n/12 + 2/3} |P|^{3n/4 - 1 + \varepsilon'} (1 + q^{2n-3} |P|^{3/2 - n/2})$. If $|r| < q^{-3} \tilde{Q}$, as in [1, Section 7.3], $K_r$ is independent of $r$. Moreover, we only get a contribution from $|\theta| < q^3 |P|^{-3}$. Thus, for $d \geq 3(n-1)/(n-3)$, we have

$$M(d) = |P|^{n-3} \mathcal{S} (Q) \tilde{J} + O(q^{11n/12 + 2/3} |P|^{3n/4 - 1}),$$

where

$$\mathcal{S} (Q) = \sum_{r \in \mathcal{O}, \ |r| \leq \tilde{Q}, \ \text{monic}} |\tilde{r}|^{-n} S_{r,t,a} (0)$$

and

$$\tilde{J} = \int_{|\varphi| < q^3} \int_{K_{\tilde{Q}}} w(tu - b) \psi (\varphi F(x)) \, dx \, d\varphi$$

is the singular integral. By taking $x_0 = t^{-1} b$ and $L = 1$ in [1 Lemma 7.5], it follows that

$$\tilde{J} = \frac{1}{q^{-n-3}}. \quad (4.1)$$

By Lemma 4.1 we can extend the summation over $r$ in $\mathcal{S} (Q)$ to infinity with acceptable error since

$$\mathcal{S} - \mathcal{S} (Q) \ll q^{\frac{2n}{3} + 4} |P|^{-\frac{n}{3} + 2 + \varepsilon} \left(1 + q^{2n + \frac{4}{3}} |P|^{2(3-n)} \right),$$

and thus,

$$|P|^{n-3} (\mathcal{S} - \mathcal{S} (Q)) \tilde{J} = |P|^{\frac{3n}{4} - 1} q^{-\frac{n}{6} + \frac{13}{3}} \left(1 + q^{2n + \frac{4}{3}} |P|^{2(3-n)} \right).$$

Then

$$\mathcal{S} = \sum_{r \in \mathcal{O}, \ r \text{ monic}} |\tilde{r}|^{-n} S_{r,t,a} (0)$$

is the absolutely convergent singular series.
Lemma 4.2. We have

\[ \mathcal{S} = q^{-n+1} + O(q^{-3n/2+3}). \]

Proof. First, recalling the definition of \( \bar{r} \), decompose \( \mathcal{S} \) into

\[ \mathcal{S} = q^{-n} \sum_{r \in \mathcal{O}, r \text{ monic}} |r|^{-n} S_{r,t,a}(0) + \sum_{r \in \mathcal{O}, r \text{ monic}} |r|^{-n} S_{r,t,a}(0). \] (4.2)

Then note that by the multiplicativity property in Lemma 5.6, given \( r = t^A \prod_{\varpi \neq t} \varpi^e \in \mathcal{O} \), where \( A \in \mathbb{Z}_{\geq 0} \) and \( \varpi \) are primes in \( \mathcal{O} \), we have

\[ S_{r,t,a}(0) = S_{t^A,t,a}(0) \prod_{\varpi \text{ prime}} S_{\varpi^e}(0), \]

where \( S_{\varpi^e}(0) = S_{\varpi^e,1,0}(0) = S_{\varpi^e,1,a}(0) \). Thus,

\[ \mathcal{S} = \left( q^{-n} + \sum_{A=1}^{\infty} q^{-An} S_{t^A,t,a}(0) \right) \prod_{\varpi \text{ prime}} \sum_{e=0}^{\infty} |\varpi|^{-en} S_{\varpi^e}(0). \]

Now, by (3.4),

\[ S_{t,t,a}(0) = \sum_{|a|<|t|}^{*} \sum_{y \in \mathcal{O}^n, |y| < |t|, y \equiv a \mod t} \psi \left( \frac{aF(y)}{t} \right) = \sum_{|a|<|t|}^{*} \psi \left( \frac{aF(a)}{t} \right) = \sum_{|a|<|t|}^{*} 1 = q - 1, \]

since \( F(a) = 0 \). Similarly, by (3.4), after making the change of variables \( y = a + tz \), we have

\[ S_{t^2,t,a}(0) = \sum_{|a|<|t|^2}^{*} \sum_{y \in \mathcal{O}^n, |y| < |t|, y \equiv a \mod t} \psi \left( \frac{az \cdot \nabla F(a)}{t} \right) = \sum_{|a|<|t|^2}^{*} q^n = 0, \]

Moreover, for \( K \geq 3 \), we have

\[ S_{t^K,t,a}(0) = \sum_{y \in \mathcal{O}^n, |y| < |t|^K, y \equiv a \mod t} \left( \sum_{|a_1|<|t|^K}^{*} \psi \left( \frac{a_1F(y)}{t^K} \right) - \sum_{|a_2|<|t|^{K-1}}^{*} \psi \left( \frac{a_2F(y)}{t^{K-1}} \right) \right) \]

\[ = q^K S_a(K) - q^{K-1+n} S_a(K - 1), \]

where \( S_a(K) = \# \{ y \in \mathcal{O}^n : |y| < |t|^K, y \equiv a \mod t, F(y) \equiv 0 \mod t^K \} \). Similarly to [S] p. 244, we have \( S_a(K) = q^{n-1} S_a(K - 1) \). Thus, for \( K \geq 3 \),

\[ S_{t^K,t,a}(0) = 0. \]
It remains to analyse $S_{\mathbf{w}^e}(0)$. By [3,4], we have $S_1(0) = 1$. Moreover, by [1] (5.2), (5.3), we have $S_{\mathbf{w}^e}(0) \ll |\mathbf{w}|^{\frac{3}{2}} + 1$ and $S_{\mathbf{w}^z}(0) \ll |\mathbf{w}|^{n+2}$. Also, [1] Lemma 5.3 implies that $S_{\mathbf{w}^3}(0) \ll |\mathbf{w}|^{2n+3}$ and $S_{\mathbf{w}^4}(0) \ll |\mathbf{w}|^{3n+3}$. By similar arguments as above, for $e \geq 5$ we have

$$S_{\mathbf{w}^e}(0) = |\mathbf{w}|^e S_0(\mathbf{w}^e) - |\mathbf{w}|^{e-1+n} S_0(\mathbf{w}^{e-1}),$$

where $S_0(\mathbf{w}^e) = \# \{ \mathbf{x} \in \mathcal{O}^n : |\mathbf{x}| < |\mathbf{w}|^e, F(\mathbf{x}) \equiv 0 \mod \mathbf{w}^e \}$, as in the proof of Lemma 3.4. Then, by (3.8) – (3.11), it follows that for $e = 3k + l \geq 5$, where $l \in \{0, 1, 2\}$, we have

$$S_{\mathbf{w}^e}(0) = |\mathbf{w}|^{e+2n(k-1)} \left\{ \begin{array}{ll}
(S_0(\mathbf{w}^4) - |\mathbf{w}|^{n-1} S_0(\mathbf{w}^2)), & \text{if } l = 0, \\
(S_0(\mathbf{w}^4) - |\mathbf{w}|^{n-1} S_0(\mathbf{w}^3)), & \text{if } l = 1, \\
(S_0(\mathbf{w}^5) - |\mathbf{w}|^{n-1} S_0(\mathbf{w}^4)), & \text{if } l = 2, 
\end{array} \right.$$  

$$= |\mathbf{w}|^{e+2nk} \left\{ \begin{array}{ll}
1 - |\mathbf{w}|^{-1}, & \text{if } e = 3k, \\
S_1(\mathbf{w}) + 1 - |\mathbf{w}|^{n-1}, & \text{if } e = 3k + 1, \\
|\mathbf{w}|^{n} - |\mathbf{w}|^{n-1}, & \text{if } e = 3k + 2.
\end{array} \right.$$  

Thus, $|\mathbf{w}|^{-en} S_{\mathbf{w}^e}(0) \ll |\mathbf{w}|^{-2n+5}$ for $e \geq 5$. Putting everything together, we obtain

$$\mathcal{G} = q^{-n+1} \prod_{\mathbf{w} \text{ prime} \atop \mathbf{w} \neq t} \left( 1 + O(|\mathbf{w}|^{-\frac{2}{3}+1}) \right),$$

where the implied constant is independent of $q$.

Then, there exists a constant $c$, that is independent of $q$, such that

$$\log q^{n-1} \mathcal{G} = \sum_{\mathbf{w} \text{ prime} \atop \mathbf{w} \neq t} \log \left( 1 + \frac{c}{\mathbf{w}^{\frac{2}{3}-1}} \right) = \sum_{d \geq 1} \sum_{\mathbf{w} \equiv q^d \atop \mathbf{w} \neq t} \sum_{m \geq 1} \frac{1}{m} \left( \frac{c}{\mathbf{w}^{\frac{2}{3}-1}} \right)^m = O(q^{2-\frac{2}{3}}),$$

by the same argument as in (2,1). Since $\exp(z) = 1 + O(|z|)$, we have $q^{n-1} \mathcal{G} = 1 + O(q^{2-\frac{2}{3}})$, which concludes the proof.  

Thus for $n \geq 10$, we have

$$M(d) = \mathcal{G} \mathfrak{J} |P|^{n-3} + O(q^{11n/12+2/3}|P|^{3n/4-1}),$$

where $\mathcal{G}$ and $\mathfrak{J}$ are given by Lemma 4.2 and (4.11), respectively. Note that the error term is satisfactory for Theorem 1.1.
5. Error term

There is a satisfactory contribution to $N(d)$ from $|\theta| < \hat{Q}^{-5}$, since by (3.3), such terms contribute

$$< \sum_{\substack{r \text{ monic} \\ |r| \leq \hat{Q}}} \hat{Q} \cdot \hat{Q}^{-5} |P|^n < \hat{Q}^{-3} |P|^n < |P|^{n-9/2} < |P|^{n-3}.$$  

Thus, we focus on the contribution from $|\theta| \geq \hat{Q}^{-5}$. As in [1], let $Y, \Theta \in \mathbb{Z}$ be such that

$$0 \leq Y \leq Q, \quad -5Q \leq \Theta < -(Y + Q). \quad (5.1)$$

We will analyse the contribution to $N(d)$ coming from $c \neq 0$ and $r, \theta$ such that $|r| = \hat{Y}$ and $|\theta| = \hat{\Theta}$. Denote this contribution by $E(d) = E(d; Y, \Theta)$. This section is similar to [1, Section 7.4] and [1, Section 8], however we need to consider separately the cases when $t | r$ and $t \nmid r$. Thus, let

$$B = \begin{cases} 0, & \text{if } t | r, \\ 1, & \text{if } t \nmid r. \end{cases}$$

Moreover, note that Lemma 3.1 imposes a constraint on $|c|$. More precisely,

$$|c| \leq \hat{C} = q|\tilde{r}| |P|^{-1} J(\Theta) = q^{B+1} |r| |P|^{-1} J(\Theta),$$

where $\tilde{r} = rt^B$ and

$$J(\Theta) = \max \left\{ 1, \hat{\Theta} |P|^3 \right\}. \quad (5.2)$$

Then $E(d)$ is given by

$$\sum_{c \in \mathcal{O}^n \atop 0 < |c| \leq q^{B+1} \hat{Y} |P|^{-1} J(\Theta)} |P|^n \sum_{r \in \mathcal{O} \atop |r| = Y \atop r \text{ monic}} q^{-Bn} |r|^{-n} \int_{|\theta| = \hat{\Theta}} S_{r, t, a}(c) I_r(\theta; c) \, d\theta.$$

By Lemma 3.3, $|I_r(\theta; c)| \ll L(\Theta)$, where

$$L(\Theta) = \min \left\{ q^{-n}, q^{\hat{\Theta} - \frac{n}{2}} |P|^{-\frac{3n}{2}} \right\}. \quad (5.3)$$

Thus, $E(d)$ is

$$\ll |P|^n \sum_{c \in \mathcal{O}^n \atop c \neq 0 \atop |c| \leq q^{B+1} \hat{Y} |P|^{-1} J(\Theta)} \sum_{r \in \mathcal{O} \atop |r| = Y \atop r \text{ monic}} q^{-Bn} |r|^{-n} \int_{|\theta| = \hat{\Theta}} |S_{r, t, a}(c)| L(\Theta) \, d\theta. \quad (5.4)$$

Moreover, since we must have $c \neq 0$ and $c \in \mathcal{O}^n$, we get the following bound

$$\hat{Y} \geq \frac{|P|}{J(\Theta) q^{B+1}}. \quad (5.5)$$
Let $S$ be a set of finite primes to be decided upon in due course but which contains $t$. Any $r \in \mathcal{O}$ can be written as $r = b_1 b''_1 r_2$, where $b_1'$ is square free such that $\varpi \mid b_1' \Rightarrow \varpi \in S$ and $b_1''$ is square-free coprime to $S$. According to Lemma $5.6$ there exist $M_1, M_2 \in \{1, t\}$ such that $M_1 \mid b_1'$ and $M_2 \mid r_2$, together with $b_1, b_2 \in (\mathcal{O}/t\mathcal{O})^n$ such that

$$|S_{r,t,a}(\mathcal{C})| = |S_{b_1',1,0}(\mathcal{C}) S_{b_1',M_1,b_1}(\mathcal{C}) S_{r_2,M_2,b_2}(\mathcal{C})|. \quad (5.6)$$

Clearly, $t \nmid b_1''$ for any $r$ and

$$\begin{cases} M_1 = t, M_2 = 1, & \text{if } t \mid r, \\ M_1 = 1, M_2 = t, & \text{if } t^2 \mid r, \\ M_1 = 1, M_2 = 1, & \text{otherwise.} \end{cases}$$

Moreover, by [1] Lemma 2.2 we have

$$\int_{[\varpi] = \hat{\Theta}} d\varpi = \hat{\Theta} + 1 - \hat{\Theta} \leq \hat{\Theta} + 1.$$ 

Let $\mathcal{O}^d = \{ b \in \mathcal{O} : b$ is monic and square-free$\}$. There exist $b_1, b_2 \in (\mathcal{O}/t\mathcal{O})^n$ such that the bound for $E(d)$ in (5.4) becomes

$$\ll \sum_{c \in \mathcal{O}^n} \frac{|P|^{n\hat{\Theta} + 1L(\hat{\Theta})}}{q^B n Y^{\frac{n-1}{2}}} \sum_{r_2 \in \mathcal{O}} \sum_{b_1' \in \mathcal{O}^d} \frac{|S_{b_1',M_1,b_1}(\mathcal{C}) S_{r_2,M_2,b_2}(\mathcal{C})|}{|b_2'|^{\frac{n+1}{2}}} \cdot S,$$

where

$$S = \sum_{b_1' \in \mathcal{O}^d} \frac{S_{b_1',1,0}(\mathcal{C})}{|b_1'|^{\frac{n+1}{2}}}.$$

From now put $b = b''_1$, and $d = b_1' r_2$, for simplicity. Also, write $S_b(\mathcal{C}) = S_{b,1,0}(\mathcal{C})$. Moreover, take

$$S = \begin{cases} \{ \varpi : \varpi \mid t F^*(\mathcal{C}) \}, & \text{if } F^*(\mathcal{C}) \neq 0, \\ \{ \varpi : \varpi \mid t \}, & \text{otherwise.} \end{cases}$$

**Lemma 5.1.** We have

$$S \ll \begin{cases} \hat{Y} \frac{\hat{Y}}{|d|}, & \text{if } F^*(\mathcal{C}) \neq 0, \\ \hat{Y} \frac{1+1/2,}{|d|} & \text{otherwise,} \end{cases}$$

where $F^*$ is the dual form of $F$. \

Proof. Recall that in our case $|\Delta_F| = 1$. Furthermore, by [8 Lemma 12] and [10 Lemma 60], there exists a constant $A(n) > 0$ depending only on $n$ such that for a prime $\varpi$ we have

$$S_{\varpi}(c) \leq A(n)|\varpi|^{n+1} |(\varpi,F^*(c))|^{1/2}. \quad (5.7)$$

By Lemma 5.6, $S_b(c)$ is a multiplicative function of $b$. Thus, by (5.7) and Lemma 2.4 we have

$$S = \sum_{b \in \mathcal{O} \atop (b,S) = 1} \frac{|S_b(c)|}{|b|^{n+1}} \ll \sum_{b \in \mathcal{O} \atop (b,S) = 1} |(b,F^*(c))|^{1/2} |b|^\varepsilon \ll \sum_{b \in \mathcal{O} \atop (b,S) = 1} |(b,F^*(c))|^{1/2}. \quad (5.8)$$

The definition of $S$ and the constraint that $(b,S) = 1$ imply that

$$(b,F^*(c)) = \begin{cases} 1, & \text{if } F^*(c) \neq 0, \\ b, & \text{otherwise,} \end{cases}$$

and this concludes the proof. \qed

Thus, we will consider separately the case when $F^*(c) \neq 0$ and the case when $F^*(c) = 0$. Denote the contributions to $E(d)$ coming from $c$ such that $F^*(c) \neq 0$, respectively $F^*(c) = 0$, by $E_1(d)$, respectively $E_2(d)$.

5.1. Treatment of the generic term. Suppose $F^*(c) \neq 0$. Then, by the first part of Lemma 5.1 we have

$$E_1(d) \ll \frac{|P^n \Theta + 1L(\Theta)\hat{Y}^{\frac{4n}{3} + \varepsilon}}{q^{Bn}} \sum_{e \in \mathcal{O}^n \atop c \neq 0 \atop |c| \leq q^{B+1}\hat{Y}|P|^{-1}J(\Theta)} R_1(c),$$

where

$$R_1(c) = \sum_{r_2 \in \mathcal{O} \atop |r_2| \leq \frac{1}{|b'_1|}} \sum_{b'_1 \in \mathcal{O}^2 \atop \varpi |b'_1 \Rightarrow \varpi \in S} \frac{|S_{b'_1,M_1,b_1}(c)S_{r_2,M_2,b_2}(c)|}{|b'_1 r_2|^{n+1}}.$$

Then, (5.7) implies that

$$R_1(c) \ll \sum_{b'_1 \in \mathcal{O}^2 \atop \varpi |b'_1 \Rightarrow \varpi \in S} \frac{1}{|b'_1|^{1-\varepsilon}} \sum_{r_2 \in \mathcal{O} \atop |r_2| \leq \hat{Y}} \frac{|S_{r_2,M_2,b_2}(c)|}{|r_2|^{n+1/2}} \ll \sum_{r_2 \in \mathcal{O} \atop |r_2| \leq \hat{Y}} \frac{|S_{r_2,M_2,b_2}(c)|}{|r_2|^{n+1/2}}.$$
since by Lemma 2.4 we have
\[
\sum_{\omega|\omega' = \omega \in S} \frac{1}{|\omega'|^{1-\epsilon}} = \prod_{\omega \in S} \left(1 - \frac{1}{|\omega|^{1-\epsilon}}\right)^{-1} = \prod_{\omega \in S} \sum_{k=0}^{\infty} \frac{1}{\omega k(1-\epsilon)} \leq \prod_{\omega \in S} C_\epsilon \ll |P|^{\epsilon}.
\]

Decompose \( r_2 \) as \( b_2 r_3 \). Then, by the multiplicativity property in Lemma 5.6 we have \( |S_{r_2,M_2,b_2}(c)| \leq |S_{b_2,M'_2,b'_2}(c)S_{r_3,M_3,b_3}(c)| \), for appropriate \( M'_2, M_3 \in \{1, t\} \) and \( b'_2, b_3 \in (\mathcal{O}/t\mathcal{O})^n \). Thus,
\[
\sum_{r_2 \in \mathcal{O}} \sum_{\omega|\omega' = \omega \in Y} \frac{|S_{r_2,M_2,b_2}(c)|}{|r_2|^{\frac{n+3}{2}}} \leq \sum_{b_2 r_3 \in \mathcal{O}} \sum_{\omega|\omega' = \omega \in Y} \frac{|S_{b_2,M'_2,b'_2}(c)S_{r_3,M_3,b_3}(c)|}{|b_2 r_3|^{\frac{n+3}{2}}}.
\]

Moreover, applying Lemma 3.3 with \( |M_3| \leq q \),
\[
\sum_{\omega|\omega' = \omega \in Y} |S_{r_3,M_3,b_3}(c)| \ll q^n |P|^{\epsilon} \left(\hat{Y}^{n/3-1/6} + \hat{C}^n\right).
\]

**Lemma 5.2.** For \( M'_2 \) and \( b'_2 \) as above and \( Y \in \mathbb{Z} \), there exists some \( \epsilon > 0 \) such that
\[
\sum_{b_2 \in \mathcal{O}} \frac{|S_{b_2,M'_2,b'_2}(c)|}{|b_2|^{\frac{n+3}{2}}} \ll q^{-2} |P|^{\epsilon},
\]
for any \( c \in \mathcal{O}^n \).

**Proof.** Suppose \( M'_2 = 1 \) so that \( S_{b_2,M'_2,b'_2}(c) = S_{b_2,1,0}(c) \). By \( \prod_{\omega}(5.3) \), together with the fact that in our case \( |\Delta_F| = 1 \), there exists a constant \( A(n) > 0 \) depending only on \( n \) such that
\[
S_{\omega^2}(c) \leq A(n)|\omega|^{n+1} |(\omega, F^*(c))|.
\]
Then, by the multiplicativity property in Lemma 5.6 and by (5.9),
\[
\sum_{b_2 \in \mathcal{O}} \frac{|S_{b_2,1,0}(c)|}{|b_2|^{\frac{n+3}{2}}} = \sum_{k_2 \in \mathcal{O}} \frac{|S_{k_2,1,0}(c)|}{|k_2|^{n+3}} \ll \sum_{k_2 \in \mathcal{O}} \frac{|A(n)|^{\omega(k_2)} |k_2|^{n+1} |(k_2, F^*(c))|}{|k_2|^{n+3}}.
\]
It follows from Lemmas 2.4 and 2.5 that this can be bounded by
\[
\ll |P|^{\epsilon} \sum_{k_2 \in \mathcal{O}} \frac{|(k_2, F^*(c))|}{|k_2|^2} \ll |P|^{\epsilon} \sum_{k_2 \in \mathcal{O}} \frac{1}{|k_2|^2} \ll |P|^{\epsilon},
\]
as required.
We now consider the case $M'_2 = t$. First, we need to bound the sum

$$S_{t^2, t, b'_2}(c) = \sum_{|a| < |t|^2} \sum_{y \in \mathcal{O}^n \mid y \equiv b'_2 \mod t} \psi \left( \frac{aF(y) - c \cdot y}{t^2} \right).$$

Making a change of variables $y = b'_2 + tz$, we get

$$S_{t^2, t, b'_2}(c) = \psi \left( \frac{-c \cdot b'_2}{t^2} \right) \sum_{|a| < |t|^2} \psi \left( \frac{aF(b'_2)}{t^2} \right) \sum_{z \in \mathcal{O}^n \mid |z| < |t|} \psi \left( \frac{a \cdot z \cdot F(b) - c \cdot z}{t} \right).$$

But Lemma 2.1 implies that

$$\sum_{z \in \mathcal{O}^n \mid |z| < |t|} \psi \left( \frac{a \cdot z \cdot F(b) - c \cdot z}{t} \right) = \begin{cases} q^n, & \text{if } |a \cdot \nabla F(b) - c| < 1, \\ 0, & \text{otherwise}, \end{cases}$$

and hence

$$|S_{t^2, t, b'_2}(c)| \leq q^n \left| \mathcal{O} : |a| < |t|^2 : |a \cdot \nabla F(b) - c| < 1 \right| \leq q^n. \quad (5.10)$$

By the definition of $S_{b'_2, M'_2, b'_2}$ and the multiplicativity property in Lemma 5.6, we can write it as $S_{b'_2, t, b'_2}(c) = S_{t^2, t, b'_2}(c) S_{(k_2/t)^2, 1, 0}(c)$. The second sum is well understood and can be bounded using (5.9), giving

$$|S_{(k_2/t)^2, 1, 0}(c)| \ll |P|^\varepsilon |b_2|^{\frac{n+2}{q^{n+2}}.}$$

Then, by (5.10) we have $|S_{b'_2, t, b'_2}(c)| \ll q^{-2} |P|^\varepsilon |b_2|^{\frac{n+2}{q^{n+2}}}$. Hence,

$$\sum_{b_2 \in \mathcal{O} \mid |b_2| \leq \hat{Y}} \frac{|S_{b'_2, t, b'_2}(c)|}{|b_2|^{\frac{n+2}{q^{n+2}}}} \ll q^{-2} |P|^\varepsilon \sum_{b_2 \in \mathcal{O} \mid |b_2| \leq \hat{Y}} \frac{1}{|b_2|^{\frac{2}{q}}} = q^{-2} |P|^\varepsilon \sum_{k_2 \in \mathcal{O} \mid |k_2| \leq \hat{Y}^{1/2}} \frac{1}{|k_2|} \ll q^{-2} |P|^\varepsilon,$$

by Lemma 2.5.

Thus, putting everything together,

$$\sum_{c \in \mathcal{O}^n \mid |c| \leq q^{B+1} \hat{Y} \cdot J(\Theta)/|P|} R_1(c) \ll q^{n-2} |P|^\varepsilon \left( \hat{Y}^{n/3-1/6} + \left( q^{B+1} \hat{Y} |P|^{-1} J(\Theta) \right)^n \right),$$

and hence,

$$E_1(d) \ll q^{n-2} |P|^\varepsilon \hat{Y}^{1/2} + L(\Theta) \left( \frac{|P|^n}{q^{Bn} \hat{Y}^{n/2}} + q^n J(\Theta) \hat{Y}^{n+2} \right).$$
Then, by (5.3) and (5.5), the first term is
\[ \ll q^{2n - \frac{7}{6} - B \left( \frac{1}{2} - \delta \right)} P^{\frac{2n}{3} + \frac{1}{6} + \varepsilon} J(\Theta)^{\frac{1}{3} - \frac{1}{2}} \min \left\{ q^{-n}, q^n \hat{\Theta}^{-\frac{2}{3}} |P|^{-\frac{2n}{3}} \right\}. \]
Noting that \( B \in \{0, 1\} \) and \( \min\{X, Z\} \leq X^u Z^v \) for any \( u, v \geq 0 \) such that \( u + v = 1 \), by (5.2), we obtain
\[ \ll q^{\frac{7n - 7}{6} - nu + nv} \hat{\Theta}^{1 - \frac{nu}{2}} |P|^\frac{3n}{6} + \frac{1}{3} - \frac{3nu}{2} + \varepsilon \max \left\{ 1, \hat{\Theta} |P|^3 \right\}^{\frac{n}{6} - \frac{4}{3}}. \]
If \( \hat{\Theta} |P|^3 \leq 1 \), take \( u = 1 - \frac{2}{n} \) and \( v = \frac{2}{n} \). Then, we obtain \( \ll q^{\frac{5n - 1}{3} + \varepsilon} |P|^{\frac{3n}{6} - \frac{5}{6} + \varepsilon} \).
Otherwise, if \( \hat{\Theta} |P|^3 > 1 \), take \( u = \frac{2}{3} + \frac{2}{3n} \) and \( v = \frac{1}{3} - \frac{2}{3n} \). Then, we get
\[ \ll q^{\frac{5n - 1}{3} + \varepsilon} |P|^{\frac{3n}{6} - \frac{5}{6} + \varepsilon}. \]
Similarly, by (5.2) and (5.3), the second term is
\[ \ll q^{2n - 1 - nu + nv} \hat{\Theta}^{1 - \frac{nu}{2}} \max \left\{ 1, \hat{\Theta} |P|^3 \right\} |P|^{3n} \frac{3n}{6} - \frac{3nu}{2} + \varepsilon \hat{Y}^{\frac{n+3}{2}}, \]
for any \( u, v \geq 0 \) such that \( u + v = 1 \). If \( \hat{\Theta} |P|^3 \leq 1 \), then by (5.1), we have
\[ \ll q^{2n - 1 + \varepsilon} \hat{\Theta}^{1 - \frac{nu}{2}} |P|^{\frac{3n}{6} + \frac{1}{3} - \frac{3nu}{2} + \varepsilon}. \]
Taking \( u = 1 - \frac{2}{n} \) and \( v = \frac{2}{n} \), we get \( \ll q^{n + 3} |P|^{\frac{3n}{6} - \frac{5}{6} + \varepsilon} \). Otherwise, if \( \hat{\Theta} |P|^3 > 1 \), we have
\[ \ll q^{2n - 1 - nu + nv} \hat{\Theta}^{1 + \nu} \frac{n}{2} |P|^{3n} \frac{3n}{6} - \frac{3nu}{2} + \varepsilon \hat{Y}^{\frac{n+3}{2}}. \]
On noting that the exponent of \( \hat{\Theta} \) is strictly positive for any \( v \leq 1 \), by (5.1), we have
\[ \ll q^{2n - 1 - nu + nv} |P|^{\frac{3n}{6} - \frac{3nu}{2} + \varepsilon} \hat{Y}^{\frac{1}{2} + \frac{1}{3} + \frac{n}{6}}. \]
Taking \( u = \frac{1}{n} \) and \( v = 1 - \frac{1}{n} \), we get \( \ll q^{3n - 3} |P|^{\frac{3n}{6} - \frac{5}{6} + \varepsilon} \).
Thus, for \( n \geq 10 \) we have \( E_1(d) \ll q^\frac{5n}{6} - \frac{11}{12} |P|^{\frac{5n}{6} - \frac{5}{6} + \varepsilon} + q^{3n - 3} |P|^{\frac{3n}{6} - \frac{3}{4} + \varepsilon}. \)
Hence,
\[ E_1(d) \ll |P|^\varepsilon \left( q^{\frac{3d + 4}{6} - \frac{5d + 16}{12}} + q^{\frac{3d + 5n}{6} - \frac{3d + 5}{4}} \right), \]
which is satisfactory for Theorem 1.1

5.2. Treatment of \( E_2(d) \). Suppose \( F^*(c) = 0 \). Denote the contribution to \( E(d) \) coming from \( c \) such that \( F^*(c) = 0 \) by \( E_2(d) \). Then, by the second part of Lemma 5.1 we have
\[ E_2(d) \ll \frac{|P|^n \hat{\Theta} + 1 \hat{Y}^{2 - \frac{5}{6} + \varepsilon} L(\Theta)}{q^B n} \sum_{\substack{c \in \mathcal{O}^n \\text{c.f.} \\mathcal{O}^n \\text{c.f.} \hat{\Theta} \\text{c.f.} \mathcal{O}^n}} R_2(c), \]
where \( |c| \ll q^{p+1} |P|^{-1} J(\Theta) \).
where
\[ R_2(c) = \sum_{r_2 \in O} \sum_{b_2' \in O} \frac{|S_{b_1, M_1, b_1}(c) S_{r_2, M_2, b_2}(c)|}{|b'_2 r_2|^\frac{3}{2} + 2}. \]

As in Section 5.1, decompose \( r_2 \) as \( b_2 r_3 \). Then, using Lemma 3.4 and \( |M_3| \leq q \), we get
\[ \sum_{c \in O^n} \sum_{r_3 \in O} \frac{|S_{r_3, M_3, b_3}(c)|}{|r_3|^\frac{3}{2} + 2} \ll q \frac{2n-5}{3} + \varepsilon R^{\frac{n}{3} - \frac{1}{2} + \varepsilon} \left( 1 + \hat{C} \right)^{n - \frac{3}{2} + \varepsilon}. \]

Then,
\[ \sum_{c \in O^n} \sum_{r_3 \in O} \frac{|S_{r_3, M_3, b_3}(c)|}{|r_3|^\frac{3}{2} + 2} \ll q \frac{2n-5}{3} + \varepsilon Y^{\frac{n}{2} - \frac{1}{2} + \varepsilon} \left( 1 + \hat{C}^{n - \frac{3}{2} + \varepsilon} \right) \quad (5.11) \]

Moreover, Lemma 5.2 implies that
\[ \sum_{b_2 \in O} \frac{|S_{b_2, M_1, b_1}(c)|}{|b_2|^\frac{3}{2} + 2} \ll q^{-2} |P|^\varepsilon. \quad (5.12) \]

It follows from (5.7) that
\[ R_2(c) \ll q^{-2} \sum_{b_2' \in O} \frac{|b'_2|^\varepsilon - 1}{|b_2' r_2|^{\frac{3}{2} + 2}} \sum_{r_2 \in O} \frac{|S_{r_2, M_2, b_2}(c)|}{|r_2|^\frac{3}{2} + 2}. \]

Then, by (5.11) and (5.12), we have
\[ \sum_{c \in O^n} \frac{R_2(c)}{|c| \leq \hat{Y}} \ll q^{\frac{2n}{3} - \frac{11}{3} \hat{Y}^{\frac{n}{2} - \frac{1}{2} + \varepsilon}} \left( 1 + (q^{B+1} \hat{Y} |P|^{-1} J(\Theta))^{n - \frac{3}{2}} \right). \]

Thus, we can bound \( E_2(d) \) by
\[ q^{2n - \frac{4}{3} |P|^\varepsilon} \theta \left( \frac{|P|^n L(\Theta)}{q^{3n} \hat{Y}^{\frac{n}{2} - \frac{1}{2}}} + q^{\frac{3(B+1)}{2}} |P|^\frac{3}{2} J(\Theta)^n \hat{Y}^{\frac{n+1}{2}} \right). \]

Then, by (5.3) and (5.5), the first term is
\[ \ll q^{\frac{2n}{3} - \frac{11}{3} B^{(\frac{3n}{2} - \frac{1}{2}) \theta} \min \left\{ q^{-n}, q^n \theta^{\frac{3}{2} |P|^{-\frac{3n}{2}}} \right\}} J(\Theta)^n \hat{Y}^{\frac{n+1}{2}} |P|^{\frac{3n}{2} + \frac{3}{2} + \varepsilon}. \]
Since \( B \in \{0, 1\} \) and \( \min\{X, Z\} \leq X^u Z^v \) for any \( u, v \geq 0 \) such that \( u + v = 1 \), by (5.2), we obtain
\[
q^{\frac{5n}{6} - \frac{25}{6} - nu + nv} \sum_{\text{max} \left\{ 1, \Theta \right\}} |P|^{3 - \frac{2n}{3} + \frac{3n + 3}{6}}.
\]

If \( \Theta |P|^3 \leq 1 \), take \( u = 1 - \frac{2}{n} \) and \( v = \frac{2}{n} \). Then, we obtain \( q^{\frac{5n}{6} - \frac{25}{6} - nu + nv} |P|^{\frac{3n}{2} - \frac{3n + 3}{6} + \varepsilon} \).

Otherwise, if \( \Theta |P|^3 > 1 \), take \( u = \frac{2}{3} + \frac{1}{n} \) and \( v = \frac{1}{3} - \frac{1}{n} \). Then, we get
\[
q^{\frac{5n}{6} - \frac{25}{6} - nu + nv} |P|^{\frac{3n}{2} - \frac{3n + 3}{6} + \varepsilon}.
\]

Similarly, by (5.2), (5.3) and (5.5), the second term is
\[
q^{\frac{5n}{6} - \frac{25}{6} - nu + nv} \sum_{\text{max} \left\{ 1, \Theta \right\}} |P|^{\frac{3n}{2} - \frac{3n + 3}{6} + \varepsilon} \max \left\{ 1, \Theta |P|^3 \right\} \frac{5n}{6} + \frac{5n}{2} - \frac{3n + 3}{6} + \varepsilon.
\]

Thus, for \( n \geq 1 \), we have
\[
E_2(d) \ll q^{\frac{5n}{6} - \frac{25}{6} - nu + nv} \sum_{\text{max} \left\{ 1, \Theta \right\}} |P|^{\frac{3n}{2} - \frac{3n + 3}{6} + \varepsilon}.
\]

Hence,
\[
E_2(d) \ll |P|^\varepsilon \left( q^{\frac{(5d + 8)n}{6} - \frac{3d + 14}{3} + \frac{(5d + 11)n}{4} - \frac{3d + 10}{12}} \right),
\]

which is satisfactory for Theorem 1.1.

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