Intrinsic measure of coherence for high-dimensional quantum systems

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The need for a measure of coherence which does not change under the change of basis has been recognized long before. The resource-theoretical framework proposed by Baumgratz et al., Phys. Rev. Lett. 113, 140401, (2014) for quantifying coherence in high-dimensional quantum systems is manifestly basis-dependent and has implications incongruent with the conventional understanding of coherence. In contrast, within the conventional, optical coherence theory, the coherence of a 2-dimensional system can be quantified in terms of a basis-independent quantity called the degree of polarization \( P_2 \). The quantity \( P_2 \) has five different known physical interpretations, namely: (i) It is the Frobenius distance between the state and identity matrix (ii) it is the norm of the Bloch-vector representing the state, (iii) it is the visibility of an interference experiment, (iv) it is the maximum pairwise-coherence of the state over all orthonormal bases, and (v) it is determined by the weightage of the pure part of the state. Recently, Yao et al., Sci. Rep. 6, 32010, (2016) have constructed a basis-independent measure of coherence \( P_N \) for \( N \)-dimensional quantum states by generalizing the Frobenius distance interpretation of \( P_2 \). In this paper, we demonstrate that all the remaining interpretations of \( P_2 \) generalizes to \( P_N \) as well. Our results theoretically establish the suitability of \( P_N \) as an intrinsic measure of coherence. This measure can be used for quantifying coherence in high-dimensional quantum states in orbital-angular-momentum and photon number bases.

I. INTRODUCTION

Coherence is the physical property that is responsible for the manifestation of interference and diffraction phenomena in nature. The rigorous characterization and quantification of coherence is the subject matter of optical coherence theory [1–7]. The classical theory of coherence quantifies the coherence between fields at a pair of space-time points using correlation functions that are directly related to the visibility of interference effects [2–4]. The advent of quantum mechanics called for the reformulation of the concept of coherence. Consequently, the quantum theory of coherence was formulated where the classical fields in the classical correlation functions were replaced by quantum field operators that in general may not commute [5–7]. This theory has been phenomenally successful in explaining interference effects involving classical and non-classical light fields. Along with space-time correlation functions, coherence of polarization states of light has also been studied, and for the polarization basis, a basis-independent measure of coherence called the degree of polarization has been formulated. The degree of polarization is an intrinsic, basis-independent measure of coherence for a two-dimensional system. But, such basis-independent quantification of coherence has not been possible beyond two dimensions. The necessity of basis-independent quantification of coherence was emphasized by Glauber in his seminal paper on quantum coherence where, concerning a set of conditions for full coherence, he stated that, “It is clear, however, that these conditions do not constitute an adequate definition of coherence, since they are not, in general, invariant under the rotation of coordinate axes.” [5]. In order for coherence to be considered as an intrinsic property of a system it must be quantified in a basis independent manner. So, the first step towards the extension of the theory of coherence is to find an intrinsic measure of coherence for states with dimensionality more than two.

In recent years, there have been efforts to quantify coherence for an arbitrary finite dimensional quantum system from a resource-theoretical viewpoint [8–15]. In particular, the paper by Baumgratz et al. laid down a formalism by identifying a set of criteria that a measure of coherence ought to satisfy [9]. Following this paper, several measures of quantum coherence have been proposed based on this formalism [11–15]. However, a recent study by Chitambar and Gour [16] has shown this formalism to have serious discrepancies. In particular, a generic resource theory defines free states (states that do not possess resource) and free operations (operations which cannot generate resource). The resource theory is mathematically consistent if free operations act invariently on the set of free states. However this mathematical restriction does not necessarily imply that the free operations can indeed be physically implemented without consuming any resource. Chitambar and Gour [16] have further pointed out that a resource theory can be deemed physically consistent only if all possible free operations can be realized by implementing some physical restriction. For instance, in the resource theory of entanglement, there is a natural class of free operations termed as local operations and classical communications (LOCC), which also turn out to be physically consistent because all LOCC operations can be realized by prohibiting the use of quantum channels. However, the class of free operations defined by Baumgratz et al. has not been shown to be physically consistent.

In addition, we identify some other features of the formalism that demand closer scrutiny, namely: (i) it is
manifestly basis-dependent, and therefore cannot provide an intrinsic measure of the coherence of a system. (ii) it defines free states as states whose off-diagonal elements are all zero in a chosen basis representation. While such a state has zero pairwise-coherence between any two basis vectors, it is not obvious why the entire state should be regarded as incoherent. This issue needs to be carefully addressed because the resource theory of coherence has been formulated in direct correspondence with the resource theory of entanglement [17–19], wherein it is known that the absence of pair-wise entanglement in a multipartite system does not necessarily imply that the full state is separable [20–22]. (iii) Even a measure, such as the $l_1$-norm of coherence defined as $\sum_{i\neq j}|\rho_{ij}|$ for a state $\rho$ [9] that satisfies all the criteria laid down by the formalism, does not regard all pure states to be equally coherent. Moreover, it sometimes ascribes a mixed state with more coherence than that of the pure states. This is not congruent with the understanding within the optical coherence theory that the coherence of a mixed state must always be lower than that of a pure state [1].

In view of these issues associated with the resource-theoretical approach, we adopt an alternative approach for quantifying coherence in high-dimensional quantum systems that traces its roots back to the optical coherence theory [1]. In particular, the optical coherence theory provides an excellent basis-independent measure of coherence, namely, the degree of polarization $P_2$, for two-dimensional (2D) polarization states of light [4]. The measure $P_2$ quantifies the intrinsic amount of polarization coherence of a partially polarized field, and can be viewed from a number of different interpretations. We believe that a systematic generalization of $P_2$ to higher dimensions that preserves all the physical interpretations would serve as an ideal measure of coherence for high-dimensional quantum systems. Recently, Yao et al. have taken precisely this approach and, by generalizing one such interpretation, namely, the Frobenius-distance interpretation, of $P_2$ to $N$ dimensions, have proposed an analogous measure, denoted as $P_N$ [23]. In this paper, we consider all the known physical interpretations of $P_2$, generalize each of the interpretations to $N$ dimensions, and show that all the interpretations identically yield $P_N$ as the generalized measure for $N$-dimensional (ND) states.

II. DEGREE OF POLARIZATION:

The polarization state of an arbitrary light field can be represented by a positive-semidefinite $2 \times 2$ Hermitian matrix, termed as the coherence matrix, that is defined as [1],

$$\rho = \begin{bmatrix} \langle E_x E_y^* \rangle & \langle E_x E_y^\dagger \rangle \\ \langle E_x^\dagger E_y \rangle & \langle E_x^\dagger E_y^\dagger \rangle \end{bmatrix},$$  \hspace{1cm} (1)

where $E_x$ and $E_y$ denote the electric field components along two orthonormal polarizations, and $\langle \cdots \rangle$ denotes an ensemble average over many realizations. The degree of polarization $P_2$ of $\rho$ can be computed as [3],

$$P_2 = \sqrt{1 - \frac{4 \det \rho}{|\text{Tr} \rho|^2}}. \hspace{1cm} (2)$$

Since the trace and the determinant are invariant under unitary operations, $P_2$ is a basis-independent quantity. Furthermore, $0 \leq P_2 \leq 1$ with $P_2 = 1$ only when $\rho$ is a perfectly-polarized field (pure state) and $P_2 = 0$ only when $\rho$ is the completely-unpolarized field represented by the identity matrix. The quantity $P_2$ is an intrinsic measure of the polarization coherence of the field. We note that since the polarization coherence matrix is formally identical to a 2D quantum density matrix, the measure $P_2$ can be used for quantifying the intrinsic coherence of any (2D) quantum state.

III. INTERPRETATIONS AND GENERALIZATIONS:

We now consider the five known interpretations of $P_2$ that justify its suitability as an intrinsic coherence measure in 2D. Following the description for each interpretation, we present its generalization to higher dimensions and obtain $P_N$ as the $N$-dimensional (ND) analog of $P_2$. In order to make the correspondence evident, we use the same symbol $\rho$ to denote a $2 \times 2$ or an $N \times N$ physical density matrix depending on the context. In both cases, we assume $\text{Tr} \rho = 1$. Also, $I_2$ and $I_N$ denote the $2 \times 2$ and $N \times N$ identity matrices respectively.

A. Frobenius Distance Interpretation

2D states

$P_2$ can be viewed as the Frobenius-distance between $\rho$ and the completely-incoherent state $I_2/2$ in the space of density matrices. Specifically,

$$\sqrt{2}\|\rho - \frac{I_2}{2}\|_F = \sqrt{2 \left( \text{Tr}(\rho^2) - \frac{1}{2} \right)} = P_2, \hspace{1cm} (3)$$

where the distance is quantified using the Frobenius-norm, defined as $\|A\|_F = \sqrt{\text{Tr}(A^\dagger A)}$. The preceding normalization-factor ensures that $0 \leq P_2 \leq 1$. It is straightforward to show that Eq.(2) and Eq.(3) are equivalent.

ND states

Following up on previous extensions by Setälä et al. to 3D states [24, 25], and Luís to 4D states [26], Yao et al. have generalized the Frobenius-distance interpretation...
of $P_2$ to $N$-dimensional states [23]. This study directly extended Eq. (3) to ND states and constructed,

$$P_N \equiv \sqrt{\frac{N}{N-1}} ||\rho - \frac{1}{N}||_F = \sqrt{\frac{N}{N-1} \left[ Tr(\rho^2) - \frac{1}{N} \right]}.$$  

(4)

In other words, they defined $P_N$ as the suitably normalized Frobenius-distance of the state $\rho$ from the completely-incoherent state $1_N/N$ in the space of $N \times N$ density matrices. The scaling factor in Eq. (4) is again chosen such that $0 \leq P_N \leq 1$. As seen previously for $P_2$, when $\rho$ is pure, $Tr(\rho^2) = 1$, which leads to $P_N = 1$, whereas when $\rho = 1_N/N$, we obtain $P_N = 0$. Interestingly, as pointed out by Yao et al., $P_N$ bears a direct relation with the square root of the Brukner-Zeilinger invariant information [27].

We now show that with the definition of the degree of coherence $P_N$ given by Yao et al. Eq. (4), all the other four interpretations of the degree of coherence $P_2$ in two dimensions generalize to $N$-dimensional systems.

### B. Bloch Vector Norm Interpretation

#### 2D states

It has been shown that any 2D state $\rho$ has a unique decomposition of the form [28],

$$\rho = \frac{1}{2} (1_2 + \sum_{i=1}^{3} r_i \sigma_i),$$  

(5)

where $\sigma_1, \sigma_2$ and $\sigma_3$ are the Pauli matrices, and the real scalar quantities $r_i$'s are referred to as the Stokes parameters of the state. Such a parametrization is possible due to the fact that $\sigma_i$'s, which are the generators of the Lie group $SU(2)$, form an orthonormal basis in the real vector space of traceless $2 \times 2$ Hermitian matrices with respect to the Hilbert-Schmidt inner-product, $(A,B) \equiv Tr (A^\dagger B)$. Consequently, the parameters $r_i$, can be regarded as the components of a 3-dimensional vector $\vec{r} \equiv (r_1, r_2, r_3)$, termed as the Bloch vector, that represents the state $\rho$ in this vector space. For a 2D density matrix $\rho$, the condition $Tr \rho^2 \leq 1$ is both necessary and sufficient to ensure positive-semidefiniteness, which in turn implies that the space of physical states is characterized by $0 \leq |\vec{r}| \leq 1$. This space can be imagined to be a closed sphere in 3 dimensions, termed as the Bloch sphere. The pure states reside on the surface of this sphere with $P_2 = |\vec{r}| = 1$, whereas the maximally incoherent state $1_2/2$ with $P_2 = |\vec{r}| = 0$ resides at the center of the sphere. In fact, from Eq. (2), it has been shown that the norm of the Bloch vector is equal to $P_2$ [28], that is,

$$|\vec{r}| = \sqrt{\sum_{i=1}^{3} |r_i|^2} = P_2.$$  

(6)

Thus, $P_2$ can be interpreted as the norm of the Bloch vector representing the state.

#### ND states

In direct correspondence to Eq. (5), it has been shown that any ND state $\rho$ can be decomposed as [29–32],

$$\rho = \frac{1}{N} (1_N + \sqrt{\frac{N(N-1)}{2}} \sum_{i=1}^{(N^2-1)} r_i \Lambda_i),$$  

(7)

where $\Lambda_i$'s are the generalized $N \times N$ Gellmann matrices, and the scalar quantities $r_i$'s are the $N$-dimensional analogs of Stokes parameters. In exact analogy with the 2D case, this parametrization is made possible by the fact that $\Lambda_i$'s, which are the $(N^2 - 1)$ generators of the Lie group $SU(N)$, form an orthonormal basis in the real vector space of traceless $N \times N$ Hermitian matrices with respect to the Hilbert-Schmidt inner-product. The parameters $r_i$ form the components of the $(N^2-1)$-dimensional Bloch vector $\vec{r}$ representing the state $\rho$. We note that in contrast with the 2D case, the condition $Tr \rho^2 \leq 1$ is not sufficient to ensure positive-semidefiniteness of ND density matrices. Consequently, only a subset of states represented by the $(N^2-1)$-dimensional sphere and defined by $0 \leq |\vec{r}| \leq 1$ correspond to physical states [30, 31]. Nevertheless, we now explicitly show that for a physical state, the norm of the Bloch vector is equal to $P_N$ of Eq. (4). The set of $(N^2 - 1)$ generalized Gellmann matrices $\Lambda_i$'s comprises of three subsets: the set $\{U\}$ of $N(N-1)/2$ symmetric matrices, the set $\{V\}$ of $N(N-1)/2$ anti-symmetric matrices, and the set $\{W\}$ of $(N-1)$ diagonal matrices. The explicit forms of these matrices in a particular orthonormal basis $\{\{i\}_N\}_{i=1}^N$ are [30],

$$U_{jk} = |j\rangle \langle k| + |k\rangle \langle j|,$$  

(8a)

$$V_{jk} = -i|j\rangle \langle k| + |k\rangle \langle j|,$$  

(8b)

$$W_l = \sqrt{\frac{2}{l(l+1)}} \left( \sum_{m=1}^{l} |m\rangle \langle m|-l(l+1)l+1 \right),$$  

(8c)

where $1 \leq j < k \leq N$ and $1 \leq l \leq (N-1)$. In terms of these definitions, we write Eq. (7) as,

$$\rho = \frac{1}{N} \left[ \frac{1}{N} + \sqrt{\frac{N(N-1)}{2}} \sum_{j=1}^{N} \sum_{k=j+1}^{N} \{u_{jk} U_{jk} + v_{jk} V_{jk} \} + \sum_{i=1}^{N-1} w_i W_i \right],$$  

(9)

where $u_{jk}, v_{jk}$ and $w_i$ are the Bloch-vector components along the Gellmann matrices $U_{jk}, V_{jk}$ and $W_i$ respectively. Here, we have relabeled the set of components
\( \{ r_i \} \) and the set of matrices \( \{ \Lambda_i \} \) of Eq. (7) by the set of parameters \( \{ \{ u_{jk} \}, \{ v_{jk} \}, \{ w_i \} \} \) and the set of matrices \( \{ \{ U_{jk} \}, \{ V_{jk} \}, \{ W_i \} \} \), respectively. The norm of the Bloch vector \( \vec{r} \) is therefore given by,

\[
|\vec{r}| = \sqrt{\sum_{i=1}^{(N^2-1)} r_i^2} = \sqrt{\sum_{j=1}^{N} \sum_{k=j+1}^{N} \left[ u_{jk}^2 + v_{jk}^2 \right] + \sum_{l=1}^{N-1} w_l^2}. \tag{10}
\]

We calculate the components \( u_{jk}, v_{jk} \) and \( w_l \) in terms of the density matrix elements and find them to be

\[
u_{jk} = \sqrt{\frac{N}{2(N-1)}} (\rho_{jk} + \rho_{kj}), \tag{11a}\]

\[
v_{jk} = i \sqrt{\frac{N}{2(N-1)}} (\rho_{jk} - \rho_{kj}), \tag{11b}\]

\[
w_l = \frac{N}{l(l+1)(N-1)} \left( \sum_{m=1}^{l} \rho_{mm} - l \rho_{l+1,l+1} \right). \tag{11c}\]

In order to evaluate \( |\vec{r}| \) using Eqs. (10) and (11), we first find that

\[
\sum_{j=1}^{N} \sum_{k=j+1}^{N} \left[ u_{jk}^2 + v_{jk}^2 \right] = \frac{2N}{N-1} \sum_{j=1}^{N} \sum_{k=j+1}^{N} |\rho_{jk}|^2. \tag{12}\]

We then evaluate the other summation in Eq. (10) to be

\[
\sum_{l=1}^{N-1} w_l^2 = \sum_{l=1}^{N-1} \frac{l(l+1)(N-1)}{\left( \sum_{m=1}^{l} \rho_{mm} - l \rho_{l+1,l+1} \right)^2} \left( \sum_{m=1}^{l} \rho_{mm} - l \rho_{l+1,l+1} \right)^2
= \frac{N}{N-1} \sum_{l=1}^{N-1} \left[ \frac{1}{l(l+1)} \sum_{m=1}^{l} \rho_{mm} + l^2 \rho_{l+1,l+1}^2 \right. \\
\left. \quad + \frac{2}{l} \sum_{m=1}^{l} \sum_{s=m+1}^{l} \rho_{mm} - (l \rho_{l+1,l+1}) \right] \sum_{m=1}^{l} \rho_{mm} + l^2 \rho_{l+1,l+1}^2 \\
= \frac{N}{N-1} \sum_{i=1}^{N} \rho_{ii} \left\{ \sum_{j=i}^{N-1} \frac{1}{j(j+1)} + \frac{i-1}{i} \right\} \\
\quad - \frac{2}{N-1} \sum_{i=1}^{N} \sum_{j=i+1}^{N} \rho_{ii} \rho_{jj} \\
= \sum_{i=1}^{N} \rho_{ii}^2 - \frac{2}{N-1} \sum_{i=1}^{N} \sum_{j=i+1}^{N} \rho_{ii} \rho_{jj}. \tag{13}\]

Substituting Eqs. (12) and (13) into Eq. (10) we obtain

\[
|\vec{r}| = \sqrt{\frac{N}{N-1} \left[ Tr(\rho^2) - \frac{1}{N} \right]} = P_N. \tag{14}\]

Thus \( P_N \), just like its two-dimensional analog, can be interpreted as the norm of the Bloch vector corresponding to the \( N \)-dimensional state.

### C. Visibility Interpretation

#### 2D states

The visibility interpretation of \( P_2 \) for a 2D state was given by Emil Wolf [4] using an experimental scheme. We will discuss the scheme with slight modifications in order to make it more amenable to generalization to higher dimensions. But the modified scheme is essentially equivalent to the one discussed in section 6.3 of Ref. [1]. Consider a light beam in the polarization state \( \rho \) as shown in Fig. 1a. The beam passes through a wave-plate (WP) that introduces a phase \( \delta \) between the horizontal and vertical polarizations, and subsequently through a rotation plate (RP) which rotates the polarization vector by an angle \( \theta \). The WP and RP together implement two-dimensional unitary transformations \( (U(2)) \) on the polarization state. Then the light passes through a polarizing beam splitter (PBS) which divides the light into two orthogonal polarizations. PBS is is a device with two output ports. We denote the intensity recorded at the two ports as \( I_1 \) and \( I_2 \). The visibility \( V \) of the interference is given by

\[
V = \max_{U \in U(2)} f(I_1, I_2) = \max_{U \in U(2)} \left| I_1 - I_2 \right| \tag{15}\]

where \( U(2) \) is the group of 2D unitary matrices representing the joint operations by WP and RP. The definition of function \( f \) given in Eq. (15) is same as the definition of Michelson’s visibility \( (I_1 - I_2)/(I_1 + I_2) \), with the following properties: (i) It is 1 if and only if one of the \( I_1 \)s is 1 and other is 0, (ii) It is 0 if and only if \( I_1 = I_2 \), (iii) It is a Schur-convex function, i.e, for any two intensity distributions \( \{ I_1, I_2 \} \) and \( \{ J_1, J_2 \} \) such that \( \{ J_1, J_2 \} \) majorizes \( \{ I_1, I_2 \} \) we have \( f(I_1, I_2) \leq f(J_1, J_2) \) [33]. The maximization involved in Eq. (15) can be carried out using Schur’s theorem which states that the measured probability distribution of a state in any basis is majorized by the eigenvalue distribution of the state \( \{ \lambda_1, \lambda_2 \} \), i.e, \( \{ I_1, I_2 \} \preceq \{ \lambda_1, \lambda_2 \} \). There always exists a unitary transformation such that \( I_1 = \lambda_1 \) and \( I_2 = \lambda_2 \), and as \( f \) is a Schur-convex function, \( f \) becomes maximum when its arguments are the eigenvalues of the 2D state \( \rho \). Therefore, we get

\[
V = f(\lambda_1, \lambda_2) = \left| \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2} \right| = P_2, \tag{16}\]

that is, \( P_2 \) is equal to the visibility.

#### ND states

In direct analogy with scheme depicted in Fig. 1a, consider the situation shown in Fig. 1b. At first, the density matrix \( \rho \) is acted upon by a general \( N \times N \) unitary operator \( U \). Then, it passes through a \( N \)-port splitter (NPS) which divides the density matrix into \( N \) or-
thear of polarization coherence, another quantity that is often used is $\mu_2$, 

$$\mu_2 = \frac{|\rho_{12}|}{\sqrt{\rho_{11}\rho_{22}}}.$$  

Here $\rho_{ij}$ for $i, j = 1, 2$ denote the matrix elements of $\rho$ in the $\{|1\}, |2\rangle$ basis. Essentially, $\mu_2$ quantifies the degree of correlation between the field components along the pair of orthonormal polarizations corresponding to $|1\rangle$ and $|2\rangle$, and so we refer to it as the degree of pairwise coherence. $\mu_2$ is a manifestly basis-dependent quantity and it relates polarization correlation to the non-diagonal elements of a density matrix. Using Eqs. (2) and (20), it can be shown that $0 \leq \mu_2 \leq P_2$. In fact, $\mu_2$ can be shown to attain its maximum value $\mu_2 = P_2$ when the basis vectors $\{|1\}, |2\rangle \rangle$ are chosen such that $\rho_{11} = \rho_{22}$. Thus, we have

$$\max_{\{|1\}, |2\rangle \rangle} \mu_2 = P_2, \quad (21)$$

i.e., $P_2$ can also be interpreted as the maximum value attainable by the degree of pair-wise coherence $\mu_2$ over the set $S$ of all orthonormal bases.

**D. Maximum Pairwise Coherence Interpretation**

### 2D states

In the context of polarization coherence, another quantity that is often used is $\mu_2$, 

$$\mu_2 = \frac{|\rho_{12}|}{\sqrt{\rho_{11}\rho_{22}}}.$$  

Here $\rho_{ij}$ for $i, j = 1, 2$ denote the matrix elements of $\rho$ in the $\{|1\}, |2\rangle$ basis. Essentially, $\mu_2$ quantifies the degree of correlation between the field components along the pair of orthonormal polarizations corresponding to $|1\rangle$ and $|2\rangle$, and so we refer to it as the degree of pairwise coherence. $\mu_2$ is a manifestly basis-dependent quantity and it relates polarization correlation to the non-diagonal elements of a density matrix. Using Eqs. (2) and (20), it can be shown that $0 \leq \mu_2 \leq P_2$. In fact, $\mu_2$ can be shown to attain its maximum value $\mu_2 = P_2$ when the basis vectors $\{|1\}, |2\rangle \rangle$ are chosen such that $\rho_{11} = \rho_{22}$. Thus, we have

$$\max_{\{|1\}, |2\rangle \rangle} \mu_2 = P_2, \quad (21)$$

i.e., $P_2$ can also be interpreted as the maximum value attainable by the degree of pair-wise coherence $\mu_2$ over the set $S$ of all orthonormal bases.

**ND states**

We now generalize the pair-wise correlation function $\mu_2$ of Eq. (20) for N-dimensional systems. For this we need to find the N-D analog of $\mu_2$ which must reduce to $\mu_2$ for $N = 2$. Moreover, in analogy with 2-dimensional case, the N-dimensional quantity should have the non-diagonal terms in numerator and product of diagonal terms in denominator. We define

$$\mu_N = \sqrt{\frac{\sum_{i=1}^{N} \sum_{j=i+1}^{N} |\rho_{ij}|^2}{\sum_{i=1}^{N} \sum_{j=i+1}^{N} \rho_{ii}\rho_{jj}}}, \quad (22)$$

The maximum attainable value of $\mu_N$ is then $P_N$ by virtue of Schur’s theorem [34], which states that $\{\rho_1, \rho_2, ..., \rho_N\} = \{\lambda_1, \lambda_2, ..., \lambda_N\}$. Consequently,

$$V = \max_{U \in U(N)} f(I_1, I_2, ..., I_N) = \frac{\sum_{i=1}^{N} \sum_{j=i+1}^{N} (\lambda_i - \lambda_j)^2}{(N-1)(\sum_{i=1}^{N} \lambda_i)^2} = \frac{N}{N-1} \left[ \frac{\sum_{i=1}^{N} \lambda_i^2}{\sum_{i=1}^{N} \lambda_i} - \frac{1}{N} \left( \sum_{i=1}^{N} \lambda_i \right)^2 \right] = \frac{N}{N-1} \left[ Tr(\rho^2) - \frac{1}{N} \right], \quad (19)$$

Thus, we have shown that $P_N$ can also be interpreted as the N-dimensional visibility of the intensity distribution.
where \( \rho_{ij} \) are the matrix elements of the state \( \rho \) in an orthonormal basis \( \{|1\}, \{|2\}, \ldots, \{|N\} \rangle \). The quantity \( \mu_N \) reduces to \( \mu_2 \) for \( N = 2 \) and it is a basis-dependent measure that quantifies the total amount of pairwise coherence across all possible pairs of basis vectors, and can be considered the \( N \)-dimensional analog of \( \mu_2 \). We now show that for a general state, \( \mu_N \) satisfies \( 0 \leq \mu_N \leq P_N \). From Eq. (4) and Eq. (22),

\[
1 - P_N^2 = 1 - \frac{N}{N-1} \left[ \text{Tr} \rho^2 - \frac{1}{N} \right] \\
= 1 - \frac{N}{N-1} \sum_{i=1}^{N} \rho_{ii}^2 - \frac{2N}{N-1} \sum_{i=1}^{N} \sum_{j=i+1}^{N} \left[ \rho_{ij}^2 - \frac{\rho_{ii} \rho_{jj}}{N} \right] \\
= \frac{2N}{N-1} (1 - \mu_N^2) \sum_{i=1}^{N} \rho_{ii}^2 - \frac{2N^2}{N(N-1)} \sum_{i=1}^{N} \sum_{j=i+1}^{N} \rho_{ii} \rho_{jj} \\
= \frac{2N}{N-1} (1 - \mu_N^2) \left[ 2 + \frac{\sum_{i=1}^{N} \rho_{ii}^2}{\sum_{i=1}^{N} \sum_{j=i+1}^{N} \rho_{ii} \rho_{jj}} \right]^{-1}.
\]

(23)

Now the inequality \( \sum_{i=1}^{N} \sum_{j=i+1}^{N} (\rho_{ii} - \rho_{jj})^2 \geq 0 \) implies that,

\[
\frac{\sum_{i=1}^{N} \rho_{ii}^2}{\sum_{i=1}^{N} \sum_{j=i+1}^{N} \rho_{ii} \rho_{jj}} \geq \frac{2}{N-1}.
\]

(24)

We substitute this relation in Eq. (23), and obtain \( \mu_N \leq P_N \). Moreover, \( \mu_N \) takes on the maximum value \( \mu_N = P_N \), when \( \rho_{ii} = \rho_{jj} \) for all \( i, j \). Therefore, in direct correspondence with Eq. (21), we have

\[
\max_{\{|\psi_i\rangle\} \in S} \mu_N = P_N,
\]

i.e., \( P_N \) can also be interpreted as the maximum value attainable by the degree of pair-wise coherence \( \mu_N \) over the set \( S \) of all orthonormal bases in the \( N \)-dimensional Hilbert space.

### E. Weightage of Pure Part Interpretation

#### 2D states

It has been shown that any 2D state \( \rho \) can be uniquely decomposed into a weighted mixture of a perfectly-coherent state and the completely-incoherent state \( \{|1\}, \{|3\} \rangle \), i.e.,

\[
\rho = s_1 |\psi_1\rangle \langle \psi_1| + (1 - s_1) \frac{1}{2},
\]

(26)

where \( |\psi_1\rangle \) is a pure state with weightage \( s_1 \). If we assume that the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) of \( \rho \) satisfy \( \lambda_1 > \lambda_2 \), it can be shown that \( |\psi_1\rangle \) is the eigenstate of \( \rho \) corresponding to the eigenvalue \( \lambda_1 \). We can express Eq. (26) in terms of the eigenvalues as,

\[
\rho = (\lambda_1 - \lambda_2) |\psi_1\rangle \langle \psi_1| + (2\lambda_2) \frac{1}{2},
\]

(27)

Comparing the above equation with Eq. (26) and Eq. (16), we find that weightage of the pure portion \( s_1 \) is given by,

\[
s_1 = \lambda_1 - \lambda_2 = P_2.
\]

(28)

Thus, \( P_2 \) has the interpretation that it is the weightage of the pure portion of the state.

#### ND states

We will now generalize this interpretation of \( P_2 \) to higher dimensions. Previous studies have shown that a decomposition of the form Eq. (26) involving a single pure state is in general not possible for higher dimensions \( [36] \). However, we now show that a generalized decomposition of the form,

\[
\rho = \sum_{i=1}^{N-1} s_i |\psi_i\rangle \langle \psi_i| + \left(1 - \sum_{i=1}^{N-1} s_i \right) \frac{1}{N}
\]

(29)

exists. Here the states \( \{|\psi_i\rangle\} \) are pure and orthonormal and the corresponding weightages \( s_i \)'s are real and non-negative. In order to ensure a unique decomposition for every physical density matrix, it must be verified that the number of independent parameters are identical on the two sides of Eq. (29). On the left side, the density matrix \( \rho \) has \( (N^2 - 1) \) free parameters. On the right side, we have \( \sum_{i=1}^{N-1} s_i = (N-1) \) \( s_i \)'s (ii) each of the \( (N-1) \) \( |\psi_i\rangle \)'s has \( 2(N-1) \) free parameters, and (iii) the mutual orthogonality between \( |\psi_i\rangle \)'s would introduce \( (N-1)(N-2) \) constraints. These conditions imply \( (N^2 - 1) \) free parameters on the right-hand side as well. If we introduce an additional vector \( |\psi_N\rangle \) to the set of \( (N-1) \) \( |\psi_i\rangle \)'s to form an orthonormal and complete basis, then the right-hand side of Eq. (29) is completely diagonal in this basis. This implies that the representation of \( \rho \) on the left-hand side must also be diagonal in this basis, i.e., \( |\psi_i\rangle \)'s have to be eigenvectors of \( \rho \). If we denote the corresponding eigenvalues as \( \lambda_i \) and assume \( \lambda_1 > \lambda_2 > \ldots > \lambda_N \), then Eq. (29) takes the form,

\[
\rho = \sum_{i=1}^{N-1} (\lambda_i - \lambda_N) |\psi_i\rangle \langle \psi_i| + \left(N \lambda_N\right) \frac{1}{N}
\]

(30)

In order for the weightages \( s_i = (\lambda_i - \lambda_N) \) to be all positive, the above decomposition is necessarily unique. In terms of these weightages, \( P_N \) takes the form,

\[
P_N = \sqrt{\sum_{i=1}^{N-1} s_i^2 + \frac{2N}{N-1} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N-1} (s_i - s_j)^2} \\
= \sqrt{\left(\sum_{i=1}^{N-1} s_i^2\right) - \frac{2N}{N-1} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N-1} s_is_j} \leq \sum_{i=1}^{N-1} s_i
\]

(31)
Thus, for $N$-dimensional states the total pure state weightage $\sum_{i=1}^{N-1} s_i$ determines an upper bound on $P_N$. The bound is saturated when only one of the $(N-1)$ $s_i$’s is non-zero, which happens when the state has only two non-zero eigenvalues, just as in the case of 2D states.

IV. CONCLUSION AND DISCUSSION

We have shown that generalization of all the known interpretation of degree of polarization $P_2$ for 2-dimensional state leads to $P_N$, the measure of coherence for a $N$-dimensional density matrix. We demonstrated that for an $N$-dimensional state $\rho$, the norm of the $(N^2 - 1)$-dimensional Bloch vector representing $\rho$ is equal to $P_N$. We then generalized the visibility interpretation of $P_2$, and showed that $P_N$ can be interpreted as the $N$-dimensional visibility. Thereafter, we defined a basis-dependent measure of the total pairwise coherence between pairs of basis vectors, and showed that the maximum pairwise coherence over all possible orthonormal basis representations is equal to $P_N$. Finally, we demonstrated that an $N$-dimensional state $\rho$ can be decomposed as a weighted sum of $(N-1)$ orthonormal states and the maximally-incoherent identity state and that $P_N$ is bounded from above by the total weight of the $(N-1)$ pure states. Therefore, our results establish the suitability of $P_N$ as an intrinsic measure of coherence for high-dimensional quantum systems.

We expect the measure $P_N$ to be directly relevant for photonic quantum states in two important high-dimensional bases. One is the infinite-dimensional orbital angular momentum basis, which has become extremely important for quantum information and quantum communication [37–40], and the other one is the infinite-dimensional photon-number basis, which has been explored extensively for studying multiphoton states such as squeezed states, thermal states and coherent states [41–45]. Moreover, we expect $P_N$ to be useful in studying decoherence processes, such as light propagation through noisy environments.

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