Two and three loops computations of renormalization constants for lattice QCD

F. Di Renzo\textsuperscript{a}, A. Mantovi\textsuperscript{a}, V. Miccio\textsuperscript{a}, L. Scorzano\textsuperscript{b}, C. Torrero\textsuperscript{a}

\textsuperscript{a}Dipartimento di Fisica, Università di Parma and INFN, Gruppo Collegato di Parma, Italy

\textsuperscript{b}Institüt für Physik, Humboldt Universität, Berlin, Germany

Renormalization constants can be computed by means of Numerical Stochastic Perturbation Theory to two/three loops in lattice perturbation theory, both in the quenched approximation and in the full (unquenched) theory. As a case of study we report on the computation of renormalization constants of the propagator for Wilson fermions. We present our unquenched ($N_f = 2$) computations and compare the results with non perturbative determinations.

\section*{INTRODUCTION & MOTIVATION}

Numerical Stochastic Perturbation Theory (NSPT) is a numerical method \cite{1} to perform Lattice Perturbation calculations. It has been shown to be very powerful in the quenched approximation, but the technique is quite general and can be in principle applied to any theory \cite{2}. In particular, it can be generalized to actually carry on unquenched QCD calculations \cite{3}. The aim is to use NSPT to look at interesting quantities like improvement and renormalization coefficients (quark propagator, currents, etc...), up to 2-3 loops and potentially for any fermionic action. In these pages we will present our results about the critical mass and the field renormalization constant (up to 3 and 2 loops respectively).

\section*{1. COMPUTATIONAL SETUP}

For a comprehensive account of the strategy of unquenched NSPT we refer the reader to \cite{3} and references therein.

Simulations are performed on a \textit{APEmille} crate (128 FPUs for 64 GFlops of peak performance), for $N_f = 0, 2, 3$, massless quarks Wilson action on a $32^4$ lattice; gauge is fixed to the Landau condition using a Fourier accelerated algorithm. Stability with respect to single/double precision has been tested, as well as finite size effects are checked looking at smaller lattices on a PC cluster. So far we reached quite a wide configurations database (some hundreds, 1.8GB each) ready for any other observable.

\section*{2. RESULTS}

Both critical mass and field renormalization constant are extracted from measurements of the quark propagator. Actually one averages $S$ on the configurations and then gets $\Gamma_2$ by inversion. The structure of the result is as follow

\begin{equation}
\Gamma_2(p^2, m) = S(p^2, m)^{-1} = i\not{p} - m - \Sigma(p^2, m) \tag{1}
\end{equation}

where

\begin{equation}
\Sigma(p^2, m) = \Sigma_c + m \Sigma_s(p^2, m) + i\not{p} \Sigma_v(p^2, m) \tag{2}
\end{equation}

So, projecting our numerical results onto the $\gamma$-matrices one recovers the last term in the sum and, from it, one can reach the field renormalization constant; in the same way, projecting onto the $\gamma$-identity one recovers the other two pieces of $\Sigma$, which in (our) massless case reduce just to $\Sigma_c$, the so-called critical mass: it is the additive renormalization for the mass one has to face for having broken chiral symmetry on the lattice.

We present results only for the $N_f = 2$ case.
2.1. Critical mass

Perturbative expansion for $\Sigma_c$ is already analytically known up to 2 loops \cite{4,5}. If we write

$$-\Sigma_c = \Sigma^{(1)}_c \beta^{-1} + \Sigma^{(2)}_c \beta^{-2} + \Sigma^{(3)}_c \beta^{-3} + \ldots,$$

for $N_f = 2$ we have $\Sigma^{(1)}_c = 2.6057$ and $\Sigma^{(2)}_c = 4.293$. In order to go beyond these coefficients and get $\Sigma^{(3)}_c$, one must plug them in as counterterms in the simulation because they contribute to higher loops results. So, if one looks at these terms in our results, one actually sees the additive renormalized coefficients. This is just what is shown in Fig. 1. The non-smoothness of these curves is a consequence of having the continuous euclidean $O(4)$ symmetry broken on the lattice: at finite lattice spacing $\Sigma$ is not simply a function of $a^2 p^2$ but it takes also corrections from other lattice invariants one can construct as powers of $ap$. Taking into account both the leading $a^2 p^2$ and higher order invariants, one can extrapolate the $a \to 0$ limit value one is interested in. These values are just zero because of the counterterms subtraction.

Fig. 2 shows the same result, but for the 3-loop term. For it there are no counterterms plugged in the simulations (this is precisely the result one is looking for), and so the extrapolation is not zero. The cross represents the extrapolated ($a \to 0$) value, which reads

$$\Sigma^{(3)}_c = 11.79^{(2)}_{(5)}$$

in which the errors are estimated by looking at the stability of the fitted result with respect to varying the number of points included in the fit and the higher orders lattice invariants taken into account.

2.2. Field renormalization constant

As said, the field renormalization constant $Z_q$ can be extracted looking at $S^{-1}$ along the $\gamma$-matrices

$$Z_q = -\frac{i}{4} \text{Tr}(\not{p} S^{-1})$$

$$= 1 + Z^{(1)}_q \beta^{-1} + Z^{(2)}_q \beta^{-2} + \ldots$$

In contrast with the case of the critical mass, in this case we have to face logarithmic divergences, $\log(a^2 p^2)$, coming from Renormalization Group Theory. These divergences are ruled by anomalous dimensions and so, because we are in Landau gauge, we have just no log-terms at 1-loop and only a simple log-term at 2-loops (i.e. without $\log^2$-term).

So, at 1-loop one has no divergences to subtract and one can extrapolate directly to the $a \to 0$
limit. The result we found is shown in Fig. 3 and reads
\[ Z_q^{(1)} = -0.843(2) \]  \hspace{1cm} (6)

This result is in perfect agreement with the known value \( Z_q^{(1)} = -0.843 \ldots \)

At 2-loops one has to subtract the known log-term before taking the continuum limit. In Fig. 4 we plot both the crude data and the data with the log divergence subtracted: the extrapolation for the latter reads
\[ Z_q^{(2)} = -1.34(2) \]  \hspace{1cm} (7)

We can compare this new result with a non-perturbative value [8], which at \( \beta = 5.8 \) and \( a\mu = 1 \) reads \( Z_q^{(N-P)} = 0.786(5) \). Our 2-loop expansion add up to \( Z_q^{(P)} = 0.815 \). So the discrepancy reduces from 9% (1-loop) to 3-4%. In order to improve convergence, one can switch from a bare to a better-converging coupling expansion (such as in the so-called Boosted Perturbation Theory). On the other hand we already have the signal for the 3-loop coefficient.

3. CONCLUSION & PERSPECTIVES

We are currently refining the statistic in order to reduce statistical errors and to see the 3-loop coefficient for \( Z_q \).

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