NOTE ON A NOTE OF GOLDSTON AND SURIAJAYA

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Abstract: We show that the assumption of a weak form of the Hardy-Littlewood conjecture on the Goldbach problem suffices to disprove the possible existence of exceptional zeros of Dirichlet $L$-functions. This strengthens a result of the authors named in the title.

1. Introduction

A fundamental problem in analytic number theory is that of establishing zero-free regions for Dirichlet $L$-functions. In case the corresponding character $\chi \mod q$ is complex, or alternatively, in the case of complex zeros $\rho = \beta + i\gamma$ with $\gamma \neq 0$, one has long known how to produce zero-free regions of the type

\begin{equation}
\sigma \geq 1 - c/\log q(|t| + 2)
\end{equation}

where $s = \sigma + it$. In the case both $\chi$ and $s$ are real much less is known, nothing more recent than a famous “ineffective” estimate of Siegel which replaces $\log q$ by $q^\varepsilon$. This exponentially weaker result has been a serious impediment to progress in many basic questions.

In the absence of a solution to the problem of these exceptional zeros there are some useful results showing their rarity. There have also been attempts to relate the question to other very difficult problems. One famous example, by now folklore, shows that the non-existence of such zeros would follow from improvements seductively small in the Brun-Titchmarsh theorem which gives uniform upper bounds for the number of primes in an arithmetic progression.

In more recent years some results have been obtained by showing how relatively good bounds for exceptional zeros would follow from assumptions about the seemingly less directly related Goldbach conjecture. The latter famous statement predicts that every even integer exceeding two can be written as the sum of two primes. Hardy and Littlewood [HL] put forth a conjectured asymptotic formula for the number of representations of $n$ as the sum of two primes. Following the normal practice in the subject, we find it simpler to consider a

* Supported in part by NSERC grant A5123.
weighted sum over the representations involving the von Mangoldt function, one which leads to an entirely equivalent conjecture. Let

\[ G(n) = \sum_{\substack{m_1 + m_2 = n \\text{ and} \ \gcd(m_1, m_2) = 1}} A(m_1)A(m_2). \]  

The Hardy-Littlewood conjecture predicts that, for \( n \) even, we have \( G(n) \sim \mathcal{G}^n(n) \) where \( \mathcal{G}(n) \) is a certain positive product over the primes, defined in (6.2) to (6.4) and easily large enough to imply Goldbach for all large \( n \).

A rather weakened, but still formidable, form of the Hardy-Littlewood conjecture which features in this (and in earlier) work is as follows.

**Weak Hardy-Littlewood-Goldbach Conjecture:** For all sufficiently large even \( n \) we have

\[ \delta \mathcal{G}(n)n < G(n) < (2 - \delta)\mathcal{G}(n)n, \]

for some fixed \( 0 < \delta < 1 \).

Despite having a slightly different name, this is the same conjecture employed in [GS]. We are going to prove the following result.

**THEOREM 1.** Assume that the Weak Hardy-Littlewood-Goldbach Conjecture (1.3) holds for all sufficiently large even \( n \). Then there are no zeros of any Dirichlet \( L \)-function in the region (1.1) with a positive constant \( c \) which is now allowed to depend on \( \delta \).

Of course, as remarked above, we are allowed to restrict the proof to the case of real zeros of \( L \)-functions with real characters. We can also assume that the modulus \( q \) is large since if there were only a finite number of exceptional zeros we could reach a region of type (1.1) simply by adjusting the constant \( c \) (since \( L(1, \chi) > 0 \)).

Precursors of this work include [Fe], [BHMS], [BH], [Ji] and [GS]. A discussion of their results can be found in [GS]. In those works the results led to an interval of width \( c/(\log q)^2 \) rather than \( c/\log q \). Aside from an only slightly different approach, our main innovation is the use of a theorem of Bombieri, described in Section 4, which allows us to incorporate the zero repulsion principle used, for example, in connection with Linnik’s theorem on the least prime in an arithmetic progression.

**Acknowledgement** We were completely motivated to consider this problem by the paper [GS] of Goldston and Suriajaya and by the beautiful lecture of Dan Goldston delivered to the web seminar of the American Institute of Mathematics on May 4, 2021. We are also grateful to the Institute for having provided this venue.
2. A Goldbach number generating series

We let $N$ and $q$ be given positive integers. We consider the sum
\begin{equation}
S(q) = \sum_{n \equiv 0 \pmod{q}} G(n)e^{-n/N}
\end{equation}
where $G(n)$ is given by (1.2). We are going to estimate $S(q)$ in two different ways and then compare the results.

3. Zeros of $L$-functions and an explicit formula

By the orthogonality property for the characters we write
\begin{equation}
S(q) = \sum_{m_1 + m_2 \equiv 0 \pmod{q}} \Lambda(m_1) \Lambda(m_2)e^{-(m_1 + m_2)/N}
\end{equation}
\begin{equation}
= \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \chi(-1) \sum_m \left( \chi(m) \Lambda(m)e^{-m/N} \right)^2 + O\left(N(\log q)^2\right)
\end{equation}
where the error term accounts for a trivial estimation of the contribution to the sum from terms $m_1, m_2$ with $(m_1m_2, 2q) \neq 1$.

Let $N \geq q$. The principal character contributes to $S(q)$ an amount
\begin{equation}
S_0(q) = \frac{1}{\varphi(q)} \left( \sum_{(m,q)=1} \Lambda(m)e^{-m/N} \right)^2 = \frac{N^2}{\varphi(q)} \left(1 + O\left(\frac{1}{\log N}\right)\right)
\end{equation}
The exceptional character $\chi_1 \pmod{q}$ contributes
\begin{equation}
S_1(q) = \frac{\chi_1(-1)}{\varphi(q)} P(\chi_1)^2
\end{equation}
where for any $\chi$ we have
\begin{equation}
P(\chi) = \sum_m \chi(m) \Lambda(m)e^{-m/N}.
\end{equation}
We have the trivial bound $P(\chi) \ll N$. Hence, all the other characters $\chi \neq \chi_0, \chi_1$ contribute at most
\begin{equation}
S_\infty(q) \ll \frac{N}{\varphi(q)} \sum_{\chi \neq \chi_0, \chi_1} |P(\chi)|.
\end{equation}
We evaluate $P(\chi)$ for each $\chi \neq \chi_0$ as follows:
\begin{equation}
P(\chi) = \frac{1}{2\pi i} \int_{(2)} \Gamma(s)N^s \frac{-L'}{L}(s)ds
\end{equation}
\begin{equation}
= -\sum_{\rho} \Gamma(\rho)N^\rho + O\left(N^{\frac{3}{4}}(\log N)^2\right),
\end{equation}
on moving the line of integration to \( \text{Res} = \frac{1}{2} \). Here, \( \rho = \beta + i\gamma \) runs over the zeros of \( L(s, \chi) \) with \( \frac{1}{2} \leq \beta < 1 \), giving the explicit formula for the sum \( P(\chi) \) in (3.3).

4. Linnik’s zero repulsion à la Bombieri

We have the following result.

**Proposition 4.1.** Suppose there is a real character \( \chi_1(\text{mod } q) \) with a real zero \( \beta_1 \) of \( L(s, \chi_1) \) in the segment

\[
1 - \frac{c}{\log q} \leq \beta_1 < 1
\]

where \( c \) is a small positive constant. Let \( N(\alpha, T) \) denote the number of zeros, \( \rho = \beta + i\gamma, \rho \neq \beta_1 \) of \( L(s, \chi) \) for all \( \chi(\text{mod } q) \), counted with multiplicity, in the region

\[
\alpha \leq \beta < 1, \quad |\gamma| \leq T.
\]

Then, for \( \frac{1}{2} \leq \alpha \leq 1, T \geq 2 \)

\[
N(\alpha, T) \ll (1 - \beta_1)(\log q)q^{b(1-\alpha)}T^3
\]

where \( b \) and the implied constant are absolute.

This proposition follows from Théorème 14 of Bombieri [B] which states the following.

**Proposition 4.2.** For \( \frac{1}{2} \leq \alpha \leq 1 \) and \( T \geq 2 \), we have

\[
\sum_{q \leq T} \sum_{\chi(\text{mod } q)}^* N(\alpha, T, \chi) \ll (1 - \beta_1)(\log T)T^{b(1-\alpha)}
\]

where \( \chi \) runs over primitive characters and \( N(\alpha, T, \chi) \) is the number of zeros \( \rho = \beta + i\gamma \), of \( L(s, \chi) \) counted with multiplicity, in the region \( \big[1/2, 1\big] \), apart from one real and simple zero \( \beta_1 \geq 1 - c/\log T \).

To deduce Proposition 4.1 we first consider the case \( T \leq q \). Now, our sum \( N(\alpha, T) \) satisfies

\[
N(\alpha, T) \leq N(\alpha, q) \leq \sum_{\chi(\text{mod } q)} N(\alpha, q, \chi) = \sum_{k|q}^* \sum_{\chi(\text{mod } k)} N(\alpha, q, \chi)
\]

\[
\leq \sum_{k \leq q} \sum_{\chi(\text{mod } k)}^* N(\alpha, q, \chi) \ll (1 - \beta_1)(\log q)q^{b(1-\alpha)},
\]

giving the result in this case.

On the other hand, if \( T > q \) the result follows from the classical bound \( N(T, \chi) \ll T \log qT \).
We remark that the factor $T^3$ in (1.3) is wasteful but simplifies the proof and is unimportant for our application.

5. First estimation of $S(q)$

We are now ready to complete the estimation of $S(q)$ by means of Dirichlet characters. We input the result of Proposition 4.1. We derive by partial summation that

$$
\sum_{\chi \pmod{q}} \sum_{\rho \neq \beta_1} |\Gamma(\rho) N^\rho| \ll N(1 - \beta_1) \log q,
$$

provided $N \geq q^{b+1}$. Hence, by (3.4) and (3.5), we have

$$
S_\infty(q) \ll \frac{N^2}{\phi(q)}(1 - \beta_1) \log q.
$$

On the other hand, taking only the exceptional character rather than the others in (5.1), we get

$$
P(\chi_1) = -\Gamma(\beta_1) N^{\beta_1} + O\left(N(1 - \beta_1) \log q\right)
$$

and so

$$
P(\chi_1)^2 = \Gamma(\beta_1)^2 N^{2\beta_1} + O\left(N^2(1 - \beta_1) \log q\right).
$$

Finally, adding the above estimates we obtain

$$
S(q) = S_0(q) + \frac{\chi_1(-1)}{\phi(q)} N^{2\beta_1} + O\left(\frac{N^2}{\phi(q)}(1 - \beta_1) \log q\right)
$$

(5.5)

$$
= \left(1 + \chi_1(-1) + \varepsilon(q, N)\right) \frac{N^2}{\phi(q)},
$$

where

$$
\varepsilon(q, N) \ll (1 - \beta_1) \log N + (\log N)^{-1}
$$

if $N \geq q^{b+1}$. The implied constant is absolute.

6. A model of $S(q)$

In this section we consider the a model for $S(q)$ to be given by (2.1).

Although our computations in this section, as for those earlier, are unconditional, the closeness of the model to the sum $S(q)$ is not. That will rest on the assumption in the next section.

The Hardy-Littlewood conjecture for the Goldbach problem asserts that, for $n$ even,

$$
G(n) \sim \mathcal{G}(n)n,
$$

(6.1)
where
\begin{equation}
\mathcal{G}(n) = 2CH(n),
\end{equation}
and, in turn, $C$ is the positive absolute constant
\begin{equation}
C = \prod_{p > 2} \left(1 - \frac{1}{(p-1)^2}\right),
\end{equation}
and $H(n)$ is the multiplicative function
\begin{equation}
H(n) = \prod_{p \mid n, p > 2} \left(1 + \frac{1}{p-2}\right).
\end{equation}

**Lemma 6.1.** We have
\begin{equation}
\sum_{n \equiv 0 (\text{mod } q) \atop n \text{ even}} \mathcal{G}(n) e^{-n/N} = \frac{N^2}{\varphi(q)} \left(1 + O\left(\frac{q}{N}\right)^\frac{1}{2}\right).
\end{equation}

**Proof.** Pulling out the constant factor $2C$ we evaluate the sum
\begin{equation}
\sum_{n \equiv 0 (\text{mod } q) \atop n \text{ even}} H(n) e^{-n/N} = \frac{N}{2\pi i} \int_{(2)} \Gamma(s+1)N^s Z_q(s) ds
\end{equation}
where
\[ Z_q(s) = \sum_{n \equiv 0 (\text{mod } k)} H(n) n^{-s} \]
and $k = [2, q] = 2\varphi(q)/2$. We have
\[
Z_q(s) = H(k) k^{-s} \sum_{\ell=1}^{\infty} \prod_{p \mid \ell \atop p \not\mid k} \left(1 + \frac{1}{p-2}\right) \ell^{-s}
\]
\[
= H(k) k^{-s} \prod_{p \mid k} \left(1 - \frac{1}{p^s}\right) \prod_{p \mid k} \left(1 + \left(1 + \frac{1}{p-2}\right) \frac{1}{p^s} \left(1 - \frac{1}{p^s}\right)^{-1}\right)
\]
\[
= H(k) k^{-s} \zeta(s) \prod_{p \mid k} \left(1 + \frac{1}{p^s(p-2)}\right).
\]
This has a simple pole at $s = 1$ with residue
\begin{equation}
\text{res}_{s=1} Z_q(s) = \frac{H(k)}{k} \prod_{p \mid k} \left(1 + \frac{1}{p(p-2)}\right) = \frac{1}{2C \varphi(q)}.
\end{equation}
Moving the line of integration in (6.6) to $\text{Re } s = \frac{1}{2}$, we obtain (6.5).
7. Completion

In this section we begin by completing our second estimation for $S(q)$ by comparing its model. In this we need to draw on an unproved assumption. We assume the following.

**Weak Hardy-Littlewood-Goldbach Conjecture:** For all sufficiently large even $n$ we have

\[(7.1) \quad \delta \mathcal{S}(n)n < G(n) < (2 - \delta)\mathcal{S}(n)n,\]

where $0 < \delta < 1$ is fixed.

From (7.1) and Lemma 6.1 we see that, with $N \geq q^2$,

\[(7.2) \quad \delta \frac{N^2}{\varphi(q)} \left(1 + O\left(\frac{1}{\log N}\right)\right) < S(q) < (2 - \delta)\frac{N^2}{\varphi(q)} \left(1 + O\left(\frac{1}{\log N}\right)\right).\]

We now combine our two estimations for $S(q)$. By (7.2) and (5.5) we have

\[(7.3) \quad \delta < 1 + \chi_1(-1) + \varepsilon(q, N) < 2 - \delta\]

if $N \geq q^{b+1}$, where (a now slightly different) $\varepsilon(q, N)$ still satisfies (5.6). Taking the constant $c$ in (4.11) to be sufficiently small, we see from (5.6) that

\[(7.4) \quad |\varepsilon(q, N)| \leq \frac{\delta}{2}.\]

Since $1 + \chi_1(-1) = 0$ or $= 2$, this is not possible.

**Conclusion:** The assumption (7.1) with some positive $\delta$ implies that there is a real zero-free interval $s \geq 1 - c(\delta)/\log q$, (and hence a zero-free region in the plane $\sigma \geq 1 - c(\delta)/\log \sqrt{|t| + 2}$ for all $L(s, \chi)$ to all moduli $q$). In the case that $\chi_1(-1) = 1$ only the upper bound in (7.1) is required and in the other case, that $\chi_1(-1) = -1$, only the lower bound is required.

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