Signed Young Modules and Simple Specht Modules
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Abstract

Let $F$ be a field of characteristic $p$. By a result of Hemmer, when $p \geq 3$, every simple Specht module of a finite symmetric group over $F$ is a signed Young module. While Specht modules are parametrized by partitions, indecomposable signed Young modules are parametrized by certain pairs of partitions. The main result of this article establishes the signed Young module labels of simple Specht modules. Along the way we prove a number of results concerning indecomposable signed Young modules that are of independent interest. In particular, when $p \geq 3$, we determine the label of the indecomposable signed Young module obtained by tensoring a given indecomposable signed Young module with the sign representation. As consequences, when $p$ is positive, we obtain the Green vertices, Green correspondents, cohomological varieties, and complexities of all simple Specht modules and a class of simple modules of symmetric groups over $F$, and extend the results of Gill on periodic Young modules to periodic indecomposable signed Young modules.

1 Introduction

Young modules and Specht modules are of central importance for the representation theory of symmetric groups, and have been studied intensively for more than a century. Given a field $F$ and the symmetric group $S_n$ of degree $n \geq 0$, both the isomorphism classes of Young $F S_n$-modules and Specht $F S_n$-modules are well known to be labelled by the partitions of $n$. If $F$ has characteristic 0 then the Specht $F S_n$-modules are precisely the simple $F S_n$-modules, otherwise every simple $F S_n$-module occurs as a quotient of a suitable Specht module. In 2001 Donkin introduced the notion of indecomposable signed Young modules (see [8]). Donkin proved that the isomorphism classes of indecomposable signed Young $F S_n$-modules are parametrized by certain pairs of partitions.

A tight connection between the indecomposable signed Young modules and simple Specht modules has been established by the result of Hemmer [18]. Suppose now that $F$ has odd characteristic. Hemmer showed that every simple Specht $F S_n$-module is isomorphic to an indecomposable signed Young module. However, to our knowledge, it has so far been an open problem to determine the signed Young module label of a given simple Specht module (see [18, Problem 5.2]). A conjecture concerning this labelling was first put up by the first author in [3, Vermutung 5.4.2] and later, independently, by the second author [28, Conjecture 8.2] and Orlob [39, Vermutung A.1.10]. In an unpublished note, the second author and Orlob verified the conjecture in the case when the simple Specht modules are labelled by hook partitions. The main result of the present article, Theorem 5.1, confirms this conjecture in general.

Our strategy towards the proof of Theorem 5.1 follows the ideas used by Fayers in [12] and by Hemmer in [18] as follows. Given a simple Specht $F S_n$-module $S^\lambda$, there is always some $m \geq n$ and some simple Specht $F S_m$-module $S^\mu$ belonging to a Rouquier block such that $S^\mu$ is isomorphic to a direct summand of the induced module $\text{Ind}_{S_n}^{S_m}(S^\lambda)$ and $S^\lambda$ is isomorphic to a direct summand of the restricted module $\text{Res}_{S_n}^{S_m}(S^\mu)$. The structure of Rouquier blocks is generally much better understood than that of arbitrary blocks of symmetric groups. We prove our main result, Theorem 5.1, in two steps: first we reduce the proof to the Rouquier block case in Proposition 5.4 and then verify our result for this special case.
Altogether our proof of the main result requires three key ingredients: firstly, we need a ‘twisting formula’ for indecomposable signed Young modules, that is, given an indecomposable signed Young $F\mathfrak{S}_n$-module $Y$, we determine the label of the signed Young $F\mathfrak{S}_n$-modules obtained by tensoring $Y$ with the sign representation. This formula is given in Theorem 3.18 and should be of independent interest.

Secondly, we shall exploit the theory of Young vertices and a generalized version of Green correspondence, which we shall refer to as Young–Green correspondence. This theory has been introduced by Grabmeier [16] in order to generalize Green’s classical vertex theory of finite groups. Donkin has shown in [8] that the indecomposable signed Young $F\mathfrak{S}_n$-modules are precisely the indecomposable $F\mathfrak{S}_n$-modules with linear Young source; in particular, they have trivial Green sources. Donkin has also described explicitly the Young vertices and Young–Green correspondents of the indecomposable signed Young $F\mathfrak{S}_n$-modules.

Lastly, we shall make use of a recent generalization of the well-known Young Rule. This generalization is due to the second author and Tan [31], and it describes the multiplicities of Specht $F\mathfrak{S}_n$-modules appearing as factors of a particular filtration of signed Young permutation $F\mathfrak{S}_n$-modules (see Theorem [3.6]).

Some immediate consequences can be drawn from the twisting and labelling formulae in Theorem 3.18 and Theorem 5.1 respectively. In particular, we are able to describe the Green vertices, Green correspondents, cohomological varieties, and complexities of all simple Specht modules and a class of simple modules of symmetric groups over any (algebraically closed) field of positive characteristic. In doing so, we recover a result of Wildon [42] on simple Specht modules labelled by hook partitions, and extend the results of Gill [15] on periodic Young modules to periodic indecomposable signed Young modules.

The present paper is organized as follows. In Section 2 we set up the notation needed throughout, and we give a summary of some well-known basic properties of symmetric groups and their modules. In Section 3 we first review the theory of Young vertices and Young–Green correspondence, and then prove the twisting formula Theorem 3.18. Along the way, we obtain the Green correspondents, with respect to any subgroup containing the normalizers of their respective Green vertices, and the Broué correspondents of indecomposable signed Young modules. In Section 4 we analyze the set of partitions that label the simple Specht modules in odd characteristic $p$. The analysis is based on Fayers’ characterization of these partitions [12, Proposition 2.1]. We show that the $p$-cores of such partitions can be obtained by the independent procedures of stripping off horizontal and vertical $p$-hooks only. Moreover, we analyze the relation between two such partitions that are adjacent in the sense of Fayers (see [4.11]). The latter analysis is particularly important for the proof of the reduction of Theorem 5.1 to the Rouquier block case in Proposition 5.4. Section 5 is devoted to the proof of the labelling formula of a simple Specht module as an indecomposable signed Young module. In Section 6 we apply our main results to derive the above-mentioned information concerning Green vertices, Green correspondents, cohomological varieties and complexities of all simple Specht modules and a class of simple modules of the symmetric groups. Furthermore, using Gill’s result [15], we describe the periods and, in the case of blocks of weight 1, we also give a minimal projective resolution of non-projective periodic indecomposable signed Young modules.

Since our notation is slightly different and there are some subtleties regarding sign representations of various permutation groups involved, in Appendix A we follow closely the argument given by Donkin in [8, 5.2] to give a proof of his characterization of the Young–Green correspondents of indecomposable signed Young modules. In Appendix B to make our paper self-contained, we present a signed version of Young’s Rule following...
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2 Preliminaries

We begin by fixing the notation used throughout this article. We also collect some well-known facts about the representation theory of symmetric groups that we shall need repeatedly. For further details, we refer the reader to [22, 23, 37].

2.1 Notation. Suppose that \( G \) is a finite group and that \( F \) is a field. By an \( FG \)-module we always understand a finite-dimensional left \( FG \)-module. If \( M_1 \) and \( M_2 \) are \( FG \)-modules such that \( M_1 \) is isomorphic to a direct summand of \( M_2 \) then we write \( M_1 \mid M_2 \).

If \( H \leq G \) and if \( M \) and \( N \) are \( FG \)-module and \( FH \)-module, respectively, then we denote by \( \text{Res}^G_H(M) \) the restriction of \( M \) to \( H \) and by \( \text{Ind}^G_H(N) \) the induction of \( N \) to \( G \).

For subgroups \( H \) and \( K \) of \( G \), we write \( H \leq_G K \) if \( H \) is \( G \)-conjugate to a subgroup of \( K \), and we write \( H =_G K \) if \( H \) is \( G \)-conjugate to \( K \). For \( g \in G \), we set \( ^gH := gHg^{-1} \).

For two finite groups \( G \) and \( H \), an \( FG \)-module \( M \) and an \( FH \)-module \( N \), we denote by \( M \otimes_F N \) the outer tensor product of \( M \) and \( N \), which is naturally a module of \( F\mathbb{G} \times H \).

2.2. Partitions, Young diagrams and abaci. Denote by \( \mathbb{N} \) and \( \mathbb{Z}^+ \) the sets of non-negative and positive integers, respectively. Let \( n \in \mathbb{N} \), and let \( p \) be a prime.

(a) Let \( \lambda = (\lambda_1, \ldots, \lambda_r) \) be a partition of \( n \), where we shall mostly assume \( \lambda_r > 0 \). In this case, we write \( |\lambda| = n \) or \( \lambda \vdash n \). The unique partition of 0 is denoted by \( \emptyset \). The partition \( \lambda \) is identified with its Young diagram \( [\lambda] \). The elements of \( [\lambda] \) are called the nodes of \( [\lambda] \). If \( m \in \{0, 1, \ldots, n\} \) and if \( \mu \) is a partition of \( m \) such that \( \mu_i \leq \lambda_i \), for all \( i \geq 1 \), then \( \mu \) is called a subpartition of \( \lambda \). One then has \( |\mu| \subseteq |\lambda| \), and one calls \( |\lambda| \setminus |\mu| \) a skew-diagram.

(b) The \( p \)-core \( \tilde{\lambda} \) of \( \lambda \) is the partition obtained from \( \lambda \) by successively removing all skew \( p \)-hooks from \( [\lambda] \). The \( p \)-core of \( \lambda \) is independent of the procedure of skew \( p \)-hooks removal. The \( p \)-weight of \( \lambda \) is the total number of such skew \( p \)-hooks removed to reach its \( p \)-core. Thus the \( p \)-weight of \( \lambda \) is \( \frac{1}{p}(|\lambda| - |\tilde{\lambda}|) \).

The hook length \( h_\lambda(a,b) \) of a node \((a,b)\) of a diagram \( [\lambda] \) is the total number of nodes \((x,b)\) and \((a,y)\) of \([\lambda]\) satisfying \( x \geq a \) and \( y \geq b \), respectively.

The \( p \)-residue diagram of \( \lambda \) is obtained by replacing each node \((i,j)\) of the Young diagram \([\lambda]\) by the residue of \( j-i \) modulo \( p \).

The partition \( \lambda \) is called a \( p \)-singular partition if

\[
\lambda_i+1 = \lambda_i+2 = \cdots = \lambda_i+p \geq 1,
\]

for some \( i \in \mathbb{N} \). A partition that is not \( p \)-singular is called \( p \)-regular. If \( \lambda \) is \( p \)-regular then the conjugate partition \( \lambda' \) is called \( p \)-restricted. Note that the \( p \)-core of a partition is both \( p \)-regular and \( p \)-restricted.

We denote the set of all partitions and \( p \)-restricted partitions of \( n \) by \( \mathcal{P}(n) \) and \( \mathcal{R} \mathcal{P}(n) \), respectively.
(c) By Lemma 7.5, every partition $\lambda$ of $n$ can be written as $\lambda = \sum_{i=0}^{s} p^i \cdot \lambda(i)$, for some $s \in \mathbb{N}$ and uniquely determined $p$-restricted partitions $\lambda(1), \ldots, \lambda(s)$. One calls this the $p$-adic expansion of $\lambda$.

(d) An abacus (display) consists of $p$ runners, labelled from $0$ to $p-1$ from left to right, with positions $(i - 1)p + j$ belonging to the $i$th row and runner $j$ such that each position of the abacus is either vacant or occupied by a bead. The position $k$ is said to be higher (respectively, lower) than the position $l$ if $k < l$ (respectively, $k > l$). Every partition $\lambda$ can be associated to an abacus in the following way. Suppose that $\lambda = (\lambda_1, \ldots, \lambda_r)$ is a partition. Let $s \geq r$. A sequence of $\beta$-numbers is a sequence of non-negative integers $(\beta_1, \ldots, \beta_s)$ such that $\beta_i = \lambda_i - i + s$, which is also called an $s$-element $\beta$-set. The abacus associated to these $\beta$-numbers is the abacus whose position $(i - 1)p + j$ is occupied if and only if, for some $1 \leq k \leq s$, $\beta_k = (i - 1)p + j$. Clearly, the abacus display of $\lambda$ depends on the choice of $s$.

Conversely, given an abacus with $p$ runners and a finite number of beads, we can read off the partition it represents in the following way. For each bead on the abacus, we count the number of vacant positions that are higher than the position the bead occupies on the abacus. Suppose that the numbers obtained in this manner are $\lambda_1 \geq \lambda_2 \geq \cdots$. Then the partition associated to this abacus is $\lambda = (\lambda_1, \lambda_2, \ldots)$.

Moreover, we obtain the $p$-core of $\lambda$ as the partition obtained from an abacus of $\lambda$ by moving all beads on the abacus as high up as possible along every runner. The $p$-weight of $\lambda$ is precisely the total number of such bead movements.

Given an abacus of a partition $\lambda$ with $p$ runners, for each $0 \leq i \leq p - 1$, let $\lambda_j^{(i)}$ be the number of unoccupied positions higher than the $j$th lowest occupied position on runner $i$. Then we obtain the partition $\lambda^{(i)} = (\lambda_1^{(i)}, \lambda_2^{(i)}, \ldots)$ of $\lambda$ with respect to the given abacus. The $p$-quotient of $\lambda$ with respect to the abacus is the sequence $$(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(p-1)}).$$

The $p$-quotient of $\lambda$ is unique up to cyclic place permutations. In particular, the $p$-weight of $\lambda$ is the sum of the sizes of the partitions $\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(p-1)}$. We demonstrate this by an example.

2.3 Example. Suppose that we are given an abacus as follows. For each bead, we record the number of vacant positions that are higher than the position the bead occupies. We obtain the following configuration.

```
  0 0 0
 1 1 1
1
5 5 5
```

The partition this abacus represents is $\lambda = (6, 5, 5, 1, 1, 1)$. As we move the beads on every runner as high up as possible, we obtain the following abacus.

```
  0 0 0
 1 1 1
1
5 5 5
```
2.5. Blocks of $F$ each of the group algebra partition $\lambda$ set of $p$ so that $D^\mu$ have $F$ or each $\lambda$. By convention, $\mathfrak{S}_0$ is the trivial group.

2.6. Remark. For every $\lambda \in \mathcal{P}(n)$, one can also parametrize the isomorphism classes of simple $F\mathfrak{S}_n$-modules by $S_\lambda$, one has

$$S_\lambda := (S^\lambda)^* \cong S^{\lambda'} \otimes \text{sgn}(n),$$

where $\lambda' \in \mathcal{P}(n)$ is the conjugate partition of $\lambda$ and $\text{sgn}(n)$ denotes the sign representation of $F\mathfrak{S}_n$.

Recall that $\mathcal{R}(n)$ is the subset of $\mathcal{P}(n)$ consisting of $p$-restricted partitions. Using the isomorphism [11], one can also parametrize the isomorphism classes of simple $F\mathfrak{S}_n$-modules by the $p$-restricted partitions of $n$; namely, we set $D_\mu := \text{Soc}(S^\mu)$, for each $\mu \in \mathcal{R}(n)$.

The connection between these two labellings of simple $F\mathfrak{S}_n$-modules is given by

$$D_\lambda \cong D^{\lambda'} \otimes \text{sgn}(n),$$

for every $\lambda \in \mathcal{R}(n)$.

For each $\lambda \in \mathcal{R}(n)$, the $F\mathfrak{S}_n$-module $D_\lambda \otimes \text{sgn}(n)$ is of course simple again, and we have $D_\lambda \otimes \text{sgn}(n) \cong D_m(\lambda)$, where $m : \mathcal{R}(n) \to \mathcal{R}(n)$ is the Mullineux map on the set of $p$-restricted partitions of $n$.

Lastly, recall that, whenever $\lambda \in \mathcal{P}(n)$ is such that $\lambda = \tilde{\lambda}$, then $\lambda$ is both $p$-regular and $p$-restricted. Furthermore, in this case, we have $S^\lambda \cong D^\lambda \cong D_\lambda$ and $S^{\lambda'} \cong D^{\lambda'} \cong D_{\lambda'}$, so that $D_\lambda \otimes \text{sgn}(n) \cong S^\lambda \otimes \text{sgn}(n) \cong S^{\lambda'} \cong D_{\lambda'}$, and thus $m(\lambda) = \lambda'$.

2.5. Blocks of $F\mathfrak{S}_n$. By the Nakayama Conjecture, proved by Brauer and Robinson, the blocks of the group algebra $F\mathfrak{S}_n$ are labelled by the $p$-cores of the partitions of $n$. Given $\lambda \in \mathcal{P}(n)$, we shall denote by $b_\lambda$ the block of $F\mathfrak{S}_n$ associated to $\lambda$. With this notation, each of the $F\mathfrak{S}_n$-modules $S^\lambda, S_\lambda, D^\lambda$ and $D_\lambda$ belongs to the block $b_\lambda$, for every admissible partition $\lambda$.

2.6. Remark. An $F\mathfrak{S}_n$-module $M$ is said to admit a Specht filtration if there is a sequence of $F\mathfrak{S}_n$-submodules

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_r \subset M_{r+1} = M$$
and partitions $\varrho_1, \ldots, \varrho_{r+1}$ of $n$ such that $M_i/M_{i-1} \cong S^{\varrho_i}$, for $i = 1, \ldots, r + 1$. However, in general, $M$ may have several Specht filtrations, and if $p \in \{2, 3\}$ then the number of factors isomorphic to a given Specht $FG$-module $S^\lambda$ depends on the chosen filtration; this has been shown by Hemmer and Nakano in [20]. In view of these obstacles, Lemma 2.8 below will be important for the proof of our main result in Section 3.

2.7 Lemma. Let $G$ be any finite group, let $N$ be an FG-module, and let $D$ be a simple FG-module. Let further $U$ be an FG-submodule of $M := D \oplus N$. If $U \nsubseteq N$ then $D \mid U$. In particular, if $U$ is indecomposable then either $U \subseteq N$ or $U = D$.

Proof. Suppose that $U \nsubseteq N$. Consider the canonical projection $e_D : M \to D$. Then $e_D(U) \subseteq D$, so that either $e_D(U) = \{0\}$ or $e_D(U) = D$. Also, $e_D(U) \not\cong U/N \cap N$. Since $U \nsubseteq N$, we cannot have $e_D(U) = \{0\}$. Hence $e_D(U) = D \subseteq U$, and the restriction of $e_D$ to $U$ is an idempotent endomorphism. Hence we get $D = e_D(U) \mid U$ as desired. \hfill $\square$

2.8 Lemma. Suppose that $p \neq 2$, and let $M$ be an $FG_n$-module that admits a Specht filtration. Suppose further that $S^\lambda$ is a simple Specht $FG_n$-module such that $S^\lambda \mid M$. Then every Specht filtration of $M$ has a factor isomorphic to $S^\lambda$.

Proof. We may identify $S^\lambda$ with a submodule of $M$ and write $M = S^\lambda \oplus N$, for some $FG_n$-module $N$. Let

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_r \subset M_{r+1} = M \tag{2}$$

be a Specht filtration of $M$, and let $\varrho_1, \ldots, \varrho_{r+1} \in \mathcal{P}(n)$ be such that $M_i/M_{i-1} \cong S^{\varrho_i}$, for $i = 1, \ldots, r + 1$. We have to show that $S^{\varrho_j} \cong S^\lambda$, for some $j \in \{1, \ldots, r + 1\}$. Recall from [22] Corollary 13.18 that, since $p \neq 2$, every Specht $FG_n$-module is indecomposable.

We show, by induction on $i$, that if, for all $1 \leq j \leq i$, we have $S^{\varrho_j} \not\cong S^\lambda$ then $M_j \subseteq N$. If $i = 1$ then $S^{\varrho_1} \cong M_1 \subseteq M$ is indecomposable. Since we are assuming $S^{\varrho_0} \not\cong S^\lambda$, Lemma 2.7 forces $M_1 \subseteq N$. Now let $i \geq 2$. Since we are assuming $S^{\varrho_j} \not\cong S^\lambda$, for all $1 \leq j \leq i$, we also have $S^{\varrho_i} \not\cong S^\lambda$, for all $1 \leq j \leq i - 1$. Thus, by induction, we know that $M_{i-1} \subseteq N$. So $S^{\varrho_i} \cong M_i/M_{i-1} \subseteq M/M_{i-1} = (S^\lambda + M_{i-1})/M_{i-1} \cong (N/M_{i-1}) \cong S^\lambda \oplus (N/M_{i-1})$. Since $M_i/M_{i-1}$ is indecomposable and not isomorphic to $S^\lambda$, we have $M_i/M_{i-1} \subseteq N/M_{i-1}$, by Lemma 2.7. Thus also $M_i \subseteq N$.

Consequently, since $N$ is a proper submodule of $M$, there must be some $j \in \{1, \ldots, r + 1\}$ with $S^{\varrho_j} \cong S^\lambda$ as claimed. \hfill $\square$

2.9. Young subgroups, wreath products, Sylow $p$-subgroups of $\mathfrak{S}_n$. Let $n \in \mathbb{Z}^+$.

(a) For a partition, or, more generally, a composition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of $n$, we denote by $\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_k}$ the corresponding standard Young subgroup of $\mathfrak{S}_n$.

(b) Let $G$ be any finite group, and let $H \leq \mathfrak{S}_n$. We have the wreath product $G \wr H := \{(g_1, \ldots, g_n; \sigma) : g_1, \ldots, g_n \in G, \sigma \in H\}$, whose multiplication is given by

$$(g_1, \ldots, g_n; \sigma) \cdot (h_1, \ldots, h_n; \pi) = (g_1 h_{\sigma^{-1}(1)}, \ldots, g_n h_{\sigma^{-1}(n)}; \sigma \pi),$$

for $g_1, \ldots, g_n, h_1, \ldots, h_n \in G$ and $\sigma, \pi \in H$. We denote by $G^n$ the base group $G^n := \{(g_1, \ldots, g_n) : g_1, \ldots, g_n \in G\}$ of $G \wr H$. If $U \leq H$ then we set $U^G := \{(1, \ldots, 1; \sigma) : \sigma \in U\} \leq G \wr H$.

Whenever we have subgroups $H_1$ and $H_2$ of $H$ with $H_1 \times H_2 \leq H$, we may identify the group $G \wr (H_1 \times H_2)$ with the group $G \wr H_1 \times G \wr H_2$.

Lastly, if $G \leq \mathfrak{S}_m$, for some $m \geq 1$, then we identify $G \wr H$ with a subgroup of the symmetric group $\mathfrak{S}_m$ in the usual way: namely, $G^n$ is identified with a subgroup of the
Young subgroup $\mathfrak{S}_{(m^n)}$ and $H^i$ is identified with the subgroup of $\mathfrak{S}_{mn}$ that permutes $n$ successive blocks of size $m$ according to $H$.

(c) We denote by $P_n$ a Sylow $p$-subgroup of $\mathfrak{S}_n$. Recall from [23, 4.1.22, 4.1.24] that, if $n$ has $p$-adic expansion $n = \sum_{i=0}^{r} a_ip^i$, for $a_0, \ldots, a_r \in \{0, \ldots, p-1\}$, we may choose $P_n$ as a Sylow $p$-subgroup of the Young subgroup $\prod_{i=0}^r (\mathfrak{S}_{p^i})^a_i$. Moreover, a Sylow $p$-subgroup $P_{p^i}$ of $\mathfrak{S}_{p^i}$ is isomorphic to the $i$-fold wreath product $C_{p^i} \wr \cdots \wr C_{p^i}$.

The proof of the next lemma is straightforward, and is thus left to the reader.

2.10 Lemma. Let $G$ be a finite group, and let $N \leq H \leq G$ be such that $N \leq G$. Moreover, let $V$ be an $F[H/N]$-module. Then one has

$$\text{Inf}^G_{G/N} (\text{Ind}^{G/N}_{H/N}(V)) \cong \text{Ind}^G_H (\text{Inf}^G_{H/N}(V))$$

as $FG$-modules.

2.11 Remark. Suppose that $G$ is a finite group. Let $N \leq G$, and let $K \leq H \leq \mathfrak{S}_n$. Then we have $N^n \leq G \wr K \leq G \wr H$, $N^n \leq G \wr H$, and we have natural group isomorphisms $(G \wr H)/N^n \cong (G/N) \wr K$ and $(G \wr K)/N^n \cong (G/N) \wr K$. If $V$ is an $F[(G/N) \wr K]$-module then Lemma 2.10 applies, and we get an isomorphism

$$\text{Inf}^{GH}_{(G/N)K} (\text{Ind}^{(G/N)H}_{(G/N)K}(V)) \cong \text{Ind}^{GK}_{GK} (\text{Inf}^{GK}_{(G/N)K}(V))$$

of $F[G \wr H]$-modules.

3 Signed Young Modules, Vertices, Green Correspondents, and Young-Green Correspondents

In this section, let $n \in \mathbb{N}$, and let $F$ be a field of characteristic $p \geq 3$. For convenience, suppose that $F$ is algebraically closed. We collect the basic properties of indecomposable signed Young modules and prove our main result, Theorem 3.18 concerning the twisting of indecomposable signed Young modules by the sign representation.

In the sequel, we shall have to deal with sign representations of various permutation groups. In order to always indicate which permutation group we are working with, we shall introduce the following notation.

3.1 Notation. Let $\text{sgn}(n)$ be the sign $F\mathfrak{S}_n$-module. For every subgroup $H \leq \mathfrak{S}_n$, we denote by $\text{sgn}(H)$ the sign representation of $H$ over $F$, that is, $\text{sgn}(H) = \text{Res}_{H}^{S_n}(\text{sgn}(n))$. In the case when $H = \mathfrak{S}_\lambda$, for some composition $\lambda$ of $n$, we also write $\text{sgn}(\lambda)$ instead of $\text{sgn}(\mathfrak{S}_\lambda)$. Thus, if $\lambda = (\lambda_1, \ldots, \lambda_k)$, we have $\text{sgn}(\lambda) = \text{sgn}(\lambda_1) \boxtimes \cdots \boxtimes \text{sgn}(\lambda_k)$.

Similarly, we denote by $F(n)$ the trivial $F\mathfrak{S}_n$-module and, for $H \leq \mathfrak{S}_n$, we denote by $F(H)$ the trivial $FH$-module. If $H = \mathfrak{S}_\lambda$ then we set $F(\lambda) := F(H)$.

3.2. Signed Young permutation and indecomposable signed Young modules. Let $\mathcal{P}^2(n)$ be the set consisting of all pairs $(\lambda|\zeta)$ of partitions $\lambda, \zeta$ such that $|\lambda| + |\zeta| = n$. We allow that $\lambda$ or $\zeta$ be the empty partition $\emptyset$. For $(\lambda|\zeta), (\alpha|\beta) \in \mathcal{P}^2(n)$, one says that $(\lambda|\zeta)$ dominates $(\alpha|\beta)$, and writes $(\lambda|\zeta) \trianglerighteq (\alpha|\beta)$, if, for all $k \geq 1$, one has

(a) $\sum_{i=1}^k \lambda_i \geq \sum_{i=1}^k \alpha_i$, and
(b) $|\lambda| + \sum_{i=1}^k \zeta_i \geq |\alpha| + \sum_{i=1}^k \beta_i$.

This gives rise to a partial order $\trianglerighteq$ on $\mathcal{P}^2(n)$, which is called the dominance order. Sometimes we shall also write $(\alpha|\beta) \trianglerighteq (\lambda|\zeta)$ instead of $(\lambda|\zeta) \trianglerighteq (\alpha|\beta)$.
Suppose that \((\lambda|\zeta) \triangleright (\alpha|\beta)\). We have both \(|\lambda| \geq |\alpha|\) and \(|\zeta| \leq |\beta|\). Furthermore, in the case when \(|\lambda| = |\alpha|\), one also has \(|\beta| = |\zeta|\), and hence \(\lambda \triangleright \alpha\) and \(\zeta \triangleright \beta\), where here \(\triangleright\) are the usual dominance orders on the sets of partitions of \(|\alpha|\) and \(|\beta|\), respectively.

For each \((\lambda|\zeta) \in \mathcal{P}^2(n)\), one has the signed Young permutation \(F\mathfrak{S}_n\)-module

\[
M(\lambda|\zeta) := \text{Ind}_{\mathfrak{S}_n \times \mathfrak{S}_n}^{\mathfrak{S}_n} (F(\lambda) \boxtimes \text{sgn}(\zeta)).
\]

In the case when \(\zeta = \emptyset\), one obtains the usual Young permutation module \(M^\lambda = M(\lambda|\emptyset)\).

An indecomposable summand of a signed Young permutation module is called an indecomposable signed Young module. Following Donkin [8], the isomorphism classes of indecomposable signed Young \(F\mathfrak{S}_n\)-modules are labelled by the pairs of partitions of the form \((\lambda|\zeta) \in \mathcal{P}^2(n)\), where each pair of \(\zeta\) is divisible by \(p\), that is, \(\zeta = p\mu = (p\mu_1, p\mu_2, \ldots)\), for some partition \(\mu\). Given \((\lambda|\mu) \in \mathcal{P}^2(n)\), one denotes the corresponding indecomposable signed Young \(F\mathfrak{S}_n\)-module by \(Y(\lambda|\mu)\). By [8] 2.3(8), one has

\[
M(\lambda|\mu) \cong Y(\lambda|\mu) \oplus C(\lambda|\mu),
\]

where \(C(\lambda|\mu)\) is isomorphic to a direct sum of indecomposable signed Young modules \(Y(\alpha|\beta)\) such that \((\alpha|\beta) \triangleright (\lambda|\mu)\). Furthermore, whenever \(\zeta = \emptyset\), one recovers the usual Young module \(Y^\lambda \cong Y(\lambda|\emptyset)\). By [19] Corollary 5.2.9, \(Y(\lambda|\mu)\) belongs to the block \(b_\lambda^\mu\) of \(F\mathfrak{S}_n\).

More generally, following [8] 2.3(7), for an arbitrary pair \((\lambda|\zeta) \in \mathcal{P}^2(n)\), the signed Young permutation module \(M(\lambda|\zeta)\) is isomorphic to a direct sum of some signed Young \(F\mathfrak{S}_n\)-modules \(Y(\alpha|\beta)\) such that \((\alpha|\beta) \triangleright (\lambda|\zeta)\).

3.3 Lemma. Let \(n \in \mathbb{N}\), and let \((\alpha|\beta) \in \mathcal{P}^2(n)\).

(a) For every positive integer \(m > n\), one has

\[
\text{Ind}^{\mathfrak{S}_m}_{\mathfrak{S}_n} (M(\alpha|\beta)) \cong M(\alpha\#(1^{m-n})|\beta),
\]

where \(\alpha\#(1^{m-n})\) is the concatenation of the partitions \(\alpha\) and \((1^{m-n})\).

(b) For every non-negative integer \(m < n\), the restriction \(\text{Res}^{\mathfrak{S}_m}_{\mathfrak{S}_n} (M(\alpha|\beta))\) is isomorphic to a direct sum of copies of signed Young permutation modules of the form \(M(\delta|\delta)\), where \((\delta|\delta)\) is the pair of partitions obtained from the pairwise rearrangement of some pair of compositions obtained from \((\alpha|\beta)\) by removing \(n - m\) nodes.

Proof. As for part (a), we have

\[
\text{Ind}^{\mathfrak{S}_m}_{\mathfrak{S}_n} (M(\alpha|\beta)) = \text{Ind}^{\mathfrak{S}_m}_{\mathfrak{S}_n \times \mathfrak{S}_n \times \mathfrak{S}_n \times \mathfrak{S}_n} (F(\alpha) \boxtimes \text{sgn}(\beta) \boxtimes F(\mu))
\cong \text{Ind}^{\mathfrak{S}_m}_{\mathfrak{S}_n \times \mathfrak{S}_n \times \mathfrak{S}_n \times \mathfrak{S}_n} (F(\alpha) \boxtimes \text{sgn}(\beta) \boxtimes F((1^{m-n}))) \cong M(\alpha\#(1^{m-n})|\beta).
\]

For part (b), we use the Mackey Formula to get

\[
\text{Res}^{\mathfrak{S}_m}_{\mathfrak{S}_n} (M(\alpha|\beta)) \cong \bigoplus_g \text{Ind}^{\mathfrak{S}_m}_{\mathfrak{S}_n \times \mathfrak{S}_n \times \mathfrak{S}_n \times \mathfrak{S}_n} \left( \text{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_n \times \mathfrak{S}_n \times \mathfrak{S}_n \times \mathfrak{S}_n} (F(\alpha) \boxtimes \text{sgn}(\beta)) \right)_{\mathfrak{S}_m} (g(N(g))_1).
\]

where \(g\) varies over a set of representatives of the double cosets \(\mathfrak{S}_m \setminus \mathfrak{S}_n / (\mathfrak{S}_n \times \mathfrak{S}_n)\).

Each direct summand \(N(g)\) is induced from some subgroup of the form \(g(\mathfrak{S}_n \times \mathfrak{S}_n) \cap \mathfrak{S}_m = g(\mathfrak{S}_\delta \times \mathfrak{S}_\delta)\), for a pair of compositions \((\delta|\delta)\) obtained from the pair \((\alpha|\beta)\) by removing some \(n - m\) nodes. The subgroup \(g(\mathfrak{S}_n \times \mathfrak{S}_n)\) is conjugate to the direct product of a pair of Young subgroups of \(\mathfrak{S}_m\). More precisely, after rearranging the parts of \(\delta\) and \(\hat{\delta}\), respectively, we obtain a pair of partitions \((\delta|\hat{\delta})\) such that \(M(\delta|\hat{\delta})\) is isomorphic to \(N(g)\). \(\square\)

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3.4 Notation. For the remainder of this section, we fix $\lambda|p\mu \in \mathcal{P}^2(n)$. Moreover, we fix the $p$-adic expansions

$$\lambda = \sum_{i=0}^{r_\lambda} p^i \cdot \lambda(i) \quad \text{and} \quad \mu = \sum_{i=0}^{r_\mu} p^i \cdot \mu(i)$$

(3)

of $\lambda$ and $\mu$, respectively, where $\lambda(r_\lambda) \neq \emptyset \neq \mu(r_\mu)$. In the case when $\lambda = \emptyset = \lambda(0)$, we set $r_\lambda := 0$, and in the case when $\mu = \emptyset = \mu(0)$, we set $r_\mu := 0$. Set $r := \max\{r_\lambda, r_\mu + 1\}$.

We define the following composition $\rho$ of $n$:

$$\rho := (1^{\lambda(0)}, p^{\lambda(1)}|\mu(0)}, (p^2)^{\lambda(2)}|\mu(1)}, \ldots, (p^r)^{\lambda(r)}|\mu(r-1)})).$$

(4)

3.5 Remark. Consider the indecomposable signed Young $F\mathfrak{S}_n$-module $Y(\lambda|p\mu)$ as introduced in 3.2 Since $Y|\lambda|p\mu) \mid M(\lambda|p\mu)$, the $F\mathfrak{S}_n$-module $Y(\lambda|p\mu) \otimes \text{sgn}(n)$ is an indecomposable direct summand of the signed Young permutation module

$$M(\lambda|p\mu) \otimes \text{sgn}(n) \cong \text{Ind}_{\mathfrak{S}_\lambda \times \mathfrak{S}_\mu}^{\mathfrak{S}_n}((F(\lambda) \boxtimes \text{sgn}(p\mu)) \otimes (\text{sgn}(\lambda) \boxtimes \text{sgn}(p\mu))) \cong M(p\mu|\lambda).$$

Thus $Y(\lambda|p\mu) \otimes \text{sgn}(n)$ is again an indecomposable signed Young $F\mathfrak{S}_n$-module. In Theorem 3.18, we shall determine the indecomposable signed Young module label of $Y(\lambda|p\mu) \otimes \text{sgn}(n)$ by exploiting the theory of Young vertices and the generalized Green correspondence in the sense of Grabmer [16] and Donkin [8]; we shall refer to the latter as Young–Green correspondence. We shall heavily use [8] 5.2. Some arguments there are quite subtle, since one has to be careful with various sign representations involved in the definition of the Young–Green correspondents of indecomposable signed Young modules. Therefore, for the reader’s convenience and to make our arguments as self-contained as possible, in Appendix A we follow the arguments in [8] 5.2 to present the results using our set-up.

3.6. Young vertices and Green vertices. To this end we begin by recalling some known facts from [8] [10].

(a) Let $G$ be any finite group, and let $M$ be an $FG$-module. If $H \leq G$ is such that $M \mid \text{Ind}_G^H(\text{Res}_H^G(M))$ then $M$ is called relatively $H$-projective. In the case when $M$ is an indecomposable $FG$-module and $Q$ is minimal such that $M$ is relatively $Q$-projective, $Q$ is called a Green vertex of $M$. By [17], the Green vertices of $M$ form a $G$-conjugacy class of $p$-subgroups of $G$. Moreover, if $Q$ is a Green vertex of $M$ then there is an indecomposable $FQ$-module $S$, unique up isomorphism and $N_G(Q)$-conjugation, such that $M \mid \text{Ind}_Q^G(S)$. One calls $S$ a Green $Q$-source of $M$.

(b) There is a generalized vertex theory in terms of Mackey systems. For details, we refer the reader to [16]. We briefly record the results related to the symmetric groups that we shall need in this paper. Suppose that $M$ is an indecomposable $F\mathfrak{S}_n$-module, and let $H \leq \mathfrak{S}_n$ be a Young subgroup such that $M$ is relatively $H$-projective but is not relatively $K$-projective for any proper Young subgroup $K$ of $H$. We call $H$ a Young vertex of $M$. Young vertices of $M$ are uniquely determined up to $\mathfrak{S}_n$-conjugation. If $H$ is a Young vertex of $M$ then there is some indecomposable $FH$-module $L$ such that $M \mid \text{Ind}_H^\mathfrak{S}_n(L)$. This module $L$ is called a Young $H$-source of $M$, and is uniquely determined up to isomorphism and $N_{\mathfrak{S}_n}(H)$-conjugation. In the case when $L$ is one-dimensional, one calls $M$ a linear-source module. By [8], the indecomposable linear-source $F\mathfrak{S}_n$-modules with respect to the Mackey system $\mathcal{Y}$ of Young subgroups are precisely the indecomposable signed Young modules.

Whenever $U \leq \mathfrak{S}_n$, one considers the set $\mathcal{Y} \downarrow U := \{H \cap U : H \in \mathcal{Y}\}$ of subgroups of $U$. One then has an analogous notion of $(\mathcal{Y} \downarrow U)$-vertices and sources of indecomposable
FU-modules. We shall then speak of Young vertices and Young sources of indecomposable FU-modules as well. Note that if \( U \) is a Young subgroup of \( \mathfrak{S}_n \) then \( \mathcal{Y} \downarrow U \subseteq \mathcal{Y} \).

(c) Suppose that \( U \leq \mathfrak{S}_n \), and let \( M \) be an indecomposable FU-module with Green vertex \( Q \). Then, by [16] Lemma 3.9,

\[
H := \bigcap_{K \in \mathcal{Y} \downarrow U} K
\]

is a Young vertex of \( M \). Moreover, one has \( N_U(Q) \leq N_U(H) \).

The next theorem describes the Young vertices and sources as well as the Green vertices and sources of indecomposable signed Young modules. Part (a) is due to Donkin, see [8, 5.1]. The assertion of part (b) is certainly also well known, and a proof in the case when \( \mu = \emptyset \) can, for instance, be found in [9]. We include a short proof of part (b) for convenience.

3.7 Theorem. Let \((\lambda|\mu)\in \mathcal{P}^2(n)\), and let \( \rho \) be the composition of \( n \) as in [37].

(a) The indecomposable signed Young module \( Y(\lambda|\mu) \) has Young vertex \( \mathfrak{S}_\rho \) and linear Young source.

(b) Any Sylow \( p \)-subgroup of \( \mathfrak{S}_\rho \) is a Green vertex of the indecomposable signed Young module \( Y(\lambda|\mu) \). Moreover, \( Y(\lambda|\mu) \) has trivial Green sources.

Proof. Set \( Y := Y(\lambda|\mu) \). As mentioned above, part (a) was proved in [8, 5.1(3)].

To prove part (b), set \( H := \mathfrak{S}_\rho \), and let \( L \) be a linear Young \( H \)-source of \( Y \). Let \( R \) be a Sylow \( p \)-subgroup of \( H \), and let \( Q \) be a Green vertex of \( Y \). Since \( Y \) is relatively \( H \)-projective, there is an indecomposable \( FH \)-module \( L' \) such that \( L' \mid \text{Res}_H^R(Y) \) and \( Y \mid \text{Ind}_R^H(L') \). By definition, \( L' \) is also a Young \( H \)-source of \( Y \). Thus, by [16] Lemma 2.10], \( L' \cong g L, \) for some \( g \in N_{\mathfrak{S}_n}(H) \). In particular, \( L' \) has dimension 1 as well. Therefore, \( L' \) has Green vertex \( R \) and trivial Green \( R \)-source \( F(R) \), so that \( L' \mid \text{Ind}_R^H(F(R)) \) and \( F(R) \mid \text{Res}_R^H(L') \). This implies \( Y \mid \text{Ind}_R^H(F(R)) \) and \( F(R) \mid \text{Res}_R^H(Y) \). So, by [37, §4, Lemma 3.4], we get \( Q \leq \mathfrak{S}_n R \) as well as \( R \leq \mathfrak{S}_n Q \). Hence \( Q = \mathfrak{S}_n R \) is a Green vertex of \( Y \), and \( F(Q) \) is a Green \( Q \)-source of \( Y \).

3.8. Young–Green correspondents and Green correspondents. Recall, for instance from [37, Section 4.4], the notion of Green correspondence. There is an analogous notion of Green correspondence with respect to a Mackey system \( \mathcal{Y} \) of subgroups of a given finite group (see [16, Section 3]). In the case of the symmetric groups, we shall consider the Mackey system of Young subgroups, and from now on refer to the resulting generalized Green correspondence as Young–Green correspondence.

Suppose that \( M \) is an indecomposable \( F\mathfrak{S}_n \)-module with Young vertex \( H \), and let \( N \) be a subgroup of \( \mathfrak{S}_n \) such that \( N_{\mathfrak{S}_n}(H) \leq N \leq \mathfrak{S}_n \). Let \( M' \) be the Young–Green correspondent of \( M \) with respect to \( N \). By [16] Satz 3.7], one has \( M' \mid \text{Res}_N^M(M) \) and \( M \mid \text{Res}_N^M(M') \). Moreover, \( H \) is also a Young vertex of \( M' \), and \( M \) and \( M' \) share a common Young \( H \)-source. In particular, \( M \) is a linear-source module if and only if \( M' \) is. The Young–Green correspondence induces a bijection between the isomorphism classes of indecomposable \( F\mathfrak{S}_n \)-modules with Young vertex \( H \) and the isomorphism classes of indecomposable \( FN \)-modules with Young vertex \( H \). Furthermore, \( M \) and \( M' \) must have a common Green vertex and a common Green source.

3.9 Lemma. Let \( M \) be an indecomposable \( F\mathfrak{S}_n \)-module, and let \( H \) be a Young vertex of \( M \). Then
Thus we may view both modules as modules of the quotient group. Retain the notation as in 3.10 above.

It is understood that, whenever \(k \geq 2\), the outer tensor product \(\text{sgn}(k) \circ m\) is a one-dimensional module of the base group of \(S_k \wr S_m\), and extends to a one-dimensional \(F[S_k \wr S_m]-\)module via tensor induction, that is,

\[
(g_1, \ldots, g_m; \sigma) \cdot x_1 \otimes \cdots \otimes x_m := g_1 x_{\sigma^{-1}(1)} \otimes \cdots \otimes g_m x_{\sigma^{-1}(m)}
\]

for \(x_1, \ldots, x_m \in \text{sgn}(k)\) and \((g_1, \ldots, g_m; \sigma) \in S_k \wr S_m\). Thus the group \(S_m^2 \leq S_k \wr S_m\) acts trivially on the resulting \(F[S_k \wr S_m]-\)module, which, by abuse of notation, we denote by \(\text{sgn}(k) \circ m\) again. On the other hand, \(\text{sgn}(k) \circ m\) has another extension, namely,

\[
\text{sgn}(k) \circ m := \text{sgn}(k) \circ m \otimes \text{Inf}^{S_k \wr S_m}(\text{sgn}(m))
\]

So the action of \(S_k \wr S_m\) on \(\text{sgn}(k) \circ m\) is given by

\[
(g_1, \ldots, g_m; \sigma) * x_1 \otimes \cdots \otimes x_m = \text{sgn}(\sigma) \cdot \text{sgn}(g_1) \cdots \text{sgn}(g_m) \cdot x_1 \otimes \cdots \otimes x_m
\]

for \(x_1, \ldots, x_m \in \text{sgn}(k)\) and \((g_1, \ldots, g_m; \sigma) \in S_k \wr S_m\).

Let \(m_1, m_2 \in \mathbb{N}\) be such that \(m = m_1 + m_2\). Let further \(\alpha \in \mathcal{R} \mathcal{P}(m_1)\) and \(\beta \in \mathcal{R} \mathcal{P}(m_2)\). Following [3], we construct an \(F[S_k \wr S_m]-\)module \(R_k(\alpha|\beta)\) as follows. Let \(P(\alpha)\) be the projective cover of the simple \(F S_{m_1}-\)module \(D_\alpha\), and let \(P(\beta)\) be the projective cover of the simple \(F S_{m_2}-\)module \(D_\beta\). Set \(\hat{P}(\alpha) := \text{Inf}^{S_k \wr S_m}_{S_{m_1}}(P(\alpha))\) and \(\hat{P}(\beta) := \text{Inf}^{S_k \wr S_m}_{S_{m_2}}(P(\beta))\). With the above notation, we obtain the \(F[S_k \wr S_m]-\)module

\[
R_k(\alpha|\beta) := \text{Ind}^{S_k \wr S_m}_{S_k \wr S_{m_2}}(\hat{P}(\alpha) \boxtimes (\hat{P}(\beta) \circ m))
\]

(5)

It is understood that, whenever \(m_1 = 0\) and \(m_2 > 0\) (respectively, \(m_1 > 0\) and \(m_2 = 0\)), we have \(R_k(\alpha|\beta) = \hat{P}(\beta) \circ m\) (respectively, \(R_k(\alpha|\beta) = \hat{P}(\alpha)\)). Furthermore, \(R_k(\emptyset|\emptyset)\) is the trivial \(F S_0\)-module if \(m = 0\).

3.11 Remark. Retain the notation as in 3.10 above.

(a) Note that, in the case when \(k\) is odd, we have \(\text{sgn}(k) \circ m \cong \text{sgn}(S_k \wr S_{m_2})\). But if \(k\) is even then \(\text{sgn}(k) \circ m_2 \cong \text{sgn}(S_k \wr S_{m_2})\).

(b) Let \(A_k \leq S_k \wr S_{m_2}\) be the alternating group of degree \(k\). The wreath product \(S_k \wr S_{m_2}\) contains the normal subgroup \(A_k \wr S_{m_2}\), which acts trivially on both \(\text{sgn}(k) \circ m_2\) and \(\text{sgn}(k)\).

Thus we may view both modules as modules of the quotient group \((S_k \wr S_{m_2})/A_k \wr S_{m_2} \cong (S_k/A_k) \wr S_{m_2} \cong S_2 \wr S_{m_2}\). Via the latter group isomorphism, we also have \(\text{sgn}(k) \circ m_2 \cong \text{Inf}^{S_2 \wr S_{m_2}}(\text{sgn}(2) \circ m_2)\) and \(\text{sgn}(k) \circ m_2 \cong \text{Inf}^{S_2 \wr S_{m_2}}(\text{sgn}(2) \circ m_2)\). However, \(\text{sgn}(S_k \wr S_{m_2}) \cong \text{sgn}(2) \circ m_2 \neq \text{sgn}(2)\).

(c) By part (b) and Lemma 2.10, we have \(R_k(\alpha|\beta) \cong \text{Inf}^{S_2 \wr S_{m_2}}(R_2(\alpha|\beta))\) via the epimorphism \(S_k \wr S_{m_2} \to (S_k \wr S_{m_2})/A_k \wr S_{m_2} \cong S_2 \wr S_m\).
3.12 Lemma. Let $k \in \mathbb{N}$ be odd. With the notation as in 3.11, we have

$$R_k(\alpha|\beta) \otimes \text{sgn} (\mathcal{G}_k \wr \mathcal{G}_m) \cong R_k(\beta|\alpha).$$

Proof. By Remark 3.11(a) and the Frobenius Formula, we have

$$R_k(\alpha|\beta) \otimes \text{sgn} (\mathcal{G}_k \wr \mathcal{G}_m)$$

$$= \text{Ind}_{(\mathcal{G}_k \wr \mathcal{G}_m) \times (\mathcal{G}_k \wr \mathcal{G}_m)} \left( \left( \tilde{P}(\alpha) \boxtimes \tilde{P}(\beta) \otimes \text{sgn} (\mathcal{G}_k \wr \mathcal{G}_m) \right) \otimes \text{sgn} (\mathcal{G}_k \wr \mathcal{G}_m) \right)$$

$$\cong \text{Ind}_{(\mathcal{G}_k \wr \mathcal{G}_m) \times (\mathcal{G}_k \wr \mathcal{G}_m)} \left( \left( \tilde{P}(\alpha) \boxtimes \tilde{P}(\beta) \otimes \text{sgn} (\mathcal{G}_k \wr \mathcal{G}_m) \right) \otimes \text{sgn} (\mathcal{G}_k \wr \mathcal{G}_m) \right) \otimes \tilde{P}(\beta)$$

$$\cong \text{Ind}_{(\mathcal{G}_k \wr \mathcal{G}_m) \times (\mathcal{G}_k \wr \mathcal{G}_m)} \left( \left( \tilde{P}(\alpha) \otimes \text{sgn} (\mathcal{G}_k \wr \mathcal{G}_m) \right) \otimes \tilde{P}(\beta) \right) \cong R_k(\beta|\alpha).$$

\[\square\]

3.13 Notation. Again, following the notation as in 3.4, let $(\lambda|\mu) \in \mathcal{S}^2(n)$, let $\lambda = \sum_{i=0}^{r_\lambda} p^i \cdot \lambda(i)$ and $\mu = \sum_{i=0}^{r_\mu} p^i \cdot \mu(i)$ be the $p$-adic expansions of $\lambda$ and $\mu$ respectively, let $r := \max\{r_\lambda, r_\mu + 1\}$, and let $\rho$ be the composition of $n$ defined in [4]. Set $n_0 := |\lambda(0)|$ and, for $i = 1, \ldots, r$, set $n_i := |\lambda(i)| + |\mu(i) - 1|$. By Theorem 3.7(a), the Young subgroup

$$\mathcal{G}_p = (\mathcal{G}_1)^{n_0} \times (\mathcal{G}_p)^{n_1} \times \cdots \times (\mathcal{G}_P)^{n_r}$$

of $\mathcal{G}_n$ is a Young vertex of $Y(\lambda|\mu)$; here the first non-trivial direct factor $\mathcal{G}_p$ is supposed to act on the set $\{n_0 + 1, \ldots, n_0 + p\}$, and so on.

In the sequel, we shall denote the normalizer $N_{\mathcal{G}_n}(\mathcal{G}_p)$ by $N(\rho)$ and identify $N(\rho)$ with the direct product

$$\mathcal{G}_{n_0} \times (\mathcal{G}_p \wr \mathcal{G}_{n_1}) \times \cdots \times (\mathcal{G}_p \wr \mathcal{G}_{n_r}).$$

For $i = 0, \ldots, r$, let $P_i$ be a Sylow $p$-subgroup of $\mathcal{G}_p$, so that $P_i := (P_i)^{n_0} \times (P_i)^{n_1} \times \cdots \times (P_i)^{n_r}$ is a Sylow $p$-subgroup of $\mathcal{G}_p$. Now observe that

$$N_{\mathcal{G}_n}(P_i) = \mathcal{G}_{n_0} \times N_{\mathcal{G}_{n_1}}((P_i)^{n_1}) \times \cdots \times N_{\mathcal{G}_{n_r}}((P_i)^{n_r})$$

$$= \mathcal{G}_{n_0} \times (N_{\mathcal{G}_p}(P_i) \wr \mathcal{G}_{n_1}) \times \cdots \times (N_{\mathcal{G}_p}(P_i) \wr \mathcal{G}_{n_r}) \subseteq N(\rho).$$

With the above notation, we can now state Donkin’s result on Young–Green correspondents of indecomposable signed Young modules (see also Appendix A).

3.14 Theorem (3.5 (2))). Let $(\lambda|\mu) \in \mathcal{S}^2(n)$, let $\lambda = \sum_{i=0}^{r_\lambda} p^i \cdot \lambda(i)$ and $\mu = \sum_{i=0}^{r_\mu} p^i \cdot \mu(i)$ be the $p$-adic expansions of $\lambda$ and $\mu$, respectively, let $r$ be the integer, and let $\rho$ be the composition of $n$ as in 3.4. Then the $\text{FN}(\rho)$-module

$$R(\lambda|\mu) := P(\lambda(0)) \boxtimes R_{p}(\lambda(1)|\mu(0)) \boxtimes \cdots \boxtimes R_{p^r}(\lambda(r)|\mu(r-1))$$

is the Young–Green correspondent of $Y(\lambda|\mu)$ with respect to $N(\rho)$.

As we have seen in Theorem 3.7(b), any Sylow $p$-subgroup of the Young subgroup $\mathcal{G}_\rho$ of $\mathcal{G}_n$ is a Green vertex of the indecomposable signed Young module $Y(\lambda|\mu)$. As an easy consequence of Theorem 3.14 we get the following corollary.

3.15 Corollary. Let $(\lambda|\mu) \in \mathcal{S}^2(n)$, and let $P_\rho$ be a Sylow $p$-subgroup of $\mathcal{G}_\rho$ as in 3.14. Suppose that $N_{\mathcal{G}_n}(P_\rho) \leq H \leq N(\rho)$. Then $\text{Res}_H^{N(\rho)}(R(\lambda|\mu))$ is the Green correspondent of $Y(\lambda|\mu)$ with respect to the subgroup $H$. 
Proof. It suffices to show that $R(\lambda|\rho\mu)$ restricts indecomposably to $H$ and that the indecomposable restriction has Green vertex $P_{\rho}$. First we shall prove the corollary in the case when $H = N_{\mathfrak{S}_{n}}(P_{\rho})$. For this it further suffices to show that, for $i = 1, \ldots, r$, the $F[\mathfrak{S}_{p_{i}} \wr \mathfrak{S}_{n_{i}}]$-module $R_{\rho_{i}}(\lambda(i)\mu(i-1))$ restricts indecomposably to $N_{\mathfrak{S}_{p_{i}}}(P_{\rho_{i}})\wr \mathfrak{S}_{n_{i}}$ and that the restriction has Green vertex $(P_{\rho_{i}})_{m_{i}}$. Fix $i \in \{1, \ldots, r\}$ for the remainder of this proof, and set $m_{1} := |\lambda(i)|$, $m_{2} := |\mu(i-1)|$ and $N := N_{\mathfrak{S}_{p_{i}}}(P_{\rho_{i}})$. Recall from (5) that

$$R_{\rho_{i}}(\lambda(i)\mu(i-1)) = \text{Ind}_{\mathfrak{S}_{p_{i}} \wr \mathfrak{S}_{n_{i}}}^{\mathfrak{S}_{p_{i}} \wr \mathfrak{S}_{n_{i}}}(\hat{P}(\lambda(i)) \boxtimes (\hat{P}(\mu(i-1)) \otimes \text{sgn}(p'))^{\otimes m_{2}})).$$

Here we have used that $\text{sgn}(k)^{\otimes m_{2}} \cong \text{sgn}(k)^{\otimes m_{2}} \otimes \text{Ind}_{\mathfrak{S}_{m_{2}}}^{\mathfrak{S}_{p_{i}} \wr \mathfrak{S}_{m_{2}}}(\text{sgn}(m_{2}))$ as $F[\mathfrak{S}_{p_{i}} \wr \mathfrak{S}_{m_{2}}]$-modules and that $\text{sgn}(m_{2}) \otimes P(\mu(i-1)) \cong P(\mu(|\mu(i-1)|))$ as $F[\mathfrak{S}_{m_{2}}]$-modules. To determine $\text{Res}^{\mathfrak{S}_{m_{1}} \wr \mathfrak{S}_{n_{i}}}_{N \wr \mathfrak{S}_{n_{i}}}(R_{\rho_{i}}(\lambda(i)\mu(i-1)))$, we first apply the Mackey Formula. Every element $x \in \mathfrak{S}_{p_{i}} \wr \mathfrak{S}_{n_{i}}$ can be written as a product $xy$, for uniquely determined elements $y \in \mathfrak{S}_{m_{i}} \leqslant N \wr \mathfrak{S}_{n_{i}}$ and $z \in \mathfrak{S}_{m_{2}} \leqslant (\mathfrak{S}_{p_{i}} \wr \mathfrak{S}_{m_{2}}) \wr (\mathfrak{S}_{p_{i}} \wr \mathfrak{S}_{m_{2}})$. Therefore, there is precisely one double coset in $(N \wr \mathfrak{S}_{n_{i}})(\mathfrak{S}_{p_{i}} \wr \mathfrak{S}_{m_{2}})/(\mathfrak{S}_{p_{i}} \wr \mathfrak{S}_{m_{2}}) \wr (\mathfrak{S}_{p_{i}} \wr \mathfrak{S}_{m_{2}})$, and the Mackey Formula gives

$$\text{Res}^{\mathfrak{S}_{m_{1}} \wr \mathfrak{S}_{n_{i}}}_{N \wr \mathfrak{S}_{n_{i}}}(R_{\rho_{i}}(\lambda(i)\mu(i-1))) = \text{Ind}^{\mathfrak{S}_{m_{1}} \wr \mathfrak{S}_{n_{i}}}_{(N \wr \mathfrak{S}_{n_{i}}) \times (N \wr \mathfrak{S}_{n_{i}})}(\hat{P}(\lambda(i)) \boxtimes (\hat{P}(\mu(i-1)) \otimes \text{sgn}(p'))^{\otimes m_{2}})).$$

where $\hat{P}(\lambda(i))$ denotes the inflation of the $F[\mathfrak{S}_{m_{1}} \wr \mathfrak{S}_{m_{2}}]$-module $P(\lambda(i))$ to $N \wr \mathfrak{S}_{m_{1}}$ and $\hat{P}(\mu(i-1))$ denotes the inflation of the $F[\mathfrak{S}_{m_{2}} \wr \mathfrak{S}_{m_{2}}]$-module $P(\mu(i-1))$ to $N \wr \mathfrak{S}_{m_{2}}$. Recall that $P(\lambda(i)) \boxtimes P(\mu(i-1)))$ is an indecomposable projective $F[\mathfrak{S}_{m_{1}} \wr \mathfrak{S}_{m_{2}}]$-module and that both $\text{sgn}(p')$ and the trivial $F[\mathfrak{S}_{m_{2}}]$-module have Green vertex $P_{\rho_{i}}$. Thus [27, Proposition 5.1] implies that $X$ is an indecomposable $F[N \wr \mathfrak{S}_{n_{i}}]$-module with Green vertex $(P_{\rho_{i}})^{m_{1}+m_{2}} = (P_{\rho_{i}})^{n_{i}}$ as claimed.

Suppose now that $H$ is any subgroup satisfying $N_{\mathfrak{S}_{n}}(P_{\rho}) \leqslant H \leqslant N(\rho)$. Since $R(\lambda|\rho\mu)$ restricts indecomposably to $N_{\mathfrak{S}_{n}}(P_{\rho})$, it also restricts indecomposably to $H$. Let $Q$ be a Green vertex of $\text{Res}^{N(\rho)}_{H}(R(\lambda|\rho\mu))$. Then $P_{\rho} \leqslant_{H} Q$, by [37, §4, Lemma 3.4]. On the other hand, $\text{Res}^{N(\rho)}_{H}(R(\lambda|\rho\mu)) \mid \text{Res}^{\mathfrak{S}_{n}}_{H}(Y(\lambda|\rho\mu))$ has also Green vertex $P_{\rho}$, by Theorem [3.7(b)]. Thus $Q \leqslant_{\mathfrak{S}_{n}} P_{\rho}$, by [37, §4, Lemma 3.4] again. This implies $Q = H P_{\rho}$, so that $P_{\rho}$ is indeed a Green vertex of $\text{Res}^{N(\rho)}_{H}(R(\lambda|\rho\mu))$. Hence $\text{Res}^{N(\rho)}_{H}(R(\lambda|\rho\mu))$ is the Green correspondent of $Y(\lambda|\rho\mu)$ with respect to $H$.

3.16. Broué correspondents. We refer the reader to [2] for the details of the Broué correspondence for $p$-permutation modules. For any finite group $G$, a $p$-permutation $FG$-module is a direct sum of some indecomposable $FG$-modules with trivial Green sources. Given a $p$-subgroup $P$ of a finite group $G$, Broué correspondence gives a one-to-one correspondence between the isomorphism classes of indecomposable trivial-Green-source $FG$-modules with Green vertex $P$ and the isomorphism classes of indecomposable projective $F[N_{G}(P)/P]$-modules. More precisely, an indecomposable $FG$-module $M$ with Green vertex $P$ and trivial Green $P$-source is sent to its Brauer quotient $M(P)$. The latter carries the structure of an $FN_{G}(P)$-module on which $P$ acts trivially. Moreover, $M(P)$ is isomorphic to the Green correspondent of $M$ with respect to the subgroup $N_{G}(P)$.

Since indecomposable signed Young modules have trivial Green sources, by Theorem [3.7(b)], they are indecomposable $p$-permutation modules. Thus Corollary [3.15] gives us the Broué correspondents of indecomposable signed Young modules as follows.
3.17 Corollary. Let \((\lambda|\mu)\) \(\in \mathcal{P}^2(n)\), and let \(P_\rho\) be a Sylow \(p\)-subgroup of \(S_p\) as in \([3.14]\). Then \(P_\rho\) acts trivially on the Green correspondent \(\text{Res}^{N(\rho)}_{N_{\mathfrak{S}_n}(p)}(R(\lambda|\mu))\) of \(Y(\lambda|\mu)\) with respect to \(N_{\mathfrak{S}_n}(P_\rho)\). Moreover, viewed as \(F[N_{\mathfrak{S}_n}(P_\rho)/P_\rho]\)-module, the restriction \(\text{Res}^{N(\rho)}_{N_{\mathfrak{S}_n}(p)}(R(\lambda|\mu))\) is the Broué correspondent of \(Y(\lambda|\mu)\).

We can now state our main result of this section.

3.18 Theorem. Let \((\lambda|\mu)\) \(\in \mathcal{P}^2(n)\). Then one has an isomorphism

\[
Y(\lambda|\mu) \otimes \text{sgn}(n) \cong Y(\mathbf{m}(\lambda(0)) + p\mu|\lambda - \lambda(0))
\]

of \(F\mathfrak{S}_n\)-modules.

Proof. Recall the notation in \([3.4]\) and \([3.13]\). By Theorem \([3.7](a)\) and Lemma \([3.9](a)\), both \(Y(\lambda|\mu)\) and \(Y(\lambda|\mu) \otimes \text{sgn}(n)\) have Young vertex \(\mathfrak{S}_p\), where

\[
\rho = (1^{n_0}, p^{n_1}, (p^2)^{n_2}, \ldots, (p^r)^{n_r}),
\]

\(n_0 := |\lambda(0)|\), and \(n_i := |\lambda(i)| + |\mu(i-1)|\), for all \(i = 1, \ldots, r\). Next we show that \(Y(\mathbf{m}(\lambda(0)) + p\mu|\lambda - \lambda(0))\) also has Young vertex \(\mathfrak{S}_p\). Set \(\alpha := \mathbf{m}(\lambda(0)) + p\mu\) and \(\beta := \lambda - \lambda(0)\).

Since both \(\lambda(0)\) and \(\mathbf{m}(\lambda(0))\) are \(p\)-restricted, \(\alpha\) has \(p\)-adic expansion \(\alpha = \mathbf{m}(\lambda(0)) + p\mu = p^0 \cdot \mathbf{m}(\lambda(0)) + \sum_{i=1}^r p^i \cdot \mu(i-1)\). Moreover, \(\beta\) has \(p\)-adic expansion \(\beta = \sum_{i=1}^r p^{i-1} \lambda(i)\).

Hence \(|\alpha(0)| = |\mathbf{m}(\lambda(0))| = |\lambda(0)| = n_0\), and \(|\alpha(i)| + |\beta(i-1)| = |\mu(i-1)| + |\lambda(i)| = n_i\), for every \(i = 1, \ldots, r\). Thus, by Theorem \([3.7](a)\), \(Y(\alpha|\beta)\) has Young vertex \(\mathfrak{S}_p\) as desired.

Now, in order to prove the isomorphism \(Y(\lambda|\mu) \otimes \text{sgn}(n) \cong Y(\alpha|\beta)\), it suffices to verify that the Young–Green correspondents of the modules with respect to \(N(\rho)\) are isomorphic. By Theorem \([3.14]\) and Lemma \([3.9](b)\), \(Y(\lambda|\mu) \otimes \text{sgn}(n)\) has Young–Green correspondent \(R(\lambda|\mu) \otimes \text{sgn}(N(\rho)) = (P(\lambda(0)) \boxtimes R_{\mathfrak{S}_p}(\lambda(1)|\mu(0)) \boxtimes \cdots \boxtimes R_{\mathfrak{S}_p}(\lambda(r)|\mu(r-1))) \otimes \text{sgn}(N(\rho))\). Furthermore, together with Lemma \([3.12]\) we have

\[
R(\lambda|\mu) \otimes \text{sgn}(N(\rho)) \cong (P(\lambda(0)) \otimes \text{sgn}(n_0)) \boxtimes (R_{\mathfrak{S}_p}(\lambda(1)|\mu(0)) \otimes \text{sgn}(S_{p^e} \leq S_n)) \boxtimes \cdots \boxtimes (R_{\mathfrak{S}_p}(\lambda(r)|\mu(r-1)) \otimes \text{sgn}(S_{p^e} \leq S_n))
\]

\[
\cong P(\mathbf{m}(\lambda(0))) \boxtimes R_{\mathfrak{S}_p}(\mu(0)|\lambda(1)) \boxtimes \cdots \boxtimes R_{\mathfrak{S}_p}(\mu(r-1)|\lambda(r))
\]

\[
= P(\alpha(0)) \boxtimes R_{\mathfrak{S}_p}(\alpha(1)|\beta(0)) \boxtimes \cdots \boxtimes R_{\mathfrak{S}_p}(\alpha(r)|\beta(r-1)) = R(\alpha|\beta),
\]

which, by Theorem \([3.14]\), is the Young–Green correspondent of \(Y(\alpha|\beta)\) with respect to \(N(\rho)\). Consequently, \(Y(\lambda|\mu) \otimes \text{sgn}(n) \cong Y(\alpha|\beta)\), by \([16]\) Satz 3.7.

4 Simple Specht Modules

Let \(F\) be a field of characteristic \(p > 0\). In this section, we collect some combinatorial properties of the JM-partitions labelling the simple Specht modules for \(p > 2\), and we recall the procedure of inducing any simple Specht module successively to obtain a simple Specht module belonging to a Rouquier block. We introduce the function \(\Phi: \mathcal{P}(n) \rightarrow \mathcal{P}^2(n)\), which will eventually give us the correct labelling of a simple Specht module as an indecomposable signed Young module (see Definition \([4.7]\) and Theorem \([4.1]\)). We study the effects of \(\Phi\) on JM-partitions and pairs of adjacent JM-partitions.

In \([24]\), James and Mathas established a characterization of simple Specht \(F\mathfrak{S}_n\)-modules in the case when \(p = 2\), and conjectured a characterization for \(p > 2\). The conjecture was proved by the work of Fayers \([11, 12]\) and Lyle \([23]\). We shall now describe the characterization in the case when \(p\) is odd. For the remainder of this section, let \(p \geq 3\).

For an integer \(m \geq 1\), we denote by \(\nu_p(m)\) the largest non-negative integer \(l\) such that \(p^l\) divides \(m\).
4.1 Definition. A partition $\lambda \in \mathcal{P}(n)$ is called a JM-partition if there are no nodes $(a, b)$, $(a, y)$ and $(x, b)$ in the Young diagram $[\lambda]$ such that
(a) $\nu_p(h_\lambda(a, b)) > 0$, and
(b) $\nu_p(h_\lambda(x, b)) \neq \nu_p(h_\lambda(a, b)) \neq \nu_p(h_\lambda(a, y))$.
We denote the subset of $\mathcal{P}(n)$ consisting of JM-partitions by $\text{JM}(n)$.

As an immediate consequence of the above definition, one has:

4.2 Lemma. Let $\lambda \in \mathcal{P}(n)$. Then $\lambda \in \text{JM}(n)$ if and only if $\lambda' \in \text{JM}(n)$.

For the purpose of our paper, we need the following characterization of JM-partitions proved by Fayers.

4.3 Proposition ([12] Proposition 2.1]). Let $\lambda \in \mathcal{P}(n)$. Then $\lambda \in \text{JM}(n)$ if and only if, for every abacus display of $\lambda$, there exist some integers $i, j \in \{0, \ldots, p - 1\}$ satisfying the following conditions:
(F1) $\lambda^{(k)} = \emptyset$, for all $k \in \{0, \ldots, p - 1\}$ such that $i \neq k \neq j$;
(F2) if a position $i + ap$ on runner $i$ is unoccupied then every position $b > i + ap$ not on runner $i$ is unoccupied;
(F3) if a position $j + cp$ on runner $j$ is occupied then every position $d < j + cp$ not on runner $j$ is occupied;
(F4) $\lambda^{(i)}$ is a $p$-regular partition, and
(F5) $\lambda^{(j)}$ is a $p$-restricted partition.

4.4 Remark. We again emphasize that the abacus display of a partition $\lambda$ depends on the length of the chosen sequence of $\beta$-numbers. In particular, if $\lambda$ is a JM-partition then the numbers $i$ and $j$ in Proposition 4.3 depend on the chosen $\beta$-numbers. However, the partitions $\lambda^{(i)}$ and $\lambda^{(j)}$ do not depend on the choice of $\beta$-numbers. In other words, $\lambda^{(i)}$ and $\lambda^{(j)}$ are uniquely determined by the JM-partition $\lambda$.

4.5 Theorem ([11] [12] [33]). Let $\lambda \in \mathcal{P}(n)$. Then the Specht $F \mathfrak{S}_n$-module $S^\lambda$ is simple if and only if $\lambda \in \text{JM}(n)$.

We shall investigate JM-partitions in more detail. More generally, we are mostly interested in partitions with abacus displays satisfying the conditions (F1), (F2) and (F3) in Proposition 4.3 for some runners $i$ and $j$.

4.6 Lemma. Let $\lambda \in \mathcal{P}(n)$, and let $i, j \in \{0, \ldots, p - 1\}$ be integers satisfying the conditions (F1), (F2) and (F3) in Proposition 4.3 for some abacus display of $\lambda$. If $\tilde{\lambda} \neq \lambda$ then $i \neq j$.

Proof. Assume that $i = j$ and $\tilde{\lambda} \neq \lambda$. By Proposition 4.3(F1), runner $i$ is the only runner such that $\lambda^{(i)} \neq \emptyset$. Since $\tilde{\lambda} \neq \lambda$, there are some integers $0 \leq a < b$ such that the positions $i + ap$ and $i + bp$ are unoccupied and occupied, respectively. The position $i + ap + 1$ is not on runner $i$. By Proposition 4.3(F2), the position $i + ap + 1$ is unoccupied. On the other hand, by Proposition 4.3(F3), the position $i + ap + 1$ is occupied, a contradiction. \hfill \Box

The next definition will be crucial to state Theorem 5.1, our main result of this paper.

4.7 Definition. Let $\Phi$ be the map $\Phi : \mathcal{P}(n) \to \mathcal{P}^2(n)$ defined by
$$\Phi(\lambda) = (\alpha|p\beta),$$
where $p\beta = \lambda' - \lambda'(0)$ and $\alpha = (\lambda'(0))'$.
4.8 Remark. In order to obtain \( \Phi(\lambda) \) for a given \( \lambda = (\lambda_1, \ldots, \lambda_k) \in \mathcal{P}(n) \), one proceeds as follows. Successively remove all possible vertical rim p-hooks from the rightmost, that is, the \( \lambda_1 \)th column of \( [\lambda] \), and let \( \beta_{\lambda_1} \) be the total number of vertical p-hooks removed. Next remove all possible vertical rim p-hooks from the \((\lambda_1 - 1)\)st column of the Young diagram of the resulting partition, and let \( \beta_{\lambda_1 - 1} \) be the total number of these vertical p-hooks. Proceeding in this way from right to left, we end up with the partition \( \alpha \). Moreover, \( \beta = (\beta_1, \ldots, \beta_{\lambda_1}) \) counts the number of vertical p-hooks removed from each column of the initial Young diagram \([\lambda]\).

We shall be mostly interested in \( \Phi(\lambda) \) in the case when \( \lambda \) is a JM-partition. The next lemma shows, in particular, how \( \Phi(\lambda) \) can be read off from the abacus display in Proposition 4.3 when \( \lambda \) is a JM-partition.

4.9 Lemma. Let \( \lambda \in \mathcal{P}(n) \), and let \( i, j \in \{0, \ldots, p - 1\} \) be integers satisfying the conditions \((F1), (F2) \) and \((F3) \) in Proposition 4.3 for some abacus display of \( \lambda \). Then the p-core of \( \lambda \) is obtained by the independent procedures of stripping off all horizontal and vertical rim p-hooks of \( \lambda \). Furthermore, one has

\[
\Phi(\lambda) = (\tilde{\lambda} + p\lambda(1)\,|p(\lambda(2))\ldots\,= (\tilde{\lambda} + \lambda - \lambda(0)|\lambda' - \lambda(0)).
\]

Proof. Let \( B \) be the abacus display in the statement. If \( \tilde{\lambda} = \lambda \) then there is nothing to show. Suppose that \( \tilde{\lambda} \neq \lambda \). By Lemma 4.6 \( i \neq j \). Label all the beads of \( B \) by \( B_1, \ldots, B_s \) such that their positions are \( b_1 > \cdots > b_s \), respectively. Recall how one can read off the partition \( \lambda \) from \( B \) as explained in 2.2(i). For each \( k \in \{1, \ldots, s\} \), there are exactly \( \lambda_k \) unoccupied positions \( t \) such that \( t < b_k \), and let this number be \( u_B(B_k) \). So we get \( \lambda = (u_B(B_1), u_B(B_2), \ldots) \).

Suppose that \( \lambda(0) \neq \emptyset \), and let \( i + a_0p \) be the highest unoccupied position on runner \( i \). Let \( k_i \) be the unique largest number such that \( i + a_0p < b_{k_i} \). Then \( B_1, \ldots, B_{k_i} \) belong to runner \( i \), by Proposition 4.3(F2). Moving a bead \( B_k \), where \( 1 \leq k \leq k_i \), to the vacant position \( b_k - p \) yields the abacus \( B' \), where \( u_{B'}(B_k) = u_B(B_k) - p \), and \( u_{B'}(B_l) = u_B(B_l) \) whenever \( l \neq k \). This is equivalent to stripping off a horizontal p-hook from the \( k \)th row of \( \lambda \). If \( \lambda(0) \) is empty then, inductively, we obtain the p-core of \( \lambda \) by stripping off all horizontal p-hooks from \( \lambda \).

Let \( k_j \) be the unique smallest number such that \( B_{k_j} \) belongs to runner \( j \); namely \( B_{k_j} \) is the lowest bead on runner \( j \). Furthermore, if \( B_k \) belongs to runner \( j \) then, by Proposition 4.3(F3), we have

\[
b_{k+1} = b_k - 1, \ldots, b_{k+p-1} = b_k - (p - 1),
\]

and thus \( u_B(B_k) = u_B(B_{k+1}) = \cdots = u_B(B_{k+p-1}) \). Suppose that \( \lambda(0) \neq \emptyset \). Moving a bead \( B_k \) on runner \( j \) to the vacant position \( b_k - p \) is equivalent to moving the beads \( B_i \), one for each \( k \leq l \leq k + p - 1 \), from the position \( b_l \) to the position \( b_l - 1 \), and this yields the abacus \( B'' \), where \( u_{B''}(B_l) = u_B(B_l) - 1 = u_B(B_k) - 1 \) if \( k \leq l \leq k + p - 1 \), and \( u_{B''}(B_l) = u_B(B_l) \) if \( l \not\in \{k, k+1, \ldots, k+p-1\} \). This is equivalent to stripping off a vertical p-hook from the \( \lambda_k \)th column of \( \lambda \). If \( \lambda(0) \) is empty then, inductively, we obtain the p-core of \( \lambda \) by stripping off all vertical p-hooks from \( \lambda \).

Suppose that \( \lambda(0) \neq \emptyset \neq \lambda(0) \). Since \( B_1, \ldots, B_{k_i} \) belong to runner \( i \) and \( i \neq j \), we have \( k_j > k_i \). Furthermore, \( i + a_0p > b_{k_j} \) by Proposition 4.3(F3). Thus stripping off a horizontal p-hook involves a bead of \( B_1, \ldots, B_{k_i} \), and positions \( t \) such that \( t \geq i + a_0p \), and, on the other hand, stripping off a vertical p-hook involves \( p \) beads \( B_{k_1}, B_{k+1}, \ldots, B_{k+p-1} \) such that \( k \geq k_j \) and positions \( t \) such that \( t < i + a_0p \). Thus horizontal stripping involves the first \( k_i \) rows, vertical stripping involves the \((k_j + 1)\)st and lower rows, and so these two stripping procedures are independent of each other.
We shall now prove that \( \Phi(\lambda) \) has the desired form by using induction on the \( p \)-weight of \( \lambda \). We deal with the case when \( \lambda^{(i)} \neq \emptyset \) (the other case is similar). Move every bead \( B_k \) of \( B_k, \ldots, B_1 \) in turn from the position \( b_k \) to \( b_k - p \). This is equivalent to stripping off \( k_i \) horizontal \( p \)-hooks, one for each row, from the first \( k_i \) rows of \( \lambda \). Let the abacus obtained be \( B'' \). The abaci \( B, B'' \) have exactly the same configuration except on runner \( i \). The abacus \( B'' \) represents the partition \( \mu = \lambda - (p^{k_i}) \), where \( (p^{k_i}) \) is the partition \( (p, \ldots, p) \vdash k_i p \). Furthermore, we have

\[
\mu^{(i)} = \lambda^{(i)} - (1^{k_i})
\]

and \( \mu^{(k)} = \lambda^{(k)} \), for all \( k \neq i \). Also, by the previous paragraph, we have that

\[
\mu' - \mu'(0) = \lambda' - \lambda'(0).
\]

We claim that \( \mu \) is again a JM-partition. Observe that

(i) \( \mu^{(k)} = \emptyset \) whenever \( i \neq k \neq j \);
(ii) suppose that a position \( i+ap \) is unoccupied on runner \( i \) of \( B'' \). Then \( i+ap \geq i+a0p \).
If a position \( b > i + ap \) is not on the runner \( i \) of \( B'' \) then the position \( b \) is unoccupied in \( B \), and hence is also unoccupied in \( B'' \);
(iii) suppose that a position \( j + cp \) is occupied in \( B'' \). Then \( j + cp \leq b_{k_j} \). If a position \( d < j + cp \) is not on runner \( j \) then the position is occupied in \( B \), and hence is also occupied in \( B'' \);
(iv) since \( \lambda^{(i)} \) has exactly \( k_i \) nonzero parts and \( \lambda^{(i)} \) is \( p \)-regular, we have that \( \mu^{(i)} = \lambda^{(i)} - (1^{k_i}) \) is also \( p \)-regular;
(v) \( \mu^{(j)} \) is \( p \)-restricted, since \( \mu^{(j)} = \lambda^{(j)} \).

By induction,

\[
\Phi(\mu) = (\tilde{\mu} + p\mu^{(i)}|p(\mu^{(j)})') = (\tilde{\mu} + \mu - \mu(0)|\mu' - \mu'(0)).
\]

Let \( \Phi(\lambda) = (\alpha|p\beta) \). Then

\[
\alpha = \tilde{\mu} + p\mu^{(i)} + (p^{k_i}) = \tilde{\mu} + p\lambda^{(i)} = \tilde{\lambda} + p\lambda^{(i)}.
\]

The case when \( \lambda^{(j)} \neq \emptyset \) can be treated similarly by considering \( \lambda' \).

Together with Theorem \( 3.18 \) we obtain the following useful corollary.

**4.10 Corollary.** Suppose that \( \lambda \in \text{JM}(n) \). We have

\[
Y(\Phi(\lambda)) \otimes \text{sgn}(n) \cong Y(\Phi(\lambda')).
\]

**Proof.** By Lemma \( 1.2 \) the conjugate partition \( \lambda' \) is also a JM-partition of \( n \). Set \( p\sigma := \lambda - \lambda(0) \) and \( pr := \lambda' - \lambda'(0) \). Then, by Proposition \( 1.3 \) and Lemma \( 1.9 \) we have \( \Phi(\lambda) = (\tilde{\lambda} + p\sigma|pr) \) and \( \Phi(\lambda') = (\tilde{\lambda}' + pr|p\sigma) \). Since \( \tilde{\lambda} \) is a \( p \)-core, we have \( m(\tilde{\lambda}) = \tilde{\lambda}' \), by \( 2.4 \) and also \( (\tilde{\lambda} + p\sigma)(0) = \tilde{\lambda} \). So, by Theorem \( 3.18 \) we get

\[
Y(\Phi(\lambda)) \otimes \text{sgn}(n) = Y(\tilde{\lambda} + p\sigma|pr) \otimes \text{sgn}(n) \cong Y(m(\tilde{\lambda}) + p\sigma|p\sigma) = Y(\tilde{\lambda}' + pr|p\sigma) = Y(\Phi(\lambda')).
\]
4.11. Rouquier blocks and adjacent JM-partitions. Recall from [23] that, for \( \lambda \in \mathcal{P}(n) \), we denote by \( b_{\lambda} \) the block of \( F\mathfrak{S}_n \) labelled by the p-core \( \lambda \).

(a) Suppose that \( b_{\lambda} \) has p-weight \( w \). Following the notation used by Fayers in [12], one calls \( b_{\lambda} \) a Rouquier block if \( \lambda \) (and thus also \( \lambda \)) admits an abacus display such that the number of beads on runner \( i + 1 \) exceeds the number of beads on runner \( i \) by at least \( w - 1 \), for all \( 0 \leq i \leq p - 2 \).

(b) We shall follow Fayers’s idea [12, §3] to induce a simple Specht \( F\mathfrak{S}_n \)-module \( S^\lambda \) to a simple Specht module lying in a Rouquier block as follows.

Suppose that \( \lambda \in \text{JM}(n) \), and consider an abacus display \( B \) of \( \lambda \). Suppose that \( B \) is obtained from an \( s \)-element \( \beta \)-set. Suppose that, for some \( 1 \leq l \leq p - 1 \), there are \( r \geq 1 \) more beads on runner \( l - 1 \) than on runner \( l \) and that whenever there is a bead in position \( ap + l \) there is also a bead in position \( ap + l - 1 \). Putting this differently, the Young diagram \( [\lambda] \) has precisely \( r \) addable nodes of \( p \)-residue \( l - s \) and no removable nodes of this \( p \)-residue, where \( l - s \in \{0, \ldots, p - 1\} \) is the residue of \( l - s \) modulo \( p \). Swapping runners \( l - 1 \) and \( l \) of \( B \) yields an abacus display of a JM-partition \( \mu \in \text{JM}(n + r) \), and the Young diagram \( [\mu] \) is obtained by adding all addable nodes of \( p \)-residue \( l - s \) to \( [\lambda] \). One calls \( \lambda \) and \( \mu \) a pair of adjacent JM-partitions.

4.12 Lemma. If \( \lambda \in \text{JM}(n) \) and \( \mu \in \text{JM}(n + r) \) form a pair of adjacent JM-partitions then so do \( \lambda' \) and \( \mu' \).

Proof. Since \( \lambda \) and \( \mu \) form a pair of adjacent partitions, there is some \( t \in \{0, \ldots, p - 1\} \) such that \([\lambda]\) has precisely \( r \) addable nodes of \( p \)-residue \( t \) and has no removable nodes of \( p \)-residue \( t \). Equivalently, \( \lambda' \) has precisely \( r \) addable nodes of \( p \)-residue \( p - t \) and has no removable nodes of \( p \)-residue \( p - t \). By Lemma 4.2, both \( \lambda' \) and \( \mu' \) are JM-partitions. Moreover, since \([\mu]\) is obtained by adding all \( t \)-addable nodes to \([\lambda]\), the diagram \([\mu']\) is obtained by adding all addable nodes of \( p \)-residue \( p - t \) to \([\lambda']\). Thus \( \lambda' \) and \( \mu' \) form a pair of adjacent partitions. \( \square \)

4.13 Notation. Suppose that \( G \) is a finite group and \( H \leq G \). Let \( B \) be a block of \( FG \) with block idempotent \( e_B \), and let \( b \) be a block of \( FH \) with block idempotent \( e_b \). Given an \( FG \)-module \( M \) and an \( FH \)-module \( N \), we then get the \( FH \)-module \( M \uparrow^H_i := e_b \cdot \text{Res}_H^G(M) \) and the \( FG \)-module \( N \uparrow^B := e_B \cdot \text{Ind}_H^G(N) \), respectively.

4.14 Proposition. Let \( \lambda \in \text{JM}(n) \). Then there is a sequence of JM-partitions \( \lambda = \varrho_1, \varrho_2, \ldots, \varrho_t \) of \( n = n_1 < n_2 < \cdots < n_t \), respectively, such that, for each \( 1 \leq i \leq t - 1 \), one has the following:

(a) \( \varrho_i \) and \( \varrho_{i+1} \) form a pair of adjacent JM-partitions,
(b) \( S^{\varrho_i} \uparrow^B_{\varrho_{i+1}} \cong \bigoplus_{(n_{i+1} - n_i)!} S^{\varrho_{i+1}} \),
(c) \( S^{\varrho_{i+1}} \downarrow^B_{\varrho_i} \cong \bigoplus_{(n_{i+1} - n_i)!} S^{\varrho_i} \),
(d) \( b_{\varrho_i} \) is a Rouquier block.

Proof. Following [12, Lemma 3.1, Lemma 3.3], there is a sequence of JM-partitions \( \lambda = \varrho_1, \varrho_2, \ldots, \varrho_t \) of \( n = n_1 < n_2 < \cdots < n_t \), respectively, satisfying parts (a) and (d) and such that \( S^{\varrho_i} \uparrow^B_{\varrho_{i+1}} \) has a filtration of \( (n_{i+1} - n_i)! \) copies of the simple Specht module \( S^{\varrho_{i+1}} \) and \( S^{\varrho_{i+1}} \downarrow^B_{\varrho_i} \) has a filtration of \( (n_{i+1} - n_i)! \) copies of the simple Specht module \( S^{\varrho_i} \), for \( 1 \leq i \leq t - 1 \). In particular, \( S^{\varrho_i} \uparrow^B_{\varrho_{i+1}} \neq \{0\} \) and \( S^{\varrho_{i+1}} \downarrow^B_{\varrho_i} \neq \{0\} \). Fix \( i \in \{1, \ldots, t - 1\} \), and set \( r := n_{i+1} - n_i \). Moreover, let \( s \in \{0, \ldots, p - 1\} \) be such that \( [\varrho_{i+1}] \) is obtained by adding \( r \) nodes of \( p \)-residue \( s \) to \([\varrho_i] \). By Kleshchev’s modular branching rules [23].
Theorems 11.2.10–11.2.11], we know that there are an \(F\mathfrak{S}_n\)-module \(N\) and an \(F\mathfrak{S}_{n+1}\)-module \(M\) such that
\[
S_\theta \uparrow ^b_{\theta \downarrow ^b} \cong \bigoplus _{r !} M \quad \text{and} \quad S_\theta \downarrow ^{\mu} \downarrow ^{\lambda} \cong \bigoplus _{r !} N.
\]
Since \(M \neq \{0\}\), \(M\) has some composition factor \(D\). But we already know that \(S_\theta \uparrow ^b_{\theta \downarrow ^b} \) has a composition series all of whose factors are isomorphic to \(S_\theta \downarrow ^{\mu} \downarrow ^{\lambda}\). This forces \(S_\theta \downarrow ^{\mu} \cong M\). Analogously, we get \(N \cong S_\theta\).

4.15 Remark. In fact, [12 Lemma 3.1, Lemma 3.3] and the proof of Proposition 4.14 say that, as soon as we have a pair of adjacent \(JM\)-partitions \(\lambda, \mu\) such that \(|\mu| > |\lambda|\), we have both \(S_\lambda \uparrow ^{\mu} \cong \bigoplus _{r !} S_\mu \) and \(S_\mu \downarrow ^{\lambda} \cong \bigoplus _{r !} S_\lambda\), where \(r = |\mu| - |\lambda|\).

For our purpose, we need to analyze the effect of the function \(\Phi\) on a pair of adjacent \(JM\)-partitions.

4.16 Lemma. Suppose that \(\lambda \in JM(n)\) and \(\mu \in JM(n + r)\) form a pair of adjacent \(JM\)-partitions, for some \(r > 0\), and let \(B_\mu\) be the abacus display of \(\mu\) obtained from an abacus display \(B_\lambda\) of \(\lambda\) by exchanging runners \(l - 1\) and \(l\). Let \(i_\lambda, j_\lambda, i_\mu, j_\mu \in \{0, \ldots, p - 1\}\) be numbers satisfying the conditions (F1)–(F5) in Proposition 4.3 with respect to \(B_\lambda\) and \(B_\mu\), respectively. Then one has \(\Phi(\lambda) = (\lambda + p\sigma|p\beta|)\) and \(\Phi(\mu) = (\mu + p\sigma|p\beta|)\), where \(\sigma = \lambda(i_\lambda) = \mu(i_\mu)\) and \(\beta = \lambda(j_\lambda)\).

Proof. We shall justify that \(\lambda(i_\lambda) = \mu(i_\mu)\) and \(\lambda(j_\lambda) = \mu(j_\mu)\). Once we have done that, we obtain our desired result, since, by Lemma 4.9, we have \(\Phi(\lambda) = (\lambda + p\lambda(i_\lambda)|p(\lambda(j_\lambda)))\) and \(\Phi(\mu) = (\mu + p\mu(i_\mu)|p(\mu(j_\mu)))\).

If \(\lambda\) is a \(p\)-core then \(\mu\) is a \(p\)-core. In this case, \(\lambda(i_\lambda) = \mu(i_\mu)\) and \(\lambda(j_\lambda) = \mu(j_\mu)\) are both \(p\)-cores. Suppose that \(\lambda\) is not a \(p\)-core, in which case \(\mu\) is not a \(p\)-core either. Suppose further that \(\lambda(i_\lambda) \neq \varnothing\); the case when \(\lambda(j_\lambda) \neq \varnothing\) can be treated similarly. Let \(pa_0 + i_\lambda\) be the highest unoccupied position on runner \(i_\lambda\) of \(B_\lambda\). If \(i_\lambda \notin \{l - 1, l\}\) then we may choose \(i_\mu = i_\lambda\), so that \(\lambda(i_\lambda) = \mu(i_\mu)\). Once \(\lambda(i_\lambda)\) (respectively, \(\mu(i_\mu)\)) is identified, \(\lambda(j_\lambda)\) (respectively, \(\mu(j_\mu)\)) is uniquely determined. Thus \(\lambda(j_\lambda) = \mu(j_\mu)\).

Next suppose that \(i_\lambda \in \{l - 1, l\}\). Since \(\lambda(i_\lambda) \neq \varnothing\), there is some \(c > a_0\) such that \(cp + i_\lambda\) is occupied. If \(i_\lambda = l\) then position \(cp + l - 1\) would have to be both occupied by \(4.11(b)\), and unoccupied, by Proposition 4.3 F2, a contradiction. Thus we must have \(i_\lambda = l - 1\).

Let \(i_\lambda = l - 1\). Then, by Proposition 4.3 F2, for any \(b \geq a_0\), the position \(l + bp\) is unoccupied. Since \(\lambda(i_\lambda) \neq \varnothing\), we have a bead at the position \(l + 1 + cp\), for some \(c > a_0\). Exchanging runners \(l - 1\) and \(l\) of \(B_\lambda\) we obtain that the position \(l + 1 + cp\) is unoccupied, while position \(l + cp\) is occupied in \(B_\mu\). This shows that \(j_\mu \neq l\). Since \(\mu(l) \neq \varnothing\), we necessarily have that \(l = i_\mu\). So we obtain that \(\mu(i_\mu) = \mu(l) = \lambda(l - 1) = \lambda(i_\lambda)\), and hence \(\mu(j_\mu) = \lambda(j_\lambda)\).

4.17 Remark. The previous lemma shows that if \(\lambda \in JM(n)\) and \(\mu \in JM(n + r)\) form a pair of adjacent \(JM\)-partitions then the horizontal and vertical \(p\)-hooks removed from \([\lambda]\) and \([\mu]\) to obtain their respective \(p\)-cores are identical. Furthermore, the \(p\)-core \(\tilde{\mu}\) is obtained by adding \(r\) nodes of a particular \(p\)-residue to \([\lambda]\).

5 Labelling Simple Specht Modules as Signed Young Modules

Throughout this section, let again \(F\) be a field of characteristic \(p > 2\), and let \(n \in \mathbb{Z}^+\). In [8], Hemmer proved that every simple Specht \(F\mathfrak{S}_n\)-module is isomorphic to a signed
Young $FG_n$-module. So, given a simple Specht $FG_n$-module $S^\lambda$, how does one determine $(\alpha|p\beta) \in \mathcal{B}^2(n)$ satisfying $Y(\alpha|p\beta) \cong S^\lambda$? This was posed as an open problem in [15, Problem 5.2]. A conjecture concerning the correct labelling was put forward by the first author in [3, Vermutung 5.4.2] and, independently, by the second author [28, Conjecture 8.2], and Orlob [39, Vermutung A.1.10]. In this section, we confirm the conjecture, by proving the following theorem. Recall the map $\Phi$ defined in Definition 4.7.

5.1 Theorem. Suppose that $\lambda \in JM(n)$. The simple Specht $FG_n$-module $S^\lambda$ is isomorphic to $Y(\Phi(\lambda))$.

Our strategy of proving Theorem 5.1 is similar to that employed by Fayers in [12] and by Hemmer in [18]. More precisely, we shall first reduce the verification of Theorem 5.1 to the case when $b_\lambda$ is a Rouquier block and then show that Theorem 5.1 holds true for simple Specht modules belonging to Rouquier blocks.

5.2 Lemma. Suppose that $\lambda \in JM(n)$ and $\mu \in JM(n + r)$ form a pair of adjacent $JM$-partitions, for some $r > 0$. Suppose that $S^\mu \cong Y(\Phi(\mu))$ and $S^\lambda \cong Y(\alpha|p\beta)$. Then $\Phi(\lambda) = (\gamma|p\beta)$ for some partition $\gamma$. Moreover, $p\beta = \lambda' - \lambda'(0)$.

Proof. Let $m = n + r$, and suppose that $\Phi(\mu) = (\xi|p\zeta)$. Recall that $Y(\alpha|p\beta)$ is a direct summand of $M(\alpha|p\beta)$. By Proposition 4.11(b), we have $S^\mu \mid \text{Ind}_{S_n}^S(\lambda^\mu) \cong \text{Ind}_{S_n}^S(\alpha|p\beta)) \mid M(\alpha\#(1^{m - n})|p\beta)$, where $\alpha\#(1^{m - n})$ denotes the concatenation of the partitions $\alpha$ and $(1^{m - n})$. Therefore, we have

$$Y(\xi|p\zeta) \cong S^\mu \mid \text{Ind}_{S_n}^S(\lambda^\mu) \cong \text{Ind}_{S_n}^S(\alpha|p\beta)) \mid M(\alpha\#(1^{m - n})|p\beta)$$

by [8, 2.3(8)].

On the other hand, by Proposition 4.11(c), we have $S^\lambda \mid \text{Res}_{S_n}^S(S^\mu)$, and hence $S^\lambda \mid \text{Res}_{S_n}^S(M(\xi|p\zeta))$. By Lemma 3.3(b), $S^\lambda \cong Y(\alpha|p\beta)$ is isomorphic to a direct summand of $M(\delta|\partial)$ for suitable partitions $\delta$ and $\partial$. Therefore, we have

$$(\alpha|p\beta) \cong (\delta|\partial)$$

by [8, 2.3(8)] again. By (3) and (7), we have $p|\zeta| \leq p|\beta| \leq |\partial|$. By Lemma 3.3(b), $\partial$ is the partition of the rearrangement of a composition obtained from $p\zeta$ after removing some nodes, so that we necessarily have $|\partial| \leq p|\zeta|$. This shows that $|\partial| = p|\beta| = p|\zeta|$, and hence we have removed no node from $p\zeta$ to obtain $\partial$, that is, $\partial = p\zeta$. We must also have $|\alpha| = |\delta|$, $|\zeta| = |\beta|$, and $|\xi| = |\alpha\#(1^{m - n})|$. So, from (3) and (7), we get $p\zeta \cong p\beta \cong \partial = p\xi$. Thus $\beta = \zeta$, and there is some partition $\gamma$ with $\Phi(\lambda) = (\gamma|p\beta)$, by Lemma 4.16. By the definition of $\Phi$, we have also $p\beta = \lambda' - \lambda'(0)$. \(\square\)

5.3 Lemma. If $S^\lambda$ is a simple Specht $FG_n$-module and if $(\alpha|p\beta) \in \mathcal{B}^2(n)$ is such that $S^\lambda \cong Y(\alpha|p\beta)$ then $\lambda = \alpha(0) = \lambda$.

Proof. Let $w \geq 0$ be the $p$-weight of $\lambda$, so that, by [23, Theorem 6.2.45, 4.1.22, 4.1.24], the Sylow $p$-subgroups of $G_{pw}$ are defect groups of the block containing $Y(\alpha|p\beta)$. Let $Q \leq G_{pw}$ be a Green vertex of $Y(\alpha|p\beta)$. In consequence of Knörr’s Theorem [26] and [38, Proposition 1.4], $Q$ does not have any fixed points on $\{1, \ldots, pw\}$ (see also [4, Proposition 3.4]).

By Theorem 5.1(a), the indecomposable signed Young module $Y(\alpha|p\beta)$ has Young vertex $G_{pw}$, where $\rho$ is the composition

$$\rho = (1^{\alpha(0)}, p^{\alpha(1)+|\beta(0)|}, (p^2)^{\alpha(2)+|\beta(1)|}, \ldots)$$

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of \( n \). Moreover, by Theorem \[3.7\] (b), every Sylow \( p \)-subgroup \( P \) of \( \mathfrak{S}_p \) is a Green vertex of \( Y(\alpha|p\beta) \). Thus \( P = \mathfrak{S}_n \). Clearly, \( P \) fixes exactly \( |\alpha(0)| \) numbers in \( \{1, \ldots, n\} \), which implies that
\[
wp = n - |\alpha(0)|.
\] (8)

On the other hand, recall from \[2.3\] and \[2.2\] that both \( S^\lambda \) and \( Y(\alpha|p\beta) \) lie in the block \( b_\lambda = b_{\tilde{\alpha}} \) of \( F\mathfrak{S}_n \), so that \( \tilde{\lambda} = \tilde{\alpha} \). Since \( \alpha(0) \) is obtained from \( \alpha \) by stripping off horizontal \( p \)-hooks only, we have \( |\alpha(0)| \geq |\tilde{\alpha}| = |\tilde{\lambda}| = n - wp \). Using equation (5), we have \( |\alpha(0)| = |\tilde{\alpha}| \), and hence we conclude that \( \alpha(0) = \tilde{\alpha} = \tilde{\lambda} \).

5.4 Proposition. Theorem \[5.1\] holds true if it holds true for simple Specht modules belonging to Rouquier blocks.

Proof. Assume that Theorem \[5.1\] holds true for simple Specht modules of symmetric groups belonging to Rouquier blocks. Let \( S^\lambda \) be any simple Specht \( F\mathfrak{S}_n \)-module. By Proposition \[2.14\] there is a sequence of JM-partitions \( \lambda = \varnothing_1, \ldots, \varnothing_t \) of natural numbers \( n = n_1 < n_2 < \cdots < n_t \) such that \( \varnothing_i \) and \( \varnothing_{i+1} \) are adjacent, for every \( i = 1, \ldots, t - 1 \), and such that \( b_{\varnothing_t} \) is a Rouquier block. We argue by reverse induction on \( t \) to show that \( S^{\varnothing_i} \cong Y(\Phi(\varnothing_j)), \) for all \( i = 1, \ldots, t \). If \( i = t \) then we are done. Suppose that \( i < t \) and that we have already proved \( S^{\varnothing_i} \cong Y(\Phi(\varnothing_j)), \) for all \( j \geq i + 1 \). By Corollary \[4.10\] we then also have
\[
S^{\varnothing_i} \cong S^{\varnothing_i+1} \otimes \text{sgn}(n_{i+1}) \cong Y(\Phi(\varnothing_{i+1})) \otimes \text{sgn}(n_{i+1}) \cong Y(\Phi(\varnothing_{i+1})).
\]

Lemma \[4.12\] guarantees that \( \varnothing_i \) and \( \varnothing_{i+1} \) also form a pair of adjacent JM-partitions. Applying Lemma \[4.2\] to both of the pairs \( \varnothing_i, \varnothing_{i+1} \) and \( \varnothing'_i, \varnothing'_{i+1} \), we have \( S^{\varnothing_i} \cong Y(\alpha|\varnothing'_i - \varnothing'_i(0)) \) and \( S^{\varnothing'_i} \cong Y(\xi|\varnothing_i - \varnothing_i(0)) \), for suitable partitions \( \alpha \) and \( \xi \). By Lemma \[5.3\], \( \tilde{\alpha} = \alpha(0) = \tilde{\varnothing}_i \), and, by Theorem \[3.18\]
\[
Y(\xi|\varnothing_i - \varnothing_i(0)) \cong S^{\varnothing_i+1} \otimes \text{sgn}(n_i) \cong Y(\alpha|\varnothing'_i - \varnothing'_i(0)) \otimes \text{sgn}(n_i)
\]
\[
\cong Y(\Phi(\varnothing_i)) + \varnothing'_i - \varnothing'_i(0) \alpha - \alpha(0).
\]

The equation above implies, in particular, that \( \alpha = \alpha(0) + (\varnothing_i - \varnothing_i(0)) = \tilde{\varnothing}_i + (\varnothing_i - \varnothing_i(0)). \) Applying Lemma \[4.9\] we have \( \Phi(\varnothing_i) = (\tilde{\varnothing}_i + (\varnothing_i - \varnothing_i(0))|\varnothing'_i - \varnothing'_i(0)) \). Therefore, \( S^{\varnothing_i} \cong Y(\Phi(\varnothing_i)) \) as required. \( \square \)

5.5 Remark. Let \( \lambda \in JM(n) \) be such that the Specht module \( S^\lambda \) lies in a Rouquier block. Furthermore, let \( i, j \in \{0, \ldots, p - 1\} \) be integers satisfying the conditions (F1)–(F5) in Proposition \[1.3\] for some abacus display \( B \) of \( \lambda \). By Lemma \[1.9\] we have \( \Phi(\lambda) = (\tilde{\lambda} + p\sigma|\tau\rho) \), where \( \lambda^{(i)} = \sigma \) and \( (\lambda^{(i)})' = \tau \). The partition \( \tilde{\lambda} + p\sigma \) is obtained from \( B \) by moving all beads on runner \( j \) as high as possible, which is equivalent to removing all vertical \( p \)-hooks from \( [\lambda] \) (see the proof of Lemma \[1.9\]). Let \( B' \) be the abacus display obtained. So \( \tilde{\lambda} + p\sigma \) is a \( p \)-regular JM-partition with the same runners \( i \) and \( j \) satisfying conditions (F1)–(F5) for the abacus \( B' \). The is again a simple Specht module. We conclude that \( S^{\lambda + p\sigma} \cong Y^{\lambda + p\sigma} \cong Y(\tilde{\lambda} + p\sigma|\mathfrak{O}) \) (see, for instance, \[18\] Proposition 1.1]). In particular, the \( F\mathfrak{S}_n \)-module
\[
T := \text{Ind}_{\mathfrak{S}_n|p\sigma|p\tau}^{\mathfrak{S}_n}(S^{\lambda + p\sigma} \boxtimes \text{sgn}(p\tau))
\] (9)
is isomorphic to a direct summand of the signed Young permutation module \( \mathfrak{M}(\tilde{\lambda} + p\sigma|p\tau) \). In \[18\], Hemmer examined the module \( T \) and, in particular, showed that \( S^\lambda \) is isomorphic to a direct summand of \( T \) (see \[18\] proof of Theorem 4.2).

We are now in the position to prove Theorem \[5.1\].
Proof of Theorem 5.1. By Proposition 5.4, it suffices to consider the Rouquier block case. Suppose that $\lambda$ is a JM-partition of $n$ such that the Specht $F\mathfrak{S}_n$-module $S^\lambda$ belongs to some Rouquier block. By Lemma 5.9, we have $\Phi(\lambda) = (\lambda + p\sigma|p\tau)$, for some partitions $\sigma$ and $\tau$. Suppose that $(\alpha|p\beta) \in \mathfrak{R}^2(n)$ is such that $S^\lambda \cong Y(\alpha|p\beta)$. By Lemma 5.3, we have $\hat{\alpha} = \alpha(0) = \lambda$. We aim to prove that $(\alpha|p\beta) = \Phi(\lambda)$.

Consider the $F\mathfrak{S}_n$-module $T$ defined as in (9). As mentioned in Remark 5.3, we have $Y(\alpha|p\beta) \cong S^\lambda | T | M(\lambda + p\sigma|p\tau)$, so that $(\alpha|p\beta) \cong (\lambda + p\sigma|p\tau)$, by [5, 2.3(8)]. In particular, we have $p|\beta| \leq p|\tau|$. We shall show that, in fact, $p|\beta| = p|\tau|$.

The Specht $F\mathfrak{S}_n$-module $S^\lambda$ is also simple, lies in a Rouquier block, and satisfies

$$S^\lambda \cong (S^\lambda)^* \otimes \text{sgn}(n) \cong S^\lambda \otimes \text{sgn}(n) \cong Y(\alpha|p\beta) \otimes \text{sgn}(n) \cong Y(\mathfrak{m}(\alpha(0)) + p\beta|\alpha - \alpha(0)),$$

by Theorem 5.18.

Now, replacing $\lambda$ by $\lambda'$ in Remark 5.3, we also deduce that $S^\lambda'$ is isomorphic to a direct summand of the signed permutation module $M(\lambda' + p\tau|p\sigma)$. Thus, since $\mathfrak{m}(\lambda) = \lambda'$, we have

$$(\mathfrak{m}(\alpha(0)) + p\beta|\alpha - \alpha(0)) = (\mathfrak{m}(\lambda) + p\beta|\alpha - \lambda) \cong (\lambda' + p\tau|p\sigma).$$

In particular, we have $|\mathfrak{m}(\lambda)| + p|\beta| \geq |\lambda'| + p|\tau|$, and hence $p|\beta| \geq p|\tau|$. This shows that $p|\beta| = p|\tau|$, and thus $|\alpha| = |\lambda| + p|\sigma|$. The condition $(\alpha|p\beta) \cong (\lambda + p\sigma|p\tau)$ is thus equivalent to $\alpha \cong \lambda + p\sigma$ and $p\beta \cong p\tau$. The next aim is to show that we indeed have $\alpha = \lambda + p\sigma$ and $p\beta = p\tau$, which will then complete the proof of the theorem.

Since $S^\lambda \cong Y(\alpha|p\beta) | M(\alpha|p\beta)$, we know, by Lemma 2.3, that $S^\lambda$ occurs with non-zero multiplicity in every Specht filtration of $M(\alpha|p\beta)$. By Theorem 5.18, there is some Specht filtration of $M(\alpha|p\beta)$ in which $S^\lambda$ occurs as a factor with multiplicity equal to the number of semistandard $\lambda$-tableaux of type $(\alpha|p\beta)$. Hence there must exist a semistandard $\lambda$-tableau of type $(\alpha|p\beta)$. Let $\sigma = (\sigma_1, \ldots, \sigma_r)$ be such that $\sigma_i \neq 0$. Then $\lambda_i = \lambda_i + p\sigma_i$, for all $1 \leq i \leq r$. Recall that $\alpha \cong \lambda + p\sigma$. If $\alpha_1 > \lambda_1$ then there would be no semistandard $\lambda$-tableau of type $(\alpha|p\beta)$, since the $\alpha_1$ copies of the number 1 would have to be assigned to the first row of $[\lambda]$ and clearly there would not be enough nodes in the first row of $[\lambda]$. Thus $\alpha_1 = \lambda_1$. Arguing inductively we obtain that $\lambda_i = \alpha_i$, for all $1 \leq i \leq r$, that is, $\alpha_i = \lambda_i + p\sigma_i$, for all $1 \leq i \leq r$. Using the fact that $\alpha(0) = \lambda$, we have $\alpha - \alpha(0) = (p\sigma_1, \ldots, p\sigma_r, \ldots)$, and hence $\alpha - \alpha(0) = (p\sigma_1, \ldots, p\sigma_r)$, since $|\alpha| - |\alpha(0)| = p|\sigma|$. This shows that $\alpha = \lambda + p\sigma$.

Repeating the argument in the previous paragraph with the Specht factor $S^\lambda'$ in the signed Young permutation module $M(\lambda' + p\tau|p\sigma)$, we deduce that $\mathfrak{m}(\alpha(0)) + p\beta = \lambda' + p\tau$. Since $\mathfrak{m}(\alpha(0)) = \mathfrak{m}(\lambda) = \lambda'$, we obtain $p\beta = p\tau$. Altogether, this gives

$$(\alpha|p\beta) = (\lambda + p\sigma|p\tau) = \Phi(\lambda).$$

The proof is now complete.
6.1 Notation. For the theory and notation of cohomological variety and complexity, we refer the reader to 
[1] §5, and briefly summarize here the notation we shall use in the following.

Suppose that $G$ is any finite group. Then $H(G, F)$ denotes the cohomology ring $H^*(G, F)$ if $p = 2$, and it denotes the subring generated by elements of even degrees if $p \geq 3$. If $H$ is a subgroup of $G$ then one has a restriction map $\text{Res}_{G,H} : H(G, F) \rightarrow H(H, F)$, which then induces a map $\text{Res}_{G,H}^* : V_H \rightarrow V_G$ on the level of cohomological varieties of $H$ and $G$.

Let $M$ be an $FG$-module. The cohomological variety $V_G(M)$ of $M$ is the maximal ideal spectrum of $V_G$ containing a certain ideal. Furthermore, we denote by $c_G(M)$ the complexity of $M$, which is the rate of growth of a minimal projective resolution of $M$, or equivalently, the dimension of the cohomological variety $V_G(M)$. For precise definitions of $V_G(M)$ and $c_G(M)$, see [1] §5.

The cohomological variety $V_G(M)$ and complexity $c_G(M)$ of an indecomposable $FG$-module $M$ are related to its Green vertices and sources as follows.

6.2 Lemma ([28] Proposition 2.15]). Let $M$ be an indecomposable $FG$-module, let $Q$ be a Green vertex, and let $S$ be a Green $Q$-source of $M$. Then one has $V_G(M) = \text{Res}_{G,Q}^*(V_Q(S))$. Moreover, the complexity $c_G(M)$ of $M$ equals the complexity $c_Q(S)$ of $S$. If the dimension of $S$ is coprime to $p$ then $c_Q(S)$ equals the $p$-rank of $Q$.

Proof. Since $M \mid \text{Ind}_Q^G(S)$, we have $V_G(M) \subseteq \text{Res}_{G,Q}^*(V_Q(S))$, by [1] Proposition 5.7.5 and [10] Proposition 8.2.4. Conversely, since $S \mid \text{Res}_G^Q(M)$, we have $\text{Res}_{G,Q}^*(V_Q(S)) \subseteq V_G(M)$, by [1] Proposition 5.7.5.

By [28] Proposition 2.12(vii) and (ix)), we have $c_G(M) = c_Q(S)$. The last statement follows from [1] Corollary 5.8.5.

Applying the general theory to indecomposable signed Young modules, we obtain the following proposition.

6.3 Proposition. Let $p \geq 3$, let $(\lambda|\mu) \in \mathcal{P}^2(n)$, and let $p$ be the composition of $n$ as in 5.4. The cohomological variety of the indecomposable signed Young module $Y(\lambda|\mu)$ is

$$\text{Res}_{G_n,P_p}^*(V_{P_p}(F)),$$

where $P_p$ is a Sylow $p$-subgroup of the Young vertex $G_n$. In particular, $Y(\lambda|\mu)$ has complexity $|\mu| + \frac{1}{p}|(\lambda| - |\lambda(0)|)|$.

Proof. By Theorem 3.7(b), $Y(\lambda|\mu)$ has Green vertex $P_p$ and trivial Green $P_p$-source $F$. By Lemma 5.2 the first assertion is justified. Since the trivial $FP_p$-module $F$ has dimension coprime to $p$, the complexity of $Y(\lambda|\mu)$ is the $p$-rank of $P_p$, which is

$$|\lambda(1)| + |\mu(0)| + p(|\lambda(2)| + |\mu(1)|) + p^2(|\lambda(3)| + |\mu(2)|) + \cdots = |\mu| + \frac{|\lambda| - |\lambda(0)|}{p}.$$  

Recall that, for every finite group $G$, an $FG$-module has complexity 0 if and only if it is projective, and has complexity 1 if and only if it is non-projective periodic. Applying Proposition 6.3 we obtain the following corollary. Note that the assertion in part (b) can also be found in the paper of Hemmer and Nakano [21] Corollary 3.3.3] in the case of Young modules.
Let $\alpha \in \mathcal{P}(n)$. Recall from [22 Corollary 13.18] that, in the case when $p$ is odd, or $p = 2$ and $\alpha$ is 2-regular, the Specht $F\mathfrak{S}_n$-module $S^\alpha$ is indecomposable. If $S^\alpha$ is simple then $S^\alpha$ is isomorphic to $D^{\alpha^R}$, where $\alpha^R$ is the $p$-regularization of the partition $\alpha$ (see [23 6.3.59]). In the case when $p$ is odd, Hemmer’s result in [18] implies that a simple Specht $F\mathfrak{S}_n$-module $S^\alpha$ and the associated simple module $D^{\alpha^R}$ have trivial Green sources. Our result Theorem 5.1 allows us to describe the Green vertices, Green correspondents, cohomological varieties and complexities for all simple Specht $F\mathfrak{S}_n$-modules. Recall the map $\Phi : \mathcal{P}(n) \to \mathcal{P}^2(n)$ in Definition 4.7.

6.5 Corollary. Let $p \geq 3$, and suppose that $S^\alpha$ is a simple Specht $F\mathfrak{S}_n$-module. Let $\Phi(\alpha) = (\lambda|\mu)$, let $\rho$ be the composition of $n$ as in 3.4, and let $N(\rho)$ be as in 3.13. Then the $F\mathfrak{S}_n$-module $S^\alpha \cong D^{\alpha^R}$ has

- (a) Green vertex a Sylow $p$-subgroup $P_\rho$ of $\mathfrak{S}_\rho$,
- (b) Green correspondent $\text{Res}_{N(\rho)}^n(H(\mu|\lambda))$ with respect to $N_{\mathfrak{S}_n}(P_\rho) \leq H \leq N(\rho)$,
- (c) cohomological variety $\text{Res}_{\mathfrak{S}_n, P_\rho}^n(V_{P_\rho}(F))$, and
- (d) complexity equal to the $p$-weight of $\alpha$.

6.6 Remark. Corollary 6.5 thus reveals the class of all simple Specht $F\mathfrak{S}_n$-modules and the class of simple $F\mathfrak{S}_n$-modules

$$\{D^{\alpha^R} : \alpha \in JM(n)\},$$

whose Green vertices, Green correspondents, cohomological varieties and complexities are known. In the literature, there are other special classes of Specht or simple $F\mathfrak{S}_n$-modules whose Green vertices or complexities are computed: results concerning Green vertices and complexities can, for instance, be found in [14, 28, 29, 30, 36] and [42, 43] for Specht modules, and in [4, 5, 6, 7, 21, 32, 35] and [42] for simple modules.

In fact, Corollary 6.5 recovers Wildon’s result [42] when $p$ is odd.

6.7 Corollary (42 Theorem 2]). Suppose that $p \geq 3$. If $p$ does not divide $n$ then, for any $r \in \{0, \ldots, n-1\}$, a Sylow $p$-subgroup of $\mathfrak{S}_{n-r-1} \times \mathfrak{S}_r$ is a Green vertex of the Specht module $S^{(n-r, r')}$.

Proof. By [40], the Specht module $S^{(n-r, r')}$ is simple. Let $n - r - 1 = \sum_{i \geq 0} s_i p^i$ and $r = \sum_{i \geq 0} r_i p^i$ be the $p$-adic expansions of the integers $n - r - 1$ and $r$, respectively. Then Lemma 4.9 gives

$$\Phi((n - r, r')) = \left(1 + s_0, 1^{\alpha_0} + \sum_{i \geq 1} p^i(s_i)\sum_{i \geq 1} p^i(r_i)\right).$$

Moreover, $S^{(n-r, r')} \cong Y(\Phi((n-r, r')))$. By Theorem 5.1. By Theorem 3.7, $Y(\Phi((n-r, r'))) has Young vertex $\mathfrak{S}_\rho$, where $\rho = (1^{1+s_0+r_0}, p^{s_1+r_1}, (p^2)^{s_2+r_2}, \ldots)$. By Corollary 6.5, the
Thus we only need to consider the signed Young modules. Since the same holds true for \(\mathfrak{S}_p\), the signed Young modules and the class of simple modules mentioned in Remark 6.6. By [15, Theorem 1], a minimal projective resolution in the weight 1 case, and hence deduce similar results for all simple Specht modules.

Proof. Suppose that \(\mathfrak{S}_p\) is a Sylow \(p\)-subgroup of \(\mathfrak{S}_n\). By Lemma 6.2, the cohomological variety of \(\mathfrak{S}_p\)-modules is \(\text{Res}_{\mathfrak{S}_4}^{\mathfrak{S}_4}(V_E(F))\), and hence the complexity of \(\mathfrak{S}_p\) is 2.

Together with Corollary 6.5, we have a complete understanding of the properties we desired about all simple Specht modules over fields of positive characteristics.

6.8 Remark. Corollary 6.4(b) classifies the class of non-projective periodic signed Young modules. In what follows, we make use of Gill’s result [13] to obtain the periods of non-projective periodic indecomposable signed Young modules and their minimal projective resolutions in the weight 1 case, and hence deduce similar results for all simple Specht modules and the class of simple modules mentioned in Remark 6.6. By [13, Theorem 1], a non-projective periodic Young module \(Y^\lambda\) has period \(2p - 2\) when \(p \geq 3\), and has period 1 when \(p = 2\). Moreover, if \(Y^\lambda\) belongs to a block of weight 1 then one can describe a minimal projective resolution of \(Y^\lambda\) [15, §4.1].

6.10 Corollary. Suppose that \(p \geq 3\). Then every non-projective periodic indecomposable signed Young \(F\mathfrak{S}_n\)-module has period \(2p - 2\).

Proof. Let \((\lambda|\mu) \in \mathfrak{P}^2(n)\) be such that the signed Young module \(Y(\lambda|\mu)\) is non-projective and periodic. By Corollary 6.4 there are only two possibilities. In the case when \(\mu = \varnothing\), the signed Young module \(Y(\lambda|\mu)\) is isomorphic to the Young module \(Y^\lambda\). Thus we only need to consider the signed Young \(F\mathfrak{S}_n\)-modules \(Y(\lambda|p(1))\), where \(\lambda\) is a \(p\)-restricted partition. By Theorem 3.18

\[ Y(\lambda|p(1)) \otimes \text{sgn}(n) \cong Y(m(\lambda) + p(1)|\varnothing) . \]

Since \(Y(m(\lambda) + p(1)|\varnothing) \cong Y^{m(\lambda) + p(1)}\), it has period \(2p - 2\), by [15, Theorem 1]. Hence the same holds true for \(Y(\lambda|p(1))\).
6.11 Corollary. Let \( p \geq 2 \) be any prime number, and let \( S^\alpha \) be a simple Specht \( F\mathfrak{S}_n \)-module. Suppose that \( S^\alpha \) is non-projective and periodic. If \( p \geq 3 \) then \( S^\alpha \cong D^\alpha R \) has period \( 2p - 2 \). If \( p = 2 \) then \( S^\alpha \cong D^\alpha R \) has period 1.

Proof. Suppose first that \( p \geq 3 \), so that \( S^\alpha \cong Y(\Phi(\alpha)) \) by Theorem 3.18. So \( S^\alpha \) has period \( 2p - 2 \), by Corollary 6.10. If \( p = 2 \) then, as we have seen in Remark 6.8, the result of James and Mathas [24] implies that \( S^\alpha \) is isomorphic to one of the indecomposable Young \( F\mathfrak{S}_n \)-modules \( Y^\alpha \) or \( Y^{\alpha'} \), or that \( \alpha = (2, 2) \). By Remark 6.8, the Specht \( F\mathfrak{S}_4 \)-module \( S^{(2,2)} \) has complexity 2, and hence is not periodic. By [15] Theorem 1, every non-projective indecomposable Young \( F\mathfrak{S}_n \)-module that is periodic has period 1. Thus the assertion also holds true when \( p = 2 \).

6.12. Minimal projective resolutions. Suppose that \( p \geq 3 \). A minimal projective resolution of a periodic non-projective Young \( F\mathfrak{S}_n \)-module belonging to a block of \( p \)-weight 1 is known, by the result of Gill [15] §4.1. We make use of this result to obtain a minimal projective resolution of a periodic non-projective indecomposable signed Young \( F\mathfrak{S}_n \)-module of the form \( Y(\lambda p(1)) \) that belongs to a block of \( p \)-weight 1. Note that \( \lambda \) must then be a \( p \)-core, which will be fixed from now on. Recall from [24] that \( m(\lambda) = \lambda' \).

Suppose that \( \alpha \) be a simple Specht \( F\mathfrak{S}_n \)-module belonging to a block of \( p \)-regular, \( \alpha = (\lambda, 1) \). The assertion also holds true when \( p = 2 \).

By [15] §4.1, the Young module \( Y^{\lambda'} + p(1) = Y^{\varrho_0} \) has a minimal projective resolution of the form

\[
\cdots \to Y^{\varrho_0} \to Y^{\varrho_1} \to Y^{\varrho_2} \to \cdots \to Y^{\varrho_0} \to Y^{\varrho_1} \to Y^{\varrho_2} \to \cdots \to Y^{\varrho_0} \to Y^{\varrho_1} \to \cdots \to \{0\}.
\]

(10)

Since the partitions \( \varrho_0, \ldots, \varrho_{p-2} \) are \( p \)-restricted, we have \( \varrho_i = \varrho_i(0) \), for all \( i = 0, \ldots, p-2 \). Theorem 3.18 implies \( Y^{\varrho_i} \otimes \text{sgn}(n) \cong Y^{m(\varrho_i)} \), for \( i = 0, \ldots, p-2 \), and, after tensoring (10) by \( \text{sgn}(n) \), we obtain a minimal projective resolution

\[
\cdots \to Y^{m(\varrho_0)} \to Y^{m(\varrho_1)} \to \cdots \to Y^{m(\varrho_0)} \to Y^{m(\varrho_1)} \to \cdots \\
\to Y^{m(\varrho_{p-2})} \to Y(\lambda p(1)) \to \{0\}
\]

(11)

of \( Y^{\lambda'+p(1)} \otimes \text{sgn}(n) \cong Y(\lambda p(1)) \). The partitions \( \varrho_1, \ldots, \varrho_{p-1}, \varrho_0, \ldots, \varrho_{p-2} \) are \( p \)-restricted, and the partitions \( \varrho_0, \ldots, \varrho_{p-2}, \varrho_1, \ldots, \varrho_{p-1} \) are \( p \)-regular. The structures of both of the Specht modules \( S^{\varrho_i} \) and \( S^{\varrho'_i} \) are well known, for every \( i = 0, \ldots, p-1 \). In fact, every block of \( F\mathfrak{S}_n \) of \( p \)-weight 1 is Scopes equivalent to the principal block of \( F\mathfrak{S}_p \) (see [31]). The Specht modules in the principal block of \( F\mathfrak{S}_p \) are labelled by the hook partitions of \( p \), whose module structures have been determined by Peel [30]. With our notation, we get \( S^{\varrho_0} \cong D^{\varrho_0}, \ S^{\varrho_{p-1}} \cong D^{\varrho_{p-1}}, \ S^{\varrho_0} \cong D^{\varrho_1}, \ S^{\varrho_{p-1}} \cong D^{\varrho_{p}} \). Furthermore, for \( i = 1, \ldots, p-2 \), the Specht modules \( S^{\varrho_i} \) and \( S^{\varrho'_i} \) both have composition length 2 and Loewy structures

\[
S^{\varrho_i} = \begin{bmatrix} D^{\varrho_i} \\ D^{\varrho_{i+1}} \end{bmatrix} \quad \text{and} \quad S^{\varrho'_i} = \begin{bmatrix} D^{\varrho'_i} \\ D^{\varrho_{i+1}} \end{bmatrix},
\]

26
respectively. Hence, by \[27\] we get \( D_{\mathcal{G}_i} \cong \text{Soc}(S_{\mathcal{G}_i}) \cong D_{\mathcal{G}_{i+1}} \), and thus

\[
D_m(\mathcal{G}_i) \cong D_{\mathcal{G}_i} \otimes \text{sgn}(n) \cong D_{\mathcal{G}_{i+1}} \otimes \text{sgn}(n) \cong \text{Hd}(S_{\mathcal{G}_{i+1}} \otimes \text{sgn}(n))
\]

\[
\cong \text{Hd}((S_{\mathcal{G}_{i+1}})^*) \cong \text{Soc}(S_{\mathcal{G}_{i+1}}) \cong D_{\mathcal{G}_{i+1}},
\]

for all \( i = 0, \ldots, p - 2 \). This implies \( m(\mathcal{G}_i') = \mathcal{G}_{i+1} \), for \( i = 0, \ldots, p - 2 \). Together with \[11\], this gives the minimal projective resolution

\[
\cdots \rightarrow Y^e_{p-1} \rightarrow Y^e_{p-2} \rightarrow \cdots \rightarrow Y^e_1 \rightarrow Y^e_0 \rightarrow Y^e_2 \rightarrow \cdots \rightarrow Y^e_{p-1} \rightarrow Y(\lambda|p(1)) \rightarrow \{0\}
\]
of \( Y(\lambda|p(1)) \).

In \cite{15} §4.1 Gill also determines the Heller translates \( \Omega^i(Y^e_{p-1}) \) in terms of their Loewy structures, for all \( i = 0, \ldots, 2p - 2 \). Since \( \Omega^i(Y^e_{p-1}) \otimes \text{sgn}(n) \cong \Omega^i(Y^e_{p-1} \otimes \text{sgn}(n)) \cong \Omega^i(Y(\lambda|p(1))) \), for all integers \( i \), our previous considerations and Gill’s result yield the following Loewy structures of the Heller translates of \( Y(\lambda|p(1)) \):

\[
\Omega^i(Y(\lambda|p(1))) \cong \begin{cases} 
D_{\mathcal{G}_{i+1}} & \text{if } 1 \leq i \leq p - 2, \\
D_{\mathcal{G}_{i+1}} & \text{if } i = p - 1, \\
D_{\mathcal{G}_{i+1}} & \text{if } p - 1 \leq i \leq 2p - 3, \\
Y(\lambda|p(1)) & \text{if } i = 2p - 2.
\end{cases}
\]

## Appendix: Proof of Theorem \[3.14\]

Throughout this appendix, let \( p \geq 3 \) be a prime, and let \( F \) be an algebraically closed field of characteristic \( p \). In the following, we shall establish a proof of Theorem \[3.14\] following the arguments given by Donkin in \cite{8} 5.2. To this end, let \( n \geq 1 \), and fix a pair \( (\lambda|\mu) \in \mathcal{B}^2(n) \).

As in \cite{8} 3.13 let \( \lambda = \sum_{i=0}^{r_\lambda} p^i \cdot \lambda(i) \) and \( \mu = \sum_{i=0}^{r_\mu} p^i \cdot \mu(i) \) be the \( p \)-adic expansions of \( \lambda \) and \( \mu \), respectively, let \( r := \max\{r_\lambda, r_\mu + 1\} \), let \( n_0 = |\lambda(0)| \), \( n_i = |\lambda(i)| + |\mu(i - 1)| \), for every \( i = 1, \ldots, r \), and let \( \rho = (1^{n_0}, p^{n_1}, \ldots, (p^r)^{n_r}) \) be the composition of \( n \) as in \[11\]. We identify the normalizer \( N_{\mathcal{S}_n}(\mathcal{S}_\rho) \) with \( N(\rho) = \mathcal{S}_{n_0} \times (\mathcal{S}_p \wr \mathcal{S}_{n_1}) \times \cdots \times (\mathcal{S}_p \wr \mathcal{S}_{n_r}) \).

As we have seen in Theorem \[5.7\] (a), the signed Young \( F\mathcal{S}_n \)-module \( Y(\lambda|\mu) \) has Young vertex \( \mathcal{S}_\rho \). Let

\[
A(\rho) := (\mathcal{S}_1)^{n_0} \times (\mathcal{S}_p)^{n_1} \times \cdots \times (\mathcal{S}_p)^{n_r} \leq N(\rho),
\]

so that \( N(\rho)/A(\rho) \cong \mathcal{S}_{n_0} \times (\mathcal{S}_2 \wr \mathcal{S}_{n_1}) \times \cdots \times (\mathcal{S}_2 \wr \mathcal{S}_{n_r}) \). By \cite{8} 1.1(1)], the isomorphism classes of indecomposable signed Young \( F\mathcal{S}_n \)-modules with Young vertex \( \mathcal{S}_\rho \) are in bijection with the isomorphism classes of indecomposable projective \( F[N(\rho)/A(\rho)] \)-modules.

More precisely, this bijection is induced by the Young–Green correspondence with respect to \( N(\rho) \). The next proposition together with the observation in Remark \[3.11\] will imply that the Young–Green correspondent of \( Y(\lambda|\mu) \) with respect to \( N(\rho) \) must be isomorphic to one of the \( FN(\rho) \)-modules \( R(\nu|\rho\delta) \), where \( (\nu|\rho\delta) \in \mathcal{B}^2(n) \).

### A.1 Proposition

Let \( m \in \mathbb{N} \), and let \( P \) be the set of pairs \( (\alpha, \beta) \) of \( p \)-restricted partitions such that \( |\alpha| + |\beta| = m \). Then, as \( (\alpha, \beta) \) varies over \( P \), the \( F[\mathcal{S}_2 \wr \mathcal{S}_{m}] \)-module \( R_2(\alpha|\beta) \) varies over a set of representatives of the isomorphism classes of indecomposable projective \( F[\mathcal{S}_2 \wr \mathcal{S}_{m}] \)-modules.
Proof. Fix $m_1, m_2 \in \mathbb{N}$ such that $m_1 + m_2 = m$. For $\alpha \in \mathcal{R}(m_1)$ and $\beta \in \mathcal{R}(m_2)$, we consider the $F[\mathcal{S}_2 \wr \mathcal{S}_m]$-module

$$D_2(\alpha|\beta) := \text{Ind}_{\mathcal{S}_2 \wr \mathcal{S}_{m_1}}^{\mathcal{S}_2 \wr \mathcal{S}_{m_2}}((F(2)^{\otimes m_1} \boxtimes \text{sgn}(2)^{\otimes m_2}) \otimes \text{Ind}_{\mathcal{S}_1 \wr \mathcal{S}_{m_1}}^{\mathcal{S}_2 \wr \mathcal{S}_{m_2}}(D_\alpha \boxtimes D_{m(\beta)})).$$

Recall from Remark 3.11 that $\text{sgn}(2)^{\otimes m_2} = \text{sgn}(\mathcal{S}_2 \wr \mathcal{S}_m)$, and $F(2)^{\otimes m_1}$ is of course the trivial $F[\mathcal{S}_2 \wr \mathcal{S}_{m_1}]$-module. As $\alpha$ varies over $\mathcal{R}(m_1)$, the module $D_\alpha$ varies over a set of representatives of the isomorphism classes of simple $F\mathcal{S}_{m_1}$-modules, and as $\beta$ varies over $\mathcal{R}(m_2)$, both $D_\beta$ and $D_{m(\beta)}$ vary over a set of representatives of the isomorphism classes of simple $F\mathcal{S}_{m_2}$-modules.

Thus, by [23, Theorem 4.3.34], the modules $D_2(\alpha|\beta)$, as $(\alpha, \beta)$ varies over $P$, vary over a set of representatives of the isomorphism classes of simple $F[\mathcal{S}_2 \wr \mathcal{S}_m]$-modules. Given $(\alpha, \beta) \in P$, we have $D_\alpha \boxtimes D_{m(\beta)} \cong \text{Hd}(P(\alpha) \boxtimes P(\beta))$ as $F[\mathcal{S}_{m_1} \times \mathcal{S}_{m_2}]$-modules. Hence $D_2(\alpha|\beta)$ is isomorphic to a quotient module of $R_2(\alpha|\beta)$. Since $\text{char}(F) \neq 2$, both the $F\mathcal{S}_2$-modules $F(2)$ and $\text{sgn}(2)$ are projective. Therefore, $R_2(\alpha|\beta)$ is a projective indecomposable $F[\mathcal{S}_2 \wr \mathcal{S}_m]$-module, for every $(\alpha, \beta) \in P$; a proof of this can, for instance, be found in [27, Proposition 5.1]. This forces that $R_2(\alpha|\beta)$ is a projective cover of $D_2(\alpha|\beta)$. So, altogether, we obtain that $R_2(\alpha|\beta)$, as $(\alpha, \beta)$ varies over $P$, varies over a set of representatives of the isomorphism classes of indecomposable projective $F[\mathcal{S}_2 \wr \mathcal{S}_m]$-modules.

A.2 Corollary. Given the composition $\rho = (1^{n_0}, p^{m_1}, \ldots, (p^r)^{n_r})$ of $n$, let $L \subseteq \mathcal{P}(n)$ be the set of pairs $(\nu|\rho\delta)$ such that $\nu$ and $\delta$ have $p$-adic expansions satisfying $r = \max\{r_\nu, r_\delta + 1\}$, $n_0 = |\nu(0)|$, and $|\nu(i)| + |\delta(i - 1)| = n_i$, for every $i = 1, \ldots, r$. Then the $F(N(\rho))/A(\rho)$-modules $R(\nu|\rho\delta)$, as $(\nu|\rho\delta)$ varies over the set $L$, form a set of representatives of the isomorphism classes of Young–Green correspondence of indecomposable signed Young $F\mathcal{S}_n$-modules with Young vertex $\mathcal{S}_\rho$.

Proof. Recall that $N(\rho)/A(\rho) \cong \mathcal{S}_{n_0} \times (\mathcal{S}_2 \wr \mathcal{S}_{n_1}) \times \cdots \times (\mathcal{S}_2 \wr \mathcal{S}_{n_r})$. Via this isomorphism, in consequence of Proposition A.1, the modules

$$P(\nu(0)) \boxtimes R_2(\nu(1)|\delta(0)) \boxtimes \cdots \boxtimes R_2(\nu(r)|\delta(r - 1)),$$

as $(\nu|\rho\delta)$ varies over the set $L$, form a set of representatives of the isomorphism classes of indecomposable projective $F[N(\rho)]/A(\rho)]$-modules. Note that, for some $0 \leq i \leq r$, the integers $n_i$ may be 0. In the latter case, $\mathcal{S}_2 \wr \mathcal{S}_{n_i}$ is just the trivial group and $R_2(\nu(i)|\delta(i - 1)) = R_2(\otimes_2)$ is the trivial $F[\mathcal{S}_2 \wr \mathcal{S}_{n_i}]$-module, by definition. So, by [3, 1.1(1)], the isomorphism classes of indecomposable signed Young $F\mathcal{S}_n$-modules with Young vertex $\mathcal{S}_\rho$ are, via Young–Green correspondence, in bijection with the inflations

$$\text{Inf}^{N(\rho)}_{N(\rho)/A(\rho)}(P(\nu(0)) \boxtimes R_2(\nu(1)|\delta(0)) \boxtimes \cdots \boxtimes R_2(\nu(r)|\delta(r - 1)))$$

$$\cong P(\nu(0)) \boxtimes R_{p^\nu(\nu)}(\nu(1)|\delta(0)) \boxtimes \cdots \boxtimes R_{p^\nu(\nu)}(\nu(r)|\delta(r - 1)) = R(\nu|\rho\delta),$$

as $(\nu|\rho\delta)$ varies over $L$. □

A.3 Notation. (a) Let $m_1, m_2 \in \mathbb{N}$, and let $m := m_1 + m_2$. Let further $k \geq 2$. Given partitions $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathcal{P}(m_1)$ and $\beta = (\beta_1, \ldots, \beta_t) \in \mathcal{P}(m_2)$, we denote by $I_k(\alpha|\beta)$ the following $F[\mathcal{S}_k \wr (\mathcal{S}_\alpha \times \mathcal{S}_\beta)]$-module:

$$I_k(\alpha|\beta) := F(\mathcal{S}_k \wr \mathcal{S}_\alpha) \boxtimes \text{sgn}(k)^{\otimes \beta_1} \boxtimes \cdots \boxtimes \text{sgn}(k)^{\otimes \beta_t}. $$

Note that the trivial $F[\mathcal{S}_k \wr \mathcal{S}_\alpha]$-module $F(\mathcal{S}_k \wr \mathcal{S}_\alpha)$ is also isomorphic to the inflation of the $m_1$-fold outer tensor product $F(k)^{\otimes m_1}$ of $F(k)$ to $\mathcal{S}_k \wr \mathcal{S}_\alpha$. Moreover, if $k$ is odd then $\text{sgn}(k)^{\otimes \beta_1} \boxtimes \cdots \boxtimes \text{sgn}(k)^{\otimes \beta_t} \cong \text{sgn}(\mathcal{S}_k \wr \mathcal{S}_\beta)$ (see Remark 3.11).
Hence, applying the Frobenius Formula and Lemma 2.10, we get

Now set

\[ J_k(\alpha|\beta) := \begin{cases} \text{Ind}_{\mathfrak{S}_k(\mathfrak{S}_m \times \mathfrak{S}_\beta)}(I_k(\alpha|\beta)) & \text{if } m > 0, \\ F(k) & \text{if } m = 0. \end{cases} \]

(b) With the above notation we obtain the \( F(\rho) \)-module

\[ J(\lambda|\mu) := M^{\lambda(0)} \boxtimes J_{\rho}(\lambda(1)|\mu(0)) \boxtimes \cdots \boxtimes J_{\rho'}(\lambda(r)|\mu(r-1)). \]

A.4 Lemma. With the notation as in A.3, suppose further that both \( \alpha \) and \( \beta \) are \( p \)-restricted. Then the \( F[\mathfrak{S}_k \wr \mathfrak{S}_m] \)-module \( R_k(\alpha|\beta) \) is isomorphic to a direct summand of \( J_k(\alpha|\beta) \).

Proof. If \( m = 0 \) then the assertion is clear. Suppose that \( m \geq 1 \). Recall that \( R_k(\alpha|\beta) = \text{Ind}_{\mathfrak{S}_k(\mathfrak{S}_m)}(\tilde{P}(\alpha) \boxtimes (\tilde{P}(\beta) \otimes \text{sgn}(k)^{\otimes m_2})) \), where \( \tilde{P}(\alpha) = \text{Ind}_{\mathfrak{S}_m}^{\mathfrak{S}_k(\mathfrak{S}_m)}(P(\alpha)) \) and \( \tilde{P}(\beta) = \text{Ind}_{\mathfrak{S}_m}^{\mathfrak{S}_k(\mathfrak{S}_m)}(P(\beta)) \). Since

\[ J_k(\alpha|\beta) = \text{Ind}_{\mathfrak{S}_k(\mathfrak{S}_m)}(I_k(\alpha|\beta)) \cong \text{Ind}_{\mathfrak{S}_k(\mathfrak{S}_m)}^{\mathfrak{S}_k(\mathfrak{S}_m \times \mathfrak{S}_\beta)}(I_k(\alpha|\beta)) \]

it suffices to show that \( \tilde{P}(\alpha) \boxtimes (\tilde{P}(\beta) \otimes \text{sgn}(k)^{\otimes m_2}) \) is isomorphic to a direct summand of \( \text{Ind}_{\mathfrak{S}_k(\mathfrak{S}_m \times \mathfrak{S}_\beta)}^{\mathfrak{S}_k(\mathfrak{S}_m \times \mathfrak{S}_\beta)}(I_k(\alpha|\beta)) \).

Firstly, by Lemma 2.10, we have

\[
\begin{align*}
\text{Ind}_{\mathfrak{S}_k(\mathfrak{S}_m)}^{\mathfrak{S}_k(\mathfrak{S}_m \times \mathfrak{S}_\beta)}(I_k(\alpha|\beta)) & \cong \text{Ind}_{\mathfrak{S}_k(\mathfrak{S}_m)}^{\mathfrak{S}_k(\mathfrak{S}_m \times \mathfrak{S}_\beta)}(F(\mathfrak{S}_k \wr \mathfrak{S}_\alpha)) \boxtimes \text{Ind}_{\mathfrak{S}_k(\mathfrak{S}_m)}^{\mathfrak{S}_k(\mathfrak{S}_m \times \mathfrak{S}_\beta)} \left( \text{sgn}(k)^{\otimes \beta_1} \boxtimes \cdots \boxtimes \text{sgn}(k)^{\otimes \beta_t} \right) \\
& \cong \text{Ind}_{\mathfrak{S}_k(\mathfrak{S}_m)}^{\mathfrak{S}_k(\mathfrak{S}_m \times \mathfrak{S}_\beta)}(\text{Ind}_{\mathfrak{S}_m}^{\mathfrak{S}_k(\mathfrak{S}_m)}(F(\alpha))) \boxtimes \text{Ind}_{\mathfrak{S}_k(\mathfrak{S}_m \times \mathfrak{S}_\beta)}^{\mathfrak{S}_k(\mathfrak{S}_m \times \mathfrak{S}_\beta)} \left( \text{sgn}(k)^{\otimes \beta_1} \boxtimes \cdots \boxtimes \text{sgn}(k)^{\otimes \beta_t} \right) \\
& \cong \text{Ind}_{\mathfrak{S}_m}^{\mathfrak{S}_k(\mathfrak{S}_m)}(M(\alpha|\beta)) \boxtimes \text{Ind}_{\mathfrak{S}_k(\mathfrak{S}_m \times \mathfrak{S}_\beta)}^{\mathfrak{S}_k(\mathfrak{S}_m \times \mathfrak{S}_\beta)} \left( \text{sgn}(k)^{\otimes \beta_1} \boxtimes \cdots \boxtimes \text{sgn}(k)^{\otimes \beta_t} \right).
\end{align*}
\]

Secondly, consider the \( F[\mathfrak{S}_k \wr \mathfrak{S}_\beta] \)-module \( \text{sgn}(k)^{\otimes \beta_1} \boxtimes \cdots \boxtimes \text{sgn}(k)^{\otimes \beta_t} \), which is isomorphic to \( \text{Ind}_{\mathfrak{S}_m}^{\mathfrak{S}_k(\mathfrak{S}_m)}(\text{sgn}(\beta)) \boxtimes (\text{sgn}(k)^{\otimes \beta_1} \boxtimes \cdots \boxtimes \text{sgn}(k)^{\otimes \beta_t}) \). As well, the \( F[\mathfrak{S}_k \wr \mathfrak{S}_\beta] \)-module \( \text{sgn}(k)^{\otimes \beta_1} \boxtimes \cdots \boxtimes \text{sgn}(k)^{\otimes \beta_t} \) is just the restriction of the \( F[\mathfrak{S}_k \wr \mathfrak{S}_{m_2}] \)-module \( \text{sgn}(k)^{\otimes m_2} \). Hence, applying the Frobenius Formula and Lemma 2.10, we get

\[
\begin{align*}
\text{Ind}_{\mathfrak{S}_m}^{\mathfrak{S}_k(\mathfrak{S}_m)} \left( \text{sgn}(k)^{\otimes \beta_1} \boxtimes \cdots \boxtimes \text{sgn}(k)^{\otimes \beta_t} \right) & \cong \text{Ind}_{\mathfrak{S}_m}^{\mathfrak{S}_k(\mathfrak{S}_m)}(\text{Ind}_{\mathfrak{S}_m}^{\mathfrak{S}_k(\mathfrak{S}_m)}(\text{sgn}(\beta))) \boxtimes \text{sgn}(k)^{\otimes m_2} \\
& \cong \text{Ind}_{\mathfrak{S}_m}^{\mathfrak{S}_k(\mathfrak{S}_m)}(\text{Ind}_{\mathfrak{S}_m}^{\mathfrak{S}_k(\mathfrak{S}_m)}(\text{sgn}(\beta))) \boxtimes \text{sgn}(k)^{\otimes m_2} \\
& \cong \text{Ind}_{\mathfrak{S}_m}^{\mathfrak{S}_k(\mathfrak{S}_m)}(M(\alpha|\beta)) \boxtimes \text{sgn}(k)^{\otimes m_2} \\
& \cong \text{Ind}_{\mathfrak{S}_m}^{\mathfrak{S}_k(\mathfrak{S}_m)}(M(\alpha|\beta)) \boxtimes \text{Ind}_{\mathfrak{S}_m}^{\mathfrak{S}_k(\mathfrak{S}_m)}(\text{sgn}(m_2)) \boxtimes \text{sgn}(k)^{\otimes m_2} \\
& \cong \text{Ind}_{\mathfrak{S}_m}^{\mathfrak{S}_k(\mathfrak{S}_m)}(M(\alpha|\beta)) \boxtimes \text{Ind}_{\mathfrak{S}_m}^{\mathfrak{S}_k(\mathfrak{S}_m)}(\text{sgn}(m_2)) \boxtimes \text{sgn}(k)^{\otimes m_2} \\
& \cong \text{Ind}_{\mathfrak{S}_m}^{\mathfrak{S}_k(\mathfrak{S}_m)}(M(\alpha|\beta)) \boxtimes \text{Ind}_{\mathfrak{S}_m}^{\mathfrak{S}_k(\mathfrak{S}_m)}(\text{sgn}(m_2)) \boxtimes \text{sgn}(k)^{\otimes m_2} \\
& \cong \text{Ind}_{\mathfrak{S}_m}^{\mathfrak{S}_k(\mathfrak{S}_m)}(M(\alpha|\beta)) \boxtimes \text{Ind}_{\mathfrak{S}_m}^{\mathfrak{S}_k(\mathfrak{S}_m)}(\text{sgn}(m_2)) \boxtimes \text{sgn}(k)^{\otimes m_2} \\
& \cong \text{Ind}_{\mathfrak{S}_m}^{\mathfrak{S}_k(\mathfrak{S}_m)}(M(\alpha|\beta)) \boxtimes \text{Ind}_{\mathfrak{S}_m}^{\mathfrak{S}_k(\mathfrak{S}_m)}(\text{sgn}(m_2)) \boxtimes \text{sgn}(k)^{\otimes m_2}.
\end{align*}
\]
Therefore,
\[
\text{Ind}_{\mathcal{E}_k(\mathcal{E}_m \times \mathcal{E}_n)}(I_k(\alpha|\beta)) \cong \text{Inf}_{\mathcal{E}_m}^{\mathcal{E}_k}(M(\alpha|\emptyset)) \boxtimes \left( \text{Inf}_{\mathcal{E}_n}^{\mathcal{E}_m}(M(\beta|\emptyset)) \otimes \operatorname{sgn}(k) \right).
\]

Of course we have \( Y^\alpha \mid M^\alpha = M(\alpha|\emptyset) \) and \( Y^\beta \mid M^\beta = M(\beta|\emptyset) \). Since both \( \alpha \) and \( \beta \) are \( p \)-restricted, the Young modules \( Y^\alpha \) and \( Y^\beta \) are projective modules of \( F \mathcal{E}_m \) and \( F \mathcal{E}_n \), respectively. Since \( S^\alpha \) is an \( F \mathcal{E}_m \)-submodule of \( Y^\alpha \) with \( \operatorname{Soc}(S^\alpha) \cong D_\alpha \) and since \( D_\alpha \) is self-dual, we must have \( Y^\alpha \cong P(\alpha) \). Analogously, \( Y^\beta \cong P(\beta) \). This shows that \( \text{Ind}_{\mathcal{E}_k(\mathcal{E}_m \times \mathcal{E}_n)}(I_k(\alpha|\beta)) \) has a direct summand isomorphic to \( P(\alpha) \boxtimes (P(\beta) \otimes \operatorname{sgn}(k)) \) as claimed.

**A.5 Remark.** In the proof of the next lemma we shall have to determine the structure of the intersections of \( N(\rho) \) with \( \mathcal{E}_n \)-conjugates of the Young subgroup \( \mathcal{E}_\lambda \times \mathcal{E}_\mu \). To this end, suppose that \( \lambda = (\lambda_1, \ldots, \lambda_u) \) and \( \mu = (\mu_1, \ldots, \mu_v) \), for some \( u, v \in \mathbb{N} \). Given the \( p \)-adic expansions \( \{\lambda\} \) of \( \lambda \) and \( \mu \), we thus have \( \lambda_j = \sum_{i=0}^{r} p^i \cdot \lambda(i)_j \) as well as \( \mu_{\ell} = \sum_{i=1}^{v} p^\ell \cdot \mu(i-1)_\ell \) for \( j = 1, \ldots, u \) and \( \ell = 1, \ldots, v \), where here we set \( \lambda(i)_j := \emptyset \), for \( i > r \lambda \), and \( \mu(k)_\ell := \emptyset \), for \( k > \mu \). Therefore, we have
\[
H := \prod_{j=1}^{u} (\mathcal{E}_{\lambda(0)} \times \prod_{i=1}^{r} \mathcal{E}_{p^i}) \times \prod_{\ell=1}^{v} \prod_{i=1}^{u} \mathcal{E}_{p^i} \leq \prod_{j=1}^{u} \mathcal{E}_{\lambda_j} \times \prod_{\ell=1}^{v} \mathcal{E}_{p^\ell} = \mathcal{E}_\lambda \times \mathcal{E}_\mu \leq \mathcal{E}_n.
\]

Now let \( g \in \mathcal{E}_n \) be such that
\[
gH = \mathcal{E}_\lambda(0) \times \prod_{i=1}^{r} \left( \prod_{j=1}^{u} \mathcal{E}_{p^i} \mathcal{E}_{p^i} \right) \leq \mathcal{E}_\lambda \times \mathcal{E}_\mu \times \cdots \times \mathcal{E}_\mu = g(\mathcal{E}_\lambda \times \mathcal{E}_\mu).
\]
Then also \( gH \leq N(\rho) \), since \( gH \leq \mathcal{E}_\lambda(0) \times \mathcal{E}_\mu \times \cdots \times \mathcal{E}_\mu \). On the other hand, every element of \( N(\rho) \cap g(\mathcal{E}_\lambda \times \mathcal{E}_\mu) \) fixes the orbits of \( g(\mathcal{E}_\lambda \times \mathcal{E}_\mu) \) and permutes the orbits of \( gH \). This implies that
\[
N(\rho) \cap g(\mathcal{E}_\lambda \times \mathcal{E}_\mu) = \mathcal{E}_\lambda(0) \times \prod_{i=1}^{r} \mathcal{E}_{p^i} \leq (\mathcal{E}_\lambda(i) \times \mathcal{E}_{\mu(i-1)}).
\]

We illustrate this by an example as below.

**A.6 Example.** Consider the case when \( p = 3 \), \( \lambda = (14, 3) \) and \( \mu = (10, 1) \). Then \( \lambda = (2) + 3 \cdot (1, 1) + 3^2 \cdot (1) \) and \( \mu = (1, 1) + 3^2 \cdot (1) \), so that \( r = 3 \) and \( \lambda(0) = (2) \), \( \lambda(1) = (1, 1) \), \( \lambda(2) = (1) \), \( \mu(0) = (1, 1) \), and \( \mu(2) = (1) \). Also, \( n_0 = 2 \), \( n_1 = 2 + 2 = 4 \), \( n_2 = 1 = n_3 \). Hence \( \rho = (1^2, 3^4, 9, 27) \) and \( N(\rho) = \mathcal{E}_2 \times (\mathcal{E}_3 \mathcal{E}_4 \mathcal{E}_9 \mathcal{E}_{27}) \). On the other hand, the group called \( H \) in Remark \( \text{A.5} \) above has the form
\[
H = \mathcal{E}_2 \times \mathcal{E}_3 \times \mathcal{E}_3 \times \mathcal{E}_3 \times \mathcal{E}_3 \times \mathcal{E}_3 \times \mathcal{E}_3 \times \mathcal{E}_3 \leq \mathcal{E}_{14} \times \mathcal{E}_3 \times \mathcal{E}_{30} \times \mathcal{E}_3 = \mathcal{E}_\lambda \times \mathcal{E}_\mu.
\]
So we take \( g \in \mathcal{E}_{50} \) to be the following permutation
\[
g := \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & \cdots & 14 & 15 & \cdots & 20 & 21 & \cdots & 47 & 48 & 49 & 50 \\
2 & 3 & 4 & 5 & 15 & \cdots & 23 & 6 & \cdots & 11 & 12 & \cdots & 13 & 14
\end{pmatrix}
\]

to get
\[
gH = \mathcal{E}_2 \times \mathcal{E}_3 \times \mathcal{E}_3 \times \mathcal{E}_3 \times \mathcal{E}_3 \times \mathcal{E}_9 \times \mathcal{E}_{27}
\]
\[
\leq \mathcal{E}_{\{1, \ldots, 5, 15, \ldots, 23\}} \times \mathcal{E}_{\{6, 7, 8\}} \times \mathcal{E}_{\{9, 10, 11, 12, \ldots, 50\}} \times \mathcal{E}_{\{12, 13, 14\}}
\]
\[
= g(\mathcal{E}_\lambda \times \mathcal{E}_\mu)
\]
and \( N(\rho) \cap g(\mathcal{E}_\lambda \times \mathcal{E}_\mu) = \mathcal{E}_2 \times (\mathcal{E}_3)^4 \times \mathcal{E}_9 \times \mathcal{E}_{27} \).
A.7 Lemma. For $(\lambda|\mu_1) \in S^2(n)$, one has
\[
R(\lambda|\mu_1) \mid J(\lambda|\mu_1) \mid Res^S_{N(\rho)}(M(\lambda|\mu_1))
\]
as $FN(\rho)$-modules.

Proof. Since $R(\lambda|\mu_1) = P(\lambda(0)) \boxtimes R_p(\lambda(1)|\mu(0)) \boxtimes \cdots \boxtimes R_p(\lambda(r)|\mu(r-1))$ and $J(\lambda|\mu_1) = P(\lambda(0)) \boxtimes J_p(\lambda(1)|\mu(0)) \boxtimes \cdots \boxtimes J_p(\lambda(r)|\mu(r-1))$, we obtain $R(\lambda|\mu_1) \mid J(\lambda|\mu_1)$, by Lemma A.3.

As explained in Remark A.3, there is some $g \in S_\alpha$ such that $N(\rho) \cap g(S_\lambda \times S_{\mu_1}) = S_\lambda(0) \times \prod_{i=1}^r S_{\mu_1}$. By the Mackey Formula,
\[
\text{Ind}^N_{N(\rho)}(S_\lambda \times S_{\mu_1})(F(\lambda_0) \boxtimes \text{sgn}(\lambda_{\mu_1})) \mid \text{Res}^S_{N(\rho)}(M(\lambda|\mu_1)).
\]
Furthermore, we have
\[
\text{Res}^S_{N(\rho)}(S_\lambda \times S_{\mu_1})(F(\lambda_0) \boxtimes \text{sgn}(\lambda_{\mu_1})) \cong F(\lambda(0)) \boxtimes \left(\bigotimes_{i=1}^r F(S_{\mu_1} \cap S_{\lambda(i)}) \boxtimes \text{sgn}(S_{\mu_1} \cap S_{\lambda(i-1)})\right),
\]
when identifying each wreath product $S_{\mu_1}(S_{\lambda(i)} \times S_{\mu_1})$ with $(S_{\mu_1}(S_{\lambda(i)}) \times (S_{\mu_1}(S_{\lambda(i)})))$ as usual. For each $i = 1, \ldots, r$, let $\mu(i-1) = (\mu(i-1)_1, \ldots, \mu(i-1)_t)$; here $\mu(i-1)_t$ may be 0. Since $p$ is odd, we have $\text{sgn}(S_{\mu_1} \cap S_{\lambda(i-1)}) = \text{sgn}(p_1) \boxtimes \cdots \boxtimes \text{sgn}(p_t)$, by Remark A.11. This shows that
\[
\text{Ind}^N_{N(\rho)}(S_\lambda \times S_{\mu_1})(F(\lambda_0)) \boxtimes \left(\bigotimes_{i=1}^r \text{Ind}^S_{S_{\lambda(i)}}(S_{\mu_1}) \right) \text{Ind}^S_{S_{\lambda(i)}}(F(\lambda(i)|\mu(i-1))) \cong M(\lambda(0)) \boxtimes J_p(\lambda(1)|\mu(0)) \boxtimes \cdots \boxtimes J_p(\lambda(r)|\mu(r-1)) = J(\lambda|\mu_1),
\]
which completes the proof of the lemma. \hfill \Box

To conclude this appendix, we shall now prove Theorem A.14.

Proof of Theorem A.14. Let $L \subseteq S^2(n)$ be the set of pairs of partitions defined as in Corollary A.2. That is, $(\nu|\rho) \in L$ if and only if $r = \max\{r_\nu, r_\rho+1\}$, $n_0 = |\nu(0)|$, and $|\nu(i)| + |\rho(i-1)| = n_i$, for every $i = 1, \ldots, r$. By Corollary A.2, there is a bijection $\psi: L \rightarrow L$ induced by the Young–Green correspondence with respect to $N(\rho)$ of indecomposable signed Young $F\tilde{S}_n$-modules with Young vertex $S_\rho$. We have to show that $\psi(\alpha|\beta) = (\alpha|\beta)$, for all $(\alpha|\beta) \in L$.

Let $(\alpha|\beta) \in L$, and assume that $(\nu|\rho) = \psi(\alpha|\beta)$. By [10] Satz 6.3, the Krull–Schmidt multiplicity of the indecomposable signed Young module $Y(\alpha|\beta)$ as a direct summand of $M(\nu|\rho)$ equals the Krull–Schmidt multiplicity of the Young–Green correspondent $R(\nu|\rho)$ of $Y(\alpha|\beta)$ as a direct summand of $\text{Res}^S_{N(\rho)}(M(\nu|\rho))$. By Lemma A.7, we know that $R(\nu|\rho) \mid \text{Res}^S_{N(\rho)}(M(\nu|\rho))$, so that also $Y(\alpha|\beta) \mid M(\nu|\rho)$. By [8] 2.3([8]), this implies $\psi(\alpha|\beta) = (\nu|\rho) \cong (\alpha|\beta)$. Now we argue by induction on the length of $(\alpha|\beta)$ in the poset $(L, \leq)$.

Suppose that $(\alpha|\beta)$ is a minimal element in $(L, \leq)$, that is, has length 0 in $(L, \leq)$. Then $\psi(\alpha|\beta) \cong (\alpha|\beta)$ forces $\psi(\alpha|\beta) = (\alpha|\beta)$. Now suppose that $(\alpha|\beta)$ has some length $\ell > 1$ in $(L, \leq)$ and that $(\zeta|\xi) = \psi(\zeta|\xi)$, for all $(\zeta|\xi) \in L$ of length less than $\ell$. Then $\psi(\alpha|\beta) \cong (\alpha|\beta)$, and if $(\zeta|\xi) \prec (\alpha|\beta)$ then $(\zeta|\xi) = \psi(\zeta|\xi)$ by induction. Since $\psi$ is injective, this implies $\psi(\alpha|\beta) = (\alpha|\beta)$, so that $Y(\alpha|\beta)$ has Young–Green correspondent $R(\alpha|\beta)$ as desired. \hfill \Box
B Signed Young Rule

Given a Young permutation module, Young’s Rule (see [22 14.1, 17.14]) describes the multiplicities of Specht modules occurring as factors of certain Specht series of the Young permutation module. More precisely, given a partition $\alpha \in \mathcal{P}(n)$, the Young permutation $F\mathfrak{S}_n$-module $M^\alpha = M(\alpha|\emptyset)$ admits a Specht filtration in which a Specht $F\mathfrak{S}_n$-module $S^\lambda$ occurs as a factor with multiplicity equal to the number of semistandard $\lambda$-tableaux of type $\alpha$, where here $\lambda \in \mathcal{P}(n)$ is arbitrary.

In this appendix, following [31], we describe the multiplicities of the Specht $F\mathfrak{S}_n$-modules occurring as factors of certain Specht series of a signed Young permutation $F\mathfrak{S}_n$-module. Theorem B.6 can thus be seen as a generalization of Young’s Rule.

To generalize the notion of a semistandard tableau of a particular shape and type, consider the set of ‘primed’ positive integers $\{1', 2', 3', \ldots\}$. We define the obvious total ordering on the set $\{1', 2', 3', \ldots\}$, by declaring

$$1' < 2' < 3' < \cdots.$$ 

B.1 Definition ([34 §2.4]). Let $n \in \mathbb{Z}^+$, and let $(\alpha|\beta) \in \mathcal{P}^2(n)$. Moreover, let $\lambda \in \mathcal{P}(n)$. A semistandard $\lambda$-tableaux $t$ of type $(\alpha|\beta)$ is obtained by filling the Young diagram $[\lambda]$ with $\alpha_i$ entries equal to $i$ and $\beta_j$ entries equal to $j'$, for $i, j \in \mathbb{Z}^+$, such that the following conditions are satisfied.

(a) The sub-tableau $s$ of $t$ occupied by the non-primed integers is a semistandard $\mu$-tableaux of type $\alpha$ for some sub-partition $\mu$ of $\lambda$, that is, the non-primed integers are strictly increasing along every column from top to bottom and are weakly increasing along every row from left to right.

(b) The skew tableau $t \setminus s$ occupied by the primed integers forms a conjugate semistandard skew $\lambda/\mu$-tableau, that is, the primed integers are weakly increasing along every column from top to bottom and strictly increasing along every row from left to right.

In the case when $\beta = \emptyset$, one recovers the usual semistandard $\lambda$-tableau $t$ of type $\alpha$.

B.2 Example. Let $\lambda = (4, 3, 2, 2)$, and let $(\alpha|\beta) = ((3, 3, 1)|(2, 2))$. There are a total of five semistandard $\lambda$-tableaux of type $(\alpha|\beta)$ as below.

\[
\begin{array}{ccc}
1 & 1 & 2 \\
3 & 1' \\
1' & 2'
\end{array}
\quad
\begin{array}{ccc}
1 & 1 & 2 \\
2 & 2 & 3 \\
1' & 2'
\end{array}
\quad
\begin{array}{ccc}
1 & 1 & 3 \\
2 & 2 & 3 \\
1' & 2'
\end{array}
\quad
\begin{array}{ccc}
1 & 1 & 1' \\
2 & 2 & 2 \\
1' & 2'
\end{array}
\quad
\begin{array}{ccc}
1 & 1 & 1' \\
2 & 2 & 2 \\
3 & 2'
\end{array}
\quad
\begin{array}{ccc}
1 & 1 & 1' \\
3 & 3' \\
1' & 2'
\end{array}
\]

B.3 Notation. Suppose that $\lambda \in \mathcal{P}(n)$, and let $\gamma$ and $\xi$ be partitions such that $|\gamma| + |\xi| = n$. Denote by $c^\lambda_{\gamma, \xi}$ the Littlewood–Richardson coefficient in the sense of [22 §16]. By [23 2.8.13], the $F\mathfrak{S}_n$-module $\text{Ind}_{\mathfrak{S}_{\gamma} \times \mathfrak{S}_{\xi}}^{\mathfrak{S}_n}(S^\gamma \boxtimes S^\xi)$ admits a Specht filtration such that $S^\lambda$ occurs with multiplicity $c^\lambda_{\gamma, \xi}$ in this filtration.

Moreover, if $\alpha \in \mathcal{P}(n)$ then, by Young’s Rule, the permutation $F\mathfrak{S}_n$-module $M^\alpha = M(\alpha|\emptyset)$ admits a Specht filtration in which $S^\lambda$ occurs with multiplicity equal to the number of semistandard $\lambda$-tableaux of type $\alpha$, and we denote this multiplicity by $y_{\alpha, \lambda}$.

By convention, we set $c^\alpha_{\emptyset, \xi} := 1$ and $y_{\emptyset, \xi} := 1$.

B.4 Lemma. Let $m \in \mathbb{Z}^+$, and let $\beta \in \mathcal{P}(m)$. Then the signed Young permutation $F\mathfrak{S}_m$-module $M(\emptyset|\beta)$ has a Specht series in which, for each $\xi \in \mathcal{P}(m)$, the Specht $F\mathfrak{S}_m$-module $S^\xi$ occurs as a factor with multiplicity $y_{32, \xi}$.
Proof. By Young’s Rule [22, 17.14], the Young permutation \( F \mathfrak{S}_m \)-module \( M^\beta = M(\beta|\emptyset) \) has a Specht filtration in which the Specht \( F \mathfrak{S}_m \)-module \( S^\xi \) occurs with multiplicity \( y_{\beta, \xi} \).

Now recall from 2.4 that we have \( S^\xi \cong (S^{\xi'})^* \otimes \text{sgn}(m) \) and \( (M(\emptyset|\emptyset))^* \otimes \text{sgn}(m) \cong M(\emptyset|\emptyset) \otimes \text{sgn}(m) \cong M(\emptyset|\beta) \). Consequently, \( M(\emptyset|\beta) \) admits a Specht filtration in which \( S^\xi \) occurs as a factor with multiplicity \( y_{\beta, \xi} \).

The proof of the following lemma requires the notion of the rectification of a skew tableau. Since we do not need this notion elsewhere in our paper, we refer the reader to the reference [13, §5.1] for the necessary definitions and notation that we follow.

**B.5 Lemma.** Let \( \lambda \in \mathcal{P}(n) \), and let \( (\alpha|\beta) \in \mathcal{P}^2(n) \). The number of semistandard \( \lambda \)-tableaux of type \( (\alpha|\beta) \) is

\[
\sum_{\gamma \vdash |\alpha|, \xi \vdash |\beta|} y_{\alpha, \gamma} y_{\beta, \xi} c_{\gamma, \xi}^\lambda.
\]

Proof. There is a one-to-one correspondence between the set of semistandard \( \lambda \)-tableaux of type \( (\alpha|\beta) \) and the set of pairs consisting of a semistandard \( \gamma \)-tableau of type \( \alpha \) and a conjugate semistandard skew \( (\lambda/\gamma) \)-tableau of type \( \beta \), as \( \gamma \) varies over the set of subpartitions of \( \lambda \) of size \( |\alpha| \). Thus, it suffices to show that, for each such \( \gamma \), the number of conjugate semistandard skew \( (\lambda/\gamma) \)-tableau of type \( \beta \) is

\[
\sum_{\xi \vdash |\beta|} y_{\beta, \xi} c_{\gamma, \xi}^\lambda.
\]

Fix a partition \( \xi \) of \( |\beta| \) and a semistandard \( \xi' \)-tableau \( t' \) of type \( \beta \). Let \( S((\lambda/\gamma)', t') \) be the set consisting of semistandard skew \( (\lambda/\gamma)' \)-tableaux whose rectification is \( t' \) (see [13, §5.1]). By [13, §5.1 Corollary 2(iii)], the set \( S((\lambda/\gamma)', t') \) has the same cardinality as \( S(\lambda/\gamma, t) \), which is also the same as the Littlewood–Richardson coefficient \( c_{\gamma, \xi}^\lambda \). This shows that, for each conjugate semistandard \( \xi \)-tableau \( t \) of type \( \beta \), and hence every semistandard \( \xi' \)-tableau \( t' \) of type \( \beta \), we get all semistandard skew \( (\lambda/\gamma) \)-tableaux whose rectification is \( t \). Hence we get a total of \( y_{\beta, \xi} c_{\gamma, \xi}^\lambda \) conjugate semistandard skew \( (\lambda/\gamma) \)-tableau of type \( \beta \).

**B.6 Theorem** (Signed Young Rule [31]). Let \( (\alpha|\beta) \in \mathcal{P}^2(n) \). The signed Young permutation \( F \mathfrak{S}_n \)-module \( M(\alpha|\beta) \) has a Specht series in which, for each \( \lambda \in \mathcal{P}(n) \), the Specht module \( S^\lambda \) occurs as a factor with multiplicity equal to the number of semistandard \( \lambda \)-tableaux of type \( (\alpha|\beta) \).

Proof. Suppose that \( n_1 = |\alpha| \) and \( n_2 = |\beta| \). By Young’s Rule and Lemma B.5 we obtain Specht series of \( M(\alpha|\emptyset) \) and of \( M(\emptyset|\beta) \) in which the Specht \( F \mathfrak{S}_{n_1} \)-module \( S^\gamma \) and the Specht \( F \mathfrak{S}_{n_2} \)-module \( S^\xi \) occur as factors with multiplicities \( y_{\alpha, \gamma} \) and \( y_{\beta, \xi} \), respectively. Moreover, we have \( M(\alpha|\emptyset) \cong \text{Ind}_{\mathfrak{S}_{n_1}}^{\mathfrak{S}_n} (M(\alpha|\emptyset) \otimes M(\emptyset|\beta)) \). Thus, by [23, 2.8.13], \( M(\alpha|\beta) \) has a Specht filtration in which \( S^\lambda \) occurs as a factor with multiplicity

\[
\sum_{\gamma \vdash n_1, \xi \vdash n_2} y_{\alpha, \gamma} y_{\beta, \xi} c_{\gamma, \xi}^\lambda.
\]

The result now follows from Lemma B.5.

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