Another Family of Permutations Counted by the Bell Numbers

Fufa Beyene $^{1a}$, Roberto Mantaci$^b$

$^a$Addis Ababa University, Addis Ababa, Ethiopia and CoRS; email: fufa.beyene@aau.edu.et

$^b$IRIF, Université de Paris, Paris, France and CoRS; email: mantaci@irif.fr

Abstract

Using a permutation code introduced by Rakotondrajao and the second author, we associate with every set partition of $[n]$ a permutation over $[n]$, thus defining a class of permutation whose size is the $n$-th Bell number. We characterize the permutations belonging to this class and we study the distribution of weak exceedances over these permutations, which turns out to be enumerated by the Stirling numbers of the second kind. We provide a direct bijection between our class of permutations and another equisized class of permutations introduced by Poneti and Vajnovszki.

Keywords: Permutations, Set Partitions, Codes, Subexceedant functions, Exceedances, Bell numbers, Stirling numbers of the second kind.

1. Introduction

Permutations and set partitions are among the richest objects in combinatorics, they have been enumerated according to several criteria of interest.

There is a plethora of studies of statistics on permutations, many of which are counted by eulerain or macmahonian numbers, such as descents, exceedances, right-to-left minima or maxima, inversions, etc. On the other hand, we recall the two most basic enumerations for set partitions : the total number of set partitions of $[n]$ is the Bell number $\text{[Ro]}$ and the number of set partitions of $[n]$ with $k$ blocks is the Stirling number of the second kind, as given in $\text{[Bo, St1]}$.  

$^1$Corresponding author.
Both permutations and set partition can be coded by subexceedant functions, that is, functions $f$ over $[n]$ such that $1 \leq f(i) \leq i$ for all $i \in [n]$ (in some contexts it is rather required that $0 \leq f(i) \leq i - 1$).

Some permutation codes with subexceedant functions are very well known (Lehmer code or inversion table, Denert code, ...). On the other hand, a way to code set partitions with subexceedant functions is provided by Mansour’s definition of canonical form for a set partition $\pi$ (Ma). In this form any integer $i \in [n]$ is coded with the number of the block of $\pi$ where $i$ belongs when $\pi$ is written in standard form, that is, the elements in each block are arranged increasingly and the blocks are arranged in increasing order of their first elements. In fact, canonical forms of set partitions are restricted growth functions (RGF), a particular case of subexceedant functions.

Several properties of set partitions or permutations can easily be read on their corresponding codes, this allows to prove elegantly some results by reasoning on the codes rather than on the coded objects themselves, see for instance the nice article of Baril and Vajnovszki (Ba-Va), and also the article of Foata and Zeilberger (Fo-Ze).

Furthermore, these codes are also useful to implement efficient algorithms for the exhaustive generation of the corresponding class of objects. For instance, M. Orlov (Or) used the representation of set partitions as RGFs to implement an algorithm to generate all set partitions in constant space and amortized time complexity.

F. Rakotondrajao and the second author in (Ma-Ra) defined a new way of coding permutations with subexceedant functions. This code associates with a subexceedant function $f$ with the permutation $\sigma = \hat{\phi}(f)$ defined by the product of transpositions (the leftmost always acts first):

$$
\sigma = (n \ f(n)) \ (n - 1 \ f(n - 1)) \ \cdots \ (1 \ f(1)).
$$

We studied further this code in (Be-Ma), where we gave for it a new interpretation based on the action and the cycle structure of the permutation. In that work we also introduced the definition of inom, which will be reminded in the Section 2 and by which we propose to call this code “inom code”.

In this paper we associate bijectively a permutation on $[n]$ with every set partition of $[n]$ by composing the canonical form and the inom code. We obtain this way a family of permutations counted by the Bell numbers and we present some properties and some recurrence relations satisfied by these objects.
We show in particular in Section 3 that the permutations belonging to this class can be characterised combinatorially and that the distribution of the weak exceedances statistic in this class is the same as the distribution of number of blocks in the set of set partitions, and therefore is given by the Stirling numbers of the second kind.

In Section 4, we provide a direct bijection between each partition and its corresponding permutation (without passing through canonical form and inom code).

Finally, in Section 5, we provide a bijection between our family of permutations and another Bell-counted class of permutations introduced by Poneti and Vajnovszki [Po-Va].

2. Definitions, Notations and Preliminaries

2.1. Subexceedant functions

**Definition 2.1.** A function $f$ over $[n]$ is said to be subexceedant if $1 \leq f(i) \leq i$ for all $i$, $1 \leq i \leq n$ (in some contexts it is required that $0 \leq f(i) \leq i - 1$).

**Notation 2.1.** We denote by $F_n$ the set of all subexceedant functions over $[n]$, and if $f$ is a function over $[n]$, we often denote by $f_i$ the value $f(i)$ and write $f$ as the word $f_1 f_2 \ldots f_n$.

Subexceedant functions can be used to code permutations, the Inversion table or the Denert table are two examples. F. Rakotondrajao and the second author defined in ([Ma-Ra]) a new bijection $\phi$ associating to each subexceedant function $f$ a permutation $\sigma = \phi(f)$ defined as the product of transpositions (the product is from left to right):

$$\sigma = (1 \: f_1) \: (2 \: f_2) \: \cdots \: (n \: f_n).$$

If $f_i = i$, then $(i \: f_i) = (i)$ does not represent a transposition but the identity permutation.

Further, a variation of this bijection associates with a subexceedant function $f$ the permutation $\sigma = \tilde{\phi}(f)$ defined by the product of transpositions (the product is from left to right):

$$\sigma = (n \: f_n) \: (n - 1 \: f_{n-1}) \: \cdots \: (1 \: f_1).$$

For example, take $f = 121132342$. Then $\phi(f) = 568179342$ while $\tilde{\phi}(f) = 497812536$.

In this paper we will work with the variation $\tilde{\phi}$.

In a previous work ([Be-Ma]) we gave the following :
Definition 2.2. Let $\sigma = \sigma(1)\sigma(2) \cdots \sigma(n) \in \mathfrak{S}_n$. Then the inverse nearest orbital minorant (inom) of $i \in [n]$ is the integer $j = \sigma^{-t}(i) \leq i$ with $t \geq 1$ chosen as small as possible.

Example 2.1. Let $\sigma = 10 \ 6 \ 8 \ 5 \ 1 \ 4 \ 9 \ 3 \ 2 \ 7 = (1 \ 10 \ 7 \ 9 \ 2 \ 6 \ 4 \ 5)(3 \ 8)$. Then

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|---|---|---|---|---|---|---|---|---|----|
| inom($x$) | 1 | 1 | 3 | 2 | 4 | 2 | 1 | 3 | 7 | 1 |

This can be clearer with a picture where the permutation is represented as a union of cyclic graphs. The blue continuous arcs $(i, \sigma(i))$ represent the action of the permutation, the red dashed arcs $(i, \text{inom}(i))$ represent the action of the corresponding subexceedant function.

Definition 2.3. We call inom code the bijection $\tilde{\phi}^{-1}$.

In ([Be-Ma]), we also proved the following:

Theorem 2.1. If $\tilde{\phi}^{-1}(\sigma) = f = f_1f_2 \cdots f_n$, then $f(i) = \text{inom}(i)$.

Notation 2.2. We denote by $F_n$ the set of all subexceedant functions over $[n]$, and $f = f_1f_2 \cdots f_n \in F_n$, where $f_i = f(i), i \in [n]$.

Let $f = f_1f_2 \cdots f_n \in F_n$. Then the set of images of $f$ is denoted by $\text{Im}(f)$ and its cardinality by $\text{IMA}(f)$. For instance, in $f = 121132342$, then $\text{Im}(f) = \{1, 2, 3, 4\}, \text{IMA}(f) = 4$. 

Figure 1: $\sigma$ and inom
2.2. Set Partitions

**Definition 2.4.** Let $S = [n]$, the set of the first $n$ positive integers. A set partition $\pi$ of $S$ is defined as a collection $B_1, \ldots, B_k$ of nonempty disjoint subsets such that $\bigcup_{i=1}^{k} B_i = S$. The subsets $B_i$ will be referred to as "blocks".

The block representation of a set partition $\pi$ is: $\pi = B_1/B_2/\ldots/B_k$.

**Definition 2.5.** The block representation of a set partition is said to be standard if the blocks $B_1, \ldots, B_k$ are sorted in such way that $\min(B_1) < \min(B_2) < \cdots < \min(B_k)$ and if the elements of every block are arranged in an increasing order.

We consider set partitions only in their standard representation. The condition on the order of the blocks and the arrangement of the integers in each block implies that the standard representation of a set partition is unique and that two different set partitions have different standard representations.

**Remark 2.1.** Note that in the standard representation, the integer 1 is always in the first block, 2 is in one of the first two blocks, and 3 is in any one of the first three blocks. Indeed, it is easy to show that in the standard representation of a set partition $\pi$ of $[n]$,

$$\text{every element } i \text{ of } [n] \text{ is necessarily in one of the first } i \text{ blocks.} \quad (1)$$

**Definition 2.6.** The canonical form of a set partition of $[n]$ is a $n$-tuple indicating the block of the standard representation in which each integer occurs, that is, $f = (f_1, f_2, \ldots, f_n)$ such that $j \in B_{f_j}$ for all $j$ with $1 \leq j \leq n$.

**Example 2.2.** The sequences of canonical forms corresponding to the set partitions of $[3]$ are: $\langle 1, 1, 1 \rangle, \langle 1, 1, 2 \rangle, \langle 1, 2, 2 \rangle, \langle 1, 2, 1 \rangle, \langle 1, 2, 3 \rangle$.

Note that canonical form of a set partition is a subexceedant function. But, not all subexceedant functions are canonical forms of set partitions.

The set partitions of $[n]$ having exactly $k$ blocks are counted by the Stirling numbers of the second kind ([BB, St1]) denoted $S(n, k)$, which satisfy the recurrence relation:

$$S(n, k) = kS(n - 1, k) + S(n - 1, k - 1)$$

On the other hand, the number of all set partitions over $[n]$ is counted by the Bell numbers, denoted $B(n)$ and satisfying:

$$B(n) = \sum_{k=0}^{n} S(n, k) \quad \text{and} \quad B(n + 1) = \sum_{i=0}^{n} \binom{n}{i} B(i), n \geq 0, \text{ with } B(0) = 1.$$
We denote the set of all set partitions of $[n]$ by $\mathcal{P}(n)$, and its cardinality by $B_n$, (equal to the $n$-th Bell number $B(n)$) with $B_0 = 1$ (as there is only one set partition of the empty set).

In Subsection 3.1 we will associate each set partition of $[n]$ with a permutation of the symmetric group $\mathfrak{S}_n$ having certain properties.

### 2.3. Statistics on permutations

Let $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n) \in \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the symmetric group of permutations over $[n]$.

Recall that a weak exceedance of $\sigma$ is a position $i$ such that $\sigma(i) \geq i$; the set of weak exceedances of $\sigma$ is $w-\text{Exc}(\sigma) = \{i : \sigma(i) \geq i\}$. The values of weak exceedances are said to be weak exceedance letters and the subword of $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n)$ including all weak exceedance letters is denoted by $w-\text{ExcL}(\sigma)$. An anti-exceedance of $\sigma$ is a position $i$ such that $\sigma(i) \leq i$, and its value is called an anti-exceedance letter.

**Example 2.3.** Let $\sigma = 435129678$. Then $w-\text{Exc}(\sigma) = \{1, 2, 3, 6\}$. The set of weak exceedance letters of $\sigma$, $w-\text{ExcL}(\sigma) = \{\sigma(i) : i \in w-\text{Exc}(\sigma)\} = \{4, 3, 5, 9\}$.

In [Ma-Ra] it is proved that if $\sigma = \phi(f)$ then the elements in $IMA(f)$ correspond to the anti-exceedance letters of $\sigma$, while in [Be-Ma], it is proved that if $\sigma = \tilde{\phi}(f)$ then the elements in $IMA(f)$ correspond to the weak exceedences (positions) of $\sigma$.

**Remark 2.2.** It is also proved that $\sigma$ has an anti-exceedance at $i$ if and only if at position $i$ there is the rightmost occurrence of the integer $k = f(i)$ in $f = \phi^{-1}(\sigma)$. Analoguously $\sigma$ has a weak exceedance at $i$ if and only if at position $k$ there is the rightmost occurrence of an integer $i = f(k)$ in $f = \tilde{\phi}^{-1}(\sigma)$.

### 3. Inom code and Canonical Forms

In this section we will compare the inom code for permutations with the canonical form of set partitions of $[n]$ and we define a family of permutations counted by the Bell numbers.

M. Orlov in [Or] used the following characterization of canonical forms of set partitions to implement an efficient algorithm for the exhaustive generation of all set partitions for a given $n$, with constant space and amortised constant time complexity.
Lemma 3.1. ([Or]) There is a bijection between all set partitions of the set \([n]\) and the set

\[
\{ (1, k_2, \ldots, k_n) : k_i \in \mathbb{N} \text{ and for all } i \text{ with } 2 \leq i \leq n \text{ one has } 1 \leq k_i \leq 1 + \max_{1 \leq j < i} k_j \} \tag{2}
\]

for all \(n \in \mathbb{N}\).

The \(n\)–tuples satisfying condition \((2)\) are called Restricted Growth Functions (RGF).

The canonical form of a set partition of \([n]\) is indeed a subexceedant function: we noted before that in the standard representation of a set partition, an integer \(i\) is always in one of the \(i\) first blocks so one has \(f_i \leq i\) for all \(i\) in \([n]\).

We have noted before, not all subexceedant functions are RGF and therefore canonical forms of set partitions.

We want to study the subset of \(F_n\) corresponding to \(\Psi(n)\), as well as the permutations whose inom codes is one of these subexceedant functions.

This set of permutations is counted by the Bell numbers, therefore we will call these objects “Bell permutations of the second kind” (M. Poneti and V. Vajnovski in [Po-Va] already introduced another family of permutations counted by the Bell numbers that they called “Bell permutations”).

We will first reformulate the condition expressed in equation \((2)\) of Lemma 3.1 in such a manner that is more useful for our purposes.

Proposition 3.1. Let \(f = f_1 f_2 \ldots f_n\) be a subexceedant function, then \(f\) satisfies the condition expressed in equation \((2)\) of Lemma 3.1 if and only if for all \(i \in [n]\) one has \(\{f_1, f_2, \ldots, f_i\} = \{1, 2, \ldots, p\} = [p]\) for a certain \(p\). In other terms, for all \(i \in [n]\), the set \(\{f_1, f_2, \ldots, f_i\}\) is an integer interval with minimum value 1.

Proof. Suppose that \(f\) satisfies the condition expressed in equation \((2)\) of Lemma 3.1. We use induction on \(i\).

For \(i = 1\), \(f_1 = 1\) and \(\{1\}\) is an interval.

Suppose \(\{f_1, f_2, \ldots, f_{i-1}\}\) is an interval \([p]\). Then the condition expressed in \((2)\) implies \(f_i \leq 1 + \max \{f_1, \ldots, f_{i-1}\} \leq p + 1\).

So \(\{f_1, f_2, \ldots, f_i\} = [p] \cup \{f_i\}\) is either the interval \([p]\) or the interval \([p + 1]\).

Conversely, suppose that for all \(i \in [n]\) the set \(\{f_1, f_2, \ldots, f_i\}\) is an integer interval \([k]\) for some \(k\), let us prove that for all \(i\), \(f_1 \leq f_i \leq \max_{1 \leq j < i} \{f_j\} + 1\).

This is obviously true if \(i = 1\).
Let \( i \geq 2 \). Let \( \{f_1, f_2, ..., f_i\} = [k_1] \) and \( \{f_1, f_2, ..., f_{i-1}\} = [k_2] \) for some integers \( k_1 \) and \( k_2 \). Then there are only two possibilities: either \( k_1 = k_2 \) or \( k_1 = k_2 + 1 \).

1. **Case** \( k_1 = k_2 \) : this implies that \( f_i \leq \text{Max}_{1 \leq j < i}\{f_j\} \leq \text{Max}_{1 \leq j < i}\{f_j + 1\} \).
2. **Case** \( k_1 = k_2 + 1 \) : this implies that \( f_i = k_1 = k_2 + 1 = \text{Max}_{1 \leq j < i}\{f_j + 1\} \).

\[
\square
\]

### 3.1. Bell Permutations of the second kind

In this subsection, we will present the class permutations associated to RGFs under \( \hat{\phi} \).

**Definition 3.1.** We define Bell permutation of the second kind a permutation \( \sigma \in S_n \) whose corresponding subexceedant function \( f = f_1 f_2 ... f_n = (\hat{\phi})^{-1}(\sigma) \) satisfies: the set \( \{f_1, f_2, ..., f_i\} \) is an integer interval for all \( i \in [n] \).

We denote by \( BP_2(n) \) the set of Bell permutations of the second kind over \( [n] \) and by \( b(n) \) its cardinality, the \( n \)-th Bell number.

**Example 3.1.** There are five Bell permutations of the second kind on \( [3] \). These are all the permutations of \( S_3 \) except the permutation 213. Note that \( f = (\hat{\phi})^{-1}(213) = 113 \) and \( \text{Im}(f) = \{f_1, f_2, f_3\} = \{1, 3\} \) is not an integer interval.

**Remark 3.1.** If \( \sigma = \sigma(1) \sigma(2) ... \sigma(n) \in S_n \) is a Bell permutation of the second kind, then its inom code \( f \) has \( \text{IMA}(f) = \{1, 2, ..., p\} \) for a certain \( p \), therefore \( \sigma \) has weak exceedances exactly at \( \{1, 2, ..., p\} \).

**Definition 3.2.** Let \( \sigma = \sigma(1) \sigma(2) ... \sigma(n) \in S_n \). Then the increasing integer sequence \( \text{Seq}(\sigma) \) associated to \( \sigma \) is given as follows:

For \( x = 1, 2, ..., n \), we add \( x \) to \( \text{Seq}(\sigma) \) if \( \text{inom}(x) = y \) is not the inom of some integer smaller than \( x \).

**Example 3.2.** Let \( \sigma = 435129678 = (1 4)(2 3 5)(6 9 8 7) \). Then \( \text{Seq}(\sigma) = \langle 1, 2, 5, 6 \rangle \).

**Remark 3.2.** The cardinality of \( \text{Seq}(\sigma) \), \( \sigma \in S_n \) is equal to the cardinality of \( w-\text{Exc}(\sigma) \). That is, if \( f = \hat{\phi}^{-1}(\sigma) \), then \( i \in \text{Seq}(\sigma) \) if and only if at the position \( i \) there is the leftmost occurrence of the integer \( k = f(i) \) in \( f \).

The characterisation of RGFs as expressed in Proposition 3.1 implies a characterization for Bell permutations of the second kind.

8
**Theorem 3.1.** Let $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n) \in \mathfrak{S}_n$ with the set of weak exceedance letters $w-\text{ExcL}(\sigma) = \langle \alpha_1, \alpha_2, \ldots, \alpha_k \rangle$ and $\text{Seq}(\sigma) = \langle \gamma_1, \gamma_2, \ldots, \gamma_k \rangle$. Then $\sigma$ is a Bell-permutation of the second kind if and only if

1. the set of the weak exceedances of $\sigma$ is exactly the interval $\{1, 2, \ldots, k\}$, and
2. \[ \gamma_i \leq \alpha_i, \text{ for all } i = 1, 2, \ldots, k. \] (3)

**Proof.** Let $f \in F_n$ be the code of the permutation $\sigma$ via the bijection $(\tilde{\phi})^{-1}$. As a special case of Proposition 3.1 for $i = n$, we have $\text{Im}(f) = \{f_1, f_2, \ldots, f_n\}$ is an integer interval $[p]$ for some $p$. But on the other hand $\text{Im}(f)$ coincides with the set of the weak exceedances of $\sigma$ whose cardinality is $k$ (Remark 2.2). Therefore $p = k$ and the set of the weak exceedances of $\sigma$ is exactly $\{1, 2, \ldots, k\}$.

Further, for $\gamma_i \in \text{Seq}(\sigma)$ at position $\gamma_i$ in $f$ we have the value $f_{\gamma_i}$ is the leftmost occurrence in $f$. Because $\text{Seq}(\sigma)$ only includes the positions of the smallest values of all among the values of $\text{inoms}$ ($f_j = \text{inom}(j)$) and for $\alpha_i \in w-\text{ExcL}(\sigma)$ we have $f_{\alpha_i}$ is the rightmost occurrence in $f$ (Remark 2.2). From this we see that $f = f_1f_2\ldots f_n$ is an integer interval for each $j \in [n]$ if and only if $w-\text{Exc}(\sigma) = [k]$ and the leftmost occurrences of $f$ are increasing. Hence $\gamma_i \leq \alpha_i$ for all $i \in [k]$.

**Example 3.3.**

1. Let $\sigma = 763592148 = (1\ 7)(2\ 6)(3)(4\ 5\ 9\ 8)$. Then $\text{Seq}(\sigma) = \langle 1, 2, 3, 4, 8 \rangle$ and $w-\text{ExcL}(\sigma) = \langle 7, 6, 3, 5, 9 \rangle$. Thus the two sequences satisfy (3) and hence $\sigma$ is a Bell-permutation of the second kind.

2. Let $\sigma = 245987316 = (1\ 2\ 4\ 9\ 6\ 7\ 3\ 5\ 8)$. Then $\text{Seq}(\sigma) = \langle 1, 3, 5, 6, 7, 8 \rangle$ and $w-\text{ExcL}(\sigma) = \langle 2, 4, 5, 9, 8, 7 \rangle$. Observe that $\gamma_6 = 8 > 7 = \alpha_6$ and hence $\sigma$ is not a Bell permutation of the second kind.

The following proposition gives a recursive procedure to check if a permutation is a Bell Permutation of the second kind, for its proof it is convenient to give first the following.

**Lemma 3.2.** Let $\tilde{\phi}(f) = \sigma \in \mathfrak{S}_n$, where $f$ is a subexceedant function over $[n]$ obtained from the subexceedant function $f'$ over $[n-1]$ by concatenating some $j \in [n]$ at its end, where $\tilde{\phi}(f') = \sigma'$. Then $\sigma$ is obtained from $\sigma'$ by replacing the integer $n$ by the integer $\sigma'(j)$ in $\sigma'$ and appending the integer $\sigma'(j)$ at the end.

If $j = n$, then $\sigma$ is obtained from $\sigma'$ by appending $n$ at the end of $\sigma'$.

**Proposition 3.2.** A permutation $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n) \in \mathfrak{S}_n$ whose set of weak exceedances is an integer interval $[k]$ is in $BP_2(n)$ if and only if the permutation $\sigma' \in \mathfrak{S}_{n-1}$ obtained from $\sigma$ by exchanging the integer $n$ by $\sigma(n)$ in the word $\sigma(1)\sigma(2)\ldots\sigma(n-1)$ is in $BP_2(n-1)$. 

9
Proof. According to Lemma 3.2, for all permutations $\sigma$, if $f = f_1 \cdots f_n = (\tilde{\phi})^{-1}(\sigma)$ and $\sigma' \in S_{n-1}$ is the permutation obtained from $\sigma$ by replacing the integer $n$ by $\sigma(n)$, then the subexceedant function associated with $\sigma'$ is $f' = f_1 f_2 \cdots f_{n-1}$.

In the hypothesis that $w\text{-Exc}(\sigma) = \{f_1, \ldots, f_n\}$ is an integer interval $[k]$, the two following conditions become trivially equivalent:

1. for all $i \in [n]$, the set $\{f_1, f_2, \ldots, f_i\}$ is an integer interval with minimum value 1.
2. for all $i \in [n-1]$, the set $\{f_1, f_2, \ldots, f_i\}$ is an integer interval with minimum value 1.

That is, according to Proposition 3.1, $\sigma$ is Bell if and only if $\sigma'$ is Bell. 

Example 3.4. We will look at the three different cases:

1. Consider $\sigma = 7156432$. We have $w\text{-Exc}(\sigma) = \{1, 3, 4\}$, which is not an integer interval, thus $\sigma$ is not a Bell permutation of the second kind.
2. As we have already observed, if one considers $\sigma = 2431$, this permutation has set of weak exceedances $\{1, 2, 3\} = [3]$ but it is not a Bell permutation of the second kind as $2431 \rightarrow 213$ and $213$ is not Bell permutation of the second kind because its set of weak exceedances, $\{1, 3\}$ is not an interval.
3. Let $\sigma = 7245613$. We have $w\text{-Exc}(\sigma) = [5]$, so $\sigma$ may be a Bell permutation of the second kind. We apply Proposition 3.2: $7245613 \rightarrow 324561 \rightarrow 32451 \rightarrow 321$. Since $321$ is a Bell permutation of the second kind of $S_3$ we can conclude that $\sigma$ and those permutations obtained in the process are Bell permutations of the second kind.

3.2. The distribution of weak exceedances on Bell Permutations of the second kind

We denote by $BP_2(n, k)$ the set of Bell permutations of the second kind over $[n]$ having $k$ weak exceedances and by $b(n, k)$ their cardinalities. We can use Proposition 3.2 to give a direct, constructive proof that the cardinalities of the sets $BP_2(n, k)$ equal the Stirling numbers of the second kind $S(n, k)$.

Accordingly, we define the following operation to construct a Bell permutation of the second kind $\sigma$ in $BP_2(n)$ starting from a Bell-permutation of the second kind $\sigma' \in BP_2(n-1)$ (and an integer):

Let $\sigma' \in BP_2(n-1, k)$ and $i \in [k+1],$

then $\sigma$ is obtained by replacing $\sigma'(i)$ by $n$ and then appending $\sigma'(i)$ at the end. 

Proposition 3.3. The number of Bell permutations of the second kind over \([n]\) having \(k\) weak exceedances equals the Stirling number of the second kind \(S(n,k)\).

Proof. We prove that the numbers \(b(n,k)\) satisfy the same recurrence relation as the Stirling number of the second kind:

\[
b(n,k) = kb(n-1,k) + b(n-1,k-1).
\]

Observe that the operation defined in (4) either preserves the number of weak exceedances or increases it by 1. Therefore, any Bell permutation of the second kind in \(BP_2(n,k)\) can uniquely be obtained from a permutation \(\sigma' \in BP_2(n-1,k)\) or from a permutation \(\sigma' \in BP_2(n-1,k-1)\). More precisely:

1. if \(w-Exc(\sigma') = [k]\), \(i \in [k]\) and \(\sigma\) is obtained from \(\sigma'\) by the operation defined in (4), then \(\sigma \in BP_2(n,k)\). There are \(b(n-1,k)\) possible choices for \(\sigma'\) and \(k\) possible choices for \(i\) hence this contributes \(kb(n-1,k)\) to \(b(n,k)\).

2. if \(w-Exc(\sigma') = [k-1]\), \(i = k\) and \(\sigma\) is obtained from \(\sigma'\) by the operation defined in (4), then \(\sigma \in BP_2(n,k)\). This contributes \(b(n-1,k-1)\) to the number \(b(n,k)\).

This completes the proof.

Example 3.5. Note that \(b(4,2) = 2b(3,2) + b(3,1) = 2 \times 3 + 1 = 6 + 1 = 7\). So, from each of the elements of \(BP_2(3,2) = \{132, 231, 321\}\) we obtain 2 Bell permutations of the second kind in \(BP_2(4,2)\) by the first operation described in the proof of Proposition 3.3. That is, take \(2 \times \{132, 231, 321\}\) and apply the first operation to get \(\{4321, 1423, 4312, 2413, 4213, 3412\} \subseteq BP_2(4,2)\).

Again from \(BP_2(3,1) = \{312\}\) we obtain 1 Bell permutation of the second kind in \(BP_2(4,2)\) by the second operation. That is, take 312 and \(k = 1 + 1 = 2\). Then by the second operation of the proof of Proposition 3.3 we have \(\{3421\} \subseteq BP_2(4,2)\).

4. A bijection between Bell permutations of the second kind and set partitions

Here we give a direct bijection between Bell permutations of the second kind in \(BP_2(n)\) and set partitions in \(\mathfrak{P}(n)\).

Let \(\pi = B_1/B_2/\cdots/B_k\) be a set partition of \([n]\) having \(k\) blocks. Then by (1) each element of \(B_i\) is greater than or equal to \(i\), for all \(i \in [k]\).
Let $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ be a Bell permutation of the second kind with $k$ weak exceedances, then the set of these weak exceedances is $w-Exc(\sigma) = [k]$.

Define a map $\lambda : BP_2(n) \to \mathfrak{P}(n)$ by $\lambda(\sigma) = \pi$ provided that:

1. for all weak exceedances $(i, \sigma(i))$ with $i \geq \sigma(i)$, insert $\sigma(i)$ in the $i$-th block, and
2. for all non weak exceedance letters $i$, taken in decreasing order, insert $i$ at the beginning of the $inom(i)$-th block.

**Example 4.1.** Let $\sigma = 45213 \in \mathfrak{B}(5, 2)$. Then $\lambda(\sigma) = \pi$ has two blocks. That is, we have $4/5$. Observe that $inom(3) = 2$, $inom(2) = 2$ and $inom(1) = 1$. So $3$ and $2$ are in the second block, and $1$ is in the first block. Thus we have $\pi = 14/235 \in \mathfrak{P}(5, 2)$.

**Remark 4.1.** If $\lambda(\sigma) = \pi = B_1/\ldots/B_k$, where $w-Exc(\sigma) = [k]$, then $\max(B_i) = \sigma(i), i = 1, \ldots, k$.

**Proposition 4.1.** The map $\lambda$ is a bijection from $BP_2(n)$ onto $\mathfrak{P}(n)$.

**Proof.** We will prove that $\lambda$ is indeed the composition of two bijections: the restriction of the permutation code $(\tilde{\phi})^{-1}$ to $BP_2(n)$ and the bijection associating to a canonical form its correspondent partition.

For any $\sigma \in BP_2(n)$, let $\pi \in \mathfrak{P}(n)$ with $\pi = \lambda(\sigma)$, let $f_\sigma = (\tilde{\phi})^{-1}(\sigma)$ and $f_\pi$ is the canonical form of $\pi$.

It suffices to prove that for all Bell-permutations of the second kind $\sigma$ one has $f_\sigma = f_\pi = f_{\lambda(\sigma)}$. Now the definition of $\tilde{\phi}$ implies easily that every integer $i$ is placed in $f_\sigma(i)$-th block of $\pi$ and hence $f_\pi(i) = f_\sigma(i)$ for all $i$. □

**Example 4.2.** Take $\sigma = 36821457 \in BP_2(8)$. Then $\lambda(\sigma) = \pi = 13/246/578$.

By applying $\tilde{\phi}^{-1}$ to $\sigma$ we get $\tilde{\phi}^{-1}(\sigma) = f_\sigma = 12123233 = f_\pi$.

We now give the direct inverse of the bijection $\lambda$.

Let us define a mapping $\chi : \mathfrak{P}(n) \mapsto BP_2(n)$ such that $\chi(\pi) = \sigma$, where $\sigma$ is obtained from $\pi = B_1/B_2/\ldots/B_k \in \mathfrak{P}(n)$ as follows. Let $m_i = \max(B_i)$ and $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n)$. Then

1. For $i = 1, 2, \ldots, k$ set $\sigma(i) = m_i$.
2. For $j \in [n]\{m_1, \ldots, m_k\}$, in decreasing order, set $j = \sigma(\sigma^t(r))$, where $r$ is the number of the block where $j$ is and $t$ is the smallest positive integer such that $\sigma^t(r)$ has not yet been received an image.

**Lemma 4.1.** For all $\pi \in \mathfrak{P}(n)$ the image $\sigma = \chi(\pi) \in BP_2(n)$. 

12
Proof. Let us start by proving $w-Exc(\sigma) = [k]$, where $k$ is the number of blocks of $\pi$. Note that every element of block $B_i$ of $\pi$ is greater than or equal to $i$. Hence $\sigma(i) = m_i \geq i$, for $i = 1, \ldots, k$ and $i \in [k]$ is a weak exceedance of $\sigma$.

Take an integer $p > k$. We want prove that $q = \sigma(p) < p$. Let $q \in B_r$ for some $r \leq k$. Then $p = \sigma^t(r)$ for a certain $t$.

If $t = 1$, then $p = m_r$ and hence $p > q$ since $q$ is in the $r$-th block.

If $t > 1$, then the position $\sigma^{t-1}(r)$ had already received $\sigma(\sigma^{t-1}(r)) = \sigma^t(r) = p$ as image. Since the integers are inserted in decreasing order we have $p > q$.

Therefore, $w-Exc(\sigma) = [k]$ and $w-ExcL(\sigma) = (\alpha_1, \ldots, \alpha_k)$, where $\alpha_i = m_i$ and $m_i$ is the largest integer having inom equal to $i$.

By the definition of $Seq(\sigma) = \langle \gamma_1, \ldots, \gamma_k \rangle$, the integer $\gamma_i$ is the smallest integer having inom equal to $i$. Therefore, $\gamma_i \leq \alpha_i$ for all $i$.

Hence $\sigma \in BP_2(n)$.

\[ \square \]

Proposition 4.2. The map $\chi$ is the inverse of the map $\lambda$.

Proof. Let $\chi(\pi) = \sigma$ and $\lambda(\sigma) = \pi'$, where $\pi \in \mathcal{P}(n, k)$. Then we show that $\pi = \pi'$.

Since $w-Exc(\sigma) = [k]$ we have that $\pi' \in \mathcal{P}(n, k)$. That is, $\pi'$ has $k$ blocks.

Let $x$ be in the $i$-th block of $\pi$.

If $x = m_i$, then $\sigma(i) = x$ and hence $x$ is in the $i$-th block of $\pi'$ under $\lambda$. This is because $i \in w-Exc(\sigma)$.

If $x \neq m_i$, then there is the smallest integer $t$ such that $x = \sigma(\sigma^t(i))$. This means that $\sigma^{-1}(x) = i$ and hence inom$(x) = i$ and $x$ is in the $i$-th block of $\pi'$ under $\lambda$. Thus $\pi = \pi'$.

\[ \square \]

Example 4.3. Let $\pi = 1 4 2 9 3 5 10 6 8 \in \mathcal{P}(10)$. Then $\sigma = \sigma(1)\sigma(2) \ldots \sigma(n)$, where : $\sigma(1) = 7, \sigma(2) = 9, \sigma(3) = 10, \sigma(4) = 8$, and $6 = \sigma^2(4) = \sigma(8), 5 = \sigma^2(3) = \sigma(10), 4 = \sigma^2(1) = \sigma(7), 3 = \sigma^3(3) = \sigma(5), 2 = \sigma^2(2) = \sigma(9), 1 = \sigma^5(1) = \sigma(6)$. Thus, $\sigma = \chi(\pi) = 7 9 10 8 3 1 4 6 2 5 \in BP_2(10)$.

Our bijection $\lambda$ allows us to characterise and enumerate some subclasses of $BP_2(n)$ that are associated to some remarkable subclasses of $\mathcal{P}(n)$. We give two exemples in the following corollaries.

Corollary 4.1. The number of Bell permutations of the second kind in $BP_2(n, k)$ with $\sigma(k) = n$ and $\sigma(n) < k, k \neq 1, n$ equals the total number of Bell permutations of the second kind in $BP_2(n-1, k-1)$. That is,

\[ b(n - 1, k - 1) = \# \{ \sigma \in BP_2(n, k) : \sigma(k) = n, k \neq 1, \sigma(n) < k < n \} \]
Proof. We prove that the integer \( n \) forms a singleton block in the set \( \mathfrak{p}(n,k) \) if and only if the corresponding Bell permutation of the second kind in \( BP_2(n,k) \) satisfies condition \( \ref{condition5} \).

Let \( \pi \in \mathfrak{p}(n,k) \) and \( n \) forms a singleton block in \( \pi \). Then deleting this block we get \( \pi' \in \mathfrak{p}(n-1,k-1) \).

Let \( \lambda^{-1}(\pi) = \sigma \) and \( \lambda^{-1}(\pi') = \sigma' \). Then \( \sigma \in BP_2(n,k) \) and \( \sigma' \in BP_2(n-1,k-1) \) and \( \sigma'(k) < k \) since \( \sigma' \) has only \( k-1 \) weak exceedances. So by Proposition \( \ref{prop3.3} \) \( \sigma \in BP_2(n,k) \) satisfies condition \( \ref{condition5} \) if and only if \( \sigma = (k \cdot n) \cdot \sigma' \). That is, \( \sigma(k) = n \) and \( \sigma(n) = \sigma'(k) < k \).

Note that the number of set partitions in \( \mathfrak{p}(n,k) \) in which \( n \) forms a singleton block equals \( b(n-1,k-1) \), and hence the proof. \( \square \)

Remark 4.2. The equation in \( \ref{condition5} \) is also true for \( k = 1 \) or \( k = n \). That is, there is only one set partition with \( n \) blocks which corresponds to the identity permutation, and only one set partition with one block which corresponds to \( 1 2 \ldots n-1 \). So for all possible \( k \) satisfying the condition of the previous corollary we have all Bell permutations of the second kind over \( [n-1] \).

Example 4.4. There are five Bell permutations of the second kind over \( [4] \) that satisfy the condition of Corollary \( \ref{cor4.1} \). These are: 4123, 2341, 3241, 1342, 1234. Their corresponding set partitions under \( \lambda \), respectively, are: 123/4, 12/3/4, 13/2/4, 1/23/4, 1/2/3/4. Removing the singleton block of 4 from each of them we get all set partitions over \( [3] \).

Further, by the corollary we have 4123 ↔ 312, 2341 ↔ 231, 3241 ↔ 321, 1342 ↔ 132, and 1234 ↔ 123.

Corollary 4.2. The number of Bell permutations of the second kind of size \( n \) having \( n-1 \) weak exceedances is equal to the number \( b(n,n-1) \) of set partitions over \( [n] \) having \( n-1 \) blocks, which, as commonly known, is equal to the sum of the first \( n-1 \) positive integers \( \binom{n}{2} = \frac{n(n-1)}{2} \).

Proof. The second part can be easily demonstrated. Let \( \pi \in \mathfrak{p}(n) \) with \( n-1 \) blocks. Then there is some \( i \in [n-1] \) such that the \( i \)-th block contains exactly two elements. Since every element \( j \in [n] \) must be in any one of the first \( j \) blocks, then for all \( j \in [i] \) the integer \( j \) is in the \( j \)-th block, hence the second element of block \( i \) can only be chosen from \([i+1,n]\), therefore we have \( n-i \) possible ways to select the other element in to the \( i \)-th block. This implies that there are \( (n-1) + (n-2) + \cdots + 2 + 1 = \binom{n}{2} = \frac{n(n-1)}{2} \) total such set partitions.

Further we have observed that set partitions with \( n-1 \) blocks correspond to Bell permutations of the second kind with \( n-1 \) weak exceedances. Then the number of Bell-permutations of the second kind over \( [n] \) with \( n-1 \) weak exceedances also equals the sum of the first \( n-1 \) positive integers.

Note that set partitions with \( n-1 \) blocks in which the block containing two elements is the \( i \)-th block correspond to Bell-permutations of the second kind \( \sigma = \sigma(1)\sigma(2)\ldots\sigma(n) \) with \( n-1 \) weak exceedances, where \( \sigma(n) = i \). \( \square \)
5. A bijection between Bell permutations of the first and the second kind

In this part we present a bijection between the set $BP_1(n)$ of Bell permutations over $[n]$ introduced by M. Poneti and V. Vajnovszki ([Po-Va]) (which we will call Bell permutations of the first kind) and the set $BP_2(n)$ of Bell permutations of the second kind over $[n]$.

First, we recall the definition of Bell permutations of the first kind.

Let $\pi = B_1/B_2/\ldots/B_k \in \mathfrak{P}(n)$ a set partition in its standard representation and let $\mu : \mathfrak{P}(n) \mapsto BP_1(n)$, where the permutation $\mu(\pi)$ is constructed as follows:

- reorder all integers in each block $B_i$ in decreasing order;
- transform each of these blocks into a cycle.

**Example 5.1.** Let $\pi = 1279/356/48$. Then $\mu(\pi) = (9721)(653)(84)$.

**Remark 5.1.** By definition of Bell permutations of the first kind and by definition of inom code, if $\sigma \in BP_1(n)$ and $f = \tilde{\phi}^{-1}(\sigma)$ is its inom code, then for all $i \in [n]$,

$$f(i) = \text{inom}(i) = \text{minimum of the block containing } i.$$

Recall also that by definition of Bell permutations of the second kind, if $\sigma \in BP_2(n)$ and $f = \tilde{\phi}^{-1}(\sigma)$ is its inom code, then for all $i \in [n]$,

$$f(i) = \text{inom}(i) = \text{number of the block containing } i.$$

Let us define a map $\beta : BP_1(n) \mapsto BP_2(n)$.

Let $\sigma_1 = C_1C_2\ldots C_k \in BP_1(n)$ be a permutation of $BP_1(n)$ written as a product of disjoint cycles, where the cycles have been ordered with their respective minima increasing. Then $\sigma_2 = \beta(\sigma_1)$ is constructed from $\sigma_1$ according to the rule:

for $i = k, \ldots, 2$, if the integer $i$ is not in the cycle $C_i$, then insert the cycle $C_i$ after $i$ in the cycle containing $i$.

**Example 5.2.** Let $\sigma_1 = (9721)(653)(84)$. Then $\sigma_2$ is obtained as:

$$\sigma_1 = (9721)(653)(84) \longrightarrow (9721)(65384) \longrightarrow (972653841) = \sigma_2$$

We start by proving that $\beta(BP_1(n)) \subseteq BP_2(n)$. 

15
Lemma 5.1. Let $\sigma_1 \in BP_1(n)$ with $k$ cycles. If $\sigma_2 = \beta(\sigma_1)$, then $w-Exc(\sigma_2) = \{1, 2, \ldots, k\}$.

Proof. Let $\sigma_1 = C_1C_2 \ldots C_k$, where $k$ is the number of cycles of $\sigma_1$.

The operation obviously constructs a weak exceedance at each $p \leq k$, because it inserts after $p$ a sequence of integers all larger than $p$. It remains to be proved that if $p > k$ then $p$ is a strict anti-exceedance for $\sigma_2$, that is $\sigma_2(p) < p$.

If $p > k$ and $p$ is a strict anti-exceedance for $\sigma_1$, that is if $\sigma_1(p) < p$, then the construction never inserts anything between $p$ and $\sigma_1(p)$, therefore $\sigma_2(p) = \sigma_1(p) < p$.

Let then $p > k$ and $\sigma_1(p) \geq p$. Note that this can only happen if $p$ is the minimum of its cycle of $\sigma_1$, say $C_t$.

Let $\sigma_1(p) \ldots p$ be the sequence of elements of the cycle $C_t$ of $\sigma_1$, where $p = \min(C_t)$ and $\sigma_1(p) = \max(C_t)$. Then $t \leq k < p$ and $t$ is not in the cycle $C_t$ because $p$ is the minimum in $C_t$, hence there is some integer $t_1 < t$ such that $t \in C_{t_1}$.

For $i = k, \ldots, t+1$ the procedure has not yet modified $C_t = (\sigma_1(p) \ldots p)$ since all of its elements are greater than $k$.

Now for $i = t$, the operation inserts $\sigma_1(p) \ldots p$ in to $C_{t_1}$ after $t$ (the following diagram shows this).

\[\begin{array}{c}
\cdots (\ldots t \ \sigma_1(p) \ldots ) \ldots \\
C_{t_1}
\end{array}\]

If $t \neq \min(C_{t_1})$, then $t$ is a strict anti-exceedance for $\sigma_1$, that is, $\sigma_1(t) < t$. The steps of the transformation for $k, k - 1, \ldots, t + 1$ have not inserted any integer between $t$ and $\sigma_1(t)$. When we insert $C_t$ in between these two integers, $\sigma_1(t)$ become the image of $p$, then we have $\sigma_2(p) = \sigma_1(t) < t$.

Suppose $t = \min(C_{t_1})$, then $t_1 \not\in C_{t_1}$ for otherwise $t_1 \geq t$ and hence a contradiction.

Thus, there is some integer $t_2 < t_1$ such that $t_1 \in C_{t_2}$.

For $i = t_1$, the operation inserts the sequence of the elements of $C_{t_1}$ in to $C_{t_2}$ after $t_1$.

\[\begin{array}{c}
\cdots (\ldots t_1 \ldots t \ \sigma_1(p) \ldots ) \ldots \\
C_{t_2}
\end{array}\]

By the same argument as above, if $t_1 \neq \min(C_{t_2})$, then we have $\sigma_2(p) < t_1$.

Suppose $t_1 = \min(C_{t_2})$, then $t_2 \not\in C_{t_2}$ for otherwise $t_2 \geq t_1$ and hence a contradiction.

Thus, repeating the same procedure a finite number of times we must eventually find some integer $t_s$ such that $1 \leq t_s < t_{s-1} \neq \min(C_{t_s})$ and $t_s \in C_{t_s}$. Then $\sigma_2(p) < t_{s-1}$.

Therefore, $p$ is not a weak exceedance of $\sigma_2$. Hence $w-Exc(\sigma_2) = [k]$. \qed
Lemma 5.2. For all \( \sigma_1 \in BP_1(n) \), \( \beta(\sigma_1) = \sigma_2 \in BP_2(n) \).

Proof. Let \( \sigma_1 = C_1C_2 \ldots C_k \). By Lemma 5.1 we have \( w-Exc(\sigma_2) = [k] \) for a certain integer \( k \). Then \( w-ExcL(\sigma_2) = \langle \alpha_1, \ldots, \alpha_k \rangle \), where \( \alpha_i = \sigma_2(i), i = 1, 2, \ldots, k \).

Let \( Seq(\sigma_2) = \langle \gamma_1, \ldots, \gamma_k \rangle \). Then note that for all \( i \in [k] \), the integers \( \gamma_i \) and \( \alpha_i \), are respectively the minimum and maximum integers having \( \text{inom} \) equal to \( i \). Thus \( \gamma_i \leq \alpha_i \) for all \( i \).

Therefore, by Theorem 3.1 \( \sigma_2 \in BP_2(n) \). \(\square\)

Theorem 5.1. The map \( \beta \) is a bijection between \( BP_1(n) \) and \( BP_2(n) \).

Proof. We deduce the result from the fact that the following diagram is commutative where :

- \( \tilde{\phi} \) denotes the \( \text{inom} \) code;
- \( \tau \) denotes the bijection associating each partition \( \pi \) with its canonical form;
- \( \mu \) denotes the bijection introduced by Poneti and Vajnovszki;
- \( \nu \) denotes the transformation that normalizes any \( f \in \tilde{\phi}^{-1}(BP_1(n)) \) via an order-preserving bijection of \( \text{Im}(f) \) into \( [IMA(f)] \);
- \( \zeta \) denotes the transformation that replaces every integer in any \( f = f_1f_2 \ldots f_n \in \tilde{\phi}^{-1}(BP_2(n)) \) with the leftmost position where such integer occurs.

![Figure 2](image)

From the remark 5.1 it is easy to see that the \( \text{inom} \) code of the permutation \( \beta(\sigma_1) \) is obtained by applying \( \nu \) to the \( \text{inom} \) code of \( \sigma_1 \) and that the \( \text{inom} \) code of \( \sigma_1 \) is obtained by applying \( \zeta \) to the \( \text{inom} \) code of \( \beta(\sigma_1) \), (the maps \( \mu \) and \( \zeta \) are the inverse of each other when restricted to \( \tilde{\phi}^{-1}(BP_1(n)) \) and \( \tilde{\phi}^{-1}(BP_2(n)) \) respectively).

So we have \( \tilde{\phi}^{-1} \circ \beta = \nu \circ \tilde{\phi}^{-1} \), or equivalently \( \beta = \tilde{\phi} \circ \nu \circ \tilde{\phi}^{-1} \) and therefore \( \beta \) is a bijection. \( \square \)
We can also define directly the inverse of $\beta$, a map $\vartheta : BP_2(n) \mapsto BP_1(n)$ such that $\vartheta(\sigma_2) = \sigma_1$, where $\sigma_1$ is obtained as follows.

Take $\sigma_2 \in BP_2(n)$ and let $C_1C_2 \ldots C_t$ be its cycle decomposition. Recall we showed that the set of weak-exceedances of a Bell permutation of the second kind is an interval $[k]$.

For $i = 2, \ldots, k$, if $i$ is not the minimum of its own cycle $C_j$, then form a new cycle by taking out of $C_j$ the longest sequence of integers greater than $i$ starting immediately after $i$.

**Example 5.3.** Let $\sigma_2 = 468912357 = (1497358)(26)$ in cycle notation and with the weak exceedances in bold. Then $\sigma_1$ is obtained as:

\[
\sigma_2 = (1497358)(26) \rightarrow (14973)(26)(58) \rightarrow (143)(26)(58)(97) \rightarrow (143)(26)(5)(97)(8) = \sigma_1
\]

**Lemma 5.3.** For all $\sigma_2 \in BP_2(n)$, $\vartheta(\sigma_2) = \sigma_1 \in BP_1(n)$.

**Proof.** The construction produces a permutation with $k$ cycles in which elements of each cycle are decreasing. \hfill $\square$

**Proposition 5.1.** The map $\vartheta$ is the inverse of $\beta$.

**Proof.** Let $\sigma_2 \in BP_2(n)$, $w-\text{Exc}(\sigma_2) = [k]$ and let $\beta(\vartheta(\sigma_2)) = \beta(\sigma_1) = \sigma'_2$. Then we show that $\sigma_2 = \sigma'_2$. Since $\sigma_1$ has $k$ cycles, $\sigma'_2$ also has weak-exceedances at $\{1, 2, \ldots, k\}$, as well as $\sigma_2$.

If $i > k$, then $\sigma_2(i) < i$ and therefore the construction never changes the image of $i$ and we have $\sigma_2(i) = \sigma_1(i) = \sigma'_2(i)$.

Assume that $\sigma_2(i) \neq \sigma'_2(i)$ for some $i \in [k]$.

Note that if $\vartheta(\sigma_2) = \sigma_1$, then for every weak exceedance $i$ of $\sigma_2$, one has that $\sigma_2(i)$ is equal to the maximum element of the $i$-th cycle of $\sigma_1$. On the other hand, if $\sigma'_2 = \beta(\sigma_1)$ then for every weak exceedance $i$ of $\sigma'_2$, one also has that $\sigma'_2(i)$ is equal to the maximum element of the $i$-th cycle of $\sigma_1$. This is a contradiction.

Therefore $\sigma_2 = \sigma'_2$ and hence $\vartheta = \beta^{-1}$. \hfill $\square$

**Remark 5.2.** Under the above bijection $\beta : \sigma_1 \mapsto \sigma_2$, the number of cycles of $\sigma_1$ is equal to the number of weak exceedances of $\sigma_2$. 

18


Acknowledgements

Both authors are members of the project CoRS (Combinatorial Research Studio), supported by the Swedish government agency SIDA. The most significant advances of this research work have been made during two visits of the first author to IRIF. The first visit was entirely supported by IRIF, the second visit was substantially supported by ISP (International Science Programme) of Uppsala University (Sweden) and partially supported by IRIF. The authors are deeply grateful to these two institutions. We also thank our colleagues from CoRS for valuable discussions and comments.

References

[Ba-Va] J. Baril and V. Vajnovszki, A permutation code preserving a double Eulerian bistatistic, Discrete Applied Mathematics, Vol. 224, 9-15, (2017).

[Be-Ma] F. Beyene and R. Mantaci, Investigations on a Permutation Code, Submitted to Electronic Journal of Combinatorics, (2020)

[Bo] M. Bona, Introduction to Enumerative Combinatorics, The McGraw Hill Companies, (2007).

[Du-Vi] D. Dumont and G. Viennot, A combinatorial interpretation of the Seidel generation of Genocchi numbers, Ann. Discrete Math. 6, 77-87, (1980).

[Fo-Ze] D. Foata and D. Zeilberger, Denert’s Permutation Statistic is indeed Euler-Mahonian, Studies in Applied Mathematics, 31-59, (1990).

[Le] D. H. Lehmer, Teaching combinatorial tricks to a computer, Proc. Sympos. Appl. Math. Combinatorial Analysis, Amer. Math. Soc., vol. 10, p. 179-193, (1960).

[Ma] T. Mansour. Combinatorics of set partitions. Taylor & Francis Group, LLC, (2013).

[Ma-Ra] R. Mantaci and F. Rakotondrajao. A permutation representation that knows what “Eulerian” means. Discrete Mathematics and Theoretical Computer Science 4, 101-108, (2001).

[Or] M. Orlov, Efficient Generation of Set Partitions, (2002)

[Po-Va] M. Poneti and V. Vajnovszki, Generating restricted classes of involutions, Bell and Stirling permutations, European Journal of Combinatorics 31, 553-564, (2010).
[Ro] G. Rota, *The number of partitions of a set*, Amer. Math. Monthly 71, 498–504, (1964).

[St1] R. P. Stanley, *Enumerative Combinatorics*, Vol. 1, 2nd ed, Cambridge Studies of Advanced Mathematics, Cambridge University Press, (2011).

[St2] R. P. Stanley. *Enumerative combinatorics*. Vol. 2. Cambridge Studies of Advanced Mathematics, Cambridge University Press, (1999).