1 Introduction and abstract

Over the course of the last 50 years, many questions in the field of computability were left surprisingly unanswered. One example is the question of $P$ vs $NP \cap co-NP$. It could be phrased in loose terms as “If a person has the ability to verify a proof and a disproof to a problem, does this person know a solution to that problem?”.

When talking about people, one can of course see that the question depends on the knowledge the specific person has on this problem. Our main goal will be to extend this observation to formal models of set theory $ZFC$: given a model $M$ and a specific problem $L$ in $NP \cap co-NP$, we can show that the problem $L$ is in $P$ if we have “knowledge” of $L$.

In this paper, we’ll define the concept of knowledge and elaborate why it agrees with the intuitive concept of knowledge. Next we will construct a model in which we have knowledge on many functions. From the existence of that model, we will deduce that in any model with a worldly cardinal we have knowledge on a broad class of functions.

As a result we show that if we assume a worldly cardinal exists, then the statement “a given definable language which is provably in $NP \cap co-NP$ is also in $P$ “ is provable.

Assuming a worldly cardinal, we show by a simple use of these theorems that one can factor numbers in poly-logarithmic time.

This article won’t solve the $P$ vs $NP \cap co-NP$ question, but its main result brings us one step closer to deciding that question.

2 Preliminaries

Before I begin, and since the proofs use a few known theorems and basic definitions, I shall quote the theorems which I’ll use later.

2.1 Definitions and notations

- Unless specifically mentioned otherwise $\mathbb{N}$ will denote the set of natural numbers $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$ which includes zero.
  - As, strictly speaking, $\mathbb{N}$ can’t be defined I will not use the notation $\mathbb{N}$ except for intuitions.
For the following definitions we will assume we have one universal Turing machine (i.e a coding method of Turing machines) by which we measure the length of other Turing machines

- The Kolmogorov Complexity of a natural number $x$ is denoted by $\text{Kol}(x)$.

**Definition (text to integer coding):** Given a finite set of letters $\Sigma$ and string of text $T \in \Sigma^*$ one can code $T$ into a number in the following way:

1. fix a numbering on the set of letters
   - for example if $\Sigma = \{a, b, c, \ldots\}$ set $\#a = 1, \#b = 2, \ldots$
2. replace every letter in $T$ with its numbering, and get a number of $|T|$ digits in base $|\Sigma|$
   - for example if $\Sigma = \{a, b, c, \ldots, z\}$ and $T = "cbac"$ then the coding is
     
     \[
     \#c + \#b \cdot 26 + \#a \cdot 26^2 + \#c \cdot 26^3 = \\
     2 + 1 \cdot 26 + 0 \cdot 26^2 + 2 \cdot 26^3 = 17,604
     \]

Such a coding is called **text to integer coding**.

**Observation:** text to integer coding can be performed using only the arithmetic operations addition, multiplication, exponentiation along with the numbering on the set of letters (in the above example $\#a = 0, \#b = 1, \ldots$)

**Definition:** The language of set theory is a single two place relation along with the symbol of equality, that is $\{\in, =\}$

**Assumption:** We will fix a numbering on the alphabet of first order logic along with the language of set theory i.e a numbering for the alphabet

\[
\{"("", ")", "\neg", "\forall", "\exists", "\land", "\lor", "\in", ",", ") = "\}
\]

**Notation:** A Turing machine will refer to a RAM computational machine.

**Definition:** A **structure** for the language of set theory is a set $A$ along with a two place relation $\in \subset A^2$.

I.e. a set of elements $A$ along with a subset $R \subset A^2$ which we will denote by $a \in b \iff aRb$
Definition (ZF): the set of axioms on the language of set theory called ZF (Zermelo–Fraenkel axiomatic set theory) is the following set:

1. Axiom of extensionality \( \forall x \forall y ((\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)) \)
2. Axiom of regularity \( \forall x (\exists a (a \in x) \rightarrow \exists y (y \in x \land \neg \exists z (z \in y \land z \in x))) \)
3. Axiom schema of specification \( \forall z \forall w_1 \forall w_2 \ldots \forall w_n \exists y \forall x (x \in y \leftrightarrow (x \in z \land \phi (w_1, \ldots, w_n, x))) \)
4. Axiom of pairing \( \forall x \forall y \exists z (x \in z \land y \in z) \)
5. Axiom of union \( \forall F \exists A \forall Y \forall x ((x \in Y \land Y \in F) \rightarrow x \in A) \)
6. Axiom schema of replacement
\[ \forall A \forall w_1 \forall w_2 \ldots \forall w_n [\forall x (x \in A \rightarrow \exists ! y \phi (w_1, \ldots, w_n, x, y)) \rightarrow \exists B \forall x (x \in A \rightarrow \exists y (y \in B \land \phi (w_1, \ldots, w_n, x, y)))] \]
7. Axiom of infinity \( \exists X (\emptyset \in X \land \forall y (y \in X \rightarrow S (y) \in X)) \)
8. Axiom of power set \( \forall x \exists y \forall z (z \subseteq x \rightarrow x \in y) \)

- Definition: The axiom of choice (AC) is the following statement:
\[ \forall X [\emptyset \notin X \Rightarrow \exists f : X \rightarrow \bigcup X \ \forall a \in X (f (a) \in a)] \]
- Definition: the set of axioms ZF with AC is called ZFC axioms set.
- Definition: the set of axioms ZF without regularity is called ZF−

Unless mentioned otherwise - a language will refer only to a definable language.

Definition: A model of ZF/ZFC/ ZF− etc is a structure for the language of set theory which satisfies the appropriate axioms of ZF/ZFC/ ZF− respectively.

Definition and notation of \( \omega \): Within a model \( M \) of ZF the standard way to construct the “natural numbers” within the model is to define 0 to be \( \emptyset \) and \( S (x) = x \cup \{x\} \) (successor operation). So 1 = \( \{\emptyset\} \) and 2 = \( \{\{\emptyset\}, \emptyset\} \) and so on. The minimal set which contains \( \emptyset \) and is closed under the operation \( S \) is called \( \omega \) and the its existence is guaranteed by the axiom of infinity. However, and even though we conceive \( \omega \) as \( \mathbb{N} \), they may be very different objects in some models. If in a certain model, the set \( \omega \) differs from our regular notion of natural numbers, we can consider such a model as having “non-standard” arithmetic.

- The notation \( \omega \) will be a more suitable notation than the imprecise notation \( \mathbb{N} \).

Notation:
- Given a model \( M \) of ZF denote \( \omega (M) \) to be the \( \omega \) of the model \( M \).
- Given an \( n \in \omega (M) \) denote
\[ [n] \triangleq \{i \in \omega (M) | i \leq n\} = \{0, 1, 2, \ldots, n\} \]
Definitions of consistencies:

- for a given $j \in \omega$, $j - \text{con}(ZF)$ is defined by the following statement:
  Given a model $M$ of ZF, one can find a sequence $M_1, M_2, M_3, ..., M_j$ within $M$ such that $M_1$ is a model of ZF and a set in $M$ and for every $i < j$ $M_{i+1}$ is a set of $M_i$ and a model of ZF and $\epsilon_{i+1}$ (as a subset of $M_{i+1} \times M_{i+1}$) is a set of $M_i$.

- the statement $\omega - \text{con}(ZF)$ is the following statement:
  Given a model $M$ of ZF for all $j \in \omega (M)$, one can find a sequence of models of ZF $M_1, M_2, M_3, ..., M_j$ within $M$ such that $M_1$ is a set in $M$ and for every $i < j$ $M_{i+1}$ is a set of $M_i$ and $\epsilon_{i+1}$ (as a subset of $M_{i+1} \times M_{i+1}$) is a set of $M_i$.

- the statement $(\omega + 1) - \text{con}(ZF)$ is the following statement:
  Given a model $M$ of ZF, one can find a model $M_0$ of ZF within $M$ s.t $M_0$ satisfies $\omega - \text{con}(ZF)$.

- The same definitions $j - \text{con}(ZFC)$ apply for $j \in \omega$ or $j = \omega$ or $j = \omega + 1$.

Observation:

- For a model $M$ of ZF and for $j \in \omega (M)$ one can define the formula $j - \text{con}(ZFC)$ within the model $M$ arithmetic. This applies to the case where $M$ has “non-standard” arithmetic and $j$ is a non-standard number as well. The same holds for $j - \text{con}(T)$ and for effective theories $T$.

Definition (Von Neumann universe): Within a model $V, \in$ of ZFC define:

- $V_0 = \emptyset$
- $V_{\alpha+1} = P(V_\alpha)$ where $P(X)$ is the power set of $X$
- $V_\alpha = \cup_{\beta < \alpha} V_\beta$ for $a = \cup_{\beta < \alpha} \epsilon_\beta$ limit ordinal.

Definition (transitive set): Within a model $V, \in V$ of ZF a set $A$ is called transitive if for every $x, y$ sets of $V$ if $x \in V A$ and $y \in V x$ then $y \in V A$.

Definition (set model): Given a ZF model $V_1, \in V_1$ another model $V_2, \in V_2$ of ZF is said to be inside $V_1$ (or a set of $V_1$) if the following hold:

- $V_2$ is a set within $V_1$
- all sets of $V_2$ are also sets in $V_1$.
- $\epsilon_{V_2}$ as a set of 2-topuls (i.e as a subset of $(V_2)^2$) is a set of $V_1$

Lemma: Let $V_2$ and $V_1$ be ZF models if $V_2$ is a set of $V_1$, $a$ is a set of $V_2$, $b \in V_2 a$, then $b$ is a set of $V_2$ and $b$ is a set of $V_1$. 

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Proof: As $\in_{V_2}$ is a subset of $(V_2)^2$, $b \in_{V_2}$ a means $b$ is a set of $V_2$. As $V_2$ is inside $V_1$, this means $b$ is also a set of $V_1$. ■

2.1.1 Worldly cardinal

Definition: A cardinal $k$ is called worldly if $(V_k, \in)$ is a ZFC model.

2.1.2 Mostowski’s Collapsing Theorem

Theorem:

1. If $E$ is a well-founded and extensional relation on a class $P$, then there is a transitive class $M$ and an isomorphism $\pi$ between $(P, E)$ and $(M, \in)$. The transitive class $M$ and the isomorphism $\pi$ are unique.

2. In particular, every extensional class $P$ is isomorphic to a transitive class $M$. The transitive class $M$ and the isomorphism $\pi$ are unique.

3. In case (2), if $T \subset P$ is transitive, then $\pi x = x$ for every $x \in T$.

Proof reference: Please refer to [I] Chapter 6 “The Axiom of Regularity” (pg. 69) theorem 6.15.

2.1.3 Lowenheim Skolem theorem

Theorem: Every infinite model for a countable language has a countable elementary sub model.

Proof reference: Please refer to [I] Chapter 12 “Models of Set Theory” (pg. 157) theorem 12.1.

2.2 Forcing

Definition: A partial order set (POS) is a triple $(P, \leq_P, 0_P)$ s.t $P$ is a set, $\leq_P$ is a partial order on $P$ and $0_P$ is a minimal element in $P$.

Let $(P, \leq_P, 0_P)$ be a POS. Define:

- a set $D$ is dense with respect to $P$ if
  $$\forall p \in P \exists q \in D \; q \leq_P p$$

- $G \subset P$ is called a filter on $P$ if it satisfies the following three conditions:
  - $0_P \in G$
  - $\forall p, q \in P \; (q \leq_P p \land p \in G) \rightarrow q \in G$
  - $\forall p, q \in G \; \exists r \in G \; r \geq_P p \land r \geq_P q$
• For a collection of sets $M$ (which may be a model of $ZF$) and a filter $G$ on $P$, we say that $G$ generic over $M$ if for every dense set $D \in M$ we have $G \cap D \neq \emptyset$.

• Given $E \subset P$ and $p \in P$, we say that $E$ is dense above $p$ if
  $\forall q \geq p \exists r \geq q \ (r \in E)$

• $\tau$ is a P name if $\tau$ is a relation and
  $\forall (\sigma, p) \in \tau \ [\tau$ is a P name $\land p \in P]$

• For $M$ a model of $ZF$ the P names in $M$ are
  $M^P = \{ \tau \in M \mid \tau$ is a P name in $M \}$

• For $M$ a model of $ZF$ and $G$ a filter on $P$, the valuation of a name is
  $Val(\tau, G) = \tau_G = \{ Val(\sigma, G) \mid \exists p \in G \ (\sigma, p) \in \tau \}$

• For $M$ a model of $ZF$ and $G$ a filter on $P$ define
  $M[G] = \{ \tau_G \mid \tau \in M^P \}$

Remark:
• $P$ names and valuations are both defined recursively.

Notation:
• for $p, r \in P$ denote $p \perp r$ if $\neg \exists q \in P \ (q \leq p \land r \leq p)$

Theorems:
1. Let $P$ be a POS and $M$ a countable collection of sets and let $p \in P$. Then there exists a generic filter $G$ over $M$ s.t $p \in G$.

2. If $M$ is a transitive model of $ZFC$ and $G$ is a generic filter over $M$ and $P \in M$ is a POS s.t
  $\forall p \in P \exists q, r \in P \ (p \leq q \land p \leq r \land q \perp r)$
  then $G \not\in M$.

3. Given a countable transitive model $M$ of $ZFC$ and $P \in M$ and a a generic filter $G$ over $M$ and $p \in G$ and $E \in M$. If $E \subset P$ is such that $E$ is dense above $p$, then $G \cap E \neq \emptyset$.

4. If $M$ is a transitive model of $ZFC$ and $P \in M$ and $G$ is a generic filter over $M$, then $G \in M[G]$.

5. If $M$ is a countable transitive model of $ZFC$ and $P \in M$ and $G$ is a generic filter over $M$, then $M[G]$ is a countable transitive model of $ZFC$. 

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Proof references: Please refer to Chapter 7 “Forcing”:

1. pg. 186 Lemma 2.3.
2. pg. 187 Lemma 2.4.
3. pg. 192 Lemma 2.19 (ii).
4. pg. 190 Lemma 2.13
5. pg. 201 Theorem 4.2 for the fact that \( M[G] \) holds ZFC. \( M[G] \) is countable (as a countable union of countable sets) and transitive by it’s construction.

2.3 Basic Theorems

2.3.1 Chaitin’s incompleteness theorem (1971)

Theorem: Let \( V \) be a model of \( ZF \). Let \( T \) be an effective consistent set of axioms. Then, there exist \( L \in \omega(V) \) (which depends on the set of axioms) such that for every \( x \in \omega(V) \) the statement \( L \leq Kol(x) \) can’t be proven from the set \( T \).

For completeness of the article I’ll add proof to this theorem:

The reader may want to consider the case of \( \omega(V) = \mathbb{N} \) at first.

Proof: Within \( V \), any proof of claim \( \phi \) from \( T \) is a number within \( \omega(V) \). Denote \( w \in \omega(V) \) to be the first proof of a claim in \( \{ L \leq Kol(x) \mid x \in \omega(V) \} \) and let \( x' \in \omega(V) \) be the number s.t \( T \vdash L \leq Kol(x') \) (the proof \( w \) proves that \( L \leq Kol(x') \)). Create a Turing machine \( M \) that goes over \( y \in \omega(V) \) (in the regular order) and checks if \( y \) is a proof of the statement \( L \leq Kol(x) \) and halts if it is and prints \( x \). The size of the TM \( M \) (as TM could be coded as natural numbers too) is \( \log(L) + C \) for a fixed \( C \in \omega(V) \). As, the decimal representation of \( L \) is \( \log(L) \) and \( C \) is the extra size required to represent the theory \( T \) and the operation of \( M \).

The TM \( M \) is a TM that prints \( x' \) and its size is \( \log(L) + C \) and as \( L \leq Kol(x') \) it follows that \( L \leq \log(L) + C \) (as we’ve just shown a TM of size \( \log(L) + C \)). the inequality \( L \leq \log(L) + C \) can’t hold for a sufficiently large \( L \).

Notation: given an effective consistent set of axioms \( T \) define \( L(T) \) to be the minimal number s.t \( L > \log(L) + C \).

Remarks:

• Notice that \( L(T) \leq 2C \).
• The notation \( L(T) \) doesn’t mention \( V \) because the value doesn’t depend on \( V \) to a large extent (and it will later be stated and proved).
• here we abuse the notation somewhat, as $T$ is not a set of axioms, but also a coding of a TM that identifies said set axioms and as such the value of $\mathcal{L}(T)$ as defined above depends on the representation of the machine.

2.3.2 Compactness theorem

**Theorem:** Assume $T$ is a set of axioms such that every finite subset $A \subset T$, $A$ is consistent, then $T$ is consistent.

**Proof reference:** Please refer to [2] Corollary 3.8 in chapter 3 pg. 27.

2.3.3 Binary tree construction

**Theorem:** Let $M$ be a model of ZF let $\tilde{z} \in \omega(M)$ be a fixed number, $a \in \omega(M)\{\tilde{z}\}$ is any sequence of numbers of length $\tilde{z}$ within the model.

Then exist $t \in \omega(M)$, which encodes a Turing machine that given $k \leq \tilde{z}$ in binary representation calculates $a_k$ in $[\log_2(k+1) + \log_2(a_k) + 1]$ computational steps.

**Intuition:** The key idea here is to build a binary tree with the values of $a_k$ and create a Turing machine which fetch $a_n$ using that binary tree. Then the access time is the size of the representation of the number $k$ which is $[\log_2(k+1) + 1]$ and along with the time to write the result is $[\log_2(k+1) + \log_2(a_k) + 1]$.

We will assume w.o.l.g that traveling through one of the edges of the tree takes one computation step

**Proof:** Let $z \in \omega(M)$ be such that $\tilde{z} \leq 2^z$. Create the following data structure:

• a binary tree where every node has a value (in $\omega(M)$)

• The depth of the tree is $z$.
  – the value of $a_0$ will be held in a special place in memory.

• The root will hold the value of $a_1$ (a number in $\omega(M)$)

• by induction, if a node $k$ held the value of $a_n$ then:
  – The left son of $k$ will hold the value of $a_{2n}$
  – The right son of $k$ will hold the value of $a_{2n+1}$

The Turing machine will go over the data structure from the root based on the binary representation of $k$ (right to left), for every 1 it would go to the right son and for 0 it would go to the left son.

For example, if we want to fetch $a_{13}$ as $13 = 1101_2$ then for the root we would go right then right then left and finally right to get to $a_{13}$. We will
assume w.o.l.g that traveling through one of the edges of the tree takes one computation step (because, for example, such an operation is implemented in the hardware) and thus this process takes the size of the representation of $k$ and the time to write the data $a_k$ which is at most $\lceil \log_2 (k + 1) + \log_2 (a_k) + 1 \rceil$ steps.

Please note that using this construction as $\widetilde{z}$ gets larger and larger so does $t$.

3 Extension and Lattices - first part.

On this part I’ll describe the principle of numbers extensions and show that given a definable function ,which satisfies certain conditions, one can build a sequence of arithmetic models in which a sequence of bounded Turing machines exist that calculates this function in logarithmic time to an increasingly larger numbers (i.e a model of $ZFC$ exist such that it’s $\omega$ has a Turing machine that calculates in logarithmic time...)

3.0.1 Definable function

Definition: Assume $\phi (n_1, n_2)$ is a two variable formula in $ZF$ language and let $V$ be a model of $ZF$. We say that a set $A^2 \subset \omega (V)^2$ in model $V$ is definable using formula $\phi$ if these conditions hold

$$A = \{ (x, y) \in \omega (V)^2 | \phi (x, y) \}$$

Examples:

1. The set of numbers $n, k$ s.t $n$ is divisible by $k$ is definable using the formula $\phi (a, b) = (\exists c \in \omega) (b \cdot c = a)$.

2. Given a lexicographic coding of 3-sat formulas. Denote the formula $s_3sat (n, k)$ “the formula $\#n$ is a 3-sat formula that can be satisfy using the assignment $\#k$”

3. Given a coding of Turing machines, the set of Turing machines which halts by the $k$ step is definable using the formula $\phi (n, k) "n$ is a TM that halts by the $k$ step”.

Definition: Within a ZF model $V$. A two variable formula $\phi (n, k)$ defines a function in $V$ if

$$V \models (\forall x) (\exists ! y \in \omega) \phi (x, y)$$

Definition: A two variable formula $\phi (n, k)$ defines a function in $ZF$ if

$$ZF \vdash (\forall x) (\exists ! y \in \omega) \phi (x, y)$$
3.0.2 Definition of model extension

Definition: Given two models of \( ZF \), \( V_1, V_2 \) we say that \( V_2 \) number extends \( V_1 \) using mapping \( f : \omega (V_1) \to \omega (V_2) \) if the arithmetic operations of \( V_1, V_2 \) i.e.
\[ +_1, +_2, \cdot_1, \cdot_2 \land_1, \land_2 \]
accordingly are respected by \( f \). Meaning:

1. \( \forall a,b \in \omega (V_1) \) \( f (a +_1 b) = f (a) +_2 f (b) \)
2. \( \forall a,b \in \omega (V_1) \) \( f (a \cdot_1 b) = f (a) \cdot_2 f (b) \)
3. \( \forall a,b \in \omega (V_1) \) \( f (a^b) = f (a)^{f(b)} \)
4. \( f (0_{V_1}) = 0_{V_2} \) and \( f (1_{V_1}) = 1_{V_2} \)

I’ll denote it by \( V_1 \xrightarrow{f} V_2 \).

Remark:

- Please note that there is no assumption that inductive arguments on elements of \( \omega (V_1) \) are “transferred” (in some way or another) into \( \omega (V_2) \cap f (\omega (V_1)) \).
- The function \( f \) itself isn’t assumed to be a set of either \( V_1 \) or \( V_2 \).

3.0.3 Definition of model initial segment extension

Definition: If \( V_1 \xrightarrow{f} V_2 \), and the extension satisfies
\[ \forall b \in \omega (V_1) \forall a \in \omega (V_2) \exists c \in \omega (V_1) (a < f (b) \rightarrow a = f (c)) \]
the extension is said to be initial segment extension (i.s extension) and will be denoted by \( V_1 \xleftarrow{f} V_2 \).

Please note:

1. The intuitive meaning of i.s. extension is that the model \( V_2 \) has “more numbers” than \( V_1 \) but the numbers \( V_1 \) “behave the same” on \( V_2 \) “lower parts”.
2. The function \( f \) is usually undefinable from the models themselves.
3. The fact that \( V_1 \xrightarrow{f} V_2 \) for two models doesn’t imply that the two models are elementary equivalent. Nor does it mean that they agree on \( \omega \) attributed formulas.
3.0.4 Definition of the propriety condition

Definition: Given a two variable formula $\phi(n,k)$ (in the language of set theory) we say that the set defined by $\phi$ satisfies the propriety condition if for every two models $V_1, V_2$ of ZF such that $V_2$ is a set of $V_1$ (and thus $V_1 \xrightarrow{f} V_2$) it holds true that

$$(\forall n, k \in \omega ) (V_1 \models \phi(n,k) \iff V_2 \models \phi(f(n), f(k)))$$

Please note:

- The meaning of the propriety condition is that one can determine if a certain $a, b$ is a member of the set only by looking at arithmetic operation on it an initial segment of the model.
  - One may think of the propriety condition of a function as a “uniformly recursive” function

3.1 Basic properties of extension

Lemma: Given two models of ZF, $V_1, V_2$ s.t

- $V_2$ is a set within $V_1$
- $\in_2$ is a set (of pairs) within $V_1$

then a mapping $f \in V_1, f : \omega(V_1) \to \omega(V_2)$ exist s.t $V_1 \xrightarrow{f} V_2$.

Proof: The construction of $f$ is by induction:

- The base case: define $f(0_{V_1}) = 0_{V_2}$.
- For $a = b + 1$ where $a, b \in V_1 \omega(V_1)$ define $f(a) = f(b + 1) = f(b) +_{V_2} 1$
- As every natural number is either 0 or $b + 1$ for another natural number $b$, $f$ is defined.

Recall that the definitions of $+, \times, \land$ are inductive as well. For $a, c \in V_1 \omega(V_1)$

- If $c = 0_{V_1}$ then by definition:
  - $a + c = a + 0 = a$
  - $a \cdot c = a \cdot 0 = 0$
  - $a^c = a^0 = 1$ and if $a \neq 0$ then $c^a = 0^a = 0$
- Therefore if $c = 0$, $f$ holds the equalities in 3.0.2
- $f(a + 1) = f(a) = f(a + 2)\ 0_{V_2}$.
- $f(a\cdot 1) = f(a\cdot 0_{V_2}) = f(0_{V_2}) = f(a\cdot 2)\ 0_{V_2} = f(a\cdot 2)\ f(c)$
- $f(a^c) = f(a^b) = f(1_{V_1}) = 1_{V_2} = f(a)^0_{V_2} = f(a)^f(c)$
- If $a \neq 0$ then $f(a^c) = f(0^a) = f(0_{V_1}) = 0_{V_2} = f(c)^f(a)$

- For $c \neq 0$ exist $b \in V_1\ \omega(V_1)$ s.t $c = b + 1$ and thus by definition:
  - $a + c = a + (b + 1) = (a + 1) + b$
  - $a \cdot c = a \cdot (b + 1) = (a \cdot b) + a$
  - $a^c = a^{b+1} = a \cdot (a^b)$

- Therefore if $c \neq 0$, $f$ holds the equalities in $\mathbb{N}$.

So far we’ve seen that $V_1 \xrightarrow{f} V_2$.
This $f$ also holds the extension property as well:
As $V_2$ is a set of $V_1$, in $V_1$ one can define the following set

$$S = \{ b \in \omega(V_1) \ | \ \exists a \in \omega(V_2) \forall c \in \omega(V_1) ((a < f(b)) \land (a \neq f(c))) \}$$

if $S$ isn’t empty and as $S$ is a set of natural numbers in $\omega(V_1)$ it has a minimum.
Denote

$$s = \min S$$

notice that $s \neq 0$ as $f(0_{V_2}) = 0_{V_2}$ and $\neg\exists a \in \omega(V_2) \ (a < 0_{V_2} = f(0_{V_2}))$. So $s > 0$ and as such $s = m + 1$ for some $m \in V_1\ \omega(V_1)$. Therefore $f(s) = f(m) + 2_{1_{V_2}}$.
As $s$ was minimal it holds that $m \notin V_1\ S$. and thus for $m$

$$\forall a \in \omega(V_2) \exists c \in \omega(V_1) \ (a < f(m) \rightarrow a = f(c)) \tag{1}$$

as $s \in S$ we know

$$\exists a \in \omega(V_2) \forall c \in \omega(V_1) \ ((a < f(s)) \land (a \neq f(c)))$$

let $\exists a \in \omega(V_2)$ be constant and receive

$$\forall c \in \omega(V_1) \ ((a < f(s)) \land (a \neq f(c))) \tag{2}$$

if $a < f(m)$ then by (1) we know that $\exists c \in \omega(V_1)$ that violates (2).
Otherwise if $a = f(m)$ condition (2) is violated with $c = m$. Lastly if $a > f(m)$ it holds that $a \geq f(m) + 1_{V_2} = f(s)$ which is a contradiction to $a < f(s)$ in (2). As we’ve received a contradiction it must be the case that $S$ above is empty. Thus

$$\forall b \in \omega(V_1) \forall a \in \omega(V_2) \exists c \in \omega(V_1) \ (a < f(b) \rightarrow a = f(c))$$
Remark: Please note that the condition $V_2$ is a set of $V_1$ gives us $V_1 \xrightarrow{f} V_2$. But in fact it's a much much stronger assertion, as in this case $f$ is also in $V_1$.

Specifically, one could use this fact to define induction on $[a]$ for $a \in \omega(V_2) \cap \text{Im}(f)$ and pull back the argument by $f^{-1}$ to an induction on $\omega(V_1)$. And, use this fact to use inductive arguments on $\text{Im}(f)$ as seen above.

3.1.1 Overspill principle

Theorem: Let $V, \in V$ be a $ZF$ model and $M, \in M$ and another ZF let $a_1, \ldots, a_n \in M$ and $\phi$ be a formula set theory language. s.t

- $M$ is a set model of $V$
- $\omega(M) \neq \omega(V)$
- $V \xrightarrow{f} M$
- for every $x \in \omega(V)$ it holds $M \models \phi(f(x), a_1, \ldots, a_n)$

then exactly one of the following holds:

- $\forall n \in \omega(M)$ it holds $M \models \phi(n, a_1, \ldots, a_n)$
- exist $k \in \omega(M)$ which is non standard w.r.t $V$ s.t $\forall n < k$ it holds $M \models \phi(n, a_1, \ldots, a_n)$

Proof: Within $M$ look at the set $A = \{n \in \omega(M) \mid \neg \phi(n, a_1, \ldots, a_n)\}$. If the set $A$ is empty then $\forall n \in \omega(M)$ it holds $M \models \phi(n, a_1, \ldots, a_n)$ and the theorem holds true with the first condition. Otherwise $A$ isn't empty and as a subset of natural numbers, it has a minimum. Let $k = \min A$ as for every $x \in \omega(V)$ it holds $M \models \phi(f(x), a_1, \ldots, a_n)$ it follows that $k$ must be non standard w.r.t $V$. As $k$ was the minimum of $A$ it holds $\forall n < k$ it holds $M \models \phi(n, a_1, \ldots, a_n)$ and the theorem holds true with the second condition. ■

3.1.2 Non-standard number definition

Definition: For a set model $M$ of $V$ we know by 3.1 that $f : \omega(V) \to \omega(M)$ exist s.t $V_1 \xrightarrow{f} V_2$. $j \in \omega(M)$ is called non-standard w.r.t $V$ if $j \not\in f(\omega(V))$.

Definition: For a set model $M$ of $V$ we know by 3.1 that $f : \omega(V) \to \omega(M)$ exist s.t $V_1 \xrightarrow{f} V_2$. A model $M$ of ZFC in $V$ has a standard $\omega$ w.r.t $V$ if $f(\omega(V)) = \omega(M)$. Or, in other words, $M$ has no non-standard numbers.
3.1.3 Absoluteness of Turing machines in sub-models

**Theorem:** Let $V_1, V_2$ be two models of $ZF$ s.t $V_2$ is a set of $V_1$ $V_1 \xrightarrow{f} V_2$ and let $T_1 \in \omega(V_1)$ represent a coding of a Turing machine. Let $z \in \omega(V_1)$ be any number.

Denote $R(T_1, z) \in \omega(V_1)$ to be the coded state of the machine $T_1$ after $z$ steps (in $V_1$) and $R(f(T_1), f(z))$ be the coded state of the machine $f(T_1)$ after $f(z)$ steps (in $V_1$).

Then

$$f(R(T_1, z)) = R(f(T_1), f(z))$$

**Proof (sketch):** Denote $J$ to be the operation of running the machine in a specific status one more step, i.e., the function that

$$J(R(T, z)) = R(T, z + 1)$$

for any TM $T \in \omega(V_1)$ and any number $z \in \omega(V_1)$. The function $J$ can be expressed using arithmetic operations. The proof is done by induction on $\omega(V_1)$:

The base case is $R(T_1, 0_{V_1})$ as $T_1$ is a machine that is coded by a text to integer coding, it holds that $f(T_1)$ is a text to integer coding of the same machine in $\omega(V_2)$. Thus we get

$$f(R(T_1, 0_{V_1})) = R(f(T_1), f(0_{V_1})) = R(f(T_1), 0_{V_2})$$

for $z + 1$ we recall that $J$ is an arithmetic function and thus $f(J(x)) = J(f(x))$ for any $x \in \omega(V_1)$ and thus

$$f(R(T, z + 1)) = f(J(R(T_1, z)))$$

$$= f(J(R(T_1, z))) = J(f(R(T_1, z)))$$

$$= J(R(f(T_1), f(z))) = R(f(T_1), f(z) + 1)$$

$$= R(f(T_1), f(z + 1))$$

Now for the induction, denote the set

$$S = \{z \in \omega(V_1) \mid f(R(T_1, z)) \neq R(f(T_1), f(z))\}$$

the set $S$ is a subset of natural numbers definable in $V_1$ (as $V_2$ is a set of $V_1$). If $S$ isn’t empty, it must have a minimum let $z'$ be that minimum. $z' \neq 0_{V_1}$ as we’ve shown that

$$f(R(T_1, 0_{V_1})) = R(f(T_1), 0_{V_2})$$

if $z' = z'' + 1$ then $z'' \notin S$ and thus

$$f(R(T_1, z'')) = R(f(T_1), f(z''))$$

and thus we get that $z' = z'' + 1 \notin S$ as

$$f(R(T_1, z'' + 1)) = R(f(T_1), f(z'' + 1))$$
Therefore, $S$ above must be empty and thus

$$f(R(T_1, z)) = R(f(T_1), f(z))$$

for every $z \in \omega(V_1)$.\[\Box\]

**Remark:**

1. Please note that $V_2$ may may more numbers which are not in $Im(f)$. In this case, it may be that for a TM $T_1$, $T_1$ will not halt in $V_1$ but will halt in $V_2$. How every for any step in $Im(f)$ the running in $V_1$ and $V_2$ will agree.

2. The “sketch” part of this proof is the fact that the coding of $R$ and the $J$ operation wasn’t fully defined. As I trust the reader is familiar with such constructions, I don’t see added value in elaborating.

3. Please note the importance of the assumption: $V_2$ is a set of $V_1$. The fact $V_1 \xrightarrow{f} V_2$ alone isn’t enough for this proof (as we need to use induction on $\omega(V_1)$). However, the above theorem still holds true in the case of $V_1 \xrightarrow{f} V_2$ but as it won’t be used it isn’t shown.

### 3.1.4 Absoluteness of $\mathcal{L}(T)$

**Theorem:** Let $V_1, V_2$ be two models of ZF s.t $V_2$ is a set of $V_1$, $V_1 \xrightarrow{f} V_2$ and let $T_1 \in \omega(V_1)$ be a coding of a TM the recognizes $T$, an effective consistent set of axioms (we assume here that $T$ is consistent according to both $\omega(V_1)$ and $\omega(V_2)$). Let $\mathcal{L}_1(T_1), \mathcal{L}_2(f(T_1))$ be $\mathcal{L}(T)$ computed within $V_1, V_2$ respectively. Then $f(\mathcal{L}_1(T_1)) = \mathcal{L}_2(f(T_1))$.

**Intuition:** Recall the definition of $\mathcal{L}(T)$ given in 2.3.1. $\mathcal{L}(T)$ to be the minimal number s.t $L > \log(L) + C$. Therefore, as long as $C$ is interpreted the same in both models, $\mathcal{L}(T)$ will also be the same.

Moreover, $C$ contains:

1. Representation of a TM which computes $T$ (which is assumed to be the same).

2. A representation of the machine $M$ which goes over the proofs from $T$ and finds the first proof of the claim in $\{L \leq Kol(x) | x \in \omega(V)\}$ (for a given fixed $L$).

As both of these are absolute in sub-models (as seen in 3.1.3) $C$ must be interpreted the same.
**Proof (sketch):** The reader may want to think of the case $\omega(V_1) = N$ at first. The idea of this proof is that all of 2.3.1 construction could be done in a bounded set of $\omega(V_1)$ and $\omega(V_2)$ behaves “the same” within on such subsets (lower parts). Here are some details:

Recall the proof of 2.3.1 the proof creates a Turing machine $M$ which goes over all elements of $\omega(V_1)$ until it finds the first element $w$ which proves a claim in the set $\{L \leq Kol(x) \mid x \in \omega(V)\}$.

- Proofs are a list of statements in first order logic, each statement can be either an axiom or derived from the previous statements.
- Such a proof can be coded into an integer be text to integer coding. Such a coding require only the operations of addition, multiplication, exponentiation.
- The proof could be verified by a Turing machine, which runs the same (by 3.1.3) in both $V_1$ and $V_2$. Therefore the proofs interprets the same in both $V_1$ and $V_2$.
- i.e if $T \models \phi$ in $V_1$ then $f(T) \models f(\phi)$ in $V_2$, where $T, p, \phi$ are coding of a theory, proof and a statement respectively.
- Let $w \in \omega(V_1)$ be the first proof of a statement in $\{L \leq Kol(x) \mid x \in \omega(V_1)\}$ in $V_1$. The proof $f(w)$ is also a proof of a statement in $\{L \leq Kol(x) \mid x \in \omega(V_2)\}$ in $V_2$ and due to the one to one correspondence of $f$, $f(w)$ must be the first such proof in $\omega(V_2)$ as well.
- Therefore the operation of machine $M$ will work “the same” and return $x$ in $V_1$ and $f(x)$ in $V_2$.
- for $L_1 \in \omega(V_1), L_2 \in \omega(V_2)$ we’ve created two Turing machines $M_1$ in $V_1$ and $f(M_1)$ in $V_2$ that computes $x'$, $f(x')$ in $V_1, V_2$ respectively. Therefore in both $V_1, V_2$ the two inequalities must hold:
  \[
  L_1 \leq \log(L_1) + C \\
  L_2 \leq \log(L_2) + f(C)
  \]
  and this the minimum number that violates them must be “the same” i.e
  \[
  f(L_1(T_1)) = L_2(f(T_1))
  \]

### 3.1.5 Reduction of machine number

**Theorem:** Let $T$ be a effective consistent theory containing the axioms of ZF ($T$ may contain some more consistencies axioms as well). Let $C$ be the size of the universal TM.

Assume
• $M$ is a model of $T \cup con - (T)$

• $\phi(n,k)$ is a two variable formula that defines a function in ZF and holds the propriety condition

• $t \in \omega(M)$

• $x \in \omega(M)$ codes a TM which computes the mapping $n \to k$ s.t $\phi(n,k)$ holds for every $n < t$ in $[\log(n+1) + \log(k+1) + 1]$ computational steps.

• let $\bar{o} \in \omega(M)$ s.t $\bar{o} \in \mathcal{L}(T) + C$.

Then, a model $N$ of $T$ exist s.t:

• $N$ is a set of $M$

• $M \xrightarrow{f} N$ for a function $f : \omega(M) \to \omega(N)$

• $\exists y, c \in \omega(N)$ s.t $y \leq f(\bar{o}Z)$ and $y$ codes a TM which computes for every $n < f(t)$ the value of $k$ s.t $\phi(n,k)$ in $[\log(n+1) + \log(k+1) + c]$ steps.

Proof: Let $M, \phi, t, x, \bar{o}Z$ be as in the theorem. If $x < \bar{o}Z$ then $M = N$ and $y = x$ and $c = 1$ holds the conclusions of the theorem. Otherwise assume $x \geq \bar{o}Z$. As $\bar{o}Z > \mathcal{L}(T)$ and $x \geq \bar{o}Z$ we get $x > \mathcal{L}(T)$. As $M$ is a model of $T \cup con - (T)$ and by Chaitin’s incompleteness theorem\[2.3.1\] (on $M$) we know that the statement “$Kol(x) > \mathcal{L}(T)$” can’t be proven from $T$. For that reason the set of axioms $T \cup \{"Kol(x) \leq \mathcal{L}(T)"\}$ is a consistent set of axioms (within $M$) thus $M$ has a model of $T \cup \{"Kol(x) \leq \mathcal{L}(T)"\}$. Let $N$ be this model . As $N$ is a set model of $M$ by \[3.1\] we know $M \xrightarrow{f} N$. As $N$ holds ”$Kol(x) \leq \mathcal{L}(T)$” In $\omega(N)$ exist $y' \in_N \omega(N)$ s.t $y' \leq f(\bar{o}Z)$ and $y'$ codes a TM that calculates $x$. The TM $y$ receives $n < f(t)$ first calculates $x$ using $y'$ and then execute $x$ on $y$. The size of $y$ is at most $y'$ and the size of a universal TM on it’s output and hence $y < f(\bar{o}Z)$. The running time of $y$ is the same as $x$ up to a constant hence for every $n < f(t)$ $y$ calculates the value of $k$ s.t $\phi(n,k)$ in $[\log(n+1) + \log(k+1) + c]$ steps for some constant $c$. ■

4 What is knowledge?

In this section we break the sequence of the construction in order to discuss the implications of theorem \[3.1.5\] and how it leads to a definition of knowledge\[4\]

\[1\] The reader who wishes to skip this section may jump to section \[5\] which is a sequel to section \[4\].
Let’s look at theorems 3.1.5 and 2.3.3. Together they state that given an arbitrary sequence of natural numbers of arbitrary length, one can construct a model in which the sequence is computable in linear time using a machine of bounded size. It is of crucial importance to emphasize that the size of the machine bound is independent of the length and the numbers of the chosen sequence. It is philosophically unacceptable that by pure coincidence it just “happens” that we can compute the chosen sequence using a machine of a small size for every choice on the sequence. The construction of theorem 2.3.3 alone gave us a machine which represented a table of values and depended on the length and the numbers on the sequence. Therefore there is no cognitive dissonance when we conceive it as having full knowledge on the sequence. This way of looking at things is incompatible with 3.1.5: it is inconceivable that the machine number still holds full knowledge on the sequence while the size of the machine is bounded. We must conclude that the knowledge on the sequence got transferred to the structure of $\omega$ of the new model. So we must conclude that the new model in 3.1.5 has “learned” the knowledge hidden in the sequence.

The following question suggests itself naturally: can we make a construction similar to the one in theorem 3.1.5 for an infinite sequence? Let’s assume that we’ve successfully done this. An infinite sequence is a function from $\omega$ to itself. Let’s further assume that this function is definable (see 3.0.1) as it must have an interpretation in different models. Let’s also assume that the definition satisfies the propriety condition (see 3.0.4) because the interpretation of the function in our construction is the correct interpretation of the function in the new model.

So, given a two variable formula $\phi(n, k)$ in the language of set theory which defines a function in $ZF$ and which satisfies the propriety condition and given an $x \in \omega$, we take the sequence $(a_n)_{n=0}^x$ s.t

$$\forall n \ 0 \leq n \leq x \rightarrow \phi(n, a_n)$$

and use 3.1.5 to create a model $M_x$ in which the function is “known” up to $x$. Next we must “tie” or “combine” all these models together in order to create an all encompassing model $\bigoplus_x M_x$. This object isn’t defined yet, but we wish the model to be such as to have a constant $\tilde{Z}$ which bounds all TMs that compute an element in one of the sequences. We also want every $n$ in $\omega$ to be contained in at least one sequence. Formally we define:

$$\exists \tilde{Z} \in \omega \ \forall n \in \omega \ \exists x, k \in \omega \ \left( \phi(n, k) \land Run(x, n) = k \land x < \tilde{Z} \right)$$  \hspace{1cm} (3)

where $Run(x, n)$ denotes the function that returns the value returned by the TM numbered $x$ on input $n$.

Note that the critical demand of running time is omitted from definition 3. Recall that the TMs in 3.1.5 worked using linear time computation on “legal” inputs (i.e halted and gave the right answer). We won’t make any demands
regarding running time (or even halting) on other inputs. So, a coding \( x \) of a TM is of the right running time if

\[
(\exists c \in \omega)(\forall n', k' \in \omega)(\phi(n', k') \land \text{Run}(x, n') = k' \rightarrow (\text{Time}(x, n') = \lceil \log (n' + 1) + \log (k' + 1) + c \rceil))
\]

Therefore, the full definition of “knowledge on the function \( \phi \)” will be

\[
(\exists Z \in \omega)(\forall n \in \omega)(\exists x, k, c \in \omega)(\phi(n, k) \land \text{Run}(x, n) = k \land x < Z \land (\forall n', k' \in \omega)(\phi(n', k') \land \text{Run}(x, n') = k' \rightarrow (\text{Time}(x, n') = \lceil \log (n' + 1) + \log (k' + 1) + c \rceil))
\]

where \( \text{Run}(x, n) \) denotes the function that returns the value returned by the TM numbered \( x \) on input \( n \) and \( \text{Time}(x, n) \) returns the number of steps done in the calculation of \( x \) on input \( n \). As this definition is inspired by an extrapolation of 3.1.5 we expect that definition (4) will be held by at least some models of ZFC.

Please note:

• The definition of knowledge is a computation definition which is different from the more common definition of “computational decision”.
  
  – every function with a bounded image is clearly known, even these function which aren’t computable \ aren’t computable in linear time \ aren’t computable in efficient time (in whichever definition of efficient we may use).

• Unlike in the traditional definition, the running time comes “baked in” this definition and must be always linear.
  
  – In traditional definition, the larger the running time bound the more languages one can decide using such a running time. We don’t expect the same to be the case in knowledge as we expect linear running time to be enough.

• Unlike in the traditional definition, the running time must be equal to linear and not just linearly bounded.
  
  – As the running time was dictated from the running time of 2.3.3 with an addition of a uniform constant in 3.1.5. A running time of equal or less than linear will introduce other TMs that work in a different fashions (which isn’t our intent).

Given such \( \phi \) the question of whether or not \( \phi \) is known in a model (or even if there is a model where \( \phi \) is known) is a \textit{percolation} conjecture. In 7 we will show that in a model with a worldly cardinal \( \phi \) is known for any \( \phi \) two variable formula \( \phi(n, k) \) (in the language if set theory) which defines a function in \( ZF \) and which holds the propriety condition.
5 Extension and Lattices. Second part.

5.1 Base model definitions

Now we will start to define the models in question. We start with $W$, a model of $ZFC$ with a worldly cardinal. Within $W$ exist a countable transitive model of $ZFC + (\omega - \text{con}(ZFC))$ called $V$. Within $V$ exist a countable (non-transitive) model $M_1$ s.t $j \in \omega(M_1)$ exist where $j$ isn’t standard and $M_1$ is a model of $ZFC + j - \text{con}(ZFC)$. These will be our base models and base on them we will use forcing in the next section.

Notation: Let $W$ be a model of $ZFC$ with a worldly cardinal.

Theorem: $W$ holds $\omega - \text{con}(ZFC)$

Proof: Denote for $k \in ON(W)$ denote $V^W_k$ to be the Von-neumann universe of $W$ and let $k' \in ON(W)$ be worldly (i.e $V^W_k, \in_W$ is a ZFC model). We will prove $\forall t \in \omega(W) \ W \models t - \text{con}(ZFC)$ by induction over $t$.

• Base case $t = 1$. As $V^W_k, \in_W$ is a ZFC model $W$ has a set model of ZFC and as such can’t prove a contradiction from ZFC and thus in $W$, ZFC is consistent and so $W$ holds $1 - \text{con}(ZFC)$.

• Assume that $W$ holds $t - \text{con}(ZFC)$. As the property $t - \text{con}(ZFC)$ can be expressed as a property of natural numbers (the set of axioms can’t prove a contradiction) and as $\omega(W) = \omega(V^W_k)$ we get that $V^W_k$ as a set model also hold $t - \text{con}(ZFC)$. As such $V^W_k$ has a a sequence of ZFC models $M'_1, M'_2, M'_3, ..., M'_t$ within $V^W_k$ s.t $M'_1$ is a set of $V^W_k$ and each model is a set of its previous. Thus $W$ has a sequence $V^W_k, M'_1, M'_2, M'_3, ..., M'_t$ of ZFC models s.t $V^W_k$ is a set of $W$ and every model is a set of its previous. Thus $W \models (t + 1) - \text{con}(ZFC)$

As we got that for all $t \in \omega(W) \ W \models t - \text{con}(ZFC)$ we know by definition that $W$ holds $\omega - \text{con}(ZFC)$.■

Corollary: As $W$ holds $\omega - \text{con}(ZFC)$ and as $\omega - \text{con}(ZFC)$ is a property that can be expressed as a property of natural numbers we get from the same argument that $V^W_k$ also holds $\omega - \text{con}(ZFC)$.

5.1.1 $V$ construction

Theorem: $W$ has a countable transitive set model of $ZFC + (\omega - \text{con}(ZFC))$.

Proof: As seen in previously in $\text{[5.1]}$ $ZFC + (\omega - \text{con}(ZFC))$ axioms are consistent in $W$ and have a model $V^W_k$ of them. As the language of set theory is countable by Lowenheim Skolem theorem $\text{[2.1.3]}$ exists $X$ a countable elementary sub model.
X, as a subset of \( V^W_k \), is also well founded w.r.t \( \in W \). X is countable but might not be transitive. By Mostowski’s collapsing theorem 2.1.2, we know that X can be collapsed to a transitive set V and so \( V, \in W \) is a countable transitive set model of \( ZFC + (\omega - \text{con}(ZFC)) \). ■

**Notation:** Let V be the model a countable transitive set model \( ZFC + (\omega - \text{con}(ZFC)) \) within W.

**Lemma:** It holds that \( \omega(V) = \omega(W) \).

**Proof:** As V is a set model of W we know by 3.1 that \( W \xrightarrow{f} V \) and as \( \emptyset \) and the successor operation interprets the same in W and V we know that \( \omega(W) \subseteq \omega(V) \). As V is transitive, if \( \omega(W) \subseteq \omega(V) \) then in W the set \( \omega(W) \setminus \omega(V) \) must contain an infinite decreasing sequence as for every number \( x \in W \omega(W) \setminus \omega(V) \) the number \( x - k \) for \( k \in \omega(W) \) is also in \( \omega(W) \setminus \omega(V) \). As W is a ZFC model it can’t contain an infinite decreasing sequence and thus \( \omega(V) = \omega(W) \). ■

### 5.1.2 \( M_1 \) construction

**Theorem:** \( V \) has a countable model \( M_1, j \in \omega(M_1) \) non standard w.r.t \( V \) and \( M_1 \) is a model of \( ZFC + (j - \text{con}(ZFC)) \) but not of \( ZFC + ((j + 1) - \text{con}(ZFC)) \)

**Proof:** Extend the language of set theory to include one more variable \( j' \). Build the following set of axioms over the extended language:

1. all axioms of \( ZFC \)
2. for every number \( n \in \omega(V) \) add the following axioms:
   (a) \( j' > n \)
   (b) \( (j' - \text{con}(ZFC)) \)
3. \( \exists n' \in \omega - (n' - \text{con}(ZFC)) \)

As every finite set of these axioms is consistent (with a choice of a large enough \( j' \) from \( \omega(V) \)) we know by the compactness theorem 2.5.2 that the whole set of axioms is consistent. As the axioms set is consistent let \( M'_1 \) be a model. As the extended language is countable, by Lowenheim Skolem theorem 2.1.3 we know that a countable model \( M_1 \) exist in which \( j' \in \omega(M_1) \) exist s.t for every \( n \in \omega(V) \) \( j' > n \) and \( M_1 \) holds \( (j' - \text{con}(ZFC)) \). As the axiom \( \exists n' \in \omega - (n' - \text{con}(ZFC)) \) holds we now that \( M_1 \) doesn’t hold \( \omega - \text{con}(ZFC) \) denote \( j \) to be the maximal \( j' \) s.t \( (j' - \text{con}(ZFC)) \) holds in \( M_1 \). ■

**Notation:** Denote the above model \( M_1 \). Denote the collection of ZFC model within \( M_1 \) to be \( \text{Models}_{M_1}(ZFC) \).
Corollary: As $M_1$ is a countable set of $V$ and as being a model is a property (which some sets in $M_1$ have and some don’t) it following that $Models_{M_1}(ZFC)$ is a countable set in $V$.

5.2 Forcing POS over $V$

Now given $W, V, M_1$ we define the forcing conditions and generic filters over $V$. $W$ will be used as a would model on which we will build our generic filters and $V$ will be the model being extended. $M_1$ will be a prat of the conditions.

Notation: For a ZFC model $V'$ denote the power set of a set $A$ in $V'$ to be $POW_V(A)$.

Definition: For a ZFC model $V'$ a finite ordered set is a function $f$ with domain $[a] = \{i \in \omega(V') \mid i \leq a\}$ where $a \in \omega(V')$.

Definition: Define the following POS within $V$:

$$P_0 = \left\{ (f_1, f_2, f_3) \mid \begin{array}{l}
\text{f}_1 : Models_{M_1}(ZFC) \to Pow_V(\omega(V)) \\
\text{f}_2 : Models_{M_1}(ZFC) \to Pow_V(\omega(V)) \\
\text{f}_3 : Models_{M_1}(ZFC) \to Pow_V(\omega(V)) \\
\text{f}_1, \text{f}_2, \text{f}_3 \text{ are partial functions with a finite (by V) domain.} \\
\text{Dom}(\text{f}_1) = \text{Dom}(\text{f}_2) = \text{Dom}(\text{f}_3).
\end{array} \right\}$$

along with the partial order $f \leq_{P_0} g$ if $\text{Dom}(f_1) \subset \text{Dom}(g_1)$ and $g_1 \upharpoonright \text{Dom}(f) = f_1$ and $g_2 \upharpoonright \text{Dom}(f) = f_2$ and $g_3 \upharpoonright \text{Dom}(f) = f_3$.

Explanation: $P_0$ is a set of partial functions. A function $f$ is a function from a finite subset of $Models_{M_1}(ZFC)$ that returns for each $M$ in its domain:

- a finite (by $V$) set of elements of $M$.
- a finite (by $V$) set of formulas in the language of set theory with free variables corresponding to the above set.
- as, by text to integer coding, every formula can by coded by an integer and thus the function is into $\omega(V)$.

- please note that the formulas must by in $V$.
- please note that there is no consistency requirement of these formulas.
Lemma:

1. Within $\mathcal{W}$ a filter $G_0 \subset P_0$ exist which is a generic filter over $V$

2. $V[G_0]$ is a countable transitive model of $ZFC + (\omega - \text{con}(ZFC))$.

Proof: By application of theorem 1 in 2.2 we get the filter $G_0$ and by applying theorem 4 we get that $V[G_0]$ is a countable transitive model of $ZFC$. As $V[G_0]$ is transitive $\omega(V[G_0]) = \omega(V)$ and as $(\omega - \text{con}(ZFC))$ is a set of formulas that can be expressed as natural numbers and as $V$ holds $ZFC + (\omega - \text{con}(ZFC))$ we get that $V[G_0]$ is a countable transitive model of $ZFC + (\omega - \text{con}(ZFC))$.

\[\blacksquare\]

Notation remark: For a model $A$, $\in_A$ in $M_1$ we will use the notation $G_0(A)$ to symbolize the value of $G_0$ as a function on $A$. As $G_0$ is a collection of function (and not just a single function) it may be unclear. However, as every two function in $G_0$ that have $A$ in their domain must agree on their assigned value on $A$ we can view $\cup G_0$ as one big function that gives the value $G_0(A)$.

Remark: Given a model $A$ in $M_1$, $G_0(A)$ is defined. As the set $E$ of partial function in $P_0$ which are defined on $A$ is a dense set in $V$ and as such $G_0$ as a generic filter must intersect it. As such, $G_0(A)$ is defined.

5.2.1 Formula reduction definition

Definition: Let $M$ be a model of $ZFC$ which is a set of $V$. For a set of formulas $A$ in $V$ (which may have free variables in them) and $k \in \omega(V)$

- we say that $A$ is $k$ consistent in $M$ from $ZFC$ if , $M$’s arithmetic holds

\[ k - \text{con}(ZFC \cup A) \]

In other words, $M$ has a sequence of $k$ models, each model is a set of its previous and all hold $ZFC \cup A$.

Please note that unlike $\{n - \text{con}(ZFC) \mid n \in \omega(V)\}$ the set $ZFC \cup n - \text{con}(ZFC)$ for $n \in \omega(V)$ can be defined in $M$.

Definition: Let a model $M$ of $ZFC \cup \{n - \text{con}(ZFC) \mid n \in \omega(V)\}$, let $a, b, c \in \omega(M)$ and let a set of constant $\{e_i\}_{i \in [c]}$ and two function $f_1 : [a] \rightarrow \omega(M)$ and $f_2 : [b] \rightarrow \omega(M)$ which represent a text to integer coding of formulas in $M$ with the constants in $\{e_i\}_{i \in [c]}$. Assume that the formulas in $f_1$ are $k$ consistent in $M$ from $ZFC$ for every $k \in \omega(V)$ (i.e

\[ k - \text{con}(ZFC \cup f([a])) \]

in $M$ for every $k \in \omega(V)$). Define the reduction function by recursion over $b$ by the following recursive algorithm:
1. If \( f_2 \) isn’t an empty function, look at the formula \( f_2(0) \):

   (a) If
   \[
   k - \text{con}(\text{ZFC} \cup f([a]) \cup f_2(0))
   \]
   in \( M \) for every \( k \in \omega(V) \) define \( f_{1}^{\text{new}} = f_{1}^{\text{old}} \cup (\text{Dom}(f_1), f_2(0)) \) and for \( 0 \leq n \leq b - 1 \) define \( f_{2}^{\text{new}}(n) = f_{2}^{\text{old}}(n + 1) \). Go to back to step (1) with \((f_{1}^{\text{new}}, f_{2}^{\text{new}})\).

   (b) Otherwise, define \( f_{1}^{\text{new}} = f_{1}^{\text{old}} \cup (\text{Dom}(f_1), \neg f_2(0)) \) and for \( 0 \leq n \leq b - 1 \) define \( f_{2}^{\text{new}}(n) = f_{2}^{\text{old}}(n + 1) \). Go to back to step (1) with \((f_{1}^{\text{new}}, f_{2}^{\text{new}})\).

2. If \( f_2 \) is the empty function define \( \text{Reduce}_M(f_1, f_2) = f_1 \).

**Explanation:** The process of \( \text{Reduce} \) takes two lists of axioms. Where the first list \( f_1 \) is assumed to be consistent. It adds axioms from \( f_2 \), one at a time by their order. As axioms can be added as long as it doesn’t create a contradiction. If the axiom does create a contradiction the negation of the axiom is added and the process goes on to the next axiom in the list of \( f_2 \). By the end of this process (as the lists are finite) we get a consistent list of axioms called \( \text{Reduce}_M(f_1, f_2) \) and every axiom in \( f_2 \) is either listed in \( \text{Reduce}_M(f_1, f_2) \) or its negation is listed there.

### 5.3 Lattice construction

In this part, within \( V[G_0] \), we will now construct the set of models \( \{M_i\}_{i \in \omega(V)} \) on which we will later define a limit model. Please recall that \( M_1 \) is already constructed along with \( j \in \omega(M) \) which is non standard w.r.t \( V \) and \( M_1 \) holds \( j - \text{con}(\text{ZFC}) \). As \( M_1 \) is a countable model (and therefore countable set), fix a numbering (in \( V \) of \( M_1 \).

#### 5.3.1 \( M_i \)'s construction

The definition of \( M_i \) will be by induction. For \( i \in \omega(V) \) s.t \( i \geq 1 \) we assume we’ve defined \( M_i, \text{var}_i, \text{Formulas}_i \) s.t

- \( M_i \) is a \( \text{ZFC} \) model.
- \( M_i \) is a set model of \( V \).
- \( M_i \) holds \( \text{ZFC} \cup \{n - \text{con}(\text{ZFC}) \mid n \in \omega(V)\} \).
- \( M_i \) holds \( \text{ZFC} \cup \text{Formulas}_i \cup \{n - \text{con}(\text{ZFC} \cup \text{Formulas}_i) \mid n \in \omega(V)\} \).
- for \( i > 1 \) \( M_i \) is an element \( M_1 \)
- \( \text{var}_i \) is a finite (by \( V \)) set of variables of set in \( M_i \).
  - For the case \( i = 1 \) define \( \text{var}_i \) to be the empty set.
– As \( \text{var}_i \) is a finite set (by \( V \)) of elements in \( M_i \), \( \text{var}_i \) is also a set of \( M_i \).

- \( \text{Formulas}_i \) is a finite (by \( V \)) set of formulas (in \( V \)) with variables in \( \text{var}_i \).
  - For the case \( i = 1 \) define \( \text{Formulas}_i \) to be an empty set of formulas.
  - As \( \text{Formulas}_i \) is a finite set (by \( V \)) of elements in \( M_i \) and each formula is a number in \( \omega (V) \), \( \text{Formulas}_i \) is also a set of \( M_i \).

From the generic filter \( G_0 \) we receive:

- \((G_0 (M_i))_1\) is a finite (by \( V \)) set of variables of set in \( M_i \).
- \((G_0 (M_i))_2\) is a finite set of formulas (in \( V \)) with variables in \((G_0 (M_i))_1\).
  Each formula is a formula in \( V \).
- \((G_0 (M_i))_3\) is a number in \( \omega (M_i) \).

Definitions: Given \( i + 1 \) define the following:

Denote

- \( \text{var}_{i+1} = \text{var}_i \cup (G_0 (M_i))_1 \).
- \( \text{Formulas}_{i+1} = \text{Reduce}_{M_i} (\text{Reduce}_{M_i} (\emptyset, \text{Formulas}_i), (G_0 (M_i))_2) \).
  Where \( \text{Reduce}_{M_i} \) is the reduction function defined in 5.2.1.

Axioms of \( M_{i+1} \): We define the extended language to be the language of set theory along with constants for each variable in \( \text{var}_{i+1} \). The following list of axioms is

We demand the following axioms:

1. \( ZFC \)

2. All formulas in \( \text{Formulas}_{i+1} \) must hold (as formulas with the appropriate constant in \( \text{var}_{i+1} \)). and

3. for every \( j \in \omega (V) \)

\[ j \cong (ZFC \cup \text{Formulas}_{i+1}) \]

must hold.

(a) Denote \( k \) to be the maximal number s.t

\[ k \cong (ZFC \cup \text{Formulas}_{i+1}) \]

is consistent in \( M_i \). Such \( k \) must be non standard w.r.t \( V \).
Remark:
1. As $M_i$ is consistent with
   \[ ZFC \cup \text{Formulas}_i \cup \left\{ n - \text{con} (ZFC \cup \text{Formulas}_i) \mid n \in \omega(V) \right\} \]
   and as $\text{Formulas}_{i+1}$ was constructed to be consistent with
   \[ ZFC \cup \text{Formulas}_{i+1} \cup \left\{ n - \text{con} (ZFC \cup \text{Formulas}_{i+1}) \mid n \in \omega(V) \right\} \]
   in $M_i$ we know that if $k$ is the maximal number s.t
   \[ k - \text{con} (ZFC \cup \text{Formulas}_{i+1}) \]
   is consistent in $M_i$, such a $k$ must be non standard w.r.t $V$ by the overspill principle [3.1.3] and such $k$ must exist as $M_1$ didn’t hold $\omega - \text{con} (ZFC)$ (and consequently all $M_i$ won’t hold $\omega - \text{con} (ZFC)$).

2. The demand every $j \in \omega(V)$
   \[ j - \text{con} (ZFC \cup \text{Formulas}_{i+1}) \]
   must hold can’t be stated inside $M_i$ (as $M_i$ doesn’t have access to $\omega(V)$) but given that $k$ of axioms 3 is non standard w.r.t $V$ (which it is by our construction) we can demand
   \[ k - \text{con} (ZFC \cup \text{Formulas}_{i+1}) \]
   and the demand that for every $j \in \omega(V)$
   \[ j - \text{con} (ZFC \cup \text{Formulas}_{i+1}) \]
   follows that statement.

Lemma: The axioms 1+2+3 above can be demanded in $M_i$ and are consistent in it.

Proof: Let $k$ be the maximal number s.t
   \[ k - \text{con} (ZFC \cup \text{Formulas}_{i+1}) \]
is consistent in $M_i$. The model $M_i$ is assumed to be consistent with
   \[ j - \text{con} (ZFC \cup \text{Formulas}_i) \]
for every $j \in \omega(V)$. $\text{Formulas}_{i+1}$ is a $j$ consistent set of axioms for every $j \in \omega(V)$ (as $\text{Formulas}_{i+1}$ was chosen such). Thus, for every $j \in \omega(V)$
   \[ j - \text{con} (ZFC \cup \text{Formulas}_{i+1}) \]
is consistent in $M_i$. Such a $k$ therefore must be non-standard w.r.t $V$.

By previous remark in order to show that we can demand axiom 3 in $M_i$ (as $\omega(V)$ can’t be defined within $M_i$) it is suffice to show that
   \[ k - \text{con} (ZFC \cup \text{Formulas}_{i+1}) \]
is consistent in $M_i$ and that $k$ is non standard w.r.t $V$.■
5.3.2 $M_{i+1}$ definition

Definitions:

1. Define the model $M_{i+1}$ to be the minimal model within $M_i$ that holds axioms 1-3 above. Where the minimum is taken using the numbering of elements of $M_1$ in $V$.

2. As every variable in $\text{var}_{i+1}$ has an interpretation in $M_{i+1}$ and as $\text{var}_i$ were elements of $M_i$ and as $\text{var}_i \subset \text{var}_{i+1}$ define the function

$$I_i : \text{var}_i \rightarrow M_{i+1}$$

to be the mapping between $\text{var}_i$ as elements of $M_i$ and the corresponding elements in $M_{i+1}$.

3. As $M_{i+1}$ is a set model of $M_i$ by 3.1 we know that $f_i : \omega(M_i) \rightarrow \omega(M_{i+1})$ exist s.t $M_i \xrightarrow{f_i} M_{i+1}$. Define $f_i$ to be that function (i.e $f_i$ maps the omega of $M_i$ to the omega of $M_{i+1}$).

5.3.3 Knowledge in $M_i$

Let $\phi(n,k)$ be a two variable formula (in $V$) which defines a function in ZFC and holds the propriety condition 3.0.4 in $V[G_0]$.

Lemma: Within $M_i$ denote the axioms 1+2+3 (as interpreted in $M_i$) by the set $T$ and let $C$ be the size of the universal TM. Then

$$10k > \mathcal{L}(T) + C$$

where $k$ is

$$k = \max_{k' \in \omega(M_i)} \{ M_i \vdash k' \text{ - con (ZFC)} \}$$

Proof: Recall the composition of $T$:

- The axioms of ZFC can be coded using a number $C_1 \in \omega(V)$ (i.e $C_1$ is a coding of a TM that identifies the axioms of ZFC).

- The axioms of $\text{Formulas}_{i+2}$ being a finite set in $V$ can be coded using a number $C_2 \in \omega(V)$.

- for a given $k' \in \omega$ the axioms of $k' \text{ - con (ZFC } \cup \text{ Formulas}_{i+1})$ can be coded using a number $C_3 \in \omega(V)$. This is a coding of a TM which takes two inputs $n', k'$ and returns true if $n$ is an axioms of $k' \text{ - con (ZFC } \cup \text{ Formulas}_{i+1})$

- The number $k$ is a natural number in $\omega(M_i)$. 

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In order to identify axioms of $T$ one needs the coding of $C_1, C_2, C_3$ above and $k$ defined in axioms 2. As all $C_1, C_2, C_3$ were standard numbers w.r.t $V$ and $k$ was non standard numbers w.r.t $V$ we get that

$$C_1 < k \land C_2 < k \land C_3 < k$$

and thus the coding of $T$ is smaller than

$$C_1 + C_2 + C_3 + k \leq k + k + k + 4k = 4k$$

As $L(T)$ is smaller than $2|T|$ (recall [2.3.1] first remark) we know that

$$L(T) < 8k$$

and as $C$ the size of the universal TM is also a standard number w.r.t $V$ we get that $C < k$ and as such

$$L(T) + C < 9k < 10k$$

which is the assertion of the lemma. ■

**Theorem:** $M_i$ hold this axiom: For every $j \in \omega(V)$ it holds

$$j - \text{con} \left( ZFC \cup \text{Formulas}_{i+1} \cup \phi \right)$$

where $\phi$ denote the axiom $\exists y, c \in \omega \text{ s.t } y \leq 10^k$ and $y$ codes a TM which computes for every $n < (G_0(M_i))_3$ the value of $k' \in \omega \text{ s.t } \phi(n, k')$ in $\lceil \log(n + 1) + \log(k' + 1) + c \rceil$ steps and $k$ is

$$k = \max_{k' \in \omega(M_i)} \{ M_i \models k' - \text{con}(ZFC) \}.$$  

**Remark:** Please note that the statement $\phi$ in the lemma isn’t a formula in $V$ (as it has the value of $k$ and of $(G_0(M_{i+1}))_3$ in it, and both aren’t a standard number w.r.t $V$) and as such it and its negation may not appear in $\text{Formulas}_{i'}$ for any $i' \in \omega(V)$. However, we still may ask the question of "$\phi$ is $j$ consistent with the previous statements or not?" which the above theorem answers.

**Proof:** First by previous lemma we know that axioms 1-3 are consistent in $M_i$ as $M_i$ held

$$j - \text{con} \left( ZFC \cup \text{Formulas}_{i+1} \right)$$

for every $j \in \omega(V)$. Denote $T$ to be the set of axioms 1-3 in $M_i$. Let $t = (G_0(M_i))_3 \in \omega(M_i)$, by the binary tree construction [2.3.3] we know that exist $x \in \omega(M_i)$ codes a TM which computes the mapping $n \rightarrow k''$ s.t $\phi(n, k'')$ holds for every $n < t$ in $\lceil \log(n + 1) + \log(k'' + 1) + 1 \rceil$ computational steps. Let $k$ be the number above.

Let $\tilde{Z} = 10^k$ and let $C$ be the size of the universal TM, then we know by previous lemma that $\tilde{Z} > L(T) + C$. By reduction of machine number [2.4.5]
we know that a model $N$ in $M'$ of axioms 1-3 exists that holds $\varphi$ of the axiom. Thus axioms 1-3 and $\varphi$ have a model in $M'$ and therefore are consistent. Let $j \in \omega(V)$, as $N$ is a model of

$$j \rightarrow \text{con}(ZFC \cup \text{Formulas}_{i+1})$$

we get that exist in $N$ a sequence of models $N_1, \ldots, N_j$ each is a set of its previous and all holds $ZFC \cup \text{Formulas}_{i+1}$. As $\varphi$ of axiom is absolute (the numbers $y, c \in \omega(N)$ exist also in $N_i$ for $1 \leq i' \leq j$ and holds the same condition due to the absoluteness to TM in sub-model 3.1.1 and the fact that $\phi$ holds the propriety condition 3.0.4). As such, $N$ holds for every $j \in \omega(V)$

$$j \rightarrow \text{con}(ZFC \cup \text{Formulas}_{i+1} \cup \varphi)$$

Thus $M'$ holds

$$j \rightarrow \text{con}(ZFC \cup \text{Formulas}_{i+1} \cup \varphi)$$

with the models $N_1, \ldots, N_j$. As $j \in \omega(V)$ was chosen arbitrarily the statement holds for every $j \in \omega(V)$ and therefore $M_i$ holds the axiom.

5.3.4 Lattice definition

As $M_i$ was constructed for every $i \in \omega(V)$, the set $\{M_i\}_{i \in \omega(V)}$ exists in $V$.

Please note that indeed every $M_i$ separately is a set of $M_1$ (and moreover is a set of every $M_t$ for $t < i$) but as $M_1$ doesn’t have access to $\omega(V)$ the collection $\{M_i\}_{i \in \omega(V)}$ can’t be defined in $M_1$.

Definition: The lattice is the following collection defined in $V[G_0]$:

- The set of models $\{M_i\}_{i \in \omega(V)}$
- The set of variables $\{\text{var}_i\}_{i \in \omega(V)}$ within each model.
- The set of formulas $\{\text{Formulas}_i\}_{i \in \omega(V)}$ of each model.
- The mappings $\{I_i\}_{i \in \omega(V)}$ from variables of one model to the next.
- The mappings $\{f_i\}_{i \in \omega(V)}$ from $\omega$ of one model to the next.

5.3.5 Definition of the lattice within $V$

Theorem: Let $k \in \omega(V)$, the lattice up to $k$ (i.e the set $(M_i, \text{var}_i, \text{Formulas}_i, I_i, f_i)_{1 \leq i \leq k}$) can be defined within $V$.

Proof: Please note that for $i \in \omega(V)$ the construction above of $M_{i+1}, I_i, f_i, \text{var}_{i+1}, \text{Formulas}_{i+1}$ doesn’t depend on $G_0$ entirely. It only depends on the values of $G_0(M_i)$. In other words, in order to define $M_{i+1}$ we don’t need to know all of $G_0$ but we only need to know a partial function on $P_0$ which contains $\{M_k \mid k \leq i\}$ in its domain. As the restriction of $G_0$ to the set $\{M_k \mid k \leq i\}$ gives us a partial function in $P_0$ and as such partial function are assumed to be in $V$ we get that the set $(M_i, \text{var}_i, \text{Formulas}_i, I_i, f_i)_{1 \leq i \leq k}$ can be defined within $V$. ■
5.3.6 independence of the lattice from \((G_0)_3\)

**Observation:** Recall that the demands on \(M_{i+1}\) were axioms 1-3 and recall that axioms 1-3 didn’t use the values of \((G_0)_3\). Therefore the lattice is constructed independently of the values \((G_0)_3\). Or, in other words, if the values in \((G_0)_3\) were to be change the lattice under the above construction would remain the same.

6 Limit of the lattice

In this part we will use the construction of the lattice defined above to create a “limit” model \(M_{\text{Limit}}\) of models \(\{M_i\}_{i \in \omega(V)}\) as a type of straight limit over the sets \(\{\text{var}_i\}_{i \in \omega(V)}\) and the mappings \(\{I_i\}_{i \in \omega(V)}\) in such a way that, the limit model will hold \(ZFC\) and will have good knowledge properties (we will define them in 6.3).

6.1 Limit model definition

**Definition:** Denote the set

\[
Lace_i = \left\{(a_k)_{k \in \omega(V)} \land k \geq 1 \mid \forall k \geq i \ a_{k+1} = I_k(a_k)\right\}
\]

In other words, as every element in \(\text{var}_i\) is mapped into \(\text{var}_{i+1}\) we denote their trajectories by \(Lace_i\).

**Definition:** Denote

\[
Lace = \bigcup_{i \in \omega(V)} Lace_i
\]

**Lemma:** If \(a \in Lace_{i_1}\) and \(b \in Lace_{i_2}\) let \(k \geq \max \{i_1, i_2\}\) s.t \(a_k = b_k\). Then for all \(k' \geq k\) it holds \(a_{k'} = b_{k'}\).

**Proof:** By induction over \(k' \geq k\): The base case is \(a_k = b_k\) and is assumed true. Assume \(a_{k'} = b_{k'}\) for \(k' \geq k\) then \(a_{k'+1} = I_{k'}(a_{k'}) = I_{k'}(b_{k'}) = b_{k'+1}\)

**Definition:** Define the following equivalence relation on elements of \(Lace\): for \(a \in Lace_{i_1}\) and \(b \in Lace_{i_2}\) we denote \(a \sim b\) if exist \(k \geq \max \{i_1, i_2\}\) s.t \(a_k = b_k\).

6.1.1 Limit model set definition on

**Definition:** Define

\[
M_{\text{Limit}} = Lace/ \sim
\]

the equivalence class of \(Lace\) under \(\sim\) above.
6.1.2 Limit model

Definition: For $a \in \text{Lace}_{i_1}$ and $b \in \text{Lace}_{i_2}$ we denote $a \in_{\text{Lace}} b$ if exist $k \geq \max \{i_1, i_2\}$ s.t $\forall k' \geq k$ it holds that $a_{k'} \in_{M_k} b_{k'}$.

Definition: For $a, b \in M_{\text{Limit}}$ define $a \in_{\text{Limit}} b$ if for all $a', b'$ being representatives of the equivalent classes $a, b$ respectively it holds $a' \in_{\text{Lace}} b'$.

Remark: In the following lemma we will prove that $a \in_{\text{Limit}} b$ is independent of the choice of representatives, so this definition could have been define as exist $a', b'$ representatives s.t $a' \in_{\text{Lace}} b'$.

Lemma: Let $a, b \in M_{\text{Limit}}$ then $a \in_{\text{Limit}} b$ if and only if exist $a'', b''$ representatives of the equivalent classes $a, b$ respectively that holds $a'' \in_{\text{Lace}} b''$.

Proof: First assume that exist $a'', b''$ representatives s.t $a'' \in_{\text{Lace}} b''$ and let $a', b'$ be another pair of representatives of $a, b$. Then (as they represent the same equivalent class) $a' \sim a''$ and $b' \sim b''$. By the definition of $\sim$ exist $k_1, k_2 \in \omega (V)$ s.t $\forall k > k_1 \ a'_k = a''_k$ and $\forall k > k_2 \ b'_k = b''_k$. As $a'' \in_{\text{Lace}} b''$ by definition exist $k_3 \in \omega (V)$ s.t $\forall k > k_3 \ a''_k \in_{M_k} b''_k$ denote $k' = \max \{k_1, k_2, k_3\}$ then for $k > k'$ it holds that

$$a'_k \in_{M_k} b''_k = b'_k \Rightarrow a'_k \in_{M_k} b'_k$$

and therefore $a' \in_{\text{Lace}} b'$ and hence as $a', b'$ were any pair of representatives of $a, b$ we get $a \in_{\text{Limit}} b$.

The other direction is trivial: If $a \in_{\text{Limit}} b$ then by definition exist $a', b'$ representatives of $a, b$ and as $a'' \in_{\text{Lace}} b''$ for any two representatives $a'', b''$ we get $a' \in_{\text{Lace}} b'$ for the case of $a', b'$ as well. \square

6.2 Stabilization theorems

6.2.1 Appearance of formulas in $\text{Formulas}_k$

Theorem: Let $\psi(x_1, \ldots, x_n)$ be a formula of the language of set theory (in $V$) with $n \in \omega (V)$ free variables and let $a_1, \ldots, a_n \in \text{Lace}$ be elements of $\text{Lace}$ then exist $k \in \omega (V)$ s.t either the formula $\psi (a_{1,k}, \ldots, a_{n,k})$ or its negation is in $\text{Formulas}_k$. Where $a_{i,k}$ is the element $a_i$ at $\text{var}_k$.

Note: Please note that in the above theorem we didn’t argue that the truth value of $\psi$ is stabilized from a certain point on, this will be proven later. This theorem shows that the formula $\psi$ itself is to be found within $\text{Formulas}_k$.

Proof: Let $k \in \omega (V)$ be large enough s.t all $a_{1,i}, \ldots, a_{n,i}$ are to be found in $\text{var}_k$. Recall that by Proposition 3.5 $k \in \omega (V)$ we can define $(M_i, \text{var}_i, \text{Formulas}_i, I_i, f_i)_{1 \leq i \leq k}$ within $V$. Define the following set $E$ in $V$: $E$ is the set of partial function $f$ in $P_0$ that agreed with $G_0$ on $(M_i)_{1 \leq i \leq k}$ and such that if we continue the construction
of the lattice according to $f$ we will get the formula $\psi$ (or its negation) in some later construction of $\text{Formulas}_{k'}$ according to $f$. As every partial function that extends $G_0 \upharpoonright_{(M_i)_{1 \leq i \leq k}}$ can be extended into a function in $E$ we get that $E$ is a set in $V$ that is dense over $G_0 \upharpoonright_{(M_i)_{1 \leq i \leq k}}$ and thus by 2.2 theorem 3 we get that $E \cap G_0 \neq \emptyset$. That is, exist $k' \in \omega(V)$ s.t $\psi$ (or its negation) appear in $\text{Formulas}_{k'}$ (and now the construction is done according to $G_0$, the usual way).

6.2.2 Stabilization in $\text{Formulas}_k$

Lemma: Let $(x_1, \ldots, x_n)$ be a formula of the language of set theory (in $V$) with $n \in \omega(V)$ free variables and let $a_1, \ldots, a_n \in \text{Lace}$ be elements of Lace then exist $k \in \omega(V)$ s.t exactly one of the following holds:

1. for all $k' \in \omega(V)$ s.t $k' > k$ the formula $\psi(a_{1,k'}, \ldots, a_{n,k'})$ is in $\text{Formulas}_{k'}$.
2. for all $k' \in \omega(V)$ s.t $k' > k$ the formula $\neg \psi(a_{1,k'}, \ldots, a_{n,k'})$ is in $\text{Formulas}_{k'}$.

Terminology remark: If for all $k' \in \omega(V)$ s.t $k' > k$ the formula $\psi(a_{1,k'}, \ldots, a_{n,k})$ is in $\text{Formulas}_{k'}$ we say that the formula $\psi$ had stabilized. This is different than the assertion in 6.2.1 as in that section we only argued that the formula $\psi$ or $\neg \psi$ appears in $\text{Formulas}_{k'}$, but perhaps it may be the case that for even $k$'s $\psi$ is in $\text{Formulas}_{k'}$ and for odd $k$'s $\neg \psi$ is in $\text{Formulas}_{k'}$. Stabilization is the assertion that such phenomena doesn’t happen as either $\psi$ appears in $\text{Formulas}_{k'}$ from one point on or $\neg \psi$ appears in $\text{Formulas}_{k'}$ from one point on.

Proof: By previous theorem 6.2.1 we know that exist $k_0 \in \omega(V)$ s.t either the formula $\psi(a_{1,k_0}, \ldots, a_{n,k_0})$ or it negation is in $\text{Formulas}_{k_0}$. Recall that $\text{Formulas}_k$ is a function (i.e an ordered set of formulas) the proof is by induction over the place number of $\psi$ in the list. Recall the definition of $\text{Formulas}_{k_0+1}$:

$$\text{Formulas}_{k_0+1} = \text{Reduce}_{M_{k_0}} \left( \text{Reduce}_{M_{k_0}} \left( \emptyset, \text{Formulas}_{k_0} \right), \left( G_3 (M_{k_0}) \right)_2 \right)$$

and as such every $\psi$ that appear in $\text{Formulas}_{k_0}$ must appear in $\text{Formulas}_{k_0+1}$ as the formula itself or its negation. By induction we receive that for all $k' > k_0$ the formula $\psi$ or its negation appear in $\text{Formulas}_{k'}$. And moreover, the if the formula or its negation held place $i$ in $\text{Formulas}_{k_0}$, the same formula or its negation will have place $i$ in $\text{Formulas}_{k'}$. Now we will prove the lemma by induction over the place number of $\psi$ in the list.

Base case: The formula $\psi$ appear as first formula in the list:

If for all $k' > k_0$ the formula $\psi$ (and not $\neg \psi$) appears in $\text{Formulas}_{k'}$ then the lemma holds for $k = k_0$ with condition (1). Otherwise exist $k_1 > k_0$ s.t $\neg \psi$ appear in $\text{Formulas}_{k_1}$. By the definition of $\text{Reduce}_{M_{k_0}}$ it means that the formula $\psi$ can be proven false from a finite (by $M_{k_1}$) subset of the axioms of $

\{ s - \text{con}(\text{ZFC}) \mid s \in \omega(V) \}$

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For $k' > k_1$, as the set

$$\{ s - \text{con} (ZFC) \mid s \in \omega (V) \}$$

doesn’t change and as every proof in $M_{k_1}$ is also a proof in $M_{k'}$ it must be the case that $\neg \psi$ appear in $\mathit{Formulas}_{k'}$ as well. In this case, the lemma holds for $k = k_1$ with condition (2).

**Step case:** The formula $\psi$ appear as $t + 1$ formula in the list for $t \in \omega (V)$.

In this case let by the induction’s assumption we know that for every formula up to $t$ exist $k' \geq k_1$ s.t for all $k' > k_1$ the formula stabilizes in $\mathit{Formulas}_{k'}$. Denote $k_0 = \max \{ k_0', \ldots , k_1' \}$. We know that all formulas in $\mathit{Formulas}_{k_0}$ up to $\psi$ had stabilized. If for all $k' > k_0$ the formula $\psi$ (and not $\neg \psi$) appears in $\mathit{Formulas}_{k'}$ then the lemma holds for $k = k_0$ with condition (1). Otherwise exist $k_1 > k_0$ s.t $\neg \psi$ appear in $\mathit{Formulas}_{k_1}$. By the definition of $\text{Reduce}_{M_1}$ it means that the formula $\psi$ can be proven false from a finite (by $M_{k_1}$) subset of the axioms of

$$\{ s - \text{con} (ZFC \cup A) \mid s \in \omega (V) \}$$

where $A$ is the set of formulas that appear before $\psi$ and $t \in \omega (V)$. For $k' > k_1$, as the set

$$\{ s - \text{con} (ZFC \cup A) \mid s \in \omega (V) \}$$

doesn’t change, as all formulas in $A$ stabilized up to $z_0$ and as every proof in $M_{k_1}$ is also a proof in $M_{k'}$ it must be the case that $\neg \psi$ appear in $\mathit{Formulas}_{k'}$ as well. In this case, the lemma holds for $k = k_1$ with condition (2).

**Remark:** As it may be unclear, for a set $B$ of formulas that exist in $M_i$ if $B$ is inconsistent, then a contradiction can be proven from a finite set of statements in $B$. The same proof is valid in any set model of $M_i$ as proofs are verifiable using a TM and due to absoluteness of Turing machines in sub-models $\mathbb{3}$.4.3

### 6.2.3 Stabilization of formulas in $\mathit{Lace}$

**Theorem:** Let $n \in \omega (V)$ and $a_1, \ldots , a_n \in \mathit{Lace}$ s.t $a_k \in \mathit{Lace}_{i_k}$. Let $\psi$ be a formula (in $V$) of the language of set theory with $n$ free variables. then exist $k \in \omega (V)$, $k \geq \max \{ i_1, i_2, \ldots , i_n \}$ s.t exactly one of the following holds:

1. for all $k' \in \omega (V)$ s.t $k' > k$ $M_{k'} \models \psi (a_{1,k}, \ldots , a_{n,k})$. Where $a_{i,k}$ is the element $a_i$ at $\text{var}_k$.

2. for all $k' \in \omega (V)$ s.t $k' > k$ $M_{k'} \models \neg \psi (a_{1,k}, \ldots , a_{n,k})$. Where $a_{i,k}$ is the element $a_i$ at $\text{var}_k$.

**Proof:** By 6.2.2 we know that exist $k_0 \in \omega (V)$ s.t $\phi$ stabilizes from the $k_0$ place onward on $\mathit{Formulas}_k$. Denote $k = k_0 + 1$.

If for $k' \in \omega (V)$ s.t $k' > k_0$ the formula $\psi (a_{1,k'}, \ldots , a_{n,k'})$ is in $\mathit{Formulas}_{k'}$. Then, as $k' \geq k_0 + 1$ we know that formula $\psi (a_{1,k'-1}, \ldots , a_{n,k'-1})$ is in $\mathit{Formulas}_{k'-1}$

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and as such all formulas in \( \text{Formulas}_{k'-1} \) are axioms of \( M_{k'} \) if follows that \( M_{k'} \models \psi(a,_{1,k'},\ldots,a,_{n,k'}) \) and the theorem holds with condition (1).

Otherwise, If for \( k' \in \omega(V) \) s.t \( k' > k_0 \) the formula \( \neg \psi(a,_{1,k'},\ldots,a,_{n,k'}) \) is in \( \text{Formulas}_{k'} \) Then, as \( k' \geq k_0 + 1 \) we know that formula \( \neg \psi(a,_{1,k'-1},\ldots,a,_{n,k'-1}) \) is in \( \text{Formulas}_{k'-1} \) and as such all formulas in \( \text{Formulas}_{k'-1} \) are axioms of \( M_{k'} \) if follows that \( M_{k'} \models \neg \psi(a,_{1,k'},\ldots,a,_{n,k'}) \) and the theorem holds with condition (2). ■

6.2.4 Witnesses of formulas in Lace

**Theorem:** Let \( n \in \omega(V) \) and \( a,_{1},\ldots,a,_{n} \in \text{Lace} \) s.t \( a,_{k} \in \text{Lace}_{i_{k}} \). Let \( \psi \) be a formula (in \( V \)) of the language of set theory with \( n + 1 \) free variables. Assume that exist \( k \in \omega(V), k \geq \max\{i,_{1},i,_{2},\ldots,i,_{n}\} \) s.t for all \( k' \in \omega(V) \) s.t \( k' > k \) \( M_{k'} \models \exists y (y,a,_{1,k'},\ldots,a,_{n,k'}) \). Then, exist \( z \in \text{Lace} \) and \( k'' > k \) s.t for all \( k' \in \omega(V) \) s.t \( k' > k'' \) \( M_{k'} \models \psi(z,k',\ldots,a,_{n,k'}) \).

**Proof:** Let \( k_0 \in \omega(V) \) be a number s.t \( \exists y \psi(y,a,_{1,k'},\ldots,a,_{n,k'}) \) stabilizes and all previous formulas up to it had stabilized as well. Let the set \( E \) be the set of partial functions \( g = (g_1,g_2) \) extending \( G_0 \) on \( \{M_i\}_{i=2}^{k_0} \). Let for every finite sequence of models \( \{M_i\}_{i \in [a]} \) \( (a \in \omega(V)) \) in \( \text{Dom}(g) \) with \( M_0 = M_{k_0} \) if the construction of the lattice was made using \( M_1,\ldots,M_{k_0} = M_0, M'_1,\ldots,M'_a \) then exists \( i \leq a \) s.t if \( M'_i \models \exists y \psi(y,a'_1,i,\ldots,a'_{n,i}) \) with \( a'_1,i,\ldots,a'_{n,i} \) the appropriate variables in \( \text{var} \) of \( M'_i \) then \( y' \) is chosen to be a variable in \( g_1 \) that isn’t in \( \text{Var}_{i-1} \) and the formula \( \psi(y',a'_1,i,\ldots,a'_n,i) \) was chosen as a formula in \( g_2 \).

As \( \text{Var}_{i-1} \) is a finite (by \( V \)) set there is \( y' \) that wasn’t chosen. partial function can be extended to include \( y' \) and the formula \( \psi \), the set \( E \) is dense above the partial function of \( G_3 \) on \( \{M_i\}_{i=2}^{k_0} \) in \( V \). So it must be the case that \( E \cap G_0 \neq \emptyset \).

Therefore, exist \( k_1 > k_0 \) s.t an element \( y' \) exists in \( \text{var}_{k_1} \) but not in \( \text{var}_{k_1-1} \) and the formula \( \psi(y',a'_1,i,\ldots,a'_n,i) \) is in \( \text{Formulas}_{k_1} \). As consistency, \( y' \) didn’t appear in \( \text{var}_{k_1-1} \) so the above formula is equivalent to \( \exists y \psi(y,a'_1,i,\ldots,a'_n,i) \) (as \( y' \) may be mapped to any element, as no other axioms are demanded on it). We know that the formula \( \exists y \psi(y,a'_1,i,\ldots,a'_n,i) \) is consistent with the previous formulas (as it holds true from \( M_{k_1+1} \) and the latter is a set of \( M_{k_1} \)). Let \( k' > k_1 \), as \( M_{k'+1} \models \exists y \psi(y,a,_{1,k'},\ldots,a,_{n,k'}) \) we get that \( \exists y \psi(y,a'_1,i,\ldots,a'_n,i) \) is consistent with the previous formulas and as \( y' \) appear first in \( \psi(y',a_1,i,\ldots,a_n,i) \) we know that it is equivalent to \( \exists y \psi(y,a'_1,i,\ldots,a'_n,i) \) hence \( M_{k'} \models \psi(y',a,_{1,k'},\ldots,a,_{n,k'}) \). Let \( z = y,_{k_1},y,_{k_1+1},y,_{k_1+2},\ldots \) be that element in \( \text{Lace} \) and \( k'' = k_1 \). We get that for all \( k' \in \omega(V) \) s.t \( k' > k'' \) \( M_{k'} \models \psi(z,k',\ldots,a,_{n,k'}) \). ■

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6.2.5 Truth value in $M_{\text{Limit}}$

**Theorem:** Let $n \in \omega(V)$ and $a_1, \ldots, a_n \in M_{\text{Limit}}$. Let $\psi$ be a formula (in $V$) of the language of set theory with $n$ free variables. Then $M_{\text{Limit}} \models \psi(a_1, \ldots, a_n)$ if and only if exist $a_1, \ldots, a_n \in \text{Lace}$ s.t $a_k \in \text{Lace}_{i_k}$ which are representatives of $a_1, \ldots, a_n \in M_{\text{Limit}}$ respectively. And exist $k \in \omega(V)$, $k \geq \text{max}\{i_1, i_2, \ldots, i_n\}$ s.t for all $k' \in \omega(V)$ s.t $k' > k$ $M_{k'} \models \psi(a_{1,k'}, \ldots, a_{n,k'})$. Where $a_{i,k}$ is the element $a_i$ at $\text{var}_k$.

**Remark:** As the truth value doesn’t depend on the representative we could have said “... if and only if for all $a_1, \ldots, a_n \in \text{Lace}$ s.t $a_k \in \text{Lace}_{i_k}$ which are representatives of $a_1, \ldots, a_n \in M_{\text{Limit}}$ respectively...”.

**Proof:** The proof is by induction over the structure of the formula $\psi$ where the formula is in $V$ and the induction is done in $V$ as well:

Base cases:

- For the basic case $\psi(a', b') = "a \in b"$ if for all $k' \in \omega(V)$ s.t $k' > k$ $M_{k'} \models (a' \in a, b')$ then by definition of $\varepsilon_{\text{Limit}}$ (see 6.1.2) it holds $M_{\text{Limit}} \models a \in b$. On the other hand, if $M_{\text{Limit}} \models a \in \varepsilon_{\text{Limit}} b$ then by definition exist $k \geq \text{max}\{i_1, i_2\}$ s.t $\forall k' \geq k$ it holds that $a_k \in a, b$.

- For the basic case of $\psi(a', b') = "a = b"$ if exist $k' \in \omega(V)$ s.t $M_{k'} \models (a_k = b_k)$ the mapping $I_i$ will map both $a_k, b_k$ to the same element in $M_{k+1}$ and therefore both laces $a', b'$ are equivalent under $\sim$ and as such represent the same element in $M_{\text{Limit}}$ (i.e. the are in the same equivalence class) and as such $M_{\text{Limit}} \models (a = b)$. On the other hand, if $M_{\text{Limit}} \models (a = b)$ then by definition of equality $a, b$ represent the same element in $M_{\text{Limit}}$ choose $a', b' \in \text{Lace}$ which represent that (same) element and as such $M_{k'} \models (a_k = b_k)$ for $k'$ large enough and on.

Composite case:

- For the case of $\psi(a_1, \ldots, a_n) = \psi_1(a_1, \ldots, a_n) \land \psi_2(a_1, \ldots, a_n)$, as exist $k \in \omega(V)$, $k \geq \text{max}\{i_1, i_2, \ldots, i_n\}$ s.t for all $k' \in \omega(V)$ s.t $k' > k$ $M_{k'} \models \psi(a_{1,k'}, \ldots, a_{n,k'})$ it follows that $M_{k'} \models \psi_1(a_{1,k'}, \ldots, a_{n,k'})$ and $M_{k'} \models \psi_2(a_{1,k'}, \ldots, a_{n,k'})$. As $\psi_1, \psi_2$ were formulas of lower depth we get from the induction’s assumption that $M_{\text{Limit}} \models \psi_1(a_1, \ldots, a_n)$ and $M_{\text{Limit}} \models \psi_2(a_1, \ldots, a_n)$ and therefore $M_{\text{Limit}} \models \psi(a_1, \ldots, a_n)$. On the other hand, if $M_{\text{Limit}} \models \psi(a_1, \ldots, a_n)$ then we know that $M_{\text{Limit}} \models \psi_1(a_1, \ldots, a_n)$ and $M_{\text{Limit}} \models \psi_2(a_1, \ldots, a_n)$ and therefore exists $k_1, k_2$ s.t for all $k' \in \omega(V)$ s.t $k' > \text{max}\{k_1, k_2\}$ it holds $M_{k'} \models \psi_1(a_{1,k'}, \ldots, a_{n,k'})$ and $M_{k'} \models \psi_2(a_{1,k'}, \ldots, a_{n,k'})$ it follows that $M_{k'} \models \psi(a_{1,k'}, \ldots, a_{n,k'})$ as needed.

- The case $\psi(a_1, \ldots, a_n) = \psi_1(a_1, \ldots, a_n) \lor \psi_2(a_1, \ldots, a_n)$ is analogues to the previous case.
For the case \( \psi(a_1, \ldots, a_n) = \neg \psi_1(a_1, \ldots, a_n) \), as \( \psi_1 \) formula of lower depth we get from the induction’s assumption that exist \( k \in \omega(V) \), \( k \geq \max \{i_1, i_2, \ldots, i_n\} \) s.t. for all \( k' \in \omega(V) \) s.t. \( k' > k \) \( M_{k'} \models \psi_1(a_{1,k'}, \ldots, a_{n,k'}) \) if and only if \( M_{Limit} \models \neg \psi_1(a_1, \ldots, a_n) \). As such, the same assertion is true of the negation of the formula, we get that exist \( k \in \omega(V) \), \( k \geq \max \{i_1, i_2, \ldots, i_n\} \) s.t. for all \( k' \in \omega(V) \) s.t. \( k' > k \) \( M_{k'} \models \neg \psi_1(a_{1,k'}, \ldots, a_{n,k'}) \) if and only if \( M_{Limit} \models \neg \psi_1(a_1, \ldots, a_n) \).

quantifier case:

For the case \( \psi(a_1, \ldots, a_n) = \exists y \psi_1(y, a_1, \ldots, a_n) \): If exist \( k \in \omega(V) \), \( k \geq \max \{i_1, i_2, \ldots, i_n\} \) s.t. for all \( k' \in \omega(V) \) s.t. \( k' > k \) \( M_{k'} \models \exists y \psi_1(y, a_{1,k'}, \ldots, a_{n,k'}) \) it follows by the existence of witnesses in \( \text{Lace}\) (see \( 6.2.4 \)) that exist \( z' \in Lace \) and \( k'' > k \) s.t. for all \( k' \in \omega(V) \) s.t. \( k' > k'' \) \( M_{k''} \models \psi_1(z'_k, a_{1,k'}, \ldots, a_{n,k'}) \). As \( \psi_1 \) is a formula of less depth we get that \( M_{Limit} \models \psi_1(z, a_1, \ldots, a_n) \) for \( z \in M_{Limit} \) being the equivalence class of \( z' \). As such \( z \in M_{Limit} \) s.t. \( M_{Limit} \models \psi_1(z, a_1, \ldots, a_n) \) it holds \( M_{Limit} \models \exists y \psi_1(y, a_1, \ldots, a_n) \). On the other hand, if \( M_{Limit} \models \exists y \psi_1(y, a_1, \ldots, a_n) \) then exist \( z \in M_{Limit} \) s.t. \( M_{Limit} \models \psi_1(z, a_1, \ldots, a_n) \) then as \( \psi_1 \) is a formula of less depth we know that exist \( z' \in Lace \) and exist \( k \in \omega(V) \), \( k \geq \max \{i_1, i_2, \ldots, i_n, i_{n+1}\} \) s.t. for all \( k' \in \omega(V) \) s.t. \( k' > k \) \( M_{k'} \models \psi_1(z'_k, a_{1,k'}, \ldots, a_{n,k'}) \) as \( z'_k \in M_{k'} \) exist s.t. \( \psi_1(z'_k, a_{1,k'}, \ldots, a_{n,k'}) \) we get that \( M_{k'} \models \exists y \psi_1(y, a_{1,k'}, \ldots, a_{n,k'}) \).

For the case \( \psi(a_1, \ldots, a_n) = \forall y \psi_1(y, a_1, \ldots, a_n) \): The case is equivalent to \( \psi(a_1, \ldots, a_n) = \neg \exists y \psi_1(y, a_1, \ldots, a_n) \) which is composed of formulas of the previous cases.

6.2.6 ZFC holds in M_{Limit}

**Theorem:** The set \( M_{Limit} \) along with \( \in_{Limit} \) holds the axioms of ZFC and as such \( (M_{Limit}, \in_{Limit}) \) is a ZFC model.

**Proof:** Let \( \psi \) be a formula (in \( V \)) which is an axioms of ZFC (such a formula must be without free variables). For all \( k' \in \omega(V) \) as \( M_{k'} \) is a ZFC model we know \( M_{k'} \models \psi \). By \( 6.2.3 \) we know that if \( M_{k'} \models \psi \) holds from a certain \( k \) and on then \( M_{Limit} \models \psi \). As such \( M_{Limit} \) holds all axioms of ZFC and as such \( (M_{Limit}, \in_{Limit}) \) is a ZFC model.

6.3 Knowledge in M_{Limit}

6.3.1 \( M_{Limit} \) knows \( \phi \) function

Let \( \phi(n, k) \) be a two variable formula (in \( V \)) which defines a function in ZFC and holds the propriety condition \( 6.0.1 \) in \( V[G_0] \). Recall that \( M_{Limit} \) was a limit of models \( M_k \). We now are ready to show that the model \( M_{Limit} \) holds that exists \( \hat{Z} \in_{Limit} \omega(M_{Limit}) \) and for all elements \( n \in_{Limit} \omega(M_{Limit}) \) exists
x < 2 and c ∈ Limit ω(MLimit) s.t x represents a TM which calculates the value k ∈ Limit ω(MLimit) s.t MLimit ⪰ φ(n, k) in ⌊log(n + 1) + log(k + 1) + c⌋ steps and c depends only on x (and not on n). In other words MLimit will hold

\[\exists Z \in \omega \ \forall n \in \omega \ \exists x, k, c \in \omega \]

\[
\begin{cases}
\phi(n, k) \land \text{Run}(x, n) = k \land x < Z \\
(\forall n', k' \in \omega) (\phi(n', k') \land \text{Run}(x, n') = k' \rightarrow (\text{Time}(x, n') = \lceil \log(n' + 1) + \log(k' + 1) + c \rceil))
\end{cases}
\]

where Run(x, n) denote that function that return the value return by the TM numbered x on input n and Time(x, n) return the number of steps done in the calculation of x on input n.

6.3.2 Consistency number in MLimit

Definition: Define the consistency number

\[cn = (cn_i)_{i \in \omega(V)} \in \Pi_{i \in \omega(V)} \omega(M_i)\]

to be, for a given i ∈ ω(V) the number cn_i ∈ M_i, ω(M_i) is the only number s.t

\[M \models (cn_i) \land \text{con}(ZFC)\]

but not

\[M \not\models (cn_i + 1) \land \text{con}(ZFC)\]

Remark: In other words cn_i is defined as

\[cn_i = \max_{k' \in \omega(M_i)} \{M_i \models k' \land \text{con}(ZFC)\}\]

as every M_i is a set of M_1 and as M_1 doesn’t hold \omega \land \text{con}(ZFC) such cn_i must exist in every M_i

Definition: For k ∈ ω(V) denote cn \[<k to be

\[cn \[<k = (cn_i)_{i \in \omega(V)} \in \Pi_{i \in \omega(V)} \omega(M_i) \quad i > k \]

Lemma: Exist k ∈ ω(V) s.t cn \[<k ∈ Lace_k

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Proof: Denote by $\psi(x)$ the formula, in the language of set theory, that states “$x \in \omega$ and $\neg((x + 1) - \text{con}(ZFC))$”. Define the following set $E$ in $V$:

$E$ is the set of partial functions $f$ in $P_0$ such that if we continue the construction of the lattice according to $f$ we will get $k' \in \omega(V)$ s.t $k' > 1$ and we will get a variable $x$ that didn’t appear in $\text{var}_{k'-1}$ and $\psi(x)$ (or its negation) appear in $\text{Formulas}_{k'}$. As every partial function can be extended into a function in $E$ we get that $E$ is a set in $V$ that is dense and thus by 2.2 theorem 3 we get that $E \cap G_0 \neq \emptyset$. Therefore, a $k' \in \omega(V)$ exist s.t the variable $x$ is a new variable in $\text{Var}_{k'}$ and $\psi$ appears in $\text{Formulas}_{k'}$. Thus, as $\psi(x)$ is consistent with the construction (as for every $M_i$, the model has such $x$) the formula $\psi(x)$ (and not it’s negation) appears in $\text{Formulas}_{k'}$. Let $x_t$ for $t > k'$ denote the trajectory of $x$ under $I_t$. Thus by definition $cn_t = x_t$ for every $t > k'$ and as such $cn_{<k} \in \text{Lace}_k$ for $k = k' + 1$. ■

Corollaries:

1. As $cn$ is in $\text{Lace}$ from a certain $k$ and on, there is an element $cn \in M_{\text{Limit}}$ that is represented by $cn_{<k} \in \text{Lace}_k$ for the appropriate $k \in \omega(V)$.

2. As $cn \in M_{\text{Limit}}$ and as $cn_i \in \omega(M_i)$ ($cn_i$ was a number) so must $cn \in \omega(M_{\text{Limit}})$ be a number.

Notation: As $cn \in \omega(M_{\text{Limit}})$ is a number the notation of $10 \cdot cn$ is the number $cn$ times the number 10.

6.3.3 Formulas definition

Definition Let $\phi(n,k)$ be a two variable formula (in $V$) which defines a function in $ZFC$ and holds the propriety condition 3.0.4 in $V$. Define that following formula:

$$\zeta(Z,n) = \exists x, k, c \in \omega \left( \phi(n,k) \land \text{Run}(x,n) = k \land x < Z \land (\forall n', k' \in \omega)(\phi(n',k') \land \text{Run}(x,n') = k') \rightarrow (\text{Time}(x,n') = \lceil \log(n' + 1) + \log(k' + 1) + c \rceil) \right)$$

Explanation: The formula $\zeta$ states that for given $Z, n$ the number $Z$ is large enough so that the value of the function $\phi$ on the input $n$ is computed by a machine of size smaller than $Z$ and which runs in linear time. Our aim will be to prove that $M_{\text{Limit}} \models (\forall k \in \omega) \zeta(10 \cdot cn, k)$ where $cn \in M_{\text{Limit}}$ was defined in 6.3.2.

6.3.4 Knowledge is consistent in $M_{\text{Limit}}$

Theorem: Let $k' \in \omega(V)$ and let $a \in \text{Lace}_{k'}$. Assume that:
• $M_{Limit} \models a \in \omega$ (where $a$ is used here as also the appropriate equivalence class).

• $cn \upharpoonright_{< k'} \in Lace_{k'}$

• both $cn$ and $a$ are in $\var_k$

• The formula $\zeta (10 \cdot cn, a)$ or its negation is in $\text{Formulas}_{k+1}$

• $M_{k'} \models a_{k'} < (G_0 (M_{k}))_3$

Then the formulas $\zeta (10 \cdot cn, a)$ (and not its negation) is in $\text{Formulas}_{k+1}$

Proof: Recall that

$$cn_{k'} = \max_{t \in \omega (M_{k'})} \{ M_{k'} \models t - \text{con} (ZFC) \}$$

and denote $\varphi$ the axiom “$\exists y, c \in \omega$ s.t $y \leq 10 \cdot cn_{k'}$ and $y$ codes a TM which computes for every $n < (G_0 (M_{k}))_3$ the value of $k'' \in \omega$ s.t $\phi (n, k'')$ in $[\log (n + 1) + \log (k'' + 1) + c]$ steps”. Let $j \in \omega (V)$. Recall that by the theorem of knowledge in $M_{5.3.3}$

$$M_{k'} \models j - \text{con} (ZFC \cup \text{Formulas}_{k'+1} \cup \varphi)$$

and notice that $\zeta (cn_{k'}, a_{k'})$ follows from $\varphi$ (as such $y$ in $\varphi$ will be the $x$ in $\zeta (cn_{k'}, a)$ that give us the coding of the TM). Therefore, as

$$M_{k'} \models j - \text{con} (ZFC \cup \text{Formulas}_{k'+1} \cup \varphi)$$

we know that $M_{k'}$ has a sequence of $j$ models, each a set of its previous, and each holds $\varphi$. As $\zeta (cn_{k'}, a_{k'})$ follows from $\varphi$ it must be the case that these models also hold $\zeta (cn_{k'}, a_{k'})$ and as such

$$M_{k'} \models j - \text{con} (ZFC \cup \text{Formulas}_{k'+1} \cup \zeta (cn_{k'}, a_{k'}))$$

holds. If $\neg \zeta (cn_{k'}, a_{k'})$ was in $\text{Formulas}_{k'+1}$ it would be the case that a formula and its negation are consistent, which is a contradiction to $M_{k'} \models \text{con} (ZFC)$ (which we assumed). Thus, $\zeta (cn_{k'}, a_{k'})$ was in $\text{Formulas}_{k'+1}$.  

6.3.5 $M_{Limit}$ knows $\phi$ on every number

Lemma: Let $k' \in \omega (V)$ and let $a \in Lace_{k'}$ s.t for all $i \in \omega (V)$ where $i > k'$, it holds $M_{k'} \models a_{k'} \in \omega (M_{k'})$.

Then, for infinitely many $i \in \omega (V)$ where $i > k'$, it holds that $M_i \models a_i < (G_0 (M_i))_3$

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Proof: Assume that $k > k'$, we will show the existence of an $i > k$ such that $M_i \models a_i \prec (G_0 (M_i))_3$. Denote $E \subseteq P_0$ to be the set of partial function $f$ in $V$ which agree with $G_0$ on $(M_i)_i$ and such that if we continue the construction according to $f$ we will get $i > k$ such that $M_i \models a_i \prec (f (M_i))_3$.

Please note that, as the construction of $M_i$ depended on $f_1, f_2$ alone and didn’t depend on $f_3$ (see 5.3.4) any partial function $f'$ could be extended into a partial function in $E$. As one could simply change the value of $(f (M_i))_3$, for $i$ not yet defined on, to be larger than $a_i$ (and as $M_i$ or $a_i$ didn’t depend on this value, they remain the same). Thus as every partial function in $P_0$ can be extended to a function in $E$ and thus $E$ is a set in $V$ that is dense over $G_0 | (M_i)_{i \leq k}$ and thus by 2.2 theorem 3 we get that $E \cap G_0 \neq \emptyset$. As such an $i > k$ exists such that $M_i \models a_i \prec (G_0 (M_i))_3$. □

Theorem: Let $a \in M_{\text{Limit}}$ s.t $M_{\text{Limit}} \models a \in \omega$ then $M_{\text{Limit}} \models \zeta (10 \cdot cn, a)$

Proof: Recall that by theorem 6.2.2 exist a $k \in \omega (V)$ s.t either for all $k' \in \omega (V)$ s.t $k' > k$ the formula $\zeta (10 \cdot cn_k, a_k)$ is in Formulas$_k$. Or, for all $k' \in \omega (V)$ s.t $k' > k$ the formula $\neg \zeta (10 \cdot cn_k, a_k)$ is in Formulas$_k$.

By 6.2.1 the formula $\zeta (10 \cdot cn_k, a_k)$ or its negation must appear in Formulas$_k$ from a certain point on.

By the lemma above we know that for infinitely many $i \in \omega (V)$ where $i > k'$, it holds that $M_i \models a_i \prec (G_0 (M_i))_3$ (where $k'$ is large enough to include $cn$ and $a$ variables).

By 6.3.4 we know that when $M_i \models a_i \prec (G_0 (M_i))_3$ it must be the case that $\zeta (10 \cdot cn, a)$ (and not its negation) is in Formulas$_{i+1}$. As the formula $\zeta (10 \cdot cn, a)$ appears infinitely many times in Formulas$_i$ for different $i$ and as such a formula must stabilize we get that $\zeta (10 \cdot cn_k, a_k)$ appears on Formulas$_k$ from certain point on and as such by 6.2.3 we get that $M_{\text{Limit}} \models \zeta (10 \cdot cn, a)$. □

6.3.6 $M_{\text{Limit}}$ knows $\phi$

Theorem: It holds that $M_{\text{Limit}} \models (\forall k \in \omega) \zeta (10 \cdot cn, k)$

Proof: Let $k \in M_{\text{Limit}}$ s.t $M_{\text{Limit}} \models \forall k \in \omega$ by previous theorem in 6.3.4 we know that $M_{\text{Limit}} \models \zeta (10 \cdot cn, k)$ and as such $M_{\text{Limit}} \models (\forall k \in \omega) \zeta (10 \cdot cn, k)$. □

Theorem: It holds that $M_{\text{Limit}} \models \left( \exists \alpha \in \omega \right) (\forall k \in \omega) \zeta (\alpha, \alpha, k)$

Proof: By the previous theorem the assertion hold with $Z = 10 \cdot cn$ and where $cn \in M_{\text{Limit}}$. □
7 Percolation theorems

Notation: Let $W$ be a ZFC model with a worldly cardinal and let $\phi(x,y)$ be a two variable formula (in $W$) which defines a function in ZFC and holds the propriety condition 3.0.4. Denote the formula

\[\text{know}_\phi = \exists Z \in \omega \forall n \in \omega \exists x, k, c \in \omega \ \phi(n, k) \land \text{Run}(x, n) = k \land x < Z \land \left((\forall n', k' \in \omega) \left((\phi(n', k') \land \text{Run}(x, n') = k') \rightarrow (\text{Time}(x, n') = \lceil \log(n' + 1) + \log(k' + 1) + c \rceil)\right)\right)\]

where $\text{Run}(x, n)$ denote that function that return the value return by the TM numbered $x$ on input $n$ and $\text{Time}(x, n)$ return the number of steps done in the calculation of $x$ on input $n$.

7.1 First percolation theorem

Theorem: Let $W$ be a ZFC model with a worldly cardinal and let $\phi(x,y)$ be a two variable formula (in $W$) which defines a function in ZFC and holds the propriety condition 3.0.4. Then $\text{know}_\phi$ is consistent with ZFC.

Proof: Use 5.1 to construct $V, M_1$ then use 5.2 to define $P_0$ and then construct the lattice 5.3. Define $M_{\text{Limit}}$ as in 6.1 and thus $M_{\text{Limit}}$ is a ZFC model which holds $\text{know}_\phi$. Thus there is a model of ZFC that holds $\text{know}_\phi$, the formula $\text{know}_\phi$ is consistent with ZFC.

7.2 Second percolation theorem

Theorem: Let $W$ be a ZFC model with a worldly cardinal and let $\phi(x,y)$ be a two variable formula (in $W$) which defines a function in ZFC and holds the propriety condition 3.0.4. Then

$W \models \text{know}_\phi$

Proof: Assume by negation that

$W \not\models \text{know}_\phi$

Then we can define a new axiom set

$ZFC^{\text{new}} = ZFC \cup \lnot \text{know}_\phi$

as $W$ is a model of $ZFC^{\text{new}}$ with a worldly cardinal we can construct the same construction of 7.1 when using $ZFC^{\text{new}}$ instead of the regular ZFC. Thus we will get a model of $ZFC^{\text{new}}$, $M_{\text{limit}}$ in which $\text{know}_\phi$ is held. But, $M_{\text{limit}}$ also hold $\lnot \text{know}_\phi$ as a model of $ZFC^{\text{new}}$ which is a contradiction. Thus we get

$W \models \text{know}_\phi$
Additional explanation: In light of the above proof. We’d like to offer additional explanation on key points of the construction done under $ZFC_{\text{new}}$ of $M_{\text{Limit}}$:

1. Recall that we start with $W$ a model with a worldly cardinal in both cases in 5.1 and in 7.2.

2. As the formula $\neg\text{know}_\psi$ is $\omega$ attributed it must be held by any model with the same $\omega$. As such both $W$ and $V^W_k$ hold $\omega - \text{con} (ZFC_{\text{new}})$ by the theorem in 6.1. As such $V$ will hold $\omega - \text{con} (ZFC_{\text{new}})$ by the theorem in 5.1. As such $V$ will hold it as an elementary countable submodel.

3. $M_1$ is chosen the same as in 5.1.2 with $ZFC_{\text{new}}$ replacing $ZFC$.

4. The forcing POS is done the same way and by the forcing theorem $V[G_0]$ is a $ZFC$ model. But as $\omega(V) = \omega(V[G_0])$ the model $V[G_0]$ also hold $\omega - \text{con} (ZFC_{\text{new}})$.

5. The rest of the construction of the lattice is done the same with $ZFC_{\text{new}}$ replacing $ZFC$. And the construction of $M_{\text{Limit}}$ is done the same.

6. As each $M_i$ held $ZFC_{\text{new}}$, by the same proof as in 6.2.6 the model $M_{\text{Limit}}$ holds $ZFC_{\text{new}}$.

7. The same proofs of the lemma and the theorem of knowledge in $M_i$ of 5.3.3 applies with $ZFC_{\text{new}}$ replacing $ZFC$.

8. The definition of the consistency number in 6.3.2 is done the same with $ZFC_{\text{new}}$ replacing $ZFC$.

9. The same proofs of 6.3.4 and 6.3.5 holds the same with with $ZFC_{\text{new}}$ replacing $ZFC$.

8. $P$ vs $NP \cap co - NP$ and other uses cases

In this section we’ll demonstrate the usage of section 7 in order to show that for a model $w$ with a worldly cardinal for any language $L \in NP \cap co - NP$ provably, it is the case that $L \in P$. In other words

\[ P = NP \cap co - NP \]

assuming knowledge. Additionally, I’ll give another use case of number factoring.

8.1 Basic definition

Definition: Let $\psi(x)$ be a formula with a single free variable. The set

\[ L_\psi = \{ x \in \omega | \psi(x) \} \]

is called the language defined by $\psi$. 42
Definition: Let $V, \in V$ be a ZFC model. Let $\psi(x)$ be a formula in $V$ with a single free variable. $\psi$ is said to define a language in $NP \cap co-NP$ uniformly in $V$ if

$m_1, m_2, k, s, t \in V \omega(V)$ exist s.t the following statements are provable from ZFC within $V$:

- $m_1$ and $m_2$ codes TM
  - $m_1$ will be the positive verifier and $m_2$ will be the negative verifier.
- the running time of $m_1$ on input $(x,y)$ and $m_2$ on input $(x,y)$ is at most $s \cdot \log^k x + t$ (for any input $x,y$)
- for any $x \in \omega$:
  - if $\psi(x)$ then:
    * exist $y \in \omega$ s.t $y \leq s \cdot x^k + t$ and $[m_1](x,y)$ accepts.
    * for any $z \in \omega$ s.t $z \leq s \cdot x^k + t$, $[m_2](x,z)$ rejects.
  - if $\neg \psi(x)$ then:
    * exist $y \in \omega$ s.t $y \leq s \cdot x^k + t$ and $[m_2](x,y)$ accepts.
    * for any $z \in \omega$ s.t $z \leq s \cdot x^k + t$, $[m_1](x,z)$ rejects.

Remarks:

1. The reader may want to think of the case $\omega(V) = N$ at first.
2. the notation $[m](x,y)$ is the run of $m$ on the inputs $x,y$
3. Please notice that we ask that the statements are provable, meaning that all set models of $V$ will hold these conditions with the “same” $m_1, m_2, k, s, t$ constants.
4. For the reader who is familiar with the $NP$ and $co-NP$ “regular” definitions, the above is the equivalent definitions using verifiers.
   
   (a) The “regular” definition allows for different polynomials that bounds the running time of $[m_1], [m_2]$ and that bound the maximal length of the proofs. As we don’t want to use many constants for this (it’ll make the notation even more cumbersome). One can take the maximum of the powers and the maximum of the free constant in order to get one bounding polynomial.

8.2 construction’s definitions and lemmas

Definition: Let $V, \in V$ be a ZFC model. Let $\psi(x)$ be a formula in $V$ that define a language in $NP \cap co-NP$ uniformly. Let $m_1, m_2, s, k, t \in V \omega(V)$ be the constants as in 8.1. Define the following function

$$g(x) = \begin{cases} 
1, y' & \psi(x) \\
0, y'' & \neg\psi(x)
\end{cases}$$
where \( y' \) is the minimal number s.t \([m_1](x, y')\) accepts (positive proof) and \( y'' \) is the minimal number s.t \([m_2](x, y'')\) accepts (negative proof). The numbers 0, 1 are just to indicate that we code a negative \ positive proof.

Define \( \phi_\psi(x, y) \) to be the formula \( y = g(x) \) for the above \( g(x) \).

**Lemma:** The formula \( \phi_\psi(x, y) \) defines a function in \( ZF \) and holds the propriety condition.

**Proof:** The fact that \( \phi_\psi(x, y) \) defines a function is obvious, as \( \phi_\psi(x, y) \) it true iff \( y = g(x) \) for a function \( g \). One needs only prove that \( \phi_\psi \) holds the propriety condition. Let \( V_1 \xrightarrow{f} V_2 \) s.t \( V_2 \) is a set of \( V_1 \) and assume \( x, y \in \omega(V_1) \) s.t \( V_1 \models \phi_\psi(x, y) \) in which case \( y \) is either 1, \( y' \) s.t \([m_1](x, y')\) accepts or 0, \( y'' \) s.t \([m_2](x, y'')\) accepts. By \( 8.1.3 \) we know that if \([m_1](x, y')\) accepts then so \([f(m_1)](f(x), f(y'))\) and the same for \([m_2](x, y'')\). Therefore, if \( x, y \in \omega(V_1) \) s.t \( V_1 \models \phi(x, y) \) then \( V_2 \models \phi_\psi(f(x), f(y)) \). On the other hand if \( x, y \in \omega(V_1) \) are s.t \( V_1 \not\models \phi_\psi(x, y) \) then either

1. \( y \) isn’t of the form \( 1, y' \) or \( 0, y'' \)
2. \( y \) is of the form \( 1, y' \) but \([m_1](x, y')\) rejects
3. \( y \) is of the form \( 0, y'' \) but \([m_2](x, y'')\) rejects

In all cases \( f(y) \) will hold:

1. \( f(y) \) isn’t of the form \( 1, f(y') \) or \( 0, f(y'') \)
2. \( f(y) \) is of the form \( 1, f(y') \) and \([f(m_1)](f(x), f(y'))\) rejects
3. \( f(y) \) is of the form \( 0, f(y'') \) and \([f(m_1)](f(x), f(y''))\) rejects.

and in all cases \( V_2 \models \neg \phi_\psi(f(x), f(y)) \). \( \blacksquare \)

### 8.3 Given knowledge, \( P = NP \cap co - NP \) non-uniformly

**Theorem:** \( M, \in_M \) be a model with a worldly cardinal. Let \( \psi(x) \) be a formula in \( M \) that define a language in \( NP \cap co - NP \) uniformly. Then a the language

\[ \{x \in_M \omega(M) \mid M \models \psi(x)\} \subset \omega(M) \]

is decidable in poly-logarithmic time in \( M \).

**Proof:** First we may assume that \( k(n) \) s.t \( \phi_\psi(n, k(n)) \). Let \( m_1, m_2, s, k, t \in_M \omega(M) \) be the constants of \( \psi(x) \) in \( M \).

In \( M \) it holds that \( m_1, m_2 \) are positive and negative verifies respectively and each run at at most \( s \cdot \log^* x + t \) steps at most, as this was proved from \( ZFC \).
In $M$, by 7.2, it holds

$$M \models \exists \bar{Z} \in \omega \ \forall n \in \omega \ \exists x, k, c \in \omega \ \begin{aligned} &\phi_\psi (n, k) \land \text{Run} (x, n) = k \land x < \bar{Z} \land \\
&(\forall n', k' \in \omega) \left( (\phi_\psi (n', k') \land \text{Run} (x, n') = k') \rightarrow \\
&\text{Time} (x, n') = \lceil \log (n' + 1) + \log (k' + 1) + c \rceil \right) \end{aligned}$$

where $\text{Run} (x, n)$ denote that function that return the value return by the TM numbered $x$ on input $n$ and $\text{Time} (x, n)$ return the number of steps done in the calculation of $x$ on input $n$.

Define the following TM $T$ in $M$, given $n \in_M \omega$:

- Until a $k$ s.t $\phi_\psi (n, k)$ is found:
  1. run all TMs coded by numbers $y <_M \bar{Z}$ one more step.
  2. for each $y$, a TM that halted on the last step, check:
     (a) if the result of the calculation is of the form $1, y'$ for a number $y' \in \omega$ which hold $y' \leq s \cdot n^k + t$ check if $[m_1] (n, y')$. If so. then $k = \langle 1, y' \rangle$ halt $T$ and return true (as we’ve found that $\psi (n)$).
     (b) if the result of the calculation is of the form $0, y''$ for a number $y'' \in \omega$ which hold $y'' \leq s \cdot n^k + t$ check if $[m_2] (n, y'')$. If so. then $k = \langle 0, y'' \rangle$ halt $T$ and return false (as we’ve found that $\neg \psi (n)$).
- if both conditions (a) + (b) failed return to (1).

The TM $T$ calculates $k$ s.t $\phi_\psi (n, k)$ (due to the fact that $\text{know}_\phi$ holds in $M$). The question now is its running time.

**Running time analysis:** The following running time analysis is done within $M$:

- each step of (1) takes $\bar{Z}$ steps (assuming simulating a TM one step takes also one step, if it takes $z'$ steps then step (1) $z' \cdot \bar{Z}$ steps).
- 2a takes at most $s \cdot \log^n y' + t$ steps.
- 2b takes at most $s \cdot \log^n y'' + t$ steps.
- both $y', y''$ are bounded by $n^k + t$.
- exist $y <_M \bar{Z}$ that codes a TM which computes $k (n)$ in $\log (n + 1) + \log (k + 1) + c_y$ computing steps. Therefore, the total number of iterations of step (1) is bounded by
  $$\bar{Z} \cdot \left( \log (n + 1) + \log (k + 1) + \max_{0 \leq y \leq \bar{Z}} c_y \right)$$
  when the maximum $\max_{0 \leq y \leq \bar{Z}} c_y$ is taken within $M$. 45
• So, the total running time of this algorithm is poly-logarithmic time bounded.

Additional explanation of $\max_{0 \leq y \leq \bar{Z}} c_y$: The reader may be baffled by the idea that $\max_{0 \leq y \leq \bar{Z}} c_y$ can be taken. However, $c_y$ can be defined within the model $M$, as for each $y$ if $y$ codes a TM which computes $k(n)$ s.t $\phi(n,k)$ for every $n \leq t$ (for some $t$) in $\log(n+1) + \log(k+1) + c_y$ steps we can know the value $c_y$. If $y$ doesn’t code such TM we can define $c_y$ to be 0. And thus the maximum $\max_{0 \leq y \leq \bar{Z}} c_y$ is a maximum of $\bar{Z}$ numbers in $M$.

Remark: The term non-uniformly reminds us that the language we’ve started with is provable in $NP \cap co-NP$. Thus, this assertion is different from the assertion $P = NP \cap co-NP$ as in the latter, one must show that for any language in $NP \cap co-NP$ (regardless of provability) that language is in $P$. Of course, we haven’t showed the latter statement in this paper.

8.4 Number factorization
In this subsection, I’ll describe the usage of computational knowledge to factor natural numbers.

8.4.1 Construction definition

Definitions:
• for $n,k \in \omega$ the notation “$n \mod (k,n) = 0$” denotes that $k$ divides $n$.
• Let $n \in \omega$
  - $n$ is called a composite if $n = a \cdot b$ for two numbers $a, b > 1$.
  - let $n$ be a composite number. A number $b$ is called non-trivial divisor of $n$ if $\mod (b,n) = 0$ and $1 < b < n$.

Definition: Let $V, \in V$ be a ZFC model. define the minimal divisor function by

$$g(n) = \min_{1 < x \leq n} \{x \mid \mod (x,n) = 0\}$$

Define $\phi_{div}(x,y)$ to be the formula $y = g(x)$ for the above $g(x)$.

8.4.2 Basic lemma

Lemma: The formula $\phi_{V}(x,y)$ defines a function in ZFC and holds the property condition
Proof: The fact that $\phi_{\text{div}}(x, y)$ defines a function is obvious, as $\phi_{\text{div}}(x, y)$ it true iff $y = g(x)$ for a function $g$. One needs only prove that $\phi_{\text{div}}$ holds the propriety condition. Let $V_1 \xrightarrow{f} V_2$ s.t $V_2$ is a set of $V_1$ and assume $x, y \in \omega(V_1)$ s.t $V_1 \models \phi_{\text{div}}(x, y)$ then $V_1 \models \text{mod } (y, x) = 0$ and as such

$$V_2 \models \text{mod } (f(y), f(x)) = 0$$

(as $f$ is arithmetic) and as $V_1 \models \forall 1 < k < y \text{ mod } (f(x) \neq 0$ it is the case that

$$V_1 \models \forall 1 < k < f(y) \text{ mod } (f(k), f(x)) = 0$$

(as $f$ is arithmetic). And thus $V_2 \models \phi_{\text{div}}(f(x), f(y))$ ■

8.4.3 Given knowledge, factoring in in $P$

Theorem: $M, \in M$ be a model with a worldly cardinal then in $M$ then exists, within $M$, a TM that given $n$ a composite number, $T_1$ computes a non-trivial divisor.

Proof: In $M$, by [7.2] it holds

$$M \models \exists \bar{Z} \in \omega \forall n \in \omega \exists x, k, c \in \omega$$

$$\begin{cases} \phi_{\text{div}}(n, k) \land Run(x, n) = k \land x < \bar{Z} \land \\ (\forall n', k' \in \omega) \left( \phi_{\text{div}}(n', k') \land Run(x, n') = k' \rightarrow \\ \left( Time(x, n') = \lceil \log (n' + 1) \rceil + \lceil \log (k' + 1) \rceil + c \right) \right) \end{cases}$$

where $Run(x, n)$ denote that function that return the value return by the TM numbered $x$ on input $n$ and $Time(x, n)$ return the number of steps done in the calculation of $x$ on input $n$.

Define the following TM $T_1$ in $M$, given $n \in M \omega$ composite number to find a non trivial divisor of $n$:

- Until a $k$ s.t $1 < k < n$ and $\text{mod } (k, n) = 0$ is found:
  1. run all TMs coded by numbers $y \leq M \bar{Z}$ one more step.
  2. for each $y$, a TM that halted on the last step, check if the result of the calculation $z'$:
     (a) check if $1 < z' < n$ if so, go to (b) otherwise return to (1).
     (b) check if $\text{mod } (z', n) = 0$ if so, halt and return $z'$ otherwise return to (1).

due to the fact that $\text{know}_{\phi_{\text{div}}}$ holds in $M$, we know that for some $y \leq M \bar{Z}$ the answer of $g(n)$ will be given. We can’t know that $g(n)$ will be the first divisor to show up in the process but the TM $T_1$ calculates a divisor, i.e a $k$ s.t $1 < k < n$ and $\text{mod } (k, n) = 0$.

The question now is its running time.
Running time analysis: The following running time analysis is done within $M$:

- each step of (1) takes $\tilde{O} Z$ steps (assuming simulating a TM one step takes also one step, if it takes $z'$ steps then step (1) $z' \cdot \tilde{O} Z$ steps).
- step 2a is a simple comparison that takes at most $\log(n)$
- step 2b takes at most poly-logarithmic time in the output of the machine, as any candidate must be smaller than $n$ and division of two numbers is done in poly-logarithmic time.
- exist $y <_M \tilde{O} Z$ that codes a TM which computes $g(n)$ in $\log(n+1) + \log(k+1)+c_y$ computing steps. Therefore, the total number of iterations of step (1) is bounded by
  \[
  \tilde{O} Z \cdot \left( \log(n+1) + \log(k+1) + \max_{0 \leq y \leq \tilde{O} Z} c_y \right)
  \]
  where $k$ is the divisor returned and when the maximum $\max_{0 \leq y \leq \tilde{O} Z} c_y$ is taken within $M$.
- So, the total running time of this algorithm is poly-logarithmic time bounded.

Theorem: $M, \in_M$ be a model with a worldly cardinal then in $M$ one can factor numbers in poly-logarithmic time.

Proof: In $M$ use the following algorithm $T$, given $n \in \omega$:

1. verify that $n$ is a composite. If $n$ is prime, return $n$ as farther factoring can’t be done.
2. for $n$ a composite number run $T_1$ of the above theorem and find a non-trivial factor $k$.
3. divide $n$ by $k$ and receive $\frac{n}{k}$, return the pair $n, \frac{n}{k}$.

As by [7] we know that given a number $n$, testing if $n$ is prime or not is done in poly-logarithmic time. We know by the theorem above that step 2 can be done in poly-logarithmic time and we know that step 3 can be done in poly-logarithmic time and as such one can factor a number in poly-logarithmic time.

■
References

[1] “SET THEORY” by Kenneth Kunen, ISBN: 0 444 86839 9. Elsevier science publishers (1980).

[2] “Notes on logic and set theory” by P. T. Johnstone, Cambridge university press 1987, ISBN: 0521336929

[3] “Set Theory” by Thomas Jech - 3rd Millennium ed Springer (2002) ISBN 3-540-44085-2.

[4] “Models and Ultraproducts: An Introduction (reprint of 1974 ed.)” by Bell, John Lane; Slomson, Alan B. (2006) [1969] (ISBN 0-486-44979-3)

[5] “The Higher Infinite” by Akihiro Kanamori ISBN: 978-3-540-88866-6. Springer-Verlag Berlin Heidelberg (2009)

[6] Models of Peano arithmetic by Richard Kaye. Clarendon Press Oxford. 1991 (ISBN 0-19-853213)

[7] Agrawal, Manindra; Kayal, Neeraj; Saxena, Nitin (2004). "PRIMES is in P". Annals of Mathematics. 160 (2): 781–793. doi:10.4007/annals.2004.160.781. JSTOR 3597229.