Polynomials of Degree-Based Indices for Swapped Networks Modeled by Optical Transpose Interconnection System

ALI AHMAD, ROSLAN HASNI, KASHIF ELAHI, AND MUHAMMAD AHSAN ASIM

1College of Computer Science and Information Technology, Jazan University, Jazan 45142, Saudi Arabia
2School of Informatics and Applied Mathematics, Universiti Malaysia Terengganu, Kuala Terengganu 21030, Malaysia
3Deanship of E-learning and Information Technology, Jazan University, Jazan 45142, Saudi Arabia

Corresponding author: Roslan Hasni (hroslan@umt.edu.my)

ABSTRACT The Optical Transpose Interconnection System (OTIS) has applications in parallel processing, distributed processing, routing, and networks. It is used for efficient usage of multiple parallel algorithms or parallel systems, with different global interconnections in a network as it is an optoelectronic (combination of light signals and electronics). In chemical graph theory, topological indices are used to study characteristics of the chemical structures or biological activities. Topological indices are sometimes studied with the assistance to their polynomial. In this article, polynomials of degree-based topological indices for OTIS and swapped networks have been studied. Results can be used to compute any degree-based topological polynomials for OTIS swapped network.

INDEX TERMS Topological polynomials, degree-based index, optical transpose interconnection system.

I. INTRODUCTION AND PRELIMINARY RESULTS

Graph theory has been proved as a vast field by solving problems in multiple fields, like chemistry, physics, computer science, statistics, robotics, networks and routing. Graph can be represented numerically in different forms of data container like matrices, vectors and polynomials. Different real life problems can be represented with the help of a graph. Graph theory provides further operations to find the solution of a problem. Chemical graph theory has further branches quantitative structure-property relationship (QSPR) and quantitative structure-activity relationship (QSAR), which are essential part in studying the characteristics or chemical properties of molecules and atoms [1]–[4].

Topological index is a single value used to represent the characteristics of the graph. It is invariant under graph automorphism. Topological indices have played an important part in the study of chemical properties under the branches of graph theory QSPR and QSAR. They are used to correlate biological activity or other properties of molecules with their chemical structure. Weiner Index was the first topological index, introduced by Harry Weiner in 1947. Topological indices can be categorized on the bases of their calculation mechanism. Degree based topological indices involves degree of vertices of graph in the calculation. Randić index, Zagreb index, harmonic index, atom bond connectivity and geometric-arithmetic index are some known degree-based topological indices and polynomials [5]–[17].

OTIS is an optoelectronic. In a network, it provides competency in both optical and electronic technologies. Multiple groups are connected efficiently, electronic connections are used within the same group while optical links are used to communicate between different groups. Multiple algorithms are used for routing, image processing, parallel processing, matrix multiplication (hybrid network model), sorting, selecting and fourier transform (sound and signals) [18]–[20]. A network can be represented graphically. Servers and processors can be represented by vertices while the connections between them can be represented by edges. The number of links on servers or processors are degree of vertices. The maximum distance between two network heads is grid diameter [21]–[23].

II. DEGREE-BASED INDICES AND THEIR POLYNOMIALS

Let $G$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. The degree $d_v$ of a vertex $v \in V(G)$ is the number of neighbours of $v$. The most general indices based on degrees
are the general Randić index of a graph $G$, 
\[ R_\alpha(G) = \sum_{uv \in E(G)} (d_u d_v)^\alpha, \]  
the general sum-connectivity index 
\[ \chi_\alpha(G) = \sum_{uv \in E(G)} (d_u + d_v)^\alpha, \]  
and the generalized Zagreb index 
\[ GZ_{\alpha,\beta}(G) = \sum_{uv \in E(G)} d_u^\alpha d_v^\beta + d_u^\beta d_v^\alpha. \]

Note that the third redefined Zagreb index is defined as 
\[ ReZ(G) = \sum_{uv \in E(G)} d_u(d_u + d_v), \]

the harmonic index is defined as 
\[ H(G) = \sum_{uv \in E(G)} \frac{2}{d_u + d_v}. \]

the third Zagreb index 
\[ M_3(G) = \sum_{uv \in E(G)} (d_u - d_v) \]

the fourth Zagreb index 
\[ M_4(G) = \sum_{uv \in E(G)} d_u(d_u + d_v) \]

and the fifth Zagreb index 
\[ M_5(G) = \sum_{uv \in E(G)} d_u d_v \]

Let us introduce a general invariant for polynomials of above mentioned topological indices. 
\[ P(G, x) = \sum_{uv \in E(G)} x^{\varphi(d_u, d_v)}, \]

where $\varphi(d_u, d_v)$ is a function of $d_u$ and $d_v$ such that $\varphi(d_u, d_v) = \varphi(d_v, d_u)$.

- If $\varphi(d_u, d_v) = (d_u d_v)^\alpha$, where $\alpha$ is a positive integer, then $P(G, x)$ is the general Randić polynomial of $G$. Moreover, $P(G, x)$ is the second Zagreb polynomial if $\alpha = 1$.
- If $\varphi(d_u, d_v) = (d_u + d_v)^\alpha$, where $\alpha$ is a positive integer, then $P(G, x)$ is the general sum-connectivity polynomial of $G$. Furthermore, $P(G, x)$ is the first Zagreb polynomial for $\alpha = 1$ and the hyper-Zagreb polynomial for $\alpha = 2$.
- If $\varphi(d_u, d_v) = d_u^\alpha d_v^\beta + d_u^\beta d_v^\alpha$, where $\alpha$ is a positive integer and $\beta$ is a non-negative integer, then $P(G, x)$ is the generalized Zagreb polynomial of $G$. Moreover, $P(G, x)$ is the forgotten polynomial if $\alpha = 2$ and $\beta = 0$.
- If $\varphi(d_u, d_v) = d_u d_v(d_u + d_v)$, then $P(G, x)$ is the third redefined Zagreb polynomial of $G$.
- If $\varphi(d_u, d_v) = d_u + d_v - 1$, then $P(G, x)$ is one half of the harmonic polynomial $H(G, x)$ of $G$. Note that the harmonic polynomial is defined differently from the other polynomials.
- If $\varphi(d_u, d_v) = |d_u - d_v|$, then $P(G, x)$ is the third Zagreb polynomial of $G$.
- If $\varphi(d_u, d_v) = d_u(d_u + d_v)$, then $P(G, x)$ is the fourth Zagreb polynomial of $G$.
- If $\varphi(d_u, d_v) = d_u d_v$, then $P(G, x)$ is the fifth Zagreb polynomial of $G$.

So the general Randić polynomial of any graph $G$ is defined as 
\[ R_\alpha(G, x) = \sum_{uv \in E(G)} x^{\varphi(d_u, d_v)}, \]

the general sum-connectivity polynomial is 
\[ \chi_\alpha(G, x) = \sum_{uv \in E(G)} x^{\varphi(d_u, d_v)}, \]

the generalized Zagreb polynomial of any graph $G$, 
\[ GZ_{\alpha,\beta}(G, x) = \sum_{uv \in E(G)} x^{\varphi(d_u, d_v)}, \]

the third redefined Zagreb polynomial is defined as 
\[ ReZ(G, x) = \sum_{uv \in E(G)} x^{\varphi(d_u, d_v)}, \]

the harmonic polynomial is 
\[ H(G, x) = 2 \sum_{uv \in E(G)} x^{d_u + d_v - 1}. \]

the third Zagreb polynomial 
\[ M_3(G, x) = \sum_{uv \in E(G)} x^{\varphi(d_u, d_v)} \]

the fourth Zagreb polynomial 
\[ M_4(G, x) = \sum_{uv \in E(G)} x^{\varphi(d_u, d_v)} \]

and the fifth Zagreb polynomial 
\[ M_5(G, x) = \sum_{uv \in E(G)} x^{\varphi(d_u, d_v)} \]

III. OPTICAL TRANPOSE INTERCONNECTION SYSTEM SWAPPED NETWORK

Considered a graph $G$ having vertex set $V(G)$ and edge set $E(G)$, OTIS swapped network $O_G$ can be defined as below: $V(O_G) = \{(x, y) | x, y \in V(G), E(O_G) = \{(x, y_1), (x, y_2) | x \in V(G), (y_1, y_2) \in E(G), U \cup \{(x, y), (y, x) | x, y \in E(G), x \neq y \} \}$. [24].

For the mutual OTIS network $O_G$, the graph $G$ is called the factor of the graph or grid. If there are $m$ primary network $G$. So, $O_G$ consists of a separate subnet from the $m$ node groups are called, and they are similar to $G$ [24]. The node name $(x, y)$ in $O_G$ select the y node handle in the group $[18]$–[20], [25]–[27]. Next some some degree based topological polynomials of swapped networks are calculated. For a given path $P_m$ on $m$ vertices and $O_P_m$, as its OTIS swapped network with basis network $P_m$ is shown in Figure 1.
IV. RESULTS FOR OTIS SWAPPED NETWORK $OP_m$

We present results, which can be used to compute any degree-based topological polynomials. Our results generalize known results in the area. We give exact values of the most well-known degree-based polynomials for optical transpoe interconnection system $OP_m$. Vetrík [28] introduced a new method to calculate the topological indices and also in [29], we follow the same technique in this paper. Let us give a formula, which can be used to obtain any polynomial of indices based on degrees for optical transpoe interconnection system $OP_m$.

**Lemma 1:** Let $OP_m$ be a optical transpoe interconnection system. Then $P(OP_m, x) = \frac{3m^2}{2}x^{\lambda(3,3)} + m\left\{6x^{\lambda(2,3)} - \frac{15}{2}x^{\lambda(3,3)}\right\} + 2x^{\lambda(1,3)} + 3x^{\lambda(2,2)} - 14x^{\lambda(2,3)} + 9x^{\lambda(3,3)}$.

**Proof 1:** The graph $OP_m$ contains $m^2$ vertices and $\frac{3m^2 - 3m}{2}$ edges. Each vertex of $OP_m$ has degree 1, 2, or 3, vertices of $OP_m$ can be partitioned according to their degrees. Let $V_j = \{e \in V(OP_m) : d_e = j\}$. This means that the set $V_j$ contains the vertices of degree $j$. The set of vertices with respect to their degrees are as follows:

$V_1 = \{e \in V(OP_m) : d_e = 1\}$
$V_2 = \{e \in V(OP_m) : d_e = 2\}$
$V_3 = \{e \in V(OP_m) : d_e = 3\}$

Since, $|V_1| = 2$, $|V_2| = 3m - 4$ and $|V_3| = |V(OP_m)| - |V_1| - |V_2| = m^2 - 2 - (3m - 4) = m^2 - 3m + 2$. Let us divide the edges of $OP_m$ into partition sets according to the degree of its end vertices. Let

$\mathcal{E}_{1,3} = \{e \in E(OP_m) : d_e = 1, d_v = 3\}$
$\mathcal{E}_{2,2} = \{e \in E(OP_m) : d_e = 2, d_v = 2\}$
$\mathcal{E}_{2,3} = \{e \in E(OP_m) : d_e = 2, d_v = 3\}$
$\mathcal{E}_{3,3} = \{e \in E(OP_m) : d_e = 3, d_v = 3\}$.

Note that $E(OP_m) = \mathcal{E}_{1,3} \cup \mathcal{E}_{2,2} \cup \mathcal{E}_{2,3} \cup \mathcal{E}_{3,3}$. The number of edges incident to one vertex of degree 1 and another vertex of degree 3 is 2, so $|\mathcal{E}_{1,3}| = 2$. The number of edges incident to two vertices of degree 2 is 3, so $|\mathcal{E}_{2,2}| = 3$. The number of edges incident to one vertex of degree 2 and other vertex of degree 3 are $6m - 14$, so $|\mathcal{E}_{2,3}| = 6m - 14$. Now, the remaining number of edges are those edges which are incident to two vertices of degree 3, i.e $\mathcal{E}_{3,3} = |E(OP_m)| - \mathcal{E}_{1,3} - \mathcal{E}_{2,2} - \mathcal{E}_{2,3} = \frac{3m^2 - 3m}{2} - 2 - 3 - (6m - 14) = \frac{3m^2 - 15m + 18}{2}$.

Hence, $P(OP_m, x) = \sum_{e \in \mathcal{E}_{1,3}} x^{\lambda(d_e, d_v)} + \sum_{e \in \mathcal{E}_{2,2}} x^{\lambda(2,2)} + \sum_{e \in \mathcal{E}_{2,3}} x^{\lambda(2,3)} + \sum_{e \in \mathcal{E}_{3,3}} x^{\lambda(3,3)}$. After simplification, we get $P(OP_m, x) = \frac{3m^2}{2}x^{\lambda(3,3)} + m\left\{6x^{\lambda(2,3)} - \frac{15}{2}x^{\lambda(3,3)}\right\} + 2x^{\lambda(1,3)} + 3x^{\lambda(2,2)} - 14x^{\lambda(2,3)} + 9x^{\lambda(3,3)}$.

Now we present polynomials of the best-known degree based polynomials of optical transpose interconnection system in the following theorem.

**Theorem 1:** For the optical transpose interconnection system $OP_m$, we have the general Randić polynomial of $OP_m$, $R_\alpha(OP_m, x) = \frac{3}{2}m^2x^{\alpha} + m(6x^{\alpha} - \frac{15}{2}x^{\alpha'}) + 2x^{\alpha'} + 3x^{\alpha' - 14x^{\alpha'} + 9x^{\alpha''}}$.

the second Zagreb polynomial of $OP_m$, $R_1(OP_m, x) = \frac{3}{2}m^2x^{\alpha} + m(6x^{\alpha} - \frac{15}{2}x^{\alpha'}) + 2x^{\alpha'} + 3x^{\alpha' - 14x^{\alpha'} + 9x^{\alpha''}}$.

**Proof 2:** For $R_\alpha(OP_m, x)$ which is the general Randić polynomial of $OP_m$, we have $\lambda(d_e, d_v) = (d_e + d_v)\alpha$, therefore $\lambda(1, 3) = (3)^\alpha$, $\lambda(2, 2) = (4)^\alpha$, $\lambda(2, 3) = (6)^\alpha$ and $\lambda(3, 3) = (9)^\alpha$. Thus by Lemma 1, $R_\alpha(OP_m, x) = \frac{3}{2}m^2x^{\alpha} + m(6x^{\alpha} - \frac{15}{2}x^{\alpha'}) + 2x^{\alpha'} + 3x^{\alpha' - 14x^{\alpha'} + 9x^{\alpha''}}$.

For $\alpha = 1$, the second Zagreb polynomial is $R_1(OP_m, x) = \frac{3}{2}m^2x^{\alpha} + m(6x^{\alpha} - \frac{15}{2}x^{\alpha'}) + 2x^{\alpha'} + 3x^{\alpha' - 14x^{\alpha'} + 9x^{\alpha''}}$.

In the next theorem, we determined general sum-connectivity polynomial, first Zagreb polynomial and hyper-Zagreb polynomial of the optical transpose interconnection system $OP_m$.

**Theorem 2:** For the optical transpose interconnection system $OP_m$, we have

the general sum-connectivity polynomial of $OP_m$, $\chi_\alpha(OP_m, x) = \frac{3}{2}m^2x^{\alpha} + m(6x^{\alpha} - \frac{15}{2}x^{\alpha'}) + 5x^{\alpha''} - 14x^{\alpha'} + 9x^{\alpha''}$.

the first Zagreb polynomial of $OP_m$, $\chi_1(OP_m, x) = \frac{3}{2}m^2x^{\alpha} + m(6x^{\alpha} - \frac{15}{2}x^{\alpha'}) + 5x^{\alpha''} - 14x^{\alpha'} + 9x^{\alpha''}$.

the hyper-Zagreb polynomial of $OP_m$, $\chi_2(OP_m, x) = \frac{3}{2}m^2x^{\alpha} + m(6x^{\alpha} - \frac{15}{2}x^{\alpha'}) + 5x^{\alpha''} - 14x^{\alpha'} + 9x^{\alpha''}$.

**Proof 3:** For $\chi_\alpha(OP_m, x)$ which is the general sum-connectivity polynomial of $OP_m$, we have $\lambda(d_e, d_v) = (d_e + d_v)\alpha$, therefore $\lambda(1, 3) = (4)^\alpha$, $\lambda(2, 2) = (4)^\alpha$, $\lambda(2, 3) = (5)^\alpha$ and $\lambda(3, 3) = (6)^\alpha$. Thus by Lemma 1, $\chi_\alpha(OP_m, x) = \frac{3}{2}m^2x^{\alpha} + m(6x^{\alpha} - \frac{15}{2}x^{\alpha'}) + 5x^{\alpha''} - 14x^{\alpha'} + 9x^{\alpha''}$.
For $\alpha = 1$, the first Zagreb polynomial is $\chi_1(O_{pa}, x) = \frac{3}{2} m^2 x^2 + m(6 x^3 - \frac{15}{2} x^5) + 5 x^6 - 14 x^8 + 9 x^{10}$.

For $\alpha = 2$, the hyper-Zagreb polynomial is $\chi_2(O_{pa}, x) = \frac{3}{2} m^2 x^3 + m(6 x^{25} - \frac{15}{2} x^{36}) + 5 x^{16} - 14 x^{25} + 9 x^{36}$.

In the following theorem, we determined generalized Zagreb polynomial and forgotten polynomial of the optical transpose interconnection system $O_{pa}$.

**Theorem 3:** For the optical transpose interconnection system $O_{pa}$, we have

the generalized Zagreb polynomial of $O_{pa}$,

\[
G_{Z\alpha, \beta}(O_{pa}, x) = \frac{3}{2} m^2 x^{(3^{\alpha} 3^{\beta} + 3^{\beta} 3^{\alpha})} + m(6 x^{(2^{\alpha} 3^{\beta} + 3^{\beta} 2^{\beta})} - \frac{15}{2} x^{(3^{\alpha} 3^{\beta} + 3^{\beta} 3^{\alpha})}) + 2 x^{(\alpha + \beta + 1) 3^{\alpha} 3^{\beta}} + 3 x^{(3^{\alpha} 3^{\beta} + 3^{\beta} 3^{\alpha})} - 14 x^{(2^{\alpha} 3^{\beta} + 3^{\beta} 2^{\beta})} + 9 x^{(3^{\alpha} 3^{\beta} + 3^{\beta} 3^{\alpha})},
\]

the forgotten polynomial of $O_{pa}$,

\[
G_{F\alpha, \beta}(O_{pa}, x) = \frac{3}{2} m^2 x^{(3^{\alpha} 3^{\beta} + 3^{\beta} 3^{\alpha})} + m(6 x^{(2^{\alpha} 3^{\beta} + 3^{\beta} 2^{\beta})} - \frac{15}{2} x^{(3^{\alpha} 3^{\beta} + 3^{\beta} 3^{\alpha})}) + 2 x^{(\alpha + \beta + 1) 3^{\alpha} 3^{\beta}} + 3 x^{(3^{\alpha} 3^{\beta} + 3^{\beta} 3^{\alpha})} - 14 x^{(2^{\alpha} 3^{\beta} + 3^{\beta} 2^{\beta})} + 9 x^{(3^{\alpha} 3^{\beta} + 3^{\beta} 3^{\alpha})},
\]

for $\alpha, \beta = 0$.

**Proof 4:** For $G_{Z\alpha, \beta}(O_{pa}, x)$ which is the generalized Zagreb polynomial of $O_{pa}$, we have $\lambda(d_e, d_v) = (d_e + d_v)^\alpha$, therefore $\lambda(1, 3) = (4)^\alpha$, $\lambda(2, 3) = (5)^\alpha$ and $\lambda(3, 3) = (6)^\alpha$. Thus by Lemma 1,

\[
G_{Z\alpha, \beta}(O_{pa}, x) = \frac{3}{2} m^2 x^{(3^{\alpha} 3^{\beta} + 3^{\beta} 3^{\alpha})} + m(6 x^{(2^{\alpha} 3^{\beta} + 3^{\beta} 2^{\beta})} - \frac{15}{2} x^{(3^{\alpha} 3^{\beta} + 3^{\beta} 3^{\alpha})}) + 2 x^{(\alpha + \beta + 1) 3^{\alpha} 3^{\beta}} + 3 x^{(3^{\alpha} 3^{\beta} + 3^{\beta} 3^{\alpha})} - 14 x^{(2^{\alpha} 3^{\beta} + 3^{\beta} 2^{\beta})} + 9 x^{(3^{\alpha} 3^{\beta} + 3^{\beta} 3^{\alpha})}.
\]

In the following theorem, we determined the third redefined Zagreb polynomial and harmonic polynomial of the optical transpose interconnection system $O_{pa}$.

**Theorem 4:** For the optical transpose interconnection system $O_{pa}$, we have

the third redefined Zagreb polynomial of $O_{pa}$,

\[
ReZ(O_{pa}, x) = \frac{3}{2} m^2 x^{3} + m(6 x^3 - \frac{15}{2} x^5) + 2 x + 3 x^6 - 14 x^{10} + 9 x^{18},
\]

the harmonic polynomial of $O_{pa}$,

\[
H(O_{pa}, x) = \frac{3}{2} m^2 x^{3} + m(6 x^3 - \frac{15}{2} x^5) + 5 x^3 - 14 x^7 + 9 x^{10}.
\]

**Proof 5:** For $ReZ(O_{pa}, x)$ which is the third redefined Zagreb polynomial of $O_{pa}$, we have $\lambda(d_e, d_v) = d_e + d_v - 1$, therefore $\lambda(1, 3) = (3)^\alpha$, $\lambda(2, 2) = (3)^\alpha$, $\lambda(2, 3) = (4)^\alpha$ and $\lambda(3, 3) = (5)^\alpha$. Thus by Lemma 1,

\[
ReZ(O_{pa}, x) = \frac{3}{2} m^2 x^{3} + m(6 x^3 - \frac{15}{2} x^5) + 5 x^3 - 14 x^7 + 9 x^{10}.
\]

In the following theorem, we determined the third Zagreb polynomial, fourth Zagreb polynomial and fifth Zagreb polynomial of the optical transpose interconnection system $O_{pa}$.

**Theorem 5:** For the optical transpose interconnection system $O_{pa}$, we have

the third Zagreb polynomial of $O_{pa}$,

\[
M_3(O_{pa}, x) = \frac{3}{2} m^2 x + m(6 x^3 - \frac{15}{2} x^5) + 2 x^2 - 14 x + 12.
\]

the fourth Zagreb polynomial of $O_{pa}$,

\[
M_4(O_{pa}, x) = \frac{3}{2} m^2 x^{18} + m(6 x^{10} - \frac{15}{2} x^{12}) + 2 x^4 + 3 x^8 - 14 x^{10} + 9 x^{18}.
\]

the fifth Zagreb polynomial of $O_{pa}$,

\[
M_5(O_{pa}, x) = \frac{3}{2} m^2 x^{18} + m(6 x^{15} - \frac{15}{2} x^{17}) + 2 x^{12} + 3 x^8 - 14 x^{15} + 9 x^{18}.
\]

**Proof 6:** For $M_3(O_{pa}, x)$ which is the third Zagreb polynomial of $O_{pa}$, we have $\lambda(d_e, d_v) = |d_e - d_v|$, therefore $\lambda(1, 3) = (2)^\alpha$, $\lambda(2, 2) = (0)^\alpha$, $\lambda(2, 3) = (1)^\alpha$ and $\lambda(3, 3) = (0)^\alpha$. Thus by Lemma 1,

\[
M_3(O_{pa}, x) = \frac{3}{2} m^2 x + m(6 x^3 - \frac{15}{2} x^5) + 2 x^2 - 14 x + 12.
\]

For $M_4(O_{pa}, x)$ which is the fourth Zagreb polynomial of $O_{pa}$, we have $\lambda(d_e, d_v) = d_e(d_e + d_v)$, therefore $\lambda(1, 3) = (4)^\alpha$, $\lambda(2, 2) = (8)^\alpha$, $\lambda(2, 3) = (10)^\alpha$ and $\lambda(3, 3) = (18)^\alpha$. Thus by Lemma 1,

\[
M_4(O_{pa}, x) = \frac{3}{2} m^2 x^{18} + m(6 x^{10} - \frac{15}{2} x^{12}) + 2 x^4 + 3 x^8 - 14 x^{10} + 9 x^{18}.
\]

For $M_5(O_{pa}, x)$ which is the fifth Zagreb polynomial of $O_{pa}$, we have $\lambda(d_e, d_v) = d_e(d_e + d_v)$, therefore $\lambda(1, 3) = (18)^\alpha$, $\lambda(2, 2) = (8)^\alpha$, $\lambda(2, 3) = (10)^\alpha$ and $\lambda(3, 3) = (18)^\alpha$. Thus by Lemma 1,

\[
M_5(O_{pa}, x) = \frac{3}{2} m^2 x^{18} + m(6 x^{15} - \frac{15}{2} x^{17}) + 2 x^{12} + 3 x^8 - 14 x^{15} + 9 x^{18}.
\]

**V. OTIS SWAPPED NETWORK $O_{Km}$**

The complete graph denoted by $K_m$ with $m$ vertices and $O_{Km}$ be the OTIS swapped network for $O_{K^4}$ as shown in Figure 2.

**FIGURE 2. OTIS swapped network $O_{K^4}$**

**Lemma 2:** Let $O_{K_m}$ be a. Then $P(O_{K_m}, x) = \frac{m^2}{2} x^{(\lambda(m, m) - \lambda(m, m - 1))} + \frac{m^3}{2} x^{(\lambda(m, m - 1) - \lambda(m, m - 2))}$.

**Proof 7:** The graph $O_{K_m}$ contains $m^2$ vertices and $\frac{m^3 - m}{2}$ edges. Each vertex of $O_{K_m}$ has degree $m - 1$ or $m$, vertices of $O_{K_m}$ can be partitioned according to their degrees. Let

\[
V_{j} = \{ e \in V(O_{K_m}) : d_e = j \}.
\]

This means that the set $V_{j}$ contains the vertices of degree $j$. The set of vertices with respect to their degrees are as follows:

\[
V_{m-1} = \{ e \in V(O_{K_m}) : d_e = m - 1 \}
\]

\[
V_{m} = \{ e \in V(O_{K_m}) : d_e = m \}.
\]
Since $|V_{m-1}| = m$ and $|V_m| = |V(O_{K_m})| - |V_{m-1}| = m^2 - m$. Let us divide the edges of $O_{K_m}$ into partition sets according to the degree of its end vertices. Let

$$
\Sigma_{m,m-1} = \{ e \in E(O_{K_m}) : d_e = m, d_v = m-1 \}
$$
$$
\Sigma_{m,m} = \{ e \in E(O_{K_m}) : d_e = m, d_v = m \}.
$$

Note that $E(O_{K_m}) = \Sigma_{m,m-1} \cup \Sigma_{m,m}$. The number of edges incident to one vertex of degree $m-1$ and other vertex of degree $m$ is $m^2 - m$, so $|\Sigma_{m,m-1}| = m^2 - m$. Now, the remaining number of edges are those which are incident to two vertices of degree $m$, i.e. $E(O_{K_m}) - \Sigma_{m,m-1} = m^3 - m - (m^2 - m) = m^3 - 2m^2 + m = m(m-1)(m-2)$.

Hence,

$$
P(O_{K_m}, x) = \sum_{e \in E(O_{K_m})} x^{d_e} x^{d_v} = \sum_{e \in \Sigma_{m,m-1}} x^{\lambda(m,m-1)} + \sum_{e \in \Sigma_{m,m}} x^{\lambda(m,m)}.
$$

After simplification, we get

$$
P(O_{K_m}, x) = \frac{m^3}{2} x^{\lambda(m,m)} + m^2 x^{(m^2-m)} - \frac{1}{2} m(m-1)(m-2).
$$

**Theorem 6:** For the optical transpose interconnection system swapped network $O_{K_m}$, we have the general Randić polynomial of $O_{K_m}$,

$$
R_s(O_{K_m}, x) = \frac{m^3}{2} x^{(m^2)} + m^2 x^{(m^2-m)} - x^{(m^2)} + m\left(\frac{1}{x^{(m-2)}} - x^{(-m^2)}\right).
$$

The second Zagreb polynomial of $O_{K_m}$,

$$
R_1(O_{K_m}, x) = \frac{m^3}{2} x^{(m^2)} + m^2 x^{(m^2-m)} - x^{(m^2)} + m\left(\frac{1}{x^{(m-2)}} - x^{(-m^2)}\right).
$$

**Proof 8:** For $R_s(O_{K_m}, x)$ which is the general Randić polynomial of $O_{K_m}$, we have $\lambda(d_e, d_v) = (d_e - d_v)^\alpha$, therefore $\lambda(m, m-1) = (m^2 - m)^\alpha$ and $\lambda(m, m) = (m^2)^\alpha$. Thus by Lemma 2,

$$
R_s(O_{K_m}, x) = \frac{m^3}{2} x^{(m^2)} + m^2 x^{(m^2-m)} - x^{(m^2)} + m\left(\frac{1}{x^{(m-2)}} - x^{(-m^2)}\right).
$$

For $\alpha = 1$, the second Zagreb polynomial is

$$
R_1(O_{K_m}, x) = \frac{m^3}{2} x^{(m^2)} + m^2 x^{(m^2-m)} - x^{(m^2)} + m\left(\frac{1}{x^{(m-2)}} - x^{(-m^2)}\right).
$$

In the next theorem, we determined general sum-connectivity polynomial, first Zagreb polynomial and hyper-Zagreb polynomial of the optical transpose interconnection system swapped network $O_{K_m}$.

**Theorem 7:** For the optical transpose interconnection system swapped network $O_{K_m}$, we have the general sum-connectivity polynomial of $O_{K_m}$,

$$
\chi_s(O_{K_m}, x) = \frac{m^3}{2} x^{(m^2)} + m^2 x^{(m^2-m)} - x^{(m^2)} + m\left(\frac{1}{x^{(m-2)}} - x^{(-m^2)}\right).
$$

The first Zagreb polynomial of $O_{K_m}$,

$$
\chi_1(O_{K_m}, x) = \frac{m^3}{2} x^{(m^2)} + m^2 x^{(m^2-m)} - x^{(m^2)} + m\left(\frac{1}{x^{(m-2)}} - x^{(-m^2)}\right).
$$

The hyper-Zagreb polynomial of $O_{K_m}$,

$$
\chi_2(O_{K_m}, x) = \frac{m^3}{2} x^{(m^2)} + m^2 x^{(m^2-m)} - x^{(m^2)} + m\left(\frac{1}{x^{(m-2)}} - x^{(-m^2)}\right).
$$

**Proof 9:** For $\chi_s(O_{K_m}, x)$ which is the general sum-connectivity polynomial of $O_{K_m}$, we have $\lambda(d_e, d_v) = (d_e + d_v)^\alpha$, therefore $\lambda(m, m-1) = (2m - 1)^\alpha$ and $\lambda(m, m) = (m^2)^\alpha$. Thus by Lemma 2,

$$
\chi_s(O_{K_m}, x) = \frac{m^3}{2} x^{(m^2)} + m^2 x^{(m^2-m)} - x^{(m^2)} + m\left(\frac{1}{x^{(m-2)}} - x^{(-m^2)}\right).
$$

For $\alpha = 1$, the first Zagreb polynomial is

$$
\chi_1(O_{K_m}, x) = \frac{m^3}{2} x^{(m^2)} + m^2 x^{(m^2-m)} - x^{(m^2)} + m\left(\frac{1}{x^{(m-2)}} - x^{(-m^2)}\right).
$$

For $\alpha = 2$, the hyper-Zagreb polynomial is

$$
\chi_2(O_{K_m}, x) = \frac{m^3}{2} x^{(m^2)} + m^2 x^{(m^2-m)} - x^{(m^2)} + m\left(\frac{1}{x^{(m-2)}} - x^{(-m^2)}\right).
$$

Theorem 8: For the optical transpose interconnection system swapped network $O_{K_m}$, we have the generalized Zagreb polynomial of $O_{K_m}$,

$$
G_{\alpha, \beta}(O_{K_m}, x) = \frac{m^3}{2} x^{(m^2)} + m^2 x^{(m^2-m)} - x^{(m^2)} + m\left(\frac{1}{x^{(m-2)}} - x^{(-m^2)}\right).
$$

**Proof 10:** For $G_{\alpha, \beta}(O_{K_m}, x)$ which is the generalized Zagreb polynomial of $O_{K_m}$, we have $\lambda(d_e, d_v) = (d_e + d_v)^\alpha$, therefore $\lambda(m, m-1) = (2m - 1)^\alpha$ and $\lambda(m, m) = (m^2)^\alpha$. Thus by Lemma 2,

$$
G_{\alpha, \beta}(O_{K_m}, x) = \frac{m^3}{2} x^{(m^2)} + m^2 x^{(m^2-m)} - x^{(m^2)} + m\left(\frac{1}{x^{(m-2)}} - x^{(-m^2)}\right).
$$

For $\alpha = 2$, $\beta = 0$, the forgotten polynomial is

$$
G_{2,0}(O_{K_m}, x) = \frac{m^3}{2} x^{(m^2)} + m^2 x^{(m^2-m)} - x^{(m^2)} + m\left(\frac{1}{x^{(m-2)}} - x^{(-m^2)}\right).
$$

Theorem 9: For the optical transpose interconnection system swapped network $O_{K_m}$, we have the third redefined Zagreb polynomial of $O_{K_m}$,

$$
\chi_s(O_{K_m}, x) = \frac{m^3}{2} x^{(m^2)} + m^2 x^{(m^2-m)} - x^{(m^2)} + m\left(\frac{1}{x^{(m-2)}} - x^{(-m^2)}\right).
$$

**Proof 11:** For $\chi_s(O_{K_m}, x)$ which is the third redefined Zagreb polynomial of $O_{K_m}$, we have $\lambda(d_e, d_v) = (d_e + d_v)^\alpha$, therefore $\lambda(m, m-1) = (2m - 1)^\alpha$ and $\lambda(m, m) = (m^2)^\alpha$. Thus by Lemma 2,

$$
\chi_s(O_{K_m}, x) = \frac{m^3}{2} x^{(m^2)} + m^2 x^{(m^2-m)} - x^{(m^2)} + m\left(\frac{1}{x^{(m-2)}} - x^{(-m^2)}\right).
$$
For $H(OK_n, x)$ which is the harmonic polynomial of $OK_n$, we have $\lambda(d_e, d_v) = d_e + d_v - 1$, therefore $\lambda(m, m - 1) = x^{2(m-1)}$, and $\lambda(m, m) = x^{(m^2-1)}$. Thus by Lemma 2, $H(OK_n, x) = m\frac{x^{2m}}{2} + m^2(x - 1) + m\frac{1}{2} - x$. The fourth Zagreb polynomial of $OK_n$, $M_4(OK_n, x) = m\frac{x^{2m}}{2} + m^2[x^{(m+2m)}] - x^{(2m^2)} + m\{x^{(2m^2)} - x^{(m-1)(2m-1)}\}$. The fifth Zagreb polynomial of $OK_n$, $M_5(OK_n, x) = m\frac{x^{2m}}{2} + m^2[x^{(m-1)(2m-1)}] - x^{(2m^2)} + m\{x^{(2m^2)} - x^{(m-1)(2m-1)}\}$.

Proof 12: For $M_3(OK_n, x)$ which is the third Zagreb polynomial of $OK_n$, we have $\lambda(d_e, d_v) = |d_e - d_v|$, therefore $\lambda(m; m - 1) = x$ and $\lambda(m, m) = 1$. Thus by Lemma 2, $M_3(OK_n, x) = m\frac{x^{2m}}{2} + m^2(x - 1) + m\frac{1}{2} - x$. For $M_4(OK_n, x)$ which is the fourth Zagreb polynomial of $OK_n$, we have $\lambda(d_e, d_v) = d_e + d_v$, therefore $\lambda(m, m - 1) = x^{(m+2m)}$ and $\lambda(m, m) = x^{(2m^2)}$. Thus by Lemma 2, $M_4(OK_n, x) = m\frac{x^{2m}}{2} + m^2[x^{(m+2m)}] - x^{(2m^2)} + m\{x^{(2m^2)} - x^{(m-1)(2m-1)}\}$. For $M_5(OK_n, x)$ which is the fifth Zagreb polynomial of $OK_n$, we have $\lambda(d_e, d_v) = d_e + d_v$, therefore $\lambda(m, m - 1) = x^{(m-1)(2m-1)}$ and $\lambda(m, m) = x^{(2m^2)}$. Thus by Lemma 2, $M_5(OK_n, x) = m\frac{x^{2m}}{2} + m^2[x^{(m-1)(2m-1)}] - x^{(2m^2)} + m\{x^{(2m^2)} - x^{(m-1)(2m-1)}\}$.

VI. CONCLUSION

Optical Transpose Interconnection Systems (OTIS) swapped networks are optoelectronic and have been used in efficient parallel processing and services in large global networks. Topological indices are often studied with the assistance of their polynomials. Formulae for degree-based topopolical polynomials for Optical Transpose Interconnection Systems swapped network have been derived. Results can be used to compute any degree-based topological polynomials for OTIS swapped network. These results will help in the future research of networks, mechanics, computer science, and chemistry.

REFERENCES
[1] F. M. Brückler, A. Graovac, and I. Gutman, “On a class of distance-based molecular structure descriptors,” Chem. Phys. Lett., vol. 593, nos. 4–6, pp. 336–338, Feb. 2011.
[2] H. Gonzalez-Diaz, S. Villar, L. Santana, and E. Uriarte, “Medicinal chemistry and bioinformatics—current trends in drug discovery with networks topological indices,” Current Topics Medicinal Chem., vol. 7, no. 10, pp. 1015–1029, May 2007.
[3] I. Gutman, “Selected properties of the schultz molecular topological index,” J. Chem. Inf. Model., vol. 34, no. 5, pp. 1087–1089, Sep. 1994.
[4] W. Imrich and S. Klavžar, Product Graphs: Structure and Recognition. New York, NY, USA: Wiley, 2000.
[5] A. Ahmad, “On the degree based topological indices of benzene ring embedded in P-type-surface in 2D network,” Hacetette J. Math. Statist., vol. 49, no. 4, Jan. 2017.
[6] M. Baia, J. Horváthová, M. Mokriová, and A. Suhányiová, “On topological indices of a carbon nanotube network,” Can. J. Chem., vol. 93, no. 10, pp. 1157–1160, Oct. 2015.
[7] M. Baia, J. Horváthová, M. Mokriová, and A. Suhányiová, “On topological indices of fullerene,” Appl. Math. Comput., vol. 251, pp. 154–161, Jan. 2015.
[8] M. R. Farahani, “Some connectivity indices and Zagreb index of polyhex nanotubes,” Acta Chimica Slovenica, vol. 59, no. 4, pp. 779–783, 2012.
[9] A. Ahmad, “Computation of certain topological properties of para-line graph of honeycomb networks and graphene,” Discrete Math., Algorithms Appl., vol. 9, no. 05, Oct. 2017. Art. no. 1750064.
[10] S. Akhter and M. Imran, “On molecular topological properties of Benzenoid structures,” Can. J. Chem., vol. 94, no. 8, pp. 687–698, Aug. 2016.
[11] M. Ajmal, W. Nazeer, M. Munir, S. M. Kang and Y. C. Kwon, “Some algebraic polynomials and topological indices of generalized prism and toroidal polyhex networks,” Symmetry, vol. 9, pp. 1–5, Dec. 2017.
[12] M. R. Farahani, W. Gao, M. R. K. Rana, R. P. Kumar, and J.-B. Liu, “General randi, sum-connectivity, hyper-zagreb and harmonic indices, and harmonic polynomial of molecular graphs,” Adv. Phys. Chem., vol. 2016, pp. 1–6, Sep. 2016.
[13] M. I. Javid, I.-B. Liu, M. A. Rehman, and S. Wang, “On the certain topological indices of Titania Nanotube TiO2[n,m],” Zeitschrift Für Naturforschung, vol. 72, no. 7, pp. 647–654, 2017.
[14] M. I. Javid, M. U. Rehman, and J. Cao, “Topological indices of rhombus type silicate and oxide networks,” Can. J. Chem., vol. 95, no. 2, pp. 134–143, Feb. 2017.
[15] M. K. Siddiqui, M. Imran and A. Ahmad, “On Zagreb indices, Zagreb polynomials of some nanostar dendrimers,” Appl. Math. Comput., vol. 280, pp. 132–139, Apr. 2016.
[16] T. Vetrik, “Degree-based topological indices of hexagonal nanotubes,” J. Appl. Math. Comput., vol. 58, nos. 1–2, pp. 111–124, Oct. 2018.
[17] G. Hong, Z. Gu, M. Javid, H. M. Awais, and M. K. Siddiqui, “Degree-based topological invariants of metal-organic networks,” IEEE Access, vol. 8, pp. 68288–68300, 2020.
[18] C.-F. Wang and S. Sahni, “Image processing on the OTIS-mesh optoelectronic computer,” IEEE Trans. Parallel Distrib. Syst., vol. 11, no. 2, pp. 97–109, Dec. 2000.
[19] C.-F. Wang and S. Sahni, “Matrix multiplication on the OTIS-mesh optoelectronic computer,” IEEE Trans. Comput., vol. 50, no. 7, pp. 635–646, Jul. 2001.
[20] C.-F. Wang and S. Sahni, “Basic operations on the OTIS-mesh optoelectronic computer,” IEEE Trans. Parallel Distrib. Syst., vol. 9, no. 12, pp. 1226–1236, 1998.
[21] G. Marsden, P. Marchand, P. Harvey, and S. Esener, “Optical transpose interconnection system architecture,” Opt. Lett., vol. 18, pp. 1083–1085, Aug. 1993.
[22] H. Wiener, “Structural determination of paraffin boiling points,” J. Amer. Chem. Soc., vol. 69, no. 1, pp. 17–20, Jan. 1947.
[23] W. Xiao, W. Chen, M. He, W. Wei, and B. Parhami, “Biswapped networks and their topological properties,” in Proc. 8th ACIS Int. Conf. Softw. Eng., Artif. Intell., Neww., Parallel/Disrib. Comput. (SNPD), Jul. 2007, pp. 1–4.
[24] N. Zahra, M. Ibrahim, and M. K. Siddiqui, “On topological indices for swapped networks modeled by optical transpose interconnection system,” IEEE Access, vol. 8, pp. 200091–200099, 2020, doi: 10.1109/ACCESS.2020.3034443.
[25] K. Day, “Optical transpose k-ary n-cube networks,” J. Syst. Archit., vol. 50, no. 11, pp. 697–705, Nov. 2004.
[26] K. Day and A.-E. Al-Ala’yyoub, “Topological properties of OTIS-networks,” IEEE Trans. Parallel Distrib. Syst., vol. 13, no. 4, pp. 359–366, Apr. 2002.
[27] P. K. Jana, “Polynomial interpolation and polynomial root finding on OTIS-mesh,” Parallel Comput., vol. 32, no. 4, pp. 301–312, Apr. 2006.
[28] T. Vetrik, “Polynomials of degree-based indices for hexagonal nanotubes,” U.P.B. Sci. Bull., Series B, vol. 81, no. 1, pp. 109–120, 2019.
[29] A. Ahmad, “Topological properties of Sodium chloride,” U.P.B. Sci. Bull., Series B, vol. 82, no. 1, pp. 35–46, 2020.
ALI AHMAD received the M.Sc. degree in mathematics from Punjab University, Lahore, Pakistan, in 2000, the M.Phil. degree in mathematics from Bahauddin Zakariya University, Multan, Pakistan, in 2005, and the Ph.D. degree in mathematics from the Abdus Salam School of Mathematical Sciences, GC University, Lahore, in 2010. He is currently an Assistant Professor with the College of Computer Science and Information Technology, Jazan University, Saudi Arabia. His research interests include graph labeling, metric dimension, minimal doubly resolving sets, distances in graphs, and topological indices of graphs.

ROSLAN HASNI received the M.Sc. degree from Universiti Kebangsaan Malaysia (UKM), Selangor, Malaysia, and the Ph.D. degree in pure mathematics (graph theory) from Universiti Putra Malaysia (UPM), Selangor, in 2005. He was formerly attached with the School of Mathematical Sciences, Universiti Sains Malaysia (USM) Penang, Malaysia, from 2005 to 2012. He is currently an Associate Professor with Universiti Malaysia Terengganu (UMT), Kuala Terengganu, Malaysia. His research interests include chromaticity in graphs, domination graph theory, graph labeling, and chemical graph theory.

KASHIF ELAHI received the master’s degree in information and operational management from the University of the Punjab, Lahore, Pakistan, in 2005. He is currently working as a Lecturer with Jazan University, Jazan, Saudi Arabia. His research interests include algorithms, graph theory, distance in graph, software engineering, and topological indices.

MUHAMMAD AHSAN ASIM received the master’s degree from the Blekinge Institute of Technology (BTH), Sweden, and the Ph.D. degree in mathematical sciences from University Malaysia Terengganu (UMT), Malaysia, in 2018. He is currently working as a Lecturer with the Department of Information Technology and Security, Jazan University, Saudi Arabia. His research interests include graph theory, graph algorithms, and experimental mathematics.

***