Simulations of a magnetic fluctuation driven large-scale dynamo and comparison with a two-scale model

Kiwan Park* and E. G. Blackman*

Department of Physics and Astronomy, University of Rochester, Rochester, NY 14627, USA

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ABSTRACT
Models of large-scale (magnetohydrodynamic) dynamos (LSDs) which couple large-scale field growth to total magnetic helicity evolution best predict the saturation of LSDs seen in simulations. For the simplest so-called ‘α2’ LSDs in periodic boxes, the electromotive force driving LSD growth depends on the difference between the time-integrated kinetic and current helicity associated with fluctuations. When the system is helically kinetically forced (KF), the growth of the large-scale helical field is accompanied by growth of small-scale magnetic (and current) helicity which ultimately quench the LSD. Here, using both simulations and theory, we study the complementary magnetically forced (MF) case in which the system is forced with an electric field that supplies magnetic helicity. For this MF case, the kinetic helicity and turbulent diffusion terms comprise the backreaction that saturates the LSD. Simulations of both MF and KF cases can be approximately modelled with the same equations of magnetic helicity evolution, but with complementary initial conditions. A key difference between KF and MF cases is that the helical large-scale field in the MF case grows with the same sign of injected magnetic helicity, whereas the large- and small-scale magnetic helicities grow with opposite sign for the KF case. The MF case can arise even when the thermal pressure is approximately smaller than the magnetic pressure, and requires only that helical small-scale magnetic fluctuations dominate helical velocity fluctuations in LSD driving. We suggest that LSDs in accretion discs and Babcock models of the solar dynamo are actually MF LSDs.

Key words: Sun: dynamo – planets and satellites: magnetic fields – ISM: magnetic fields.

1 INTRODUCTION
Understanding the magnetohydrodynamics (MHD) of large-scale magnetic field generation in turbulent astrophysical rotators is the enterprise of large-scale dynamo (LSD) theory. The LSD problem is usually posed in kinetically forced (KF) circumstances, whereby an initially weak field is subject to relatively strong hydrodynamic forcing. LSDs then describe the growth or sustenance of fields on spatial or time-scales large compared to the largest scale of the underlying turbulent forcing. Small-scale dynamos (SSDs), in contrast, describe field growth on scales at or below the turbulent forcing scale. LSDs and SSDs are contemporaneous and interacting. Understanding that interaction and how KF LSDs saturate has been the subject of much research.

The LSDs of classic 20th century textbooks on the subject do not predict LSD saturation as they focus on the kinematic theory where the velocity flows are prescribed and the linear growth of the mean field is studied (Moffatt 1978; Parker 1979; Krause & Rädler 1980; Zeldovich 1983). A related fact is that these textbook LSDs do not conserve magnetic helicity, the inclusion of which has proven effective at improving the prediction of LSD saturation. Progress in understanding principles of KF LSD saturation has emerged from simple numerical experiments to compare with theoretical predictions of mean field models. The simplest experiments are those of the so-called α2 dynamo in which isotropically driven kinetic helicity is injected into a periodic box at intermediate wavenumber $k = 5$ and the $k = 1$ large-scale field grows (Brandenburg 2001). The saturation of the LSD in such simulations is best modelled by theories that solve for the coupled evolution of large-scale helical magnetic field and the small-scale helical magnetic field. In such models, the electromotive force (EMF, $\mathbf{E}$) of the mean field theory emerges to be the difference between kinetic helicity and current helicity, each multiplied by a corresponding correlation time. This difference was first proposed spectrally in Frisch et al. (1975) and derived and solved dynamically as a two-scale mean field theory in Blackman & Field (2002). The results of the dynamical mean field theory agree with the saturation seen in the simulation of Brandenburg (2001). Recently, the three-scale version of the theory was also compared to numerical simulations (Park & Blackman 2012). Further extensions and approaches for understanding LSDs via tracking
magnetic helicity evolution and flow for more realistic open systems that include boundary flux terms are a fruitful ongoing enterprise.

The basic insight gained from the basic periodic box KF \( \alpha^2 \) LSD studies is that as the kinetic helicity is forced and drives large-scale helical magnetic fields, the small-scale helical field grows because magnetic helicity is largely conserved. The build-up of the small-scale current helicity associated with the small-scale field then offsets the driving kinetic helicity and quenches the LSD. This physical mechanism and the equations that describe it also motivate consideration of the complementary case of driving the initial system with current helicity rather than kinetic helicity. This was investigated analytically in Blackman & Field (2004), where it was found that indeed driving with current helicity produces a magnetically forced (MF) analogue to the \( \alpha^2 \) dynamo. In this paper, we present simulations of this MF analogue to the \( \alpha^2 \) dynamo and compare the results with a two-scale mean field theory. We force the system in the induction equation with an electric field that drives small-scale magnetic helicity. Because the MF system is driven magnetically, there is no ‘kinematic’ regime in the sense that the magnetic field is strong from the outset. However, the initial velocity fluctuations are weak. Thus, the analogue of the kinematic regime for the MF \( \alpha^2 \) case is a ‘magnematic’ regime where the small-scale velocity fluctuations are small. As we will see, both simulations and theory show that it is indeed the build-up of the kinetic helicity and diffusion terms that ends the magnematic regime and quenches the MF analogue of the \( \alpha^2 \) dynamo.

There are a few previous simulations of the MF case (Brandenburg, Dobler & Subramanian 2002; Brandenburg & Subramanian 2005) but the results were not compared to a mean field theory. A more involved study of the inverse cascade of magnetic helicity for the MF was pursued in Alexakis, Mininni & Pouquet (2006) but with a focus on non-local transfer functions in the interpretation of the inverse cascade, and also without a dynamical theoretical model in terms of a mean field theory. Here we will present new simulations, and compare the simulations to mean field theory. We also address the astrophysical relevance of MF LSDs.

In Section 2, we more specifically discuss the basic problem to be solved and the computational methods. In Section 3, we present the basic results of the simulations. In Section 4, we develop the theory in more detail and discuss the comparison between theory and simulations. In Section 5, we discuss why MF LSDs are in fact of basic conceptual relevance to LSDs of accretion discs, Babcock-type stellar dynamos and magnetic relaxation of astrophysical coronae and laboratory fusion plasma configurations. We also discuss some basic open questions. We conclude in Section 6.

### 2 PROBLEM TO BE STUDIED AND METHODS

We used the high-order finite difference PENCIL CODE (Brandenburg 2001) along with the message passing interface for parallelization. The equations solved by the code are

\[
\frac{\partial \rho}{\partial t} = -\rho \nabla \cdot u, \tag{1}
\]

\[
\frac{\partial u}{\partial t} = -c_s^2 \nabla \log \rho + \frac{J \times B}{\rho} + \frac{\mu}{\rho} \left( \nabla^2 u + \frac{1}{3} \nabla \nabla \cdot u \right), \tag{2}
\]

\[
\frac{\partial A}{\partial t} = u \times B - \eta J + f. \tag{3}
\]

where \( \rho \) is the density, \( u \) is the velocity, \( B \) is the magnetic field, \( A \) is the vector potential (\( B = \nabla \times A \)), \( J \) is the current density, \( \eta = \partial / \partial t + u \cdot \nabla \) is the advective derivative, ‘\( \eta \)’ is magnetic diffusivity, ‘\( \mu \)’ is viscosity and \( c_s \) is the sound speed, and \( f \) is forcing function. The default units of the PENCIL code are ‘cgs’.

We employ a periodic box of dimensionless spatial volume \((2\pi)^3\) and a mesh size of \(216^3\). No large-scale velocity forcing is imposed. Instead, our forcing function \( f \) is placed in the magnetic induction equation and is a fully helical, Gaussian random force-free function \((\nabla \times f \propto f)\) given by \( f(x, t) = N f_x(t) \exp[i k(t) \cdot x + i\phi(t)] \), where \( N \) is a normalization factor and \( k(t) \) is the forcing wavenumber with \(|k(t)| \sim 5 \) \((5 < |k(t)| < 5.5)\), and \( |f_x| = 0.03 \) \((|f_x| = 0.07 \) was used in Park & Blackman (2012)\). This forcing function can be thought of as imposing a small-scale external electric field \( e_{\text{ext}} \) such that \( f = e_{\text{ext}} \). It is a source term for magnetic helicity for the small-scale field proportional to \( (e_{\text{ext}} \cdot B) \).

From the solutions of the equations, we compute a range of quantities including the box-averaged spectra of kinetic energy (\( E_k \)), magnetic energy (\( E_B \)), kinetic helicity (\( \langle v \cdot \nabla \times v \rangle \)) and magnetic helicity (\( \langle A \cdot B \rangle \)). From these quantities, we further construct mean quantities for comparisons with a two-scale theoretical model as discussed in subsequent sections. We note that although our system is magnetically driven, the ‘plasma beta’ (i.e. \( \beta_p \equiv \text{ratio of magnetic energy density to thermal energy density} \)) satisfies \( \beta_p \gtrsim 1 \) most of the time in our simulations.

In the main body of this paper comparing theory and simulation, we use a Prandtl number of unity, for two different values of the resistivity. In Appendix A, we discuss the effect of varying Prandtl number on the saturation of large-scale field growth in the simulations.

In some of the analysis of our equations, we need to integrate over all wavenumbers and it will be noted that in both cases we take \( k_{\text{max}} \) to be 107. For the two values of \( \eta \) that we choose, namely \( \eta = 0.0025 \) and 0.006, we find that \( k_{\text{dis}} \sim 107 \) for \( \eta = 0.006 \) and \( k_{\text{dis}} \leq k_{\text{max}} \) for \( \eta = 0.0025 \) but that very little power in any of the relevant quantities appears above \( k = 107 \) for either case. Thus, we typically used the same upper bound in \( k \) for both cases.

For the data analysis, we used Interactive Data Language (IDL).

### 3 SIMULATION RESULTS

#### 3.1 Basic results: growth of large-scale field and saturation

Figs 1(a)–2(b) show a comparison between our simulation results and two-scale theory (discussed further in Section 4) for the dimensionless large- and small-scale current helicities (\( k_1^2 h_1 \) and \( k_2^2 h_2 \), respectively), kinetic energy (\( e \)), kinetic helicity (\( h_k \)) and EMF (\( \equiv E = \langle v \times b \rangle \)). The simulation data for the large-scale field correspond to quantities at \( k_1 = 1 \) and the simulation data for the small-scale field correspond to the sum of quantities for \( 2 < k < k_{\text{max}} \). The analytic two-scale model (discussed later in more detail) takes \( k_1 = 1 \) for the large scale and the forcing scale \( k_2 = 5 \) for the small scale.

Roughly, the large-scale current helicity growth in Figs 1(a)–(c) resembles that of the current helicity studied in the KF case (Park & Blackman 2012). Unlike the KF case, the large-scale magnetic (and current) helicity has the same sign as that on the small scale for the present MF case. This is because our forcing in the induction equation injects magnetic helicity of a fixed sign into the system and magnetic helicity is conserved until resistive terms kick in. The MF \( \alpha^2 \) LSD thus acts as a dynamical relaxation process: the injection of small magnetic helicity drives the system away from equilibrium.
and the LSD competes with this injection by converting much of the small-scale injected magnetic helicity into large-scale magnetic helicity of the same sign. The magnetic energy of a fixed amount of magnetic helicity is less on large scales than on small scales and much of the energy lost in the relaxation goes into kinetic energy. Note from Figs 2(a) and (b) that a sizeable fraction of the kinetic energy gained is helical.

As we will see in the next subsection, most of kinetic helicity is on the forcing and smaller scales. The growing kinetic helicity ultimately quenches the LSD in the non-linear regime. This mechanism of quenching is supported by a comparison between curves in Figs 1(a) and (b) with curves in Figs 2(a) and (b).

In this context, note that Fig. 1(c) shows that the higher magnetic Reynolds number (equal to magnetic Reynolds in this paper) case takes longer for the large-scale field to saturate. This is analogous to the KF case where the higher magnetic Reynolds number case also takes longer to saturate. The overall saturation is less dependent on Reynolds number. This is discussed further in Appendix A, where we present simulations for a few different values of the magnetic Prandtl number \( Pr_M = \nu/\eta \) for two values of the resistivity. More detailed study of \( Pr_M \) dependence is beyond the scope of the present paper.

### 3.2 Time evolution of energy and helicity spectra

For the MF case, Figs 3(a)–4(c) show the time evolution of the magnetic helicity, magnetic energy and current helicity spectra as the simulation for the two different values of diffusivities used. Note the strong early peak at the forcing scale and the ultimate transfers to the \( k = 1 \) scale.

Figs 5(a)–6(b) show the corresponding spectral evolution of the kinetic helicity and kinetic energy. Note that the kinetic energy is less peaked than the magnetic energy at the forcing scale for this MF case and that there is some growth of kinetic energy at the large \( k = 1 \) scale. As we discuss later, this is due to the Lorentz force at \( k = 1 \) before the large-scale field becomes force free.

To study the spectral evolution of various quantities and their migration between scales, it is useful to define average wavenumbers with the different bases defined below (Davidson 2004; Park & Blackman 2012) as seen below:

\[
\langle k \rangle_A B = \frac{\int_{k_{\min}}^{k_{\max}} k A \cdot B \, dk}{\int_{k_{\min}}^{k_{\max}} A \cdot B \, dk} \approx \frac{\sum_{h_{\min}}^{h_{\max}} k A \cdot B}{\sum_{h_{\min}}^{h_{\max}} A \cdot B},
\]
Figure 3. Time evolution for the spectra of (a) magnetic helicity, (b) magnetic energy and (c) current helicity for the $\eta = 0.006$ run.

Figure 4. Time evolution for the spectra of (a) magnetic helicity, (b) magnetic energy and (c) current helicity for the $\eta = 0.0025$ run.

Figure 5. Time evolution for the spectra of (a) kinetic helicity and (b) kinetic energy for the $\eta = 0.006$ run.

Figure 6. Time evolution for the spectra of (a) kinetic helicity and (b) kinetic energy for the $\eta = 0.0025$ run.
Figure 7. Mean kinetic wavenumbers in the different kinetic bases for the two different resistivity cases for MF case ($\eta = \nu = 0.0025$). Simulations used a forcing parameter $f_0 = 0.03$. 

Figure 8. Mean kinetic wavenumbers in the different kinetic bases for the two different resistivity cases for KF case. Simulations used a forcing parameter $f_0 = 0.07$. 

\[
\langle k \rangle_{E\text{mag}} = \int_{k_i}^{k_{\max}} k B \cdot B \, dk \
\int_{k_i}^{k_{\max}} B \cdot B \, dk \simeq \sum_{k_i}^{k_{\max}} k B \cdot B \sum_{k_i}^{k_{\max}} B \cdot B ,
\]  

where $k_i$ will be taken either as $k_i = 1$ or 2 depending on whether we compute the averages for all wavenumbers or extract the large-scale $k = 1$ in order to compute small-scale averages.

Analogously, for $\langle k \rangle_{v \cdot v}$, $\langle k \rangle_{v \cdot \omega}$, and $\langle k \rangle_{\omega \cdot \omega}$,

\[
\langle k \rangle_{v \cdot v} = \int_{k_i}^{k_{\max}} k v \cdot v \, dk \
\int_{k_i}^{k_{\max}} v \cdot v \, dk \simeq \sum_{k_i}^{k_{\max}} k v \cdot v \sum_{k_i}^{k_{\max}} v \cdot v ,
\]  

\[
\langle k \rangle_{v \cdot \omega} = \int_{k_i}^{k_{\max}} k v \cdot \omega \, dk \
\int_{k_i}^{k_{\max}} v \cdot \omega \, dk \simeq \sum_{k_i}^{k_{\max}} k v \cdot \omega \sum_{k_i}^{k_{\max}} v \cdot \omega ,
\]  

and

\[
\langle k \rangle_{\omega \cdot \omega} = \int_{k_i}^{k_{\max}} k \omega \cdot \omega \, dk \
\int_{k_i}^{k_{\max}} \omega \cdot \omega \, dk \simeq \sum_{k_i}^{k_{\max}} k \omega \cdot \omega \sum_{k_i}^{k_{\max}} \omega \cdot \omega ,
\]  

The time evolution of the average wavenumbers in the kinetic bases is shown in Figs 7(a) and (b) for the MF case, which can be compared with the KF case (Park & Blackman 2012) of Figs 8(a) and (b). The time evolution of the average wavenumbers in the magnetic bases for the MF case is shown in Figs 9(a) and (b) which can be compared with KF case in Figs 10(a) and (b). Each of the figures shows two kinds of averages: the ‘total average’ for which the integration range extends from $k_i = 1$ to $k_{\max} = 107$ and the ‘small scale’ for which the integration range extends from $k_i = 2$ to $k_{\max} = 107$. To distinguish these, we use the subscript ‘tot’ to indicate averages defined by the former and ‘s’ by the latter. For example, we write $k_{J\cdot B\text{tot}}$ or $k_{J\cdot B\cdot s}$ or $k_{tot}$ and $k_{s}$ when not specifying the basis.

Using $k = 1$ for the large scale and the range $k = [2, 107]$ for the small scale, Figs 11(a) and (b) highlight the increase of small-scale magnetic energy and current helicity at early times that results from the MF case followed by the longer term growth of the large-scale field. Eventually the magnetic energy is dominated by that on the large scale. Figs 11(c) and (d) show the analogous curve for the kinetic energy and kinetic helicity. Although the kinetic energy and kinetic helicity also grow in response to the MF, the kinetic energy is dominated by small-scale contributions.

Figs 5(b) and 6(b) show that most of the kinetic energy remains on small scales during the simulations but there is some growth of large-scale kinetic energy. There is no independent inverse cascade of velocity in 3D, only in 2D (Davidson 2004), so the growth of
any large-scale velocity in our case is expected to be the direct result of the Lorentz force on those scales. Indeed, according to Figs 7(a) and (b), we can see that the average wavenumbers \( k_{2,\text{tot}} \) and \( k_{2,s} \) are unequal, highlighting that there is some energy in the large-scale \( k = 1 \) velocity. This inequality disappears as time passes and coincidence occurs approximately when the magnetic field becomes fully helical (Figs 12a, b, 13a and b). Since the magnetic field can only transfer energy to velocity fields when the Lorentz force \( J \times B \), equation (18) \( \) is finite on this scale, no energy can be transferred to the kinetic eddy once the field becomes fully helical. The evolution of the helicity fractions of both the kinetic and magnetic helicity fractions as a function of wavenumber and time are shown in Figs 12(a) and (b). The time evolution of the helicity fractions for \( k = 1 \) are shown in Fig. 13(a) and (b).

The evolution of the mean wavenumbers seen in Figs 9(a) and (b) highlights the evolution of the magnetic energy and helicity spectra in MF case. At very early times, \( k_{B,\text{tot}} = k_{B,s} = k_{A,B,\text{tot}} = k_{J,B,\text{tot}} = k_{B,2} = k_{A,B,2} = k_{A,J,2} = 5 \), i.e. \( k_{\text{tot}} \approx k_1 \) in all bases. This means the magnetic energy resides almost exclusively at the forcing scale \( k_1 \) in this early-time regime. From equation (4), the average \( k \) at this state is simply \( k A \cdot B / A \cdot B \big|_{k=k_1} \), which is \( '5' \). Since magnetic field and current helicity are helical \( J = k B = k^2 A \), the basis independent profile in the early time is explained.

By \( t \sim 50 \), the energy begins to cascade both forward (to small scales) and inversely (to large scales). For \( 50 < t < 100 \), we still have \( k_{\text{tot}} \approx k_1 \) in all bases (since all six curves of e.g. Fig. 9a continue to overlap in this time regime) but \( k_{A,B} \), \( k_{B,B} \) and \( k_{J,B} \) split. The fact that the quantities all decrease together in this regime shows that although there magnetic energy is still overwhelmingly contained in the small scales, the quantities are evolving towards larger scales but they have not crossed over to the large scale yet. The two regimes just discussed can be referred to as ‘pre-LSD I’ and ‘pre-LSD II’ regimes because there is little growth of magnetic energy at \( k = 1 \).

Beyond \( t \sim 100 \), Figs 9(a) and (b) show the influence of the LSD. Note that the evolution of \( k_s \) and \( k_{\text{tot}} \) for each quantity now deviate from each other. That is, calculating the \( k \) averages is dramatically affected by changing the lower integration bound from \( k = 1 \) to \( 2 \). By this time, \( B, v \) and EMF have grown and driven growth of the large-scale \( k = 1 \) field. Also at this time, the sign of \( \text{d}k_s / \text{d}t \) changes from negative to positive. For example, \( (k)_{A,B,s} \) of \( \eta = \nu = 0.0025 \) (Fig. 9b), transitions from negative to positive growth over the regime \( \sim 100 < t < \sim 200 \). The reason for this change is twofold: (i) the LSD takes inversely transferred magnetic helicity from the small scales to \( k = 1 \), thus dropping magnetic helicity and magnetic energy out of the bins that contribute to \( k \) (equation (11),

\[ <k> = \langle n \rangle = 0.006 \]

\[ <k> = \langle n \rangle = 0.0025 \]

\[ <k> = \langle n \rangle = 0.006 \]

\[ <k> = \langle n \rangle = 0.0025 \]

**Figure 9.** Mean magnetic wavenumbers in the different magnetic bases for the two different resistivity cases for MF case. Simulations used a forcing parameter \( f_0 = 0.03 \).

**Figure 10.** Mean magnetic wavenumbers in the different magnetic bases for the two different resistivity cases for KF case. Simulations used a forcing parameter \( f_0 = 0.07 \).
and (ii) the small helicity is continuously driven at the forcing scale $k = 5$.

The build-up of kinetic helicity eventually slows the LSD. Already around $t \sim 200$, the slopes of the curves in Figs 9(a) and (b) flatten, seemingly consistent with the time at which $(\mathbf{v} \cdot \nabla \times \mathbf{v})$ approaches saturation and approximately equals $(\mathbf{j} \cdot \mathbf{b})$. That marks the end of the kinematic regime, as discussed in the previous subsection. Eventually, the system reaches a steady state.
3.2.1 Relation between \( k_{\text{tot}} \) and \( k_x \)

We can mathematically relate \( k_{\text{tot}} \) and \( k_x \) for each basis. For example, for the case of \( k_{A,B,\text{tot}} \) and \( k_{A,B,x} \) (we let \( \sum \equiv \sum_{k=0}^{k_{\text{tot}}} \) for convenience in this subsection), we have

\[
k_{A,B,\text{tot}} = \sum_{k=0}^{k_{\text{tot}}} k \cdot a \cdot b + 1 \cdot \bar{A} \cdot \bar{B}, \quad k_{A,B,x} = \sum_{k=0}^{k_{\text{tot}}} k \cdot a \cdot b
\]

\[
\Rightarrow k_{A,B,x} \sum a \cdot b + \bar{A} \cdot \bar{B} = k_{A,B,\text{tot}} \left( \sum a \cdot b + \bar{A} \cdot \bar{B} \right)
\]

\[
\Rightarrow (k_{A,B,x} - k_{A,B,\text{tot}}) \sum a \cdot b = (k_{A,B,\text{tot}} - 1) \bar{A} \cdot \bar{B}
\]

\[
\Rightarrow \frac{\bar{A} \cdot \bar{B}}{\sum a \cdot b} = \frac{k_{A,B,x} - k_{A,B,\text{tot}}}{k_{A,B,\text{tot}} - 1}.
\]  

(10)

When \( k_{A,B,x} = k_{A,B,\text{tot}} \), the numerator on the right-hand side is 0. This means most of the magnetic helicity is in the small scale. In contrast, when \( k_{A,B,\text{tot}} \sim 1 \), most of the magnetic helicity is in the large scale. Generally, the time rate of change of the large scale over small-scale helicities is given by

\[
\frac{d}{dt} \left( \frac{\bar{A} \cdot \bar{B}}{\sum a \cdot b} \right)
\]

\[
= \frac{(k_{A,B,x} - k_{A,B,\text{tot}})(k_{A,B,\text{tot}} - 1)}{(k_{A,B,\text{tot}} - 1)^2}.
\]  

(11)

In the early-time regimes I and II, \( k_{A,B,x} = k_{A,B,\text{tot}} \) and \( k_{A,B,\text{tot}} = k_{A,B,x} \), so \( d(\bar{A} \cdot \bar{B} / \sum a \cdot b) / dt \) is zero. Once \( k_{A,B,x} \neq k_{A,B,\text{tot}} \) and \( k_{A,B,\text{tot}} < k_{A,B,x} \), this quantity becomes positive, i.e. the inverse cascade sets in.

The equations of this subsection can also be applied to \( k_{J,B,\text{tot}} \) and \( k_{J,B,x} \), and can be used to quantify the shape of the curves in e.g. Figs 9(a) and (b). For example, when \( k_{A,B,\text{tot}} < 0 \) and \( k_{A,B,x} = 0 \), the inverse cascade begins to accelerate.

4 THEORETICAL TWO-SCALE MODEL

4.1 Basic equations

The equations in this paper were modified from the theory in Blackman & Field (2004) with respect to the external force term. In that paper, the forcing was imposed such that the small-scale magnetic helicity was kept constant from \( t = 0 \). In the present case, to better match what was done in our simulations, we impose the force in the induction equation.

The modified magnetic induction equation becomes

\[
\frac{\partial \bar{B}}{\partial t} = \nabla \times (\bar{V} \times \bar{B} - \eta J + F),
\]  

(12)

where \( F \) is the forcing function. We divide the system into two scales (wavenumber \( k_1 \), or \( k_2 \)) and small scale (wavenumber \( k_2 \) or \( k_3 \)).

We use an overbar to indicate the mean and lower case letters to indicate fluctuations. We assume that the forcing applies only to the small scale so that \( F = f \) (i.e. \( \bar{F} = 0 \)), and is helical such that \( |\nabla \times f| = |k_2 f| \). We also assume the mean velocity is zero such that \( \bar{V} = \bar{v} + v = \bar{v} \). We can then write the remaining MHD quantities as \( E = \bar{E} + e, B = \bar{B} + b, J = \bar{J} + j \). The electric field \( E = -\bar{V} \times B + \eta J - F \) in large and small scales are

\[
\bar{E} = -\bar{E} + \eta \bar{J}
\]

(13)

and

\[
e = -v \times B + \bar{\varepsilon} + \eta j - f,
\]

(14)

where \( E \equiv v \times \bar{B} \).

The magnetic helicities in the large and small scale are

\[
\frac{\partial H_{\text{tot}}^L}{\partial t} = -2\langle \bar{E} \cdot \bar{B} \rangle = 2\langle \bar{E} \cdot \bar{B} \rangle - 2\eta \langle \bar{J} \cdot \bar{B} \rangle
\]

(15)

and

\[
\frac{\partial H_{\text{tot}}^N}{\partial t} = -2\langle e \cdot b \rangle \sim -2\langle \bar{E} \cdot \bar{B} \rangle + 2\eta \langle j \cdot b \rangle + 2\langle b \cdot f \rangle.
\]

(16)

The equation for \( E \cdot \bar{B} / \langle \bar{B} \rangle \) can be obtained using

\[
\frac{\partial \varepsilon}{\partial t} = \frac{\partial}{\partial t} \left( \bar{V} \times B + v \times \frac{\partial B}{\partial t} \right) \cdot \frac{\bar{B}}{\langle \bar{B} \rangle} + \bar{v} \times \frac{\partial}{\partial t} \left( \frac{\bar{B}}{\langle \bar{B} \rangle} \right),
\]

(17)

which is derived analogously to that in Blackman & Field (1999, 2002) from the small-scale momentum and magnetic induction equations given by

\[
\frac{\partial v}{\partial t} = v \times (\nabla \times v) - \nabla \times (\bar{V} \times v) - \nabla P_{\text{eff}} + \bar{J} \times \bar{B}
\]

(18)

and

\[
\frac{\partial b}{\partial t} = \bar{B} \cdot \nabla v - v \cdot \bar{B} + b \cdot \nabla v - \nabla \times \bar{E} + \lambda \nabla^2 b + k_2 f.
\]

(19)
The equation for the EMF $\mathcal{E}$ becomes

$$\frac{\partial \mathcal{E}_i}{\partial t} = \frac{1}{3} \left( b \cdot \nabla \times b - v \cdot \nabla \times v \right) \frac{\mathcal{B}^2}{|\mathcal{B}|} - \frac{1}{3} v^2 \nabla \times \mathcal{B} - \frac{2}{3} \epsilon_{ij}^{\parallel} \mathcal{E}_i + k_2 v^2 + \frac{1}{3} k_2 \nabla \times \mathcal{B} \frac{\mathcal{B}^2}{|\mathcal{B}|} \epsilon (20)$$

Here the term involving $c k_2 v^{3/2} \mathcal{E}_i$ is the result of the ‘minimal tau' closure, used in e.g. Blackman & Field (2002) that incorporates correlations ‘$T$’ as well as microscopic magnetic diffusivity ‘$\eta$’, kinetic viscosity ‘$v$’, triple and the last term (equation 6 in Blackman & Field 2002). The quantity $c$ is a constant that will later be determined empirically.

The mean kinetic energy per unit mass ($\langle v^2 \rangle / 2$) can be obtained from the momentum equation

$$\frac{\partial \langle v^2 \rangle}{\partial t} \sim 2 \langle v \cdot j \rangle + 2 \langle v \cdot \mathcal{J} \rangle + 2 \langle v \cdot \nabla \times \mathcal{B} \rangle - 2 \langle \nabla \times (v \mathcal{B}) \rangle - 2 \langle \nabla (\mathcal{B}^2) \rangle \epsilon . \quad (21)$$

Using vector identities and defining the helicity ratios (Park & Blackman 2012),

$$f_{mi} = \frac{\langle k_i A_i \cdot B_i \rangle}{|B|^2} = \frac{\langle J_i \cdot B_i \rangle}{k_i |B|^2} \quad (i = 1, 2),$$

we find the equation for $\langle v^2 \rangle$ to be

$$\frac{\partial \langle v^2 \rangle}{\partial t} \sim 2 \langle f_{m2} k_2 - f_{m1} k_1 \rangle \langle \mathcal{E} \cdot \mathcal{B} \rangle - 2 v k_2^2 \langle v \rangle . \quad (24)$$

We non-dimensionalize the equation with the scalings below:

$$H^2 = H^1 \times H^2 \Rightarrow H_{2s} \equiv H_{2s} \times \tau \equiv t k_2^2 / \sqrt{H_{2s}}, \quad Q \equiv \mathcal{E}^2 / k H_{2s}, \quad \varepsilon \equiv u / k_{01} H_{2s}, \quad R_{\text{m}} \equiv H_{2s} / \eta k^{1/2}, \quad R_{\text{m}} \equiv H_{2s} / \eta k^{1/2}. \quad (25)$$

The use of $k_2$ and rms small-scale magnetic field $h_{\text{rms}}(\equiv b_h)$ in these normalizations is natural since the external MF was applied to $k = k_2 = 5$, and the rms small-scale field is steadily injected. The ordinary differential equations to be simultaneously solved are the dimensionless versions of equations (15), (16), (20) and (21), which are given, respectively, by

$$\frac{d h_1}{d \tau} = 2 \frac{k_1}{k_2} \frac{1}{k_1 \frac{1}{2}} Q h_1^{1/2} - 2 \frac{k_1}{k_2} \frac{1}{b_r k_1}, \quad (26)$$

$$\frac{d h_2}{d \tau} = 2 \frac{k_1}{k_2} \frac{1}{k_1 \frac{1}{2}} Q h_1^{1/2} - 2 \frac{k_1}{b_r k_1} h_2^{1/2} + 2 \frac{k_1}{b_r k_1} h_2^{1/2} f', \quad (27)$$

$$\frac{d Q}{d \tau} = \frac{1}{3} f_{m1}^2 \left( h_2 - f_{m2} \epsilon \right) \frac{k_1}{k_2} h_1^{1/2} - \frac{1}{3} f_{m1} \frac{1}{k_1 k_2} \epsilon h_1^{1/2}$$

and

$$\frac{\partial \epsilon}{\partial \tau} = \frac{2}{f_{m1}^2} \left( f_{m2} - f_{m1} \right) \frac{k_1}{k_2} \frac{1}{h_1^{1/2} \epsilon} Q - \frac{2}{b_r} \frac{k_1}{h_2^{1/2} \epsilon} . \quad (29)$$

In our analytic two-scale model, we have assumed that the kinetic helicity of the fluctuations satisfies $\langle v \cdot \nabla \times v \rangle = f_{m2} k_2 \langle v^2 \rangle$, with $f_{m2}$ fixed and $k_2 = k_1$. Here the sign of the small-scale kinetic helicity is positive, which is consistent with our simulations expected when $H_{2s}^2$ is initially positive from the imposed forcing. Non-dimensionally, we have $h_2 = f_{m2} \epsilon$. Although the simulations show that $f_{m2}$ at $k = 5$ is not strictly constant at all times during the simulations, the assumption of such is not too far off. In more accurate treatments, a separate dynamical equation for $h_2$ should be included.

Note also that the term labelled 5 on the right-hand side of equation (27) resulted from $(b \cdot f)$ and was not solved for dynamically. We empirically determined the magnitude of $f'$, the magnitude of the forcing used in the theoretical equations to best match the simulations which employed a forcing magnitude $f_0$. The magnitude of forcing used in simulation $f_0 = 0.03$ is slightly larger than the values of $f'$ employed in the two-scale model (see Table 1).

The term labelled 9 in equation (28), namely $(b \times f) \cdot \mathcal{B}$, was taken to be zero: it requires a correlation between two nearly isotropic functions. This assumption is consistent with the simulations as the quantity is measured to be even smaller than $(b \times b)$, which also requires a correlation between two nearly isotropic functions.

### 4.2 Discussion of solutions

We numerically solve the ordinary set of differential equations (26)–(29) for $h_1, h_2, Q$ and $\epsilon$ (from which we obtain $h_3$). Table 1 shows the parameters defined in the previous section that best match the simulation. The dotted lines in Figs 1(a)–(c), 2(a) and (b) represent the solutions with these parameter choices.

From equation (28), for $Q$, we see that the $\alpha$ coefficient (term 6) increases at first due to $h_2$ supplied by the forcing. This drives the growth of $h_1$ of the same sign and the kinetic energy $\epsilon$ also grows. Some of this kinetic energy is helical and so $h_1$ grows. This $h_2$ grows with the same sign of $h_2$, so when the former is subtracted from the latter in $\alpha$, the overall $\alpha$ effect is reduced. Thus, $h_2$ acts as the backreaction in $\alpha$.

The associated build-up of $\epsilon$ also increases the turbulent diffusion of $Q$ (7) and the dissipation term (8). Note that the growth of the turbulent diffusion coefficient as a backreaction in the MF case differs from that of the KF where the driving itself directly supplies steady turbulent diffusion coefficient from the outset. The turbulent diffusion has no magnetic component and so must wait for the velocity field to build up. This delay in growth of the turbulent diffusion is consistent with the difference between the early-time evolution of the mean magnetic basis wavenumbers in the MF case for Figs 9(a) and (b) compared to KF case of Figs 10(a) and (b). In the latter, the KF immediately leads to a turbulent cascade available to induce scale separation in the magnetic quantities.

As the backreaction from the kinetic energy and kinetic helicity grow, $dQ/d\tau$ converges to zero and $Q$ becomes saturated at small positive value after the initial bump (Figs 2a and b). The right-hand side of equation (26) (terms 1+2) then also converges to zero and $h_1$ saturates. The behaviour of $h_2$ can be explained similarly.

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Complete saturation occurs when all of the terms on the right-hand sides of the dimensionless equations (26)–(29) equal zero. The sum of the kinetic helicity and turbulent diffusion terms for the MF case represents the backreaction that ultimately leads to the dissipation-dominated saturation regime. At early times $h_2 \gg h_1$, but by during saturation $h_1$ is within a factor of $<3$ of $h_2$ as seen in Figs 1(b) and 2(b). The late-time evolution is limited by primarily resistivity rather than viscosity (see Appendix A). Fig. 1(c) shows that the lower $R_{\text{EM}}$ case approaches saturation earlier than the larger $R_{\text{EM}}$ case.

At present, the theory we have generated does reasonably well to match the simulations at late times, but less well at early times. It is likely that what is needed is a separate evolution equation for kinetic helicity and kinetic energy; in the present case, we have assumed a simple proportionality. This proportionality really needs to be treated as a dynamical variable at early times. This is a topic for future work.

4.3 Analytic simplifications

Although equations (26)–(29) require a numerical solution, some insight can be gained by an analytic approximation. The $h_1$ equation has the form of a Bernoulli equation and can be linearized by changing variables. If we use $'u_1'(= h_1^{1/2})$, the differential equation for $h_1$ becomes

$$\frac{\partial u_1}{\partial \tau} = -\eta \frac{k_1}{b_1 k_2} u_1 + \frac{1}{f_{m_1}} \left( \frac{k_1}{k_2} \right)^{1/2} Q \equiv -P_1 u_1 + Q_1,$$

where $P_1 \equiv \frac{\eta k_1^2}{b_1 k_1^2}$, $Q_1 \equiv \frac{1}{f_{m_1}} \left( \frac{k_1}{k_2} \right)^{1/2} Q$.

(30)

The solution of this standard inhomogeneous differential equation is known. Using the same method as in Park & Blackman (2012), we find

$$u_1(t) = e^{-\frac{t}{P_1}} P_1 \left[ \int_0^t e^{\frac{t}{P_1}} P_1 \, Q_1(t') \, dt' + \text{constant} \right]$$

$$= e^{-\frac{t}{P_1}} \left[ \int_0^t \left( e^{\frac{t}{P_1}} \right)^{\frac{Q_1(t)}{P_1}} \, dt' + \text{constant} \right]$$

$$\sim \frac{Q_1(t)}{P_1} + \left( \text{constant} - \frac{Q_1(0)}{P_1} \right) e^{\frac{t}{P_1}}.$$

(31)

As $t \rightarrow \infty$, $u_1(t)(= h_1^{1/2})$ approaches $Q_1(t)/P_1$, which can be calculated by setting $dh_1/d\tau$ to be zero. This approximate value $Q_1(t)/P_1$ matches the numerically solved $h_1^{1/2}$ after $t \sim 1500$, for $\eta = v = 0.006$.

The equation for $h_2^{1/2}$ cannot be linearized. Instead, setting $dh_2/d\tau = 0$ and $u_2 \equiv h_2^{1/2}$, we obtain a quadratic equation:

$$u_2(\tau) = \frac{1}{2} \left[ \frac{f'}{\eta k_2 b_1 f_{m_2}^{1/2}} - \left( \frac{f'}{\eta k_2 b_1 f_{m_2}^{1/2}} \right)^2 - \frac{4b_1 Q_1 h_1^{1/2}}{\eta f_{m_2}^{1/2}} \left( \frac{k_1}{k_2} \right)^{1/2} \right].$$

(32)

Except the early-time regime ($t \lesssim 130$, $\eta = v = 0.006$), this solution fits the numerical result well. These approximate equations explain how the saturated magnetic helicities in large and small scale depend on $'\eta'$.

5 ASTROPHYSICAL RELEVANCE OF MAGNETICALLY FORCED LARGE-SCALE DYNAMOS

The MF LSD that we have discussed in this paper maintains a plasma $\beta_p$ greater than unity for most of the time. Although we consider the simplest such MF LSD in that we employ periodic boundaries and no rotation or shear, the basic concept of an LSD that is driven by small-scale magnetic fluctuation rather than kinetic fluctuation in $\beta_p > 1$ environments has important conceptual relevance to both solar dynamos and dynamos in accretion disks.

In the solar context, helical LSD models have been separated into two classes: (1) helical forcing is primarily kinetic helicity driven by thermal convection and (2) flux transport models where helical fields are the result of magnetic instabilities (cf. Charbonneau 2007). We suggest that the second class of dynamos can be viewed as an MF LSD, albeit with more complexity than the simple version we consider in the present paper. Magnetic instability driven dynamos in the radiative zone (Spruit 2002) also seem to fit into this category.

For accretion discs, LSDs are now commonly seen in shearing box simulations (e.g. Brandenburg et al. 1995; Davis, Stone & Pessah 2010; Gressel 2010; Kápyá & Korpi 2011). The magnetorotational instability operating in these simulations produces turbulence that drives an LSD. But the magnetic fluctuations typically exceed the kinetic fluctuations and it seems that the best agreement between mean field theory and simulation requires that the driver of the LSD growth is not the kinetic helicity term but the current helicity term (Gressel 2010). Thus, accretion discs are another environment where more general $\beta_p > 1$ MF LSD dynamos operate.

Although the calculations of the present paper are in the $\beta_p \gg 1$ limit, we note that for the $\beta_p \ll 1$ limit, MF LSDs have long been studied in the laboratory context (e.g. Ji & Prager 2002). The direct analogy to these low $\beta$ MF LSDs may occur in astrophysical coronae (Blackman 2007).

6 CONCLUSION

We performed numerical simulations of the analogue of an $\alpha^2$ LSD in a periodic box when the system is driven with magnetic (or current) helicity rather than kinetic helicity. The simulations indeed show that LSD action results from the injection of small-scale magnetic helicity, analogously to the KF LSD action from injection of kinetic helicity. This analogy is expected because the growth driver is proportional to the time integral difference between current helicity and kinetic helicity.

We compared the simulation results with a two-scale theory and found general consistency with respect to the basic mechanism of
large-scale field growth and saturation. When the system is magnetically driven at $k = 5$, the large-scale $k = 1$ helical magnetic field grows in the MF with the same sign as that on the driving scale. Injecting $k = 5$ magnetic helicity drives the system away from its natural relaxed state and the LSD evolves the magnetic helicity to the large scale where the same amount of magnetic helicity has lower energy. Eventually, the LSD saturates because the growth driver is a difference between current helicity and kinetic helicity + diffusion terms (the sum of the latter terms being the backreaction that causes the quenching). This situation complements the KF case where the LSD drives oppositely signed growth of large- and small-scale magnetic helicities and the current helicity quenches the LSD.

We presented the time evolution of the spectra of magnetic energy, magnetic helicity and current helicity from the simulations. Taken together, these spectra exhibit the expected inverse cascade of magnetic helicity that is at the heart of LSD action, and the absence of an inverse cascade of kinetic helicity. Future work is needed to better understand the early-time growth regime and more comprehensively study the dependence on magnetic Prandtl number.

The MF LSD studied herein was for MHD plasmas with ratios of thermal-to-magnetic pressure overall larger than unity. We checked that this was the case at all times during the simulations. We discussed that such an MF LSD may ultimately be involved in producing the LSD action in MRI simulations and may also be away to distinguish Babcock-type solar dynamo models from KF kinetic helicity driven solar dynamo models.

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APPENDIX A: \((J \cdot B)\) AND \((\nu \cdot \omega)\) FOR \(\Pr_{M} = 1\)

In the bulk of this paper, we have focused on the MF case for magnetic Prandtl number $\Pr_M = 1$. Our results for that case are consistent with the interpretation that saturation of the MF LSD occurs as the combination of the kinetic helicity and the turbulent diffusion terms grows and together acts as the backreaction against the current helicity driving term in the growth equation for $h_1$. To determine whether the evolution to saturation depends more on the magnetic Reynolds number or the hydrodynamic Reynolds number, we performed four additional simulations corresponding to $\Pr_M < 1$ and $\Pr_M > 1$ each for two values of resistivity $\eta$. The results are shown in Figs A1(a)–(d).

In contrast to the plots in the main body of the paper, in these plots we have left the plotted quantities in dimensional cgs units. The figures show that the asymptotic saturation curves are more sensitive to the magnetic Reynolds number than the hydrodynamic Reynolds number. There is a very mild sensitivity to the magnetic Prandtl number in the sense that the larger the magnetic Prandtl number, the larger the saturation value of the large-scale current helicity and the smaller the saturation value of the small-scale kinetic helicity. This sensitivity is very weak.

The weak sensitivity to $\Pr_M$ arises because of a competition between two effects. An increased viscosity $\nu$ ($\sim \Pr_M$) for fixed $\Rey$ somewhat suppresses the kinetic helicity because the velocity is subject to extra damping. This is evident in Fig. A1(d). The turbulent diffusion term in $\mathcal{E}_f$ (or equivalently, the term 7 in equation 28) is also suppressed. Both of these two velocity terms act as a backreaction to the current helicity driving term (the $h_1$ term in equation 28, or dimensionally, $H^2_{\text{M}}$) supplied by the forcing. If $H^2_{\text{M}}$ were kept exactly constant by the forcing for different $\Pr_M$ runs, then this reduction of the velocity backreaction terms would be the strongly dominant effect of increasing $\Pr_M$ and a much larger growth of $H^2_{\text{M}}$ would result in the higher $\Pr_M$ case. However, because we solve dynamically the coupled equations for $H^2_{\text{M}}$ and $H^2_{\text{H}}$, and keep the magnitude of the forcing in the induction equation the same for all runs, $H^2_{\text{M}}$ evolves slightly differently for the different $\Pr_M$ cases. In particular, a faster growth of $h_1$ that might arise by reducing the backreaction from velocity terms is weakened because this faster growth of $H^2_{\text{M}}$ means a faster depletion of $H^2_{\text{H}}$ (as evidenced by equations 26 and 27) and this feedback effect whereby $H^2_{\text{H}}$ slightly decreases in cohort with a reduced velocity backreaction leaves the difference between the current helicity growth driver $k^2 H^2_{\text{M}}$ and the backreaction terms relatively constant. This explains the weak dependence on $\Pr_M$ of the saturation curves.

The above explanation is highlighted by the relatively modest deviations in the difference between the small-scale current helicity and kinetic helicity for the various cases shown in Figs A1(a)–(d).

The weak dependence on $\Pr_M$ shown here contrasts that which results from the equations in the analytic study of Blackman & Field (2004). There, instead of using a forcing function in the induction equation, $h_1$ was enforced to be a constant in the analogous-driven MF case studied therein. There the equation for $h_1$ dropped out of the evolution for the MF-driven case. In our present paper, we would have to adjust our forcing function between runs to mimic that circumstance. Thus, for the driven subcase of Blackman & Field (2004), only the first effect discussed in the previous paragraph is present – namely the decrease in velocity backreaction terms, leading to a more dramatic increase in saturation value of $H^2_{\text{M}}$.
Figure A1. Plots showing the effect of varying $Pr_M$ on various quantities for two values of resistivity and two values of $Pr_M$. (a) Large-scale current helicity; (b) and (c) small-scale current helicity; (d) kinetic helicity for all four cases. In these simulations, the mesh for $\eta = \nu = 0.006$ is $216^3$, and that for the other cases is $256^3$. $Re_M$ is 48 ($\eta = \nu = 0.006$), 53 ($\eta = 0.006, \nu = 0.0025$), 128 ($\eta = 0.0025, \nu = 0.006$) and 141 ($\eta = \nu = 0.0025$). In panel (d), a solid line is used for $Pr_M = 1$.

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