Analytical model of the single emitter laser with modulated coupling constant

R Khachatrian and E Popov
Peter the Great St. Petersburg Polytechnic University, 195251, St. Petersburg, Russia
E-mail: ab0baiih@gmail.com

Abstract. Using the master equation for the density operator of the laser field, we constructed an analytic model of a single emitter laser under a strong atom-field coupling in the non-stationary regime, that is obtained by coupling constant modulation at the defined frequency. The model demonstrated the emergence of a harmonic force, that partially compensates for relaxation processes in the quantum system. The mechanism of force occurrence, is described by changing the coupling constant.

1. Introduction
Correct description of the single emitter laser is an important problem of the quantum optics [1,2] and a famous fundamental task. Any two-level system can be considered as single emitter. The stationary theory of a single emitter laser has been studied for a long time. There were a lot of experimental and mathematical researches in the works [3-11], but the dynamics of single emitter laser are not fully covered. In this work we suggest to develop the theory for a non-stationary regime of generation, which obtained by modulation of the field-atom coupling constant. The motivation of this study is observation of the transient processes in a single emitter laser, during which the degree of light squeezing is greatly increased. The task would be to obtain solution by two methods: numerically and analytically. The first approach can give primary information about dynamics of generation, but don’t allow to explain it. Therefore we aim to use the second approach, that can give us the simple mathematical dependencies for qualitative understanding of the generating mechanisms under modulation of the coupling constant in the non-stationary mode.

Modulation of the atom-field coupling constant can be implemented by different methods. One of them is forced oscillations of the resonator mirrors. A qualitative estimate shows that for observing the effect considered in the work, the oscillation frequency of mirrors less than 10^8 Hz is sufficient, and such a frequency can be achieved, for example, using piezoelectric manipulators. Another way is to excite forced oscillations of a single emitter in an optical lattice, the potential well of which is governed by the parameters of the confining rays.

2. Mathematical model
A single emitter laser is described by the equation [2]

\[ \frac{\partial \rho}{\partial t} = g(\hat{a}^+ \hat{\sigma} \hat{\rho} - \hat{\sigma}^+ \hat{a} \hat{\rho} - \hat{\rho} \hat{a}^+ \hat{\sigma} + \hat{\rho} \hat{\sigma}^+ \hat{a}) + \frac{\kappa}{2}(2\hat{a} \hat{\rho} \hat{a}^+ - \hat{a}^+ \hat{\rho} \hat{a} - \hat{\rho} \hat{a}^+ \hat{a}) + \frac{\gamma}{2}(2\hat{\sigma} \hat{\rho} \hat{\sigma}^+ - \hat{\sigma}^+ \hat{\rho} \hat{\sigma} - \hat{\rho} \hat{\sigma}^+ \hat{\sigma}) + \frac{\Gamma}{2}(2\hat{\sigma}^+ \hat{\rho} \hat{\sigma} - \hat{\sigma}^+ \hat{\sigma} \hat{\rho} - \hat{\rho} \hat{\sigma}^+ \hat{\sigma}), \]  

(1)
where $\Gamma$ is the incoherent pumping rate from the lower level $|a>$ to the upper level $|b>$; $\frac{2}{\gamma}$ is the decay rate due to spontaneous emission; $\frac{2}{\gamma}$ is the field decay rate in the cavity; $g$ is the field-atom coupling constant, $\rho$ is the atom-field density matrix of quantum system, $a$ and $a^+$ are photon annihilation and photon creation operators, $\sigma$ and $\sigma^+$ are atomic polarization operators.

Equations on matrix elements look like this:

$$\dot{x}_n = g\sqrt{n}(w_{n-1} + y_{n-1}) + \kappa(n + 1)x_{n+1} - (\kappa n + \Gamma)x_n + \gamma z_n$$

$$\dot{y}_n = g\sqrt{n+1}(z_n - x_{n+1}) + \kappa\sqrt{(n + 1)(n + 2)y_{n+1}} - \frac{\kappa(2n+1) + \Gamma + \gamma}{2} y_n$$

$$\dot{w}_n = g\sqrt{n+1}(z_n - x_{n+1}) + \kappa\sqrt{(n + 1)(n + 2)w_{n+1}} - \frac{\kappa(2n+1) + \Gamma + \gamma}{2} w_n$$

$$\dot{z}_n = -g\sqrt{n+1}(w_n + y_n) + \kappa(n + 1)z_{n+1} - (\kappa n + \gamma)z_n + \Gamma x_n.$$  \hspace{1cm} (5)

In this system of equations, nonzero elements of density matrix are $\rho_{nn}^{aa}$, $\rho_{nn}^{bb}$, $\rho_{nn+1}^{ab}$, $\rho_{n+1n}^{ba}$, $n$ is the number of photons in the resonator. The notation is introduced $\rho_{nn}^{aa} = x_n$, $\rho_{nn}^{bb} = z_n$, $\rho_{n+1n}^{ab} = y_n$, $\rho_{n+1n}^{ba} = w_n$, with $y_n^* = w_n$; where $x_n$ is probability that emitter is in the lower state $|a>$ and there are $n$ photons in the cavity; $z_n$ is probability that emitter is in the upper state $|b>$ and there are $n$ photons in the cavity; $y_n$ and $w_n$ are coherences between the states $x_{n+1}$ and $z_n$. The system can be simplified by making the replacement $b_n = w_n + y_n$, $c_n = w_n - y_n$, then instead of the equations (3) and (4) we get

$$\dot{b}_n = 2g\sqrt{n+1}(z_n - x_{n+1}) + \kappa\sqrt{(n + 1)(n + 2)b_{n+1}} - \frac{\kappa(2n+1) + \Gamma + \gamma}{2} b_n$$

$$\dot{c}_n = 0.$$ \hspace{1cm} (7)

So $c_n$ is determined simply from the initial conditions, and remains the system of equations (2), (6) and (5).

As suggested earlier we will modulate the coupling constant:

$$g(t) = g_0 + g_1 \cos(\Omega t) \hspace{0.5cm} g_0 \gg g_1, \Gamma, \kappa;$$  \hspace{1cm} (8)

The condition of smallness for $g_1$ allows us to get some approximations:

$$x_n \gg x_{n+1} \hspace{0.5cm} b_n \gg b_{n+1} \hspace{0.5cm} z_n \gg z_{n+1}.  \hspace{1cm} (9)$$

The degree of light squeezing is determined by the Mandel Q-parameter

$$Q = \frac{\langle n^2 \rangle - \langle n \rangle^2}{\langle n \rangle^2} - 1 \hspace{1cm} (10)$$

If Q-parameter less than zero, the light becomes squeezed. In this system there is little excitation, so we can restrict ourselves to considering possible only 0, 1, 2, 3 photon states, thus, only $x_0, x_1, x_2, x_3, z_0, z_1, z_2$ will be involved into calculation of average values.

We will find the solution of the system (2), (6), (5) by the perturbation method, representing $x_n(t)$, $z_n(t)$, $b_n(t)$ as iterative series,

$$x_n(t) = \sum_{i=0}^{\infty} x_n^{(i)}(t) \hspace{1cm} (11)$$

$$b_n(t) = \sum_{i=0}^{\infty} b_n^{(i)}(t) \hspace{1cm} (12)$$

$$z_n(t) = \sum_{i=0}^{\infty} z_n^{(i)}(t), \hspace{1cm} (13)$$
where \( x^{(0)}_n, b^{(0)}_n, z^{(0)}_n \) solutions of a stationary system that are independent of time, and \( x^{(i)}_n \gg x^{(i+1)}_n \), similarly for \( b^{(i)}_n \) and \( z^{(i)}_n \). Substitution of these expressions into equations (2), (6), (5) with consideration of conditions (8) and (9), gives us equation:

\[
\begin{align*}
\dot{b}^{(1)}_n &= 2g_0\sqrt{n + 1}(z^{(1)}_n - x^{(1)}_{n+1}) - \frac{\kappa(2n+1) + \Gamma + \gamma}{2} b^{(1)}_n + \\
&\quad + 2g_1\sqrt{n + 1}(z^{(0)}_n - x^{(0)}_{n+1}) \cos(\Omega t) \\
\dot{z}^{(1)}_n &= -g_0\sqrt{n + 1}b^{(1)}_n - g_1\sqrt{n + 1}b^{(0)}_n \cos(\Omega t) \\
\dot{x}^{(1)}_{n+1} &= g_0\sqrt{n + 1}b^{(1)}_n + g_1\sqrt{n + 1}b^{(0)}_n \cos(\Omega t) \\
\dot{x}^{(1)}_0 &= 0.
\end{align*}
\]

The second term on the right side is left in the equation (14), since the field relaxation, pumping and spontaneous decay strongly influence the coherence between the levels, and in the future this term will give us the width of the resonance curve. The infinite set of equations decomposes into an infinite set of three-equation systems. In all following calculations, we will omit the superscript ”(1)” and understand by ”\( a^{(1)}_n \)” , unless otherwise specified. Simplifying the system of equations (16), (14), (15), we arrive at non-stationary equations:

\[
\begin{align*}
x^{(i+1)}_n &= -z^{(i)}_n \\
b^{(i)}_n &= -\frac{\dot{z}^{(i)}_n}{g_0\sqrt{n + 1}} - \frac{g_1b^{(0)}_n}{2g_0} \cos(\Omega t) \\
\ddot{z}^{(i)}_n + 2\beta_n \dot{z}^{(i)}_n + \Omega^2 z^{(i)}_n &= \Omega G_n \sin(\Omega t) + F_n \cos(\Omega t),
\end{align*}
\]

where \( \beta_n = \frac{1}{4}(\kappa(2n+1) + \Gamma + \gamma) \), \( \Omega_n = 2g_0\sqrt{n + 1} \), \( G_n = g_1\sqrt{n + 1}b^{(0)}_n \), \( F_n = -2g_1\sqrt{n + 1}(\beta_n b^{(0)}_n + g_0\sqrt{n + 1}(z^{(0)}_n - x^{(0)}_{n+1})) \). From the equation (20) we get:

\[
\dot{z}^{(i)}_n = \frac{F_n(\Omega^2_n - \Omega^2) - 2\beta_n \Omega^2 G_n}{(\Omega^2_n - \Omega^2)^2 + 4\beta^2_n \Omega^2} \cos(\Omega t) + \frac{\Omega G_n(\Omega^2_n - \Omega^2) - 2\beta_n \Omega F_n}{(\Omega^2_n - \Omega^2)^2 + 4\beta^2_n \Omega^2} \sin(\Omega t).
\]

We proceed to the calculation of the Mandel parameter. To begin with, we will write general formulas, and only then we restrict the basis.

\[
\begin{align*}
\langle n \rangle &= \sum_{n=1}^{\infty} n(x^{(0)}_n + x^{(1)}_n + z^{(0)}_n + z^{(1)}_n) = \sum_{n=1}^{\infty} n(x^{(0)}_n + z^{(0)}_n) + \\
&\quad + \sum_{n=1}^{\infty} n(x^{(1)}_n + z^{(1)}_n) = \langle n_0 \rangle + \langle n_1 \rangle \\
\langle n^2 \rangle &= \sum_{n=1}^{\infty} n^2(x^{(0)}_n + x^{(1)}_n + z^{(0)}_n + z^{(1)}_n) = \sum_{n=1}^{\infty} n^2(x^{(0)}_n + z^{(0)}_n) + \\
&\quad + \sum_{n=1}^{\infty} n^2(x^{(1)}_n + z^{(1)}_n) = \langle n_0^2 \rangle + \langle n_1^2 \rangle.
\end{align*}
\]

From to the equation (18), expressions for \( \langle n_1 \rangle \) and \( \langle n_1^2 \rangle \) become

\[
\begin{align*}
\langle n_1 \rangle &= \sum_{n=1}^{\infty} n(x^{(1)}_n + z^{(1)}_n) = \sum_{n=1}^{\infty} x^{(1)}_n \\
\langle n_1^2 \rangle &= \sum_{n=1}^{\infty} n^2(x^{(1)}_n + z^{(1)}_n) = \sum_{n=1}^{\infty} (2n - 1)x^{(1)}_n.
\end{align*}
\]
In our case

\[
\langle n_1 \rangle = x_1^{(1)} + x_2^{(1)} + x_3^{(1)}
\]

\[
\langle n_1^2 \rangle = x_1^{(1)} + 3x_2^{(1)} + 5x_3^{(1)}.
\]

We write out the formula for the correction to the Mandel parameter [1], taking into account only the first-order corrections

\[
Q^{(1)} = \frac{\langle n_1^2 \rangle - \langle n_1 \rangle}{\langle n_0 \rangle} - 2 < n_1 > - Q^{(0)} \frac{\langle n_1 \rangle}{\langle n_0 \rangle}
\]

3. Analysis of results

Notice the formulas (18-20), which describes the dynamics of the first-order corrections to the solution for the density matrix elements. Their form corresponds to a damped oscillator, that is excited by an external harmonic force. The solution of such equations is well known. From equation (20) it follows that the modulation of the coupling constant \( g \) leads to the appearance of some effective external force, which leads to steady-state oscillations of the probabilities \( \rho_{n+1,n+1}^{aa}, \rho_{n,n}^{bb} \), as well as coherence between these states \( \rho_{n,n+1}^{ba} \). The force is proportional to the modulation amplitude of the coupling constant \( g_1 \), as shown in the comments to formula (20). The frequency of free oscillations is determined by the number of photons. It suggests the existence of several resonances for different numbers of photons in the resonator.

In order to verify the reliability of the proposed analytical model, we compared it with numerical calculation. The numerical calculation was carried out on the basis of the general equation (1). Notice limited basis of the number of photons \( n = 30 \) was the single approximation in the calculation. The comparison is shown in Figure 1. It can be concluded that the analytical model is reliable and allows us to explain the nature of the parametric resonance at the modulation frequency of the coupling constant \( \Omega = 2g_0 \).

![Figure 1](image-url)

**Figure 1.** Minimal value of the Q-parameter for the period of coupling constant modulation. Parameters of calculation: \( \Gamma/g_0 = 0.08, \kappa/g_0 = 0.2, \gamma/g_0 = 0.01, g_1/g_0 = 0.1. \) Solid line – analytical function (28), dashed line – numerical calculation.

Using the suggested analytic model we can explain appearance of series of the picks in Figure 1. From formula (21) it follows that parametric resonance can be obtained at the some Rabi frequencies \( \Omega_n = 2g_0 \sqrt{n+1}. \) It leads to the idea about effective control of the quantum statistic of the single emitter laser by several harmonics modulations in (8).
4. Conclusion
In this paper, we proposed a simple analytical model of squeezed light generation by single emitter laser in non-stationary mode. Within the framework of the model, the linear dependence of the first-order correction to the elements of the density matrix on the amplitude of the modulation of the coupling constant was explained. Numerical calculations showed the reliability of the analytical model and the physical adequacy of the approximations made.

Acknowledgment
This work was supported by the Russian Foundation for Basic Research (grants 18-32-00250).

References
[1] Mu Y and Savage C 1992 Phys. Rev. A. 46 5944.
[2] Larionov N and Kolobov M 2013 Phys. Rev. A. 88 013843
[3] McKeever J, Boca A and Boozer A, Buck J and Kimble H 2003 Phys. Rev. A 425 268
[4] Nomura N, Kumagai N, Iwamoto S, Ota Y and Arakawa Y 2009 Opt. Express 17 15975
[5] Dubin F, Russo C, Barros H, Stute A, Becher C, Schmidt P and Blatt R 2010 Nature Physics 6 350
[6] Kozlovski A and Oraevski A 1999 JETP 115 1210
[7] Karlovich T and Kilin S 2001 Opt. Spectr. 91 343-351
[8] Kilin S and Karlovich T 2002 JETP 122 933-949
[9] Karlovich T and Kilin S 2007 Opt. Spectr. 103 260
[10] Karlovich T and Kilin S 2008 Laser Physics 18 783
[11] Popov E and Larionov N 2016 Proc. SPIE 9917 1-6