AN APPLICATION TO THE STUDY OF POLYNOMIAL AUTOMORPHISMS VIA A NONUNIFORM GLOBAL STABILITY PROBLEM

ÁLVARO CASTAÑEDA, IGNACIO HUERTA, AND GONZALO ROBLEDO

Abstract. Firstly, we introduce a nonautonomous nonuniform Markus-Yamabe Conjecture, namely, a global problem on nonuniform asymptotic stability for nonautonomous differential systems, whose restriction to the autonomous case is related to the classical Markus–Yamabe Conjecture. Secondly, we propose a couple of definitions of injectivity for a parametrized family of maps and study its link with the above stated nonautonomous conjecture. This relation allow us to study a particular family of parametrized polynomial automorphisms and to prove that they have polynomial inverse for certain parameters, which is reminiscent to the Jacobian Conjecture.

1. INTRODUCTION

1.1. Preliminaries. The Markus-Yamabe Conjecture (MYC) is a problem of global asymptotic stability for continuous autonomous dynamical systems on finite dimension, introduced in 1960 by L. Markus and H. Yamabe [23], which states that if the ODE system

\[ \dot{x} = f(x) \]  \hspace{1cm} (1)

where \( f : \mathbb{R}^n \to \mathbb{R}^n \) of class \( C^1 \), \( f(0) = 0 \) and is a Hurwitz vector field, that is, the eigenvalues of the Jacobian matrix of \( f \) have negative real part at any \( x \in \mathbb{R}^n \), or equivalently \( Jf(x) \) is a Hurwitz matrix for any \( x \), then the origin is globally asymptotically stable. A careful reading of the hypotheses and the statement of the conjecture show two features: i) the global and nonlinear stability of (1) is deduced via the study of the stability for a family of linear systems \( \dot{z} = Jf(x)z \) for any \( x \in \mathbb{R}^n \) and ii) the stability is in fact the global uniform asymptotical stability. Additionally, this conjecture was proved to be false for \( n \geq 3 \) by A. Cima et al. in [7] and is true for \( n \leq 2 \), the planar case was verified independently by C. Gutiérrez [16], R. Fessler [13] and A. A. Glutsyuk [14].

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1.2. **Global nonuniform stability problem.** The main idea of this article is to settle a global nonuniform stability problem for a nonlinear nonautonomous system

\[ \dot{x} = f(t, x). \]  

While the autonomous dynamical systems depend only on the time elapsed from the initial time \( t_0 \), the nonautonomous dynamical systems are also dependent on the initial time \( t_0 \) itself, which has several consequences to characterize limiting objects. As it was pointed out in [21, Ch.2] and [22, Sec.2.2], the dynamics arising from nonautonomous ODE (2) can be formally described by two approaches where the above mentioned \( t_0 \) plays a key role: the skew product semiflows and the process formalism, also known as the two parameter \((- (t, t_0))\)-semigroups.

Let us recall that the **MYC** is stated in terms of the negativeness of the real part of the eigenvalues of \( Jf(x) \) and the global attractiveness of the origin has a behavior described by the uniform asymptotic stability. Contrarily, it is known that the local stability of a linear nonautonomous system cannot be always determined by the eigenvalues, see e.g. [23, p.310]. However, we point out about the existence of several spectral theories based either on characteristic exponents (Lyapunov, Perron and Bohl exponents) or dichotomies [12], which have associated a wide range of asymptotic stabilities for nonautonomous linear systems, being the uniform asymptotic stability only a particular case.

In this article we will work with the global nonuniform asymptotic stability (GNUAS) to settle a nonautonomous and nonuniform Markus–Yamabe conjecture (NNMYC) and prove it for \( n = 1 \) (Theorem 1). Notice that, in the linear case, the GNUAS is consequence of the nonuniform exponential stability, which can be described in terms of the nonuniform exponential dichotomy. We emphasize that this property of dichotomy is associated to a spectral theory, which will allow us to emulate the notion of Hurwitz vector fields to the nonautonomous framework.

1.3. **Injectiveness of a parametrized family of vector fields.** A remarkable result from Van den Essen [11, p.177] states that if **MYC** would be true, then the vector field \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is injective. By using this result, G. Fournier and M. Martelli (see [11, p.175] and [24] for a detailed description) proved that if **MYC** would be true for polynomial vector fields \( f(x) = x + H(x) \) in any dimension, where \( H \) is an homogeneous polynomial with degree \( \leq 3 \) and \( JH \) is nilpotent, then \( f(x) \) has a polynomial inverse and the Jacobian Conjecture would be true. Let us recall that the Jacobian Conjecture states that if a polynomial function \( P : \mathbb{K}^n \rightarrow \mathbb{K}^n \) (where \( \mathbb{K} \) is a field with characteristic zero) has Jacobian determinant which is a non-zero constant, then \( P \) has a polynomial inverse. This conjecture was introduced by O.H. Keller in 1939 [19] and it is still open even in dimension two.

In spite that the approach of Fournier and Martelli has a basis problem since **MYC** is not true, the general idea is a good example about how to
address a conjecture by proving its equivalence or its implication to another one. In this context, a second goal of this article will be to enquire about the following problem: If we assume that NNMYC is true, what can be said about the injectiveness of the family of maps associated to (2)?

Note that $f(t, x)$ can be seen as a family of maps $x \mapsto F_t(x) = f(t, x)$ parametrized by $t$. We point out that there exist several ways to define injectivity for this family. In this article we will propose two notions: partial injectivity and $\sigma$–uniform injectivity. In addition, our second result (Theorem 2) proves that if NNMYC is true for any dimension, then $x \mapsto F_t(x)$ is partially injective. Roughly speaking, the map $F_t$ is injective for a set of parameters $t$.

1.4. An application to the study of polynomial automorphisms. Our last result (Theorem 3) is concerned with systems (2) of the form $f(t, x) = x + H(t, x)$, where the coordinates of $x \mapsto H(t, x)$ are homogeneous polynomials map of degree 3 for any $t \geq 0$, while the Jacobian $JH(t, x)$ is nilpotent for any $t \geq 0$ and verifies a smallness condition for bigger values of this parameter. We prove that if NNMYC is verified for this class of systems in any dimension, then Theorem 2 implies that the family $x \mapsto x + H(t, x)$ will be partially injective. At this point, for any $t \geq 0$ such that $x \mapsto x + H(t, x)$ is injective in the classical sense, it will be proved that the Jacobian Conjecture is satisfied for these maps.

We emphasize that for any fixed $t$, the previous maps are the key tool of the reduction theorems obtained by H. Bass et al. [3] and A.V. Yagzhev [29]. These results in [3, 29] establish that, when addressing the Jacobian Conjecture, it is sufficient to verify it on those maps, for all dimension $n \geq 1$.

Theorem 2 can be seen in the spirit of Fourier and Martelli approach in the sense that intend to address a conjecture by proving another one. Nevertheless, our result is partial since we only proved that the Jacobian Conjecture is verified for a subset of parameters $t$.

Structure of the article. The section 2 provides a short review about the theory of nonuniform asymptotic stability for nonlinear and linear nonautonomous systems. The linear case is related to the nonuniform exponential dichotomy, which allows to construct a spectral theory for these systems. The nonuniform nonautonomous Markus–Yamabe Conjecture encompasses all these nonautonomous tools and is formally stated in terms of the above mentioned spectrum and nonuniform stability. The section 3 introduces two notions of injectivity for a parametrized family of maps, namely, partial injectivity and $\sigma$–uniform injectivity, also provides illustrative examples for both concepts and explain why the partial injectivity is related with the nonautonomous nonuniform conjecture. It is proved that if a differential system satisfies the nonuniform nonautonomous Markus–Yamabe Conjecture, then its related family of maps parametrized by $t$ is partially injective. The section 4 is an application of the results obtained in the previous sections to
the study of a family of polynomial automorphisms. In fact, we show a vector field satisfying the conditions and conclusion of the nonuniform nonautonomous Markus–Yamabe Conjecture whose corresponding parametrized vector field has explicit polynomial inverse.

**Notations.** In this paper $M_n(\mathbb{R})$ is the set of $n \times n$ matrices over $\mathbb{R}$, $I_n$ is the identity matrix and we will use $\text{Diag}\{\lambda\}$ to denote $\lambda I_n$, namely, the diagonal matrix with terms $\lambda$. The matrix norm induced by the euclidean vector norm $|\cdot|$ will be denoted by $||\cdot||$. Finally, we consider $\mathbb{R}^+ = [0, +\infty)$.

### 2. Nonuniform Nonautonomous Markus-Yamabe Conjecture

In this section we establish a nonuniform version of the nonautonomous Markus–Yamabe Conjecture whose uniform version was introduced in [5]. This new conjecture has the same structure that the previously mentioned reference but it will be fashioned along the nonuniform exponential dichotomy (NUED) instead the uniform one, namely, the problem will be stated in terms of a spectral theory arising from NUED, which allow an alternative characterization of the global nonuniform asymptotic stability for linear systems. For this purpose, we will recall the property of GNUAS and its relation with the NUED.

We emphasize that in [9, p.540] is stated that the nonuniform exponential dichotomy is admitted by any linear system with nonzero Lyapunov exponents, moreover in [1, Proposition 2.3] it is showed a example of a linear nonautonomous system that admits this dichotomy but not the uniform one.

#### 2.1. Nonuniform asymptotic stability of nonlinear systems.

Let us consider the nonlinear system

$$\dot{x} = g(t, x),$$

where $g: \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is such that the existence, uniqueness and unbounded forward continuability of the solutions is ensured. The solution of \( \text{(3)} \) with initial condition $x_0$ at $t_0$ will be denoted by $x(t, t_0, x_0)$.

It will be assumed that the origin is an equilibrium, that is, $g(t, 0) = 0$ for any $t \geq 0$. The stability of the origin in \( \text{(3)} \) will be addressed with the comparison functions [20]:

- A function $\alpha: \mathbb{R}^+ \to \mathbb{R}^+$ is a $K$ function if $\alpha(0) = 0$ and it is nondecreasing.
- A function $\alpha: \mathbb{R}^+ \to \mathbb{R}^+$ is a $K_\infty$ function if $\alpha(0) = 0$, $\alpha(t) \to +\infty$ as $t \to +\infty$ and it is strictly increasing.
- A function $\alpha: \mathbb{R}^+ \to (0, +\infty)$ is a $N$ function if it is nondecreasing.
- A function $\alpha(t, s): \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a $K\mathcal{L}$ function if $\alpha(t, \cdot) \in K$ and $\alpha(\cdot, s)$ is decreasing with respect to $s$ and $\lim_{s \to +\infty} \alpha(t, s) = 0$.

Now we define the type of asymptotic stability, which will be the central focus of this work.
Definition 1. The equilibrium \( x = 0 \) of (3) is globally nonuniformly asymptotically stable if, for any \( \eta > 0 \), there exists a \( \delta(t_0, \eta) > 0 \) such that
\[
|x_0| < \delta(t_0, \eta) \Rightarrow |x(t, t_0, x_0)| < \eta \quad \forall t \geq t_0
\]
and for any \( x_0 \in \mathbb{R}^n \) it follows that \( \lim_{t \to +\infty} x(t, t_0, x_0) = 0 \).

The comparison functions allow an alternative characterization for global nonuniform asymptotic stability.

Proposition 1. [18, Prop. 2.5] The origin \( x = 0 \) of (3) is globally nonuniformly asymptotically stable if and only if there exists \( \beta \in \mathcal{KL} \) and \( \theta \in \mathbb{N} \) such that for any \( x(t_0) \in \mathbb{R}^n \) it follows that
\[
|x(t, t_0, x_0)| \leq \beta(\theta(t_0)|x_0|, t - t_0) \quad \forall t \geq t_0.
\]

Remark 1. Observe that:

i) Definition 1 considers initial conditions \( x_0 \) inside a ball having radius dependent of the initial time \( t_0 \). If \( \delta \) is not dependent of \( t_0 \), it is said (see e.g. [20]) that \( x = 0 \) is globally uniformly asymptotically stable.

ii) If \( \theta(\cdot) \equiv 1 \) in Proposition 1 we also recover the characterization of the global uniform asymptotic stability by comparison functions.

iii) In the vast majority of the literature, Definition 1 is referred as global asymptotic stability instead of GNUAS.

2.2. Nonuniform exponential stability. Let us consider the nonautonomous linear system
\[
\dot{x} = A(t)x,
\]
where \( x \in \mathbb{R}^n \), \( A: \mathbb{R}^+ \to M_n(\mathbb{R}) \) is a locally integrable matrix function. A basis of solutions or fundamental matrix of (5) is denoted by \( \Phi(t) \), which satisfies \( \dot{\Phi}(t) = A(t)\Phi(t) \) and its corresponding transition matrix is \( \Phi(t, s) = \Phi(t)\Phi^{-1}(s) \), then the solution of (5) with initial condition \( x_0 \) at \( t_0 \) verifies \( x(t, t_0, x_0) = \Phi(t, t_0)x_0 \).

Definition 2. ([1], [9], [30]) The system (3) has a nonuniform exponential dichotomy (NUED) on a subinterval \( J \subseteq \mathbb{R}^+ \) if there exist a projector \( P(\cdot) \), constants \( K \geq 1 \), \( \alpha > 0 \) and \( \varepsilon \in [0, \alpha) \) such that for any \( t, s \in J \) we have
\[
\begin{align*}
P(t)\Phi(t, s) &= \Phi(t, s)P(s), \\
\|\Phi(t, s)P(s)\| &\leq Ke^{-\alpha(t-s)+\varepsilon s}, \quad t \geq s, \\
\|\Phi(t, s)(I_n - P(s))\| &\leq Ke^{-\alpha(s-t)+\varepsilon s}, \quad t \leq s.
\end{align*}
\]

Remark 2. The above definition deserves a few comments:

i) A consequence of the first equation of (7) is that \( \dim \ker P(t) = \dim \ker P(s) \) for all \( t, s \in J \); this motives that, in the literature, the projector \( P(\cdot) \) is known as invariant projector.

ii) If \( \varepsilon = 0 \), we recover the classical exponential dichotomy, also called uniform exponential dichotomy [21, 28].
A special case of the nonuniform exponential dichotomy is given when the projector is the identity and deserves special attention.

**Definition 3.** The linear system (5) is nonuniformly exponentially stable if and only if there exist constants \( K \geq 1, \alpha > 0 \) and \( \varepsilon \in [0, \alpha) \) such that
\[
\|\Phi(t, s)\| \leq Ke^{-\alpha(t-s) + \varepsilon s} \text{ for any } t \geq s, \text{ with } t, s \in J = [T, +\infty).
\]

Moreover, if (5) is nonuniformly exponentially stable, then this property can be preserved for any perturbed system
\[
\dot{x} = [A(t) + B(t)]x,
\]
where \( B \) is small in a sense that will be described in the next result:

**Proposition 2.** [2, Th. 1] If we assume that the system (5) is nonuniformly exponentially stable in \([T, +\infty)\) and \(\|B(t)\| \leq \delta e^{-\varepsilon t} \text{ for } t \in [T, +\infty), \) with \( \delta < \frac{\alpha}{K} \). Then the system (7) is nonuniformly exponentially stable, i.e.,
\[
\|\Phi_{A+B}(t, s)\| \leq Ke^{-(\alpha-\delta K)(t-s)+\varepsilon s} \text{ for any } t \geq s, \text{ with } t, s \in [T, +\infty).
\]

The above Definition and Proposition have been stated considering an interval \([T, +\infty)\). In the next result we will see that the nonuniform exponential stability can be extended to \(\mathbb{R}^+\).

**Lemma 1.** If the system
\[
\dot{x} = C(t)x
\]
has nonuniform exponential dichotomy with projector \( P(\cdot) \) on \([T, +\infty)\), then it also has nonuniform exponential dichotomy in \(\mathbb{R}^+\) with the same projector.

**Proof.** We denote \( \Phi_C(t, s) \) as the transition matrix of the system (8). If this system admits nonuniform exponential dichotomy on \([T, +\infty)\), then we have the following estimate for the projector \( P(\cdot) \):
\[
\|\Phi_C(t, s)P(s)\| \leq Ke^{-\alpha(t-s) + \varepsilon s} \text{, with } t \geq s \geq T.
\]

In order to complete the proof, we will consider two cases for the parameters \( t, s \), namely, \( 0 \leq s \leq t \leq T \) and \( 0 \leq s \leq T \leq t \). For the first case, due that the transition matrix and the projector are continuous, we have that
\[
\|\Phi_C(t, s)P(s)\| \leq L = Le^{-\alpha(t-s) + \varepsilon s}e^{\alpha(t-s)-\varepsilon s} \leq Le^{\alpha T}e^{-\alpha(t-s)+\varepsilon s}
\]
and for the second one, we use the hypothesis and properties of the transition matrix:
\[
\|\Phi_C(t, s)P(s)\| = \|\Phi_C(t, T)\Phi_C(T, s)P(s)\| \leq \|\Phi_C(t, T)\|\|\Phi_C(T, s)P(s)\| \leq LKe^{-\alpha(t-T)+\varepsilon T}.
\]

In summary, if \( 0 \leq s \leq t \) we prove that
\[
\|\Phi_C(t, s)P(s)\| \leq LKe^{\alpha T}e^{-\alpha(t-s)+\varepsilon s},
\]
and this same reasoning will make it possible to show the estimation associated with the projector \( I - P(\cdot) \) and \( 0 \leq t \leq s \). \( \Box \)
The previous Lemma extends a previous result made by W. Coppel [8, p.13] in the uniform context. A direct consequence of the above Lemma is that we can extend the interval \([T, +\infty)\) to \(\mathbb{R}^+\) in the Definition 3 and also in Proposition 2 without considering additional conditions for \(B(t)\) on the interval \([0, T]\).

It is also important to emphasize that the nonuniform exponential stability is also a particular case of GNUAS as states the following result.

**Lemma 2.** If the linear system (5) is nonuniformly exponentially stable then it is globally nonuniformly asymptotically stable.

**Proof.** As the linear system is nonuniformly exponentially stable, that is

\[ |x(t, t_0, x(t_0))| = |\Phi(t, t_0)x(t_0)| \leq Ke^{\varepsilon t_0}e^{-\alpha(t-t_0)}|x(t_0)|, \]

clearly the inequality (4) is verified with the functions \(\theta(t_0) = e^{\varepsilon t_0}\) and \(\beta(r, t - t_0) = K r e^{-\alpha(t-t_0)}\) and the result follows from Proposition 1. □

**Remark 3.** Let us recall that in the uniform framework, namely, when \(\varepsilon = 0\), the Lemma 2 also has a converse statement and there exists an equivalence between uniform exponential stability and global uniform asymptotical stability. We refer to [20, pp.156–157] for details.

**Remark 4.** The NUED on \(\mathbb{R}^+\) with a non trivial projector \(P(\cdot)\) implies that any non zero solution \(t \mapsto x(t, t_0, \xi) = \Phi(t, t_0)\xi\) can be splitted into

\[ t \mapsto x^+(t, t_0, \xi) = \Phi(t, t_0)P(t_0)\xi \quad \text{and} \quad t \mapsto x^-(t, t_0, \xi) = \Phi(t, t_0)[I-P(t_0)]\xi \]

such that \(x^+(t, t_0, \xi)\) converges nonuniformly exponentially to the origin when \(t \to +\infty\) while \(x^-(t, t_0, \xi)\) has a nonuniform exponential growth. This asymptotic behavior justifies the name of nonuniform exponential dichotomy.

2.3. The nonuniform exponential dichotomy spectrum.

**Definition 4.** ([9], [30]) The nonuniform spectrum (also called nonuniform exponential dichotomy spectrum) of (7) is the set \(\Sigma(A)\) of \(\lambda \in \mathbb{R}\) such that the system

\[ \dot{x} = [A(t) - \lambda I_n]x \quad (9) \]

does not have a NUED on \(\mathbb{R}^+\) stated in Definition 2.

**Remark 5.** The construction of \(\Sigma(A)\) has been carried out by J. Chu et al. [9] and X. Zhang [30] by emulating the work developed by S. Siegmund [28] in order to provide a friendly and simple presentation of the uniform exponential dichotomy spectrum, which backs to the seminal work of R.J. Sacker and G. Sell [27].

The following result establishes simple conditions ensuring that \(\Sigma(A)\) is a non empty and compact set.
Proposition 3. ([9], [21], [28], [30]). If the transition matrix $\Phi(t, s)$ of (7) has a nonuniformly bounded growth, namely, there exist constants $K_0 \geq 1$, $a \geq 0$ and $\bar{\varepsilon} \geq 0$ such that
$$
\|\Phi(t, s)\| \leq K_0 e^{a|t-s|+\bar{\varepsilon}s}, \quad t, s \in \mathbb{R}^+,
$$
its nonuniform spectrum $\Sigma(A)$ is the union of $m$ compact intervals where $0 < m \leq n$, that is,
$$
\Sigma(A) = \bigcup_{i=1}^{m} [a_i, b_i],
$$
with $-\infty < a_1 \leq b_1 < \ldots < a_m \leq b_m < +\infty$.

The intervals $\rho_1(A) = (-\infty, a_1)$, $\rho_{m+1}(A) = (b_m, +\infty)$ and $\rho_{i+1}(A) = (b_i, a_{i+1})$ for $i = 1, \ldots, m-1$ are called spectral gaps. By Definition 3, for any $\lambda \in \rho_j(A)$, the system (9) has a nonuniform exponential dichotomy with $P_j := P_j(\cdot)$. It can be proved, see e.g. [9], that $P_1 = 0$, $P_{m+1} = I_n$ and
$$
\dim \ker P_i < \dim \ker P_{i+1}
$$
for any $i = 1, \ldots, m$.

which allows an alternative characterization of the nonuniform exponential stability of the system (5) in terms of $\Sigma(A)$:

Lemma 3. The system (5) is nonuniformly exponentially stable if and only if $\Sigma(A) \subset (-\infty, 0)$.

Proof. If $\Sigma(A) \subset (-\infty, 0)$, it follows that $0 \in \rho_{m+1}(A)$ and [9] with $\lambda = 0$ coincides with (5), which has a nonuniform exponential dichotomy with projector $P_{m+1} = I_n$ and the nonuniform exponential stability follows from Definition 3. If (5) is nonuniform exponentially stable, it follows by Definition 3 that (5) has a nonuniformly exponential dichotomy with the identity as projector, which is equivalent to $\lambda = 0 \in \rho_{m+1}(A) = (b_m, +\infty)$ and implies that $\Sigma(A) \subset (-\infty, 0)$. \hfill $\square$

2.4. Statement of the conjecture. As we have set forth the premises now we are able to state our main result of this section.

Conjecture 1 (Nonuniform Nonautonomous Markus–Yamabe Conjecture (NNMYC)). Let us consider the nonlinear system
$$
\dot{x} = f(t, x)
$$
where $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$. If $f$ satisfies the following conditions

(G1) $f$ is continuous in $\mathbb{R}^+ \times \mathbb{R}^n$ and $C^1$ with respect to $x$. Moreover, $f$ is such that the forward solutions are defined in $[t_0, +\infty)$ for any $t_0 \geq 0$.

(G2) $f(t, x) = 0$ if $x = 0$ for all $t \geq 0$.

(G3) For any bounded piecewise continuous function $t \mapsto \omega(t)$, the linear system
$$
\dot{\varrho} = Jf(t, \omega(t))\varrho,
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$$
\dot{\varrho} = Jf(t, \omega(t))\varrho,
$$
where \( Jf(t, \cdot) \) is the jacobian matrix of \( f(t, \cdot) \), has a nonuniform exponential dichotomy spectrum satisfying

\[
\Sigma(Jf(t, \omega(t))) \subset (-\infty, 0).
\]

(G4) For any piecewise continuous function \( t \mapsto \omega(t) \), each one of linear systems of the family (11) has a transition matrix associated \( \Phi_{\omega(t)}(t, s) \) with nonuniformly bounded growth.

Then the trivial solution of the nonlinear system (10) is globally nonuniformly asymptotically stable.

The properties (G1) and (G4) are essentially technical. In fact, (G1) implies the existence, uniqueness and infinite forward continuability of the solutions while (G4) and Proposition 3 imply the compactness of the spectra stated in (G3). On the other hand, (G2) and (G3) emulates the classical Markus–Yamabe conjecture since (G2) recalls that the origin is an equilibrium while (G3) combined with Lemma 3 say that the linearization of the vector field \( x \mapsto f(t, x) \) around any bounded piecewise continuous function \( \omega(t) \) is nonuniform exponentially stable and mimics the property of Hurwitz vector fields stated in the conjecture.

It is important to emphasize that the restriction of (10) to the autonomous case is not equivalent to MYC, this would be the case only if (G3) considers constant functions instead of bounded piecewise continuous ones. On the other hand MYC and NNMYC are formulated in terms of spectra which are not coincident. Some examples of systems \( \dot{x} = A(t)x \) verifying \( \Sigma(A) \subset (-\infty, 0) \) and having eigenvalues with positive real part are shown in [17, p.158].

Remark 6. The assumptions (G2) and (G3) have subtle differences with those stated on [5], where it was assumed that \( x = 0 \) is the unique equilibrium of (10) and \( t \mapsto \omega(t) \) was considered only a measurable function. A careful reading of the next section and the next result will show us that we can weaken our assumption about equilibrium while demanding stronger conditions for \( t \mapsto \omega(t) \).

In order to show that NNMYC is well posed, we prove that it is true for scalar equations.

Theorem 1. The NNMYC is verified for dimension \( n = 1 \).

Proof. Let us consider the nonlinear scalar equation

\[
\dot{x} = g(t, x),
\]

with the initial condition \( x_0 \neq 0 \) at time \( t = t_0 \). Without loss of generality, it will be supposed that \( x_0 > 0 \). By uniqueness of the solution it follows that \( x(t) > 0 \) for any \( t \geq t_0 \).
Notice that any solution of (12) can be written as follows

\[ g(t, x(t)) = \frac{\partial g}{\partial x}(t, \omega(t))x(t) \quad \text{with} \quad 0 < \omega(t) < x(t) \]

\[ \dot{x}(t) = \frac{\partial g}{\partial x}(t, \omega(t))x(t) \quad \text{with} \quad 0 < \omega(t) < x(t). \]

For any fixed \( t > 0 \), the existence of \( \omega(t) \) is ensured by the mean value Theorem. In addition, by using the Implicit function Theorem, it can be proved that \( \omega(t) \) is continuous in a neighborhood of \( t \) and to conclude that \( t \mapsto \omega(t) \) is piecewise continuous. Then we have that

\[ \int_{t_0}^{t} \dot{x}(\tau) \, d\tau = \int_{t_0}^{t} \frac{\partial g}{\partial x}(\tau, \omega(\tau))x(\tau) \, d\tau \]

\[ x(t) = x_0 + \int_{t_0}^{t} \frac{\partial g}{\partial x}(\tau, \omega(\tau))x(\tau) \, d\tau, \]

where \( 0 < \omega(\tau) < x(\tau) \). The above inequality is equivalent to

\[ \frac{d}{dt} \ln \left( x_0 + \int_{t_0}^{t} \frac{\partial g}{\partial x}(\tau, \omega(\tau))x(\tau) \, d\tau \right) = \frac{\partial g}{\partial x}(t, \omega(t)) \quad \text{with} \quad 0 < \omega(\tau) < x(t). \]

We integrate between \( t_0 \) and \( t \) obtaining that

\[ x(t) = x_0 \exp \left( \int_{t_0}^{t} \frac{\partial g}{\partial x}(\tau, \omega(\tau)) \, d\tau \right) \quad \text{with} \quad 0 < \omega(\tau) < x(\tau), \]

that is, the solutions of (12) can be seen as solutions of the linear equation

\[ \dot{z} = \frac{\partial g}{\partial x}(t, \omega(t))z, \quad (13) \]

for some bounded piecewise continuous function \( t \mapsto \omega(t) \). Now, the assumption

\[ \Sigma \left( \frac{\partial g}{\partial x}(t, \omega(t)) \right) \subset (-\infty, 0) \quad (14) \]

and the nonuniform asymptotic stability is a consequence of Lemma 3.

\[ \square \]

**Remark 7.** The condition \((G4)\) is established as consequence of \((14)\) which is equivalent to the system \((13)\) has nonuniform dichotomy with projector \( P(\cdot) = I_1 \).

3. **Nonautonomous injectivies and Markus–Yamabe Conjecture**

In this section we intend to explore the consequences of the NNM\( Y\) on the injectivity properties of the family of maps \( x \mapsto F_1(x) := f(t, x) \) associated to (10). Our interest is motivated by the Fournier and Martelli
result, which stated that if MYC would be true for certain autonomous differential systems, then the Jacobian Conjecture is true.

As the injectivity is a property for a single map, we will address the injectiveness for a family of maps parametrized by $t$ as above, by proposing the following definitions:

**Definition 5.** A family of maps $F_t : \mathbb{K}^n \rightarrow \mathbb{K}^n$ is:

i) Partially injective if

\[ [(\exists \tau \geq 0) (\forall t \geq \tau) (F_t(x) = F_t(y))] \Rightarrow (x = y), \]

ii) $\sigma$–uniformly injective if

\[ (\exists \sigma \geq 0) (\forall t \geq \sigma) (F_t(x) = F_t(y) \Rightarrow x = y). \]

**Remark 8.** The partial injectivity stated in Definition 5 deserves some comments:

a) An equivalent expression for (i) in Definition 5 is the following:

\[ (x \neq y) \Rightarrow [(\forall \tau \geq 0) (\exists t_{\tau} \geq \tau) (F_{t_{\tau}}(x) \neq F_{t_{\tau}}(y))], \]

that means there exists an unbounded function $\tau \mapsto t_{\tau}$ of real numbers such that $F_{t_{\tau}}(\cdot)$ is injective.

b) If $t \mapsto F_t(x)$ is continuous for any $x$, it can be proved that for any $t_{\tau}$, there exists $\delta_{\tau} > 0$ such that $F_t(\cdot)$ is injective for all $t \in (t_{\tau} - \delta_{\tau}, t_{\tau} + \delta_{\tau})$.

c) The partial injectivity of the family $F_t$ suggest to introduce the non empty set

\[ \mathcal{PI}(F) := \{ t \geq 0 : F_t \text{ is injective} \}. \]

**Remark 9.** An equivalent expression for $\sigma$–uniformly injectivity is the following:

\[ (\exists \sigma \geq 0) (\forall t \geq \sigma) (x \neq y \Rightarrow F_t(x) \neq F_t(y)), \]

that means, there exists an interval $J = [\sigma, \infty)$ such that $F_t(\cdot)$ is injective for any $t \in J$. We can see that $J \subset \mathcal{PI}(F)$ due to the possible existence of $0 \leq \ell < \sigma$ such that $F_\ell(\cdot)$ is injective.

**Remark 10.** The relation between previous definitions is an elusive problem. However, when the family $F_t$ is $0$–uniformly injective then it is partially injective. Note that in this case we have $J = [0, +\infty) = \mathcal{PI}(F)$.

In order to illustrate the above defined injectivities for the family $F_t$ and its relations, we will consider the following examples:

**Example 1.** The family $F_t : \mathbb{K}^3 \rightarrow \mathbb{K}^3$ given by

\[ F_t(x, y, z) = (-x + e^{-t}(x + y)^3, -y + e^{-t}[x + z]^3 - (x + y)^3, -z - e^{-t}(x + y)^3) \]

is $0$–uniformly injective. In fact, we can find an explicit the inverse for this map for each $t \geq 0$. Namely, $F_t^{-1}(x, y, z) = (G_1, G_2, G_3)_t(x, y, z)$ where
\[
G_{1t} = -x - e^{-t}(x+y)^3(1+e^{-t}(x+y)^2)^3 \\
G_{2t} = -y - e^{-t}(x+y)^3 - (x+y)^3(1+e^{-t}(x+y)^2)^3 \\
G_{3t} = -z + e^{-t}(x+y)^3(1+e^{-t}(x+y)^2)^3.
\]

The above example is inspired in the classification of the nilpotent maps achieved in [6, Theorem 1].

**Example 2.** The family \( F_t : \mathbb{R}^n \to \mathbb{R}^n \) given by \( F_t(x) = f(t)g(x) \), where \( f : \mathbb{R}^+ \to \mathbb{R} \) is a polynomial of degree \( k \) and \( g : \mathbb{R}^n \to \mathbb{R}^n \) is an injective function (in the usual sense), is \( \sigma \)-uniformly injective where \( \sigma = \max \{ t \in \mathbb{R}^+ : f(t) = 0 \} \). In addition, it is easy to see that this family is partially injective. Nevertheless and contrarily to the previous example, the partial injectivity is not a consequence of the \( \sigma \)-uniform one, due to the existence of \( s < \sigma \) such that the map \( F_s \) is injective in the usual sense. In addition, note that the set
\[ \mathcal{P}J(F) = \{ t \in \mathbb{R}^+ : f(t) \neq 0 \} \]
has a finite complement.

**Example 3.** Given \( \lambda_0 \) and \( a \) such that \( \lambda_0 < a < 0 \), the family of maps \( F_t : \mathbb{R} \to \mathbb{R} \) defined by \( F_t(x) = [\lambda_0 + at \sin(t)]x \) is partially injective due to the set
\[ \mathcal{P}J(F) = \{ t \in \mathbb{R}^+ : \lambda_0 + at \sin(t) \neq 0 \} \]
has a countable complement described by
\[ \{ t \in \mathbb{R}^+ : \lambda_0 + at \sin(t) = 0 \}, \]
which is upperly unbounded and its elements are isolated points. On the other hand, the family cannot be \( \sigma \)-uniformly injective for none \( \sigma \). In fact, given \( \sigma \geq 0 \), we can choose
\[ t_\sigma = \min \{ t > \sigma : \lambda_0 + at \sin(t) = 0 \} \]
and \( F_{t_\sigma}(x) = F_{t_\sigma}(y) \) is verified for any \( x, y \in \mathbb{R}^n \) with \( x \neq y \).

Now we will introduce a Weak Markus-Yamabe Conjecture in a nonautonomous context, which will allow us to connect the Nonuniform Nonautonomous Markus-Yamabe Conjecture and the Jacobian Conjecture for a parametrized family of maps. Let us recall that the Jacobian Conjecture is stated for a single map \( F \) while in our framework we revisit it in terms of a family \( F_t \).

**Conjecture 2 (Nonautonomous Weak Markus–Yamabe Conjecture).** Let \( f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n \) as in the equation (10), which satisfies (G1), (G3) and (G4), then the family of maps \( t \mapsto F_t(x) = f(t,x) \) is partially injective.

The following proposition relates the Nonuniform Nonautonomous Markus-Yamabe Conjecture and Weak Markus-Yamabe Conjecture.

**Theorem 2.** If NNMYC is satisfied then the Nonautonomous Weak Markus-Yamabe Conjecture is true.
Proof. The proof will be made by contradiction by assuming that the family of maps \( F_t(x) := f(t, x) \) satisfies (G1)–(G4) but not verify the definition of partial injectivity: The negation of the statement a) of Remark 8 is
\[
(x \neq y) \land \left( \exists \tau \geq 0 \right) \left( \forall t \geq \tau \right) (F_t(x) = F_t(y)),
\]
which allows to construct an auxiliary map \( G : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n \) defined by
\[
G(t, z) := G_t(z) = F_t(z + x) - F_t(x).
\]
Notice that \( G(t, 0) = G_t(0) = 0 \) for any \( t \geq 0 \) and we can verify that the differential system
\[
\dot{z} = G(t, z)
\]
(15)
satisfies (G1), (G3) and (G4) as a consequence of the nonlinear system \( \dot{z} = f(t, x) \) also satisfies them, thus the Nonuniform Nonautonomous Markus-Yamabe Conjecture assures that the origin is globally nonuniformly asymptotically stable for (15).

On the other hand, note that the initial value problem
\[
\begin{cases}
\dot{z} = G(t, z) \\
z(\tau) = z_0
\end{cases}
\]
with \( z_0 = y - x \), has a constant solution \( z(t, \tau, z_0) = y - x \neq 0 \) for all \( t \geq \tau \), which does not converge to 0 when \( t \to \infty \), therefore we obtain a contradiction. Finally the family of maps \( F_t \) is partially injective and therefore the Nonautonomous weak Markus–Yamabe Conjecture follows. \( \square \)

Remark 11. The proof of the Theorem 2 is inspired by \cite{11, p.177}. That is, in an autonomous context is proved that if a \( C^1 \) vector field satisfies the hypothesis of the MYC then the vector field is injective. As stated in the introduction, MYC is true when \( n = 2 \), which was proved independently by C. Gutiérrez \cite{16}, R. Feßler \cite{13} and A. A. Glutsyuk \cite{14}; who used the fact that hypothesis of this problem, in dimension two, is equivalent to the map is injective in the usual sense (see \cite{25}).

Remark 12. Note that the map \( F_t \) studied in the Example 3 is associated to the differential equation
\[
\dot{x} = [\lambda_0 + at \sin(t)]x,
\]
which is a well known case of nonuniform asymptotic stability and satisfies the conditions of NNMYC. As \( F_t \) is not \( \sigma \)-uniformly injective, this shows that the NNMYC cannot implies this type of injectivity.

4. An application to the study of polynomial automorphisms

This section introduces an application of the NNMYC to the study of the partial injectiveness for the following family of polynomial maps parametrized by \( t \), defined as \( M : \mathbb{R}^+ \times \mathbb{C}^n \to \mathbb{C}^n \) with
\[
(t, x) \mapsto M(t, x) = M_t(x) = (M_1(t, x), \ldots, M_n(t, x)),
\]
\[
= (\lambda x_1 + H_1(t, x), \ldots, \lambda x_n + H_n(t, x)),
\]
(16)
where $\lambda < 0$ and $x \mapsto M_t(x)$ is a polynomial for any fixed $t$ such that:

(i) $M$ is continuous with respect to $t$.

(ii) for all $t \geq 0$ fixed, $JH_t(x)$ is nilpotent and $(H_i)_t$ is zero or homogeneous of degree 3 for $i = 1, \ldots, n$.

In the autonomous case, the above family plays a key role in the study of the Jacobian Conjecture. In fact, H. Bass et al. [3] and A.V. Yagzhev [29] proved that it is sufficient to focus on maps, for all dimension $n \geq 1$, having the form $X + H$ where $X = (x_1, \ldots, x_n)$ and $H$ is a homogeneous polynomial of degree 3 and its Jacobian $JH$ is nilpotent, in order to prove the Jacobian Conjecture; this approach was improved by M. de Bondt and A. van den Essen in [4] who show that it is sufficient to investigate the Jacobian conjecture for all maps of the form $(x_1 + f x_1, \ldots, x_n + f x_n)$ where $f$ is a homogeneous polynomial of degree 4, $f_{x_i}$ denotes the partial derivatives of $f$ with respect to $x_i$ and $n \geq 1$.

An interesting property of the family of maps $x \mapsto H(t, x)$ is given by the following result:

**Lemma 4.** For any family $x \mapsto H(t, x)$ satisfying the property (ii) there exists a continuous function $a(t)$ and a positive constant $C$ such that the Euclidean norm of $H(t, x)$ verifies:

$$||H(t, x)|| \leq C a(t) ||x||^3$$

(17)

*Proof.* Note that

$$||H(t, x_1, \ldots, x_n)|| = \sqrt{\sum_{i=1}^{n} H_i^2(t, x_1, \ldots, x_n)},$$

then (17) is equivalent to

$$\sum_{i=1}^{n} H_i^2(t, x_1, \ldots, x_n) \leq C^2 a^2(t) (x_1^2 + \cdots + x_n^2)^3.$$

The result follows if we prove that for any homogeneous polynomial $H_i(t, x_1, \ldots, x_n)$ of degree 3, there exist $D_i > 0$ and $\alpha_i(t)$ such that

$$H_i^2(t, x_1, \ldots, x_n) \leq D_i \alpha_i(t) (x_1^2 + \cdots + x_n^2)^3.$$  

(18)

By using the identity

$$(x_1^2 + \cdots + x_n^2)^3 = \sum_{i=1}^{n} x_i^6 + 3 \sum_{i=1}^{n} \sum_{j=1}^{i-1} x_i^4 x_j^2 + 3 \sum_{i=1}^{n} \sum_{j=1}^{i-1} x_i^2 x_j^4 + 6 \sum_{i=1}^{n} \sum_{j=1}^{i-1} \sum_{k=1}^{j-1} x_i^2 x_j^2 x_k^2,$$

we can see that (18) will be verified if the indeterminates of $H_i^2(t, x_1, \ldots, x_n)$ are always bounded by expressions as $x_i^6$, $x_i^4 x_j^2$, $x_i^2 x_j^4$ or $x_i^2 x_j^2 x_k^2$. 


As the polynomial $H_\ell(t, x_1, \ldots, x_n)$ has the representation

$$H_\ell(t, x_1, \ldots, x_n) = \sum_{i=1}^{n} \alpha_i(t)x_i^3 + \sum_{i=1}^{n} \left[ \sum_{j=1, j\neq i}^{n} \alpha_{ij}(t)x_i^2x_j \right] + \sum_{i\neq j\neq k}^{n} \alpha_{ijk}(t)x_ix_jx_k,$$

we can see that its square contains indeterminates which are present in explicit representation of $(x_1^3 + \cdots + x_n^3)^3$ as: $x_1^6$, $x_1^3x_2^3$, $x_1^3x_3^3$ and $x_1^2x_2^2x_3^2$.

We will see that indeterminates of $H_\ell^2(t, x_1, \ldots, x_n)$ which are not present $(x_1^3 + \cdots + x_n^3)^3$ can be upperly bounded by indeterminates which are present: note that by Young’s inequality we have that

$$x_1^3x_3^3 \leq \frac{1}{2}(x_1^6 + x_3^6), \quad x_1^4x_jx_k \leq \frac{1}{2}(x_1^2x_3^2 + x_4^2x_6^2), \quad x_1^3x_2^3x_k \leq \frac{1}{2}(x_1^6 + x_5^4x_6^2)$$

and

$$x_1^5x_j \leq \frac{x_5^p}{p} + \frac{x_j^q}{q}, \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1.$$ 

Then by considering $p = \frac{6}{5}$ and $q = 6$, we have

$$x_1^5x_j \leq \frac{5}{6}x_6^6 + \frac{1}{6}x_6^6$$

and the estimation (18) can be obtained. Then, we can see that

$$\sum_{\ell=1}^{n} H_\ell^2(t, x_1, \ldots, x_n) \leq \max_{\ell=1, \ldots, n} \{D_\ell \alpha_\ell(t)\}(x_1^2 + \cdots + x_n^2)^3.$$ 

□

Last but not least, note that the jacobian of any polynomial $x \mapsto M_t(x)$ described by (16) and satisfying (i)–(ii) has constant determinant $\lambda < 0$. This arises the question: Are the polynomials $M_t$ invertible with polynomial inverse?. Our next result gives a partial answer provided that the NNMYC is true.

**Theorem 3.** If for all $n \geq 1$ NNMYC is true for any family $x \mapsto M_t(x)$ defined by (16), which satisfies the properties (i), (ii) and

(iii) the continuous function $a(t)$ present in (17) is upperly bounded.

(iv) for any $\varepsilon \in (0, 1)$ and any bounded piecewise continuous map $t \mapsto \omega(t)$, there exists an interval $[T_\omega, +\infty)$ and $\delta < -\lambda$ such that

$$\|JH(t, \omega(t))\| \leq \delta e^{-\varepsilon t} \quad \text{for any} \quad t \geq T_\omega,$$ 

(19)

then the map $x \mapsto M_t(x)$ has a polynomial inverse when $t \in P\mathcal{M}(\mathcal{M})$.

**Proof.** The result follows if we prove that the nonautonomous vector field $\overline{M}(t, x): \mathbb{R}^+ \times \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ defined by

$$\overline{M}(t, x) := (\text{Re } M_1(t, x), \text{Im } M_1(t, x), \ldots, \text{Re } M_n(t, x), \text{Im } M_n(t, x)),$$

satisfies the Nonuniform Nonautonomous Markus-Yamabe Conjecture. This proof will be made in several steps.
Step 1: The nonautonomous vector field $\overrightarrow{M}(t, x)$ verifies (G1)–(G2). As we know that $H(t, x)$ is continuous, to prove (G1) we only need to verify that the solutions of

$$\dot{x} = \lambda x + H(t, x)$$

are defined in $[t_0, +\infty)$ for any $t_0 \geq 0$. This proof will be made by contradiction by supposing that there exists a forward solution $t \mapsto x(t)$ of (20) passing by $x_0$ at $t = t_0$ having a bounded maximal domain $(t_0, T)$, which implies that $\lim_{t \to T^-} ||x(t)|| = +\infty$.

Now, the scalar product of (20) with $x \neq 0$ followed by its division by its euclidean norm $||x||$, give us

$$\frac{d}{dt} ||x|| = \lambda ||x|| + \frac{\langle H(t, x), x \rangle}{||x||}.$$ 

By the Cauchy–Schwarz inequality combined with Lemma 4 and (iii) we can deduce that any nontrivial solution of (20) has a euclidean norm satisfying the scalar differential inequality

$$\frac{d}{dt} ||x|| \leq \lambda ||x|| + C a(t) ||x||^3.$$ 

By using a technical result (see for example [15, Th.4.1, Ch.4]) for scalar differential inequalities, we can compare the solutions of the above inequality with the solutions of

$$\dot{v} = \lambda v + C ||a||_\infty v^3 \quad \text{with} \quad v(t_0) = ||x_0||$$

and to deduce that $||x(t)|| \leq v(t)$ for any $t \in (t_0, T) \cap I$, where $I$ is the domain of the solution of (21).

It is straightforward to verify that the solution of (21) is defined on $[t_0, +\infty)$. Finally, as $||x(t)|| \leq v(t)$ for any $t \in (t_0, T)$ and $v$ is upperly bounded on $(t_0, T]$, we obtain a contradiction and (G1) is verified.

The condition (G2) is verified due to the fact that $H$ is an homogeneous map and it is continuous with respect to $t$.

Step 2: The nonautonomous vector field $\overrightarrow{M}(t, x)$ verifies (G3)–(G4). In fact, let us consider any bounded piecewise continuous function $t \mapsto \omega(t)$ and the $2n -$dimensional linear system

$$\dot{\vartheta} = \text{Diag}\{\lambda\} \vartheta + J\overrightarrow{H}(t, \omega(t)) \vartheta.$$ 

It is clear that the system $\dot{\vartheta} = \text{Diag}\{\lambda\} \vartheta$ has nonuniform exponential dichotomy on any unbounded interval $J \subseteq \mathbb{R}^+$ with projector $P(t) = I_n$ for any $t \in J$. In addition, by using Proposition 2 combined with the property (iv), it follows from (19) that the system (22) has a nonuniform exponential dichotomy on $J_\omega = [T_\omega, +\infty)$ with projector $P(t) = I_n$, for any $t \in J_\omega$.

Now, the Lemma 1 states that the system (22) has in fact a nonuniform exponential dichotomy on $\mathbb{R}^+$ with projector $P(t) = I_n$, for any $t \geq 0$, or
equivalently, is nonuniformly exponentially stable in the sense of Definition 3 with \( J = \mathbb{R}^+ \). Finally, Lemma 3 assures that

\[
\Sigma(\text{Diag}\{\lambda\} + J\overline{H}(t, \omega(t))) \subset (-\infty, 0)
\]

for any \( \omega(\cdot) \) and then \((G3)-(G4)\) hold.

**Step 3:** As \( \text{NNMYC} \) is assumed to be true for all \( n \geq 1 \), Theorem 2 says that the family of maps \( M_t(x) : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n} \) is partially injective, and therefore the family of maps \( M_t : \mathbb{C}^n \rightarrow \mathbb{C}^n \) is also partially injective.

Finally, for any \( t \in \mathcal{P}(M) \), a result of Cynk [10, Theorem 2.2] says that \( M_t \) has a polynomial inverse and the result follows.

\[ \square \]

### 4.1. Theorem 3 and the Jacobian Conjecture.

The statement and the proof of the Theorem 3 are inspired on an idea of G. Fournier and M. Martelli, who proved that if the Markus-Yamabe Conjecture is true for autonomous systems \( \dot{x} = M(x) \), where \( M \) is a polynomial vector field satisfying properties similar to (i) and (ii) in an autonomous framework, then the Jacobian Conjecture is true. This result is deduced by using the reduction result aforementioned. Unfortunately, the idea of Fournier and Martelli becomes obsolete due to the Markus–Yamabe conjecture was proved to be false for \( n \geq 3 \) by A. Cima et al. in [7] in the autonomous case.

Additionally, in a similar way to the Theorem 2, the proof of above theorem follows the steps done by van den Essen in [11], where the Cynk’s result [10, Theorem 2.2] plays an important role. However, in contrast with the previous approach is the use of a nonautonomous spectral theory instead of the eigenvalues spectrum, which induces differences beyond the formal and requires the mastery of several tools which can be pigeonholed in the nonautonomous linear algebra (see [26, p.423]).

### 4.2. An illustrative example.

The following example shows a nonautonomous polynomial map \( M(t, \cdot) \) which satisfies conditions (i)–(iii) and then also verifies \((G1)-(G4)\) by following the lines of proof previous theorem. Moreover, for each \( t \in \mathcal{P}(M) \), the map \( u \mapsto M_t(u) \) has polynomial inverse; in particular we can find explicitly its inverse.

Let us consider the nonautonomous map

\[
M(t, x, y, z) = (\lambda x + e^{-t}y^3, \lambda y + e^{-t}(x + z)^3, \lambda z - e^{-t}y^3), \lambda < 0.
\]

It is easy to see that \( M \) satisfies (i) and (ii). In fact, note that \( u \mapsto H_i(t, u) \) is homogeneous of degree 3 for \( i = 1, 2, 3 \). In addition, we have

\[
JH(t, x, y, z) = 3e^{-t}\begin{pmatrix}
0 & y^2 & 0 \\
(x + z)^2 & 0 & (x + z)^2 \\
0 & -y^2 & 0
\end{pmatrix},
\]

and it is easy to verify that is a nilpotent matrix for any fixed \( t > 0 \).

The property (iii) is verified since \( e^{-t} \leq 1 \) for any \( t \geq 0 \). In order to verify the property (iv), we note that for any bounded piecewise continuous
map \( t \mapsto \omega(t) = (\omega_1(t), \omega_2(t), \omega_3(t)) \) we have that

\[
JM(t, \omega(t)) = \text{Diag}\{\lambda\} + 3e^{-t} \left( \begin{array}{ccc} 0 & \omega_2(t)^2 & 0 \\ (\omega_1(t) + \omega_3(t))^2 & 0 & (\omega_1(t) + \omega_3(t))^2 \\ 0 & -\omega_2(t)^2 & 0 \end{array} \right),
\]

thus we have

\[
\|JH(t, \omega(t))\| = \sqrt{18}e^{-t} \max \{\omega_2(t)^2, (\omega_1(t) + \omega_3(t))^2\}.
\]

Now, as \( t \mapsto \omega(t) \) is bounded and piecewise continuous, the number

\[
L_\omega := \sup_{t \geq 0} \max \{\omega_2(t)^2, (\omega_1(t) + \omega_3(t))^2\}
\]

is well defined. If we fix \( \delta < -\lambda \) and assume without loss of generality that \( -\lambda < \sqrt{18L_\omega} \), we can deduce that (iv) is verified for any \( t \geq T_\omega = \frac{1}{2(\varepsilon - 1)} \ln(\delta^2/18L_\omega^2) \).

An interesting fact of this example is that the NNMYC can be verified explicitly. In fact let \( t \mapsto (x(t), y(t), z(t)) \) be a solution of \( \dot{z} = M(t, z) \) with initial condition \( u_0 = (x_0, y_0, z_0) \) at time \( t_0 \). Notice that \( \dot{x}(t) + \dot{z}(t) = \lambda [x(t) + z(t)] \) and consequently, if \( t > t_0 \) it follows that

\[
x(t) + z(t) = e^{\lambda(t-t_0)}(x_0 + z_0).
\]

Upon inserting this term in the second equation, we have

\[
\dot{y} = \lambda y + e^{-t}e^{3\lambda(t-t_0)}(x_0 + z_0)^3
\]

and it can be proved that \( |y(t)| \leq \beta(||u_0||)e^{\lambda(t-t_0)} \) for any \( t \geq t_0 \), where \( \beta \in \mathcal{K}_\infty \). Upon inserting this solution on the first and third equations, we obtain

\[
\dot{x} = \lambda x + e^{-t}y(t) \quad \text{and} \quad \dot{z} = \lambda z + e^{-t}y(t)
\]

and it can be proved similarly that \( |x(t)| \leq \beta_1(||u_0||)e^{\lambda(t-t_0)} \) and \( |z(t)| \leq \beta_2(||u_0||)e^{\lambda(t-t_0)} \) for any \( t \geq t_0 \) with \( \beta_1, \beta_2 \in \mathcal{K}_\infty \). Then, the uniform asymptotic stability (which is a particular case of the nonuniform one) is verified.

As done in Example 1, it can be proved directly, that is without using Theorem 2 that the family of maps \( M_t(\cdot) \) is partially injective where \( \mathcal{P}(M) = \mathbb{R}^+ \). Therefore, \( M_t \) satisfies the Jacobian Conjecture for any \( t \in \mathbb{R}^+ \) since we can find explicitly the inverse of \( M_t(\cdot) \) for each \( t \geq 0 \). Namely, \( M_t^{-1}(x, y, z) = (N_1, N_2, N_3)(x, y, z) \) where

\[
N_{1t}(x, y, z) = \frac{1}{\lambda} \left( x - e^{-t} \left[ \frac{1}{\lambda} \left( y - e^{-t}(\frac{x+z}{\lambda})^3 \right) \right]^3 \right)
\]

\[
N_{2t}(x, y, z) = \frac{1}{\lambda} \left( y - e^{-t}(\frac{x+z}{\lambda})^3 \right)
\]

\[
N_{3t}(x, y, z) = \frac{1}{\lambda} \left( z + e^{-t} \left[ \frac{1}{\lambda} \left( y - e^{-t}(\frac{x+z}{\lambda})^3 \right) \right]^3 \right).
\]
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