Convergence of spherical averages for actions of free groups

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1. Introduction

Let \((X, \nu)\) be a probability space and suppose a free group \(F_m\) with \(m\) generators acts on \((X, \nu)\) by measure-preserving transformations. Let \(\{a_1, \ldots, a_m\}\) be a set of free generators for \(F_m\) and let \(T_1, \ldots, T_m : X \to X\) be transformations corresponding to the generators. Write \(T^{-i} = T_{-1}^i\) for \(i = 1, \ldots, m\), and set \(A = \{-m, \ldots, -1, 1, \ldots, m\}\). We also have the action \(F_m\) on \(L_1(X, \nu)\), defined by \(T_g \varphi = \varphi \circ T_{g^{-1}}\), \(g \in F_m\).

Consider the set \(W_A\) of all finite words over the alphabet \(A\):
\[
W_A = \{w = w_1w_2\ldots w_n \mid w_i \in A\}.
\]
For each \(w \in W_A\), \(w = w_1 \ldots w_n\), define a transformation
\[
T_w = T_{w_1}T_{w_2}\ldots T_{w_n}.
\]

Let \(\Pi\) be a stochastic \(2m \times 2m\) matrix, whose rows and columns are indexed by elements of \(A\), that is, \(\Pi = (p_{ij}), i, j \in A\). Assume that \(\Pi\) has a unique stationary distribution \((p_{-m}, \ldots, p_{-1}, p_1, \ldots, p_m)\) and that \(p_i > 0\) for all \(i \in A\).

For \(w \in W_A\), \(w = w_1 \ldots w_n\), denote
\[
p(w) = p_{w_nw_{n-1}}p_{w_{n-1}w_{n-2}}\ldots p_{w_2w_1}, \quad \pi(w) = p_{w_n}p(w).
\]
Consider the operators
\[
s_n^\Pi = \sum_{|w| = n} \pi(w)T_w.
\]

In this paper, we investigate convergence of this sequence of operators.

Definition 1. We shall say that the matrix \(\Pi\) generates the free group if \(p_{ij} = 0\) is equivalent to \(i + j = 0\).
We shall need the symmetry condition
\[ p_i = p_{-i}, \quad p_{-i,-j} = \frac{p_{ij}}{p_i}. \]
Relation (3) is equivalent to saying that all operators \( s_n^\Pi \) are self-adjoint.

Let \( F^2_m \) be the subgroup of words of even length in \( F_m \), that is, the subgroup generated by \( a_i a_j, i,j \in \{1, \ldots, m\} \).
Recall that \( L \log L(X,\nu) = \{ \varphi \in L_1(X,\nu) : \int_X |\varphi| \log^+ |\varphi| d\nu < \infty \} \).

**Theorem 1.** Let \((X,\nu)\) be a Lebesgue probability space. Assume the matrix \( \Pi \) generates the free group and satisfies (3). Then for any \( \varphi \in L \log L(X,\nu) \), the sequence \( s_{2n}^\Pi \varphi \) converges as \( n \to \infty \) both \( \nu \)-almost everywhere and in \( L_1(X,\nu) \) to an \( F^2_m \)-invariant function.

**Remark.** The sequence \( s_{2n+1}^\Pi \varphi \) also converges. The sequence \( s_n^\Pi \) need not converge, however, because the action of \( F_m \) might have an eigenfunction with eigenvalue \(-1\), that is, a nonzero function \( \psi \in L_1(X,\nu) \) such that \( T_i \psi = -\psi \) for all \( i \in A \) (for the same reason, the limit in Theorem 1 must be \( F^2_m \)-invariant but need not be \( F_m \)-invariant). If the action does not have eigenfunctions with eigenvalue \(-1\) then for any \( \varphi \in L \log L(X,\nu) \) the sequence \( s_{2n}^\Pi \varphi \) converges as \( n \to \infty \) both \( \nu \)-almost everywhere and in \( L_1(X,\nu) \) to an \( F_m \)-invariant limit.

Averages \( s_{2n}^\Pi \) converge under weaker assumptions on the matrix \( \Pi \) than in Theorem 1.

**Definition 2.** A matrix \( \Pi \) with nonnegative entries will be called irreducible if for some \( n > 0 \) all entries of the matrix \( \Pi + \Pi^2 + \ldots + \Pi^n \) are positive (if \( \Pi \) is stochastic then this is equivalent to saying that in the corresponding Markov chain any state is attainable from any other state).

**Definition 3.** A matrix \( \Pi \) with nonnegative entries will be called strictly irreducible if \( \Pi \) is irreducible and \( \Pi \Pi^T \) is irreducible (here \( \Pi^T \) stands for the transpose of \( \Pi \)).

Clearly, a matrix generating the free group is strictly irreducible.

**Theorem 2.** Let \((X,\nu)\) be a Lebesgue probability space and let \( p > 1 \). Assume the matrix \( \Pi \) is strictly irreducible and satisfies (3). Then for any \( \varphi \in L_p(X,\nu) \), the sequence \( s_{2n}^\Pi \varphi \) converges as \( n \to \infty \) both \( \nu \)-almost everywhere and in \( L_p \) to an \( F^2_m \)-invariant function.

**2. History**

First ergodic theorems for actions of arbitrary countable groups were obtained by V. I. Oseledets [17] in the following setting.

Let \( \Gamma \) be a countable group that acts by measure-preserving transformations of a probability space \((X,\nu)\), and for \( g \in \Gamma \) let \( T_g \) be the corresponding
transformation. Let $\mu$ be a probability measure on $\Gamma$ satisfying the condition $\mu(g^{-1}) = \mu(g)$. Let $\mu^{(n)}$ be the $n$-th convolution of $\mu$. The ergodic theorem of Oseledets states that for $\varphi \in L \log L(X, \nu)$, the averages

$$A_{2n}\varphi = \sum_{g \in \Gamma} \mu^{(2n)}(g)T_g \varphi$$

converge almost everywhere. The proof is based on consideration of the self-adjoint Markov operator $Q = \sum_{g \in \Gamma} \mu(g)T_g$.

In 1969 Y. Guivarc'h [9] (motivated by the work of Arnold and Krylov [1]) considered uniform spherical averages on the free group; that is,

$$s_n = \frac{1}{2m(2m-1)^{n-1}} \sum_{g:|g|=n} T_g$$

and proved that for $\varphi \in L_2(X, \nu)$ the sequence $s_{2n}\varphi$ converges in $L_2$ to an $F^2_m$-invariant function.

In 1986 R. I. Grigorchuk [6] (see also [7]) announced pointwise convergence for the averages

$$C_N = \frac{1}{N} \sum_{n=0}^{N-1} s_n.$$

In 1994 Nevo and Stein proved:

**Theorem 3** (Nevo and Stein [15]). Let $p > 1$. Then for any $\varphi \in L_p(X, \nu)$ the sequence $s_{2n}\varphi$ converges as $n \to \infty$ both $\nu$-almost everywhere and in $L_p$ to an $F^2_m$-invariant function.

The Nevo-Stein theorem is a particular case of Theorem 1; we shall however consider it separately in Section 4 in order to illustrate the ideas of the proof of Theorem 1.

### 3. The Markov operator

Recall that if $(Z, \mu)$ is a probability space then a linear operator $Q$ on $L_1(Z, \mu)$ is called a *measure-preserving Markov operator* if it preserves the cone of nonnegative functions, $L_1$-norm, and $L_\infty$-norm.

Let $p = \{p_-m, \ldots, p_{-1}, p_1, \ldots, p_m\}$ be the stationary distribution of the matrix $\Pi$.

Consider the space $Y = X \times A$ with the measure $\eta = \nu \times p$ and a Markov operator $P$ on $L_1(Y, \eta)$ given by

$$P\varphi(x, i) = \sum_{j \in A} p_{ij} \varphi(T_ix, j).$$

$P$ is a measure-preserving Markov operator on $L_1(Y, \eta)$. It was introduced by R. I. Grigorchuk [7], J.-P. Thouvenot (oral communication), and myself [3].
For \( n > 1 \) we have

\[
P^n \varphi(x, i) = \sum_{w \in W(n-1), j \in A} p_{iw_{n-1}} p(w) \varphi(T_w T_i x, j),
\]

which implies:

**Proposition 1.** Let \( \psi \in L_1(X, \nu) \). Let \( \varphi \in L_1(Y, \eta) \) be given by \( \varphi(x, a) = \psi(x) \). Then

\[
s_n \psi = \sum_{i \in A} p_i P^n \varphi(x, i).
\]

To prove Theorem 1, it suffices to prove the following:

**Lemma 1.** Suppose \( \Pi \) generates the free group and satisfies the symmetry condition (3). Suppose the action of \( F_m^2 \) on \( (X, \nu) \) is ergodic. Then for any \( \varphi \in L \log L(Y, \eta) \),

\[
P^n \varphi \to \int_Y \varphi d\eta
\]

both \( \eta \)-almost everywhere and in \( L_1(Y, \eta) \).

First we discuss ergodicity of \( P \) and \( P^2 \).

**Lemma 2.** If the action of \( F_m \) on \( (X, \nu) \) is ergodic and \( \Pi \) is strictly irreducible, then \( P \) is ergodic.

**Definition 4.** A function \( \varphi \in L_1(Y, \eta) \) does not depend on \( A \) if there exists \( \psi \in L_1(X, \nu) \) such that \( \varphi(x, a) = \psi(x) \) for all \( a \in A \).

**Definition 5.** A subset of \( A \) of \( Y \) will be called \( P \)-invariant if \( P \chi_A = \chi_A \) (where \( \chi_A \) stands for the characteristic function of \( A \)).

Ergodicity of a measure-preserving Markov operator is equivalent to the absence of nontrivial invariant subsets (see [20]). Lemma 2 follows now from:

**Proposition 2.** Suppose that \( \Pi \) is strictly irreducible. Then \( A \subset Y \) is \( P \)-invariant if and only if \( \chi_A \) does not depend on \( A \) and is \( F_m \)-invariant.

**Proof.** If \( p_{kl} > 0 \) then \( \chi_A(T_k x, l) = \chi_A(x, k) \). If \( (\Pi^T \Pi)_{ij} > 0 \) then there exists \( k \in A \) such that \( p_{ki} > 0, p_{kj} > 0 \). Therefore, \( \chi_A(x, k) = \chi_A(T_k x, i) = \chi_A(T_k x, j) \), which implies \( \chi_A(x, i) = \chi_A(x, j) \) and proves that \( \chi_A \) does not depend on \( A \). The equality \( \chi_A(T_i x, j) = \chi_A(x, i) \), true when \( p_{ij} > 0 \), and the irreducibility of \( \Pi \) imply group-invariance of \( \chi_A \).

**Lemma 3.** Suppose that \( \Pi \) is strictly irreducible and \( F_m^2 \) acts ergodically on \( (X, \nu) \). Then the operator \( P^2 \) is ergodic.
By Lemma 2, \( P \) is ergodic. If \( P^2 \) is not ergodic, then there exists a nonconstant function \( \psi \in L_1(Y, \eta) \) such that \( P\psi = -\psi \). Arguing in the same way as in Proposition 2, we obtain that \( \psi \) does not depend on \( A \), in other words, there exists \( \phi \in L_1(X, \nu) \) such that \( \psi(x, a) = \phi(x) \). The relation \( P\psi = -\psi \) implies \( T_i \phi = -\phi \) for all \( i \in A \), whence \( T_g \phi = \phi \) for all \( g \in F_m^2 \), and the Lemma is proved.

**Remark.** The Kakutani-Hopf ergodic theorem for Markov operators immediately implies that if the action of \( F_m \) on \( (X, \nu) \) is ergodic then for any \( \phi \in L_1(X, \nu) \),

\[
\frac{1}{N} \sum_{n=0}^{N-1} s_n^\Pi \phi \rightarrow \int_X \phi \, d\nu
\]

both \( \nu \)-almost everywhere and in \( L_1(X, \nu) \) as \( N \rightarrow \infty \) (see [7], [3]).

The operator adjoint to \( P \) is given by

\[
P^* \phi(x, i) = \sum_{j \in A} p^{-i, -j} \phi(T_{-j} x, j).
\]

Consider a unitary operator \( U \) given by

\[
U \phi(x, i) = \phi(T_i x, -i).
\]

Clearly, \( U^2 = \text{Id} \).

**Proposition 3.** Suppose the matrix \( \Pi \) satisfies the symmetry condition (3). Then \( P = UP^* U \).

Indeed, using (3), we can write

\[
P^* \phi(x, i) = \sum_{j \in A} p_{-i, -j} \phi(T_{-j} x, j) = UPU \phi(x, i).
\]

**4. Uniform spherical averages**

In this section, we illustrate the method of the proof of Theorem 1, by deducing the Nevo-Stein theorem from Rota’s “Alternierende Verfahren” theorem [19] applied to the Markov operator (5).

Consider uniform spherical averages (4). They are a particular case of the averages \( s_n^\Pi \) for \( \Pi \) defined by \( p_{ij} = 1/(2m - 1) \) for \( i + j \neq 0 \) and \( p_{ij} = 0 \) for \( i + j = 0 \).

For \( \Pi \) thus defined, the Markov operator (5) takes the form

\[
P \phi(x, i) = \frac{1}{2m - 1} \sum_{j: i + j \neq 0} \phi(T_i x, j)
\]
and its adjoint is given by

\[ P^* \varphi(x, i) = \frac{1}{2^{m+1}} \sum_{j + j \neq 0} \varphi(T_{-j}x, j). \]

**Lemma 4.** For \( P \) given by (9) and \( U \) given by (8),

\[ P^*P = \frac{2m - 2}{2m - 1}UP + \frac{1}{2m - 1} \text{Id}. \]

**Proof.** We have

\[ UP\varphi(x, i) = P\varphi(T_i x, -i) = \frac{1}{2^{m+1}} \sum_{k:k \neq i} \varphi(x, k) \]

and

\begin{align*}
P^*P\varphi(x, i) & = \frac{1}{2^{m+1}} \sum_{j + j \neq 0} P\varphi(T_{-j}x, j) \\
& = \frac{1}{(2m - 1)^2} \sum_{j + j \neq 0 \land k + k \neq 0} \sum_{k:k \neq i} \varphi(x, k) \\
& = \frac{2m - 2}{(2m - 1)^2} \sum_{k:k \neq i} \varphi(x, k) + \frac{1}{2^{m+1}} \varphi(x, i) \\
& = \frac{(2m - 2)}{2^{m+1}} UP + \frac{1}{2^{m+1}} \text{Id}\varphi(x, i).
\end{align*}

From Lemma 4 and Proposition 3, by induction, we obtain

\[ (P^*)^n P^n = \frac{2m - 2}{2m - 1} UP^{2n-1} + \frac{1}{2^{m+1}} (P^*)^{n-1} P^{n-1} \]

or

\[ P^{2n-1} = \frac{2m - 1}{2m - 2} UP(P^*)^n P^{n-1} - \frac{1}{2^{m+1}} UP(P^*)^{n-1} P^{n-1}. \]

The Nevo-Stein theorem easily follows now from the Alternierende Verfahren theorem of Gian-Carlo Rota [19]:

**Theorem 4 (Rota [19]).** Let \((Z, \mu)\) be a probability space. Let \(Q\) be a measure-preserving Markov operator on \(L_1(Z, \mu)\). Then for any \(\varphi \in L \log L(Z, \mu)\) the sequence \((Q^*)^n Q^n \varphi\) converges \(\mu\)-almost everywhere and in \(L_1\) as \(n \to \infty\).

Theorem 4 generalizes Stein’s theorem [21] on convergence of powers of self-adjoint operators and easily follows from the Martingale convergence theorem; we recall its proof in Section 6. Ornstein’s counterexample [16] shows that neither Stein’s nor Rota’s theorem holds for \(\varphi \in L_1\).

The equation (10) and Theorem 4 yield the convergence of \(P^{2n}\varphi\) for \(\varphi \in L \log L(Y, \eta)\). Lemma 3 implies \(F_{m^2}\)-invariance of the limit. The Nevo-Stein theorem is proved.
5. Proof of Lemma 1

**Lemma 5.** Suppose $\Pi$ generates the free group and satisfies the symmetry condition (3). Then there exists a positive constant $c$ depending only on $\Pi$ such that for any nonnegative $\varphi \in L_1(Y, \eta)$ and any $n > 0$,

\[(P^*)^n P^n \varphi \geq c U P^{2n-1} \varphi.\]

**Proof.** We first prove the statement for $n = 1$:

\[P^* P \varphi \geq c U P \varphi.\]

Now,

\[P^* P \varphi (x, i) = \sum_{j, k \in A} \frac{p_{ji} p_{jk}}{p_i} p_{jk} \varphi (x, k).\]

If $\Pi$ generates the free group, then for any $i, k \in A$ we have $\sum_{j \in A} \frac{p_{ji} p_{jk}}{p_i} p_{jk} > 0$. Since

\[U P (x, i) = \sum_k p_{-i, k} \varphi (x, k),\]

(12) is proved; in view of Proposition 3, (11) follows by induction, and the lemma is proved.

Now we prove $L_1$-convergence of the powers $P^n$. The following proposition is well known (see, for example, [11]).

**Proposition 4.** Let $Q$ be a measure-preserving Markov operator on a probability space $(Z, \mu)$. Then the tail sigma-algebra of $Q$ is trivial if and only if for any $\varphi \in L_1(Z, \mu)$, $(Q^*)^n \varphi \to \int_Z \varphi d\mu$ in $L_1(Z, \mu)$ as $n \to \infty$.

Since $P = UP^* U$, triviality of the tail sigma-algebra of $P$ is equivalent to the triviality of that of $P^*$. To establish this triviality, we shall use the following version of the 0-2 law for Markov operators.

**Lemma 6.** Let $Q$ be an arbitrary measure-preserving Markov operator on a probability space $(Z, \mu)$.

If the tail sigma-algebra of $Q$ is trivial then for any $\varphi, \psi \in L_2(Z, \mu)$

\[\int_Z (Q^*)^n \varphi \cdot (Q^*)^n \psi d\mu \to \int_Z \varphi d\mu \int_Z \psi d\mu\]

as $n \to \infty$.

If the tail sigma-algebra of $Q$ is nontrivial then for any $\varepsilon > 0$ there exist positive functions $\varphi, \psi \in L_\infty(Z, \mu)$ of integral 1 such that

\[\limsup_{n \to \infty} \int_Z (Q^*)^n \varphi \cdot (Q^*)^n \psi d\mu < \varepsilon.\]
The proof of Lemma 6 closely models Vadim A. Kaimanovich’s proof of the 0-2 law [11] and will be given in Section 6.

**Lemma 7.** Under assumptions of Lemma 1, for any $\varphi, \psi \in L_2(Y, \eta),$

$$\int_Y P^n \varphi \cdot \psi d\eta \to \int_Y \varphi d\eta \int_Y \psi d\eta.$$

This follows from the $K$-property for the operator $P$, which we prove in Section 7 (Lemma 10).

Lemma 6, Lemma 7, and the inequality (11) easily imply triviality of the tail sigma-algebra of $P$.

Indeed, for any positive $\varphi, \psi \in L_\infty(Y, \eta),$ we have

$$\int_Y P^n \varphi \cdot P^n \psi d\eta = \int_Y (P^*)^n P^n \varphi \cdot \psi d\eta \geq c \int_Y U P^{2n-1} \varphi \cdot \psi d\eta \to c \int_Y \varphi d\eta \int_Y \psi d\eta$$

as $n \to \infty$. In view of Lemma 6, this relation implies that $P^*$ (and hence also $P$, since $P = UP^*U$) has trivial tail sigma-algebra.

Proposition 4 yields that for any $\varphi \in L_1(Y, \eta),$

$$P^n \varphi \to \int_Y \varphi d\eta$$

in $L_1$ as $n \to \infty$.

Now we establish pointwise convergence of $P^n \varphi$ for $\varphi \in L \log L(Z, \mu)$.

Recall that if $(Z, \mu)$ is an arbitrary probability space then the Orlicz norm (see [24], [22]) on the space $L \log L(Z, \mu)$ can be introduced, for example, by putting

$$||\varphi||_{L \log L} = \inf\{c : \int_Z \frac{|\varphi|}{c} \cdot \log\left(\frac{|\varphi|}{c} + 2\right) d\mu \leq 1\}.$$

**Lemma 8.** Let $(Z, \mu)$ be a probability space and let $Q$ be a measure-preserving Markov operator on $L_1(Z, \mu)$.

For any $p > 1$ there exists a constant $A_p > 0$ such that for any $\varphi \in L_p(Z, \mu)$ we have

$$\|\sup_n (Q^*)^n Q^n \varphi\|_{L_p} \leq A_p \|\varphi\|_{L_p}.$$  

There exists a constant $A > 0$ such that for any $\varphi \in L \log L(Z, \mu),$

$$\|\sup_n (Q^*)^n Q^n \varphi\|_{L_1} \leq A \|\varphi\|_{L \log L}.$$  

Lemma 8 will be proved in Section 6.

Lemmas 5, 8 yield:
Lemma 9. Let $p > 1$. Then there exists a constant $p > 1$ such that for any $\varphi \in L_p(Y, \eta)$,
\[ \| \sup_n P^{2n} \varphi \|_{L_p} \leq A_p \| \varphi \|_{L_p}. \]
There exists a constant $A > 0$ such that for any $\varphi \in L \log L(y, \eta)$,
\[ \| \sup_n P^{2n} \varphi \|_{L_1} \leq A \| \varphi \|_{L \log L}. \]

Proposition 5. Let $Q$ be a measure-preserving Markov operator on a probability space $(Z, \mu)$. If the tail sigma-algebra of $Q^*$ is trivial then for any $\varphi \in L_2(Z, \mu)$ we have $Q^n \varphi \to \int \varphi$ in $L_2$ as $n \to \infty$.

The proof is given in Section 6.

Now let $\varphi \in L_2(Y, \eta)$, $\int_Y \varphi d\eta = 0$. Then $\|P^n \varphi\|_{L_2} \to 0$ as $n \to \infty$ by Proposition 5. By Lemma 9, for any positive integer $k$, we have
\[ \| \sup_n P^{2n+2k} \varphi \|_{L_2} \leq A_2 \| P^{2k} \varphi \|_{L_2}, \]
and the right part of the inequality tends to 0 as $k \to \infty$. This implies pointwise convergence of $P^{2n} \varphi$ for $\varphi \in L_2(Y, \eta)$, and, since we have $L_1$-convergence for the whole sequence $P^n \varphi$, we also have pointwise convergence for $P^n \varphi$ with $\varphi \in L_2(Y, \eta)$.

Since $L_2$ is dense in $L \log L$, pointwise convergence of $P^n \varphi$ for $\varphi \in L_2$ and the $L \log L$-maximal inequality of Lemma 9 yield pointwise convergence of $P^n \varphi$ for any $\varphi \in L \log L$.

To complete the proof of Lemma 1 and Theorem 1, it only remains to prove Lemmas 6,7, 8 and Proposition 5. We do so in the following two sections.

6. Proofs of Lemmas 6, 8 and of Proposition 5

Let $(Z, \mu)$ be a probability space and let $Q$ be an arbitrary measure-preserving Markov operator on $L_1(Z, \mu)$. Let
\[ Z^Z = \{ z = (z_n), n \in \mathbb{Z}, z_n \in Z \} \]
be the space of bi-infinite sequences of elements of $Z$ and let $Q_\mu$ be the Markov measure on $Z^Z$ corresponding to the operator $Q$ and the stationary distribution $\mu$. Let $\sigma_Q$ be the shift on $(Z^Z, Q_\mu)$ given by $(\sigma_Q(z))_n = (z)_{n+1}$; clearly, $\sigma_Q$ preserves the measure $Q_\mu$.

For any $k, m \in \{-\infty\} \cup \mathbb{Z} \cup \{ \infty \}$, $k \leq m$, denote by $\mathcal{F}_k^m$ the sigma-algebra on $Z^Z$ generated by the random variables $z_l$, $k \leq l \leq m$. In particular, $\mathcal{F}_k$ is the sigma-algebra generated by $z_k$. We shall sometimes write $\mathcal{F}_{\geq k}$ for $\mathcal{F}_k^\infty$ and $\mathcal{F}_{\leq k}$ for $\mathcal{F}_{-\infty}^k$. 
If \( \varphi \in L_1(Z, \mu) \) and \( \Phi \in L_1(Z^\mathbb{Z}, Q_\mu) \) is given by \( \Phi(z) = \varphi(z_0) \), then
\[
E\left(\Phi(z) | F_{-n}^\infty\right) = Q^n \varphi(z_{-n}), \quad \text{and} \quad E\left(E\left(\Phi(z) | F_{-n}^\infty\right) | F_0\right) = (Q^*)^n Q^n \varphi(z_0).
\]

Rota's theorem (Theorem 4) and Lemma 8 immediately follow now from the inverted Martingale dominated convergence theorem and the corresponding maximal inequalities (see [13, Chap. IV, Props. 2-8, 2-10]). This argument implies, moreover, the following:

**Proposition 6.** Suppose that the tail sigma-algebra of \( Q^* \) is trivial. Then for all \( \varphi \in L \log L(Z, \mu), \lim_{n \to \infty} (Q^*)^n Q^n \varphi = f \varphi d\mu. \)

Proposition 6 implies Proposition 5, because if the tail sigma-algebra of a Markov operator is trivial then for any \( \varphi \in L_2(Z, \mu) \) satisfying \( \int_z \varphi d\mu = 0 \), we have
\[
\int_Z (Q^n \varphi)^2 d\mu = \int_Z (Q^*)^n Q^n \varphi \cdot \varphi d\mu \to 0
\]
as \( n \to \infty \), by Proposition 6.

Now we prove Lemma 6. The proof closely models Kaimanovich's proof of the 0-2 law for Markov operators [11].

The first part of the lemma is a corollary of Proposition 5. To prove the second part, let \( F_\infty \) be the tail sigma-algebra of \( Q \), that is, \( F_\infty = \bigwedge_{k>0} F_{\geq k} \), and assume there exists \( A \in F_\infty \) such that \( 0 < Q_\mu(A) < 1 \). Set
\[
\Phi(z) = \chi_A(z) / Q_\mu(A), \quad \Psi(z) = \chi_{(Z^\mathbb{Z} \setminus A)}(z) / Q_\mu(Z^\mathbb{Z} \setminus A).
\]
Then \( \Phi, \Psi \) are positive, bounded, tail-measurable, \( \int \Phi dQ_\mu = \int \Psi dQ_\mu = 1 \), \( \Phi \cdot \Psi = 0 \). Let \( M \) be a constant such that \( M > \Phi, M > \Psi \). Set \( \varphi_k(z_k) = E(\Phi(z)|F_{\leq k}), \psi_k(z_k) = E(\Psi(z)|F_{\leq k}) \). Clearly, \( \varphi_k, \psi_k \) are positive and bounded from above by \( M \). By the Martingale convergence theorem, \( \varphi_k(z_k) \to \Phi(z), \psi_k(z_k) \to \Psi(z) \) both \( Q_\mu \)-almost everywhere and in \( L_1(Z^\mathbb{Z}, Q_\mu) \) as \( k \to \infty \).

Since \( \Phi, \Psi \) are \( F_\infty \)-measurable, we have
\[
E(\varphi_k(z_k)|F_\infty) \to \Phi, \quad E(\psi_k(z_k)|F_\infty) \to \Psi
\]
both \( Q_\mu \)-almost everywhere and in \( L_1(Z^\mathbb{Z}, Q_\mu) \) as \( k \to \infty \).

Choose \( k \) in such a way that
\[
\int_{Z^\mathbb{Z}} E(\varphi_k(z_k)|F_\infty) E(\psi_k(z_k)|F_\infty) dQ_\mu < \varepsilon.
\]
Clearly,
\[
E(\varphi_k(z_k)|F_{\geq n+k}) = (Q^*)^n \varphi(z_{n+k}), \quad E(\psi_k(z_k)|F_{\geq n+k}) = (Q^*)^n \psi(z_{n+k}).
\]

Therefore, as \( n \to \infty \),
\[
\int_Z (Q^*)^n \varphi_k(z) \cdot (Q^*)^n \psi_k(z) d\mu = \int_{Z^\mathbb{Z}} E(\varphi_k(z_k)|F_{\geq n+k}) E(\psi_k(z_k)|F_{\geq n+k}) dQ_\mu
\]
\[
\to \int_{Z^\mathbb{Z}} E(\varphi_k(z_k)|F_\infty) E(\psi_k(z_k)|F_\infty) dQ_\mu < \varepsilon,
\]
and Lemma 6 is proved.
7. K-property and the proof of Lemma 7

Let $Y^Z$ be the space of biinfinite sequences of elements of $Y$:

$$Y^Z = \{ y : y = (y_n), n \in \mathbb{Z}, y_n \in Y \}.$$

Let $P_\eta$ be the measure corresponding to the operator $P$ and the stationary distribution $\eta$, and let $\sigma_P$ be the shift on $(Y^Z, P_\eta)$. In order to prove Lemma 7, it suffices to show that $\sigma_P$ is mixing. To do so, we establish the following

**Lemma 10.** Assume $F^2_m$ acts ergodically on $(X, \nu)$ and assume the matrix $\Pi$ is strictly irreducible. Then the system $(Y^Z, P_\eta, \sigma_P)$ has K-property.

The proof is based on the Rohlin-Sinai theorem [18]. First, we give another realization of $\sigma_P$.

Let $\Sigma_A$ be the space of bi-infinite sequences of symbols of $A$:

$$\Sigma_A = \{ \omega : \omega = (\omega_n), n \in \mathbb{Z}, \omega_n \in A \}.$$

Let $\sigma_A : \Sigma_A \rightarrow \Sigma_A$ be the shift on $\Sigma_A$. Let $\mu_\Pi$ be the $\sigma_A$-invariant Markov measure on $\Sigma_A$ corresponding to the matrix $\Pi$ and its stationary distribution $p$. Consider the map $T : \Sigma_A \times X \rightarrow \Sigma_A \times X$ given by the formula

$$T(\omega, x) = (\sigma_A \omega, T_{\omega_0} x).$$

Clearly, the map $T$ preserves the measure $\mu_\Pi \times \nu$.

**Lemma 11.** The systems $(\Sigma_A \times X, \mu_\Pi \times \nu, T)$ and $(Y^Z, P_\eta, \sigma_P)$ are isomorphic.

**Proof.** Let $y \in Y^Z$. Then $y = (y_n)$, where $y_n \in Y$; that is, $y_n = (i_n, x_n)$, $i_n \in A$, $x_n \in X$. Set $\omega(y) = (i_n)$, $n \in \mathbb{Z}$ and $x(y) = x_0$. The map $F : Y^Z \rightarrow \Sigma_A \times X$ given by $F(y) = (\omega(y), x(y))$ produces the desired isomorphism ($F$ is invertible because for $P_\eta$-almost all $y \in Y^Z$, we have $x_1 = T_{i_0} x_0$, $x_2 = T_{i_1} x_1$, $x_{-1} = T_{-i_{-1}} x_0$, etc.)

Now we establish the K-property for the system $(\Sigma_A \times X, \mu_\Pi \times \nu, T)$. The proof follows the method of Oseledets [17].

As in last section, write $F^m(Y^Z)$ for the $\sigma$-algebra in $Y^Z$ generated by the random variables $y_l$, $k \leq l \leq m$; write $F^m(\Sigma_A)$ for the $\sigma$-algebra in $\Sigma_A$ generated by the random variables $\omega_l$, $k \leq l \leq m$; write $F_{\geq k}$ instead of $F^k$, $F_{\leq k}$ instead of $F^k$, and $F_k$ instead of $F^k$; finally, denote by $B(X)$ the $\sigma$-algebra of all $\nu$-measurable subsets of $X$, by $B(\Sigma_A \times X)$ the $\sigma$-algebra of all $\mu_\Pi \times \nu$-measurable subsets of $\Sigma_A \times X$.

Let $\pi(T)$ be the Pinsker $\sigma$-algebra of $T$. We shall use the Rohlin-Sinai theorem [18] to prove the triviality of $\pi(T)$, and therefore the K-property of $T$. 
Consider the σ-algebra $\mathcal{G}_+ = \mathcal{F}_{\geq 0}(\Sigma_A) \times \mathcal{B}(X)$ (the future of our Markov process). Clearly, $T_\mathcal{G}_+ \supset \mathcal{G}_+$, and $\forall k \in \mathbb{Z} T^k \mathcal{G}_+ = \mathcal{B}(\Sigma_A \times X)$; by the Rohlin-Sinai theorem [18], $\mathcal{G}_+ \supset \pi(T)$. Let $\mathcal{G}_- = \mathcal{F}_{\leq 0}(\Sigma_A) \times \mathcal{B}(X)$ (the past of our Markov process). Clearly, $T^{-1} \mathcal{G}_- \supset \mathcal{G}_-$, and $\forall k \in \mathbb{Z} T^k \mathcal{G}_- = \mathcal{B}(\Sigma_A \times X)$. By the Rohlin-Sinai theorem, $\mathcal{G}_- \supset \pi(T^{-1}) = \pi(T)$. We have, therefore, $\pi(T) \subset \mathcal{G}_+ \land \mathcal{G}_-$. It is easy to check that $(\mu \land \nu)$ almost surely we have

\[(14) \quad (\mathcal{F}_{\geq 0}(\Sigma_A) \times \mathcal{B}(X)) \land (\mathcal{F}_{\leq 0}(\Sigma_A) \times \mathcal{B}(X)) = \mathcal{F}_0(\Sigma_A) \times \mathcal{B}(X).\]

For $k \in \mathbb{Z}$, let $\mathcal{G}_k = \mathcal{F}_k(\Sigma_A) \times \mathcal{B}(X)$ (the moment $k$ of our Markov process). By (14), $\pi(T) \subset \mathcal{G}_0$. Since $T \pi(T) = \pi(T)$ and $T^k \mathcal{G}_0 = \mathcal{G}_k$, $\pi(T) \subset \land_{k \in \mathbb{Z}} \mathcal{G}_k$.

Now let $\varphi : \Sigma_A \times X \to \mathbb{R}$ be $\pi(T)$-measurable. Then for any $k \in \mathbb{Z}$ there exists $\psi : Y \to \mathbb{R}$ such that $\varphi(\omega, x) = \psi_k(\omega_k, x)$.

Since for all $k \in \mathbb{Z}$ we have $E(E(\varphi|\mathcal{G}_k)\mathcal{G}_0) = \varphi$, we obtain

\[(15) \quad (P^*)^k P^k \varphi_0 = P^k (P^*)^k \varphi_0 = \varphi_0 \quad \text{for all} \quad k \in \mathbb{N}.\]

To prove the triviality of $\pi(T)$, it remains to prove that a function $\varphi_0 : Y \to \mathbb{R}$, satisfying (15), is a constant.

**Proposition 7.** Suppose $\Pi$ is strictly irreducible, $\varphi \in L_1(Y, \eta)$. Then a set $A$ is $P^* P$-invariant if and only if $\chi_A$ does not depend on $A$.

Indeed,

\[P^* P \chi_A(x, i) = \sum_{k,l} \frac{p_{k,l} o_k}{p_i} p_{k,l} \chi_A(x, l)\]

and, for $i, l$ fixed, we have $\sum_k \frac{p_{k,l} o_k}{p_i} p_{k,l} > 0$ if and only if $(\Pi^T \Pi)_{i,l} > 0$, which implies the proposition.

**Lemma 12.** Suppose the matrix $\Pi$ is strictly irreducible. Suppose a set $A \subset Y$ is both $P^* P$ and $(P^*)^2 P^2$ invariant. Then $\chi_A$ does not depend on $A$ and is $F^2_m$-invariant.

By the previous proposition, $\chi_A$ does not depend on $A$. Write

\[\chi_A(x, i) = (P^*)^2 P \chi_A(x, i) = \sum \frac{p_{j,l} o_j}{p_i} p_{k,l} \chi_A(T_i T_{-j} x, m).\]

We have, then, $\chi_A(x) = \chi_A(T_i T_{-j} x)$ for all $j, l$ such that $(\Pi^T \Pi)_{i,l} > 0$. Since the matrix $\Pi$ is strictly irreducible, the claim is proved.

Lemma 10 is proved and it implies, in particular, that $\sigma_P$ is mixing, which yields Lemma 7.

The proof of Theorem 1 is complete.
Remark 1. Let \((Z, \mu)\) be a Lebesgue probability space, \(Q\) a measure-preserving Markov operator on \(L_1(Z, \mu)\), \((Z^Z, Q_\mu)\) the space of trajectories of \(Q\), \(\sigma_Q\) the corresponding shift, and \(\pi(\sigma_Q)\) the Pinsker sigma-algebra of \(\sigma_Q\). Then we have:

**PROPOSITION 8.** Note that \(\pi(\sigma_Q) \subset \mathcal{F}_0(Z^Z)\). If \(C \subset Z\) and the set 
\[ \{z : z_0 \in C\} \in \pi(\sigma_Q) \text{ then } \chi_C = Q^k(Q^*)^k \chi_C = (Q^*)^kQ^k \chi_C \text{ for any } k \in \mathbb{N}. \]

The proof is the same as that of Lemma 10: first, the Rohlin-Sinai theorem gives that \(\pi(\sigma_Q) \subset \mathcal{F}_{\geq 0}(Z^Z) \cap \mathcal{F}_{\leq 0}(Z^Z) = \mathcal{F}_0(Z^Z)\), then the \(\sigma_Q\)-invariance of \(\pi(\sigma_Q)\) implies that \(\pi(\sigma_Q) \subset \wedge_{k \in \mathbb{Z}} \mathcal{F}_k(Z^Z)\), which implies the proposition.

**Remark 2.** Let \(\mu\) be an arbitrary Borel probability \(\sigma_A\)-invariant measure on \(\Sigma_A\). Clearly, the map \(T\), defined by (13), preserves the measure \(\mu \times \nu\).

Let \(B = (B_{ij})\), \(i, j \in A\), be a \(0-1\) matrix, and let \(\mu\) be a Gibbs measure (in the sense of Bowen [2]) on the subshift of \(\Sigma_A\) given by the matrix \(B\).

Arguing in the same way as in the proof of Lemma 10, we see that if \(B\) is strictly irreducible and the action of \(F_\mu^2\) on \(X\) is ergodic, then the system \((\Sigma_A \times X, \mu \times \nu, T)\) has the \(K\)-property.

**8. Proof of Theorem 2**

**LEMMA 13.** Suppose that \(\Pi\) satisfies (3) and that all entries of the matrix \(\Pi I + (\Pi I)^2 + \cdots + (\Pi I)^k\) are positive. Then there exists a constant \(c > 0\) such that for any nonnegative \(\varphi \in L_1(Y, \eta)\),
\[ ((P^*)^n P^n + (P^*)^n PP^*P^n + \cdots + (P^*)^n PP^*P^n)^n \varphi \geq cU P^{2n-1} \varphi \]
almost everywhere.

The proof is the same as that of Lemma 5.

**LEMMA 14.** Let \((Z, \mu)\) be a probability space, let \(Q\) be a measure-preserving Markov operator on \(L_1(Z, \mu)\), let \(p > 1\) and let \(k\) be a positive integer. Then for any \(\varphi \in L_p(X, \nu)\) the sequence \(Q^n(Q^*Q)^k(Q^*)^n \varphi\) converges \(\mu\)-almost everywhere in \(L_p\) as \(n \to \infty\).

Moreover,
\[ \|\sup_n (Q^*Q)^k Q^n \varphi\|_{L_p} \leq A_p \|
\phi\|_{L_p}. \]

Let \(\varphi \in L_1(Z, \mu)\) and define \(\Phi \in L_1(Z^Z, Q_\mu)\) by \(\Phi(z) = \varphi(z_0)\). Set
\[ \Phi_0^n(z) = E(\Phi(z) | \mathcal{F}_n^\infty) = (Q^*)^n \varphi(z_n) \]
and for \(i \geq 1\) let
\[ \Phi_i^n(z) = E(E(\Phi_{i-1}^n(z) | \mathcal{F}_{n-1}) | \mathcal{F}_n). \]
Clearly, 
\[ \Phi_i^i(z) = (Q^*Q)^i(Q^*)^n\phi(z_n), \]
and
\[ E(\Phi_i^i(z)|F_0) = Q^n(Q^*Q)^i(Q^*)^n\phi(z_n). \]

The statement of the proposition follows now from Rota’s theorem (Theorem 4) and the \( L_p \) maximal inequality for martingales (see [13, Prop. IV-2-8]) by induction on \( i \).

In a similar fashion, Lemma 6 implies:

**Lemma 15.** Let \( k \) be a nonnegative integer. If the tail sigma-algebra of \( Q \) is trivial then for any \( \phi, \psi \in L_2(Z, \mu) \)
\[ \int Q^k(Q^*)^n\phi \cdot Q^k(Q^*)^n\psi d\mu \rightarrow \int \phi d\mu \int \psi d\mu \]
as \( n \rightarrow \infty \).

If the tail sigma-algebra of \( Q \) is nontrivial then for any \( \varepsilon > 0 \) there exist positive functions \( \phi, \psi \in L_\infty(Z, \mu) \) of integral 1 such that
\[ \limsup_{n \rightarrow \infty} \int Q^k(Q^*)^n\phi \cdot Q^k(Q^*)^n\psi d\mu < \varepsilon. \]

The rest of the proof goes the same way as that of Theorem 1, with Lemma 14 being used instead of Lemma 8 and Lemma 15 assuming the role of Lemma 6.

### 9. A conjecture

Theorem 2 can be applied to obtain spherical convergence for actions of some classes of Markov groups (in the sense of Gromov [8]).

Let \( \Gamma \) be a Markov group. Its elements can then be coded by admissible words in a topological Markov chain. Assume that the matrix \( A \) of the chain is irreducible and let \( \Pi \) be the matrix of the Parry measure (in other words, the measure of maximal entropy) corresponding to \( A \). If \( \Pi \) is strictly irreducible and satisfies the symmetry condition (3), then Theorem 2 is applicable. The spherical averages \( s_{\Pi}^f \) for \( \Pi \) thus chosen can easily be reduced to uniform spherical averages in \( \Gamma \) (see [4]). Theorem 2 then yields convergence of uniform spherical averages for the group \( \Gamma \). For example, this takes place for Vershik’s locally finite groups [23].

Gromov [8] proved that Gromov hyperbolic groups are Markov. If the coding satisfied the assumptions of Theorem 2, then Theorem 2 would yield the following:
Conjecture 1. Let $\Gamma$ be a Gromov hyperbolic group, let $S$ be a symmetric set of generators, and denote by $\Gamma^2$ the subgroup generated by elements that have a geodesic representation of even length over the alphabet $S$. Let $p > 1$. Suppose $\Gamma$ acts on a probability space $(X, \nu)$ by measure-preserving transformations.

Then for any $\varphi \in L^p(X, \nu)$ the sequence

$$s_{2n}\varphi = \frac{1}{\# \{ g : |g|_S = 2n \}} \sum_{g : |g|_S = 2n} T_g \varphi$$

converges as $n \to \infty$ almost everywhere and in $L_p$ to a $\Gamma^2$-invariant function.

Assuming exponential mixing, Fujiwara and Nevo [5] obtained a convergence theorem for Cesaro averages of the spherical averages for Gromov hyperbolic groups.

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