Provably Efficient Lifelong Reinforcement Learning with Linear Function Approximation

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Abstract

We study lifelong reinforcement learning (RL) in a regret minimization setting of linear contextual Markov decision process (MDP), where the agent needs to learn a multi-task policy while solving a streaming sequence of tasks. We propose an algorithm, called UCB Lifelong Value Distillation (UCBlvd), that provably achieves sublinear regret for any sequence of tasks, which may be adaptively chosen based on the agent’s past behaviors. Remarkably, our algorithm uses only sublinear number of planning calls, which means that the agent eventually learns a policy that is near optimal for multiple tasks (seen or unseen) without the need of deliberate planning. A key to this property is a new structural assumption that enables computation sharing across tasks during exploration. Specifically, for $K$ task episodes of horizon $H$, our algorithm has a regret bound $\tilde{O}((d^3 + d'd)H^4K)$ based on $O(dH \log(K))$ number of planning calls, where $d$ and $d'$ are the feature dimensions of the dynamics and rewards, respectively. This theoretical guarantee implies that our algorithm can enable a lifelong learning agent to accumulate experiences and learn to rapidly solve new tasks.

1 Introduction

Recently there has been a surging interest in designing lifelong learning agents that can continuously learn to solve multiple sequential decision making problems in its lifetime [Thrun and Mitchell, 1995, Silver et al., 2013, Xie and Finn, 2021]. This scenario is in particular motivated by building multi-purpose embodied intelligence [Roy et al., 2021], such as robots working in a weakly structured environment. Typically, curating all tasks beforehand for such problems is nearly infeasible, and the problems the agent is tasked with may be adaptively selected based on the agent’s past behaviors. Consider household robot as an example. Because each household is unique, it is difficult to anticipate upfront all scenarios the robot would encounter. Moreover, the tasks the robot faces are not independent and identically distributed (i.i.d.). Instead, what the robot has done before can affect the next task and its starting state; e.g., if the robot fails to bring a glass of water and breaks it, then the user is likely to command the robot to clean up the mess. Therefore, it is critical that the agent can continuously improve and generalize learned abilities to different tasks, regardless of their order.

In this work, we study lifelong reinforcement learning (RL) theoretically in a regret minimization setting [Thrun and Mitchell, 1995, Ammar et al., 2015], where the agent needs to solve a sequence of tasks using rewards in an unknown environment while balancing exploration and exploitation. Motivated by the embodied intelligence scenario, we suppose that tasks differ in rewards, but share the same state and action spaces and the transition dynamics [Xie and Finn, 2021]. To be realistic, we make no assumptions on how the tasks and initial states are selected.\footnote{We adopt a stricter definition of lifelong RL here to distinguish it from multi-task RL, while there are existing works on lifelong RL (e.g. [Brunskill and Li, 2014, Lecarpentier et al., 2021]) assuming i.i.d. tasks.} Generally, we allow them to be chosen from a continuous set...
potentially by an adversary based on the agent’s past behaviors. Once a task is specified, the agent has one chance to complete the task and then the next task is revealed.

The goal of the agent is to perform near optimally for the tasks it faces, despite the online nature of the problem. For simplicity, we assume that there is no memory constraint; this is usually the case for robotics applications where real-world interactions are the main bottleneck [Xie and Finn, 2021]. Nonetheless, the agent should eventually learn to make decisions without frequent deliberate planning, because planning is time consuming and creates undesirable wait time for user-interactive scenarios. In other words, the agent needs to learn a multi-task policy, generalizing from not only past samples but also past computation, to solve new tasks.

Formally, we consider an episodic setup based on the framework of contextual Markov decision process (MDP) [Abbasi-Yadkori and Neu, 2014, Hallak et al., 2015]. It repeats the following steps: 1) At the beginning of an episode, the agent is set to an initial state and receives a context specifying the task reward, both of which can be arbitrarily chosen. 2) When needed, the agent uses its past experiences to plan for the current task. 3) The agent runs a policy in the environment for a fixed horizon in an attempt to solve the assigned task and gains experiences from its policy execution. The agent’s performance is measured as the regret with respect to the optimal policy of the corresponding task. We require that, for any task sequence, both the agent’s overall regret and number of planning calls to be sublinear in the number of episodes.

While lifelong RL is not new, the need of simultaneously achieving 1) sublinear regret and 2) sublinear number of planning calls for 3) a potential adversarial sequence of tasks and initial states makes the setup considered here particularly challenging. To our knowledge, existing works only address a strict subset of these requirements; especially, the computation aspect is often ignored. Most provable works in lifelong RL make the assumption that the tasks are finitely many [Ammar et al., 2015, Ammar et al., 2014, Zhan et al., 2017, Brunskill and Li, 2015], or are i.i.d. [Brunskill and Li, 2014, Abel et al., 2018a, Abel et al., 2018b, Lecarpentier et al., 2021], while others considering similar setups to ours do not provide regret guarantees [Isele et al., 2016, Xie and Finn, 2021]. On the technical side, we found that the closest works are [Modi and Tewari, 2020, Abbasi-Yadkori and Neu, 2014, Hallak et al., 2015, Modi et al., 2018, Kakade et al., 2020] for contextual MDP and [Wu et al., 2021, Abels et al., 2019] for the dynamic setting of multi-objective RL, which study the sample complexity of learning for arbitrary task sequences; however, they either assume the problem is tabular or require a model-based planning oracle with unknown complexity. Importantly, none of the existing works properly address the need of sublinear number of planning calls, which creates a large gap between the abstract setup and practice need.

In this paper, we propose the first provably efficient lifelong RL algorithm, **UCB Lifelong Value Distillation (UCBvlvd, pronounced as “UC Boulevard”),** that possesses all three desired qualities. Under the assumption of contextual MDP with linear features [Yang and Wang, 2019, Jin et al., 2020] and a new completeness-style assumption, UCBvlvd achieves sublinear regret for any online sequence of tasks while using sublinear number of planning calls. Specifically, for \( K \) episodes of horizon \( H \), we prove a regret bound \( \tilde{O}(\sqrt{d^3 + d'd}H^4K) \) based on \( \tilde{O}(dH \log(K)) \) number of planning calls, where \( d \) and \( d' \) are the feature dimensions of the dynamics and rewards, respectively.

From a high-level viewpoint, UCBvlvd uses the linear structure to identify what to transfer and operates by interleaving 1) independent planning for a set of representative task contexts and 2) distilling the planned results into a multi-task value-based policy. UCBvlvd also constantly monitors whether the new experiences it gained is sufficiently significant, based on a doubling schedule, to avoid unnecessary planning. The design of UCBvlvd is inspired by single-task LSVI-UCB [Jin et al., 2020] but we introduce a novel distillation step, along with a new completeness assumption, to enable computation sharing across tasks; in addition, we extend the low-switching cost technique [Abbasi-Yadkori et al., 2011, Gao et al., 2021] for single-task RL to the lifelong setup to achieve sublinear number of planning calls.
2 Preliminaries

Notation. Throughout the paper, we use lower-case letters for scalars, lower-case bold letters for vectors, and upper-case bold letters for matrices. The Euclidean-norm of a vector $\mathbf{x}$ is denoted by $\|\mathbf{x}\|_2$. We denote the transpose of a vector $\mathbf{x}$ by $\mathbf{x}^\top$. For any vectors $\mathbf{x}$ and $\mathbf{y}$, we use $\langle \mathbf{x}, \mathbf{y} \rangle$ to denote their inner product. We denote the Kronecker product by $\mathbf{A} \otimes \mathbf{B}$. Let $\mathbf{A} \in \mathbb{R}^{d \times d}$ be a positive definite and $\mathbf{v} \in \mathbb{R}^d$. The weighted 2-norm of $\mathbf{v}$ with respect to $\mathbf{A}$ is defined by $\|\mathbf{v}\|_{\mathbf{A}} := \mathbf{v}^\top \mathbf{A} \mathbf{v}$. For a positive integer $n$, $[n]$ denotes the set $\{1, 2, \ldots, n\}$. For a real number $\alpha$, we denote $\{\alpha\}^+ = \max\{\alpha, 0\}$. Finally, we use the notation $\tilde{O}$ for big-O notation that ignores logarithmic factors.

2.1 Problem Formulation

We formulate lifelong RL as a regret minimization problem in contextual MDP [Abbasi-Yadkori and Neu, 2014, Hallak et al., 2015] with adversarial context and initial state sequences. We suppose that a context determines the reward but does not affect the dynamics. Such a context dependency is common for the lifelong learning scenario where an embodied agent consecutively solves multiple tasks. Below we give the formal problem definition.

Finite-horizon Contextual MDP. We consider a finite-horizon contextual MDP denoted by $M = (\mathcal{S}, \mathcal{A}, \mathcal{W}, H, \mathbb{P}, r)$, where $\mathcal{S}$ is the state space, $\mathcal{A}$ is the action space, $\mathcal{W}$ is the task context space, $H$ is the horizon (length of each episode), $\mathbb{P} = \{\mathbb{P}_h\}_{h=1}^H$ are the transition probabilities, and $r = \{r_h\}_{h=1}^H$ are the reward functions. We allow $\mathcal{S}$ and $\mathcal{W}$ to be continuous or infinitely large, while we assume $\mathcal{A}$ is finite such that $\max_{a \in \mathcal{A}}$ can be performed easily. For $h \in [H]$, $r_h(s, a, w)$ denotes the reward function whose range is assumed to be in $[0, 1]$, and $\mathbb{P}_h(s'|s, a, w)$ denotes the probability of transitioning to state $s'$ upon playing action $a$ at state $s$. In short, a contextual MDP can be viewed as an MDP with state space $\mathcal{S} \times \mathcal{W}$ and action space $\mathcal{A}$ where the context part of the state remains constant in an episode.\(^2\) To simplify the notation, for any function $f$, we write $\mathbb{P}_h[f](s, a) := \mathbb{E}_{s' \sim \mathbb{P}_h(., s, a)}[f(s')]$.

Policy and Value Functions. In a finite-horizon contextual MDP, a policy $\pi = \{\pi_h\}_{h=1}^H$ is a sequence where $\pi_h : \mathcal{S} \times \mathcal{W} \rightarrow \mathcal{A}$ determines the agent’s action at time-step $h$. Given $\pi$, we define its state value function as $V_\pi^h(s, w) := \mathbb{E}[\sum_{t=h}^H r_t \left(s_t, \pi_{t}(s_t, w), w\right) | s_0 = s]$ and its action-value function as $Q_\pi^h(s, a, w) := r_h(s, a, w) + \mathbb{P}_h[V_{\pi_{h+1}}^*(., w)](s, a)$, where $Q_{\pi_{H+1}}^* = 0$. We denote the optimal policy as $\pi^*_h(s, w) := \sup_{\pi} V_{\pi}^h(s, w)$, and let $V_{\pi}^* := V_{\pi^*}^*$ and $Q_{\pi}^* := Q_{\pi^*}^*$ denote the optimal value functions. Lastly, we recall the Bellman equation of the optimal policy:

$$Q^*_h(s, a, w) = r_h(s, a, w) + \mathbb{P}_h[V^*_{\pi_{h+1}}(., w)](s, a), \quad V^*_h(s, w) = \max_{a \in \mathcal{A}} Q^*_h(s, a, w),$$

(1)

Interaction Protocol of Lifelong RL. The agent interacts with a contextual MDP $M$ in episodes. For presentation simplicity, we assume that the reward functions $r$ are known, while the transition probabilities $\mathbb{P}$ are unknown and must be learned online; we will discuss how reward learning can be naturally incorporated in Section 4.3. At the beginning of episode $k$, the agent receives a task context $w^k \in \mathcal{W}$ and is set to an initial state $s^k_1$, both of which can be adversarially chosen. The agent can use past experiences to plan for the current task, if needed. Then the agent executes its policy $\pi^k$: at each time-step $h \in [H]$, it observes the state $s^k_h$, plays an action $a^k_h = \pi^k_h(s^k_h, w^k)$, observes a reward $r^k_h := r_h(s^k_h, a^k_h, w^k)$, and goes to the next state $s^k_{h+1}$ according to $\mathbb{P}_h(., s^k_h, a^k_h)$. Let $K$ be the total number of episodes. The agent’s goal is to achieve sublinear regret, where the regret is defined as

$$R_K := \sum_{k=1}^K V^*_1(s^k_1, w^k) - V^*_1(s^k_1, w^k).$$

(2)

As the comparator policy above (namely $\pi^*$ that defines $V^*_1$) also knows the task context, achieving sublinear regret implies that the agent would attain near task-specific optimal performance on average.\(^2\)

\(^2\)In general, a context-dependent dynamics would take the form $\mathbb{P}_h(s'|s, a, w)$.
2.2 Assumptions

Throughout the paper, we rely on the following assumptions.

**Assumption 1** (Linear MDP). \( M = (S, A, H, \mathbb{P}, r, W) \) is a linear MDP with feature maps \( \phi : S \times A \to \mathbb{R}^{d} \) and \( \psi : S \times A \times W \to \mathbb{R}^{d'} \). That is, for any \( h \in [H] \), there exist a vector \( \eta_h \) and \( d \) measures \( \mu_h := [\mu_h^{(1)}, \ldots, \mu_h^{(d)}]^\top \) over \( S \) such that \( \mathbb{P}_h(\cdot|s, a) = \langle \mu_h^{(1)}, \phi(s, a) \rangle \) and \( r_h(s, a, w) = \langle \eta_h, \psi(s, a, w) \rangle \), for all \( (s, a, w) \in S \times A \times W \).

**Assumption 2** (Boundedness). Without loss of generality, \( \|\phi(s, a)\|_2 \leq 1, \|\psi(s, a, w)\|_2 \leq 1, \|\mu_h(S)\|_2 \leq \sqrt{d}, \) and \( \|\eta_h\|_2 \leq \sqrt{d'} \) for all \( (s, a, w, h) \in S \times A \times W \times [H] \).

**Example 1** (Weighted Rewards). An interesting and common special case is \( \psi(s, a, w) = \phi(s, a) \otimes \rho(w) \), for some mapping \( \rho : W \to \mathbb{R}^m \). In this case, it holds that \( d' = md \) and \( r_h(s, a, w) = \langle \rho(w), r_h(s, a) \rangle \), where \( r_h(s, a) = A_h \phi(s, a) \in \mathbb{R}^m \), for some \( A_h \in \mathbb{R}^{m \times d} \), is the vector reward functions at time-step \( h \). We can view \( r_h(s, a, w) \) as a weighted reward with weights \( \rho(w) \) that depend on task \( w \). This setting is closely related to Multi-Objective RL studied for tabular case in [Wu et al., 2021], which studies the case where \( \rho(w) = w \in \mathbb{R}^m \) along with tabular \( S \) and \( A \).

3 A Warm-up Algorithm for Lifelong RL

We first present a warm-up algorithm, termed Lifelong Least-Squares Value Iteration (Lifelong-LSVI), in Algorithm 1. Lifelong-LSVI extends the single-task LSVI-UCB algorithm proposed by [Jin et al., 2020] to the lifelong learning setting. It runs LSVI-UCB as a subroutine for each task by leveraging the structure in Assumption 1: as the transition dynamics is context independent, dynamics samples collected during solving other tasks can be relabeled with the current reward to plan for the current task. The motivation of this warm-up algorithm is to give intuitions on how the problem structure in Assumption 1 can be used to achieve small regret in lifelong learning. We will show that Lifelong-LSVI has a sublinear regret bound, which matches the minimax optimal rate in the special case studied by [Wu et al., 2021] in terms of number of objectives, \( m \) (see Example 1).

However, we will also show that Lifelong-LSVI is not computationally efficient, in the sense that the number of planning calls it requires grows linearly with the number of episodes. This is because the agent never learns to internalize the task solving skills but requires going though all past experiences for planning every time a new task arrives. Moreover, we will discuss why it cannot be made computationally efficient in an easy manner. This drawback motivates our main algorithm, UCBlvd, in Section 4, which is provably efficient in both regret and number of planning calls.

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**Algorithm 1:** Lifelong-LSVI

| Input: \( \mathcal{A}, \lambda, \delta, H, K, \beta \) |
| Set: \( Q^k_{H+1}(\ldots) = 0, \forall k \in [K] \) |
| for episodes \( k = 1, \ldots, K \) do |
| Observe the initial state \( s^k_1 \) and the task context \( w^k \). |
| for time-steps \( h = H, \ldots, 1 \) do |
| Compute \( \tilde{\theta}^k_{h}(w^k) \) as in (6) using \( Q^k_{h+1} \) defined in (8). |
| for time-steps \( h = 1, \ldots, H \) do |
| Compute \( Q^k_h(s^k_h, a, w^k) \) for all \( a \in \mathcal{A} \) as in (8). |
| Play \( a^k_h = \arg\max_{a \in \mathcal{A}} Q^k_h(s^k_h, a, w^k) \) and observe \( s^k_{h+1} \) and \( r^k_h \). |
3.1 Algorithmic Notations

To begin, we introduce the template and the notations that will be used commonly in presenting the warm-up algorithm, Lifelong-LSVI, and our main algorithm, UCBlvd. For each algorithm, first we will define an algorithm-specific action-value function $Q_k^h : \mathcal{S} \times \mathcal{A} \times \mathcal{W} \to \mathbb{R}$, which determines the agent’s policy at time-step $h$ in episode $k$; then we present the full algorithm and its analyses using the quantities below, which are defined with respect to each algorithm’s definition of $Q_k^h$.

Given $\{Q_k^h\}_{h \in [H]}$, we define state value functions and their backups as

$$V_k^h(s, w) := \min \left\{ \max_{a \in A} Q_k^h(s, a, w), H \right\},$$

$$\theta_k^h(w) := \int_{\mathcal{S}} V_k^h(s', w) d\mu_h(s'),$$

Thanks to the linear MDP structure in Assumption 1, it holds that

$$\mathbb{P}_h \left[ V_k^h(\cdot, w) \right] (s, a) = \left\langle \theta_k^h(w), \phi(s, a) \right\rangle.$$

Let $\lambda > 0$ be a constant. We define the $\lambda$-regularized least squares estimator of $\theta_k^h(w)$ as

$$\hat{\theta}_k^h(w) := \left( \Lambda_k^h \right)^{-1} \sum_{\tau=1}^{k-1} \phi_h V_k^{\tau}(s_h^{\tau+1}, w)$$

$$\Lambda_k^h := \lambda I_d + \sum_{\tau=1}^{k-1} \phi_h^T \phi_h.$$

where $\theta_k^h(w)$ is the solution to $min_{\theta \in \mathbb{R}^d} \sum_{\tau=1}^{k-1} \left\langle \theta, \phi(s_h^{\tau}, a_h^{\tau}) \right\rangle - V_k^{\tau}(s_h^{\tau+1}, w) \right\rangle^2 + \lambda \left\| \theta \right\|^2_2, \phi_h^r := \phi(s_h^{\tau}, a_h^{\tau}),$ and $I_d \in \mathbb{R}^{d \times d}$ is the identity matrix.

3.2 Details of Lifelong-LSVI and Theoretical Analysis

We define the upper confidence bound (UCB) style action-value function of Lifelong-LSVI as follows:

$$Q_k^h(s, a, w) := r_h(s, a, w) + \left\langle \hat{\theta}_h^k(w), \phi(s, a) \right\rangle + \beta \left\| \phi(s, a) \right\|_2 (\Lambda_k^h)^{-1},$$

where $Q_{k+1}^h(\cdot, \cdot, \cdot) = 0$ and $\hat{\theta}_h^k(w)$ and $\Lambda_k^h$ are defined in (6) and (7), respectively. Here, $\beta$ is an exploration factor that is passed as an input of Lifelong-LSVI and will be appropriately chosen in Theorem 1. At episode $k$, given $w^k$, Lifelong-LSVI first performs planning backward in time based on past data to compute $\hat{\theta}_h^k(w^k)$ in (6) using $Q_{k+1}^h$ defined in (8) (Lines 4-5). Then, in execution, it uses $\hat{\theta}_h^k(w^k)$ to compute $Q_k^h(s, a, w^k)$ for the current state and all $a \in A$ (Line 7) and executes the action with the highest value (Line 8).

We show that Lifelong-LSVI achieves sublinear regret for our lifelong RL setup. The complete proof is reported in Appendix A, which follows the ideas of LSVI-UCB [Jin et al., 2020].

**Theorem 1.** Let $T = KH$. Under Assumptions 1 and 2, there exists an absolute constant $c > 0$ such that for any fixed $\delta \in (0, 0.5)$, if we set $\lambda = 1$ and $\beta = cH \left( d + \sqrt{d^3} \right) \sqrt{\log(dd'\bar{T}/\delta)}$ in Algorithm 1, then with probability at least $1 - 2\delta$, it holds that

$$R_K \leq 2H \sqrt{T \log(dT/\delta)} + 2H \beta \sqrt{2dK \log(K)} \leq \tilde{O} \left( \sqrt{d^3 + dd'}H^3T \right).$$

Before moving forward to our main algorithm in Section 4, we make a few remarks on the regret and number of planning calls of Lifelong-LSVI. First, Theorem 1 implies that for the special case studied by
**Algorithm 2: UCBlvd (UCB Lifelong Value Distillation)**

| Input: $A$, $\lambda$, $\delta$, $H$, $K$, $\beta$ |
| Set: $Q^k_{H+1}(\ldots) = 0$, $\forall k \in [K]$, $\tilde{k} = 1$ |
| for episodes $k = 1, \ldots, K$ do |
| Observe the initial state $s_1^k$ and the task context $w^k$. |
| if $\exists h \in [H]$ such that $\log \det \Lambda_h - \log \det \Lambda_h^k > 1$ then |
| for time-steps $h = H, \ldots, 1$ do |
| Compute $\tilde{s}_h^k$ as in (11). |
| for time-steps $h = 1, \ldots, H$ do |
| Compute $Q_h^k(s_h^k, a, w^k)$ for all $a \in A$ as in (10). |
| Play $a_h^k = \arg \max_{a \in A} Q_h^k(s_h^k, a, w^k)$ and observe $s_{h+1}^k$ and $r_h^k$. |

[Wu et al., 2021] (summarized in Example 1), the regret bound of Lifelong-LSVI becomes $O(\sqrt{md^3H^3T})$. This rate is optimal in terms of its dependency on $m$, as shown in [Wu et al., 2021], for this specific reward structure. Furthermore, this rate matches the regret dependencies on $d$ and $H$ of LSVI-UCB’s for the single-task setting [Jin et al., 2020].

While Lifelong-LSVI has a decent regret guarantee, we observe that it requires computing $\hat{\theta}_h^k(w^k)$ for all $h \in [H]$, whenever a distinct new task $w^k$ arrives. Since the number of unique tasks may be as large as $K$, the total number of planning calls required in Lifelong-LSVI is $K$ in the worst case. Unfortunately, the number of planning calls of Lifelong-LSVI cannot be easily improved due to the nonlinearity of $Q_h^k(s, a, w)$ on $w$ through $\hat{\theta}_h^k(w)$ in (8), which could lead to a covering number no less than $K$ in general. In particular, it is also hopeless to employ low switching cost techniques like [Abbasi-Yadkori et al., 2011] to reduce the number of planning calls, because we always need to recalculate $\hat{\theta}_h^k(w)$ for every new task.

In the next section, we discuss how placing a completeness-style assumption would help circumvent the issue of non-linear dependency of the action-value functions on $w$, and consequently would enable computation sharing to decrease the number of planning calls to $O(dH \log (1 + K/d\lambda))$.

## 4 UCB Lifelong Value Distillation (UCBlvd)

In this section, we present our main algorithm, **UCB Lifelong Value Distillation (UCBlvd)**, in Algorithm 2. Under extra structural assumptions we will introduce in Section 4.1, UCBlvd shares the same regret bound as Lifelong-LSVI but reduces the number of planning calls to be sublinear. In contrast to Lifelong-LSVI which learns individual action-value function for each $w^k$ using $\phi(s, a)$, UCBlvd learns a single action-value function for all $w \in W$ based on $\psi(s, a, w)$ to enable computation sharing across tasks. In general, directly extending Lifelong-LSVI to use feature $\psi(s, a, w) \in \mathbb{R}^{d'}$ with $d' \geq d$ would increase the regret from that with $\phi(s, a) \in \mathbb{R}^d$, because the latter can exploit the context-independent dynamics structure. UCBlvd maintains the same order of regret as Lifelong-LSVI by separating the planning into a novel two-step process: 1) independent planning with $\phi$ for a set of representative task contexts and 2) distilling the planned results into a multi-task value function parameterized by $\psi$. In addition, UCBlvd runs a doubling schedule to decide whether replanning is necessary, which makes the total number of planning calls sublinear. Below we give the details of UCBlvd.
4.1 Enabling Computation Sharing

First, we introduce two extra assumptions needed by UCBlvd to share computation across tasks. We will discuss how these assumptions can be relaxed in Section 4.3.

The first assumption is a new completeness-style assumption.

**Assumption 3 (Completeness).** Given feature maps \( \phi : S \times A \to \mathbb{R}^d \) and \( \psi : S \times A \times W \to \mathbb{R}^{d'} \) in Assumption 1, consider the function class

\[
\mathcal{F} = \left\{ f : f(s, w) = \min \left\{ \max_{a \in A} \left\{ \langle \nu, \phi(s, a) \rangle + \beta \| \phi(s, a) \|_{A_2} \right\}, H \right\}, \nu \in \mathbb{R}^d, A \in \mathbb{S}_{++}^d, \beta \in \mathbb{R} \right\}.
\]

For any \( f \in \mathcal{F} \) and \( h \in [H] \), there exists a vector \( \xi^f_h \in \mathbb{R}^{d'} \) with \( \| \xi^f_h \| \leq H \sqrt{d'} \) such that

\[
\mathbb{P}_h \left[ f(., w) \right](s, a) = \langle \xi^f_h, \psi(s, a, w) \rangle.
\]

It says the backup of functions in \( \mathcal{F} \) should be captured by the feature \( \psi \) with bounded parameters. The definition of \( \mathcal{F} \) models closely the structure of action-value function used by Lifelong-LSVI in (8), except \( \langle \theta_h^k(w), \phi(s, a) \rangle \) there is replaced by functions linear in \( \psi(s, a, w) \). We will see that the action-value function used by UCBlvd defined in the next section is contained in \( \mathcal{F} \).

We introduce an extra structure on \( \psi \) inspired by Example 1.

**Assumption 4 (Mappings).** We assume \( \psi(s, a, w) = \phi(s, a) \otimes \rho(w) \), for some mapping \( \rho : W \to \mathbb{R}^m \), i.e., \( d' = md \). We assume that there is a known set \( \{w^{(1)}, w^{(2)}, \ldots, w^{(n)}\} \) of \( n \leq m \) task contexts such that \( \rho(w) \in \text{Span}(\{\rho(w^{(j)})\}_{j \in [n]}) \) for all \( w \in W \). That is, for any \( w \in W \), there exist coefficients \( \{c_j(w)\}_{j \in [n]} \) such that

\[
\rho(w) = \sum_{j \in [n]} c_j(w) \rho(w^{(j)}).
\]

We assume \( \sum_{j \in [n]} |c_j(w)| \leq L \) for all \( w \in W \) and some \( L < \infty^3 \).

4.2 Details of UCBlvd

We define the UCB style action-value function of UCBlvd as follows:

\[
Q^k_h(s, a, w) := r_h(s, a, w) + \left\langle \xi^k_h, \psi(s, a, w) \right\rangle + 2L\beta \| \phi(s, a) \|_{(A^k_h)} \tag{10}
\]

The parameter \( \xi^k_h \) is computed by solving the convex quadratically constrained quadratic program (QCQP) in (11) below, which is defined on a set of representative task contexts \( \{w^{(1)}, w^{(2)}, \ldots, w^{(n)}\} \) in Assumption 4 and state-action pairs \( D := \{(s, a) : \phi(s, a) \text{ are } d \text{ linearly independent vectors}\} \).

\[
\xi^k_h, \{\theta_h^{(j)}\}_{j \in [n]} = \arg \min_{\xi, \{\theta^{(j)}\}_{j \in [n]} : (s, a) \in D} \sum_{s, a \in D} \left( \langle \theta^{(j)}, \phi(s, a) \rangle - \langle \xi, \psi(s, a, w^{(j)}) \rangle \right)^2 \tag{11}
\]

s.t. \( \| \theta^{(j)} - \theta_h^k(w^{(j)}) \|_{A^k_h} \leq \beta, \forall j \in [n] \) and \( \| \xi \|_2 \leq H \sqrt{md} \).

where \( \theta_h^k(w) \) and \( A^k_h \) are defined in (6) and (7), respectively. We will show later in Lemma 3 that the action-value function in (10) is an optimistic estimate of the optimal action-value function.

---

3Such set \( \{\rho(w^{(j)})\}_{j \in [n]} \) always exists for finite-dimensional problems. We assume that this set is known to the algorithm.
UCBlvd also uses the linear dependency of $Q_h^k$ on $\psi$ to reduce calls of the planning step in (11). The agent triggers replanning only when it has gathered enough new information compared to the last update at episode $\tilde{k}$. This is measured by tracking the variations in the gram matrices $\{\Lambda_h^k\}_{h \in [H]}$ (Line 4 for Algorithm 2). Finally, when executing the policy at episode $k$, the agent chooses the action according to $Q_h^k$ in Line 10.

4.3 Theoretical Analysis of UCBlvd

We present the main theoretical result which shows UCBlvd achieves sublinear regret in lifelong RL using sublinear number of planning calls, for any sequence of tasks.

**Theorem 2.** Let $T = KH$. Under Assumptions 1, 2, 3, and 4, the number of planning calls in Algorithm 2 is at most $dH \log(1 + \frac{K}{d})$, and there exists an absolute constant $c > 0$ such that for any fixed $\delta \in (0, 0.5)$, if we set $\lambda = 1$ and $\beta = cH(d + \sqrt{md})\sqrt{\log(mdT/\delta)}$ in Algorithm 2, then with probability at least $1 - 2\delta$, it holds that

$$R_K \leq 2H \sqrt{T \log(dT/\delta)} + 8HL\beta \sqrt{2dK \log(K)} \leq \tilde{O}\left(L \sqrt{(d^3 + md^2)H^3T}\right).$$

Theorem 2 shows that UCBlvd has the same regret bound as Lifelong-LSVI in Theorem 1, but reduces the number of planning calls from $K$ to $dH \log(1 + \frac{K}{d})$. As we discussed before, this is made possible by the unique QOCQP-based distillation step of UCBlvd in (11). If we were to simply perform least-squares regression to fit $\langle \psi(s, a, w), \tilde{\xi}_k^k \rangle$ to $\{\langle \phi(s, a), \tilde{\phi}_h^k(w(j)) \rangle\}_{j \in [n]}$ for distillation, we cannot guarantee the required optimism, because $\tilde{\phi}_h^k(w)$ computed based on finite samples can be an irregular function that cannot be modelled by $\psi(s, a, w)$.

**Remark 1.** We can extend our results to learn unknown rewards, i.e., $\eta_h$ in Assumption 1. This can be done by introducing a slightly different completeness assumption with an additional exploration bonus in terms of $\psi$, and then combining tools from linear bandits [Abbasi-Yadkori et al., 2011] and our analysis for proving Theorem 2. Because reward learning affects the radius of our high probability confidence intervals for $\tilde{\phi}_h^k(w)$, the number of planning calls and regret would increase by factors of $O(m)$ and $O(\sqrt{m})$ respectively, compared to those in Theorem 2. See Appendix C for details.

**Remark 2.** It is possible to eliminate the assumption that $\psi(s, a, w) = \phi(s, a) \otimes \rho(w)$. In this case, our analysis requires a set $\{w^{(1)}, w^{(2)}, \ldots, w^{(n)}\}$ of $n$ tasks such that $\psi(s, a, w) \in \text{Span}\{\langle \psi(s, a, w^{(j)}), H(k) \rangle\}_{j \in [n]}$ for all $(s, a, w) \in S \times A \times W$. In Appendix D, we provide details of this relaxation, and show that the corresponding modified version of UCBlvd still enjoys planning calls and regret of the same order as those of UCBlvd.

**Remark 3.** We can eliminate Assumptions 1 and 4 and instead design a computation-sharing version of Lifelong-LSVI by a slightly different completeness assumption with an exploration bonus $\beta \|\psi(s, a, w)\|_{\tilde{\Lambda}^{-1}}$. This version would use $Q_h^k(s, a, w) := \{r_h(s, a, w) + \langle \tilde{\nu}_h^k, \psi(s, a, w) \rangle + \beta \|\psi(s, a, w)\|_{(\tilde{\Lambda}_h^k)^{-1}}\}^+$, where $\tilde{\nu}_h^k = (\tilde{\Lambda}_h^k)^{-1} \sum_{r=1}^{k-1} \psi_h^r, \min\{\max_{a \in A} Q_h^r(s_{r+1}^k, a, w), H\}$, $\tilde{\Lambda}_h^k = \lambda I_d + \sum_{r=1}^{k-1} \psi_h^r \psi_h^{r\top}, \psi_h^r = \psi(s_h^r, a_h^r, w)$, and $\beta = \tilde{O}(d')$. In Appendix E, we show how this change results in $O(mdH)$ number of planning calls and a regret scaling with $\tilde{O}(\sqrt{m^3d^3})$ for settings with $\psi(s, a, w) = \phi(s, a) \otimes \rho(w)$. These are worse than the number of planning calls and regret in Theorem 2 of UCBlvd by a factor of $O(m)$.

4.4 Proof Sketch of Theorem 2

The complete proof of Theorem 2 is reported in Appendix B. Here we provide a sketch. Because the proof for the bound on the number of planning calls follows standard arguments in low switching cost analysis [Abbasi-Yadkori et al., 2011], in this section, we focus on the proof sketch for the regret bound.

---

4While for both settings in this remark and Remark 3, the action-value functions contain exploration bonus in terms of $\psi$, the regret here is better by a factor of $\sqrt{m}$ and this is because the multiplicative factor $\beta$ here saves a factor $\sqrt{m}$ compared to that in Remark 3.
We start by introducing the following lemma of a high probability event $\mathcal{E}_1$, which is the foundation of the analysis.

**Lemma 1.** Follow the setting of Theorem 2. The event

$$
\mathcal{E}_1(w) := \left\{ \left\| \theta^k_h(w) - \hat{\theta}^k_h(w) \right\|_{\mathcal{A}_h^k} \leq \beta, \forall (h,k) \in [H] \times [K] \right\}.
$$

(12)

holds with probability at least $1 - \delta$ for a fixed $w$.

The following lemma highlights the importance of the carefully designed planning step in (11). In particular, it emphasizes how this step paired with the choice of set $\mathcal{D}$, Assumptions 3 and 4 leads to good estimators for $\xi_{h+1}$, without the need of the bonus term $\|\psi(s,a,w)\|_{(\mathcal{A}_h^k)^{-1}}$, that the alternate extension of Lifelong-LSVI in Remark 3 has. This step saves a factor of $O(m)$ in the number of planning calls and regret.

**Lemma 2.** Let $\mathcal{W} = \{w \uparrow : \tau \in [K]\} \cup \{w^{(j)} : j \in [n]\}$. Conditioned on events $\{\mathcal{E}_1(w)\}_{w \in \mathcal{W}}$ defined in (12), for all $(s,a,w,h,k) \in S \times A \times W \times [H] \times [K]$, it holds that

$$
\left\| \xi_h, \psi(s,a,w) \right\| - \mathbb{P}[V_{h+1}^k(,w)(s,a)] \leq 2L\beta\|\phi(s,a)\|_{(\mathcal{A}_h^k)^{-1}}.
$$

As the final step in the regret analysis, we state the following lemma which uses Lemma 2 to prove the optimistic nature of UCBlvd. Then following the standard analysis of single-task LSVI-UCB we derive the regret bound in Theorem 2.

**Lemma 3.** Let $\mathcal{W} = \{w \uparrow : \tau \in [K]\} \cup \{w^{(j)} : j \in [n]\}$. Conditioned on events $\{\mathcal{E}_1(w)\}_{w \in \mathcal{W}}$ defined in (12), and with $Q_h^k$ computed as in (10), it holds that $Q_h^k(s,a,w) \geq Q_h^k(s,a,w)$ for all $(s,a,w,h,k) \in S \times A \times W \times [H] \times [K]$.

5 Related Work

We consider the regret minimization setup of lifelong RL under the contextual MDP framework, where the agent receives tasks specified by contexts in sequence and needs to achieve a sublinear regret for any task sequence. Below, we contrast our work with related work in the literature.

**Lifelong RL** Generally lifelong RL studies how to learn to solve a streaming sequence of tasks using rewards. While it was originally motivated by the need of endless learning of robots [Thrun and Mitchell, 1995], historically many works on lifelong RL [Brunskill and Li, 2014, Abel et al., 2018a, Abel et al., 2018b, Lecarpentier et al., 2021] assume that the tasks are i.i.d. (similar to multi-task RL; see below). There are works for adversarial sequences, but most of them assume finite number of tasks [Ammar et al., 2014, Brunskill and Li, 2015, Ammar et al., 2015, Zhan et al., 2017] or are purely empirical [Xie and Finn, 2021]. The work by [Isele et al., 2016] uses contexts to enable zero-shot learning like here, but it (as well as most works above) do not provide formal regret guarantees. [...]

[Ammar et al., 2015] give regret bounds but only for linearized value difference; [Brunskill and Li, 2015] show regret bounds only for finite number of tasks.
Contextual MDP and Multi-objective RL  Our setup is closely related to the exploration problem studied in the contextual MDP literature, though contextual MDP is originally not motivated from the lifelong learning perspective. A similar mathematical problem appears in the dynamic setup of multi-objective RL [Wu et al., 2021, Abels et al., 2019], which can be viewed as a special case of contextual MDP where the context linearly determines the reward function but not the dynamics. Most contextual MDP works allow adversarial contexts and initial states, but a majority of them focuses on the tabular setup [Abbasi-Yadkori and Neu, 2014, Hallak et al., 2015, Modi et al., 2018, Modi and Tewari, 2020, Levy and Mansour, 2022, Wu et al., 2021], whereas our setup allows continuous states. [Kakade et al., 2020, Du et al., 2019] allow continuous state and actions, but the former assumes a planning oracle with unclear computational complexity and the latter focuses on only LQG problems. While generally contextual MDP allows both the reward and the dynamics to vary with contexts, we focus on the effects of context-independent dynamics similar to [Kakade et al., 2020, Wu et al., 2021]. In particular, the recent work of [Wu et al., 2021] is the closest to ours, but they study the sample complexity in the tabular setup with linearly parameterized rewards. In view of Example 1, their proposed algorithm has a regret bound $\tilde{O}(\sqrt{\min\{m,|S|\}H|S||A|K})$. However, they need linear number of planning calls. On the contrary, our algorithm, UCBlvd, allows continuous states, nonlinear context dependency, and has both sublinear regret and number of planning calls.

Multi-Task RL  Another closely related line of work is multi-task RL. Compared to our setting, multi-task RL assumes that there are beforehand known finite tasks and/or they are i.i.d samples from a fixed distribution. For example, in [Yang et al., 2020, Hessel et al., 2019, Brunskill and Li, 2013, Fifty et al., 2021, Zhang and Wang, 2021, Sodhani et al., 2021], tasks are assumed to be chosen from a known finite set, and in [Yang et al., 2020, Wilson et al., 2007, Brunskill and Li, 2013, Sun et al., 2021], tasks are sampled from a fixed distribution. By contrast, our setting provides guarantees on regret and number of planning calls for adversarial task sequences.

6 Discussion

In this paper, we make a link between lifelong RL and contextual MDPs. We propose UCBlvd, an algorithm that simultaneously satisfies the need of achieving 1) sublinear regret and 2) sublinear number of planning calls for 3) a potential adversarial sequence of tasks and initial states. Specifically, for $K$ task episodes of horizon $H$, we proved that UCBlvd has a regret bound $\tilde{O}(\sqrt{(d^2 + d')H^4K})$ based on $\tilde{O}(dH \log(K))$ number of planning calls, where $d$ and $d'$ are the feature dimensions of the dynamics and rewards, respectively. We believe that our results would inspire several research directions in the literature of CMDP and multi-objective RL, as existing work to our knowledge does not cover the computation complexity sharing aspect. That said, our work’s limitations motivate further investigations in the following directions: 1) extension to more general class of MDPs, potentially using general function approximation tools, 2) establishing an information-theoretic lower bound on the number of planning calls/computation complexity.

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References

[Abbasi-Yadkori and Neu, 2014] Abbasi-Yadkori, Y. and Neu, G. (2014). Online learning in mdps with side information. arXiv preprint arXiv:1406.6812.

[Abbasi-Yadkori et al., 2011] Abbasi-Yadkori, Y., Pál, D., and Szepesvári, C. (2011). Improved algorithms for linear stochastic bandits. In Advances in Neural Information Processing Systems, pages 2312–2320.

[Abel et al., 2018a] Abel, D., Arumugam, D., Lehnert, L., and Littman, M. (2018a). State abstractions for lifelong reinforcement learning. In International Conference on Machine Learning, pages 10–19. PMLR.

[Abel et al., 2018b] Abel, D., Jinnai, Y., Guo, S. Y., Konidaris, G., and Littman, M. (2018b). Policy and value transfer in lifelong reinforcement learning. In International Conference on Machine Learning, pages 20–29. PMLR.

[Abels et al., 2019] Abels, A., Roijers, D., Lenaerts, T., Nowé, A., and Steckelmacher, D. (2019). Dynamic weights in multi-objective deep reinforcement learning. In International Conference on Machine Learning, pages 11–20. PMLR.

[Ammar et al., 2014] Ammar, H. B., Eaton, E., Ruvolo, P., and Taylor, M. (2014). Online multi-task learning for policy gradient methods. In International conference on machine learning, pages 1206–1214. PMLR.

[Ammar et al., 2015] Ammar, H. B., Tutunov, R., and Eaton, E. (2015). Safe policy search for lifelong reinforcement learning with sublinear regret. In International Conference on Machine Learning, pages 2361–2369. PMLR.

[Brunskill and Li, 2013] Brunskill, E. and Li, L. (2013). Sample complexity of multi-task reinforcement learning. In Proceedings of the Twenty-Ninth Conference on Uncertainty in Artificial Intelligence, pages 122–131.

[Brunskill and Li, 2014] Brunskill, E. and Li, L. (2014). Pac-inspired option discovery in lifelong reinforcement learning. In International conference on machine learning, pages 316–324. PMLR.

[Brunskill and Li, 2015] Brunskill, E. and Li, L. (2015). The online coupon-collector problem and its application to lifelong reinforcement learning. arXiv preprint arXiv:1506.03379.

[Du et al., 2019] Du, S. S., Wang, R., Wang, M., and Yang, L. F. (2019). Continuous control with contexts, provably. arXiv preprint arXiv:1910.13614.

[Fifty et al., 2021] Fifty, C., Amid, E., Zhao, Z., Yu, T., Anil, R., and Finn, C. (2021). Efficiently identifying task groupings for multi-task learning. Advances in Neural Information Processing Systems, 34.

[Gao et al., 2021] Gao, M., Xie, T., Du, S. S., and Yang, L. F. (2021). A provably efficient algorithm for linear markov decision process with low switching cost. arXiv preprint arXiv:2101.00494.

[Hallak et al., 2015] Hallak, A., Di Castro, D., and Mannor, S. (2015). Contextual markov decision processes. arXiv preprint arXiv:1502.02259.

[Hessel et al., 2019] Hessel, M., Soyer, H., Espeholt, L., Czarnecki, W., Schmitt, S., and van Hasselt, H. (2019). Multi-task deep reinforcement learning with popart. In Proceedings of the AAAI Conference on Artificial Intelligence, volume 33, pages 3796–3803.

[Isele et al., 2016] Isele, D., Rostami, M., and Eaton, E. (2016). Using task features for zero-shot knowledge transfer in lifelong learning. In IJCAI, volume 16, pages 1620–1626.

[Jin et al., 2020] Jin, C., Yang, Z., Wang, Z., and Jordan, M. I. (2020). Provably efficient reinforcement learning with linear function approximation. In Conference on Learning Theory, pages 2137–2143.

[Kakade et al., 2020] Kakade, S., Krishnamurthy, A., Lowrey, K., Ohnishi, M., and Sun, W. (2020). Information theoretic regret bounds for online nonlinear control. Advances in Neural Information Processing Systems, 33:15312–15325.
Lecarpentier et al., 2021] Lecarpentier, E., Abel, D., Asadi, K., Jinnai, Y., Rachelson, E., and Littman, M. L. (2021). Lipschitz lifelong reinforcement learning. In Proceedings of the AAAI Conference on Artificial Intelligence, volume 35, pages 8270–8278.

Levy and Mansour, 2022] Levy, O. and Mansour, Y. (2022). Learning efficiently function approximation for contextual mdp. arXiv preprint arXiv:2203.00995.

Modi et al., 2018] Modi, A., Jiang, N., Singh, S., and Tewari, A. (2018). Markov decision processes with continuous side information. In Algorithmic Learning Theory, pages 597–618. PMLR.

Modi and Tewari, 2020] Modi, A. and Tewari, A. (2020). No-regret exploration in contextual reinforcement learning. In Conference on Uncertainty in Artificial Intelligence, pages 829–838. PMLR.

Roy et al., 2021] Roy, N., Posner, I., Barfoot, T., Beaudoin, P., Bengio, Y., Bohg, J., Brock, O., Depatie, I., Fox, D., Koditschek, D., et al. (2021). From machine learning to robotics: Challenges and opportunities for embodied intelligence. arXiv preprint arXiv:2110.15245.

Silver et al., 2013] Silver, D. L., Yang, Q., and Li, L. (2013). Lifelong machine learning systems: Beyond learning algorithms. In 2013 AAAI spring symposium series.

Sodhani et al., 2021] Sodhani, S., Zhang, A., and Pineau, J. (2021). Multi-task reinforcement learning with context-based representations. In International Conference on Machine Learning, pages 9767–9779. PMLR.

Sun et al., 2021] Sun, Y., Yin, X., and Huang, F. (2021). Temple: Learning template of transitions for sample efficient multi-task rl. In Proceedings of the AAAI Conference on Artificial Intelligence, volume 35, pages 9765–9773.

Thrun and Mitchell, 1995] Thrun, S. and Mitchell, T. M. (1995). Lifelong robot learning. Robotics and autonomous systems, 15(1-2):25–46.

Wilson et al., 2007] Wilson, A., Fern, A., Ray, S., and Tadepalli, P. (2007). Multi-task reinforcement learning: a hierarchical bayesian approach. In Proceedings of the 24th international conference on Machine learning, pages 1015–1022.

Wu et al., 2021] Wu, J., Braverman, V., and Yang, L. (2021). Accommodating picky customers: Regret bound and exploration complexity for multi-objective reinforcement learning. Advances in Neural Information Processing Systems, 34:13112–13124.

Xie and Finn, 2021] Xie, A. and Finn, C. (2021). Lifelong robotic reinforcement learning by retaining experiences. arXiv preprint arXiv:2109.09180.

Yang and Wang, 2019] Yang, L. and Wang, M. (2019). Sample-optimal parametric q-learning using linearly additive features. In International Conference on Machine Learning, pages 6995–7004. PMLR.

Yang et al., 2020] Yang, R., Xu, H., Wu, Y., and Wang, X. (2020). Multi-task reinforcement learning with soft modularization. Advances in Neural Information Processing Systems, 33:4767–4777.

Zhan et al., 2017] Zhan, Y., Ammar, H. B., and Taylor, M. E. (2017). Scalable lifelong reinforcement learning. Pattern Recognition, 72:407–418.

Zhang and Wang, 2021] Zhang, C. and Wang, Z. (2021). Provably efficient multi-task reinforcement learning with model transfer. Advances in Neural Information Processing Systems, 34.
A Proofs of Section 3

To prove Theorem 1, we will use the high probability event $\mathcal{E}_2$ defined in Lemma 5 to prove the UCB nature of Lifelong-LSVI in Lemma 6, which is the key to controlling the regret. We first state the following lemma that will be used in the proof of Lemma 5.

**Lemma 4.** Under the setting of Theorem 1, let $c_\beta$ be the constant in the definition of $\beta$. Then, for a fixed $w$, there is an absolute constant $c_0$ independent of $c_\beta$, such that for all $(h, k) \in [H] \times [K]$, with probability at least $1 - \delta$ it holds that

$$
\left\| \sum_{\tau=1}^{k-1} \phi^\tau_h \left( V_{h+1}^k(s_{h+1}^\tau, w) - \mathbb{P}_h[V_{h+1}^k(\cdot, w)](s_h^\tau, a_h^\tau) \right) \right\|_{(A^k_h)^{-1}} \leq c_0 H \left( d + \sqrt{d} \right) \sqrt{\log((c_\beta + 1)dd'T/\delta)},
$$

where $c_0$ and $c_\beta$ are two independent absolute constants.

**Proof.** We note that $\|\eta_h\|_2 \leq \sqrt{T}$ (Assumption 2), $\|\theta^k_h(w)\|_2 \leq H\sqrt{T}$ (Lemma 16), and $\left( A^k_h \right)^{-1} \leq \frac{1}{\lambda}$. Thus, Lemmas 17 and 19 together imply that for all $(h, k) \in [H] \times [K]$, with probability at least $1 - \delta$ it holds that

$$
\left\| \sum_{\tau=1}^{k-1} \phi^\tau_h \left( V_{h+1}^k(s_{h+1}^\tau, w) - \mathbb{P}_h[V_{h+1}^k(\cdot, w)](s_h^\tau, a_h^\tau) \right) \right\|_{(A^k_h)^{-1}}^2 \leq 4H^2 \left( \frac{d}{2} \log \left( \frac{k + \lambda}{\lambda} \right) + d\log(1 + 4d'/\epsilon) + d\log(1 + 4Hd'/\epsilon) + d^2 \log \left( \frac{1 + 8B^2\sqrt{d}}{\lambda \epsilon^2} \right) + \log \left( \frac{1}{\delta} \right) \right) + \frac{8k^2\epsilon^2}{\lambda}.
$$

If we let $\epsilon = \frac{dH}{\sqrt{T}}$ and $\beta = c_\beta \left( d + \sqrt{d} \right) H \sqrt{\log(d'T/\delta)}$, then, there exists an absolute constant $C > 0$ that is independent of $c_\beta$ such that

$$
\left\| \sum_{\tau=1}^{k-1} \phi^\tau_h \left( V_{h+1}^k(s_{h+1}^\tau, w) - \mathbb{P}_h[V_{h+1}^k(\cdot, w)](s_h^\tau, a_h^\tau) \right) \right\|_{(A^k_h)^{-1}}^2 \leq C(d' + d^2) H^2 \log \left( (c_\beta + 1)dd'T/\delta \right).
$$

□

**Lemma 5.** Let the setting of Theorem 1 holds. The event

$$
\mathcal{E}_2(w) := \left\{ \left\| \theta^k_h(w) - \tilde{\theta}^k_h(w) \right\|_{A^k_h} \leq \beta, \forall (h, k) \in [H] \times [K] \right\}.
$$

holds with probability at least $1 - \delta$ for a fixed $w$.

**Proof.**

$$
\theta^k_h(w) - \tilde{\theta}^k_h(w) = \theta^k_h(w) - \left( A^k_h \right)^{-1} \sum_{\tau=1}^{k-1} \phi^\tau_h V_{h+1}^k(s_{h+1}^\tau, w)
$$

$$
= \left( A^k_h \right)^{-1} \left( A^k_h \theta^k_h(w) - \sum_{\tau=1}^{k-1} \phi^\tau_h V_{h+1}^k(s_{h+1}^\tau, w) \right)
$$

$$
= \lambda \left( A_h^k \right)^{-1} \theta^k_h(w) - \left( A_h^k \right)^{-1} \sum_{\tau=1}^{k-1} \phi^\tau_h \left( V_{h+1}^k(s_{h+1}^\tau, w) - \mathbb{P}_h[V_{h+1}^k(\cdot, w)](s_h^\tau, a_h^\tau) \right).
$$
Thus, in order to upper bound \( \| \boldsymbol{\theta}_h^k(w) - \bar{\boldsymbol{\theta}}_h^k(w) \|_{\Lambda_h^k} \), we bound \( \| q_1 \|_{\Lambda_h^k} \) and \( \| q_2 \|_{\Lambda_h^k} \) separately.

From Lemma 16, we have

\[
\| q_1 \|_{\Lambda_h^k} = \lambda \| \boldsymbol{\theta}_h^k(w) \|_{(\Lambda_h^k)^{-1}} \leq \sqrt{\lambda} \| \boldsymbol{\theta}_h^k(w) \|_2 \leq H \sqrt{\lambda d}. \tag{14}
\]

Thanks to Lemma 4, for all \((w, h, k)\), with probability at least \(1 - \delta\), it holds that

\[
\| q_2 \|_{\Lambda_h^k} \leq \sum_{t=1}^{k-1} \phi_h^w \left( V_h^{k-1}(s_{h+1}, w) - \mathbb{E}_h [V_h^{k-1}(, w)](s_{h+1}, a_{h+1}) \right) \| \Lambda_h^k \|_{(\Lambda_h^k)^{-1}} \leq c_0 H \left( d + \sqrt{d'} \right) \sqrt{\log(c_\beta + 1) dd'T/\delta}, \tag{15}
\]

where \( c_0 \) and \( c_\beta \) are two independent absolute constants.

Combining (14) and (15), for all \((w, h, k)\), with probability at least \(1 - \delta\), it holds that

\[
\left\| \boldsymbol{\theta}_h^k(w) - \bar{\boldsymbol{\theta}}_h^k(w) \right\|_{\Lambda_h^k} \leq cH \left( d + \sqrt{d'} \right) \sqrt{\log(dd'T/\delta)}
\]

for some absolute constant \( c > 0 \).

\[ \square \]

**Lemma 6.** Let \( \widetilde{W} = \{w^1, w^2, \ldots, w^K\} \). Conditioned on events \( \{ \mathcal{E}_2(w) \}_{w \in \widetilde{W}} \) defined in (13), and with \( Q_h^k \) computed as in (8), it holds that \( Q_h^k(s, a, w) \geq Q_h^k(s, a, w) \) for all \((s, a, w, h, k) \in \mathcal{S} \times \mathcal{A} \times \widetilde{W} \times [H] \times [K] \).

**Proof.** We first note that conditioned on events \( \{ \mathcal{E}_2(w) \}_{w \in \widetilde{W}} \), for all \((s, a, w, h, k) \in \mathcal{S} \times \mathcal{A} \times \widetilde{W} \times [H] \times [K] \), it holds that

\[
\left| r_h(s, a, w) + \left( \boldsymbol{\theta}_h^k(w), \phi(s, a) \right) - Q_h^k(s, a, w) - \mathbb{E}_h \left[ V_h^{k-1}(, w) - V_h^{k-1}(, w) \right] (s, a) \right|
\]

\[
= \left| r_h(s, a, w) + \left( \boldsymbol{\theta}_h^k(w), \phi(s, a) \right) - r_h(s, a, w) - \mathbb{E}_h \left[ V_h^{k-1}(, w) \right] (s, a) \right|
\]

\[
= \left| \left( \boldsymbol{\theta}_h^k(w), \phi(s, a) \right) - \mathbb{E}_h \left[ V_h^{k-1}(, w) \right] (s, a) \right|
\]

\[
= \left| \left( \boldsymbol{\theta}_h^k(w) - \bar{\boldsymbol{\theta}}_h^k(w), \phi(s, a) \right) \right|
\]

\[
\leq \frac{\beta}{\| \phi(s, a) \|_{(\Lambda_h^k)^{-1}}}, \quad \text{(Lemma 5)}
\]

for any policy \( \pi \).

Now, we prove the lemma by induction. The statement holds for \( H \) because \( Q_{H+1}^k(, \ldots, ) = Q_{H+1}^k(, \ldots, ) = 0 \) and thus conditioned on events \( \{ \mathcal{E}_2(w) \}_{w \in \widetilde{W}} \), defined in (13), for all \((s, a, w, k) \in \mathcal{S} \times \mathcal{A} \times \widetilde{W} \times [K] \), we have

\[
\left| r_H(s, a, w) + \left( \boldsymbol{\theta}_H^k(w), \psi(s, a) \right) - Q_H^k(s, a, w) \right| \leq \beta \| \phi(s, a) \|_{(\Lambda_h^k)^{-1}}.
\]
Therefore, conditioned on events \( \{ \mathcal{E}_2(w) \} \) for all \((s, a, w, k) \in \mathcal{S} \times \mathcal{A} \times \tilde{\mathcal{W}} \times [K] \), we have

\[
Q_H^k(s, a, w) \leq r_H(s, a, w) + \left\langle \theta_H^k(w), \phi(s, a) \right\rangle + \beta \| \phi(s, a) \|_{(\mathcal{A}_H^k)^{-1}} = Q_H^k(s, a, w).
\]

Now, suppose the statement holds at time-step \( h + 1 \) and consider time-step \( h \). Conditioned on events \( \{ \mathcal{E}_2(w) \} \), for all \((s, a, w, h, k) \in \mathcal{S} \times \mathcal{A} \times \tilde{\mathcal{W}} \times [H] \times [K] \), we have

\[
0 \leq r_h(s, a, w) + \left\langle \theta_h^k(w), \phi(s, a) \right\rangle - Q_h^k(s, a, w) - \mathbb{P}_h \left[ V_{h+1}^k(s, a, w) - V_{h+1}^\pi(s, a, w) \right] (s, a) + \beta \| \phi(s, a) \|_{(\mathcal{A}_h^k)^{-1}}
\]

\[
\leq r_h(s, a, w) + \left\langle \theta_h^k(w), \phi(s, a) \right\rangle - Q_h^k(s, a, w) + \beta \| \phi(s, a) \|_{(\mathcal{A}_h^k)^{-1}}. \tag{Induction assumption}
\]

Therefore, conditioned on events \( \{ \mathcal{E}_2(w) \} \), for all \((s, a, w, h, k) \in \mathcal{S} \times \mathcal{A} \times \tilde{\mathcal{W}} \times [H] \times [K] \), we have

\[
Q_h^k(s, a, w) \leq r_h(s, a, w) + \left\langle \theta_h^k(w), \phi(s, a) \right\rangle + \beta \| \phi(s, a) \|_{(\mathcal{A}_h^k)^{-1}} = Q_h^k(s, a, w).
\]

This completes the proof.

\[\square\]

### A.1 Proof of Theorem 1

Let \( \delta_h^k = V_h^k(s^k_h, w^k) - V_h^\pi(s^k_h, w^k) \) and \( \xi_{h+1}^k = \mathbb{E} \left[ \delta_{h+1}^k | s_h^k, a_h^k \right] - \delta_{h+1}^k \). Conditioned on events \( \{ \mathcal{E}_2(w) \} \), for all \((s, a, w, h, k) \in \mathcal{S} \times \mathcal{A} \times \tilde{\mathcal{W}} \times [H] \times [K] \), we have

\[
Q_h^k(s, a, w) - Q_h^\pi(s, a, w) = r_h(s, a, w) + \left\langle \theta_h^k(w), \phi(s, a) \right\rangle - Q_h^k(s, a, w) + \beta \| \phi(s, a) \|_{(\mathcal{A}_h^k)^{-1}}
\]

\[
\leq \mathbb{P}_h \left[ V_h^k(s, a, w) - V_h^\pi(s, a, w) \right] (s, a) + 2\beta \| \phi(s, a) \|_{(\mathcal{A}_h^k)^{-1}}. \tag{16}
\]

Note that \( \delta_h^k \leq \mathbb{E} \left[ \delta_{h+1}^k | a_h^k, w_h^k \right] - \delta_h^k \). Thus, combining (16), Lemma 5, and a union bound over \( \widetilde{\mathcal{W}} \), we conclude that for all \((h, k) \in [H] \times [K] \), with probability at least \( 1 - \delta \), it holds that

\[
\delta_h^k \leq \xi_{h+1}^k + 2\beta \| \phi(s_h^k, a_h^k) \|_{(\mathcal{A}_h^k)^{-1}}.
\]

Now, we complete the regret analysis

\[
R_K = \sum_{k=1}^K V_1^k(s_1^k, w^k) - V_1^\pi(s_1^k, w^k)
\]

\[
\leq \sum_{k=1}^K V_1^k(s_1^k, w^k) - V_1^\pi(s_1^k, w^k) \tag{Lemma 6}
\]

\[
= \sum_{k=1}^K \delta_1^k
\]

\[
\leq \sum_{k=1}^K \sum_{h=1}^H \xi_h^k + 2\beta \sum_{k=1}^K \sum_{h=1}^H \| \phi(s_h^k, a_h^k) \|_{(\mathcal{A}_h^k)^{-1}}
\]

\[
\leq 2H \sqrt{T \log(dT/\delta)} + 2H \beta \sqrt{2dK \log(1 + K/\lambda)}
\]

\[
\leq \tilde{O} \left( \sqrt{N(d^3 + dd^r)H^3T} \right).
\]
The third inequality is true because of the following: we observe that \( \{ \xi^k_h \} \) is a martingale difference sequence satisfying \( |\xi^k_h| \leq 2H \). Thus, thanks to Azuma-Hoeffding inequality, we have

\[
\mathbb{P}\left( \sum_{k=1}^{K} \sum_{h=1}^{H} \xi^k_h \leq 2H \sqrt{T \log(dT/\delta)} \right) \geq 1 - \delta. \tag{17}
\]

In order to bound \( \sum_{k=1}^{K} \sum_{h=1}^{H} \| \phi^k_h \| (A^k_h)^{-1} \), note that for any \( h \in [H] \), we have

\[
\sum_{k=1}^{K} \| \phi^k_h \| (A^k_h)^{-1} \leq \sqrt{K} \sum_{k=1}^{K} \| \phi^k_h \|^2 (A^k_h)^{-1} \leq 2K \log \left( \frac{\det(A^k_h)}{\det(A^k_h)} \right) \leq 2dK \log \left( 1 + \frac{K}{d} \right). \tag{18}
\]

In inequality (18), we used the standard argument in regret analysis of linear bandits [Abbasi-Yadkori et al., 2011] (Lemma 11) as follows:

\[
\sum_{t=1}^{n} \min \left( \| y_t \|^2, 1 \right) \leq 2 \log \frac{\det V_{n+1}}{\det V_1} \text{ where } V_n = V_1 + \sum_{t=1}^{n-1} y_t y_t^T. \tag{20}
\]

In inequality (19), we used Assumption 2 and the fact that \( \det(A) = \prod_{i=1}^{d} \lambda_i(A) \leq (\text{trace}(A)/d)^d \).

**B Proofs of Section 4**

**B.1 Proof of Lemma 1**

First, we state the following lemma that will be used in the proof of Lemma 1.

**Lemma 7.** Under the setting of Lemma 1, let \( c_\beta \) be a constant in the definition of \( \beta \). Then, for a fixed \( w \), there is an absolute constant \( c_0 \) independent of \( c_\beta \), such that for all \( (h, k) \in [H] \times [K] \), with probability at least \( 1 - \delta \) it holds that

\[
\left\| \sum_{i=1}^{k-1} \phi^k_h \left( V^k_{h+1}(s^k_{h+1}, w) - \mathbb{P}_h[V^k_{h+1}(., w)](s^k_{h+1}, a^k_h) \right) \left( A^k_h \right)^{-1} \right\| \leq c_0 H \left( d + \sqrt{md} \right) \sqrt{\log((c_\beta + 1)mdT/\delta)},
\]

where \( c_0 \) and \( c_\beta \) are two independent absolute constants.

**Proof.** We note that \( \left\| \eta_h + \xi^k_h \right\|_2 \leq (1 + H)\sqrt{md} \) and \( \left\| (A^k_h)^{-1} \right\|_2 \leq \frac{1}{\lambda} \). Thus, Lemmas 17 and 20 together imply that for all \( (h, k) \in [H] \times [K] \), with probability at least \( 1 - \delta \) it holds that

\[
\left\| \sum_{i=1}^{k-1} \phi^k_h \left( V^k_{h+1}(s^k_{h+1}, w) - \mathbb{P}_h[V^k_{h+1}(., w)](s^k_{h+1}, a^k_h) \right) \right\|^2 \left( A^k_h \right)^{-1} \leq 4H^2 \left( \frac{d}{2} \log \left( \frac{k + \lambda}{\lambda} \right) + md \log(1 + 8H\sqrt{md}/\epsilon) + d^2 \log \left( \frac{1 + 32L^2\beta^2}\lambda \right) + \log \left( \frac{1}{\delta} \right) \right) + \frac{8k^2\epsilon}{\lambda}.
\]
If we let $\epsilon = \frac{dT}{\delta}$ and $\beta = c_\beta(d + \sqrt{md})H\sqrt{\log(dT/\delta)}$, then, there exists an absolute constant $C > 0$ that is independent of $c_\beta$ such that

$$
\left\| \sum_{\tau=1}^{k-1} \phi^*_h \left( V^k_{h+1}(s^*_{h+1}, w) - P_h[V^k_{h+1}(., w)](s^*_h, a^*_h) \right) \right\|_2^2 \leq C(md + d^2)H^2 \log((c_\beta + 1)md/\delta).
$$

\[\square\]

Now, we begin the formal proof of Lemma 1:

$$
\theta^k_h(w) - \tilde{\theta}^k_h(w) = \theta^k_h(w) - \left( \Lambda^k_h \right)^{-1} \sum_{\tau=1}^{k-1} \phi^*_h V^k_{h+1}(s^*_{h+1}, w)
$$

$$
= \left( \Lambda^k_h \right)^{-1} \left( \lambda \Lambda^k_h \theta^k_h(w) - \sum_{\tau=1}^{k-1} \phi^*_h V^k_{h+1}(s^*_{h+1}, w) \right)
$$

$$
= \lambda \left( \Lambda^k_h \right)^{-1} \theta^k_h(w) - \left( \Lambda^k_h \right)^{-1} \sum_{\tau=1}^{k-1} \phi^*_h \left( V^k_{h+1}(s^*_{h+1}, w) - P_h[V^k_{h+1}(., w)](s^*_h, a^*_h) \right).
$$

Thus, in order to upper bound $\left\| \theta^k_h(w) - \tilde{\theta}^k_h(w) \right\|_{\Lambda^k_h}$, we bound $\|q_1\|_{\Lambda^k_h}$ and $\|q_2\|_{\Lambda^k_h}$ separately.

From Lemma 16, we have

$$
\|q_1\|_{\Lambda^k_h} = \lambda \left\| \theta^k_h(w) \right\|_{(\Lambda^k_h)^{-1}} \leq \sqrt{\lambda} \left\| \theta^k_h(w) \right\|_2 \leq H \sqrt{\lambda}.
$$

(21)

Thanks to Lemma 7, for all $(w, h, k)$, with probability at least $1 - \delta$, it holds that

$$
\|q_2\|_{\Lambda^k_h} \leq \left\| \sum_{\tau=1}^{k-1} \phi^*_h \left( V^k_{h+1}(s^*_{h+1}, w) - P_h[V^k_{h+1}(., w)](s^*_h, a^*_h) \right) \right\|_{(\Lambda^k_h)^{-1}}
$$

$$
\leq c_0 H \left( d + \sqrt{md} \right) \sqrt{\log((c_\beta + 1)md/\delta)},
$$

(22)

where $c_0$ and $c_\beta$ are two independent absolute constants.

Combining (21) and (22), for all $(h, k) \in [H] \times [K]$, with probability at least $1 - \delta$, it holds that

$$
\left\| \theta^k_h(w) - \tilde{\theta}^k_h(w) \right\|_{\Lambda^k_h} \leq c H \left( d + \sqrt{md} \right) \sqrt{\lambda \log(md/\delta)}
$$

for some absolute constant $c > 0$.

**B.2 Proof of Lemma 2**

Thanks to Assumption 3 and conditioned on events $\{E_1(w)\}_{w \in \tilde{W}}$, one set of solution for (11) is $\left\{ \theta^k_h(w^{(j)}) \right\}_{j \in [n]}$ and $\xi^{V^k_{h+1}}$ with corresponding zero optimal objective value. Therefore, it holds that

$$
\left\langle \theta^k_h, \phi(s, a) \right\rangle = \left\langle \xi^k_h, \psi(s, a, w^{(j)}) \right\rangle, \quad \forall (j, (s, a)) \in [n] \times D.
$$

(23)
Let \( (s^{(i)}, a^{(i)}) \) be the \( i \)-th element of \( D \) and \( \{c'_i(s, a)\}_{i \in [d]} \) be the coefficients such that

\[
\phi(s, a) = \sum_{i \in [d]} c'_i(s, a) \phi\left( s^{(i)}, a^{(i)} \right).
\]

For any triple \((s, a, j) \in S \times A \times [n]\), we have

\[
\langle \xi_h^k, \psi\left( s, a, w^{(j)} \right) \rangle = \langle \xi_h^k, \phi(s, a) \otimes \rho\left( w^{(j)} \right) \rangle = \langle \xi_h^k, \sum_{i \in [d]} c'_i(s, a) \phi\left( s^{(i)}, a^{(i)} \right) \otimes \rho\left( w^{(j)} \right) \rangle \tag{Assumption 4}
\]

\[
= \sum_{i \in [d]} c'_i(s, a) \langle \xi_h^k, \psi\left( s^{(i)}, a^{(i)}, w^{(j)} \right) \rangle \tag{Eqn. (23)}
\]

\[
= \sum_{i \in [d]} c'_i(s, a) \langle \theta_h^{k(j)}, \phi\left( s^{(i)}, a^{(i)} \right) \rangle.
\tag{24}
\]

For any \((s, a, w) \in S \times A \times W\), it holds that

\[
\mathbb{P}_h \left[ V_{h+1}^k(., w) \right] (s, a) = \langle \theta_h^{k}(w), \phi(s, a) \rangle \tag{Eqn. (5)}
\]

\[
= \langle \xi_h^{V_{h+1}^k}, \psi(s, a, w) \rangle \tag{Assumption 3}
\]

\[
= \sum_{j \in [n]} c_j(w) \langle \xi_h^{V_{h+1}^k}, \psi\left( s, a, w^{(j)} \right) \rangle \tag{Eqn. (9)}
\]

\[
= \sum_{j \in [n]} c_j(w) \mathbb{P}_h \left[ V_{h+1}^k(., w^{(j)}) \right] (s, a) \tag{Assumption 3}
\]

\[
= \sum_{j \in [n]} c_j(w) \langle \theta_h^{k}(w^{(j)}), \phi(s, a) \rangle. \tag{25}
\]
Finally, conditioned on events \(\{\mathcal{E}_1(w)\}_{w \in \mathcal{W}}\), for all \((s, a, w, h, k) \in \mathcal{S} \times \mathcal{A} \times \mathcal{W} \times [H] \times [K]\), it holds that

\[
\begin{align*}
\left| \langle \xi_h, \psi(s, a, w) \rangle - \mathbb{P}_h \left[ V_{h+1}^k(., w) \right] (s, a) \right| \\
= \left| \langle \xi_h, \psi(s, a, w) \rangle - \langle \theta_h^k (w), \phi(s, a) \rangle \right| \\
= \sum_{j \in [n]} c_j(w) \left| \langle \xi_h, \psi(s, a, w^{(j)}) \rangle - \langle \theta_h^k (w^{(j)}), \phi(s, a) \rangle \right| \\
\leq \sum_{j \in [n]} c_j(w) \left| \langle \xi_h, \psi(s, a, w^{(j)}) \rangle - \langle \theta_h^{(j)} (w^{(j)}), \phi(s, a) \rangle \right| \\
\quad + \sum_{j \in [n]} c_j(w) \left| \langle \theta_h^k (w^{(j)}), \phi(s, a) \rangle - \langle \theta_h^k (w^{(j)}), \phi(s, a) \rangle \right| \\
\leq \sum_{j \in [n]} c_j(w) \left| \langle \xi_h, \psi(s, a, w^{(j)}) \rangle - \langle \theta_h^{(j)} (w^{(j)}), \phi(s, a) \rangle \right| \\
\quad + \sum_{j \in [n]} c_j(w) \left| \langle \theta_h^k (w^{(j)}), \phi(s, a) \rangle - \langle \theta_h^k (w^{(j)}), \phi(s, a) \rangle \right| \\
\leq 2L \beta \| \phi(s, a) \| (\Lambda_h^k)^{-1}.
\end{align*}
\]  

(Eqns. (9) and (25))

**B.3 Proof of Lemma 3**

We first note that conditioned on events \(\{\mathcal{E}_1(w)\}_{w \in \mathcal{W}}\), for all \((s, a, w, h, k) \in \mathcal{S} \times \mathcal{A} \times \mathcal{W} \times [H] \times [K]\), it holds that

\[
\begin{align*}
\left| r_h(s, a, w) + \langle \xi_h, \psi(s, a, w) \rangle - Q_h^\pi(s, a, w) - \mathbb{P}_h \left[ V_{h+1}^k(., w) - V_{h+1}^\pi(., w) \right] (s, a) \right| \\
= \left| r_h(s, a, w) + \langle \xi_h, \psi(s, a, w) \rangle - r_h(s, a, w) - \mathbb{P}_h \left[ V_{h+1}^k(., w) \right] (s, a) \right| \\
= \left| \langle \xi_h, \psi(s, a, w) \rangle - \mathbb{P}_h \left[ V_{h+1}^k(., w) \right] (s, a) \right| \\
\leq 2L \beta \| \phi(s, a) \| (\Lambda_h^k)^{-1},
\end{align*}
\]

for any policy \(\pi\).

Now, we prove the lemma by induction. The statement holds for \(H\) because \(Q_{H+1}^h(\ldots) = Q_{H+1}^\pi(\ldots) = 0\) and thus conditioned events \(\{\mathcal{E}_1(w)\}_{w \in \mathcal{W}}\), defined in (12), for all \((s, a, w, k) \in \mathcal{S} \times \mathcal{A} \times \mathcal{W} \times [K]\), we have

\[
\begin{align*}
\left| r_H(s, a, w) + \langle \xi_h, \psi(s, a, w) \rangle - Q_h^\pi(s, a, w) \right| \leq 2L \beta \| \phi(s, a) \| (\Lambda_h^k)^{-1}.
\end{align*}
\]
Therefore, conditioned on events \( \{ \mathcal{E}_1(w) \}_{w \in \tilde{W}} \), for all \( (s, a, w, k) \in \mathcal{S} \times \mathcal{A} \times \tilde{W} \times [K] \), we have
\[
Q^*_H(s, a, w) \leq r_H(s, a, w) + \left\langle \xi^k_H, \psi(s, a, w) \right\rangle + 2L\beta\|\phi(s, a)\|_{(A^k_h)}^{-1}
\]
\[
= \left\{ r_H(s, a, w) + \left\langle \xi^k_H, \psi(s, a, w) \right\rangle + 2L\beta\|\phi(s, a)\|_{(A^k_h)}^{-1} \right\}^+
\]
\[
= Q^k_H(s, a, w),
\]
where the first equality follows from the fact that \( Q^*_H(s, a, w) \geq 0 \). Now, suppose the statement holds at time-step \( h + 1 \) and consider time-step \( h \). Conditioned on events \( \{ \mathcal{E}_1(w) \}_{w \in \tilde{W}} \), for all \( (s, a, w, h, k) \in \mathcal{S} \times \mathcal{A} \times \tilde{W} \times [H] \times [K] \), we have
\[
0 \leq r_h(s, a, w) + \left\langle \xi^k_h, \psi(s, a, w) \right\rangle - Q^*_h(s, a, w) - \mathbb{P}_h \left[ V^k_{h+1}(\cdot, w) - V^*_h(\cdot, w) \right] (s, a) + 2L\beta\|\phi(s, a)\|_{(A^k_h)}^{-1}
\]
\[
\leq r_h(s, a, w) + \left\langle \xi^k_h, \psi(s, a, w) \right\rangle - Q^*_h(s, a, w) + 2L\beta\|\phi(s, a)\|_{(A^k_h)}^{-1}.
\]
(Induction assumption)

Therefore, conditioned on events \( \{ \mathcal{E}_1(w) \}_{w \in \tilde{W}} \), for all \( (s, a, w, h, k) \in \mathcal{S} \times \mathcal{A} \times \tilde{W} \times [H] \times [K] \), we have
\[
Q^*_h(s, a, w) \leq r_h(s, a, w) + \left\langle \xi^k_h, \psi(s, a, w) \right\rangle + 2L\beta\|\phi(s, a)\|_{(A^k_h)}^{-1}
\]
\[
= \left\{ r_h(s, a, w) + \left\langle \xi^k_h, \psi(s, a, w) \right\rangle + 2L\beta\|\phi(s, a)\|_{(A^k_h)}^{-1} \right\}^+
\]
\[
= Q^k_h(s, a, w),
\]
where the first equality follows from the fact that \( Q^*_h(s, a, w) \geq 0 \). This completes the proof.

### B.4 Proof of Theorem 2

First, we bound the number of times Algorithm 2 updates \( \xi^k \), i.e., number of planning calls. Let \( P \) be the total number of updates and \( k_p \) be the episode at which the agent did replanning for the \( p \)-th time. Note that \( \det A^k_h = \lambda^d \) and \( \det A^K_h \leq \text{trace}(A^K_h/d)^d \leq \left( \lambda + \frac{K}{n} \right)^d \), and consequently:
\[
\frac{\det A^K_h}{\det A^1_h} = \prod_{p=1}^P \frac{\det A^k_{h_p}}{\det A^{k_{p-1}}_{h_p}} \leq \left( 1 + \frac{K}{d\lambda} \right)^d,
\]
and therefore
\[
\prod_{h=1}^H \frac{\det A^K_h}{\det A^1_h} = \prod_{h=1}^H \prod_{p=1}^P \frac{\det A^k_{h_p}}{\det A^{k_{p-1}}_{h_p}} \leq \left( 1 + \frac{K}{d\lambda} \right)^{dH}.
\]
(27)

Since \( 1 \leq \frac{\det A^k_{h_p}}{\det A^{k_{p-1}}_{h_p}} \) for all \( p \in [P] \), we can deduce from (27) that
\[
\exists h \in [H] \text{ such that } e < \frac{\det A^k_{h}}{\det A^k_{h-1}} \text{ happens for at most } dH \log \left( 1 + \frac{K}{d\lambda} \right) \text{ number of episodes } k \in [K]. \text{ This concludes that the number of planing calls in UCB1vd is } dH \log \left( 1 + \frac{K}{d\lambda} \right).
Now, we prove the regret bound. Let \( \delta^k_h = V^k_h(s^k_h, w^k) - V^\pi_h(s^k_h, w^k) \) and \( \xi^k_h = \mathbb{E} \left[ \delta^k_{h+1} | s^k_h, a^k_h \right] - \delta^k_h \). Conditioned on events \( \{ \mathcal{E}_1(w) \}_{w \in \mathbb{W}} \) for all \((s, a, w, h, k) \in \mathcal{S} \times \mathcal{A} \times \mathbb{W} \times [H] \times [K] \), we have

\[
Q^k_h(s, a, w) - Q^\pi_h(s, a, w) = r_h(s, a, w) + \left( \xi^k_h, \psi(s, a, w) \right) - Q^\pi_h(s, a, w) + 2L\beta \| \phi(s, a) \|_{(\Lambda^k_h)^{-1}} \\
\leq \mathbb{P}_h \left[ V^k_{h+1}(., w) - V^\pi_{h+1}(., w) \right] (s, a) + 4L\beta \| \phi(s, a) \|_{(\Lambda^k_h)^{-1}}.
\]

(28)

Note that \( \delta^k_h \leq Q^k_h(s^k_h, a^k_h, w^k) - Q^\pi_h(s^k_h, a^k_h, w^k) \). Thus, combining (28), Lemma 1, and a union bound over \( \mathcal{W} \), we conclude that for all \((h, k) \in [H] \times [K] \), with probability at least \( 1 - \delta \), it holds that gives

\[
\delta^k_h \leq \xi^k_h + \delta^k_{h+1} + 4L\beta \| \phi(s^k_h, a^k_h) \|_{(\Lambda^k_h)^{-1}}.
\]

Note that for any positive semi-definite matrices \( \mathbf{A} \), \( \mathbf{B} \), and \( \mathbf{C} \) such that \( \mathbf{A} = \mathbf{B} + \mathbf{C} \), we have:

\[
\text{det}(\mathbf{A}) \geq \text{det}(\mathbf{B}), \quad \text{det}(\mathbf{A}) \geq \text{det}(\mathbf{C}),
\]

and for any \( x \neq 0 \) ([Abbasi-Yadkori et al., 2011, Lemm. 12]):

\[
\frac{\| x \|^2}{\| x \|^2_B} \leq \frac{\text{det}(\mathbf{A})}{\text{det}(\mathbf{B})} \quad \text{and} \quad \frac{\| x \|^2}{\| x \|^2_A} \leq \frac{\text{det}(\mathbf{A})}{\text{det}(\mathbf{B})}.
\]

(30)

Now, we complete the regret analysis following similar steps as those of Theorem 1’s proof:

\[
R_K = \sum_{k=1}^K V^*_1(s^k_1, w^k) - V^\pi_1(s^k_1, w^k)
\]

\[
\leq \sum_{k=1}^K V^k_1(s^k_1, w^k) - V^\pi_1(s^k_1, w^k) \quad \text{(Lemma 3)}
\]

\[
= \sum_{k=1}^K \delta^k_h
\]

\[
\leq \sum_{k=1}^K \sum_{l=1}^H \xi^k_h + 4L\beta \sum_{k=1}^K \sum_{l=1}^H \| \phi(s^k_h, a^k_h) \|_{(\Lambda^k_h)^{-1}}
\]

\[
\leq \sum_{k=1}^K \sum_{l=1}^H \xi^k_h + 4L\beta \sum_{k=1}^K \sum_{l=1}^H \| \phi(s^k_h, a^k_h) \|_{(\Lambda^k_h)^{-1}} \sqrt{\frac{\text{det}(\Lambda^k_h)}{\text{det}(\Lambda^k_h)}}
\]

\[
\leq 2H \sqrt{T \log(dT/\delta)} + 8HL\beta \sqrt{2dK \log(1 + K/\lambda)}
\]

\[
\leq \tilde{O} \left( L \sqrt{\lambda (d^3 + md^2)H^3T} \right).
\]

C UCBlvd with Unknown Rewards

In order for our analysis to go through, we need a slightly different completeness assumption as below:

**Assumption 5.** Given feature maps \( \phi: \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}^d \) and \( \psi: \mathcal{S} \times \mathcal{A} \times \mathcal{W} \rightarrow \mathbb{R}^{d'} \), consider function class

\[
\mathcal{F} = \left\{ f: f(s, w) = \min_{a \in \mathcal{A}} \left\{ \max_{\nu \in \mathbb{R}^d} \left\{ \left( \nu, \psi(s, a, w) \right) + \beta \| \phi(s, a) \|_{\Lambda^{-1}} + \tilde{\beta} \| \psi(s, a, w) \|_{\tilde{\Lambda}^{-1}} \right\} + H \right\} , H \right\},
\]

\( \nu \in \mathbb{R}^{d'}, \Lambda \in \mathbb{S}^{d'+1}, \tilde{\Lambda} \in \mathbb{S}^{d'+1}, \beta, \tilde{\beta} \in \mathbb{R} \).
Algorithm 3: UCBBlvd with Unknown Rewards

**Input:** $A, \lambda, \delta, H, K, \beta, \hat{\beta}$

1. Set: $Q_{H+1}(\ldots) = 0, \forall k \in [K], \tilde{k} = 1$

2. for episodes $k = 1, \ldots, K$ do
   1. Observe the initial state $s_k^1$ and the task context $w_k$.
   2. if $\exists h \in [H]$ such that $\frac{\det \Lambda_k^h}{\det \tilde{\Lambda}_k} > e$ or $\frac{\det \Lambda_k^\tau}{\det \tilde{\Lambda}_k^\tau} > e$ then
      1. $\tilde{k} = k$
   3. for time-steps $h = H, \ldots, 1$ do
      1. Compute $\xi_h$ as in (33).
   4. for time-steps $h = 1, \ldots, H$ do
      1. Compute $Q_h^k(s_k^h, a, w_k)$ for all $a \in A$ as in (31).
      2. Play $a_k^h = \arg\max_{a \in A} Q_h^k(s_k^h, a, w_k)$ and observe $s_{h+1}^k$ and $r_h^k$.

Then for any $f \in F$, and $h \in [H]$, there exists a vector $\xi_h^f \in \mathbb{R}^d$ with $\|\xi_h^f\| \leq H\sqrt{d}$ such that

\[ \mathbb{P}_h[f(., w)](s, a) = \langle \xi_h^f, \psi(s, a, w) \rangle. \]

**C.1 Overview**

Let $\psi^*_h = \psi(s_h^*, a_h^*, w^*)$. UCBBlvd with unknown rewards works with the following action-value functions:

\[ Q_h^k(s, a, w) = \left( \tilde{\eta}_h^k + \xi_h^k, \phi(s, a) \right) + \beta \|\phi(s, a)\|_{(\Lambda_h^k)^{-1}} + \hat{\beta} \|\psi(s, a, w)\|_{(\tilde{\Lambda}_h^k)^{-1}} \]

where

\[ \tilde{\eta}_h^k = \left( \tilde{\Lambda}_h^{-1} \right) \sum_{\tau=1}^{k-1} \psi_h^\tau r_h^\tau \quad \text{and} \quad \tilde{\Lambda}_h = \lambda I_{md} + \sum_{\tau=1}^{k-1} \psi_h^\tau \psi_h^\tau \top, \]

and

\[ \hat{\xi}_h^k, \hat{\theta}_h^{k(j)} = \arg\min_{\xi, \theta} \sum_{j \in [n]} \sum_{(s, a) \in D} \left( \theta^{(j)}, \phi(s, a) \right) - \left( \theta^{(j)}, \psi(s, a, w^{(j)}) \right)^2 \]

s.t. $\|\theta^{(j)} - \hat{\theta}_h^{k(j)}(w^{(j)})\|_{\Lambda_h^k} \leq \beta, \forall j \in [n]$ and $\|\xi\|_2 \leq H \sqrt{md},$

$D = \{(s, a) : \phi(s, a) \text{ are } d \text{ linearly independent vectors.}\}$, and $\hat{\theta}_h^{k(j)}(w)$ and $\Lambda_h^k$ are defined in (6) and (7), respectively.

We note that compared to (10), action-value function defined in (31) involves an extra term $\langle \tilde{\eta}_h^k, \psi(s, a, w) \rangle + \hat{\beta} \|\psi(s, a, w)\|_{(\tilde{\Lambda}_h^k)^{-1}}$. This term is in fact an upper bound on $r_h(s, a, w)$. Specifically, from Theorem 2 in [Abbasi-Yadkori et al., 2011], we know that for $\hat{\beta} = \sqrt{\lambda md}$, it holds that

\[ \|\eta_h - \tilde{\eta}_h^k\|_{\tilde{\Lambda}_h^k} \leq \hat{\beta}, \forall (h, k) \in [H] \times [K]. \]

**Theorem 3.** Let $T = KH$. Under Assumptions 1, 2, 4, and 5, the number of planning calls in Algorithm 3 is at most $dH \log \left(1 + \frac{K}{\lambda^2}\right) + mdH \log \left(1 + \frac{K}{md}\right)$, and there exists an absolute constant $c > 0$ such that for
any fixed $\delta \in (0, 0.5)$, if we set $\lambda = 1$, $\beta = cH (md) \sqrt{\log(mdT/\delta)}$ and $\tilde{\beta} = \sqrt{md}$ in Algorithm 3, then with probability at least $1 - 2\delta$, it holds that

$$R_K \leq 2H \sqrt{T \log(dT/\delta)} + 4H \sqrt{K \left( L\beta \sqrt{2d \log(1 + K/\lambda)} + \tilde{\beta} \sqrt{2m d \log(1 + K/\lambda)} \right)} \leq \tilde{O} \left( L \sqrt{m^2 d^3 H^3 T} \right).$$

C.2 Necessary Analysis for the Proof of Theorem 3

**Lemma 8.** Let $c_{\beta}$ be a constant in the definition of $\beta$. Then, under Assumptions 1, 2, 4, and 5, for a fixed $w$, there is an absolute constant $c_0$ independent of $c_{\beta}$, such that for all $(h, k) \in [H] \times [K]$, with probability at least $1 - \delta$ it holds that

$$\left\| \sum_{\tau=1}^{k-1} \phi_h^\tau \left( V_{h+1}^k(s_{h+1}, w) - \mathbb{P}_h[V_{h+1}^k(., w)](s_h, a_h) \right) \right\|_{(\Lambda_h^k)^{-1}} \leq c_0 m d H \sqrt{\log((c_{\beta} + 1) mdT/\delta)},$$

where $c_0$ and $c_{\beta}$ are two independent absolute constants.

**Proof.** We note that $\left\| \eta_h^k + \xi_h^k \right\|_2 \leq H \sqrt{md} + K/\lambda$ and $\left\| \Lambda_h^k \right\|^{-1} \leq \frac{1}{\lambda}$ and $\left\| \Lambda_h^k \right\|^{-1} \leq \frac{1}{\lambda}$. Thus, Lemmas 17 and 21 together imply that for all $(h, k) \in [H] \times [K]$, with probability at least $1 - \delta$ it holds that

$$\left\| \sum_{\tau=1}^{k-1} \phi_h^\tau \left( V_{h+1}^k(s_{h+1}, w) - \mathbb{P}_h[V_{h+1}^k(., w)](s_h, a_h) \right) \right\|_{(\Lambda_h^k)^{-1}}^2 \leq 4H^2 \left( \frac{d}{2} \log \left( \frac{k + \lambda}{\lambda} \right) + md \log(1 + 8H \sqrt{md}/\epsilon) + d^2 \log \left( \frac{1 + 32L^2 \beta^2 \sqrt{d}}{\lambda \epsilon^2} \right) 
+ m d^2 \log \left( \frac{1 + 8\tilde{\beta}^2 \sqrt{md}}{\lambda \epsilon^2} \right) + \log \left( \frac{1}{\delta} \right) \right) + \frac{8k^2 \epsilon^2}{\lambda}.$$

If we let $\epsilon = \frac{dH}{4m^2}$ and $\beta = c_{\beta}(md) H \sqrt{\log(mdT/\delta)}$, then, there exists an absolute constant $C > 0$ that is independent of $c_{\beta}$ such that

$$\left\| \sum_{\tau=1}^{k-1} \phi_h^\tau \left( V_{h+1}^k(s_{h+1}, w) - \mathbb{P}_h[V_{h+1}^k(., w)](s_h, a_h) \right) \right\|_{(\Lambda_h^k)^{-1}}^2 \leq C(m^2 d^2) H^2 \log \left( (c_{\beta} + 1) mdT/\delta \right).$$

\[\square\]

**Lemma 9.** Under Assumptions 1, 2, 4, and 5, if we let $\beta = c_{\beta}(md) H \sqrt{\log(mdT/\delta)}$ with an absolute constant $c > 0$, then the event

$$\mathcal{E}_\delta(w) := \left\{ \left\| \theta_h^k(w) - \tilde{\theta}_h^k(w) \right\|_{(\Lambda_h^k)^{-1}} \leq \beta, \ \forall (h, k) \in [H] \times [K] \right\}.$$

holds with probability at least $1 - \delta$ for a fixed $w$.

**Proof.** The proof follows the same steps as those of Lemma 1, except that it uses Lemma 8 instead of Lemma 7 due to different structure of action-value functions $Q_h^k$ in this section. \[\square\]
Lemma 10. Let \( \tilde{W} = \{ w^\tau : \tau \in [K] \} \cup \{ w^{(j)} : j \in [n] \} \). Conditioned on events \( \{ E_3(w) \}_{w \in \tilde{W}} \) defined in (35), for all \((s, a, w, h, k) \in S \times A \times \tilde{W} \times [H] \times [K]\), it holds that
\[
\left| \left\langle \tilde{\eta}^k_H + \xi^k_H, \psi(s, a, w) \right\rangle - \mathbb{P}_h \left[ V^k_{h+1}(., w) \right] (s, a) \right| \leq 2L\beta \| \phi(s, a) \|_{(\Lambda^k_h)^{-1}}. \tag{34}
\]
Proof. The proof follows the exact same steps as those of Lemma 2's proof.

Lemma 11. Let \( \tilde{W} = \{ w^\tau : \tau \in [K] \} \cup \{ w^{(j)} : j \in [n] \} \). Conditioned on events \( \{ E_3(w) \}_{w \in \tilde{W}} \) defined in (35), and with \( Q^k_H \) computed as in (31), it holds that \( Q^k_H(s, a, w) \geq Q^k_H(s, a, w) \) for all \((s, a, w, h, k) \in S \times A \times \tilde{W} \times [H] \times [K]\).

Proof. We first note that conditioned on events \( \{ E_3(w) \}_{w \in \tilde{W}} \), for all \((s, a, w, h, k) \in S \times A \times \tilde{W} \times [H] \times [K]\), it holds that
\[
\left| \left\langle \tilde{\eta}^k_H + \xi^k_H, \psi(s, a, w) \right\rangle - \mathbb{P}_h \left[ V^k_{h+1}(., w) \right] (s, a) \right| \leq 2L\beta \| \phi(s, a) \|_{(\Lambda^k_h)^{-1}} + \beta \| \psi(s, a, w) \|_{(\Lambda^k_h)^{-1}}. \tag{35}
\]

for any policy \( \pi \).

Now, we prove the lemma by induction. The statement holds for \( H \) because \( Q^k_{H+1}(., ., .) = Q^k_{H+1}(., ., .) = 0 \) and thus conditioned events \( \{ E_3(w) \}_{w \in \tilde{W}} \), defined in (35), for all \((s, a, w, k) \in S \times A \times \tilde{W} \times [K]\), we have
\[
\left| \left\langle \tilde{\eta}^k_H + \xi^k_H, \psi(s, a, w) \right\rangle - \mathbb{P}_h \left[ V^k_{h+1}(., w) \right] (s, a) \right| \leq 2L\beta \| \phi(s, a) \|_{(\Lambda^k_h)^{-1}} + \beta \| \psi(s, a, w) \|_{(\Lambda^k_h)^{-1}}. \tag{36}
\]
Therefore, conditioned on events \( \{ E_3(w) \}_{w \in \tilde{W}} \), for all \((s, a, w, k) \in S \times A \times \tilde{W} \times [K]\), we have
\[
Q^k_H(s, a, w) \leq \left\langle \tilde{\eta}^k_H + \xi^k_H, \psi(s, a, w) \right\rangle + 2L\beta \| \phi(s, a) \|_{(\Lambda^k_h)^{-1}} + \beta \| \psi(s, a, w) \|_{(\Lambda^k_h)^{-1}} \]
\[
= \left\{ \left\langle \tilde{\eta}^k_H + \xi^k_H, \psi(s, a, w) \right\rangle + 2L\beta \| \phi(s, a) \|_{(\Lambda^k_h)^{-1}} + \beta \| \psi(s, a, w) \|_{(\Lambda^k_h)^{-1}} \right\}^+ \]
\[
= Q^k_H(s, a, w),
\]
where the first equality follows from the fact that \( Q^k_H(s, a, w) \geq 0 \). Now, suppose the statement holds at time-step \( h + 1 \) and consider time-step \( h \). Conditioned on events \( \{ E_3(w) \}_{w \in \tilde{W}} \), for all \((s, a, w, h, k) \in S \times A \times \tilde{W} \times [H] \times [K]\), we have
\[
0 \leq \left\langle \tilde{\eta}^k_H + \xi^k_H, \psi(s, a, w) \right\rangle - Q^k_H(s, a, w) - \mathbb{P}_h \left[ V^k_{h+1}(., w) - V^*_{h+1}(., w) \right] (s, a)
+ 2L\beta \| \phi(s, a) \|_{(\Lambda^k_h)^{-1}} + \beta \| \psi(s, a, w) \|_{(\Lambda^k_h)^{-1}}
\leq \left\langle \tilde{\eta}^k_H + \xi^k_H, \psi(s, a, w) \right\rangle - Q^k_H(s, a, w) + 2L\beta \| \phi(s, a) \|_{(\Lambda^k_h)^{-1}} + \beta \| \psi(s, a, w) \|_{(\Lambda^k_h)^{-1}}. \tag{Induction assumption}
\]
Therefore, conditioned on events \(\{E_3(w)\}_{w \in \overrightarrow{W}}\), for all \((s, a, w, h, k) \in \mathcal{S} \times \mathcal{A} \times \overrightarrow{W} \times [H] \times [K]\), we have

\[
Q_h^k(s, a, w) \leq \left( \tilde{\eta}_h^k + \xi_h^k, \psi(s, a, w) \right) + 2L\beta \| \phi(s, a) \|_{(A_h^k)^{-1}} + \tilde{\beta} \| \psi(s, a, w) \|_{(A_h^k)^{-1}}
\]

\[
= \left\{ \tilde{\eta}_h^k + \xi_h^k, \psi(s, a, w) \right\} + 2L\beta \| \phi(s, a) \|_{(A_h^k)^{-1}} + \tilde{\beta} \| \psi(s, a, w) \|_{(A_h^k)^{-1}}
\]

\[
= Q_h^k(s, a, w),
\]

where the first equality follows from the fact that \(Q_h^k(s, a, w) \geq 0\). This completes the proof.

\[\square\]

### C.3 Proof of Theorem 3

First, we bound the number of times Algorithm 3 updates \(\tilde{\xi}_h^k\), i.e., number of planning calls. Let \(P\) be the total number of policy updates and \(k_p\) be the episode at the agent did replanning for the \(p\)-th time. Note that \(\det A_h^k = \lambda^d\) and \(\det A_h^K \leq \text{trace}(A_h^K/d) \leq \left(\lambda + \frac{K}{d}\right)^d\), and consequently:

\[
\frac{\det A_h^K}{\det A_h^1} = \prod_{p=1}^P \frac{\det A_h^{k_p}}{\det A_h^{k_{p-1}}} \leq \left(1 + \frac{K}{d\lambda}\right)^d,
\]

and therefore

\[
\prod_{h=1}^H \frac{\det A_h^K}{\det A_h^1} = \prod_{h=1}^H \prod_{p=1}^P \frac{\det A_h^{k_p}}{\det A_h^{k_{p-1}}} \leq \left(1 + \frac{K}{d\lambda}\right)^{dH}. \tag{37}
\]

We similarly have

\[
\prod_{h=1}^H \frac{\det A_h^K}{\det A_h^1} = \prod_{h=1}^H \prod_{p=1}^P \frac{\det A_h^{k_p}}{\det A_h^{k_{p-1}}} \leq \left(1 + \frac{K}{md\lambda}\right)^{mdH}. \tag{38}
\]

Since \(1 \leq \frac{\det A_h^{k_p}}{\det A_h^{k_{p-1}}}\) for all \(p \in [P]\), we can deduce from (37) and (38) that

\[
\exists h \in [H] \text{ such that } e < \frac{\det A_h^k}{\det A_h^K} \text{ or } e < \frac{\det A_h^k}{\det A_h^K} \tag{39}
\]

happens for at most \(dH \log \left(1 + \frac{K}{d\lambda}\right) + mdH \log \left(1 + \frac{K}{md\lambda}\right)\) number of episodes \(k \in [K]\). This concludes that number of planning calls in Algorithm 3 is at most \(dH \log \left(1 + \frac{K}{d\lambda}\right) + mdH \log \left(1 + \frac{K}{md\lambda}\right)\).

Now, we prove the regret bound. Let \(\delta_h^k = V_h^{k}(s_h^k, w^k) - V_h^{k}(s_h^k, w^k)\) and \(\xi_h^{k+1} = E \left[ \delta_h^{k+1} | s_h^k, a_h^k \right] - \delta_h^{k+1} \). Conditioned on events \(\{E_3(w)\}_{w \in \overrightarrow{W}}\), for all \((s, a, w, h, k) \in \mathcal{S} \times \mathcal{A} \times \overrightarrow{W} \times [H] \times [K]\), we have

\[
Q_h^k(s, a, w) - Q_h^k(s, a, w) = \left( \tilde{\eta}_h^k + \xi_h^k, \psi(s, a, w) \right) - Q_h^k(s, a, w) + 2L\beta \| \phi(s, a) \|_{(A_h^k)^{-1}} + \tilde{\beta} \| \psi(s, a, w) \|_{(A_h^k)^{-1}}
\]

\[
\leq P_{h} \left[ V_{h+1}^{k}(s, w) - V_{h+1}^{k}(s, w) \right] (s, a) + 4L\beta \| \phi(s, a) \|_{(A_h^k)^{-1}} + 2\tilde{\beta} \| \psi(s, a, w) \|_{(A_h^k)^{-1}}. \tag{40}
\]
Note that $\delta_h^k \leq Q_h^k(s_h^k, a_h^k, w^h) - Q_h^k(s_h^k, a_h^k, w^h)$. Thus, combining (40), Lemma 9, and a union bound over $\mathcal{W}$, we conclude that for all $(h, k) \in [H] \times [K]$, with probability at least $1 - \delta$, it holds that gives

$$\delta_h^k \leq \epsilon_h^k + \delta_h^k + 4L\beta \left\| \phi(s_h^k, a_h^k) \right\|_{(A_h^k)^{-1}} + 2\beta \left\| \psi(s_h^k, a_h^k, w^h) \right\|_{(A_h^k)^{-1}}.$$ 

Now, we complete the regret analysis following similar steps as those of Theorem 1’s proof:

$$R_K = \sum_{k=1}^{K} V_1^* (s_1^k, w^k) - V_1^* (s_1^k, w^k)$$

$$\leq \sum_{k=1}^{K} V_1^* (s_1^k, w^k) - V_1^* (s_1^k, w^k) \quad \text{(Lemma 11)}$$

$$= \sum_{k=1}^{K} \delta_h^k$$

$$\leq \sum_{k=1}^{K} \sum_{h=1}^{H} \epsilon_h^k + 4L\beta \sum_{k=1}^{K} \sum_{h=1}^{H} \left\| \phi(s_h^k, a_h^k) \right\|_{(A_h^k)^{-1}} + 2\beta \sum_{k=1}^{K} \sum_{h=1}^{H} \left\| \psi(s_h^k, a_h^k, w^h) \right\|_{(A_h^k)^{-1}}$$

$$\leq \sum_{k=1}^{K} \sum_{h=1}^{H} \epsilon_h^k + 4L\beta \sum_{k=1}^{K} \sum_{h=1}^{H} \left\| \phi(s_h^k, a_h^k) \right\|_{(A_h^k)^{-1}} \sqrt{\frac{\det A_h^k}{\det A_h}} + 2\beta \sum_{k=1}^{K} \sum_{h=1}^{H} \left\| \psi(s_h^k, a_h^k, w^h) \right\|_{(A_h^k)^{-1}} \sqrt{\frac{\det A_h^k}{\det A_h}}$$

$$\leq 2H \sqrt{T \log(dT/\delta)} + 4H \sqrt{K \left( L\beta \sqrt{2d \log(1 + K/\lambda)} + \beta \sqrt{2md \log(1 + K/\lambda)} \right)}$$

$$\leq \tilde{O} \left( L\sqrt{\lambda m^2 d^2 H^2 T} \right).$$

## D Relaxation of Assumption 4

In this section, we replace Assumption 4 with the following assumption:

**Assumption 6.** There is a known set $\{w^{(1)}, w^{(2)}, \ldots, w^{(n)}\}$ of $n \leq d'$ tasks such that $\psi(s, a, w) \in \text{Span} \left( \left\{ \psi(s, a, w^{(j)}) \right\}_{j \in [n]} \right)$ for all $(s, a, w) \in S \times A \times \mathcal{W}$. This implies that for any $(s, a, w) \in S \times A \times \mathcal{W}$, there exist coefficients $\{c_j(s, a, w)\}_{j \in [n]}$ such that

$$\psi(s, a, w) = \sum_{j \in [n]} c_j(s, a, w) \psi \left( s, a, w^{(j)} \right). \quad \text{(41)}$$

Moreover, $\sum_{j \in [n]} |c_j(s, a, w)| \leq L$ for all $(s, a, w) \in S \times A \times \mathcal{W}$.

Define the concatenated mapping $\hat{\psi} : S \times A \times \mathcal{W} \rightarrow \mathbb{R}^{d+d'}$ such that $\hat{\psi}(s, a, w) = [\phi(s, a)^\top, \psi(s, a, w)^\top]^\top$. For any $w \in \mathcal{W}$, define $\mathcal{D}(w) = \left\{ (s, a) : \hat{\psi}(s, a, w) \text{ are } d + d' \text{ linearly independent vectors} \right\}$. Given Assumption 6, we modify the planning step of UCBblvd to the following:

$$\hat{\xi}_h, \left\{ \hat{\theta}_h^k(j) \right\}_{j \in [n]} = \arg \min_{\xi, \left\{ \theta(j) \right\}_{j \in [n]}} \sum_{j \in [n]} \sum_{(s, a) \in \mathcal{D}(w^{(j)})} \left( \left\langle \theta(j), \phi(s, a) \right\rangle - \left\langle \xi, \psi \left( s, a, w^{(j)} \right) \right\rangle \right)^2$$

$$\text{s.t.} \left\| \theta(j) - \hat{\theta}_h^k (w^{(j)}) \right\|_{A_h^k} \leq \beta, \ \forall j \in [n] \text{ and } \|\xi\|_2 \leq H \sqrt{d^{'}}. \quad \text{(42)}$$

The only change we make in Algorithm 2 is in Line 9, in which $\hat{\xi}_h^k$ is now computed as defined in (42). We present this modification in Algorithm 4 for completeness.
Algorithm 4: Modified UCBlvd

Input: $\mathcal{A}, \lambda, \delta, H, K, \beta$

1. Set $Q_{H+1}^k(0, \ldots, 0) = 0, \forall k \in [K], \tilde{k} = 1$
2. for episodes $k = 1, \ldots, K$ do
   3. Observe the initial state $s_1^k$ and the task context $w^k$.
   4. if $\exists h \in [H]$ such that $\frac{\det \Delta_{k}^h}{\det \Delta_{k}} > e$ then
      5. $\tilde{k} = k$
      6. for time-steps $h = H, \ldots, 1$ do
         7. Compute $\xi_h^k$ as in (42).
      8. for time-steps $h = 1, \ldots, H$ do
         9. Compute $Q_h^k(s_h^k, a, w^k)$ for all $a \in \mathcal{A}$ as in (10).
     10. Play $a_h^k = \arg \max _{a \in \mathcal{A}} Q_h^k(s_h^k, a, w^k)$ and observe $s_{h+1}^k$ and $r_h^k$.

Theorem 4. Let $T = KH$. Under Assumptions 1, 2, 3, and 6, the number or planning calls in Algorithm 4 is at most $dH \log (1 + \frac{KH}{\delta})$ and there exists an absolute constant $c > 0$ such that for any fixed $\delta \in (0, 0.5)$, if we set $\lambda = 1$ and $\beta = cH \left( d + \sqrt{d'} \right) \sqrt{\lambda \log (d'd'/\delta)}$ in Algorithm 4, then with probability at least $1 - 2\delta$, it holds that

$$R_K \leq 2H \sqrt{T \log (d'd'/\delta)} + 8HL\beta \sqrt{2dK \log (K)} \leq 3 \hat{O} \left( L \sqrt{(d^3 + dd')H^2T} \right).$$

Proof of Theorem 4 follows exactly the same steps as those of Theorem 2. The only difference is the proof of Lemma 2, which we clarify in the proof of following lemma.

Lemma 12. Let $\mathcal{W} = \{ w^r : r \in [K] \} \cup \{ w^{(j)} : j \in [n] \}$. Under Assumptions 1, 2, 3, and 6, if we let $\beta = cH \left( d + \sqrt{d'} \right) \sqrt{\lambda \log (d'd'/\delta)}$ with an absolute constant $c > 0$, then for all $(s, a, w, h, k) \in \mathcal{S} \times \mathcal{A} \times \mathcal{W} \times [H] \times [K]$ with probability at least $1 - \delta$, it holds that

$$\left| \left\langle \xi_h^k, \psi(s, a, w) \right\rangle - \mathbb{P}_h[V_{h+1}^k(\cdot, w)](s, a) \right| \leq 2L\beta \| \phi(s, a) \| (k^h)^{-1}.$$
Moreover, for any triple \((s, a, j) \in \mathcal{S} \times \mathcal{A} \times [n]\), we have

\[
\langle \xi_h^k, \psi(s, a, w^{(j)}) \rangle = \sum_{i \in [d+d']} c_i \langle s, a, w^{(j)} \rangle \langle \xi_h^k, \psi_i \rangle (w^{(j)}) \tag{Eqn. (44)}
\]

\[
= \sum_{i \in [d+d']} c_i \langle s, a, w^{(j)} \rangle \langle \theta_h^{k(j)}, \phi_i \rangle \tag{Eqn. (45)}
\]

\[
= \langle \theta_h^{k(j)}, \phi(s, a) \rangle. \tag{46}
\]

For any \((s, a, w) \in \mathcal{S} \times \mathcal{A} \times \mathcal{W}\), it holds that

\[
\mathbb{P}_h \left[ V_{h+1}^k(., w) \right](s, a) = \left\langle \theta_h^k(w), \phi(s, a) \right\rangle \tag{Eqn. (5)}
\]

\[
= \langle \xi_h^{k+1}, \psi(s, a, w) \rangle \tag{Assumption 3}
\]

\[
= \sum_{j \in [n]} c_j(s, a, w) \left\langle \xi_h^{k+1}, \psi(s, a, w^{(j)}) \right\rangle \tag{Eqn. (41)}
\]

\[
= \sum_{j \in [n]} c_j(s, a, w) \mathbb{P}_h \left[ V_{h+1}^k(., w^{(j)}) \right](s, a) \tag{Assumption 3}
\]

\[
= \sum_{j \in [n]} c_j(s, a, w) \left\langle \theta_h^k \left( w^{(j)} \right), \phi(s, a) \right\rangle. \tag{47}
\]

Finally, conditioned on events \(\{\xi_i(w)\}_{w \in \mathcal{W}}\), for all \((s, a, w, h, k) \in \mathcal{S} \times \mathcal{A} \times \mathcal{W} \times [H] \times [K]\), it holds that

\[
\left| \langle \xi_h^k, \psi(s, a, w) \rangle - \mathbb{P}_h \left[ V_{h+1}^k(., w) \right](s, a) \right| \leq \sum_{j \in [n]} c_j(s, a, w) \left| \left\langle \xi_h^k, \psi(s, a, w^{(j)}) \right\rangle - \left\langle \theta_h^k \left( w^{(j)} \right), \phi(s, a) \right\rangle \right| \tag{Eqns. (41) and (25)}
\]

\[
+ \sum_{j \in [n]} c_j(s, a, w) \left| \left\langle \theta_h^{k(j)} - \theta_h^k \left( w^{(j)} \right), \phi(s, a) \right\rangle \right| \tag{Eqn. (24)}
\]

\[
+ \sum_{j \in [n]} c_j(s, a, w) \left| \left\langle \theta_h^k \left( w^{(j)} \right) - \theta_h^k \left( w^{(j)} \right), \phi(s, a) \right\rangle \right| \tag{Lemma 1}
\]

\[
\leq 2L\beta \|\phi(s, a)\| (\Lambda_h^k)^{-1}.
\]

**E Standard Lifelong-LSVI with Computation Sharing**

In this section, we only rely on the following two assumptions:
Algorithm 5: Standard Lifelong-LSVI with Computation Sharing

\begin{algorithm}
  \textbf{Input:} \( A, \lambda, \delta, \beta, H, K \)
  \begin{algorithmic}[1]
    \State \textbf{Set:} \( Q^k_{H+1}(\ldots) = 0, \forall k \in [K], \tilde{k} = 1 \)
    \For {episodes \( k = 1, \ldots, K \)}
      \State Observe the initial state \( s^k_1 \) and the task context \( w^k \).
      \If {\( \exists h \in [H] \) such that \( \frac{\det \Lambda^k_h}{\det \Lambda^k} > e \)}
        \State \( \tilde{k} = k \)
      \EndIf
      \For {time-steps \( h = H, \ldots, 1 \)}
        \State Compute \( \tilde{\nu}^k_h \) as in (52).
      \EndFor
      \For {time-steps \( h = 1, \ldots, H \)}
        \State Compute \( Q^k_h(s^k_h, a, w^k) \) for all \( a \in A \) as in (51).
      \EndFor
      \State Play \( a^k_h = \arg \max_{a \in A} Q^k_h(s^k_h, a, w^k) \) and observe \( s^k_{h+1} \) and \( r^k_h \).
    \EndFor
  \end{algorithmic}
\end{algorithm}

Assumption 7. Given a feature map \( \psi : S \times A \times W \to \mathbb{R}^{d'} \), consider function class

\[ F = \left\{ f : f(s, w) = \min \left\{ \max_{a \in A} \left\langle \nu, \psi(s, a, w) \right\rangle + \beta \| \psi(s, a, w) \|_{\Lambda^{-1}} \right\}, \nu \in \mathbb{R}^{d'}, \beta \in \mathbb{R}, \Lambda \in S^{d'+1} \right\}. \]

(49)

Then for any \( f \in F \) and \( h \in [H] \), there exists a vector \( \nu^f_h \in \mathbb{R}^{d'} \) with \( \| \nu^f_h \|_2 \leq H \sqrt{d'} \) such that

\[ \mathbb{P}_h [f(., w)] (s, a) = \langle \psi(s, a, w), \nu^f_h \rangle. \]

(50)

Moreover, for every \( h \in [H] \), there exists a vector \( \eta_h \) such that \( r_h(s, a, w) = \langle \eta_h, \psi(s, a, w) \rangle \).

Assumption 8. Without loss of generality, \( \| \psi(s, a, w) \|_2 \leq 1 \) for all \( (s, a, w) \in S \times A \times W \), and \( \| \eta_h \|_2 \leq \sqrt{d'} \) for all \( h \in [H] \).

E.1 Overview

Let \( \psi_h^* = \psi(s^*_h, a^*_h, w^*_h) \). Standard Lifelong-LSVI with computation sharing works with the following action-value functions:

\[ Q^k_h(s, a, w) = r_h(s, a, w) + \left\langle \tilde{\nu}^k_h, \psi(s, a, w) \right\rangle + \beta \| \psi(s, a, w) \|_{(\Lambda^k_h)^{-1}} \right\}^+, \]

(51)

where

\[ \tilde{\nu}^k_h = \left( \Lambda^k_h \right)^{-1} \sum_{\tau=1}^{k-1} \psi^*_\tau \min \left\{ \max_{a \in A} Q^k_{h+1}(s^*_\tau, a, w^*_\tau), H \right\} \quad \text{and} \quad \Lambda^k_h = \lambda I_{d'} + \sum_{\tau=1}^{k-1} \psi^*_\tau \psi^*_\tau^T. \]

(52)

Theorem 5. Let \( T = KH \). Under Assumptions 7 and 8, the number of planning calls in 5 is at most \( d' H \log \left( 1 + \frac{\beta}{\delta K^2} \right) \) and there exists an absolute constant \( c > 0 \) such that for any fixed \( \delta \in (0, 0.5) \), if we set \( \lambda = 1 \) and \( \beta = cd' H \sqrt{\log(d'T/\delta)} \) in Algorithm 5, then with probability at least \( 1 - 2\delta \), it holds that

\[ R_K \leq 2H \sqrt{T \log(d'T/\delta)} + 4H\beta \sqrt{2d'K \log(K)} \leq \hat{O} \left( \sqrt{d^3H^3T} \right). \]
E.2 Necessary Analysis for the Proof of Theorem 5

Thanks to Assumption 7, we have

$$\mathbb{P}_h \left[ V^k_{h+1}(s, w) \right](s, a) = \left\langle \nu_h^k, \psi(s, a, w) \right\rangle,$$

where $\nu_h^k = \nu_h^{V_{h+1}^k}$.

**Lemma 13.** Let $c_\beta$ be a constant in the definition of $\beta$. Then, under Assumption 8, there is an absolute constant $c_0$ independent of $c_\beta$, such that for all $(h, k) \in [H] \times [K]$, with probability at least $1 - \delta$ it holds that

$$\left\| \sum_{\tau=1}^{k-1} \phi_h^\tau \left( V^k_{h+1}(s^\tau_{h+1}, w^\tau) - \mathbb{P}_h[V^k_{h+1}(s^\tau_{h+1}, w^\tau)](s^\tau_h, a^\tau_h) \right) \right\|_{(\Lambda_h^k)^{-1}} \leq c_0 \epsilon H \sqrt{\log((c_\beta + 1)T/\delta)},$$

where $c_0$ and $c_\beta$ are two independent absolute constants.

**Proof.** We note that $\left\| \eta_h + \nu_h^k \right\|_2 \leq (1 + H)\sqrt{d'}$ and $\left( \Lambda_h^k \right)^{-1} \leq \frac{1}{\lambda}$. Thus, Lemmas 17 and 22 together imply that for all $(h, k) \in [H] \times [K]$, with probability at least $1 - \delta$ it holds that

$$\left\| \sum_{\tau=1}^{k-1} \phi_h^\tau \left( V^k_{h+1}(s^\tau_{h+1}, w^\tau) - \mathbb{P}_h[V^k_{h+1}(s^\tau_{h+1}, w^\tau)](s^\tau_h, a^\tau_h) \right) \right\|_{(\Lambda_h^k)^{-1}} \leq 4H^2 \left( \frac{d'}{2} \log \left( \frac{k + \lambda}{\lambda} \right) + d' \log(1 + 8H\sqrt{d'/\epsilon}) + d'^2 \log \left( \frac{1 + 2L^2 \beta^2 \sqrt{d'}}{\epsilon c^2} \right) + \log \left( \frac{1}{\delta} \right) \right) + \frac{8k^2 \epsilon^2}{\lambda}.$$

If we let $\epsilon = \frac{dH}{k}$ and $\beta = c_\beta(d' + \sqrt{d'})H \sqrt{\log(d'T/\delta)}$, then, there exists an absolute constant $C > 0$ that is independent of $c_\beta$ such that

$$\left\| \sum_{\tau=1}^{k-1} \phi_h^\tau \left( V^k_{h+1}(s^\tau_{h+1}, w^\tau) - \mathbb{P}_h[V^k_{h+1}(s^\tau_{h+1}, w^\tau)](s^\tau_h, a^\tau_h) \right) \right\|_{(\Lambda_h^k)^{-1}} \leq C(d' + d'^2)H^2 \log \left( (c_\beta + 1)T/\delta \right).$$

**Lemma 14.** Under Assumptions 7 and 8, if we let $\beta = cd'H \sqrt{\log(d'T/\delta)}$ with an absolute constant $c > 0$, then the event

$$\mathcal{E}_4 := \left\{ \left\| \nu_h^k - \tilde{\nu}_h^k \right\|_{\Lambda_h^k} \leq \beta, \forall (h, k) \in [H] \times [K] \right\}.$$  

holds with probability at least $1 - \delta$. 

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Proof.

\[
\nu_h^k - \tilde{\nu}_h^k = \nu_h^k - \left( \bar{A}_h^k \right)^{-1} \sum_{\tau=1}^{k-1} \psi_h^\tau V_{h+1}^k (s_{h+1}^\tau, w^\tau)
\]

\[
= \left( \Lambda_h^k \right)^{-1} \left( \bar{A}_h^k \nu_h^k - \sum_{\tau=1}^{k-1} \psi_h^\tau V_{h+1}^k (s_{h+1}^\tau, w^\tau) \right)
\]

\[
= \lambda \left( \bar{A}_h^k \right)^{-1} \left( \nu_h^k - \left( \bar{A}_h^k \right)^{-1} \frac{1}{\lambda} \sum_{\tau=1}^{k-1} \psi_h^\tau \left( V_{h+1}^k (s_{h+1}^\tau, w^\tau) - \bar{P}_h [V_{h+1}^k (., w^\tau) | (s_h^\tau, a_h^\tau)] \right) \right) \quad \text{(Eqn. (53))}
\]

Thus, in order to upper bound \( \| \nu_h^k - \tilde{\nu}_h^k (w) \|_{\bar{A}_h^k} \), we bound \( \| q_1 \|_{\Lambda_h^k} \) and \( \| q_2 \|_{\Lambda_h^k} \) separately.

From Assumption 8, we have

\[
\| q_1 \|_{\Lambda_h^k} = \lambda \| \nu_h^k \|_{(\Lambda_h^k)^{-1}} \leq \sqrt{\lambda} \| \nu_h^k \|_{2} \leq H \sqrt{d'}. \quad \text{(55)}
\]

Thanks to Lemma 13, for all \( (h, k) \in [H] \times [K] \), with probability at least \( 1 - \delta \), it holds that

\[
\| q_2 \|_{\Lambda_h^k} \leq \left\| \frac{1}{\lambda} \sum_{\tau=1}^{k-1} \psi_h^\tau \left( V_{h+1}^k (s_{h+1}^\tau, w^\tau) - \bar{P}_h [V_{h+1}^k (., w^\tau) | (s_h^\tau, a_h^\tau)] \right) \right\|_{(\Lambda_h^k)^{-1}} \leq c_0 d' H \sqrt{\log((c_\beta + 1)d'T/\delta)}, \quad \text{(56)}
\]

where \( c_0 \) and \( c_\beta \) are two independent absolute constants.

Combining (55) and (56), for all \( (h, k) \in [H] \times [K] \), with probability at least \( 1 - \delta \), it holds that

\[
\| \nu_h^k - \tilde{\nu}_h^k \|_{\bar{A}_h^k} \leq c d' H \sqrt{\lambda \log(d'T/\delta)}
\]

for some absolute constant \( c > 0 \).

Lemma 15. Let the setting of Lemma 14 holds. Conditioned on events \( \mathcal{E}_4 \) defined in (54), and with \( Q_h^k \) computed as in (51), it holds that \( Q_h^k (s, a, w) \geq Q_h^k (s, a, w) \) for all \( (s, a, w, h, k) \in \mathcal{S} \times \mathcal{A} \times \mathcal{W} \times [H] \times [K] \).

Proof. We first note that conditioned on the event \( \mathcal{E}_4 \), for all \( (s, a, w, h, k) \in \mathcal{S} \times \mathcal{A} \times \mathcal{W} \times [H] \times [K] \), it holds that

\[
\begin{align*}
\left| r_h(s, a, w) + \left\langle \tilde{\nu}_h^k, \psi(s, a, w) \right\rangle - Q_h^k (s, a, w) - \bar{P}_h \left[ V_{h+1}^k (., w) - V_{h+1}^k (., w) \right] (s, a) \right| \\
= \left| r_h(s, a, w) + \left\langle \tilde{\nu}_h^k, \psi(s, a, w) \right\rangle - r_h(s, a, w) - \bar{P}_h \left[ V_{h+1}^k (., w) \right] (s, a) \right| \\
= \left| \left\langle \tilde{\nu}_h^k, \psi(s, a, w) \right\rangle - \bar{P}_h \left[ V_{h+1}^k (., w) \right] (s, a) \right| \\
= \left| \tilde{\nu}_h^k - \nu_h^k, \psi(s, a, w) \right| \\
\leq \| \nu_h^k - \tilde{\nu}_h^k \|_{\Lambda_h^k} \| \psi(s, a, w) \|_{(\Lambda_h^k)^{-1}} \\
\leq \beta \| \psi(s, a, w) \|_{(\Lambda_h^k)^{-1}}, \quad \text{(Lemma 14)}
\end{align*}
\]
for any policy \( \pi \).

Now, we prove the lemma by induction. The statement holds for \( H \) because \( Q_H^k(\ldots) = Q_H^{\ast}(\ldots) = 0 \) and thus conditioned on the event \( E_4 \), defined in (54), for all \((s, a, w, k) \in S \times A \times W \times [K] \), we have

\[
\left| r_h(s, a, w) + \langle \nu_H^k, \psi(s, a, w) \rangle - Q_H^*(s, a, w) \right| \leq \beta \| \psi(s, a, w) \|_{(\Lambda_H^k)^{-1}}.
\]

Therefore, conditioned on the event \( E_4 \), for all \((s, a, w, k) \in S \times A \times W \times [K] \), we have

\[
Q_H^*(s, a, w) \leq r_h(s, a, w) + \langle \nu_H^k, \psi(s, a, w) \rangle + \beta \| \psi(s, a, w) \|_{(\Lambda_H^k)^{-1}}
\]

\[
= \left\{ r_h(s, a, w) + \langle \nu_H^k, \psi(s, a, w) \rangle + \beta \| \psi(s, a, w) \|_{(\Lambda_H^k)^{-1}} \right\}^+ = Q_H^k(s, a, w),
\]

where the first equality follows from the fact that \( Q_H^*(s, a, w) \geq 0 \). Now, suppose the statement holds at time-step \( h + 1 \) and consider time-step \( h \). Conditioned on events \( E_4 \), for all \((s, a, w, h, k) \in S \times A \times W \times [H] \times [K] \), we have

\[
0 \leq r_h(s, a, w) + \langle \nu_H^k, \psi(s, a, w) \rangle - Q_H^k(s, a, w) - \nabla_h \left[ V_H^k(\ldots), w \right] \langle \psi^*, (a, w) \rangle + \beta \| \psi(s, a, w) \|_{(\Lambda_H^k)^{-1}}
\]

\[
\leq r_h(s, a, w) + \langle \nu_H^k, \psi(s, a, w) \rangle - Q_H^*(s, a, w) + \beta \| \psi(s, a, w) \|_{(\Lambda_H^k)^{-1}}. \quad \text{(Induction assumption)}
\]

Therefore, conditioned on events \( E_4 \), for all \((s, a, w, h, k) \in S \times A \times W \times [H] \times [K] \), we have

\[
Q_H^*(s, a, w) \leq r_h(s, a, w) + \langle \nu_H^k, \psi(s, a, w) \rangle + \beta \| \psi(s, a, w) \|_{(\Lambda_H^k)^{-1}}
\]

\[
= \left\{ r_h(s, a, w) + \langle \nu_H^k, \psi(s, a, w) \rangle + \beta \| \psi(s, a, w) \|_{(\Lambda_H^k)^{-1}} \right\}^+ = Q_H^k(s, a, w),
\]

where the first equality follows from the fact that \( Q_H^*(s, a, w) \geq 0 \). This completes the proof.

\[ \square \]

### E.3 Proof of Theorem 5

First, we bound the number of times Algorithm 5 updates \( \nu_H^k \). Let \( P \) be the total number of updates and \( k_p \) be the episode at which the agent did replanning for the \( p \)-th time. Note that \( \det \Lambda_h^1 = \lambda^{d'} \) and

\[
\det \Lambda_h^K \leq \text{trace}(\Lambda_h^K / d')^d \leq \left( \lambda + \frac{K}{d'} \right)^{d'},
\]

and consequently:

\[
\frac{\det \Lambda_h^K}{\det \Lambda_h^1} = \prod_{p=1}^{P} \frac{\det \Lambda_h^{k_p}}{\det \Lambda_h^{k_{p-1}}} \leq \left( 1 + \frac{K}{d' \lambda} \right)^{d'},
\]

and therefore

\[
\prod_{h=1}^{H} \prod_{p=1}^{P} \frac{\det \Lambda_h^{k_p}}{\det \Lambda_h^{k_{p-1}}} \leq \left( 1 + \frac{K}{d' \lambda} \right)^{d' H}.
\]
Since $1 \leq \frac{\det \tilde{\Lambda}_h^{kp}}{\det \Lambda_h^{ kp - 1}}$ for all $p \in [P]$, we can deduce from (57) that

$$\exists h \in [H] \text{ such that } \epsilon < \frac{\det \tilde{\Lambda}_h}{\det \Lambda_h}$$

happens for at most $d'H \log \left(1 + \frac{K}{d'H}\right)$ number of episodes $k \in [K]$. This concludes that number of planning calls in Algorithm 5 is at most $d'H \log \left(1 + \frac{K}{d'H}\right)$.

Now, we prove the regret bound. Let $\delta_h^k = V^k_h(s_h^k, w^k) - V^*_{h+1}(s_h^k, w^k)$ and $\xi_h^k = E \left[\delta_h^{k+1}| s_h^k, a_h^k \right] - \delta_h^k$. Conditioned on $E$, for all $(s, a, w, h, k) \in \mathcal{S} \times \mathcal{A} \times \mathcal{W} \times [H] \times [K]$, we have

$$Q_h^k(s, a, w) - Q_h^*(s, a, w) = r_h(s, a, w) + \langle \theta_h^k, \psi_h(s, a, w) \rangle - Q_h^*(s, a, w) + \beta \|\psi_h(s, a, w)\|_2 \tilde{\Lambda}_h^{-1} \delta_h^k \leq \mathbb{P}_h \left[V_{h+1}^k(., w) - V_{h+1}^k(., w) \right](s, a) + 2\beta \|\psi_h(s, a, w)\|_2 \tilde{\Lambda}_h^{-1}.$$  \hfill (58)

Note that $\delta_h^k \leq Q_h^k(s_h^k, a_h^k, w^k) - Q_h^*(s_h^k, a_h^k, w^k)$. Thus, (58) and Lemma 14 imply that for all $(h, k) \in [H] \times [K]$, it holds that

$$\delta_h^k \leq \xi_h^{k+1} + \delta_h^{k+1} + 2\beta \|\psi_h(s_h^k, a_h^k, w^k)\|_2 \tilde{\Lambda}_h^{-1}.$$  \hfill (Eqn. (30))

Now, we complete the regret analysis following similar steps as those of Theorem 1’s proof:

$$R_K = \sum_{k=1}^{K} V^*_1(s_1^k, w^k) - V^*_1(s_1^k, w^k)$$  \hfill (Lemma 15)

$$= \sum_{k=1}^{K} \delta_h^k$$

$$\leq \sum_{k=1}^{H} \sum_{h=1}^{K} \xi_h^k + 2\beta \sum_{k=1}^{K} \sum_{h=1}^{H} \|\psi_h(s_h^k, a_h^k, w^k)\|_2 \tilde{\Lambda}_h^{-1}$$

$$\leq 2H\sqrt{T \log(d'H/\delta)} + 4H\beta \sqrt{2\lambda d'H \log(1 + K/\lambda)}$$

$$\leq \hat{O} \left(\sqrt{\lambda d^3 H^3 T} \right).$$

F Auxiliary Lemmas

Notations. $\mathcal{N}_\epsilon(V)$ denotes the $\epsilon$-covering number of the class $V$ of functions mapping $\mathcal{S}$ to $\mathbb{R}$ with respect to the distance $\text{dist}(V, V') = \sup_s |V(s) - V'(s)|$.

**Lemma 16 (Bound on Weights $\theta_h^k(w)$).** Under Assumption 1, for any set of action-value functions $\{Q_h^k\}_{h \in [H]}$, and $(w, h, k) \in \mathcal{W} \times [H] \times [K]$, it holds that

$$\|\theta_h^k(w)\|_2 \leq H\sqrt{d}.$$
Proof. Recall that $V_k^s(s, w) = \min \{\max_{a \in A} Q_k^s(s, a, w), H\}$ and $\theta_k^s(w) := \int_S V_{k+1}^s(s', w) d\mu_h(s')$. Thus, we have
\[
\|\theta_k^s(w)\|_2 = \left\| \int_S V_{k+1}^s(s', w) d\mu_h(s') \right\| \leq H \sqrt{d}.
\]

Lemma 17 (Lemma D.4 in [Jin et al., 2020]). Let $\{s_t\}_{t=1}^{\infty}$ be a stochastic process on state space $S$ with corresponding filtration $\{\mathcal{F}_t\}_{t=0}^{\infty}$. Let $\{\phi_t\}_{t=0}^{\infty}$ be an $\mathbb{R}^d$-valued stochastic process where $\phi_t \in \mathcal{F}_{t-1}$, and $\|\phi_t\| \leq 1$. Let $A_k = A_1 + \sum_{t=1}^{k} \phi_t \phi_t^\top$. Then with probability at least $1 - \delta$, for all $k \geq 0$ and $V \in \mathcal{V}$ such that $\sup_{s \in S} |V(s)| \leq H$, we have
\[
\left\| \sum_{t=1}^{k} \phi_t \cdot \left( V(s_t) - \mathbb{E}_s \left[ V(s_t) | \mathcal{F}_{t-1} \right] \right) \right\|^2 \leq 4H^2 \left( \frac{d}{2} \log \left( \frac{k + \lambda}{\lambda} \right) + \log \left( \frac{N_\infty(V)}{\delta} \right) \right) + \frac{8k^2 e^2}{\lambda}.
\]

Lemma 18. For any $\epsilon > 0$, the $\epsilon$-covering number of the Euclidean ball in $\mathbb{R}^d$ with radius $R > 0$ is upper bounded by $(1 + 2R/e)^d$.

Lemma 19. For a fixed $w$, let $V$ denote a class of functions mapping from $S$ to $\mathbb{R}$ with following parametric form
\[
V(.) = \min \left\{ \max_{a \in A} \langle z, \psi(s, a, w) \rangle + \langle y, \phi(s, a) \rangle + \beta \sqrt{\phi(s, a)^\top Y \phi(s, a)}, H \right\},
\]
where the parameters $\beta \in \mathbb{R}$, $z \in \mathbb{R}^d$, $y \in \mathbb{R}^d$, and $Y \in \mathbb{R}^{d \times d}$ satisfy $0 \leq \beta \leq B$, $\|z\| \leq z$, $\|y\| \leq y$, and $\|Y\| \leq \lambda^{-1}$. Assume $\|\phi(s, a)\| \leq 1$ and $\|\psi(s, a, w)\| \leq 1$ for all $(s, a, w) \in S \times A \times \mathcal{W}$. Then
\[
\log (N_\epsilon(V)) \leq d' \log(1 + 4z/e) + d \log(1 + 4y/e) + d^2 \log \left( \frac{1 + 8B^2 \sqrt{d}}{\lambda e^2} \right).
\]

Proof. First, we reparametrize $\mathcal{V}$ by letting $\breve{Y} = \beta^2 Y$. We have
\[
V(.) = \min \left\{ \max_{a \in A} \langle z, \psi(s, a, w) \rangle + \langle y, \phi(s, a) \rangle + \sqrt{\phi(s, a)^\top \breve{Y} \phi(s, a)}, H \right\},
\]
for $\|z\| \leq z$, $\|y\| \leq y$, and $\|\breve{Y}\| \leq \frac{B^2}{\lambda}$. For any two functions $V_1, V_2 \in \mathcal{V}$ with parameters $(z^1, y^1, \breve{Y}^1)$ and $(z^2, y^2, \breve{Y}^2)$, respectively, we have
\[
\text{dist}(V_1, V_2) \leq \sup_{(s, a) \in S \times A} \left\| \begin{bmatrix} z^1, \psi(s, a, w) \\ y^1, \phi(s, a) \end{bmatrix} + \sqrt{\phi(s, a)^\top \breve{Y}^1 \phi(s, a)} \right\|
\]
\[
- \left\| \begin{bmatrix} z^2, \psi(s, a, w) \\ y^2, \phi(s, a) \end{bmatrix} + \sqrt{\phi(s, a)^\top \breve{Y}^2 \phi(s, a)} \right\|
\]
\[
\leq \sup_{\psi \|\psi\| \leq 1, \phi \|\phi\| \leq 1} \left\| \begin{bmatrix} z^1, \psi \\ y^1, \phi \end{bmatrix} + \sqrt{\phi^\top \breve{Y}^1 \phi} - \begin{bmatrix} z^2, \psi \\ y^2, \phi \end{bmatrix} - \sqrt{\phi^\top \breve{Y}^2 \phi} \right\|
\]
\[
\leq \sup_{\psi \|\psi\| \leq 1} \left\| \begin{bmatrix} z^1 - z^2, \psi \\ y^1 - y^2, \phi \end{bmatrix} \right\| + \sup_{\phi \|\phi\| \leq 1} \left\| \begin{bmatrix} y^1 - y^2, \phi \end{bmatrix} \right\| + \sup_{\phi \|\phi\| \leq 1} \sqrt{\phi^\top \begin{bmatrix} \breve{Y}^1 - \breve{Y}^2 \end{bmatrix} \phi}
\]
\[
= \|z^1 - z^2\| + \|y^1 - y^2\| + \sqrt{\|\breve{Y}^1 - \breve{Y}^2\|^2}
\]
\[
\leq \|z^1 - z^2\| + \|y^1 - y^2\| + \sqrt{\|\breve{Y}^1 - \breve{Y}^2\|^2}.
\] (59)
Let $\mathcal{C}_z$ and $\mathcal{C}_y$ be $\epsilon/2$-covers of $\{z \in \mathbb{R}^d : \|z\| \leq z\}$ and $\{y \in \mathbb{R}^d : \|y\| \leq y\}$, respectively, with respect to the 2-norm, and $\mathcal{C}_y$ be an $\epsilon^2/4$-cover of $\{Y \in \mathbb{R}^{d \times d} : \|Y\|_F \leq \frac{B^2 \sqrt{d}}{\lambda^2}\}$, with respect to the Frobenius norm. By Lemma 18, we know

$$|\mathcal{C}_z| \leq (1 + 4z/\epsilon)^d, \quad |\mathcal{C}_y| \leq (1 + 4y/\epsilon)^d, \quad |\mathcal{C}_Y| \leq \left(\frac{1 + 8B^2 \sqrt{d}}{\lambda \epsilon^2}\right)^d.$$ 

According to (59), it holds that $\mathcal{N}_s(V) \leq |\mathcal{C}_z| |\mathcal{C}_y| |\mathcal{C}_Y|$, and therefore

$$\log(\mathcal{N}_s(V)) \leq d' \log(1 + 4z/\epsilon) + d \log(1 + 4y/\epsilon) + d^2 \log\left(\frac{1 + 8B^2 \sqrt{d}}{\lambda \epsilon^2}\right).$$

Lemma 20. For a fixed $w$, let $V$ denote a class of functions mapping from $S$ to $\mathbb{R}$ with following parametric form

$$V(.) = \min \left\{ \max_{a \in A} \left\{ (z, \psi(., a, w)) + 2L\beta \sqrt{\phi(., a)\top Y \phi(., a)} \right\}_+, H \right\},$$

where the parameters $\beta \in \mathbb{R}$, $z \in \mathbb{R}^d$ and $Y \in \mathbb{R}^{d \times d}$ satisfy $0 \leq \beta \leq B$, $\|z\| \leq z$, and $\|Y\| \leq \lambda^{-1}$. Assume $\|\phi(s, a)\| \leq 1$ and $\|\psi(s, a, w)\| \leq 1$ for all $(s, a, w) \in S \times A \times W$. Then

$$\log(\mathcal{N}_s(V)) \leq d' \log(1 + 4z/\epsilon) + d^2 \log\left(\frac{1 + 8B^2 \sqrt{d}}{\lambda \epsilon^2}\right).$$

Proof. First, we reparametrize $V$ by letting $\tilde{Y} = \beta^2 Y$. We have

$$V(.) = \min \left\{ \max_{a \in A} \left\{ (z, \psi(., a, w)) + \sqrt{\phi(., a)\top \tilde{Y} \phi(., a)} \right\}_+, H \right\},$$

for $\|z\| \leq z$, and $\|\tilde{Y}\| \leq \frac{B^2}{\lambda^2}$. For any two functions $V_1, V_2 \in V$ with parameters $(z^1, \tilde{Y}_1^1)$ and $(z^2, \tilde{Y}_1^2)$, respectively, we have

$$\text{dist}(V_1, V_2) \leq \sup_{(s, a) \in S \times A} \left| \left( \langle z^1, \psi(s, a, w) \rangle + \sqrt{\phi(s, a)\top \tilde{Y}_1 \phi(s, a)} \right) - \left( \langle z^2, \psi(s, a, w) \rangle + \sqrt{\phi(s, a)\top \tilde{Y}_2 \phi(s, a)} \right) \right| \leq \sup_{\psi: \|\psi\| \leq 1, \phi: \|\phi\| \leq 1} \left| \left( \langle z^1, \psi \rangle + \sqrt{\phi\top \tilde{Y}_1^1 \phi} \right) - \left( \langle z^2, \psi \rangle + \sqrt{\phi\top \tilde{Y}_2^2 \phi} \right) \right|$$

$$\leq \sup_{\psi: \|\psi\| \leq 1} \left| \langle z^1 - z^2, \psi \rangle \right| + \sup_{\phi: \|\phi\| \leq 1} \left| \sqrt{\phi\top \tilde{Y}_1^1 \phi} - \sqrt{\phi\top \tilde{Y}_2^2 \phi} \right|$$

$$\leq \left| z^1 - z^2 \right| + \left| \sqrt{\tilde{Y}_1^1 - \tilde{Y}_2^2} \right|$$

(because $\sqrt{a} - \sqrt{b} \leq \sqrt{|a - b|}$ for $a, b \geq 0$)

$$= \left| z^1 - z^2 \right| + \left| \sqrt{\tilde{Y}_1^1 - \tilde{Y}_2^2} \right| \leq \left| z^1 - z^2 \right| + \sqrt{\|\tilde{Y}_1^1 - \tilde{Y}_2^2\|_F}. \quad (60)$$

Let $\mathcal{C}_z$ be an $\epsilon/2$-cover of $\{z \in \mathbb{R}^d : \|z\| \leq z\}$ with respect to the 2-norm, and $\mathcal{C}_y$ be an $\epsilon^2/4$-cover of $\{Y \in \mathbb{R}^{d \times d} : \|Y\|_F \leq \frac{B^2 \sqrt{d}}{\lambda^2}\}$, with respect to the Frobenius norm. By Lemma 18, we know

$$|\mathcal{C}_z| \leq (1 + 4z/\epsilon)^d, \quad |\mathcal{C}_y| \leq \left(\frac{1 + 8B^2 \sqrt{d}}{\lambda \epsilon^2}\right)^d.$$
According to (60), it holds that \( \mathcal{N}_e(\mathcal{V}) \leq |\mathcal{C}_z||\mathcal{C}_Y| \), and therefore

\[
\log (\mathcal{N}_e(\mathcal{V})) \leq d' \log(1 + 4z/\epsilon) + d^2 \log \left( \frac{1 + 8B^2\sqrt{d}}{\lambda\epsilon^2} \right). 
\]

\[ \square \]

**Lemma 21.** For a fixed \( w \), let \( \mathcal{V} \) denote a class of functions mapping from \( \mathcal{S} \) to \( \mathbb{R} \) with following parametric form

\[
V(.) = \min \left\{ \max_{a \in A} \left\{ (z, \psi(s,a,w)) + 2L\beta_1 \sqrt{\langle \phi(s,a) \rangle^T Y \phi(s,a)} + \beta_2 \sqrt{\langle \phi(s,a,w) \rangle^T \bar{Y} \phi(s,a,w)} \right\}, H \right\},
\]

where the parameters \( \beta, \beta_1, \beta_2 \in \mathbb{R}, \ z \in \mathbb{R}^{d'}, \ Y \in \mathbb{R}^{d \times d'} \) and \( \bar{Y} \in \mathbb{R}^{d \times d'} \) satisfy \( 0 \leq \beta \leq B, \ 0 \leq \beta_1 \leq \hat{B} \|z\| \leq z, \ \|Y\| \leq \lambda^{-1} \) and \( \|\bar{Y}\| \leq \lambda^{-1} \). Assume \( \|\phi(s,a)\| \leq 1 \) and \( \|\psi(s,a,w)\| \leq 1 \) for all \( (s,a,w) \in \mathcal{S} \times A \times \mathcal{W} \). Then

\[
\log (\mathcal{N}_e(\mathcal{V})) \leq d' \log(1 + 4z/\epsilon) + d^2 \log \left( \frac{1 + 8B^2\sqrt{d}}{\lambda\epsilon^2} \right) + d^2 \log \left( \frac{1 + 8\hat{B}^2\sqrt{d}}{\lambda\epsilon^2} \right).
\]

**Proof.** First, we reparametrize \( \mathcal{V} \) by letting \( Z = \beta^2 Y \) and \( \bar{Z} = \hat{\beta}^2 \bar{Y} \). We have

\[
V(.) = \min \left\{ \max_{a \in A} \left\{ (z, \psi(s,a,w)) + \sqrt{\langle \phi(s,a) \rangle^T Z \phi(s,a)} + \sqrt{\langle \phi(s,a,w) \rangle^T \bar{Z} \phi(s,a,w)}, H \right\} \right\},
\]

for \( \|z\| \leq z, \ \|Z\| \leq \frac{B^2}{\lambda^2} \), and \( \|\bar{Z}\| \leq \frac{\hat{B}^2}{\lambda^2} \). For any two functions \( V_1, V_2 \in \mathcal{V} \) with parameters \( (Z^1, Z^1, \bar{Z}^1) \) and \( (Z^2, Z^2, \bar{Z}^2) \), respectively, we have

\[
\text{dist}(V_1, V_2) \leq \sup_{(s,a) \in \mathcal{S} \times A} \left| \left( \langle Z^1, \psi(s,a,w) \rangle + \sqrt{\langle \phi(s,a) \rangle^T Z^1 \phi(s,a)} + \sqrt{\langle \psi(s,a,w) \rangle^T \bar{Z}^1 \psi(s,a,w)} \right) 
\right.

\[ - \left( \langle Z^2, \psi(s,a,w) \rangle + \sqrt{\langle \phi(s,a) \rangle^T Z^2 \phi(s,a)} + \sqrt{\langle \psi(s,a,w) \rangle^T \bar{Z}^2 \psi(s,a,w)} \right) \right| 
\]

\[
\leq \sup_{\psi \|\psi\| \leq 1} \left| \left( \langle z^1, \psi \rangle + \sqrt{\langle \phi \rangle^T Z^1 \phi} + \sqrt{\psi^T \bar{Z}^1 \psi} \right) - \left( \langle z^2, \psi \rangle + \sqrt{\langle \phi \rangle^T Z^2 \phi} + \sqrt{\psi^T \bar{Z}^2 \psi} \right) \right|
\]

\[
\leq \sup_{\psi \|\psi\| \leq 1} \left| \left( z^1 - z^2, \psi \right) \right| + \sup_{\phi \|\phi\| \leq 1} \sqrt{\langle \phi \rangle^T (Z^1 - Z^2) \phi} + \sup_{\psi \|\psi\| \leq 1} \sqrt{\psi^T (\bar{Z}^1 - \bar{Z}^2) \psi}
\]

(because \( \sqrt{a} - \sqrt{b} \leq \sqrt{|a - b|} \) for \( a, b \geq 0 \))

\[
= \left\| z^1 - z^2 \right\| + \sqrt{\left\| Z^1 - Z^2 \right\|^2} + \sqrt{\left\| \bar{Z}^1 - \bar{Z}^2 \right\|^2}
\]

\[
\leq \left\| z^1 - z^2 \right\| + \sqrt{\left\| Z^1 - Z^2 \right\|^2} + \sqrt{\left\| \bar{Z}^1 - \bar{Z}^2 \right\|^2}. \quad (61)
\]

Let \( \mathcal{C}_z \) be an \( \epsilon/2 \)-cover of \( \{ z \in \mathbb{R}^{d'} : \|z\| \leq z \} \) with respect to the 2-norm, \( \mathcal{C}_Z \) be an \( \epsilon^2/4 \)-cover of \( \{ Z \in \mathbb{R}^{d \times d'} : \|Z\| \leq \frac{B^2\sqrt{d}}{\lambda} \} \), and \( \mathcal{C}_{\bar{Z}} \) be an \( \epsilon^2/4 \)-cover of \( \{ \bar{Z} \in \mathbb{R}^{d \times d'} : \|\bar{Z}\| \leq \frac{\hat{B}^2\sqrt{d}}{\lambda} \} \) with respect to the Frobenius norm.

By Lemma 18, we know

\[
|\mathcal{C}_z| \leq (1 + 4z/\epsilon)^{d'}, \quad |\mathcal{C}_Z| \leq \left( \frac{1 + 8B^2\sqrt{d}}{\lambda\epsilon^2} \right)^{d^2}, \quad |\mathcal{C}_{\bar{Z}}| \leq \left( \frac{1 + 8\hat{B}^2\sqrt{d}}{\lambda\epsilon^2} \right)^{d^2}.
\]
According to (61), it holds that $\mathcal{N}_e(\mathcal{V}) \leq |\mathcal{C}_z||\mathcal{C}_Y|$, and therefore

$$
\log (\mathcal{N}_e(\mathcal{V})) \leq d' \log(1 + 4z/\epsilon) + d^2 \log \left( \frac{1 + 8B^2 \sqrt{d}}{\lambda \epsilon^2} \right) + d'^2 \log \left( \frac{1 + 8\tilde{B}^2 \sqrt{d'}}{\lambda \epsilon^2} \right).
$$

\[ \square \]

**Lemma 22.** Let $\mathcal{V}$ denote a class of functions mapping from $\mathcal{S}$ to $\mathbb{R}$ with following parametric form

$$
V(\cdot, \cdot) = \min \left\{ \max_{a \in \mathcal{A}} \left\{ \langle \mathbf{z}, \psi(\cdot, a, \cdot) \rangle + 2L\beta \sqrt{\psi(\cdot, a, \cdot)^T \mathbf{Y} \psi(\cdot, a, \cdot)} \right\}^+, H \right\},
$$

where the parameters $\beta \in \mathbb{R}$, $\mathbf{z} \in \mathbb{R}^{d'}$ and $\mathbf{Y} \in \mathbb{R}^{d' \times d'}$ satisfy $0 \leq \beta \leq B$, $\|\mathbf{z}\| \leq z$, and $\|\mathbf{Y}\| \leq \lambda^{-1}$. Assume $\|\psi(s, a, w)\| \leq 1$ for all $(s, a, w) \in \mathcal{S} \times \mathcal{A} \times \mathcal{W}$. Then

$$
\log (\mathcal{N}_e(\mathcal{V})) \leq d' \log(1 + 4z/\epsilon) + d^2 \log \left( \frac{1 + 8B^2 \sqrt{d'}}{\lambda \epsilon^2} \right).
$$

**Proof.** First, we reparametrize $\mathcal{V}$ by letting $\tilde{\mathbf{Y}} = \beta^2 \mathbf{Y}$. We have

$$
V(\cdot, \cdot) = \min \left\{ \max_{a \in \mathcal{A}} \left\{ \langle \mathbf{z}, \psi(\cdot, a, \cdot) \rangle + \sqrt{\psi(\cdot, a, \cdot)^T \tilde{\mathbf{Y}} \psi(\cdot, a, \cdot)} \right\}, H \right\},
$$

for $\|\mathbf{z}\| \leq z$, and $\|\tilde{\mathbf{Y}}\| \leq \frac{B^2}{\lambda}$. For any two functions $V_1, V_2 \in \mathcal{V}$ with parameters $(\mathbf{z}_1, \tilde{\mathbf{Y}}_1)$ and $(\mathbf{z}_2, \tilde{\mathbf{Y}}_2)$, respectively, we have

$$
dist(V_1, V_2) \leq \sup_{(s, a, w) \in \mathcal{S} \times \mathcal{A} \times \mathcal{W}} \left| \left\langle \mathbf{z}_1, \psi(s, a, w) \right\rangle + \sqrt{\psi(s, a)^T \tilde{\mathbf{Y}}_1 \psi(s, a)} \right| - \left| \left\langle \mathbf{z}_2, \psi(s, a, w) \right\rangle + \sqrt{\psi(s, a)^T \tilde{\mathbf{Y}}_2 \psi(s, a)} \right|
$$

$$
\leq \sup_{\mathbf{z} : \|\mathbf{z}\| \leq 1} \left| \left\langle \mathbf{z}_1, \psi \right\rangle + \sqrt{\psi^T \tilde{\mathbf{Y}}_1 \psi} - \left\langle \mathbf{z}_2, \psi \right\rangle + \sqrt{\psi^T \tilde{\mathbf{Y}}_2 \psi} \right|
$$

$$
\leq \sup_{\mathbf{z} : \|\mathbf{z}\| \leq 1} \left| \left\langle \mathbf{z}_1 - \mathbf{z}_2, \psi \right\rangle \right| + \sup_{\mathbf{z} : \|\mathbf{z}\| \leq 1} \sqrt{\psi^T \left( \tilde{\mathbf{Y}}_1 - \tilde{\mathbf{Y}}_2 \right) \psi} (\text{because } |\sqrt{a} - \sqrt{b}| \leq \sqrt{|a - b|} \text{ for } a, b \geq 0)
$$

$$
= \|\mathbf{z}_1 - \mathbf{z}_2\| + \|\tilde{\mathbf{Y}}_1 - \tilde{\mathbf{Y}}_2\|.
$$

Let $\mathcal{C}_z$ be an $\epsilon/2$-cover of $\{\mathbf{z} \in \mathbb{R}^{d'} : \|\mathbf{z}\| \leq z\}$ with respect to the 2-norm, and $\mathcal{C}_Y$ be an $\epsilon^2/4$-cover of $\{\mathbf{Y} \in \mathbb{R}^{d' \times d'} : \|\mathbf{Y}\|_F \leq \frac{B^2 \lambda^{1/2}}{\epsilon}\}$, with respect to the Frobenius norm. By Lemma 18, we know

$$
|\mathcal{C}_z| \leq (1 + 4z/\epsilon)^{d'}, \quad |\mathcal{C}_Y| \leq \left( \frac{1 + 8B^2 \sqrt{d'}}{\lambda \epsilon} \right)^{d^2}.
$$

According to (62), it holds that $\mathcal{N}_e(\mathcal{V}) \leq |\mathcal{C}_z||\mathcal{C}_Y|$, and therefore

$$
\log (\mathcal{N}_e(\mathcal{V})) \leq d' \log(1 + 4z/\epsilon) + d^2 \log \left( \frac{1 + 8B^2 \sqrt{d'}}{\lambda \epsilon^2} \right).
$$

\[ \square \]