High Energy Particle Collisions in Superspinning Kerr Geometry

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We investigate here the particle acceleration and high energy collision in the Kerr geometry containing a naked singularity. We show that the center of mass energy of collision between two particles, dropped in from a finite but arbitrarily large distance along the axis of symmetry is arbitrarily large, provided the deviation of the angular momentum parameter from the mass is very small for the Kerr naked singularity. The collisions considered here are between particles, one of them ingoing and the other one being initially ingoing but which later emerges as an outgoing particle, after it suffers a reflection from a spatial region which has a repulsive gravity in the vicinity of the naked singularity. High energy collisions take place around a region which marks a transition between the attractive and repulsive regimes of gravity. We make a critical comparison between our results and the BSW acceleration mechanism [M. Banados, J. Silk, and S. M. West, Phys. Rev. Lett. 103, 111102 (2009).] for extremal Kerr blackholes, and argue that the scenario we give here has certain distinct advantages. If compact objects exist in nature with exterior Kerr superspinning geometry then such high energy collisions would have a significant impact on the physical processes occurring in its surrounding and could possibly lead to their own observational signatures. As an aside we also suggest a curious Gedanken collider physics experiment which could in principle be constructed in this geometry.

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I. INTRODUCTION

Various terrestrial particle collider experiments such as the Large Hadron Collider probe physics upto 10 TeV. This energy scale is almost 15 orders of magnitude smaller than the Planck scale. Particle physics models in such very high energy regime remain completely unexplored and untested by means of any terrestrial collider physics experiment at the current epoch due to various limitations of technology available to us. High precision cosmic microwave background experiments might shed some light on the new physics at high energies in near future.

An alternative intriguing possibility to study such a new physics would be to make use of various naturally occurring extreme gravity astrophysical objects in our surrounding universe. In this spirit, it was suggested recently [1], that the blackholes that are either extremal or very close to being extremal, could be used as particle accelerators to probe new physics all the way upto Planck scale. In that case, the particles thrown in from infinity could interact with divergent center of mass energies near the event horizon of the extremal blackhole, provided that certain fine-tuning conditions were imposed on the angular momentum of one of the colliding particles.

In this work we shall show that the Kerr naked singularities can as well act as particle accelerators to arbitrarily high energies in the limit where the deviation of angular momentum parameter $\bar{\alpha} = \frac{a}{M}$ is sufficiently small from unity. The mechanism we propose here has a distinct advantage over the blackhole case, necessarily arising from the absence of an event horizon, and due to the presence of a repulsive gravity regime in the vicinity of the naked singularity.

Thus the Kerr naked singularity, if it occurs in nature provides a suitable environment where ultrarelativistic collisions could take place at extremely large center of mass energies. The interaction between the particles in this case could be dominated by new channels of reactions dictated by the beyond standard model physics. This could provide an excellent laboratory to study high energy physics unexplored at various terrestrial accelerator experiments. Unlike the blackhole case, due to the absence of an event horizon in these models, a large fraction of the high energy particles produced in these collisions would either escape away to infinity and get detected on earth, or they could interact with the surrounding gas and leave behind their characteristic astrophysical imprint. This might lead to either direct or indirect observational signatures of the Kerr superspinning objects.

We also propose here a rather efficient and economical curious Gedanken collider experiment in the environment of the naked singularity along the axis of symmetry of the Kerr geometry. However, this requires an extreme finetuning of a Kerr spin parameter close to unity.

Our consideration of Kerr naked singular geometries is motivated by recent theoretical developments in string theory which suggest that the timelike naked singularities are naturally resolved and pathological features associated with them like causality violation are avoided by high energy modifications to classical general relativity [2], thus transcending the classical cosmic censorship.

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conjecture [3]. This is consistent with the thinking that in general the spacetime singularities will be resolved when we use a correct quantum gravity theory to study the ultra-high and extreme gravity phenomena. Earlier, the phenomena of naked singularity resolution is also reported within the framework of the loop quantum gravity theory [4].

The point here is that, even when the final spacetime singularity may be resolved by the quantum gravity effects which will be operative at the Planck length in the vicinity of the singularity, even then the presence or otherwise of an event horizon, faraway from the singularity in the spacetime, will make a very significant difference as far as the physical effects are concerned within either the blackhole or naked singularity spacetime geometry.

In general, either a blackhole or a naked singularity will arise as the endstate of gravitational collapse of a massive star, which shrinks and collapses continually under the force of its own gravity towards the end of its life cycle on exhausting the nuclear fuel within. The formation of blackholes as well as naked singularity as an endstate of such a gravitational collapse has been rigorously investigated in recent years. There are several spherically symmetric models of matter collapse with realistic equation of states, which lead to the formation of naked singularities in a gravitational collapse, starting from a regular initial data specified on an initial hypersurface. The genericity as well as stability of these models have also been investigated [5]. Basically, the blackhole or naked singularity formation as the final state of the star collapse is decided by the matter initial data in terms of the density and pressure and velocity profiles of the collapsing matter, and the allowed dynamical evolutions, as permitted by the Einstein equations. For example, considering a pressure free dust collapse evolution, the cloud will evolve to a blackhole with the spacetime singularity hidden within the event horizon, in the case when the density is taken to be homogeneous always and when the velocity profile for the collapsing shells have a specific fine-tuned form. This is the well-known model case studied by Oppenheimer and Snyder [6]. On the other hand, when the initial density profile is allowed to be inhomogeneous, say higher at the center of the star and decreasing slowly with increasing radius, a naked singularity develops as the collapse end product [7]. Collapse models with non-vanishing pressures have also been studied in much detail in past years and it is seen that it is the initial data and allowed dynamical evolutions in the above sense that decide the blackhole or naked singularity final states for the collapsing matter cloud.

In general, it is expected that a collapsing star would give rise to either a Kerr blackhole or spinning naked singular configuration as the endstate of a continual gravitational collapse. We note that the formation of either of the above in a gravitational collapse with angular momentum has not been demonstrated conclusively. There has been some progress in this direction using numerical techniques [8]. There has also been an investigation demonstrating formation of rotating naked singularities in shell collapse in 2+1 dimensions [9]. The stability and genericity of this process is yet to be explored. The formation of superspinning compact objects, whose size is much smaller than the gravitational radius, via accretion onto slowly rotating compact objects, which possess not only mass and angular momentum but also an anomalous quadruple moment has also been demonstrated [10].

On the observational front, the existence of event horizon of supermassive and stellar mass blackhole candidates has not been shown conclusively, although it is widely believed and most likely that these objects are indeed Kerr blackholes. The measured values of spin parameter by X-ray continuum emission line shape fitting method is dependent on the assumptions regarding accretion onto these objects, modifying which could yield inferred spin values larger than unity [11]. Keeping this in mind there has been a significant effort to design strategies to distinguish blackholes from naked singularities. The silhouette of Kerr naked singularity is shown to differ significantly from the blackhole case in the gravitational lensing studies [12]. Also, the accretion around naked singularities is shown to differ significantly from the blackholes [13].

We note that the Kerr naked singularity is not a unique Vacuum, axisymmetric, asymptotically flat solution to Einstein equations in the absence of an event horizon. The most general solution in this case happens to be the Tomimatsu-Sato spacetime [14]. In this paper we focus only on the Kerr naked singular solution for simplicity and clarity. Also the stability of a Kerr superspinning configuration after the singularity is resolved by quantum gravitational corrections, would be an interesting issue for investigation in future.

The next Section II discusses the Kerr geometry in a proper coordinate setting, and in Section III the geodesic motion is examined with a particular focus on the axis of symmetry. Then Sections IV and V discuss the particle acceleration and a collider thought experiment using the Kerr naked singularity. We then make a comparison of the naked singularity case with the blackhole case in Section VI, and certain possible observational implications of the Kerr naked singularity as related to the particle acceleration processes are indicated in Section VII. Finally Section VIII gives concluding remarks.

II. KERR SPACETIME IN KERR-SCHILD COORDINATES

The Kerr metric [15], [16], [17] is characterized by two parameters, namely mass $M$ and angular momentum per unit mass $a = J/M$. When $a < M$ the Kerr metric represents a blackhole, whereas $a > M$ stands for a naked singularity without an event horizon. We focus here on the particles following geodesic motion along the axis of symmetry of Kerr spacetime with $a > M$. Thus we use the Kerr-Schild (KS) coordinate system $(t, x, y, z)$, which
is well-behaved around axis of symmetry \[17, 18\].

The Kerr spacetime geometry is given by,

\[
\begin{align*}
    ds^2 &= -dt^2 + dx^2 + dy^2 + dz^2 + \frac{2Mr^3}{r^4 + a^2z^2} \left[ dt + \frac{zdz}{r} + \frac{r(xdx + ydy) - a(xdy - ydx)}{r^2 + a^2} \right]^2 
\end{align*}
\]  (1)

where \(r(x, y, z)\) is a solution to the equation

\[
    r^4 - \left(x^2 + y^2 + z^2 - a^2\right)r^2 - a^2z^2 = 0
\]

We make a further coordinate transformation and introduce a new time coordinate \(T(t, x, y, z)\) as,

\[
    dT = dt - \beta dz
\]  (2)

where

\[
    \beta = -\frac{\frac{z}{r} \frac{2Mr^3}{r^4 + a^2z^2}}{-1 + \frac{2Mr^3}{r^4 + a^2z^2}}
\]

The metric to leading order, in the spacetime region close to the symmetry axis can be written as

\[
    ds^2 = \left(1 - \frac{2Mrz}{r^2 + a^2} \right) dT^2 + dx^2 + dy^2 + \left(1 - \frac{2Mz}{z^2 + a^2} \right)^{-1} dz^2
\]

which is well behaved and regular metric around the axis, which we use below.

III. GEODESIC MOTION ALONG AXIS OF SYMMETRY

In this section we investigate motion of the particles following timelike geodesics along the axis of symmetry, i.e. along \(z\)-axis.

Since the metric coefficients are independent of \(T\), the spacetime admits a Killing vector field \(\xi = \partial_T\). For a par-
particle following geodesic motion, the quantity $E = -\xi^\mu U_\mu$ is then conserved, $U$ being the velocity of the particle, and $E$ is interpreted as the conserved energy per unit mass of the particle. The equation depicting conserved energy $E = -\xi^\mu U_\mu$ and the normalization $U^\mu U_\mu = -1$, together with (5) and $U^\mu = (U^T, 0, 0, U^z)$, allows components of velocity of the particle to be written as follows,

$$U^T = \frac{E}{f}$$

$$\left(U^z\right)^2 + f = E^2$$

$$U^z = \pm \sqrt{E^2 - f}$$

where

$$f = \left(1 - \frac{2Mz}{z^2 + a^2}\right)$$

Here ± correspond to the outgoing and ingoing geodesics respectively. By analogy in Newtonian mechanics, the function $f$ in (6) can be thought of as an effective potential for a motion along z-axis.

The effective potential $f$ takes a maximum value at $z = 0$ and as $z \to \infty$. It takes a minimum value at an intermediate point $z = a$. The behavior of effective potential is shown in Fig. 1. Maximum and minimum values are given by

$$f_{\text{max}} = f(z = 0) = f(z \to \infty) = 1$$

$$f_{\text{min}} = f(z = a) = \left(1 - \frac{M}{a}\right) = \epsilon > 0$$

(8)

Since we are dealing with the Kerr solution which is a naked singular spacetime, the minimum value of $f$ is strictly larger than zero. The parameter $\epsilon > 0$ we have introduced in the above indicates the deviation of the Kerr from the extremal case where $a = M, \epsilon = 0$. It follows from (5) that the particle with conserved energy per unit mass $E < 1$ will be confined between the values of $z$ as given by,

$$z_- = \frac{M - \sqrt{M^2 - (1 - E^2) a^2}}{1 - E^2}$$

$$z_+ = \frac{M + \sqrt{M^2 - (1 - E^2) a^2}}{1 - E^2}$$

(9)

which are the turning points where $U^z = 0$.

For an infalling particle, $U^z$ goes on increasing when $z > a$, indicating the attractive nature of gravity. The same quantity goes on decreasing when $z < a$, and eventually it stops and turns back, thus indicating the ‘repulsive nature’ of gravity in this regime. All stationary spacetimes admitting naked singularities are found to exhibit such a repulsive gravity effect in the close neighborhood of singularity [19, 20]. In Kerr spacetimes, the attractive or repulsive nature of gravity is roughly determined by whether or not $(r^2 - a^2 \sin^2 \theta) > 0$ or $< 0$, respectively (when expressed in the Boyer-Lindquist coordinates [21]). A particle with $E = \sqrt{1 - \frac{M^2}{a^2}}$, stays at rest at $z = a$, which marks a transition between attractive and repulsive regimes of gravity.

**IV. PARTICLE ACCELERATION BY KERR NAKED SINGULARITY**

We consider a collision of two particles, each of mass $m$ and conserved energy of per unit mass $E$. One of the particles is taken to be ingoing and the other one is outgoing. The center of mass energy $E_{c.m.}$ of collision between two such particles with velocities $U^1, U^2$ is given by [1],

$$E_{c.m.}^2 = 2m^2 \left(1 - g_{\mu \nu} U^1 \mu U^2 \nu\right)$$

(10)

Thus from (5), (6), (7), (10), the center of mass energy of collision in this case would be,

$$E_{c.m.}^2 = \frac{4m^2 E^2}{f}$$

(11)

From (5) it can be seen that the center of mass energy will be maximum if the collision happens at $z = a$, which is given by,

$$E_{c.m., max}^2 = \frac{4m^2 E^2}{\epsilon}$$

(12)

Thus it is seen from the expression above that the center of mass energy of collision between ingoing and outgoing particles will be extremely large if the $\epsilon$, which
initially speeds up when \( z > a \) as it falls in. Its speed \( U^z \) is maximum when it is at \( z = a \), which is the minimum of the effective potential \( f \). It then slows down and turns back at \( z = \frac{a^2}{z_{in}} \). It speeds up again but now in the outward direction, speed being maximum at \( z = a \) in the outward direction. We make this particle collide with the second incoming particle at near \( z = a \), when its speed is maximum in the inwards direction. The center of mass energy of collision in this process is given by,

\[
E_{c.m.} = \frac{2m\sqrt{f(z_{in})}}{\epsilon}
\]

which is arbitrarily large for small enough values of \( \epsilon \). Also, the desired energy of collision can be obtained or tuned by making an appropriate choice for the initial point along the axis \( z = z_{in} \) from which the particles are dropped, thus allowing us to probe new physics at a range of different very high energy scales.

The particle detector is placed at \( (x = 0, y = 0, z = a) \). Since this is a point at the interface of the attractive and repulsive regimes of gravity, the detector would stay there at rest on its own, without the need of any rockets. However some rocket support might be required to stabilize its motion along \( x \) and \( y \) directions. The measurements from the detector placed at the site of collision are used to unravel the new physics. There is no substantial power consumption required to either accelerate the particles, or maintain the location of detector in space, making it a very efficient arrangement to perform particle collider experiments at arbitrarily large energies. In contrast, if we want to use the Kerr blackhole as particle accelerator, much effort and energy will be needed to stabilize the detector near the event horizon.

We note, however, that for the collider experiment suggested above to work in order to probe the Planck scale physics, the Kerr spin parameter would require an extreme fine-tuning which has to be very close to unity. If the colliding particles are neutrons with rest mass \( m_n = 940\, MeV \approx 1\, GeV \), the center of mass energy of collision would be comparable to the Planck scale \( E_{pl} \approx 10^{19}\, GeV \), provided the deviation of spin from unity is \( \epsilon \approx 10^{-38} \). It would be probably unreasonable to expect an occurrence in nature of Kerr super-spinning configurations with extreme fine-tuning of Kerr parameter such as above. Thus it may not be possible to actually realize such a collider experiment in practice. However, it still continues to be an example of a curious Gedanken experiment, the likes of which over a period of time have played an important role in the development of relativity theory and theoretical physics in general.

\section{VI. COMPARISON WITH BSW ACCELERATION MECHANISM}

We now compare these results with BSW particle acceleration mechanism in the case of extremal or near extremal blackholes \cite{1, 22, 23}. The BSW mechanism...
deals with collision between two infalling particles, which collide near the event horizon of near extremal Kerr blackhole. Although the horizon is an infinite blue-shift surface, since infalling particles arrive almost perpendicularly, their relative velocity is small. Thus the center of mass energy of collision would be finite. For divergence of center of mass energy, the fine-tuning of angular momentum of one of the infalling particles is necessary. It must have largest possible angular momentum that still allows it to reach horizon. This restriction demands that near the horizon, \( \dot{r} = \dot{\phi} = 0 \), where the dot denotes the derivative with respect to the affine parameter, and \( \dot{r} \) is the Boyer-Lindquist radial coordinate \([21]\). The condition above implies that the amount of proper time required for the particle to reach horizon and participate in collision is infinite. However, in the case of Kerr naked singularities, due to the absence of an event horizon and the transition of nature of gravity from being attractive to repulsive, we can consider the collision between an ingoing and the other outgoing particles which have extremely large relative velocity at the point of collision. Also, since both the conditions \( \dot{z} = 0, \dot{\bar{z}} = 0 \) are not realized simultaneously anywhere along the geodesic, the proper time required for the collision to happen is finite.

In the BSW mechanism, the maximum center of mass energy of collision grows as the blackhole approaches extremality, \(-\epsilon = M - a \to 0\), because \( E_{c.m.,\text{max}} \sim \left(\frac{1}{\epsilon}\right)^{\gamma/2} \). In our case, because the extremality is approached from the higher side of the parameter \( a \), the maximum center of mass energy grows twice as fast (as compared to the BSW mechanism), on a logarithmic scale \( E_{c.m.,\text{max}} \sim \left(\frac{1}{\epsilon}\right)^{\gamma/2} \). Thus to probe Planck scale physics using Kerr blackholes as particle accelerators, the spin parameter must be tuned to unity one part in \( 10^{76} \). Whereas for Kerr naked singularities the required fine-tuning is brought down to \( 10^{18} \).

**VII. POSSIBLE OBSERVATIONAL IMPLICATIONS OF KERR NAKED SINGULARITY ARISING FROM PARTICLE ACCELERATION PROCESS**

As we have shown above, the Kerr naked singularities with spin parameter close to unity provide an environment where collisions with large center of mass energies can take place. In a typical astrophysical setting it would be unreasonable to expect that the fine-tuning necessary for collisions with center of mass energy comparable to Planck scales can be actually realized. However, it would be possible to have collisions with reasonably large center of mass energies with a reasonable fine-tuning of the spin parameter close enough to unity as we show below.

We consider a hypothetical situation where a supermassive astrophysical blackhole candidate, say at the center of the galaxy, with \( M \approx 10^8 M_\odot \) is modeled by a Kerr naked singular geometry. Here \( M_\odot = 2.10^{30} \text{kg} \) is mass of the sun. Let us assume that the Kerr configuration under consideration is about one \( M_\odot \) away from the extremality. That is, upon the addition of one solar mass the Kerr naked singularity will turn itself into an extremal Kerr blackhole. In this case we have \( \epsilon \approx 10^{-8} \) and the center of mass energy of collision could be as high as \( E_{c.m.} \approx 10^4 m \), \( m \) being the mass of the colliding particles. If the colliding particles are assumed to be either protons or neutrons with mass \( m \approx 1 \text{GeV} \), the center of mass energy of collisions between them would be \( E_{c.m.} \approx 10^4 \text{GeV} \approx 10^3 \text{TeV} \). This is the energy scale at which protons collide at LHC. If the colliding particles are hypothetical dark matter particles with mass \( m \approx 100 \text{GeV} \), then the center of mass energy of collision would be \( E_{c.m.} \approx 10^6 \text{GeV} \approx 10^5 \text{TeV} \), which is 100 times larger than the LHC energy scale. The cross-section of interaction between the particles is dependent on the center of mass energy of collisions and typically it is large at the larger values of center of mass energy. It is possible that new reactions channels for the interactions might also be available at larger energies. Various high energy particles with smaller rest masses would be produced in the interactions. Due to the absence of the event horizon, majority of those particles might escape to infinity unlike the blackhole case where they are absorbed by the event horizon.

Since the galaxies are formed in the dark matter halos, the central supermassive object, modeled by a Kerr naked singularity in this case would be surrounded by and accrete the dark matter. Due to the enhanced density in the high gravity region and due to the large annihilation cross-section at large center of mass energy of collisions, dark matter particles would annihilate to ordinary standard model particles with high energies, majority of which eventually escape to infinity. These particles could be possibly detected on earth. It is also quite likely that the high energy particles would interact with the surrounding gas in the galaxy and would dissipate their energy as thermal energy. The heating up of the cloud would then affect the star formation rate in the surroundings of the galactic center. Thus, if the Kerr naked singularities exist in nature, they would leave an imprint on the astrophysical phenomena.

A rigorous analysis will be necessary to come up with concrete predictions, which will require both the detailed modeling of the astrophysical scenario as well as the understanding of the particle physics processes at the required high energies. This is beyond the scope of the present paper and will be discussed elsewhere in future.

**VIII. CONCLUDING REMARKS**

In this paper we showed that it is possible to have particle collisions with arbitrarily high center of mass energy in the vicinity of Kerr naked singularity, with the angular momentum parameter exceeding unity by a vanishingly small amount, even if the particles are sent in from infinity from rest. We note that in this analysis, we have used
the test particle approximation, neglecting the self-force and backreaction.

We focussed our attention to geodesics that are restricted along the axis of symmetry. But we also expect such high energy collisions to take place in the region around the $z$-axis. The high energy collisions were essentially a consequence of the fact that the metric coefficient $g_{zz}^{-1} = f$ is vanishingly small at $z = a$, and because there is a transition from attractive to repulsive gravity regime, which allowed us to have collisions between the ingoing and outgoing particles.

Since, by continuity, both the conditions hold good in the region nearby the $z$-axis, such collisions would be realized there as well. We shall present a detailed analysis elsewhere. Our purpose here has been essentially to demonstrate the intriguing possibility of having high energy collisions of particles, for the Kerr spacetime without an event horizon.

The high energy particles produced in these collisions could be detected at infinity or these would interact with the gas in the surroundings. This could in principle lead to specific observational signatures of the Kerr naked singularities if they exist in nature.

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