Einstein-Chern-Simons equations on the 3-brane world

F. Izaurieta¹, P. Salgado² and R. Salgado¹
¹Departamento de Física, Universidad de Concepción
Casilla 160-C, Concepción, Chile
²Instituto de Ciencias Exactas y Naturales (ICEN)
Facultad de Ciencias, Universidad Arturo Prat
Avda. Arturo Prat 2120, Iquique, Chile

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Abstract

In this article it is studied the 3-brane world in the context of five-dimensional Einstein-Chern-Simons gravity. We started by considering Israel’s junction condition for AdS-Chern-Simons gravity. Using the S-expansion procedure, we mapped the AdS-Chern-Simons junction conditions to Einstein-Chern-Simons gravity, allowing us to derive effective four-dimensional Einstein-Chern-Simons field equations.

1 Introduction

The observations and experiments show that General Relativity and the Standard Model provide the current understanding of the natural phenomena. From a theoretical point of view, however, the Standard Model are gauge theories, i.e., they are theories whose fundamental field, is a connection, while General Relativity is a theory whose fundamental field is a metric.

A gauge theory for the gravitational field requires a fundamental field given by a connection. An action for gravity fulfilling these conditions is the Chern-Simons gravity action, which was proposed long ago by Chamseddine [1], [2], [3].

This Chern-Simons gravity is a well-defined gauge theory, but the presence of higher powers of the curvature makes its dynamics very remote from that for standard Einstein Hilbert gravity. However in Refs. [4], [5], [6] was shown that the standard, five-dimensional General Relativity can be obtained from Chern-Simons gravity theory for the Lie algebra $\mathfrak{so}_5$, whose generators $\{J_{AB}, P_A, Z_{AB}, Z_A\}$ satisfy the commutation relationships

\[
\begin{align*}
[J_{AB}, J_{CD}] &= \eta_{CB}J_{AD} - \eta_{CA}J_{BD} + \eta_{DB}J_{CA} - \eta_{DA}J_{CB} \\
[J_{AB}, P_C] &= \eta_{CB}P_A - \eta_{CA}P_B
\end{align*}
\]
\[ [P_A, P_B] = Z_{AB} \]
\[ [J_{AB}, Z_{CD}] = \eta_{CB} Z_{AD} - \eta_{CA} Z_{BD} + \eta_{DB} Z_{CA} - \eta_{DA} Z_{CB} \]
\[ [J_{AB}, Z_C] = \eta_{CB} Z_A - \eta_{CA} Z_B \]
\[ [Z_{AB}, P_C] = \eta_{CB} Z_A - \eta_{CA} Z_B, \]

which can be obtained from the AdS algebra by means of the S-expansion procedure introduced in Refs. [7], [8], [9], [10]. An expansion is, in general, an algebra dimension-changing process, i.e., is a way to obtain new algebras of increasingly higher dimensions from a given one. A physical motivation for increasing the dimension of Lie algebras is that increasing the number of generators of an algebra is a non-trivial way of enlarging spacetime symmetries. Examples of this can be found in Refs. [11], [12], where applications of Maxwell’s algebra in gravity were studied (This algebra is a modification to the Poincaré symmetries and can be obtained, via S-expansion, from the anti-de Sitter (AdS) which is also known as \( \mathfrak{B}_4 \) algebra). Another interesting modification to the Poincaré symmetries are the so-called generalized Poincaré algebras [13] of which the \( \mathfrak{B}_5 \) algebra is an example.

For this reason the Chern-Simons gravity theory for the \( \mathfrak{B}_5 \) Lie algebra, known as Einstein-Chern-Simons gravity [14] can be understood as a theory that could allow us to know that if the space-time has (or not) more symmetry than those usually described by Poincaré or (A)dS algebras. This can be achieved by studying cosmological and black hole solutions such as that found in Refs. [15], [16], [17], [18], [19], [20], [21]. In particular in [21] EChS gravity was considered instead of General Relativity to describe the expansion of a flat 5-dimensional universe, where a cosmological analysis was performed. The \( k^{ab} \)-field was assumed null by virtue of the gauge freedom and the \( h^a \) field was represented as a perfect fluid. Was found an accelerating Dirac-Milne universe and a fluid, that early behaves like dark matter but later behaves like stiff matter. In the same reference was carry out a compactification 5D to 4D and was found that in a 4D context, it is possible to conjecture the existence of accelerated solutions, eventually driven by the \( h \)-field. The \( h \)-field was associated with a scalar field, exhibiting the behavior of cosmological constant (dark energy).

In order to write down a Chern–Simons lagrangian for the \( \mathfrak{B}_5 \) algebra, we start from the one-form gauge connection

\[ A = \frac{1}{2} \omega^{AB} J_{AB} + \frac{1}{l} e^A P_A + \frac{1}{2} k^{AB} Z_{AB} + \frac{1}{l} h^A Z_A, \]

and the two-form curvature

\[ F = \frac{1}{2} R^{AB} J_{AB} + \frac{1}{l} T^A P_A + \frac{1}{2} \left( D_\omega k^{AB} + \frac{1}{l^2} e^A e^B \right) Z_{AB} + \frac{1}{l} \left( D_\omega h^A + k^A e^B \right) Z_A. \]

In this point, it might be of interest to remember that: (i) clearly \( l \) could be eliminated by absorbing it in the definition of the vielbein, but then the
space-time metric $g_{\mu\nu}$ would no longer be related to $e^a$ through the relation $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$; (ii) the interpretation of the $l$ parameter as a parameter related to the radius of curvature of the $AdS$ space-time, could be inherited for the space-time whose symmetries are described by the $B_5$ generalized Poincaré algebra. This can be seen by recalling that the commutation relation $[P_a, P_b] = \frac{1}{l^2} Z_{ab}$ is obtained, using the expansion method, from the commutation relation $[\tilde{P}_a, \tilde{P}_b] = \frac{1}{l^2} J_{ab}$ for $AdS$ translations,

$$[P_a, P_b] = \lambda_2 \left[ \tilde{P}_a, \tilde{P}_b \right] = \frac{1}{l^2} \lambda_2 J_{ab} = \frac{1}{l^2} Z_{ab}$$

where $\lambda_2$ is an element of the semigroup $S_E^{(3)}$.

In Ref. [23] the $l$ parameter was interpreted as proportional to the Planck length $l_P$, which provides an intuitive way to understand the translation group as a non-abelian group on the Planck scale ($\sim 10^{-33}$cm) which appears as an abelian group on larger scales such as the scale of elementary particles ($\sim 10^{-13}$cm). In this same reference it was found that for that a gravitational theory, understood as a gauge theory of the de-Sitter group, contain the Einstein-Hilbert-Cartan action, it is necessary that

$$l^2 = \frac{16\pi G}{c^4}.$$  

(4)

This means that if the structure of space-time at the microscopic scale is governed by the de-Sitter group, then the constant $l$ appears naturally as a constant associated with gravitational interaction. From (4) and from the definition of the Planck length it is straightforward to see that $l$ is proportional to the Planck length.

Consistency with the dual procedure of $S$-expansion in terms of the Maurer-Cartan forms [8] demands that $h^A$ inherits the unit of length from the fünfbein. That is why it is necessary to introduce the $l$ parameter again, this time associated with $h^A$. Could be interesting to observe that $J_{AB}$ are still Lorentz generators, but $P_A$ are no longer $AdS$ boosts. In fact, $[P_A, P_B] = Z_{AB}$. However $e^A$ still transform as a vector under Lorentz transformations, as it must, in order to recover gravity in this scheme.

A Chern-Simons lagrangian in $d = 5$ dimensions is defined to be the following local function of a one-form gauge connection $A$:

$$L_{ChS}^{(5)}(A) = C \left< AF^2 - \frac{1}{2} A^3 F + \frac{1}{10} A^5 \right>,$$

(5)

where $\langle \cdots \rangle$ denotes a invariant tensor for the corresponding Lie algebra, $F = dA + AA$ is the corresponding the two-form curvature and $C$ is a constant [24].

Using theorem VII.2 of Ref. [7], it is possible to show that the only non-
vanishing components of an invariant tensor for the \( \mathfrak{B}_5 \) algebra are given by

\[
\langle J_{A_1} J_{A_2} J_{A_3} A_4 P_{A_5} \rangle = \frac{4l^3}{3} \varepsilon_{A_1 \cdots A_5},
\]

\[
\langle J_{A_1} J_{A_2} J_{A_3} A_4 Z_{A_5} \rangle = \frac{4l^3}{3} \varepsilon_{A_1 \cdots A_5},
\]

\[
\langle J_{A_1} J_{A_2} Z_{A_3} A_4 P_{A_5} \rangle = \frac{4l^3}{3} \varepsilon_{A_1 \cdots A_5},
\]

where \( \alpha_1 \) and \( \alpha_3 \) are arbitrary independent constants of dimensions \([\text{length}]^{-3}\).

Using the extended Cartan’s homotopy formula as in Ref. [22], and integrating by parts, it is possible to write down the Chern-Simons Lagrangian in five dimensions for the \( \mathfrak{B}_5 \) algebra as [4, 18]

\[
L^{(5)}_{\text{EChS}} = \alpha_1 l^2 \varepsilon_{ABCDE} e^A R^{BC} R^{DE}
\]

\[
\quad + \alpha_3 \varepsilon_{ABCDE} \left( \frac{2}{3} R^{AB} e^C e^D e^E + l^2 R^{AB} R^{CD} h^E + 2l^2 k^{AB} R^{CD} T^E \right)
\]

\[
\quad + d\hat{B}^{(4)}_{\text{EChS}},
\]

with

\[
\hat{B}^{(4)}_{\text{EChS}} = \varepsilon_{abcde} \left\{ \alpha_1 l^2 e^a \omega^{bc} \left( \frac{2}{3} \omega^{de} + \frac{1}{2} \omega^d j \omega f \right) + \right.
\]

\[
\quad + \alpha_3 l^2 \left( h^a \omega^{bc} + k^{ab} e^c \right) \left( \frac{2}{3} \omega^{de} + \frac{1}{2} \omega^d j \omega f \right) +
\]

\[
\left. + \left( k^{ab} \omega^{cd} \left( \frac{2}{3} \omega^{de} + \frac{1}{2} \omega^d j \omega f \right) + \frac{\alpha_3}{6} e^a e^b e^c \omega^{de} \right) \right\}
\]

where \( \alpha_1, \alpha_3 \) are parameters of the theory, \( l \) is a coupling constant, \( R^{AB} = d\omega^{AB} + \omega^A \omega^B + \omega^C \omega^{CB} \) and \( T^A = D e^A \) correspond to the curvature 2-form and the torsion 2-form respectively in the first-order formalism related to the spin connection 1-form and \( e^A \), \( h^A \) and \( k^{AB} \) are others gauge fields presents in the theory. It should be noted that the kinetic terms for the \( h^A \) and \( k^{AB} \) fields are present only in the surface term of the Lagrangian shown in (7).

It is also interesting to note that when the constant \( \alpha_1 \) vanishes, the lagrangian (6) almost exactly matches the one given in Ref. [5], the only difference being that in our case the coupling constant \( l^2 \) appears explicitly in the last two terms. The presence or absence of the coupling constant \( l \) in the lagrangian could seem like a minor or trivial matter, but it is not. As the authors of Ref. [5] clearly state, the presence of the Einstein–Hilbert term in this kind of action does not guarantee that the dynamics will be that of general relativity. In general, extra constraints on the geometry do appear, even around a “vacuum” solution with \( k^{ab} = h^a = 0 \). In fact, the variation of the lagrangian, modulo
boundary terms, can be written as

$$\delta L^{(5)}_{CS} = \varepsilon_{abcde} \left( 2\alpha_3 R^{ab} e^e d^e + \alpha_1 l^2 R^{ab} R^{cd} + 2\alpha_3 l^2 D_{c_9} k^{ab} R^{cd} \right) \delta e^e$$

$$+ \alpha_3 l^2 \varepsilon_{abcde} R^{ab} R^{cd} \delta h^e + 2\varepsilon_{abcde} \delta \omega^{ab} \left( \alpha_1 l^2 R^{cd} T^e + \alpha_3 l^2 D_{f_9} T^e + \alpha_3 e^e d^e T^e \right)$$

$$+ \alpha_3 l^2 R^{cd} D_{h^e} + \alpha_3 l^2 R^{cd} k^{e} f^{f} + 2\alpha_3 l^2 \varepsilon_{abcde} \delta k^{ab} R^{cd} T^e \right). \quad (8)$$

This means that when the condition $\alpha_1 = 0$ is chosen, the torsionless condition imposed, and a solution without matter ($k^{ab} = h^a = 0$) is picked out, we are left with

$$\delta L^{(5)}_{CS} = 2\alpha_3 \varepsilon_{abcde} R^{ab} e^e d^e e^e + \alpha_3 l^2 \varepsilon_{abcde} R^{ab} R^{cd} \delta h^e. \quad (9)$$

In this way, besides general relativity equations of motions $\varepsilon_{abcde} R^{ab} e^e d^e = 0$, the equations of motion of pure Gauss-Bonnet theory $\varepsilon_{abcde} R^{ab} R^{cd} = 0$ do also appear as an anomalous constraint on the geometry. It is at this point where the presence of the coupling constant $l$ makes the difference. In the present approach, it does play the role of a coupling constant between geometry and “matter”. For this reason, in this case the limit $l \to 0$ leads to the Einstein–Hilbert term in the lagrangian,

$$L^{(5)}_{CS} = \frac{2}{3} \alpha_3 \varepsilon_{abcde} R^{ab} e^e d^e e^e. \quad (10)$$

In the same way, when we impose the weak limit of coupling constant, $l \to 0$, the extra constraints just vanish, and $\delta L^{(5)}_{CS} = 0$ lead us to just the Einstein–Hilbert dynamics in the vacuum,

$$\delta L^{(5)}_{CS} = 2\alpha_3 \varepsilon_{abcde} R^{ab} e^e d^e e^e + 2\alpha_3 \varepsilon_{abcde} \delta \omega^{ab} e^e d^e T^e \right). \quad (11)$$

However, the Einstein-Chern-Simons gravity is valid only in odd dimensions and in order to have a well defined four-dimensional theory is necessary to carry out a dimensional reduction.

The aim of the present work is to derive the effective Einstein-Chern-Simons equations on the 3-brane following the procedure used in Ref. [25]. For simplicity the bulk spacetime is assumed to have 5 dimensions. In the beginning we do not assume any conditions on the bulk spacetime. Later, we assume the $Z_2$ symmetry and confinement of the matter energy momentum tensor on the brane, in accordance with the brane world scenario.

This work is organized as follows: in section 2 we briefly review the Izrael’s junction condition for Lovelock and AdS-Chern-Simons gravity. In section 3 we will study the junction conditions of Israel for the case of the Einstein-Chern-Simons equations. In section 4 we will obtain 4-dimensional effective gravitational equations on the brane. Conclusions and discussion are presented in Sect. 5. Appendices present details omitted in the main text.
Israel’s junction condition for Lovelock and AdS-Chern-Simons gravity

To study the braneworld in the context of Einstein-Chern-Simons gravity it is necessary first to know the junction conditions in the context of the Lovelock gravity theory [26], which allow us to find the appropriate junction conditions for AdS-Chern-Simons gravity.

In Refs. [27], [28], [29], the study of the general theory of relativity was generalized to the case of an edged manifold. This has consequences for the application of the variational principle. In fact, the total derivative term in the Euler-Lagrange variation leads to an boundary term over \( \partial M \).

The variation of an action of the form

\[
S = S(g_{AB}, \partial_C g_{AB}, \partial_D \partial_C g_{AB})
\]

leads to [30]

\[
\delta S = \int d^D x \left\{ \left[ \frac{\partial L}{\delta g_{AB}} - \frac{\partial L}{\partial (\partial_C g_{AB})} \right] \delta g^{AB} + \partial_D \partial_C \left( \frac{\partial L}{\partial (\partial_D \partial_C g^{AB})} \right) \delta g^{AB} + \partial_D \delta g^{AB} \frac{\partial L}{\partial (\partial_D \partial_C g^{AB})} \right\}.
\]

(12)

The last term corresponds to a boundary term. If there were only terms proportional to \( \delta g^{\mu \nu} \) and proportional to the derivative of \( \delta g^{\mu \nu} \) at the boundary, there would be no problem. The problem arises when there are normal derivatives of \( \delta g^{\mu \nu} \) at the boundary coming from the last term of (12). In other words, the problematic terms are the ones proportional to \( \partial_D \delta g^{AB} \) at the boundary because they cannot be overridden by fixing the induced metric on the hypersurface. For this reason, it is necessary to add a term to the Lagrangian that cancels the contribution of the term proportional to \( \partial_D \delta g^{AB} \) at the edge, i.e., it cancels the normal derivatives of the metric variation. In Lovelock gravity, R. C. Myers proposed an appropriate boundary term in Ref. [29], and Willinson and Gravanis generalized it in Refs. [30, 31]. The term they proposed corresponds to

\[
S_{\Sigma} = \sum_n n \beta_n \int_0^1 dt \epsilon_{A_1 A_2 \ldots A_{2n+1}} \Theta^{A_1 A_2} R^{A_3 A_4} \ldots R^{A_{2n-1} A_{2n}} \bar{e}^{A_{2n+1}} \ldots \bar{e}^{A_d},
\]

(13)

where \( \beta_n \) are arbitrary constants, \( R^{AB} = d\omega^{AB} + \omega^A_{\ (i)} \omega^B_{\ (i)} \) is the curvature 2-form, \( \omega^{AB} = \bar{\omega}^{AB} + t \Theta^{AB} \) is the 1-form that interpolates between the bulk spin connection and the one of the brane; here \( \Theta^{AB} = \omega^{AB} - \bar{\omega}^{AB} \).

2.1 Boundary term for AdS-Chern-Simons Lagrangian

In references [32, 33, 34, 35, 36, 37], was constructed a boundary term that regularizes the action for AdS Chern-Simons gravity in \((2n+1)\)-dimensions.
In Chern-Simons AdS gravity, the Lagrangian is constructed from Euler’s topological invariant for the group $SO(2n, 2)$, so that the action is given by

$$S_{2n+1}^A = \int_{M_{2n+1}} L^A_{2n+1} + \int_{\Sigma = \partial M_{2n+1}} B_{2n}$$

(14)

where

$$L^A_{2n+1} = \int_0^1 dt \langle F^n_t e \rangle = \int_0^1 dt \varepsilon_{A_1 \cdots A_{2n+1}} F^{A_1 A_2}_t F^{A_3 A_4}_t \cdots F^{A_{2n-1} A_{2n}}_t e^{A_{2n+1}}$$

$$B_{2n} = -n(n+1) \int_0^1 ds \int_0^1 dt \langle A_t \Theta \left(s F_t + s(s-1)A^2_t\right)^{n-1}\rangle$$

(15)

with

$$A_t = tA + (1-t)\bar{A} = \bar{A} + t\Theta, \text{ with } \Theta = A - \bar{A}$$

(16)

$$F_t = dA_t + A^2_t = tF + (1-t)\bar{F} - (1-t)\Theta^2$$

$$= dA + (t-1)d\Theta + \frac{1}{2}[A, A] + (t-1)[A, \Theta] + \frac{(t-1)^2}{2} [\Theta, \Theta]$$

(17)

where $A$ corresponds to the 1-form gauge potential of the hypersurface $\Sigma$ and $A$ is the 1-form gauge potential of the bulk.

Let us now consider the 5-dimensional case. In this case we have

$$S_5^A = \int_{M_{2n+1}} L^A_5 + \int_{\partial M_{2n+1}} B_4,$$

(18)

where $L^A_5$ is given by the first equation in (15) when $n = 2$ and $B_4$ is given by

$$B_4 = -6 \int_0^1 dt \int_0^1 ds \langle A_t A_t \left(s^2 F_t + s^2(s-1)A^2_t\right)\rangle$$

$$= \int_0^1 dt \left\langle \frac{1}{2} \Theta A^3_t - 2\Theta A_t F_t \right\rangle.$$

(19)

Introducing (16,17) in (19) we find

$$B_4 = \left\langle \Theta A \left( -\frac{3}{2} F + d\Theta - \frac{1}{2} dA + \frac{3}{4} [A, \Theta] - \frac{1}{4} [\Theta, \Theta] \right) + \Theta \Theta \left( \frac{3}{4} F - \frac{2}{3} d\Theta + \frac{3}{16} [\Theta, \Theta] \right) \right\rangle,$$

(20)

where the gauge potential 1-form as well as their corresponding curvature 2-form
for the AdS algebra, are given by

\[ A = \frac{1}{l} e^A \tilde{P}_A + \frac{1}{2} \omega^{AB} \tilde{J}_{AB}, \]
\[ F = \frac{1}{l} T^A \tilde{P}_A + \frac{1}{2} \left( R^{AB} + \frac{1}{l^2} e^A e^B \right) \tilde{J}_{AB}, \]
\[ \bar{A} = \frac{1}{\bar{l}} \bar{e}^A \bar{P}_A + \frac{1}{2} \bar{\omega}^{AB} \bar{J}_{AB}, \]
\[ \bar{F} = \frac{1}{\bar{l}} \bar{T}^A \bar{P}_A + \frac{1}{2} \left( \bar{R}^{AB} + \frac{1}{\bar{l}^2} \bar{e}^A \bar{e}^B \right) \bar{J}_{AB}. \]

(21)

Here \( e^A \) and \( \bar{e}^A \) are the vierbeins, \( \omega^{AB} \) and \( \bar{\omega}^{AB} \) are the spin connections, \( R^{AB} = d\omega^{AB} + \omega^A \omega^C_{\;\;AB} \) and \( \bar{R}^{AB} = d\bar{\omega}^{AB} + \bar{\omega}^A \bar{\omega}^C_{\;\;AB} \) are the curvature 2-forms. \( \tilde{P}_A \) and \( \tilde{J}_{AB} \) are the generators of the AdS algebra.

Taking into account that considering the non-zero components of the invariant tensor for the AdS algebra in 5 dimensions are proportional to the Levi-Civita symbol, namely,

\[ \langle \tilde{J}_{AB} \tilde{J}_{CD} \tilde{P}_E \rangle = \frac{4k}{3} \varepsilon_{ABCDE} \]

(22)

we have that terms of order greater than or equal to 2 in \( \Theta \) will be null, so that (19) takes the form

\[ B_4 = \left\langle \Theta A \left( -\frac{3}{2} F - \frac{1}{2} dA + \frac{3}{4} [A, \Theta] - \frac{1}{4} [\Theta, \Theta] \right) \right\rangle, \]

(23)

where, using the commutation relations of the AdS algebra, we find

\[ [\Theta, A] = \omega^A \varepsilon^{LB} J_{AB} + \frac{2}{l} \Theta^A \varepsilon^L P_A \]
\[ [\Theta, \Theta] = \Theta^A \varepsilon^L J_{AB}. \]

(24)

Introducing (24) in (23) and using (22) we obtain

\[ B_4^{AdS} = \frac{k}{3l} \varepsilon_{ABCDE} \left\{ -\frac{1}{2} \Theta^{AB} \omega^{CD} T^E + \frac{3}{2} D \Theta^{AB} \omega^{CD} e^E - \frac{7}{2} \Theta^{AB} R^{CD} e^E + \right. \]
\[ -\omega^A \varepsilon^L \varepsilon^{LB} \omega^{CD} e^E + \frac{3}{2} \Theta^{AB} \omega^C \varepsilon^L \varepsilon^{LD} e^E - \frac{1}{2} \Theta^{AB} \Theta^C \varepsilon^L \varepsilon^{LD} e^E \left\} \right. \]

(25)

where \( T^E \) is the torsion 2-form.

In the braneworld context the \( A \) indices run from 0 to 4 and the \( a \) indices run from 0 to 3. This means that \( \omega^{AB} = (\omega^{ab}, \omega^{a4}), e^A = (e^a, e^4), \omega^{ab} = \bar{\omega}^{ab} \) and \( e^a = \bar{e}^a \). So keeping in mind that \( \Theta^{AB} = \omega^{AB} - \bar{\omega}^{AB} \) and \( \Theta^A = e^A - \bar{e}^A \) we can write

\[ \Theta = \frac{1}{l} \varepsilon^{AB} J_{AB} + \frac{1}{l} \varepsilon^A \bar{P}_A = \omega^{aA} J_{4a} + \frac{1}{l} e^A \bar{P}_A. \]

(26)
Introducing these results in (25) we find

\[ B_{4}^{\text{AdS}} = \kappa \varepsilon_{abcd} \left\{ -\frac{1}{3!} \omega^{4a} \omega^{bc} T^{d} - \frac{7}{3!} \omega^{4a} \left( R^{bc} + \frac{1}{l^2} e^{b} e^{c} \right) e^{d} + \frac{4}{3!} \omega^{4a} e^{b} e^{c} e^{d} + \right. \]

\[ + \frac{1}{l} R^{4a} \omega^{bc} e^{d} - \frac{1}{3} \omega^{a} \omega^{4} \omega^{4} \omega^{bc} e^{d} \left. \right\} \]  

(27)

3 Israel junction condition for Einstein-Chern-Simons gravity

3.1 Boundary term for Einstein-Chern-Simons Lagrangian

In the introduction we noted that the so-called Einstein-Chern-Simons Lagrangian (6) was obtained from the AdS-Chern-Simons Lagrangian by means of the expansion procedure. The corresponding boundary term for the EChS Lagrangian can be obtained from the boundary term of the AdS-Chern-Simons Lagrangian following the same procedure. Indeed, making use of the dual S-expansion procedure [8], it is found that the boundary term for the Einstein-Chern-Simons Lagrangian (6), and that must be added to it, is given by

\[ B_{4}^{\text{EChS}} = \varepsilon_{abcd} \left\{ \alpha_{1} l^{2} \left( -\frac{1}{3} \omega^{4a} \omega^{bc} T^{d} - \frac{7}{3} \omega^{4a} R^{bc} e^{d} + R^{4a} \omega^{bc} e^{d} - \frac{1}{3} \omega^{a} \omega^{4} \omega^{4} \omega^{bc} e^{d} \right) + \right. \]

\[ + \alpha_{3} l^{2} \left( -\frac{1}{3} \omega^{4a} \omega^{bc} D h^{d} - \frac{7}{3} \omega^{4a} R^{bc} h^{d} + R^{4a} \omega^{bc} h^{d} - \frac{1}{3} \omega^{a} \omega^{4} \omega^{4} \omega^{bc} h^{d} \right) + \]

\[ - \alpha_{3} \omega^{4a} e^{b} e^{c} e^{d} \left. \right\}, \]  

(28)

where we have used \( k^{AB} = 0 \) and \( h^{4} = 0 \).

This means that the action for so-called Einstein-Chern-Simons gravity in five dimensions is given by

\[ S_{\text{EChS}}^{(5)} = \int_{M} \varepsilon_{ABCDE} \left[ \alpha_{1} l^{2} R^{AB} R^{CD} e^{E} + \alpha_{3} \left( \frac{2}{3} R^{AB} e^{C} e^{D} e^{E} + l^{2} R^{AB} R^{CD} h^{E} \right) \right] \]

\[ + \int_{S} \varepsilon_{abcd} \left\{ \alpha_{1} l^{2} \left( 4 K^{a} R^{bc} e^{d} + \frac{1}{3} K^{a} K^{b} K^{c} e^{d} \right) + \right. \]

\[ + \alpha_{3} l^{2} \left( 4 K^{a} R^{bc} h^{d} + \frac{1}{3} K^{a} K^{b} K^{c} h^{d} \right) + \frac{4}{3} \alpha_{3} K^{a} e^{b} e^{c} e^{d} \left. \right\}, \]  

(29)

where we have used the fact that when the torsion is zero, it is possible to write the normal component of the spin connection in the form

\[ \omega^{4a} = -K^{a} = -K^{a} l e^{l}, \]  

(30)

where \( K^{a} l \) corresponds to the extrinsic curvature.

Taking the limit \( l \to 0 \) (low energy limit) we find
\[ S^{(5)}_{\text{EChS}} = \frac{2}{3} \alpha_3 \int_M \varepsilon_{ABCDE} R^{AB} e^C e^D e^E + \frac{4}{3} \alpha_3 \int_{\Sigma} \varepsilon_{abcd} K^a e^b e^c e^d, \]  

which corresponds to the Einstein-Hilbert term plus the Gibbons-Hawking-York boundary term, that is, in the limit \( l \to 0 \) general relativity is recovered.

### 3.2 Einstein-Chern-Simons junction

Since the brane divides the space \( M \) into two spaces, \( M^+ \) with metric \( g^{+}_{\mu\nu} \) and \( M^- \) with metric \( g^{-}_{\mu\nu} \). Each of these spaces induces a metric \( q^{\pm}_{ab} \) on the brane, which has associated a normal vector pointing from \( M^- \) to \( M^+ \). Due to this division the total action can be written in the form

\[
S^{EChS(5)}_{\text{total}} = S^{(5)}_{EChS^+} - S^{(5)}_{EChS^-} + \int_M L_M
\]

\[
= \int_{M^+} \varepsilon_{ABCDE} \left( \alpha_1 l^2 R^+_{AB} R^+_{CD} e^E + \alpha_3 \left( \frac{2}{3} R^+_+ e^C e^D e^E + l^2 R^+_+ R^+_+ h^E \right) \right)
\]

\[
+ \int_{M^-} \varepsilon_{ABCDE} \left( \alpha_1 l^2 R^-_{AB} R^-_{CD} e^E + \alpha_3 \left( \frac{2}{3} R^-_{+} e^C e^D e^E + l^2 R^-_{+} R^-_{+} h^E \right) \right)
\]

\[
+ \int_{\Sigma} \varepsilon_{abcd} \left\{ \alpha_1 l^2 \left( 4 [K^a] R^{bc} e^d + \frac{1}{3} [K^a K^b K^c] e^d \right) + \right.
\]

\[
+ \alpha_3 l^2 \left( 4 [K^a] R^{bc} h^d + \frac{1}{3} [K^a K^b K^c] h^d \right) + \frac{4}{3} \alpha_3 [K^a] e^b e^c e^d \}
\]

\[
+ \kappa \int_M L_M
\]

where \( S^{EChS\pm}_{\text{total}} \) corresponds to the action of the bulk in each part of the space, \( L_M \), correspond to matter contribution to the action and \([X] = X^+ - X^-\). In the previous action we have considered that the metric of the space is continuous on \( \Sigma \) in such a way that the connection is well defined from the metric. We have also considered that the field \( h^A \) is different from zero only in the brane, that is

\[
h^3 = 0; \quad h^a = h^a.
\]
Varying \( S_{\text{total}}^{EChS(5)} \) we have

\[
\delta S_{\text{total}}^{EChS(5)} = \int_{M^+} \varepsilon_{ABCDE} \left\{ \left( \alpha_1 t^2 R_{\alpha}^{AB} R_{\alpha}^{CD} + 2 \alpha_3 R_{\alpha}^{AB} e_{\alpha}^{C,D} \right) \delta e_{\alpha}^{E} + \alpha_3 t^2 R_{\alpha}^{AB} R_{\alpha}^{CD} \delta h_{\alpha}^{E} + 2 \alpha_3 R_{\alpha}^{AB} D h^{E} \delta \omega^{DE} \right\} + \int_{M^-} \varepsilon_{ABCDE} \left\{ \left( \alpha_1 t^2 R_{\alpha}^{AB} R_{\alpha}^{CD} + 2 \alpha_3 R_{\alpha}^{AB} e_{\alpha}^{C,D} \right) \delta e_{\alpha}^{E} + \alpha_3 t^2 R_{\alpha}^{AB} R_{\alpha}^{CD} \delta h_{\alpha}^{E} + 2 \alpha_3 R_{\alpha}^{AB} D h^{E} \delta \omega^{DE} \right\} + \int \varepsilon_{abcd} \left\{ \left( 4 \alpha_1 t^2 [K^a] \bar{R}^{bc} + \frac{13}{3} \alpha_1 t^2 [K^a K^b K^c] \right) + \frac{4}{3} \alpha_3 [K^a] e^b e^c \delta e^d + \alpha_3 t^2 \left( 4 [K^a] \bar{R}^{bc} + \frac{13}{3} [K^a K^b K^c] \right) \delta h^d + \left( 9 t^2 K^a K^b (\alpha_1 e^c + \alpha_3 h^c) + 12 \alpha_3 e^a e^b e^c \right) \delta K^d + \left( 9 t^2 K^a K^b (\alpha_1 e^c + \alpha_3 h^c) + 12 \alpha_3 e^a e^b e^c \right) \delta K^d + \kappa \int \left( T_E \delta e^E + T^{(h)}_E \delta h^E \right) \right\} = 0, \tag{34}
\]

where \( T_E = -\frac{1}{4} \varepsilon_{ABCDE} T^S E e^a e^b e^c e^D \) and \( T^{(h)}_E = -\frac{1}{4} \varepsilon_{ABCDE} T^{(h)S} E e^a e^b e^c e^D \) corresponds to the forms energy-momentum associated to \( e^a \chi h^A \), respectively.

The previous structure allows to separate the energy-momentum tensors in the form

\[
T_E = Q_+ \Theta (\chi) + Q_- \Theta (-\chi) + \delta (\chi) \bar{T}_E, \tag{35}
\]

where \( \chi \) is a coordinate defined in the direction normal to \( \Sigma \) such that the brane is located at \( \chi = 0 \). So \( \bar{T}_d = -\frac{1}{4} \varepsilon_{abcd} \bar{T}^a e^b e^c \), is the part of energy-momentum tensor on the brane and \( Q_E \) the part external to the brane (bulk).

Bearing this in mind, the equation \( \tag{34} \) leads to the following field equations

\[
4 \kappa_5 Q_{\pm E} = \varepsilon_{ABCDE} \left( \alpha_1 t^2 R_{\pm}^{AB} R_{\pm}^{CD} + 2 \alpha_3 R_{\pm}^{AB} e_{\pm}^{C,D} \right), \tag{36}
\]

\[
8 \kappa_5 Q_{\pm E} = \alpha_3 t^2 \varepsilon_{ABCDE} R_{\pm}^{AB} R_{\pm}^{CD}, \tag{37}
\]

\[
0 = 2 \alpha_3 \varepsilon_{ABCDE} R_{\pm}^{AB} D h^{C}, \tag{38}
\]

\[
\kappa \bar{T}_d = \varepsilon_{abcd} \left( 4 \alpha_1 t^2 [K^a] \bar{R}^{bc} + \frac{13}{3} \alpha_1 t^2 [K^a K^b K^c] \right) + \frac{4}{3} \alpha_3 [K^a] e^b e^c, \tag{39}
\]

\[
\kappa \bar{T}^{(h)}_d = \alpha_3 t^2 \varepsilon_{abcd} \left( 4 [K^a] \bar{R}^{bc} + \frac{13}{3} [K^a K^b K^c] \right), \tag{40}
\]

\[
0 = \varepsilon_{abcd} \left( 9 t^2 K_{\pm}^a K_{\pm}^b (\alpha_1 e^c + \alpha_3 h^c) + 12 \alpha_3 e^a e^b e^c \right), \tag{41}
\]

with \( \kappa_5 = \kappa/8 \alpha_3 \).
Note that the equations (36-38) correspond to the five-dimensional EChS field equations and the equations (39-41) are the conditions that must be satisfied for the curvature to be well defined at the junction.

Replacing (40) in (39) we obtain

\[- \varepsilon_{abcd} [K^a] e^b e^c = 2 \kappa_5 \tilde{T}_d,\]  

which in tensor language takes the form

\[[K] \delta^m_d - [K^m]_d = \kappa_5 \tilde{T}_d.\]  

Contracting the indices \(m\) and \(d\) we find that \([K] = \kappa_5 \tilde{T}/3\). So that

\[[K^m]_d = -\kappa_5 \left( \tilde{T}_m - \frac{1}{3} \tilde{T}_\delta^m \right),\]  

which coincides with the usual Israel's junction condition as long as \(\kappa = \kappa_5\). This means that the Lanczos equation for Einstein-Chern-Simons gravity is an Israel-type join condition, even though the Lagrangian contains quadratic terms on the curvature. Unlike the case of General Relativity, the extrinsic curvature is subject to the condition on both faces of the brane.

### 4 3-brane in Einstein-Chern-Simons gravity

It is possible to study the 3-brane world in Einstein-Chern-Simons gravity using the results of the previous section and the procedure of Ref. \[25\]. We must consider the induced metric \(q_{ab}\) on the brane \(\Sigma\) and a normal vector \(n^\mu\) embedded in the five-dimensional bulk \(M\) with metric \(g_{\mu
\nu}\).

The action (6) gives rise to the field equations (38)

\[\varepsilon_{ABCDE} R^{AB} e^C e^D = 4k_5 \left( T_E + \alpha T_E(h) \right),\]  

\[\frac{I^2}{8k_5} \varepsilon_{ABCDE} R^{AB} R^{CD} = T_E(h),\]  

\[\varepsilon_{ABCDE} R^{CD} D h^E = 0,\]  

where the capital letters denote the bulk indices \(\{0, 1, 2, 3, 4\}\), while the lower-case letters will denote brane indices \(\{0, 1, 2, 3\}\). The matter Lagrangian gives rise to two stress-energy tensors, \(T_E = \delta L_M/\delta e^E\) and \(T_E(h) = \delta L_M/\delta h^E\).
To study the brane, we must consider the interior derivative concerning the
normal vector to the brane \( i_A \). Since \( i_a e^b = \delta^b_a \) and \( i_4 R^{ab} e^4 = 0 \), eq. (47) takes the form
\[
\varepsilon_{a4bcd} \left( i_4 R^{a4} e^c - R^{ac} i_4 e^4 \right) e^d = 2k_5 i_4 \left( T_e + \alpha T_e^{(h)} \right).
\]
(50)

Here \( i_4 R^{a4} = -\tilde{E}^a m e^m \) and \( i_4 R^{ac} = -\tilde{B}^{ac m} e^m \), where \( \tilde{E}^a m \) and \( \tilde{B}^{ac m} \) are the so-called electric and magnetic parts of the Riemann tensor.

For the second member of (50),
\[
i_4 T_e = -\frac{1}{3!} \varepsilon_{abcd} T^{m} d e^a b e^c,
\]
(51)
and similarly for \( T_e^{(h)} \). Therefore, eq. (50) takes the form
\[
2k_5 i_4 \hat{T}_d = \varepsilon_{abcd} \left( \tilde{E}^a m e^m d e^b \right) + \tilde{R}^{ab} - K^a b e^c m e^m \),
\]
(52)
where we have used the Gaussian equation.

Now, let us consider eq. (48). We have that
\[
8k_5 i_4 T_e^{(h)} = 4l^2 \varepsilon_{abcd} \left( R^{ab} d e^m m e^4 R^{cd} \right) + R^{4b} R^{cd} 4m e^m \),
\]
(53)
and using the Gauss-Codazzi equations,
\[
-2k_5 i_4 T_d^{(h)} = \varepsilon_{abcd} \left( \tilde{E}^a m e^m \left( \tilde{R}^{bc} - K^b c e^e m e^m \right) + R^{4a} \tilde{B}^{bc} m e^m \right).
\]
(54)
In the same way, from (49) we see that
\[
0 = l^2 \varepsilon_{abcd} \left( \tilde{E}^c I d l e^d + R^{4c} D_d l e^d \right).
\]
(55)

On the other hand, from the decomposition of the curvature tensor
\[
R^{AB}_{\ CD} = C^{AB}_{\ CD} + \frac{2}{3} \delta^A_{[C} R^B_{\ D]} - \delta^B_{[C} R^A_{\ D]} - \frac{1}{6} \delta^A_{[C} \delta^B_{D]} R,
\]
(56)
we can find \( \tilde{E}^a m \) and \( \tilde{B}^{ab m} \) as a function of the electric part \( E^a m \) and the magnetic part \( B^{ab m} \) of the Weyl tensor. We have that \( E^a m = R^{4a} 4m \), \( \tilde{B}^{ab m} = R^{ab m} \), \( E^a m = C^{4a} 4m \), and \( B^{ab m} = C^{ab m} \), and therefore,
\[
\tilde{E}^a m = E^a m + \frac{1}{3} \left( R^b d + R^{4b} 4d \right) - \frac{1}{12} R^b a,
\]
(57)
\[
\tilde{B}^{ab m} = B^{ab m} + \frac{1}{3} \left( \delta^b c R^c d + \delta^b c R^a d \right).
\]
(58)
Since
\[
R = -\frac{2}{3} k_5 \left( T + \alpha T^{(h)} \right),
\]
(59)
\[
R^b d = k_5 \left( T^b d + \alpha T^{(h)} b d - \frac{1}{3} \left( \alpha T^{(h)} + T \right) \delta^b d \right),
\]
(60)
we have that the relations between the electrical parts of the Riemann tensor \( \tilde{E}^a_l \) and the Weyl tensor \( E^a_l \) are
\[
\tilde{E}^a_l = E^a_l + \frac{k_5}{3} \left[ \tilde{T}^a_l + \delta^a_l \left( \tilde{T}^4_4 - \frac{\tilde{T}}{2} \right) \right],
\]
moreover, the relations between the corresponding magnetic parts are
\[
\tilde{B}^{ab}_l = B^{ab}_l + \frac{2}{3} k_5 \delta^{[a}_{l} \tilde{T}^{b]}_4.
\]
Replacing (61) and (62) in the equations of motion, we have
\[
\varepsilon_{abcd} R^{ab}_{\ e} e^c = -2 k_5 i_4 \tilde{T}_{d} - \varepsilon_{abcd} \left\{ \left( E^a_m e^m + \frac{k_5}{3} \left[ \tilde{T}^a_l e^l + \left( \tilde{T}^4_4 - \frac{\tilde{T}}{2} \right) e^a \right] \right) e^b 
- K^a_l K^b_m e^m \right\} e^c
\]
\[
2 k_5 i_4 \tilde{T}^{(b)}_d = -l^2 \varepsilon_{abcd} \left( E^a_m e^m + \frac{k_5}{3} \left[ \tilde{T}^a_l e^l + \left( \tilde{T}^4_4 - \frac{\tilde{T}}{2} \right) e^a \right] \right) R^{bc}_{\ e} + 
-l^2 \varepsilon_{abcd} \left\{ R^{ab}_{\ c} e^c - e^a \right\} \tilde{T}^4_4 e^e + 
- k_5 \left[ \tilde{T}^a_l e^l + \left( \tilde{T}^4_4 - \frac{\tilde{T}}{2} \right) e^a \right] K^b_l K^c_m e^m \right\}
\]
\[
0 = l^2 \varepsilon_{abcd} \left\{ R^{4c}_{\ d} D_4 h^d + \left( E^c_m e^m + \frac{k_5}{3} \tilde{T}^c_l e^l + \tilde{T}^4_4 + \frac{\tilde{T}}{2} \right) e^c \right\}
\]

The brane equations of motion are subject to the conditions given by the normal component of the equations of motion in bulk. In the equations (47-49) we make the free index equal to “4” and then we apply the interior derivative with respect to the normal,
\[
l^2 \varepsilon_{abcd} B^{ab}_{\ i} \tilde{R}^{cd} e^j = 8 k_5 i_4 T^{(h)}_d - l^2 \varepsilon_{abcd} \left( 2 \tilde{k}_3 \tilde{T}^b_4 e^a K^c_m K^d_m e^m + 
+ \frac{2}{3} k_5 \tilde{T}^4_4 e^a \tilde{R}^{cd} + B^{ab}_{\ i} K^c m e^m \right)
\]
\[
l^2 \varepsilon_{abcd} R^{bc}_{\ d} D_4 h^d = -l^2 \varepsilon_{abcd} \left( B^{bc}_{\ i} D_4 h^d + 
+ B^{bc}_{\ i} D_4 h^d + \frac{2}{3} k_5 \tilde{T}^4_4 e^b D_4 h^d \right)
\]
Following Ref. \[25\] we define the coordinate $\chi$, such that the brane is at $\chi = 0$. Therefore, at the brane we have

$$n_\mu dx^\mu = d\chi,$$

(68)

and the bulk metric at the brane takes the form,

$$ds^2 = d\chi^2 + \eta_{ab} e^a e^b.$$

(69)

Using the normal coordinate, we separate the stress-energy tensors in the same way as in Ref. \[25\],

$$T_{AB} = -\Lambda \eta_{AB} + \bar{T}_{AB} \delta(\chi)$$

(70)

$$T_{ab} = -\lambda \eta_{ab} + \bar{\tau}_{ab}$$

(71)

$$T_{(h)}_{AB} = \bar{T}_{(h)}_{AB} \delta(\chi),$$

(72)

where $\lambda$ is the brane tension, $\tau_{ab}$ is the stress-energy tensor of the matter on the brane, and $T_{(h)}_{ab}$ is the stress-energy tensor associated with the $h^a$ field in the brane. By imposing the symmetry $Z_2$, the juncture condition takes the form

$$K^k_d = -\frac{k_5}{2} \left( \bar{T}^k_d - \frac{1}{3} \bar{T}^k_d \right)$$

(73)

On the other side from the contracted Codazzi equation, we find

$$0 = -\frac{k_5}{2} \bar{D}_s \bar{\tau}^s_d - \frac{k_5}{2} \alpha \bar{D}_s \bar{T}_{(h)}^s_d,$$

(74)

where, as we have previously seen, $\alpha = -\lambda_1/\lambda_3$. Since $\lambda_1$ and $\lambda_3$ are independent, it implies

$$\bar{D}_s \bar{\tau}^s_d = 0, \quad \bar{D}_s \bar{T}_{(h)}^s_d = 0,$$

(75)

that in the language of forms takes the form

$$\bar{D} \star \bar{T}^a = 0, \quad \bar{D} \star \bar{T}_{(h)}^a = 0,$$

(76)

where

$$\bar{T}^a = \frac{\bar{\tau}^a}{6} e^a - \frac{1}{2} \bar{x}^a_l e^l,$$

(77)

$$\bar{T}_{a(h)} = \frac{\bar{T}_{(h)}^a}{6} e^a - \frac{1}{2} \bar{\tau}_{a(h)}^l e^l.$$

(78)

Using these results in the equations of motion \[63\]-\[65\] for the brane, we find,

$$\varepsilon_{abcd} \bar{R}^{ab} e^c = \varepsilon_{abcd} \left\{ \frac{\lambda_3}{3} e^a e^b - \frac{k_5^2}{8} \left( \bar{\Pi}^{ab} + \alpha^2 \bar{\Pi}^{ab(h)} \right) - 16\pi G_N \left( \bar{T}^a + \alpha \bar{T}_{a(h)} \right) e^b + \frac{k_5^2}{4} \alpha \bar{F}^{ab} - E^m e^m e^b \right\} e^c$$

(79)
\[ 2k_5i_4 \tilde{T}^{(h)}_d = - l^2 \varepsilon_{abcd} \left( \left( E^a \epsilon^m + \frac{k_5}{6} \Lambda e^a \right) \left( R^{bc} + \frac{k_5^2}{36} \lambda^2 \epsilon^b \epsilon^c - \frac{k_5^2}{8} \right) (\tilde{\Pi}^{bc} + \right. \\
+ \left. \alpha^2 \tilde{\Pi}^{bc(h)} - 16\pi G_N \left( \tilde{T}^a + \alpha \tilde{T}^{(h)(a)} \right) e^c + \frac{k_5^2}{4} \alpha \tilde{T}^{bc} \right) + \tilde{D} K^{a b c d} e_d \right) \] 

\[ (80) \]

\[ l^2 \varepsilon_{abcd} \left( E^a \epsilon^m + \frac{k_5}{6} \Lambda e^a \right) \left( \tilde{D} \epsilon^d - \frac{k_5}{6} \lambda e^d - k_5 \tilde{T}^d \right) + l^2 \varepsilon_{abcd} \tilde{D} K^{a b c d} h^d = 0, \]

where,

\[ I^{ab} = 2 \tilde{T}^{(h)(a)} \tilde{T}^{(h)(b)} \epsilon^a \epsilon^b - \frac{2}{9} \tilde{T}^{(h)(a)} e^a \epsilon^b - \frac{2}{3} \tilde{T}^{(h)(a)} \epsilon^a \epsilon^b - \frac{2}{3} \tilde{T}^{(h)(a)} \epsilon^a e^b, \]

\[ (82) \]

\[ \tilde{\Pi}^{ab} = \frac{1}{3} \tilde{T}^{(h)(a)} \tilde{T}^{(h)(b)} \epsilon^m \epsilon^m - \frac{1}{18} \varepsilon^a \epsilon^b, \]

\[ (83) \]

\[ \tilde{\Pi}^{ab(h)} = \frac{1}{3} \tilde{T}^{(h)(a)} \tilde{T}^{(h)(b)} \epsilon^m \epsilon^m - \frac{1}{18} \tilde{T}^{(h)(a)} \tilde{T}^{(h)(b)} \epsilon^m \epsilon^m - \frac{1}{18} \varepsilon^a \epsilon^b, \]

\[ (84) \]

In the normal direction the equations take the form

\[ l^2 \varepsilon_{abcd} B^{ab} \epsilon^c e^d \epsilon^l = 8k_5i_4 \tilde{T}^{(h)}_4 - l^2 \varepsilon_{abcd} \left( B^{ab} \epsilon^c \epsilon^d \epsilon^d l \right) \left( \frac{k_5^2}{36} \lambda^2 \epsilon^c \epsilon^d - \frac{k_5^2}{8} \left( \tilde{\Pi}^{cd} + \right. \\
+ \left. \alpha^2 \tilde{\Pi}^{cd(h)} - 16\pi G_N \left( \tilde{T}^c + \alpha \tilde{T}^{(h)(c)} \right) e^d + \right. \\
+ \left. \frac{k_5^2}{4} \alpha \tilde{T}^{cd} \right) \right) \]

\[ (85) \]

\[ l^2 \varepsilon_{abcd} R^{bcde} \epsilon^d d \epsilon^a d = - l^2 \varepsilon_{abcd} \left( \frac{k_5^2}{36} \lambda^2 \epsilon^c \epsilon^d - \frac{k_5^2}{8} \left( \tilde{\Pi}^{cd} + \alpha^2 \tilde{\Pi}^{cd(h)} - 16\pi G_N \left( \tilde{T}^c + \right. \\
+ \left. \alpha \tilde{T}^{(h)(c)} \right) e^d + \frac{k_5^2}{4} \alpha \tilde{T}^{cd} \right) \right) \]

\[ (86) \]

So far, we have obtained the equations of motion (79) and (81) for the brane, besides the junction conditions (73), and conditions (85) and (86). The equation (79) contains terms of order one and quadratics in both types of matter, \( T^a \), \( \Pi^{ab} \), \( T^{(h)a} \), and \( \Pi^{(h)ab} \), as well as an interaction term \( I^{ab} \) between both types of matter. If the interaction term, the quadratic terms and the simple terms in \( T^{(h)ab} \) are of the same or smaller order of magnitude, they can be neglected if \( \alpha \) is small.

The system of equations given by (79, 80) and (81) cannot be solved unless we also consider the equations (85) and (86) in addition to the equations describing both parts of the Weyl tensor in bulk given in the appendix to (25).
Taking the exterior covariant derivative of the equation (74), and using the Bianchi identity, we obtain

$$\varepsilon_{abcd} \bar{\nabla} E^a \varepsilon^m e^b e^c = \varepsilon_{abcd} \left\{ -\frac{k_5^2}{8} \left( \bar{\nabla} \Pi^{ab} + \alpha^2 \bar{\nabla} \Pi^{ab(h)} \right) + \frac{k_5^2}{4} \alpha^2 \bar{\nabla} \Pi^{ab} \right\} e^c \quad (87)$$

In contrast to the case of Ref. [25], here the electrical part of the Weyl tensor is also restricted by $T^{a(h)}_{ab}$ and the interaction $I^{ab}$.

Let us compare the new terms in the equations to the stress-energy tensor $\tau^a_{\, \, b}$, in a way similar to Ref. [25]. We set the scales of the constants

$$k_5 = \frac{1}{M_G}, \quad \lambda = M_\lambda^4 \quad (88)$$

$$|\bar{\tau}^a_{\, b}| = M^4, \quad |\bar{\tau}^{a(h)}_{\, b}| = M^4(h) \quad (89)$$

where $M_G$ and $M_\lambda$ are larger than the characteristic energy scales $M$ and $M(h)$, with $M$ being of the same or greater order of magnitude than $M(h)$.

$$k_5^2 \alpha^2 \frac{\varepsilon_{abcd} \Pi^{ab(h)} e^c}{G_N \varepsilon_{lmsn} \bar{T}^{em(e)n}} \sim \alpha^2 \frac{M^8(h)}{M_\lambda^4 M^4} \quad (90)$$

$$\alpha \frac{G_N \varepsilon_{abcd} \bar{T}^{a(h)} e^b e^c}{G_N \varepsilon_{lmsn} \bar{T}^{e(m)e(n)}} \sim \alpha \frac{M^4(h)}{M^4} \quad (91)$$

$$\alpha \frac{k_5^2 \varepsilon_{abcd} \bar{T}^{ab} e^c}{G_N \varepsilon_{lmsn} \bar{T}^{em(e)n}} \sim \alpha \frac{M^4(h)}{M_\lambda^4} \quad (92)$$

From the equations (90)-(92), we can see, including the case $M = M(h)$, that, when $\alpha$ is little, these terms are negligible compared to the tensor $\tau^{ab}$ components.

It is useful to separate $E_{ab}$ into two parts, namely, into a transverse part (no trace), $E_{(TT)}$, and a longitudinal part, $E_{(L)}$, where only the latter is determined by the matter, since $E_{(TT)}$ corresponds to the part that interacts between the brane and the bulk.

The longitudinal part of $E_{ab}$ is restricted, as can be seen in the equation (87), both by the quadratic term in $\tau_{ab}$, and by $T^{(h)}$.

Comparing $E_{ab}$ with $\tau_{ab}$ we find

$$\frac{\varepsilon_{abcd} E^{a(l)e} \varepsilon^m e^b e^c}{G_N \varepsilon_{lmsn} \bar{T}^{em(e)n}} \sim \frac{1}{G_N \varepsilon_{\tau_{ab}}} \left[ G_N \alpha T^{(h)}_{ab} + k_5^2 \left( \tau_{al} \tau^l_{\, b} + \ldots \right) + k_5^2 \alpha^2 \left( T^{(h)}_{a(l)h} + \ldots \right) \right], \quad (93)$$
where
\[ \varepsilon_{abcd} E^a_{(l)m} \epsilon^m e^b e^c \] \[ \sim \frac{\alpha}{M^4} \left( \frac{M^4}{M^4} + \frac{\alpha}{M^4} \right) + \alpha \frac{M^8}{M^4} \lambda + \alpha \frac{M^4}{M^4} \lambda, \]
i.e., the electrical part of the Weyl tensor would be negligible as long as \( \alpha \) and \( l \) are small.

It is interesting to notice that when the \( R^{ab} \) components are small, then the limit \( \alpha \rightarrow 0 \) and \( l \rightarrow 0 \) lead to the known results of [25]. Indeed, the equations of motion take the form,

\[ \varepsilon_{abcd} \bar{R}^{ab} e^c e^d = \varepsilon_{abcd} \left\{ \frac{\Lambda_4}{3} e^a e^b - \frac{k_2}{M^4} \Pi^{ab} - 16\pi G_N \bar{T}^a e^b - E^a e^m e^b \right\} e^c, \]

\[ \varepsilon_{abck} \bar{T}^{(h)k} d e^a e^b e^c = 0. \] (94)

In this limit equation for \( \omega^{ab} \) is identically null as well as the tensor \( \bar{T}^{(h)k} \). The equation (94) matches the equation (17) of the reference [25]. Note also that in this limit the equation (94) implies the conservation of \( \tau^{a \ b} \).

5 Concluding remarks

This article shows that it is possible to obtain both the Lagrangean and the equations of motions for a 3-brane in 5-dimensional Einstein-Chern-Simons gravity.

We constructed the Einstein-Chern-Simons gravity juncture conditions starting from the Lovelock boundary term of Ref. [31], using AdS-Chern-Simons as an intermediate step. The S-expansion procedure closes the gap between both Chern-Simons theories, mapping one boundary term into the other. The key is to consider the extra \( h^A \) field as a matter field.

The new junction condition obtained for the extrinsic curvature corresponds to the Darmois-Israel joint condition plus a correction, which corresponds to the matter \( T^{ab} \), which vanishes at the low energy limit.

The procedure described in Ref. [25] lead in this case to the effective equations of motion for a 3-brane embedded in a five-dimensional space obeying the Einstein-Chern-Simons field equations with \( T^a = 0 \) and \( k_{ab} = 0. \)

The imposition of the mentioned junction conditions leads to equations for the brane, showing new terms corresponding to the new type of matter and the interaction between it and the usual matter \( \tau^a \ b \).

These terms disappear in the limit \( l \rightarrow 0 \), leading to the former case studied in the reference [25]. The cosmological implications of these new terms will be studied elsewhere (work in progress).

The generalized Poincaré algebra \( \mathfrak{g}_n \) [13], [4], [6], can be obtained from the anti-de-Sitter algebra and the semigroup \( S^{2n-1} = \{ \lambda_0, \cdots , \lambda_{2n} \} \) whose multiplication law is given by \( \lambda_\alpha \lambda_\beta = \lambda_{\alpha+\beta} \) when \( \alpha + \beta \leq 2n \) and \( \lambda_\alpha \lambda_\beta = \lambda_\beta \) when \( \alpha + \beta > 2n \), where \( \lambda_{2n} \) corresponds to the zero element of the semigroup. The
generators of $\mathcal{B}_n$ denoted by $\left(P_a, J_{ab}, Z_{ab}^{(i)}, Z_a^{(i)}\right)$ satisfy the following commutation relations

$$[P_a, P_b] = \Lambda Z_{ab}^{(1)}, \quad [J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b,$$

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} + \eta_{ad} J_{bc} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac},$$

$$J_{ab}, Z_c^{(i)} = \eta_{bc} Z_a^{(i)} - \eta_{ac} Z_b^{(i)},$$

$$Z_{ab}^{(i)}, P_c = \eta_{bc} Z_a^{(i)} - \eta_{ac} Z_b^{(i)},$$

$$Z_{ab}^{(i)}, Z_c^{(j)} = \eta_{bc} Z_a^{(i+j)} - \eta_{ac} Z_b^{(i+j)},$$

$$J_{ab}, Z_{cd}^{(i)} = \eta_{bc} Z_{ad}^{(i)} + \eta_{ad} Z_{bc}^{(i)} - \eta_{ac} Z_{bd}^{(i)} - \eta_{bd} Z_{ac}^{(i)},$$

$$Z_{ab}^{(i)}, Z_{cd}^{(j)} = \eta_{bc} Z_{ad}^{(i+j)} + \eta_{ad} Z_{bc}^{(i+j)} - \eta_{ac} Z_{bd}^{(i+j)} - \eta_{bd} Z_{ac}^{(i+j)},$$

$$P_a, Z_b^{(i)} = Z_{ab}^{(i+1)},$$

$$Z_a^{(i)}, Z_b^{(j)} = Z_{ab}^{(i+j+1)}$$

(95)

where, $\tilde{J}_{ab}$ and $\tilde{P}_a$ are the generators of the anti-de-Sitter algebra and $J_{ab} = \lambda_0 \otimes \tilde{J}_{ab}, Z_{ab}^{(i)} = \lambda_2 \otimes \tilde{J}_{ab}, P_a = \lambda_1 \otimes \tilde{P}_a$ and $Z_a^{(i)} = \lambda_2 \otimes \tilde{P}_a$, with $i, j = 0, 1, \cdots, n-1$ are the generators of the $B_n$ algebra.

This means that the results obtained so far can be generalized to the case of generalized Poincare algebras $\mathcal{B}_n$ where $n$ can be either even (\(\mathcal{B}_{2m}\)) or odd (\(\mathcal{B}_{2m+1}\)) and subsequently study their respective applications in black holes and in cosmology. Works in this direction are in progress.

6 Appendix 1: Gauss-Codazzi equations in the Cartan formalism

Let us consider an $n$-dimensional manifold $\Sigma$ immersed into an $m$-dimensional manifold $M$.

For the $m$-dimensional manifold $M$, at each point $P$ we define a cotangent space $T^*_P(M)$, and a local coordinate system $y^\mu$. It allows us to define a coordinate base $\{dx^\mu\}_{\mu=1}^m$, with $\mu, \nu, \cdots = 1, \cdots, m$, and an orthonormal basis $e^A = e^A_\mu dx^\mu$, with $A, B, \cdots = 1, \cdots, m$, so that $e^A \cdot e^B = \eta^{AB}$ and and $g^{\mu\nu} = dx^\mu \cdot dx^\nu$. Similarly, for the $n$-dimensional manifold, at each point $P$ we define a cotangent space $T^*_P(\Sigma)$ and a local coordinate system $x^i$. It allows us to define a coordinate base $\{dx^i\}_{i=1}^n$, with $i, j, \cdots = 1, \cdots, n$, and an orthonormal basis $\tilde{e}^a = \tilde{e}^a dx^i$, with $a, b, \cdots = 1, \cdots, n$, such that $\tilde{e}^a \cdot \tilde{e}^b = \eta^{ab}$ and and $g^{ij} = dx^i \cdot dx^j$. We will use the indices $r, s, \cdots = n + 1 = m$ for a base $M$ orthogonal to $\Sigma$.

Let us consider the application of the structure equations to the spaces $\Sigma$ and $M$ with the null torsion condition. We have $\tilde{e}^a = \tilde{e}^a, e^{n+1} = 0$, and $\omega^a_c = \omega^a_c$ when pulled back to $T^*_P(\Sigma)$ From the first equation of structure $T^A =$
\[ \text{de}^A + \omega^A_C e^C = 0, \]

we have that \( \mathbf{T}^a = \text{de}^a + \bar{\omega}^a e^c = 0 \) and \( \omega^{n+1} e^c = 0 \).

From the Cartan lemma we can write that \( \omega^{n+1} e_c = K_{ac} e^a \), that is, \( K_{ac} = \omega^{n+1} a \cdot e_c \)

which corresponds to the Gaussian-Weingarten equations, where \( K_{ac} \) is the extrinsic curvature.

From the second structure equation over \( M \), \( R^A_B = d\omega^A_B + \omega^A_C \omega^C_B \) it is straightforward to see that for the indices \( a, b \)

we have \( R^a_b = \mathbf{R}^a_b + K^a_n K^b_m e^n e^m \)

and for the indices \( n+1, b \) we have \( R^{n+1}_b = (dK^{(n+1)}_{bf} - K^{(n+1)}_{cf} \omega^c_b - K^{(n+1)}_{sf} \omega^f_b) e^f \),

which corresponds to the Codazzi equation.

## 7 Appendix 2: Israel Junction Conditions

The presence of a hypersurface \( \Sigma \), in a manifold \( M \) divides space-time into two regions \( M^+ \) and \( M^- \) that have \( \Sigma \) as boundary, and \( g^+_{\alpha\beta} \) and \( g^-_{\alpha\beta} \) respectively as metrics. Let us call \( \xi^a \), \( a = 1, 2, 3 \) to the "intrinsic" coordinates on both faces of the hypersurface \( \Sigma \), and \( x^\alpha_\pm \), \( \alpha = 0, 1, 2, 3 \) to the coordinates of the varieties \( M^\pm \).

Let us define a normal vector \( N_\alpha = \varepsilon \partial_\alpha \ell \) to the hypersurface, such that \( N^\alpha N_\alpha = \varepsilon \) and they point from \( M^- \) to \( M^+ \) [39], where \( \ell \) denotes the proper distance along the geodesics, so that \( \ell = 0 \) when the geodesics traverses the hypersurface.

The step function \( \Theta (\ell) \) is defined, equal to +1 if \( \ell > 0 \), 0 if \( \ell < 0 \), such that

\[ \Theta^2 (\ell) = \Theta (\ell), \quad \Theta (\ell) \Theta (-\ell) = 0, \quad \frac{d}{d\ell} \Theta (\ell) = \delta (l), \]

where \( \delta (l) \) is the Dirac distribution. We will denote with the symbol [] the "jump" of a tensor quantity \( \Omega \) through the hypersurface \( \Sigma \)

\[ [\Omega] = \Omega (M^+)\big|_\Sigma - \Omega (M^-)\big|_\Sigma, \]

where \( \Omega \) is defined on both sides of the hypersurface. The equation \([N^\alpha] = 0\) follows from the relation \( N_\alpha = \varepsilon \partial_\alpha \ell \) and the continuity of both \( \ell \) and \( x^\alpha \) through \( \Sigma \). The equation \([X^\alpha_\pm] = 0\) follows from the fact that the coordinates \( \xi^a \) are the same on both sides of the hypersurface [39].

We can write the metric \( g_{\alpha\beta} \) as

\[ g_{\alpha\beta} = \Theta (\ell) g^+_{\alpha\beta} + \Theta (-\ell) g^-_{\alpha\beta} \]

where \( g^+_{\alpha\beta} \) is the metric in \( M^\pm \) expressed in the coordinates \( x^\alpha \). From here it is direct to see that

\[ g_{\alpha\beta/\gamma} = \Theta (\ell) g^+_{\alpha\beta/\gamma} + \Theta (-\ell) g^-_{\alpha\beta/\gamma} + \varepsilon \delta (l) [g_{\alpha\beta}] N_\gamma, \]

where the last term is singular. To solve this problem we must impose the continuity of the metric through the hypersurface, \([g_{\alpha\beta}] = 0\), a condition that can be rewritten in the form

\[ [g_{\alpha\beta}] X^\alpha_a X^\beta_b = [g_{\alpha\beta}] X^\alpha_a X^\beta_b = [\gamma_{ab}] = 0 \]

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and that is known as the first joint condition. A direct calculation shows that
the Riemann tensor is given by

\[ R^{\alpha}_{\beta\gamma\delta} = \Theta (\ell) R_{\beta\gamma\delta}^{\alpha} + \Theta (-\ell) R_{\beta\gamma\delta}^{-\alpha} + \delta (\ell) A_{\beta\gamma\delta}^{\alpha}, \]  

(98)

where

\begin{align*}
R^{\alpha}_{\beta\gamma\delta} & = \Gamma^{\alpha}_{\beta\delta/\gamma} + \Gamma^{\alpha}_{\mu\gamma} \Gamma^{\mu}_{\beta\delta} - \Gamma^{\alpha}_{\mu\delta} \Gamma^{\mu}_{\beta\gamma} \\
R^{-\alpha}_{\beta\gamma\delta} & = \Gamma^{-\alpha}_{\beta\delta/\gamma} - \Gamma^{-\alpha}_{\mu\gamma} \Gamma^{-\mu}_{\beta\delta} - \Gamma^{-\alpha}_{\mu\delta} \Gamma^{-\mu}_{\beta\gamma} \\
A_{\beta\gamma\delta}^{\alpha} & = \varepsilon \left( [\Gamma_{\beta\delta}^\alpha] N_\gamma - [\Gamma_{\beta\gamma}^\alpha] N_\delta \right). 
\end{align*}

(99)

An explicit expression for the tensor \( A_{\beta\gamma\delta}^{\alpha} \) can be obtained taking into
account that the metric is continuous through \( \Sigma \). If \( g_{\alpha\beta/\gamma} \) were discontinuous,
then the discontinuity must be directed along the normal vector \( N_{\alpha} \), which
implies that there must exist a field \( \kappa_{\alpha\beta} \) such that \( [g_{\alpha\beta/\gamma}] = \kappa_{\alpha\beta} N_\gamma \), and
therefore

\[ [\Gamma_{\beta\gamma}^\alpha] = \frac{1}{2} \left( \kappa_{\beta\gamma} N_\alpha + \kappa_{\alpha\gamma} N_\beta - \kappa_{\beta\gamma} N_\alpha \right). \]

This implies that

\begin{align*}
A_{\alpha\beta} & = A^{\mu}_{\alpha\mu\beta} = \frac{\varepsilon}{2} \left( \kappa_{\mu\alpha} N_\mu N_\beta + \kappa_{\mu\beta} N_\mu N_\alpha - \kappa N_\alpha N_\beta - \varepsilon \kappa_{\alpha\beta} \right), \\
A & = A^\mu_{\alpha\mu} = \varepsilon \left( \kappa_{\mu\nu} N_\mu N_\nu - \varepsilon \kappa \right). 
\end{align*}

(100)

(101)

where \( \kappa = \kappa^\mu_\mu \).

The stress-energy tensor corresponds to \([39]\)

\[ T_{\alpha\beta} = \Theta (\ell) T^{+}_{\alpha\beta} + \Theta (-\ell) T^{-}_{\alpha\beta} + \delta (\ell) S_{\alpha\beta}, \]  

(102)

where \( T^{+}_{\alpha\beta} \) and \( T^{-}_{\alpha\beta} \) are the stress-energy tensors of the \( M^+ \) and \( M^- \) regions,
and \( S_{\alpha\beta} \) is the stress-energy tensor associated with \( \Sigma \).

In the Einstein’s field equations

\[ G_{\alpha\beta} = \kappa T_{\alpha\beta}, \]

we have

\[ G_{\alpha\beta} = \Theta (\ell) G^{+}_{\alpha\beta} + \Theta (-\ell) G^{-}_{\alpha\beta} + \delta (\ell) G^{\Sigma}_{\alpha\beta}, \]  

(103)

and \( T_{\alpha\beta} \) corresponds to \([102]\). Therefore

\begin{align*}
G^{+}_{\alpha\beta} & = R^{+}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}^{+} R^{+} = \kappa T^{+}_{\alpha\beta} \\
G^{-}_{\alpha\beta} & = R^{-}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}^{-} R^{+} = \kappa T^{-}_{\alpha\beta} \\
G^{\Sigma}_{\alpha\beta} & = A_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} A = \kappa S_{\alpha\beta},
\end{align*}

\[ \text{21} \]
where $S_{ab} = S_{\alpha \beta} X_\alpha^a X_\beta^b$ takes the form

$$S_{ab} = -\frac{\varepsilon}{\kappa} ([K_{ab}] - \gamma_{ab} [K]), \quad (104)$$

$$[K_{ab}] = -\varepsilon \kappa \left( S_{ab} - \frac{1}{2} \gamma_{ab} S \right), \quad (105)$$

equation known as the Lanczos equation [39].

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The name “Einstein-Chern-Simons gravity” comes from the fact that General Relativity is recovered as a low energy limit of the theory. The name was to make the difference with the “Chern-Simons gravity theory” based on AdS algebra, which does not lead to any limit in general relativity. In hindsight, an alternative name might be “$B_5$-Chern-Simons gravity”, in analogy to the well-known AdS-Chern-Simons gravity.

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