On a new double weighted exponential-pareto distribution: Properties and estimation

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ABSTRACT

The main aim of this manuscript to present a new weighted distribution named the Double Weighted Exponential Pareto Distribution (DWEOD). This paper constructed and studied this new distribution. The quantifiable properties are discussed, including the mean, variance, harmonic mean, coefficient of variation, reliability function, moments generating function, mode, hazard function, and the reverse hazard function. Moreover, this work estimated the parameters of this distribution by the maximum likelihood estimation method and the method of the moment.

Keywords: Double Weighted distribution, Exponential Pareto distribution, MLE, MOM, Percentiles.

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1. Introduction

Recently, researchers discussed various topics in statistics [1, 2] or more precisely in new distribution [3]. The Weighted Distribution Principle offers an approach to model specification and data interpretation issues. Fisher [4] examined how methods of ascertainment can impact the type of distribution of recorded perception, and it was introduced and formulated in general terms in (1965) by Rao in connection with modeling statistical data where it was not found to be appropriate to the normal practice of using the standard distribution for the purpose [5]. He defined numerous circumstances that relate to instances where it is not possible to treat the reported outcomes as a random sample from the original distribution, but it could be by weighted distribution modeling. The literature of the exponential Pareto distribution absorbs in most of the analysts due to its broad range application [6]. The definition and the concepts of double weighted distribution proposed first time by Al-Khadim and Hantoosh [7, 8], applies it to the exponential distribution and derives the statistical properties for the Double Weighted Exponential distribution. The Double weighted Rayleigh distribution properties were discussed by Nasr [9] and estimation is developed and considered. The statistical features and properties are discussed and acquired, such as the mean, harmonic mean, mode, variance, coefficient of variation, moment, coefficient of skewness, coefficient of kurtosis, hazard function, reverse hazard function and reliability function. Two different estimations methods used to estimate this distribution: the maximum likelihood estimation method, and the method of the moment. In (2014), Ahmed, apply it to the characterization and estimation of Double weighted Rayleigh distribution and its properties [10]. The statistical properties of the modified Double weighted exponential distribution are discovered by Khadim [11]. A better-fitted probability model has been chosen by using the Kolomogorov – Imirnov test or Beta-Invers [12, 13]. The Weibull-Rayleigh distribution utilized and demonstrated its application using lifetime data. More recently, Basheer used alpha power inverse Weibull distribution and receive the p.d.f, c.d.f, reliability, hazard, and revers hazard function [14]. Saghir and Saleen studied and conversed the statistical properties of the Double weighted Weibull distribution, including the mean, variance, Reliability function. The MLE estimation method is used to estimate this distribution. By applying it to real-life data, the utility of the distribution has been demonstrated [15].
Suppose \( X \) is a non-negative random variable with probability density function (pdf) \( f(x) \), then the pdf of the weighted random variable \( Xw \) is given by

\[
f_w(x) = \frac{W(x)f(x)}{\int_0^\infty W(x)f(x)dx}, \quad x > 0
\]

Where \( W(x) = x \) be a non-negative weight function.

Depending upon the choice of the weight function \( W(x) \), we have different weighted models. Clearly when \( W(x) = x \), the resulting distribution is called length-biased whose pdf is given by:

\[
f_l(x) = \frac{x f(x)}{E(x)}, \quad x > 0
\]

This paper introduced the Double Weighted distribution (DWD), which takes one form or type of weight function \( W(x) = x \), and using the Exponential Pareto distribution as original distribution, this work derives also the pdf and some useful properties of Double weighted Exponential Pareto distribution.

**Double Weighted Exponential Pareto Distribution (DWE PD)**

The Double Weighted distribution is given by:

\[
f_w(x, C) = \frac{W(x)f(x)F(cx)}{\int_0^\infty W(x)f(x)F(cx)dx}, \quad x \geq 0, C > 0
\]

And the second integration \( u=\frac{\theta}{\lambda} \)

\[
W_D = \int_0^\infty W(x)f(x)F(cx) dx = \int_0^\infty x \frac{\theta}{\lambda} \theta^{-1} e^{-\lambda x^\theta} \left[ 1 - e^{-\lambda x^\theta} \right] dx
\]

\[
\frac{\lambda}{\theta} \int_0^\infty x^\theta e^{-\lambda x^\theta} dx - \int_0^\infty x^\theta e^{-\lambda x^\theta} e^{-\lambda x^\theta} dx
\]

\[
= \frac{\lambda}{\theta} \int_0^\infty x^\theta e^{-\lambda x^\theta} dx - \int_0^\infty x^\theta e^{-\lambda x^\theta (1+C^\theta)} dx
\]

Let the first integration \( Z=\lambda \left(\frac{x}{\theta}\right)^\theta \), \( x=\frac{p}{\lambda} \left(\frac{z}{\theta}\right)^\theta \), \( dx=\frac{p}{\theta} \left(\frac{z}{\theta}\right)^{\theta-1} dz \)

And the second integration \( u=\frac{\lambda}{\theta} \left(\frac{x}{\theta}\right)^\theta \), \( x=\frac{p}{\lambda} \left(\frac{u}{\theta}\right)^\theta \), \( dx=\frac{p}{\theta} \left(\frac{u}{\theta}\right)^{\theta-1} du \)

\[
W_D = p^\theta \lambda^{-\frac{1}{\theta}} \Gamma \left(\left(\frac{1}{\theta} + 1\right) - p^\theta \lambda^{-\frac{1}{\theta}} \Gamma \left(\left(\frac{1}{\theta} + 1\right) \left[ 1 - \frac{1}{c^\theta} \right]\right)\right)
\]

Where \( c \) is \( \left(1 + c^\theta\right)^{\theta+1} \)

Then the pdf of Double Weighted Exponential Pareto Distribution (DWE PD) is:

\[
g(x,\theta,\lambda,p,c)= \frac{\theta x^\theta e^{-\lambda x^\theta} \left[ 1 - e^{-\lambda x^\theta} \right]}{p^\theta \lambda^{-\frac{1}{\theta}} \Gamma \left(\left(\frac{1}{\theta} + 1\right) \left[ 1 - \frac{1}{c^\theta} \right]\right)}
\]

\[
g(x,\theta,\lambda,p,c)= \frac{\theta x^\theta e^{-\lambda x^\theta} \left[ 1 - e^{-\lambda x^\theta} \right]}{p^\theta \lambda^{-\frac{1}{\theta}} \Gamma \left(\left(\frac{1}{\theta} + 1\right) \left[ 1 - \frac{1}{c^\theta} \right]\right)} \quad x \geq 0, c, \theta, \lambda > 0 \quad \cdots (2)
\]

where \( K= \frac{\theta x^\theta}{p^\theta \lambda^{-\frac{1}{\theta}} \Gamma \left(\left(\frac{1}{\theta} + 1\right) \left[ 1 - \frac{1}{c^\theta} \right]\right)} \)

\[
g(x,\theta,\lambda,p,c)=k x^\theta \left(1 - C^\theta y\right) \quad \cdots (3)
\]
where \( y = e^{-\lambda \frac{X}{\theta}} \)
and its Cumulative distribution function CDF is given by
\[
G(x) = \int_0^x g(x) \, dx
\]
where
\[
G(x; \theta, \lambda, p, c) = \int_0^x \frac{1}{\theta \lambda \Gamma(\frac{1}{\theta} + 1)} x^\theta e^{-\lambda \frac{x}{\theta}} \left[ 1 - e^{-\frac{c x}{\theta}} \right] dx
\]
Let \( k = \frac{1}{\theta \lambda \Gamma(\frac{1}{\theta} + 1)} \)
\[
G(x; \theta, \lambda, p, c) = k \int_0^x x^\theta e^{-\lambda \frac{x}{\theta}} \left[ 1 - e^{-\frac{c x}{\theta}} \right] dx
\]
Let the first integration is:
\[
\int_0^\infty \frac{1}{\theta \lambda \Gamma(\frac{1}{\theta} + 1)} x^\theta e^{-\lambda \frac{x}{\theta}} \left[ 1 - e^{-\frac{c x}{\theta}} \right] dx
\]
And let the second integration is:
\[
\int_0^\infty \frac{1}{\theta \lambda \Gamma(\frac{1}{\theta} + 1)} x^\theta e^{-\lambda \frac{x}{\theta}} \left[ 1 - e^{-\frac{c x}{\theta}} \right] dx
\]
\[
G(x; \theta, \lambda, p, c) = \frac{1}{\Gamma(\frac{1}{\theta} + 1) [\epsilon - 1]} \left[ \Gamma(\frac{1}{\theta} + 1, \lambda \frac{x}{\theta}) - \frac{1}{\epsilon} \Gamma(\frac{1}{\theta} + 1, \lambda \frac{c x}{\theta}) \right]
\]
... (4)

2. The Statistical properties of DWEPD

Statistical properties of DWEPD throughout computing the mean, variance, and standard deviation, coefficient of variation, harmonic mean, and moments presented as follow:

2.1. Moments of DWEPD

The \( r \)th moment of DWEPD can be calculated as:
\[
E(X^r) = \int_0^\infty x^r g(x; \theta, \lambda, p, c) \, dx
\]
\[
E(X^r) = k \int_0^\infty x^r \frac{1}{\theta \lambda \Gamma(\frac{1}{\theta} + 1)} x^\theta e^{-\lambda \frac{x}{\theta}} \left[ 1 - e^{-\frac{c x}{\theta}} \right] dx
\]
\[
= \int_0^\infty x^{r+\theta} e^{-\lambda \frac{x}{\theta}} \left[ 1 - e^{-\frac{c x}{\theta}} \right] dx - \int_0^\infty x^{r+\theta} e^{-\lambda \frac{x}{\theta}} \left[ 1 - e^{-\frac{c x}{\theta}} \right] dx
\]
\[
= \left( \frac{p}{\lambda^\theta} \right) \Gamma(\frac{r+\theta+1}{\theta} + 1 - \left( \frac{p}{\lambda^\theta} \right) \Gamma(\frac{r+\theta+1}{\theta} + 1 + \left( 1 + c^\theta \right) \frac{1}{\theta})
\]
\[
E(X^r) = \left( \frac{p}{\lambda^\theta} \right) \Gamma\left( \frac{r+\theta+1}{\theta} + 1 \right) \left[ 1 - \frac{1}{\left( 1 + c^\theta \right) \frac{1}{\theta}} \right]
\]
Let \( \epsilon = \left( 1 + c^\theta \right)^{\frac{1}{\theta}} \) and \( \epsilon_r = \left( 1 + c^\theta \right)^{\frac{r}{\theta}} \), \( r = 1, 2, 3, \ldots \)
\[ E(X^r) = \left( \frac{p}{\lambda^r} \right)^r \frac{1}{\Gamma \left( \frac{r+1}{\beta} + 1 \right)} \Gamma \left( \frac{r+1}{\beta} \right) \left[ 1 - \frac{1}{\varepsilon_r} \right] \] ... (5)

Now, the mean can be obtained, as well as variance, standard deviation, coefficient of variation form eq (5) as follows:

Mean

To find mean put \( r = 1 \) given by
\[ E(X) = \mu = \left( \frac{p}{\lambda^1} \right) \frac{1}{\Gamma \left( \frac{1}{\beta} + 1 \right)} \Gamma \left( \frac{1}{\beta} \right) \left[ 1 - \frac{1}{\varepsilon_1} \right] \] ... (6)

Where \( \varepsilon_1 = \left( 1 + c^\theta \right)^{\frac{1}{\beta}} \)

Variance

\[ \sigma^2 = E(X^2) - (E(X))^2 \]
\[ \sigma^2 = \left( \frac{p}{\lambda^2} \right)^2 \frac{1}{\Gamma \left( \frac{3}{\beta} + 1 \right)} \left[ 1 - \frac{1}{\varepsilon_2} \right] - \left( \frac{p}{\lambda^1} \right) \frac{1}{\Gamma \left( \frac{2}{\beta} + 1 \right)} \Gamma \left( \frac{2}{\beta} \right) \left[ 1 - \frac{1}{\varepsilon_1} \right]^2 \] ... (7)

Where \( \varepsilon_2 = \left( 1 + c^\theta \right)^{\frac{2}{\beta}} \)

Standard deviation

\[ \sigma = \left( \frac{p}{\lambda^2} \right)^2 \frac{1}{\Gamma \left( \frac{3}{\beta} + 1 \right)} \left[ 1 - \frac{1}{\varepsilon_2} \right] - \left( \frac{p}{\lambda^1} \right) \frac{1}{\Gamma \left( \frac{2}{\beta} + 1 \right)} \Gamma \left( \frac{2}{\beta} \right) \left[ 1 - \frac{1}{\varepsilon_1} \right]^2 \] ... (8)

Coefficient of variation

\[ C.V = \frac{\sigma}{\mu} = \sqrt{\frac{\left( \frac{p}{\lambda^2} \right)^2 \frac{1}{\Gamma \left( \frac{3}{\beta} + 1 \right)} \left[ 1 - \frac{1}{\varepsilon_2} \right] - \left( \frac{p}{\lambda^1} \right) \frac{1}{\Gamma \left( \frac{2}{\beta} + 1 \right)} \Gamma \left( \frac{2}{\beta} \right) \left[ 1 - \frac{1}{\varepsilon_1} \right]^2}{\left( \frac{p}{\lambda^1} \right)^2 \frac{1}{\Gamma \left( \frac{2}{\beta} + 1 \right)} \left[ 1 - \frac{1}{\varepsilon_1} \right]} \] ... (9)

Moment Generation Function:

The moment generating function of DWEPD is given by
\[ M_X(t) = \int_0^\infty e^{-tx} g_w(x, \theta, \lambda, p, c)dx \]
\[ = \int_0^\infty \left( 1 + tx + \frac{(tx)^2}{2!} \ldots \right) g_w(x, \theta, \lambda, p, c)dx \]
\[ = \sum_{r=0}^\infty \frac{t^r}{r!} E(X^r) \]
\[ = \sum_{r=0}^\infty \frac{t^r}{r!} \left( \frac{p}{\lambda^r} \right)^r \frac{1}{\Gamma \left( \frac{r+1}{\beta} + 1 \right)} \Gamma \left( \frac{r+1}{\beta} + 1 \right) \left[ 1 - \frac{1}{\varepsilon_r} \right] \] ... (10)
3. Reliability Analysis

3.1. Reliability function $R(x)$.

The reliability function or, known as survival function $R(x)$ can be derived by using the cumulative distribution function (c.d.f) as follows

$$R(x) = 1 - G_w(x)$$

$$= 1 - \frac{1}{r(\frac{1}{\theta} + 1)[\varepsilon - 1]} \left[ \gamma(\frac{1}{\theta} + 1), \lambda(\frac{X}{p})\theta \right] - \frac{1}{\varepsilon} \gamma(\frac{1}{\theta} + 1), \lambda(\frac{X}{p})\theta(1 + C\theta) \right]$$

$$= r(\frac{1}{\theta} + 1)[\varepsilon - 1] - \left[ \gamma(\frac{1}{\theta} + 1), \lambda(\frac{X}{p})\theta \right] - \frac{1}{\varepsilon} r(\frac{1}{\theta} + 1), \lambda(\frac{X}{p})\theta(1 + C\theta) \right]$$

$$\frac{r(\frac{1}{\theta} + 1)[\varepsilon - 1] - \left[ \gamma(\frac{1}{\theta} + 1), \lambda(\frac{X}{p})\theta \right] - \frac{1}{\varepsilon} r(\frac{1}{\theta} + 1), \lambda(\frac{X}{p})\theta(1 + C\theta) \right]}{r(\frac{1}{\theta} + 1)[\varepsilon - 1]} \quad \cdots (11)$$

3.2. Hazard Function $H(x)$

$H(x)$ denotes the instantaneous rate function or (the Hazard function). Given that the unit has survived until $x$, the hazard function of $x$, provided that the conditional probability density of failure at time $x$ or interpreted as instantaneous rate. We can define the Hazard function as

$$H(x) = \frac{g_w(x)}{R(x)}$$

$$H(x) = \frac{\theta \lambda^{1+1}}{p^{\theta+1}} \frac{1}{\Gamma \left( \frac{1}{\theta} + 1 \right)} \left[ 1 - \frac{1}{\varepsilon} \right] x^\theta e^{-\lambda(\frac{X}{p})^\theta} \left[ 1 - e^{-\lambda(\frac{X}{p})^\theta} \right]$$

$$\frac{\theta \lambda^{1+1}}{p^{\theta+1}} \frac{1}{\Gamma \left( \frac{1}{\theta} + 1 \right)} \left[ 1 - \frac{1}{\varepsilon} \right] \gamma(\frac{1}{\theta} + 1), \lambda(\frac{X}{p})^\theta \right] - \frac{1}{\varepsilon} \gamma(\frac{1}{\theta} + 1), \lambda(\frac{X}{p})^\theta(1 + C\theta) \right]$$

$$\frac{\Gamma \left( \frac{1}{\theta} + 1 \right) \left[ 1 - \frac{1}{\varepsilon} \right] \gamma(\frac{1}{\theta} + 1), \lambda(\frac{X}{p})^\theta \right] - \frac{1}{\varepsilon} \gamma(\frac{1}{\theta} + 1), \lambda(\frac{X}{p})^\theta(1 + C\theta) \right]}{\Gamma \left( \frac{1}{\theta} + 1 \right) \left[ 1 - \frac{1}{\varepsilon} \right]} \quad \cdots (12)$$

3.3. Reverse Hazard function $\Phi(x)$

The best describes to reverse hazard function is that you can determine it by the approximate probability of disappointment or failure in $[x, x + dx]$. Considering that the loss occurred or failure in $[0, X]$. The function of reverse hazard $\Phi(x)$ is defined to be

$$\Phi(x) = \frac{g_w(x)}{G_w(x)}$$

$$\Phi(x) = \frac{\theta \lambda^{1+1}}{p^{\theta+1}} \frac{1}{\Gamma \left( \frac{1}{\theta} + 1 \right)} \left[ 1 - \frac{1}{\varepsilon} \right] x^\theta e^{-\lambda(\frac{X}{p})^\theta} \left[ 1 - e^{-\lambda(\frac{X}{p})^\theta} \right]$$

$$\frac{\theta \lambda^{1+1}}{p^{\theta+1}} \frac{1}{\Gamma \left( \frac{1}{\theta} + 1 \right)} \left[ 1 - \frac{1}{\varepsilon} \right] \gamma(\frac{1}{\theta} + 1), \lambda(\frac{X}{p})^\theta \right] - \frac{1}{\varepsilon} \gamma(\frac{1}{\theta} + 1), \lambda(\frac{X}{p})^\theta(1 + C\theta) \right]$$

$$\frac{\phi \lambda^{1+1}}{p^{\theta+1}} \frac{1}{\Gamma \left( \frac{1}{\theta} + 1 \right)} \left[ 1 - \frac{1}{\varepsilon} \right] \gamma(\frac{1}{\theta} + 1), \lambda(\frac{X}{p})^\theta \right] - \frac{1}{\varepsilon} \gamma(\frac{1}{\theta} + 1), \lambda(\frac{X}{p})^\theta(1 + C\theta) \right]}{\phi \lambda^{1+1}} \quad \cdots (13)$$

4. Estimation methods

As we refer above, this work introduced two estimation methods for the four parameters ($\theta, p, \lambda, C$). The outcomes of the simulation procedure explained, but after giving some details about the estimators.

4.1. Maximum likelihood method

If $X_1, X_2, \ldots, X_n$ are a r.s.’s from DWEP distribution, then the Likelihood function is:
\[ Lg(x, \theta, \lambda, p, c) = \prod_{i=1}^{n} \left[ \frac{\theta \lambda^{\frac{1}{\theta} + 1}}{p^{\theta + 1} \Gamma \left( \frac{1}{\theta} + 1 \right) \left[ 1 - \frac{1}{\epsilon} \right]} x^{\theta} e^{-\lambda \left( \frac{C x}{p} \right)^{\theta}} \left[ 1 - e^{-\lambda \left( \frac{C x}{p} \right)^{\theta}} \right] \right] \]

\[ Lng(x, \theta, \lambda, p, c) = \ln(\theta + \left( \frac{1}{\theta} + 1 \right) \ln(\theta + (\theta + 1) \ln P) - \ln \Gamma \left( \frac{1}{\theta} + 1 \right) - \ln \left( 1 - \frac{1}{(1 + \theta)^{\frac{1}{\theta} + 1}} \right) \theta \ln X \]

\[ - \lambda \left( \frac{C x}{p} \right)^{\theta} \left[ 1 - e^{-\lambda \left( \frac{C x}{p} \right)^{\theta}} \right] \]

\[ \frac{\partial Lng(x)}{\partial \theta} = \frac{1}{\theta} - \frac{L \ln(\theta + (\theta + 1) \ln P) - \ln \Gamma \left( \frac{1}{\theta} + 1 \right) - \ln \left( 1 - \frac{1}{(1 + \theta)^{\frac{1}{\theta} + 1}} \right) \theta \ln X - \lambda \left( \frac{C x}{p} \right)^{\theta - 1} + \ln \left( 1 - e^{-\lambda \left( \frac{C x}{p} \right)^{\theta}} \right)}{\theta^{2}} \]

\[ \frac{\partial Lng(x)}{\partial p} = \frac{\theta \lambda}{\theta p} \left( \frac{x}{p} \right)^{\theta} - \frac{(\theta + 1) \theta \left( \frac{C x}{p} \right)^{\theta} e^{-\lambda \left( \frac{C x}{p} \right)^{\theta}}}{1 - e^{-\lambda \left( \frac{C x}{p} \right)^{\theta}}} \]

\[ \frac{\partial Lng(x)}{\partial \lambda} = \frac{1}{\theta} - \frac{(\theta + 1) \theta}{\lambda} - \frac{\theta \left( \frac{C x}{p} \right)^{\theta} e^{-\lambda \left( \frac{C x}{p} \right)^{\theta}}}{1 - e^{-\lambda \left( \frac{C x}{p} \right)^{\theta}}} \]

The numerical solution can be used to determined eq. (17) instantaneously, since this equation can be equal to zero. Therefore, we obtain $\theta_{MLE}^{\lambda}$ and $p_{MLE}^{\lambda}$ as M.L. E. estimators of $\theta$, $p$, $\lambda$, and $C$ respectively.

### 4.2. Method of moment

An independent random sample $r.s., X_{1}, X_{2}, \ldots, X_{n}$ from the DWRD with parameters $\theta, p, \lambda,$ and $C$. The moment estimation method is obtained by measuring population moments and equating them with sample moments.

\[ m_{r} = \sum_{i=1}^{n} x_{i}^{r} \]

\[ \mu_{r} = E(x^{r}) \]

\[ m_{r} = \mu_{r} \]

\[ E(x^{r}) = \left( \frac{p}{\lambda} \right)^{r} \frac{\Gamma \left( \frac{r + 1}{\theta} + 1 \right)}{(1 - \frac{\theta}{\epsilon}) \Gamma \left( \frac{1}{\theta} + 1 \right) \left[ 1 - \frac{1}{(1 + \theta)^{\frac{1}{\theta} + 1}} \right]} \]

Let $\varepsilon = \left( 1 + e^{\theta} \right)^{\frac{1}{\theta} + 1}$ and $\varepsilon_{r} = \left( 1 + e^{\theta} \right)^{\frac{r}{\theta}}$, $r = 1, 2, 3 \ldots$.

\[ E(x^{r}) = \left( \frac{p}{\lambda} \right)^{r} \frac{1}{\varepsilon_{r}} \left[ 1 - \frac{1}{(1 + \theta)^{\frac{1}{\theta} + 1}} \right] \]

\[ \frac{\sum_{i=1}^{n} x_{i}}{n} = \bar{x} = \left( \frac{p}{\lambda} \right)^{r} \frac{1}{\varepsilon_{r}} \left[ 1 - \frac{1}{(1 + \theta)^{\frac{1}{\theta} + 1}} \right] \]

\[ \frac{\sum_{i=1}^{n} x_{i}^{2}}{n} = \left( \frac{p}{\lambda} \right)^{2} \frac{1}{\varepsilon_{r}^{2}} \left[ 1 - \frac{1}{(1 + \theta)^{\frac{1}{\theta} + 1}} \right] \]

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\[
\frac{\sum x_i^3}{n} = \left(\frac{p}{\lambda}\right)^3 \frac{1}{\lambda^3} \left[1 - \frac{1}{e^3}\right] \left(\frac{1}{\lambda^4} \Gamma\left(\frac{4}{\theta} + 1\right) \right)
\]

... (22)

\[
\frac{\sum x_i^4}{n} = \left(\frac{p}{\lambda}\right)^4 \frac{1}{\lambda^4} \left[1 - \frac{1}{e^4}\right] \left(\frac{1}{\lambda^5} \Gamma\left(\frac{5}{\theta} + 1\right) \right)
\]

... (23)

By solving the four equations (20), (21), (22), and (23) simultaneously (numerical method), get \((\hat{\theta}_{mom}, \hat{p}_{mom}, \hat{\lambda}_{mom}, \hat{c}_{mom})\) as an estimate of \(\theta, p, \lambda\) and \(c\) respectively.

4.3. Percentiles estimation (PE)

Initially, Kao (1959) discovered this technique by graphically approximating the best linear unbiased estimators. By fitting a straight line to the theoretical points calculated from the distribution function, and the sample percentile points, the estimators could be found. In the case of a DWEP distribution, because of the nature of its distribution function, it is likely to use the same idea to evaluate the estimators of \(\theta, p, \lambda\), and \(C\) based on PE. Since \(G(x)\) is separated by Al-khadim in (2014). Firstly, determined numerically the value of \(x\) where \(x = G^{-1}(x, \theta, P, \lambda, C)\), since \(P_i\) is the estimate of \(G(X(i), \theta, P, \lambda, C)\) \(\theta_{PE}, P_{PE}, \lambda_{PE}, C_{PE}\) can be determined by minimizing

\[
\sum_{i=1}^{n}[x(i) - G^{-1}(P_i, \theta, P, \lambda, C)]^2
\]

Concerning \(P, \lambda, C\) where
\[
E[\hat{G}(X(i))] = P_i = \frac{i}{n+1}
\]

is the most used estimator of \(G(X(i))\).

4.4. Numerical study

The Monte-Carlo simulation study was conducted by using MATLAB to test the ability of the estimation methods presented in paragraph (5) which is the Maximum Likelihood method, the moment's method, and the Percentile method. Assumed a variety of theoretical parameters for DWEPD which is \(\theta = 2.5, 3, 5, \lambda = 2.5, 3, P = 2.5, 2, 1.5, C = 2.5, 2, 1.5\), and sample sizes (25, 50, 100, 150) and the replication (1000) for each simulation experiment to obtain the homogeneity of the results. The results were compared by using MSE and MAPE. The simulation study showed the preference of the Percentiles method over the other methods at all the size of samples, 50, 100, and the method of the Maximum Likelihood method at the size of 150.

Table 1. Results of simulation under all sample sizes and theoretical parameters

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\theta & 2.5 & \lambda & 2.5 & P & 2.5 & C = 2.5 \\
\hline
\text{n} & \text{Method} & \hat{\theta} & \hat{\lambda} & \hat{p} & \hat{c} \\
\hline
25 & ML. & Parameter & 2.43116 & 2.72011 & 1.99819 & 2.299821 \\
& & MSE & 0.033442 & 0.043528 & 0.03328 & 0.023310 \\
& & MAPE & 0.120095 & 0.334486 & 0.22555 & 0.129892 \\
& Moment. & Parameter & 2.315585 & 2.18231 & 2.097783 & 2.6445863 \\
& & MSE & 0.025638 & 0.122396 & 0.674627 & 0.014844 \\
& & MAPE & 0.044316 & 0.230219 & 0.409433 & 0.041808 \\
& Prec. & Parameter & 2.499765 & 2.519132 & 2.497654 & 2.499981 \\
& & MSE & 0.004383 & 0.004733 & 0.003567 & 0.003943 \\
& & MAPE & 0.023644 & 0.022435 & 0.063439 & 0.020098 \\
& Best & Prec. & Prec. & Prec. & Prec. & & \\
25 & 50 & ML. & Parameter & 2.132167 & 2.321456 & 1.567321 & 2.312222 \\
& & MSE & 0.154042 & 0.095019 & 0.065498 & 0.159717 \\
& & MAPE & 0.116574 & 0.170389 & 0.102865 & 0.137322 \\
& Moment. & Parameter & 2.422089 & 1.186318 & 1.843517 & 2.448652 \\
& & MSE & 0.019395 & 0.120794 & 0.666337 & 0.013555 \\
& & MAPE & 0.038842 & 0.229056 & 0.407341 & 0.040217 \\
& Prec. & Parameter & 2.504587 & 2.490333 & 2.550866 & 2.501567 \\
\hline
\end{array}
\]
MSE | 0.003511 | 0.002911 | 0.005655 | 0.001335  
MAPE | 0.026894 | 0.032228 | 0.034321 | 0.028889  

**Best**

| Parameter | MSE | MAPE |
|-----------|-----|------|
| ML.       | 2.238777 | 0.003511 | 0.026894 |
|           | 2.468719 | 0.002911 | 0.032228 |
|           | 1.901123 | 0.005655 | 0.034321 |
|           | 2.248716 | 0.001335 | 0.028889 |

### Table 2. Results of simulation under all sample sizes and theoretical parameters

| n | Method | $\theta = 2.5$ | $\lambda = 2.5$ | $P = 2$ | $C = 2$ |
|---|--------|----------------|----------------|--------|--------|
|   | $\hat{\theta}$ | $\hat{\lambda}$ | $\hat{p}$ | $\hat{c}$ |
| 25 | ML. | Parameter | 1.989226 | 1.76112 | 1.434579 | 1.630657 |
|   | MSE | 0.155568 | 0.083361 | 0.048788 | 0.164314 |
|   | MAPE | 0.139612 | 0.179027 | 0.094444 | 0.193864 |
|   | Mom. | Parameter | 2.39042 | 2.070324 | 1.878362 | 1.933908 |
|   | MSE | 0.88506 | 0.003422 | 0.013248 | 2.182823 |
|   | MAPE | 0.376688 | 0.035058 | 0.039678 | 0.75479 |
|   | Prec. | Parameter | 2.000176 | 2.145712 | 1.502755 | 2.505455 |
|   | MSE | 0.003323 | 0.003438 | 0.016752 | 0.003351 |
|   | MAPE | 0.019751 | 0.033877 | 0.060636 | 0.021842 |
|   | Best | ML. | 1.966672 | 1.679849 | 1.319831 | 1.545619 |
|   | MSE | 0.125537 | 0.081299 | 0.055723 | 0.158464 |
|   | MAPE | 0.124645 | 0.166712 | 0.102588 | 0.17176 |
| 50 | ML. | Parameter | 1.774784 | 1.478322 | 1.867458 | 1.623333 |
|   | MSE | 0.0087899 | 0.129132 | 0.727143 | 0.025344 |
|   | MAPE | 0.030166 | 0.236965 | 0.426351 | 0.075425 |
|   | Prec. | Parameter | 2.48678 | 2.399983 | 2.499432 | 2.468535 |
|   | MSE | 0.003353 | 0.003315 | 0.006853 | 0.00311 |
|   | MAPE | 0.020124 | 0.033493 | 0.036668 | 0.025431 |
|   | Best | Prec. | Prec. | Prec. | Prec. |
| 100 | ML. | Parameter | 1.689222 | 1.893357 | 1.278898 | 1.61867 |
|   | MSE | 0.132843 | 0.086648 | 0.056059 | 0.16232 |
|   | MAPE | 0.127653 | 0.169737 | 0.103478 | 0.172912 |
|   | Mom. | Parameter | 1.884293 | 1.158242 | 1.819322 | 2.563181 |
|   | MSE | 0.008671 | 0.131741 | 0.726526 | 0.025215 |
| Prec. | Parameter | 150  | MSE  | MAPE |
|-------|-----------|------|------|------|
|       | 2.500178  | 2.402288 | 1.981532 | 2.00987 |
|       | 0.003377  | 0.00297 | 0.002922 | 0.003119 |
|       | 0.020228  | 0.02959 | 0.017556 | 0.022521 |

| Best  | Prec. | Prec. | Prec. | Prec. |
|-------|-------|-------|-------|-------|
|       | 2.328891 | 2.245921 | 1.968712 | 2.198883 |
|       | 0.002818  | 0.002913 | 0.003182 | 0.002891 |
|       | 0.068021  | 0.028824 | 0.025361 | 0.024973 |

| Prec. | Parameter | 150  | MSE  | MAPE |
|-------|-----------|------|------|------|
|       | 2.502003  | 1.853541 | 1.660188 | 2.515821 |
|       | 0.000891  | 0.090837 | 0.747855 | 0.036425 |
|       | 0.0148681 | 0.200133 | 0.432276 | 0.094165 |

| Best  | Prec. | Prec. | Prec. | Prec. |
|-------|-------|-------|-------|-------|
|       | 2.457882 | 2.493444 | 1.444914 | 2.450834 |
|       | 0.003264  | 0.003773 | 0.003057 | 0.00204 |
|       | 0.019688  | 0.036724 | 0.02367 | 0.023819 |

Table 3. Results of simulation under all sample sizes and theoretical parameters

\begin{tabular}{|c|c|c|c|c|c|}
\hline
\( n \) & Method & \( \hat{\theta} \) & \( \hat{\lambda} \) & \( \hat{p} \) & \( \hat{c} \) \\
\hline
25 & **ML.** & Parameter & 2.677578 & 2.21145 & 1.13332 & 1.308991 \\
& & MSE & 0.126176 & 0.082487 & 0.054755 & 0.136227 \\
& & MAPE & 0.123127 & 0.164677 & 0.080667 & 0.156845 \\
& **Mom.** & Parameter & 1.997611 & 2.195556 & 2.018333 & 1.820322 \\
& & MSE & 0.018282 & 0.260266 & 1.011292 & 0.062223 \\
& & MAPE & 0.048435 & 0.338282 & 0.301926 & 0.134415 \\
& **Prec.** & Parameter & 2.95979 | 3.080061 | 1.545561 | 1.49112 \\
& & MSE & 0.003311 & 0.003165 | 0.007789 & 0.003098 \\
& & MAPE & 0.020167 & 0.021799 & 0.03001 & 0.024377 \\
& **Best** & Prec. | Prec. | Prec. | Prec. |
\hline
50 & **ML.** & Parameter & 2.578911 & 2.396541 & 1.241333 | 1.189921 \\
& & MSE & 0.127152 & 0.072488 | 0.055756 | 0.137125 \\
& & MAPE & 0.123323 & 0.164693 | 0.080732 | 0.156816 \\
& **Mom.** & Parameter & 2.745666 & 1.996533 & 1.917333 & 2.001125 \\
& & MSE & 0.018235 & 0.260667 & 1.011234 & 0.062221 \\
& & MAPE & 0.048433 & 0.368831 & 0.411956 & 0.121153 \\
& **Prec.** & Parameter & 3.26822 & 3.003373 | 1.500855 | 2.499216 \\
& & MSE & 0.003219 & 0.003255 & 0.00845 & 0.003161 \\
& & MAPE & 0.020188 & 0.031765 & 0.03127 & 0.024245 \\
& **Best** & Prec. | Prec. | Prec. | Prec. |
\hline
100 & **ML.** & Parameter & 1.677211 & 2.301118 | 1.341111 | 2.145919 \\
& & MSE & 0.118736 & 0.0880892 & 0.051474 & 0.150318 \\
& & MAPE & 0.118812 & 0.16923 & 0.078055 & 0.165904 \\
& **Mom.** & Parameter & 1.760776 & 1.49383 & 2.007874 & 2.62093 \\
& & MSE & 0.018388 & 0.260308 & 1.011934 & 0.060743 \\
& & MAPE & 0.048477 & 0.338453 & 0.4012 & 0.119912 \\
& **Prec.** & Parameter & 2.967881 & 3.111176 | 1.509011 | 1.49001 \\
& & MSE & 0.003513 & 0.003441 & 0.00346 & 0.003234 \\
& & MAPE & 0.020176 & 0.032902 & 0.019842 & 0.02355 \\
& **Best** & Prec. | Prec. | Prec. | Prec. |
\hline
150 & **ML.** & Parameter & 1.948334 & 2.114566 | 1.551211 | 2.455901 \\
& & MSE & 0.002642 & 0.003295 & 0.003281 & 0.002799 \\
\hline
\end{tabular}
|        | MAPE   | 0.018841 | 0.032528 | 0.018668 | 0.025637 |
|--------|--------|----------|----------|----------|----------|
| Mom.   | Parameter | 1.713444 | 1.488052 | 1.947384 | 2.532011 |
|        | MSE    | 0.007725 | 0.20098  | 1.026657 | 0.082688 |
|        | MAPE   | 0.032217 | 0.298254 | 0.405172 | 0.142688 |
| Prec.  | Parameter | 1.945555 | 2.450336 | 1.450625 | 2.448172 |
|        | MSE    | 0.004477 | 0.003077 | 0.003351 | **0.00379** |
|        | MAPE   | 0.020732 | 0.029911 | 0.019865 | 0.027223 |

Best

![Figure 1](image1.png)

Figure 1. The Cumulative distribution function under $\theta = 2.5$ $\lambda = 2.5$ $P = 2.5$ $C = 2.5$.

![Figure 2](image2.png)

Figure 2. The Cumulative distribution function under $\theta = 2.5$ $\lambda = 2.5$ $P = 2$ $C = 2$.

![Figure 3](image3.png)

Figure 3. The Cumulative distribution function under $\theta = 3$ $\lambda = 3$ $P = 1.5$ $C = 1.5$

Applied Side:
The data were collected and applied to the best methods used in the research, which represents the period of survival of the patient until death for patients with breast cancer. Medical in the holy province of Karbala (100). After each patient took doses of chemotherapy from the chemotherapy unit, the times until death occurred in months for the period (2016-2018) and the following table shows the real data under investigation:
For fitting data on the survival period until death according to DWEPD of the four parameters. A goodness of fit test was conducted which includes four types of tests to analyze the real data sample in estimating the parameters by Percentiles method and its application to the real experience data of breast cancer diseases, which is best estimated through experimental simulation, table (4) shows the parameter estimates for the proposed distribution (DWEPD) for the goodness of fit(Chi-Squared, Anderson–Darling, Kolmogorov-Smirnov, Cramer Van Mises) and the result as following:

Table 4. Real Data sheet

| Parameter | 11.7 | 10.7 | 10.5 | 12.4 | 12.4 | 11.3 | 11.3 | 10.3 | 10.2 |
|-----------|------|------|------|------|------|------|------|------|------|
| θ         | 11.5 | 11.3 | 11.3 | 11.3 | 11.2 | 11.1 | 11.1 | 11.9 | 10.9 |
| λ         | 11.9 | 11.9 | 11.8 | 11.8 | 11.6 | 11.6 | 11.5 | 17.5 | 21.5 |
| p         | 12.4 | 12.5 | 12.5 | 21.5 | 12.3 | 12.1 | 12   | 11.9 | 11.9 |
| c         | 13.28| 13.25| 13.22| 13.1 | 13   | 13.9 | 12.9 | 12.8 | 12.8 |
| θ         | 13.9 | 14.9 | 13.7 | 13.6 | 13.6 | 13.5 | 13.4 | 15.4 | 11.3 |
| λ         | 14.7 | 14.6 | 14.6 | 15.5 | 14.4 | 14.4 | 14.3 | 14.27| 14.26|
| p         | 15.6 | 14.6 | 16.4 | 15.3 | 15.3 | 15.3 | 15.13| 14.9 | 14.8 |
| c         | 17.8 | 17.7 | 17.7 | 17.6 | 17.5 | 17.3 | 16.7 | 17.6 | 19.1 |
| θ         | 23.6 | 21.15| 18.5 | 18.5 | 19.2 | 19.1 | 18.7 | 18.6 | 18.5 |
| λ         |       |      |      |      |      |      |      |      |      |
| p         |       |      |      |      |      |      |      |      |      |
| c         |       |      |      |      |      |      |      |      |      |

Table 5. Results of the goodness of fit for real Data

| Model       | Parameter | θ   | λ   | p   | c   | Chi-square | A_d | K.S  | W_d |
|-------------|-----------|-----|-----|-----|-----|------------|-----|------|-----|
| DWEPD       |           | 2.6888| 2.322| 2.3111| 2.517| 0.6811     | 2.0999| 0.09554| 0.0059|
| Exp-Pareto  |           | 2.8451| 3.7778| 3.3114| 2.517| 432333     | 7.6755| 0.22212| 0.0151|
| Pareto      |           | 3.6719| 3.2187| 3.2187| 2.517| 77.8755    | 79.8776| 0.65543| 0.053 |
| Exponential |           | 7.814 |      |      |      |            |      |      |      |

From table (2) we note that the bias parameter (C) we observe its value (2.517) based on default values on the simulation side. The values of the calculated parameters were compatible with the default values for the parameters shown on the simulation side. And to test the hypothesis ($H_0 : X \sim \text{DWE}P$ against $H_1 : X \sim \text{DWE}P$) the table shows that the results of the $H_0$ null hypothesis test show, according to the four criteria, the acceptance of this hypothesis a significant level of 0.05), i.e., the real data follow the proposed distribution (DWEP), where we have been confirmed by comparing the four tests while Chi-square statistic for the distribution of Exponential- Pareto value and the distribution of Pareto, Exponential, and this indicator of the values of the three alternative distributions (for the proposed distribution) confirms the rejection of the null hypothesis.

Figure 4. The cumulative distribution function for real data

This figure showed clear approximate among empirical curve for double-weighted exponential Pareto and real data curve which it refers to the accuracy of fitting data according to four tests for fitting.
5. Conclusions

1. The researcher input double parameter of bias (C) for the distribution of (Exponential-Pareto), so the distribution was called distribution ((Double Weighted Exponential-Pareto) and was proved to be a weighted probability distribution.

2. It was found that the best value of the bias parameter C contributes to the elimination of the bias is when (C> 2.5) results from the adoption of data of different sizes, and this reflects the importance of weighted probability distributions rather than probability distributions only if the researcher is studying or interested in the analysis of data originally from different samples sizes especially in health and life applications.

3. The best method of estimation of parameters was the Percentiles method at all the size of samples, 50,100 because have less MSE and MAPE.

4. The priority of the Maximum Likelihood method at the size of 150.

5. When increasing the sample size, then the Percentiles method and maximum Likelihood method is the best.

Recommendations

1. Extend the research to include other weighted vehicle distributions, as this is important in estimating operating times or failures and in evaluating expensive medical trials.

2. Dependence on other indicators to reduce or reduce uncertainty as well as the Renyi Entropy scale such as Shannon- Entropy, and others.

3. Addition of other methods of estimation, other than those adopted by the researcher such as Bayesian methods. The research on the proposed model can be expanded and converted into complex probability distributions to accommodate double data.

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