Time-varying state-feedback stabilisation of stochastic feedforward nonlinear systems with unknown growth rate

Ticao Jiao\textsuperscript{a,b}, Wei Xing Zheng\textsuperscript{b} and Shengyuan Xu\textsuperscript{a}

\textsuperscript{a}School of Automation, Nanjing University of Science and Technology, Nanjing, Jiangsu, P.R. China; \textsuperscript{b}School of Computing, Engineering and Mathematics, Western Sydney University, Sydney, Australia

\textbf{ABSTRACT}
We consider the time-varying state-feedback stabilisation problem for a class of stochastic feedforward nonlinear systems with unknown growth rate in this paper. A new LaSalle-type theorem for stochastic time-varying systems is firstly established by using the generalized weakly positive definite function. As an application, to deal with serious uncertainties in the unknown growth rate, a time-varying approach, rather than an adaptive one, is adopted to design the scheme of a state-feedback controller for stochastic feedforward systems. Based on the established LaSalle-type theorem, it is shown that all signals of the resulting closed-loop system converge to zero almost surely. Illustrative examples are given to verify the theoretical findings.

\textbf{1. Introduction}
Since stochastic integral was first introduced by Itô in 1949, there has been a quick development of the theory of stochastic differential equations. The powerful Lyapunov method has been used to deal with stochastic stability by many authors, see, e.g., Khasminskii (2011), Kushner (1972), Deng, Krstić, and Williams (2001), Wu, Cui, Shi, and Karimi (2013), Chen, Zheng, and Shen (2009), Wu, Zheng, and Gao (2013), Kang, Zhai, Liu, Zhao, and Zhao (2014), Chen and Zheng (2016), Kang, Zhai, Liu, and Zhao (2016) and the references therein. Note that these results are only applied to time-invariant systems. Recently, a new type of stability theorem for stochastic systems with arbitrary variability in the time was established in Zhao and Deng (2015) by the extension of positive definite functions. However, these results cannot be used to locate the limit sets of stochastic systems and the definition of weakly positive definite functions seems to be restrictive. This problem was firstly dealt with in Mao (1999), where the LaSalle-type theorem for stochastic delay systems was proposed. Then, better results were obtained in Yu, Xie, and Duan (2010), Li and Mao (2012) by removing the linear growth condition in Mao (1999).

Feedforward systems, also called upper-triangular systems, have been widely applied to model many physical devices, such as the cart-pendulum system (Mazenc & Bowong, 2003) and the induction heater circuit system (Jo, Choi, & Lim, 2014). Based on the fundamental stability theory, the controller design and analysis for feedforward systems have been a hot topic recently, see, e.g., Teel (1992), Mazenc and Praly (1994), Tsinias and Tzamtzi (2001), Zhang, Liu, Baron, and Boukas (2011), Du, Qian, and Li (2013), Qian and Du (2012) and the references therein. More precisely, the problem of asymptotic tracking for feedforward systems was addressed in Mazenc and Praly (2000) by constructing time-varying state feedback. In Shang, Liu, & Zhang (2015), Ye and Unbehauen (2004), the adaptive stabilizing problem for feedforward systems with unknown linear growth rates was investigated by introducing a dynamic gain and a switching logic, respectively. By introducing the dynamic gain approach to time-delay systems, constructive control techniques were proposed for controlling feedforward nonlinear time-delay systems in Zhang, Baron, Liu, & Boukas (2011), Zhang, Liu, et al. (2011). Although noticeable progresses have been made for deterministic feedforward systems (see, e.g., Frye, Trevino, & Qian, 2007; Tsinias & Tzamtzi, 2001; Zhang, Baron, et al., 2011), up to now there has been very few literature considering stabilisation of stochastic feedforward systems. In Liu and Xie (2013), a state-feedback controller was designed to globally stabilise a class of stochastic feedforward nonlinear systems. Subsequently, for stochastic high-order feedforward systems, Zhao and Xie (2014) examined the problem of state controller design and then (Jiao et al., 2014) studied the problem of decentralised stabilisation for a class of large-scale cases. However, these works are limited in the sense that the nonlinearities depending on
the states grow at a known constant rate. For stochastic feedforward systems with unknown growth rates, how to design a stabilizing controller as in Shang et al. (2015), Zhang, Baron, et al. (2011) still remains as an open and challenging problem.

The purpose of this paper is to develop a global time-varying stabiliser for a class of stochastic feedforward systems possessing an unknown growth rate. The main contributions of this paper are highlighted as follows:

1. In almost all the literature, the global stability analysis of controller designs of stochastic nonlinear systems heavily relies on the stochastic theories in Deng et al. (2001), Khasminskii (2011). However, it is not applicable to more general stochastic systems, particularly to stochastic time-varying cases. On the other hand, although Zhao and Deng (2015) provided a good solution to this problem, the condition of weakly positive definition in Zhao and Deng (2015) is still restrictive (see the motivating example in the next section). In this paper, we will generalise the work of Zhao and Deng (2015) under more common assumptions. Thus, a new definition of the weakly positive definite function is given, based on which we provide a new LaSalle-type theorem for stochastic time-varying systems.

2. With the stochastic LaSalle-type theorem, a time-varying state-feedback controller (instead of an adaptive one) for stochastic feedforward nonlinear systems with an unknown growth rate is designed to guarantee that all signals of the closed-loop system are bounded almost surely and the system states converge to zero almost surely.

The remainder of this paper is organised as follows. Section 2 introduces some preliminaries on stochastic theory and gives a LaSalle-type theorem. In Section 3 a time-varying state-feedback design scheme is developed for a class of stochastic feedforward systems with an unknown growth rate. Section 4 illustrates our proposed control design through one simulation example and one practical example, after which the paper is concluded in Section 5.

Notation. The following standard notation is used throughout this paper. For a vector $x$, $|x|$ stands for its usual Euclidean norm and $x^T$ denotes its transpose. $\|A\|$ represents the 2-norm of a matrix $A$. $EX(t)$ denotes the expectation of the stochastic process $X(t)$. $\mathcal{K}$ stands for the set of all functions $\alpha(s) : \mathbb{R}_+ \to \mathbb{R}_+$, which are continuous, strictly increasing and vanishing at zero; $\mathcal{K}_{\infty}$ denotes the set of all functions which are of class $\mathcal{K}$ and unbounded; $\mathcal{KL}$ denotes the set of all functions $\beta(s, t) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ which are of class $\mathcal{K}$ for each fixed $t$, and decrease to zero as $t \to \infty$ for each fixed $s$. $L^1(\mathbb{R}_+; \mathbb{R}_+)$ denotes the family of all Borel measurable functions $l : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_{0}^{\infty} l(s)ds < \infty$. $\Psi(\mathbb{R}_+; \mathbb{R}_+)$ represents the family of function $\sigma(t)$ satisfying that $\sum_{k=1}^{\infty} \int_{t_k}^{t_k+\delta} \sigma(s) ds = \infty$ for any $\delta > 0$ and any increasing sequence $\{t_k\}_{k \geq 1}$ ($t_k \to \infty$ as $k \to \infty$).

2. Mathematical preliminaries

In this section, the LaSalle-type theorem for stochastic time-varying systems is presented.

Consider the following stochastic nonlinear system

\[
\begin{align*}
\dot{x}(t) &= f(x(t), t) dt + g^T(x(t), t) dB(t), \\
x(0) &= x_0 \in \mathbb{R}^n,
\end{align*}
\]

(1)

where $x(t) \in \mathbb{R}^n$ denotes the state vector, $x_0$ is a deterministic initial value, $B(t)$ is an $m$-dimensional standard Wiener process defined on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ with $\Omega$ being a sample space, $\mathcal{F}$ being a $\sigma$-field, $\{\mathcal{F}_t\}_{t \geq 0}$ being a filtration, and $P$ being a probability measure. Functions $f(\cdot) \in \mathbb{R}^n$ and $g(\cdot) \in \mathbb{R}^{m \times n}$ satisfy the following assumption.

Assumption 2.1 (Zhao & Deng, 2015): The generalised local Lipschitz condition. For all $K > 0$, there exists a continuous function $L(K, t) > 0$ such that, $\forall \ t \geq 0$ and $\forall \ |x| \sqrt{|y|} \leq K$,

\[|f(x, t) - f(y, t)| + |g(x, t) - g(y, t)| \leq L(K, t)|x - y|.
\]

Let us give an example to show the necessity of introducing LaSalle-type theorem for stochastic time-varying systems.

A motivating example: Consider a simple one-dimensional stochastic system

\[
\begin{align*}
\begin{cases}
\dot{x}(t) &= -4 \cos^2(t)x^3(t) dt + 2 \cos(t)x^2(t) dB(t), \\
x(0) &= x_0 \in \mathbb{R}^n.
\end{cases}
\end{align*}
\]

(2)

By choosing the Lyapunov function $V(x) = \frac{1}{2}x^2$, from (2) one has

\[
\mathcal{L}V(x(t)) = -4 \cos^2(t)x^4(t) + \cos^2(t)x^4(t) = -3 \cos^2(t)x^4(t).
\]

(3)

From the fact that system (2) and function $-3 \cos^2(t)x^4(t)$ are time varying, it follows that only by means of (3) and Lemma 1 in Yu et al. (2010), one cannot obtain the global asymptotic stability in probability of system (2). On the other hand, although Zhao and Deng (2015) proposed a
new type of stability theorem for stochastic time-varying systems, many functions do not belong to the family of weakly positive definite functions defined in Zhao and Deng (2015), for example, \( \cos^2(t) \), \( \sin^2(t) \) and \( \lvert \cos(t) \rvert \), etc. From the viewpoint of control theory, it is interesting and necessary to extend the conclusion of Theorem 1 in Zhao and Deng (2015) for more general class of stochastic time-varying systems. The main aim of this section is to give a very positive answer.

**Definition 2.1** (Yu & Xie, 2010): A stochastic process \( \{\xi(t)\}_{t \geq 0} \) is said to be bounded almost surely if \( \sup_{t \geq 0} \lvert \xi(t) \rvert < \infty \).

**Definition 2.2** (Deng et al., 2001): The equilibrium \( x = 0 \) of system (1) is said to be

- globally stable in probability if \( \forall \epsilon \), there exists a class \( K \) function \( \gamma(\cdot) \) such that
  \[
  P(|x(t)| \leq \gamma(|x(0)|)) \geq 1 - \epsilon, \quad \forall t \geq 0, \quad \forall x(0) \in \mathbb{R}^n \setminus \{0\};
  \]
- globally asymptotically stable in probability if it is globally stable in probability and
  \[
  P\left( \lim_{t \to \infty} |x(t)| = 0 \right) = 1.
  \]

**Definition 2.3**: A time-varying function \( W(x, t) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+ \) is said to be generalised weakly positive definite, if for any \( \delta > 0 \) and any increasing sequence \( \{t_k\}_{k \geq 1} \) (\( t_k \to \infty \) as \( k \to \infty \)), there holds \( \sum_{k=1}^{\infty} \int_{t_k}^{t_k+\delta} W_k(s)ds = \infty \), where \( W_k(t) = \inf_{|x| \leq l} W(t, x) \).

**Remark 2.1**: Obviously, the generalised weakly positive definite function covers that defined in Zhao and Deng (2015). For the special case \( W(x, t) = \sigma(t)\varphi(x) \), where \( \varphi(\cdot) \) is positive definite, the generalised weakly positive definiteness of \( W(t, x) \) can be described by \( \sigma(t) \in \Psi(\mathbb{R}_+; \mathbb{R}_+) \).

We now state the following LaSalle-type theorem.

**Lemma 2.1** (LaSalle-type Theorem): Consider the stochastic system (1). Assume that there are functions \( V(x(t), t) \in \mathcal{C}^{1,1}, \varrho, \bar{\varrho} \in \mathcal{K}_\infty, l \in \mathcal{L}^1(\mathbb{R}_+; \mathbb{R}_+), \rho \in \Psi(\mathbb{R}_+; \mathbb{R}_+) \), continuous nonnegative functions \( W(x(t), t) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+ \) and \( w(x) : \mathbb{R}^n \to \mathbb{R}_+ \) such that

\[
\varrho(|x(t)|) \leq V(x(t), t) \leq \bar{\varrho}(|x(t)|), \quad (4)
\]
\[
\mathcal{L}V(x(t), t) \leq l(t) - W(x(t), t), \quad (5)
\]

Then

(i) There exists a unique and bounded almost surely strong solution to system (1) in \([0, \infty)\);

(ii) When \( W(x, t) = \rho(t)w(x) \), \( P(\lim_{t \to \infty} d(x(t), Ker(\omega)) = 0) = 1 \), where \( Ker(\omega) = \{x \in \mathbb{R}^n : \omega(x) = 0\} \);

(iii) When \( W(x(t), t) \) is generalised weakly positive definite, \( f(0, t) = 0, g(0, t) = 0, \) and \( l(t) = 0 \), the equilibrium \( x = 0 \) is globally asymptotically stable in probability.

**Proof**: For any \( r > 0 \), define the stopping time as \( \zeta_r = \inf\{t \geq 0 : |x(t)| \geq r\} \) with \( \inf \phi = \infty \). By Dynkin’s formula, one has

\[
EV(x(t \wedge \zeta_r), t \wedge \zeta_r) = V(x(0)) + E\left( \int_0^{t \wedge \zeta_r} \mathcal{L}V(x(s), s)ds \right) \leq V(x(0)) + E\int_0^{t \wedge \zeta_r} l(s)ds - E\int_0^{t \wedge \zeta_r} W(x(s), s)ds \leq V(x(0)) + \int_0^\infty l(s)ds - \int_0^\infty W(x(s), s)ds.
\]

(6)

Thus, there exists a constant \( M > 0 \) such that \( EV(x(t \wedge \zeta_r), t \wedge \zeta_r) \leq M \) for \( \forall r, t \geq 0 \), which leads to the first conclusion i) by Lemma 2 in Yu and Xie (2010).

Next, we give a proof of conclusion ii). When \( W(x, t) = \rho(t)w(x) \), from (6), by letting \( r \to \infty, t \to \infty \) and applying Fatou’s lemma, one gets

\[
E\int_0^\infty \rho(s)w(x(s))ds \leq V(x(0)) + \int_0^\infty l(s)ds < \infty.
\]

(7)

Based on the proof of Theorem 2.1 in Li and Mao (2012), it follows from (7) that

\[
P\left( \lim_{t \to \infty} \omega(x(t)) = 0 \right) = 1.
\]

Observe from conclusion i) that there is an \( \Omega_0 \subseteq \Omega \) with \( P(\Omega_0) = 1 \), such that

\[
\lim_{t \to \infty} \omega(x(t, \omega)) = 0, \quad \text{and} \quad \sup_{t \geq 0} |x(t, \omega)| < \infty \quad \text{for} \quad \forall \omega \in \Omega_0.
\]

Fix any \( \omega_0 \in \Omega_0 \). Then \( \{x(t, \omega_0)\}_{t \geq 0} \) is bounded, so there is an increasing sequence \( \{t_i\}_{i \geq 1} \) such that \( \{x(t_i, \omega_0)\}_{i \geq 1} \) converges, i.e.

\[
\lim_{i \to \infty} x(t_i, \omega_0) = y.
\]
Since $\omega$ is a continuous function, we have
\[
\lim_{t \to \infty} \omega(x(t, \sigma_0)) = \omega(y) = 0,
\]
from which we obtain that $y \in \text{Ker}(\omega)$ and $\text{Ker}(\omega)$ contains all of the limit points of $x(t, \sigma_0)$. By the continuity of $x(t; \sigma_0)$, we arrive at
\[
\lim_{t \to \infty} d(x(t, \sigma_0), \text{Ker}(\omega)) = 0,
\]
which, together with $P(\Omega_0) = 1$, implies conclusion ii).

It follows from (4)–(5) and $l(t) = 0$ that the equilibrium $x = 0$ is globally stable in probability by Theorem 2.1 in Deng et al. (2001).

Next we show that
\[
P\left( \lim_{t \to \infty} |x(t)| = 0 \right) = 1.
\]
To this end, we first prove that
\[
P\left( \lim_{t \to \infty} V(x(t), t) = 0 \right) = 1.
\]
If this is not true, then we can find a pair of positive constants $\kappa, \varepsilon$ such that
\[
P(\Omega_1) \geq \varepsilon,
\]
where $\Omega_1 = \{\sigma \in \Omega : \lim \inf_{t \to \infty} V(x(t, \sigma), t) \geq \kappa\}$.

According to the continuity of $V(x(t), t)$ and the generalised weak positive definiteness of $W(x(t), t)$, we know that there exists an instant $t^* > 0$ such that, $\forall t > t^*$,
\[
V(x(t, \sigma), t) \geq \kappa, \quad W(x(t, \sigma), t) \geq 0, \forall \sigma \in \Omega_1.
\]

Then, by (4), we obtain
\[
|x(t, \sigma)| \geq \bar{\alpha}^{-1}(V(x(t, \sigma), t)) \geq \bar{\alpha}^{-1}(\kappa) \triangleq L, \quad \forall t > t^*, \forall \sigma \in \Omega_1.
\]

Then, by (6), (8) and the weakly positive definiteness of $W(x(t), t)$, by choosing $t_k = t^* + k - 1$ and $\delta = 1$, we get
\[
V(x(0)) \geq \int_0^\infty W(x(s), s)ds = \int_0^t W(x(s), s)ds + \int_t^\infty W(x(s), s)ds \geq \int_0^t W(x(s), s)ds + \sum_{k=1}^{\infty} \int_{t^*+k-1}^{t^*+k} W_x(s)ds = \infty, \quad \forall \sigma \in \Omega_1.
\]

This leads to a contradiction, thus it follows that
\[
P\left( \lim_{t \to \infty} V(x(t), t) = 0 \right) = 1.
\]

By combining this with $\varphi(|x(t)|) \leq V(x(t), t)$, we must have
\[
P\left( \lim_{t \to \infty} |x(t)| = 0 \right) = 1.
\]

Hence, by Definition 2.2, the equilibrium $x = 0$ is globally asymptotically stable in probability.

Remark 2.2: Given that the LaSalle theorems are powerful in the study of stability for locating limit sets of time-invariant systems, here we have presented a new LaSalle-type theorem for time-varying systems, which not only can be applied much more easily but also can cover a much wider class of stochastic time-varying systems. To see this point, let us return our attention to the motivating example (2). It is easy to see that $\cos^2(t) \in \Psi([0, \infty))$. By Lemma 2.1, we can conclude that the equilibrium $x = 0$ of system (2) is globally asymptotically stable in probability. Figure 1 shows the simulation result of the system state with $x(0) = 2$, which also implies the effectiveness of the proposed result in Lemma 2.1.

3. Control design and stability analysis with an unknown growth rate

3.1 Problem formulation

In this paper, consider the following stochastic feedforward nonlinear system
\[
\begin{aligned}
\dot{x}_i &= x_{i+1} + f_i(x, u, t) dt + g_i^T(x, u, t) dB(t), \\
\dot{x}_n &= u dt + f_n(u, t) dt + g_n^T(u, t) dB(t),
\end{aligned}
\]
where $x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$ denotes the measurable state vector and $u \in \mathbb{R}$ is the control input. $B(t)$ is an $m$-dimensional standard Wiener process defined on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ with $\Omega$ being a sample space, $\mathcal{F}$ being a $\sigma$-field, $\{\mathcal{F}_t\}_{t \geq 0}$ being a filtration, and $P$ being a probability measure. $f_i(\cdot) \in \mathbb{R}^n$, $g_i(\cdot) \in \mathbb{R}^{m \times n}$ are generalised locally Lipschitz functions and satisfy $f_i(0, t) = 0, g_i(0, t) = 0$.

For system (9), the following assumption is needed:
Figure 1. Response of the motivating system state.

Assumption 3.1: There is an unknown constant \( \theta > 0 \), such that

\[
|f_i(x, t)| \leq \theta \left( \sum_{k=i+2}^{n+1} |x_k(t)| + |u| \right),
\]

\[
|g_i(x, t)| \leq \theta \left( \sum_{k=i+2}^{n+1} |x_k(t)| + |u| \right), \quad i = 1, \ldots, n - 1,
\]

\[
|f_n(u, t)| \leq \theta |u|,
\]

\[
|g_n(u, t)| \leq \theta |u|,
\]

where \( x_{n+1}(t) = 0 \).

Remark 3.1: It follows from Assumption 3.1 that system (9) is indeed a stochastic feedforward nonlinear system possessing linear state-dependent growth with unknown growth rates. The system considered in the paper is more common than some existing ones, for instance, the diffusion terms in Du, Qian, He, and Cheng (2013), Du et al. (2013), Zhang, Baron, et al. (2011), Ye and Unbehauen (2004) are identically equal to zero. In addition, even though the nonlinearities of systems in Frye et al. (2007), Zhao and Xie (2014), Jiao et al. (2014), Liu and Xie (2013) are stronger than that of system (9), the growth rates in Assumption 3.1 of Frye et al. (2007) and Assumption 1 of Zhao and Xie (2014), Liu and Xie (2013), Jiao et al. (2014) are all limited to known constants. It is worth noting that the growth rate is unknown in our work, which indicates the presence of completely unknown parameters and implies the essential difference from those in Frye et al. (2007), Du et al. (2013), Zhao and Xie (2014), Liu and Xie (2013), Jiao et al. (2014).

This paper aims to find a time-varying state-feedback control \( u = \mu(x, t) \) such that the resulting closed-loop system has a unique and almost surely bounded strong solution \( x(t) \) in \([0, \infty)\), and \( P(\lim_{t \to \infty} |x(t)| = 0) = 1 \).

3.2 Time-varying state-feedback controller

In what follows, we present a time-varying state-feedback controller design that globally stabilises system (9), which consists of the following two parts.

3.2.1 Transformation of coordinates

First, let us introduce the following time-scaling coordinate change to transform system (9) into a system with a time gain in the nonlinearities:

\[
\eta_i = x_i(t + b)^{i-1}, \quad v = u(t + b)^n, \quad i = 1, \ldots, n,
\]

where \( b > 0 \) is a constant.

Hence, by (10), system (9) can be expressed as

\[
\begin{align*}
\dot{\eta}_i &= \frac{\eta_{i+1}}{t + b} dt + \frac{i - 1}{t + b} \eta_i dt + \tilde{f}_i(x, u, t) dt \\
&\quad + \tilde{g}_i^T(x, u, t) dB(t), \quad i = 1, \ldots, n - 1, \\
\dot{\eta}_n &= \frac{v}{t + b} dt + \frac{n - 1}{t + b} \eta_n dt + \tilde{f}_n(u, t) dt \\
&\quad + \tilde{g}_n^T(u, t) dB(t),
\end{align*}
\]

where \( x_{n+1}(t) = 0 \).
where

\[
|\tilde{f}(x, t)| = |\tilde{f}(x, t)(t + b)^{i-1}|
\leq \theta(t + b)^{i-1} \left( \sum_{k=i+2}^{n+1} |x_k(t)| + |u| \right)
= \theta(t + b)^{i-1} \left( \sum_{k=i+2}^{n+1} \frac{\eta_k}{(t + b)^{i+1}} + \frac{v}{(t + b)^{i+1}} \right)
\leq \frac{\theta}{(t + b)^2} \left( \sum_{k=i+2}^{n+1} |\eta_k| + |v| \right),
\]

|\tilde{g}(x, t)| = |\tilde{g}(x, t)(t + b)^{i-1}|
\leq \frac{\theta}{(t + b)^2} \left( \sum_{k=i+2}^{n+1} |\eta_k| + |v| \right),

|\tilde{f}(u, t)| \leq \frac{\theta}{(t + b)^2} |v|,

|\tilde{g}(u, t)| \leq \frac{\theta}{(t + b)^2} |v|,

and \( \eta_{n+1}(t) = 0 \).

### 3.2.2 State-feedback controller design of the nominal system of system (11)

We design a linear state-feedback controller for the nominal system of the transformed system:

\[
\begin{cases}
   d\eta_i = \frac{\eta_{i+1}}{(t + b)} dt + \frac{i - 1}{t + b} \eta_i dt, & i = 1, \ldots, n - 1, \\
   d\eta_n = \frac{v}{(t + b)} dt + \frac{n - 1}{t + b} \eta_n dt.
\end{cases}
\]

**Step 1:** Define \( \varphi_1 = \eta_1, \varphi_2 = 0 \), and construct the Lyapunov function \( V_1(\varphi_1) = \frac{1}{2} \eta_1^2 \). Differentiating \( V_1 \) along the solution of (11) leads to

\[
\mathcal{L}V_1(\varphi_1) = \frac{\varphi_1 \eta_2}{t + b}.
\]

Then, choosing the first virtual controller

\[
\eta_2^* = -c_{11} \varphi_1 \triangleq -\alpha_1 \varphi_1, \quad c_{11} > 0,
\]

we have

\[
\mathcal{L}V_1(\varphi_1) \leq -\frac{c_{11}}{t + b} \varphi_1^2 + \frac{\varphi_1}{t + b} (\eta_2 - \eta_2^*).
\]  

**Step 2:** Let us consider the second Lyapunov function

\[
V_2(\varphi_1, \varphi_2) = V_1 + \frac{1}{2} \varphi_2^2 \text{ with } \varphi_2 = \eta_2 - \eta_2^*.
\]

Then, from (11) and (11), it follows that

\[
\mathcal{L}V_2(\varphi_1, \varphi_2) \leq -\frac{c_{11}}{t + b} \varphi_1^2 + \frac{\varphi_1}{t + b} \varphi_2^2 + \varphi_2 \left[ \frac{\eta_3}{t + b} + 1 \right] \frac{1 + \frac{1}{t + b} \eta_j}{j}
\]

By using Young's inequality, one has

\[
\frac{V_1 \varphi_2}{t + b} \leq \frac{l_{211}}{t + b} \varphi_1^2 + \frac{\rho_{21}}{t + b} \varphi_2^2,
\]

where \( l_{211}, l_{212} \) are positive constants.

Substituting (13) and (14) into (12) yields

\[
\mathcal{L}V_2(\varphi_1, \varphi_2) \leq -\frac{c_{11}}{t + b} \varphi_1^2 + \frac{\varphi_1}{t + b} \varphi_2^2 + \frac{l_{211}}{t + b} \varphi_1^2 + \frac{\rho_{22}}{t + b} \varphi_2^2.
\]

With the choice of the virtual controller

\[
\eta_3^* = -(c_{22} + \rho_{21} + \rho_{22}) \varphi_2 \triangleq -\alpha_2 \varphi_2, \quad \alpha_2 > 0,
\]

we arrive at

\[
\mathcal{L}V_2(\varphi_1, \varphi_2) \leq -\sum_{j=1}^{k-1} \frac{c_{2j}}{t + b} \varphi_j^2 + \frac{\varphi_k}{t + b} (\eta_3 - \eta_k^*),
\]

where \( c_{21} = c_{11} - l_{211} - l_{212} > 0 \).

**Recursive Step k (k = 3, \cdots, n):** Suppose that steps 1, \ldots, \( k - 1 \) have been completed and there is a \( C^2 \), positive definite and proper Lyapunov function \( V_{k-1}(\varphi_1, \cdots, \varphi_{k-1}) \) satisfying

\[
\mathcal{L}V_{k-1}(\varphi_1, \cdots, \varphi_{k-1}) \leq -\sum_{j=1}^{k-1} \frac{c_{2j}}{t + b} \varphi_j^2 + \frac{\varphi_k}{t + b} (\eta_k - \eta_k^*),
\]

(17)

where

\[
\eta_j^* = -\alpha_{j-1} \varphi_{j-1}, \quad \varphi_j = \eta_j - \eta_j^*,
\]

\( \alpha_0 = 0, \quad \alpha_j > 0, \quad c_{k-1,j} > 0, \quad j = 1, \ldots, k. \)
Next we show that a similar conclusion still holds true for Step $k$. Let $\psi_k = \eta_k - \eta^*_k$ and then define the $k$-th Lyapunov function $V_k = V_{k-1} + \frac{1}{2}\psi_k^2$. It follows from (11) and (17) that

$$\mathcal{L}V_k(\varphi_1, \ldots, \varphi_k) \leq -\sum_{j=1}^{k-1} \frac{c_{k-1,j}}{t+b} \psi_j^2 + \frac{\psi_k}{t+b} \left[ \eta_{k+1} + \frac{1}{t+b} \eta_k \right] + \frac{1}{t+b} \psi_k \left[ \eta_{j+1} + \frac{j-1}{t+b} \eta_j \right].$$

With the aid of Young’s inequality, we have

$$\frac{1}{t+b} \left| \varphi_k \eta_k - \varphi_k \sum_{j=1}^{k-1} \frac{\partial \eta_{k}^*}{\partial \eta_{j}} \left[ \eta_{j+1} + (j-1)\eta_j \right] \right| \leq \frac{1}{t+b} \left| \varphi_k (\varphi_k - \alpha_{k-1} \varphi_{k-1}) \right| + \left| \varphi_k \sum_{j=1}^{k-1} \alpha_{k-1} \cdots \alpha_{j} (\varphi_{j+1} - \varphi_j) \right| + (j-1)(\varphi_j - \alpha_{j-1} \varphi_{j-1}) \right| \leq \frac{1}{t+b} \sum_{j=1}^{k-1} l_{j2} \psi_j^2 + \frac{\rho_{k2}}{t+b} \psi_k^2.$$

where $l_{k1} > 0$, $l_{j2} > 0$ are constants.

Combining (18) and (19) with (20) results in

$$\mathcal{L}V_k(\varphi_1, \ldots, \varphi_k) \leq -\sum_{j=1}^{k-1} \frac{c_{k-1,j}}{t+b} \psi_j^2 + \frac{l_{k1}}{t+b} \psi_k^2 + \frac{\rho_{k1}}{t+b} \psi_k^2 + \psi_k \eta_{k+1} + \frac{l_{j2}}{t+b} \psi_j^2 + \frac{\rho_{k2}}{t+b} \psi_k^2.$$

By choosing the following virtual controller

$$\eta^*_{k+1} = -(c_{kk} + \rho_{k1} + \rho_{k2}) \varphi_k \triangleq -\alpha_k \varphi_k,$$

and defining

$$c_{kj} = \begin{cases} c_{k-1,j} - l_{j2} > 0, & j = 1, \ldots, k-2, \\ c_{k-1,j} - l_{k1} - l_{j2} > 0, & j = k-1, \\ \epsilon_j > 0, & j = k, \end{cases}$$

we have

$$\mathcal{L}V_k(\varphi_1, \ldots, \varphi_k) \leq -\sum_{j=1}^{k} \frac{c_{k,j}}{t+b} \psi_j^2 + \frac{\psi_k}{t+b} (\eta_{k+1} - \eta^*_{k+1}).$$

Hence, from the last step of the above design procedure, we obtain the actual controller

$$v = -\alpha_n \varphi_n = -\sum_{i=1}^{n} \bar{\alpha}_i \eta_i,$$

which leads to

$$\mathcal{L}V_n(\varphi_1, \ldots, \varphi_n) \leq -\sum_{j=1}^{n} \frac{c_{n,j}}{t+b} \psi_j^2,$$

where $V_n(\varphi_1, \ldots, \varphi_n) = \sum_{i=1}^{n} \frac{1}{2}\psi_k^2$, and $\bar{\alpha}_i = \alpha_n \cdots \alpha_i$ and $c_{n,j}$ are positive constants.

### 3.3 Stability analysis

Now we are ready to state the main result.

**Theorem 3.1:** Under Assumption 3.1, there is a global unique solution for the closed-loop system consisting of the system (9) and the time-varying state-feedback controller

$$u = \frac{v}{(t+b)^2}.$$

Moreover, $P(\lim_{t \to \infty} x(t) = 0) = 1$.

**Proof:** The proof can be divided into the following two steps.

**Step 1:** We first prove that the closed-loop system consisting of (11) and (22) has a global unique solution, which converges to zero almost surely.

For illustration convenience, the closed-loop system (11) and (22) is written in the following compact form as

$$d\eta = E(\eta)dt + \tilde{F}(\eta, t)dt + \tilde{G}^T(\eta, t)dB(t),$$

where

$$E(\eta) = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_n \end{bmatrix}, \quad \eta = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_n \\ \eta_n \end{bmatrix}, \quad \tilde{F}(\eta, t) = \begin{bmatrix} \tilde{f}_1(\eta, t) \\ \vdots \\ \tilde{f}_{n-1}(\eta, t) \\ \tilde{f}_n(\eta, t) \end{bmatrix}, \quad \tilde{G}(\eta, t) = \begin{bmatrix} \tilde{g}_1(\eta, t) \\ \vdots \\ \tilde{g}_{n-1}(\eta, t) \\ \tilde{g}_n(\eta, t) \end{bmatrix}.$$
Obviously, the controller v defined in (22) is $C^1$. On the other hand, the nonlinear functions $f_j$ and $\tilde{g}_i$ are generalised locally Lipschitz. Therefore, the generalised locally Lipschitz condition of the closed-loop system (24) can be guaranteed.

For system (24), by constructing the Lyapunov function $V(\varphi) = V_n(\varphi_1, \ldots, \varphi_n)$, it follows from (23) that

$$LV(\varphi) \leq -\sum_{j=1}^n \frac{c_{n,j}}{t+b} \varphi_j^2 + \frac{\partial V(\varphi)}{\partial \eta} \tilde{F}(\eta, t) + \frac{1}{2} \operatorname{Tr} \left\{ \tilde{G}(\eta, t) \frac{\partial^2 V(\varphi)}{\partial \eta^2} \tilde{G}^T(\eta, t) \right\},$$  \hspace{1cm} (25)

where $\varphi = [\varphi_1, \ldots, \varphi_n]^T$.

From the definitions of $V(\varphi)$ and $\tilde{F}(\eta, t)$, we have

$$\left| \frac{\partial V(\varphi)}{\partial \eta} \tilde{F}(\eta, t) \right| = \sum_{j=1}^n \left| \frac{\partial V(\varphi)}{\partial \eta_j} f_j(\eta, t) \right|$$

$$\leq \frac{\theta}{(t+b)^2} \sum_{j=1}^n \left( |\varphi_j| + \sum_{k=j+1}^n \alpha_k \cdots \alpha_j |\varphi_k| \right)$$

$$\times \left( \sum_{j=1}^n |\varphi_j - \alpha_{j-1}| \varphi_{j-1} + \alpha_n |\varphi_n| \right)$$

$$\leq \frac{c_{02}}{(t+b)^2} \sum_{j=1}^n \varphi_j^2,$$  \hspace{1cm} (26)

where $c_{02} > 0$ is unknown.

Similarly, there exist $b_1 > 0$, $b_2 > 0$ and an unknown positive constant $c_{03}$ such that

$$\left| \frac{1}{2} \operatorname{Tr} \left\{ \tilde{G}(\eta, t) \frac{\partial^2 V(\varphi)}{\partial \eta^2} \tilde{G}^T(\eta, t) \right\} \right|$$

$$\leq b_1 \sum_{i,j=1}^n \left| \frac{\partial^2 V(\varphi)}{\partial \eta_i \partial \eta_j} \tilde{g}_i(\eta, t) |\tilde{g}_j(\eta, t)| \right|$$

$$\leq b_2 \sum_{i=1}^n \left( \frac{\theta}{(t+b)^2} \left( \sum_{k=i+1}^{n+1} \eta_k + \alpha_n |\varphi_n| \right) \right)^2$$

$$\leq \frac{c_{03}}{(t+b)^4} \sum_{j=1}^n \varphi_j^2.$$  \hspace{1cm} (27)

With the help of (26) and (27), one has

$$LV(\varphi) \leq -\sum_{j=1}^n \frac{c_{n,j}}{t+b} \varphi_j^2 + \frac{c_{02}}{(t+b)^2} \sum_{j=1}^n \varphi_j^2 + \frac{c_{03}}{(t+b)^4} \sum_{j=1}^n \varphi_j^2$$

$$\leq -\sum_{j=1}^n \frac{c_{n,j}}{t+b} \varphi_j^2 + \frac{c_{02}}{(t+b)^2} \sum_{j=1}^n \varphi_j^2 + \frac{c_{03}}{(t+b)^4} \sum_{j=1}^n \varphi_j^2$$

$$= -\frac{c_{01}}{t+b} V(\varphi) + \left( -\frac{c_{01}}{t+b} + \frac{2c_{02}}{(t+b)^2} + \frac{2c_{03}}{(t+b)^4} \right) V(\varphi),$$

where $c_{01} = \min_{j=1, \ldots, n} \{ c_{n,j} \}$.

From (28), we obtain

$$LV(\varphi) \leq (2c_{02} + 2c_{03}) V(\varphi),$$  \hspace{1cm} (29)

which implies the existence of global solution for system (24) by Theorem 3.5 in Khasminskii (2011).

We then prove $\lim_{t \to \infty} |\eta(t)| = 0$ a.s. By (28), there is a sufficiently large $T^* > 0$ such that

$$LV(\varphi) \leq -\frac{c_{01}}{t+b} V(\varphi), \forall t > T^*.$$  \hspace{1cm} (30)

On the other hand,

$$\int_0^\infty \frac{c_{01}}{t+b} \, dt = \infty,$$

from which it yields that $\frac{c_{01}}{t+b} \in \Psi(\mathbb{R}_+; \mathbb{R}_+)$. 

Combining this with the definition of $V(\varphi)$ and Lemma 2.1 produces

$$P\left( \lim_{t \to \infty} |\varphi(t)| = 0 \right) = 1.$$  \hspace{1cm} (31)

Noting that $\eta_k = \varphi_k - \alpha_k \cdots \alpha_1 \varphi_1, k = 2, \ldots, n$, we derive that

$$P\left( \lim_{t \to \infty} |\eta(t)| = 0 \right) = 1.$$  \hspace{1cm} (32)

**Step 2:** From (10), we see that the coordinate transformation is equivalent. Hence, the closed-loop system (9) with $u = \frac{v}{(t+b)^\alpha}$ has the same properties as systems (11) and (22). Therefore, Theorem 3.1 holds true. 

**Remark 3.2:** Compared with the work by Liu and Xie (2013) and Zhao and Xie (2014), we have only got the almost sure convergence of the closed-loop system, rather than the asymptotic stability. In fact, since $c_{01}$, $c_{02}$ and $c_{03}$ in (28) are dependent on the unknown parameter $\theta$, the right-hand side of (28) may be positive during a small initial time interval. Hence, the asymptotic stability cannot be deduced for the closed-loop system from (28).
**Remark 3.3:** We would like to point out that although there are fruitful works on the deterministic feedforward nonlinear systems with unknown growth rates (see, e.g. Shang et al., 2015; Zhang, Baron et al., 2011; Zhang, Liu et al., 2011), those proposed results by using the dynamic gain control approach cannot be extended to system (9) with an unknown growth rate. The main reason lies in the fact that to now, there has been no effective proof method guaranteeing the monotonic property and the almost sure boundedness of the dynamic gain in the context of stochastic cases. Due to the incapability of the existing time-invariant and adaptive methods, in this paper, we have developed a new time-varying scheme to globally stabilise system (9).

### 4. Simulation studies

In this section, two examples are presented to validate the proposed results.

**Example 4.1:** Consider a stochastic time-varying feedforward system described by

\[
\begin{align*}
\dot{x}_1 &= x_2 + a_1(1 + \sin(t))x_3 + a_2 \cos(t)x_3 \, dB(t), \\
\dot{x}_2 &= x_3, \\
\dot{x}_3 &= u \, dt,
\end{align*}
\]

where \(a_1\) and \(a_2\) are unknown constants. It is easy to check that Assumption 3.1 is satisfied.

By the design method proposed in Section 3.2, the controller design procedure for system (31) is in detail presented in the following.

**Coordinate change:** We introduce the following time-scaling coordinate change:

\[
\begin{align*}
\eta_1 &= x_1, \\
\eta_1 &= x_2(t + b), \\
\eta_3 &= x_3(t + b)^2, \\
v &= u(t + b)^3,
\end{align*}
\]

where \(b > 0\) is a designed constant. Then, system (31) becomes

\[
\begin{align*}
\dot{\eta}_1 &= \frac{\eta_2}{t + b} dt + a_1(1 + \sin(t))x_3 dt + a_2 \cos(t)x_3 dB(t), \\
\dot{\eta}_2 &= \frac{\eta_3}{t + b} dt + \frac{\eta_2}{t + b} dt, \\
\dot{\eta}_3 &= \frac{\eta_3}{t + b} dt + \frac{2\eta_3}{t + b} dt.
\end{align*}
\]

**Controller design:** We first consider the following subsystem:

\[
\begin{align*}
d\eta_1 &= \frac{\eta_2}{t + b} dt, \\
d\eta_2 &= \frac{\eta_3}{t + b} dt + \frac{\eta_2}{t + b} dt, \\
d\eta_3 &= \frac{\eta_3}{t + b} dt + \frac{2\eta_3}{t + b} dt. \\
\end{align*}
\]

To begin with, let \(\varphi_1, \varphi_1^* = 0\), and \(V_1(\varphi_1) = \frac{1}{2} \eta_1^2\). Then,

\[
\mathcal{L}V_1(\varphi_1) = \frac{\varphi_1 \eta_2}{t + b}.
\]

By defining

\[
\eta_2^* = -c_{11} \varphi_1 \triangleq -\alpha_1 \varphi_1, \quad c_{11} > 0,
\]

we get

\[
\mathcal{L}V_1(\varphi_1) \leq -\frac{c_{11}}{t + b} \varphi_1^2 + \frac{\varphi_1}{t + b} (\eta_2 - \eta_2^*).
\]

Consider \(V_2(\varphi_1, \varphi_2) = V_1 + \frac{1}{2} \varphi_2^2\) with \(\varphi_2 = \eta_2 - \eta_2^*\).

It follows from (33) and (34) that

\[
\mathcal{L}V_2(\varphi_1, \varphi_2) \leq -\frac{c_{11}}{t + b} \varphi_1^2 + \frac{\varphi_1}{t + b} \varphi_2
\]

\[
+ \varphi_2 \left[ -\frac{\eta_3}{t + b} + \frac{1}{t + b} \eta_2 \right]
\]

\[
- \varphi_2 \sum_{j=1}^{1} \frac{1}{\eta_j} \eta_{j+1} + \frac{j - 1}{t + b} \eta_j \right].
\]

By using Young’s inequality, we have

\[
\begin{align*}
\left| \varphi_1 \varphi_2 \right| \leq & \frac{l_{211}}{t + b} \varphi_1^2 + \frac{1}{4l_{211}(t + b)} \varphi_2^2 \\
\leq & \frac{l_{211}}{t + b} \varphi_1^2 + \frac{\rho_{21}}{t + b} \varphi_2^2,
\end{align*}
\]

\[
\begin{align*}
\frac{1}{t + b} \varphi_2 \eta_2 - \varphi_2 \sum_{j=1}^{1} \frac{1}{\eta_j} \eta_{j+1} \right| \\
\leq & \frac{1 + \alpha_1}{t + b} (\varphi_2^2 + \alpha_1 \varphi_3 \varphi_2) \\
\leq & \frac{l_{212}}{t + b} \varphi_1^2 + \frac{(\alpha_1 + \alpha_1^2)}{4l_{212}(t + b)} \varphi_2^2 + \frac{1 + \alpha_1}{t + b} \varphi_2^2 \\
\leq & \frac{l_{212}}{t + b} \varphi_1^2 + \frac{\rho_{22}}{t + b} \varphi_2^2,
\end{align*}
\]

where \(l_{211} > 0, l_{212} > 0\) are constants.
Substituting (36) and (37) into (35) leads to

\[
\mathcal{L}V_2(\varphi_1, \varphi_2) \leq -\frac{c_{11} - l_{311} - l_{312}}{t + b} \varphi_1^2 + \frac{1}{t + b} \left( \eta_3 + (\rho_{21} + \rho_{22}) \varphi_2 \right) \varphi_2. \\
(38)
\]

By choosing

\[
\eta_3^* = -(c_{22} + \rho_{21} + \rho_{22}) \varphi_2 \triangleq -\alpha_2 \varphi_2, \quad c_{22} > 0,
\]

it yields

\[
\mathcal{L}V_2(\varphi_1, \varphi_2) \leq -\sum_{j=1}^{2} \frac{c_{2j}}{t + b} \varphi_j^2 + \frac{\varphi_2}{t + b} (\eta_3 - \eta_3^*),
\]

where \( c_{21} = c_{11} - l_{311} - l_{312} > 0. \)

Define \( \varphi_3 = \eta_3 - \eta_3^* \) and \( V_3(\varphi_1, \varphi_2, \varphi_3) = V_2 + \frac{1}{3} \varphi_3^2. \)

From (33) and (34), we have

\[
\mathcal{L}V_3(\varphi_1, \varphi_2, \varphi_3) \leq -\sum_{j=1}^{2} \frac{c_{3j}}{t + b} \varphi_j^2 + \frac{\varphi_2}{t + b} (\eta_3 - \eta_3^*) + \varphi_3 \left[ \frac{\nu}{t + b} + \frac{2}{t + b} \eta_3 \right] - \varphi_3 \sum_{j=1}^{2} \frac{\partial \eta_3^*}{\partial \eta_j} \left[ \frac{\eta_{j+1}}{t + b} + \frac{j - 1}{t + b} \eta_j \right].
\]

Thus, by choosing

\[
\nu = -(c_{33} + \rho_{31} + \rho_{32}) \varphi_3 \\
\triangleq -\alpha_3 \varphi_3 \\
= -\left( \alpha_3 \eta_3 + \alpha_3 \alpha_2 \eta_2 + \alpha_3 \alpha_2 \alpha_1 \eta_1 \right),
\]

\( c_{33} > 0, \)

(40) becomes

\[
\mathcal{L}V_3(\varphi_1, \varphi_2, \varphi_3) \leq -\sum_{j=1}^{3} \frac{c_{3j}}{t + b} \varphi_j^2,
\]

where \( c_{31} = c_{21} - l_{311} > 0, c_{32} = c_{22} - l_{312} - l_{322} > 0. \)

Finally, from (32), it is concluded that system (31) can be globally stabilised by the time-varying state-feedback control

\[
u = -(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3).
\]

For simulation purposes, the parameters are chosen as \( a_1 = -12, a_2 = -6, b = 6, c_{11} = 0.5, c_{22} = 0.7, c_{33} = 0.8, l_{311} = l_{312} = l_{311} = 0.1, l_{312} = l_{322} = 0.3, \) the sampling period 0.001 s and the initial conditions \( x_{10} = 0.2, x_{20} = -3.5, x_{30} = 0.5. \) Simulation results plotted in Figures 2 and 3 show the usefulness of the obtained results.

Example 4.2: Consider an induction heater circuit system (Landers, 1987) (see Figure 4). It is known that the input voltage \( V \) may change due to the external noises. That is, as proposed in Ugrinovskii and Petersen (1999), one may suppose that the input voltage \( V \) is given by \( V = V_1 + aV_1B(t) \), where \( B(t) \) is the noise and \( a \) is an unknown constant. Thus, by defining \( x_1 = V_1, x_2 = i_L \), the state equation of the circuit system can be expressed as

\[
\begin{aligned}
\dot{x}_1 &= -\frac{1}{R_1} + \frac{d}{C} x_1 - \frac{1}{C} x_2, \\
\dot{x}_2 &= \frac{1}{L} x_1 - \frac{R_2}{C} x_2 - \frac{1}{C} (V_1 + aV_1B(t)).
\end{aligned}
\]

(41)

Then, if we choose the parameters \( L = 1H, C = 1F, d = 2A/V, R_1 = R_2 = 1\Omega \), and the unit of the current is A, the unit of the voltage is V, by the invertible transformation \( \tilde{x}_1 = x_1, \tilde{x}_2 = x_1 - x_2, u = V_1 \), system (41) is rewritten as

\[
\begin{aligned}
d\tilde{x}_1 &= \tilde{x}_2 dt, \\
d\tilde{x}_2 &= u + au dB(t).
\end{aligned}
\]

(42)

Obviously, system (42) satisfies Assumption 3.1.
Figure 2. Responses of the system states.

Figure 3. Trajectory of the controller.

Figure 4. Induction heater circuit system.
Now we are in a position to present the controller design procedure for system (42).

**Coordinate change:** Introduce the following time-scaling coordinate change:

\[
\eta_1 = \tilde{x}_1, \quad \eta_2 = \tilde{x}_2(t + b), \quad v = u(t + b)^2, \tag{43}
\]

where \(b > 0\) is a designed constant. Then, system (42) becomes

\[
\begin{aligned}
&d\eta_1 = \frac{\eta_2}{t + b} dt, \\
&d\eta_2 = \frac{\eta_2}{t + b} dt + \frac{\eta_2}{t + b} dt + \frac{av}{t + b} dB(t).
\end{aligned}
\]

**Controller design:** We first consider the following subsystem:

\[
\begin{aligned}
&d\eta_1 = \frac{\eta_2}{t + b} dt, \\
&d\eta_2 = \frac{v}{t + b} dt + \frac{\eta_2}{t + b} dt. \tag{44}
\end{aligned}
\]

Define \(\varphi_1 = \eta_1\), \(\varphi_1^* = 0\), and \(V_1(\varphi_1) = \frac{1}{2}\eta_1^2\), then we have

\[
L V_1(\varphi_1) = \frac{\varphi_1 \eta_2}{t + b}.
\]

By choosing

\[
\eta_2^* = -c_{11} \varphi_1 \triangleq -\alpha_1 \varphi_1, \quad c_{11} > 0,
\]

we get

\[
L V_1(\varphi_1) \leq -\frac{c_{11}}{t + b} \varphi_1^2 + \frac{\varphi_1}{t + b} (\eta_2 - \eta_2^*). \tag{45}
\]

Consider \(V_2(\varphi_1, \varphi_2) = V_1 + \frac{1}{2} \varphi_2^2\) with \(\varphi_2 = \eta_2 - \eta_2^*\).

From (44) and (45), it follows that

\[
L V_2(\varphi_1, \varphi_2) \leq -\frac{c_{11}}{t + b} \varphi_1^2 + \frac{\varphi_1}{t + b} \varphi_2 \\
+ \varphi_2 \left[ \frac{v}{t + b} + \frac{1}{t + b} \eta_2 \right] \\
- \varphi_2 \left[ \frac{1}{t + b} + \frac{\rho_{21}}{t + b} \eta_j \right]. \tag{46}
\]

With the aid of Young’s inequality, we have

\[
\left| \frac{\varphi_1 \varphi_2}{t + b} \right| \leq \frac{l_{11}}{t + b} \varphi_1^2 + \frac{1}{4l_{11}(t + b)} \varphi_2^2 \\
\triangleq \frac{l_{11}}{t + b} \varphi_1^2 + \frac{\rho_{21}}{t + b} \varphi_2^2, \tag{47}
\]

\[
\left| \frac{1}{t + b} \varphi_2 \eta_2 - \varphi_2 \sum_{j=1}^{J} \frac{\partial \eta_2^*}{\partial \eta_j} \right| \eta_{j+1} \leq \frac{l_{11}}{t + b} \varphi_1^2 + \frac{\rho_{22}}{t + b} \varphi_2^2, \tag{48}
\]

where \(l_{11} > 0, l_{12} > 0\) are constants.

Substituting (47) and (48) into (46) arrives at

\[
L V_2(\varphi_1, \varphi_2) \leq -\frac{c_{11} - l_{11} - l_{12}}{t + b} \varphi_1^2 \\
+ \frac{1}{t + b} (v + (\rho_{21} + \rho_{22}) \varphi_2) \varphi_2.
\]
Finally, by choosing
\[ v = -(c_{22} + \rho_{21} + \rho_{22})\varphi_2 \triangleq -\alpha_2 \varphi_2, \quad c_{22} > 0, \]
it yields
\[ \mathcal{L}V_2(\varphi_1, \varphi_2) \leq -\sum_{j=1}^{2} \frac{c_{2j}}{t+b} \varphi_j^2, \]
where \( c_{2j} = c_{j1} - l_{211} - l_{212} > 0. \)

From (43), system (41) can be globally stabilised by the time-varying state-feedback control
\[ V_1 = -\left( \frac{\bar{\alpha}_1 x_1}{(t+b)^2} + \frac{\bar{\alpha}_2 (x_1 - x_2)}{(t+b)} \right), \]
where \( \bar{\alpha}_1 = \alpha_1 \alpha_1, \bar{\alpha}_2 = \alpha_2. \)

In the simulation, the parameters are chosen as \( a = 0.5, b = 10, c_{11} = 3, c_{22} = 2, l_{211} = l_{212} = 1, \) the sampling period 0.001 s and the initial conditions \( x_{10} = 2.2 \text{ V}, x_{20} = 1.8 \text{ A}. \) Figures 5 and 6 plot the responses of the closed-loop system, which implies the validity of the proposed control strategy.

\section{5. Concluding remarks}
In this paper, the stabilisation problem has been studied for a class of stochastic feedforward nonlinear systems whose growth rate is not necessarily known. To effectively compensate the serious uncertainty, we have proposed a time-varying state-feedback controller. With the help of the new type of LaSalle-type theorem, it has been proved that the closed-loop system has a global unique solution, which converges to zero almost surely. Two examples have been provided to show the effectiveness of the proposed control scheme. Finally, our future research will seek to investigate the stochastic feedforward nonlinear system with state-dependent growth rates, which will be an interesting and challenging topic.

\section{Disclosure statement}
No potential conflict of interest was reported by the authors.

\section{Funding}
This work was supported in part by the Australian Research Council [grant number DP120104986].

\section{References}
Chen, W., Zheng, W.X., & Shen, Y. (2009). Delay-dependent stochastic stability and \( H_\infty \)-control of uncertain neutral stochastic systems with time delay. \textit{IEEE Transactions on Automatic Control}, 54(7), 1660–1667.

Chen, Y., & Zheng, W.X. (2016). Stability analysis and control for switched stochastic delay systems. \textit{International Journal of Robust and Nonlinear Control}, 26(2), 303–328.

Deng, H., Krstić, M., & Williams, R.J. (2001). Stabilization of stochastic nonlinear systems driven by noise of unknown covariance. \textit{IEEE Transactions on Automatic Control}, 46(8), 1237–1253.

Du, H., Qian, C., He, Y., & Cheng, Y. (2013). Global sampled-data output feedback stabilisation of a class of upper-triangular systems with input delay. \textit{IET Control Theory and Applications}, 7(10), 1437–1446.

Du, H., Qian, C., & Li, S. (2013). Global stabilization of a class of uncertain upper-triangular systems under sampled-data
control. *International Journal of Robust and Nonlinear Control*, 23(6), 620–637.

Frye, M.T., Trevino, R., & Qian, C. (2007). Output feedback stabilization of nonlinear feedforward systems using low gain homogeneous domination. Proceedings of the 2007 IEEE international conference on control and automation (pp. 422–427). Guangzhou, China: IEEE.

Jiao, T., Xu, S., Wei, Y., Chu, Y., & Zou, Y. (2014). Decentralized global stabilization for stochastic high-order feedforward nonlinear systems with time-varying delays. *Journal of the Franklin Institute*, 351(10), 4872–4891.

Jo, H.W., Choi, H.L., & Lim, J.T. (2014). Observer-based output feedback control. *IEEE Transactions on Industrial Electronics*, 23(6), 203–210.

Kang, Y., Zhai, D.H., Liu, G.P., & Zhao, Y.B. (2016). Stability analysis of a class of hybrid stochastic retarded systems under asynchronous switching. *IEEE Transactions on Automatic Control*, 59(6), 1511–1523.

Khasminskii, R. (2011). *Stochastic stability of differential equations* (Vol. 66). Berlin: Springer Science and Business Media.

Kushner, H.J. (1972). *Stochastic stability*. Berlin: Springer.

Lander, C.W. (1987). *Power electronics*. New York, NY: McGraw-Hill.

Li, X., & Mao, X. (2012). The improved LaSalle-type theorems for stochastic differential delay equations. *Stochastic Analysis and Applications*, 30(4), 568–589.

Liu, L., & Xie, X. (2013). State feedback stabilization for stochastic feedforward nonlinear systems with time-varying delay. *Automatica*, 49(4), 936–942.

Mao, X. (1999). LaSalle-type theorems for stochastic differential delay equations. *Journal of Mathematical Analysis and Applications*, 236(2), 350–369.

Mazenc, F., & Bowong, S. (2003). Tracking trajectories of the cart-pendulum system. *Automatica*, 39(4), 677–684.

Mazenc, F., & Praly, L. (1994). *Adding an integration and global asymptotic stabilization of feedforward systems*. Proceedings of the 33rd IEEE conference on decision and control (Vol. 1, pp. 121–126). Lake Buena Vista, FL: IEEE.

Mazenc, F., & Praly, L. (2000). Asymptotic tracking of a reference state for systems with a feedforward structure. *Automatica*, 36(2), 179–187.

Qian, C., & Du, H. (2012). Global output feedback stabilization of a class of nonlinear systems via linear sampled-data control. *IEEE Transactions on Automatic Control*, 57(11), 2934–2939.

Shang, F., Liu, Y., & Zhang, G. (2015). Adaptive stabilization for a class of feedforward systems with zero-dynamics. *Journal of Systems Science and Complexity*, 28(2), 305–315.

Teel, A. (1992). Using saturation to stabilize a class of single-input partially linear composite systems. Proceedings of the 2nd IFAC symposium on nonlinear control systems (pp. 379–384). Bordeaux, France.

Tsias, J., & Tzamtzi, M. (2001). An explicit formula of bounded feedback stabilizers for feedforward systems. *Systems and Control Letters*, 43(4), 247–261.

Ugrinovskii, V., & Petersen, I. (1999). Absolute stabilization and minimax optimal control of uncertain systems with stochastic uncertainty. *SIAM Journal on Control and Optimization*, 37(4), 1089–1122.

Wu, L., Zheng, W.X., & Gao, H. (2013). Dissipativity-based sliding mode control of switched stochastic systems. *IEEE Transactions on Automatic Control*, 58(3), 785–791.

Wu, Z., Cui, M., Shi, P., & Karimi, H.R. (2013). Stability of stochastic nonlinear systems with state-dependent switching. *IEEE Transactions on Automatic Control*, 58(8), 1904–1918.

Ye, X., & Unbehauen, H. (2004). Global adaptive stabilization for a class of feedforward nonlinear systems. *IEEE Transactions on Automatic Control*, 49(5), 768–792.

Yu, X., & Xie, X. (2010). Output feedback regulation of stochastic nonlinear systems with stochastic iISS inverse dynamics. *IEEE Transactions on Automatic Control*, 55(2), 304–320.

Yu, X., Xie, X., & Duan, N. (2010). Small-gain control method for stochastic nonlinear systems with stochastic iISS inverse dynamics. *Automatica*, 46(11), 1790–1798.

Zhang, X., Baron, L., Liu, Q., & Boukas, E.K. (2011). Design of stabilizing controllers with a dynamic gain for feedforward nonlinear time-delay systems. *IEEE Transactions on Automatic Control*, 56(3), 692–697.

Zhang, X., Liu, Q., Baron, L., & Boukas, E.K. (2011). Feedback stabilization for high order feedforward nonlinear time-delay systems. *Automatica*, 47(5), 962–967.

Zhao, C., & Xie, X. (2014). Global stabilization of stochastic high-order feedforward nonlinear systems with time-varying delay. *Automatica*, 50(1), 203–210.

Zhao, X., & Deng, F. (2015). A new type of stability theorem for stochastic systems with application to stochastic stabilization. *IEEE Transactions on Automatic Control*, 61(1), 240–245.