Analytic Subordination Consequences of Free Markovianity

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1. Introduction

In [4], under some easy to remove genericity assumptions, we proved the analytic sub-
ordination of the Cauchy transforms of distributions $G_{\mu X+Y}$ and $G_{\mu X}$ for a pair of freely
independent self-adjoint random variables $X, Y$. We used this to obtain inequalities among
$p$-norms of densities of distributions, free entropies and Riesz energies. This was followed by
P.Biane’s discovery [1] that subordination extends, roughly speaking, to the resolvents, i.e. is
an operator-valued occurrence. He also showed that this implies a noncommutative Markov-
transitions property for free increment processes and that there are similar subordination
results for multiplicative processes.

The analytic function approach in [4] and the combinatorial one in [1] did not shed
much light on why analytic subordination appears in this context. In [6] we found a simple
explanation of this phenomenon based on the coalgebra structure associated with the free
difference quotient derivation. In the additive case this also led to a far reaching generaliza-
tion of the analytic subordination result to the $B$-valued case, i.e. to free independence with
amalgamation over an algebra $B$.

Here, based on simple operator-valued analytic function considerations, we build further
on the result in [6]. We derive more general analytic subordination results for a freely Marko-
vian triple $A, B, C$. This also includes the $B$-valued extension of the result for multiplication
of free unitary variables.

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2. Notation and Preliminaries

Throughout, \((M, \tau)\) will denote a tracial \(W^*\)-probability space, i.e. a von Neumann algebra endowed with a normal faithful trace state. A triple of von Neumann subalgebras \(A, B, C\) contained in \(M\) and containing \(C\) is freely Markovian (see [5] or [7]) if \(A\) and \(C\) are \(B\)-free in \((M, E_B)\) where \(E_B\) is the canonical conditional expectation of \(M\) onto \(B\) (see [7] or [8]). If \(A, B\) are von Neumann subalgebras of \(M\) containing 1, we shall denote by \(A \vee B\) or \(W^*(A \cup B)\) the von Neumann subalgebra generated by \(A \cup B\). If \(X = X^* \in M\) and \(1 \in B \subset M\) is a \(*\)-subalgebra we shall denote by \(B[X]\) the \(*\)-subalgebra generated by \(B\) and \(X\) and we shall denote by \(E_{B[X]}\) the conditional expectation onto the von Neumann subalgebra \(W^*(B[X])\). For several selfadjoint elements \(X, Y, Z, \ldots\) (noncommuting) we shall also use notations \(B[X, Y, Z], E_{B[X, Y, Z]}, \ldots\)

If \(A\) is a unital \(C^*\)-algebra we denote by \(\mathbb{H}_+(A) = \{T \in A \mid \text{Im} T \geq \varepsilon 1\text{ for some }\varepsilon > 0\}\) the upper half-plane of \(A\) and by \(\mathbb{H}_-(A) = -\mathbb{H}_+(A)\) the lower half-plane. Note that \(T \in \mathbb{H}_+(A) \iff T^{-1} \in \mathbb{H}_-(A)\), in particular elements in \(\mathbb{H}_\pm(A)\) are invertible. Also, \(A_{sa}\) will denote the selfadjoint elements in \(A\).

The open unit ball will be denoted \(\mathbb{D}(A) = \{T \in A \mid \|T\| < 1\}\). For balls of radius \(R\) we shall also use the notations \(\mathbb{D}_R(A) = R\mathbb{D}(A)\).

3. Analytic Subordination

We begin by recalling the result ([6] Theorem 3.8) which serves as our starting point.

3.1 Theorem. Let \(1 \in B \subset M\) be a \(W^*\)-subalgebra and let \(X = X^* \in M, Y = Y^* \in M\). Assume \(X, Y\) are \(B\)-free in \((M, E_B)\). Then there is a holomorphic map \(F: \mathbb{H}_+(B) \to \mathbb{H}_+(B)\) such that

\[
E_{B[X]}((X + Y) - b)^{-1} = (X - F(b))^{-1}
\]

if \(b \in \mathbb{H}_+(B)\).

3.2 Proposition. Let \(1 \in B \subset M\) and \(1 \in A \subset M\) be \(W^*\)-subalgebras and let \(X = X^* \in M\). Assume \(A\) and \(X\) are \(B\)-free in \((M, E_B)\). Then there is a holomorphic map \(F: \mathbb{H}_+(A) \to \mathbb{H}_+(B)\) such that

\[
E_{B[X]}(a - X)^{-1} = (F(a) - X)^{-1}
\]
Proof. It is easily seen that

\[ F_1(a) = (E_{B[X]}(a - X)^{-1})^{-1} + X \]

is a well-defined holomorphic map

\[ \mathbb{H}_+(A) \to \mathbb{H}_+(W^*(B[X])) . \]

We must show that \( F_1(\mathbb{H}_+(A)) \subset B \). By Theorem 3.1,

\[ F_1(i\varepsilon I + A_{sa}) \subset \mathbb{H}_+(W^*(B[X])) \]

if \( \varepsilon > 0 \). Indeed, if \(-Y \in A_{sa}\) then Proposition 3.2 gives

\[ E_{B[X]}(i\varepsilon I - Y - X)^{-1} = (b - X)^{-1} \]

for some \( b \in \mathbb{H}_+(B) \) which means \( F_1(i\varepsilon I - Y) = b \).

Since \( \varepsilon I + A_{sa} \) is a uniqueness set for holomorphic functions on \( \mathbb{H}_+(A) \), the inclusion \( F_1(i\varepsilon I + A_{sa}) \subset B \) implies \( F_1(\mathbb{H}_+(A)) \subset B \) (use functionals \( f \in M_*, \ f|B = 0 \) to transform the given inclusion into \( (f \circ F_1) \mid i\varepsilon I + A_{sa} = 0 \)).

\[ \square \]

3.3 Proposition. Let \( A, B, C \) be a freely Markovian triple of \( W^*\)-subalgebras in \( (M, \tau) \).
Then there is a holomorphic map

\[ F : \mathbb{H}_+(A) \times \mathbb{H}_+(C) \to B \]

such that

\[ (E_{AVB}(a + c)^{-1})^{-1} = a + F(a, c) \]

if \( a \in \mathbb{H}_+(A), \ c \in \mathbb{H}_+(C) \).

Proof. Clearly, we have \( a + c \in \mathbb{H}_+(M), \ (a + c)^{-1} \in \mathbb{H}_-(M), \ E_{AVB}(a + c)^{-1} \in \mathbb{H}_-(M), \)

\[ (E_{AVB}(a + c)^{-1})^{-1} \in \mathbb{H}_+(M) . \]

Hence

\[ F_1 : \mathbb{H}_+(A) \times \mathbb{H}_+(C) \to A \vee B \vee C \]

given by

\[ F_1(a, c) = (E_{AVB}(a + c)^{-1})^{-1} - a \]
is well-defined and holomorphic. Thus, the proof reduces to showing that
\[ F_1(H_+ (A) \times H_+ (C)) \subseteq B. \]
Since \((i \varepsilon I + A_{sa}) \times (i \varepsilon I + C_{sa})\), where \(\varepsilon > 0\) is a set of uniqueness
for holomorphic maps on \(H_+ (A) \times H_+ (C)\), it suffices to show that
\[
(E_{AVB}((X + Y) + 2\varepsilon i))^{-1} - X \in B
\]
when \(X \in A_{sa}, Y \in C_{sa}\). Since \((X + Y + 2\varepsilon i)^{-1} \in W^*(B[X]) \vee C\) and \(A\) and \(C\) are \(B\)-free
in \((M, E_B)\) we have
\[
E_{AVB}(X + Y + 2\varepsilon i) = E_B[X](X + Y + 2\varepsilon i)^{-1}.
\]
By Theorem 3.1 we have
\[
E_B[X](X + Y + 2\varepsilon i)^{-1} = (X + b)^{-1}
\]
for some \(b \in H_+(B)\). In particular
\[
(E_{AVB}(X + Y + 2\varepsilon i)^{-1})^{-1} - X \in B
\]

3.4 Lemma. If \(x \in A\), where \(A\) is a unital \(C^*\)-algebra, the following are equivalent:

(i) \(|x| < 1\)
(ii) \(1 - x\) is invertible and \(2 \text{Re}(1 - x)^{-1} \geq (1 + \varepsilon)\) for some \(\varepsilon > 0\).

Proof. This fact, not new, is for instance an immediate consequence of unitary dilation 
[3] and of the corresponding fact when \(A = \mathbb{C}\). For the reader’s convenience, here is a direct 
proof.

Both (i) and (ii) imply \(1 - x\) is invertible, in which case we have:
\[
(1 - x)^{-1} + (1 - x^*)^{-1} = 1 + (1 - x)^{-1}(1 - xx^*)^{-1}(1 - x^*)^{-1}.
\]
To prove (i) \(\Rightarrow\) (ii) remark that \(1 - xx^* \geq (1 - \|x\|^2)\) and that
\[
\|(1 - x^*)^{-1} \xi\| \geq \|1 - x^*\|^{-1}\|\xi\| \geq 2^{-1}\|\xi\|
\]
when \(A\) acts on a Hilbert space \(\mathcal{H}\) and \(\xi \in \mathcal{H}\). This gives \((1 - x)^{-1}(1 - xx^*)(1 - x^*)^{-1} \geq \n
To show (ii) \(\Rightarrow\) (i) note that (ii) implies \((1 - x)^{-1}(1 - xx^*)(1 - x^*)^{-1} \geq \varepsilon 1\) so that
\(1 - xx^* \geq 0\) and \(1 - xx^*\) is invertible, which gives (i). □
3.5 Proposition. Let $A, B, C$ be a freely Markovian triple of $W^*$-algebras in $(M, \tau)$ and let $\Omega = \{(a, c) \in A \times C \mid a$ invertible, $\|a^{-1}c\| < 1\}$. Then there is a holomorphic map $\Phi : \Omega \to B$ such that if $(a, c) \in \Omega$, then $\|a^{-1}\Phi(a, c)\| < 1$ and $E_{AVB}(a - c)^{-1} = (a - \Phi(a, c))^{-1}$.

Proof. Let $(a, c) \in \Omega$. Then $a - c$ is invertible, the inverse being $(1 - a^{-1}c)^{-1}$. Using Lemma 3.41 we have

$$E_{AVB}(1 - a^{-1}c)^{-1} \in \frac{1}{2} - i\mathbb{H}_+(M)$$

and hence

$$E_{AVB}(1 - a^{-1}c)^{-1} = (1 - x)^{-1}$$

for some $x \in \mathbb{D}(M)$. Moreover there is a holomorphic map $\Psi : \Omega \to \mathbb{D}(M)$ such that $\Psi(a, c) = 1 - (E_{AVB}(1 - a^{-1}c)^{-1})^{-1}$.

Let $\Phi_1 : \Omega \to M$ be defined by $\Phi_1(a, c) = a\Psi(a, c)$. Clearly $\Phi_1$ is holomorphic, $\|a^{-1}\Phi_1(a, c)\| < 1$ and

$$E_{AVB}(a - c)^{-1} = (E_{AVB}(1 - a^{-1}c)^{-1})a^{-1} = (1 - \Psi(a, c))^{-1}a^{-1} = (a - \Phi_1(a, c))^{-1}.$$ Thus the proof reduces to showing that $\Phi_1(a, c) \in B$.

The open set $\Omega$ is connected. Indeed, if $(a, c) \in \Omega$ then $(\lambda a, c) \in \Omega$ for all $\lambda \in \mathbb{C}$, $|\lambda| \geq 1$. In particular there is a segment in $\Omega$ connecting $(a, c)$ and $(Na, c)$ where $N \geq 1$ is such that $\|(Na)^{-1}\| \leq (1 + \|c\|)^{-1}$. Then $(Na, tc + (1 - t)1) \in \Omega$ for $0 \leq t \leq 1$ and we have a path from $(a, c)$ to $(Na, 1)$. If $Na = u(|N|a)$ is the polar decomposition, then $N|a| \geq (1 + \varepsilon)1$ for some $\varepsilon > 0$ and the elements $(u((1 - t)N|a| + t(1 + \varepsilon)1, 1), 0 \leq t \leq 1$, are in $\Omega$ connecting $(Na, 1)$ and $((1 + \varepsilon)u, 1)$. Finally $u = \exp(ith)$ for some $h = h^* \in A$ and $((1 + \varepsilon)\exp(ith), 1) \in \Omega$ for $0 \leq t \leq 1$ continue our path to $((1 + \varepsilon)1, 1)$, etc.

Since $\Omega$ is connected, the set

$$\omega = \{(a, c) \in A \times C \mid \|a - 3i1\| < 1, \ \text{Im} \ c \leq -\frac{1}{2}1, \ \|c\| < 1\} \subset \mathbb{H}_+(A) \times \mathbb{H}_-(C)$$

and $\omega \subset \Omega$ is a uniqueness set for holomorphic functions in $\Omega$. Moreover, using Proposition 3.3 we have $\Phi(a, c) = F(a, -c) \in B$ where $(a, c) \in \omega$ and $F$ is the function in Proposition 3.3.

3.6 Theorem. Let $1 \in B$, $1 \in C$ be $W^*$-algebras in $(M, \tau)$ and let $u$ be a unitary element in $M$. Assume $C$ and $\{u, u^*\}$ are $B$-free in $(M, E_B)$. Then there is a holomorphic map

$$G : \mathbb{D}(C) \to \mathbb{D}(B)$$
such that
\[ E_{B(u,u^{-1})}(u - c)^{-1} = (u - G(c))^{-1} \]
if \( c \in \mathbb{D}(B) \).

**Proof.** This follows immediately from Proposition 3.5 since \((u, c) \in \Omega\) if \( c \in \mathbb{D}(C) \) and \( \|u^{-1}\Phi(u, c)\| < 1 \iff \|\Phi(u, c)\| < 1 \) and we take \( G(c) = \Phi(u, c) \).

\[ \square \]

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