On the Topological Complexity of Maps

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Abstract

We define and develop a homotopy invariant notion for the topological complexity of a map \( f : X \rightarrow Y \), denoted \( \text{TC}^p f \), that interacts with \( \text{TC}^p X \) and \( \text{TC}^p Y \) in the same way \( \text{cat} f \) interacts with \( \text{cat} X \) and \( \text{cat} Y \). Furthermore, \( \text{TC}^p f \) and \( \text{cat} f \) satisfy the same inequalities as \( \text{TC}^p X \) and \( \text{cat} X \). We compare it to other invariants defined in the papers [15, 16, 17, 18, 20]. We apply \( \text{TC}^p f \) to studying group homomorphisms \( f : H \rightarrow G \).

1 Introduction

Let \( X \) denote the configuration space of a mechanical system, then a navigation algorithm of the system is a map \( \sigma : X \times X \rightarrow X^I, \; I = [0, 1] \), to the path space such that \( \sigma(x_0, x_1)(0) = x_0 \) and \( \sigma(x_0, x_1)(1) = x_1 \). Clearly, if there exists a continuous navigation algorithm, then \( X \) is contractible. Since configuration spaces are rarely contractible, one has to do partitioning of \( X \times X \) into pieces such that on each piece there is a continuous navigation. M. Farber called the minimal number of such pieces \([7]\) (also see \([10]\)) the topological complexity of \( X \) and denoted it by \( \text{TC}^p X \). Formally, \( \text{TC}^p X \) is the minimal number \( k \) such that \( X \times X \) can be covered by open sets \( U_0, \ldots, U_k \) such that the path space fibration \( p : X^I \rightarrow X \times X \) admits a section on each \( U_i \). Here we gave the definition of the reduced topological complexity which differs by one from the original. This invariant was intensively studied in the last 20 years.

When the mechanical system is a robotic arm the \( \text{TC}^p X \) input on the navigation problem is not quite satisfactory. In the case of robotic arm one has a map \( f : X \rightarrow Y \) of the configuration space \( X \) to the work space \( Y \). In that case in his lectures at the workshop on Applied Algebraic Topology in Castro Urdiales (Spain, 2014) A. Dranishnikov suggested to study the topological complexity of maps \( \text{TC} f \) and defined it as the minimal number \( k \) such that \( X \times Y \) can be covered by open sets \( U_0, \ldots, U_k \) such that the over each \( U_i \) there is a section to the map \( \text{q} : X^I \rightarrow X \times Y \) defined as \( \text{q}(\phi) = (\phi(0), f \phi(1)) \) (see \([16]\)). The drawback of this definition is that it can be applied only to maps \( f : X \rightarrow Y \) that have the path lifting property. All path fibrations and left divisor of path fibrations have this property, but it is unclear if the forward kinematic maps for general robotic arms are left divisors of fibrations. In \([17, 18]\) P. Pavesic modified this definition to cover general maps, but still his definition lacks one important feature: It is not a homotopy invariant.

Topological complexity is a homotopy invariant similar to LS-category, in that both can be characterized as the sectional category of a particular fibration, i.e., the topological complexity
of a path-connected space $X$ is the sectional category of the path space fibration $X^I \to X \times X$, and the LS-category of $X$, denoted $\text{cat}(X)$, is the sectional category of the based path space fibration $P_0X \to X$. Both of these invariants are notoriously difficult to compute, with lower bounds usually being the more difficult problem.

In the case of LS-category, the notion was extended to maps $f : X \to Y$ satisfying the inequality $\text{cat}(f) \leq \text{cat}(X), \text{cat}(Y)$. It proved to be helpful in computing the LS-category of a space [22] and it is homotopy invariant. The goal of this paper is to define and study analogous an invariant $\text{TC}(f)$ for topological complexity. There was no invariant occupying this theoretical niche. Only very recently when this work was in progress A. Murillo and J. Wu defined their own notion of $\text{TC}(f)$ in [15], which I will later prove is equivalent to the notion presented here. Notably, with the perspective Murillo and Wu took, they were unable to prove the inequality $\text{TC}(f) \leq \text{TC}(Y)$.

Since $\text{cat}$ and $\text{TC}$ are homotopy invariants, they produce numerical invariant of discrete groups. In the case of the category by the theorem of Eilenberg and Ganea $\text{cat}(G)$ is just the cohomological dimension of a group $G$. An algebraic description of $\text{TC}(G)$ of a group $G$ is one of the most important problems in the area [11]. In this paper I applied both invariants $\text{cat}$ and $\text{TC}$ to group homomorphisms. Even in the case of $\text{cat}$, despite on easy answer for groups, there is no easy answer for homomorphisms. In Section 6, we present a conjecture on that (Conjecture 6.6) which grew out of discussion of the problem with A. Dranishnikov. The conjecture is proved in the paper for free groups and free abelian groups. In the case of topological complexity I don’t have any conjecture of what $\text{TC}(f)$ is for a group homomorphism $f : H \to G$, since there is no corresponding theorem for the topological complexity of groups, $\text{TC}(G)$. Again, when both $G$ and $H$ are free or free abelian we give the answer in the paper.

The paper is organized as follows. In section 3, we give a definition for $\text{TC}(f)$ that takes inspiration from the fact that $\text{cat}(f)$ is equal to the sectional category of the pullback fibration $f^*\pi^Y$, where $\pi^Y : P_0Y \to Y$ is the based path space fibration (see Theorem 19' of [19]). Thus, we can define $\text{TC}(f)$ to be the sectional category of the pullback of the path space fibration $Y^I \to Y \times Y$ by $f \times f$. As this definition is a bit theoretical in nature, we take inspiration from the various formulations of $\text{TC}(X)$ to define $f$-motion planners and instead start section 3 with this version of the definition. Here we prove various properties for $\text{TC}(f)$, including homotopy invariance, inequalities mirroring those for $\text{cat}(f)$, and inequalities that extend the interaction between $\text{cat}(X)$ and $\text{TC}(X)$ to maps.

In section 4, we consider instead the fibration obtained as the pullback of the path space fibration by $Id_Y \times f$. In [18], Pavešič proved that the sectional category of this fibration is equal to his complexity, denoted here as $\text{TC}^r(f)$, if $f$ is a fibration. Here we study this sectional category without that assumption as an intermediary step to get to the pullback by $f \times f$. We call it the mixed topological complexity of $f$, denoted $\text{TC}^m(f)$. Section 4 is structured similarly to section 3, but at the end we prove the inequalities $\text{TC}(f) \leq \text{TC}^m(f) \leq \text{TC}^r(f)$.

The use of $f$-motion planners as a means of defining $\text{TC}(f)$ inspires definitions for symmetric and monoidal $\text{TC}$ of maps, so section 5 of this paper proves some basic properties for the latter and leaves the former for future work.

The final section of this paper uses the homotopy invariance of $\text{TC}(f)$ to define the topological complexity of group homomorphisms, as discussed above.
2 Preliminaries

2.1 Sectional Category

**Definition 2.1.** Let \( p : E \to B \) be a fibration. Then the *sectional category* of \( p \), denoted \( \text{secat}(p) \), is the smallest integer \( k \) such that \( B \) can be covered by \( k + 1 \) open sets \( U_0, \ldots, U_k \) that each admit a partial section \( s_i : U_i \to E \) of \( p \). If no such integer \( k \) exists, then we set \( \text{secat}(p) = \infty \).

**Proposition 2.2.** Let \( p : E \to B \) and \( p' : E' \to B' \) be fibrations. If there is a homotopy commutative diagram

\[
\begin{array}{ccc}
E & \xrightarrow{p} & E' \\
p \downarrow & & \downarrow p' \\
B & \xrightarrow{h} & B'
\end{array}
\]

such that \( h \) is a homotopy domination, then \( \text{secat}(p') \leq \text{secat}(p) \).

**Proof.** Let \( g \) be the right homotopy inverse of \( h \), and let \( \text{secat}(p) = k \) so that there is an open cover \( U_0, \ldots, U_k \) of \( B \) and partial sections \( s_i : U_i \to E \) of \( p \). Let \( V_i = g^{-1}(U_i) \), which forms an open cover of \( B' \). Now define \( s'_i : V_i \to E' \) by \( s'_i = \overline{h}s_i|_{V_i} \). Then

\[
p's'_i = p'lhs_i|_{V_i} \simeq hps_i|_{V_i} = h(Id_B|_{U_i})g|_{V_i} = hg|_{V_i}
\]

but \( hg \) is homotopic to \( Id_{B'} \) so that \( p's'_i \) is homotopic to the inclusion map \( V_i \to B' \); hence, \( s'_i \) is a homotopy section of \( p' \). Therefore, \( \text{secat}(p') \leq \text{secat}(p) \). \( \square \)

The following is proven in Proposition 7 of [19]:

**Proposition 2.3.** Let \( p : E \to B \) be a fibration, let \( f : B' \to B \) be any map, and consider the pullback diagram

\[
\begin{array}{ccc}
f^*E & \xrightarrow{f^*p} & E \\
\downarrow f^* & & \downarrow p \\
B & \xrightarrow{f} & B'
\end{array}
\]

Then \( \text{secat}(f^*p) \leq \text{secat}(p) \).

**Proof.** Suppose \( \text{secat}(p) = k \) and let \( U_0, \ldots, U_k \) be an open cover of \( B \) with sections \( s_i : U_i \to E \) of \( p \). Let \( V_i = f^{-1}(U_i) \). Using the pullback diagram, there are maps \( s'_i : V_i \to f^*E \) such that
the diagrams

commute. Therefore, each \( s_i' \) is a section of \( f^*p \) on \( V_i \) so that \( \text{secat}(f^*p) \leq \text{secat}(p) \).

The following product formula is Proposition 22 of [19], but a more understandable proof is given by Theorem 11 of [7] for the case of topological complexity. The conditions here are weaker than those of [7] and [19], but the only necessary condition for the proof is that of normality as it ensures the existence of partitions of unity subordinate to finite open covers.

**Proposition 2.4.** Let \( p : E \to B \) and \( p' : E' \to B' \) be fibrations such that \( B \) and \( B' \) are normal. Then \( \text{secat}(p \times p') \leq \text{secat}(p) + \text{secat}(p') \).

### 2.2 The Fiberwise Join

For \( 1 \leq i \leq n \), let \( p_i : E_i \to B \) be a fibration with fiber \( F_i \). The fiberwise join of the total spaces \( E_i \) over the base \( B \) is defined to be the subspace of the usual join \( E_1 * \ldots * E_n \) given by

\[
E_1 *_B \ldots *_B E_n = \{ t_1e_1 + \ldots + t_ne_n \in E_1 * \ldots * E_n \mid t_i, t_j \neq 0 \implies p_i(e_i) = p_j(e_j) \}.
\]

Then the fiberwise join of the fibrations \( p_i \) is the map

\[
p_1 *_B \ldots *_B p_n : E_1 *_B \ldots *_B E_n \to B
\]

given by

\[
t_1e_1 + \ldots + t_ne_n \mapsto p_i(e_i)
\]

for any \( i \) such that \( t_i \neq 0 \). This map is a well defined fibration by the definition of the space \( E_1 *_B \ldots *_B E_n \). This operation is called the fiberwise join as the fiber of \( p_1 *_B \ldots *_B p_n \) is the join of the fibers \( F_1 * \ldots * F_n \).

If \( p_i = p : E \to B \) for all \( i \), then we denote the fibration \( p_1 *_B \ldots *_B p_n \) by \( *^n_B p \) and we denote its total space \( E_1 *_B \ldots *_B E_n \) by \( *^n_B E \).

The following theorem encapsulates the Ganea-Schwarz approach of using fiberwise joins to calculate sectional category:

**Theorem 2.5** ([19]). Let \( p : E \to B \) be a fibration with \( B \) normal. Then \( \text{secat}(p) \leq n \) if and only if \( *^{n+1}_B p \) admits a section.
2.3 LS-Category

Definition 2.6. The (reduced) Lusternik-Schnirelmann category of a space $X$, denoted $\text{cat} (X)$, is defined to be the smallest integer $n \geq -1$ such that $X$ admits an open cover by $n + 1$ sets $U_0, \ldots, U_n$ that are each contractible in $X$.

If $X$ is nonempty and path-connected, then the sectional category of the based path space fibration $\pi : P_0(X) \to X$ is exactly $\text{cat} (X)$ so that LS-category can be thought of as a special case of sectional category. Moreover, we call $G_n(X) = *^{n+1}_X P_0(X)$ the $n$-th Ganea space of $X$ and $p_n^X = *^{n+1}_X \pi$ the $n$-th Ganea fibration of $X$.

Properties 2.7. Let $X$ and $Y$ be normal and path-connected spaces.

- If $X$ is $(r - 1)$-connected, then $\text{cat} (X) \leq \frac{\dim X}{r}$.
- $\text{cat} (X \times Y) \leq \text{cat} (X) + \text{cat} (Y)$.
- If $X$ is an ANR, then $\text{cup} \big( \tilde{H}^*(X) \big) \leq \text{cat} (X)$.
- $\text{cat} (X) \leq n$ if and only if $p_n^X$ admits a section.

Definition 2.8. The (reduced) Lusternik-Schnirelmann category of a map $f : X \to Y$, denoted $\text{cat} (f)$, is defined to be the smallest integer $n \geq -1$ such that $X$ admits an open cover by $n + 1$ sets $U_0, \ldots, U_n$ such that each restriction $f|_{U_i}$ is nullhomotopic.

Properties 2.9. Let $f : X \to Y$ and $g : Z \to W$ be maps between normal and path-connected spaces.

- $\text{cat} (f) \leq \text{cat} (X), \text{cat} (Y)$.
- $\text{cat} (f \times g) \leq \text{cat} (f) + \text{cat} (g)$.
- $\text{cup} (\ker f^*) \leq \text{cat} (f)$.
- $\text{cat} (f) = \text{secat} (f^* p_0^Y)$.
- $\text{cat} (f) \leq n$ if and only if there is a lift of $f$ with respect to $p_n^Y$.

2.4 Topological Complexity

Definition 2.10. Let $X$ be a topological space.

1. A motion planner on a subset $Z \subset X \times X$ is a map $s : Z \to X^I$ such that $s(x_0, x_1)(0) = x_0$ and $s(x_0, x_1)(1) = x_1$.

2. A motion planning algorithm is a cover of $X \times X$ by sets $Z_0, \ldots, Z_k$ such that on each $Z_i$ there is some motion planner $s_i : Z_i \to X^I$. 
3. The topological complexity of a space $X$, denoted $\text{TC}(X)$, is the least $k$ such that $X \times X$ can be covered by $k + 1$ open subsets $U_0, ..., U_k$ on which there are motion planners. If no such $k$ exists, we define $\text{TC}(X) = \infty$.

4. Given a subset $A \subset X \times X$, the relative topological complexity of $A$ in $X$, denoted $\text{TC}_X(A)$, is the least $k$ such that $A$ can be covered by $k + 1$ sets $U_0, ..., U_k$ open in $A$ on which there are motion planners $U_i \to X^I$ for $X$. As before, if no such $k$ exists, we define $\text{TC}_X(A) = \infty$.

If $X$ is nonempty and path-connected, then the sectional category of the path space fibration $\Delta^X_0 : P(X) \to X \times X$ is exactly $\text{TC}(X)$ so that topological complexity can also be thought of as a special case of sectional category. We use the notation $\Delta^X_0$ for the path space fibration since it is a fibration replacement of the diagonal map $\Delta : X \to X \times X$. Moreover, we denote the corresponding fiberwise joins as $\Delta^n(X) = *_{X \times X}^n P(X)$ and $\Delta^X_n = *_{X \times X}^n \Delta^X$.

Properties 2.11. Let $X$ and $Y$ be normal and path-connected spaces.

- $\text{cat}(X) \leq \text{TC}(X) \leq \text{cat}(X \times X)$.
- $\text{TC}(X \times Y) \leq \text{TC}(X) + \text{TC}(Y)$.
- If $X$ is an ANR, then $\text{cup} \{ \ker \Delta^n \} \leq \text{TC}(X)$.
- $\text{TC}(X) \leq n$ if and only if $\Delta^n_0$ admits a section.
- $\text{TC}_X(X \times X) = \text{TC}(X)$.
- If $A \subset B \subset X \times X$, then $\text{TC}_X(A) \leq \text{TC}_X(B)$.

3 The Pullback TC of a Map

3.1 Definitions

Definition 3.1. Let $f : X \to Y$ be a map.

1. An $f$-motion planner on a subset $Z \subset X \times X$ is a map $f_Z : Z \to Y^I$ such that $f_Z(x_0, x_1)(0) = f(x_0)$ and $f_Z(x_0, x_1)(1) = f(x_1)$.

2. An $f$-motion planning algorithm is a cover of $X \times X$ by sets $Z_0, ..., Z_k$ such that on each $Z_i$ there is some $f$-motion planner $f_i : Z_i \to Y^I$.

3. The (pullback) topological complexity of $f$, denoted $\text{TC}(f)$, is the least $k$ such that $X \times X$ can be covered by $k + 1$ open subsets $U_0, ..., U_k$ on which there are $f$-motion planners. If no such $k$ exists, we define $\text{TC}(f) = \infty$.

Remark 3.2. If $f$ doesn’t map $X$ into a single path component of $Y$, then we always have $\text{TC}(f) = \infty$ by this definition, so to avoid this one would have to add up the TC for each path component $f$ maps into. For the sake of this work, all spaces are assumed to be path-connected.
Examples 3.3.

1. \( \text{TC}(f) = -1 \) if and only if \( f \) is the empty map.

2. If \( i : A \to X \) is an inclusion map, then \( \text{TC}(i) = \text{TC}_X (A \times A) \).

3. \( \text{TC}(\text{Id}_X) = \text{TC}(X) \) for any space \( X \).

The following theorem shows that the notion of an \( f \)-motion planner can be substituted for sections of a fibration or a deformation into the diagonal as is the case with the topological complexity of a space. Additionally, the last item in the following equivalence shows an equivalence with the notion of TC studied in [15].

**Theorem 3.4.** Let \( f : X \to Y \) be a map and let \( Z \subset X \times X \). The following are equivalent:

1. there is a partial section \( s : Z \to (f \times f)^*Y^I \) of the pullback fibration \( (f \times f)^* \Delta^Y_0 \);
2. there is an \( f \)-motion planner \( f_Z : Z \to Y^I \);
3. \( (f \times f)|_Z \) can be deformed into \( \Delta Y \)
4. there is a map \( h : Z \to X \) such that \( (f \times f)\Delta h \) is homotopic to \( (f \times f)|_Z \).

**Proof.** (1 \( \implies \) 2) Let \( s : Z \to (f \times f)^*Y^I \) be a partial section of the pullback fibration \( (f \times f)^* \Delta^Y_0 \) and consider the pullback diagram of the maps \( f \times f \) and \( \Delta^Y_0 \)

\[
\begin{array}{ccc}
(f \times f)^*Y^I & \xrightarrow{f \times f} & Y^I \\
\downarrow (f \times f)|_Z & & \downarrow \Delta^Y_0 \\
X \times X & \xrightarrow{f \times f} & Y \times Y
\end{array}
\]

where the \( (f \times f)^*Y^I \) is given by

\[
(f \times f)^*Y^I = \{(x_0, x_1, \alpha) \in X \times X \times Y^I \mid \alpha(0) = f(x_0) \text{ and } \alpha(1) = f(x_1)\}
\]

and \( (f \times f)^* \Delta^Y_0 \) and \( f \times f \) are the obvious projection maps. Define the \( f \)-motion planner \( f_Z : Z \to Y^I \) by \( f_Z = (f \times f)s \). Then it follows that

\[
\Delta^Y_0 f_Z = \Delta^Y_0 (f \times f)s = (f \times f)((f \times f)^* \Delta^Y_0) s = (f \times f)\text{Id}_Z = (f \times f)|_Z
\]

so that applying both sides to some \( (x_0, x_1) \in Z \) gives

\[
(f_Z(x_0, x_1)(0), f_Z(x_0, x_1)(1)) = (\Delta^Y_0 f_Z)(x_0, x_1) = (f \times f)|_Z(x_0, x_1) = (f(x_0), f(x_1)).
\]

Therefore, \( f_Z \) is an \( f \)-motion planner on \( Z \).
(2 $\implies$ 1) Let $f_z : Z \to Y^I$ be an $f$-motion planner. Since $(f \times f)^*Y^I$ is the pullback of $f \times f$ and $\Delta^Y_0$, there is some $s : Z \to (f \times f)^*Y^I$ such that the diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{s} & (f \times f)^*Y^I \\
\downarrow & & \downarrow \quad f \times f \\
X \times X & \xrightarrow{f \times f} & Y \times Y
\end{array}
\]

commutes where $i : Z \to X \times X$ is the inclusion map. Therefore, $((f \times f)^*\Delta^Y_0)s = i$ so that $s$ is a partial section of $(f \times f)^*\Delta^Y_0$ on $Z$.

(2 $\implies$ 3, 4) Let $f_z : Z \to Y^I$ be an $f$-motion planner. Now define a homotopy $H : X \times X \times I \to Y \times Y$ by

\[H(x_0, x_1, t) = (f_z(x_0, x_1)(t), f(x_1)),\]

which is a deformation of $(f \times f)|_Z$ into $\Delta Y$, showing (3). Also observe that

\[H(x_0, x_1, 1) = (f(x_1), f(x_1)) = (f \times f)(x_1, x_1) = (f \times f)\Delta(x_1) = (f \times f)\Delta(\text{proj}_1)|_Z(x_0, x_1)\]

where $\text{proj}_1 : X \times X \to X$ is the projection map into the second coordinate. This proves (4) by setting $h = (\text{proj}_1)|_Z$.

(3 $\implies$ 2) Now let $H : Z \times I \to Y \times Y$ be a deformation of $(f \times f)|_Z$ into $\Delta Y$. Define a map $f_z : Z \to Y^I$ by

\[f_z(x_0, x_1)(t) = \begin{cases} 
\text{proj}_0(H(x_0, x_1, 2t)) & \text{if } 0 \leq t \leq \frac{1}{2} \\
\text{proj}_1(H(x_0, x_1, 2 - 2t)) & \text{if } \frac{1}{2} \leq t \leq 1
\end{cases}\]

where $\text{proj}_0, \text{proj}_1 : Y \times Y \to Y$ are the projection maps into the corresponding coordinates. Then $f_z$ is well-defined and continuous since $H(x_0, x_1, t) \in \Delta Y$ for all $(x_0, x_1) \in Z$. Moreover,

\[f_z(x_0, x_1)(0) = \text{proj}_0(H(x_0, x_1, 0)) = \text{proj}_0(f(x_0), f(x_1)) = f(x_0)\]

and

\[f_z(x_0, x_1)(1) = \text{proj}_1(H(x_0, x_1, 0)) = \text{proj}_1(f(x_0), f(x_1)) = f(x_1)\]

so that $f_z$ is an $f$-motion planner on $Z$.

(4 $\implies$ 3) Note that the image of $(f \times f)\Delta h$ will always be contained in $\Delta Y$. \hfill \Box

This theorem implies that $\text{TC}(f)$ is the minimum $k$ such that there is an open cover $U_0, ..., U_k$ on each of which at least one of the above equivalent properties hold. Moreover, if $X$ and $Y$ are ANRs, then any cover will do as we will see in the following theorem.
Definition 3.5. Let $p : E \to B$ be a fibration. Then the generalized sectional category of $p$, denoted $\text{secat}_g(p)$, is the smallest integer $k$ such that $B$ can be covered by $k + 1$ sets $Z_0, \ldots, Z_k$ that each admit a partial section $s_i : Z_i \to E$ of $p$. If no such integer $k$ exists, then we set $\text{secat}_g(p) = \infty$.

Theorem 3.6. If $f : X \to Y$ is a map between ANRs, then $\text{TC}(f) = \text{secat}_g((f \times f)^*\Delta^Y_0)$.

Proof. Since $X$ and $Y$ are ANRs, it follows that $X \times X$, $Y \times Y$, and $Y^I$ are all ANRs. Since $\Delta^Y_0$ is a fibration, it follows that $(f \times f)^*Y^I$ is an ANR (see Theorem 2.2 of [14]). Thus, $(f \times f)^*\Delta^Y_0$ is a fibration between ANRs so that $\text{secat}((f \times f)^*\Delta^Y_0) = \text{secat}_g((f \times f)^*\Delta^Y_0)$ (see [12] and [21]). Thus, $\text{TC}(f) = \text{secat}_g((f \times f)^*\Delta^Y_0)$ by Theorem 3.3.

Corollary 3.7. Let $f : X \to Y$ be any map with $X \times X$ normal. Then $\text{TC}(f) \leq k$ if and only if there is a lift of $f \times f$ with respect to $\Delta^Y_{k+1}$.

Proof. Consider the pullback diagram

\[
\begin{array}{ccc}
(f \times f)^*Y^I & \xrightarrow{f \times f} & Y^I \\
\downarrow (f \times f)^*\Delta^Y_0 & & \downarrow \Delta^Y_0 \\
X \times X & \xrightarrow{f \times f} & Y \times Y
\end{array}
\]

as usual. Note that taking the self fiberwise join of a fibration commutes with taking pullbacks, so we get the pullback diagram

\[
\begin{array}{ccc}
(f \times f)^*\Delta_k(Y) & \xrightarrow{f \times f} & \Delta_k(Y) \\
\downarrow (f \times f)^*\Delta^Y_k & & \downarrow \Delta^Y_k \\
X \times X & \xrightarrow{f \times f} & Y \times Y
\end{array}
\]

where $\ast_{X \times X}^{k+1}(f \times f)^*Y^I = (f \times f)^*\Delta_k(Y)$ and $\ast_{X \times X}^{k+1}(f \times f)^*\Delta^Y_0 = (f \times f)^*\Delta^Y_k$. Thus, by Theorem 2.5, this corollary follows immediately.

3.2 Basic Inequalities

Proposition 3.8. Let $f : X \to Y$ be a map. Then

1. $\text{TC}(f) \leq \min\{\text{TC}(X), \text{TC}(Y)\}$;
2. $\text{cat}(f) \leq \text{TC}(f) \leq \text{cat}(f \times f)$.

Proposition 2.2 of [15] proves all of these inequalities with the exception of $\text{TC}(f) \leq \text{TC}(Y)$, but that inequality is relatively easy now that we have the perspective of pullbacks.
proof. (1) Note that TC \( f \leq TC(Y) \) is immediate by Proposition 2.3 since TC \( f = \text{secat} ((f \times f)^* \pi^Y) \). Now suppose TC \( (X) = k \) and let \( U_0, ..., U_k \) be an open cover of \( X \times X \) with motion planners \( s_i : U_i \rightarrow X^I \). Define \( f_i : U_i \rightarrow Y^I \) by \( f_i = f_s s_i \). Then
\[
 f_i(x_0, x_1)(0) = f(s_i(x_0, x_1)(0)) = f(x_0)
\]
and similarly
\[
 f_i(x_0, x_1)(1) = f(s_i(x_0, x_1)(1)) = f(x_1).
\]
so that the \( f_i \) are \( f \)-motion planners and TC \( f \leq k \).

(2) Note that if \( X \) is empty, then cat \( f = TC(f) = cat (f \times f) = -1 \), so it suffices to assume \( X \) is nonempty.

First we prove cat \( f \leq TC(f) \). Suppose TC \( (f) = k \) and let \( U_0, ..., U_k \) be an open cover of \( X \times X \) with \( f \)-motion planners \( f_i : U_i \rightarrow Y^I \). Let \( b \) be a choice of base point for \( X \) and define \( V_i = \{ x \mid \langle b, x \rangle \in U_i \} \). Then \( V_0, ..., V_k \) is clearly an open cover for \( X \). Now we define a nullhomotopy \( H_i : V_i \times I \rightarrow Y \) of \( f \) on \( V_i \) by
\[
 H_i(x, t) = f(b, x)(t).
\]
Therefore, cat \( f \leq k \).

Finally, we show that TC \( f \leq cat (f \times f) \). Suppose cat \( (f \times f) = k \) and let \( U_0, ..., U_k \) be open sets covering \( X \times X \) such that each \( (f \times f)|_{U_i} \) is nullhomotopic. Let \( H_i : U_i \times I \rightarrow Y \times Y \) be a nullhomotopy of \( (f \times f)|_{U_i} \) to some \( (y^i_0, y^i_1) \in Y \times Y \). Since \( Y \) is path-connected, let \( \gamma_i \) be some path from \( y^i_0 \) to \( y^i_1 \). Now define \( f_i : U_i \rightarrow Y^I \) by
\[
 f_i(x_0, x_1)(t) = \begin{cases} 
 p_0 (H_i(x_0, x_1, 3t)) & \text{if } 0 \leq t \leq \frac{1}{3} \\
 \gamma_i(3t - 1) & \text{if } \frac{1}{3} \leq t \leq \frac{2}{3} \\
 p_1 (H_i(x_0, x_1, 3 - 3t)) & \text{if } \frac{2}{3} \leq t \leq 1
\end{cases}
\]
where \( p_0, p_1 : Y \times Y \rightarrow Y \) are projections into the first and second coordinates, respectively. Then \( f_i \) is continuous since \( H_i(x_0, x_1)(1) = (y^i_0, y^i_1) \). Furthermore,
\[
 f_i(x_0, x_1)(0) = p_0 (H_i(x_0, x_1, 3 \cdot 0)) = p_0 ((f \times f)(x_0, x_1)) = f(x_0)
\]
and
\[
 f_i(x_0, x_1)(1) = p_1 (H_i(x_0, x_1, 3 \cdot 3 \cdot 1)) = p_1 ((f \times f)(x_0, x_1)) = f(x_1)
\]
so that the \( f_i \) are \( f \)-motion planners. Therefore, TC \( f \leq k \). \( \square \)

The following corollary corresponds to Corollary 2.3 of [15]:

**Corollary 3.9.** Let \( f : X \rightarrow Y \) be any map. Then TC \( f = 0 \) if and only if \( f \) is nullhomotopic.

**Proof.** First suppose TC \( f = 0 \). Then
\[
 0 \leq \text{cat} (f) \leq \text{TC} (f) = 0
\]
so that cat \( f = 0 \) and \( f \) is nullhomotopic.
Now suppose \( f \) is nullhomotopic. Then \( f \times f \) is nullhomotopic so that
\[
0 \leq \text{TC}(f) \leq \text{cat}(f \times f) = 0
\]
and hence \( \text{TC}(f) = 0 \). \( \square \)

**Proposition 3.10.** Let \( f : X \to Z \) and \( g : Y \to W \) be any maps such that \( X \times X \) and \( Y \times Y \) are both normal. Then \( \text{TC}(f \times g) \leq \text{TC}(f) + \text{TC}(g) \).

**Proof.** This is immediate by Proposition 2.4. \( \square \)

The cup length lower bound was proven in [15] for CW-complexes with coefficients in a constant ring. With an identical proof, we can switch CW-complexes for \( X \) an ANR and use coefficients in any \( \pi_1(Y) \times \pi_1(Y) \) module:

**Theorem 3.11.** Let \( f : X \to Y \) be any map with \( X \) an ANR. Suppose that \( u_i \in \tilde{H}^*(X \times X; \mathbb{A}_i) \) are such that \( u_i \in \ker \Delta^* \cap \text{im}(f \times f)^* \) and \( u_0 \cup \ldots \cup u_k \neq 0 \), where each \( \mathbb{A}_i \) are \( \pi_1(Y) \times \pi_1(Y) \) modules. Then \( \text{TC}(f) \geq k + 1 \). In other words,
\[
\text{cup} \left( \ker \Delta^* \cap \text{im}(f \times f)^* \right) \leq \text{TC}(f).
\]

**Proof.** Suppose \( \text{TC}(f) \leq k \) so that there are open sets \( U_0, ..., U_k \) covering \( X \times X \) such that on each \( U_i \) there is a map \( h : U_i \to X \) satisfying \( (f \times f)\Delta h \simeq (f \times f)|_{U_i} \). Thus, for each \( i \) we have a commutative diagram
\[
\begin{array}{cccccc}
... & \rightarrow & H^n(X \times X; U_i; \mathbb{A}_i) & \overset{q_i^*}{\longrightarrow} & H^n(X \times X; \mathbb{A}_i) & \overset{j_i^*}{\rightarrow} & H^n(U_i; \mathbb{A}_i) & \rightarrow & ...
\end{array}
\]

where the top row is the long exact sequence of the pair \( (X \times X, U_i) \), and the maps \( q_i \) and \( j_i \) are the obvious ones.

Now let \( u_0, ..., u_k \in \ker \Delta^* \cap \text{im}(f \times f)^* \) where each \( u_i \) is taken in coefficients \( \mathbb{A}_i \) and has degree at least 1. Then there is some \( v_i \in \ker \tilde{H}^*(Y \times Y; \mathbb{A}_i) \) such that \( (f \times f)^* v_i = u_i \). Since each \( u_i \in \ker \Delta \), it follows that
\[
j_i^* u_i = j_i^* (f \times f)^* v_i = h_i^* \Delta^* (f \times f)^* v_i = h_i^* \Delta^* u_i = h_i^* 0 = 0
\]
so that \( u_i \in \ker j_i^* \). By exactness, there is some \( w_i \in H^*(X \times X, U_i; \mathbb{A}_i) \) such that \( q_i^* w_i = u_i \). Now consider \( w_0 \cup \ldots \cup w_k \), which lies in
\[
H^*(X \times X, U_0 \cup \ldots \cup U_k; \mathbb{A}_0 \otimes \ldots \otimes \mathbb{A}_k) = H^*(X \times X, X \times X; \mathbb{A}_0 \otimes \ldots \otimes \mathbb{A}_k) = 0
\]
so that \( w_0 \cup \ldots \cup w_k = 0 \); hence,
\[
u_0 \cup \ldots \cup u_k = q_0^* w_0 \cup \ldots \cup q_k^* w_k = q^* (w_0 \cup \ldots \cup w_k) = 0
\]
where \( q : X \times X \to (X \times X, X \times X) \) is the obvious map of pairs. Thus, the theorem follows. \( \square \)
3.3 Homotopy Invariance

Proposition 3.12. If \( f, g : X \to Y \) are homotopic, then \( TC(f) = TC(g) \).

Proof. By symmetry, it suffices to prove \( TC(g) \leq TC(f) \). Suppose \( TC(f) = k \) so that there is an open covering \( U_0, \ldots, U_k \) of \( X \times X \) such that each restriction \( (f \times f)|_{U_i} \) can be deformed into \( \Delta Y \). Since \( g \) is homotopic to \( f \), it follows that \( (g \times g)|_{U_i} \) is homotopic to \( (f \times f)|_{U_i} \); hence, \( (g \times g)|_{U_i} \) can also be deformed into \( \Delta Y \). Therefore, \( TC(g) \leq k = TC(f) \). \( \square \)

Proposition 3.13. \( TC(gf) \leq \min\{TC(g), TC(f)\} \) for any maps \( f : X \to Y \) and \( g : Y \to Z \).

Proof. Suppose that \( TC(f) = k \) so that there is an open cover \( U_0, \ldots, U_k \) of \( X \times X \) such that each restriction \( (f \times f)|_{U_i} \) can be deformed into \( \Delta Y \) via some homotopy \( F_i : U_i \times I \to Y \times Y \). Now define a homotopy \( H_i : U_i \times I \to Z \times Z \) by \( H_i = (g \times g)F_i \), which defines a deformation of \( (gf \times gf)|_{U_i} \) into \( \Delta Z \); hence, \( TC(gf) \leq k = TC(f) \).

Now suppose that \( TC(g) = k \) so that there is an open cover \( V_0, \ldots, V_k \) of \( Y \times Y \) such that each restriction \( (g \times g)|_{V_i} \) can be deformed into \( \Delta Z \) via some homotopy \( G_i : V_i \times I \to Z \times Z \). Now let \( U_i = (f \times f)^{-1}(V_i) \) and define a homotopy \( H_i : U_i \times I \to Z \times Z \) by
\[
H_i(x_0, x_1, t) = G_i(f(x_0), f(x_1), t),
\]
which defines a deformation of \( (gf \times gf)|_{U_i} \) into \( \Delta Z \); hence, \( TC(gf) \leq k = TC(g) \). \( \square \)

Corollary 3.14. Let \( h : X \to Y \) be any map.

1. If \( h \) has a right homotopy inverse, then \( TC(h) = TC(Y) \).

2. If \( h \) has a left homotopy inverse, then \( TC(h) = TC(X) \).

3. If \( h \) is a homotopy equivalence, then \( TC(h) = TC(X) = TC(Y) \).

Proof. (1) Let \( g : Y \to X \) be the right homotopy inverse of \( h \). Then \( hg \) is homotopic to \( Id_Y \), so that \( TC(hg) = TC(Id_Y) \). Thus, we have
\[
TC(Y) = TC(Id_Y) = TC(hg) \leq TC(h) \leq TC(Y)
\]
so that \( TC(h) = TC(Y) \).

(2) Let \( g : Y \to X \) be the left homotopy inverse of \( h \). Then \( gh \) is homotopic to \( Id_X \), so that \( TC(gh) = TC(Id_X) \). Thus, we have
\[
TC(X) = TC(Id_X) = TC(gh) \leq TC(h) \leq TC(X)
\]
so that \( TC(h) = TC(X) \). \( \square \)

Proposition 3.15. Suppose we have a diagram of the form
\[
\begin{array}{ccc}
X & \xrightarrow{h} & X' \\
\downarrow f & & \downarrow f' \\
Y & & 
\end{array}
\]
such that \( h \) is a homotopy domination. Then \( TC(f) = TC(f') \).
Proof. It’s immediate that \( TC(f) = TC(f'h) \leq TC(f') \), so it suffices to prove \( TC(f') \leq TC(f) \). Since \( h \) is a homotopy domination, it has a right homotopy inverse, i.e., a map \( g : X' \to X \) such that \( hg \) is homotopic to \( Id_{X'} \). Then \( fg = f'hg \), so that \( fg \) is homotopic to \( f' \); hence, \( TC(f') = TC(fg) \leq TC(f) \).

\[ \text{Proposition 3.16. Suppose we have a diagram of the form} \]

\[ \begin{array}{ccc}
  & X & \\
  f & \downarrow & f' \\
  Y & \longrightarrow & Y'
\end{array} \]

\[ \text{such that } h \text{ has a left homotopy inverse. Then } TC(f) = TC(f'). \]

Proof. It’s immediate that \( TC(f') = TC(hf) \leq TC(f) \), so it suffices to prove \( TC(f) \leq TC(f') \). Let \( g : Y' \to Y \) be the left homotopy inverse of \( h \) so that \( gh \) is homotopic to \( Id_Y \). Then \( gf' = ghf \), so that \( gf' \) is homotopic to \( f \); hence, \( TC(f) = TC(gf') \leq TC(f') \).

\[ \text{Remark 3.17. Propositions 3.15 and 3.16 imply that } f \text{ can always be replaced by either a fibration or a cofibration for the purposes of computing } TC(f). \]

3.4 Examples

Examples 3.18. Fiber bundles of projective spaces:

1. Let \( p : S^n \to \mathbb{R}P^n \) be the obvious covering map. Consider the sets

\[ F_0 = \{(x, y) \in S^n \times S^n \mid x \neq -y\} \]

and

\[ F_1 = \{(x, -x) \mid x \in S^n\}, \]

i.e., the sets of nonantipodal and antipodal pairs respectively. On the set of nonantipodal pairs, we can map pairs \((x, y)\) to the unique geodesic from \( x \) to \( y \) and compose that with \( p \) to get a \( p \)-motion planner \( p_0 : F_0 \to (\mathbb{R}P^n)^{p} \). Since antipodal pairs map to the same equivalence class in \( \mathbb{R}P^n \), it follows that mapping \((x, -x)\) to the constant path at \( p(x) \) is a \( p \)-motion planner on \( F_1 \). Thus, \( TC(p) = 1 \) since \( p \) is not nullhomotopic.

2. \( TC(p : S^{2n+1} \to \mathbb{C}P^n) = 1 \) since \( TC(p) \leq TC(S^{2n+1}) = 1 \) and \( p \) is not nullhomotopic.

3. Similarly, \( TC(S^{4n+3} \to \mathbb{H}P^n) = 1 \) and \( TC(S^{8n+7} \to \mathbb{O}P^n) = 1 \).

Definition 3.19. An H-space is a topological space \( X \) with a map \( \mu : X \times X \to X \) and an element \( e \in X \) such that \( \mu_i_0 \) and \( \mu_i_1 \) are homotopic to the identity map \( Id_X \), where \( i_0, i_1 : X \to X \times X \) are the inclusion maps given by \( i_0(x) = (x, e) \) and \( i_1(x) = (e, x) \).

For an H-space \( X \), there is an equality \( TC(X) = \text{cat}(X) \). This is proven in [1] and also in [8] for the special case of topological groups. We extend this theorem to maps when the domain is an H-space:
**Theorem 3.20.** Let \( f : X \to Y \) be any map such that \( X \) is an H-space. Then \( \text{TC}(f) = \text{cat}(f) \).

**Proof.** Let \( \mu, \iota_0, \iota_1 \), and \( e \) be as in definition 3.19 for the H-space \( X \). Note that it is sufficient to prove \( \text{TC}(f) \leq \text{cat}(f) \), so let \( \text{cat}(f) = k \) and let \( U_0, \ldots, U_k \) be an open cover of \( X \) such that each restriction \( f|_{U_i} \) is nullhomotopic. Define \( V_i \) by \( V_i = \mu^{-1}(U_i) \), let \( H_i : X \times I \to Y \) be a nullhomotopy of \( f|_{U_i} \), and let \( F_0, F_1 : X \times I \to X \) be homotopies of \( \mu \iota_0 \) to \( \text{Id}_X \) and of \( \mu \iota_1 \) to \( \text{Id}_X \) respectively. Now define an \( f \)-motion planner \( f_i : V_i \to Y \) on \( V_i \) by

\[
f_i(x_0, x_1)(t) =
\begin{cases}
H_i(F_0(x_0, 1 - 2t), 2t) & 0 \leq t \leq \frac{1}{2} \\
H_i(F_1(x_1, 2t - 1), 2 - 2t) & \frac{1}{2} \leq t \leq 1
\end{cases}
\]

so that \( \text{TC}(f) \leq \text{cat}(f) \). \hfill \Box

**4 The Mixed TC of a Map**

In a series of three papers [16, 17, 18], the (reduced) topological complexity of a map \( f : X \to Y \) is defined to be the least integer \( n \) such that there is a sequence of closed sets

\[
\emptyset = C_{-1} \subset C_0 \subset \ldots \subset C_n = Y \times X
\]

such that each difference \( C_i \setminus C_{i-1} \) admits a partial section of the map \( \pi_f : X^I \to Y \times X \) given by \( \alpha \mapsto (f(\alpha(0)), \alpha(0)) \). In this paper, we will call this invariant the **robotic topological complexity** of the map \( f \), denoted \( \text{TC}^{ra}(f) \), in order to differentiate it from the definition in the previous section. This name is inspired by the fact that it was initially defined for applications involving robotic arms.

**Theorem 4.1 ([18]).** Let \( f : X \to Y \) be a map. Then \( \pi_f \) is a fibration if and only if \( f \) is a fibration. Moreover, if \( f \) is a fibration, then \( \text{TC}^{ra}(f) = \text{secat}(\pi_f) = \text{secat}((\text{Id}_Y \times f)^*\Delta^Y_0) \).

In light of this theorem, the main focus of this section is to study a notion for the topological complexity of a map using the following pullback diagram:

\[
\begin{array}{ccc}
(Id_Y \times f)^*Y^I & \xrightarrow{T_{fY} \times f} & Y^I \\
\downarrow{(Id_Y \times f)^*\Delta^Y_0} & & \downarrow{\Delta^Y_0} \\
Y \times X & \xrightarrow{Id \times f} & Y \times Y
\end{array}
\]

**4.1 Definitions**

**Definition 4.2.** Let \( f : X \to Y \) be a map.

1. A **mixed \( f \)-motion planner** on a subset \( Z \subset Y \times X \) is a map \( f_Z : Z \to Y^I \) such that \( f_Z(y, x)(0) = y \) and \( f_Z(x_0, x_1)(1) = f(x_1) \).
2. A mixed $f$-motion planning algorithm is a cover of $Y \times X$ by sets $Z_0, \ldots, Z_k$ such that on each $Z_i$ there is some mixed $f$-motion planner $f_i : Z_i \to Y^I$.

3. The mixed topological complexity of $f$, denoted $\text{TC}^\sharp(f)$, is the least $k$ such that $Y \times X$ can be covered by $k + 1$ open subsets $U_0, \ldots, U_k$ on which there are mixed $f$-motion planners. If no such $k$ exists, we define $\text{TC}^\sharp(f) = \infty$.

Remark 4.3. If $Y$ isn’t path connected, $\text{TC}^\sharp(f)$ will always be $\infty$, so to avoid this one would need to add up $\text{TC}^\sharp(f)$ for each path component of $Y$. Thus, as with the previous section, we assume all spaces to be path-connected.

Examples 4.4.

1. $\text{TC}^\sharp(f) = -1$ if and only if $f$ is the empty map.

2. If $i : A \to X$ is an inclusion map, then $\text{TC}^\sharp(i) = \text{TC}^\sharp_X(X \times A)$.

3. $\text{TC}^\sharp(\text{Id}_X) = \text{TC}(X)$ for any space $X$.

As with (pullback) TC, there are versions of Theorems 3.4 and 3.6 and Corollary 3.7 for mixed TC as well, though we will skip the proofs as they are identical.

Theorem 4.5. Let $f : X \to Y$ be a map and let $Z \subset Y \times X$. The following are equivalent:

1. there is a section $s : Z \to (\text{Id}_Y \times f)^*Y^I$ of the pullback fibration $(\text{Id}_Y \times f)^*\Delta^Y_0$;

2. there is a mixed $f$-motion planner $f_Z : Z \to Y^I$;

3. $(\text{Id}_Y \times f)|_Z$ can be deformed into $\Delta Y$.

Theorem 4.6. If $f : X \to Y$ is a map between ANRs, then $\text{TC}^\sharp(f) = \text{secat}_g((\text{Id}_Y \times f)^*\Delta^Y_0)$.

Corollary 4.7. Let $f : X \to Y$ be any map with $Y \times X$ normal. Then $\text{TC}^\sharp(f) \leq k$ if and only if there is a lift of $\text{Id} \times f$ with respect to $\Delta^Y_{k+1}$.

4.2 Basic Inequalities

Proposition 4.8. Let $f : X \to Y$ be a map. Then

1. $\text{cat}(Y) \leq \text{TC}^\sharp(f) \leq \text{TC}(Y)$;

2. $\text{cat}(f) \leq \text{TC}^\sharp(f) \leq \text{cat}(\text{Id}_Y \times f)$.

Proof. The proofs for (2) and for the inequality $\text{TC}^\sharp(f) \leq \text{TC}(Y)$ are the same as in Proposition 3.8. Then the proof of $\text{cat}(\text{Id}_Y) \leq \text{TC}^\sharp(f)$ is the same as the proof of $\text{cat}(f) \leq \text{TC}^\sharp(f)$, so that $\text{cat}(Y) \leq \text{TC}^\sharp(f)$. □

Corollary 4.9. Let $f : X \to Y$ be any map. If $f$ is nullhomotopic, then $\text{TC}^\sharp(f) = \text{cat}(Y)$. 15
Example 4.10. Let $Y$ be a non-contractible topological group, e.g., $S^1$. Then we have $TC^\frac{1}{2}(Id_Y) = TC(Y) = \text{cat}(Y)$, but $Id_Y$ isn’t nullhomotopic since $Y$ isn’t contractible.

Corollary 4.11. Let $f : X \to Y$ be any map. Then $TC^\frac{1}{2}(f) = 0$ if and only if $Y$ is contractible.

Proof. If $TC^\frac{1}{2}(f) = 0$, then $\text{cat}(Y) \leq TC^\frac{1}{2}(f) = 0$ so that $Y$ is contractible. Conversely, if $Y$ is contractible, then $f$ is nullhomotopic so that $TC^\frac{1}{2}(f) = \text{cat}(Y) = 0$.

Proposition 4.12. Let $f : X \to Z$ and $g : Y \to W$ be any maps such that $Z \times X$ and $W \times Y$ are both normal. Then $TC(f \times g) \leq TC(f) + TC(g)$.

Proof. This is immediate by Proposition 2.4.

4.3 Homotopy Invariance

Proposition 4.13. If $f, g : X \to Y$ are homotopic, then $TC^\frac{1}{2}(f) = TC^\frac{1}{2}(g)$.

Proof. This follows immediately from property (3) of Theorem 4.5.

Proposition 4.14. $TC^\frac{1}{2}(gf) \leq TC^\frac{1}{2}(g)$ for any maps $f : X \to Y$ and $g : Y \to Z$.

Proof. Let $TC^\frac{1}{2}(g) = k$ so that there is an open cover $V_0, \ldots, V_k$ of $Z \times Y$ such that each restriction $(Id_Z \times g)|_{V_i}$ can be deformed into $\Delta Z$ via some homotopy $G_i : V_i \times I \to Z \times Z$. Now let $U_i = (Id_Z \times f)^{-1}(V_i)$ and define a homotopy $H_i : U_i \times I \to Z \times Z$ by

$$H_i(z, x, t) = G_i(z, f(x), t),$$

which defines a deformation of $(Id_Z \times gf)|_{U_i}$ into $\Delta Z$; hence, $TC^\frac{1}{2}(gf) \leq k = TC^\frac{1}{2}(g)$.

Unfortunately, unlike the (pullback) TC of a map, it is possible for $TC^\frac{1}{2}(f)$ to be smaller than $TC^\frac{1}{2}(gf)$:

Example 4.15. Consider any maps $f : X \to Y$ and $g : Y \to Z$ such that $f$ is nullhomotopic and $\text{cat}(Y) < \text{cat}(Z)$. Then $gf$ is also nullhomotopic so that $TC^\frac{1}{2}(f) = \text{cat}(Y)$ and $TC^\frac{1}{2}(gf) = \text{cat}(Z)$ by Corollary 4.9, hence, $TC^\frac{1}{2}(f) < TC^\frac{1}{2}(gf)$. E.g., take $X = Y = \ast$ and $Z = S^1$ with $f$ and $g$ the obvious maps so that $TC^\frac{1}{2}(f) = 0$ and $TC^\frac{1}{2}(gf) = 1$. 16
Corollary 4.16. Let \( h : X \to Y \) be any map.

1. If \( h \) has a right homotopy inverse, then \( \text{TC}^\frac{1}{2} (h) = \text{TC} (Y) \).

2. If \( h \) is a homotopy equivalence, then \( \text{TC}^\frac{1}{2} (h) = \text{TC} (X) = \text{TC} (Y) \).

Proof. (1) Let \( g : Y \to X \) be the right homotopy inverse of \( h \). Then \( hg \) is homotopic to \( \text{Id}_Y \), so that \( \text{TC}^\frac{1}{2} (hg) = \text{TC}^\frac{1}{2} (\text{Id}_Y) \). Thus, we have

\[
\text{TC} (Y) = \text{TC}^\frac{1}{2} (\text{Id}_Y) = \text{TC}^\frac{1}{2} (hg) \leq \text{TC}^\frac{1}{2} (h) \leq \text{TC} (Y)
\]

so that \( \text{TC}^\frac{1}{2} (h) = \text{TC} (Y) \).

Unlike Corollary 3.14 for \( \text{TC} (h) \), it does not follow that \( \text{TC}^\frac{1}{2} (h) = \text{TC} (X) \) when \( h \) has a left homotopy inverse. See the following counterexample:

Example 4.17. Let \( h : \ast \to S^1 \) be an inclusion map. Then \( h \) has the obvious left (homotopy) inverse \( g : S^1 \to \ast \). Since \( h \) is nullhomotopic, \( \text{TC}^\frac{1}{2} (h) = \text{cat} (S^1) = 1 \) but unfortunately \( \text{TC} (\ast) = 0 < \text{TC}^\frac{1}{2} (h) \).

Proposition 4.18. Suppose we have a diagram of the form

\[
\begin{array}{ccc}
X & \xrightarrow{h} & X' \\
\downarrow^{f} & & \downarrow^{f'} \\
Y & \xleftarrow{f} & Y'
\end{array}
\]

such that \( h \) is a homotopy domination. Then \( \text{TC}^\frac{1}{2} (f) = \text{TC}^\frac{1}{2} (f') \).

Proof. It’s immediate that \( \text{TC}^\frac{1}{2} (f) = \text{TC}^\frac{1}{2} (f'h) \leq \text{TC}^\frac{1}{2} (f') \), so it suffices to prove \( \text{TC}^\frac{1}{2} (f') \leq \text{TC}^\frac{1}{2} (f) \). Since \( h \) is a homotopy domination, it has a right homotopy inverse, i.e., a map \( g : X' \to X \) such that \( hg \) is homotopic to \( \text{Id}_{X'} \). Then \( fg = f'hg \), so that \( fg \) is homotopic to \( f' \); hence, \( \text{TC}^\frac{1}{2} (f') = \text{TC}^\frac{1}{2} (fg) \leq \text{TC}^\frac{1}{2} (f) \).

Proposition 4.19. Suppose we have a diagram of the form

\[
\begin{array}{ccc}
X & \xrightarrow{h} & Y' \\
\downarrow^{f} & & \downarrow^{f'} \\
Y & \xleftarrow{f} & Y'
\end{array}
\]

such that \( h \) is a homotopy equivalence. Then \( \text{TC}^\frac{1}{2} (f) = \text{TC}^\frac{1}{2} (f') \).
Proof. Let \( g : Y' \to Y \) be the homotopy inverse of \( h \) and note that the commutative triangle above induces the following commutative square:

\[
\begin{array}{ccc}
Y \times X & \xrightarrow{h \times Id_X} & Y' \times X \\
\downarrow{Id_Y \times f} & & \downarrow{Id_{Y'} \times f'} \\
Y \times Y & \xrightarrow{h \times h} & Y' \times Y'
\end{array}
\]

First let \( TC^\sharp (f) = k \) and let \( U_0, ..., U_k \) be an open cover of \( Y \times X \) such that each restriction \( (Id_Y \times f)|_{U_i} \) can be deformed into \( \Delta Y \) via some homotopy \( H_i : U_i \to Y \times Y \). Define an open cover \( V_i = (g \times Id_X)^{-1}(U_i) \) of \( Y' \times X \). Then it follows that

\[
Id_{Y'} \times f' \simeq h g \times f' = h g \times f.
\]

Now define a deformation \( G_i : V_i \to Y \times Y \) of \( (g \times f)|_{V_i} \) into \( \Delta Y \) given by \( G_i(y', x, t) = H_i(g(y'), x, t) \); hence, \( (h g \times h f)|_{V_i} \) can be deformed into \( \Delta Y' \). Therefore, \( TC^\sharp (f') \leq k \).

Now let \( TC^\sharp (f') = k \) and let \( V_0, ..., V_k \) be an open cover of \( Y' \times X \) such that each restriction \( (Id_{Y'} \times f')|_{V_i} \) can be deformed into \( \Delta Y' \) via some homotopy \( G_i : V_i \to Y' \times Y' \). Define an open cover \( U_i = (h \times Id_X)^{-1}(V_i) \) of \( Y \times X \). Then it follows that

\[
Id_Y \times f \simeq g h \times g f' = g h \times g f.'
\]

Now define a deformation \( H_i : U_i \to Y' \times Y' \) of \( (h \times f')|_{U_i} \) into \( \Delta Y' \) given by \( H_i(y, x, t) = G_i(h(y), x, t) \); hence, \( (g h \times g f')|_{U_i} \) can be deformed into \( \Delta Y \). Therefore, \( TC^\sharp (f) \leq k \).

Remark 4.20. Propositions 4.18 and 4.19 imply that \( f \) can always be replaced by either a fibration or a cofibration for the purposes of computing \( TC^\sharp (f) \). In fact, many of the bounds for mixed \( TC \) that have been presented here can also be obtained by taking a fibration replacement and then applying the results of [18].

4.4 Cohomological Lower Bound

Theorem 4.21. Let \( f : X \to Y \) be any map such that \( X \times Y \) is an ANR. Suppose that \( u_i \in H^\ast(Y \times X; A_i) \) are such that \( u_i \in \ker(f, Id_X)^\ast \) and \( u_0 \cup ... \cup u_k \neq 0 \), where each \( A_i \) are \( \pi_1(Y) \times \pi_1(X) \) modules. Then \( TC^\sharp (f) \geq k + 1 \). In other words,

\[
\cup(\ker(f, Id_X)^\ast) \leq TC^\sharp (f).
\]

Proof. First note that we can assume \( f \) is a fibration by taking a fibration replacement and applying Proposition 4.18 so that \( \pi_f \) is a fibration with \( \secat(\pi_f) = TC^\sharp (f) = TC'^\ast (f) \) by Theorem 4.1. From here one need only apply the usual proof for the cup product lower bound for the sectional category of the fibration \( \pi_f \) (see [19]), or one could note that the cup product lower bound in [18] also works more generally for ANRs and coefficients in \( \pi_1(Y) \times \pi_1(X) \) modules.
4.5 Comparisons

So far this paper has mentioned 3 potential notions for the topological complexity of a map, two of which are the homotopy invariant notions that are studied in depth in sections 3 and 4. These notions are defined as pullbacks, but one might also consider relative notions such as \( \text{TC}(C_f) \) where \( C_f \) is the mapping cone of \( f \); or \( \text{TC}_{M_f}(X \times X) \) where \( M_f \) is the mapping cylinder; or the relative topological complexity of the pair \( \text{TC}(M_f, X) \), whose general notion was introduced in [20].

**Example 4.22.** If \( f = \text{Id}_{S^1} \), then \( C_f \) is contractible so that \( \text{TC}(C_f) = 0 \), but \( \text{TC}(S^1) = 1 \). This shows that \( \text{TC}(C_f) \) wouldn’t be a generalization of the topological complexity of a space, so it isn’t considered here.

**Definition 4.23.** Let \( (X, A) \) with \( A \subset X \) be a pair of topological spaces. Then the relative topological complexity of \( (X, A) \), denoted \( \text{TC}(X, A) \), is defined to be \( \text{secat} \left( (i \times \text{Id}_X)^* \Delta_0^X \right) \) where \( i : A \to X \) is the inclusion map. (See [20] for a full introduction of this invariant.)

**Remark 4.24.** By Theorem 4.5, \( \text{TC}(X, A) \) is the same as \( \text{TC}^\downarrow (i) \).

We now have a list of reasonable definitions for the topological complexity of a map:

1. \( \text{TC}(f) \)
2. \( \text{TC}_{M_f}(X \times X) \)
3. \( \text{TC}^\downarrow (f) \)
4. \( \text{TC}(M_f, X) \)
5. \( \text{TC}^\rightarrow (f) \)

The following theorem describes their relationships:

**Theorem 4.25.** Let \( f : X \to Y \) be any map. Then

\[
\text{TC}(f) = \text{TC}_{M_f}(X \times X) \leq \text{TC}^\downarrow (f) = \text{TC}(M_f, X) \leq \text{TC}^\rightarrow (f).
\]

**Proof.** First note that Propositions 3.16 and 4.19 allow us to replace \( f \) with the cofibration \( i : X \to M_f \) for the purposes of computing \( \text{TC}(f) \) and \( \text{TC}^\downarrow (f) \) respectively. Therefore, \( \text{TC}(f) = \text{TC}_{M_f}(X \times X) \) and \( \text{TC}^\downarrow (f) = \text{TC}(M_f, X) \).

The inequality \( \text{TC}^\downarrow (f) \leq \text{TC}^\rightarrow (f) \) follows by taking a fibration replacement \( p_f \) of \( f \) and applying Theorem 4.4 of [18] to get that \( \text{TC}^\downarrow (f) = \text{TC}^\downarrow (p_f) = \text{TC}^\rightarrow (p_f) \). Then [18] proves that fibration replacements give the inequality \( \text{TC}^\rightarrow (p_f) \leq \text{TC}^\rightarrow (f) \).

All that remains is to prove \( \text{TC}(f) \leq \text{TC}^\downarrow (f) \). Note that \( \text{TC}(f) \) and \( \text{TC}^\downarrow (f) \) are defined as the sectional categories of \( (f \times f)^* \Delta_0^Y \) and \( (\text{Id}_Y \times f)^* \Delta_0^Y \) respectively. Furthermore, it follows that

\[
(f \times f)^* \Delta_0^Y = (f \times \text{Id}_X)^* (\text{Id}_Y \times f)^* \Delta_0^Y
\]

so that \( \text{TC}(f) \leq \text{TC}^\downarrow (f) \) by Proposition 2.3. \( \square \)
Corollary 4.26. Let \( f : X \to Y \) be a fibration. Then \( \text{TC}^{\frac{1}{2}}(f) = \text{TC}(M_f, X) = \text{TC}^{ra}(f) \).

Proof. This is immediate from the previous theorem and Theorem 4.1. \( \square \)

In [18], it’s shown that \( \text{TC}^{\frac{1}{2}}(f) \) and \( \text{TC}^{ra}(f) \) can be arbitrarily far apart when \( f \) is not a fibration. Example 4.17 also shows that \( \text{TC}(f) \) and \( \text{TC}^{\frac{1}{2}}(f) \) can differ, so that each of these three notions of TC are distinct. In fact, it seems to be the case that \( \text{TC}^{\frac{1}{2}}(f) \) is significantly more dependent on the codomain than \( \text{TC}(f) \), e.g., \( \text{TC}^{\frac{1}{2}}(f : X \to T^n) = n \) independent of \( X \) or \( f \).

5 Monoidal TC of a Map

This section will show how \( f \)-motion planners can be used to extend the definition of monoidal TC to maps. A similar adjustment can also be used to define the symmetric TC of maps, but that will be omitted for purposes of this work.

For this section, denote by \( \Delta_f \) the set \((f \times f)^{-1}(\Delta Y)\).

Definition 5.1. Let \( f : X \to Y \) be any map.

1. A reserved \( f \)-motion planner on a subset \( Z \subset X \times X \) is an \( f \)-motion planner \( f_Z \) on \( Z \) such that \( f_Z\Delta_f = i_Y(f \times f)|_{Z \cap \Delta_f} \).

2. A reserved \( f \)-motion planning algorithm is a cover of \( X \times X \) by sets \( Z_0, ..., Z_k \) such that on each \( Z_i \) there is some reserved \( f \)-motion planner \( f_i : Z_i \to Y^I \).

3. The monoidal topological complexity of \( f \), denoted \( \text{TC}^M(f) \), is the least \( k \) such that \( X \times X \) can be covered by \( k + 1 \) open subsets \( U_0, ..., U_k \) on which there are reserved \( f \)-motion planners. If no such \( k \) exists, then define \( \text{TC}^M(f) = \infty \).

Observe that we recover the usual definition \( \text{TC}^M(Id_X) = \text{TC}^M(X) \) for any space \( X \). Furthermore, \( \text{TC}(f) \leq \text{TC}^M(f) \) for any map \( f : X \to Y \). Thus, \( \text{TC}^M(f) \) is a good candidate for generalizing the modoidal topological complexity of a space. As we’ll see in what follows, many of the basic facts for spaces generalize to maps as well.

Unfortunately, monoidal TC isn’t a homotopy invariant for maps since the counterexample for spaces applies to \( \text{TC}^M(Id_X) \) (see [12]). Although this means it can’t be characterized as sectional category, like regular TC is, the reserved \( f \)-motion planners can still be characterized by a deformation property as is done for spaces in [3].

Theorem 5.2. Let \( f : X \to Y \) be a map and let \( Z \subset X \times X \). The following are equivalent:

1. there is a reserved \( f \)-motion planner \( f_Z : Z \to Y^I \);

2. \((f \times f)|_Z \) can be deformed into \( \Delta Y \) rel \( \Delta_f \cap Z \).
Proof. (1 \implies 2) Let \( f_Z : Z \to Y^I \) be a reserved \( f \)-motion planner. Now define a homotopy \( H : X \times X \times I \to Y \times Y \) by

\[
H(x_0, x_1, t) = (f_Z(x_0, x_1)(t), f(x_1)),
\]

which is a deformation of \((f \times f)|_Z\) into \(\Delta Y \cap \Delta_f \cap Z\).

(2 \implies 1) Now let \( H : Z \times I \to Y \times Y \) be a deformation of \((f \times f)|_Z\) into \(\Delta Y \cap \Delta_f \cap Z\). Define a map \( f_Z : Z \to Y^I \) by

\[
f_Z(x_0, x_1)(t) = \begin{cases} 
\text{proj}_0(H(x_0, x_1, 2t)) & \text{if } 0 \leq t \leq \frac{1}{2} \\
\text{proj}_1(H(x_0, x_1, 2t)) & \text{if } \frac{1}{2} < t \leq 1
\end{cases}
\]

where \(\text{proj}_0, \text{proj}_1 : Y \times Y \to Y\) are the projection maps into the corresponding coordinates. Then \( f_Z \) is well-defined and continuous since \( H(x_0, x_1, t) \in \Delta Y \) for all \((x_0, x_1) \in Z\). Moreover,

\[
f_Z(x_0, x_1)(0) = \text{proj}_0(H(x_0, x_1, 0)) = \text{proj}_0(f(x_0), f(x_1)) = f(x_0)
\]

and

\[
f_z(x_0, x_1)(1) = \text{proj}_1(H(x_0, x_1, 2)) = \text{proj}_1(H(x_0, x_1, 0)) = \text{proj}_1(f(x_0), f(x_1)) = f(x_1).
\]

Therefore, \( f_Z \) is an \( f \)-motion planner on \( Z \). Furthermore, if \( f(x_0) = f(x_1) \) for \((x_0, x_1) \in Z\), then \( H(x_0, x_1, t) = f(x_0) \) since \( H \) is a deformation rel \( \Delta_f \cap Z \); hence, \( \overline{f} \) is reserved. \(\square\)

In [3] and [13], it was shown for spaces that \(\text{TC} (X) \leq \text{TC}^M (X) \leq \text{TC} (X) + 1\) when \( X \) is an ENR. This generalizes to maps as follows:

**Theorem 5.3.** Let \( f : X \to Y \) be any map with \( X \) an ENR and \( Y \) Hausdorff. Then \( \text{TC} (f) \leq \text{TC}^M (f) \leq \text{TC} (f) + 1\).

**Proof.** Let \( \text{TC} (f) = k \) and let \( U_0, \ldots, U_k \) be an open cover of \( X \times X \) such that each restriction \((f \times f)|_{U_i}\) has a deformation \( H_i : U_i \times I \to Y \times Y \) into \( \Delta Y \). Consider the open sets \( V_i = U_i \cap \Delta_f \), on which there is a deformation of \((f \times f)|_{V_i}\) rel \( \Delta_f \cap V_i = \emptyset \), namely \( H_i|_{V_i} \). Note that \( \Delta_f \) is an ENR since \( X \) is an ENR and \( Y \) is Hausdorff; hence, there is an open neighborhood \( W \) of \( \Delta_f \) and a homotopy \( H : W \times I \to X \times X \) such that \( H(\cdot, 0) \) is the inclusion map and \( H(\cdot, 1) \) is a retraction. Then \((f \times f)H\) is a deformation of \((f \times f)|_W\) into \( \Delta Y \) rel \( \Delta_f \). Thus, \( \text{TC}^M (f) \leq \text{TC} (f) + 1 \) since \( V_0, \ldots, V_k, W \) is an open cover of \( X \times X \). \(\square\)

**Proposition 5.4.** Let \( f : X \to Y \) be any map and consider the diagram:

\[
\begin{array}{ccc}
X \times X & \xrightarrow{f \times f} & Y \times Y \\
\downarrow q_X & & \downarrow q_Y \\
(X \times X)/\Delta X & \xrightarrow{f_\Delta} & (Y \times Y)/\Delta Y \\
\end{array}
\]

where \( q_X \) and \( q_Y \) are the obvious quotient maps and \( f_\Delta \) is the induced map on the quotient spaces. Then \( \text{cat} (q_Y (f \times f)) \leq \text{TC} (f) \) and \( \text{cat} (f_\Delta) \leq \text{TC}^M (f) \).
Proof. Let $TC(f) = k$ so that there is an open cover $U_0, ..., U_k$ of $X \times X$ such that each restriction $(f \times f)|_{U_i}$ can be deformed into $\Delta Y$ via some homotopy $H_i : U_i \times I \to Y \times Y$. Now define $\overline{H}_i : X \times X \to (Y \times Y)/\Delta Y$ by $\overline{H}_i = q_Y H_i$, which is clearly a nullhomotopy of $q_Y(f \times f)$ on $U_i$; hence, $\text{cat}(q_Y(f \times f)) \leq TC(f)$.

Now let $TC^M(f) = k$. Then there are open sets $U_0, ..., U_k$ such that each restriction $(f \times f)|_{U_i}$ can be deformed into $\Delta Y \cap \Delta f \cap U_i$. Since the deformation on $(f \times f)|_{U_i}$ is done rel $\Delta f \cap U_i$, it induces a nullhomotopy on each $f_{\Delta} \cap q_Y(U_i)$ so that $\text{cat}(f_{\Delta}) \leq k$.

**Corollary 5.5.** Let $f : X \to Y$ be any map with $X$ an ENR and $Y$ Hausdorff. Then

$$\text{cat}(f_{\Delta}) \leq TC(f) + 1.$$ 

**Proof.** This is immediate by combining Theorem 5.3 and Proposition 5.4. 

6 Discrete Group Homomorphisms

**Definition 6.1.** Given a group $\pi$, an Eilenberg-MacLane space $K(\pi, 1) = B\pi$ is defined to be a space such that $\pi_1(B\pi) = \pi$ and $\pi_k(B\pi) = 0$ for $k \neq 1$.

Given $\pi$, an Eilenberg-MacLane space $B\pi = K(\pi, 1)$ is unique up to homotopy equivalence so that we can define $\text{cat}(\pi) = \text{cat}(B\pi)$ and $\text{TC}(\pi) = \text{TC}(B\pi)$. The case of LS-category is known:

**Theorem 6.2** ([6]). If $\pi$ is a group, then $\text{cat}(\pi) = \text{cd}(\pi)$, where $\text{cd}(\pi)$ is the cohomological dimension of $\pi$.

Since the cohomological dimension of groups with torsion is infinite, we consider only torsion free groups in this paper.

Unfortunately, the case of topological complexity is more complicated since $\text{TC}(\pi)$ can be anywhere between $\text{cd}(\pi)$ and $2\text{cd}(\pi)$. If $\pi$ is abelian, then $B\pi$ is a topological group so that $\text{TC}(\pi) = \text{cd}(\pi)$. If $\pi$ is a finitely generated torsion free hyperbolic group, then $\text{TC}(\pi) = 2\text{cd}(\pi)$ (see [4] and [11]).

The goal of this section is to introduce the LS-category and topological complexity of a group homomorphism. The former seems to have not been considered yet in the literature, which is likely because the LS-category of a group is known, but it is helpful in computing the latter.

Due to the homotopy invariance of Eilenberg MacLane spaces, there is a one-to-one correspondence between homomorphisms $f : G \to H$ and homotopy classes of continuous maps $Bf : BG \to BH$ that induce $f$ on the fundamental group. Therefore, the following definitions are well-defined:

**Definition 6.3.** Let $f : G \to H$ be a group homomorphism.

1. The LS-category of $f$, denoted $\text{cat}(f)$, is defined to be $\text{cat}(Bf)$.

2. The topological complexity of $f$, denoted $\text{TC}(f)$, is defined to be $\text{TC}(Bf)$.
6.1 LS-Category

Let $\pi$ be a group, and $I(\pi) \subset \mathbb{Z}[\pi]$ its augmentation ideal. Then

$$H^1(\pi; I(\pi)) \cong \text{Hom}_{\mathbb{Z}[\pi]}(I(\pi), I(\pi)).$$

The Bernstein-Schwarz class for $\pi$, denoted $\beta_n$, is defined to be the cohomology class on the left hand side that corresponds to $Id_{\mathbb{Z}[\pi]}$ on the right hand side.

**Theorem 6.4** ([5, 19]). Let $\pi$ be a group. Then

$$\text{cd}(\pi) = \max\{k \mid \beta^k_n \neq 0\}.$$  

In light of this fact, comes the following conjecture preceded by a useful lemma:

**Lemma 6.5.** Let $f : G \to H$ be a homomorphism between discrete groups. If $f^*\beta^k_n = 0$, then $f^*u = 0$ for all $u \in H^k(H; A)$ where $A$ is any $\mathbb{Z}H$-module.

**Proof.** Note that there is a $\mathbb{Z}H$-module homomorphism $\mu : I(H)^k \to A$ such that $\mu_*\beta^k_n = u$ (see Proposition 34 of [19] and Corollary 3.5 of [5]). Therefore,

$$f^*u = f^*\mu_*\beta^k_n = \mu_*f^*\beta^k_n = \mu_*0 = 0.$$  

$\square$

**Conjecture 6.6.** Let $f : G \to H$ be a homomorphism between discrete groups. Then

$$\text{cat}(f) = \max\{k \mid f^*\beta^k_n \neq 0\}.$$  

Unfortunately, all that has been produced so far is a partial proof, which is provided below:

**Proof.** The inequality $\text{cat}(f) \geq \max\{k \mid f^*\beta^k_n \neq 0\}$ is immediate by the cup length lower bound for $\text{cat}(f)$. Thus, suppose that $f^*\beta^{n+1}_n = 0$. Using the Ganea-Schwarz approach to prove $\text{cat}(f) \leq n$, it is sufficient to find a lift $\lambda$ for the diagram below:

\[
\begin{array}{c}
G_n(BH) \\
\downarrow \text{p}_n^{BH} \\
BG \\
\downarrow Bf \\
BH
\end{array}
\]

The fiber $*^{n+1}\Omega BH$ of $p_n^{BH}$ is $(n-1)$-connected so that $\lambda$ can immediately be extended to the $n$-skeleton of $BG$. Now let $\theta^{n+1} \in H^{n+1}(G; \pi_n(*^{n+1}\Omega BH))$ be the primary obstruction to extending the lift $\lambda$ to the $(n+1)$-skeleton of $BG$, and let $\phi^{n+1} \in H^{n+1}(H; \pi_n(*^{n+1}\Omega BH))$ be the primary obstruction to extending a section of $p_n^{BH}$ to the $(n+1)$-skeleton of $BH$. Then

$$\theta^{n+1} = f^*\phi^{n+1} = 0$$  

by Lemma 6.5 and the naturality of the obstruction cocycle. Therefore, we can extend the lift $\lambda$ to the $(n+1)$-skeleton of $BG$.  

$\square$
Proof. Consider the following pullback diagrams:

Let $f : G = \hat{H}$ be a homomorphism between discrete groups. If $f^*\beta_n \neq 0$ and $f^*\beta_{n+1} = 0$ with $\dim BG \leq n + 1$, then $\text{cat}(f) = n$.

Corollary 6.7. Let $f : G 	o H$ be a homomorphism between discrete groups such that $G$ is free. Then $\text{cat}(f) = 1$ if and only if $f^*\beta_n \neq 0$, i.e., Conjecture [??] holds.

Additionally, we can show the case of free abelian groups:

Theorem 6.9. Let $f : \mathbb{Z}^n \to \mathbb{Z}^m$ be a group homomorphism. Then

$$\text{cat}(f) = \max\{k \mid f^*\beta_k^B \neq 0\} = \text{rank}(f).$$

Proof. As before the inequality $\text{cat}(f) \geq \max\{k \mid f^*\beta_k^B \neq 0\}$ is immediate by the cup length lower bound for category. Let $j \leq m$ be the rank of $\text{im}(f)$ and note that $\mathbb{Z}^n = \ker(f) \oplus \text{im}(f)$; hence, we can break up $f$ into parts $f = f' \oplus g \oplus h$ where $f' : \mathbb{Z}^j \to \mathbb{Z}^j$, $g : \mathbb{Z}^{n-j} \to 0$, and $h : 0 \to \mathbb{Z}^{m-j}$ where $f'$ is injective. For all three of these maps we can easily calculate their $\text{LS}$-categories: $\text{cat}(f') = j$, $\text{cat}(g) = 0$, and $\text{cat}(h) = 0$. Therefore,

$$j = \text{cat}(f') \leq \text{cat}(f) \leq \text{cat}(f') + \text{cat}(g) + \text{cat}(h) = j$$

so that $\text{cat}(f) = j = \text{rank}(f)$.

Now suppose that $f^*\beta_{n+1}^B = 0$. To finish the proof, it suffices to show that $j \leq k$. Note that we can choose generators $b_1, \ldots, b_k$ of $\mathbb{Z}^n$ and $c_1, \ldots, c_m$ of $\mathbb{Z}^m$ and integers $0 \neq \alpha_1, \ldots, \alpha_j \in \mathbb{Z}$ such that $f(b_i) = \alpha_ic_i$ for $i \leq j$ and $f(b_i) = 0$ for $i > j$. Since $\mathbb{Z}^n$ and $\mathbb{Z}^m$ are abelian, it follows that $Bf_* = f$ on $H_1$. Thus, we can choose the dual elements $\hat{c}_1, \ldots, \hat{c}_j \in H^1(\mathbb{Z}^m)$ of $c_1, \ldots, c_j \in H_1(\mathbb{Z}^m)$. Next note that $f^*(\hat{c}_1 \cup \ldots \hat{c}_j) \neq 0$ because

$$f_*((b_1 \otimes \ldots \otimes b_j) \cap f^*(\hat{c}_1 \cup \ldots \cup \hat{c}_j)) = f_*((b_1 \otimes \ldots \otimes b_j) \cap (\hat{c}_1 \cup \ldots \cup \hat{c}_j)) = (\alpha_1c_1 \otimes \ldots \otimes \alpha_jc_j) \cap (\hat{c}_1 \cup \ldots \cup \hat{c}_j) = \alpha_1 \ldots \alpha_j \neq 0.$$ 

Then $j \leq k$ by Lemma [5.9] since $0 \neq f^*(\hat{c}_1 \cup \ldots \hat{c}_j) \in H^1(\mathbb{Z}^n)$ and $f^*\beta_{k+1}^B = 0$. \hfill \Box

Lemma 6.10. Let $f : G \to H$ be a group homomorphism. Then $\text{cat}(f) = \text{secat}((Bf)^*u_H)$, where $u_H : EH \to BH$ is the universal cover of $BH$.

Proof. Consider the following pullback diagrams:

$$
\begin{array}{ccc}
(Bf)^*EH & \longrightarrow & EH \\
\downarrow (Bf)^*u_H & & \downarrow u_H \\
BG & \longrightarrow & BH \\
\end{array}
\quad
\begin{array}{ccc}
(Bf)^*PBH & \longrightarrow & PBH \\
\downarrow (Bf)^*p_0^{BH} & & \downarrow p_0^{BH} \\
BG & \longrightarrow & BH \\
\end{array}
$$

Note that $\text{cat}(Bf) = \text{secat}((Bf)^*p_0^{BH})$ and our goal is to prove that $\text{cat}(Bf) = \text{secat}((Bf)^*u_H)$, so by Proposition [2.2] it’s sufficient to show that $(Bf)^*p_0^{BH}$ has a homotopy lift with respect to $(Bf)^*u_H$ and vice-versa. Then by the universal property of pullbacks, it suffices to prove
that $p_0^{BH}$ has a homotopy lift with respect to $u_H$ and vice-versa. Since $P_0BH$ is contractible, $p_0^{BH}$ trivially satisfies the lifting criterion for the covering map $u_H$ so that there is some lift $L$ of $p_0^{BH}$ with respect to $u_H$. Since $P_0BH$ and $EH$ are both contractible, $L$ must be a homotopy equivalence with some homotopy inverse $L$. Then

$$p_0^{BH}L = (u_H L)\sim u_H \text{Id}_{EH} = u_H$$

so that $L$ is a homotopy lift of $u_H$ with respect to $p_0^{BH}$. Therefore, $\text{cat}(Bf) = \text{secat}((Bf)^* u_H)$.

Now with this Lemma, if we assume $f$ is injective, then we get the following:

**Theorem 6.11.** Let $f : H \to G$ be an injective group homomorphism with $\text{cd}(H) < \infty$. Then $\text{cat}(f) = \text{cat}(H)$ and so Conjecture 6.6 holds.

**Proof.** We immediately have $\text{cat}(f) \leq \text{cat}(H)$, so it’s sufficient to prove $\text{cat}(H) \leq \text{cat}(f)$.

By Lemma 6.10, we know that $\text{cat}(f) = \text{secat}((Bf)^* u_G)$. By applying Proposition 2.2, we can reduce proving $\text{cat}(H) \leq \text{cat}(f)$ to proving that there is a homotopy lift of $(Bf)^* u_G$ with respect to the covering map $u_H : EH \to BH$, i.e., we need to prove the dashed arrow exists in the following diagram:

\[
\begin{array}{ccc}
EH & \xleftarrow{(Bf)^* EG} & EG \\
\downarrow{u_H} & \downarrow{(Bf)^* u_G} & \downarrow{u_G} \\
BH & \xleftarrow{Bf} & BG
\end{array}
\]

By the construction of $K(\pi, 1)$-spaces, it follows that $Bf$ can be taken to be injective since $f$ is injective. Thus, the map $Bf : (Bf)^* EG \to EG$ is injective. Since $EH$ and $EG$ are both contractible, it follows that $Ef$ is a homotopy equivalence with some homotopy inverse $L : EG \to EH$. Therefore, restricting $L$ to $(Bf)^* EG$, i.e., the map $LBf$, gives the homotopy lift we’re looking for. Thus, $\text{cat}(H) \leq \text{cat}(f)$. 

6.2 TC

When the domain is an abelian group, the topological complexity reduces to the LS-category:

**Theorem 6.12.** Let $f : G \to H$ be a homomorphism with $G$ abelian. Then $\text{cat}(f) = \text{TC}(f)$.

**Proof.** Note that $BG$ is an H-space since $G$ is abelian. Therefore $\text{cat}(f) = \text{TC}(f)$ by Theorem 3.20.

**Corollary 6.13.** Let $f : \mathbb{Z}^n \to \mathbb{Z}^m$ be a homomorphism. Then $\text{cat}(f) = \text{rank}(f)$.

**Proof.** This is immediate by Theorems 6.12 and 6.9.
Recall that thus, it’s easy to compute that the complexity of a map can be characterized by what spaces the map can factor through up to homotopy, i.e., a point when $TC(f) = 0$ and $S^1$ when $TC(f) \leq 1$.

**Theorem 6.14.** Let $f : F(n) \to F(m)$ be a homomorphism between free groups. Then

$$TC(f) = \begin{cases} 
0 & \text{if } f = 0 \\
1 & \text{if } \text{im}(f) \cong \mathbb{Z} \\
2 & \text{otherwise}
\end{cases}$$

**Proof.** It’s immediate that $TC(f) = 0$ if and only if $f = 0$, so it suffices to show that $TC(f) = 1$ if and only if $\text{im}(f) \cong \mathbb{Z}$.

First suppose that $\text{im}(f) \cong \mathbb{Z}$. Then $f$ is not nullhomotopic so that $TC(f) \geq 1$. Note that we can factor $f$ into $if'$ where $i : \text{im}(f) \to F(m)$ is the inclusion map and $f' : F(n) \to \text{im}(f)$ is the map restricting the codomain of $f$ to $\text{im}(f)$. Then it follows that

$$TC(f) \leq TC(f') \leq TC(\mathbb{Z}) = TC(S^1) = 1$$

so that $TC(f) = 1$.

Now suppose that $\text{im}(f)$ has at least two distinct generators, say $h_1$ and $h_2$, and let $g_1$ and $g_2$ be such that $f(g_i) = h_i$. Then $\langle g_1, g_2 \rangle$ is a free group on two generators and

$$TC(f|_{\langle g_1, g_2 \rangle}) \leq TC(f) \leq TC(F(n)) \leq 2$$

so that it suffices to prove $TC(f) = 2$. Consider the cohomology class

$$\nu_m \in H^1(F(m) \times F(m); I(F(m)))$$

given by the representative map $(g, h) \mapsto gh^{-1} - 1$. Since $f$ is injective, it follows that $m \geq 2$ so that $\nu_m^2 \neq 0$ by [2]. Note that $\nu_m \in \ker \Delta_2^1$ so that it suffices to prove that $(f \times f)^* \nu_m^2 \neq 0$ by Theorem 3.11. Note that we define $\nu_2$ in the same way as $\nu_m$ and similarly $\nu_2^2 \neq 0$. Since $f$ is injective, it induces an inclusion map $I(f) : I(F(2)) \to I(F(m))$, and on cohomology $f \times f$ simply induces a restriction on the representatives

$$[h : F(m) \times F(m) \to I(F(m))] \mapsto [h|_{\text{im}(f \times f)}].$$

Thus, it’s easy to compute that

$$(f \times f)^* \nu_m = I(f) \nu_2$$

so that

$$(f \times f)^* \nu_m^2 = (I(f) \otimes I(f)) \nu_2^2.$$

Recall that $I(F(2))$ is a free $\mathbb{Z}$-module so that, in particular, it is a flat $\mathbb{Z}$-module. Therefore, $I(f) \otimes I(f)$ is injective since $I(f)$ is injective. Thus, $(f \times f)^* \nu_m^2 \neq 0$ since $\nu_2^2 \neq 0$, so that $TC(f) = 2$. \hfill \Box

**Remark 6.15.** For 1-dimensional CW-complexes, Theorem 6.14 implies that the topological complexity of a map can be characterized by what spaces the map can factor through up to homotopy, i.e., a point when $TC(f) = 0$ and $S^1$ when $TC(f) \leq 1$. 

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In [8] and [9], it was shown that the topological complexity of a finite connected graph is the minimum of its first betti number and 2. Since graphs are aspherical spaces with free fundamental group, it follows that $TC(F(n)) = \min\{2, n\}$ where $F(n)$ is the free group on $n$-generators. For homomorphisms between free groups, the picture is a little bit more complicated but still achievable, but oddly enough the result still depends on the number of generators of the image.
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