INDUCED CONNECTIONS IN FIELD THEORY: 
THE ODD-DIMENSIONAL YANG-MILLS CASE

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Abstract
We consider $SU(N)$ Yang-Mills theories in $(2n+1)$-dimensional Euclidean space-time, where $N \geq n+1$, coupled to an even flavour number of Dirac fermions. After integration over the fermionic degrees of freedom the wave functional for the gauge field inherits a non-trivial $U(1)$-connection which we compute in the limit of infinite fermion mass. Its Chern-class turns out to be just half the flavour number so that the wave functional now becomes a section in a non-trivial complex line bundle. The topological origin of this phenomenon is explained in both the Lagrangean and the Hamiltonian picture.

Introduction
Induced connections can arise in any theory whose classical configuration space displays a certain topological richness. If these connections have non-trivial curvature – the case we are interested in – the precise condition is that the second cohomology $H^2(Q,\mathbb{Z})$ of the classical configuration space $Q$ must have a non-trivial free part (i.e. factors of $\mathbb{Z}$). Rather than outlining the general theory (which is basically the classification of $U(1)$-principal bundles over $Q$, lucidly explained e.g. in Ref. 1) we try to develop some feeling for the underlying mechanism and assumptions, by first discussing a simple finite dimensional toy model which mimics exactly the essential features without the analytic complications. Similar toy models have been used extensively throughout the literature in explaining the geometric and topological origin of anomalies and Berry-phases. But eventually we are interested in

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field theory. There we wish to understand in detail how the mere possibility of induced connections – established by pure topological arguments – is actualized by concretely given dynamical laws. The purpose of this article is to present such an example in which this process can be studied in detail. Similar to so many contributions to the understanding of anomalies, it will once more show a deep link between topology, geometry and dynamics.

Section 1: A Toy Model

Consider the 2-dimensional Hilbert space, \( C^2 \), and a Hamiltonian

\[
H = x \cdot \tau ,
\]

parameterized by the 2-sphere \( S^2 = \{ x \in \mathbb{R}^3 / \| x \| = 1 \} \). \( \tau = (\tau_1, \tau_2, \tau_3) \) are the Pauli matrices. We think of \( S^2 \) as the position-space of some particle. If we calculate the eigenvectors of \( H \), we find that there is no global phase choice to make them well defined over the whole of the 2-sphere. To circumvent this, we think of \( S^2 \) as the quotient of \( S^3 \approx SU(2) \) via the action of \( U(1)_R \), the group of right translations generated by \( \frac{i}{2} \tau_3 \). Let \( g \) be a general element of \( SU(2) \). The quotient map (the Hopf map) is then given by

\[
\pi : S^3 \to S^2 \\
g \mapsto g\tau_3 g^{-1} = x \cdot \tau
\]

such that the Hamiltonian is now given by \( g\tau_3 g^{-1} \), which is clearly invariant under \( U(1)_R \). Its eigenvalues are \( \pm 1 \) with eigenvectors, \( |\pm, g\rangle \), given by

\[
|\pm, g\rangle = g \cdot e_{\pm}, \quad e_{+} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_{-} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

That is, the positive eigenvector is given by the first column of the matrix \( g \), the negative one by the second. We write this as

\[
|\pm, g\rangle = \{ g_{i1} \}, \quad |-, g\rangle = \{ g_{i2} \},
\]

such that

\[
|\pm, g \exp(\pm i\frac{1}{2} \alpha \tau_3)\rangle = \exp(\pm i\frac{1}{2} \alpha)| \pm, g\rangle,
\]
where the last equation specifies the representations, \( \rho_{\pm} \), of \( U(1)_R \). Infinitesimally, the \( g \)-dependence of \( |g, \pm\rangle \) can be written as

\[
d g_{lk} = \Theta^i_k g_{in}
\]

where \( \Theta \) is the matrix of left invariant 1-forms, \( \{\sigma^i\} \), on \( SU(2) \):

\[
\Theta = g^{-1} d g = i 2 \sigma \cdot \tau.
\]

The eigenvectors, \( |+\rangle \) and \(-\rangle \), are equivariant functions on \( S^3 \) into the Hilbert-eigenspaces \( \mathcal{H}_\pm \) (here 1-dimensional) carrying the representation, \( \rho_+ \), and its complex conjugate, \( \rho_- \), respectively. This can be expressed by the commutative diagram:

\[
\begin{array}{ccc}
S^3 & \xrightarrow{|\pm\rangle} & \mathcal{H}_\pm \\
\downarrow^{U(1)_R} & & \downarrow^{\rho_{\pm}} \\
S^3 & \xrightarrow{|\pm\rangle} & \mathcal{H}_\pm \\
\end{array}
\]

If we now want to quantize the \( S^2 \)-degree of freedom as well (i.e. the particle motion) we can instead use \( S^3 \) as enlarged configuration space with one redundant (gauge) degree of freedom. We then have a wave function

\[
\psi : S^3 \longrightarrow \mathcal{H}_+ \oplus \mathcal{H}_-
\]

which, in the adiabatic approximation, (i.e., under the hypothesis of a slowly moving particle) can be split into \( \psi_{\pm} \), each mapping into the 1-dimensional \( \mathcal{H}_\pm \) only. The redundant degree of freedom, which is not to be quantized, is then taken care of by imposing the “Gauss constraint” which just expresses the adiabaticity condition in the requirement that \( \psi_{\pm} \) be an equivariant, \( \mathcal{H}_{\pm} \)-valued function on \( S^3 \), or equivalently, a section in a nontrivial line bundle over \( S^2 \), associated to the Hopf bundle (1.2) in the representation \( \rho_{\pm} \). According to (1.7) the Gauss constraint thus reads

\[
X_3 \psi_{\pm} = \pm i 2 \psi_{\pm},
\]

with \( X_3 \) being the left invariant vector field dual to \( \sigma_3 \). The last step is now to implement (1.9) dynamically, i.e. to find a Lagrangean for the particle on \( S^3 \) which
has $p_3 = \pm \frac{1}{2}$ as a constraint. To do this, we write down the standard line element on $S^2$ in terms of the 1-forms $\{\sigma^i\}$

$$ds^2 = \frac{1}{2} \left[ \sigma_1^2 + \sigma_2^2 \right],$$

(1.10)

which is $SU(2)_L \times U(1)_R$ invariant. The unique term with the same symmetries that gives the desired constraint is $\pm \frac{1}{2} \sigma_3$. The “effective” Lagrangean can then be locally projected onto $S^2$ and reads in the coordinate system that covers the 2-sphere except at the north pole

$$L = \frac{1}{2}m|\dot{x}|^2 \pm \frac{1}{2}(1 + \cos \theta)\dot{\phi}.$$ (1.11)

But this is just the Lagrangean of an electrically charged particle of unit charge in the background of a magnetic monopole of strength $g = \frac{1}{2}$. Recall that the connection (gauge potential) has been deduced under the explicit assumption of slow motion (hypothesis of adiabaticity). It might well be called the adiabatic connection, and its holonomies are just the celebrated Berry-Phases. For the field theoretic model of the next section it will be useful to have the following correspondences in mind:

$$\mathcal{H} = C^2 \longleftrightarrow \text{fermionic Hilbert space}$$
$$S^3 \longleftrightarrow \text{space of gauge potentials: } \mathcal{A}$$
$$S^2 \longleftrightarrow \text{space of gauge orbits: } \mathcal{A}/\mathcal{G} = \mathcal{Q}.$$ (1.12)

In this model massive fermions will induce a connection on the effective gauge theory which will be the adiabatic connection in the limit of infinite fermion mass. In the toy model we had $H^2(S^2, \mathbb{Z}) = \mathbb{Z}$ and the “magnetic field of the monopole” represented a non-trivial class therein (using DeRahm’s construction). The same picture arises in the infinite dimensional model. (See Ref. 2 for a general discussion of how this topological class is also generally responsible for a specific type of anomalies.)
Section 2: The 2n+1-Dimensional Yang-Mills Case

In this section we consider 2n+1 dimensional Euclidean Yang Mills fields coupled to Dirac fermions. The gauge group is taken to be $G = SU(N)$ where $N \geq 2$. Since Euclidean space is contractible, the bundle is necessarily trivial. Elements of the group of gauge-transformations are given by ordinary $SU(N)$-valued functions. By standard arguments gauge transformations are restricted to be the identity at infinity. This holds for space-time gauge transformations, as well as just spatial ones. In the Hamiltonian formulation we go into the $A_0 = 0$ gauge and consider instead a $G$-bundle over the $t =\text{const.}$ slices (called $\Sigma$). In both cases, the group of gauge transformations may be identified with the function space of the form

$$(R^N, \infty) \rightarrow (G, e),$$

where $N$ is either $2n + 1$ or $2n$ and $e$ is the identity in $G$. In the first case we call it $G^{2n+1}$, in the second $G^{2n}$. The symbol $\infty$ denotes the infinity in Euclidean space and the gauge transformations map the point at infinity to the identity $e$. The spaces of gauge potentials are called $A^{2n+1}$ and $A^{2n}$ respectively. For statements which are true in either case, we shall omit the superscripts.

$G$ acts on $A$ via

$$g \times A \mapsto g^{-1}(A + d)g.$$  \hfill (2.2)

If $A \in A$ is fixed by the action of $G$, it obeys $Dg = 0$, where $D$ is the exterior covariant derivative. With the imposed boundary conditions it is easy\(^a\) to see that this implies $g \equiv e$. $G$ therefore acts freely and $A$ can be given the structure of a principal fibre bundle (see Ref. 3)

$$G \rightarrow A \xrightarrow{\tau} Q.$$  \hfill (2.3)

\(^a\) The best way to see this is not to look at the gauge potentials but at the connection 1-forms on the principal bundle, the former being pull-backs of the latter by some local sections. On the total space $P$, gauge transformations are given by bundle automorphisms projecting to the identity, or, equivalently, by $G$ valued functions on $P$ which are $Ad$-equivariant under the right action of $G$ on $P$. Covariant constancy now means that the differential of this matrix valued function is zero if restricted to horizontal subspaces. The boundary conditions at infinity then force it to be the unit matrix.
where we have in fact two spaces, $Q^{2n+1}$ and $Q^{2n}$. Only the latter acts as configuration space in the canonical formulation and we will simply call it the configuration space. Since $A$ is an affine space, we have from the associated exact homotopy sequence

$$
\pi_k(Q^{2n}) = \pi_{k-1}(G^{2n}) = \pi_{2n+k-1}(G) \tag{2.4}
$$
$$
\pi_k(Q^{2n+1}) = \pi_{k-1}(G^{2n+1}) = \pi_{2n+k}(G). \tag{2.5}
$$

In particular we have, now specializing $^b N$ to $N \geq n + 1$,

$$
\pi_0(G^{2n}) = \pi_1(Q^{2n}) = 1 = \pi_1(G^{2n+1}) = \pi_2(Q^{2n+1}) \tag{2.6}
$$
$$
\pi_0(G^{2n+1}) = \pi_1(Q^{2n+1}) = Z = \pi_1(G^{2n}) = \pi_2(Q^{2n}), \tag{2.7}
$$

which also implies $H^2(Q^{2n}) = Z$ and hence the possibility of monopoles in the configuration space $Q^{2n}$. It is the purpose of the rest of this paper to demonstrate that the interaction with matter (here massive Dirac fermions) causes the wave function for the Yang Mills field to actualize this topological possibility. In Ref. 4 the possibility of induced connections has been anticipated and their consequences for the equal-time commutation relations discussed. In our derivation we follow the spirit of Ref. 2 and make use of both, the Lagrangean formulation, where gauge fields are defined over space time $M$, and the Hamiltonian formulation, where via the gauge condition $A_0 = 0$ one has a gauge theory over the spatial sections $\Sigma$. Note that a gauge transformation in $A^{2n+1}$ is given by a function $g : [0, 1] \times S^{2n} \to G$, such that $g_1 = g_0 \equiv e$ and $g_t(\infty) = e \forall t$, which at the same time defines an element of $\pi_1(G^{2n})$. Therefore, given a non-closed path in $A^{2n+1}$ which connects two different components of $G^{2n+1}$ in such a way that it projects to a loop in $Q^{2n+1}$ which generates $\pi_1(Q^{2n+1})$, one has at the same time found a generator of $\pi_1(G^{2n})$. In $A^{2n}$ this generator is the boundary of a 2-disk whose image (under the quotient map $A^{2n} \to Q^{2n}$) is a non-contractible 2-sphere generating $\pi_2(Q^{2n})$.

Finally, let us note that the bundle (2.3) with group $G^{2n}$ total space $A^{2n}$ and base $Q^{2n}$ can be given a natural connection once the metric on the spatial slices has been specified. Tangent vectors in $A \in A^{2n}$ are Lie algebra-valued one forms which under gauge transformations transform with the inverse\(^c\) adjoint representation. We

\(^b\) Bott-periodicity implies $\pi_{2n}(SU(N)) = 0$ and $\pi_{2n+1}(SU(N)) = Z$ for $N \geq n + 1$.
\(^c\) In our convention gauge transformations are right actions.
call this space $\Lambda^1(LieG)$. Let $T_A(A^{2n}) = V_A \oplus H_A$ be an orthogonal decomposition of the tangent space at $A$. $V_A$ is the vertical space spanned by vectors of the form $X^\omega_A = D_A\omega$, where $D_A$ is the covariant derivative at $A$ and $\omega$ is an element of $\Lambda^0(LieG)$, the Lie algebra of $G^{2n}$. $H_A$ is by definition the orthogonal complement of $V_A$ using the metric $r_A$, defined by ($\ast$ is the Hodge-duality map)

$$r_A(Y_A, Z_A) := \int_\Sigma \text{tr} (Y_A \wedge \ast Z_A).$$

(2.8)

$r$ is invariant under the action of $G^{2n}$ and hence $H_A$ defines a connection. Locally $H_A$ can be expressed as the kernel of the operator $D_A^\dagger$, which is the adjoint of $D_A$ with respect to $r$. The connection 1-form can then be written as (see Ref. 5)

$$C_A := G_A \circ D_A^\dagger,$$

(2.9)

where $G_A = (D_A^\dagger D_A)^{-1}$. It annihilates elements of $H_A$, transforms in the appropriate form under gauge transformations in $G^{2n}$, and, when acting upon vertical vector fields $X^\omega_A$, one has

$$C_A(X^\omega_A) = \omega \in \Lambda^0(LieG)$$

(2.10)

as required for connections. The metric (2.8) and the connection (2.9) have already been used in attempts to geometrically understand anomalies and also to formulate a Riemannian geometry of $Q^{2n}$ (Refs. 5,6,7).

The Euclidean action for Yang-Mills coupled to Dirac fermions is given by

$$S_E = \int_M d^{2n+1}x \left[ \frac{1}{4} tr(F_{\mu\nu}F^{\mu\nu}) - \bar{\psi}(i\not{D} - m)\psi \right],$$

(2.11)

where the fermions carry in addition to their spin and Lie group index also a flavour index of dimension $f$. The path integral over the fermions defines an effective action for the gauge field $A$ by

$$\int dA \int d\bar{\psi} d\psi e^{-S_E[A, \bar{\psi}, \psi]} =: \int dA e^{-S_{\text{eff}}[A]}.$$  

(2.12)

We wish to determine $W[A] = \ln Z[A]$, where

$$Z[A] := \int d\bar{\psi} d\psi e^{\int \bar{\psi}(i\not{D} - m)\psi}.$$  

(2.13)

\textsuperscript{d} We use the convention $\{\gamma^\mu, \gamma^\nu\} = -2\delta^{\mu\nu}$. The $\gamma^\mu$'s are thus anti-Hermitean $2^n \times 2^n$-matrices.
We expand the connection in terms of Hermitean basis matrices $\{T_1, \ldots, T_k\}$, $k = \text{dim} \text{SU}(N)$, so that $A = iT_p A_\mu^p \, dx^\mu$, and define the current by

$$I^\mu_p[A] = \frac{\delta}{\delta A^p_\mu} W[A].$$  \hfill (2.14)

By construction only the exponential of $W[A]$ is expected to give a well defined function on $\mathbb{Q}^{2n+1}$. A method to obtain local expressions for $W[A]$ is to calculate the one-form $\delta W[A]$ at a preferred point $A$ and then integrate this expression within a simply connected neighbourhood of $A$. We shall follow this strategy in the appendix. The obstruction to extend this to a globally defined function $W[A]$ is given by the cohomology class in $H^1(\mathbb{Q}^{2n+1})$ generated by the one form $\delta W$.

Using known techniques, we calculate $I^\mu_A$ in a $\frac{1}{m}$ - expansion. The zeroth order term (i.e. the $m \to \infty$ limit) then gives us the adiabatic connection. The calculation, which we defer to the appendix, yields for the zeroth order term (compare formula (A.14) from the appendix)

$$I^0_p[A] = -F \frac{i}{2} \frac{1}{n!} \left( \frac{i}{2\pi} \right)^n \frac{1}{2^{n}} \varepsilon^{0i_1 \cdots i_{2n}} \text{tr} \left( T_p F_{i_1i_2} \cdots F_{i_{2n-1}i_{2n}} \right).$$ \hfill (2.15)

Here $T$ is a basis element of $\text{Lie} G$ and the trace is taken over the Lie algebra indices. It follows that $I^0_A$ transforms with the inverse co-adjoint representation under gauge transformations which it should do being an element of the dual of the Lie algebra of $G^{2n}$. We shall use this fact later in the canonical picture. Here we shall follow the original plan and insert the result in (2.14) to obtain

$$\delta W[A] = 2\pi i \frac{1}{2} \frac{1}{n!} \left( \frac{i}{2\pi} \right)^{n+1} \int_M \text{tr} \left( \delta A F^n[A] \right).$$ \hfill (2.16)

According to the discussion following (2.7) we now want to integrate (2.16) along a generator of $\pi_1(\mathbb{Q}^{2n+1})$, i.e. along a curve that starts and ends in different components of $G^{2n+1}$. For this we can take any starting point $A$ in $A^{2n+1}$ since the result does not depend on it. Following the standard notations (see e.g. Ref. 8), we call

$$\omega_{2n+1}^1[A] := (n + 1) \int_M \text{tr} \left( \delta A F^n[A] \right)$$ \hfill (2.17)

$$\Omega_{2n+1}^1(0, A) := \int_{\gamma(0,A)} \omega_{2n+1}^1,$$ \hfill (2.18)
where in the first equation we have defined a gauge invariant closed 1-form in \( A^{2n+1} \), which defines therefore a closed 1-form in \( Q^{2n+1} \), and in the second equation we integrated this 1-form over a straight path from 0 to \( A \) which we denoted by \( \gamma(0, A) \). \( \Omega(0, A) \) is also known as the first Chern-Simons form.

We now integrate the 1-form \( \omega_{2n+1}^1 \) along the edges of two different triangles in \( A^{2n+1} \). The first one has vertices \((0, A, A^g)\), the second \((0, g^{-1}dg, A^g)\). Since in \( A^{2n+1} \) closed forms are necessarily exact, the two integrals are zero. We thus arrive at the two relations

\[
\Omega_{2n+1}^1(A, A^g) = \Omega_{2n+1}^1(0, A^g) - \Omega_{2n+1}^1(0, A) \\
\Omega_{2n+1}^1(0, g^{-1}dg) = \Omega_{2n+1}^1(0, A^g) - \Omega_{2n+1}^1(g^{-1}dg, A^g).
\]  

(2.19)  

(2.20)

Invariance of \( \Omega_{2n+1}^1 \) under simultaneous gauge transformations in both arguments implies equality of the last terms in each line, and hence equality of the expressions on the left sides of (2.19) and (2.20). Since \( \Omega_{2n+1}^1(A, A^g) \) is the line integral of \( \omega_{2n+1}^1 \) from \( A \) to \( A^g \), it also represents the loop integral of the projected 1-form on \( Q^{2n+1} \). This 1-form generates \( H^1(Q^{2n+1}) \) if its integral along a generator of \( \pi_1(Q^{2n+1}) \) gives the result 1. For this, \( g \) has to be a gauge transformation of unit winding number. The desired expression for this integral is now seen to be given by the integral along the straight path between 0 and \( g^{-1}dg \) which is independent of \( A \) as required. Elementary integration yields

\[
\int_{\gamma(0, g^{-1}dg)} \omega_{2n+1}^1 = (-1)^n \frac{n!(n+1)!}{(2n+1)!} \int_M \text{tr}(g^{-1}dg)^{2n+1}.
\]  

(2.21)

On the other hand, the integer-valued winding number \( w(g) \) of \( g \) is given by the expression (see Ref. 9)

\[
w[g] = \left( \frac{i}{2\pi} \right)^{n+1} \frac{n!}{(2n+1)!} \int_M \text{tr}(g^{-1}dg)^{2n+1}
\]  

(2.22)

so that we finally arrive at the following expression for the line integral of \( W[A] \) along a generator of \( w(g) \cdot Z \)

\[
\oint \delta W[A] = (-1)^n 2\pi i \frac{f}{2} w[g].
\]  

(2.23)

We notice that \( \exp(W[A]) \) will only be a well defined function on \( Q^{2n+1} \) if \( f \) is even. In this case, \( \delta W[A] \) represents the integer class \( f/2 \) in \( H^1(Q^{2n+1}) = Z \).
Let us now turn to the Hamiltonian picture. For this, we go back to (2.15). There we expect to find encoded the same information in $H^2(Q^{2n})$ represented by some curvature 2-form. The infinitesimal holonomy enters the physical picture by anomalous commutators (Schwinger terms). Let us try to explain this in the geometric picture developed so far.

If the action (2.11) is put into canonical form, the first class constraint associated with the gauge freedom in $A^{2n}$ appears as Gauss' law

$$D_k \pi^k = \bar{\psi} \gamma^0 T \psi, \quad \pi^k = F^{0k}. \quad (2.24)$$

In an effective theory for the gauge field the right hand side is replaced by its expectation value $I^0_A$. Quantizing the gauge field in the Schrödinger picture involves the constraint (here the dot $\cdot$ represents summation and integration)

$$[X^\omega - i \omega \cdot I^0] \psi[A] = 0, \quad (2.25)$$

where $X^\omega[A] = D_A \omega \cdot \frac{\delta}{\delta A}$

are the fundamental vector fields on the principal bundle $A^{2n}$. It is easy to verify that the map $\omega \mapsto X^\omega$ furnishes a homomorphism from the Lie algebra of $G^{2n}$ into the Lie algebra of vector fields on $A^{2n}$. With the aid of the connection $C_A$ from equation (2.9) and the charge density $I^0_A$ we can form the $G^{2n}$-invariant 1-form on $A^{2n}$

$$\Omega = I^0 \cdot C, \quad (2.27)$$

satisfying $\Omega_A(X^\omega_A) = \omega \cdot I^0_A$, \quad (2.28)

so that (2.25) can now be written in the form of a parallel transportation law along all vertical directions with respect to the $U(1)$-connection $-i\Omega$, where, as usual, we separated an imaginary unit to have $\Omega$ real valued. The corresponding covariant derivative operator is called $\nabla$:

$$[X^\omega - i \Omega(X^\omega)] \psi[A] =: \nabla_{X^\omega} \psi[A] = 0. \quad (2.29)$$

In quantum field theory, an anomalous commutator is defined as the deficiency term that prevents the mapping $\omega \mapsto \nabla_{X^\omega}$ from the Lie algebra of $G^{2n}$ to the commutator algebra of linear operators on quantum states from being a homomorphism of Lie algebras:

$$[\nabla_{X^\omega}, \nabla_{X^\eta}]_{\text{anom}} := [\nabla_{X^\omega}, \nabla_{X^\eta}] - \nabla_{[X^\omega, X^\eta]}, \quad (2.30)$$
where we used that $X^{[\omega,\eta]} = [X^\omega, X^\eta]$. But the right hand side is just the curvature two-form –denoted by $K$– for the $U(1)$-connection $\Omega$, evaluated on the fundamental vector fields $X^\omega$ and $X^\eta$. It satisfies

$$K(X,Y) = d\Omega(X,Y) := X[\Omega(Y)] - Y[\Omega(X)] - \Omega([X,Y]) .$$

(2.31)

Using (2.15), we write the connection as follows:

$$\Omega_A = -\frac{f^2}{2} 2\pi \frac{1}{n!} \left( \frac{i}{2\pi} \right)^{n+1} \int_\Sigma \text{tr} \left( C_A F^n[A] \right) .$$

(2.32)

Integrating $K$ over a 2-sphere in $Q^{2n}$ can be done by integrating our expression for $K$ over a disc in $A^{2n}$ with boundary in a fibre $G^{2n}$, or, equivalently, by integrating $\Omega$ over the boundary circle. To do this, let $g(t)$ be a loop in $G^{2n}$ and $\gamma(t) := A^g(t)$ the associated loop through $A$ in $A^{2n}$. The generating vector field is given by

$$\gamma_\ast \frac{d}{dt} = D_{\gamma(t)} \left( g^{-1} \frac{d}{dt} g \right) .$$

(2.33)

We now integrate $\Omega$ along this vector field and obtain:

$$\oint \gamma_\ast \Omega = -2\pi f \frac{1}{n!} \left( \frac{i}{2\pi} \right)^{n+1} \int_{S^1} dt \int_\Sigma \text{tr} \left( g^{-1}(t) \frac{dg(t)}{dt} F^n[\gamma(t)] \right) .$$

(2.34)

which is just $i$ times the expression for the integral (2.16), if written in the (time dependent) gauge where $A_0 = 0$. The integration thus leads to $i$ times the right hand side of (2.23), where $w[g]$ is now the integer in $\pi_1(G^{2n})$ represented by the loop along which we just integrated. For $-i\Omega$ to be a $U(1)$ connection, the result of the integration must be $2\pi$ times an integer (see e.g. Ref. 1) which again leads to the condition of $f$ being even. The integer is then known as the Chern-class of the $U(1)$ bundle, which here represents an element in $H^2(Q^{2n}, R)$, the second DeRahm cohomology group.
We wish to calculate $\delta W[A]$. For this we calculate the variation $\delta W[A]$ at a particular point $A$ and obtain the function $W[A]$ in a simply connected neighbourhood by integration. For simplicity we chose as evaluation point $A$ a static ($\partial_0 A = 0$) pure magnetic field ($F_{0k} = 0$) in the gauge $A_0 = 0$. \{$T_1, \ldots, T_k$\}, $k = \text{dim} SU(N)$, denote Hermitian matrices such that the matrices $iT_p$ form a basis for the Lie algebra of $SU(N)$. Flavour and spinor indices on the fermionic fields will not be displayed.

We then have

\[
\frac{\delta}{\delta A^p} W[A] = \frac{\delta}{\delta A^p} \ln \left[ \int \bar{\psi} d\psi \ e^{\int \bar{\psi} (\mathcal{D} - m) \psi} \right] 
= \frac{\delta}{\delta A^p} \ln \text{DET}(\mathcal{D} - m) 
= \frac{\delta}{\delta A^p} \ln \text{DET}(\mathcal{D} - m + i\gamma^0 A_0)(-i\mathcal{D} - m) 
= \frac{\delta}{\delta A^p} \ln \text{DET} \left[ \mathcal{D}^2 + m^2 - i\gamma^0 A_0(i\mathcal{D} + m) \right] \quad (A.1)
\]

where $\mathcal{D} := \mathcal{D}|_{A_0 = 0} = \gamma^0 \partial_0 + \mathcal{D}_{2n}$. We now regard the third term in the last expression of (A.1) as a perturbation to $X = \mathcal{D}^2 + m^2$. The determinant itself is defined via $\zeta$-function regularisation:

\[
\zeta_X(s) := \sum_n \lambda_n^{-s} 
\text{det} X := \exp(-\zeta'_X(0)) \quad (A.2)
\]

Here $\lambda_n$ is the n’th eigenvalue of the operator $X$. If $|\phi_n\rangle$ are the corresponding eigenvectors, we have

\[
\delta \lambda_n = \langle \phi_n | \delta (\mathcal{D}^2 + m^2) | \phi_n \rangle = \langle \phi_n | -i\gamma^0 \delta A_0 (i\mathcal{D} + m) | \phi_n \rangle. \quad (A.3)
\]

This gives for the variation $\delta \ln \text{DET}$ on the right hand hand side of (A.1)

\[
-\frac{d}{ds} \bigg|_{s=0} \delta \zeta_X(A_0) = \frac{d}{ds} \bigg|_{s=0} \sum_n s \delta \lambda_n \lambda_n^{-(s+1)} 
= f \frac{d}{ds} \bigg|_{s=0} \text{TR}_{\phi \lambda} \left[ \delta A_0^p T_p \gamma^0 s(i\mathcal{D} + m)(\mathcal{D}^2 + m^2)^{-(s+1)} \right] \quad (A.4)
\]
where $\text{TR}_{\phi\lambda\sigma}$ denotes the trace operation over the space-time functions ($\phi$), the Lie algebra ($\lambda$) and the spinor space ($\sigma$). The factor $f$ results from having already taken the trace over the $f$-dimensional flavour space. Negative powers of positive operators are defined via

$$X^{-(s+1)} := \frac{1}{\Gamma(s+1)} \int_{0}^{\infty} dt \, t^{s} \exp(-tX), \quad (A.5)$$

so that after some rearrangements we obtain for the expression in $(A.4)$

$$fm \left| \frac{d}{ds} \right|_{s=0} \frac{s}{\Gamma(s+1)} \int_{R^{2n+1}} d^{2n+1}x \, \text{TR}_{\lambda\sigma} \left[ T_{p} \gamma_{0} \delta A_{0}^{p} \int_{R^{2n+1}} \frac{d^{2n+1}k}{(2\pi)^{2n+1}} \exp(-ik_{\mu}x^{\mu}) \right. \times \left. (i\gamma_{0} \partial_{0} + i\mathcal{D}_{2n} + m) \int_{0}^{\infty} dt \, t^{s} \exp(-t(-\partial_{0}^{2} + \mathcal{D}_{2n}^{2} + m^{2})) \exp(\frac{1}{D}k_{\mu}x^{\mu}) \right] \quad (A.6)$$

where we explicitly expressed the trace over the space-time functions in a plane wave basis $\exp(ik_{\mu}x^{\mu})$. We write $k_{0}$ for the time-component and $\vec{k}$ for the collection of space-components $k_{i}$ of $k_{\mu}$. Now, the first two terms in the round bracket do not contribute since

$$i\partial_{0}e^{-t(\cdots)}e^{ikz} = e^{ikz}i(\partial_{0} + ik_{0})e^{-t(\cdots)} \quad (A.7)$$

which vanishes upon $k_{0}$ integration and our staticity requirement, and

$$\text{TR}_{\sigma}(\gamma_{0}\mathcal{D}_{2n}e^{-t(\cdots)}\gamma_{0}) = -\text{TR}_{\sigma}(\mathcal{D}_{2n}e^{-t(\cdots)}\gamma_{0}) = 0, \quad (A.8)$$

since there are an even number of spatial $\gamma^{i}$'s in $(\cdots)$. Further, we have

$$\mathcal{D}^{2}_{2n} = D_{k}D^{k} + \frac{1}{2}\gamma^{k}\gamma^{l}F_{kl}, \quad (A.9)$$

and can thus write $(A.1)$

$$\delta A_{0}^{p} \bigg|_{\lambda_{0}=0} \frac{s}{\Gamma(s+1)} W[A] =$$

$$\left. fm \frac{d}{ds} \right|_{s=0} \frac{s}{\Gamma(s+1)} \text{TR}_{\lambda\sigma} \left[ T_{p} \gamma_{0} \int_{0}^{\infty} dt \, t^{s} \int_{R^{2n}} \frac{d^{2n}k}{(2\pi)^{2n}} \exp(-tk^{2}) \times \int_{-\infty}^{\infty} \frac{dk_{0}}{2\pi} e^{-t(k_{0}^{2} + m^{2})} \exp(-t(D_{k}D^{k} + 2ik_{l}D_{l} + \frac{1}{2}\gamma^{l}\gamma^{k}F_{kl})) \right]. \quad (A.10)$$
In the 2n-dimensional spatial space, $\gamma^{2n+1}$ ("gamma five") is given by $\gamma^{2n+1} = i^n \gamma^1 \cdots \gamma^{2n}$ which is Hermitean and squares to one. Also, $\text{TR}_\sigma (\gamma^{2n+1} \gamma^{i_1} \cdots \gamma^{i_{2n}}) = (-i)^n 2^n \varepsilon^{i_1 \cdots i_{2n}}$. So if we choose $\gamma^0 = -i \gamma^{2n+1}$, we obtain

$$\text{TR}_\sigma (\gamma^0 \gamma^{i_1} \cdots \gamma^{i_{2n}}) = (-i)^n 2^n \varepsilon^{i_1 \cdots i_{2n}}, \quad (A.11)$$

and have for the first non-vanishing contribution from the exponential in (A.10)

$$(-i)^{n+1} \frac{t^n}{2^n n!} \text{TR}_\lambda \sigma \left[ T_{\mu} \gamma^0 \gamma^{i_1} \cdots \gamma^{i_{12n}} F_{i_1 i_2} \cdots F_{i_{2n-1} i_{2n}} \right]$$

$$= -i \frac{t^n}{n! 2^n} 2^n \varepsilon^{i_1 \cdots i_{2n}} \text{TR}_\lambda (T_{\mu} i F_{i_1 i_2} \cdots i F_{i_{2n-1} i_{2n}})$$

$$= -it^n 2^n \left[ \ast \text{TR}_\lambda \{ T_{\mu} \exp(iF) \} \right]^0$$ \quad (A.12)

where from the first to the second line we have performed the trace over the $2^n$-dimensional spinor space. The last expression is meant to be the 0-component of the 1-form in curly brackets, where $\ast$ denotes the Hodge-duality operator with respect to the $2n + 1$-dimensional metric $\delta_{\mu \nu}$. Performing the $dk$-integration yields a factor of $(4\pi t)^{-n}$ so that we can write expression (A.12) as follows:

$$-ifm \frac{d}{ds} \bigg|_{s=0} \frac{s}{\Gamma(s + 1)} \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \int_0^\infty dt \, t^n e^{-t(k_0^2 + m^2)} \left[ \ast \text{TR}_\lambda \{ T_{\mu} \exp \left( \frac{i}{2\pi} F \right) \} \right]^0$$

$$= -ifm \left[ \ast \text{TR}_\lambda \{ T_{\mu} \exp \left( \frac{i}{2\pi} F \right) \} \right]^0 \int_{-\infty}^{\infty} \frac{dk_0}{2\pi (k_0^2 + m^2)}$$

$$= -if \frac{1}{2} \left[ \ast \text{TR}_\lambda \{ T_{\mu} \exp \left( \frac{i}{2\pi} F \right) \} \right]^0. \quad (A.13)$$

Although we had selected the 0-th component to arrive at this expression, relativistic covariance tells us that the corresponding relations hold for any component (this is also apparent from the derivation, where any other component could have been preferred).

If we now include higher powers $t^{n+r}$ from the exponential in (A.10) the $k$-integration deletes again $n$ of them so that we are left with an integral of the form

$$\frac{s}{\Gamma(s + 1)} \int_0^\infty dt \, t^{r+s} e^{-t(k_0^2 + m^2)} = \frac{\Gamma(s + 1 + r)}{\Gamma(s + 1)} \left( \frac{s}{(k_0^2 + m^2)^{s+r+1}} \right) \quad (A.14)$$
which, when acted upon by $\frac{d}{ds}|_{s=0}$, gives a term $\propto (k_0^2 + m^2)^{-r-1}$, and after $k^0$-integration a term $\propto m^{-2r-1}$.

So, finally, writing $\text{tr}$ for $\text{TR}_\lambda$, we arrive at the compact formula

$$\delta W[A] = -\frac{f}{2} * \text{tr} \left\{ \delta A \exp \left( \frac{i}{2\pi} F \right) \right\} + \text{terms } \propto \frac{1}{m}. \quad (A.15)$$
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