The convergence of a numerical method for total variation flow

Qianying Hong\textsuperscript{1}, Ming-jun Lai\textsuperscript{2} and Jingyue Wang\textsuperscript{1}

Abstract
We present a convergence analysis for a finite difference scheme for the time dependent partial differential equation called gradient flow associated with the Rudin-Osher-Fatemi model. We devise an iterative algorithm to compute the solution of the finite difference scheme and prove the convergence of the iterative algorithm. Finally computational experiments are shown to demonstrate the convergence of the finite difference scheme.

Keywords
Rudin-Osher-Fatemi, model, Total Variation flow, finite difference method, numerical analysis

Received 15 November 2020; revised 27 February 2021; accepted 1 April 2021

Introduction
The well-known ROF model may be approximated in the following way

\[
\min_{u \in BV(\Omega)} \int_{\Omega} \sqrt{\epsilon + |\nabla u|^2} \, dx + \frac{1}{2\lambda} \int_{\Omega} |u - f|^2 \, dx
\]

When \( \epsilon = 0 \), it is called TV flow. When using ROF for the purpose of denoising image, \( f \) is a given noised image, and \( u \) is the denoised image. Similar partial differential equations also appear in geometry analysis. See references, e.g., Lichenewsky and Temam,\textsuperscript{1} Gerhardt,\textsuperscript{2} Andreu et al.,\textsuperscript{3–5} and the references therein. The existence, uniqueness, stability of the weak solutions to these time dependent PDE were studied in the literature mentioned above. Numerical solution of the PDE (3) using finite elements has been discussed in Feng and Prohl\textsuperscript{6} and Feng et al.\textsuperscript{7} In particular, the researchers showed that the finite element solution exists, is unique, is convergent to the weak solution of the PDE (3), the rate of convergence under some sufficient conditions, and the computation is stable. Numerical solution to the weak solution using bivariate spline method and its convergence was studied in Hong et al.\textsuperscript{8} A fixed point iterative algorithm for the associated system of nonlinear equations was discussed in Vogel and Oman\textsuperscript{9} and its convergence was studied in

\[
\frac{d}{dt} u = \text{div} \left( \frac{\nabla u}{\sqrt{\epsilon + |\nabla u|^2}} \right) - \frac{1}{\lambda} (u - f), \quad \epsilon > 0, \quad \Omega_T = [0, T] \times \Omega, \quad \text{div} \frac{\partial u}{\partial n} = 0, \quad u(\cdot, 0) = u_0(\cdot)
\]

where \( \Omega_T = [0, T] \times \Omega \), \( \frac{\partial}{\partial n} \) is the outward normal derivative operator. It is called the gradient flow of (1).
Dobson and Vogel. An algorithm based on the split Bregman iterations was discussed in Li et al. In Khan et al., gave an a mesh-free algorithm to solve the ROF model. Although the finite difference solution of the time dependent PDE (3) has been the method of choice for image denoising (e.g. See Vese and Osher), no convergence of the finite difference solution to the weak solution of the PDE has been established in the literature so far to the best of the authors’ knowledge Lai.

The purpose of this paper is to establish the convergence of the discrete solution obtained from a finite difference scheme for (3) to the weak solution. See our Theorem 3.4. Then we discuss how to numerically solve the time dependent PDE (3) by using our finite difference scheme. As the finite difference scheme is a system of nonlinear equations, we shall derive an iterative algorithm and show that the iterative solutions are convergent.

For convenience, let \( \Omega = [0, 1] \times [0, 1] \). We let \( N > 0 \) be a positive integer and divide \( \Omega \) to solve the time dependent \( x_i = ih \) and \( y_j = jh \) for \( 0 \leq i, j \leq N - 1 \) where \( h = 1/N \). For any \( f(x, y) \) defined on \( \Omega \) let \( \phi_{ij} = f(x_i, y_j) \). We shall use two different divided differences \( \nabla^+ \) and \( \nabla^- \) to approximate the gradient operator. That is,

\[
\nabla^+ \phi_{ij} = \left( \frac{\phi_{i+1,j} - \phi_{ij}}{h}, \frac{\phi_{ij+1} - \phi_{ij}}{h} \right)
\]

and

\[
\nabla^- \phi_{ij} = \left( \frac{\phi_{i,j-1} - \phi_{ij}}{h}, \frac{\phi_{ij} - \phi_{i-1,j}}{h} \right)
\]

for all \( 0 \leq i, j \leq N - 1 \) with \( \phi_{i-1,j} = \phi_{i,N}, \phi_{i,j} = \phi_{N,j-1} \) for all \( i \) and \( \phi_{j-1} = \phi_{0,j}, \phi_{j,N} = \phi_{N,j} \) for all \( j \). Furthermore, we define discrete divergence operators \( \nabla^+ \) and \( \nabla^- \) to approximate the continuous divergence operator, i.e., suppose the grid size is \( h > 0 \),

\[
(\nabla^+ p)_{ij} = \begin{cases} 
\frac{p_{i+1,j}}{h}, & i = 0, \\
\frac{p_{i,j+1}}{h}, & i = 0, \\
\frac{p_{i,j}}{h}, & 0 < i < N - 1, \\
\frac{p_{i-1,j+1}}{h}, & i = N - 1 
\end{cases}
\]

\[
(\nabla^- p)_{ij} = \begin{cases} 
\frac{p_{i,j-1}}{h}, & j = 0, \\
\frac{p_{i,j}}{h}, & 0 < j < N - 1, \\
\frac{p_{i+1,j}}{h}, & j = N - 1, \\
\frac{p_{i,j+1}}{h}, & j = 0 
\end{cases}
\]

for all \( 0 \leq i, j \leq N - 1 \) and similarly for \( \nabla^- \). By their definitions, we have for every \( p \in \mathbb{R}^{N \times N} \times \mathbb{R}^{N \times N} \) and \( u \in \mathbb{R}^{N \times N} \)

\[
\langle -\nabla^+ p, u \rangle = \langle p, \nabla^+ u \rangle, \quad \langle -\nabla^- p, u \rangle = \langle p, \nabla^- u \rangle
\]

With these notations, we are able to define a finite difference scheme for numerical solution of the time dependent PDE (3).

\[
\frac{d}{dt} u_{ij} = \frac{1}{2} \text{div}^+ \left( \frac{\nabla^+ u_{ij}}{\sqrt{\epsilon + |\nabla^+ u_{ij}|^2}} \right) + \frac{1}{2} \text{div}^- \left( \frac{\nabla^- u_{ij}}{\sqrt{\epsilon + |\nabla^- u_{ij}|^2}} \right) - \frac{1}{\lambda} (u_{ij} - f_{ij})
\]

\[
0 \leq i, j \leq N - 1, \quad t \in [0, T],
\]

\[
\frac{\partial}{\partial n} u_{ij} = 0
\]

\[
i = 0, N, 0 \leq j \leq N - 1; \quad j = 0, N, 0 \leq i \leq N - 1;
\]

\[
u(x_i, y_j, 0) = u_0(x_i, y_j), \quad 0 \leq i, j \leq N - 1
\]

Next we discretize the time domain \([0, T] \) by equally-spaced points \( t_k = k\Delta t \), \( \Delta t = T/M \). We approximate the \( \frac{d}{dt} u_{ij} \) by \( (u_{ij}^k - u_{ij}^{k-1})/\Delta t \) to have the fully discrete version of finite difference scheme:

\[
\frac{1}{\Delta t} (u_{ij}^k - u_{ij}^{k-1}) = \frac{1}{2} \text{div}^+ \left( \frac{\nabla^+ u_{ij}^k}{\sqrt{\epsilon + |\nabla^+ u_{ij}^k|^2}} \right) + \frac{1}{2} \text{div}^- \left( \frac{\nabla^- u_{ij}^k}{\sqrt{\epsilon + |\nabla^- u_{ij}^k|^2}} \right) - \frac{1}{\lambda} (u_{ij}^k - f_{ij}),
\]

\[
0 \leq i, j \leq N - 1, 1 \leq k \leq M
\]

\[
\frac{\partial}{\partial n} u_{ij}^k = 0,
\]

\[
i = 0, N, 0 \leq j \leq N - 1; \quad j = 0, N, 0 \leq i \leq N - 1,
\]

\[
0 \leq k \leq M
\]

\[
u(x_i, y_j, 0) = u_0(x_i, y_j), \quad 0 \leq i, j \leq N - 1
\]

Remark 1.1. In our numerical scheme, the discrete variation for any array \( u^k := \{u^k_{ij}, 0 \leq i, j \leq N \} \) is in fact defined by

\[
|u^k|_{DBV} = \frac{1}{2} \sum_{ij} \sqrt{\epsilon + |\nabla^+ u^k_{ij}|^2} + \frac{1}{2} \sum_{ij} \sqrt{\epsilon + |\nabla^- u^k_{ij}|^2}
\]

This way of defining discrete variation makes it possible to connect discrete and continuous variations by the observation that \( |U|_{BV} = |u^0|_{DBV} \) where \( U \) is a piecewise linear function obtained by interpolating \( u^k \) over grids on
piecewise linear function obtained by interpolating et discrete and continuous variations by the observation that the scheme is a system of nonli

\[ \arg \min_{E_k(v)} \]

for each step \( k \) where

\[ E_k(v) = \frac{1}{2} \sum_{ij} \sqrt{\epsilon + |v_{ij}^h|^2}^2 h^2 + \frac{1}{2} \sum_{ij} \sqrt{\epsilon + |v_{ij}^h|^2}^2 h^2 + \frac{1}{2} \sum_{ij} (v_{ij} - f_{ij}^h)^2 + \frac{1}{2} \sum_{ij} (v_{ij} - u_{ij}^{k-1})^2 h^2 \]

for all arrays \( \{v_{ij}\}, 0 \leq i, j \leq N - 1 \).

We shall first show that the above scheme has a uniqueness solution. Then we shall show the solution in (5) converges to the weak solution of time dependent PDE (3). These will be done in the next 2 sections. Next we shall explain how to numerically solve this system of nonlinear equations in the Section 4. We report our computational results in the Section 5. Finally we give a few remarks the last section.

**Preliminary results**

We introduce a weak formulation of PDE (3) that is suggested by Feng and Prohl.\(^6\)

**Definition 2.1.** We say that \( u \in L^1([0, T], BV(\Omega)) \) is a weak solution of (3) if \( u \) satisfies the initial value and boundary conditions in (3) and for any \( w \in L^1([0, T], W^{1,1}(\Omega)) \) with \( \frac{\partial}{\partial t} w(x, t) = 0 \) for all \( (t, x) \in [0, T] \times \partial \Omega \),

\[ \begin{align*}
\int_0^T \int_\Omega \frac{d}{dt} u w dx dt + \int_0^T \int_\Omega \nabla u \cdot \nabla w dx dt + \\
\frac{1}{2} \int_0^T \int_\Omega (u - f) w dx dt = 0
\end{align*} \]

for any \( s \in (0, T] \).

It is known (cf. Feng and Prohl\(^6\)) there exists a unique weak solution \( U^* \) satisfying the above weak formulation. \( U^* \) is in fact in \( L^\infty([0, T], BV(\Omega)) \) if \( u^0 \in BV(\Omega) \) and \( f \in L^2(\Omega) \). Following the ideas in Lichnewsky and Temam,\(^1\) the researchers in Feng and Prohl\(^6\) further showed the weak solution can be characterized by the following inequality.

**Theorem 2.1.** Let \( u \) be a weak solution as in Definition 2.1. Then \( u \) satisfies the following inequality: for any \( s \in (0, T] \),

\[ \begin{align*}
\int_0^s \int_\Omega \frac{d}{dt} (v - u) dx dt + \int_0^s (J(v) - J(u)) dx dt &
\geq \\
\frac{1}{2} \left[ \int_\Omega (v(x, s) - u(x, s))^2 dx - \int_\Omega (v(x, 0) - u_0(x, 0))^2 dx \right]
\end{align*} \]

for all \( v \in L^1([0, T], W^{1,1}(\Omega)) \) with \( \frac{\partial}{\partial t} v(x, t) = 0 \) for all \( (t, x) \in [0, T] \times \partial \Omega \), where

\[ J(u) = \int_\Omega \sqrt{\epsilon + \nabla u(x, t)^2} + \\
\frac{1}{2} \int_\Omega \sqrt{\epsilon + \nabla u(x, t)^2} dx 
\]

On the other hand, if a function \( u \in L^1([0, T], BV(\Omega)) \) satisfies the above inequality (7), then \( u \) is a weak solution.

Regarding to the solution of finite difference scheme (5), we prove some basic properties in this section. To this end, we assume that the initial data for our numerical scheme \{\( f_{ij}^h, 0 \leq i, j \leq N - 1 \)\} is a discretization of the initial data for PDE (3). Specifically assuming the region \( \Omega = [0, 1] \times [0, 1] \) is partitioned evenly into \( N \) by \( N \) grids with a grid size of \( h = 1/N \), we suppose

\[ f_{ij}^h = \frac{1}{h^2} \int_{ih}^{(i+1)h} \int_{jh}^{(j+1)h} f(x) dx, \quad 0 \leq i, j \leq N - 1 \]

i.e., the pixel value on each grid with index \( (i, j) \) is \( f_{ij}^h \). In the next section we sometimes denote by \( P_{nf} \) a related piecewise constant function defined on \( \Omega \) for which

\[ (P_{nf})(x) = f_{ij}^h, \quad x \in [ih, (i+1)h] \times [jh, (j+1)h]. \]

Let us start with the following existence and uniqueness theorem.

**Theorem 2.2.** Fix \( N > 0 \) and \( M > 0 \). There exists a unique array \( u_{ij}^h, 0 \leq i, j \leq N - 1, 0 \leq k \leq M \) satisfying the above system of nonlinear equations.

Proof. We define a variation functional that is a discretized version of (8)

\[ J^h(v) = \frac{1}{2} \sum_{ij} \sqrt{\epsilon + |v_{ij}^h|^2}^2 h^2 + \\
\frac{1}{2} \sum_{ij} \sqrt{\epsilon + |v_{ij}^h|^2}^2 h^2 + \frac{1}{2} \sum_{ij} (v_{ij} - f_{ij}^h)^2 h^2 
\]

and the discrete energy functional

\[ E^h(v) = J^h(v) + \frac{1}{2} \sum_{ij} (v_{ij} - u_{ij}^{k-1})^2 h^2 
\]

for all arrays \( v_{ij}, 0 \leq i, j \leq N - 1 \).

The Euler-Lagrange equation for the following minimization problem

\[ \min_v E^h(v) \]
is

\[
\frac{u_{i,j}^k - u_{i,j}^{k-1}}{\Delta} t - \frac{1}{2} \text{div}^+ \left( \frac{\nabla^+ u_{i,j}^k}{\sqrt{\epsilon + |\nabla^+ u_{i,j}^k|^2}} \right) \leq \\
- \frac{1}{2} \text{div}^- \left( \frac{\nabla^- u_{i,j}^k}{\sqrt{\epsilon + |\nabla^- u_{i,j}^k|^2}} \right) + \frac{1}{\lambda} (u_{i,j}^k - f_{i,j}^k) = 0,
\]

\(0 \leq i, j \leq N - 1, 1 \leq k \leq M\)

The existence and uniqueness of \(u_{i,j}^k\) follows from the strict convexity of the functional \(J^h\) and \(E^h\).

The following property is a characterization of the discrete solution of (5).

**Lemma 2.1.** Suppose that array \(\{u_{i,j}^k, 0 \leq i, j \leq N - 1, 0 \leq k \leq M\}\) is a solution of the finite difference scheme (5). Then \(u_{i,j}^k\) satisfies the following inequality

\[
\sum_{ij} \frac{u_{i,j}^k - u_{i,j}^{k-1}}{\Delta} t(v_{i,j} - u_{i,j}^k) + \frac{1}{2} \left( \sum_{ij} \sqrt{\epsilon + |\nabla^+ v_{i,j}|^2} - \sum_{ij} \sqrt{\epsilon + |\nabla^+ u_{i,j}^k|^2} \right) \leq \frac{1}{2} \left( \sum_{ij} \sqrt{\epsilon + |\nabla^- v_{i,j}|^2} - \sum_{ij} \sqrt{\epsilon + |\nabla^- u_{i,j}^k|^2} \right) + \frac{1}{2\lambda} \sum_{ij} (v_{i,j} - f_{i,j}^k)^2 
\]

\[
\geq 0
\]

for all arrays \(v_{i,j}\) that satisfies the Neumann boundary condition. On the other hand, if an array \(\{u_{i,j}^k, 0 \leq i, j \leq N - 1, 0 \leq k \leq M\}\) satisfies the above inequality for all \(v_{i,j}\) satisfying the discrete Neumann boundary condition in (5), then array \(\{u_{i,j}^k, 0 \leq i, j \leq N - 1\}\) is a solution of (5).

**Proof.** From the Euler-Lagrange equation (13),

\[
\frac{u_{i,j}^{k-1} - u_{i,j}^k}{\Delta} t = \partial J^h(u_{i,j}^k) \\
= - \frac{1}{2} \text{div}^+ \left( \frac{\nabla^+ u_{i,j}^k}{\sqrt{\epsilon + |\nabla^+ u_{i,j}^k|^2}} \right) \\
- \frac{1}{2} \text{div}^- \left( \frac{\nabla^- u_{i,j}^k}{\sqrt{\epsilon + |\nabla^- u_{i,j}^k|^2}} \right) + \frac{1}{\lambda} (u_{i,j}^k - f_{i,j}^k)
\]

The result follows from the definition of subgradient \(\partial J^h(u^k)\).

The following result shows that the computation of finite difference scheme (5) is stable. For the simplicity of the notations, we define the discrete \(L^2\) norms in analogue of standard \(L^2\) norms. Assuming \(\{u_{i,j}\}\) is an array, we define

\[
||u|| := \left\{ \sum_{ij} (u_{i,j})^2 h^2 \right\}^{1/2}
\]

**Theorem 2.3.** Let \(\{u_{i,j}^h, 0 \leq k \leq M\}\) be the solution of the system of nonlinear equations (5) associated with \(f^h\) with initial value \(u_{i,j}^0\). Similarly, let \(\{u_{i,j}^g, 0 \leq k \leq M\}\) be the corresponding solution of (5) associated with \(g^h\) with initial value \(u_{i,j}^0\). Then

\[
||u_{i,j}^h - u_{i,j}^g|| \leq \max\{||u_{i,j}^0 - u_{i,j}^0||, ||f^h - g^h||\}, 1 \leq k \leq M
\]

**Proof.** We prove by induction. It is obvious true for \(k = 0\). Assume the inequality holds for \(k\). Assume the ineqau \(L^2\) terms in (12). We have \(u_{i,j}^k\) is the minimizer of the following problem.

\[
\min_v \frac{h^2}{2} \sum_{ij} \sqrt{\epsilon + |\nabla^+ v_{i,j}|^2} + \frac{h^2}{2} \sum_{ij} \sqrt{\epsilon + |\nabla^- v_{i,j}|^2} + (\mu_1 + \mu_2)||v - (k_1 f^h + k_2 u_{i,j}^{k-1})||^2
\]

\[
\frac{1}{2} \left( (v_{i,j} - f_{i,j}^k)^2 \right) \leq 0
\]

According to the standard stability property of the minimization problem (16) (cf. Lucier and Wang)

\[
||u_{i,j}^h - u_{i,j}^g|| \leq ||(k_1 f^h + k_2 u_{i,j}^{k-1}) - (k_1 g^h + k_2 u_{i,j}^{k-1})||
\]

\[
\leq k_1 ||f^h - g^h|| + k_2 ||u_{i,j}^{k-1} - u_{i,j}^g||
\]

\[
\leq \max\{||f^h - g^h||, ||u_{i,j}^{k-1} - u_{i,j}^g||\}
\]

\[
\leq \max\{||f^h - g^h||, ||u_{i,j}^0 - u_{i,j}^0||\}
\]

**Remark 2.2.** As a direct deduction, if \(g^h = u_{i,j}^0 = 0\), the solution \(u_{i,j}^h\) is also zero for all \(k\), then

\[
||u_{i,j}^h|| \leq \max\{||u_{i,j}^0||, ||f^h||\}, 1 \leq k \leq M
\]

**Main result and its proof**

In this section, we shall show that the solution of the finite difference scheme (5) converges to the solution of the gradient flow (3). We suppose that the array \(\{u_{i,j}^k, 0 \leq i, j \leq N - 1, 0 \leq k \leq M\}\) is the solution of (5).

We first define a mapping of the array \(\{u_{i,j}^k, 0 \leq i, j \leq N - 1, 0 \leq k \leq M\}\) in the form of a piecewise linear interpolant of \(u_{i,j}^k\).
Let $t_h$ be the following type of triangulation of $\Omega \times [0, 1]$, with vertices $(i + 1/2, j, j + 1/2)$, $0 \leq i, j \leq N - 1$. Suppose the base functions of the finite element space $S^1(\Delta) \equiv \{ \phi_{ij}(x), (i, j) \in \mathbb{Z}^2 \}$ where $\phi_{ij}$ is a scaled and translated standard box spline function $\phi(x)$ with $\phi_{ij}(x) := \phi(x/h - (i + 1/2, j + 1/2))$ for any $(i, j) \in \mathbb{Z}^2$.

For any $k$, we define piecewise linear function $U_{N,M}(x, t_h)$ on def

$$U_{N,M}(x, t_h) = \sum_{i,j=0}^{N-1} u^{i,j}_k \phi_{ij}(x)$$

Having defined $U_{N,M}(\cdot, t_h)$ for $k = 0, \ldots, M$ on $g$ defined cewise lin$U_{N,M}(\cdot, t)$ for $t_k \leq t \leq t_{k+1}$ by linear interpolating $U_{N,M}(\cdot, t_{k-1})$ and $U_{N,M}(\cdot, t_k)$ on interval $[t_{k-1}, t_k]$, we define piecewise linear function $U_{N,M}(\cdot, t)$ on $g$ defined cewise lin$U_{N,M}(\cdot, t)$ for $t_k \leq t \leq t_{k+1}$ by linear interpolating $U_{N,M}(\cdot, t_{k-1})$ and $U_{N,M}(\cdot, t_k)$ on interval $[t_{k-1}, t_k]$.

$$U_{N,M}(\cdot, t) = \frac{t - t_{k-1}}{\Delta} t U_{N,M}(\cdot, t_k) + \frac{t_{k+1} - t}{\Delta} t U_{N,M}(\cdot, t_{k-1})$$

We need a sequence of useful lemmas in order to show that the solution of finite difference scheme (5) converges to the weak solution (6). We first show that the interpolant $U_{N,M}(\cdot, t)$ is TV monotone (We abuse a bit the notation of TV here since our $J(\cdot)$ includes not only a variation term but also an extra $L^2$ term).

**Lemma 3.2.**

$$J(U_{N,M}(\cdot, t_h)) \leq J(U_{N,M}(\cdot, t)), \quad t_{k-1} \leq t \leq t_k$$

**Proof.** The proof is straightforward. First, it is easy to verify that the continuous variation $J(U_{N,M}(\cdot, t_h))$ equals the discrete variation $J^h(u^h)$. Then

$$J(U_{N,M}(\cdot, t_h)) = J^h(u^h) \leq J^h(u^{h-1}) = J(U_{N,M}(\cdot, t_{k-1}))$$

Define

$$u(t) = \frac{t - t_{k-1}}{\Delta} t u_k + \frac{t_{k+1} - t}{\Delta} t u_{k-1}, \quad t_{k-1} \leq t \leq t_k$$

To prove (19) is equivalent to prove

$$J^h(u^h) \leq J^h(u(t)), \quad t_{k-1} \leq t \leq t_k$$

Since $u^h$ is the minimizer of the following functional

$$E^h(\nu) = J^h(\nu) + \frac{1}{2\Delta t} ||u^{k-1} - \nu||^2$$

we have

$$J^h(u^h) + \frac{1}{2\Delta t} ||u^{k-1} - u^h||^2 \leq J^h(u(t)) + \frac{1}{2\Delta t} ||u^{k-1} - u(t)||^2.$$  

For each term in the summation of the $L^2$ square term

$$||u^{k-1} - u(t)||^2 = \left| ||u^{k-1} - \frac{t - t_{k-1}}{\Delta} t u_k + \frac{t_{k+1} - t}{\Delta} t u_{k-1}|| \right|$$

$$= \frac{t - t_{k-1}}{\Delta} t \leq u^k - u^{k-1} \leq |u^k - u^{k-1}|$$

Then

$$\frac{1}{2\Delta t} ||u^{k-1} - u(t)||^2 \leq \frac{1}{2\Delta t} ||u^{k-1} - u^{k-1}||^2$$

We conclude from (20) that

$$J^h(u^h) \leq J^h(u(t))$$

**Lemma 3.3.** Suppose $u^0 \in W^{1,1}(\Omega), f \in L^2(\Omega)$. Then $||\frac{1}{\Delta t} U_{N,M}||_{L^2(\Omega)} \leq C$ for a positive constant $C$ only depending on $u^0$ and $f$.

**Proof.** From the Euler-Lagrange equation (13)

$$\frac{u^{k-1} - u^k}{\Delta} t = \partial J^h(u^h)$$

The equation holds element-wise at each index $(i, j)$. For the equation at each index $(i, j)$, we multiply both sides by $u^{k-1}_j - u^k_j$ and add the equations for all $(i, j)$. We use inner product notation to write the result in a concise way:

$$\left\langle \frac{u^{k-1} - u^k}{\Delta} t, u^{k-1} - u^k \right\rangle = \left\langle \partial J^h(u^h), u^{k-1} - u^k \right\rangle$$

By the definition of sub-differential $\partial J^h(u^h)$

$$\left\langle \frac{u^{k-1} - u^k}{\Delta} t, u^{k-1} - u^k \right\rangle = \left\langle \partial J^h(u^h), u^{k-1} - u^k \right\rangle$$

$$\leq J^h(u^{k-1}) - J^h(u^k).$$

Note that

$$\frac{dU_{N,M}}{dt} = \frac{u^{k-1} - u^k}{\Delta} t, \quad t_{k-1} < t < t_k$$

We have

$$\frac{1}{\Delta} t ||u^{k-1} - u^k||^2 \leq J^h(u^{k-1}) - J^h(u^k), \quad 1 \leq k \leq M$$

Add the above inequalities for $k = 1, \ldots, M$,

$$\sum_{k=1}^M \frac{1}{\Delta} t ||u^{k-1} - u^k||^2 \leq J^h(u^0) - J^h(u^M)$$  

(21)
It equals
\[ \left\| \frac{dU_{N,M}}{dt} \right\|_{L^2(\Omega_t)}^2 \leq \mathcal{J}^k(u^0) - \mathcal{J}^k(u^M) \leq \mathcal{J}^k(u^0) \]

This completes the proof.

**Lemma 3.4.** Suppose \( u^0, f \in L^2(\Omega) \). Then 
\[ \|U_{N,M}\|_{L^2(\Omega_t)} \leq C \] for a constant \( C \) only dependent on \( f \) and \( u^0 \).

**Proof.** We use (17) to bound \( U_{N,M} \). Recall \( u_j^0 = u^0 \). Letting \( C = \max\{\|u_j^0\|, \|\mathcal{J}^k\|\} \), we have 
\[ \|U_{N,M}\|_{L^2(\Omega_t)}^2 = \int_0^T \|U_{N,M}(t)\|_{L^2(\Omega)}^2 \, dt \]
\[ = \sum_{k=1}^M \int_0^T \left\| \left( t - t_{k-1} \right) U_{N,M}(\cdot, t) + (t - t_k - t_{k-1}) U_{N,M}(\cdot, t_{k-1}) \right\|_{L^2(\Omega)}^2 \, dt \]
\[ \leq \sum_{k=1}^M \int_0^T \left\| U_{N,M}(\cdot, t) \right\|_{L^2(\Omega)}^2 + \left\| U_{N,M}(\cdot, t_{k-1}) \right\|_{L^2(\Omega)}^2 \, dt \]
\[ \leq \sum_{k=1}^M \int_0^T \|u_j^k\|^2 + \|u_j^{k-1}\|^2 \, dt \leq 2TC^2 \]

This completes the proof.

In image analysis, the input image usually does not have much regularity. For example, most natural images do not even have weak derivatives. Therefore, to model images, we introduce the notation of Lipschitz space, and treat images as elements in this space.

**Definition 3.2.** Let \( \alpha \in [0, 1] \) be a real number. A function \( f \in \text{Lip}(\alpha, L^2(\Omega)) \) if \( f \in L^2(\Omega) \) and the following quantity
\[ \|f\|_{\text{Lip}(\alpha, L^2(\Omega))} := \sup_{|h| \leq 1} \left\| \frac{|f(\cdot) - f(\cdot + h)|}{|h|^\alpha} \right\|_{L^2(\Omega)} \] (22)
is finite, where \( \Omega_h := \{ x \in \Omega, x + th \in \Omega, \forall t \in [0, 1] \} \). We let \( \|f\|_{\text{Lip}(\alpha, L^2(\Omega))} = \|f\|_{L^2(\Omega)} + \|f\|_{\text{Lip}(\alpha, L^2(\Omega))}\).

The parameter \( \alpha \) is related to the quantityon mage usually does not have much regularity. For example, most natural ischit spaces with larger \( \alpha \) values.

**Lemma 3.5.** Define translation operators \( T_{1,0} \) and \( T_{0,1} \) by
\[ (T_{1,0}u)^k_{ij} = u^k_{i+1,j}, \quad 0 \leq i, j \leq N - 1, \]
\[ (T_{0,1}u)^k_{ij} = u^k_{i,j+1}, \quad 0 \leq i, j \leq N - 1 \]
Then
\[ \|T_{1,0}u^k - u^k\| \leq (\|u^0\|_{\text{Lip}(\alpha, L^2(\Omega))} + \|f\|_{\text{Lip}(\alpha, L^2(\Omega))})h^2 \]
and similarly
\[ \|T_{0,1}u^k - u^k\| \leq (\|u^0\|_{\text{Lip}(\alpha, L^2(\Omega))} + \|f\|_{\text{Lip}(\alpha, L^2(\Omega))})h^2 \]

**Proof.** We only prove the first inequality. Recall the Euler-Lagrange equation that
\[ \frac{u^{k-1} - u^k}{\Delta t} = \partial \mathcal{J}^k(u^k) \]
We write the equation element-wisely as
\[ \frac{u^k_{i+1,j} - u^k_{ij}}{\Delta t} = \frac{1}{2} \text{div}^+ \left( \frac{\nabla^+ u^k_{ij}}{\sqrt{1 + |\nabla^+ u^k_{ij}|^2}} \right) + \frac{1}{2} \text{div}^- \left( \frac{\nabla^- u^k_{ij}}{\sqrt{1 + |\nabla^- u^k_{ij}|^2}} \right) - \frac{1}{2} (u^k_{ij} - f^k_{ij}) \]
Then subtract the equation at index \((i + 1, j)\) from the same equation at index \((i, j)\) for \( 0 \leq i \leq N - 2, \)
\[ u^k_{i+1,j} - u^k_{ij} = \frac{\Delta t}{\Delta t} \left( u^k_{i+1,j} - u^k_{ij} \right) = \Delta \left( F(\nabla^+ u^k_{i+1,j}, \nabla^+ u^k_{ij}) + F(\nabla^- u^k_{i+1,j}, \nabla^- u^k_{ij}) \right) \]
\[ - \frac{1}{2} (u^k_{i+1,j} - u^k_{ij}) + \frac{1}{2} (f^k_{i+1,j} - f^k_{ij}) \] (23)
where \( F(\nabla^+ u^k_{i+1,j}, \nabla^+ u^k_{ij}) \) is defined by
\[ F(\nabla^+ u^k_{i+1,j}, \nabla^+ u^k_{ij}) = \frac{1}{2} \text{div}^+ \left( \frac{\nabla^+ u^k_{i+1,j}}{\sqrt{1 + |\nabla^+ u^k_{ij}|^2}} \right) + \frac{1}{2} \text{div}^- \left( \frac{\nabla^- u^k_{ij}}{\sqrt{1 + |\nabla^- u^k_{ij}|^2}} \right) \]
Equation (23) only holds for \( 0 \leq i \leq N - 2, 0 \leq j \leq N - 1 \). Although equation (23) is not defined for \( i = N - 1 \), we can set \( u^k_{N+1,j} = u^k_{N,j} \) and \( f^k_{N+1,j} = f^k_{N,j} \), and equation (23) still holds. We multiply (23) by \( u^k_{i+1,j} - u^k_{ij} \) and add all resulting equations for \( 0 \leq i, j \leq N - 1 \), then
\[ \frac{1}{\Delta t} \sum_{i,j=0}^{N-1} (u^k_{i+1,j} - u^k_{ij})^2 \]
\[ = \frac{1}{\Delta t} \sum_{i,j=0}^{N-1} (u^{k-1}_{i+1,j} - u^{k-1}_{ij})(u^k_{i+1,j} - u^k_{ij}) \]
\[ + \sum_{i,j=0}^{N-1} F(\nabla^+ u^k_{i+1,j}, \nabla^+ u^k_{ij})(u^k_{i+1,j} - u^k_{ij}) \]
\[ + \sum_{i,j=0}^{N-1} F(\nabla^- u^k_{i+1,j}, \nabla^- u^k_{ij})(u^k_{i+1,j} - u^k_{ij}) \]
\[ - \sum_{i,j=0}^{N-1} \frac{1}{2} (u^k_{i+1,j} - u^k_{ij})^2 \]
\[ + \sum_{i,j=0}^{N-1} \frac{1}{2} (f^k_{i+1,j} - f^k_{ij})(u^k_{i+1,j} - u^k_{ij}) \]
We show next that the second term is no greater than zero. The third term can be proved to be nonpositive similarly. By definition of $F$,

$$
\begin{align*}
\sum_{i,j=0}^{N-1} F(\nabla^+ u_{i+1,j}^k, \nabla^+ u_{i,j}^k)(u_{i+1,j}^k - u_{i,j}^k) \\
= \sum_{i,j=0}^{N-1} 2 \text{div}^+ \left( \frac{\nabla^+ u_{i+1,j}^k}{\sqrt{\epsilon + |\nabla^+ u_{i+1,j}^k|^2}} \right) (u_{i+1,j}^k - u_{i,j}^k)
- \sum_{i,j=0}^{N-1} 2 \text{div}^+ \left( \frac{\nabla^+ u_{i,j}^k}{\sqrt{\epsilon + |\nabla^+ u_{i,j}^k|^2}} \right) (u_{i+1,j}^k - u_{i,j}^k)
\end{align*}
$$

Using summation by part,

$$
\begin{align*}
\sum_{i,j=0}^{N-1} F(\nabla^+ u_{i+1,j}^k, \nabla^+ u_{i,j}^k)(u_{i+1,j}^k - u_{i,j}^k) \\
= \frac{1}{2} \sum_{i,j=0}^{N-1} (u_{i+1,j}^k - u_{i,j}^k) \\
\cdot \left( \text{div}^+ \left( \frac{\nabla^+ u_{i+1,j}^k}{\sqrt{\epsilon + |\nabla^+ u_{i+1,j}^k|^2}} \right) - \text{div}^+ \left( \frac{\nabla^+ u_{i,j}^k}{\sqrt{\epsilon + |\nabla^+ u_{i,j}^k|^2}} \right) \right) \\
= \frac{1}{2} \sum_{i,j=0}^{N-1} \left( \frac{\nabla^+ u_{i+1,j}^k}{\sqrt{\epsilon + |\nabla^+ u_{i+1,j}^k|^2}} - \frac{\nabla^+ u_{i,j}^k}{\sqrt{\epsilon + |\nabla^+ u_{i,j}^k|^2}} \right) \\
\cdot (\nabla^+ u_{i+1,j}^k - \nabla^+ u_{i,j}^k) - \frac{1}{2} \sum_{j=0}^{N-1} \frac{|\nabla^+ u_{i,j}^k|^2}{\sqrt{\epsilon + |\nabla^+ u_{i,j}^k|^2}}
\end{align*}
$$

Each term in the first sum is non-negative due to the fact that for any $x, y \in \mathbb{R}^2$,

$$
\frac{x}{\sqrt{\epsilon + |x|^2}} - \frac{y}{\sqrt{\epsilon + |y|^2}} \cdot (x - y) \geq 0
$$

By similar arguments, one has

$$
\sum_{i,j=0}^{N-1} F(\nabla^- u_{i+1,j}^k, \nabla^- u_{i,j}^k)(u_{i+1,j}^k - u_{i,j}^k) \leq 0
$$

It follows that

$$
\begin{align*}
\frac{1}{\Delta} \sum_{i,j=0}^{N-1} (u_{i+1,j}^k - u_{i,j}^k)^2 \\
\leq \frac{1}{\Delta} \sum_{i,j=0}^{N-1} (u_{i+1,j}^{k-1} - u_{i,j}^{k-1})(u_{i+1,j}^k - u_{i,j}^k) \\
- \sum_{i,j=0}^{N-1} \frac{1}{2} (u_{i+1,j}^k - u_{i,j}^k)^2 + \sum_{i,j=0}^{N-1} \frac{1}{2} (f_{i+1,j}^k - f_{i,j}^k)(u_{i+1,j}^k - u_{i,j}^k)
\end{align*}
$$

We rewrite the sums in form of discrete integrals and discrete inner products, and apply Cauchy-Schwartz inequality

$$
\begin{align*}
\frac{1}{\Delta} \int |T_{1,0} u^k - u^k|^2 \\
\leq \frac{1}{\Delta} \int (T_{1,0} u^{k-1} - T_{1,0} u^k - u^k) \\
- \frac{1}{\Delta} \int (T_{1,0} u^k - u^k)^2 + \frac{1}{\Delta} \int (T_{1,0} f - T_{1,0} u^k - u^k) \\
\leq \frac{1}{2 \Delta} \|T_{1,0} u^{k-1} - T_{1,0} u^k - u^k\|^2 + \frac{1}{\Delta} \|T_{1,0} u^k - u^k\|^2 \\
- \frac{1}{2 \Delta} \|T_{1,0} f - T_{1,0} u^k - u^k\|^2 + \frac{1}{\Delta} \|T_{1,0} f - f\|^2
\end{align*}
$$

Rearrange and combine similar terms to have

$$
\left(\frac{1}{\Delta} + \frac{1}{\Delta} \right) \|T_{1,0} u^k - u^k\|^2 \leq \frac{1}{\Delta} \|T_{1,0} u^{k-1} - u^k\|^2 + \frac{1}{\Delta} \|T_{1,0} f - f\|^2
$$

We now prove the following inequality by induction

$$
\|T_{1,0} u^k - u^k\|^2 \leq \max\{\|T_{1,0} u^0 - u^0\|^2, \|T_{1,0} f - f\|^2\}
$$

It is obvious true for $k = 0$. Assuming the inequality holds for $k$. Assuming the inequality holds for $k$ by (24). Therefore, one has

$$
\|T_{1,0} u^k - u^k\| \leq \|T_{1,0} u^0 - u^0\| + \|T_{1,0} f - f\| \\
\leq (\|u^0\|_{L^p(x,L^2)} + \|f\|_{L^p(x,L^2)})\Delta
$$

This completes the proof.

We also define a piecewise constant function $\bar{U}_{N,M}(\cdot, t)$ in a similar way to the definition of $U_{N,M}(\cdot, t)$. First we define for $k = 0, \ldots, M$

$$
\bar{U}_{N,M}(x, t_k) = u_{i,j}^k, \quad \forall x \in [ih, (i+1)h] \times [jh, (j+1)h].
$$

Then we define $\bar{U}_{N,M}(\cdot, t)$ for $t_k - 1 \leq t \leq t_k$ by interpolating $\bar{U}(\cdot, t_{k-1})$ and $\bar{U}(\cdot, t_k)$:

$$
\bar{U}(\cdot, t) = \frac{t - t_{k-1}}{\Delta} \bar{U}(\cdot, t_k) + \frac{t_k - t}{\Delta} \bar{U}(\cdot, t_{k-1})
$$

We are now ready to show

**Lemma 3.6.** Suppose $f, u^0 \in \text{Lip}(x,L^2(\Omega))$,

$$
\|U_{N,M} - \bar{U}_{N,M}\|_{L^2(\Omega)} \\
\leq C\sqrt{T}(\|u^0\|_{L^p(x,L^2)} + \|f\|_{L^p(x,L^2)})\Delta
$$

for a positive constant $C$ dependent only on $f$ and $u^0$. 
Proof. Let \( g(x, t) = U_{N,M}(x, t) - \bar{U}_{N,M}(x, t) \). For any \( x \), \( g(x, t) \) is a linear function of \( t \). A direct calculation shows

\[
\int_{t_k}^{t_{k+1}} \| g(x, t) \|_{L^2(\Omega)}^2 \, dt \leq \frac{1}{2} \left( \| g(x, t_k) \|_{L^2(\Omega)}^2 + \| g(x, t_{k-1}) \|_{L^2(\Omega)}^2 \right) (t_k - t_{k-1})
\]

Adding the inequality for \( k = 1, \cdots, M \), we have

\[
\int_0^T \| g(x, t) \|_{L^2(\Omega)}^2 \, dt \leq \Delta t \sum_{k=0}^M \| g(x, t_k) \|_{L^2(\Omega)}^2 \tag{26}
\]

Then we only need to bound \( \| g(x, t_k) \| \). We note that \( g(x, t) \) is a piecewise linear function of \( x \) on each sub-grid \( \Omega_{ij} := [j_i, (i+1)h] \times [j_j, (j+1)h], 0 \leq i, j \leq N - 1 \) for any \( t \). Tedious calculation gives

\[
\| g(x, t_k) \|_{L^2(\Omega)}^2 = \sum_{i,j} \int_{\Omega_{ij}} |U_{N,M}(x, t_k) - U_{N,M}(x, t_{k-1})|^2 \leq C \left( \| T_{1,0} u^k - u^k \|_{L^2(\Omega)}^2 + \| T_0 u^k - u^k \|_{L^2(\Omega)}^2 \right) 
\]

The last line follows from Lemma 3.5. We substitute the bound for \( \| g(x, t_k) \|_{L^2(\Omega)}^2 \) in inequality (26) to complete the proof.

Finally we are ready to prove the main result of this section.

Theorem 3.4. Suppose that \( u_0 \in W^{1,1}(\Omega), f \in L^2(\Omega) \). Furthermore, suppose that \( f \in \operatorname{Lip}(\Omega, L^2(\Omega)) \). If we choose \( \Delta t = o(h^2) \), then there exists a function \( U^* \) in \( L^2(\Omega_T) \) so that \( U_{N,M} \) converge to \( U^* \) weakly as \( N, M \to \infty \) and \( U^* \) is the weak solution of (3).

Proof. By Theorem 3.4, there exists a weakly convergent subsequence of \( \{ U_{N,M}, N \geq 1, M \geq 1 \} \) in \( L^2(\Omega_T) \). For convenience, we assume the whole sequence converges to \( U^* \in L^2(\Omega_T) \) weakly. We now show \( U^* \) is the weak solution of the gradient flow (2.1). As the weak solution is unique, the whole sequence \( \{ U_{N,M}, N \geq 1, M \geq 1 \} \) converges weakly to \( U^* \).

Let us outline the main ideas of the proof. By using Theorem 2.1, we need to show that \( U^* \) satisfies the following inequality

\[
\int_0^T \int_{\Omega} \left( \frac{d}{dt} v - J(U^*) \right) \, dx \, dt + \int_0^T (J(v) - J(U^*)) \, dt \geq \frac{1}{2} \int_{\Omega} (v(x, s) - U^*(x, s))^2 \, dx - \frac{1}{2} \int_{\Omega} (v(x, 0) - u_0(x, 0))^2 \, dx \tag{27}
\]

for all \( v \in L^1([0, T], W^{1,1}(\Omega)) \) with \( \frac{d}{dt} v(x, t) = 0 \) for all \( (t, x) \in [0, T] \times \partial \Omega \), where

\[
J(u) = \int_{\Omega} \sqrt{1 + |\nabla u(x, t)|^2} \, dx + \frac{1}{2} \int_{\Omega} (u(t) - v)^2 \, dx
\]

By the lower semi-continuity of \( J \), we have

\[
\int_0^T \int_{\Omega} \frac{d}{dt} v(x, t) \, dx dt + \int_0^T (J(v) - J(U^*)) \, dt \geq \liminf_{N,M \to \infty} \left[ \int_0^T \int_{\Omega} \frac{d}{dt} v(x, t) \, dx dt + \int_0^T (J(v) - J(U_{N,M})) \, dt \right]
\]

and

\[
\liminf_{N,M \to \infty} \frac{1}{2} \int_{\Omega} (v(x, s) - U_{N,M}(x, s))^2 \, dx
\]

for all \( v \in L^1([0, T], W^{1,1}(\Omega)) \) with \( \frac{d}{dt} v(x, t) = 0 \) for all \( (t, x) \in [0, T] \times \partial \Omega \), where

\[
\frac{d}{dt} v(x, t) = 0
\]

and

\[
\limsup_{N,M \to \infty} \frac{1}{2} \int_{\Omega} (v(x, s) - U_{N,M}(x, s))^2 \, dx
\]

We now prove the following inequality to finish the proof.

\[
\int_0^T \int_{\Omega} \frac{d}{dt} v(x, t) \, dx dt + \int_0^T (J(v) - J(U_{N,M})) \, dt \geq \frac{1}{2} \int_{\Omega} (v(x, s) - U_{N,M}(x, s))^2 \, dx - \int_{\Omega} (v(x, 0) - u_0(x, 0))^2 \, dx
\]

We now prove the following inequality to finish the proof.

\[
\int_0^T \int_{\Omega} \frac{d}{dt} v(x, t) \, dx dt + \int_0^T (J(v) - J(U_{N,M})) \, dt \geq \frac{1}{2} \int_{\Omega} (v(x, s) - U_{N,M}(x, s))^2 \, dx - \int_{\Omega} (v(x, 0) - u_0(x, 0))^2 \, dx
\]

where \( \text{Error}_{N,M} > 0 \) is an error term that goes to zero as \( N, M \to \infty \). Itan error term that goes to \( z \) (cf. Feng and Prohl), that the above inequality is equivalent to

\[
\int_0^T \int_{\Omega} \frac{d}{dt} U_{N,M}(v - u) \, dx dt + \int_0^T (J(v) - J(U_{N,M})) \, dt \geq -\text{Error}_{N,M}
\]

Since \( U_{N,M} \) is in the finite element space \( S_1^0(\Delta_N) \) associated with triangulation \( \Delta_N \), we replace the original \( W^{1,1} \) test function \( v' \) in (30) with a test function \( v'(\cdot, t) \) that is in \( L^1([0, T], S_1^0(\Delta_N)) \) and introduce an error \( e_{N,M} \).

\[
e_{N,M} = \int_0^T \int_{\Omega} \frac{d}{dt} U_{N,M}(v - v') + (J(v) - J(v'))
\]
It is easy to show \( e_{N,M} \) tends to zero as \( N \) goes to infinity by standard density argumentation. Thus we only need to prove
\[
\int_0^\infty \left[ \int_\Omega \frac{d}{dt} U_{N,M}(v - U_{N,M}) \, dx + (J(v) - J(U_{N,M})) \right] \, dt \\
\geq -\text{Error}_{N,M}
\]
(31)

for all test functions \( v \) in \( L^1([0, T], S^1_t(\Delta_N)) \) where \( \text{Error}_{N,M} \) tends to zero as \( N, M \rightarrow \infty \).

Let us verify the key inequality (31). We first prove the inequality for \( t = t_k \). Consider the integrand of the left side of (31) for \( t = t_k \). For a continuous piecewise linear function \( v(\cdot, t_k) \in S^1_t(\Delta_N) \), assuming \( v(\cdot, t_k) = \sum v_{ij}^k \phi_{ij}^k \), we have
\[
J(v(\cdot, t_k)) = \frac{h^2}{2} \sum_{ij} \sqrt{\epsilon + |\nabla^+ v_{ij}^k|^2}^2 + \frac{h^2}{2} \sum_{ij} \sqrt{\epsilon + |\nabla^- v_{ij}^k|^2}^2 \\
+ \frac{1}{2\beta} \int_\Omega (v(\cdot, t_k) - f)^2
\]
(32)

We need to replace the continuous integral in (32) by a discrete summation with some error. Let \( \tilde{P}_Nv(\cdot, t_k) \) be a piecewise constant function defined by
\[
\tilde{P}_Nv(x, t_k) = v_{ij}^k, \quad x \in [ih, (i+1)h] \times [jh, (j+1)h]
\]
and \( P_Nf \) be the piecewise constant projection of \( f \) as defined in (9). Replacing \( f \) and \( v \) by \( P_Nf \) and \( P_Nv \) respectively, we have
\[
\frac{1}{2\beta} \int_\Omega (v(\cdot, t_k) - f)^2 \\
= \frac{1}{2\beta} \sum_{ij} (v_{ij}^k - f_{ij}^h)^2 h^2 \\
+ \frac{1}{2\beta} \int_\Omega ((v(\cdot, t_k) - f)^2 - (\tilde{P}_Nv(\cdot, t_k) - P_Nf)^2)
\]
It is a standard analysis that the second term on the right-hand side converges to zero as \( P_Nf \rightarrow f \) and \( P_Nv \rightarrow v \) in \( L^2(\Omega) \). We omit the detail here. Then
\[
\frac{1}{2\beta} \int_\Omega (v(\cdot, t_k) - f)^2 = \frac{1}{2\beta} \sum_{ij} (v_{ij}^k - f_{ij}^h)^2 h^2 + \text{Err}_1
\]
where \( \text{Err}_1 \) denotes the error dependent on \( N \) that is convergent to zero when \( N \rightarrow \infty \).

Similarly, we have
\[
\int_\Omega (U_{N,M}(\cdot, t_k) - f)^2 = \sum_{ij} (u_{ij}^k - f_{ij}^h)^2 h^2 + \text{Err}_2
\]
with error term \( \text{Err}_2 \) that converges to zero as \( N \rightarrow \infty \) by Lemma 3.6.

We remind the reader that that for \( t \in (t_{k-1}, t_k) \),
\[
\frac{d}{dt} U_{N,M}(\cdot, t) = \frac{U_{N,M}(\cdot, t_k) - U_{N,M}(\cdot, t_{k-1})}{\Delta t}
\]
(33)

Then
\[
\int_\Omega \frac{d}{dt} U_{N,M}(v - U_{N,M}) \, dx \\
= \int_\Omega \frac{U_{N,M}(\cdot, t_k) - U_{N,M}(\cdot, t_{k-1})}{\Delta t} (v(\cdot, t_k) - U_{N,M}(\cdot, t_k)) \, dx
\]
Replacing \( v, U_{N,M} \) by \( \tilde{P}_Nv, \tilde{U}_N \) respectively, we have
\[
\int_\Omega \frac{U_{N,M}(\cdot, t_k) - U_{N,M}(\cdot, t_{k-1})}{\Delta t} (v(\cdot, t_k) - U_{N,M}(\cdot, t_k)) \\
= \int_\Omega \frac{\tilde{U}_N(\cdot, t_k) - \tilde{U}_N(\cdot, t_{k-1})}{\Delta t} (\tilde{P}_Nv(\cdot, t_k)) \\
- \frac{\tilde{U}_N(\cdot, t_k)}{\Delta t} (\text{Err}_1 + \text{Err}_2)
\]
where \( \text{Err}_3 \) stands for another error term that can be bounded by Lemma 3.6. Note that we have to use Cauchy-Schwartz inequality to show that \( \text{Err}_3 \) goes to zero at the rate of \( h^2 \). By one of the assumptions, we know \( \text{Err}_1/\Delta t \rightarrow 0 \) when \( N \rightarrow \infty \).

We put all the estimates above together to have
\[
\int_\Omega \frac{d}{dt} U_{N,M}(\cdot, t_k) (v(\cdot, t_k) - U_{N,M}(\cdot, t_k)) \, dx \\
+ J(v(\cdot, t_k)) - J(U_{N,M}(\cdot, t_k)) \\
= \sum_{ij} (u_{ij}^k - u_{ij}^{k-1}) (v_{ij}^k - f_{ij}^h) h^2 \\
+ \frac{1}{2} \sum_{ij} \sqrt{\epsilon + |\nabla^+ v_{ij}^k|^2}^2 h^2 + \frac{1}{2} \sum_{ij} \sqrt{\epsilon + |\nabla^- v_{ij}^k|^2}^2 h^2 \\
+ \frac{1}{2} \sum_{ij} (v_{ij}^k - f_{ij}^h)^2 h^2 - \frac{1}{2} \sum_{ij} (u_{ij}^k - f_{ij}^h)^2 h^2 \\
+ \frac{1}{2} \sum_{ij} \sqrt{\epsilon + |\nabla^- u_{ij}^k|^2}^2 h^2 - \frac{1}{2} \sum_{ij} (u_{ij}^k - f_{ij}^h)^2 h^2 \\
+ \text{Err}_3 + \text{Err}_1 - \text{Err}_2
\]
Note that the first 7 terms on the right-hand side above are nonnegative by inequality (14) in Lemma 2.1. Then
\[
\int_\Omega \frac{d}{dt} U_{N,M}(\cdot, t_k) (v(\cdot, t_k) - U_{N,M}(\cdot, t_k)) \, dx \\
+ J(v(\cdot, t_k)) - J(U_{N,M}(\cdot, t_k)) \\
\geq \text{Err}_3 + \text{Err}_1 - \text{Err}_2
\]
Thus inequality (31) for \( t = t_k \) is proved.
Now we consider the inequality when \( t \in (t_{k-1}, t_k) \). Note that

\[
U_{N,M}(\cdot, t) = U_{N,M}(\cdot, t_{k-1})(t_k - t) / \Delta t + U_{N,M}(\cdot, t_k)(t - t_k) / \Delta t
\]

For \( t \in (t_{k-1}, t_k) \), we use (33) to have

\[
\int_0^T \int_\Omega \frac{d}{dt} U_{N,M}(v(\cdot, t) - U_{N,M}(\cdot, t)) \, dx \, dt
\]

\[
= \sum_{k=1}^M \int_{t_{k-1}}^{t_k} \int_\Omega \frac{U_{N,M}(t, t_k) - U_{N,M}(t, t_{k-1})}{\Delta t} \cdot (v(\cdot, t) - U_{N,M}(\cdot, t)) \, dx \, dt
\]

\[
= \sum_{k=1}^M \int_{t_{k-1}}^{t_k} \int_\Omega \frac{U_{N,M}(t, t_k) - U_{N,M}(t, t_{k-1})}{\Delta t} \cdot (v(\cdot, t) - U_{N,M}(\cdot, t)) \, dx \, dt + \text{Err}_4
\]

We bound \( \text{Err}_4 \) by

\[
|\text{Err}_4| \leq \sum_{k=1}^M \int_{t_{k-1}}^{t_k} \left| \int_\Omega \frac{U_{N,M}(t, t_k) - U_{N,M}(t, t_{k-1})}{\Delta t} \cdot (v(\cdot, t) - U_{N,M}(\cdot, t)) \, dx \right| \, dt
\]

\[
\leq \sum_{k=1}^M \int_{t_{k-1}}^{t_k} \left| \int_\Omega \frac{U_{N,M}(t, t_k) - U_{N,M}(t, t_{k-1})}{\Delta t} \cdot (v(\cdot, t) - U_{N,M}(\cdot, t)) \, dx \right| \, dt
\]

\[
= \frac{1}{2} \Delta t \left\| \frac{dU_{N,M}}{dt} \right\|_{L^2(\Omega)}
\]

The last line comes from Lemma 3.3.

For the variation term, we have

\[
\int_0^T J(v(\cdot, t)) - J(U_{N,M}(\cdot, t))
\]

\[
= \sum_{k=1}^M \int_{t_{k-1}}^{t_k} J(v(\cdot, t)) - J(U_{N,M}(\cdot, t)) \, dt
\]

\[
= \sum_{k=1}^M \int_{t_{k-1}}^{t_k} J(v(\cdot, t)) - J(U_{N,M}(\cdot, t_k)) \, dt + \text{Err}_5
\]

To bound \( \text{Err}_5 \), we use the monotonicity of the variation term in Lemma 3.2.

\[
|\text{Err}_5| = \sum_{k=1}^M \int_{t_{k-1}}^{t_k} \left| J(U_{N,M}(\cdot, t)) - J(U_{N,M}(\cdot, t_k)) \right| \, dt
\]

By the convexity of \( J \)

\[
|\text{Err}_5| \leq \sum_{k=1}^M \int_{t_{k-1}}^{t_k} \left( \frac{t - t_{k-1}}{\Delta} t \right) J(U_{N,M}(\cdot, t_k))
\]

\[
+ \frac{t_k - t}{\Delta} J(J_{N,M}(\cdot, t_k)) - J(U_{N,M}(\cdot, t_k)) \right) \, dt
\]

\[
= \sum_{k=1}^M \int_{t_{k-1}}^{t_k} \left( J(U_{N,M}(\cdot, t_k)) - J(U_{N,M}(\cdot, t_k)) \right) \, dt
\]

\[
\leq \Delta t J(U_{N,M}(\cdot, t_k)) = \Delta t J(U^0)
\]

We conclude that \( \text{Err}_5 \) tends to zero as \( \Delta t \) approaches zero. Collecting these results together, we proved inequality (31) for \( t_{k-1} \leq t \leq t_k \).

**Numerical solution of our finite difference scheme**

The system (5) of nonlinear equations has been solved by many methods as explained in Vogel and Oman. In Dobson and Vogel, the researchers provided an analysis of a fixed point method proposed in Vogel and Oman based on auxiliary variable and functionals and proved that the iterative method converges. In this section, we mainly present another method to show the convergence of the fixed point method. From notation simplicity, we assume the grid size \( h = 1 \) in this section that has no influence in the convergence analysis of our algorithm.

First of all, let us explain the fixed point method. Recall that we need to solve \( \{t_{ij}^k, 0 \leq i, j \leq N - 1\} \) from the following equations

\[
\frac{u_{ij}^k - u_{ij}^{k-1}}{\Delta} = - \frac{1}{2} \text{div}^{+} \left( \sqrt{\epsilon + |\nabla u_{ij}^k|^2} \right)
\]

\[
- \frac{1}{2} \text{div}^{-} \left( \frac{\nabla^{-} u_{ij}^k}{\sqrt{\epsilon + |\nabla^{-} u_{ij}^k|^2}} \right) + \frac{1}{\lambda} (u_{ij}^k - f_{ij}^k) = 0,
\]

\[
0 \leq i, j \leq N - 1
\]

assuming that we have the solution \( \{u_{ij}^{k-1}, 0 \leq i, j \leq N - 1\} \). Let us define an iterative algorithm to compute \( u_{ij}^k \).

**Algorithm 4.1.** Starting with \( v_{ij}^0 = u_{ij}^{k-1}, 0 \leq i, j \leq N - 1, \) for \( \ell = 1, 2, \cdots \), we compute

\[
\frac{v_{ij}^\ell - u_{ij}^{k-1}}{\Delta} = - \frac{1}{2} \text{div}^{+} \left( \sqrt{\epsilon + |\nabla v_{ij}^\ell|^2} \right)
\]

\[
- \frac{1}{2} \text{div}^{-} \left( \frac{\nabla^{-} v_{ij}^\ell}{\sqrt{\epsilon + |\nabla^{-} v_{ij}^\ell|^2}} \right) + \frac{1}{\lambda} (v_{ij}^\ell - f_{ij}^\ell) = 0,
\]

\[
0 \leq i, j \leq N - 1
\]
\begin{equation}
\frac{1}{2} \text{div}^+ \left( \frac{\nabla^- v_{ij}^{\ell}}{\sqrt{\epsilon + |\nabla^- v_{ij}^{\ell}|^2}} \right) - \frac{1}{\lambda} (v_{ij}^{\ell} - f_{ij}^{\ell}), \quad 0 \leq i, j \leq N - 1
\end{equation}

(35)

together with boundary conditions in (5).

We now show that the iterative solutions \( \{v_{ij}^{\ell}, 0 \leq i, j \leq N - 1\}, \ell \geq 0 \) converge. Indeed, we first have

**Lemma 4.7.** There exists a positive constant \( C \) dependent only on \( f \) and initial values \( v_{ij}^{k-1} \) such that

\begin{equation}
||v^{\ell}||^2 := \sum_{ij} |v_{ij}^{\ell}|^2 \leq C
\end{equation}

(36)

for all \( \ell \geq 1 \).

**Proof.** Multiplying \( v_{ij}^{\ell} \) to the equation (34) and summing over \( i, j = 0, \ldots, N - 1 \), we have

\[
\left| v_i^{\ell} \right|^2 = \frac{1}{\Delta} \sum_{ij} u_{ij}^{k-1} v_{ij}^{\ell} - \frac{1}{\Delta} \sum_{ij} \frac{\nabla^+ v_{ij}^{\ell} \nabla^+ v_{ij}^{\ell}}{\sqrt{\epsilon + |\nabla^+ v_{ij}^{\ell}|^2}} - \frac{1}{\lambda} \sum_{ij} \frac{\nabla^- v_{ij}^{\ell} \nabla^- v_{ij}^{\ell}}{\sqrt{\epsilon + |\nabla^- v_{ij}^{\ell}|^2}} - \frac{1}{\lambda} \sum_{ij} \frac{\nabla^+ v_{ij}^{\ell} \nabla^+ v_{ij}^{\ell}}{\sqrt{\epsilon + |\nabla^+ v_{ij}^{\ell}|^2}}.
\]

By using the Cauchy-Schwartz equality, it follows that

\[
\left( \frac{\Delta}{\lambda} + \frac{1}{\lambda} \right) \left| v_i^{\ell} \right|^2 \leq \frac{1}{\Delta} \sum_{ij} \left| u_{ij}^{k-1} \right| |v_{ij}^{\ell}|^2 + \frac{1}{\lambda} \sum_{ij} \frac{\nabla^+ v_{ij}^{\ell} \nabla^+ v_{ij}^{\ell}}{\sqrt{\epsilon + |\nabla^+ v_{ij}^{\ell}|^2}}.
\]

Hence, \( ||v^{\ell}|| \) is bounded by a constant \( C \) independent of \( \ell \).

It follows that the sequence of vectors \( \{v_{ij}^{\ell}, 0 \leq i, j \leq N - 1\}, \ell \geq 1 \) contains a convergent subsequence. Let us say the vectors \( v_{ij}^{\ell}, 0 \leq i, j \leq N - 1 \) converge to \( v_{ij}^{\ast}, 0 \leq i, j \leq N - 1 \). Next we claim that the whole sequence converges. To prove this claim, we recall the energy functional

\begin{equation}
E^\ast(v) = J^\ast(v) + \frac{1}{2\lambda} \sum_{ij} (v_{ij} - u_{ij}^{k-1})^2.
\end{equation}

(37)

where

\begin{equation}
J^\ast(v) = \frac{1}{2} \sum_{ij} \sqrt{\epsilon + |\nabla^+ v_{ij}|^2} + \frac{1}{2} \sum_{ij} \sqrt{\epsilon + |\nabla^- v_{ij}|^2} + \frac{1}{2\lambda} \sum_{ij} (v_{ij} - f_{ij})^2
\end{equation}

(38)

Let us prove the following lemma

**Lemma 4.8.** For all \( \ell \geq 1 \), we have

\[
\frac{1}{2\lambda} \left| v^{\ell} - v^{\ell-1} \right|^2 \leq E(v^{\ell-1}) - E(v^{\ell}).
\]

**Proof.** Fix \( \ell \geq 1 \). For the terms in \( E(v^{\ell-1}) - E(v^{\ell}) \), we first consider

\[
\frac{1}{2\Delta} \sum_{ij} (v_{ij}^{\ell-1} - u_{ij}^{k-1})^2 - \frac{1}{2\Delta} \sum_{ij} (v_{ij}^{\ell} - u_{ij}^{k-1})^2 = \frac{1}{2\Delta} \sum_{ij} (v_{ij}^{\ell-1} - v_{ij}^{\ell})^2 + \frac{1}{2\Delta} \sum_{ij} (v_{ij}^{\ell} - u_{ij}^{k-1})(v_{ij}^{\ell-1} - v_{ij}^{\ell}).
\]

(39)

To estimate the second term on the right-hand side of the equation above, we multiply \( v_{ij}^{\ell-1} - v_{ij}^{\ell} \) to the equation (34) and sum over \( i, j = 0, \ldots, N - 1 \) to have

\[
\frac{1}{\Delta} \sum_{ij} (v_{ij}^{\ell} - u_{ij}^{k-1})(v_{ij}^{\ell-1} - v_{ij}^{\ell})
\]

\[
= \frac{1}{2\Delta} \sum_{ij} \frac{\nabla^+ v_{ij}^{\ell} \nabla^+ (v_{ij}^{\ell-1} - v_{ij}^{\ell})}{\sqrt{\epsilon + |\nabla^+ v_{ij}^{\ell}|^2}} - \frac{1}{2\Delta} \sum_{ij} \frac{\nabla^- v_{ij}^{\ell} \nabla^- (v_{ij}^{\ell-1} - v_{ij}^{\ell})}{\sqrt{\epsilon + |\nabla^- v_{ij}^{\ell}|^2}} - \frac{1}{2\Delta} \sum_{ij} \frac{\nabla^+ v_{ij}^{\ell} \nabla^+ (v_{ij}^{\ell-1} - v_{ij}^{\ell})}{\sqrt{\epsilon + |\nabla^+ v_{ij}^{\ell}|^2}}.
\]

Note that it is easy to see

\[
-\frac{1}{2\Delta} \sum_{ij} \frac{\nabla^+ v_{ij}^{\ell} \nabla^+ (v_{ij}^{\ell-1} - v_{ij}^{\ell})}{\sqrt{\epsilon + |\nabla^+ v_{ij}^{\ell}|^2}}
\]

\[
\geq -\frac{1}{4\Delta} \sum_{ij} \sqrt{\epsilon + |\nabla^+ v_{ij}^{\ell}|^2} + \frac{1}{4\Delta} \sum_{ij} \frac{\nabla^+ v_{ij}^{\ell} \nabla^+ v_{ij}^{\ell}}{\sqrt{\epsilon + |\nabla^+ v_{ij}^{\ell}|^2}}.
\]

Similar for other term involving \( \nabla^- \).

Next we consider

\[
\frac{1}{2\lambda} \sum_{ij} (v_{ij}^{\ell-1} - f_{ij}^{\ell})^2 - \frac{1}{2\lambda} \sum_{ij} (v_{ij}^{\ell} - f_{ij}^{\ell})^2
\]

\[
= \frac{1}{2\lambda} \sum_{ij} (v_{ij}^{\ell-1} - v_{ij}^{\ell})(v_{ij}^{\ell-1} + v_{ij}^{\ell} - 2f_{ij}^{\ell})
\]

\[
= \frac{1}{2\lambda} \sum_{ij} (v_{ij}^{\ell-1} - v_{ij}^{\ell})^2 + \frac{1}{2\lambda} \sum_{ij} (v_{ij}^{\ell} - f_{ij}^{\ell})(v_{ij}^{\ell-1} - v_{ij}^{\ell}).
\]

(40)
Proof. We have already shown that the iterative solutions defined in Algorithm 4.1 converge to the solution of (5) for any fixed $\Delta t > 0$, $h > 0$, and $k \geq 1$.

We are now ready to prove the main result in this subsection.

**Theorem 4.5.** The iterative solutions defined in Algorithm 4.1 converge to the solution of (5) for any fixed $\Delta t > 0$, $h > 0$, and $k \geq 1$.

**Proof.** We have already shown that the iterative solution vectors \( \{v_{i,j}^k\}, 0 \leq i, j \leq N - 1 \) have a convergent subsequence \( \{v_{i,j}^{k_n}\}, 0 \leq i, j \leq N - 1 \), for each $n = 1, 2, \ldots$ to a vector $v'$. It is easy to see that the energies $E(v^{k_n})$ are decreasing for all $n$ and hence, $E(v^{k_n+1})$ decrease to $E(v')$. By using Lemma 4.8, we know that energies $E(v^{k_n})$ are decreasing for all $\ell$ and hence, $E(v^{k_n+1})$ decrease to $E(v')$. By using Lemma 4.8 again, we see $\|v^{k_n+1} - v^{k_n}\|^2 \leq 2\lambda(E(v^{k_n}) - E(v^{k_n+1})) \to 0$. Thus, $v^{k_n+1}, n \geq 1$ are also convergent to $v'$. The uniqueness of the solution of (5) implies that $v'$ is the solution vector $\{v_{i,j}^*, 0 \leq i, j \leq N - 1\}$.

**Numerical experiments**

In this section we carry out two numerical experiments. First we demonstrate the convergence of our Algorithm 4.1. Second we apply our algorithm to approximate the solution of (3) on a given time interval. We have implemented the algorithms in MATLAB.

**Convergence of Algorithm 4.1**

According to Theorem 4.5, given any $\Delta t > 0$, $h > 0$, \( \{v_{i,j}^{k-1}\} \), and \( \{f_{i,j}\}, 0 \leq i, j \leq N - 1 \), Algorithm 4.1 will generate a convergence sequence \( \{v_{i,j}\}, k \) which converges \( \{v_{i,j}^k\} \) of (5).

We first generate two random arrays $u^{k-1} = \{u_{i,j}^{k-1}, 0 \leq i, j \leq 50 - 1\}$ and $u^k = \{u_{i,j}^k, 0 \leq i, j \leq 50 - 1\}$ using Matlab function `rand()`. Then according to (5) we can calculate $f^k = \{f_{i,j}, 0 \leq i, j \leq 50 - 1\}$. With $u^{k-1}$ and $f^k$ at hands, we iterate $v^* = \{v_{i,j}^*, 0 \leq i, j \leq 50 - 1\}$ according to Algorithm 4.1, with $v_0^* = u^{k-1}$. We show that $v^*$ converges to $u^k$, as $l \to \infty$, in a proper norm.

To show that the convergence does not occur imate small $\Delta t$, we use the following parameters: $h = 1$, $\Delta t = 2$, $\epsilon = 0.1$, $\lambda = 2$.

The log of the errors $||v' - u^k||_2$ and $|\rho(v' - u^k)|$ are plotted against the iteration counts $l = 0, 1, \ldots, 40$ in Figure 2.

**Approximation of the solution of (3)**

One major difficulty to approximate the solution of (3) is, while our algorithm converges to $u^k$ of (5) for each $k$, the solutions of (5) and (3) are different. We have proved in Theorem 3.4 that, when the grid size of space domain $h \to 0$ and $\Delta t = o(h^2)$, the discrete solution of (5) converges to the weak solution of (3). However, in reality, it is not easy to obtain data of arbitrary small $h$. For example, if $f^k$ is a discrete image, then $h$ is fixed to be 1. Therefore, in this experiment, we fix $h = 1$, and make $\Delta t$ go to 0.

Consider the exact solution of (3)

$$u(x, y, t) = \frac{100\cos(\pi x / 50)\cos(\pi y / 50)}{1 + t}$$

for $(x, y) \in [0, 50] \times [0, 50]$ and $t \in [0, 10]$. It is straightforward to calculate from the time dependent PDE (3) the corresponding function $f$ with $\lambda > 0$.

We use a uniform mesh $\Delta x = \Delta y = 1$ (Figure 1) to discretize the space domain, leading to a set of 51 dent PDE (3) the

$$\hat{\Omega} = \{(x_i, y_j) \mid x_i = i, y_j = j, 0 \leq i, j \leq 50\}$$

On the time domain, we use a uniform step size $\Delta t_M = 10/M$, and denote $T_M = \{t_k = k\Delta t\}_k^{M=0}$. We

![Figure 1. A triangulation.](image_url)
set \( u(x, y, 0) = 100\cos(\pi x/50)\cos(\pi y/50) \) as an initial value. Evolving (5) according to Algorithm 4.1, we get an approximation \( \hat{u}_M(x_i, y_j, t_k) \) on grid points \( \Omega \times T_M \).

We compare our approximation with the exact solution of (3) at time \( t = 10 \), that is, compare \( \hat{u}_M(x_i, y_j, 10) \) with
\[
u(x, y, 10) = \frac{100\cos(\pi x/50)\cos(\pi y/50)}{11} \]

To test the convergence of the algorithm as \( \Delta t \to 0 \), we compute on a sequence of decreasing time step sizes \( \Delta t_M = 10/M \), where \( M = 10, 11, \ldots, 40 \). The errors are measured in terms of the following norm:
\[
\| u - \hat{u}_M \|_{2, \Omega} \triangleq \sqrt{\sum_{(x_i, y_j) \in \Omega} |u(x_i, y_j, t) - \hat{u}_M(x_i, y_j, t)|^2}
\]

Theoretically, for \( t > 0 \), the error \( \| u - \hat{u}_M \|_{2, \Omega} \) will not go to zero, because according to Theorem 4.5, as \( M \to \infty, \hat{u}_M \) goes to the discrete solution of (5) rather than \( u \), which is the exact solution of (3). However, by Theorem 3.4, we should observe that \( \| u - \hat{u}_M \|_{2, \Omega} \) converges to a relatively small positive number. It would be ideal, if we could establish an upper bound of the difference between the solutions of (5) and (3). It needs some further study on this matter.

The results of our experiments are illustrated in Figure 3, where the log of the errors \( \| u - \hat{u}_M \|_{2, \Omega} \) and \( \| u \|_{2, \Omega} \) are plotted as functions of \( M \) respectively. We observe that our algorithm converges quickly after \( M = 30 \), in all measures.

**Conclusions**

We end this paper with a few remarks.
Remark 6.3. P. Perona and J. Malik proposed a non-stationary PDE model in Parona and Malik\(^1\) to remove noises by using anisotropic diffusion. For a given noised image \(f\), we find an improved image \(u\) by solving the following non-stationary PDE model with initial value \(f\) over time \(t \in [0, T]\):

\[
\begin{align*}
\frac{\partial u}{\partial t} & = \text{div}(c(|\nabla u|^2) \nabla u), \quad \text{in } \Omega \times (0, T) \\
\frac{\partial u}{\partial n} & = 0, \quad \text{on } \partial \Omega \times (0, T), \\
\quad u(0, x) & = f(x), \quad \text{in } \Omega.
\end{align*}
\]

(43)

where \(c(s): [0, +\infty) \rightarrow [0, +\infty)\) is a diffusive function which is a decreasing function satisfying \(c(0) = 1\) and \(\lim_{s \to +\infty} c(s) = 0\). The following is a list of commonly used diffusive functions:

- \(c(s) = 1/\sqrt{1 + s^2}\) which is called Charbonnier diffusivity.
- \(c(s) = 1/(1 + s/\lambda)\) which was used in Parona and Malik.\(^6\) We may call it Perona-Malik diffusivity.
- \(c(s) = \exp(-c/\lambda)\) which is the standard Gaussian diffusivity function.
- \(c(s) = (1 + s/\beta)^{2 - 1/2}\) for \(\beta \in (0, 1/2)\).

For a fixed \(c(s)\), we solve \(u(T, x)\) for a large \(T\) such that the restored image \(u(T, x)\) is a satisfactory one. If we let \(T\) sufficiently large, \(u(T, x)\) starts a degradation such as some edges are lost or severely blurred.

It is clear when using the Charbonnier diffusivity, i.e., \(c(s) = 1/\sqrt{\varepsilon + s}\), the PDE in (43) is very similar to the one in (3) with two distinct differences: one is \(T\) sufficient\(\infty\) and the other one is to use the noised image \(f\) as an initial value. Our convergence analysis discussed in the previous two sections can be applied to the PM model with the special diffusive function \(c(s)\). In addition, the convexity of anti-derivative of \(c(s)\) plays a significant role in our analysis. For other diffusive functions, e.g., Perona-Malik diffusive function, we notice that the function \(C(s)\) such that \(C(s) = c(s) = 1/(1 + s/\lambda)\) is not convex when \(s > \lambda\). When \(s \leq \lambda\), i.e., \(|\nabla u| \leq C\) for \(t \in [0, T]\) for some \(T > 0\), \(C(s)\) is convex and our analysis can be used to show that the corresponding finite difference method is convergent.

Remark 6.4. Our convergence analysis is independent of \(c\). Thus, we can let \(c = 0\). Also, we can replace the integral with coefficient \(1/(2\varepsilon)\) by the boundary integral. Then the time dependent PDE is associated with evolutionary surfaces with prescribed mean curvature as in Lichnewsky and Temam\(^1\) and Gerhardt.\(^2\) Our analysis can be used to show that the corresponding finite difference method for evolutionary surfaces of prescribed mean curvature is convergent to the pseudo-solution.

Remark 6.5. It is interesting to know the convergence rate of the finite difference solution to the weak solution of (3). The convergence rate of the fully discrete finite element solution was established in Feng et al.\(^7\) under a high regularity assumption on the noised image \(f\), i.e., \(f \in L^\infty((0, T); W^{1,\infty}(\Omega))\) and a very high regularity condition on domain \(\Omega\) image \(\partial \Omega \in C^3\). In general, an image function may not have such a high regularity. We hope to reduce the assumption on the regularities and give an estimate of convergence rate for the finite element solutions. These have to be left to the interested reader.

Acknowledgements

We sincerely thank our anonymous reviewers, whose reading and suggestions helped improve and clarify this manuscript a lot. The work described in this paper is supported by the Educational Research Project for Young and Middle-aged Teachers of Fujian Provincial Department of Education (Grant No. J180042) and the National Natural Science Foundations of China (Grant No. 11771084, 61771141). This paper does not necessarily reflect the views of the funding agencies.

Declaration of Conflicting Interests

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding

The author(s) disclosed receipt of the following financial support for the research, authorship, and/or publication of this article: The work described in this paper is supported by the Educational Research Project for Young and Middle-aged Teachers of Fujian Provincial Department of Education (Grant No. J180042) and the National Natural Science Foundations of China (Grant No. 11771084, 61771141). This paper does not necessarily reflect the views of the funding agencies.

ORCID iD

Qianying Hong  
https://orcid.org/0000-0001-9627-5399

References

1. Lichnewsky A and Temam R. Pseudo-solution of the time dependent minimal surface problem. J Differ Equ 1978; 30: 340–364.
2. Gerhardt C. Evolutionary surfaces of prescribed mean curvature. J Differ Equ 1980; 36: 139–172.
3. Andreu F, Ballester C, Caselles V, et al. The Dirichlet problem for the total variation flow. J Funct Anal 2001; 180: 347–403.
4. Andreu F, Ballester C, Caselles V, et al. Minimizing total variation flow. Differ Integral Equ 2001; 14: 321–360.
5. Andreu F, Caselles V, Díaz JJ, et al. Some qualitative properties for the total variation flow. J Funct Anal 2002; 188: 516–547.
6. Feng X and Prohl A. Analysis of total variation flow and its finite element approximations. * ESAim Math Model Numer Anal* 2003; 37: 533–556.
7. Feng X, von Oehsen M and Prohl A. Rate of convergence of regularization procedures and finite element approximations for the total variation flow. * Numer Math* 2005; 100: 441–456.
8. Hong Q, Lai M, Matamba L, et al. Galerkin method with splines for total variation minimization. * J Algorithm Comput Technol* 2019; 13: 1–16.
9. Vogel CR and Oman ME. Iterative methods for total variation denoising. * SIAM J Sci Comput* 1996; 17: 227–238.
10. Dobson DC and Vogel CR. Convergence of an iterative method for total variation denoising. * SIAM J Numer Anal* 1997; 34: 1779–1791.
11. Li M, Han C, Wang R, et al. Shrinking gradient descent algorithms for total variation regularized image denoising. * Comput Optim Appl* 2017; 68: 643–660.
12. Khan MA, Chen W, Ullah A, et al. A mesh-free algorithm for ROF model. * EURASIP J Adv Signal Process* 2017; 53. https://link.springer.com/article/10.1186/s13634-017-0488-6#citeas
13. Vese L and Osher S. Numerical methods for p-harmonic flows and applications to image processing. * SIAM J Numer Anal* 2002; 40: 2085–2104.
14. Lucier B and Wang J. Error bounds for finite-difference methods for Rudin-Osher-Fatemi image smoothing. * SIAM J Numer Anal* 2011; 49: 845–868.
15. Parona P and Malik J. Scale-space and edge detection using anisotropic diffusion. * IEEE Trans. Pattern Anal. Mach. Intell* 1990; 12: 629–639.
16. Acar R and Vogel CR. Analysis of bounded variation penalty methods for ill-posed problems. * Inverse Probl* 1994; 10: 1217–1229.
17. Feng X and Yoon M. Finite element approximation of the gradient flow for a class of linear growth energies with applications to color image denoising. * Int J Numer Anal Model* 2009; 6: 389–340.
18. Chen D, Chen YQ and Xue D. Fractional-Order total variation image restoration based on primal-dual algorithm. * Abstr Appl Anal* 2013; 2013: 10.