A BUFFER HAWKES PROCESS FOR LIMIT ORDER BOOKS

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Abstract. We introduce a Markovian single point process model, with random intensity regulated through a buffer mechanism and a self-exciting effect controlling the arrival stream to the buffer. The model applies the principle of the Hawkes process in which point process jumps generate a shot-noise intensity field. Unlike the Hawkes case, the intensity field is fed into a separate buffer, the size of which is the driving intensity of new jumps. In this manner, the intensity loop portrays mutual-excitation of point process events and buffer size dynamics. This scenario is directly applicable to the market evolution of limit order books, with buffer size being the current number of limit orders and the jumps representing the execution of market orders. We give a branching process representation of the point process and prove that the scaling limit is Brownian motion with explicit volatility.

1. Introduction

The self-exciting point process known as the Hawkes process is a highlighted model of choice in a number of recent mathematical studies of limit order books (LOB) [2]. It represents the dependence of the interarrival times of market events in order to match the empirical order flow and spread dynamics [17]. The point of view behind the approach is that the execution events of bids and calls when orders are removed from the book trigger an increase of the rate at which new orders enter the limit order book. In this paper, we construct and analyze a more general self-exciting process which aims at targeting more directly the basic dynamics of limit orders and market orders. It involves the Markovian Hawkes process in part, and captures mutual-excitation observed in limit order books in a nonstandard way compared with multidimensional Hawkes processes.

A large and growing literature is devoted to application of Hawkes processes in finance [4]. The Markovian Hawkes process is a process \((\Lambda_t, N_t)\), such that \(N_t\) is a counting process and \(\Lambda_t\) has exponentially decreasing paths and positive jumps generated by \(N_t\) according to

\[
\Lambda_t = \lambda_0 + \int_0^t ae^{-b(t-s)} dN_s, \quad t \geq 0,
\]

Date: October 10, 2017.

2010 Mathematics Subject Classification. 60G55, 91G80, 60J80.

Key words and phrases. Self-exciting, market price, order book, branching process.

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with \( \lambda_0 > 0 \) and \( a, b, 0 < a < b \), parameters, and furthermore, \( \Lambda_t \) is the stochastic intensity of \( N_t \). The cluster representation due to Hawkes and Oakes [18] reveals the close link between self-exciting processes and branching models. In this view, \( \lambda_0 \) is the intensity of arriving immigrants and \( \xi(ds) = ae^{-bs} ds \) is the intensity of a Poisson offspring distribution on the positive half line of a subcritical branching process with mean offspring \( \nu_0 = \int_{\mathbb{R}_+} \xi(ds) = a/b < 1 \), which counts additional events due to the internal feedback. The Hawkes process is the aggregate overlay of independent branching processes initiated by each immigrant at the time of arrival.

In finance and other applications, processes are studied in greater generality than the exponential shot noise in (1). Intensity processes of the form

\[
\Lambda_t = \Lambda_0 + \int_0^t \phi(t-s) \, dN_s = \Lambda_0 + \sum_{s_i \leq t} \phi(t-s_i),
\]

where \( \phi \) is a suitable nonnegative function and \( \Lambda_0 \) is random or constant, defines a wider class of linear Hawkes models with self-excited event times \( s_1, s_2, \ldots \), [6]. Order book modeling and financial data analysis as a rule apply multivariate point processes \( \{(s_i, d_i), i \geq 1\} \), where \( d_i \) is the component index of a vector valued counting process \( N_t = (N_1^1, \ldots, N_n^n) \) with intensity \( \Lambda_t = (\Lambda_1^1, \ldots, \Lambda_n^n) \) such that

\[
\Lambda_t^j = \Lambda_0^j + \sum_{i=1}^n \int_0^t \phi_{ij}(t-s) \, dN_s^i, \quad 1 \leq j \leq n.
\]

In LOB context, typically, the Hawkes point process \( N^j \) would count limit order arrivals or market order executions of asset \( j \) while the intensities \( \Lambda^j \) control the rate of those orders. An enlargement of this model involves another set of Hawkes processes for the limit orders of assets \( j = 1, \ldots, n \) and allows for mutual excitation between different types of orders as well, see e.g. [2]. As for the long time behavior of the univariate model the strong law of large numbers is \( N_t/t \to (1 - \nu_0)^{-1} \), almost surely, as \( t \to \infty \) and the functional central limit theorem is the weak convergence

\[
\frac{1}{\sqrt{m}} \left( N_{mt} - \frac{1}{1 - \nu_0} \right) \to \sigma B_t, \quad m \to \infty, \quad \sigma^2 = \frac{1}{(1 - \nu_0)^3},
\]

where \( B \) is Brownian motion. Analogous results for multivariate Hawkes models are established in [5] under the assumption that the matrix \( (\Phi_{ij}) \), \( \Phi_{ij} = \int_0^\infty \phi_{ij}(t) \, dt \), has spectral radius strictly less than 1.

An important feature captured by Hawkes processes is clustering in time, which is a well observed empirical fact for order arrivals [8, 11]. This is due to the self-exciting property, that is, the jump intensity increases with the process itself. Another feature of order flows as demonstrated by several studies is mutual excitation. Market orders excite limit orders, and limit orders that change the price
excite market orders \[14, 18\]. Aiming at a mechanism for reproducing such stylized facts, we propose an intertwined model of

- market order arrivals \(N\), with intensity proportional to existing limit orders \(\Gamma\);
- limit order arrivals \(L\), with intensity \(\Lambda\) excited by market orders;
- cancellation of an arriving limit order,

in the LOB for a single asset. We show that all these events come together in a coherent and analytically tractable way. The model represents the contagion of the limit orders and the market orders from each other.

Explicitly, our approach is to consider a Markov process

\[ X_t = (\Lambda_t, \Gamma_t, N_t) \]

where \(\Lambda\) and \(N\) are still linked by relation (1), but where \(\Lambda\) is no longer the stochastic intensity of the market order process \(N\). In contrast, \(\Lambda\) is the stochastic intensity of an auxiliary process \(L\) which is the arrival process of an LOB infinite server buffer for limit orders. The arrival events are placements of new limit orders while departures from the service system are limit order cancellations or market order executions, and it is (a multiple of) the resulting buffer size \(\Gamma\) which is now the stochastic intensity of \(N\). The interpretation of \(X\) is that \(\Lambda\) is the intensity process for limit orders entering the LOB, \(\Gamma\) is the current size of the LOB, and \(N\) counts the accumulated number of executed market orders. In this model, the number of entries in the LOB arises as the net balance of limit orders either arriving, triggered by market order executions, or departing, as the result of cancellation or execution, and therefore we refer to \(X\) as a buffer-Hawkes process. We derive explicit formulas for the first and second order properties of buffer-Hawkes process. We demonstrate its branching process representation, which reveals the clustering caused by mutual excitation, and in turn, the dependence of the increments. The diffusion limit is found as a result of long time scaling of the process.

We consider the market impact through a mid-price model

\[ S_t = S_0 + \left( N_t^+ - N_t^- \right) \frac{\alpha}{2}, \]

where \(N^+\) and \(N^-\) are market price counts, + stands for up, that is buy, and − stands for down, that is sell, \(\alpha > 0\) is a tick price parameter and \(S\) is the price of the asset, as in [11] where this set-up is called the toy-model. We obtain the scaling limit of \(S\) on the basis of those results for each market order assuming \(N^+\) and \(N^-\) are independent. Martingale machinery is crucial for obtaining tightness in the functional central limit theorem even in the independent case. The asymptotic volatility is obtained explicitly as a function of the model parameters. A simple model is useful for example in analyzing the optimal high-frequency trading strategy. A similar market-impact model is proposed for obtaining a dynamic optimal execution framework in [1] where the price expression includes a
multiple of the market orders that obey a Hawkes process. The market orders affect the drift of the mid-price in [7], which aims at a control strategy on the basis of self-exciting buy and sell orders.

Related work on Hawkes processes and limit order books shed light on future work with the model of the present paper. Multidimensional processes to include several assets have been considered and long-time behavior is established in e.g. [3]. The queues at different tick levels on the ask and bid sides of a limit order book are considered in [10]. For theoretical study of more general Hawkes processes, a functional central limit theorem is proven in the context of queues in [16]. In [22], bid and ask prices are studied using Hawkes processes with additional constraints. In addition to endogenous effects modeled by Hawkes mechanism, exogenous effects are also appended to the intensity process in [12] to model the contagion impact from various factors of the underlying system. For a vast list of further references, we refer to [4, 5]. Most recently, in [19] an integral process is shown to arise as the limit of Hawkes processes and to exhibit, by their nature, the self-exciting property.

Organization of the paper is as follows. In Section 2, we motivate a preliminary version and then construct the buffer-Hawkes process for limit order books. We also compute the first and second order moments. In Section 3, the branching representation of the process is given. The diffusion scaling as a long time behavior is studied in Section 4, where both the market order process and the price process are considered.

### 2. Model of Market and Limit Orders

We develop a model for the limit order book step by step considering the events that excite one another. First, the market arrival process $N$ is constructed as a self-exciting process by a process $\Gamma$. Then, the latter is linked to the limit orders so that mutual excitation between market and limit orders is captured.

Our approach follows the construction in [9, Thm.VI.6.11]. We begin with a probability space $(\Omega, \mathcal{H}, \mathbb{P})$ and the measurable space $(\mathbb{R}_+ \times \mathbb{R}_+, \mathcal{B})$ equipped with a filtration $\mathcal{F} = (\mathcal{F}_t)$ adapted to the Borel product $\sigma$-algebra $\mathcal{B} = (\mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{B}_{\mathbb{R}_+})$. Let $M : \Omega \times \mathcal{B} \mapsto [0, +\infty)$ be a Poisson random measure on $(\mathbb{R}_+ \times \mathbb{R}_+, \mathcal{B})$ with Lebesgue intensity measure $\mu(ds, dz) = ds\,dz$ relative to $\mathcal{F}$. This means that $\omega \mapsto M(\omega, B) = M(B)$ is a random variable for each $B \in \mathcal{B}$ and $B \mapsto M(\omega, B)$ is a measure on $(\mathbb{R}_+ \times \mathbb{R}_+, \mathcal{B})$ for each $\omega$. Moreover, for each $B \in \mathcal{B}$ the random variable $M(B)$ is Poisson distributed with expectation $\mu(B) = \int_B \mu(ds, dz)$ and for $B_1, \ldots, B_n \in \mathcal{B}$, $n \geq 2$, the variables $M(B_1), \ldots, M(B_n)$ are independent. Furthermore, $M(B) \in \mathcal{F}_t$ for every $B \in \mathcal{B}_{[0,t]} \otimes \mathcal{B}_{\mathbb{R}_+}$, and the measure $M$ restricted to the set $(t, \infty) \times \mathbb{R}_+$ is independent of $\mathcal{F}_t$.

Let $\Gamma = (\Gamma_t)_{t \geq 0}$ be a nonnegative process with càdlàg paths adapted to $\mathcal{F}$ and let $(\Gamma_{t-})_{t \geq 0}$ be the left-continuous and hence $\mathcal{F}$-predictable version of $\Gamma$. Suppose
that $N$ is a counting process that satisfies

$$N_t = \int_0^t \int_0^\infty 1_{(0,\Gamma_s-]}(z) M(ds, dz).$$

Then, the process

$$N_t - \int_0^t \Gamma_s ds = \int_0^t \int_0^\infty 1_{(0,\Gamma_s-]}(z) (M(ds, dz) - ds dz)$$

is an $\mathcal{F}$-martingale. Its quadratic variation is given by

$$\int_0^t \int_0^\infty 1_{(0,\Gamma_s-]}(z) dsdz = \int_0^t \Gamma_s ds$$

as $1^2 = 1$. Here, $(\Gamma_{t-})$ is the stochastic intensity process for $N$. To add flexibility to the model it is convenient to include a further parameter $c > 0$ controlling the rate of impact of $\Gamma$ on $N$. To do so, replace $M(ds, dz)$ by a Poisson random measure $M_c(ds, dz)$ with intensity measure $\mu_c(ds, dz) = c ds dz$, or replace $\Gamma$ in the definition of $N$, by $c \Gamma$, so that

(3) $$N_t = \int_0^t \int_0^\infty 1_{(0,c\Gamma_s-]}(z) M(ds, dz).$$

With this extension the compensated process $N_t - c \int_0^t \Gamma_s ds, t \geq 0$, is an $\mathcal{F}$-martingale, and $c$ controls the strength of the feedback mechanism by which jumps in $N$ influence the intensity of additional jumps later on. Next, introduce $\Lambda_t$ to be the shot noise process generated by $N$ for a given shot profile function $\phi$, as defined in (2).

The Hawkes process arises by the choice $\Gamma = \Lambda$, hence postulating that the intensity process of $N$ is the predictable version $(\Lambda_{t-})_{t \geq 0}$ of the self-referential process $\Lambda$. The setting in (2) yields a general class of non-Markovian Hawkes processes whereas the case $\Lambda_0 = \lambda_0$ and $\phi(t) = ae^{-bt}$ is the classical Hawkes process (cf. [9, pg.311], [4]). In the latter case $\Lambda_t$ is the base level-reverting shot-noise process with exponential pulse function defined in (1), such that $d\Lambda_t = -b(\Lambda_t - \lambda_0) dt + a dN_t$ and $\Lambda_t \geq \lambda_0$. In terms of the underlying Poisson random measure $M_c$, these relations show further that $\Lambda_t$ is a solution of the stochastic integral equation

$$\Lambda_t = \lambda_0 + a \int_0^t \int_0^{\Lambda_{s-}} e^{-b(t-s)} M_c(ds, du),$$

as discussed in [19]. For this classical case with fixed $a$ and $b$, and taking into account the parameter $c$, it is well-known that $\Lambda_t$ has a stationary distribution for $ac < b$. Under this assumption, by taking expectations in the previous displayed relation, or relation (1),

$$E(\Lambda_t) = \lambda_0 + \int_0^t ae^{-b(t-s)} c E(\Lambda_s) ds,$$
and hence \( EA_\infty = \lambda_0 b / (b - ac) \).

The infinitesimal generator \( \mathcal{L} f(\lambda, n) = \frac{\partial}{\partial \lambda} \mathbb{E}[f(\Lambda_t, N_t) | \Lambda_0 = \lambda, N_0 = n]_{t=0} \), of the continuous time Markov process \((\Lambda_t, N_t)\) satisfies for suitable functions \( f \)

\[
\mathcal{L} f(\lambda, n) = b(\lambda_0 - \lambda) \frac{\partial f}{\partial \lambda}(\lambda, n) + c\gamma(f(\lambda + a, n + 1) - f(\lambda, n)).
\]

The cluster representation is used systematically in the present work. We recall the basic case here in a setting that will be extended to \( X_t \). The parameters \( \lambda_0 > 0 \) and \( a, b, 0 < a < b \) are fixed. Let \( \mathcal{N}(ds) \) and \( \mathcal{M}(ds) \) be independent Poisson random measures on the positive real line with intensity measures \( n(ds) = \lambda_0 ds \) for \( \mathcal{N} \) and \( \xi(ds) = ae^{-bs} ds \) for \( \mathcal{M} \). Let \( Z_t \) be the subcritical branching process with Poisson offspring intensity \( m(ds) \) and mean offspring \( \nu_0 = \int_{\mathbb{R}_+} \xi(ds) = a/b < 1 \), which satisfies the branching relation

\[
Z_t = 1 + \int_{\mathbb{R}_+} 1_{\{s \leq t\}} Z_{t-s}^{(s)} \mathcal{M}(ds)
\]

where \( \{Z^{(s)}\} \) are independent copies of \( Z_t \). The Hawkes process is the aggregate of independent branching processes which is generated by a sequence of immigrants that arrive according to \( \mathcal{N} \), that is

\[
N_t = \int_{\mathbb{R}_+} 1_{\{s \leq t\}} Z_{t-s}^{(s)} \mathcal{N}(ds).
\]

2.1. Buffer-regulated basic counting process. In this section, we discuss buffer mechanisms driving the intensity process of \( N \). For the simplest instance, let \( \Lambda_0 = \lambda_0 \) be constant and take \( a = 0 \) so that the effect of the Hawkes mechanism is turned off and the intensity process is trivial, \( \Lambda_t = \lambda_0 \). Still, \( \lambda_0 \) determines the Poisson rate of arriving limit orders. Let \( L_t \) be the corresponding Poisson process with intensity \( \lambda_0 \) and let \( N_t \) be defined by (3) with

\[
\Gamma_t = (L_t - N_t)^+,
\]

which is the buffer size of an M/M/\( \infty \) queue with parameters \( \lambda \) and \( c \) [9, Exer.VI.6.53]. Moreover, since \( \Gamma_t = 0 \) when \( L_{t^-} \leq N_{t^-} \), we have \( N_t \leq L_t \) for all \( t \). This fact and \( M \) and \( L \) being independent from each other help to show that \( (\Gamma_t, N_t) \) is Markov. By the same arguments \( \Gamma_t \) is also Markov. The resulting counting process \( N \) is self-regulating in the sense that if \( N \) happens to have many jumps in a given time interval then its intensity process is reduced accordingly, causing the further accumulation of jumps to slow down.

To make this example more relevant for the LOB context we add cancellations. Suppose each limit order is cancelled at a constant rate \( d \geq 0 \). Put

\[
K_t = \text{number of non-cancelled limit orders arrived by } t,
\]

so that \( K \) is an M/M/\( \infty \) buffer process of limit orders with arrival rate \( \lambda_0 \) and cancellation rate \( d \). Again let \( N \) be the counting process with intensity \( (c \Gamma_{t^-}) \),
but now let $\Gamma$ be the dynamic storage process

$$\Gamma_t = \sup_{r \leq t} (K_t - K_r - (N_t - N_r)) = K_t - N_t + \sup_{r \leq t} \{-(K_r - N_r)\}, \quad t \geq 0.$$  

The continuous time Markov process $(\Gamma_t, N_t)$ has infinitesimal generator $Lf(\gamma, n)$, given by

$$Lf(\gamma, n) = \lambda_0 (f(\gamma + 1, n) - f(\gamma, n)) + d\gamma (f(\gamma - 1, n) - f(\gamma, n)) + c\gamma (f(\gamma - 1, n + 1) - f(\gamma, n)).$$  

The counting process $N$ causes $\Gamma$ to decrease with rate $c$ as well, so marginally $\Gamma$ is distributed as the $M/M/\infty$ buffer with parameters $\lambda_0$ and $c + d$ as evident from the generator $L$.

### 2.2. Buffer-Hawkes process.

We are now prepared to construct what we call a buffer-Hawkes process by combining the self-exciting Hawkes mechanism with the self-regulating buffer mechanism. Let $S = \mathbb{R}_+ \times \mathbb{N} \times \mathbb{N}$. The buffer-Hawkes process is a Markov process $X_t = (\Lambda_t, \Gamma_t, N_t)$ with càdlàg realizations on the state space $S$, with $\Lambda_t$ standing for the intensity of limit orders entering the limit order book, $\Gamma_t$ standing for the size of the limit order book, and $N_t$ standing for the number of executed market orders. The set of parameters in the model are $\lambda_0 > 0$, $a \geq 0$, $b > 0$, $c > 0$, $d \geq 0$ for which we assume the stability condition

$$ac < b(c + d),$$

which is imposed from now on.

We assume that $\Gamma$ and $N$ are related by (3), that is,

$$N_t = \int_0^t \int_0^\infty 1_{(0, c\Gamma_s]}(z) M(ds, dz)$$

so that $N_t$ is a pure jump process with compensator $c\int_0^t \Gamma_s ds$, and that $\Lambda_t$ is obtained from the jumps of $N_t$ as defined in (1), that is,

$$\Lambda_t = \lambda_0 + \int_0^t ae^{-b(t-s)} dN_s.$$  

Let $\Xi(ds, du)$ be a Cox random measure on $\mathbb{R}_+ \times \mathbb{R}_+$ with stochastic intensity measure $\Lambda_{t-} dt d e^{-du} du$, that is, a conditionally Poisson random measure given $\Lambda$ [9 pg.262]. Then

$$L_t = \int_{\mathbb{R}_+^2} 1_{\{s \leq t\}} \Xi(ds, du)$$

is the pure jump process with compensator $\int_0^t \Lambda_s ds$ and the nonnegative integer-valued process

$$K_t = \int_{\mathbb{R}_+^2} 1_{\{s \leq t \leq s+u\}} \Xi(ds, du)$$

is the corresponding $L_t/M/\infty$ buffer process with arrival process $L_t$ and exponential cancellation times of rate $d$. The special choice $a = 0$ is the previously
studied case when $L_t$ is a Poisson process with intensity $\lambda_0$ and $K_t$ is the M/M/$\infty$ process with parameters $\lambda_0$ and $d$. Finally, to complete the construction of $X$, let $\Gamma_t$ be the buffer process which results from the net input $K_t - N_t$, as defined in (5), that is,

$$\Gamma_t = \sup_{r \leq t} (K_r - K_t - (N_t - N_r)), \quad t \geq 0.$$ 

Note that $\Lambda_t$ and $\Gamma_t$ correspond to the arrival intensity of the limit and market orders, respectively, in the limit order book.

The construction of $X_t$ implies that $\Gamma_t$ is a birth-death process with births given by $L_t$ and deaths shared with the downward jumps of $K_t$ and the upward jumps of $N_t$, as long as the buffer is non-empty. The simultaneous jumps structure is the key to writing down the Markov generator for $X$. For $x = (\lambda, \gamma, n) \in S$ and functions $f$ on $S$ we denote $E_x f(X_t) = E(f(X_t)|X_0 = x)$. The generator $\mathcal{L}$ of $X$ defined by $\mathcal{L}f(x) = \frac{d}{dt}E_x f(X_t)|_{t=0}$, for sufficiently regular $f$, has the form

$$\mathcal{L}f(x) = b(\lambda_0 - \lambda)\frac{\partial f}{\partial \lambda}(x) + \lambda(f(\lambda, \gamma + 1, n) - f(x))$$

$$+ d\gamma(f(\lambda, \gamma - 1, n) - f(x)) + c\gamma(f(\lambda + a, \gamma - 1, n + 1) - f(x)).$$

Consequently, for such $f$,

$$\tilde{X}_t[f] = f(X_t) - f(x) - \int_0^t \mathcal{L}f(X_s) \, ds, \quad t \geq 0,$$ 

is a zero-mean $\mathcal{F}$-martingale. In particular, using $f_1(x) = \lambda$, $f_2(x) = \gamma$, $f_3(x) = n$, this yields the semimartingale representations

$$\Lambda_t = \lambda_0 + \int_0^t (b(\lambda_0 - \lambda_s) + ac\Gamma_s) \, ds + \tilde{X}_t[f_1]$$

$$\Gamma_t = \int_0^t (\Lambda_s - (c + d)\Gamma_s) \, ds + \tilde{X}_t[f_2]$$

$$N_t = \int_0^t c\Gamma_s \, ds + \tilde{X}_t[f_3].$$

To have these relations consistent with the existence of finite expected values in equilibrium as $t \to \infty$, one needs $b\lambda_0 - bE(\Lambda_\infty) + acE(\Gamma_\infty) = 0$ and $E(\Lambda_\infty) = (c + d)E(\Gamma_\infty)$. Hence $E(\Gamma_\infty) = \lambda_0 b/(b(c + d) - ac) < \infty$, due to the stability condition (7).

To close this subsection, we comment on the two-dimensional marginals of $X$. The marginal distribution $(\Lambda_t, N_t)$ is not Markovian, neither is $(\Gamma_t, N_t)$ except for the case $a = 0$ in (6). The marginal distribution $(\Lambda_t, \Gamma_t)$ is a Markov process with generator

$$\mathcal{L}f(\lambda, \gamma) = b(\lambda_0 - \lambda)\frac{\partial f}{\partial \lambda}(x) + \lambda(f(\lambda, \gamma + 1) - f(\lambda, \gamma))$$

$$+ d\gamma(f(\lambda, \gamma - 1) - f(\lambda, \gamma)) + c\gamma(f(\lambda + a, \gamma - 1) - f(\lambda, \gamma)).$$
This bivariate process is a self-exciting modification of the M/M/∞ model where each departure from the service system independently with probability \( c/(c + d) \) triggers a novel shot-noise contribution to the intensity of subsequent customer arrivals.

2.3. First and second order moments. We derive expressions for the first and second order moments of the buffer-Hawkes process in terms of the parameters \( a, b, c, d \) and \( \lambda_0 \). Put

\[
Q = \sqrt{(b - c - d)^2 + 4ac}, \quad q_- = (b + c + d - Q)/2, \quad q_+ = (b + c + d + Q)/2.
\]

**Proposition 2.1.** Denote \( \mathbb{E}(X_t) = (\ell_t, g_t, m_t) \). We have

\[
\ell_t = \frac{\lambda_0}{q_+ - q_-} \left( q_+ - q_- + ac \int_0^t (e^{-q_- s} - e^{-q_+ s}) \, ds \right)
\]

\[
g_t = \frac{\lambda_0}{q_+ - q_-} \left( e^{-q_- t} - e^{-q_+ t} + b \int_0^t (e^{-q_- s} - e^{-q_+ s}) \, ds \right)
\]

\[
m_t = c \int_0^t g_s \, ds.
\]

**Proof.** Using Dynkin’s formula for the generator introduced in (8),

\[
(9) \quad \mathbb{E}_x f(X_t) = f(x) + \mathbb{E}_x \int_0^t \mathcal{L} f(X_s) \, ds, \quad X_0 = x,
\]

it is straightforward to derive coupled systems of ODE’s for the first and second order moments, cf. [11, Chp.2]. Then, using (9),

\[
\begin{bmatrix}
\ell'_t \\
g'_t \\
m'_t
\end{bmatrix} +
\begin{bmatrix}
b & -ac & 0 \\
-1 & c + d & 0 \\
0 & -c & 0
\end{bmatrix}
\begin{bmatrix}
\ell_t \\
g_t \\
m_t
\end{bmatrix} =
\begin{bmatrix}
b\lambda_0 \\
0 \\
0
\end{bmatrix}
\]

In view of (7),

\[
q_+ \geq q_- > 0, \quad q_+ - q_- = Q \geq 0, \quad q_- q_+ = b(c + d) - ac > 0.
\]

For \( ac > 0 \) we have \( Q > 0 \). The solutions of the linear ODE system yield the result. \( \square \)

Asymptotically as \( t \to \infty \), we get

\[
\ell_t \to \ell_\infty = \frac{b(c + d)\lambda_0}{b(c + d) - ac}, \quad g_t \to g_\infty = \frac{\ell_\infty}{c + d}, \quad m_t \sim \frac{c\ell_\infty}{c + d} t
\]

Note that the case \( a = 0 \) with \( c > 0 \) is the basic buffer model in Section 2.1 for which \( q_- = (c + d) \land b, q_+ = (c + d) \lor b, \) and \( Q = |b - c - d| \). Then, for any \( b \),

\[
\ell_t = \lambda_0, \quad g_t = \frac{\lambda_0}{c + d} (1 - e^{-(c + d)t}) \to \frac{\lambda_0}{c + d}, \quad m_t = \frac{c\lambda_0}{c + d} \int_0^t (1 - e^{-(c + d)s}) \, ds \sim \frac{c\lambda_0}{c + d} t.
\]

We compute all second order moments in pursuit of asymptotic variance of \( N_t \) as given next.
Proposition 2.2. For large $t$, we have

$$V(N_t) \sim \frac{\lambda_0 c}{c + d} \left( \frac{b(c + d)}{b(c + d) - ac} \right)^3 t.$$ 

Proof. Using \((\mathfrak{2})\), the second order moments

$$p_t = \mathbb{E}(\Lambda_t \Gamma_t), \quad q_t = \mathbb{E}(\Lambda^2_t), \quad r_t = \mathbb{E}(\Gamma^2_t), \quad u_t = \mathbb{E}(\Lambda_t N_t), \quad v_t = \mathbb{E}(\Gamma_t N_t), \quad w_t = \mathbb{E}(N^2_t)$$

satisfy the system of equations

$$\begin{bmatrix}
    p'_t \\
    q'_t \\
    r'_t \\
    u'_t \\
    v'_t
\end{bmatrix}
+ \begin{bmatrix}
    b + c + d & -1 & -ac & 0 & 0 \\
    -2ac & 2b & 0 & 0 & 0 \\
    -2 & 0 & 2(c + d) & 0 & 0 \\
    -c & 0 & 0 & b & -ac \\
    0 & 0 & -c & -1 & c + d
\end{bmatrix}
\begin{bmatrix}
    p_t \\
    q_t \\
    r_t \\
    u_t \\
    v_t
\end{bmatrix}
= \begin{bmatrix}
    b\lambda_0 - ac \\
    c + d \\
    ac \\
    -c
\end{bmatrix}
\begin{bmatrix}
    g_t + 1 \\
    \ell_t
\end{bmatrix}
+ \begin{bmatrix}
    0 \\
    2b\lambda_0 \\
    0 \\
    b\lambda_0
\end{bmatrix}
\begin{bmatrix}
    0 \\
    0 \\
    0 \\
    m_t
\end{bmatrix},$$

and, furthermore, $w'_t = 2cv_t + cg_t$. Rewriting, the functions

$$\bar{p}_t = C(\Lambda_t, \Gamma_t) = p_t - g_t \ell_t, \quad \bar{q}_t = V(\Lambda_t) = q_t - \ell^2_t, \quad \bar{r}_t = V(\Gamma_t) = r_t - g^2_t,$$

solve

$$\begin{bmatrix}
    \bar{p}'_t \\
    \bar{q}'_t \\
    \bar{r}'_t
\end{bmatrix}
+ \begin{bmatrix}
    b + c + d & -1 & -ac \\
    -2ac & 2b & 0 \\
    -2 & 0 & 2(c + d)
\end{bmatrix}
\begin{bmatrix}
    \bar{p}_t \\
    \bar{q}_t \\
    \bar{r}_t
\end{bmatrix}
= \begin{bmatrix}
    -ac \\
    ca^2 \\
    c + d
\end{bmatrix}
\begin{bmatrix}
    g_t \\
    \ell_t
\end{bmatrix}
+ \begin{bmatrix}
    0 \\
    0 \\
    1
\end{bmatrix},$$

while $\bar{u}_t = C(\Lambda_t, N_t), \bar{v}_t = C(\Gamma_t, N_t)$ and $\bar{w}_t = V(N_t)$ are the solutions of

$$\begin{bmatrix}
    \bar{u}'_t \\
    \bar{v}'_t
\end{bmatrix}
+ \begin{bmatrix}
    b & -ac \\
    -1 & c + d
\end{bmatrix}
\begin{bmatrix}
    \bar{u}_t \\
    \bar{v}_t
\end{bmatrix}
= \begin{bmatrix}
    ac \\
    -c
\end{bmatrix}
\begin{bmatrix}
    g_t + c \\
    \bar{p}_t
\end{bmatrix}$$

and $\bar{w}'_t = 2c\bar{v}_t + cg_t$. In the limit $t \to \infty$,

$$\bar{p}_\infty = \frac{ca^2(c + d)g_\infty}{2(b + c + d)(b(c + d) - ac)}, \quad \bar{q}_\infty = \frac{ca^2((c + d)(b + c + d) - ac)g_\infty}{2(b + c + d)(b(c + d) - ac)},$$

and

$$\bar{r}_\infty = g_\infty + ca^2 \frac{g_\infty}{2(b + c + d)(b(c + d) - ac)},$$

Moreover,

$$\begin{bmatrix}
    \bar{u}_\infty \\
    \bar{v}_\infty
\end{bmatrix}
= \frac{1}{b(c + d) - ac}
\begin{bmatrix}
    c + d & ac \\
    1 & b
\end{bmatrix}
\begin{bmatrix}
    acg_\infty + \bar{p}_\infty \\
    c(\bar{r}_\infty - g_\infty)
\end{bmatrix}$$

$$= \begin{bmatrix}
    c + d + \frac{ac((c+d)^2-ac)}{2(b+c+d)(b(c+d)-ac)} \\
    1 + \frac{ac(c+d)^2-ac}{2(b(c+d)-ac)}
\end{bmatrix}
\begin{bmatrix}
    acg_\infty \\
    b(c+d) - ac
\end{bmatrix}.$$
Finally,

\[ V(N_t) \sim c(2\overline{\nu}_\infty + \overline{\gamma}_\infty) t = \frac{\lambda_0 c}{c + d} \left( \frac{b(c + d)}{b(c + d) - ac} \right)^3 t. \]

\[ \square \]

3. BRANCHING PROCESS REPRESENTATION

The market orders component \( N_t \) in the buffer-Hawkes process admits a branching process representation analogous to that of the original Hawkes process as in \( \mathbb{H} \). The branching representation emphasizes the clustering in time, which occurs as a result of self-excitation, or mutual excitation as modelled above. It also helps to give a straight proof of the diffusion limit for finite dimensional distributions.

To derive the branching representation we use the same probabilistic setting as detailed in the construction of the Markov process \( (X_t) \) but now suppose that \( \mathcal{N}(ds, du) \) and \( \mathcal{M}(ds, du) \) are Poisson measures on \( \mathbb{R}_+ \times \mathbb{R}_+ \) with intensity measures \( n(ds, du) \) and \( m(ds, du) \), respectively, given by

\[ n(ds, du) = \lambda_0 ds \ e^{-(c+d)u} du, \quad m(ds, du) = ae^{-bs}ds \ e^{-(c+d)u} du. \]  

To see why these intensities reflect the dynamics of \( X \), consider a time \( t \) when the size of the LOB is some number \( \Gamma_t = \gamma \). Then the rate of execution is \( c\gamma \) and the rate of cancellation is \( d\gamma \). In other words, each single limit order has execution rate \( c \) and cancellation rate \( d \) and hence remains in the book during an exponentially distributed time with total intensity \( c + d \). After independent thinning with probability \( c/(c + d) \) one obtains the intensity \( n(ds, du) \) to have among the regular arrivals of limit orders at rate \( \lambda_0 \) an entry at time \( s + u \). Similarly, \( m(ds, du) \) is the Poisson intensity for the corresponding event to occur as the result of a separate contribution \( ae^{-bs} \), \( s \geq 0 \), to the arrival intensity of limit orders. Consequently,

\[ N_t(0) = \int_{\mathbb{R}_+ \times \mathbb{R}_+} 1_{\{s+u \leq t\}} N(ds, du) = \int_0^t \int_0^{t-s} N(ds, du) \]

is the number of market orders in \([0, t]\) generated by the base level intensity \( \lambda_0 \). At the time of execution each of these events independently adds to the intensity of limit orders. Due to the Hawkes mechanism the new intensity contributions are governed by the Poisson measure \( \mathcal{M} \) and come in the shape of shot-noise profiles of height \( a \) and removal rate \( b \). Each new market order executed as a result of the added intensity repeats the procedure independently, and therefore the total number of market orders grow as the total number of progeny in a branching process. Let \( Z_t, t \geq 0 \), denote the total number of individuals at time \( t \) in a continuous time branching process with \( Z_0 = 1 \) and offspring distribution on \( \mathbb{R}_+ \) consisting of one offspring unit at birth time \( s + u \) for each Poisson point \((s, u)\)
of \( \mathcal{M} \). Because of (7) the branching process is subcritical with mean offspring
\[
(11) \quad \nu = \int_{\mathbb{R}_+ \times \mathbb{R}_+} m(ds, du) = \int_0^\infty \int_0^\infty a e^{-bs} c e^{-(c+d)u} du ds = \frac{ac}{b(c+d)} < 1.
\]

Let \( Z^{(s,u)}, (s,u) \in \mathcal{M} \), denote a collection of independent copies of \( Z \). The branching property relation is
\[
(12) \quad Z_t = 1 + \int_0^t \int_0^{t-s} Z_{t-s-u}^{(s,u)} \mathcal{M}(ds, du).
\]

In conclusion, the combined number of market orders arise as
\[
(13) \quad N_t = \int_0^t \int_0^{t-s} Z_{t-s-u}^{(s,u)} \mathcal{N}(ds, du),
\]
where the \( Z \)-processes are independent of \( \mathcal{N} \). Thus, the component \( (N_t) \) of \( X \) is also a branching process with immigration, such that single immigrants arrive at times \( s + u \) for each Poisson point \( (s,u) \) of \( \mathcal{N} \) and each immigrant generate independent offspring according to \( Z \). It is sometimes convenient to split up the immigrants counted by \( N_t^{(0)} \) from the further jumps of \( N_t \) along trajectories of \( Z_t \) due to the Hawkes feedback, that is
\[
(14) \quad N_t = N_t^{(0)} + \int_0^t \int_0^{t-s} (Z_{t-s-u}^{(s,u)} - 1) \mathcal{N}(ds, du).
\]

### 3.1. Properties of the subcritical branching process

Since the underlying branching mechanism is subcritical, as \( t \to \infty \) the almost surely nondecreasing process \( Z_t \) reaches a limit \( Z_\infty < \infty \), \( \mathbb{P} \)-a.s., attaining the distribution of the ultimate total progeny in a Galton Watson process with Poisson(\( \nu \)) offspring distribution starting from a single individual. It is well-known that \( Z_\infty \) has a Borel distribution with mean \( (1 - \nu)^{-1} < \infty \). These facts are recalled next as we derive bounds for the the moment generating functions \( \mathbb{E}[e^{\theta Z_t}] \) and \( \mathbb{E}[e^{\theta N_t}] \). By (12), for all \( \theta \leq 0 \) at least,
\[
\ln \mathbb{E}[e^{\theta Z_t}] = \theta + \int_0^t \int_0^{t-s} \left( \mathbb{E}[e^{\theta Z_{t-s-u}}] - 1 \right) m(ds, du).
\]

Let
\[
V_t(\theta) = \ln \mathbb{E}[e^{\theta Z_t}], \quad V_\infty(\theta) = \ln \mathbb{E}[e^{\theta Z_\infty}],
\]
for each \( \theta \) where these functions exist finitely. The above integral equation,
\[
V_t(\theta) = \theta + \int_0^t \int_0^{t-s} (e^{V_{t-s-u}(\theta)} - 1) a e^{-b(t-s)} du c^{-c+d} du,
\]
demonstrates that \( V_t(\theta) \) is differentiable in \( t \) and solves the nonlinear ODE
\[
(15) \quad V_t''(\theta) + (b+c+d)V_t'(\theta) + b(c+d)V_t(\theta) = b(c+d)\theta + ac(e^{V_t(\theta)} - 1), \quad V_0(\theta) = \theta, \quad V_0'(\theta) = 0.
\]
By monotone convergence,
\[ V_\infty(\theta) = \theta + \nu(e^{V_\infty(\theta)} - 1). \]

Equivalently
\[-(V_\infty(\theta) - \theta + \nu) \exp\{-(V_\infty(\theta) - \theta + \nu)\} = -\nu e^{\theta - \nu}. \]

For each \( \theta \) such that \(-e^{-1} \leq -\nu e^{\theta - \nu} < 0\), this equation has two solutions given by the two branches \( W_0 \) and \( W_{-1} \) of the real valued Lambert-W function. The secondary branch \( W_{-1} \) is excluded since the property \( W_{-1}(x) \leq -1, e^{-1} \leq x < 0 \), would imply \( V_\infty(\theta) > \theta \) for all \( \theta \), which is not the case. The relevant solution in terms of the primary branch \( W_0 \) of the Lambert-W function is
\[ V_\infty(\theta) = \theta - \nu - W_0(-\nu e^{\theta - \nu}), \quad \theta < \theta_0 = -\ln \nu + \nu - 1. \]

Since \( \theta_0 > 0 \) for \( 0 < \nu < 1 \) and \( W_0(e^{-1}) = -1 \), the (logarithmic) moment generating functions \( V_t(\theta) \) and \( V_\infty(\theta) \) exist finitely for \( \theta \) in an open interval containing 0, namely
\[ V_t(\theta) \leq V_\infty(\theta) \leq V_\infty(\theta_0) = -\ln \nu, \quad \theta \leq \theta_0. \]

In particular, the moments of any order \( n \) are finite,
\[ \mathbb{E}Z^n_t \leq \mathbb{E}Z^n_\infty < \infty, \quad t > 0. \]

The defining property \( W(x)e^{W(x)} = x \) of the Lambert-W function implies
\[ \mathbb{E}[e^{\theta Z_\infty}] = e^{\theta - \nu} e^{-W_0(-\nu e^{-\nu} e^\theta)} = \frac{W_0(-\nu e^{-\nu} e^\theta)}{-\nu}, \quad \theta \leq \theta_0. \]

An application of the Taylor series of \( W_0 \) around 0,
\[ W_0(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} x^n, \quad |x| \leq e^{-1}, \]

reveals the Borel distribution
\[ \mathbb{P}(Z_\infty = k) = \frac{(k\nu)^{k-1}}{k!} e^{-\nu k}, \quad k = 1, 2 \ldots \]

It is seen from (15) that the mean and variance functions
\[ x_t = \mathbb{E}Z_t = \frac{d}{d\theta} V_t(\theta)|_{\theta=0}, \quad y_t = \text{Var}Z_t = \frac{d^2}{d\theta^2} V_t(\theta)|_{\theta=0}, \]
satisfy
\[ x_t'' + (b + c + d)x_t' + (b(c + d) - ac)x_t = b(c + d), \quad x_0 = 1, \ x_0' = 0 \]
and
\[ y_t'' + (b + c + d)y_t' + (b(c + d) - ac)y_t = acx_t^2, \quad y_0 = 0, \ y_0' = 0. \]

The solutions are
\[ x_t = 1 + ac \int_0^t \frac{e^{-q_s} - e^{-q+s}}{q_+ - q_-} \, ds. \]
and

\[ y_t = ac \int_0^t x_s^2 \frac{e^{-g_-(t-s)} - e^{-g_+(t-s)}}{q_+ - q_-} \, ds \]

Moreover, as \( t \to \infty \),

\[ x_t \to x_\infty = E[Z_\infty] = \frac{b(c+d)}{b(c+d) - ac} = \frac{1}{1 - \nu}, \]

\[ y_t \to y_\infty = \text{Var}Z_\infty = (x_\infty - 1)x_\infty^2 = \frac{\nu}{(1 - \nu)^2}. \]

As for properties of the increments of \( Z_t \) we mention the following property which will be used to obtain the covariance of a stationary increments version of the buffer-Hawkes process.

**Proposition 3.1.** The squared increment expectation

\[ w_r(t) = \mathbb{E}[(Z_{r+t} - Z_r)^2], \quad t \geq 0, \]

is integrable over \( r \), such that

\[
\int_0^\infty w_r(t) \, dr = \frac{1}{1 - \nu} \int_0^\infty (x_{r+t} - x_r)^2 \, dr + \frac{\nu}{1 - \nu} \int_0^t \frac{(c+d)e^{-b(t-u)} - be^{-(c+d)(t-u)}}{c+d-b} (x_u^2 + y_u) \, du < \infty
\]

is uniformly bounded in \( t \).

**Proof.** Using

\[ Z_{r+t} - Z_r = \int 1_{0<s+u<r}(Z_{s+u}^{(s,u)} - Z_{r-s-u}^{(s,u)}) \mathcal{M}(ds,du) + \int 1_{r<s+u<r+t}(Z_{r+t-s-u}^{(s,u)}) \mathcal{M}(ds,du), \]

it follows

\[ w_r(t) = \mathbb{E}[Z_{r+t} - Z_r]^2 + \int 1_{0<s+u<r} \mathbb{E}[(Z_{r+t-s-u} - Z_{r-s-u})^2] \, m(ds,du) + \int 1_{r<s+u<r+t} \mathbb{E}[Z_{r+t-s-u}^2] \, m(ds,du) + \int 1_{r<s+u<r+t} (x_{r+t-s-u}^2 + y_{r+t-s-u}) \, m(ds,du). \]
An integration of this relation over \( r \) yields
\[
\int_0^\infty w_r(t) \, dr = \int_0^\infty (x_{r+t} - x_r)^2 \, dr + \frac{ac}{b(c+d)} \int_0^\infty w_r(t) \, dr \\
+ \frac{ac}{b} \int_0^t \frac{(c+d)e^{-(t-u)} - be^{-(c+d)(t-u)}}{(c+d)(c+d-b)}(x_u^2 + y_u) \, du,
\]
which yields the stated integral expression. It is straightforward to use the explicit representations for \( x_t \) and \( y_t \) to check the uniform boundedness in \( t \). \( \square \)

3.2. **Further properties of \( N_t \) derived from cluster representation.** Using (13),
\[
\ln \mathbb{E}[e^{\theta N_t}] = \int_0^t \int_s^t (e^{V_{t-s-u}(\theta)} - 1)\lambda_0 ce^{-(c+d)u} \, duds \\
= \frac{\lambda_0 c}{c+d} \int_0^t (e^{V_u(\theta)} - 1)(1 - e^{-(c+d)(t-u)}) \, du,
\]
so by (16)
\[
\ln \mathbb{E}[e^{\theta N_t}] \leq \frac{\lambda_0 c}{(c+d)^2} \frac{1 - \nu}{\nu} ((c+d)t - 1 + e^{-(c+d)t}), \quad \theta \leq \theta_0.
\]
The mean \( m_t = \mathbb{E}N_t \) derived in Proposition 2.1, is further related to \( x_t = \mathbb{E}Z_t \) via the ODE
\[
m_t'' + (c + d)m_t' = c(g_t' + (c + d)g_t) = \lambda_0 c x_t.
\]
The solution of this equation with \( m_0 = m_0' = 0 \) is
\[
\mathbb{E}N_t = \int_0^t \int_s^t \mathbb{E}Z_{t-s-u} n(ds, du) \\
= \frac{\lambda_0 c}{c+d} \int_0^t x_u(1 - e^{-(c+d)(t-u)}) \, du,
\]
which is also immediate from (18) by differentiation with respect to \( \theta \). The same method allows us to derive an expression for the variance of \( N_t \), which is more convenient than the previous \( \bar{w}_t \), namely
\[
\text{Var}N_t = \mathbb{E}[Z^2_{t-s-u}] n(ds, du) \\
= \frac{\lambda_0 c}{c+d} \int_0^t (y_u + x_u^2)(1 - e^{-(c+d)(t-u)}) \, du.
\]
Using a scaling parameter \( m \), the asymptotic rate of growth of market orders as \( m \to \infty \) is
\[
\frac{1}{m} \mathbb{E}N_{mt} \to \frac{\lambda_0 c}{c+d} x^\infty t = \frac{\lambda_0 c}{c+d} \frac{1}{1 - \nu} t.
\]
with
\[ \frac{1}{m} \text{Var} N_{mt} \to \frac{\lambda_0 c}{c + d} (x_\infty^2 + y_\infty) t = \frac{\lambda_0 c}{c + d (1 - \nu)^3} t, \]
in agreement with Proposition 2.2. Putting these relations together we obtain the weak law of large numbers. On the other hand, the strong law given by
\[ \frac{N_{mt}}{m} \to \frac{\lambda_0 c}{c + d} \frac{1}{1 - \nu} t, \quad m \to \infty, \]
follows from the ergodic theorem in view of the existence of a stationary increments version of \( N \) as given in subsection 4.3 below. Note that stationarity is based on the stability assumption \( \nu < 1 \), and ergodicity is implied by that of the time shifts of a Poisson random measure.

4. DIFFUSION LIMIT

In this section, we consider the long-time scaling of the market orders to obtain a diffusion limit. This component of buffer-Hawkes process is in our focus because the price formation will be based on the market orders in the sequel.

4.1. Diffusion scaling of the market order process. Let us introduce the centered and scaled market orders \( N_t^{(m)} \) by putting
\[ \bar{N}_t = N_t - \mathbb{E}N_t, \quad N_t^{(m)} = \frac{\bar{N}_{mt}}{\sqrt{m}}, \quad t \geq 0. \]
We need the following tightness result to prove functional convergence of the scaled process.

**Lemma 4.1.** The sequence of processes \( \{N_t^{(m)}\}_{m \geq 1} \) is tight.

**Proof.** Let
\[ M_t = N_t - c \int_0^t \Gamma_s \, ds, \quad A_t = c \int_0^t (\Gamma_s - g_s) \, ds, \quad B_t = c^2 \int_0^t \Gamma_s \, ds. \]
Then \( M_t \) is a \((\mathbb{P}, \mathcal{F})\)-martingale, \( A_t \) is the drift and \( \langle M, M \rangle_t = B_t \) is the quadratic variation in the semimartingale decomposition of \( \bar{N}_t \), given by \( \bar{N}_t = A_t + M_t, \quad t \geq 0. \) Let \( (\tau_m)_{m \geq 1} \) be a family of \((\mathcal{F})\)-stopping times all bounded by some constant \( T \), \( \sup_{m \geq 1} \tau_m \leq T \). To verify the Aldous-Rebolledo tightness criterion (see e.g. [13]) we will show that for each \( \epsilon > 0 \) there exist \( \delta > 0 \) and an integer \( m_0 \) such that
\[ \sup_{m \geq m_0} \sup_{h \in [0, \delta]} \mathbb{P} \left( \left| \frac{1}{\sqrt{m}} A_{m(\tau_m + h)} - A_{m\tau_m} \right| > \epsilon \right) \leq \epsilon, \]
and furthermore that the same boundedness property holds when \( A_t \) is replaced by \( B_t \) in (19).
By Chebyshev's inequality,
\[
\mathbb{P}\left( \left| A_{m(\tau_m+h)} - A_{m\tau_m} \right| \geq \epsilon \sqrt{m} \right) \leq \frac{c^2}{\epsilon^2} \mathbb{E}\left[ \left( \int_{\tau_m}^{\tau_m+h} (\Gamma_{ms} - g_{ms}) \, ds \right)^2 \right]
\]
Without restricting the scope of the proof we may take \( h \leq 1 \). Put \( T' = T + 1 \).
By Hölder’s inequality,
\[
\left( \int_{\tau_m}^{\tau_m+h} (\Gamma_{ms} - g_{ms}) \, ds \right)^2 \leq \int_0^{T'} (\Gamma_{ms} - g_{ms})^2 \, ds \cdot \int_{\tau_m}^{\tau_m+h} \, ds = \int_0^{T'} (\Gamma_{ms} - g_{ms})^2 \, ds \cdot h,
\]
and combining the two previous bounds
\[
\mathbb{P}\left( \left| A_{m(\tau_m+h)} - A_{m\tau_m} \right| \geq \epsilon \sqrt{m} \right) \leq \frac{c^2}{\epsilon^2} T' \bar{r}_{\text{sup}} \delta.
\]
Take \( \delta = \epsilon^3/(c^2(T+1)\bar{r}_{\text{sup}}) \) to obtain (19) for the drift process \( A_t \). The same arguments using \( r_{\text{sup}} = \sup_{t \geq 0} \mathbb{E}[\Gamma_t^2] < \infty \) instead of \( \bar{r}_{\text{sup}} \) shows that (19) holds for the quadratic variation process \( B_t \).

Now, we are ready to prove the diffusion limit of \( N^{(m)} \) in the following theorem where asymptotic variance is found in terms of the parameters of the buffer-Hawkes process.

**Theorem 4.2.** \( \{N_t^{(m)}\}_{t \geq 0} \) converges weakly as \( m \to \infty \) in the space \( D([0,\infty), \mathbb{R}) \) of real-valued càdlàg processes to Brownian motion with variance coefficient
\[
\sigma^2 = \frac{\lambda_0 c}{c + d(1-\nu)^2}.
\]

**Proof.** We show convergence of the finite-dimensional distributions. Put
\[
\Phi(x) = e^x - 1 - x.
\]
The cumulant function of \( \bar{N}_t \),
\[
\ln \mathbb{E}\exp\{\theta \bar{N}_t\} = \int_{\mathbb{R}^2_+} 1_{\{s+u \leq t\}} \mathbb{E}\left[ \Phi(\theta Z_{t-s-u}) \right] n(ds, du)
\]
exists finitely for each \( \theta \leq \theta_0 \), due to (16), (17). More generally,
\[
\ln \mathbb{E}\exp\left\{ \sum_{i=1}^n \theta_i \bar{N}_{t_i} \right\} = \int_{\mathbb{R}_+^n \times \mathbb{R}_+^n} \mathbb{E} \left[ \Phi\left( \sum_{i=1}^n \theta_i Z_{t_i-s-u} 1_{\{s+u \leq t_i\}} \right) \right] n(ds, du)
\]
is well-defined for \( 0 \leq t_1 \leq \cdots \leq t_n \) and \( \theta_1, \ldots, \theta_n, n \geq 1 \), with \( \sum_{k=1}^{n} \theta_k \leq \theta_0 \).

Under scaling with scaling parameter \( m \to \infty \), Hlder’s inequality with \( p, q > 1 \), \( \frac{1}{p} + \frac{1}{q} = 1 \) applies to control the remainder term using

\[
m \mathbb{E} \left[ \Phi\left( \frac{X}{\sqrt{m}} \right) - \frac{1}{2} X^2 \right] \leq \frac{1}{\sqrt{m}} \mathbb{E} \left[ |X|^3 \mathbf{e}^{|X|} \right] \leq \frac{1}{\sqrt{m}} \mathbb{E} \left[ |X|^3 \right]^{1/p} \mathbb{E} \left[ \mathbf{e}^{|X|} \right]^{1/q},
\]

for a generic random variable \( X \). Indeed, with

\[
0 < \theta' < \theta_0, \quad \sum_{k=1}^{n} \theta_k \leq \theta' \quad q = \frac{\theta_0}{\theta'} > 1,
\]

and using (17), for large \( m \),

\[
\ln \mathbb{E} \exp \left\{ i \sum_{i=1}^{n} \theta_i N_{t_i}^{(m)} \right\}
= \frac{1}{2m} \int_{\mathbb{R}_+ \times \mathbb{R}_+} \mathbb{E} \left[ \left( \sum_{i=1}^{n} \theta_i Z_{mt_i-s-u} 1_{\{s+u \leq mt_i\}} \right)^2 \right] n(ds, du) + O(\frac{1}{\sqrt{m}})
\sim \frac{1}{2} \sum_{1 \leq i, j \leq n} \theta_i \theta_j \int_{\mathbb{R}_+ \times \mathbb{R}_+} \mathbb{E} \left[ Z_{m(t_i-s)-u} Z_{m(t_j-s)-u} 1_{\{s+u \leq mt_i \land mt_j\}} \right] n(ds, du).
\]

As \( m \to \infty \), the leading term on the right hand side converges to

\[
-\frac{1}{2} \sum_{1 \leq i, j \leq n} \theta_i \theta_j \int_{0}^{t_i \land t_j} \int_{0}^{\infty} \mathbb{E}(Z_{\infty}^2) n(ds, du) = -\frac{\sigma^2}{2} \sum_{1 \leq i, j \leq n} \theta_i \theta_j (t_i \land t_j)
\]

and hence

\[
\ln \mathbb{E} \exp \left\{ i \sum_{i=1}^{n} \theta_i N_{t_i}^{(m)} \right\} \to -\frac{\sigma^2}{2} \sum_{1 \leq i, j \leq n} \theta_i \theta_j (t_i \land t_j).
\]

In view of Lemma 4.1, the proof is complete. \( \square \)

4.2. Price process and its long-time scaling. Suppose \( N^+ \) and \( N^- \) are copies of the above buffer-Hawkes process, representing market call order and market bid order executions, respectively. These two counting processes are naturally associated with up and down movements of the trading price of the underlying asset. The simplest trading price is the midprice formed as in [11, pg.74] by

\[
S_t = \left( N_t^+ - N_t^- \right) \frac{\alpha}{2},
\]

where \( + \) stands for up, and \( - \) stands for down, and \( \alpha > 0 \) is a tick price parameter, and we have taken \( S_0 = 0 \) for simplicity.

When \( N^+ \) and \( N^- \) are independent from each other and identical in distribution, consider \( S_t^{(m)} = S_{mt}/\sqrt{m} \). As a result of the diffusion limit for each \( N^+ \) and \( N^- \), we have that \( \{S_t^{(m)}\}_{t \geq 0} \) converges weakly as \( m \to \infty \) in the space \( D([0, \infty), \mathbb{R}) \) of real valued càdlàg processes to Brownian motion with variance...
coefficient $\beta$, namely, the asymptotic volatility. For the proof of this fact, we observe the semi-martingale decomposition

$$N^+_t - N^-_t = N^+_t - c \int^t_0 \Gamma^+_s \, ds - N^-_t + c \int^t_0 \Gamma^-_s \, ds + A_t$$

where the drift is given by

$$A_t = c \int^t_0 \Gamma^+_s \, ds - c \int^t_0 \Gamma^-_s \, ds,$$

and the quadratic variation of the martingale part of $N^+ - N^-$ is

$$B_t := c^2 \int^t_0 \Gamma^+_s \, ds + c^2 \int^t_0 \Gamma^-_s \, ds.$$  

Therefore, the tightness proof is similar to that of $N^+$ or $N^-$. We get the asymptotic volatility as

$$\beta := \frac{\alpha^2 \sigma^2}{2} = \frac{\alpha^2 \lambda_0 c}{2(c + d)(1 - \nu)^3}$$

where $\nu = \frac{ac}{(c + d)} < 1$ by (11). Note that the volatility increases as $\nu$ increases due to an increase in either $a/b$ or $c/d$ in the buffer-Hawkes model, whereas the volatility solely depends on $a/b$ in the Markovian Hawkes process [11, pg.75].

4.3. Stationary increments version of $N_t$ and a covariance formula. The model considered so far starts with an empty limit order book at time zero, $\Gamma_0 = 0$. The transition of $\Gamma_t$ during a start-up phase from its initial value to approaching the steady-state $\Gamma_\infty$ imposes a non-stationary behavior to $N$. In this section, we construct a version of $N_t$ with stationary increments. Of course, the diffusion approximation will be the same as for the original process. Apart from achieving linear increase in $t$ of the expected value, the main advantage is an explicit expression for the covariance along the paths of the $N$ process.

Recall that in the original model each market order occurs at a time $s + u$, where $s > 0$ is the entry time of a limit order in the LOB and $u$ is the occupation time in the book until the order is executed. The process $N_t$ counts all orders with $s + u \leq t$, including $N_t^{(0)}$ as well as additional market orders accounted for by $Z_{t-s-u}^{(s,u)} - 1$ in (14). To obtain a version $\tilde{N}_t$ of $N_t$ with stationary increments we extend the construction of the buffer-Hawkes process $X$ using a Poisson measure $M$ defined on $\mathbb{R} \times \mathbb{R}_+$ in (3), make the corresponding adjustments elsewhere whenever needed, and move the initiation time of the limit orders backwards to include orders placed at times $s < 0$. For such an $s$, if $s + u \leq 0$ then the resulting number of market orders in $[0, t]$ is given by $Z_{t-s-u}^{(s,u)} - Z_{0-s-u}^{(s,u)}$. If $0 < s + u \leq t$ then as before the contribution to $\tilde{N}_t$ is $Z_{t-s-u}^{(s,u)}$. Upon summing over all Poisson points in $\mathcal{N}$ we define

$$\tilde{N}_t = \int_{\mathbb{R} \times \mathbb{R}_+} \{1_{s+u \leq 0}(Z_{t-s-u}^{(s,u)} - Z_{0-s-u}^{(s,u)}) + 1_{0<s+u\leq t}Z_{t-s-u}^{(s,u)}\} \mathcal{N}(ds, du), \quad t \geq 0.$$
Here, re-ordering terms,

\[
\tilde{N}_{r+t} - \tilde{N}_r = \int_{\mathbb{R} \times \mathbb{R}_+} \{1_{\{s+u \leq 0\}}(Z_{r+t-s-u}^{(s,u)} - Z_0^{(s,u)}) + 1_{\{0 < s+u \leq r+t\}} Z_{r+t-s-u}^{(s,u)}\} \mathcal{N}(ds, du)
- \int_{\mathbb{R} \times \mathbb{R}_+} \{1_{\{s+u \leq 0\}}(Z_{r-s-u}^{(s,u)} - Z_0^{(s,u)}) + 1_{\{0 < s+u \leq r\}} Z_{r-s-u}^{(s,u)}\} \mathcal{N}(ds, du)
= \int_{\mathbb{R} \times \mathbb{R}_+} \{1_{\{s+u \leq r\}}(Z_{r+t-s-u}^{(s,u)} - Z_{r-s-u}^{(s,u)}) + 1_{\{r < s+u \leq r+t\}} Z_{r+t-s-u}^{(s,u)}\} \mathcal{N}(ds, du)
\]

and hence by the shift-invariance of \(\mathcal{N}(ds, du)\) with respect to \(s\) we obtain the desired stationary increments property

\[
\tilde{N}_{r+t} - \tilde{N}_r \overset{d}{=} \tilde{N}_t.
\]

The stationary mean value is obtained as

\[
\tilde{m}_t = \mathbb{E}(\tilde{N}_t) = m_t + \int_{\mathbb{R} \times \mathbb{R}_+} 1_{\{s \leq 0, 0 \leq s+u \leq t\}} x_{t-s-u} n(ds, du)
+ \int_{\mathbb{R} \times \mathbb{R}_+} 1_{\{s+u \leq 0\}}(x_{t-s-u} - x_{0-s-u}) n(ds, du)
= \frac{\lambda_0 c}{c + d} \int_0^t x_s \, ds + \frac{\lambda_0 c}{c + d} \int_0^t \frac{ac}{q_- - q_+} e^{-q_- s} - e^{-q_+ s} \, ds
= \cdots = \frac{\lambda_0 c b t}{b(c + d) - ac}
\]

and the variance is

\[
\text{Var}\tilde{N}_t = \int_{\mathbb{R} \times \mathbb{R}_+} 1_{\{0 \leq s+u \leq t\}} \mathbb{E}[Z_{t-s-u}^2] n(ds, du)
+ \int_{\mathbb{R} \times \mathbb{R}_+} 1_{\{s+u \leq 0\}} \mathbb{E}[(Z_{t-s-u} - Z_{0-s-u})^2] n(ds, du)
= \frac{\lambda_0 c}{c + d} \int_0^t \mathbb{E}[Z_s^2] \, ds + \frac{\lambda_0 c}{c + d} \int_0^\infty \mathbb{E}[(Z_{t+s} - Z_s)^2] \, ds.
\]

By Proposition 3.1, the second term in

\[
\text{Var}\tilde{N}_t = \frac{\lambda_0 c}{c + d} \int_0^t (x_n^2 + y_n) \, ds + \frac{\lambda_0 c}{c + d} \int_0^\infty w_r(t) \, dr
\]

vanishes in the scaling limit, and

\[
\frac{1}{m} \text{Var}\tilde{N}_{mt} \to \frac{\lambda_0 c}{c + d} (x_\infty^2 + y_\infty) = \frac{\lambda_0 c}{c + d} x_\infty^3 t.
\]
Since the increments are stationary, for $s \leq t$,
\[
\text{Cov}(\tilde{N}_s, \tilde{N}_t - \tilde{N}_s) = \frac{1}{2} (\text{Var}\tilde{N}_t - \text{Var}\tilde{N}_s - \text{Var}\tilde{N}_{t-s}) = \lambda_0 c^2 (c + d) \left( \int_s^t (x_u^2 - x_{u-s}^2) \, du + \int_s^t (y_u - y_{u-s}) \, du + \int_0^\infty (w_r(t) - w_r(s) - w_r(t-s)) \, dr \right).
\]

Clearly, $\text{Cov}(\tilde{N}_{ms}, \tilde{N}_{mt} - \tilde{N}_{ms})/m \to 0$, $m \to \infty$, in agreement with the Brownian motion scaling limit.

5. Conclusions and Outlook

With the emphasis in our model on self-exciting features and mutual excitation of market events, it is natural to look further at the implications of these mechanisms for applications in financial mathematics and quantitative finance computations. To indicate potential use of the model developed in this paper, we conclude by mentioning a few examples.

Parameter estimation. Suppose we extract trading data for a single asset and tentatively apply the Markov model $(X_t)$ to capture its evolution over time. The parameter ratio $c/(c + d)$ corresponds to the ratio of executed versus cancelled limit orders, which is a directly observable quantity. Also available is the average number of registered limit orders, represented by $E\Gamma_\infty = \lambda_0 (c + d)^{-1}(1 - \nu)^{-1}$. The coefficient of variation, that is, the ratio of sample variance to sample mean, of many observed $N_{t+1} - N_t$, say, would moreover give a point estimate of $\nu$, and hence of $a/b$. It would be a separate investigation to look into efficient and reliable estimation procedures set up along these lines.

Price formation process. During trading, at the time $s$ of a market order execution there is a supply of $\Gamma_s$ limit orders in the LOB, each equipped with a bid or call price. The market price change, up or down, is determined by the most favorable offer in the LOB, which we may think of as a minimum of the current limit order entries. It is reasonable, therefore, that the change in price will be inversely proportional to $\Gamma_s$. Thus, a more elaborate attempt to model the link between the number of market orders and the trading price could lead to price processes of the type
\[
S_t = \left( \int_0^t \frac{1}{\Gamma_{s-}} 1_{\{x_{s-}^+ \geq 1\}} \, dN_{s+}^+ - \int_0^t \frac{1}{\Gamma_{s-}} 1_{\{x_{s-}^- \geq 1\}} \, dN_{s-}^- \right)^{\alpha/2}.
\]

The moments and long time behavior of the price can be studied under this model.
**Geometric buffer-Hawkes process.** In parallel with geometric Poisson processes, see e.g. [20] Ch. 11, consider a price process of the form

\[ S_t = S_0(1 + \sigma) e^{\alpha t - \sigma \int_0^t c \Gamma_s \, ds}, \]

where \( \sigma > -1 \) is a constant. Then \( S_s - S_- = \sigma S_- \, dN_s \), and hence the discounted price process \( \bar{S}_t = e^{-\alpha t} S_t \) satisfies

\[
\bar{S}_t = \bar{S}_0 - \sigma c \int_0^t \bar{S}_s \Gamma_s \, ds + \sigma \int_0^t \bar{S}_s \, dN_s \\
= S_0 + \sigma \int_0^t \bar{S}_s \, (dN_s - c \Gamma_s \, ds)
\]

which shows that \( \bar{S}_t \) is a martingale with \( \mathbb{E}\bar{S}_t = S_0 \) and mean return \( \alpha \).

**Non-Markovian extensions.** While our construction concerned the Markov process \( (X_t) \), of course the point process component \( (N_t) \) is non-Markovian in its own sake. The branching representation of \( N \) directly provides further extensions in analogy to those in [2]. For example, power-law kernels are used in current studies of high-frequency trading data [21]. In (12), consider a non-negative function \( \phi \) on \( \mathbb{R}_+ \), let \( \mathcal{M} \) be a Poisson measure with intensity process \( m(ds, du) = \phi(s) e^{-(c+d)u} du \), and replace (11) by the condition

\[ \nu = \frac{c}{c + d} \int_0^\infty \phi(s) \, ds < 1. \]

Now, let \( Z \) be the subcritical branching process related to \( \mathcal{M} \) via (12) and, as before, generate \( N \) as in (13). Some of the properties of \( N \) we have studied have direct counterparts in this more general situation, some would need to be studied in greater detail.

**Multi-variate extensions.** Multidimensional versions can be considered for modeling several assets as in e.g. [15, 3]. Here, we mention the two-dimensional case with two buffer-Hawkes processes for buying and selling orders, \( X_t^+ = (\Lambda_t^+, \Gamma_t^+, N_t^+) \) and \( X_t^- = (\Lambda_t^-, \Gamma_t^-, N_t^-) \), respectively. The component processes are defined as previously except that the execution of buy orders trigger the arrival of sell orders, and vice versa, which suggests the semimartingale relations

\[
\begin{cases}
    d\Lambda_t^+ = b(\lambda_0 - \Lambda_t^+) \, dt + ac\Gamma_t^- \, dt + d\tilde{X}_t^+ [f_1] \\
    d\Gamma_t^+ = (\Lambda_t^+ - (c + d)\Gamma_t^+) \, dt + d\tilde{X}_t^+ [f_2] \\
    dN_t^+ = c\Gamma_t^+ \, dt + d\tilde{X}_t^+ [f_3]
\end{cases}
\]

and

\[
\begin{cases}
    d\Lambda_t^- = b(\lambda_0 - \Lambda_t^-) \, dt + ac\Gamma_t^+ \, dt + d\tilde{X}_t^- [f_1] \\
    d\Gamma_t^- = (\Lambda_t^- - (c + d)\Gamma_t^-) \, dt + d\tilde{X}_t^- [f_2] \\
    dN_t^- = c\Gamma_t^- \, dt + d\tilde{X}_t^- [f_3],
\end{cases}
\]

with \( (\tilde{X}^+[\cdot], \tilde{X}^-[\cdot]) \) a six-component martingale, compare section 2.2.
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