A new example on the $f(R)$ and Brans-Dicke theories correspondence

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Abstract

The equivalence between metric $f(R)$ theories and Brans-Dicke theory has led to some confusion in the first years of $f(R)$ theories. The Schwarzschild-de Sitter solution, for example, has been found in the context of $f(R)$ theories, and the same solution is known to have been ruled out by the Brans-Dicke equations of motion in vacuum. As long as the debate continues, we expect to contribute to the subject. We present a new example that confirm the equivalence between $f(R)$ theories and Brans-Dicke theory. Using the Brans-Dicke equations of motion, we find a new solution which corresponds to one recently found in the context of $f(R)$ theories.

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1 Introduction

Since the discovery that the universe is currently in an acceleration expansion era, several theories have been proposed to explain such an unexpected fact [1]. One of these are the so-called $f(R)$ theories, where the Einstein-Hilbert action is replaced by a more general action involving a generic function of the Ricci scalar (for a review, [2]). In this context, one of the first solutions found was the Schwarzschild-de Sitter metric [3].

However, almost at the same time, it has been shown that metric $f(R)$ theories (where the connection does depend of the metric) are in fact equivalent to Brans-Dicke theory with $\omega = 0$ [4], and it is well known that Brans-Dicke theory is incompatible with solar-system experiments for $\omega \geq 50.000$. These results are contradicatory, because the Schwarzschild-de Sitter metric is fully compatible with solar-system tests.

After some debate, the issue has been resolved by showing that the Schwarzschild-de Sitter metric is a solution of the equations of motion, but does not satisfy the boundary conditions [5]. The solution compatible with the boundary conditions is in fact equivalent to the Brans-Dicke solution, and so the $f(R)$ theories are really incompatible with the solar-system tests.

Even being incompatible with our real world, $f(R)$ theories continue to be an active area of research, partially because it helps us better understand the characteristics of General Relativity and partially due to the hope that some underlying mechanism (such as the chameleon effect [6]) can prevent the theory from being ruled out by the solar-system tests.

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The search for new exact solutions of metric $f(R)$ theories is still an open subject, and in the last years some new solutions have been found. Taking into account the equivalence with $\omega = 0$ Brans-Dicke theory it is expected that any solution of this theory has at least one correspondence with the $f(R)$ context.

Our aim in this work is to present a new example on the correspondence between $f(R)$ and Brans-Dicke theories, and try to clarify that metric $f(R)$ theories are not a constrained Brans-Dicke theory in the sense that the $\omega$ parameter is fixed. In fact, the price to be paid by the Brans-Dicke theory to be equivalent to $f(R)$ theories is to have a free scalar potential, which will compensate our freedom to choose the $f(R)$ functional form.

2 The metric $f(R)$/Brans-Dicke correspondence

From the Einstein-Hilbert action to the more general $f(R)$ action we just need to replace the Ricci scalar by a generic function of it. When varying this new action with respect to the metric we get the following equations of motion,

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - [\nabla_{\mu}\nabla_{\nu} - g_{\mu\nu}\square]f'(R) = 0,$$

where the $'$ represents the derivative with respect to the Ricci scalar. Taking the trace of the above equation we get

$$f'(R)R - 2f(R) + 3\square f'(R) = 0$$

where we should note that $f'(R)$ obeys a dynamical equation instead of an algebraic one. Therefore, we can consider $f'(R)$ a field in the coordinates $x^\mu$, which in principle can assume any form. Let’s consider an example. We wish to use the $f(R)$ theories to find as a solution the metric

$$ds^2 = (1 - Ar)dt^2 - (1 - Ar)^{-1}dr^2 - r^2d\theta^2 - r^2 \sin(\theta)^2d\phi^2,$$

where $r$, $\theta$ and $\phi$ are the usual spherical coordinates, and $A$ is some parameter.

Following [7], it can be shown that this metric must obey the equation

$$f'(R) = ar + b.$$  \hspace{1cm} (2.4)

The Ricci scalar for this metric is $r = -6A/R$, and if we substitute it in (2.4) and integrate, we find

$$f(R) = bR - (6aA)\ln(R),$$  \hspace{1cm} (2.5)

which resembles the solution for the global monopole in $f(R)$ theories found in [8]. Using this method we can find practically any desired solution in a ‘reconstructive’ approach. This reconstructive method has been vastly used in cosmology to model the currently and past accelerated phases of the universe. However, the function form of these $f(R)$ theories can be very complicated, and hardly represents a fundamental or even an effective potential coming from a deeper theory.

On the other side of the correspondence there is the Brans-Dicke theory with the equations of motion in vacuum given by

$$\phi R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\phi = \frac{\omega}{\phi}(\partial_{\mu}\phi\partial_{\nu}\phi - \frac{1}{2}g_{\mu\nu}\partial_{\rho}\phi\partial^{\rho}\phi) + (\nabla_{\mu}\nabla_{\nu} - g_{\mu\nu}\square)\phi$$

and

$$\square\phi = 0.$$  \hspace{1cm} (2.7)
If we take $\omega = 0$ in (2.6), it assumes a functional form which resembles (2.1). Let’s suppose that $\phi(x^\mu)$ can be taken equal to $f'(R(x^\mu))$. Then, in order that (2.6) be equivalent to (2.1), we need to replace the second term on the LHS by $\frac{1}{2}g_{\mu\nu}f(R)$. This can be achieved by the introduction of a potential for the scalar field,

$$V(\phi) = R\phi - f(R(\phi)).$$

(2.8)

It is worth calling attention to the fact that it is exactly this potential that will be responsible by the mapping between both theories.

3 From f(R) solutions to Brans-Dicke solutions

In this section, we will assume that $V(\phi)$ has different functional forms and discuss the implications on the Brans-Dicke solutions.

3.1 $V(\phi)$ is constant

Using a constant potential, we have

$$\Lambda = f'(R)R - f(R),$$

(3.1)

which has as solution $f(R) = \phi R - \Lambda$, where $\Lambda$ is a constant. This relation is obvious, since if we impose that the potential assumes a constant value, we are in fact imposing that the scalar field assumes a constant value, and the Brans-Dicke action becomes

$$S = \int d^4x \sqrt{-g}(\alpha R - V(\alpha)),$$

(3.2)

where both $\alpha$ and $V(\alpha)$ are constants. This is the same as the Einstein-Hilbert action with a cosmological constant term, and we can say that we have found the Schwarzschild-de Sitter metric as a solution to Brans-Dicke theory with no surprises.

Let us clarify this subject further: taking the trace of (2.2), which give us $f'(R) = f'(R)R/2$ and, and substituting in the field equations given by (2.1), we get

$$f'(R)R_{\mu\nu} - \frac{1}{4}f'(R)g_{\mu\nu}R = 0,$$

(3.3)

which resembles the Einstein’s vacuum equations. As long as $f'(R) \neq 0$, the only difference is the factor on the second term, $1/4$, replacing the usual $1/2$ of Einstein’s equations. This factor is crucial, because the trace of the above equation gives us $R - R = 0$, avoiding a constraint in the Ricci scalar. Any other numerical factor would led us to conclude that $R = 0$ and thus the impossibility of the Ricci scalar to be a constant.

This could lead us to the search of a fundamental theory that replaces Einstein’s equations by the above equations in vacuum. However the above equations violate WEP in any other case than constant $R$.

3.2 $V(\phi)$ is spherically symmetric

The $f(R)$ theories can guide us to new exact solutions of Brans-Dicke theory with less effort, by using the equivalence between these theories. We will now find a solution for Brans-Dicke with $\omega = 0$ which corresponds to one already obtained in the scope of $f(R)$ theories few years ago. We will first derive it using an ansatz, and then show that it could be easily mapped from the $f(R)$ solution already obtained [7].

Let’s start with a spherically symmetric metric
Using the Ricci scalar to link both theories, we have
\[ ds^2 = a(r)dt^2 - a(r)^{-1}dr^2 - r^2d\theta^2 - r^2\sin(\theta)^2d\phi^2, \] (3.4)
and put it in the Brans-Dicke equations in the vacuum, with \( \omega = 0 \) and a generic potential \( V(\phi) \). These equations are given by

\[ \phi R_{\mu\nu} - \frac{1}{2}\phi g_{\mu\nu}R = \nabla_{\mu}\nabla_{\nu}\phi - \frac{1}{6}V(\phi)g_{\mu\nu} + \frac{1}{3}g_{\mu\nu} \frac{\partial V(\phi)}{\partial \phi} \phi, \] (3.5)

\[ \Box \phi = \frac{2}{3}V(\phi) - \frac{1}{3} \phi \frac{\partial V(\phi)}{\partial \phi}. \] (3.6)

Thus, using the above metric we get the following set of equations:

\[ 6\phi(r)a'(r) - 6\phi(r) + 6\phi(r)a(r) - 3\phi'(r)a'(r)r^2 - V(\phi)r^2 + 2\phi(r) \frac{\partial V(\phi)}{\partial \phi} = 0 \] (3.7)

\[ 6\phi(r)a'(r) - 6\phi(r) + 6\phi(r)a(r) - 3\phi'(r)a'(r)r^2 - V(\phi)r^2 + 2\phi(r) \frac{\partial V(\phi)}{\partial \phi} - 6r^2 \phi''(r)a(r) = 0 \] (3.8)

\[ 6\phi(r)a'(r) + 3\phi(r)a''(r) - 6\phi'(r)a(r) - V(\phi)r + 2\phi(r) \frac{\partial V(\phi)}{\partial \phi}r = 0 \] (3.9)

\[ 3\phi'(r)a'(r)r + 3\phi''(r)a(r)r + 2V(\phi)r - \phi(r) \frac{\partial V(\phi)}{\partial \phi}r = 0 \] (3.10)

where now the \( ' \) means a derivative with respect to the radial coordinate.

Comparing (3.7) and (3.8) we find that \( \phi(r) = \alpha r + \beta \) and, for simplicity, we will choose \( \beta = 0 \). We can then use equations (3.9) and (3.10) to find \( V(\phi) \) and \( \frac{\partial V(\phi)}{\partial \phi} \). Now, let us verify if they are consistent with each other. Explicitly, \( V(\phi) \) and \( \frac{\partial V(\phi)}{\partial \phi} \) are given by

\[ V(r) = \frac{\alpha}{r^2}(4a'(r)r + 2a(r) + a''(r)r^2), \] (3.11)

\[ \frac{\partial V(\phi)}{\partial \phi} = \frac{-1}{r^2}(5ra'(r) - 2a(r) + 2a''(r)r^2). \] (3.12)

Thus, substituting these equations in (3.7), we have the following to solve

\[ -a''(r)r^2 - ra'(r) - 2 + 4a(r) = 0. \] (3.13)

The solution of the above equation is

\[ a(r) = \frac{1}{2}(1 - \frac{C_1}{r^2} + C_2r^2), \] (3.14)

and we need to check if our result for \( V(\phi) \) is consistent with its derivative. Replacing \( a(r) \) in (3.11) and (3.12), we get

\[ V(r) = -6C_2 \alpha r - \frac{\alpha}{r} = -6C_2 \phi - \frac{\alpha^2}{\phi}, \] (3.15)

and

\[ \frac{\partial V(\phi)}{\partial \phi} = -6C_2 + \frac{1}{r^2} = -6C_2 + \frac{\alpha^2}{\phi^2} \] (3.16)

which is consistent with our choice \( \phi(r) = \alpha r \).

Let’s now compare our result with the result obtained by Sebastiani and Zerbini \cite{7} in the context of \( f(R) \) theories. They started with the same ansatz we used for the metric and found that the same result reconstructs the theory \( f(R) = \frac{\alpha}{2}\sqrt{R + 6C_2} \), together with \( f'(R) = \alpha r \). Using the Ricci scalar to link both theories,
we have

\[ V(\phi) = f'(R)R - f(R) = \alpha \sqrt{\frac{1}{R + 6C_2}} R - \frac{\alpha}{2} \sqrt{R + 6C_2} \]  \hspace{1cm} (3.18)

\[ = \alpha r - \frac{\alpha}{2r} = f'(R) - \frac{\alpha^2}{2f'(R)}, \]  \hspace{1cm} (3.19)

with the same form we have found for the potential when \( f'(R)(r) = \phi(r) \).

4 Conclusions

The \( f(R) \) theories and Brans-Dicke theory with a potential are both open theories where we can find several new solutions not allowed by Einstein’s theory. This possibility results from the arbitrariness of our choice of the functional form in \( f(R) \) theories and the scalar potential in Brans-Dicke theory. In the spherically symmetric case, as long as we can write our Ricci scalar and the function \( f'(R) \) as functions of the radial coordinate and invert it, we can in principle map all \( f(R) \) solutions in the corresponding Brans-Dicke ones. If these solutions have some physical meaning, it is not well known and it requires further investigations.

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