Anisotropic bulk viscous string cosmological models of the Universe under a time-dependent deceleration parameter

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Abstract

We investigate a new class of LRS Bianchi type-II cosmological models by revisiting in the paper of Mishra et al (2013) by considering a new deceleration parameter (DP) depending on the time in string cosmology for the modified gravity theory suggested by Sáez & Ballester (1986). We have considered the energy-momentum tensor proposed by Letelier (1983) for bulk viscous and perfect fluid under some assumptions. To make our models consistent with recent astronomical observations, we have used scale factor \((a(t) = \exp\left(\frac{1}{3}\sqrt{2\beta t + k}\right)\) where \(\beta\) and \(k\) are positive constants and it provides a time-varying DP. By using the recent constraints \((H_0 = 73.8, q_0 = -0.54)\) from SN Ia data in combination with BAO and CMB observations (Giostri et al, arXiv:1203.3213v2[astro-ph.CO]), we affirm \(\beta = 0.0062\) and \(k = 0.00016\). For these constraints, we have substantiated a new class of cosmological transit models for which the expansion takes place from early decelerated phase to the current accelerated phase. Also, we have studied some physical, kinematic and geometric behavior of the models, and have found them consistent with observations and well established theoretical results. We have also compared our present results with those of Mishra et al (2013) and observed that the results in this paper are much better, stable under perturbation and in good agreement with cosmological reflections.

Keywords: String cosmology; Sáez-Ballester theory; Bulk viscosity; Transit Universe.
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1 Introduction

The current astronomical reflexions, modern experimental data from SNe Ia [1–6]; CMBR [7, 8]; WMAP [9–12] have established two main characteristics of the universe: (a) the existence of the anisotropic universe at the early stage of the evolution, which in due course of time attains isotropy, and (b) the current universe is not only expanding but also the rate of expansion is increasing (i.e. accelerating universe). The SNe Ia measurements indicate a universe which undergoes through a transition from past decelerating to present accelerating expansion. So, it is a challenge for theoreticians to provide satisfactory theoretical support to these observations.
Friedmann-Robertson-Walker (FRW) spacetime describes spatially homogenized and isotropic universes, which can be appropriate for the contemporary universe, however since they have higher symmetries, thus it doesn’t provide a correct matter description of the early universe and presents the poor approximations for an early universe. Therefore, those models are additional applications for the outline of the whole evolution of the universe, that have an anisotropic nature in early time and approaches to isotropy at late times. Bianchi spacetimes offer a decent framework for this. Out of all, Bianchi type-II (B-type-II) frame of reference plays a very important affirm in making models for the measurements of flour-ish of the universe throughout its early phase. Moreover, B-type-II line-element yields an anisotropic spatial curvature. Recently, Asseo and Sol [13] and Roy & Banerjee [14] stressed the importance of B-type-II and proscribed LRS cosmological model. Kumar and Akarsu [15] mentioned B-type-II universe with anisotropic dark energy and perfect fluid. Wang [16]−[18] has investigated the models of Letelier-type within the theoretical account of LRS B-type-II. In the context of massive string, Pradhan et al. [19] have analyzed LRS Bianchi type-II spacetime. B-type-II frame of reference is employed to analyze dark energy models within the new role of time-dependent DP by Maurya et al [19]. Within the present study, we tend to look into LRS B-type-II string models of the universe for perfect and viscous fluid beneath three conditions.

Next, although Einstein’s general theory of gravity (GR) explains a large number of the astrophysical phenomenon, it fails to describe some, for instance, the expanding and late time accelerated the expansion of the universe. To deal with these, many alternative theories are proposed, out of which, Brans & Dicke [21] and Sáez & Ballester [22] scalar-tensor theories are of significant involvement. In the present paper, we have studied the Sáez-Ballester modified theory of gravity. In this theory, Einstein’s field equations have been modified by incorporating a dimensionless scalar field $\phi$ coupled with the metric $g_{ij}$ in a simple manner. This modification satisfactorily describes the weak field in which an accelerated expansion regime appears. This theory also advises an answer to the question of disappeared matter in a non-flat FRW universe.

In recent years, string cosmology is gaining significant interest. Cosmic strings are topologically stable objects, which could be shaped throughout a phase transition within the early universe. Cosmic strings make for a significant role to study in the early universe. It is assumed that cosmic strings bring about to density perturbations, that cause the formation of galaxies or cluster of galaxies [23]. One more necessary feature of the string is that the string tension provides rise to an efficient anisotropic pressure. Also, the stress-energy of the string coupled with the gravitational field is also used to elucidate several alternative cosmological phenomena. The pioneer works in string theory were done by many authors [24, 25]. LRS B-type-II cosmological models have been discussed [26−29] in different context. Recently, Pradhan et al [30] have looked into string models of accelerated expansion in $f(R, T)$-gravity with a magnetic field.

Also, the dissipation effect together with bulk viscosity presents another model of dark energy. Relaxation processes related to bulk viscosity effectively reduce the pressure in an expanding system, in comparison the worth prescribed by the equation of state $p = \omega \rho$. The effective pressure becomes negative for a sufficiently large viscosity that could imitate a dark energy behavior. The idea that the bulk viscosity drives the acceleration of the universe is
mentioned in [31, 32].

In recent years, many researchers [33]−[38] and references therein have investigated the cosmological universes in Saez-Ballester modified gravity theory in various contexts. Under above-discussed perspective, the S´alez & Ballester field equations have been solved in an LRS B-type-II space-time in the presence of a cloud of massive string and bulk viscous fluid, under some physically and geometrically viable assumptions. In the present paper, we are revisiting the solutions obtained by Mishra et al [39], by assuming a scale factor $a(t) = \exp[\frac{1}{\beta^2} \sqrt{2\beta^2 t + k}]$ which resulting into a time-dependent DP having a transition from the decelerating universe to presently accelerating universe.

The plan of the manuscript is the following. Section 2 contains definitions and theoretical calculations. Subsec. 2.1 deals with the metric and field equations are mentioned. Subsec. 2.2 deals with assumptions and under these assumptions, the solution of the field equations are found. In Section 3, we have derived the solutions of field equations for three different cases, Subsections 3.1, 3.2 & 3.3. Results and discussions are given in Sec. 4. Stability of corresponding solutions is analyzed in Sect. 5. Finally, conclusions are summarized in Sec. 6.

2 Definitions and Theoretical Calculations

2.1 Metric and Field equations

We consider an LRS B-type-II space-time [39]:

$$ds^2 = -dt^2 + X^2dx^2 + Y^2dy^2 + 2X^2xdydz + (Y^2x^2 + X^2)dz^2$$

where $X = X(t), Y = Y(t)$.

The field equation (in gravitational units $8\pi G = 1$) proposed by S´alez & Ballester [22]:

$$G_{ij} - \omega \phi r(\phi, \phi) - \frac{1}{2} g_{ij} \phi, k \phi^k = -T_{ij}.$$  

(2)

Here $G_{ij} = R_{ij} - \frac{1}{2} R g_{ij}$ and $T_{ij}$ stands for the energy-momentum tensor and $\phi$ for the scalar field satisfying the equation

$$r \phi^{r-1} \phi, k \phi^k + 2 \phi^r \phi_i^i = 0$$

(3)

Here $\omega$ and $r$ stand for a dimensionless coupling and arbitrary constant respectively. A comma denotes the partial derivative whereas a semi-colon denotes partial covariant differentiation w. r. to $t$.

$T_{ij}$, for a cloud of massive string & bulk viscous fluid, reads:

$$T_{ij} = \overline{p} g_{ij} - \lambda x_i x_j + (\rho + \overline{p}) v_i v_j,$$

(4)

where

$$\overline{p} = p - 3H\xi$$

(5)
In above Eqs. (4) and (5) the different quantities have their usual meaning as already described in [39]. The four velocity of the particles \( v^i = (0, 0, 0, 1) \) and a unit space-like vector \( x^i \) representing the direction of string satisfy \( g_{ij}v^iv^j = -g_{ij}x^ix^j = -1, v^ix_i = 0 \). In LRS Bianchi type-II metric, the mean Hubble parameter \( H \) can be defined as

\[
H = \frac{\dot{a}}{a} = \frac{1}{3} \left( 2\frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} \right) = \frac{1}{3}(2H_1 + H_2). \tag{6}
\]

Here \( H_1 = \frac{\dot{X}}{X} \) and \( H_2 = \frac{\dot{Y}}{Y} \) are directional Hubble parameters in the directions of \( x \) and \( y \) axes respectively. Here \( a = a(t) \) is average scale factor, which, for LRS B-type-II model, is written as

\[
a(t) = (X^2Y)^{\frac{1}{3}} \tag{7}
\]

The particle density denoted by \( \rho_\rho \) follows the relation

\[
\rho = \rho_\rho + \lambda \tag{8}
\]

For the Metric, (1), the Sáez-Ballester field equations (2) & (3), along with energy-momentum tensor given by (4), we obtain the following system of field equations

\[
\frac{\ddot{X}}{X} + \frac{\ddot{Y}}{Y} + \frac{\dot{X} \dot{Y}}{XY} + \frac{1}{4} \frac{Y^2}{X^4} = \frac{1}{2} \omega \phi \dot{\phi} = \lambda - p \tag{9}
\]

\[
\frac{2}{X} \frac{\dddot{X}}{X} + \frac{\dddot{Y}}{Y} - \frac{3}{4} \frac{Y^2}{X^4} - \frac{1}{2} \omega \phi \dot{\phi} = -p \tag{10}
\]

\[
\frac{\dot{X}^2}{X^2} + \frac{\dot{Y}^2}{Y^2} - \frac{1}{4} \frac{Y^2}{X^4} - \frac{1}{2} \omega \phi \dot{\phi} = \rho \tag{11}
\]

\[
\ddot{\phi} + \frac{\dot{\phi}}{\dot{\phi}} \left( 2\frac{\dddot{X}}{X} + \frac{\dddot{Y}}{Y} \right) + \frac{r}{2} \frac{\dot{\phi}^2}{\phi} = 0 \tag{12}
\]

In the usual notation, expansion scalar \( \theta \) and the shear scalar \( (\sigma) \) are defined and given as

\[
\theta = v^i_{;j} = 3 \frac{\ddot{a}}{a} = 2 \frac{\dddot{X}}{X} + \frac{\dddot{Y}}{Y} \tag{13}
\]

and

\[
\sigma^2 = \frac{1}{2} \frac{\sigma_{ij} \sigma^{ij}}{\dot{\theta}^2} = \frac{1}{2} \left[ 2 \frac{\dot{X}^2}{X^2} + \frac{\dot{Y}^2}{Y^2} \right] - \frac{1}{6} \dot{\theta}^2 \tag{14}
\]

where

\[
\sigma_{ij} = v_{ij} + \frac{1}{2} (v_{i;k} v^k v_j + v_{j;k} v^k v_i) + \frac{1}{2} \dot{\theta} (g_{ij} + v_i v_j)
\]

The anisotropy parameter \( (A_m) \) is defined as

\[
A_m = 6 \left( \frac{\sigma}{\dot{\theta}} \right)^2 = \frac{2 \sigma^2}{3 H^2} \tag{15}
\]
2.2 Assumptions

There are four equations (9)-(12) having seven unknowns $X, Y, \phi, p, \rho, \xi$ and $\lambda$. For deterministic solutions of this system, we have to take three more equations, which relates these parameters.

As suggested by Thorne [40] and followed by many researchers [41, 42], we first, assume $\theta$ is proportional to $\sigma$ which gives

$$\frac{1}{\sqrt{3}} \left( \frac{2\dot{X}}{X} - \frac{\dot{Y}}{Y} \right) = \ell \left( \frac{2\ddot{X}}{X} + \frac{\ddot{Y}}{Y} \right)$$

(16)

where $\ell$ is the constant of proportionality. This yields

$$\frac{\dot{X}}{X} = m \frac{\dot{Y}}{Y},$$

(17)

where $m = \frac{\sqrt{3+\ell}}{\ell-2\sqrt{3}}$. We have select $m > 0$ for anisotropic universe, provided $m \neq 1$, as the study presents a picture of FRW model for $m = 1$. Integrating Eq. (17) and we get

$$X = c_1 (Y)^m,$$

(18)

where $c_1$ is a constant of integration. Any loss of generality and for simplicity, $c_1 = 1$ is considered. Hence Eq. (18) is reduced to

$$X = (Y)^m$$

(19)

Secondly, we consider $q$ as linear function of Hubble parameter [43 – 46]:

$$q = -\frac{a\ddot{a}}{a^2} = \beta H + \alpha = \beta \frac{\dot{a}}{a} + \alpha.$$  

(20)

Here $\alpha$, and $\beta$ stand for arbitrary constants. Eq. (20) renders as $\frac{a\ddot{a}}{a^2} + \beta \frac{\dot{a}}{a} + \alpha = 0$, which by solving proceeds as

$$a = \exp \left[ -\frac{(1 + \alpha)}{\beta} t - \frac{1}{(1 + \alpha)} + \frac{l}{\beta} \right], \text{ provided } \alpha \neq -1.$$  

(21)

Here $l$ is a constant of integration.

From Eq. (21), we calculate

$$\dot{a} = - \left( \frac{1 + \alpha}{\beta} \right) \exp \left[ - \left( \frac{1 + \alpha}{\beta} \right) t \right] - \frac{1}{(1 + \alpha)} + \frac{l}{\beta},$$

$$\ddot{a} = \left( \frac{1 + \alpha}{\beta} \right)^2 \exp \left[ - \left( \frac{1 + \alpha}{\beta} \right) t - \frac{1}{(1 + \alpha)} + \frac{l}{\beta} \right].$$

(22)

Eqs. (20) and (22) render the value of DP as $q = -1$. We also observed the same value of DP for $\alpha = 0$. 

5
For $\alpha = -1$, Eq. (20) is changed into the form:

$$q = -\frac{a\ddot{a}}{a^2} = -1 + \beta H,$$

(23)

Eq. (23) reproduced the following differential equation:

$$\frac{a\ddot{a}}{a^2} + \beta \frac{\dot{a}}{a} - 1 = 0.$$  

(24)

The solution of above equation is found to be (Sharma et al. 2019, Garg et al. 2019)

$$a = \exp \left[ \frac{1}{\beta} \sqrt{2\beta t + k} \right],$$  

(25)

where $k$ is an integrating constant. Eq. (25) is recently used by different authors [43]–[46] in different contexts.

For the study of cosmic decelerated-accelerated expansion of the universe, we only consider the case $\alpha = -1$.

For the scale factor (25), the DP $(q)$ and Hubble parameter $H$ is given as

$$q = -1 + \frac{\beta}{\sqrt{2\beta t + k}}, \quad H = \frac{1}{\sqrt{2\beta t + k}}.$$  

(26)

From Eq. (26), we observe that $q > 0$ for $t < \frac{\beta^2 - k}{2\beta}$ and $q < 0$ for $t > \frac{\beta^2 - k}{2\beta}$.

3 Solution of the field Equation

By using Eqs. (19), (25) and (7), we obtain:

$$X = (e^{\frac{1}{\rho} \sqrt{2\beta t + k}})^{\frac{3m}{2m+1}},$$  

(27)

$$Y = (e^{\frac{1}{\rho} \sqrt{2\beta t + k}})^{\frac{3}{2m+1}}.$$  

(28)

From Eqs. (12), (27) and (28), we evaluate scalar field $(\phi)$ as

$$\phi(t) = \left[ \frac{r + 2}{2} \left( \phi_0 \int \frac{dt}{(e^{\frac{1}{\rho} \sqrt{2\beta t + k}})} + \phi_1 \right) \right]^{\frac{2}{r+2}},$$  

(29)

where $\phi_0$ and $\phi_1$ are integrating constants.

Solving Eqs. (9)-(11) by using Eqs. (27)-(29), we obtain energy density $\rho$, effective pressure $\pi$ and string tension density $\lambda$ as

$$\rho = \left[ \frac{9m(m+2)}{(2m+1)^2(2\beta t + k)} - \frac{1}{4} (e^{\frac{1}{\rho} \sqrt{2\beta t + k}})^{\frac{6-12m}{2m+1}} + \frac{1}{2} \left( e^{\frac{\omega}{\beta} \sqrt{2\beta t + k}} \right)^2 \right] + \omega \phi_0^2.$$  

(30)
\[ p = \left[ -\frac{27m^2}{(2m+1)^2(2\beta t + k)} + \frac{6m\beta}{(2m+1)}(2\beta t + k)^{\frac{3}{2}} + \frac{3}{4}(e^{\frac{1}{2}\sqrt{2\beta t + k}})^{\frac{6-12m}{2m+1}} + \frac{1}{2}(\frac{\omega\phi_0^2}{e^{\frac{1}{2}\sqrt{2\beta t + k}}}) \right] \]  
\[ \lambda = \left[ -\frac{18m^2 + 9m + 9}{(2m+1)^2(2\beta t + k)} - \frac{3\beta(m-1)}{(2m+1)(2\beta t + k)^{\frac{3}{2}}} + (e^{\frac{1}{2}\sqrt{2\beta t + k}})^{\frac{6-12m}{2m+1}} \right] \]  

Accordingly, the particle density \( \rho_p \) is obtained as

\[ \rho_p = \left[ \frac{(27^2 + 9m - 9)}{(2m+1)^2(2\beta t + k)} - \frac{5}{4}(e^{\frac{1}{2}\sqrt{2\beta t + k}})^{\frac{6-12m}{2m+1}} + \frac{1}{2}(\frac{\omega\phi_0^2}{e^{\frac{1}{2}\sqrt{2\beta t + k}}}) - \frac{3\beta(m-1)}{(2m+1)(2\beta t + k)^{\frac{3}{2}}} \right] \]  

For calculating the other parameters, we shall consider the following three cases.

**3.1 Case I: Bulk Viscous Model with \( p = \alpha \rho \)**

Considering perfect gas equation of state as:

\[ p = \alpha \rho, \]  

where \( \alpha (0 \leq \alpha \leq 1) \) is a constant. For the various values of \( \alpha \), we will get three types of models:

(i) if \( \alpha = 0 \), we tend to get matter dominant model.

(ii) if \( \alpha = \frac{1}{3} \), we tend to get radiation dominant model.

if \( \alpha = 1 \), we get \( \rho = p \) which is termed as Zel’dovich fluid or stiff fluid model [47].

Therefore by Eqs. (31) and (33), we can directly calculate the following values of \( (p) \) and \( (\xi) \):

\[ p = \left[ \frac{9ma(m+2)}{(2m+1)^2(2\beta t + k)} - \frac{\alpha}{4}(e^{\frac{1}{2}\sqrt{2\beta t + k}})^{\frac{6-12m}{2m+1}} + \frac{\alpha}{2}(\frac{\omega\phi_0^2}{e^{\frac{1}{2}\sqrt{2\beta t + k}}}) \right] \]  

\[ \xi = \left[ \frac{(3m^2\alpha + 6ma + 9m^2)}{(2m+1)^2\sqrt{2\beta t + k}} - \frac{\alpha + 3}{12}(e^{\frac{1}{2}\sqrt{2\beta t + k}})^{\frac{6-12m}{2m+1}}\sqrt{2\beta t + k} - \frac{2m\beta}{(2m+1)(2\beta t + k)} + \frac{(\alpha - 1)\omega\phi_0^2\sqrt{2\beta t + k}}{(e^{\frac{1}{2}\sqrt{2\beta t + k}})} \right] \]
3.2 Case II: Bulk Viscous Model with $\xi = \xi_0 \rho^n$

For most of the investigations, we found that the coefficient of bulk viscosity $\xi$ is considered as a simple power function of energy density and it depends on time. It is assumed,

$$\xi = \xi_0 \rho^n,$$

where $\xi_0$ and $n$ are real constants [18 - 50]. For small density and radiative fluid, $n$ may be equal to 1 [51, 52]. For $(0 \leq n \leq 1/2)$ is good assumption to obtain realistic results, as given by Belinskii and Khalatnikov (1975).

Using Eqs. (5), (30), (31) and (37), the expressions for $\xi$ and $p$ are given as:

$$\xi = \xi_0 \left[ \frac{9m(m+2)}{(2m+1)^2(2\beta t+k)} - \frac{1}{4} \left( e^{\frac{1}{2} \sqrt{2\beta t+k}} \right)^{ \frac{6-12m}{2m+1} } + \frac{1}{2} \frac{\omega \phi_0^2}{2 \left( e^{\frac{1}{2} \sqrt{2\beta t+k}} \right)} \right]^n$$

$$p = \left[ \frac{3\xi_0}{\sqrt{2\beta t+k}} - \frac{9m(m+2)}{(2m+1)^2(2\beta t+k)} - \frac{1}{4} \left( e^{\frac{1}{2} \sqrt{2\beta t+k}} \right)^{ \frac{6-12m}{2m+1} } + \frac{1}{2} \frac{\omega \phi_0^2}{2 \left( e^{\frac{1}{2} \sqrt{2\beta t+k}} \right)} \right]^n - \frac{27m^2}{(2m+1)^2(2\beta t+k)} + \frac{6m\beta}{(2m+1)(2\beta t+k)} \frac{e^{\frac{1}{2} \sqrt{2\beta t+k}}}{2}$$

$$+ \frac{3}{4} \left( e^{\frac{1}{2} \sqrt{2\beta t+k}} \right)^{ \frac{6-12m}{2m+1} } + \frac{1}{2} \frac{\omega \phi_0^2}{2 \left( e^{\frac{1}{2} \sqrt{2\beta t+k}} \right)}$$

3.3 Case III: Perfect Fluid Model with $\xi = 0$

For perfect fluid, the coefficient of bulk viscosity is assumed to be zero. The rest of six unknowns $X, Y, \phi, p, \rho$ and $\lambda$ can be directly calculated from the field Eqs. (9)-(12). For $\xi = 0$, Eq. (5) gives $p = p$ i.e. effective pressure equals to isotropic pressure, and the expression is given by

$$p = \left[ - \frac{27m^2}{(2m+1)^2(2\beta t+k)} + \frac{6m\beta}{(2m+1)(2\beta t+k)} \frac{e^{\frac{1}{2} \sqrt{2\beta t+k}}}{2} + \frac{3}{4} \left( e^{\frac{1}{2} \sqrt{2\beta t+k}} \right)^{ \frac{6-12m}{2m+1} } + \frac{1}{2} \frac{\omega \phi_0^2}{2 \left( e^{\frac{1}{2} \sqrt{2\beta t+k}} \right)} \right]$$

4 Interpretation of the Results

From Eq. (26), the present value of declaration parameter can be taken as $q_0 = -1 + \beta H_0 = -1 + \frac{\beta}{\sqrt{2\beta t_0+k}}$, where $H_0$ and $t_0$ have their usual meaning.

By using the recent constraints ($H_0 = 73.8$, and $q_0 = -0.54$) from SN Ia data in combination BAO and CMB observations [54], we concentrate the values of $\beta = 0.0062$ and $k = 0.000016$. We have used these values in formulating and drawing the different figures to
analyze the nature of physical quantities.

We have plotted the variation of deceleration parameter $q$ with respect to cosmic time $t$ in Figure 1(a), and observed that deceleration parameter is positive at early time and negative at present time indicating that our models are evolving from decelerating phase ($q > 0$) to accelerating phase ($q < 0$), and the models show a phase transition from positive to negative for DP $q$ for $k = 0.000016$ and $\beta = 0.0062$. The critical time at which the phase transition took place is given by $t_c = \frac{\beta^2 - k}{2\beta}$. Also, when $t \to \infty$, $q \to -1$. According to SNe Ia observation, the universe is accelerating at present and the value of DP lies in the range $-1 < q < 0$. So our models show consistency with recent observations.

Figure 1(b) shows the variation of Hubble parameter $H$ with respect to cosmic time $t$ as per Eq. (26). We see that $H$ is a positive, decreasing function of time, and tends to zero as $t \to \infty$, which totally agrees with the established theories.

The average scale factor $a(t)$ in terms of redshift $z$ is given by $a(t) = \frac{a_0}{1+z}$, where $a_0$ is the present value of the average scale factor $a(t)$.

From Eq. (25), we can get $a_0 = \exp \left[ \frac{1}{\beta} \sqrt{2\frac{\beta}{H_0} + k} \right]$. Using the values of $\beta = 0.0062$, $k = 0.000016$ and $H_0 = 73.8$ we get $a_0 = 8.6021$. We have used these values to draw the graph.

From Eq. (25), we get $\sqrt{\frac{2\beta + k}{\beta}} = \ln(a)$. Also from $a(t) = \frac{a_0}{1+z}$, we have $\ln(a) = \ln(a_0) - \ln(1 + z)$.

Substituting the above in Eq. (21), we get

$$q(z) = -1 + \frac{1}{\ln(a_0) - \ln(1 + z)}$$

Figure 1: (a) The plot DP $q$ versus $t$, (b) The plot of $H$ versus $t$, Here $\beta = 0.0062$ and $k = 0.000016$. 


Figure 2: (a) The plot of redshift $z$ versus cosmic time $t$, (b) The plot of deceleration parameter $q$ versus redshift $z$.

Figure 2(a) shows the fluctuation of redshift $z$ with cosmic time $t$ for our derived models. From the figure, we see that the redshift $z$ is a monotonic decreasing function of cosmic time $t$ for the present value $a_0 = 8.6021$. Also, $z$ starts with a small positive value 5.16 at $t = 0$ and $z \to -1$ as $t \to \infty$ for our derived models. So, we can say that $t \to \infty$ corresponds to $z \to -1$.

In Figure 2(b), we have shown the fluctuation of $q$ concerning for redshift $z$ as per Eq. (4). From this figure, we see that as the redshift $z$ decreases, the DP $q$ is changing its phase from positive (decelerating phase) to negative (accelerating phase) and $q \to -1$ as $z \to -1$. Recently, [55, 56] have studied the transition redshift in $f(T)$ cosmology and observational constraints and cosmographic bounds on the cosmological deceleration-acceleration transition redshift in $F(R)$ gravity respectively.

It was found in its analysis that the SNe data favor current acceleration ($z < 0.5$) and past deceleration ($z > 0.5$). Recently, according to the High-z Supernova Search (HZSNS) team $z_t = 0.46 \pm 0.13$ at (1 $\sigma$) c.l. [3] which has been further analyzed to $z_t = 0.43 \pm 0.07$ at (1 $\sigma$) c.l. [3]. According to SNLS [57], as well as the one recently compiled by [58], yield a transition redshift $z_t \sim 0.6(1 \sigma)$ in better agreement with the flat $\Lambda$CDM model ($z_t = (2\Omega_\Lambda/\Omega_m)^{1/3} - 1 \sim 0.66$). Another limit is $0.60 \leq z_t \leq 1.18$ (2$\sigma$, joint analysis) [59]. Further, the transition redshift for our derived model comes to be $z_t \cong 1.965$ (see Fig. 2b) which is in good agreement with the Type Ia supernovae observations, including the farthest known supernova SNI997ff at $z \approx 1.7$ [2] and [60]. We see that the variation of $q$ versus $z$ obtained in our model is compatible with the results obtained in the above references.
Figure 3: (a) Plot of energy densities versus cosmic time $t$, (b) The plot of $\frac{\rho_p}{\lambda}$ versus cosmic time $t$, (c) The plot of effective pressure $\overline{p}$ versus cosmic time $t$. Here $\omega = \phi_0 = 1, m = 0.5, \beta = 0.0062$ and $k = 0.000016$.

In figure 3(a), we have shown three curves of total energy density $\rho$, particle energy density $\rho_p$, and string tension density $\lambda$. We see that all the energy densities are positive decreasing functions of time showing expanding universe. All the energy densities approach to zero as $t \to \infty$, meaning that the universe will keep on expanding forever. Also, we see that $\lambda < \rho_p$ for an early phase of the evolution i.e., particle dominates over the string, and then $\lambda > \rho_p$ in due course of evolution i.e., the string dominates over the particle thereafter.

The comparative behavior of particle density $\rho_p$ and string tension density $\lambda$ is also studied in figure 3(b). From figure 3(b) we see that the ratio $\frac{\rho_p}{\lambda}$ is greater than 0 throughout. At the early phase of the evolution, the ratio $\frac{\rho_p}{\lambda}$ is greater than 1, indicating that $\rho_p > \lambda$ i.e. the particle dominated phase. But, as the time progresses, the ratio falls below 1 indicating the string dominated phase. These observations are supported by Krori [61] and Kibble [62].

In Figure 3(c), we have plotted the variation of the effective pressure $\overline{p}$ concerning cosmic time $t$ as per Eq. (31). We see that $\overline{p}$ is negative at present, which may be seed for current accelerated expansion of the universe.

In Figure 4(a) we have plotted the behavior of isotropic pressure $p$ for case-I when $p = \alpha \rho$ for three scenarios $\alpha = 0$ (dust filled), $\alpha = 1/3$ (radiation dominated) and $\alpha = 1$ (stiff matter filled) universe. In all the cases we find that the isotropic pressure is a positive decreasing function of time.

In Figure 4(b) we consider case-II when $\xi = \xi_0 p^n$ and plotted $p(t)$ for three values of $n = 0, 1/2$ and 1. We observed that for all the three values of $n$, the isotropic pressure $p$ is again a positive decreasing function of time. We also observed in both the case-I and case-II
that \( p \to 0 \) when \( t \to \infty \).

In case-III, when \( \xi = 0 \) (i.e., in the absence of viscous effect) the isotropic and effective pressures become equal. The behavior of the effective pressure is graphed in Figure 4(c). We see that in the absence of viscosity the effective pressure becomes highly negative at the early time then increases and tends to a small negative value at late time.

![Figure 4](image1.png)

Figure 4: (a) Plot of isotropic pressure \( p \) versus \( t \) for case I, (b) Plot of isotropic pressure \( p \) versus \( t \) for case II, (c) Plot of isotropic pressure \( p \) versus \( t \) for case III. Here \( \omega = \phi_0 = 1, m = 0.5, \beta = 0.0062 \) and \( k = 0.000016 \).

In Figure 5(a) we have plotted the of bulk viscosity coefficient \( \xi \) for case-I when \( p = \alpha \rho \) for three scenarios \( \alpha = 0, 1/3 \& 1 \). In all the cases we find that \( \xi \) is a positive decreasing function of time. In the early universe, it was high and after that it reduces gradually and tends to zero as \( t \to \infty \). So, we can say that the nature of the fluid was highly viscous at the time of the early universe which tends to reduce and vanish in due course of time. In Figure 5(b) we consider case-II when \( \xi = \xi_0 \rho^n \) and plotted \( \xi(t) \) for three values of \( n = 0, 1/2 \& 1 \). Here also, we observe the same behavior for the two values of \( n = 1/2 \) and \( 1 \), whereas for \( n = 0 \) the viscous effect vanishes throughout the evolution of the universe.
Other physical parameters expansion scalar($\theta$), Volume scalar ($V$), shear scalar $\sigma$ and anisotropy parameter($A_m$) and directional Hubble parameters ($H_1$ and $H_2$) are obtained as

\[
\theta = 3H = \frac{3}{\sqrt{2\beta t + k}} 
\]  
(42)

\[
V = A^2 B = e^{\frac{3}{2}\sqrt{2\beta t + k}} 
\]  
(43)

\[
\sigma^2 = -\frac{1}{2} \left[ \frac{9(2m^2 + 1)}{(2m + 1)^2(2\beta t + k)} \right] - \frac{3}{2\sqrt{2\beta t + k}} 
\]  
(44)

\[
A_m = \frac{2m^2 - 4m + 2}{(2m + 1)^2} 
\]  
(45)

\[
H_1 = mH_2 = \frac{3m}{(2m + 1)\sqrt{2\beta t + k}} 
\]  
(46)

In Big Bang scenario all the parameters like shear scalar ($\sigma$), expansion scalar ($\theta$), and Hubble parameter ($H$) are finite. From Eq. (43), Spatial volume ($V$) is zero at $t = 0$. As $t \to \infty$, $V$ becomes infinite whereas $\theta$, $H_1$, and $\sigma$ approach to zero.

5 The Model Stability

We have tested the stability of the background solution w.r.to perturbations of the metric. For the study, we adopt the notation $a_i$ for the metric potentials. (ie. $a_1 = A$ and $a_2 = B$).

The stability analysis is performed against the perturbations of all possible fields. The stability of the solution has been first discussed by Chen and Kao [63]. Here perturbation will be considered for the two expansion factor $a_i$ via

\[
a_i \to a_{B_i} + \delta a_i = a_{B_i}(1 + \delta b_i). 
\]  
(47)

where $\delta a_i = a_{B_i}\delta b_i$. 

Figure 5: (a) Plot of viscosity parameter $\xi$ versus $t$ for case I, (b) Plot of viscosity parameter $\xi$ versus $t$ for case II. Here $\omega = \phi_0 = 1$, $m = 0.5$, $\beta = 0.0062$ and $k = 0.000016$. 

Accordingly, the perturbations of the volume scalar, directional Hubble factors, the mean Hubble parameter are shown as follows:

\[
V \rightarrow V_B + V_B \sum_{i=1}^{3} \delta b_i, \quad H_i \rightarrow H_{B_i} + \dot{\delta b}_i, \\
H \rightarrow H_B + \frac{1}{3} \sum_{i=1}^{3} \delta b_i, \quad \sum_{i=1}^{3} H_i^2 \rightarrow \sum_{i=1}^{3} H_{B_i}^2 + 2 \sum_{i=1}^{3} H_{B_i} \delta b_i
\]  

(48)

It can be derived that metric perturbations \( \delta b_i \) is to linear order in \( \delta b_i \) obey the following equations

\[
\Sigma \delta \dddot{b}_i + 2 \Sigma H_{B_i} \dot{\delta b}_i = 0 
\]

(49)

\[
\Sigma \delta \dddot{b}_i + 2 \frac{\dot{V}_B}{V_B} \delta b_i + \Sigma \delta \dot{b}_j H_{B_i} = 0
\]

(50)

\[
\Sigma \delta \dot{b}_j = 0
\]

(51)

From the above three equations, It can easily be seen that

\[
\delta \dddot{b}_i + \frac{\dot{V}_B}{V_B} \delta b_i = 0
\]

(52)

where \( V_B \) is the background volume scalar and in this model, it is given by

\[
V_B = e^{\frac{2}{3} \sqrt{2\beta t + k}}
\]

(53)

We can calculate the \( \delta b_i \) with the help of Eq. (51), then we get

\[
\delta b_i = e^{\frac{2}{3} \sqrt{2\beta t + k}} \left( -\frac{1}{3} \sqrt{2\beta t + k} + \frac{1}{9} \beta \right) + c
\]

(54)

where \( c \) is constant of integration. Therefore the actual fluctuations for each expansion factor \( \delta a_i = a_{B_i} \delta b_i \) are given by

\[
\delta a_i = a_{B_i} e^{\frac{2}{3} \sqrt{2\beta t + k}} \left( -\frac{1}{3} \sqrt{2\beta t + k} + \frac{1}{9} \beta \right) + c
\]

(55)

From Eq. (55) and Figure (6), we observed that for positive value of \( \beta \) and \( k \), \( \delta a_i \) approaches to zero for large \( t \) i.e. \( t \rightarrow \infty, \delta a_i \rightarrow 0 \). Consequently, the background solution is stable against the metric perturbation.

If \( \delta b_i \) tends to zero, from Eq. (48), we see that \( V \rightarrow V_B, \ H_i \rightarrow H_{B_i}, \ H \rightarrow H_B \) so we can say that our solution is stable against the perturbation of volume scale, directional Hubble and average Hubble parameters also.
6 Conclusion

The present study contributes to the exact solutions of the scalar-tensor theory of gravitation described by Sáez & Ballester. It is worth mentioned here that the scalar field $\phi$ plays a significant role in the expression for the physical quantities $\overline{p}, \rho, p, \xi$ and $p_p$. We find a point type singularity in the derived models as $p, \rho, \lambda, p_p$ diverge at $t \to \infty$.

The model shows a phase transition from an early decelerating to present the accelerating expansion of the universe. The phase transition took place at $z = 1.965 \approx 2$. Recently, Hayes et al [65] and Dunlop [65] use the comparison of Lyman-\(\alpha\) and H-\(\alpha\) luminosity functions to deduce the range of redshift, which currently is feasible at $z \approx 2$. Thus, $z = 1.965$ in our derived models is consistent with observational value [64, 65].

Also, our derived models are stable under perturbations.

So, we may conclude that our models are improved from earlier works and it presents a better picture of the universe. So it deserves attention.

From Figures 4(a), (b) and (c) we observed that the isotropic pressure, in the presence of the bulk viscosity $\lambda$ is a decreasing function of time $t$ and approaches to zero at late time but in the absence of bulk viscosity the presence is always negative and tends to zero at present time. Thus, we see the role of bulk viscosity for the evolution of the universe.

Lastly, we conclude that our derived models deserve attention and show a better shape of the universe.

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