A general criterion for nonclassicality from a signaling perspective

S. Aravinda

*Poornaprajna Institute of Scientific Research, Sadashivnagar, Bangalore, India*

R. Srikanth

*Poornaprajna Institute of Scientific Research, Sadashivnagar, Bangalore, India and Raman Research Institute, Sadashivnagar, Bangalore, India.*

We argue that the essence of nonclassicality of a bipartite correlation is a positive signal deficit—the communication cost excess over the available signaling. By this criterion, while violations of Bell-type and contextuality inequalities are necessarily non-classical, some violations of the Leggett-Garg inequality are classical. Further, signaling tends to diminish nonclassical properties, such as intrinsic randomness, no-cloning, complementarity, etc. Signal deficit is shown to have its ultimate origin in intrinsic randomness. A possible analogy of nonclassicality to the metamathematical concept of Gödel incompleteness is noted.

*Introduction.* What exactly makes quantum mechanics (QM) nonclassical? Traditionally this question has been answered in different ways in quantum optics, in the foundations of QM, etc. In quantum information theory, we associate nonclassicality with features like fundamental indeterminism, Heisenberg uncertainty, monogamy and privacy of correlations, and the impossibility of perfect cloning. In terms of bi-partite correlations, it turns out that all these features derive from just two assumptions [1]. The first is no-signaling, which stipulates that signals cannot propagate except through material mediation. The second is nonlocality, whereby multipartite correlations can violate Bell-type inequalities [2, 3]. It is known that a wider class of generalized probability theories, e.g., the Popescu-Rohrlich (PR) box [4], satisfy these postulates.

Yet, nonclassicality is arguably strictly weaker. Local correlations in QM, such as that between sequential measurements on a qubit, can violate the temporal equivalent of the Bell-type inequality, namely the Leggett-Garg (LG) inequality [5], but such local correlations involve ‘signaling in time’, so that the above postulates do not cover such phenomena. We consider in this article the problem of refining the above criterion of non-classicality by relaxing the no-signaling condition. (This does not violate the relativistic prohibition on superluminal signaling, because the signaling correlations considered here are between timelike separated events occuring on the same particle, and thus just consistute memory.)

Since classical models that simulate nonlocal, non-signaling correlations entail non-vanishing communication cost [6], it is natural to suppose that the weakening of nonclassicality through signaling can be characterized in terms a signaling deficit: i.e., the excess of communication cost for a given correlation inequality (Bell, contextuality or LG) above the available signal. The remaining article is devoted to the elucidation of this point.

*Bell and LG inequalities.* The LG inequality is satisfied by all noninvasive-realist theories (For related variants, cf. Ref. [7]). ‘Realism’ is the assumption that the given system Q possesses determinate properties prior to measurement. ‘Noninvasiveness’ is the assumption of measurability of a system without disturbing the subsequent evolution of its possessed value. Thus, measurement only reveals a pre-existing value.

Suppose the quantities labeled $a = 0, 1$ are measured at time $t_A$ and those labeled $b = 0, 1$ at time $t_B > t_A$, both yielding respective outcomes $x, y = \pm 1$. Then under the stated assumptions, any correlation $P \equiv P(x, y|a, b)$ satisfy:

$$\Lambda(P) \equiv |\sum_{a,b} (-1)^a \Lambda(xy)| \leq 2,$$

which is the LG inequality in its 2-time ($t_A, t_B$) variant [8]. At the microscopic scale, because quantum measurement is invasive, the LG inequality was originally proposed for macroscopic systems, which can in principle be measured non-invasively. When $a$ and $b$ are interpreted as observables belonging to spatially separated particles, Eq. (1) is the Clauser-Horne-Shimony-Holt (CHSH) inequality [2], a Bell-type inequality.

The assumptions behind the derivation of a Bell-type inequality are localism and realism. As a classical theory is necessarily local and realist, a violation of Bell’s inequality implies non-classicality. Likewise, as a classical theory is necessarily non-contextual and realist, a violation of a contextuality inequality [9] also implies non-classicality. However, the violation of the LG inequality does not entail non-classicality, since a classical theory is not necessarily non-invasive.

Invasiveness implies a signal [10] carried forward in time from one measurement to another on the same particle, such that the probability distribution of a subsequent measurement depends on the choice made earlier [11]. For sufficiently large signal, an invasive-realist classical mechanism can presumably be used to violate the LG inequality. Then, the violation of the LG inequality is not non-classical unless the degree of violation is shown to be larger than can be explained by a classical mechanism that uses the signaling in the correlations.
A set of correlations $\mathbf{P} \equiv P(x,y|a,b)$ between two parties $A$ and $B$ can be described by a joint distribution if and only if it has a deterministic hidden variable (HV) description $P(x,y|a,b) = \int \rho(\lambda)P(x|a,\lambda)P(y|b,\lambda)\,d\lambda$. For the case where $a,b = 0,1$ and $x,y = \pm 1$, this is equivalent to satisfying correlation inequality \[ (1) \] (no matter whether it is spatial, temporal or context-based).

Violation of a correlation inequality constitutes what we call a general disturbance. Nonlocality or contextuality is one type of disturbance. Disturbance may or may not be signaling. In QM, if the joint measurements $a$ and $b$ commute (as in Bell-type or contextuality inequality), then the disturbance is non-signaling.

\[ S(\mathbf{P}) = \max_{\delta,\alpha} [H(\alpha) + \alpha P^b_0 \log(\alpha P^b_0/\mathbf{P}) + \beta P^b_1 \log(\beta P^b_1/\mathbf{P}) + \alpha Q^b_0 \log(\alpha Q^b_0/\mathbf{Q}) + \beta Q^b_1 \log(\beta Q^b_1/\mathbf{Q})]. \]  

\[ (2) \]

**Communication cost.** Disturbance implies that the communication cost, the information about $a$ required to output $y$ (or about $b$ to output $x$) for simulating $\mathbf{P}$, is greater than 0. Following Ref. [13], we consider the general $\mathbf{P}$ as a mixture of deterministic strategies. Examples of such strategies are deterministic local correlations like $d^{ab}$ defined by $P(x,y|a,b) = \delta^a_0 \delta^b_0$ and $d^{b\prime} = \delta^a_\alpha \delta^b_\beta$. For these the signal $(S)$ and the communication cost $(C)$ are identically zero. Correlations for which $S(\mathbf{P}) = C(\mathbf{P}) = 1$ bit are the deterministic 1-bit strategies $d^{0\prime}$ defined by $P(x,y|a,b) = \delta^a_0 \delta^b_\beta$ and its signal complement $d^{b\prime \prime} \equiv \delta^a_\alpha \delta^b_\beta$; a convex combination of these two reduces the signal at a fixed communication cost of 1 bit, and at fixed maximal inequality violation of $\Lambda(\mathbf{P}) = 4$ in Eq. \[ (1) \]. Indeed, $S(d^{0\prime}) = 1$ bit, being the maximal signal from Alice to Bob, when she chooses $a = 0$ or $a = 1$ with equal probability and Bob always chooses $b = 1$. Further, $C(d^{0\prime}) = 1$ since Bob needs log $|a|$ bits (here: 1 bit) of communication specifying $a$, in order to output $y = a \cdot b$.

For Eq. \[ (1) \], it can be shown (cf. the Appendix) that the average communication cost
\[ C(\mathbf{P}) = \max (C(\Lambda(\mathbf{P}), S(\mathbf{P}))), \]  

\[ (3) \]
where $C(\Lambda(\mathbf{P})) \equiv \frac{1}{2} \Lambda(\mathbf{P}) - 1$ is referred to as the disturbance cost. This generalizes the result of Ref. [13], where $C(\Lambda(\mathbf{P}))$ is the general lower bound on average communication cost. For maximal quantum violation of the inequality \[ (1) \], which is $2\sqrt{2}$, $C(\mathbf{P}) = \sqrt{2} - 1 \approx 0.41$ bits.

For a probabilistic $\mathbf{P}$ obtained by mixing the local and 1-bit deterministic strategies, by virtue of positivity of $C$ and convexity of $S$, we have:

\[ S(\mathbf{P}) \leq C(\mathbf{P}), \]  

\[ (4) \]

**Signaling.** Let $P^b_0$ (resp.) represent the probability that Bob, measuring observable $b$, finds $y = +1$ when Alice measures observable $a = 0$ (resp.). A qualitative indication of signal strength is $s = \max_{\beta} |P^b_0 - P^b_1|$, where $b$. More quantitatively, the signalled information $S(\mathbf{P})$ received by Bob is quantified by the mutual information $I(A : Y)$ maximized over Alice’s and Bob’s choices. Letting $\alpha (\beta \equiv 1 - \alpha)$ denote the probability with which Alice chooses $a = 0$ (resp.), $Q^b_0 \equiv 1 - P^b_1$, $\mathbf{P} \equiv \alpha P^b_0 + \beta P^b_1$ and $\mathbf{Q} \equiv \alpha Q^b_0 + \beta Q^b_1$, it is straightforward to find (cf. Appendix)

This is also generally true, since $S(\mathbf{P}) \equiv I(a : y) \leq H(a) \equiv C(\mathbf{P})$, where $I$ is mutual information and $H$ binary entropy. A uniform mixture consisting only of $d^{b\prime}$ and $d^{b\prime \prime}$ yields the PR box [2], for which $C(\mathbf{P}) = 1$, but $S(\mathbf{P}) = 0$.

**Quantum correlations.** In what follows, we may denote the observables $a \equiv 0,1 \text{ (and } b \equiv 0,1)$ by $\hat{a}, \hat{a}^\prime (\hat{b}, \hat{b}^\prime)$. For a qubit in QM, the correlations for sequential measurements $a$ then $b$ are given by $P(x,y|\hat{a},\hat{b}) = \frac{1}{2} (I + \hat{a}^\dagger \hat{b} \rho \hat{b}^\dagger \hat{a} + \frac{1}{2} \text{Tr} (\hat{a} \rho \hat{b}^\dagger))$ and its signal complement $P(x,y|\hat{a}^\prime,\hat{b}^\prime) \equiv \frac{1}{2} (I + \hat{a}^\dagger \hat{b} \rho \hat{b}^\dagger \hat{a} - \frac{1}{2} \text{Tr} (\hat{a} \rho \hat{b}^\dagger))$. Given that Alice measures $\hat{a}$ or $\hat{a}^\prime$, the condition for signaling is $s \equiv |P^b_0 - P^b_1| = \frac{1}{2} \text{Tr} (\frac{1}{2} \text{Tr} \frac{1}{2} \text{Tr} (\hat{a}^\dagger \hat{b} \rho \hat{b}^\dagger)) > 0$, which is bounded above in QM as: $s \leq \frac{1}{2} \text{Tr} (\hat{a}^\dagger \hat{b} \rho \hat{b}^\dagger) \leq \frac{1}{2} \text{Tr} (\hat{a}^\dagger \hat{b}^\dagger \rho \hat{b}^\dagger) \leq \frac{1}{2}$. Thus, $|\hat{a}, \hat{b}^\prime \equiv 0 \text{ and/or } |\hat{a}^\prime, \hat{b}\rangle \not\equiv 0$ is necessary for signaling.

Let the density operator obtained by measuring $\hat{a}$ (resp.) on $|\psi\rangle$ be $\rho_0 (\rho_1)$. Intuitively, we expect to maximize signaling $s$ when $|\psi\rangle$ is an eigenstate of $\hat{a}$, and $\hat{a}^\prime$ maximally fails to commute with $\hat{a}$ (i.e., the two observables form a pair of mutually unbiased bases). By a similar argument as used to bound $s$, trace distance $\tau$ satisfies $\tau \geq \frac{1}{2} ||\rho_0 - \rho_1|| \leq \frac{1}{2}$. This inequality, as well as that for $s$ above, are saturated for the settings $\Sigma : |\psi\rangle = |0\rangle; \hat{a} = \sigma_x, \hat{a}^\prime = \sigma_x; \hat{b} = \sigma_z$. It may be noted that there is no backward signaling from Bob to Alice, i.e., Alice’s outcome probabilities as derived from $P(x,y|\hat{a},\hat{b})$ above are independent of Bob’s settings, as expected.

Now $P(x,y|\hat{a},\hat{b})$ yields the correlator $\langle \hat{a} \hat{b} \rangle = \sum_{x,y} xy P(x,y) = \frac{1}{2} \text{Tr} (\hat{a} \hat{b} \rho) = \hat{a} \cdot \hat{b}$, where $\hat{a} = \hat{a} \cdot \hat{\sigma}$ and $\hat{b} = \hat{b} \cdot \hat{\sigma}$. This correlator (for qubits) is independent of temporal order, though there is a signaling from Al-
ice to Bob. These temporal correlations are the same as that obtained by von Neumann measurements on singlets. Both CHSH and LG inequalities are microscopically violated, with the Cirelson bound \( |C| \) being the same in both cases \([10]\).

If the state is the maximally mixed \( I/2 \), then \( s = \tau = 0 \), since \( \hat{a}^2 = (\hat{a}')^2 = I \) for any qubit observable with spectrum \( \pm 1 \). Interestingly, because the correlators \( \langle \hat{a}\hat{b} \rangle \) are state-independent, this state will nevertheless violate the LG inequality maximally.

**Non-classicality.** We propose that a bipartite correlation \( \mathbf{P} \) is classical precisely if

\[
S(\mathbf{P}) = C(\mathbf{P}),
\]

(5)

for in this case the available signal can be used in a classical mechanism to simulate the correlation, and conversely a classical scheme for simulation exists only if this equality holds (cf. Eq. \([4]\)). The \( d^{10} \) strategies, for which \( S(\mathbf{P}) = C(\mathbf{P}) = 0 \), are trivially classical. Thus, correlation \( \mathbf{P} \) is nonclassical if \( S(\mathbf{P}) < C(\mathbf{P}) \). In conjunction with Eq. \([5]\), a necessary and sufficient condition for classicality is

\[
S(\mathbf{P}) \geq C_\Lambda(\mathbf{P})
\]

(6)

applicable generally to spatial, temporal and contextual correlations. In the spatial (i.e., Bell) case, by no-signaling \( S(\mathbf{P}) = 0 \), so that Eq. \([6]\) is equivalent to \( C_\Lambda(\mathbf{P}) \leq 0 \), which is just the usual CHSH inequality. Thus, any violation of the CHSH inequality is necessarily non-classical. More generally, the form \([6]\) coincides with the usual forms of Bell-type and contextuality inequalities, but deviates for LG inequalities on account of the non-commutativity of the correlated terms. A nontrivial instance of this can be seen in Figure 1 where \( \hat{a}, \hat{b}, \hat{a}' \) and \( \hat{b}' \) are oriented in the \( xy \) plane, and separated by angular intervals \( \theta \); the initial state is the \( +1 \) eigenstate of \( \hat{a} \). The range \( \theta > 1.08 \) is classical by criterion \([6]\) even though the LG inequality is violated.

A PR box maximally violates \([6]\) while \( d^{20}_1 \equiv \delta_\theta^1 \delta_\varphi^0 \), which is 1-bit signaling but does not violate inequality \([6]\), maximally satisfies it. We define the **signal deficit**, \( \eta \equiv C(\mathbf{P}) - S(\mathbf{P}) \), which vanishes for classical correlations. If inequality \([6]\) is violated, then \( \eta = C_\Lambda(\mathbf{P}) - S(\mathbf{P}) \) by virtue of Eq. \([4]\), and represents the number of bits on average necessary over the available signaling to simulate the disturbance. For a given correlation inequality, it quantifies the degree of nonclassicality.

For sequential measurements on a qubit, \( S(\mathbf{P}) \) in Eq. \([2]\) is bounded above by the Holevo quantity \( \chi = S(\alpha\rho_0 + \beta\rho_1) - \alpha S(\rho_0) - \beta S(\rho_1) \). Direct substitution in Eq. \([2]\) using settings \( \Sigma \) yields \( S(\mathbf{P}) = H(\alpha) + \alpha \log \left( \frac{2\alpha}{1+\alpha} \right) + \frac{1-\alpha}{2} \log \left( \frac{1-\alpha}{1+\alpha} \right) \), which, when maximized, yields \( S(\mathbf{P}) = \mu_s \equiv \log(5) - 2 \approx 0.32 \) bits at \( \alpha = 3/5 \). This is the maximum signaling quantumly possible from Alice to Bob \([10]\), and saturates the Holevo bound. Since \( C_\Lambda \approx 0.41 \) bits for maximal quantum violation of inequality \([1]\) and hence also the LG inequality, the violation of the inequality \([6]\) follows. Thus maximal quantum violation of inequality \([1]\) is indeed non-classical. It entails a signal deficit of \( \eta = 0.41 - 0.32 = 0.09 \) bits.

Setting \( S(\mathbf{P}) = \mu_s \) in Eq. \([6]\), we find that

\[
\Lambda(\mathbf{P}) \leq 2(\mu_s + 1) \approx 2.64,
\]

(7)

as the signal-corrected version of the LG inequality \([1]\), whose violation gives a sufficient condition for nonclassicality. That its violation is not necessary is clear from Figure 1 in the approximate range \( \theta \in [1.0, 1.03] \).

By Eq. \([1]\), the maximally nonlocal and signaling correlations \( d^{11}_1 \) and \( d^{12}_1 \) are classical, but cannot be accessed deterministically because intrinsic randomness ensures that complements in a signaling pair occur always in tandem. Since this coupling behavior is the cause of nonvanishing signal deficit, intrinsic randomness emerges as the ultimate origin of nonclassicality.

For the maximal quantum violation of the CHSH and LG inequalities, \( \eta \approx 0.41 \) and 0.09, indicating that the maximal quantum violation of the CHSH inequality is in a sense more nonclassical than the maximal quantum violation of the LG inequality. Interestingly, as we saw earlier, for the maximally mixed state \( S(\mathbf{P}) = 0 \). Thus the maximal violation of the LG inequality with an initial maximally mixed state is more non-classical than that by an initial pure state according to this criterion.

Intuitively, a qubit is a non-classical object. However, Bell showed that the outcome of a single projective measurement on any qubit state can be classically simulated with a HV model \([15]\). Further, a proof of non-classicality via contextuality requires dimension greater than 2, which thus does not apply to a qubit subjected to

![FIG. 1: For the range of angles \( \theta \) given, the LG inequality is always violated, with normalized LG inequality violation \( (\Lambda(\mathbf{P})/\Lambda_0, small-dashed line) exceeding 1. In the region where signaling \( S(\mathbf{P}) \) with maximization over \( b \) restricted to \( \{b, b'\} \), large-dashed line exceeds the disturbance cost \( (C_\Lambda, dotted line) \), above \( \theta \geq 1.08 \), the LG inequality violations in QM are classical, according to Eq. \([6]\). The plain line marks the Holevo bound (asymptotically accessible signal).](image-url)
projective measurements. However, our discussion above establishes non-classicality of a qubit by demonstrating temporal correlations for which \( \eta > 0 \).

**Intrinsic randomness, no-cloning, etc.** It can be shown that nonclassical features like randomness, no-cloning, privacy and monogamy of correlations, and the impossibility of perfect cloning, etc., which occur in non-signaling nonlocal theories \[1\], also occur in signaling correlations that are nonclassical according to the criterion \[9\], though their strength tends to diminish in the measure that signaling in the correlations increases. These details will be presented elsewhere, and here we indicate the general sense in which signaling undermines nonclassicality. Suppose Alice and Bob share an ‘unbalanaced PR box’, i.e., a correlation \( Q(p) = p d_0 + (1-p)d_1 \) by combination of two signal complements. Defining intrinsic randomness by \( I(Q(p)) = \min\{p, 1-p\} \), which we take to be \( p \) without loss of generality. Thus, we find \( s + 2I = 1 \), implying a trade-off between signaling and randomness. More generally, \( s + 2I \geq 1 \) if we include \( d_0 \) strategies; cf. Appendix and also Ref. \[10\]. When a system is nonclassical, then \( s < 1 \) and intrinsic randomness is necessarily non-vanishing.

Irrespective of \( p \), \( Q(p) \) satisfies \( a \cdot \vec{b} = x \oplus y \). With perfect cloning by Bob, one has \( a \cdot \vec{b} = x \oplus y' \), so that: \( a \cdot (\vec{b} \oplus \vec{b'}) = y \oplus y' \), from which Bob deterministically obtains Alice’s input. On the other hand, \( S(Q)(p)) = 1 - 2p \). Generalizing no-cloning to the stipulation of ‘no-cloning above available signaling’, we find that perfect cloning violates the generalized no-cloning by \( 1 - (1 - 2p) = 2p \) bits. Thus the violation is maximum (and hence most ‘forbidden’) when the correlations are most nonclassical, viz. \( Q(\frac{1}{2}) \), with \( \eta = 1 \); and least when the system is classical, viz. \( Q(0) \) with \( \eta = 0 \).

**Nonclassicality and metamathematical incompleteness.** We now mention briefly the similarity of our main result to a theorem in metamathematics, the study of mathematics using mathematical tools. (More generally, a *meta-theory* is a theory about a theory). At the moment, the analogy is admittedly very sketchy, but we feel that it offers potential clarity in the way we think about physical theories by taking on an ‘outside view’. A well-known metamathematical result is that of incompleteness due to Gödel \[17\], according to which given any axiomatization \( A \) of arithmetic, if it is consistent, then it is incomplete, in the sense of there being truths (theorems) expressible in \( A \) but not provable within \( A \). An existential proof of the result is that the set of theorems in \( A \) has the cardinality of the continuum, i.e., it is uncountably large, whilst the number of proofs is only countably infinite.

Strictly speaking, while signaling \( S(P) \) is a concept in the base theory (QM), communication cost \( C(P) \) is meta-theoretic, as therefore are Eqs. \[4\] and \[5\], which are statements about QM ‘from the outside’ rather than from within QM. Thus the notion of non-classicality as developed here is also meta-theoretic. Given theory \( T \), predictions in it are, technically, theorems, while signaling indicates a sequence of causes and effects, which is like a train of logical inferences, and thus is like a proof. Accordingly: completeness (resp., consistency) entails that all (resp. only) predicted effects in the theory are explainable via the available signaling. Thus nonclassicality corresponds to incompleteness in that there exist effects not attributable to signals. A little thought shows that it is related to EPR incompleteness \[18\]. The analogy of nonclassicality to metamathematical incompleteness gives the sense that nonclassicality is generic rather than pathological.

**Discussion and conclusions.** Our work provides a new characterization of nonclassicality of bipartite correlations, and thus of a theory that allows it, in terms of the signaling available within. We attribute non-classicality to a single unified postulate: that of signal deficit (instead of nonlocality and no-signaling), which in turn arises ultimately from intrinsic randomness. Our work furnishes a framework to go beyond the no-signaling paradigm, leading to new questions, such as “Why are signaling correlations always local in non-relativistic QM?”, and to make a clear separation between theory and metatheory within physics, highlighting the possible primacy of information as a fundamental resource of Nature.

**Acknowledgments.** SA acknowledges support through the INSPIRE fellowship [IF120025] by the Department of Science and Technology, Govt. of India.

---

* Electronic address: srik@poornaprajna.org

[1] L. Masanes, A. Acin, and N. Gisin, Phys. Rev. A 73, 012112 (2006).
[2] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. 23, 880 (1969).
[3] J. Bell, Physics 1, 195 (1964).
[4] S. Popescu and D. Rohrlich, Found. Phys. 24, 379 (1994).
[5] A. J. Leggett and A. Garg, Phys. Rev. Lett. 54, 857 (1985).
[6] B. F. Toner and D. Bacon, Phys. Rev. Lett. 91, 187904 (2003).
[7] R. Lapiedra, EPL (Europhysics Letters) 75, 202 (2006).
[8] C. Brukner, S. Taylor, S. Cheung, and V. Vedral, arXiv:quant-ph/0402127v1.
[9] P. Badziag, I. Bengtsson, A. Cabello, and I. Pitowsky, Phys. Rev. Lett. 103, 050401 (2009).
[10] T. Fritz, New Journal of Physics 12, 083055 (2010).
[11] J. Kofler and C. Brukner, arXiv:quant-ph/1207.3666.
[12] A. Fine, Phys. Rev. Lett. 48, 291 (1982).
[13] S. Pironio, Phys. Rev. A 68, 062102 (2003).
[14] B. S. Cirelson, Lett. Math. Phys. 4, 93 (1980).
[15] J. S. Bell, Rev. Mod. Phys. 38, 447 (1966).
[16] M. J. W. Hall, Phys. Rev. A 82, 062117 (2010).
[17] K. Gödel, Mon. für Math. und Phys. 38, 173 (1931).
[18] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47, 777 (1935).
Appendix of supplementary material

Derivation of Eq. (2).

We have:

\[ I(A : Y) = H(\alpha) - H(A | Y) \]
\[ = H(\alpha) - \sum_{a=0,1} \sum_{y=\pm} P(a, y) \log(P(a | y)), \quad (8) \]

where \( P(a, y) = P(y | a) P(a) \) and \( P(a | y) = P(y | a) P(a) / P(y) \). We let \( P^y_0 = P(y = + | a = 0), P^y_1 = P(y = + | a = 1), Q^y_0 = P(y = - | a = 0) \), and \( Q^y_1 = P(y = - | a = 1) \). Further, \( P \equiv P(y = +) = \alpha P^0_y + \beta P^1_y \) and \( Q \equiv P(y = -) = \alpha Q^0_y + \beta Q^1_y \). Substituting these values in Eq. (8), and maximizing over \( \alpha \) and settings \( b \), we obtain Eq. (2).

Communication cost of signaling correlations.

Claim. For any bipartite two-settings, two-outcomes correlation \( P \), \( C(P) = \max (S(P), C_\lambda(P)) \).

Proof. Ref. [13] proves that \( C(P) \geq C_\lambda(P) \), and furthermore that \( C(P) = C_\lambda(P) \) if \( S(P) = 0 \). Here we generalize this result by first showing that \( C(P) = C_\lambda(P) \) if \( S(P) \leq C_\lambda(P) \). Consider the most general strategy with which \( P \) can be constructed by using the eight deterministic local strategies \( d^{00} \) and the eight deterministic 1-bit strategies \( d^{11} \) strategy, given in Eqs. [9] and [10] in the notation of Ref. [13]. The local strategies are:

\[ d^{00} = \delta_0^x \delta_0^y; \quad d^{11} = \delta_1^x \delta_1^y \]
\[ d^{10} = \delta_0^x \delta_1^y; \quad d^{01} = \delta_1^x \delta_0^y \]
\[ d^{40} = \delta_1^x \delta_1^y; \quad d^{04} = \delta_0^x \delta_1^y \]
\[ d^{60} = \delta_0^x \delta_0^y; \quad d^{06} = \delta_1^x \delta_0^y \]

and the 1-bit strategies that violated inequality [1] by 4 are:

\[ d^{01} = \delta_0^x \delta_0^y; \quad d^{31} = \delta_1^x \delta_1^y; \quad (10a) \]
\[ d^{11} = \delta_0^x \delta_0^y; \quad d^{21} = \delta_0^x \delta_1^y; \quad (10b) \]
\[ d^{41} = \delta_1^x \delta_0^y; \quad d^{71} = \delta_1^x \delta_1^y; \quad (10c) \]
\[ d^{51} = \delta_0^x \delta_0^y; \quad d^{61} = \delta_0^x \delta_1^y, \quad (10d) \]

Given \( P \equiv P(x, y | a, b) \), we wish to construct a protocol

\[ P(x, y | a, b) = \sum_{\lambda_a = 0}^7 q_{\lambda_a} d^{\lambda_a}_{xy} | a b \quad \sum_{\lambda_1 = 0}^7 q_{\lambda_1} d^{\lambda_1}_{yx} | a b \]
\[ \equiv q_0 + q_1. \quad (11) \]

where \( q_{jk} \) are the probabilities for the strategy \( d^{jk} \). The probabilities have to satisfy normalization constraints \( \sum_{x,y} P(x, y | a, b) = 1 \).

The no-signalling conditions are given as

\[ \sum_y P(x, y | a, b) = \sum_y P(x, y | a, b') \quad \forall b, b' \]
\[ \sum_x P(x, y | a, b) = \sum_x P(x, y | a', b) \quad \forall a, a' \quad (12) \]

For the two-settings two-output case, taking into consideration the normalization conditions, there are only 4 independent no-signaling conditions. Allowing for their general violation, these 4 conditions are:

\[ P(-1, y | 0, 0) + P(+1, y | 0, 0) = P(-1, y | 1, 0) + P(+1, y | 1, 0) \pm \delta_{03} \quad (13a) \]
\[ P(-1, y | 0, 0) + P(+1, y | 0, 0) = P(-1, y | 1, 1) + P(+1, y | 1, 1) \pm \delta_{12} \quad (13b) \]
\[ P(x, -1 | 1, 0) + P(x, +1 | 1, 0) = P(x, -1 | 1, 1) + P(x, +1 | 1, 1) \pm \delta_{47} \quad (13c) \]
\[ P(x, -1 | 0, 0) + P(x, +1 | 0, 0) = P(x, -1 | 0, 1) + P(x, +1 | 0, 1) \pm \delta_{56} \quad (13d) \]
where the \( \delta_j \)'s \( (j \in T = \{03, 12, 47, 56\}) \) are positive quantities that quantify violation of the no-signaling condition and satisfy \( \sum_{j \in T} \delta_j \leq 1 \).

Eqs. \[13a\] and \[13b\] represent signaling from Alice to Bob while Eqs. \[13c\] and \[13d\] that from Bob to Alice. Each such violation of an independent no-signaling condition corresponds to a pair of \( d^i \) strategies of Eqs. \[10\] which we call a signaling pair. The two members of a given pair are signal complements of each other. E.g., the violation of no-signaling in Eq. \[13a\] is produced by having the complements in Eq. \[10a\] occur with unequal probabilities. More generally, the violation of any of the 4 independent no-signaling conditions arises when the complements in any of the signal pairs are imbalanced. Thus:

\[
\begin{align*}
\delta_{03} & \equiv |q_{01} - q_{31}|; \quad \delta_{12} \equiv |q_{11} - q_{21}|; \\
\delta_{47} & \equiv |q_{41} - q_{71}|; \quad \delta_{56} \equiv |q_{51} - q_{61}|. \\
\end{align*}
\tag{14}
\]

If we use only the local and 1-bit strategies of Eqs. \[9\] and \[10\], then

\[
S(P) = \max_{j \in T} \delta_j,
\tag{15}
\]

and

\[
C_A(P) = \sum_{\lambda = 0}^{7} q_{\lambda 1},
\tag{16}
\]

It follows from Eqs. \[15\], \[14\] and \[10\] that

\[
S(P) \leq C_A(P).
\tag{17}
\]

Thus we must have

\[
\forall j \delta_j \leq C_A(P).
\tag{18}
\]

We will return to the more general case than Eq. \[16\] later.

To show that \( C(P) = C_A(P) \) for any signaling \( P \) such that \( S(P) \leq C_A \), we will provide an explicit protocol that realizes \( P \) using the deterministic strategies \( d^{00} \) and \( d^{01} \) a correlation \( P \) which violates precisely one of the four independent no-signaling conditions \[12\], say Eq. \[13a\]. The idea straightforwardly generalizes to the general case of all four no-signaling conditions being violated. In particular, let \( \delta_{03} > 0 \) but the other \( \delta_j \)'s vanish.

The contribution of the negative sign for \( A(P) \) is only from the \( d^{00} \) strategies, and fixes the eight \( q_{0j} \)'s. For example, \( q_{10} = P(1, 0|1, 1) \), and so on \[13\]. The positive terms are constructed with both \( d^{00} \) and \( d^{01} \) deterministic strategies. For example, using Eqs. \[9\] and \[10\],\footnote{The result of Ref. \[13\] for the case of no-signaling correlations is obtained as a special case by setting \( \delta_3 = \sigma_3 = 0 \).}

\[
\begin{align*}
q_{10} &= \frac{1}{2} \left( \frac{C_A + 3\sigma_{03}}{4} \right) \pm \frac{\delta_{03}}{2}, \\
q_{12} &= \frac{1}{2} \left( \frac{C_A + 3\sigma_{03}}{4} \right) \mp \frac{\delta_{03}}{2}, \\
q_{41} &= q_{21} = q_{51} = q_{61} = q_{71} = \frac{1}{2} \left( \frac{C_A - \sigma_{03}}{4} \right),
\end{align*}
\tag{19}
\]

where \( 0 \leq \sigma_{03} \leq C_A \) in order to guarantee positivity of \( q_j \) \( (j = 1, 2, 4, 5, 6, 7) \) and \( 0 \leq \delta_{03} \leq \frac{C_A}{4} + 3\sigma_{03} \) to guarantee positivity of \( q_{01} \) and \( q_{31} \). The result of Ref. \[13\] for the case of no-signaling correlations is obtained as a special case by setting \( \delta_{03} = \sigma_{03} = 0 \).

More generally, any of the other three no-signaling conditions can be violated, and we can define \( \sigma_j \) \( (j \in \{12, 47, 56\}) \) as above to determine the total probability of a signal complement, and the above result can be generalized to an arbitrary signaling distribution for which Eq. \[17\] holds. The communication cost associated with this protocol with signalling is given, in view of Eqs. \[16\] and \[19\], by \( C(P) = C_A(P) \).

Since for the strategies \( d^{00} \) and \( d^{01} \), Eq. \[17\] holds, it therefore follows that if \( S(P) > C_A(P) \), then deterministic strategies that are signaling but produce no violation of inequality (1), such as \( d^{00} \equiv \delta_0^a \delta_0^b \), should be used with total probability \( q_0' = S(P) - C_A(P) \). In simulating the protocol, Bob receives with probability \( S(P) = q_1 + q_1' \) a 1-bit message. To simulate \( P \), he implements the above protocol with probability \( q_1/S(P) \), and chooses a 1-bit Bell non-violating strategy \( d^{01} \) with probability \( q_1'/S(P) \), to ensure that there is no violation beyond \( C_A(P) \). The general super-disturbance signaling correlation can thus be simulated with \( S(P) \) bits on average, so that \( C(P) = S(P) \).

Combining the results for the above sub-disturbance and super-disturbance cases of the signal, we obtain the required result.