Shadow cast of non-commutative black holes in Rastall gravity

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Abstract. We study the shadow and energy emission rate of a spherically symmetric non-commutative black hole in Rastall gravity. Depending on the model parameters, the non-commutative black hole can reduce to the Schwarzschild black hole. Since the non-vanishing Rastall parameter affects the formation of event horizon, the visibility of the resulting shadow depends on the Rastall parameter. The obtained sectional shadows respect the unstable circular orbit condition, which is crucial for physical validity of the black hole image model. Effect of the Rastall parameter on the energy emission rate is also discussed.

Keywords: Black hole shadow; non-commutative black hole; Rastall gravity; Energy emission rate
1 Introduction

The defining feature of a black hole (BH) is the event horizon, the boundary from within which a particle cannot escape. Often, this is described as the boundary within which the BH’s escape velocity is greater than the speed of light. In other words, the gravitational pull becomes so large that none of the physical particles can escape, including light. On the other hand, outside the event horizon, light can escape from a BH. The surrounding matter pulled by BH is called accretion. The accretion overheats in time because of viscous dissipation. Thus, it radiates very brightly at different frequencies including the radio waves, which can be detected by radio telescopes. This shining material accreting onto the BH crosses the event horizon, resulting in a dark area over a bright background: this is the so-called BH shadow (BHS). Although the concept of BHS was known since seventies, the idea to image the BHS locating in the center of our Milky Way was first considered by Falcke et al (2000) [1]. Recently, Event Horizon Telescope has observed the image of BHS of Messier 87 galaxy [2]. With this observation, the shadow of BHS has become a very popular subject in today’s literature [3–71].

Despite intense efforts over the last years, it is still far from being understood what a consistent theory of quantum gravity will look like and what its main features will be. There are various number of proposals to find an effective quantum gravity theory such as loop quantum gravity, string theory, non-commutative geometry etc. On the other hand, BHs have curvature singularities and to heal this problem, quantum gravity can be used where general relativity not anymore working well at small scales [72, 73]. Non-commutative (NC) theory can modify the structure of spacetime by assuming universal minimal length scale where quantization of spacetime [74–77] needs non commute of its coordinates :

\[ [x^A, x^B] = i\vartheta^{AB}, \]

in which \( \vartheta^{AB} \) is a real-valued anti-symmetric matrix with \( \vartheta^{AB} = \vartheta\text{diag}(\epsilon_{ij}, \epsilon_{ij}, ...) \), \( \vartheta \) is a constant of dimension \([\text{Length}]^2\). In the limit \( \vartheta \to 0 \), the ordinary spacetime is recovered. Physically, \( \vartheta^{AB} \) represents a small patch in \( AB \)-plane of the observable area as the Planck’s constant \( (\hbar) \) illustrates the smallest fundamental cell of the observable phase space in quantum mechanics.

Using the NC spacetimes, one can remove the singularities from the BHs and construct regular BH. Afterwards, many other important aspects of spacetime non-commutativity have been discussed in literature [see for example [78–89]].
In this paper, we study the shadow of charged non-rotating NC geometry inspired Schwarzschild BH (NCGiSBH) belonging to the Rastall gravity. To this end, we consider both equatorial and non-equatorial planes and depict the shadow with the help of celestial coordinates. Furthermore, we compute the energy emission rate of the NC BH and discuss the energy influence from the plasma of accretion on the shadow.

The paper is organized as follows. In the next section, we introduce the NC BH in the Rastall gravity. Section 3 is devoted to study the apparent shape of the NCGiSBH geometry by finding its shadow. In Sect. 4, we investigate the influence of plasma on the shadows of the NCGiSBH. The results are concluded in the last section.

2 NC BH spacetime in Rastall gravity

NC BH arises as solutions of Einstein equation where the coordinates are NC and satisfy the following algebra

\[ [x^\mu, x^\nu] = i \theta^{\mu\nu} \]

in which \( \theta^{\mu\nu} \) represents an anti-symmetric tensor encoded the discretization of the spacetime.

Using the NC algebra, one can construct a spacetime, having Schwarzschild BH limit \[78\] as follows:

\[
ds^2 = -f(r) \, dt^2 + \frac{dr^2}{f(r)} + r^2 \left[ d\theta^2 + \sin^2(\theta) \, d\phi^2 \right],
\] (2.1)

where,

\[
f(r) = 1 - \frac{4 M}{r} \frac{\sqrt{\pi}}{\gamma \left( \frac{3}{2}, \frac{r^2}{4} \right)},
\] (2.2)

with the gamma \( \gamma \) function:

\[
\gamma \left( \frac{3}{2}, \frac{r^2}{4\epsilon} \right) \equiv \int_0^{r^2/4\epsilon} \sqrt{t} \, e^{-t} \, dt.
\] (2.3)

Here, \( \sqrt{\epsilon} \) stands for the width of the Gaussian mass-energy. The NC BH reduces to the Schwarzschild BH as \( \frac{\sqrt{\epsilon}}{r} \to \infty \). The horizon of the BH is conditional on \( f(r_h) = 0 \):

\[
r_h = 4 \frac{M}{\sqrt{\pi}} \gamma \left( \frac{3}{2}, \frac{r_h^2}{4} \right).
\] (2.4)

Note that the NC effects are dominant at short distances, on the other hand, at large distances, they become non-effective.

In Rastall gravity theory, the source of matter diffuses on the region. In this theory, the mass density of the smeared matter distribution and the corresponding ansatz of the NC BH metric are given by \[88\]

\[
\rho = -T^t_t = \frac{M}{(4\pi\theta)^{3/2}} \exp \left( -\frac{r^2}{4\theta} \right),
\] (2.5)

\[
f(r) = B_1 - B_0 \frac{r}{r} - \frac{2GM}{r\sqrt{\pi}} \gamma \left( \frac{1}{2}, \frac{r^2}{4\theta} \right).
\] (2.6)

It is noted that the gamma function seen in Eq. (2.6) is given by

\[
\gamma \left( \frac{1}{2}, \frac{r^2}{4\theta} \right) = \int_0^{r^2/4\theta} dt \, t^{-1/2} e^{-t}.
\] (2.7)
The integration constants are fixed at $B_1 = 1$ and $B_0 = 0$ so that metric function becomes

$$f(r) \equiv f = 1 - \frac{2GM}{r \sqrt{\pi}} \gamma \left( \frac{1}{2}, \frac{r^2}{4\theta} \right),$$

(2.8)

where the solution (2.8) asymptotically approaches to the Schwarzschild BH since $\gamma(1/2, r^2/4\theta) \to \sqrt{\pi})$. For this reason, we call the metric (2.1) with Eq. (2.8) as the NCGiSBH (in the Rastall gravity). The NCGiSBH solution is finite at the origin:

$$f = 1 - \frac{2GM}{\sqrt{\pi} \theta} + \frac{GMr^2}{6\sqrt{\pi} \theta^{3/2}} + O(r^4),$$

(2.9)

but it is a singular BH, which can be best seen from its Ricci scalar:

$$R = \frac{GM e^{-\frac{r}{4\theta}} (4\theta - r^2)}{r^{3/2} \theta^{1/2}}.$$

(2.10)

When $M/\sqrt{\theta} < \sqrt{\pi}/2$, the NCGiSBH does not possess an event horizon. On the other hand, it has an event horizon $r_H$ when $M/\sqrt{\theta} > \sqrt{\pi}/2$. The Hawking temperature [91] of the NCGiSBH can be found as

$$T_H = \frac{f'(r_H)}{4\pi} = \frac{1}{4\pi r_H} \left[ 1 - \frac{r_H e^{-\frac{r}{4\theta}}}{\sqrt{\theta} \gamma \left( 1/2, \frac{r_H^2}{4\theta} \right)} \right].$$

(2.11)

It is worth noting that as it was stated in [88], while $r_H/\sqrt{\theta}$ gets large values, the temperature of the NCGiSBH almost matches with the temperature of the Schwarzschild BH. It was thoroughly discussed that although the NCGiSBH of GR theory admits a zero-temperature remnant with radius $r_H = 3\sqrt{\theta}$, the NCGiSBH of the Rastall gravity evaporates by ending up a point-like ($r_H = 0$) massive remnant. From now on, in our computations we shall focus on the NCGiSBH of the Rastall gravity.

3 Null Geodesics and Shadow Casts of NCGiSBH in Rastall gravity

For studying the BHS, it is necessary to consider the geodesics of a test particle in the associated BH geometry. To this end, we use the Hamilton-Jacobi and Carter’s constant separable methods for the NCGiSBH geometry.

The relativistic Hamilton-Jacobi equation reads [90]

$$\frac{\partial S}{\partial \sigma} = -\frac{1}{2} g^{\mu \nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu},$$

(3.1)

where $S$ is called the Jacobi action and $\sigma$ denotes the affine parameter along the geodesics. The separable solution for the Jacobi action $S$ can be introduced as

$$S = \frac{1}{2} m^2 \sigma - Et + L\phi + S_r(r) + S_\theta(\theta),$$

(3.2)

where $m$, $E$, and $L$ are the test particle’s mass, energy, and angular momentum, with respect to the rotation axis, respectively. $S_r(r)$ and $S_\theta(\theta)$ are functions of $r$ and $\theta$, respectively.
Inserting Eq. (3.2) into Eq. (3.1) and applying the method of separation of variables, one can obtain the null geodesic equations for a test particle around the NCGiSBH as follows

\[ \Sigma \frac{dt}{d\sigma} = \frac{1}{f} (E r^2), \]  
\[ \Sigma \frac{dr}{d\sigma} = \sqrt{\mathcal{R}}, \]  
\[ \Sigma \frac{d\theta}{d\sigma} = \sqrt{\Theta}, \]  
\[ \Sigma \frac{d\phi}{d\sigma} = \left( \frac{L}{\sin^2 \theta} \right), \]  

(3.3)  
(3.4)  
(3.5)  
(3.6)

where \( \mathcal{R}(r) \) and \( \Theta(\theta) \) take the following forms

\[ \mathcal{R}(r) = E^2 r^4 - r^2 f [m^2 r^2 + L^2 + K], \]  
\[ \Theta(\theta) = K - \left( \frac{L^2}{\sin^2 \theta} \right) \cos^2 \theta, \]  

(3.7)  
(3.8)

in which \( K \) is the Carter constant [15]. The above geodesic equations (3.3-3.6) fully describe the dynamics of the test particle around the NCGiSBH. The boundary of the BHS is mainly determined by the unstable circular orbit. Here, we consider the case of photons \((m = 0)\).

For an observer afar off a BH, the photons arrive the BH near the equatorial plane \((\theta = \pi/2)\).

The unstable circular orbits satisfy the following condition:

\[ \mathcal{R} = \frac{\partial \mathcal{R}}{\partial r} = 0. \]  

(3.9)

Recalling Eq. (3.7) and by introducing the two impact parameters \( \xi \) and \( \eta \):

\[ \xi = L/E, \quad \eta = K/E^2, \]  

(3.10)

we obtain

\[ r^4 - [\eta + \xi^2] r^2 f(r) = 0, \]  
\[ 4r^3 - [\eta + \xi^2] [2rf(r) + r^2 f'(r)] = 0. \]  

(3.11)  
(3.12)

Combining Eqs. (3.11-3.12) and making some algebra, \( \xi^2 + \eta \) yields

\[ \xi^2 + \eta = 2 \frac{r^2 f^2}{(rf' + 2 f)^2} - 8 \frac{r^2 f^2}{(rf' + 2 f)^2} + 16 \frac{r^2 f}{(rf' + 2 f)^2}, \]  
\[ = \frac{2 r^2 \left( -2r \sqrt{\vartheta} e^{-1/4} \pi \left( \text{erf} \left( \frac{r}{2\sqrt{\vartheta}} \right) - 1 \right) \sqrt{\pi} + e^{-\frac{r^2}{8\vartheta}} \pi \theta \left( r^2 - 3 \left( \text{erf} \left( \frac{r}{2\sqrt{\vartheta}} \right) \right)^2 + 6 \text{erf} \left( \frac{r}{2\sqrt{\vartheta}} \right) - 3 \right) \right)}{\left( e^{-\frac{r^2}{8\vartheta}} r + \sqrt{\vartheta} \sqrt{\pi} \left( \text{erf} \left( \frac{r}{2\sqrt{\vartheta}} \right) + r - 1 \right) \right)^2}. \]  

(3.13)  
(3.14)

To determine the shape of the BHS, we introduce the following celestial coordinates \( \alpha \) and \( \beta \):

\[ \alpha = \lim_{r_o \to \infty} \left( -r_o^2 \sin \theta_o \frac{d\phi}{dr} \right), \]  
\[ \beta = \lim_{r_o \to \infty} \left( r_o^2 \frac{d\theta}{dr} \right). \]  

(3.15)  
(3.16)
where $r_o$ is the distance between the BH and the observer, $\theta_o$ is the angle between the rotation axis of the BH and the line of sight of the observer (i.e., inclination angle) as shown in Fig. 1. $\alpha$ measures the apparent perpendicular distance of the shadow as seen from the axis of symmetry, and $\beta$ reads the apparent perpendicular distance of the shadow as seen from its projection on the equatorial plane.

\[ \alpha = -\frac{\xi}{\sin \theta}, \]
\[ \beta = \pm \sqrt{\eta - \xi^2 \cot^2 \theta}. \]

Using the null geodesic equations (3.3-3.6), we can obtain the relations between celestial coordinates and impact parameters $\xi$ and $\eta$ as

\[ \alpha^2 + \beta^2 = r_o^2 \left( \frac{r_0^6 + 4\rho^3 e^{\frac{r_0^2}{49}} \left( \pi r_0^2 - 12\gamma \left( \frac{3}{2}, \frac{r_0^2}{49} \right)^2 \right) - 8\rho^3/2 e^{\frac{r_0^2}{49}} \gamma \left( \frac{3}{2}, \frac{r_0^2}{49} \right)}{r_0^3 - 2\rho^3/2 e^{\frac{r_0^2}{49}} \left( \sqrt{\pi} r_0 - 2\gamma \left( \frac{3}{2}, \frac{r_0^2}{49} \right) \right)^2} \right). \]

The radii of the BHS are presented in Fig. (2).
Figure 2: The case of $r = M_0 = 1$: the green line represents $\vartheta = 0.001$, navy line is for $\vartheta = 0.3$, red and black lines are for $\vartheta = 0.999$ and the Schwarzschild BH ($f = 1 - \frac{2M}{r}$), respectively.

It is noted that small value of $\sqrt{\vartheta}/M_0$, the radius becomes $R_s = 3\sqrt{3}M_0$ for same Schwarzschild BH. For an observer located at an infinite distance, the black hole shadow corresponds to the high energy absorption cross section, the latter oscillating around a limiting constant value $\sigma_{\text{lim}}$ for a spherical symmetric black hole. $\sigma_{\text{lim}}$ is approximately equal to the geometrical cross section of the photon sphere

$$\sigma_{\text{lim}} \approx \pi R_s^2,$$

with $R_s$ the radius of the black hole shadow. The energy emission rate of the black hole is therefore

$$\frac{d^2E(\omega)}{d\omega dt} = \frac{2\pi^2 \sigma_{\text{lim}}}{e^{\omega/T_H} - 1} \omega^3,$$

where $\omega$ is the frequency of photon and recall that $T_H$ is the Hawking temperature (2.11) of the NCGiSBH. In Fig. (2), we depict the energy emission rate with frequency $\omega$ for the NCGiSBH. It can be seen that, the peak value of that Gaussian type plot decreases when $\vartheta$ gets higher values. The latter remark indicates also that the NCGiSBH’s area will also decrease for the higher values of $\vartheta$. 

$$-6-$$
4 Conclusions

In this study, we have studied the shadow of the NCGiSBH in the Rastall gravity and revealed the physical differences between it and the Schwarzschild BHs. We have shown that for the NCGiSBH, the shape of the BHS is nothing but a ring and its radius depends on $\sqrt{\vartheta M_0^{-1}}$. When $\sqrt{\vartheta M_0^{-1}}$ gets higher values, it was shown in Fig. 2 that the radius $R_s$ approaches to the result of the Schwarzschild BH. When $\sqrt{\vartheta M_0^{-1}}$ gets smaller values, $R_s$ tends to expand. Therefore, the effect of the Rastall gravity varies the size of the shadow. Thus, this property will enable us to distinguish the NCGiSBH from a Schwarzschild BH’s shadow. We have also discussed the energy emission rate for the NCGiSBH. As can be seen from Fig. (2), the obtained Gaussian type graph decreases its peak value as $\sqrt{\vartheta M_0^{-1}}$ increases. For this reason, the area of NCGiSBH shrinks with increasing $\sqrt{\vartheta M_0^{-1}}$ value.

The observational study of the BHS has just initiated with EHT. In fact, the image of EHT does not depict the BH’s event horizon, but a shadow cast by the light around it due to the unstable orbits of photons around the central object. We believe that in the near future many other images belonging to the various BHSs will be available by developing technology and science. In summary, we are at the forefront of the BHS and this subject will be a test center for GR in the forthcoming years.

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