AN INFINITE TORUS BRAID YIELDS A CATEGORIZED JONES-WENZL PROJECTOR

LEV ROZANSKY

ABSTRACT. A sequence of Temperley-Lieb algebra elements corresponding to torus braids with growing twisting numbers converges to the Jones-Wenzl projector. We show that a sequence of categorification complexes of these braids also has a limit which may serve as a categorification of the Jones-Wenzl projector.

Contents

1. Introduction 1
2. Notations and results 3
3. Elementary homological calculus 12
4. A direct system of categorification complexes of torus braids 16
5. A categorified Jones-Wenzl projector 21
6. The other projector 22
References 23

1. Introduction

A Jones-Wenzl projector $P_n$ is a special idempotent element of the $n$-strand Temperley-Lieb algebra $\text{TL}_n$, whose defining property is the annihilation of cap and cup tangles. The coefficients in its expression in terms of Temperley-Lieb tangles are rational (rather than polynomial) functions of $q$. This suggests that the categorification $P_n$ of $P_n$ in the universal tangle category $\text{TL}_n$ constructed by D. Bar-Natan [BN05] should be presented by a semi-infinite chain complex. In fact, there are two mutually dual categorifications: the complex $P_n^-$ which is bound from above and the complex $P_n^+$ which is bound from below. We will consider only $P_n^-$ in detail, since the story of $P_n^+$ is totally similar.

The work of L.R. was supported in part by the NSF grant DMS-0808974.
The construction of $P^+_n$ by B. Cooper and S. Krushkal [CK] is based upon the Frenkel-Khovanov formula for $P_n$ and requires the invention of morphisms between constituent TL tangles as well as non-trivial ‘thickening’ of the complex. An alternative ‘representation-theoretic’ approach to the categorification of the Jones-Wenzl projector is developed by Igor Frenkel, Catharina Stroppel, and Joshua Sussan [FSS].

Our approach is rather straightforward: the categorified projector $P^-_n$ is a direct limit of appropriately shifted categorification complexes of torus braids (i.e. braid analogs of torus links) with high clockwise twist (the other projector $P^+_n$ comes from high counterclockwise twists). The limit $P^-_n$ can be presented as a cone:

$$P^-_n \sim \text{Cone} \left( O^h_{-}(2m(n-1)) \longrightarrow \langle \langle \mathcal{X}_{n} \rangle \rangle^{s} \right),$$

where $\mathcal{X}_{n}$ is a torus braid with $m$ full clockwise rotations of $n$ strands, $\langle \langle - \rangle \rangle^{s}$ is the categorification complex with a special grading shift, and $O^h_{-}(k)$ denotes a chain complex which ends at the homological degree $-k$. Theorem 2.8 imposes even stronger restrictions on the complex $O^h_{-}(2m(n-1))$ in eq. (1.1).

The advantage of our approach is that one can use torus braids with high twist as approximations to $P^-_n$ in a computation of Khovanov homology of a spin network which involves Jones-Wenzl projectors: if a spin network $\nu$ is constructed by connecting $P_n$ to an $(n,n)$-tangle $\tau$ such that $\langle \langle \tau \rangle \rangle \sim O^h_{-}(k)$, while a spin network $\nu_m$ is constructed by replacing $P_n$ in $\nu$ with $\mathcal{X}_{n}$, then the homology of $\langle \langle \nu \rangle \rangle$ coincides with the shifted homology of $\langle \langle \nu_m \rangle \rangle$ in all homological degrees $i$ such that $i > -k - 2m(n-1)$. Thus one may say that there is a stable limit

$$\langle \langle \nu \rangle \rangle = \lim_{m \to +\infty} \langle \langle \nu_m \rangle \rangle^{s}. \quad (1.2)$$

We will define homological limits more precisely in subsection 2.2.2.

The practical importance of the relation between $\langle \langle \nu \rangle \rangle$ and $\langle \langle \nu_m \rangle \rangle$ stems from the fact that $\nu_m$ is an ordinary link and its homology can be computed with the help of existing efficient computer programs even for high values of $m$.

The simplest example of a spin network is the unknot ‘colored’ by the $(n+1)$-dimensional representation of SU(2) with the help of the projector $P_n$. Its Khovanov homology is approximated by the homology of torus links $T_{n,-m}$, which appear as cyclic closures of $\mathcal{X}_{n}$. The Khovanov homology of torus links has been studied by Marko Stosic [Sto07], who observed that it stabilizes at lower degrees as $m$ grows. This is a particular case of the ‘stable limit’ (1.2).
In Section 2 we explain all notations and conventions which are used in the paper. In particular, in subsection 2.1.4 we define a non-traditional grading of Khovanov homology, which is convenient for our computations. Then we formulate our results.

In Section 3 we review basic facts about homological ‘calculus’ required to work with limits of sequences of complexes in a homotopy category. In Section 4 we construct a sequence of categorification complexes of torus braids related by special chain morphisms. This sequence yields $P_n^-$ as its direct limit. In Section 5 we use homological calculus of Section 3 in order to prove that $P_n^-$ is a categorification of the Jones-Wenzl projector.

Acknowledgements. This paper is a spinoff of a joint project with Mikhail Khovanov [KR] which is dedicated to the study of categorification complexes of torus braids and their relation to the categorification of the Witten-Reshetikhin-Turaev invariant of links in $S^1 \times S^2$. I am deeply indebted to Mikhail for numerous discussions and suggestions.

I would like to thank Slava Krushkal for sharing the results of his ongoing research. I am also indebted to organizers of the M.S.R.I. workshop ‘Homology Theories of Knots and Links’ which stimulated me to write this paper.

This work is supported by the NSF grant DMS-0808974.

2. Notations and results

2.1. Notations.

2.1.1. Tangles and Temperley-Lieb algebra. All tangles in this paper are framed and we assume the blackboard framing in pictures. We use the symbol $\bigcirc_k$ to indicate an addition of $k$ framing twists to a tangle strand:

\[
\bigcirc = \bigcirc_1
\]  

(2.1)

A tangle is called planar if it can be presented by a diagram without crossings. A planar tangle is called connected or Temperley-Lieb (TL) if it does not contain disjoint circles. Let $Tng$ denote the set of all framed tangles, $Tng_{m,n}$ – the set of $(m,n)$-tangles and $Tng_n$ – the set of $(n,n)$-tangles. We adopt similar notations for the set $TL$ of TL-tangles.

We use the symbol $\circ$ to denote the composition of tangles: $\tau_1 \circ \tau_2$. The same symbol is used to denote the multiplication in Temperley-Lieb algebra and the composition bifunctor in the category TL.
A Temperley-Lieb algebra $TL$ over the ring of Laurent polynomials $\mathbb{Z}[q, q^{-1}]$ is a quiver ring. The vertices $v_n$ of the quiver are indexed by non-negative integers $n$ and each pair of vertices $v_m, v_n$, such that $m - n$ is even, is connected by an edge $e_{mn}$. To a vertex $v_n$ we associate a ring $TL_{n,n}$ (also denoted as $TL_n$) and to an edge $e_{mn}$ we associate a $TL_n \otimes TL_{m,n}^\text{op}$-module $TL_{m,n}$. As a module, $TL_{m,n}$ is generated freely by elements $\langle \lambda \rangle$ corresponding to $TL_{(m,n)}$-tangles $\lambda$, while ring and module structures come from the composition of tangles modulo the relation

$$\langle \bigcirc \rangle = -(q + q^{-1}), \quad (2.2)$$

which is needed to remove disjoint circles that may appear in the composition of Temperley-Lieb tangles.

The map $\text{Tng} \xrightarrow{\langle - \rangle} TL$ associates an element $\langle \tau \rangle$ to a tangle $\tau$ with the help of eq. (2.2) and the Kauffman bracket relation

$$\langle \bigotimes \rangle = q^{\frac{1}{2}} \langle \bigotimes \rangle + q^{-\frac{1}{2}} \langle \bigotimes \rangle. \quad (2.3)$$

This relation removes crossings and disjoint circles from the diagram of $\tau$, hence

$$\langle \tau \rangle = \sum_{\lambda \in TL_n} a_\lambda(\tau) \langle \lambda \rangle, \quad a_\lambda(\tau) = \sum_{i \in \mathbb{Z}} a_{\lambda,i}(\tau) q^i \quad (2.4)$$

with only finitely many coefficients $a_{\lambda,i}(\tau)$ being non-zero.

If two tangles differ only by the framing of their strands, then the corresponding algebra elements differ by the $q$ power factor coming from the following relation associated with the first Reidemeister move:

$$\langle \bigotimes_1 \rangle = -q^{\frac{1}{2}} \langle \bigotimes_1 \rangle \quad (2.5)$$

A $(0,0)$-tangle $L$ is a framed link, so $\langle L \rangle$ is the framing dependent Jones polynomial defined by the Kauffman bracket.

We use the notations $QTL$ and $TL^+$ for Temperley-Lieb algebras defined over the field $Q(q)$ of rational functions of $q$ and over the field $\mathbb{Z}[[q, q^{-1}]]$ of Laurent power series. A sequence of injective homomorphisms $\mathbb{Z}[q, q^{-1}] \rightarrow Q(q) \rightarrow \mathbb{Z}[[q, q^{-1}]]$, the latter one generated by the expansion in powers of $q$, produce a sequence of injective homomorphisms of the corresponding Temperley-Lieb algebras.

\footnote{It is clear from our normalization of the Kauffman bracket relation (2.3) that we should rather use the ring $\mathbb{Z}[q^{1/2}, q^{-1/2}]$. However, in all expressions in this paper the half-integer power of $q$ appears only as a common factor, so the terms with integer and half-integer powers of $q$ do not mix. Hence we refer to $\mathbb{Z}[q, q^{-1}]$, while keeping in mind that $q^{1/2}$ may appear as a common factor is some expressions.}
2.1.2. The Jones-Wenzl projector. Let $\begin{tikzpicture} \draw[thick] (0,0) -- (0.5,0) -- (0.5,0.5) -- (0,0.5) -- cycle; \end{tikzpicture}$ $n \in TL_{n-2,n}$ and $\begin{tikzpicture} \draw[thick] (0,0) -- (0.5,0) -- (0.5,0.5) -- (0,0.5) -- cycle; \end{tikzpicture}$ $\begin{tikzpicture} \draw[thick] (0,0) -- (0.5,0) -- (0.5,0.5) -- (0,0.5) -- cycle; \end{tikzpicture}$ $i \in TL_{n,n-2}, 1 \leq i \leq n - 1$, denote the following TL tangles:

\[ \begin{tikzpicture} \draw[thick] (0,0) -- (0.5,0) -- (0.5,0.5) -- (0,0.5) -- cycle; \end{tikzpicture} \begin{array}{c} \cdots \\ i+1 \end{array} n, \quad \begin{tikzpicture} \draw[thick] (0,0) -- (0.5,0) -- (0.5,0.5) -- (0,0.5) -- cycle; \end{tikzpicture} \begin{array}{c} \cdots \\ i+1 \\ \cdots \\ 1 \end{array} i \begin{tikzpicture} \draw[thick] (0,0) -- (0.5,0) -- (0.5,0.5) -- (0,0.5) -- cycle; \end{tikzpicture} \begin{array}{c} \cdots \\ n \end{array} \]

Their compositions $U_{n,i} = \begin{tikzpicture} \draw[thick] (0,0) -- (0.5,0) -- (0.5,0.5) -- (0,0.5) -- cycle; \end{tikzpicture} \begin{array}{c} \cdots \\ i+1 \end{array} n \circ \begin{tikzpicture} \draw[thick] (0,0) -- (0.5,0) -- (0.5,0.5) -- (0,0.5) -- cycle; \end{tikzpicture} \begin{array}{c} \cdots \\ i+1 \\ \cdots \\ 1 \end{array} i \begin{tikzpicture} \draw[thick] (0,0) -- (0.5,0) -- (0.5,0.5) -- (0,0.5) -- cycle; \end{tikzpicture} \begin{array}{c} \cdots \\ n \end{array}$ are standard generators of the Temperley-Lieb algebra $TL_n$.

The Jones-Wenzl projector $P_n \in QTL_n$ is the unique non-trivial idempotent element satisfying the condition

\[ \langle \begin{tikzpicture} \draw[thick] (0,0) -- (0.5,0) -- (0.5,0.5) -- (0,0.5) -- cycle; \end{tikzpicture} \begin{array}{c} \cdots \\ i+1 \\ \cdots \\ 1 \end{array} i \begin{tikzpicture} \draw[thick] (0,0) -- (0.5,0) -- (0.5,0.5) -- (0,0.5) -- cycle; \end{tikzpicture} \begin{array}{c} \cdots \\ n \end{array} \rangle \circ P_n = 0, \quad 1 \leq i \leq n - 1. \tag{2.6} \]

The Jones-Wenzl projector also satisfies the relation

\[ P_n \circ \langle \begin{tikzpicture} \draw[thick] (0,0) -- (0.5,0) -- (0.5,0.5) -- (0,0.5) -- cycle; \end{tikzpicture} \begin{array}{c} \cdots \\ i+1 \\ \cdots \\ 1 \end{array} i \begin{tikzpicture} \draw[thick] (0,0) -- (0.5,0) -- (0.5,0.5) -- (0,0.5) -- cycle; \end{tikzpicture} \begin{array}{c} \cdots \\ n \end{array} \rangle = 0, \quad 1 \leq i \leq n - 1. \tag{2.7} \]

We denote the idempotent element of $TL_n^+$ corresponding to $P_n$ as $P_n^+$.  

2.1.3. Basic notions of homological algebra. Let $\text{Ch}(A)$ be a category of chain complexes associated with an additive category $A$. An object of $\text{Ch}(A)$ is a chain complex

\[ A = (\cdots \to A_i \overset{d_i}{\to} A_{i+1} \to \cdots), \]

and a morphism between two chain complexes is a chain morphism defined as a multi-map

\[ \begin{array}{cccccccc} A & \cdots & \overset{d_{i-1}}{\to} & A_i & \overset{d_i}{\to} & A_{i+1} & \overset{d_{i+1}}{\to} & \cdots \\ \downarrow f & \downarrow & \downarrow & \downarrow & \downarrow f_i & \downarrow f_{i+1} & \downarrow & \cdots \\ B & \cdots & \overset{d'_{i-1}}{\to} & B_i & \overset{d'_i}{\to} & B_{i+1} & \overset{d'_{i+1}}{\to} & \cdots \end{array} \tag{2.8} \]

which commutes with the chain differential: $d'_i f_i = f_{i+1} d_i$ for all $i$. The cone of a chain morphism $A \overset{f}{\to} B$ is a complex

\[ \text{Cone}(f) = \left( \begin{array}{cccccccc} \cdots & \overset{-d_i}{\to} & A_i & \overset{-f_i}{\to} & A_{i+1} & \cdots \\ \downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \cdots \\ \cdots & \overset{d'_{i-1}}{\to} & B_{i-1} & \overset{d'_i}{\to} & B_i & \cdots \end{array} \right) \]
in which the object $A_{i+1} \oplus B_i$ has the homological degree $i$. There are two special chain morphisms $B \xrightarrow{\iota_f} \text{Cone}(f)$ and $\text{Cone}(f)[1] \xrightarrow{\delta_f} A$ associated to the cone:

$$
\begin{array}{cccccccc}
B & \cdots & B_i & B_{i+1} & \cdots \\
\downarrow{\iota_f} & & \downarrow{0 \oplus 1} & & \downarrow{0 \oplus 1} \\
\text{Cone}(f) & \cdots & A_{i+1} \oplus B_i & A_{i+2} \oplus B_{i+1} & \cdots \\
\downarrow{\delta_f} & & \downarrow{1 \oplus 0} & & \downarrow{1 \oplus 0} \\
A[-1] & \cdots & A_{i+1} & A_{i+2} & \cdots 
\end{array}
$$

These complexes and chain morphisms form a distinguished triangle:

$$
A \xrightarrow{f} B \xrightarrow{\iota_f} \text{Cone}(f) \xrightarrow{\delta_f} A[-1].
$$

(2.9)

The homotopy category of complexes $\mathbf{K}(A)$ has the same objects as $\text{Ch}(A)$ and the morphisms are the morphisms of $\text{Ch}(A)$ modulo homotopies. We denote homotopy equivalence by the sign $\sim$. The notion of a cone extends to $\mathbf{K}(A)$ and there are additional relations in that category: $\text{Cone}(\iota_f) \sim A[-1]$ and $\text{Cone}(\delta_f) \sim B[-1]$, so all vertices of a distinguished triangle have equal properties.

2.1.4. *A triply graded categorification of the Jones polynomial.* In his famous paper [Kho00], M. Khovanov introduced a categorification of the Jones polynomial of links. To a diagram $L$ of a link he associates a complex of graded modules

$$
\langle\langle L \rangle\rangle = (\cdots \to C_i \to C_{i+1} \to \cdots)
$$

(2.10)

so that if two diagrams represent the same link then the corresponding complexes are homotopy equivalent, and the graded Euler characteristic of $\langle\langle L \rangle\rangle$ is equal to the Jones polynomial of $L$.

Thus, overall, the complex (2.10) has two gradings: the first one is the grading related to powers of $q$ and the second one is the homological grading of the complex itself, the corresponding degree being equal to $i$. In this paper we adopt a slightly different convention which is convenient for working with framed links and tangles. It is inspired by matrix factorization categorification [KR08] and its advantage is that it is no longer necessary to assign orientation to link strands in order to obtain the grading of the categorification complex (2.10) which would make it invariant under the second Reidemeister move.

To a framed link diagram $L$ we associate a $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$-graded complex (2.10) with degrees $\deg_h$, $\deg_q$ and $\deg_2$. The first two gradings are of the same nature as in [Kho00] and, in particular, $\deg_h C_i = i$. The third grading is an inner grading of chain modules defined modulo 2 and of homological nature, that is, the homological parity of an element of $\langle\langle L \rangle\rangle$,
which affects various sign factors, is the sum of $\deg_h$ and $\deg_2$. Both homological degrees are either integer or half-integer simultaneously, so the homological parity is integer and takes values in $\mathbb{Z}_2$. The $q$-degree $\deg_q$ may also take half-integer values.

Let $[m, l, n]$ denote the shift of three degrees by $l$, $m$ and $n$ units respectively. We use abbreviated notations

$$[l, m] = [l, m, 0], \quad [m]_q = [0, m, 0]$$

as well as the following ‘power’ notation:

$$[l, m, n]^k = [kl, km, kn].$$

With new grading conventions, the categorification formulas of $[\text{Kho00}]$ take the following form: the module associated with an unknot is still $\mathbb{Z}[x]/(x^2)$ but with a different degree assignment:

$$\langle O \rangle = \mathbb{Z}[x]/(x^2) [0, -1, 1], \quad (2.11)$$

$$\deg_q 1 = 0, \quad \deg_q x = 2, \quad \deg_h 1 = \deg_h x = \deg_2 1 = \deg_2 x = 0, \quad (2.12)$$

and the categorification complex of a crossing is the same as in $[\text{Kho00}]$ but with a different degree shift:

$$\langle X \rangle = \left( \langle \circ \circ \rangle \left[ -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right] \xrightarrow{f} \langle \circ \circ \circ \rangle \left[ \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right] \right), \quad (2.13)$$

where $f$ is either a multiplication or a comultiplication of the ring $\mathbb{Z}[x]/(x^2)$ depending on how the arcs in the r.h.s. are closed into circles. The resulting categorification complex $(2.10)$ is invariant up to homotopy under the second and third Reidemeister moves, but it acquires a degree shift under the first Reidemeister move:

$$\langle \ | \ \rangle = \langle \ | \ \rangle \left[ -\frac{1}{2}, \frac{3}{2}, -\frac{1}{2} \right]. \quad (2.14)$$

It is easy to see that the whole categorification complex $(2.10)$ has a homogeneous degree $\deg_2$.

\[\text{Our degree shift is defined in such a way that if an object } M \text{ has a homogeneous degree } n, \text{ then the shifted object } M[1] \text{ has a homogeneous degree } n + 1.\]
2.1.5. A universal categorification of the Temperley-Lieb algebra. D. Bar-Natan [BN05] described the universal category $\mathcal{TL}$, whose Grothendieck $K_0$-group is $\mathcal{TL}$ considered as a $\mathbb{Z}[q, q^{-1}]$-module. We will use this category with obvious adjustments required by the new grading conventions.

Let $\mathcal{TL}$ be an additive category whose objects are in one-to-one correspondence with Temperley-Lieb tangles, morphisms being generated by tangle cobordisms (see [BN05] for details). The universal category $\mathcal{TL}$ is the homotopy category of bounded complexes associated with $\mathcal{TL}$. In other words, an object of $\mathcal{TL}$ is a complex

$$\mathbf{C} = (\cdots \to C_i \to C_{i+1} \to \cdots), \quad C_i = \bigoplus_{\lambda \in \mathcal{TL}_n} \bigoplus_{j, \mu} c_{i,j,\mu}^{\lambda} \langle \langle \lambda \rangle \rangle [0, j, \mu],$$

(2.15)

where non-negative integers $c_{i,j,\mu}^{\lambda}$ are multiplicities; since the complex is bounded, they are non-zero for only finitely many values of $i$.

A categorification map $\mathcal{Tng} \langle \langle - \rangle \rangle \rightarrow \mathcal{TL}$ turns a framed tangle diagram $\tau$ into a complex $\langle \langle \tau \rangle \rangle$ according to the rules (2.11) and (2.13), the morphism $f$ in the complex (2.13) being the saddle cobordism. A composition of tangles becomes a composition bi-functor $\mathcal{TL} \times \mathcal{TL} \rightarrow \mathcal{TL}$ if we apply the categorified version of the rule (2.2) in order to remove disjoint circles:

$$\langle \langle \bigcirc \bigcirc \rangle \rangle = \langle \langle \lambda_\emptyset \rangle \rangle [0, 1, 1] + \langle \langle \lambda_\emptyset \rangle \rangle [0, -1, 1],$$

(2.16)

where $\lambda_\emptyset$ is the empty TL $(0, 0)$-tangle.

A complex $\langle \langle \tau \rangle \rangle$ associated to a tangle $\tau$ is defined only up to homotopy. We use a notation $\langle \langle \tau \rangle \rangle_\sharp$ for a particular complex with special properties which represents $\langle \langle \tau \rangle \rangle$.

Overall, we have the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{Tng} & \langle \langle \cdot \rangle \rangle & \mathcal{TL} \\
\downarrow & \downarrow K_0 & \downarrow \\
\mathcal{TL} & \langle \langle \cdot \rangle \rangle & \mathcal{TL} \\
\end{array}
\]

where the map $K_0$ turns the complex (2.15) into the sum (2.4):

$$K_0(\mathbf{C}) = \sum_{\lambda \in \mathcal{TL}_n} \sum_j a_{\lambda,j} q^j \langle \langle \lambda \rangle \rangle, \quad a_{\lambda,j} = \sum_{i, \mu} (-1)^{i+\mu} c_{i,j,\mu}^{\lambda}.$$  

(2.18)

Since the complex is bounded, the sum in the expression for $a_{\lambda,j}$ is finite.

In addition to $\mathcal{TL}$ we consider a category $\mathcal{TL}^{-}$ of complexes bounded from above, that is, the multiplicity coefficients in the sum (2.15) are zero if $i$ is greater than certain value. Define the $q^+$ order of a chain ‘module’ $C_i$: $|C_i|_q = \inf \{ j : \exists \mu : c_{i,j,\mu}^{\lambda} \neq 0 \}$. A complex $\mathbf{C}$
in $\text{TL}^{-}$ is $q^{+}$-bounded if $\lim_{i \to \infty} |C_{-i}|_q = +\infty$. For a $q^{+}$-bounded complex, the sum in the expression (2.18) for $a_{\lambda,j}$ is finite, hence the element $K_{0}(C)$ is well defined.

2.2. Results.

2.2.1. Infinite torus braid as a Jones-Wenzl projector in a Temperley-Lieb algebra. A braid with $n$ strands is a particular example of a $(n, n)$-tangle. A torus braid is a braid that can be drawn on a cylinder $S^1 \times [0, 1]$ without intersections. In fact, all torus braids have the form $\beta_{\text{cyl}, n}^m$, $m \in \mathbb{Z}$, where $\beta_{\text{cyl}, n}$ is the elementary clockwise winding torus braid:

\[
\beta_{\text{cyl}, n} = \begin{array}{c}
\cdots \\
1 \quad 2 \\
1 \quad n-1 \\
n
\end{array}
\]

We introduce a special notation for the torus braid which corresponds to $m$ full rotations of $n$ strands:

\[
\begin{array}{c}
\cdots \\
1 \quad 2 \\
1 \quad n-1 \\
n
\end{array}^{m} = \beta_{\text{cyl}}^{mn}.
\]

Let $O_+ (q^m)$ denote any element of $\text{TL}^+$ of the form $\sum_{\lambda \in \text{TL}_n} \sum_{j \geq m} a_{\lambda,j} q^j \langle \lambda \rangle$. We define a $q$-order of an element $\alpha \in \text{TL}^+$ as $|\alpha|_q = \sup \{ m : \alpha = O_+ (q^m) \}$.

Definition 2.1. A sequence of elements $\alpha_1, \alpha_2, \ldots \in \text{TL}^+$ has a limit $\lim_{k \to \infty} \alpha_k = \beta$, if $\lim_{i \to \infty} |\beta - \alpha_k|_q = +\infty$.

The following theorem may be known, so we do not claim credit for it. It appears here as a by-product and it is an easy corollary of eq. (2.26).

Theorem 2.2. The $\text{TL}$ element corresponding to the infinite torus braid equals the Jones-Wenzl projector:

\[
\lim_{m \to +\infty} q^{\frac{1}{2} m n (n-1)} \langle \begin{array}{c}
\cdots \\
1 \quad 2 \\
1 \quad n-1 \\
n
\end{array}^{m} \rangle = P_{n}^{+},
\]

(2.20)

where $P_{n}^{+} \in \text{TL}_{n}^{+}$ corresponds to the Jones-Wenzl projector $P_{n} \in \text{QTL}_{n}$.

In fact, a more general statement is also true:

\[
\lim_{m \to +\infty} q^{\frac{1}{2} m n (n-1)} \langle \beta_{\text{cyl}, n}^{m} \rangle = P_{n}^{+},
\]

(2.21)

but its proof is more technical and we omit it here.
2.2.2. A bit of homological calculus. Let $K(A)$ denote the homotopy category of complexes associated with an additive category $A$ (we have in mind a particular case of $K(A) = TL^-$).

A chain complex is considered ‘homologically small’ if it ends at a low (that is, high negative) homological degree. Let $O_h(m)$ denote a complex which ends at $(-m)$-th homological degree: $O_h(m) = (\cdots A_{-m-1} \to A_{-m})$. We define a homological order of a complex $A$ as $|A|_h = \sup \{ m : A \sim O_h(m) \}$.

Two complexes connected by a chain morphism: $A \xrightarrow{f} B$ are considered ‘homologically close’ if Cone($f$) is homologically small.

A direct system is a sequence of complexes connected by chain morphisms:

$$\mathcal{A} = (A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \cdots).$$

**Definition 2.3.** A direct system $\mathcal{A}$ is Cauchy if $\lim_{i \to \infty} |\text{Cone}(f_i)|_h = \infty$.

**Definition 2.4.** A direct system has a limit $\mathcal{A}$ if and only if it is Cauchy.

A direct system $\mathcal{A}$ has a limit $\mathcal{A} = A$, where $A$ is a chain complex, if there exist chain morphisms $A_i \xrightarrow{\tilde{f}_i} A$ such that they form commutative triangles

$$A_i \xrightarrow{f_i} A_{i+1} \xrightarrow{f_{i+1}} A, \quad \tilde{f}_i \sim \tilde{f}_{i+1} f_i \quad (2.22)$$

and $\lim_{i \to \infty} |\text{Cone}(\tilde{f}_i)|_h = \infty$.

In Section 3 we prove the following homology versions of standard theorems about limits (Propositions 3.7, 3.12 and 3.13):

**Theorem 2.5.** A direct system $\mathcal{A}$ has a limit if and only if it is Cauchy.

**Theorem 2.6.** The limit of a direct system is unique up to homotopy equivalence.

2.2.3. Infinite torus braid as a Jones-Wenzl projector in the universal category. For a tangle diagram $\tau$ let $\langle\langle \tau \rangle\rangle^s$ denote the categorification complex $\langle\langle \tau \rangle\rangle$ with a degree shift proportional to the number $n_\times(\tau)$ of crossings in the diagram $\tau$:

$$\langle\langle \tau \rangle\rangle^s = \langle\langle \tau \rangle\rangle \left[ \begin{array}{c} -1/2, 1/2, 1/2 \end{array} \right]^{n_\times(\tau)}. \quad (2.23)$$

---

This definition differs from the standard categorical definition of a direct limit, however Theorem 3.9 indicates that our definition implies the standard one. We expect that both definitions are equivalent.
In subsection 4.2 we define a direct system of categorification complexes of torus braids connected by special chain morphisms

\[ \mathcal{B}_n = \left( \langle \ \begin{array}{c} \vdots \\ \vdots \end{array} \cdot n \rangle \rightarrow \langle \begin{array}{c} 1 \\ \vdots \end{array} \cdot n \rangle \right)^s \rightarrow f_1 \rightarrow \ldots \]

\[ \ldots \rightarrow \langle \begin{array}{c} m \\ \vdots \end{array} \cdot n \rangle \rightarrow \langle \begin{array}{c} m+1 \\ \vdots \end{array} \cdot n \rangle \rightarrow \ldots \right) \quad (2.24) \]

We prove that |\text{Cone}(f_m)| \geq 2m(n - 1) + 1, so \( \mathcal{B}_n \) is a Cauchy system and by Theorem 2.5 it has a unique limit: \( \lim_{n \to} \mathcal{B}_n = \mathcal{P}_n^{-} \in \text{TL}_n^{-} \).

**Theorem 2.7.** The limiting complex \( \mathcal{P}_n^{-} \) has the following properties:

1. A composition of \( \mathcal{P}_n^{-} \) with cap- and cup-tangles is contractible:

\[ \langle \langle \begin{array}{c} i \\ \vdots \end{array} \cdot n \rangle \rangle \circ \mathcal{P}_n^{-} \sim \mathcal{P}_n^{-} \circ \langle \begin{array}{c} i \\ \vdots \end{array} \cdot n \rangle \rangle \sim 0. \]

2. The complex \( \mathcal{P}_n^{-} \) is idempotent with respect to tangle composition: \( \mathcal{P}_n^{-} \circ \mathcal{P}_n^{-} \sim \mathcal{P}_n^{-} \).

We provide a glimpse into the structure of \( \mathcal{P}_n^{-} \). A complex \( \mathcal{C} \) in \( \text{TL}_n \) is called 1-cut if \( \vdots \cdot n \) never appears in chain ‘modules’ \( \mathcal{C}_i \). A complex \( \mathcal{C} \) in \( \text{TL}_n \) is called angle-shaped if the multiplicities \( c^\lambda_{i,j,\mu} \) of eq. (2.15) satisfy the property

\[ c^\lambda_{i,j,\mu} = 0 \quad \text{if } i < 0, \text{ or } j < i, \text{ or } j > 2i. \quad (2.25) \]

Let \( \langle \begin{array}{c} m \\ \vdots \end{array} \cdot n \rangle \rangle \rightarrow \mathcal{P}_n^{-} \) be chain morphisms associated with the limit \( \lim_{n \to} \mathcal{B}_n = \mathcal{P}_n^{-} \) in accordance with Definition 2.4.

**Theorem 2.8.** There exist 1-cut angle-shaped complexes \( \tilde{\mathcal{C}}_{m,n} \) such that

\[ \text{Cone}(\tilde{f}_m) \sim \tilde{\mathcal{C}}_{m,n} \left[ -n + 1, n \right]^{2m} [-1, 1]. \]

In other words, there exists a distinguished triangle

\[ \tilde{\mathcal{C}}_{m,n} \left[ -n + 1, n \right]^{2m} [1] \rightarrow \langle \begin{array}{c} m \\ \vdots \end{array} \cdot n \rangle \rangle \rightarrow \mathcal{P}_n^{-} \rightarrow \tilde{\mathcal{C}}_{m,n} \left[ -n + 1, n \right]^{2m} [-1, 1] \]

so there is a presentation

\[ \mathcal{P}_n^{-} \sim \text{Cone} \left( \tilde{\mathcal{C}}_{m,n} \left[ -n + 1, n \right]^{2m} [1] \rightarrow \langle \begin{array}{c} m \\ \vdots \end{array} \cdot n \rangle \rangle \right), \quad (2.26) \]

where the complex \( \tilde{\mathcal{C}}_{m,n} \) is 1-cut and angle-shaped.

At \( m = 0 \) the formula (2.26) becomes

\[ \mathcal{P}_n^{-} \sim \text{Cone} \left( \tilde{\mathcal{C}}_{0,n} \left[ 1 \right] \rightarrow \langle \begin{array}{c} \vdots \\ \vdots \end{array} \cdot n \rangle \rangle \right), \quad (2.27) \]
where the complex \( \tilde{C}_{0,n} \) is 1-cut and angle-shaped.

Since \( \tilde{C}_{0,n} \) is angle-shaped, the complex \( \Cone(\tilde{f}_0) \) is also angle-shaped and consequently \( q^+ \)-bounded. Hence \( K_0(P_n^-) \) is well-defined. Also \( K_0(P_n^-) \neq 0 \), because it contains \( \langle \tilde{f}_0 \rangle \) with coefficient 1. Theorem 2.7 indicates that \( K_0(P_n^-) \) satisfies defining properties of the Jones-Wenzl projector, hence by uniqueness it is the Jones-Wenzl projector:

**Corollary 2.9.** The complex \( P_n^- \) categorifies the Jones-Wenzl projector in \( TL^+ \):

\[
K_0(P_n^-) = P_n.
\] (2.28)

3. Elementary homological calculus

3.1. Limits in the category of complexes. Consider a category \( \text{Ch}(A) \) of chain complexes associated with an additive category \( A \). An \( i \)-th truncation \( t_{\leq i} A \) of a chain complex \( A \) is the chain complex \( A \rightarrow A_{-i} \rightarrow A_{-i+1} \rightarrow \cdots \). An \( i \)-th truncation of a chain morphism \( f \) is defined similarly.

Define an isomorphism order \( \| f \|_\approx \) of a chain map \( A \rightarrow B \) as the largest number \( i \) for which a truncated chain morphism \( t_{\leq i} f \) is an isomorphism of truncated complexes.

**Remark 3.1.** Consider a distinguished triangle \( (2.9) \). If \( A \sim O^h(m) \), then \( \| f \|_\approx \geq m - 1 \).

**Definition 3.2.** A direct system \( \mathcal{A} = (A_1 \rightarrow f_1 \rightarrow A_2 \rightarrow f_2 \rightarrow \cdots) \) is stabilizing if \( \lim_{i \to \infty} \| f_i \|_\approx = \infty \).

**Definition 3.3.** A direct system \( \mathcal{A} \) has a chain limit \( \lim_{\text{Ch}} \mathcal{A} = A \) if there exist chain morphisms \( A_i \rightarrow f_i \rightarrow A \) such that \( f_i = f_{i+1} f_i \) and \( \lim_{i \to \infty} \| f_i \|_\approx = \infty \).

The following two theorems are easy to prove:

**Theorem 3.4.** A direct system has a chain limit if and only if it is stabilizing. If a chain limit exists then it is unique.

**Theorem 3.5.** Suppose that \( \lim_{\text{Ch}} \mathcal{A} = A \). Then for a complex \( B \) and chain morphisms \( A_i \rightarrow g_i \rightarrow B \) such that \( g_i = g_{i+1} f_i \), there exists a unique chain morphism \( A \rightarrow g \rightarrow B \) such that \( g_i = g f_i \).

**Definition 3.6.** A sequence of chain morphisms \( A \rightarrow f_0, f_1, \ldots \rightarrow B \) has a chain limit \( \lim_{i \to \infty} f_i = f \) if for any \( N \) there exists \( N' \) such that \( t_{\leq N} f_i = t_{\leq N'} f \) for any \( i \geq N' \).
3.2. Limits in the homotopy category. Definitions 2.3 and 2.4 extend the notion of a stabilizing direct system and its limit to the homotopy category $K(A)$: obviously, a stabilizing direct system is Cauchy, while $\lim_\mapsto \Ch A = A$ implies $\lim A = A$.

**Proposition 3.7.** A Cauchy system has a limit.

**Proof.** Consider a Cauchy system $A$. We construct a special chain complex $A$ such that $\lim_\mapsto A = A$ in accordance with Definition 2.4. Roughly speaking, we take $A_0$ and attach to it the cones $\Cone(f_i)$ represented by homologically small complexes, one by one. The result is a sequence $A = A_{f,0}, A_{f,1}, \ldots$ of stabilizing complexes $A_{f,i}$ such that $A_{f,i} \sim A_i$, and $A = \lim \Ch A$ is their chain limit.

Here is a detailed explanation. By Definition 2.3, there exist complexes $C_i$ such that $\Cone(f_i) \sim C_i[1] = O^h(m_i)$, $\lim_{i \to \infty} m_i = +\infty$. (3.1)

The complexes $A_i, A_{i+1}$ and $C_i$ form exact triangles:

$$
C_i \xrightarrow{\delta_i} A_i \xrightarrow{f_i} A_{i+1} \xrightarrow{C_i 
\delta_i} C_i[-1]
$$

and $A_{i+1} \sim \Cone(\delta_i)$. We define recursively a new sequence of complexes $A_{f} = (A_{f,0} \xrightarrow{f_{0}} A_{f,1} \xrightarrow{f_{1}} \cdots)$ by the relations $A_{f,0} = A_0, A_{f,i} \sim A_i$ and $A_{f,i+1} = \Cone(g_i)$, where the chain morphism $C_i \xrightarrow{g_i} A_{f,i}$ is homotopy equivalent to the chain morphism $\delta_i$. In other words,

$$
A_{f,i} = \Cone(C_i \xrightarrow{g_i} \Cone(C_{i-1} \xrightarrow{g_{i-1}} \cdots \xrightarrow{g_2} \Cone(C_1 \xrightarrow{g_1} \Cone(C_0 \xrightarrow{\delta_{0}} A_0))})
$$

(3.2)

According to Remark 3.1, $|g_i| \geq m_i$, hence the sequence $A_{f}$ is stabilizing, so there exists a chain limit $\lim_\Ch A = A_f$ and consequently there is a limit $\lim_\mapsto A = A_f$. $\Box$

Simply saying, the complex $A_f$ is an infinite multi-cone extension of the complex (3.2):

$$
A_f = \cdots \xrightarrow{g_3} \Cone(C_2 \xrightarrow{g_2} \Cone(C_1 \xrightarrow{g_1} \Cone(C_0 \xrightarrow{\delta_{0}} A_0))).
$$

(3.3)

For our applications it is important to express $\Cone(\tilde{f}_0)$ in terms of complexes $C_i$. This can be done by rearranging the infinite multi-cone (3.3) with the help of associativity of cone formation, which exists even within the category $\Ch(A)$:

$$
A_f = \Cone(\tilde{C} \xrightarrow{g} A_0), \quad \tilde{C} = \cdots \xrightarrow{h_2} \Cone(C_2[1] \xrightarrow{h_1} \Cone(C_1[1] \xrightarrow{h_0} C_0)),
$$

(3.4)

so that $\tilde{f}_0 \sim \tilde{g}_i$, and $\Cone(\tilde{f}_0) \sim \tilde{C}[-1]$ is expressed in terms of complexes $C_i$ arranged into an infinite multi-cone $\tilde{C}$. Here is a more formal statement.
Theorem 3.8. For a Cauchy system $\mathcal{A}$ there exists another Cauchy system $\mathcal{C} = (\mathcal{C}_0 \xrightarrow{h_0} \mathcal{C}_1 \xrightarrow{h_1} \cdots)$ and chain morphisms $\mathcal{C}_i[1] \xrightarrow{h_i} \mathcal{C}_i$ such that $\text{Cone}(h_i) = \mathcal{C}_{i+1}$, $h' = \iota_n$, and for the limiting complex $\mathcal{C} = \lim_{\text{Ch}} \mathcal{C}$ there exists a chain morphism $\mathcal{C} \xrightarrow{g} \mathcal{A}_0$ such that $\mathcal{A}_i = \text{Cone}(g)$, $\mathcal{f}_0 \sim \iota_g$ and consequently $\text{Cone}(\mathcal{f}_0) \sim \mathcal{C}[-1]$.

Proof. Let us recall the associativity of cones in a general setting. For a chain morphism $\mathcal{A} \xrightarrow{f} \mathcal{B}$, a chain morphism $\mathcal{C} \xrightarrow{g} \text{Cone}(\mathcal{f})$ is a sum: $g = g_{\mathcal{A}} \oplus g_{\mathcal{B}}$

\[
\begin{array}{c}
\mathcal{A} \xrightarrow{f} \mathcal{B} \\
\mathcal{C} \xrightarrow{g} \text{Cone}(\mathcal{f})
\end{array}
\]

where $\mathcal{A} \xrightarrow{g_{\mathcal{A}}} \mathcal{A}[-1]$ is a chain morphism and $\mathcal{C} \xrightarrow{g_{\mathcal{B}}} \mathcal{B}$ is a multi-map. Now it is obvious that

\[
\text{Cone}(\mathcal{C} \xrightarrow{g} \text{Cone}(\mathcal{A} \xrightarrow{f} \mathcal{B})) = \text{Cone}(\text{Cone}(\mathcal{C}[1] \xrightarrow{g_{\mathcal{A}}} \mathcal{A}) \xrightarrow{g_{\mathcal{B}} \oplus f} \mathcal{B}).
\]  

(3.5)

We apply the associativity relation (3.5) to multi-cones (3.2) consecutively for $i = 1, 2, \ldots$ in order to rearrange them, so that $\mathcal{A}_{\mathcal{C}_i} = \text{Cone}(\mathcal{C}_i \xrightarrow{g_i} \mathcal{A}_0)$, while the complexes $\mathcal{C}_i$ and chain morphisms $\mathcal{g}_i$ are defined recursively: $\mathcal{C}_0 = \mathcal{C}_0$, $\mathcal{g}_0 = \delta_{\mathcal{C}_0}$, $\mathcal{C}_{i+1} = \text{Cone}(\mathcal{h}_i)$, while the chain morphisms $\mathcal{C}_i[1] \xrightarrow{h_i} \mathcal{C}_i$ and $\mathcal{C}_{i+1} \xrightarrow{g_{i+1}} \mathcal{A}_0$ are defined by applying the associativity relation (3.5) to the double cone on the second line of the following equation:

\[
\begin{aligned}
\mathcal{A}_{\mathcal{C}_{i+1}} &= \text{Cone}(\mathcal{C}_i \xrightarrow{g_i} \mathcal{A}_{\mathcal{C}_i}) \\
&= \text{Cone}(\text{Cone}(\mathcal{C}_i \xrightarrow{h_i} \mathcal{C}_i) \xrightarrow{g_{i+1}} \mathcal{A}_0)) \\
&= \text{Cone}(\mathcal{C}_{i+1} \xrightarrow{g_{i+1}} \mathcal{A}_0).
\end{aligned}
\]  

(3.6)

Distinguished triangles

\[
\begin{array}{c}
\mathcal{C}_i[1] \xrightarrow{h_i} \mathcal{C}_i \xrightarrow{\iota_{h_i}} \mathcal{C}_{i+1} \xrightarrow{\iota_{\mathcal{C}_i}} \mathcal{C}_i
\end{array}
\]

determine chain morphisms $h'_i = \iota_{h_i}$ of the direct system $\mathcal{C} = (\mathcal{C}_0 \xrightarrow{h'_0} \mathcal{C}_1 \xrightarrow{h'_1} \cdots)$. By Remark 3.1 it has a chain limit: $\lim_{\text{Ch}} \mathcal{C} = \mathcal{C}$, which is an infinite multi-cone:

\[
\mathcal{C} = \cdots \xrightarrow{h_2} \text{Cone}(\mathcal{C}_2[1] \xrightarrow{h_1} \text{Cone}(\mathcal{C}_1[1] \xrightarrow{h_0} \mathcal{C}_0)).
\]

The chain morphisms $\mathcal{C}_i \xrightarrow{h'_i} \mathcal{C}_{i+1}$ satisfy a relation $\mathcal{g}_i = \mathcal{g}_{i+1} h'_i$, so by Theorem 3.5 there exists a unique chain morphism $\mathcal{C} \xrightarrow{g} \mathcal{A}_0$ such that $\mathcal{g}_i = \mathcal{g}_i h'_i$. It is easy to show that $\mathcal{A}_i = \text{Cone}(\mathcal{C} \xrightarrow{g} \mathcal{A}_0)$, and $\mathcal{f}_0 = \iota_g$, hence $\text{Cone}(\mathcal{f}_0) \sim \mathcal{C}$.

\[\Box\]

It is easy to prove the analog of Theorem 3.5.
Theorem 3.9. For a complex $B$ and chain morphisms $A_i \xrightarrow{g_i} B$ such that $g_i \sim g_{i+1} f_i$, there exists a unique (up to homotopy) chain morphism $A_i \xrightarrow{g} B$ which forms commutative triangles

$$A_i \xrightarrow{g_i} A_{i+1} \xrightarrow{g} B, \quad g_i \sim g_{i+1} f_i.$$  

(3.7)

In order to complete the proof of Theorems 2.5 and 2.6, we need two simple propositions. The first one establishes a triangle inequality for homological orders of cones.

Proposition 3.10. If three chain morphisms form a commutative triangle

$$A \xrightarrow{f_{AB}} B \xrightarrow{f_{BC}} C, \quad f_{AC} \sim f_{BC} f_{AB}.$$  

(3.8)

then the homological orders of their cones satisfy the inequalities

$$|\text{Cone}(f_{AB})|_h \geq \min \left( |\text{Cone}(f_{AC})|_h, |\text{Cone}(f_{BC})|_h - 1 \right),$$

(3.9)

$$|\text{Cone}(f_{BC})|_h \geq \min \left( |\text{Cone}(f_{AB})|_h + 1, |\text{Cone}(f_{AC})|_h \right).$$

(3.10)

Proof. If chain morphisms form a commutative triangle (3.8), then their cones form a distinguished triangle

$$\text{Cone}(f_{AB}) \xrightarrow{g_1} \text{Cone}(f_{AC}) \xrightarrow{g_2} \text{Cone}(f_{BC}) \xrightarrow{g_3} \text{Cone}(f_{AB})[1],$$

so the first inequality follows from the relation $\text{Cone}(f_{AB}) \sim \text{Cone}(g_2)[1]$ and the second inequality follows from the relation $\text{Cone}(f_{BC}) \sim \text{Cone}(g_1)$. \qed

The second proposition says that if a complex is homologically infinitely small then it is contractible.

Proposition 3.11. If $|A|_h = +\infty$ then $A$ is contractible.

Proof. Since $|A|_h = +\infty$, there exist complexes $A_i \sim A$, such that $A_i = O^b(m_i)$ and $\lim_{i \to \infty} m_i = +\infty$. Consider a sequence of chain morphisms establishing homotopy equivalence between the complexes:

$$A \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \equiv \cdots \equiv A_i \xrightarrow{f_i} A_{i+1} \equiv \cdots,$$  

where $1_{A_i}$ is the identity chain morphism of $A_i$, while $A_i[1] \xrightarrow{h_i} A_i$ is a homotopy chain morphism (it does not commute with the chain differential $d_i$ in the complex $A_i$).

Consider the compositions $\hat{f}_i = f_i \cdots f_1 f_0$, $\hat{g}_i = g_0 g_1 \cdots g_i$, and $\hat{h}_i = g_{i-1} h_i \hat{f}_i$. It is easy to see that $\hat{g}_i \hat{f}_i - \hat{g}_i \hat{f}_i = d h_i + h_i d_i$, hence $1_A - \hat{g}_i \hat{f}_i = d h_i + h_i d_i$, where $h_i =$
\( h_0 + h_1 + \cdots + h_i \). There is a limit (cf. Definition 3.6) \( \lim_{i \to \infty} h_i = h \), while \( \lim_{i \to \infty} g_i f_i = 0 \), hence \( 1_A = d h + h d \) which means that the complex \( A \) is contractible. \( \square \)

**Proposition 3.12.** If a direct system \( A \) has a limit, then it is Cauchy.

**Proof.** The inequality (3.9) applied to the commutative triangle (2.22) says that
\[
|\text{Cone}(f_i)|_h \geq \min \left( |\text{Cone}(\tilde{f}_i)|_h, |\text{Cone}(\tilde{f}_{i+1})|_h - 1 \right),
\]
hence the limit \( \lim_{i \to \infty} |\text{Cone}(\tilde{f}_i)|_h = +\infty \) implies the Cauchy property of \( A \).

**Proposition 3.13.** If a direct system \( A \) has a limit then it is unique.

**Proof.** If \( A \) has a limit then by Proposition 3.12 it is Cauchy. Hence it has a special limit \( A^*_z \) described in the proof of Proposition 3.7. If \( A \) has another limit \( A' \) with chain morphisms \( A_i \overset{f_i}{\to} A' \) then by Theorem 3.9 there is a chain morphism \( A^*_z \overset{g}{\to} A' \) with commutative triangles (3.7). The inequality (3.10) says
\[
|\text{Cone}(g)|_h \geq \min \left( |\text{Cone}(\tilde{f}_i)|_h + 1, |\text{Cone}(g_i)|_h \right).
\]
Since both cones in the r.h.s. become homologically infinitely small at \( i \to +\infty \), the cone \( \text{Cone}(g) \) is also homologically infinitely small. Then Proposition 3.11 says that \( \text{Cone}(g) \) is contractible and as a result \( A' \sim A^*_z \). \( \square \)

We end this section with a theorem which follows easily from Definition 2.4.

**Theorem 3.14.** If a direct system \( A \) satisfies the property \( \lim_{i \to \infty} |A_i|_h = +\infty \) then its limit is contractible: \( \lim_{\to} A = 0 \).

4. A Direct System of Categorification Complexes of Torus Braids

4.1. A special categorification complex of a negative braid. Let \( \sigma_i \) denote an elementary negative \( n \)-strand braid:
\[
\sigma_i = \begin{array}{c}
\vdots \\
1 & \cdot & \cdot & \cdots & i & i+1 & \cdots & n
\end{array}
\]

**Theorem 4.1.** If an \( n \)-strand braid \( \beta \) can be presented as a product of elementary negative braids: \( \beta = \sigma_{i_k} \cdots \sigma_{i_2} \sigma_{i_1} \), then its categorification complex has a special presentation \( \langle\langle \beta \rangle\rangle \):
\[
\langle\langle \beta \rangle\rangle^s = \left( \ldots \to C_{-2} \to C_{-1} \to \langle\langle \vdots : n \rangle\rangle \right)
\]
such that the complex
\[
C = (\ldots \to C_{-2} \to C_{-1}) [1, -1]
\]
is 1-cut and angle-shaped.

More abstractly, the theorem says that there exists a 1-cut and angle-shaped complex $C$ and a chain morphism $C \to \langle \langle \vdots \ n \rangle \rangle$ such that $\langle \langle \beta \rangle \rangle_s^\times \sim \text{Cone} \left( C[1] \to \langle \langle \vdots \ n \rangle \rangle \right)$.

**Remark 4.2.** Theorem 4.1 implies that the special complex $\langle \langle \beta \rangle \rangle_s^\times$ is angle-shaped.

**Proof of Theorem 4.1.** Let $\lambda$ be a TL $(n, n)$-tangle. Fix $i$ such that $1 \leq i \leq n - 1$. If the composition $\circ^i \lambda$ does not contain a disjoint circle, then, in accordance with eq. (2.13), we define the special categorification complex of $\circ^i \lambda$ as

$$\langle \langle \circ^i \lambda \rangle \rangle^\times_s = \left( \langle \langle U_{n, i} \circ \lambda \rangle \rangle [-1, 1, 1] \to \langle \langle \lambda \rangle \rangle \right)$$

(4.3)

If $\circ^i \lambda$ contains a disjoint circle, then $\lambda$ must have the form $\circ^i \lambda$. Hence $\circ \lambda = \circ \lambda$. The tangle $\circ \lambda$ is the same as $\circ \lambda$ with a positive framing twist, so according to eq. (2.14), $\langle \langle \circ \lambda \rangle \rangle = \langle \langle \lambda \rangle \rangle_{[-1, 3, 1]}$. Hence in this case we define the special categorification complex of $\circ \lambda$ simply as shifted $\langle \langle \lambda \rangle \rangle$:

$$\langle \langle \circ \lambda \rangle \rangle^\times_s = \langle \langle \lambda \rangle \rangle_{[-1, 2, 0]}.$$  

(4.4)

Now we define a recursive algorithm for constructing the complex $\langle \langle \beta \rangle \rangle^\times_s$. For $\beta = \circ n$ we define $\langle \langle \beta \rangle \rangle^\times_s = \langle \langle \vdots \ n \rangle \rangle$. Let $\beta = \circ \circ \cdots \circ \circ$ and suppose that we have defined its special complex $\langle \langle \beta \rangle \rangle^\times_s$. We define the special categorification complex of a braid $\beta' = \circ \circ$ by applying the rules (4.3) and (4.4) to all constituent tangles $\lambda$ in the complex $\langle \langle \beta \rangle \rangle^\times_s$ (see the formula (2.15)).

We prove the properties of $\langle \langle \beta \rangle \rangle^\times_s$ by induction over $k$. If $k = 0$ then $\beta = \circ n$ and the properties of $\langle \langle \beta \rangle \rangle^\times_s$ are obvious.

Suppose that the special categorification complex $\langle \langle \beta \rangle \rangle^\times_s$ of a braid $\beta = \circ \circ \cdots \circ \circ$ has the form (4.1) and its tail (4.2) is 1-cut and angle-shaped. Consider a longer braid $\beta' = \circ \circ \circ$. The object $\langle \langle \vdots \ n \rangle \rangle$ may appear in $\langle \langle \beta' \rangle \rangle^\times_s$ if and only if $\lambda = \circ \circ n$ and the extra crossing $\circ \circ$ is negatively spliced in eq. (4.3), hence $\langle \langle \beta' \rangle \rangle^\times_s$ has the form (4.1) and its tail (4.2) is 1-cut.

If the negative crossing $\circ \circ$ is composed with the head $\langle \langle \vdots \ n \rangle \rangle$ of the complex (4.1), then the formula (4.3) applies and the tangle $U_{n, ik+1}$ appearing in the tail of $\langle \langle \beta' \rangle \rangle^\times_s$ satisfies the property (2.25).
If the crossing $\sigma_{i,k+1}$ is composed with a TL tangle $\lambda$ from the $(-i)$-th chain ‘module’ $C_{-i}$ (see eq. (2.15)) in the tail of the complex $\langle \langle \beta \rangle \rangle_s$ with the $q$-degree shift $j$ satisfying the inequality $i - 1 \leq j - 1 \leq 2(i - 1)$, then the shifted objects in the r.h.s. of eqs. (4.3) and (4.4) also satisfy this inequality. \hfill \Box

The picture (2.19) presents a torus braid as a product of negative crossings, hence

**Corollary 4.3.** A torus braid $m$ has a special angle-shaped categorification complex $\langle \langle \mathcal{X} \mathcal{X}^m n \rangle \rangle_s$. In particular, for $m = 1$

$$\langle \langle \mathcal{X} \mathcal{X}^m n \rangle \rangle_s = \text{Cone} \left( C_{1,n} [1]_q \rightarrow \langle \langle \mathcal{X} \rangle \rangle_s \right),$$  \hfill (4.5)

where the complex $C_{1,n}$ is 1-cut and angle-shaped.

### 4.2. Special morphisms between torus braid complexes.

Relation (4.5) indicates that there is a distinguished triangle

$$C_{1,n} [1]_q \rightarrow \langle \langle \mathcal{X} \rangle \rangle_s \xrightarrow{f_1} \langle \langle \mathcal{X} \mathcal{X}^m n \rangle \rangle_s \rightarrow C_{1,n} [-1, 1]$$

and

$$\text{Cone}(f_1) \sim C_{1,n} [-1, 1].$$  \hfill (4.6)

Composing both sides of the morphism $f_1$ with the torus braid complex $\langle \langle \mathcal{X} \mathcal{X}^m n \rangle \rangle_s$, we get a morphism

$$\langle \langle \mathcal{X} \mathcal{X}^m n \rangle \rangle_s \xrightarrow{f_m} \langle \langle \mathcal{X} \mathcal{X}^m n \rangle \rangle_s$$

such that

$$\text{Cone}(f_m) \sim \text{Cone}(f_1) \circ \langle \langle \mathcal{X} \mathcal{X}^m n \rangle \rangle_s.$$  \hfill (4.7)

**Theorem 4.4.** The cone (4.7) can be presented by a shifted complex

$$\text{Cone}(f_m) \sim C_{m,n} [-n + 1, n]^{2m} [-1, 1],$$

such that $C_{m,n}$ is 1-cut and angle-shaped.

The proof is based on a simple geometric lemma:

**Lemma 4.5.** For $n \geq 2$, the following two compositions of framed tangles are isotopic:

$$\mathcal{C}_n \circ \mathcal{X} \mathcal{X}^i n = \mathcal{X} \mathcal{X}^{n-2} \circ 2 \mathcal{C}_n$$  \hfill (4.8)
where \( \bigcirc n \) is the tangle \( \bigcirc n \) with double framing twist on the cap:

\[
k \bigcirc n = \bigcirc 1 \cdots \bigcirc i \bigcirc i+1 \cdots \bigcirc n
\]

**Proof.** This lemma is geometrically obvious: a cap on a pair of adjacent strands slides down through the torus braid to the bottom. \( \square \)

An immediate corollary of eq. (4.8) and of the framing change formula (2.14) is the following relation:

\[
\langle \bigcirc n \rangle^{m} \sim \langle \bigcirc n \bigcirc n-2 \bigcirc \bigcirc n \rangle^{-s} [ s - n + 1, n ]^{2m}.
\]

(4.9)

In order to prove Theorem 4.4, we need three simple propositions. For a positive integer \( d \leq \frac{n}{2} \), let \( I = (i_1, \ldots, i_d) \) be a sequence of positive integer numbers such that \( i_k < n - 2k + 2 \) for all \( k \in \{1, \ldots, d\} \). A cap-tangle \( \bigcirc n \) is a \( (n, n - 2d) \)-tangle which can be presented as a product of \( d \) tangles of the form \( \bigcirc m \):

\[
\bigcirc n = i_d \bigcirc n-2d+2 \cdots \bigcirc i_2 \bigcirc n-2 \bigcirc i_1 \bigcirc n.
\]

A cup-tangle \( \bigcirc n \) is defined similarly:

\[
\bigcirc n = i_1 \bigcirc n-2 \cdots \bigcirc i_d \bigcirc n-2d+2.
\]

The first proposition is obvious:

**Proposition 4.6.** Every TL \( (n, n) \)-tangle \( \lambda \) has a presentation

\[
\lambda = n \bigcirc \bigcirc n, \quad |I| = |I'|.
\]

(4.10)

The number \( d_\lambda = |I| = |I'| \) is determined by the tangle \( \lambda \) and we call it the cap-degree (or cup-degree) of \( \lambda \).

The second proposition is also obvious:

**Proposition 4.7.** If at least one of two complexes \( C_1 \) and \( C_2 \) in \( \text{TL}_n \) is 1-cut then their composition \( C_1 \circ C_2 \) is 1-cut.

Note that even if both complexes are angle-shaped, then their composition is not necessarily angle-shaped. Indeed, in contrast to the homological degree, the \( q \)-degree is not additive with respect to the composition of tangles: if the composition of two TL tangles contains a disjoint circle then the \( q \)-degree shifts of the rule (2.16) violate additivity. However, if the
upper tangle in the composition has no caps or the lower tangle has no cups then no circles are created and the angle shape is maintained:

**Proposition 4.8.** If a complex \( C \) in \( TL_{n-2d_\lambda} \) is angle-shaped, then the complexes \( \langle \langle \tilde{I} \rangle \rangle \circ C \) and \( C \circ \langle \langle \tilde{I} \rangle \rangle \) are also angle-shaped.

**Proof of Theorem 4.4.** In order to construct the 1-cut and angle-shaped complex \( C_{m,n} \), we use the presentation

\[
\text{Cone}(\mathfrak{f}_m) \sim C_{1,n} \circ \langle \langle X_n \rangle \rangle^s [-1, 1],
\]

which follows from eqs. (4.7) and (4.6). We construct \( C_{m,n} \) by simplifying the complexes \( \langle \langle \lambda \circ X_n \rangle \rangle^s \) for TL \((n, n)\)-tangles \( \lambda \) appearing in the chain ‘modules’ of \( C_{1,n} \), with the help of the relation (4.9), thus creating necessary degree shifts, and then using Corollary 4.3 which says that emerging torus braids have angle-shaped categorification complexes.

Let \( \langle \langle \lambda \rangle \rangle [-i, j] \) be an object appearing in the \((-i)\)-th chain ‘module’ of \( C_{1,n} \) with a non-zero multiplicity (we made its homological degree explicit by including \(-i\) in the shift). We apply eq. (4.9) consequently to every cap \( \tilde{k}_n \) appearing in the cap-tangle \( \tilde{I} \langle \langle \lambda \rangle \rangle \langle \langle \tilde{I} \rangle \rangle \langle \langle \tilde{I} \rangle \rangle \) in the presentation (4.11) of \( \lambda \):

\[
\langle \langle \lambda \rangle \rangle [-i, j] \circ \langle \langle X_n \rangle \rangle^s \sim \left( \langle \langle \tilde{I} \rangle \rangle \circ \langle \langle X_n \rangle \rangle^s \circ \langle \langle \tilde{I} \rangle \rangle \right) [-b_\lambda, a_\lambda]^{2m} [-i, j] \sim \langle \langle \tilde{I} \rangle \rangle \langle \langle X_n \rangle \rangle^s [\sim -b_\lambda, a_\lambda]^{2m} [-i, j],
\]

where

\[
a_\lambda = \sum_{k=1}^{d_\lambda-1} (n - 2k), \quad b_\lambda = \sum_{k=1}^{d_\lambda-1} (n - 2k - 1).
\]

The object \( \langle \langle \lambda \rangle \rangle \) comes from the 1-cut complex \( C_{1,n} \), hence \( d_\lambda > 0 \) and the complex in big brackets in the r.h.s. of eq. (4.12) is 1-cut in view of Proposition 4.7. Proposition 4.8 implies that the complex \( \langle \langle \tilde{I} \rangle \rangle \circ \langle \langle X_n \rangle \rangle^s \circ \langle \langle \tilde{I} \rangle \rangle \) is also angle-shaped. Since \( \langle \langle \lambda \rangle \rangle \) comes from the angle-shaped complex \( C_{1,n} \), the numbers \( i \) and \( j \) satisfy inequalities \( i \geq 0 \) and \( i \leq j \leq 2i \). It is easy to check that the numbers \( a_\lambda \) and \( b_\lambda \) of eq. (4.13) satisfy the same inequalities: \( b_\lambda \geq 0, b_\lambda \leq a_\lambda \leq 2b_\lambda \), hence the complex in big brackets in the r.h.s. of eq. (4.12) is also angle-shaped. The complex \( C_{1,n} \circ \langle \langle X_n \rangle \rangle^s \) in the r.h.s. of eq. (4.11) is composed of complexes (4.12), hence Theorem 4.4 is proved. \( \square \)
5. A categorified Jones-Wenzl projector

Consider the direct system (2.24). Theorem 4.4 implies that $|\text{Cone}(f_m)|_h \geq 2m(n-1)+1$, hence $B_n$ is Cauchy and it has a unique limit $\lim_{\rightarrow} B_n = P_n^- \in TL_n^-$. 

Now we prove Theorems 2.7 and Theorem 2.8 which describe the properties of $P_n^-$. 

**Proof of Theorem 2.8** Consider the direct system (2.24) truncated from below: 

$$B_{m,n} = \left( \langle \bigotimes_{n}^{m} \right) s \frac{f_m}{\langle \bigotimes_{n}^{m+1} \rangle s} \frac{f_{m+1}}{\cdots} \longrightarrow P_n^-.$$ 

According to Theorem 3.8, the limit $P_n^-$ can be presented as a cone (2.26), where $\tilde{C}_{m,n} = \tilde{C}_{m,n}[-n+1,n]^{2m}$ and $\tilde{C}_{m,n}$ is an infinite multi-cone:

$$C_{m,n} = \cdots \rightarrow \text{Cone}(C_{m+k,n}[-2k(n-1)+1,2kn]) \rightarrow \cdots$$

$$\cdots \rightarrow \text{Cone}(C_{m+1,n}[-2n+3,2n] \rightarrow C_{m,n}))$$

with 1-cut and angle-shaped complexes $C_{m,n}$ introduced in Theorem 4.4. Hence the complex $\tilde{C}_{m,n}$ itself is 1-cut and angle-shaped. 

**Remark 5.1.** The contractibility of $P_n^- \circ \langle \bigotimes_{n}^{i} \rangle$ is proved similarly. 

**Corollary 5.2.** If $C$ is a 1-cut complex in $TL_n^-$, then $C \circ P_n^-$ is contractible.
Proof of part 2 of Theorem 2.7. According to eq. (2.27),

\[ P_n - n \circ P_n - n \sim \text{Cone} \left( \hat{C}_{0,n} [1]_q \longrightarrow \left\langle \begin{array}{c} n \\ \end{array} \right\rangle \right) \circ P_n^- \]

\[ \sim \text{Cone} \left( \hat{C}_{0,n} \circ P_n^- [1]_q \longrightarrow \left\langle \begin{array}{c} n \\ \end{array} \right\rangle \circ P_n^- \right) \sim P_n^-, \]

where we used the fact that \( \hat{C}_{0,n} \) is 1-cut and Corollary 5.2 in order to establish the last homotopy equivalence.

\[ \square \]

Proof of Theorem 2.2. The complexes \( P_n^-, \hat{C}_{m,n} \) and \( \left\langle \begin{array}{c} m \\ \end{array} \right\rangle \) in eq. (2.26) are angle-shaped, hence they are \( q^+ \)-bounded and their \( K_0 \) images are well-defined. Applying \( K_0 \) to this equation and taking into account eq. (2.28) and the definition (2.23), we find

\[ P_n = q^{\frac{1}{2} mn(n-1)} \left\langle \begin{array}{c} m \\ \end{array} \right\rangle - q^{2mn+1} K_0(\hat{C}_{m,n}). \]

The complex \( \hat{C}_{m,n} \) is angle-shaped, so \( \left| K_0(\hat{C}_{m,n}) \right|_q \geq 0 \) and by Definition 2.1 there is a limit (2.20).

6. THE OTHER PROJECTOR

A dual of an \((m,n)\)-tangle \( \tau \) is the \((n,m)\)-tangle tangle \( \tau^\vee \) which is its mirror image. Duality extends to an isomorphism \( TL \overset{\vee}{\rightarrow} TL^{\text{op}} \) combined with the isomorphism of the ground ring \( \mathbb{Z}[q,q^{-1}] \overset{\vee}{\rightarrow} \mathbb{Z}[q,q^{-1}] \), such that \( q^\vee = q^{-1} \). Furthermore, duality establishes an isomorphism \( TL^+ \overset{\vee}{\rightarrow} (TL^-)^{\text{op}} \), where \( TL^- \) is the analog of \( TL^+ \) constructed over the ring \( \mathbb{Z}[[q^{-1},q]] \) of Laurent series in \( q^{-1} \).

Since the relations (2.6) and (2.7) are dual to each other, while the idempotency condition \( P_n \circ P_n = P_n \) is duality invariant, the uniqueness of the Jones-Wenzl projector implies that it is duality invariant: \( P_n^\vee = P_n \). Hence the corresponding idempotents \( P_n^+ \in TL^+ \) and \( P_n^- \in TL^- \) are also dual to each other: \( P_n^- = (P_n^+)^\vee \). Taking the dual of eq. (2.20) we find that \( P_n^- \) is the limit of torus braids with high positive (that is, counterclockwise) twist:

\[ \lim_{m \to +\infty} q^{-\frac{1}{2} mn(n-1)} \left\langle \begin{array}{c} m \\ \end{array} \right\rangle = P_n^-, \]

which is motivated by (6.1)

Duality extends to a contravariant equivalence functor \( TL \overset{\vee}{\rightarrow} TL^{\text{op}} \), where \( TL^{\text{op}} \) is the same category as \( TL \), except that the composition of tangles is performed in reversed order. The
AN INFINITE TORUS BRAID AND THE JONES-WENZL PROJECTOR

functor ∨ also switches the signs of all three gradings of TL. Applying the duality functor to the construction of \( P_n^- \) we find that there exists a direct system

\[
\mathcal{B}_n^\vee = \left( \left\langle \tilde{\cdots} \hspace{1cm} n \right\rangle \leftarrow \left\langle 1 \hspace{1cm} \cdots \hspace{1cm} n \right\rangle \leftarrow \left\langle 1 \hspace{1cm} \cdots \hspace{1cm} n \right\rangle^{-s} \leftarrow \left\langle 1 \hspace{1cm} \cdots \hspace{1cm} n \right\rangle \leftarrow \cdots \right),
\]

where \(-s\) denotes the grading shift which is opposite to \( (2.23) \). The system \((6.2)\) is dual to the system \((2.24)\) and it has an inductive limit \( \lim_{\leftarrow} \mathcal{B}_n^\vee = P_n^+ \), which satisfies projector properties

\[ P_n^+ \circ P_n^+ \sim P_n^+, \hspace{1cm} \left\langle \left\langle \hspace{1cm} n \right\rangle \left. \right. \left\langle \hspace{1cm} \cdots \hspace{1cm} n \right\rangle \right\rangle \circ P_n^+ \sim P_n^+ \circ \left\langle \left\langle \hspace{1cm} n \right\rangle \left. \right. \left\langle \hspace{1cm} \cdots \hspace{1cm} n \right\rangle \right\rangle \sim 0 \]

and has a presentation

\[
P_n^- \sim \text{Cone} \left( \tilde{C}_{m,n}^\vee \left[ n - 1, n \right]^{2m} [\cdots] q \rightarrow \left\langle 1 \hspace{1cm} \cdots \hspace{1cm} n \right\rangle^{-s} \right),
\]

where the complex \( \tilde{C}_{m,n} \) is 1-cut and angle-shaped. In particular, at \( m = 0 \) we get the dual of presentation \((2.27)\):

\[
P_n^+ \sim \text{Cone} \left( \tilde{C}_{0,n}^\vee \left[ -1 \right] q \rightarrow \left\langle \hspace{1cm} n \right\rangle \right),
\]

where the complex \( \tilde{C}_{0,n} \) is 1-cut and angle-shaped.

REFERENCES

[BN05] Dror Bar-Natan, Khovanov's homology for tangles and cobordisms, Geometry and Topology 9 (2005), 1443-1499, available at \texttt{arXiv:math.GT/0410495}

[CK] Ben Cooper and Slava Krushkal, Categorification of the Jones-Wenzl projectors, in preparation.

[FSS] Igor Frenkel, Catharina Stroppel, and Joshua Sussan. in preparation.

[Kho00] Mikhail Khovanov, A categorification of the Jones polynomial, Duke Journal of Mathematics 101 (2000), 359-426, available at \texttt{arXiv:math.QA/9908171}

[KR08] Mikhail Khovanov and Lev Rozansky, Matrix factorizations and link homology, Fundamenta Mathematicae 199 (2008), 1-91, available at \texttt{arXiv:math.QA/0401268}

[KR] ______. in preparation.

[Sto07] Marko Stosic, Homological thickness and stability of torus knots, Algebraic and Geometric Topology 7 (2007), 261-284, available at \texttt{arXiv:math.GT/0511532}

L. ROZANSKY, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH CAROLINA AT CHAPEL HILL, CB # 3250, PHILLIPS HALL, CHAPEL HILL, NC 27599

E-mail address: rozansky@math.unc.edu