Designing Universal Causal Deep Learning Models: The Case of Infinite-Dimensional Dynamical Systems from Stochastic Analysis

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Abstract Causal operators (CO), such as various solution operators to stochastic differential equations, play a central role in contemporary stochastic analysis; however, there is still no canonical framework for designing Deep Learning (DL) models capable of approximating COs. This paper proposes a “geometry-aware” solution to this open problem by introducing a DL model-design framework that takes suitable infinite-dimensional linear metric spaces as inputs and returns a universal sequential DL model adapted to these linear geometries. We call these models Causal Neural Operators (CNOs). Our main result states that the models produced by our framework can uniformly approximate on compact sets and across arbitrarily finite-time horizons Hölder or smooth trace class operators, which causally map sequences between given linear metric spaces. Our analysis uncovers new quantitative relationships on the latent state-space dimension of CNOs which even have new implications for (classical) finite-dimensional Recurrent Neural Networks (RNNs). We find that a linear increase of the CNO’s (or RNN’s) latent parameter space’s dimension and of its width, and a logarithmic increase of its depth imply an exponential increase in the number of time steps for which its approximation remains valid. A direct consequence of our analysis shows that RNNs can approximate causal functions using exponentially fewer parameters than ReLU networks.

Keywords Universal Approximation, Causality, Operator Learning.

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1 Introduction

Infinite-dimensional (non-linear) dynamical systems play a central role in several sciences, especially for disciplines driven by stochastic analytic modeling. However, despite this fact, the causal neural network approximation theory for most relevant dynamical systems in stochastic analysis remain largely misunderstood. Indeed, we currently only comprehend neural network approximations of Stochastic Differential Equations (SDEs) with deterministic coefficients (e.g., [33]) and time-invariant random dynamical systems with the fading memory and echo state property/unique solution property (e.g., [63,34]). A significant problem is causal neural network approximation of solution operators to non-Markovian SDEs.

Moreover, the understanding of how sequential deep learning models work is still not fully developed, even in the classical finite-dimensional setting. For instance, the seemingly elementary empirical fact that a sequential DL model’s expressiveness increases when one utilizes a high-dimensional latent state space is understood qualitatively for general dynamical systems on Euclidean spaces (as in the reservoir computing literature (e.g., [31])).
However, the quantitative understanding of the relationship between a sequential learning model’s state and its expressiveness remains an open problem. One notable exception to this fact is the approximation of linear state-space dynamical systems by a stylized class of Recurrent Neural Networks (RNNs, henceforth); see [45,61].

**Our contribution.** Our paper provides a simple quantitative solution to a far reaching generalization of the above problem of constructing neural network approximation of infinite-dimensional (generalized) dynamical systems on “good” linear metric spaces. More precisely, we construct a neural network approximation of any function \( f \) that “causally” and “regularly” maps sequences \((x_t)_{n=\infty}^{\infty}\) to sequences \((y_t)_{n=\infty}^{\infty}\), where each \(x_t\) and every \(y_t\) lives in a suitable linear metric space. In particular, we construct our causal neural network approximation framework on the following desiderata:

1. **(D1)** Predictions are causal, i.e., each \(y_t\) is predicted independently of \((x_m)_{m>n}\).
2. **(D2)** Each \(y_t\) is predicted with a small neural network specialized at time \(t_n\).
3. **(D3)** Only one of these specialized networks is stored in working memory at a time.

We first begin by describing our causal neural network model’s design. Subsequently, we will discuss our approximation theory’s implications in computational stochastic analysis.

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**Fig. 1:** The Causal Neural Operator Model:

**Summary:** An efficient universal approximator of causal sequences of operators between well-behaved Fréchet spaces.

**Overview:** The model successively applies a “universal” neural filter (see Figure 2) on consecutive time-windows; the internal parameters of this neural filter are evolve according to a latent dynamical system on the neural filter’s parameter space; implemented by a deep ReLU network called a hypernetwork.

Our neural network model, which we call the **Causal Neural Operator** (CNO, henceforth) is illustrated in Figure 1 and works in the following way. At any given time \(t_n\), it predicts an instance of the output time-series at that time \(t_n\) using an immediate time-window from the input time-series (e.g., it predicts each \(y_t\) using only \((x_i)_{i=n-10}\)). At each time \(t_n\), this prediction is generated by a non-linear operator defined by a finitely parameterized neural network model, called a **neural filter** (the vertical black arrows in Figure 1). Our neural network model stores only one neural filter’s parameters in working memory at the current time by using an auxiliary ReLU neural network, called a **hypernetwork** in the machine learning literature (e.g., [37,80]), to generate the next neural filter specialized at \(t_{n+1}\) using only the parameters of the current “active” neural filter specialized at time \(t_n\) (the blue box in Figure 1). Thus, a dynamical system (i.e., the hypernetwork) on the neural filter’s parameter space interpolating between each neural filter’s parameters encodes our entire model.

The principal approximation-theoretic advantage of this approach lies in the fact that the hypernetwork is not designed to approximate anything, but rather, it only needs to memorize/interpolate a finite number of finite-dimensional (parameter) vectors. Since memorization (e.g., [79,54]) requires only a polynomial number of parameters, while approximation [84,55,85,56] requires an exponential number of them, then, if memorization can be leveraged, the constructed neural network model is exponentially more efficient. Thus, this neural network design allows us to successfully encode all the parameters required to approximate long stretches of time \(\{t_0, \ldots, t_N\}\) (for large \(N\)) with far fewer parameters (i.e., at the cost of \(O(\log(N))\) additional layers in the hypernetwork). Thus, we successfully achieve desiderata (D1)–(D3) provided that each neural filter relies on only a small number of
parameters. We show that this is the case whenever $f$ is “sufficiently smooth”; the rigorous formulation of all these outlined ideas are expressed in Lemma 5 and Theorem 2.

![Neural Filter](image)

**Fig. 2:** The Neural Filter  

**Summary:** An efficient universal approximator between any well-behaved Fréchet spaces.

**Overview:** The neural filter first encodes inputs from a (possibly infinite-dimensional) linear space by approximate representing the input as coefficients of an sparse (Schauder) basis. These basis coefficients are then transformed by a deep ReLU network and the network’s outputs are decoded by into coefficients of a sparse basis representation of an element of the output linear space. Assembling the basis using the outputted coefficients produces the neural filter’s output.

Though we are focused on the approximation theoretic properties of our modeling framework, we have designed our CNO by considering practical considerations. Namely, we intentionally designed the CNO model so that, like transformer networks [78], it can be trained non-recursively (via our federated training algorithm, see Algorithm 1 below). This design choice is motivated by the main reasons why the transformer network model (e.g., [78]) has replaced residual (e.g., [39]) and RNN (especially Long Short-Term Memory (LSTMs, henceforth) [41]) counterparts in practice (e.g., [42,82]); namely, not back-propagating through time during training. The reason is that omitting any recurrence relation between a model’s prediction in sequential prediction tasks, at-least during the model’s construction, has been empirically confirmed to yield more reliable and accurate models trained faster and without vanishing or exploding gradient problems; see, e.g., [40,70]. Nevertheless, our model does ultimately reap the benefits of recursive models even if we construct it non-recursively, using our parallelizable training procedure.

The neural filter, illustrated in Figure 2, is a neural operator with quantitative universal approximation guarantees far beyond the Hilbert space setting. It works by first encoding infinite-dimensional problems into finite-dimensions problems, as the Fourier Neural Operator (FNO, henceforth) of [60], using a predetermined truncated Schauder basis. It then predicts outputs by passing the truncated basis coefficients through a feed-forward neural network with trainable (P)ReLU activation function. Finally, it reassembles them in the output space by interpreting that network’s outputs as the coefficients of a pre-specified Schauder basis or if both spaces are reproducing kernel Hilbert spaces then the first few basis functions can learned from data using principal component analysis\(^1\), e.g. as with PCA–Net [58]. Similarly, instead complete a set of parameterized function (learned linearly independent) learned during training, as is implicitly the case with the DeepONet architecture [62]; which are universal when inputs and outputs belong to certain Sobolev Spaces in which the parameterized (neural network) functions are universal approximators [44].

Our “static” efficient approximation theorems provides quantitative approximation guarantees for several “neural operators” used in practice, especially in the numerical Partial Differential Equations (PDEs) (e.g., [49]) and the inverse-problem literature (e.g., [2,15,3,16,22]). Notable examples are the FNO (see [53] for a qualitative universal result and quantitative guarantees for approximation of the Darcy flow and incompressible Navier-Stokes PDEs [53, Theorem 26 and 32], respectively), the wavelet neural operator recently introduced in the numerical PDE literature ([77]), and several other neural operators who now have quantitative universal approximation guarantees thanks to our “static” universal approximation result, when the target map is assumed to be sufficiently regular (for a more

\(^{1}\) Or a robust version thereof, e.g. [30] and then normalizing and orthogonalizing via Gram-Schmidt.
precise statement refer to Definition 5 and 6). The same argument is valid also for the general qualitative theorems of [75,11].

We now describe more in detail the different areas in which the present paper contributes.

**Our contribution in the Approximation Theory of Neural Operators.** Our results provide the first set of quantitative approximation guarantees for generalized dynamical systems evolving on general infinite-dimensional spaces. By refining the memorizing hypernetwork argument of [1], together with our general solution to the static universal approximation problem, in the class of Hölder functions, we are able to confirm a well-known folklore approximation of dynamical systems literature. Namely, that increasing a sequential neural operator’s latent space’s dimension by a positive integer $Q$ and our neural network’s depth by $\tilde{O}(T^{-Q} \log(T^{-Q}))$ and width by $\tilde{O}(QT^{-Q})$ implies that we may approximate $O(T)$ more time-steps in the future with the same prescribed approximation error.

To the best of our knowledge, our dynamic result is the only quantitative universal approximation theorem guaranteeing that a recurrent neural network model can approximate any suitably regular infinite-dimensional non-linear dynamical systems. Likewise, our static result is to the best of our knowledge the only general infinite-dimensional guarantee showing that a neural operator enjoys favourable approximation rates when the target map is smooth enough.

**Our contribution in the Approximation Theory of RNNs** In the finite-dimensional context, CNOs become strict sub-structures of full RNNs, where the internal parameters are updated/generated via an auxiliary hypernetwork. Noticing this structural inclusion, our results rigorously justify the folklore that RNNs can approximate causal maps strictly more efficiently than feedforward neural network (FFNN, henceforth), see Section 5.

**Technical contributions:** Our results apply to sequences of non-linear operators between any “good linear” metric spaces. By “good linear” metric space we mean any Fréchet space admitting Schauder basis. This includes many natural examples (e.g., the sequence space $\mathbb{R}^\infty$ with its usual metric) outside the scope of the Banach, Hilbert spaces, we note that these conditions are automatically satisfied in that setting.

**Organization of our paper** This research project answers theoretical deep learning questions by combining tools from approximation theory, functional analysis, and stochastic analysis. Therefore, we provide a concise exposition of each of the relevant tools from these areas in our “preliminaries” Section 2.

Section 3 contains our quantitative universal approximation theorems. In the static case, we derive expression rates for the static component of our model, namely the neural filters, which depend on the regularity of the target operator being approximated; from Hölder trace-class to smooth trace-class and on the usual quantities. Our main approximation theorem in the dynamic case additionally encodes the target causal map’s memory decay rate.

Section 4 applies our main results to derive approximation guarantees for the solution operators of a broad range of SDEs with stochastic coefficients, possibly having jumps (“stochastic discontinuities”) at times on a pre-specified time-grid and with initial random noise. Section 5, examines the implication of our approximation rates for RNNs, in the finite-dimensional setting, where we find that RNNs are strictly more efficient than FFNN when approximating causal maps. Section 6 concludes. Finally, Appendix A contains any background material required in the derivations of our main results whose derivations are relegated to Appendix B.

1.1 Notation

For the sake of the reader, we collect and define here the notations we will use in the rest of the paper, or we indicate the exact point where the first appearance of a symbol occurs:

1. $\mathbb{N}_+ := \{1, 2, 3, \cdots\}$.
2. $\mathbb{N} := \{0, 1, 2, 3, \cdots\}$.
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4. Given two topological vector spaces \((E, \sigma)\) and \((F, \tau)\), \(L(E, F)\) denotes the space of continuous linear operators from \(E\) into \(F\); if \(E = F\), then we will write \(L(E) = L(E, E)\).

5. Given a Fréchet space \(F\), we use \((\cdot, \cdot)\) to denote the canonical pairing of \(F\) with its topological dual \(F'\).

6. We denote the open ball of radius \(r > 0\) about a point \(x\) in a metric space \((X, d)\) by \(\text{Ball}_{(X, d)}(x, r) \defeq \{ u \in X : d(x, u) < r \}\).

7. We denote the closure of a set \(A\) in a metric space \((X, d)\) by \(\overline{A}\).

8. \(P, p_k\): 2.1

9. \(\Phi\): (2)

10. \(p_k^F\) with \(F =\) Fréchet space: (7)

11. \(d_{F, n}\) with \(F =\) Fréchet space: (8)

12. \([d], P([d]): 2.3\)

13. \(P_{F,n}, I_{F,n}\) with \(F =\) Fréchet space: (12) and (13)

14. \(C_{tr}^k(K, B)\) and \(C_{\alpha tr}^k(K, B): 5\) and 6

15. \(\psi_n\) and \(\varphi_n\): (15) (16)

16. The canonical projection onto the \(n^{th}\) coordinate of an \(x \in \prod_{n \in \mathbb{Z}} X_n\) is denoted by \(x_n\); where each \(X_n\) is an arbitrary non-empty set.

In particular, if \(f : A \to \prod_{n \in \mathbb{Z}} X_n\), with \(A\) an arbitrary non-empty set, then \(f(x)_n\) denotes the projection of \(f(x) \in \prod_{n \in \mathbb{Z}} X_n\) onto the \(n^{th}\) coordinate,

17. \(\mathcal{N}_{F[n]}^{[P]\text{ReLU}}\): The set of neural filters from \(B\) to \(E\).

18. \(V\): the “special function”, defined as the inverse of the map \({}^6 u \mapsto u^4 \log_4(u + 2)\) on \([0, \infty)\).

2 Preliminaries

In this section, we remind some preparatory material for the derivations of the main results of this paper. Finally, we remark that the notation in each of the subsequent subsections is self-contained and it is the one used on the cited paper: it will be up to the reader to contextualize it in the next sections.

2.1 Fréchet spaces

The main references for this subsection are the following ones: [38], Part I; [19] Chapter IV; [73], Chapter III and the working paper of [12]; all the vector spaces we will deal with will be vector spaces over \(\mathbb{R}\). Before defining a Fréchet space, we remind that a locally convex topological vector space, say \((F, \tau)\), is a topological vector space whose topology \(\tau\) arises from a collection of seminorms \(\mathcal{P}\). When clear from the context, we will write \(F\) instead of \((F, \tau)\). The topology is Hausdorff if and only if for every \(x \in F\) with \(x \neq 0\) there exists a \(p \in \mathcal{P}\) such that \(p(x) > 0\). On the other hand, the topology is metrizable if and only if it may be induced by a countable collection \(\mathcal{P} = \{p_k\}_{k \in \mathbb{N}_+}\) of seminorms, which we may assume to be increasing, namely \(p_k(\cdot) \leq p_{k+1}(\cdot), k \in \mathbb{N}_+\).

Definition 1 (Fréchet space) A Fréchet space \(F\) is a complete metrizable locally convex topological vector space.

Evidently, every Banach space \((F, \|\cdot\|_F)\) is a Fréchet space; in this case, simply \(\mathcal{P} = \{\|\cdot\|_F\}\).

A canonical choice for the metric on a Fréchet space \(F\) (that generates the pre-existing topology) is given by:

\[
d_F(x, y) \defeq \sum_{k=1}^{\infty} 2^{-k} \Phi(p_k(x - y)), \quad x, y \in F, \tag{1}
\]

where

\[
\Phi(t) \defeq \frac{t}{1 + t}, \quad t \geq 0. \tag{2}
\]

We now remind the concept of directional derivative of a function between two Fréchet spaces. This notion of differentiation is significantly weaker than the concept of the derivative of a function between two Banach spaces. Nevertheless, it is the weakest notion of differentiation for which many of the familiar theorems from calculus hold. In particular, the chain rule is true (cfr. [38]). Let \(F\) and \(G\) be Fréchet spaces, \(U\) an open subset of \(F\), and \(P : U \subseteq F \to G\) a continuous map.

Definition 2 (Directional Derivative) The derivative of \(P\) at the point \(x \in U\) in the direction \(h \in F\) is defined by:

\[
DP(x)h = \lim_{t \to 0} \frac{P(x + th) - P(x)}{t}. \tag{3}
\]

\(^6\) The map \(u \mapsto u^4 \log_4(u + 2)\) is a continuous and strictly increasing surjection of \([0, \infty)\) onto itself; whence, \(V\) is well-defined.
In particular, $P$ is said to be differentiable at $x$ in the direction $h$ if the previous limit exists. $P$ is said to be $C^1$ on $U$ if the limit in Equation (3) exists for all $x \in U$ and all $h \in F$, and $DP : (U \subseteq F) \times F \rightarrow G$ is continuous (jointly as a function on a subset of the product).

As anticipated, the Definition 2 of a $C^1$ map disagrees with the usual definition for a Banach space in the sense that the derivative will be the same map, but the continuity requirement is weaker. The previous definition can be generalized and applied to higher-order derivatives. For instance, if $P : U \subseteq F \rightarrow G$ we require $D^2P$ to be continuous jointly as a function on the product space

$$D^2P : (U \subseteq F) \times F \times F \rightarrow G.$$  

Similarly, the $k$-th derivative $D^kP(x)\{h_1,h_2,\ldots,h_k\}$ will be regarded as a map $D^kP : (U \subseteq F) \times F \times \ldots \times F \rightarrow G$. $P$ is of class $C^k$ on $U$ if $D^kP$ exists and is continuous. If $x$ is of class $C^k$ on $U$ then:

$$D^kP(x) \{h_1,h_2,\ldots,h_k\} \text{ will be regarded as a map}$$

Remark 1 We will say that $P$ is $C^k$-Dir if $P$ satisfies the previous definition.

Next, we introduce the concept of Schauder basis ([64]). Let $F$ be a Fréchet space. A sequence $(f_k)_{k \in \mathbb{N}_+} \subset F$ is called a Schauder basis if every $x \in F$ has a unique representation

$$x = \sum_{k=1}^{\infty} x_k f_k,$$

where the series converges in $F$ (in the ordinary sense). It is immediate to see from the definition that the maps

$$F \ni x \mapsto x_k, \quad k \in \mathbb{N}_+$$

are continuous linear functionals. We remark that if a Fréchet space admits a Schauder basis, it is separable. However, the converse does not hold in general; whether every separable Banach space has a basis appeared in 1931 for the first time in the Polish edition of Banach's book ([6]) and was solved in the negative by Enflo ([25]).

We now state and prove the following auxiliary lemma.

**Lemma 1** Let $F$ be a separable Fréchet space admitting a Schauder basis $(f_k)_{k \in \mathbb{N}_+}$, and $d_F$ a metric on $F$ compatible with the pre-existing topology (see Equation (1)). Fix $n \in \mathbb{N}_+$ and define on $\mathbb{R}^n$ the following metric:

$$d_{F,n}(x,y) \overset{\text{def.}}{=} d_F \left( \sum_{k=1}^{n} x_k f_k, \sum_{k=1}^{n} y_k f_k \right), \quad x,y \in \mathbb{R}^n.$$  

Then, the topology induced on $\mathbb{R}^n$ by this metric is the standard one.

**Proof** First, notice that $d_{F,n}$ is a metric on $F$. This follows directly from the fact that $d_F$ is a metric\footnote{The only non trivial thing to prove is the identity of indiscernibles, i.e. that $d_{F,n}(x,y) = 0 \iff x = y$. But this fact follows directly from the fact that $d_F$ is a metric and from the definition of Schauder basis $(f_k)_{k}$; see Subsection 2.1.}. Now, let $x^{(J)} \overset{\text{def.}}{=} (x_1^{(J)},\ldots,x_n^{(J)})$, $J \in \mathbb{N}$ and $x \overset{\text{def.}}{=} (x_1,\ldots,x_n)$ such that

$$x^{(J)} \overset{d_{F,n}}{\rightarrow} x.$$  

This means in particular that

$$d_F \left( \sum_{k=1}^{n} x_k^{(J)} f_k, \sum_{k=1}^{n} x_k f_k \right) \overset{J \rightarrow \infty}{\rightarrow} 0, \text{ i.e., } \sum_{k=1}^{n} x_k^{(J)} f_k \overset{F}{\rightarrow} \sum_{k=1}^{n} x_k f_k.$$  

Now, let $(\beta_k^F)_{k \leq n}$ be the unique sequence in the topological dual of $F$, say $F'$, such that each $f \in F$ has the following representation $f = \sum_{k=1}^{\infty} (\beta_k^F, f) f_k$. Because $(\beta_k^F)_{k \leq n}$ are continuous and linear, we clearly get that $x_k^{(J)} \overset{J \rightarrow \infty}{\rightarrow} x_k$ for each $k \in \{1,\ldots,n\}$. This implies that

$$\left[ \sum_{k=1}^{n} |x_k^{(J)} - x_k|^2 \right]^{1/2} \overset{J \rightarrow \infty}{\rightarrow} 0, \text{ i.e., } x^{(J)} \overset{\|\cdot\|}{\rightarrow} x.$$
Vice-versa, let \( x^{(j)} \defeq (x_1^{(j)}, \ldots, x_n^{(j)}) \) and \( x \defeq (x_1, \ldots, x_n) \) such that \( x^{(j)} \xrightarrow{\ell_2} x \). This implies that \( \sum_{k=1}^n |x_k^{(j)} - x_k| \xrightarrow{j \to \infty} 0 \). We pick an arbitrary continuous seminorm \( p \in \mathcal{P} \). It holds for all \((t_1, \ldots, t_n) \in \mathbb{R}^n\) that
\[
p \left( \sum_{k=1}^n t_k f_k \right) \leq \sum_{k=1}^n |t_k| p(f_k) \leq \max_{k=1, \ldots, n} p(f_k) \sum_{k=1}^n |t_k|.
\]
This shows that
\[
p \left( \sum_{k=1}^n x_k^{(j)} f_k - \sum_{k=1}^n x_k f_k \right) \xrightarrow{j \to \infty} 0
\]
for all \( p \in \mathcal{P} \). This means in particular that
\[
dF \left( \sum_{k=1}^n x_k^{(j)} f_k, \sum_{k=1}^n x_k f_k \right) \xrightarrow{j \to \infty} 0, \text{ i.e., } dF_n(x^{(j)}, x) \xrightarrow{j \to \infty} 0.
\]
Since the metric spaces \((\mathbb{R}^n, d_{F,n})\) and \((\mathbb{R}^n, \|\cdot\|_2)\) enjoy the same converging sequences, the topology must be the same.

2.2 Generalized inverses

[24] wrote a thorough paper about generalized inverses and their properties. Analogously to [24], we understand increasing in the weak sense, that is, \( T : \mathbb{R} \to \mathbb{R} \) is increasing if \( T(x) \leq T(y) \) for all \( x < y \). Also, we remind the notion of an inverse for such functions.

**Definition 3 (Generalized Inverse)** For an increasing function \( T : \mathbb{R} \to \mathbb{R} \) with \( T(-\infty) \defeq \lim_{x \downarrow -\infty} T(x) \) and \( T(\infty) \defeq \lim_{x \uparrow \infty} T(x) \), the generalized inverse \( T^- : \mathbb{R} \to [T(\infty), T(-\infty)] \) is defined by
\[
T^-(y) \defeq \inf \{ x \in \mathbb{R} : T(x) \geq y \}, \quad y \in \mathbb{R},
\]
with the convention that \( \inf \emptyset = \infty \).

To keep our manuscript self-contained, we list some properties of generalized inverses which can be found in ([24], cfr. Proposition 1). We denote the range of a map \( T : \mathbb{R} \to \mathbb{R} \) by \( T \defeq \{ T(x) : x \in \mathbb{R} \} \).

**Proposition 1 (Properties of Generalized Inverses)** Let \( T \) be as in Definition 3 and let \( x, y \in \mathbb{R} \). Then,

1. \( T^-(y) = -\infty \) if and only if \( T(x) \geq y \) for all \( x \in \mathbb{R} \). Similarly, \( T^+(y) = \infty \) if and only if \( T(x) < y \) for all \( x \in \mathbb{R} \).
2. \( T^- \) is increasing. If \( T^- \infty \in (\infty, \infty) \), \( T^- \) is left-continuous at \( y \) and admits a limit from the right at \( y \).
3. \( T^- (T(x)) \leq x \). If \( T \) is strictly increasing, \( T^- (T(x)) = x \).
4. \( T \) is right-continuous. Then \( T^-(y) < \infty \) implies \( T(T^- (y)) = y \). Furthermore, \( y \in \text{ran} T \cup \{ \text{inf ran} T, \text{sup ran} T \} \) implies \( T(T^- (y)) = y \). Moreover, if \( y < \text{inf ran} T \) then \( T(T^- (y)) > y \) and if \( y > \text{sup ran} T \) then \( T(T^- (y)) < y \).

2.3 Feedforward Neural Networks with ReLU and PReLU activation functions

We give the definition of feed-forward neural networks with ReLU activation function (ReLU FFNNs, henceforth) and with a trainable Parametric ReLU activation function (PReLU FFNNs, henceforth). Interestingly, Proposition 1 in [84] shows that using a ReLU activation function is not much different from using a PReLU activation function, in the sense that it is possible to replace a ReLU FFNN with a PReLU FFNN while only increasing the number of units and weights by constant factors. However, the main advantage of using a PReLU FFNN with respect to a ReLU FFNN is that the former can synchronize the depth of several functions realized by ReLU FFNNs, a fact that will be extremely important in the derivation of Theorem 2. In particular, a PReLU activation function is any map \( \sigma : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \), \( (\alpha, x) \to \sigma_{\alpha}(x) \defeq \max \{ x, \alpha x \} \); the parameter \( \alpha \) is called slope. Notice that for \( \alpha = 0 \) one obtains the ReLU activation function. As it is customary in the literature, in what follows we will often by applying the (P)ReLU activation function component-wise. More precisely, for any \( \alpha \in \mathbb{R} \) and an \( x \in \mathbb{R}^N \), \( N \in \mathbb{N}_+ \), we have
\[
\sigma_{\alpha} \cdot x \defeq (\sigma_{\alpha}(x_i))_{i=1}^N.
\]
Fix \( J \in \mathbb{N}_+ \) and a multi-index \( |d| \defeq (d_0, \ldots, d_J) \), and let \( P(|d|) \defeq J + \sum_{j=0}^{J-1} d_j (d_{j+1} + 1) + d_J \). Weights, biases, and slopes are identified in a unique parameter \( \theta \in \mathbb{R}^{P(|d|)} \) with
\[
\mathbb{R}^{P(|d|)} \ni \theta \iff ((A(j), b(j), \alpha(j))_{j=0}^{J-1}, c), \quad (A(j), b(j), \alpha(j)) \in \mathbb{R}^{d_{j+1} \times d_j} \times \mathbb{R}^{d_j} \times \mathbb{R}, \quad c \in \mathbb{R}^{d_J}.
\]
With the previous identification, the recursive representation function of a \([d]\)-dimensional deep feed-forward network is given by

\[
\mathbb{R}^{P([d])} \times \mathbb{R}^{d_0} \ni (\theta, x) \mapsto \hat{f}_\theta(x) \overset{\text{def.}}{=} x^{(J)} + c, \\
x^{(j+1)} \overset{\text{def.}}{=} A^{(j)} \sigma_{\alpha^{(j)}} \left( x^{(j)} + b^{(j)} \right) \quad \text{for } j = 0, \ldots, J - 1, \\
x^{(0)} \overset{\text{def.}}{=} A^{(0)} x.
\]

We will refer to \(J\) as \(\hat{f}_\theta\)'s depth. We will denote by \(\mathcal{N}^{[d]}_{[d]}\) a deep ReLU FFNN with complexity \([d]\).

### 3 Main Results

#### 3.1 Static Case: Efficient Universal Approximation

We begin by treating the “static case” wherein we show that CNO’s neural filters, illustrated in Figure 3, are universal approximators of (non-linear) Hölder class operators between “good” linear spaces. We note that the application of the CNO only requires us to customize its neural filters to the relevant input and outputs’ geometries.

We first fix our working setting for this section

(A1) Let \(N, M \in \mathbb{N}_+ \cup \{\infty\}\). Let \(E\) and \(B\) be two separable Fréchet spaces admitting Schauder bases \((e_h)_{h \leq N}\) and \((b_h)_{h \leq M}\). Let \(E'\) and \(B'\) be the topological dual of \(E\) and \(B\) respectively. Let \((\beta^E_h)_{h \leq N}\) (resp. \((\beta^B_h)_{h \leq M}\)) be the unique sequence in \(E'\) (resp. \(B'\)) such that each \(e \in E\) (resp. each \(b \in B\)) has the following representation

\[
e = \sum_{h=1}^{N} \langle \beta^E_h, e \rangle e_h, \quad \text{(resp. } b = \sum_{h=1}^{M} \langle \beta^B_h, b \rangle b_h),
\]

where \(\langle \cdot, \cdot \rangle\) is the canonical pairing between \(E'\) and \(E\) (resp. between \(B'\) and \(B\)). For each \(n \in \mathbb{N}_+\), we denote by \(P_{E,n} : (E, d_E) \to (\mathbb{R}^n, d_{E,n})\) the function defined as

\[
P_{E,n} : (E, d_E) \to (\mathbb{R}^n, d_{E,n}), \quad e \mapsto (\langle \beta^E_1, e \rangle, \langle \beta^E_2, e \rangle, \ldots, \langle \beta^E_n, e \rangle)^T,
\]

where \(d_{E,n}\) is the metric defined in Lemma 1. Moreover, \(I_{E,n} : (\mathbb{R}^n, d_{E,n}) \to (E, d_E)\) is the function defined as

\[
I_{E,n} : (\mathbb{R}^n, d_{E,n}) \to (E, d_E), \quad \beta \mapsto \sum_{h=1}^{n} \beta_h e_h.
\]

Analogous definitions hold for \(P_{B,n}\) and \(I_{B,n}\).

Before proceeding, we make the following trivial, yet useful remark
At this point, some remarks are in order. In general, the problem of identifying when a map belongs to \( C^k \) is a well-studied and independent area of research dating back to the beginning of the previous century (e.g., recently in a series of articles starting with [28]). The interested reader may consult [14] where the \( C^k \)-stability, \( k \in \mathbb{N} \), of a non-linear operator mapping from a Frechet space \( E \) to a Frechet space \( B \).

**Definition 4 (\( C^k \)-Stability)** Let \( E \) and \( B \) be two Frechet spaces. A (non-linear) operator \( f : E \to B \) is called \( C^k \)-stable if for every \( m, n \in \mathbb{N} \), and every pair of continuous and linear maps \( \tilde{I} : (\mathbb{R}^n, \| \cdot \|_2) \to (E, d_E) \) and \( \tilde{P} : (B, d_B) \to (\mathbb{R}^m, \| \cdot \|_2) \) the following composition

\[
\tilde{P} \circ f \circ \tilde{I} : \mathbb{R}^n \to \mathbb{R}^m,
\]

is of class \( C^k \) in the usual sense.

We now state and prove the following lemma.

**Lemma 2** Let \( E \) and \( B \) be two Frechet spaces. Let \( f : E \to B \) be a (non-linear) operator between these two spaces which is \( C^k \)-Dir. (see Subsection 2.1, below Equation (5)). Then, \( f \) is \( C^k \) stable as in Definition 4.

**Proof** See Appendix B, Subsection B.1

The restriction of any \( C^k \)-stable (non-linear) operator \( f : E \to B \) between two Frechet spaces \( E \) and \( B \) to any non-empty compact subset \( K \subseteq E \) extends to a \( C^k \)-stable (non-linear) operator defined on all \( E \), namely the function \( f \) itself. However, because our approximation theorems will hold for a pair \((f, K)\) of a (non-linear) operator \( f : E \to B \) and compact set \( K \), then \( f \) does not need to be smooth on \( K \) but only indistinguishable from a smooth operator on \( K \). That is, our main results focus on non-linear operators belonging to the following trace class.

**Definition 5 (Trace Class)** Let \( E \) and \( B \) be two Frechet spaces and let \( \lambda > 0 \) be a constant. Let \( K \subseteq E \) be a non-empty compact set. We say that a (non-linear and possibly discontinuous) operator \( f : E \to B \) belongs to the trace class \( C_{tr}^{k,\lambda}(K, B) \) if there exists a \( \lambda \)-Lipschitz\(^a\) \( C^k \)-stable (non-linear) operator \( F : E \to B \) satisfying

\[
F(x) = f(x)
\]

for every \( x \in K \).

The following Example 1, pictorially represented in Figure 4, highlights our main interest in trace class maps. Precisely, these maps can be globally poorly behaved, even discontinuous, but indistinguishable from smooth functions “locally” (i.e. on a particular compact subset of the input space \( E \)).

![Fig. 4: Pictorial representation of the fact that the indicator function of the interval \([0, 1]\) belongs to \( C_{tr}^{k,\lambda}([0, 1], \mathbb{R}) \) for all \( k \in \mathbb{N} \) and \( \lambda > 0 \); see Example 1.](image)

**Example 1 (The indicator of the unit interval is in \( C_{tr}^{k,\lambda}(K, B) \)).** Let \( E = B = (\mathbb{R}, | \cdot |) \), \( K = [0, 1] \cup [2, 3] \), and \( f = I_{[0, 1]} \), i.e. the indicator function of the interval \([0, 1]\). Then, by means of a bump function, we immediately see that for every \( k \in \mathbb{N} \) and \( \lambda > 0 \), \( f \in C_{tr}^{k,\lambda}(K, B) \).

At this point, some remarks are in order. In general, the problem of identifying when a map belongs to \( C_{tr}^{k,\lambda}(K, B) \) is a well-studied and independent area of research dating back to the beginning of the previous century (e.g., [81]). Nonetheless, by virtue of Lemma 2 a full characterization of the pairs of functions and sets \((f, K)\) that belongs to \( C_{tr}^{k,\lambda}(K, B) \) in the special case that \( E = B \) are Euclidean spaces has been derived only (relatively) recently in a series of articles starting with [28]. The interested reader may consult [14] where the \( C_{tr}^{1,\lambda}(K, B) \) case is treated in the case that \( B \) is Banach and \( K \) is finite-dimensional (in a suitable metric-theoretic sense), for some \( \lambda > 0 \) depending on \( K \) and on \( f \). The case where \( K \) is a subset of a separable Hilbert space is explicitly solved in [5].

Moreover, we provide results for the following trace class.

---

\(^a\) By \( \lambda \)-Lipschitz we mean that the optimal Lipschitz constant is \( \lambda \). Notice that the case \( \lambda = 0 \) corresponds to the trivial case of a constant \( f \) which is not treated in the present work.
Theorem 1 (Neural Filters Efficiently Approximate of Non-Linear Operators)

Set of all neural filters with representation (17) is denoted by \( \text{NF} \), the statement of the theorem, we give here some definitions. First, for any \( n \in \mathbb{N}_+ \), we will use \( \psi_n \) and \( \varphi_n \) to denote the following two set-theoretic maps:

\[
\psi_n : (\mathbb{R}^n, d_{E,n}) \to (\mathbb{R}^n, \| \cdot \|_2), \quad z \mapsto \psi_n(z),
\]

\[
\varphi_n : (\mathbb{R}^n, \| \cdot \|_2) \to (\mathbb{R}^n, d_{B,n}), \quad z \mapsto \varphi_n(z).
\]

When it is clear from the context, we suppress the index \( n \) and write \( \psi_n \) instead of \( \psi \) (resp. \( \varphi_n \) instead of \( \varphi \)).

Definition 7 (Neural Filters)

For every “encoding error” \( \epsilon_D \) > 0 and every “approximation error” \( \epsilon_A \) > 0 there exist \( \hat{f} \in \mathcal{NF}^{(P)\text{ReLU}}_{[n]} \) satisfying

\[
\max_{x \in K} d_B(f(x), \hat{f}(x)) \leq \epsilon_D + \epsilon_A,
\]

where \( [n_{\epsilon_D}] = (d_0, \ldots, d_j) \) is a multi-index such that \( d_0 = n_{\epsilon_D}^{in} \) and \( d_j = n_{\epsilon_D}^{out} \) defined as in Table 1.

Theorem 1 (Neural Filters Efficiently Approximate of Non-Linear Operators)

Assume setting (A1). Fix a compact subset \( K \subseteq E \) with at-least two points, \( k \in \mathbb{N}_+, \alpha \in (0, 1], \lambda > 0 \) and a (non-linear) operator \( f : E \to B \) belonging to either the trace-class \( C^L_{\alpha,\tr}(K, B) \) or to the trace-class \( C^L_{\alpha,\tr}(K, B) \).

For every “encoding error” \( \epsilon_D \) > 0 and every “approximation error” \( \epsilon_A \) > 0 there exist \( \hat{f} \in \mathcal{NF}^{(P)\text{ReLU}}_{[n_{\epsilon_D}]} \) satisfying

\[
\max_{x \in \hat{f}} d_B(f(x), \hat{f}(x)) \leq \epsilon_D + \epsilon_A,
\]

where \( [n_{\epsilon_D}] = (d_0, \ldots, d_j) \) is a multi-index such that \( d_0 = n_{\epsilon_D}^{in} \) and \( d_j = n_{\epsilon_D}^{out} \) defined as in Table 1.

Proof See Appendix B, Subsection B.2.

The rates in Table 1 are optimal for finite-dimensional Banach space input spaces and one-dimensional output space. To see this, we only need to consider the case where \( E \) is a finite-dimensional Euclidean space and \( B \) is the real-line with Euclidean distance. In this setting, neural filter model is a deep feedforward neural network with ReLU activation function. In which case, a direct inspection of the approximation rates in Table 1 reveal that they coincide with the approximation rates for Hölder functions derived in [85] which are optimal, as they achieve the Vapnik–Chervonenkis (VC) lower-bound on a real-valued model class’ approximation rate (see [85, Theorem 2.4]) determined by its VC-dimension\(^{10}\).

Remark 3 (Technicalities in Table 1) We emphasize that in the following, \( \langle \cdot, \cdot \rangle \) denotes the Euclidean inner product\(^{11}\). In particular, in the first column of Table 1, the functions \( \hat{f}_i \) are defined by

\[
\hat{f}_i \defeq \langle \varphi \circ P_{B,n_{\epsilon_D}^{out}} \circ F \circ I_{E,n_{\epsilon_D}^{in}} \circ \psi^{-1} \circ W^{-1}, \bar{e}_i \rangle \defeq \langle \hat{f} \circ W^{-1}, \bar{e}_i \rangle,
\]
When \( f \) belongs to the \( C^0 \)-trace class: the constants in Table 1 are \( C_1 = 3^{n_{F,D}} + 3 \) and \( C_2 = 18 + 2n_{F,D} \).

When \( f \) is belongs to the \( C^{k,\lambda} \)-trace class: then \( C_1 = 17k^{n_{F,D}} + 13n_{F,D} n_{F,D}^2, C_2 = 18k^2 = \max_{i=1 \ldots n_{F,D}} \|f_i\|_{C^{k,\lambda}([0,1]^{n_{F,D}})} \).

### Table 1: Optimal Approximation Rates - Neural Filter with ReLU activation function

| Hyperpar. | Exact Quantity - High Regularity - \( C^{k,\lambda}(K, B) \) |
|-----------|-----------------------------------------------------------|
| \( n_{0,F} \) | \( \inf \{ n \in \mathbb{N} : \max_{x \in K} d_E(A_{E,n}(x), y) \leq \frac{1}{2} \omega_{A,E} \left( \frac{\|V\|}{\lambda} \right)^{\frac{1}{\alpha}} \} \) |
| \( n_{0,D} \) | \( \inf \{ n \in \mathbb{N} : \max_{x \in K} d_B(A_{B,n}(x), y) \leq \frac{1}{2} \omega_{A,E} \left( \frac{\|V\|}{\lambda} \right)^{\frac{1}{\alpha}} \} \) |
| Width | \( n_{0,F}(n_{0,F} - 1) + C_1 \max \{ \left[ \frac{1}{2} \omega_{A,E} \left( \frac{\|V\|}{\lambda} \right)^{-\frac{1}{\alpha}} \right] \epsilon D \} + 2 \) |
| Depth | \( n_{0,D} \left( 1 + 2 \epsilon D \right) \) |

for \( i \in \mathbb{N} \), where the function \( W : (\mathbb{R}^{n_{F,D}}, \| \cdot \|_2) \rightarrow (\mathbb{R}^{n_{F,D}}, \| \cdot \|_2) \) is defined as:

\[
W : (\mathbb{R}^{n_{F,D}}, \| \cdot \|_2) \rightarrow (\mathbb{R}^{n_{F,D}}, \| \cdot \|_2) \rightarrow \mathbb{R}^{n_{D}}, \quad x \rightarrow W(x) \overset{\text{def.}}{=} (2r_K)^{-1}(x - x_0) + \frac{1}{2}.
\]

In the previous expression, we have \( x_0 \in \mathbb{R}^{r_K} \), \( \epsilon \in \{1, \ldots, 1\} \in \mathbb{R}^{r_K} \) and \( r_K \) is a constant that depends on the compact \( K \). Moreover, in Table 1 we use the abbreviated notation \( A_{E,n} \equiv I_{E,n_{F,D}} \circ P_{E,n_{F,D}}, A_{B,n} \equiv I_{B,n_{F,D}} \circ P_{B,n_{F,D}} \), and \( \omega_{A,E} \) is a modulus of continuity of the maps \( (A_{E,n})_{n \in \mathbb{N}} \) realizing the bounded approximation property on \( E \) and \( \omega_{A,E} \) denotes the generalized inverse of \( \omega_{A,E} \).

### Obstructions to Universal Approximation of Continuous Functions in Infinite-Dimensions

The inability to extend higher-regularity (Lipschitz or smooth) functions while preserving their regularity, is precisely the obstruction lying at the heart of any quantitative approximation theorem between general infinite-dimensional Fréchet spaces. More precisely, a qualitative guarantee for continuous functions would require a version of McShane’s extension theorem [9] for \( B \)-valued continuous maps but, to the best of our knowledge, such a result is only available when both \( E \) and \( B \) are separable Hilbert space [10, Theorem 1.12]. However, such a result would not provide control on the target function’s regularity. Thus, without assuming that the target function belongs to a given trace-class, e.g. Hölder or smooth trace classes, as considered here, there is no a-priori way to clearly relate the complexity of a deep learning model, which depend on the regularity of the extension of regularity to the deep learning model restricted to \( K \).

Even in finite-dimensions highly-regular extensions, such as smooth extensions, see [81] and [28], need not exist. Moreover, it is not even clear if a uniformly continuous function can be extended to a uniformly continuous function with a proportional modulus of continuity (see [36] for details).

### 3.2 Dynamic Case: Efficient Universal Approximation

Theorem 1 was a static result certifying that suitable non-linear operators between infinite-dimensional linear metric spaces can be efficiently approximated by our “neural filter” operator network. By training several neural filters, independently on separate time-windows, and then re-assembling them via a “central” hypernetwork we can causally approximate “any” (generalized) dynamical system between such infinite-dimensional spaces.

The construction of a finitely-parameterized causal neural network approximator for these types of dynamical systems is our main result, and the main focus of this section. However, our construction is not only a certificate that causal operator network model can approximate suitable infinite-dimensional dynamical systems, nor only that we can estimate the required number of parameters for this to happen. Rather our argument shows how one can algorithmically construct such causal neural operators in the idealized setting, familiar to universal approximation theory [50, 84, 55], where one has complete access to a target function evaluated at all points in the input space, unobscured by any noise, as well as a perfect optimization algorithm which can always identify a...
minimizer to any optimization problem. In this idealized setting, where we can distill the approximation theoretic capabilities of our DL model apart from optimization or statistical learning question, we are able to explain its construction algorithmically.

We now present this idealized CNO construction algorithm, Algorithm 1. Our main result (Theorem 2) effectively certifies its ability to construct a CNO approximating any noiseless target function in this idealized approximation-theoretic framework. By a \( \delta \)-packing of a set, we mean the maximum number of points which can be placed in that set which are each at a distance of \( \delta \) > 0 apart\(^{14}\).

---

**Algorithm 1: Construct CNO**

```
Require: Causal map \( f : \mathcal{X} \to \mathcal{Y} \), errors: encoding \( \epsilon_D > 0 \) and approximation \( \epsilon_A > 0 \), hyperparameters: latent code complexity \( Q \in \mathbb{N}_+ \) and depth hyperparameter \( \delta \in \mathbb{N}_+ \)
/* Initialize CNO’s hyperparameters */
Viable time-steps: \( I_{k,Q} \) \( \triangleq [\delta-Q] \)
Memory: \( M = O(\epsilon_A^2) \)
Set \( [d] \) as in Table 2
Get \( \delta \)-packing \( \{z_{i}\}_{i=0}^\infty \) of \( \text{Ball}_{L_2}(0,1) \)
/* Nodes optimize neural filters on individual time windows in parallel */
For \( 1 \leq i \leq I_{k,Q} \) in parallel
  \( J_{th_i} \leftarrow \arg\min_{f \in \mathcal{N}_{\mathcal{X}(d)}(\mathbb{R})} d_{B_{\mathcal{X}(d)}}(f_t(x_{t,i-\Delta t}),x_{t,i}) < \epsilon_A + \epsilon_D \)
  \( z_{t,i} \leftarrow (\theta_{t,i},z_{i}) \)
/* Create Recurrence/ Encode Causality via Hypernetwork */
\( \hat{h} \leftarrow \arg\min_{h \in \mathcal{N}_{\mathcal{X}(d)}(\mathbb{R})} \sum_{1 \leq i \leq I_{k,Q}-1} \|h(z_{t,i}) - z_{t,i+1}\|_2 = 0 \)
/* Server receives parameters of optimized neural filters for each time window */
\( L : \mathbb{R}^{P(d)} \times \mathbb{R} \to \mathbb{R}^{P(d)} \) projection onto first component
return Trained CNO: \( (f,z_0) \)
```

---

**Remark 4 (Algorithm 1 Is Federated)** Algorithm 1 is a federated training algorithm\(^{15}\). In it, every neural filter acts as a nodes, which is trained independently from one another. Once optimized, these nodes send their parameters to the hypernetwork, which acts as a server synchronizing each of nodes into a central DL model.

We henceforth fix a non-degenerate time grid (cfr. Assumption 4.1 in [1]), by which we mean a sequence \( (t_n)_{n \in \mathbb{Z}} \subset \mathbb{R} \) satisfying the following structural properties.

**Assumptions 11 (Time Grid)** The time-grid \( (t_n)_{n \in \mathbb{Z}} \) is assumed to satisfy

1. \( t_0 = 0 \);
2. \( 0 < \inf_{n \in \mathbb{Z}} \Delta t_n \leq \sup_{n \in \mathbb{Z}} \Delta t_n < \infty \);
3. \( \inf_{n \in \mathbb{Z}} t_n = -\infty \) and \( \sup_{n \in \mathbb{Z}} t_n = \infty \).

In what follows, we will refer to each element \( t_n \) in the non-degenerate time grid as “time”. We give now the following

**Definition 8 (Path Space)** Let \( (t_n)_{n \in \mathbb{Z}} \) be a fixed non-degenerate time grid. For every \( n \in \mathbb{Z} \), let \( E_{t_n} \) be a separable Fréchet space carrying a Schauder basis \( (e_h^{(n)})_{h \in \mathbb{N}_+} \), and let \( \mathcal{X}_n \) be a non-empty closed subset of \( E_{t_n} \). The topological product \( X \triangleq \prod_{n \in \mathbb{Z}} \mathcal{X}_n \) is called path-space. The path space \( X \) is called linear if \( \mathcal{X}_n = E_{t_n}, n \in \mathbb{Z}, \) i.e. if \( X = \prod_{n \in \mathbb{Z}} E_{t_n} \).

Before proceeding, we introduce the following notation. For any \( n, m \in \mathbb{Z} \) with \( n < m \) and \( x \in \mathcal{X} \) we denote by \( x_{(t_n,t_m]} \triangleq x_{(t_n,\cdots, t_m]} \) and by \( \mathcal{X}_{(t_n,t_m]} \triangleq \prod_{n=\infty}^{m} \mathcal{X}_n \). From Tychonoff’s theorem\(^{16}\) we know that an arbitrary product of compact spaces is compact in the product topology. Therefore, a path space \( X \) is compact in the product topology if and only if each \( \mathcal{X}_n \) is a compact subset of \( E_{t_n}, n \in \mathbb{Z} \). We will study causal maps between path spaces. Briefly, what we mean with this statement is that we will analyze maps between path spaces that respect the causal forward-flow of information in time. Said differently, we will analyze maps for which, at any given time, the output must not depend on any future inputs. Because we are interested in efficient approximation results, rather than approximation guarantees via models whose number of parameters depends exponentially on the “encoding error” or on the “approximation error” (see Theorem 1), we will focus on the class of maps in the

---

\(^{14}\) See Appendix A.2 for details.

\(^{15}\) See for example [59] for further details on federated learning algorithms.

\(^{16}\) See Theorem 3.1 in [65].
subsequent Definition 9, which are the analogue of the $C_{k}^{r, \lambda}(K, B)$ and $\lambda_{r, \lambda}(K, B)$ maps introduced in Definition 5 and 6, respectively. Notice that Definition 9 makes sense thanks to Lemma 6, which states that the finite\textsuperscript{17} Cartesian product of Fréchet spaces with Schauder basis is a Fréchet space with a Schauder basis.

**Definition 9 (Causal Maps of Finite Virtual Memory)** Let $\mathcal{X} = \prod_{n \in \mathbb{Z}} \mathcal{X}_{n}$ be a compact path-space according to Definition 8. Let also $\mathcal{Y} = \prod_{n \in \mathbb{Z}} B_{n}$ be a linear path-space; in particular, each $B_{n}$ is a separable Fréchet space with a Schauder basis. A map $f : \mathcal{X} \to \mathcal{Y}$ is called a causal map with virtual memory $r \geq 0$, if for every “memory compression level” $\varepsilon > 0$ and each “time-horizon” $I \in \mathbb{N}^{+}$ there are $M = M(\varepsilon, I) \in \mathbb{N}$ with $M(\varepsilon, I) \in O(\varepsilon^{-r})$, and there are functions $f_{i} \in C(\mathcal{X}_{(t_{i}, \ldots, t_{i}+M)}, B_{i}), i \in [|I|]$ satisfying

$$\max_{i \in [|I|]} d_{B_{i}}(f(x)_{t_{i}}, f_{i}(x(t_{i}, \ldots, t_{i}+M))) < \varepsilon. \quad (19)$$

We will typically require our causal maps to possess a certain degree of regularity to deduce efficient approximation rates. The most regular maps considered in this manuscript are those causal maps of finite virtual memory which smooth trace-class maps can efficiently approximate at each instance in time.

**Definition 10 (Smooth Causal Maps of Finite Virtual Memory)** Let $f : \mathcal{X} \to \mathcal{Y}$ be a causal map, in the notation of Definition 9. If there exists a positive integer $k$ and a $\lambda > 0$ such that $f_{i} \in C_{k}^{r, \lambda}(\mathcal{X}_{(t_{i}, \ldots, t_{i}+M)}, B_{i}), i \in [|I|]$, then we say that the causal map $f$ is $(r, k, \lambda)$-smooth. If, moreover, the functions $f_{i}$ belong to $C_{k}^{r, \lambda}(\mathcal{X}_{(t_{i}, \ldots, t_{i}+M)}, B_{i})$ for every $k \in \mathbb{N}^{+}$ then we will say that $f$ is $(r, \infty, \lambda)$-smooth.

We also derive approximation guarantees for the low-regularity analogue of smooth causal maps.

**Definition 11 (Hölder-Causal Maps of Finite Virtual Memory)** Let $f : \mathcal{X} \to \mathcal{Y}$ be an causal map, in the notation of Definition 9. If there are an $\alpha \in (0, 1]$ and a $\lambda > 0$ such that $f_{i} \in C_{1, \alpha}^{r, \lambda}(\mathcal{X}_{(t_{i}, \ldots, t_{i}+M)}, B_{i}), i \in [|I|]$, then we say that $f$ is $(r, \alpha, \lambda)$-Hölder.

We now present the main result of the paper. Our causal universal approximation theorem guarantees that the CNO model can approximate any causal map while “preserving its forward flow of information through time”. The quantitative approximation rates, describing the complexity of the CNO model implementing the approximation are recorded in Table 2 below.

**Theorem 2 (CNOs are Efficient Universal Approximators of Causal Maps)** Let $\mathcal{X} = \prod_{n \in \mathbb{Z}} \mathcal{X}_{n}$ be a compact path space, $\mathcal{Y} = \prod_{n \in \mathbb{Z}} B_{n}$ a linear path space\textsuperscript{18}, and $f : \mathcal{X} \to \mathcal{Y}$ either a $(r, k, \lambda)$-smooth or a $(r, \alpha, \lambda)$-Hölder causal map\textsuperscript{19}. Fix “hyperparameters” $Q \in \mathbb{N}^{+}$ and $0 < \delta < 1$. For every “encoding error” $\varepsilon_{D} > 0$, every “approximation error” $\varepsilon_{A} > 0$, and every “time-horizon” $I \in \mathbb{N}^{+}$ with $I_{S,Q} \overset{\text{def.}}{=} \lfloor \delta^{-Q} \rfloor \geq 1$ then there is an integer $M \leq \varepsilon_{A}^{-1}$, a multi-index $[d]$, a “latent code” $z_{n} \in \mathbb{R}^{P([d])+Q}$, a linear readout map $L : \mathbb{R}^{P([d])+Q} \to \mathbb{R}^{P([d])}$, and a (“hypernetwork”) ReLU FFNN $\hat{h} : \mathbb{R}^{P([d])+Q} \to \mathbb{R}^{P([d])+Q}$ such that the sequence of parameters $\theta_{t_{i}} \in \mathbb{R}^{P([d])}$, defined recursively by

$$\begin{align*}
\theta_{t_{i}} & \overset{\text{def.}}{=} L(z_{t_{i}}), \\
z_{t_{i+1}} & \overset{\text{def.}}{=} \hat{h}(z_{t_{i}}),
\end{align*}$$

with $i \in [|I|] \cup \{0\}$, satisfies the following uniform spatio-temporal estimate:

$$\max_{i \in [|I|]} d_{B_{i}}(\hat{f}_{i}(x(t_{i}, \ldots, t_{i}+M)), f(x)_{t_{i}}) < \varepsilon_{A} + \varepsilon_{D},$$

where\textsuperscript{20} $\hat{f}_{i} \in \mathcal{N}_{[n_{D}]}^{P, \text{ReLU}}$ and $\hat{f}_{i} = I_{B_{i}} \circ \varphi_{n_{i}^{out}} \circ \theta_{t_{i}} \circ \psi_{n_{i}^{out}} \circ P_{E(t_{i}, \ldots, t_{i}+M)} \circ n_{D}^{i}$, where each $\hat{f}_{i}$ is a $(P)\text{ReLU}$ FFNN in $\mathcal{N}_{[n_{D}]}^{P, \text{ReLU}}$ with multi-index $[n_{D}^{i}] = (d_{0}, \ldots, d_{j})$ with $d_{0} = n_{i}^{in}$ and $d_{j} = n_{i}^{out}$ defined as in Table 1.

The model complexity of the hypernetwork $\hat{h}$ is recorded in Table 2.

**Proof** See Appendix B, Subsection B.4

For brevity, we do not repeat the complexities of the neural filters approximating the target function on any time window and recall that the neural filters’ approximation rates have previously been recorded in Table 1.

\textsuperscript{17} We remark that the countably infinite direct product of Fréchet spaces each admitting a Schauder basis does itself admit a Schauder basis and the proof of this fact is similar but, due to its length, we do not include it in our manuscript.

\textsuperscript{18} See Definition 8.

\textsuperscript{19} See Definition 9.

\textsuperscript{20} See Definition 7.
Table 2: Causal Approximation Rates - (CNO) Causal Neural Operator: The model complexity estimates of the hypernetwork $\hat{h}$ defining the CNO in Theorem 2, as a function of the target causal maps $f$’s regularity, the amount of memory allocated to the hypernetwork's latent space $Q \in \mathbb{N}_+$, and the length of the time-horizon the approximation is required to hold on $I \in \mathbb{N}_+$.

| Hyperparam.                        | Upper Bound                                                                 |
|------------------------------------|------------------------------------------------------------------------------|
| Width - Hyper. Net. ($\hat{h}$)    | $(P([d])) + Q)I_{\delta,Q} + 12$                                           |
| Depth - Hyper. Net. ($\hat{h}$)    | $O\left(I_{\delta,Q} + (1 + \sqrt{I_{\delta,Q}} \log(I_{\delta,Q})) \left(1 + \frac{\log(2)}{\log(I_{\delta,Q})} \left(C + \left(\frac{\log(I_{\delta,Q}^{1/2} - \log(\delta))}{\log(I_{\delta,Q})}\right)^2\right)\right)\right)$ |
| N. Param. - Hyper. Net. ($\hat{h}$) | $O\left(I_{\delta,Q}^3 (P([d])) + Q)^2 \left(1 + (P([d]) + Q) \sqrt{I_{\delta,Q} \log(I_{\delta,Q})} \left(1 + \frac{\log(2)}{\log(I_{\delta,Q})} \left(C_d + \left(\frac{\log(I_{\delta,Q}^{1/2} - \log(\delta))}{\log(I_{\delta,Q})}\right)^2\right)\right)\right)\right)$ |
| Memory - Neural Filters ($M$)       | $O(c_{\delta}^{-1})$                                                        |
| Complexity - Neural Filters        | Table 1                                                                     |
| Constant ($C_d$)                   | $(P([d])) + Q)I_{\delta,Q} + 12$                                           |

We now apply our results to efficiently approximate solution operators arising in stochastic analysis.

4 Applications to Stochastic Analysis

We apply our results to show that several solution operators from stochastic analysis can be approximated by the CNO. Our neural network model can approximate stochastic processes without assuming strong structural conditions describing their evolution. We illustrate our result’s implications for obtaining numerical solutions to SDEs, and we discuss the implications for more general stochastic processes, e.g., processes with jumps, towards the end of this section.

![Diagram](image)

Fig. 5: Illustration of our “static” operator network in Definition 7 specialized to the geometry of the input space $L^2(\Omega, G_T, P)$ and the output space $L^2(\Omega, F_t, P)$; for $\sigma$ algebras $G$ and $F$ on a sample space $\Omega$. The network is works in three phases. 1) First inputs are encoded as finite-dimensional Euclidean data by mapping them to their truncated (Schauder) basis coefficients in the input space $E$. 2) Next these coefficients are transformed by a ReLU FFNN. 3) The outputs of ReLU FFNN’s output are interpreted as coefficients a Wiener Chaos expansion a truncated (Schauder) basis in the output space $F$. 

4.1 A primer on Wiener Chaos

We fix a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) supporting a standard one dimensional Brownian motion \((B_t)_{t \geq 0}\) and let \(\mathbb{F} \triangleq (\mathcal{F}_t)_{t \geq 0}\) denote the complete and right-continuous enlargement of the filtration generated by \((B_t)_{t \geq 0}\). We recall that the Itô (stochastic) integral of a (deterministic) simple function \(f = \sum_{i=1}^k \beta_i I_{[0,t_i]}\) in \(L^2([0,t])\), where \(0 \leq t_1 < \cdots < t_k \leq t\) is the Gaussian random variable

\[
\int_0^t f(s) \, dB_s \triangleq \sum_{i=1}^k \beta_i (B_{t_i} - B_{t_{i-1}}).
\]

More generally, the Itô integral of any function \(f \in L^2([0,t])\) is defined as the limit in \(L^2(\Omega, \mathcal{F}_t, \mathbb{P})\) of a sequence \(\{f_k(s) \, dB_s\}_{k=1}^\infty\) where the \(\{f_k\}_{k=1}^\infty\) is any choice of simple integrands converging to \(f\) in \(L^2([0,t])\). Thus, \(\int_0^t f(s) \, dB_s\) is a centered normal random variable with variance \(\int_0^t f^2(s) \, ds\). We also note that such a sequence always exists and \(\int_0^t f(s) \, dB_s\) is independent of the particular choice of the approximating sequence \(\{f_k\}_{k=1}^\infty\).

Using tools common to (Malliavin) stochastic calculus we may exhibit an orthonormal basis of \(L^2(\Omega, \mathcal{F}_t, \mathbb{P})\). We refer the interested reader to [67] for a more detailed discussion on this construction. This construction relies on a system of orthogonal polynomials \(\{h_k\}_{k=1}^\infty\) known as Hermite polynomials and defined by the recurrence relation

\[h_{k+1}(x) = xh_k(x) - h'_k(x),\]

where \(h_0(x) \triangleq 1\). For instance, \(h_1(x) = x\), \(h_2(x) = x^2 - 1\), and so on.

By means of the Itô stochastic integral and the Hermite polynomials we may define the \(q\)th Wiener Chaos to be the subspace \(\mathcal{H}_q^2\) of \(L^2(\Omega, \mathcal{F}_t, \mathbb{P})\) spanned by the random variables of the form

\[P^q(f) \triangleq h_q \left( \int_0^t f(s) \, dB_s \right),\]

where \(f \in L^2([0,t])\), \(q \in \mathbb{N}_+,\) and \(\mathcal{H}_q^{2} \triangleq \mathbb{R}\). The Wiener chaos \((\mathcal{H}_q^{2})_{q=0}^\infty\) produces an orthogonal decomposition, given in [67, Theorem 1.1.1], of \(L^2(\Omega, \mathcal{F}_t, \mathbb{P})\), meaning that for each pair of random variables \(Y_q \in \mathcal{H}_q^2\) and \(Y_\tilde{q} \in \mathcal{H}_{\tilde{q}}^2\) are orthogonal in \(L^2(\Omega, \mathcal{F}_t, \mathbb{P})\) whenever \(q \neq \tilde{q}\); every random variable \(Y \in L^2(\Omega, \mathcal{F}_t, \mathbb{P})\) can uniquely be decomposed as

\[Y = \sum_{q=0}^\infty Y_q,\]

where \(Y_q \in \mathcal{H}_q^{2}\) for each \(q \in \mathbb{N}\) and where the sum converges in \(L^2(\Omega, \mathcal{F}_t, \mathbb{P})\).

Since the Wiener Chaos is an orthogonal decomposition of \(L^2(\Omega, \mathcal{F}_t, \mathbb{P})\) then the union of any set of orthogonal bases of each \(\mathcal{H}_q^2\) is an orthogonal basis of \(L^2(\Omega, \mathcal{F}_t, \mathbb{P})\) itself. Therefore, we only need to exhibit an orthogonal basis of each \(\mathcal{H}_q^2\) for \(q \in \mathbb{N}_+\).

We leverage the symmetrized tensor product of elements \(f_1, \ldots, f_q \in L^2([0,t])\) defined by

\[
\text{sym} \left( f_1 \otimes \cdots \otimes f_q \right) \triangleq \frac{1}{q!} \sum_{\pi \in S^q} f_{\pi(1)} \otimes \cdots \otimes f_{\pi(q)},
\]

where \(S^q\) is the set of permutations of the indices \(\{1, \ldots, q\}\). More concretely, the Hilbert space generated by the symmetrized tensor product is identified with the set of symmetric functions in \(L^2([0,t]^q)\) which we denote by \(L^2_{\text{sym}}([0,t]^q)\). Since the \(q\)-fold symmetrized tensor product is a subspace of the (usual) \(q\)-fold tensor product then the identification of the \(q\)-fold symmetric tensor product of \(L^2([0,t])\) with \(L^2_{\text{sym}}([0,t]^q)\) may be further simplified to

\[
\text{sym} \left( f_1 \otimes \cdots \otimes f_q \right) \leftrightarrow \frac{1}{q!} \sum_{\pi \in S^q} \prod_{i=1}^q f_{\pi(i)}(s_i).
\]

The connection between the symmetrized tensor product and the \(q\)th Wiener Chaos is that the \(q\)th Wiener Chaos \(\mathcal{H}_q^2\) is structurally identical to \(L^2_{\text{sym}}([0,t]^q)\) (identified with the \(q\)-fold symmetrized tensor product of \(L^2_{\text{sym}}([0,t])\) with itself). The map realizing this identification sends any \(f \in L^2_{\text{sym}}([0,t]^q)\) to its \(q\)-fold multiple stochastic integral

\[
f \mapsto \int_0^{t_1} \cdots \int_0^{t_q} f(s_1, \ldots, s_q) \, dB_{s_1} \cdots dB_{s_q}.
\]

\[\text{See } [13, \text{Chapter IV page } 43].\]

\[\text{See } [71, \text{Lemma } 8.4.2].\]

\[A \ "function" \ f \in L^2([0,t]^q) \ is \ symmetric \ if \ f(s_1, \ldots, s_q) = f(s_{\pi(1)}, \ldots, s_{\pi(q)}), \ for \ all \ \pi \in S^q, \ outside \ a \ set \ of \ q\text{-dimensional Lebesgue measure } 0.\]
Moreover, the map (21) is linear isometric isomorphism preserving inner products. Consequentially, any orthogonal basis of $L^2_{\text{sym}}([0,t]^q)$ is sent to an orthogonal basis of $\mathcal{H}_t^q$ under this identification. Since an orthogonal basis of $L^2_{\text{sym}}([0,t]^q)$ is given by the set

$$\text{sym} \{ f_1 \otimes \cdots \otimes f_q \}$$

where $\{f_j\}_{j=1}^{\infty}$ is an orthogonal basis of $L^2([0,t])$, then the identification (21) implies that the corresponding set of random variables

$$\int_0^{t_1} \cdots \int_0^{t_1} \text{sym} \{ f_1 \otimes \cdots \otimes f_q \}(s_1, \ldots, s_q) \ dB_{s_1} \ldots dB_{s_q},$$

is an orthogonal basis of the $q^{th}$ Wiener Chaos $\mathcal{H}_t^q$. Such an orthogonal basis of $L^2([0,t])$ is given by the Fourier basis whose elements are

$$f_{j,i}(x) \defeq \begin{cases} \sqrt{\frac{2}{\pi}} \sin \left( \frac{\pi x}{t} \right) & \text{if } i = 0, \\ \sqrt{\frac{2}{\pi}} \cos \left( \frac{(i-1)\pi x}{t} \right) & \text{if } i = 1, \end{cases}$$

where $j \in \mathbb{N}_+$ and $i \in \{0,1\}$. For convenience, with some abuse of notation, we denote an enumeration of $\{f_{j,i}\}_{j \in \mathbb{N}, i \in \{0,1\}}$ by $\{f_k\}_{k=1}^{\infty}$. Consequentially, an orthogonal basis of $L^2(\Omega, \mathcal{F}_t, \mathbb{P})$ is given by the countable family of random variables

$$Z_{(k_1, \ldots, k_q)} \defeq \frac{1}{q!} \sum_{\pi \in S_q} \int_0^{t_1} \cdots \int_0^{t_1} \prod_{r=1}^{q} f_{k_{\pi(r)}}(s_r) \ dB_{s_1} \ldots dB_{s_q},$$

where $(k_1, \ldots, k_q)$ is a multi-index belonging to $A \defeq \bigcup_{q=0}^{\infty} \mathbb{N}^q$; we also make the convention that $Z_{\varnothing} \defeq 1$, and we have used the linearity of the Ito (stochastic) integral in conjunction with the above considerations.

### 4.2 Simultaneous Approximation of SDEs with Different Initial Conditions using CNOs

Monte Carlo methods allow for the efficient solution to stochastic differential equations (SDEs) with a convergence rate of $O(1/\sqrt{T})$ to the true solution, where $T$ is the number of samples, plus a comparable discretization error when resorting to a tamed Euler scheme [46]. It is known that deep learning can provide a suitable alternative to Monte Carlo schemes by learning the SDE’s solution map given deterministic initial conditions, for a fixed terminal time, by efficiently approximating the solutions to their associated PDEs [8] given by the Feynman-Kac Theorem.

In this section, we show how a single CNO can be used for efficient simultaneously solving SDEs with various noisy initial conditions across different time-horizons, by simultaneously approximately learning solve a family of stochastic differential equations with many different stochastic initial conditions and different initial times.

This section’s application shows that the CNO can approximate causal maps with stochastic inputs on arbitrarily long time horizons. This extends the known guarantees for recurrent neural networks, specifically reservoir computers, which can approximate time-invariant causal maps [31].

We are given a non-degenerate time grid $(t_n)_{n \in \mathbb{Z}}$ as in Assumption 11, and $\alpha$ and $\beta$ in $C([0,\infty) \times \mathbb{R}, \mathbb{R})$ such that there exists $M > 0$ such that for all $t \geq 0$ and all $x_1, x_2 \in \mathbb{R}$, we have

$$|\beta(t, x_1) - \beta(t, x_2)|^2 + |\alpha(t, x_1) - \alpha(t, x_2)|^2 \leq M^2|x_1 - x_2|$$

and

$$|\beta(t, x_1)|^2 + |\alpha(t, x_1)|^2 \leq M^2(1 + |x_1|^2).$$

Theorem 8.7 in [20] guarantees that for all $i \in \mathbb{N}_+$, under the growth conditions (23) and (24), for $\eta \in L^2(\Omega, \mathcal{F}_t, \mathbb{P})$ there exists a unique $X \in C([t_i, t_{i+1}]; L^2(\Omega, \mathcal{F}_{t_{i+1}}, \mathbb{P}))$ which satisfies $\mathbb{P}$-a.s.

$$X_{t_{i+1}} = \eta + \int_{t_i}^{t_{i+1}} \alpha(s, X_s) \ ds + \int_{t_i}^{t_{i+1}} \beta(s, X_s) \ dB_s,$$

where we set $X_{t_i} = \eta$; in what follows, we will indicate the explicit dependence on $\eta$ in $X_{t_{i+1}}$, i.e. $X_{t_{i+1}}^{\eta}$. Therefore, for all $i \in \mathbb{N}_+$ the following (non-linear) solution operator

$$\text{SDE-Solve}_{t_i, t_{i+1}} : L^2(\Omega, \mathcal{F}_{t_{i+1}}, \mathbb{P}) \to L^2(\Omega, \mathcal{F}_{t_{i+1}}, \mathbb{P}), \eta \to X_{t_{i+1}}^{\eta}$$

is well defined. To see that each of the maps SDE-Solve$_{t_i, t_{i+1}}$ satisfies the assumptions of our theorems, it is sufficient to note that under (23) and (24), the operator SDE-Solve$_{t_i, t_{i+1}}$ is Lipschitz and, in view of [20, Proposition 8.15], it belongs to the trace-class $C_{\lambda,t_i}(K, L^2(\Omega, \mathcal{F}_{t_{i+1}}, \mathbb{P}))$ for all compact subsets $K$ of $L^2(\Omega, \mathcal{F}_{t_{i+1}}, \mathbb{P})$, since

$$\|X_{t_{i+1}}^{\eta} - X_{t_{i+1}}^{\bar{\eta}}\|_{L^2(\Omega, \mathcal{F}_{t_{i+1}}, \mathbb{P}; \mathbb{R})} \leq \sqrt{3}\mathbb{E}^{\frac{1}{2}} M^2(t_{i+1} - t_i)(t_{i+1} - t_i) \|\hat{\eta} - \tilde{\eta}\|_{L^2(\Omega, \mathcal{F}_{t_{i+1}}, \mathbb{P}; \mathbb{R})} \leq \sqrt{3}\mathbb{E}^{\frac{1}{2}} M^2(\Delta^* + 1)\Delta \|\tilde{\eta} - \bar{\eta}\|_{L^2(\Omega, \mathcal{F}_{t_{i+1}}, \mathbb{P}; \mathbb{R})}.$$
with \( \lambda \leq \sqrt{3\varepsilon^2 M^2(\Delta^+ + 1) \Delta^+} \) and \( \Delta^+ \overset{\text{def}}{=} \sup_{t \in \mathbb{Z}} \Delta t_i < \infty \) as in Assumption 11.

We consider the causal map

\[
\text{SDE-Solve} : \left( \prod_{i \in \mathbb{Z}, t_i < 0} \{0\} \right) \times \prod_{i \in \mathbb{Z}, t_i \geq 0} L^2(\Omega, \mathcal{F}_{t_i}, \mathbb{P}) \rightarrow \left( \prod_{i \in \mathbb{Z}, t_i < 0} \{0\} \right) \times \prod_{i \in \mathbb{Z}, t_i \geq 0} L^2(\Omega, \mathcal{F}_{t_i}, \mathbb{P}),
\]

\( (\eta_i)_{i \in \mathbb{Z}} \mapsto \text{SDE-Solve} [(\eta_i)_{i \in \mathbb{Z}}] \),

(28)

where each \( \text{SDE-Solve}_{t_i, t_{i+1}}(\eta_i) \) is defined as in Equation (26). The typical example which we have in mind, in the following, are input sequences which are orbits of square-integrable random variables under the an SDE’s solution operator; i.e.

\[
(\text{SDE-Solve} [(\eta_i)_{i \in \mathbb{Z}}])_j = \begin{cases} 0, & \text{if } t_j < 0 \\ \text{SDE-Solve}_{t_i, t_{i+1}}(\eta_i), & \text{if } t_j \geq 0, \end{cases}
\]

(29)

for some \( X \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}) \). Thus, approximating SDE-Solve and applying it to any compact subset of the path-space comprised of elements of the form (30) corresponds to simultaneously solving an SDE for several random initial conditions across arbitrarily time-intervals beginning at several initial times.

By Equation (27), SDE-Solve is a causal map as in Definition (9), since in this case we can simply take \( r = 0, \alpha = 1, M = 1, f_{t_i} = \text{SDE-Solve}_{t_i, t_{i+1}}, \) and \( \lambda \leq \sqrt{3\varepsilon^2 M^2(\Delta^+ + 1) \Delta^+} \) holds for any \( i \in \mathbb{N}_+ \). Theorem 2 guarantees that there exists a CNO which approximates the map in Equation (28), as soon as we confine ourselves on a compact path space. Let us summarize our findings in

Corollary 1 (Causal Universal Approximation of SDEs with Stochastic Dynamics) Consider the setting of this section and fix the path space

\[
\mathcal{X} \overset{\text{def}}{=} \left( \prod_{i \in \mathbb{Z}, t_i < 0} \{0\} \right) \times \prod_{i \in \mathbb{Z}, t_i \geq 0} \mathcal{X}_i,
\]

where each \( \mathcal{X}_i \) is a compact subset of \( L^2(\Omega, \mathcal{F}_{t_i}, \mathbb{P}) \). Then the operator SDE-Solve

\[
\text{SDE-Solve} : \left( \prod_{i \in \mathbb{Z}, t_i < 0} \{0\} \right) \times \prod_{i \in \mathbb{Z}, t_i \geq 0} \mathcal{X}_i \rightarrow \left( \prod_{i \in \mathbb{Z}, t_i < 0} \{0\} \right) \times \prod_{i \in \mathbb{Z}, t_i \geq 0} L^2(\Omega, \mathcal{F}_{t_i}, \mathbb{P})
\]

is \((0, 1, \sqrt{3\varepsilon^2 M^2(\Delta^+ + 1) \Delta^+})\)-Hölder.

Given \( Q, \delta \in \mathbb{N}_+ \), an “encoding error” \( \varepsilon > 0 \) and an “approximation error” \( \varepsilon_A > 0 \) there exist a multi-index \([d]\), a “latent code” \( z_0 \in \mathbb{R}^{P([d]) + Q} \), a linear readout map \( L : \mathbb{R}^{P([d]) + Q} \rightarrow \mathbb{R}^{P([d])} \), and a ReLU FFNN \( \hat{h} : \mathbb{R}^{P([d]) + Q} \rightarrow \mathbb{R}^{P([d]) + Q} \) such that the sequence of parameters \( \theta_{t_i} \in \mathbb{R}^{P([d])} \) defined recursively

\[
\theta_{t_i} \overset{\text{def}}{=} L(z_{t_i}), \quad z_{t_{i+1}} \overset{\text{def}}{=} \hat{h}(z_{t_i}),
\]

with \( i \) belongs to \([\lfloor I \rfloor] \cup \{0\}\) provided by the definition of causal maps \(28\), satisfies to the following uniform estimates:

\[
\max_{i \in [\lfloor I \rfloor]} \sup_{X \in \mathcal{X}} \| \hat{f}_{t_i} (X_{(t_{i-1}, t_i)}) - \text{SDE-Solve}(X)_{t_i} \|_{L^2} < \varepsilon_A + \varepsilon_D,
\]

where \( \hat{f}_{t_i} \in \mathcal{N}^{\mathcal{F}^{P}_{[\lfloor n_D \rfloor]}} \). Moreover, for the hyperparameter \( n_{\varepsilon_D}^{in} \) it holds

\[
n_{\varepsilon_D}^{in} = \inf \left\{ n \in \mathbb{N}_+: \max_{x \in \mathcal{X}} d_E(A_{E_{n}}(x), x) \leq \frac{\varepsilon_D}{2A} \right\}
\]

where we have set \( E \overset{\text{def}}{=} \Pi_{i \in \mathbb{Z}} L^2(\Omega, \mathcal{F}_{t_i}, \mathbb{P}) \).

\(28\) See Definition 9.

\(29\) We recall, Definition 7, stating that \( \hat{f}_{t_i} \overset{\text{def}}{=} IB_{n_{\varepsilon_D}^{in}} \circ \varphi_{n_{\varepsilon_D}^{in}} \circ \hat{f}_{t_i} \circ \psi_{n_{\varepsilon_D}^{in}} \circ P_{E_{n_{\varepsilon_D}^{in}}} \).
4.3 Discussion - Corollary 1: Jumps, Path-Dependence, and Accelerated Approximation Rates Under Smoothness

We briefly discuss some points surrounding Corollary 1. For instance, how the result allows for stochastic discontinuity-type jumps. We also discuss how the scope of Theorem 1 allows for Corollary 1 to be easily generalized; but we opt not to do that in this manuscript, rather opting for a less technical illustration of our general framework.

Improved Approximation Rates for SDEs Driven by Smooth Coefficients If, in addition to conditions (24) and (23), the drift and diffusion coefficients $\alpha$ and $\beta$ are sufficiently differentiable$^{30}$, then [72, Theorem 3.9] implies that each of the maps SDE-Solve$_{t_i,t_{i+1}}$ are $C^k$. Whence, the operator SDE-Solve is a smooth causal map of finite virtual memory. Thus, in this case, Theorem 2 implies improved approximation rates by the CNO model.

Stochastic Discontinuities at Time-Grid Points We highlight that the adapted map SDE-Solve does accommodate jumps but only if those jumps occur on the fixed time-grid points $\{t_i\}_{i \in \mathbb{N}}$. Such constructions have recently appeared in the rough path literature [4] and the causal/functional Itô calculus literature [18].

In financial applications, the possibility of a stochastic process’ to jump at predetermined times (called stochastic discontinuities in that context) are an essential ingredient of accurately modeling interest rates; for example, European reference interest rates typically exhibit jumps directly after monetary policy meetings of ECB [29].

Path Dependent Dynamics One could consider SDEs driven with path dependant random drift and diffusion coefficients, since all that is needed to apply Theorem 2 is the regularity of the SDE-Solve operator; which is guaranteed by results such as [21] or [72]. However, we instead opted for a simple first presentation, explicitly illustrating the scope of our results in this easier case.

5 The Benefit of Causal Approximation: Super-Optimal Approximation Rates for Causal Maps

We now illustrate the quantitative advantage of causal approximation, i.e. using our CNO architecture, when the target function is causal. For illustrative purposes, we consider the simplest case where all involved spaces are finite-dimensional and Euclidean. By considering this setting, we can juxtapose our approximation rates derived from Theorem 2 against the best rates for ReLU networks [85] which are optimal, as shown in the constructive approximation literature [23, 26]. Therefore, when the target function has a causal structure, “super-optimal uniform approximation rates” can be achieved only if one encodes that structure into the neural network model; as in the case with the CNO. Throughout this section, we consider the integer time-grid $\{t_i\}_{i \in \mathbb{Z}} = \{t\}_{t \in \mathbb{Z}}$; which we note satisfies the non-degeneracy condition in Assumption 11.

5.1 In the Euclidean Case, CNOs are a simple class of RNNs which are universal dynamical systems

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$^{30}$ The precise conditions are formalized in [72, Assumption 3.7].
In [51], the authors investigate the problem of approximating a dynamical system on a Euclidean space by a RNN. In their most general form, RNNs – sometimes also called “fully RNN”, or fRNNs - are given for times $t > 0$ by

$$
y_t \overset{\text{def.}}{=} \hat{f}_{\theta_t}(y_{t-1}, x_t), \quad y_0 \overset{\text{def.}}{=} y,$$

(31)

where $y_t$ is the state of the system, $x_t$ is an external input, $y$ the initial state, and $\hat{f}_{\theta_t}$ are (possibly deep) FFNNs with a priori no relationship among their parameters $(\theta_t)_{t \in \mathbb{N}^+}$. In particular, each FFNNs may have different depth and/or width. However, in practice, restrictions are put on the sequence of networks $(\hat{f}_{\theta_t})_{t \in \mathbb{N}^+}$; precisely, it is usually required that they all have the same complexity, and each $\theta_{t+1}$ is recursively determined from the pair $(\theta_t, x_t)$. For instance, if it is only assumed that each FFNNs in Equation (31) has the same complexity, then the classical result of [74] shows that one may simulate all Turing Machines by fRNNs with rational weights and biases. Although this result is promising for the expressive power of fRNNs, it is far removed from any practical model since it places absolutely no restriction on how the sequence $(\theta_t)_{t \in \mathbb{N}^+}$ is determined. As a consequence, the model in Equation (31) is not implementable since it depends on an infinite number of parameters, as there is no relationship between $\theta_t$ and any $\theta_s$ for all past times $s < t$. On the other extreme, a very recent paper [45] prove that a RNN with a single hidden layer and with $\theta_t = \theta_0$, for all $t \in \mathbb{N}_+$, can approximate linear time-invariant dynamical systems quantitatively.

Still, surprisingly, many questions surrounding the approximation power of more sophisticated but implementable RNNs remain open. For instance, the ability of such RNNs to approximate non-linear dynamical systems, quantitatively, and the quantitative role of the hidden state space/latent code’s dimension are still open problems in the neural network literature. This subsection, addresses these open problems as a simple and direct consequence of Theorem 2.

This is because if $E = B = \mathbb{R}^d$, (with $\mathbb{R}^d$ equipped with the Euclidean distance), then our CNO model defines a very simple RNN. In order to see this, let $(e_i)_{i=1}^d$ be the standard basis of $\mathbb{R}^d$, which is trivially a Schauder basis for the latter. Requiring that the encoding and the decoding dimensions of our CNO model are at least $d$, we have that the latter is given by $^{31}:

$$
\left\{ \begin{array}{ll}
y_t & \overset{\text{def.}}{=} \hat{f}_{\theta_t}(x_t), \\
\theta_t & \overset{\text{def.}}{=} L(z_t), \\
z_{t+1} & \overset{\text{def.}}{=} \hat{h}(z_t).
\end{array} \right.
$$

(32)

Moreover, by pre-composing each $\hat{f}_{\theta_t}$ in Equation (32) with the following linear projection

$$
A : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N, \quad (y, x) \mapsto x,
$$

and by noting that $\hat{f}_{\theta_t} \circ A$ is a FFNN because of the invariance with respect the pre-composition by affine functions, we have that the CNO becomes

$$
\left\{ \begin{array}{ll}
y_t & \overset{\text{def.}}{=} \hat{f}_{\theta_t}(y_{t-1}, x_t), \\
\theta_t & \overset{\text{def.}}{=} L(z_t), \\
z_{t+1} & \overset{\text{def.}}{=} \hat{h}(z_t),
\end{array} \right.
$$

(33)

where with a minor abuse of notation we keep using $\hat{f}_{\theta_t}$ instead of $\hat{f}_{\theta_t} \circ A$. By comparing Equations (31) and (33), we see that the CNO model is a RNN whose weights and biases do not depend upon the input sequence $(x_t)_{t \in \mathbb{N}^+}$, and are determined recursively by the hypernetwork $\hat{h}$, as in [37]. Therefore, our CNO is essentially the classical Elman RNN of [27] with $\hat{f}_{\theta_t}$ and $\hat{h}$ having several, instead of one, hidden layer.

We now illustrate the expressive power of the CNO model in Equation (33). For simplicity, we consider the case of dynamical system defined on a smooth compact sub-manifold $\mathcal{M}$ of $\mathbb{R}^d$, possibly with boundary; these types of dynamical systems arise often in physics [76,68] and are actively studied in the reservoir computing literature [35]. We let $(g_t)_{t \in \mathbb{N}}$ be a sequence of smooth functions from $\mathbb{R}^d$ to itself which fix the manifold $\mathcal{M}$, namely, $g_t(\mathcal{M}) \subseteq \mathcal{M}$ for every $n \in \mathbb{N}$. We further require that the family $(g_t)_{t \in \mathbb{N}}$ has uniformly bounded gradient on $\mathcal{M}$; meaning that for some $\lambda \geq 0$ it holds

$$
\sup_{t \in \mathbb{N}} \max_{x \in \mathcal{M}} \| \nabla g_t(x) \| \leq \lambda.
$$

NB, this is of-course satisfied by any autonomous dynamical system; namely when $g_t = g_0$ for all integers $t$, with $g_0$ smooth.

Then the restriction of each $g_t$ to $\mathcal{M}$ defines a dynamical system and we can express the causal structure in the orbit of any initial state $x_0 \in \mathcal{M}$ evolving under $g$ as a smooth causal map$^{32}$. To see this, consider the path space

$^{31}$ See Theorem 2 for the precise notation.
$^{32}$ See Definitions 9.
\( X \) whose elements are sequences \( x : [0, 1]^T \) of the form
\[
x^{(t)} \coloneqq \begin{cases} g_t \circ \ldots \circ g_0(x_0) & \text{if } t > 0 \\ x_0 & \text{if } t \leq 0.
\end{cases}
\]

Now, let \( Y \coloneqq (\mathbb{R}^d)^\infty \). Then, by construction, we immediately deduce that the operator \( f : X \to Y \) defined as
\[
f(x)_t \coloneqq \begin{cases} g_t(x^{(t)}) & \text{if } t > 0 \\ x_0 & \text{if } t \leq 0,
\end{cases}
\]
defines a \((0, \infty, \lambda)\)-smooth causal map.

**The Quantitative Advantage of the Hypernetwork for Approximating Causal Maps**

We fix a positive integer \( T \) and a 1-Lipschitz function \( G : \mathbb{R}^2 \to [0, 1] \). For any input sequence \( (z_t)_{t=1}^T \in [0, 1]^T \) define the output sequence \( (z^{(t)})_{t=1}^T \) by
\[
z^{(t)} \coloneqq G(z_t, z^{(t-1)}), \quad t = 1, \ldots, T,
\]
where we set \( z^{(0)} \coloneqq 0 \). We define the map \( f : [0, 1]^T \to \mathbb{R} \) as follows
\[
f(z_1, \ldots, z_T) \coloneqq z^{(T)} = G(z_T, z^{(T-1)}).
\]

Evidently, \( f \) is causal, whence, it can be approximated both by the CNO model or by a neural filter (which in this setting reduces to a deep ReLU FFNN). Comparing the approximation rates in either case in Tables 2 and 1 we see that an approximation by a deep ReLU network (i.e., a neural filter in this case) requires a depth of \( \tilde{O}(\epsilon_A^{-2/T}) \) and a width of \( \tilde{O}(\epsilon_A^{-2/T}) \) to approximate \( f \) uniformly on \([0, 1]^T\) to a maximal error of \( \epsilon_A \). In contrast, a CNO model only requires a latent state dimension \( P([d]) + Q = \tilde{O}(\epsilon_A^{-6} - \log_{1/2}(T - 1)) \) with hypernetwork \( h \) of depth \( \tilde{O}(T^{3/2}) \) and width \( \tilde{O}(\epsilon_A^{-6} - \log_{1/2}(T - 1)T) \) in order to achieve the same uniform approximation of \( f \) on \([0, 1]^T\) with a maximal error of \( \epsilon_A \).

As shown in [85, Theorem 2.4], the ReLU feedforward networks achieve the optimal approximation rates when approximating arbitrary Lipschitz functions, then, our rates in Theorem 2 imply that the CNO achieves super-optimal rates when approximating generic Lipschitz functions of the form in (35). Moreover, a direct examination of the above rates shows that the CNO is not cursed by dimensionality when measured in the number of time steps one wishes the uniform approximation to hold for, while deep ReLU FFNNs are. Consequently, this shows that CNOs are highly advantageous for (causal) sequential learning tasks from the approximation theoretic perspective.

6 Conclusion

We presented a first universal approximation theorem which is both causal, quantitative, compatible with infinite-dimensional operator learning, and which is not restricted to “function spaces” but is compatible with general “good” infinite-dimensional linear metric spaces. Our main contributions, Theorem 1 and Theorem 2, provided approximation guarantees for any smooth or Hölder (non-linear) operator between Fréchet spaces in the “static” or “causal” case, where temporal structure is or is not present in the approximation problem, respectively.

We showed how the CNO model can approximate a variety of solution operators, and infinite dimensional dynamical systems, arising in stochastic analysis. Moreover, in the Euclidean case, we showed that our neural filter’s approximation rates are optimal. We then showed that, when the target operator being approximated is a dynamical system, then the CNO’s approximation rates are super-optimal. Optimality is quantified in terms of the number of parameters required to approximate any arbitrary map belonging to some broad class as in constructive approximation theory of [23].

We believe the observations made in this work open up avenues for future literature. As a prime example, we would like to further optimize our CNO for the stochastic filtering problem assuming additional structural conditions. As future work, we aim to build on these results in the context of robust finance.

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A Background material for proofs

In an effort to keep the paper as self-contained as possible, this appendix contains any background material required in the derivations of our main results but not required for their formulation. We cover various properties of deep ReLU neural networks, covering and packing results, and we overview some properties of finite-dimensional “linear dimension reduction” techniques in well-behaved Fréchet spaces. We also include a list of some useful properties of generalized inverses.

A.1 Neural Network Regressors

This section contains auxiliary results on neural network approximation, parallelization, and memorization.

A.1.1 DNN Approximation for Smooth and Hölder Functions

Theorem 1.1 in [47] proves that ReLU FFNs with width \( O(N \log(N)) \) and depth \( O(L \log(L) + d) \) can approximate a function \( f \in C^s([0,1]^d) \) with a nearly optimal approximation complexity \( O(\|f\|_{C^s([0,1]^d)} N^{-2s/d} L^{-2s/d}) \), where the norm \( \| \cdot \|_{C^s([0,1]^d)} \) is defined as:

\[
\| f \|_{C^s([0,1]^d)} \overset{\text{def.}}{=} \max \{ \| \partial^\alpha f \|_{L^\infty([0,1]^d)} : \| \alpha \| \leq s, \alpha \in \mathbb{N}^d \}, \quad f \in C^s([0,1]^d).
\]  

More precisely, they state and prove the following

**Theorem 3 ([47])** Given a function \( f \in C^s([0,1]^d, \mathbb{R}) \) with \( s \in \mathbb{N}_+ \), for any \( N, L \in \mathbb{N}_+ \), there exists a function \( \varphi \) implemented by a ReLU FFN with width \( C_1 (N + 2) \log_2 (8N) \) and depth \( C_2 (L + 2) \log_2 (4L) + 2d \) such that

\[
\| \varphi - f \|_{L^\infty([0,1]^d)} \leq C_3 \| f \|_{C^s([0,1]^d)} N^{-2s/d} L^{-2s/d},
\]  

where \( C_1 = 17d^{d+1}3^dd, \ C_2 = 18s^2 \) and \( C_3 = 85(s + 1)^3d^4 \).

In particular, note that the previous result does not privilege the width to the depth and vice versa because the exponent for both \( N \) and \( L \) on the right-hand side of Equation (37) is \(-2s/d\).

On the other hand, [85], as a consequence of their main theorem for explicit error characterization, state and prove the following:

**Theorem 4 ([85])** Given a Hölder continuous function on \([0,1]^d\) of order \( \alpha \in [0,1) \) with Hölder constant \( \lambda > 0 \), i.e., \( f \in C^\alpha([0,1]^d, \mathbb{R}) \), then for any \( N \in \mathbb{N}_+ \), \( L \in \mathbb{N}_+ \) and \( p \in [1, \infty) \), there exists a function \( \varphi \) implemented by a ReLU network with width \( C_1 \max\{d, N^{\lambda/d}\}, \ N + 2 \} \) and depth \( 11L + C_2 \) such that

\[
\| f - \varphi \|_{L^p([0,1]^d)} \leq 131\lambda^2d^2(2L^2 \log_3(N + 2))^{-\alpha/d},
\]  

where \( C_1 = 16 \) and \( C_2 = 18 \) if \( p \in [1, \infty) \); \( C_1 = 3^d + 3 \) and \( C_2 = 18 + 2d \) if \( p = \infty \).

A.1.2 Efficient parallelization of ReLU neural networks ([17])

[17] propose an efficient parallelization of neural networks with different depths for a special class of activation functions, namely the ones that have the so-called \( c \)-identity requirements. Before giving a formal definition of such activation functions, we remind some quantities introduced in [17]. More precisely, \( N \) denotes the set of neural network skeletons, i.e.,

\[
N = \bigcup_{D \in \mathbb{N}^+} \bigcup_{(I_1, \ldots, I_D) \in \mathbb{N}^{D+1}} \bigcup_{k=1}^{D} \left( \mathbb{R}^{l_k \times t_{k-1}} \times \mathbb{R}^{l_k} \right),
\]  

where we follow the convention that the empty Cartesian product is the empty set. For \( \varphi \in N \), the quantity \( D(\varphi) = D \) indicates the depth of \( \varphi \), \( t_k^D = l_k \) the number of neurons in the \( k \)th layer, \( k \in \{0, \ldots, D\} \), and \( P(\varphi) = \sum_{k=1}^{D} l_k(l_{k-1} + 1) \) the number of network parameters.

If \( \varphi \in N \) is given by \( \varphi = \left([V_0, h_1], \ldots, [V_D, h_D]\right), A_k^\varphi \in C(\mathbb{R}^{t_{k-1}}, \mathbb{R}^{t_k}), k \in \{1, \ldots, D\}, \) denotes the affine function \( x \mapsto V_k x + h_k \). In addition, \( a : \mathbb{R} \mapsto \mathbb{R} \) indicates a continuous activation function which can be naturally extended to a function from \( \mathbb{R}^d \) to \( \mathbb{R}^d \), \( d \in \mathbb{N}_+ \) applying \( a \) component-wise. Finally, the \( a \)-realization of \( \varphi \in N \) is the function \( A^\varphi_a \in C(\mathbb{R}^D, \mathbb{R}^D) \) given by:

\[
A^\varphi_a = A^\varphi_0 \circ \cdots \circ A^\varphi_{D-1} \circ a \circ A^\varphi_D.
\]  

We give now the following definition (cfr. [17], Definition 4):

**Definition 12** A function \( a \in C(\mathbb{R}, \mathbb{R}) \) fulfills the \( c \)-identity requirement for a number \( c \geq 2 \) if there exists \( I \in N \) such that \( D(I) = 2 \), \( t_k^I \leq c \), and \( R_k^I = 1d^k \).

For our scopes, we note that the ReLU activation fulfills the 2-identity requirement with \( I = \{(1 - 1)^T, (0 0)^T, (1 - 1), 0\} \). In addition, the following proposition hold (cfr. [17], Proposition 5):

**Proposition 2** Assume that \( a \in C(\mathbb{R}, \mathbb{R}) \) fulfills the \( c \)-identity requirement for a number \( c \geq 2 \) with \( I \in N \). Then, the parallelization \( p_I : \bigcup_{n \in \mathbb{N}} N^N \to N \) satisfies:

\[
P(p_I(\varphi_1, \ldots, \varphi_n)) \leq \left( \frac{11}{16} c^2 d^2 n^2 - 1 \right) \sum_{j=1}^{n} P(\varphi_j)
\]  

for all \( n \in \mathbb{N} \) and \( \varphi_1, \ldots, \varphi_n \in N \), where \( l = \max_{j \in \{1, \ldots, n\}} \max\{t_j^I, l_j^I\} \). In particular, \( p_I(\varphi_1, \ldots, \varphi_n) \) denotes the parallelization of \( \varphi_1, \ldots, \varphi_n \).
A.1.3 Memory Capacity of Deep ReLU regressor ([54])

We here report a very recent lemma appearing in the deep metric embedding paper of [54]; see Lemma 20 in the just cited reference. For the sake of completeness, we remind that the aspect-ratio of the finite metric space \( (X_N, \| \cdot \|_2) \) is defined as the ratio of the maximum distance between any two points therein over the minimum separation between any two distinct points, i.e.:

\[
\text{aspect}(X_N, \| \cdot \|_2) \overset{\text{def.}}{=} \frac{\max_{x_i, x_j \in X_N} \| x_i - x_j \|_2}{\min_{x_i, x_j \in X_N, x_i \neq x_j} \| x_i - x_j \|_2}.
\]

(42)

We notice that [57] introduce the notion of an aspect ratio of a measure space as the ratio of total mass over the minimum mass at any point. The relevance of the aspect ratio to our analysis is that it quantifies the difficulty to memorize a dataset. This is because finite subset of a Euclidean space with large aspect ratio are logarithmically (in the aspect ratio) more difficult to memorize than subsets with a small aspect ratio.

Lemma 3 Let \( n, d, N \in \mathbb{N}_+ \), let \( f : \mathbb{R}^n \to \mathbb{R}^d \) be a function, and consider pair-wise distinct \( x_1, \ldots, x_N \in \mathbb{R}^n \). There exists a deep ReLU networks \( NN : \mathbb{R}^n \to \mathbb{R}^d \) satisfying

\[
NN(x_i) = f(x_i),
\]

for every \( i = 1, \ldots, N \). Furthermore, the following quantitative "model complexity estimates" hold

(i) **Width**: \( NN \) has width \( n(N - 1) + \max(d, 12) \),

(ii) **Depth**: \( NN \) has depth of the order of

\[
O\left( N \left( 1 + \frac{\log(2)}{\log(N)} \left( C_d + \frac{\log \left( N^2 \text{aspect}(X_N, \| \cdot \|_2) \right)}{\log(2)} \right) \right) \right),
\]

where \( X_N \overset{\text{def.}}{=} \{ x_1, \ldots, x_N \} \).

(iii) **Number of non-zero parameters**: The number of non-zero parameters in \( NN \) is at most

\[
O\left( N \left( \frac{11}{4} \max(n, d) N^2 - 1 \right) \left( d + \sqrt{\frac{N \log(N)}{\log(2)}} \left( C_d + \frac{\log \left( N^2 \text{aspect}(X_N, \| \cdot \|_2) \right)}{\log(2)} \right) \right) \left( \max(d, 12) \left( \max(d, 12) + 1 \right) \right) \right).
\]

The “dimensional constant” \( C_d \) is defined by

\[
C_d \overset{\text{def.}}{=} \frac{2 \log(5\sqrt{2}\pi) + 3 \log(d) - \log(d + 1)}{2 \log(2)}.
\]

A.2 Covering and packing numbers

We remind here the concept of covering and packing; we refer to the Lecture 14 of the Lecture notes of [83].

**Definition 13** \((c\text{-covering})\) Let \( (V, \| \cdot \|_1) \) be a normed space, and \( \Theta \subset V \) a subset. \( \{ V_1, \ldots, V_N \} \subset V \) is an \( c \)-covering of \( \Theta \) if \( \Theta \subset \bigcup_{i=1}^N \text{Ball}_{(V, \| \cdot \|_1)}(V_i, c) \), or equivalently, for any \( \theta \in \Theta \) there exists \( i \) such that \( \| \theta - V_i \|_1 \leq c \).

**Definition 14** \((c\text{-packing})\) Let \( (V, \| \cdot \|_1) \) be a normed space, and \( \Theta \subset V \) a subset. \( \{ \theta_1, \ldots, \theta_M \} \subset \Theta \) is an \( c \)-packing of \( \Theta \) if \( \min_{i \neq j} \| \theta_i - \theta_j \|_1 > c \) (notice the inequality is strict), or equivalently, \( \bigcap_{i=1}^M \text{Ball}_{(V, \| \cdot \|_1)}(\theta_i, c/2) = \emptyset \).

Linked to the previous definitions we have the following ones:

**Definition 15** (Covering number)

\[
N(\Theta, \| \cdot \|_1, c) \overset{\text{def.}}{=} \min\{ n : \exists c\text{-covering over } \Theta \text{ of size } n \}
\]

**Definition 16** (Packing number) If \( \#(\Theta) \geq 2 \) then we define

\[
M(\Theta, \| \cdot \|_1, c) \overset{\text{def.}}{=} \max\{ m : \exists c\text{-packing over } \Theta \text{ of size } m \}.
\]

If \( \#(\Theta) = 1 \) then, \( N(\Theta, \| \cdot \|_1, c) \overset{\text{def.}}{=} 1 \) and \( M(\Theta, \| \cdot \|_1, c) = 0 \).

When \( (V, \| \cdot \|_1) \) is the \( d \)-dimensional Euclidean space, the following theorem gives us the relation between the packing number and the covering number.

**Theorem 5** Let \( \Theta \subset V = \mathbb{R}^d \) such that \( \text{vol}(\Theta) \neq 0 \) where \( \text{vol}(\cdot) \) indicates the volume with respect to the Lebesgue measure. Set for brevity \( B = \text{Ball}_{(\mathbb{R}^d, \| \cdot \|_1)}(0, 1) \) and \( \frac{1}{2} B = \text{Ball}_{(\mathbb{R}^d, \| \cdot \|_1)}(0, \sqrt{2}/2) \), and let + denote the Minkowski sum. Then

\[
\left( \frac{1}{c} \right)^d \frac{\text{vol}(\Theta)}{\text{vol}(B)} \leq N(\Theta, \| \cdot \|_1, c) \leq M(\Theta, \| \cdot \|_1, c) \leq \frac{\text{vol}(\Theta + \frac{1}{2} B)}{\text{vol}(B)} \leq \left( \frac{3}{c} \right)^d \frac{\text{vol}(\Theta)}{\text{vol}(B)}.
\]

(43)

\[33\ [54, \text{Lemma 20}].\]
A.3 Bounded Approximation Property in Fréchet spaces with Schauder bases

We now remind the following important definition (cfr. [12] Definition 1.6) and proposition (cfr. [12] Proposition 1.16 (2)).

Definition 17 (Bounded Approximation Property) A locally convex space $E$ has the bounded approximation property (BAP, henceforth) if there exists an equi-continuous net $(A_j)_{j \in I} \in L(E)$, with $\operatorname{dim}(A_j(E)) < \infty$ for every $j \in I$ and $\lim_{j \in I} A_j(x) = x$ for every $x \in E$. In other words, the net $(A_j)_{j \in I}$ converges to the identity for the topology of point-wise or simple convergence. In all the previous expressions, $I$ denotes a generic directed indexing set.

Proposition 3 If $F$ is a barreled locally convex space with a Schauder basis, then $F$ has the BAP.

Since every Fréchet space $F$ is barreled34, then $F$ will enjoy the BAP as soon as it admits a Schauder basis. We also have the following:35 if $(A_j)_{j \in I}$ is a sequence of continuous linear operators from $E$ onto itself such that $A_0(x) \overset{\text{def.}}{=} \lim_{n \to \infty} A_j(x)$ exists for every $x \in E$, then $(A_j)_{j \in I}$ is equicontinuous by the Banach-Steinhaus36 theorem for Fréchet spaces, $A_0$ is a continuous linear operator, and the sequence $(A_j)_{j \in I}$ converges to $A_0$ uniformly on the compact subsets of $E$.

Also, we have the following proposition regarding finite-dimensional topological vector spaces:

Proposition 4 A finite-dimensional vector space $F$ can have just one vector space topology up to homeomorphism.

Remark 5 We observe the following characterization for an equi-continuous family $H \subset L(E, F)$, with $E, F$ Fréchet spaces.

- $H \subset L(E, F)$ is an equi-continuous family if and only if
- for any $V \subset F$ open neighborhood of the origin, $\cap_{x \in U} T^{-1}(V)$ is an open neighborhood of the origin ([12] page 1), if and only if
- for any $V \subset F$ open neighborhood of the origin, there exists $U \subset E$ open neighborhood of the origin such that $U \cap T(U) \subset V$.

In this last case, we call the family $H$ uniformly equi-continuous (see [52], page 169).

B Proofs

B.1 Proof of Lemma 2

Proof By assumption, $f : E \to B$ is $C^k$-Dir. This means that $D^k f : E \times E^k \to B, \ (x, h_1, \ldots, h_k) \to D^k f(x)(h_1, \ldots, h_k)$ is continuous, jointly as a function on the product space. Moreover, an arbitrary linear and continuous operator $T : E \to B$ between two Fréchet spaces is trivially $C^k$-Dir, for any $k$. By implication, $I$ and $P$ are $C^k$-Dir. By Theorem 3.6.4 in [38] (chain rule), $P \circ f \circ I$ is $C^k$-Dir. In other words,

$D^k(P \circ f \circ I) : \mathbb{R}^n \times (\mathbb{R}^n)^k \to \mathbb{R}^m, \ (x, h_1, \ldots, h_k) \mapsto D^k(P \circ f \circ I)(x)(h_1, \ldots, h_k)$

is jointly continuous in the product space. To conclude the proof, it is sufficient to choose as directions $h_1, \ldots, h_k$ in the previous expression the ones: $h_1 = e_{j_1}, \ldots, h_k = e_{j_k}$, being $\{e_1, \ldots, e_n\}$ the canonical basis of $\mathbb{R}^n$. In this case, we obtain:

$D^k(P \circ f \circ I)(x)(h_1, \ldots, h_k) = \partial_{i_1 \ldots i_k}(P \circ f \circ I)(x)$,

which is, as a function of $x$ only, continuous. Thus, we see that all the partial derivatives of order $k$ of $(P \circ f \circ I)$ are continuous on $\mathbb{R}^n$, and so $(P \circ f \circ I)$ is $C^k$ in the usual sense: Namely, $f$ is $C^k$ stable.

Before proceeding, we state and prove the following Lemma.

Lemma 4 Let $(X, d)$ and $(Y, g)$ be two metric spaces and let $F \subset C(X, Y)$ be a family of maps from $X$ to $Y$ such that $\forall \epsilon > 0 \ \exists \delta > 0 : d(x, x') \leq \delta \rightarrow \epsilon(g(f(x)), f(x')) \leq \epsilon, \ f \in F$. Then, the family $F$ has a common modulus of continuity.

Proof Let $\omega : [0, \infty) \to [0, \infty]$ be defined as:

$\delta \overset{\text{def.}}{=} \sup\{\epsilon(f(x), f(x')) : d(x, x') \leq \delta, \ f \in F\}$.

It holds that: (i) $\omega(0) = 0$; (ii) $\omega(\delta) \in [0, +\infty]$, $\delta > 0$, but $\omega(0) = 0$ in a neighborhood of 0; (iii) $\omega$ is non decreasing; (iv) continuity at 0: it holds that $\lim_{\delta \to 0^+} \omega(\delta) = \inf_{\delta > 0} \omega(\delta) = \epsilon \in [0, +\infty)$. In order to prove the statement, we have to prove that $\epsilon = 0$. Assume by contradiction that $\epsilon > 0$ and let $(\delta_n)_{n \in \mathbb{N}}$ a decreasing sequence to zero such that $\omega(\delta_n)$ converges toward $\epsilon$ from above. By definition of sup, $\exists x_n, x'_n \in X : d(x_n, x'_n) \leq \delta_n$ and $f_n \in F : \epsilon(f_n(x_n), f_n(x'_n)) > \epsilon/2, n \in \mathbb{N}$. Now, set $\epsilon = \epsilon/4$ in the definition of uniform continuity and choose $\delta > 0$ accordingly, i.e.,

$d(x, x') \leq \delta \rightarrow \epsilon(f(x), f(x')) \leq \epsilon/4, \ f \in F$.

Now, pick a $\delta_n < \delta$. Because $d(x_n, x'_n) \leq \delta_n < \delta$, we have that the following inequality holds $\epsilon(f_n(x_n), f_n(x'_n)) \leq \epsilon/4$, which is a contradiction. Finally, given $z, z' \in X, z \neq z'$, by definition it holds that:

$\epsilon(f(z), f(z')) \leq \omega(d(z, z'))$, for any $x, x' : d(x, x') \leq d(z, z')$, $f \in F$.

In particular it holds for $x = z$ and $x' = z'$, i.e. $\epsilon(f(z), f(z')) \leq \omega(d(z, z'))$, $f \in F$. Notice that if $z = z'$, than the statement is trivial.

Remark 6 We observe that, in view of Remark 5 and the fact that the metric of a Fréchet space is translation-invariant, an equi-continuous family $H \subset L(E, F)$, with $E, F$ Fréchet spaces, satisfies the assumption of Lemma 4.

34 See [69, Theorem 4.5].
35 All the authors warmly thank Prof. José Bonet for providing us a precise reference on the following fact.
36 See, e.g., [52], Result 39.1 Page 141).
B.2 Proof of Theorem 1

"Proof" In order to outline the ideas behind Theorem 1, we draw the diagram chase in Figure 7. Moreover, in order not to burden the notations, we will use the following abbreviations for any "encoding error" $\varepsilon_\mathcal{E}$: $n^{\text{in}} \overset{\text{def}}{=} n^{\text{in}}_D$ and $n^{\text{out}} \overset{\text{def}}{=} n^{\text{out}}_D$. In what follows, we detail the proof for the case that $f \in C^{A,0}_0(K,B)$. The case where $f$ belongs to $C^{A,0}_0(K,B)$ will be treated at the end of the Proof for the sake of clarity, and we will highlight the main differences with respect to the $C^{A,0}_0(K,B)$ case.

By assumption, $f : K \to B$ belongs to the trace-class $C^{A,0}_0(K,B)$. Therefore, there exists a $\lambda$-Lipschitz $C^k$-stable (non-linear) operator $F : E \to B$ such that $F(x) = f(x)$ for every $x \in K$. Whence, it is sufficient to approximate $F$, and then restrict $F$ to $K$ to deduce an estimate on $f$. Without loss of generality, we can assume that the function $f$ is not constant.

To shorten the notation, we now set for $\omega_{A,B} : [0,\infty) \to [0,\infty)$ the following estimation holds:

$$
\omega_{A,B}(t) \overset{\text{def}}{=} \int_0^t \lambda \cdot \psi(e_{-1}(\partial B, x)) \, dx,
$$

where $\lambda$ is the unique real sequence satisfying the following equality $x = \sum_{k=1}^n (\beta_k B, x) e_k$. It is manifest that these maps $\omega_{A,B}$ are linear, continuous, with finite dimensional range, and converging to the identity of $E$ as $n \to \infty$, i.e. they are equi-continuous.

Let define $\omega_{A,E} : [0,\infty) \to [0,\infty)$ the modulus of continuity of the family $(A_{E,n})_{n \in \mathbb{N}}$, which we get from Lemma 4 and Remark 6. Since $\omega_{A,E}$ might be not non-decreasing, with a slight abuse of notation we re-define it as $\frac{1}{\lambda} \int_0^t \lambda \cdot \omega_{A,E}(s) \, ds$, obtaining now the sought non-decreasing property. Moreover, let $\omega_{A,E}$ be the generalized inverse of $\omega_{A,E}$; see Subsection 2.2. A similar reasoning done into the Fréchet space $B$ with $A_{B,n}$ defined similarly to $A_{E,n}$, leads to the existence of a continuous non-decreasing modulus of continuity $\omega_{A,B} : [0,\infty) \to [0,\infty)$, whose generalized inverse will be denoted as $\omega_{A,B}^*$.

Because of the equi-continuity of $(A_{E,n})_{n \in \mathbb{N}}$, for any "encoding error" $\varepsilon_\mathcal{E}$ there exists $n^* \in \mathbb{N}$ such that, if $n \geq n^*$, then the following estimation holds: $\max_{x \in K} d_E(A_{E,n}(x), x) \leq \frac{\varepsilon_\mathcal{E}}{\lambda}$; see the argument below Proposition 3 for a precise reference of the previous fact.

Moreover, analogously as above, we derive the following inequality, because $F(K)$ is compact: $\max_{x \in F(K)} d_B(A_{B,n}(x), x) \leq \frac{\varepsilon_\mathcal{E}}{\lambda}.

Thus, the following positive integers

$$
n^{\text{in}} \overset{\text{def}}{=} \inf \left\{ n \in \mathbb{N} : \max_{x \in K} d_E(A_{E,n}(x), x) \leq \frac{1}{\lambda} \omega_{A,B}(\frac{\varepsilon_\mathcal{E}}{2}) \right\},
$$

$$
n^{\text{out}} \overset{\text{def}}{=} \inf \left\{ n \in \mathbb{N} : \max_{y \in F(K)} d_B(A_{B,n}(y), y) \leq \frac{\varepsilon_\mathcal{E}}{\lambda} \right\},
$$

are finite. At this point, we remind that $\psi$ and $\varphi$ are the following two set-theoretic identity maps

$$
\psi : (\mathbb{R}^{n^{\text{in}}, d_{E,n^{\text{in}}}}) \to (\mathbb{R}^{n^{\text{in}}}, \| \cdot \|_2),
\varphi : (\mathbb{R}^{n^{\text{out}}, \| \cdot \|_2}) \to (\mathbb{R}^{n^{\text{out}}}, d_{B,n^{\text{out}}}).
$$

and we define the following map $F : (\mathbb{R}^{n^{\text{in}}, \| \cdot \|_2}) \to (\mathbb{R}^{n^{\text{out}}, \| \cdot \|_2})$ by $F \overset{\text{def}}{=} \varphi^{-1} \circ P_{B,n^{\text{out}}} \circ F \circ I_{E,n^{\text{in}}} \circ \psi^{-1}$. Notice that since $\varphi \circ P_{B,n^{\text{out}}}$ and $I_{E,n^{\text{in}}} \circ \psi^{-1}$ are continuous linear maps and $F$ is $C^{k,\lambda}$-stable by assumption, then $F \in C^{k,\lambda}(\mathbb{R}^{n^{\text{in}}}, \mathbb{R}^{n^{\text{out}}})$.

Now, let $f_B \in \mathcal{N}_{\mathcal{E}}$ a deep ReLU core network having complexity $[d] \overset{\text{def}}{=} (d_0, \ldots, d_J)$ for a multi-index $[d]$ and $A \in \mathbb{N}$ such that $d_0 = n^{\text{in}}$ and $d_J = n^{\text{out}}$. Moreover, in order not to burden the notation, we set for $k \in \{E, B\}$ and $l \in \{\text{in, out}\}$, $I_k \overset{\text{def}}{=} I_{k,n^{l}}, P_k \overset{\text{def}}{=} P_{k,n^{l}}$ and, as before, $A_k \overset{\text{def}}{=} I_{k,n^{l}} \circ P_k$. Then, the following estimate holds:

$$
\max_{x \in K} d_B(I_B \circ \varphi \circ f_B \circ \psi \circ P_E(x), f(x))
$$

$$
\leq \max_{x \in K} d_B(I_B \circ \varphi \circ f_B \circ \psi \circ P_E(x), F(x))
$$

$$
\leq \max_{x \in K} \left( d_B(I_B \circ \varphi \circ f_B \circ \psi \circ P_E(x), I_B \circ \varphi \circ \psi^{-1} \circ P_B \circ f \circ I_E \circ \psi^{-1} \circ \psi \circ P_E(x))
\right)
$$

$$
+ \max_{x \in K} \left( d_B(I_B \circ \varphi \circ \psi^{-1} \circ P_B \circ f \circ I_E \circ \psi^{-1} \circ \psi \circ P_E(x), I_B \circ \varphi \circ \psi^{-1} \circ P_B \circ f \circ I_E \circ \psi^{-1} \circ \psi \circ P_E(x))
\right)
$$

$$
+ \max_{x \in K} d_B(I_B \circ \varphi \circ \psi^{-1} \circ P_B \circ f \circ I_E \circ \psi^{-1} \circ \psi \circ P_E(x), F(x))
$$

$$
\leq \max_{x \in K} d_B(I_B \circ \varphi \circ f_B \circ \psi \circ P_E(x), I_B \circ \varphi \circ f \circ P_E(x))
$$

$$
+ \max_{y \in F(K)} d_B(I_B \circ P_B \circ f \circ I_E \circ P_E(x), I_B \circ P_B \circ f \circ I_E \circ P_E(x))
$$

$$
+ \max_{y \in F(K)} d_B(I_B \circ P_B(y), y).
$$

\[37\] See Definition 5.
where the equality in Equation (47) follows from the fact that on the compact $K$ the maps $f$ and $F$ coincide, the inequality in Equation (48) follows from the triangular inequality by using the diagram chase in Figure 7, and the equality in Equation (49) from the definition of $F$. We now bound each of the above terms (49), (50) and (51). We start from the last one: it is controlled, by using the definition of $n^{out}$ as:

$$
\max_{y \in f(K)} d_B(I_B \circ P_B(y), y) < \frac{\varepsilon_D}{2}.
$$

We now bound the second term, i.e., the term $\max_{x \in K} d_B(I_B \circ P_B \circ F \circ P_E(x), I_B \circ P_B \circ F(x))$. Recall that $F$ is $\lambda$-Lipschitz. By using the definition of $n^{in}$ in (44), we have for $x \in K$:

$$
d_B(I_B \circ P_B \circ F \circ P_E(x), I_B \circ P_B \circ F(x)) \\
\leq \omega_{A,B} \left( d_B(F \circ P_E(x), F(x)) \right) \\
\leq \omega_{A,B} \left( \lambda d_E(I_E \circ P_E(x), x) \right) \\
\leq \omega_{A,B} \left( \frac{1}{\lambda} d_{E,A,B} \left( \frac{\varepsilon_D}{2} \right) \right) = \frac{\varepsilon_D}{2},
$$

and hence $\max_{x \in K} d_B(I_B \circ P_B \circ F \circ P_E(x), I_B \circ P_B \circ F(x)) \leq \varepsilon_D/2$.

We now control the term (49). In order to do so, we make the following observations: (1) $(R^{n^{in}}, d_{E,n^{in}})$ is a topological vector space in which the topology coincides with the standard one; see Lemma 1; (2) therefore, the identity map and its inverse are continuous. (3) Being linear, it is also uniform continuous; see [73], Page 74. These observations allow us to define $\omega_p : [0, +\infty) \to [0, +\infty)$ the modulus of continuity of the map $\varphi$ which we may assume to be, without loss of generality,38 continuous and strictly monotone; $\omega_p^*$ will denote, as usual, its generalized inverse. This allows us to compute:

$$
\max_{x \in K} d_B(I_B \circ \varphi \circ f_B \circ \psi \circ P_E(x), I_B \circ \varphi \circ F \circ \psi \circ P_E(x)) \\
\leq \max_{x \in K} \omega_{p^{*}} \left( \varphi \circ f_B \circ \psi \circ P_E(x), \varphi \circ F \circ \psi \circ P_E(x) \right) \\
\leq \max_{x \in K} \omega_{p^{*}} \left( \left\| f_B \circ \psi \circ P_E(x) - F \circ \psi \circ P_E(x) \right\|_2 \right) \\
\leq \max_{x \in K} \omega_{p^{*}} \left( \left\| f_B \circ \psi \circ P_E(x) - F \circ \psi \circ P_E(x) \right\|_2 \right) \\
= \omega_{p^{*}} \left( \max_{u \in \psi \circ P_E(K)} \left\| f_B(u) - F(u) \right\|_2 \right),
$$

where the second line of (54) holds since $I_B$ is an isometric embedding, and thus in particular $\text{Lip}(I_B) = 1$.

We now remind that $F \in C^{k,\lambda}(R^{n^{in}},R^{n^{out}})$; by Theorem 3, we can pick the above-mentioned ReLU network $f_B$ in such a way that

$$
\max_{u \in \psi \circ P_E(K)} \left\| f_B(u) - F(u) \right\|_2 \leq \omega_{p^{*}}(\varepsilon_A) =: \delta,
$$

where $\varepsilon_A$ is the “approximation error” as in the statement of the theorem; we will prove later on the existence of such $f_B$. Meanwhile, we note that the bound in Equation (54) becomes:

$$
\max_{x \in K} d_B(I_B \circ \varphi \circ f_B \circ \psi \circ P_E(x), I_B \circ \varphi \circ F \circ \psi \circ P_E(x)) \leq \omega_{p^{*}} \left( \omega_{p}(\varepsilon_A) \right) \leq \varepsilon_A.
$$

Putting together the previous equation with the estimates in Equations (52) and (53), we have that:

$$
\max_{x \in K} d_B(I_B \circ \varphi \circ f_B \circ \psi \circ P_E(x), f(x)) \leq \varepsilon_D + \varepsilon_A.
$$

Finally, we demonstrate the existence of a map $f_B$, which “depends upon some parameters” and that satisfies the estimates in Equation (55). Before proceeding, we make the following considerations: (1) $F \in C^{k,\lambda}(R^{n^{in}},R^{n^{out}})$, where $R^{n^{in}}$ and $R^{n^{out}}$ are endowed with the Euclidean topology. (2) We can define, by using a reasoning similar to the one used for $\omega_p$; $\omega_p : [0, +\infty) \to [0, +\infty)$ the modulus of continuity of the map $\psi$ which we may assume to be continuous and strictly monotone; $\omega_p^*$ will denote its generalized inverse. (3) Moreover, the following estimates hold true:

$$
d_{E,n^{in}}(P_E(x), P_E(y)) = d_E \left( \sum_{h=1}^{n^{in}} h \beta_{k}^E, x \right) e_h \left. \right| \sum_{h=1}^{n^{in}} h \beta_{k}^E, y \right) e_k \\
= d_E(A_E(x), A_E(y)) = \omega_{A,E}(d_E(x,y)) \quad \forall x, y \in E.
$$

Now, let $\text{diam}_{E}(-)$, $\text{diam}_{2}(-)$ and $\text{diam}_{E,n^{in}}(-)$ denote the diameter computed with respect to the metric $d_E$, the Euclidean distance and the distance $d_{E,n^{in}}$, respectively. It holds that:

$$
d_{E,n^{in}}(P_E(x), P_E(y)) \leq \omega_{A,E}(d_E(x,y)) \leq \omega_{A,E}(\text{diam}_{E}(K)), \quad \forall x, y \in K.
$$

Moreover, it follows that:

$$
\left\| \psi \circ P_E(x) - \psi \circ P_E(y) \right\|_2 \leq \omega_p(d_{E,n^{in}}(P_E(x), P_E(y))) \leq \omega_p(\omega_{A,E}(\text{diam}_{E}(K))), \quad \forall x, y \in K.
$$

In particular, it holds that:

$$
\text{diam}_{2}(\psi \circ P_E(K)) \leq \omega_p(\omega_{A,E}(\text{diam}_{E}(K))).
$$

We now identify a hypercube “nesting” $\psi \circ P_{E,n^{in}}(K)$, and we explicit the dependence on $n^{in}$. To this end, let

$$
r_K \overset{\text{def}}{=} \omega_p(\omega_{A,E}(\text{diam}_{E}(K))) \sqrt{\frac{n^{in}}{2(n^{in} + 1)}},
$$

38 See the argument done above for $\omega_{A,E}$.
By Jung’s Theorem\(^{39}\), there exists \(x_0 \in \mathbb{R}^{n^m}\) such that the closed Euclidean ball \(\text{Ball}_{\mathbb{R}^{n^m}}(1/2, x_0 + K)\) contains \(\psi \circ P_{E,n^m}(K)\). Now, set, for rotational convenience, \(\tilde{I} \triangleq (1, \ldots, 1) \in \mathbb{R}^{n^m}\), and define the the following affine function \(W : (\mathbb{R}^{n^m}, \|\cdot\|_2) \to (\mathbb{R}^{n^m}, \|\cdot\|_2)\):
\[
W : (\mathbb{R}^{n^m}, \|\cdot\|_2) \to (\mathbb{R}^{n^m}, \|\cdot\|_2) \quad x \to W(x) \triangleq (2r_K)^{-1}(x - x_0) + \frac{1}{2} \tilde{I},
\]
which is well-defined and invertible, and maps \(\psi \circ P_{E,n^m}(K)\) to \([0,1]^{n^m}\). In particular, the map
\[
\tilde{F} : W^{-1} : (\mathbb{R}^{n^m}, \|\cdot\|_2) \to (\mathbb{R}^{n^mout}, \|\cdot\|_2)
\]
is of class \(C^{k,\lambda}\); indeed, we already know that \(\tilde{F}\) is \(C^{k,\lambda}\)-pre-composing \(\tilde{F}\) with the smooth map \(W^{-1}\) clearly produces an object of class \(C^{k,\lambda}\). As a consequence, if we denote by \((\tilde{e}_i)_{i=1}^{n^out}\) the standard orthonormal basis of \((\mathbb{R}^{n^mout}, \|\cdot\|_2)\), then the maps \(\tilde{f}_i \triangleq (\tilde{F} \circ W^{-1}, \tilde{e}_i), \quad i \in [n^{out}]\), are of class \(C^{k,\lambda}\); where here, \((\cdot, \cdot)\) is the standard Euclidean scalar product. Moreover, by construction, for each \(x \in \mathbb{R}^{n^m}\) it holds that
\[
\sum_{i=1}^{n^out} \tilde{f}_i(x) = 1.
\]
By Jung’s Theorem\(^{39}\), there exists \(x_0 \in \mathbb{R}^{n^m}\) such that the closed Euclidean ball \(\text{Ball}_{\mathbb{R}^{n^m}}(1/2, x_0 + K)\) contains \(\psi \circ P_{E,n^m}(K)\). Now, set, for rotational convenience, \(\tilde{I} \triangleq (1, \ldots, 1) \in \mathbb{R}^{n^m}\), and define the the following affine function \(W : (\mathbb{R}^{n^m}, \|\cdot\|_2) \to (\mathbb{R}^{n^m}, \|\cdot\|_2)\):
\[
W : (\mathbb{R}^{n^m}, \|\cdot\|_2) \to (\mathbb{R}^{n^m}, \|\cdot\|_2) \quad x \to W(x) \triangleq (2r_K)^{-1}(x - x_0) + \frac{1}{2} \tilde{I},
\]
which is well-defined and invertible, and maps \(\psi \circ P_{E,n^m}(K)\) to \([0,1]^{n^m}\). In particular, the map
\[
\tilde{F} : W^{-1} : (\mathbb{R}^{n^m}, \|\cdot\|_2) \to (\mathbb{R}^{n^mout}, \|\cdot\|_2)
\]
is of class \(C^{k,\lambda}\); indeed, we already know that \(\tilde{F}\) is \(C^{k,\lambda}\)-pre-composing \(\tilde{F}\) with the smooth map \(W^{-1}\) clearly produces an object of class \(C^{k,\lambda}\). As a consequence, if we denote by \((\tilde{e}_i)_{i=1}^{n^out}\) the standard orthonormal basis of \((\mathbb{R}^{n^mout}, \|\cdot\|_2)\), then the maps \(\tilde{f}_i \triangleq (\tilde{F} \circ W^{-1}, \tilde{e}_i), \quad i \in [n^{out}]\), are of class \(C^{k,\lambda}\); where here, \((\cdot, \cdot)\) is the standard Euclidean scalar product. Moreover, by construction, for each \(x \in \mathbb{R}^{n^m}\) it holds that
\[
\sum_{i=1}^{n^out} \tilde{f}_i(x) = 1.
\]

\(^{39}\) See \[48\].

\(^{40}\) See Definition 12.
The $C^\lambda_{\alpha, t^i}(K, B)$ Case: We report to the reader the main changes of the proof.

(i) The quantity $n^n$ in Equation (44) is instead given by:

$$n^n \overset{\text{def}}{=} \inf \left\{ n \in \mathbb{N}_+ : \max_{x \in K} d_{E}(A_{E,n}(x), x) \leq \left( \frac{1}{\lambda} \omega_{A,B} \left( \frac{\varepsilon D}{2} \right) \right)^{1/\alpha} \right\}.$$ 

In this way, the estimate in Equation (53) continues to hold with $F \in C^\lambda_{\alpha, t^i}(K, B)$.

(ii) The inequality in Equation (55) is now guaranteed by Theorem 4, instead of by Theorem 3. Note, that the pre/post-composition of an $\alpha$-Hölder function with a Lipschitz function is again an $\alpha$-Hölder function.

(iii) The function $F \circ W^{-1}$ in Equation (57) is $C^\lambda_{\alpha, t^i}(K, B)$, and so, we may apply Theorem 4 to deduce that there are $n^n$ ReLU FFNNs satisfying the estimates in Equation (99).

(iv) Note that the map $u \mapsto t^i \log_2(u + 2)$ is strictly increasing on $[0, \infty)$ and surjectively maps $[0, \infty)$ onto itself. The width and the depth of each $f^i_0$ are thus provided by Theorem 4. Setting $N = L$ in that result yields

(i) **Width**: 

$$C_1 \max\left\{ n^n \left( (\omega_{A}(\varepsilon A))^{{n\alpha}} V((131 \lambda)^{n\alpha / \alpha} (n^m n^{\alpha m})^{n^{\alpha / \alpha}}) + 2 \right) \right\}$$

with $C_1 = 3^{n^n} + 3$.

(ii) **Depth**:

$$11 \left( (\omega_{A}(\varepsilon A))^{{n\alpha}} V((131 \lambda)^{n\alpha / \alpha} (n^m n^{\alpha m})^{n^{\alpha / \alpha}}) + C_2 \right)$$

with $C_2 = 18 + 2 n^n$.

(iii) The considerations on the existence of an "efficient parallelization" continue to hold with the width and depth appropriately defined by using (v).

B.3 The Dynamic Weaving Lemma

We now present our main technical tool for "weaving together" several neural filters approximating a causal map on distinct time windows. The key technical insight here is that, each neural filter approximated while the hypernetwork "weaving together" these neural filter memories, and memorization requires exponentially fewer parameters than does approximation.

**Lemma 5 (Dynamic Weaving Lemma)** Let $\|d\| = (d_0, \ldots, d_J)$, $J \in \mathbb{N}_+$, be a multi-index such that $P(\|d\|) = \sum_{j=0}^{J-1} d_j + 1 + 2 + d_J \geq 1$, and let $(f_{\theta_j})_{j \in \mathbb{N}_+}$ a sequence in $\mathcal{NN}(\|d\|)^{P(\|d\|) \times \text{ReLU}}$. Then, for every "latent code dimension" $Q \in \mathbb{N}_+$ with $Q + P(\|d\|) \geq 22$ and every "coding complexity parameter" $\delta > 0$, there is a ReLU FFNN $\tilde{h} : \mathcal{P}(\|d\|)^{Q} \to \mathcal{P}(\|d\|)^{Q} + \mathcal{Q}$, an "initial latent code" $z_0 \in \mathcal{P}(\|d\|)^{Q}$, and a linear map $L : \mathcal{P}(\|d\|)^{Q} \to \mathcal{P}(\|d\|)^{Q}$ satisfying

$$\tilde{f}_{\lambda}(z_{\ast}) = f_{\theta_1}, \qquad z_{\ast+1} = \tilde{h}(z_{\ast}).$$

for every "time" $t = 0, \ldots, |\delta^{-1}| =: T_{\delta^{-1}}$. Moreover, the "model complexity" of $\tilde{h}$ is specified by

(i) **Width**: $\mathcal{NN}$ has width at-most $(P(\|d\|) + Q)T + 12$.

(ii) **Depth**: $\mathcal{NN}$ has depth at-most of the order of

$$O\left(T \left( 1 + \sqrt{T \log(T)} \left( \frac{C}{T^2} \log(T) \right)^{\left( \left( \frac{\log(T^2 2^{1/2}) - \log(T)}{2 \log(2)} \right) \right)} \right) \right).$$

(iii) **Number of non-zero parameters**: The number of non-zero parameters in $\mathcal{NN}$ is at-most

$$O\left(T^3 (P(\|d\|) + Q)^2 \left( 1 + \left( P(\|d\|) + Q \right) \sqrt{T \log(T)} \left( \frac{\log(T^2 2^{1/2}) - \log(T)}{2 \log(2)} \right)^{\left( \left( \frac{\log(T^2 2^{1/2}) - \log(T)}{2 \log(2)} \right) \right)} \right) \right),$$

where the constant $C_d > 0$ is defined by

$$C_d \overset{\text{def}}{=} \frac{2 \log(\sqrt{2} T^2) + 3 \log(P(\|d\|) + Q) - \log(P(\|d\|) + Q + 1)}{2 \log(2)}.$$ 

In the previous expressions (i), (ii) and (iii) we set, for simplicity of notation, $T \overset{\text{def}}{=} T_{\delta^{-1}} - 1$.

**Proof** Set $P \overset{\text{def}}{=} P(\|d\|)$, and let $Q \in \mathbb{N}_+$ such that $P + Q \geq 12$. Moreover, let $R > 0$ such that $0 < \delta < R$; the precise value of $R$ will be derived below. Now, let $(\theta_t)_{t \in \mathbb{N}_+}$ be a sequence in $\mathcal{P}(\mathbb{R}^P)$ (P defined at the beginning of the proof), and let, for every $T \in \mathbb{N}_+$, $M_T$ be the constant defined as:

$$M_T \overset{\text{def}}{=} \max\{ 1, \max_{x \in \mathbb{R}^T, 0 \leq t \leq T} \| \theta_t - \theta_0 \| \} \tag{66}$$

Now, let $\mathcal{B}_{\mathbb{R}^Q, \|\cdot\|_2}(0, R) \subset \mathbb{R}^Q$ be the closed Euclidean ball centered in zero and with radius $R$. Because $\delta < R$ and because of the geometry of the Euclidean ball, there exists an integer $T_{R, \delta, Q} \geq 1$ such that $\{ z_0, \ldots, z_{T_{R, \delta, Q} - 1} \}$ is a $\delta$-packing of $\mathcal{B}_{\mathbb{R}^Q, \|\cdot\|_2}(0, R)$ meaning that $\min_{i, j = 0, \ldots, T_{R, \delta, Q} - 1 : i \neq j} \| z_i - z_j \|_2 > \delta$. It holds that:

$$\left( \frac{R}{\delta} \right)^Q \leq T_{R, \delta, Q}.$$
At this point, we define the sequence \((z_t)_{t \in \mathbb{N}} \in \mathbb{R}^{P+Q}\) in the following way:

\[
z_t \stackrel{\text{def}}{=} \begin{cases} \left( \frac{1}{P} \theta_t, \tilde{z}_t \right) & : t < T_{R,\delta,Q} \\ \left( \theta_{T_{R,\delta,Q}}, \mathbf{0}_Q \right) & : t \geq T_{R,\delta,Q} 
\end{cases}
\]

(67)

where \(\mathbf{0}_Q \stackrel{\text{def}}{=} (0, \ldots, 0) \in \mathbb{R}^Q\).

At this point, we use the (multi-dimensional) Pythagorean theorem and by construction of the sequence \((z_t)_{t \in \mathbb{N}} \in \mathbb{R}^{P+Q}\) each \(z_0, \ldots, z_{T_{R,\delta,Q}-1}\) is distinct from each other and the aspect ratio, see Equation (42), of the finite metric space \((Z_{T_{R,\delta,Q}}, \| \cdot \|_2)\), where \(Z_{T_{R,\delta,Q}} \stackrel{\text{def}}{=} \{z_0, \ldots, z_{T_{R,\delta,Q}-1}\}\), is bounded above by:

\[
\text{aspect}(Z_{T_{R,\delta,Q}}, \| \cdot \|_2) = \frac{\max_{t=0, \ldots, T_{R,\delta,Q}-1} \| z_t - z_{t+1} \|}{\min_{j=0, \ldots, T_{R,\delta,Q}-1; \neq j} \| z_t - z_{j} \|} \leq \frac{\max_{t=0, \ldots, T_{R,\delta,Q}-1} \| \theta_t - \theta_s \|_2 + \max_{k,l=0, \ldots, T_{R,\delta,Q}-1} \| z_k - z_l \|_2^{1/2}}{\min_{j=0, \ldots, T_{R,\delta,Q}-1; \neq j} \| z_t - z_{j} \|_2} \leq \frac{1 + 4R^2}{\delta}^{1/2}.
\]

(68)

Therefore, we can apply Lemma 3 to say that there exists a deep ReLU network \(\hat{h} : \mathbb{R}^{P+Q} \rightarrow \mathbb{R}^{P+Q}\) satisfying

\[z_{t+1} = \hat{h}(z_t),\]

for every \(t = 0, \ldots, T_{R,\delta,Q} - 1\). Furthermore, the following quantitative “model complexity estimates” hold

(i) **Width**: \(\hat{h}\) has width \((P + Q)T_{R,\delta,Q} + 12\).

(ii) **Depth**: \(\hat{h}\) has depth of the order of

\[\mathcal{O}\left( T_{R,\delta,Q} \left( 1 + \sqrt{T_{R,\delta,Q} \log(T_{R,\delta,Q})} \left( 1 + \frac{\log(2)}{\log(T_{R,\delta,Q})} \left[ C_d + \frac{\log \left( 2R^2 \left( 1 + 4R^2 \right)^{1/2} - \log(\delta) \right)}{\log(2)} \right] \right) \right) \right)\]

(iii) **Number of non-zero parameters**: The number of non-zero parameters in \(NN\) is at most

\[\mathcal{O}\left( T_{R,\delta,Q}(P + Q)^2 \left( 1 + (P + Q)\sqrt{T_{R,\delta,Q} \log(T_{R,\delta,Q})} \left( 1 + \frac{\log(2)}{\log(T_{R,\delta,Q})} \left[ C_d + \frac{\log \left( 2R^2 \left( 1 + 4R^2 \right)^{1/2} - \log(\delta) \right)}{\log(2)} \right] \right) \right) \right)\]

The “dimensional constant” \(C_d > 0\) is defined by

\[C_d \stackrel{\text{def}}{=} \frac{2\log(5\sqrt{2\pi}) + 3\log(P + Q) - \log(P + Q + 1)}{2\log(2)}.
\]

At this point, define the map \(\hat{h} : \mathbb{R}^{P+Q} \rightarrow \mathbb{R}^{P+Q}\) by

\[\hat{h} \stackrel{\text{def}}{=} \hat{h} \circ L_2\]

where \(L_2 : \mathbb{R}^{P+Q} \rightarrow \mathbb{R}^{P+Q}\) maps any \((\vartheta, z) \in \mathbb{R}^{P+Q}\) to \(\left( \frac{1}{M_{T_{R,\delta,Q}}} \vartheta, z \right)\). Since every linear map is affine and the composition of affine maps are again affine then \(\hat{h}\) is itself a deep ReLU network with depth, width, and number of non-zero parameters equal to that of \(\hat{h}\), respectively. Define the linear map \(L_1 : \mathbb{R}^{P+Q} \rightarrow \mathbb{R}^P\) as sending any \((\vartheta, z) \in \mathbb{R}^P \times \mathbb{R}^Q\) to \(M_{R,\delta,Q}\). By construction we have that for every \(t = 0, \ldots, T_{R,\delta,Q} - 1\)

\[\theta_{t+1} = L_1 \circ \hat{h}(z_t),\]

for every \(t = 0, \ldots, T_{R,\delta,Q}\). Setting \(R \stackrel{\text{def}}{=} 1\) and \(T \stackrel{\text{def}}{=} T_{R,\delta,Q}\) we conclude.

B.4 Proof of Theorem 2

We first introduce the following “zero-padding” notation, where \(A \oplus B\) denotes the direct sum between two matrices \(A\) and \(B\). For any \(k, s \in \mathbb{N}_+,\) we denote by \(0_{k \times s}\) the \(k \times s\) zero-matrix and by \(0_d\) the column zero-vector in \(\mathbb{R}^d\). Instead, for any non-positive integers \(k, s\), we define \(A \oplus 0_{k \times s} \stackrel{\text{def}}{=} A\), for any matrix \(A\), and \(b \oplus 0_d \stackrel{\text{def}}{=} b\), for any vector column vector \(b\). As in Theorem 1, we will detail the proof for the case that \(f\) is \((r, \kappa, \lambda)\)-smooth; the case in which \(f\) is \((r, \alpha, \lambda)\)-Hölder is analogous.

Let \(\varepsilon_A > 0\) be a given “approximation error” and a “time horizon” \(I \in \mathbb{N}_+\) satisfying \(I \leq |\delta^{-1/2}|\). By assumption, \(f : X \rightarrow Y\) is \((r, \kappa, \lambda)\)-smooth, \(X\) is compact and \(Y\) is linear. Therefore, there exists \(M\) such that for every \(i \in [I]\) there is a \(f_i \in C_{r,\kappa,\lambda}(X_{t_i-M,t_i}; B_{1,t_i})\) which satisfies the following inequality:

\[
\max_{x \in [I]} \sup_{t \in X} \| d_{B_{1,t_i}}(f_i(x_{t_i-M,t_i}), f(x)_{t_i}) \|_1 < \varepsilon_A \big/ 2,
\]

(69)

where \(M = M(\varepsilon_A, I) = O(\varepsilon_A^{-1})\). Now, for every \(i \in [I]\), for a fixed “encoding error” \(\varepsilon_D > 0\) and “approximation error” \(\varepsilon_A\), Theorem 1 ensures the existence of a neural filter \(f_i \in \mathcal{N}_{F_{R,\delta,Q}}\) satisfying the following uniform estimates:

\[
\max_{x \in [I]} \sup_{u \in X_{t_i-M,t_i}} \| d_{B_{1,t_i}}(f_i(u), f_i(u)) \|_1 < \varepsilon_D + \frac{\varepsilon_A}{2}.
\]

\[\text{See Definition 9.}\]

\[\text{See Definition 7.}\]
In particular, with the previous definition we ensure that each matrix endowed with the product topology is still a Fréchet space carrying a Schauder basis: a canonical choice for this one is provided by $d_{j}^{(i)}$. Now, for every $d_{j}^{(i)}$ dimensional, instead of being $d_{j}^{(i)} × d_{j}^{(i)}$-dimensional. For each $i ∈ [[I]]$ and $j ∈ [[J^{∗}]]$, set

$$\|I\|_{j} = \{i ∈ [[I]] : d_{j}^{(i)} \text{ and } j ≤ J\}$$

and $d_{j}^{*}$ be the maximum width among the $j^{th}$ layers, i.e. $d_{j}^{*} = \max_{i ∈ [[I]]} d_{j}^{(i)}$. Define $A ⊕ 0 = A$ for any matrix $A$. Let $d_{j}^{*} = (d_{j}^{[1]},...,d_{j}^{[I]}).$ Now, for each $i ∈ [[I]]$ and $j ∈ [[d_{j}^{*},I]]$ we define:

$\tilde{A}_{j}^{(i)} = \begin{cases} A_{j}^{(i)} ⊕ 0 (d_{j}^{i+1} − d_{j}^{(i)}) × (d_{j}^{i+1} − d_{j}^{(i)}) : & \text{if } j ≤ J_{(i)} \\ I d_{j}^{i} ⊕ 0 (d_{j}^{i+1} − d_{j}^{(i)}) × (d_{j}^{i+1} − d_{j}^{(i)}) : & \text{if } J_{(i)} < j ≤ J^{∗} \end{cases}$

$\tilde{b}_{j}^{(i)} = \begin{cases} b_{j}^{(i)} ⊕ 0 (d_{j}^{i+1} − d_{j}^{(i)}) : & \text{if } j ≤ J_{(i)} \\ 0 & \text{if } J_{(i)} < j ≤ J^{∗} \end{cases}$

$\tilde{d}_{j}^{(i)} = \begin{cases} 0 & \text{if } J_{(i)} < j ≤ J^{∗} \end{cases}$

In particular, with the previous definition we ensure that each matrix $\tilde{A}_{j}^{(i)}$ is $d_{j}^{i+1} × d_{j}^{(i)}$-dimensional, instead of being $d_{j}^{i+1} × d_{j}^{(i)}$. Now, for every $i ∈ [[I]]$ we define $\theta_{j}^{(i)}$ by $\theta_{j}^{(i)} = \tilde{A}_{j}^{(i)} \tilde{b}_{j}^{(i)} \tilde{d}_{j}^{(i)}$. Instead, for every $i > I$ we set $\theta_{j}^{(i)} = \tilde{d}_{j}^{(i)}$. Notice that by construction

$$\hat{f}_{k}^{(i)} \hat{h}_{k} \hat{I}_{k} = (\hat{f}_{k}^{(i)} \hat{h}_{k} \hat{I}_{k})_{k}$$

is a sequence in $\mathcal{N} \mathcal{A}_{k}^{(d)}$. We therefore apply Lemma 5. In particular, for every there is a (P)ReLU FFNN $\hat{h} : \mathbb{R}^{P([d^{*}] + Q)} \rightarrow \mathbb{R}^{P([d^{*}] + Q)}$, with $P([d^{*}]) = \sum_{j=0}^{j^{∗}−1} d_{j}^{(i)} (d_{j}^{i+1} + 2) + d_{j+1} \geq 1$, an "initial latent code" $\hat{z} ∈ \mathbb{R}^{P([d^{*}] + Q)}$, and a linear map $L : \mathbb{R}^{P([d^{*}] + Q)} \rightarrow \mathbb{R}^{P([d^{*}] + Q)}$, satisfying

$$\hat{f}_{k}^{(i)} \hat{h}_{k} \hat{I}_{k} = \hat{f}_{k}^{(i)} \hat{h}_{k} \hat{I}_{k} \hat{z} \hat{t}_{k+1} = \hat{h}(\hat{z}_{k})$$

for every "time" $i = 1, ..., I_{k} Q, 1$, where $I_{k} Q = \{\delta^{−Q}\}$. The depth and the width of the network are provided by the same lemma with $T_{k} Q = I_{k} Q$. Equations (71) and (72) imply that

$$\hat{f}_{k}^{(i)} \hat{h}_{k} \hat{I}_{k} = \hat{f}_{k}^{(i)} \hat{h}_{k} \hat{I}_{k} \hat{z} \hat{t}_{k+1} = \hat{h}(\hat{z}_{k})$$

for every $i ∈ [[I]]$. At this point, combining Equations (69) and (70), we have:

$$\max_{i ∈ [[I]]} \sup_{x ∈ X} d_{B_{i}}(f_{i}(x_{(i−M,t_{i})}), f(x_{i})) ≤ \max_{i ∈ [[I]]} \sup_{x ∈ X} d_{B_{i}}(f_{i}(x_{(i−M,t_{i})}), f(x_{i})) + \max_{i ∈ [[I]]} \sup_{x ∈ X} d_{B_{i}}(f_{i}(x_{(i−M,t_{i})}), f_{i}(x_{(i−M,t_{i})}))$$

which concludes the proof.

C Technical Lemmata

Lemma 6 Let $(E, (p_{e})_{e=1}^{∞}, (e)_{e=1}^{∞})$ (respectively $(F, (q_{m})_{m=1}^{∞}, (f)_{m=1}^{∞})$) be a Fréchet space with seminorms $(p_{e})_{e=1}^{∞}$ (respectively $(q_{m})_{m=1}^{∞}$) and Schauder basis $(e_{k})_{k}$ (respectively $(f_{k})_{k}$). Then the Cartesian product

$$G = E × F$$

endowed with the product topology is still a Fréchet space carrying a Schauder basis: a canonical choice for this one is provided by $(b_{t})_{t=1}^{∞} ⊂ G$, where

$$\begin{cases} b_{2t−1} = (e_{t}, 0), & t = 1, 2, ... \\ b_{2t} = (0, f_{t}), & t = 1, 2, ... \end{cases}$$

43 Refer to equation (17)
Proof From elementary results from functional analysis and topology, it is clear that $G$ endowed with the product topology is a topological vector space. This topology can be induced also by a metric, e.g.,

$$d : G \times G \to [0, \infty)$$

$$d((e, f), (e', f')) \stackrel{\text{def.}}{=} d_E(e, e') + d_F(f, f'), \quad (e, f), (e', f') \in G,$$

where $d_E$ (respectively $d_F$) is a compatible metric for $E$ (respectively $F$). Evidently, $(G, d)$ is also complete. This topology is locally convex because it can be induced by the following countable collection of seminorms

$$\gamma_{\ell, m}(e, f) \stackrel{\text{def.}}{=} p_{\ell}(e) + q_m(f), \quad \ell, m \in \mathbb{N}_+, e, f \in F.$$

Define the following elements of $G$:

$$\begin{align*}
&b_{2t-1} \stackrel{\text{def.}}{=} (e_t, 0), \quad t = 1, 2, \ldots \\
&b_{2t} \stackrel{\text{def.}}{=} (0, f_t), \quad t = 1, 2, \ldots
\end{align*}$$

We claim that $\{b_t\}_{t=1}^{\infty}$ is a Schauder basis for $G$. Indeed, let $x = (e, f)$, with

$$e = \sum_{k=1}^{\infty} \beta^E_k(e) e_k, \quad f = \sum_{k=1}^{\infty} \beta^F_k(f) f_k.$$

Let $\varepsilon > 0$ be arbitrary. Since $(e_k)_k$ and $(f_k)_k$ are Schauder basis, it follows that there exists $N_\varepsilon$ such that for all $N \geq N_\varepsilon$

$$d_E \left( \sum_{k=1}^{N} \beta^E_k(e) e_k, e \right) < \varepsilon/2,$$

$$d_F \left( \sum_{k=1}^{N} \beta^F_k(f) f_k, f \right) < \varepsilon/2.$$

Set $T_\varepsilon = 2N_\varepsilon$ and consider $T \in \mathbb{N}_+$ with $T \geq T_\varepsilon$. Set

$$x^T \stackrel{\text{def.}}{=} \beta^E_T(e) b_1 + \beta^F_T(f) b_2 + \beta^E_{T-1}(e) e_{T-1} + \beta^F_{T-1}(f) f_{T-1} + \cdots + ub_T \in G$$

whereas

$$u = \begin{cases} 
\beta^E_{T+1/2}(f), & \text{if } T \text{ even} \\
\beta^E_{(T+1)/2}(e), & \text{if } T \text{ odd}.
\end{cases}$$

Thus, for $T$ odd, we have

$$d(x^T, x) = d_E(\beta^E_T(e) e_1 + \cdots + \beta^E_{T+1/2}(e) e_{T+1/2}, e)$$

$$+ d_F(\beta^F_T(f) f_1 + \cdots + \beta^F_{T-1/2}(f) f_{T-1/2}, f)$$

and, for $T$ even,

$$d(x^T, x) = d_E(\beta^E_T(e) e_1 + \cdots + \beta^E_{T+1/2}(e) e_{T+1/2}, e)$$

$$+ d_F(\beta^F_T(f) f_1 + \cdots + \beta^F_{T-1/2}(f) f_{T-1/2}, f).$$

In both cases, we deduce by construction that

$$d(x^T, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon, \quad T \geq T_\varepsilon,$$

namely $x^T \to x$ as $T \to \infty$. This proves that any $x \in G$ can be written as

$$x = \sum_{t=1}^{\infty} x_t b_t$$

with

$$x_t = \begin{cases} 
\beta^E_{T+1/2}(f), & \text{if } t \text{ even} \\
\beta^E_{(T+1)/2}(e), & \text{if } t \text{ odd}.
\end{cases}$$

(75)

In order to prove that such decomposition is unique, suppose that there exists $x \in G$ such that

$$\sum_{t=1}^{\infty} x_t b_t = x = \sum_{t=1}^{\infty} \tilde{x}_t b_t$$

with $x_t$ defined as in (75) and with $x_t \neq \tilde{x}_t$ for some $t$. Let $t_0$ be one of those coefficients, and suppose wlog that $t_0 = 2j$: the odd-case is similar and it will not be treated. By projecting on the factor $F$ we obtain ($H_F =$canonical projection)

$$H_F \sum_{t=1}^{\infty} x_t b_t = H_F \sum_{t=1}^{\infty} \tilde{x}_t b_t$$

$$\sum_{t=1}^{\infty} x_t H_F b_t = \sum_{t=1}^{\infty} \tilde{x}_t H_F b_t$$

$$\sum_{t=1}^{\infty} x_{2t} f_t = \sum_{t=1}^{\infty} \tilde{x}_{2t} f_t$$

and $x_{2j} \neq \tilde{x}_{2j}$, contradicting the fact that $(f_t)_t$ is a Schauder basis. Therefore, the expansion (74) is unique, and this concludes the proof.
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