U-Duality and Integral Structures

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ABSTRACT

We analyze the U-duality group for the case of a type II superstring compactified to four dimensions on a K3 surface times a torus. The various limits of this theory are considered which have interpretations as type IIA and IIB superstrings, the heterotic string, and eleven-dimensional supergravity, allowing all these theories to be directly related to each other. The integral structure which appears in the Ramond-Ramond sector of the type II superstring is related to the quantum cohomology of general Calabi-Yau threefolds which allows the moduli space of type II superstring compactifications on Calabi-Yau manifolds to be analyzed.

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1 Introduction

Consider a theory of supergravity in $d$ dimensions with $N$ supersymmetries. Given $d$, for a suitably large $N$ one expects the local geometry of the space parametrized by the massless scalars to be tightly constrained. This was shown, for example, for the case of maximal $N$ in $[1, 2]$ and for $N = 4, d = 4$ in $[3]$. In each case the parameter space is of the form $G/G_c$ where $G_c$ is the maximal compact subgroup of the Lie group $G$.

If such a theory is derived as the low energy limit of a superstring theory one generally expects that different values of the parameters derive from inequivalent string vacua. However, some parameter values may correspond to equivalent theories. One therefore expects that the moduli space of such string theories takes the form $U\backslash G/G_c$, where $U$ is some discrete subgroup of $G$. The group $U$ is a purely stringy phenomenon and cannot be determined from considerations of supergravity alone.

If one envisions the effective theory of supergravity to have arisen as the compactification of some string theory on some compact space $X$, then some of the elements of $U$ may be identified. At the simplest level, the classical description of the moduli space of $X$ will lead to some of these identifications. The “T-duality” elements arise from a conformal field theory description of $X$ — two geometrically distinct $X$’s may give rise to the same conformal field theory. The “S-duality” elements arise from an $SL(2, \mathbb{Z})$ acting on the axion-dilaton system (in ten dimensions) and include a $\mathbb{Z}_2$ element that exchanges strong and weak string-coupling. It would be unreasonable to expect conformal field theory to “see” such a symmetry directly. In addition, the S-duality and T-duality generators need not commute (as was shown in the example studied in $[4]$, for example) and so may generate a larger group. In the cases of interest to us this larger group will be $U$, the group of “U-dualities”.

The group $U$ was first analyzed in $[5]$ by making use of some conjectures regarding the spectrum of solitons in the theory. In this letter we will be concerned with U-duality in four-dimensional theories. In particular we will focus on the case of the type I (A or B) superstring compactified on K3 times a torus. Rather than use the approach of $[5]$, we will generally try to avoid using any results requiring a knowledge of solitons (although there has been some recent progress in this subject $[6, 7, 8]$). Instead, we will follow more closely the logic of using the simpler structure of various weak coupling limits as was done in $[9]$. We examine how the discrete group $U$ may be derived purely from a knowledge of conformal field theory (and hence T-duality) and the hypothesized S-duality on the type IIB superstring in ten dimensions.

Unfortunately, for realistic models the value of $N$ is smaller than what is required for the above scenario to work. In these cases the T-duality of conformal field theory moduli space does not provide as much information and a phase structure becomes important $[10, 11]$. We must therefore expect the U-duality picture to become much more subtle. However, the breakdown of the T-duality method has not prevented the moduli spaces of general $N = 2$ superconformal field theories from being determined to a large extent. In many cases mirror
symmetry can effectively solve the problem as was done in [12] (for a review see [13]). We will see that the integral structures coming from the U-duality picture of the simple cases analyzed generalizes naturally to the general Calabi–Yau manifold and this likewise allows for a determination of the moduli space of type II superstring compactifications.

2 U-duality for K3 Surfaces

In order to explain the tools we will use for the desired case of a K3 surface times a torus we will explicitly review the simpler case of a type IIB superstring compactified on a K3 surface [9].

Given a compactified string theory, one might hope to find a more conventional physical description of the theory in a limit where some parameter tends to zero or infinity, i.e., by going out to the boundary of the moduli space. This idea was analyzed and applied in [9] and we review the method quickly here.

Consider some semi-simple Lie group G. Take the Dynkin diagram for the algebra associated to the group and remove one vertex. The resultant diagram may be associated to a subgroup H of G. This procedure determines a maximal embedding

\[ G \supset H \times J, \]

where J is a continuous abelian group of dimension 1. Clearly J must be U(1) or the group \( \mathbb{R}_+ \) of positive real numbers under multiplication. We will be interested in the case \( J \cong \mathbb{R}_+ \). For a given real form of a Lie group one may label the dots in the Dynkin diagram according to whether the resultant J will be compact or not. For the case studied in [9], \( G \cong E_{7(7)} \) (which is a maximally noncompact form) and any dot leads to \( J \cong \mathbb{R}_+ \).

Dividing both sides of (1) in the case \( J \cong \mathbb{R}_+ \) by the maximal compact subgroup leads in general to a decomposition of the following sort

\[ \frac{G}{G_c} \cong \frac{H}{H_c} \times \mathbb{R}_+ \times \mathbb{R}^M, \]

where \( \mathbb{R}^M \) is some linear vector space which comes equipped\(^1\) with a representation of H. The parameter describing \( \mathbb{R}_+ \) can be taken to infinity to define the boundary required. Thus each dot associated to a noncompact J in the Dynkin diagram defines some limit in the moduli space.

Compactifying the type IIB string on K3 to \( d = 6 \) gives the supergravity studied in [14] leading to the result that \( G \cong O(5, 21) \). This has the following Dynkin diagram

\[ \text{Diagram} \]

\[ e_1 \quad e_2 \quad e_3 \quad e_4 \quad e_5 \]

\(^1\)In fact, the subgroup of G which preserves the decomposition (2) contains both H and a translation subgroup isomorphic to \( \mathbb{R}^M \); the representation of H on \( \mathbb{R}^M \) intertwines the two.
where the black dots represent roots associated to noncompact generators. Thus we may obtain 5 limits by removing each of \(e_1, \ldots, e_5\) in turn. First, in order to make contact with the geometry we understand, we should go to the large radius limit of the compact K3 surface. The corresponds to removing \(e_2\) giving the following decomposition

\[
\frac{O(5, 21)}{O(5) \times O(21)} \cong \frac{O(3, 19)}{O(3) \times O(19)} \times \frac{\text{Sl}(2)}{U(1)} \times \mathbb{R}_+ \times \mathbb{R}^{45}.
\]

Up to discrete group identifications, the first factor on the right-hand side is the moduli space of Ricci-flat metrics on a K3 surface of fixed volume \([15]\) (including orbifold metrics \([16, 17]\)). The second factor is the moduli space of type IIB strings in ten dimensions and the \(\mathbb{R}_+\) is the volume of the K3 surface. The 45 other fields consist of 22 \(B\)-field deformations and 23 \(R\)-\(R\) moduli. (There is one additional \(R\)-\(R\) field within the \(\text{Sl}(2)/U(1)\) factor.) We may use this knowledge as follows to build up part of the U-duality group on the left-hand side.

The group \(O(5, 21)\) acts naturally on the space \(\mathbb{R}^{5,21}\) into which we can embed a unique even self-dual lattice \(\Lambda^{5,21}\). The subgroup \(O(\Lambda^{5,21}) \subset O(5, 21)\) preserving this lattice is a maximal discrete group \([18]\). That is to say, if we divide \(O(5, 21)\) by a group containing \(O(\Lambda^{5,21})\) as a proper subgroup the resulting quotient will not be Hausdorff. We will embed discrete group actions on the right-hand side of (4) into \(O(\Lambda^{5,21})\) and see how much of \(O(\Lambda^{5,21})\) we can generate.

The inner product on \(\mathbb{R}^{5,21}\) may be applied to the generators of \(\Lambda^{5,21}\) to form the matrix \((-E_8)^{\oplus 2} \oplus H^{5,5}\), where \(E_8\) is the Cartan matrix of \(E_8\) and

\[
H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

The discrete group action by which \(O(3, 19)/(O(3) \times O(19))\) should be divided to obtain the moduli space of Ricci-flat metrics is\(^2\) \(O(\Lambda^{3,19})\). In this action, \(\Lambda^{3,19}\) comes from the second cohomology group \(H^2(K3, \mathbb{Z})\) and has intersection form \((-E_8)^{\oplus 2} \oplus H^{3,3}\). It is clear how we embed \(O(\Lambda^{3,19})\) into \(O(\Lambda^{5,21})\).

The conjecture for U-duality for the type IIB superstring in ten dimensions tells us that the group \(\text{Sl}(2)/U(1)\) should be divided by \(\text{Sl}(2, \mathbb{Z})\) in the second factor of the right-hand side of (4). Since this part of the discrete group acts on a separate factor of the decomposition, we must embed this \(\text{Sl}(2, \mathbb{Z})\) into \(O(\Lambda^{5,21})\) in such a way that it commutes with \(O(\Lambda^{3,19})\). This can be done as follows. Take the lattice orthogonal to \(\Lambda^{3,19} \subset \Lambda^{5,21}\) with intersection form \(H \oplus H\). Let \(\alpha_1, \alpha_2\) be two generators of this lattice such that \(\langle \alpha_1, \alpha_1 \rangle = \langle \alpha_1, \alpha_2 \rangle = \langle \alpha_2, \alpha_2 \rangle = 0\) (i.e., one generator comes from each \(H\) factor). The group \(\text{Sl}(2, \mathbb{Z})\) may then be taken to act on this pair to form the required embedding.

\(^2\)In order to avoid cluttering notation we are being a little careless with some \(\mathbb{Z}_2\) factors. If these are properly taken into account, our results are not affected.
Thus far we have built $O(\Lambda^{3,19}) \times Sl(2,\mathbb{Z})$ as part of the U-duality group. To proceed further we go to another limit point. In the weak string-coupling limit we expect to recover the conformal field theory picture of the moduli space. In order to analyze this limit we use the root labeled $e_1$ in (3). This leads to the decomposition

$$\frac{O(5,21)}{O(5) \times O(21)} \cong \frac{O(4,20)}{O(4) \times O(20)} \times \mathbb{R}_+ \times \mathbb{R}^{24},$$

where the first factor on the right is the moduli space of $N = 4$ superconformal field theories corresponding to K3 surfaces [19], the second factor is the dilaton and the third factor comes from the 24 R-R fields. It was shown in [20] that the discrete group action on the first factor should be $O(\Lambda^{4,20})$. In the decomposition (6), the group $O(4,20)$ acts on the space $\mathbb{R}^{24}$ as the group of rotations with respect to an inner product of signature $(4,20)$. We will denote this space by $\mathbb{R}^{4,20}$ and note that it forms the standard vector representation of the rotation group $O(4,20)$. Another subgroup of $O(5,21)$ (isomorphic to $\mathbb{R}^{4,20}$) acts as translations on this space. One can show that the intersection of this translation group with the discrete group $O(\Lambda^{5,21})$ gives the group of translations by the lattice $\Lambda^{4,20}$. By combining these facts with our earlier decomposition we find a much larger structure within $O(\Lambda^{5,21})$.

The classical moduli space of K3 surfaces embeds nicely into the moduli space of $N = 4$ conformal field theories showing that $\Lambda^{3,19} \subset \Lambda^{4,20}$. That is, $\Lambda^{4,20}$ can be taken to be $\Lambda^{3,19}$ plus one of the two $H$ sublattices mentioned earlier on which $Sl(2,\mathbb{Z})$ acts. Consider first the generator “$\tau \rightarrow \tau + 1$” of $Sl(2,\mathbb{Z})$. This generates translations in $\mathbb{R}^{4,20}$ by one of the unit vectors. Since $O(\Lambda^{4,20})$ acts within this space, we may generate the whole $\Lambda^{4,20}$ lattice of translations in $\mathbb{R}^{4,20}$. Now consider the other generator of $Sl(2,\mathbb{Z})$, “$\tau \rightarrow -1/\tau$”. This exchanges the two $H$ sublattices of $\Lambda^{5,21}$ that were orthogonal to $\Lambda^{3,19}$. In fact, it swaps the $H$ lattice “outside” $\Lambda^{4,20}$ with one of the ones inside. We thus have three sets of generators to generate a subgroup of $O(\Lambda^{5,21})$:

1. The group $O(\Lambda^{4,20})$.

2. The group of translations $\Lambda^{4,20}$.

3. The $\mathbb{Z}_2$ exchanging two $H$ sublattices one of which is the orthogonal complement to $\Lambda^{4,20}$.

This is precisely equivalent to the situation studied in [20] in which the space of conformal field theories on K3 surfaces was analyzed. In the latter case the three sets of generators understood “classically” were

1. The group $O(\Lambda^{3,19})$ of classical automorphisms of a K3 surface.

2. The group of translations $\Lambda^{3,19}$ of the $B$-field.
3. The $\mathbb{Z}_2$ exchanging two $H$ sublattices one of which is the orthogonal complement to $\Lambda^{3,19}$ (up to an $O(\Lambda^{3,19})$ rotation) — namely mirror symmetry.

Just as in this latter case, the generators we have specified within $O(\Lambda^{5,21})$ are sufficient to generate the entire group. That is, the U-duality group for IIB superstrings on a K3 surface is $O(\Lambda^{5,21})$ as stated in [9].

Note the structure of the R-R part of the moduli space we have uncovered without really thinking directly about these fields. Firstly we have shown that they are periodic thanks to the division by the $\Lambda^{4,20}$ lattice (that is, they live on a 24-dimensional torus). Secondly we see that these fields transform as a vector of $O(4,20)$ and so transform in the cohomology of the K3 surface. We will have more to say about these points later.

3 The Case $K3 \times T^2$

The example we wish to study in detail concerns a type II superstring compactified to four dimensions on the product of a K3 surface and a 2-torus. For definiteness we will first specify that we have a type IIB string. This gives an $N = 4$ supergravity theory in four dimensions with 22 vector multiplets. The results of [3] tell us that for the moduli space in question $G \cong O(6,22) \times Sl(2)$. We now wish to study limits in the moduli space which may allow us to identify parts of the discrete group $U$.

The group $G \cong O(6,22) \times Sl(2)$ has Dynkin diagram

\[ e_7 \rightarrow e_6 \rightarrow e_5 \rightarrow e_4 \rightarrow e_3 \rightarrow e_2 \rightarrow e_1 \]  

Let us now identify some of the limits we can obtain. Firstly let us go to the large area limit of the torus. This is achieved by decomposing over $e_2$:

\[
\frac{O(6,22)}{O(6) \times O(22)} \times \frac{Sl(2)}{U(1)} \cong \frac{O(5,21)}{O(5) \times O(21)} \times \frac{Sl(2)}{U(1)} \times \mathbb{R}_+ \times \mathbb{R}^{26}. 
\]  

The first factor is clearly the space of type IIB strings on a K3 surface as we might expect, the second factor is the complex structure of the torus and the third factor is its area. The group $O(\Lambda^{5,21})$ acts on the first factor as explained above and $Sl(2,\mathbb{Z})$ acts on the second factor as is well-known.

Now let us go to the weak string-coupling limit. This is achieved by decomposing over $e_3$:

\[
\frac{O(6,22)}{O(6) \times O(22)} \times \frac{Sl(2)}{U(1)} \cong \frac{O(4,20)}{O(4) \times O(20)} \times \frac{Sl(2)}{U(1)} \times \mathbb{R}_+ \times \mathbb{R}^{49}. 
\]  

The first factor is the moduli space of conformal field theories on a K3 surface, the second and third factors give the space of conformal field theories on a torus and the fourth factor...
is the dilaton. The groups $O(\Lambda^{4,20})$, $Sl(2,\mathbb{Z})$ and $Sl(2,\mathbb{Z})$ are known to act respectively on the first three factors. We may use the same trick as in the previous section to use these two limits to build the U-duality group. The result is $U \cong O(\Lambda^{6,22}) \times Sl(2,\mathbb{Z})$, in agreement with the conjecture in [3]. Again, no larger group than this may act on the “Teichmüller space” $G/G_c$ without destroying some nice properties of the moduli space. Note that the $Sl(2,\mathbb{Z})$ which appears as a separate factor in the U-duality group comes from the moduli space of complex structures on the torus.

It is interesting to compare this with the moduli space of heterotic strings compactified toroidally to four dimensions. In fact, that compactified weakly-coupled heterotic string is the limiting theory corresponding to the decomposition over $e_1$. From our analysis we recover the precise structure expected from string-string duality, except that the $Sl(2,\mathbb{Z})$ now acts on the four-dimensional axion-dilaton system. This model was thoroughly analyzed in [21]. An exchange of the rôles of $Sl(2,\mathbb{Z})$ groups between the type II superstring and the heterotic string in four dimensions was discussed in [22, 9] although the two $Sl(2,\mathbb{Z})$’s exchanged there were not the same ones we have here. To understand the picture fully we need to look at mirror symmetry.

Consider the superconformal field theory description of a $d$-dimensional Minkowski space $\mathbb{R}^{d-1,1}$, where $d$ is even. For simplicity we go to the light-cone gauge in which the target space fermions transform in spinor representations of $SO(d-2)$. The superconformal field theory has an $N = 2$ structure where the $\hat{u}(1)$ affine algebra from the $N = 2$ algebra lies in the affine algebra $\hat{so}(d-2)$.

Let $C$ be a $\mathbb{Z}_2$ automorphism of the weight space of $so(d-2)$ such that $C : \lambda \rightarrow -\lambda$ for any $\lambda$ in the weight space. If $R$ is a spinor representation of $so(d-2)$ then $C$ is a symmetry of the corresponding set of weights if and only if $d \in 4\mathbb{Z} + 2$. If on the other hand $d \in 4\mathbb{Z}$, the action $C$ takes each spinor to the spinor of opposite chirality (the spinor is in a complex representation). Note that $C$ changes the sign of the $U(1)$ charge of the weights for any embedding $U(1) \subset SO(d-2)$.

The mirror map takes an $N = (2,2)$ superconformal field theory and changes the sign of the $U(1)$ charges for the left sector and leaves the right sector untouched. Thus, for spinors in $d$-dimensional Minkowski space it will leave everything invariant if $d \in 4\mathbb{Z} + 2$ and change the relative chirality of the left and right spinors if $d \in 4\mathbb{Z}$.

Therefore, in $d = 10$, the type IIA string is self-mirror and the type IIB string is self-mirror. Indeed mirror symmetry had better not identify these two theories since their moduli spaces are quite different. Similarly the IIA and IIB string are quite different when compactified on a K3 surface. However, when we compactify on a space of 2 or 6 real dimensions we expect that the type IIA string compactified on such a space should be mirror to the IIB string compactified on the mirror.\(^3\) For the 2-torus the mirror map is a special kind of “$R \leftrightarrow 1/R$” duality and this mirror statement between IIA and IIB string compactifica-

\(^3\)The similarities between IIA and IIB theories on Calabi–Yau threefolds were studied some time ago [23], and the connection to mirror symmetry has been pointed out in [24, 25].
tions was effectively shown in \cite{26, 27}. For compactifications on Calabi–Yau threefolds this statement is potentially more powerful. We will discuss mirror symmetry in more detail in section 5.

We may apply mirror symmetry to the decomposition (9). The mirror map acts within the group $O(\Lambda^{4,20})$ for a K3 surface \cite{20} but for the torus it exchanges the two $SL(2)/U(1)$ factors \cite{28} and hence the corresponding $SL(2,\mathbb{Z})$'s. Thus our statements regarding the IIB string compactified on a K3 times a torus are precisely the same as those for the IIA string so long as we reverse the rôles of the $SL(2)/U(1)$ factors and the $SL(2,\mathbb{Z})$'s which act upon them. The moduli space of type IIA strings has a U-duality group $O(\Lambda^{6,22}) \times SL(2,\mathbb{Z})$ where now the $SL(2,\mathbb{Z})$ acts on the complex Kähler form of the torus. Thus we see a kind of “triality” in which this $SL(2,\mathbb{Z})$ has three different rôles according to whether we are talking about the IIA, IIB or heterotic string.

4 Exotic Limits

It is interesting to explore the other possible decompositions of the diagram (7). For $e_4$ we obtain

$$\frac{O(6,22)}{O(6) \times O(22)} \times \frac{SL(2)}{U(1)} \cong \frac{O(3,19)}{O(3) \times O(19)} \times \frac{SL(3)}{SO(3)} \times \frac{SL(2)}{U(1)} \times \mathbb{R}_+ \times \mathbb{R}^{69}. \quad (10)$$

The first factor is the space of Ricci-flat metrics on a K3 surface of fixed volume. The second factor is the moduli space of a 3-torus $T^3$. The third factor we associate to a four-dimensional axion-dilaton system. From \cite{9} we thus see that this is eleven-dimensional supergravity compactified down to four dimensions on K3$ \times T^3$. We have thus been able to identify limits corresponding to (at least as far as their low-energy effective theories are concerned) type IIA and type IIB superstrings, the heterotic string and eleven-dimensional supergravity all in the same moduli space!

Further interesting features are found at other limit points. Consider $e_5$. This gives

$$\frac{O(6,22)}{O(6) \times O(22)} \times \frac{SL(2)}{U(1)} \cong \frac{O(2,18)}{O(2) \times O(18)} \times \frac{SL(4)}{SO(4)} \times \frac{SL(2)}{U(1)} \times \mathbb{R}_+ \times \mathbb{R}^{86}. \quad (11)$$

The first factor appears to be related to the moduli space of K3 surfaces. The space $O(3,19)/(O(3) \times O(19))$ may be thought of as the Grassmanian of space-like 3-planes in $\mathbb{R}^{3,19}$. To identify this as the space of metrics on a K3, we take $\mathbb{R}^{3,19}$ to be $H^2(K3, \mathbb{R})$ equipped with the natural metric from the cup product; the 3-plane is spanned by the real and imaginary parts of the holomorphic 2-form $\Omega$ and the Kähler form $J$ \cite{15}. Usually one normalizes $J$ such that $\langle J, J \rangle = 1$ which is equivalent to fixing the volume of the K3 surface to 1 since

$$Vol(K3) = \int J \wedge J = \langle J, J \rangle. \quad (12)$$
The volume of the K3 surface may then be represented by a separate $\mathbb{R}_+$ factor in the moduli space. Embedded in the space $\mathbb{R}^{3,19}$ we have the lattice $H^2(K3, \mathbb{Z}) \cong \Lambda_{3,19}$. On the boundary of this moduli space, the 3-plane will acquire a null direction which we assume to be along a generator of an $H$ sublattice of $H^2(K3, \mathbb{Z})$. This gives a natural embedding of $O(2,18)/(O(2) \times O(18))$ in the boundary of the moduli space of K3 surfaces. It would thus appear natural to identify the first factor on the left-hand side of (11) with some degenerated K3 surface.

Let us try to identify such a degeneration. We are free to specify that the null direction in the 3-plane is given by the $J$-direction without any loss of generality. Note however that this means that we cannot use the usual normalization conventions of $\langle J, J \rangle = 1$ since now we have $\langle J, J \rangle = 0$. Thus the volume of the limiting K3 surface is zero. The volume element on a smooth surface is an everywhere positive quantity and so this degeneration of the metric we are studying must have caused the effective volume to shrink down to zero at every point in the K3 surface. That is, the effective dimension of the space must have decreased. Since we are studying a generic degeneration of this type it seems reasonable to expect that this “squashed” K3 surface is now an object of real dimension 3.

The second factor is the moduli space of a 4-torus. We therefore claim that this limit in the moduli space represents eleven-dimensional supergravity compactified down to four dimensions by compactifying on a squashed K3 surface times $T^4$. Recall however that the string theory in question could also be considered as the heterotic string compactified on a 6-torus which is $T^4 \times T^2$. By taking the large radius limit of these $T^4$’s we see that the heterotic string compactified on a 2-torus is “equivalent” (in suitable limits) to eleven-dimensional supergravity compactified on the squashed K3 surface. Actually we could have also seen this more directly by applying a similar construction to the observation in [9] that eleven-dimensional supergravity compactified on a K3 surface appears to be “equivalent” to the heterotic string on a 3-torus.

The decomposition on $e_6$ is similar. Now we have supergravity compactified on a K3 surface squashed down to 2 dimensions times a 5-torus. It can be shown that in this case the degeneration in question may be achieved through a deformation of complex structure. This allows algebro-geometric techniques to be employed [29]. The typical degeneration in algebraic geometry is to a limiting complex surface with two components which meet on an elliptic curve (i.e., $T^2$), along which there are 16 singular points. To get a complete picture of the degeneration it is still necessary to carefully consider the way in which the Ricci-flat metric affects the geometry near the degeneration, which we have not done here. However, combining the algebro-geometric analysis with our expectation that the limit is essentially a 2-dimensional object suggests that the limit can be regarded as roughly being a 2-torus with 16 special points. We thus claim that eleven-dimensional supergravity compactified on this object should be “equivalent” to the heterotic string compactified on a circle.

The decomposition on $e_7$ is perhaps the most interesting one. The first factor in the decomposition is $O(16)/O(16)$ which is of course trivial. Note however that this decomposition
may be done in two ways. When we reduce the K3 surface moduli space in a degeneration we are effectively removing some \( H \) sublattices from the \( H^2(K3,\mathbb{Z}) \) lattice. In the present case we need to remove three \( H \) sublattices and there are two choices, depending on whether the 16-dimensional even self-dual lattice we leave behind is \( \Lambda^8 \oplus \Lambda^8 \) or the Barnes-Wall lattice \( \Lambda^{16} \). The second factor will be the moduli space of 6-tori. These two decompositions thus represents ways of squashing a K3 surface down to one of two 1-dimensional structures \( \Xi_1 \) or \( \Xi_2 \) and the theory in question is now eleven-dimensional supergravity compactified on a 6-torus times \( \Xi_i \). We claim that the heterotic string in ten dimensions should be “equivalent” to eleven-dimensional supergravity compactified on \( \Xi_i \).

The two choices of \( \Xi_1 \) and \( \Xi_2 \) for compactifying the supergravity theory will presumably lead to the \( E_8 \times E_8 \) heterotic string and the \( \text{Spin}(32)/\mathbb{Z}_2 \) heterotic string, respectively. Clearly the \( \Xi_i \) spaces are not manifolds since the only compact 1-dimensional manifold is a circle and eleven-dimensional supergravity on a circle is expected to give the type IIA superstring \([30,9]\). The \( \Xi_i \) spaces are probably 1-skeletons of some polytopes. It would be interesting, but perhaps rather difficult, to establish their shape.

5 Integral Structures

Now let us consider some general facts about the moduli space of type II string theories. As discussed above, when we decomposed the space of type IIB strings on a K3 surface by taking the large dilaton limit, we were able to identify the R-R fields as living on a 24-dimensional torus. This torus can also be identified as \( H^*(K3,\mathbb{R})/H^*_Q(K3,\mathbb{Z}) \). The group \( H^*_Q(K3,\mathbb{Z}) \) is the lattice of integral quantum cohomology which coincides with the group \( H^*(K3,\mathbb{Z}) \) in some large radius limit. Away from such a limit, the generators of \( H^*_Q(K3,\mathbb{Z}) \) cannot be associated with pure \( p \)-forms but will have other degrees mixed in. The precise way in which this happens is known from conformal field theory and mirror symmetry \([20]\).

A similar phenomenon can also be deduced from the above calculation for a IIA superstring compactified on a K3 times a torus. In the weak coupling limit we have

\[
\frac{O(6,22)}{O(6) \times O(22)} \times \frac{Sl(2)}{U(1)} \cong \frac{O(4,20)}{O(4) \times O(20)} \times \left( \frac{Sl(2)}{U(1)} \right)_{1} \times \left( \frac{Sl(2)}{U(1)} \right)_{2} \times \mathbb{R}_+ \times \mathbb{R}^{49},
\]

where the \((Sl(2)/U(1))_2\) factor is to be identified with the \( Sl(2)/U(1) \) on the left and so provides the “area+B-field” degrees of freedom of the torus on which we are compactifying. Analysis of the decomposition (13) shows that the group \( O(4,20) \times Sl(2) \) acts on the \( \mathbb{R}^{49} \) space through a representation of the form \((24,2)+(1,1)\). Let us concentrate on the \( \mathbb{R}^{48} \) subspace of R-R fields forming the nontrivial irreducible representation (the other field is simply the four-dimensional axion that was partnered with the dilaton). From our knowledge of the K3 quantum cohomology and the fact that the group \( Sl(2,\mathbb{Z})_1 \) exchanges 1-cycles on the torus we see that \( \mathbb{R}^{48} \cong H^{\text{odd}}(K3 \times T^2,\mathbb{R}) \). In the large radius limit of the compact
space this is consistent with the counting of the R-R fields coming from the reduction of the 1-form and 3-form R-R fields of the ten-dimensional type IIA string. The knowledge of the complete U-duality group for this case also allows us to find the discrete identifications on the R-R moduli space. The result is that the R-R moduli space becomes

\[ \mathcal{M}_A \cong \frac{H^{\text{odd}}(X, \mathbb{R})}{H^{\text{odd}}_Q(X, \mathbb{Z})}, \]  

where \( X \) is the space on which are compactifying. Similarly for the IIB string we obtain the result that the 48 R-R scalars parameterize

\[ \mathcal{M}_B \cong \frac{H^{\text{even}}(X, \mathbb{R})}{H^{\text{even}}_Q(X, \mathbb{Z})}. \]  

We conjecture that this structure survives in the case that \( X \) is a more general Calabi–Yau manifold. We should add some cautionary notes to this statement however. The situation is best compared to that of the rôle the \( B \)-fields play in the moduli space of \( N = 2 \) superconformal field theories. In terms of the non-linear \( \sigma \)-model with target space \( X \), it is easy to see that the \( B \)-fields parameterize the space \( H^2(X, \mathbb{R})/H^2(X, \mathbb{Z}) \). In terms of the exact conformal field theory one sees this by putting a system of “flat” coordinates in the neighbourhood of the large radius limit. This picture is only natural in the Calabi–Yau phase of the theory (in the sense of [10, 11]). In other phases one gets quite different pictures. The same is probably true of the R-R system we are considering here and we only have a right to expect the behaviour we claim in some neighbourhood of weak string-coupling. In particular our conjecture is not incompatible with the local analysis of the moduli space done in [23, 31] where it was discovered that the R-R fields along with the axion-dilaton system parameterize a space locally of the form \( SU(1, n)/(U(1) \times SU(n)) \). Let us also note that the full U-duality picture is probably less useful in this general Calabi–Yau case in the same way that T-duality becomes less useful in this more general case (see, for example [13]).

In our picture the moduli space of compactifications of type II superstrings has the form of a fibration where the base space is the moduli space of \( N = 2 \) superconformal field theories given by NS-NS moduli (including the axion-dilaton) and the fiber is a torus of the form (14) or (15). Each fiber also has a natural complex structure making it into the “intermediate Jacobian” of the Calabi–Yau manifold, and the total space of the fibration is well-understood [32].

If we consider \( X \) at large radius limit, then simple dimensional reduction of the 1-form and 3-form of the IIA superstring and 0-form, 2-form and “anti-self-dual” 4-form of the IIB superstring tell us that at least the dimension counting in equations (14) and (15) is correct. Assuming that \( h^{1,0}(X) = 0 \) which is a reasonable assumption for a generic Calabi–Yau manifold, \( H^3(X) \) provides all the odd cohomology and so \( H^{\text{odd}}_Q(X, \mathbb{Z}) \cong H^3(X, \mathbb{Z}) \). Thus we may move away from the large radius limit of \( X \) and still understand the denominator of (14).
Given a mirror pair $X$ and $Y$, part of the mirror map conjecture (which has yet to be proven in full generality) is that the “horizontal” integral structure $H^3(Y, Z)$ should somehow be equivalent to the “vertical” integral structure $H^0(X, Z) \oplus H^2(X, Z) \oplus H^4(X, Z) \oplus H^6(X, Z)$ in the large radius limit \[33, 34\]. If this is true, we may determine the moduli space of R-R fields in the IIB superstring compactification by using the fact that $H^0_Q(X, Z) \cong H^0_Q(Y, Z)$. The assumption of S-duality for the IIB superstring goes some way towards proving our conjecture about the moduli space. The ten-dimensional axion, $a$, in the R-R sector is associated to $H^0(X)$ when $X$ is at large radius. The assumption of S-duality for the type IIB string gives $a \cong a + 1$. This part of the moduli space may be rather trivially identified as $H^0(X, \mathbb{R})/H^0(X, \mathbb{Z})$. When $X$ is at large radius $Y$ will be at “large complex structure”, i.e., it will be degenerating. The direction $H^0(X)$ is “mirror” to the direction $H^3(Y)$ in this limit \[33\]. In this way, the axion periodicity for the IIB superstring corresponds to an element of $H^3(Y, Z)$ for the IIA compactification. Now we may consider the monodromy of $H^3(Y, Z)$ as we move around the moduli space of complex structures of $Y$. This will take the single element we have and produce more elements. In fact we can generate enough elements this way to build at least a finite index sublattice of $H^3(Y, Z)$. (The fact that we do not necessarily generate the whole $H^3(Y, Z)$ means that our argument falls short of a full proof of the conjecture.) Taking the mirror map to go back to $X$ we see that we are naturally forced to consider not only shifts in $H^0(X)$ but all of the even cohomology of $X$.

It is interesting to note that this picture of the moduli space of type II compactifications cannot be seen purely in terms of world-sheet (i.e., conformal field theory) considerations or target space effective field theory considerations (i.e., explicit compactification down from ten to four dimensions). The R-R sector of the moduli space is rather trivial in the conformal field theory point of view since all the fields there do not couple to any R-R charges. On the other hand the target space point of view knows only about classical cohomology and cannot describe a shift in quantum cohomology unless one is near the large radius limit where such objects can be identified with classical forms of definite degree. It would appear that we are in some way probing the effects of nonperturbative phenomena in the, as yet unknown, correct formulation of string theory.

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