The Jiang–Su Absorption for Inclusions of Unital C*-algebras

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Abstract. We introduce the tracial Rokhlin property for a conditional expectation for an inclusion of unital C*-algebras $P \subset A$ with index finite, and show that an action $\alpha$ from a finite group $G$ on a simple unital C*-algebra $A$ has the tracial Rokhlin property in the sense of N. C. Phillips if and only if the canonical conditional expectation $E: A \to A^G$ has the tracial Rokhlin property. Let $C$ be a class of infinite dimensional stably finite separable unital C*-algebras that is closed under the following conditions:

1. If $A \in C$ and $B \cong A$, then $B \in C$.
2. If $A \in C$ and $n \in \mathbb{N}$, then $M_n(A) \in C$.
3. If $A \in C$ and $p \in A$ is a nonzero projection, then $pAp \in C$.

Suppose that any C*-algebra in $C$ is weakly semiprojective. We prove that if $A$ is a local tracial C*-algebra in the sense of Fan and Fang and a conditional expectation $E: A \to P$ is of index-finite type with the tracial Rokhlin property, then $P$ is a unital local tracial C*-algebra.

The main result is that if $A$ is simple, separable, unital nuclear, Jiang–Su absorbing and $E: A \to P$ has the tracial Rokhlin property, then $P$ is Jiang–Su absorbing. As an application, when an action $\alpha$ from a finite group $G$ on a simple unital C*-algebra $A$ has the tracial Rokhlin property, then for any subgroup $H$ of $G$ the fixed point algebra $A^H$ and the crossed product algebra $A \rtimes_{\alpha,H} H$ is Jiang–Su absorbing. We also show that the strict comparison property for a Cuntz semigroup $W(A)$ is hereditary to $W(P)$ if $A$ is simple, separable, exact, unital, and $E: A \to P$ has the tracial Rokhlin property.

1 Introduction

The purpose of this paper is to introduce the tracial Rokhlin property for an inclusion of separable simple unital C*-algebras $P \subset A$ with finite index in the sense of [38], and prove theorems of the following type. Suppose that $A$ belongs to a class of C*-algebras characterized by some structural property, such as tracial rank zero in the sense of [20]. Then $P$ belongs to the same class. The classes we consider include:

- simple C*-algebras with real rank zero or stable rank one,
- simple C*-algebras with tracial rank zero or tracial rank less than or equal to one,
- simple C*-algebras with Jiang–Su algebra absorption,
- simple C*-algebras for which the order on projections is determined by traces,
- simple C*-algebras with the strict comparison property for the Cuntz semigroup.

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The third and fifth conditions are important properties related to Toms and Winter’s conjecture, that is, the properties of strict comparison, finite nuclear dimension, and \( Z \)-absorption are equivalent for separable simple infinite-dimensional nuclear unital \( \text{C}^{*} \)-algebras ([36, 39]).

We show that an action \( \alpha \) from a finite group \( G \) on a simple unital \( \text{C}^{*} \)-algebra \( A \) has the tracial Rokhlin property in the sense of [30] if and only if the canonical conditional expectation \( E : A \to A^{G} \) has the tracial Rokhlin property for an inclusion \( A^{G} \subset A \). When an action \( \alpha \) from a finite group on a (not necessarily simple) unital \( \text{C}^{*} \)-algebra has the Rokhlin property in the sense of [13], all of the above results are proved in [28, 29].

The essential observation was made in the proof of [30, Theorem 2.2] the crossed product \( A \rtimes_{\alpha} G = C^{*}(G, A, \alpha) \) in [30]) has a local approximation property by \( \text{C}^{*} \)-algebras stably isomorphic to homomorphic images of \( A \). Since the Jiang–Su algebra \( Z \) belongs to classes of direct limits of semiprojective building blocks in [16], technical difficulties arise because we must treat arbitrary homomorphic images in the approximation property. (Homomorphic images of semiprojective \( \text{C}^{*} \)-algebras need not be semiprojective.) In [40] they introduced the unital local tracial \( \mathcal{C} \) property, which generalizes one of a local \( \text{C} \) property in [26], for proving that a \( \text{C}^{*} \)-algebra with a local approximation property by homomorphic images of a suitable class \( \mathcal{C} \) of semiprojective \( \text{C}^{*} \)-algebras can be written as a direct limit of algebras in the class. When each homomorphism is injective, the unital local tracial \( \mathcal{C} \) property is equivalent to the tracial approximation property in [7]. Note that when an action \( \alpha \) from a finite group \( G \) on a simple unital \( \text{C}^{*} \)-algebra \( A \), the tracial approximation property for \( A \) is inherited to the crossed product algebra \( A \rtimes_{\alpha} G \) (see [40] and Theorem 3.3).

We know of several results like those above for tracial approximation in the literature: for stable rank one ([7, Theorem 4.3] and [9]), for real rank zero ([9]), for the \( Z \)-absorption ([11, Corollary 5.7]), for the order on projections determined by traces ([7, Theorem 4.12]).

The paper is organized as follows. In Section 2 we introduce the notion of a unital local tracial \( \mathcal{C} \)-algebra and a tracial approximation class (TAc class), and we show in Section 3 that when an action \( \alpha \) from a finite group \( G \) on a simple unital \( \text{C}^{*} \)-algebra \( A \) has the tracial Rokhlin property, the crossed product algebra \( A \rtimes_{\alpha} G \) belongs to the class TAc for \( A \) in TAc. In Section 4 we introduce the tracial Rokhlin property for an inclusion \( P \subset A \) of unital \( \text{C}^{*} \)-algebras and show that if \( A \) is a simple local tracial \( \mathcal{C} \)-algebra, then so is \( P \) (Theorem 4.11). In particular, if \( A \) has tracial topological rank zero (resp. less than or equal to one), so does \( P \) (Corollary 4.12). In Section 5 we present the main theorem: given an inclusion \( P \subset A \) of separable simple nuclear unital \( \text{C}^{*} \)-algebras of finite index type with the tracial Rokhlin property, if \( A \) is \( Z \)-absorbing, then so is \( P \) (Theorem 5.4). As an application, any fixed point algebra \( A^{H} \) for any subgroup \( H \) of a finite group \( G \) is \( Z \)-absorbing under the assumption that there exists an action \( \alpha \) from \( G \) on a simple nuclear unital \( \text{C}^{*} \)-algebra \( A \) such that \( A \) is \( Z \)-absorbing (Corollary 5.5). Before treating the strict comparison for a Cuntz semigroup, we consider the Cuntz equivalent for positive elements and show that under the assumption that an inclusion \( P \subset A \) of unital \( \text{C}^{*} \)-algebras has the tracial Rokhlin property, for \( n \in \mathbb{N} \) and positive elements \( a, b \in M_{n}(P) \), if \( a \) is subequivalent to \( b \)
in \( M_n(A) \), then \( a \) is subequivalent to \( b \) in \( M_n(P) \) (Proposition 6.2). Finally, we consider the strict comparison property for a Cuntz semigroup and show that the strict comparison property is inherited to \( P \) when an inclusion \( P \subset A \) of simple separable exact unital C*-algebras has the tracial Rokhlin property and \( A \) has the strict comparison (Theorem 7.2). Using a similar argument we show that if an inclusion \( P \subset A \) of separable simple unital C*-algebras has the tracial Rokhlin property and the order on projections in \( A \) is determined by traces, then the order on projections in \( P \) is determined by traces (Corollary 7.3).

2 Local Tracial C\( Claref{\mathcal{C}} \)-algebra

We recall the definition of the local \( \mathcal{C} \)-property in [26, Definition 1.1].

**Definition 2.1** Let \( \mathcal{C} \) be a class of separable unital C*-algebras. Then \( \mathcal{C} \) is **finitely saturated** if the following closure conditions hold:

(i) if \( A \in \mathcal{C} \) and \( B \cong A \), then \( B \in \mathcal{C} \);

(ii) if \( A_1, A_2, \ldots, A_n \in \mathcal{C} \), then \( \bigoplus_{k=1}^n A_k \in \mathcal{C} \);

(iii) if \( A \in \mathcal{C} \) and \( n \in \mathbb{N} \), then \( M_n(A) \in \mathcal{C} \);

(iv) if \( A \in \mathcal{C} \) and \( p \in A \) is a nonzero projection, then \( pAp \in \mathcal{C} \).

Moreover, the **finite saturation** of a class \( \mathcal{C} \) is the smallest finitely saturated class that contains \( \mathcal{C} \).

**Definition 2.2** Let \( \mathcal{C} \) be a class of separable unital C*-algebras. A **unital local \( \mathcal{C} \)-algebra** is a separable unital C*-algebra \( A \) such that for every finite set \( F \subset A \) and every \( \varepsilon > 0 \), there is a C*-algebra \( B \) in the finite saturation of \( \mathcal{C} \) and a unital *-homomorphism \( \varphi: B \to A \) (not necessarily injective) such that \( \text{dist}(x, \varphi(B)) < \varepsilon \) for all \( x \in F \).

When \( B \) in Definition 2.2 is non-unital, we perturb the condition as follows.

**Definition 2.3** Let \( \mathcal{C} \) be a class of separable unital C*-algebras.

(i) A unital C*-algebra \( A \) is said to be a **unital local tracial \( \mathcal{C} \)-algebra** if for every finite set \( \mathcal{F} \subset A \) and every \( \varepsilon > 0 \), and any non-zero \( a \in A^* \), there exist a non-zero projection \( p \in A \), a C*-algebra \( B \in \mathcal{C} \), and *-homomorphism \( \varphi: B \to A \) such that \( \varphi(1_B) = p \), and for all \( x \in \mathcal{F} \):

(a) \( |xp - px| < \varepsilon \),

(b) \( \text{dist}(pxp, \varphi(B)) < \varepsilon \),

(c) \( 1 - p \) is Murray–von Neumann equivalent to a projection in \( aAa \).

(ii) A unital C*-algebra \( A \) is said to belong to the **class TA\( \mathcal{C} \)** if for every finite set \( \mathcal{F} \subset A \) and every \( \varepsilon > 0 \), and any non-zero \( a \in A^* \), there exist a non-zero projection \( p \in A \) and a sub C*-algebra \( B \subset A \) such that \( B \in \mathcal{C} \), \( 1_B = p \), and for all \( x \in \mathcal{F} \):

(a) \( |xp - px| < \varepsilon \),

(b) \( \text{dist}(pxp, B) < \varepsilon \),

(c) \( 1 - p \) is Murray–von Neumann equivalent to a projection in \( aAa \).
Note that (i) comes from [40, Definition 2.13], and (ii) is [7, Definition 2.2].

**Remark 2.4**  
(i) When a unital C*-algebra $A$ is a unital local tracial C*-algebra and each $\phi(C) \in \mathcal{C}, A$ belongs to the class $\text{TAC}$.  
(ii) If $\mathcal{C}$ is the class of finite dimensional C*-algebras $\mathcal{F}$, then a local $\text{TAF}$-algebra belongs to the class of tracially AF C*-algebras ([20]).  
(iii) If $\mathcal{C}$ is the class of interval algebras $\mathcal{I}$, then a local $\text{TAF}$-algebra belongs to the class of C*-algebras of tracial topological one (TAF-algebras) ([23]) in the sense of Lin.

Recall that a C*-algebra $A$ is said to have Property (SP) if any nonzero hereditary C*-subalgebra of $A$ has a nonzero projection.

We have the following relation between the local $\mathcal{C}$ property and the local tracial approximational $\mathcal{C}$ property.

**Proposition 2.5**  
Let $\mathcal{C}$ be a finitely saturated class and let $A$ be a local tracial $\mathcal{C}$-algebra. Then $A$ has the property (SP) or $A$ is a local $\mathcal{C}$-algebra.

**Proof**  
Suppose that $A$ does not have the Property (SP). Then there is a positive element $a \in A$ such that $\overline{aa^*}$ has no non zero projection. Since $A$ is a local $\text{TAC}$-algebra, for every finite set $\mathcal{F} \subset A$ and every $\varepsilon > 0$, we conclude that there are a unital C*-algebra $C$ in the class $\mathcal{C}$ and a unital *-homomorphism $\phi: C \to A$ such that $\mathcal{F}$ can be approximated by a C*-algebra $\phi(C)$ to within $\varepsilon$. Hence, $A$ is a local $\mathcal{C}$-algebra.

### 3 Tracial Rokhlin Property for Finite Group Actions

Inspired by the concept of the tracial AF C*-algebras in [20] Phillips defined the tracial Rokhlin property for a finite group action in [30, Lemma 1.16] as follows.

**Definition 3.1**  
Let $\alpha$ be the action of a finite group $G$ on a unital infinite dimensional finite simple separable unital C*-algebra $A$. The action $\alpha$ is said to have the **tracial Rokhlin property** if for every finite set $F \subset A$, every $\varepsilon > 0$, and every nonzero positive $x \in A$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

(i) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$;  
(ii) $\|e_{g^{-1}a} - ae_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$;  
(iii) with $e = \sum_{g \in G} e_g$, the projection $1 - e$ is Murray–von Neumann equivalent to a projection in the hereditary subalgebra of $A$ generated by $x$.

It is obvious that the Rokhlin property is stronger than the tracial Rokhlin property. As pointed out in [13], the Rokhlin property gives rise to several $K$-theoretical constrains. For example, there is no action with the Rokhlin property on the noncommutative 2-torus. On the contrary, if $A$ is a simple higher dimensional noncommutative torus with standard unitary generators $u_1, u_2, \ldots, u_n$, then the automorphism that sends $u_k$ to $\exp(2\pi/n)u_k$, and fixes $u_j$ for $j \neq k$, generates an action $\mathbb{Z}/n\mathbb{Z}$ and has the tracial Rokhlin property, but for $n > 1$ never has the Rokhlin property ([30]).
Lemma 3.2 ([26, Theorem 3.2], [1, Lemma 3.1]) Let $A$ be an infinite-dimensional, stably finite, simple, unital $C^*$-algebra with Property (SP) such that the order on projections over $A$ is determined by traces. Let $G$ be a finite group of order $n$ and let $\alpha: G \to \text{Aut}(A)$ be an action of $G$ with the tracial Rokhlin property. Then for any $\epsilon > 0$, any finite set $\mathcal{T} \subset A \rtimes_{\alpha} G$, any $N \in \mathbb{N}$, and any non-zero $z \in (A \rtimes_{\alpha} G)^+$, there exist a non-zero projection $e \in A \subset A \rtimes_{\alpha} G$, a unital $C^*$-subalgebra $D \subset e(A \rtimes_{\alpha} G)e$, a projection $f \in A$ and an isomorphism $\phi: M_n \otimes fAf \to D$, such that the following hold.

(i) With $(e_{g})$ for $g, h \in G$ being a system of matrix units for $M_n$, we have $\phi(e_{11} \otimes a) = a$ for all $a \in fAf$ and $\phi(e_{gg} \otimes 1) \in A$ for $g \in G$.

(ii) With $(e_{gg})$ as in (i), we have $\|\phi(e_{gg} \otimes a) - a\| \leq \epsilon |a|$ for all $a \in fAf$.

(iii) For every $\alpha \in \mathcal{T}$, there exist $b_1, b_2 \in D$ such that $\|ea - b_1\| < \epsilon$, $\|ae - b_2\| < \epsilon$ and $\|b_1\|, \|b_2\| \leq \|a\|$.

(iv) $e = \sum_{g \in G} \phi(e_{gg} \otimes 1)$.

(v) $1 - e$ is Murray-von Neumann equivalent to a projection in $z(A \rtimes_{\alpha} G)z$.

(vi) There are $N$ mutually orthogonal projections $f_1, f_2, \ldots, f_N \in eD_0$, each of which is Murray-von Neumann equivalent in $A \rtimes_{\alpha} G$ to $1 - e$.

Proof In [1] the author assumed that $A$ has real rank zero. But since $A$ is simple and $A$ has Property (SP), any nonzero positive element $z \in A \rtimes_{\alpha} G$ [14, Theorem 4.2] (with $N = 1$) supplies a nonzero projection $q \in A$ that is Murray-von Neumann equivalent in $A \rtimes_{\alpha} G$ to a projection in $z(A \rtimes_{\alpha} G)z$. Moreover, [22, Lemma 3.5.7] provides nonzero orthogonal Murray-von Neumann equivalent projections $q_0, q_1, \ldots, q_{2N} \in qAq$.

Therefore, the statement comes from the same argument as in [1, Lemma 3.1].

Theorem 3.3 Let $\mathcal{C}$ be a class of infinite dimensional stably finite separable unital $C^*$-algebras that is closed under the following conditions:

(i) $A \in \mathcal{C}$ and $B \cong A$, then $B \in \mathcal{C}$.

(ii) If $A \in \mathcal{C}$ and $n \in \mathbb{N}$, then $M_n(A) \in \mathcal{C}$.

(iii) If $A \in \mathcal{C}$ and $p \in A$ is a nonzero projection, then $pAp \in \mathcal{C}$.

Let $A \in \mathcal{TAC}$ be a simple $C^*$-algebra such that the order on projections over $A$ is determined by traces. If $\alpha$ is an action of a finite group $G$ on $A$ with the tracial Rokhlin property, then $A \rtimes_{\alpha} G$ belongs to the class $\mathcal{TAC}$.

Proof Since $\alpha$ is outer by [30, Lemma 1.5], $A \rtimes_{\alpha} G$ is simple by [18, Theorem 3.1]. By [30, Lemma 1.13], $A$ has Property (SP) or $\alpha$ has the strict Rokhlin property.

Let $\mathcal{T} \subset A \rtimes_{\alpha} G$ be a finite set and let $z$ be a positive nonzero element of $A \rtimes_{\alpha} G$ with $\|z\| \leq 1$ and $\epsilon > 0$.

If $A$ has the strict Rokhlin property, then there are $n \in \mathbb{N}$, a projection $f \in A$, and a unital homomorphism $\phi: M_n \otimes fAf \to A \rtimes_{\alpha} G$ such that $\text{dist}(a, \phi(M_n \otimes fAf)) < \epsilon$ for all $a \in \mathcal{T}$ by [26, Theorem 3.2]. Since $M_n \otimes fAf \in \mathcal{C}$, from the simplicity of $M_n \otimes fAf$ we know $\phi(M_n \otimes fAf) \in \mathcal{C}$. Hence, $A \rtimes_{\alpha} G$ is unital local $\mathcal{C}$-algebra; that is, $A \rtimes_{\alpha} G$ belongs to $\mathcal{TAC}$.

Next, suppose that $A$ has Property (SP). Then there exists a non-zero projection $q \in A$ that is Murray-von Neumann equivalent in $A \rtimes_{\alpha} G$ to a projection in $z(A \rtimes_{\alpha} G)z$ by [25, Theorem 2.1]. Since $A$ is simple, take orthogonal nonzero projections $q_1, q_2$
with \( q_1, q_2 \leq q \) by [22, Lemma 3.5.7]. Set \( n = \text{card}(G) \), and set \( \varepsilon_0 = \frac{1}{12} \varepsilon \). By Lemma 3.2 for \( n \) as given, for \( \varepsilon \) in place of \( \varepsilon \), and for \( q_1 \) in place of \( z \) there exist a non-zero projection \( e \in A \subset A \rtimes_n G \), a unital \( C^* \)-subalgebra \( D \subset e(A \rtimes_n G)e \), a projection \( f \in A \), and an isomorphism \( \phi: M_n \otimes fAf \to D \), such that the following hold:

(a) With \((e_{gh})\) for \( g, h \in G \) being a system of matrix units for \( M_n \), we have \( \phi(e_{11} \otimes a) = a \) for all \( a \in fAf \) and \( \phi(e_{g \varepsilon} \otimes 1) \in A \) for \( g \in G \); (b) with \((e_{g \varepsilon})\) as in (a), we have \( |\phi(e_{g \varepsilon} \otimes a) - \alpha_{\varepsilon}(a)| \leq \varepsilon_0 \) for all \( a \in fAf \); (c) for every \( a \in F \) there exist \( d_1, d_2 \in D \) such that \( |e a - d_1| < \varepsilon_0 \), \( |a e - d_2| < \varepsilon_0 \) and \( |d_1|, |d_2| \leq 1 \); (d) \( e = \sum_{g \in G} \phi(e_{g \varepsilon} \otimes 1) \);

(e) \( 1 - e \) is Murray-von Neumann equivalent to a projection in \( q_1(A \rtimes_n G)q_1 \).

We note that there is a finite set \( T \) in the closed unit ball of \( M_n \otimes fAf \) such that for every \( a \in F \) there are \( b_1, b_2 \in T \) such that \( |e a - \phi(b_1)| < \varepsilon_0 \) and \( |a e - \phi(b_2)| < \varepsilon_0 \). Moreover, \( |e a e - a e| < 8\varepsilon_0 \). Indeed, the condition that \( |e a - \phi(b_1)| < \varepsilon_0 \) and \( \phi(b_1) e = \phi(b_1) \) implies that \( |e a e - \phi(b_1)| < \varepsilon_0 \). Similarly, the condition that \( |a e - \phi(b_2)| < \varepsilon_0 \) implies that \( |e a^* e - a e| < 2\varepsilon_0 \). Hence, \( |e a e - a e| < 2\varepsilon_0 \).

Since \( A \) is simple and has Property (SP), we choose equivalent nonzero projections \( f_1, f_2 \in A \) such that \( f_1 \leq f \) and \( f_2 \leq q_2 \) by [22, Lemma 3.5.6]. Since \( M_n \otimes fAf \in TAE \), by [7, Lemma 2.3], there is a projection \( p_0 \in M_n \otimes fAf \) and a \( C^* \)-subalgebra \( \subset M_n \otimes fAf \) such that \( C \subset \mathcal{C} \), \( \mathcal{C} = p_0 \) such that \( |p_0 b - b p_0| < \frac{1}{4} \varepsilon \) for all \( b \in T \), such that for every \( b \in T \), there exists \( c \in C \) with \( |p_0 b - c| < \frac{1}{4} \varepsilon \), and such that \( 1 - p_0 \leq e_{11} \otimes f_1 \) in \( M_n \otimes fAf \).

Set \( p = \phi(p_0) \), and set \( E = \phi(C) \), which is a unital subalgebra of \( p(A \rtimes_n G) p \) and belongs to \( E \). Note that \( e - p = \phi(1 - p_0) \leq \phi(e_{11} \otimes f_1) = f_1 \).

Let \( a \in F \). Then we can take \( b \in T \) such that \( |\phi(b) - e a e| < \frac{1}{4} \varepsilon \). Indeed, since condition (c) implies that there is an element \( b \in T \) such that \( |e a - \phi(b)| < \varepsilon_0 \) and \( |e a e - e a| < 2\varepsilon_0 \), we have

\[
|\phi(b) - e a e| = |\phi(b) - e a + e a e - e a e| < |\phi(b) - e a| + |e a e - e a| < (2 + 1)\varepsilon_0 = \frac{3}{12} \varepsilon = \frac{1}{4} \varepsilon.
\]

Then, using \( p e = e p = p \),

\[
|p a - a p| \leq 2|e a - a e| + |p e a e - e a p| \leq 2|e a - a e| + 2|e a e - \phi(b)| + |p_0 b - b p_0| < 4\varepsilon_0 + 6\varepsilon_0 + \varepsilon_0 = 11\varepsilon_0 < \varepsilon.
\]

Choosing \( c \in C \) such that \( |p_0 b - c| < \frac{1}{4} \varepsilon \), the element \( \phi(c) \) is in \( E \) and satisfies

\[
|p a p - \phi(c)| = |p e a e p - \phi(c)| = |p(e a e - \phi(b))p + p\phi(b)p - \phi(c)| \leq |e a e - \phi(b)| + |p_0 b p_0 - c| < \frac{1}{4} \varepsilon + \frac{1}{4} \varepsilon < \varepsilon.
\]

Finally, in \( A \rtimes_n G \) we have

\[
1 - p = (1 - e) + (e - p) \leq q_1 + q_2 \leq q.
\]
and $q$ is Murray–von Neumann equivalent to a projection in $z(A \rtimes_a G)z$. ■

By using Theorem 3.3 we will provide a new proof of [9, Theorem 3.1].

**Theorem 3.4** ([9, Theorem 3.1]) Let $A$ be an infinite dimensional simple separable unital $C^*$-algebra with stable rank one and let $\alpha : G \to \text{Aut}(A)$ be an action of a finite group $G$ with the tracial Rokhlin property. Then $A \rtimes_a G$ has stable rank one.

**Proof** Let $\mathcal{C}$ be the set of unital $C^*$-algebras with stable rank one. Then $\mathcal{C}$ is closed under three conditions in Theorem 3.3 from [3], Theorem 3.3 and [2, Theorem 4.5]. Then from Theorem 3.3 $A \rtimes_a G$ belongs to the class $\mathcal{T}_a \mathcal{C}$.

Hence from [7, Theorem 4.3], $A \rtimes_a G$ has stable rank one. ■

**Theorem 3.5** Let $\mathcal{C}$ be the class of unital separable $C^*$-algebras with real rank zero. Then any simple unital stably finite $C^*$-algebra in the class $\mathcal{T}_a \mathcal{C}$ has real rank zero.

**Proof** We can deduce this from the same argument as in the proof of [7, Theorem 4.3]. ■

Using Theorems 3.3 and 3.5 we will provide a new proof of [9, Theorem 3.2].

**Corollary 3.6** ([9, Theorem 3.2]) Let $A$ be an infinite dimensional simple separable unital $C^*$-algebra with real rank zero and let $\alpha : G \to \text{Aut}(A)$ be an action of a finite group $G$ with the tracial Rokhlin property. Then $A \rtimes_a G$ has real rank zero.

**Proof** Let $\mathcal{C}$ be the set of unital $C^*$-algebras with real rank zero. Then $\mathcal{C}$ is closed under the three conditions in Theorem 3.3, from [4, Corollary 2.8 and Theorem 2.10]. Then from Theorem 3.3, $A \rtimes_a G$ belongs to the class $\mathcal{T}_a \mathcal{C}$.

Hence from Theorem 3.5, $A \rtimes_a G$ has real rank zero. ■

**Theorem 3.7** Let $A$ be an infinite-dimensional simple separable unital $C^*$-algebra such that the order on projections over $A$ is determined by traces, and let $\alpha : G \to \text{Aut}(A)$ be an action of a finite group $G$ with tracial Rokhlin property. Then the order on projections over $A \rtimes_a G$ is determined by traces.

**Proof** Let $\mathcal{C}$ be the set of unital $C^*$-algebras such that the order on projections over them is determined by traces. Then $\mathcal{C}$ is closed under three conditions in Theorem 3.3. Indeed, conditions (i) and (ii) are obvious from the definition. We will check condition (iii). Let $r$ be a projection in $A$ and suppose that the order of projections on $A$ is determined by traces over $A$. Let $p, q$ be projections in $rAr$ and assume for any tracial state $\tau$ on $rAr$, that $\tau(p) < \tau(q)$. Then, for any tracial state $\rho$ on $A$, the restriction $\rho(r)^{-1}\rho_{|rAr}$ of $\rho$ on $rAr$ is also a tracial state on $rAr$. Hence,

$$\rho(p) = \rho(r)\{\rho(r)^{-1}\rho_{|rAr}(p)\} < \rho(r)\{\rho(r)^{-1}\rho_{|rAr}(q)\} = \rho(q).$$

Since the order of projections on $A$ is determined by traces, $p \leq q$ in $A$. That is, there is a partial isometry $u \in A$ such that $u^*u = p$ and $uu^* \leq q$. Set $w = rur$. Then $w \in rAr$, and $w^*w = ru^*rur = ru^*rur \leq ru^*ur = rpr = p$. Since $q \leq r$, $u^*qu \leq u^*ru$. Hence

$$p = u^*(uu^*)u \leq u^*qu \leq u^*ru.$$
Therefore, \( p = rpr \leq ru^*ru = w^*w \). Then we have, \( w^*w = p \).

On the contrary,
\[
ww^* = ru r^*u \leq ru r^*u = qr = q.
\]
This implies that \( p \leq q \) in \( rAr \). Hence, the order of projections on \( rAr \) is determined by traces over \( rAr \). That is, \( \mathcal{C} \) satisfies condition (iii).

Then from Theorem 3.3 \( A \rtimes_a G \) belongs to the class \( \mathcal{T} \mathcal{C} \).

Hence, from [7, Theorem 4.12], the order on projections over \( A \rtimes_a G \) is determined by traces.

\[\text{Definition 3.8} \quad ([21, \text{Theorem 6.13}]) \quad \text{Let } \mathcal{T}^{(0)} \text{ be the class of all finite-dimensional } C^*\text{-algebras and let } \mathcal{T}^{(k)} \text{ be the class of all } C^*\text{-algebras with the form } pM_n(C(X))p, \text{ where } X \text{ is a finite CW complex with dimension } k \text{ and } p \in M_n(C(X)) \text{ is a projection.} \]

A simple unital \( C^* \)-algebra \( A \) is said to have tracial topological rank no more than \( k \) if for any set \( \mathcal{T} \subset A \), and \( \varepsilon > 0 \) and any nonzero positive element \( a \in A \), there exists a \( C^* \)-subalgebra \( B \subset A \) with \( B \in \mathcal{T}^{(k)} \) and \( \text{id}_B = p \) such that

(i) \( \|xp - px\| < \varepsilon \) for all \( x \in \mathcal{T} \),
(ii) \( xp \in B \), for all \( x \in \mathcal{T} \),
(iii) \( 1 - p \) is Murray–von Neumann equivalent to a projection in \( aAa \).

The following is proved in [27], but we will provide its proof.

\[\text{Theorem 3.9} \quad ([27]) \quad \text{Let } A \text{ be an infinite-dimensional simple unital } C^*\text{-algebra with tracial topological rank no more than (resp. equal to) } k, \text{ and let } \alpha: G \to \text{Aut}(A) \text{ be an action of a finite group } G \text{ with tracial Rokhlin property. Then } A \rtimes_a G \text{ has tracial topological rank more than (resp. equal to) } k.\]

\[\text{Proof} \quad \text{Let } \mathcal{C} \text{ be the set } \mathcal{T}^{(k)}. \text{ Then } \mathcal{T}^{(k)} \text{ is closed under the three conditions in Theorem 3.3 from [21, Remark 3.6, Theorems 5.3 and 5.8]. Then from Theorem 3.3, } A \rtimes_a G \text{ belongs to the class } \mathcal{T} \mathcal{C}. \text{ This means that } A \rtimes_a G \text{ has tracial topological rank no more than (respectively equal to) } k \text{ from the Definition 3.8.}\]

4 The Tracial Rokhlin Property for an Inclusion of Unital \( C^* \)-algebras

Let \( P \subset A \) be an inclusion of unital \( C^* \)-algebras and let \( E: A \to P \) be a conditional expectation of index-finite as defined in [38, Definition 1.2.2]. Note that \( E \) is faithful and satisfies that

\[
E(b_1ab_2) = b_1E(a)b_2
\]

for any \( a \in A \) and \( b_1, b_2 \in P \).

As in the case of the Rokhlin property in [19, Definition 3.1], we can define the tracial Rokhlin property for a conditional expectation for an inclusion of unital \( C^* \)-algebras.

Recall that an inclusion of unital \( C^* \)-algebras \( P \subset A \) with a conditional expectation \( E \) from \( A \) to \( P \) has finite index in the sense of Watatani [38] if there is a finite set
\((\{u_i, v_i\})_{i=1}^{n} \subset A \times A\) such that for every \(a \in A\),
\[
a = \sum_{i=1}^{n} u_i E(v_i a) = \sum_{i=1}^{n} E(a u_i) v_i.
\]
Set \(E = \sum_{i=1}^{n} u_i v_i\).

We give several remarks about the above definitions.

(a) Index \(E\) does not depend on the choice of the quasi-basis in the above formula, and it is a central element of \(A\) [38, Proposition 1.2.8].

(b) Once we know that there exists a quasi-basis, we can choose one of the form \(\{(w_i, w'_i)\}_{i=1}^{m}\), which shows that Index \(E\) is a positive element [38, Lemma 2.1.6].

(c) By the above statements, if \(A\) is a simple \(C^*\)-algebra, then Index \(E\) is a positive scalar.

(d) If Index \(E < \infty\), then \(E\) is faithful; that is, \(E(x^*x) = 0\) implies \(x = 0\) for \(x \in A\).

Let \(A_p = A\) be a pre-Hilbert module over \(P\) with a \(P\)-valued inner product
\[
\langle x, y \rangle_p = E(x^* y), \quad x, y \in A_p.
\]
We denote by \(E_E\) and \(\eta_E\) the Hilbert \(P\)-module completion of \(A\) by the norm \(\|x\|_p = \|\langle x, x \rangle_p \|^\frac{1}{2}\) for \(x \in A\) and the natural inclusion map from \(A\) into \(E_E\). Then \(E_E\) is a Hilbert \(C^*\)-module over \(P\). Since \(E\) is faithful, the inclusion map \(\eta_E\) from \(A\) to \(E_E\) is injective. Let \(L_p(E_E)\) be the set of all (right) \(P\)-module homomorphisms \(T : E_E \to E_E\) with an adjoint right \(P\)-module homomorphism \(T^* : E_E \to E_E\) such that
\[
\langle T \xi, \zeta \rangle = \langle \xi, T^* \zeta \rangle, \quad \xi, \zeta \in E_E.
\]
Then \(L_p(E_E)\) is a \(C^*\)-algebra with the operator norm \(\|T\| = \sup\{\|T \xi\| : \|\xi\| = 1\}\).

There is an injective \(*\)-homomorphism \(\lambda : A \to L_p(E_E)\) defined by
\[
\lambda(a) \eta_E(x) = \eta_E(ax)
\]
for \(x \in A_p\) and \(a \in A\), so that \(A\) can be viewed as a \(C^*\)-subalgebra of \(L_p(E_E)\). Note that the map \(e_p : A_p \to A_p\) defined by
\[
e_p \eta_E(x) = \eta_E(E(x)), \quad x \in A_p
\]
is bounded, and thus it can be extended to a bounded linear operator, denoted by \(e_p\) again, on \(E_E\). Then \(e_p \in L_p(E_E)\) and \(e_p^2 = e_p\); that is, \(e_p\) is a projection in \(L_p(E_E)\). A projection \(e_p\) is called the Jones projection of \(E\).

The \((reduced)\, C^*\)-basic construction is a \(C^*\)-subalgebra of \(L_p(E_E)\) defined as
\[
C^*_p(A, e_p) = \text{span}\{\lambda(x) e_p \lambda(y) \in L_p(E_E) : x, y \in A\}.
\]
If Index \(E\) is finite, \(C^*_p(A, e_p)\) has the certain universalitiy ([38, Proposition 2.2.9]), so we call it the \(C^*\)-basic construction and denote it by \(C^*(A, e_p)\) by identifying \(\lambda(A)\) with \(A\) in \(C^*\langle A, e_p \rangle\); that is,
\[
C^*(A, e_p) = \left\{ \sum_{i=1}^{n} x_i e_p y_i : x_i, y_i \in A, n \in \mathbb{N} \right\}.
\]

Note that by [38, Lemma 2.1.1],
\[
e_p a e_p = E(a) e_p
\]
for any \(a \in A\).
Then there exists a dual conditional expectation $\tilde{E}: C^*(A, e_P) \to A$ such that
\begin{equation}
\tilde{E}(xe_P y) = (\text{Index } E)^{-1}xy
\end{equation}
and $\text{Index } \tilde{E} = \text{Index } E$ ([38, Proposition 2.3.4]). Note that the basic construction $C^*(A, e_P)$ is isomorphic to $qM_n(P)q$ for some $n \in \mathbb{N}$ and a projection $q \in M_n(P)$ ([38, Lemma 3.3.4]).

For a $C^*$-algebra $A$, we set
\[
c_0(A) = \left\{ (a_n) \in l^\infty(\mathbb{N}, A) : \lim_{n \to \infty} \|a_n\| = 0 \right\},
\]
\[
A^\infty = l^\infty(\mathbb{N}, A)/c_0(A).
\]
We identify $A$ with the $C^*$-subalgebra of $A^\infty$ consisting of the equivalence classes of constant sequences and set $A_{\infty} = A^\infty \cap A'$. For an automorphism $\alpha \in \text{Aut}(A)$, we denote by $\alpha^\infty$ and $\alpha_{\infty}$ the automorphisms of $A^\infty$ and $A_{\infty}$ induced by $\alpha$, respectively.

**Example 4.1** Let $A$ be a unital $C^*$-algebra and let $\alpha$ be an action from a finite group $G$ on $\text{Aut}(A)$.

(a) An inclusion of $A \subset A \rtimes_\alpha G$ is of index-finite type and $\text{Index } F = |G|$, where $F$ is a canonical conditional expectation from $A \rtimes_\alpha G$ onto $A$ such that $F(\sum_{g \in G} a_g u_g) = a_e$. Indeed, $\{ (u^*_g, u_g) \}_{g \in G}$ is a quasi-basis for $F$ and $\text{Index } F = \sum_{g \in G} u^*_g u_g = |G|$.

(b) If $A$ is simple and $\alpha$ is outer, then an inclusion $A^\alpha \subset A$ is of index-finite type and $\text{Index } E = |G|$, where $E$ is the canonical conditional expectation from $A$ onto $A^\alpha$ such that $E(a) = \frac{1}{|G|} \sum_{g \in G} a_g(a)$. Indeed, since $A$ is simple and $\alpha$ is outer, $\alpha$ is saturated by [15, Remark 4.6]. Then by [15, Theorem 4.1] an inclusion $A^\alpha \subset A$ is of index-finite type and $\text{Index } E = |G|$. Note that the crossed product $A \rtimes_\alpha G$ is equal to the basic construction $C^*(A^\alpha, e_P)$, where $e_P = \frac{1}{|G|} \sum_{g \in G} u_g$. See the detail in [15, Sections 3 and 4]. Note that $A \rtimes_\alpha G$ is isomorphic to $pM_{|G|}(A^\alpha)p$ for some projection $p \in M_{|G|}(A^\alpha)$ by [38, Lemma 3.3.4].

**Definition 4.2** Let $P \subset A$ be an inclusion of unital $C^*$-algebras and let $E: A \to P$ be a conditional expectation of index-finite type. We denote by $E^\infty$ the canonical conditional expectation from $A^\infty$ to $P^\infty$ induced by $E$. A conditional expectation $E$ is said to have the tracial Rokhlin property if for any nonzero positive $z \in A^\infty$ there exists a projection $e \in A' \cap A^\infty$ satisfying that $(\text{Index } E^\infty(e)) = g$ is a projection, and $1 - g$ is Murray–von Neumann equivalent to a projection in the hereditary subalgebra of $A^\infty$ generated by $z$, and a map $A \ni x \mapsto xe$ is injective. We call $e$ a Rokhlin projection.

As in the case of an action with the tracial Rokhlin property ([30, Lemma 1.13]), if $E: A \to P$ is a conditional expectation of index-finite type for an inclusion of unital $C^*$-algebras $P \subset A$ and $E$ has the tracial Rokhlin property, then $A$ has Property (SP) or $E$ has the Rokhlin property; that is, there is Rokhlin projection $e \in A' \cap A^\infty$ such that $(\text{Index } E^\infty(e)) = 1$.

**Lemma 4.3** Let $P \subset A$ be an inclusion of unital $C^*$-algebras and let $E: A \to P$ be a conditional expectation of index-finite type. Suppose that $E$ has the tracial Rokhlin property; then $A$ has Property (SP) or $E$ has the Rokhlin property.
Proof If \(A\) does not have Property (SP), then \(A^\infty\) does not have Property (SP); that is, there is a nonzero positive element \(x \in A^\infty\) that generates a hereditary subalgebra that contains no nonzero projection. Since \(E\) has the tracial Rokhlin property, there exists a projection \(e \in A_{\infty}\) such that \(1 - (\text{Index}E)E^\infty(e)\) is equivalent to some projection in \(\overline{xA^\infty x}\). Hence, \(1 - (\text{Index}E)E^\infty(e) = 0\). This implies that \(E\) has the Rokhlin property. 

Remark 4.4 

(i) A projection \(g\) in Definition 4.2 is not zero, because that \(E^\infty\) is faithful. 

(ii) A projection \(g\) in Definition 4.2 satisfies that \(g \in P' \cap P^\infty\). Indeed, for any \(x \in P\), since \(ex = xe\) and \(E^\infty\) has norm one, we have

\[
x g = x(\text{Index}E)E^\infty(e) = (\text{Index}E)x(E(e_1), E(e_n), \ldots) \quad (e = (e_n)) 
= (\text{Index}E)(xe_1, xe_2, \ldots) = (\text{Index}E)(E(xe_1), E(xe_2), \ldots) 
= (\text{Index}E)E^\infty(xe) = (\text{Index}E)E^\infty(ex) 
= (\text{Index}E)(E(e_1), E(e_2), \ldots) = (\text{Index}E)(E(e_1)x, E(e_2)x, \ldots) 
= (\text{Index}E)(E(e_1), E(e_2), \ldots) x = (\text{Index}E)E^\infty(e)x = gx.
\]

Remark 4.5 In Definition 4.2 when \(A\) is simple, the following hold: 

(i) We do not need the injectivity of the map \(A \ni x \mapsto xe\).

(ii) We have \(ege = e\). Indeed, since \(A\) is simple, \(\text{Index}E\) is scalar by [38, Remark 2.3.6] and from [38, Lemma 2.1.5 (2)], there exists a constant \(C > 0\) such that

\[
E(e) \geq \frac{C}{(\text{Index}E)^2}e.
\]

Thus, \(g \geq e\), which implies that \(e \in gA^\infty g\). Therefore, \((1 - g)e = 0\); that is, \((4.4)\)

\[
ge e = e.
\]

Hence, \(e = ege\).

Lemma 4.6 Let \(E: A \rightarrow P\) be of index-finite type with the tracial Rokhlin property and consider the basic extension \(P \subset A \subset B\). Then the Rokhlin projection \(e \in A' \cap A^\infty\) satisfies \(eBe = Ae\).

Proof Let \(e_p\) be the Jones projection for the inclusion \(A \ni P\). Then, since \(g = (\text{Index} E)E^\infty(e)\) is a projection such that \(ge = e\) by (4.4) we have

\[
f^2 = (\text{Index} E)^2 e e_p e e_p e = (\text{Index} E)^2 e e_p (e_n) e e_p e = (e_n) \quad (e = (e_n)) 
= (\text{Index} E)^2 e (e_p e_n) e = (\text{Index} E)^2 e (e(n)_e) e_p e \quad (4.2) 
= (\text{Index} E)^2 e (E(e_n)) e_p e = (\text{Index} E)^2 e E^\infty(e) e_p e = (\text{Index} E) e e_p e \quad (4.4) 
= (\text{Index} E) e e_p e = (4.4)
\]

\[
f = f.
\]
Let $\widehat{E}$ be the dual conditional expectation for $E$. Using [19, 2.3 (4)],
\[
\widehat{E}^{\infty}(e - f) = e - \text{Index} E \widehat{E}^{\infty}(ee_p e) = e - e = 0.
\]
Thus, since $\widehat{E}$ is faithful, we have $e = f = (\text{Index} E) ee_p e$; that is,
\[
(4.5) \quad ee_p e = (\text{Index} E)^{-1} e.
\]
Then since we have for any $x, y \in A$,
\[
e(x e_p y) e = x ee_p ey = (\text{Index} E)^{-1} x ey = (\text{Index} E)^{-1} xy e \in Ae.
\]
Since $B$ is the linear span of $\{xe_p y | x, y \in A\}$, we have $eBe \subset Ae$. Conversely, since $A \subset B$, $Ae \subset eBe$, and we conclude that $eBe = Ae$.

The following is the heredity of Property (SP) for an inclusion of unital $C^*$-algebras.

**Proposition 4.7** Let $P \subset A$ be an inclusion of unital $C^*$-algebras with index-finite type. Suppose that $A$ is simple and $E : A \to P$ has the tracial Rokhlin property. Then we have that

(i) $P$ is simple;

(ii) $A$ has Property (SP) if and only if $P$ has Property (SP).

**Proof** (i): Let $e$ be a Rokhlin projection for $E$ and $P \subset A \subset B$ be the basic extension. Since $P$ is stably isomorphic to $B$ by [38, Lemma 3.3.4], we will show that $B$ is simple. By [12, Theorem 3.3] $B$ can be written as finite direct sums of simple $C^*$-algebras. Moreover, each simple $C^*$-subalgebra has the form of $Bz$ for some projection $z \in B \cap B'$. To show the simplicity of $B$ it is enough to show that $B' \cap B = \mathbb{C}$.

Since $e = [(e_n)] \in A' \cap A^\infty$, for any $x \in A' \cap B$, we have
\[
ex = [(e_n)] x = [(e_n x)] = [(xe_n)] = xe.
\]
We can assume that $x = a_1 e_p a_2$, where $e_p$ is the Jones projection for $E$. Then
\[
xe = e x e = e(a_1 e_p a_2) e = a_1 ee_p e a_2
\]
\[
= (\text{Index} E)^{-1} a_1 a_2 e \quad (4.5)) = \widehat{E}(x) e,
\]
where $\widehat{E} : B \to A$ be the dual conditional expectation of $E$. Note that $\widehat{E}(x) \in A'$. Hence, we have $xe \in (A' \cap A)e$. Therefore, $(A' \cap B)e \subset (A' \cap A)e$.

Since $A$ is simple and $(B' \cap B)e \subset (A' \cap B)e$, we have $(B' \cap B)e = \mathbb{C}e$. Since the map $\rho : A' \cap B \to (A' \cap A)e$ by $\rho(x) = xe$ is an isomorphism, $B' \cap B = \mathbb{C}$; that is, $B$ is simple, and $P$ is simple.

(ii) This follows from [25, Corollary, Section 5].

**Proposition 4.8** Let $G$ be a finite group, $\alpha$ an action of $G$ on an infinite dimensional finite simple separable unital $C^*$-algebra $A$, and $E$ the canonical conditional expectation from $A$ onto the fixed point algebra $P = A^\alpha$ defined by
\[
E(x) = \frac{1}{|G|} \sum_{g \in G} \alpha_g(x) \quad \text{for} \ x \in A,
\]
where $|G|$ is the order of $G$. Then $\alpha$ has the tracial Rokhlin property if and only if $E$ has the tracial Rokhlin property.
Proof Suppose that \( \alpha \) has the tracial Rokhlin property. Since \( A \) is separable, there is an increasing sequence of finite sets \( \{ F_n \}_{n \in \mathbb{N}} \subset A \) such that \( \bigcup_{n \in \mathbb{N}} F_n = A \). Let any nonzero positive element \( x = (x_n) \in A^{\infty} \). Then we can assume that each \( x_n \) is a nonzero positive element. The simplicity of \( A \) implies that the map \( A \ni x \mapsto xe \) is injective.

Since \( \alpha \) has the tracial Rokhlin property, for each \( n \) there are mutually orthogonal projections \( \{ e_{g,n} \}_{g \in G} \) such that the following hold:

(i) \[ \| \alpha_h(e_{g,n}) - e_{h,g,n} \| < \frac{1}{n} \text{ for all } g, h \in G, \]

(ii) \[ \| e_{g,n} \| < \frac{1}{n} \text{ for all } g \in G \text{ and } a \in F_n \text{ with } |a| \leq 1, \]

(iii) \[ 1 - \sum_{g \in G} e_{g,n} \text{ is equivalent to a projection } q_n \text{ in } x_n A x_n. \]

Set \( e_g = (e_{g,n}) \in A^{\infty} \) for \( g \in G \). Then for all \( g, h \in G \),

\[ \| \alpha_h^\infty(e_g) - e_{h,g} \| = \limsup \| \alpha_h(e_{g,n}) - e_{h,g,n} \| = 0; \]

hence, \( \alpha_h^\infty(e_g) = e_{h,g} \) for all \( g, h \in G \).

For all \( a \in \bigcup_{n \in \mathbb{N}} F_n \) with \( |a| \leq 1 \) and all \( g \in G \), we have

\[ \| e_{g,n} \| = \limsup \| e_{g,n} \| = 0; \]

hence, \( e_g \in A_{\infty} \) for all \( g \in G \).

Set \( q = (q_n) \in A^{\infty} \). Then \( q \) is a projection in \( x A^{\infty} x \) and

\[ 1 - \sum_{g \in G} e_g = \left( 1 - \sum_{g \in G} e_{g,n} \right) \sim (q_n) = q. \]

Therefore, if we set \( e = e_1 \) for the identity element \( 1 \) in \( G \), then \( e \in A' \cap A^{\infty} \) and

\[ E^{\infty}(e) = \frac{1}{|G|} \sum_{g \in G} \alpha_h^\infty(e) = \frac{1}{|G|} \sum_{g \in G} e_g, \]

\[ 1 - |G| E^{\infty}(e) = 1 - \sum_{g \in G} e_g = 1 - q \in x A^{\infty} x. \]

Note that Index \( E = |G| \) by Example 4.1. It follows that \( E \) has the tracial Rokhlin property.

Conversely, suppose that \( E \) has the tracial Rokhlin property. From Lemma 4.3 \( A \) has Property (SP) or \( E \) has the Rokhlin property. If \( E \) has the Rokhlin property, then \( \alpha \) has the Rokhlin property by \([19, Proposition 3.2]\); hence, \( \alpha \) has the tracial Rokhlin property from the definition.

We can assume that \( A \) has Property (SP). Then for any finite set \( F \subset A, \varepsilon > 0 \), and any nonzero positive element \( x \in A \) there is a projection \( e \in A_{\infty} \) such that \( |G| E^{\infty}(e)(= g) \) is a projection and \( 1 - g \) is equivalent to a projection \( q \in x A^{\infty} x \). We note that \( g \neq 0 \) by Remark 4.4, and \( e \neq 0 \). When we write \( e = (e_n) \) and \( q = (q_n) \), we can assume that for each \( n \in \mathbb{N} \) \( e_n \) is projection and \( 1 - e_n \) is equivalent to \( q_n \).

Define \( e_g = \alpha_h^\infty(e) \in A_{\infty} \) for \( g \in G \); write \( e_g = (\alpha_{g}(e_n)) = (e_{g,n}) \) for \( g \in G \). Then since we have

\[ \sum_{g \in G} e_g = \sum_{g \in G} \alpha_h(e) = |G| E^{\infty}(e) = g \]

and \( g \) is projection, we can assume that \( \{ e_{g,n} \}_{g \in G} \) are mutually orthogonal projections for each \( n \in \mathbb{N} \) by \([22, Lemma 2.5.6]\).
Then \( \alpha_h^\infty(e) = e_{hg} \) for all \( g, h \in G \), \( \|e_{x}, a\| = 0 \) for all \( a \in F \) and all \( g \in G \), and
\[
1 - \sum_{g \in G} e_{g} = 1 - \sum_{g \in G} \alpha_g^\infty(e) = 1 - |G|E^\infty(e) = 1 - g \sim q \in xA^\infty x,
\]
Then there exists \( n \in \mathbb{N} \) such that \( \|\alpha_n(e_{g,n}) - e_{hg,n}\| < \varepsilon \) for all \( g, h \in G \), \( \|e_{g,n}, a\| < \varepsilon \) for all \( a \in F \) and \( g \in G \), and
\[
1 - \sum_{g \in G} e_{g,n} \sim q_n \in xAx.
\]
Set \( f_g = e_{g,n} \) for \( g \in G \); then we have
\[
\|\alpha_n(f_g) - f_{hg}\| < \varepsilon,
\]
for all \( g, h \in G \), \( \|f_g, a\| < \varepsilon \) for all \( a \in F \) and \( g \in G \), and
\[
1 - \sum_{g \in G} f_g \sim q_n \in xAx.
\]
Hence, \( \alpha \) has the tracial Rokhlin property.

The following lemma is key to proving the main theorem in this section.

Lemma 4.9 Let \( \Lambda \supset P \) be an inclusion of unital C*-algebras and let \( E \) be a conditional expectation from \( A \) onto \( P \) with index-finite type. Suppose that \( A \) is simple. If \( E \) has the tracial Rokhlin property with a Rokhlin projection \( e \in A_{\infty} \) and a projection \( g = (\text{Index}E)E^\infty(e) \), then there is a unital linear map \( \beta: A^\infty \rightarrow P^\infty \) such that for any \( x \in A^\infty \) there exists the unique element \( y \) of \( P^\infty \) such that \( xe = ye = \beta(x)e \) and \( \beta(A' \cap A^\infty) \subset P' \cap P^\infty \). In particular, \( \beta_n \) is a unital injective *-homomorphism and \( \beta(x) = xg \) for all \( x \in P \).

Proof Since \( E \) has the tracial Rokhlin property, \( A \) has Property (SP) or \( E \) has the Rokhlin property by Lemma 4.3. If \( E \) has the Rokhlin property, then the conclusion comes from \([28, \text{Lemma 2.5}]\) with \( g = 1 \). Therefore, we can assume that \( A \) has Property (SP).

Since \( A \) has Property (SP), \( g \) and \( e \) are nonzero projections by Remark 4.4. As in the same argument in the proof of \([28, \text{Lemma 2.5}]\), we have for any element \( x \) in \( A^\infty \) there exists a unique element \( y = (\text{Index}E)E^\infty(xe) \in P^\infty \) such that \( xe = ye \). Indeed, by Lemma 4.6 we have \( epee = (\text{Index}E)^{-1}e \). Then
\[
xe = (\text{Index}E)E^\infty(e_pxe) = (\text{Index}E)^2E^\infty(e_pexe_p) = (\text{Index}E)E^\infty(xe)e,
\]
where \( E \) is the dual conditional expectation for \( E \). Put \( y = (\text{Index}E)E^\infty(xe) \in P^\infty \). Then we have \( xe = ye \). Note that since \( eg = e \) by Remark 4.5, we have
\[
yg = (\text{Index}E)E^\infty(xe)g = (\text{Index}E)E^\infty(xeg) (g \in P^\infty) = (\text{Index}E)E^\infty(xe) = y.
\]

Therefore, we can define a unital map \( \beta: A^\infty \rightarrow P^\infty \) \( \beta(x) = (\text{Index}E)E^\infty(xe) \) such that \( xe = ye = \beta(x)e \) and \( \beta(A' \cap A^\infty) \subset P' \cap P^\infty \). Indeed, from the definition
of \( \beta \) we know that \( \beta(A' \cap A^{\infty}) \subset P^\infty g \). On the contrary, for any \( x \in A' \cap A \) and \( a \in P \) we have

\[
\begin{align*}
  a\beta(x) &= a(\text{Index}E)E^\infty(xe) = \text{Index}E)E^\infty(axe) \\
               &= \text{Index}E)E^\infty(xea) = \text{Index}E)E^\infty(xe)a \\
               &= \beta(x)a
\end{align*}
\]

(4.1)

Hence \( \beta(A' \cap A^{\infty}) \subset P' \). Therefore, \( \beta(A' \cap A^{\infty}) \subset P' \cap P^\infty g \).

Note that \( \beta \) is injective. Indeed, if \( \beta(x) = 0 \) for \( x \in A \), then \( xe = 0 \). Hence, from the definition of the tracial Rokhlin property for \( E, x = 0 \).

Since for any \( x \in A \)

\[
\beta(x)g = (\text{Index}E)E^\infty(xe)g = (\text{Index}E)E^\infty(xeg)
\]

\[
= (\text{Index}E)E^\infty(ex) = (\text{Index}E)E^\infty(gex)
\]

\[
= g(\text{Index}E)E^\infty(xe) = g\beta(x),
\]

we know that \( \beta|_A \) is a unital *-homomorphism from \( A \) to \( gP^\infty g \) from the same argument as in the proof of [28, Lemma 2.5]. In particular for any \( x \in P \), we have

\[
\beta(x) = (\text{Index}E)E^\infty(xe) = x(\text{Index}E)E^\infty(e) = xg = gx.
\]

The following lemma is important to prove the heredity of the local tracial \( C^* \)-property for an inclusion of unital \( C^* \)-algebras.

**Lemma 4.10** Let \( P \subset A \) be an inclusion of unital \( C^* \)-algebras with index-finite type, and \( E: A \to P \) has the tracial Rokhlin property. Suppose that projections \( p, q \in P^\infty \) satisfy \( ep = pe \) and \( q \leq ep \) in \( A^{\infty} \), where \( e \) is a Rokhlin projection for \( E \). Then \( q \leq p \) in \( P^\infty \).

**Proof** Let \( s \) be a partial isometry in \( A^{\infty} \) such that \( s^*s = q \) and \( ss^* \leq ep \).

Set \( v = (\text{Index}E)^{1/2}E^{\infty}(s) \). Then

\[
\begin{align*}
  v^*ve_p &= (\text{Index}E)E^\infty(s^*E^\infty(s)e_p = (\text{Index}E)E^\infty(s^*e)e^\infty(es)e_p \\
          &= (\text{Index}E)e_p s^*es_p e_p e_p = e_p s^*es_p \quad ((\text{Index}E)e_p e_p = e) \quad (4.5)
\end{align*}
\]

\[
= E^\infty(s^*es)e_p = E^\infty(s^*s)e_p = E^\infty(q)e_p = qe_p.
\]

Hence,

\[
\bar{E}^{\infty}(v^*ve_p) = \bar{E}^{\infty}(qe_p),
\]

\[
(\text{Index}E)^{-1}v^*v = (\text{Index}E)^{-1}q,
\]

\[
v^*v = q.
\]

Since

\[
pv = p(\text{Index}E)^{1/2}E^\infty(s) = (\text{Index}E)^{1/2}E^\infty(ps) = (\text{Index}E)^{1/2}E^\infty(s) = v,
\]

we have \( q \leq p \) in \( P^\infty \).

**Theorem 4.11** Let \( \mathcal{C} \) be a class of weakly semiprojective \( C^* \)-algebras satisfying conditions in Theorem 3.3. Let \( A \supset P \) be an inclusion of unital \( C^* \)-algebras and \( E \) a conditional
expectation from $A$ onto $P$ with index-finite type. Suppose that $A$ is a simple, local tracial $C^*$-algebra and $E$ has the tracial Rohlin property. Then $P$ is a local tracial $C^*$-algebra.

**Proof** We will prove that for every finite set $F \subset P$, every $\varepsilon > 0$, and $x \in P^* \setminus 0$ there are $C^*$-algebra $Q \in \mathcal{C}$ with $q = 1_Q$ and $*$-homomorphism $\pi : Q \to A$ such that $\|\pi(q)x - x\pi(q)\| < \varepsilon$ for all $x \in F$, $\pi(q)S\pi(q) \subset \pi(Q)$, and $1 - \pi(q)$ is equivalent to some non-zero projection in $\overline{zPz}$.

Since $E$ has the tracial Rohlin property, $A$ has Property (SP) or $E$ has the Rohlin property by Lemma 4.3.

Suppose that $E$ has the Rohlin property. Then we have from [28, Lemma 2.5] that there is a unital $*$-homomorphism $\beta : A \to P^\infty$ such that $\beta(x) = x$ for all $x \in P$. Since $A$ is a local tracial $C^*$-algebra, there are an algebra $B \in \mathcal{C}$ with $1_B = p$ and a $*$-homomorphism $\pi : B \to A$ such that $\|x\pi(p) - \pi(p)x\| < \varepsilon$ for all $x \in F$, $\pi(p)F\pi(p) \subset \pi(B)$, such that $1 - \pi(p)$ is equivalent to a non-zero projection $q = \overline{zPz}$. Since $E$ has the Rohlin property, there exists a non-zero projection $e \in A' \cap A^\infty$ such that $E^\infty(e) = \frac{1}{\text{Index}\pi}$.

Since $B$ is weakly semiprojective, there exists $k \in \mathbb{N}$ and $\overline{\beta \circ \pi : B \to \prod_{n=k}^\infty P}$ such that $\beta \circ \pi = \pi_k \circ \overline{\beta \circ \pi}$, where $\pi_k((b_k, b_{k+1}, \ldots)) = (0, \ldots, 0, b_k, b_{k+1}, \ldots)$. For each $l \in \mathbb{N}$ with $l \geq k$ let $\beta_l$ be a $*$-homomorphism from $B$ to $P$ such that $\beta_l \circ \pi = (\beta_l(b))_{n=k}^\infty$ for $b \in B$. Then $\beta_l \circ \pi(b) = (0, \ldots, 0, \beta_k(b), \beta_{k+1}(b), \ldots) + C_0(P)$ for $b \in B$, and $\beta_l$ is a $*$-homomorphism for $l \geq k$.

Since $1 - p - q \in \overline{zPz}$,

$$1 - \beta_l \circ \pi(p) = 1 - \beta_l(1 - \pi(p)) = \beta_l(1 - \pi(p)) \sim \beta_l \circ \pi(q) \in \beta_l(\overline{zPz}),$$

$$[(1 - \beta_k(\pi(p)))] \sim [(q_k)] \in \overline{zP^\infty z},$$

where each $q_k$ are projections in $P$. Taking the sufficient large $k$, since

$$\lim_k \|\beta_k(x) - x\| = 0$$

for $x \in P$, we have

(a) $\|x\beta_k(p) - \beta_k(p)x\| < 2\varepsilon$ for any $x \in F$,

(b) $\beta_k(p)F\beta_k(p) \subset \beta_k(p)\beta_k(B)\beta_k(p)$,

(c) $1 - \beta_k(p) = \beta_k(1 - p) \sim q_k \in \overline{zPz}$.

Hence, $P$ is a local tracial $C^*$-algebra.

Suppose that $A$ has Property (SP). Since $A$ is simple, from Proposition 4.7, $P$ also has Property (SP). Let $F \subset P$ be a finite set, $\varepsilon > 0$, and $x \in P^* \setminus 0$. Since $P$ is simple and has Property (SP), there are orthogonal non-zero projections $r_1, r_2 \in \overline{zPz}$.

Since $A$ is a local tracial $C^*$-algebra, there are an algebra $B \in \mathcal{C}$ with $1_B = p$ and a $*$-homomorphism $\pi : B \to A$ such that $\|x\pi(p) - \pi(p)x\| < \varepsilon$ for all $x \in F$, $\pi(p)F\pi(p) \subset B$, and $1 - \pi(p)$ is equivalent to a non-zero projection $q \in r_1Ar_1$. Since $E$ has the tracial Rohlin property, there exist the Rohlin projection $e' \in A' \cap A^\infty$. Take another Rohlin projection $e \in A' \cap A^\infty$ for a projection $e'r_2$ such that $g = \text{IndexEE}(e)$ satisfies $1 - g$ is equivalent to a projection $e'r_2A^\infty e'r_2$; that is, $1 - g \leq e'r_2$ in $A^\infty$. Then
by Lemma 4.10, we know that \(1 - g \leq r_2 \in P^\infty\); that is, there is a projection \(s \leq r_2 \in P^\infty\) such that \(1 - g \sim s\).

Write \(g = \left[ (g_n) \right]\) for some projections \(\{g_k\}_{k \in \mathbb{N}} \subset P\). From Lemma 4.9, there exists an injective *-homomorphism \(\beta : A \to gP^\infty g\) such that \(\beta(x) = xg\) for all \(x \in P\), and \(\overline{\beta \circ \pi} : B \to \Pi_{n=k}^\infty P\) such that \(\beta \circ \pi = \pi_k \circ \overline{\beta \circ \pi}\), where

\[
\pi_k(b, b_1, 1, \ldots) = (0, \ldots, 0, b_k, b_{k+1}, \ldots) + C_0(P).
\]

For each \(l \in \mathbb{N}\) with \(l \geq k\), let \(\beta_l\) be a map from \(B\) to \(g_lPg_l\), so that \(\overline{\beta \circ \pi}(b) = (\beta_l(b))_{l=k}^\infty\) for \(b \in B\). Then \(\beta \circ \pi(b) = (0, \ldots, 0, \beta_k(b), \beta_{k+1}(b), \ldots) + C_0(P)\) for all \(b \in B\) and \(\beta_l\) is a *-homomorphism for \(l \geq k\).

Since \(1 - \pi(p) \sim q \in r_1A_F\),

\[
1 - (\beta \circ \pi)(p) = 1 - g + g - \beta(\pi(p)) = 1 - g - \beta(1 - \pi(p))
\]

\[
\sim s + \beta(q) \in r_2P^\infty r_2 + s = \beta(r_1A_F)
\]

\[
\subset r_2P^\infty r_2 + r_1P^\infty r_1 \subset zP^\infty z,
\]

we have \(\left[(1 - \beta_k(p))\right] \sim [(q_k)] \in zP^\infty z\), where each \(q_k\) is projection in \(P\). Taking a sufficiently large \(k\), since \(\lim_k \|\beta_k(x) - x\| = 0\) for \(x \in P\), we have

(a) \(\|x\beta_k(p) - \beta_k(p)x\| < 2\|x\|\) for any \(x \in F\),
(b) \(\beta_k(p)F\beta_k(p) \subset \beta_k(p)\beta_k(B)\beta_k(p)\),
(c) \(1 - \beta_k(p) \sim q_k \in zPz\).

Hence, \(P\) is a local tracial C*-algebra.

\[\text{Corollary 4.12} \quad \text{Let } P \subset A \text{ be an inclusion of unital C*-algebras and let } E \text{ be a conditional expectation from } A \text{ onto } P \text{ with index-finite type. Suppose that } A \text{ is an infinite-dimensional simple C*-algebra with tracial topological rank zero (resp. less than or equal to one) and } E \text{ has the tracial Rokhlin property. Then } P \text{ has tracial rank zero (resp. less than or equal to one).}\]

\[\text{Proof} \quad \text{Since the classes } \mathfrak{T}(k) \quad (k = 0, 1) \text{ are semiprojective with respect to a class of unital C*-algebras [24] and finitely saturated [26, Examples 2.1 & 2.2, and Lemma 1.6], the conclusion comes from Theorem 4.11 and Definition 3.8.}\]

Finally, in this section we give the heredity of stable rank one and real rank zero for an inclusion of unital C*-algebras.

\[\text{Proposition 4.13} \quad \text{Let } P \subset A \text{ be an inclusion of unital C*-algebras with index finite-type. Suppose that } E : A \to P \text{ has the tracial Rokhlin property and } A \text{ is simple with } \text{tsr}(A) = 1. \text{ Then } \text{tsr}(P) = 1.\]

\[\text{Proof} \quad \text{Since } E \text{ has the tracial Rokhlin property, } E \text{ has the Rokhlin property or } A \text{ has Property (SP) by Lemma 4.3. If } E \text{ has the Rokhlin property, we conclude that } \text{tsr}(P) = 1 \text{ by [19, 5.9]. Therefore, we assume that } A \text{ has the Property (SP). Then we know that } P \text{ is simple and has Property (SP) by Proposition 4.7.}\]

\(\text{Note that } \text{tsr}(A) = 1, \text{ and } A \text{ is stably finite by [31, Theorem 3.3]. Since an inclusion } P \subset A \text{ is of index-finite type, } P \text{ is stably finite. Hence, using the idea in [32] we}\]
The Jiang–Su Absorption for Inclusions of Unital C*-algebras

have only to show that any two sided zero divisor in \( P \) is approximated by invertible elements in \( P \).

Let \( x \in P \) be a two sided zero divisor. From [32, Lemma 3.5] we can assume that there is a positive element \( y \in P \) and a unitary \( u \in P \) such that \( yu x = 0 = ux y \). If we show that \( ux \) can be approximated by invertible elements, so does \( x \). Hence, we can assume that \( yx = 0 = xy \). Since \( P \) has Property (SP), there is a non-zero projection \( e \in \overline{yPy} \). Since \( P \) is simple, we can take orthogonal projections \( e_1 \) and \( e_2 \) in \( P \) such that \( e = e_1 + e_2 \) and \( e_2 \leq e_1 \) by [22, Lemma 3.5.6 (2)]. Note that \( x \in (1 - e_1)A(1 - e_1) \). Since \( \text{tr}(1 - e_1)A(1 - e_1) = 1 \), there is an invertible element \( b \) in \( (1 - e_1)A(1 - e_1) \) such that \( |x - b| < \frac{1}{\varepsilon} \).

Since \( E \) has the tracial Rokhlin property, there is a projection \( g \in P' \cap P^\infty \) such that \( 1 - g \leq e_2 \). That is, there is a partial isometry \( w \in P^\infty \) such that \( w^* w = 1 - g \) and \( w w^* \leq e_1 \) by Lemma 4.10 (see also Corollary 6.3.). Moreover, \( |\beta(x) - \beta(b)| < \frac{1}{3} \varepsilon \) by Lemma 4.9. Note that \( \beta(b) \) is invertible in \( (1 - e_1)P^\infty P(1 - e_1)g \).

Set \( v = w(1 - e_1) \). Then
\[
\begin{align*}
v^* v & = (1 - e_1)w^* w (1 - e_1) = (1 - g)(1 - e_1), \\
v v^* & = w(1 - e_1)w^* \leq w w^* \leq e_1.
\end{align*}
\]
Set
\[
z = \frac{\varepsilon}{3}(e_1 - v v^*) + \frac{\varepsilon}{3} v + \frac{\varepsilon}{3} v^* + (1 - g)x(1 - g).
\]
Hence, \( z \) is invertible in \( e_1 P^\infty e_1 + (1 - g)(1 - e_1)P^\infty (1 - e_1)(1 - g) \) and \( |z - (1 - g)x(1 - g)| < \frac{\varepsilon}{3} \).

Then \( \beta(b) + z \in P^\infty \) is invertible and
\[
|z - (\beta(b) + z)| = |z - \beta(x)| + |(1 - g)x(1 - g) - z| \\
\leq |z - \beta(x)| + |\beta(x) - \beta(b) + (1 - g)x(1 - g) - z| \\
\leq |\beta(x) - \beta(b)| + |(1 - g)x(1 - g) - z| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \leq \varepsilon.
\]

Write \( \beta(b) + z = (y_n) \) such that \( y_n \) is invertible in \( P \). Therefore, there is a \( y_n \) such that \( |x - y_n| < \varepsilon \), and we conclude that \( \text{tr}(P) = 1 \).

**Proposition 4.14** Let \( P \subset A \) be an inclusion of unital C*-algebras with index-finite type and \( E : A \to P \) has the tracial Rokhlin property. Suppose that \( A \) is simple, stably finite, with real rank zero. Then \( P \) has real rank zero.

**Proof** Let \( x \in P \) be a self-adjoint element and \( \varepsilon > 0 \). Consider a continuous real valued function \( f \) defined by \( f(t) = 1 \) for \( |t| \leq \frac{\varepsilon}{12} \), \( f(t) = 0 \) if \( |t| \geq \frac{\varepsilon}{6} \), and \( f(t) \) is linear if \( \frac{\varepsilon}{12} \leq |t| \leq \frac{\varepsilon}{6} \). We may assume that \( f(x) \neq 0 \). Note that \( |y x| < \frac{\varepsilon}{6} \) for any \( y \in f(x)Pf(x) \) with \( |y| \leq 1 \).

Since \( A \) is simple and has Property (SP), \( P \) has Property (SP) by Proposition 4.7; that is, there is a non-zero projection \( e \in \overline{f(x)Pf(x)} \). Moreover, there are orthogonal projections \( e_1 \) and \( e_2 \) such that \( e = e_1 + e_2 \) such that \( e_2 \sim e_1 \) by [22, Lemma 3.5.7]. Then
\[
|y - (1 - e_1)x(1 - e_1)| = |e_1 x e_1 + e_1 x (1 - e_1) + (1 - e_1)x e_1)| < \frac{\varepsilon}{12} = \frac{\varepsilon}{4}.
\]
As in the same step in the argument in Proposition 4.13, we have there is an invertible self-adjoint element \( z \in P \) such that \( \|(1 - e_1)x(1 - e_1) - z\| < \frac{\varepsilon}{3} \). Hence, we have \( \|x - z\| < \varepsilon \), and we conclude that \( P \) has real rank zero.

\[\text{\ }\]

5 The Jiang–Su Absorption

In this section we discuss about the heredity for the Jiang–Su absorption for an inclusion of unital \( C^* \)-algebras with the tracial Rokhlin property.

**Definition 5.1** ([11, Definition 2.1]) A unital \( C^* \)-algebra \( A \) is said to be tracially \( \mathcal{Z} \)-absorbing if \( A \not\cong \mathbb{C} \) and for any finite set \( F \subset A \) and non-zero positive element \( a \in A \) and \( n \in \mathbb{N} \) there is an order zero contraction \( \phi: M_n \to A \) such that the following hold:

(i) \( 1 - \phi(1) \leq a \);
(ii) for any normalized element \( x \in M_n \) and any \( y \in F \), we have \( \|\phi(x), y\| < \varepsilon \).

**Theorem 5.2** (II, Theorem 4.1) Let \( A \) be a unital, separable, simple, nuclear \( C^* \)-algebra. If \( A \) is tracially \( \mathcal{Z} \)-absorbing, then \( A \cong A \otimes \mathcal{Z} \).

Note that for a simple unital \( C^* \)-algebra \( A \), if \( A \) is \( \mathcal{Z} \)-absorbing, then \( A \) is tracially \( \mathcal{Z} \)-absorbing ([II, Proposition 2.2]).

**Theorem 5.3** Let \( P \subset A \) be an inclusion of unital \( C^* \)-algebra and let \( E \) be a conditional expectation from \( A \) onto \( P \) with index-finite type. Suppose that \( A \) is simple, separable, unital, tracially \( \mathcal{Z} \)-absorbing, and that \( E \) has the tracial Rokhlin property. Then \( P \) is tracially \( \mathcal{Z} \)-absorbing.

**Proof** Take any finite set \( F \subset P \) and non-zero positive element \( a \in P \) and \( n \in \mathbb{N} \). Since \( E: A \to P \) has the tracial Rokhlin property, \( E \) has the Rokhlin property or \( A \) has Property (SP). If \( E \) has the Rokhlin property, then \( P \) is \( \mathcal{Z} \)-absorbing ([28, Theorem 3.3]), and we are done. Hence, we can assume that \( A \) has Property (SP).

Since \( A \) is simple and has Property (SP), \( P \) has Property (SP) by Proposition 4.7. Then there exist orthogonal projections \( p_1, p_2 \) in \( aPa \).

Since \( A \) is tracially \( \mathcal{Z} \)-absorbing, there is an order zero contraction \( \phi: M_n \to A \) such that the following hold:

(a) \( 1 - \phi(1) \leq p_1 \);
(b) For any normalized element \( x \in M_n \) and any \( y \in F \) we have \( \|\phi(x), y\| < \varepsilon \).

Since \( E: A \to P \) has the tracial Rokhlin property, there is a projection \( e \in A' \cap A^\infty \) such that \( (\text{Index } E)E^\infty (e) = g \) is a projection and \( 1 - g \leq p_2 \). Moreover, by Lemma 4.9 there is an injective *homomorphism \( \beta \) from \( A \) into \( gP^\infty g \) such that \( \beta(1) = g \) and \( \beta(a) = ag \) for \( a \in P \).

(a) Then the function \( \beta \circ \phi(= \psi): M_n \to P^\infty \) is an order zero map such that

\[
1 - \psi(1) = 1 - (\beta \circ \phi)(1) = 1 - g + \beta(1 - \phi(1)) \\
\leq p_2 + \beta(p_1) = p_2 + p_1 \beta(1) \leq a,
\]

that is, \( 1 - \psi(1) \leq a \) in \( P^\infty \).
(b) For any normalized element \( x \in M_n \) and \( y \in F \),
\[
\| [\psi(x), y] \| = \| [\beta(\phi(x)), y] \| = \| \beta(\phi(x)) y - y \beta(\phi(x)) \| \\
= \| \beta(\phi(x)) \beta(y) - \beta(y) \beta(\phi(x)) \| \\
= \| \beta(\phi(x) y - y \phi(x)) \| \leq \| \phi(x) y - y \phi(x) \| < \varepsilon.
\]

Since \( C^*(\phi(M_n)) \) is semiprojective in the sense of [24, Definition 14.13], there is a \( k \in \mathbb{N} \) and a \(*\)-homomorphism \( \widetilde{\beta} : C^*(\phi(M_n)) \to \Pi P / \oplus_{i=1}^k P \to P^\infty \) such that \( \pi_k \circ \widetilde{\beta} = \beta \), where \( \pi_k \) is the canonical map from \( \Pi P / \oplus_{i=1}^k P \) to \( P^\infty \). Write \( \widetilde{\beta}(x) = (\widetilde{\beta}_l(x)) + \oplus_{i=1}^k P \) and \( g = (g_l) \) for some projections \( g_l \in P \) for \( l \in \mathbb{N} \). Then we have for sufficiently large \( l \) the order zero map \( \widetilde{\beta}_l \circ \phi : M_n \to P \) satisfies
\[
1 - \widetilde{\beta}_l \circ \phi(1) = 1 - \widetilde{\beta}_l(\phi(1)) = 1 - g_l + g_l - \widetilde{\beta}_l(\phi(1)) \\
= 1 - g_l + \widetilde{\beta}_l(1 - \phi(1)) (\widetilde{\beta}_l(1) = g_l) \\
\leq 1 - g_l + \widetilde{\beta}_l(p_l) \leq p_2 + p'_1,
\]
where \( p'_1 \in p_1 P p_1 \) is projection such that \( \widetilde{\beta}_l(p_l) \sim p'_1 \). Note that since \( \| \widetilde{\beta}_l(p_l) - p_1 g_l \| \) is very small (if \( \| \widetilde{\beta}_l(p_l) - p_1 g_l \| < \frac{1}{2} \) is enough), there are projections \( p'_1 \in p_1 P p_1 \) such that \( \widetilde{\beta}_l(p_l) \sim p'_1 \). Since \( p_2 \perp p_1, 1 - g_l \leq p_2, \) and \( \widetilde{\beta}_l(p_l) \sim p'_1 \), from [6, 11 Proposition] we have \( 1 - g_l + \widetilde{\beta}_l(p_l) \leq p_2 + p'_1 \). Hence, we have
\[
1 - \widetilde{\beta}_l \circ \phi(1) \leq p_2 + p'_1 \leq a
\]
This implies that \( P \) is tracially \( \mathbb{Z} \)-absorbing.

The following is the main theorem in this paper.

**Theorem 5.4** Let \( P \subset A \) be an inclusion of unital \( C^* \)-algebras and let \( E \) be a conditional expectation from \( A \) onto \( P \) with index-finite type. Suppose that \( A \) is simple, separable, nuclear, \( \mathbb{Z} \)-absorbing, and that \( E \) has the tracial Rokhlin property. Then \( P \) is \( \mathbb{Z} \)-absorbing.

**Proof** This follows from Theorems 5.3 and 5.2.

**Corollary 5.5** Let \( A \) be an infinite dimensional simple, unital, simple, nuclear \( C^* \)-algebra and let \( \alpha : G \to \text{Aut}(A) \) be an action of a finite group \( G \) with the tracial Rokhlin property. Suppose that \( A \) is \( \mathbb{Z} \)-absorbing. Then we have the following:

(i) the fixed point algebra \( A^\alpha \) and the crossed product \( A \rtimes_\alpha G \) are \( \mathbb{Z} \)-absorbing ([11]);

(ii) for any subgroup \( H \) of \( G \) the fixed point algebra \( A^H \) is \( \mathbb{Z} \)-absorbing.

**Proof** (i) Since the canonical conditional expectation \( E : A \to A^\alpha \) has the tracial Rokhlin property by Proposition 4.8, \( A^\alpha \) is \( \mathbb{Z} \)-absorbing, by Theorem 5.4.

Let \( |G| = n \). Then \( A \rtimes_\alpha G \) is isomorphic to \( p M_n (A^\alpha) p \) for some projection \( p \in M_n (A^\alpha) \) by Example 4.1(ii). Since \( A^\alpha \) is \( \mathbb{Z} \)-absorbing, \( p M_n (A^\alpha) p \) is \( \mathbb{Z} \)-absorbing by [37, Corollary 3.1], hence \( A \rtimes_\alpha G \) is \( \mathbb{Z} \)-absorbing.

(ii) Since \( \alpha_{\|H|} : H \to \text{Aut}(A) \) has the tracial Rokhlin property by [8, Lemma 5.6], we know that \( A^H \) is \( \mathbb{Z} \)-absorbing, by (i).
6 Cuntz-equivalence for Inclusions of C*-algebras

In this section we study the heredity for Cuntz equivalence for an inclusion of unital C*-algebras with the tracial Rokhlin property.

Let $M_\infty(A)^+$ denote the disjoint union $\bigcup_{n=1}^\infty M_n(A)^+$. For $a \in M_n(A)^+$ and $b \in M_m(A)^+$ set $a \oplus b = \text{diag}(a, b) \in M_{n+m}(A)^+$, and write $a \preceq b$ if there is a sequence $\{x_k\}$ in $M_{m,n}(A)$ such that $x_k^* b x_k \to a$. Write $a \sim b$ if $a \preceq b$ and $b \preceq a$. Put $W(A) = M_\infty(A)^+ / \sim$, and let $\langle a \rangle \in W(A)$ be the equivalence class containing $a$. Then $W(A)$ is a positive ordered abelian semigroup equipped with the relations

$$\langle a \rangle + \langle b \rangle = \langle a \oplus b \rangle, \quad \langle a \rangle \leq \langle b \rangle \iff a \preceq b, \quad a, b \in M_\infty(A)^+.$$  

We call $W(A)$ the Cuntz semigroup.

Lemma 6.1 Let $P \subset A$ be an inclusion of unital C*-algebras with index-finite type and let $E : A \to P$ have the tracial Rokhlin property. Suppose that positive elements $a, b \in P^\infty$ satisfy $eb = be$ and $a \preceq eb$ in $A^\infty$, where $e$ is a Rokhlin projection for $E$. Then $a \preceq b$ in $P^\infty$.

Proof Since $a \preceq eb$ in $A^\infty$, there is a sequence $\{v_n\}_{n \in \mathbb{N}}$ in $A^\infty$ such that

$$\|a - v_n^* eb v_n\| \to 0 \quad (n \to \infty).$$

Let $E : A \to P$ be a conditional expectation of index-finite type. Set

$$w_n = \text{Index } E \frac{1}{2} E^\infty(ev_n)$$

for each $n \in \mathbb{N}$. Then, since

$$w_n^* b w_n e_p = (\text{Index } E) E^\infty(v_n^* e)b E^\infty(ev_n) e_p$$

$$\quad = (\text{Index } E) E^\infty(v_n^* eb) E^\infty(ev_n) e_p \quad (4.1)$$

$$\quad = (\text{Index } E) e_p v_n^* eb e_p v_n e_p \quad (4.2)$$

$$\quad = (\text{Index } E) e_p v_n^* beeb e_p v_n e_p \quad (4.5)$$

$$\quad = E^\infty(v_n^* eb v_n) e_p \quad (4.2),$$



$$w_n^* b w_n = E^\infty(v_n^* eb v_n)$$

from [38, Lemma 2.1.4]. Therefore,

$$\|a - w_n^* b w_n\| = \|a - E^\infty(v_n^* eb v_n)\| = \|E^\infty(a - v_n^* eb v_n)\|$$

$$\quad \leq \|a - v_n^* eb v_n\| \to 0 \quad (n \to \infty).$$

This implies that $a \preceq b$ in $P^\infty$. \qed

Proposition 6.2 Let $P \subset A$ be an inclusion of unital C*-algebras with index-finite type. Suppose that $E : A \to P$ has the tracial Rokhlin property. If two positive elements $a, b \in P$ satisfy $a \preceq b$ in $A$, then $a \preceq b$ in $P$.

Proof Let $a, b \in P$ be positive elements such that $a \preceq b$ in $A$ and $\varepsilon > 0$. Since for any constant $K > 0$ $a \preceq Kb$ is equivalent to $Ka \preceq Kb$, we can assume that $a$ and $b$ are contractive. If $b$ is invertible, then $a = (a^{1/2} b^{-1/2}) b (a^{1/2} b^{-1/2})^*$, and $a \preceq b$ in $P$. Hence, we may assume that $b$ has $0$ in its spectrum.
Since \( a \leq b \) in \( A \), for every \( \varepsilon > 0 \) there is \( \delta > 0 \) and \( r \in A \) such that \( f_\varepsilon(a) = rf_\delta(b) = 1 \) by [33, Proposition 2.4], where \( f_\varepsilon: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is given by

\[
f_\varepsilon(t) = \begin{cases} 
0, & t \leq \varepsilon, \\
\varepsilon^{-1}(t - \varepsilon), & \varepsilon \leq t \leq 2\varepsilon, \\
1, & t \geq 2\varepsilon.
\end{cases}
\]

Set \( a_0 = f_\delta(b)^{1/2}r^*rf_\delta(b)^{1/2} \). Then \( f_\varepsilon(a) \sim a_0 \) by [5, 1.5]. Set a continuous function \( g_\delta(t) \) on \([0, 1]\) by

\[
g_\delta(t) = \begin{cases} 
\delta^{-1}(\delta - t), & 0 \leq t \leq \delta, \\
0, & \delta \leq t \leq 1.
\end{cases}
\]

Since \( b \) has 0 in its spectrum, \( g_\delta(b) \neq 0 \) and \( g_\delta(b)^2f_\delta(b) = 0 \), which implies that \( a_0g_\delta(b) = 0 \). Note that \( g_\delta(b)(c) \) and \( a_0 \) belong to \( bPb \). Therefore, the positive elements \( a_0 \) in \( bAb \) and \( c \) in \( bPb \) satisfy \( (a - \varepsilon)_+ \leq a_0 + c \in A \). Indeed,

\[
(a - \varepsilon)_+ \leq f_\varepsilon(a) \sim a_0 \leq a_0 + c \quad ([6, Proposition 1.1]).
\]

Take a Rokhlin projection \( e \in A' \cap A^\infty \) for \( E \). Then there is a projection \( g \in P' \cap P^\infty \) such that \( (1 - g) \leq ec \). Hence, \( (a - \varepsilon)_+(1 - g) \leq ec \in A^\infty \). Note that since \( e \in P \), we have \( ec = ce \). By Lemma 6.1, \( (a - \varepsilon)_+(1 - g) \leq c \) in \( P^\infty \).

Then we have in \( P^\infty \)

\[
(a - \varepsilon)_+ = (a - \varepsilon)_+g + (a - \varepsilon)_+(1 - g)
\]

\[
= \beta((a - \varepsilon)_+) + (a - \varepsilon)_+(1 - g)
\]

\[
\leq \beta(a_0) + (a - \varepsilon)_+(1 - g) \quad ([6, Proposition 1.1])
\]

\[
\leq \beta(a_0) + c \in bP^\infty b,
\]

where \( \beta: A \rightarrow bP^\infty b \) is defined as in Lemma 4.9. Hence, \( (a - \varepsilon)_+ \leq b \) in \( P^\infty \).

Since \( \varepsilon > 0 \), we have \( a \leq b \) in \( P^\infty \), and \( a \leq b \) in \( P \).

The following result implies that the canonical inclusion from \( K_0(P) \) into \( K_0(A) \) is injective.

**Corollary 6.3** Under the same assumption in Proposition 6.2, if two projections \( p, q \in P \) satisfy \( p \leq q \) in \( A \), then \( p \leq q \) in \( P \).

### 7 The Strict Comparison Property

In this section we study the strict comparison property for a Cuntz semigroup and show that for an inclusion \( P \subset A \) of exact, unital \( C^* \)-algebras with the tracial Rokhlin property if \( A \) has strict comparison, then so does \( P \). When \( E: A \rightarrow P \) has the Rokhlin property, the statement is proved in [29].

A **dimension function** on a \( C^* \)-algebra \( A \) is a function \( d: M_\infty(A)^+ \rightarrow \mathbb{R}^+ \) that satisfies \( d(a \oplus b) = d(a) + d(b) \), and \( d(a) \leq d(b) \) if \( a \leq b \) for all \( a, b \in M_\infty(A)^+ \). If \( \tau \) is a positive trace on \( A \), then

\[
d_\tau(a) = \lim_{n \rightarrow \infty} \tau(a^n)^{1/n} = \lim_{t \rightarrow 0^+} \tau(f_t(a)), \quad a \in M_\infty(A)^+
\]
defines a dimension function on $A$. Every lower semicontinuous dimension function on an exact $C^*$-algebra arises in this way ([3, Theorem II.2.2], [10], [17]). For the Cuntz semigroup $W(A)$ an additive order preserving mapping $\hat{d}: W(A) \to \mathbb{R}^+$ is given by $\hat{d}(a) = d(a)$ from a dimension function $d$ on $A$. We use the same symbol to denote the dimension function on $A$ and the corresponding state on $W(A)$.

Recall that an $C^*$-algebra $A$ has strict comparison if, whenever $x, y \in W(A)$ are such that $d(x) < d(y)$ for every dimension function $d$ on $A$, we have $x \leq y$. If $A$ is simple, exact and unital, then the strict comparison property is equivalent to the strict comparison property by traces; that is, for all $x, y \in W(A)$ one has that $x \leq y$ if $d_\tau(x) < d_\tau(y)$ for all tracial states $\tau$ on $A$ ([34, Corollary 4.6]).

Let $T(A)$ be the set of all traces on a $C^*$-algebra $A$.

**Theorem 7.1** Let $P \subseteq A$ be an inclusion of unital $C^*$-algebras with index-finite type. Suppose that $A$ is simple and exact and $A$ has strict comparison and $E: A \to P$ has the tracial Rokhlin property. Then $P$ has strict comparison.

**Proof** Since $E: A \to P$ is of index-finite type and $A$ is simple and exact, $P$ is exact and simple by Proposition 4.7 (i). Note that the strict comparison property is equivalent to the the strict comparison property given by traces, i.e., for all $x, y \in W(A)$ one has that $x \leq y$ if $d_\tau(x) < d_\tau(y)$ for all tracial states $\tau$ on $A$ (see [35, Remark 6.2] and [34, Corollary 4.6]).

Since $E \otimes \text{id}: A \otimes M_n \to P \otimes M_n$ is of index-finite type and has the tracial Rokhlin property, it suffices to verify the condition that whenever $a, b \in P$ are positive elements such that $d_\tau(a) < d_\tau(b)$ for all $\tau \in T(P)$, then $a \leq b$.

Let $a, b \in P$ be positive elements such that $d_\tau(a) < d_\tau(b)$ for all $\tau \in T(P)$. Then for any tracial state $\tau \in T(A)$ the restriction $\tau_{|P}$ belongs to $T(P)$. Hence, we have $d_\tau(a) < d_\tau(b)$ for all tracial states $\tau \in T(A)$. Since $A$ has strict comparison, $a \leq b$ in $A$. Therefore, by Proposition 6.2, $a \leq b$ in $P$, and $P$ has strict comparison.

**Corollary 7.2** Let $P \subseteq A$ be an inclusion of unital $C^*$-algebras of index-finite type. Suppose that $A$ is simple, the order on projections on $A$ is determined by traces, and $E: A \to P$ has the tracial Rokhlin property. Then the order on projections on $P$ is determined by traces.

**Proof** Since $E \otimes \text{id}: A \otimes M_n \to P \otimes M_n$ is of index-finite type and has the tracial Rokhlin property, it suffices to verify the condition that whenever $p, q \in P$ are projections such that $\tau(p) < \tau(q)$ for all $\tau \in T(P)$, $p$ is Murray–von Neumann equivalent to a subprojection of $q$ in $P$.

Let $p, q \in P$ be projections such that $\tau(p) < \tau(q)$ for all $\tau \in T(P)$. Since for any tracial state $\tau \in T(A)$ the restriction $\tau_{|P}$ belongs to $T(P)$, we have $\tau(p) < \tau(q)$ for all tracial states $\tau \in T(A)$. Since the order on projections on $A$ is determined by traces, $p$ is Murray–von Neumann subequivalent to $q$ in $A$, and $p \leq q$ in $A$ by [33, Proposition 2.1]. Therefore, by Proposition 6.2 $p \leq q$ in $P$, and so $p$ is Murray–von Neumann equivalent to a subprojection of $q$ in $P$. Hence, the order on projections on $P$ is determined by traces.
The following is well known, but there is no direct proof, so we present it for convenience of the reader.

**Lemma 7.3** Let $A$ be an exact C*-algebra and let $p$ be a projection of $A$. Suppose that $A$ has strict comparison. Then so does $pAp$.

**Proof** Since for each $n \in \mathbb{N}$ $M_n(A)$ is exact and has strict comparison, we have only to show that whenever $a, b \in pAp$ are positive elements such that $d_\tau(a) < d_\tau(b)$ for all tracial states $\tau \in T(pAp)$; then $a \leq b$.

Let $a, b \in pAp$ be positive elements such that $d_\tau(a) < d_\tau(b)$ for all tracial states $\tau \in T(pAp)$. Then for any tracial state $\rho \in T(A)$ the restriction $\rho|_{pAp}$ belongs to $T(pAp)$. Since $d_\rho(a) = d_{\rho|_{pAp}}(a) < d_{\rho|_{pAp}}(b) = d_\rho(b)$, we have $a \leq b$ in $A$ by the assumption. Hence there is a sequence $\{x_n\} \subset A$ such that $x_n^*bx_n - a \to 0$. Set $y_n = px_n p \in pAp$ for each $n \in \mathbb{N}$, then

$$
\|y_n^*by_n - a\| = \|px_n^*pbx_n p - pap\| = \|px_n^*bx_n p - pap\| \\
\leq \|x_n^*bx_n - a\| \to 0,
$$

and $a \leq b$ in $pAp$. Therefore, $pAp$ has strict comparison. \[\Box\]

**Corollary 7.4** Let $A$ be an infinite-dimensional, simple, separable, unital C*-algebra and let $\alpha: G \to \text{Aut}(A)$ be an action of a finite group $G$ with the tracial Rokhlin property. Suppose that $A$ is exact and has strict comparison.

(i) The fixed point algebra $A^\alpha$ and the crossed product $A \rtimes_\alpha G$ have strict comparison.

(ii) For any subgroup $H$ of $G$ the fixed point algebra $A^H$ has strict comparison.

**Proof** (i) Since the canonical conditional expectation $E: A \to A^\alpha$ has the tracial Rokhlin property by Proposition 4.8, $A^\alpha$ has strict comparison by Theorem 7.1.

Let $|G| = n$. Then $A \rtimes_\alpha G$ is isomorphic to $pM_n(A^\alpha)p$ for some projection $p \in M_n(A^\alpha)$ by Example 4.1(ii). Since $A^\alpha$ has strict comparison, $pM_n(A^\alpha)p$ has strict comparison by Lemma 7.3, hence $A \rtimes_\alpha G$ has strict comparison.

(ii) Since $\alpha_H: H \to \text{Aut}(A)$ has the tracial Rokhlin property by [8, Lemma 5.6], we know that $A^H$ has strict comparison by (i). \[\Box\]

Similarly, we have the following corollary.

**Corollary 7.5** Let $A$ be an infinite dimensional simple separable unital C*-algebra and let $\alpha: G \to \text{Aut}(A)$ be an action of a finite group $G$ with the tracial Rokhlin property. Suppose that the order on projections on $A$ is determined by traces.

(i) The order on projections on the fixed point algebra $A^\alpha$ and the crossed product $A \rtimes_\alpha G$ is determined by traces.

(ii) For any subgroup $H$ of $G$ the order on projections on the fixed point algebra $A^H$ is determined by traces.

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