ON SOME NEW HADAMARD-TYPE INEQUALITIES FOR CO-ORDINATED QUASI-CONVEX FUNCTIONS

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Abstract. In this paper, we establish some Hadamard-type inequalities based on co-ordinated quasi-convexity. Also we define a new mapping associated to co-ordinated convexity and we prove some properties of this mapping.

1. INTRODUCTION

Let \( f : I \subset \mathbb{R} \rightarrow \mathbb{R} \) be a convex function on the interval of \( I \) of real numbers and \( a, b \in I \) with \( a < b \). The following double inequality

\[
\frac{f\left(\frac{a+b}{2}\right)}{f(a)} \leq \frac{\int_{a}^{b} f(x)dx}{b-a} \leq \frac{f(a) + f(b)}{2}
\]

is well-known in the literature as Hadamard’s inequality. We recall some definitions;

Definition 1. (See [4]) A function \( f : [a, b] \rightarrow \mathbb{R} \) is said quasi-convex on \([a, b]\) if

\[
f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}, \quad (QC)
\]

holds for all \( x, y \in [a, b] \) and \( \lambda \in [0, 1] \).

Clearly, any convex function is quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex. In [5], Dragomir defined convex functions on the co-ordinates as following:

Definition 2. Let us consider the bidimensional interval \( \Delta = [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b, c < d \). A function \( f : \Delta \rightarrow \mathbb{R} \) will be called convex on the co-ordinates if the partial mappings \( f_y : [a, b] \rightarrow \mathbb{R}, f_y(u) = f(u, y) \) and \( f_x : [c, d] \rightarrow \mathbb{R}, f_x(v) = f(x, v) \) are convex where defined for all \( y \in [c, d] \) and \( x \in [a, b] \). Recall that the mapping \( f : \Delta \rightarrow \mathbb{R} \) is convex on \( \Delta \) if the following inequality holds,

\[
f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x, y) + (1-\lambda)f(z, w)
\]

for all \( (x, y), (z, w) \in \Delta \) and \( \lambda \in [0, 1] \).

In [5], Dragomir established the following inequalities of Hadamard’s type for co-ordinated convex functions on a rectangle from the plane \( \mathbb{R}^2 \).
Theorem 1. (see [5], Theorem 1) Suppose that $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ is convex on the co-ordinates on $\Delta$. Then one has the inequalities:

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy \right]$$

(1.2)

The above inequalities are sharp.

Similar results for co-ordinated $m$–convex and $(\alpha, m)$–convex functions can be found in [2]. In [2], Dragomir considered a mapping which closely connected with above inequalities and established main properties of this mapping as following:

Now, for a mapping $f : \Delta := [a, b] \times [c, d] \to \mathbb{R}$ is convex on the co-ordinates on $\Delta$, we can define the mapping $H : [0, 1]^2 \to \mathbb{R}$,

$$H(t, s) := \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\left(tx + (1-t)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right) dxdy$$

Theorem 2. Suppose that $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ is convex on the co-ordinates on $\Delta = [a, b] \times [c, d]$. Then:

(i) The mapping $H$ is convex on the co-ordinates on $[0, 1]^2$.

(ii) We have the bounds

$$\sup_{(t, s) \in [0, 1]^2} H(t, s) = \int_a^b \int_c^d f(1, 1) = H(1, 1)$$

$$\inf_{(t, s) \in [0, 1]^2} H(t, s) = f \left(\frac{a+b}{2}, \frac{c+d}{2}\right) = H(0, 0)$$

(iii) The mapping $H$ is monotonic nondecreasing on the co-ordinates.

Definition 3. Consider a function $f : V \to \mathbb{R}$ defined on a subset $V$ of $\mathbb{R}_n$, $n \in \mathbb{N}$. Let $L = (L_1, L_2, ..., L_n)$ where $L_i \geq 0$, $i = 1, 2, ..., n$. We say that $f$ is $L$–Lipschitzian function if

$$|f(x) - f(y)| \leq \sum_{i=1}^n L_i |x_i - y_i|$$

for all $x, y \in V$.

In [3], Özdemir et al. defined quasi-convex function on the co-ordinates as following:
Definition 4. A function \( f : \Delta = [a, b] \times [c, d] \rightarrow \mathbb{R} \) is said quasi-convex function on the co-ordinates on \( \Delta \) if the following inequality
\[
f (\lambda x + (1 - \lambda) z, \lambda y + (1 - \lambda) w) \leq \max \{ f (x, y), f (z, w) \}
\]
holds for all \((x, y), (z, w) \in \Delta \) and \( \lambda \in [0, 1] \).

Let consider a bidimensional interval \( \Delta := [a, b] \times [c, d] \), \( f : \Delta \rightarrow \mathbb{R} \) will be called co-ordinated quasi-convex functions on the co-ordinates if the partial mappings
\[
f_y : [a, b] \rightarrow \mathbb{R}, \quad f_y (u) = f (u, y)
\]
and
\[
f_x : [c, d] \rightarrow \mathbb{R}, \quad f_x (v) = f (x, v)
\]
are convex where defined for all \( y \in [c, d] \) and \( x \in [a, b] \). We denote by \( QC(\Delta) \) the class of quasi-convex functions on the co-ordinates on \( \Delta \).

In \([1]\), Sarıkaya et al. proved following Lemma and established some inequalities for co-ordinated convex functions.

Lemma 1. Let \( f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( \frac{\partial^2 f}{\partial t \partial s} \in L(\Delta) \), then the following equality holds:
\[
\frac{f (a, c) + f (a, d) + f (b, c) + f (b, d)}{4} + \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f (x, y) \, dy \, dx
\]
\[
- \frac{1}{2} \left[ \frac{1}{b - a} \int_a^b [f (x, c) + f (x, d)] \, dx + \frac{1}{d - c} \int_c^d [f (a, y) + f (b, y)] \, dy \right]
\]
\[
= \frac{(b - a)(d - c)}{4} \int_0^1 \int_0^1 (1 - 2t) (1 - 2s) \frac{\partial^2 f}{\partial t \partial s} (ta + (1 - t)b, sc + (1 - s)d) \, dt \, ds.
\]

The main purpose of this paper is to obtain some inequalities for co-ordinated quasi-convex functions by using Lemma 1 and elementary analysis.

2. MAIN RESULTS

Theorem 3. Let \( f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( \frac{\partial^2 f}{\partial t \partial s} \) is quasi-convex on the co-ordinates on \( \Delta \), then one has the inequality:
\[
\left| \frac{f (a, c) + f (a, d) + f (b, c) + f (b, d)}{4} + \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f (x, y) \, dy \, dx - A \right|
\]
\[
\leq \frac{(b - a)(d - c)}{16} \max \left\{ \left| \frac{\partial^2 f}{\partial t \partial s} (a, b) \right|, \left| \frac{\partial^2 f}{\partial t \partial s} (c, d) \right| \right\}
\]
where
\[
A = \frac{1}{2} \left[ \frac{1}{b - a} \int_a^b [f (x, c) + f (x, d)] \, dx + \frac{1}{d - c} \int_c^d [f (a, y) + f (b, y)] \, dy \right].
\]
Proof. From Lemma 1, we can write
\[
\left| f(a, c) + f(a, d) + f(b, c) + f(b, d) \right| \\
+ \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) \, dy \, dx - A \\
\leq \frac{(b - a)(d - c)}{4} \\
\times \int_0^1 \int_0^1 |(1 - 2t)(1 - 2s)| \left| \frac{\partial^2 f}{\partial t \partial s}(ta + (1 - t)b, sc + (1 - s)d) \right| \, dt \, ds.
\]
Since \( \left| \frac{\partial^2 f}{\partial t \partial s} \right| \) is quasi-convex on the co-ordinates on \( \Delta \), we have
\[
\left| f(a, c) + f(a, d) + f(b, c) + f(b, d) \right| \\
+ \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) \, dy \, dx - A \\
\leq \frac{(b - a)(d - c)}{4} \\
\times \int_0^1 \int_0^1 |(1 - 2t)(1 - 2s)| \max \left\{ \left| \frac{\partial^2 f}{\partial t \partial s}(a, b) \right|, \left| \frac{\partial^2 f}{\partial t \partial s}(c, d) \right| \right\} \, dt \, ds.
\]
On the other hand, we have
\[
\int_0^1 \int_0^1 |(1 - 2t)(1 - 2s)| \, dt \, ds = \frac{(b - a)(d - c)}{16}.
\]
The proof is complete. \( \Box \)

**Theorem 4.** Let \( f : \Delta \subset \mathbb{R}^2 \to \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( \left| \frac{\partial^2 f}{\partial t \partial s} \right|^q \), \( q > 1 \), is quasi-convex function on the co-ordinates on \( \Delta \), then one has the inequality:
\[
\left| f(a, c) + f(a, d) + f(b, c) + f(b, d) \right| \\
+ \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) \, dy \, dx - A \\
\leq \frac{(b - a)(d - c)}{4(p + 1)^{\frac{1}{q}}} \left( \max \left\{ \left| \frac{\partial^2 f}{\partial t \partial s}(a, b) \right|^q, \left| \frac{\partial^2 f}{\partial t \partial s}(c, d) \right|^q \right\} \right)^{\frac{1}{q}}
\]
where
\[
A = \frac{1}{2} \left[ \frac{1}{b - a} \int_a^b [f(x, c) + f(x, d)] \, dx + \frac{1}{d - c} \int_c^d [f(a, y) + f(b, y)] \, dy \right]
\]
and \( \frac{1}{p} + \frac{1}{q} = 1 \).
Proof. From Lemma 1 and using Hölder inequality, we get

\[
\frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} + \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) \, dy \, dx - A \leq \frac{(b - a)(d - c)}{4} \\
\times \left( \int_0^1 \left( (1 - 2t)(1 - 2s) \right)^{\frac{1}{p}} \, dt \right) \left( \int_0^1 \left( (1 - 2t)(1 - 2s) \right)^{\frac{1}{q}} \, ds \right)^{\frac{1}{q}}.
\]

Since \( \frac{\partial^2 f}{\partial t \partial s} \) is quasi-convex on the co-ordinates on \( \Delta \), we have

\[
\frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} + \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) \, dy \, dx - A \leq \frac{(b - a)(d - c)}{4} \\
\times \left( \int_0^1 \left( (1 - 2t)(1 - 2s) \right)^{\frac{1}{p}} \, dt \right) \left( \int_0^1 \left( (1 - 2t)(1 - 2s) \right)^{\frac{1}{q}} \, ds \right)^{\frac{1}{q}} \max \left\{ \left| \frac{\partial^2 f}{\partial t \partial s}(a, b) \right|^q, \left| \frac{\partial^2 f}{\partial t \partial s}(c, d) \right|^q \right\}^{\frac{1}{q}}
\]

So, the proof is complete. \( \square \)

Corollary 1. Since \( \frac{1}{q} < \frac{1}{p + 1} \frac{1}{q} < 1 \), for \( p > 1 \), we have the following inequality:

\[
\frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} + \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) \, dy \, dx - A \leq \frac{(b - a)(d - c)}{4} \max \left\{ \left| \frac{\partial^2 f}{\partial t \partial s}(a, b) \right|^q, \left| \frac{\partial^2 f}{\partial t \partial s}(c, d) \right|^q \right\}^{\frac{1}{q}}.
\]

Theorem 5. Let \( f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( \left| \frac{\partial^2 f}{\partial t \partial s} \right|^q \geq 1 \), is quasi-convex function.
on the co-ordinates on Ω, then one has the inequality:

\[
\left| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) \, dy \, dx - A \right| \\
\leq \frac{(b-a)(d-c)}{16} \left( \max \left\{ \left| \frac{\partial^2 f}{\partial t \partial s}(a,b) \right|^q, \left| \frac{\partial^2 f}{\partial t \partial s}(c,d) \right|^q \right\} \right)^{\frac{1}{q}}
\]

where

\[
A = \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b [f(x,c) + f(x,d)] \, dx + \frac{1}{d-c} \int_c^d [f(a,y) + f(b,y)] \, dy \right].
\]

Proof. From Lemma 1 and using Power Mean inequality, we can write

\[
\left| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) \, dy \, dx - A \right| \\
\leq \frac{(b-a)(d-c)}{4} \\
\times \int_0^1 \int_0^1 |(1-2t)(1-2s)| \left( \int_0^1 \left( \int_0^1 \left| \frac{\partial^2 f}{\partial t \partial s}(ta+(1-t)b, sc+(1-s)d) \right|^q \, dt \, ds \right)^{-\frac{1}{q}} \right) \\
\leq \frac{(b-a)(d-c)}{4} \left( \int_0^1 \int_0^1 |(1-2t)(1-2s)| \, dt \, ds \right)^{1-\frac{1}{q}} \\
\times \left( \int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial t \partial s}(ta+(1-t)b, sc+(1-s)d) \right|^q \, dt \, ds \right)^{\frac{1}{q}} \\
= \frac{(b-a)(d-c)}{16} \left( \max \left\{ \left| \frac{\partial^2 f}{\partial t \partial s}(a,b) \right|^q, \left| \frac{\partial^2 f}{\partial t \partial s}(c,d) \right|^q \right\} \right)^{\frac{1}{q}}
\]

which completes the proof. □
Remark 1. Since \( \frac{1}{4} < \frac{1}{(p+1)p} < 1 \), for \( p > 1 \), the estimation in Theorem 4 is better than Theorem 3.

Now, for a mapping \( f : \Delta := [a, b] \times [c, d] \to \mathbb{R} \) is convex on the co-ordinates on \( \Delta \), we can define the mapping \( G : [0, 1]^2 \to \mathbb{R} \),

\[
G(t, s) := \frac{1}{4} \left[\begin{array}{l}
f \left( ta + (1-t) \frac{a+b}{2}, sc + (1-s) \frac{c+d}{2} \right) \\
+ f \left( tb + (1-t) \frac{a+b}{2}, sc + (1-s) \frac{c+d}{2} \right) \\
+ f \left( ta + (1-t) \frac{a+b}{2}, sd + (1-s) \frac{c+d}{2} \right) \\
+ f \left( tb + (1-t) \frac{a+b}{2}, sd + (1-s) \frac{c+d}{2} \right)
\end{array}\right]
\]

We will give following theorem which contains some properties of this mapping.

**Theorem 6.** Suppose that \( f : \Delta \subset \mathbb{R}^2 \to \mathbb{R} \) is convex on the co-ordinates on \( \Delta = [a, b] \times [c, d] \). Then:

(i) The mapping \( G \) is convex on the co-ordinates on \([0, 1]^2\).

(ii) We have the bounds

\[
\inf_{(t, s) \in [0, 1]^2} G(t, s) = f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) = G(0, 0)
\]

\[
\sup_{(t, s) \in [0, 1]^2} G(t, s) = \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} = G(1, 1)
\]

(iii) If \( f \) is satisfy Lipschitzian conditions, then the mapping \( G \) is \( L \)-Lipschitzian on \([0, 1] \times [0, 1]\).

(iv) Following inequality holds;

\[
\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) dxdy \\
\leq \frac{1}{4} \left[ f(a, c) + f(a, d) + f(b, d) + f(bracket) \right]
\]
Proof. (i) Let $s \in [0,1]$. For all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and $t_1, t_2 \in [0,1]$, then we have

$$G(\alpha t_1 + \beta t_2, s)$$

$$= \frac{1}{4} \left[ f \left( (\alpha t_1 + \beta t_2) a + (1 - (\alpha t_1 + \beta t_2)) \frac{a + b}{2}, sc + (1 - s) \frac{c + d}{2} \right) + f \left( (\alpha t_1 + \beta t_2) b + (1 - (\alpha t_1 + \beta t_2)) \frac{a + b}{2}, sc + (1 - s) \frac{c + d}{2} \right) + f \left( (\alpha t_1 + \beta t_2) a + (1 - (\alpha t_1 + \beta t_2)) \frac{a + b}{2}, sd + (1 - s) \frac{c + d}{2} \right) + f \left( (\alpha t_1 + \beta t_2) b + (1 - (\alpha t_1 + \beta t_2)) \frac{a + b}{2}, sd + (1 - s) \frac{c + d}{2} \right) \right].$$

Using the convexity of $f$, we obtain

$$G(\alpha t_1 + \beta t_2, s) \leq \frac{1}{4} \left[ \alpha \left( f \left( t_1 a + (1 - t_1) \frac{a + b}{2}, sc + (1 - s) \frac{c + d}{2} \right) + f \left( t_1 b + (1 - t_1) \frac{a + b}{2}, sc + (1 - s) \frac{c + d}{2} \right) + f \left( t_1 a + (1 - t_1) \frac{a + b}{2}, sd + (1 - s) \frac{c + d}{2} \right) + f \left( t_1 b + (1 - t_1) \frac{a + b}{2}, sd + (1 - s) \frac{c + d}{2} \right) \right) + \beta \left( f \left( t_2 a + (1 - t_2) \frac{a + b}{2}, sc + (1 - s) \frac{c + d}{2} \right) + f \left( t_2 b + (1 - t_2) \frac{a + b}{2}, sc + (1 - s) \frac{c + d}{2} \right) + f \left( t_2 a + (1 - t_2) \frac{a + b}{2}, sd + (1 - s) \frac{c + d}{2} \right) + f \left( t_2 b + (1 - t_2) \frac{a + b}{2}, sd + (1 - s) \frac{c + d}{2} \right) \right) \right] \right].$$

If $s \in [0,1]$. For all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and $t_1, t_2 \in [0,1]$, then we also have:

$$G(t, \alpha s_1 + \beta s_2) \leq \alpha G(t, s_1) + \beta G(t, s_2)$$

and the statement is proved.
By using the triangle inequality, we get

$$\inf_{(t,s) \in [0,1]^2} G(t,s) = f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) = G(0,0)$$

$$\sup_{(t,s) \in [0,1]^2} G(t,s) = \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} = G(1,1).$$

(iii) Let \( t_1, t_2, s_1, s_2 \in [0,1] \), then we have

$$|G(t_2, s_2) - G(t_1, s_1)|$$

By using the triangle inequality, we get

$$\leq \frac{1}{4} \left| f \left( t_2 a + (1 - t_2) \frac{a + b}{2}, s_2 c + (1 - s_2) \frac{c + d}{2} \right) - f \left( t_1 a + (1 - t_1) \frac{a + b}{2}, s_1 c + (1 - s_1) \frac{c + d}{2} \right) \right|$$

$$+ f \left( t_2 a + (1 - t_2) \frac{a + b}{2}, s_2 d + (1 - s_2) \frac{c + d}{2} \right) - f \left( t_1 a + (1 - t_1) \frac{a + b}{2}, s_1 d + (1 - s_1) \frac{c + d}{2} \right)$$

$$+ f \left( t_2 b + (1 - t_2) \frac{a + b}{2}, s_2 c + (1 - s_2) \frac{c + d}{2} \right) - f \left( t_1 b + (1 - t_1) \frac{a + b}{2}, s_1 c + (1 - s_1) \frac{c + d}{2} \right)$$

$$+ f \left( t_2 b + (1 - t_2) \frac{a + b}{2}, s_2 d + (1 - s_2) \frac{c + d}{2} \right) - f \left( t_1 b + (1 - t_1) \frac{a + b}{2}, s_1 d + (1 - s_1) \frac{c + d}{2} \right).$$

By using the \( f \) is satisfy Lipschitzian conditions, then we obtain

$$\leq \frac{1}{4} \left| |L_1(b-a)| t_2 - t_1| + L_2(d-c)| s_2 - s_1| + L_3(b-a)| t_2 - t_1| + L_4(d-c)| s_2 - s_1|$$

$$+ L_5(b-a)| t_2 - t_1| + L_6(d-c)| s_2 - s_1| + L_7(b-a)| t_2 - t_1| + L_8(d-c)| s_2 - s_1|$$

$$= \frac{1}{4} |(L_1 + L_2 + L_3 + L_4)(b-a)| t_2 - t_1| + (L_5 + L_6 + L_7 + L_8)(d-c)| s_2 - s_1|$$

this imply that the mapping \( G \) is \( L \)-Lipschitzian on \([0,1] \times [0,1] \).
(iv) By using the convexity of \( G \) on \([0, 1] \times [0, 1]\), we have
\[
\begin{align*}
&f \left( ta + (1 - t) \frac{a + b}{2}, sc + (1 - s) \frac{c + d}{2} \right) + f \left( tb + (1 - t) \frac{a + b}{2}, sd + (1 - s) \frac{c + d}{2} \right) \\
&+ f \left( ta + (1 - t) \frac{a + b}{2}, sd + (1 - s) \frac{c + d}{2} \right) + f \left( tb + (1 - t) \frac{a + b}{2}, sc + (1 - s) \frac{c + d}{2} \right) \\
&\leq tsf(a, c) + t(1 - s) f \left( a, \frac{c + d}{2} \right) + (1 - t) sf(\frac{a + b}{2}, c) + (1 - t)(1 - s) f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \\
&+ tsf(b, c) + t(1 - s) f \left( b, \frac{c + d}{2} \right) + (1 - t) sf(\frac{a + b}{2}, c) + (1 - t)(1 - s) f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \\
&+ tsf(a, d) + t(1 - s) f \left( a, \frac{c + d}{2} \right) + (1 - t) sf(\frac{a + b}{2}, d) + (1 - t)(1 - s) f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \\
&+ tsf(b, d) + t(1 - s) f \left( b, \frac{c + d}{2} \right) + (1 - t) sf(\frac{a + b}{2}, d) + (1 - t)(1 - s) f \left( \frac{a + b}{2}, \frac{c + d}{2} \right).
\end{align*}
\]

By integrating both sides of the above inequality and by taking into account the change of the variables, we obtain
\[
\begin{align*}
&\frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) dy dx \\
&\leq \frac{1}{4} \left[ f(a, c) + f(b, c) + f(a, d) + f(b, d) \\
&+ f \left( \frac{a + b}{2}, c \right) + f \left( \frac{a + b}{2}, d \right) + f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \right].
\end{align*}
\]

Which completes the proof. \( \square \)

**References**

[1] M.Z. Sarıkaya, E. Set, M.E. Özdemir and S.S. Dragomir, New Some Hadamard’s type inequalities for co-ordinated convex functions, Accepted.

[2] M.E. Özdemir, E. Set, M.Z. Sarıkaya, Some new Hadamard’s type inequalities for co-ordinated m-convex and (\( a, m \))-convex functions, Accepted.

[3] M.E. Özdemir, A.O. Akdemir and Ç. Yıldız, On co-ordinated quasi-convex functions, Submitted.

[4] J. Pečarić, F. Proschan and Y.L. Tong, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press (1992), Inc.

[5] S.S. Dragomir, On the Hadamard’s inequality for convex functions on the co-ordinates in a rectangle from the plane, *Taiwanese Journal of Mathematics*, 5 (2001), no. 4, 775-788.

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