Fine-tuning in Federated Learning: 
a simple but tough-to-beat baseline

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Abstract
We study the performance of federated learning algorithms and their variants in an asymptotic framework. Our starting point is the formulation of federated learning as a multi-criterion objective, where the goal is to minimize each client’s loss using information from all of the clients. We analyze a linear regression model, where, for a given client, we theoretically compare the performance of various algorithms in the high-dimensional asymptotic limit. This asymptotic multi-criterion approach naturally models the high-dimensional, many-device nature of federated learning and suggests that personalization is central to federated learning. In this paper, we investigate how some sophisticated personalization algorithms fare against simple fine-tuning baselines. In particular, our theory suggests that Federated Averaging with client fine-tuning is competitive than more intricate meta-learning and proximal-regularized approaches. In addition to being conceptually simpler, our fine-tuning-based methods are computationally more efficient than their competitors. We corroborate our theoretical claims with extensive experiments on federated versions of the EMNIST, CIFAR-100, Shakespeare, and Stack Overflow datasets.

1 Introduction
In Federated learning (FL), a collection of client machines, or devices, collect data and coordinate with a central server to fit machine-learned models, where communication and availability constraints add challenges [KMA+19]. A natural formulation here, assuming a supervised learning setting, is to assume that among \( m \) distinct clients, each client \( i \) has distribution \( P_i \), draws observations \( Z \sim P_i \), and wishes to fit a model—which we represent abstractly as a parameter vector \( \theta \in \Theta \)—to minimize a risk, or expected loss, \( L_i(\theta) := \mathbb{E}_{P_i}[\ell(\theta; Z)] \), where the loss \( \ell(\theta; z) \) measures the performance of \( \theta \) on example \( z \). Thus, at the most abstract level, the federated learning problem is to solve the multi-criterion problem

\[
\text{minimize } (L_1(\theta_1), \ldots, L_m(\theta_m)).
\] (1)

The formulation (1) highlights the centrality of personalization in federated learning [KMA+19]: we wish to find a parameter \( \theta_i \) for device \( i \) that does as well as possible on its population \( P_i \).

At this level, problem (1) is thus both trivial—one should simply minimize each risk \( L_i \) individually—and impossible, as no individual machine has enough data locally to effectively minimize \( L_i \). Consequently, methods in federated learning typically take various departures from the multicriterion objective (1) to provide more tractable problems. Many approaches build off of the empirical risk minimization principal [Vap92, Vap95, HTF09], where we seek a single parameter \( \theta \) that does well across all machines and data, minimizing the average (sometimes a weighted average) loss

\[
\frac{1}{m} \sum_{i=1}^{m} L_i(\theta)
\] (2)

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over $\theta \in \Theta$. This collapsed “zero personalization” approach has the advantage that data is (relatively) plentiful, and has led to a substantial literature. Much of this line of work focuses on developing efficient first- and higher-order methods that limit possibly expensive and unreliable communication between centralized servers and distributed devices [HM19, RCZ+21, MMR+17, KKM+20, MSS19, LSZ+20]. Given (i) the challenges of engineering such large-scale systems [MMR+17, KMA+19], (ii) the evident success of large-scale machine learning models even without personalization, and (iii) the plausibility of individual devices having reasonably similar distributions $P_i$, the zero personalization approach is natural (moreover, models may incorporate local contextual clues in predictions, though this appears mostly unexplored [KMA+19]). However, as distributions across individual devices are typically not completely identical, it is of interest to develop methods that more closely attempt the full problem (1); here we advocate beginning the design process of federated algorithms with problem (1).

Related Work. There is a growing line of research in Federated learning that addresses related questions of data and other types of device heterogeneity. The tried-and-true method of adapting to new data distributions is fine-tuning [HR18]. In Federated learning, this broadly corresponds to fine-tuning a global model (e.g., trained via FedAvg) on a user’s local data [WMK+19, YBS20, LHBS21]. In spite of its simplicity and practical efficacy, to our knowledge, little theoretical analysis has been done to model the fine-tuning procedure.

Another major direction is to design some type of surrogate, personalization-incentivizing objective. [SCST17], for example, build out of the long literature on multitask and transfer learning [Car97, PY09, e.g.] to formulate a multi-task learning objective, treating each machine as an individual task; this and other papers [FMO20, MMRS20, DTN20] show corresponding rates of convergence for optimization methods on these surrogates. These methods use the heuristic that personalized, local models should lie “close” to one another, and the authors provide empirical evidence for their improved performance. Yet it is not always clear what conditions are necessary (or sufficient) for these specialized personalization methods to outperform naive zero collaboration—fully local training on available data on each individual device—and zero personalization (averaged) methods. In a related vein, meta-learning approaches [FAL17, FMO20, JKRK19] seek a global model that can quickly “adapt” to local distributions $P_i$, typically by using a few gradient steps. This generally yields a quite non-convex objective, making it hard to give even heuristic generalization guarantees, and leads authors instead to emphasize worst-case convergence rate guarantees of iterative algorithms to (near) stationary points.

Other methods of personalization have also been proposed. In contrast to using a global model to help train the local model, [MMRS20, ZMM+20] use a mixture of global and local models to incorporate personalized features. Recently, [CZL+21] propose to evaluate federated algorithms via the formulation (1), and give minimax bounds to distinguish situations in which zero collaboration and zero personalization (averaged) methods (2) are respectively worst-case optimal.

Our approach. In this paper, we focus on the multi-criterion objective (1), proposing a stylized linear regression model to allow us to model and analyze various personalized Federated learning algorithms. We consider a high-dimensional asymptotic model, where clients solve statistically related linear regression problems, and each client $i \in [m]$ has a local dataset size $n_i$ smaller than (but comparable to) the dimension $d$ of the problem. This choice models the empirical fact that the data on a single client is typically small relative to model dimension (e.g., even training the last layer of a deep neural network). In particular we consider the high dimensional limit as the dimension $d$ and sample sizes $n_i$ grow together as $m \to \infty$, where $d \to \infty$ and $d/n_i \to \gamma_i > 1$ for each $i \in [m]$. Our theory suggests that Fine-tuned Federated Averaging (FTFA), i.e., Federated Averaging followed by local training, and the ridge regularized variant Ridge-tuned Federated Averaging (RTFA) are competitive with more sophisticated meta-learning (MAML-FL [FMO20]) and proximal-regularized approaches (pFedMe [DTN20]). We note that the idea of using fine-tuning is not new; we define and use the terms FTFA and RTFA largely for referential clarity. Instead our focus is to explicitly compare the performance of these methods against one another. We give concrete conditions for when these algorithms outperform naive zero-personalization and zero-collaboration algorithms, and we show that simple Federated Averaging [MMR+17], followed by ridge-regularized local training (RTFA) should obtain lower risk than the meta-learning approach and match the risk of the prox-regularized approach.
Yet given the stylized model, this would all be meaningless if it gave no predictions for real-world data; therefore, we put our (theoretical) analyses to the test: empirical results on several datasets corroborate the theoretical predictions. In all, our results suggests that FTFA and RTFA, which require training only a single global model followed by completely local computation—and therefore are no more complex than FedAvg itself—may be preferable to more complex and computationally expensive options, such as MAML-FL and pFedMe. More broadly, our results support our thesis that fine-tuning is a tough-to-beat benchmark in federated learning.

2 Stylized Linear Model

We now present the Bayesian linear model we use in Sections 3 and 4. Let there be $m$ clients, where each client $i \in [m]$ uses an overparameterized linear regression problem to recover an unknown parameter $\theta_i^* \in \mathbb{R}^d$. Client $i$ has $n_i$ i.i.d. observations $(x_{i,k}, y_{i,k}) \in \mathbb{R}^d \times \mathbb{R}$,

$$y_{i,k} = x_{i,k}^T \theta_i^* + \xi_{i,k}, \quad x_{i,k} \overset{iid}{\sim} P_x^i \quad \text{and} \quad \xi_{i,k} \overset{iid}{\sim} P_{\xi_i}^i.$$  

We make the routine assumption that the features are centered, with finite variance, i.e., $\mathbb{E}[x_{i,k}] = 0$ and $\text{Cov}(x_{i,k}) = \Sigma_i$. We also assume that the noise is centered with finite variance, i.e., $\mathbb{E}[\xi_{i,k}] = 0$ and $\text{Var}(\xi_{i,k}) = \sigma_i^2$. For convenience, we let $X_i \in \mathbb{R}^{n_i \times d}$ and $y_i \in \mathbb{R}^{n_i}$ denote client $i$’s data, and $X := [X_1^T, \ldots, X_m^T]^T$. We also let $N := \sum_{j=1}^{m} n_j$.

A prior $P_0^i$ on the parameter $\theta_i^*$ relates tasks on each client, where conditional on $\theta_i^*$, $\theta_i^*$ is supported on $r_i \mathbb{S}^{d-1} + \theta_0^*$—the sphere of radius $r_i$ (bounded by a constant for all $i \in [m]$) centered at $\theta_0^*$—with $\mathbb{E}[\theta_i^*] = \theta_0^*$. The variation between clients is captured by differences in $r_i$ (label shift) and $\Sigma_i$ (covariate shift), while the similarity is captured by the shared center $\theta_0^*$. Intuitively, data from client $j$ is useful to client $i$ as it provides information on the possible location of $\theta_0^*$. Lastly, we assume that the distributions of $x$, $\theta^*$, and $\xi$ are independent of each other and across clients.

Every client $i$ seeks to minimize its local population loss—the squared prediction error of a new example, such as MAML-FL and pFedMe. More broadly, our results support our thesis that fine-tuning is a tough-to-beat benchmark in federated learning.

For analysis purposes, we will often consider the equivalent bias-variance decomposition:

$$L_i(\hat{\theta}_i|X) = \mathbb{E}[\|\hat{\theta}_i - \theta_i^*\|^2 | X] + \text{tr}(\text{Cov}(\hat{\theta}_i|X)).$$

$$L_i(\hat{\theta}_i|X) := \mathbb{E}[\|\hat{\theta}_i - \theta_i^*\|^2 | X] = \mathbb{E}[\|\hat{\theta}_i - \theta_i^*\|^2 | X],$$

where the expectation is taken over $(x_{i,0}, \theta_i^*, \xi_i) \sim P_x^i \times P_{\theta_i}^i \times P_{\xi_i}^i$. It is essential here that we focus on per client performance: the ultimate goal is to improve performance on any given client, as per eq. (1). For analysis purposes, we will often consider the equivalent bias-variance decomposition:

$$L_i(\hat{\theta}_i|X) = \mathbb{E}[\|\hat{\theta}_i|X - \theta_i^*\|^2 | X] + \text{tr}(\text{Cov}(\hat{\theta}_i|X)).$$

Our main asymptotic assumption, which captures the high-dimensional and many-device nature central to modern federated learning problems, follows:

**Assumption A1.** As $m \to \infty$, both $d = d(m) \to \infty$ and $n_j = n_j(m) \to \infty$ for $j \in [m]$, and $\lim_m \frac{d}{n_j} = \gamma_j$. Moreover, $1 < \gamma_{\text{min}} \leq \lim_m \inf_{j \in [m]} \frac{d}{n_j} \leq \lim_m \sup_{j \in [m]} \frac{d}{n_j} \leq \gamma_{\text{max}} < \infty$.

Importantly, individual devices are overparameterized: we always have $\gamma_j > 1$, as is common, when the dimension of models is large relative to local sample sizes, but may be smaller than the (full) sample. Intuitively, $\gamma_j$ captures the degree of overparameterization of the network for user $j$. We also need an assumption controlling the eigenspectrum of our data (same as Assumption 1 of [HMRT19]). We first define the empirical distribution of a matrix $\Sigma$ and then outline the assumption after.

**Definition 2.1.** Let $\mu(\cdot; \Sigma) : \mathbb{R}^+ \to \mathbb{R}^+$ denote the empirical distribution of the eigenvalues of $\Sigma$:

$$\mu(s; \Sigma) := \frac{1}{d} \sum_{j=1}^{d} 1 \{s \geq s_j\},$$

where $s_1 \geq s_2 \geq \cdots \geq s_d$ are the eigenvalues of $\Sigma$.
Assumption A2. For each user $i$, each data point $x \sim P_i$ is of the form $x = \Sigma_i z$ and we have

(a) The vector $z = (z_1, z_2, \ldots, z_d)$ has independent entries with $E[z_i] = 0$, $E[z_i^2] = 1$, and $E[|z_i|^{2q}] \leq \kappa_q < \infty$ for some $q > 2$.

(b) $s_1 = \|\Sigma_i\|_{op} \leq M$, $s_d = \lambda_{\min}(\Sigma_i) \geq 1/M$, and $\int s^{-1}d\mu(s; \hat{\Sigma}_i) < M$.

(c) $\mu(s; \hat{\Sigma}_i)$ converges weakly to $\nu_i$ ($\mu(s; \hat{\Sigma}_i) \Rightarrow \nu_i$).

3 Using FedAvg to Warm Start

In this section, we describe and analyze algorithms (FTFA and RTFA) which use the FedAvg solution as a warm start to find personalized models. We compare the test loss of these algorithms with naive, zero personalization and zero collaboration approaches. Among other things, we show that RTFA is able to outperform FTFA and all of the naive algorithms.

3.1 Fine-tuned FedAvg (FTFA)

FTFA attempts to minimize the multi-criterion loss (1) using a two-step procedure described in Algorithm 1 (a detailed pseudocode can be found in Section 7). Let $S_i$ denote the set of all samples of client $i$. The idea is to do FedAvg in the first step on the empirical risk $\hat{L}_i(\theta) := \frac{1}{n_i} \sum_{z \in S_i} \ell(\theta; z)$ to leverage the similarity between clients and to use the FedAvg solution as a warm-start for local training in the second step. Intuitively, FTFA interpolates between zero collaboration and zero personalization algorithms. We note that it is reasonable to assume that each client has the ability to locally train their model to convergence, since each client already has to do multiple passes over its local data in FedAvg. The fact that the second step is fully parallel, only serves to make FTFA more feasible.

Algorithm 1: FTFA & RTFA (details in appendix)

1. The server coordinates using FedAvg to find a global model using data from all clients. FedAvg solves the following problem:

   $$\hat{\theta}_0^{FA} = \arg\min_{\theta} \sum_{j=1}^{m} p_j \hat{L}_j(\theta),$$

   where $\hat{\theta}_0^{FA}$ denotes the global model and $p_j \in (0, 1)$ are weights such that $\sum_{j=1}^{m} p_j = 1$. The server then broadcasts $\hat{\theta}_0^{FA}$ to all clients.

2.a. FTFA: Each client $i$ learns a model, $\hat{\theta}_i^{FA}$, by optimizing its own empirical risk, $\hat{L}_i(\cdot)$, using the Stochastic Gradient Method (SGM) with $\hat{\theta}_0^{FA}$ as the initial point.

2.b. RTFA: Each client $i$ learns a model, $\hat{\theta}_i^{R}(\lambda)$, by minimizing the regularized empirical risk:

   $$\hat{\theta}_i^{R}(\lambda) = \arg\min_{\theta} \hat{L}_i(\theta) + \frac{\lambda}{2} \|\theta - \hat{\theta}_0^{FA}\|_2^2.$$

For the linear model described in Section 2, FTFA first uses FedAvg to minimize the average weighted loss $\sum_{j=1}^{m} p_j \frac{1}{2n_j} \|X_j^i \theta - y_j\|_2^2$ over all clients; we note that FedAvg does in fact converge to the minimizer for general convex losses [LHY+20, WPS20]. Then, each client $i \in [m]$ runs SGM initialized at the FedAvg solution to minimize $\frac{1}{2n_i} \|X_i^i \theta - y_i\|_2^2$. Since we are solving an overparameterized linear regression problem in the second step, this, with appropriate step size, corresponds to solving a minimum $\ell_2$ norm regression problem (see Theorem 1 [GLSS18]). To summarize, on our linear regression problem,
for appropriately chosen step size and as the steps taken goes to infinity, FTFA is equivalent to the following two step procedure:

\[
\hat{\theta}_0^{FA} = \arg \min_{\theta} \sum_{j=1}^{m} p_j \frac{1}{2n_j} \|X_j \theta - y_j\|_2^2
\]
\[
\hat{\theta}_i^{FA} = \arg \min_{\theta} \|\hat{\theta}_0^{FA} - \theta\|_2 \quad s.t. \quad X_i \theta = y_i,
\]

where FTFA outputs the model \( \hat{\theta}_i^{FA} \) for client \( i \). Before giving our result, we make an additional assumption regarding the asymptotics of the number of clients and the dimension of the data.

**Assumption A3.** \((\log d)^c \sum_{j=1}^{m} p_j^{q/2+1} n_j \rightarrow 0 \) as \( m, d, n_j \rightarrow \infty \), where \( q > 2 \), and \( c \) is a constant.

In this assumption, \( p_j \) is the weight associated with the loss of \( j^{th} \) client when finding the global model using federated averaging. To ground this assumption, consider two particular cases of interest: (i) \( p_j = 1/m \), when every client is weighted equally, and (ii) \( p_j = n_j/N \), when each data point is weighted equally. When \( p_j = 1/m \), we have \((\log d)^c \sum_{j=1}^{m} p_j^{q/2+1} n_j = \frac{N}{m^{q/2+1}} \). When \( p_j = n_j/N \), using Assumption A1 we have \((\log d)^c \sum_{j=1}^{m} p_j^{q/2+1} n_j = (\log d)^c \sum_{j=1}^{m} n_j^{q/2} \), \( \leq \left( \frac{\max_{j \in [m]} n_j}{\min_{j \in [m]} n_j} \right)^{q/2} < \frac{N}{m^{q/2+1}} \).

Thus, in both the cases, ignoring the polylog factors, if we have \( \frac{N}{m^{q/2+1}} \rightarrow 0 \), i.e., \( mn^2 \) grows faster than the average client sample size, \( N/m \), then Assumption A3 holds. With these assumptions defined, we are able to compute the asymptotic test loss of FTFA.

**Theorem 1.** Consider the observation model described in section 2 and the estimator \( \hat{\theta}_i^{FA} \) defined by eq. (7). Let Assumption A1 hold, and let Assumption A2 and A3 hold with \( c = 2 \) and the same \( q > 2 \). Additionally, assume that for any given \( m \), for all \( j \in [m] \), \( \left\| E(\hat{\Sigma}_j^2) \right\|_{op} \leq \tau_2 \), where \( \tau_2 \) is an absolute constant.

For client \( i \), when \( d, m, n_i \rightarrow \infty \) according to Assumption A1, the asymptotic prediction bias and variance of FTFA are given by

\[
\lim_{m \rightarrow \infty} B_i(\hat{\theta}_i^{FA}|X) \overset{p}{=} \lim_{m \rightarrow \infty} \|\Pi_i[\hat{\theta}_0^i - \theta^*_i]\|_{\hat{\Sigma}_i}^2 \quad \lim_{m \rightarrow \infty} V_i(\hat{\theta}_i^{FA}|X) \overset{p}{=} \lim_{m \rightarrow \infty} \frac{\sigma_i^2}{n_i} \text{tr}(\hat{\Sigma}_i^2 \Sigma_i),
\]

where \( \Pi_i := I - \hat{\Sigma}_i^2 \hat{\Sigma}_i \) and \( \|z\|_{\hat{\Sigma}_i}^2 := z^T \Sigma_i z. \) The exact expressions of these limits for general choice \( \Sigma \) in the implicit form can be found in the appendix. For the special case when \( \Sigma_i = I \), the closed form limits are given by

\[
B_i(\hat{\theta}_i^{FA}|X) \overset{p}{\rightarrow} r_i^2 \left( 1 - \frac{1}{\gamma_i} \right) \quad V_i(\hat{\theta}_i^{FA}|X) \overset{p}{\rightarrow} \frac{\sigma_i^2}{\gamma_i - 1},
\]

where \( \overset{p}{\rightarrow} \) denotes convergence in probability.

### 3.2 Ridge-tuned FedAvg (RTFA)

Minimum-norm results provide insight into the behavior of popular algorithms including SGM and mirror descent. Having said that, we can also analyze ridge penalized versions of FTFA. In this algorithm, the server finds the same global model as FTFA, but then each client uses a regularized objective to find a local personalized model as described in 2b of Algorithm 1. More concretely, in the linear regression setup, for appropriately chosen step size and as the number of steps taken goes to infinity, this corresponds to the two step procedure with the first step given by eq. (6) and the following second step:

\[
\hat{\theta}_i^R(\lambda) = \arg \min_{\theta} \frac{1}{2n_i} \|X_i \theta - y_i\|_2^2 + \frac{\lambda}{2} \|\hat{\theta}_0^{FA} - \theta\|_2^2,
\]

where RTFA outputs the model \( \hat{\theta}_i^R(\lambda) \) for client \( i \). The first step can again be solved using FedAvg, while the second step can be solved using SGM with a ridge penalty. Under the same assumptions as Theorem 1, we can again calculate the asymptotic test loss.
Theorem 2. Consider the observation model described in section 2 and the estimator $\hat{\theta}_i^R(\lambda)$ defined by eq. (8). Let Assumption A1 hold, and let Assumption A2 and A3 hold with $c = 2$ and the same $q > 2$. Additionally, assume that $\|E[\hat{\Sigma}]\|_{op} \leq \tau_2$, where $\tau_2$ is an absolute constant. For client $i$, when $d, m, n_i \to \infty$ according to Assumption A1, the asymptotic prediction bias and variance of RTFA are given by

$$
\lim_{m \to \infty} B_i(\hat{\theta}_i^R|X) \overset{p}{=} \lim_{m \to \infty} \lambda^2 \|\hat{\Sigma}_i^{-1}(\hat{\theta}_i^R - \theta)_i\|^2_{\Sigma_i},
$$

$$
\lim_{m \to \infty} V_i(\hat{\theta}_i^R|X) \overset{p}{=} \lim_{m \to \infty} \frac{\sigma_i^2}{n_i} \text{tr}(\hat{\Sigma}_i^{-1}(\lambda I + \hat{\Sigma}_i)^{-2}),
$$

The exact expressions of these limits for general choice $\Sigma$ in the implicit form can be found in the appendix. For the special case when $\Sigma_i = I$, the closed form limits are given by

$$
B_i(\hat{\theta}_i^R|X) \overset{p}{=} r_i^2 \lambda^2 m'_i(-\lambda),
$$

$$
V_i(\hat{\theta}_i^R|X) \overset{p}{=} \sigma_i^2 \gamma_i m_i(-\lambda) - \lambda m'_i(-\lambda),
$$

where $m_i(z)$ is the Stieltjes transform of the spectral measure of the sample covariance matrix [DW18, BS10]. When $\text{Cov}(x_{ij, k}) = I$, $m_i(z) = \left(1 - \gamma_i - z - \sqrt{(1 - \gamma_i - z)^2 - 4\gamma_i z}\right)/2\gamma_i z$. For each client $i \in [m]$, when $\lambda$ is set to be the minimizing value $\lambda^*_i = \sigma_i^2 \gamma_i/r_i^2$, the expression simplifies to $L_i(\hat{\theta}_i^R(\lambda^*_i)|X) \to \sigma_i^2 \gamma_i m_i(-\lambda^*_i)$.

We remark that with the optimal choice of hyperparameter $\lambda$, RTFA has lower test loss than FTFA. This should not be surprising, as RTFA is a generalization of FTFA. Indeed, recall that ridge regression with $\lambda \to 0$ recovers minimum $\ell_2$-norm regression.

3.3 Comparison to Naive Estimators

Three natural baselines to compare FTFA and RTFA to are the zero personalization estimator $\hat{\theta}_0^{FA}$, the zero collaboration estimator

$$
\hat{\theta}_i^N = \arg\min_{\theta} \|\theta\|_2 \quad \text{s.t.} \quad X_i \theta = y_i,
$$

(9)

and the ridge-penalized, zero-collaboration estimator

$$
\hat{\theta}_i^N(\lambda) = \arg\min_{\theta} \frac{1}{2n_i} \|X_i \theta - y_i\|_2^2 + \frac{\lambda}{2} \|\theta\|_2^2.
$$

(10)

In fact, as with FTFA and RTFA, we can compute the asymptotic test loss explicitly. We give the test loss expressions only for the identity covariance case for ease of exposition, similar results and comparisons hold for general covariance matrices.

Corollary 3.1. Consider the observation model described in section 2 and the estimator $\hat{\theta}_0^{FA}$ defined by eq. (6) with $\Sigma_i = I$ for all users. For client $i$, as $n_i, d, m \to \infty$, under Assumption A1, we have

$$
B_i(\hat{\theta}_0^{FA}|X) \overset{p}{=} r_i^2 \quad V_i(\hat{\theta}_0^{FA}|X) \overset{p}{=} 0.
$$

Consider the estimator $\hat{\theta}_i^N$ defined by eq. (9). In addition to the above conditions, suppose that $\theta^*_i$ is drawn such that $\|\theta^*_i\|_2 = \rho_i \in \mathbb{R}$ is a constant with respect to $m$; we note that $\rho_i \geq r_i$ by construction. Further suppose that for some $q > 2$, for all $j \in [m]$ and $k, l \in [n_j] \times [d]$, $E[(x_{j, k})_i^{2q}] \leq \kappa_q$, where $\kappa_q$ is an absolute constant. Then, we have

$$
B_i(\hat{\theta}_i^N|X) \overset{p}{=} \rho_i^2 \left(1 - \frac{1}{\gamma_i}\right) \quad V_i(\hat{\theta}_i^N|X) \overset{p}{=} \frac{\sigma_i^2}{\gamma_i - 1}.
$$
Consider the estimator $\hat{\theta}_i^N(\lambda)$ defined by eq. (10). Under all the conditions specified above, we have

$$B_i(\hat{\theta}_i^N(\lambda)|X) \xrightarrow{p} \rho_i^2 \lambda^2 m_i'(-\lambda) \quad V_i(\hat{\theta}_i^N(\lambda)|X) \xrightarrow{p} \sigma_i^2 \gamma_i m_i(-\lambda) - \lambda m_i'(-\lambda)).$$

Moreover, if $\lambda$ is set to be the loss-minimizer $\lambda_i^* = \sigma_i^2 \gamma_i / \rho_i^2$, then $L_i(\hat{\theta}_i^N(\lambda_i^*); \theta^*)|X \xrightarrow{p} \sigma_i^2 \gamma_i m_i(-\lambda_i^*)$.

Even though FTFA has uniformly larger test loss than optimally-tuned RTFA, given the ubiquity of unregularized SGM, we compare FTFA to the naive estimators. Its straightforward to see that FTFA outperforms FedAvg, $\hat{\theta}_0^{FA}$, if and only if $\sigma_i^2 < r_i^2 \gamma_i - 1 / \gamma_i$. This intuitively makes sense: if the noise is too large, then local tuning is fitting mostly to noise. Furthermore, FTFA always outperforms the ridgeless zero-collaboration estimator, $\hat{\theta}_i^N$, because $\rho_i \geq r_i$ is always true.

The weaknesses of FTFA are addressed by RTFA. Unlike FTFA, in the large noise regime, RTFA is able to choose a large value of $\lambda$ to mitigate the effects of the noise. Thus, formally, we are able to show that RTFA with the optimal hyperparameter always outperforms the zero-personalization estimator, $\hat{\theta}_0^{FA}$ (see appendix). Furthermore, since $\rho_i \geq r_i$, its straightforward to see that RTFA outperforms ridgeless zero-collaboration estimator $\hat{\theta}_i^N$, and the ridge-regularized zero-collaboration estimator $\hat{\theta}_i^N(\lambda^*)$ as well.

4 Meta learning and Proximal Regularized Algorithms

Recent work [FMO20, DTN20] have proposed other ways of personalizing FL models, among them are meta-learning and proximal-regularized approaches. Convergence rates have been shown for both of these methods, but to our knowledge, there has not been any theory developed for comparing the next-sample-prediction performance of these methods. In this section, we show that these more sophisticated approaches perform no better, in our asymptotic framework, than the FTFA and RTFA algorithms we outlined in Section 3.

4.1 Model-Agnostic Meta-Learning

Model-Agnostic Meta-Learning (MAML) [FAL17] was developed as a method which learns models that could easily adapt to related tasks. In contrast to the $\min_{\theta} \sum_{j=1}^{m} \hat{L}_i(\theta)$-type objective used in FTFA, MAML optimizes an augmented objective designed to enhance personalization performance. Recently, [FMO20] adapt MAML to the federated setting; we will call this federated version MAML-FL. We describe their two step procedure in Algorithm 2 (a detailed pseudocode can be found in Section 7). Algorithm 2 has two variants [FMO20]; in one, the Hessian term is ignored, and in the other, the Hessian is approximated using finite differences. [FMO20] showed that these these algorithms converge to a stationary point of eq. (11) (with $p_j = 1/m$) for general non-convex smooth functions.

**Algorithm 2**: MAML-FL (details in appendix)

1. Server coordinates with the clients to solve

$$\hat{\theta}_0^M(\alpha) = \arg \min_{\theta} \sum_{j=1}^{m} p_j \hat{L}_j(\theta - \alpha \nabla \hat{L}_j(\theta),$$

where $\hat{\theta}_0^M(\alpha)$ denotes the global model, $p_j \in (0, 1)$ are weights such that $\sum_{j=1}^{m} p_j = 1$ and $\alpha$ denotes stepsize. Server then broadcasts $\hat{\theta}_0^M(0)$ to clients.

2. Each client $i$ then learns a model, $\hat{\theta}_i^M(\alpha)$, by optimizing its empirical risk, $\hat{L}_i(\cdot)$, using SGM with $\hat{\theta}_0^M(\alpha)$ as the initial point.

7
In our linear model, for appropriately chosen hyperparameters and as the number of steps taken goes to infinity, this personalization method corresponds to the following two step procedure:

\[
\hat{\theta}_0^M(\alpha) = \arg\min_{\theta} \sum_{j=1}^{m} \frac{p_j}{2n_j} \|X_j \left[\theta - \frac{\alpha}{n_j} X_j^T (X_j \theta - y_j)\right] - y_j\|_2^2 \tag{12}
\]

\[
\hat{\theta}_i^M(\alpha) = \arg\min_{\theta} \|\hat{\theta}_0^M(\alpha) - \theta\|_2 \quad \text{s.t.} \quad X_i \theta = y_i \tag{13}
\]

Here, we assume that the personalization scheme has each machine running SGM initialized at \( \hat{\theta}_0^M(\alpha) \) to convergence. We can again analyze the test loss of this personalization scheme in our asymptotic framework.

**Theorem 3.** Consider the observation model described in section 2 and the estimator \( \hat{\theta}_i^M(\alpha) \) defined by eq. (13). Let Assumption A1 hold, and let Assumption A2 hold with \( \Pi \) in the implicit form can be found in the appendix. For the special case when \( \text{limits are given by } \sum \text{ Theorem 3. } \) Consider the observation model described in section 2 and the estimator \( \hat{\theta}_i^M(\alpha) \) defined by eq. (13). Let Assumption A1 hold, and let Assumption A2 hold with \( \Pi \) in the implicit form can be found in the appendix. For the special case when \( \text{limits are given by } \sum \text{ Theorem 3. } \)

\[
\lim_{m \to \infty} B_i(\hat{\theta}_i^M(\alpha)|X) \xrightarrow{P} \lim_{m \to \infty} \|\Pi_i[\theta_0 - \theta_i^*]\|_i^2, \quad \lim_{m \to \infty} V_i(\hat{\theta}_i^M(\alpha)|X) \xrightarrow{P} \lim_{m \to \infty} \frac{\sigma_i^2}{n_i} \text{tr}(\tilde{\Sigma}_i \Sigma_i),
\]

where \( \Pi_i := I - \tilde{\Sigma}_i \Sigma_i \) and \( \|z\|_{\tilde{\Sigma}_i}^2 := z^T \Sigma_i z \). The exact expressions of these limits for general choice \( \Sigma \) in the implicit form can be found in the appendix. For the special case when \( \Sigma_i = I \), the closed form limits are given by

\[
B_i(\hat{\theta}_i^M(\alpha)|X) \xrightarrow{P} r_i^2 \left(1 - \frac{1}{\gamma_i}\right), \quad V_i(\hat{\theta}_i^M(\alpha)|X) \xrightarrow{P} \frac{\sigma_i^2}{\gamma_i - 1}.
\]

In short, the asymptotic test loss for MAML-FL match that of FTFA (see Theorem 1). We note that even though MAML-FL is convex in our linear regression model in eq. (12), in general, the MAML-FL objective in eq. (11) is often non-convex even when \( \tilde{L}_j \) is convex. Additionally, MAML-FL is an expensive second-order method. Even the hessian free and first order versions of MAML-FL still use four times and two times the gradient computations respectively compared to FTFA; see appendix for discussion. Thus, our theory suggests that the added complexity and potential non-convexity of MAML-FL has no payoff relative to FTFA.

**Remark** We note that the algorithm given in [FMO20] suggests performing one SGM step for personalization. For our analysis, we instead decide to locally train to convergence (step 2 of Algorithm 2) for a couple of reasons. First, the experiments of [JKRK19] and our experiments in Figures 5 and 6 empirically show more steps of personalization is better. Furthermore, as mentioned earlier, performing personalization SGM steps asynchronously is not more expensive than running the first step of Algorithm 2. Finally, locally training to convergence presents a more fair comparison between the algorithms we consider.

### 4.2 Proximal-Regularized Approach

Instead of a sequential, fine-tuning approach, an alternative paradigm of personalization involves jointly optimizing local parameters. Recently, [DTN20] proposed the pFedMe algorithm (details in appendix) to solve the following coupled optimization problem to find personalized models for each client:

\[
\hat{\theta}_0^P(\lambda), \hat{\theta}_1^P(\lambda), \ldots, \hat{\theta}_m^P(\lambda) = \arg\min_{\theta_0, \theta_1, \ldots, \theta_m} \sum_{j=1}^{m} p_j \left(\tilde{L}_i(\theta_j) + \frac{\lambda}{2} \|\theta_j - \theta_0\|_2^2\right).
\]
The proximal penalty encourages the local models $\theta_i$ to be close to one another. In our linear model, for appropriately chosen hyperparameters and as the number of steps taken goes to infinity, the proposed optimization problem simplifies to

$$\hat{\theta}^p_0(\lambda), \hat{\theta}^p_1(\lambda), \ldots, \hat{\theta}^p_m(\lambda) = \arg\min_{\theta_0, \theta_1, \ldots, \theta_m} \sum_{j=1}^m p_j \left(\frac{1}{2m_j} \|X_j \theta_j - y_j\|^2_2 + \frac{\lambda}{2} \|\theta_j - \theta_0\|^2_2\right),$$

(14)

where $\hat{\theta}^p_0(\lambda)$ denotes the global model and $\hat{\theta}^p_i(\lambda)$ denote the local models. We can again use our asymptotic framework to analyze the test loss of this scheme. Note that for this result, we use an additional condition on $\sup_{j \in [m]} P(\lambda_{\max}(\hat{\Sigma}_j) > R)$ that gives us uniform control over the eigenvalues of all the users.

**Theorem 4.** Consider the observation model described in section 2 and the estimator $\hat{\theta}^p_0(\lambda)$ defined by eq. (14). Let Assumption A1 hold, and let Assumption A2 and A3 hold with $c = 2$ and the same $q > 2$. Additionally, assume that $E \left[ \left\| \hat{\Sigma}^2_j \right\|_{\text{op}} \right] \leq \tau_3$, where $\tau_3$ is an absolute constant. Further suppose that there exists $R \geq M$ such that $\lim_{m \to \infty} \sup_{j \in [m]} P(\lambda_{\max}(\hat{\Sigma}_j) > R) \leq \frac{1}{16M^2 \tau_3}$. For client $i$, when $d, m, n_i \to \infty$ according to Assumption A1, the asymptotic prediction bias and variance of pFedMe are given by

$$\lim_{m \to \infty} B_i(\hat{\theta}^p_0(\lambda)|X) \overset{P}{=} \lim_{m \to \infty} \lambda^2 \left\| (\hat{\Sigma}_i + \lambda I)^{-1} (\theta^*_0 - \theta^*_i) \right\|^2_{\Sigma_i},$$

$$\lim_{m \to \infty} V_i(\hat{\theta}^p_0(\lambda)|X) \overset{P}{=} \lim_{m \to \infty} \frac{\sigma^2_i}{n_i} \text{tr}(\Sigma_i \hat{\Sigma}_i (\lambda I + \hat{\Sigma}_i)^{-2}).$$

The exact expressions of these limits for general choice $\Sigma$ in the implicit form can be found in the appendix. For the special case when $\Sigma_i = I$, the closed form limits are given by

$$B_i(\hat{\theta}^p_0(\lambda)|X) \overset{P}{=} r^2_i \lambda^2 m_i(-\lambda) \quad V_i(\hat{\theta}^p_0(\lambda)|X) \overset{P}{=} \sigma^2_i \gamma(m_i(-\lambda) - \lambda m_i(-\lambda),$$

where $m(z)$, when $\text{Cov}(x_{j,k}) = I$, is given in Theorem 2. For each client $i \in [m]$, when $\lambda$ is set to be the minimizing value $\lambda^*_i = \sigma^2_i \gamma/r^2_i$, the expression simplifies to $L_i(\hat{\theta}^p_0(\lambda^*_i)|X) \to \sigma^2_i \gamma m_i(-\lambda^*_i)$.

The asymptotic test loss of the proximal-regularized approach matches that of RTFA (Theorem 2). We note that [DTN20]'s algorithm to optimize eq. (14) is sensitive to hyperparameter choice, meaning significant hyperparameter tuning may be needed for good performance. Moreover, a local update step in pFedMe requires approximately solving a proximal-regularized optimization problem, as opposed to taking a single stochastic gradient step. This can make pFedMe much more computationally expensive depending on the properties of $\hat{L}$. Thus, again, our theory suggests a simpler, more computationally efficient fine-tuning approach suffices in capturing the advantages of a proximal-regularized scheme.

## 5 Experiments

**Setup** We perform experiments on federated versions of the Shakespeare [MMR +17], CIFAR-100 [KH09], EMNIST [CATvS17], and Stack Overflow [MRR +19] datasets; dataset statistics and details of how the data is divided into users are given in Section 8. For each dataset, we compare the performance of the following algorithms: Zero Communication (Local Training), Zero Personalization (FedAvg), FTFA, RTFA, MAML-FL, and pFedMe [DTN20]. For each classification task, we use each FL algorithm to train the last layer of a pre-trained neural network. We run each algorithm for 400 communication rounds, and we compute the test accuracy, i.e., the fraction of total test data points across machines which were classified correctly, every 50 communication rounds. FTFA, RTFA, and MAML-FL each do 10 epochs of local training for each client before the evaluation of test accuracy. For each client, pFedMe uses the respective trained local models to compute test accuracy. We first hyperparameter tune each method using training and validation splits; details on this are given in Section 8. Then, we track the test accuracy of each tuned method over 11 trials using two different kinds of randomness:
Different seeds: We run each hyperparameter-tuned method on 11 different seeds. This captures how different initializations and batching affect accuracy.

2. Different training-validation splits: We generate 11 different training and validation splits (the test split stays the same) and run each hyperparameter-tuned method on each split. This captures how variations in user data affect test accuracy.

Results In Figures 1 to 4, we plot the test accuracy against communication rounds. The performance of MAML-FL is similar to that of FTFA and RTFA. In fact, on the Stack Overflow and EMNIST datasets, where the total dataset size is much bigger than the other datasets, the accuracies of MAML-FL, FTFA and RTFA are nearly identical. This is consistent with our theoretical claims. The performances of the naive, zero communication and zero personalization algorithms are worse than that of FTFA, RTFA and MAML-FL in all figures. This is also consistent with our theoretical claims. The performance of pFedMe in Figures 1 to 3 is worse than that of FTFA, RTFA and MAML-FL.

In Figures 5 and 6, we plot the test accuracy of FTFA and MAML-FL and vary the number of personalization steps each algorithm takes. We notice that in both plots, the global model performs the worst, and performance improves monotonically as we increase the number of personalization steps. Since personalization steps, as mentioned earlier, are not that expensive to take relative to the centralized training procedure, this suggests that clients should locally train to convergence.

Limitations We acknowledge that our experiments should ideally performed on full sized networks. However, given that resource constraints preclude our training models on Google-scale, to maintain fidelity to true deployments of federated learning, we focus on models that can achieve reasonable test accuracy; trends observed in low test accuracy setting (e.g., by training smaller models without pretraining) may not translate to the higher test accuracy setting. Instead of training a larger model from scratch, we prioritized rigorous hyperparameter tuning for all algorithms to fairly compare “best possible” versions of each algorithm (a 25x to 75x increase in the number of runs depending on the
Moreover, we wanted confidence intervals for more robust experiments (a 20x increase in the number of runs needed). The energy expenditures alone for training full models over this many iterations would be too substantial, in our view, to justify.
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6 Proofs

6.1 Additional Notation

To simplify notation, we define some aggregated parameters, \( X_i := [x_{i,1}, \ldots, x_{i,n_i}]^T \in \mathbb{R}^{n_i \times d}, y_i = [y_{i,1}, \ldots, y_{i,n_i}]^T \in \mathbb{R}^{n_i}, X := [X_1^T, \ldots, X_m^T]^T \in \mathbb{R}^{N \times d}, \) and \( y := [y_1^T, \ldots, y_m^T]^T \in \mathbb{R}^N. \) Additionally, we define \( \hat{\Sigma} := X_i^T X_i/n_i \in \mathbb{R}^{d \times d}. \) We use the notation \( a \lesssim b \) to denote \( a \leq Kb \) for some absolute constant \( K. \)

6.2 Useful Lemmas

Lemma 6.1. Let \( x_j \) be vectors in \( \mathbb{R}^d \) and let \( \zeta_j \) be Rademacher \((\pm 1)\) random variables. Then, we have

\[
\mathbb{E} \left[ \left\| \sum_{j=1}^m \zeta_j x_j \right\|_2^p \right] \leq \sqrt{p-1} \left( \sum_{j=1}^m \|x_j\|_2^2 \right)^{\frac{p}{2}},
\]

where the expectation is over the Rademacher random variables.

Proof. Using Theorem 1.3.1 of [dlPnG99], we have

\[
\mathbb{E} \left[ \left\| \sum_{j=1}^m \zeta_j x_j \right\|_2^p \right] \leq \sqrt{p-1} \mathbb{E} \left[ \left\| \sum_{j=1}^m \zeta_j x_j \right\|_2^{2p} \right]^{\frac{1}{2}} = \sqrt{p-1} \mathbb{E} \left[ \sum_{i,j=1}^m \langle \zeta_i \zeta_j^T x_i x_j \rangle \right]^{\frac{1}{2}} = \sqrt{p-1} \left( \sum_{j=1}^m \|x_j\|_2^2 \right)^{\frac{p}{2}}.
\]

\[\square\]

Lemma 6.2. For all clients \( j \in [m], \) let the data \( x_{j,k} \in \mathbb{R}^d \) for \( k \in [n_i] \) be such that \( x_{j,k} = \Sigma_j^{1/2} z_{j,k} \) for some \( \Sigma_j, z_{j,k}, \) and \( p > 2 \) that satisfy Assumption A2. Let \( (x_{j,k})_l \in \mathbb{R} \) denote the \( l \in [d] \) entry of the vector \( x_{j,k} \in \mathbb{R}^d. \) Define \( \hat{\Sigma}_j = \frac{1}{n_j} \sum_{k[n_j]} x_{j,k} x_{j,k}^T. \) Then, we have

\[
\mathbb{E} \left[ \left\| \hat{\Sigma}_j \right\|_{\text{op}}^p \right] \leq K(e \log d)^p n_j,
\]

where the inequality holds up to constant factors for sufficiently large \( m. \)

Proof. We first show a helpful fact that \( \mathbb{E}[\|z_{j,k}\|_2^{2p}] \leq \kappa_p < \infty \implies \mathbb{E}[\|x_{j,k}\|_2^{2p}]^{1/(2p)} \leq \sqrt{d}. \) For any \( j \in [m], \) we have by Jensen’s inequality

\[
\mathbb{E}[\|x_{j,k}\|_2^{2p}] \leq M^{2p} \mathbb{E}[\|z_{j,k}\|_2^{2p}] = M^{2p} \mathbb{E} \left[ \left( \frac{1}{d} \sum_{l=1}^d (z_{j,k})_l^2 \right)^p \right] \leq M^{2p} \frac{1}{d} \sum_{l=1}^d \mathbb{E}[\|z_{j,k}\|_2^{2p}] \leq M^{2p} \kappa_p d^p.
\]
We define some constant $C_4 > M^{2p} \kappa_p$. With this fact and Theorem A.1 from [CGT12], we have

$$
\mathbb{E} \left[ \left\| \hat{\Sigma}_j \right\|_{op}^p \right] = \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{k=1}^n \frac{1}{n} x_{j,k} x_{j,k}^T \right\|_{op}^p \right] \leq 2^{p-1} \left( \left\| \Sigma_j \right\|_{op}^p + \frac{(e \log d)^p}{n_j} \mathbb{E} \left[ \max_k \left\| x_{j,k} x_{j,k}^T \right\|_{op}^p \right] \right) 
$$

$$
\leq 2^{p-1} \left( C + \frac{(e \log d)^p}{n_j} \mathbb{E} \left[ \left\| x_{j,k} \right\|_{op}^{2p} \right] \right) 
$$

$$
\leq 2^{p-1} \left( C + C_4 \frac{(e \log d)^p d^p}{n_j^{p-1}} \right) 
$$

Now, $2^{p-1} \left( C + C_4 \frac{(e \log d)^p d^p}{n_j^{p-1}} \right) \leq K (e \log d)^p n_j$ for some absolute constant $K$ since $\frac{d}{n_j} \rightarrow \gamma$. \hfill \Box

**Lemma 6.3.** For all clients $j \in [m]$, let the data $x_{j,k} \in \mathbb{R}^d$ for $k \in [n]$ be such that $x_{j,k} = \Sigma_j^{1/2} z_{j,k}$ for some $\Sigma_j$, $z_{j,k}$, and $q' > 2$ that satisfy Assumption A2. Further let $q' = pq$ where $p \geq 1$ and $q \geq 2$. Let $\hat{\Sigma}_j = \frac{1}{n_j} \sum_{k \in [n_j]} x_{j,k} x_{j,k}^T$ and $\mu_j = \mathbb{E}[\hat{\Sigma}_j^p]$. Additionally assume that $\left\| \mathbb{E}[\hat{\Sigma}_j^p] \right\|_{op} \leq C_3$ for some constant $C_3$. Let $d, n_j$ grow as in Assumption A1. Then we have for sufficiently large $m$,

$$
P \left( \left\| \sum_{j=1}^m p_j \left( \hat{\Sigma}_j^p - \mu_j \right) \right\|_{op} > t \right) \leq \frac{2^{q-1} C_4}{t^q} \left[ (e \log d)^q \sum_{j=1}^m p_j^{q/2+1} + (e \log d)^{pq+q} \sum_{j=1}^m p_j^q n_j \right] .
$$

Further supposing that $(e \log d)^{pq+q} \sum_{j=1}^m p_j^q n_j \rightarrow 0$, we get that $\left\| \sum_{j=1}^m p_j \left( \hat{\Sigma}_j^p - \mu_j \right) \right\|_{op} \rightarrow 0$ .

**Proof** Using Markov’s inequality, Jensen’s inequality, and symmeterization, we have with $\zeta_j$ iid Rademacher

$$
P \left( \left\| \sum_{j=1}^m p_j \left( \hat{\Sigma}_j^p - \mu_j \right) \right\|_{op} > t \right) \leq \mathbb{E} \left[ \left\| \sum_{j=1}^m p_j \left( \hat{\Sigma}_j^p - \mu_j \right) \right\|_{op}^q \right] \leq 2^q \mathbb{E} \frac{\left\| \sum_{j=1}^m p_j \left( \hat{\Sigma}_j^p - \mu_j \right) \right\|_{op}^q}{t^q} .
$$

We use the second part of Theorem A.1 with $q \geq 2$ from [CGT12] to bound the RHS.

$$
\mathbb{E} \left[ \left\| \sum_{j=1}^m p_j \hat{\Sigma}_j^p \zeta_j \right\|_{op}^q \right] \leq \left( \sqrt{e \log d} \left\| \mathbb{E} \left[ \sum_{j=1}^m p_j \tilde{\Sigma}_j^p \right] \right\|_{op} \right)^q + (e \log d) \mathbb{E} \left[ 2^{q/2} \left\| \sum_{j=1}^m p_j \tilde{\Sigma}_j^p \right\|_{op} \right] \right) \left\| \mathbb{E} \left[ \max_j p_j \tilde{\Sigma}_j^p \right] \right\|_{op}^{1/q} 
$$

$$
\leq 2^{q-1} (e \log d)^{q/2} \left\| \mathbb{E} \left[ \sum_{j=1}^m p_j \tilde{\Sigma}_j^p \right] \right\|_{op}^{q/2} + 2^{q-1} (e \log d)^q \mathbb{E} \left[ \max_j p_j^{1/2} \left\| \tilde{\Sigma}_j^p \right\|_{op} \right] 
$$

$$
\leq 2^{q-1} (e \log d)^{q/2} \left\| \mathbb{E} \left[ \sum_{j=1}^m p_j \tilde{\Sigma}_j^{2p} \right] \right\|_{op}^{q/2} + 2^{q-1} (e \log d)^q \mathbb{E} \left[ \sum_{j=1}^m p_j^{q/2} \left\| \tilde{\Sigma}_j^{pq} \right\|_{op} \right] 
$$

Now we bound the RHS of this quantity using the first part of Theorem A.1. For each $j \in [m]$, we have by Lemma 6.2 for sufficiently large $m$,

$$
\mathbb{E} \left[ \left\| \hat{\Sigma}_j \right\|_{op}^{pq} \right] \leq K (e \log d)^{pq} n_j ,
$$

for some absolute constant $K$. Supposing that $\left\| \mathbb{E} \left[ \hat{\Sigma}_j^{2p} \right] \right\|_{op} \leq C_3$ exist for all $j$. Combining all the inequalities, we have for sufficiently large $m$, 

15
\[ \mathbb{E} \left[ \left\| \sum_{j=1}^{m} p_j \hat{y}_{j} \xi_j \right\|_q \right] \leq 2^{q-1}(e \log d)^{q/2} \left( \sum_{j=1}^{m} p_j^{q/2+1} \mathbb{E} \left[ \hat{\Sigma}_j \right] \right)^{q/2} \]

\[ + 2^{q-1}(e \log d)^q \sum_{j=1}^{m} p_j^q K(e \log d)^{pq} n_j \]

\[ \leq C_2 \left[ (\log d)^{q/2} \sum_{j=1}^{m} p_j^{q/2+1} + (\log d)^{pq+q} \sum_{j=1}^{m} p_j^q n_j \right], \]

where in the first term of the first inequality, we use Jensen’s inequality to pull out \( \sum_{j=1}^{m} p_j \) of the expectation.

To prove the second part of the lemma, we observe that if \( (\log d)^{(p+1)q} \sum_{j=1}^{m} p_j^q n_j \to 0 \) as \( m \to \infty \) such that \( d/n_i \to \gamma_i > 1 \) for all devices \( i \in [m] \), then \( (\log d)^{q/2} \sum_{j=1}^{m} p_j^{q/2+1} \to 0 \). To see this, we first observe

\[ (\log d)^{(p+1)q} \sum_{j=1}^{m} p_j^q n_j \geq (\max_{j \in [m]} p_j (\log d)^{(p+1)q}), \]

so we know that \( \max_{j \in [m]} p_j (\log d)^{(p+1)q} \to 0 \). Further, by Holder’s inequality, we know that

\[ (\log d)^{q/2} \sum_{j=1}^{m} p_j^{q/2+1} \leq (\max_{j \in [m]} p_j \log d)^{q/2}. \]

By the continuity of the \( q/2 \) power, we get the result.

\[ \square \]

**Lemma 6.4.** Let \( U \in \mathbb{R}^{d \times d} \) and \( V \in \mathbb{R}^{d \times d} \) be positive semidefinite matrices such that \( \lambda_{\min}(U) \geq \lambda_0 \) for some constant \( \lambda_0 \). Let \( d, n_j, m \to \infty \) as in Assumption A1. Suppose \( \|V - U\|_{\text{op}} \overset{P}{\to} 0 \), then \( \|V^{-1} - U^{-1}\|_{\text{op}} \overset{P}{\to} 0. \)

**Proof** For any \( t > 0 \), we have by Theorem 2.5 (from Section III) of [SS90]

\[ P \left( \|V^{-1} - U^{-1}\|_{\text{op}} > t \right) \]

\[ \leq P \left( \|V^{-1} - U^{-1}\|_{\text{op}} > t \cap \|V - U\|_{\text{op}} < \frac{1}{\|V^{-1}\|_{\text{op}}} \right) + P \left( \|V - U\|_{\text{op}} \geq \lambda_0 \right) \]

\[ \leq P \left( \|U^{-1}(V - U)\|_{\text{op}} > \frac{t}{\|U^{-1}\|_{\text{op}}} \right) + o(1) \]

\[ \leq P \left( \|V - U\|_{\text{op}} > \frac{t \lambda_0}{t + \lambda_0^{-1}} \right) + o(1) \]

We know this quantity goes to 0 by assumption.

\[ \square \]

### 6.3 Some useful definitions from previous work

In this section, we recall some definitions from [HMRT19] that will be useful in finding the exact expressions for risk. The expressions for asymptotic risk in high dimensional regression problems (both ridge and ridgeless) are given in an implicit form in [HMRT19]. It depends on the geometry of the covariance matrix \( \Sigma \) and the true solution to the regression problem \( \theta^\star \). Let \( \Sigma = \sum_{i=1}^{d} s_i v_i^T v_i^T \) denote
the eigenvalue decomposition of $\Sigma$ with $s_1 \geq s_2 \cdots \geq s_d$, and let $(c, \ldots, v_d^T \theta^*)$ denote the inner products of $\theta^*$ with the eigenvectors. We define two probability distributions which will be useful in giving the expressions for risk:

$$\hat{H}_n(s) := \frac{1}{d} \sum_{i=1}^{d} 1\{s \geq s_i\}, \quad \hat{G}_n(s) := \frac{1}{\|\theta^*\|_2^2} \sum_{i=1}^{d} (v_i^T \theta^*)^2 1\{s \geq s_i\}.$$ 

Note that $\hat{G}_n$ is a reweighted version of $\hat{H}_n$ and both have the same support (eigenvalues of $\Sigma$).

**Definition 6.1.** For $\gamma \in \mathbb{R}^+$, let $c_0 = c_0(\gamma, \hat{H}_n)$ be the unique non-negative solution of

$$1 - \frac{1}{\gamma} = \int \frac{1}{1 + c_0 \gamma s} d\hat{H}_n(s),$$

the predicted bias and variance is then defined as

$$\mathcal{B}(\hat{H}_n, \hat{G}_n, \gamma) := \|\theta^*\|_2^2 \left\{ 1 + c_0 \gamma \int \frac{s^2}{(1+c_0 \gamma s)} d\hat{H}_n(s) \right\} \cdot \int \frac{s}{(1+c_0 \gamma s)} d\hat{G}_n(s),$$

$$\mathcal{V}(\hat{H}_n, \gamma) := \sigma^2 \gamma \int s^2 \frac{1}{(1+c_0 \gamma s)} d\hat{H}_n(s) \int \frac{s}{(1+c_0 \gamma s)} d\hat{G}_n(s).$$

**Definition 6.2.** For $\gamma \in \mathbb{R}^+$ and $z \in \mathbb{C}_+$, let $m_n(z) = m(z; \hat{H}_n, \gamma)$ be the unique solution of

$$m_n(z) := \int \frac{1}{s[1 - \gamma - \gamma zm_n(z)] - z} d\hat{H}_n(s).$$

Further, define $m_{n,1}(z) = m_{n,1}(z; \hat{H}_n, \gamma)$ as

$$m_{n,1}(z) := \int \frac{s^2[1-\gamma - \gamma zm_n(z)]}{[s][1 - \gamma - \gamma zm_n(z)] - z^2} d\hat{H}_n(s).$$

The definitions are extended analytically to $\text{Im}(z) = 0$ whenever possible, the predicted bias and variance are then defined by

$$\mathcal{B}(\lambda; \hat{H}_n, \hat{G}_n, \gamma) := \lambda^2 \|\theta^*\|_2^2 \frac{1}{(1 + \gamma m_n(\lambda))} \int \frac{s}{[\lambda + (1 - \gamma + \gamma \lambda m_n(\lambda))]s^2} d\hat{G}_n(s),$$

$$\mathcal{V}(\lambda; \hat{H}_n, \gamma) := \sigma^2 \gamma \int \frac{s^2[(1 - \gamma + \gamma \lambda m_n'(\lambda))]}{[\lambda + (1 - \gamma + \gamma \lambda m_n(\lambda))]s^2} d\hat{H}_n(s).$$

### 6.4 Proof of Theorem 1

On solving (6) and (7), the closed form of the estimators $\hat{\theta}_0^F$ and $\hat{\theta}_i^F$ is given by

$$\hat{\theta}_0^F = \arg\min_{\theta} \sum_{j=1}^{m} p_j \frac{1}{2n_j} \|X_j \theta - y_j\|_2^2 = \left( \sum_{j=1}^{m} p_j \hat{S}_j \right)^{-1} \sum_{j=1}^{m} p_j \frac{X_j^T y_j}{n_j}$$

$$= \left( \sum_{j=1}^{m} p_j \hat{S}_j \right)^{-1} \sum_{j=1}^{m} p_j \hat{S}_j \theta^* + \left( \sum_{j=1}^{m} p_j \hat{S}_j \right)^{-1} \sum_{j=1}^{m} p_j \frac{X_j^T \xi_j}{n_j}$$

and

$$\hat{\theta}_i^F = (I - \hat{\Sigma}_i^T \hat{\Sigma}_i) \hat{\theta}_0^F + X_i^T y_i = (I - \hat{\Sigma}_i^T \hat{\Sigma}_i) \hat{\theta}_0^F + \hat{\Sigma}_i^T \hat{\Sigma}_i \theta^* + \frac{1}{n_i} \hat{\Sigma}_i^T X_i^T \xi_i$$

$$= \Pi_i \left( \sum_{j=1}^{m} p_j \hat{S}_j \right)^{-1} \sum_{j=1}^{m} p_j \hat{S}_j \theta^*_i + \left( \sum_{j=1}^{m} p_j \hat{S}_j \right)^{-1} \sum_{j=1}^{m} p_j \frac{X_j^T \xi_j}{n_j} + \hat{\Sigma}_i^T \hat{\Sigma}_i \theta^*_i + \frac{1}{n_i} \hat{\Sigma}_i^T X_i^T \xi_i$$

17
The idea is to show that the second term goes to 0 as the probability of bias. For simplicity, we let \( B \) and variance.

Bias:

\[
B_i(\hat{\theta}^{FA}|X) := \left\| \mathbb{E}[\hat{\theta}^{FA}|X] - \theta_i^* \right\|_{\Sigma_i}^2 = \left\| \Pi_i \left[ \left( \sum_{j=1}^{m} p_j \hat{\Sigma}_j \right)^{-1} \sum_{j=1}^{m} p_j \hat{\Sigma}_j (\theta_j^* - \theta_i^*) \right] \right\|_{\Sigma_i}^2
\]

\[
= \left\| \Sigma_i^{1/2} \Pi_i \left[ \theta_0^* - \theta_i^* + \left( \sum_{j=1}^{m} p_j \hat{\Sigma}_j \right)^{-1} \sum_{j=1}^{m} p_j \hat{\Sigma}_j (\theta_j^* - \theta_0^*) \right] \right\|_2^2
\]

The idea is to show that the second term goes to 0 and use results from [HMRT19] to find the asymptotic bias. For simplicity, we let \( \Delta_j := \theta_j^* - \theta_0^* \), and we define the event:

\[
B_i := \left\{ \left\| \left( \sum_{j=1}^{m} p_j \hat{\Sigma}_j \right)^{-1} - \left( \sum_{j=1}^{m} p_j \Sigma_j \right)^{-1} \right\|_{op} > t \right\}
\]

\[
A_i := \left\{ \left\| \left( \sum_{j=1}^{m} p_j \hat{\Sigma}_j \right)^{-1} \sum_{j=1}^{m} p_j \hat{\Sigma}_j \Delta_j \right\|_{\Sigma_i} > t \right\}
\]

The proof proceeds in the following steps:

**Bias Proof Outline**

**Step 1.** We first show for any \( t > 0 \), the \( \mathbb{P}(B_i) \to 0 \) as \( d \to \infty \)

**Step 2.** Then we show for any \( t > 0 \), the \( \mathbb{P}(A_i) \to 0 \) as \( d \to \infty \)

**Step 3.** We show that for any \( t \in (0,1] \) on event \( A_i \), \( B_i(\hat{\theta}^{FA}|X) \leq \| \Pi_i[\theta_0^* - \theta_i^*]\|_{\Sigma_i}^2 + ct \) and \( B_i(\hat{\theta}^{FA}|X) \geq \| \Pi_i[\theta_0^* - \theta_i^*]\|_{\Sigma_i}^2 - ct \)

**Step 4.** Show that \( \lim_{d \to \infty} \mathbb{P}(|B_i(\hat{\theta}^{FA}|X) - \| \Pi_i[\theta_0^* - \theta_i^*]\|_{\Sigma_i}^2 | \leq \varepsilon) = 1 \)

**Step 5.** Finally, using the asymptotic limit of \( \| \Pi_i[\theta_0^* - \theta_i^*]\|_{\Sigma_i}^2 \) from Theorem 1 of [HMRT19], we get the result.

**Step 1** Since we have \( \lambda_{\min}(\sum_{j=1}^{m} p_j \Sigma_j) > 1/M > 0 \), it suffices to show by Lemma 6.4 that the probability of

\[
C_i := \left\{ \left\| \sum_{j=1}^{m} p_j \hat{\Sigma}_j - \sum_{j=1}^{m} p_j \Sigma_j \right\|_{op} > t \right\}
\]

goes to 0 as \( d, m \to \infty \) (obeys Assumption A1). Using Lemma 6.3 with \( p = 1 \), we have that

\[
\mathbb{P}(C_i) \leq \frac{2^{q-1} C_2}{1 q} \left[ (\log d)^{q/2} \sum_{j=1}^{m} p_j^{q/2+1} + (\log d)^{2q} \sum_{j=1}^{m} p_j^n \right]
\]

Since \( (\log d)^{2q} \sum_{j=1}^{m} p_j^n \to 0 \), this quantity goes to 0.
Step 2  Fix any \( t > 0 \),
\[
\mathbb{P}(A_t) \leq \mathbb{P}\left( \left\{ \left\| \left( \sum_{j=1}^{m} p_j \hat{\Sigma}_j \right)^{-1} \sum_{j=1}^{m} p_j \hat{\Sigma}_j \Delta_j \right\|_{\Sigma_i} > t \right\} \cap B_{c_1}^c \right) + \mathbb{P}(B_{c_1})
\]
\[
\leq \mathbb{P}\left( M(c_1 + M) \left\| \sum_{j=1}^{m} p_j \hat{\Sigma}_j \Delta_j \right\|_{2} > t \right) + \mathbb{P}(B_{c_1})
\]

By Step 1, we know that \( \mathbb{P}(B_{c_1}) \to 0 \). The second inequality comes from \( \|Ax\|_2 \leq \|A\|_{op} \|x\|_2 \) and triangle inequality. Now to bound the first term, we use Markov and a Khintchine inequality (Lemma 6.1). We have that
\[
\mathbb{P}\left( M(c_1 + M) \left\| \sum_{j=1}^{m} p_j \hat{\Sigma}_j \Delta_j \right\|_{2} > t \right) \leq \frac{(M(c_1 + M))^q E \left[ \left\| \sum_{j=1}^{m} p_j \hat{\Sigma}_j \Delta_j \right\|_{2}^q \right]}{t^q}
\]
\[
\leq \frac{(2M(c_1 + M)^{\sqrt{q}} q/2 E \left[ \left\| \sum_{j=1}^{m} p_j \hat{\Sigma}_j \Delta_j \right\|_{2}^{q/2} \right]}{t^q}
\]

Using Jensen’s inequality and the definition of operator norm, we have
\[
(2M(c_1 + M)^{\sqrt{q}} q/2 E \left[ \left\| \sum_{j=1}^{m} p_j \hat{\Sigma}_j \Delta_j \right\|_{2}^{q/2} \right]) = \frac{(2M(c_1 + M)^{\sqrt{q}} q/2 E \left[ \left\| \sum_{j=1}^{m} p_j \hat{\Sigma}_j \Delta_j \right\|_{2}^{q/2} \right]}{t^q}
\]
\[
\leq \frac{(2M(c_1 + M)^{\sqrt{q}} q/2 E \left[ \left\| \sum_{j=1}^{m} p_j \hat{\Sigma}_j \Delta_j \right\|_{2}^{q/2} \right]}{t^q}
\]

Lastly, we can bound this using Lemma 6.2 as follows.
\[
\mathbb{P}\left( M(c_1 + M) \left\| \sum_{j=1}^{m} p_j \hat{\Sigma}_j \Delta_j \right\|_{2} > t \right) \leq \frac{K (2M(c_1 + M)^{\sqrt{q}} q/2 E \left[ \left\| \sum_{j=1}^{m} p_j \hat{\Sigma}_j \Delta_j \right\|_{2}^{q/2} \right]}{t^q}
\]

Using \( (\log d)^q \sum_{j=1}^{m} p_j^{q/2+1} n_j r_j^q \to 0 \).

Step 3  For any \( t \in (0, 1] \), on the event \( A_t^c \), we have that
\[
B(\hat{\theta}_t - \theta_t + E) = \| \Pi_i [\theta_0^* - \theta_t^* + E] \|_{\Sigma_i}^2,
\]
for some vector \( E \) where we know \( \|E\|_2 \leq t \)(which means \( \|E\|_2 \leq t \sqrt{M} \)).

Thus, we have
\[
\| \Pi_i [\theta_0^* - \theta_t^* + E] \|_{\Sigma_i}^2 \leq \| \Pi_i [\theta_0^* - \theta_t^*] \|_{\Sigma_i}^2 + \| \Pi_i E \|_{\Sigma_i}^2 + 2 \| \Pi_i E \|_{\Sigma_i} \| \Pi_i [\theta_0^* - \theta_t^*] \|_{\Sigma_i}
\]
\[
\leq \| \Pi_i [\theta_0^* - \theta_t^*] \|_{\Sigma_i}^2 + M^{2t^2} + 2 t M^{3/2} r_t^2
\]
\[
\| \Pi_i [\theta_0^* - \theta_t^* + E] \|_{\Sigma_i}^2 \geq \| \Pi_i [\theta_0^* - \theta_t^*] \|_{\Sigma_i}^2 + \| \Pi_i E \|_{\Sigma_i}^2 - 2 \| \Pi_i E \|_{\Sigma_i} \| \Pi_i [\theta_0^* - \theta_t^*] \|_{\Sigma_i}
\]
\[
\geq \| \Pi_i [\theta_0^* - \theta_t^*] \|_{\Sigma_i}^2 - 2 t M^{3/2} r_t^2
\]

Since \( t \in (0, 1] \), we have that \( t^2 \leq t \) and thus we can choose \( c = M^{2} + 2 M^{3/2} r_t^2 \).
Step 4  Reparameterizing $\varepsilon := ct$, we have that for any $\varepsilon > 0$

$$\lim_{n \to \infty} \mathbb{P}(\|B_i(\hat{\theta}_i^{FA}) - \Pi_i[\theta_i^0 - \theta_i^*]\|_{\Sigma_i} \leq \varepsilon) \geq \lim_{n \to \infty} \mathbb{P}(\|B_i(\hat{\theta}_i^{FA}) - \Pi_i[\theta_i^0 - \theta_i^*]\|_{\Sigma_i}^2 \leq \varepsilon \wedge c) \geq \lim_{n \to \infty} \mathbb{P}(A_{c}^{\varepsilon}) = 1$$

Step 5  Using Theorem 3 of [HMRT19], as $d \to \infty$, such that $\frac{d}{n} \to \gamma_i > 1$, we know that the limit of $\|\Pi_i[\theta_i^0 - \theta_i^*]\|_{\Sigma_i}^2$ is given by (15) with $\gamma = \gamma_i$ and $\hat{H}_n, \hat{G}_n$ be the empirical spectral distribution and weighted empirical spectral distribution of $\Sigma_i$ respectively.

In the case when $\Sigma_i = I$, using Theorem 1 of [HMRT19] we have $B_i(\hat{\theta}_i^{FA}) = \Pi_i[\theta_i^0 - \theta_i^*]\|_{2} \to r_i^2 \left(1 - \frac{1}{n_i}\right)$.

Variance:

We let $\xi_i = [\xi_{i,1}, \ldots, \xi_{i,n}]$ denote the vector of noise.

$$V_i(\hat{\theta}_i^{FA}|X) = \text{tr}(\text{Cov}(\hat{\theta}_i^{FA}|X)\Sigma_i) = E \left[ \left\| \hat{\theta}_i^{FA} - E[\hat{\theta}_i^{FA}|X] \right\|_{\Sigma_i}^2 |X \right]$$

$$= E \left[ \left\| \Pi_i \left[ \left( \sum_{j=1}^{m} p_j \hat{\Sigma}_j \right)^{-1} \sum_{j=1}^{m} p_j \frac{X_i^{T} \xi_i}{n_i} + \frac{1}{n_i} \hat{\Sigma}_i^{T} X_i^{T} \xi_i \right] \right\|_{\Sigma_i}^2 |X \right]$$

$$= \sum_{j=1}^{m} \frac{p_j^2}{n_j} \text{tr} \left( \Pi_i \Sigma_i \Pi_i \left[ \left( \sum_{j=1}^{m} p_j \hat{\Sigma}_j \right)^{-1} \hat{\Sigma}_j \left( \sum_{j=1}^{m} p_j \hat{\Sigma}_j \right)^{-1} \right] \sigma_j^2 + 2 \text{tr} \left( \Sigma_i \hat{\Sigma}_i^{T} X_i \left[ \left( \sum_{j=1}^{m} p_j \hat{\Sigma}_j \right)^{-1} \right] \Pi_i \right) \sigma_i^2 \right)$$

We now study the asymptotic behavior of each of the terms (i), (ii) and (iii) separately.

(i)  Using the Cauchy Schwartz inequality on Schatten $p$-norms and using the fact that the nuclear norm of a projection matrix is at most $d$, we get

$$\sum_{j=1}^{m} \frac{p_j^2\sigma_j^2}{n_j} \text{tr} \left( \Pi_i \Sigma_i \Pi_i \left[ \left( \sum_{j=1}^{m} p_j \hat{\Sigma}_j \right)^{-1} \hat{\Sigma}_j \left( \sum_{j=1}^{m} p_j \hat{\Sigma}_j \right)^{-1} \right] \right)$$

$$\leq \sum_{j=1}^{m} \frac{p_j^2\sigma_j^2}{n_j} \|\Pi_i\|_1 \|\Sigma_i\|_{\text{op}} \left\| \left( \sum_{j=1}^{m} p_j \hat{\Sigma}_j \right)^{-1} \right\|_{\text{op}} \left\| \hat{\Sigma}_j \right\|_{\text{op}} \left\| \left( \sum_{j=1}^{m} p_j \hat{\Sigma}_j \right)^{-1} \right\|_{\text{op}}$$

$$\leq C_3 \sigma_i^2 \max \gamma_{\max} \left( \sum_{j=1}^{m} p_j^2 \left\| \hat{\Sigma}_j \right\|_{\text{op}} \right), \quad (20)$$
where the last inequality holds with probability going to 1 for some constant $C_3$ because $P(B_t) \to 0$.

Lastly, we show that $P\left(\sum_{j=1}^{m} p_j^2 \|\hat{\Sigma}_j\|_{\text{op}} > t\right) \to 0$. Using Markov’s and Jensen’s inequality, we have

$$P\left(\sum_{j=1}^{m} p_j^2 \|\hat{\Sigma}_j\|_{\text{op}} > t\right) \leq \frac{\mathbb{E}\left[\sum_{j=1}^{m} p_j^2 \|\hat{\Sigma}_j\|_{\text{op}}^q\right]}{t^q} \leq \frac{\sum_{j=1}^{m} p_j^{q+1} \mathbb{E}\left[\|\hat{\Sigma}_j\|_{\text{op}}^q\right]}{t^q}$$

Using Lemma 6.2, we have

$$P\left(\sum_{j=1}^{m} p_j^2 \|\hat{\Sigma}_j\|_{\text{op}} > t\right) \leq K \frac{\sum_{j=1}^{m} p_j^{q+1}(e \log d)^{q}n_j}{t^q}$$

Finally, since we know that $\sum_{j=1}^{m} p_j^{q+1}(e \log d)^{q}n_j \to 0$, we have $\sum_{j=1}^{m} p_j^2 \|\hat{\Sigma}_j\|_{\text{op}} \overset{P}{\to} 0$. Thus,

$$\sum_{j=1}^{m} \frac{p_j^2 \sigma_j^2}{n_i} \text{tr} \left(\Pi_i \Sigma_i \Pi_i \left(\sum_{j=1}^{m} p_j \hat{\Sigma}_j\right)^{-1}\hat{\Sigma}_j \left(\sum_{j=1}^{m} p_j \hat{\Sigma}_j\right)^{-1}\right)^{\frac{p_j}{n_i}} \overset{P}{\to} 0$$

(ii) Using the Cauchy Schwartz inequality on Schatten $p$-norms and using the fact that the nuclear norm of a projection matrix is $d - n$, we get

$$\frac{2p_i \sigma_i^2}{n_i} \text{tr} \left(\Pi_i \Sigma_i \Sigma_i^T \Sigma_i \left(\sum_{j=1}^{m} p_j \hat{\Sigma}_j\right)^{-1}\right) \leq \frac{2p_i \sigma_i^2}{n_i} \text{tr} \left(\Pi_i \Sigma_i \Sigma_i^T \Sigma_i \left(\sum_{j=1}^{m} p_j \hat{\Sigma}_j\right)^{-1}\right) \leq C_4 p_i,$$

where the last inequality holds with probability going to 1 for some constant $C_4$ because $P(B_t) \to 0$ and using Assumption A2. Since $p_i \to 0$, we have

$$\frac{2p_i \sigma_i^2}{n_i} \text{tr} \left(\Pi_i \Sigma_i \Sigma_i^T \Sigma_i \left(\sum_{j=1}^{m} p_j \hat{\Sigma}_j\right)^{-1}\right) \to 0$$

(iii)

$$\frac{1}{n_i} \text{tr}(\hat{\Sigma}_i^T X_i^T X_i \hat{\Sigma}_i) \sigma_i^2 = \frac{1}{n_i} \text{tr}(\hat{\Sigma}_i^T \Sigma_i) \sigma_i^2$$

Using Theorem 3 of [HMRT19], as $d \to \infty$, such that $d_{n_i} \to \gamma_i > 1$, we know that the limit of $\frac{\sigma_i^2}{n_i} \text{tr}(\hat{\Sigma}_i^T \Sigma_i)$ is given by (16) with $\gamma = \gamma_i$ and $\hat{H}_n, \hat{G}_n$ be the empirical spectral distribution and weighted empirical spectral distribution of $\Sigma_i$ respectively.

In the case when $\Sigma_i = I$, using Theorem 1 of [HMRT19] we have $V_i(\hat{\theta}_i^F \mid X) = \frac{\sigma_i^2}{n_i} \text{tr}(\hat{\Sigma}_i) \to \frac{\sigma_i^2}{\gamma_i - 1}$.
6.5 Proof of Theorem 2

We use the global model from (6) and the personalized model from (8). The closed form of the estimators \( \hat{\theta}^{FA}_0 \) and \( \hat{\theta}^R_i(\lambda) \) is given by

\[
\hat{\theta}^{FA}_0 = \arg\min_{\theta} \sum_{j=1}^{m} p_j \frac{1}{2n_j} \| X_j \theta - y_j \|_2 = \left( \sum_{j=1}^{m} p_j \hat{\Sigma}_j \right)^{-1} \sum_{j=1}^{m} p_j \frac{X_j y_j}{n_j}
\]

and

\[
\hat{\theta}^R_i(\lambda) = \arg\min_{\theta} \frac{1}{2n_i} \| X_i \theta - y_i \|_2^2 + \frac{\lambda}{2} \| \hat{\theta}^{FA}_0 - \theta \|_2^2 = (\hat{\Sigma}_i + \lambda I)^{-1} \left( \lambda \hat{\theta}^{FA}_0 + \hat{\Sigma}_i \theta^* + \frac{1}{n_i} X_i^T \xi_i \right)
\]

We now calculate the risk by splitting it into two parts as in (3), and then calculate the asymptotic bias and variance.

**Bias:**

\[
B(\hat{\theta}^R_i(\lambda)|X) := \left\| \mathbb{E}[\hat{\theta}^R_i(\lambda)|X] - \theta^*_i \right\|_{\Sigma_i}^2 = \lambda^2 \left\| (\hat{\Sigma}_i + \lambda I)^{-1} \left[ \sum_{j=1}^{m} p_j \hat{\Sigma}_j \right]^{-1} \sum_{j=1}^{m} p_j \hat{\Sigma}_j (\theta^*_j - \theta^*_0) \right\|_{\Sigma_i}^2
\]

The idea is to show that the second term goes to 0 and use results from [HMRT19] to find the asymptotic bias. For simplicity, we let \( \Delta_j := \theta^*_j - \theta^*_0 \), and we define the event:

\[
B_t := \left\{ \left\| \left( \sum_{j=1}^{m} p_j \hat{\Sigma}_j \right)^{-1} - \left( \sum_{j=1}^{m} p_j \Sigma_j \right)^{-1} \right\|_{op} > t \right\}
\]

\[
A_t := \left\{ \left\| \left( \sum_{j=1}^{m} p_j \hat{\Sigma}_j \right)^{-1} \sum_{j=1}^{m} p_j \hat{\Sigma}_j \Delta_j \right\|_{\Sigma_i} > t \right\}
\]

The proof proceeds in the following steps:
Bias Proof Outline

**Step 1.** We first show for any \( t > 0 \), the \( \mathbb{P}(B_t) \to 0 \) as \( d \to \infty \).

**Step 2.** We show for any \( t > 0 \), the \( \mathbb{P}(A_t) \to 0 \) as \( d \to \infty \).

**Step 3.** We show that for any \( t \in (0, 1] \) on event \( A_t^c \) where \( T_t^{-1} = (\hat{\Sigma}_t + \lambda I)^{-1} \), we have that

\[
B(\hat{\theta}_R^*(\lambda)|X) = \lambda^2 \left\| T_t^{-1}[\theta_0^* - \theta_1^* + E]\right\|^2_{\Sigma_t}
\]

for some vector \( E \) where we know \( \|E\|_{\Sigma_t} \leq t \) (which means \( \|E\|_2 \leq t\sqrt{M} \)).

We can form the bounds

\[
\left\| T_t^{-1}[\theta_0^* - \theta_1^* + E]\right\|^2_{\Sigma_t} \leq \left\| T_t^{-1}[\theta_0^* - \theta_1^*]\right\|^2_{\Sigma_t} + \left\| T_t^{-1}E\right\|^2_{\Sigma_t} + 2 \left\| T_t^{-1}E\right\|_{\Sigma_t} \left\| T_t^{-1}[\theta_0^* - \theta_1^*]\right\|_{\Sigma_t}
\]

\[
\leq \left\| T_t^{-1}[\theta_0^* - \theta_1^*]\right\|^2_{\Sigma_t} + M^2 \lambda^{-2}t^2 + 2M^3/2t\lambda^{-2}r_t^2
\]

\[
\left\| T_t^{-1}[\theta_0^* - \theta_1^* + E]\right\|^2_{\Sigma_t} \geq \left\| T_t^{-1}[\theta_0^* - \theta_1^*]\right\|^2_{\Sigma_t} + \left\| T_t^{-1}E\right\|^2_{\Sigma_t} - 2 \left\| T_t^{-1}E\right\|_{\Sigma_t} \left\| T_t^{-1}[\theta_0^* - \theta_1^*]\right\|_{\Sigma_t}
\]

\[
\geq \left\| T_t^{-1}[\theta_0^* - \theta_1^*]\right\|^2_{\Sigma_t} - 2M^2 t \lambda^{-2}r_t^2.
\]

Since \( t \in (0, 1] \), we have that \( t^2 \leq t \) and thus we can choose \( c = \lambda^{-2}(M^2 + 2M^3/2r_t^2) \).

**Step 4** Reparameterizing \( \varepsilon := c t \), we have that for any \( \varepsilon > 0 \)

\[
\lim_{n \to \infty} \mathbb{P}(\|B(\hat{\theta}_R^*(\lambda)|X) - \lambda^2 \left\| (\hat{\Sigma}_t + \lambda I)^{-1}[\theta_0^* - \theta_1^*]\right\|^2_{\Sigma_t} | \leq \varepsilon)
\]

\[
\geq \lim_{n \to \infty} \mathbb{P}(\|B(\hat{\theta}_R^*(\lambda)|X) - \lambda^2 \left\| (\hat{\Sigma}_t + \lambda I)^{-1}[\theta_0^* - \theta_1^*]\right\|^2_{\Sigma_t} | \leq \varepsilon \land c)
\]

\[
\geq \lim_{n \to \infty} \mathbb{P}(A_t^c \land 1) = 1.
\]

**Step 5** Using Theorem 6 of [HMRT19], as \( d \to \infty \), such that \( \frac{d}{n} \to \gamma_i > 1 \), we know that the limit of \( \lambda^2 \left\| (\hat{\Sigma}_t + \lambda I)^{-1}[\theta_0^* - \theta_1^*]\right\|^2_{\Sigma_t} \) is given by (17) with \( \gamma = \gamma_i \) and \( \hat{H}_n, \hat{G}_n \) be the empirical spectral distribution and weighted empirical spectral distribution of \( \Sigma_i \) respectively.

In the case when \( \Sigma_i = I \), using Corollary 5 of [HMRT19] we have \( B_i(\hat{\theta}_R^*(\lambda)|X) = \|\Pi_i[\theta_0^* - \theta_1^*]\|_2^2 \to r_t^2 \lambda^2 m_t^2(\lambda) \).
We let $\xi_i = [\xi_{i,1}, \ldots, \xi_{i,n}]$ denote the vector of noise. Substituting in the variance formula and using $E[\xi_i, \xi_i^T] = 0$ and $E[\xi_i, \xi_i^T] = \sigma^2 I$, we get

\[
\text{Var}(\hat{\theta}_i^R(\lambda)|X) = E \left[ \left( \tilde{\Sigma}_i + \lambda I \right)^{-1} \left( \frac{1}{n_i} X_i^T \xi_i + \lambda \left( \sum_{j=1}^{m} p_j \tilde{\Sigma}_j \right) \right)^{-1} \sum_{j=1}^{m} p_j X_j \xi_i \right]_i^2 |X \right]
\]

\[
= \sum_{j=1}^{m} \frac{\lambda^2 p_j^2 \sigma^2}{n_j} \text{tr} \left( (\tilde{\Sigma}_i + \lambda I)^{-1} \Sigma (\tilde{\Sigma}_i + \lambda I)^{-1} \left( \sum_{j=1}^{m} p_j \tilde{\Sigma}_j \right) \right)^{-1} \sigma_j^2
\]

\[
+ 2\lambda p_i \text{tr} \left( \frac{X_i^T X_i}{n_i^2} \left( \sum_{j=1}^{m} p_j \tilde{\Sigma}_j \right)^{-1} \left( \tilde{\Sigma}_i + \lambda I \right)^{-1} \Sigma (\tilde{\Sigma}_i + \lambda I)^{-1} \right) \sigma_i^2
\]

\[
+ \text{tr} \left( (\tilde{\Sigma}_i + \lambda I)^{-1} \Sigma (\tilde{\Sigma}_i + \lambda I)^{-1} \right) \frac{\sigma_i^2}{n_i}
\]

We now study the asymptotic behavior of each of the terms (i), (ii) and (iii) separately.

(i) Using the Cauchy Schwartz inequality on Schatten $p$-norms, we get

\[
\sum_{j=1}^{m} \frac{p_j^2 \lambda^2 \sigma^2}{n_j} \text{tr} \left( (\tilde{\Sigma}_i + \lambda I)^{-1} \Sigma (\tilde{\Sigma}_i + \lambda I)^{-1} \left( \sum_{j=1}^{m} p_j \tilde{\Sigma}_j \right) \right)^{-1} \sigma_j^2
\]

\[
\leq \sum_{j=1}^{m} \frac{\lambda^2 p_j^2 \sigma^2}{n_j} \left\| (\tilde{\Sigma}_i + \lambda I)^{-1} \Sigma \right\|_1 \left\| \Sigma \right\|_{op} \left\| (\tilde{\Sigma}_i + \lambda I)^{-1} \right\|_{op} \left\| \left( \sum_{j=1}^{m} p_j \tilde{\Sigma}_j \right) \right\|_{op} \left\| \sigma_j \right\|_{op} \left\| \left( \sum_{j=1}^{m} p_j \tilde{\Sigma}_j \right)^{-1} \right\|_{op}
\]

\[
\leq C_5 \sigma_{max}^2 \gamma_{max} \left( \sum_{j=1}^{m} p_j^2 \left\| \tilde{\Sigma}_j \right\|_{op} \right),
\]

where the last inequality holds with probability going to 1 for some constant $C_5$ because $P(B_i) \to 0$. Note that this expression is same as (20) and hence the rest of the analysis for this term is same as the one in the proof of FTFA (Section 6.4).

(ii) Using the Cauchy Schwartz inequality on Schatten $p$-norms, we get

\[
\frac{2p_i \lambda \sigma^2}{n_i} \text{tr} \left( (\tilde{\Sigma}_i + \lambda I)^{-1} \tilde{\Sigma}_i \left( \sum_{j=1}^{m} p_j \tilde{\Sigma}_j \right) \right)^{-1} \left( \tilde{\Sigma}_i + \lambda I \right)^{-1} \Sigma
\]

\[
\leq \frac{2p_i \lambda \sigma^2}{n_i} \left\| (\tilde{\Sigma}_i + \lambda I)^{-1} \right\|_1 \left\| \tilde{\Sigma}_i \right\|_{op} \left\| \left( \sum_{j=1}^{m} p_j \tilde{\Sigma}_j \right)^{-1} \right\|_{op} \left\| \Sigma \right\|_{op} \left\| (\tilde{\Sigma}_i + \lambda I)^{-1} \right\|_{op}
\]

\[
\leq \frac{C_6 \sigma_{max}^2 \gamma_{max}}{\lambda n_i},
\]
where $C_6$ is an absolute constant which captures an upper bound on the operator norm of the sample covariance matrix $\hat{\Sigma}$ using Bai Yin Theorem \cite{BY93}, and an upper bound on the operator norm of 
\[
\left(\sum_{j=1}^{m} p_j \hat{\Sigma}_j\right)^{-1},
\] which follows from $\mathbb{P}(B_t) \to 0$. Since $p_t \to 0$, we have
\[
\frac{2p_t \lambda \sigma^2}{n_i} \text{tr}\left((\hat{\Sigma}_i + \lambda I)^{-1} \hat{\Sigma}_i \left(\sum_{j=1}^{m} p_j \hat{\Sigma}_j\right)^{-1} (\hat{\Sigma}_i + \lambda I)^{-1} \Sigma\right) \xrightarrow{p} 0.
\]

(iii) Using Theorem 3 of \cite{HMRT19}, as $d \to \infty$, such that $\frac{d}{n_i} \to \gamma_i > 1$, we know that the limit of 
\[
\text{tr}((\hat{\Sigma}_i + \lambda I)^{-2} \hat{\Sigma}_i \Sigma_i) \sigma^2_{n_i}^2
\] is given by (18) with $\gamma = \gamma_i$ and $\tilde{H}_n, \tilde{G}_n$ be the empirical spectral distribution and weighted empirical spectral distribution of $\Sigma_i$ respectively.

In the case when $\Sigma_i = I$, using Theorem 1 of \cite{HMRT19} we have $V_i(\hat{\theta}^R_i(\lambda)|X) = \frac{\sigma^2}{n_i} \text{tr}((\hat{\Sigma}_i + \lambda I)^{-2} \hat{\Sigma}_i \Sigma_i) \xrightarrow{p} \frac{\sigma^2}{\gamma_i - 1}$.

6.6 Proof of Theorem 3

On solving (12) and (13), the closed form of the estimators $\hat{\theta}_0^M(\alpha)$ and $\hat{\theta}_i^M(\alpha)$ is given by

\[
\hat{\theta}_0^M(\alpha) := \arg\min_\theta \sum_{j=1}^{m} \frac{p_j}{2n_j} \left\| X_j \left[ \theta - \frac{\alpha}{n_j} X_j^T (X_j^i \theta - y_j) \right] - y_j \right\|^2_2
\]

\[
= \arg\min_\theta \sum_{j=1}^{m} \frac{p_j}{2n_j} \left\| (I_n - \frac{\alpha}{n_j} X_j X_j^T) (X_j^i \theta - y_j) \right\|^2_2
\]

\[
= \left( \sum_{j=1}^{m} \frac{p_j}{n_j} X_j^T W_j^2 X_j \right)^{-1} \left( \sum_{j=1}^{m} \frac{p_j}{n_j} X_j^T W_j^2 y_j \right)
\]

where $W_j := I - \frac{\alpha}{n_j} X_j X_j^T$ and

\[
\hat{\theta}_i^M(\alpha) := \arg\min_\theta \left\| \hat{\theta}_0^M(\alpha) - \theta \right\|_2 \quad \text{s.t.} \quad X_i \theta = y_i
\]

\[
= (I - \hat{\Sigma}_i \hat{\Sigma}_i) \hat{\theta}_0^M(\alpha) + \hat{\Sigma}_i \hat{\Sigma}_i \theta^* + \frac{1}{n_i} \hat{\Sigma}_i \xi_i
\]

We now calculate the risk by splitting it into two parts as in (3), and then calculate the asymptotic bias and variance.

Bias:

\[
B(\hat{\theta}_i^M(\alpha)|X) := \left\| \mathbb{E}[\hat{\theta}_i^M(\alpha)|X] - \theta_i^* \right\|_{\hat{\Sigma}_i}^2
\]

\[
= \left\| \Pi_i \left[ \sum_{j=1}^{m} \frac{p_j}{n_j} X_j^T W_j^2 X_j \right]^{-1} \sum_{j=1}^{m} \frac{p_j}{n_j} X_j^T W_j^2 (\theta_j^* - \theta_j^*) \right\|_{\hat{\Sigma}_i}^2
\]

\[
= \left\| \Pi_i \left[ \theta_i^* - \theta_i^* + \left( \sum_{j=1}^{m} \frac{p_j}{n_j} X_j^T W_j^2 X_j \right) \sum_{j=1}^{m} \frac{p_j}{n_j} X_j^T W_j^2 X_j (\theta_j^* - \theta_j^*) \right] \right\|_{\hat{\Sigma}_i}^2
\]
For simplicity, we let $\Delta_j := \theta_j^* - \theta_0^*$, and we define the events:

$$
B_t := \left\{ \left\| \left( \sum_{j=1}^{m} \frac{p_j}{n_j} X_j^T W_j X_j \right)^{-1} - \mathbb{E} \left[ \sum_{j=1}^{m} \frac{1}{n_j} X_j^T W_j X_j \right] \right\|_{op} > t \right\} \quad (21)
$$

$$
A_t := \left\{ \left\| \left( \sum_{j=1}^{m} \frac{p_j}{n_j} X_j^T W_j X_j \right)^{-1} \sum_{j=1}^{m} \frac{p_j}{n_j} X_j^T W_j^2 X_j \Delta_j \right\|_{\Sigma_i} > t \right\} \quad (22)
$$

The proof proceeds in the following steps:

**Bias Proof Outline**

**Step 1.** We first show for any $t > 0$, the $\mathbb{P}(B_t) \to 0$ as $d \to \infty$

**Step 2.** Then, we show for any $t > 0$, the $\mathbb{P}(A_t) \to 0$ as $d \to \infty$

**Step 3.** We show that for any $t \in (0,1]$ on event $A_i^c$, $B(\hat{\theta}_i^M(\alpha) | X) \leq \|\Pi_i [\theta_0^* - \theta_*^*] \|^2_{\Sigma_i} + ct$ and $B(\hat{\theta}_i^M(\alpha) | X) \geq \|\Pi_i [\theta_0^* - \theta_*^*] \|^2_{\Sigma_i} - ct$

**Step 4.** Show that $\lim_{d \to \infty} \mathbb{P}(\|B(\hat{\theta}_i^M(\alpha) | X) - \|\Pi_i [\theta_0^* - \theta_*^*] \|^2_{\Sigma_i} | \leq \varepsilon) = 1$

**Step 5.** Finally, using the asymptotic limit of $\|\Pi_i [\theta_0^* - \theta_*^*] \|^2_{\Sigma_i}$ from Theorem 1 of [HMRT19], we get the result.

We now give the detailed proof:

**Step 1** Since $\lambda_{\min} \left( \mathbb{E} \left[ \frac{1}{n_j} X_j^T W_j^2 X_j \right] \right) \geq \lambda_0$, it suffices to show by Lemma 6.4 that the probability of

$$
C_t := \left\{ \left\| \sum_{j=1}^{m} p_j \left( \frac{1}{n_j} X_j^T W_j^2 X_j - \mathbb{E} \left[ 1 \right. \frac{1}{n_j} X_j^T W_j^2 X_j \right] \right\|_{op} > t \right\}
$$

goes to 0 as $d,m \to \infty$ under Assumption A1.

$$
\mathbb{P}(C_t) = \mathbb{P} \left( \left\| \sum_{j=1}^{m} p_j \left( \hat{\Sigma}_j - 2\alpha^2 \hat{\Sigma}_j^2 + \alpha^2 \hat{\Sigma}_j^3 \right) - \mathbb{E} \left[ \frac{1}{n_j} X_j^T W_j^2 X_j \right] \right\|_{op} > t \right) \\
\leq \mathbb{P} \left( \left\| \sum_{j=1}^{m} p_j \left( \hat{\Sigma}_j - \mu_{\Sigma,j} \right) \right\|_{op} > t/3 \right) \\
+ \mathbb{P} \left( 2\alpha \sum_{j=1}^{m} p_j \left( \hat{\Sigma}_j^2 - \mu_{\Sigma,j}^2 \right) \right|_{op} > t/3 \right) + \mathbb{P} \left( \left\| \alpha^2 \sum_{j=1}^{m} p_j \left( \hat{\Sigma}_j^3 - \mu_{\Sigma,j}^3 \right) \right\|_{op} > t/3 \right), \quad (23)
$$

where $\mu_{p,j} := \mathbb{E}[\hat{\Sigma}_j^p]$. We repeatedly apply Lemma 6.3 for $p = 1,2,3$ to bound each of these three terms. It is clear that if $\mathbb{E} \|X_k \|^2 \leq \mathbb{E} \|\Sigma_k\|_{op} \leq C_3$ for some constant $C_3$, $(\log d)^{4q} \sum_{j=1}^{m} p_j n_j \to 0$, and $(\log d)^{q/2} \sum_{j=1}^{m} p_j^{q/2+1} \to 0$, then (23) goes to 0.

**Step 2**

$$
\mathbb{P}(A_t) \leq \mathbb{P}(A_t \cap B_t^c) + \mathbb{P}(B_t^c) \\
\leq \mathbb{P} \left( M \left( c_1 + \frac{1}{\lambda_0} \right) \left\| \sum_{j=1}^{m} \frac{p_j}{n_j} X_j^T W_j^2 X_j \Delta_j \right\|_2 > t \right) + \mathbb{P}(B_t^c)
$$
From Step 1, we know that \( \lim_{n \to \infty} P(B_{c_i}) = 0 \). The second inequality comes from the fact that \( \|Ax\|_2 \leq \|A\|_{op} \|x\|_2 \). To handle the first term, we use Markov’s inequality.

\[
P \left( c_2 \left\| \sum_{j=1}^{m} \frac{p_j}{n_j} X_j^T W_j^2 X_j \Delta_j \right\|_2 > t \right) \leq \frac{(2c_2 \sqrt{q})^q}{t^q} \frac{m}{2^q} \sum_{j=1}^{m} \left( \frac{p_j}{n_j} \right) \left( \frac{1}{n_j} \right) \left\| X_j^T W_j^2 X_j \Delta_j \right\|_2^{q/2} \]

\[
\leq \frac{(2c_2 \sqrt{q})^q}{t^q} \frac{m}{2^q} \sum_{j=1}^{m} \left( \frac{p_j}{n_j} \right) \left( \frac{1}{n_j} \right) \left\| X_j^T W_j^2 X_j \Delta_j \right\|_2^{q/2} \]

\[
= \frac{(2c_2 \sqrt{q})^q}{t^q} \frac{m}{2^q} \sum_{j=1}^{m} \left( \frac{p_j}{n_j} \right) \left( \frac{1}{n_j} \right) \left\| X_j^T W_j^2 X_j \Delta_j \right\|_2^{q/2} \]

\[
\leq \frac{(8c_2 \sqrt{q})^q}{2t^q} \frac{m}{2^q} \sum_{j=1}^{m} \left( \frac{p_j}{n_j} \right) \left( \frac{1}{n_j} \right) \left\| X_j^T W_j^2 X_j \Delta_j \right\|_2^{q/2} \]

where the last step follows from Lemma 6.2 and the final expression goes to 0 since \((\log d)^{3q} \sum_{j=1}^{m} p_j^{q/2+1} n_j r_j^q \to 0\).

**Step 3, 4 and 5** are same as the bias calculation of proof of Theorem 1.

**Variance:**

We let \( \xi_i = [\xi_{i,1}, \ldots, \xi_{i,n}] \) denote the vector of noise.

\[
V_i(\hat{\theta}_i^M(\alpha); \theta_i^*|X) = \text{tr}(\text{Cov}(\hat{\theta}_i^M(\alpha)|X) \Sigma) = \mathbb{E} \left[ \left\| \hat{\theta}_i^M(\alpha) - \mathbb{E}[\hat{\theta}_i^M(\alpha)|X] \right\|_{\Sigma_i}^2 |X \right] \\
= \mathbb{E} \left[ \left\| \Pi_i \left[ \sum_{j=1}^{m} \frac{p_j}{n_j} X_j^T W_j^2 X_j \right]^{-1} \sum_{j=1}^{m} \frac{p_j}{n_j} X_j^T W_j^2 \xi_j \right\|_{\Sigma_i}^2 |X \right] \\
= \sum_{j=1}^{m} \frac{p_j^2}{n_j^2} \text{tr} \left[ \Pi_i \Sigma_i \Pi_i \left[ \sum_{j=1}^{m} \frac{p_j}{n_j} X_j^T W_j^2 X_j \right]^{-1} X_j^T W_j^2 X_j \left( \sum_{j=1}^{m} \frac{p_j}{n_j} X_j^T W_j^2 X_j \right)^{-1} \right] \sigma_i^2 + \\
\sum_{j=1}^{m} \frac{p_j^2}{n_j^2} \text{tr} \left[ \Pi_i \Sigma_i \Pi_i \left[ \sum_{j=1}^{m} \frac{p_j}{n_j} X_j^T W_j^2 X_j \right]^{-1} X_j^T W_j^2 X_j \left( \sum_{j=1}^{m} \frac{p_j}{n_j} X_j^T W_j^2 X_j \right)^{-1} \right] \sigma_i^2 + \frac{1}{n_i^2} \text{tr}(\Sigma_i \Pi_i \Sigma_i \Pi_i \Sigma_i \Pi_i) \sigma_i^2 \\
\]

We now study the asymptotic behavior of each of the terms \((i), (ii)\) and \((iii)\) separately.
We have

Using Lemma 6.2 and Markov’s inequality, we have

Finally, since we know that \( \sum_{j=1}^{m} p_j^{q+1} (\log d) 5q n_j \rightarrow 0 \), we have \( \mathbb{P} \left( \sum_{j=1}^{m} p_j^2 \left\| \hat{\Sigma}_j (I - \alpha \hat{\Sigma}_j) \right\|_{op} > t \right) \rightarrow 0.

(ii) Using the Cauchy Schwartz inequality on Schatten \( p \)-norms and using the fact that the nuclear norm of a projection matrix is \( d - n \), we get

\[
\sum_{j=1}^{m} p_j^2 \sigma_j^2 \frac{1}{n_j} \text{tr} \left( \Pi_i \Sigma_i \left( \sum_{j=1}^{m} \frac{p_j}{n_j} X_j^T W_j^2 X_j \right) \right) \leq \sum_{j=1}^{m} p_j^2 \left\| \Pi \Sigma \right\|_{op} \left\| \sum_{j=1}^{m} \frac{p_j}{n_j} X_j^T W_j^2 X_j \right\|_{op} \leq C_7 \sigma_{\text{max}} \gamma_{\text{max}} \left( \sum_{j=1}^{m} p_j^2 \left\| \hat{\Sigma}_j (I - \alpha \hat{\Sigma}_j) \right\|_{op} \right),
\]

where the last inequality holds with probability going to 1 for some constant \( C_7 \) because \( \mathbb{P}(C_t) \rightarrow 0 \).

Lastly, we show that \( \mathbb{P} \left( \sum_{j=1}^{m} p_j^2 \left\| \hat{\Sigma}_j (I - \alpha \hat{\Sigma}_j) \right\|_{op} > t \right) \rightarrow 0 \). Using Markov’s and Jensen’s inequality, we have

\[
\mathbb{P} \left( \sum_{j=1}^{m} p_j^2 \left\| \hat{\Sigma}_j (I - \alpha \hat{\Sigma}_j) \right\|_{op} > t \right) \leq \frac{E \left( \sum_{j=1}^{m} p_j^2 \left\| \hat{\Sigma}_j (I - \alpha \hat{\Sigma}_j) \right\|_{op}^q \right)}{t^q} \leq \frac{\sum_{j=1}^{m} p_j^{q+1} E \left( \left\| \hat{\Sigma}_j (I - \alpha \hat{\Sigma}_j) \right\|_{op}^q \right)}{t^q} \leq \frac{\sum_{j=1}^{m} p_j^{q+1} E \left( \left\| \hat{\Sigma}_j \right\|_{op} \left( \left\| I \right\|_{op} + \alpha \left\| \hat{\Sigma}_j \right\|_{op} \right)^{4q} \right)}{t^q} \leq \frac{\sum_{j=1}^{m} p_j^{q+1} E \left( \left\| \hat{\Sigma}_j \right\|_{op} + \alpha \left\| \hat{\Sigma}_j \right\|_{op}^{5q} \right)}{t^q} \leq 2^{4q-1} \frac{\sum_{j=1}^{m} p_j^{q+1} E \left( \left\| \hat{\Sigma}_j \right\|_{op} + \alpha \left\| \hat{\Sigma}_j \right\|_{op}^{5q} \right)}{t^q}.
\]

Using Lemma 6.2 and Markov’s inequality, we have

\[
\mathbb{P} \left( \sum_{j=1}^{m} p_j^2 \left\| \hat{\Sigma}_j \right\|_{op} > t \right) \leq 2^{4q-1} \left( K_1 \frac{\sum_{j=1}^{m} p_j^{q+1} (e \log d) n_j}{t^q} + K_2 \alpha^4 \frac{\sum_{j=1}^{m} p_j^{q+1} (e \log d)^5 q n_j}{t^q} \right) .
\]

Finally, since we know that \( \sum_{j=1}^{m} p_j^{q+1} (\log d)^{5q} n_j \rightarrow 0 \), we have \( \mathbb{P} \left( \sum_{j=1}^{m} p_j^2 \left\| \hat{\Sigma}_j (I - \alpha \hat{\Sigma}_j) \right\|_{op} > t \right) \rightarrow 0.\)
The solution to this minimization problem in (14) is given by

\[ 2 \text{tr} \left( \hat{\Sigma}_i X_i^T W_i^2 X_i \left( \sum_{j=1}^m \frac{p_j}{n_j} X_j^T W_j^2 X_j \right)^{-1} \Pi_i \Sigma_i \right) \sigma_i^2 \]

\[ = 2 \text{tr} \left( \Pi_i \Sigma_i \hat{\Sigma}_i (I - \hat{\Sigma}_i)^2 \left( \sum_{j=1}^m \frac{p_j}{n_j} X_j^T W_j^2 X_j \right)^{-1} \right) \sigma_i^2 \]

\[ \leq \frac{2p_i \sigma_i^2}{n_i} \| \Pi_i \| \| \Sigma_i \|_{op} \| \hat{\Sigma}_i \|_{op} \| (I - \hat{\Sigma}_i)^2 \|_{op} \left( \sum_{j=1}^m \frac{p_j}{n_j} X_j^T W_j^2 X_j \right)^{-1} \|_{op} \]

\[ \leq C_4 p_i, \]

where the last inequality holds with probability going to 1 for some constant \( C_4 \) because \( P(B_i) \to 0 \) and using Assumption A2. Since \( p_i \to 0 \), we have

\[ 2 \text{tr} \left( \hat{\Sigma}_i X_i^T W_i^2 X_i \left( \sum_{j=1}^m \frac{p_j}{n_j} X_j^T W_j^2 X_j \right)^{-1} \Pi_i \Sigma_i \right) \sigma_i^2 \to 0 \]

(iii)

\[ \frac{1}{n_i^2} \text{tr}(\hat{\Sigma}_i^j X_i^T X_i \hat{\Sigma}_i^j \Sigma_i) \sigma_i^2 = \frac{1}{n_i} \text{tr}(\hat{\Sigma}_i^j \Sigma_i) \sigma_i^2 \]

Using Theorem 3 of [HMRT19], as \( d \to \infty \), such that \( \frac{d}{n_i} \to \gamma_i > 1 \), we know that the limit of \( \frac{n_i^2}{n_i^2} \text{tr}(\hat{\Sigma}_i^j \Sigma_i) \) is given by (16) with \( \gamma = \gamma_i \) and \( \hat{H}_n, \hat{G}_n \) be the empirical spectral distribution and weighted empirical spectral distribution of \( \Sigma_i \) respectively.

In the case when \( \Sigma_i = I \), using Theorem 1 of [HMRT19] we have \( V_i(\hat{\theta}_i^M(\alpha)|X) = \frac{2\sigma_i^2}{n_i} \text{tr}(\hat{\Sigma}_i^j) \to \frac{\sigma_i^2}{\gamma_i-1} \).

### 6.7 Proof of Theorem 4

The solution to this minimization problem in (14) is given by

\[ \hat{\theta}_i^P(\lambda) = \theta_0^* + Q^{-1} \left( \sum_{j=1}^m p_j T_j^{-1} \hat{\Sigma}_j \Delta_j + \sum_{j=1}^m p_j T_j^{-1} \frac{1}{n_j} X_j^T \xi_j \right), \]

where \( \Delta_j = \theta_j^* - \theta_0^* \), \( T_j = \hat{\Sigma}_j + \lambda I \) and \( Q = I - \lambda \sum_{j=1}^m p_j T_j^{-1} \). The personalized solutions are then given by

\[ \hat{\theta}_i^P(\lambda) = T_i^{-1} \left( \lambda \hat{\theta}_i^P(\lambda) + \hat{\Sigma}_i \theta_i^* + \frac{1}{n_i} X_i^T \xi_i \right) \]

We now calculate the risk by splitting it into two parts as in (3), and then calculate the asymptotic bias and variance.

**Bias:**

Let \( \Delta_j := \theta_j^* - \theta_0^* \), then we have

\[ B(\hat{\theta}_i^P(\lambda)|X) := \left\| T_i^{-1} \left( \lambda \theta_0^* - \lambda \theta_i^* + \lambda Q^{-1} \left( \sum_{j=1}^m p_j T_j^{-1} \hat{\Sigma}_j \Delta_j \right) \right) \right\|_{\Sigma_i}^2 \]
The idea is to show that the second term goes to 0 and use results from [HMRT19] to find the asymptotic bias. To do this, we first define the events:

\[
C_t := \left\{ \left\| \sum_{j=1}^{m} p_j(T_j^{-1} - \mathbb{E}[T_j^{-1}]) \right\|_{op} > t \right\}
\]

\[
A_t := \left\{ \left| Q^{-1} \sum_{j=1}^{m} p_j T_j^{-1} \hat{\Sigma}_j \Delta_j \right| > t \right\}
\]

The proof proceeds in the following steps:

**Bias Proof Outline**

**Step 1.** We first show for any \( t > 0 \), the \( \mathbb{P}(C_t) \to 0 \) as \( d \to \infty \).

**Step 2.** Then, we show for any \( t > 0 \), the \( \mathbb{P}(A_t) \to 0 \) as \( d \to \infty \).

**Step 3.** We show that for any \( t \in (0, 1] \), \( B(\hat{\theta}_i^P(\lambda)|X) \leq \left\| T_i^{-1}[\lambda \theta_0^* - \lambda \theta_i^*] \right\|_2^2 + ct \) and \( B(\theta, X) \geq \left\| T_i^{-1}[\lambda \theta_0^* - \lambda \theta_i^*] \right\|_2^2 - ct \).

**Step 4.** Show that \( \lim_{d \to \infty} \mathbb{P}(|B(\hat{\theta}_i^P(\lambda)|X) - \left\| T_i^{-1}[\lambda \theta_0^* - \lambda \theta_i^*] \right\|_2^2 | \leq \varepsilon) = 1 \).

**Step 5.** Finally, using the asymptotic limit of \( \left\| T_i^{-1}[\lambda \theta_0^* - \lambda \theta_i^*] \right\|_2^2 \) from Corollary 5 of [HMRT19], we get the result.

We now give the detailed proof:

**Step 1**

\[
\mathbb{P}(C_t) = \mathbb{P}\left( \left\| \sum_{j=1}^{m} p_j(T_j^{-1} - \mathbb{E}[T_j^{-1}]) \right\|_{op} > t \right) \leq \frac{2^q \mathbb{E}\left[ \left\| \sum_{j=1}^{m} \xi_j p_j T_j^{-1} \right\|_{op}^q \right]}{t^q},
\]

We use Theorem A.1 from [CGT12] to bound this object.

\[
\mathbb{E}\left[ \left\| \sum_{j=1}^{m} \xi_j T_j^{-1} \right\|_{op}^q \right] \leq \left[ e \log d \left( \left\| \sum_{j=1}^{m} p_j^2 \mathbb{E}[T_j^{-1}] \right\|_{op}^{1/2} \right)^2 + (e \log d)(\max_j \left\| p_j T_j^{-1} \right\|_{op}^{1/2}) \right]^q.
\]

\[
\leq 2^{q-1} \sqrt{e \log d^q \left( \sum_{j=1}^{m} \mathbb{E}[T_j^{-1}] \right) \left( \sum_{j=1}^{m} p_j^2 \mathbb{E}[T_j^{-1}] \right)^{1/2}} + (e \log d)^q (\max_j \left\| p_j T_j^{-1} \right\|_{op}^{1/2}) \]

\[
\leq 2^{q-1} \left( e \log d^q \left( \sum_{j=1}^{m} \mathbb{E}[T_j^{-1}] \right) \left\| p_j \right\|_{op}^{q/2} + (e \log d)^q \max_j \left\| p_j \right\|_{op}^q \right) \]

\[
\leq 2^{q-1} \left( e \log d^q \sum_{j=1}^{m} \mathbb{E}[T_j^{-1}] \right)^{q/2} \left\| \mathbb{E}[T_j^{-1}] \right\|_{op}^{q/2} + \frac{1}{\lambda^q} (e \log d)^q \sum_{j=1}^{m} p_j^{q/2} \]

\[
\leq 2^{q-1} \frac{1}{\lambda^q} \left( e \log d \right)^{q/2} \sum_{j=1}^{m} p_j^{q/2+1} + (e \log d)^q \sum_{j=1}^{m} p_j^q \right).
\]
where we use the fact that $\|T_j^{-1}\|_{op} = \| (\hat{\Sigma}_j + \lambda I)^{-1} \|_{op} \leq \frac{1}{\lambda}$ since $\hat{\Sigma}_j$ is always positive semidefinite.

Since $(\log d)^{q/2} \sum_{j=1}^m p_j^{q/2+1}$ and $(\log d)^q \sum_{j=1}^m p_j^q$, we get that $\mathbb{P}(C_t) \to 0$ for all $t > 0$.

**Step 2** To prove this step, we will first use a helpful lemma,

**Lemma 6.5.** Suppose that $\Sigma = \mathbb{E}[\hat{\Sigma}] \in \mathbb{R}^{d \times d}$ has a spectrum supported on $[a, b]$ where $0 < a < b < \infty$. Further suppose that $\mathbb{E}\left[\|\hat{\Sigma}\|_{op}^2\right] \leq \tau$ and there exists an $R \geq b$ such that $\mathbb{P}(\lambda_{\text{max}}(\hat{\Sigma}) > R) \leq \frac{a^2}{8\tau}$, then

$$\mathbb{E}\left[\|\hat{\Sigma} + \lambda I\|_{op}^{-1}\right]_{op} \leq \frac{1}{\lambda} \left(1 - \frac{a^3}{16\tau(R + \lambda)}\right) \leq \frac{1}{\lambda}$$

**Proof** Fix an arbitrary vector $u \in \mathbb{R}^d$ with unit $\ell_2$ norm. We fix $\delta = a/2 > 0$, we define the event $A := \{u^T \hat{\Sigma} u \geq \delta\}$ and $B := \{\lambda_{\text{max}}(\hat{\Sigma}) \leq R\}$

$$u^T \mathbb{E}[\hat{\Sigma} + \lambda I]^{-1} u \leq \mathbb{E}\{1 \{A \cap B\} u^T (\hat{\Sigma} + \lambda I)^{-1} u\} + \frac{1}{\lambda} (1 - \mathbb{P}(A \cap B))$$

Let $\sigma^2_i$ and $v_i$ denote the $i$th eigenvalue and eigenvector of $\hat{\Sigma}$ respectively sorted in descending order with respect to eigenvalue ($\sigma^2_1 \geq \sigma^2_2 \geq \ldots \geq \sigma^2_d$). On the event $A$, we have that $u^T (\hat{\Sigma} + \lambda I)^{-1} u$ has value no larger than

$$\max_{\alpha \in \mathbb{R}^d} \sum_{i=1}^d \frac{1}{\sigma_i^2 + \lambda} \alpha_i$$

s.t. $\alpha \geq 0$

$$1^T \alpha = 1$$

$$\sum_{i=1}^d \sigma_i^2 \alpha_i \geq \delta$$

The dual of this problem is

$$\min_{\theta} \max_{\beta \in [0]} \left\{\theta \sigma^2_j + \frac{1}{\sigma^2_j + \lambda} \right\} - \theta \delta$$

s.t. $\theta \geq 0$

It suffices to demonstrate that there exists a $\theta$ which satisfies the constraints of the dual and has objective value less than $\frac{1}{\lambda}$. We can verify that selecting $\theta = \frac{1}{\lambda(\sigma^2_1 + \lambda)}$ has an objective value of

$$\frac{1}{\lambda} - \frac{\delta}{\lambda(\sigma^2_1 + \lambda)}$$

which is less than the desired $\frac{1}{\lambda}$. All that remains is to lower bound $\mathbb{P}(A \cap B) \geq \mathbb{P}(A) - \mathbb{P}(B^c)$. We know by Paley-Zygmund

$$\mathbb{P}(A) \geq \mathbb{P}\left(u^T \hat{\Sigma} u \geq \frac{\delta}{a} u^T \Sigma u\right) \geq \mathbb{P}\left(u^T \hat{\Sigma} u \geq \frac{1}{2} u^T \Sigma u\right) \geq \frac{(u^T \Sigma u)^2}{4\mathbb{E}[(u^T \Sigma u)^2]} \geq \frac{a^2}{4\tau}$$

Note that $a^2/4\tau < 1$ because the second moment of a random variable is no smaller than the first moment squared of the random variable. Moreover, by construction, $R$ is large enough such that $\mathbb{P}(B^c) \leq \mathbb{P}(A)/2$, thus,

$$u^T \mathbb{E}[\hat{\Sigma} + \lambda I]^{-1} u \leq \frac{1}{\lambda} \left(1 - \frac{a}{2(R + \lambda)}\right) \frac{a^2}{8\tau} + \frac{1}{\lambda} \left(1 - \frac{a^2}{8\tau}\right)$$

$$= \frac{1}{\lambda} \left(1 - \frac{a^3}{16\tau(R + \lambda)}\right)$$

31
Recall that we have the assumptions that for sufficiently large \( m \), for all \( j \in [m] \) we have \( \Sigma_j \) has a spectrum supported on \([a, b] \) where \( a = 1/M \) and \( b = M \) and \( \mathbb{E} \left[ \left\| \Sigma_j^2 \right\|_{op} \right] \leq \tau_3 \). Moreover, since we have the assumption that there exists an \( R \geq b \) such that \( \limsup_{m \to \infty} \sup_{j \in [m]} P_{\lambda}(\lambda_{max}(\Sigma_j) > R) \leq \frac{\sigma^2}{16 \log_3} \), by Lemma 6.5 there exists and \( 1 > \varepsilon > 0 \) such that for sufficiently large \( m \), for all \( j \in [m] \), \( \left\| \mathbb{E}[(\Sigma_j + \lambda I)^{-1}] \right\|_{op} \leq \frac{1-\varepsilon}{\lambda} \).

\[
P(A_t) \leq P(A_t \cap C_{c_1}^c) + P(C_{c_1})
\]

Since we know \( P(C_{c_1}) \to 0 \), it suffices to bound the first term.

\[
P(A_t \cap C_{c_1}^c) \leq \mathbb{P} \left( \sqrt{M} \left\| Q^{-1} \right\|_{op} \left\| \sum_{j=1}^{m} p_j T_j^{-1} \hat{\Sigma}_j \Delta_j \right\|_2 > t \cap C_{c_1}^c \right) = \mathbb{P} \left( \sqrt{M} \left( 1 - \left\| \lambda \sum_{j=1}^{m} p_j T_j^{-1} \right\|_{op} \right)^{-1} \left\| \sum_{j=1}^{m} p_j T_j^{-1} \hat{\Sigma}_j \Delta_j \right\|_2 > t \cap C_{c_1}^c \right)
\]

\[
\leq \mathbb{P} \left( \sqrt{M} \left( 1 - \left\| \lambda \sum_{j=1}^{m} p_j \| E[T_j^{-1}] \|_{op} - c_1 \right\|_{op} \right)^{-1} \left\| \sum_{j=1}^{m} p_j T_j^{-1} \hat{\Sigma}_j \Delta_j \right\|_2 > t \right)
\]

where we used Jensen’s inequality in the last step. \( E_{c_1} \) is a matrix error term which on the event \( C_{c_1}^c \) has operator norm bounded by \( c_1 \). As discussed, we have that \( \sum_{j=1}^{m} p_j \| E[T_j^{-1}] \|_{op} \) is less than \( \frac{1-\varepsilon}{\lambda} \), which shows there exists a constant \( c_2 \), such that \( \sqrt{M} (1 - \lambda \sum_{j=1}^{m} p_j \| E[T_j^{-1}] \|_{op} - c_1)^{-1} \leq 2 \). Now, we have, using Lemma 6.1,

\[
P \left( c_2 \left\| \sum_{j=1}^{m} p_j T_j^{-1} \hat{\Sigma}_j \Delta_j \right\|_2 > t \right) \leq \frac{\mathbb{E} \left[ \sum_{j=1}^{m} p_j T_j^{-1} \hat{\Sigma}_j \Delta_j \right]^{q/2}}{t/q} \leq \frac{(2c_2 \sqrt{q})^q \mathbb{E} \left[ \sum_{j=1}^{m} p_j T_j^{-1} \hat{\Sigma}_j \Delta_j \right]^{q/2}}{t/q}
\]

Using Jensen’s inequality and the definition of operator norm, we have

\[
\frac{(2c_2 \sqrt{q})^q \mathbb{E} \left[ \sum_{j=1}^{m} p_j T_j^{-1} \hat{\Sigma}_j \Delta_j \right]^{q/2}}{t/q} = \frac{(2c_2 \sqrt{q})^q \mathbb{E} \left[ \sum_{j=1}^{m} p_j T_j^{-1} \hat{\Sigma}_j \Delta_j \right]^{q/2}}{t/q} \leq \frac{(2c_2 \sqrt{q})^q \sum_{j=1}^{m} p_j^{q/2+1} \mathbb{E} \left[ \left\| T_j^{-1} \hat{\Sigma}_j \Delta_j \right\| \right]^{q}}{t/q}
\]

Lastly, we can bound this using Lemma 6.2 as follows and using the fact that \( \| T_j^{-1} \|_{op} \leq \frac{1}{\lambda} \).

\[
P \left( c_2 \left\| \sum_{j=1}^{m} p_j T_j^{-1} \hat{\Sigma}_j \Delta_j \right\|_2 > t \right) \leq \frac{(2c_2 \sqrt{q})^q \sum_{j=1}^{m} p_j^{q/2+1} Kn_j (\epsilon \log d)^q \delta_j^q}{\lambda^q t q} \to 0,
\]

32
using $(\log d)^a \sum_{j=1}^m n_j p_j^{q/2+1} \rightarrow 0$

**Step 3** For any $t \in (0,1]$, on the event $A_i^c$, we have that

$$B(\hat{\theta}_i^P(\lambda)|X) = \|T_i^{-1}[\lambda \theta_0^* - \lambda \theta_i^* + E]\|_{\Sigma_i}^2$$

for some vector $E$ where we know $\|E\|_{\Sigma_i} \leq t$ (which means $\|E\|_2 \leq t\sqrt{M}$).

We can form the bounds

$$\|T_i^{-1}[\lambda \theta_0^* - \lambda \theta_i^* + E]\|_{\Sigma_i}^2 \leq \lambda^2 \|T_i^{-1}[\theta_0^* - \theta_i^*]\|_{\Sigma_i}^2 + \|T_i^{-1}E\|_{\Sigma_i}^2 + 2\lambda \|T_i^{-1}E\|_{\Sigma_i} \|T_i^{-1}[\theta_0^* - \theta_i^*]\|_{\Sigma_i}$$

$$\leq \lambda^2 \|T_i^{-1}[\theta_0^* - \theta_i^*]\|_{\Sigma_i}^2 + \lambda^{-2} t^2 M^2 + 2\lambda^{-1} r_i^2 M^{3/2}$$

$$\|T_i^{-1}[\lambda \theta_0^* - \lambda \theta_i^* + E]\|_{\Sigma_i}^2 \geq \lambda^2 \|T_i^{-1}[\theta_0^* - \theta_i^*]\|_{\Sigma_i}^2 + \|T_i^{-1}E\|_{\Sigma_i}^2 + 2\lambda \|T_i^{-1}E\|_{\Sigma_i} \|T_i^{-1}[\theta_0^* - \theta_i^*]\|_{\Sigma_i}$$

$$\geq \lambda^2 \|T_i^{-1}[\theta_0^* - \theta_i^*]\|_{\Sigma_i}^2 - 2\lambda^{-1} r_i^2 M^{3/2}$$

Since $t \in (0,1]$, we have that $t^2 \leq t$ and thus we can choose $c = \lambda^{-2} M^2 + 2r_i^2 \lambda^{-1} M^{3/2}$

**Step 4** Reparameterizing $\varepsilon := ct$, we have that for any $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} P(|B(\hat{\theta}_i^P(\lambda), X) - \|T_i^{-1}[\theta_0^* - \theta_i^*]\|_{\Sigma_i}^2| \leq \varepsilon) \geq \lim_{n \rightarrow \infty} P(|B(\hat{\theta}_i^P(\lambda), X) - \|T_i^{-1}[\theta_0^* - \theta_i^*]\|_{\Sigma_i}^2| \leq \varepsilon \wedge c)$$

$$\geq \lim_{n \rightarrow \infty} P(A_i^c \wedge 1) = 1$$

**Step 5** Using Theorem 6 of [HMRT19], as $d \rightarrow \infty$, such that $\frac{d}{n_i} \rightarrow \gamma_i > 1$, we know that the limit of $\lambda^2 \|\hat{\Sigma}_i + \lambda I\|^{-1} [\theta_0^* - \theta_i^*]\|_{\Sigma_i}$ is given by (17) with $\gamma = \gamma_i$ and $\hat{H}_n, \hat{G}_n$ be the empirical spectral distribution and weighted empirical spectral distribution of $\Sigma_i$ respectively.

In the case when $\Sigma_i = I$, using Corollary 5 of [HMRT19] we have $B_i(\hat{\theta}_i^P(\lambda), X) = \|\Pi_i[\theta_0^* - \theta_i^*]\|_2^2 \rightarrow r_i^2 \lambda^2 m_i^2(-\lambda)$.

**Variance**

$$\text{Var}(\hat{\theta}_i^P(\lambda)|X) = E\left[\left\|T_i^{-1}\left(\lambda Q^{-1}\left[\sum_{j=1}^m p_j T_j^{-1}\frac{1}{n} X_j^T \xi_j + \frac{1}{n_i} X_i^T \xi_i\right]\right)\right\|_{\Sigma_i}^2\right]$$

$$= \sum_{j=1}^m \frac{\lambda^2 p_j^2}{n_j} \text{tr}\left(T_i^{-1}\Sigma_i T_i^{-1}Q^{-1}T_j \hat{\Sigma}_j T_j Q^{-1}\right) \sigma_j^2$$

$$+ \frac{2\lambda \sigma_i^2 p_i}{n_i} \text{tr}\left(\Sigma_i T_i^{-1}Q^{-1}T_i^{-1}\hat{\Sigma}_i T_i^{-1}\right)$$

$$+ \text{tr}\left(T_i^{-1}\Sigma_i T_i^{-1}\hat{\Sigma}_i\right) \frac{\sigma_i^2}{n_i}$$

We now study the asymptotic behavior of each of the terms $(i),(ii)$ and $(iii)$ separately. In these steps, we will have to bound $\|Q^{-1}\|_{op}$. To do this, we observe that there exists a sufficiently large constant $t$
such that the following statement is true.

\[
\mathbb{P}(\|Q^{-1}\|_{\text{op}} > t) = \mathbb{P}(\|Q^{-1}\|_{\text{op}} > t \land C_{c_1}^c) + \mathbb{P}(C_{c_1})
\]

\[
= \mathbb{P}\left(1 - \left\| \lambda \sum_{j=1}^{m} p_j T_j^{-1}\right\|_{\text{op}}^{-1} > t \land C_{c_1}^c\right) + o(1)
\]

\[
\leq \mathbb{P}\left(1 - \left\| \lambda \sum_{j=1}^{m} p_j E[T_j^{-1}] + E_{c_1}\right\|_{\text{op}}^{-1} > t\right) + o(1)
\]

\[
\leq \mathbb{P}\left(1 - \lambda \sum_{j=1}^{m} p_j \|E[T_j^{-1}]\|_{\text{op}} - c_1\right) > t) + o(1)
\]

\[
\leq o(1).
\]

This is true because of Lemma 6.5.

(i) Using the Cauchy Schwartz inequality on Schatten p–norms and using the high probability bounds from the bias proof, we get that for some constant \(C_p\), the following holds with probability going to 1.

\[
\sum_{j=1}^{m} \frac{\lambda^2 \sigma_j^2}{n_j} \text{tr} (T_i^{-1} \Sigma_i T_i^{-1} Q^{-1} T_j \hat{\Sigma}_j T_j Q^{-1}) \sigma_j^2
\]

\[
\leq \sum_{j=1}^{m} \frac{\sigma_j^2 \lambda^2 \sigma_j^2}{n_j} \|T_i^{-1}\|_1 \|T_j^{-1}\|_{\text{op}} \|\Sigma_i\|_{\text{op}} \|\Sigma_j\|_{\text{op}} \|Q^{-1}\|_{\text{op}} \|T_j\|_{\text{op}} \|\hat{\Sigma}_j\|_{\text{op}} \|\Sigma_j\|_{\text{op}}
\]

\[
\leq \sum_{j=1}^{m} \frac{MC_p \sigma_j^2 \lambda^2 \sigma_j^2 d}{n_j} \|\hat{\Sigma}_j\|_{\text{op}}
\]

\[
\leq \gamma_{\text{max}}MC_p \sigma_j^2 \lambda^2 \sum_{j=1}^{m} \frac{p_j}{n_j} \|\Sigma_j\|_{\text{op}}
\]

\(\|Q^{-1}\|_{\text{op}}\) is upper bounded by some constant as shown above. We use the same technique as in proof of the variance of Theorem 1 calculation from Section 6.4 to show that \(\sum_{j=1}^{m} p_j^2 \|\hat{\Sigma}_j\|_{\text{op}} \xrightarrow{p} 0\).

(ii) Using the Cauchy Schwartz inequality on Schatten p–norms and using the high probability bounds from the bias proof, we get that for some constant \(C_p\), the following holds with probability going to 1.

\[
\frac{2\lambda \sigma_j^2}{n_i} \text{tr} (T_i^{-1} \Sigma_i T_i^{-1} \hat{\Sigma}_i T_i^{-1}) \leq \frac{2\lambda \sigma_j^2}{n_i} \|T_i^{-1}\|_1 \|T_i^{-1}\|_{\text{op}} \|\Sigma_i\|_{\text{op}} \|\Sigma_i\|_{\text{op}} \|T_i^{-1}\|_{\text{op}} \|\hat{\Sigma}_i\|_{\text{op}}
\]

\[
\leq \frac{MC_p \sigma_j^2 d}{\lambda n_i}.
\]

\(\|Q^{-1}\|_{\text{op}}\) is upper bounded by some constant as shown above. Moreover, since \(p_i \to 0\), we have \(\frac{2\lambda \sigma_j^2}{n_i} \text{tr} (T_i^{-1} \Sigma_i T_i^{-1} \hat{\Sigma}_i T_i^{-1}) \xrightarrow{p} 0\).

(iii) Using Theorem 3 of [HMRT19], as \(d \to \infty\), such that \(\frac{d}{n_i} \to \gamma_i > 1\), we know that the limit of \(\text{tr}((\hat{\Sigma}_i + \lambda I)^{-2} \Sigma_i \Sigma_i)^{\sigma_i^2}_{n_i}\) is given by (18) with \(\gamma = \gamma_i\) and \(\hat{H}_n, \hat{G}_n\) be the empirical spectral distribution and weighted empirical spectral distribution of \(\Sigma_i\) respectively.

In the case when \(\Sigma_i = I\), using Theorem 1 of [HMRT19] we have \(V_i(\hat{\theta}^p(\lambda)|X) = \frac{\sigma_i^2}{n_i} \text{tr}((\hat{\Sigma}_i + \lambda I)^{-2} \Sigma_i)^{\sigma_i^2}_{n_i} \xrightarrow{p} \frac{\sigma_i^2}{\gamma_i - 1}\).

34
6.8 Proof of Corollary 3.1

We first prove that \( \rho_i \geq r_i \). If we let \( \omega \in \Omega \) be the probability space associated with \( \theta^*_\omega \), we claim that \( \langle \theta^*_\omega \rangle \neq \theta^*_0 \) for all (not just a.s.) \( \omega \in \Omega \). For the sake of contradiction, suppose that there exists \( \omega' \) such that \( \langle \theta^*_\omega \rangle \neq \theta^*_0 \) for all \( \omega \). The variance proof of Theorem 1, again treating \( \Pi_i \), we know that this quantity converges in probability to \( r^2_i \). To show the result for \( \hat{\theta}^F \), recall eq. (19). The bias associated with this estimator is

\[
B_i(\hat{\theta}^F_i) = \left\| \theta^*_\omega - \theta^*_0 \right\|^2 + \left\| \theta^*_0 \right\|^2 + 2\langle \theta^*_\omega, \theta^*_0 \rangle = \left\| \theta^*_\omega \right\|^2 = \rho_i^2 = \left\| \theta^*_\omega \right\|^2
\]

In looking at the first and last term, we see that this implies \( \langle \theta^*_\omega \rangle \neq \theta^*_0 \), which is a contradiction. Consider the other possibility that for all \( \omega' \in \Omega \), \( \langle \theta^*_\omega \rangle \neq \theta^*_0 \). This would imply that \( E[(\theta^*_\omega - \theta^*_0)] > 0 \) which is a contradiction, since \( E[\theta^*_\omega] = \theta^*_0 \). Now since \( \langle \theta^*_\omega, \theta^*_0 \rangle = 0 \), we have that \( \rho_i^2 = r^2_i + \| \theta^*_0 \|^2 \geq r^2_i \).

To show the result for \( \hat{\theta}^F \), recall eq. (19). The bias associated with this estimator is

\[
B_i(\hat{\theta}^F_i) = \left\| \theta^*_\omega - \theta^*_0 \right\|^2 + \left\| \theta^*_0 \right\|^2 + 2\langle \theta^*_\omega, \theta^*_0 \rangle = \left\| \theta^*_\omega \right\|^2 = \rho_i^2 = \left\| \theta^*_\omega \right\|^2
\]

Using the bias proof of Theorem 1, treating \( \Pi_i \) as the identity, we know that this quantity converges in probability to \( r^2_i \). Showing the variance of \( \hat{\theta}^F \) of goes to 0 follows directly from part (i) of the variance proof of Theorem 1, again treating \( \Pi_i \) as the identity.

The result regarding the estimator \( \hat{\theta}^N \) is a direct consequence of Theorem 1 from [HMRT19]. The result regarding the estimator \( \hat{\theta}^N_i = \hat{\theta}^F_i \) is a direct consequence of Corollary 5 from [HMRT19].

6.9 Proof that RTFA has lower risk than FedAvg

We show that RTFA with optimal hyperparameter has lower risk than FedAvg by using the fact that \((1 - 1/\gamma)^2 \leq (1 + 1/\gamma)^2 \) for \( \gamma \geq 1 \) and completing the square:

\[
L_i(\hat{\theta}^F_i; \theta^*_\omega | X) = \frac{1}{2} \left[ r^2_i \left( 1 - \frac{1}{\gamma} \right) - \sigma^2_i + \sqrt{r^4_i \left( 1 - \frac{1}{\gamma} \right)^2 + \sigma^4_i + 2\sigma^2_i r^2_i \left( 1 + \frac{1}{\gamma} \right)} \right]
\]

\[
\leq r^2_i = L_i(\hat{\theta}^F_i; \theta^*_0 | X).
\]
7 Algorithm implementations

In this section, we give all steps of the exact algorithms used to implement all algorithms in the experiments section.

Algorithm 3: Naive local training

**Data:** m: number of users, K: epochs

**Result:** \( \hat{\theta}^N \): Estimator

for \( i \leftarrow 1 \) to \( m \) do
  Each client runs K epochs of SGM with personal stepsize \( \alpha \)
end

Algorithm 4: Federated Averaging [MMR\textsuperscript{+}17]

**Data:** \( R \): Communication Rounds, 
\( D \): Number of users sampled each round,
\( K \): Number of local update steps,
\( \hat{\theta}^{FA}_{0,0} \): Initial iterate for global model

**Result:** \( \hat{\theta}^{FA}_{0,R} \): Estimator

for \( r \leftarrow 0 \) to \( R - 1 \) do
  Server samples a subset of clients \( \mathcal{S}_r \) uniformly at random such that \(|\mathcal{S}_r| = D\)
  Server sends \( \hat{\theta}^{FA}_{0,r} \) to all clients in \( \mathcal{S}_r \)
  for \( i \in \mathcal{S}_r \) do
    Sample a batch \( D^i_k \) of size \( B \) from user \( i \)'s data \( D^i \)
    Compute Stochastic Gradient \( g(\hat{\theta}^{FA}_{i,r+1,k-1}; D^i_k) = \frac{1}{n_i} \sum_{S \in D^i_k} \nabla F(\hat{\theta}^{FA}_{i,r+1,k-1}; S) \)
    Set \( \hat{\theta}^{FA}_{i,r+1,k} \leftarrow \hat{\theta}^{FA}_{i,r+1,k-1} - \eta g(\hat{\theta}^{FA}_{i,r+1,k-1}; D^i_k) \)
  end
  Client \( i \) sends \( \hat{\theta}^{FA}_{i,r+1,K} \) back to the server.
end
  Server updates the central model using \( \hat{\theta}^{FA}_{0,r+1} = \sum_{j=1}^{D} \frac{n_j}{\sum_{j=1}^{D} n_j} \hat{\theta}^{FA}_{i,r+1,K} \).
end
return \( \hat{\theta}^{FA}_{0,R} \)

Algorithm 5: FTFA

**Data:** \( P \): Personalization iterations

**Result:** \( \hat{\theta}^{FA}_{i,P} \): Estimator

Server sends \( \hat{\theta}^{FA}_{0} = \hat{\theta}^{FA}_{0,R} \) (using Algorithm 4 with stepsize \( \eta \)) to all clients
for \( i \leftarrow 1 \) to \( m \) do
  Run \( P \) steps of SGM on \( \hat{L}_i(\cdot) \) using \( \hat{\theta}^{FA}_0 \) as initial point with learning rate \( \alpha \) and output \( \hat{\theta}^{FA}_{i,P} \)
end
return \( \hat{\theta}^{FA}_{i,P} \)
Algorithm 6: RTFA

**Data:** \(P\): Personalization iterations

**Result:** \(\hat{\theta}_{i,P}^R(\lambda)\): Estimator

Server sends \(\hat{\theta}_{0,R}^{FA} = \hat{\theta}_{0,R}^{FA}\) (using Algorithm 4) to all clients

for \(i \leftarrow 1\) to \(m\) do

- Run \(P\) steps of SGM on \(\hat{L}_i(\theta) + \frac{1}{2} \left\| \theta - \hat{\theta}_{0}^{FA} \right\|_2^2\) with learning rate \(\alpha\) and output \(\hat{\theta}_{i,P}^{FA}\)

end

return \(\hat{\theta}_{i,P}^R(\lambda)\)

Algorithm 7: MAML-FL-HF [FMO20]

**Data:** \(R\): Communication Rounds,
\(D\): Number of users sampled each round,
\(K\): Number of local update steps,
\(\hat{\theta}_{0,0}(\alpha)\): Initial iterate for global model

**Result:** \(\{\hat{\theta}_{i}(\alpha)\}_{i=1}^m\): Estimators

Server sends \(\hat{\theta}_{0,r}^{M} = \hat{\theta}_{0,r}^{M}\) to all clients

for \(r \leftarrow 0\) to \(R - 1\) do

- Server samples a subset of clients \(S_r\) uniformly at random such that \(|S_r| = D\)
- Server sends \(\hat{\theta}_{0,r}^{M}\) to all clients in \(S_r\)

for \(i \in S_r\) do

- Set \(\hat{\theta}_{i,r+1,0}(\alpha) \leftarrow \hat{\theta}_{0,r}^{M}(\alpha)\)

  for \(k \leftarrow 1\) to \(K\) do

  - Sample a batch \(\mathcal{D}_k^i\) of size \(B\) from user \(i\)'s data \(\mathcal{D}_i\)
  - Compute Stochastic Gradient \(g(\hat{\theta}_{i,r+1,k-1}^M(\alpha); \mathcal{D}_k^i) = \frac{1}{B} \sum_{S \in \mathcal{D}_k^i} \nabla F(\hat{\theta}_{i,r+1,k-1}^M(\alpha); S)\)
  - Set \(\hat{\theta}_{i,r+1,k}^M(\alpha) \leftarrow \hat{\theta}_{i,r+1,k-1}^M(\alpha) - \alpha g(\hat{\theta}_{i,r+1,k-1}^M(\alpha); \mathcal{D}_k^i)\)

  - Sample batches \(\mathcal{D}_k^{i,j}\) and \(\mathcal{D}_k^{i,j}\) of size \(B\) from user \(i\)'s data \(\mathcal{D}_i\)
  - Compute Stochastic Gradient \(g(\hat{\theta}_{i,r+1,k}^M(\alpha); \mathcal{D}_k^{i,j})\) and hessian \(H(\hat{\theta}_{i,r+1,k-1}^M(\alpha); \mathcal{D}_k^{i,j})\)
  - Set \(\hat{\theta}_{i,r+1,k}^M(\alpha) \leftarrow \hat{\theta}_{i,r+1,k-1}^M(\alpha) - \eta (I - \alpha H(\hat{\theta}_{i,r+1,k-1}^M(\alpha); \mathcal{D}_k^{i,j}))g(\hat{\theta}_{i,r+1,k}^M(\alpha); \mathcal{D}_k^{i,j})\)

end

Client \(i\) sends \(\hat{\theta}_{i,r+1,K}(\alpha)\) back to the server.

end

Server updates the central model using \(\hat{\theta}_{0,r+1}^M(\alpha) = \sum_{j=1}^D \frac{n_j}{n} \hat{\theta}_{i,r+1,K}(\alpha)\).

Server sends \(\hat{\theta}_{0,R}^M(\alpha)\) to all clients

for \(i \leftarrow 1\) to \(m\) do

- Run \(P\) steps of SGM on \(\hat{L}_i(\cdot)\) using \(\hat{\theta}_{0}^M(\alpha)\) as initial point with learning rate \(\alpha\) and output \(\hat{\theta}_{i,P}^M(\alpha)\)

end

return \(\hat{\theta}_{i,P}^M(\alpha)\)
Algorithm 8: pFedMe [DTN20]

**Data:** $R$: Communication Rounds,
$D$: Number of users sampled each round,
$K$: Number of local update steps,
$\theta^{P}_{0,0}(\lambda)$: Initial iterate for global model

**Result:** $\hat{\theta}^P(\lambda)$: Estimator

```plaintext
for $r \leftarrow 0$ to $R - 1$ do
    Server samples a subset of clients $S_r$ uniformly at random such that $|S_r| = D$
    Server sends $\hat{\theta}^P_0(\lambda)$ to all clients in $S_r$
    for $i \in S_r$ do
        Set $\hat{\theta}^P_{i,r+1,0}(\lambda) \leftarrow \hat{\theta}^P_{i,r}(\lambda)$
        for $k \leftarrow 1$ to $K$ do
            Sample a batch $D^i_k$ of size $B$ from user $i$’s data $D_i$
            Compute $\theta_i(\hat{\theta}^P_{i,r+1,k-1}(\lambda)) = \arg\min_\theta \frac{1}{D} \sum_{S \in D^i_k} \nabla F(\theta; S) + \frac{\lambda}{2} \| \theta - \hat{\theta}^P_{i,r+1,k-1}(\lambda) \|^2_2$
            Set $\hat{\theta}^P_{i,r+1,k}(\lambda) \leftarrow \hat{\theta}^P_{i,r+1,k-1}(\lambda) - \eta \lambda (\theta_i(\hat{\theta}^P_{i,r+1,k-1}(\lambda)) - \theta_i(\hat{\theta}^P_{i,r+1,k-1}(\lambda)))$
        end
        Client $i$ sends $\hat{\theta}^P_{i,r+1,K}(\lambda)$ back to the server.
    end
    Server updates the central model using
    $\hat{\theta}^P_{0,r+1}(\lambda) = (1 - \beta)\hat{\theta}^P_{0,r}(\lambda) + \beta \sum_{j=1}^{D} \frac{n_j}{\sum_{j=1}^{D} n_j} \hat{\theta}^P_{i,r+1,K}(\lambda)$.
end
return $\hat{\theta}^P_{0,R}(\lambda)$
```
8 Experimental Details

8.1 Dataset Details

In this section, we provide detailed descriptions on datasets and how they were divided into users. We perform experiments on federated versions of the Shakespeare [MMR+17], CIFAR-100 [KH09], EMNIST [CATvS17], and Stack Overflow [MRR+19] datasets. We download all datasets using FedML APIs [HLS+20] which in turn get their datasets from [MRR+19]. For each dataset, for each client, we divide their data into train, validation and test sets with roughly a 80%, 10%, 10% split. The information regarding the number of users in each dataset, dimension of the model used, and the division of all samples into train, validation and test sets is given in Table 1.

| Dataset          | Users | Dimension | Train | Validation | Test  | Total Samples |
|------------------|-------|-----------|-------|------------|-------|---------------|
| CIFAR 100        | 600   | 51200     | 48000 | 6000       | 6000  | 60000         |
| Shakespeare      | 669   | 23040     | 33244 | 4494       | 5288  | 43026         |
| EMNIST           | 3400  | 31744     | 59552 | 76062      | 77483 | 749068        |
| Stackoverflow-nwp| 300   | 960384    | 155702| 19341      | 19736 | 194779        |

Table 1: Dataset Information

Shakespeare  Shakespeare is a language modeling dataset built using the works of William Shakespeare and the clients correspond to a speaking role with at least two lines. The task here is next character prediction. The way lines are split into sequences of length 80, and the description of the vocabulary size is same as [RCZ+21] (Appendix C.3). Additionally, we filtered out clients with less than 3 sequences of data, so as to have a train-validation-test split for all the clients. This brought the number of clients down to 669. More information on sample sizes can be found in Table 1. The models trained on this dataset are trained on two Tesla P100-PCIE-12GB GPUs.

CIFAR-100  CIFAR-100 is an image classification dataset with 100 classes and each image consisting of 3 channels of 32x32 pixels. We use the clients created in the Tensorflow Federated framework [MRR+19] — client division is described in Appendix F of [RCZ+21]. Instead of using 500 clients for training and 100 clients for testing as in [RCZ+21], we divided each clients’ dataset into train, validation and test sets and use all the clients’ corresponding data for training, validation and testing respectively. The models trained on this dataset are trained on two Titan Xp GPUs.

EMNIST  EMNIST contains images of upper and lower characters of the English language along with images of digits, with total 62 classes. The federated version of EMNIST partitions images by their author providing the dataset natural heterogeneity according to the writing style of each person. The task is to classify images into the 62 classes. As in other datasets, we divide each clients’ data into train, validation and test sets randomly. The models trained on this dataset are trained on two Tesla P100-PCIE-12GB GPUs.

Stack Overflow  Stack Overflow is a language model consisting of questions and answers from the StackOverflow website. The task we focus on is next word prediction. As described in Appendix C.4 of [RCZ+21], we also restrict to the 10000 most frequently used words, and perform padding/truncation to ensure each sentence to have 20 words. Additionally, due to scalability issues, we use only a sample of 300 clients from the original dataset from [MRR+19] and for each client, we divide their data into train, validation and test sets randomly. The models trained on this dataset are trained on two Titan Xp GPUs.
8.2 Hyperparameter Tuning Details

8.2.1 Pretrained Model

We now describe how we obtain our pretrained models. First, we train and hyperparameter tune a neural net classifier on the train and validation sets in a non-federated manner. The details of the hyperparameter sweep are as follows:

**Shakespeare** For this dataset we use the same neural network architecture as used for Shakespeare in [MMR+17]. It has an embedding layer, an LSTM layer and a fully connected layer. We use the StepLR learning rate scheduler of PyTorch, and we hyperparameter tune over the step size $[0.0001, 0.001, 0.01, 0.1, 1]$ and the learning rate decay gamma $[0.1, 0.3, 0.5]$ for 25 epochs with a batch size of 128.

**CIFAR-100** For this dataset we use the Res-Net18 architecture [HZRS16]. We perform the standard preprocessing for CIFAR datasets for train, validation and test data. For training images, we perform a random crop to shape $(32, 32, 3)$ with padding size 4, followed by a horizontal random flip. For all training, validation and testing images, we normalize each image according to their mean and standard deviation. We use the hyperparameters specified by [wei20] to train our nets for 200 epochs.

**EMNIST** For this dataset, the architecture we use is similar to that found in [RCZ+21]; the exact architecture can be found in our code. We use the StepLR learning rate scheduler of PyTorch, and we hyperparameter tune over the step size $[0.0001, 0.001, 0.01, 0.1, 1]$ and the learning rate decay gamma $[0.1, 0.3, 0.5]$ for 25 epochs with a batch size of 128.

**Stackoverflow** For this dataset we use the same neural network architecture as used for Stack Overflow next word prediction task in [RCZ+21]. We use the StepLR learning rate scheduler of PyTorch, and we hyperparameter tune over the step size $[0.0001, 0.001, 0.01, 0.1, 1]$ and the learning rate decay gamma $[0.1, 0.3, 0.5]$ for 25 epochs with a batch size of 128.

8.2.2 Federated Last Layer Training

After selecting the best hyperparameters for each net, we pass our data through said net and store their representations (i.e., output from penultimate layer). These representations are the data we operate on in our federated experiments.

Using these representations, we do multi-class logistic regression with each of the federated learning algorithms we test; we adapt and extend this code base [DTN20] to do our experiments. For all of our algorithms, the number of global iterations $R$ is set to 400, and the number of local iterations $K$ is set to 20. The number of users sampled at global iteration $r$, $D$, is set to 20. The batch size per local iteration, $B$, is 32. The random seed is set to 1. For algorithms FTFA, RTFA, MAML-FL-FO, and MAML-FL-HF, we set the number of personalization epochs $P$ to be 10. We fix some hyperparameters due to computational resource restrictions and to avoid conflating variables; we choose to fix these ones out of precedence, see experimental details of [RCZ+21]. We now describe what parameters we hyperparameter tune over for each algorithm.

**Naive Local Training** This algorithm is described in Algorithm 3. We hyperparameter tune over the step size $\alpha$ $[0.0001, 0.001, 0.01, 0.1, 1, 10]$.

**FedAvg** This algorithm is described in Algorithm 4. We hyperparameter tune over the step size $\eta$ $[0.0001, 0.001, 0.01, 0.1, 1, 10]$.

**FTFA** This algorithm is described in Algorithm 1. We hyperparameter tune over the step size of FedAvg $\eta$ $[0.0001, 0.001, 0.01, 0.1, 1]$, and the step size of the personalization SGM steps $\alpha$ $[0.0001, 0.001, 0.01, 0.1, 1]$. 

40
RTFA  This algorithm is described in Algorithm 6. We hyperparameter tune over the step size of FedAvg $\eta$ [0.0001, 0.001, 0.01, 0.1, 1], the step size of the personalization SGM steps $\alpha$ [0.0001, 0.001, 0.01, 0.1, 1], and the ridge parameter $\lambda$ [0.001, 0.01, 0.1, 1, 10].

MAML-FL-HF  This is the hessian free version of the algorithm, i.e., the hessian term is approximated via finite differences (details can be found in [FMO20]). This algorithm is described in Algorithm 7. We hyperparameter tune over the step size $\eta$ [0.0001, 0.001, 0.01, 0.1, 1], the step size of the personalization SGM steps $\alpha$ [0.0001, 0.001, 0.01, 0.1, 1], and the hessian finite-difference-approximation parameter $\delta$ [0.001, 0.00001]. We used only two different values of $\delta$ because the results of preliminary experiments suggested little change in accuracy with changing $\delta$.

MAML-FL-FO  This is the first order version of the algorithm, i.e., the hessian term is set to 0 (details can be found in [FMO20]). This algorithm is described in Algorithm 7. We hyperparameter tune over the step size $\eta$ [0.0001, 0.001, 0.01, 0.1, 1], the step size of the personalization SGM steps $\alpha$ [0.0001, 0.001, 0.01, 0.1, 1].

pFedMe  This algorithm is described in Algorithm 8. We hyperparameter tune over the step size $\eta$ [0.0005, 0.005, 0.05], and the weight $\beta$ [1, 2]. The proximal optimization step size, hyperparameter $K$, and prox-regularizer $\lambda$ associated with approximately solving the prox problem is set to 0.05, 5, and 15 respectively. We chose these hyperparameters based on the suggestions from [DTN20]. We were unable to hyperparameter tune pFedMe as much as, for example, RTFA because each run of pFedMe takes significantly longer to run. Additionally, for this same reason, we were unable to run pFedMe on the Stack Overflow dataset. While we do not have wall clock comparisons (due to multiple jobs running on the same gpu), we have observed that pFedMe, with the hyperparameters we specified, takes approximately 20x the compute time to complete relative to FTFA, RTFA, and MAML-FL-FO.

The ideal hyperparameters for each dataset can be found in the tables below:

| Algorithm     | $\eta$ | $\alpha$ | $\lambda$ | $\delta$ | $\beta$ |
|---------------|--------|----------|-----------|----------|---------|
| Naive Local   | -      | 0.1      | -         | -        | -       |
| FedAvg        | 0.1    | -        | -         | -        | -       |
| FTFA          | 1      | 0.1      | -         | -        | -       |
| RTFA          | 1      | 0.1      | 0.1       | -        | -       |
| MAML-FL-HF    | 1      | 0.1      | -         | 0.00001  | -       |
| MAML-FL-FO    | 1      | 0.1      | -         | -        | -       |
| pFedMe        | 0.05   | -        | -         | -        | 2       |

Table 2: Shakespeare Best Hyperparameters

| Algorithm     | $\eta$ | $\alpha$ | $\lambda$ | $\delta$ | $\beta$ |
|---------------|--------|----------|-----------|----------|---------|
| Naive Local   | -      | 0.1      | -         | -        | -       |
| FedAvg        | 0.01   | -        | -         | -        | -       |
| FTFA          | 0.001  | 0.1      | -         | -        | -       |
| RTFA          | 0.001  | 0.1      | 0.1       | -        | -       |
| MAML-FL-HF    | 0.001  | 0.01     | -         | 0.001    | -       |
| MAML-FL-FO    | 0.001  | 0.01     | -         | -        | -       |
| pFedMe        | 0.05   | -        | -         | -        | 1       |

Table 3: CIFAR-100 Best Hyperparameters
| Algorithm          | $\eta$ | $\alpha$ | $\lambda$ | $\delta$ | $\beta$ |
|--------------------|--------|----------|-----------|----------|--------|
| Naive Local        | -      | 0.001    | -         | -        | -      |
| FedAvg             | 0.01   | -        | -         | -        | -      |
| FTFA               | 0.1    | 0.01     | -         | -        | -      |
| RTFA               | 0.1    | 0.01 0.1 | -         | -        | -      |
| MAML-FL-HF         | 0.1    | 0.01     | -         | 0.00001  | -      |
| MAML-FL-FO         | 0.1    | 0.01     | -         | -        | -      |
| pFedMe             | 0.05   | -        | -         | -        | 2      |

Table 4: EMNIST Best Hyperparameters

| Algorithm          | $\eta$ | $\alpha$ | $\lambda$ | $\delta$ | $\beta$ |
|--------------------|--------|----------|-----------|----------|--------|
| Naive Local        | -      | 0.1      | -         | -        | -      |
| FedAvg             | 1      | -        | -         | -        | -      |
| FTFA               | 1      | 0.1      | -         | -        | -      |
| RTFA               | 1      | 0.1 0.001| -         | -        | -      |
| MAML-FL-HF         | 1      | 0.1      | -         | 0.00001  | -      |
| MAML-FL-FO         | 1      | 0.1      | -         | -        | -      |

Table 5: Stack Overflow Best Hyperparameters

### 8.3 Additional Results

In this section, we add additional plots from the experiments we conducted, which were omitted from the main paper due to length constraints. In essence, these plots only strengthen the claims made in the experiments section in the main body of the paper.

**Figure 7.** CIFAR-100. Best-average-worst intervals created from different random seeds.

**Figure 8.** EMNIST. Best-average-worst intervals created from different train-val splits.
Figure 9: Shakespeare. Best-average-worst intervals created from different random train-val splits.

Figure 10. EMNIST. Gains of personalization for FO-MAML-FL

Figure 11. CIFAR. Gains of personalization for FO-MAML-FL

Figure 12. EMNIST. Gains of personalization for HF-MAML-FL

Figure 13. CIFAR. Gains of personalization for FTFA