Λ-adic modular symbols and several variable $p$-adic $L$-functions over totally real fields

B. Balasubramanyam and M. Longo

Contents

1 Introduction 1

2 Hida’s theory for Hilbert modular forms 7
  2.1 Hilbert modular forms . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 7
  2.2 Hecke operators . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 8
  2.3 Nearly ordinary Hecke algebras . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 10

3 Λ-adic modular symbols 13
  3.1 Classical modular symbols . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 13
  3.2 Λ-adic modular symbols . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 14
  3.3 The Control Theorem . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 18
  3.4 Description of Ker($\rho_\kappa$) . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 19
  3.5 Lifting system of eigenvalues . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 20
  3.6 Proof of the Control Theorem . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 23

4 $p$-adic $L$-functions 24
  4.1 Complex $L$-functions . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 24
  4.2 Special values . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 25
  4.3 Interpolation formulas for classical modular symbols . . . . . . . . . . . . . . . . . . . . . . . 26
  4.4 $p$-adic $L$-functions of Λ-adic modular symbols . . . . . . . . . . . . . . . . . . . . . . . . . . 29
  4.5 Relations with classical $p$-adic $L$-functions . . . . . . . . . . . . . . . . . . . . . . . . . . . . 35

1 Introduction

Let $F/\mathbb{Q}$ be a totally real field of degree $d$. Denote by $\mathfrak{o}$ its ring of integers. For any ring $A$, define $\hat{A} := A \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$, where $\hat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$. Fix a compact open subgroup $S \subseteq \text{GL}_2(\hat{F})$ such that $U_0(\mathfrak{n}) \supseteq S \supseteq U_1(\mathfrak{n})$ for some integral ideal $\mathfrak{n}$ of $F$, where $U_0(\mathfrak{n})$ (respectively, $U_1(\mathfrak{n})$) are the usual
congruence groups defined in §2.1 Equations (6) and (7). Let $p$ be a rational prime prime to $2n$ which does not ramify in $F$ and define $S(p^n) := S \cap U(p^n)$ for any non negative integer $\alpha$, where $U(p^n)$ is defined in §2.1 Equation (8).

Fix an embedding $\iota : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$ and a finite extension $K$ of $\mathbb{Q}_p$ containing $\iota \circ \mu(F)$ for all archimedean places $\mu$ of $F$. Denote by $\mathcal{O}$ the ring of integers of $K$.

Let $I$ denote the set of embeddings of $F$ into $\mathbb{C}$. Let $n, v \in \mathbb{Z}[I]$ be fixed weight vectors such that $n + 2v \equiv 0 \pmod{Zt}$, where $t = (1,...,1) \in \mathbb{Z}[I]$ and let $k := n + 2t$ and $w := v + k - t$. Following [5], we denote by $h_{k,w}^{n,\text{ord}}(S(p^n), \mathcal{O})$ the Hecke algebra over $\mathcal{O}$ for the space of $p$-nearly ordinary Hilbert cusp forms of weight $(k,w)$ and level $S(p^n)$. In §2.3 we recall Hida’s construction of the universal $p$-ordinary Hecke algebra

$$R \simeq \lim_{\alpha} h_{2t,t}^{n,\text{ord}}(S(p^n), \mathcal{O}).$$

This Hecke algebra is universal in the sense that each nearly ordinary Hecke algebra $h_{k,w}^{n,\text{ord}}(S_0(p^n), \epsilon, K)$ acting on the $K$-vector space of cusp forms of weight $(k,w)$, level $S_0(p^n)$ and finite order character $\epsilon : S_0(p^n)/S_1(p^n) \rightarrow \mathbb{C}$ is isomorphic to a residue algebra of $R$. Here $S_0(p^n) := S \cap U_0(p^n)$ and $S_1(p^n) := S \cap U_1(p^n)$. More precisely, let

$$G := \lim_{\alpha} S_0(p^n)r^\times/S(p^n)r^\times,$$

where $S_0(p^n) := S \cap U_0(p^n)$ and denote by $W$ the free part of $G$. Then $R$ has a natural structure of $\tilde{\Lambda}$-algebra, where $\tilde{\Lambda} := \mathcal{O}[G]$ is the Iwasawa algebra of $G$ and there are isomorphisms:

$$R_P/P \rightarrow h_{k,w}^{n,\text{ord}}(S_0(p^n), \epsilon, K)$$

for suitable ideals $P$ of $\tilde{\Lambda}$. See [5] Theorem 2.4 or §2.3 for details.

Denote $\mathcal{L} := \text{Frac}(\Lambda)$, the field of fractions of $\mathcal{L}$ and by $\tilde{\mathcal{L}}$, its algebraic closure. Define

$$\mathcal{X}(\tilde{\mathcal{L}}) := \text{Hom}_{\text{cont}}^\mathcal{O}(\mathcal{R}, \tilde{\mathcal{L}})$$

and say that a point $\kappa \in \mathcal{X}(\tilde{\mathcal{L}})$ is arithmetic if its restriction to $G$ has kernel equal to $P$ for some of the ideals $P$ appearing in [1]. See [5] pages 150-152 or §2.3 for a precise notion of arithmetic point. As a corollary of [4] Theorem 2.4 is that points in correspond to $p$-adic families of Hilbert modular forms interpolating classical Hilbert modular forms of level $S(p^n)$. More precisely, fix a point $\theta \in \mathcal{X}(\tilde{\mathcal{L}})$. By [5] Theorem 2.4, the image of $\theta$ is contained in a finite extension $K$ of $\mathcal{L}$. Let $\mathcal{I}$ denote the integral closure of $\Lambda$ in $K$. Then
each arithmetic point $\kappa \in \text{Hom}^{O_{\text{cont}}}_{\text{algebras}}(I, \mathbb{Q}_p)$ corresponds to a classical $p$-nearly ordinary Hilbert modular form $f_\kappa$ of suitable weight $(k, w)$, level $S_0(p^\alpha)$ and character $\epsilon$. See [5, Corollary 2.5] for details. In particular, to any arithmetic point $\kappa$ is associated a weight $(n, v)$ (or $(k, w)$), a level $S_0(p^\alpha)$ and a character $\epsilon$. Finally, introduce the notion of primitive arithmetic point, which will be needed in the statement of Theorem 1.3 as in [4, pages 317, 318].

Remark 1.1. Note that the Iwasawa algebra $\Lambda := O[[W]]$ of $W$ is isomorphic to the formal power series ring in $s$ variables $O[[X_1, \ldots, X_s]]$, where $s = d+1+\delta_F$ and $\delta_F$ is the non negative integer appearing in Leopold’s conjecture. An other way to state [5, Corollary 2.5] is that any $f \in S_{k,w}(U_1(np^\alpha), \mathcal{K})$ has $s$-dimensional $p$-adic deformations over $O$ (in the sense of [5, pages 152-153]).

To discuss the main result of this paper, we need some technical assumptions. Denote by $F_\mathbb{A}$ the adele ring of $F$. By the Strong Approximation Theorem, choose $t_\lambda \in \text{GL}_2(F_\mathbb{A})$ for $\lambda \in \{1, \ldots, h(p^\alpha)\}$ and a suitable integer $h(p^\alpha)$ depending on $\alpha$ with $(t_\lambda)_{np} = (t_\lambda)_\infty = 1$ and such that there is the following disjoint union decomposition:

$$\text{GL}_2(F_\mathbb{A}) = \bigsqcup_{\lambda=1}^{h(p^\alpha)} \text{GL}_2(F) t_\lambda S R_+^\times$$

where $R_+$ is the set of positive real numbers and $(t_\lambda)_{np}$ (respectively, $(t_\lambda)_\infty$) is the $np$-part (respectively, the archimedean part) of $t_\lambda$. If $\alpha = 0$ write $h$ for $h(1)$. Define the following arithmetic groups depending on $S$:

$$\Gamma^\lambda(p^\alpha) := x_\lambda S(p^\alpha)x_\lambda^{-1} \cap \text{GL}_2^+(F)$$

and $\overline{\Gamma}^\lambda(p^\alpha) := \Gamma^\lambda(p^\alpha)/(\Gamma^\lambda(p^\alpha) \cap F^\times)$,

where $\text{GL}_2^+(F)$ is the subgroup of $\text{GL}_2(F)$ consisting of matrices with totally positive determinant.

Assumption 1.2. The groups $\overline{\Gamma}^\lambda := \overline{\Gamma}^\lambda(1)$ are torsion free for all $\lambda = 1, \ldots, h$.

See [4, Lemma 7.1] for conditions under which condition 1.2 is verified. In particular, there are infinitely many square-free integers for which $\overline{\Gamma}^\lambda(U_0(N))$ is torsion-free for all $\lambda$. Under this assumption, for any $\Gamma^\lambda$-module $E$, there is a canonical isomorphism $H^d(\Gamma^\lambda \backslash \mathcal{S}^d, \overline{E}) \cong H^d(\overline{\Gamma}^\lambda, E)$, where $\mathcal{S}$ is the complex upper half plane and $\overline{E}$ is the coefficient system associated to $E$.

The aim of this work is to combine Hida theory in the totally real case and Greenberg-Stevens theory of $\Lambda$-adic modular symbols to obtain a $p$-adic $L$-function attached to a Hida family of nearly ordinary Hilbert cusp
forms. This $p$-adic $L$-function interpolates $p$-adic $L$-functions attached to classical forms $f_\kappa$ in the Hida families as in [3, Theorem 5.15]. See also the work of Kitagawa [7] which inspired the construction of the $p$-adic $L$-function in [3]. The main difference with respect to the case of [3] relies on the fact that, as noticed earlier, the Iwasawa algebra $\Lambda$ in this case is isomorphic to a power series ring in at least $d+1$-variables over $O$, while in the rational case it is just isomorphic to a power series ring in one variable. In [4] Hida constructs for each weight $v$ as above a nearly ordinary universal Hecke algebra $R_v = h_n,v^{\text{ord}}(U_1(p\infty),O)$ such that each $h_{k,w}^{n,v}(U_1(p^\alpha),O)$ with $k$ parallel to $v$ can be obtained as a residue algebra of $R_v$. These Hecke algebras $R_v$ are endowed with a structure of $O[X_1, \ldots, X_{1+\delta_p}]$-algebra. The Iwasawa algebra $\Lambda$ considered here has more variables in order to unify these various Hecke algebras as $v$ and the character $\epsilon$ vary.

A consequence of that fact that the Iwasawa algebra considered here is bigger than that in [3] is that the role played by the set of primitive vectors $(Z_p^2)'$ in [3] will be played in this context by the $p$-adic space

$$X := NC/\text{GL}_2(\mathfrak{r}_p) \cong \lim_{\alpha} S(p^\alpha)\mathfrak{r}^\times \backslash \text{St}^\times,$$

which has a greater rank. Here $N$ is the standard lower unipotent subgroup of $\text{GL}_2(\mathfrak{r}_p)$ and $C$ is the closure of $\mathfrak{c} := \text{St} \cap F$ embedded diagonally in $\text{GL}_2(\mathfrak{r}_p)$, where $\mathfrak{r}_p := \mathfrak{r} \otimes \mathbb{Z}_p$. A similar $p$-adic space has been defined and studied in [2]. The action of the Hecke operators $U_p$ for prime ideals $p | p$ on $X$ is similar to that considered in [2]. To describe $X$ more precisely, for any prime ideal $p | p$ of $F$, let $(\mathfrak{r}_p^2)'$ denote the set of primitive vectors of $\mathfrak{r}_p^2$, i.e. the set of elements $(x, y) \in \mathfrak{r}_p^2$ such that at least one of $x$ and $y$ does not belong to $p$. Set $(\mathfrak{r}_p^2)' := \prod_p (\mathfrak{r}_p^2)'$. Denote by $\mathfrak{r}$ the closure of $\text{St}^\times \in \mathfrak{r}_p^\times$. Then $X$ can be identified via the map

$$\gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \mapsto ((a, b), \det(\gamma))$$

with $\mathfrak{r} \setminus ((\mathfrak{r}_p^2)' \times \mathfrak{r}_p^\times)$. Hence $X$ may be viewed as an analogue of the primitive vectors appearing in [3]. Elements of $X$ will be denoted by $(x, y, z)$.

Following [3], define $D_X$ to be the space of $O$-valued measures on $X$. This space is endowed with $\tilde{\Lambda}$ and $\Lambda$-algebra structures. Let $D_X$ denote the local coefficient system on $X_S$ associated to $D_X$. We define the space of $\Lambda$-adic modular symbols in this context to be

$$\mathbb{W} := H^d_{\text{cpt}}(X_S, D_X)$$

the $d$-th cohomology with compact supports of the Hilbert modular variety $X_S$ associated to $S$ with coefficients in $D_X$. It follows from [1, Proposition 4.2]
that this definition of $\Lambda$-adic modular symbols is consistent with the analogous definition in [3]. The group $W$ is endowed with a structure of $\tilde{\Lambda}$-module. It is also endowed with an action of the involution $((1 0, 0 -1), \ldots, (1 0, 0 -1)) \in \text{GL}_2(F)^d$, so there is a notion of sgn-eigenspace in $W$ for each choice of $\text{sgn} \in \{\pm 1\}^d$.

Let $\text{Sym}^n(K)$ be the space of homogeneous polynomials with coefficients in $K$ in $2d$ variables $X_{\sigma}, Y_{\sigma}, \sigma \in I$, of degree $n_{\sigma}$ in $(X_{\sigma}, Y_{\sigma})$. For any primitive arithmetic point $\kappa$ of weight $(n, v)$ and character $\epsilon$, define a specialization map $\rho_{\kappa} : \mathbb{D}_X \to \text{Sym}^n(K)$ by

$$\mu \mapsto \rho_{\kappa}(\mu) := \int_{X'} \epsilon(x) z^v (xY - yX)^n d\mu(x, y, z),$$

where $X'$ the subset of $X$ consisting of elements $((x, y), z)$ such that $x \in r \times p$. The map $\rho_{\kappa}$ induces a map on the cohomology groups which we again denote by the same symbol

$$\rho_{\kappa} : \mathbb{W} \otimes_{\Lambda} \mathcal{R}_P \to H^d_{\text{cpt}}(X_{S_0(p^m)}, \text{Sym}^n(K)).$$

Here $P := \kappa|_{\Lambda}$ and $\mathcal{R}_P$ is the localization of $\mathcal{R}$ at the kernel of $P : \Lambda \to \mathbb{Q}_p$ (compare with (1)) and $S_0(p^m)$ is the level of the modular form $f_{\kappa}$ associated to the arithmetic point $\kappa$. By following [3] and [2], a corresponding control theorem for these specialization maps is stated in Theorem 3.4 and proved in §3.6. This result states that, for fixed $\theta : \mathcal{R} \to I$ and $\kappa_0 \in \mathcal{X}(I)$ an arithmetic point and for each choice of sign $\text{sgn} \in \{\pm 1\}^d$, there exists $\Phi \in (\mathbb{W} \otimes_{\Lambda} \mathcal{R}_P)_{\text{sgn}}$ (with $\kappa_0|_{\Lambda} = P_0$) such that

$$\rho_{\kappa_0}(\Phi) = \Psi_{f_{\kappa_0}}^{\text{sgn}},$$

where $\Psi_{f_{\kappa_0}}^{\text{sgn}}$ is the $\text{sgn}$-classical modular symbol associated to the cusp form $f_{\kappa_0}$ (see §3.1 for details and definitions). The interpolation formulas satisfied by the classical modular symbols, which are recalled in §4.3, lead to the following result:

**Theorem 1.3.** Let $\theta, \kappa_0$ and $\Phi$ as above (hence, (2) holds). Choose a sign $\epsilon \in \{\pm 1\}^d$. There exists a $p$-adic analytic function $I_p^{\text{sgn}}(\Phi, \theta, \kappa, \sigma)$ in the variables $\kappa \in \mathcal{X}(I)$ and $\sigma \in \mathcal{X}(r_p^\times) = \text{Hom}_{\text{cont}}(r_p^\times, \mathbb{Q}_p^\times)$ such that for any primitive arithmetic point $\kappa \in \mathcal{X}(I)$ of weight $k$ and character $\epsilon$, save possibly a finite number, any positive integer $m$ in the critical strip for the modular form $f_{\kappa}$ defined in (23) and any primitive character $\chi$ of the ideal class group of $F$ of conductor $p^m$ and of sign $\text{sgn}$:

$$I_p^{\text{sgn}}(\Phi, \theta, \kappa, \chi \chi_{\text{cyc}}^{m-1}) = \left(1 - \chi(p) \chi_{\text{cyc}}^{m-1}(p) \right) \frac{\Omega(\Phi, \theta, \kappa) \Omega_{\text{sgn}}(f, \chi) a_p(\theta, \kappa)}{\Omega_{\text{sgn}}(f, \chi)^a_p(\theta, \kappa)^m} \Lambda(f_{\kappa}, \chi, m),$$

5
where $\chi_{cyc}$ is the cyclotomic character, $\Lambda(f, \chi, m)$ is the complex $L$-function defined in $§4.3$, Equations (25) and (26), $a_p(\theta, \kappa) := \kappa \circ \theta(T(p))$ and finally $\Omega(\Phi, \theta, \kappa)$ and $\Omega_{sgn}(f, \chi)$ are suitable periods defined in (33) and (28).

Let $f$ be an ordinary Hilbert modular form in the sense of Panchishkin $[11, \S 8]$. It is well known that $f$ is nearly ordinary in the sense of Hida. For any sign $\text{sgn} \in \{\pm 1\}^d$, denote by $L_{\text{sgn}}^p(f, s)$ the $p$-adic $L$-function attached to $f$ and $\text{sgn}$ constructed by Manin in $[8]$. This function can be characterized by its interpolation property: for any integer $m$ in the critical strip of $f$,

$$L_{\text{sgn}}^p(f, m) = \left(1 - \frac{\chi(p)^m - 1}{a_p(\theta, \kappa)}\right) \frac{\Lambda(f, \chi, m)}{\Omega_{\text{sgn}}(f, \chi) a_p(\theta, \kappa)^m}.$$  (3)

This formula can be found in a less explicit form in $[8, \S 5]$ or in $[11$, Theorem 8.2] in a form closest to this. The interpolation formulas in Theorem 1.3 and (3), the analyticity of the $p$-adic $L$-functions and the fact that $L_{\text{sgn}}^p(f, \chi)$ is uniquely determined by (3) by $[11$, Theorem 8.2 (iii)] lead to the following:

**Corollary 1.4.** Let $\Phi, \theta$ and $\kappa$ be as in Theorem 1.3. Suppose that $f_\kappa$ is ordinary in the sense of $[11]$. Then there is an equality of Iwasawa functions in $\sigma$:

$$L_{\text{sgn}}^p(\Phi, \theta, \kappa, \sigma) = \Omega(\Phi, \theta, \kappa) L_{\text{sgn}}^p(f_\kappa, \sigma).$$

In the spirit of $[3]$, the two variable $L$-function is a tool to prove the exceptional zero conjecture for the derivative of the one variable $p$-adic $L$-function associated to an elliptic curve $E/\mathbb{Q}$ stated by Mazur, Tate and Teitelbaum in $[10]$. The natural development of this work is to investigate the analogue conjecture for elliptic curves $E/F$ over totally real fields. An interesting and new feature of the totally real context is that in this case it may be possible to calculate partial derivatives $\partial/\partial_\sigma$ with respect to each $\sigma \in I$. It is expected that the parallel derivative $\prod_\sigma \partial/\partial_\sigma$ will play the role of the weight derivative in the context of $[3]$. It may also be interesting to investigate the meaning of the non parallel derivative operators and their connections with the geometry of the elliptic curve $E/F$. We hope to come to these questions in a future work.

**Acknowledgments.** This paper is part of the Ph.D thesis of the first author. The first author would like to thank sincerely his advisor Prof. F. Diamond for his help and encouragement. Both authors would like to thank Prof. H. Darmon for suggesting the problem and for his advice and support during their visits to McGill University.
2 Hida’s theory for Hilbert modular forms

2.1 Hilbert modular forms

We recall results from [13], [4] and [5]. Let $F$ be a totally real field and denote by $\mathfrak{r}$ its ring of integers. Denote by $F_\mathfrak{r}$ the adele ring of $F$ and by $\hat{F}$ the ring of finite adeles. For any place $v$ of $F$ and any $x \in \text{GL}_2(F_\mathfrak{r})$, denote by $x_v$ the $v$-component of $x$. Fix an open compact subgroup $U$ of $\text{GL}_2(\hat{F})$. By the Strong Approximation Theorem, choose $t_\lambda \in \text{GL}_2(F_\mathfrak{r})$ for $\lambda \in \{1, \ldots, h(U)\}$ and a suitable integer $h(U)$ depending on $U$ with $(t_\lambda)_q = (t_\lambda)_v = 1$ for any prime ideal $q$ dividing the adelized determinant $\hat{\det}(U)$ of $U$ and any archimedean place $v$, and such that there is the following disjoint union decomposition:

$$\text{GL}_2(F_\mathfrak{r}) = \prod_{\lambda=1}^{h(U)} \text{GL}_2(F)t_\lambda U S_\infty$$

where $S_\infty := (\text{SO}_2(\mathbb{R})\mathbb{R}_+)^d$ and $\mathbb{R}_+$ is the group of positive real numbers. Define the following arithmetic groups depending on $U$:

$$\Gamma^\lambda(U) := x_\lambda(U)x^{-1}_\lambda \cap \text{GL}_2^+(F)$$

and

$$\overline{\Gamma}^\lambda(U) := \Gamma^\lambda(U)/(\Gamma^\lambda(U) \cap F^\times),$$

where $\text{GL}_2^+(F)$ is the subgroup of $\text{GL}_2(F)$ consisting of matrices with totally positive determinant. For any $\lambda = 1, \ldots, h(U)$, denote by $S_{k,w}(\Gamma^\lambda(U), \mathbb{C})$ the $\mathbb{C}$-vector space of Hilbert cusp forms with respect to the automorphic factor $j(\gamma, z) := \frac{\det(\gamma)^w}{(cz + d)^k}$ for $\gamma \in \Gamma^\lambda(U)$, where the usual multi-index notations are used: for $z = (z_\sigma)_{\sigma \in I}$ and $z' = (z'_\sigma)_{\sigma \in I} \in \mathfrak{H}[I]$ ($\mathfrak{H}$ is the complex upper half plane) and $m = (m_\sigma)_{\sigma \in I} \in \mathbb{Z}[I]$, $z^m := \prod_{\sigma \in I} z_\sigma^{m_\sigma}$ and $z + z' = (z_\sigma + z'_\sigma)_{\sigma \in I}$. Explicitly, for any $\gamma \in \Gamma^\lambda(U)$:

$$(f|\gamma)(z) := \det(\gamma)^w(cz + d)^{-k}f(\gamma(z)) = f(z),$$

where the action of $\gamma$ on $\mathfrak{H}[I]$ is given by composing the usual action $z \mapsto \alpha(z)$ on $\mathfrak{H}$ by fractional linear transformations of the group $\text{GL}_2^+(\mathbb{R})$ of real matrices with positive determinant with the injections $\sigma : \text{GL}_2^+(F) \hookrightarrow \text{GL}_2^+(\mathbb{R})$ deduced from $\sigma \in I$; more precisely, $\gamma((z_\sigma)_\sigma) = (\sigma(\gamma)(z_\sigma))_\sigma$. Define

$$S_{k,w}(S(U), \mathbb{C}) := \prod_{\lambda=1}^{h(U)} S_{k,w}(\Gamma^\lambda(U), \mathbb{C}).$$
For any ideal \( m \subseteq r \) define the following open compact semigroups of \( \text{GL}_2(\hat{F}) \):

\[
\Delta_0(m) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\hat{F}) \cap M_2(r) : c \equiv 0 \pmod{m} \right\},
\]

\[
\Delta_1(m) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta_0(m) : a_q - 1 \in m_q \text{ for all prime ideals } q | m \right\}.
\]

\[
U_0(m) := \{ \gamma \in \Delta_0(m) : \det(\gamma_q) \in \hat{r}^\times_q, \text{ for all prime ideals } q \subseteq r \},
\]

\[
U_1(m) := U_0(m) \cap \Delta_1(m),
\]

\[
U(m) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_1(m) : d_q - 1 \in m_q \text{ for all prime ideals } q | m \right\}.
\]

where \( \hat{r} := r \) is the profinite completion of \( r \). Fix an ideal \( n \subseteq r \), a rational prime \( p \) prime to \( 2n \) and not ramified in \( F \) and an open compact subgroup \( S \subseteq \text{GL}_2(\hat{F}) \) such that \( U_0(n) \supseteq S \supseteq U_1(n) \) and the \( p \)-component \( S_p \) of \( S \) is \( \text{GL}_2(r_p) \). For any positive integer \( \alpha \), set

\[
S_0(p^\alpha) := S \cap U_0(p^\alpha), \quad S_1(p^\alpha) := S \cap U_1(p^\alpha), \quad S(p^\alpha) := S \cap U(p^\alpha).
\]

Denote by \( \Gamma^\lambda(p^\alpha) \) and \( \overline{\Gamma}^\lambda(p^\alpha) \) (respectively, \( \Gamma^\lambda_0(p^\alpha) \) and \( \overline{\Gamma}^\lambda_1(p^\alpha) \)) the arithmetic groups associated as in [5] to \( S(p^\alpha) \) (respectively, \( S_0(p^\alpha); S_1(p^\alpha) \)).

Any modular form \( f_\lambda \in S_{k,w}(\Gamma^\lambda(p^\alpha), \mathbb{C}) \) has a Fourier expansion of the form

\[
f_\lambda(z) = \sum_{\xi \in \mathfrak{t}_\lambda, \xi \gg 0} a_\lambda(\xi) e^{2\pi i (\xi \cdot z)},
\]

where the notations are as follows: \( \mathfrak{t}_\lambda \) is an ideal represented by \( t_\lambda \), \( \mathfrak{d} \) is the different ideal of \( F/\mathbb{Q} \), \( \xi \gg 0 \) if and only if, by definition, \( \xi \) is totally positive, and \( (\xi \cdot z) := \sum_{\sigma \in I} \sigma(\xi) z_\sigma \) is the scalar product. For details, see [4] Corollary 4.3.

### 2.2 Hecke operators

The right action on \( S_{k,w}(S(p^\alpha), \mathbb{C}) \) of the Hecke algebra \( R(S(p^\alpha), \Delta_0(np^\alpha)) \), which is by definition the free \( \mathbb{Z} \)-module generated by double cosets

\[
T(x) := S(p^\alpha)xS(p^\alpha)
\]
for $x \in \Delta_0(np^\alpha)$ with multiplication defined by
\[
\sum_i a_i T(x) \cdot \sum_j b_j T(y) := \sum_{i,j} a_i b_j T(x_i y_j),
\]
can be described as follows. Fix $x \in \Delta_0(np^\alpha)$ and $\lambda \in \{1, \ldots, h\}$. Let $\mu \in \{1, \ldots, h\}$ such that $\det(x) t_\lambda t_{-\mu}^{-1}$ is trivial in the strict class group of $F$. Let $\alpha_\lambda$ such that $S(p^\alpha) x S(p^\alpha) = S(p^\alpha) x^{\lambda^{-1}} \alpha_\lambda x_{\mu} S(p^\alpha)$ and form the finite disjoint coset decomposition $\Gamma_\lambda \alpha_\lambda \Gamma_\mu = \sum_j \Gamma_\lambda \alpha_{\lambda,j}$ where $\alpha_{\lambda,j} \in \text{GL}_2(F) \cap x_\lambda \Delta_0(np^\alpha) x_\mu^{-1}$. Define
\[
g_\mu := \sum_j f_j | \alpha_{\lambda,j} \quad \text{and} \quad f | T(x) := (g_1, \ldots, g_h).
\]

Denote by $\tilde{F}$ the composite field of all the images $\sigma(F)$ of $F$ under the elements $\sigma \in I$ and by $\tilde{r}$ its ring of integers. Fix an $\tilde{r}$-algebra $A \subseteq \mathbb{C}$ such that for every $x \in \tilde{F}^\times$ and every $\sigma \in I$, the $A$-ideal $x^\sigma A$ is generated by a single element of $A$. For any prime ideal $q \subseteq \tilde{r}$, choose a generator $\{q^\sigma\} \in A$ of the principal ideal $q^\sigma A$. Define $\{q\}^v := \prod_{\sigma \in I} (q^\sigma)^{v_\sigma}$. Write a fractional ideal $m$ of $\tilde{r}$ as a product of prime ideals $m = \prod_q q^{m(q)}$ and define $\{m\}^v := \prod_q (\{q\}^v)^{m(q)}$. For any element $x \in \tilde{F}^\times$, denote by $m_x$ the fractional $\tilde{r}$-ideal corresponding to $x$ and define $\{x\}^v := \{m_x\}^v$. Modify the Hecke operators $T(x) \in R(S(p^\alpha), \Delta_0(np^\alpha))$ by setting:
\[
T_0(x) := \{(x)^v\}^{-1} T(x).
\]

Denote by $h_{k,w}(S(p^\alpha), A)$ the $A$-subalgebra of $\text{End}(S_{k,w}(S(p^\alpha), \mathbb{C}))$ generated over $A$ by operators $T_0(x)$ for $x \in \Delta_0(np^\alpha)$. By [5, Proposition 1.1], $h_{k,w}(S(p^\alpha), A)$ is commutative.

For $x \in F$ and $m = (m_\sigma)_\sigma \in \mathbb{Z}[I]$, set $x^m := \prod_{\sigma \in I} \sigma(x)^{m_\sigma}$. For any integral ideal $m$, choose $\lambda = \lambda(m)$ such that $m$ is equivalent to $t_\lambda d$ in the strict ideal class group of $F$ and let $\xi_m \in t_\lambda d$ such that $\xi_m \gg 0$ and $m = \xi_m (t_\lambda d)^{-1}$. Define the modified Fourier coefficients as in [4, Corollary 3.4]:
\[
C(m, f) := \frac{a_\lambda(\xi_m) \xi_m^v}{b_{v,\lambda}}, \quad \text{where} \quad b_{v,\lambda} := \frac{N(t_\lambda)}{(t_\lambda d)^v}. \tag{9}
\]

Suppose that $f \in S_{k,w}(S(p^\alpha), \mathbb{C})$ is an eigenform for $h(S(p^\alpha), A)$ such that $C(t, f_\lambda) = 1$ for all $\lambda = 1, \ldots, h$ (call such a form normalized). Then by [4, Corollary 4.2]:
\[
f | T(m) = C(m, f).
The group $G_\alpha := S_0(p^\alpha)\mathfrak{r}_\alpha / S(p^\alpha)\mathfrak{r}_\alpha$ acts on $S_{k,w}(S(p^\alpha), \mathbb{C})$ via the operator $\omega(q_+^{n+2q})^{-1}T(x)$ for $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S_0(p^\alpha)$, where $\omega$ is the Teichmuller character.

For any prime ideal $q \subseteq \mathfrak{r}$, choose an element $q \in T^\infty$ such that $q\mathfrak{r} = \hat{q}$ and the $\mathfrak{r}$-component $q_1$ of $q$ is equal to 1 for all prime ideals $\mathfrak{l} \subseteq \mathfrak{r}$, $\mathfrak{l} \neq q$. Define $T(q) := T\left(\begin{array}{cc} 1 & 0 \\ 0 & q \end{array}\right)$ for all prime ideals $q \subseteq \mathfrak{r}$ and $T(q,q) := T\left(\begin{array}{cc} q & 0 \\ 0 & q \end{array}\right)$ for prime ideals $q \subseteq \mathfrak{r}$ such that $q \nmid n$. By [5, Proposition 1.1], if $\mathfrak{S} \supseteq U_1(\mathfrak{a})$, then $h_{k,w}(S(p^\alpha), \mathfrak{A})$ is generated over $\mathfrak{A}$ by the operators induced from the action of $G_\alpha$ and $T_0(q)$ for all prime ideals $q$.

If $f$ is an eigenform for the Hecke algebra $h_{k,w}(S(p^\alpha), \mathfrak{A})$, then its eigenvalues are algebraic numbers. If $k \sim t$, then they are algebraic integers. Define $S_{k,w}(S(p^\alpha), \mathfrak{A}) \subseteq S_{k,w}(S(p^\alpha), \mathbb{C})$ to be the $\mathfrak{A}$-module consisting of forms whose Fourier expansion has coefficients in $\mathfrak{A}$. The $\mathfrak{A}$-module $S_{k,w}(S(p^\alpha), \mathfrak{A})$ is stable under the action of $h_{k,w}(S(p^\alpha), \mathfrak{A})$.

### 2.3 Nearly ordinary Hecke algebras

Choose an embedding $i : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, so that any algebraic number is equipped with a $p$-adic valuation. Fix a ring of integers $\mathcal{O}$ of a finite extension of the completion of $i(\hat{F})$. After choosing an embedding $i : \overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$, the $\mathfrak{r}$-algebra $\mathcal{O}$ satisfies the conditions of [22].

By [5, Lemma 2.2], $h_{k,w}(S(p^\alpha), \mathcal{O})$ is free of finite rank over $\mathcal{O}$ and hence can be decomposed as a direct sum:

$$h_{k,w}(S(p^\alpha), \mathcal{O}) = h^{\text{ord}}_{k,w}(S(p^\alpha), \mathcal{O}) \oplus h^{\text{ns}}_{k,w}(S(p^\alpha), \mathcal{O})$$

such that the image of $T_0(p)$ in the first factor, the nearly ordinary part, is a unit while its image in the other factor is topologically nilpotent.

For any pair of non negative integers $\beta \geq \alpha$ the map $T_0(x) \mapsto T_0(x)$ for $x \in \Delta_0(np^\alpha)$ induces a surjective ring homomorphism $\rho^\beta_\alpha : h_{k,w}(S(p^\beta), \mathcal{O}) \rightarrow h_{k,w}(S(p^\alpha), \mathcal{O})$. Define:

$$h_{k,w}(S(p^\infty), \mathcal{O}) := \varprojlim h_{k,w}(S(p^\alpha), \mathcal{O})$$

and

$$h^{\text{ord}}_{k,w}(S(p^\infty), \mathcal{O}) := \varprojlim h^{\text{ord}}_{k,w}(S(p^\alpha), \mathcal{O})$$

where the inverse limits are with respect to the maps $\rho^\beta_\alpha$. By [5, Theorem 2.3], for any weight $(k, w)$ there is an isomorphism $h_{k,w}(S(p^\infty), \mathcal{O}) \simeq h_{2t,\ell}(S(p^\infty), \mathcal{O})$ which takes $T(q)$ to $T(q)$ and $T(q,q)$ to $T(q,q)$ for all prime

10
ideals \( q \nmid p \). This isomorphism induces an isomorphism between the nearly ordinary parts

\[
h^{\text{n,ord}}_{k,w}(S(p^\infty), \mathcal{O}) \simeq \mathcal{R} := h^{\text{n,ord}}_{2t,t}(S(p^\infty), \mathcal{O}).
\]

Set \( S_F := S \cap \hat{F}^\times \), \( S_F(p^\alpha) := S(p^\alpha) \cap \hat{F}^\times \), \( \overline{Z}_\alpha := S_F \mathfrak{z}^x / S_F(p^\alpha) \mathfrak{z}^x \) and \( \overline{Z}_\infty := \lim \overline{Z}_\alpha \). By [5, Lemma 2.1] the map \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \mapsto (a_{p}^{-1}d_p, a) \) induces an isomorphism \( G_\alpha \simeq (\mathfrak{r}/p^\alpha)^\times \times \overline{Z}_\alpha \) and hence an isomorphism

\[
G := \lim_{\leftarrow} G_\alpha \simeq \mathfrak{r}_p^\times \times \overline{Z}_\infty.
\]

Since \( h_{k,w}(S(p^\alpha), \mathcal{O}) \) is a \( \mathcal{O}[G_\alpha] \)-algebra and this algebra structure is compatible with the maps \( \rho_\alpha \), it follows that \( h^{\text{n,ord}}_{k,w}(S(p^\infty), \mathcal{O}) \) is a \( \widetilde{\Lambda} := \mathcal{O}[G] \)-algebra. Write \( G \simeq W \times G^{\text{tors}} \) where \( G^{\text{tors}} \) is the torsion subgroup of \( G \) and \( W \) is torsion-free. Note that \( W \) is well determined only up to isomorphism; fix from now on a choice of \( W \). The ring \( \Lambda := \mathcal{O}[W] \) is isomorphic to the power series ring \( \mathcal{O}[X_1, \ldots, X_s] \) in \( d < s < 2d \) variables, where \( d := [F : \mathbb{Q}] \).

If Leopold’s conjecture holds, then \( s = d + 1 \). By [5, Theorem 2.4], the nearly ordinary Hecke algebra \( \mathcal{R} = h^{\text{n,ord}}_{2t,t}(S(p^\infty), \mathcal{O}) \) is a torsion-free \( \Lambda \)-module of finite type.

For any finite order character \( \epsilon : S_0(p^\alpha)/S_1(p^\alpha) \to \overline{\mathbb{Q}}^\times \), define characters \( \epsilon_\lambda : \Gamma_0^\epsilon(p^\alpha)/\Gamma_1^\epsilon(p^\alpha) \to \mathbb{C}^\times \) by setting \( \epsilon_\lambda(\gamma) := \epsilon(x_\lambda^{-1} \gamma x_\lambda) \). Denote by \( S_{k,w}(S_0(p^\alpha), \epsilon, \mathbb{C}) \) the \( \mathbb{C} \)-subspace of \( S_{k,w}(S_0(p^\alpha), \mathbb{C}) \) consisting of forms \( f = (f_1, \ldots, f_h) \) such that for any \( \lambda = 1, \ldots, h \) and any \( \gamma \in \Gamma_0^\epsilon(p^\alpha) \), \( (f_\lambda|\gamma)(z) = \epsilon_\lambda^{-1}(\gamma)f(z) \). Suppose that \( \mathcal{O} \) contains the values of \( \epsilon \) and set \( S_{k,w}(S_0(p^\alpha), \epsilon, \mathcal{O}) := S_{k,w}(S_0(p^\alpha), \epsilon, \mathbb{C}) \cap S_{k,w}(S_0(p^\alpha), \mathcal{O}) \). Denote by

\[
h_{k,w}(S_0(p^\alpha), \epsilon, \mathcal{O})
\]

the \( \mathcal{O} \)-subalgebra of \( \text{End}(S_{k,w}(S_0(p^\alpha), \epsilon, \mathbb{C})) \) generated over \( \mathcal{O} \) by operators \( T_0(x) \) for \( x \in \Delta_0(p^\alpha) \). Define \( h^{\text{n,ord}}_{k,w}(S_0(p^\alpha), \epsilon, \mathcal{O}) \) to be the maximal factor of \( h_{k,w}(S_0(p^\alpha), \epsilon, \mathcal{O}) \) such that the image of \( T_0(p) \) is a unit in that factor. (In the following the \( \epsilon_\lambda \)'s will often be simply denoted by \( \epsilon \).

**Definition 2.1.** An eigenform \( f \in S_{k,w}(S_0(p^\alpha), \epsilon, \mathbb{C}) \) for the Hecke algebra \( h_{k,w}(S_0(p^\alpha), A) \) is said to be \( p \)-nearly ordinary if the eigenvalue of \( T_0(p) \) is a \( p \)-adic unit.

Denote by \( \mathfrak{r}^x_p \) the group of totally positive units of \( \mathfrak{r} \). Define

\[
Z_\alpha := S_F \mathfrak{r}^x_p / S_F(p^\alpha) \mathfrak{r}^x_p
\]

(12)
and \( Z_\infty := \varprojlim Z_\alpha \). The kernel of the natural surjection map \( Z_\infty \to \mathbb{Z}_\infty \) is finite and annihilated by a power of 2. Denote by \( \chi_{\text{cyc}} : Z_\infty \to \mathbb{Z}_p^\times \) the cyclotomic character defined by \( \chi_{\text{cyc}}(x) = x^t = \prod_{\sigma \in \Gamma} \sigma(x) = N(x) \). Let \( \epsilon : Z_\infty \to \mathcal{O}_\mathbb{Z}^\times \) be a character factoring through \( Z_\alpha \). Suppose that \( \epsilon_{\text{cyc}}^{n+2v} \) factors through \( \mathbb{Z}_\infty \), where if \( n+2v = mt \) with \( m \in \mathbb{Z} \), then \( \chi_{\text{cyc}}^{n+2v} \) is by definition \( \chi_{\text{cyc}}^m \). Let

\[
P_{n,v,\epsilon} : G \cong \mathfrak{g}_p^\times \times \mathbb{Z}_\infty \to \mathcal{O}_\mathbb{Z}^\times
\]

be the character defined by \( P_{n,v,\epsilon}(a,z) := \epsilon \chi_{\text{cyc}}^{n+2v}(z)a^v \). Denote by the same symbol the homomorphism \( \hat{\Lambda} \to \mathcal{O} \) deduced from \( P_{n,v,\epsilon} \) by extension of scalars. The kernel of this homomorphism is a prime ideal of \( \hat{\Lambda} \), denoted be the same symbol \( P_{n,v,\epsilon} \). To simplify notations, set \( P := P_{n,v,\epsilon}. \) Let \( \hat{\Lambda}_p \) (respectively, \( \mathcal{R}_p \)) denote the localization of \( \hat{\Lambda} \) (respectively, \( \mathcal{R} \)) in \( P \). Then \( \mathcal{R}_p \) is free of finite rank over \( \hat{\Lambda}_p \) and the natural surjective morphism \( \mathcal{R} \to h_{k,w}(S_0(p^\alpha), \epsilon, \mathcal{O}) \) induces by \([5, \text{Theorem 2.4}]\) an isomorphism:

\[
\mathcal{R}_p/\mathcal{P}\mathcal{R}_p \cong h_{k,w}(S_0(p^\alpha), \epsilon, \mathcal{K}),
\]

where \( K := \text{Frac}(\mathcal{O}) \) is the fraction field of \( \mathcal{O} \). In particular, the dimension of the \( K \)-vector space \( h_{k,w}(S_0(p^\alpha), \epsilon, \mathcal{K}) \) does not depend on \( \epsilon \) and \( (k, w) \) and is equal to the \( \Lambda_p \)-rank of \( \mathcal{R}_p \).

Let \( \mathcal{L} := \text{Frac}(\Lambda) \) be the fraction field of \( \Lambda \) and fix an algebraic closure \( \overline{\mathcal{L}} \) of \( \mathcal{L} \). Let \( \theta : \mathcal{R} \to \overline{\mathcal{L}} \) be a \( \Lambda \)-algebra homomorphism. The image \( \text{Im}(\theta) \) of \( \theta \) is finite over \( \Lambda \). Denote by \( \mathcal{I} \) the integral closure of \( \text{Im}(\theta) \) in the fraction field \( \mathcal{K} := \text{Frac}(\text{Im}(\theta)) \). Define

\[
\mathcal{X}(\mathcal{I}) := \text{Hom}_{\mathcal{O}_{\text{alg}}}(\mathcal{I}, \overline{\mathcal{O}}_p)
\]

and denote by \( \mathcal{A}(\mathcal{I}) \) the subset of \( \kappa \in \mathcal{X}(\mathcal{I}) \) consisting of points whose restriction to \( \Lambda \) coincide with the restriction to \( \Lambda \) of some character \( P_{n(\kappa),v(\kappa),\epsilon(\kappa)} \) as above. Points in \( \mathcal{A}(\mathcal{I}) \) are called arithmetic. In this case, denote \( P_{n(\kappa),v(\kappa),\epsilon(\kappa)} \) by \( P_\kappa \) and set \( k(\kappa) := n(\kappa) - 2t \) and \( w(\kappa) := k(\kappa) + v(\kappa) - t \). Let \( C(\kappa) \) denote the conductor of \( \epsilon \) restricted to the torsion free part \( W \) of \( \mathbb{Z}_\infty \) and \( \epsilon_W \) the restriction of \( \epsilon \) to \( W \). Let \( \mathbb{Z}_\infty^{\text{tors}} \) denote the maximal torsion subgroup of \( \mathbb{Z}_\infty \) and let \( \psi : \mathbb{Z}_\infty^{\text{tors}} \to \overline{\mathcal{L}} \) be the composite of \( \theta \) with the natural map \( \mathbb{Z}_\infty^{\text{tors}} \to \mathcal{R}_p^{\times} \) induced by the action of \( G \) on \( \mathcal{R} \). Define \( r_p^{\text{tors}} \) denote the maximal torsion subgroup of \( r_p^{\times} \). For any \( \kappa \in \mathcal{X}(\mathcal{I}) \), define

\[
\theta_\kappa := \kappa \circ \theta : \mathcal{R} \to \overline{\mathcal{O}}_p.
\]

By \([5, \text{Corollary 2.5}]\), if \( \kappa \in \mathcal{A}(\mathcal{I}) \) and \( \theta \) restricted to \( r_p^{\text{tors}} \) is the character \( x \mapsto x^{\nu(\kappa)} \), then \( \theta_\kappa(T(q)) \) are algebraic numbers for all prime ideals \( q \) and
there exists a unique up to constant factors $p$-nearly ordinary eigenform
\[ f_\kappa \in S_{k_\kappa,w_\kappa}(U_0(nC_\kappa)), \epsilon_W \psi^{-\omega(n(\kappa)+2v(\kappa))}, \mathbb{C} \]
such that $f_\kappa|T(q) = \theta_\kappa(T(q))f_\kappa$, where $\omega$ is the Teichmüller character and if $n(\kappa) + 2v(\kappa) = mt$ with $m \in \mathbb{Z}$, then $\omega^{-\omega(n(\kappa)+2v(\kappa))} := \omega^{-m}$. Conversely, if $\alpha > 0$ and $f \in S_{k,w}(U_1(np^\alpha), \mathbb{C})$ is a $p$-nearly ordinary eigenform, then there exists $\kappa \in \mathcal{A}(\mathcal{I})$ and $\theta$ as above such that $f$ is a constant multiple of $f_\kappa$.

3 $\Lambda$-adic modular symbols

3.1 Classical modular symbols
Define the Hilbert variety associated to an open compact subgroup $U$ of $GL_2(\hat{F})$ to be the complex variety:
\[ X_U := GL_2(F) \setminus GL_2(F_\kappa)/U \cdot S_\infty. \]
By strong approximation,
\[ X_U \simeq \prod_{\lambda=1}^{h(U)} \Gamma^\lambda(U) \setminus \mathfrak{g}_d^\lambda. \]
Suppose that $E$ is a (right or left) $\Gamma^\lambda(U)$-module for all $\lambda$. Denote by $\mathcal{E}$ the coefficient system on $X_U$ associated to $E$. Then
\[ H^d(X_U, \mathcal{E}) \simeq \oplus_{\lambda=1}^{h(U)} H^d(\Gamma^\lambda(U) \setminus \mathfrak{g}_d^\lambda, \mathcal{E}) \simeq \oplus_{\lambda=1}^{h(U)} H^d(\Gamma^\lambda(U), E). \]
For any $\omega \in H^d(X_U, \mathcal{E})$, write $\omega_\lambda$ for its projection to $H^d(\Gamma^\lambda(U) \setminus \mathfrak{g}_d^\lambda, \mathcal{E})$.

**Definition 3.1.** The group of *modular symbols* on $X(U)$ associated to $E$ is the group $H^d_{\text{cpt}}(X_U, \mathcal{E})$ of cohomology with compact support.

Suppose that $E$ is a right $t_\lambda \Delta_0(np^\alpha)t_\lambda^{-1} \cap GL_2(F)$ for all $\lambda$. Define an action of the Hecke algebra $R(S(p^\alpha), \Delta_0(np^\alpha))$ by the formula:
\[ \omega_\lambda(z)|T(x) := \sum_j \omega_\lambda(\alpha_{\lambda,j} z)|\alpha_{\lambda,j} \in H^d(\Gamma^\mu(p^\alpha) \setminus \mathfrak{g}_d^\mu, \mathcal{E}) \]
(same notations as in §2.2). Equivalently, identifying $\omega_\lambda$ with a $d$-cocycle by the above isomorphism, $T(x)$ can be defined as in [2] by the formula:
\[ (\omega_\lambda)|T(x)(\gamma_0, \ldots, \gamma_d) := \sum_j \omega_\lambda(t_j(\gamma_0), \ldots, t_j(\gamma_d))|\alpha_{\lambda,j}, \]
where \( t_j : \Gamma^u(p^\alpha) \rightarrow \Gamma^\lambda(p^\alpha) \) are defined for \( \gamma \in \Gamma^u(p^\alpha) \) by the equations \( \Gamma^\lambda \alpha_j \gamma = \Gamma^\lambda \alpha_j \) and \( \alpha_j \gamma = t_j(\gamma) \alpha_j \).

The group of modular symbols \( H^d_{\text{cpt}}(X_{S(p^\alpha)}, \mathcal{E}) \) is an \( R(S(p^\alpha), \Delta_0(np^\alpha)) \)-module if \( E \) is.

The modular symbol \( \omega(f) \) associated to \( f \in S_{k,w}(S(p^\alpha), \mathbb{C}) \) can be described as follows. For any ring \( R \), let \( L(n, R) \) be the \( R \)-module of homogeneous polynomials in \( 2 \) variables \( X = (X_\sigma)_{\sigma \in I} \) and \( Y = (Y_\sigma)_{\sigma \in I} \) of degree \( n_\sigma \) in \( X_\sigma, Y_\sigma \). Denote by \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \) the main involution of \( M_2(R) \). Define a right action of \( \text{GL}_2(F) \) on \( L(n, \mathbb{C}) \) by \( (P[\gamma](X,Y) := \det(\gamma)^n P((X,Y)\gamma^*) \), where \( (X,Y)\gamma^* \) is matrix multiplication. Denote by \( L(n,v,\mathbb{C}) \) the right representation of \( \text{GL}_2(F) \) thus obtained. The differential form \( \omega(f_\lambda)(z) := f(z)/(zX + Y)^{-2}dz \) (usual multi-index notations) satisfies the transformation formula for any \( \gamma \in \Gamma^\lambda(p^\alpha) \):

\[
(\omega(f_\lambda)(\gamma(z))|\gamma = \omega(f_\lambda)(z)
\]

hence, by \( \mathcal{H} \), \( \omega(f_\lambda) \in H^d(\Gamma^\lambda(p^\alpha)\backslash \mathfrak{S}^d, L(n,v,\mathbb{C})) \), where \( \mathcal{L}(n,v,\mathbb{C}) \) is the coefficient system on \( \Gamma^\lambda(p^\alpha)\backslash \mathfrak{S}^d \) associated to \( L(n,v,\mathbb{C}) \). Since \( f_\lambda \) is a cuspidal form, it can be proved that \( \omega(f_\lambda) \) has compact support. Hence, \( \omega(f) := (\omega(f_1), \ldots, \omega(f_h)) \in H^d_{\text{cpt}}(X_{S(p^\alpha)}, \mathcal{L}(n,v,\mathbb{C})) \).

For any character \( \epsilon \) as above, write \( \hat{L}(n,v,\epsilon,R) \) for the \( \Delta_0(\mathfrak{n}) \)-module \( L(n,v,R) \) with the action of \( \Delta_0(\mathfrak{n}) \) twisted by \( \epsilon \), that is, denoting by \( |_{\epsilon} \) this new action: \( P|_{\epsilon} \gamma := \epsilon(\gamma)P|\gamma \) for \( \gamma \in \Delta_0(\mathfrak{n}) \). If \( f_\lambda \in S_{k,w}(S_0(p^\alpha), \epsilon, \mathbb{C}) \), then \( \omega(f_\lambda) \in H^d(\Gamma_0^\lambda(p^\alpha), \mathcal{L}(n,v,\epsilon,\mathbb{C})) \), where \( \mathcal{L}(n,v,\epsilon,\mathbb{C}) \) is the coefficient system associated to \( L(n,v,\epsilon,\mathbb{C}) \). Hence \( \omega(f) \in H^d_{\text{cpt}}(X_{S_0(p^\alpha)}, \mathcal{L}(n,v,\epsilon,\mathbb{C})) \).

A straightforward calculation shows that the map \( f \mapsto \omega(f) \) is equivariant for the action of \( R(S(p^\alpha), \Delta_0(np^\alpha)) \).

### 3.2 \( \ell \)-adic modular symbols

Define \( \epsilon = \epsilon_S := S \cap F^\times \). Then \( \overline{Z}_\alpha = (\mathfrak{r}_p/p^\alpha \mathfrak{r}_p)^\times /\epsilon \), so \( G = \mathfrak{r}_p^\times \times \mathfrak{r}_p^\times /\overline{\epsilon} \), where \( \overline{\epsilon} \) is the closure of \( \epsilon \) in \( \mathfrak{r}_p^\times \). It follow that \( G \cong (\mathfrak{r}_p^\times \times \mathfrak{r}_p^\times) /\overline{\epsilon} \) via the map \( (x,y) \mapsto (xy,y) \). Embed diagonally \( \overline{\epsilon} \) in \( \text{GL}_2(\mathfrak{r}_p) \) and call \( C \) the image. Let \( N \) be the standard lower unipotent subgroup of \( \text{GL}_2(\mathfrak{r}_p) \). Define:

\[
X := NC\backslash \text{GL}_2(\mathfrak{r}_p) \cong \lim_{\longrightarrow} X_\alpha, \text{ where } X_\alpha := S(p^\alpha)\mathfrak{r}^\times \backslash \mathfrak{S}^\times.
\]

Let \( F_p = F \otimes_{\mathbb{Q}} Q_p = \prod_{p \mid \ell} F_p \) and \( Y := N(F_p)C\backslash \text{GL}_2(F_p) \), where \( N(F_p) \) is the group of lower triangular matrices with entries in \( F_p \) and diagonal entries 1. Write \( \mathfrak{r}_p = \prod_{p \mid \ell} \mathfrak{r}_p \), where \( \mathfrak{r}_p \) is the completion of \( \mathfrak{r} \) at \( p \). Define \( (\mathfrak{r}_p^\times)^\prime \) to be the set of primitive vectors of \( \mathfrak{r}_p^2 \), that is, the pair of elements \( (a,b) \in \mathfrak{r}_p^2 \)
such that at least one of $a$ and $b$ does not belong to $p$. Set $(r_p^2)' := \prod_{p|p}(r_p^2)'$.

The map $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto ((a, b), \det(g))$ defines a bijection between $X$ and $\mathfrak{r}((r_p^2)^{-1} \times r_p^\infty)$, where the action of $e \in \mathfrak{r}$ is $e \cdot ((x, y), z) = ((ex, ey), e^2z)$.

Define $\pi_p$ to be the element in $\text{GL}_2(\hat{F})$ whose $p$-component is $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ and is 1 outside $p$. Note that $\pi_p$ normalizes $N(F_p) = \prod_{p'|p} N(F_{p'})$ because $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ normalizes $N(F_p)$. Hence, it is possible to define an action of $\pi_p$ on $Y$ by letting $\pi_p$ act on its $p$-component as:

$$N(F_p)g * \pi_p = N(F_p)\pi_p^{-1}g\pi_p.$$ 

Identify $Y$ with $\mathfrak{r}((F_p^2)^\times \times F_p^\infty)$, where $(F_p^2)' := \{(x, y) \in F_p^2 : xy \neq 0\}$ via the map $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto ((a, b), \det(\gamma))$. Then

$$(x, y, z) * \pi_p = (x, py, z),$$

where for any $y = \prod_{p'|p} y_{p'}$, write $py := \prod_{p'|p, p' \neq p} y_{p'} \times py_p$. In particular, $\pi_p$ does not affect the determinant of the matrix.

Let $G^\pi$ be the semigroup generated by $\text{GL}_2(r_p)$ and $\pi_p$ for all divisors $p$ of $p$. Using that any element $s \in G^\pi$ can be expressed as a word in terms of $\text{GL}_2(r_p)$ and $\pi_p$, and that the actions of the $\pi_p$’s commute, extend the $*$ action to $G^\pi$ by letting any $\pi_p$ act through $*$ and elements of $\text{GL}_2(r_p)$ through right multiplication, so that

$$N(F_p)g * s = N(F_p)\prod_{p|p} \pi_p^{-c(p)}gs$$

for any $N(F_p)g \in Y$ and $s \in G^\pi$, where $c(p)$ is the number of times $\pi_p$ appears in the expression of $s$. Since this number does not depend on the specific expression we chose, the action of $G^\pi$ on $Y$ is well defined.

Let $Y'$ denote the smallest subset of $Y$ containing $X$ and stable under $G^\pi$. Define $\mathcal{D}_X$ (respectively, $\mathcal{D}_{Y'}$) to be the $\mathcal{O}$-module of $\mathcal{O}$-values measures on $X$ (respectively, on $Y'$). For $s \in G^\pi$ and $\mu \in \mathcal{D}_{Y'}$, define $\mu * s$ by the integration formula:

$$\int_{Y'} \varphi(\eta)d(\mu * s)(\eta) := \int_{Y'} \varphi(\eta * s)d\mu(\eta),$$

where $\varphi$ is any $\mathbb{C}_p$-valued step function on $Y'$. Denote by $\mathcal{D}_X \xrightarrow{i} \mathcal{D}_{Y'}$ the canonical inclusion defined by extending measures by zero and by $\mathcal{D}_{Y'} \xrightarrow{\tilde{p}} \mathcal{D}_X$. 

15
the canonical projection map. If \( \mu \in \mathbb{D} \) and \( s \in G^\pi \), define
\[
\mu \ast s = p(i(\mu) \ast s).
\]
Since by [2] Lemma 3.1 the kernel of \( p \) is stable under \( G^\pi \), the action is well defined.

By the choice of \( t_\lambda \) in [2] \( \Gamma^\lambda(p^\alpha) \subseteq \text{GL}_2(\mathfrak{t}_p) \) for all \( \lambda \). Hence, \( \mathbb{D}_X \) is a right \( \Gamma^\lambda(p^\alpha) \)-module for all \( \lambda \). Denote by \( \mathbb{D}_X \) the coefficient system associated to \( \mathbb{D}_X \) and set:
\[
\mathbb{W} := H_{cpt}^d(X_S, \mathbb{D}_X).
\]

**Remark 3.2.** Since \( \text{SO}_2(\mathbb{R}) \) is compact and isomorphic to the unit circle \( \mathbb{C}^1 \) and the \( \Gamma^\lambda \) are discrete, the stabilizer \( (\Gamma^\lambda)_{z_0} \) of any element \( z_0 \in \mathcal{S}_d^\mathfrak{t} \) is a finite cyclic group. Since the groups \( \Gamma^\lambda \) are torsion-free, it follows that \( \gamma \) is in the center of \( \Gamma^\lambda \) and hence acts trivially on \( \mathbb{D}_X \). Hence the sheaf \( \mathbb{D}_X \) is well-defined.

Since \( t_\lambda \Delta_0(n)t_\lambda^{-1} \subseteq G^\pi \) for all \( \lambda = 1, \ldots, h \), it follows that \( \mathbb{W} \) is an \( R_{\mathcal{O}}(S, \Delta_0(n)) := \mathcal{O} \otimes_{\mathbb{Z}} R(S, \Delta_0(n)) \)-module. Let \( G \) act on \( X \) by left multiplication. Define \( G' \) to be the multiplicative subset of \( (r \times r)/r \times r \) consisting of pair of elements \( (x, y) \) such that \( x \) and \( y \) are prime to \( p \). The map \( (a, d) \mapsto \omega(a^{n+2v})^{-1}T\left(\begin{array}{cc} a & 0 \\ 0 & d \end{array}\right) \) for \( (a, d) \in G' \) considered in [2.2] is multiplicative, hence extends to a \( \mathcal{O} \)-algebra homomorphism \( \mathcal{O}[G'] \to R_{\mathcal{O}}(S, \Delta_0(n)) \). On the other hand, \( G' \subseteq G \), hence \( \mathcal{O}[G'] \) embeds naturally on \( \tilde{\Lambda} = \mathcal{O}[G] \). Form the \( \tilde{\Lambda} \)-algebra
\[
\mathcal{H} := R_{\mathcal{O}}(S, \Delta_0(n)) \otimes_{\mathcal{O}[G']} \tilde{\Lambda}.
\]
Since the action of \( G' \) on \( \mathbb{W} \) extends to a continuous action of \( G \), it follows that \( \mathbb{W} \) is an \( \mathcal{H} \)-module.

From the fact that \( h_{2t,\epsilon}(S(p^\infty), \mathcal{O}) \) is generated over \( \mathcal{O} \) by \( T(q) \) for all prime ideals \( q \) and those operators coming from the action of \( G_\alpha \), it follows that there is are surjective homomorphisms of \( \tilde{\Lambda} \)-algebras \( \mathcal{H} \to h_{2t,\epsilon}(S(p^\infty), \mathcal{O}) \) and \( \mathcal{H} \to \mathcal{R} \).

Define a subset \( X' \) of \( X \) as follows:
\[
X' = \left\{ x = NC\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in X \left| \left. a \in \mathfrak{r}_p^\times \right) \right. \right\}.
\] (15)

It is easy to check that the definition does not depend on the choice of the representative matrix used to define it and that \( X' \) can be identified with the set \( \overline{\mathfrak{r} \setminus (\mathfrak{r}_p^\times \times \mathfrak{r}_p^\times \times \mathfrak{r}_p^\times)} \) under the above identification between \( X \) and \( \overline{\mathfrak{r} \setminus ((\mathfrak{r}_p^2)^t \times \mathfrak{r}_p^\times)} \). From now on denote elements of \( X \) by \( ((x, y), z) \), where \( (x, y) \) is the first arrow on the matrix and \( z \) is its determinant.
Let $\kappa \in \mathcal{A}(\mathcal{I})$ be an arithmetic point of weight $(n,v)$ and character $\epsilon$ factoring through $\mathbb{Z}_\alpha$. Define the specialization map $\rho_\kappa : \mathbb{D}_X \to L(n,v,\epsilon,\mathcal{O})$ at $P$ by:

$$\mu \mapsto \rho_\kappa(\mu) := \int_{X'} z^v \epsilon(x)(xY - yX)^n d\mu(x,y,z).$$

Suppose that the conductor of $\epsilon$ is $p^\alpha$ for some non negative integer $\alpha$. A simple computation shows that

$$\rho_\kappa(\mu \ast \gamma) = \rho_\kappa(\mu)|\epsilon\gamma$$

for $\gamma \in \text{GL}_2(\mathfrak{r}_p) \cap \Delta_0(p^\alpha)$. It follows that the specialization map $\rho_\kappa$ is $\text{GL}_2(\mathfrak{r}_p) \cap \Delta_0(p^\alpha)$-equivariant. Letting $K := \text{Frac}(\mathcal{O})$, there are $\text{GL}_2(\mathfrak{r}_p) \cap \Delta_0(p^\alpha)$-equivariant maps:

$$\rho_\kappa : \mathbb{W} \to \mathbb{W}_\kappa := H^d_{\text{cpt}}(X_{S(p^\alpha)}, L(n,v,\epsilon,\mathcal{O})).$$

**Proposition 3.3.** Let $\Phi \in \mathbb{W}$.

1. For any prime ideal $q$ of $\mathfrak{r}$ prime to $p$: $\rho_\kappa(\Phi \ast T(q)) = (\rho_\kappa(\Phi))|T(q)$.

2. $\rho_\kappa(\Phi \ast T(p)) = (\rho_\kappa(\Phi))|T_0(p)$.

**Proof.** The equivariance for the action of Hecke operators $T(q)$ in the first statement is immediate because of the $\text{GL}_2(\mathfrak{r}_p) \cap \Delta_0(p^\alpha)$-equivariance of $\rho_\kappa$. It remains to check the action of $T(p)$. To see this, write $\Gamma^\lambda \pi \Gamma^\lambda = \bigcup_t \Gamma^\lambda \alpha_{\lambda,t}$ and note that

$$\rho_\kappa(\Phi \ast T_0(p)) = \int_{X'} \sum_t z^v \epsilon(x)(xY - yX)^n d(\Phi \ast \alpha_{\lambda,t})$$

$$= \int_{X'} \sum_t z^v \epsilon(x)(xY - yX)^n d(\pi^{-1}\Phi \alpha_{\lambda,t})$$

$$= \sum_t \int_{X'} \{p^v\}^{-1} z^v \epsilon(x)(xY - yX)^n |\alpha_{\lambda,t} d(\Phi)$$

$$= \rho_\kappa(\Phi)|T_0(p).$$

This proves the second formula. \qed

Note that

$$\mathbb{W} = \prod_{\lambda=1}^h \mathbb{W}^\lambda, \text{ where } \mathbb{W}^\lambda = H^d_{\text{cpt}}(\Gamma^\lambda \backslash S^d, \mathcal{D}_X)$$

17
\[ \mathbb{W}_\kappa = \prod_{\lambda=1}^{h} \mathbb{W}_\lambda^\lambda, \quad \text{where} \quad \mathbb{W}_\lambda^\lambda := H_{\text{opt}}^d (\tilde{\Gamma}_0^\lambda (p^\alpha) \backslash \mathcal{A}^d, \mathcal{L}(n, v, \epsilon, \mathcal{O})). \]

Any element \( \Phi \in \mathbb{W} \) will be written as \((\Phi_\lambda)_{\lambda=1,...,h}\) while any element of \( \omega \in \mathbb{W}_\kappa \) will be denoted as \((\omega_\lambda)_{\lambda=1,...,h}\). Define:

\[ \Phi_{\lambda, \theta} := \rho_\kappa (\Phi_\lambda) \in \mathbb{W}_\lambda^\lambda. \] (16)

### 3.3 The Control Theorem

Fix \( \theta : \mathcal{R} \to \overline{\mathcal{L}} \) (where \( \mathcal{L} = \text{Frac}(\Lambda) \)) and denote as in §2.3 by \( \mathcal{I} \) the integral closure of \( \Lambda \) in \( \text{Im}(\theta) \). Recall the specialization map \( \theta_\kappa := \kappa \circ \theta : \mathcal{R} \to \mathcal{O}_{p} \) which corresponds to an eigenform \( f_\kappa \). The map \( \theta_\kappa \) extends to the localization \( \mathcal{R}_{P_\kappa} \) of \( \mathcal{R} \) at \( P_\kappa \) and one can intertwine \( \theta_\kappa \) with the map \( \rho_\kappa \) defining

\[ \rho_\kappa : \mathbb{W} \times \mathcal{R}_{P_\kappa} \to \mathbb{W}_\kappa \] (17)

by

\[ \sum_\lambda \Phi_\lambda \times r_\lambda \mapsto \sum_\lambda \rho_\kappa (\Phi_\lambda) \cdot \theta_\kappa (r_\lambda). \]

This is well defined because, if \( g \) belongs to the free part \( W \) of \( G \) and is represented by the matrix \( \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in \text{GL}_2(\mathfrak{p}_p) \), then we have for \( \mu \in \mathbb{D}_X \),

\[ \rho_\kappa (g \mu) = \rho_\kappa (g) \rho_\kappa (\mu). \]

Since \( \rho_\kappa (\phi g, r) = \rho_\kappa (\phi, gr) \) for \( g \in W \) and by continuity the same is true for any element in \( \Lambda \), the map (17) induces a homomorphism

\[ \rho_\kappa : \mathbb{W} \otimes \Lambda \mathcal{R}_{P_\kappa} \to \mathbb{W}_\kappa \]

which is Hecke equivariant.

For any \( \mathcal{H} \)-module \( M \), let \( M^{\text{ord}} \) denote its ordinary part, that is, the maximal subspace of \( M \) on which the \( T(p) \) operator acts as a unit. Let \( h : \mathcal{H} \to \mathcal{R} \) be the natural map obtained by the action of Hecke operators on \( \Lambda \)-adic cusp forms. For any arithmetic point \( \kappa \), let \( h_\kappa \) be the composition of \( h \) with the localization morphism \( \mathcal{R} \to \mathcal{R}_{P_\kappa} \). For any \( \mathcal{H} \otimes \Lambda \mathcal{R}_{P_\kappa} \)-module \( M \), let

\[ M^{h_\kappa} = \{ m \in M \mid (T(q) \otimes 1)m = (1 \otimes h_\kappa (T(q))) \cdot m \text{ for all prime ideals } q \text{ in } \mathfrak{p} \} \]

denote the \( h_\kappa \)-eigenspace of \( M \). If \( f_\kappa \) is a classical eigenform for an arithmetic point \( \kappa \), let

\[ \mathbb{W}_{f_\kappa} = \{ \phi \in \mathbb{W}_\kappa \mid T_0(q) \phi = a_q(g) \phi \} \]
denote the $f_\kappa$-eigenspace of $\mathbb{W}_\kappa$, where $a_q(g)$ is the eigenvalue of the $T_0(q)$ operator on $f_\kappa$. Hence, there is a map:

$$\rho_\kappa : (\mathbb{W} \otimes_\Lambda \mathcal{R}_{P_\kappa})^{h_\kappa} \to \mathbb{W}_\kappa^{f_\kappa}.$$ 

The action of the involution

$$\tau := \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \in \text{GL}_2(\mathfrak{t}_p) \cap \Delta_0(p^\alpha)$$

on $\text{GL}_2(\mathfrak{t}_p) \cap \Delta_0(p^\alpha)$-modules gives rise to $2^d$ eigenspaces indexed by $\text{sgn} \in \{\pm\}^d$. For each $\text{GL}_2(\mathfrak{t}_p)$-module $M$, let $M^{\text{sgn}}$ denote the corresponding eigenspace.

Note that, since $S \supseteq U_1(n)$, there is a notion of primitive homomorphisms which can be introduced as in [4, pages 317, 318]. Say that an arithmetic point $\kappa$ is primitive if both $\theta$ and $b_\kappa$ are primitive characters. Note that in particular $f_\kappa$ is a primitive form of level $U_1(n)$ (see [4 (3.10b)]) and that, by [4 Corollary 3.7],

$$\mathcal{R} \otimes_\Lambda \mathcal{K} \cong \mathcal{K} \oplus \mathcal{B}$$

as an algebra direct sum such that the projection to $\mathcal{K} = \text{Frac}(\text{Im}(\theta))$ coincides with $\theta$ on $\mathcal{R}$.

**Theorem 3.4.** Let $\kappa \in A(I)$ be a primitive arithmetic point of weight $(n_\kappa, v_\kappa)$ and character $\epsilon_\kappa$. For each $\text{sgn} \in \{\pm\}^d$ the map

$$\rho_\kappa : (\mathbb{W} \otimes_\Lambda \mathcal{R}_{P_\kappa})^{h_\kappa, \text{sgn}} / P_\kappa(\mathbb{W} \otimes_\Lambda \mathcal{R}_{P_\kappa})^{h_\kappa, \text{sgn}} \to \mathbb{W}_\kappa^{f_\kappa, \text{sgn}}$$

is an isomorphism.

The proof of this Theorem will be given in §3.6. Before of explaining the proof, we need some preliminary results, stated in §3.4 and §3.5.

### 3.4 Description of $\text{Ker}(\rho_\kappa)$

**Proposition 3.5.** The group $\mathbb{W}_\text{ord}$ of ordinary $\Lambda$-adic modular symbols is a free $\Lambda$-module of finite rank. The kernel of $\rho_\kappa$ is equal to $P_\kappa \mathbb{W}_\text{ord}$.

**Proof.** This is [2 Theorem 5.1], so only a sketch of the proof will be given. First prove the equality $P_\kappa \mathbb{W}_\text{ord} = \text{Ker}(\rho_\kappa)$.

1. $\text{Ker}(\rho_\kappa) \supseteq P_\kappa \mathbb{W}_\text{ord}$: Let $\Phi \in P_\kappa \mathbb{W}_\text{ord}$ and write $\Phi = (\Phi_1, \ldots, \Phi_h)$. Fix $\lambda$ and represent $\Phi_\lambda$ by a cocycle $z$ as above. It follows from [2 Lemma 6.3](19)
that \( \int_X \varphi^{(m)} dz(f) = 0 \) for all \( f \in F_d^\lambda \) and all characteristic functions \( \varphi^{(m)} \). Since the function

\[
((x, y), z) \mapsto \epsilon(x) z^v (xY - yX)^k
\]

appearing in the specialization map \( \rho_\kappa \) can be written as an uniform limit of functions \( \varphi^{(m)} \), the inclusion follows.

2. \( \text{Ker}(\rho_\kappa) \subseteq P_\kappa \mathbb{W}_{\text{ord}} \): Let \( c \in \text{Hom}_{\Gamma_\lambda}(F_k^\lambda, \mathbb{D}_X) \) and choose \( b \) such that \( c = T(p^m)b \): this is possible because \( T(p) \) induces an isomorphism on \( \mathbb{W}_{\text{ord}} \) and, since \( p \) is a principal ideal of \( r \), the \( T(p) \) operator preserves each of the cohomology groups \( H^d(\Gamma^\lambda, \mathbb{D}_X) \). Set \( \pi := \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \).

Since \( X_m \cap X \neq \emptyset \) by \[ \text{Lemma 6.6} \], the above sum is equal to

\[
\int_{X_m} db(f \gamma_{\lambda,t}^{-1})(y) = \rho_\kappa^*(\Phi_\lambda)(f \gamma_{\lambda,t}^{-1}) = 0.
\]

From \[ \text{Lemma 6.3} \] it follows that \( b \) takes values in \( P_\kappa \mathbb{D}_X \). Hence, \( b \) belongs to the image of \( H^d(\Gamma_\lambda, P_\kappa \mathbb{D}_X) \) in \( H^d(\Gamma^\lambda, D_X) \) which, by \[ \text{Lemma 1.2} \], is equal to \( P_\kappa H^d(\Gamma^\lambda, D_X) \).

Finally, from the equality \( P_\kappa \mathbb{W}_{\text{ord}} = \text{Ker}(\rho_\kappa) \) and a compact version of Nakayama’s lemma, it follows that \( \mathbb{W}_{\text{ord}} \) is a \( \Lambda \)-module of finite type.

### 3.5 Lifting system of eigenvalues

Define:

\[
\mathcal{V}_\alpha := H^d_{\text{par}}(X_{S(p\alpha)}, K/\mathcal{O}), \quad \mathcal{V}_\infty := \lim_{\alpha} \mathcal{V}_\alpha,
\]

\[
\mathcal{V}_\alpha^* := \text{Hom}_\mathcal{O}(\mathcal{V}_\alpha, K/\mathcal{O}), \quad \mathcal{V}_\infty^* := \text{Hom}_\mathcal{O}(\mathcal{V}_\infty, K/\mathcal{O})
\]

where the direct limit is computed with respect to the projection maps \( X_{S(p^{\beta})} \to X_{S(p^\alpha)} \) for \( \beta \geq \alpha \). The Hecke algebras \( h_{k,w}(S_0(p^\infty), \epsilon, \mathcal{O}) \) defined in \[ \text{§2.3} \] can be equivalently introduced as \( \lim\limits_{\alpha} h_{k,w}^\alpha(S_0(p^\alpha), \epsilon, \mathcal{O}) \) of the algebra generated over \( \mathcal{O} \) by the Hecke operators; the same observation holds for the ordinary parts: see \[ \text{§3} \] for details.
Let \( P_\kappa \) be an arithmetic point corresponding to a primitive form \( f_\kappa \) of tame level \( S \). Let \( \mathcal{R}_{P_\kappa} \) denote the localization of \( \mathcal{R} \) at \( P_\kappa \). Set \( \mathcal{K}_{P_\kappa} := \text{Frac}(\mathcal{R}_{P_\kappa}) \). Let \( \mathcal{V}_{\infty, \text{ord}}^* \) denote the ordinary submodule of \( \mathcal{V}^* \). For any arithmetic character \( P_{n,v,\epsilon} \) which factors through \( \mathcal{R}_{P_\kappa} \), let and \( \mathcal{V}_{\infty, P_{n,v,\epsilon}}^* \) denote the localization of \( \mathcal{V}_{\infty, \text{ord}}^* \) at \( P_{n,v,\epsilon} \).

**Proposition 3.6.** Let \( d \) be odd. Then \( \mathcal{V}_{\infty, P_{n,v,\epsilon}}^* \) is free of rank \( 2d \) over \( \mathcal{R}_{P_\kappa} \) and for each sign \( \text{sgn} \in \{\pm 1\}^d \), its \( \text{sgn} \)-eigenmodule is free of rank one.

**Proof.** It is enough to prove the second statement. First note that as in \([6, \text{page } 1032]\) there is an isomorphism:

\[
(\mathcal{V}_{\infty, P_{n,v,\epsilon}}^* / \mathcal{V}_{\infty, P_{n,v,\epsilon}}^* / \mathcal{V}_{\infty, P_{n,v,\epsilon}}^* / \text{sgn}) \cong k_{\text{ord}}(S_0(p^\alpha), \epsilon, K)
\]

for any arithmetic point \( P_{n,v,\epsilon} \). By \([13]\), \( k_{\text{ord}}(S_0(p^\infty), \epsilon, K) \) is free of rank one over \( \mathcal{R}_{P_\kappa} / \mathcal{V}_{\infty, P_{n,v,\epsilon}}^* / \text{sgn} \), and hence, if \( \mathfrak{m} \) is the maximal ideal of \( \mathcal{R}_{P_\kappa} \), if follows that \( (\mathcal{V}_{\infty, P_{n,v,\epsilon}}^* / \mathcal{V}_{\infty, P_{n,v,\epsilon}}^* / \mathcal{V}_{\infty, P_{n,v,\epsilon}}^* / \text{sgn}) / \mathfrak{m}(\mathcal{V}_{\infty, P_{n,v,\epsilon}}^* / \text{sgn}) \) is free of rank one over \( \mathcal{R}_{P_\kappa} / \mathfrak{m} \). The result follows from \([5, \text{Lemma } 3.10]\) and its proof.

Recall the following:

**Lemma 3.7.** Let \( Z \) be a topological space that is an inverse limit of finite discrete topological spaces \( Z_\alpha \) for \( \alpha \) in some indexing set. Then the space of \( \mathcal{O} \)-valued measures on \( Z \) is isomorphic to \( \lim \leftarrow \text{Fns}(Z_\alpha, \mathcal{O}) \) where \( \text{Fns}(Z_\alpha, \mathcal{O}) = \mathcal{O}^Z_\alpha \) is the space of continuous \( \mathcal{O} \)-valued functions on \( Z_\alpha \).

**Proof.** If \( \phi \in \lim \leftarrow \text{Fns}(Z_\alpha, \mathcal{O}) \), then \( \mu \) can be written as a compatible sequence of the form \( \{\sum_{x \in Z_\alpha} a_x \cdot x\}_\alpha \). Let \( p_\alpha : Z \to Z_\alpha \) be the natural projection map and for each \( x \in Z_\alpha \) set \( U_{\alpha,x} = p_\alpha^{-1}(x) \). Then this is isomorphic to the space of \( \mathcal{O} \)-valued measures on \( Z \) by the map

\[
\phi \mapsto \mu \text{ such that } \mu(U_{\alpha,x}) = a_x.
\]

This defines a measure due to the compatibility of the sequence. \( \square \)

For each \( \alpha \), let \( p_\alpha : X_{S(p^\alpha)} \to X_S \). Let \( \mathcal{F}_\alpha = p_{\alpha*}\mathcal{O} \) be the direct image of the constant sheaf \( \mathcal{O} \) on \( X_S \). Fix a point \( x \in X_S \) and define \( Y_\alpha = Y_{\alpha,x} \) to be the fiber \( p_\alpha^{-1}(x) \) of \( x \) under \( p_\alpha \). By Lemma \([37, \text{Lemma } 3.17]\), \( \lim_{\text{proj}} \mathcal{O}^\alpha \) is the space of \( \mathcal{O} \)-valued measures on the space \( \lim_{\text{proj}} Y_\alpha \). Now, for the double coset \( \text{GL}_2(F)_{xSS_\infty} \) in \( X_S \), there is a natural map

\[
S/S(p^\alpha) = \text{GL}_2(F)/S(p^\alpha) \to Y_\alpha
\]

given by

\[
zS(p^\alpha) \mapsto \text{GL}_2(F)_{xzS(p^\alpha)S_\infty}.
\]

21
This map induces a bijection from $\text{GL}_2(\mathfrak{t}_p)/S(p^\alpha)(\mathfrak{t}^\times \cap SS_\infty)$ to $Y_\alpha$. Hence the inverse limit $\lim_{\leftarrow} Y_\alpha$ can be identified with $\text{GL}_2(\mathfrak{t}_p)/NC$ which is finally identified with $X$.

**Lemma 3.8.** The sheaves $\mathcal{D}_X$ and $\lim_{\alpha} \mathcal{F}_\alpha$ on $X_S$ are isomorphic.

**Proof.** Let $U \in X_S$ be an open set. For each $\alpha$ and $w \in Y_\alpha$, let $X_{\alpha,w} \subset X$ be the inverse image of $w$ under the natural projection map $X \to Y_\alpha$. Let $u \in U$, then choose $x \in \text{GL}_2(F_A)$ such that $u = \text{GL}_2(F)xSS_\infty$. If $s \in \mathcal{D}_X(U)$ is a section in $\mathcal{D}_X(U)$, we can express $s(u) = \text{GL}_2(F)(x, \mu(x))(SS_\infty)$ for some $\mu \in \mathcal{D}_X$, depending on $s$. In fact, since $s$ is a locally constant section, this expression is valid in a neighborhood $U_u$ of $u$. Then define a map:

$$\mathcal{D}_X(U) \xrightarrow{\phi(U)} \mathcal{F}_\alpha(U) = \mathcal{O}(p_\alpha^{-1}(U)) \quad u \mapsto \sum_{w \in Y_\alpha} \mu(x)(X_{\alpha,w}).$$

This map is independent of the choice of $x$ since a different representative $x'$ of the double coset $u$ would yield the same measure $\mu(x)$ since $s$ is a section on $X_S$.

The compatible maps $\phi_\alpha$ then give rise to a map

$$\phi(U) : \mathcal{D}_X(U) \to (\lim_{\alpha} \mathcal{F}_\alpha)(U).$$

At the level of the stalks this map is the isomorphism in (18). It follows that $\phi$ is an isomorphism of sheaves. \qed

By Lemma 3.8,

$$\mathbb{W} = H^d_{\text{cpt}}(X_S, \mathcal{D}_X) \simeq H^d_{\text{cpt}}(X_S, \lim_{\alpha} \mathcal{F}_\alpha).$$

Since

$$H^d_{\text{cpt}}(X_S, \mathcal{F}_\alpha) = H^d_{\text{cpt}}(X_S, p_{\alpha*} \mathcal{O}) \simeq H^d_{\text{cpt}}(X_{S(p^\alpha)}, \mathcal{O}),$$

there is a surjective map

$$\mathbb{W} \to \lim_{\alpha} H^d_{\text{cpt}}(X_{S(p^\alpha)}, \mathcal{O}).$$

(19)

By Poincaré duality:

$$H^d_{\text{cpt}}(X_{S(p^\alpha)}, \mathcal{O}) \simeq H_d(X_{S(p^\alpha)}, \mathcal{O}).$$

(20)

By Pontryagin duality there is a canonical isomorphism:

$$\text{Hom}_\mathcal{O}(H^d(X_{S(p^\alpha)}, K/\mathcal{O}), K/\mathcal{O}) \simeq H_d(X_{S(p^\alpha)}, \mathcal{O}),$$

(21)
where \( K := \text{Frac}(\mathcal{O}) \).

The injection
\[
\mathcal{V}_\alpha \hookrightarrow H^d(X_{S(p^\alpha)}, K/\mathcal{O})
\]
induces an injection passing to the direct limits:
\[
\mathcal{V}_\infty \hookrightarrow \lim_{\alpha} H^d(X_{S(p^\alpha)}, K/\mathcal{O})
\]
and hence there is a surjective map:
\[
\text{Hom}_\mathcal{O}(\lim_{\alpha} H^d(X_{S(p^\alpha)}, K/\mathcal{O}), K/\mathcal{O}) \to \mathcal{V}_\infty^*.
\] (22)

By composing the maps (19), (20), (21), (22), we get a surjective map:
\[
\mathbb{W} \to \mathcal{V}_\infty^*.
\]

Note that this map is also equivariant for the action of the Hecke operators and of the involution \( \tau = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \ldots, \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \).

**Corollary 3.9.** The \( h_\kappa \)-eigenspace of \( \mathbb{W}_L \otimes_L \mathcal{K}_{P_\kappa} \) has dimension at least \( 2^d \) over \( \mathcal{K}_{P_\kappa} \).

**Proof.** For each sign sgn there is an Hecke and \( \tau \) equivariant map of finite dimensional \( \mathcal{K}_{P_\kappa} \)-vector spaces
\[
\mathbb{W}_L \otimes_L \mathcal{K}_{P_\kappa} \to \mathcal{V}_{\infty,\text{ord}}^{*,\text{ord}} \otimes_{\mathcal{R}_{P_\kappa}} \mathcal{K}_{P_\kappa}.
\]

Since \( \mathcal{H}[\tau] \) is commutative and the \((h_\kappa, \text{sgn})\)-eigenspace of \( \mathcal{V}_{\infty,\text{ord}}^{*,\text{ord}} \otimes_{\mathcal{R}_{P_\kappa}} \mathcal{K}_{P_\kappa} \) in non-trivial, being of dimension 1 over \( \mathcal{K}_{P_\kappa} \), it follows that the \((h_\kappa, \text{sgn})\)-eigenspace in \( \mathbb{W}_L \otimes_L \mathcal{K}_{P_\kappa} \) is non-zero too (for this linear algebra argument, see [12, Lemma 5.10]). Since this holds for all \( \text{sgn} \in \{\pm 1\}^d \), the result follows.

### 3.6 Proof of the Control Theorem

Before proving Theorem 3.4, recall the following Lemma. Let \( I_h := \text{Ker}(h) \) be the kernel of the canonical map \( h : \mathcal{H} \to \mathcal{R} \).

**Lemma 3.10.** Let \( M \) be an \( \mathcal{H} \)-module and \( P \) be an ideal in \( \Lambda \). Suppose that \( P \) is generated by an \( M \)-regular sequence \( (x_1, \ldots, x_r) \). Then the image of the map
\[
i_* : \text{Ext}_{\mathcal{H}}^*(\mathcal{H}/I_h, PM) \to \text{Ext}_{\mathcal{H}}^*(\mathcal{H}/I_h, M)
\]
induced by the inclusion \( i : PM \to M \) is equal to \( P\text{Ext}_{\mathcal{H}}^*(\mathcal{H}/I_h, M) \).
Proof. The proof of this lemma is a slight modification of that of [2, Lemma 1.2] and will be omitted.

Remark 3.11. In order to apply this lemma in the proof of the main theorem, we first note that for any \( H \)-module \( M \), the Hecke eigenspace \( M^{h_0} = \text{Hom}_H(\mathcal{R}, M) = \text{Ext}^0_H(\mathcal{R}, M) \) and that \( \mathbb{W}^{\text{ord}} \) is a \( H \) direct summand of \( \mathbb{W} \). This follows from the fact that the generators of \( P_\kappa \) are a regular sequence for \( \Lambda_\pi \) and that \( \text{Ord} \) is a \( H \)-direct summand of \( \text{Ord} \).

Finally, we need to show that the generators of \( P_\kappa \) is a \( \text{Ord} \)-regular sequence. This follows from the fact that the generators of \( P_\kappa \) are a regular sequence for \( \Lambda_\pi \) and that \( \text{Ord} \) is a free module of finite rank over \( \Lambda_\pi \).

Now the proof of the Control Theorem follows [3] and can be described as follows. Recall that we have to show that for each primitive arithmetic point \( \kappa \) of weight \((n(\kappa), v(\kappa))\) and character \( \epsilon(\kappa) \), and for choice of sign \( \text{sgn} \in \{\pm\}^d \) the map

\[
\rho_\kappa : \left( \mathbb{W} \otimes_\Lambda \mathcal{R}_{P_\kappa} \right)^{h_\kappa, \text{sgn}} / P_\kappa \left( \mathbb{W} \otimes_\Lambda \mathcal{R}_{P_\kappa} \right)^{h_\kappa, \text{sgn}} \rightarrow \mathbb{W}_\kappa
\]

is an isomorphism.

Since \( h_\kappa(T(p)) \) is a unit in \( \mathcal{R} \), the module \( \left( \mathbb{W} \otimes_\Lambda \mathcal{R}_{P_\kappa} \right)^{h_\kappa} \) is contained in the nearly ordinary part \( \mathbb{W}^{\text{ord}} \otimes_\Lambda \mathcal{R}_{P_\kappa} \). Since, by Proposition 3.5, \( \mathbb{W}^{\text{ord}} \) is a free \( \Lambda_\pi \)-module of finite rank, it follows that \( \mathbb{W}^{\text{ord}} \otimes_\Lambda \mathcal{R}_{P_\kappa} \) is a free \( \mathcal{R}_{P_\kappa} \)-module of finite rank. By Proposition 3.5 and the fact that \( \mathcal{R}_{P_\kappa} \) is unramified over \( \Lambda_\pi \), it follows that the kernel of the map \( \mathbb{W}^{\text{ord}} \otimes_\Lambda \mathcal{R}_{P_\kappa} \rightarrow \mathbb{W}_0^{\text{ord}} \) is \( P_\kappa \left( \mathbb{W} \otimes_\Lambda \mathcal{R}_{P_\kappa} \right)^{\text{ord}} \). This is the same as \( P_\kappa \left( \mathbb{W} \otimes_\Lambda \mathcal{R}_{P_\kappa} \right)^{h_\kappa} \) by Lemma 3.10 and the remark following it. Hence, we get an injective map

\[
\left( \mathbb{W} \otimes_\Lambda \mathcal{R}_{P_\kappa} \right)^{h_\kappa} / P_\kappa \left( \mathbb{W} \otimes_\Lambda \mathcal{R}_{P_\kappa} \right)^{h_\kappa} \rightarrow \mathbb{W}_\kappa^{f_\kappa}.
\]

Since \( \mathbb{W}_\kappa^{f_\kappa} \) is \( 2^d \)-dimensional, to prove the surjectivity of the map it suffices to show that \( \left( \mathbb{W} \otimes_\Lambda \mathcal{R}_{P_\kappa} \right)^{h_\kappa} \) has \( \mathcal{R}_{P_\kappa} \)-rank at least \( 2^d \). Recall that by Corollary 3.9 the \( h_\kappa \)-eigenspace of \( \mathbb{W} \otimes_\Lambda \mathcal{K}_{P_\kappa} \) has dimension \( 2^d \) over \( \mathcal{K}_{P_\kappa} \). The intersection of this eigenspace with \( \mathbb{W} \otimes_\Lambda \mathcal{R}_{P_\kappa} \) is a \( \mathcal{R}_{P_\kappa} \)-submodule of \( \left( \mathbb{W} \otimes_\Lambda \mathcal{R}_{P_\kappa} \right)^{h_\kappa} \) of rank \( 2^d \). The surjectivity of the above map follows.

Since specialization commutes with the action of the \( 2^d \)-complex conjugations, the result follows.

4 \( p \)-adic L-functions

4.1 Complex L-functions

Let \( f \in S_{k, \psi}(S(p^\alpha), \mathbb{C}) \) for some \( \alpha \geq 0 \). Recall the modified Fourier coefficients \( C(m, f) \) defined in [3]. Let \( I_\ell \) be the the group of fractional ideals of \( \mathfrak{r} \).
prime to \( c \) and fix a character \( \chi : I_c \to \mathbb{C} \) of sign \( \text{sgn}(\chi) \). For integral ideals \( m \) which are not prime to \( c \), let \( \chi(m) := 0 \). Define the Dirichlet series:

\[
L(f, \chi, s) := \sum_m \chi(m)C(m, f) N(m)^s,
\]

where the sum is over all integral ideals of \( \mathfrak{r} \) and \( N(m) \) is the norm of \( m \). By [13], this \( L \)-series converges if \( \Re(s) \) is sufficiently large and can be continued analytically to an entire function on \( \mathbb{C} \). Choose a finite index subgroup \( U \subseteq \mathfrak{r}^x \) such that \( a_\lambda(\epsilon \xi) = \epsilon^{k/2} a_\lambda(\epsilon \xi) \) for all \( \epsilon \in U \), \( \xi \in t_\lambda \mathfrak{d} \) and \( \lambda \). The choice of \( U \) is possible by [13, 1.8]. Define

\[
L(f_\lambda, \chi, s) = [r^x \mathfrak{r}]^{-1} \sum_{\xi \in \tau_\lambda \mathfrak{d} / U, \xi \gg 0} \chi(\xi(t_\lambda \mathfrak{d})^{-1}) a_\lambda(\xi) \xi^u b_{v,\lambda} N(\xi(t_\lambda \mathfrak{d})^{-1})^s.
\]

(Recall that \( b_{v,\lambda} \) is defined in (9).) Then

\[
L(f, \chi, s) = \sum_{\lambda = 1}^{h(\rho^u)} L(f_\lambda, \chi, s).
\]

### 4.2 Special values

Let \( \mathfrak{r}_+^x \) the group of totally positive units in \( \mathfrak{r}^x \). For any element \( x \in F \), choose a subgroup \( \mathfrak{r}_x \subseteq \mathfrak{r}_+^x \) of finite index such that \( ex \equiv x \pmod{t_\mathfrak{r}^{-1}} \) for all \( e \in \mathfrak{r}_x \). The group \( \mathfrak{r}_+^x \) act on \( \mathbb{R}_+[I] \) by

\[
x = \sum_{\sigma \in I} x_\sigma \sigma \mapsto ex := \sum_{\sigma \in I} \sigma(e)x_\sigma \sigma.
\]

Let \( D := \{ x \in \mathbb{R}_+[I] : \prod_{\sigma \in I} x_\sigma = 1 \} \). Since \( \prod_{\sigma \in I} \sigma(e) = 1 \) for all \( e \in \mathfrak{r}_x^x \), the action of \( \mathfrak{r}_+^x \) on \( \mathbb{R}_+[I] \) restricts to an action on \( D \). The logarithm map \( \log : \mathbb{R}_+[I] \to \mathbb{R}[I] \) defined by \( \log(\sum_{\sigma \in I} x_\sigma \sigma) := \sum_{\sigma \in I} \log(x_\sigma) \sigma \) is an isomorphism and the action of \( \mathfrak{r}_+^x \) on the left corresponds to the translation by the sublattice \( \log(\mathfrak{r}_+^x) \) of rank \( d - 1 \) on the right. Choose a fundamental parallelogram of this lattice and denote by \( B(x) \) its inverse image under \( \log \). Define the cone \( C(x) \) with vertex \( 0 \in \mathbb{R}[I] \) and base \( B(x) \) to be

\[
C(x) := \left\{ t \xi = \sum_{\sigma \in I} t \xi_\sigma \sigma : 0 < t < \infty, \xi = \sum_{\sigma \in I} \xi_\sigma \sigma \in B(x) \right\}.
\]

Then \( C(x) \) is a fundamental domain in \( \mathbb{R}_+[I] \) for \( \mathbb{R}_+[I] / \mathfrak{r}_x \).
Let \( E \) be a \( \Gamma^\lambda(p\alpha) \)-module. Suppose that \( E \) is also a \( \mathbb{Z}[T_x, x \in F] \)-module, where \( T_x := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \). For any modular symbol \( \omega \in H^d_{\text{cpt}}(\Gamma^\lambda(p\alpha) \backslash \mathcal{H}, \mathcal{E}) \), where \( \mathcal{E} \) is the coefficient system associated to \( E \), and any \( x \in F \), define

\[
L(\omega, C(x)) := \frac{\int_{x+iC(x)} \omega|_E T_x}{|r^*_+ : r^-_x|}.
\]

In the above formula \(|_E\) is the action of \( T_x \) on \( E \). For \( x = 0 \), write simply \( L(\omega) \) for \( L(\omega, C(0)) \).

For any finite order character \( I_{\epsilon} \rightarrow \mathbb{C}^\times \), define:

\[
L(\omega, \chi) := \sum_{a \in \epsilon^{-1}I_{\lambda}^{-1}/I_{\lambda}^{-1}} \text{sgn}(a)\text{sgn}(\chi)\chi(\alpha t_{\lambda}) L(\omega, C(a)).
\]

Remark 4.1. Note that in general \( L(\omega, C(x)) \) depends on the choice of the fundamental domain \( C(x) \). However, the values \( L(\omega, C(x)) \) appearing in the arithmetic applications do not depend on the choice of \( C(x) \), as will be clarified in the following.

4.3 Interpolation formulas for classical modular symbols

Let \( f = (f_\lambda)_{\lambda=1}^{h(p\alpha)} \in S_{k,w}(S(p\alpha), \mathbb{C}) \) for some \( \alpha \geq 0 \). Use the following notations: for any \( r \in \mathbb{Z}[I] \) with \( 0 \leq r_\sigma \leq n_\sigma \), set \( \binom{n}{r} := \prod_{\sigma \in I} \binom{n_\sigma}{r_\sigma} \) and \( |r| := \sum_\sigma n_\sigma \). Write

\[
L(\omega(f_\lambda), C(x)) = \sum_{r=0}^{n} \binom{n}{r} (-1)^{r+t} L(\omega(f_\lambda), C(x), r) X^r Y^{n-r}.
\]

For \( x = 0 \), write simply \( L(\omega(f_\lambda), r) \) for \( L(\omega(f_\lambda), C(0), r) \). Likewise, write

\[
L(\omega(f_\lambda), \chi) = \sum_{r=0}^{n} \binom{n}{r} (-1)^{r+t} L(\omega(f_\lambda), \chi, r) X^r Y^{n-r}.
\]

In the above formulas, the sum is over all \( r \in \mathbb{Z}[I] \) such that \( 0 \leq r_\sigma \leq n_\sigma \).

Set \( k_0 := \max\{k_\sigma, \sigma \in I\} \) and \( k_1 := \min\{k_\sigma, \sigma \in I\} \). Define the critical strip for \( f \) to be the set of complex numbers \( s \in \mathbb{C} \) such that

\[
k_* := \frac{k_0 - k_1}{2} + 1 \leq \Re(s) \leq k^* := \frac{k_0 + k_1}{2} - 1.
\]
It is easy to show that for any integer $m$ in the critical strip, the values $L_{m-v-1}(\omega(f_\lambda), C(x))$ does not depend on the choice of the fundamental domain for $\mathbb{R}_+\llbracket I \rrbracket / \mathfrak{r}_k^\times$ (see [8, §3.4]).

Set:

$$L(\omega(f), r) := \sum_{\lambda=}^{\text{h}(p^n)} L(\omega(f_\lambda), r)N(t_\lambda \mathfrak{d})^{-r+v+1}b_{v,\lambda}.$$  

For any $n = \sum n_\sigma \sigma \in \mathbb{Z}[I]$, define $\Gamma_F(n) := \prod_{\sigma \in \mathcal{I}} \Gamma(n_\sigma)$, where $\Gamma$ is the usual complex $\Gamma$-function.

**Proposition 4.2.** For any integer $k_* \leq m \leq k^*$ in the critical strip, the following interpolation formulas hold:

$$L(\omega(f_\lambda), m - v - 1) = c(m, v, \lambda)L(f_\lambda, m)\Gamma_F(m - v)$$

where $c(m, v, \lambda) := i^{m-v-t}(2\pi)^{|v-m|}N(t_\lambda \mathfrak{d})^m b_{v,\lambda}$, and

$$L(\omega(f), m - v - 1) = \frac{i^{m-v-1}L(f, m)\Gamma_F(m - v)}{(2\pi)^{|m-t|}}.$$  

**Proof.** It is enough to show the first formula, which can be obtained by following [13, §4]. Choose $U$ a finite index subgroup of $\mathfrak{r}_k^\times$ as in §4.1 such that $a_{\lambda}(\epsilon_\xi) = e^{k/2}a_{\lambda}(\epsilon_\xi)$ for all $\epsilon \in U$ and all $\lambda$. First note that

$$\int_{C(0)} f_{\lambda}(iy)y^{m-v-t}dy = \int_{\mathbb{R}_+^d / U} \sum_{\epsilon \in U} \sum_{\xi \in t_\lambda \mathfrak{d} / U, \xi \gg 0} a_{\lambda}(\epsilon_\xi)e^{-2\pi(\epsilon_\xi y)}y^{m-v-t}dy$$

(here quotient measures are implicitly used). Set $a'_{\lambda}(\xi) := a_{\lambda}(\xi)x^v$. Since $a'_{\lambda}(\epsilon_\xi) = a'_{\lambda}(\xi)$ for all $\epsilon \in U$, it follows that:

$$\int_{C(0)} f_{\lambda}(iy)y^{mt-v-t}dy = \sum_{\xi \in t_\lambda \mathfrak{d} / U, \xi \gg 0} \frac{a'_{\lambda}(\xi)}{\xi} \int_{\mathbb{R}_+^d} e^{-2\pi(\xi y)}(\xi y)^{mt-v-t}dy.$$  

Changing variables $s := 2\pi(\xi y_\sigma)_{\sigma}$ yields

$$\int_{\mathbb{R}_+^d} e^{-2\pi(\xi y)}(\xi y)^{mt-v-t}dy = \int_{\mathbb{R}_+^d} e^{-s} s^{mt-v-t} ds \frac{1}{(2\pi)^{|mt-v-t|}(2\pi)^{|t|\xi}}$$

hence:

$$\int_{C(0)} f_{\lambda}(iy)y^{m-v-t}dy = (2\pi)^{|v-mt|} \sum_{\xi \in t_\lambda \mathfrak{d} / U, \xi \gg 0} \frac{a'_{\lambda}(\xi)}{\xi} \Gamma_F(m - v).$$

The result follows. \qed
Fix a character \( \chi \) with sign \( \text{sgn}(\chi) \) as in [\ref{4.1}]. Assume that \( \chi \) is primitive modulo \( c \). Form the Gauss sum:

\[
\tau(\chi) := \sum_{u \in c^{-1}/O_F} \text{sgn}(u)^{\text{sgn}(\chi)} \chi(uc)e^{2\pi i(u \cdot t)}.
\]

If \( \chi \) is not primitive modulo \( c \), let \( \chi' \) be the primitive associated character and define \( \tau(\chi) := \tau(\chi') \). Define:

\[
f^\chi_\lambda(z) := \sum_{x \in t \lambda d, \xi \gg 0} \chi(x(t \lambda d)^{-1})a_\lambda(x)e^{2\pi i(x \cdot z)}.
\]

By [\cite{13} Proposition 4.4], \( f^\chi_\lambda \) is a modular form for the same congruence subgroup and of the same weight as \( f_\lambda \), with character \( \epsilon \chi^{-2} \), and

\[
f^\chi_\lambda(z) = \tau(\chi)^{-1} \sum_{u \in c^{-1}t_\lambda^{-1}/t_{\lambda'}^{-1}} \text{sgn}(u)^{\text{sgn}(\chi)} \chi(uct_\lambda)f_\lambda(z + u).
\]

Combining this expression with the definition of twisted \( L \)-function and the interpolation formula described above yields:

\[
\sum_{u \in c^{-1}t_\lambda^{-1}/t_{\lambda'}^{-1}} \text{sgn}(u)^{\text{sgn}(\chi)} \chi(uct_\lambda)L(\omega(f_\lambda), C(u), m - v - 1) = \frac{c(m, v, \lambda)}{\tau(\chi)} \Gamma_F(m - v)L(f_\lambda, \chi, m).
\]

(24)

**Remark 4.3.** In the above formula we understand that a finite index subgroup \( I \subseteq \mathfrak{k}_r^\chi \) has been chosen such that \( I \subseteq \mathfrak{t}_u \) for all \( u \). Use the independence of \( L(\omega(f_\lambda), C(u)) \) on the choice of \( I \) as above to recover formula (24). We will not mention this point hereafter.

Setting

\[
L(\omega(f), \chi, r) := \sum_{\lambda=1}^{h(p^\alpha)} L(\omega(f_\lambda), \chi, r)N(t_\lambda)^{-m+v+1}b_{v,\lambda}
\]

\[
\Lambda(f_\lambda, \chi, m) := \frac{c(m, v, \lambda)}{\tau(\chi)} \Gamma_F(m - v)L(f_\lambda, \chi, m)
\]

(25)

\[
\Lambda(f, \chi, m) := \sum_{\lambda=1}^{h(p^\alpha)} \Lambda(f_\lambda, \chi, m)N(t_\lambda)^{-r+v+1}b_{v,\lambda}
\]

(26)
yields formulas in a more compact form:

\[
L(\omega(f), \chi, m - v - 1) = \Lambda(f, \chi, m) \quad \text{and} \quad L(\omega(f), \chi, m - v - 1) = \Lambda(f, \chi, m).
\]

(27)

Fix a sign \(\text{sgn} = \{\text{sgn}_\sigma, \sigma \in I\}\), a character \(\chi\) of sign \(\text{sgn}(\chi) = \text{sgn}\) and an eigenform \(f \in S_{k,w}(S(p^n), K_f)\) with eigenvalues in \(K_f\). Denote by \(K_f(\chi)\) the extension of \(K_f\) generated by the values of \(\chi\). By [13], for any sign \(\text{sgn}' \in \{\pm 1\}\) there exists periods \(\Omega_{\text{sgn}}^f(f)\) such that for any integer \(m\) in the critical strip for the weight of \(\kappa\):

\[
\frac{L(\omega(f), \chi, m - v - 1)}{\Omega_{\text{sgn}(m)}(f, \chi)} \in K(\chi),
\]

(28)

where \(\text{sgn}(m) := (-1)^m\) (see [11, §8.1] for a result stated in a form close to this). Define the normalized modular symbols associated to \(f\) to be

\[
\Psi_{f,\chi}^{\text{sgn}} := \omega(f)/\Omega_{\text{sgn}}^f(f, \chi),
\]

so that

\[
L(\Psi_{f,\chi}^{\text{sgn}}, \chi, m - v - 1) = \frac{\Lambda(f, \chi, m)}{\Omega_{\text{sgn}}(f, \chi)},
\]

(29)

where \(L(\Psi_{f,\chi}^{\text{sgn}}, \chi, r) := \frac{L(\omega(f, \chi)^{\text{sgn}}, r)}{\Omega_{\text{sgn}}^f(f, \chi)}\) for \(r = m - v - 1\) and \(m\) in the critical strip. Setting \(L(\Psi_f^{\text{sgn}}, \chi, r) := \sum_{\lambda} L(\Psi_{f,\chi}^{\text{sgn}}, \chi, r)N(t_\lambda)^{-r+v+1}b_v,\lambda\) as usual for \(r\) as above yields:

\[
L(\Psi_f^{\text{sgn}}, \chi, m - v - 1) = \frac{\Lambda(f, \chi, m)}{\Omega_{\text{sgn}}(f, \chi)}.
\]

(30)

### 4.4 \(p\)-adic \(L\)-functions of \(\Lambda\)-adic modular symbols

The map \(z \mapsto iz\) defines a map: \(\Delta_\lambda^1 : C(x) \to X_S\) which induces a corresponding map on the cohomology \(H^d_{\text{cpt}}(X_S, D_X) \to H^d_{\text{cpt}}(C(x), \Delta_\lambda^1 D_X)\), denoted by the same symbol. For any \(\Phi \in \mathbb{W}\), write \(\Phi = (\Phi_1, \ldots, \Phi_h)\) and define the special value of \(\Phi\) to be \(L(\Phi) = (L(\Phi_1), \ldots, L(\Phi_h))\) where for each \(\lambda = 1, \ldots, h:\)

\[
L(\Phi_\lambda, C(0)) = L(\Phi_\lambda) := \int_{C(0)} \Delta_\lambda^1 \Phi_\lambda \in D.
\]

Define:

\[
X'' := NC \backslash \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in X' : d \in r_p^x \right\}.
\]

Then \(X''\) is identified with \(\mathbf{\overline{r}} \backslash (r_p^x \times r_p^x \times r_p^x)\) under the bijection between \(X\) and \(\mathbf{\overline{r}} \backslash ((r_p^x)^x \times r_p^x)\).
For any \( p \)-adic group \( \Pi \), set \( \mathcal{X}(\Pi) := \text{Hom}_{\text{cont}}(\Pi, \overline{\mathbb{Q}}^\times) \). Any \( P \in \mathcal{X}(G) \) induces a character on \( \overline{\mathbb{Q}}^\times/(\mathfrak{r}_p^\times \cdot \mathfrak{r}_p^\times) \) via the isomorphism \( G \simeq (\mathfrak{r}_p^\times \cdot \mathfrak{r}_p^\times)/\mathfrak{r} \). In particular, if \( \kappa \in \mathcal{A}(L) \) has weight \((n, v)\) and character \( \epsilon \),

\[
\kappa((x, z)) = \epsilon x^n z^v.
\]

Fix \( \Phi \in \mathcal{W} \). For \( (\kappa, \sigma) \in \mathcal{X}(G) \times \mathcal{X}(\mathfrak{r}_p^\times) \), define the standard several variables \( p \)-adic \( L \)-function of \( \Phi_\lambda \):

\[
L_p(\Phi_\lambda, \kappa, \sigma) = \int_{\mathcal{X}^n} \kappa(x, z/x^2) \sigma(y/x) \, dL(\Phi_\lambda)(x, y, z).
\]

Fix two multi-integers \( r = (r_\sigma)_\sigma \in \mathbb{Z}[I] \) and \( m = (m_\sigma)_\sigma \in \mathbb{Z}[I] \) with all \( r_\sigma \) and \( m_\sigma \) non negative. Write \( p^m \) for \( \prod_{\sigma \in I} p_\sigma^{m_\sigma} \), where \( p_\sigma \) is the prime divisor of \( \mathfrak{m} \sigma(p) \). Say that a point \( \sigma \in \mathcal{X}(\mathfrak{r}_p^\times) \) is arithmetic if there exist a character \( \chi : I_{p^m} \to \overline{\mathbb{Q}}^\times_p \) such that \( \sigma(x) = \sigma_{\chi, r}(x) = \chi(\overline{x})x^r \), where \( x \mapsto \overline{x} \) is the projection map \( \mathfrak{r}_p^\times \to \mathfrak{r}^\times/p^m \) and \( m, r \) are as above. Denote by \( \mathcal{A}(\mathfrak{r}_p^\times) \) the set of arithmetic characters. Define the sign of \( \sigma_{\chi, m} \) to be the sign of \( \chi \).

If \( \sigma_{\chi, r} \in \mathcal{A}(\mathfrak{r}_p^\times) \), then it can be extended to a character \( \sigma \in \text{Hom}_{\text{cont}}(\mathfrak{r}_p, \overline{\mathbb{Q}}_p) \) by setting \( \sigma(x) = x^r \) if \( \chi \) is the trivial character and \( \sigma(x) := 0 \) for \( x \notin \mathfrak{r}_p^\times \) otherwise. For \( (\kappa, \sigma) \in \mathcal{X}(G) \times \mathcal{A}(\mathfrak{r}_p^\times) \), define the improved several variables \( p \)-adic \( L \)-function of \( \Phi_\lambda \):

\[
L^*_p(\Phi_\lambda, \kappa, \sigma) = \int_{X^*} \kappa(x, z/x^2) \sigma(y/x) \, dL(\Phi_\lambda)(x, y, z).
\]

It follows from the definitions that \( L_p(\Phi_\lambda, \kappa, \sigma) \) is analytic in \( (\kappa, \sigma) \in \mathcal{X}(G) \times \mathcal{X}(\mathfrak{r}_p^\times) \) and \( L^*_p(\Phi_\lambda, \kappa, \sigma) \) is analytic in \( (\kappa, \sigma) \in \mathcal{X}(G) \times \mathcal{A}(\mathfrak{r}_p^\times) \). Define homomorphism of \( \widetilde{\Lambda} \)-algebras

\[
L(\cdot, \sigma) : \mathcal{W} \to \widetilde{\Lambda} \quad \text{and} \quad L^*(\cdot, \sigma) : \mathcal{W} \to \widetilde{\Lambda}
\]

by requiring that

\[
\kappa(L_p(\Phi_\lambda, \sigma)) = L_p(\Phi_\lambda, \kappa, \sigma) \quad \text{and} \quad \kappa(L^*_p(\Phi_\lambda, \sigma)) = L^*_p(\Phi_\lambda, \kappa, \sigma).
\]

For any continuous \( \widetilde{\Lambda} \)-algebra \( R \), extending by \( R \)-linearity the homomorphisms \( L(\cdot, \sigma) \) and \( L^*(\cdot, \sigma) \) yields homomorphisms of \( R \)-algebras

\[
L_p(\cdot, \sigma)_R : \mathcal{W}_R \to R \quad \text{and} \quad L^*_p(\cdot, \sigma)_R : \mathcal{W}_R \to R.
\]

For any \( \Phi_\lambda \in \mathcal{W}_R \), define standard and improved \( p \)-adic \( L \)-functions:

\[
L_p(\Phi_\lambda, \kappa, \sigma) := \kappa(L_p(\Phi_\lambda, \sigma)_R), \quad \text{for} \quad (\kappa, \sigma) \in \mathcal{X}(R) \times \mathcal{X}(\mathfrak{r}_p^\times),
\]

30
\[ L_p^*(\Phi\lambda, \kappa, \sigma) := \kappa(L_p^*(\Phi\lambda, \sigma)_R), \text{ for } (\kappa, \sigma) \in X'(R) \times A(r^\infty) \]

Fix a prime divisor \( p \) of \( p \) and an index \( \lambda \in \{1, \ldots, h\} \). Let \( \mu \) such that \( \mathfrak{p}t_\lambda t_\lambda^{-1} \) is trivial in the strict class group of \( F \). Let \( \alpha \) such that \( S\pi_p S = Sx_\lambda^{-1} \alpha x_\mu S \) and form the coset decomposition \( \Lambda^\lambda p^m \Gamma^\mu = \bigsqcup I_{\alpha,\lambda,t} \). The matrices \( \alpha_{\lambda,t} \) can be written in the form \( \alpha_\infty := \begin{pmatrix} \alpha^m & 0 \\ 0 & 1 \end{pmatrix} \) or \( \alpha_a := \begin{pmatrix} 1 & a \\ 0 & \alpha^m \end{pmatrix} \)

with \( \text{val}_p(\alpha) = 1 \) and \( a \) varying over a set \( \Sigma(p^m) \) of representatives of \( t_\lambda^{-1}/p^m t_\lambda^{-1} \). Since \( t_\lambda \) is prime to \( p \), \( \alpha_\infty \) and \( \alpha_a \) for \( a \) as above all belong to \( M_2(\mathfrak{p}_R) \). Let \( U(t, p^m) := X \cap Y'^* \alpha_t \) for \( t \in \Sigma(p^m) \cup \{\infty\} \) and denote by \( \mathcal{D}(t, p^m) \) the subset of \( \mathcal{D} \) consisting of those measures supported on \( U(t, p^m) \). Then \( \Phi_\lambda * T(p)^m = \sum_t \Phi_{\mu,t} \) where \( \Phi_{\mu,t} \in H^d_{\text{cpt}}(\Lambda, \mathcal{D}(t, p^m)) \). Let \( \Sigma^\times(p^m) \) denote the subset of \( \Sigma(p^m) \) consisting of elements which are prime to \( p \).

**Lemma 4.4.** Let \( (\Phi_\lambda)_\lambda \in \mathbb{W}_R \) for a continuous \( \tilde{\Lambda} \)-algebra \( R \). Then:

\[
L_p(\Phi_\lambda * T(p)^m, \kappa, \sigma) = \sum_{a \in \Sigma(p^m)} \int_{X'} \kappa(x, z/x^2) \sigma(a + \alpha^m y/x) L(\Phi_{\mu,a})(x, y)
\]

\[
L_p^*(\Phi_\lambda * T(p)^m, \kappa, \sigma) = \sum_{a \in \Sigma(p^m)} \int_{X'} \kappa(x, z/x^2) \sigma(a + \alpha^m y/x) L(\Phi_{\mu,a})(x, y).
\]

**Proof.** Note that \( L(\Phi_\lambda * T(p)^m) = \sum_t L(\Phi_{\mu,t}) \) where \( t \in \Sigma(p^m) \cup \{\infty\} \). Since the standard and improved \( L \)-functions are defined as integrals over \( X' \) and \( X^\infty \), \( \alpha_\infty \) does not contribute to the integral. A simple computation shows that the action of \( \begin{pmatrix} 1 & a \\ 0 & \alpha^m \end{pmatrix} \) on the characteristic function of \( U(a, p^m) \) is the characteristic function of \( X' \). The result follows. \( \square \)

**Proposition 4.5.** Let \( (\Phi_\lambda)_\lambda \in \mathbb{W}_R \) for a continuous \( \tilde{\Lambda} \)-algebra \( R \). Then:

\[
L_p(\Phi_\lambda * T(p), \kappa, \sigma) = L_p^*(\Phi_\lambda * T(p), \kappa, \sigma) - \sigma(p) L_p^*(\Phi_\mu, \kappa, \sigma).
\]

**Proof.** Put \( m = 1 \) in the formulas of Lemma 4.4 and note that the only term surviving in the difference is that for \( a = 0 \). \( \square \)

Fix \( \theta : \mathcal{R} \to \mathcal{I} \) and \( \kappa \in A(\mathcal{I}) \) an arithmetic point. Let \( P := \kappa|_{\lambda} \). Recall the notations

\[
\Phi_\lambda,\theta_\kappa = \rho_\kappa(\Phi_\lambda)
\]

introduced in (16), where \( \rho_\kappa : \mathbb{W} \otimes_\Lambda \mathcal{R}_P \to \mathbb{W}_\kappa \) is the specialization map. Let \( \chi \) be a finite order character of \( I(p^m) \) and write

\[
L(\Phi_\lambda,\theta_\kappa, \chi) = \sum_{r=0}^{n} \binom{n}{r} (-1)^r L(\Phi_\lambda,\theta_\kappa, \chi, r) X^r Y^{n-r}.
\]
Lemma 4.6. Notations as above. Let $\sigma = \sigma_{\chi,r}$ with $r \in \mathbb{Z}[I]$ and $\chi$ a finite order character of $I_{p^m}$. Suppose that $r_p \leq n_\sigma$ for all $\sigma$ associated to $p$. Fix $\lambda$ and let $\mu$ defined as before Lemma 4.4. Then

$$L^*(\Phi_{\lambda} * T(p^m), \kappa, \sigma) = L(\Phi_{\mu,\theta_\kappa}, \chi, r).$$

Proof. To simplify notations set $\chi'(a) := \text{sgn}(a) \text{sgn}(\chi) \chi(at_\lambda)$. Compute from the equation of Lemma 4.4:

$$L^*_p(\Phi_{\lambda} * T(p^m), \kappa, \sigma) = \sum_{a \in \Sigma(p^m)} \chi'(a) \int_{X'} z^n \epsilon(x)x^{n-r}(ax + \alpha^m y)^r dL(\Phi_{\mu,a})(x, y, z).$$

Hence

$$\sum_{r=0}^n \binom{n}{r} (-1)^r L^*_p(\Phi_{\lambda} * T(p^m), \kappa, \sigma) X^r Y^{k-2-r} = \sum_{a \in \Sigma(p^m)} \chi'(a) \int_{X'} z^n \epsilon(x)(xY - aX) - y\alpha^m X)^n dL(\Phi_{\mu,a})(x, y, z).$$

On the other hand, since

$$L(\Phi_{\mu,\theta_\kappa}, \chi, r) = \sum_{a \in \Sigma(p^m)} \chi'(a) L(\Phi_{\mu,\theta_\kappa}, 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} a \\ \alpha^m \end{pmatrix},$$

it follows that

$$\sum_{r=0}^n \binom{n}{r} (-1)^r L(\Phi_{\mu,\theta_\kappa}, \chi, r) X^r Y^{k-2-r} = \sum_{a \in \Sigma(p^m)} \chi'(a) \int_{X'} z^n \epsilon(x)(xY - y\alpha^m X)^n dL(\Phi_{\mu,a})(x, y, z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} a \\ \alpha^m \end{pmatrix}.$$ 

Comparing the two displayed quantities yields the result. \(\square\)

Proposition 4.7. Notations as above. Let $\sigma = \sigma_{\chi,r}$ with $r \in \mathbb{Z}[I]$ and $\chi$ a finite order character of $I_{p^m}$ for some $m \in \mathbb{Z}[I]$. Suppose that $r_p \leq n_\sigma$ for all $\sigma$ associated to $p$. Let $\lambda$ and $\mu$ be as in Lemma 4.6. Then

$$L^*(\Phi_{\lambda} * T(p^m), \kappa, \sigma) = L(\Phi_{\mu,\theta_\kappa}, \chi, r).$$

Proof. Follows from Lemma 4.6 \(\square\)
Recall the notations of $\mathcal{X} = \mathcal{H}_{\text{ord}}(\text{Spec}(\mathcal{O}_0), \mathcal{O})$ and for any $\kappa \in \mathcal{X}(\mathcal{I})$ with $\kappa|_{\mathcal{I}} =: P$, $\mathcal{R}_P$ is the localization of $\mathcal{R}$ at the kernel of $P$. Set $\mathcal{K} := \mathcal{R} \otimes \Lambda$ and $\mathcal{K}_P := \text{Frac}(\mathcal{R}_P)$ (fraction field). Since $\mathcal{K}_P$ is a direct factor of $\mathcal{K}$, it follows that $\mathcal{W}_{\mathcal{K}_P}$ is a direct factor of $\mathcal{W}_{\mathcal{K}}$. Say that $\Phi \in \mathcal{W}_{\mathcal{K}}$ is regular at $\kappa$ if the projection of $\Phi$ to $\kappa$ and $\mathcal{K}_P$ follows that $\mathcal{W}_{\mathcal{K}_P}$ with $\kappa$ be the set of a character of sign $\text{sgn}$ as above and $\chi$ such that for any $\kappa \in U(\Phi, \theta_{\kappa_0})$, $\chi$ a character of sign $\text{sgn}$ as above and $m$ an integer, define:

$$L_p^\text{sgn}(\Phi, \theta, \kappa, \chi, m) := L_p(\Phi, \kappa, \chi \chi^m \gamma_{\text{cyc}})$$

$$L_p^\text{sng}(\Phi, \theta, \kappa, \chi, m) := L_p^*(\Phi, \kappa, \chi \chi^m \gamma_{\text{cyc}}).$$

Here by abuse of notations, $\chi^m \gamma_{\text{cyc}}$ denotes the character $x \mapsto x^r$ for any $r \in \mathbb{Z}[I]$. Although these two functions are defined for $m \in \mathbb{Z}$, they can be extended in a unique way to functions $L_p^\text{sgn}(\Phi, \theta, \kappa, \sigma)$ for $\sigma \in \mathcal{X}(r^\infty) \cap U(\Phi, \theta_{\kappa_0})$ and $L_p^\text{sng}(\Phi, \theta, \kappa, \sigma)$ for $\sigma \in \mathcal{A}(r^\infty) \cap U(\Phi, \theta_{\kappa_0})$ such that for $\sigma = \chi \chi^m$:

$$L_p^\text{sgn}(\Phi, \theta, \kappa, \sigma) = L_p^\text{sgn}(\Phi, \theta, \kappa, \chi, m)$$

$$L_p^\text{sng}(\Phi, \theta, \kappa, \sigma) = L_p^\text{sng}(\Phi, \theta, \kappa, \chi, m).$$

Let $\theta$ and $\kappa_0$ be fixed as above. Fix a sign $\text{sgn} \in \{\pm 1\}^d$. Choose by the Control Theorem $3.3$ a modular symbol $\Phi \in \mathcal{W}_{\mathcal{R}_0}^{\text{sgn}}$ such that

$$\rho_{\kappa_0}(\Phi) = \Psi_{\text{sgn}}^{\mathcal{R}_0}. \tag{32}$$

The Control Theorem $3.3$ implies the existence of a unique period $\Omega(\Phi, \theta, \kappa)$ such that

$$\Phi_{\lambda, \theta_{\kappa}} = \Omega(\Phi, \theta, \kappa) \Psi_{\text{sgn}}^{\mathcal{R}_0, \lambda}. \tag{33}$$

**Proposition 4.8.** Let $\theta$, $\kappa_0$ and $\Phi$ be fixed as above such that $(32)$ holds. For any $\lambda = 1, \ldots, h$, any arithmetic character $\kappa \in \mathcal{A}(\mathcal{I}) \cap U(\Phi, \theta_{\kappa_0})$, any character $\chi$ of sign $\text{sgn}$ as above and any positive integer $m$ in the critical strip for the weight of $f_\kappa$:

$$L_p^\text{sgn}(\Phi_{\lambda, \theta}, \kappa, \chi, m) = e(p, \chi, m) L_p^\text{sng}(\Phi_{\lambda, \theta}, \kappa, \chi, m),$$

where

$$e(p, \chi, m) := \prod_{p | p} \left(1 - \frac{\chi(p) \chi^m / \gamma_{\text{cyc}}(p)}{a_p(\theta, \kappa)}\right)$$

and $a_p(\theta, \kappa) := \theta_{\kappa}(T(p))$.  

33
**Proof.** This follows from Proposition 4.5 and behavior of the specialization map described in Proposition 3.3, part 2 (note that $\Phi_\lambda \ast T(p) = a_p(\theta, \kappa)\Phi_{\mu}$).

**Proposition 4.9.** Notations and assumptions as in Proposition 4.8. Then

$$L(\Phi_\lambda, \theta, \kappa, \chi, m) = \Omega(\Phi, \theta, \kappa)\Omega_{\text{sgn}}(f_{\kappa, \chi})a_p(\theta, \kappa)^m \Lambda(f_{\kappa, \chi}, \chi, m).$$

**Proof.** This formula follows by combining equations (29) and (33).

**Theorem 4.10.** Notations and assumptions as in Proposition 4.8. Then

$$L^s_{\text{sgn}}(\Phi_\lambda, \theta, \kappa, \chi, m) = \frac{\Omega(\Phi, \theta, \kappa)}{\Omega_{\text{sgn}}(f_{\kappa, \chi})a_p(\theta, \kappa)^m} \Lambda(f_{\kappa, \chi}, \chi, m),$$

where $a_p(\theta, \kappa)^m \coloneqq \prod_{p|\kappa} a_p(\theta, \kappa)^{m_p}$.

**Proof.** This follows by combining Proposition 4.7 with Proposition 4.9.

**Theorem 4.11.** Notations and assumptions as in Proposition 4.8. Then

$$L^s_{\text{sgn}}(\Phi_\lambda, \theta, \kappa, \chi, \sigma) = e(p, \chi, m)\Omega(\Phi, \theta, \kappa)\Omega_{\text{sgn}}(f_{\kappa, \chi})a_p(\theta, \kappa)^m \Lambda(f_{\kappa, \chi}, \chi, m).$$

**Proof.** This follows from Theorem 4.10 and Equation (26).

**Theorem 4.12.** Notations and assumptions as in Proposition 4.8. Then

$$L^s_{\text{sgn}}(\Phi_\lambda, \theta, \kappa, \chi_{\text{cyc}}, m) = e(p, \chi, m)\frac{\Omega(\Phi, \theta, \kappa)}{\Omega_{\text{sgn}}(f_{\kappa, \chi})a_p(\theta, \kappa)^m} \Lambda(f_{\kappa, \chi}, \chi, m).$$

**Proof.** Follows from Theorem 4.11 and Equation (26).
4.5 Relations with classical $p$-adic $L$-functions

Recall Panchishkin’s notion of ordinary form in [11]. Let $g$ be a cusp form in $S_{k,w}(S(p^s), \epsilon, \mathbb{C})$ and write its $p$-Hecke polynomial for $p | p$:

$$1 - C(p, f)X + \epsilon(p)N(p)^{k_0-1}X^2 = (1 - \alpha(p)X)(1 - \alpha'(p)X).$$

Order $\alpha(p)$, $\alpha'(p)$ so that $\text{ord}_p \alpha(p) \leq \text{ord}_p \alpha'(p)$, where $\text{ord}_p(p)$ is normalized so that $\text{ord}_p(p) = [F_p : \mathbb{Q}_p]$. For any $\sigma \in I$, denote by $p(\sigma)$ the prime divisor of $\iota_p \circ \mu(p)$.

**Definition 4.13.** The modular form $g$ is said to be $p$-ordinary if

$$\text{ord}_p \alpha(p) = (k_0 - k_\sigma)/2$$

for all prime divisors $p$ of $p$, where $\sigma$ is chosen so that $p = p(\sigma)$ is associated to $\sigma$.

It is clear that if $g$ is ordinary in the sense of Definition 4.13 then it is nearly ordinary in the sense of Definition 2.1. For parallel weights, the two notions coincide. For any ordinary eigenform $g$ (in the sense of Definition 4.13) and any sign $\text{sgn}$, denote by $L_{p}^{\text{sgn}}(g, \chi_{\text{cyc}}^{m})$ the classical $p$-adic $L$-function attached to $g$ constructed by Manin in [8]. This function can be characterized by requiring it to satisfy the following interpolation property: for any integer $m$ in the critical strip of $f$,

$$L_p^{\text{sgn}}(f, m) = \frac{e(p, \chi, m)\Lambda(f, \chi, m)}{\Omega^{\text{sgn}}(f, \chi)a_p(\theta, \kappa)^m}. \quad (36)$$

This formula can be found in a less explicit form in [8, §5] or in [11] Theorem 8.2 in a form closest to this.

**Corollary 4.14.** Notations and assumptions as in Proposition 4.8. Suppose further that $f_\kappa$ is ordinary in the sense of Definition 4.13. Then

$$L_p^{\text{sgn}}(\Phi, \theta, \kappa, \sigma) = \Omega(\Phi, \theta, \kappa)L_p^{\text{sgn}}(f_\kappa, \sigma).$$

**Proof.** This result follows by comparing the interpolation formulas (35) and (36), after noticing that, by [11] Theorem 8.2 (iii)], $L_p^{\text{sgn}}(f, \chi)$ is uniquely determined by (36).

**References**

[1] A. Ash, G. Stevens. Modular forms in characteristic $l$ and special values of their $L$-functions. Duke Math. J. 53 (1986), no. 3, 849–868.
[2] A. Ash, G. Stevens. $p$-adic deformations of cohomology classes of subgroups of $GL(n, \mathbb{Z})$. Collect. Math. 48, 1-2 (1997), 1-30.

[3] R. Greenberg, G. Stevens. $p$-adic $L$-functions and $p$-adic periods of modular forms. Invent. Math. 111 (1993), no. 2, 407–447.

[4] H. Hida. On $p$-adic Hecke algebras for $GL_2$ over totally real fields. Ann. of Math. (2) 128 (1988), no. 2, 295–384.

[5] H. Hida. On nearly ordinary Hecke algebras for $GL(2)$ over totally real fields. Algebraic number theory, 139–169, Adv. Stud. Pure Math., 17, Academic Press, Boston, MA, 1989.

[6] H. Hida. $p$-adic ordinary Hecke algebras for $GL(2)$. Ann. Inst. Fourier (Grenoble) 44 (1994), no. 5, 1289–1322.

[7] K. Kitagawa. On standard $p$-adic $L$-functions of families of elliptic cusp forms. $p$-adic monodromy and the Birch and Swinnerton-Dyer conjecture (Boston, MA, 1991), 81–110, Contemp. Math., 165, Amer. Math. Soc., Providence, RI, 1994.

[8] J. Manin. Non-Archimedean integration and $p$-adic Jacquet-Langlands $L$-functions. (Russian) Uspehi Mat. Nauk 31 (1976), no. 1(187), 5–54.

[9] Y. Matsushima, G. Shimura: On the cohomology groups attached to certain vector valued differential forms on the product of the upper half planes. Ann. of Math. (2) 78, 1963, 417–449.

[10] B. Mazur, J. Tate, J. Teitelbaum. On $p$-adic analogues of the conjectures of Birch and Swinnerton-Dyer. Invent. Math. 84 (1986), no. 1, 1–48.

[11] A. Panchishkin. Motives over totally real fields and $p$-adic $L$-functions. Ann. Inst. Fourier (Grenoble) 44 (1994), no. 4, 989–1023.

[12] R. Pollack, G. Stevens. Explicit computations with overconvergent modular symbols.

[13] G. Shimura. The special values of the zeta functions associated with Hilbert modular forms. Duke Math. J. 45 (1978), no. 3, 637–679.