TRIPARTITE UNIONS

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Abstract. This note provides conditions under which the union of three well-founded binary relations is also well-founded.

This note concerns conditions under which the union of several well-founded (binary) relations is also well-founded.

To garner insight, we tackle just three relations, $A$, $B$, and $C$, over some underlying set $V$. Let

$$\{A|B\}$$

denote $A \cup B$, and so on for other unions of relations. And let juxtaposition indicate composition of relations and superscript $*$ signify transitive closure. We'll refer to the relations as "colors".

**Theorem 1** (Ramsey). The union $\{A|B|C\}$ is well-founded if

$$\{A|B|C\}\{A|B|C\} \subseteq \{A|B|C\}$$

**Proof.** The infinite version of Ramsey’s Theorem applies when the union is transitive, so that every two (distinct) nodes within an infinite chain in the union of the colors has a colored (directed) edge. Then, there must lie an infinite monochrome subchain within any infinite chain, contradicting the well-foundedness of each color alone. □

Only three of the nine cases are actually needed for the limited outcome that we are seeking (an infinite monochromatic path, rather than a clique—as in Ramsey’s Theorem), as we observe next.

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1By well-founded, we mean the absence of infinite forward-pointing paths. For some of the history of well-foundedness based on Ramsey’s Theorem, see Pierre Lescanne’s *Rewriting List*, contributions 38–41 at [http://www.ens-lyon.fr/LIP/REWRITING/CONTRIBUTIONS](http://www.ens-lyon.fr/LIP/REWRITING/CONTRIBUTIONS) and Andreas Blass and Yuri Gurevich, “Program Termination and Well Partial Orderings”, *ACM Transactions on Computational Logic* 9 (3), 2008 (available at [http://research.microsoft.com/en-us/um/people/gurevich/Opera/178.pdf](http://research.microsoft.com/en-us/um/people/gurevich/Opera/178.pdf)).

2See Alfons Geser, *Relative Termination*, Ph.D. dissertation, Fakultät für Mathematik und Informatik, Universität Passau, Germany, 1990 (Report 91-03, Ulmer Informatik-Berichte, Universität Ulm, 1991; available at [http://homepage.cs.uiowa.edu/~astump/papers/geser_dissertation.pdf](http://homepage.cs.uiowa.edu/~astump/papers/geser_dissertation.pdf)).
Theorem 2. The union \{A|B|C\} is well-founded if
\[ BA \cup CA \cup CB \subseteq \{A|B|C\}. \]

Proof. When the union is not well-founded, there is an infinite path \( X = \{x_i\} \), with each edge from \( x_i \) to \( x_{i+1} \) one of \( A, B, \) or \( C \). Extract a maximal subsequence \( \{x_{i_j}\} \) of \( X \) such that \( x_{i_j} A x_{i_{j+1}} \) for each \( j \). If it’s finite, then repeat at the first opportunity in the tail. If any is infinite, we have our contradiction. If they’re all finite, then consider the first occurrence of \( x \{B|C\} y \) \( A \) \( z \). Since we could not take an \( A \)-step from \( x \), or we would have, the conditions tell us that \( x \{B|C\} z \). Swallowing up all such (non-initial) \( A \)-steps in this way, we are left with an infinite chain in \( B \cup C \), for which we also know that no \( A \)-steps are possible anywhere. Now extract maximal \( B \)-chains and then erase them, replacing \( x C y B z \) with \( x C z \) (\( A \)- and \( B \)-steps having been precluded), leaving an infinite chain colored purely \( C \). □

Corollary 1. If \( A, B, \) and \( C \) are transitive and
\[ BA \cup CA \cup CB \subseteq \{A|B|C\}, \]
then, whenever there is an infinite path in the union \{A|B|C\}, there is an infinite monochromatic clique.

We can do considerably better than the previous theorem:

Theorem 3 (Tripartite). The union \{A|B|C\} is well-founded if
\[ \{B|C\} A \subseteq A\{A|B|C\}^* \cup B \cup C \]
\[ CB \subseteq A\{A|B|C\}^* \cup BB^* \cup C. \]

Let’s call the existence of an infinite outgoing chain in the union \{A|B|C\} immortality.

Proof. We first construct an infinite chain \( X = \{x_i\} \), in which an \( A \)-step is always preferred over \( B \) or \( C \), as long as immortality is maintained. To do this, we start with an immortal element \( x_0 \in V \). At each stage in the construction, if the chain so far ends in \( x_i \), we look to see if there is any \( y \) such that \( x_i A y \) and from which proceeds some infinite chain in the union, in which case \( y \) is chosen to be \( x_{i+1} \). Otherwise, \( x_{i+1} \) is any immortal element \( z \) such that \( x_i B z \) or \( x_i C z \).

If there are infinitely many \( B \)-’s and/or \( C \)-’s in \( X \), use them—by means of the first condition—to remove all subsequent \( A \)-steps, leaving only \( B \)- and \( C \)-steps going out of points from which \( A \) leads of necessity to mortality. From what remains, if there is any \( C \)-step at a point where one could take one or more \( B \)-steps to anyplace later in the chain, take the latter route instead. What remains now are \( C \)-steps at points...
where $BB^*$ detours are also precluded. If there are infinitely many such $C$-steps, then applying the condition for $CB$ will result in a pure $C$-chain, because neither $A\{A|B|C\}^*$ nor $BB^*$ are options.

Dropping $C$ from the conditions of the previous theorem, one gets the jumping criterion for well-foundedness of the union of two well-founded relations $A$ and $B$:

$$BA \subseteq A\{A|B\}^* \cup B .$$

Applying this criterion twice, one gets somewhat different (incomparable) conditions for well-foundedness.

**Theorem 4** (Jumping). The union $\{A|B|C\}$ is well-founded if

$$BA \subseteq A\{A|B\}^* \cup B \quad \text{and} \quad \{B|C\}A \subseteq A\{A|B|C\}^* \cup B \cup C . \quad (4)$$

Proof. The first inequality is the jumping criterion. The second is the same with $C$ for $B$ and $\{A|B\}$ in place of $A$. □

For two relations, jumping provides a substantially weaker criterion for well-foundedness than does the appeal to Ramsey. But for three, whereas jumping allows more than one step for $BA$ (in essence, $AA^*B^*$), it doesn’t allow for $C$, which Ramsey does.

Switching roles, start with jumping for $\{B|C\}$ before combining with $A$, we get slightly different conditions yet:

**Theorem 5** (Jumping). The union $\{A|B|C\}$ is well-founded if

$$CB \subseteq B\{B|C\}^* \cup C \quad \text{and} \quad \{B|C\}A \subseteq A\{A|B|C\}^* \cup B \cup C . \quad (5)$$

Both this version of jumping and our tripartite condition allow

$$\{B|C\}A \subseteq A\{A|B|C\}^* \cup B \cup C$$

$$CB \subseteq BB^* \cup C .$$

They differ in that jumping also allows

$$CB \subseteq B\{B|C\}^* .$$

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3See Henk Doornbos and Burghard von Karger, “On the Union of Well-Founded Relations”, *Logic Journal of the IGPL* 6(2), pp. 195–201, 1998 (available at [http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.28.8953&rep=rep1&type=pdf](http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.28.8953&rep=rep1&type=pdf)). The property is called “jumping” in Nachum Dershowitz, “Jumping and Escaping: Modular Termination and the Abstract Path Ordering”, *Theoretical Computer Science* 464, pp. 35–47, 2012 (available at [http://nachum.org/papers/Toyama.pdf](http://nachum.org/papers/Toyama.pdf)).
whereas tripartite has
\[ CB \subseteq A\{A|B|C\}^* \]
instead.

Sadly, we cannot have the best of both worlds. Let’s color edges \( A \), \( B \), and \( C \) with (solid) azure, (dotted) black, and (dashed) crimson ink, respectively. The graph

\[
\begin{align*}
\bullet & \quad \bullet \\
\bullet & \quad \bullet \\
\end{align*}
\]

only has multicolored loops despite satisfying
\[
\begin{align*}
\{B|C\}A \subseteq C \\
CB \subseteq A \cup B\{B|C\}^* .
\end{align*}
\]

Even
\[
\begin{align*}
\{B|C\}A \subseteq C \\
CB \subseteq B\{A|B\}^* .
\end{align*}
\]
doesn’t work. To wit, the double loop in

\[
\begin{align*}
\bullet & \quad \bullet \\
\bullet & \quad \bullet \\
\end{align*}
\]

harbors no monochrome subchain. By the same token,

\[
\begin{align*}
\bullet & \quad \bullet \\
\bullet & \quad \bullet \\
\end{align*}
\]

counters the putative hypothesis
\[
\begin{align*}
BA \cup CB \subseteq C \\
CA \subseteq BA^* .
\end{align*}
\]