Necessary Optimality Conditions for Implicit Control Systems with Applications to Control of Differential Algebraic Equations

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Abstract In this paper we derive necessary optimality conditions for optimal control problems with nonlinear and nonsmooth implicit control systems. Implicit control systems have wide applications including differential algebraic equations (DAEs). The challenge in the study of implicit control system lies in that the system may be truly implicit, i.e., the Jacobian matrix of the constraint mapping may be singular. Our necessary optimality conditions hold under the so-called weak basic constraint qualification plus the calmness of a perturbed constraint mapping. Such constraint qualifications allow for singularity of the Jacobian and hence are suitable for implicit systems. Specifying these results to control of semi-explicit DAEs we obtain necessary optimality conditions for control of semi-explicit DAEs with index higher than one.

Keywords Necessary optimality conditions · Optimal control · Implicit control systems · Differential algebraic equations · Calmness · Variational analysis

Mathematics Subject Classification (2010) 45K15 · 49K21 · 49J53

Dedicated to the memory of Jonathan Michael Borwein

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1 Introduction

Given a time interval \([t_0, t_1] \subseteq \mathbb{R}\), very often, the dynamic behavior of a system is most naturally modeled as an implicit control system:

\[
\text{ICS} \quad \varphi(x(t), u(t), \dot{x}(t)) \in K_\varphi \quad \text{a.e.} \ t \in [t_0, t_1],
\]

\[
u(t) \in U \quad \text{a.e.} \ t \in [t_0, t_1],
\]

\[(x(t_0), x(t_1)) \in S,
\]

where \(\varphi : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_x} \to \mathbb{R}^m\), \(K_\varphi \subseteq \mathbb{R}^m\), \(U \subseteq \mathbb{R}^{n_u}\), \(S \subseteq \mathbb{R}^{n_x} \times \mathbb{R}^{n_x}\).

A particular case of the implicit control system is described by scalar equations, namely, differential algebraic equations (DAEs):

\[
\text{DAE} \quad \varphi(x(t), u(t), \dot{x}(t)) = 0 \quad \text{a.e.} \ t \in [t_0, t_1],
\]

\[
u(t) \in U \quad \text{a.e.} \ t \in [t_0, t_1],
\]

\[(x(t_0), x(t_1)) \in S.
\]

A very popular model of a DAE is the so-called semi-explicit DAE:

\[
\text{seDAE} \quad \dot{x}(t) = \phi(x(t), y(t), u(t)) \quad \text{a.e.} \ t \in [t_0, t_1],
\]

\[0 = h(x(t), y(t), u(t)) \quad \text{a.e.} \ t \in [t_0, t_1],
\]

\[(x(t_0), x(t_1)) \in S,
\]

where \(\phi : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}\), \(h : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_y}\).

In the past couple decades, DAEs have become a very important generalization of ordinary differential equations (ODEs) and have numerous applications in mathematical modeling of various dynamical processes; see e.g. [3, 5, 13, 31] and the references therein.

In this paper we study the optimal control problem of an implicit system:

\[
(P_{ICS}) \quad \min \ J(x, u) := \int_{t_0}^{t_1} F(x(t), u(t), \dot{x}(t)) dt + f(x(t_0), x(t_1))
\]

\[s.t. \quad \varphi(x(t), u(t), \dot{x}(t)) \in K_\varphi \quad \text{a.e.} \ t \in [t_0, t_1],
\]

\[
u(t) \in U \quad \text{a.e.} \ t \in [t_0, t_1],
\]

\[(x(t_0), x(t_1)) \in S,
\]

where \(F : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_x} \to \mathbb{R}\), \(f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \to \mathbb{R}\). Our basic assumptions for problem \((P_{ICS})\) are very general. We assume all sets involved are closed and all functions involved are locally Lipschitz continuous.

To our knowledge, there is very little done for implicit control problems stated in such a general form as in \((P_{ICS})\). In [12, Theorem 1.1], for problem \((P_{ICS})\) with free end point, Devdariani and Ledyaev derived a necessary optimality condition in a form that closely resembles the classical Pontryagin maximum principle with an implicitly defined Hamiltonian. For control of semi-explicit DAEs, de Pinho and Vinter [11] derived a strong maximum principle under the assumption that the velocity set is convex and a weak maximum principle without the convexity assumption. Moreover a counter example in [11] shows that the strong maximum principle may not hold if the velocity set is nonconvex. The assumption on the convexity of the velocity set in [11, Theorem 3.1] was relaxed for the Bolza problem in [31]. A key assumption for the maximum principles in [11] to hold is that the Jacobian matrix \(\nabla_y h\) must be nonsingular along the optimal pair. This means that the maximum principles derived in [11] can only be applied to control of seDAEs with index one. Recently some necessary optimality conditions for control of DAEs with higher indexes have been derived [13, 25, 31].
In this paper, we aim at deriving necessary optimality conditions for a (weak) local minimum of radius \( R(\cdot) \) for nonsmooth problems (PICS) in the following sense. A control or control function \( u(\cdot) \) is a measurable function on \([t_0, t_1]\) such that \( u(t) \in U \) for almost every \( t \in [t_0, t_1] \). The state or state trajectory, corresponding to a given control \( u(\cdot) \), refers to an absolutely continuous function \( x(\cdot) \) which together with \( u(\cdot) \) satisfying all conditions in (ICS). We call such a pair \((x(\cdot), u(\cdot))\) an admissible pair. For simplicity we may omit the time variable and write \( x, u \) instead of \( x(t), u(t) \), respectively. Let \( R(t) : [t_0, t_1] \rightarrow (0, +\infty] \) be a radius function. We say that \((x_*, u_*)\) is a local minimum of radius \( R(\cdot) \) for problem (PICS) if \((x_*, u_*)\) minimizes the value of the cost function \( J(x, u) \) over all admissible pairs \((x, u)\) which satisfies

\[
\begin{align*}
|x(t) - x_*(t)| &\leq \varepsilon \text{ a.e. } t \in [t_0, t_1], \\
\int_{t_0}^{t_1} |\dot{x}(t) - \dot{x}_*(t)| \, dt &\leq \varepsilon, \\
|(u(t), \dot{x}(t)) - (u_*(t), \dot{x}_*(t))| &\leq R(t) \text{ a.e. } t \in [t_0, t_1].
\end{align*}
\] (1.1)

This local minimum concept is even weaker than the so-called \( W^{1,1} \) local minimum which is the case when \( R(t) \equiv \infty \), because of the additional restriction (1.1) stemming from the radius function. Note that \( W^{1,1} \) local minimum is known to be weaker than the classical strong local minimum which has only the restriction that \( |x(t) - x_*(t)| \leq \varepsilon \text{ a.e. } \).

In [8, Theorem 6.1], Clarke and de Pinho obtained a set of necessary optimality conditions for problem (PICS) with \( K_\varphi = \{0\} \) under the above concept of weak local minimum. In [10, Theorem 2.1], this result is extended to the problem (PICS) without the restriction of \( K_\varphi = \{0\} \) under the classical strong local minimum concept. Moreover the result for the smooth case is further investigated in [10]. These necessary optimality conditions, however, require the calibrated constraint qualification (CCQ) which is stronger than the classical Mangasarian-Fromovitz constraint qualification (MFCQ) in mathematical programming. The main purpose of this paper is to derive necessary optimality conditions in the form of [8, Theorem 6.1] and [10, Theorem 2.1] under weaker constraint qualifications.

Following the same strategy as proposed in [8, 10], by introducing a vector variable \( v(t) := \dot{x}(t) \), we transform (PICS) into the following equivalent problem:

\[
(P_{ECS}) \quad \min J(x, u) := \int_{t_0}^{t_1} F(x(t), u(t), v(t)) \, dt + f(x(t_0), x(t_1)) \\
\text{s.t.} \quad \dot{x}(t) = v(t) \quad \text{a.e. } t \in [t_0, t_1], \\
\varphi(x(t), u(t), v(t)) \in K_\varphi \quad \text{a.e. } t \in [t_0, t_1], \\
u(t) \in U \quad \text{a.e. } t \in [t_0, t_1], \\
(x(t_0), x(t_1)) \in S,
\]

obtain a set of necessary optimality conditions for problem (P_{ECS}) and then transform back to the one for the original problem (PICS). Problem (P_{ECS}) belongs to the class of optimal control problems with mixed state and control constraints. A set of necessary optimality conditions for a local minimum of radius \( R(\cdot) \) for this class of problems has been developed in Clarke and de Pinho [8, Theorem 4.3] under the CCQ. Motivated by the recent progress in mathematical programming toward deriving necessary optimality conditions for mathematical programs under constraint qualifications such as the calmness condition...
which is weaker than MFCQ, Li and Ye [26] proposed the so-called weak basic constraint qualification (WBCQ) plus the calmness of the perturbed constraint mapping

\[
M_\varphi(\Theta) := \left\{ (x, u, v) \in \mathbb{R}^{n_x} \times U \times \mathbb{R}^{n_x} : \varphi(x, u, v) + \Theta \in K_\varphi \right\},
\]

and obtained necessary optimality conditions for a local minimum of radius \( R(\cdot) \) for the optimal control problem with mixed state and control constraints. Note that the concept of a local minimum of radius \( R(\cdot) \) is slightly stronger than the one defined as in (1.1). In this paper we first show that result of [26, Theorem 4.2] remains true for the weaker local optimality concept in this paper and apply it to (P_ECS) to obtain necessary optimality conditions of (P_IICS) under the desired constraint qualification.

In the case of DAEs with optimal controls lying in the interior of the control set, MFC is equivalent to the maximum rank of the Jacobian matrix \( \nabla \varphi \) and in the case of semi-explicit DAEs, it amounts to that the problem is index one. Applying our results for the control of DAEs to the optimal control of semi-explicit DAEs, we derive necessary optimality conditions for control of semi-explicit DAEs with index higher than one. In our necessary optimality conditions, the form of the maximum principle for control of semi-explicit DAEs is the weak maximum principle as in [11, Theorem 3.2] plus some extra condition called the Weierstrass condition. Hence in the autonomous case, our necessary optimality condition is a maximum principle stronger than [11, Theorem 3.2] under weaker constraint qualifications.

The paper is organized as follows. Section 2 contains preliminaries on variational analysis. In Section 3, we derive necessary optimality conditions for an autonomous optimal control problems with mixed state and control constraints. In Section 4, we derive necessary optimality conditions for the optimal control of an implicit control system. Optimal control of semi-explicit systems are studied in Section 5. In Section 6 we give verifiable sufficient conditions for the constraint qualifications required in the paper. The proof of the main result in Section 3 is given in Appendix.

2 Background in variational analysis

In this section we present preliminaries on variational analysis that will be needed in this paper. We give only concise definitions and conclusions that will be needed in the paper. For more detailed information on the subject we refer the reader to [6, 9, 26, 27, 30].

Throughout the paper, \(| \cdot |\) denotes the Euclidean norm, \( B \) and \( B(x, \delta) \) the open unit ball and the open ball centered at \( x \) with radius \( \delta > 0 \), respectively. Unless otherwise specified, the closure, the convex hull and the closure of the convex hull of a subset \( \Omega \subseteq \mathbb{R}^n \) are denoted by \( \bar{\Omega}, \text{co} \Omega, \) and \( \text{co} \Omega \), respectively. For a set \( \Omega \subseteq \mathbb{R}^n \) and a point \( x \in \mathbb{R}^n \), \( d(x, \Omega) \) is the distance from point \( x \) to set \( \Omega \). For any \( a, b \in \mathbb{R}^n \), \( \langle a, b \rangle \) denotes the inner product of vectors \( a \) and \( b \). Given a mapping \( \psi : \mathbb{R}^n \to \mathbb{R}^m \) and a point \( x \in \mathbb{R}^n \), \( \nabla \psi(x) \in R^{m \times n} \) stands for the Jacobian of \( \psi(\cdot) \) at \( x \). Given a function \( f : \mathbb{R}^n \to \mathbb{R} \), \( \nabla^2 f(x) \) is the Hessian matrix. For a set-valued map \( \Psi : \mathbb{R}^n \Rightarrow \mathbb{R}^q \), \( \text{gph} \Psi := \{(x, y) : y \in \Psi(x)\} \) is its graph, \( \Psi^{-1}(y) := \{x : y \in \Psi(x)\} \) is its inverse.

Let \( S \subseteq \mathbb{R}^n \). The tangent cone to \( S \) at \( \bar{x} \) is defined by

\[
T_S(\bar{x}) := \{w \in \mathbb{R}^n : \exists t_k \downarrow 0, w_k \to w \text{ with } \bar{x} + t_k w_k \in S, \forall k\}.
\]
The Fréchet normal cone to $S$ at $\bar{x} \in S$ is defined by

$$\hat{N}_S(\bar{x}) := \{ v^* \in \mathbb{R}^n : \limsup_{x \rightarrow \bar{x}} \frac{\langle v^*, x - \bar{x} \rangle}{|x - \bar{x}|} \leq 0 \},$$

where $x_i \xrightarrow{S} \bar{x}$ means that $x_i \in S$ and $x_i \rightarrow \bar{x}$. The limiting normal cone $N_S(\bar{x})$ to $S$ is defined by

$$N_S(\bar{x}) := \left\{ \lim \xi_i : \xi_i \in \hat{N}_S(x_i), x_i \xrightarrow{S} \bar{x} \right\}.$$

$S$ is said to be normally regular if $\hat{N}_S(\bar{x}) = N_S(\bar{x})$ for all $\bar{x} \in S$. Recently Gfrerer [15] introduced the concept of the directional limiting normal cone. The limiting normal cone to $S$ in direction $w \in \mathbb{R}^n$ at $\bar{x}$ is defined by

$$N_S(\bar{x}; w) := \{ v^* \in \mathbb{R}^n : \exists t_k \downarrow 0, w_k \rightarrow w, v^*_k \rightarrow v^* \text{ s.t. } v^*_k \in \hat{N}_S(\bar{x} + t_kw_k), \forall k \}.$$

Consider a lower semicontinuous function $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ and a point $\bar{x} \in \mathbb{R}^n$ where $f$ is finite. A vector $\xi \in \mathbb{R}^n$ is called a proximal subgradient of $f$ at $\bar{x}$ provided that there exist $\sigma, \delta > 0$ such that

$$f(x) \geq f(\bar{x}) + \langle \xi, x - \bar{x} \rangle - \sigma |x - \bar{x}|^2, \forall x \in B(\bar{x}, \delta).$$

The set of such $\xi$ is denoted $\partial^P f(\bar{x})$ and referred to as the proximal subdifferential. The limiting subdifferential of $f$ at $\bar{x}$ is the set

$$\partial f(\bar{x}) := \left\{ \lim \xi_i : \xi_i \in \partial^P f(x_i), x_i \rightarrow \bar{x}, f(x_i) \rightarrow f(\bar{x}) \right\}.$$

For a locally Lipschitz function $f$ on $\mathbb{R}^n$, the generalized gradient $\partial^C f(\bar{x})$ coincides with $co\partial f(\bar{x})$; further the associated Clarke normal cone $N^C_S(\bar{x})$ at $\bar{x} \in S$ coincides with $\overline{co}N_S(\bar{x})$.

We now review some concepts of Lipschitz continuity of set-valued maps.

**Definition 2.1** [28] A set-valued map $\Psi : \mathbb{R}^n \rightrightarrows \mathbb{R}^q$ is said to be upper-Lipschitz at $\bar{x}$ if there exist $\mu \geq 0$ and a neighborhood $U(\bar{x})$ of $\bar{x}$ such that

$$\Psi(x) \subseteq \Psi(\bar{x}) + \mu|x - \bar{x}|B, \forall x \in U(\bar{x}).$$

**Definition 2.2** [27, Definition 1.40] A set-valued map $\Psi : \mathbb{R}^n \rightrightarrows \mathbb{R}^q$ is said to be pseudo-Lipschitz (or locally Lipschitz like or has the Aubin property) around $(\bar{x}, \bar{y}) \in \text{gph}\Psi$ if there exist $\mu \geq 0$ and neighborhoods $U(\bar{x}), U(\bar{y})$ of $\bar{x}$ and $\bar{y}$, respectively, such that

$$\Psi(x) \cap U(\bar{y}) \subseteq \Psi(x') + \mu|x - x'|B, \forall x, x' \in U(\bar{x}).$$

Equivalently, $\Psi$ is pseudo-Lipschitz around $(\bar{x}, \bar{y})$ if there exist $\mu \geq 0$ and neighborhoods $U(\bar{x}), U(\bar{y})$ of $\bar{x}$ and $\bar{y}$, respectively, such that

$$d(y, \Psi(x')) \leq \mu d(x', \Psi^{-1}(y)) \quad \forall x' \in U(\bar{x}), y \in U(\bar{y}).$$

**Definition 2.3** [30, 35] A set-valued map $\Psi : \mathbb{R}^n \rightrightarrows \mathbb{R}^q$ is said to be calm (or pseudo upper-Lipschitz continuous) at $(\bar{x}, \bar{y}) \in \text{gph}\Psi$ if there exist $\mu \geq 0$ and neighborhoods $U(\bar{x}), U(\bar{y})$ of $\bar{x}$ and $\bar{y}$, respectively, such that

$$\Psi(x) \cap U(\bar{y}) \subseteq \Psi(\bar{x}) + \mu|x - \bar{x}|B, \forall x \in U(\bar{x}).$$

Equivalently, $\Psi$ is calm around $(\bar{x}, \bar{y})$ if there exist $\mu \geq 0$ and a neighborhood $U(\bar{y})$ of $\bar{y}$ such that

$$d(y, \Psi(\bar{x})) \leq \mu d(\bar{x}, \Psi^{-1}(y)) \quad \forall y \in U(\bar{y}).$$
Definition 2.4 [23] A set-valued map $\Sigma : \mathbb{R}^q \rightrightarrows \mathbb{R}^n$ is said to be metrically subregular at $(\bar{y}, \bar{x}) \in \text{gph} \Sigma$ if there exist $\mu \geq 0$ and a neighborhood $U(\bar{y})$ of $\bar{y}$ such that

$$d(y, \Sigma^{-1}(\bar{x})) \leq \mu d(\bar{x}, \Sigma(y)) \quad \forall y \in U(\bar{y}).$$

From definition, it is easy to see that a set-valued map $\Sigma$ is metrically subregular at $(\bar{y}, \bar{x}) \in \text{gph} \Sigma$ if and only if its inverse map $\Sigma^{-1}$ is calm at $(\bar{x}, \bar{y}) \in \text{gph} \Sigma^{-1}$.

In this paper we are mostly interested in the calmness of a set-valued map defined as the perturbed constrained system:

$$M(\Theta) := \{(x, u) \in \mathbb{R}^{nx} \times U : \Phi(x,u) + \Theta \in \Omega\},$$

(2.1)

where $\Phi : \mathbb{R}^{nx} \times \mathbb{R}^{nu} \to \mathbb{R}^d$ and $U \subseteq \mathbb{R}^{nu}$, $\Omega \subseteq \mathbb{R}^d$.

We now summarize some constraint qualifications that will be used in the paper.

Definition 2.5 Let $(\bar{x}, \bar{u}) \in M(0)$, $\Phi$ is Lipschitz continuous at $(\bar{x}, \bar{u})$ and $U$, $\Omega$ are closed.

- ([8]) We say the calibrated constraint qualification (CCQ) holds at $(\bar{x}, \bar{u})$ if there exists $\mu > 0$ such that

$$\left\{ (\alpha, \beta) \in \partial \langle \lambda, \Phi \rangle(\bar{x}, \bar{u}) + \{0\} \times N_U(\bar{u}), \lambda \in N_{\Omega}(\Phi(\bar{x}, \bar{u})) \right\} \implies |\lambda| \leq \mu |\beta|.$$

- ([8] We say the MFC holds at $(\bar{x}, \bar{u})$ if

$$\left\{ (\alpha, 0) \in \partial \langle \lambda, \Phi \rangle(\bar{x}, \bar{u}) + \{0\} \times N_U(\bar{u}), \lambda \in N_{\Omega}(\Phi(\bar{x}, \bar{u})) \right\} \implies \lambda = 0.$$

- ([27]) We say the no nonzero abnormal multiplier constraint qualification (NNAMCQ) holds at $(\bar{x}, \bar{u})$ if

$$\left\{ (0, 0) \in \partial \langle \lambda, \Phi \rangle(\bar{x}, \bar{u}) + \{0\} \times N_U(\bar{u}), \lambda \in N_{\Omega}(\Phi(\bar{x}, \bar{u})) \right\} \implies \lambda = 0.$$

- ([26]) We say the weak basic constraint qualification (WBCQ) holds at $(\bar{x}, \bar{u})$ if

$$\left\{ (\alpha, 0) \in \partial \langle \lambda, \Phi \rangle(\bar{x}, \bar{u}) + \{0\} \times N_U(\bar{u}), \lambda \in N_{\Omega}(\Phi(\bar{x}, \bar{u})) \right\} \implies \alpha = 0.$$

It is easy to check that the following implications hold:

$$\text{CCQ} \implies \text{MFC} \iff \text{WBCQ+NNAMCQ} \implies \text{WBCQ + Calmness of } M,$$

and the WBCQ+Calmness of $M$ may not imply NNAMCQ (see [26, Example 2.1]). Although in general CCQ is stronger than MFC, if MFC holds for every point in certain compact set, then it implies CCQ for every point in the same compact set under certain assumptions; see [8, Proposition 4.6] for details.
3 Optimal Control Problems with Mixed State and Control Constraints

In this section, we consider the following autonomous optimal control problem in which the state and control variables are subject to mixed state and control constraints:

\[(P) \min J(x, u) := \int_{t_0}^{t_1} F(x(t), u(t))dt + f(x(t_0), x(t_1))
\]

s.t. \[\dot{x}(t) = \phi(x(t), u(t)) \quad a.e. \in [t_0, t_1],\]
\[\Phi(x(t), u(t)) \in \Omega \quad a.e. \in [t_0, t_1],\]
\[u(t) \in U \quad a.e. \in [t_0, t_1],\]
\[(x(t_0), x(t_1)) \in S,\]

where \(F : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}, f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}, \phi : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}, \Phi : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^d \) and \(U \subseteq \mathbb{R}^{n_u}, \Omega \subseteq \mathbb{R}^d, S \subseteq \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}. \) Unless otherwise stated, in this section we assume that \(F, f, \phi, \Phi\) are locally Lipschitz continuous, and the sets \(U, \Omega, S\) are closed.

Let \(R : [t_0, t_1] \to (0, +\infty)\) be a given measurable radius function. As in [8], we say that an admissible pair \((x_*, u_*)\) is a local minimum of radius \(R(\cdot)\) for problem \((P)\) if it minimizes the value of the cost function \(J(x, u)\) over all admissible pairs \((x, u)\) which satisfies

\[|x(t) - x_*(t)| \leq \varepsilon, \quad |u(t) - u_*(t)| \leq R(t) \quad a.e., \int_{t_0}^{t_1} |\dot{x}(t) - \dot{x}_*(t)|dt \leq \varepsilon.\]

For any given \(\varepsilon > 0\) and a given radius function \(R(\cdot)\), define

\[\tilde{S}_*^{\varepsilon, R}(t) := \{(x, u) \in \tilde{B}(x_*(t), \varepsilon) \times U : \Phi(x, u) \in \Omega, |u - u_*(t)| \leq R(t)\},\]
\[\tilde{C}_*^{\varepsilon, R} := cl\{(t, x, u) \in [t_0, t_1] \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} : (x, u) \in \tilde{S}_*^{\varepsilon, R}(t)\},\]

where \(cl\) denotes the closure. In the case where the control set \(U\) is closed, the optimal control \(u_*(t)\) is continuous and the radius function \(R(t)\) is either identical to \(\infty\) or continuous, the closure operation is superfluous and hence can be removed. A sufficient condition for the compactness of the set \(\tilde{C}_*^{\varepsilon, R}\) is that \(\varepsilon < \infty\) and either \(U\) is compact or \(u_*(t)\) is continuous and \(R(t)\) is either identical to \(\infty\) or continuous.

The main result of this section is the following theorem whose proof can be found in the appendix.

**Theorem 3.1** Let \((x_*, u_*)\) be a local minimum of radius \(R(\cdot)\) for \((P)\). Suppose that there exists \(\delta > 0\) such that \(R(t) \geq \delta\). Suppose that \(\tilde{C}_*^{\varepsilon, R}\) is compact and for all \((t, x, u) \in \tilde{C}_*^{\varepsilon, R}\) the WBCQ holds:

\[\begin{aligned}
&\{0, 0\} \in \partial(\langle \lambda, \Phi(x, u) \rangle + \{0\} \times N_U(u),
&\lambda \in N_{\Omega}(\Phi(x, u)) \implies \alpha = 0 \quad (3.1)
\end{aligned}\]

and the mapping \(M\) defined as in (2.1) is calm at \((0, x, u)\). Then there exist an arc \(p\) and a number \(\lambda_0\) in \([0, 1]\), satisfying the nontriviality condition \((\lambda_0, p(t)) \neq 0, \forall t \in [t_0, t_1]\), the transversality condition

\[\langle p(t_0), -p(t_1) \rangle \in \lambda_0 \partial f (x_*(t_0), x_*(t_1)) + N_S(x_*(t_0), x_*(t_1)),\]

and the Euler adjoint inclusion for almost every \(t\):

\[\begin{aligned}
&\{\hat{p}(t), 0\} \in \partial \{\langle -p(t), \phi \rangle + \lambda_0 F \} (x_*(t), u_*(t)) + \{0\} \times N_{\Omega}(u_*(t))
&+ co\{\partial \langle \lambda, \Phi(x_*(t), u_*(t)) \rangle : \lambda \in N_{\Omega}(\Phi(x_*(t), u_*(t)))\}, \quad (3.2)
\end{aligned}\]
as well as the Weierstrass condition of radius $R(\cdot)$ for almost every $t$:
\[
\Phi(x_s(t), u) \in \Omega, u \in U, \quad |u - u_*(t)| < R(t) \implies (p(t), \phi(x_s(t), u)) - \lambda_0 F(x_s(t), u) \leq (p(t), \phi(x_s(t), u_*)) - \lambda_0 F(x_s(t), u_*).
\]
Moreover in the case of free end point, $\lambda_0$ can be taken as 1.

For the autonomous control problem $(P)$, the conclusions of Theorem 3.1 are exactly the same as those in Clarke and de Pinho [8, Theorem 4.3] except that the Weierstrass condition holds only on the open ball $B(u_*(t), R(t))$ instead of the closed ball $\bar{B}(u_*(t), R(t))$. However our assumption that the WBCQ plus the calmness condition is weaker than the calibrated constraint qualification in [8, Theorem 4.3], which is even stronger than the MFC. In fact, the Weierstrass conditions in [7, 8] can only hold on the open ball $B(u_*(t), R(t))$ instead of the closed ball $\bar{B}(u_*(t), R(t))$. This imprecision was spotted and remedied in [4]. Moreover the authors in [4] introduced a notion of radius multifunction and used it to consider a more general concept of a local minimum and necessary optimality conditions.

The Euler adjoint inclusion (3.2) in Theorem 3.1 is in an implicit form. In the case where $\Phi$ is smooth, one can find a measurable multiplier $\lambda(t) \in N^C_{\Omega} (\Phi(x_s(t), u_*(t)))$ such that the Euler adjoint inclusion takes an explicit multiplier form by using the measurable selection theorem.

To give an estimate for the multiplier $\lambda$ we need to use the following result.

**Proposition 3.1** [17, Proposition 4.1] Let $\Psi : \mathbb{R}^n \rightrightarrows \mathbb{R}^q$ be a set-valued map with closed graph. Given $(\bar{x}, \bar{y}) \in gph\Psi$, assume that $\Psi$ is metrically subregular at $(\bar{x}, \bar{y})$ with modulus $\kappa$. Then
\[
N_{\Psi^{-1}(\bar{y})}(\bar{x}) \subseteq \{\gamma : \exists \lambda \in \kappa | \bar{B} : (\gamma, \lambda) \in N_{gph\Psi}(\bar{x}, \bar{y})\}.
\]

We are now in a position to give the Euler adjoint inclusion an explicit multiplier form when $\Phi$ is smooth. Moreover in the case where $\Phi$ smooth and $u_*$ is in the interior of $U$ for almost all $t$, we show that a multiplier can be chosen such that an estimate in terms of adjoint arc holds as in [8, Theorems 4.3]. Our result improves the corresponding result in [8, Theorems 4.3] in that for the autonomous case, the estimate holds under the WBCQ plus the calmness condition which is weaker than the calibrated constraint qualification required in [8, Theorems 4.3].

**Theorem 3.2** In additions to the assumptions of Theorem 3.1, suppose that $\Phi$ is strictly differentiable. Then the Euler adjoint inclusion can be replaced by the one in the explicit multiplier form, i.e., there exists a measurable function $\lambda : [t_0, t_1] \to \mathbb{R}^d$ with $\lambda(t) \in N^C_{\Omega} (\Phi(x_s(t), u_*(t)))$ for almost every $t \in [t_0, t_1]$ satisfying
\[
\begin{align*}
\langle \dot{p}(t), 0 \rangle & \in \partial^C \{ - p(t), \phi \} + \lambda_0 F(x_s(t), u_*) \\
\nabla \Phi(x_s(t), u_*)^T \lambda(t) & + \{0\} \times N^C_{U} (u_*(t)).
\end{align*}
\]
Moreover if $N^C_{U} (u_*(t)) = \{0\}$ and $\Omega$ is normally regular, then the multiplier $\lambda(t)$ can be chosen such that the following estimate holds:
\[
|\lambda(t)| \leq \kappa \{(k + k\phi) | p(t) | + \lambda_0 kF\} \quad a.e.
\]
for some positive constants $k, \kappa, k\phi, kF$, where $k\phi, kF$ are the Lipschitz coefficients of $\phi, F$ on set $D$ defined as in (3.5) respectively.
Proof By [30, Theorem 14.26], one can easily get the measurability of the mapping \( \lambda : t \rightarrow N_\Omega^C(\Phi(x_*(t), u_*(t))) \). The Euler adjoint inclusion in the explicit multiplier form can be easily verified in (3.2) when \( \Phi \) is strictly differentiable.

We now prove the estimate for \( \lambda(t) \) in (3.4). Since the set-valued map \( M \) is calm at \((0, x_*(t), u_*(t))\), it is equivalent to saying that the set-valued map \( M^{-1}(x, u) := \Phi(x, u) - \Omega \) is metrically subregular at \((x_*(t), u_*(t), 0)\). Since the set

\[
D := cl \left\{ \bigcup_{t \in [0, t_1]} (x_*(t), u_*(t)) \right\} 
\]

is compact, one can find a constant \( \kappa > 0 \) such that the set-valued map \( M^{-1} := \Phi(x, u) - \Omega \) is metrically subregular at \((x_*(t), u_*(t), 0)\) for all \((x_*(t), u_*(t)) \in D \) with the same modulus \( \kappa > 0 \). We get by Proposition 3.1 that

\[
N_M(x_*(t), u_*(t)) 
\leq \{(\alpha, \beta) : \exists -\lambda \in \kappa |(\alpha, \beta)| \tilde{B} \ s.t. (\alpha, \beta, -\lambda) \in N_{gphM^{-1}}(x_*(t), u_*(t), 0)\}.
\]

Since \( gphM^{-1} = \{(x, u, v) : v \in \Phi(x, u) - \Omega = \{(x, u, v) : \Phi(x, u) - v \in \Omega\} \) follows from [30, Exercise 6.7] that

\[
N_{gphM^{-1}}(x_*(t), u_*(t), 0) = \{(\alpha, \beta, -\lambda) : (\alpha, \beta) = \nabla \Phi(x_*(t), u_*(t))^T \lambda, \lambda \in N_\Omega(\Phi(x_*(t), u_*(t)))\}.
\]

Therefore

\[
N_M(x_*(t), u_*(t)) 
\leq \{(\alpha, \beta) : \exists \lambda \in \kappa |(\alpha, \beta)| \tilde{B} \cap N_\Omega(\Phi(x_*(t), u_*(t)))\}, (\alpha, \beta) = \nabla \Phi(x_*(t), u_*(t))^T \lambda \in N_\Omega(\Phi(x_*(t), u_*(t)))\}
\]

(3.6)

Since the proof of Theorem 3.1 is based on Proposition 6.3 which is [26, Theorem 4.2] whose proof is based on transforming the optimal control problem to a differential inclusion problem with a pseudo-Lipschitz set-valued map, we can obtain that \(|\dot{\lambda}(t)| \leq k|p(t)|\) where constant \( k > 0 \) is the pseudo-Lipschitz module of the set-valued map. Moreover since \( \Omega \) is normally regular, the limiting normal cone coincides with the Clarke normal cone to \( \Omega \). Hence from the proof of [26, Theorems 4.1 and 4.2], if we use the estimate in (3.6) to replace the estimate for \( N_M(x_*(t), u_*(t))\), then for almost every \( t \), we can find \( \lambda(t) \in \kappa |\nabla \Phi(x_*(t), u_*(t))^T \tilde{\lambda}(t)| \tilde{B} \cap N_\Omega(\Phi(x_*(t), u_*(t)))\) satisfying the Euler’s inclusion:

\[
(\dot{\kappa}(t), 0) \in \partial^C \{-p(t, \phi) + \lambda_0 F(x_*(t), u_*(t)) + \nabla \Phi(x_*(t), u_*(t))^T \tilde{\lambda}(t)\}.
\]

From this Euler’s inclusion, we may choose

\[
(\xi(t), \eta(t)) \in \partial^C \{-p(t, \phi) + \lambda_0 F(x_*(t), u_*(t))\}
\]

satisfying \((\dot{\kappa}(t), 0) - (\xi(t), \eta(t)) = \nabla \Phi(x_*(t), u_*(t))^T \tilde{\lambda}(t)\). In view of the Lipschitz assumption on \( \phi, F \) and the compactness of set \( D \), we get that \(|(\xi(t), \eta(t))| \leq k^\phi |p(t)| + \lambda_0 k^F\), where \( k^\phi, k^F \) are the Lipschitz coefficients of \( \phi, F \). With respect to \((x, u)\) on set \( D \) respectively. It follows that

\[
|\tilde{\lambda}(t)| \leq \kappa |\nabla \Phi(x_*(t), u_*(t))^T \tilde{\lambda}(t)| 
\leq \kappa |(\dot{\kappa}(t), 0) - (\xi(t), \eta(t))| 
\leq \kappa k|p(t)| + \kappa(k^\phi |p(t)| + \lambda_0 k^F) \leq \kappa \{(k + k^\phi)|p(t)| + \lambda_0 k^F\}, \text{ a.e.}
\]
The constraint qualification imposed in Theorem 3.1 is required to hold for points in a neighborhood of the optimal process \((x_*, u_*)\). It is natural to ask whether this condition can be imposed only along the optimal process \((x_*, u_*)\). In order to answer this question we first introduce the following concept.

**Definition 3.1** [8, Definition 4.7] We say that \((t, x_*(t), u)\) is an admissible cluster point of \((x_*, u_*)\) if there exists a sequence \(t_i \in [t_0, t_1]\) converging to \(t\) and \(\Phi(x_i, u_i) \in \Omega, u_i \in U\) such that \(\lim x_i = x_*(t)\) and \(\lim u_i = \lim u_*(t_i) = u\).

We now derive a similar result as Clarke and de Pinho [8, Theorem 4.8] under the WBCQ plus the calmness of \(M\) which is weaker than MFC required by [8, Theorem 4.8]. Note that in the case where \(u_*(t)\) is continuous, the only admissible cluster point of \((x_*, u_*)\) is \((t, x_*(t), u_*(t))\) and hence the constraint qualification is only needed to be verified along the optimal process \((x_*, u_*)\).

**Theorem 3.3** Let \((x_*, u_*)\) be a local minimum of constant radius \(R\) for \((P)\). Suppose that the optimal control \(u_*\) is bounded. Assume that for every \((x_*(t), u)\) such that \((t, x_*(t), u)\) is an admissible cluster point of \((x_*, u_*)\), the WBCQ holds:

\[
\begin{align*}
(\alpha, 0) \in \partial \langle \lambda, \Phi \rangle(x_*(t), u) + \{0\} \times N_U(u), \\
\lambda \in N_{\Omega}(\Phi(x_*(t), u)) 
\end{align*}
\]

and the map \(M\) defined as in (2.1) is calm at \((0, x_*(t), u)\). Then the necessary optimality conditions of Theorem 3.1 hold as stated with some radius \(\eta \in (0, R)\): for some \(\eta \in (0, R)\), for \(t\ a.e.,\)

\[
\Phi(x_*(t), u) \in \Omega, u \in U, \ |u - u_*(t)| < \eta \implies
\langle p(t), \phi(x_*(t), u) \rangle - \lambda_0 F(x_*(t), u) \leq \langle p(t), \phi(x_*(t), u_*(t)) \rangle - \lambda_0 F(x_*(t), u_*(t)).
\]

Moreover if \(u_*(\cdot)\) is continuous, then the WBCQ and the calmness condition are only required to hold along \((x_*(t), u_*(t))\).

Moreover if \(\Phi\) is strictly differentiable, then the Euler adjoint inclusion can be replaced by the one in the explicit multiplier form (3.3) and if \(N_C^C(u_*(t)) = \{0\}\) and \(\Omega\) is normally regular, then the estimate for the multiplier \(\lambda(t)\) in (3.4) also holds.

The proof of Theorem 3.3 uses the following result.

**Proposition 3.2** [26, Theorem 4.3] Let \((x_*, u_*)\) be a \(W^{1,1}\) local minimum of constant radius \(R\) for \((P)\). Suppose that there exists \(\delta > 0\) such that \(R(t) \geq \delta\). Moreover suppose that for all \((x_*(t), u)\) such that \((t, x_*(t), \phi(x_*(t), u))\) is an admissible cluster point of \(x_*\) in the sense of [26, Definition 4.1], the WBCQ holds:

\[
\begin{align*}
(\alpha, 0) \in \partial \langle \lambda, \Phi \rangle(x_*(t), u) + \{0\} \times N_U(u), \\
\lambda \in N_{\Omega}(\Phi(x_*(t), u)) 
\end{align*}
\]

and the mapping \(M\) defined as in (2.1) is calm at \((0, x_*(t), u)\). Then the necessary optimality conditions of Proposition 6.3 holds as stated with some radius \(\eta \in (0, R)\). Moreover if \(\dot{x}_*(\cdot)\) is continuous, then the WBCQ and the calmness condition are only required to hold along \((x_*(t), u_*(t))\).
Proof of Theorem 3.3 The proof is similar to the one in Theorem 3.1. The only difference is that instead of using Proposition 6.3, we use Proposition 3.2. The last statement of Theorem 3.3 follows from Theorem 3.2. □

4 Optimal Control Problems with Implicit Control Systems

The main purpose of this section is to derive necessary optimality conditions for problem \( (P_{ICS}) \). As commented in Section 1, we can transform \( (P_{ICS}) \) into the equivalent problem \( (PECS) \) by introducing a vector variable \( v(t) := ˙x(t) \). The problem \( (PECS) \) is a special case of problem \( (P) \) studied in Section 3 with \( \phi := v \). Unless otherwise specified, in this section we assume that \( F, f, ϕ \) are locally Lipschitz continuous, and the sets \( U, K_ϕ, S \) are closed. It is easy to check that the concept of a local minimum of radius \( R(·) \) for the implicit control problem \( (P_{ICS}) \) defined as in the introduction coincides with the definition of a local minimum of radius \( R(·) \) for problem \( (P) \).

Define
\[
S_{ϕ}^{c,R}(t) := \{(x, u, v) ∈ M_ϕ(0) : |x − x_⋆(t)| ≤ ε, |(u, v) − (u_⋆(t), ˙x_⋆(t))| ≤ R(t)\},
\]
\[
C_{ϕ}^{c,R} := cl \left\{ (t, x, u, v) ∈ [t_0, t_1] × N^{u} × N^{u} : (x, u, v) ∈ S_{ϕ}^{c,R}(t) \right\},
\]
where the set-valued map \( M_ϕ(θ) \) is defined as in (1.2). With these identifications, the following results follow immediately from Theorems 3.1, 3.2, 3.3 and the calculus rule for normal cones.

Theorem 4.1 Let \((x_⋆, u_⋆)\) be a local minimum of radius \( R(·) \) for \((P_{ICS})\). Suppose that there exists \( δ > 0 \) such that \( R(t) ≥ δ \). Suppose further that \( C_{ϕ}^{c,R} \) is compact and for all \((t, x, u, v) ∈ C_{ϕ}^{c,R}\) the WBCQ holds:
\[
\begin{align*}
(α, 0, 0) & ∈ ∂(λ_ϕ, ϕ)(x, u, v) + [0] × N^{u}(u) × [0], & ⇒ α = 0
\end{align*}
\]
and the mapping \( M_ϕ \) defined as in (1.2) is calm at \((0, x, u, v)\). Then there exist an arc \( p \) and a number \( λ_0 \) in \([0, 1]\), satisfying the nontriviality condition \((λ_0, p(t)) \neq 0, ∀t ∈ [t_0, t_1]\), the transversality condition
\[
(p(t_0), −p(t_1)) ∈ λ_0∂f(x_⋆(t_0), x_⋆(t_1)) + N_{S}(x_⋆(t_0), x_⋆(t_1)),
\]
and the Euler adjoint inclusion for almost every \( t \):
\[
\begin{align*}
(&p(t), −μ(t), p(t)) ∈ λ_0∂^C F(x_⋆(t), u_⋆(t), ˙x_⋆(t)) \\
+&c_0[∂μ_ϕ(x_⋆(t), u_⋆(t), ˙x_⋆(t)) : λ_ϕ ∈ N_{K_ϕ}(ϕ(x_⋆(t), u_⋆(t), ˙x_⋆(t)))],
\end{align*}
\]
where \( μ(·) \) is a measurable function satisfying \( μ(t) ∈ N^{U}_{U}(u_⋆(t)) a.e. \) as well as the Weierstrass condition of radius \( R(·) \) for almost every \( t \):
\[
(x_⋆(t), u, v) ∈ M_ϕ(0), |(u, v) − (u_⋆(t), ˙x_⋆(t))| ≤ R(t) \implies
\]
\[
⟨p(t), v⟩ − λ_0 F(x_⋆(t), u, v) ≤ ⟨p(t), ˙x_⋆(t)⟩ − λ_0 F(x_⋆(t), u_⋆(t), ˙x_⋆(t)).
\]
Moreover if either \( K_ϕ ⊆ R^{m} \) or \( ϕ \) is strictly differentiable, then the Euler adjoint inclusion can be replaced by the one in the explicit multiplier form, i.e., there exists measurable functions \( λ_φ : [t_0, t_1] → R^m_+, \mu : [t_0, t_1] → R^{ns} \) with \( λ_ϕ(t) ∈ N^{C}_{K_ϕ}(ϕ(x_⋆(t), u_⋆(t), ˙x_⋆(t))), \mu(t) ∈ N^{C}_{U}(u_⋆(t)) a.e. \) satisfying
\[
(\dot{p}(t), −μ(t), p(t)) ∈ λ_0∂^C F(x_⋆(t), u_⋆(t), ˙x_⋆(t)) + ∂^C ϕ(x_⋆(t), u_⋆(t), ˙x_⋆(t))^T λ_ϕ(t) a.e. \]
If $N^C_U(u_*(t)) = \{0\}$, $K_\varphi$ is normally regular and $\varphi$ is strictly differentiable, then the estimate for the multiplier $\lambda_\varphi(t)$ in (3.4) also holds, namely,
\[
|\lambda_\varphi(t)| \leq \kappa [k|p(t)| + \lambda_0 k^F] \quad a.e.
\]
for some positive constants $k, \kappa, k^F$, where $k^F$ is the Lipschitz coefficients of $F$ on set $D$ defined as in (3.5) respectively. Moreover if $u_*(\cdot)$ is continuous, then the WBCQ and the calmness condition are only required to hold along $(x_*(t), u_*(t))$. In the case of free end point, $\lambda_0$ can be taken as $1$.

A special case of the optimal control of implicit systems is the following problem
\[
(P_{DAE}) \quad \min J(x, u) := \int_{t_0}^{t_1} F(x(t), u(t), \dot{x}(t))dt + f(x(t_0), x(t_1)),
\]
s.t. $\varphi(x(t), u(t), \dot{x}(t)) = 0,
\]
$u(t) \in U \quad a.e. t \in [t_0, t_1],
\]
$(x(t_0), x(t_1)) \in S.$

This problem was studied in [8, Section 6] with a time dependent control set $U(t)$. Applying Theorem 4.1 with $K_\varphi = \{0\}$, we immediately have the following result.

**Corollary 4.1** Let $(x_*, u_*)$ be a local minimum of radius $R(\cdot)$ for $(P_{DAE})$. Suppose that there exists $\delta > 0$ such that $R(t) \geq \delta$. Suppose further that $C^e, R$ as defined in (4.1) with $K_\varphi = \{0\}$ is compact and for all $(t, x, u, v) \in C^e, R$ the WBCQ holds:
\[
\lambda_\varphi \in \mathbb{R}^m, \quad (\alpha, 0, 0) \in \partial (\lambda_\varphi, \varphi)(x, u, v) + \{0\} \times N_U(u) \times \{0\} \implies \alpha = 0
\]
and the mapping $M_\varphi$ as defined in (1.2) with $K_\varphi = \{0\}$ is calm at $(0, x, u, v)$. Then there exist an arc $p$ and a number $\lambda_0$ in $[0, 1]$, satisfying the nontriviality condition $(\lambda_0, p(t)) \neq 0, \forall t \in [t_0, t_1]$, the transversality condition
\[
(p(t_0), -p(t_1)) \in \lambda_0 \partial f(x_*(t_0), x_*(t_1)) + N_S(x_*(t_0), x_*(t_1)),
\]
and the Euler adjoint inclusion for almost every $t$:
\[
(\dot{p}(t), -\mu(t), p(t)) \in \lambda_0 \partial^C F(x_*(t), u_*(t), \dot{x}_*(t))
\]
\[
+ \cos \{\partial (\lambda_\varphi, \varphi)(x_*(t), u_*(t), \dot{x}_*(t)) : \lambda_\varphi \in \mathbb{R}^m\},
\]
where $\mu(\cdot)$ is a measurable function satisfying $\mu(t) \in N^C_U(u_*(t))$ a.e., as well as the Weierstrass condition of radius $R(\cdot)$ for almost every $t$:
\[
u \in U, \varphi(x_*(t), u, v) = 0, |(u, v) - (u_*(t), \dot{x}_*(t))| < R(t) \implies \langle p(t), v \rangle - \lambda_0 F(x_*(t), u, v) \leq \langle p(t), \dot{x}_*(t) \rangle - \lambda_0 F(x_*(t), u_*(t), \dot{x}_*(t)).
\]
Suppose further that $\varphi$ is strictly differentiable, then the Euler adjoint inclusion can be expressed in the explicit form: there exists measurable functions $\lambda_\varphi : [t_0, t_1] \rightarrow \mathbb{R}^m, \mu : [t_0, t_1] \rightarrow \mathbb{R}^m$ with $\mu(t) \in N^C_U(u_*(t))$ a.e. such that
\[
(\dot{p}(t), -\mu(t), p(t)) \in \lambda_0 \partial^C F(x_*(t), u_*(t), \dot{x}_*(t)) + \nabla \varphi(x_*(t), u_*(t), \dot{x}_*(t)) \lambda_\varphi(t) a.e.
\]
If $N^C_U(u_*(t)) = \{0\}$, then the estimate for the multiplier $\lambda_\varphi(t)$ in (3.4) also holds:
\[
|\lambda_\varphi(t)| \leq \kappa [k|p(t)| + \lambda_0 k^F] \quad a.e.
\]
for some positive constants $k, \kappa, k^F$, where $k^F$ is the Lipschitz coefficients of $F$ on set $D$ defined as in (3.5) respectively. Moreover if $u_*(\cdot)$ is continuous, then the WBCQ and the
calmness condition are only required to hold along \((x_*(t), u_*(t), \dot{x}_*(t))\). In the case of free end point, \(\lambda_0\) can be taken as 1.

Note that in [8, Theorem 6.1 and Corollary 6.2], a similar result is obtained. Their results allow for the dynamic system to be nonautonomous but they require the calibrated constraint qualification or MFC to hold which are stronger than WBCQ+calmness.

5 Optimal Control of Semi-Explicit DAEs

In this section we consider the following optimal control problem of semi-explicit DAEs:

\[
\begin{align*}
(P_{seDAE}) \quad & \min_{x, y, u} J(x, y, u) := \int_{t_0}^{t_1} F(x(t), y(t), u(t)) dt + f(x(t_0), x(t_1)), \\
& \text{s.t. } \dot{x}(t) = \phi(x(t), y(t), u(t)) \quad \text{a.e. } t \in [t_0, t_1], \\
& \quad 0 = h(x(t), y(t), u(t)) \quad \text{a.e. } t \in [t_0, t_1], \\
& \quad u(t) \in U \quad \text{a.e. } t \in [t_0, t_1], \\
& \quad (x(t_0), x(t_1)) \in S,
\end{align*}
\]

where \( F : \mathbb{R}^{nx} \times \mathbb{R}^{ny} \times \mathbb{R}^{nu} \to \mathbb{R}, \phi : \mathbb{R}^{nx} \times \mathbb{R}^{ny} \times \mathbb{R}^{nu} \to \mathbb{R}^{nx}, h : \mathbb{R}^{nx} \times \mathbb{R}^{ny} \times \mathbb{R}^{nu} \to \mathbb{R}^{ny}, \) the others are the same as in (P). In this section, unless otherwise specified we assume that \( F, f, \phi, h \) are locally Lipschitz continuous.

The dynamic is said to have “index \(k\)” if one needs to differentiate the algebraic part \((k-1)\)-times in time to get the underlying system of ODE [19]. The main restriction on the necessary optimality condition of the optimal control problem of semi-explicit DAEs is the assumption that the dynamics have “index one” (see e.g. [8, 11, 24]), i.e., the Jacobian matrix \( \nabla_y h(x_*(t), y_*(t), u_*(t)) \) has full rank, or equivalently

\[
\det \nabla_y h(x_*(t), y_*(t), u_*(t)) \neq 0.
\]

In the index one case, by using the implicit function theory, the variable \(y(t)\) can be solved locally and hence the system behaves like an ODE. Derivation of optimality conditions for higher index problems is a challenging area.

We take two approaches to study the problem. In the first approach we treat \(y\) as a control and explore the consequences of Theorem 3.1 and in the second approach we treat \(y\) as a state and explore the consequences of Corollary 4.1. Both approaches allow us to derive necessary optimality conditions without the assumption that the problem is of index one. Such approaches have also been taken in [24] to specialize the results of [8] to the control of semi-explicit DAEs. But their results can only be applied to problem of index one.

If we treat \(y\) as a control, then both \(u(\cdot)\) and \(y(\cdot)\) are measurable functions on \([t_0, t_1]\) such that \(u(t) \in U\) for almost every \(t \in [t_0, t_1]\). The state corresponding to a given control \((u(\cdot), y(\cdot))\), refers to an absolutely continuous function \(x(\cdot)\) which together with \(u(\cdot), y(\cdot)\) satisfying all the constraints of the problem \(P_{seDAE}\). We call such a pair \((x(\cdot), y(\cdot), u(\cdot))\) an admissible pair. Let \(R : [t_0, t_1] \to (0, +\infty)\) be a radius function. We say that \((x_*, y_*, u_*)\) is a local minimum of radius \(R(\cdot)\) for \(P_{seDAE}\) if it minimizes the value of the cost function \(J(x, y, u)\) over all admissible pairs \((x, y, u)\) which satisfies

\[
|x(t) - x_*(t)| \leq \varepsilon, \quad |(y(t), u(t)) - (y_*(t), u_*(t))| \leq R(t) \quad \text{a.e.}, \quad \int_{t_0}^{t_1} |\dot{x}(t) - \dot{x}_*(t)| dt \leq \varepsilon.
\]
Define a set-valued map as the perturbed constrained system:
\[ M_h(\Theta) := \{(x, y, u) \in \mathbb{R}^{ nx } \times \mathbb{R}^{ ny } \times U : h(x, y, u) + \Theta = 0\} \] (5.1)
and
\[ S^C_{h,R}(t) := \{(x, y, u) \in M_h(0) : |x - x_a(t)| \leq \varepsilon, |(y, u) - (y_a(t), u_a(t))| \leq R(t)\}, \]
\[ C^C_{h,R} := cl\{(t, x, y, u) \in [t_0, t_1] \times \mathbb{R}^{ nx } \times \mathbb{R}^{ ny } \times U : (x, y, u) \in S^C_{h,R}(t)\}. \]

A simple application of Theorem 3.1 yields the following results.

**Theorem 5.1** Let \((x_a, y_a, u_a)\) be a local minimum of radius \(R(\cdot)\) for \((P_{\varepsilon\text{DAE}})\). Suppose that \(C^C_{h,R}\) is compact, and there exists \(\delta > 0\) such that \(R(t) \geq \delta\). Suppose further that, for all \((t, x, y, u) \in C^C_{h,R}\) the WBCQ holds:
\[ \lambda \in \mathbb{R}^{ ny }, (\alpha, 0, 0) \in \partial(\lambda, h)(x, y, u) + \{(0, 0)\} \times N_U(u) \implies \alpha = 0, \] (5.2)
and the mapping \(M_h\) is calm at \((0, x, y, u)\). Then there exist an arc \(p\) and a number \(\lambda_0\) in \([0, 1]\), satisfying the nontriviality condition \((\lambda_0, p(t)) \neq 0, \forall t \in [t_0, t_1]\), the transversality condition
\[ (p(t_0), -p(t_1)) \in \lambda_0 \partial f(x_a(t_0), x_a(t_1)) + N_S(x_a(t_0), x_a(t_1)), \]
and the Euler adjoint inclusion for almost every \(t\):
\[ (\dot{p}(t), 0, -\mu(t)) \in \partial^C \{(p(t), \phi) + \lambda_0 F(x_a(t), y_a(t), u_a(t))\}
+ \text{co}[\partial(\lambda, h(x_a(t), y_a(t), u_a(t)))) : \lambda \in \mathbb{R}^{ ny }], \]
where \(\mu(\cdot)\) is a measurable function satisfying \(\mu(t) \in N_U^C(u_a(t))\) a.e., as well as the Weierstrass condition of radius \(R(\cdot)\) for almost every \(t\):
\[ u \in U, h(x_a(t), y, u) = 0, |(y, u) - (y_a(t), u_a(t))| < R(\cdot) \implies \]
\[ (p(t), \phi(x_a(t), y, u)) - \lambda_0 F(x_a(t), y, u) \leq (p(t), \phi(x_a(t), y_a(t), u_a(t)))
- \lambda_0 F(x_a(t), y_a(t), u_a(t)). \]

Moreover if we assume further that \(h\) is strictly differentiable, then the Euler adjoint inclusion can be replaced by the one in the explicit multiplier form, i.e., there exist measurable functions \(\lambda_h : [t_0, t_1] \to \mathbb{R}^{ ny }, \mu : [t_0, t_1] \to \mathbb{R}^{ nu }\) with \(\mu(t) \in N_U^C(u_a(t))\) a.e. satisfying
\[ (\dot{p}(t), 0, -\mu(t)) \in \partial^C \{(p(t), \phi) + \lambda_0 F(x_a(t), y_a(t), u_a(t))\}
+ \nabla h(x_a(t), y_a(t), u_a(t))^T \lambda_h(t), \ a.e. \ t \in [t_0, t_1]. \]

If \(N_U(u_a(t)) = \{0\}\), then the estimate for the multiplier \(\lambda_h(t)\) in (3.4) also holds:
\[ |\lambda_h(t)| \leq \kappa |k| p(t)| + \lambda_0 k^F \]
a.e.
for some positive constants \(k, \kappa, k^F\), where \(k^F\) is the Lipschitz coefficients of \(F\) on set \(D\) defined as in (3.5) respectively. Moreover if \(u_a(\cdot)\) is continuous, then the WBCQ and the calmness condition are only required to hold along \((x_a(t), y_a(t), u_a(t))\). In the case of free end point, \(\lambda_0\) can be taken as 1.

Note that our necessary optimality condition is not the so-called strong maximum principle as in [11, Theorem 3.1]. It was shown in [11] by using the following example that that a strong maximum principle may not hold if the velocity set is nonconvex. But the conclusion of our necessary optimality condition is more than just weak maximum principle as in [11, Theorem 3.2]. In fact only the nontriviality condition, the transversality condition and the Euler adjoint inclusion alone constitute the weak maximum principle, let alone the extra
Necessary optimality conditions for implicit control systems. Weierstrass condition. A consequence is that we derive the weak maximum principle under the WBCQ plus calmness condition which allows application to problems with index higher than one.

Example 5.1 [11].

\[
\begin{align*}
\text{min} & \quad -x(1) \\
\text{s.t.} & \quad \dot{x}(t) = (u(t) - y(t))^2 \quad a.e. t \in [0, 1], \\
& \quad 0 = u(t) - y(t) \quad a.e. t \in [0, 1], \\
& \quad u(t) \in [-1, 1], \\
& \quad x(0) = 0.
\end{align*}
\]

In this example, the function \( h \) is independent of \( x \) and is affine. In fact if \( h \) is independent of \( x \) and is affine, by [26, Proposition 2.2], \( M_h \) is calm. Consequently the WBCQ plus calmness condition holds automatically. Then the following results follow from Theorem 5.1.

Corollary 5.1 Let \((x_*, y_*, u_*)\) be a local minimum of radius \( R(\cdot) \) for \((P_{seDAE})\). Suppose that \( F, f, \phi \) are locally Lipschitz continuous, \( h \) is independent of the variable \( x \) and is affine and \( U \) is a union of finitely many polyhedral sets. Suppose further that \( C^R_h \) is compact, and there exists \( \delta > 0 \) such that \( R(t) \geq \delta \). Then the conclusions of Theorem 5.1 hold with the explicit Euler adjoint inclusion

\[
(\dot{p}(t), 0, -\mu(t)) \in \partial C \left\{ (-p(t), \phi) + \lambda_0 F \right\}(x_*(t), y_*(t), u_*(t)) + \nabla h(x_*(t), y_*(t), u_*(t))^T \lambda_h(t), \quad a.e. \ t \in [t_0, t_1].
\]

If \( N^C_U(u_*(t)) = \{0\} \), then the estimate for the multiplier \( \lambda_h(t) \) in (3.4) also holds:

\[
|\lambda_h(t)| \leq \kappa \{ k |p(t)| + \lambda_0 k^F \} \quad a.e.
\]

for some positive constants \( k, \kappa, k^F \), where \( k^F \) is the Lipschitz coefficients of \( F \) on set \( D \) defined as in (3.5) respectively.

Taking \( \varepsilon > 0 \) to be finite and \( R(t) = \infty \), it is obvious that \((x_*, y_*, u_*) = (0, 0, 0)\) is a local minimum of radius \( R \) for the problem in Example 5.1, the set

\[
C^R_h := \{(t, x, y, u) \in [0, 1] \times R \times R \times [-1, 1] : y = u, |x| \leq \varepsilon \}
\]

is compact. Hence all assumptions in Corollary 5.1 holds. Since it is a free end-point problem, \( \lambda_0 = 1 \). It is easy to show that all conditions of the necessary optimality conditions hold with \( p(t) \equiv 1 \), \( \lambda_h(t) \equiv 0 \).

Now we take the second approach by considering \( z = (x, y) \) as the state variable. We consider the problem \( P_{seDAE} \) as the following implicit control problem:

\[
(P_{DAE}) \min \ J(z, u) := \int_{t_0}^{t_1} F(z(t), u(t)) dt + f(x(t_0), x(t_1)),
\]

s.t. \( \varphi(z(t), u(t), \dot{z}(t)) = 0, \)

\( u(t) \in U \quad a.e. t \in [t_0, t_1], \)

\( (x(t_0), x(t_1)) \in S, \)
with \( z = (x, y) \) and
\[
\varphi(z, u, v) := (\phi(z, u) - v_1, h(z, u))^T, \quad v := (v_1, 0)
\]
\( v_1 \in \mathbb{R}^{n_x} \) and apply Corollary 4.1. The state corresponding to a given control \( u(\cdot) \), refers to an absolutely continuous function \((x(\cdot), y(\cdot))\) which together with \( u(\cdot) \) satisfying all conditions in \((P_{\text{DAE}})\). Let \( R : [t_0, t_1] \rightarrow (0, +\infty] \) be a radius function. We say that \((x_*, y_*, u_*)\) is a local minimum of radius \( R(\cdot) \) for \((P_{\text{DAE}})\) if it minimizes the value of the cost function \( J(x, y, u) \) over all admissible pairs \((x, y, u)\) which satisfies
\[
|\langle x(t), y(t) \rangle - \langle x_*(t), y_*(t) \rangle | \leq \varepsilon, \quad |\langle u(t), \dot{x}(t), \dot{y}(t) \rangle - \langle u_*(t), \dot{x}_*(t), \dot{y}_*(t) \rangle | \leq R(t) \text{ a.e.,}
\]
\[
\int_{t_0}^{t_1} |\langle \dot{x}(t), \dot{y}(t) \rangle - \langle \dot{x}_*(t), \dot{y}_*(t) \rangle | \, dt \leq \varepsilon.
\]
Let \( z_* := (x_*, y_*) \). Define a set-valued map as the perturbed constrained system:
\[
M_\varphi(\Theta) := \left\{ (x, y, u, v) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times U \times \mathbb{R}^{n_x+n_y} : \varphi(x, y, u, v) + \Theta = 0 \right\}, \quad (5.3)
\]
and
\[
S_\varphi \Gamma_1^R(t) := \left\{ (z, u, v) \in M_\varphi(0) : |z - z_*(t)| \leq \varepsilon, |\langle u, v \rangle - \langle u_*(t), \dot{z}_*(t) \rangle | \leq R(t) \right\},
\]
\[
C_\varphi \Gamma_1^R := \text{cl} \{ (t, z, u, v) \in [t_0, t_1] \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times U \times \mathbb{R}^{n_x+n_y} : (z, u, v) \in S_\varphi \Gamma_1^R(t) \}.
\]
With these identifications, we can apply Corollary 4.1 and obtain the results as follows.

**Theorem 5.2** Let \((x_*, y_*, u_*)\) be a local minimum of radius \( R(\cdot) \) for \((P_{\text{DAE}})\) in the above sense. Suppose that there exists \( \delta > 0 \) such that \( R(t) \geq \delta \). Suppose further that \( C_\varphi \Gamma_1^R \) is compact and for all \((t, z, u, v) \in C_\varphi \Gamma_1^R\) the WBCQ holds:
\[
\lambda \in \mathbb{R}^{n_x}, (\alpha_1, \alpha_2, 0) \in \partial(\lambda, h)(x, y, u) + \{(0, 0)\} \times NU(u) \implies \alpha_1 = 0, \alpha_2 = 0 \quad (5.4)
\]
and the mapping \( M_\varphi \) defined as in \((5.3)\) is calm at \((0, x, y, u) \). Then there exist an arc \( p \) and a number \( \lambda_0 \) in \([0, 1] \), satisfying the nontriviality condition \((\lambda_0, p(t)) \neq 0, \forall t \in [t_0, t_1] \), the transversality condition
\[
(p(t_0), -p(t_1)) \in \lambda_0 \partial f(x_*(t_0), x_*(t_1)) + NS(x_*(t_0), x_*(t_1)),
\]
and the Euler adjoint inclusion for almost every \( t \):
\[
(\dot{p}(t), 0, -\mu(t)) \in \lambda_0 \partial^C F(x_*(t), y_*(t), u_*(t))
\]
\[+\text{co}\{\partial((\lambda_0, \mu) + (\lambda_h, h))(x_*(t), y_*(t), u_*(t)) : \lambda_0 \in \mathbb{R}^{n_x}, \lambda_h \in \mathbb{R}^{n_y}\},
\]
where \( \mu(\cdot) \) is a measurable function satisfying \( \mu(t) \in N^C_U(u_*(t)) \) a.e., as well as the Weierstrass condition of radius \( R(\cdot) \) for almost every \( t \):
\[
\phi(x_*(t), y_*(t), u) - v_1 = 0, h(x_*(t), y_*(t), u) = 0, |\langle u, v \rangle - \langle u_*(t), \dot{z}_*(t) \rangle | < R(t),
\]
\[u \in U \implies (p(t), v - \dot{z}_*(t)) \leq \lambda_0 (F(x_*(t), y_*(t), u) - F(x_*(t), y_*(t), u_*(t))).
\]
Suppose further that \( \phi, h \) are strictly differentiable, then the Euler adjoint inclusion can be expressed in the explicit form: there exist measurable functions \( \lambda_h : [t_0, t_1] \rightarrow \mathbb{R}^{n_y}, \mu : [t_0, t_1] \rightarrow \mathbb{R}^{n_x} \) with \( \mu(t) \in N^C_U(u_*(t)) \) a.e. such that
\[
(\dot{p}(t), 0, -\mu(t)) \in \lambda_0 \partial^C F(x_*(t), y_*(t), u_*(t))
\]
\[+\nabla \phi(x_*(t), y_*(t), u_*(t))^T p(t) + \nabla h(x_*(t), y_*(t), u_*(t))^T \lambda_h(t).
\]
In the case of free end point, \( \lambda_0 \) can be taken as 1.
Proof  By Corollary 4.1, if for any \((t, z, u, v) \in C^{c,R}_\varphi\), the WBCQ holds:
\[
\begin{align*}
(\alpha, 0, 0) & \in \partial_{z,u} \{ (\lambda_1, \phi) + (\lambda_2, h) \} (z, u) \times \{ 0 \} \\
\lambda_1 & \in \mathbb{R}^{n_x}, \lambda_2 \in \mathbb{R}^{n_y}
\end{align*}
\]
and the mapping \(M_\varphi\) is calm at \((0, x, y, u, v)\), then there exist arcs \(p_x, p_y\) and \(\lambda_0 \in [0, 1]\), satisfying the nontriviality condition \((\lambda_0, p_x(t), p_y(t)) \neq 0, \forall t \in [t_0, t_1]\), the transversality condition
\[
(p_x(t_0), -p_x(t_1)) \in \lambda_0 \partial f(x_*(t_0), x_*(t_1)) + N_S(x_*(t_0), x_*(t_1));
\]
and the Euler adjoint inclusion for almost every \(t\):
\[
(\dot{p}_x(t), \dot{p}_y(t), -\mu(t), p_x(t), p_y(t)) \in
\lambda_0 \partial^C F(x_*(t), y_*(t), u_*(t)) \times \{ 0, 0 \}
\]
where \(\mu(\cdot)\) is a measurable function satisfying \(\mu(t) \in N^C_U(u_*(t))\) a.e., as well as the Weierstrass condition of radius \(R(\cdot)\) for almost every \(t\):
\[
\phi(x_*(t), y_*(t), u) - v_1 = 0, h(x_*(t), y_*(t), u) = 0, \| (u, v) - (u_*(t), \dot{z}_*(t)) \| < R(t),
\]
Suppose further that \(\phi, h\) are strictly differentiable, then the Euler adjoint inclusion can be expressed in the explicit form: there exist measurable functions \(\lambda_\varphi : [t_0, t_1] \to \mathbb{R}^{n_x}\), \(\lambda_h : [t_0, t_1] \to \mathbb{R}^{n_y}\), \(\mu : [t_0, t_1] \to \mathbb{R}^{n_u}\) with \(\mu(t) \in N^C_U(u_*(t))\) a.e. such that
\[
(\dot{p}_x(t), \dot{p}_y(t), -\mu(t), p_x(t), p_y(t)) \in
\lambda_0 \partial^C F(x_*(t), y_*(t), u_*(t)) \times \{ 0, 0 \}
\]
It is easy to see that the WBCQ (5.5) is equivalent to the WBCQ (5.4) and hence all the conclusions above hold. From the above Euler adjoint inclusion we get \(p_y(t) = 0\). In the case where \(\phi, h\) are strictly differentiable, we also get \(p_x(t) = \lambda_\varphi(t)\) a.e.. Hence by taking \(p(t) = p_x(t)\), the conclusions follow.

We now compare Theorem 5.1 (treating \(y\) as a control variable) with Theorem 5.2 (treating \(y\) as a state variable). It is obvious that the WBCQ in (5.4) implies (5.2) and so the WBCQ required for treating \(y\) as a control variable is weaker. In the case where \(\phi, h\) are strictly differentiable, all conclusions except the Weierstrass condition are the same. The Weierstrass condition for treating \(y\) as control is stronger since it implies the one for treating \(y\) as a state variable. In summary, treating \(y\) as control gives stronger necessary optimality conditions under weaker constraint qualifications. But this is not surprising since treating \(y\) as state variables requiring \(y\) to be absolutely continuous while treating \(y\) as control only requires \(y\) to be weaker, i.e., only measurable.

6 Discussion of Constraint Qualifications

In this session we discuss sufficient conditions for constraint qualifications required in Theorems 4.1 and 5.1 to hold. The sufficient conditions for constraint qualifications required in other necessary optimality conditions are similar.
We first discuss sufficient conditions for constraint qualifications for Theorem 4.1 to hold. The constraint qualifications involve the WBCQ plus the calmness of the set-valued map \( M_\varphi \) defined as in (1.2).

It is easy to check that the calmness condition of \( M_\varphi \) at \((0, \bar{x}, \bar{u}, \bar{v})\) holds if and only if the system defining the set \( M_\varphi(0) \) has a local error bound at \((\bar{x}, \bar{u}, \bar{v})\) (see e.g. [22]). There are many sufficient conditions under which the local error bound holds (see e.g. Wu and Ye [32–34]). However not many of them are easy to verify. Two easiest criteria for

There are many sufficient conditions under which the local error bound holds (see e.g. Wu [1, 2, 14, 18, 20]). For convenience, we summarize some prominent verifiable sufficient conditions for the WBCQ plus the calmness of \( M_\varphi \) as follows.

**Proposition 6.1** Let \((\bar{x}, \bar{u}, \bar{v}) \in M_\varphi(0)\), \( \varphi \) is Lipschitz continuous at \((\bar{x}, \bar{u}, \bar{v})\) and \( U, K_\varphi \) are closed. Then the WBCQ

\[
\lambda \in N_{K_\varphi}(\varphi(\bar{x}, \bar{u}, \bar{v})), (\alpha, 0, 0) \in \partial \langle \lambda, \varphi \rangle(\bar{x}, \bar{u}, \bar{v}) + \{0\} \times N_U(\bar{u}) \times \{0\} \implies \alpha = 0 \quad (6.1)
\]

and the set-valued map \( M_\varphi \) defined as in (1.2) is calm at \((0, \bar{x}, \bar{u}, \bar{v})\) if one of the following conditions holds:

(i) The WBCQ (6.1) and the linear constraint qualification (Linear CQ) holds: \( \varphi \) is affine and \( U, K_\varphi \) are the union of finitely many polyhedral sets.

(ii) The CCQ holds at \((\bar{x}, \bar{u}, \bar{v})\): there exists \( \mu > 0 \) such that

\[
\lambda \in N_{K_\varphi}(\varphi(\bar{x}, \bar{u}, \bar{v})), (\alpha, \beta, \gamma) \in \partial \langle \lambda, \varphi \rangle(\bar{x}, \bar{u}, \bar{v}) + \{0\} \times N_U(\bar{u}) \times \{0\} \implies |\lambda| \leq \mu |(\beta, \gamma)|.
\]

(iii) The MFC holds at \((\bar{x}, \bar{u}, \bar{v})\):

\[
\lambda \in N_{K_\varphi}(\varphi(\bar{x}, \bar{u}, \bar{v})), (\alpha, 0, 0) \in \partial \langle \lambda, \varphi \rangle(\bar{x}, \bar{u}, \bar{v}) + \{0\} \times N_U(\bar{u}) \times \{0\} \implies \lambda = 0.
\]

(iv) The NNAMCQ holds at \((\bar{x}, \bar{u}, \bar{v})\):

\[
\lambda \in N_{K_\varphi}(\varphi(\bar{x}, \bar{u}, \bar{v})), (0, 0, 0) \in \partial \langle \lambda, \varphi \rangle(\bar{x}, \bar{u}, \bar{v}) + \{0\} \times N_U(\bar{u}) \times \{0\} \implies \lambda = 0.
\]

(v) The WBCQ (6.1) and the Quasinormality holds at \((\bar{x}, \bar{u}, \bar{v})\):

\[
\exists (x^k, u^k, v^k, y^k, \lambda^k) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{K}_\varphi \times \mathbb{R}^{n_v} \implies (\bar{x}, \bar{u}, \bar{v}, \varphi(\bar{x}, \bar{u}, \bar{v}), \lambda)
\]

such that for each \( \lambda_i \neq 0 \implies \lambda_i(\varphi_i(x^k, u^k, v^k) - y^k_i) > 0
\]

\[
\implies \lambda = 0.
\]

(vi) The WBCQ (6.1) and the first order sufficient condition for metric subregularity (FOSCMS) at \((\bar{x}, \bar{u}, \bar{v})\): \( \varphi \) is differentiable at \((\bar{x}, \bar{u}, \bar{v})\), and for every \( 0 \neq d := (d_1, d_2, d_3) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_v} \) with \( \nabla \varphi(\bar{x}, \bar{u}, \bar{v}) d \in T_{K_\varphi}(\varphi(\bar{x}, \bar{u}, \bar{v})) \), one has

\[
\{ 0, 0, 0 \in \nabla \varphi(\bar{x}, \bar{u}, \bar{v})^T \lambda + \{0\} \times N_U^T(\bar{u}; d_2) \times \{0\}, \}
\]

\[
\lambda \in N_{K_\varphi}(\varphi(\bar{x}, \bar{u}, \bar{v}); \nabla \varphi(\bar{x}, \bar{u}, \bar{v}) d)
\]

\[
\implies \lambda = 0.
\]

(vii) The WBCQ (6.1) and the second order sufficient condition for metric subregularity (SOSCMS) at \((\bar{x}, \bar{u}, \bar{v})\): \( \varphi \) is twice Fréchet differentiable at \((\bar{x}, \bar{u}, \bar{v})\) and \( K_\varphi, U \) are the union of finitely many convex polyhedra sets, and for every \( 0 \neq d :=
(\(d_1, d_2, d_3\)) \in \mathbb{R}^{n_s} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_s} \text{ with } \nabla \varphi(\bar{x}, \bar{u}, \bar{v})d \in T_{K_{\varphi}}(\varphi(\bar{x}, \bar{u}, \bar{v})) , d_2 \in T_U(\bar{u}) \text{ one has}

\[
\begin{align*}
(0, 0, 0) & \in \nabla \varphi(\bar{x}, \bar{u}, \bar{v})^T \lambda + \{0\} \times N_{U}^{\perp}(\bar{u}; d_2) \times \{0\}, \\
\lambda & \in N_{K_{\varphi}}(\varphi(\bar{x}, \bar{u}, \bar{v}); \nabla \varphi(\bar{x}, \bar{u}, \bar{v})/d), \\
d^T \nabla^2(\lambda, \varphi)(\bar{x}, \bar{u}, \bar{v})d & \geq 0
\end{align*}
\]

(iii) The WBCQ (6.1) and the relaxed constant positive linear dependence (RCPLD) holds at \((\bar{x}, \bar{u}, \bar{v})\): \(\varphi\) is differentiable at \((\bar{x}, \bar{u}, \bar{v})\), \(U = \mathbb{R}^{n_u}, K_{\varphi} = \mathbb{R}^{m_1} \times \mathbb{R}^{m_{\text{m} - m_1}}, J \subseteq \{1, \ldots, m_1\}\) is such that \(\{\nabla \varphi_j(\bar{x}, \bar{u}, \bar{v})\}_{j \in J}\) is a basis for the span\(\{\nabla \varphi_j(\bar{x}, \bar{u}, \bar{v})\}_{j = 1}^{m_1}\) and there exists \(\delta > 0\) such that

\[
- \{\nabla \varphi_j(x, u, v)\}_{j = 1}^{m_1}
\]

\(\text{has the same rank for each } (x, u, v) \in B((\bar{x}, \bar{u}, \bar{v}), \delta)\);

- For every \(I \subseteq I(\bar{x}, \bar{u}, \bar{v}) := \{i \in \{m_1 + 1, \ldots, m\} : \varphi_i(\bar{x}, \bar{u}, \bar{v}) = 0\}\), if there exists \(\{\lambda_j\}_{j \in J} \cup I\) with \(j \geq 0 \forall j \in I\) not all zero such that

\[
\sum_{j \in J \cup I} \lambda_j \nabla \varphi_j(\bar{x}, \bar{u}, \bar{v}) = 0,
\]

then \(\{\nabla \varphi_j(x, u, v)\}_{j \in J \cup I}\) is linearly dependent for each \((x, u, v) \in B((\bar{x}, \bar{u}, \bar{v}), \delta)\).

Proof (i) Under Linear CQ, the set-valued map \(M_{\varphi}\) is a polyhedral multifunction and hence upper Lipschitz continuous as shown by Robinson [29]. The results follows from the fact that the upper Lipschitz continuity implies the calmness.

(ii)-(v) By definition, it is easy to see that

\[
\text{CCQ} \Rightarrow \text{MFC} \Rightarrow \text{NNAMCQ} \Rightarrow \text{WBCQ and NNAMCQ} \Rightarrow \text{Quasinormality.}
\]

By [20, Theorem 5.2], the quasinormality implies the calmness.

(vi) Let \(q(x, u) := (\varphi(x, u, v), u) \in \Gamma := \Omega \times U\). Note that the calmness of the set-valued map \(M_{\varphi}()\) at \((0, \bar{x}, \bar{u}, \bar{v})\) is equivalent to the metric subregularity of the set-valued map \(\Sigma(x, u, v) := q(x, u, v) - \Gamma (\bar{x}, \bar{u}, \bar{v}, 0)\). By [16, 1. of Corollary 1], it suffices to show that for every \(0 \neq w \in \nabla q(\bar{x}, \bar{u}, \bar{v})\) one has

\[
\nabla q(\bar{x}, \bar{u}, \bar{v})^T \eta = 0, \eta \in N_{\Gamma}(q(\bar{x}, \bar{u}, \bar{v})); \nabla q(\bar{x}, \bar{u}, \bar{v})w = \eta = 0.
\]

By [36, Proposition 3.3], we have

\[
T_{\Gamma}(q(\bar{x}, \bar{u}, \bar{v})) \subseteq T_{K_{\varphi}}(\Phi(\bar{x}, \bar{u}, \bar{v})) \times T_{U}(\bar{u}),
\]

\[
N_{\Gamma}(q(\bar{x}, \bar{u}, \bar{v})); \nabla q(\bar{x}, \bar{u}, \bar{v})u \subseteq N_{\Omega}(\varphi(\bar{x}, \bar{u}, \bar{v})); \nabla \varphi(\bar{x}, \bar{u}, \bar{v})u) \times N_{U}(\bar{u}; d_2),
\]

and the equality holds if at most one of the sets \(K_{\varphi}, U\) is directionally regular. Hence the FOSCMS defined as in (vi) is stronger than the condition required above and the calmness holds.

(vii) By the same arguments as above, we can verify that the SOSCMS satisfies the condition of [16, 2. of Corollary 1]. So the result holds.

(viii) follows from [21, Theorem 4.2].

Now we discuss sufficient conditions for constraint qualifications for Theorem 5.1 to hold. The constraint qualifications involve the WBCQ plus the calmness of the set-valued map \(M_{\kappa}\) defined as in (5.1) where we treat \(y\) as a control. The proof of the results are similar to Proposition 6.1 and hence we omit it.
Proposition 6.2 Let \((\bar{x}, \bar{y}, \bar{u}) \in M_h(0)\), \(h\) is Lipschitz continuous at \((\bar{x}, \bar{y}, \bar{u})\) and \(U\) is closed. Then the WBCQ

\[
\lambda \in \mathbb{R}^{n_x}, (\alpha, 0, 0) \in \partial(\lambda, h)(\bar{x}, \bar{y}, \bar{u}) + \{(0, 0)\} \times N_U(\bar{u}) \implies \alpha = 0
\]

and the set-valued mapping \(M_h\) defined as in (5.1) is calm at \((0, \bar{x}, \bar{y}, \bar{u})\) if one of the following conditions holds:

(i) The WBCQ (6.2) and the linear constraint qualification (Linear CQ) holds: \(h\) is affine and \(U\) is the union of finitely many polyhedral sets.

(ii) The CCQ holds at \((\bar{x}, \bar{y}, \bar{u})\): there exists \(\mu > 0\) such that

\[
\lambda \in \mathbb{R}^{n_x}, (\alpha, \beta, \gamma) \in \partial(\lambda, h)(\bar{x}, \bar{y}, \bar{u}) + \{(0, 0)\} \times N_U(\bar{u}) \implies |\lambda| \leq \mu |(\beta, \gamma)|.
\]

(iii) The MFC holds at \((\bar{x}, \bar{y}, \bar{u})\):

\[
\lambda \in \mathbb{R}^{n_x}, (\alpha, 0, 0) \in \partial(\lambda, h)(\bar{x}, \bar{y}, \bar{u}) + \{(0, 0)\} \times N_U(\bar{u}) \implies \lambda = 0.
\]

(iv) The NNAMCQ holds at \((\bar{x}, \bar{y}, \bar{u})\):

\[
\lambda \in \mathbb{R}^{n_x}, (0, 0, 0) \in \partial(\lambda, h)(\bar{x}, \bar{y}, \bar{u}) + \{(0, 0)\} \times N_U(\bar{u}) \implies \lambda = 0.
\]

(v) The WBCQ (6.2) and the quasinormality holds at \((\bar{x}, \bar{y}, \bar{u})\):

\[
\begin{align*}
0, 0, 0 & \in \partial(\lambda, h)(\bar{x}, \bar{y}, \bar{u}) + \{(0, 0)\} \times N_U(\bar{u}), \\
\exists (x^k, y^k, u^k, \lambda^k) & \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^d \to (\bar{x}, \bar{y}, \bar{u}, \lambda) \\
\text{such that for each } k, \lambda_k & \neq 0 \implies \lambda_k h_1(x^k, y^k, u^k) > 0
\end{align*}
\]

\[
\implies \lambda = 0.
\]

(vi) The WBCQ (6.2) and the FOSCMS at \((\bar{x}, \bar{y}, \bar{u})\): \(h\) is differentiable at \((\bar{x}, \bar{y}, \bar{u})\), and for every \(0 \neq d := (d_1, d_2) \in \mathbb{R}^{n_x+n_y} \times \mathbb{R}^{n_u}\) with \(\nabla h(\bar{x}, \bar{y}, \bar{u})d = 0\), \(d_2 \in T_U(\bar{u})\) one has

\[
\lambda \in \mathbb{R}^{n_y}, (0, 0, 0) \in \nabla h(\bar{x}, \bar{y}, \bar{u})^T \lambda + \{(0, 0)\} \times N_U^L(\bar{u}; d_2) \implies \lambda = 0.
\]

(vii) The WBCQ (6.2) and SOSCMS at \((\bar{x}, \bar{y}, \bar{u})\): \(h\) is twice Fréchet differentiable at \((\bar{x}, \bar{y}, \bar{u})\), \(U\) is the union of finitely many convex polyhedral sets, and for every \(0 \neq d := (d_1, d_2) \in \mathbb{R}^{n_x+n_y} \times \mathbb{R}^{n_u}\) with \(\nabla h(\bar{x}, \bar{y}, \bar{u})d = 0\), \(d_2 \in T_U(\bar{u})\) one has

\[
\begin{align*}
0, 0, 0 & \in \nabla h(\bar{x}, \bar{y}, \bar{u})^T \lambda + \{(0, 0)\} \times N_U^L(\bar{u}; d_2), \\
d^T \nabla^2 h(\lambda, h)(\bar{x}, \bar{y}, \bar{u})d & \geq 0, \lambda \in \mathbb{R}^{n_x}
\end{align*}
\]

\[
\implies \lambda = 0.
\]

(viii) The WBCQ (6.2) and the constant rank constraint qualification (CRCQ) at \((\bar{x}, \bar{y}, \bar{u})\): suppose \(h\) is differentiable around \((\bar{x}, \bar{y}, \bar{u})\) and \(U = \mathbb{R}^{n_u}\), there exists \(\delta > 0\) such that \(\{\nabla h_j(x, y, u)\}_{j=1}^n\) has the same rank for each \((x, y, u) \in B((\bar{x}, \bar{y}, \bar{u}), \delta)\).

To compare with [10, Section 4], next we consider a special case of \((P_{DAE})\) with

\[
\varphi(x, u, v) := Ev - g(x, u) \text{ and } K\varphi = [0]
\]

\[
\begin{align*}
(P_{DAE}') & \min f(x(t_0), x(t_1)) \\
\text{s.t.} & \quad E\dot{x}(t) - g(x(t), u(t)) = 0, \\
& \quad u(t) \in U \quad a.e. t \in [t_0, t_1], \\
& \quad (x(t_0), x(t_1)) \in S,
\end{align*}
\]

where \(E\) is a \(m \times n_x\) matrix with \(\text{rank}(E) = r\), \(g : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^m\) is strictly differentiable. Depending on the rank of the matrix \(E\), the following three cases are considered in [10, Section 4].
Case (A) $E$ is of full row rank;
Case (B) $E$ is of full column rank;
Case (C) $E$ is of neither of full row rank nor of column rank.

Note that [10] allows for the dynamic system to be nonautonomous but the matrix $E$ is required to have some special forms. For those special matrix $E$, depending on the cases, de Pinho [10] augmented the system and transform the original problem to the one that may be easier to analyze.

In case (A), we obtain the following results as a corollary of Corollary 4.1.

**Corollary 6.1** Let $(x^*, u^*)$ be a local minimum of radius $R(\cdot)$ for $(P_{DAE})$. Suppose that $E$ is of full row rank and that there exists $\delta > 0$ such that $R(t) \geq \delta$. Suppose further that $C^R_{\varphi}$ as defined in (4.1) with $K_{\varphi} = \{0\}$ is compact. Then there exist an arc $p$, a number $\lambda_0$ in $\{0, 1\}$ and a measurable function $\mu : [t_0, t_1] \to \mathbb{R}^{nu}$ with $\mu(t) \in N^C_U(u_\varphi(t))$ a.e. satisfying the nontriviality condition $\langle \lambda_0, p(t) \rangle \neq 0, \forall t \in [t_0, t_1]$, the transversality condition

\[
(p(t_0), -p(t_1)) \in \lambda_0 \partial f(x_\varphi(t_0), x_\varphi(t_1)) + N^C_S(x_\varphi(t_0), x_\varphi(t_1)),
\]

and the Euler adjoint inclusion for almost every $t$:

\[
\begin{aligned}
\dot{p}(t) &= -\nabla_x g(x_\varphi(t), u_\varphi(t))^T (EE^T)^{-1} Ep(t), \text{ a.e.}, \\
\mu(t) &= \nabla_u g(x_\varphi(t), u_\varphi(t))^T (EE^T)^{-1} Ep(t), \text{ a.e.,}
\end{aligned}
\]

(6.3)
as well as the Weierstrass condition of radius $R(\cdot)$ for almost every $t$:

\[
u \in U, Ev = g(x_\varphi(t), u), |(v, v) - (u_\varphi(t), \dot{x}_\varphi(t))| < R(t) \implies \langle p(t), v - \dot{x}_\varphi(t) \rangle \leq 0.
\]

In the case of free end point, $\lambda_0$ can be taken as 1.

**Proof** Since $\nabla \varphi = E$ is of full row rank, the NNAMCQ holds automatically at any feasible point. By Proposition 6.1(iv), WBCQ plus the calmness of $M_\varphi$ holds. Hence all the assumptions in Corollary 4.1 are satisfied. By Corollary 4.1, there exist an arc $p$, a number $\lambda_0$ in $\{0, 1\}$, and measurable functions $\lambda_\varphi : [t_0, t_1] \to \mathbb{R}^m$, $\mu : [t_0, t_1] \to \mathbb{R}^{nu}$ satisfying the nontriviality condition, the transversality condition, the Euler adjoint inclusion and the Weierstrass condition. We only need to prove the Euler adjoint inclusion (6.3). By the Euler adjoint inclusion in Corollary 4.1, we have

\[
p(t) = \nabla \varphi^T \lambda_\varphi = E^T \lambda_\varphi.
\]

Since $E$ is of full row rank, we can solve $\lambda_\varphi = (EE^T)^{-1} Ep(t)$ from the above linear system and hence the proof is completed.

If $E = \begin{pmatrix} E_a & 0 \end{pmatrix}$ where $E_a$ is a $m \times m$ nonsingular matrix, the results obtained for the case (A) are the same as that of [10, Corollary 4.1] but without requiring the restriction for the function $f$.

In case (B) and (C), we obtain the following results as a corollary of Corollary 4.1.

**Corollary 6.2** Let $(x_\varphi, u_\varphi)$ be a local minimum of radius $R(\cdot)$ for $(P_{DAE})$. Suppose that $E$ is not of full row rank but one of assumptions in Proposition 6.1(i)(v)(vi)(vii)(viii) holds. Suppose further that $C^R_{\varphi}$ as defined in (4.1) with $K_{\varphi} = \{0\}$ is compact and there exists $\delta > 0$ such that $R(t) \geq \delta$. Then there exist an arc $p$, a number $\lambda_0$ in $\{0, 1\}$ and measurable functions $\lambda_\varphi : [t_0, t_1] \to \mathbb{R}^m$, $\mu : [t_0, t_1] \to \mathbb{R}^{nu}$ with $\mu(t) \in N^C_U(u_\varphi(t))$ a.e.
satisfying the nontriviality condition, the transversality condition, the Weierstrass condition as in Corollary 6.1 and the Euler adjoint inclusion for almost every $t$:

$$
\begin{align*}
\dot{p}(t) &= -\nabla_x g(x_*(t), u_*(t))^T \lambda_\varphi(t), \text{ a.e.,} \\
\mu(t) &= \nabla_u g(x_*(t), u_*(t))^T \lambda_\varphi(t), \text{ a.e.,} \\
p(t) &= E^T \lambda_\varphi(t).
\end{align*}
$$

(6.4)

If $N_C^U(u_*(t)) = \{0\}$, then the estimate for the multiplier $\lambda_\varphi(t)$ also holds:

$$
|\lambda_\varphi(t)| \leq k|p(t)| \text{ a.e.}
$$

for some positive constant $k > 0$. In the case of free end point, $\lambda_0$ can be taken as 1.

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**Appendix: Proof of Theorem 3.1**

Our proof is based on the following result.

For convenience, we first recall the following result from [26]. For any given $\varepsilon > 0$ and a given radius function $R(t)$, define

$$
S^{\varepsilon,R}_*(t) := \{(x, u) \in \bar{B}(x_*(t), \varepsilon) \times U : \Phi(x, u) \in \Omega, |\phi(x, u) - \dot{x}_*(t)| \leq R(t)\},
$$

$$
C^{\varepsilon,R}_* = cl\{(t, x, v) \in [t_0, t_1] \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} : v = \phi(x, u), (x, u) \in S^{\varepsilon,R}_*(t)\},
$$

where $cl$ denotes the closure.

**Proposition 6.3** [26, Theorem 4.2] Let $(x_*, u_*)$ be a $W^{1,1}$ local minimum of radius $R(\cdot)$ for $(P)$ in the sense that that $(x_*, u_*)$ minimizes $J(x, u)$ over all admissible pairs $(x, u)$ which satisfies both $|x(t) - x_*(t)| \leq \varepsilon, |\dot{x}(t) - \dot{x}_*(t)| \leq R(t)$ a.e. and $\int_{t_0}^{t_1} |\dot{x}(t) - \dot{x}_*(t)| dt \leq \varepsilon$.

Suppose that there exists $\delta > 0$ such that $R(t) \geq \delta$. Moreover suppose that $C^{\varepsilon,R}_*$ is compact and that for all $(t, x, u)$ with $(t, x, \phi(x, u)) \in C^{\varepsilon,R}_*$, the WBCQ holds:

$$
\begin{align*}
&\{(\alpha, 0) \in \partial \langle \lambda, \Phi \rangle(x, u) + \{0\} \times N_U(u) \} \\
&\lambda \in N^L_{\Omega}(\Phi(x, u)) \\
\implies & \alpha = 0
\end{align*}
$$

and the mapping $M$ defined as in (2.1) is calm at $(0, x, u)$. Then the transversality condition, the Euler adjoint inclusion in Theorem 3.1 hold and the Weierstrass condition of radius $R(\cdot)$ holds for almost every $t$:

$$
\Phi(x_*(t), u) \in \Omega, u \in U, |\phi(x_*(t), u) - \phi(x_*(t), u_*(t))| < R(t) \implies (p(t), \phi(x_*(t), u) - \phi(x_*(t), u_*(t))) - \lambda_0 F(x_*(t), u_*(t)).
$$

We now use Proposition 6.3 to prove Theorem 3.1.

Define $y_*(t) = \rho \int_{t_0}^{t} u_*(s) ds$ as well as a radius function $R_\rho(t) := \rho R(t)$ with $\rho > 1$. 

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We claim that \((x^*, y^*, u^*)\) is a \(W^{1,1}\) local minimum with radius \(R_\rho(\cdot)\) for the following problem:

\[
(P_\rho) \quad \min_{(x,u)} J(x,u) := \int_{t_0}^{t_1} F(x(t), u(t)) \, dt + f(x(t_0), x(t_1)),
\]

s.t.

\[
\dot{x}(t) = \phi(x(t), u(t)) \quad a.e. \, t \in [t_0, t_1],
\]
\[
\dot{y}(t) = \rho u(t) \quad a.e. \, t \in [t_0, t_1],
\]
\[
\Phi(x(t), u(t)) \in \Omega \quad a.e. \, t \in [t_0, t_1],
\]
\[
u(t) \in \mathcal{U} \quad a.e. \, t \in [t_0, t_1],
\]
\[
(x(t_0), x(t_1), y(t_0)) \in S \times \{0\}.
\]

Let \((x, y, u)\) be an admissible pair for problem \((P_\rho)\) satisfying

\[
|(\dot{x}(t), \dot{y}(t)) - (\dot{x}^*(t), \dot{y}^*(t))| \leq R_\rho(t) \text{ a.e.,}
\]
\[
|(x(t), y(t)) - (x^*(t), y^*(t))| \leq \varepsilon \text{ a.e.,}
\]
\[
\int_{t_0}^{t_1} |(\dot{x}(t), \dot{y}(t)) - (\dot{x}^*(t), \dot{y}^*(t))| \, dt \leq \varepsilon.
\]

Then it is obvious that \((x(t), u(t))\) is an admissible pair for \((P)\) with

\[
|u(t) - u^*(t)| \leq R(t), |x(t) - x^*(t)| \leq \varepsilon \text{ a.e.,}
\]
\[
\int_{t_0}^{t_1} |\dot{x}(t) - \dot{x}^*(t)| \, dt \leq \varepsilon.
\]

It follows by the fact that \((x^*, u^*)\) is a local minimum of radius \(R(\cdot)\) for \((P)\) that

\[
\int_{t_0}^{t_1} F(x^*(t), u^*(t)) \, dt + f(x(t_0), x(t_1)) \leq \int_{t_0}^{t_1} F(x(t), u(t)) \, dt + f(x(t_0), x(t_1)).
\]

Since \((6.7)\) holding for all admissible pair \((x, y, u)\) satisfying \((6.5)-(6.6)\), \((x^*, y^*, u^*)\) is a \(W^{1,1}\) local minimum of radius \(R_\rho(\cdot)\) for \((P_\rho)\).

Denote by

\[
\tilde{S}^{x,R_\rho}_x(t) := \left\{ (x, y, u) \in \tilde{B}((x^*(t), y^*(t)), \varepsilon) \times \mathcal{U} : \Phi(x, u) \in \Omega, |(\phi(x, u) - \dot{x}^*(t), \rho u - \rho u^*(t))| \leq \rho R(t) \right\},
\]
\[
C^{x,R_\rho}_x := cl\{ (t, x, y, \phi(x, u), \rho u) : (t, x, u) \in \tilde{C}^{x,R_\rho}_x(t) \}.
\]

It is obvious that the compactness of \(\tilde{C}^{x,R}_x\) implies the compactness of \(C^{x,R_\rho}_x\). It is also obvious that \((t, x, y, \phi(x, u), \rho u) \in C^{x,R_\rho}_x\) implies that \((t, x, u) \in \tilde{C}^{x,R}_x\). Moreover the mixed constraint \(\Phi(x, u) \in \Omega\) is independent of \(y\). Hence the WBCQ in Proposition 6.3 and the calmness condition hold. By Proposition 6.3, there exist an arc \((p, q)\) such that the nontriviality condition \((\lambda_0, p(t), q(t)) \neq 0, \forall t \in [t_0, t_1]\) holds, the transversality condition as in Theorem 3.1 holds, the Euler adjoint inclusion in the form

\[
(\dot{\psi}(t), q(t), 0) \in \partial C \{(−p(t), \phi) + \lambda_0 F\}(x^*(t), y^*(t), u^*(t)) + \{(0, 0)\} \times N^c_C(u^*(t))
\]
\[
+ \text{co}\{\partial \langle \lambda, \Phi \rangle(x^*(t), u^*(t)) : \lambda \in N_\Omega(\Phi(x^*(t), u^*(t))) \} \text{ a.e.}
\]

\[(6.8)\]
holds, and the Weierstrass condition of radius $R_{\rho}(\cdot)$ holds in the form that for almost every $t$:

\[
(x_*, t, u) \in M(0), \quad |\langle p(t), \phi(x_*, t, u) \rangle - \lambda_0 F(x_*, t, u) | \leq |\langle p(t), \phi(x_*, t, u^*) \rangle - \lambda_0 F(x_*, t, u^*) |.
\]

(6.9)

Because $\phi, F, \Phi$ are independent of $y$, it follows from (6.8) that $\dot{q}(t) \equiv 0$ a.e.. Together with $q(t_1) = 0$ implies that $q(t) \equiv 0$. Hence (6.8) implies the Euler adjoint inclusion (3.2) and the nontriviality condition as in Theorem 3.1.

Since $\tilde{C}^{e;R}$ is compact, the set

\[ C := cl \left\{ \bigcup_{t \in [t_0, t_1]} (x_*(t), u) \in M(0) : |u - u_*(t)| \leq R(t) \right\} \]

is compact as well. Since $\phi(x, u)$ is locally Lipschitz continuous and $C$ is compact, one can find a positive constant $k_{\phi}^u$ such that

\[ |\phi(x_*(t), u_1) - \phi(x_*(t), u_2)| \leq k_{\phi}^u |u_1 - u_2| \quad \forall (x_*(t), u_1), (x_*(t), u_2) \in C. \]

Let $(x_*(t), u) \in M(0), |u - u_*(t)| < R(t)$. Then $(x_*(t), u), (x_*(t), u_*(t)) \in C$ and hence

\[ |\langle p(t), \phi(x_*(t), u) \rangle - \lambda_0 F(x_*(t), u) | \leq |\langle p(t), \phi(x_*, t, u^*) \rangle - \lambda_0 F(x_*, t, u^*) | \leq \max \{ k_{\phi}^u, \rho \} |u - u_*(t)| \leq \max \{ k_{\phi}^u, \rho \} R(t). \]

Take a special $\rho > k_{\phi}^u$. Then $\max \{ k_{\phi}^u, \rho \} = \rho$ and hence (6.9) implies that the Weierstrass condition in Theorem 3.1 holds. Moreover as discussed in [26, Remark 3.1], $\lambda_0$ can be chosen as 1 in the case of free end point.

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