Disturbance Enhanced Uncertainty Relations

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Uncertainty and disturbance are two most fundamental properties of a quantum measurement and they are usually separately studied in terms of the preparation and the measurement uncertainty relations. Here we shall establish an intimate connection between them that goes beyond the mentioned two kinds of uncertainty relations. Our basic observation is that the disturbance of one measurement to a subsequent measurement, which can be quantified based on observed data, sets lower-bounds to uncertainty. This idea can be universally applied to various measures of uncertainty and disturbance, with the help of data processing inequality. The obtained relations, referred to as disturbance enhanced uncertainty relations, immediately find various applications in the field of quantum information. They ensure preparation uncertainty relation such as novel entropic uncertainty relations independent of the Maassen and Uffink relation. And they also result in a simple protocol to estimate coherence. We anticipate that this new twist on uncertainty principle may shed new light on quantum foundations and may also inspire further applications in the field of quantum information.

I. INTRODUCTION

Uncertainty principle, one of fundamental traits of quantum mechanics, is quantified by two kinds of uncertainty relations, namely, for preparation and for measurement. They are commonly investigated in terms of the uncertainties [1–6] and the error-disturbance [7–13], respectively. Till now, these two fundamental properties have gained a relative well understanding and have been harnessed for applications such as the security proof of quantum communication [5, 14–16], witnessing entanglement [17–18], bounding quantum correlations [19–21], and quantum metrology [22–26]. As two basic properties, the disturbance and uncertainty present themselves in a quantum measurement process simultaneously. Frequently, they are studied separately within one of the two types of uncertainty relations.

A physical state, which may be prepared in a coherent superposition of classically distinguishable ones, undergoes a sudden random collapse after a measurement. The post-measurement state and subsequent measurements are therefore disturbed. Intuitively, with suitable measures, the less uncertainty a measurement exhibits, the less it will disturb quantum state and the subsequent measurements. Therefore the measurement uncertainty should impose some constraints on its disturbance effect, and vice versa. For a given outcome of measurement, Winter’s gentle measurement lemma states that the post-measurement state is disturbed slightly if the associated outcome happens with a high probability (exhibiting small uncertainty) [27] and this special link has already led to some important applications such as channel coding theorem [28, 29]. By using a specific distance measure to quantify disturbance, a trade-off between uncertainty and disturbance has been established, which, though dealing with local properties, can be used to bound nonlocality [30]. These results suggest an intrinsic link between uncertainty and disturbance, which, if existing, shall bring a deeper insight on uncertainty principle and may find novel applications in the fields of quantum information sciences.

In this paper, we present such a link that is referred to as disturbance enhanced uncertainty relation, exploiting the data processing inequality and the fact that the disturbance caused by measurement to the state can be quantified by physical measures based on observed data. We establish a universal relation between uncertainty and
disturbance, valid for all disturbance measures that are induced by state distance measures with suitable properties (see Eq. (1) below). Specifically we also derive corresponding uncertainty-disturbance trade-offs for some widely used distance measures in various quantum informational scenarios. As applications, our relations, which impose constraints on the intrinsic "spreads" of incompatible quantities, can be seen as preparation uncertainty relations, giving rise to novel entropic uncertainty relations that are independent of the Maassen and Uffink’s [6]. Moreover, our disturbance enhanced uncertainty relations are suitable for coherence detection, providing operational upper and lower bounds for coherence measures based on distance. Therefore we anticipate this new perspective of uncertainty principle may promise far-reaching impact on quantum information science and quantum foundations.

II. UNCERTAINTY-DISTURBANCE RELATIONS

We consider a sequential measurement $\hat{A} \rightarrow \hat{B}$ of two incompatible observables $\hat{A} = \{\hat{A}_i\} = \Pi_A^i$ and $\hat{B} = \{\hat{B}_j\} = \Pi_B^j$, where $\{\hat{A}_i\}$ and $\{\hat{B}_j\}$ are two sets of orthonormal bases. The intrinsic distributions of $\hat{A}$ and $\hat{B}$ on quantum state $\rho$ are given by probability vectors $p = \{p_i = \text{tr}(\rho \Pi_A^i)\}$ and $q = \{q_j = \text{tr}(\rho \Pi_B^j)\}$. Measuring $\hat{A}$ on the state $\rho$ leaves a disturbed state $\rho_A = \Phi_A(\rho) = \sum_ip_i\Pi_A^i$, where $\Phi_A(\cdot) = \sum_i\Pi_A^i\Pi_A^i$ and similarly $\Phi_B(\cdot) = \sum_i\Pi_B^i\Pi_B^i$ are two complete-positive-trace-preserving (CPTP) operations. A subsequent measurement $\hat{B}$ on the state $\rho_A$ would yield a disturbed distribution $q' = \{q'_j\}$ with $q'_j = \text{tr}(\rho_A \Pi_B^j) = \sum_i c_{ij}p_i = [C[p]]_j$, where the overlaps of eigenstates $\{c_{ij} = |\langle A_i | B_j \rangle|^2\}$ form a unitary stochastic matrix [31, 32], denoted by $C$, which depends only on the choices of $A$ and $B$ and characterizes their incompatibility.

Here we shall quantify the uncertainty of $\hat{A}$ by some measure $\delta_A$, which is expected to respect Shur concavity as a function of probability distribution and may depend on which state distance that is used to quantify the distinguishability of states (as shown below). We denote by $D_{A \rightarrow B}$ the disturbance introduced by $\hat{A}$ to $\hat{B}$ which is quantified by the distance between $q$ and $q'$ and depends also on the state distance we choose. We note that the disturbance is estimated directly from experimental data, i.e., the undisturbed and disturbed statistics of $\hat{B}$.

Two key tools developed in the field of quantum information turn out to be very useful to explore the relation between $\delta_A$ and $D_{A \rightarrow B}$. One tool is the state distance, specified by $D(\cdot, \cdot)$, which quantifies how distinguishable two quantum states are. Widely used measures of state distance are trace distance [33], Rényi divergence [34, 35], Tsallis relative entropy [36], and infidelity. The state distance we choose should be non-increasing under at least CPTP operations $\Phi_B$. The other tool is the data processing inequality, i.e., two quantum states become less distinguishable after a general CPTP operation $D(\rho_1, \rho_2) \geq D[\Phi(\rho_1), \Phi(\rho_2)]$. The data processing inequality has been used to derive entropic uncertainty relations [37, 39]. Here, we shall see that it can lead to more fruitful results. In what follows we shall present two kinds of characterization of uncertainty and disturbance trade-off, a universal one and a distance measure dependent one.

For the purpose of a universal relation we shall make two reasonable assumptions about the state distance measure we choose besides its monotonicity under CPTP operations. First, we assume that the distance between two pure states, say $|\phi\rangle$ and $|\phi_A\rangle$, is an increasing function [56]

$$D(\phi, \phi_A) = G_D \left(\text{IF}(\phi, \phi_A)\right), \quad (1)$$

of their infidelity $\text{IF}(\phi, \phi_A)$, where $G_D(x)$ is an increasing single variable function for $x \geq 0$, depending on the distance measure we choose, and the infidelity for two general density matrices is $\text{IF}(\rho_1, \rho_2) = \sqrt{1 - F(\rho_1, \rho_2)^2}$ with $F(\rho_1, \rho_2) = \text{tr}(\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_2}})$ being the quantum fidelity. For two pure states, their infidelity becomes simply their non-overlap $\text{IF}(\phi, \phi_A) = \sqrt{1 - |\langle \phi | \phi_A \rangle|^2}$. This assumption is reasonable since the more distinguishable the states the larger the infidelity and therefore the larger the distance should be. For each distance measure satisfying above property Eq. (1), we can introduce a gauged distance measure

$$\tilde{D}(\rho_1, \rho_2) = G_D^{-1} \left(\tilde{D}(\rho_1, \rho_2)\right) \quad (2)$$

which also satisfies the data processing inequality as function $G_D$ is monotonous. We shall refer to such kinds of distance measures as gaugeable. As can be seen in Table I, almost all commonly used distance measures are gaugeable. Second, we assume the state distance is unitary invariant, i.e., $D(\rho_1, \rho_2) = D(U_1 \rho_1 U_1, U_2 \rho_2 U_1)$, which is satisfied by all distance measures considered below. As a result we can write $D(\Phi_B(\rho), \Phi_B(\rho_A)) = D(q, q')$ since these two density matrices are diagonal in the same basis. For the state distance measure assuming above two assumptions, we have an universal disturbance-uncertainty relation (see Methods).

Theorem 1 (Uncertainty-Disturbance relation). Performing sharp measurements in order $\hat{A} \rightarrow \hat{B}$ on a system in state $\rho$, resulting in data $p, q$ and $q'$, it holds

$$\delta_A(\rho) \geq \max_D \tilde{D}(q, q') := \tilde{D}_{A \rightarrow B}(\rho), \quad (3)$$

where the maximization is taken over all monotonous gaugeable distance measures.

In general, if we can bound a given specific distance measure $D(\rho, \rho_A)$ from above by a Schur concave function $\delta_A^D$ of $p$, the data processing inequality will lead
to a specific pairwise definitions of uncertainty $\delta_A^D$ and disturbance $D_{A\rightarrow B}$ via a distance-measure dependent uncertainty-disturbance relation

$$\delta_A^D(\rho) \geq D(\rho, \rho_A) = D(\phi_B(\rho), \phi_B(\rho_A)) = D_{A\rightarrow B}(\rho),$$

(4)

where $D(\rho, \rho_A)$ specifies disturbance in $\rho$, and $D_{A\rightarrow B}$ specifies disturbance in measurement. For a given distance measure, the specific uncertainty-disturbance relation Eq. (4) may be different from the universal one, i.e., Eq. (3).

In Table 1, we have summarized uncertainty-disturbance relations, universal Eq. (3) or distance-Eq(3). These distance measures with proofs present in Methods. These uncertainty-disturbance relations, universal Eq.(3) or distance measure, the specific uncertainty-disturbance relation can be obtained. Fourth, when considering non entropy, a uncertainty-disturbance relations include all $\delta_A^D$ and $D_{A\rightarrow B}$, and not a general CPTP operation [40], a uncertainty-disturbance relations, in which a lower bound on overlaps of eigenstates of $A$ and $B$. Provided $C$, the relations, which are actually formulated in terms of intrinsic distributions $p$ and $q$, can constrain their uncertainties and capture the spirit of preparation uncertainty.

As an example, we shall derive a novel entropic uncertainty relation from our uncertainty-disturbance relation. From uncertainty-disturbance relation $U_{tr}$, arising form Tsallis relative entropy, it follows the following novel entropic uncertainty relation

$$\frac{1}{2-\alpha} H_{2-\alpha}(q) + H_{\alpha}(p) \geq -\log c,$$

(5)

where $0 \leq \alpha < 1$. This is due to the fact that $q'_i = \sum_j c_{ij} p_j \leq \max_{ij} c_{ij} \equiv c$ so that

$$\frac{1}{2-\alpha} H_{2-\alpha}(p) \geq \frac{1}{\alpha-1} \log \sum_i q_i^\alpha c^{1-\alpha} = -\log c - H_{\alpha}(p).$$

A dual of Eq (5) can be readily obtained by swapping $p$ and $q$. In comparison, the well-known Maassen-Uffink (MU) entropic uncertainty relation reads

$$H_{\alpha}(p) + H_{\beta}(q) \geq -\log c,$$

(6)

where $\alpha, \beta \geq \frac{1}{2}$ satisfying $\frac{\alpha}{\alpha-1} + \frac{\beta}{\beta-1} = 2$ are referred to as conjugated indices. The MU relation have found numerous applications in many information tasks and have been generalized in various scenarios [5, 37, 39, 42, 43]. Generalizing entropic uncertainty relation to the cases of nonconjugated indices is a topic that has sustained interests [44].

We shall see that uncertainty relations Eq.(3) and Eq.(6) are independent. In fact, the factor $(2-\alpha)^{-1} \leq 1$ in Eq. (5) allows a possible strengthening over MU uncertainty Eq. (6). To see this we assume that $p$ is uniform so that $H_{\alpha}(p) = \log d$ for arbitrary $\alpha$ and the indices pair for Eq. (5) is $\{\beta, 2-\beta\}$, then and $\log d + \frac{1}{\beta} H_{\beta}(q) \leq \log d + H_{\beta}(q)$ when $1 \leq \beta$. Therefore, Eq. (5) indicates an alternative approach of generalization, namely, by introducing modifying fractions, which is made possible by considering disturbance.

In MU entropic uncertainty relations, the incompatibility is characterized by a single number $c$ instead of the whole set of $\{c_{ij}\}$ or the transition matrix $C$. This simplification leads to neat representations of uncertainty relations while sacrifices their tightness. In contrast, our uncertainty-disturbance trade-offs include all $\{c_{ij}\}$ and therefore are tighter, justifying the name disturbance enhanced uncertainty relations. For a visualized comparison, we present two case studies of the uncertainty-disturbance type uncertainty relations regarding to the tightness in the case of qubit ($d = 2$) and qudit ($d = 3$) as illustrated in Table 2.
For classical Rényi divergence. For universal trade-offs we list only the disturbance arising from gauged distance measures in

\[ U_D : \delta_A (\rho) \geq \sqrt{1 - \left( \sum_i q_i q'_i \right)^\alpha} \]

\[ \frac{1}{2} \left( \| p \|_2 - 1 \right) \geq |q - q'| \]

\[ S(\rho || \rho_A) : H(p) \geq H(q || q') \]

\[ \sqrt{\text{Tr}(\rho - \rho_A)^2} \]

Table I: A list of universal and distance-measure dependent uncertainty-disturbance relations arising from commonly used distance measures. For convenience we have denoted \( \alpha = 1 - \alpha \) with \( \frac{1}{2} \leq \alpha < 1 \) Rényi divergence and \( 0 \leq \alpha < 1 \) for Tsallis relative entropy. Moreover, \( |q - q'| = \frac{1}{2} \sum_i |q_i - q'_i| \), \( H_\alpha(p) = \frac{1}{\alpha} \log \| p \|_\alpha \) for Rényi entropy, and \( D_\alpha(q || q') = \frac{1}{\alpha} \log \sum_i q_i^\alpha q'_i^{1-\alpha} \) for classical Rényi divergence. For universal trade-offs we list only the disturbance arising from gauged distance measures in last column.

| Relations | Volume(d=2) | Volume(d=3) |
|-----------|-------------|-------------|
| \( U_{tr} \) | 0.930 | 0.94675 |
| \( U'_{tr} \) | 0.705 | 0.94682 |
| \( t^{rd}_{0.5} \) | 0.787 | 0.917 |
| \( U_{re} \) | 0.770 | 0.905 |
| \( U_{ts}^{0.5} \) | 0.814 | 0.937 |
| \( U_{hs} \) | 0.705 | 0.887 |
| Maassen-Uffink | 0.974 | 0.999 |

Table II: Computed volumes according to different uncertainty-disturbance relations and and their duals in pure states. The difference between the computed volumes via \( U_{tr} \), \( U'_{tr} \) for qutrit states are small but nonvanishing.

Figure 1: Geometrical visualization of the constraints of uncertainty relations imposed on the observed data. In qubit case, the distributions from measuring \( \hat{A} \) and \( \hat{B} \) can be specified respectively by \( p_0 \) and \( q_0 \), and their incompatibility is characterized by \( c_{00} \), which are constrained by various uncertainty relations. Plots (a-e) correspond to \( U_{tr}, U'_{tr}, U_{ts}^{0.5}, U_{ts}^{rd}, U_{re}, \) and the MU relation for measurements \( \hat{A} \rightarrow \hat{B} \) together with their dual inequalities for measurements \( \hat{B} \rightarrow \hat{A} \) enclose a region shown in Fig.1 and their volumes are computed in Table 2. Hence, the diagrams and the volumes present a direct comparison regarding the tightness of these uncertainty relations. The smaller the volume, the tighter the constraint. The volume of parameter space is normalized to 1 if without any other constraints. Under constraint of our disturbance enhanced uncertainty relations, the volumes are in the range \([0.705, 0.930]\), which are significantly smaller than the volume \((=0.974)\) for the MU relation \((\alpha = 1)\).

For the qutrit case, the hypersurfaces corresponding to the uncertainty-disturbance relations enclose some regions in the parameter space spanned by \( p, q, \) and \( c \). These conditions respectively yield the same computed volumes (see Table 2). The corresponding volumes are in the range \([0.887, 0.947]\), which are also significantly smaller than the volume \((=0.999)\) for the MU relation (see Table 2). Therefore, we have shown that the uncertainty-disturbance relations impose stronger constraints than the MU relation on the observed data, since in our cases we have used all elements of matrix \( C \) in the analysis.
IV. DETECTING COHERENCE WITH UNCERTAINTY-DISTURBANCE RELATION.

Coherence is a fundamental quantum feature that finds many applications ranging from quantum information science and quantum foundations \[45\] to quantum biology\[46, 47\]. In practice detecting coherence is a crucial but complicated task. As another application, our disturbance enhanced uncertainty relation can provide an efficient detection of coherence.

Coherence is characterized with respect to some computation basis, for example, \{\{A_i\}\} and a coherence measure quantifies how much coherence contained in a quantum state \[48\]. A widely used measure of coherence \[45, 49, 50\] is defined by the minimum distance \(C_D = \min \mathbb{D}(\rho, \sigma)\) to the set of all incoherent states \((\sigma)\), which are diagonal states in the given basis. Here \(D\) is some suitable state distance measure. Naturally, an operational upper-bound \(C_D \leq \mathbb{D}(\rho, \rho_A) \leq \delta^\alpha_D(\rho)\) in terms of the uncertainty follows from uncertainty disturbance trade-off Eq.\[1\] since \(\rho_A\) is an incoherent state. We note that both uncertainty and coherence are quantities defined with respect to computation basis, and the former relates to the diagonal terms of density matrix while the latter relates to the off-diagonal terms. The upper bound above provide a connection between them. For those distance measures whose nearest incoherent state is \(\rho_A\), our uncertainty disturbance trade-off Eq.\[1\] also provides an operational lower-bound \(D_{A\rightarrow B} \leq C_D\), in terms of disturbance.

As a case illustration, we take the relative entropy as distance measure and we have relative entropy measure of coherence \(C_r(\rho) = S(\rho||\rho_A)\), which quantifies the distillable coherence from a state \(\rho\) as well as extractable quantum randomness \[51, 52\]. In this case the nearest incoherent state to \(\rho\) is exactly \(\rho_A\). By the above analysis, one immediate has operational bounds as

\[
H(\rho) \geq C_r(\rho) \geq H(q||q'). \quad (7)
\]

One needs to independently measure \(\hat{A} \rightarrow \hat{B}\) and observable \(\hat{B}\), which yield distributions \(\rho, q\) and \(q' = C \cdot p\). When \(q' = q\), the observable \(\hat{B}\) yields a trivial estimation and these failing cases compose a set of measure zero. Therefore, it is almost impossible to yield a trivial estimation even only one additional measurement is employed. For comparison, in a previous approach based on the theory of majorization \[53, 54\], where a nontrivial estimation of the spectrum of \(\rho\) require that the distribution from a test measurement majorize the diagonal parts of the state, one may need several measurements so as to obtain such a distribution.

V. DISCUSSIONS

To summarize, we have established a quantitative link, a universal one (Theorem 1) and a distance-measure dependent one (for commonly used distance measures), between disturbance and uncertainty in sequential sharp measurements, uncovering a facet of uncertainty principle that has not been covered by previous approaches via preparation and the measurement uncertainty relations. Our uncertainty-disturbance trade-offs involve many basic concepts such as \(\alpha\)-Rényi entropy, Tsallis relative entropy, etc., which have nice properties and are frequently employed in the field of quantum information. Thus, the reported relations naturally find corresponding applications such as in deriving novel uncertainty relations and detecting coherence. This new twist on uncertainty principle promises potential further applications in quantum information science and may shed new light in quantum foundations. One next immediate task is to generalize the uncertainty disturbance relations to general measurements find possible applications in tasks where uncertainty relation plays a key role, e.g., the detection of coherence quantified in other measures and the certification of quantum randomness. Moreover our relations, as generalization of Winter’s gentle measurement lemma, may inspire further applications in channel coding theorem.

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SUPPLEMENTARY MATERIALS

Proof of Theorem 1.

Our main tool is the data processing inequality. Consider an arbitrary monotonous (under CPTP) and gaugeable distance (satisfying assumption Eq. (1)), we have

$$\delta_A(\rho) := \sqrt{1 - \|p\|_2^2} = \sqrt{1 - \text{tr}(\rho \rho_A)} \geq \text{IF}(\rho, \rho_A) = \text{IF}(\phi, \phi_A) = \tilde{D}(\phi, \phi_A) \geq \tilde{D}(\rho_B(\rho), \rho_B(\rho_A)) = \tilde{D}(q, q')$$

In the first line above we have denoted $$\|p\|_2 = (\sum_i p_i^2)^{\frac{1}{2}}$$ and $$\sqrt{1 - \|p\|_2^2}$$ is Shur concave and therefore a well-defined uncertainty measure. In the second line the inequality is due to the fact $$\text{F}(\rho_1, \rho_2)^2 \geq \text{tr}(\rho_1 \cdot \rho_2)$$ for two general density matrices $$\rho_1$$ and $$\rho_2$$ and the equality is because for two mixed states, say $$\rho$$ and $$\rho_A$$, there are different purifications and the optimal ones $$|\phi\rangle$$ and $$|\phi_A\rangle$$ give the quantum fidelity $$|\langle \phi | \phi_A \rangle| = \text{F}(|\rho|, |\rho_A|)$$. In the third line the equality is due to the definition of gauged distance measure and the inequality is due to data processing inequality applied to $$\phi, \phi_A$$ under partial trace. In the fourth line data processing inequality is employed again for $$\rho, \rho_A$$ under $$\Phi_B$$. In the last line that we have noted that $$\Phi_B(\rho)$$ and $$\Phi_B(\rho_A)$$ are diagonal states in the same basis and the distance is basis-independent due to the unitary invariance, therefore the distance is the function of diagonal terms, namely, statistics arising from measuring $$B$$ on $$\rho$$ and $$\rho_A$$.

Proof of uncertainty-disturbance $$U_{tr}$$ and $$U'_{tr}$$

As the first example we employ the trace distance $$D_{tr}(\rho, \rho_A) = \frac{1}{2} \text{tr} |\rho - \rho_A|$$ as the distance measure where $$\text{tr} |N| = \text{tr} \sqrt{NN^T}$$. For two pure states we have $$D_{tr}(\phi, \phi_A) = \text{IF}(\phi, \phi_A)$$ and therefore $$G_{tr}(x) = x$$, i.e., the gauged distance is identical to the original one. Similarly by using data processing inequality as above we have

$$\delta_A(\rho) := \sqrt{1 - \|p\|_2^2} \geq \frac{1}{2} \text{tr} |\rho - \rho_A| \geq \frac{1}{2} \sum_i |q_i - q'_i| := D_{tr}^A(\rho),$$

where $$D_{tr}^A(\rho) \equiv \frac{1}{2} \sum_i |q_i - q'_i|$$ is $$l_1$$ or the Kolmogorov distance commonly used disturbance measure. From an alternative upper bound of trace distance, i.e.,

$$\frac{1}{2} (\|p\|_2 - 1) = \sum_{i>j} \sqrt{p_i p_j} \geq \sum_{i>j} |\rho_{ij}| \geq \frac{1}{2} \text{tr} |\rho - \rho_A|,$$

where $$\{\rho_{ij}\}$$ are elements of density matrix $$\rho$$ we obtain $$U'_{tr}$$ which was first derived in a previous work [30].

Proof of uncertainty-disturbance relations $$U_{rel}^{\alpha_1, \alpha_2}, \tilde{U}^{\alpha_1, \alpha_2}, U_{if}, \tilde{U}_{if},$$ and $$U_{ro}$$

We present below the variants associated with the distance measures of $$\alpha$$–Rényi divergence and relative entropy. Employ Rényi divergence

$$D_{\alpha}(\rho|\rho_A) = \frac{1}{\alpha - 1} \log \text{tr}(\rho_A^{\frac{1-\alpha}{\alpha}} \rho^{\frac{1-\alpha}{\alpha}}),$$

with $$1/2 \leq \alpha < 1$$, as state distance measure. It is obviously invariant under unitaries and satisfying data processing inequality. For two pure states we have $$D_{\alpha}(\phi|\phi_A) = \frac{\alpha}{1 - \alpha} \log(1 - \text{IF}(\phi, \phi_A)^2)$$ and $$G_{D}(x) = \frac{\alpha}{1 - \alpha} \log(1 - x^2)$$ which is a reversible and monotonously increasing function for $$1/2 \leq \alpha < 1$$. The corresponding gauged distance measure reads

$$\tilde{D}_{\alpha}(\rho|\rho_A) = \sqrt{1 - \left(\text{tr}(\rho_A^{\frac{1-\alpha}{\alpha}} \rho^{\frac{1-\alpha}{\alpha}})^{\alpha}\right)^\frac{1}{2}}.$$
There are also other ways of bounding the Rényi divergence. We note that for a positive Hermitian operator $O$ it holds $\text{Tr} \ O^\alpha \geq (\text{Tr} \ O)^\alpha$ for $\alpha < 1$. As a result we have

$$D_\alpha(\rho||\rho_A) \leq \frac{\alpha}{\alpha - 1} \log \text{tr}(\rho^{1-\alpha}_A \rho \rho_A^{1-\alpha}) = \frac{\alpha}{\alpha - 1} \log \|p\|_\alpha,$$

which gives rise to trade-off

$$H_\alpha(p) \geq D_\alpha(q||q') := \frac{1}{\alpha - 1} \log \sum_i q_i^{\alpha} q_i^{1-\alpha},$$

with Rényi entropy defined as

$$H_\alpha(p) = \frac{\alpha}{1-\alpha} \log \|p\|_\alpha$$

for the uncertainty measure. We note that the uncertainty in the first variant Eq. (5) is equivalent to the collision entropy $H_2(p) = -\log(1 - \delta_2^{2}(\rho))$.

Specifically, the $\alpha$-Rényi divergence becomes relative entropy when $\alpha$ tends to 1, namely, $D_{\alpha \rightarrow 1}(\rho||\rho_A) = S(\rho||\rho_A) = \text{tr}(\rho \log \rho) - \text{tr}(\rho \log \rho_A)$. We note that in this case the distance measure does not satisfy condition Eq. (1), i.e., $G_D$ is not well-defined. However, we still have similar uncertainty-disturbance trade-off following from the data processing inequality

$$H(p) \geq S(\rho||\rho_A) \geq H(q||q'),$$

where the Shannon entropy $H(p) \equiv -\sum_i p_i \log p_i$ and classical relative entropy $H(q||q') \equiv \sum_i q_i \log \frac{q_i}{q_i'}$ quantify uncertainty and disturbance, respectively.

Take infidelity $IF(\rho, \rho_A)$ as the distance measure, we have $U_{it}$ as a special case of Eq. (10) for $\alpha = \frac{1}{2}$, i.e.,

$$H_2(p) \geq D_{\frac{1}{2}}(q||q'),$$

where $\delta_A(\rho) = H_2(p)$ and $D_{A \rightarrow B} = D_{\frac{1}{2}}(q||q')$. It turns out that it is equivalent to $U_{it}$ and is tighter than the first variant Eq. (5).

**Proof of $U_{it}^\alpha$ and $\tilde{U}_{it}^\alpha$**

Let us now employ Tsallis relative entropy as the state distance measure

$$T_\alpha(\rho, \rho_A) = \frac{1}{1-\alpha} [1 - \text{Tr}(\rho^\alpha \rho_A^{1-\alpha})],$$

with $0 \leq \alpha < 1$. The assumption Eq. (1) is satisfied as $T_\alpha(\phi, \phi_A) = \frac{1}{1-\alpha} \text{IF}(\phi, \phi_A)^2$ and we have a monotonous increasing function $G_D(x) = \frac{1}{1-\alpha} x^2$, which gives an identical universal uncertainty disturbance trade-off as Eq. (9) from Rényi divergence. As an alternative way of bounding the Tsallis distance we note first that

$$\text{tr}(\rho^\alpha \rho_A^{1-\alpha}) \geq \text{tr}(\rho \rho_A^{1-\alpha}) = \|p\|^{2-\alpha}_{2-\alpha}$$

for $\alpha < 1$ so that, by formulating in a similar way as uncertainty Eq. (10), we have

$$\frac{1}{2-\alpha} H_{2-\alpha}(p) \geq D_\alpha(q||q'),$$

where $\frac{1}{2-\alpha} H_{2-\alpha}(p)$ and $D_\alpha(q||q')$ quantify uncertainty and disturbance, respectively. We note that uncertainty relation above is stronger than Eq. (10) and also that from quantum Rényi divergence we can also obtain this stronger version of trade-off by considering Araki-Lieb-Thirring inequality, in the same manner.
\textbf{Proof of $U_{hs}$}

In fact our method can be also slightly generalized to cover more distance measures. As Eq.(4) only requires the monotonicity of distance measure under the dephasing operation $\Phi_B(\cdot)$, the uncertainty-disturbance trade-off applies also to the Hilbert-Schmidt distance $D_{HS}(\rho,\rho_A) \equiv \sqrt{\text{tr}(\rho - \rho_A)^2}$ which is monotonous under dephasing operation (but not a general under CPTP \([40]\)). It follows from the data processing inequality that

\[ \delta_A(\rho) = \sqrt{1 - \|p\|_2^2} \geq \sqrt{\text{tr}\rho^2 - \|p\|_2^2} \]

\[ = D_{HS}(\rho,\rho_A) \geq D_{HS}(\Phi_B(\rho),\Phi_B(\rho_A)) \]

\[ = \sqrt{\sum_i(q_i - q_i')^2} := D_{A \rightarrow B}^{HS}(\rho). \]  \hspace{1cm} (14)