A BACKWARD UNIQUENESS RESULT
FOR THE WAVE EQUATION
WITH ABSORBING BOUNDARY CONDITIONS

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Abstract. We consider the wave equation $u_{tt} = \Delta u$ on a bounded domain
$\Omega \subset \mathbb{R}^n$, $n > 1$, with smooth boundary of positive mean curvature. On the
boundary, we impose the absorbing boundary condition $\frac{\partial u}{\partial \nu} + u_t = 0$. We prove
uniqueness of solutions backward in time.

1. Introduction. Well-posed initial-value problems for partial differential equations need not satisfy backward uniqueness, for instance, for the problem
$$u_t + u_x = 0, \quad u(0, t) = 0, \quad u(x, 0) = u_0(x),$$
posed on the unit interval, all solutions become equal to zero in finite time. It is
therefore of interest to characterize problems for which backward uniqueness holds.
The property of backward uniqueness, aside from being of interest in its own right,
plays a role in a number of results on unique continuation and control. For instance,
approximate controllability follows from null controllability if backward uniqueness
holds for an adjoint problem. Also, for many systems, results on approximate
controllability can be deduced from backward uniqueness in conjunction with a
characteristics argument.

A number of methods exist for proving backward uniqueness. Backward uniqueness
is obvious for $C_0$-groups and analytic semigroups. Other approaches include
the logarithmic convexity method and Carleman estimates. In this work, we shall
use the method introduced in [4], which is based on the Phragmen-Lindelöf theorem.
This method was applied in [4] to thermoelastic plates. It has since been used for a
number of other systems which also involve a coupling of hyperbolic and parabolic
equations [1, 2, 3]. These examples are in a sense perturbations of a system (the
uncoupled hyperbolic and parabolic equations) for which backward uniqueness is
easy, and the task is to show that the perturbation imposed by coupling preserves
this property. In a recent paper by the author [5], it was shown that the method
can also be applied to certain problems where the “unperturbed” problem does
not have backward uniqueness. The examples discussed in [5] included a linearized
problem for one-dimensional compressible flow and the damped wave equation on
an interval with an absorbing boundary condition. Note that, with this boundary
condition, backward uniqueness does not hold for the undamped wave equation.
The damping term, although it is a lower order perturbation, is responsible for the
uniqueness property.

The abstract result for backward uniqueness, as proved in [5], is the following.

**Theorem 1.** Let $A$ be the infinitesimal generator of a $C_0$-semigroup on a Banach
space $X$. Suppose that there exists $\theta$ with $\pi/2 < \theta < 3\pi/2$ and $R$ such that for
$\lambda = r \exp(i\theta)$ and $r > R$ we have the resolvent estimate

$$
\|(A - \lambda I)^{-1}\| \leq M \exp(Cr^\rho),
$$

where $\rho < 1$. Then $\exp(At)$ is injective for every $t > 0$.

In this paper, we consider the wave equation in more than one space dimension.
In this case, it turns out that backward uniqueness actually holds without damping.
This is because “absorbing” boundary conditions are actually not exactly absorbing:
A radially propagating wave is not exactly of the form $u(r-t)$ and therefore does not
satisfy a boundary condition $u_r + u_t = 0$ on the boundary of a ball. The curvature
of the boundary will play the same role as damping plays in one dimension. Let $\Omega$
be a bounded domain in $\mathbb{R}^n$ with a smooth boundary of positive mean curvature.
We consider the following initial boundary value problem:

$$
\begin{align*}
  u_{tt} &= \Delta u, & x &\in \Omega, \\
  \frac{\partial u}{\partial \nu} + u_t &= 0, & x &\in \partial \Omega, \\
  u(\cdot,0) &= u_0 \in H^1(\Omega), & u_t(\cdot,0) &= u_1 \in L^2(\Omega).
\end{align*}
$$

(3)

Here $\partial u/\partial \nu$ denotes the derivative in the direction of the outer normal.

Our goal is the following result:

**Theorem 2.** Let $u$ be a solution of (3). If, for any time $T > 0$, $u(\cdot, T) = u_t(\cdot, T) = 0$, then $u = 0$.

In the usual fashion, we associate our problem with a $C_0$-semigroup on $H^1(\Omega) \times L^2(\Omega)$. Let $u_t = v$, and define

$$
A(u, v) = (v, \Delta u),
$$

$$
D(A) = \{(u, v) \in H^2(\Omega) \times H^1(\Omega) \mid \frac{\partial u}{\partial \nu} + v = 0 \text{ on } \partial \Omega\}.
$$

(4)

In order to apply Theorem 1, we need to consider the resolvent problem $(A - \lambda I)(u, v) = (f, g)$. We can combine this problem into the single equation and boundary condition

$$
\Delta u - \lambda^2 u = g + \lambda f, \quad \frac{\partial u}{\partial \nu} + \lambda u = -f \text{ on } \partial \Omega.
$$

(5)

In view of Theorem 1, our result follows from the following.

**Theorem 3.** Let $\theta$ be any angle in $(\pi/2, \pi)$. For $\lambda = r \exp(i\theta)$ and $r$ sufficiently
large, a unique solution of (5) exists and satisfies a bound of the form

$$
\|u\|_{H^1(\Omega)} + |\lambda|\|u\|_{L^2(\Omega)} \leq C(\|f\|_{H^1(\Omega)} + \|g\|_{L^2(\Omega)}).
$$

(6)
2. Iterative construction. We shall construct the solution of the resolvent problem \((5)\) in the form of an infinite series \(u = \sum_{n=0}^{\infty} \tilde{u}_n\). At each step of the iteration, we start from a problem

\[
\Delta u_n - \lambda^2 u_n = g_n + \lambda f_n, \quad \frac{\partial u_n}{\partial \nu} + \lambda u_n = -f_n \text{ on } \partial \Omega.
\]

We then find an approximate solution \(\tilde{u}_n\) for \(u_n\) and end up with a new problem

\[
\Delta u_{n+1} - \lambda^2 u_{n+1} = g_{n+1} + \lambda f_{n+1}, \quad \frac{\partial u_{n+1}}{\partial \nu} + \lambda u_{n+1} = -f_{n+1} \text{ on } \partial \Omega
\]

for the residual \(u_{n+1} = u_n - \tilde{u}_n\). The convergence of the iteration follows from estimates of the form

\[
\|\tilde{u}_n\|_{H^1(\Omega)} + |\lambda| \|\tilde{u}_n\|_{L^2(\Omega)} \leq C(\|f_n\|_{H^1(\Omega)} + \|g_n\|_{L^2(\Omega)}),
\]

and

\[
\|f_{n+1}\|_{H^1(\Omega)} + \|g_{n+1}\|_{L^2(\Omega)} \leq \gamma(\|f_n\|_{H^1(\Omega)} + \|g_n\|_{L^2(\Omega)}),
\]

where \(\gamma < 1\). At the beginning, we set \(f_0 = f\), \(g_0 = g\).

We now describe the construction of \(\tilde{u}_n\). For convenience, we suppress the index \(n\), i.e. \(f\) shall stand for \(f_n\) etc. Let \(G\) denote a smooth bounded domain containing \(\Omega\) in its interior. Let \(E\) denote an extension operator which is bounded from \(H^1(\Omega)\) to \(H^0_0(G)\) and from \(L^2(\Omega)\) to \(L^2(G)\). On \(G\), we can solve the Dirichlet problem

\[
\Delta v - \lambda^2 v = Eg + \lambda Ef, \quad v = 0 \text{ on } \partial G.
\]

If \(\lambda\) is on a ray in the complex plane other than the imaginary axis, it is easy to show that

\[
\|v\|_{H^2(G)} + |\lambda| \|v\|_{H^1(G)} + \|\lambda^2 v + \lambda Ef\|_{L^2(G)} \leq C(\|f\|_{H^1(\Omega)} + \|g\|_{L^2(\Omega)}).
\]

Next, we set \(u = v + w\), so that the problem for \(w\) becomes

\[
\Delta w - \lambda^2 w = 0, \quad \frac{\partial w}{\partial \nu} + \lambda w = b := -f - \frac{\partial v}{\partial \nu} - \lambda v \text{ on } \partial \Omega.
\]

From the estimate \((12)\), it follows that

\[
\|b\|_{H^{1/2}(\partial \Omega)} + |\lambda|^{1/2} \|b\|_{L^2(\partial \Omega)} \leq C(\|f\|_{H^1(\Omega)} + \|g\|_{L^2(\Omega)}).
\]

The idea in finding an approximation for \(w\) is that, for large \(|\lambda|\), \(w\) will be essentially confined to a small neighborhood of the boundary \(\partial \Omega\). We shall need a cut-off function \(\psi\) which is smooth, taking values between 0 and 1, having support in a sufficiently small neighborhood of \(\partial \Omega\), and equal to 1 in an even smaller neighborhood of \(\partial \Omega\). Below we shall construct an operator \(L\) which in a sense approximates the Laplacian in a neighborhood of \(\partial \Omega\). We approximate \(w\) by \(\tilde{w}\) where \(\tilde{w}\) satisfies

\[
L \tilde{w} - \lambda^2 \tilde{w} = 0,
\]

on the support of \(\psi\) and

\[
\frac{\partial \tilde{w}}{\partial \nu} + \lambda \tilde{w} = b,
\]

on \(\partial \Omega\). Finally, we set \(\tilde{u} = v + \psi \tilde{w}\). For the next iterate, we then have \(f_{n+1} = 0\), and

\[
g_{n+1} = -\Delta(\psi \tilde{w}) + \lambda^2 \psi \tilde{w} = \psi (L \tilde{w}) - \Delta(\psi \tilde{w}) = \psi \Delta \tilde{w} - \Delta(\psi \tilde{w}) + \psi (L \tilde{w} - \Delta \tilde{w}).
\]

To complete our argument, we need to show an estimate of the form

\[
\|g_{n+1}\|_{L^2(\Omega)} \leq \gamma \|b\|_{H^{1/2}(\partial \Omega)},
\]
where $\gamma \to 0$ as $|\lambda| \to \infty$. We note that, from (17), we find

$$
\|g_{n+1}\|_{L^2(\Omega)} \leq C(\|L\tilde{w} - \Delta\tilde{w}\|_{L^2(\text{supp } \psi)} + \|\tilde{w}\|_{H^1(\{0<\psi<1\})}).
$$

(19)

3. **Construction of a boundary layer approximation.** To construct the operator $L$, we make a change of coordinates near $\partial \Omega$. Let $\xi$ denote the distance of a point from $\partial \Omega$, and let $\eta$ be a surface coordinate identifying the nearest point on $\partial \Omega$. In these coordinates, the domain $\Omega$ corresponds to $\xi > 0$, and we have

$$
\Delta \tilde{w} = \frac{\partial^2 \tilde{w}}{\partial \xi^2} - \kappa(\eta) \frac{\partial \tilde{w}}{\partial \xi} + \Delta_S \tilde{w} + \xi Q \tilde{w},
$$

(20)

where $\kappa$ denotes the sum of principal curvatures of the boundary, $\Delta_S$ is the surface Laplacian on $\partial \Omega$, and $Q$ is a differential operator which is first order in $\xi$, and second order in $\eta$. We now set

$$
L = e^{\kappa \xi/2} \left( \frac{\partial^2}{\partial \xi^2} + \Delta_S \right) e^{-\kappa \xi/2}.
$$

(21)

We verify that

$$
L = \frac{\partial^2}{\partial \xi^2} - \kappa(\eta) \frac{\partial}{\partial \xi} + \frac{\kappa(\eta)^2}{4} + \xi R,
$$

(22)

where $R$ is a differential operator of first order in $\eta$. In summary, we have

$$
\Delta - L = \xi(Q - R) - \frac{\kappa^2}{4}.
$$

(23)

As a consequence, we have

$$
\|\Delta \tilde{w} - L\tilde{w}\|_{L^2} \leq C(\|\tilde{w}\|_{L^2} + \|\xi \tilde{w}\|_{H^1} + \|\xi \Delta_S \tilde{w}\|_{L^2}).
$$

(24)

We now consider the solution of

$$
L\tilde{w} - \lambda^2 \tilde{w} = 0,
$$

(25)

for $\xi > 0$, subject to the boundary condition

$$
- \frac{\partial \tilde{w}}{\partial \xi} + \lambda \tilde{w} = b(\eta)
$$

(26)

for $\psi = 0$. We substitute $\tilde{w} = \exp(\kappa(\eta)\xi/2)\hat{w}$. Consequently, we obtain the equation

$$
\frac{\partial^2 \hat{w}}{\partial \xi^2} + \Delta_S \hat{w} = \lambda^2 \hat{w},
$$

(27)

with the boundary condition

$$
- \frac{\partial \hat{w}}{\partial \xi} + \lambda \hat{w} - \frac{\kappa(\eta)}{2} \hat{w} = b(\eta).
$$

(28)

Let $\mu_n$ denote the eigenvalues of $-\Delta_S$, and let $\phi_n$ denote the corresponding eigenfunctions, normalized in $L^2(\partial \Omega)$. We can represent $\hat{w}$ as

$$
\hat{w} = \sum_{n=1}^{\infty} a_n \exp(-\sqrt{\lambda^2 + \mu_n \xi})\phi_n(\eta).
$$

(29)

Here the branch of the square root is the one taking values in the right half plane. The boundary condition becomes

$$
\sum_{n=1}^{\infty} (\sqrt{\lambda^2 + \mu_n} + \lambda)a_n \phi_n(\eta) - \frac{\kappa(\eta)}{2} \sum_{n=1}^{\infty} a_n \phi_n(\eta) = b(\eta) = \sum_{n=1}^{\infty} b_n \phi_n(\eta).
$$

(30)
Let \( \alpha_n = |\lambda| + \sqrt{\mu_n} \) and \( q_n = \sqrt{\lambda^2 + \mu_n} + \lambda \). Up to equivalence of norms, we have the following:

\[
\|b\|_{H^{1/2}(\partial \Omega)}^2 + |\lambda|\|b\|_{L^2(\Omega)}^2 = \sum_{n=1}^{\infty} |b_n|^2 \alpha_n,
\]

\[
\|\hat{w}\|_{L^2}^2 + |\xi\hat{w}|_{H^1}^2 = \sum_{n=1}^{\infty} |a_n|^2 \alpha_n,
\]

\[
|\xi\hat{w}|_{L^2}^2 + |\lambda^2|\|\hat{w}\|_{L^2}^2 = \sum_{n=1}^{\infty} \alpha_n |a_n|^2.
\]

\[
|\xi\Delta_S \hat{w}|_{L^2}^2 = \sum_{n=1}^{\infty} |a_n|^2 \mu_n^2 \alpha_n.
\] (31)

We fix integers \( m \) and \( M \) (depending on \( \lambda \)) such that \( m = \beta_1 |\lambda| \) \( M = \beta_2 |\lambda|^2 \), with \( \beta_1, \beta_2 \) chosen sufficiently small but independent of \( \lambda \) so that in particular \( \Re q_n < \frac{1}{4} \min_{\partial \Omega} \kappa \) for \( \mu_n < m \). We divide the natural numbers into three subsets:

\[
I_1 = \{ n \in \mathbb{N} \mid \mu_n < m \},
\]

\[
I_2 = \{ n \in \mathbb{N} \mid m \leq \mu_n \leq M \},
\]

\[
I_3 = \{ n \in \mathbb{N} \mid M < \mu_n \}.
\] (32)

Recall that \( |\lambda| \) is assumed large, and \( \pi/2 < \arg \lambda < \pi \). We note the following facts about \( q_n \):

1. On \( I_1 \), \( |q_n| \) is small of order \( m/|\lambda| \).
2. On \( I_2 \), the leading contribution to \( q_n \) is \( -\mu_n/(2\lambda) \). In particular, this implies that real and imaginary parts of \( q_n \) have the same order of magnitude.
3. On \( I_3 \), the real part of \( q_n \) is positive and dominates over the imaginary part.
4. The imaginary part of \( q_n \) is always positive.

We introduce the notation

\[
h(\eta) = \hat{w}(0, \eta) = \sum_{n=1}^{\infty} a_n \phi_n(\eta).
\] (33)

Due to the positivity of mean curvature, there are constants such that

\[
C_1 \int_{\partial \Omega} \kappa(\eta)|h(\eta)|^2 \, d\eta \leq \sum_{n=1}^{\infty} |a_n|^2 \leq C_2 \int_{\partial \Omega} \kappa(\eta)|h(\eta)|^2 \, d\eta.
\] (34)

Moreover, we note for the following that, for some constant,

\[
\left| \int_{\partial \Omega} (\Delta_S \overline{h(\eta)}) \kappa(\eta) h(\eta) \, d\eta \right| \leq C \|h\|_{L^2(\Omega)}^2 \leq C \sum_{n=1}^{\infty} \max(1, \mu_n) |a_n|^2,
\]

\[
|\Im \left( \int_{\partial \Omega} (\Delta_S \overline{h(\eta)}) \kappa(\eta) h(\eta) \, d\eta \right)| \leq C \|\nabla h\|_{L^2(\partial \Omega)} \|h\|_{L^2(\partial \Omega)}
\]

\[
\leq C \left( \sum_{n=1}^{\infty} \mu_n |a_n|^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} |a_n|^2 \right)^{1/2}.
\] (35)

Below, we also need to consider

\[
\int_{\partial \Omega} (\Delta_S \overline{h_3(\eta)}) \kappa(\eta) h(\eta) \, d\eta,
\] (36)
where we set
\[ h_k = \sum_{I_k} a_n \phi_n. \] (37)

As before, we have
\[ \left| \int_{\partial \Omega} \Delta_S h_3(\eta) \kappa(\eta) (h_2(\eta) + h_3(\eta)) \, d\eta \right| \leq C \left( \sum_{I_2 \cup I_3} \mu_n |a_n|^2 \right)^{1/2} \left( \sum_{I_3} \mu_n |a_n|^2 \right)^{1/2}. \] (38)

For the remaining term, however, we first integrate by parts:
\[ \int_{\partial \Omega} \Delta_S h_3(\eta) \kappa(\eta) h_1(\eta) \, d\eta = \left| \int_{\partial \Omega} \overline{h_3(\eta)} \Delta_S (\kappa(\eta) h_1(\eta)) \, d\eta \right| \]
\[ \leq C \| h_3 \|_{L^2(\partial \Omega)} \| h_1 \|_{L^2(\partial \Omega)} \leq C \left( \sum_{I_3} |a_n|^2 \right)^{1/2} \left( \sum_{I_3} \max(1, \mu_n^2) |a_n|^2 \right)^{1/2} \]
\[ \leq C \left( \sum_{I_3} \mu_n |a_n|^2 \right)^{1/2} \left( \sum_{I_3} |a_n|^2 \right)^{1/2} \frac{m}{\sqrt{M}}. \] (39)

Here we have used that \( \mu_n \leq m \) on \( I_1 \) and \( \mu_n \geq M \) on \( I_3 \).
We now multiply (30) by \( h(\eta) \), integrate over \( \partial \Omega \) and isolate the term \( -\int_{\partial \Omega} \kappa |h|^2 / 2 \, d\eta \) on the left side. The result is an estimate of the form
\[ \sum_{I_1} |\lambda| |a_n|^2 \leq C \left( \sum_{n=1}^{\infty} \alpha_n |b_n|^2 + \sum_{I_2} \mu_n |a_n|^2 + \sum_{I_3} |a_n|^2 |\lambda| \mu_n / \alpha_n \right). \] (40)

Next, we write the equation (30) in the form
\[ \sum_{n=1}^{\infty} q_n a_n \phi_n = \sum_{n=1}^{\infty} b_n \phi_n + \frac{\kappa}{2} \sum_{n=1}^{\infty} a_n \phi_n. \] (41)

We multiply by \( -\Delta_S h(\eta) \), integrate, and then take the imaginary part. Since the imaginary part of \( q_n \) is always positive, we find
\[ \sum_{I_2} \mu_n |a_n|^2 (\text{Im} \, q_n) \leq \text{Im} \left( \int_{\partial \Omega} \frac{\kappa}{2} (\Delta_S h) \, d\eta + \sum_{n=1}^{\infty} b_n \mu_n \overline{a_n} \right). \] (42)

This results in a bound of the form
\[ \sum_{I_2} \frac{\mu_n^2}{|\lambda|} |a_n|^2 \leq C \left( \frac{1}{K} \sum_{n=1}^{\infty} \mu_n |a_n|^2 + K \sum_{n=1}^{\infty} |a_n|^2 \right) \]
\[ + K \sum_{n=1}^{\infty} \alpha_n |b_n|^2 + \frac{1}{K} \sum_{I_1 \cup I_3} \mu_n^2 / \alpha_n |a_n|^2 \right). \] (43)

Here we are free to choose \( K \) as large as we wish. Finally, we multiply (41) by \( -\Delta_S h_3(\eta) \) and integrate, resulting in
\[ \sum_{I_3} q_n |a_n|^2 \mu_n = -\int_{\partial \Omega} \frac{\kappa}{2} (\Delta_S h_3) \, d\eta + \sum_{I_3} b_n \mu_n \overline{a_n}. \] (44)

Taking the real part, we find a bound of the form
\[ \sum_{I_3} \frac{\mu_n^2}{\alpha_n} |a_n|^2 \leq \sum_{n=1}^{\infty} \alpha_n |b_n|^2 + K \sum_{I_3} \mu_n |a_n|^2 + \frac{1}{K} \sum_{I_2} \mu_n |a_n|^2 + \frac{m}{\sqrt{MK}} \sum_{I_3} |a_n|^2. \] (45)
We now note the assumptions on the size of $\mu_n$ on the subsets $I_k$. If we choose $K$ suitably large, we can combine the inequalities (40), (43) and (45) to obtain
\begin{equation}
|\lambda| \sum_{I_1} |a_n|^2 + \sum_{I_2 \cup I_3} \frac{\mu_n^2}{\alpha_n} |a_n|^2 \leq C \sum_{n=1}^{\infty} \alpha_n |b_n|^2.
\end{equation}
(46)
The rest now follows from this bound.

Remarks.

1. The argument given above proves that the resolvent is bounded for large $|\lambda|$. Hence, in contrast to the application to linearized compressible flow in [5], the full force of the abstract result proved there is not needed; the theorem from [4] is sufficient. However, I note that the norm of the resolvent does not decay at a rate of $|\lambda|^{-1}$.

2. The argument above works just as well if a component of the boundary has uniformly negative rather than positive mean curvature. Thus the result can easily be extended to domains with holes or exterior domains.

3. The argument can easily be modified to cover a damped wave equation $u_{tt} + \gamma u_t = \Delta u$. In that case the role of $\kappa$ in the analysis above is taken over by $\kappa + \gamma$. Thus the theorem holds under the assumption that $\kappa + \gamma$ is either uniformly positive or uniformly negative on each component of the boundary.

4. In [5], Theorem 1 is also applied to linearized one-dimensional compressible flow. In a way, this problem is more interesting than the wave equation, since the resolvent along a ray in the left half plane is actually not bounded, but grows like $\exp(C|\lambda|^{1/2})$. Unfortunately, this result does not seem to extend to higher dimensions. A case amenable to analysis is that of a parallel strip, where separation of variables can be done. In this case, backward uniqueness holds, since each transverse wave number can be analyzed separately, and that analysis is analogous to the one-dimensional case. However, I have done a formal analysis of the eigenvalues which shows that we cannot expect any sector in the left half plane to be free from eigenvalues. Thus a direct application of Theorem 1 to the two-dimensional problem is not possible.

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