A Fast accurate heuristic algorithm for the consensus ranking problem.

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Abstract. Preference rankings virtually appear in all field of science (political sciences, behavioral sciences, machine learning, decision making and so on). The well-know social choice problem consists in trying to find a reasonable procedure to use the aggregate preferences expressed by subjects (usually called judges) to reach a collective decision. This problem turns out to be equivalent to the problem of estimating the consensus (central) ranking from data that is known to be a NP-hard Problem. Emond and Mason in 2002 proposed a branch and bound algorithm to calculate the consensus ranking given \( n \) rankings expressed on \( m \) objects. Depending on the complexity of the problem, there can be multiple solutions and then the consensus ranking may be not unique. We propose a new algorithm to find the consensus ranking that is equivalent to Emond and Mason’s algorithm in terms of at least one of the solutions reached, but permits a really remarkable saving in computational time.

Keywords: Preference rankings, Consensus ranking, Kemeny distance, Social choice problem, Branch and bound algorithm
1 Introduction

The consensus ranking problem, also known as social choice problem, arises any time \( n \) subjects (or judges) are asked to express their preferences on a set of \( m \) objects. These objects are placed in order by each subject (where 1 represents the best and \( m \) the worst) without any attempt to describe how much one differs from the others or whether any of the alternatives is good or acceptable. Every independent observation is a permutation of \( m \) distinct positive integer numbers. To be more specific, when the subject assigns the integer values from 1 to \( m \) to all the \( m \) items we have a **complete** (or full) ranking. Whenever instead the judge fails to distinguish between two or more items and assigns to them the same integer number (expressing indifference to the relative order of this set of items), we deal with **tied** (or weak) rankings. Moreover we have a **partial** ranking when judges are asked to rank a subset of the entire set of objects (e.g. pick the three most favourite items out of a set of five). Rankings are by nature peculiar data in the sense that the sample space of \( m \) objects can be only visualized in a \((m-1)\)-dimensional hyperplane by a discrete structure that is called the **permutation polytope**, \( S_m \). A polytope is a convex hull of a finite set of points in \( \mathbb{R}^m \) (Thompson, 1993; Heiser, 2004). For example the space considering 4 objects with all possible ties is a truncated octahedron as visualized in Figure 1 (Heiser and D’Ambrosio, 2013). As we already pointed out, the permutation polytope is inscribed in a \((m-1)\)-dimensional subspace, hence, for \( m > 4 \), such structures are impossible to visualize.

The permutation polytope is the natural space for ranking data. To define it no data are required, it is completely determined by the number of items involved in the preference choice; data add only information on which rankings occur and with what frequency they occur. This space is discrete and finite. It is characterized by symmetries and it is endowed with a graphical metric.

The problem of combining rankings to obtain a ranking representative of the group has been studied by numerous researchers in several areas, e.g. voting systems, economics, machine learning, psychology, political sciences, for more than two century. In the framework of distance-based models for rankings, searching for consensus ranking is a very important step in modeling the ranking process (Marden, 1995). These models are usually exponential family models (Diaconis, 1988) and they are completely specified by two parameters, a dispersion parameter and a consensus (central) ranking. Max-
Imum likelihood estimates of the dispersion parameter assume the knowledge of the central ranking. When the consensus ranking is not known it should be estimated. Unfortunately, even if there are close formulas for this estimation they are not feasible because of the complexity of the problem (Critchlow, 1980; Fligner and Verducci, 1986, 1988; Diaconis, 1988; Critchlow, Fligner and Verducci, 1991). Several methods to aggregate individual preference rankings have been proposed since the works of Borda (1781); Condorcet (1785); Black (1958); Arrow (1951); Goodman and Markowitz (1952); Coombs (1964); Davis et al. (1972); Bogart (1973); Cook and Seiford (1978), Barthélemy and Monjardet (1981), Emond and Mason (2002); Meilă et al. (2007).

In this paper we propose two accurate heuristic algorithms to derive a consensus ranking from the aggregation of individuals preferences within the Kemeny and Snell axiomatic framework that is equivalent to the one proposed by Emond and Mason (2002). The algorithm is empirically tested and
its results are reported to indicate the algorithm efficiency.
The rest of the paper is organized as follow. In Section 2 we briefly present
some of the proposed approaches to aggregate preference rankings and derive
a consensus. In Section 3 we describe the branch and bound algorithm by
Emond and Mason. Section 4 is devoted to describe the proposed algorithms,
then in sections 5 and 6 we present a simulation study and application on
real data to evaluate both the accuracy and the efficiency of our proposal.
Concluding remarks are then found in section 7.

2 Finding the consensus ranking, some approaches.

How to aggregate subjects preferences to create a consensus is a problem
that goes back to 1770 when Borda formulated the method of marks (also
known as Borda’s count) for determining the winner in elections with more
than 2 candidates. This method is quite simple and it is based on calculating
the total rank for each alternative. For example, if we consider the rankings
in Table 1

| Alternatives | # voters | A | B | C |
|--------------|----------|---|---|---|
|              | 12       | 2 | 1 | 3 |
|              | 5        | 1 | 2 | 3 |
|              | 7        | 3 | 2 | 1 |

the total rank for each alternative is given by:

- \( A = 12 \times 2 + 5 \times 1 + 7 \times 3 = 50, \)
- \( B = 12 \times 1 + 5 \times 2 + 7 \times 2 = 36, \)
- \( C = 12 \times 3 + 5 \times 3 + 7 \times 1 = 58, \)

resulting in the consensus (213). Borda’s method of marks was criticized
by Condorcet, which proposed to use the majority rule on all the pairwise
comparisons between alternatives. Condorcet’s solution for the rankings reported in Table 1 can be obtained by calculating the support obtained by every pairwise comparison between options, reported in Table 2.

|   | A  | B  | C  |
|---|---|---|---|
| A | -5| 17|   |
| B | 19| - | 17|
| C | 7 | 7 | - |

From Table 2 we can deduce that $B \succ A$, $B \succ C$ and $A \succ C$, resulting also in the consensus ranking (213). In applying this method, unfortunately, one problem can be encountered, i.e. if intransitive preferences occur the simple majority procedure breaks down (paradox of voting (Arrow, 1951), according to which a set of transitive preferences can generate a global intransitive preference as group preference).

In the last century the rank aggregation problem has been approached from a statistical perspective. Kendall (1938) was the first to propose a method to aggregate input rankings to find a consensus. He studied the consensus problem as a problem of estimation and he proposed to rank items according to the mean of the ranks assigned, thus proposing a method perfectly equivalent to Borda’s one. Moreover he suggested to consider the Spearman rank correlation coefficient $\rho$, defined as:

$$\rho = 1 - \frac{6 \sum_{i=1}^{n} (d_{sp})^2}{n^2(n-1)},$$  \hspace{1cm} (1)

where $d_{sp}(R, R^*) = \sum_{j=1}^{m} (R_j - R_j^*)^2$. The Spearman’s $\rho$ is equivalent to the product moment correlation coefficient and it treats rankings as they are scores summing the square of ranked differences.

Kendall (1938) proposed his own correlation coefficient, named after him as Kendall $\tau$, by introducing the concept of ranking matrices. The ranking matrix associated with the ranking $R_i$ of $m$ objects, is a $m \times m$ matrix $\{a_{ij}\}$ whose elements are defined as

$$a_{ij} = \begin{cases} 
1 & \text{if object } i \text{ is ranked ahead of object } j \\
-1 & \text{if object } i \text{ is ranked behind object } j \\
0 & \text{if the objects are tied, or if } i = j 
\end{cases}$$  \hspace{1cm} (2)
The Kendall correlation coefficient $\tau$ between two rankings, $R$ with score matrix $\{a_{ij}\}$ and $R^*$ with score matrix $\{b_{ij}\}$, can be then defined as the generalized correlation coefficient:

$$
\tau(R, R^*) = \frac{\sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij}b_{ij}}{\sqrt{\sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij}^2 \sum_{i=1}^{m} \sum_{j=1}^{m} b_{ij}^2}} .
$$

In the same period Kemeny (1959) and Kemeny and Snell (1962) proposed and proved an axiomatic approach to find a unique distance measure for rankings and define a consensus ranking. They introduced four axioms, reported in Table 3, that should apply to any distance measure between two rankings.

Table 3: Kemeny and Snell axioms

1. Axiom 1.1: Positivity.
   
   $d(R_1, R_2) \geq 0$, with equality if and only if $R_1 \equiv R_2$.

2. Axiom 1.2: Symmetry
   
   $d(R_1, R_2) = d(R_2, R_1)$.

3. Axiom 1.3: Triangular inequality
   
   $d(R_1, R_3) \leq d(R_1, R_2) + d(R_2, R_3)$ for any three rankings $R_1$, $R_2$, $R_3$, with equality holding if and only if ranking $R_2$ is between $R_1$ and $R_3$.

4. Axiom 2: Invariance
   
   $d(R_1, R_2) = d(R'_1, R'_2)$, where $R'_1$ and $R'_2$ result from $R_1$ and $R_2$ respectively by the same permutation of the alternatives.

5. Axiom 3: Consistency in measurement
   
   If two rankings $R_1$ and $R_2$ agree except for a set $S$ of $k$ elements, which is a segment of both, then $d(R_1, R_2)$ may be computed as if these $k$ objects were the only objects being ranked.

6. Axiom 4: Scaling
   
   The minimum positive distance is 1.

They also proved the existence of a distance metric that satisfies all these axioms, known as Kemeny distance, and its uniqueness. By using the score
matrices as defined by Kendall, Kemeny’s distance between two rankings $R$ (with score matrix $\{a_{ij}\}$) and $R^*$ (with score matrix $\{b_{ij}\}$) is defined as:

$$d_{Kem}(R, R^*) = \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} |a_{ij} - b_{ij}| . \quad (4)$$

Kemeny and Snell then suggested the idea to use this distance function to construct the consensus ranking. According to their definition the consensus ranking is the point in the ranking space that shows the best agreement with the set of input rankings. So, given a set of $n$ independent input rankings $\{R_i\}_{i=1}^{n}$, the consensus ranking, $\hat{S}$, is the ranking that represents them in the best way in the sense that it maximizes the agreement between itself and all the rankings that belong to that set. Kemeny and Snell suggested two reasonable criteria to select this point: the median ranking and the mean ranking defined as

- median ranking - the point (or the points) for which $\sum_{i=1}^{n} d(R_i, S)$ is a minimum
- mean ranking - the point (or the points) for which $\sum_{i=1}^{n} d(R_i, S)^2$ is a minimum.

They left the problem of which criterion to choose unsolved. For an extensive discussion on the reasons why to choose median ranking over mean ranking we refer to Young and Levenglick (1978) and Monjardet (2008). Bogart (1973, 1975) generalized the Kemeny and Snell approach by considering both transitive and intransitive preferences. Following the Kemeny and Snell approach the research of the median (consensus) ranking requires searching the space of all possible rankings of $m$ object. Given a set of $n$ independent input rankings the problem consists in finding the ranking $\hat{S}$ that best represents the combined preferences of the judges. This is a NP-hard problem, i. e. it is not possible to find the consensus ranking in polynomial time even when $m = 4$ (Barthelemy et al., 1989).

Cook and Saipe (1976) proposed a branch and bound algorithm to determine the median ranking out of a set of $n$ independent preference rankings with the restriction on full rankings, in which they used a $L1$-norm distance computationally similar to Spearman distance. Cook et al. (2007) presented a branch and bound algorithm for finding the consensus ranking in presence of partial rankings, but not allowing for ties.
Emond and Mason (2002) proposed a new rank correlation coefficient called \( \tau_x \) that is equivalent to the Kemeny and Snell distance metric. They defined the score matrices in a slightly different way respect to the Kendall’s representation: \( a_{ij} = 1 \) if object \( i \) is either ranked ahead or tied with object \( j \), and \( a_{ij} = 0 \) only if \( i = j \). Using the score matrices modified in this way, they defined the rank correlation coefficient as:

\[
\tau_x = \frac{\sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} b_{ij}}{m(m - 1)}
\]

(5)

Note that \( \tau_x \) is perfectly equivalent to Kendall’s \( \tau \) when ties are not allowed.

By using this correlation coefficient they proposed a branch and bound algorithm to deal with the consensus ranking problem when the number of object \( m \) is at most equal to 20 in a reasonable computing time. Given \( n \) weak orderings of \( m \) objects, \( R_1, \ldots, R_n \), where each ordering carries a positive weight \( w_k \), consensus ranking \( \hat{S} \) is the one (or the ones) that maximizes the weighted average correlation with the \( n \) input rankings or, equivalently, is the one (or the ones) that minimizes the weighted average Kemeny distance to the \( n \) input rankings,

\[
\max \sum_{k=1}^{m} \frac{w_k \tau_x(S, R^{(k)})}{\sum_{k=1}^{w_k}}
\]

(6)

Indicating as \( \{s_{ij}\} \) and \( \{r_{ij}\}^{(k)} \) are the scoring matrices for \( S \) and the \( k \)th ordering \( R, k = 1 \ldots, n \), the problem is:

\[
\max \sum_{k=1}^{m} w_k \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} s_{ij} r_{ij}^{(k)} \right\} = \max \sum_{i=1}^{n} \sum_{j=1}^{n} s_{ij} c_{ij},
\]

(7)

where \( c_{ij} = \sum_{k=1}^{n} w_k r_{ij}^{(k)} \).

The score matrix \( \{c_{ij}\} \) was called by Emond and Mason Combined Input Matrix (CI) because it is the result of a summation of each input ranking. Defined in this way, it summarizes the rankings information in a single matrix.

Emond and Mason conceived a branch-and-bound algorithm to maximize equation [7] by defining an upper limit on the value of that dot product. This limit, considering that the score matrix consists only of the values 1, 0 and \(-1\), is given by the sum of absolute values of its elements:

\[
V = \sum_{i=1}^{n} \sum_{j=1}^{n} |c_{ij}|.
\]
3 Emond and Mason’s branch and bound algorithm

If a weak ordering of \( m \) objects is given as initial solution, it is possible to compute the associated score matrix \( \{s_{ij}\} \) and evaluate the value of expression 7. Then it is possible to define an initial penalty \( P \) by subtracting this value from \( V \). The problem is to search the set of all weak orderings of \( m \) objects to find those with the minimum penalty. This set can be divided into three mutually exclusive branches based on the relative position of the first two objects in the ordering represented in the initial solution, labeled as \( i \) and \( j \). An incremental penalty for each of the branches can be calculated, by considering the corresponding elements \( c_{ij} \) and \( c_{ji} \) of the CI matrix, as specified in Table 4.

Table 4: Penalty computation in the branch and bound algorithm

| Branch | Conditions | Incremental Penalty |
|--------|------------|---------------------|
| 1      | \( c_{ij} > 0 \) and \( c_{ji} < 0 \) | \( \delta P = 0 \) |
|        | \( c_{ij} > 0 \) and \( c_{ji} > 0 \) | \( \delta P = c_{ji} \) |
|        | \( c_{ij} < 0 \) and \( c_{ji} > 0 \) | \( \delta P = c_{ji} - c_{ij} \) |
| 2      | \( c_{ij} > 0 \) and \( c_{ji} < 0 \) | \( \delta P = -c_{ij} \) |
|        | \( c_{ij} > 0 \) and \( c_{ji} > 0 \) | \( \delta P = 0 \) |
|        | \( c_{ij} < 0 \) and \( c_{ji} > 0 \) | \( \delta P = c_{ij} \) |
| 3      | \( c_{ij} > 0 \) and \( c_{ji} < 0 \) | \( \delta P = c_{ij} - c_{ji} \) |
|        | \( c_{ij} > 0 \) and \( c_{ji} > 0 \) | \( \delta P = c_{ij} \) |
|        | \( c_{ij} < 0 \) and \( c_{ji} > 0 \) | \( \delta P = 0 \) |

If the incremental penalty for a branch is greater than the initial penalty, then we do not consider it any longer because all orderings in that branch will have a penalty larger than the initial one. If the incremental penalty of a branch is smaller (or equal) than the initial one, then...
penalty, we then consider the next object in the initial solution and create new branches by placing this object in all possible positions relative to the objects already considered.

The algorithm continues in an iterative way by including all other objects until all the branches to be considered are checked. This branch and bound algorithm works with complete, incomplete and partial rankings. It deals with incomplete rankings thanks to the convention that unranked objects do not add anything in forming the combined input matrix. Emond and Mason stated that the computation time needed to reach a solution(s) depends both on the inherent degree of consensus in the sample of judges and on the quality of the initial solution used to initialize the algorithm. For an extensive discussion on the branch and bound algorithm we refer to Emond and Mason (2000, 2002).

4 Two accurate heuristic algorithms

The first element to be evaluated in developing our algorithm is the combined input matrix. This matrix contains all the information about the rankings expressed by all the subjects and, if it is a valid score matrix, then the consensus ranking can be immediately found. Unfortunately such a situation rarely happens. But by evaluating the CI with more detailed attention it is possible to identify a good candidate to be the consensus ranking that can be used as an input in the algorithm. Let $S = 1$ be a vector of ones of size $m$. Let $\{c_{ij}\}$ be the $m \times m$ combined input matrix. As $\{c_{ij}\}$ is not necessarily symmetric each pair of objects $i$ and $j$ is evaluated $m(m-1)/2$ times. A moderately accurate first candidate to be the consensus ranking can be computed as follow:

If $\text{sign } c_{ij} = 1 \& \text{sign } c_{ji} = -1$, then $S_i = S_i + 1$;
If $\text{sign } c_{ij} = -1 \& \text{sign } c_{ji} = 1$, then $S_j = S_j + 1$;
If $\text{sign } c_{ij} = 1 \& \text{sign } c_{ji} = 1$, then $S_i = S_i + 1$, $S_j = S_j + 1$.

In this way, we obtain the updated rank vector $S$ containing the number of times each object is preferred to the others in the pairwise comparisons. This vector is the starting point of our algorithm. The first step is to compute $\{s_{ij}\}$, namely the score matrix associated with $S$. Then we compute the associated penalty as:

$$P = V - \sum_{ij} c_{ij} s_{ij}$$  (8)
After this step we take into account the object in S ranked at the second position, and we evaluate equation 8 by placing that object in all possible positions relative to the object ranked ahead, including ties. Once the penalties are computed, we update the candidate consensus by selecting the ranking that is associated with the minimum penalty. Subsequently we add the object ranked in the third position in the initial S vector, and again we compute the values of equation 8 by placing that object in all possible positions relative to the objects already ranked ahead, including all possible ties. As before, we update the candidate consensus ranking by selecting the one that minimizes the penalty. We continue in this way until all the objects are processed and we reach a possible solution.

We use, then, the obtained solution as starting point for a new complete loop. The overall procedure is repeated again by considering also the reverse ranking of the initial S vector as candidate consensus ranking. The complete algorithm is summarized in Box 1.

Box 1 Quick algorithm for the consensus ranking problem

- input $\{c_{ij}\}, S$
- initialize: fix the rank of the first ranked object in S

(1.) consider the next ranked object in S

(2.) evaluate eq. 8 for all the rankings obtained by placing that object in all possible positions wrt the fixed ranked objects

(3.) store only the ranking associated with minimum value of eq. 8

(4.) fix the rank of the processed object and return to step (1.) until all objects in S are processed

Obtain the update ranking $CR$, and repeat all previous steps by replacing S with $CR$

output: $CR =$ consensus ranking.

Note that when we evaluate the penalty, we consider all the objects in the ranking that is considered. This is a fundamental difference with the original algorithm, because Emond and Mason calculate the penalty values by only considering the elements of the combined input matrix associated with the processed objects, and updating the penalty by adding up these partial values as depicted in Table 4. Indeed, we never use this system of penalty update.
We call this algorithm “quick” because it is able to reach at least one solution, or a solution really close to the true one, in few seconds even when working with a huge number of objects. In our experience, by using our definition of starting point $S$, at least one solution is found. But, sometimes, solutions were also reached with random starting points. For this reason, we decided to use the quick algorithm as building block of our accurate FAST heuristic algorithm for the consensus ranking problem, whose pseudo-code is shown in Box 2. Of course, our FAST algorithm is useful when the complexity of the problem is really intractable, e.g. when the number of objects to be ranked is higher than 15, up to a reasonable upper limit.

**Box 2 Accurate FAST heuristic algorithm**

```plaintext
input \{c_{ij}\}
for iter=1:maxiter do
  if iter=1 then
    CR=quick(S,\{c_{ij}\}), with $S$ as defined before
    store CR
  else
    S=random permutation of $m$ objects
    CR=quick(S,\{c_{ij}\})
    store CR
output: CR=CR,$\tau_x=\max$
```

Among the solutions returned by the quick algorithm, the consensus rankings are those showing the highest value of the average $\tau_x$ rank correlation coefficient.

## 5 Simulation study

We implemented the branch and bound algorithm by Emond and Mason, as well as both the quick and fast algorithms in MatLab and in R environments. The reported results are based on codes written in MatLab language. A beta version of the R `ConsRank` package is available upon request to the authors, as well as the MatLab codes. Analysis were made by using a Computer Intel Core i5-3317U 1.70 GHz and 4GB of RAM.

To evaluate the performance of our algorithms in terms of accuracy and
efficiency, we performed a simulation study. Firstly, ranking data were simulated according to a distance-based model by selecting three different levels of the dispersion parameter $\theta$, which governs the degree of consensus in the sample of rankings. In the distance-based models framework, for a given consensus, $S$, a distance function, $d$, and some real parameter $\theta$, the density with respect to the Uniform distribution is

$$f_0(a; S) = C(\theta) \exp(-\theta d(S, a)),$$

where $a$ is a ranking and $C(\theta)$ is a normalizing constant. For more details on distance-based models we refer to Marden (1995), Feigin and Cohen (1978) and Critchlow et al. (1991).

The three chosen levels of $\theta$ were 0.7, 0.4 and 0.1, the distance used was the Kemeny distance. We decided to consider 4 different levels for $m$: 4, 9, 15 and 20. In the case of 4 and 9 objects, we repeated the experiment both considering only complete rankings and the full space of complete and tied rankings, while in the case of 15 and 20 objects we decided to limit the experiment only to complete rankings sampled from a limited sub-population of size 10 millions. These sub-populations were generated from the full rankings space of 10 objects by adding the remaining objects in such a way that they were at first ranked below, later ranked ahead, and then randomly ranked in a middle position. Sample size was always equal to 200. Another experiment involved incomplete rankings. We chose a scheme of the type “pick k out of m”, and precisely: pick 2 out of 4, pick 5 out of 9 and pick 10 out of 15. Rankings were sampled in this way: first we extracted a random number of rankings (from a minimum of 15 to a maximum of 30) according to the uniform distribution by setting $\theta = 0$ from the corresponding spaces, then we generated the weights from a normal distribution with means randomly generated between 10 and 30 and standard deviations randomly generated between 2.5 and 9. After we normalized the weights and multiplied them by the total sample size to have data sets approximatively of size 200. Each experiment was repeated ten times, for globally 240 data sets. Table 5 summarizes the experimental design.

For each data set we ran the branch and bound, the quick and the FAST heuristic algorithms. We checked the consensus rankings found by the three algorithms as well as the elapsed time in seconds in reaching the solutions. We used the branch and bound algorithm as benchmark to check the accuracy of our heuristic algorithms in terms of solutions.

Table 6 shows in the first column a summary of the solutions reached by
Table 5: Experimental factors by levels

| Objects | Rankings | \( \theta \) |
|---------|----------|------------|
| 4       | Full     | 0.7        |
|         |          | 0.4        |
|         |          | 0.1        |
|         | Tied     | 0.7        |
|         |          | 0.4        |
|         |          | 0.1        |
| 9       | Full     | 0.7        |
|         |          | 0.4        |
|         |          | 0.1        |
|         | Tied     | 0.7        |
|         |          | 0.4        |
|         |          | 0.1        |
| 15      | Full     | 0.7        |
|         |          | 0.4        |
|         |          | 0.1        |
| 20      | Full     | 0.7        |
|         |          | 0.4        |
|         |          | 0.1        |
| pick 2 out of 4 | Incomplete | Normal Uniform |
| pick 5 out of 9 | Incomplete | Normal Uniform |
| pick 10 out of 20 | Incomplete | Normal Uniform |
Table 6: Summary measures of the number of solutions reached by BB algorithm and of the proportion of solutions found by Quick and Fast with respect to BB solutions by number of objects

|          | BB solutions | % Quick | % FAST |
|----------|--------------|---------|--------|
| 4 objects|              |         |        |
| Mean     | 1.150        | 0.969   | 0.989  |
| Median   | 1.000        | 1.000   | 1.000  |
| Minimum  | 1.000        | 0.500   | 0.667  |
| Maximum  | 3.000        | 1.000   | 1.000  |
| 9 objects|              |         |        |
| Mean     | 1.233        | 0.959   | 0.993  |
| Median   | 1.000        | 1.000   | 1.000  |
| Minimum  | 1.000        | 0.222   | 0.556  |
| Maximum  | 9.000        | 1.000   | 1.000  |
| 15 objects|             |         |        |
| Mean     | 2.567        | 0.820   | 0.919  |
| Median   | 1.000        | 1.000   | 1.000  |
| Minimum  | 1.000        | 0.056   | 0.333  |
| Maximum  | 18.000       | 1.000   | 1.000  |
| 20 objects|             |         |        |
| Mean     | 2.600        | 0.818   | 0.919  |
| Median   | 1.000        | 1.000   | 1.000  |
| Minimum  | 1.000        | 0.222   | 0.444  |
| Maximum  | 9.000        | 1.000   | 1.000  |

the branch and bound algorithm and in the second and in the third columns respectively shows the summary of the proportion of solutions returned by the Quick and FAST algorithms with respect to the ones handed back by the Emond and Mason’s one. Note that always both Quick and FAST algorithm could find at least one solution, and the proportion of solutions found by the FAST algorithm is always higher (or equal) to the one returned by the Quick. There were no relevant differences among the factors of the experimental design except, as expected, that the lower was $\theta$ the higher was the number of solutions identified. This was due to the fact that in this particular experiment, even when $\theta$ was set equal to 0.01, in all generated data sets there was a moderate internal degree of consensus present in the data sample.

Table 7 reports the solutions returned by the branch and bound algorithm and the proportion of them recovered by the quick and by the FAST algorithms in the experiment with incomplete rankings. In this case, due to the
Table 7: Summary measures of the number of solutions reached by BB algorithm and of the proportion of solutions found by Quick and Fast with respect to BB solutions

|          | BB solutions | % Quick | % FAST |
|----------|--------------|---------|--------|
| 2 out of 4 | 1.550        | 0.892   | 0.983  |
| Mean     | 1.000        | 1.000   | 1.000  |
| Median   | 1.000        | 0.500   | 0.667  |
| Minimum  | 3.000        | 1.000   | 1.000  |
| Maximum  | 5 out of 9   | 7.350   | 0.534  |
| Mean     | 4.000        | 0.548   | 0.646  |
| Median   | 1.000        | 0.105   | 0.353  |
| Minimum  | 31.000       | 1.000   | 1.000  |
| Maximum  | 10 out of 15 | 451.050 | 0.295  |
| Mean     | 8.000        | 0.171   | 0.522  |
| Median   | 1.000        | 0.000   | 0.013  |
| Minimum  | 7761.000     | 1.000   | 1.000  |
| Maximum  |

sampling procedure, the internal degree of consensus in the data sets was quite poor. The complexity of the problem is more evident by looking at the experiments with 9 and 15 objects, that respectively count a maximum number of solutions equal to 31 and 7761. In one case the Quick algorithm failed to find a solution, but it did not happen with the FAST algorithm. This particular case can help to understand why we called this algorithm “FAST”. The branch and bound algorithm found 25 solutions in 24240.054 seconds (~ 6.733 hours), each one reaching an average $\tau_x = 0.106$. The FAST algorithm could find 6 of the 25 solutions in 64.932 seconds. The two solutions found by the Quick algorithm were found in 0.693 seconds and were really close to be real solutions because they were characterized by an average $\tau_x = 0.104$. This was the unique case in which the Quick algorithm failed in finding a correct solution.

Figures 2, 3 and 4 show the distribution of working time of both branch and bound and Quick algorithms. We decided to no show the box-plots relative to the FAST heuristic algorithm because its computing time was approximately equal to the number of iterations multiplied by the computing time of the Quick algorithm. As it can be noticed, the Quick algorithm is on average faster than the branch and bound algorithm, and the variability of
the computing time increases as the value of \( \theta \) decreases.

Figure 2: Working time in second. The first row of box-plots refers to complete rankings, the second row refers to tied and complete rankings

Table 8 summarizes the computing time for the experiment involving incomplete rankings. In this case it can be noted that the computation time for the Quick algorithm has not a considerable variability while, especially in the case of 15 objects, it shows higher variability in the case of the branch and bound algorithm.

6 Real data applications

The first real data application is about the data reported by Emond and Mason (2000, pag. 28) which are shown in Table 9. The first 15 columns
Figure 3: Working time in second. The first row of box-plots refers to complete rankings, the second row refers to tied and complete rankings

represent the objects to be ranked with labels in the first row, while the last column reports the weight associated with every ranking.

By using the branch and bound algorithm we obtained exactly the following solutions (as also reported by Emond and Mason, 2000, page 29), with a $\tau_x$ equal to 0.166:

1. $<D\ L\ (E-M)\ (A-B)\ I\ P\ (C-N)\ H\ F\ G\ (O-Q)>$
2. $<D\ L\ (E-M)\ (A-B-P)\ (C-N)\ I\ H\ F\ G\ (O-Q)>$
3. $<D\ L\ (E-M)\ (B-P)\ A\ (C-N)\ I\ H\ F\ G\ (O-Q)>$

Computing time was equal to 5113.608 seconds. We ran the Quick algorithm on these data obtaining solution number 3 in a computing time of 0.155 sec-
The second data set used to compare the computing time of the algorithms is the famous data set about voters for the 1980 election of American Psychological Association president (Diaconis, 1989; Murphy and Martin, 2003). This data set contains the rankings expressed by 15,449 psychologists on five candidates: A = Bevan, B = Iscoe, C = Kiesler, D = Siegle and E = Wriths. Of these rankings only 5,738 are complete while the remaining are partial rankings. As shown in table 10 all the algorithms reach the same unique solution. The third data set used is known as the Sports data set and it comes from Louis Roussos (Marden, 1996). In this data 130 student of the University of Illinois...
Table 8: Summary measures of elapsed times in finding the solutions

|                | BB  | Quick | FAST |
|----------------|-----|-------|------|
| **2 out of 4** |     |       |      |
| Mean           | 0.031 | 0.012 | 0.337|
| Median         | 0.012 | 0.010 | 0.318|
| Minimum        | 0.009 | 0.008 | 0.261|
| Maximum        | 0.097 | 0.027 | 0.595|
| **5 out of 9** |     |       |      |
| Mean           | 0.282 | 0.170 | 14.328|
| Median         | 0.287 | 0.185 | 16.278|
| Minimum        | 0.218 | 0.063 | 7.788 |
| Maximum        | 0.378 | 0.219 | 16.398|
| **10 out of 15** |     |       |      |
| Mean           | 1967.438 | 0.745 | 65.910|
| Median         | 255.663 | 0.686 | 66.103|
| Minimum        | 0.745  | 0.660 | 64.413|
| Maximum        | 24240.054 | 1.343 | 68.537|

were asked to rank seven sports according to their preference of participating in. The sports considered were: A = baseball, B = football, C = basketball, D = tennis, E = cycling, F = swimming and G = jogging. Also in this case there is a unique solution, and the results are reported in Table [11]. Also in this case all the algorithms reach the same unique solution, as reported in Table [11].

The forth data set is a subset of rankings collected by Kathleen O’Leary Morgan and Scott Morgan (2010) on the 50 American States. The number of items (the number of American States) is equal to 50, and the number of rankings is equal to 104. It was unfeasible to run the Emond and Mason’s algorithm on this data. The orderings corresponding to the three solutions found by the FAST heuristic algorithm are reported in table [12]. These solutions are obtained in 1177.274 seconds (∼ 19 minutes) with the algorithm set with 1000 iterations. The quick algorithm found 1 solution (solution 2 in Table [12]) in 16.384 seconds.

7 Concluding remarks

In this work we proposed two heuristic algorithms for solving the consensus ranking problem, namely the Quick heuristic and the FAST heuristic. Our
Table 9: Emond and Mason’s data

|     | A | B | C | D | E | F | G | H | I | L | M | N | O | P | Q | w_k |
|-----|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|-----|
| 1   | 6 | 4 | 5 | - | 1 | 2 | 7 | 3 | 1 | 5 | 2 | 6 | 5 | 5 | - | 4  |
| 11  | 10| 4 | 8 | 9 | 1 | 7 | 12| 2  | 3 | 2 | 6 | 13| 5 | 14| 4 |    |
| 11  | 12| 3 | 11| 7 | 1 | 4 | 5 | 12| 2 | 6 | 10| 11| 8 | 9 | 4 |    |
| 2   | 4 | 3 | 3 | 11| 8 | 10| 9 | 6 | 10| 5 | 1 | 5 | 7 | 5 | 5 |    |
| 2   | 8 | 4 | 8 | 7 | 1 | 2 | 5 | 2 | 3 | 6 | 7 | 8 | - | - | 4 |    |
| 2   | 9 | 5 | 1 | 4 | 3 | 2 | 7 | 3 | 1 | 8 | 6 | 3 | 4 | 8 | 5 |    |
| 3   | 9 | 7 | 1 | 2 | 8 | 13| 6 | 1 | 10| 5 | 11| 9 | 4 | 14| 5 |    |
| 4   | 2 | 9 | 1 | 3 | 12| 6 | 10| 13| 14| 11| 9 | 7 | 8 | 5 | 5 |    |
| 4   | 3 | 5 | 11| 12| 10| 13| 7 | 6 | 8 | 2 | 1 | 9 | 9 | 11| 7 |    |
| 4   | 7 | 8 | 6 | 13| 2 | 3 | 12| 9 | 1 | 5 | 10| 5 | 11| 11| 4 |    |
| 6   | 1 | 3 | 3 | 6 | 2 | 6 | 5 | 4 | 5 | 1 | 1 | 2 | 1 | 1 | 5 |    |
| 6   | 10| 14| 5 | 7 | 1 | 8 | 3 | 2 | 3 | 4 | 11| 13| 12| 9 | 4 |    |
| 6   | 6 | 8 | 1 | 1 | 3 | 5 | 1 | 10| 7 | 2 | 10| 9 | 4 | 6 | 7 |    |
| 7   | 2 | - | 1 | 2 | 10| 5 | 3 | 9 | 8 | 6 | 7 | 7 | 6 | 4 | 5 |    |
| 7   | 4 | 6 | 1 | 5 | 14| 10| 12| 15| 3 | 13| 9 | 8 | 2 | 11| 5 |    |
| 7   | 8 | 4 | 5 | 7 | 1 | 6 | 5 | 3 | 2 | 7 | 9 | 10| 11| 12| 4 |    |
| 8   | 4 | 7 | 2 | 1 | 11| 4 | 6 | 3 | 12| 6 | 10| 13| 5 | 9 | 7 |    |
| 9   | 8 | 7 | 6 | 3 | 4 | - | 2 | 5 | 1 | 3 | 7 | 6 | 4 | 6 | 7 |    |
| -   | - | 3 | 1 | 1 | 5 | 5 | 4 | 5 | 2 | 4 | 2 | 6 | 7 | 8 | 7 |    |
| -   | - | 4 | 7 | 2 | 10| 11| 5 | 8 | 8 | 9 | 1 | 2 | 3 | 6 | 7 |    |
| -   | - | 5 | 6 | 12| 9 | 10| 8 | 2 | 11| 1 | 4 | 7 | 2 | 3 | 7 |    |

Table 10: APA data set

| Algorithm | solution       | elapsed time | replications |
|-----------|----------------|--------------|--------------|
| EM        | <C A E D B>    | 1.033        | -            |
| Quick     | <C A E D B>    | 0.764        | -            |
| Fast      | <C A E D B>    | 27.814       | 50           |

approach to find the consensus ranking lies into the Kemeny and Snell theoretical framework and it is closely related to the branch and bound algorithm by Emond and Mason (2002). Both our algorithms can easily deal with complete and tied rankings as well as with partial (or incomplete) rankings. We illustrated the performance of both these algorithms in terms of accuracy and efficiency via simulated and real data sets comparing them with the results and the execution time (in elapsed seconds) needed to obtain them.

The Emond and Mason’s branch and bound algorithm is obviously a good algorithm that permits to explore efficiently all the possible solutions, but as the number of items increases, it shows a highly variable computation time to reach the final solution(s), e.g. from less than 1 seconds up to 24.000
Table 11: Sports data set

| Algorithm | solution          | elapsed time | replications |
|-----------|------------------|--------------|--------------|
| EM        | <E F C A D B G>   | 0.076        | -            |
| Quick     | <E F C A D B G>   | 0.084        | -            |
| Fast      | <E F C A D B G>   | 3.592        | 50           |

seconds in our experiments, also depending on the internal degree of consensus present in the data sample. On the other hand our algorithms can deal with a quite large number of objects and they result to be, as shown by the simulation study and the real data example, very accurate since they reach at least one solution in a reasonable amount of time. An important remark about the Quick heuristic algorithm is that it can find a solution in really few seconds with a high probability. When the reached solution does not coincide with a global solution, our algorithms anyway seem to be quite accurate since the find solution(s) really close to the real one(s). But even in this case, we are confident that the FAST heuristic reaches a correct solution. As can be noticed by the consensus rankings found in the real data applications, if multiple consensuses are present they are really similar since almost always the same objects are ranked in the first positions and the same happens for the objects ranked at last positions while little modifications are present in the middle positions. For this reason multiple consensuses can be considered mutually coherent. Of course, only through the branch and bound algorithm or an exhaustive research we can be absolutely sure that we have found the consensus ranking.

As already reported in Section 2, other branch and bound algorithms were proposed to solve the consensus ranking problem over the years, but they are mostly based on other paradigms and for this reason not comparable with ours.

References

Arrow, K. J. (1951). Social choice and individual values. Wiley, New York

Barthélemy, J. P., and Monjardet, B. (1981). The median procedure in cluster analysis and social choice theory. Mathematical social sciences,
Barthélemy, J. P., Guénoche, A., and Hudry, O. (1989). Median linear orders: heuristics and a branch and bound algorithm. *European Journal of Operational Research, 42*(3), 313-325.

Black, D. (1958). *The Theory of Committees and Elections*, Cambridge Univ. Press, Cambridge.

Bogart, K. P. (1973). Preference structures I: Distances between transitive preference relations. *Journal of Mathematical Sociology, 3*(1), 49-67.

Bogart, K. P. (1975). Preference Structures. II: Distances Between Asymmetric Relations. *SIAM Journal on Applied Mathematics, 29*(2), 254-262.

Borda, de, J. C. (1781). *Mémoire sur les élections au scrutin*, Historie de l’académie royale des sciences. Paris, France.

Condorcet, JANdC (1785). *Essai sur l’application de l’analyse a la probabilité des décisions rendues a la pluralité des voix*. Paris: De l’Imprimerie royale.

Cook, W. D., and Saipe, A. L. (1976). Committee approach to priority planning: the median ranking method. *Cahiers du Centre d’Études de Recherche Opérationnelle, 18*(3), 337-351.

Cook, W. D., and Seiford, L. M. (1978). Priority ranking and consensus formation. *Management Science, 24*(16), 1721-1732.

Cook, W. D., Golany, B., Penn, M., and Raviv, T. (2007). Creating a consensus ranking of proposals from reviewers partial ordinal rankings. *Computers & Operations Research, 34*(4), 954-965.

Coombs, C. H. (1964). *A theory of data*. New York: Wiley.

Critchlow, D. E. (1980). *Metric Methods for Analyzing Partially Ranked Data*. in Lecture Notes in Statistics 34. Springer-Verlag.

Critchlow, D. E., Fligner, M. A., and Verducci, J. S. (1991). Probability models on rankings. *Journal of Mathematical Psychology, 35*, 294318.
Davis, O. A., DeGroot, M. H., and Hinich, M. J. (1972). Social preference orderings and majority rule. *Econometrica: Journal of the Econometric Society*, 147-157.

Diaconis, P. (1988). *Group representations in probability and statistics*. Institute of Mathematical Statistics, Hayward, CA.

Emond, E.J., Mason, D.W. (2000), A new technique for high level decision support. *ORD project Report PR2000/13* Department of National Defence, Canada.

Emond, E.J., Mason, D.W. (2002), A new rank correlation coefficient with application to the consensus ranking problem. *Journal of Multi-Criteria Decision Analysis*, 11, 17-28.

Feigin, P.D., Cohen, A. (1978), On a model for concordance between judges. *Journal of the Royal Statistical Society*, Series B (Methodological), 40, 203-213 (2002).

Fligner, M. A., and Verducci, J. S. (1986). Distance based ranking models. *Journal of the Royal Statistical Society*, Series B, 48, 359-369.

Fligner, M.A., and Verducci, J.S. (1988). Multistage rankings models. *Journal of the American Statistical Association*, 83, 892-901.

Goodman, L. A., and Markowitz, H. (1952). Social welfare functions based on individual rankings. *American Journal of Sociology*, 257-262.

Kemeny, J. G. (1959). Mathematics without numbers. *Daedalus*, 88(4), 577-591.

Kemeny, J. G. and Snell, L. (1962), *Mathematical Models in the Social Sciences*. Ginn and Company.

Kendall, M. G. (1938). A new measure of rank correlation. Biometrika, 81-93.

Heiser W.J. (2004), Geometric representation of association between categories. *Psychometrika*, 69(4), 513-545
Heiser, W.J., and D’Ambrosio, A. (2013). Clustering and prediction of rankings within a Kemeny distance framework, in Lausen, B, Van den Poel, D. and Ultsch, A. (EDS.), Algorithms from and for Nature and Life, Springer series in Studies in Classification, Data Analysis, and Knowledge Organization, 19-31, Springer International Publishing Switzerland.

Marden, J.I. (1995), Analyzing and modelling rank data. Chapman & Hall, London.

Meilă, M., Phadnis, K., Patterson, A., and Bilmes, J. (2007). Consensus ranking under the exponential model. Department Technical Report nr. 515, Statistics Department University of Washington.

Monjardet, B. (2008). “Mathématique Sociale” and Mathematics. A case study: Condorcet’s effect and medians. Electronic Journal for History of Probability and Statistics, 4(1), 1-26.

Kathleen O’Leary Morgan and Scott Morgan (Eds.), (2010). State Rankings 2010: A Statistical View of America; Crime State Ranking 2010: Crime Across America; Health Care State Rankings 2010: Health Care Across America. State Fact Finder Series, Washington: CQ Press (Sage).

Thompson, G. L. (1993). Generalized permutation polytopes and exploratory graphical methods for ranked data. The Annals of Statistics, 1401-1430.

Young, H. P., and Levenglick, A. (1978). A consistent extension of Condorcet’s election principle. SIAM Journal on Applied Mathematics, 35(2), 285-300.
Table 12: Consensus ranking found by Fast heuristic algorithm, American states data

|       | solution 1 | solution 2 | solution 3 |
|-------|------------|------------|------------|
| 1     | CA         | CA         | CA         |
| 2     | NY         | NY         | NY         |
| 3     | FL         | FL         | FL         |
| 4     | MD         | MD         | MD         |
| 5     | LA         | LA         | LA         |
| 6     | NM         | NM         | NM         |
| 7     | DE         | TX         | DE         |
| 8     | TX         | IL         | TX         |
| 9     | IL         | DE         | IL         |
| 10    | PA         | PA         | PA         |
| 11    | MI         | MI         | MI         |
| 12    | GA         | GA         | GA         |
| 13    | NC         | NC         | NC         |
| 14    | NJ         | NJ         | NJ         |
| 15    | MA         | MA         | MA         |
| 16    | WA         | WA         | WA         |
| 17    | OH         | OH         | OH         |
| 18    | VA         | VA         | VA         |
| 19    | TN         | TN         | TN         |
| 20    | NV         | NV         | NV         |
| 21    | AZ         | AZ         | AZ         |
| 22    | MO         | MO         | MO         |
| 23    | IN         | IN         | IN         |
| 24    | AK         | AK         | AK         |
| 25    | WI         | WI         | WI         |
| 26    | CO         | CO         | CO         |
| 27    | CT         | CT         | CT         |
| 28    | MN         | MN         | MN         |
| 29    | AL         | AL         | AL         |
| 30    | SC         | SC         | SC         |
| 31    | OR         | OR         | OR         |
| 32    | OK         | OK         | OK         |
| 33    | MS         | MS         | KY         |
| 34    | AR         | AR         | MS         |
| 35    | HI         | HI         | AR         |
| 36    | KY         | KY         | HI         |
| 37    | (KS - RI)  | (KS - RI)  | (KS - RI)  |
| 39    | UT         | UT         | UT         |
| 40    | (IA - NE)  | (IA - NE)  | (IA - NE)  |
| 42    | WY         | WY         | WY         |
| 43    | WV         | WV         | WV         |
| 44    | ID         | ID         | ID         |
| 45    | ME         | ME         | ME         |
| 46    | MT         | MT         | MT         |
| 47    | NH         | NH         | NH         |
| 48    | SD         | SD         | SD         |
| 49    | VT         | VT         | VT         |
| 50    | ND         | ND         | ND         |

$\tau_s = 0.298$ $\tau_s = 0.298$ $\tau_s = 0.298$