All solutions for geodesic anisotropic spherical collapse with shear and heat radiation

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We introduce a physically important object, called the horizon function, in the study of geodesic collapse. It is closely related to the stellar characteristics and satisfies a simple Riccati equation. This equation is integrated and all of its solutions are found in terms of some generating function. Previous solutions are regained and further investigated.

Subject headings: geodesic fluid, gravitational collapse
1. Introduction

Gravitational collapse is an important issue in relativistic astrophysics. There are many indications that the collapsing fluid in the star models is anisotropic (Herrera and Santos 1997). In addition, this process is highly dissipative, required to account for the enormous binding energy of the resulting object (Herrera et al 2006). Thus a realistic scenario is the collapse with heat flow (Bonnor et al 1989) or pure radiation (Tewari 2010). Spherical collapse is described in the general case by a diagonal metric with three independent components. For simplicity, shearless fluid is used quite often, which reduces the metric components to two. Even in the isotropic case the amount of interior solutions is enormous (Ivanov 2012). The exterior solution is the Vaidya shining star (Vaidya 1951). The main junction condition states that at the star surface the radial pressure should equal the heat flux. This gives a non-linear differential equation in partial derivatives (along radius and time) and follows from the matching of the second fundamental forms. In the streaming approximation (pure radiation) it leads to the vanishing of the radial pressure.

In the shearless case the differential equation involves only the two metric components $g_{00}$ and $g_{rr}$. When shear is present, the general three component metric should be used, which further complicates the differential equation. One can reduce in a different way the metric components to two, $g_{rr}$ and $g_{\theta\theta}$, by studying the geodesic case, $g_{00} = 1$ (Kolassis et al 1988).

Interior anisotropic geodesic solutions with shear and without radiation have been discussed in (Ivanov 2011). No matching to the exterior Schwarzschild solution was done. The same problem in non-comoving coordinates, but with matching was solved in (Herrera et al 2002).

The first exact solution with radiation was obtained by (Naidu et al 2006). After
that (Rajah and Maharaj 2008) noticed that the junction condition is a Riccati equation for $g_{rr}$. They found two simple regular solutions in separated variables. The solution of (Naidu et al 2006) is regained when certain parameters are set to zero.

Later, (Thirukkanesh and Maharaj 2010) found even more general exact solutions depending on arbitrary functions of the coordinate radius. They encompass the previous solutions.

In a recent development (Abebe et al 2014) further expanded the realm of analytic solutions by studying the Lie point symmetries of the boundary condition. Generalized travelling waves and self-similar solutions were found.

The junction condition has been investigated also in the non-geodesic case. For example a number of conformally flat solutions have been found (Herrera et al 2004), (Maharaj and Govender 2005), (Herrera et al 2006)

The main idea of the present paper is to find and integrate an equation for a physically meaningful object, which we call the horizon function. It is directly related to the redshift and the formation of a horizon, which means the appearance of a black hole as the end product of collapse. It enters the expression for the mass of the star, the heat flow and the luminosity at infinity. The equation is so simple that it is easily integrated with the help of a generating function.

In Sect. 2 we present the Einstein equations, which in the anisotropic case are expressions for the energy density, the radial and the tangential pressure and the heat flow. The definition of the shear, the expansion, the horizon function and the redshift are given. The relation between the mass of the star and the horizon function is clarified. The main results of the matching to the exterior Vaidya solution are shown. The most important of them is a differential equation, involving the metric components. We show that it is the
same in the diffusion and in the streaming approximations. The other stellar characteristics are also given. In Sect. 3, modifying the method of \cite{Thirukkanesh and Maharaj 2010}, the junction equation is written as a Riccati equation for the horizon function with simple coefficients. This allows to derive all geodesic solutions from two generating functions. Simple expressions are given for the mass and its time derivative. It is shown how the horizon function and even the redshift may be taken as alternative generating functions. In Sect. 4 previously found solutions with radiation are derived from the general solution and further investigated. In Sect. 5 we regain the solutions without radiation. Sect. 6 contains conclusions.

2. Stellar characteristics

The collapse of an anisotropic geodesic fluid sphere with shear is described by the following metric

\[ ds^2 = -dt^2 + B^2 dr^2 + R^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right), \]

(1)

where \( B \) and \( R \) are independent functions of time \( t \) and the radius \( r \). The spherical coordinates are numbered as \( x^0 = t, \ x^1 = r, \ x^2 = \theta \) and \( x^3 = \varphi \). The energy-momentum tensor, describing dissipation through heat flow and null fluid, reads

\[ T_{ik} = (\mu + p_t) u_i u_k + p_t g_{ik} + (p_r - p_t) \chi_i \chi_k + q_i u_k + u_i q_k + \varepsilon l_i l_k. \]

(2)

Here \( \mu \) is the energy density, \( p_r \) is the radial pressure, \( p_t \) is the tangential pressure, \( u^i \) is the four-velocity of the fluid, \( \chi^i \) is a unit spacelike vector along the radial direction, \( q^i \) is the heat flow vector, also in the radial direction, \( \varepsilon \) is the energy density of the radiated null fluid and the vector \( l^i \) is null. In comoving coordinates we have

\[ u^i = \delta^i_0, \quad \chi^i = B^{-1} \delta^i_1, \quad q^i = q \chi^i, \quad l^i = u^i + \chi^i. \]

(3)
The Einstein field equations become (Thirukkanesh and Maharaj 2010), (Ivanov 2010)

\[
\mu + \varepsilon = \left( \frac{2\dot{B}}{B} + \frac{\ddot{R}}{R} \right) \frac{\dot{R}}{R} - \frac{1}{B^2} \left( \frac{2R''}{R} + \frac{R'^2}{R^2} - \frac{2BR'}{BR} - \frac{B^2}{R^2} \right),
\]

(4)

\[
p_r + \varepsilon = \frac{2\ddot{R}}{R} - \frac{\ddot{R}^2}{R^2} + \frac{R'^2}{B^2R^2} - \frac{1}{R^2},
\]

(5)

\[
p_t = - \left( \frac{\ddot{B}}{B} + \frac{\ddot{R}}{R} + \frac{\dot{B}R}{BR} \right) + \frac{1}{B^2} \left( \frac{R''}{R} - \frac{B'R'}{BR} \right),
\]

(6)

\[
qB + \varepsilon = - \frac{2}{B} \left( \frac{\dot{B}R'}{BR} - \frac{\dot{R}'}{R} \right).
\]

(7)

Here the dot means a time derivative, while the prime stands for a radial derivative.

For the line element (1) the four-acceleration vanishes, while the shear and the expansion scalars are given by

\[
\sigma = \frac{1}{3} \left( \frac{\dot{R}}{R} - \frac{\dot{B}}{B} \right),
\]

(8)

\[
\Theta = \frac{2\ddot{R}}{R} + \frac{\dot{B}}{B}.
\]

(9)

Next, we introduce the important object \( H \), which we call ”the horizon function” for reasons to become clear later:

\[
H = \frac{R'}{B} + \dot{R}.
\]

(10)

The mass entrapped within radius \( r \) is given by the expression (Cahill and McVittie 1970)

\[
m = \frac{R}{2} \left[ 1 + \dot{R}^2 - \left( \frac{R'}{B} \right)^2 \right].
\]

(11)

On the stellar surface \( \Sigma \) it becomes the mass of the star. The compactness parameter reads \( u = m/R \). Eq (11) can be rewritten using \( H \)

\[
\frac{2m}{R} = 1 + 2H\dot{R} - H^2.
\]

(12)

The exterior spacetime is given by the Vaidya shining star solution

\[
ds^2 = - \left[ 1 - \frac{2M(v)}{\rho} \right] dv^2 - 2dv d\rho + \rho^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right),
\]

(13)
where \( M(v) \) is the mass of the star measured at time \( v \) by an observer at infinity, while \( \rho \) is the exterior coordinate radius. Both solutions should be joined smoothly at \( \Sigma \), which leads to the following junction conditions:

\[
R_{\Sigma} = \rho_{\Sigma}(v),
\]

\[
m_{\Sigma} = M_{\Sigma},
\]

\[
(p_r)_{\Sigma} = (qB)_{\Sigma}.
\]

Eq (16) should be satisfied by \( R \) and \( B \) while the other equations are definitions of different stellar characteristics. When \( q \) vanishes, the radial pressure should vanish at the surface, but when \( \varepsilon \neq 0 \) Eqs. (5,7) show that in terms of the metric Eq (16) is restored. This condition was used in many works with null fluid radiation (Tewari 2010). When \( \varepsilon \) also vanishes we get collapse without radiation and the exterior solution is the Schwarzschild vacuum solution. In the following we set \( \varepsilon = 0 \).

Some important stellar characteristics are also defined on the surface of the star. These are the redshift \( z_{\Sigma} \)

\[
z_{\Sigma} = \frac{1}{H_{\Sigma}} - 1,
\]

the surface luminosity \( \Lambda_{\Sigma} \) and the luminosity at infinity \( \Lambda_{\infty} \)

\[
\Lambda_{\Sigma} = \left( \frac{1}{2} qBR^2 \right)_{\Sigma},
\]

\[
\Lambda_{\infty} = H_{\Sigma}^2 \Lambda_{\Sigma}.
\]

The temperature at the surface is given by

\[
T_{\Sigma}^4 = \frac{(qB)_{\Sigma}}{8\pi\delta},
\]

where \( \delta \) is some constant.
It is seen that the star properties have simpler expressions when written in terms of \( H \). The redshift is positive during collapse. Then Eq (17) shows that \( 0 \leq H_{\Sigma} \leq 1 \). When \( H_{\Sigma} = 0 \) we obtain from Eq (12) and the junction conditions

\[
\left( 1 - \frac{2M(v)}{\rho} \right)_{\Sigma} = 0. \tag{21}
\]

This signals the appearance of a horizon and a black hole within it, which is the typical end of gravitational collapse. This explains the name of \( H \). The redshift becomes infinite, while the luminosity at infinity drops to zero. The point in time when collapse starts is taken usually as \(-\infty\). There \( H_{\Sigma} \) should have some positive value less or equal to 1. Thus during the collapse the horizon function decreases to zero and \( \dot{H}_{\Sigma} \leq 0 \). Other scenarios will be discussed in the following.

3. Solution of the junction equation

With the help of Eqs. (5,7) the main junction condition (16) becomes

\[
2R\dddot{R} + \dddot{R}^2 + 1 - \frac{R'^2}{B^2} + \frac{2R\dot{R}'}{B} - \frac{2\dot{B}RR'}{B^2} = 0. \tag{22}
\]

It and the following equations hold on the surface. This equation coincides with Eq (9) from (Thirukkanesh and Maharaj 2010) and determines the evolution of a radiating geodesic and anisotropic star with shear. It is a highly nonlinear differential equation in partial derivatives. It is simplified when one introduces the function \( Z \)

\[
Z = \frac{B}{\dot{R}'} \tag{23}
\]

and becomes

\[
\dot{Z} = \frac{1}{2R} \left( FZ^2 - 1 \right), \tag{24}
\]

where

\[
F = 2R\dddot{R} + \dddot{R}^2 + 1. \tag{25}
\]
Eq (24) is separable and integrable as long as $F$ does not depend on time. The latter condition leads to an equation for $R$, which can be solved and special solutions for $R$ and $B$ may be obtained.

In the general case, let us express $Z$ in terms of the horizon function. Eqs (10, 23) yield

$$\frac{1}{Z} = H - \dot{R}. \quad (26)$$

Now we insert this expression in Eq (24) and obtain after some calculations and cancellations

$$2R\dot{H} = H^2 - 2\dot{R}H - 1. \quad (27)$$

Eq (27) is much simpler than Eq (22). It involves only first time derivatives and the physical quantities $R$ (the radius of the star as seen from an exterior observer) and the horizon function $H$ which controls the black hole formation, the redshift, the mass and the luminosity of the star. It is a Riccati equation with respect to $H$ and a linear one with respect to $R$. We can do even better by defining the non-negative function $D = RH$. Then Eq (27) transforms into

$$\dot{D} = \frac{1}{2R^2}D^2 - \frac{1}{2}. \quad (28)$$

$D$ still satisfies a Riccati equation, but $R$ enters in an algebraic way. Thus we get an expression for $R$ in terms of $D$

$$R = \frac{D}{\sqrt{2D + 1}}. \quad (29)$$

In addition to $D \geq 0$, $D$ should satisfy $\dot{D} > -1/2$.

Eq (29) holds on $\Sigma$, that is, $r = r_\Sigma$, which is some constant. We can take any reasonable $D(t)$ and promote the constants in it to arbitrary functions of the radius. This situation is similar to the solution of the isotropic equation for perfect fluid without shear, when the metric can be written in isotropic coordinates (which contains only radial derivatives) (Ivanov 2012).
One difference is that the junction equation holds for any fluid, either perfect or imperfect, as long as there is a clear boundary between the interior and the exterior solution.

Another difference is that we can add to the r.h.s. of Eq (29) the function \( P = g(r) F(t,r) \) where \( g \) and \( F \) are arbitrary up to ensuring that \( R \) is positive. In addition \( f(r_{\Sigma}) = 0 \) and the term \( P \) does not show on the surface (zero constant there). Thus the continuation of \( R \) in the bulk of the star depends on a second generating function \( P \). For the moment we set \( g = 0 \) and shall comment on it at the end of the section. Hence, Eq (29) and the following equations hold in the bulk too.

From Eq (29) and the definition of \( D \) we obtain an expression for \( H \)

\[
H = \sqrt{2\dot{D} + 1}.
\]  

(30)

Then Eq (10) gives an expression for \( B \)

\[
B = \frac{R'}{H - \dot{R}}.
\]  

(31)

where we should insert the expressions for \( R \) and \( H \) in terms of \( D \). In the process of collapse \( R \) decreases, hence \( \dot{R} < 0 \) and the denominator in Eq (31) is positive. Then it follows that \( R' > 0 \). This is exactly the condition for the absence of shell crossing singularities (Malafarina and Joshi 2011).

We have determined the metric components in terms of \( D \) and have solved the junction equation. The arbitrary function \( D(t,r) \) plays the role of a generating function. All stellar characteristics become functions of it and its time derivatives.

Eqs (29,30) lead to the useful formulas

\[
\dot{R} = \frac{(2\dot{D} + 1)\ddot{D} - D\dddot{D}}{(2\dot{D} + 1)^{3/2}},
\]  

(32)
\[ H - \dot{R} = \frac{(2\ddot{D} + 1)(\dot{D} + 1) + D\dddot{D}}{(2\dot{D} + 1)^{3/2}}, \]  
\[ \dot{H} = \frac{\ddot{D}}{\sqrt{2\dot{D} + 1}}, \]

hence, \( \ddot{D} < 0 \). A combination of Eqs (12) and (27) gives a simple formula for the mass function

\[ m = -R^2 \dot{H}. \]  
(35)

In terms of \( D \) it becomes

\[ m = -\frac{D^2 \ddot{D}}{(2\dot{D} + 1)^{3/2}}. \]  
(36)

Thus, while \( H = 0 \) describes the formation of a black hole, \( \dot{H} = 0 \) describes the burning out of the mass due to the radiation, which is another reason for the end of collapse. The question is which one happens first. A third scenario is the "eternal collapse" when \( 2m/R \) is a constant, smaller than 1 so that a horizon never develops. It is a well-known option in the shearless case (Tewari and Charan 2015).

Let us take next Eq (7), the expression for \( qB \). It may be transformed into

\[ qB = -\frac{2R'}{BR} \left( \ln \frac{B}{B'} \right)' = \frac{2}{R} \left( \frac{R'}{B} \right)' \]  
(37)

which can be expressed through \( H \)

\[ qB = \frac{2}{\dot{R}} \left( H - \dot{R} \right)'. \]  
(38)

The l.h.s. is positive because \( q > 0 \) and the energy is radiated out.

On the other side Eq (35) yields

\[ \dot{m} = -R \left( 2\dot{R}\dot{H} + R\dddot{H} \right). \]  
(39)
The time derivative of the junction equation, however, gives

\[ 2 \dot{R} \dot{H} + R \ddot{H} = H \left( H - \dot{R} \right). \tag{40} \]

Comparing this to Eq \((38)\) yields finally

\[ 2 \dot{m} = -qBR^2 H. \tag{41} \]

This relation shows how the decrease in mass is governed by the heat flow. In terms of the generating function \(D\) we have

\[ qB = 2 \left( \frac{2 \dot{D} + 1}{(2 \dot{D} + 1)^2} \right) \left( 2 \dot{D} \ddot{D} + DD\cdots \right) - 3D \dot{D}^2, \tag{42} \]

\[ \dot{m} = \frac{D \left[ 3D \ddot{D}^2 - \left( 2 \dot{D} + 1 \right) \left( 2 \dot{D} \ddot{D} + DD\cdots \right) \right]}{(2 \dot{D} + 1)^{5/2}}, \tag{43} \]

Knowing \(qB\) and \(H\) we get expressions for the redshift, the two luminosities and the surface temperature from Eqs \((17-20)\). We have explained that the junction equation holds for all \(r\), so it gives

\[ p_r = qB \tag{44} \]

and \(p_r > 0\) everywhere inside the star.

The other characteristics of the fluid will be given for simplicity in terms of \(R, H\) and sometimes \(B\). The shear and the expansion become

\[ \sigma = \frac{1}{3} \left[ \ln \frac{R}{R'} \left( H - \dot{R} \right) \right], \tag{45} \]

\[ \Theta = \left( \ln \frac{R^2 R'}{H - \dot{R}} \right). \tag{46} \]

The energy density \(\mu\) is related to the mass

\[ \mu = \frac{2m}{R^3} + \frac{2}{BR} \left[ \dot{B} \dot{R} - \left( H - \dot{R} \right)^2 \right], \tag{47} \]
while the tangential pressure is

\[ p_t = \frac{1}{2} \left( \frac{2m}{R^3} - \mu \right) - \frac{1}{BR} \left( \ddot{R}B + R\ddot{B} \right), \]  

(48)

The function \( D \) is not the only generation function. One can use \( H \) instead. Eq (30) may be integrated to give

\[ 2D = \int H^2 dt - t. \]  

(49)

Inserting the definition \( D = HR \) we obtain an expression for \( R \)

\[ R = \frac{1}{2H} \left( \int H^2 dt - t \right). \]  

(50)

Taking the time derivative of this equation and using the junction equation (27) we get

\[ \dot{R} = -\frac{\dot{H}}{H} R + \frac{H^2 - 1}{2H}, \]  

(51)

\[ H - \dot{R} = \frac{\dot{H}}{H} \dot{R} + \frac{H}{2} + \frac{1}{2H}. \]  

(52)

With the help of Eqs (50-52) we can reduce all expressions above in \( R \) and \( H \) to functions of \( H \) and its derivatives, without going to the \( D \) level.

The initial junction equation (16) is a relation between physical characteristics of the model - the radial pressure and the heat flow. We can do the same for the transformed equation (27), which already contains the luminosity radius \( R \). With the help of the Eq (17), where the constants are promoted to functions of \( r \) we can pass from \( H \) to the redshift function \( z(t, r) \) which on the surface is a physical characteristic of the model. The result is once again a Riccati equation for \( z \) and the analogue of Eq (50) for \( R \)

\[ \dot{z} = \frac{1}{2R} z^2 + \frac{1 + \dot{R}}{R} z + \frac{\dot{R}}{R}, \]  

(53)

\[ R = 2 \left( 1 + z \right) \left( \int \frac{dt}{(1 + z)^2} - t \right). \]  

(54)
One should use reasonable values for $z$ which are limited from above by the energy conditions (Ivanov 2002).

What happens when $g \neq 0$? As long as a stellar characteristic depends on $D$ and on $R$ or/and its time derivatives, $P = 0$ on the surface where many of the important stellar characteristics are defined. It is seen that almost all of the above expressions are of this type. The metric component $B$ is an exception, but it has no physical meaning. Nevertheless, let us see how the above formulas change in this case. We have

$$R = R_0 + P$$

(55)

where $R_0$ is the star radius given by Eq (29),

$$H = H_0 \frac{R_0}{R_0 + P}$$

(56)

$$\frac{1}{2} qBR = H_0 \frac{\dot{R}_0 P - R_0 \dot{P}}{(R_0 + P)^2} + \frac{\dot{H}_0 R_0}{R_0 + P} - \ddot{R}_0 - \dddot{P}$$

(57)

and so on.

### 4. Previous solutions with heat radiation

Now one can take different functions $D$, $H$ or $z$ and try to find realistic stellar models. It is interesting to obtain in the first place the different particular solutions found in the past. The most general of them were presented by (Abebe et al 2014), using the Lie symmetry method. In this way generalised travelling waves and self-similar solutions have been found, which depend on an arbitrary function. In our approach we take $D = D(x)$ where $x$ is

$$x = \int \frac{dr}{f(r)} - \frac{t}{a}$$

(58)

and $a$ is a constant, while $f(r)$ is an arbitrary function. When $f(r) = 1$ we have a travelling wave with speed $1/a$. One easily finds that $\dot{D} = \dot{D}(x)$ and Eq (29) shows that $R = R(x)$. 
Also $H = H(x)$ and from Eq (31)

$$B = \frac{R_x}{f(r)(H + R_x/a)} = \frac{h(x)}{f(r)}. \quad (59)$$

We obtain exactly the first class of solutions in (Abebe et al 2014). Here $h$ satisfies a Riccati equation because $B$ does so, as seen from Eq (22).

Now let us take $D$ in the form

$$D = D_1(x) t, \quad x = t \exp \left( - \int \frac{dr}{af(r)} \right). \quad (60)$$

When $f = r/a, x = t/r$, that is, $x$ is a self-similar variable. Now $\dot{D} = \dot{D}(x)$ and $R = g(x) t$, where $g$ is arbitrary. Next $H = H(x)$ and

$$B = \frac{-xR_x}{af(H - xg_x - g_x)} = \frac{h(x)}{f(r)} \exp \int \frac{dr}{af(r)}, \quad (61)$$

where $h$ satisfies a different Riccati equation. Thus we obtain the second class of (Abebe et al 2014) solutions.

Next, let us go back to Eqs (24,25). Eq (25) may be written as

$$F = \left( \frac{R\dot{R}^2}{\dot{R}} \right) + 1. \quad (62)$$

When $F = 1 + R_1(r)^2$, with $R_1(r)$ arbitrary, the solution of Eq (62) is

$$R = R_1 t + R_2, \quad (63)$$

where $R_2(r)$ is another arbitrary function of $r$ (Thirukkanesh and Maharaj 2010). This means that $\ddot{R} = 0$. Since $H$ decreases, $\dot{H} \leq 0$ and, e.g., the mass function (35) is positive. But then Eq (38) shows that $q \leq 0$, therefore the star absorbs radiation from outside, which is not realistic. Consequently, the linear in time solution for $R$ is not physical.

When $F = 1$ Eq (62) yields

$$R = (R_1 t + R_2)^{2/3}. \quad (64)$$
Now Eq (38) reads
\[ qBR = -\frac{2\dot{Z}}{Z^2} \] (65)
and thus \( \dot{Z} < 0 \). Then Eq (24) shows that \( Z^2 < 1 \). It is easily seen (Thirukkanesh and Maharaj 2010) that
\[ Z = \frac{1 - y}{1 + y}, \] (66)
where
\[ y = -f(r) \exp \left[ 3 \left( R_1 t + R_2 \right)^{1/3} / R_1 \right]. \] (67)
The above condition for \( Z^2 \) demands that \( y \) is positive and, hence, \( f \) is negative. Usually, collapse is supposed to take place for \(-\infty < t \leq 0 \) and at \( t = 0 \) a black hole appears. Therefore we take on the surface \( R_1 < 0 \) and \( R_2 > 0 \). Then \( \dot{R}_\Sigma \) is really negative and the star shrinks. As we have mentioned before, the condition for a black hole formation is \( H_{\Sigma 0} = 0 \). Eq (64) leads to
\[ \dot{R} = \frac{2 R_1}{3 \sqrt{R}}, \quad \ddot{R} = -\frac{2 R_1^2}{9 R^2}. \] (68)

Then Eq (26) gives
\[ \frac{1 + \gamma_{\Sigma 0}}{1 - \gamma_{\Sigma 0}} = c, \quad c = \left( -\frac{2 R_1}{3 R_2^{1/3}} \right)_\Sigma > 0, \] (69)
which is a linear equation for \( \gamma_{\Sigma 0} \) with solution
\[ \gamma_{\Sigma 0} = -f(r_\Sigma) \exp \left( -\frac{2}{c} \right) = \frac{c - 1}{c + 1}. \] (70)
This defines \( f(r_\Sigma) \) as long as the constant \( c > 1 \). At \( t = -\infty \) we have \( R = \infty, \ Z = 1, \ H = 1 \), while at \( t = 0 \) we find \( R_0 = R_2^{2/3}, \ Z_{\Sigma 0} = 1/c, \ H_{\Sigma 0} = 0 \). In fact, the collapse of the star may start at some finite negative time from a static model, which develops for some reasons non-trivial heat flow.

Is it possible that the end state is not a black hole but Minkowski spacetime, because all the mass has been burnt out and we get \( m_{\Sigma} = 0 \)? This situation occurs in the non-geodesic
collapse with shear when $R_\Sigma = 0$ (Pinheiro and Chan 2011). Eq (35) shows that in the geodesic case this may happen when $R_\Sigma = 0$ or $\dot{H}_\Sigma = 0$. Differentiating Eq (26) and using Eq (24) we obtain

$$\dot{H} = \frac{1}{2R} \frac{1 - Z^2}{Z^2} + \ddot{R}. \quad (71)$$

Inserting this in the mass formula and using Eq (68) yields

$$m = R \frac{Z^2 - 1}{Z^2} + \frac{2}{9} R_1^2. \quad (72)$$

When $t = -(R_2/R_1)_\Sigma$, which is positive, $R_\Sigma = 0$. Then Eqs (66, 67) show that $Z_\Sigma$ becomes a finite constant, while Eq (68) gives $\dot{R}_\Sigma = -\infty$. Thus Eq (26) gives $H_\Sigma = -\infty$, well beyond its range of $[0, 1]$. Eq (17) yields for the redshift $z_\Sigma = -1$, which is completely unrealistic. In addition, Eq (72) shows that the mass does not vanish for $R_\Sigma = 0$. Hence, the scenario of (Pinheiro and Chan 2011) is not realized here. In fact, $H_\Sigma$, being a continuous function in time, will pass through its zero before going to negative infinity and there a black hole will be formed, stopping the collapse.

Let us explore the other possibility, namely, $\dot{H}_\Sigma = 0$ when $t = 0$. When $\dot{H} = 0$, Eq (71) gives

$$Z^2 \left(1 - 2R\ddot{R}\right) = 1. \quad (73)$$

Going to the star surface and setting $t = 0$ we obtain

$$y_{\Sigma 0} = -f(r_\Sigma) e^{-\frac{2}{R}} = \frac{\sqrt{1 + c^2} - 1}{\sqrt{1 + c^2} + 1}. \quad (74)$$

which is the analogue of Eq (70) and fixes $f(r_\Sigma)$. Eq (26) gives

$$H_{\Sigma 0} = \sqrt{1 + c^2} - c \in (0, 1]. \quad (75)$$

Hence, the star burns out ($m_{\Sigma 0} = 0$) before the formation of horizon and a black hole ($H_{\Sigma 0} > 0$) and the process of collapse stops. The interior solution becomes Minkowski
spacetime with radius \( R_{\Sigma}^{2/3} \), and it is joined smoothly to the exterior solution, which is again Minkowski spacetime (\( M = 0 \)).

Eq (26) gives general expressions for the two generating functions \( H \) and \( D \):

\[
H = \frac{1}{Z} + \dot{R} \tag{76}
\]

\[
D = \frac{R}{Z} + R\dot{R} \tag{77}
\]

When \( F = 1 \), \( H \) becomes due to Eqs. (64, 66)

\[
H = \frac{1 + y}{1 - y} + \frac{2}{3} \dot{R}_1 (R_1 t + R_2)^{-1/3} \tag{78}
\]

where \( y \) is given by Eq. (67). A similar expression holds for \( D \).

When \( F = 1 + R_1^2 \), \( H \) reads

\[
H = \sqrt{R_1^2 + \frac{1 + l}{1 - l} + R_1} \tag{79}
\]

\[
l = -f(r) (R_1 t + R_2) \sqrt{R_1^2 + 1/R_1} \tag{80}
\]

where the formula for \( l \) has been taken from (Thirukkanesh and Maharaj 2010). Similar formula holds for \( D \).

Finally, there is a solution with vanishing luminosity \( \Lambda_{\Sigma} \), even though the heat flux does not vanish in the bulk (but vanishes on \( \Sigma \) so that Eq (18) holds). It is called the generalized LTB solution (Herrera et al 2010b). In it \( q \) is invoked wholly by the term \( P \). As seen from Eq (57) \( q \sim g(r) \) when \( q_0 \) (given by Eq (38)) is zero and vanishes on the surface, but not in the bulk of the star.

5. Previous solutions without heat radiation

Now let us make the passage to the case with \( q = 0 \). Then Eqs (37, 65, 76) yield

\[
\dot{Z} = 0, \quad B = R'/a(r), \quad H = a + \dot{R} \tag{81}
\]
where \(a(r)\) is some positive function of integration. Plugging this form of \(H\) into its Riccati equation (27) we get an equation for \(R\)

\[
2R\ddot{R} + \dot{R}^2 + 1 = a^2
\]

(82)

A look at Eq(5) (with \(\varepsilon = 0\)) shows that this gives the vanishing of the radial pressure \(p_r = 0\). This also follows from the main form of the junction condition, Eq (16) when \(q = 0\). It naturally gives \(F = a^2\), which also follows from Eq (24) when \(\dot{Z} = 0\). Using the expression for \(F\) from Eq (59) we transform Eq (82) into

\[
\left( R\dot{R}^2 \right)' = (a^2 - 1) \dot{R}
\]

(83)

Integration of this equation leads to

\[
\dot{R}^2 = \frac{2m}{R} + k
\]

(84)

Here \(m(r)\) is another function of integration and we have introduced \(k = a^2 - 1\). The meaning of \(m\) is seen from Eq (12). When \(q = 0\), Eq (41) shows that the mass is a function of \(r\) only. Plugging \(H\) from Eq (81) into Eq (12) results into Eq (84) with \(m\) being the mass function of the star.

Interior anisotropic geodesic solutions with shear and without radiation have been discussed in (Ivanov 2011). Eq (82) follows from Eq (19) in this reference when \(p_r = 0\). The same problem in non-comoving coordinates, but with matching was solved in (Herrera et al 2002). Eq (81) for \(B\) and Eq (84) characterize the LTB solution (Herrera et al 2010b), which has three cases, but here it arises from an anisotropic source. When the fluid is perfect \((p_r = p_t)\), both pressures vanish and the general dust solution is obtained.

Let us consider now the so called Euclidean star solutions (Herrera and Santos 2010). They satisfy the constraint \(B = R'\). Then Eq (10) becomes

\[
H - \dot{R} = 1.
\]

(85)
Now Eq (28) yields $q = 0$, hence, the geodesic Euclidean stars are non-radiative. This also follows from Eq (39) in the previous reference. They fall in the parabolic LTB class because $a = 1$ and hence $k = 0$.

The generating function of the LTB solutions is rather simple

$$D = R \left( \sqrt{1 + k + \dot{R}} \right)$$

where $R$ satisfies Eq (84).

6. Conclusions

In this paper we have integrated the main junction condition (16, 22) for geodesic anisotropic spherical collapse and expressed all possible solutions through the generating functions $D = RH$ and $P$. We have also shown that $H$ or the redshift may play the role of alternative generating functions. For this purpose we have introduced the physically important object $H$, called the horizon function. Fortunately, it satisfies a Riccati equation, simpler than the previous such equations for $B$ and $Z$. It rules the appearance of a black hole ($H = 0$) or the vanishing of the star’s mass due to radiation ($\dot{H} = 0$).

All previous exact solutions have been regained. Those obtained by the Lie symmetry method (Abebe et al 2014) are based on $D$ depending on some function of $t, r$.

The solution with linear in time physical radius $R$ leads in the shearless case to infinite collapse without black hole (Banerjee et al 2002), (Ivanov 2012), (Tewari and Charan 2015). We have shown that when shear is present it has negative heat flow and is physically unrealistic.

Another class of solutions of (Thirukkanesh and Maharaj 2010) appears to be realistic. Depending on the integration constants their collapse may lead to a black hole or Minkowski
spacetime due to the burning away of the star’s mass.

We have also regained the LTB solutions, the generalised LTB solutions (Herrera et al 2010b) and the geodesic Euclidean stars (Herrera and Santos 2010), which must be non-radiative.

Certainly, the use of generating functions will lead in the future to the discovery of other realistic star models, describing anisotropic geodesic collapse with radiation.
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