HOMOGENEOUS ALGEBRAS

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Abstract

Various concepts associated with quadratic algebras admit natural generalizations when the quadratic algebras are replaced by graded algebras which are finitely generated in degree 1 with homogeneous relations of degree $N$. Such algebras are referred to as homogeneous algebras of degree $N$. In particular it is shown that the Koszul complexes of quadratic algebras generalize as $N$-complexes for homogeneous algebras of degree $N$.

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1 Introduction and Preliminaries

Our aim is to generalize the various concepts associated with quadratic algebras as described in [27] when the quadratic algebras are replaced by the homogeneous algebras of degree \( N \) with \( N \geq 2 \) (\( N = 2 \) is the case of quadratic algebras). Since the generalization is natural and relatively straightforward, the treatment of [26], [27] and [25] will be directly adapted to homogeneous algebras of degree \( N \). In other words we dispense ourselves to give a review of the case of quadratic algebras (i.e. the case \( N = 2 \)) by referring to the above quoted nice treatments. In proceeding to this adaptation, we shall make use of the following slight elaboration of an ingredient of the elegant presentation of [25].

**Lemma 1** Let \( A \) be an associative algebra with product denoted by \( m \), let \( C \) be a coassociative coalgebra with coproduct denoted by \( \Delta \) and let \( \text{Hom}_K(C, A) \) be equipped with its structure of associative algebra for the convolution product \( (\alpha, \beta) \mapsto \alpha * \beta = m(\alpha \otimes \beta) \circ \Delta \). Then one defines an algebra-homomorphism \( \alpha \mapsto d_\alpha \) of \( \text{Hom}_K(C, A) \) into the algebra \( \text{End}_A(A \otimes C) = \text{Hom}_A(A \otimes C, A \otimes C) \) of endomorphisms of the left \( A \)-module \( A \otimes C \) by defining \( d_\alpha \) as the composite

\[
A \otimes C \xrightarrow{I_A \otimes \Delta} A \otimes C \otimes C \xrightarrow{I_A \otimes \alpha \otimes I_C} A \otimes A \otimes C \xrightarrow{m \otimes I_C} A \otimes C
\]

for \( \alpha \in \text{Hom}_K(C, A) \).

The proof is straightforward, \( d_\alpha \circ d_\beta = d_{\alpha * \beta} \) follows easily from the coassociativity of \( \Delta \) and the associativity of \( m \). As pointed out in [25] one obtains a graphical version (“electronic version”) of the proof by using the usual graphical version of the coassociativity of \( \Delta \) combined with the usual graphical version of the associativity of \( m \). The left \( A \)-linearity of \( d_\alpha \) is straightforward.
In the above statement as well as in the following, all vector spaces, algebras, coalgebras are over a fixed field $\mathbb{K}$. Furthermore unless otherwise specified the algebras are unital associative and the coalgebras are counital coassociative. For instance in the previous case, if $1_l$ is the unit of $A$ and $\varepsilon$ is the counit of $C$, then the unit of $\text{Hom}_\mathbb{K}(C, A)$ is the linear mapping $\alpha \mapsto \varepsilon(\alpha)1_l$ of $C$ into $A$. In Lemma 1 the left $A$-module structure on $A \otimes C$ is the obvious one given by

$$x(a \otimes c) = (xa) \otimes c$$

for any $x \in A$, $a \in A$ and $c \in C$.

Besides the fact that it is natural to generalize for other degrees what exists for quadratic algebras, this paper produces a very natural class of $N$-complexes which generalize the Koszul complexes of quadratic algebras [26], [27], [33], [25], [19] and which are not of simplicial type. By $N$-complexes of simplicial type we here mean $N$-complexes associated with simplicial modules and $N$-th roots of unity in a very general sense [12] which cover cases considered e.g. in [28], [20], [13], [14], [21] the generalized homology of which has been shown to be equivalent to the ordinary homology of the corresponding simplicial modules [12]. This latter type of constructions and results has been recently generalized to the case of cyclic modules [35]. In spite of the fact that they compute the ordinary homology of the simplicial modules, the usefulness of these $N$-complexes of simplicial type comes from the fact that they can be combined with other $N$-complexes [17], [18]. In fact the BRS-like construction [4] of [18] shows that spectral sequences arguments (e.g. in the form of a generalization of the homological perturbation theory [31]) are still working for $N$-complexes. Other nontrivial classes of $N$-complexes which are not of simplicial type are the universal construction of [16] and the
$N$-complexes of [14], [13] (see also in [13] for a review). It is worth noticing here that elements of homological algebra for $N$-complexes have been developed in [21] and that several results for $N$-complexes and more generally $N$-differential modules like Lemma 1 of [12] have no nontrivial counterpart for ordinary complexes and differential modules. It is also worth noticing that besides the above mentioned examples, various problems connected with theoretical physics implicitly involve exotic $N$-complexes (see e.g. [23], [24]).

In the course of the paper we shall point out the possibility of generalizing the approach based on quadratic algebras of [27] to quantum spaces and quantum groups by replacing the quadratic algebras by $N$-homogeneous ones. Indeed one also has in this framework internal end, etc. with similar properties.

Finally we shall revisit in the present context the approach of [8], [4] to Koszulity for $N$-homogeneous algebras. This is in order since as explained below, the generalization of the Koszul complexes introduced in this paper for $N$-homogeneous algebras is a canonical one. We shall explain why a definition based on the acyclicity of the $N$-complex generalizing the Koszul complex is inappropriate and we shall identify the ordinary complex introduced in [8] (the acyclicity of which is the definition of Koszulity of [8]) with a complex obtained by contraction from the above Koszul $N$-complex. Furthermore we shall show the uniqueness of this contracted complex among all other ones. Namely we shall show that the acyclicity of any other complex (distinct from the one of [8]) obtained by contraction of the Koszul $N$-complex leads for $N \geq 3$ to an uninteresting (trivial) class of algebras.
Some examples of Koszul homogeneous algebras of degree $> 2$ are given in [8], including a certain cubic Artin-Schelter regular algebra [1]. Recall that Koszul quadratic algebras arise in several topics as algebraic geometry [22], representation theory [5], quantum groups [26], [27], [33], [34], Sklyanin algebras [30], [32]. A classification of the Koszul quadratic algebras with two generators over the complex numbers is performed in [7]. Koszulity of non-quadratic algebras and each of the above items deserve further attention.

The plan of the paper is the following. In Section 2 we define the duality and the two (tensor) products which are exchanged by the duality for homogeneous algebras of degree $N$ ($N$-homogeneous algebras). These are the direct extension to arbitrary $N$ of the concepts defined for quadratic algebras ($N = 2$), [26], [27], [25] and our presentation here as well as in Section 3 follows closely the one of reference [27] for quadratic algebras.

In Section 3 we elaborate the categorical setting and we point out the conceptual reason for the occurrence of $N$-complexes in the framework of $N$-homogeneous algebras. We also sketch in this section a possible extension of the approach of [27] to quantum spaces and quantum groups in which relations of degree $N$ replace the quadratic ones.

In Section 4 we define the $N$-complexes which are the generalizations for homogeneous algebras of degree $N$ of the Koszul complexes of quadratic algebras [26], [27]. The definition of the cochain $N$-complex $L(f)$ associated with a morphism $f$ of $N$-homogeneous algebras follows immediately from the structure of the unit object $\wedge_N\{d\}$ of one of the (tensor) products of $N$-homogeneous algebras. We give three equivalent definitions of the chain $N$-complex $K(f)$: A first one by dualization of the definition of $L(f)$, a
second one which is an adaptation of [25] by using Lemma 1, and a third one which is a component-wise approach. It is pointed out in this section that one cannot generalize naively the notion of Koszulity for \(N\)-homogeneous algebras with \(N \geq 3\) by the acyclicity of the appropriate Koszul \(N\)-complexes. In Section 5, we recall the definition of Koszul homogeneous algebras of [8] as well as some results of [8], [9] which justify this definition. It is then shown that this definition of Koszulity for homogeneous \(N\)-algebras is optimal within the framework of the appropriate Koszul \(N\)-complex.

Let us give some indications on our notations. Throughout the paper the symbol \(\otimes\) denotes the tensor product over the basic field \(K\). Concerning the generalized homology of \(N\)-complexes we shall use the notation of [20] which is better adapted than other ones to the case of chain \(N\)-complexes, that is if \(E = \bigoplus_n E_n\) is a chain \(N\)-complex with \(N\)-differential \(d\), its generalized homology is denoted by \(pH(E) = \bigoplus_{n \in \mathbb{Z}} pH_n(E)\) with

\[
pH_n(E) = \text{Ker}(d^p : E_n \to E_{n-p})/\text{Im}(d^{N-p} : E_{n+N-p} \to E_n)
\]

for \(p \in \{1, \ldots, N-1\}, (n \in \mathbb{Z})\).

2 Homogeneous algebras of degree \(N\)

Let \(N\) be an integer with \(N \geq 2\). A \textit{homogeneous algebra of degree} \(N\) or \textit{\(N\)-homogeneous algebra} is an algebra of the form

\[
\mathcal{A} = A(E, R) = T(E)/(R)
\]

(1)

where \(E\) is a finite-dimensional vector space (over \(K\)), \(T(E)\) is the tensor algebra of \(E\) and \((R)\) is the two-sided ideal of \(T(E)\) generated by a linear subspace \(R\) of \(E^{\otimes N}\). The homogeneity of \((R)\) implies that \(\mathcal{A}\) is a graded algebra \(\mathcal{A} =\)
⊕_{n \in \mathbb{N}} \mathcal{A}_n \text{ with } \mathcal{A}_n = E^\otimes n \text{ for } n < N \text{ and } \mathcal{A}_n = E^\otimes n / \sum_{r+s=n-N} E^\otimes r \otimes R \otimes E^\otimes s \text{ for } n \geq N \text{ where we have set } E^\otimes 0 = \mathbb{K} \text{ as usual. Thus } \mathcal{A} \text{ is a graded algebra which is connected } (\mathcal{A}_0 = \mathbb{K}) \text{, generated in degree 1 } (\mathcal{A}_1 = E) \text{ with the ideal of relations among the elements of } \mathcal{A}_1 = E \text{ generated by } R \subset E^\otimes N = (\mathcal{A}_1)^\otimes N.

A morphism of \(N\)-homogeneous algebras \(f : A(E, R) \to A(E', R')\) is a linear mapping \(f : E \to E'\) such that \(f^\otimes N(R) \subset R'\). Such a morphism is a homomorphism of unital graded algebras. Thus one has a category \(\mathcal{H}_N\text{Alg}\) of \(N\)-homogeneous algebras and the forgetful functor \(\mathcal{H}_N\text{Alg} \to \text{Vect}, A \mapsto E\), from \(\mathcal{H}_N\text{Alg}\) to the category \(\text{Vect}\) of finite-dimensional vector spaces (over \(\mathbb{K}\)).

Let \(A = A(E, R)\) be a \(N\)-homogeneous algebra. One defines its dual \(A^!\) to be the \(N\)-homogeneous algebra \(A^! = A(E^*, R^\perp)\) where \(E^*\) is the dual vector space of \(E\) and where \(R^\perp \subset E^* \otimes N = (E^\otimes N)^*\) is the annihilator of \(R\) i.e. the subspace \(\{ \omega \in (E^\otimes N)^* | \omega(x) = 0, \ \forall x \in R\}\) of \((E^\otimes N)^*\) identified with \(E^* \otimes N\). One has canonically

\[
(\mathcal{A}^!)^! = \mathcal{A}
\]

and if \(f : A \to A' = A(E', R')\), is a morphism of \(\mathcal{H}_N\text{Alg}\), the transposed of \(f : E \to E'\) is a linear mapping of \(E'^*\) into \(E^*\) which induces the morphism \(f^! : (\mathcal{A}'^!)^! \to \mathcal{A}^!\) of \(\mathcal{H}_N\text{Alg}\) so \((A \mapsto \mathcal{A}^!, f \mapsto f^!\) is a contravariant (involutive) functor.

Let \(\mathcal{A} = A(E, R)\) and \(\mathcal{A}' = A(E', R')\) be \(N\)-homogeneous algebras; one defines \(\mathcal{A} \circ \mathcal{A}'\) and \(\mathcal{A} \bullet \mathcal{A}'\) by setting

\[
\mathcal{A} \circ \mathcal{A}' = A(E \otimes E', \pi_N(R \otimes E'^\otimes N + E^\otimes N \otimes R'))
\]

7
\[ \mathcal{A} \cdot \mathcal{A}' = A(E \otimes E', \pi_N(R \otimes R')) \]

where \( \pi_N \) is the permutation

\[
(1, 2, \ldots, 2N) \mapsto (1, N + 1, 2, N + 2, \ldots, k, N + k, \ldots, N, 2N) \quad (3)
\]

belonging to the symmetric group \( S_{2N} \) acting as usually on the factors of the tensor products. One has canonically

\[
(A \circ A')^! = A^! \cdot A'^!, \quad (A \cdot A')^! = A' \circ A'^! \quad (4)
\]

which follows from the identity \( \{ R \otimes E^{\otimes N} + E^{\otimes N} \otimes R' \}^! = R^! \otimes R'^! \). On the other hand the inclusion \( R \otimes R' \subset R \otimes E^{\otimes N} + E^{\otimes N} \otimes R' \) induces an surjective algebra-homomorphism \( p : \mathcal{A} \cdot \mathcal{A}' \to \mathcal{A} \circ \mathcal{A}' \) which is of course a morphism of \( H_N \text{Alg} \).

It is worth noticing here that in contrast with what happens for quadratic algebras if \( \mathcal{A} \) and \( \mathcal{A}' \) are homogeneous algebras of degree \( N \) with \( N \geq 3 \) then the tensor product algebra \( \mathcal{A} \otimes \mathcal{A}' \) is no more a \( N \)-homogeneous algebra. Nevertheless there still exists an injective homomorphism of unital algebra \( i : \mathcal{A} \circ \mathcal{A}' \to \mathcal{A} \otimes \mathcal{A}' \) doubling the degree which we now describe. Let \( \tilde{i} : T(E \otimes E') \to T(E) \otimes T(E') \) be the injective linear mapping which restricts as

\[
\tilde{i} = \pi^{-1}_n : (E \otimes E')^\otimes \rightarrow E^\otimes \otimes E'^\otimes
\]

on \( T^n(E \otimes E') = (E \otimes E')^\otimes \) for any \( n \in \mathbb{N} \). It is straightforward that \( \tilde{i} \) is an algebra-homomorphism which is an isomorphism onto the subalgebra \( \oplus_n E^\otimes \otimes E'^\otimes \) of \( T(E) \otimes T(E') \). The following proposition is not hard to verify.
PROPOSITION 1 Let $\mathcal{A} = A(E, R)$ and $\mathcal{A}' = A(E', R')$ be two $N$-homogeneous algebras. Then $\tilde{i}$ passes to the quotient and induces an injective homomorphism $i$ of unital algebras of $\mathcal{A} \circ \mathcal{A}'$ into $\mathcal{A} \otimes \mathcal{A}'$. The image of $i$ is the subalgebra $\oplus_n \mathcal{A}_n \otimes \mathcal{A}'_n$ of $\mathcal{A} \otimes \mathcal{A}'$.

The proof is almost the same as for quadratic algebras [27].

Remark. As pointed out in [27], any finitely related and finitely generated graded algebra (so in particular any $N$-homogeneous algebra) gives rise to a quadratic algebra. Indeed if $\mathcal{A} = \oplus_{n \geq 0} \mathcal{A}_n$ is a graded algebra, define $\mathcal{A}^{(d)}$ by setting $\mathcal{A}^{(d)} = \oplus_{n \geq 0} \mathcal{A}_{nd}$. Then it was shown in [3] that if $\mathcal{A}$ is generated by the finite-dimensional subspace $\mathcal{A}_1$ of its elements of degree 1 with the ideal of relations generated by its components of degree $\leq r$, then the same is true for $\mathcal{A}^{(d)}$ with $r$ replaced by $2 + (r - 2)/d$.

3 Categorical properties

Our aim in this section is to investigate the properties of the category $\mathbf{H}_N \mathbf{Alg}$. We follow again closely [27] replacing the quadratic algebras considered there by the $N$-homogeneous algebras.

Let $\mathcal{A} = A(E, R)$, $\mathcal{A}' = A(E', R')$ and $\mathcal{A}'' = A(E'', R'')$ be three homogeneous algebras of degree $N$. Then the isomorphisms $E \otimes E' \simeq E' \otimes E$ and $(E \otimes E') \otimes E'' \simeq E \otimes (E' \otimes E'')$ of $\mathbf{Vect}$ induce corresponding isomorphisms $\mathcal{A} \circ \mathcal{A}' \simeq \mathcal{A}' \circ \mathcal{A}$ and $(\mathcal{A} \circ \mathcal{A}') \circ \mathcal{A}'' \simeq \mathcal{A} \circ (\mathcal{A}' \circ \mathcal{A}'')$ of $N$-homogeneous algebras (i.e. of $\mathbf{H}_N \mathbf{Alg}$). Thus $\mathbf{H}_N \mathbf{Alg}$ endowed with $\circ$ is a tensor category [10] and furthermore to the 1-dimensional vector space $\mathbb{K}t \in \mathbf{Vect}$ which is a unit object of $(\mathbf{Vect}, \otimes)$ corresponds the polynomial algebra $\mathbb{K}[t] = A(\mathbb{K}t, 0) \simeq T(\mathbb{K})$ as unit object of $(\mathbf{H}_N \mathbf{Alg}, \circ)$. In fact the
isomorphisms $\mathbb{K}[t] \circ A \simeq A \simeq A \circ \mathbb{K}[t]$ are obvious in $\mathbf{H}_N \mathbf{Alg}$. Thus one has Part (i) of the following theorem.

**THEOREM 1** The category $\mathbf{H}_N \mathbf{Alg}$ of $N$-homogeneous algebras has the following properties (i) and (ii)

(i) $\mathbf{H}_N \mathbf{Alg}$ endowed with $\circ$ is a tensor category with unit object $\mathbb{K}[t]$.

(ii) $\mathbf{H}_N \mathbf{Alg}$ endowed with $\bullet$ is a tensor category with unit object $\wedge_N \{d\} = \mathbb{K}[t]^!$.

Part (ii) follows from (i) by the duality $A \mapsto A^!$. In fact (i) and (ii) are equivalent in view of (2) and (4).

The $N$-homogeneous algebra $\wedge_N \{d\} = \mathbb{K}[t]^! \simeq T(\mathbb{K})/\mathbb{K}^N$ is the (unital) graded algebra generated in degree one by $d$ with relation $d^N = 0$. Part (ii) of Theorem 1 is the very reason for the appearance of $N$-complexes in the present context, remembering the obvious fact that graded $\wedge_N \{d\}$-module and $N$-complexe are the same thing.

**THEOREM 2** The functorial isomorphism in $\mathbf{Vect}$

$$\text{Hom}_\mathbb{K}(E \otimes E', E'') = \text{Hom}_\mathbb{K}(E, E'^* \otimes E'')$$

induces a corresponding functorial isomorphism

$$\text{Hom}(A \bullet B, C) = \text{Hom}(A, B^! \circ C)$$

in $\mathbf{H}_N \mathbf{Alg}$, (setting $A = A(E, R)$, $B = A(E', R')$ and $C = A(E'', R'')$).

Again the proof is the same as for quadratic algebras [27]. It follows that the tensor category $(\mathbf{H}_N \mathbf{Alg}, \bullet)$ has an internal $\text{Hom}$ [10] given by

$$\text{Hom}(B, C) = B^! \circ C$$

(5)
for two \( N \)-homogeneous algebras \( B \) and \( C \). Setting \( A = A(E, R) \), \( B = A(E', R') \) and \( C = A(E'', R'') \) one verifies that the canonical linear mappings \((E^* \otimes E') \otimes E \to E'\) and \((E^* \otimes E'') \otimes (E^* \otimes E') \to E^* \otimes E''\) induce products

\[
\mu : \text{Hom}(A, B) \otimes A \to B \quad (6)
\]

\[
m : \text{Hom}(B, C) \otimes \text{Hom}(A, B) \to \text{Hom}(A, C) \quad (7)
\]

these internal products as well as their associativity properties follow more generally from the formalism of tensor categories \([10]\).

Following \([27]\), define \( \text{hom}(A, B) = \text{Hom}(A^!, B^! \setminus A^! \bullet B \). Then one obtains by duality from \((8)\) and \((7)\) morphisms

\[
\delta : B \to \text{hom}(A, B) \circ A \quad (8)
\]

\[
\Delta : \text{hom}(A, C) \to \text{hom}(B, C) \circ \text{hom}(A, B) \quad (9)
\]
satisfying the corresponding coassociativity properties from which one obtains by composition with the corresponding homomorphisms \( i \) the algebra homomorphisms

\[
\delta : B \to \text{hom}(A, B) \otimes A \quad (10)
\]

\[
\Delta : \text{hom}(A, C) \to \text{hom}(B, C) \otimes \text{hom}(A, B) \quad (11)
\]

**THEOREM 3** Let \( A = A(E, R) \) be a \( N \)-homogeneous algebra. Then the \((N\text{-homogeneous})\) algebra \( \text{end}(A) = A^! \bullet A = \text{hom}(A, A) \) endowed with the coproduct \( \Delta \) becomes a bialgebra with counit \( \varepsilon : A^! \bullet A \to \mathbb{K} \) induced by the duality \( \varepsilon = \langle \cdot, \cdot \rangle : E^* \otimes E \to \mathbb{K} \) and \( \delta \) defines on \( A \) a structure of left \( \text{end}(A) \)-comodule.
4 The $N$-complexes $L(f)$ and $K(f)$

Let us apply Theorem 2 with $A = \wedge_N \{d\}$ and use Theorem 1 $(ii)$. One has

$$\text{Hom}(B, C) = \text{Hom}(\wedge_N \{d\}, B^i \circ C)$$

and we denote by $\xi_f \in B^i \circ C$ the image of $d$ corresponding to the morphism $f \in \text{Hom}(B, C)$. One has $(\xi_f)^N = 0$ and by using the injective algebra-homomorphism $i : B^i \circ C \to B^i \otimes C$ of Proposition 1 we let $d$ be the left multiplication by $i(\xi_f)$ in $B^i \otimes C$. One has $d^N = 0$ so, equipped with the appropriate graduation, $(B^i \otimes C, d)$ is a $N$-complex which will be denoted by $L(f)$. In the case where $A = B = C$ and where $f$ is the identity mapping $I_A$ of $A$ onto itself, this $N$-complex will be denoted by $L(A)$. These $N$-complexes are the generalizations of the Koszul complexes denoted by the same symbols for quadratic algebras and morphisms [27]. Note that $(B^i \otimes C, d)$ is a cochain $N$-complex of right $C$-modules, i.e. $d : B^i_n \otimes C \to B^i_{n+1} \otimes C$ is $C$-linear.

Similarly the Koszul complexes $K(f)$ associated with morphisms $f$ of quadratic algebras generalize as $N$-complexes for morphisms of $N$-homogeneous algebras. Let $B = A(E, R)$ and $C = A(E', R')$ be two $N$-homogeneous algebras and let $f : B \to C$ be a morphism of $N$-homogeneous algebras ($f \in \text{Hom}(B, C)$). One can define the $N$-complex $K(f) = (C \otimes B^i, d)$ by using partial dualization of the $N$-complex $L(f)$ generalizing thereby the construction of [26] or one can define $K(f)$ by generalizing the construction of [27], [25].

The first way consists in applying the functor $\text{Hom}_C(\_ , C)$ to each right $C$-module of the $N$-complex $(B^i \otimes C, d)$. We get a chain $N$-complex of left $C$-modules. Since $B^i_n$ is a finite-dimensional vector space, $\text{Hom}_C(B^i_n \otimes C, C)$ is canonically identified to the left module $C \otimes (B^i_n)^*$. Then we get the $N$-
complex \( K(f) \) whose differential \( d \) is easily described in terms of \( f \). In the case \( A = B = C \) and \( f = I_A \), this complex will be denoted by \( K(A) \).

We shall follow hereafter the second more explicit way. Let us associate with \( f \in \text{Hom}(B, C) \) the homogeneous linear mapping of degree zero \( \alpha : (B^i)^* \to C \) defined by setting \( \alpha = f : E \to E' \) in degree 1 and \( \alpha = 0 \) in degrees different from 1. The dual \((B^i)^*\) of \( B^i \) defined degree by degree is a graded coassociative counital coalgebra and one has \( \alpha^{*N} = \alpha \cdots \alpha = 0 \). Indeed it follows from the definition that \( \alpha^{*N} \) is trivial in degrees \( n \neq N \). On the other hand in degree \( N \), \( \alpha^{*N} \) is the composition

\[
R \xrightarrow{f^{\otimes N}} E^{*\otimes N} \to E'^{\otimes N} / R'
\]

which vanishes since \( f^{\otimes N}(R) \subset R' \). Applying Lemma 1 it is easily checked that the \( N \)-differential

\[
d_\alpha : C \otimes B^{i*} \to C \otimes B^{i*}
\]

coincides with \( d \) of the first way.

Let us give an even more explicit description of \( K(f) \) and pay some attention to the degrees. Recall that by \((B^i)^*\) we just mean here the direct sum \( \oplus_n (B^i_n)^* \) of the dual spaces \((B^i_n)^*\) of the finite-dimensional vector spaces \( B^i_n \). On the other hand, with \( B = A(E, R) \) as above, one has

\[
B^i_n = E^{*\otimes n} \text{ if } n < N
\]

and

\[
B^i_n = E^{*\otimes n} / \sum_{r+s=n-N} E^{*\otimes r} \otimes R^\perp \otimes E^{*\otimes s} \text{ if } n \geq N.
\]
So one has for the dual spaces

\[(B_n^!)^* \cong E^{\otimes n} \text{ if } n < N \tag{13}\]

and

\[(B_n^!)^* \cong \bigcap_{r+s=n-N} E^{\otimes r} \otimes R \otimes E^{\otimes s} \text{ if } n \geq N. \tag{14}\]

In view of (13) and (14), one has canonical injections

\[(B_n^!)^* \hookrightarrow (B_k^!)^* \otimes (B_\ell^!)^* \]

for \(k + \ell = n\) and one sees that the coproduct \(\Delta\) of \((B^!)^*\) is given by

\[\Delta(x) = \sum_{k+\ell=n} x_{k\ell}\]

for \(x \in (B_n^!)^*\) where the \(x_{k\ell}\) are the images of \(x\) into \((B_k^!)^* \otimes (B_\ell^!)^*\) under the above canonical injections.

If \(f : B \to C = A(E', R')\) is a morphism of \(H_N\textbf{Alg}\), one verifies that the \(N\)-differential \(d\) of \(K(f)\) defined above is induced by the linear mappings

\[c \otimes (e_1 \otimes e_2 \otimes \cdots \otimes e_n) \mapsto cf(e_1) \otimes (e_2 \otimes \cdots \otimes e_n) \tag{15}\]

of \(C \otimes E^{\otimes n}\) into \(C \otimes E^{\otimes n-1}\). One has \(d \left( C_s \otimes (B_r^!)^* \right) \subset C_{s+1} \otimes (B_{r-1}^!)^*\) so the \(N\)-complex \(K(f)\) splits into subcomplexes

\[K(f)^n = \oplus_m C_{n-m} \otimes (B_m^!)^*, \quad n \in \mathbb{N}\]

which are homogeneous for the total degree. Using (13), (14), (15) one can describe \(K(f)^0\) as

\[\cdots \to 0 \to \mathbb{K} \to 0 \to \cdots \tag{16}\]
and $K(f)^n$ as

$$
\cdots \to 0 \to E_0 \xrightarrow{f \otimes I_{E_0}^{N-1}} E' \otimes E_0^{N-1} \to \cdots \tilde{\otimes} f \xrightarrow{\otimes f} E' \otimes E_0 \to 0 \to \cdots (17)
$$

for $1 \leq n \leq N - 1$ while $K(f)^N$ reads

$$
\cdots \to R \xrightarrow{f \otimes I_{E'}^{N-1}} E' \otimes E' \otimes E_0^{N-1} \to \cdots \tilde{\otimes} E' \otimes \tilde{\otimes} E_0 \to \mathcal{C}_N \to 0 \cdots (18)
$$

where $can$ is the composition of $I_{E'}^{N-1} \otimes f$ with canonical projection of $E' \otimes E_0$ onto $E' \otimes E_0 / R' = \mathcal{C}_N$.

Let us seek for conditions of maximal acyclicity for the $N$-complex $K(f)$. Firstly, it is clear that $K(f)^0$ is not acyclic, one has $\rho H_0(K(f)^0) = \mathbb{K}$ for $p \in \{1, \ldots, N - 1\}$. Secondly if $N \geq 3$, it is straightforward that if $n \in \{1, \ldots, N - 2\}$ then $K(f)^n$ is acyclic if and only if $E = E' = 0$. Next comes the following lemma.

**LEMMA 2** The $N$-complexes $K(f)^{N-1}$ and $K(f)^N$ are acyclic if and only if $f$ is an isomorphism of $N$-homogeneous algebras.

**Proof.** First $K(f)^{N-1}$ is acyclic if and only if $f$ induces an isomorphism $f : E \cong E'$ of vector spaces as easily verified and then, the acyclicity of $K(f)^N$ is equivalent to $f^{\otimes N}(R) = R'$ which means that $f$ is an isomorphism of $N$-homogeneous algebras. $\square$

It is worth noticing here that for $N \geq 3$ the nonacyclicity of the $K(f)^n$ for $n \in \{1, \ldots, N - 2\}$ whenever $E$ or $E'$ is nontrivial is easy to understand and to possibly cure. Let us assume that $K(f)^{N-1}$ and $K(f)^N$ are acyclic. Then by identifying through the isomorphism $f$ the two $N$-homogeneous algebras, one can assume that $\mathcal{B} = \mathcal{C} = \mathcal{A} = A(E, R)$ and that $f$ is the identity.
mapping \( L_A \) of \( A \) onto itself, that is with the previous notation that one is dealing with \( K(f) = K(A) \). Trying to make \( K(A) \) as acyclic as possible one is now faced to the following result for \( N \geq 3 \).

**Proposition 2** Assume that \( N \geq 3 \), then one has

\[
\text{Ker}(d^{N-1} : A_2 \otimes (A^{1}_{N-1})^* \to A_{N+1}) = \text{Im}(d : A_1 \otimes (A^1_N)^* \to A_2 \otimes (A^1_{N-1})^*)
\]

if and only if either \( R = E \otimes^N \) or \( R = 0 \).

**Proof.** One has

\[
A_2 \otimes (A^1_{N-1})^* = E \otimes^2 \otimes E \otimes^{N-1} \simeq E \otimes^{N+1}, A_{N+1} \simeq E \otimes^{N+1} / E \otimes R + R \otimes E
\]

and \( d^{N-1} \) identifies here with the canonical projection

\[
E \otimes^{N+1} \to E \otimes^{N+1} / E \otimes R + R \otimes E
\]

so its kernel is \( E \otimes R + R \otimes E \). On the other hand one has \( A_1 \otimes (A^1_N)^* = E \otimes R \) and \( d : E \otimes R \to E \otimes^{N+1} \) is the inclusion. So \( \text{Im}(d) = \text{Ker}(d^{N-1}) \) is here equivalent to \( R \otimes E = E \otimes R + R \otimes E \) and thus to \( R \otimes E = E \otimes R \) since all vector spaces are finite-dimensional. It turns out that this holds if and only if either \( R = E \otimes^N \) or \( R = 0 \) (see the appendix). \( \square \)

**Corollary 1** Assume that \( N \geq 3 \) and let \( A = A(E, R) \) be a \( N \)-homogeneous algebra. Then the \( K(A)^n \) are acyclic for \( n \geq N - 1 \) if and only if either \( R = 0 \) or \( R = E \otimes^N \).

**Proof.** In view of Proposition 2, \( R = 0 \) or \( R = E \otimes^N \) is necessary for the acyclicity of \( K(A)^{N+1} \); on the other hand if \( R = 0 \) or \( R = E \otimes^N \) then the acyclicity of the \( K(A)^n \) for \( n \geq N - 1 \) is obvious. \( \square \)
Notice that $R = 0$ means that $\mathcal{A}$ is the tensor algebra $T(E)$ whereas $R = E^\otimes N$ means that $\mathcal{A} = T(E^*)^!$. Thus the acyclicity of the $K(\mathcal{A})^n$ for $n \geq N - 1$ is stable by the duality $\mathcal{A} \mapsto \mathcal{A}^!$ as for quadratic algebras ($N = 2$). However for $N \geq 3$ this condition does not lead to an interesting class of algebras contrary to what happens for $N = 2$ where it characterizes the Koszul algebras \cite{29}. This is the very reason why another generalization of Koszulity has been introduced and studied in \cite{8} for $N$-homogeneous algebras.

5 Koszul homogeneous algebras

Let us examine more closely the $N$-complex $K(\mathcal{A})$: $$\cdots \rightarrow \mathcal{A} \otimes (\mathcal{A}_1^*)^* \xrightarrow{d} \mathcal{A} \otimes (\mathcal{A}_{i-1}^*)^* \rightarrow \cdots \rightarrow \mathcal{A} \otimes (\mathcal{A}_1^*)^* \xrightarrow{d} \mathcal{A} \rightarrow 0.$$ The $\mathcal{A}$-linear map $d : \mathcal{A} \otimes (\mathcal{A}_1^*)^* \rightarrow \mathcal{A} \otimes (\mathcal{A}_{i-1}^*)^*$ is induced by the canonical injection (see in last section)

$$(\mathcal{A}_1^*)^* \hookrightarrow (\mathcal{A}_1^*)^* \otimes (\mathcal{A}_{i-1}^*)^* = \mathcal{A}_1 \otimes (\mathcal{A}_{i-1}^*)^* \subset \mathcal{A} \otimes (\mathcal{A}_{i-1}^*)^*.$$ The degree $i$ of $K(\mathcal{A})$ as $N$-complex has not to be confused with the total degree $n$. Recall that, when $N = 2$, the quadratic algebra $\mathcal{A}$ is said to be Koszul if $K(\mathcal{A})$ is acyclic at any degree $i > 0$ (clearly it is equivalent to saying that each complex $K(\mathcal{A})^n$ is acyclic for any total degree $n > 0$).

For any $N$, it is possible to contract the $N$-complex $K(\mathcal{A})$ into $(2)$-complexes by putting together alternately $p$ or $N - p$ arrows $d$ in $K(\mathcal{A})$. The complexes so obtained are the following ones

$$\cdots \xrightarrow{d^{N-p}} \mathcal{A} \otimes (\mathcal{A}_{N+r}^*)^* \xrightarrow{d^p} \mathcal{A} \otimes (\mathcal{A}_{N-p+r}^*)^* \xrightarrow{d^{N-p}} \mathcal{A} \otimes (\mathcal{A}_r^*)^* \xrightarrow{d^p} 0,$$

which are denoted by $C_{p,r}$. All the possibilities are covered by the conditions $0 \leq r \leq N - 2$ and $r + 1 \leq p \leq N - 1$. Note that the complex $C_{p,r}$ at degree
$i$ is $\mathcal{A} \otimes (\mathcal{A}_k^1)^*$, where $k = jN + r$ or $k = (j+1)N - p + r$, according to $i = 2j$ or $i = 2j+1$ ($j \in \mathbb{N}$).

In [8], the complex $C_{N-1,0}$ is called the Koszul complex of $\mathcal{A}$, and the homogeneous algebra $\mathcal{A}$ is said to be Koszul if this complex is acyclic at any degree $i > 0$. A motivation for this definition is that Koszul property is equivalent to a purity property of the minimal projective resolution of the trivial module. One has the following result [8], [9]:

**PROPOSITION 3** Let $\mathcal{A}$ be a homogeneous algebra of degree $N$. For $i = 2j$ or $i = 2j+1$, $j \in \mathbb{N}$, the graded vector space $\text{Tor}^\mathcal{A}_i(\mathbb{K}, \mathbb{K})$ lives in degrees $\geq jN$ or $\geq jN + 1$ respectively. Moreover, $\mathcal{A}$ is Koszul if and only if each $\text{Tor}^\mathcal{A}_i(\mathbb{K}, \mathbb{K})$ is concentrated in degree $jN$ or $jN+1$ respectively (purity property).

When $N = 2$, it is exactly Priddy’s definition [29]. Another motivation is that a certain cubic Artin-Schelter regular algebra has the purity property, and this cubic algebra is a good candidate for making non-commutative algebraic geometry [1], [2]. Some other non-trivial examples are contained in [8].

The following result shows how the Koszul complex $C_{N-1,0}$ plays a particular role. Actually all the other contracted complexes of $K(\mathcal{A})$ are irrelevant as far as acyclicity is concerned.

**PROPOSITION 4** Let $\mathcal{A} = A(E, R)$ be a homogeneous algebra of degree $N \geq 3$. Assume that $(p, r)$ is distinct from $(N-1,0)$ and that $C_{p,r}$ is exact at degree $i = 1$. Then $R = 0$ or $R = E \otimes^N$. 

**Proof.** Assume $r = 0$, hence $1 \leq p \leq N-2$. Regarding $C_{p,0}$ at degree 1 and total degree $N + 1$, one gets the exact sequence

$$E \otimes R \xrightarrow{dp} E \otimes^{N+1} d^{N-p} E \otimes^{N+1} / E \otimes R + R \otimes E,$$
where the maps are the canonical ones. Thus $E \otimes R = E \otimes R + R \otimes E$, leading to $R \otimes E = E \otimes R$. This holds only if $R = 0$ or $R = E^\otimes N$ (Appendix).

Assume now $1 \leq r \leq N - 2$ (hence $r + 1 \leq p \leq N - 1$). Regarding $C_{p,r}$ at degree 1 and total degree $N + r$, one gets the exact sequence

$$(A_{N+r}^1)^* \xrightarrow{d^p} E^\otimes N + r \xrightarrow{d^{N-p}} E^\otimes N + r / R \otimes E^\otimes r,$$

where the maps are the canonical ones. Thus $(A_{N+r}^1)^* = R \otimes E^\otimes r$, and $R \otimes E^\otimes r$ is contained in $E^\otimes r \otimes R$. So $R \otimes E^\otimes r = E^\otimes r \otimes R$, which implies again $R = 0$ or $R = E^\otimes N$ (Appendix).□

It is easy to check that, if $R = 0$ or $R = E^\otimes N$, any $C_{p,r}$ is exact at any degree $i > 0$. On the other hand, for any $R$, one has

$$H_0(C_{p,r}) = \oplus_{0 \leq j \leq N - p - 1} E^\otimes j \otimes E^\otimes r,$$

which can be considered as a Koszul left $A$-module if $A$ is Koszul.

### 6 Appendix : a lemma on tensor products

**Lemma 3** Let $E$ be a finite-dimensional vector space. Let $R$ be a subspace of $E^\otimes N$, $N \geq 1$. If $R \otimes E^\otimes r = E^\otimes r \otimes R$ holds for an integer $r \geq 1$, then $R = 0$ or $R = E^\otimes N$.

**Proof.** Fix a basis $X = (x_1, \ldots, x_n)$ of $E$, ordered by $x_1 < \cdots < x_n$. The set $X^N$ of the words of length $N$ in the letters $x_1, \ldots, x_n$ is a basis of $E^\otimes N$ which is lexicographically ordered. Denote by $S$ the $X^N$-reduction operator of $E^\otimes N$ associated to $R$ [8], [9]. This means the following properties:

(i) $S$ is an endomorphism of the vector space $E^\otimes N$ such that $S^2 = S$,
(ii) for any $a \in X^N$, either $S(a) = a$ or $S(a) < a$ (the latter inequality means $S(a) = 0$, or otherwise any word occurring in the linear decomposition of $S(a)$ on $X^N$ is $< a$ for the lexicographic ordering),

(iii) $\text{Ker}(S) = R$.

Then $S \otimes I_{E^\otimes r}$ and $I_{E^\otimes r} \otimes S$ are the $X^{N+r}$-reduction operators of $E^\otimes N+r$, respectively associated to $R \otimes E^\otimes r$ and $E^\otimes r \otimes R$. By assumption these endomorphisms are equal. In particular, one has

$$\text{Im}(S) \otimes E^\otimes r = E^\otimes r \otimes \text{Im}(S).$$

But the subspace $\text{Im}(S)$ is monomial, i.e. generated by words. So it suffices to prove the lemma when $R$ is monomial.

Assume that $R$ contains the word $x_{i_1} \ldots x_{i_N}$. For any letters $x_{j_1}, \ldots, x_{j_r}$, the word $x_{i_1} \ldots x_{i_N} x_{j_1} \ldots x_{j_r}$ belongs to $E^\otimes r \otimes R$. Thus $x_{i_{r+1}} \ldots x_{i_N} x_{j_1} \ldots x_{j_r}$ belongs to $R$. Continuing the process, we see that any word belongs to $R$. □
References

[1] M. Artin, W. F. Schelter. Graded algebras of global dimension 3. *Adv. Math.* **66** (1987) 171-216.

[2] M. Artin, J. Tate, M. Van den Bergh. Some algebras associated to automorphisms of elliptic curves. The Grothendieck Festschrift (P. Cartier et al. eds), Vol. I. Birkhäuser 1990.

[3] J. Backelin, R. Fröberg. Koszul algebras, Veronese subrings and rings with linear resolutions. *Revue Roumaine de Math. Pures et App.* **30** (1985) 85-97.

[4] C. Becchi, A. Rouet, R. Stora. Renormalization models with broken symmetries. in “Renormalization Theory”, Erice 1975, G. Velo, A.S. Wightman Eds, Reidel 1976.

[5] A.A. Beilinson, V. Ginzburg, W. Soergel. Koszul duality patterns in representation theory, *J. Am. Math. Soc.* **9** (1996) 473-527.

[6] R. Berger. Confluence and Koszulity. *J. Algebra* **201** (1998) 243-283.

[7] R. Berger. Weakly confluent quadratic algebras. *Algebras and Representation Theory* **1** (1998) 189-213.

[8] R. Berger. Koszulity for nonquadratic algebras. *J. Algebra* **239** (2001) 705-734.

[9] R. Berger. La catégorie graduée. preprint, 2002.

[10] P. Deligne, J. Milne. Tannakian categories. Lecture Notes in Math. **900**, p.101-228. Springer-Verlag 1982.
[11] M. Dubois-Violette. Generalized differential spaces with $d^N = 0$ and the $q$-differential calculus. *Czech J. Phys.* **46** (1997) 1227-1233.

[12] M. Dubois-Violette. $d^N = 0$: Generalized homology. *K-Theory* **14** (1998) 371-404.

[13] M. Dubois-Violette. Lectures on differentials, generalized differentials and some examples related to theoretical physics. [math.QA/0005256](https://arxiv.org/abs/math.QA/0005256).

[14] M. Dubois-Violette, M. Henneaux. Generalized cohomology for irreducible tensor fields of mixed Young symmetry type. *Lett. Math. Phys.* **49** (1999) 245-252.

[15] M. Dubois-Violette, M. Henneaux. Tensor fields of mixed Young symmetry type and $N$-complexes. [math.QA/0110088](https://arxiv.org/abs/math.QA/0110088).

[16] M. Dubois-Violette, R. Kerner. Universal $q$-differential calculus and $q$-analog of homological algebra. *Acta Math. Univ. Comenian.* **65** (1996) 175-188.

[17] M. Dubois-Violette, I.T. Todorov. Generalized cohomology and the physical subspace of the $SU(2)$ WZNW model. *Lett. Math. Phys.* **42** (1997) 183-192.

[18] M. Dubois-Violette, I.T. Todorov. Generalized homology for the zero mode of the $SU(2)$ WZNW model. *Lett. Math. Phys.* **48** (1999) 323-338.

[19] A. Guichardet. Complexes de Koszul. (Version préliminaire 2001).

[20] M.M. Kapranov. On the $q$-analog of homological algebra. Preprint Cornell University 1991; [q-alg/9611005](https://arxiv.org/abs/q-alg/9611005).
[21] C. Kassel, M. Wambst. Algèbre homologique des $N$-complexes et homologies de Hochschild aux racines de l’unité. *Publ. RIMS, Kyoto Univ.* **34** (1998) 91-114.

[22] G.R. Kempf. Some wonderful rings in algebraic geometry. *J. Algebra* **134** (1990) 222-224.

[23] R. Kerner. $\mathbb{Z}_3$-graded algebras and the cubic root of the Dirac operator. *J. Math. Phys.* **33** (1992) 403-411.

[24] R. Kerner. The cubic chessboard. *Class. Quantum Grav.* **14 (1A)** (1997) A203-A225.

[25] J.L. Loday. Notes on Koszul duality for associative algebras. Homepage [http://www-irma.u-strasbg.fr/~loday/](http://www-irma.u-strasbg.fr/~loday/).

[26] Yu.I. Manin. Some remarks on Koszul algebras and quantum groups. *Ann. Inst. Fourier, Grenoble* **37** (1987) 191-205.

[27] Yu.I. Manin. Quantum groups and non-commutative geometry. CRM Université de Montréal 1988.

[28] M. Mayer. A new homology theory I, II. *Ann. of Math.* **43** (1942) 370-380 and 594-605.

[29] S.B. Priddy. Koszul resolutions. *Trans. Amer. Math. Soc.* **152** (1970) 39-60.

[30] S.P. Smith, J.T. Stafford. Regularity of the four dimensional Sklyanin algebra. *Compos. Math.* **83** (1992) 259-289.

[31] J.D. Stasheff. Constrained hamiltonians: A homological approach. *Suppl. Rendiconti del Circ. Mat. di Palermo* (1987) 239-252.
[32] J. Tate, M. Van den Bergh. Homological properties of Sklyanin algebras. *Invent. Math.* **124** (1996) 619-647.

[33] M. Wambst. Complexes de Koszul quantiques. *Ann. Inst. Fourier, Grenoble* **43** (1993) 1089-1156.

[34] M. Wambst. Hochschild and cyclic homology of the quantum multiparametric torus. *Journal of Pure and Applied Algebra* **114** (1997) 321-329.

[35] M. Wambst. Homologie cyclique et homologie simpliciale aux racines de l’unité. *K-Theory* **23** (2001) 377-397.