A Two-Loop Five-Gluon Helicity Amplitude in QCD

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ABSTRACT: We compute the planar part of the two-loop five gluon amplitude with all helicities positive. To perform the calculation we develop a $D$-dimensional generalized unitarity procedure allowing us to reconstruct the amplitude by cutting into products of six-dimensional trees. We find a compact form for the integrand which only requires topologies with six or more propagators. We perform cross checks of the universal infra-red structure using numerical integration techniques.

KEYWORDS: QCD, Amplitudes, Higher Orders
1 Introduction

Precision QCD looks set to play a leading role in the next phase of LHC operations and forthcoming analyses. Next-to-leading order (NLO) and in some cases next-to-next-to-leading order (NNLO) are extremely important for accurate modelling of QCD backgrounds to new physics searches as well as measurements of standard model parameters such as the strong coupling, $\alpha_s$. Predictions at this level of accuracy require complicated loop amplitude computations which have always been a bottleneck using the traditional Feynman diagram approach. The number of terms appearing in intermediate expressions grows extremely fast with additional external particles and especially when increasing the loop order. Unitarity and on-shell methods [1–3] have been developed to allow computations using only the physical degrees of freedom, reducing the complexity. The principle of the generalized unitarity method [4–8] is to reduce the loop amplitudes to the sum over multiple cuts where each cut factorizes into the product of tree-level amplitudes. The integrand reduction method of Ossola, Papadopoulos and Pittau (OPP) [9] shows how to systematically remove singularities from previously computed cuts so a systematic, top-down approach can be taken. The complete method yields a completely algebraic approach to the computation of one-loop amplitudes and has been automated in several numerical algorithms [10–19].

Multi-loop methods for highly symmetric theories such as $\mathcal{N} = 4$ super Yang-Mills theory are by now extremely advanced [20–24]. Similar progress in multi-leg QCD computations has however not been possible, mainly due to the much larger set of master integrals appearing in the amplitudes. State of the art computations in QCD have been completed for most $2 \to 2$ scattering processes where a Feynman diagram approach combined with integration-by-parts (IBP) identities [25] has been successful [26–32]. Unitarity based methods have also played an important role in computations of a similar level of complexity [33–38]. Recently it has also been possible to obtain the first genuine NNLO QCD corrections to $2 \to 2$ scattering after cancelling the infra-red divergences and performing the phase-space integration [39–41].

Motivated by the successes of one-loop techniques, there has been recent progress in extending generalized unitarity and integrand reduction methods for applications in multi-leg two-loop amplitudes. The IBP identities have been understood in a unitarity compatible form which has shed light on the integral basis [42]. Following this direction, the maximal unitarity method has been developed for maximal cuts of two-loop massless and massive amplitudes [43–48]. Integrand reduction methods have also been developed to the two-loop level [49, 50] via the polynomial fitting and Gram matrix constraints. Furthermore, the integrand reduction method was systematically generalized to all loop-orders using computational algebraic geometry [51, 52]. This approach has been applied to a number of two-loop [53, 54] and three-loop [55] examples. These approaches offer the benefit that they apply to arbitrary gauge theories and are not limited to super-symmetric amplitudes. Understanding the role of algebraic geometry in these methods has been particularly important and some of the more formal mathematical aspects have also been recently explored [56].
The aim of this paper is to generalize the integrand reduction method to dimensionally regulated amplitudes in a way compatible with generalized unitarity cuts. Though the planar part of the five-gluon amplitude with all positive helicities formally contributes at \( N^3_{\text{LO}} \) it is, to the best of our knowledge, the first computation of a five-point amplitude in a non-super-symmetric theory. In order to perform the multiple cuts in \( D \) dimensions it is necessary to consider tree-level amplitudes in a minimum six dimensions. We make use of the six-dimensional spinor-helicity formalism \([57]\) which has been used previously for generalized cuts at one-loop \([58, 59]\). The algebraic steps with five scale kinematics can be treated efficiently when written in terms of momentum twistors \([60]\) enabling the final result to be written in a particularly compact form which we present in eq. (4.4).

Our paper is organized as follows: In section 2 we outline the generalization of the integrand reduction methods to \( D \) dimensions. We prove that the approach is compatible with the fitting of each integrand from the product of six-dimensional tree-level amplitudes via generalized unitarity cuts. As the first non-trivial application of the integrand reduction procedure at two-loops we present the planar five-gluon amplitude with all positive helicities in section 4. We present a numerical evaluation of the amplitude in Section 5 and check the universal infra-red properties before presenting our conclusions. We include an appendix describing the explicit parametrization of the kinematics in terms of momentum twistors used to simplify the computation, and one listing the Feynman rules we use for tree-level calculations.

### 1.1 Notation

The paper will adopt a fairly conventional approach to the spinor products and Lorentz products, nevertheless we outline them here for clarity. External momenta will be denoted \( p_i^\mu \) with the usual short-hand notation for their sums and invariants,

\[
p_{ij} = p_i + p_j, \quad s_{ij} = p_{ij}^2. \tag{1.1}
\]

Spinor products are constructed from holomorphic \((\lambda_\alpha)\) and anti-holomorphic \((\tilde{\lambda}_\dot{\alpha})\) two component Weyl spinors

\[
\langle ij \rangle = \lambda_\alpha(p_i)\lambda^\alpha(p_j), \quad [ij] = \tilde{\lambda}^\dot{\alpha}(p_i)\tilde{\lambda}_\dot{\alpha}(p_j), \tag{1.2}
\]

such that \( \langle ij \rangle [ji] = s_{ij} \). We find that the amplitudes are conveniently written in terms of traces over \( \gamma \)-matrices:

\[
\text{tr}_\pm(abcd) = \frac{1}{2} \text{tr}((1 \pm \gamma_5)\slashed{p}_a\slashed{p}_b\slashed{p}_c\slashed{p}_d), \tag{1.3}
\]

where the parity odd contracted anti-symmetric tensor \( \text{tr}_5 = 4i\varepsilon_{\mu_1\mu_2\mu_3\mu_4}p_1^{\mu_1}p_2^{\mu_2}p_3^{\mu_3}p_4^{\mu_4} \) is constructed by the linear combination,

\[
\text{tr}_5 = \text{tr}(\gamma_5\slashed{p}_1\slashed{p}_2\slashed{p}_3\slashed{p}_4) = [12][23][34][41] - [12][23][34][41]. \tag{1.4}
\]
The two-loop integrals (and integrands) appearing in the paper will be written as:

\[
I^{[D]}_{n_1 n_2 n_{12}; P}[N] = \int \frac{d^D k_1 \ d^D k_2}{(2\pi)^D \ (2\pi)^D} \frac{N}{\prod_{i=1}^{n_1} (k_1 - P_i)^2 \ \prod_{j=1+n_1}^{n_1+n_2} (k_2 - P_j)^2 \ \prod_{h=1+n_1+n_2}^{n_1+n_2+n_{12}} (k_1 + k_2 - P_h)^2},
\]

where \( P \) contains any additional information necessary to specify the topology, namely the configuration of external momenta flowing along each propagator which defines the set \( \{P_i\} \). We will specify a shorthand for \( P \) on a case by case basis rather than opting for a more general notation. Topologies for which \( n_{12} = 0 \) will be referred to as butterfly-type topologies. Figure 1 gives a pictorial representation of the planar topologies considered in the rest of the paper.

**Figure 1:** A pictorial representation of the planar two-loop topology denoted \( I^{[D]}_{n_1 n_2 n_{12}; P} \)

2 Integrand reduction and generalized unitarity in \( D \) dimensions

In this section we develop a multi-loop integrand reduction procedure valid in \( D \) dimensions. The main aim is to obtain a formalism that allows the integrand form of the amplitude to be computed from the product of tree-level amplitudes by applying generalized unitarity cuts. Section 2.1 we will review integrand reduction for multi-loop amplitudes and emphasize the special features which appear when applying the method to dimensionally regulated amplitudes. The extra dimensional parts of the amplitude can be expressed using three mass-like parameters which can be effectively embedded in six dimensions, as described in section 2.2. In order to do calculations in the six-dimensional space, the six-dimensional spinor-helicity formalism developed by Cheung and O’Connell [57], which is described in section 2.3, proves itself useful. In section 2.4 we will describe how to do generalized unitarity cuts in \( D \) dimensions in the context of our specific parametrization, illustrated by a specific \( 2 \to 2 \) example. In section 2.5 we will comment on how to reproduce the \( 6D \) set-up from Feynman diagrams.
2.1 \textit{D}-dimensional integrand reduction

A generic loop diagram (with \(L\) loops and \(P\) propagators) can be written as

\[
I = \int \frac{d^D k_1}{(2\pi)^D} \cdots \frac{d^D k_L}{(2\pi)^D} \frac{N}{D_1 \cdots D_P}. \tag{2.1}
\]

The numerator function \(N\) can be a function of the loop momenta \(k_i\) only through scalar products of the form \(k_i \cdot k_j\), \(k_i \cdot p_j\), or \(k_i \cdot \omega_j\), where \(p_i\) are the momenta of the external particles, and \(\omega_i\) are vectors constructed to be perpendicular to all the \(p_i\). Some of the scalar products can be expressed in terms of the propagators \(D_i\), giving

\[
N = \Delta + \sum_{i=1}^{P} \kappa_i D_i, \tag{2.2}
\]

where \(\Delta\) can be expressed polynomially in terms of the remaining scalar products \((x_1, \ldots, x_n)\), known as irreducible scalar products (ISPs):

\[
\Delta = \sum_{i_1 \ldots i_n} c_{i_1 \ldots i_n} x_1^{i_1} \cdots x_n^{i_n}. \tag{2.3}
\]

The upper limits in the sum are determined by the division over the Gröbner basis, as explained below.

It is required that the reduction to the irreducible numerator \(\Delta\) is maximal, i.e. if \(N\) can be written as a combination of \(D_i\)'s, then \(\Delta\) must be zero. This process is known as integrand reduction, and the result will be an amplitude split up into a set of topologies, each characterized by a set of propagators and a corresponding irreducible numerator.

Explicitly, the integrand reduction is achieved by the \textit{Gröbner basis} method and \textit{synthetic polynomial division}. [51, 52] (For mathematical details, see Chapter 2 of [61].) The denominators \(D_1, \ldots, D_P\) generate an ideal,

\[
I = \langle D_1, \ldots, D_P \rangle, \tag{2.4}
\]

and we can calculate the Gröbner basis \(G(I) = \{g_1, \ldots, g_m\}\) of \(I\) using a monomial ordering. Then the polynomial division over \(G(I)\) is performed,

\[
N = \Delta + \sum_{i=1}^{P} q_i g_i, \tag{2.5}
\]

where \(\Delta\) is the integrand basis, while \(\sum_{i=1}^{k} q_i g_i \in I\) contributes to diagrams with fewer propagators. The Gröbner basis ensures that this reduction is maximal. This algebraic geometry approach works for any number of loops, and both in an integer and a dimensionally regulated number of dimensions.

In practice, if the explicit form of \(N\) is known from Feynman rules, then the division in eq. (2.5) over \(G(I)\) directly determines \(\Delta\). Alternatively, we can fit coefficients in eq. (2.3) by the \textit{generalized unitarity method}, which puts all the propagating momenta on shell,

\[
D_1 = \ldots = D_P = 0, \tag{2.6}
\]
imposing a set of constraints on the loop momenta $k_i$. The solution to the equation system (2.6) may have several branches. Mathematically eq. (2.6) defines an algebraic set, which decomposes as the union of several affine varieties. In that case the ideal $I$ decomposes as the intersection of several primary ideals $I_j$,

$$I = I_1 \cap \ldots \cap I_n,$$  

(2.7)

where each $I_j$ is a primary ideal corresponding to one branch of the solution. For each branch, the freedom remaining after imposing the unitarity cut constraints can be parametrized by a set of parameters $\tau_i$, giving

$$\Delta|_{\text{cut}} = \sum_{j_1 \ldots j_m} d_{j_1 \ldots j_m} \tau_{j_1}^1 \ldots \tau_{j_m}^m,$$  

(2.8)

where $\Delta|_{\text{cut}}$ can be found as a product of tree amplitudes (see section 2.4).

Inserting the constrained loop-momenta into eq. (2.3) allows us to set up a linear relation between the two kinds of coefficients,

$$d = Mc,$$  

(2.9)

with $c$ and $d$ being vectors of the coefficients from eq. (2.3) and (2.8) respectively. Solving eq. (2.9) for $c$ allows us to determine the irreducible numerator straight from unitarity cuts.

One subtle question is whether eq. (2.9) has a unique solution. More explicitly, is there any polynomial $f$ such that $f$ vanishes at the unitarity cut, but $f \not\in I$? In that case, the term $cf$ in the numerator contributes to the integrand basis. However, since $cf$ vanishes at the unitarity cut, the value of $c$ cannot be fixed by polynomial fitting and the solution of eq. (2.9) is not unique.

We show that this problem can be avoided if the ideal $I$ is radical. The radical of $I$ is defined as the ideal,

$$\sqrt{I} = \{ f | f^n \in I, n \in \mathbb{N} \},$$  

(2.10)

where $I$ is a subset of $\sqrt{I}$. If $I = \sqrt{I}$, then we say that $I$ is radical. Hilbert’s Nullstellensatz [62] states that $\sqrt{I}$ is the set of all polynomials vanishing on the cut, and hence if $I$ is radical, then all the coefficients of the integrand basis $\Delta$ can be extracted from unitarity cuts.

In this paper we focus on two-loop $D$-dimensional integrand reduction. Specifically we will be using the four-dimensional helicity scheme (FDH) which consists of leaving the external particles and all polarizations in four dimensions, but shifting the loop-momenta to ($D = 4 - 2\epsilon$) dimensions [63].

We will handle the $D$-dimensional loop-momenta by splitting them into four-dimensional and higher dimensional components:

$$k_i = \bar{k}_i + k_i^{[-2\epsilon]}, \quad i = 1, 2.$$  

(2.11)

By the symmetry of the higher-dimensional space, the amplitudes can depend on $k_i^{[-2\epsilon]}$ only through the three scalar products,

$$\mu_{11} = -(k_1^{[-2\epsilon]} \cdot k_1^{[-2\epsilon]}), \quad \mu_{22} = -(k_2^{[-2\epsilon]} \cdot k_2^{[-2\epsilon]}), \quad \mu_{12} = -2(k_1^{[-2\epsilon]} \cdot k_2^{[-2\epsilon]}).$$  

(2.12)

The $D$-dimensional integrand reduction has several good properties:
• The ideal $I$ is radical, so all coefficients in the integrand basis can be fixed. This can be proved as follows: At two loop order there are two types of ISPs, namely $m$ ISPs \{${k_1, \ldots, k_m}$\} of the form $k_i \cdot p_j$ or $k_i \cdot \omega_j$, and the remaining three $\mu_{11}$, $\mu_{12}$, and $\mu_{22}$. For a diagram with a $(k_1 + k_2)$ internal leg and $P$ propagators, cut equations can be rewritten as the three quadratic equations $k_1^2 = k_2^2 = (k_1 + k_2)^2 = 0$ and $P - 3$ linear equations. These linear equations determine $(P - 3)$ reducible scalar products (RSPs), which always have the form $k_i \cdot \mu_j$. So $m = 8 - (P - 3) = 11 - P$. After eliminating these RSPs, we get the ideal of cut equations:

$$I = \langle \mu_{11} - f_1(x_1, \ldots, x_m), \mu_{12} - f_2(x_1, \ldots, x_m), \mu_{22} - f_3(x_1, \ldots, x_m) \rangle. \quad (2.13)$$

We then have the following map,

$$\phi : \mathbb{C}[x_1, \ldots, x_n, \mu_{11}, \mu_{12}, \mu_{22}] / I \rightarrow \mathbb{C}[x_1, \ldots, x_n], \quad (2.14)$$

with $\mu_{11} \rightarrow f_1(x_1, \ldots, x_m)$, $\mu_{12} \rightarrow f_2(x_1, \ldots, x_m)$ and $\mu_{22} \rightarrow f_3(x_1, \ldots, x_m)$. It is clear that $\phi$ is an isomorphism, and since $\mathbb{C}[x_1, \ldots, x_n]$ is a domain, $I$ is a prime ideal. A prime ideal must be radical, which proves the proposition. Similarly, for a butterfly diagram without any $(k_1 + k_2)$ internal leg, $I = \langle \mu_{11} - f_1(x_1, \ldots, x_m), \mu_{22} - f_3(x_1, \ldots, x_m) \rangle$, $m = 10 - P$ and the proof is similar.

• Since $I$ is prime, the primary decomposition is trivial and there is only one branch of the unitarity cut.

• The unitary cut solution always has $11 - P$ degrees of freedom. For a diagram with a $(k_1 + k_2)$ internal leg and $P$ propagators, $m = 11 - P$, so by the isomorphism $\phi$,

$$\dim \mathcal{Z}(I) = \dim \mathbb{C}[x_1, \ldots, x_m, \mu_{11}, \mu_{12}, \mu_{22}] / I = \dim \mathbb{C}[x_1, \ldots, x_m] = 11 - P, \quad (2.15)$$

where $\mathcal{Z}(I)$ is the zero locus of $I$ [62]. Similarly, for a butterfly diagram with $P$ propagators, $m = 10 - P$, so

$$\dim \mathcal{Z}(I) = \dim \mathbb{C}[x_1, \ldots, x_m, \mu_{12}] = m + 1 = 11 - P. \quad (2.16)$$

This conclusion implies that a diagram and its parent diagram must have different cut solutions, since the solutions have different dimensions. In the $4D$ case, there are examples in which a diagram and its parent diagram have the same cut solutions, and it is then difficult to carry out the subtraction. However, for cuts in $D = 4 - 2\epsilon$ dimensions, this difficulty is avoided.

### 2.2 Tree-level amplitudes with the six-dimensional spinor-helicity formalism

From eq. (2.12), we saw that the set of loop-momenta gets three extra components $\mu_{11}$, $\mu_{22}$, and $\mu_{12}$ in dimensional reduction, and to embed those we need at least a six-dimensional space. A specific embedding is

$$K_1 = (k_1, m_1 \cos \theta_1, m_1 \sin \theta_1), \quad K_2 = (k_2, m_2 \cos \theta_2, m_2 \sin \theta_2), \quad (2.17)$$
with \( m_1^2 = \mu_{11} \), \( m_2^2 = \mu_{22} \), and the angles being related by

\[
\cos(\theta_2 - \theta_1) = \frac{\mu_{12}}{2m_1 m_2}.
\]

(2.18)

The fourth degree of freedom is left free, but due to the Lorentz symmetry of the extra dimensions final results will be independent of its value.

Thus we need to calculate a numerator with the loop-momenta living in \( D \geq 6 \) dimensions but with the polarizations of gluons circulating in the loops living in \( D_s = 4 \) dimensions. We will now for a moment treat \( D_s \) as a free parameter. If \( D_s > D \), a careful consideration \([7, 63]\) shows that each of the \((D_s - D)\) higher dimensional components will act as a scalar-like particle, which behaves just like the coloured scalars known from \( \mathcal{N} = 4 \) super Yang-Mills. This means that in addition to the three-point gluon-scalar-scalar-interaction, it has four-point vertices with gluons and with other scalars. All the Feynman rules for gluons and scalars are listed in appendix B.

Expressed in terms of the scalar particle, we get that for pure Yang-Mills theory

\[
\Delta^{[D_s]}_g = \Delta^{[D]}_g + (D_s - D)\Delta^{[D]}_s + (D_s - D)^2 \Delta^{[D]}_2,
\]

(2.19)

where \( \Delta_s \) and \( \Delta_{2s} \) are the contributions from diagrams with respectively one and two scalar loops. We will perform the computation using \( D = 6 \) and analytically continue eq. (2.19) to \( D_s < 6 \). We note that taking \( D_s \rightarrow 4, D \rightarrow 4 - 2\epsilon \) in the final results corresponds to the FDH scheme while \( D_s = D \rightarrow 4 - 2\epsilon \) corresponds to the ’t Hooft-Veltman scheme \([63]\).

### 2.3 The six-dimensional spinor-helicity formalism

To calculate the six-dimensional numerators, we will be dealing with six-dimensional momenta and polarizations, and a convenient way to handle those is the six-dimensional spinor-helicity formalism developed by Cheung and O’Connell \([57]\). In six dimensions, Weyl-spinors are defined as \( \Lambda^{Aa} \) and \( \tilde{\Lambda}^{\hat{A}} \hat{a} \), with \( A \) being a spinor-index running from 1 to 4, and \( a \) and \( \hat{a} \), each running from 1 to 2, being indices of the little group.

Adopting a notation where \( \Lambda \) is denoted by an angle bracket and \( \tilde{\Lambda} \) by a square bracket

\[
\Lambda^{Aa} = \langle p^a \rangle, \quad \tilde{\Lambda}_{A\hat{a}} = [p_{\hat{a}}],
\]

(2.20)

the six-dimensional spinors obey a set of relations similar to Weyl-spinors in four dimensions,

\[
\Lambda^{Aa} \Lambda^{Bb} = P_{\mu} \tilde{\Sigma}^{\mu AB}, \quad \tilde{\Lambda}_{A\hat{a}} \tilde{\Lambda}^{\hat{b}b} = P_{\mu} \Sigma^{\mu AB}, \quad \mu = 0, \ldots, 5,
\]

(2.21)

and

\[
P^{\mu} = -\frac{1}{4} \langle i^a \Sigma^\mu_{ia} \rangle = -\frac{1}{4} [i_{\hat{a}} \tilde{\Sigma}^\mu \hat{a}],
\]

(2.22)

with \( \Sigma^\mu, \tilde{\Sigma}^\mu \) being the generators of the Clifford algebra, \( \Sigma^\mu \tilde{\Sigma}^\nu + \Sigma^\nu \tilde{\Sigma}^\mu = 2g^{\mu\nu} \). Additionally one can construct spinor products \( \langle i_{a} j_{\hat{b}} \rangle \) with the property

\[
\text{det} \left( \langle i_{a} j_{\hat{b}} \rangle \right) = (P_i + P_j)^2,
\]

(2.23)
where the determinant is taken over the little group indices.

Also mirroring the four-dimensional case, we can construct a set of polarization vectors valid for massless six-momenta

$$\varepsilon^\mu_{\alpha\dot{\alpha}}(P, K) = -\frac{1}{\sqrt{2}} \frac{\langle P_a \Sigma^b K^b \rangle}{2P \cdot K} = \frac{1}{\sqrt{2}} \frac{\langle P_a K_b \rangle [K^b \Sigma^a P_a]}{2P \cdot K},$$

(2.24)

where $K$ is an arbitrarily chosen reference vector. The polarization vectors obey $P \cdot \varepsilon_{\alpha\dot{\alpha}} = K \cdot \varepsilon_{\alpha\dot{\alpha}} = 0$ and the relations

$$\varepsilon_{11} \cdot \varepsilon_{22} = -1, \quad \varepsilon_{12} \cdot \varepsilon_{21} = 1, \quad \text{other combinations} = 0.$$

(2.25)

The six-dimensional helicity sum reads,

$$\varepsilon^\mu_{\alpha\dot{\alpha}} \varepsilon^{\nu\dot{\alpha}b} = \varepsilon^\mu_{11} \varepsilon^{\nu22} + \varepsilon^\mu_{22} \varepsilon^{\nu11} - \varepsilon^\mu_{12} \varepsilon^{\nu21} - \varepsilon^\mu_{21} \varepsilon^{\nu12} = -g^{\mu\nu} + \frac{P^\mu K^\nu + K^\mu P^\nu}{P \cdot K},$$

(2.26)

The four possible combinations of little-group indices on the polarization vectors in eq. (2.24) correspond to the four polarization directions, or helicities, available in six dimensions.

### 2.4 Generalized unitarity in $D$ dimensions

![Figure 2: Conventions for the momentum flow in the four-point double box, (331), and the butterfly topology, (330).](image)

In this section we will illustrate our method by reproducing a known result, the two-loop correction to the all-plus helicity amplitude for $2 \rightarrow 2$ gluon scattering, first calculated in [33]. The parent topology for this amplitude is the double-box, which can be written as

$$I_{331}^{[D]} = \int \frac{d^D k_1}{(2\pi)^D} \int \frac{d^D k_2}{(2\pi)^D} \frac{\Delta_{331}^{[D]}}{\prod_{i=1}^7 l_i^2},$$

(2.27)

where, as shown in figure 2, the propagators are,

$$l_1 = k_1, \quad l_2 = k_1 - p_1, \quad l_3 = k_1 - p_2 - p_3, \quad l_4 = p_3 + p_4 - k_2,$$

$$l_5 = p_4 - k_2, \quad l_6 = -k_2, \quad l_7 = -k_1 - k_2.$$

(2.28)
As the topology has three propagators parametrized only by $k_1$, three propagators parametrized only by $k_2$, and one parametrized in terms of both, we denote this topology (331). The only other topology contributing to the amplitude is a four leg butterfly, which we denote (330).

Using constraints from the division over the Gröbner basis, as automated by the Mathematica package {	extsc{BasisDet}} [51], we get that the irreducible numerator $\Delta_{331}$ can be expressed as a sum of at most 160 coefficients multiplying various powers of the ISPs

\[(k_1 \cdot \omega), \ (k_2 \cdot \omega), \ (k_1 \cdot p_4), \ (k_2 \cdot p_1), \ \mu_{11}, \ \mu_{12}, \ \mu_{22}, \] (2.29)

with

\[\omega^\mu = \frac{\langle 23 | 31 | 1 \rangle \gamma^\mu [2]}{\langle 13 | 32 | 2 \rangle \gamma^\mu [1]} \] (2.30)

is an auxiliary vector perpendicular to all the external momenta. (Since all four external momenta are in 4D and linear dependent, there exists such an $\omega$ in 4D which can be explicitly constructed from the 4D spinor helicity formalism instead of the 6D formalism.) That is

\[\Delta_{331} = \sum_{i=1}^{160} c_i (k_1 \cdot \omega)^{n_{1,i}} (k_2 \cdot \omega)^{n_{2,i}} (k_1 \cdot p_4)^{n_{3,i}} (k_2 \cdot p_1)^{n_{4,i}} \mu_{11}^{n_{5,i}} \mu_{12}^{n_{6,i}} \mu_{22}^{n_{7,i}}. \] (2.31)

Such a parametrization can be found for each of the terms contributing to the regulated amplitude in eq. (2.19), $\Delta_{6}^{[6]}$, $\Delta_{8}^{[6]}$, and $\Delta_{22}^{[6]}$, and the specific expressions can be found using generalized unitarity.

Each of the three terms appearing in eq. (2.19) can be fit to the above parametrization using generalized unitarity cuts. In this case we have seven on-shell constraints,

\[\{l_i^2 = 0\}, \] (2.32)

which define a system with four degrees of freedom. As proven in section 2.1, this system has exactly one solution, which we choose to parametrize in terms of four free variables $\tau_1, \ldots, \tau_4$,

\[\bar{k}_1^\mu = p_1^\mu + \tau_1 \frac{\langle 23 | 1 \rangle \gamma^\mu [2]}{\langle 13 | 32 \rangle \gamma^\mu [1]} + \tau_2 \frac{\langle 23 | 2 \rangle \gamma^\mu [1]}{\langle 13 | 32 \rangle \gamma^\mu [1]} , \]
\[\bar{k}_2^\mu = p_4^\mu + \tau_3 \frac{\langle 41 | 3 \rangle \gamma^\mu [4]}{\langle 31 | 42 \rangle \gamma^\mu [3]} + \tau_4 \frac{\langle 41 | 4 \rangle \gamma^\mu [3]}{\langle 31 | 42 \rangle \gamma^\mu [3]} . \] (2.33)

The three extra-dimensional parameters are determined by substituting the above expressions into,

\[\mu_{11} = \bar{k}_1^2, \quad \mu_{22} = \bar{k}_2^2, \quad \mu_{12} = 2 \bar{k}_1 \cdot \bar{k}_2 . \] (2.34)

Inserting eq. (2.33) into eq. (2.31), we get the expansion

\[\Delta_{331}|_{\text{cut}} = \sum_{j=1}^{160} d_j \tau_1^{n_{1,j}} \tau_2^{n_{2,j}} \tau_3^{n_{3,j}} \tau_4^{n_{4,j}} . \] (2.35)
for each of $\Delta_g^{[6]}$, $\Delta_s^{[6]}$, and $\Delta_{2s}^{[6]}$, where the coefficients are related by

$$d = M c$$

where the matrix $M$ is square and non-singular. $\Delta_{331}\big|_{cut}$ can be found as a product of trees, as in the four-dimensional case [4, 6, 9, 50]. Whenever a cut gluon propagator appears, we need to sum over all possible values of its helicity: \{11\}, \{12\}, \{21\}, and \{22\}. However, in order to correctly reproduce the spin sum of eq. (2.26), we need to multiply by $-1$ whenever the propagating gluon has helicity \{12\} or \{21\}.

For the $\Delta_g^{[6]}$-contribution we get for instance

$$\Delta_{g,331}^{[6]}\big|_{cut} = \sum_{\{h_1,\ldots,h_7\} \in \{11,12,21,22\}} \langle\sigma_{h_1,\ldots,h_7} \rangle \langle\sigma_{h_1,\ldots,h_7} \rangle \langle\sigma_{h_1,\ldots,h_7} \rangle \langle\sigma_{h_1,\ldots,h_7} \rangle$$

$$\langle\sigma_{h_1,\ldots,h_7} \rangle \langle\sigma_{h_1,\ldots,h_7} \rangle \langle\sigma_{h_1,\ldots,h_7} \rangle \langle\sigma_{h_1,\ldots,h_7} \rangle \langle\sigma_{h_1,\ldots,h_7} \rangle \langle\sigma_{h_1,\ldots,h_7} \rangle$$

with

$$\sigma_{h_1,\ldots,h_n} = \prod_{i=1}^{n} (2\delta_{a_i d_i} - 1),$$

and with the \{11\} on the external gluons corresponding to helicity ‘plus’ in four dimensions.

$\Delta_{s,331}^{[6]}$ contains three contributions, each having one complete scalar loop as shown in fig. 3. Each term can be constructed similarly to eq. (2.37), with no subtleties arising from cut scalars. $\Delta_{2s}^{[6]}$ is zero for the (331) topology. Having extracted the values of the coefficients $d_j$ in eq. (2.35) for both contributions, we are able to solve the linear system in eq. (2.36) and compute $\Delta_{331}$ from eq. (2.19):

$$\Delta_{g,331}^{[D_s]} = \Delta_{g,331}^{[6]} + (D_s - 6)\Delta_{s,331}^{[6]}.$$

Following this procedure, one can check the known result of Ref. [33]:

$$\Delta_{331} = \frac{-is^2_{12}s_{14}F_1(D_s, \mu_{11}, \mu_{22}, \mu_{12})}{(12)(23)(34)(45)(51)}.$$
where

$$F_1(D_s, \mu_{11}, \mu_{22}, \mu_{12}) = (D_s - 2) (\mu_{11} \mu_{22} + \mu_{11} \mu_{33} + \mu_{22} \mu_{33}) + 4 (\mu_{12}^2 - 4 \mu_{11} \mu_{22}) ,$$  \hspace{1cm} (2.41)

and \( \mu_{33} = \mu_{11} + \mu_{22} + \mu_{12} \).

For the butterfly topology (330) all the above can be repeated. The solution to the on-shell constraints are identical to eqs. (2.33) and (2.34), with the exception that no constraint fixes \( \mu_{12} \), making it a fifth free parameter \( \mu_{12} = s_{12} \tau_5 \), where \( s_{12} \) has been inserted to make \( \tau_5 \) dimensionless. The ISPs are the same as for the (331)-case, though in this case the most general ISP basis has 146 terms.

When performing this sub-maximal cut we must remove the previously computed leading singularity, (331), using the OPP subtraction procedure. For \( \Delta_{g}^{[6]} \) the cut integrand is simply,

$$\Delta_{g, 330}^{[6]}_{\text{cut}} = \sum_{\{h_1, \ldots, h_6\}} \sigma_{h_1, \ldots, h_6} A(-l_1^{-h_1}, p_{1}^{(11)}, i_{2}^{h_2}) A(-l_2^{-h_2}, p_2^{(11)}, i_3^{h_3}) A(-l_3^{-h_3}, p_4^{(11)}, -l_6^{-h_6}, i_1^{h_1})$$

$$A(-l_4^{-h_4}, p_3^{(11)}, i_5^{h_5}) A(-l_5^{-h_5}, p_4^{(11)}, i_6^{h_6}) - \frac{i}{(l_6 - l_4)^2} \Delta_{g, 331}^{[6]}.$$

\( \Delta_{g, 330}^{[6]} \) will have the same three contributions as the (331)-topology, but in addition it will get a fourth contribution coming from the four-point scalar \( ss's's' \)-vertex (see appendix B), which can be interpreted as a scalar loop forming a figure eight, shown in figure 4. \( \Delta_{2s, 330}^{[6]} \) is non-zero and comes from the \( sss's's' \)-vertex, shown in figure 5.

After solving the linear system to find the integrand coefficients, the full result, via eq. 

![Figure 4: The flavour contributions to \( \Delta_{s, 330}^{[6]} \).](image)

![Figure 5: The flavour contribution to \( \Delta_{2s, 330}^{[6]} \).](image)
(2.19), can be shown to reproduce the known result [33],
\[
\Delta_{330} = -is_{12}s_{14} \frac{2(D_s - 2)(\mu_{11} + \mu_{22})\mu_{12} + (D_s - 2)^2\mu_{11}\mu_{22}((k_1 + k_2)^2 + s_{12})/s_{12}}{(12)(23)(34)(45)(51)}.
\]

(2.42)

2.5 Feynman diagram set-up for the FDH scheme

The input for the reduction can be generated from Feynman diagrams using a Feynman gauge for the internal propagators and using,

\[
g_{\mu}^{\mu} = D_s,
\]

(2.43)

where \(D_s > D = 4 - 2\epsilon\). To correctly reproduce the results of the four-dimensional helicity scheme obtained from the six-dimensional formalism via eq. (2.19), a four ghost interaction is necessary in butterfly-type topologies [63]. We find the momentum twistor parametrization outlined in appendix A particularly useful when dealing with the large intermediate expressions that arise from this approach.

3 Integrand reduction for the five gluon amplitude

In this section we summarize the generalized unitarity cuts for all topologies contributing to the all-plus helicity two-loop gluon amplitude in Yang-Mills theory. It turns out that only eight different topologies, each with six or more propagators, are necessary to write down the complete amplitude (fig. 6). In addition to those described in detail below, we have computed a selection of other four, five and six propagator cuts which evaluate to zero. These cuts have been important in determining the exact form of the tensor integrals. All the eight topologies are descended from the same maximum cut parent topology (431), (fig. 7) which has eight propagators:

\[
\begin{align*}
l_1 &= k_1, & l_2 &= k_1 - p_1, & l_3 &= k_1 - p_1 - p_2, \\
l_4 &= k_1 - p_1 - p_2 - p_3, & l_5 &= -k_2 + p_4 + p_5, & l_6 &= -k_2 + p_5, \\
l_7 &= -k_2, & l_8 &= -k_1 - k_2.
\end{align*}
\]

(3.1)
meaning that the propagators of each topology are a subset of the eight above.

Only six of the topologies are independent, as the $M_1$ and $M_2$ topologies are related by

$$
\Delta_{M_1}(k_1, k_2, p_1, p_2, p_3, p_4, p_5) = -\Delta_{M_2}(-p_{45} - k_1, p_{45} - k_2, p_3, p_2, p_1, p_5, p_4), \quad (3.2)
$$

and in the following sections we will present the on-shell cut solutions and integrand parametrizations for these six topologies, along with a list of ISPs and the allowed number of ISP monomials.

3.1 The pentagon-box: (431)

The maximum eight-fold cut can be parametrized using three free parameters $\tau_1, \tau_2$ and $\tau_3$, as

$$
\begin{align*}
\bar{k}_1^\mu &= a_1 p_1^\mu + a_2 p_2^\mu + a_3 \frac{(25)}{2[13]} (1|\gamma^\mu|2) + a_4 \frac{(25)}{2[13]} (2|\gamma^\mu|1), \\
\bar{k}_2^\mu &= b_4 p_4^\mu + b_5 p_5^\mu + b_3 \frac{(51)}{2[43]} (4|\gamma^\mu|5) + b_4 \frac{(51)}{2[43]} (5|\gamma^\mu|4),
\end{align*}
$$

where

$$
\begin{align*}
a_1 &= 1, & a_2 &= 0, & a_3 &= \tau_1, & a_4 &= 1 - \tau_1, \\
b_1 &= 0, & b_2 &= 1, & b_3 &= \tau_2, & b_4 &= \tau_3, \\
\mu_{11} &= \bar{k}_1^2, & \mu_{22} &= \bar{k}_2^2, & \mu_{12} &= 2(\bar{k}_1 \cdot \bar{k}_2).
\end{align*}
$$

The general integrand has 79 coefficients in terms of the ISPs

$$
(k_1 \cdot p_5), (k_2 \cdot p_2), (k_2 \cdot p_1), \mu_{11}, \mu_{12}, \mu_{22}, \quad (3.5)
$$

the form of which can easily be obtained using the BasisDet package [51]. We choose to prefer the monomials in $\mu_{ij}$ over the higher powers of $(k_i \cdot p_j)$ which would be preferred by the polynomial division. This is important in order to make the four dimensional limit manifest.

Figure 7: Conventions for the momentum flow in the pentagon-box parent topology, (431).
For the all-plus helicity configuration of the five gluon amplitude we find:

\[
\Delta_{431}(1^+, 2^+, 3^+, 4^+, 5^+) = \\
- i s_{12} s_{23} s_{45} F_1(D_s, \mu_{11}, \mu_{22}, \mu_{12}) \\
\frac{1}{(12)(23)(34)(45)(51) \text{tr}_5} (\text{tr}_+(1345)(k_1 + p_5)^2 + s_{15} s_{34} s_{45}),
\]

(3.6)

where

\[
F_1(D_s, \mu_{11}, \mu_{22}, \mu_{12}) = (D_s - 2) (\mu_{11} \mu_{22} + \mu_{11} \mu_{33} + \mu_{22} \mu_{33}) + 4 \left( \mu_{12}^2 - 4 \mu_{11} \mu_{22} \right),
\]

(3.7)

with \( \mu_{33} = \mu_{11} + \mu_{22} + \mu_{12} \), just like for the \( 2 \to 2 \) case in eq. (2.41).

### 3.2 The massive double-box: \((331; M_2)\)

This topology can be parametrized by

\[
\tilde{k}_1^\mu = a_1 p_1^\mu + a_2 p_2^\mu + a_3 \frac{s_{23}}{s_{12} + s_{13}} (1 | \gamma^\mu | \bar{p}_{23}) + a_4 \frac{s_{23}}{s_{12} + s_{13}} (\bar{p}_{23} | \gamma^\mu | 1),
\]

\[
\tilde{k}_2^\mu = b_1 p_4^\mu + b_2 p_5^\mu + b_3 \frac{s_{51}}{s_{23}} (4 | \gamma^\mu | 5) + b_4 \frac{s_{51}}{s_{23}} (5 | \gamma^\mu | 4),
\]

(3.8)

where

\[
a_1 = 1, \quad a_2 = 0, \quad a_3 = \tau_1, \quad a_4 = \tau_2, \]

\[
b_1 = 0, \quad b_2 = 1, \quad b_3 = \tau_3, \quad b_4 = \tau_4,
\]

\[
\mu_{11} = \tilde{k}_1^2, \quad \mu_{22} = \tilde{k}_2^2, \quad \mu_{12} = 2 (\tilde{k}_1 \cdot \tilde{k}_2),
\]

(3.9)

and

\[
p_{23}^\mu = p_{2}^\mu + p_{3}^\mu + \frac{s_{23}}{s_{12} + s_{13}} p_{1}^\mu.
\]

(3.10)

The general integrand has 160 coefficients in terms of the ISPs

\[
(k_1 \cdot \omega), \ (k_2 \cdot \omega), \ (k_1 \cdot p_5), \ (k_2 \cdot p_1), \ \mu_{11}, \ \mu_{12}, \ \mu_{22},
\]

(3.11)

with \( \omega^\mu \) being a vector which can be constructed to be perpendicular to \( p_1, p_4, p_5, \) and \( p_{23} \), the specific form of which is not important.

We find the result

\[
\Delta_{331; M_2}(1^+, 2^+, 3^+, 4^+, 5^+) = \\
- i s_{15} s_{23}^2 \text{tr}_- (1234) F_1(D_s, \mu_{11}, \mu_{22}, \mu_{12}) \\
\frac{1}{(12)(23)(34)(45)(51) \text{tr}_5}.
\]

(3.12)

### 3.3 The five-legged double-box: \((331; 5L)\)

This topology can be parametrized by

\[
\tilde{k}_1^\mu = a_1 p_1^\mu + a_2 p_2^\mu + a_3 \frac{23}{213} (1 | \gamma^\mu | 2) + a_4 \frac{23}{213} (2 | \gamma^\mu | 1),
\]

\[
\tilde{k}_2^\mu = b_1 p_4^\mu + b_2 p_5^\mu + b_3 \frac{51}{214} (4 | \gamma^\mu | 5) + b_4 \frac{51}{214} (5 | \gamma^\mu | 4),
\]

(3.13)
This topology can be parametrized by

\[
\begin{align*}
  a_1 & = 1, \quad a_2 = 0, \quad a_3 = \tau_1, \quad a_4 = \tau_2, \\
  b_1 & = 0, \quad b_2 = 1, \quad b_3 = \tau_3, \quad b_4 = \tau_4, \\
  \mu_{11} & = \tilde{k}_1^2, \quad \mu_{22} = \tilde{k}_2^2, \quad \mu_{12} = 2(\tilde{k}_1 \cdot \tilde{k}_2).
\end{align*}
\]  

(3.14)

The general integrand has 160 coefficients in terms of the ISPs

\[
(k_1 \cdot p_5), \quad (k_1 \cdot p_4), \quad (k_2 \cdot p_2), \quad (k_2 \cdot p_1), \quad \mu_{11}, \quad \mu_{12}, \quad \mu_{22},
\]

(3.15)

and we find the result

\[
\Delta_{331;5L}(1^+, 2^+, 3^+, 4^+, 5^+) = \frac{i s_{12}s_{23}s_{34}s_{45}s_{15} F_1(D_s, \mu_{11}, \mu_{22}, \mu_{12})}{(12)\langle 23 \rangle\langle 34 \rangle\langle 45 \rangle\langle 51 \rangle \text{tr}_5}.
\]  

(3.16)

3.4 The box-triangle butterfly: (430)

This topology can be parametrized by

\[
\begin{align*}
  \tilde{k}_1^1 & = a_1 p_1^\mu + a_2 p_2^\mu + a_3 \frac{23}{21} \langle 1 | \gamma^\mu | 2 \rangle + a_4 \frac{23}{21} \langle 2 | \gamma^\mu | 1 \rangle, \\
  \tilde{k}_2^1 & = b_1 p_4^\mu + b_2 p_5^\mu + b_3 \frac{51}{32} \langle 4 | \gamma^\mu | 5 \rangle + b_4 \frac{51}{32} \langle 5 | \gamma^\mu | 4 \rangle,
\end{align*}
\]  

(3.17)

where

\[
\begin{align*}
  a_1 & = 1, \quad a_2 = 0, \quad a_3 = \tau_1, \quad a_4 = 1 - \tau_1, \\
  b_1 & = 0, \quad b_2 = 1, \quad b_3 = \tau_2, \quad b_4 = \tau_3, \\
  \mu_{11} & = \tilde{k}_1^1, \quad \mu_{22} = \tilde{k}_2^1, \quad \mu_{12} = s_{4574}.
\end{align*}
\]  

(3.18)

The general integrand has 85 coefficients in terms of the ISPs

\[
(k_1 \cdot \omega_{123}), \quad (k_2 \cdot \omega_{45+}), \quad (k_2 \cdot \omega_{45-}), \quad \mu_{11}, \quad \mu_{12}, \quad \mu_{22},
\]

(3.19)

with

\[
\omega_{123}^\mu = \frac{(23) \langle 31 \rangle}{s_{12}} \frac{1 | \gamma^\mu | 2}{2} - \frac{(13) \langle 32 \rangle}{s_{12}} \frac{1 | \gamma^\mu | 2}{2},
\]  

(3.20)

defined as in the 2 \to 2 case in eq. (2.30), to be perpendicular to \(p_1, p_2, \) and \(p_3, \) \(\omega_{45-}\) and \(\omega_{45+}\) are defined to be perpendicular to \(p_4, p_5,\) and to each other. The final expression does, however, turn out to simplify a lot by expressing the part of the result proportional to \((D_s - 2)^2\) in terms of

\[
(k_1 + k_2)^2 = k_1^2 + k_2^2 + 2 \tilde{k}_1 \cdot \tilde{k}_2 - \mu_{12},
\]  

(3.21)

rather than \(\mu_{12},\) a simplification which also takes place for the other butterfly-type topologies.

The result is

\[
\Delta_{430}(1^+, 2^+, 3^+, 4^+, 5^+) = - \frac{i s_{12} \text{tr}_+(1345)}{2(12)\langle 23 \rangle\langle 34 \rangle\langle 45 \rangle\langle 51 \rangle s_{13}} (2(k_1 \cdot \omega_{123}) + s_{23}) \times \\
\left( 2(D_s - 2)(\mu_{11} + \mu_{22})\mu_{12} + (D_s - 2)^2 \mu_{11} \mu_{22} \frac{(k_1 + k_2)^2 + s_{45}}{s_{45}} \right).
\]  

(3.22)
3.5 The massive double-triangle butterfly: \((330; M_2)\)

This topology can be parametrized by

\[
\begin{align*}
\vec{k}_1^\mu &= a_1 p_1^\mu + a_2 p_2^\mu + a_3 p_23^\mu + a_4 p_23^\mu \frac{1}{2}[\gamma^\mu |p_23|], \\
\vec{k}_2^\mu &= b_1 p_1^\mu + b_2 p_5^\mu + b_3 p_23^\mu \frac{1}{2}[\gamma^\mu |p_23|] + b_4 p_23^\mu \frac{1}{2}[\gamma^\mu |p_23|],
\end{align*}
\]

where

\[
\begin{align*}
a_1 &= 1, & a_2 &= 0, & a_3 &= \tau_1, & a_4 &= \tau_2, \\
b_1 &= 0, & b_2 &= 1, & b_3 &= \tau_3, & b_4 &= \tau_4, \\
\mu_{11} &= \vec{k}_1^2, & \mu_{22} &= \vec{k}_2^2, & \mu_{12} &= s_{45} \tau_5.
\end{align*}
\]

The general integrand has 146 coefficients in terms of the ISPs

\[(k_1 \cdot \omega_{1b}^-), (k_2 \cdot \omega_{45}^-), (k_1 \cdot \omega_{1b}^+), (k_2 \cdot \omega_{45}^+), \mu_{11}, \mu_{12}, \mu_{22},\]

where \(\omega_{1b}^-\) and \(\omega_{1b}^+\) are defined to be perpendicular to \(p_1, p_23\), and to each other, while \(\omega_{45}^-\) and \(\omega_{45}^+\) are defined as above.

The result is

\[
\begin{align*}
\Delta_{330;M_2}(1^+, 2^+, 3^+, 4^+, 5^+) &= \\
&= \frac{i \text{tr}_+(1345)}{2[12] \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle} \frac{s_{45} - s_{23} \times}{s_{13}} \\
&\quad \left(2(D_s - 2)(\mu_{11} + \mu_{22})\mu_{12} + (D_s - 2)^2\mu_{11}\mu_{22} \frac{(k_1 + k_2)^2 + s_{45}}{s_{45}} \right). 
\end{align*}
\]

3.6 The five-leg double-triangle butterfly: \((330; 5L)\)

This topology can be parametrized by

\[
\begin{align*}
\vec{k}_1^\mu &= a_1 p_1^\mu + a_2 p_2^\mu + a_3 p_23^\mu + a_4 p_23^\mu \frac{1}{2}[\gamma^\mu |p_23|], \\
\vec{k}_2^\mu &= b_1 p_1^\mu + b_2 p_5^\mu + b_3 p_23^\mu \frac{1}{2}[\gamma^\mu |p_23|] + b_4 p_23^\mu \frac{1}{2}[\gamma^\mu |p_23|],
\end{align*}
\]

where

\[
\begin{align*}
a_1 &= 1, & a_2 &= 0, & a_3 &= \tau_1, & a_4 &= \tau_2, \\
b_1 &= 0, & b_2 &= 1, & b_3 &= \tau_3, & b_4 &= \tau_4, \\
\mu_{11} &= \vec{k}_1^2, & \mu_{22} &= \vec{k}_2^2, & \mu_{12} &= s_{45} \tau_5.
\end{align*}
\]

The general integrand has 146 coefficients in terms of the ISPs

\[(k_1 \cdot \omega_{123}), (k_1 \cdot p_3), (k_2 \cdot \omega_{453}), (k_2 \cdot p_3), \mu_{11}, \mu_{12}, \mu_{22},\]
with \( \omega_{123}^\mu \) and \( \omega_{453}^\mu \) defined in analogy with eq. (3.20). We pick this basis rather than a completely spurious one, as it makes the result simplify:

\[
\Delta_{330;5L}(1^+,2^+,3^+,4^+,5^+) = -\frac{i}{(12)(23)(34)(45)(51)} \times \\
\left( \frac{1}{2} \left( \text{tr}_+(1245) - \text{tr}_+(1345) \text{tr}_+(1235) \right) \left( 2(D_s - 2)(\mu_{11} + \mu_{22})\mu_{12} + (D_s - 2)^2 \mu_{11}\mu_{22} \left( \left( k_1 + k_2 \right)^2 s_{15} \right. \right. \right. \\
\left. \left. \left. + \text{tr}_+(1325) \left( \frac{1}{2} k_1 \cdot p_3 \right) \left( k_1 \cdot p_3 \right) + \left( k_1 + k_2 \right)^2 (s_{12} + s_{45} + s_{12}s_{45}) \right) + (D_s - 2)^2 \mu_{11}\mu_{22} \left( \left( k_1 + k_2 \right)^2 s_{15} \right. \right. \right. \\
\left. \left. \left. + \text{tr}_+(1325) \left( \frac{1}{2} k_1 \cdot p_3 \right) \left( k_1 \cdot p_3 \right) \left( 1 + \frac{2(k_2 \cdot \omega_{453})}{s_{35}} + \frac{s_{12} - s_{45}(k_2 - p_5)^2}{s_{35}s_{45}} \right) \right) \right) \right) \right) \right) \\
(3.30)
\]

The two terms proportional to \( (k_1 - p_1)^2 \) and \( (k_2 - p_5)^2 \), which vanish on the cut, are included in order to absorb some terms which would otherwise appear in lower point cuts.

In addition to those outlined above we have computed the cuts (421), (321; 5L), (420), (320; 5L), and (220; 5L), which all vanish using the integrands presented above as subtraction terms.

4 The planar five gluon two-loop amplitude

Since we only have the planar part of the primitive amplitude, we are restricted to the leading colour part of the full amplitude.

\[
A_5(1^+,2^+,3^+,4^+,5^+) |\text{leading colour} = \\
g_5^7 N_c^2 \sum_{\sigma \in S_5} \text{tr} (T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} T^{a_{\sigma(3)}} T^{a_{\sigma(4)}} T^{a_{\sigma(5)}}) A_5^{(2)}(\sigma(1)^+,\sigma(2)^+,\sigma(3)^+,\sigma(4)^+,\sigma(5)^+),
\]

(4.1)

where \( T^a \) are the generators of \( SU(N_c) \), \( g_5 \) is the strong coupling constant, and

\[
A_5^{(2)}(1^+,2^+,3^+,4^+,5^+) = A_5^{(2),\text{bare}}(1^+,2^+,3^+,4^+,5^+), \quad A_5^{(2),\text{bare}}(1^+,2^+,3^+,4^+,5^+) = -\frac{11}{36} A_5^{(1)}(1^+,2^+,3^+,4^+,5^+),
\]

(4.2)

and

\[
A_5^{(2),\text{bare}}(1^+,2^+,3^+,4^+,5^+) = \sum_{i=1}^{5} A_5^{(P)}(1^+,2^+,3^+,4^+,5^+),
\]

(4.3)

where we have included the appropriate UV counter-term. The primitive amplitude \( A_5^{(P)} \) can be written in terms of the eight integral families for which the coefficients can be determined from the cuts of the previous section.
After eliminating spurious terms which integrate to zero, the final result is

\[
A_s^{[P]}(1^+, 2^+, 3^+, 4^+, 5^+) = \frac{i}{(\sqrt{2})^4(\sqrt{45})^5(511)} (c_{431} I_{431} [F_1] \\
+ c_{431}^T I_{431} [F_1 (k_1 + p_5)^2] + c_{331:M_1} I_{331:M_1} [F_1] + c_{331:M_2} I_{331:M_2} [F_1] + c_{331:5L} I_{331:5L} [F_1] \\
+ c_{430} (s_{23} I_{330} [(k_1 + k_2)^2 + s_{45}] + I_{430} [(k_1 + k_2)^2 + s_{45}] 2(k_1 \cdot \omega_{123})) \\
+ c_{330:M_1} I_{330:M_1} [F_3 ((k_1 + k_2)^2 + s_{45})] + c_{330:M_2} I_{330:M_2} [F_3 ((k_1 + k_2)^2 + s_{45})] \\
+ c_{330:5L} I_{330:5L} [F_3 N_1(k_1, k_2, 1, 2, 3, 4, 5)] + c_{330:5L}^b I_{330:5L} [F_3 N_2(k_1, k_2, 1, 2, 3, 4, 5)] \\
+ c_{330:5L}^d I_{330:5L} [F_3 N_2(k_2, k_1, 5, 4, 3, 2, 1)] + c_{330:5L}^d I_{330:5L} [F_3 (k_1 + k_2)^2]) \),
\]

(4.4)

where

\[
c_{431} = \frac{-s_{12}s_{23}s_{34}s_{45}s_{15}}{t_5}, \quad c_{331:M_1} = \frac{-s_{34}s_{45}^2 t_{1235}}{t_5}, \quad c_{331:M_2} = \frac{-s_{15}s_{45}^2 t_{1234}}{t_5},
\]

\[
c_{331:5L} = \frac{-s_{12}s_{23}s_{34}s_{45}s_{15}}{t_5}, \quad c_{330:M_1} = \frac{-s_{45} - s_{12}}{2s_{13}s_{45}}, \quad c_{330:M_2} = \frac{-s_{45} - s_{23}}{2s_{13}s_{45}},
\]

\[
c_{330:5L}^b = \frac{t_{1235}}{2s_{13}s_{45}}, \quad c_{330:5L}^c = \frac{t_{1235}}{2s_{13}s_{45}}, \quad c_{330:5L}^d = \frac{t_{1235}}{2s_{13}s_{45}}.
\]

\[
c_{330:5L}^a = \frac{1}{2} \left( t_{1235} - \frac{t_{1235}}{s_{13}s_{45}} \right),
\]

and

\[
F_1 = (D_s - 2)(\mu_{11}\mu_{22} + \mu_{11}\mu_{33} + \mu_{22}\mu_{33}) + 4(\mu_{12}^2 - 4\mu_{11}\mu_{22}),
\]

\[
F_3 = (D_s - 2)^2\mu_{11}\mu_{22},
\]

\[
N_1(k_1, k_2, 1, 2, 3, 4, 5) = \frac{1}{s_{12}s_{45}} (4(k_1 \cdot p_3)(k_2 \cdot p_3) + s_{12}s_{45}),
\]

\[
N_2(k_1, k_2, 1, 2, 3, 4, 5) = \frac{2}{s_{45}} (k_1 \cdot p_3) (s_{35}s_{45} - (s_{12} - s_{45})2(k_2 \cdot p_5)).
\]

(4.6)

For performing the integrals over the higher-dimensional $\mu$-parameters, we use the Schwinger parametrization technique described in [37]. Defining

\[
\int d\mu = \int \frac{d^{-2\epsilon} k_1}{(2\pi)^{-2\epsilon}} \int \frac{d^{-2\epsilon} k_2}{(2\pi)^{-2\epsilon}},
\]

(4.7)

to be the higher-dimensional part of the integral, the result for the specific insertions is
that
\[
\int d\mu \left( \mu_1^2 - 4\mu_1\mu_2 \right) I_{A}^{[4-2\epsilon]} = -2\epsilon(2\epsilon - 1) I_{A}^{[6-2\epsilon]},
\]
\[
\int d\mu \left( \mu_{11}\mu_{22} + \mu_{11}\mu_{33} + \mu_{22}\mu_{33} \right) I_{A}^{[4-2\epsilon]} = 3\epsilon^2 I_{A}^{[6-2\epsilon]} + 2\epsilon(\epsilon - 1) \sum_{i,j \in P(A)} i^+ j^+ I_{A}^{[8-2\epsilon]},
\]
\[
\int d\mu \mu_{11}\mu_{22} I_{B}^{[4-2\epsilon]} = \epsilon^2 I_{B}^{[6-2\epsilon]},
\]
\[
\int d\mu \left( \mu_{11} + \mu_{22} \right) I_{B}^{[4-2\epsilon]} = 0,
\]

where \( A = \{(431), (331; M_1), (331; M_2), (331; 5L)\} \) and \( B = \{(430), (330; M_1), (330; M_2), (330; 5L)\} \). The last identity has already been applied in eq. (4.4). The set \( P(A) \) includes all possible ways to increase the power of any two propagators along a given branch \((k_1, k_2, \text{or } k_1 + k_2)\) of a topology \( A \). Explicitly we can write the sum as,
\[
\sum_{i,j \in P(n_1 n_2 n_{12}; x)} = \sum_{i=1}^{n_1} \sum_{j=i}^{n_1} + \sum_{i=n_1 + 1}^{n_1 + n_2} \sum_{j=i}^{n_1 + n_2} + \sum_{i=n_1 + n_2 + 1}^{n_1 + n_2 + n_{12}} \sum_{j=i}^{n_1 + n_2 + n_{12}},
\]

where \( n_1 \) denotes the number of propagators on the \( k_1 \)-branch, \( n_2 \) the number of propagators on the \( k_2 \)-branch, and \( n_{12} \) the number of propagators on the \( k_1 + k_2 \)-branch. For example the \((331; x)\) topology \( I_{331; x}^{[8-2\epsilon]}\) \((1,1,1,1,1,1)\) expand to
\[
\sum_{i,j \in P(331; x)} i^+ j^+ I_{331; x}^{[8-2\epsilon]}(1,1,1,1,1,1) =
\]
\[
\begin{align*}
I_{331; x}^{[8-2\epsilon]}(1,1,1,1,1,1) + I_{331; x}^{[8-2\epsilon]}(2,1,1,1,1,1) + I_{331; x}^{[8-2\epsilon]}(1,1,1,2,1,1) + I_{331; x}^{[8-2\epsilon]}(1,1,1,1,2,1) + I_{331; x}^{[8-2\epsilon]}(1,1,1,1,1,2) + \\
I_{331; x}^{[8-2\epsilon]}(1,1,1,1,1,1) + I_{331; x}^{[8-2\epsilon]}(1,2,1,1,1,1) + I_{331; x}^{[8-2\epsilon]}(1,1,1,2,1,1) + I_{331; x}^{[8-2\epsilon]}(1,1,1,1,2,1) + I_{331; x}^{[8-2\epsilon]}(1,1,1,1,1,2) + \\
I_{331; x}^{[8-2\epsilon]}(1,1,1,1,1,1) + I_{331; x}^{[8-2\epsilon]}(1,1,1,1,2,1) + I_{331; x}^{[8-2\epsilon]}(1,1,1,2,1,1) + I_{331; x}^{[8-2\epsilon]}(1,1,1,2,1,1) + I_{331; x}^{[8-2\epsilon]}(1,1,1,1,2,1) + \\
I_{331; x}^{[8-2\epsilon]}(1,1,1,1,1,1) + I_{331; x}^{[8-2\epsilon]}(1,1,1,1,1,1) + I_{331; x}^{[8-2\epsilon]}(1,1,1,1,1,1) + I_{331; x}^{[8-2\epsilon]}(1,1,1,1,1,1).
\end{align*}
\]

4.1 Analytical result for the butterfly-type topologies

As the integrals for the butterfly topologies are products of one-loop integrals, they can easily be found analytically. As the six-dimensional boxes, triangles and bubbles diverge as mostly \(1/\epsilon\), the \(\epsilon^2\) from the third identity in eq. (4.8) ensures that the result is finite. Writing the butterfly contribution to the primitive amplitude in eq. (4.4) as
\[
A_{5}^{[P]}(1^+, 2^+, 3^+, 4^+, 5^+)) = \frac{i(Ds - 2)^2}{(12)(23)(34)(45)(51)} \times
\]
\[
\left( c_{430}(s_{23}I_{430}^{[R]} + t_{430}^{[R]}) + c_{330; M_1}I_{330; M_1} + c_{330; M_2}I_{330; M_2} + c_{330; 5L}I_{330; 5L} + c_{330; 5L}^{(4)}I_{330; 5L}^{(4)} + c_{330; 5L}^{(5)}I_{330; 5L}^{(5)} + c_{330; 5L}^{(6)}I_{330; 5L}^{(6)} \right),
\]

(4.11)
the integrals are

\begin{align*}
I_{430}^a &= \frac{1}{4} + \mathcal{O}(\epsilon), \\
I_{430}^b &= \frac{\text{tr}_5}{\text{36} s_{12}} + \mathcal{O}(\epsilon), \\
I_{330;M_1} &= \frac{s_{34} - 2 s_{12} + 5 s_{45}}{36} + \mathcal{O}(\epsilon), \\
I_{330;M_2} &= \frac{s_{15} - 2 s_{23} + 5 s_{45}}{36} + \mathcal{O}(\epsilon), \\
I_{330;5L}^a &= \frac{(s_{23} - 2 s_{45})(s_{34} + 2 s_{45}) + s_{12}(2 s_{34} + 17 s_{45} - 2 s_{23} - 4 s_{12})}{36 s_{12} s_{45}} + \mathcal{O}(\epsilon), \\
I_{330;5L}^b &= \frac{(2 s_{35} - s_{34})(2 s_{13} + s_{23})}{36} + \mathcal{O}(\epsilon), \\
I_{330;5L}^c &= \frac{(2 s_{13} - s_{23})(2 s_{35} + s_{34})}{36} + \mathcal{O}(\epsilon), \\
I_{330;5L}^d = \frac{-2 s_{12} + s_{15} + s_{23} + s_{34} - 2 s_{45}}{36} + \mathcal{O}(\epsilon).
\end{align*}

An important cross check on this part of the amplitude comes from the cancellation of un-physical poles in \(s_{13}\) and \(s_{35}\) appearing in coefficients of eq. (4.5). We find that after summing over the five cyclic permutations of all such poles vanish in the partial amplitude \(A_5^{(2)}(1^+, 2^+, 3^+, 4^+, 5^+)\). The finite remainder can be written,

\begin{equation}
A_5^{(2)} \text{ butterfly} = \frac{i(D_s - 2)^2}{(12)(23)(34)(45)(51)} \frac{-1}{s_{12}s_{23}s_{34}s_{45}s_{15}} \sum_{\text{cyclic}} \ X(1, 2, 3, 4, 5) ,
\end{equation}

where

\begin{equation}
X(1, 2, 3, 4, 5) = s_{12}^2 \left( s_{23}(s_{12}s_{15}s_{23}s_{34} + s_{15}s_{23}s_{34} - 2 s_{12}s_{15}s_{23}s_{34} - 2 s_{12}s_{15}s_{23}s_{45} + s_{12}s_{23}s_{45} + s_{15}s_{23}s_{34}s_{45} + 12 s_{15}s_{23}s_{34}s_{45} - 2 s_{23}s_{34}s_{45} + 2 s_{15}s_{34}s_{45} - 2 s_{23}s_{34}s_{45} + 2 s_{23}s_{34}s_{45}^2 + 2 s_{34}s_{45}^2 - (s_{23}s_{45} + s_{15}s_{34}s_{45} + + s_{23}s_{34}s_{45} - s_{15}s_{23}s_{34} - s_{15}s_{23}s_{45}) \text{ tr}_5 \right).
\end{equation}

\section{5 Numerical evaluation}

Three of the integrals appearing in the amplitude have five scales and are yet unknown analytically. We therefore opt for a numerical evaluation in order to check the universal infra-red pole structure:

\begin{equation}
A_5^{(2)}(1^+, 2^+, 3^+, 4^+, 5^+) = -\left( \frac{1}{\epsilon^2} \sum_{i=1}^{5} \left( \frac{\mu_R^2}{s_{i,i+1}} \right)^{\epsilon} \right) \epsilon + \frac{11}{6 \epsilon} \right) A_5^{(1)}(1^+, 2^+, 3^+, 4^+, 5^+) + \mathcal{O}(\epsilon),
\end{equation}

where the \(D\)-dimensional one-loop amplitude is given by \([64, 65],\)

\begin{equation}
A_5^{(1)}(1^+, 2^+, 3^+, 4^+, 5^+) = \frac{-i\epsilon(1-\epsilon)}{(12)(23)(34)(45)(51)} \left( s_{12}s_{23} I_{4,1234}^{8-2\epsilon} [1] + s_{23}s_{34} I_{4,2345}^{8-2\epsilon} [1] \\
+ s_{34}s_{45} I_{4,3451}^{8-2\epsilon} [1] + s_{45}s_{15} I_{4,4512}^{8-2\epsilon} [1] + s_{15}s_{12} I_{4,5123}^{8-2\epsilon} [1] + 2(2-\epsilon) \text{ tr}_5 I_{4,12345}^{10-2\epsilon} [1] \right).
\end{equation}
We use two techniques for the numerical integration. Firstly we use the Mellin-Barnes approach with the help of AMBRE [66], M. Czakon’s MB.m [67], A. V. Smirnov’s MBsolve.m [68] and D. A. Kosower’s barnesroutines.m. The second approach uses sector decomposition via the FIESTA Mathematica package [69, 70]. The results of the numerical evaluation are shown in table 1 using the phase-space point:

\[
\begin{align*}
    p_1 &= (8/3, 1/2, i/2, 8/3), \\
    p_2 &= (0, 1/2, -i/2, 0), \\
    p_3 &= (-1, 1, 2i, 2), \\
    p_4 &= (61.23163693, -59.08662701, 76.08662701, 77.76412206), \\
    p_5 &= (-62.89830359, 57.08662701, -78.08662701, -82.43078872).
\end{align*}
\]

Though the configuration is complex, it has been constructed so that the exact kinematics can be obtained using the following values of the invariants:

\[
\begin{align*}
    s_{12} &= -1, & s_{23} &= -3, & s_{34} &= -11, & s_{45} &= -19, & s_{15} &= -31, & \text{tr}_5 &= -\sqrt{154429}.
\end{align*}
\]

We can see from the values in table 1 that the two-loop amplitude is in agreement with the IR pole structure within the numerical integration errors.

6 Conclusions

In this paper we have described how $D$-dimensional integrand reduction and generalized unitarity cuts are efficient methods for two-loop amplitude computations.

\[\footnote{We thank Tristan Dennen for providing a copy of his private implementation based on the AMBRE algorithm.}\]
The methods presented provide a general algorithm for the reduction of any loop amplitude. Though the procedure does not change a lot from the four-dimensional case already presented [43, 49–51], we find that the $D$-dimensional case resolves some difficulties that can occur otherwise. We have shown that all ideals defined by the propagators are radical ideals. The first consequence of this is that all ideals are prime ideals and that each set of propagators admits only a single branch of the solution to the on-shell constraints. We have also shown that the dimension of each ideal is $11 - P$ for a $P$ propagator system, a condition which is not always satisfied in four dimensions.

As a non-trivial application of the method, we have computed the first five-point two-loop amplitude in QCD: the planar all-plus helicity amplitude. The final result obtained has a remarkably simple and compact form. We learned important lessons about the benefits of choosing a good basis of ISPs for the integrand: Firstly, we required the four dimensional limit of the integrand basis to be manifest, a condition which is not satisfied when using a standard ordering for the polynomial division. Secondly, we found that correctly identifying spurious directions in the bow-tie topologies led to significant simplification in the final coefficients. Following these guidelines we were able to find an integrand representation with only six or higher propagator topologies with all remaining cuts evaluating to zero. As a result, we did not require integration-by-parts to simplify the computation at any stage, since the all-plus helicity configuration is already extremely simple at the integrand level.

A particularly interesting feature of the amplitude is the close relation to the known result in $\mathcal{N} = 4$ super Yang-Mills [71, 72]. At one-loop it is known that there is a close relation between MHV amplitudes in $\mathcal{N} = 4$ super Yang-Mills and self-dual Yang-Mills [65]:

$$A_n^{(1),4-2\epsilon}(1^+,\cdots,n^+) = -2(4\pi)^2\epsilon(1-\epsilon)\langle 12 \rangle^{-4} A^{(1),[N=4],8-2\epsilon}(1^-,2^-,3^+,\cdots,n^+). \quad (6.1)$$

At the integrand level this corresponds to,

$$\Delta_n^{(1)}(1^+,\cdots,n^+)=\langle 12 \rangle^{-4}(D_s-2)\mu_{11}^2\Delta_n^{(1),[N=4]}(1^-,2^-,3^+,\cdots,n^+). \quad (6.2)$$

Though the relation is not so simple at two-loops, we observe for $n = 4, 5$,

$$\Delta_n^{(2)}(1^+,\cdots,n^+) = F_1\langle 12 \rangle^{-4}\Delta^{(2),[N=4]}(1^-,2^-,3^+,\cdots,n^+) + \text{butterfly topologies.} \quad (6.3)$$

where $F_1 = (D_s - 2) (\mu_{11}\mu_{22} + \mu_{11}\mu_{22} + \mu_{22}\mu_{33}) + 4 (\mu_{12}^2 - 4\mu_{11}\mu_{22})$ as given in eq. (3.7). As noted in section 4.1, all the remaining butterfly contributions are finite, though we clearly see from eq. (4.8) that the dimension shifting formula no longer holds. It would certainly be interesting if there was a better way of understanding this rather unexpected connection between the two theories.

The main prospects for future research would be to complete the computation of the other helicity configurations for 5-gluon scattering. We would like to investigate extensions of the integrand reduction method to the non-planar where some recent developments using the colour-kinematics duality may be extremely helpful [73, 74].
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A Momentum twistor parametrization of the kinematics

An analytical calculation of a scattering amplitude usually results in a huge function of the scalar products $s_{ij}$, the antisymmetric product $\text{tr}_5$ defined in eq. (1.4), and spinor products $\langle ij \rangle$ and $\lbrack ij \rbrack$. These quantities are not independent, because of energy-momentum conservation and algebraic constraints like the Schouten identity, but to automate the reduction to some minimal set is hard beyond $2 \to 2$ kinematics.

In this paper, we use the momentum twistor parametrization for the analytical computations, in which the kinematics can be represented by momentum twistors $Z_i(\lambda_i, \tilde{\mu}_i)$ for each momentum [60].

The advantage of momentum twistor variables is that all identities like the Schouten identity, energy-momentum conservation, etc. are satisfied automatically, making the final simplification process straightforward. It is also easy to convert the momentum-twistor variables back to the traditional kinematic variables.

The first two components $\lambda_i$ of a momentum twistor $Z_i(\lambda_i, \tilde{\mu}_i)$ are the holomorphic spinors, whereas the anti-holomorphic spinors are obtained via

$$\tilde{\lambda}_i = \frac{\langle i i + 1 \rangle \tilde{\mu}_{i-1} + \langle i + 1 i - 1 \rangle \tilde{\mu}_i + \langle i - 1 i \rangle \tilde{\mu}_{i+1}}{\langle i i + 1 \rangle \langle i + 1 i - 1 \rangle \langle i - 1 i \rangle},$$

(A.1)

where $\langle ij \rangle$ denotes the usual spinor products.

We are able to use the symmetries of $Z$ to reduce the number of independent kinematic quantities.

The momentum twistor $Z_i(\lambda_i, \tilde{\mu}_i)$ has the following symmetries,

- Poincaré symmetry.

- $U(1)$ symmetry for each particle: $\lambda_i \to e^{i\theta_i} \lambda_i$, $\mu_i \to e^{i\theta_i} \mu_i$.

For a $n$-particle process, there will be $4n$ momentum twistor components, but the symmetries reduce this number to $4n - 10 - n = 3n - 10$ free components. In practice many different ways of choosing those free components are available.
A.1 Four-point momentum twistors

The four-point kinematics are very simple, so as a warm-up example we will derive the four-point momentum twistors.

We choose the momentum-twistor parametrization to be,

\[
Z = \begin{pmatrix}
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\
\mu_1 & \mu_2 & \mu_3 & \mu_4
\end{pmatrix} = \begin{pmatrix}
1 & 0 & -\frac{1}{s} & -\frac{1}{s} - \frac{1}{t} \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

(A.2)

For this choice all the spinors \(\lambda_i\) and \(\tilde{\lambda}_i\) and the momenta \(p_i\) are rational functions of the independent momentum-twistor variables \(s\) and \(t\).

In this formalism \(s_{12} = \langle 12 \rangle[21] = s\), and \(s_{14} = \langle 14 \rangle[41] = t\), so the twistor variables in the four-point case are just the Mandelstam variables. Any physical expression without an overall helicity factor is calculable directly using twistors. As a simple example, the ratio \((\langle 12 \rangle \langle 34 \rangle) / (\langle 13 \rangle \langle 24 \rangle)\) is helicity-free, so by inserting eq. (A.1) and (A.2) we get that \(\langle 12 \rangle \rightarrow -1\), \(\langle 34 \rangle \rightarrow -1/t\), \(\langle 13 \rangle \rightarrow -1\), and \(\langle 24 \rangle \rightarrow -1/s - 1/t\), and thus

\[
\frac{\langle 12 \rangle \langle 34 \rangle}{\langle 13 \rangle \langle 24 \rangle} = \frac{(-1)(-1/t)}{(-1)(-1/s - 1/t)} = \frac{s}{s + t}.
\]

(A.3)

Note that this calculation is straightforward, without invoking energy-momentum conservation or the Schouten identity, as these constraints are imposed automatically by the twistor formalism.

A.2 Five-point momentum twistors

There are five free components in five-point momentum twistors. The explicit form of five parameters is not unique, and it is hard to say whether or not there is an ideal choice. One practical version is

\[
Z = \begin{pmatrix}
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\
\mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5
\end{pmatrix} = \begin{pmatrix}
1 & 0 & \frac{1}{x_1} & \frac{1}{x_1} + \frac{1}{x_2} & \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & x_4 & 1 \\
0 & 0 & 0 & 1 & \frac{x_5}{x_4}
\end{pmatrix}.
\]

(A.4)

Again, all the spinors \(\lambda_i\) and \(\tilde{\lambda}_i\) and the momenta \(p_i\) are rational functions of the independent momentum-twistor variables \(x_1, x_2, x_3, x_4\) and \(x_5\).

The physical five-point amplitude, after striping out the overall helicity factor, should, by Lorentz symmetry, be a function of kinematic invariants \(s_{ij}\) and \(tr_5\) only.

\[
A(1, 2, 3, 4, 5) = h \cdot f(s_{ij}, tr_5),
\]

(A.5)

where \(h\) contains all the helicity information of the amplitude. In practice, if the corresponding tree amplitude is non-zero, we can choose \(h = A_{\text{tree}}(1, 2, 3, 4, 5)\). For the all-plus amplitude, for which the tree amplitude vanishes, we can simply choose \(h =\)
\(1/(12 \langle 23 \langle 34 \langle 45 \langle 51 \rangle)\). Only five of the \(s_{ij}\)-variables are independent, and we choose to pick \(s_{12}, s_{23}, s_{34}, s_{45}, \) and \(s_{15}\). Note that \(\text{tr}_5\) is not completely independent, as

\[
(\text{tr}_5)^2 = \det G \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ p_1 & p_2 & p_3 & p_4 \end{pmatrix} = G_4(s_{12}, s_{23}, s_{34}, s_{45}, s_{15}). \tag{A.6}
\]

These variables can be written as functions of the momentum-twistor variables

\[
s_{12} = x_1,
\]
\[
s_{23} = x_2 x_4,
\]
\[
s_{34} = \frac{x_1 (x_3 (x_4 - 1) + x_2 x_4) + x_2 x_3 (x_4 - x_5)}{x_2},
\]
\[
s_{45} = x_2 (x_4 - x_5),
\]
\[
s_{15} = -x_3 (x_5 - 1),
\]
\[
\text{tr}_5 = x_1 (x_3 (x_1 (x_5 - 2) + 1) + x_2 x_4 (x_5 - 1)) + x_2 x_3 (x_5 - x_4), \tag{A.7}
\]

and inversely, the momentum twistor variables can be uniquely expressed as rational functions of \(s_{ij}\) and \(\text{tr}_5\):

\[
x_1 = s_{12},
\]
\[
x_2 = \frac{s_{12} (s_{23} - s_{15}) + s_{23} s_{34} + s_{15} s_{45} - s_{34} s_{45} - \text{tr}_5}{2 s_{34}},
\]
\[
x_3 = \frac{(s_{23} - s_{45}) (s_{23} s_{34} + s_{15} s_{45} - s_{34} s_{45} - \text{tr}_5) + s_{12} (s_{15} - s_{23}) s_{23} + s_{12} (s_{15} + s_{23}) s_{45}}{2 (s_{12} + s_{23} - s_{45}) s_{45}},
\]
\[
x_4 = -\frac{s_{12} (s_{23} - s_{15}) + s_{23} s_{34} + s_{15} s_{45} - s_{34} s_{45} + \text{tr}_5}{2 s_{12} (s_{15} - s_{23} + s_{45})},
\]
\[
x_5 = \frac{(s_{23} - s_{45}) (s_{12} (s_{23} - s_{15}) + s_{23} s_{34} + s_{15} s_{45} - s_{34} s_{45} + \text{tr}_5)}{2 s_{12} s_{23} (-s_{15} + s_{23} - s_{45})}. \tag{A.8}
\]

Mathematically, the kinematic space of five-particle massless scattering is defined by the affine variety \(V_5 = Z(\text{tr}_5^2 - G_4(s_{12}, s_{23}, s_{34}, s_{45}, s_{15}))\) in \(\mathbb{C}^6\), spanned by the variables \(s_{12}, s_{23}, s_{34}, s_{45}, s_{15},\) and \(\text{tr}_5\). This affine variety is birationally equivalent to \(\mathbb{C}^5\) spanned by \(x_1, x_2, x_3, x_4\) and \(x_5\) via the maps (A.7) and (A.8).

Note that momentum-twistor variables \(x_1, x_2, x_3, x_4\) and \(x_5\), are only equivalent to the kinematics variables \(s_{ij}\) and \(\text{tr}_5\), but not to the spinor products \(\langle ij \rangle\) or \([ij]\), as they contain the additional phase information.

Hence our strategy for calculations of five-point massless amplitudes using the momentum-twistor formalism can be summarized as follows:

- Calculate the amplitude \(A(1, 2, 3, 4, 5)\) in momentum-twistor variables and simplify the result.
- Isolate the helicity factor \(h\), as \(A(1, 2, 3, 4, 5) = h \cdot f\).
- Use eq. (A.8) to rewrite \(f(x_1, x_2, x_3, x_4, x_5)\) as \(f(s_{ij}, \text{tr}_5)\), where eq. (A.6) ensures us that the result can be expressed as mostly linear in \(\text{tr}_5\).
B Feynman rules for gluons and scalars

The tree-level amplitudes in this paper have been computed using the following colour ordered Feynman rules,

\[ \text{prop}_{g}^{\mu_\nu} = \frac{-ig^{\mu_\nu}}{p^2}, \quad \text{prop}_{s} = \frac{i}{p^2}, \] (B.1)

\[ V_{ggg}^{\mu_1\mu_2\mu_3} = \frac{i}{\sqrt{2}} \left( g^{\mu_2\mu_3} (p_2 - p_3)^{\mu_1} + g^{\mu_3\mu_1} (p_3 - p_1)^{\mu_2} + g^{\mu_1\mu_2} (p_1 - p_2)^{\mu_3} \right), \] (B.2)

\[ V_{ggg}^{\mu_1\mu_2\mu_3\mu_4} = ig^{\mu_1\mu_3} g^{\mu_2\mu_4} - \frac{i}{2} (g^{\mu_1\mu_2} g^{\mu_3\mu_4} + g^{\mu_1\mu_4} g^{\mu_2\mu_3}), \] (B.3)

\[ V_{sgs}^{\mu} = \frac{i}{\sqrt{2}} (p_1 - p_3)^{\mu}, \] (B.4)

\[ V_{ggs}^{\mu} = \frac{i}{2} g^{\mu}, \quad V_{gsg}^{\mu} = -ig^{\mu}, \]

\[ V_{ss's'} = -\frac{i}{2}, \quad V_{ss's} = i. \] (B.5)

where the subscribed $g$ refers to gluons, and where the scalars $s$ and $s'$ may have different flavours. The rules in this appendix are consistent with those listed in [75].

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