Magnetic and electric properties of a quantum vacuum

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Received 10 May 2012, in final form 5 October 2012
Published 14 December 2012
Online at stacks.iop.org/RoPP/76/016401

Abstract
In this report we show that a vacuum is a nonlinear optical medium and discuss what the optical phenomena are that should exist in the framework of the standard model of particle physics. We pay special attention to the low energy limit. The predicted effects for photons of energy smaller than the electron rest mass are of such a level that none have yet been observed experimentally. Progress in field sources and related techniques seem to indicate that in a few years vacuum nonlinear optics will be accessible to human investigation.

The article was invited by George T Gillies.

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1. Introduction

Since ancient times the existence of the vacuum has been one of the most fundamental problems in science. In Aristotle’s Physics one reads that ‘the investigation of questions on the vacuum must be held to belong to the physicist—namely whether or not it exists, how it exists and what it is’ [1]. In his work, Aristotle defines what place, time and vacuum are, which ‘must, if they exist, be places deprived of body’ [1].

Although discussing the circular arguments on the concept of the vacuum in the history of physics is beyond the scope of this report, we must at least define what we mean by ‘vacuum’ in the following. We are interested in the electromagnetic properties of a vacuum, so our definition is deeply related to electromagnetism: a vacuum is a region of space in which a monochromatic electromagnetic plane wave propagates at a velocity that is equal to \(c\). Following special relativity, this velocity is independent both of the source’s motion and of the observer’s inertial frame of reference. Such a vacuum must not be empty. For example, in 19th century classical electrodynamics, a vacuum was thought to be filled with ether. Our definition is essentially a phenomenological one. In principle it provides a way to test if a region is a vacuum or not.

In classical electrodynamics, vacuum electromagnetic properties are simply represented by two fundamental constants: the vacuum permittivity \(\varepsilon_0\) and the vacuum permeability \(\mu_0\). These constants are linked to the velocity of light in vacuum \(c\) thanks to the fundamental relation \(c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}\). They describe respectively the proportionality factor between \(D\) and \(E\) and between \(B\) and \(H\) in a vacuum: \(D = \varepsilon_0 E\) and
\[ B = \mu_0 H, \text{ where } D \text{ is the electric displacement vector, } E \text{ is the electric field vector, } B \text{ is the magnetic induction vector and } H \text{ is the magnetic-field vector. } B \text{ is also called the magnetic-field vector as well as } H \text{ when there is no risk of confusion} [2]. \]

We conform to this common terminology in the following.

Any variation of the velocity of light with respect to \( c \) is ascribed to the fact that light propagates in a medium, i.e. not in a vacuum. To describe such a phenomenon one must introduce the constants \( \epsilon \) and \( \mu \) which characterize the medium itself: \( D = \epsilon E \) and \( B = \mu H \). The velocity of light in a medium is less than the velocity of light in vacuum by a factor \( n \), and the index of refraction equal to \( n = \sqrt{\frac{\epsilon\mu}{\epsilon\mu_0}} \). A vacuum is therefore the medium with which, in classical electrodynamics, one associates an index of refraction \( n \) exactly equal to 1, and therefore in a vacuum \( \epsilon = 1 \) and \( \mu = 1 \).

Since the middle of 19th century, when Faraday discovered that an external magnetic field could change the polarization of light propagating in matter because of a magnetically induced circular birefringence [3], it has been known that the presence of electrostatic fields induces a polarization of light propagating in matter because of a field sources and related techniques seem to indicate that in a few years vacuum nonlinear optics will be accessible to human investigation.

### 2. Theory

#### 2.1. General formalism

In a medium the excitation due to light and external fields produces a polarization \( P \) and a magnetization \( M \). Both of them depend on the electromagnetic fields, and hence the response of the medium to the excitation is nonlinear. To describe this nonlinear interaction, one uses the constitutive equations of the medium giving the relationship between \( P \) and \( (E, B) \) and between \( M \) and \( (E, B) \), and Maxwell’s equations [5]. When no charge density or current density are present, Maxwell’s equations can be written in SI units as:

\[ \nabla \times E = -\frac{\partial B}{\partial t}, \]
\[ \nabla \times H = \frac{\partial D}{\partial t}, \]  
\[ \nabla \cdot D = 0, \]
\[ \nabla \cdot B = 0, \]

with

\[ H = \frac{1}{\mu_0} B - M, \]
\[ D = \epsilon_0 E + P. \]  

Thanks to Maxwell’s equations one can fully determine the wave propagation.

The constitutive equations can be obtained by the following relations [6]:

\[ D = \frac{\partial L}{\partial E}, \]  
\[ H = -\frac{\partial L}{\partial B}, \]

where \( L \) is the effective Lagrangian representing the interaction of electromagnetic fields in vacuum.

The mathematical expression of the effective Lagrangian \( L \) is essentially determined by the fact that it has to be relativistic invariant and therefore it can only be a function of the Lorentz invariants \( F \) and \( G \) [6]:

\[ F = \left( \epsilon_0 E^2 - \frac{B^2}{\mu_0} \right), \]
\[ G = \frac{\epsilon_0}{\sqrt{\mu_0}} (E \cdot B). \]

The general expression can be therefore written as:

\[ L = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{i,j} F^i G^j. \]
which at the lowest orders gives
\[
D = 2\varepsilon_0 c_{1,0} E + \sqrt{\frac{\varepsilon_0}{\mu_0}} c_{0,1} B + 2\varepsilon_0 c_{1,1} G E
+ \sqrt{\frac{\varepsilon_0}{\mu_0}} c_{1,1} F B + 4\varepsilon_0 c_{2,0} G E + 2\varepsilon_0 c_{0,2} G B, 
\]
(10)
\[
H = 2\varepsilon_{1,0} \frac{B}{\mu_0} - \sqrt{\frac{\varepsilon_0}{\mu_0}} c_{0,1} E + 2\varepsilon_{1,1} \frac{B}{\mu_0} - \sqrt{\frac{\varepsilon_0}{\mu_0}} c_{1,1} F E
+ 4\varepsilon_{2,0} \frac{B}{\mu_0} - 2\sqrt{\frac{\varepsilon_0}{\mu_0}} c_{0,2} G E. 
\]
(11)

The classical equations \(D = \varepsilon_0 E\) and \(H = \frac{B}{\mu_0}\) are recovered at the lowest order in the fields by imposing \(c_{1,0} = \frac{1}{2}\) and \(c_{0,1} = 0\).

In the case of a plane wave propagating in a vacuum, both \(F\) and \(G\) are equal to zero and therefore \(L = 0\) as well. This means that, because of Lorentz invariance, the propagation of a plane wave in vacuum cannot be affected by any nonlinear interactions. It can be shown that \(L\) is also equal to 0 in the case of two copropagating \(k_1 = k_2\) plane waves of different polarization \((E_1 \neq E_2)\). The simplest cases in which \(L \neq 0\), giving rise to nonlinear effects in a vacuum, are those of a plane wave propagating in the presence of external static electric or magnetic fields \((E_0, B_0)\), and that of two plane waves of the same polarization \((E_1 = E_2)\) but of different wavevectors \((k_1 \neq k_2)\). This is true in particular for counter-propagating plane waves \((k_1 = -k_2)\).

It is worth stressing that, as far as Lorentz invariance holds, our phenomenological definition of a vacuum also holds even in the presence of nonlinear interactions. A plane wave velocity different to \(c\) can be ascribed to the presence of matter, as in classical electrodynamics, and/or to the presence of electromagnetic fields.

Since \(L\) can only be the sum of terms containing powers of \(F\) and \(G\), Lagrangian terms containing a product of an odd number of electromagnetic fields are not allowed in a vacuum (see equation (7)). This means that not all the nonlinear effects existing in a standard medium exist in a vacuum. The form of Lagrangian \(L\) also indicates that the \(B\) and \(E\) fields of waves play an equivalent role as far as nonlinear effects in a vacuum are concerned. In standard media the \(B\) field is usually neglected and all the effects are ascribed to an \(\epsilon\) function only of \(E\), while \(\mu\) is assumed equal to \(\mu_0\) [5]. In this sense a vacuum can be considered to be a magnetic medium.

Finally, the energy density \(U\) can be written as [6]:
\[
U = E \frac{\partial L}{\partial E} - L
= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{i,j} (2\varepsilon_0 F^{(i)} E^2 + (j - 1) F^j G^j).
\]
(12)

Taking into account that \(c_{0,0} = 0\), \(c_{1,0} = \frac{1}{2}\) and \(c_{0,1} = 0\), the term \((i = 1, j = 0)\) gives the classical energy density \(U_0\),
\[
U_0 = \frac{1}{2} \left(\varepsilon_0 E^2 + \frac{B_0^2}{\mu_0}\right). 
\]
(13)

This expansion of the electromagnetic energy density in a vacuum can be compared with the one given by Buckingham and Pople [7] in the case of a molecule in the presence of external electromagnetic fields. One can find a direct correspondence between the vacuum terms and the molecular ones, showing once more that a vacuum behaves as a standard medium. In the case of molecular energy density each term represents a specific microscopical property of a single molecule, like polarizability, while for a vacuum we are dealing with macroscopical properties. To pass from microscopical to macroscopical properties in the case of molecules one has to take into account the molecular density, a concept that has no equivalence for a vacuum.

In the different theoretical frameworks one can find predictions for the \(c_{i,j}\) coefficients introduced in equation (7). Quantum electrodynamics (QED) provides the most complete theoretical treatment.

2.2. QED effective Lagrangian

In 1933 the calculations of gamma-ray absorption due to the formation of electron–positron pairs by Oppenheimer and Plesset [8] gave a striking confirmation of Dirac’s theory of positrons as holes in a sea of negative energy states, setting up a new picture of the vacuum in which all negative energy states are occupied and all positive energy states are unoccupied. Dirac gave a clear overview of his model in his 1934 contribution to the Solvay workshop [9]. It was immediately clear that an important prediction of Dirac’s theory, which could in principle be experimentally tested, was the existence of photon–photon scattering [10, 11].

A first theoretical formulation of optical nonlinearities in a vacuum at the lowest orders in electromagnetic fields was published in 1935 by Euler and Kochel [12]. The details of their calculation can be found in [13]. In the 1936 paper by Heisenberg and Euler [14] a complete theoretical study of the phenomena related to the fact that electromagnetic radiation can be transformed into matter and vice versa can be found. The authors’ starting point was that it was no more possible to separate processes in the vacuum from those involving matter since electromagnetic fields can create matter if they are strong enough. Moreover, even if they are not strong enough to create matter, they polarize the vacuum because of the virtual possibility of creating matter, essentially with electron–positron pairs, and therefore they change the constitutive equations [14].

The resulting effective Lagrangian of the field reads [14]:
\[
L_{\text{HE}} = \frac{1}{2} \left(\varepsilon_0 E^2 - \frac{B^2}{\mu_0}\right) + \alpha \int_0^\infty e^{-\eta} \frac{d\eta}{\eta^3}
\times \left\{ \left(\frac{1}{2\eta^2} \sqrt{\frac{\varepsilon_0}{\mu_0}} (E \cdot B) - \sqrt{\frac{\varepsilon_0 E_{\text{cr}}}{\mu_0}} \right)^2 \cos \left(\frac{\eta}{\varepsilon_0 E_{\text{cr}}} \sqrt{C}\right) + \text{conj.}
\right.
\]
\[
\left. + \frac{\varepsilon_0 E_{\text{cr}}}{\mu_0} + \frac{\eta^2}{3} \left(\varepsilon_0 E^2 - \frac{B^2}{\mu_0}\right) \right\},
\]

with \(C = \left(\varepsilon_0 E^2 - \frac{B^2}{\mu_0}\right) + 2\varepsilon_0 E_{\text{cr}} (E \cdot B)\),

where \(\alpha = e \sqrt{\frac{\varepsilon_0}{4\pi}}\) is the fine structure constant, \(e\) the elementary charge, \(\hbar\) the Planck constant \(\hbar\) divided by \(2\pi\),
and \( \eta \) is the integration variable. \( E_{cr} = \frac{m_e c^2}{\hbar} \) is a quantity obtained by combining the fundamental constant \( m_e \), the electron mass, with \( e \) and \( \hbar \). \( E_{cr} \) has the dimensions of an electric field, and is called the critical electric field. Its value is \( E_{cr} = 1.3 \times 10^{15} \text{ V m}^{-1} \). It corresponds to the field one needs to get an energy \( e E_{cr} L \) equal to an electron rest mass \( m_e c^2 \) over a length \( L \) equal to the reduced electron Compton wavelength \( \hbar \frac{c}{2\pi} = \frac{\hbar}{m_e} \).

A critical magnetic field can also be defined in the same manner, \( B_{cr} = \frac{eB_{cr}}{c} \). The cyclotron pulsation for an electron in a \( B_{cr} \) field, \( \omega_c = \frac{eB_{cr}}{m_e} \), is such that the associated energy \( h\omega_c \) is equal to its rest mass \( m_e c^2 \). \( L_{nHE} \) is valid in the approximation that the fields vary very slowly over a length equal to the reduced electron Compton wavelength during a time \( t_e = \frac{\hbar}{m_e c^2} \), which corresponds to:

\[
\frac{\hbar}{m_e c^2} \left| \nabla E(B) \right| \ll E(B),
\]

\[
\frac{\hbar}{m_e c^2} \left| \frac{\partial E(B)}{\partial t} \right| \ll E(B).
\]

A study of the vacuum electrodynamics based on the quantum theory of the electron can also be found in the 1936 paper by Weisskopf [15], in which a simplified method to obtain \( L_{nHE} \) is shown.

In general the QED Lagrangian \( L_{nHE} \) can be expanded as indicated in equation (7). Thanks to the symmetry properties of the \( E \) and \( B \) fields (see table 1), \( F \) and \( G \) are also CPT invariant, but while \( F \) is \( C, P \) and \( T \) invariant, \( G \) violates \( P \) and \( T \).

Table 1. Symmetry properties of the electromagnetic fields.

| \( C \) | \( P \) | \( T \) | \( CPT \) |
|---|---|---|---|
| \( E \) | \( - \) | \( - \) | \( + \) | \( + \) |
| \( B \) | \( - \) | \( + \) | \( - \) | \( + \) |
| \( F \) | \( + \) | \( + \) | \( + \) | \( + \) |
| \( G \) | \( + \) | \( - \) | \( - \) | \( - \) |

Following the developments in QED in the 1940s by Tomonaga et al. [17], a quantum field theory has been fully established and Dirac’s model of a vacuum has become a somewhat obsolete and unnecessary concept. In the field-theory perspective, the vacuum becomes the ground state of the quantum field. This theoretical definition of a vacuum is in agreement with our own, which refers to macroscopical phenomenological electromagnetic properties such as \( \epsilon \) and \( \mu \) [18].

A modern confirmation of the Euler and Kochel result was given by Karplus and Neuman in 1950 [19], and the Heisenberg–Euler Lagrangian was validated by Schwinger in 1951 [20]. The lowest order in the nonlinear effect given by the Euler–Kochel Lagrangian can be represented by the Feynman diagram of figure 1 [19].

An analytic form for the Heisenberg–Euler Lagrangian can be found in [21], but for most of the practical cases, effects can be calculated using the first terms given by \( L_{EK} \). In fact, equation (14) can be expanded in terms of reduced Lorentz invariants \( F' \) and \( G' \)

\[
F' = \left( \frac{E^2}{E_{cr}^2} - \frac{B^2}{B_{cr}^2} \right),
\]

\[
G' = \left( \frac{E}{E_{cr}} \frac{B}{B_{cr}} \right),
\]

Next term coefficients can be found in [14–16].

\[
c_{2,0} = \frac{32\pi^2 e^3 h^3}{315 m_e^2 c^10} = \frac{2\alpha}{315\pi} \frac{1}{6} E_{cr}^2
\]

\[
\simeq \frac{2\alpha}{315\pi} \frac{1}{6} E_{cr}^2 \simeq 6.2 \times 10^{-56} \left[ \frac{m_e^7}{J^2} \right],
\]

(20)

\[
c_{1,2} = \frac{13}{2} c_{3,0},
\]

(21)

\[
c_{4,0} = \frac{3568\pi^2 e^5 h^6}{945 m_e^2 c^{15}} = \frac{223\alpha}{3780 \pi} \frac{1}{6} E_{cr}^4
\]

\[
\simeq \frac{223\alpha}{3780 \pi} \frac{1}{6} E_{cr}^4 \simeq 4.4 \times 10^{-80} \left[ \frac{m_e^9}{J^2} \right],
\]

(22)

\[
c_{2,2} = \frac{402}{223} c_{4,0},
\]

(23)

\[
c_{0,4} = \frac{304}{223} c_{4,0},
\]

(24)

Figure 1. Feynman diagram for the lowest order in the nonlinear effect in a vacuum. Solid circular lines represent the electron–positron loops, wavy lines the photons and wavy lines with crossed ends the external fields.
which means that higher and higher order terms corresponding to higher and higher powers of $F'$ and $G'$ give smaller and smaller contributions when $\frac{\mu_0}{\epsilon_0}$ and $\frac{\mu_0}{\epsilon}$ are much smaller than 1.

After 75 years the impact of the Heisenberg–Euler Lagrangian in fundamental physics is still very important as discussed in [22]. Our review is essentially based on its predictions.

The Heisenberg–Euler Lagrangian does not take into account all of the microscopic phenomena related to the photon–photon interaction in a vacuum. Corrections to the value of the coefficients $c_{1,j}$ obtained using $L_{\text{HE}}$ can be calculated taking into account the change induced by the external fields in the radiative interactions of the vacuum electrons. In particular Ritus [23] published in 1975 the corrections to $c_{2,0}$ and $c_{0,2}$. The lowest-order radiative corrections can be represented by the Feynman diagram of figure 2, and the corresponding effective Lagrangian can be written following [23] as

$$L_R = \frac{\alpha^2 \hbar^3}{81 \pi m_e^2 c^5} \left( 16 F^2 + \frac{263}{2} G^2 \right),$$

which gives a 1.0% correction to $c_{2,0}$:

$$c_{2,0}^R = \frac{2 \alpha^2 \hbar^3}{45 m_e^2 c^5} \left( 1 + \frac{40 \alpha}{9 \pi} \right)$$

and a 1.2% correction to $c_{0,2}$:

$$c_{0,2}^R = \frac{14 \alpha^2 \hbar^3}{45 m_e^2 c^5} \left( 1 + \frac{1315 \alpha}{252 \pi} \right).$$

The Ritus corrections to $c_{2,0}$ and $c_{0,2}$ are about $\alpha$ times smaller than the Euler–Kochel values for these two coefficients. Nevertheless, these corrections are more important than the next terms in the expansion of $L_{\text{HE}}$ corresponding to the coefficients $c_{1,2}, c_{3,0}$ when $E \ll E_{\text{cr}}$ and $B \ll B_{\text{cr}}$.

Following equation (12), the energy density $U$ when $E \ll E_{\text{cr}}$ and $B \ll B_{\text{cr}}$ can be written as [15]:

$$U = \frac{1}{2} \left( \epsilon_0 E^2 + \frac{B^2}{\mu_0} + c_{2,0} \epsilon_0 E^2 - \frac{B^2}{\mu_0} \right) \left( 3 \epsilon_0 E^2 + \frac{B^2}{\mu_0} \right)$$

$$+ \frac{c_{0,2} \epsilon_0}{\mu_0} (E \cdot B)^2 + c_{3,0} \left( \epsilon_0 E^2 - \frac{B^2}{\mu_0} \right)^2 \left( 5 \epsilon_0 E^2 + \frac{B^2}{\mu_0} \right)$$

$$+ \frac{c_{1,2} \epsilon_0}{\mu_0} (E \cdot B)^2 \left( 3 \epsilon_0 E^2 - \frac{B^2}{\mu_0} \right).$$

Using the relations (10), (11) and (2) one obtains the polarization and the magnetization of the vacuum at the lowest orders in the fields:

$$P = 4 c_{2,0} \epsilon_0 E F + 2 c_{0,2} \sqrt{\frac{\epsilon_0}{\mu_0}} B G,$$

$$M = - 4 c_{2,0} \frac{B}{\mu_0} F + 2 c_{0,2} \sqrt{\frac{\epsilon_0}{\mu_0}} E G.$$  

In principle, since electromagnetic fields have self-interactions, one should be able to calculate the corrections to the Maxwell classical solutions for any distribution of charges and currents [24].

The corrections to the Coulomb potential of a charge induced by vacuum polarization were calculated in 1935 by Uehling [25], together with the corresponding displacement of atomic energy levels. This is a fundamental result for the QED of bound states.

The case of a magnetic dipole has been treated in [26], where the field equations of a static magnetic field have been considered and the field of the dipole has been calculated taking into account one-loop QED corrections.

The value of the lowest-order coefficients $c_{2,0}$ and $c_{0,2}$ depends on the fourth power of the inverse of the electron mass $m_e$. Following [20] one can generalize the Heisenberg–Euler result to any spin $\frac{1}{2}$ charged field such as the one corresponding to negative and positive muons. For the sake of comparison, one can consider that the value of $B_{\text{cr}}$ sets the relative scale of different contributions coming from different fermions. The critical magnetic field for muon leptons is about $1.9 \times 10^{14}$ T and for tau leptons $5.3 \times 10^{16}$ T. The contributions coming from muon or tau fields are usually neglected.

2.3. Other contributions to the effective Lagrangian $L$

The effective Lagrangian $L$ is valid in the approximation of constant or slowly varying electromagnetic fields. As the available laser pulse intensity increases and pulse time width decreases, one needs in principle to take into account the dispersive corrections of $L$. This means that one has to take into consideration a correction to $L$ depending on the derivatives of electromagnetic fields. How to treat dispersion and absorption in the Lagrangian formalism is a debated question and has generated a large amount of literature to correctly describe quantum electromagnetism in dielectric media. From this literature it is worth mentioning the two recent works of Huttner and Barnett [27], and Philbin [28].

As far as a quantum vacuum is concerned, the authors of [29] give the effective Lagrangian corresponding to dispersion corrections, and they derive in the low energy limit ($\hbar \omega \ll m_e c^2$) the vacuum dispersion relation:

$$\omega \simeq c \kappa \left( 1 - \frac{1}{2} \zeta^2 Q^2 - \zeta^2 Q^4 k^2 \right),$$

where $\zeta$ depends on light polarization and it is equal to $4 c_{2,0}$ or $2 c_{0,2}$, and $\sigma = \frac{2 m_e c^2}{16 \pi \epsilon_0}$ is a parameter corresponding to the dispersive properties of the polarized vacuum. Its numerical value is about $1.4 \times 10^{-28}$ m$^2$ and $Q$ is a parameter depending
on the electromagnetic fields. For electromagnetic fields perpendicular to the light wavevector \( k \), \( Q \) can be written as:

\[
Q^2 = \varepsilon_0 E^2 + \frac{B^2}{\mu_0} - 2 \sqrt{\varepsilon_0 \mu_0} \vec{E} \times \vec{B}.
\] (34)

The first two terms of equation (33) correspond to a linear dispersion relation representing a vacuum velocity of light that depends slightly on the light polarization. This is an important point that will be treated in detail in the following paragraphs. The third term is the nonlinear term coming from the dispersive correction to \( L \). Its order of magnitude can be estimated as \( 8 \times 10^{-86} Q^2 k^2 \), and it appears challenging to detect in a laboratory [29].

In the framework of the standard model, contributions other than QED ones appear essentially at the QCD scale and at the electroweak scale. The QCD scale can be associated with a mass \( A_{\text{QCD}} \approx 200 \text{ MeV} \) close to the mass of the pion \( \pi^\pm \) meson. The corresponding critical magnetic field \( B_{cT}^{\text{QCD}} \) is of the order of \( 10^{15} \text{ T} \) [30]. The electroweak scale can be associated with the mass of the \( W^\pm \) boson which is about 80 GeV. The corresponding critical magnetic field \( B_{cT}^{\text{EW}} \) is of the order of \( 10^{20} \text{ T} \) [30]. It is obvious that these contributions can be neglected most of the time and there is not much literature on them.

3. Instruments

The experimental tests of the Heisenberg–Euler Lagrangian need high magnetic or electric fields. In this section, we give a short overview of the existing solutions for producing such intense fields.

3.1. Light sources

Since 1960 and the invention of the laser by Maiman [31], the available intensity has jumped to higher and higher levels. Today, a number of exawatt (\( 10^{18} \) W)-class facilities are already in the planning stage (the extreme light infrastructure (ELI) program in Europe [32] and Exawatt Laser in Japan [33], for example). High power lasers are expected in the future expected to approach the Schwinger limit corresponding to an intensity \( I \approx 10^{33} \text{ W m}^{-2} \). At such a level the electric field \( E_{cr} \) associated with the wave is of the order of the critical one, \( E_{cr} = \sqrt{\frac{\varepsilon_0 c}{2\mu_0}} \approx E_{cr} \), which allows the creation of real \( e^-e^+ \) pairs from the vacuum [20].

All facilities around the world are nowadays based on chirped pulse amplification (CPA) proposed by G Mourou in 1985 [34]. Since then the available laser power has increased rapidly. Basically, high powers are limited by the damage threshold of materials, which typically lies around \( 10^8 \text{ GW m}^{-2} \). The problem thus is to amplify a short pulse without reaching this damage intensity. The principle of CPA is the following (see figure 3): a short laser impulsion is temporally stretched before being amplified up to \( 10^4 \text{ GW m}^{-2} \). Then the pulse is temporally recompressed before being focused on a target. In this way, one can amplify the intensity (expressed in \( \text{J s}^{-1} \text{ m}^{-2} \)) of the pulse without decreasing its fluency (expressed in \( \text{J m}^{-2} \)).

The ELI project represents one of the largest laser projects in the world. Its goal is to produce a few kJ of energy in 10 fs, which means more than \( 10^{14} \) W of power with a target intensity of \( 10^{30} \text{ W m}^{-2} \) [32]. As far as we know, the highest intensity ever reached nowadays is \( 2 \times 10^{20} \text{ W m}^{-2} \) [35] at the HERCULES petawatt facility [36] in the USA.

Two ambitious projects are also in progress, the National Ignition Facility, USA [37] and the laser Megajoule, France [38], in order to create fusion ignition in a laboratory. Both of them will be able to fire an energy of about 1.8 MJ thanks to the amplification of more than 200 laser beams.

For smaller intensities, commercial sources are also available. For example, one can buy table-top sources which deliver more than 100 TW (2.5 J in 25 fs). A laser beam of this kind focused on a \( 10^{-6} \text{ m}^2 \) spot gives an intensity of \( 10^{20} \text{ W m}^{-2} \), which corresponds to an energy density \( \varepsilon_0 E_{cr}^2 + B_{cr}^2/\mu_0 \) such that the electric field \( E_{cr} \) is about \( 1.36 \times 10^{11} \text{ V m}^{-1} \) and the magnetic field \( B_{cr} \) is about \( 4.5 \times 10^7 \text{ T} \). These values are of the order of the highest static fields ever obtained in laboratories, and therefore intense lasers can also be used as sources of electromagnetic fields to induce nonlinear optics effects. We label these kinds of effects with the \textit{optical field-induced} prefix in their name.

3.2. Electrostatic fields

In principle, the external fields needed for experiments can be either magnetic or electric. In a vacuum the same level of effect is obtained in the presence of a \( B \) field or an electric field \( E \) equal to \( c B \). This means that to equalize an effect created by a magnetic field of 1 T, one has to use an electric field of \( 300 \text{ MV m}^{-1} \). It appears that, from a technological point of view, magnetic fields of several tesla are easier to produce than electric fields of about 1 GV m\(^{-1}\). Experimentalists have
therefore mostly concentrated their efforts to magnetically induced effects in a vacuum.

To produce a magnetic field, the standard method is to let a current circulate in a coil. The magnetic field obtained is proportional to the current density. This current density also creates a force density which is proportional to the product of the current density and the magnetic field, and hence to the square of the magnetic field. Moreover, the current creates losses in the conductors by Joule effect and it heats the system. These two effects have to be taken into account to reach high magnetic fields.

Several solutions exist. One is to avoid heating thanks to superconducting wires. This is the solution chosen at CERN, for example, for the Large Hadron Collider magnets [39]. The maximum field achievable depends on the critical fields of the materials above which the superconductivity property disappears. Magnets around 20 T are commercially available.

Another solution is to remove the dissipate electrical energy. This can be done with water. In this way, one can achieve a magnetic field of up to 35 T, but one needs a big electrical installation of several tens of megawatts. Only a few installations around the world have installed this electrical power: NHMFL-DC (Tallahassee, USA) [40], LNCMI-G (Grenoble, France) [41], HFML (Niemegen, Netherlands) [42], HMFL (Hefei, China) [43] and TML (Tsukuba, Japan) [44]. A third way is to produce a magnetic field with a combination of the two previous methods: superconducting and copper wires cooled with water. The record for this type of coil is 45 T for 1 h.

Another way to avoid the problem of heating is to use pulsed fields. The idea is to discharge a lot of energy for a small duration in a coil. In this way, one can reach 80 T in 10–100 ms depending on the bank of capacitors which delivers the energy. Up to date, the world record has been reached at the NHMFL-PF in Los Alamos in the United States, with 100 T. The main worldwide pulsed-field installations are NHMFL-PF (Los Alamos, USA) [45], LNCMI-T (Toulouse, France) [46], HLD (Dresden, Germany) [47], HFML (Niemegen, Netherlands) [42], WHMFC (Wuhan, China) [48] and TML (Tsukuba, Japan) [44]. The limitation of these pulsed coils is a combination of heating and magnetic pressure which can reach intensities of $10^9$ Pa.

To go further and to reach 300 T in a few microseconds, a possibility is to use a single-turn coil. This is called a megaGauss installation and only two are operational around the world (in Toulouse and Tokyo; see e.g. [49]). In this experiment, a single-turn coil is placed at the end of a generator which delivers more than 60,000 A in one microsecond. After a few microseconds the coil is destroyed but this duration is long enough to perform optical measurements.

More recently, the LNCMI-Toulouse has developed special coils and generators for bringing high magnetic fields to special external installations such as particle accelerator facilities or intense laser facilities. The generators are transportable so high magnetic fields can be moved almost everywhere (see e.g. [50]).

The highest fields that we have discussed previously are longitudinal magnetic fields. This means that the magnetic field is parallel to the optical axis, as in the case of simple solenoids [2]. This is called Faraday configuration, in reference to the Faraday effect which needs a longitudinal magnetic field with respect to the light propagation.

For some experiments, one needs a magnetic field transverse to the optical axis, as in the case of Helmholz coils [51]. This configuration is less conventional. The major difficulty lies with the possibility of putting efficient reinforcements against magnetic pressure because of the non-cylindrical symmetry of this kind of coil. Superconducting dipole magnets at CERN are able to give 10 T over 10 m. At LNCMI-Toulouse, the laboratory has developed a special coil [46] dedicated to the observation of vacuum magnetic birefringence which has already produced more than 30 T over a length of 50 cm. This equipment can be used almost everywhere thanks to the transportable generators. Transverse coils designed to deliver up to 40 T have also been developed at LNCMI-T for plasma experiments where a strong pulsed field is coupled to an intense pulsed laser at LULI (France) [52].

### 4. Phenomenology and related experiments

A very large number of phenomena are expected in a nonlinear optical medium. The nonlinear response can give rise to exchanges of energy between electromagnetic fields of different frequencies. Few of them are allowed in a quantum vacuum because of C, P, T and Lorentz invariances. The absence of linear terms in $E$ or $B$ in the energy density given in equation (30) means that the vacuum cannot have a permanent electric or magnetic dipole moment, which seems obvious. The quadratic terms in $E$ or $B$ in equation (30) correspond to the Maxwell energy density. Any ulterior term of this kind can be canceled out by a renormalization of the velocity of light.

As already stated in the introduction, we focus our attention on low energy effects that affect the propagation of light in a vacuum. We also restrict ourselves mainly to effects induced by fields that are small compared with the critical ones. Generally speaking, we treat phenomena mostly in the approximation

$$\frac{\hbar \omega}{m_c c^2} \frac{B}{B_{cr}} = \frac{\hbar \omega}{m_c c^2} \frac{E}{E_{cr}} \ll 1. \quad (35)$$

In this approximation, we restrict ourselves to the first terms of the development of the Heisenberg–Euler Lagrangian. This Lagrangian has not yet been tested experimentally and therefore any experiment whose goal is the measurement of one of the following effects tests a pure QED fundamental prediction.

#### 4.1. Three-wave mixing

Three-wave mixing indicates any term in equation (30) proportional to a product of three electromagnetic fields such as $E^3$, $E^2 B$, $E B^2$ and $B^3$. No term containing three electromagnetic fields exists in equation (30), and therefore none of these effects is allowed in a vacuum.

In a nonlinear medium, these terms are linked to the second order nonlinear susceptibility, and they include...
power from a ‘pump’ wave at $\omega_3$ by the frequencies and optical rectification, the Faraday effect, the Pockels effect, and the second harmonic generation or the parametric amplification [5].

A general three-wave mixing can be viewed as the generation of an optical wave by the combination of two other waves and vice versa (see figure 4). Let the frequencies and wavevectors of these two optical waves be $(\omega_1, k_1)$, $(\omega_2, k_2)$, then the frequency and the wavevector of the third optical wave can be written as:

$$\omega_3 = \omega_1 \pm \omega_2 \quad k_3 = k_1 \pm k_2 \quad (36)$$

The optical rectification is the generation of a dc polarization or a dc magnetization in a nonlinear medium at the passage of an intense optical beam [53] ($\omega_1 = \omega_2$, $\omega_3 = \omega_1 - \omega_2 = 0$). It is a special case of difference frequency generation [5] since it can be interpreted as initial photons forming new photons of zero energy and frequency.

The second harmonic generation, also called frequency doubling, is a process in which photons interact with a nonlinear material to form new photons of twice the energy, and therefore twice the frequency, of the initial photons [4] ($\omega_3 = 2\omega_1$). It is a special case of sum frequency generation [5].

Optical Parametric amplification involves the transfer of power from a ‘pump’ wave at $\omega_3$ to two waves at lower frequencies $\omega_1$ and $\omega_2$, with $\omega_3 = \omega_1 + \omega_2$ [54]. In particular, a photon interacting with a nonlinear material may give rise to two photons each of half the energy of the incoming one [55].

The Faraday effect is the rotation of the plane of polarization of a linearly polarized light beam which is linearly proportional to the component of the magnetic field in the direction of propagation [3]. The Faraday effect can be associated with a circular birefringence, i.e. with a difference in the index of refraction for light rightward or leftward circularly polarized with respect to the direction of the magnetic field [5].

The Pockels effect, also known as the electro-optic effect, is the linear birefringence in an optical medium induced by an electric field [56]. Linear birefringence means that the index of refraction depends on the linear polarization of light [5].

As said before, none of these effects exist in a vacuum.

### 4.2. Four-wave mixing

Four-wave mixing is a general name representing any effect due to the combination of four electromagnetic fields which can be collinear or not. Some of them can be of zero frequency i.e. electrostatic fields. It corresponds to the terms proportional to $E^4$, $E^2B^2$ and $B^4$ in equation (30). As shown in figure 5, this can be viewed as a combination of two waves to give two other waves, or a combination of three of them to give one, and vice versa. Four-wave mixing is allowed in a quantum vacuum.

#### 4.2.1. Vacuum nonlinear static polarization or magnetization.

Let us first present the case in which only static fields are involved (see figure 6). Equations (31) and (32) become:

$$P_0 = 4c_2,0\varepsilon_0 E_0 \left( \varepsilon_0 E_0^2 - \frac{B_0^2}{\mu_0} \right) + 2c_{0,2} \frac{\varepsilon_0}{\mu_0} B_0 (E_0 \cdot B_0), \quad (37)$$

$$M_0 = -4c_{2,0} \frac{B_0}{\mu_0} \left( \varepsilon_0 E_0^2 - \frac{B_0^2}{\mu_0} \right) + 2c_{0,2} \frac{\varepsilon_0}{\mu_0} E_0 (E_0 \cdot B_0),$$

where $E_0$ and $B_0$ are the static electric and magnetic fields, respectively. These formulas clearly indicate that a vacuum is polarized and magnetized by the presence of static fields. It is important to stress that $P_0$ not only depends on $E_0$ but also on $B_0$, and $M_0$ not only depends on $B_0$ but also on $E_0$. Moreover $P_0$ and $M_0$ depend as well on the angle between $E_0$ and $B_0$.

For the sake of argument, let us estimate the value of magnetization expected only when $B_0$ is present. Equation (38) gives $\mu_0 M_0 \approx 5.3 \times 10^{-24} \times B_0 (T)^3$, a value that appears out of reach since currently the best magnetometers are not able to measure less than $10^{-15} T$ in one second (see e.g. [57]).

In [30, 58], the expected magnetization in the presence of a external magnetic field is given in the framework of QCD. This value of magnetization for a magnetic field $B_0 \ll B_0^{QCD}$ scales as the fourth power of the ratio between the electron mass and the pion mass, which is about 280. A magnetization $\mu_0 M_0 \approx 10^{-33} \times B_0 (T)^3$ is predicted [30].

#### 4.2.2. Kerr effect, Cotton–Mouton effect, Jones birefringence, magneto-electric birefringence.

In this section we deal with linear birefringences in a vacuum. These birefringences can be induced by an electric field, a magnetic field or a combination of both. All these birefringences are manifestations of four-wave mixing when two of the waves are static fields.

Following equation (31), we can write:

$$P = 4c_{2,0} \varepsilon_0 (E_0^2 + E_0) \times \left( \varepsilon_0 E_0^2 - \frac{B_0^2}{\mu_0} + 2\varepsilon_0 E_0 \cdot B_0 - \frac{2B_0 \cdot B_0}{\mu_0} \right) + 2c_{0,2} \frac{\varepsilon_0}{\mu_0} (B_0 + B_0)(E_0 \cdot B_0 + E_0 \cdot B_0 + E_0 \cdot B_0),$$

(39)
induced by the combined effect of a transverse electric field when one can write that Jones birefringence the difference \( n_B - n_E = k_{ME} E_0 B_0 \). Symmetry considerations indicate that \( k_j = k_{ME} \) [64]. For both bilinear birefringences, another particularity is that \( n \) changes sign when the wavevector of light \( k \) becomes \( -k \). A medium immersed in a magnetic and electric field therefore also shows an axial birefringence since \( \Delta n_\parallel = n_{+} k - n_{-} k = k_0 E_0 B_0 \) as proved experimentally for the first time in [65].

All these phenomena are expected in a vacuum. Therefore, in the presence of electrostatic fields a quantum vacuum behaves as a uniaxial birefringent crystal. However, as is shown in [66], in the presence of both electric and magnetic fields perpendicular to each other with one of these fields parallel to the direction of light propagation, no bilinear birefringence appears, but only a birefringence due to the Kerr or the CME.

Papers denoting the value of the magnetic Cotton–Mouton birefringence and of the electric Kerr birefringence are [16, 67]. More recently, in 2000, effects due to the presence of an electric field \( E_0 \) and a magnetic field \( B_0 \) were studied in [68, 69], where the connection with the magneto-electric and Jones birefringence was presented.

Following [70], we now study the CME, which is the vacuum nonlinear optics effect that has most attracted the interest of experimentalists. This effect is due to the presence of a static transverse magnetic field \( B_0 \). Neglecting static terms, the equations (39) and (40) become:

\[
P^{\text{CM}} = -4c_2 \epsilon_0 E_0^2 B_0^2 \mu_0 + 2c_0 \frac{\epsilon_0}{\mu_0} B_0 (E_0 \cdot B_0),
\]

where the connection with the magneto-electric and Jones birefrigence is given by the electric and magnetic static fields. Jones birefringence has the particularity that the birefringence axis is given by the magnetic and electric static fields. For Jones birefringence the difference \( \Delta n_j \) between the index of refraction for light polarized at \( +45^\circ \) to the electric field \( n_+ \) and the index of refraction for light polarized at \( -45^\circ \) to the electric field \( n_- \) is proportional to \( E_0 B_0, \Delta n_j = k_j E_0 B_0 \). For magneto-electric birefringence one can write that \( \Delta n_{ME} = n_B - n_E = k_{ME} E_0 B_0 \). Symmetry considerations indicate that \( k_j = k_{ME} \) [64]. For both bilinear birefringences, another particularity is that \( n \) changes sign when the wavevector of light \( k \) becomes \( -k \). A medium immersed in a magnetic and electric field therefore also shows an axial birefringence since \( \Delta n_\parallel = n_{+} k - n_{-} k = k_0 E_0 B_0 \) as proved experimentally for the first time in [65].

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\[ \epsilon_\perp = \epsilon_0 - 4c^2 \epsilon_0 B_0^2 \mu_0, \]  
\[ \Delta \epsilon = \epsilon_1 - \epsilon_\perp = 2c_0^2 \epsilon_0 B_0^2 \mu_0, \]  
(44)  

and in the same way one also has  
\[ \mu_\parallel = \mu_0 \left( 1 + \frac{4c^2 \epsilon_0 B_0^2}{\mu_0} \right), \]  
\[ \mu_\perp = \mu_0 \left( 1 + \frac{12c^2 \epsilon_0 B_0^2}{\mu_0} \right), \]  
\[ \Delta \mu = \mu_\parallel - \mu_\perp = -8c^2 \epsilon_0 B_0^2, \]  
(45)  
(46)  
(47)  

where the symbol \( \parallel \) accompanies any quantity related to a light polarization parallel to the static field, and the symbol \( \perp \) any quantity related to a light polarization perpendicular to the static field.

Then one can calculate the refractive index of the medium:  
\[ n_\parallel = \frac{\sqrt{\epsilon_\parallel \mu_\parallel}}{\sqrt{\epsilon_0 \mu_0}} = 1 + c_0^2 B_0^2 \mu_0, \]  
\[ n_\perp = \frac{\sqrt{\epsilon_\perp \mu_\perp}}{\sqrt{\epsilon_0 \mu_0}} = 1 + 4c^2 \epsilon_0 B_0^2 \mu_0, \]  
(49)  
(50)  

Finally the anisotropy \( \Delta n \) is equal to  
\[ \Delta n_{\text{CM}} = n_\parallel - n_\perp = \frac{\sqrt{\epsilon_\parallel \mu_\parallel} - \sqrt{\epsilon_\perp \mu_\perp}}{\sqrt{\epsilon_0 \mu_0}} = (c_0^2 - 4c^2 \epsilon_0 B_0^2) \frac{B_0^2}{\mu_0}. \]  
(51)

Note that \( n_\parallel \) depends only on \( c_0^2 \) and \( n_\perp \) on \( c_0^2 \) like \( \Delta \epsilon \) and \( \Delta \mu \), respectively. Note also that, since the velocity of light has to be smaller than \( c \), \( c_0^2 \) and \( c_2^0 \) have to be positive.

The result given in the previous equation holds as far as Lorentz invariance holds. QED prediction via the Heisenberg–Euler Lagrangian is that the magneto-electric one and therefore \( \Delta \mu \) is different. This gives an axial birefringence \( \Delta n_\alpha \). One can show [75] that the axial birefringence is

\[ \Delta n_{\text{CM}} = 3c_2^0 \frac{B_0^2}{\mu_0}. \]  
(52)  

Also taking into account Ritus corrections for \( c_0^2 \) and \( c_2^0 \), one obtains a more precise result:  
\[ \Delta n_{\text{CM}} = \frac{2\alpha^2 h^3}{15 m_e^2 c^5} \left( \frac{5}{6\pi m_e^2 c^5} + \frac{5}{6\pi m_e^2 c^5} \right) B_0^2 \mu_0, \]  
\[ = \frac{2\alpha^2 h^3}{15 m_e^2 c^5} \left( 1 + \frac{25\alpha}{4\pi} \right) B_0^2 \mu_0. \]  
(53)

The Ritus term for \( \Delta n_{\text{CM}} \) corresponds to a 1.45% correction to the leading term.

Finally, using CODATA values [71] for fundamental constants one obtains \( \Delta n_{\text{CM}} = (4.0317 \pm 0.0009) \times 10^{-24} \text{T}^{-2} \), where the uncertainty is calculated assuming arbitrarily that the \( \alpha^2 \) order radiative correction, which has never been calculated, amounts to about a 1.5% correction like the \( \alpha \) order one.

In [72] and more recently in [73], the indexes of refraction \( n_\parallel \) and \( n_\perp \) are given when the static magnetic field \( B_0 \) is comparable and even greater than \( B_\text{cr} \). It is worth stressing that linear birefringence effects can be considered achromatic as long as \( \frac{\hbar \omega_\parallel}{m_e^2 c^2} = \frac{\hbar \omega_\perp}{m_e^2 c^2} \ll 1 \).

In [74] one can also find corrections to the value of \( \Delta n_{\text{CM}} \) calculated in the framework of QCD. As already discussed these corrections are negligible because of the QCD energy scale.

The values of the Cotton–Mouton, Kerr and magneto-electric birefringences are all related by the vacuum Lorentz invariance as demonstrated in [66]. The Kerr effect must have the same value for \( \Delta n_K \) as the CME but with the opposite sign for \( E_\parallel = e B_0 \).

\[ \Delta n_K = -\frac{2\alpha^2 h^3}{15 m_e^2 c^5} \left( 1 + \frac{25\alpha}{4\pi} \right) \epsilon_0 E_0^2, \]  
(54)  
\[ \Delta n_{\text{ME}} = -\frac{4\alpha^2 h^3}{15 m_e^2 c^5} \left( 1 + \frac{25\alpha}{4\pi} \right) \sqrt{\frac{\epsilon_0}{\mu_0}} \cdot (E_0 \times B_0), \]  
(55)

i.e. \( k_{\perp ME} \approx \pm 2.6 \times 10^{-32} \text{m V}^{-1} \text{T}^{-1} \). Finally Jones birefringence must have the same magnitude and sign as the magneto-electric one and therefore \( k_\parallel \approx \pm 2.6 \times 10^{-32} \text{m V}^{-1} \text{T}^{-1} \) as well.

In both magneto-electric and Jones configurations, light going back and light going forward in a vacuum do not travel at the same velocity since \( n \) is different. This gives an axial birefringence \( \Delta n_\alpha \). One can show [75] that the axial birefringence is

\[ \Delta n_\alpha = n(k) - n(-k) = -\frac{8\alpha^2 h^3}{15 m_e^2 c^5} \left( 1 + \frac{25\alpha}{4\pi} \right) \sqrt{\frac{\epsilon_0}{\mu_0}} (E_0 \times B_0), \]  
(56)

i.e. \( k_\alpha \approx 5.2 \times 10^{-32} \text{m V}^{-1} \text{T}^{-1} \).

Experiments looking for a variation in the velocity of light in the presence of a magnetic field can be traced back to the end of the 19th century when Morley, Eddy and Miller somewhat inaugurated this experimental field using a Michelson–Morley interferometer [76] in 1898 [77].

The basic idea (see figure 10) was to split in two a beam of light coming from the flame of a Bunsen burner colored by a Nicol prism [76], and to observe the interference bands due to the different optical paths in the two interferometer arms. If light velocity was affected by the presence of a 0.165 T magnetic field in one of the two arms, a displacement of the interference bands would have been observed. The displacement of the fringes was simply monitored by eye. As a matter of fact, light did not propagate in the vacuum but in liquid carbon bisulfide and air, and the magnetic field was directed parallel to the light propagation vector. Authors looked for a retardation effect which should exist together with the Faraday rotation induced by the material medium. Such an effect is difficult to explain in modern terms. Nevertheless, they reached a sensitivity of one part in a hundred million in the measurement of a possible change of the velocity of light depending on the magnetic field.
Nowadays, we know that because of the CME of a vacuum, the expected variation in light velocity is quadratic in the magnetic field and is of the order of $10^{-24}$.

Motivated again by the search for a photon magnetic moment and by Watson’s paper [78], Farr and Banwell in 1932 [79] and 1940 [80] reported measurements of the velocity of the propagation of light in a vacuum in a transverse magnetic field. Their experiment was first based on a Jamin interferometer [76] and in the 1940 version on a Michelson interferometer [76].

Light from an incandescent lamp was polarized by a Nicol prism [76], separated into two rays, and then one of the rays was passed through a magnetic field of 1.8 T over 1.125 m. Observation of a possible fringe displacement induced by the magnetic field was first observed by eye. In the 1940 version of their apparatus the detector was a photoelectric cell, the signal of which was amplified and read by a galvanometer to record it on a moving photographic strip which increased the sensitivity of the apparatus. The final result was that in a 2 T field the relative variation of light velocity was less than $2 \times 10^{-9}$ [80].

No clear theoretical predictions motivated such experiments. The first citation of the existence in a vacuum of the CME can only be found in a paper by Erber in 1961 [81], which also provides an estimate of the value of the Cotton–Mouton CME, and where a discussion of different experimental approaches can also be found. This renewed interest was motivated by the laser invention [31] and the progress in magnets providing fields of several tesla. This has raised the hope that effects due to vacuum polarization in the presence of a strong magnetic field could be soon observable.

The idea of using a Michelson–Morley interferometer to measure the variation in the velocity of light induced by a magnetic field has been proposed again several times [82–85], following the technical progresses in Michelson interferometry driven by the search for gravitational waves on earth [86, 87] using such an apparatus.

The theoretical results published in the 1970s [16, 67] raised a new interest in the field, and in 1979 Iacopini and Zavattini proposed to measure the vacuum linear magnetic birefringence via the ellipticity induced on a linearly polarized laser beam by the presence of a transverse magnetic field [88]. In fact, a linearly polarized light beam propagating in a birefringent medium becomes elliptically polarized. In general, the acquired ellipticity $\Psi$ is related to a birefringence $\Delta n$ by the formula:

$$\Psi = \pi \frac{L}{\lambda} \Delta n \sin 2\theta,$$

where $L$ is the optical path in the birefringent medium, $\lambda$ is the light wavelength and $\theta$ the angle between the light polarization and the birefringence axis [76]. If $L$ is considerably greater than $\lambda$, the measurement of $\Psi$ can become an interesting indirect method of measuring $\Delta n$. The measurement of ellipticity has been the main method of studying the CME since its discovery in 1902 [89]. Since $\Psi$ is proportional to $\Delta n$, in the case of the vacuum CME $\Psi \propto (\frac{\Delta n}{\lambda})^2$.

It is worth mentioning that the propagation in a birefringent medium not only induces an ellipticity, but also...
is put between a crossed polarizer and analyzer in order to measure the enhanced by an optical cavity because the effect can be summed polarization in an elliptical polarization. The induced ellipticity is applied. The induced birefringence (CME) transforms the linear polarization of the light. The magnetic field of the Faraday cell gives a transverse magnetic field surrounded by an optical cavity that interfering light follows two different paths.

Figure 12. Zavattini’s proposal for ellipticity measurement. A linearly polarized beam is injected into a Fabry–Perot cavity formed by the two mirrors $M_1$ and $M_2$, in which a magnetic field $B_0$ is applied. The induced birefringence (CME) transforms the linear polarization in an elliptical polarization. The induced ellipticity is enhanced by an optical cavity because the effect can be summed when light goes back and forth in the magnetic field. The apparatus is put between a crossed polarizer and analyzer in order to measure the ellipticity given by the ratio $I_\perp/I_\parallel$. Moreover one can modulate the field and use an ellipticity modulator for measuring the effect with an heterodyne detection.

rotates the light polarization direction [76]. If $\Psi \ll 1$ the rotation angle $\theta_r$ can be written as $\theta_r \leq \frac{\Psi}{2}$, therefore one can assume that $\theta_r \propto (\Delta n)^2$, and in particular for the vacuum CME $\theta_r \propto \left( \frac{B}{B_{CM}} \right)^4$. This means that this rotation can be safely neglected.

Zavattini’s proposal has been a big step forward since all the experiments performed in recent years or currently under way are based on it. Let us discuss it in detail.

In order to have a larger optical path in the field, the effect to be measured is increased using an optical cavity. Moreover, the ellipticity and the magnetic field were modulated in order to be able to use a heterodyne detection technique to increase the signal-to-noise ratio. As sketched in figure 12, a linearly polarized beam passes through an ellipticity modulator composed by a Faraday cell and a quarter wave plate properly aligned. The magnetic field of the Faraday cell gives a modulated ellipticity $\eta(t) = \eta_0 \cos(\Omega t)$, which induces an ellipticity modulation. Then the beam is injected into a transverse magnetic field surrounded by an optical cavity defined by mirrors $M_1$ and $M_2$. The magnetic field is also modulated so that the ellipticity induced by the field can be written as $\psi(t) = \psi_0 \cos(\omega t + \phi)$. At the output of the optical cavity, the light beam is analyzed with a polarizer ($A$) crossed with respect to the first one ($P$). The light is sent to two photodiodes which deliver signals proportional to $I_\parallel$ and $I_\perp$. The transmitted light after the optical cavity is:

$$I_\perp = I_0 [\sigma^2 + (\Theta + \psi(t) + \eta(t))^2] , \quad (58)$$

where $I_0$ is the beam intensity before the analyzer, $\sigma^2$ is the extinction factor of the polarizers, and $\Theta$ represents any static uncompensated ellipticity.

Developing equation (58), one sees that the transmitted light is composed of different frequency components (see table 2).

| Frequency | Intensity $I_0$ |
|-----------|-----------------|
| $\Omega_0 - \omega$ | $\eta_0 \psi_0$ |
| $\Omega_0 + \omega$ | $2\eta_0 \theta_r$ |
| $\Omega_0$ | $\eta_0$ |

Table 2. Fourier components of the transmitted light of interest for the heterodyne detection technique.

respect to the Michelson interferometer apparatus where interfering light follows two different paths.

After tests at CERN [90] in Switzerland [91], an apparatus has been set up at the Brookhaven National Laboratory [92], USA [93]. It is based on a multipass cavity for enhancing the signal by a factor of 250, and a magnetic field of up to 4 T over 8.8 m modulated at about 30 mHz. In [93], authors report a sensitivity in ellipticity of $7.9 \times 10^{-8}$ rad Hz$^{-1/2}$. This sensitivity being insufficient to detect the QED effect, authors concentrated their efforts on the search for the existence of particles beyond the standard model coupling with electromagnetic fields.

In fact, in 1986, Maiani et al [94] showed that hypothetical low mass, neutral, spinless bosons, scalar or pseudoscalar, that couple with two photons, could induce an ellipticity signal in the Zavattini apparatus similar to the one predicted by QED. Moreover, an apparent rotation of the polarization vector of the light could be observed because of the conversion of photons into real bosons, resulting in a vacuum magnetic dichroism which is absent in the framework of standard QED. The measurements of ellipticity and dichroism, including their signs, can in principle completely characterize the hypothetical boson, its mass $m_s$, the inverse coupling constant $M_s$, and the pseudoscalar or scalar nature of the particle. The Maiani, Petronzio and Zavattini paper was essentially motivated by the search for axions. These are pseudoscalar, neutral, spinless bosons introduced to solve what is called the strong CP problem, i.e. the fact that there is no experimentally known violation of CP-symmetry in quantum chromodynamics even if there is no known reason for it to be conserved in QCD specifically. A discussion about non-standard-model physics in external fields can be found in [95].

No signal was observed and the final result of the Brookhaven National Laboratory experiment was that $k_{CM} \leq 2.2 \times 10^{-19}$ T$^{-2}$.

In 1991, a new attempt to measure the vacuum magnetic birefringence was begun at the LNL [96] in Legnaro, Italy, by the PVLAS collaboration [70]. This experiment is again based on [88]. A vertical Fabry–Perot cavity is used to increase the effect to be measured, while a superconductive 5 T magnet rotates around its own axis to modulate it. To the first order, this case can be calculated in the approximation by regarding the magnetic field as fixed at its instantaneous angular orientation, using the standard vacuum birefringence formulae for a static magnetic field [97]. Results on vacuum magnetic birefringence published in 2008 [98] indicate that the apparatus had a noise level of about $1.7 \times 10^{-20}$ T$^{-2}$ for $k_{CM}$.

In the meantime a new proposal has been put forward based at National Tsing Hua University [99], Hsinchu, Taiwan,
Republic of China: the Q&A project. This project started around 1996 [100]. The experimental set up is similar to the PVLAS one but the magnetic field is produced by permanent magnets and the 3.5 m long cavity is formed by two high reflectivity mirrors suspended by two X-pendulum suspensions mounted on two isolated tables. No birefringence effect has yet been detected and the achieved sensitivity in ellipticity is $10^{-6}$ rad Hz$^{-1/2}$ [101].

Another proposal based on the use of pulsed magnets as suggested in [102] has been presented in [103]: the BMV project. This experiment is also based on the Zavattini proposal and is mounted at the LNCMI-T [46], Toulouse, France. The Fabry–Perot cavity is 2.2 m long and the cavity finesse is greater than 400 000. The novelty of this experiment is the use of pulsed magnets in order to reach higher fields. A specially designed magnet delivers more than 10 T in the first version, and 30 T has already been reached for the next generation experiment. In [104], the authors show that a single magnetic pulse is sufficient to detect birefringence signals as low as $k_{CM} \approx \frac{5 \times 10^{-20}}{T^2}$.

The use of long superconducting magnets developed for accelerator machines has also been proposed by a collaboration [105] based at Fermilab [106], USA, and more recently by a group based at CERN [90]: OSQAR [107].

Finally, a new version of the PVLAS experiment was very recently set up at the INFN [108] in Ferrara, Italy. As the Q&A experiment, it is based on the use of rotating permanent magnets. The value of the field is 2.3 T. A Fabry–Perot cavity with a finesse of about 250 000 was used to increase the optical path. This apparatus has given a limit on vacuum magnetic linear birefringence of $k_{CM} \leq 4.4 \times 10^{-21}$ T$^{-2}$ [109].

The ellipticity formula given before (equation (57)) shows that decreasing the light wavelength increases the ellipticity effect being measured. Erber discussed this experimental possibility in his 1961 paper [81], and more recently in [110] a proposal to use gamma rays to measure the vacuum magnetic birefringence can be found. Gamma rays are produced by inverse Compton scattering on an electron beam of a storage ring by a polarized laser beam. In principle the photons obtained are also polarized. After passing through a magnetic field, gamma polarization is analyzed by using the fact that shower production depends on gamma polarization in crystals.

The basic idea hidden in Watson’s apparatus of 1929 was to transform a light-velocity variation into a frequency variation. Frequency measurements are among the most precise measurements that can be performed nowadays (see e.g. [111]). The resonance frequency of optical cavities depends on the optical length of the cavity itself. Any variation in the index of refraction, for example induced by a magnetic field, will induce a variation in the resonance frequency. This gives the signal to be measured by comparing it with a frequency reference signal. For the vacuum CME this idea was first envisaged by Erber [81]. More recently, the use of a ring He–Ne laser, i.e. a laser based on a ring optical cavity, has been suggested in [112], a linear Fabry–Perot cavity in [89], and a feasibility study of a vacuum magnetic birefringence by frequency shift measurement in [113]. All these methods give beat frequencies to be measured of the order of 10 nHz corresponding to a relative frequency shift of about $10^{-22}$, which looks very challenging since nowadays the best relative frequency measurements are at the level of about $10^{-17}$ (see e.g. [111]).

As far as the Kerr effect is concerned, let us recall that experiments in the presence of a static electric field appear more difficult from the technological point of view, since one needs a 300 MV m$^{-1}$ electric field to equalize the effect of a 1 T magnetic field. This is certainly the reason why the experimental observation of the Kerr effect in a vacuum has not yet been attempted.

Recently magneto-electric birefringence effects have also attracted the attention of experimentalists. The particularity of this type of effect is that they do not accumulate in a linear cavity as do the Cotton–Mouton and Kerr effects, since the index of refraction depends on the direction of propagation with respect to the plane containing $E$ and $B$ vectors. One therefore needs a ring laser [68] or a ring cavity [69] (see figure 13). Very recently an experimental method to measure magneto-electric axial birefringence in dilute matter has been proposed [114], and a measurement performed in gas-phase molecular nitrogen has shown a noise level corresponding to a $k \approx 10^{-23}$ m V$^{-1}$ T$^{-1}$ [115]. A ring cavity is used [112], the injected laser beam is split in two, and one looks to the difference in the resonance frequency between the laser beam turning clockwise and anticlockwise in the cavity.

4.2.3. Optical field-induced birefringence. In this kind of experiment the external field is produced by an electromagnetic wave. One of the experimental challenges of this kind of measurement is to produce electromagnetic fields as high as possible and to have a long interaction region. Proposals have been published suggesting the use of energetic laser pulses for this purpose (see figure 14). As we have already discussed, in the focal spot of a powerful laser beam, electromagnetic fields may reach values exceeding those of static fields that can be found in laboratories.

A proposal to measure birefringence induced by a counter-propagating intense laser beam is reported in [116]. In [117],
birefringence measurements using two pulsed laser beams, one as a probe and the second, more intense, beam as a field source is proposed and in [118], in a similar configuration, phase-contrast Fourier imaging [119] is proposed as a detection technique. In [120, 121], a laser is again proposed as the field source while an x-ray beam is used as a probe to measure the birefringence, taking advantage of the shorter wavelength. For example, the authors of [121] show that a wave of 0.4 nm wavelength propagating over a distance of 1.5 µm where a standing wave provides a laser intensity of $10^{27}$ W m$^{-2}$ acquires an ellipticity of about $4 \times 10^{-8}$ rad. It is important to note that because of diffraction effects, the polarization direction is also rotated by an angle of the same order as the ellipticity. Further studies can also be found in [122] and [123]. Recently, very high purity polarization states of x-rays have been reported [124], which opens up the possibility of detecting vacuum magnetic birefringence with this kind of experimental setup.

**Figure 14.** Optical field-induced birefringence. The external electric or magnetic field is produced by an electromagnetic wave. In this kind of experiment, a linearly polarized beam becomes elliptically polarized passing through another electromagnetic wave.

### 4.2.4. Optical rectification induced by electrostatic fields i.e. inverse magneto-electric effects.

In a standard medium, optical rectification can be induced by a light beam in the presence of an external magnetic and/or electric field. These effects are also known as inverse effects with respect to the field-induced birefringences.

As an example, the inverse Cotton–Mouton effect (ICME) corresponds to a static magnetization induced in a medium by a non-resonant linearly polarized light beam propagating in the presence of a transverse magnetic field (see figure 15). This magnetization is proportional to the value of the magnetic field, and to the intensity of the propagating electromagnetic wave (see [5] and references therein). As stated in [5], microscopically, the induced dc magnetization arises in a standard medium because the optical field shifts the different magnetic states of the ground manifold differently, and mixes into these ground states different amounts of excited states. It appears that the inverse effects have not attracted much attention from experimentalists. The observation of the ICME has only been reported very recently in a terbium gallium garnet crystal [125]. The ICME has also been calculated for the quantum vacuum in [126].

The starting point of the calculation is equation (32). Two cases are possible ($E_\omega \parallel B_0, B_\omega \perp B_0$) or ($E_\omega \perp B_0, B_\omega \parallel B_0$).

In the first case one obtains:

$$M_{ICM\parallel} = 14c_2\varepsilon_0 E_\omega B_0 \mu_0 = 14c_2\varepsilon_0 \frac{I_\omega B_0}{c \mu_0}.$$  

(59)

In the second case one gets:

$$M_{ICM\perp} = 8c_2\varepsilon_0 \frac{B_\omega^2 B_0}{\mu_0 \mu_0} = 8c_2\varepsilon_0 \frac{I_\omega B_0}{c \mu_0}.$$  

(60)

where $I_\omega$ is the intensity associated with the electromagnetic wave, and where we have used the relation $c_{0,2} = 7c_{2,0}$. In both cases $M_{ICM}$ is parallel to $B_0$.

If a laser pulse is focused to get $I_\omega \approx 10^{26}$ W m$^{-2}$ in a vacuum where a transverse magnetic field of more than 10 T is present, the magnetization to be measured because of the ICME of the quantum vacuum is

$$M_{ICM\parallel} \approx 8 \times 10^{-11} \text{T}$$  

(61)

and

$$M_{ICM\perp} \approx 4.5 \times 10^{-11} \text{T};$$  

(62)

where we have used the relation $\mu_0 M(\text{A m}^{-1}) = M(\text{T})$. The authors of [126] suggest that these values could be reached with new laser facilities, but the very small value of the induced magnetization remains an experimental challenge (optical-field-induced magnetization as small as $10^{-10}$ T is reported in [127]).

Symmetry considerations already applied to direct effects indicate that an inverse Kerr effect and inverse magneto-electric birefringence effects exist in a vacuum as well.

**Figure 15.** Inverse magneto-electric effects. Optical rectification is induced by the presence of an external magnetic or electric field. Here the interaction between two photons of the electromagnetic wave and a photon of the static magnetic field gives rise to a magnetization $M$. This effect is known as the ICME if the static magnetic field is perpendicular to the direction of light propagation.

### 4.2.5. Optical-field-induced inverse magneto-electric effects. As in the case of magneto-electric birefringences, inverse effects might also be induced by fields associated with electromagnetic waves. As far as we know, this subject has never been treated in literature.
4.2.6. Parametric amplification induced by an electrostatic field: photon splitting. Since the seventies physicists have studied a phenomenon called photon splitting, which is the splitting of a photon propagating in a vacuum into two photons in the presence of a transverse magnetic field (figure 16). In terms of nonlinear optics textbooks this phenomenon is a parametric amplification induced by an electromagnetic field, but as far as we know it has not been observed in standard media. The incoming photon beam can be considered as the pump electromagnetic wave, and the outgoing photon beams as the signal electromagnetic wave and the idler electromagnetic wave [5]. The reference paper for this vacuum effect is [67]. The results of this work have been confirmed more recently in [129].

The calculations reported in [67] are far from straightforward and cannot be easily summarized here. In the following we summarize the main results and give some numerical estimates of such an effect.

First of all, let us consider the case of a constant and spatially uniform external magnetic field, with pump, signal and idler waves all collinear. The contribution of this diagram vanishes in a vacuum.

\[
\kappa_{\text{box}} \approx 12.0 \left( \frac{h\omega}{m_e c^2} \right)^5 \left( \frac{B_{\text{cr}}}{B_{\text{cr}}} \right)^6 \sin \theta
\]

where \( \kappa \) is the attenuation coefficient in m\(^{-1} \) and \( \theta \) the angle between the external magnetic field and the propagation wavevector.

This formula can be written as

\[
\kappa_{\text{box}} \approx 12.0 \left( \frac{h\omega}{m_e c^2} \right)^5 \left( \frac{B_{\text{cr}}}{B_{\text{cr}}} \right)^6 \sin \theta
\]

and can be further reduced to

\[
\kappa_{\text{box}} \approx 12.0 \left( \frac{h\omega}{m_e c^2} \right)^5 \left( \frac{B_{\text{cr}}}{B_{\text{cr}}} \right)^6 \sin \theta
\]

For photons of \( h\omega = 1.6 \times 10^{-19} \) J (1 eV) under a 10 T transverse field, \( \kappa \) is as small as \( 4.8 \times 10^{-80} \) m\(^{-1} \).

Taking into account higher order diagrams does not much change this result since their contributions are even smaller than that from the hexagonal diagram, as has been proved in [67]. Taking radiative corrections into account should give a correction of the order of \( \alpha \), a calculation which has not yet been performed.

If the outgoing photons are no longer collinear to the incoming photon, the box diagram no longer vanishes (figure 19). This is the case when the magnetic field is no longer spatially uniform. This means that the magnetic field can transfer momentum \( p \) that one can write as \( \frac{\vec{p}}{\vec{B}_0} \), where \( l \) is along the direction parallel to the incoming wavevector. Following [67], the ratio of the attenuation coefficient \( \kappa_{\text{box}} \) due to the box diagram to the attenuation coefficient \( \kappa_{\text{hexagon}} \) due to the hexagonal diagram can be written as

\[
\frac{\kappa_{\text{box}}}{\kappa_{\text{hexagon}}} \sim \left( \frac{B_{\text{cr}}}{B_0 \sin \theta} \right)^4 \left( \frac{2cp}{h\omega} \right)^2.
\]

If one has a 10 T transverse component and a 1000 T m\(^{-1} \) magnetic-field gradient, the \( \kappa_{\text{box}} \) contribution can be \( 10^{10} \) times greater than the \( \kappa_{\text{hexagon}} \) contribution, but the resulting
attenuation coefficient would be of the order $10^{-70} \text{m}^{-1}$, which is unmeasurable in a terrestrial laboratory. When $B_0$ approaches $B_{cr}$, $\kappa_{hexagon}$ becomes more important than $\kappa_{box}$.

A way of increasing photon-splitting probability is to use very high fields such as those existing in atoms. The probability increases even more if the photon energy also exceeds $m_e c^2$. As reported in [130], a review of photon splitting in atomic fields, the Heisenberg–Euler Lagrangian can only be used to derive the photon-splitting cross section when $\hbar \omega \ll m_e c^2$.

In this limit, the total cross section in the coulomb field is determined by the box diagram contribution since the field gradient is important, and can be written in barns ($10^{-28} \text{m}^2$) as

$$\sigma_{PSAF} = 7.4 \times 10^{-28} \left( \frac{Z^2 \alpha^3}{m_e^2 c^2} \right) \left( \frac{\hbar \omega}{m_e c^2} \right)^6$$

$$= 3.7 \times 10^{-25} Z^2 \alpha \left( \frac{\omega}{c} \right)^6 c_{2,0}^2,$$  

(67)

or

$$\sigma_{PSAF} = 2.3 \times 10^{-12} Z^2 \left( \frac{\hbar \omega}{m_e c^2} \right)^6,$$  

(68)

where $Z$ is the atomic number. Such a cross section for an optical photon of energy of about $1.6 \times 10^{-19} \text{J}$, assuming $Z = 100$, is about $10^{-23} \text{m}^2$, which does not seem measurable even using very intense laser beams.

Using high energy photons changes the experimental perspective. The already cited review [130] gives a clear overview of the field. In particular, one can find details of the reported first observation of photon splitting in an atomic field for photons of energy between 120 and 450 MeV [131]. With photons of this energy one gains about 18 orders of magnitude in the cross section with respect to that of optical photons. The authors of [131] observed about 400 photon-splitting events for $1.6 \times 10^9$ incoming photons on a Bi$_4$Ge$_3$O$_{12}$ (BGO) target. This result is at 1.6 standard deviations from the prediction of a Monte Carlo simulation based on the most precise QED cross section calculated, taking into account the contributions of terms of all orders in the parameter $Z \alpha$. If one uses cross sections calculated only at the lowest orders, the difference between simulation and experiment becomes 3.5 standard deviations. The experiment was conducted at the VEPP-4M collider at the Budker Institute of Nuclear Physics, Novosibirsk, Russia [132]. As far as we know, this is still the only observation of such a phenomenon.

4.2.7. Optical field-induced photon splitting. As for the case of birefringences, one may study the case of photon splitting being induced by an electromagnetic field, which is generated by an intense electromagnetic wave (see figure 20). Theoretical studies of such a phenomenon have been reported in [133] and more recently in [134].

The photon-splitting probability depends essentially on two Lorentz-invariant parameters [134], $\eta = \frac{\hbar \omega_{00}}{m_e c^2}$ and $\chi = \frac{\hbar \omega_{00} E_{cr}}{m_e c^2}$, where $\hbar \omega$ is the energy of the incoming photon, $\hbar \omega_{00}$ the energy of the electromagnetic wave generating the field, and $E_{cr}$ is the electric field associated with this electromagnetic wave. The case corresponding to $\eta \ll 1$ and $\chi \ll 1$ can be solved using the Heisenberg–Euler Lagrangian. The conversion rate obtained is $W_{PSLA} \propto \alpha^2 \chi^6$ as shown in [133, 134].

Rates in more general cases are given in [134]. Following the conclusions of the authors, it appears that even with fields $E_{cr}$ approaching $E_{cr}$, as in those that will be provided by very powerful lasers in the future, the conversion rate is such that an observation of optical field-induced photon splitting remains very difficult. For example, let us assume that the field is provided by a laser of intensity $10^{20} \text{W m}^{-2}$ corresponding to $E_{cr} \sim 5 \times 10^{-3} \text{V m}^{-1}$, and that the photon energy is $1.6 \times 10^{-19} \text{J}$, the pulse duration is 10 fs and the pulse repetition rate is 1 Hz. Let us imagine also that we have incoming photons of energy between 120 and 450 MeV, as in [131], for a total flux of $10^8$ photons per second. Following [134], this experimental configuration would give an event rate smaller than $6 \times 10^{-4} \text{s}^{-1}$ a rate that looks $10^6$ times smaller than the one corresponding to the observation of photon splitting in the atomic field [131].

4.2.8. Second harmonic generation induced by an electromagnetic field: photon fusion. Second harmonic generation and parametric amplification are the inverse effects of each other. If one exists in a vacuum the other exists as well. The same selection rules apply to both of them. This means that the contribution of the box diagram to the probability of conversion of two collinear incoming photons into one outgoing photon of twice the energy of one incoming photon in the presence of a transverse magnetic field is zero [135, 136]. At the lowest orders the transition probability is therefore proportional to $(\frac{E_{cr}}{E_{max}})^6$ and as for the photon splitting, photon fusion of collinear photons in a uniform magnetic field looks unmeasurable in any terrestrial laboratory. As far as we know, second harmonic generation induced by a magnetic field has never been observed even in standard media, while
second harmonic generation induced by an electric field has been observed since the 1970s [137].

The box diagram does not vanish if the incoming photons are not collinear and/or the magnetic field is not uniform. These cases are treated in [138] and more recently in [139]. In particular, the authors of these two papers treated the example of a pulsed plane wave or a Gaussian laser beam propagating in uniform or non-uniform dc fields. Following [139], an order of magnitude of the expected photon rate for a Gaussian beam in a uniform magnetic field can be obtained by the following formula:

\[
N_{\text{SHG}} = 1.2 \times 10^{-36} \rho I_\lambda B_0^2,
\]

where the numerical factor is proportional to \( e_{2,0}^2 \), \( \rho \) is the total time-averaged power of fundamental harmonics, \( \lambda \) is the wavelength of the fundamental harmonic, and \( I \) is the laser maximal intensity at focal point. Taking the values to be \( \rho = 10^5 \text{ W} \), \( I = 10^{26} \text{ W m}^{-2} \) and \( \lambda = 0.8 \times 10^{-6} \text{ m} \), as did the authors of [139], and letting \( B_0 = 50 \text{ T} \), one obtains \( N_{\text{SHG}} \approx 2.5 \times 10^{-8} \text{ s}^{-1} \), which seems very difficult to observe. For sake of argument let us recall that best single photon detectors in the near infrared region have dark count rates of the order of \( 10^{-3} \text{ s}^{-1} \) (see e.g. [140]).

4.2.9. Optical field-induced photon fusion. Photon fusion should also be induced using an electromagnetic wave as the field source, but as far as we know, no papers about this exist in literature. As for the case of photon splitting, one may guess that the optical field-induced effect should not be easier to observe than the one induced via electrostatic fields.

4.2.10. Intensity dependent refractive index. As already discussed, the index of refraction for light propagating in the presence of an electromagnetic field depends on the field amplitude. This means that if the field is not uniform but depends on spatial coordinates, light will propagate in the presence of a gradient of the index of refraction. It is also known that a light ray generally bends toward regions with a higher index of refraction. In this case the ray behavior can be described solving what is called the eikonal equation [76]. For example, light passing in a vacuum near a magnetic pole will bend toward the pole giving the impression it is attracted by the magnetic pole itself. A rough estimation of the deviation angle \( \theta_0 \) is \( n(B_0)/n(0) \), which for a vacuum reduces to \( \theta_0 \sim 7e_{2,0}^2 B_0^2/\rho \approx 9.3 \times 10^{-24} B_0^2 \) for light polarized parallel to \( B_0 \).

The magnetic deviation calculated using the eikonal equation in the case of a magnetic dipole has been given in [141, 142]. It depends on the magnetic-moment orientation with respect to the direction of light propagation, and it scales with the minimal distance of the light ray to the magnetic moment \( \rho_m \) as \( 1/\rho_m^5 \). Our rough estimation holds also for this more-complicated case.

An experiment was actually performed around 1961 to look for such a deviation, and is reported in [143]. The maximum field was about 1T and results indicated that the magnetic deviation was less than \( 5 \times 10^{-13} \text{ rad} \), which is obviously in agreement with our estimation of about \( 9 \times 10^{-24} \text{ rad} \).

It is clear that if one imagines a powerful laser beam, such that during propagation energy density corresponds to fields approaching the critical ones, self-focusing should be observed because external rays will bend toward the inner region of the beam where the index of refraction is higher. A discussion of this effect and others related to it can be found in [144]. From the experimental point of view the problem is that to obtain fields of the order of the critical ones one needs lasers with a power of the order of \( 10^{33} \text{ W m}^{-2} \), and this may well be possible only far into the future, since current projects have a target intensity of \( 10^{20} \text{ W m}^{-2} \) [32].

The QED vacuum effects at the interfaces between regions of different \( n \), like the creation of evanescent waves, have also been studied in [145].

Let us now follow the proposal in [121] of an x-ray probe beam traversing a standing electromagnetic wave. At any point of the standing wave one can define an electromagnetic field, and therefore an index of refraction, that depends on spatial coordinates. In principle the standing wave acts on the probe beam as a diffraction grating, and a diffraction by a standing wave should be observed. The calculation of the nonlinear phase shift acquired by crossing electromagnetic waves in a vacuum is also reported [146]. A more recent proposal to observe light by light diffraction in a vacuum can be found in [147]. A natural extension of this kind of apparatus is to have more than a standing wave to observe double-slit light by light interference, as proposed in [123, 148], or strong periodic fields structured [149] to induce Bragg scattering [150]. The authors of [148] argue that for experimental parameters attainable at upcoming XFEL and at ELI facilities (80 GW of x-rays in a 100 fs pulse of 0.4 nm wavelength focused in a spot of 100 \( \mu m \) radius, and ELI expected power focused in a diffraction limited spot), approximately 2 photons per laser shot can be diffracted to give a measurable signal.

4.2.11. Photon–photon scattering. Since the original paper of Euler and Kochel [12], it was clear that in the framework of QED photon–photon scattering was allowed in a vacuum. The first determination of the photon–photon scattering amplitude can be found in [12]. This result was later confirmed using Feynman diagrams [151]. Very recently, a tutorial paper showing how to compute low energy photon–photon scattering has also been published [152].

In the low energy approximation \( \hbar \omega \ll m_e c^2 \) and for unpolarized light, the total cross section can be written as

\[
\sigma_{\gamma\gamma \to \gamma\gamma} = \frac{973}{10125 \pi} \left( \frac{m_e c^2}{\hbar \omega} \right)^6 \left( \frac{\hbar \omega}{m_e c^2} \right) \left( \frac{\hbar \omega}{m_e c^2} \right)^6 e_{2,0}^2
\]

(70)

where the electron classical radius is \( r_e = a \chi \), and \( \hbar \omega \) is the energy of a photon in the reference frame of the center-of-mass\(^1\).

This means that \( \sigma_{\gamma\gamma \to \gamma\gamma} \approx 7.3 \times 10^{-70} \text{ m}^2 \) when \( \hbar \omega = 1.6 \times 10^{-19} \text{ J} \).

\(^1\) The photon center-of-mass frame (also called photon center of momentum frame) is the inertial frame where the photon total energy is the smallest one.
The cross section increases very rapidly with photon energy, reaching a maximum of $1.6 \times 10^{-34} \text{ m}^2$ when $\hbar \omega = 1.5 m_e c^2$, and then decreases as $(1/\hbar \omega)^2$ [153, 154].

As far as we know the first attempt to observe photon–photon scattering dates from 1928 [155] (see also [156]), followed in 1930 by the experiment reported in [157]. No scattered light was detected corresponding to an upper limit $\sigma_{\gamma \gamma \rightarrow \gamma \gamma} \leq 3 \times 10^{-24} \text{ m}^2$. Notwithstanding proposals to use gamma rays [158] or x-rays [159] to take advantage of the higher cross section at these energies, no other experiment was tried until 1996, when a new attempt to observe photon–photon scattering was reported in [160]. This experiment was based on two laser beams provided by LULI, France [52], colliding head-on in vacuum with a center-of-mass energy of 1.7 eV. The corresponding QED cross section was $1.6 \times 10^{-60} \text{ m}^2$. No scattered photon was observed and the authors gave an upper limit of the cross section $\sigma_{\gamma \gamma \rightarrow \gamma \gamma} \leq 9.9 \times 10^{-34} \text{ m}^2$.

In [161] photon scattering is studied in the case of three incoming beams arranged so that the energy and momentum conservation conditions $\hbar (\omega_1 + \omega_2 - \omega_3) = \hbar \omega_4$, $k_1 + k_2 - k_3 = k_4$ can be satisfied by a fourth, scattered beam. This configuration stimulates the photon emission in the fourth beam and so the matching conditions fix the direction of the generated wave (see figure 21).

This three-beam geometry is also discussed in [162]. Following these proposals, a new experiment was mounted at LULI, France [52], and the results reported in [163] in 2000.

The following geometry was chosen: $k_1 = k \hat{x}$, $k_2 = k \hat{y}$, $k_3 = \frac{1}{2} \hat{z}$ and $k_4 = k \hat{x} + k \hat{y} - \frac{1}{2} \hat{z}$. The authors chose a wavelength of 800 nm for beams 1 and 2, and 1300 nm for the third one. The generated signal was expected in the visible range at 577 nm. Thanks to this geometry, both the wavelength and the direction of the generated wave were different from the others, allowing us to easily separate photons from QED scattering from photons belonging to the incoming lasers. The energy of the two main beams at 800 nm was 0.4 J with a duration of 40 fs (full width of half maximum), while the third one was generated in an optical amplifier to shift its wavelength up to 1300 nm. The three beams were injected in a vacuum chamber with a diameter of 3 cm. The authors calibrated their apparatus by measuring the third-order susceptibility of nitrogen gas. In a vacuum no evidence for photon–photon scattering was observed, and the authors gave an upper limit of the cross section $\sigma_{\gamma \gamma \rightarrow \gamma \gamma} \leq 1.5 \times 10^{-52} \text{ m}^2$ at 18 orders of magnitude from QED prediction.

In principle a vacuum can be treated as a standard medium and, as shown in [164], a third-order nonlinear effective susceptibility $\chi^{(3)}$ can be defined as:

$$\chi^{(3)} = \frac{K}{45 \pi c^2 \alpha} \left( \frac{r_e \epsilon_0}{m_e c^2} \right)^2 = 2 \epsilon_0 K c_{2.0}. \quad (71)$$

$K$ is a factor that depends on the directions of the incident beams and of their polarization ($K < 14$) [164]. For $K = 1$, one gets $\chi^{(3)} \approx 3.0 \times 10^{-41} \text{ m}^2 \text{ V}^{-2}$.

In recent years several new proposals have been published but no new experiments have been performed. Whilst in [165] the possibility of measuring photon–photon scattering using electromagnetic modes in a waveguide is discussed, the other proposals [166–171] are mostly motivated by recent evolutions in the field of very powerful laser sources, which constitute a very important tool for fundamental physics [172, 173]. In particular the perspectives of photon–photon interaction experiments using high intensity lasers are discussed in [174].

The connection between the vacuum index of refraction $n$ in the presence of an external electromagnetic field and the photon–photon scattering amplitude in the forward direction $f_0$ can be established thanks to the optical theorem [175]

$$n = 1 + \frac{2 \pi}{k^2} N f_0, \quad (72)$$

where $N$ is the average density of centers of scattering that is proportional to the energy density of the external field and inversely proportional to the photon energy in the center-of-mass reference frame.

Photon–photon cross sections at low energy limits depend on the lowest-order coefficients of the development of the Heisenberg–Euler Lagrangian like $c_{2.0}$. This is true for any vacuum effect calculated in the same approximation, and therefore any experiment testing one of these effects and hence measuring such coefficients also indirectly tests the existence of all the other effects. For example the authors of [176] argue that attempts to measure vacuum magnetic birefringence give better indirect limits on $\sigma_{\gamma \gamma \rightarrow \gamma \gamma}$ than the one obtained by experiment, reported in [163].

### 4.3. Vacuum dichroism

In 1964 [177] it was pointed out that the Heisenberg–Euler Lagrangian has imaginary contributions due to the poles in equation (14). This means that together with a real part of the index of refraction which gives rise to vacuum electromagnetic birefringences there exists an imaginary part which corresponds to vacuum dichroism i.e. the absorption of photons in a vacuum depending on photon polarization. This absorptive index of refraction arises for the case of a pure external electric field. For the case of a pure magnetic field there is no absorptive contribution to the index of refraction. If a static external electric field $E_0$ is present, one can write [177]

$$n_\perp \approx 1 \quad (73)$$

and

$$n_\parallel \approx 1 + \frac{1}{4} i \alpha \frac{1}{e \frac{\epsilon_0}{m_e} - 1}. \quad (74)$$
where it is clear that only light polarized parallel to the external field can be absorbed. Finally, a linear absorption coefficient $\kappa_{\parallel}$ can be defined as:

$$\kappa_{\parallel} = \pi \frac{\alpha}{\lambda} \frac{1}{e^{|E_0/c\alpha|} - 1}. \quad (75)$$

In principle, the behavior of the vacuum in a strong electric field is such that an unpolarized light beam becomes partially polarized because the component of the electric vibration parallel to the electric field is absorbed, whereas the other component is not. The electric field required to observe such a vacuum dichroism is of the order of $E_{\text{cr}}$

With the advent of very powerful laser sources, fields approaching the critical ones can be achieved, and vacuum dichroism will eventually be observed by the associated pair production by a laser field [178] (see also [179] and references therein).

4.4. Delbrück scattering

Electric fields as high as the critical one can be found at the surface of atomic nuclei. To probe such a field, light has to have a wavelength of the order of the nucleus radius, which means high energy photons.

When high energy photons pass by a nucleus, they can not only be absorbed but they can also be deflected by the Coulomb field. This phenomenon is known as Delbrück scattering since Delbrück first proposed it [180] to explain the results of an experiment in which 2.615 MeV photons were scattered by lead and iron [181]. From a phenomenological point of view, in the low energy limit this effect can be viewed as a consequence of the refractive index induced by the Coulomb field. As shown in [182], in the high energy limit Delbrück scattering can be related to pair production via the optical theorem. As discussed in reviews [183, 184], comparisons between scattering experimental data and theoretical predictions of Delbrück scattering are complex because the measured total scattering cross section depends also on the contribution of the Rayleigh scattering and nuclear Compton scattering. The first clear observation of Delbrück scattering was reported by Schumacher et al. [185] using 2.754 MeV photons on lead nuclei. The energy range around 2.7 MeV is somewhat ideal to observe Delbrück scattering, since at lower energies Rayleigh scattering dominates, while at higher energies Compton scattering dominates. Experimental data cannot be explained without taking into account the Delbrück scattering, while to exactly reproduce their results further corrections to the theoretical predictions, called Coulomb corrections, have to be considered. The most recent measurements of high energy Delbrück scattering [186] were performed at the Budger Institute of Nuclear Physics [132], Novosibirsk, Russia.

5. Phenomenology in astrophysics

As we have discussed in the previous paragraphs, vacuum nonlinearities need an intense electromagnetic radiation source and strong electromagnetic fields to be detected. This demand is a real challenge in terrestrial laboratories, but there are places in the cosmos where both radiation and fields exist. The celestial bodies that have most attracted the attention of scientists to observe vacuum effects are strongly magnetized stars, neutron stars [187] and white dwarfs [188]. Both kinds of celestial objects are the remnants of the implosion of massive stars: stars exceeding the solar mass in the case of neutron stars, and stars of about the solar mass for white dwarfs. Neutron stars have magnetic fields on their surface typically of about $10^7$–$10^8$ T, but special neutron stars, called magnetars, are supposed to have surface magnetic fields exceeding $B_{\text{cr}}$. A white dwarf field typically ranges from 50 to $10^4$ T. The explanation of such high fields is that during implosion the magnetic-field flux is conserved. A star such as the sun has a field of about $5 \times 10^{-3}$ T and a radius of about $7 \times 10^5$ km, and after implosion a neutron star has a radius of about 10 km and a white dwarf of about $10^4$ km. The surface magnetic-field is therefore much higher. Stars are obviously sources of a large spectrum of radiation and the implications of QED in the observed emission of such celestial objects is a very wide and important field that has been already treated in specific reviews such as [187], and more recently [189].

Magnetic ray bending in neutron stars has been studied in [141, 142, 190]. This phenomenon is related to the fact that the refractive index depends on the field intensity. As in the 1961 terrestrial experiment already cited [143], the idea consists of observing the deflection of a light ray passing through a region where a non-uniform magnetic field is present. In [142], a specific neutron-star binary system is considered. Both stars are pulsars which means that they emit a radiation beam along their magnetic dipole direction. Periodically, the beam of one of the two stars is eclipsed by the other one, thus periodically the radiation beam of one of the two stars traverses a region where a field of several tesla exists over a distance of several hundreds of kilometers, and it should be bent in such a way that the signal observed on Earth will have a clear signature of this QED magnetically induced lensing. This kind of lensing effect is similar to the more-studied gravitational one where light is bent by the gravitational field [191].

Magnetically induced spontaneous vacuum energy emission via the production of real electron–positron pairs, which is also known as vacuum breakdown, and its implication for neutron-star emission, is the subject of [192]. The basic idea is that it is possible to release vacuum energy from the vacuum by introducing an external magnetic field, due to the difference in vacuum energy with and without the field. In the case of black holes [193], which are the remnants of the explosion of very massive stars, vacuum breakdown in an external field has also been studied [194].

QED vacuum magnetization around neutron-stars and its effect on neutron star braking is studied in [195, 196]. This phenomenon, which the authors call quantum vacuum friction, consists in the interaction between the star magnetic dipole and the magnetic dipole induced in the vacuum because of the vacuum magnetization predicted in the framework of QED. The star rotates around its own axis, thus the vacuum dipole also changes in time. The two are not collinear because the vacuum dipole, which is the sum of the magnetization at all the points around the neutron star, is due to a retarded field since
the velocity of light is not infinite. The interaction between the two dipoles gives rise to a couple that tend to brake the star. This has important consequences on the spindown evolution of neutron stars in particular in the case of magnetars. The possibility of detecting QED effects on the polarization of radiation emitted by neutron stars and white dwarfs is discussed in [197–204]. The combined effects of a QED vacuum and plasma in the neighborhood of a neutron star are studied in [205–207]. Photon–plasma interactions are also the subject of a review [174]. In general radiation emitted from the star propagates in a plasma but also in the presence of high magnetic fields. Light polarization changes during propagation so that star atmosphere emits polarized light even if the star surface radiation is not polarized. The emission also appears to vary across the surface. The polarization across the surface should also show special patterns because the rotating magnetic field twists the polarization. All these subtle effects give a clear indication of radiation observable on Earth.

Nonlinear propagation of electromagnetic waves in the magnetosphere of a magnetar, taking into account both the QED vacuum and the magnetized plasma, has been also studied non-perturbatively in [208]. The nonlinear behavior of these electromagnetic waves should play an important role in the energy transmission in pulsars and magnetars.

Since the very beginning, magnetized stars have been considered as the best choice to observe photon splitting [16, 128]. In a star magnetosphere only the splitting of a photon polarized parallel to the field into two photons polarized perpendicular to the field is allowed because the index of refraction of the parallel propagation mode is greater than the perpendicular propagation mode. This is the only process that conserves momentum. Again, magnetized stars such as neutron stars should appear as emitting polarized light. Moreover, photon splitting is more efficient for high energy photons, thus photon splitting has often been invoked to explain hard cutoffs in gamma-ray spectra from pulsars [209], but the data is difficult to interpret because one does not know where gamma rays are actually produced.

This wide field of research has triggered the construction of a special satellite, GEMS [210], the goal of which is to measure the polarization of x-ray sources. Its launch is scheduled after November 2014, and neutron stars are amongst its prime targets.

6. Non-trivial vacua

Examples of what is known as non-trivial vacua can be found in literature. This essentially means that light propagates in a vacuum in which a distribution of real or virtual photons exists. In this case one can calculate an average value for \( E^2 \) and \( B^2 \), and use this value to calculate the corresponding effects due in particular to four-wave mixing in the vacuum. For example, the velocity of light propagating perpendicular to two parallel infinite conducting plates and in the region between them was first calculated by Scharnhorst [211]. The variation of the velocity of light is due to virtual photon energy density which is also responsible for the well known Casimir effect [212]. The conceptual problem is that in the case treated by Scharnhorst, the velocity of light should exceed \( c \). This has attracted a lot of interest (see e.g. [213]), but as far as we understand, this effect is also expected to be undetectable with up-to-date apparatus. The effect of real photon radiation associated with temperature in comparison to that associated with virtual photons is also discussed in [214].

In the case of very intense photon beams, the photon density is so high that in literature the term of radiation gas is used to indicate such an environment. QED photon–photon interactions change the refractive index of radiation gas and new optical phenomena should appear: self-focusing, the formation of so-called photon bullets [215, 216], wave collapse [217, 218] and a phenomenon called photon acceleration in a vacuum [219].

7. CP-violating vacuum

In the framework of the standard model, CP violation has been observed (see e.g. [220]). In principle, CP violation in photon–photon interactions would modify the quantum vacuum behavior also in the case of light propagation. In particular, terms corresponding to coefficients \( c_{0,1} \) and \( c_{2,1} \) could exist in the Lagrangian \( L \). The phenomenology of the existence of a term proportional to \( F^2 G \) has been studied in [221, 222].

One of the main results is that a new type of Jones birefringence depending on \( B_2^5 \) should exist. Nevertheless the predicted effect in the framework of the standard model is very small compared to the QED birefringence effects, because the energy scale of such a new Lagrangian term is the QCD one [222].

8. Conclusion

From the phenomenological point of view, a quantum vacuum can be treated as a standard nonlinear optical medium. In the low photon energy limit, and for fields smaller than critical ones, the lowest-order terms of the Heisenberg–Euler Lagrangian are sufficient to give precise predictions of the effects to be measured, such as in the case of vacuum Cotton–Mouton magnetic birefringence. In principle, higher \( \alpha \) contributions could be calculated giving more and more precise predictions, to compare with experimental results as in the case of other important QED tests, such as the anomalous magnetic moment of the electron (see e.g. [223]). Our review is mainly devoted to this low energy, small field limit. It is worth stressing that vacuum Cotton–Mouton magnetic birefringence is the only field in which experiments are ongoing. Experimentalists are three orders of magnitude from the QED predictions [109] and there is hope that in a few years one of the most fundamental predictions of Heisenberg–Euler Lagrangian will be tested experimentally.

At higher photon energies and/or at higher fields, theoretical calculations become more difficult since they are no longer perturbative in \( \alpha \). A large theoretical literature exists covering a very wide spectrum of phenomena. Some of these works have been triggered by the new intense laser facilities under development. In actual fact, intense lasers
should give access to new and impressive QED effects (see e.g. reviews [224, 225]) that have been covered in this review but which certainly deserve more space. The last experiments on photon–photon scattering were 18 orders of magnitude from the QED prediction, but a large number of proposals are just waiting for the new facilities to be operational, and eventually a completely new field will be opened up.

For photons in the MeV region, both Delbrück scattering [185] and, recently, photon splitting in an atomic field [131] have been observed, confirming once more that QED is a very powerful tool for describing nature.

Last but not least, astrophysical tests of quantum vacuum properties seem very promising and, hopefully, thanks to more and more precise observational data, the Cosmos itself will become a laboratory to test QED photon–photon interactions.

Acknowledgments

We thank Mathilde Fouché and Geert Rikken for their strong support and very useful discussions. We also thank Mathilde Fouché for carefully reading the manuscript.

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