Abstract. In light of recent interest in Hadamard diagonalisable graphs (graphs whose Laplacian matrix is diagonalisable by a Hadamard matrix), we generalise this notion from real to complex Hadamard matrices. We give some basic properties and methods of constructing such graphs. We show that a large class of complex Hadamard diagonalisable graphs have vertex sets forming an equitable partition, and that the Laplacian eigenvalues must be even integers. We provide a number of examples and constructions of complex Hadamard diagonalisable graphs, including two special classes of graphs: the Cayley graphs over $\mathbb{Z}_d^n$, and the non-complete extended $p$-sum (NEPS). We discuss necessary and sufficient conditions for $(\alpha, \beta)$-Laplacian fractional revival and perfect state transfer on continuous-time quantum walks described by complex Hadamard diagonalisable graphs and provide examples of such quantum state transfer.

1. Introduction

There has been recent interest in graphs whose corresponding Laplacian matrix is diagonalisable by a Hadamard matrix [1, 15, 16]. Such graphs are said to be Hadamard diagonalisable. We generalise this notion to include complex Hadamard matrices, also called unitary type II matrices, originally discussed in [20]. Our primary motivation for considering this interesting problem is two-fold. One basic objective is to better understand the structure of graphs under non-conventional symmetry and regularity conditions by imposing particular entry-wise structure on certain eigenbases associated with their Laplacian matrices. Second, by expanding to the complex field, we can access more potential Hadamard matrices, which in turn opens the door to analysing additional graphs with even further potentially interesting properties. In this work, we study equitable partitions that are naturally associated with such structured eigenbases and we analyse graphs that are Hadamard diagonalisable for certain popular classes of complex Hadamard matrices. We then extend our analysis to Cayley graphs over $\mathbb{Z}_d^n$, and consider various related graph constructions for producing additional classes of graphs that are Hadamard diagonalisable over the complex numbers. Finally, we discuss the important notion of continuous-time quantum walks, where the graph represents a quantum spin network which allows for the transfer of information from one vertex to another. We are able to characterise when $(\alpha, \beta)$-Laplacian fractional revival and perfect state transfer occurs based on the structure of the complex Hadamard matrix involved.

The remainder of this section is devoted to definitions, notation, basic properties, examples, and graph constructions. In Section 2 we consider graphs that are diagonalisable by a (real) Hadamard matrix, or a class of complex Hadamard matrices called Turyn Hadamard matrices. Equitable partitions arise naturally, coming from the structure of these classes of complex Hadamard matrices. We are also able to make connections to the Cheeger constant. Section 3 focuses on graphs that are diagonalisable by a class of complex Hadamard matrices called Butson Hadamard matrices, which include as a subclass the Turyn Hadamard matrices. We give conditions on their Laplacian eigenvalues in terms of parity and multiplicity. In Section 4 we provide necessary and sufficient conditions characterising when $(\alpha, \beta)$-Laplacian fractional revival and perfect state transfer occurs based on the structure of the complex Hadamard matrix involved.

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1.1. Definitions and Notation. The results in this paper hold for both weighted (with non-negative weights) and unweighted graphs unless otherwise specified. We consider only simple graphs herein.

We quickly review some basic definitions. Given a weighted graph $G$ on $n$ vertices, its corresponding adjacency matrix is $A(G) = [a_{i,j}] \in M_n$, where $a_{i,j}$ represents the weight of the edge between vertices $i$ and $j$ ($a_{i,j} = 0$ when there is no edge between vertices $i$ and $j$; all non-zero $a_{i,j} = 1$ for unweighted graphs). The Laplacian matrix corresponding to a simple weighted graph $G$ is $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal degree matrix of $G$: the $(i, i)$-th entry of $D(G)$ is the weighted degree of vertex $i$, i.e. the sum of the weights of the edges incident to vertex $i$. It is well-known that $L(G)$ is positive semi-definite with the all-ones vector as a null-vector, and that the number of connected components of $G$ determines the multiplicity of the zero eigenvalue.

A Hadamard matrix is a square matrix with entries in $\{1, -1\}$ satisfying $HH^T = nI$, where $I$ is the $n \times n$ identity matrix. By pre- and post-multiplying a Hadamard matrix by diagonal $\{1, -1\}$-matrices, we may assume its first row and first column are all-ones vectors without loss of generality. Such a Hadamard matrix is called a normalised Hadamard matrix in the literature. The Hadamard matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \otimes n$ is called a standard Hadamard matrix. A graph is called Hadamard diagonalisable if its Laplacian matrix can be diagonalised by some Hadamard matrix; that is, if we can write $L = \frac{1}{n} HH^*$ where $\Lambda = \text{diag} (\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix of the eigenvalues of $L$. As such, the columns of $H$ give an orthogonal basis of eigenvectors for $L(G)$. If $H$ is normalised then $\lambda_1 = 0$, and we often perform a further permutation so that $\lambda_1 \leq \cdots \leq \lambda_n$. Many Hadamard diagonalisable graphs are given in [1].

The concept of a (real) Hadamard matrix can be generalised as follows:

**Definition 1.1.** A complex Hadamard matrix $H$ is an $n \times n$ matrix, with all its entries having moduli one, satisfying $HH^* = nI$.

Two complex Hadamard matrices $H_1$ and $H_2$ are equivalent if $H_1 = MH_2N$, where $M$ and $N$ are invertible monomial matrices (a monomial matrix is the product of an invertible diagonal matrix and a permutation matrix) whose non-zero entries have moduli one. Note that (real) Hadamard matrices are examples of complex Hadamard matrices, but we are more interested here in finding examples of and results involving bonafide complex Hadamard matrices.

Originally studied in [20], the concept of a complex Hadamard matrix has other names in the literature: it is the same as a unit Hadamard matrix defined in [11], as well as a unitary type II matrix (see e.g. [7][13]). A type II matrix is the same as an inverse orthogonal matrix. Complex Hadamard matrices can also be normalised so that the first row and first column are all-ones vectors, though in the complex setting this is known as dephased. Analogous to [1], we say that a graph is complex Hadamard diagonalisable if its Laplacian matrix can be diagonalised by some complex Hadamard matrix.

An $n \times n$ complex Hadamard matrix whose entries are all $r$-th roots of unity is called a Butson Hadamard matrix; the set of all Butson Hadamard matrices for fixed $r$ and $n$ is denoted $H(r, n)$ [1]. In the specific case when the complex Hadamard matrix consists only of $\pm 1$ and $\pm i$, the matrix is called a Turyn Hadamard matrix [21]. Many complex Hadamard matrices of small orders can be found in [9].

1.2. Basic Properties. Some results from [11] stated for Hadamard diagonalisable unweighted graphs, readily apply with little to no adaptation to the proofs, to the situation of complex Hadamard diagonalisable weighted graphs.

**Lemma 1.2.** A weighted graph $G$ is complex Hadamard diagonalisable if and only if there is a dephased complex Hadamard matrix that diagonalises $L(G)$.

The following result states that the regularity conclusion found in [1] Theorem 5] holds for complex Hadamard diagonalisable graphs. We say that a weighted graph $G$ is weighted–regular if for each vertex, the weighted degree of each vertex is the same. If the weights of the edges of a graph are all one, then weighted–regular reduces to regular.

**Theorem 1.3.** If $G$ is a weighted graph and its Laplacian matrix is complex Hadamard diagonalisable, then $G$ is weighted–regular.
with weighted–degree minimum row sum, with equality holding if and only if the row sums of \( M \) are all equal. Applying this result to \( A \), together with the above equation, it follows that each connected component of \( G \) is weighted–regular, with weighted–degree \( r \). Thus \( G \) is weighted–regular. \( \square \)

Note that \( [1] \) only discussed the case where the Laplacian matrix, rather than the adjacency matrix, associated to a graph is Hadamard diagonalisable. For a weighted–regular graph \( G \), \( A(G) \) is complex Hadamard diagonalisable by \( H \) if and only if \( L(G) \) is diagonalisable by \( H \). The following result is an adjacency matrix version of Theorem [13], which shows that the property of being weighted–regular is necessary for \( A(G) \) to be diagonalisable by \( H \). Thus this result answers (in the negative) the question of whether or not there exists a non–regular graph \( G \) such that \( A(G) \) is complex Hadamard diagonalisable.

**Theorem 1.4.** If \( G \) is a weighted graph and its adjacency matrix is diagonalisable by a complex Hadamard, then \( G \) is weighted–regular.

**Proof.** Let \( A \) be the adjacency matrix of \( G \), and denote its spectral radius by \( r \). Since \( A \) is diagonalisable by a complex Hadamard matrix, let \( v \) be an eigenvector of \( A \) corresponding to \( r \), such that each entry of \( v \) has modulus 1. For any vector \( u \), let \( |u| \) denote the nonnegative vector whose entries consist of the moduli of the entries in \( u \). Observe that by the triangle inequality, we have

\[
|rv| = |Av| \leq A|v| = A1,
\]

where \( 1 \) denotes the all–ones vector. In particular, for each connected component of \( G \), \( r \) is less than or equal to the minimum row sum of the corresponding principal submatrix of \( A \) (i.e. the minimum weighted–degree of the vertices in that component). A standard result from Perron–Frobenius theory (see \([19]\), for example) states that for an irreducible nonnegative matrix \( M \), the spectral radius is always bounded below by the minimum row sum, with equality holding if and only if the row sums of \( M \) are all equal. Applying this result to \( A \), together with the above equation, it follows that each connected component of \( G \) is weighted–regular, with weighted–degree \( r \). Thus \( G \) is weighted–regular. \( \square \)

1.3. **Examples and Graph Constructions.** A multipartite graph \( G[V_1, \ldots, V_m] \) is a graph with vertex set \( V = \bigcup_{i=1}^{m} V_i \), where the sets \( V_i \) are mutually disjoint, and where for any \( i \), no two vertices in \( V_i \) are adjacent. We denote by \( K_n \) the complete graph on \( n \) vertices and by \( K_{n_1,n_2,\ldots,n_m} \) the complete multipartite graph with \( m \) parts of sizes \( n_1, n_2, \ldots, n_m \) respectively.

Here, we provide details on how to construct graphs that are diagonalisable by a complex Hadamard matrix, completely characterising the low–dimensional cases.

As an example, consider the complete graph \( K_n \) and the complete bipartite graph \( K_{m\times m} \). Suppose \( H \) is a dephased complex Hadamard matrix of order \( n \). For \( k = 1, \ldots, n \), let \( h_k \) denote the \( k \)-th column of \( H \).

Then the space of \( n \times n \) matrices diagonalisable by \( H \) is

\[
\mathcal{S} = \text{span}\{h_kh_k^*: k = 1, \ldots, n\}.
\]

Observe that \( I \) and \( J = h_1h_1^* \), are in \( \mathcal{S} \), so \( K_n \) is diagonalisable by \( H \). Note that in the real case, the observation that \( K_n \) is diagonalisable by \( H \) can only be stated for \( n = 4k \), and assuming such a Hadamard matrix exists. However, there is far more flexibility when considering Hadamard matrices over the complex field.

Furthermore, if \( H \) has a column \( h_j \neq 1 \) whose entries are \( \pm 1 \), then

\[
\frac{n}{2}I + \frac{1}{2}(h_jh_j^* - J)
\]

is the Laplacian matrix of \( K_{m\times m} \), and is diagonalisable by \( H \).

Recall that for any finite group, we can construct a character table made up of rows corresponding to the irreducible group representations and columns corresponding to the conjugacy classes. The entries of the table are the values of the character of the representation, evaluated on a representative of the conjugacy class.

A Cayley graph is a graph \( G(C) \) over \( \mathbb{Z}_r^d \), where \( r \) is prime and \( C \subset \mathbb{Z}_r^d \setminus \{0\} \) is called the connection set of \( G(C) \), where \( 0 \) is the all–zeros vector. Let \( V = \mathbb{Z}_r^d \) be the vertex set of \( G(C) \); two vertices are adjacent if and only if their difference is in \( C \). In the case when \( r = 2 \) the graphs \( G(C) \) are known as cubelike graphs \([2][5]\), with the hypercube being the prototypical example.
Example 1.5. Let $\Gamma$ be a finite abelian group of order $n$. Then the transpose of its character table, $H$, is a dephased complex Hadamard matrix whose columns are the characters of $\Gamma$. The space $\mathcal{S}$ of matrices diagonalisable by $H$ has dimension $n$.

Take the regular representation of $\Gamma$, $\{A_g : g \in \Gamma\}$; then $\mathbf{h}_i$ is an eigenvector of $A_g$ with eigenvalue $\mathbf{h}_i(g)$. Since the set $\{A_g : g \in G\}$ spans an $n$-dimensional space, it contains all matrices diagonalisable by $H$. Hence the (directed) Cayley graphs of $\Gamma$ are the only unweighted (directed) graphs that are diagonalisable by $H$.

Let $\mathbb{Z}_2$ be the field of two elements. Let $C \subset \mathbb{Z}_2^n$ with $0 \notin C$. Since a cubelike graph is a Cayley graph of $\mathbb{Z}_2^n$, we recover Corollary 1 of [13] which states that $L(G)$ is diagonalisable by the Hadamard matrix

$$
\begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix}^d
$$

if and only if it is a cubelike graph.

The smallest non-Cayley regular graph has 8 vertices. Hence all regular graphs on 7 or fewer vertices are complex Hadamard diagonalisable.

The notation $G^c$ denotes the complement of the graph $G$. It is straightforward to check the following result, which, restricted to the real setting, appears in [11] Lemma 7.

Proposition 1.6. Let $G$ be a complex Hadamard diagonalisable unweighted graph, diagonalised by a complex Hadamard matrix $H$. Then $G^c$ is also diagonalised by $H$.

Given two graphs $G_1$ and $G_2$ on $n_1$ and $n_2$ vertices, respectively, their direct product $G_1 \times G_2$ is the graph with adjacency matrix $A(G_1) \otimes A(G_2)$.

Lemma 1.7. Let $H_1$ and $H_2$ be complex Hadamard matrices. Suppose $G_1, \ldots, G_r$ are diagonalisable by $H_1$ and $G_{r+1}, \ldots, G_s$ are diagonalisable by $H_2$. Then the weighted graph corresponding to the adjacency matrix

$$
\sum_{k=1}^r \sum_{\ell=r+1}^s w_{k,\ell} A(G_k) \otimes A(G_\ell)
$$

is diagonalisable by $H_1 \otimes H_2$, for $w_{k,\ell} \in \mathbb{R}$. \hfill \Box

Remark 1.8. Lemma [17] can be extended to any finite number of graphs: For any $d \in \mathbb{N}$, let $H_i$ be a complex Hadamard matrix for all $i = 1, \ldots, d$ and, for each $i = 1, \ldots, d$ and for each $j = 1, \ldots, r$, suppose the graph $G_{i,j}$ is diagonalisable by $H_j$. Then the weighted graph corresponding to the adjacency matrix

$$
\sum_{j=1}^r \otimes_{i=1}^d w_{i,j} A(G_{i,j})
$$

is diagonalisable by $\otimes_{i=1}^d H_i$, for $w_{i,j} \in \mathbb{R}$.

Consider the non-complete extended $p$–sum (NEPS) of the graphs $G_1, \ldots, G_d$ with basis set $\Omega \subset \mathbb{Z}_2^n \setminus \{0\}$, denoted NEPS($G_1, \ldots, G_d; \Omega$): such a graph has vertex set $V(G_1) \times \cdots \times V(G_d)$, where two vertices $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ are adjacent if and only if there exists a $\beta = (\beta_1, \ldots, \beta_d) \in \Omega$ such that $\beta_i = 0 \iff x_i = y_i$ and $\beta_i = 1 \iff x_i, y_i$ are adjacent in $G_i$. The adjacency matrix of NEPS($G_1, \ldots, G_m; \Omega$) is $A_\Omega = \sum_{\beta \in \Omega} A(G_1)_{\beta_1} \otimes \cdots \otimes A(G_m)_{\beta_d}$. Therefore, if $G_i$ is complex Hadamard diagonalisable by $H_i$, for each $i = 1, \ldots, d$, it follows that NEPS($G_1, \ldots, G_d; \Omega$) is diagonalisable by $H_\Omega = \sum_{\beta \in \Omega} H_1^{\beta_1} \otimes \cdots \otimes H_d^{\beta_d}$. Thus NEPS($G_1, \ldots, G_d; \Omega$) is complex Hadamard diagonalisable so long as each $G_i$ is complex Hadamard diagonalisable and $\Omega$ contains vectors that do not all have a zero entry in the same position. See [17][18] for work done on perfect state transfer and pretty good state transfer on NEPS.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs on disjoint sets of $n_1$ and $n_2$ vertices respectively. Then $G_1 + G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ denotes the union of the graphs $G_1$ and $G_2$ (note that $G_1 + G_2$ is a graph on $n_1 + n_2$ vertices), and $G_1 \vee G_2 = (G_1^c \cup G_2^c)^c$ denotes the join of the graphs $G_1$ and $G_2$ (the graph on $n_1 + n_2$ vertices obtained from $G_1 + G_2$ by adding new edges between each vertex of $G_1$ and each vertex of $G_2$). The Cartesian product of $G_1$ and $G_2$ is the graph $G_1 \square G_2 = (V_1 \times V_2, E_3)$, where $V_1 \times V_2$ is the Cartesian product of the two original sets of vertices, and there is an edge in $G_1 \square G_2$ between vertices $(g_1, g_2)$ and $(h_1, h_2)$ if and only if either (i) $g_1 = h_1$ and there is an edge between $g_2$ and $h_2$ in $G_2$, or (ii) $g_2 = h_2$ and there is an
edge between \(g_1\) and \(h_1\) in \(G_1\). Similarly, the Cartesian product of weighted graphs \(G_1\) and \(G_2\) \cite{15} is defined by setting (i) the weight of the edges between \((g_1, g_2)\) and \((h_1, h_2)\) in \(G_1 \Box G_2\) to be the same as the weight between \(g_2\) and \(h_2\) in \(G_2\), and (ii) the weight of the edges between \((g_1, g_2)\) and \((h_1, g_2)\) in \(G_1 \Box G_2\) to be the same as the weight between \(g_1\) and \(h_1\) in \(G_1\).

In \cite{15}, Johnston et al defined the merge of two graphs \(G_1\) and \(G_2\) of order \(n\) with respect to positive weights \(w_1\) and \(w_2\), denoted by \(G_1 \wedge w_1 \diamond w_2 G_2\), to be the graph with adjacency matrix

\[
\begin{bmatrix}
w_1 A(G_1) & w_2 A(G_2) \\
w_2 A(G_2) & w_1 A(G_1)
\end{bmatrix}
\]

If \(G_1\) and \(G_2\) are unweighted graphs on the same vertex set and \(E(G_1) \cap E(G_2) = \emptyset\) then the merge \(G_1 w_1 \wedge w_2 G_2\) with \(w_1 = w_2 = 1\) is called the double cover of the graph with adjacency matrix \(A(G_1) + A(G_2)\), denoted by \(G_1 \wedge G_2\) in \cite{10}. If \(G_1\) is empty, then \(G_1 \wedge G_2\) is called the bipartite double cover of \(G_2\). For example, the bipartite double cover of \(K_n\) is the complete bipartite graph \(K_{n,n}\) with a perfect matching removed (a perfect matching is an independent edge set for which every vertex is incident to precisely one edge). In particular, the bipartite double cover of \(K_3\) is the cycle on six vertices and the bipartite double cover of \(K_4\) is the 3–cube.

**Corollary 1.9.** If \(A(G_1)\) and \(A(G_2)\) are diagonalisable by a complex Hadamard matrix \(H\), then \(G_1 + G_2\), \(G_1 \vee G_2\) and \(G_1 \wedge w_1 \diamond w_2 G_2\) are diagonalisable by

\[
\begin{bmatrix}
H & H \\
H & -H
\end{bmatrix}
\]

**Proof.** First observe

\[
\begin{bmatrix}
H & H \\
H & -H
\end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes H.
\]

The Corollary follows from applyingLemma \cite{17} to

\[
A(G + G) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes A(G),
\]

\[
A(G \vee G) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes A(G) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes J, \text{ and}
\]

\[
A(G_1 \wedge w_1 \diamond w_2 G_2) = w_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes A(G_1) + w_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes A(G_2).
\]

The Cartesian product \(G_1 \Box G_2\) has adjacency matrix

\[A(G_1) \otimes I_{n_2} + I_{n_1} \otimes A(G_2).\]

A straightforward calculation gives the following result.

**Proposition 1.10.** Suppose \(G_1\) and \(G_2\) are diagonalisable by complex Hadamard matrices \(H_1\) and \(H_2\), respectively. Then \(G_1 \wedge G_2\) and \(G_1 \Box G_2\) are diagonalisable by \(H_1 \otimes H_2\).

## 2. Hadamard and Turyn Hadamard Matrices

Recall that a complex Hadamard matrix \(H\) is called a Turyn Hadamard matrix if its entries are in \(\{\pm 1, \pm i\}\). If a Turyn Hadamard matrix of order \(n\) exists then \(n\) is even, see \cite{25} (compare this with the real case: if a Hadamard matrix of order \(n\) exists then \(n = 2\) or \(n \equiv 0 \pmod{4}\)).

As a quick example, we note that Cayley graphs of \(\mathbb{Z}_4^d\) are diagonalisable by the Turyn Hadamard matrix

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{bmatrix}^d
\]

Since we are concerned with Laplacian matrices that are real–valued, one may ask if graphs that are Turyn–diagonalisable are in fact Hadamard diagonalisable (by a real Hadamard matrix). The following example...
answers that question in the negative: there are indeed instances of non-trivial Turyn–diagonalisability, even in the unweighted setting.

**Example 2.1.** By Corollary 2.5 (see below), both $K_2$ and $K_6$ are diagonalisable by a Turyn Hadamard matrix; invoking Proposition 1.10 we see that $K_2\boxtimes K_6$ is complex Hadamard diagonalisable. Yet, [1] Proposition 10 states that the only connected graphs of order 12 that are Hadamard diagonalisable are $K_{12}$ and $K_{6,6}$.

The graph partitioning problem concerns partitioning the vertices of a graph into a fixed number of cells containing fewer than some other fixed number of vertices, while minimising the number of cut edges (i.e., edges connecting vertices in different cells of the partition). This problem, often with focus on evenly balanced cuts in particular, is connected to the Cheeger constant, expander graphs, flow problems, and the volume of a subset of vertices in a graph (see, e.g. [9]). The problem has also been considered in the context of computer science [12].

**Definition 2.2.** (Equitable Partition) Let $G$ be an unweighted graph with Laplacian $L(G) = [\ell_{i,j}]$. A partition $\pi$ of the vertex set $V(G) = V_1 \cup \cdots \cup V_m$ is equitable with respect to $G$ if for all $i, j \in \{1, 2, \ldots, m\}$

$$\sum_{v \in V_j} \ell_{u,v} = d_{i,j}$$

is a constant $d_{i,j}$ for any $u \in V_i$.

In the above definition, $d_{i,j} \leq 0$ when $i \neq j$, while $d_{i,i} \geq 0$. Note that the definition of equitable partition is equally valid when considering the adjacency matrix $A(G)$: for all $i, j \in \{1, 2, \ldots, k\}$, there is a constant $|d_{i,j}|$ such that any vertex $u \in V_i$ has $|d_{i,j}|$ neighbours in $V_j$.

The second conclusion in [1, Theorem 5] is that all the eigenvalues of the Laplacian of a Hadamard diagonalisable unweighted graph are even integers. The theorem below generalises this to Turyn Hadamard matrices (more generally, any complex Hadamard with a column whose entries are in $\{\pm 1, \pm i\}$), while Theorems 5.3 and 5.2 give similar results for Butson Hadamard matrices.

Given a dephased complex Hadamard matrix $H$, let $G$ be an unweighted graph where $L(G) = \lambda_j h_j$ for $j = 1, \ldots, n$.

Permuting the columns of $H$ if necessary, we may assume that $L(G)h_1 = 0$ (hence $h_1 = 1$), and that

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$

Suppose $H$ has a column $h_k$, $k > 1$, whose entries are in $\{\pm 1, \pm i\}$. We note that, for the following argument, we do not require all four values to appear in $h_k$. Define the sets

$$\mathcal{R}_+ = \{ j : h_k(j) = 1 \}, \quad \mathcal{R}_- = \{ j : h_k(j) = -1 \},$$

$$\mathcal{I}_+ = \{ j : h_k(j) = i \}, \quad \mathcal{I}_- = \{ j : h_k(j) = -i \}.$$

As $h_k$ is orthogonal to $h_1 = 1$, we have

$$|\mathcal{R}_+| = |\mathcal{R}_-| \quad \text{and} \quad |\mathcal{I}_+| = |\mathcal{I}_-|.$$

For each vertex $u$, we use $\mathcal{R}_\pm(u)$ and $\mathcal{I}_\pm(u)$ to denote the set of neighbours of $u$ in $\mathcal{R}_\pm$ and $\mathcal{I}_\pm$, respectively. Note that, letting $d$ denote the degree of regularity, we have

$$|\mathcal{R}_+(u)| + |\mathcal{R}_-(u)| + |\mathcal{I}_+(u)| + |\mathcal{I}_-(u)| = d. \quad (1)$$

For $u \in \mathcal{R}_+$, the $u$–th entry of $(L(G)h_k)$ is

$$d - |\mathcal{R}_+(u)| + |\mathcal{R}_-(u)| - |\mathcal{I}_+(u)|i + |\mathcal{I}_-(u)|i = \lambda_k \cdot 1.$$

The imaginary part of the equation gives $|\mathcal{I}_+(u)| = |\mathcal{I}_-(u)|$. Together with (1), we get

$$|\mathcal{R}_-(u)| + |\mathcal{I}_-(u)| = \frac{\lambda_k}{2}.$$

For $u \in \mathcal{R}_-$, the $u$–th entry of $(L(G)h_k)$ is

$$-d - |\mathcal{R}_+(u)| + |\mathcal_-(u)| - |\mathcal{I}_+(u)|i + |\mathcal{I}_-(u)|i = \lambda_k(-1),$$

together with (1), yields

$$|\mathcal{I}_+(u)| = |\mathcal{I}_-(u)| \quad \text{and} \quad |\mathcal{R}_+(u)| + |\mathcal{I}_+(u)| = \frac{\lambda_k}{2}.$$
Similarly, for \( u \in \mathcal{I}_+ \), we have
\[
di - |R_+(u)| + |R_-(u)| - |I_+(u)|i + |I_-(u)|i = \lambda_k \cdot i
\]
which leads to
\[
|R_+(u)| = |R_-(u)| \quad \text{and} \quad |R_+(u)| + |I_+(u)|i = \frac{\lambda_k}{2}.
\]
Lastly, for \( u \in \mathcal{I}_- \), we have
\[
di - |R_+(u)| + |R_-(u)| - |I_+(u)|i + |I_-(u)|i = \lambda_k (-i)
\]
and
\[
|R_+(u)| = |R_-(u)| \quad \text{and} \quad |R_+(u)| + |I_+(u)|i = \frac{\lambda_k}{2}.
\]
We conclude that the partition \((R_+ \cup I_+, R_- \cup I_-)\) of the vertex set is equitable for which \(A(G)\) has the quotient matrix
\[
\begin{bmatrix}
d - \frac{\lambda_k}{2} & \frac{\lambda_k}{2} \\
\frac{\lambda_k}{2} & d - \frac{\lambda_k}{2}
\end{bmatrix}.
\]
We summarise the above discussion in the following theorem:

**Theorem 2.3.** Suppose \( G \) is diagonalisable by a dephased complex Hadamard matrix \( H \) of order \( n \). For each \( h_k, k > 1 \), containing entries in \( \{\pm 1, \pm i\} \), \( G \) has an equitable partition into two cells, each having exactly \( \frac{n}{2} \) vertices, that has quotient matrix
\[
\begin{bmatrix}
d - \frac{\lambda_k}{2} & \frac{\lambda_k}{2} \\
\frac{\lambda_k}{2} & d - \frac{\lambda_k}{2}
\end{bmatrix}.
\]
Moreover, \( \lambda_k \) is an even integer.

Further, we define two vectors \( h_k^R \) and \( h_k^T \) as follows
\[
h_k^R(j) = \begin{cases} h_k(j) & \text{if } h_k(j) = \pm 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad h_k^T(j) = \begin{cases} h_k(j) & \text{if } h_k(j) = \pm i \\ 0 & \text{otherwise} \end{cases}
\]
Then each of \( h_k^R \) and \( h_k^T \) is either the zero vector or is an eigenvector of \( L(G) \) corresponding to \( \lambda_k \). If \( h_j \) is not orthogonal to both \( h_k^R \) and \( h_k^T \) then \( \lambda_j = \lambda_k \).

**Corollary 2.4.** Suppose \( G \) is diagonalisable by a normalised Hadamard matrix or a dephased Turyn Hadamard matrix of order \( n \). For each \( k > 1 \), \( G \) has an equitable partition into two cells, each having exactly \( \frac{n}{2} \) vertices, that has quotient matrix
\[
\begin{bmatrix}
d - \frac{\lambda_k}{2} & \frac{\lambda_k}{2} \\
\frac{\lambda_k}{2} & d - \frac{\lambda_k}{2}
\end{bmatrix}.
\]
Moreover, the eigenvalues of \( L(G) \) are even integers. \( \square \)

**Remark 2.5.** If \( H \) is a normalised Hadamard matrix or a dephased Turyn Hadamard matrix, then every column of \( H \), except for the first one, gives an equitable partition of \( G \).

Further, if \( H \) is a Hadamard matrix, then these \((n-1)\) equitable partitions are distinct, and the intersection of the equitable partitions obtained from \( h_j \) and \( h_k \), for \( 2 \leq j < k \leq n \), consists of four cells of equal size.

### 2.1 Small Examples

Recall that a conference matrix \( C \) of order \( n \) is a symmetric or antisymmetric matrix with zeros on the diagonal, \( \pm 1 \)'s on the off–diagonals, and satisfies \( C^\top C = (n-1)I \).

**Example 2.6.** Let \( C \) be a real symmetric conference matrix of order \( n \). We obtain the matrix \( H \) by dephasing the Turyn Hadamard matrix \( I + IC \), see [21]. Then \( H \) has the form
\[
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & -1 & \pm i & \pm i & \cdots & \pm i \\
1 & \pm i & \ddots \\
1 & \pm i & & \ddots \\
\vdots & \vdots & \ddots \\
1 & \pm i & & & \ddots \\
\end{bmatrix}
\]
The Cheeger inequality states that [9, Chapter 2] \[ \frac{\lambda_2}{2} \leq h \leq \sqrt{2\lambda_2}, \]
and \( K_n \) is the only graph diagonalisable by \( H \).

**Theorem 2.7.** Let \( k \) be a positive integer and let \( G_j, j = 1, \ldots, 2k+1 \), be unweighted connected graphs of order \( n \). The graph \( G = G_1 + \cdots + G_{2k+1} \) is not diagonalisable by a Turyn Hadamard matrix, regardless of whether or not \( G_j \) is for some \( j \).

**Proof.** Suppose \( L(G) \) is diagonalisable by a complex Hadamard matrix \( H \). The eigenspace of \( L(G) \) for the eigenvalue 0 is spanned by \((2k+1) \) columns of \( H \) of the form
\[ h_j = (a_{j,1}, a_{j,2}, \ldots, a_{j,2(k+1)})^T \otimes 1_n, \quad \text{for} \ j = 1, \ldots, 2k+1. \]
The vectors \( h_1, \ldots, h_{2k+1} \) are mutually orthogonal, therefore the matrix
\[ \begin{bmatrix}
  a_{1,1} & a_{1,2} & \cdots & a_{1,2k+1} \\
  a_{2,1} & a_{2,2} & \cdots & a_{2,2k+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{2k+1,1} & a_{2k+1,2} & \cdots & a_{2k+1,2k+1}
\end{bmatrix} \]
is a complex Hadamard matrix of order \( 2k+1 \). We conclude that \( H \) is neither a Turyn Hadamard matrix nor a Hadamard matrix. \( \Box \)

**Corollary 2.8.** All the unweighted graphs on 8 or fewer vertices that are diagonalisable by a Turyn Hadamard matrix or Hadamard matrix are listed below:

- **Order 2:** \( K_2 \) and \( K_2^c \);
- **Order 4:** \( K_4, K_4^c, K_2 + K_2, \) and \( K_4^c \);
- **Order 6:** \( K_6, K_6^c \);
- **Order 8:** \( K_8, K_{2,2,2,2}, (C_4 + C_4)^c, (K_2 \square K_2)^c, K_4, K_4^c, K_2 + K_2 + K_2, K_2 + K_2^c, C_4 + C_4, \) and \( K_8^c \).

**Proof.** By Corollary 2.4, the graphs diagonalisable by a Turyn Hadamard matrix or a Hadamard matrix must be regular and with only even Laplacian eigenvalues. All regular graphs on two or four vertices are Hadamard diagonalisable. Among the regular graphs on six vertices that are listed in Observation 3 of [1], only \( K_6, K_{2,2,2}, K_2 + K_2 + K_2, \) and \( K_6^c \) have all eigenvalues even. By Theorem 2.7 we rule out \( K_2 + K_2 + K_2 \) and its complement \( K_{2,2,2} \).

For order 8, all regular graphs with only even eigenvalues are Hadamard diagonalisable; such graphs are listed on Page 1892 of [1]. \( \Box \)

2.2. **Cheeger Constant.** The Cheeger constant of a set \( S \) of vertices in an unweighted graph \( G \) is
\[ h_G(S) = \frac{|E(S, V(G) - S)|}{\min(\sum_{x \in S} \deg(x), \sum_{y \notin S} \deg(y))}, \]
where \(|E(S, V(G) - S)|\) is the number of edges in the edge–cut \((S, V - S)\). The Cheeger constant of \( G \) is
\[ h_G = \min_{S \subseteq V(G)} h_G(S). \]

The Cheeger inequality states that [9, Chapter 2]
\[ \frac{\gamma_2}{2} \leq h_G \leq \sqrt{2\gamma_2}, \]
where \( \gamma_2 \) is the second smallest eigenvalue of the normalised Laplacian matrix, \( \mathcal{L}(G) = D^{-1/2}LD^{1/2} \), of \( G \).

Since \( G \) is \( d \)-regular, its normalised Laplacian matrix \( \mathcal{L}(G) \) equals \( \frac{1}{d}L(G) \) and the second smallest eigenvalue of \( \mathcal{L}(G) \) is \( \gamma_2 = \frac{\lambda_2}{d} \).

If \( G \) is diagonalisable by a complex Hadamard matrix and the entries of \( h_2 \) are in \( \{\pm 1, \pm i\} \), then the Cheeger constant of the set \( S = R_+ \cup I_+ \) in \( G \) is
\[ h_G(S) = \frac{|S| \lambda_2}{|S|d} = \frac{\gamma_2}{2}. \]
Hence the lower bound on the Cheeger constant \( h_G \geq \frac{\gamma_2}{2} \) is tight, see [9].
Lemma 3.1. Let \( r \) entries being then graph \( G \) form as given in Theorem 2.3 where conditions labelled (A) and (B) in [13], which are equivalent to the resulting quotient matrix having the which verifies (3). Furthermore, any such graph will also satisfy additional regularity constraints (see the

Proof. First define the polynomial

\[ p(x) = a_{r-1}x^{r-1} + a_{r-2}x^{r-2} + \cdots + a_1x + a_0 - \lambda. \]

Suppose \( r \) is a prime number. The \( r \)-th cyclotomic polynomial is

\[ \Phi_r(x) = x^{r-1} + x^{r-2} + \cdots + x + 1. \]

It follows from \( p(\zeta) = 0 \) that \( \Phi_r(x) \) is a factor of \( p(x) \) and

\[ a_{r-1} = a_{r-2} = \cdots = a_1 = a_0 - \lambda. \]

Now suppose \( r = 2^m \), for some positive integer \( m \). The \( r \)-th cyclotomic polynomial is

\[ \Phi_r(x) = x^{\frac{r}{2}} + 1. \]

As \( p(\zeta) = 0 \), there exists polynomial \( \sum_{j=0}^{\frac{r}{2}-1} b_jx^j \) satisfying

\[ p(x) = \Phi_r(x) \left( \sum_{j=0}^{\frac{r}{2}-1} b_jx^j \right) = \left( \sum_{j=0}^{\frac{r}{2}-1} b_jx^j \right) + \left( \sum_{j=0}^{\frac{r}{2}-1} b_jx^{\frac{r}{2}+j} \right). \]

We conclude that \( a_j = a_{\frac{r}{2}+j} = b_j \), for \( j = 1, \ldots, \frac{r}{2} - 1 \). \( \square \)

For Theorems 3.2 and 3.3 we recall that all rational eigenvalues of an integer–valued matrix are in fact integers.

The following result generalises Corollary 2.3 for higher powers of two.
Theorem 3.2. Let $G$ be an integer–weighted graph. If $L(G)$ is diagonalisable by a Butson Hadamard matrix $H$ in $H(2^m, n)$, for some positive integer $m$, then all integer eigenvalues of $L(G)$ are even.

Proof. We may assume $H$ is dephased. Let $\lambda$ be an integer eigenvalue of $L(G)$ and $h$ be the column of $H$ satisfying $L(G)h = \lambda h$. Let $\zeta = e^{2\pi i / p}$. For $j = 0, \ldots, 2^m - 1$, let $X_j = \{ s : h(s) = \zeta^j \}$.

For $j = 0, \ldots, 2^m - 1$, let
$$a_j = \sum_{s \in X_j} L(G)_{1,s}.$$ Then the first entry of $L(G)h$ is
$$(L(G)h)_1 = \sum_{j=0}^{2^m-1} a_j \zeta^j = \lambda.$$ It follows from Lemma 3.1 that
$$\lambda = a_0 + a_{2^m - 1} \zeta^{2^m - 1} + \Phi_2(\zeta) \sum_{j=1}^{2^m-1} a_j \zeta^j = a_0 - a_{2^m - 1}.$$ Since $L(G)\mathbf{1} = 0 \cdot \mathbf{1}$, the first entry of $L(G)\mathbf{1}$ is
$$\sum_{j=0}^{2^m-1} a_j = a_0 + a_{2^m - 1} + 2 \sum_{j=1}^{2^m-1} a_j = 0.$$ We conclude that $\lambda = a_0 - a_{2^m - 1} = -2 \sum_{j=1}^{2^m-1} a_j$ is even.

In the following theorem, we give a lower bound on the multiplicity of the integer eigenvalues other than zero for a graph diagonalisable by a Butson Hadamard matrix $H$ in $H(p, n)$. We recall that the multiplicity of the zero eigenvalue for any Laplacian matrix is well–known: it is exactly the number of connected components of the graph (in particular, for a connected graph, the multiplicity of the zero eigenvalue is one). Furthermore, we show that any integer eigenvalue is a multiple of $p$.

In the real setting, if a graph is Hadamard diagonalisable, $G$ has an equitable partition corresponding to each $\{1, -1\}$–eigenvector of $L(G)$. In the case of Butson Hadamard diagonalisable graphs, the proof of Theorem 3.3 leads to a similar result: the vertex set of the corresponding graph $G$ has an equitable partition based on the given eigenvector.

Theorem 3.3. Let $G$ be an integer–weighted graph on $n$ vertices. Suppose $L(G)$ is diagonalisable by a Butson Hadamard matrix $H$ in $H(p, n)$, for some prime number $p$. If $L(G)$ has a non–zero integer eigenvalue $\lambda$, then

(i) $\lambda$ is divisible by $p$,
(ii) $\lambda$ has multiplicity at least $p - 1$, and
(iii) $G$ has an equitable partition with $p$ parts of equal size.

Proof. We define $h$ and $X_0, X_1, \ldots, X_{p-1}$ as in the proof of Theorem 3.2 and let $\zeta = e^{2\pi i / p}$. Applying Lemma 3.1 to
$$(L(G)h)_1 = \sum_{j=0}^{p-1} a_j \zeta^j = \lambda$$ yields $a_1 = \cdots = a_{p-1}$. As $L(G)$ has zero row sums, we have $a_0 = -(p - 1)a_1$ and
$$\lambda = a_0 - a_1 + a_1 \Phi_p(\zeta) = -pa_1.$$ For $j = 0, \ldots, p - 1$, let $y_j$ be the characteristic vector of the subset $X_j$ of $V(G)$ so
$$h = \sum_{j=0}^{p-1} \zeta^j y_j = \sum_{j=0}^{p-2} \zeta^j (y_j - y_{p-1})$$ with the last equality results from $\zeta^{p-1} = -\sum_{j=0}^{p-2} \zeta^j$. Now
$$(L(G) - \lambda \mathbf{1})(y_j - y_{p-1})$$
is a vector with integer entries. So each entry of $(L(G) - \lambda I)h$ is a linear combination of $1, \zeta, \zeta^2, \ldots, \zeta^{p-2}$ with integer coefficients. Since $1, \zeta, \zeta^2, \ldots, \zeta^{p-2}$ are linearly independent over $\mathbb{Q}$, we conclude that

$$(L(G) - \lambda I)(y_j - y_{p-1}) = 0, \quad \text{for } j = 0, \ldots, p - 2.$$ 

Hence the eigenspace for the eigenvalue $\lambda$ contains at least $(p-1)$ linearly independent vectors $\{y_j - y_{p-1}\}_{j=0}^{p-2}$.

For vertex $u \in X_k$, let $a_{k,j}(u)$ be the sum of the weights of edges incident with $u$ and a vertex in $X_j$. Then

$$(L(G)h)(u) = \sum_{j=0}^{p-1} a_{k,j}(u)\zeta^j = \lambda \zeta^k.$$ 

By Lemma 3.2 and the fact that $(L(G)1)(u) = \sum_{j=0}^{p-1} a_{k,j}(u) = 0$, we have

$$a_{k,j} = \begin{cases} \frac{(n-1)\lambda}{p} & \text{if } j \neq k \\ \frac{(n-1)\lambda}{p} & \text{if } j = k \end{cases}$$

which is independent of the vertex $u$. Therefore $X_0, X_1, \ldots, X_{p-1}$ is an equitable partition of $G$. 

**Remark 3.4.** Let $C_r$ be the unweighted cycle of length $r$. Then $L(C_r)$ is diagonalisable by the character table of $\mathbb{Z}_r$ which belongs to $H(r,r)$ and it has spectrum

$$\left\{ 2 - 2 \cos \left( \frac{2\ell \pi}{r} \right) \right\}_{\ell=0}^{r-1}.$$ 

There exist many integer–weighted graphs that have irrational Laplacian eigenvalues.

4. CONTINUOUS–TIME QUANTUM WALKS

Given a graph $G$, the (Laplacian) continuous–time quantum walk on $G$ is determined by the operator $e^{-itL(G)}$.

We use $e_v$ to denote the characteristic vector of $v$.

**Fractional revival** occurs from vertex $a$ to vertex $b$ in $G$ at time $\tau$ if

$$e^{-itL(G)}e_v = \alpha e_a + \beta e_b,$$

for some complex scalars $\alpha$ and $\beta \neq 0$ with $|\alpha|^2 + |\beta|^2 = 1$. We also say $(\alpha, \beta)$–fractional revival occurs. If $\alpha = 0$, then we have (Laplacian) perfect state transfer from $a$ to $b$. Analogous definitions go through for the continuous–time walk associated with the adjacency matrix and the corresponding operator $e^{-itA(G)}$.

**Theorem 4.1.** Suppose $L(G)$ is diagonalisable by a dephased complex Hadamard matrix $H$ of order $n$, say $L(G) = \frac{1}{n} H \Lambda H^*$, where $\Lambda$ is the diagonal matrix with $\Lambda_{j,j} = \lambda_j$, for $j = 1, \ldots, n$. Then $(\alpha, \beta)$–(Laplacian) fractional revival occurs from vertex $a$ to vertex $b$ in $G$ at time $\tau$ if and only if, for $j = 1, \ldots, n$,

1. $H_{a,j} = \pm H_{b,j}$, and
2. $e^{-i\tau \lambda_j} = \begin{cases} 1 & \text{if } H_{a,j} = H_{b,j}, \\
\alpha - \beta & \text{if } H_{a,j} = -H_{b,j}, \end{cases}$

In this case, there exists real number $\gamma$ such that

$$\alpha = \cos \gamma \ e^{i\gamma}, \quad \beta = -\sin \gamma \ e^{i\gamma}, \quad \text{and} \quad \alpha - \beta = e^{2i\gamma}.$$ 

**Proof.** We have

$$L(G) = \frac{1}{n} H \Lambda H^*$$

and

$$e^{-itL(G)} = \frac{1}{n} H e^{-it\Lambda} H^*.$$ 

Now $e^{-itL(G)}e_v = \alpha e_a + \beta e_b$ if and only if

$$e^{-it\Lambda} H^* e_v = \alpha H^* e_a + \beta H^* e_b,$$

Since $\lambda_1 = 0$ and $H$ is dephased, the first entry of (4) gives

$$\alpha + \beta = 1.$$ 

It follows from $|\alpha|^2 + |\beta|^2 = 1$ and $\alpha + \beta = 1$ that $\alpha = \cos \gamma \ e^{i\gamma}$ and $\beta = -\sin \gamma \ e^{i\gamma}$, for some $\gamma \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right)$. 

For $j > 1$, the $j$-th entry of (4) is

$$e^{-i\tau\lambda_j}\overline{H}_{a,j} = \alpha\overline{H}_{a,j} + \beta\overline{H}_{b,j}$$

which simplifies to

(5)

$$e^{-i\tau\lambda_j} = \beta\overline{H}_{a,j}\overline{H}_{b,j} + \alpha.$$

Since $|H_{a,j}\overline{H}_{b,j}| = 1$ and (5) implies $|\alpha + \beta (H_{a,j}\overline{H}_{b,j})| = 1$, as demonstrated in the above figure, it can be shown that

$$H_{a,j}\overline{H}_{b,j} = \pm 1$$

and

$$e^{-i\tau\lambda_j} = \begin{cases} 1 & \text{if } H_{a,j} = H_{b,j} \\ \alpha - \beta & \text{if } H_{a,j} = -H_{b,j} \end{cases}$$

for $j = 2, \ldots, n$.

**Corollary 4.2.** Suppose $L(G)$ is diagonalisable by a dephased complex Hadamard matrix $H$ of order $n$. Then $(e^{i\tau}\cos \gamma, -e^{i\tau}\sin \gamma)$- (Laplacian) fractional revival occurs from vertex $a$ to vertex $b$ in $G$ at time $\tau$ if and only if there exists $\gamma \in (-\frac{\pi}{2}, \frac{\pi}{2})$ such that, for $j = 1, \ldots, n$,

1. $H_{a,j} = \pm H_{b,j},$ and
2. $-\tau\lambda_j = \begin{cases} 0 \pmod{2\pi} & \text{if } H_{a,j} = H_{b,j}, \\ 2\gamma \pmod{2\pi} & \text{if } H_{a,j} = -H_{b,j}. \end{cases}$

**Remark 4.3.** Since a complex Hadamard diagonalisable graph $G$ is regular, we have $A(G) = dI - L(G)$ and

$$e^{-i\tau A(G)} = e^{-1id\tau e^{i\tau L(G)}} = e^{-1id\tau e^{-i\tau L(G)}}.$$ 

Hence $G$ has $(e^{i\tau}\cos \gamma, -e^{i\tau}\sin \gamma)$- (Laplacian) fractional revival from $a$ to $b$ if and only if it has $(e^{i(-d\tau - \gamma)}\cos \gamma, e^{i(-d\tau - \gamma)}\sin \gamma)$ - (adjacency) fractional revival.

It follows from Proposition 5.1 of [6] that $a$ and $b$ are strongly cospectral in $G$: that is, if $L(G)$ has spectral decomposition $\sum_{i=1}^{m} \lambda_i E_i$, then $a$ and $b$ are strongly cospectral if and only if $E_j e_a = \pm E_j e_b$ for each $j = 1, \ldots, m$. By Theorem 5.5 and Corollary 5.6 of [6] and $G$ being a regular graph, both $A(G)$ and $L(G)$ have integral eigenvalues.

The following corollary extends the proof of Theorem 4 in [15] to complex Hadamard diagonalisable graphs.

**Corollary 4.4.** Suppose $L(G)$ is diagonalisable by a dephased complex Hadamard matrix $H$ of order $n$. Then (Laplacian) perfect state transfer occurs from vertex $a$ to vertex $b$ in $G$ at time $\tau$ if and only if, for $j = 1, \ldots, n$,

1. $H_{a,j} = \pm H_{b,j},$ and
2. $-\tau\lambda_j = \begin{cases} 0 \pmod{2\pi} & \text{if } H_{a,j} = H_{b,j}, \\ \pi \pmod{2\pi} & \text{if } H_{a,j} = -H_{b,j}. \end{cases}$

We extend Theorem 2.4 of [22] to (Laplacian) fractional revival here.

**Corollary 4.5.** Let $G$ be a Cayley graph on the finite abelian group $\Gamma$ with connection set $C$. Then $(\alpha, \beta)$ - (Laplacian) fractional revival occurs in $G$ from $a$ to $b$ at time $\tau$ if and only if the following three conditions hold:
\[(1) \text{ The eigenvalues of } L(G) \text{ are integers;} \]
\[(2) a-b \text{ has order two;} \]
\[(3) e^{-i\tau \lambda_j} = \alpha + \chi_j(a-b)(1-\alpha), \text{ for } j \in \Gamma. \]

**Proof.** By Remark 4.3, \(L(G)\) has integral eigenvalues if \(G\) admits fractional revival.

For \(j \in \Gamma\), let \(\chi_j\) be the character of \(\Gamma\) indexed by \(j\). We can view \(\chi_j\) as a column of \(H\). Suppose the first column of \(H\) corresponds to the trivial character and the first row corresponds to the identity in \(\Gamma\). Condition (1) of Theorem 4.1 is equivalent to \(\chi_j(a-b) \in \{-1,1\}\) for all \(j \in \Gamma\). That is, \(a-b\) has order 2. Condition (2) of Theorem 4.1 is equivalent to

\[e^{-i\tau \lambda_j} = \alpha + \chi_j(a-b)(1-\alpha), \]

for \(j \in \Gamma. \)

**Example 4.6.** For \(n \geq 3\), the cocktail party graph \((nK_2)^c\) (i.e., the graph complement of the ladder rung graph \(nK_2\)) is diagonalisable by the character table of \(\mathbb{Z}_{2n}\). It admits (Laplacian) fractional revival from vertex \(a\) to \(a+n\) at time \(\frac{\pi}{n}\); see [3].

We apply Corollary 4.2 to \(G_1 \bowtie G_2\) where both \(G_1\) and \(G_2\) are diagonalisable by the same complex Hadamard matrix. We assume \(G_1\) and \(G_2\) have the same vertex \(V\) and use \(V \times \mathbb{Z}_2\) to denote the vertex set of \(G_1 \bowtie G_2\).

**Corollary 4.7.** Let \(G_1\) and \(G_2\) be graphs diagonalisable by a dephased complex Hadamard matrix \(H\) of order \(n\). Let

\[L(G_1)h_j = \lambda_j h_j \quad \text{and} \quad L(G_2)h_j = \mu_j h_j, \quad \text{for } j = 1, \ldots, n,\]

and let \(d_2\) be the degree of \(G_2\). Then \(G_1 \bowtie G_2\) has \((e^{i\gamma} \cos \gamma, -e^{i\gamma} \sin \gamma)\)-fractional revival from \((a,0)\) to \((a,1)\) at time \(\tau\) if and only if

\[\gamma = -d_2 \tau \quad (\text{mod } \pi)\]

and

\[\tau \lambda_j + \tau \mu_j = \tau \mu_j - \tau \mu_j = 0 \quad (\text{mod } 2\pi)\]

for \(j = 1, \ldots, n.\)

**Proof.** The Laplacian matrix of \(G_1 \bowtie G_2\)

\[
\begin{bmatrix}
L(G_1) + d_2 I & -A(G_2) \\
-A(G_2) & L(G_1) + d_2 I
\end{bmatrix}
\]

satisfies

\[
L(G_1 \bowtie G_2) \begin{bmatrix} h_j \\ h_j \end{bmatrix} = (\lambda_j + \mu_j) \begin{bmatrix} h_j \\ h_j \end{bmatrix} \quad \text{and} \quad L(G_1 \bowtie G_2) \begin{bmatrix} h_j \\ -h_j \end{bmatrix} = (\lambda_j - \mu_j + 2d_2) \begin{bmatrix} h_j \\ -h_j \end{bmatrix}.
\]

As \(L(G_1 \bowtie G_2)\) is diagonalisable by

\[
\tilde{H} = \begin{bmatrix} H & H \\
H & -H \end{bmatrix},
\]

condition (1) of Corollary 4.2 holds for \(\tilde{H}\). Condition (2) of Corollary 4.2 holds if and only if

\[
\begin{aligned}
-\tau(\lambda_j + \mu_j) = 0 \quad (\text{mod } 2\pi), \\
-\tau(\lambda_j - \mu_j + 2d_2) = 2\gamma \quad (\text{mod } 2\pi)
\end{aligned}
\]

for \(j = 1, \ldots, n.\)

Since \(\lambda_1 = \mu_1 = 0\), the second equation gives \(2d_2 \tau = -2\gamma \quad (\text{mod } 2\pi)\), hence

\[
\tau \lambda_j + \tau \mu_j = \tau \lambda_j - \tau \mu_j = 0 \quad (\text{mod } 2\pi)
\]

for all \(j.\)

**Example 4.8.** The bipartite double cover of \(K_n\) is \(K_n^c \bowtie K_n\). For \(n \geq 3\), (Laplacian) fractional revival occurs from \((a,0)\) to \((a,1)\) at time \(\tau = \frac{2\pi}{n}\) with \(\gamma = \frac{2\pi}{n}.\)
Example 4.9. Let $G_2$ be the $(2m + 1)$–cube and $G_1 = G_2^c$, for $m \geq 1$. Then both $G_1$ and $G_2$ are diagonalisable by

$$H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes (2m + 1).$$

The spectrum of $L(G_2)$ is $\{2s : s = 0, \ldots, 2m + 1\}$, see 5. For $j = 1, \ldots, 2^{m+1}$, we have $\lambda_j + \mu_j = 2^{m+1}$ and

$$\lambda_j - \mu_j \in \{2^{m+1} - 4s : s = 0, \ldots, 2m + 1\}.$$

Since $G_2$ has degree $d_2 = 2m + 1$, the double cover of $K_n$ given by $G_1 \ltimes G_2$ has (Laplacian) perfect state transfer $(\gamma = \frac{\pi}{2})$ from vertex $(a, 0)$ to $(a, 1)$ at time $\frac{\pi}{2}$.

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