BEURLING-SELBERG EXTREMIZATION FOR DUAL-BLIND DECONVOLUTION RECOVERY IN JOINT RADAR-COMMUNICATIONS

Jonathan Monsalve*, Edwin Vargas†, Kumar Vijay Mishra‡, Brian M. Sadler‡ and Henry Arguello†

†Universidad Industrial de Santander, Bucaramanga, Colombia, 680002
*Corporación Universitaria Minuto de Dios, Bogotá, Colombia, 111021
‡United States DEVCOM Army Research Laboratory, Adelphi, MD 20783 USA

ABSTRACT
Recent interest in integrated sensing and communications has led to the design of novel signal processing techniques to recover information from an overlaid radar-communications signal. Here, we focus on a spectral coexistence scenario, wherein the channels and transmit signals of both radar and communications systems are unknown to the common receiver. In this dual-blind deconvolution (DBD) problem, the receiver admits a multi-carrier wireless communications signal that is overlaid with the radar signal reflected off multiple targets. The communications and radar channels are represented by continuous-valued range-times or delays corresponding to multiple transmission paths and targets, respectively. Prior works addressed recovery of unknown channels and signals in this ill-posed DBD problem through atomic norm minimization but contingent on individual minimum separation conditions for radar and communications channels. In this paper, we provide an optimal joint separation condition using extremal functions from the Beurling-Selberg interpolation theory. Thereafter, we formulate DBD as a low-rank modified Hankel matrix retrieval and solve it via nuclear norm minimization. We estimate the unknown target and communications parameters from the recovered low-rank matrix using multiple signal classification (MUSIC) method. We show that the joint separation condition also guarantees that the underlying Vandermonde matrix is well-conditioned. Numerical experiments validate our theoretical findings.

Index Terms—Beurling-Selberg majorant, condition number, dual-blind deconvolution, joint radar-communications, passive radar.

I. INTRODUCTION

The electromagnetic spectrum is a scarce natural resource. With the advent of cellular communications and novel radar applications, the spectrum has become increasingly contested. This has led to the development of joint radar-communications (JRC) systems, which facilitate spectrum-sharing while offering benefits of low cost, compact size, and less power consumption [1–4]. The JRC design topologies broadly fall into three categories: co-design [5], cooperation [6], and co-existence [7]. The spectral co-design employs a common transmit waveform and/or hardware while the cooperation technique relies on opportunistic processing of signals from one system to aid the other. The spectral co-existence is useful for legacy systems, wherein the radar and communications transmit and access the channel independently, receive overlaid signals at the receiver, and mitigate mutual interference. In this paper, we focus on the overlaid receiver for the spectral coexistence scenario.

This research was sponsored by the Army Research Office/Laboratory under Grant Number W911NF-21-1-0099, and the VIE project entitled “Dual blind deconvolution for joint radar-communications processing”. K. V. M. acknowledges support from the National Academies of Sciences, Engineering, and Medicine via the Army Research Laboratory Harry Diamond Distinguished Fellowship.
a sliding column form \[33\] – for MUSIC-based recovery.

This paper uses boldface lowercase and uppercase for vectors and matrices, respectively. The \(i\)-th entry of the vector \(x\) is \([x]_i\), \([X]_{i,j}\) represents the \(i\)-th row and \(j\)-th column of \(X\). \(x\) is the \(i\)-th column of \(X_x\). We denote the transpose, conjugate, and Hermitian by \(\mathbf{a}^T\), \(\mathbf{a}^*\), and \(\mathbf{a}^H\) respectively. Unless otherwise specified, the integrals and summations limits are \(-\infty\) and \(\infty\). The functions \(\max(a, b)\) and \(\min(a, b)\) return, respectively, maximum and minimum of the input arguments. The expression \(\sum_{\alpha=1}^{K}\sum_{\beta=1}^{Q}\alpha_i\beta_j\mathbf{F}\) represents the continuous-time Fourier transform (CTFT); \(\ast\) denotes the convolution operation; \((\mathbf{A}, \mathbf{B}) = \text{Tr}((\mathbf{A}^\mathbf{B}))\), and \(\text{Tr}()\) is the matrix trace; \(\sigma_{\text{min}}(\mathbf{A})\) is the minimum singular value of \(\mathbf{A}\); and \(\text{rank}()\) is its matrix rank.

### II. SIGNAL MODEL

Consider a radar that transmits a time-limited baseband pulse \(g_r(t)\), whose CTFT is \(\tilde{g}_r(f) = \int g_r(t) e^{-j2\pi ft} dt\), assuming that most of the radar signal’s energy lies within the frequencies \(\pm B/2\) yields to \(g_r(f) \approx \int_{-B/2}^{B/2} g_r(t) e^{-j2\pi ft} dt\). The pulse \(g_r(t)\) is reflected back to the receiver by \(K\) targets, where the \(i\)-th target is characterized by time delay \([\tau_\mathbf{r}]_i\) and complex amplitude \([\mathbf{b}]_i\), \([\tau_\mathbf{r}]_i\) is linearly proportional to the target’s range and \([\mathbf{b}]_i\) models the path loss and reflectivity.

The radar channel is

\[ h_r(t) = \sum_{i=0}^{K-1} [\mathbf{b}]_i \delta (t - [\tau_\mathbf{r}]_i). \]

(1)

The communications transmitted signal \(g_c(t)\) is a message modeled as an orthogonal frequency-division multiplexing (OFDM) signal with bandwidth \(B\) and \(M\) equi-bandwidth sub-carriers separated by \(\Delta f\), i.e.,

\[ g_c(t) = \sum_{l=1}^{M} [\mathbf{g}]_l e^{j2\pi l\Delta ft}, \]

(2)

where \([\mathbf{g}]_l\) is the modulated symbol onto the \(l\)-th subcarrier. The message propagates through a communications channel \(h_c(t)\) that comprises \(Q\) paths characterized by their attenuation/channel coefficients \(\omega \in \mathbb{C}^Q\) and delays \([\tau_\mathbf{c}]_q\)

\[ h_c(t) = \sum_{q=1}^{Q} [\omega]_q \delta (t - [\tau_\mathbf{c}]_q). \]

(3)

The communications systems share the spectrum in a spectral coexistence scenario. The common received signal is a superposition of radar and communications signals propagated through their respective channels as

\[ y(t) = g_r(t) * h_r(t) + g_c(t) * h_c(t), \]

\[ = \sum_{i=1}^{K} [\mathbf{b}]_i g_r(t - [\tau_\mathbf{r}]_i) + \sum_{q=1}^{Q} [\omega]_q g_c(t - [\tau_\mathbf{c}]_q). \]

(4)

The CTFT of the overlaid signal is

\[ Y(f) = \sum_{i=1}^{K} [\mathbf{b}]_i e^{-j2\pi [\tau_\mathbf{r}]_i f} \tilde{g}_r(f) + \sum_{q=1}^{Q} [\omega]_q e^{-j2\pi [\tau_\mathbf{c}]_q f} \tilde{g}_c(f), \]

(5)

where \(\tilde{g}_c(f) = \mathcal{F}(g_c(t)).\) Sampling (5) at the Nyquist rate of \(\Delta f = \frac{B}{N}\), where \(n = 0, \ldots, N - 1\), yields

\[ Y(n) = \sum_{i=1}^{K} [\mathbf{b}]_i e^{-j2\pi [\tau_\mathbf{r}]_i n \Delta f} \tilde{g}_r(n\Delta f) \]

\[ + \sum_{q=1}^{Q} [\omega]_q e^{-j2\pi [\tau_\mathbf{c}]_q n \Delta f} \tilde{g}_c(n\Delta f). \]

(6)

Collect the samples of \(\tilde{g}_c(n\Delta f)\) in the vector \(\mathbf{g}_c\), i.e., \([\mathbf{g}_c]_n = \tilde{g}_c(n\Delta f)\). Similarly, \([\mathbf{g}_r]_n = \tilde{g}_r(n\Delta f)\). The samples \([y]_n = Y(n)\) of the observation vector \(y\) are

\[ [y]_n = \sum_{i=1}^{K} [\mathbf{b}]_i e^{-j2\pi [\tau_\mathbf{r}]_i n} + \sum_{q=1}^{Q} [\omega]_q e^{-j2\pi [\tau_\mathbf{c}]_q n}, \]

where \(\mathbf{g}_c\) and \(\mathbf{g}_r\) are Vandermonde matrices such that \([\mathbf{V}_c]_{i,i} = [\alpha][\tau_\mathbf{r}]_i^i\) and \([\mathbf{V}_r]_{i,i} = [\alpha][\tau_\mathbf{c}]_i^i\).

### III. BEURLING-SELBERG EXTREMAL FUNCTIONS

The guarantees to DBD recovery in previous studies \[14\] are based on the minimum separation of spikes in each channel assuming a noiseless scenario. Here, we resort to the Beurling-Selberg extremal functions to derive an optimal separation condition that imposes a joint structure on the two channels. This result follows analyzing the condition number of \(\mathbf{G}_r\) that controls the stability of the solution of (8) in the presence of additive noise.

Consider the indicator function \(I_E(t)\) defined on the interval \([0, N - 1]\), where \(I_E(t) = 1\) if \(0 < t < N - 1\) and \(0\) otherwise. We can find approximations for \(I_E(t)\) using functions \(b(t)\) and \(B(t)\) such that \(b(t)I_E(t) < B(t)\). Moreover, we seek to minimize the integrals \(\int B(t) - I_E(t)dT\) and \(\int (I_E(t) - b(t))dT\). These optimization problems are known as extremal problems, and the resulting functions \(C_E(t)\) (majorant) and \(c_E(t)\) (minorant) are considered extremal functions.

Beurling \[26, 34\] found a solution to this extremal problem for the sign function by restricting \(C_E(t)\) and \(c_E(t)\) to be real entire functions of exponential type \(2\pi\). Recall that if an entire function \(F : \mathbb{C}^N \rightarrow \mathbb{C}\) is real-valued when it is restricted to \(\mathbb{R}^N\), it is called a real entire function. Additionally, if this statement holds and the growth of \(F\) is bounded by a constant \(C\) and \(C e^{(2\pi + c)|z|}\) for every \(\epsilon > 0, z \in \mathbb{C}\), it is called a real entire function of exponential type \(2\pi\).

Selberg observed that Beurling functions can majorize and minimize the characteristic function \(I_E(t)\). The resultant majorant and minorant functions have the useful property that their Fourier transforms are continuous functions supported on
the interval $[-\Delta, \Delta]$ [24–27]. We intend to employ these functions to provide a bound on the $V_g$ condition number. Fig. 1a shows that these functions approximate and bound the indicator function as $C_E(t) \leq I_E(t) \leq c_E(t)$. Fig. 1b depicts these functions in the Fourier domain. To this end, define the separation between the delay parameters corresponding to radar-only, communications-only and radar-communications measurements as, respectively, $\gamma_{i,i'} = |\tau_{s,i} - \tau_{s,i'}|$, $\zeta_{i,j} = |\tau_{c,i} - \tau_{c,j}|$, and $\xi_{i,j} = |\tau_{r,i} - \tau_{r,j}|$. Denote $\Delta_e = \min_{i \neq i'} \gamma_{i,i'}$, $\Delta_c = \min_{i \neq j} \zeta_{i,j}$, $\Delta_r = \min_{i \neq j} \xi_{i,j}$, and $\Delta = \min(\Delta_e, \Delta_c, \Delta_r)$. The following Proposition 1 states that the condition number is governed by the number of samples and the joint minimum separation.

**Proposition 1.** The condition number $\kappa_g$ of matrix $V_g$ in (8) satisfies $\kappa_g^2 \leq (N-1/\Delta + 1)/(N-1/\Delta - 1)$.

Proof: Recall the received signal in (8). Consider the Dirac comb $h(n) = \sum_t \delta(n-t) = \sum_t e^{j2\pi t n}$. The norm of the measurements is

$$N^{-1} \sum_{n=0}^{N-1} \|y_n\|^2 = \int h(n)I_E(n)\|Y(n)\|^2 dn \leq \int h(n)C_E(n)\|Y(n)\|^2 dn = \left( \sum_{i=1}^{K} \sum_{i'=1}^{Q} [\beta_i][\beta_{i'}]^{*} \right) \int h(n)C_E(n)\|\tilde{g}_e(n)\|^2 \|\alpha^n_{i,i'}\| dn + \sum_{i,i'=1}^{K,Q} [\beta_i][\omega_{i'}]^{*} \int h(n)C_E(n)\|\alpha^n_{i,i'}\| \|\zeta_{i,i'}\| dn + \sum_{i,i'=1}^{K,Q} [\omega_i][\beta_{i'}]^{*} \int h(n)C_E(n)\|\alpha^n_{i,i'}\| \|\xi_{i,i'}\| dn + \sum_{i,i'=1}^{K,Q} [\omega_i][\omega_{i'}]^{*} \int h(n)C_E(n)\|\alpha^n_{i,i'}\| \|\gamma_{i,i'}\| dn,$$

where the inequality follows from replacing $I_E(n)$ by $C_E(n)$. Substituting the expression of the Dirac comb $h(n)$, $g_{r,c}(n) = \tilde{g}_e(n)\tilde{g}_c(n)$, and $g_{r,c}(n) = \tilde{g}_e(n)\tilde{g}_c(n)$ produces

$$\sum_{n=0}^{N-1} \|y_n\|^2 \leq \sum_{i,i'=1}^{K,Q} [\beta_i][\beta_{i'}]^{*} \int C_E(n)\|\alpha^n_{i,i'}\|^2 \|\gamma_{i,i'}\| dn + \sum_{i,i'=1}^{K,Q} [\beta_i][\omega_{i'}]^{*} \int C_E(n)\|\alpha^n_{i,i'}\|^2 \|\zeta_{i,i'}\| dn + \sum_{i,i'=1}^{K,Q} [\omega_i][\beta_{i'}]^{*} \int C_E(n)\|\alpha^n_{i,i'}\|^2 \|\xi_{i,i'}\| dn + \sum_{i,i'=1}^{K,Q} [\omega_i][\omega_{i'}]^{*} \int C_E(n)\|\alpha^n_{i,i'}\|^2 \|\gamma_{i,i'}\| dn.$$

Using the convolution theorem and $\tilde{C}_E = \mathcal{F}(C_E)$ yields

$$\sum_{n=0}^{N-1} \|y_n\|^2 \leq \sum_{i,i'=1}^{K,Q} [\beta_i][\beta_{i'}]^{*} \int \tilde{C}_E(t') \|\gamma_{i,i'}\|^2 \|\gamma_{i,i'}\| \|\gamma_{i,i'}\| dn + \sum_{i,i'=1}^{K,Q} [\beta_i][\omega_{i'}]^{*} \int \tilde{C}_E(t') \|\zeta_{i,i'}\|^2 \|\zeta_{i,i'}\| \|\zeta_{i,i'}\| dn + \sum_{i,i'=1}^{K,Q} [\omega_i][\beta_{i'}]^{*} \int \tilde{C}_E(t') \|\xi_{i,i'}\|^2 \|\xi_{i,i'}\| \|\xi_{i,i'}\| dn + \sum_{i,i'=1}^{K,Q} [\omega_i][\omega_{i'}]^{*} \int \tilde{C}_E(t') \|\gamma_{i,i'}\|^2 \|\gamma_{i,i'}\| \|\gamma_{i,i'}\| dn + \sum_{i,i'=1}^{K,Q} \left[ \sum_{t'=1}^{Q} \sum_{i,i'=1}^{Q} [\beta_i][\beta_{i'}]^{*} \tilde{C}_E(t' + \gamma_{i,i'}) \|\gamma_{i,i'}\|^2 \|\gamma_{i,i'}\| \|\gamma_{i,i'}\| dn + \sum_{i,i'=1}^{K,Q} [\beta_i][\omega_{i'}]^{*} \tilde{C}_E(t' + \zeta_{i,i'}) \|\zeta_{i,i'}\|^2 \|\zeta_{i,i'}\| \|\zeta_{i,i'}\| dn + \sum_{i,i'=1}^{K,Q} [\omega_i][\beta_{i'}]^{*} \tilde{C}_E(t' + \xi_{i,i'}) \|\xi_{i,i'}\|^2 \|\xi_{i,i'}\| \|\xi_{i,i'}\| dn + \sum_{i,i'=1}^{K,Q} [\omega_i][\omega_{i'}]^{*} \tilde{C}_E(t' + \gamma_{i,i'}) \|\gamma_{i,i'}\|^2 \|\gamma_{i,i'}\| \|\gamma_{i,i'}\| dn \right].$$

Note that $\tilde{C}_E$ is supported on $[-\Delta, \Delta]$. Hence, the first and fourth terms are non-zero only when $t' = 0$ and $i = i'$. The second and third terms are zero for all $t', i, i'$ because $|t' + \zeta_{i,i'}| > \Delta$, for all $t', i, i'$. Substituting $G_E(n) = [\beta_i]^{*}\mathcal{F}\{[\tilde{g}_e(n)]^2\alpha^n_{i,i'}\}$, $G_{c}(n) = [\zeta_{i,i'}]^{*}\mathcal{F}\{[\tilde{g}_c(n)]^2\alpha^n_{i,i'}\}$ in (12) yields

$$\|y_n\|^2 \leq \sum_{i\in \Xi} \|C_E(0) + \sum_{i} C_E(0)\|G_E(n)\|G_{c}(n)\| + \sum_{i,i'=1}^{K,Q} [\omega_i][\omega_{i'}]^{*} \int \tilde{C}_E(t' + \gamma_{i,i'}) \|\gamma_{i,i'}\|^2 \|\gamma_{i,i'}\| \|\gamma_{i,i'}\| dn,$$

From [28, Theorem 2.2], the integral of the majorant function, i.e., $\int C_E(t) = 2n + 1/\Delta$. Hence, we obtain

$$\|y_n\|^2 \leq (N-1/\Delta - 1)/(N-1/\Delta - 1).$$

From Proposition 1, the condition number depends on the minimum joint separation. This gives $N > (1/\Delta + 1)$, which shows the coupled nature of radar and communications in the PDB problem.

**IV. LOW-RANK HANKEL MATRIX RECOVERY**

The recovery of $\tau_r, \tau_c, \beta$, and $\omega$ poses challenges due to the ill-posed nature of the problem. To address this, we reasonably assume that the unknown point spread functions (PSF), $g_r$ and $g_c$, can be accurately represented within a known low-dimensional subspace. This assumption is justified by considering the rank of the Vandermonde matrices, which is influenced by the number of radar and communications delays which is usually smaller than the number of measurements [13–15, 29]. From this, we denote $g_r = B h_r$ and $g_c = B h_c$, where $B \in \mathbb{C}^{N_r \times N_r}$, $h_r \in \mathbb{C}^{N_r}$, $D \in \mathbb{C}^{N_c \times N_c}$, and $h_c \in \mathbb{C}^{N_c}$. Collect the unknown radar and communications variables, respectively, in the matrices $X_r = \sum_{k=1}^{K} [\beta_k]^{*} h_r a_{[\tau_{r_k}]^T} \in \mathbb{C}^{N_r \times N_r}$ and $X_c = \sum_{j=1}^{Q} [\omega_j]^{*} h_c a_{[\tau_{c_j}]^T} \in \mathbb{C}^{N_c \times N_c}$, with $a_{[\tau_{r_k}]} = [1, e^{-2\pi i [\tau_{r_k}]}, e^{-2\pi i [2\tau_{r_k}]}, \ldots, e^{-2\pi i [N_r \tau_{r_k}]}, (N_r-1)]$ is the vector containing all the atoms $a_{[\tau_{r_k}]}^{n_{[\tau_{r_k}]}'}$. Define $s_j = [b_j, d_j]^T$, where $b_j$ and $d_j$ are the $j$-th columns of $B^*$ and $D^*$, respectively. Rewrite (7) as

$$|y_j| = (s_j e_j^T, \mathbf{X}),$$

with $\mathbf{X} = [X_r^T, X_c^T]^T \in \mathbb{C}^{(N_r+N_c) \times N}$, and $e_j \in \mathbb{R}^N$ is the $j$-th canonical vector. For the sake of simplicity, consider $N_r = N_c = N_y$. Define a linear operator $A : \mathbb{C}^{N_r \times N} \rightarrow \mathbb{C}^{N_f}$ such that $A(s_j) = (s_j e_j^T, \mathbf{X})$. Then, (15) becomes

$$y = A(X).$$

Select $N_1, N_2 \in \mathbb{N}$ such that $N_1 + N_2 = N + 1$. The low-dimensional subspace assumption allows us to represent the unknown variables in a low-rank matrix. To construct such a matrix,
apply a linear operator $\mathcal{H}()$ to both $X_r$ and $X_c$ such that it produces a Hankel matrix of higher dimensions using the columns of the input matrix. Then, concatenating both Hankel matrices using the operator $C(X) = C([X_r^T, X_c^T]) = [\mathcal{H}(X_r), \mathcal{H}(X_c)]$ yields

$$C(X) = \begin{bmatrix} x_{r0} & \ldots & x_{rN-1}^r & x_{c0} & \ldots & x_{cN-1}^c \\ x_{r1} & \ldots & x_{rN}^r & x_{c1} & \ldots & x_{cN}^c \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{rN-1}^r & \ldots & x_{rN}^r & x_{cN-1}^c & \ldots & x_{cN}^c \end{bmatrix}. \quad (17)$$

The concatenated matrix is decomposed as

$$C(X) = V_{h,\alpha} \text{diag}(\psi) V_T^r,$$ \quad (18)

where the $(N_1 N_c) \times (K + Q)$ complex matrix $V_{h,\alpha}$ is

$$V_{h,\alpha} = \begin{bmatrix} h_r^0 & \ldots & h_r^{\tau_1} & \ldots & h_r^{\tau_K} & h_c^0 & \ldots & h_c^{\tau_1} & \ldots & h_c^{\tau_K} \\ h_{\tau_1} r_1^0 & \ldots & h_{\tau_1} r_1^{\tau_K} & \ldots & h_{\tau_1} r_K^{\tau_K} & h_{\tau_1} c_1^0 & \ldots & h_{\tau_1} c_1^{\tau_K} & \ldots & h_{\tau_1} c_K^{\tau_K} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ h_{\tau_K} r_1^0 & \ldots & h_{\tau_K} r_1^{\tau_K} & \ldots & h_{\tau_K} r_K^{\tau_K} & h_{\tau_K} c_1^0 & \ldots & h_{\tau_K} c_1^{\tau_K} & \ldots & h_{\tau_K} c_K^{\tau_K} \end{bmatrix} \in \mathbb{C}^{(K+Q) \times 2N_2},$$ \quad (19)

with $\alpha_{\tau_1}^{n_1}$ and $\alpha_{\tau_1}^{n_2}$ as defined in (8) except that $n_1 = 0, \ldots, N_1 - 1$ and $n_2 = 0, \ldots, N_2 - 1$, and

$$V_T^r = \begin{bmatrix} \alpha_0 & \alpha_2 & \ldots & \alpha_{N_2-2} & 0 & 0 & 0 & \ldots & 0 \\ 0 & \alpha_0 & \ldots & \alpha_{N_2-2} & \alpha_2 & \ldots & \alpha_{N_2-2} & \ldots & \ldots & \ldots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ldots & \ddots & \ddots \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \alpha_0 \end{bmatrix} \in \mathbb{R}^{K \times (K+Q)},$$ \quad (20)

with $\alpha_{N_2}^{\tau_1} = \left[e^{-j2\pi n_1 r_1 \gamma}, e^{-j2\pi (n_1 + 1) r_1 \gamma}, \ldots, e^{-j2\pi (N_2 - 1) r_1 \gamma}\right]^T \in \mathbb{R}^K$ and $\alpha_{N_2}^{\tau_2} = \left[e^{-j2\pi (n_1 + 1) r_2 \gamma}, e^{-j2\pi (n_1 + 2) r_2 \gamma}, \ldots, e^{-j2\pi (N_2 - 1) r_2 \gamma}\right]^T \in \mathbb{R}^Q$.

From (18), $\text{rank}(C(X)) \leq (K + Q)$. Hence, $C(X)$ is a low-rank structure. Then, the problem of recovering $X$ becomes

$$\min_{\chi \in \mathbb{C}^{N_r \times N_c}} \|C(X)\|_2 \text{ subject to } y = A(X). \quad (21)$$

An optimal solution to (21) is guaranteed [29] if $\sigma_{\text{min}}(V_{h,\alpha}) \geq N_1 / \mu$ and $\sigma_{\text{min}}(V_T^r V_T^T) \geq N_2 / \mu$ for $\mu > 1$. From Proposition 1 and [29], these conditions imply $\Delta > 2\mu / (N_1 - 1)$. Once the matrix $X$ is obtained, the delays $\tau_r, \tau_c$ are recovered via MUSIC. Finally, the waveforms $g_r, g_c$ are retrieved using the least-squares method [14].

V. NUMERICAL EXPERIMENTS

We compared our approach to DBD against SoMAN SDP in [14], which assumes a low-dimensional subspace structure of the waveform $g_r$ and message $g_c$ and additionally exploits the sparsity of the channels. The SoMAN method represents the matrices $X_r$ and $X_c$ as a linear combination of atoms given by the sets $A_r = \{ h_r a_r^{\tau_r} : \tau_r \in [0,1) \}$ and $A_c = \{ h_c a_c^{\tau_c} : \tau_c \in [0,1) \} \subset \mathbb{C}^{N_r \times N_c}$. This leads to the following formulation of atomic norms

$$\|X_r\|_{A_r} = \inf_{\{h_r a_r^{\tau_r} : \tau_r \in [0,1)\}} \sum_{\tau_r} |\beta_{\tau_r}|, \quad (22)$$

$$\|X_c\|_{A_c} = \inf_{\{h_c a_c^{\tau_c} : \tau_c \in [0,1)\}} \sum_{\tau_c} |\omega_{\tau_c}|. \quad (23)$$

The SoMAN minimization estimates $X_r$ and $X_c$ as

$$\min_{X_r, X_c} \|X_r\|_{A_r} + \|X_c\|_{A_c} \text{ s. t. } y = X_r + X_c. \quad (24)$$

where $\beta_{\tau_r}(X_r) = \text{Tr}(e_r h_r a_r^{\tau_r} X_r)$ and $\omega_{\tau_c}(X_c) = \text{Tr}(e_c h_c a_c^{\tau_c} X_c)$.

To evaluate the performance of the proposed vectorized Hankel method, we performed several computational simulations using CVX [35, 36] library in MATLAB with SDPT3 [37] solver. The delays $\tau_r$ and $\tau_c$ were drawn uniformly at random from the interval $[0,1)$; the amplitudes $\beta, \omega$ were generated following $(1 + 10^j) e^{-j\phi}$ with $\gamma$ sampled from a uniform distribution $[0, 1]$ and $\phi$ from $[0, 2\pi)$. The columns of the matrices $B$ and $D$ are random columns of the DFT matrix of the corresponding size.

Figure 2 shows a specific delay recovery with two radar targets, two communication paths, and two unknown PSFs with $N_r = N_c = 2, K = Q = 2$, and 75 samples. The pseudospectrum is computed using MUSIC [29]. The proposed method is able to recover the exact position of the delays. Additionally, the recovered radar waveform and communications message are accurately estimated with the normalized mean-squared error (NMSE) $\|g - g^*\| / \|g\|$ of 0.0098 and 2.3144 respectively. Figure 3 shows the reconstruction probability of the matrices $X_r$ and $X_c$ over 20 experiments, where success is defined if the NMSE between the truth and recovered $X$ is lower than $10^{-3}$. We observe that the phase transition curve of the proposed method is higher than the SoMAN approach.

VI. SUMMARY

We provided an optimal separation condition for the DBD problem using extremization functions. This condition depends on the separation of the radar and communications delays and, contrary to prior guarantees, is jointly imposed. Our recovery algorithm based on a modified vectorized Hankel matrix recovery outperforms SoMAN SDP. Future investigations require generalizing the proposed approach to other coexistence models considered in [14].

VII. REFERENCES

[1] K. V. Mishra, M. R. Bhavani Shankar, V. Koivunen, B. Ottersten, and S. A. Vorobyov, “Toward millimeter wave joint radar-communications: A signal processing perspective,” *IEEE Signal Processing Magazine*, vol. 36, no. 5, pp. 100–114, 2019.
[2] G. Duggal, S. Vishwakarma, K. V. Mishra, and S. S. Ram, “Doppler-resilient 802.11ad-based ultrashort range automotive joint radar-communications system,” *IEEE Transactions on Aerospace and Electronic Systems*, vol. 56, no. 5, pp. 4035–4048, 2020.

[3] A. M. Elbir, K. V. Mishra, and S. Chatzinotas, “Terahertz-band joint ultra-massive MIMO radar-communications: Model-based and model-free hybrid beamforming,” *IEEE Journal of Special Topics in Signal Processing*, vol. 15, no. 6, pp. 1468–1483, 2021.

[4] A. M. Elbir, K. V. Mishra, M. R. B. Shankar, and S. Chatzinotas, “The rise of intelligent reflecting surfaces in integrated sensing and communications paradigms,” *arXiv preprint arXiv:2204.07265*, 2022.

[5] J. Liu, K. V. Mishra, and M. Saquib, “Co-designing statistical MIMO radar and in-band full-duplex multi-user MIMO communications,” *arXiv preprint arXiv:2006.14774*, 2020.

[6] M. Bică and V. Koivunen, “Radar waveform optimization for target parameter estimation in cooperative radar-communications systems,” *IEEE Transactions on Aerospace and Electronic Systems*, vol. 55, no. 5, pp. 2314–2326, 2018.

[7] L. Wu, K. V. Mishra, M. R. Bhavani Shankar, and B. Ottersten, “Resource allocation in heterogeneously-distributed joint radar-communications under asynchronous Bayesian tracking framework,” *IEEE Journal on Selected Areas in Communications*, vol. 40, no. 7, pp. 2026–2042, 2022.

[8] S. Sedighi, K. V. Mishra, M. B. Shankar, and B. Ottersten, “Localization with one-bit passive radars in narrowband internet-of-things using multivariate polynomial optimization,” *IEEE Transactions on Signal Processing*, vol. 69, pp. 2525–2540, 2021.

[9] S. H. Dokhanichi, B. S. Mysore, K. V. Mishra, and B. Ottersten, “A mmWave automotive joint radar-communications system,” *IEEE Transactions on Aerospace and Electronic Systems*, vol. 55, no. 3, pp. 1241–1260, 2019.

[10] H. Kuschel, D. Cristallini, and K. E. Olsen, “Passive radar tutorial,” *IEEE Aerospace and Electronic Systems Magazine*, vol. 34, no. 2, pp. 2–19, 2019.

[11] A. Neskovic, N. Neskovic, and G. Paunovic, “Modern approaches in modeling of mobile radio systems propagation environment,” *IEEE Communications Surveys & Tutorials*, vol. 3, no. 3, pp. 1–2, 2000.

[12] S. Olariu and M. C. Weigle, *Vehicular networks: From theory to practice*. Chapman and Hall/CRC, 2009.

[13] P. Vouras, K. V. Mishra, A. Artusio-Glimpse, S. Pinilla, A. Xenaki, D. W. Griffith, and K. Egiaziarian, “An overview of advances in signal processing techniques for classical and quantum wideband synthetic apertures,” *arXiv preprint arXiv:2205.05602*, 2022.

[14] E. Vargas, K. V. Mishra, R. Jacome, B. M. Sadler, and H. Arguello, “Dual-blind deconvolution for overlaid radar-communications systems,” *IEEE Journal on Selected Areas in Information Theory*, 2023.

[15] ——, “Joint radar-communications processing from a dual-blind deconvolution perspective,” in *IEEE International Conference on Acoustics, Speech and Signal Processing*, 2022, pp. 5622–5626.

[16] R. Jacome, K. V. Mishra, E. Vargas, B. M. Sadler, and H. Arguello, “Multi-dimensional dual-blind deconvolution approach toward joint radar-communications,” in *IEEE International Workshop on Signal Processing Advances in Wireless Communication*, 2022, pp. 1–5.

[17] R. Jacome, E. Vargas, K. V. Mishra, B. M. Sadler, and H. Arguello, “Factor graph processing for dual-blind deconvolution at ISAC receiver,” in *IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing*, 2023, in press.

[18] S. M. Jefferies and J. C. Christou, “Restoration of astronomical images by iterative blind deconvolution,” *The Astrophysical Journal*, vol. 415, p. 862, 1993.

[19] G. Ayers and J. C. Dainty, “Iterative blind deconvolution method and its applications,” *Optics Letters*, vol. 13, no. 7, pp. 547–549, 1988.

[20] K. Abed-Meraim, W. Qiu, and Y. Hua, “Blind system identification,” *Proceedings of the IEEE*, vol. 85, no. 8, pp. 1310–1322, 1997.

[21] B. N. Bhaskar, G. Tang, and B. Recht, “Atomic norm denoising with applications to line spectral estimation,” *IEEE Transactions on Signal Processing*, vol. 61, no. 23, pp. 5987–5999, 2013.

[22] K. V. Mishra, M. Cho, A. Kruger, and W. Xu, “Spectral super-resolution with prior knowledge,” *IEEE Transactions on Signal Processing*, vol. 63, no. 20, pp. 5342–5357, 2015.

[23] W. Xu, J.-F. Cai, K. V. Mishra, M. Cho, and A. Kruger, “Precise semidefinite programming formulation of atomic norm minimization for recovering d-dimensional (d ≥ 2) off-the-grid frequencies,” in *IEEE Information Theory and Applications Workshop*, 2014, pp. 1–4.

[24] S. W. Graham and J. D. Vaaler, “A class of extremal functions for the fourier transform,” *Transactions of the American Mathematical Society*, vol. 265, no. 1, pp. 283–302, 1981.

[25] E. Carneiro, F. Littmann, and J. D. Vaaler, “Gaussian subordination for the beurling-selberg extremal problem,” *Transactions of the American Mathematical Society*, vol. 365, no. 7, pp. 3493–3534, 2013.

[26] E. Carneiro and J. D. Vaaler, “Some extremal functions in fourier analysis. ii,” *Transactions of the American Mathematical Society*, vol. 362, no. 11, pp. 5803–5843, 2010.

[27] E. Carneiro, “A survey on beurling-selberg majorants and some consequences of the riemann hypothesis,” *Matemática Contemporânea*, vol. 40, 2011.

[28] A. Moitra, “Super-resolution, extremal functions and the condition number of Vandermonde matrices,” in *Annual ACM symposium on Theory of computing*, 2015, pp. 821–830.

[29] J. Chen, W. Gao, S. Mao, and K. Wei, “Vectorized hankel lift: A convex approach for blind super-resolution of point sources,” *IEEE Transactions on Information Theory*, 2022.

[30] D. Yang, G. Tang, and M. B. Wakin, “Super-resolution of complex exponentials from modulations with unknown waveforms,” *IEEE Transactions on Information Theory*, vol. 62, no. 10, pp. 5809–5830, 2016.

[31] A. Ahmed, B. Recht, and J. Romberg, “Blind deconvolution using convex programming,” *IEEE Transactions on Information Theory*, vol. 60, no. 3, pp. 1711–1732, 2013.

[32] X. Luo and G. B. Giannakis, “Low-complexity blind synchronization and demodulation for (ultra-) wideband multi-user ad hoc access,” *IEEE Transactions on Wireless Communications*, vol. 5, no. 7, pp. 1930–1941, 2006.

[33] R. Heckel, V. I. Morgenshtern, and S. M. Soltanolkotabi, “Super-resolution radar,” *Information and Inference: A Journal of the IMA*, vol. 5, no. 1, pp. 22–75, 2016.

[34] A. Beurling, *Collected Works of Arne Buerling: Volume 2*, Harmonic Analysis. Boston, MA: Birkhäuser, 1989, pp. 351–359.

[35] M. Grant and S. Boyd, “CVX: MATLAB software for disciplined convex programming, version 2.1,” Mar. 2014.

[36] ——, “Graph implementations for smooth convex programs,” in *Recent Advances in Learning and Control*, ser. Lecture Notes in Control and Information Sciences, V. Blondel, S. Boyd, and H. Kimura, Eds. Springer-Verlag Limited, 2008, pp. 95–110.

[37] K. C. Toh, M. J. Todd, and R. H. Tutuncu, “SDPT3 - A MATLAB software package for semidefinite programming, version 1.3,” *Optimization Methods and Software*, vol. 11, no. 1-4, pp. 545–581, 1999.