A Separate Universe Approach to Quintessence Perturbations

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Abstract

There is some observational evidence that the dark energy may not be smooth on large scales. This makes it worth while to try and get as simple and as intuitive a picture of how dark energy perturbations behave so as to be able to better constrain possible models of dark energy and the generation of large scale perturbations. The separate Universe method provides an easy way to evaluate cosmological perturbations, as all that is required is an understanding of the background behavior. Here, this method is used to show how the size of the dark energy perturbations, preferred by observations, is larger than would be expected, and so some mechanism may be required to amplify them.

1 Introduction

One way to rule out a cosmological constant would be to show that the dark energy is not spatially homogenous. Recently, it was found that the low quadrupole in the WMAP data [1, 2, 3] favors perturbations in the dark energy at the two sigma level [4, 5]. The dark energy was taken to be a
canonical scalar field which is usually referred to as the ‘quintessence’. The perturbations were evaluated using the metric approach [6]. This requires specifying a foliation of the space-time into constant time hypersurfaces or equivalently the choice of a gauge. In reference [4], the Newtonian gauge is used and in reference [5] the comoving gauge.

An alternative way of treating perturbations in cosmology is the ‘separate Universe’ approach, in which the background equations can be perturbed directly to get the large scale perturbations in the flat gauge [7]. This has the advantage that the derivation and solution of the perturbation equations follow simply from the background equations. It also provides a more intuitive picture of how the perturbations behave.

In this article we study the quintessence perturbations using the separate Universe approach. There are many other treatments of quintessence perturbations, for example in the Newtonian gauge [8], in the comoving gauge [5] and in a gauge invariant formulism [9]. I hope the treatment in this article will be complementary to those, and help to provide a clearer picture on the nature of density perturbations in quintessence.

## 2 Separate Universe Method

This method provides a simple way of modeling large scale inhomogeneities. Here we give a pedagogical explanation, which emphasizes its casual underpinnings. See references [7] for more detailed derivations.

If the Universe is spatially smooth and the overall geometry is flat, then Friedmann equation relates the scale of the Universe, $a$, to its overall density, $\rho$,

$$H^2 \equiv \left( \frac{\text{d}a}{\text{d}t} \frac{1}{a} \right)^2 = \frac{1}{3M_p^2} \rho$$

(1)

where $H$ is known as the Hubble parameter, $t$ is the time and $M_p \equiv (8\pi G)^{-1/2} = 2.436 \times 10^{18}\text{GeV}$ is the reduced Planck mass. The pressure, $p$, is related to the density by the equation of state parameter

$$w \equiv p/\rho.$$  

(2)

It has the value of zero for non-relativistic matter and a third for radiation. For a canonical scalar field it is greater than or equal to minus one. Unless
otherwise specified, \( w \) will be assumed to satisfy

\[
|w| < 1. \tag{3}
\]

Einstein’s theory of general relativity also relates the acceleration of the scale factor to the matter density and pressure

\[
\frac{d^2a}{dt^2} = -\frac{1}{6M_p^2}(\rho + 3p)a. \tag{4}
\]

The continuity equation is given by

\[
\frac{d\rho}{dt} + 3H(1 + w)\rho = 0. \tag{5}
\]

Assuming the equation of state, \( w \), is constant, the solution to the above equation is

\[
\rho = \frac{\rho_0}{a^{3(1+w)}} \tag{6}
\]

where \( \rho_0 \) is the density at \( a = 1 \). The distance a signal, traveling at the speed of light, can travel is

\[
D \equiv a \int_{t_i}^{t_f} \frac{1}{a} dt \tag{7}
\]

where the units are chosen so that the speed of light is one, and \( t_i \) and \( t_f \) are the initial time and final times respectively. It is convenient to express time in terms of the number of e-folds of expansion

\[
N \equiv \log(a). \tag{8}
\]

Then from the definition of the Hubble parameter

\[
\frac{dN}{dt} = H \tag{9}
\]

which when substituted into the distance equation (Eq. (7)) gives

\[
D = a \int_0^N \frac{1}{aH} dN. \tag{10}
\]
Using the equation for the density (Eq. (6)) and the definition of efolds (Eq. (8)), the Friedmann equation (Eq. (1)) can be expressed as

$$H^2 = \frac{1}{3M_p^2} e^{-3(1+w)N}. \quad (11)$$

Substituting this into Eq. (10) and solving the integral gives

$$D = \frac{2\sqrt{3}}{1 + 3w} \frac{M_p e^N}{\sqrt{\rho_0}} \left( e^{N(1+3w)/2} - 1 \right). \quad (12)$$

The Hubble distance is defined as

$$D_H \equiv \frac{1}{H}. \quad (13)$$

Using the distance equation (Eq. (12)) and the Friedmann equation (Eq. (11)) gives

$$\frac{D}{D_H} = \frac{2}{1 + 3w} \left( 1 - e^{-\frac{1}{2}(1+3w)N} \right). \quad (14)$$

From the Friedmann equation (Eq. (11)), the distance equation (Eq. (12)) and the definition of efolds (Eq. (8))

$$\frac{D}{aD_H|_{N=0}} = \frac{2}{1 + 3w} \left( e^{\frac{1}{2}(1+3w)N} - 1 \right). \quad (15)$$

As can be seen from Eqs. (14) and (15), the value of $w = -1/3$ is special. For $w < -1/3$, the acceleration equation (Eq. (4)) and the definition of the equation of state (Eq. (2)) give an accelerating scale factor. This is thought to have occurred in the early Universe during a period known as inflation [10]. For large $N$ and $w < -1/3$, the ratio of the red shifting initial Hubble parameter to the casual distance (Eq. (15)) tends to

$$\frac{D}{aD_H|_{N=0}} \rightarrow \frac{-2}{1 + 3w}. \quad (16)$$

It follows that, points that are initially more than of order a Hubble distance apart are always out of causal contact as long as inflation lasts.
After inflation, there is the radiation dominated era with $w = 1/3$ followed by the matter dominated era with $w = 0$. When $w > -1/3$, the ratio of the casual distance to the Hubble distance (Eq. (14)) tends to

$$\frac{D}{D_H} \to \frac{2}{1 + 3w}. \quad (17)$$

Thus, scales larger than the Hubble horizon remain out of casual contact until the Hubble horizon grows to be comparable to them.

It follows that a patch of space whose density or other matter variables are different from those of the surrounding space, and whose size is larger than the Hubble distance during inflation, will evolve like a separate homogeneous Universe. It will continue to do so until the Hubble distance becomes comparable to the patch size. The difference between a matter variable in the patch and outside the patch can be evaluated by solving the background equations for the background space-time and those for the patch and then subtracting the difference between the two. The coordinate freedom of the time surfaces on which to match the patch and background is the same as the usual gauge freedom in cosmological perturbations [6]. If the coordinate system is chosen so that the patch and background have the same scale factor or equivalently the same efoldings, then, under reasonable assumptions, the difference between the matter variables in the patch and in the background space-time is the same as the perturbation in the flat gauge [7]. In a homogeneous space-time, the state of the Universe at any time can be totally specified in terms of the degrees of freedom such as the different fluids’ densities and pressures and the values and time derivatives of the scalar fields. Thus, the large scale, flat gauge perturbation, $\delta f$, of a function, $f$, of the matter degrees of freedom, $\phi_i$, can be evaluated as

$$\delta f = \sum_i \left. \frac{\partial f}{\partial \phi_i} \right|_{N=0} \delta \phi_i |_{N=0}. \quad (18)$$

This equation summarizes the separate Universe approach and will be used in evaluating matter perturbations in the rest of the paper.
3 Quintessence Background Equations

The Klein-Gordon equation for the quintessence, $Q$, is given by

$$\frac{\partial^2 Q}{\partial t^2} + 3H \frac{\partial Q}{\partial t} + \frac{\partial V}{\partial Q} = 0 \quad (19)$$

where $V$ is the quintessence potential. We can express the Klein-Gordon equation (Eq. (19)) in terms of $N$, by using the Friedmann equation (Eq. (1)) and the acceleration equation (Eq. (4)), as

$$Q'' + \frac{3}{2} (1 - w) Q' + \frac{1}{H^2} \frac{\partial V}{\partial Q} = 0 \quad (20)$$

where the prime indicates differentiation with respect to the number of eefolds, $N$, and $w$ is the total equation of state parameter (Eq. (2)), including the effects of any other matter present. If the value of $Q$ does not change much during the period of interest, then its potential may be approximated by a constant plus a linear term

$$V \approx V_* \left(1 + \frac{Q - Q_*}{M_p} \sqrt{2\epsilon_*} \right) \quad (21)$$

where $V_*$ and $\epsilon_*$ are constants and correspond to the potential and first slow roll parameter, at the point about which the expansion is taken, $Q_*$, respectively. A class of potentials that do not satisfy this criteria in general are the ‘tracking’ potentials which require [11]

$$\frac{1}{V_Q} \frac{\partial^2 V_Q}{\partial Q^2} \left( \frac{1}{V_Q} \frac{\partial V_Q}{\partial Q} \right)^{-2} \geq 1. \quad (22)$$

In the tracking regime, the solutions are insensitive to changes in the initial conditions of the quintessence. It follows, from the discussion in Sec. 2, that a patch with different initial conditions for the quintessence, will quickly approach the background solution for the quintessence, and so the perturbations will be suppressed [12, 8, 5].

Substituting the linear form of the potential (Eq. (21)) into the Klein-Gordon equation (Eq. (20)) and using the Friedmann equation (Eq. (11)) gives

$$Q'' + \frac{3}{2} (1 - w) Q' + 3\sqrt{2} \sqrt{\epsilon_*} \frac{M_p V_*}{\rho_0} e^{3(1+w)N} = 0. \quad (23)$$
Assuming the total equation of state parameter \((w)\) is a constant, less than one, and denoting the values of \(Q\) and its derivative, at \(N = 0\), by \(Q_0\) and \(Q'_0\) respectively, the solution to the Klein-Gordon equation (Eq. (23)) for large \(N\) is

\[
Q = Q_0 + \frac{2}{3(1-w)} Q'_0 - \frac{2\sqrt{2} \sqrt{\epsilon_\star}}{3(3 + 4w + w^2)} M_p \nu_* \rho_0 e^{-3(1+w)N}. \tag{24}
\]

As \(\epsilon_\star < 1\) is needed for the quintessence to cause acceleration, and \(V \lesssim \rho\) today, it follows that the last term in the above equation will be negligible until a redshift of about one. It follows that, the quintessence field is frozen until about that point. This does not apply to the case where the field starts rolling in the early Universe due to a steep potential and then rolls into an area of parameter space where the potential is shallow.

The energy density of the quintessence is

\[
\rho_Q \equiv V + \frac{1}{2} \left( \frac{\partial Q}{\partial t} \right)^2 = V + \frac{1}{2} H^2 Q^2. \tag{25}
\]

Substituting the solution for \(Q\) (Eq. (24)) and the linear potential (Eq. (21)) into the above equation gives

\[
\rho_Q = V|_{Q=Q_0} + V_\star \left( \frac{2\sqrt{2} \sqrt{\epsilon_\star}}{3(1-w) M_p} Q'_0 - \frac{8\epsilon_\star}{3(1+w)(3+w)^2} \frac{V_\star \rho_0 e^{-3(1+w)N}}{\rho_0} \right). \tag{26}
\]

This shows that the density will also be frozen until about \(z \sim 1\).

4 Perturbations

Using the separate Universe approach (Eq. (18)), the density perturbation of the quintessence, in the flat gauge and on large scales, can be written as

\[
\delta \rho_Q = \frac{\partial \rho_Q}{\partial Q_0} \delta Q_0 + \frac{\partial \rho_Q}{\partial Q'_0} \delta Q'_0 + \frac{\partial \rho_Q}{\partial \rho_0} \delta \rho_0. \tag{27}
\]

Substituting the result for the quintessence density (Eq. (26)) into the above equation and using the background density equation (Eq. (6)) gives

\[
\delta \rho_Q = V_\star \left( \frac{\sqrt{2} \sqrt{\epsilon_\star}}{3(1-w) M_p} \frac{\delta Q'_0}{M_p} + \frac{2\sqrt{2} \sqrt{\epsilon_\star}}{3(1+w)(3+w)^2} \frac{8\epsilon_\star V_\star \delta \rho_0}{\rho_0} \right). \tag{28}
\]
As can be seen from the above equation, the quintessence will acquire large scale density perturbations even if it is initially homogenous. This is in contrast to two fluid components which, as can be seen from the density equation (Eq. (6)), only depend on their own initial perturbation, and so if one is initially homogenous it will remain homogenous on large scales. The reason why the scalar field acquires a perturbation is from the coupling to the Hubble parameter in the Klein-Gordon equation (Eq. (20)). In the case of a fluid, this coupling is lost in the continuity equation (Eq. (5)) when the time parameter is converted to efolds.

Using the solution for the quintessence density (Eq. (26)) and density perturbation (Eq. (28)), it can be shown that

\[
\frac{\delta \rho_Q|_{\delta \phi_0 = 0, \delta Q'_0 = 0}}{\rho_Q} = \frac{\delta \rho_0}{\rho_0}.
\]

This shows that the perturbation in \(\rho_Q\) becomes adiabatic if the scalar field is initially unperturbed.

The magnitude of the adiabatic perturbation can be evaluated from the solution for the perturbation (Eq. (28)) and background density (Eq. (26)) to give, assuming \(\epsilon_* V_*/\rho \ll 1\),

\[
\frac{\delta \rho_Q|_{\delta \phi_0 = 0, \delta Q'_0 = 0}}{\rho_Q|_{Q = Q_*, Q'_0 = 0}} = \frac{8 \epsilon_*}{3(1 + w)(3 + w)^2} \frac{\rho_Q}{\rho} \frac{\delta \rho_0}{\rho_0}.
\]

The value of the perturbation in the matter variable, \(\rho_0\), can be evaluated by its relation to the measured value of the curvature perturbation on constant density hyper-surfaces [13]. In the flat gauge, this is given by [7]

\[
\zeta \equiv \frac{\delta \rho_0}{3(1 + w)\rho_0}.
\]

The WMAP CMB measurements constrain \(\zeta\) [2] and so give the variance of the density perturbation to be

\[
\frac{\langle \delta \rho_0^2 \rangle^{1/2}}{3(1 + w)\rho_0} \approx 5 \times 10^{-5}.
\]

Substituting the above equation into the expression for the adiabatic perturbation (Eq. (30)) gives

\[
\frac{\langle \delta \rho_Q^2 \rangle^{1/2}|_{\delta \phi_0 = 0, \delta Q'_0 = 0}}{\rho_Q|_{Q = Q_*, Q'_0 = 0}} < 4 \times 10^{-5}.
\]
Using the WMAP data, the quintessence density perturbation is measured to be \[5\]

\[
\frac{\langle \delta \rho_Q^2 \rangle^{1/2}}{\rho_Q} = (8 \pm 4) \times 10^{-4}
\] (34)

at the 68\% confidence interval. It follows that the adiabatic perturbation (Eq. (33)) is small in comparison to this value. During inflation, if the quintessence is a light field it will acquire the usual large scale perturbations which depend on the Hubble parameter,

\[
\langle \delta Q_0^2 \rangle^{1/2} = \frac{H_{\text{inf}}}{2\pi}.
\] (35)

As gravitational waves have not been detected in the WMAP data, this puts an upper limit on this quantity \[14\]

\[
\langle \delta Q_0^2 \rangle^{1/2} < 9 \times 10^{-6} M_p.
\] (36)

Using the above equation in the solution for the quintessence density perturbation and background value (Eqs. (28) and (26)) gives the non-adiabatic part, from inflation, of the quintessence density perturbation as

\[
\frac{\langle \delta \rho_Q^2 \rangle^{1/2}}{\rho_Q} \bigg|_{Q_0 = Q^*, \delta Q_0 = 0, \rho_Q \ll \rho_0} < 10^{-5}
\] (37)

which is also too small compared to the observationally preferred value (Eq. (34)).

## 5 Conclusions

Using the separate Universe approach, I have investigated the perturbations in the quintessence by perturbing the background solutions. It was shown how the quintessence perturbation and background value can be frozen from inflation until about dark energy domination. Also the way in which a homogenous quintessence field acquires an adiabatic perturbation was illuminated.

Both the adiabatic perturbation and the non-adiabatic perturbation in the quintessence from inflation were shown to be smaller than the value
preferred by observations. This problem was originally identified in references [4, 5] using the Newtonian and comoving curvature gauges.

In this paper, the adiabatic perturbation was shown to be determined by the perturbation in the non-relativistic matter and so fixed by observations. However, the non-adiabatic perturbation from $\delta Q_0$ can be made large by making the quintessence density more sensitive to the initial value of $Q_0$. This requires some evolution in $\delta Q$. A mechanism for amplifying the quintessence perturbation is given in reference [15].

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