Quantum position operator: why space-time lattice is fuzzy

Suddhasattwa Brahma,† Antonino Marcianò,† and Michele Ronco

1Center for Field Theory and Particle Physics, Fudan University, 200433 Shanghai, China
2Dipartimento di Fisica, Università di Roma “La Sapienza”, P.le A. Moro 2, 00185 Roma, Italy, EU
3INFN, Sez. Roma1, P.le A. Moro 2, 00185 Roma, Italy, EU

Working within the framework of Loop Quantum Gravity (LQG), we construct a set of three operators suitable for identifying positions on a spin-network configuration. Two different proposals are presented. The former relies on known properties of operators for angles, while the latter proposal combines both the angle and length operators, two geometric quantities which have already been discussed in the LQG literature. Both of them are defined on the kinematical Hilbert space, in a background-independent fashion. Computing their action on coherent states, we are able to study some relevant properties such as the spectra of these operators, which turn out to be discrete. In particular, we focus on the algebra generated by these operators and, remarkably, it turns out that positions do not commute for both our proposals. This provides a realization of space-time noncommutativity in the context of LQG, a finding which, taking different shapes, have appeared recursively in the literature. The semiclassical regime, necessary to make contact with coordinates on manifolds, is also explored and, specifically, is given by the large-spin limit in which commutativity can be restored. Finally, building on well-established results, we discuss how it is possible to regularize these operators.

I. INTRODUCTION

Both the logical analysis of the quantum gravity problem [11,12], as well as some technical results within formal approaches to handle it [13,14], suggest that, at a fundamental level, spacetime should have a ‘fuzzy’, quantum nature, which is rather different from the smooth continuum manifolds we are used to at the classical level. Among the studied forms of spacetime quantization, pride of place is held by the hypothesis of having a noncommutative nature of the coordinates on the manifold [15,16,42,43]. Spacetime noncommutativity turned out to be useful in the characterization of spacetime ‘fuzziness’, which one expects to become important to describe physics in higher curvature regimes. Indeed, different forms of spacetime noncommutativity can be rigorously derived from both string theory [17,18,19] and 3D quantum gravity [20,21]. On the other hand, it still remains unclear if there is room for noncommutativity of spacetime coordinates, in the context of quantization of the local gauge group of symmetries, in one of the most studied quantum-gravity approaches, namely loop quantum gravity (LQG) [22,23,44,45] Nonethelss, few encouraging steps have been taken in this direction over the last decade. For instance, in Ref. [46] it was shown that the effective dynamics of matter fields coupled to 3D quantum gravity reduces, after integration over the gravitational degrees of freedom, to a braided non-commutative quantum field theory enjoying a deformed Poincaré group of symmetries. More recently, in Ref. [47], it has been shown that a specific Planck-scale deformation of the Poincaré algebra, obtained in the zero-curvature regime of the Dirac algebra of constraints with holonomy corrections from LQG [50,51], is dual to the so-called κ-Minkowski noncommutative spacetime [48,49,50] whose coordinates close Lie algebra-type commutators. In both cases, the analysis of the symmetries is as crucial as the description of the fuzziness of the quantum geometry in terms of the non-commutativity of the space-time coordinates. Also along this direction, there has been much development in the literature, both at the level of the characterization of the deformed symmetries in terms of their associated conserved quantities [61–63,65–68], focusing on their phenomenological applications to astrophysics and cosmology [69,70], and at the level of the consequences for the Fock space of quantum field theories enjoying deformed symmetries [71,72], and the related deformation of the multi-particle states statistics [73,74].

The main implication of our analysis is that non-commutativity arises as a feature of the fuzziness of space-time at mesoscales. In other words, non-commutativity emerges in our work from coarse-graining at larger scales the microscopic texture of the geometry, which at the Planckian scale is, in turn, described non-perturbatively by quantum operators and Hilbert space states. Specifically, in the LQG approach we deploy here, information about the quantum geometry is encoded in the quantum numbers assigned to the states that form a basis in the kinematic Hilbert space. This is the basis of spin-network states supported on graphs $\Gamma$, which in turn are composed of $N$ nodes and $L$ links. The coarse-graining procedure we adopt amounts to grouping in three sets all the links emanating from a node of $\Gamma$, while the semiclassical limit, achieved by sending to infinity irreducible representations of $SU(2)$ assigned to links, necessitates the choice of the coherent spin-network states. In this sense, non-commutativity of space-time is an effective theory for the quantum geometry that is retrieved while evaluating the semi-classical limit of op-
operators defined in LQG. This procedure captures, in an intermediate regime of scales that is close enough to the Planckian regime, departures from the smooth commutativity of the space-time manifold coordinates, and thus signals modifications to the Poincaré symmetries.

The kinematical Hilbert space of LQG is constructed from abstract spin-network graphs embedded in a manifold. A variety of well-defined geometric operators can be recovered on this Hilbert space which, in turn, can be used to calculate the spectra of areas and volumes. In this way, we can regain information regarding the geometry of the spatial manifold from these abstract spin-network states. On the other hand, a gap still exists in the literature on how to regain information regarding the position of some coordinate chart on this manifold from the kinematical Hilbert space. This would require defining some position operators on the Hilbert space, which in the appropriate classical limit, would reduce to usual classical coordinates on the manifold. However, the quantum property of these operators can be significantly more exotic than their classical analogs, as we shall establish in this draft.

More concretely, we introduce an operator for position on the kinematical Hilbert space of LQG. In fact, as stated above, there is a well-known detailed analysis of the properties of geometric quantities such as areas [77][78], volumes [77][79] and also lengths [80][82], but very little is known about what happens to spacetime points or, to put it more precisely, if there exists an analogous procedure to also characterize position. A rather renowned result in the LQG literature states that areas, volumes and lengths, when realized as well-defined quantum operators, on the kinematical Hilbert space indeed have the remarkable feature of possessing discrete spectra [77][79][84]. We here propose a straightforward way of defining an operator for the three spatial coordinates. In order to define them, we use previous results [85][86] defining angle operators on spin-network states, where the angle is identified by the two links converging at the same node of the network. By using this operator for the direction of links, we are able to find a suitable definition for the position operator. Using the action on semi-classical coherent states [87][88][89], we compute the algebra generated by position operators and find out that they do not commute and, in particular, they close a sort of Lie algebroid [91]. The structure functions of the algebroid depend on half of the phase-space variables, i.e. fluxes. Following the second definition for the position operator, the algebra of coordinates is even more complicated since it also involves contributions which are non-linear in the coordinates, rendering the algebra non-associative.

The classical phase space of canonical gravity is given in terms of the couple of conjugate variables \( (A^i_a(x), E^a(x)) \) [51]. The Ashtekar-Barbero variables \( A^i_a(x) \) are \( SU(2) \)-valued connections embedded in a 3-manifold \( \Sigma \) and are conjugate to the triads \( E^a(x) \) of density weight one. Their Poisson algebra is given by

\[
\begin{align*}
\{ A^i_a(x), A^j_b(y) \} &= \{ E^a_i(x), E^b_j(y) \} = 0; \\
\{ A^i_a(x), E^b_j(y) \} &= 8\pi\gamma\delta^{ij}(x, y)\delta_a^b.
\end{align*}
\]

(1)

Here, according to the usual notation, \( \gamma \) stands for the Immirzi parameter [52][53]. Quantizing canonical gravity in Ashtekar’s formulation, it is a pre-requirement to turn to holonomies and fluxes as fundamental phase space variables. This is due to the fact that, in general, not the connection directly, but rather its parallel transport, can be represented to a well-defined operator in LQG [54]. Connections are replaced by their holonomies

\[
h_e[A] = P \exp \int_e dt \hat{e}_a A^i_a \tau_i,
\]

(2)

in which \( \tau_i = -i\sigma_i/2, \sigma_i \) denoting the Pauli matrices, and \( \hat{e}_a = d\hat{e}_a/dt \), tangent to the curve \( \hat{e}_a(t) \). Through holonomies, a group element of \( SU(2) \) is then associated to each edge \( e \). At the same time, a densitized triad \( E^a_i \) is smeared over a two surface \( S \)

\[
F_i[S] = \int_S E^a_i \epsilon_{abc} dx^b \wedge dx^c,
\]

(3)

in order to get flux variables. The introduction of fluxes and holonomies as, respectively, two and one dimensional integrals is a crucial point to have a background independent formulation in which the spacetime metric does not enter explicitly. Holonomies and fluxes can be quantized consistently and, thus, every geometric operator can then be expressed in terms of these fundamental variables of the theory. They constitute the basic ingredients of the loop quantization procedure [74][75]. The algebra of operators between a holonomy \( h_e[A] \) and a flux \( F_i(S) \) is given by

\[
\{ F_i(S), h_e[A] \} = 8\pi\gamma\tau_i h_e[A] \circ (e, S),
\]

(4)
where \( \circ(e, S) \) is equal to 0 if either the path \( e \) does not intersect \( S \) or if it lies entirely on \( S \), while it is \( \pm 1 \) if there is a single intersection where its sign depends on the mutual orientation of \( e \) and \( S \).

Taking \( N \) copies of the \( SU(2) \) group one can construct cylindrical functions

\[
\Psi_{\Gamma}[A] = \psi(h_{e_1}[A], ..., h_{e_N}[A])
\]

over the graph \( \Gamma \) composed by these \( N \) edges. These functionals of holonomies are dense in the (kinematical) Hilbert space with the Ashtekar-Lewandowski measure \([94]\). An orthonormal basis of this space is provided by spin-network states \([16, 95]\).

\[
\Psi_{\Gamma,j,i}[A] = \bigotimes_{n \in \Gamma} \psi_n \bigotimes_{e \in \Gamma} D^{(j_e)}(h_e[A]),
\]

where \( \Gamma \) is a closed graph embedded in the 3-manifold \( \Sigma \) and its edges \( e \) are labelled by irreducible representations \( j_e \) of \( SU(2) \). The edges converge in nodes \( n \) to which one associates intertwiners \( i_n \) taken in the tensor product of the representations of the edges that intersect at a given node. Each holonomy \( h_e[A] \) is written in the associated representation \( j_e \) of the group given by \( D^{(j_e)}(h_e[A]) \). Holonomies act as multiplicative operators on spin-network states, while fluxes amount to derivatives. Thanks to the self-adjointness of the kinematical

**II. ANGLE OPERATORS IN LQG**

Before we introduce an operator for position, it is useful to first review how one can associate an operator, in LQG, to angles in space. Consider a sphere around a node \( n \), with several edges emanating from it, on the spin-network graph. Let us choose three regions on the surface of a sphere and accordingly divide the edges in three sets \( S_e \) with \( e = \{1,2,3\} \) as in Fig. (4). Given this decomposition of the links, it is always possible to regard a generic node \( n \) as a trivalent node. Here \( S_1, S_2 \) and \( S_3 \) refer to the set of edges that meet at \( n \), labeled respectively by 1, 2 and 3. Suppose all the edges are outgoing and associate a flux operator \( F^e_i \) that identifies the direction of each of these sets. In other words, they are the fluxes through the surfaces dual to the (set of) links \( S_e \). In order to have null angular momentum at the node one has to impose a closure condition \( F_1^1 + F_2^2 + F_3^3 = 0 \). Then, as first recognized in \([85]\), one can define the cosine operator of the angle \( \theta \) between \( S_1 \) and \( S_2 \) as

\[
\cos \theta := \frac{F_1^1 F_2^2}{\sqrt{F_1^1 F_1^1} \sqrt{F_2^2 F_2^2}}.
\]

Analogously, one can of course define the cosine of the angle between \( S_2 \) and \( S_3 \), and for that between \( S_1 \) and \( S_3 \).

![Figure 1](image1.png)

**Figure 1.** The figure graphically represents a given spin network \( \Gamma \) made up of three nodes. The region, named \( R_n \), is the portion of space that is dual to the node \( n \). Two of the four links converging at \( n \), those labeled by \( e_1 \) and \( e_2 \), are then dual to the surfaces \( S_1 \) and \( S_2 \) respectively. The intersection of these ‘2d’ surfaces with the boundary of \( R_n \) identifies the curve \( \gamma \).

![Figure 2](image2.png)

**Figure 2.** The figure shows the way we group the links converging at a given node. We pick out three sets of links and gather them in three different “total” links, which we call \( S_1 \), \( S_2 \) and \( S_3 \). Given such a construction, it is possible to define an operator for the angle between \( S_1 \) and \( S_2 \), for the angle between \( S_2 \) and \( S_3 \), and for that between \( S_1 \) and \( S_3 \).

Its spectrum can be obtained by acting on the spin-
network state associated to the graph $\{1\}$, and by using the closure condition, and is given by
\[
\cos \theta |\Psi\rangle = \frac{j_3(j_3 + 1) - j_1(j_1 + 1) - j_2(j_2 + 1)}{\sqrt{j_1(j_1 + 1)j_2(j_2 + 1)}} |\Psi\rangle \tag{8}
\]
up to a numerical prefactor. Here $j_3$ is the total spin number labeling the group of edges $S_3$, $j_1$ is the total spin of $S_1$ and, finally, $j_2$ labels $S_2$. As already shown in [85], on taking the naive classical limit of this cosine operator, we can regain the cosine of the angle between the two surfaces.

\[\sum\]

Indeed, the edges of the vertex are distributed among three sets $S_1$, $S_2$ and $S_3$, whose total spin labels are respectively $\sum x_i$, $\sum y_j$, and $\sum z_k$. Thus, each of the sets is recast into a single edge denoted with $n_1$ for, e.g., the $S_1$ set. These three total edges now converge at a 3-vertex.

Analogously, we introduce an operator for the sine of the angle as
\[
\sin \theta := \frac{n_1\epsilon_{ijk}\hat{F}_i^1\hat{F}_j^2\hat{F}_k^3}{\sqrt{\hat{F}_1^1\hat{F}_1^1\hat{F}_2^2\hat{F}_2^2}}, \tag{9}
\]
where $n_1$ is the normal versor along the internal directions $\{j,k\}$. This operator is defined in a more natural way through the wedge product of two of the fluxes.

\[\sum\]

**III. AN OPERATOR FOR NODE POSITION: FIRST PROPOSAL**

Using the sphere described above for the angle operator, around a node, we can separate the surface of the sphere into three regions $S_e$ with $e = \{1,2,3\}$, which like before, collect the edges through each of these regions and assign a flux operator $\hat{F}_e^i$ that labels a outgoing direction for each of these regions. Then, using the outer product of two fluxes, we can define position operators, which need not be orthogonal even in the classical limit. We shall impose this condition with the help of an operator constraint, which however, is much less desirable as it can be only imposed on certain class of states. In this case, they shall be imposed on the coherent spin-network states.

The position operators are introduced as
\[
\hat{X} := r\frac{n^i\epsilon_{ijk}\hat{F}_i^1\hat{F}_j^2\hat{F}_k^3}{\sqrt{\hat{F}_1^1\hat{F}_1^1\hat{F}_2^2\hat{F}_2^2}}, \tag{10}
\]
\[
\hat{Y} := r\frac{n^i\epsilon_{ijk}\hat{F}_i^1\hat{F}_j^2\hat{F}_k^3}{\sqrt{\hat{F}_1^1\hat{F}_1^1\hat{F}_2^2\hat{F}_2^2}}, \tag{11}
\]
\[
\hat{Z} := r\frac{n^i\epsilon_{ijk}\hat{F}_i^1\hat{F}_j^2\hat{F}_k^3}{\sqrt{\hat{F}_1^1\hat{F}_1^1\hat{F}_2^2\hat{F}_2^2}}. \tag{12}
\]

Thus, space directions are identified in terms of the angular momenta of the three groups of edges converging into the same node. In particular, they are given in terms of the cross product between orthogonal flux operators identifying the three directions of space. Additionally, we impose the requirement of having orthogonal links on the operator relations $\hat{F}_k^e\hat{F}_k^{e'} = 0$ if $e \neq e'$. Imposing this condition essentially chooses the angle operators such that the spectrum of the cosine of the angle is zero whenever those between any two of the positions are calculated. These three coordinates $\hat{X}, \hat{Y}$ and $\hat{Z}$ attach a local cartesian coordinate at the position of each node. However, such conditions can only be imposed on specific set of states and would not be true in general.

Let us first note that our position operators defined in this way, is naturally regularized in a well-defined sense. Consider each of the circular regions $S_e$ to have a radius $\epsilon$. When we take the limit $\epsilon \to 0$, both the numerator and the denominator blows up but the position operator remains well behaved. To make this more precise, we define the integrated fluxes with smearing functions as done in [85]
\[
[F^e]_f = \int_{S_e} d^2S f_i^e n_a E_a^i, \tag{13}
\]
where $(e = 1,2,3)$ stands for the three surfaces. In the limit $\epsilon \to 0$, we have the test function replaced by a delta distribution. Obviously, one can immediately notice that

1. This method would work even when smearing with different test functions across the three different surfaces.
the position operators have been defined such that the dependence on the test function as well as the area of the surfaces drops out of the expressions \[ R = \frac{e^\epsilon e'' (\hat{F}^\epsilon \wedge \hat{F}^e'')}{\sqrt{(\hat{F}^\epsilon)^2(\hat{F}^e'')^2}}, \]

\( r \) being the distance of such a point on the classical smooth manifold from the node.

We can further question what might be an appropriate choice for the value of \( r \) appearing in our expressions. Given the above discussion, it should be clear that it is an arbitrary parameter with the dimension of length, whose value depends on the point we refer to. Of course, one might be worried that, being \( r \) arbitrarily large, we are introducing an unphysical noncommutativity on large scales then. From this perspective, a natural choice would be taking \( r \equiv \ell_\text{Pl} = \sqrt{\hbar G/c^3} \) — eventually dependent on the Immirzi parameter \( \gamma \) as well. However, it is worth noting that, as we discuss below, the classical limit is recovered in the large spin limit rather than naively sanding \( \ell_\text{Pl} \to 0 \). Given that, it is difficult to fix \( r \) to any such random value, we shall instead attempt, in the next section, to link this parameter to the length operator described within LQG. Thus, we shall assign well-defined length operators \([81],[82]\) to \( r \), in order to introduce the dimension of length into the position operator.

For now, having defined the operators for position coordinates in space, our next task is to compute their spectra and, finally, we want to look at their algebra. To this end, we need to act with these position operators over spin-network states \( |\Psi\rangle = |j, m\rangle \) of the geometry. Adopting the usual notation, the principal quantum number \( j \) labels the irreducible representations of the \( SU(2) \) internal gauge group, while \( m \) denotes its projection along one of the three available spin directions. Since we desire to show how the semi-classical limit of these coordinate operators can be obtained, the best option is to use the so-called coherent-picture of operators recently introduced in Ref. \([83]\). This provides a representation of operators in the basis of semi-classical state vectors. Indeed coherent states are semi-classical spin networks in the sense that they are peaked on a given classical geometry. Specifically, in spin-fmol models, it has been shown that these states exponentially dominate the partition function that sum over geometries \([87],[88]\), and can also be picked on space-time backgrounds of gravitational interest \([89]\). Another way of saying that coherent states are semi-classical is that they minimize the uncertainty of phase space operators. We will briefly comment on this below. Notice that this can be rigorously done since coherent states provide an (over-complete) basis for the kinematical Hilbert space (see e.g. \([87]\) we are interested in. Let us explicitly specify that our Hilbert space is constructed from the tensor product of three Hilbert spaces (one for each flux \( \hat{F}^\epsilon \) defined over the surface \( S^\epsilon \)), i.e. \( \mathcal{H}_{\text{tot}} := \bigotimes_{\epsilon=1}^{3} \mathcal{H}_\epsilon \) where it is useful to remind that \( \sum_{\epsilon=1}^{3} \hat{F}^\epsilon = 0 \). Consequently, our space is given by \( \mathcal{H}_{\text{tot}} = SU(2) \times SU(2) \times SU(2) \). Let us stress that we are free to choose different quantum numbers \( m \) for each of these three Hilbert spaces. Indeed, we will make use of that in order to simplify the computation of the spectrum of our position operator later in this section. The starting point is to recognize that coherent states furnish an (over) complete basis of the Hilbert space, i.e.

\[ \mathbb{I} = \int_{\Gamma} d\mu(g, \vec{\mathbf{p}}) |g, \vec{\mathbf{p}}\rangle \langle g, \vec{\mathbf{p}}| \,. \]

(14)

Here \( (g, \vec{\mathbf{p}}) \in \Gamma \) identifies a point of the phase space, \( g \) denoting a group element of \( SU(2) \) such that \( (g|j, m) = \sqrt{2j+1} D^j(g) \) and \( \vec{\mathbf{p}} \) standing for the quantum number of momenta. The explicit expression for the Haar measure \( d\mu(g, \vec{\mathbf{p}}) \) in the coherent-state expansion is given in \([87]\). We do not report it here since it will not play any role in our analysis. Using this representation of the identity matrix, any operators can be constructed in the following way

\[ \hat{O}_f = \int_{\Gamma} d\mu f(g, \vec{\mathbf{p}}) |g, \vec{\mathbf{p}}\rangle \langle g, \vec{\mathbf{p}}| \,, \]

(15)

with a proper choice of the functions \( f(g, \vec{\mathbf{p}}) \). This gives what is called the coherent-state representation of an operator.

The position operator we have introduced hitherto is defined in terms of flux operators. Thus, in order to compute the action of coherent-state position operators on spin-network states, we only need to know the action
of the flux operators. In the coherent-state picture, fluxes can be represented as
\[
\hat{F}^e_i = -i \int d\mu \, p_i |g^e, \vec{p}^e \rangle \langle g^e, \vec{p}^e | ;
\]
and their (left) action on spin-network states is [90]
\[
\hat{F}^e_i |j^e, m\rangle = \frac{i}{2} F_i (j^e) \sigma_i^{(j^e)} |j^e, m\rangle ,
\]
where — see e.g. Ref. [90] — \( F_i (j^e) \) coefficient reads
\[
F_i (j_e) = \frac{1}{2 (2 j_e + 1) (2 j_e + 3)} [j_e (t (2 j_e + 1)^2 + 2)
- \exp \left( - \frac{(2 j_e + 1)^2 t}{4} \right) \sum_s (1 + 2 s t) \exp (s^2 t)] .
\]
Here \( t \) is a parameter that controls the classicality of the coherent states, often called the Gaussian time. Small values of \( t \) correspond to states that are sharply peaked on a prescribed geometry of space. For simplicity, let us neglect the normalization in Eq. (10)-12. In this way, we are computing the action of the coherent-state cross-product operator (instead of position) on spin networks. Taking into account Eq. (17), for the cross-product operator \( \epsilon_{ijk} \hat{F}^e_i \hat{F}^e_j \) we can easily find
\[
\epsilon_{ijk} \hat{F}^e_j \hat{F}^e_k [j^e, m_j] [j^e', m_k] = -\frac{\epsilon_{ijk}}{4} \hat{F}_i (j^e) \sigma_i^{(j^e)} [j^e', m_j] \hat{F}_j (j^e') \sigma_j^{(j^e')} [j^e', m_k] .
\]
Retaining the normalization factor \( \sqrt{F^e_i F^e_j \sqrt{F^e_k F^e_l}} \), we cannot obtain an analytic expression for the action of the coordinate operators on coherent states, but we can make a numerical integration over the tensor product of the three phase space corresponding to the three links \( S_1, S_2 \) and \( S_3 \). Starting from the above formula, we can compute the algebra closed by the position operators. Again omitting the normalization part of the operators, we can calculate the action of the commutation relation
\[
\epsilon_{ijk} \epsilon_{lmn} [\hat{F}^e_j \hat{F}^e_k, \hat{F}^e'_m \hat{F}^e''_n] ,
\]
over spin-networks associated to trivalent nodes with edges colored with spins \( j^e, j^e' \) and \( j^e'' \), i.e.
\[
\epsilon_{ijk} \epsilon_{lmn} \hat{F}^e_j \hat{F}^e_k \hat{F}^e'_m \hat{F}^e''_n |j^e, m\rangle [j^e', m'], [j^e'', m''] - \langle \psi \rangle
- \frac{\epsilon_{ijk} \epsilon_{lmn}}{16} F_i (j^e) \sigma_i^{(j^e)} F_j (j^e') \sigma_j^{(j^e')} \times F_k (j^e'') \sigma_k^{(j^e'')} |\psi\rangle - \langle \psi \rangle ,
\]
where, for brevity, we rename
\[
|\psi\rangle \equiv |j^e, m\rangle [j^e', m'], [j^e'', m''] .
\]
Here, the symbol \( \langle \psi \rangle \) stands for the second term of the commutator where fluxes are exchanged, namely the operator \( \epsilon_{ijk} \epsilon_{lmn} F^e_m F^e''_n F^e'_j F^e_k \). Then, taking into account that \( [\sigma_i^{(j^e)}, \sigma_j^{(j^e')}] = 2 i \epsilon_{ijk} \sigma_k^{(j^e)} \delta_{j^e, j^e'} \), we find for the commutator
\[
\frac{\epsilon_{ijk} \epsilon_{lmn}}{8} F_i (j^e) F_j (j^e') F_k (j^e'') \sigma_i^{(j^e)} \sigma_j^{(j^e')} \sigma_k^{(j^e'')} |\psi\rangle .
\]
Reminding the definition of coordinates (10), (11), (12) and using the above calculation, we can write down the commutators between coordinate operators. We find the following algebra
\[
[\hat{X}, \hat{Y}] = i \hat{Z} \left( \frac{F^3}{(F^2)^2} \right) ; \quad [\hat{Z}, \hat{X}] = i \hat{Y} \left( \frac{F^2}{(F^2)^2} \right) ,
\]
\[
[\hat{Y}, \hat{Z}] = i \hat{X} \left( \frac{F^1}{(F^1)^2} \right) .
\]
having omitted the internal indexes. Here we have also used the fact that flux operators belonging to different edge sets commute, namely
\[
[\hat{F}^e_i, \hat{F}^e'_j] = 0 , \quad e \neq e' ,
\]
and that we are considering orthogonal edge directions
\[
\hat{F}^e_k \hat{F}^e'_k = 0 , \quad e \neq e' .
\]
We obtained a non-commutative algebra for our position operators, in which the associative property is still preserved. Indeed, we can write down the Jacobi identity:
\[
[[\hat{X}, \hat{Y}], \hat{Z}] + [[\hat{Z}, \hat{X}], \hat{Y}] + [[\hat{Y}, \hat{Z}], \hat{X}] = \left( \frac{F^3}{(F^3)^2} \right) \hat{Z} + \left( \frac{F^2}{(F^2)^2} \right) \hat{Y} + \left( \frac{F^1}{(F^1)^2} \right) \hat{X} = 0 ,
\]
where we have used the fact that \( \hat{Z} \) commutes with \( \hat{F}^3 \), since it depends only on the other two fluxes. An analogous observation applies to the other two commutators in the above expression. The first comment that is worth making at this point is that coordinate operators do not commute, as a consequence of the LQG quantization. Bearing in mind the form of coordinate operators that are expressed in terms of fluxes — namely Eqs. (10), (11) and (12) — it is possible to understand this noncommutativity as a direct consequence of having an internal SU(2) symmetry. The noncommutativity can be seen as arising from the quantization of the SU(2) Poisson brackets. Furthermore, it is worth commenting the fact that the algebra of coordinates we have derived closely resembles the commutation relations for the fuzzy sphere [91]. In fact, the above commutators can be succinctly rewritten as
\[
[\hat{X}^e, \hat{X}^e'] = i e^{e\prime} e'' \hat{X}^{e''} \left( \frac{F^{e''}}{(F^{e''})^2} \right) ,
\]
where the indexes refer to the three edge directions identified by \( S_1, S_2 \) and \( S_3 \). The main difference with respect to the standard fuzzy-sphere commutators resides in the appearance of more complicated structure
functions (rather than structure constants) in our case \cite{25}. The interest for the fuzzy sphere comes from the fact that it is the noncommutative algebra of space coordinates that arises in 3D quantum gravity \cite{10}. However, we do not obtain exactly the algebra of the fuzzy sphere due to the fact that on the right-hand side of the commutator there is still an explicit dependence on the flux. Nonetheless, our result provides a first constructive realization of space-time noncommutativity from LQG.

Finally we show that the classical commutative limit can be recovered in the large spin approximation. To this end let us compute the action of the commutator \cite{25} on a generic spin-network state associated to our 3-vertex, which we formally write as $|\Psi\rangle = |j_e, m_e\rangle |j_{e'}, m_{e'}\rangle |j_{e''}, m_{e''}\rangle$. For simplicity, let us make the case with $e = 1, e' = 2, e'' = 3$. Thus, we are taking a spin-network states given by the tensor product of three holonomies related to the three different edges of our vertex. Let us expand two holonomies in the internal $z$-direction and one on the internal $x$-direction, i.e. $|\Psi\rangle = |j_1, m_1\rangle |j_2, m_2\rangle |j_3, m_3\rangle$. Then, the action of the commutator $[\hat{X}_1, \hat{X}_2] |\Psi\rangle$ reads

$$[\hat{X}_1, \hat{X}_2] |\Psi\rangle = \frac{i\delta_{12} m_1^2 m_2^2 m_3^2}{\sqrt{j_1(j_1+1)}\sqrt{j_2(j_2+1)}\sqrt{j_3(j_3+1)}} |\Psi\rangle,$$ \hspace{1cm} (26)

having neglected numerical overall factors. From the above equation the reader can easily recognize that the classical limit coincided with the large spin limit with $j_3 \to \infty$, which restores the commutativity of coordinates. In Eq. \hspace{1cm} (26)

$$\frac{m_1 m_3 m_2}{\sqrt{j_1(j_1+1)}\sqrt{j_2(j_2+1)}\sqrt{j_3(j_3+1)}} \sim \mathcal{O}(1)$$

and, then, we have a factor $1/\sqrt{j_3(j_3+1)}$ that involves the spin on the internal edge $S_3$ shared by both $\hat{X}$ and $\hat{Y}$. The classical limit corresponds to the requirement of having large spins on the internal direction $S_3$ and, as desired, for large values of $j_3$ the right hand side of Eq. \hspace{1cm} (26) collapses to zero. The fact that the (semi-) classical limit can be obtained by taking the large spin limit lies at the very root of the role of coherent states and their role in bridging classical and quantum regimes \cite{87,90}.

IV. AN OPERATOR FOR NODE POSITION: SECOND PROPOSAL

Once again, we revert back to our definition of the sine operator \cite{49} in order to introduce operators corresponding to unit vectors along the three coordinate axes. However, we now choose a cube around the node in order to construct our operators. This is more natural since it automatically satisfies orthogonality conditions in the classical limit. We label $S_1, S_2, S_3$ the three faces of the cube through which we collect the edges passing through each of these surfaces, and associate the flux operators $\hat{F}_1, \hat{F}_2, \hat{F}_3$ respectively with each of them. Using these flux operators, we can now define operators along the unit vectors along three coordinate axes as the following

$$\hat{i}_X := \frac{n^i \epsilon_{ijk} \hat{F}_j^2 \hat{F}_k^3}{\sqrt{\hat{F}_1^2 \hat{F}_3^2 \hat{F}_2^2}},$$ \hspace{1cm} (27)

$$\hat{i}_Y := \frac{n^i \epsilon_{ijk} \hat{F}_j^3 \hat{F}_k^1}{\sqrt{\hat{F}_1^2 \hat{F}_3^2 \hat{F}_2^2}},$$ \hspace{1cm} (28)

$$\hat{i}_Z := \frac{n^i \epsilon_{ijk} \hat{F}_j^1 \hat{F}_k^2}{\sqrt{\hat{F}_1^2 \hat{F}_3^2 \hat{F}_2^2}}.$$ \hspace{1cm} (29)

Thus, space directions are identified in terms of the angular momenta of the three sets of edges emanating from the same node. In particular, they are specified in terms of the cross product between classically orthogonal flux operators identifying the three directions of space. Each of these cartesian coordinates should be thought as a unit vector along each of these (local) coordinate axes. Hence, they are defined in a dimensionless way as expected. As already pointed out, by choosing a cube, we have ensured that this local coordinate system is orthogonal in the classical limit. The requirement of having orthogonal links $\hat{F}_k^e \hat{F}_k^{e'} = 0$ if $e \neq e'$ is then automatically satisfied without having to implement such a condition on the operators anymore. This condition essentially ensures that the angle operators are such that the spectrum of the cosine of the angle is zero whenever those between any two of these positions are calculated. These three coordinates $\hat{i}_X, \hat{i}_Y$ and $\hat{i}_Z$ attach a local cartesian coordinate at the position of each node.

The domains of these operators remain as problematic as our previous definition of the position operators in Eqs. \hspace{1cm} (10)-(12). This might seem to be as restrictive at this point. However, these operators are only a stepping stone for us to define the position operators. As we shall find out, the position operators would have a much broader domain in the end. We now go on to define the position operator as follows.

Our aim is to define position operators which have dimensions of length. For this part we wish to use the expression for the length operator from LQG. Our strategy would be to put a length operator in front of each of the unit vector operators, although each of the length operators for each of the directions are different for each of the unit vector operators. We start by exploring the length operator along the $\hat{i}_X$ operator. From Ref. \cite{82}, we find that

$$L_x = \frac{\text{Ar}(S_2) \text{Ar}(S_3) \sin \theta^2}{V(\text{cube})}.$$ \hspace{1cm} (30)
Similarly, one can write down the length operators along with the sine operator, as follows. This can be easily done following the definition of the area operator in (30), in terms of our flux operators. This allows us to define a similar procedure for the other directions as well.

Then, we associate to each of these three sets a total flux $F^i$ with $i = 1, 2, 3$. However, we are here choosing a different topology for our edge decomposition. Taking a cube assures us that the coordinate basis we define is orthogonal and we won’t need to impose any orthogonality conditions on fluxes.

$S_2, S_3$ are the two faces of the cube which are orthogonal to the face $S_1$, the latter collecting all the edges corresponding to $F^k$. The volume in the denominator is that of the cube. When we want to turn the above expression into an operator on the LQG Hilbert space, we can now easily do so since we have well-defined operators corresponding to the area, the volume and the sine operator. The sine of the angle obviously goes to one in the classical limit, due to the orthogonality of the axes. However, we cannot further impose any such condition on the quantum operators, and should rather keep the expression as it is. We can now, obviously, write similar expressions for the other directions as well.

Next we proceed to recast the expression of the length operator in (30), in terms of our flux operators. This can be easily done following the definition of the area operator and the sine operator, as follows

$$L_X = \sqrt{V}^{-1} n^l \epsilon_{lmn} F^m_n F^n_l .$$

Similarly, one can write down the length operators along with the $i_Y$ and $i_Z$ directions. These would involve the other fluxes for each of the other length operators. Now we can define the position operators as

$$X := \hat{L}_X i_X = \sqrt{V}^{-1} \frac{(n^l \epsilon_{lmn} F^m_n F^n_l)^2}{\sqrt{F^2_j F^2_j \sqrt{F^3_k F^3_k}}} .$$

Similarly, we can define the $\hat{Y}$ and $\hat{Z}$ operators as well to be

$$\hat{Y} := \hat{L}_Y i_Y = \sqrt{V}^{-1} \frac{(n^l \epsilon_{lmn} F^m_n F^n_l)^2}{\sqrt{F^2_j F^2_j \sqrt{F^3_k F^3_k}}} ,$$

$$\hat{Z} := \hat{L}_Z i_Z = \sqrt{V}^{-1} \frac{(n^l \epsilon_{lmn} F^m_n F^n_l)^2}{\sqrt{F^2_j F^2_j \sqrt{F^3_k F^3_k}}} .$$

The first thing to notice is that the definition of these operators are more general than those in the previous section, since now we did not impose the orthogonality condition by hand over a specific class of states on the Hilbert space. However, the regularization is more subtle in this case, since we have not only the area operator appearing in the denominator but also the volume operator.

For the inverse volume operator, we can follow the exact steps in [82], in order to regularize this operator. We can define the inverse volume operator as

$$\sqrt{V}^{-1} := \lim_{\epsilon \to 0} \left( V^2 + \epsilon^2 \frac{\hbar}{\mu} \right)^{-1} \sqrt{V} .$$

For our definition of the position operators, we assume that there is only one node inside the cube, on which the inverse volume operator acts. This ensures that the inverse volume operator remains well defined in this case and gives a non-zero result when acting on the spin-network states. One way to extend the domain of these operators would be to apply an analogous procedure for the ‘inverse’ area operators appearing in the definition. In LQG, the area operator has a discrete spectrum with a non-zero minimum eigenvalue. However, since the denominator goes to zero if none of the edges intersect the surface corresponding to the area operator appearing in the definition, we might say that we are including zero as a discrete eigenvalue of the area operator and then regularizing this inverse operator à la Tikhonov, as for the volume operator. Of course, we have to show at the end that such a proposal is well-defined for the area operator. However, this has already been discussed in [81], demonstrating that this procedure is quite general and allows us to define a similar procedure for the area operator as

$$\hat{A}^{-1} := \lim_{\epsilon \to 0} \left( \hat{A}^2 + \epsilon^2 \frac{\hbar}{\mu} \right) \hat{A}^{-1} .$$

where $\hat{A}$ stands for any of the three area operators. Unlike the case of the inverse volume operator, which necessarily takes a non-zero eigenvalue due to the requirement of a node appearing in the fiducial cube, in this case the inverse area operator can be zero depending on whether there are edges piercing the relevant surface. This extends our definition of the position operator since now, if there are no edges coming out of $S^1$, we shall get some of our position operators to be zero whereas $X$ shall be nonzero. In this way, we do not require that there are some edges piercing all of the faces of our fiducial cube.

At this point, we can follow a procedure analogous to that one reported in the previous section, in order to
compute the algebra of these operators. After a tedious but rather straightforward calculation we obtain

\[\hat{X}, \hat{Y} = i\hat{L}_X \hat{L}_Y (\hat{L}_Z)^{-1} \hat{Z} \frac{\hat{F}^3}{(\hat{F}^3)^2} + i\hat{L}_X \hat{L}_Z \frac{\hat{F}^3}{\sqrt{(\hat{F}^2)^2}(\hat{F}^3)^2} (\hat{L}_X)^{-1} \hat{Y}\]

(37)

\[+ i\hat{L}_Y \hat{L}_Z \frac{\hat{F}^3}{\sqrt{(\hat{F}^2)^2}(\hat{F}^3)^2} (\hat{L}_X)^{-1} \hat{X}\]

\[+ i\hat{L}_Z \hat{L}_X \frac{\hat{F}^3}{\sqrt{(\hat{F}^2)^2}(\hat{F}^3)^2} (\hat{L}_Y)^{-1} \hat{Y}\]

\[+ i\hat{L}_X \hat{L}_Y \frac{\hat{F}^3}{\sqrt{(\hat{F}^2)^2}(\hat{F}^3)^2} (\hat{L}_Z)^{-1} \hat{Z}\]

(38)

and

\[\hat{Y}, \hat{Z} = i\hat{L}_Y \hat{L}_Z (\hat{L}_X)^{-1} \hat{X} \frac{\hat{F}^3}{(\hat{F}^3)^2} + i\hat{L}_Y \hat{L}_X \frac{\hat{F}^3}{\sqrt{(\hat{F}^2)^2}(\hat{F}^3)^2} (\hat{L}_Z)^{-1} \hat{Z}\]

\[+ i\hat{L}_Y \hat{L}_X \frac{\hat{F}^3}{\sqrt{(\hat{F}^2)^2}(\hat{F}^3)^2} (\hat{L}_Z)^{-1} \hat{Y}\]

\[+ i\hat{L}_Z \hat{L}_X \frac{\hat{F}^3}{\sqrt{(\hat{F}^2)^2}(\hat{F}^3)^2} (\hat{L}_Y)^{-1} \hat{Y}\]

\[+ i\hat{L}_X \hat{L}_Y \frac{\hat{F}^3}{\sqrt{(\hat{F}^2)^2}(\hat{F}^3)^2} (\hat{L}_Z)^{-1} \hat{Z}\]

(39)

and finally

\[\hat{Z}, \hat{X} = i\hat{L}_Z \hat{L}_X (\hat{L}_Y)^{-1} \hat{Y} \frac{\hat{F}^2}{(\hat{F}^2)^2} + i\hat{L}_Z \hat{L}_Y \frac{\hat{F}^2}{\sqrt{(\hat{F}^2)^2}(\hat{F}^2)^2} (\hat{L}_Y)^{-1} \hat{X}\]

\[+ i\hat{L}_Z \hat{L}_Y \frac{\hat{F}^2}{\sqrt{(\hat{F}^2)^2}(\hat{F}^2)^2} (\hat{L}_X)^{-1} \hat{X}\]

\[+ i\hat{L}_X \hat{L}_Y \frac{\hat{F}^2}{\sqrt{(\hat{F}^2)^2}(\hat{F}^2)^2} (\hat{L}_Z)^{-1} \hat{Z}\]

\[+ i\hat{L}_X \hat{L}_Y \frac{\hat{F}^2}{\sqrt{(\hat{F}^2)^2}(\hat{F}^2)^2} (\hat{L}_Z)^{-1} \hat{Z}\]

(38)

Here we used the fact that \(\hat{L}_X = (\hat{L}_X)^{-1} \hat{X}\) and similar relations for the remaining two directions. This has been done in order to express the above commutators between coordinate operators as a combination of coordinates themselves with highly non-trivial coefficients given by fluxes or functions of fluxes, whose main subtle issue is represented by ordering ambiguities as usual. Thus, also for this second choice for the position operator we find that coordinates along different directions do not commute and commutators are even more complicated. By following the steps we performed in the previous section it is possible to show that also coordinates defined as in Eqs. (32), (33), and (34) have discrete spectra, meaning that positions on a spin-network configuration can be localized only with a finite resolution given by (the inverse of) the minimum eigenvalue. Even without computing spectra explicitly, we can still say something about the behavior of these operators in the large spin \(j\) limit. As aforementioned, this is an important consistency check since, in that limit, we expect to recover standard commutative property. To this end, let us make a rough estimate of the spin order of each term appearing on the right-hand-side of Eqs. (37), (38), and (39). In particular, let us do that for the commutator \([\hat{X}, \hat{Y}]\). One can immediately realize that the following considerations apply directly also to the other two commutators. Given Eq. (31) and taking into account calculations in the previous section, it is not difficult to see that \(\hat{L}_X \sim \sqrt{j}\). Thus, it is worth saying that the length by itself is not a well-behaved operator when \(j \to \infty\). Indeed, its semi-classical limit is obtained by sending to zero the lattice parameter, i.e. when \(\ell_\text{Pl} \to 0\) — see, however, Refs. [31] [32]. Then, we have \(\hat{Z} \sim 1\) and, finally, the structure function \(\hat{F}^3/(\hat{F}^3)^2 \sim j^{-1}\). In the light of this, we have for

\[\hat{X}, \hat{Y} \sim 1/\sqrt{j} \quad \longrightarrow \quad 0.\]

As expected, as it also happened for the former choice of positions, the commutative limit coincides with the (semi-classical) large \(j\) limit. However, it is interesting to note that the two sets of coordinates do not have the same asymptotic trend. Indeed, the commutation relations decay as \(j^{-1}\) — see e.g. Eq. (26) and the discussion below — and then, roughly speaking, the classical regime is reached faster. This might be a reason to favor Eqs. (10), (11), and (12) over Eqs. (32), (33), and (34). Nonetheless, a more detailed and quantitative analysis is needed before coming to conclusions, and it will be developed elsewhere. Last thing to mention about this second construction for the quantum coordinates concerns the associativity. In the precedent section we have seen that, despite the noncommutativity, the associative property was preserved [24]. Indeed, even if often connected, these two features are independent characteristics of an algebra. By performing a tedious but straightforward calculation, one can realize that the above algebra (37)–(39) is not associative, i.e. the Jacobi identity is not verified. This can be rather rapidly recognized by focusing only on those terms on the right-hand-side of the commutators [37], [38], and [39] involving \(F/(\sqrt{F} \sqrt{F})\). We will not further develop this feature here, but it will be matter of forthcoming investigations. We only wish to stress that also the loss of associativity has been considered in some models on the quantization of the gravitational field [26] [28] for reasons that are in-principle unrelated to those motivating coordinate non-commutativity.

V. TARGET MANIFOLD AND NAIVE CLASSICAL LIMIT

In this section we wish to show, in the simplest way, how the operators for positions acting on spin-network states
can be related to usual coordinates on a smooth manifold. We have already shown how it is possible to recover the commutative property by taking the large spin $j$ limit of the expectation value of commutators calculated over the coherent states of LQG. Here we just give a naive derivation of standard positions on a manifold starting from our node position operators. In fact, according to the background-independence philosophy, Eqs. [10], [11], and [12], as well as Eqs. [32], [33], and [34] do not identify positions on a manifold but rather on an abstract spin-network graph. However, it is also well known that, in LQG, one requires a background manifold in which to embed the spin-network graphs. The full dynamics in LQG cannot be described fully just in terms of abstract graphs and combinatorial data. (The situation is different in $(2 + 1)$-dimensions as opposed to that in $(3 + 1)$-dimensions, since in the former case, one can get rid of the background manifold as well with only a combinatorial description [99].) Keeping this fact in mind, it is indeed interesting to revisit the question of background independence once more. How severe can we make our expectations on the background independence for the quantum gravity theory? Is not having any background metric structure satisfactory or should we demand that there should be no background manifold either? In this work, we have tried to answer a much less ambitious question. Assuming we have a background manifold where the spin network graphs are embedded, how can we systematically reproduce (classically) a specific aspect regarding this manifold, namely that one of a coordinate chart on this manifold, namely that one of a coordinate chart on it that tells us about the position of points on it.

If we take the classical limit na"ively and ignore all the ordering issues present in the definition of the operators, we can recover geometrical quantities defined on standard manifolds. Then, assuming that triad operators only act in a small region, we can approximate fluxes [3] as $F^a ≈ \delta^2 n^b E^a_b$, being then $E^a_b$ constant over a small surface $S \sim \delta^2$ with normal $n^a$. For the sake of brevity and simplicity, we also restrict to sufficiently small $S$ such that curvature is zero. Thus, we have simply $E^a_b = \sqrt{\rho} n^a \delta^b_i$, where $\epsilon_{abc} n^a \delta^b_i = h^{ab}$. Under these approximations, let us consider e.g. our former definition for $\hat{X}$ [27] that becomes

$$X \approx r \frac{\epsilon_{ijk} n^i_a E^j_a n^k_d E^i_d}{\sqrt{n^2_a E^j_a n^k_c E^i_c / n^3_b E^m_b n^d_m E^i_d}} \approx r \frac{\epsilon_{ijk} n^i_a n^j_b n^k_c}{\sqrt{n^2_a n^2_b n^2_c}} = r \frac{\epsilon_{abc} n^i_a}{\sqrt{n^2_a n^2_b n^2_c}} (40)$$

being $i_1$, $i_2$, and $i_3$ the orthogonal unit vectors providing the directions of the $X$, $Y$ and $Z$ axes respectively. Of course, similar conclusions apply to the operators $\hat{Y}$ and $\hat{Z}$ as defined in [11] and [12] respectively. Let us now turn to the second set of operators for coordinates we defined in the last section, in Eqs. [32], [33] and [34]. For instance, we take again $\hat{X}$ but now given by Eq. [32]. Explicitly, we have

$$X = L_X i X = V^{-1} n^i_1 \epsilon_{ijk} F^j_a F^k_b X \approx V^{-1} n^i_1 \epsilon_{ijk} \delta^i (n_1 \cdot i_1) i_1 , \quad (41)$$

where we used above results to approximate fluxes. Now, in terms of fluxes, the volume is given by $V = \sqrt{|e^{abc} \epsilon_{ijk} F^a_i F^b_j F^c_k|/(3!)}$ and, with $E^a_i \sim \delta^a_i$, we have

$$V \approx \delta^3 \sqrt{(\delta_i^1 \delta_j^1 - \delta_i^j \delta^1_i)/(3!)} \equiv \delta^3.$$

Plugging it into the above expression for $X$, we then obtain

$$X \approx \delta i_1 , \quad (42)$$

having used the fact that $||n_1|| \equiv 1$ or, equivalently, $n_1 \equiv i_1$. Thus, at least upon identifying $r \equiv \delta$, which certainly holds locally, the two definitions we proposed for quantum positions on spin-network states are equivalent if we take this classical and flat limit. What is more, in this limit, they both give us meaningful formulas for usual space-time coordinates on a manifold. Therefore, the simple derivation we showed in this section together with the commutative limit in the (semi-classical) large spin limit we studied before confirms us that we identified suitable definitions for quantum operators for positions.

VI. CONCLUSION AND OUTLOOK

Moving from a microscopic theory of background independent quantization of gravity, LQG, we found a path to derive at intermediate (mesoscopic) scales non-commutative space-times realization of the semiclassical geometry that are reminiscent of the fuzzy sphere.

Specifically, we presented two different proposals for coordinate operators in LQG. One attempt relies on some properties of operators for angles that were already established in the literature. The other one provides a proposal in which both the angle and length operators are taken into account. The definition of both the coordinate operators analyzed here has been instantiated in a background-independent fashion, and the action of the operators has been automatically specified on the kinematic Hilbert space of LQG. Thus the two sets of operators we discussed in this paper have been tailored to account for the quantum texture of the geometry at the Planck scale. The grouping of edges in a finite amount of sets, which is preliminary to the definition of these operators in our work, together with the computation of the action of these operators on coherent states, played

---

2 For instance, group field theory is another version of LQG where one truly works with abstract graphs alone [100].
a crucial role in our working strategy. Indeed these steps enabled us to develop a coarse-graining and semiclassical procedure that unveiled the noncommutativity of the spatial coordinates at mesoscopic scales. Finally, extracting the large $j$ limit out of the action of the operators on the coherent states, it has been possible to recover the coordinates’ commutativity of the space-time manifold on macroscopic scales.

There are several aspects that still need to be explored further in order to strengthen the picture we propose here. In particular it is meaningful to ask whether the algebra of coordinates we hitherto recovered may acquire a dependence on the chosen topology for the manifold our consideration starts from. Furthermore, we did not address yet in this work the reconstruction of the algebra of symmetries dual to the noncommutative version of space-time we recovered here. Because of the Hamiltonian analysis involved, we expect that at least it should be possible to look at the subgroups of translations and spatial rotations of the Poincaré group, and then comment on their (eventually expected) deformation.

We emphasize that our result, although preliminary, provides a first constructive realization of space-time noncommutativity from LQG. Noncommutative space-time can be then understood as an effective arena derived from LQG. Noncommutative space-time can be still recovered at large (macroscopic) scales.

ACKNOWLEDGEMENTS

A. M. and M. R. are grateful to Giorgio Immirzi for reading a preliminary version of this work, and acknowledge his support and encouragement. S. B. and A. M acknowledge comments and suggestions from Jerzy Lewandowski. M. R. thanks Eugenio Bianchi for his useful comments and suggestions. A. M. wishes to acknowledge support by the Shanghai Municipality, through the grant No. KBH1512299, and by Fudan University, through the grant No. JJJ1512105. The contribution of M. R. is based upon work from COST Action MP1405 QSPACE, supported by COST (European Cooperation in Science and Technology).

[1] S. Doplicher, K. Fredenhagen, J. E. Roberts, Comm. Math. Phys. 172, 187 (1995) [arXiv:hep-th/0303297].
[2] T. Padmanabhan, Class. Quantum Grav. 4, L107 (1987).
[3] L. Garay, Int. J. Mod. Phys. A 10, 145 (1995) [arXiv:gr-qc/9403006].
[4] D.V. Ahluwalia, Phys. Lett. B 339, 301 (1994) [arXiv:gr-qc/9308007].
[5] Y.T. Ng, H. Van Dam, Mod. Phys. Lett. A 9, 335 (1994).
[6] G. Amelino-Camelia, Mod. Phys. Lett. A 9, 3415 (1994) [arXiv:gr-qc/9603014].
[7] G. ’t Hooft, Class. Quantum Grav. 13, 1023 (1996) [arXiv:gr-qc/9601014].
[8] G. Veneziano, Europhys. Lett. 2, 199 (1986).
[9] K. Konishi, G. Pallutti, P. Provero, Phys. Lett. B 234, 276 (1990).
[10] J.R. Ellis, N.E. Mavromatos, D.V. Nanopoulos, Phys. Lett. B 293, 37 (1992) [arXiv:hep-th/9207103].
[11] D. Oriči (ed.), Approaches to Quantum Gravity (Cambridge University Press, Cambridge, U.K., 2009).
[12] G.F.R. Ellis, J. Murugan, A. Weltman (eds.), Noncommutative Spacetimes (Springer, Berlin, Germany, 2009).
[13] L. Smolin, arXiv:1203.3591.
[14] B. Zwiebach, A First Course in String Theory (Cambridge University Press, Cambridge, U.K., 2000).
[15] C. Rovelli, Quantum Gravity (Cambridge University Press, Cambridge, U.K., 2004).
[16] T. Thiemann, Modern Canonical Quantum General Relativity (Cambridge University Press, Cambridge, U.K., 2007) [arXiv:gr-qc/0105034].
[17] A. Perez, Living Rev. Relat. 16, 3 (2013).
[18] S. Giesel, L. Sindoni, SIGMA 12, 082 (2016) [arXiv:1602.08104].
[19] J. Ambjørn, A. Górkich, J. Jurkiewicz, R. Loll, Phys. Rep. 519, 127 (2012) [arXiv:1203.3891].
[20] F. Dowker, Gen. Relat. Grav. 45, 1651 (2013).
[21] O. Lauscher, M. Reuter, JHEP 0510 (2005) 050 [arXiv:hep-th/0509202].
[22] M. Niedermann, M. Reuter, Living Rev. Relat. 9, 5 (2006).
[23] M. Reuter, F. Saueressig, Lect. Notes Phys. 863, 185 (2013) [arXiv:1205.6432].
[24] P. Ascheri, M. Dimitrijevic, P. Kulish, F. Lizi, J. Wess, Noncommutative Spacetimes (Springer, Berlin, Germany, 2009).
[25] A.P. Balachandran, A. Ibort, G. Marmo, M. Martone, SIGMA 6, 052 (2010) [arXiv:1003.4358].
[26] E.T. Tomboulis, arXiv:hep-th/9702166.
[27] L. Modesto, Phys. Rev. D 86, 044005 (2012) [arXiv:1107.2403].
[28] S. Alexander, A. Marciano, L. Modesto Phys. Rev. D 85, 124030 (2012) [arXiv:1202.1824 [hep-th]].
[29] T. Biswas, E. Gerwick, T. Koivisto, A. Mazumdar, Phys. Rev. Lett. 108, 031101 (2012) [arXiv:1110.5269].
[30] G. Calcagni, L. Modesto, Phys. Rev. D 91, 124059 (2015) [arXiv:1404.2137].
[31] G. Amelino-Camelia, Living Rev. Relat. 16, 5 (2013) [arXiv:0806.0339].
[32] G. ’t Hooft, in Salamfestchrift, ed. by A. Ali, J. Ellis, S. Randjbar-Daemi (World Scientific, Singapore, 1990) [arXiv:gr-qc/9310226].
[33] S. Carlip, JHEP 0909, 3329.
[34] G. Calcagni, Phys. Rev. Lett. 104, 251301 (2010) [arXiv:0912.3142].
[35] G. Amelino-Camelia, G. Calcagni, M. Ronco, arXiv:1107.4376.
[36] M. Reuter, JHEP 0503 (2017) 138 [arXiv:1612.05632].
[37] S. Carlip, arXiv:1705.05417.
[38] G. Calcagni, Phys. Rev. D 95, 064057 (2017) [arXiv:1609.02776].
[39] G. Calcagni, JHEP 01 (2012) 065 [arXiv:1107.5041].
[40] G. Calcagni, Int. J. Mod. Phys. A 28, 1350092 (2013) [arXiv:1209.4376].
[41] G. Calcagni, Eur. Phys. J. C 76, 181 (2016) [arXiv:1602.01470].
[42] S. Doplicher, K. Fredenhagen, J. E. Roberts, Phys. Lett. B 331, 39 (1994).
