Sparse Bayesian factor analysis when the number of factors is unknown

Sylvia Frühwirth-Schnatter*    Darjus Hosszejni†    Hedibert Freitas Lopes‡

January 18, 2023

Abstract

There has been increased research interest in the subfield of sparse Bayesian factor analysis with shrinkage priors, which achieve additional sparsity beyond the natural parsimony of factor models. In this spirit, we estimate the number of common factors in the highly implemented sparse latent factor model with spike-and-slab priors on the factor loadings matrix. Our framework leads to a natural, efficient and simultaneous coupling of model estimation and selection on one hand and model identification and rank estimation (number of factors) on the other hand. More precisely, by embedding the unordered generalised lower triangular loadings representation into overfitting sparse factor modelling, we obtain posterior summaries regarding factor loadings, common factors as well as the factor dimension via postprocessing draws from our efficient and customised Markov chain Monte Carlo scheme.

Keywords: Hierarchical model; identifiability; point-mass mixture priors; marginal data augmentation; reversible jump MCMC; prior distribution; sparsity; Heywood problem; rotational invariance; ancillarity-sufficiency interweaving strategy; fractional priors

JEL classification: C11, C38, C63
1 Introduction

Factor analysis aims at identifying common variation in multivariate observations and relating it to hidden causes, the so-called common factors, see Thurstone (1947) and, more recently, Anderson (2003). The common setup consists of a sample \( y = \{ y_1, \ldots, y_T \} \) of \( T \) multivariate observations \( y_t = (y_{1t}, \ldots, y_{mt})' \) of dimension \( m \). For a given factor dimension \( r \), the basic factor model is defined as a latent variable model, involving the common factors \( f_t = (f_{1t}, \ldots, f_{rt})' \):

\[
f_t \sim N_r(0, I_r), \quad y_t = \Lambda f_t + \epsilon_t, \quad \epsilon_t \sim N_m(0, \Sigma_0), \quad \Sigma_0 = \text{Diag}(\sigma_1^2, \ldots, \sigma_m^2),
\]

where \( \Lambda \) is the \( m \times r \) matrix of factor loadings \( \Lambda_{ij} \) and \( \Sigma_0 \) is the covariance matrix of the idiosyncratic errors \( \epsilon_t \). Model (1.1) implies that conditional on \( f_t \) the \( m \) elements of \( y_t \) are independent and all dependence among these variables is explained through the common factors. Assuming independence of \( f_t \) and \( \epsilon_t \) implies that, marginally, \( y_t \) arises from a multivariate normal distribution, \( y_t \sim N_m(0, \Omega) \), with zero mean and a covariance matrix \( \Omega \) with the following constrained structure:

\[
\Omega = \Lambda \Lambda' + \Sigma_0.
\]

Since \( r \) typically is (much) smaller than \( m \), factor models yield a parsimonious representation of \( \Omega \) with (at most) \( m(r+1) \) instead of the \( m(m+1)/2 \) parameters of an unconstrained covariance matrix. Hence, factor models proved to be very useful for covariance estimation, especially if \( m \) is large; see Fan et al. (2008), Forni et al. (2009), Bhattacharya and Dunson (2011) and Kastner (2019), among others.

From the very beginning, the goal of factor analysis has been to extract the hidden factors to understand the driving forces behind the observed covariance. Using (1.2), this requires a decomposition of the covariance matrix \( \Omega \) into the cross-covariance matrix \( \Lambda \Lambda' \) and the covariance matrix \( \Sigma_0 \) of the uncorrelated idiosyncratic errors. With the only source of information being the observed covariance of the data, this is more challenging than estimating \( \Omega \) itself. A huge literature, going back to Koopmans and Reiersøl (1950), Reiersøl (1950) and Anderson and Rubin (1956) has addressed this problem of identification which can be resolved by imposing additional structure on the factor model, see Neudecker (1990), Geweke and Zhou (1996), Bai and Ng (2013), Chan et al. (2018), and Williams (2020), among many others. Recently, a new identification strategy based on unordered generalized lower triangular (UGLT) structures (Frühwirth-Schnatter and Lopes, 2018; Frühwirth-Schnatter et al., 2022) was introduced that allows to address not only the commonly known rotational invariance problem, but also the problem of variance identification (Anderson and Rubin, 1956) of which the literature is still less aware.

The main goal of the present paper is to foster mathematically rigorous identification in Bayesian factor analysis. The recent years have seen many contributions in the field of sparse Bayesian factor analysis which achieves additional sparsity beyond the natural parsimony of factor models through the choice of shrinkage priors. One strand of literature considers continuous shrinkage priors on the
factor loadings, often in combination with a prior that allows infinitely many columns in the loading matrix of which only a finite number is non-zero such as the Indian buffet process prior (Griffiths and Ghahramani, 2006) or the multiplicative Gamma process (Bhattacharya and Dunson, 2011), see Zhao et al. (2016), Ročková and George (2017), Kastner (2019), and Legramanti et al. (2020) for recent contributions. Alternatively, following the pioneering paper by West (2003), many authors considered point mass mixture priors on the factor loadings in basic factor models (Carvalho et al., 2008; Frühwirth-Schnatter and Lopes, 2010), in dedicated factor models with correlated factors (Conti et al., 2014) and dynamic factor models (Kaufmann and Schuhmacher, 2019).

Sparse Bayesian factor analysis with point mass mixture (also called spike-and-slab) priors allow factor loadings to be exactly zero and treat the identification of these elements as a variable selection problem. This allows to identify “simple structures” where in each row only a few nonzero loadings are present (Anderson and Rubin, 1956), a long standing issue in factor analysis addressed in Conti et al. (2014). It also allows to identify irrelevant variables \( y_{it} \) which are uncorrelated with the remaining variables in \( y_t \), since the entire row of the factor loading matrix is zero for these variables. Among other fields, this is of relevance in economics (Kaufmann and Schuhmacher, 2017), where it is common practice to include as many variables as possible in factor analysis (Stock and Watson, 2002; Boivin and Ng, 2006), and in bioinformatics (Lucas et al., 2006), where typically only a few out of potentially ten thousands of genes may be related to a certain physiological outcome.

A challenging problem in factor analysis is choosing the factor dimension \( r \) which is usually unknown, see Owen and Wang (2016) for an excellent review. Often information criteria (Bai and Ng, 2002) are used also in a Bayesian context, see e.g. Aßmann et al. (2016) and Chan et al. (2018), other authors employ marginal likelihoods (Lee and Song, 2002; Lopes and West, 2004). Learning about the factor dimension \( r \) is intrinsic in Bayesian approaches that allow infinitely many columns in the loading matrix (Griffiths and Ghahramani, 2006; Bhattacharya and Dunson, 2011; Ročková and George, 2017; Legramanti et al., 2020), even if this approach requires careful tuning of hyperparameters (Durante, 2017). A number of authors exploit variable selection in a finite-dimensional overfitting factor model (Frühwirth-Schnatter and Lopes, 2010; Conti et al., 2014; Kaufmann and Schuhmacher, 2017) and we will follow their lead in the present paper. Our approach relies on the following sparse Bayesian exploratory factor analysis (EFA) model:

\[
y_t = \beta_k t^k_t + \epsilon_t, \quad \epsilon_t \sim \mathcal{N}_m(0, \Sigma_k), \quad t^k_t \sim \mathcal{N}_k(0, I_k),
\]

(1.3)

where \( \beta_k \) is an \( m \times k \) loading matrix with elements \( \beta_{ij} \) and \( \Sigma_k \) is a diagonal matrix with strictly positive diagonal elements. Model (1.3) is potentially overfitting, since we assume a finite factor dimension \( k \) which is larger than true number of factors \( r \). We employ spike-and-slab priors, where the elements \( \beta_{ij} \) of \( \beta_k \) are allowed to be exactly zero, with the corresponding \( m \times k \) sparsity matrix with elements \( \delta_{ij} \) being denoted by \( \delta_k \). To learn the factor dimension \( r \), we exploit a finite version of the two-parameter
Beta prior (Ghahramani et al., 2007; Frühwirth-Schnatter, 2022) to define a shrinkage process prior on \( \delta_k \) that induces increasing shrinkage of the factor loadings toward zero as the column index increases, extending recent work by Legramanti et al. (2020).

To achieve identification, we impose a UGLT structure (Frühwirth-Schnatter et al., 2022) on the loading matrix \( \beta_k \) and the corresponding sparsity matrix \( \delta_k \) in the EFA model (1.3). Commonly, in machine learning and statistics, no constraints are imposed on \( \delta_k \); however, leaving the sparsity pattern unconstrained makes it more difficult to recover the true number of factors and reconstruct \( \Lambda \) from \( \beta_k \). In econometrics as well as in statistics, \( \delta_k \) is often constrained to a lower triangular matrix, however such a constraint is too restrictive (Jöreskog, 1969; Carvalho et al., 2008). The UGLT structure that we impose is a much weaker constraint which only requires the top non-zero elements in each column of the loading matrix to lie in different rows and still leaves the model unidentified. As shown in Frühwirth-Schnatter et al. (2022), on the one hand it is weak enough to ensure that any loading matrix can be rotated into a GLT representation, on the other hand it strong enough to ensure “controlled unidentifiability” which can be easily resolved.

For practical Bayesian inference, we develop a new and efficient Markov chain Monte Carlo (MCMC) procedure that delivers posterior draws from the EFA model (1.3) under point mass mixture priors, which is known to be particularly challenging (Pati et al., 2014). As part of our algorithm, we design a (simple) reversible jump MCMC sampler to navigate through the space of UGLT loading matrices of varying factor dimension. We achieve mathematically rigorous identification in the spirit of Anderson and Rubin (1956) through post-processing the posterior draws \( \left( \beta_k, \Sigma_k \right) \) from this unidentified EFA model. By ensuring variance identification through the algorithm of Hosszejni and Frühwirth-Schnatter (2022) and resolving rotational invariance during postprocessing, we are able recover the factor dimension \( r \), the idiosyncratic variances \( \Sigma_0 \) and an ordered GLT representation \( \Lambda \) of the loading matrix from the posterior draws. In this way, we add a mathematically rigorous approach to a growing literature where commonly more heuristic post-processing procedures are applied for this purpose; see (Aßmann et al., 2016; Kaufmann and Schuhmacher, 2019; Poworoznek et al., 2021; Papastamoulis and Ntzoufras, 2022).

Our sampling as well as our identification strategy work under arbitrary choices regarding the slab distribution, including fractional priors (Frühwirth-Schnatter and Lopes, 2010), the horseshoe prior (Zhao et al., 2016) and the Lasso prior (Ročková and George, 2017). In high-dimensional models, we work with structured priors with column-specific shrinkage (Legramanti et al., 2020) and employ the triple gamma prior (Cadonna et al., 2020) to achieve good separation of signal and noise.

The rest of the paper is organized as follows. Section 2 introduces sparse Bayesian EFA models with UGLT structures, while prior choices are discussed in Section 3. Section 4 introduces our MCMC sampler for this model class and Section 5 discusses post-processing MCMC draws to achieve identification. Section 6 considers applications to exchange rate data and NYSE100 returns, while Section 7 concludes.
2 Sparse Bayesian EFA models with UGLT structures

2.1 Model definition

Throughout the paper, we work with the exploratory factor analysis (EFA) model (1.3), with \( k \) potential common factors. Factor analysis based on this EFA model yields the extended variance decomposition

\[
\Omega = \beta_k \beta_k' + \Sigma_k, \tag{2.1}
\]

instead of the true variance decomposition (1.2) and the question arises how to recover \((r, \Lambda, \Sigma_0)\) from \((\beta_k, \Sigma_k)\). Without imposing constraints on \( \beta_k \), this question is not easily answered, due to the many identifiability issues in factor analysis. In recent work, Frühwirth-Schnatter et al. (2022) discuss econometric identification of \((r, \Lambda, \Sigma_0)\) from the sparse overfitting Bayesian factor model (1.3), if the loading matrices are restricted to unordered generalized lower triangular (UGLT) structures.

**Definition 1. UGLT structures.** Let \( \delta_r \) be an binary matrix of \( r \) non-zero columns. Let \( l_j \) denote the row index (also called pivot) of the top non-zero entry in the \( j \)th column of \( \delta_r \) (i.e. \( \delta_{ij} = 0, \forall i < l_j \)). \( \delta_r \) is said to be a UGLT structure, if the pivot elements \( l_r = (l_1, \ldots, l_r) \) lie in different rows. More generally, a binary matrix \( \delta_k \) with zero columns has an unordered GLT structure, if the submatrix \( \delta_r \) containing the \( r \) non-zero column of \( \delta_k \) is a UGLT structure.

To facilitate identification in sparse Bayesian factor analysis, we assume in the present paper that the factor loading matrix \( \beta_k \) and the corresponding sparsity matrix \( \delta_k \) in the EFA model (1.3) exhibit a UGLT structure. Compared to the common literature, where all elements of \( \delta_k \) are left unspecified, this imposes the constraint on \( \delta_k \) that the top non-zero element in the various (non-zero) columns lie in different rows. As discussed in Frühwirth-Schnatter et al. (2022), the (weak) UGLT constraint on \( \delta_k \) is sufficient for a mathematically rigorous identification even in an overfitting factor model, see Section 2.2 for more details. These insights will be exploited in Section 5, where the posterior draws from a sparse EFA model with UGLT structure are screened in a post-processing manner to ensure identification of the posterior draws and to learn about the unknown factor dimension \( r \), the loading matrix \( \Lambda \) as well as \( \Sigma_0 \) from the data.

2.2 Econometric identification in factor models

Consider first an EFA model (1.3) that is not overfitting, i.e. \( k = r \). A rigorous approach toward identification of factor models was first offered by Anderson and Rubin (1956) who consider identification as a two-step procedure. The first step is variance identification, i.e. identification of \( \Sigma_0 \) from the variance decomposition (1.2). Variance identification is easily violated for sparse Bayesian factor models, regardless whether \( \delta_k \) is unconstrained or exhibits a UGLT structure. Frühwirth-Schnatter et al. (2022) prove
that for UGLT structures the so-called 3579 counting rule for the elements in the $r$ non-zero columns $\delta_r$ of $\delta_k$ is sufficient for variance identification. Hosszejni and Frühwirth-Schnatter (2022) provide an efficient algorithm to verify this rule.

**Definition 2. 3579 counting rule.** A binary matrix $\delta_r$ satisfies the 3579 counting rule, if the following condition is satisfied: for each $q = 1, \ldots, r$ and for each submatrix consisting of $q$ column of $\delta_r$, the number of nonzero rows in this sub-matrix is at least equal to $2q + 1$.

The 3579 counting rule states that every column of $\delta_r$ should have at least 3, every pair of non-zero columns at least 5, every subset of 3 columns at least 7 elements and so forth. If an indicator matrix $\delta_r$ obeys the 3579 counting rule, this implies that $\Sigma_0 = \Sigma_r$ and hence, $\Lambda \Lambda' = \beta_r \beta_r'$ is identified. A necessary condition for $\delta_r$ to satisfy the 3579 counting rule is the following upper bound for $r$:

$$r \leq \frac{m - 1}{2}.$$  \hspace{1cm} (2.2)

The second step of identification is solving the rotational invariance problem provided that variance identification holds for $(\beta_r, \Sigma_r)$. Since variance identification implies that $\Lambda \Lambda' = \beta_r \beta_r'$, it follows that $\beta_r = \Lambda P$ for some orthogonal matrix $P$ (Anderson and Rubin, 1956, Lemma 5.1). The rotational identification problem is usually solved by imposing a structure on the loading matrices $\beta_r$ that ensures identification of $\Lambda$ from $\beta_r \beta_r'$ in the sense that $\beta_r = \Lambda P$ iff $P$ is equal to the identity. A weaker condition is rotational identification up to signed permutations $\beta_r = \Lambda P_{+} \rho$ where the permutation matrix $P_{+}$ corresponds to one of the $r!$ permutations and the reflection matrix $P_{+} = \text{Diag}(\pm 1, \ldots, \pm 1)$ corresponds to one of the $2^r$ ways to switch the signs of the $r$ columns of $\Lambda$. Frühwirth-Schnatter et al. (2022) show that imposing a UGLT structure on $\beta_r$ and $\Lambda$ leads to rotational identification up to signed permutations. Provided that $\beta_r$ is variance identified, $\Lambda$ is recovered from $\beta_r$ by reordering the columns of $\beta_r$ such that the pivots $l_1 < \ldots < l_r$ are increasing.

In applied factor analysis, the EFA model (1.3) is typically overfitting with $k > r$ and additional identifiability issues arise. In an overfitting EFA model, identifiability of $\Sigma_k$ and $\beta_k \beta_k'$ from (2.1) is lost and infinitely many representations $(\beta_k, \Sigma_k)$ with $\Sigma_k \neq \Sigma_0$ exist that imply the same covariance matrix $\Omega$ as $(\Lambda, \Sigma_0)$ (Geweke and Singleton, 1980; Tumura and Sato, 1980). Frühwirth-Schnatter et al. (2022) show that imposing a UGLT structure on $\beta_k$ allows to reveal $\Lambda$ from $\beta_k$ even in this case.

Consider, for illustration, an overfitting EFA model with $k = r + 1$. Factor analysis typically yields loading matrices $\beta_{r+1}$ with $r + 1$ non-zero columns rather than matrices with $r$ non-zero and a single zero column. Frühwirth-Schnatter et al. (2022, Theorem 7) prove that by imposing a UGLT structure on $\beta_{r+1}$, the additional column in the overfitting model is equal to a so-called *spurious factor* $\Xi$, e.g.:

$$\beta_{r+1} = \left( \begin{array}{cc} \Lambda & \Xi \\ \Xi^T_{sp} & 0 \end{array} \right), \quad \Xi = \left( \begin{array}{c} 0 \\ \Xi_{sp} \end{array} \right), \quad \Sigma_{r+1} = \text{Diag}(\sigma^2_1, \ldots, \sigma^2_{l_{sp}} - \Xi_{sp}^2, \ldots, \sigma^2_m),$$  \hspace{1cm} (2.3)
2.3 Relating exploratory to confirmatory Bayesian factor analysis

The sparsity matrix $\delta_k$ in the EFA model (1.3) allows to identify which factors are active (the corresponding column of $\delta_k$ has at least two non-zero loading), which factors are spurious (the corresponding column of $\delta_k$ has a single non-zero loading), and which ones are inactive (the corresponding column of $\delta_k$ is zero). This allows to split the sparsity matrix $\delta_k$ into an $m \times r$ submatrix $\delta_r$ with $r$ active columns, an $m \times r_{sp}$ submatrix $\delta_\Xi$ with $r_{sp}$ spurious columns, and a submatrix with $j_0 = k - r - r_{sp}$ zero columns. The loading matrix $\beta_k$ is split accordingly into the $m \times r$ submatrix $\beta_r$, the $m \times r_{sp}$ submatrix $\Xi$, and $j_0$ zero columns, while the factors $f^k_t$ are split into $f^r_t$, $f^{\Xi}_t$ and $f^0_t$.

Exploiting representation (2.3), we extract the following model of factor dimension $r$ which is embedded in any EFA model with UGLT structure,

$$f^r_t \sim \mathcal{N}_r \left( 0, \mathbf{I}_r \right), \quad y_t = \beta_r f^r_t + \epsilon_t, \quad \epsilon_t \sim \mathcal{N}_m \left( 0, \Sigma_r \right), \quad \Sigma_r = \Sigma_k + \Xi \Xi', \quad (2.4)$$

by moving the $r_{sp}$ spurious columns $\Xi$ to the idiosyncratic variances $\Sigma_r$. We call (2.4) the confirmatory factor analysis (CFA) model induced by the active columns of the sparsity matrix $\delta_k$ in the EFA model. The likelihood function is invariant to moving from the EFA model to the CFA model (2.4), since the implied covariance matrix $\Omega = \beta \beta' + \Sigma_k = \beta_r \beta_r' + \Sigma_r$ remains the same. On the other hand, taking the CFA model (2.4) as a starting point, we can move to the EFA model (1.3) without changing the likelihood function by adding $r_{sp}$ spurious columns $\delta_\Xi$ to $\delta_r$. The only relevant information needed for expanding the CFA model in this way is that $r_{sp}$ columns outside of $\delta_r$ have column-size one, whereas all remaining columns are zero.

For $r_{sp} = 1$, for instance, a single spurious column $\delta_\Xi$ is added to $\delta_r$ to define an EFA model with $k - r - 1$ zero columns. The position of the only non-zero indicator in column $\delta_\Xi$, denoted by $\delta_{lp}$, is not identified and can lie in any row $l_{sp}$ that is different from the pivots $l_r$ in $\delta_r$. A spurious column $\Xi$ is added to $\beta_r$ to define $\beta_k$, while the covariance matrix of the idiosyncratic errors in the EFA model is defined as $\Sigma_k = \Sigma_r - \Xi \Xi'$. The only non-zero loading $\Xi_{lp}$ in $\Xi$ can take any value such that the $l_{sp}$-th diagonal element of $\Sigma_k$ remains positive, i.e. $\Sigma_{k,l_{sp},l_{sp}} = \sigma_{l_{sp}}^2 - (\Xi_{l_{sp}})^2 > 0$. This entire move only
affects the $l_{sp}$-th row $\beta_{r,l_{sp}}$ of $\beta_r$. More specifically, for $t = 1, \ldots, T$:

$$y_{l_{sp},t} = \beta_{r,l_{sp}} f_t^r + \epsilon_{l_{sp},t}, \epsilon_{l_{sp},t} \sim \mathcal{N}\left(0, \sigma_{l_{sp}}^2\right),$$

(2.5)

$$y_{l_{sp},t} = \beta_{r,l_{sp}} f_t^r + \Xi_{l_{sp}} f_t^\Xi + \tilde{\epsilon}_{l_{sp},t}, \tilde{\epsilon}_{l_{sp},t} \sim \mathcal{N}\left(0, \left(\sigma_{l_{sp}}^2\right)_{\text{new}}\right), \left(\sigma_{l_{sp}}^2\right)_{\text{new}} = \sigma_{l_{sp}}^2 - \left(\Xi_{l_{sp}}\right)^2.$$  

By integrating model (2.5) with respect to the spurious factor $f_t^\Xi$, it can once more be verified that both models imply the same distribution $p(y_{l_{sp},t} | \beta_{r,l_{sp}}, f_t^r, \sigma_{l_{sp}}^2)$, independently of the specific value of $\Xi_{l_{sp}}$. Also for arbitrary $r_{sp} \in \{1, \ldots, k - r\}$, neither the position of the pivots $l_{sp}$ of $\delta_{l_{sp}}$ (which are the only non-zero indicator in each column) nor the non-zero elements in the corresponding spurious loading matrix $\Xi$ are identified. However, the pivots in $l_{sp}$ must lie in different rows.

Moving forth and back between the EFA model (1.3) and the CFA model (2.4) as described above is the cornerstone of an efficient MCMC algorithm developed in Section 4 that operates in the space of UGLT matrices with varying dimension without imposing further constraints. In Section 3, suitable priors are defined for a sparse EFA model with UGLT structure that are (largely) invariant to these moves.

### 3 Prior specifications

#### 3.1 Column sparsity through exchangeable shrinkage process priors

Bayesian inference is performed in the EFA model (1.3) with a finite number $k$ of potential factors. Dirac-spike-and-slab prior for the factor loadings are assumed,

$$\beta_{ij} | \tau_j \sim (1 - \tau_j) \Delta_0 + \tau_j P_{\text{slab}}(\beta_{ij}),$$

(3.1)

where the columns of the loading matrix are increasingly pulled toward 0 as the column index increases. This cumulative shrinkage is achieved indirectly by placing an exchangeable shrinkage process (ESP) prior on the slab probabilities $\tau_1, \ldots, \tau_k$:

$$\tau_j | k \sim \mathcal{B}(a_k, b_k), \quad j = 1, \ldots, k,$$

(3.2)

see (Frühwirth-Schnatter, 2022). The ESP prior turns model (1.3) into a sparse EFA model, where the number $r$ of active columns in $\delta_k$ with at least two non-zero elements is a random variable which takes values smaller than $k$ with positive probability. As recently shown by Frühwirth-Schnatter (2022), prior (3.2) has a representation as a finite cumulative shrinkage process (CUSP) prior (Legramanti et al., 2020). Prominent examples of such ESP priors are the finite one-parameter-beta (1PB) prior:

$$\tau_j | k \sim \mathcal{B}\left(\frac{\alpha}{k}, 1\right), \quad j = 1, \ldots, k,$$

(3.3)

and the finite two-parameter-beta (2PB) prior,

$$\tau_j | k \sim \mathcal{B}\left(\frac{\alpha}{k}, \gamma\right), \quad j = 1, \ldots, k.$$

(3.4)
Alternative choices are \( a_k = \alpha/k \) and \( b_k = \gamma(k-1)/k \) (Paisley and Carin, 2009). For \( k \to \infty \), prior (3.3) converges to the Indian buffet process prior (Teh et al., 2007) and has been employed by Ročková and George (2017) in sparse Bayesian factor analysis. For \( k \to \infty \), prior (3.4) converges to the two-parameter-beta prior introduced by Ghahramani et al. (2007) in Bayesian nonparametric latent feature models which can be regarded as a factor model with infinitely many columns of which only a finite number is non-zero.

As opposed to this literature, we stay within the framework of factor models with finitely many columns. While obeying the upper bound (2.2), \( k \) is selected large enough to encourage spurious columns and zero columns in \( \delta_k \). Spurious columns are essential for our strategy of recovering the factor dimension \( r \) through adding and deleting spurious columns in the overfitting EFA model (1.3). The hyperparameters \( \alpha \) and \( \gamma \) are instrumental in controlling prior column sparsity. A prior with \( \alpha < k \) and \( \gamma = 1 \), e.g., induces sparsity, since the largest slab probability \( \tau(k) \sim B(\alpha,1) \), while the smallest slab probability \( \tau(1) \sim B(\alpha/k,1) \) (Frühwirth-Schnatter, 2022). To adapt the hyperparameters to the data at hand, \( \alpha \) is assumed to be a random with prior \( \alpha \sim G(a_\alpha,b_\alpha) \), while \( \gamma \) is random with prior \( \gamma \sim G(a_\gamma,b_\gamma) \) for the 2PB-prior (3.4).

**Imposing a UGLT structure** For given numbers \( r \) and \( r_{sp} \) of, respectively, active and spurious columns, we define a prior on the pivots \( l_r = (l_1, \ldots, l_r) \) and \( l_{\Xi} = (l_{\Xi,1}, \ldots, l_{\Xi,r_{sp}}) \) such that the non-zero columns of the sparsity matrix \( \delta_k \) exhibit a UGLT structure. The prior \( p(l_r) \) in the CFA model is defined as follows. Let \( \mathcal{L}(1) = \{ i \in \{1,2,\ldots,m\} : i \notin 1 \} \). Given the pivots \( l_{r,-j} \) outside of any column \( j \), condition UGLT implies that \( l_j \) has to be different from \( l_{r,-j} \), and we assume a uniform prior distribution over all admissible pivots \( l_j \in \mathcal{L}(l_{r,-j}) \):

\[
p(l_j | l_{r,-j}) = \frac{1}{|\mathcal{L}(l_{r,-j})|} = \frac{1}{m-r+1}.
\]

(3.5)

The pivots \( l_{\Xi} \) of the spurious columns \( \delta_{\Xi} \) are assigned a uniform prior over all admissible values, given the pivots \( l_r \), using the following order-independent construction. Given \( l_r \), \( l_{\Xi,1} \) is uniform over \( \mathcal{L}(l_r) \); given \( l_{\Xi,1}, l_{\Xi,2} \) is uniform over \( \mathcal{L}(l_r \cup \{l_{\Xi,1}\}) \), and so forth.

Given the pivots \( l_j \) in all non-zero columns of \( \delta_k \), by definition \( \delta_{l_j,j} = 1 \) and \( \delta_{ij} = 0 \) for \( i < l_j \), while the \( m - l_j \) indicators \( \delta_{ij} \) below \( l_j \) are subject to variable selection with column-specific probabilities \( \tau_j \) following the ESP prior (3.2):

\[
\Pr(\delta_{ij} = 1 | l_j, \tau_j) = \begin{cases} 0, & i < l_j, \\ 1, & i = l_j, \\ \tau_j, & i = l_j + 1, \ldots, m. \end{cases}
\]

(3.6)

Let \( d_j = \sum_{i=1}^m \delta_{ij} \) be the number of non-zero indicators in columns \( j \). With \( d_j - 1 \) successes and
\(m - l_j - d_j + 1\) failures in the experiment defined in (3.6), we obtain the following prior for column \(\delta_{r,j}\):

\[
\Pr(\delta_{r,j} | l_j, \tau_j) = \tau_j^{d_j-1} (1 - \tau_j)^{m-l_j-d_j+1}.
\]  
(3.7)

If we integrate over \(\tau_j\), then we obtain:

\[
\Pr(\delta_{r,j} | l_j) = \frac{B(a_k + d_j - 1, b_k + m - l_j - d_j + 1)}{B(a_k, b_k)}.
\]  
(3.8)

### 3.2 Choosing the slab distribution

To define a prior on the loading matrix \(\beta_k\), we split \(\beta_k\) as discussed in Section 2.3 and define a prior \(p(\beta_r | \Sigma_r, \delta_r)\) on the factor loading matrix \(\beta_r\) in the CFA model (2.4), conditional on \(\Sigma_r = \text{Diag}(\sigma_1^2, \ldots, \sigma_m^2)\) and \(\delta_r\). When expanding the CFA model to an EFA model with \(r_{sp}\) columns, we define a prior \(p(\Xi | \beta_r, \Sigma_r, l_{sp})\) on the spurious loadings conditional on \(\beta_r, \Sigma_r,\) and \(l_{sp}\). The spurious factor loadings are assigned a uniform prior over all values that lead to a positive definite matrix \(\Sigma_k = \Sigma_r - \Xi \Xi'\) in the EFA model:

\[
\Xi_{l_{sp}}^2 | \sigma_{l_{sp}}^2 \sim \mathcal{U}[0, \sigma_{l_{sp}}^2].
\]  
(3.9)

This ensures for all \(l_{sp} \in 1\Xi\) that \(\Sigma_{k,l_{sp},l_{sp}} = \sigma_{l_{sp}}^2 - \Xi_{l_{sp}}^2 > 0\). By this definition, both the likelihood and the prior are invariant to moving between the EFA and the CFA model for a given number of spurious columns \(r_{sp}\), regardless of the chosen slab distribution.\(^1\)

The spike-and-slab prior (3.1) is formulated for the factor loading matrix \(\beta_r\) in the CFA model. A broad range of slab distributions has been considered for sparse Bayesian factor analysis and can be combined with the RJMCMC sampler we introduce in Section 4. Since the conditional likelihood function factors into a product over the rows of the loading matrix, prior independence across the rows is assumed. For each row \(i\) of \(\delta_r\) with \(q_i = \sum_j \delta_{ij} > 0\) nonzero elements, we consider two different prior families for the vector \(\beta_{r,i}^\delta\) of unconstrained elements in this row. First, hierarchical Gaussian priors are considered, taking the form

\[
\beta_{r,i}^\delta | \sigma_i^2 \sim N_{q_i}(0, B_{r0i}^\delta \sigma_i^2),
\]  
(3.10)

where \(B_{r0i}^\delta\) is a diagonal matrix. The variance of prior (3.10) is assumed to depend on \(\sigma_i^2\), because this allows joint drawing of \(\beta_r\) and \(\sigma_1^2, \ldots, \sigma_m^2\) and, even more importantly, sampling the sparsity matrix \(\delta_r\) without conditioning on the model parameters during MCMC estimation, see Algorithm 1. For models with a moderate number of features \(m\), the choice \(B_{r0i}^\delta = A_0 I_{q_i}\) with a fixed hyperparameter \(A_0\) is common (Lopes and West, 2004; Ghosh and Dunson, 2009; Conti et al., 2014). For high-dimensional models with larger \(m\), often a structured hierarchical prior is assumed to achieve better shrinkage of the

\(^1\)Note that this is a major improvement compared to Frühwirth-Schnatter and Lopes (2018).
factor loadings. One example are spike-and-slab priors with a column specific shrinkage parameter $\theta_j$ in the slab distribution of $\beta_{ij}$, see e.g. Kaufmann and Schuhmacher (2019), Legramanti et al. (2020) and Frühwirth-Schnatter (2022):

$$\beta_{ij}|\delta_{ij} = 1, \theta_j, \sigma_i^2, \kappa \sim N \left(0, \kappa \theta_j \sigma_i^2\right).$$

The column specific shrinkage parameters follow an exchangeable prior, either an inverse gamma prior, $\theta_j \sim G^{-1} \left(c^\theta, b^\theta\right)$ (Legramanti et al., 2020), or a triple gamma prior, $\theta_j \sim F \left(2a^\theta, 2b^\theta\right)$ (Cadonna et al., 2020). The triple gamma prior acts as a variance selection prior and puts considerable mass on small values of $\theta_j$ which simultaneously pulls the factors $f_{jt}$ toward 0 for all time points $t$. For $a^\theta = c^\theta = 0.5$, this yields a grouped version of the horseshoe prior (Zhao et al., 2016). The global shrinkage parameter follows $\kappa \sim G^{-1} \left(c^\kappa, b^\kappa\right)$ or $\kappa \sim F \left(2a^\kappa, 2b^\kappa\right)$. Prior (3.11) can be extended by assuming local shrinkage parameters $\omega_{ij}$ arising from an F-distribution for each non-zero factor loading:

$$\beta_{ij}|\delta_{ij} = 1, \omega_{ij}, \theta_j, \sigma_i^2, \kappa \sim N \left(0, \kappa \theta_j \sigma_i^2 \omega_{ij}\right), \quad \omega_{ij} \sim F \left(2a^\omega, 2b^\omega\right).$$

For $a^\omega = c^\omega = 0.5$, reduces to the horseshoe prior employed by Zhao et al. (2016).

As an alternative shrinkage prior, Frühwirth-Schnatter and Lopes (2010) introduced a conditionally conjugate fractional prior $p(\beta^\delta|\sigma_i^2, b, f_r) \propto p(\tilde{y}_i|f_r, \beta^\delta, \sigma_i^2)b$ in the spirit of O’Hagan (1995). It can be interpreted as the posterior of a non-informative prior and a small fraction $b > 0$ the conditional likelihood is derived from regression model

$$\tilde{y}_i = X^\delta_i \beta^\delta_i + \tilde{\epsilon}_i,$$

where $\tilde{y}_i = (y_{i1} \cdots y_{iT})'$, $\tilde{\epsilon}_i = (\epsilon_{i1} \cdots \epsilon_{iT})'$ and $X^\delta_i$ is a regressor matrix constructed from the latent factors $f_r = \{f_{r1}, \ldots, f_{rT}\}$ (see Appendix A.1 for details). This yields:

$$\beta^\delta_i|\sigma_i^2, b, f_r \sim N_{mT} \left(b^\delta_i T, B^\delta_i T \sigma_i^2 / b_N\right),$$

where $b^\delta_{iT} = B^\delta_{iT} (X^\delta_i)' \tilde{y}_i$ and $B^\delta_{iT} = ((X^\delta_i)'X^\delta_i)^{-1}$. Similar fractional priors have been applied for variable selection in other latent variable models (Frühwirth-Schnatter and Tüchler, 2008; Frühwirth-Schnatter and Wagner, 2010). In the spirit of Foster and George (1994), $b_N = 1/N$, since $N = mT$ observations are available to estimate $\beta_r$. The fractional prior often works better than hierarchical Gaussian priors for moderate $m$, however is not flexible enough to handle larger dimensions $m$.

### 3.3 The prior on the idiosyncratic variances

Finally, we define a prior on the idiosyncratic variances $\sigma_1^2, \ldots, \sigma_m^2$, independently of all other parameters. When estimating factor models using classical statistical methods such as ML estimation, it frequently happens that the optimal solution lies outside the admissible parameter space with one or more of
the idiosyncratic variances $\sigma_i^2$'s being negative, see e.g. (Jöreskog, 1967; Bartholomew, 1987). This difficulty became known as the Heywood problem. Using a prior on the idiosyncratic variances $\sigma_1^2, \ldots, \sigma_m^2$ within a Bayesian framework, typically chosen from the inverted Gamma family,

$$\sigma_i^2 \sim \mathcal{G}^{-1}(c_0, C_{i0}),$$

(3.15)

naturally avoids negative values for $\sigma_i^2$. Nevertheless, there exists a Bayesian analogue of the Heywood problem which takes the form of multi-modality of the posterior of $\sigma_i^2$ with one mode lying at 0. Heywood problems typically occur, if the constraint

$$\frac{1}{\sigma_i^2} \geq (\Omega^{-1})_{ii} \iff \sigma_i^2 \leq \frac{1}{(\Omega^{-1})_{ii}}$$

(3.16)

is violated, where $\Omega$ is the covariance matrix of $y_t$ defined in (1.2), see e.g. Bartholomew (1987, p. 54).

It is clear from inequality (3.16) that $1/\sigma_i^2$ has to be bounded away from 0. For this reason, improper priors on the idiosyncratic variances such as $p(\sigma_i^2) \propto 1/\sigma_i^2$ (Martin and McDonald, 1975; Akaike, 1987) are not able to prevent Heywood problems. Similarly, proper inverted Gamma priors with small degrees of freedom such as $c_0 = 1.1$ (Lopes and West, 2004) allow values too close to 0. Hence, the hyperparameters $c_0$ and $C_{i0}$ in (3.15) have to be selected in such a way that Heywood problems are avoided. In particular, $c_0$ should be chosen large enough to bound the prior away from 0, typically $c_0 = 2.5$. Regarding $C_{i0}$, the most common choice is using a fixed value $C_{i0} = C_0$.

An alternative choice has been suggested in Frühwirth-Schnatter and Lopes (2010). They reduce the occurrence probability of a Heywood problem which is equal to $\Pr(X \leq C_{i0}(\Omega^{-1})_{ii})$ where $X \sim \mathcal{G}(c_0, 1)$ through the choice of individual scalings $C_{i0}$. The smaller $C_{i0}$, the smaller is this probability.

On the other hand, a downward bias is introduced, if $C_{i0}$ is too small, since $E(\sigma_i^2) = C_{i0}/(c_0 - 1)$. Choosing $C_{i0} = (c_0 - 1)/(\hat{\Omega}^{-1})_{ii}$ as the largest value for which inequality (3.16) is fulfilled by the prior expectation $E(\sigma_i^2)$ yields the following prior:

$$\sigma_i^2 \sim \mathcal{G}^{-1}\left(c_0, (c_0 - 1)/(\hat{\Omega}^{-1})_{ii}\right),$$

(3.17)

based on an estimator $\hat{\Omega}^{-1}$ of $\Omega^{-1}$. In Frühwirth-Schnatter and Lopes (2018), a Bayesian estimator is proposed which combines the sample information with an inverted Wishart prior $\Omega^{-1} \sim \mathcal{W}_m(\nu_0, \nu_0 S_0)$:

$$\hat{\Omega}^{-1} = (\nu_0 + T/2)(\nu_0 S_0 + 0.5 \sum_{t=1}^T y_t y_t')^{-1}.$$

(3.18)

4 MCMC estimation

We use MCMC techniques to sample from the posterior distribution of the EFA model (1.3), given the priors introduced in Section 3. As noted by many authors, e.g. Pati et al. (2014), MCMC sampling for
sparse Bayesian factor models is notoriously difficult, since sampling the indicator matrix $\delta_k$ corresponds to navigating through an extremely high dimensional model space. This is even more challenging, if the sparse factor model is overfitting. In the present paper, an MCMC scheme summarized in Algorithm 1 is developed where several steps are designed specifically for sparse Bayesian factor models with UGLT structure where the factor dimension is unknown. Updating $\delta_k$ for sparse exploratory Bayesian factor analysis without imposing identification constraints on $\delta_k$ is fairly straightforward, see e.g. Carvalho et al. (2008) and Kaufmann and Schuhmacher (2019), among many others. A more refined approach is implemented in the present paper under sparse UGLT structures which will allow us to address econometric identification in a post-processing manner in Section 5.

Algorithm 1 (MCMC estimation for sparse Bayesian factor models with unordered GLT structures). Choose initial values for $\delta, f_r$ and $r_{sp}$. Iterate $M$ times through the following steps and discard the first $M_0$ draws as burn-in:

1. **(CFA)** Update all unknowns in the CFA model (2.4) corresponding to $\delta_r$:
   - (H) Update any unknown hyperparameters in the ESP prior (3.2) without conditioning on the slab probabilities $\tau_1, \ldots, \tau_r$. For $j = 1, \ldots, r$, sample $\tau_j | l_j, d_j \sim \mathcal{B}(a_k + d_j - 1, b_k + m - l_j - d_j + 1)$, where $d_j = \sum_{i=1}^m \delta_{ij}$.
   - (D) Loop over all columns of the sparsity matrix $\delta_r$ in a random order and perform variable selection by updating the $j$th column $\delta^r_{-ij}$ below the pivot, conditional on the remaining columns $\delta^r_{-j}$, the factors $f_r = (f^r_1, \ldots, f^r_T)$ and the hyperparameter $\tau_j$ without conditioning on the model parameters $\beta_r$ and $\sigma^2_1, \ldots, \sigma^2_m$:
     - (a) Sample all indicators $\delta_{ij}$ below the pivot $l_j$ from $p(\delta_{ij} | l_j, \delta^r_{-j}, f_r, \tau_j, y)$.
     - (b) If column $\delta^r_{-ij}$ is spurious after this update, increase $r_{sp}$ by one. Remove the $j$th column from $\delta_r$, the factors $f_{jt}, t = 1, \ldots, T$ from $f_r$ and $\tau_j$ from $\tau_r$ to define, respectively, $\delta_{r-1}, f_{r-1}$ and $\tau_{r-1}$ and decrease $r$ by one.
   - (L) Loop over all columns of $\delta_r$ in a random order and update the pivots without conditioning on $\beta_r$, $\sigma^2_1, \ldots, \sigma^2_m$ and the slab probabilities $\tau_r$:
     - (a) Sample a new pivot row $l_j$ in each column $j$ from $p(l_j | \delta^r_{-j}, f_r, y)$.
     - (b) If column $\delta^r_{-ij}$ is spurious after this update, increase $r_{sp}$ by one. Remove the $j$th column from $\delta_r$, the factors $f_{jt}, t = 1, \ldots, T$ from $f_r$ and $\tau_j$ from $\tau_r$ to define, respectively, $\delta_{r-1}, f_{r-1}$ and $\tau_{r-1}$ and decrease $r$ by one.
   - (P) Sample the model parameters $\beta_r$ and $\sigma^2_1, \ldots, \sigma^2_m$ jointly conditional on the sparsity matrix $\delta_r$ and the factors $f_r = (f^r_1, \ldots, f^r_T)$ from $p(\beta_r, \sigma^2_1, \ldots, \sigma^2_m | \delta_r, f_r, y)$.

\(^2\)See Appendix H for details.
(F) Sample the latent factors $\mathbf{f}_r = (f^r_1, \ldots, f^r_T)$ conditional on the model parameters $\beta_r$ and $\sigma^2_1, \ldots, \sigma^2_m$ from $p(f^r_1, \ldots, f^r_T | \beta_r, \sigma^2_1, \ldots, \sigma^2_m, y)$.

(S) For hierarchical Gaussian priors, update all unknown scaling factors, i.e. the global shrinkage parameter $\kappa$, the column-specific shrinkage parameters $\theta_j$, and the local shrinkage parameters $\omega_{ij}$.

(A) Perform a boosting step to enhance mixing.

(EFA) Move from the current CFA model to an EFA model (1.3) with $r_{sp}$ spurious columns:

(R) Use Algorithm 2 to change $r_{sp}$, while holding the number of active factors $r$ fixed. Loop over all columns of the spurious factors $\delta_\Xi$ and try to turn them into active factors.

Move from the current EFA model back to the CFA model (2.4) and preserve the updated number $r_{sp}$ of spurious columns.

Algorithm 1 consists of two main blocks. Block (CFA) operates in the confirmatory factor analysis model (2.4) corresponding to $\delta_r$. Due to the prior specification in Section 3, the number $r_{sp}$ of spurious columns is a sufficient statistic for the remaining columns in $\delta_k$ and no further information, such as the position of the pivots $l_\Xi$ in $\delta_\Xi$ or the spurious loading matrix $\Xi$, is needed to update the parameters in the CFA model, namely the inclusion probabilities $\tau_r$, the sparsity matrix $\delta_r$, the loading matrix $\beta_r$, the idiosyncratic variances $\sigma^2_1, \ldots, \sigma^2_m$ in $\Sigma_r$, the latent factors $\mathbf{f}_r = (f^r_1, \ldots, f^r_T)$, and all unknown shrinkage factors.

In Block (EFA), the sampler moves from the current CFA model to an EFA model with $r_{sp}$ spurious columns. In Step (R), dimension changing moves are performed in the much larger space underlying the EFA model. Exploiting the results of Section 2.2, spurious factors are added and deleted with the help of a reversible jump MCMC (RJMCMC) step described in more detail in Section 4.1. The sampler finally returns to a CFA model with a potentially larger number of active factors $r$.

To ensure that the loading matrix in the CFA model exhibits a UGLT structure, Step (L) performs MH steps that navigate through the space of all admissible $\delta_r$ where the pivots $l_r = (l_1, \ldots, l_r)$ lie in different rows, see Section 4.2. These steps are performed marginalized w.r.t. $\tau_r = (\tau_1, \ldots, \tau_r)$. Given the pivots $l_r$, the hyperparameters $a_k$ and $b_k$ in the ESP prior (3.2) are updated in Step (H) using an MH step without conditioning on the slab probabilities $\tau_1, \ldots, \tau_k$. Details are provided in Appendix F. The posterior $\tau_j | l_j, d_j$ is updated by combining the likelihood (3.7) with the prior $\tau_j \sim B(a_k, b_k)$ for all columns $j$. In Step (D), variable selection is performed in each column $j$ for all indicators $\delta_{ij}$ below the pivot element $l_j$. This step potentially turns an active factor into a spurious one and in this way decreases the number of active factors $r$, while increasing $r_{sp}$. All moves in Step (D) are implemented conditionally on $\tau_j$ (and all shrinkage parameters for hierarchical Gaussian priors), as this allows efficient multimove sampling of all indicators $\{\delta_{ij}, i \in \{l_j + 1, \ldots, m\}\}$, using Algorithm 4 in Appendix B.2.
The remaining steps are quite standard in Bayesian factor analysis, see Geweke and Singleton (1980) and Lopes and West (2004). In Step (F), the conditional joint posterior \( p(f_1, \ldots, f_T | \beta_r, \sigma^2_1, \ldots, \sigma^2_m, y) \) factors into \( T \) independent normal distributions given by:

\[
f_r^t | y_t, \beta_r, \Sigma_r \sim N_r \left( (I_r + \beta_r' \Sigma_r^{-1} \beta_r)^{-1} \beta_r' \Sigma_r^{-1} y_t, (I_r + \beta_r' \Sigma_r^{-1} \beta_r)^{-1} \right).
\] (4.1)

In Step (P), we use an efficient algorithm for multi-move sampling of all unknown model parameters \( \beta_r \), and \( \sigma^2_1, \ldots, \sigma^2_m \), see Appendix A.2. For the hierarchical Gaussian priors (3.11) and (3.12), all unknown shrinkage parameters are updated in Step (S), see Appendix C. Finally, the boosting Step (A) is added to improve the mixing of the MCMC scheme, see Section 4.3 and Appendix G.

4.1 Split and merge moves for overfitting models

Step (R) in Algorithm 1 is based on moving from the CFA model (2.4) to an EFA model with \( r_{sp} \) spurious factors in \( \delta_k \). As discussed in Section 2.2, spurious columns \( \Xi \) in an EFA model can be substituted by zero columns without changing the likelihood function, by adding \( \Xi \) to the covariance matrix \( \Sigma_r \) of the idiosyncratic errors in the CFA model. On the other hand, for any row \( l_{sp} \) that is not a pivot row in the sparsity matrix \( \delta_r \) of the CFA model, a fraction of the idiosyncratic variance \( \sigma^2_{l_{sp}} \) can be used to turn a zero column in \( \beta_k \) into a spurious column \( \Xi \) with a non-zero loading \( \Xi_{l_{sp}} \) without changing the likelihood function either. This is the cornerstone of the dimension changing procedure in Step (R), outlined in detail in Algorithm 2.

Algorithm 2 (Dimension changing move in an EFA model). Step (R) in Algorithm 1 is implemented in the following way:

(R-S) Perform an RJMCMC step to change the number \( r_{sp} \) of spurious columns through a split move on a zero column or a merge move on a spurious column in \( \delta_k \).

(R-L) Given \( r_{sp} \), sample the pivot rows \( l_{\Xi}|I_r \) of all \( r_{sp} \) spurious columns sequentially from the set \( L(I_r) \), where \( I_r \) are the pivot rows of the active factors \( \delta_r \). Order the spurious columns such that \( l_{\Xi,1} < \ldots < l_{\Xi,r_{sp}} \).

(R-F) Loop over all spurious columns \( j_{sp} \) and sample the factors \( f_{j_{sp}} = (f_{j_{sp},1}, \ldots, f_{j_{sp},T}) \) independently for all \( t = 1, \ldots, T \) from \( f_{j_{sp},t} | f_t, \beta_r, \sigma^2_{l_{sp}}, y_{l_{sp},t} \sim N \left( E_{j_{sp},t}, V_{j_{sp}} \right) \), where \( U_{j_{sp}} \) is a draw from a uniform distribution on [-1,1] and

\[
V_{j_{sp}} = 1 - U_{j_{sp}}^2, \quad E_{j_{sp},t} = U_{j_{sp}} \frac{y_{l_{sp},t} - \beta_{r,l_{sp}} f_t'}{\sqrt{\sigma^2_{l_{sp}}}}.
\] (4.2)

(R-H) Sample \( \tau_{j_{sp}} | l_{sp} \sim B \left( a_k, b_k + m - l_{sp} \right) \) for all spurious columns \( j_{sp} \).
(R-D) Loop over all spurious columns from the last (with the largest pivot) to the first (with the smallest pivot) and try to turn spurious columns into active ones:

(R-Da) sample all indicators $\delta_{:,j_{sp}}$ with $i \in I_{j_{sp}} = \{l_{sp} + 1, \ldots, m\}$ below the pivot $l_{sp}$ conditional on $r_{j_{sp}}$, $\delta_r$, $f_r$ and the spurious factors $f_{j_{sp}}$ without conditioning on $\beta_r$, $\Xi$ and $\sigma^2_1, \ldots, \sigma^2_m$.

(R-Db) If column $j_{sp}$ remains spurious, $r$ and $r_{sp}$ are unchanged and the EFA model is integrated over factor $j_{sp}$ by removing $\delta_{:,j_{sp}}$ from $\delta_{\Xi}$ and $f_{j_{sp}}$ from $f_{\Xi}$. Otherwise, decrease $r_{sp}$ by 1, increase $r$ by 1, add $\delta_{:,j_{sp}}$ to $\delta_r$ and $f_{j_{sp}}$ to $f_r$.

Step (R-S) in Algorithm 2 changes $r_{sp}$ by adding and deleting spurious columns in $\delta_k$. For a given $r_{sp}$, the conditional prior $p(\delta_k, \beta_k, \Sigma_k | r_{sp})$ is invariant to the specific choice of $l_{sp}$ and $\Xi$. However, the prior odds that a zero column in $\delta_k$ can be turned into an additional spurious column depends both on $r$ and $r_{sp}$ (see Appendix D for a proof):

$$O^\psi(r, r_{sp}) = \frac{a_k(m - r - r_{sp})}{b_k + m - r - r_{sp} - 1}.$$  \hfill (4.3)

Hence, simply adding or deleting spurious columns would lead to an invalid MCMC procedure and an RJMCMC step that incorporates $O^\psi(r, r_{sp})$ is performed in Step (R-S). As opposed to other applications of RJMCMC, the acceptance rate is extremely easy to compute, see (4.4) and (4.5).

At each sweep of the sampler, a split or a merge move is performed with, respectively, probability $p_{\text{split}}(r, r_{sp})$ or $p_{\text{merge}}(r, r_{sp})$. In a split move (which requires $r < k$) a randomly chosen zero column in $\delta_k$ is turned into a spurious column. The corresponding proposal density reads $q_{\text{split}}(\delta_k^\text{new} | \delta_k) = p_{\text{split}}(r, r_{sp})/\left(k - r - r_{sp}\right)$. The merge move (which requires $r_{sp} > 0$) is obtained by reversing the split move and turns one of the spurious columns into a zero column. The corresponding proposal density reads $q_{\text{merge}}(\delta_k^\text{new} | \delta_k) = p_{\text{merge}}(r, r_{sp})/r_{sp}$. A symmetric proposal is selected for all $0 \leq r_{sp} < k - r$ such that $p_{\text{split}}(r, r_{sp}) = p_{\text{merge}}(r, r_{sp} + 1) = p_{r_{sp}}$, where $p_{r_{sp}} \leq 0.5$ is a tuning parameter, while $p_{\text{merge}}(r, r_{sp}) = 0$ for $r_{sp} = 0$ and $p_{\text{split}}(r, r_{sp}) = 0$ for $r_{sp} = k - r$. We found it useful to choose a fixed probability $p_{r_{sp}} = p_s$, although other choices are possible. A split move is accepted with probability

$$A_{\text{split}}(r, r_{sp}) = \frac{q_{\text{merge}}(\delta_k^\text{new} | \delta_k)}{q_{\text{split}}(\delta_k^\text{new} | \delta_k)} O^\psi(r, r_{sp}) = \frac{a_k(m - r - r_{sp})(k - r - r_{sp})}{(r_{sp} + 1)(b_k + m - r - r_{sp} - 1)},$$  \hfill (4.4)

whereas a merge move is accepted with probability

$$A_{\text{merge}}(r, r_{sp}) = \frac{1}{A_{\text{split}}(r, r_{sp} - 1)} = \frac{r_{sp}(b_k + m - r - r_{sp})}{a_k(m - r - r_{sp} + 1)(k - r - r_{sp} + 1)}.$$  \hfill (4.5)

Very conveniently, $A_{\text{split}}(r, r_{sp})$ and $A_{\text{merge}}(r, r_{sp})$ are independent of the pivots $l_{\Xi}$ in the spurious columns. Note that there is a dynamic feature underlying this RJMCMC algorithm, with acceptance
Figure 1: MCMC moves to change the leading indices of an unordered GLT structure; from left to right: shifting the leading index, adding a new leading index, deleting a leading index and switching the leading elements depending on the number of spurious columns $r_{sp}$. For $b_r = 1$, for instance, $A_{\text{split}}(r, r_{sp})$ is monotonically decreasing and $A_{\text{merge}}(r, r_{sp})$ is monotonically increasing in $r_{sp}$.

Once $r_{sp}$ has been updated, Step (R-L) is trying to turn each spurious column into an active one. Since the likelihood is non-informative about spurious columns, pivots $l_{sp}$ are sampled from the prior $l_{\Xi|l_{r}}$, while the spurious factor loadings $\Xi_{l_{sp}}$ are sampled from the prior $(3.9)$. Given $l_{sp}$, the idiosyncratic variance $\sigma_{l_{sp}}^2$ in the CFA model is split, with the help of a random variable $U_{j_{sp}} \sim U[-1, 1]$, between $\Xi_{l_{sp}}$ and an updated idiosyncratic variance $(\sigma_{l_{sp}}^2)^{\text{new}}$. More specifically:

$$\Xi_{l_{sp}} = U_{j_{sp}} \sqrt{\sigma_{l_{sp}}^2}, \quad (\sigma_{l_{sp}}^2)^{\text{new}} = (1 - U_{j_{sp}}^2) \sigma_{l_{sp}}^2.$$  \hfill (4.6)

Given $\Xi_{l_{sp}}$ and $(\sigma_{l_{sp}}^2)^{\text{new}}$, factors $f_{j_{sp},t}$ are proposed in Step (R-F) independently for $t = 1, \ldots, T$, from the conditional density $p(f_{j_{sp},t}|f_{r,t}, \beta_r, \sigma_{l_{sp}}^2, y_{l_{sp},t})$ given in (4.2), see Appendix D for a proof. The slab probabilities $\tau_{j_{sp}}$ are sampled in Step (R-H) as in Algorithm 1, Step (H) using that $d_{j_{sp}} = 1$. Finally, in Step (R-Da), variable selection is performed in each spurious column on all indicators below $l_{sp}$ as in Step (D) of Algorithm 1, conditional on $f_{j_{sp},t}$, but marginalizing w.r.t. the idiosyncratic variances and the factor loading matrices $\beta_r$ and $\Xi$. More details about this step are provided in Appendix D. Any spurious column that is active after this step is integrated into the CFA model in Step (R-Db), increasing in this way the number of active columns $r$.

### 4.2 Special MCMC moves for unordered GLT structures

Step (L) in Algorithm 1 implements moves that explicitly change the position of the pivots $l_r = (l_1, \ldots, l_r)$ in the $r$ columns of the UGLT indicator matrix $\delta_r$. We scan all columns of $\delta_r$ in a random order and propose to change $l_j|l_{r,-j}$ given the pivots $l_{r,-j}$ in the other columns. To this aim, we use one of four MH moves, namely shifting the pivot, adding a new pivot, deleting a pivot and switching the pivots (and
additional indicators) between column \( j \) and a randomly selected column \( j' \); see Figure 1 for illustration. All moves are performed marginalized w.r.t. \( \tau_r \). Changing the pivot from \( l_j \) to \( l_j^{\text{new}} \) changes the number of unconstrained indicators, whereas the prior ratio

\[
\frac{p(l_j^{\text{new}}|l_r,j)}{p(l_j|l_r,j)} = 1,
\]

since \( l_j \) has a uniform prior over \( L(l_r,j) \), the set of admissible pivot rows in column \( j \). With \( d_j^{\text{new}} \) being the new number of non-zero elements in column \( j \), the prior ratio \( R_{\text{move}} \) can be derived from (3.8):

\[
R_{\text{move}} = \frac{\Pr(\delta_j^{\text{new}}|l_j^{\text{new}})}{\Pr(\delta_j|l_j)} = \frac{B(a_k + d_j^{\text{new}} - 1, b_k + m - l_j^{\text{new}} - d_j^{\text{new}} + 1)}{B(a_k + d_j - 1, b_k + m - l_j - d_j + 1)}.
\]

Further details are provided in Appendix E.

### 4.3 Boosting MCMC

Step (F) and Step (P) in Algorithm 1 perform full conditional Gibbs sampling for a confirmatory factor model with sparsity matrix \( \delta_r \), by sampling the factors \( f^r_t \) conditionally on the loading matrix \( \beta_r \) and the idiosyncratic covariance matrix \( \Sigma_r \) and sampling \( \beta_r \) and \( \Sigma_r \) conditionally on \( f_r = (f^r_1, \ldots, f^r_T) \). Depending on the signal-to-noise ratio of the latent variable representation, such full conditional Gibbs sampling tends to be poorly mixing. In a CFA model, where \( f^r_t \sim \mathcal{N}(0, I_r) \), the information in the data (the “signal”) can be quantified by the matrix \( \beta^r_r \Sigma_r^{-1} \beta_r \) in comparison to the identity matrix \( I_r \) (the “noise”) in the filter for \( f^r_t | y_t, \beta_r, \Sigma_r \), see (4.1). In particular for large factor models with many measurements, one would expect that the data contain ample information to estimate the factors \( f^r_t \). However, this is true only, if the information matrix \( \beta^r_r \Sigma_r^{-1} \beta_r \) increases with \( m \), hence if most of the factor loadings are nonzero. For sparse factor models many columns with quite a few zero loadings are present, leading to a low signal-to-noise ratio and, consequently, to a poorly mixing Gibbs sampler, as illustrated in the left-hand panel in Figure 2 showing posterior draws of \( \text{tr}(\beta^r_r \Sigma_r^{-1} \beta_r) \) without boosting Step (A) for the exchange data to be discussed in Section 6.1.

Hence, boosting steps are essential to obtain efficient MCMC schemes for sparse factor models. Several papers (Ghosh and Dunson, 2009; Frühwirth-Schnatter and Lopes, 2010; Conti et al., 2014; Piatek and Papaspiliopoulos, 2018) apply marginal data augmentation (MDA) in the spirit of van Dyk and Meng (2001); others (Kastner et al., 2017; Frühwirth-Schnatter and Lopes, 2018) exploit the ancillarity-sufficiency interweaving strategy (ASIS) introduced by Yu and Meng (2011). Some boosting strategies enhance mixing at the cost of changing the prior of the factor loading matrix \( \beta_r \) (Ghosh and Dunson, 2009), however, this appears undesirable in a variable selection context and is avoided by the boosting strategies applied in the present paper.
Figure 2: Exchange rate data; fractional prior. Posterior draws of $\text{tr}(\beta_r^t \Sigma_r^{-1} \beta_r^t)$ without boosting (left-hand side), boosting through ASIS with the largest loading (in absolute values) in each nonzero column serving as $\sqrt{\Psi_j}$ (middle) and boosting through MDA based on the inverted Gamma working prior $\Psi_j \sim \mathcal{G}^{-1}(1.5, 1.5)$ (right-hand side).

Boosting is based on moving from the CFA model (2.4) where $f_r^t \sim \mathcal{N}_r(0, I_r)$ to an expanded model with a more general prior:

$$y_t = \tilde{\beta}_r \tilde{f}_r^t + \epsilon_t, \quad \epsilon_t \sim \mathcal{N}_m(0, \Sigma_r), \quad \tilde{f}_r^t \sim \mathcal{N}_r(0, \Psi),$$

where $\Psi = \text{Diag}(\Psi_1, \ldots, \Psi_r)$ is a diagonal matrix. The relation between the two systems is given by the transformations $\tilde{f}_r^t = (\Psi)^{1/2} f_r^t$ and $\tilde{\beta}_r = \beta_r (\Psi)^{-1/2}$, where the nonzero elements in $\tilde{\beta}_r$ have the same position as the nonzero elements in $\beta_r$ and the sparsity matrix $\delta_r$ is not affected by the transformation. The main difference between MDA and ASIS lies in the choice of $\Psi$. While $\Psi_j$ is sampled from a working prior for MDA, $\Psi_j$ is chosen in a deterministic fashion for ASIS, see Appendix G for details. For illustration, Figure 2 shows considerable efficiency gain in the posterior draws of $\text{tr}(\beta_r^t \Sigma_r^{-1} \beta_r^t)$, when a boosting strategy such as ASIS (middle panel) or MDA (right-hand panel) is applied.

For hierarchical Gaussian priors with column specific shrinkage parameters, such as (3.11) and (3.12), we found it more useful to apply column boosting and interweave $\theta_j$ into the state equation by choosing $\Psi_j = \theta_j$. Column boosting and an additional boosting step which interweaves the global shrinkage parameter $\kappa$ into the prior of the shrinkage parameter $\theta_j$ is discussed in detail in Appendix G.

5 Post-processing posterior draws

Algorithm 1 delivers posterior draws $(\delta_r, \beta_r, \Sigma_r)$ in a CFA model with a varying number $r$ of active columns. Instead of sampling these parameters without any constraints, our sampler imposes the (mild) constraint that the pivots (the first non-zero loading in each column) lie in different rows, ensuring that posterior draws of the loading matrix exhibit a UGLT structure. While these draws are not identified in the rigorous sense discussed in Section 2.2, the UGLT structure allows identification during post-processing.
We use the 3579 counting rule and the algorithm of Hosszejni and Frühwirth-Schnatter (2022) to check for each draw $\delta_r$, if the variance decomposition is unique, and remove all posterior draws that are not variance identified. In our experience, the fraction $M_V$ of variance identified draws is relatively high for reasonably chosen priors. A low fraction of variance identified draws can be the result of poorly chosen shrinkage priors and is a hint to consider another prior family. The fractional prior on the loadings $\beta_{ij}$, for instance, tends to deliver a low fraction of variance identified draws in high-dimensional factor models, see Section 6.

Many quantities can be inferred from the variance identified posteriors draws with varying factor dimension $r$, such as the marginal covariance matrix $\Omega = \beta_r^T \beta_r + \Sigma_r$ and the idiosyncratic variances $\sigma_1^2, \ldots, \sigma_m^2$ appearing in the diagonal of $\Sigma_r$. In addition, overall sparsity in terms of the number $d = \sum_{j=1}^r \sum_{i=1}^m \delta_{ij}$ of nonzero elements in $\delta_r$, can be evaluated. Posterior draws of $d$ are particularly useful to check convergence and assessing efficiency of the MCMC sampler, as $d$ captures the ability of the sampler to move across (variance identified) UGLT factor models of different dimensions. Functionals of $\Omega$ and $\Sigma_r$, such as the trace, the log determinant, and $1_m^T \Omega^{-1} 1_m$ are further useful means to assess MCMC convergence. Furthermore, for each variable $y_{it}$, inference with respect to the proportion of the variance explained by the common factors (also known as communalities $R_{it}^2$) is possible:

$$R_{it}^2 = \sum_{j=1}^r R_{ij}^2, \quad R_{ij}^2 = \frac{\beta_{ij}^2}{\sum_{l=1}^r \beta_{il}^2 + \sigma_i^2}. \quad (5.1)$$

Most importantly, variance identified draws are instrumental for estimating the number of factors from an overfitting factor model with cumulative shrinkage process priors. As shown by Frühwirth-Schnatter et al. (2022, Theorem 7), the number $r$ of nonzero columns in $\delta_r$ is equal to the factor dimension, if the variance decomposition is unique for $r$. In this way, posterior draws of $r$ can be used for variance identified factor loading matrices to draw posterior inference on the factor dimension $r$. The mode $\tilde{r}$ of the posterior distribution $p(r|y)$, e.g., can be used to estimate the factor dimension.

Due to the UGLT structure imposed on $\beta_r$, rotational invariance reduces for variance identified draws to sign and column switching (Frühwirth-Schnatter et al., 2022, Theorem 1) and $\Lambda$ and $\delta^\Lambda$ are easily recovered. First, the columns of $\delta_r$ and $\beta_r$ are ordered such that the pivots $l_r = (l_1, \ldots, l_r)$ obey $l_1 < \ldots < l_r$; i.e. $\delta^\Lambda = \delta_r P_\rho$. Then, the sign of all columns in $\beta_r P_\rho$ is switched if the leading element is negative; i.e. $\Lambda = \beta_r P_\rho P_\pm$. In addition, the factors $f_t$ are reordered through $P_\pm P_\rho f_t$ for $t = 1, \ldots, T$. Finally, $P_\rho$ is used to reorder the draws of $\tau_r = (\tau_1, \ldots, \tau_r)$ and any local and column-specific shrinkage parameters in a hierarchical Gaussian shrinkage prior.

The posterior draws of $\delta^\Lambda$ are exploited in various ways. The highest probability model (HPM), i.e. the indicator matrix $\delta^\Lambda_H$ visited most often, its frequency $p_H$ (an estimator of the posterior probability of the HPM), its factor dimension $r_H$, its model size $d_H$, and its sequence of pivots $1_H^T$ are of interest. For each draw of $\delta^\Lambda$, the ordered pivots $l_1 < \ldots < l_r$ are draws from the marginal posterior distribution
p(l_1, \ldots, l_r | y)$, allowing additional posterior inference w.r.t. $l_r$. First of all, the posterior probability that a specific feature (row) serves as a pivot, i.e. $\Pr(i \in l_r | y)$ for $i = 1, \ldots, m$ is of interest. Second, the sequence of pivots $l^* = (l^*_1, \ldots, l^*_r)$ visited most often is determined together with its frequency $p_L$ which reflects posterior uncertainty with respect to choosing the pivots. The number $r^*$ of elements in $l^*$ provides yet another estimator of the number of factors. If the frequencies $p_H$ and $p_L$ are small, the estimator $r^*$ and $r_H$ might not coincide with the posterior mode $\hat{r}$ and $l^H$ will different from $l^*$.

To estimate the factor loading matrix for a chosen number of factors $r$, Bayesian model averaging is performed conditionally on an estimator $\hat{l}_r$ of the pivots. Averaging over all posterior draws with these specific pivots avoids column switching and provides an estimate of the factor loading matrix $\Lambda$ and the marginal inclusion probabilities $\Pr(\delta_{ij}^{\Lambda} = 1 | y, \hat{l}_r)$ for all elements of the corresponding sparsity matrix. The median probability model (MPM) $\delta_{ij}^{\Lambda}$ is obtained by setting each indicator to one whenever $\Pr(\delta_{ij}^{\Lambda} = 1 | y, \hat{l}_r) \geq 0.5$. As for any Bayesian approach, the reliability of any of these estimators might depend on how informative the data are. In settings where the data are not very informative, the chosen degree of sparsity in the prior might exhibit considerable influence on the result.

6 Applications

6.1 Sparse factor analysis for exchange rate data

As a first application, we analyze log returns from $m = 22$ exchange rates with respect to the Euro, observed for $T = 96$ months.\(^3\) A sparse EFA with UGLT structure is fitted with $k = 10$ equals the upper bound (2.2). Regarding the prior on the sparsity matrix $\delta_k$, we consider the 1PB prior (3.3) and the 2PB prior (3.4), where $\alpha \sim G(6, 3)$ and $\gamma \sim G(6, 6)$ for the 2PB prior. We combine the fractional prior (3.14) as slab distribution for the non-zero loadings in $\beta_r$ with the prior (3.17) on the idiosyncratic variances $\sigma^2_1, \ldots, \sigma^2_m$, where $c_0 = 2.5$ and $\hat{\Omega}^{-1}$ is estimated from (3.18) with $\nu_o = 3$ and $S_o = I_m$.

Algorithm 1 is run for $M = 40,000$ after a burn-in of 20,000 draws. As discussed in Section 4, this sampler navigates in the space of all unordered GLT structures with an unknown number of nonzero columns and unknown pivots without forcing any further constraint. To verify convergence, independent MCMC chains are started with, respectively, $r = 2$ and $r = 7$, and $r_{sp} = 3$ spurious columns. The sampler shows good mixing across models of different dimension, in particular for the 2PB prior, with an inefficiency factor of roughly 5. For illustration, Figure 3 shows all posterior draws of $r$ and the model size $d$, including burn-in, for the first run under the 2PB prior.

As outlined in Section 5, we resolve identification during post-processing. First, we screen for vari-

---

3The data was obtained from the European Central Bank’s Statistical Data Warehouse and ranges from January 3, 2000 to December 3, 2007. It contains the 22 exchange rates listed in Table I.1 in Appendix I from which we derived monthly returns based on the first trading day in a month. The data are demeaned and standardized.
Figure 3: Exchange rate data; posterior draws of the factor dimension \( r \) (left-hand side) and model size \( d \) (right-hand side) including burn-in, starting from \( r = 2 \) and \( r_{sp} = 3 \).

Table 1: Exchange rate data; posterior distribution \( p(r|y) \) of the number of factors and posterior mean \( E(d|y) \) of model size (based on 100\( p_V \) percent variance identified draws).

| \( p(r|y) \) | \( E(\alpha|y) \) | \( E(\gamma|y) \) | 0-2 | 3 | 4 | 5 | 6 | 7-10 | 100 \cdot p_V | E(d|y) |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| 1PB | 2.0 | - | 0.128 | 0.867 | 0.005 | \( \approx 0 \) | 0 | 93.3 | 28 |
| 2PB | 2.0 | 1.1 | 0 | 0 | 0.988 | 0.012 | 0 | 0 | 95.2 | 28 |

Note: non-zero probabilities smaller than \( 10^{-3} \) are indicated by \( \approx 0 \).

Table 2: Exchange rate data; posterior probability of the event \( \Pr(q_i = 0|y) \), where \( q_i \) is the row sum of \( \delta_r \), for various currencies.

| \( \Pr(q_i = 0|y) \) | CHF | CZK | MXN | NZD | RON | RUB | remaining |
| --- | --- | --- | --- | --- | --- | --- | --- |
| 1PB prior | 0.87 | 0.73 | 0.81 | 0.48 | 0.62 | 0.62 | 0 |
| 2PB prior | 0.88 | 0.76 | 0.82 | 0.49 | 0.62 | 0.62 | 0 |

The fraction \( p_V \) of variance identified draws, reported in Table 1, is very high for both EPS priors. For the variance identified draws, the number \( r \) of columns of the sparsity matrix \( \delta_r \) in the CFA model is regarded as a draw of the number \( r \) of factors under the specific prior choice. Table 1 reports the posterior distribution \( p(r|y) \) for both ESP priors. For the 2PB prior, this posterior is highly concentrated at a four factors. Also under the 1PB prior, the posterior mode equals 4, but also three factors receive some posterior probability. The indicator matrix \( \delta_r \) is pretty sparse, with an average posterior model size of 28.

Furthermore, the variance identified draws are used to explore if some measurements are uncorre-
Table 3: Exchange rate data; total number of visited models \( N_v \); frequency \( p_H \) (in percent), pivots \( l_H \) and model size \( d_H \) of the HPM; pivots \( l^* \) visited most often, corresponding frequency \( p_L \) (in percent) and corresponding number of factors \( r^* \); pivots \( l_H \), model size \( d_H \) and frequency \( p_H \) (in percent) of the HPM; model size \( d_M \) of the MPM.

| Prior       | \( N_v \) | 100\( p_H \) | \( l_H \) | \( d_H \) | \( l^* \) | 100\( p_L \) | \( r^* \) | \( d_M \) |
|-------------|----------|-------------|---------|--------|--------|-------------|--------|--------|
| 1PB prior   | 16508    | 4.5         | (1,2,5,7) | 26     | (1,2,5,7) | 84.8        | 4      | 26     |
| 2PB prior   | 11933    | 5.0         | (1,2,5,7) | 26     | (1,2,5,7) | 91.0        | 4      | 26     |

Figure 4: Exchange rate data; sparsity matrix \( \delta_4 \) corresponding both to the HPM and the MPM which are identical for the 1PB and the 2PB prior.

lated with the remaining measurements. This is investigated in Table 2 through the posterior probability \( \Pr(q_i = 0|y) \), where \( q_i \) is the row sum of \( \delta_r \). Regardless of the chosen prior, the Swiss franc (CHF), the Mexican peso (MXN) and the Czech koruna (CZK) have considerable probability to be uncorrelated with the rest, while the situation is less clear for the New Zealand dollar (NZD), the Romania fourth leu (RON), and the Russian ruble (RUB). The remaining currencies are clearly correlated.

To proceed with identification for all variance identified draws, rotation indeterminacy is resolved by ordering the pivots (and all column-specific variables) such that \( l_1 < \ldots < l_r \) and an ordered GLT structure is imposed. Several posterior summaries for the ordered GLT draws are reported in Table 3. For both ESP priors, \( l^* = l^H = (1, 2, 5, 7) \) is the most likely sequence of pivots. As a final step, all variance identified, ordered GLT draws where the pivots coincide with \( l^* = l^H = (1, 2, 5, 7) \) are used for both ESP priors to identify the marginal inclusion probabilities \( \Pr(\delta_{ij} = 1|y, l^*) \), the corresponding MPM, its model size \( d_M \) and the factor loading matrix \( \Lambda \) for a 4-factor model. Sign switching in the posterior draws is resolved by imposing the constraint \( \Lambda_{11} > 0, \Lambda_{22} > 0, \Lambda_{53} > 0, \) and \( \Lambda_{74} > 0 \) on \( \Lambda \). Both ESP priors yield the same MPM which coincides with the sparsity matrix of both HPMs, see Figure 4.
for illustration. Further results are reported in Appendix I. The resulting model indicates considerable sparsity, with many factor loadings being shrunk toward zero. Factor 2 is a common factor among the correlated currencies, while the remaining factors are three group specific, for the most part dedicated factors.

6.2 Sparse factor analysis for NYSE stock returns

As a second application, we consider monthly log returns from $m = 63$ firms from the NYSE observed for $T = 247$ months from February 1999 till August 2019. An EFA model with UGLT structure is fitted with $k = 31$ being equal to the upper bound (2.2). Regarding the ESP prior on the sparsity matrix $\delta_k$, we consider the 2PB prior (3.4) with $\alpha \sim \mathcal{G}(6, 12)$ and $\gamma \sim \mathcal{G}(6, 6)$. As in Section 6.1, we tried to apply a fractional prior as slab distribution, however the fraction of variance identified posterior draws was extremely low (less than 1%). Hence, the hierarchically structured Gaussian shrinkage prior (3.12) is chosen in the slab. To introduce aggressive shrinkage for the factor loadings, the local scaling parameters are assumed to follow a triple gamma prior with $\omega = \omega' = 0.2$, i.e. $\omega_{ij} \sim \mathcal{F}(0.4, 0.4)$.

The top 150 companies (as of September 13, 2019) listed on the NYSE were downloaded from Bloomberg on September 13, 2019. Since many of the companies entered the NYSE after 1990, we picked February 1999 as a starting date and, after removing all companies that were founded later, we use monthly return (determined on the last trading day in each month) for the 103 remaining companies listed on the NYSE for all $T = 247$ observation periods till August 2019. For our case study, we consider the 63 firms belonging to the following five sectors: basic industries (1-9), non-durable consumer goods (8-17), energy (18-27), finance (28-45) and health care (46-63). The data are ordered according to industry and are The data are demeaned and standardized.
Table 4: NYSE data; posterior distribution $p(r|y)$ of the number of factors and the mean and the quartiles (in parenthesis) of the posterior distribution $p(d|y)$ of model size (100$p_V$ percent variance identified draws).

| E($\alpha|y$) | E($\gamma|y$) | 0-12 | 13   | 14   | 15   | 16 | 17 | 18-31 | $p(d|y)$ |
|--------------|--------------|------|------|------|------|----|----|-------|---------|
| 2.5          | 0.8          | 0    | ≈ 0  | 0.413| 0.488| 0.098 | 0.001 | 0   | 203     |

Note: non-zero probabilities smaller than $10^{-3}$ are indicated by $\approx 0$.

Figure 6: NYSE data; posterior probability $Pr(i \in L_{r}|y)$ that a specific firm serves as pivot; the red dots indicate the first firm in each sector.

Also the column specific shrinkage parameters follow a triple gamma prior, $\theta_j \sim F(5,5)$, whereas the global shrinkage parameter follows an inverse gamma distribution, $\kappa \sim G^{-1}(10,50)$. Regarding the idiosyncratic variances, we choose the prior $\sigma_i^2 \sim G^{-1}(2.5,1.5)$. The fraction of variance identified MCMC draws under this prior is roughly 20%.

Algorithm 1 was applied to obtain $M = 50,000$ posterior draws after a burn-in of 40,000 draws, starting with $r = 10$ factors and $r_{sp} = 3$ spurious columns. The MCMC scheme shows relatively good mixing, despite the high dimensionality, as illustrated by Figure 5 showing posterior draws of the total number of non-zero columns, $r + r_{sp}$, the number of spurious columns $r_{sp}$, the extracted number of factors $r$, and the model dimension $d$. As shown in Table 4, the posterior distribution $p(r|y)$ derived from the variance identified draws yields a posterior mode of $\tilde{r} = 15$, but also 14 factors receive considerable posterior evidence.

For further inference, an ordered GLT structure is imposed on all variance identified draws with $r = 15$. Since each draw of $\Lambda$ turned out to have a different shrinkage matrix $\delta^\Lambda$, choosing a unique set
of pivots is challenging. In Figure 6, the posterior probability $\Pr(i \in l_r | y)$ that a specific firm serves as a pivot is displayed. While the first three factors use the pivots 1, 2, and 3 (as a PLT structure would do), we find that the remaining factors clearly exhibit a GLT structure and the first firm listed in a specific sectors (indicated by a red dot) typically serves as a pivot for new factor.

Since the pivot of the first factor is equal to 1 for all posterior draws, we can estimate the first column of the indicator matrix and the corresponding factor loadings as the average of all posterior draws. This factor is a market factor that loads on all 63 firms, see also Figure 8. The remaining 14 factors mainly capture industry specific correlations as well as cross-sectional correlations between specific firms, see the estimated marginal correlation matrix $\Omega^*$ that remains after extracting the first factor in Figure 7. For further illustration, factors with pivots equal to (1, 8, 18, 19, 28, 30, 46, 53) are extracted from all GLT draws with $r = 15$, where the sequence of pivots $l_r$ contains these pivots. Beyond the market factor 1, the sector specific factors 3, 6 and 7 mainly load on firms in, respectively, the energy, the finance and the health care sector. The remaining factors are weak factors with very sparse loadings.

7 Concluding remarks

We have estimated (from a Bayesian viewpoint) a fairly important and highly implemented class of sparse factor models when the number of common factors is unknown. Our framework leads to a natural, effi-
cient and simultaneous coupling of model estimation and selection on one hand and model identification and rank estimation (number of factors) on the other hand. More precisely, by combining point-mass mixture priors with overfitting sparse factor modelling, in an unordered generalised lower triangular loadings representation (Frühwirth-Schnatter et al., 2022), we obtain posterior summaries regarding factor loadings, common factors as well as the factor dimension via postprocessing draws from our highly efficient and customised MCMC scheme.

The new framework is readily available for some straightforward extensions. Relatively immediate extensions are (i) idiosyncratic errors following Student’s $t$-distributions or more general Gaussian mixtures and (ii) dynamic sparse factor models with stationary common factors; both extensions commonly found in econometrics applications, see e.g. the recent papers by Piatek and Papaspiliopoulos (2018) and Kaufmann and Schuhmacher (2019). A further interesting extension would be to design a prior on the sparsity matrix that a priori distinguishes between pervasive factors that loads on (nearly) all measurements, group specific factors that load on selected measurements and factors that mainly capture (weak) cross-sectional heterogeneity which is not built into the basic factor model. Such approximate factor models are very popular in non-Bayesian factor analysis, see e.g. Chamberlain and Rothschild (1983) and Bai and Ng (2002) and would deserve more attention from the Bayesian community. However, we leave this interesting idea for future research.

Figure 8: NYSE data; factors with pivots equal to (1, 8, 18, 19, 28, 30, 46, 53)
References

Akaike, H. (1987). Factor analysis and AIC. *Psychometrika* 52, 317–332.

Anderson, T. W. (2003). *An Introduction to Multivariate Statistical Analysis* (3 ed.). Chichester: Wiley.

Anderson, T. W. and H. Rubin (1956). Statistical inference in factor analysis. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*, Volume V, pp. 111–150.

Aßmann, C., J. Boysen-Hogrefe, and M. Pape (2016). Bayesian analysis of static and dynamic factor models: An ex-post approach toward the rotation problem. *Journal of Econometrics* 192, 190–206.

Bai, J. and S. Ng (2002). Determining the number of factors in approximate factor models. *Econometrica* 70, 191–221.

Bai, J. and S. Ng (2013). Principal components estimation and identification of static factors. *Journal of Econometrics* 176, 18–29.

Bartholomew, D. J. (1987). *Latent Variable Models and Factor Analysis*. London: Charles Griffin.

Bhattacharya, A. and D. Dunson (2011). Sparse Bayesian infinite factor models. *Biometrika* 98, 291–306.

Boivin, J. and S. Ng (2006). Are more data always better for factor analysis? *Journal of Econometrics* 132, 169–194.

Cadonna, A., S. Frühwirth-Schnatter, and P. Knaus (2020). Triple the gamma – A unifying shrinkage prior for variance and variable selection in sparse state space and TVP models. *Econometrics* 8, 20.

Carvalho, C. M., J. Chang, J. E. Lucas, J. Nevins, Q. Wang, and M. West (2008). High-dimensional sparse factor modeling: Applications in gene expression genomics. *Journal of the American Statistical Association* 103, 1438–1456.

Chamberlain, G. and M. Rothschild (1983). Arbitrage, factor structure, and mean-variance analysis on large asset markets. *Econometrica* 51, 1281–1304.

Chan, J., R. Leon-Gonzalez, and R. W. Strachan (2018). Invariant inference and efficient computation in the static factor model. *Journal of the American Statistical Association* 113, 819–828.

Conti, G., S. Frühwirth-Schnatter, J. J. Heckman, and R. Piatek (2014). Bayesian exploratory factor analysis. *Journal of Econometrics* 183, 31–57.

Durante, D. (2017). A note on the multiplicative gamma process. *Statistics and Probability Letters* 122, 198–204.
Fan, J., Y. Fan, and J. Lv (2008). High dimensional covariance matrix estimation using a factor model. *Journal of Econometrics* 147, 186–197.

Forni, M., D. Giannoni, M. Lippi, and L. Reichlin (2009). Opening the black box: Structural factor models with large cross sections. *Econometric Theory* 25, 1319–1347.

Foster, D. P. and E. I. George (1994). The risk inflation criterion for multiple regression. *The Annals of Statistics* 22, 1947–1975.

Frühwirth-Schnatter, S. (2022). Generalized cumulative shrinkage process priors with applications to sparse bayesian factor analysis.

Frühwirth-Schnatter, S., D. Hosszejni, and H. Lopes (2022). When it counts - Econometric identification of factor models based on GLT structures. *ArXiv forthcoming soon*.

Frühwirth-Schnatter, S. and H. Lopes (2010). Parsimonious Bayesian Factor Analysis when the Number of Factors is Unknown. Research report, Booth School of Business, University of Chicago.

Frühwirth-Schnatter, S. and H. Lopes (2018). Sparse Bayesian Factor Analysis when the Number of Factors is Unknown. *arXiv 1804.04231*.

Frühwirth-Schnatter, S. and R. Tüchler (2008). Bayesian parsimonious covariance estimation for hierarchical linear mixed models. *Statistics and Computing* 18, 1–13.

Frühwirth-Schnatter, S. and H. Wagner (2010). Stochastic model specification search for Gaussian and partially non-Gaussian state space models. *Journal of Econometrics* 154, 85–100.

Geweke, J. F. and K. J. Singleton (1980). Interpreting the likelihood ratio statistic in factor models when sample size is small. *Journal of the American Statistical Association* 75, 133–137.

Geweke, J. F. and G. Zhou (1996). Measuring the pricing error of the arbitrage pricing theory. *Review of Financial Studies* 9, 557–587.

Ghahramani, Z., T. L. Griffiths, and P. Sollich (2007). Bayesian nonparametric latent feature models (with discussion and rejoinder). In J. M. Bernardo, M. J. Bayarri, J. O. Berger, A. P. Dawid, D. Heckerman, A. F. M. Smith, and M. West (Eds.), *Bayesian Statistics 8*, pp. ADD–ADD. Oxford: Oxford University Press.

Ghosh, J. and D. B. Dunson (2009). Default prior distributions and efficient posterior computation in Bayesian factor analysis. *Journal of Computational and Graphical Statistics* 18, 306–320.

Griffiths, T. L. and Z. Ghahramani (2006). Infinite latent feature models and the Indian buffet process. In Y. Weiss, B. Schölkopf, and J. Platt (Eds.), *Advances in Neural Information Processing Systems*, Volume 18, pp. 475–482. Cambridge, MA: MIT Press.
Hosszejni, D. and S. Frühwirth-Schnatter (2022). Cover it up! Bipartite graphs uncover identifiability in sparse factor analysis. *arXiv 2211.00671*.

Jöreskog, K. G. (1967). Some contributions to maximum likelihood factor analysis. *Psychometrika 32*, 443–482.

Jöreskog, K. G. (1969). A general approach to confirmatory maximum likelihood factor analysis. *Psychometrika 34*, 183–202.

Kastner, G. (2019). Sparse Bayesian time-varying covariance estimation in many dimensions. *Journal of Econometrics 210*, 98–115.

Kastner, G., S. Frühwirth-Schnatter, and H. F. Lopes (2017). Efficient Bayesian inference for multivariate factor stochastic volatility models. *Journal of Computational and Graphical Statistics 26*, 905–917.

Kaufmann, S. and C. Schuhmacher (2017). Identifying relevant and irrelevant variables in sparse factor models. *Journal of Applied Econometrics 32*, 1123–1144.

Kaufmann, S. and C. Schuhmacher (2019). Bayesian estimation of sparse dynamic factor models with order-independent and ex-post identification. *Journal of Econometrics 210*, 116–134.

Koopmans, T. C. and O. Reiersøl (1950). The identification of structural characteristics. *The Annals of Mathematical Statistics 21*, 165–181.

Lee, S.-Y. and X.-Y. Song (2002). Bayesian selection on the number of factors in a factor analysis model. *Behaviormetrika 29*, 23–39.

Legramanti, S., D. Durante, and D. B. Dunson (2020). Bayesian cumulative shrinkage for infinite factorizations. *Biometrika 107*, 745–752.

Lopes, H. F. and M. West (2004). Bayesian model assessment in factor analysis. *Statistica Sinica 14*, 41–67.

Lucas, J., C. Carvalho, Q. Wang, A. Bild, J. R. Nevins, and M. West (2006). Sparse statistical modelling in gene expression genomics. In K. Do, P. Müller, and M. Vannucci (Eds.), *Bayesian Inference for Gene Expression and Proteomics*, pp. 155–176. Cambridge, UK: Cambridge University Press.

Martin, J. K. and R. P. McDonald (1975). Bayesian estimation in unrestricted factor analysis: a treatment for Heywood cases. *Psychometrika 40*, 505–517.

Neudecker, H. (1990). On the identification of restricted factor loading matrices: An alternative condition. *Journal of Mathematical Psychology 34*, 237–241.
O’Hagan, A. (1995). Fractional Bayes factors for model comparison. *Journal of the Royal Statistical Society, Ser. B* 57, 99–138.

Owen, A. B. and J. Wang (2016). Bi-Cross-Validation for Factor Analysis. *Statistical Science* 31, 119–139.

Paisley, J. W. and L. Carin (2009). Nonparametric factor analysis with beta process priors. In *Proceedings of the 26th International Conference on Machine Learning (ICML ’09)*, pp. 777–784.

Papastamoulis, P. and I. Ntzoufras (2022). On the identifiability of Bayesian factor analytic models. *Statistics and Computing* 32.

Pati, D., A. Bhattacharya, N. S. Pillai, and D. B. Dunson (2014). Posterior contraction in sparse Bayesian factor models for massive covariance matrices. *Annals of Statistics* 42, 1102–1130.

Piatek, R. and O. Papaspiliopoulos (2018). A bayesian nonparametric approach to factor analysis. Submitted.

Poworoznek, E., F. Ferrari, and D. Dunson (2021). Efficiently resolving rotational ambiguity in Bayesian matrix sampling with matching. *ArXiv 2107.13783*.

Reiersøl, O. (1950). On the identifiability of parameters in Thurstone’s multiple factor analysis. *Psychometrika* 15, 121–149.

Ročková, V. and E. I. George (2017). Fast Bayesian factor analysis via automatic rotation to sparsity. *Journal of the American Statistical Association* 111, 1608–1622.

Rue, H. and L. Held (2005). *Gaussian Markov Random Fields: Theory and Applications*, Volume 104 of *Monographs on Statistics and Applied Probability*. London: Chapman & Hall/CRC.

Stock, J. H. and M. W. Watson (2002). Macroeconomic forecasting using diffusion indexes. *Journal of Business & Economic Statistics* 20, 147–162.

Teh, Y. W., D. Görür, and Z. Ghahramani (2007). Stick-breaking construction for the Indian buffet process. In M. Meila and X. Shen (Eds.), *Proceedings of the Eleventh International Conference on Artificial Intelligence and Statistics*, Volume 2 of *Proceedings of Machine Learning Research*, San Juan, Puerto Rico, pp. 556–563. PMLR.

Thurstone, L. L. (1947). *Multiple factor analysis*. Chicago: University of Chicago.

Tumura, Y. and M. Sato (1980). On the identification in factor analysis. *TRU Mathematics* 16, 121–131.

van Dyk, D. and X.-L. Meng (2001). The art of data augmentation. *Journal of Computational and Graphical Statistics* 10, 1–50.
West, M. (2003). Bayesian factor regression models in the “large p, small n” paradigm. In J. M. Bernardo, M. J. Bayarri, J. O. Berger, A. P. Dawid, D. Heckerman, A. F. M. Smith, and M. West (Eds.), *Bayesian Statistics 7*, pp. 733–742. Oxford: Oxford University Press.

Williams, B. (2020). Identification of the linear factor model. *Econometric Reviews 39*, 92–109.

Yu, Y. and X.-L. Meng (2011). To center or not to center: That is not the question - An ancillarity-sufficiency interweaving strategy (ASIS) for boosting MCMC efficiency. *Journal of Computational and Graphical Statistics 20*, 531–615.

Zhao, S., C. Gao, S. Mukherjee, and B. E. Engelhardt (2016). Bayesian group factor analysis with structured sparsity. *Journal of Machine Learning Research 17*, 1–47.
Supplementary material for:
“Sparse Bayesian Factor Analysis when the Number of Factors is Unknown”

A Details on Step (P)

A.1 Posterior distributions in a confirmatory factor model

Step (P) of Algorithm 1 updates the parameters in the confirmatory sparse factor model factor model

\[ f^r_t \sim N_r(0, \mathbf{I}_r), \quad y_t = \beta_r f^r_t + \epsilon_t, \quad \epsilon_t \sim N_m(0, \Sigma_r), \quad \Sigma_r = \text{Diag}(\sigma^2_1, \ldots, \sigma^2_m), \quad (A.1) \]

where the indicator matrix \( \delta_r \) imposes a certain zero structure on the loading matrix \( \beta_r \). The joint posterior distribution \( p(\beta^\delta_r, \sigma^2_r | y, f_r, \delta_r) \) of the nonzero factor loadings \( \beta^\delta_r \) and the idiosyncratic variance \( \sigma^2_r \) is derived for each row \( i \) \((i = 1, \ldots, m)\) conditional on the factors \( f_r = (f^1_r, \ldots, f^r_T) \) and the indicator matrix \( \delta_r \) from the following regression model:

\[ \tilde{y}_i = X^\delta_i \beta^\delta_i + \tilde{\epsilon}_i, \quad (A.2) \]

where \( \tilde{y}_i = (y_{i1} \cdots y_{iT})' \) and \( \tilde{\epsilon}_i = (\epsilon_{i1} \cdots \epsilon_{iT})' \sim N_T(0, \sigma^2_T \mathbf{I}). \) \( X^\delta_i \) is a regressor matrix for \( \beta^\delta_i \) constructed from the \( T \times r \) dimensional latent factor matrix \( F = (f^1_r \cdots f^r_T)' \) in the following way. If no element in row \( i \) of \( \beta_r \) is restricted to 0, then \( X^\delta_i = F. \) If some elements are restricted to 0, then \( X^\delta_i \) is obtained from \( F \) by deleting all columns \( j \) where \( \delta_{ij} = 0, \) i.e. \( X^\delta_i = F \Pi^\delta_i, \) where \( \Pi^\delta_i \) is a \( r \times \sum_{j=1}^r \delta_{ij} \) selection matrix, selecting those columns \( j \) of \( F \) where \( \delta_{ij} \neq 0. \) The likelihood derived from (A.2) is combined with the inverted Gamma prior (3.15) on \( \sigma^2_{iT} \) and, respectively, the hierarchical Gaussian prior (3.10) or the fractional prior (3.14) for \( \beta^\delta_i | \sigma^2_{iT}. \) In a sparse factor model, the dimension of this posterior depends on the number of nonzero elements in the \( i \)th row of \( \beta_r, \) i.e. \( q_i = \sum_{j=1}^r \delta_{ij}. \) There are basically three types of rows, when it comes to updating the parameters: zero rows, dedicated rows and rows with multiple loadings.

For zero rows (i.e. \( q_i = 0 \)), (A.2) reduces to a “null” model without regressors \( X^\delta_i, \) that is \( \tilde{y}_i = \tilde{\epsilon}_i. \) Hence, the posterior of \( \sigma^2_{iT} \) is simply given by

\[ \sigma^2_{iT | \tilde{y}_i, f_r, \delta_r} \sim \mathcal{G}^{-1} \left( c^0_T, C^0_{iT} \right), \quad c^0_T = c_0 + \frac{T}{2}, \quad C^0_{iT} = C_{i0} + \frac{1}{2} \sum_{t=1}^T y_{it}^2. \quad (A.3) \]

For all nonzero rows (i.e. \( q_i > 0 \)), the posterior \( (\beta^\delta_i, \sigma^2_r) \) for a specific row \( i \) is given by:

\[ \sigma^2_{iT | \tilde{y}_i, f_r, \delta_r} \sim \mathcal{G}^{-1} \left( c^T, C^T_{iT} \right), \quad \beta^\delta_i | \sigma^2_{iT}, \tilde{y}_i, f_r, \delta_r \sim N_{n_i} \left( B^\delta_{iT} m^\delta_{iT}, B^\delta_{iT} \sigma^2_{iT} \right). \quad (A.4) \]
For the hierarchical Gaussian prior (3.10), the moments are given by:

\[
(B_{iT}^\delta)^{-1} = (B_{i0}^\delta)^{-1} + (X_i^\delta)'X_i^\delta, \quad m_{iT}^\delta = (X_i^\delta)'\tilde{y}_i, \quad (A.5)
\]

\[c_T = c_0 + \frac{T}{2}, \quad C_{iT}^\delta = C_{i0} + \frac{1}{2}SSR_i, \quad SSR_i = \tilde{y}_i'y_i - (m_{iT}^\delta)'B_{iT}^\delta m_{iT}^\delta.
\]

For the fractional prior (3.14), the moments are given by:

\[
(B_{iT}^\delta)^{-1} = (X_i^\delta)'X_i^\delta, \quad m_{iT}^\delta = (X_i^\delta)'\tilde{y}_i, \quad (A.6)
\]

\[c_T = c_0 + \frac{1}{2}(1-b)T, \quad C_{iT}^\delta = C_{i0} + \frac{1}{2}(1-b)SSR_i,
\]

where SSR_i is the same as in (A.5) and, for the fractional prior, identical to the residual sum of squares errors.\(^5\) For dedicated rows (i.e. \(q_i = 1\)) only a single nonzero factor loading \(\beta_{i,j_i}\) is present in a particular column \(j_i\) and the posterior given in (A.4) simplifies considerably:

\[
\sigma_i^2 y_i, f_i, \delta_i \sim \mathcal{N}^{-1}(c_T, C_{iT}), \quad \beta_{i,j_i} \sigma_i^2 y_i, f_i, \delta_i \sim \mathcal{N}(B_{iT} m_{iT}, B_{iT} \sigma_i^2).
\] \(\text{(A.7)}\)

For a hierarchical Gaussian prior, the posterior moments are given by:

\[
B_{iT} = 1/(B_{i0,j_i,j_i}^{-1} + \sum_{t=1}^{T} f_{j_i,t}^2), \quad m_{iT} = \sum_{t=1}^{T} f_{j_i,t} y_{i,t}, \quad (A.8)
\]

\[c_T = c_0 + \frac{T}{2}, \quad C_{iT} = C_{i0} + \frac{1}{2} \left(\sum_{t=1}^{T} y_{i,t}^2 - m_{iT}^2 B_{iT}\right),
\]

where \(B_{i0,j_i,j_i}\) is the prior variance of the dedicated factor loading in row \(i\), whereas for a fractional prior:

\[
B_{iT} = 1/\left(\sum_{t=1}^{T} f_{j_i,t}^2\right), \quad m_{iT} = \sum_{t=1}^{T} f_{j_i,t} y_{i,t}, \quad (A.9)
\]

\[c_T = c_0 + \frac{1}{2}(1-b)T, \quad C_{iT} = C_{i0} + \frac{1}{2}(1-b) \left(\sum_{t=1}^{T} y_{i,t}^2 - m_{iT}^2 B_{iT}\right).
\]

### A.2 Block sampling of idiosyncratic variances and factor loadings

Step (P) in Algorithm 1 could be implemented as in Lopes and West (2004), by sampling \(\beta_i^\delta\) and \(\sigma_i^2\) from the posterior distribution \(p(\beta_i^\delta, \sigma_i^2 | y, f_i, \delta_i)\) derived in Section A.1 row by row. However, an important improvement is feasible through block sampling of all idiosyncratic variances and all nonzero factor loadings, summarized in Algorithm 3.\(^6\) The use of the Cholesky decomposition of the information matrix (instead of the covariance matrix) to sample from a high-dimensional density is fashioned after Rue and Held (2005, Theorem 2.5 and Algorithm 2.5) who consider Gaussian random fields.

---

\(^5\)If the residual \(\epsilon_i = \tilde{y}_i - X_i^\delta B_{iT}^\delta m_{iT}^\delta\) is defined in the usual way, then:

\[
\epsilon_i' \epsilon_i = y_i'y_i - (m_{iT}^\delta)'B_{iT}^\delta (X_i^\delta)'y_i - y_i'X_i^\delta B_{iT}^\delta m_{iT}^\delta + (m_{iT}^\delta)'B_{iT}^\delta (X_i^\delta)'X_i^\delta B_{iT}^\delta m_{iT}^\delta
\]

\[= y_i'y_i - (m_{iT}^\delta)'B_{iT}^\delta m_{iT}^\delta - (m_{iT}^\delta)'B_{iT}^\delta m_{iT}^\delta + (m_{iT}^\delta)'B_{iT}^\delta m_{iT}^\delta = SSR_i.
\]

\(^6\)This algorithm has been implemented for the first time in the unpublished research report by Frühwirth-Schnatter and Lopes (2010).
Algorithm 3. Sampling parameters for a sparse Bayesian factor model

(P-a) For all zero rows, sample $\sigma^2_i$ from (A.3), which can be trivially vectorized.

(P-b) If the remaining rows are all dedicated with a single nonzero loading in column $j_i$ (which can be different for different rows), then sampling from (A.7) is easily vectorized, since all posterior moments are univariate.

(P-c) Even if some of the nonzero rows are not dedicated, joint sampling of all idiosyncratic variances and all factor loadings is feasible for all nonzero rows. Let $i_1, \ldots, i_n$ be the indices of all $n = m - m_0$ nonzero rows of $\beta_r$, i.e. $q_{i_l} > 0$ for $l = 1, \ldots, n$. Let $d = \sum_i q_i$ be the total number of nonzero elements in $\beta_r$ and let $\beta_r^\delta = (\beta_{i_1}^\delta, \ldots, \beta_{i_n}^\delta)'$ be the $d$-dimensional vector obtained by stacking row by row all nonzero elements in each row of $\beta_r$. To sample the idiosyncratic variances $\sigma^2_{i_1}, \ldots, \sigma^2_{i_n}$ and the nonzero factor loadings $\beta_r^\delta$ in $\beta_r$ jointly, proceed in the following way:

(P-c1) Construct the information matrix $P$ and the covector $m$ of the joint posterior

$$\beta_r^\delta | \sigma^2_{i_1}, \ldots, \sigma^2_{i_n}, f_r, y \sim N_d(P^{-1}m, P^{-1}D).$$

The matrix $D = \text{Diag}(\sigma^2_{i_1} 1_{1 \times q_{i_1}} \cdots \sigma^2_{i_n} 1_{1 \times q_{i_n}})$, with $1_{1 \times l}$ being a $1 \times l$ row vector of ones, is a $d \times d$ diagonal matrix containing the idiosyncratic variances, while the $d \times d$ matrix $P$ and the $d \times 1$ vector $m$ are given by:

$$P = \begin{pmatrix} (B_{i_1, T}^\delta)^{-1} & O & \cdots & O \\ O & (B_{i_2, T}^\delta)^{-1} & \cdots & \vdots \\ \vdots & \vdots & \ddots & O \\ O & \cdots & O & (B_{i_n, T}^\delta)^{-1} \end{pmatrix}, \quad m = \begin{pmatrix} m_{i_1, T}^\delta \\ \vdots \\ m_{i_n, T}^\delta \end{pmatrix},$$

where $(B_{i_l, T}^\delta)^{-1}$ and $m_{i_l, T}^\delta$ are the information matrix and the covector appearing in the posterior (A.4) of the nonzero elements in row $i_l$. $P$ is a sparse band matrix with maximal band width equal to $\max q_{i_l}$.

(P-c2) Compute the Cholesky decomposition $P = LL'$, where $L$ is lower triangular, using a special algorithm developed for band matrices. Next, solve $Lx = m$ for $x$ using an algorithm specially designed for triangular matrices. Evidently, $x$ is a $d \times 1$ vector.

(P-c3) Sample $\sigma^2_{i_1}, \ldots, \sigma^2_{i_n}$ jointly from (A.4). The squared sum $x'x$ can be used to vectorize the computation of $C_{i_l, T}$ for each $l = 1, \ldots, n$, since

$$x_{i_l}' x_{i_l} = (m_{i_l, T}^\delta)' B_{i_l, T}^\delta m_{i_l, T}^\delta, \quad (A.10)$$

where $x_{i_l}$ is the $q_{i_l}$-dimensional sub vector of $x$ corresponding to $\beta_{i_l}^\delta$.\pagebreak
(P-c4) Finally, define the diagonal matrix $D$ from $\sigma_{i_1}^2, \ldots, \sigma_{i_n}^2$ as described above and draw $z \sim \mathcal{N}_d(0, D)$. Solving the system

$$L' \beta^\delta_r = x + z$$

(A.11)

for $\beta^\delta_r$ leads to a draw from the joint posterior $p(\beta^\delta_r | \sigma_{i_1}^2, \ldots, \sigma_{i_n}^2, y, f_r)$.

To derive (A.10), let $L_{i_i}$ be the $q_{i_i} \times q_{i_i}$ submatrix of $L$ corresponding to $\beta^\delta_{i_i}$. Evidently, $L_{i_i}$ is equal to the Cholesky decomposition of the individual information matrix $(B_{i_i,T}^\delta)^{-1}$. Furthermore, the $q_{i_i}$-dimensional sub vector $x_{i_i}$ corresponding to $\beta^\delta_{i_i}$, satisfies $L_{i_i}x_{i_i} = m_{i_i,T}^\delta$. Therefore:

$$x_i'x_i = (m_{i_i,T}^\delta)'(L_{i_i}'L_{i_i})^{-1}m_{i_i,T}^\delta = (m_{i_i,T}^\delta)'(L_{i_i}L_{i_i})^{-1}m_{i_i,T}^\delta = (m_{i_i,T}^\delta)'B_{i_i,T}^\delta m_{i_i,T}^\delta.$$  

It is easy to prove that the solution $\beta^\delta_r$ of (A.11) is a draw from the posterior $p(\beta^\delta_r | \sigma_{i_1}^2, \ldots, \sigma_{i_n}^2, y, f)$. Note that $L'L^\delta_r = Lx + Lz = m + Lz$. Therefore

$$\beta^\delta_r = (LL')^{-1}m + (LL')^{-1}Lz = P^{-1}m + (L')^{-1}z.$$  

Evidently, $E(\beta^\delta_r) = P^{-1}m$. Since for each $l = 1, \ldots, n$, $L_{i_i}^\delta_l = \sigma_{i_l}^2 L_{i_l}$, it holds that $LD = DL$ and therefore $DL^{-1} = L^{-1}D$. Since $V(\beta^\delta_r) = (L')^{-1}DL^{-1} = (L')^{-1}L^{-1}D = P^{-1}D$, it follows that $\beta^\delta_r \sim \mathcal{N}_d(P^{-1}m, P^{-1}D)$.

B Details on Step (D)

B.1 Marginal likelihoods when the factors are known

Although we work throughout this paper with a factor model where the factors $f_r^\Gamma$ are latent, several steps of Algorithm 1 perform model selection with respect to $\delta_r$ conditional on the most recent draw of the factors $f_r = (f_r^\Gamma, \ldots, f_r^T)$ in the confirmatory factor model (A.1). Hence, to sample new indicators $\delta_r$ in row $i$, the marginal likelihood $p(\tilde{y}_i | f_r, \delta_r)$ of regression model (A.2) is needed.

If $\delta_{i_1}$ is a zero row (i.e $q_{i_1} = 0$), then the marginal likelihood simplifies to

$$p(\tilde{y}_i | f_r, \delta_r) = p(\tilde{y}_i) = \frac{\Gamma(c_T^{\delta_l})}{(2\pi)^{T/2}\Gamma(c_0^{\delta_l})} \left(C_{i_1}^{\delta_l}ight)^{c_0^{\delta_l} - c_T^{\delta_l}},$$

(B.1)

where $c_T^{\delta_l}$ and $C_{i_1}^{\delta_l}$ are the posterior moments of $\sigma_{i_1}^2$ under the “null” model given by (A.3).

If at least one element of $\delta_{i_1}$ is different from zero, then the marginal likelihood computation differs between the hierarchical Gaussian prior (3.10) and the fractional prior (3.14).
**Marginal likelihoods for a hierarchical Gaussian prior.** For a hierarchical Gaussian prior, a well-known exercise in Bayesian regression analysis yields:

$$p(\tilde{y}_i|\delta_i, \beta, \sigma_t^2) = \frac{1}{(2\pi)^{T/2} |\mathbf{B}_{\delta_t}^\delta|^1/2 \Gamma(c_T)/(C_{i0})^{c_0}} \Gamma((c_0)^c_T)$$

(B.2)

where $\mathbf{B}_{\delta_t}^\delta$, $c_T$ and $C_{i0}^{\delta}$ are the posterior moments of $p(\beta^\delta_i, \sigma_t^2|\delta_i, \tilde{y}_i, \beta_r)$ given by (A.5).

**Marginal likelihoods for a fractional prior.** For a fractional prior, the derivation of the marginal likelihood if at least one element of $\delta_i$ is different from zero, is less standard and can be obtained in a similar way as in Frühwirth-Schnatter and Wagner (2010). A fraction $b$ of the full conditional likelihood of regression model (A.2) is used to define the fractional prior $p(\beta^\delta_i, \sigma^2_t|\sigma_t^2, b, \beta_r)$ in (3.14):

$$p(\tilde{y}_i|\beta^\delta_i, \sigma_t^2) = p(\tilde{y}_i|\beta^\delta_i, \sigma_t^2) = \frac{1-b}{2} p(\tilde{y}_i|\beta^\delta_i, \sigma_t^2) \propto p(\tilde{y}_i|\beta^\delta_i, \sigma_t^2)\frac{1-b}{2} p(\beta^\delta_i|\sigma_t^2, b, \beta_r).$$

The remaining part of the likelihood, that is $p(\tilde{y}_i|\beta^\delta_i, \sigma_t^2)\frac{1-b}{2}$, is used for model selection and is combined with the prior $p(\sigma_t^2)$ defined in (3.15) and the normalized fractional prior $p(\beta^\delta_i|\sigma_t^2, b, \beta_r)$, given by:

$$p(\beta^\delta_i|\sigma_t^2, b, \beta_r) = \frac{p(\tilde{y}_i|\beta^\delta_i, \sigma_t^2)\frac{1-b}{2}}{c_i(\sigma_t^2, \beta_r, b)}.$$

The normalizing constant $c_i(\sigma_t^2, \beta_r, b)$ is given by:

$$c_i(\sigma_t^2, \beta_r, b) = \int p(\tilde{y}_i|\beta^\delta_i, \sigma_t^2)\frac{1-b}{2} d\beta^\delta_i = \frac{1}{(2\pi)^{T/2} |\mathbf{B}_{\delta_t}^\delta|^1/2 \Gamma((c_0)^c_T)\exp\left(-\frac{b}{2\sigma_t^2}\right)}$$

where $\mathbf{B}_{\delta_t}^\delta$ and $\text{SSR}_i$ are the posterior moments of $p(\beta^\delta_i, \sigma_t^2|\delta_i, \tilde{y}_i)$ given by (A.6). Integrating the fractional posterior

$$p(\tilde{y}_i|\beta^\delta_i, \sigma_t^2)\frac{1-b}{2} p(\beta^\delta_i|\sigma_t^2, b, \beta_r)p(\sigma_t^2)$$

over $\beta^\delta_i$, yields the fractional likelihood $p(\tilde{y}_i|\beta^\delta_i, \sigma_t^2, b)$:

$$p(\tilde{y}_i|\beta^\delta_i, \sigma_t^2, b) = \int p(\tilde{y}_i|\beta^\delta_i, \sigma_t^2)\frac{1-b}{2} p(\beta^\delta_i|\sigma_t^2, b, \beta_r) d\beta^\delta_i = \frac{1}{c_i(\sigma_t^2, \beta_r, b)} \int p(\tilde{y}_i|\beta^\delta_i, \sigma_t^2)\frac{1-b}{2} d\beta^\delta_i$$

$$= \left(\frac{1}{2\pi\sigma_t^2}\right)^{T(1-b)/2} \frac{1}{b^{\frac{\mathbf{B}_{\delta_t}^\delta |1/2}} \exp\left(-\frac{1-b}{2\sigma_t^2}\right)\text{SSR}_i}$$

When we combine $p(\tilde{y}_i|\beta^\delta_i, \sigma_t^2, b)$ with the prior $p(\sigma_t^2)$, then we obtain:

$$p(\tilde{y}_i|\beta^\delta_i, \sigma_t^2, b)p(\sigma_t^2) = \frac{C_{i0}^{\delta}}{\Gamma(c_0)} \left(\frac{1}{2\pi}\right)^{T(1-b)/2} \frac{1}{b^{\frac{\mathbf{B}_{\delta_t}^\delta |1/2}} \exp\left(-\frac{1-b}{2\sigma_t^2}\right)\text{SSR}_i}$$

$$= \left(\frac{1}{2\pi\sigma_t^2}\right)^{T(1-b)/2} \frac{1}{b^{\frac{\mathbf{B}_{\delta_t}^\delta |1/2}} \exp\left(-\frac{1-b}{2\sigma_t^2}\right)\text{SSR}_i}$$

5
which is the kernel of the inverted Gamma distribution in (A.4). Integrating the right hand side with respect to $\sigma_j^2$ yields the marginal likelihood under the fractional prior:

\[ p(\tilde{y}_i|\delta_j, f_r) = \frac{b^{\delta_j/2}\Gamma(c_T)(C_{i0})^{c_0}}{(2\pi)^T(1-b)/2\Gamma(c_0)(C_{iT})^{c_T}}. \]  

(B.4)

\[ p(\tilde{y}_i|f_r) = \frac{b^{\delta_j/2}\Gamma(c_T)(C_{i0})^{c_0}}{(2\pi)^T(1-b)/2\Gamma(c_0)(C_{iT})^{c_T}}. \]  

(B.4)

**B.2 Multimove sampling of a set of indicators in a column**

Another important building block of MCMC inference for sparse Bayesian factor models is sampling all indicators $\delta_{ij}$ column $j$ for a set of rows $i \in I_j \subseteq \{1, \ldots, m\}$, conditional on the factors $f_r = (f_{r1}, \ldots, f_{rp})$, the remaining indicator $\delta_{ij}^u$ and the hyperparameter $\tau_j$, without conditioning on the model parameters $\beta$ and $\sigma_j^2, \ldots, \sigma_m^2$, see Step (D) of Algorithm 1.

According to the prior (3.2), the indicators $\delta_{ij}$ are independent apriori conditional on the hyperparameter $\tau_j$, with the log prior odds $O_{ij}^{pr}$ of $\delta_{ij} = 1$ versus $\delta_{ij} = 0$ being given by:

\[ O_{ij}^{pr} = \log \frac{\Pr(\delta_{ij} = 1|\tau_j)}{\Pr(\delta_{ij} = 0|\tau_j)} = \log \frac{\tau_j}{1 - \tau_j}. \]  

(B.5)

To sample $\delta_{ij}$ conditional $\delta_{ij}^u$ and $f_r$, without conditioning on $\beta$ and $(\sigma_1^2, \ldots, \sigma_m^2)$, the log posterior odds $O_{ij}^{post}$, given by

\[ O_{ij}^{post} = \log \frac{\Pr(\delta_{ij} = 1|\delta_{ij}^u, \tau_j, \tilde{y}_i, f_r)}{\Pr(\delta_{ij} = 0|\delta_{ij}^u, \tau_j, \tilde{y}_i, f_r)} = \log \frac{p(\tilde{y}_i|\delta_{ij} = 1, \delta_{ij}^u, f_r)}{p(\tilde{y}_i|\delta_{ij} = 0, \delta_{ij}^u, f_r)} + \log \frac{\Pr(\delta_{ij} = 1|\tau_j)}{\Pr(\delta_{ij} = 0|\tau_j)} = O_{ij} + O_{ij}^{pr}, \]  

(B.6)

is required which combines the log prior odds $O_{ij}^{pr}$ given in (B.5) with the log likelihood ratio $O_{ij}$, given by:

\[ O_{ij} = \log \frac{p(\tilde{y}_i|\delta_{ij} = 1, \delta_{ij}^u, f_r)}{p(\tilde{y}_i|\delta_{ij} = 0, \delta_{ij}^u, f_r)}. \]  

(B.7)

The likelihood ratio $O_{ij}$ is easily computed from the marginal likelihoods $p(\tilde{y}_i|\delta_{ij}, \delta_{ij}^u, f_r)$ where, respectively, $\delta_{ij} = 1$ and $\delta_{ij} = 0$. As discussed in Section B.1, these marginal likelihoods are available in closed form and marginal likelihood computation can be done individually for each row $i \in I_j$, separately for $\delta_{ij} = 0$ and $\delta_{ij} = 1$.

However, this procedure is likely to be inefficient, in particular, if the set $I_j$ is large. To achieve greater efficiency, Algorithm 4 outlined below provides a technique to compute directly the log likelihood ratio $O_{ij}$ (rather than the individual marginal likelihoods) *simultaneously* for all rows $i \in I_j$. This allows joint sampling of all indicators $\delta_{ij}$ in column $j$ for all rows $i \in I_j$.

The precise form of the log likelihood ratio $O_{ij}$ of $\delta_{ij} = 1$ versus $\delta_{ij} = 0$ defined in (B.7) depends on the remaining indicators $\delta_{i,j^-}$ in row $i$. The computation of $O_{ij}$ is easily vectorized for all rows $i \in I_j$. 

6
where all elements of $\delta_{i,-j}$ are zero. In this case, a model where observation $y_{it}$ is dedicated to factor $j$ ($\delta_{ij} = 1$) is compared to a model where $y_{it}$ is uncorrelated with all remaining observations ($\delta_{ij} = 0$). In this case, $O_{ij}$ is easily obtained from the marginal likelihood of a dedicated model with $j_i = j$ and the “null” model. As shown in Algorithm 4, it is possible (but less straightforward) to vectorize the computation of the log likelihood ratio $O_{ij}$ also for the remaining rows $i \in I_j$ where at least one element of $\delta_{i,-j}$ is different from zero.

**Algorithm 4. Multimove sampling of indicators in a column.** Sample all indicators $\delta_{ij}$ in column $\delta_{.j}$ jointly for all rows $i \in I_j \subseteq \{1, \ldots, m\}$ conditional on the factors $f_r = (f_1^r, \ldots, f_T^r)$, the remaining indicators $\delta^u_r$ and the hyperparameter $\tau_j$, without conditioning on the model parameters $\beta_r$ and $\sigma^2_1, \ldots, \sigma^2_m$ using the following steps:

(I-a) Compute the log likelihood ratio $O_{ij}$ for all rows $i \in I_j$ where all elements of $\delta^u_r$ are zero as

$$O_{ij} = \log \frac{p(\tilde{y}_i|\delta_{ij} = 1, \delta^u_r, f_r)}{p(\tilde{y}_i|\delta_{ij} = 0, \delta^u_r, f_r)} = \log \frac{\Gamma(c_T)(C^u_{iT})^c}{\Gamma(c^u_T)(C^u_{iT})^c_T} + D_{ij}. \quad (B.8)$$

$c^u_T$ and $C^u_{iT}$ are the posterior moments of the null model given in (A.3). $c_T$ and $C_{iT}$ are the posterior moments of $\sigma^2_i$ for a dedicated measurement with $j_i = j$, given in (A.8) and (A.9), respectively, for a hierarchical Gaussian and a fractional prior. For a hierarchical Gaussian prior, $D_{ij} = 0.5 \log(B_{iT}/B_{i0,jj})$, where $B_{i0,jj}$ is $j$th diagonal element of the prior variance $B_{i0}$ and $B_{iT}$ is the posterior scale factor for a dedicated measurement with $j_i = j$, given in (A.8). For a fractional prior, $D_{ij} = 0.5 \log(b(2\pi)^{bT})$.

This step is trivial to vectorize.

(I-b) For all rows $i \in \{i_1, \ldots, i_n\} \subseteq I_j$ where $\delta_{i,-j}$ is non-zero, compute

$$O_{ij} = \log \frac{p(\tilde{y}_i|\delta_{ij} = 1, \delta^u_r, f_r)}{p(\tilde{y}_i|\delta_{ij} = 0, \delta^u_r, f_r)} = c_T \log \frac{C^0_{iT}}{C^u_{iT}} + D_{ij}, \quad (B.9)$$

where $c_T$ and $C^u_{iT}$ are the posterior moments of $\sigma^2_i|\delta_{ij}, \cdot$ given in (A.4) and $C^0_{iT}$ refers to a model with $\delta_{ij} = 0$, while $C^1_{iT}$ refer to a model with $\delta_{ij} = 1$. For a hierarchical Gaussian prior,

$$D_{ij} = 0.5 \log(|B^1_{iT}|/|B^0_{iT}|) - 0.5 \log(|B^1_{i0}|/|B^0_{i0}|), \quad (B.10)$$

where $B^0_{i0}$ and $B^1_{i0}$ refer to the prior and posterior moments of $\beta^0_r|\delta_{ij}, \cdot$ given in (A.5). $B^1_{iT}$ and $B^1_{iT}$ refer to the prior and posterior moments for a model where $\delta_{ij} = 1$, while $B^0_{i0}$ and $B^0_{iT}$ refer to the prior and posterior moments for a model where $\delta_{ij} = 0$. For a fractional prior, $D_{ij} \equiv 0.5 \log b$.

Use Algorithm 5 to determine $C^1_{iT}, C^0_{iT}$, as well as $D_{ij}$ for a hierarchical Gaussian prior, simultaneously for all rows $i \in \{i_1, \ldots, i_n\} \subseteq I_j$. 

7
(I-c) Determine the vector of the log posterior odds $O^\text{post}_{ij} = O_{ij} + O_{ij}^p$ for all rows $i \in I_j$. Joint sampling of $\delta_{ij}^\text{new}|\tau_j, \cdot$ is easily vectorized:

(I-c1) Propose $\delta_{ij}^\text{new} = 1 - \delta_{ij}$ for $i \in I_j$.

(I-c2) Draw a vector of $|I_j|$ random variables $U_i \sim \mathcal{U} [0,1]$, indexed by $i \in I_j$.

(I-c3) For all rows $i \in I_j$, where $\delta_{ij} = 0$, accept the proposal $\delta_{ij}^\text{new} = 1$, iff $\log U_i \leq O^\text{post}_{ij}$

(I-c4) For all rows $i \in I_j$, where $\delta_{ij} = 1$, accept the proposal $\delta_{ij}^\text{new} = 0$, iff $\log U_i \leq -O^\text{post}_{ij}$.

Using, respectively, (B.2) and (B.4), the expression for $O_{ij}$ in (B.9) is easily derived. Since the indicators in column $j$ are independent given $\tau_j$, Step (I-c) is based on $|I_j|$ independent Metropolis-Hastings (MH) steps each of which proposes to update $\delta_{ij}$ by flipping the indicator, i.e. $\delta_{ij}^\text{new} = 1 - \delta_{ij}$. It easy to verify that the acceptance rules formulated in Step (I-c3) and (I-c4) are equivalent to the more conventional rule to accept $\delta_{ij}^\text{new}$ with probability

$$\min \left\{ 1, \frac{\Pr(\delta_{ij}^\text{new} | \delta_{ij}, \tau_j, \tilde{y}_i, \tilde{f}_r)}{\Pr(\delta_{ij} | \delta_{ij}, \tau_j, \tilde{y}_i, \tilde{f}_r)} \right\} = \min \left\{ 1, \exp(\log(O^\text{post}_{ij})) \right\}.$$

Algorithm 5. To compute all relevant posterior moments in (B.9) simultaneously for all rows $\{i_1, \ldots, i_n\}$, proceed as follows:

(a) Set the indicator $\delta_{i_1,j} = 1$ in each row $i_t \in \{i_1, \ldots, i_n\}$. Reorder the columns of the factor loading matrix in such a way, that the $j$th column appears last. This is simply done by permuting the column of $F$ appropriately before defining $X^\delta_{i_t}.^8$

(b) Set up the information matrix $P$ and the covector $m$ of the corresponding joint posterior of all nonzero factor loadings in the rows $i_1, \ldots, i_n$ as described in Algorithm 3. Compute the Cholesky decomposition $L$ of $P$ and the corresponding vector $x$ solving $Lx = m$.

(c) Knowing $L$ and $x$, a vectorized computation of the log likelihood ratio (B.8) for all rows $i_t \in \{i_1, \ldots, i_n\}$ is possible. The posterior moments $C^{1}_{i_t,T}$ are directly available from the appropriate sub vectors $x_{i_t}$ of $x$, defined in (A.10). When we switch from $\delta_{i_t,j} = 1$ to a model where $\delta_{i_t,j} = 0$, then for a hierarchical Gaussian prior,

$$C^{0}_{i_t,T} = C^{1}_{i_t,T} + \frac{1}{2} (x_{i_t}^*)^2. \quad (B.11)$$

Furthermore,

$$0.5 \log(\|B^{1}_{i_t,T}\|/\|B^{0}_{i_t,T}\|) = -\log L^*_{i_t}, \quad (B.12)$$

---

$^7$Alternatively, a Gibbs step may be used, i.e. set $\delta_{ij}^\text{new} = 1$, iff $\log(\hat{U}_i/(1 - \hat{U}_i)) \leq O^\text{post}_{ij}$, otherwise $\delta_{ij}^\text{new} = 0$. However, simulation experiments indicate that the MH step is more efficient.

$^8$While the fractional prior is not affected by this, it might be necessary to reorder the prior mean and the prior covariance matrix for a hierarchical Gaussian prior.
where \( L_{ii}^* = (L_i)_{q_i,q_i} \) is the last diagonal element of the submatrix \( L_{ii} \). Therefore,

\[
D_{ij} = -\log L_{ii}^* - 0.5 \log B_{i0,jj}.
\]

For the fractional prior,

\[
C_{ii,T}^0 = C_{ii,T}^1 + \frac{1 - b}{2} (x_{i}^*)^2,
\]

(B.13)

where \( x_{i}^* = (x_{i})_{q_i} \) is the last element of \( x_{i} \).

**Derivation of Step (c).** When we switch from a model where all indicator \( \delta_{i1,j} = \ldots = \delta_{in,j} = 1 \) are equal to one to a model where all indicators \( \delta_{i1,j} = \ldots = \delta_{in,j} = 0 \) are zero, then the information matrix \( P^0 \) and the covector \( m^0 \) of the joint posterior of the remaining nonzero factor loadings is obtained from \( P \) and \( m \) simply by deleting all rows and columns corresponding to \( \delta_{i1,j}, \ldots, \delta_{in,j} \), and the Cholesky decomposition \( L^0 \) of \( P^0 \) is obtained from \( L \) in the same way. Also the vector \( x^0 \) solving \( L^0 x^0 = m^0 \) is obtained from \( x \) simply by deleting the rows corresponding to \( \delta_{i1,j}, \ldots, \delta_{in,j} \). This last result is easily seen by considering the subsystem \( L_{ii} x_{ii} = m_{ii,T}^* \) corresponding to the \( i \)th row. Because

\[
L_{ii} = \begin{pmatrix} L_{ii}^0 & \mathbf{O} \\ \mathbf{I}_{ii} & (L_i)_{q_i,q_i} \end{pmatrix} = \begin{pmatrix} L_{ii}^0 & \mathbf{O} \\ \mathbf{I}_{ii} & L_{ii}^* \end{pmatrix},
\]

(B.14)

we obtain \( L_{ii}^0 x_{ii}^0 = m_{ii,T}^0 \), where \( x_{ii}^0 \) is obtained from \( x_{ii} \) by deleting the \( q_i \)th element \( x_{ii}^* = (x_{ii})_{q_i} \). Hence, \( x_{ii}^0 \) defines the desired subvector of \( x^0 \) to compute \( C_{ii,T}^0 \) as in (A.10). Since \( (x_{ii}^0)' x_{ii}^0 = x_{ii}' x_{ii} - (x_{ii})_{q_i} x_{ii} \) we obtain from (A.4) that (B.13) holds. Note, however, that this simple relationship would not hold without reordering the columns as described above.

Finally, to compute the log likelihood ratio for a hierarchical Gaussian prior, the ratio of the determinants \( |B_{ii,T}^1| / |B_{ii,T}^0| \) is required. Since the lower triangular matrices \( L_{ii} \) and \( L_{ii}^0 \) are, respectively, the Cholesky decomposition of \( (B_{ii,T}^1)^{-1} \) and \( (B_{ii,T}^0)^{-1} \), we obtain:

\[
1 / |B_{ii,T}^1|^{1/2} = |(B_{ii,T}^1)^{-1}|^{1/2} = |L_{ii}|,
\]

(B.15)

where \( |L_{ii}| \) is the product of the diagonal elements of \( L_{ii} \). Computing \( |B_{ii,T}^0| \) in the same way and using (B.14) proves (B.12).

### C Updating shrinkage parameters in Step (S)

Under a hierarchical Gaussian prior on the factor loadings \( \beta_{ij} \), Steps (P), (D) and (L) are performed conditional on all hyperparameters of this prior. All local, column specific and global scaling parameters in the prior of the loadings \( \beta_{ij} \) are updated in Step (S) of Algorithm 1. This step relies on a representation
of an F-distribution as a Gamma scale mixture of inverse gamma distributions and avoids the GIG-
distribution which was used in Cadonna et al. (2020).  

Step (S-a). Under prior (3.12), sample the local shrinkage parameters \( \omega_{ij} \) using the representation of
the F-distribution as a scale mixture of inverse gamma distributions:

\[
\omega_{ij} | b_{ij}^\omega \sim \mathcal{G}^{-1} \left( c_{\omega}, b_{ij}^\omega \right), \quad b_{ij}^\omega | \omega_{ij} \sim \mathcal{G} \left( a_{\omega}, \frac{a_{\omega}}{c_{\omega}} \right),
\]

This yields a two step sampler, where \( b_{ij}^\omega | \omega_{ij} \) is imputed from

\[
b_{ij}^\omega | \omega_{ij} \sim \mathcal{G} \left( a_{\omega} + c_{\omega}, \frac{a_{\omega}}{c_{\omega}} + 1/b_{ij}^\omega \right), \quad (C.4)
\]

and \( \omega_{ij} | b_{ij}^\omega, \beta_{ij}, \delta_{ij} \) given \( b_{ij}^\omega \) is sampled from

\[
\omega_{ij} | b_{ij}^\omega, \beta_{ij}, \delta_{ij} \sim \mathcal{G}^{-1} \left( c_{\omega} + \delta_{ij} \frac{1}{2} b_{ij}^\omega + \delta_{ij} \frac{\beta_{ij}^2}{2 \kappa \sigma\omega^2} \right). \quad (C.5)
\]

This has to be done for the entire matrix \( \omega_r \), as \( \omega_{ij} \) is needed to compute the odd likelihood ratio \( O_{ij} \) of \( \delta_{ij} = 1 \) versus \( \delta_{ij} = 0 \) (see (B.7)) also above the current pivot.

Step (S-b). Sample the column specific shrinkage parameters \( \theta_r \) (if any). For \( \theta_j \sim \mathcal{G}^{-1} \left( c_\theta, b_\theta \right) \), this yields for all \( j = 1 \) to \( r \):

\[
\theta_j | \kappa, \beta_r, \theta_\theta \sim \mathcal{G}^{-1} \left( c_\theta + \frac{d_j}{2}, b_\theta + \frac{1}{2 \kappa} \sum_{i: \delta_{ij} = 1} \beta_{ij}^2 \right). \quad (C.6)
\]

For \( \theta_j \sim \mathcal{F} \left( 2a_\theta, 2c_\theta \right) \), we use again the representation of the F-distribution as a scale mixture of inverse
gamma distributions:

\[
\theta_j | b_j^\theta \sim \mathcal{G}^{-1} \left( c_\theta, b_j^\theta \right), \quad b_j^\theta \sim \mathcal{G} \left( a_\theta, \frac{a_\theta}{c_\theta} \right),
\]

\[9\] A r.v. \( X \sim \mathcal{F} \left( 2a, 2c \right) \) has a representation as a Gamma scale mixture of inverse gamma distributions:

\[
X | b \sim \mathcal{G}^{-1} \left( c, b \right), \quad b \sim \mathcal{G} \left( a, \frac{a}{c} \right). \quad (C.1)
\]

Hence,

\[
b_j X \sim \mathcal{G} \left( a + c, \frac{a}{c} + 1/X \right). \quad (C.2)
\]

If \( Y | X \sim \mathcal{N} \left( 0, dX \right) \), then

\[
X | b, Y \sim \mathcal{G}^{-1} \left( c + \frac{1}{2} b + \frac{Y^2}{2d} \right). \quad (C.3)
\]

10
and impute $b_j^\theta|\theta_j$ from
\[
b_j^\theta|\theta_j \sim G\left(a^\theta + c^\theta, \frac{a^\theta}{c^\theta} + \frac{1}{\theta_j}\right),
\]
(C.7)
before we update $\theta_j|\kappa, \beta_r$ from (C.6).

**Step (S-c).** If $\kappa$ is random with prior $\kappa \sim G^{-1}(c^\kappa, b^\kappa)$, then $\kappa|\beta_r, \Sigma_r, \omega_r$ is updated from
\[
kappa|\beta_r, \Sigma_r, \omega_r \sim G^{-1}\left(e^\kappa + \frac{d}{2} b^\kappa + 1/2 \sum_{i=1}^m \frac{1}{\sigma_i^2} \sum_{j=1}^r \delta_{ij} \beta_{ij}^2 \omega_j \right),
\]
(C.8)
where $d = \sum_{i=1}^m \sum_{j=1}^r \delta_{ij}$ is the total number of non-zero loadings in the CFA model. If $\kappa \sim F(2a^\kappa, 2b^\kappa)$, then using the same representation of the F-distribution, we impute $b^\kappa|\kappa$ from
\[
b^\kappa|\kappa \sim G\left(a^\kappa + c^\kappa, \frac{a^\kappa}{c^\kappa} + \frac{1}{\kappa}\right),
\]
(C.9)
before we update $\kappa|\beta_r, \Sigma_r, \omega_r$ from (C.8).

**D  Details on split and merge in Step (R)**

**Proof of (4.3).** For any zero column $j_{sp}$ in a sparsity matrix $\delta_k$ with $r$ active and $r_{sp}$ spurious columns, there are $m - (r + r_{sp})$ unrestricted elements that can be subjected to variable selection and take the value one with probability $\tau_{j_{sp}}|\kappa \sim B(a_k, b_k)$ following the prior (3.2). Hence,
\[
\Pr(d_{j_{sp}} = 0 | r, r_{sp}) = \int (1 - \tau_{j_{sp}})^{m-(r+r_{sp})} p(\tau_{j_{sp}}) d\tau_{j_{sp}} = \frac{B(a_k, b_k + m - (r + r_{sp}))}{B(a_k, b_k)}
\]
(D.1)
whereas
\[
\Pr(d_{j_{sp}} = 1 | r, r_{sp}) = (m - (r + r_{sp})) \int \tau_{j_{sp}}(1 - \tau_{j_{sp}})^{m-(r+r_{sp})-1} p(\tau_{j_{sp}}) d\tau_{j_{sp}} = \frac{(m - (r + r_{sp})) B(a_k + 1, b_k + m - (r + r_{sp}) - 1)}{B(a_k, b_k)}.
\]
(D.2)
Therefore,
\[
\frac{\Pr(d_{j_{sp}} = 1 | r, r_{sp})}{\Pr(d_{j_{sp}} = 0 | r, r_{sp})} = \frac{a_k (m - r - r_{sp})}{b_k + m - r - r_{sp} - 1}.
\]
Proposing factors in a spurious column - Proof of (4.2). Whenever a zero column is turned into a spurious column (with columns index \(j_{sp}\)), factors \(f_{j_{sp}} = (f_{j_{sp},1}, \ldots, f_{j_{sp},T})\) are proposed, while holding the factors \(f_t^r, t = 1, \ldots, T,\) in all active columns fixed. Draws of \(f_{j_{sp},t}\) could be proposed from the prior \(f_{j_{sp},t} \sim \mathcal{N}(0, 1),\) since column \(j_{sp}\) was a zero column before splitting. However, with \(l_{sp}\) being the pivot row, \(y_{l_{sp},t}\) is a measurement that contains information about \(f_{j_{sp},t}\) in a spurious column and its likelihood can be combined with the prior to define the conditional posterior density \(p(f_{j_{sp},t} | f_t^r, \beta_r, \sigma_{l_{sp}}^2, y_{l_{sp},t})\) given \(y_{l_{sp},t}\) in addition to \(f_t^r, \beta_r, \sigma_{l_{sp}}^2, y_{l_{sp},t}.\)

It is easy to verify from the filter given in (4.1) that for a spurious column \(j_{sp}\) with leading element \(\Xi_{l_{sp}},\) the conditional density \(p(f_{j_{sp},t} | f_t^r, \beta_r, \sigma_{l_{sp}}^2, y_{l_{sp},t})\) of \(f_{j_{sp},t}\) is given by:

\[
V_{j_{sp}} = \left(1 + \frac{\Xi_{l_{sp}}}{(\sigma_{l_{sp}}^2)^{\text{new}}}ight)^{-1} \quad \text{where} \quad \Xi_{l_{sp}} = \frac{\sigma_{l_{sp}}^2}{\sigma_{l_{sp}}^2} - \frac{\sigma_{l_{sp}}^2}{\sigma_{l_{sp}}^2} \tilde{y}_{l_{sp},t},
\]

where the pseudo outcome \(\tilde{y}_{l_{sp},t}\) is given by \(\tilde{y}_{l_{sp},t} = y_{l_{sp},t} - \beta_{r,l_{sp}} f_t^r.\)

Turning spurious into active columns. In Step (R-D), variable selection is performed on the spurious columns of the current EFA model by marginalizing over the idiosyncratic variances and the factor loading matrix \((\beta_r, \Xi)\) as in Step (D) of Algorithm 1. By starting with the last spurious column, we ensure that the application of Step (D) is valid. Indeed, ordering the spurious pivots in Step (L) by size guarantees that all rows of the current EFA model below \(l_{sp}\) are equal to the corresponding rows of the current CFA model in the first \(r\) columns and zero in all other columns. Hence, updating the zero column \(r + 1\) below \(l_{sp}\) can be regarded as an attempt to introduce an additional active column in the current CFA model and to increase \(r,\) while \(r_{sp}\) decreases. If a column remains spurious, then we integrate in the current EFA model immediately over the corresponding factor, while the current number of spurious columns \(r_{sp}\) is preserved.

Step (R-Da) for hierarchical Gaussian priors with unknown shrinkage factors. Step (R-Da) has to be adjusted slightly for hierarchical Gaussian priors with unknown shrinkage factors. A column specific shrinkage parameter \(\theta_{j_{sp}}\) is needed to compute the odds of turning zero loadings in the spurious column into non-zero ones. In addition, under the shrinkage prior (3.12), local shrinkage parameters \(\omega_{i,j_{sp}}\) are needed for all rows \(i = 1, \ldots, m\) in the spurious column \(j_{sp}\) (note that \(\omega_{i,j_{sp}}\) will be needed later in
In this way, we exploit the information contained in Ξ where the scale parameter is obtained from combining (4.6) and (C.6): sampling θ \( r.v. U \) where \( b \) imputed from the prior. This is achieved by sampling \( \eta | \Xi, (\sigma^2_{\text{new}}) \) from (C.6):

\[
\theta_{\text{new}} | \kappa, U_{\text{new}} \sim \mathcal{G}^{-1} \left( c^\theta + \frac{1}{2} b^\theta, \frac{U^2_{\text{new}}}{2 \kappa (1 - U_{\text{new}}^2)} \right),
\]

where the scale parameter is obtained from combining (4.6) and (C.6):

\[
\frac{\beta^2_{\text{new}, jsp}}{(\Sigma_{r+1})_{\text{new}, lsp}} = \frac{\Xi^2_{\text{new}}}{(\sigma^2_{\text{new}, jsp})} = \frac{U^2_{\text{new}}}{1 - U_{\text{new}}^2}.
\]

Under the prior \( \theta_{\text{prior}} \sim \mathcal{G}^{-1} (c^\theta, b^\theta) \), the scale \( b^\theta_{\text{prior}} = b^\theta \), whereas under the F-prior on \( \theta_{\text{prior}} \), \( b^\theta_{\text{prior}} \) is imputed from the prior. This is achieved by sampling \( \theta_{\text{prior}} | \Xi_{\text{prior}}, (\sigma^2_{\text{prior}}) \) and \( b^\theta_{\text{prior}} | \theta_{\text{prior}} \) from (C.7):

\[
b^\theta_{\text{prior}} | \theta_{\text{prior}} \sim \mathcal{G} \left( a^\theta + c^\theta, \frac{a^\theta}{c^\theta} + \frac{1}{\theta_{\text{prior}}} \right).
\]

For the shrinkage prior (3.12), first local shrinkage parameter \( \omega_{i,jsp} \) are sampled from the prior for all rows, i.e. \( \omega_{i,jsp} \sim \mathcal{F}(2a^\omega, 2c^\omega), i = 1, \ldots, m \). The information contained in \( \Xi_{\text{prior}} \) and \( (\sigma^2_{\text{prior}})_{\text{new}} \) is then exploited by sampling \( \theta_{\text{prior}} | \Xi_{\text{prior}}, (\sigma^2_{\text{prior}})_{\text{new}}, \omega_{\text{prior}, jsp} \) from (C.6):

\[
\theta_{\text{prior}} | \kappa, U_{\text{prior}}, \omega_{\text{prior}, jsp} \sim \mathcal{G}^{-1} \left( c^\theta + \frac{1}{2} b^\theta, \frac{U^2_{\text{new}}}{2 \kappa \omega_{\text{prior}, jsp} (1 - U_{\text{new}}^2)} \right),
\]

where \( b^\theta_{\text{prior}} \) is defined as for the Student-t slab prior (3.11). Finally, we resample \( \omega_{\text{prior}, jsp} \) by first imputing \( b^\omega_{\text{prior}, jsp} \) from the prior by sampling \( b^\omega_{\text{prior}, jsp} | \omega_{\text{prior}, jsp} \) from (C.4):

\[
b^\omega_{\text{prior}, jsp} | \omega_{\text{prior}, jsp} \sim \mathcal{G} \left( a^\omega + c^\omega, \frac{a^\omega}{c^\omega} + \frac{1}{\omega_{\text{prior}, jsp}} \right),
\]

(while \( \omega_{\text{prior}, jsp} \) has been sampled from the prior in the first step) and then sampling from (C.5):

\[
\omega_{\text{prior}, jsp} | b^\omega_{\text{prior}, jsp}, \theta_{\text{prior}}, \kappa, U_{\text{prior}} \sim \mathcal{G}^{-1} \left( c^\omega + \frac{1}{2} b^\omega_{\text{prior}, jsp} + \frac{U^2_{\text{prior}}}{2 \kappa \theta_{\text{prior}} (1 - U_{\text{prior}}^2)} \right).
\]

In this way, we exploit the information contained in \( \Xi_{\text{prior}} \) and \( (\sigma^2_{\text{prior}})_{\text{new}} \) for both shrinkage parameters.
E  Step (L) - Updating the pivots

This subsection provides details on Step (L) of Algorithm 1, which was briefly discussed in Section 4.2. Four local moves are applied which were illustrated in Figure 1 in Section 4.2.

**Shifting the pivot.** A shift move is selected with probability $p_{\text{shift}}$. Let $l_*$ denote the index of the first nonzero row below $l_j$ (i.e. $\delta_{l_*,j} = 1, \delta_{ij} = 0, l_j < i < l_*$). If $l_* > 2$, then a new pivot $l_j^{\new}$ is proposed by sampling uniformly from the set $\mathcal{M}(l_*, l_{r,j}) = \{1, \ldots, l_* - 1\} \cap \mathcal{L}(l_{r,j})$, where $\mathcal{L}(l_{r,j})$ is the set of pivots outside of column $j$. If $\mathcal{M}(l_*, l_{r,j})$ is empty, then no shift move is performed. Otherwise, two indicators in column $j$ are changed, namely $\delta_{ij}^{\new}$ from zero to one and $\delta_{ij}^{\new}$ from one to zero, while the remaining elements of $\delta_r$ are unchanged. The new indicator matrix $\delta_r^{\new}$ is accepted with probability $\min(1, \alpha_{\text{shift}})$, where

$$\alpha_{\text{shift}} = \exp(O_{ij}^{\new} - O_{ij}) R_{\text{shift}},$$

with $O_{ij}$ being the log likelihood ratio of $\delta_{ij} = 1$ versus $\delta_{ij} = 0$ defined in (B.7). A shift move does not change the number of nonzero elements $d_j$ in column $j$ and the prior ratio (4.7) of this move simplifies to:

$$R_{\text{shift}} = \frac{B(a_k + d_j - 1, b_k + m - l_j^{\new} - d_j + 1)}{B(a_k + d_j - 1, b_k + m - l_j - d_j + 1)}. \quad (E.1)$$

Since $l_j^{\new}$ is sampled from a set $\mathcal{M}(l_*, l_{r,j})$ that does not depend on $l_j$, the proposal density is symmetric and the proposal ratio $q(l_j^{\new}|l_*, l_{r,j})/q(l_j|l_*, l_{r,j})$ cancels from $\alpha_{\text{shift}}$.

**Switching pivots.** This move is selected with probability $p_{\text{switch}}$ and is performed only if $r > 1$. A nonzero column $\ell \neq j$ is selected randomly and all indicators between (and including) row $l_j$ and $l_\ell$ that are different are switched between the two columns, i.e. $\delta_{ij}^{\new} = 1 - \delta_{ij}$ and $\delta_{i\ell}^{\new} = 1 - \delta_{i\ell}$ for all $i \in S_{j,\ell} = \{i : \min(l_\ell, l_j) \leq i \leq \max(l_\ell, l_j), \delta_{ij} \neq \delta_{i\ell}\}$. Evidently, this move switches pivots between the two columns. Since the corresponding proposal density satisfies $q(\delta_r^{\new}|\delta_r) = q(\delta_r|\delta_r^{\new})$, $\delta_r^{\new}$ is accepted with probability $\min(1, \alpha_{\text{switch}})$, where

$$\alpha_{\text{switch}} = \prod_{i \in S_{j,\ell}} p(\tilde{y}_i|\delta_{ij}^{\new}, \delta_{i\ell}^{\new}, \delta_r^{\new}, f_r) \frac{p(\tilde{y}_i|\delta_{ij}, \delta_{i\ell}, \delta_r, f_r)}{p(\tilde{y}_i|\delta_{ij}^{\new}, \delta_{i\ell}^{\new}, \delta_r^{\new}, f_r)} R_{\text{switch}},$$

where the prior odds ratio of this move is derived from (4.7):

$$R_{\text{switch}} = \prod_{i=j,\ell} \frac{B(a_k + d_i^{\new} - 1, b_k + m - l_i^{\new} - d_i^{\new} + 1)}{B(a_k + d_i - 1, b_k + m - l_i - d_i + 1)}. \quad (E.2)$$
If \( \delta_{ij} = 0 \) (and consequently \( \delta_{i\ell} = 1 \)), then we obtain:

\[
p(\tilde{y}_i|\delta_{ij}^{\text{new}} = 1, \delta_{i\ell}^{\text{new}} = 0, \delta_{jr}^{\text{new}}, f_r) \\
p(\tilde{y}_i|\delta_{ij} = 0, \delta_{i\ell} = 1, \delta_{jr}^{\text{new}}, f_r) \\
p(\tilde{y}_i|\delta_{ij}^{\text{new}} = 1, \delta_{i\ell}^{\text{new}} = 0, \delta_{jr}^{\text{new}}, f_r) \\
p(\tilde{y}_i|\delta_{ij} = 0, \delta_{i\ell} = 1, \delta_{jr}^{\text{new}}, f_r) = \exp(O_{ij|\ell} - O_{i\ell|j}),
\]

where \( O_{i,j|j_2} \) is the log likelihood ratio of \( \delta_{i,j_1} = 1 \) versus \( \delta_{i,j_1} = 0 \) provided that the indicator \( \delta_{i,j_2} = 0 \). It can be obtained as the likelihood ratio \( O_{i,j_1} \) given in (B.7), with \( \delta_{i,j_2} = 0 \) for both models. On the other hand, if \( \delta_{ij} = 1 \) (and consequently \( \delta_{i\ell} = 0 \)), then

\[
p(\tilde{y}_i|\delta_{ij}^{\text{new}} = 0, \delta_{i\ell}^{\text{new}} = 1, \delta_{jr}^{\text{new}}, f_r) \\
p(\tilde{y}_i|\delta_{ij} = 1, \delta_{i\ell} = 0, \delta_{jr}^{\text{new}}, f_r) \\
p(\tilde{y}_i|\delta_{ij}^{\text{new}} = 0, \delta_{i\ell}^{\text{new}} = 1, \delta_{jr}^{\text{new}}, f_r) \\
p(\tilde{y}_i|\delta_{ij} = 1, \delta_{i\ell} = 0, \delta_{jr}^{\text{new}}, f_r) = \exp(O_{i\ell|j} - O_{ij|\ell}).
\]

Therefore

\[
\alpha_{\text{switch}} = R_{\text{switch}} \exp \left( \sum_{i \in S_j, \delta_{ij} = 0} (O_{ij|\ell} - O_{i\ell|j}) + \sum_{i \in S_j, \delta_{ij} = 1} (O_{i\ell|j} - O_{ij|\ell}) \right). \tag{E.3}
\]

This move allows changes in \( d_j \) and \( d_\ell \), but leaves the overall number \( d \) of nonzero elements unchanged.

**Adding or deleting a pivot.** Finally, a reversible pair of moves is selected with probability \( 1 - p_{\text{shift}} - p_{\text{switch}} \). The add move introduces a new pivot \( l_j^{\text{new}} \) in a row above the current pivot \( l_j \) which is not occupied by the pivots of the other columns. Hence, \( l_j^{\text{new}} \) is selected randomly from the set \( \mathcal{A}(l_j, 1_{r,-j}) = \{1, \ldots, l_j - 1\} \cap \mathcal{L}(1_{r,-j}) \), i.e. \( \delta_{i,j_2}^{\text{new}} = 1 \), while the remaining elements of \( \delta_r \) are unchanged (in particular \( \delta_{ij, j}^{\text{new}} = \delta_{ij, j} = 1 \)). An add move is only possible, if \( |\mathcal{A}(l_j, 1_{r,-j})| > 0 \).

The corresponding reverse move is deterministic and deletes the current pivot \( l_j \), turning \( l_j^{\text{new}} = l_* \) into the new pivot where \( l_* \) is the row index of the first nonzero element in \( \delta_r \) below \( l_j \). Hence, \( \delta_{i,j_2}^{\text{new}} = 0 \), while all other elements of \( \delta_r \) remain unchanged. A delete move is only performed, if \( l_* \) is not a pivot in any other column of \( \delta_r \) (that is \( l_* \in \mathcal{L}(1_{r,-j}) \)).

If neither an add nor a delete move is possible, then \( l_j \) remains unchanged. Otherwise, either an add or a delete move is selected with probability \( p_{\text{add}}(\delta_r) \) and \( 1 - p_{\text{add}}(\delta_r) \). If both add and delete moves are possible, then \( p_{\text{add}}(\delta_r) = p_\alpha \), with \( p_\alpha \) being a tuning parameter; if only an add move is possible, then \( p_{\text{add}}(\delta_r) = 1 \), whereas \( p_{\text{add}}(\delta_r) = 0 \), if only a delete move is possible. Note that whenever an add move is selected, the reverse delete move is always possible; and similarly, the reverse add move is

---

\(^{10}\)The number of rows in \( \mathcal{A}(l_j, 1_{r,-j}) \) is equal to \( l_j - z_j \), where \( z_j = \# \{ l_j' \in 1 : l_j' < l_j \} \) is the rank of \( l_j \) among the leading indices. Hence, an add move is possible, whenever \( l_j > z_j \).
always possible, whenever a delete move is selected. This move changes $d_j$ and increases or decreases the overall number of nonzero elements by one.

The acceptance probability for an add move is equal to $\min(1, \alpha_{\text{add}})$, with

$$\alpha_{\text{add}} = \exp(O_{ij} R_{\text{add}} \frac{|A(l_{j}, l_{r}, -j)| (1 - p_{\text{add}}(\delta_{r}^{\text{new}}))}{p_{\text{add}}(\delta_{r})}),$$

where $O_{ij}$ is the log likelihood ratio of $\delta_{ij} = 1$ versus $\delta_{ij} = 0$ defined in (B.7). Since $d_{j}^{\text{new}} = d_{j} + 1$, the prior ratio (4.7) of this move simplifies to:

$$R_{\text{add}} = \frac{B(a_{k} + d_{j} + b_{k} + m - l_{j}^{\text{new}} - d_{j})}{B(a_{k} + d_{j} - 1, b_{k} + m - l_{j} - d_{j} + 1)}.$$  \hspace{1cm} (E.4)

The acceptance probability for a delete move is equal to $\min(1, \alpha_{\text{del}})$, with

$$\alpha_{\text{del}} = \exp(-O_{ij} R_{\text{del}} \frac{p_{\text{add}}(\delta_{r}^{\text{new}})}{|A(l_{j}^{\text{new}}, l_{r}, -j)| (1 - p_{\text{add}}(\delta_{r})}).$$

Since $d_{j}^{\text{new}} = d_{j} - 1$, the prior ratio (4.7) of this move simplifies to:

$$R_{\text{del}} = \frac{B(a_{k} + d_{j} - 2, b_{k} + m - l_{j}^{\text{new}} - d_{j} + 2)}{B(a_{k} + d_{j} - 1, b_{k} + m - l_{j} - d_{j} + 1)}.$$  \hspace{1cm} (E.5)

**Tuning parameters.** These four moves involve three tuning probabilities, namely $p_{\text{shift}}$, $p_{\text{switch}}$, and $p_{a}$, with $1 - p_{\text{shift}} - p_{\text{switch}} > 0$ and $0 < p_{a} < 1$.

**F Details on Step (H)**

In this section, we provide details on updating the hyperparameters $\alpha$ and $\gamma$ for the 2PB prior (3.4) without conditioning on the slab probabilities $\tau_{1}, \ldots, \tau_{K}$. Inference is performed conditionally on the current columns sizes $d_{1}, \ldots, d_{k}$ and the number of spurious columns $r_{sp}$ in the EFA model and the current pivots in the CFA model:

$$p(\alpha, \gamma|\cdot) \propto p(\alpha)p(\gamma) \prod_{j: d_{j} > 1} \Pr(\delta_{r,j}^{\text{e}}|\alpha, \gamma, l_{j}) \prod_{j: d_{j} \leq 1} \Pr(d_{j}|\alpha, \gamma),$$

where $\alpha \sim \mathcal{G}(a^{\alpha}, b^{\alpha})$ and $\gamma \sim \mathcal{G}(a^{\gamma}, b^{\gamma})$.

The conditional posterior $p(\alpha, \gamma|\cdot)$ is based on the likelihood (3.8) for active columns, the likelihood
(D.2) for spurious columns and the likelihood (D.1) for zero columns:

\[ p(\alpha, \gamma | \cdot) \propto \frac{p(\alpha)p(\gamma)}{B(\frac{\alpha\gamma}{k}, \frac{\gamma}{k})} \cdot \prod_{j: d_j > 1} B\left(\frac{\alpha\gamma}{k} + d_j - 1, \frac{\gamma}{k} + m - l_j - d_j + 1\right) \cdot \prod_{j_{sp}=1}^{r_{sp}} B\left(\frac{\alpha\gamma}{k} + 1, \frac{\gamma}{k} + m - r - j_{sp}\right). \]

An MH-step based on a random walk proposal for, respectively, \( \log \alpha \) and \( \log \gamma \) is implemented to sample \( p(\alpha | \gamma, \cdot) \) and \( p(\gamma | \alpha, \cdot) \).

G More about boosting in Step (A)

Algorithm 6 described a generic boosting strategy in Step (A) in Algorithm 1.

**Algorithm 6 (Boosting MCMC).** Step (A) in Algorithm 1 is implemented in three steps:

(A-a) Choose a (current) value \( \Psi \) and move from the CFA model (2.4) to the expanded model (4.8) using the following transformation:

\[ \tilde{f}_r^i = (\Psi)_{1/2} f_r^i, \quad \tilde{\beta}_r = \beta_r(\Psi)^{-1/2}. \quad (G.1) \]

(A-b) Sample a new value \( \Psi^{new} \) in model (4.8) conditional on \( \tilde{f}_r = (\tilde{f}_r^1, \ldots, \tilde{f}_r^T) \) and \( \tilde{\beta}_r \) from the conditional posterior \( p(\Psi | \tilde{f}_r, \tilde{\beta}_r) \) given by

\[ p(\Psi | \tilde{f}_r, \tilde{\beta}_r) \propto p(\Psi)p(\tilde{\beta}_r | \Psi) \prod_{j=1}^{r} \Psi_j^{-T/2} \exp \left\{ -\frac{1}{2} \sum_{i=1}^{T} \tilde{f}_{jt}^2 \right\}. \quad (G.2) \]

(A-c) Move back from model (4.8) to the CFA model (2.4) using the inverse of transformation (G.1) for \( j = 1, \ldots, r \):

\[ \beta_{ij}^{new} = \beta_{ij} \sqrt{\Psi_j^{new} / \Psi_j}, \quad i = 1, \ldots, m, \quad f_{jt}^{new} = f_{jt} \sqrt{\Psi_j^{new} / \Psi_j}, \quad t = 1, \ldots, T. \]

Boosting affects the factor loading matrix \( \beta_r \) and all factors \( f_r = (f_r^1, \ldots, f_r^T) \), see Step (A-c). The main difference between ASIS and MDA lies in the choice of the current value of \( \Psi \) in Step (A-a). While \( \Psi_j \) is chosen in a deterministic fashion for ASIS, typically involving a specific factor loading \( \beta_{nj,j} \) in each column, \( \Psi_j \) is sampled from a working prior for MDA. This leads to different priors \( p(\Psi) \) and \( p(\tilde{\beta}_r | \Psi) \) in (G.2).

Note that in (D.2), \( r_{sp} \) is the number of spurious columns excluding column \( j_{sp} \). Hence, the likelihood is computed sequentially from the first to the last spurious column.
**ASIS for fractional priors.** For boosting based on ASIS, a nonzero factor loading \( \beta_{nj,j} \) is chosen in column \( j \) to define the current value of \( \Psi_j \) as \( \sqrt{\Psi_j} = \beta_{nj,j} \). This creates a factor loading matrix \( \tilde{\beta}_r \) in the expanded model (4.8) where \( \tilde{\beta}_{nj,j} = 1 \) whereas \( \tilde{\beta}_{i,j} = \beta_{ij}/\beta_{nj,j} \) for \( i \neq n_j \). \( \beta_{nj,j} \) can be chosen as the pivot element in each column, i.e. \( n_j = l_j \), or such that \( |\beta_{nj,j}| \) is maximized for all loadings in column \( j \). Apart from this choice, ASIS requires no further tuning. The implied prior of \( \Psi_j = \beta_{nj,j}^2 \) is given by \( p(\Psi_j) \propto \Psi_j^{-1/2} \), whereas \( p(\tilde{\beta}_r|\Psi_j) \propto \prod_{j=1}^r \Psi_j(d_j - 1)/2 \), since \( p(\tilde{\beta}_{ij}|\sigma_r^2, \Psi_j, \delta_{ij} = 1) \propto \Psi_j^{1/2} \) for all \( i \neq n_j \) with \( \delta_{ij} = 1 \). Hence, \( p(\Psi|\tilde{\beta}_r, \beta_{r}) \) defined in (G.2) reduces to a product of inverted Gamma distribution where for all \( j = 1, \ldots, r \):

\[
\Psi_j^{\text{new}}|\tilde{\beta}_r, \beta_r \sim G^{-1}\left(\frac{T - d_j/2}{2}, \frac{\beta_{nj,j}^2}{2} \sum_{t=1}^T f_{jt}^2\right),
\]

where \( d_j = \sum_{i=1}^m \delta_{ij} \) counts the nonzero elements in column \( j \) (which is at least equal to 2 by definition of \( \delta_r \)).

**MDA for fractional priors.** In MDA, the current value \( \Psi \) is drawn from a working prior \( p(\Psi) \) which is independent both of \( \beta_r \) and \( \Sigma_r \). This guarantees that the prior distribution of \( \beta_r \) remains unchanged, despite moving between the two models. Below, an inverted Gamma working prior \( \Psi_j \sim G^{-1}(\nu_j, q_j) \) is applied. The prior in the expanded model reads \( p(\tilde{\beta}_r|\Psi_j) \propto \prod_{j=1}^r \Psi_j^{d_j/2} \), since \( p(\tilde{\beta}_{ij}|\sigma_r^2, \Psi_j, \delta_{ij} = 1) \propto \Psi_j^{1/2} \) for all \( i \) with \( \delta_{ij} = 1 \). The two last terms in (G.2) factor into a product of independent inverted Gamma distributions which leads to an inverted Gamma posterior for each \( \Psi_j \):

\[
\Psi_j^{\text{new}}|\Psi_j, \tilde{\beta}_r \sim G^{-1}\left(\nu_j - d_j/2 + T/2, q_j + \Psi_j/2 \sum_{t=1}^T f_{jt}^2\right).
\]

**Boosting for hierarchical Gaussian priors.** Column boosting is based on interweaving \( \theta_j \) into the state equation by choosing \( \Psi_j = \theta_j \). This leads to a prior for the loadings \( \tilde{\beta}_{ij} \) which is independent of \( \theta_j \), while \( \theta_j \) acts as variance of the \( j \)th factors in the expanded model, \( \tilde{f}_{jt} / \theta_j \sim N(0, \theta_j) \). Hence, in Step (A), under the inverse gamma prior \( \theta_j \sim G^{-1}\left(c^\theta, b^\theta\right) \), \( \theta_j \) is updated in the expanded model,

\[
\theta_j^{\text{new}}|\theta_j, \tilde{\beta}_r \sim G^{-1}\left(c^\theta + T/2, B_j\right), \quad B_j = b^\theta_j + 1/2 \sum_{t=1}^T f_{jt}^2 = b^\theta_j + \theta_j/2 \sum_{t=1}^T f_{jt}^2.
\]

Moving back, all factors and factor loadings in the CFA model are updated:

\[
\beta_{ij}^{\text{new}} = \frac{\sqrt{\theta_j^{\text{new}}}}{\sqrt{\theta_j}} \tilde{\beta}_{ij}, \quad f_{jt}^{\text{new}} = \tilde{f}_{jt} / \sqrt{\theta_j^{\text{new}}} = \sqrt{\theta_j} \tilde{f}_{jt} / \sqrt{\theta_j}. \]
A boosting step that we found to be useful for hierarchical Gaussian priors in addition to column boosting is interweaving the global shrinkage parameter $\kappa$ into the prior $\theta_j \sim \mathcal{G}^{-1}\left(c^\theta, b_j^\theta\right)$ of the shrinkage parameter $\theta_j$ through the transformation $\tilde{\theta}_j = \kappa \theta_j$:

$$
\beta_{ij}\delta_{ij} = 1, \omega_{ij}, \sigma_i^2 \sim \mathcal{N}\left(0, \tilde{\theta}_j \omega_{ij} \sigma_i^2\right),
$$

$$
\tilde{\theta}_j | \kappa \sim \mathcal{G}^{-1}\left(c^\theta, \kappa b_j^\theta\right), \quad \kappa \sim \mathcal{G}^{-1}\left(c^\kappa, b^\kappa\right).
$$

The global shrinkage parameter $\kappa$ is then updated in Step (A) under this representation, combining the likelihood

$$
p(\tilde{\theta}_1, \ldots, \tilde{\theta}_r | \kappa) \propto \kappa^{r \alpha} \exp\left(-\kappa \sum_{j=1}^r \frac{b_j^\theta}{\tilde{\theta}_j}\right),
$$

which is the kernel of a gamma density in $\kappa$ with the inverse gamma prior $p(\kappa)$.

This yields a generalized inverse Gaussian distribution\footnote{The generalized inverse Gaussian distribution, $Y \sim \mathcal{GIG}(p, a, b)$, is a three-parameter family of probability distribution with support $y \in \mathbb{R}^+$. The density is given by $f(y) = \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})} y^{p-1} e^{-\left((a/2)y - b/(2y)\right)}$, where $K_p(z)$ is the modified Bessel function of the second kind, $a > 0, b > 0$ and $p$ is a real parameter.} to update $\kappa_{\text{new}}$:

$$
\kappa_{\text{new}} | \theta_1, \ldots, \theta_r, \kappa \sim \mathcal{GIG}\left(r c^\theta - c^\kappa, \frac{2}{\kappa} \sum_{j=1}^r \frac{b_j^\theta}{\theta_j}, 2b^\kappa\right).
$$

Given $\kappa_{\text{new}}$, the column specific parameters are updated as

$$
\theta_{j_{\text{new}}} = \frac{\tilde{\theta}_j}{\kappa_{\text{new}}} = \frac{\kappa}{\kappa_{\text{new}}} \theta_j, \quad j = 1, \ldots, r.
$$

**H Initialising Algorithm 1**

To check the mixing of the MCMC chain, two (or more) independent runs with different initial dimensions $r$ of the CFA model and a positive number $r_{sp}$ of spurious columns are performed. On the one hand, $r$ is chosen to be smaller than the expected number of factors, on the other hand a large value close to $k - r_{sp}$ is chosen.

Initial values for the leading indices $l_r = (l_1, \ldots, l_r)$ of the sparsity matrix $\delta_r$ are chosen by first sampling $l_1$ from $\{1, \ldots, u_1\}$, where $u_1$ is a small number, e.g. 5. Then for $j = 2, \ldots, r$, we sample $l_j$ uniformly from the set $\mathcal{L}(1-j)$ with $1-j = \{l_1, \ldots, l_{j-1}\}$. Factor loadings below the leading elements are allowed to be zero with positive probability $p_0$, e.g. $p_0 = 0.5$. We draw at most 100 initial values $\delta_r$.
Table I.1: Exchange rate data; currency abbreviations.

|   |   |   |
|---|---|---|
| 1 | AUD | Australia dollar |
| 2 | CAD | Canada dollar |
| 3 | CHF | Switzerland franc |
| 4 | CZK | Czech R. koruna |
| 5 | DKK | Denmark krone |
| 6 | GBP | UK pound |
| 7 | HKD | Hong Kong dollar |
| 8 | IDR | Indonesia rupiah |
| 9 | JPY | Japan yen |
|10 | KRW | South Korea won |
|11 | MXN | Mexican Peso |
|12 | MYR | Malaysia ringgit |
|13 | NOK | Norway krone |
|14 | NZD | New Zealand dollar |
|15 | PHP | Philippines peso |
|16 | PLN | Poland zloty |
|17 | RON | Romania fourth leu |
|18 | RUB | Russian ruble |
|19 | SEK | Sweden krona |
|20 | SGD | Singapore dollar |
|21 | THB | Thailand baht |
|22 | USD | US dollar |

...and continue...
Table I.2: Exchange rate data; inclusion probabilities for the sparsity matrix $\delta_4$ for the 2PB prior, averaged over all ordered GLT draws with pivots equal to $1^* = (1, 2, 5, 7)$.

| Currency | Factor 1 | Factor 2 | Factor 3 | Factor 4 |
|----------|----------|----------|----------|----------|
| AUD      | 1        | 0        | 0        | 0        |
| CAD      | 1        | 1        | 0        | 0        |
| CHF      | 0.01     | 0.12     | 0        | 0        |
| CZK      | 0        | 0.2      | 0        | 0        |
| DKK      | 0.01     | 1        | 1        | 0        |
| GBP      | 0.04     | 1        | 0.05     | 0        |
| HKD      | 0.01     | 1        | 0.97     | 1        |
| IDR      | 0.02     | 1        | 0.03     | 1        |
| JPY      | 0.09     | 1        | 0.01     | 0.02     |
| KRW      | 0        | 1        | 0.07     | 0.02     |
| MXN      | 0        | 0.16     | 0.01     | 0.02     |
| MYR      | 1        | 0.06     | 0.01     | 0.01     |
| NOK      | 0        | 1        | 0.01     | 0.03     |
| NZD      | 0.06     | 0.45     | 0.04     | 0.02     |
| PHP      | 0        | 1        | 0.94     | 0.04     |
| PLN      | 0        | 1        | 0.02     | 0.75     |
| RON      | 0.1      | 0.07     | 0.25     | 0.01     |
| RUB      | 0.21     | 0.1      | 0.1      | 0.16     |
| SEK      | 0        | 1        | 0.01     | 1        |
| SGD      | 0.02     | 1        | 0.03     | 0.99     |
| THB      | 0        | 1        | 0.01     | 0.02     |
| USD      | 0.01     | 1        | 1        | 0.01     |

have relatively high communalities for the different currencies, whereas for zero rows the communalities are practically equal to zero percent.
Table I.3: Exchange rate data; posterior mean of the factor loadings \( \Lambda_{ij} \), the communalities \( R_{ij}^2 \) (in percent) and the idiosyncratic variances \( \sigma_i^2 \) (2PB prior) for a 4-factor model with the GLT constraint \( I^* = (1, 3, 5, 7) \).

| Currency | \( \Lambda_{i1} \) | \( \Lambda_{i2} \) | \( \Lambda_{i3} \) | \( \Lambda_{i4} \) | \( R_{i1}^2 \) | \( R_{i2}^2 \) | \( R_{i3}^2 \) | \( R_{i4}^2 \) | \( \sigma_i^2 \) |
|----------|------------------|------------------|------------------|------------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| AUD      | 0.962            | 0                | 0                | 0                | 88.4            | 0               | 0               | 0               | 0.12            |
| CAD      | 0.391            | 0.601            | 0                | 0                | 16.9            | 38.6            | 0               | 0               | 0.42            |
| CHF      | \( \approx 0 \)  | -0.014           | 0                | 0                | 0.01            | 0.3             | 0               | 0               | 0.98            |
| CZK      | \( \approx 0 \)  | 0.034            | 0                | 0                | 0.01            | 0.8             | 0               | 0               | 0.98            |
| DKK      | \( \approx 0 \)  | 1.06             | 0.22             | 0                | \( \approx 0 \) | 95.3            | 4.1             | 0               | 0.01            |
| GBP      | 0.01             | 0.569            | -0.01            | 0                | 0.26            | 31.4            | 0.27            | 0               | 0.71            |
| HKD      | \( \approx 0 \)  | 0.502            | 0.398            | 0.762            | \( \approx 0 \) | 21.8            | 14.2            | 49.4            | 0.17            |
| IDR      | \( \approx 0 \)  | 0.8              | \( \approx 0 \)  | 0.427            | 0.05            | 57.2            | 0.06            | 16.9            | 0.28            |
| JPY      | 0.013            | 0.924            | \( \approx 0 \)  | \( \approx 0 \)  | 0.23            | 75.6            | \( \approx 0 \) | 0.01            | 0.27            |
| KRW      | \( \approx 0 \)  | 1.06             | \( \approx 0 \)  | \( \approx 0 \)  | \( \approx 0 \) | 95.9            | 0.04            | \( \approx 0 \) | 0.05            |
| MXN      | \( \approx 0 \)  | 0.025            | \( \approx 0 \)  | \( \approx 0 \)  | \( \approx 0 \) | 0.58            | 0.04            | 0.04            | 0.98            |
| MYR      | 0.793            | \( \approx 0 \)  | \( \approx 0 \)  | \( \approx 0 \)  | 61.2            | 0.04            | 0.01            | 0.01            | 0.40            |
| NOK      | \( \approx 0 \)  | 0.88             | \( \approx 0 \)  | \( \approx 0 \)  | \( \approx 0 \) | 69.7            | 0.01            | 0.05            | 0.33            |
| NZD      | 0.016            | 0.108            | -0.01            | \( \approx 0 \)  | 0.50            | 3.17            | 0.29            | 0.04            | 0.95            |
| PHP      | \( \approx 0 \)  | 0.549            | -0.415           | \( \approx 0 \)  | 0.01            | 28.7            | 17.9            | 0.17            | 0.56            |
| PLN      | \( \approx 0 \)  | 0.995            | \( \approx 0 \)  | 0.128            | \( \approx 0 \) | 85.6            | 0.01            | 2.02            | 0.14            |
| RON      | 0.027            | \( \approx 0 \)  | \( \approx 0 \)  | -0.08            | \( \approx 0 \) | 0.87            | 0.09            | 3.08            | 0.02            | 0.95 |
| RUB      | -0.067           | 0.012            | 0.029            | 0.043            | 2.37            | 0.28            | 0.95            | 1.37            | 0.94            |
| SEK      | \( \approx 0 \)  | 0.974            | \( \approx 0 \)  | 0.31             | \( \approx 0 \) | 81.8            | 0.01            | 8.7             | 0.11            |
| SGD      | \( \approx 0 \)  | 0.748            | \( \approx 0 \)  | 0.392            | 0.05            | 50.9            | 0.05            | 14.6            | 0.38            |
| THB      | \( \approx 0 \)  | 0.583            | \( \approx 0 \)  | \( \approx 0 \)  | 0.01            | 32.7            | 0.01            | 0.05            | 0.70            |
| USD      | \( \approx 0 \)  | 1.06             | 0.215            | \( \approx 0 \)  | \( \approx 0 \) | 95.3            | 4.1             | \( \approx 0 \)  | 0.01            |

Note: entries with \(|\Lambda_{ij}| < 0.01\) and entries with \(R_{ij}^2 < 0.1\) are indicated by \(\approx 0\).