Universality of hypercubic random surfaces

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Abstract

We study universality properties of the Weingarten hyper-cubic random surfaces. Since a long time the model of hypercubic random surfaces with a local restriction forbidding surface self-bendings was thought to be in a different universality class from the unrestricted model defined on the full set of surfaces. In this paper we show that both models in fact belong to the same universality class with the entropy exponent $\gamma = 1/2$ and differ by the finite size effects which are much more pronounced in the restricted model.
The model of hyper-cubic surfaces was proposed by Weingarten as a non-perturbative regularization of the world sheet of bosonic strings. It was proven that in general the string susceptibility exponent $\gamma$ is equal to $1/2$ for spherical surfaces, reflecting the fact that the ensemble is dominated by surfaces having the structure of branched polymers. It was also generally believed on grounds of the renormalization group arguments that one cannot change the exponent $\gamma$ by adding local terms to the action. Therefore it came as a surprise when results of the numerical simulations of a slightly modified model, where self-bendings surfaces were excluded from the partition function, gave $\gamma = 1/4$ in the case of a surface embedded in four dimensions.

This result was subject of much speculation, because it provided the first non-trivial value, besides the mean field result $\gamma = 1/3$, in the series of positive $\gamma$ discussed by Durhuus in [3]. On the other hand the result is not strictly in conflict with the analytic proof, as discussed in [3].

In this article we present new measurements of the exponent $\gamma$. Since the earlier measurements [3, 6] significant progress has been made both in computer speed and in the development of numerical techniques for simulations of random geometries. Combining this allows us to go to systems two orders of magnitude larger than those used in the old measurements, and to gather very good statistics. Moreover we have at our disposal a very powerful method for extracting the exponent $\gamma$ based on the analysis of the baby-universe distribution.

For this work we developed a new algorithm, which we will describe in a separate article [7]. This algorithm is capable to include self-bendings, although for most of the data presented in this letter they were suppressed by adding additional code. By reusing the $\gamma$ extraction method described in [3] we show, that the results generated by our algorithm for surfaces of the size used in that work are in agreement with the results generated by their algorithm [8]. However, as we will show our results on larger surfaces lead to completely different conclusions, namely that the exponent $\gamma$ is consistent with $1/2$, although with considerably stronger finite size corrections than in the case of unrestricted surfaces.

The model is defined by the partition function:

$$Z(\beta) = \frac{1}{S} \sum_{S} C(S) e^{-\beta A} = \sum_{A} \mathcal{N}(A) e^{-\beta A}$$

(1)

where the first sum runs over all hypercubic surfaces without self-bendings and fixed (spherical) topology, and the second runs over the number of plaquettes $A$, i.e. the surface area. The internal symmetry $C(S)$ is important mainly for very small surfaces.

In the large volume limit, the number of surfaces with a given area $A$ is expected to behave like

$$\mathcal{N}(A) \approx e^{\mu_c A} A^{\gamma - 3}$$

(2)

Our main interest is focused on the universal entropy exponent $\gamma$, which also characterizes the fractal structure related to baby-universes.
To extract this exponent, the simulations are organized in a quasi–canonical way. The area of the system fluctuates but the measurements are taken at a fixed value \(A_0\). We achieve this by adding an external quadratic potential \(\gamma (A - A_0)^2\) to the exponent in the partition function \(Z\). The optimal value of the parameter \(\beta\) is determined experimentally in an iterative procedure which after each run estimates an improved value of \(\beta\) from the \(A\) distribution.

To measure \(\gamma\) we use the baby universe method \(\cite{10}\). This method is based on the observation that the distribution of baby universes is independent of the critical value \(\mu_c\). By baby universe we mean a part of the surface which has the topology of a disc and is connected in such a way that the square of the disc perimeter \(l^2\) is much smaller than the disc area \(B\): \(l^2 \ll B\). When one looks at such a short loop from far enough away one sees it as a point to which two surfaces are pinned: one with area \(B\) and the other one, the complementary part of the whole surface, with area \(A - B\). For large \(B\) and \(A - B\) the correlations of the parts of the surface on the two sides of the loop are negligible and one can approximate the baby universe distribution by the formula \(\cite{11}\):

\[
N_A(B) \sim N_l(B)N_l(A - B)
\]

where \(N_l(A)\) is number of discs with the loop \(l\) and area \(A\). The number \(N_l(A)\) can be expressed by the number of spherical surfaces \(N(A)\). Take for example \(l\) as the perimeter of the square. We have then

\[
N_l(A) = (A + 1)N(A + 1).
\]

This means that one can make such a disk by removing one square from the lattice. Generally, for short loops one then has

\[
N_l(A) \sim AN(A)\{1 + o(1/A)\}.
\]

In our measurements we use loops of length two and four, being, respectively, the length of a double link and a square, \(i.e\). the smallest possible objects.

Using the approximate formula \(\cite{3}\) we get

\[
N_A(B) \sim (B(A - B))^{-2}
\]

for the distribution of minibus with area \(B\) in a canonical ensemble with a total area \(A\). The exponential factor \(e^{\beta B}\) becomes a normalization factor independent of the minibus area \(B\). Taking the logarithm of both sides and defining

\[
\begin{align*}
y &= \ln (B(A - B))^2 N_A(B) \\
x &= \ln B(A - B)
\end{align*}
\]

one can extract \(\gamma\) by fitting to the linear equation \(y = \gamma x + b\).

In fig(1) we show a typical minbus distribution, where we plot \(y\) versus \(x\). The total surface area is \(A = 10000\). For small values of \(B\) the data points do not lie on a single line but wiggle around. This effect comes from the combination of the symmetry factor and the surface entropy. For large values of \(B\) one can see
strong statistic noise which comes from the fact that the frequency of appearance of the large baby universes is very small. For example, for the surface $A = 10000$ one has $7 \times 10^{-4}$ baby universes of area $B = 4999$ per surface.

Due to finite size effects the curve is not quite a straight line. Therefore one actually measures an effective $\gamma_{\text{eff}}$, which should converge to $\gamma$ with larger areas $A$. When estimating the value of $\gamma$ one has to make decision which part of the distribution to use. We extract $\gamma$ from this distribution by looking at its behavior as a function of the lower cutoff, i.e. the smallest minbu-area used to fit $\gamma$. As the value of $\gamma$ we take the plateau-value, as one can see for example in figure (2) for $A = 10000$. It is important to note that the plateau value sets in for baby universes larger than 1000.

We analyzed the minbu-distribution for different canonical volumes. In figure (3) we show $\gamma_{\text{eff}}$ as a function of the canonical volume $A$. For comparison we also performed simulations for the ensemble containing self–bendings. As discussed above, one can for this case show analytically that $\gamma = \frac{1}{2}$. For the model without self–bendings (lower curve) one clearly observes stronger finite size effects. The large area behavior of $\gamma_{\text{eff}}$, however, is not affected.

This observation is different from the results of the numerical experiment reported in [3]. The reported value of $\gamma = 0.26$ can be found in figure (3) for volumes $A \approx 200$ which in fact correspond to the values used in that work. The method used there is based on the ratio of two shifted histograms of the area distribution $f(A)$ for spherical surfaces obtained for different coupling constants in the action $\beta A$. As before this eliminates the unknown parameter in the
Figure 2: The effective $\gamma_{\text{eff}}$ as a function of the lower cut-off.

Figure 3: The effective entropy exponent $\gamma_{\text{eff}}$ as a function of the canonical volume $A$ for ensembles respectively including (upper curve) or excluding (lower curve) self-bendings.
exponent which disappears as it is independent of $A$. Thus one ends up with
the formula:

$$\frac{f(A)}{f(A+\delta)} = \text{const.} \times \left(\frac{A}{A+\delta}\right)^{\gamma-3} \exp((\beta_1 - \beta_2))A$$

(8)

where the histograms were measured for two different sizes $A$ and $A+\delta$. The
method is similar to the baby universe method in the sense that one gets rid of
the inconvenient unknown exponential prefactor. Here, however, the price one
has to pay is much higher because one has to use two independent histograms,
which introduces a large statistical error.

We have repeated measurements of $\gamma$ using this method and following pa-
rameters:

$$\begin{array}{ll}
\beta & A \\
1.170 & [170, 280] \\
1.160 & [4890, 5000] \\
1.165 & [4890, 5000] \\
1.170 & [4890, 5000]
\end{array}$$

These are very similar to those used in the work \[3\]. For each coupling we
collect ten histograms, therefore we have 300 possibilities of calculating (8). The
resulting distribution of $\gamma$ - values is is quite broad and turns out to react rather
sensitive to a change of the surface–areas involved in the procedure. With the
above parameters we get $\gamma_{eff} = 0.26(10)$, which is in very good agreement with
the results presented in \[3\]. This value is different from the branched polymer
value 1/2 because $\gamma$ is measured at small volumes, where the finite size effects
hide the real value, as can be clearly seen in figure (3).

To summarize, we have measured the value of the string susceptibility ex-
ponent $\gamma$ for hypercubic surfaces embedded in $d = 4$-dimensional target space
for the ensemble of spherical surfaces with and without self–bending hinges. In
both cases the results are compatible with the branched polymer value $\gamma = 1/2$.
This means that the local infinite coupling to the external geometry, which
suppresses self–bendings, is not sufficient to modify the fractal structure of the
surface. These results do not rule out the possibility that one can influence
the fractal structure with a generalized local coupling to the external geometry.
Obviously with an infinite coupling to the external curvature the surface has to
be (up to finite size effects) entirely flat, i.e. the Hausdorff-dimension is $d_h = 2$
in this case. This value is different from $d_h = 4$, the value obtained for the
model investigated in this work. Therefore the fractal structure is affected. It
remains to find out, if a transition appears at finite couplings.

We have developed a very general implementation of our new algorithm,
which allows to study more general actions controlling self–bendings or other
local terms in the action. Moreover, the program can be run already in the
target space dimension two. We will present results of these simulations in a
forthcoming publication.

We want to thank Joachim Tabaczek and Gudmar Thorleifsson for discussion. We are grateful to the HLRZ Juelich for computer time on the PARAGON,
where parts of the simulations were performed. Furthermore we thank the DFG
under grant PE340/3–3 for support. ZB would like to thank the Polish Committee for Scientific Research under the grant no. 2P03B19609.

References

[1] D. Weingarten, Phys. Lett. B90 (1980) 280
[2] B. Durhuus, J. Fröhlich, T. Johnsson, Nucl. Phys. 240B (1980) 453
[3] B. Baumann, B. Berg, Phys. Lett. 164B (1985) 131
   B. Baumann, B. Berg, G. Münster, Nucl. Phys. B305 (1988)
[4] J. Ambjørn, Nucl. Phys. Proc. Suppl. 42 (1995) 3
[5] B. Durhuus, Nucl. Phys. B426 (1994) 203
[6] H. Kawai, Y. Okamoto, Phys. Lett. B130 (1983) 415
[7] S. Bilke, Z. Burda, In Preparation
[8] B. Berg, A. Billoire, D. Foerster, Nucl. Phys. B251 (1985) 665
   B. Baumann, Nucl. Phys. B285 (1987) 391
[9] S. Bilke, Z. Burda, B. Petersson, Phys. Lett. B395 (1997) 4
[10] J. Ambjørn, S. Jain, G. Thorleifsson, Phys. Lett. B 307 (1993)
[11] S. Jain, S. Mathur, Phys. Lett. 286B (1992) 239