BATALIN-VILKOVISKY ALGEBRA STRUCTURE ON POISSON MANIFOLDS WITH SEMI-SIMPLE MODULAR SYMMETRY

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Abstract. We study the “twisted” Poincaré duality of smooth Poisson manifolds, and show that, if the modular vector field is semi-simple (or say, diagonalizable), then there is a mixed complex associated to the Poisson complex, which, combining with the twisted Poincaré duality, gives a Batalin-Vilkovisky algebra structure on the Poisson cohomology. This generalizes the previous results obtained by Xu for uni-modular Poisson manifolds. We also show that the Batalin-Vilkovisky algebra structure is preserved under Kontsevich’s deformation quantization, and in the case of polynomial algebras it is also preserved by Koszul duality.

Keywords: modular vector field, Poincaré duality, Koszul duality, deformation quantization, Batalin-Vilkovisky

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1. Introduction

Let $X$ be a smooth, oriented Poisson manifold. Let $A$ be the algebra of smooth functions on $X$. The Poisson cohomology and homology of $A$, and hence of $X$, were introduced by Lichnerowicz [24] and Koszul [18] respectively. They were further studied by, for example, Brylinski [3], Huebschmann [15] and Xu [39]. In particular, Xu found that there is an obstruction for the existence of the Poincaré duality between the Poisson cohomology and homology of $X$. Such an obstruction lies in the first Poisson cohomology of $X$, called the modular class, and is represented by the modular vector field of the Poisson structure. If the obstruction vanishes, in which case $X$ is called uni-modular, then we have the Poincaré duality on $X$. As a corollary, he showed that there exists a Batalin-Vilkovisky algebra structure on the Poisson cohomology, which is nontrivial in general, in the sense that the Batalin-Vilkovisky operator generates the Schouten bracket.
The purpose of this paper is to generalize Xu’s result to a class of Poisson manifolds with non-trivial modular class, and then to study some algebraic structures, such as the Batalin-Vilkovisky algebra structure among others, associated to them.

1.1. Poincaré duality for Poisson manifolds. In 1998, Van den Bergh studied in \[36\] the Poincaré duality problem for associative algebras. For an associative algebras, say \(A\), Van den Bergh showed that if \(A\) is homologically smooth, then there is an isomorphism between the Hochschild cohomology of \(A\) and the Hochschild homology of \(A\) with values in \(A\) tensor with its inverse dualizing complex. If the inverse dualizing complex is trivial, in which case the algebra is called Calabi-Yau, then we have the Poincaré duality between the Hochschild cohomology and homology of \(A\).

In some cases that we are interested in, the associative algebras, such as the Artin-Schelter regular (AS-regular for short) algebras, are not Calabi-Yau, but are very close to be so. Inspired by noncommutative differential geometry Brown and Zhang studied the “twisted” Hochschild homology of an AS-regular algebra, say \(A\), and showed that Van den Bergh’s Poincaré duality has the form (see \[2\] and also \[30\])

\[
HH^\bullet(A) \cong HH_{n-\bullet}(A, A^\sigma),
\]

where \(HH^\bullet(A)\) is the Hochschild cohomology of \(A\) while \(HH_{\bullet}(A, A^\sigma)\) is the Hochschild homology of \(A\) with coefficients in \(A\) twisted with its Nakayama automorphism, and \(n\) is the global dimension of \(A\). In this case we say \(A\) admits the twisted Poincaré duality.

Going back to the Poisson algebra case, the twisted Poincaré duality was first studied by Launois and Richard \[22\] for some quadratic Poisson algebras, which was later generalized by Zhu in \[41\] and Luo, Wang and Wu in \[27\]. In 2017 Lü, Wang and Zhuang obtained in \[26\] the twisted Poincaré duality theorem for Poisson Calabi-Yau affine varieties, which covers all the above cases. It turns out this result is also a special case of the more general Poincaré duality theorem of Lie-Rinehart algebras, which was developed by Huebschmann in \[16\] (see Appendix A for more details).

**Theorem 1.1** (Theorem 2.6; see also \[16\] \[27\]). Let \(X\) be a smooth and oriented Poisson \(n\)-manifold with a fixed volume form. Let \(A\) be the ring of smooth functions on \(X\) and \(\nu\) be the modular vector field. Let \(A^\nu\) be \(A\) itself twisted with \(\nu\) (see Example 2.2(2) below for the precise definition). Then we have the twisted Poincaré duality

\[
HP^\bullet(A) \cong HP_{n-\bullet}(A, A^\nu),
\]

where \(HP^\bullet(-)\) and \(HP_{\bullet}(-)\) are the Poisson cohomology and homology functors.

1.2. The Batalin-Vilkovisky algebra structure. For an associative algebra \(A\), the Hochschild complex \((CH_\bullet(A), b, B)\) is a mixed complex, where \(b\) is the Hochschild differential and \(B\) is Connes’ cyclic operator. But for algebras with an automorphism \(\sigma\) such as AS-regular algebras as above, the Hochschild complex that we are interested in is \(CH_\bullet(A, A^\sigma)\), which does not admit a mixed complex structure since Connes’ cyclic operator does not commute with \(b\) unless \(\sigma = \text{Id}\) (that is, \(A\) is Calabi-Yau). Nevertheless, there is a special class of AS-regular algebras which do have a mixed complex structure on its
(sub but homotopy equivalent) twisted Hochschild complex. These are the AS-regular algebras whose Nakayama automorphism is diagonalizable, which is also called semi-simple in the literature. In this case Kowalzig and Krahmer showed in [19] that these algebras share even more features of Calabi-Yau algebras; for example, their Hochschild cohomology has a nontrivial Batalin-Vilkovisky algebra structure (in the Calabi-Yau case this is proved by Ginzburg in [14]). Batalin-Vilkovisky algebra arose from physics, especially from string field theory and topological conformal field theory (see, for example, [12]). They have been widely studied in recent years, by both physicists and mathematicians.

Going back to the Poisson algebra case, the situation is similar. Suppose $A$ is a Poisson algebra with nontrivial modular vector field $\nu$, then in general the twisted Poisson complex $(\text{CP}\bullet(A, A_\nu), \partial, d)$ is not a mixed complex, where $\partial$ is the Poisson boundary and $d$ is the de Rham differential. If we view the modular vector field as the infinitesimal version of the Nakayama automorphism, then in the semi-simple case, we again have a mixed complex structure on the (sub but homotopy equivalent) twisted Poisson complex. Together with the twisted Poincaré duality, the pair $(\text{HP}^\bullet(A), \text{HP}_{n-\bullet}(A, A_\nu))$ form the so-called differential calculus with duality (a notion introduced by Lambre [20] based on Tamarkin and Tsygan [34]), which leads to the following theorem and generalizes Xu’s result in [39] where only unimodular Poisson manifolds are considered.

**Theorem 1.2** (Theorems 3.10). Let $X$ be a smooth and oriented Poisson $n$-manifold with semi-simple modular vector field. Let $A$ the algebra of smooth functions on $M$. Then $\text{HP}^\bullet(A)$ has a Batalin-Vilkovisky algebra structure, whose Batalin-Vilkovisky operator generates the Schouten bracket on $\text{HP}^\bullet(A)$.

1.3. **Koszul duality, and deformation quantization.** For a quadratic Poisson polynomial algebra, Shoikhet [31] showed that its Koszul dual is graded Poisson and Tamarkin’s deformation quantizations of these two Poisson algebras, one is AS-regular and the other is Frobenius, are again Koszul dual to each other as graded associative algebras. Later this result is proved to be true for Kontsevich’s deformation quantization by Calaque et al. [5].

On the other hand, for an arbitrary Poisson polynomial algebra, Dolgushev [10] proved that its deformation quantization is an AS-regular algebra; in particular, if the Poisson algebra is unimodular, then its deformation quantization is Calabi-Yau.

Based on these results among others, it is shown in [8, 9] that for a unimodular Poisson algebra $A = \mathbb{R}[x_1, \ldots, x_n]$, if we denote by $A^!$ the Koszul dual algebra of $A$, and by $A_h$ and $A^!_h$ the deformation quantizations of $A$ and $A^!$ respectively, then the following diagram

\[
\begin{array}{ccc}
\text{HP}^\bullet(A[[\hbar]]) & \xrightarrow{\cong} & \text{HP}^\bullet(A^![[\hbar]]) \\
\downarrow & & \downarrow \\
\text{HH}^\bullet(A_h) & \xrightarrow{\cong} & \text{HH}^\bullet(A^!_h)
\end{array}
\]

is commutative as isomorphisms of Batalin-Vilkovisky algebras.

In this paper we show that the above result remains true if the modular vector field of the Poisson algebra is semi-simple.
Theorem 1.3 (Theorems 6.8). Let $A = \mathbb{R}[x_1, \cdots, x_n]$ be a Poisson algebra with semi-simple modular vector field. Let $A^!$ be the Koszul dual of $A$, and let $A_\hbar$ and $A^!_\hbar$ be the deformation quantization of $A$ and $A^!$ respectively. Then the following

\[
\begin{array}{ccc}
HP^\bullet(A[\hbar]) & \cong & HP^\bullet(A^!_\hbar) \\
\downarrow & & \downarrow \\
HH^\bullet(A_\hbar) & \cong & HH^\bullet(A^!_\hbar)
\end{array}
\]

is a commutative diagram of isomorphisms of Batalin-Vilkovisky algebras.

We remark that the Batalin-Vilkovisky algebra structures for AS-regular algebras and for Frobenius algebras with semi-simple Nakayama were independently proved by Kowalzig and Krahmer [19] and Lambre, Zhou and Zimmermann [21] respectively; their isomorphism in the Koszul case was proved by [25]. We recently learned that Wang, Wu, Zhou and Zhu have obtained the same Batalin-Vilkovisky algebra structure for Poisson algebras with a semi-simple vector vector field in an unpublished manuscript [37]. What is new in above theorem is that we study these algebraic structures in the category of Poisson algebras, and relate them via deformation quantization; it also answers a question raised in [8, §7.3] where the authors asked whether these algebraic structures exist for Poisson algebras admitting the twisted Poincaré duality.

The rest of the paper is devoted to the proof of the above theorems. It is organized as follows. In §2 we study with some details the modular vector field of Poisson algebras and then study the twisted Poincaré duality for Poisson manifolds. In §3 we study the Batalin-Vilkovisky algebra structure on the Poisson cohomology of Poisson algebras with semi-simple modular vector field. In §4 we show that the Batalin-Vilkovisky algebra structure is preserved under deformation quantization. In §5 we study the Koszul duality of quadratic Poisson algebras, which are Frobenius Poisson algebras; we then study their twisted Poincaré duality as well as their deformation quantization. In §6 we combine the above results and show Theorem 1.3. In §6.4 we also discuss an algebraic structure (the gravity algebra) on the negative cyclic homology of Poisson algebras with semi-simple modular vector fields. In the Appendix we deduce Theorem 1.1 from Huebschmann’s Poincaré duality theorem for Lie-Rinehart algebras.

Notation. Throughout this paper, $k$ denotes a field of characteristic 0. All tensors and Homs are over $k$ unless otherwise specified. All algebras (resp. coalgebras) are unital and augmented (resp. co-unital and co-augmented) over $k$. If $A$ is an associative algebra, then $A^{\text{op}}$ is its opposite and $A^e = A \otimes A^{\text{op}}$ is its envelope. All complexes are graded such that the differential has degree $-1$; for a cochain complex, it is viewed as a chain complex by negating the grading, and it is cohomology $H^\bullet(-)$ is given by $H_{-\bullet}(-)$ of its negation.

2. Modular class and the Poincaré duality

In this section, we briefly go over the modular vector fields for Poisson algebras, and discuss twisted Poincaré duality for Poisson manifolds. The main results are Theorems 2.6.
In this paper, a Poisson algebra $A$ with the Poisson structure $\pi$ is denoted by $(A, \pi)$, or by $(A, \cdot, \{-, -\})$.

**Definition 2.1.** Suppose $A$ is a Poisson $k$-algebra. A left Poisson $A$-module is a $k$-vector space $M$ endowed with two bilinear maps $\cdot$ and $\{-, -\}_M : A \otimes M \to M$ such that

1. $(M, \cdot)$ is a left module over the commutative algebra $A$;
2. $(M, \{-, -\}_M)$ is a left module over the Lie algebra $(A, \{-, -\})$;
3. $\{a, bx\}_M = \{a, b\} \cdot x + b \cdot \{a, x\}_M$ for any $a, b \in A$ and $x \in M$;
4. $\{ab, x\}_M = a \cdot \{b, x\}_M + b \cdot \{a, x\}_M$ for any $a, b \in A$ and $x \in M$.

The notion of right Poisson $A$-module is defined similarly, and is left to the reader. A left Poisson $A$-module is not necessarily a right Poisson $A$-module; however, for a right Poisson $A$-module $M$, if we denote its Lie action by $\{-, -\}_M$, then it may be equipped with a left Poisson $A$-module, whose Lie action is given by $a \otimes m \mapsto -\{m, a\}$, for all $a \in A$ and $m \in M$, and vice versa. A Poisson $A$-bimodule is both a left and a right Poisson $A$-module such that $\{a, m\}_M = -\{m, a\}_M$ for all $a \in A$ and $m \in M$. In particular, $A$ itself is automatically a Poisson $A$-bimodule.

**Example 2.2.** (1) Suppose $M$ is a right (and respectively left) Poisson module over $A$. Then its linear dual space $M^* := \text{Hom}_k(M, k)$ has a left (and respectively right) Poisson module structure over $A$, with the dot product and the bracket adjoint to the product and the bracket on $M$. In particular, $A^* := \text{Hom}_k(A, k)$ is both a right and a left Poisson $A$-module (in fact, a Poisson $A$-bimodule).

(2) Suppose $(M, \cdot, \{-, -\}_M)$ is a right Poisson $A$-module. Let $\nu \in \mathfrak{X}^1(A)$ be a Poisson derivation; that is, a derivation of $A$ which commutes with the Poisson structure. Define a new bracket $\{-, -\}_{M\nu} : M \otimes A \to M$ by

$$\{m, a\}_{M\nu} = \{m, a\}_M + m \cdot \nu(a),$$

for all $a \in A, m \in M$. Then $(M, \cdot, \{-, -\}_{M\nu})$ is again a right Poisson $A$-module, called the twisted Poisson $A$-module twisted by the Poisson derivation $\nu$; in what follows, we denote it by $M\nu$. Similarly, for a left Poisson $A$-module, we denote the corresponding twisted Poisson $A$-module by $\nu M$.

**Definition 2.3** (Lichnerowicz [24]). Suppose $(A, \pi)$ is a Poisson algebra and $M$ is a left Poisson $A$-module. Let $\mathfrak{X}^p_A(M)$ be the space of skew-symmetric multilinear maps $A^{\otimes p} \to M$ that are derivations in each argument; that is, the space of $p$-th polyvectors on $A$ with values in $M$. The Poisson cochain complex of $A$ with values in $M$, denoted by $\text{CP}^*(A, M)$, is the cochain complex

$$M = \mathfrak{X}_A^0(M) \overset{\delta}{\rightarrow} \cdots \overset{\delta}{\rightarrow} \mathfrak{X}_A^{p-1}(M) \overset{\delta}{\rightarrow} \mathfrak{X}_A^p(M) \overset{\delta}{\rightarrow} \mathfrak{X}_A^{p+1}(M) \overset{\delta}{\rightarrow} \cdots$$

where $\delta$ is given by

$$\delta^p(P)(f_0, f_1, \cdots, f_p) := \sum_{0 \leq i \leq p} (-1)^i \{f_i, P(f_0, \cdots, \hat{f}_i, \cdots, f_p)\}$$

$$+ \sum_{0 \leq i < j \leq p} (-1)^{i+j} P(\{f_i, f_j\}, f_0, \cdots, \hat{f}_i, \cdots, \hat{f}_j, \cdots, f_p),$$
and \( \sim \) means the corresponding item is omitted. The associated cohomology is called the \textit{Poisson cohomology} of \( A \) with values in \( M \), and is denoted by \( \text{HP}^\bullet(A; M) \). In particular, if \( M = A \), then the cohomology is just called the \textit{Poisson cohomology} of \( A \), and is simply denoted by \( \text{HP}^\bullet(A) \).

**Definition 2.4 (Koszul [18]).** Suppose \((A, \pi)\) is a Poisson algebra and \( N \) is a right Poisson \( A \)-module. Denote by \( \Omega^p_A(N) \) the set of \( p \)-th Kähler differential forms of \( A \) with coefficients in \( N \). Then the Poisson chain complex of \( A \) with coefficients in \( N \), denoted by \( \text{CP}^\bullet(A, N) \), is

\[
\cdots \longrightarrow \Omega^{p+1}_A(N) \overset{\partial_n}{\longrightarrow} \Omega^p_A(N) \overset{\partial_n}{\longrightarrow} \Omega^{p-1}_A(N) \overset{\partial_n}{\longrightarrow} \cdots \longrightarrow \Omega^0_A(N) = N,
\]

where \( \partial_n \) is given by

\[
\partial_n(n \otimes df_1 \wedge \cdots \wedge df_p) = \sum_{i=1}^{p} (-1)^{i-1} \{ n, f_i \} \otimes df_1 \wedge \cdots \wedge \hat{df}_i \cdots \wedge df_p + \sum_{1 \leq i < j \leq p} (-1)^{i+j} n \otimes d\{ f_i, f_j \} \otimes df_1 \wedge \cdots \wedge \hat{df}_i \cdots \wedge \hat{df}_j \cdots \wedge df_p,
\]

where \( n \in N \) and \( f_1, \ldots, f_n \in A \). The associated homology is called the \textit{Poisson homology} of \( A \) with coefficients in \( N \), and is denoted by \( \text{HP}^\bullet(A, N) \). In particular, if \( N = A \), then the homology is just called the \textit{Poisson homology} of \( A \), and is simply denoted by \( \text{HP}^\bullet(A) \).

In what follows, if \( \pi \) is clear from the text, we simply write \( \delta^\pi \) and \( \partial_\pi \) as \( \delta \) and \( \partial \) respectively. It should be noted that in both definitions, \( \delta^\pi \) and \( \partial_\pi \) are in fact the Lie derivative on a smooth Poisson manifold, or of the algebraic functions on a Poisson affine variety, then \( \text{HP}^\bullet(A) \) and \( \text{HP}^\bullet(A) \) are Poisson invariants of \( X \).

Suppose \( \nu \in \mathfrak{X}^1(A) \) is a Poisson derivation, then the chain complex \( \text{CP}^\bullet(A, A_\nu) \) has the same underlying vector space as \( \text{CP}^\bullet(A, A) \) but with the boundary, which we now denote by \( \partial_\nu \) in order to distinguish, now becomes

\[
\partial_\nu(f_0 \otimes df_1 \wedge \cdots \wedge df_n) = \partial(f_0 \otimes df_1 \wedge \cdots \wedge df_n) + \sum_{i=1}^{n} (-1)^{i-1} f_0 \nu(f_i) \otimes df_1 \wedge \cdots \wedge \hat{df}_i \cdots \wedge df_n,
\]

where \( \partial \) is the boundary on \( \text{CP}^\bullet(A, A) \).

Now suppose we have an \( n \)-form \( \eta \in \Omega^n(A) \) such that the contraction

\[
\iota_{(-)\eta} : \mathfrak{X}^\bullet_A(A) \to \Omega^{-\bullet}_A(A), \quad X \mapsto \iota_X \eta
\]

is an isomorphism, then we say \( \eta \) is a \textit{volume form} of degree \( n \). If such a form \( \eta \) exists, then we have the following diagram

\[
\begin{array}{ccc}
\mathfrak{X}^\bullet_A(A) & \overset{\iota_{(-)\eta}}{\longrightarrow} & \Omega^{-\bullet}_A(A) \\
\delta \downarrow & & \partial \\
\mathfrak{X}^{\bullet+1}_A(A) & \overset{\iota_{(-)\eta}}{\longrightarrow} & \Omega^{-\bullet-1}_A(A)
\end{array}
\]
which is not necessarily commutative, since \( \eta \) may not be a Poisson cycle. To adjust this
discrepancy, let us consider the following commutative diagram

\[
\begin{array}{ccc}
\mathfrak{X}_A^\bullet(A) & \xrightarrow{\iota(-)\eta} & \Omega^n_A^\bullet(A) \\
\text{Div} & \downarrow & \downarrow d \\
\mathfrak{X}_A^{n-1}(A) & \xrightarrow{\iota(-)\eta} & \Omega^{n-\bullet+1}_A(A),
\end{array}
\]

where \text{Div} is the divergence operator. Then

\[ \nu := -\text{Div}(\pi) \]

is a vector field, and is called the \textit{modular vector field} for \( A \). With these notations, we
have the following proposition, which is due to Xu (see [39, Proposition 4.7]):

\textbf{Proposition 2.5 (Xu).} Suppose \((A, \pi)\) is a Poisson algebra and \( \eta \) is a volume form. Then
for any \( \varphi \in \mathfrak{X}_A^p(A) \), we have

\[ (-1)^{|\varphi|-1} \partial(\iota_\varphi \eta) - \iota_\delta(\varphi) \eta = \iota_\nu(\iota_\varphi \eta). \]  

\textbf{Proof.} On one hand, if we denote \( \dagger := \iota(-)\eta \), then

\[ \iota_\nu(\iota_\varphi \eta) = \iota_\varphi(\iota_\nu \eta), \]

where the last equality holds due to the Cartan formula \( L_\pi = [\iota_\pi, d] \) and \( \eta \) being \textit{d}-closed.

On the other hand, we always have the equality

\[ (-1)^{|\varphi|-1} \partial(\iota_\varphi \eta) - \iota_\delta(\varphi) \eta = -\iota_\varphi \partial \eta \]

Plugging (8) into the above identity, we get the desired equality. \( \square \)

As an immediate corollary, we have the following “twisted Poincaré duality”:

\textbf{Theorem 2.6 (see also [16] [26]).} Let \( A \) be a Poisson algebra with a volume form of degree \( n \). Then

\[ HP^\bullet(A) \cong HP_{n-\bullet}(A, A_\nu), \]

which is called the \textbf{twisted} Poincaré duality of \( A \). In particular, if \( A \) is the set of smooth
functions on a smooth and oriented Poisson manifold, or the set of algebraic functions of
a Poisson Calabi-Yau affine variety, then (9) holds.

\textbf{Proof.} In the light of (4) and (7), we only need to show

\[ (\partial + \iota_\nu)(\omega) = \partial_\nu(\omega), \]

for any \( \omega \in \Omega^\bullet(A) \). This is a tautology by (3).

Now, since for \( A = C^\infty(X) \) of a smooth and oriented Poisson manifold, or \( A = \mathcal{O}(X) \)
of a Poisson Calabi-Yau affine variety, then the volume form of \( X \) (or say on \( A \)) always
exists, by the above argument, the theorem now follows. \( \square \)
Remark 2.7. For a smooth and oriented Poisson manifold $X$, the modular vector field $\nu$ for $A = C^\infty(X)$ is a Poisson 1-cocycle, and the cohomology class it represents does not depend on the choice of the volume form, and hence is a topological invariant of the Poisson manifold, which is usually called the modular class of $X$ (see [39] for more details). For Poisson Calabi-Yau affine varieties, if we change the volume form up to a unit, then the modular vector fields differ by a log-Hamiltonian derivation (see [10] for more details).

Remark 2.8 (Some historical remarks). (1) Xu first studied the Poincaré duality for Poisson manifolds (see [39]), where he showed that for a unimodular Poisson manifold $X$, $H^\bullet_p(X) \cong H_{n-\bullet}^\ast(X)$. Later Launois and Richard in [22], Zhu in [31], and Luo, Wang and Wu in [27] studied the twisted Poincaré duality for some special types of Poisson algebras. All these results are covered by the result of Lü, Wang and Zhuang [26, Corollary 4.4], which deals with arbitrary Poisson Calabi-Yau affine varieties, with a slightly different proof.

(2) After the first draft of the paper was finished, we learned that Huebschmann has developed in [16] a general Poincaré duality theorem for Lie-Rinehart algebras, where the Poincaré duality for Poisson algebras is a special case. It turns out that Theorem 2.6 (as well as Lü-Wang-Zhuang’s) can be deduced from Huebschmann’s theorem. In Appendix A we give a proof of this statement.

3. Modular vector fields and the Batalin-Vilkovisky structure

Xu proved in [39] that for unimodular Poisson algebras, there exists a Batalin-Vilkovisky algebra structure on its cohomology. In this section we generalize this result to Poisson algebras with semi-simple modular vector fields.

Definition 3.1 (Batalin-Vilkovisky algebra). Suppose $(V, \cdot)$ is an graded commutative algebra. A Batalin-Vilkovisky algebra structure on $V$ is the triple $(V, \cdot, \Delta)$ such that

1. $\Delta : V^i \to V^{i-1}$ is a differential, that is, $\Delta^2 = 0$; and
2. $\Delta$ is second order operator, that is,

$$\Delta(a \cdot b \cdot c) = \Delta(a \cdot b) \cdot c + (-1)^{|a|}a \cdot \Delta(b \cdot c) + (-1)^{|a|-1}|b|b \cdot \Delta(a \cdot c)$$

$$- (\Delta a) \cdot b \cdot c - (-1)^{|a|}a \cdot (\Delta b) \cdot c - (-1)^{|a|+|b|}a \cdot b \cdot (\Delta c).$$

In the above definition, if we set

$$\{-, -\} : V \otimes V \to V, (a, b) \mapsto (-1)^{|a|}(\Delta(a \cdot b) - \Delta(a) \cdot b - (-1)^{|a|}a \cdot \Delta(b)),$$

then it is direct to check that $(V, \cup, \{-, -\})$ is a Gerstenhaber algebra (see Definition 3.2 below), and we say the Gerstenhaber bracket $\{-, -\}$ is generated by the Batalin-Vilkovisky operator $\Delta$ (see Getzler [12] for more details).

Lambre observed in [20] that a lot of examples of Batalin-Vilkovisky algebras come from the structure of differential calculus, in the sense of Tamarkin and Tsygan [34], with some additional conditions. Let us recall his result first.
3.1. Differential calculus and the Batalin-Vilkovisky algebra. We start with the notion of Gerstenhaber algebras:

**Definition 3.2** (Gerstenhaber). A **Gerstenhaber algebra** is a graded $k$-vector space $A^\bullet$ endowed with two bilinear operators $\cup : A^n \otimes A^m \to A^{n+m}$ and $\{-,-\} : A^n \otimes A^m \to A^{n+m-1}$ such that: for any homogeneous elements $a, b, c \in A^\bullet$,

1. $(A^\bullet, \cup)$ is a graded commutative associative algebra, i.e.,
   
   $$a \cup b = (-1)^{|a||b|} b \cup a,$$

   satisfying associativity;

2. $(A^\bullet, \{-,-\})$ is a graded Lie algebra with the bracket $\{-,-\}$ of degree $-1$, i.e.,
   
   $$\{a, b\} = (-1)^{|a|-1} (|b|-1) \{b, a\}$$

   and

   $$\{a, \{b, c\}\} = \{(a, b), c\} + (-1)^{|a|-1} (|b|-1) \{b, \{a, c\}\};$$

3. the cup product $\cup$ and the Lie bracket $\{-,-\}$ are compatible in the sense that
   
   $$\{a, b \cup c\} = \{a, b\} \cup c + (-1)^{|a|-1} |b| \{b, \{a, c\}\}.$$

**Definition 3.3** (Tamarkin-Tsygan [34], Definition 3.2.1). Let $H^\bullet$ and $H_\bullet$ be two graded vector spaces. A **differential calculus** is a sextuple $(H^\bullet, \cup, \{-,-\}, H_\bullet, B, \cap)$, satisfying the following conditions:

1. $(H^\bullet, \cup, \{-,-\})$ is a Gerstenhaber algebra;
2. $H_\bullet$ is a graded module over $(H^\bullet, \cup)$ by the “cap action” $\cap : H^n \otimes H_m \to H_{m-n}, f \otimes \alpha \mapsto f \cap \alpha,$ i.e., $(f \cup g) \cap \alpha = f \cap (g \cap \alpha)$ for any homogeneous $f, g \in H^\bullet, \alpha \in H_\bullet$;
3. there exists a linear operator $B : H_\bullet \to H_{\bullet+1}$ such that $B^2 = 0$ and moreover, if we set $L_f(\alpha) := B(f \cap \alpha) - (-1)^{|f|} f \cap B(\alpha)$, then $L$ is a Lie algebra action of $H^\bullet$ on $H_\bullet$, that is,
   
   $$L_{\{f,g\}}(\alpha) = [L_f, L_g](\alpha),$$

   for any $f, g \in H^\bullet$ and $\alpha \in H_\bullet$.

**Example 3.4.** Let $A$ be a Poisson algebra. Then

$$(HP^\bullet(A), HP_\bullet(A, A), \cup, \{-,-\}, \cap, d)$$

form a differential calculus, where $\cup$ are $\{-,-\}$ are the wedge product and the Schouten bracket induced on the polyvectors, and $\cap$ is the contraction (also denoted by $\iota$ before), and $d$ is the de Rham differential. The key point here is to check that these operators are compatible with the Poisson boundary and coboundary maps, which, however, is a direct check; see [23, Chapter 3] for more details.
Definition 3.5 (Lambre [20]). A differential calculus \((H^*, H_*, \cup, \{-, -\}, \cap, B)\) is called a differential calculus with duality if there exists an element \(\eta \in H_n\) for some \(n\) such that
\[
\phi : H^* \to H_{n-\bullet}, \quad \varphi \mapsto \varphi \cap \eta
\]
is an isomorphism of \(H^\bullet\)-modules.

Theorem 3.6 (Lambre [20] Lemma 1.5 and Theorem 1.6). Assume \((H^*, H_*, \cup, \{-, -\}, \cap, B)\) is a differential calculus with duality. Let \(\Delta := \phi^{-1} \circ B \circ \phi\). Then \((H^\bullet, \cup, \Delta)\) is a Batalin-Vilkovisky algebra where \(\Delta\) generates the Gerstenhaber bracket.

We next apply this theorem to the case of Poisson algebras.

3.2. Poisson algebras with semi-simple modular vector field. Poisson structures with semi-simple modular vector fields are an important concept in Poisson geometry; see, for example, [11, §5.6] and [23, §8.2], for more discussions. In this subsection we show the existence of a Batalin-Vilkovisky structure on the Poisson cohomology of a Poisson algebra or a Frobenius Poisson algebra with a semi-simple modular vector field.

Definition 3.7. Suppose \(A\) is a Poisson algebra or a Frobenius Poisson algebra. The modular vector field \(\nu\) is called semi-simple if it is diagonalizable.

For a Poisson algebra with semi-simple modular vector field, we may decompose its Poisson chain and cochain complexes into the direct sum of eigenspaces, which leads to interesting results as we shall show below. We learned this idea from [19] (see also [21, 25] for some further applications).

Suppose \(A\) has a semi-simple modular vector field; then we can decompose \(A\) into the direct sum of eigenspaces of \(\nu\), namely, \(A = \bigoplus_{\lambda_i} A_{\lambda_i}\), where \(A_{\lambda_i} := \{a \in A | \nu(a) = \lambda_i a\}\). Let
\[
CP_n^\lambda(A, A_\nu) := \left\{ \sum f_0 df_1 \wedge \cdots \wedge df_n \in CP_n(A, A_\nu) \middle| f_i \in A_{\lambda_i} \text{ for some } \lambda_i, \quad i = 0, 1, \ldots, n, \sum_{i=0}^n \lambda_i = \lambda \right\}.
\]
Since \(\nu\{f, g\} = \{\nu(f), g\} + \{f, \nu(g)\}\), \(\partial_\nu\) is closed on these spaces, and hence \((CP^\bullet_n(A, A_\nu), \partial_\nu)\) is a subcomplex. We have
\[
CP^\bullet(A, A_\nu) = \bigoplus_{\lambda} CP^\lambda_n(A, A_\nu).
\]
(10)
For the Poisson cochain complex \(CP^\bullet(A, A)\), we analogously have a decomposition into the direct sum of the eigenspaces
\[
CP^\bullet(A, A) = \bigoplus_{\lambda} CP^\lambda_n(A, A) = \bigoplus_{\lambda} \bigoplus_n CP^\lambda_n(A, A),
\]
(11)
where \(CP^\lambda_n(A, A_\nu) := \{\phi \in CP^\lambda_n(A, A_\nu) | \phi(A_{\lambda_1} \otimes \cdots \otimes A_{\lambda_n}) \subset A_{\lambda_1 + \cdots + \lambda_n + \lambda}\}.

Lemma 3.8. Suppose \(A\) is a Poisson algebra with semi-simple modular vector field, then
\[
\partial_\nu \circ d + d \circ \partial_\nu = \tilde{\nu},
\]
where \(\tilde{\nu}(f_0 df_1 \wedge \cdots \wedge df_n) = \nu(f_0) df_1 \wedge \cdots \wedge df_n + \sum_{i=1}^n f_0 df_1 \wedge \cdots \wedge d(\nu(f_i)) \wedge \cdots \wedge df_n\).
Proof. For any element \( f_0 df_1 \wedge \cdots \wedge df_n \in CP_n(A, A_\nu) \), we have
\[
\partial_\nu \circ d(f_0 df_1 \wedge \cdots \wedge df_n) = \partial_\nu(d(f_0 \wedge df_1 \wedge \cdots \wedge df_n)
\]
\[
= \sum_{i=0}^{n} (-1)^i \nu(f_i) df_0 \wedge \cdots \wedge \widehat{df}_i \wedge \cdots \wedge df_n
\]
\[
+ \sum_{i=1}^{n} (-1)^i df_0 \{f_0, f_i\} \wedge df_1 \wedge \cdots \wedge \widehat{df}_i \wedge \cdots \wedge df_n
\]
\[
+ \sum_{0<i<j} (-1)^{i+j} df_0 \{f_i, f_j\} \wedge df_1 \wedge \cdots \wedge \widehat{df}_i \wedge \cdots \wedge \widehat{df}_j \wedge \cdots \wedge df_n.
\]
and
\[
d \circ \partial_\nu(f_0 df_1 \wedge \cdots \wedge df_n)
\]
\[
= d \left( \sum_{i=1}^{n} (-1)^{i-1} \{f_0, f_i\} df_1 \wedge \cdots \wedge \widehat{df}_i \wedge \cdots \wedge df_n \right)
\]
\[
+ \sum_{0<i<j} (-1)^{i+j} df_0 \{f_i, f_j\} \wedge df_1 \wedge \cdots \wedge \widehat{df}_i \wedge \cdots \wedge \widehat{df}_j \wedge \cdots \wedge df_n.
\]
Hence we have
\[
(\partial_\nu \circ d + d \circ \partial_\nu)(f_0 df_1 \wedge \cdots \wedge df_n)
\]
\[
= \nu(f_0) df_1 \wedge \cdots \wedge df_n + \sum_{i=1}^{n} f_0 df_1 \wedge \cdots \wedge df_i \wedge \cdots \wedge df_n
\]
\[
= \tilde{\nu}(f_0 df_1 \wedge \cdots \wedge df_n).
\]
This completes the proof.

Theorem 3.9. Suppose A is a Poisson algebra with semi-simple modular vector field, then
\[
HP_\bullet(A, A_\nu) = H_\bullet(CP^0_\bullet(A, A_\nu)) \quad \text{and} \quad HP^*\bullet(A, A_\nu) = H^*(CP^0_\bullet(A, A_\nu)). \tag{12}
\]
In particular,
\[
(HP^\bullet(A), HP_\bullet(A, A_\nu), \cup, \{-, -\}, \iota, d)
\]
forms a differential calculus with duality.

Proof. First, we have inclusions
\[
i : CP^0_\bullet(A, A_\nu) \to CP_\bullet(A, A_\nu) \quad \text{and} \quad i : CP^0_\bullet(A, A) \to CP^\bullet(A, A).
\]
We claim that these are homotopy equivalences of chain complexes. In fact, by (10) and (11) the homotopy inverses are given by the projections. If we denote the projections by $p$, then $p \circ i = \text{Id}$. Now by Lemma 3.8 we have

$$(\partial_{\nu} \circ d + d \circ \partial_{\nu})|_{\text{CP}^{\lambda}_{\bullet}(A, A_{\nu})} = \tilde{\nu} \cdot \text{Id}|_{\text{CP}^{\lambda}_{\bullet}(A, A_{\nu})},$$

which means for $\lambda \neq 0$, the de Rham differential $d$, up to a scalar, gives a homotopy retracting between $\text{Id}$ and $i \circ p$. This means, $i : \text{CP}^{0}_{\bullet}(A, A_{\nu}) \hookrightarrow \text{CP}^{\bullet}_{\bullet}(A, A_{\nu})$ and similarly, $i : \text{CP}^{\bullet}_{\bullet}(A, A) \hookrightarrow \text{CP}^{\bullet}_{\bullet}(A, A_{\nu})$, are equivalences of chain complexes, and (12) follows.

Observe that from (13) we also get that $$(\text{CP}^{0}_{\bullet}(A, A_{\nu}), \partial_{\nu}, d)$$ forms a mixed chain complex. Now denote by $\eta$ the volume form of $A$, which represents an $n$-class in $\text{HP}^{\lambda}_{\bullet}(A, A_{\nu})$ corresponding to eigenvalue 0. We have that the cap action $\iota(-) \eta$ preserves the eigenvalue

$$\iota(-) \eta : \text{HP}^{\bullet}_{\lambda}(A) \rightarrow \text{HP}^{\lambda}_{\bullet-n}(A, A_{\nu}).$$

Combining it with the twisted Poincare duality $\text{HP}^{\bullet}(A) \cong \text{HP}_{\bullet-n}(A, A_{\nu})$, we get that

$$(\text{HP}^{\bullet}(A), \text{HP}_{\bullet}(A, A_{\nu}), \cup, \{-,-\}, \iota, d)$$

is a differential calculus with duality. $\square$

Combining the above theorem with Lambre’s Theorem 3.6, we get the following:

**Theorem 3.10.** Suppose $A$ is a Poisson algebra with semi-simple modular vector field, then $\text{HP}^{\bullet}(A)$ has a Batalin-Vilkovisky algebra structure where the Batalin-Vilkovisky operator generates the Gerstenhaber bracket.

4. **Deformation quantization**

In this section we study the deformation quantization of Poisson algebras with nontrivial modular vector field. The ground field $k$ in this section is taken to be $\mathbb{R}$.

Suppose $A$ is a Poisson algebra; its (formal) deformation quantization, denoted by $A_{h}$, is a $k[[h]]$–linear associative product (called the star-product) on $A[[h]]$

$$a \star b = a \cdot b + B_{1}(a, b)h + B_{2}(a, b)h^{2} + \ldots, \text{ for } a, b \in A$$

such that $B_{i} : A \otimes A \rightarrow A$ are bidifferential operators, satisfying

$$B_{1}(a, b) - B_{1}(b, a) = \{a, b\}.$$

In what follows, we also write $B_{i}(a, b)$ as $\ast_{i}(a, b)$.

In [37] Kontsevich showed that there is a one-to-one correspondence between the equivalence classes of the star-products and the equivalence classes of Poisson structures $\pi_{h} = \pi + \pi_{1}h + \cdots$ on $A[[h]]$. He also constructed an explicit $L_{\infty}$-quasi-isomorphism

$$\mathcal{U} : T_{\text{poly}}(A) \rightarrow D_{\text{poly}}(A)$$
from the space of polyvector fields to the Hochschild cochain complex which acts on each component in $A$ as multi-derivations, where the first term of $U$ is the classical Hochschild-Kostant-Rosenberg quasi-isomorphism. Via this map, the Poisson bivector $\pi_\hbar$ on $A[\hbar]$ corresponds to a star-product $\star$ on $A_\hbar$. By considering the tangent map of $U$, one then gets a quasi-isomorphism

$$\text{CP}^\bullet(A[\hbar], \pi_\hbar) \cong \text{CH}^\bullet(A_\hbar, \star).$$

The reader may refer to Kontsevich’s paper [17] for a proof (see also Manchon-Torossian [29] for more details).

Later Dolgushev showed in [10] that the deformation quantization of a Poisson polynomial algebra is an AS-regular algebra; similarly, the deformation quantization of a Poisson exterior algebra is a graded Frobenius algebra.

What we are interested in now is to study the behavior of the twisted Poisson homology $\text{HP}^\bullet(A, A_\nu)$ under deformation quantization.

4.1. Deformation quantization of Poisson bimodules. We now briefly go over the deformation quantization of Poisson bimodules.

Definition 4.1 (Bursztyn-Waldmann [4]). Suppose $M$ is a Poisson $A$-bimodule. Suppose $A$ has a deformation quantization $A_\hbar$. A deformation quantization of $M$, denoted by $M_\hbar$, is $M[\hbar]$ equipped with an $A_\hbar$-bimodule structure such that

$$a \star_1 m - m \star_1 a = \hbar \{a, m\}, \quad \text{for all } a \in A, m \in M,$$

where $a \star_1 m$ and $m \star_1 a$ are the first terms in the deformations of $M$ as left and right $A$-modules:

$$a \star m = a \cdot m + a \star_1 m \hbar + \cdots \quad \text{and} \quad m \star a = m \cdot a + m \star_1 a \hbar + \cdots,$$

where $\star$ are the deformed (left and right) actions of $A_\hbar$ on $M_\hbar$.

The following theorem about deformation quantization of Poisson bimodules is proved by Chemla:

Theorem 4.2 (Chemla [7] Corollary 21). Let $A$ be the Poisson algebra of a Poisson manifold, and $M$ be a Poisson $A$-bimodule. Then

$$\text{HP}^\bullet(A[\hbar], M[\hbar]) \cong \text{HH}^\bullet(A_\hbar, M_\hbar).$$

We next apply this theorem to the case of Poisson algebras with nontrivial modular vector fields. To this end, we first have to introduce the notion of Artin-Schelter regular algebras.

4.2. Artin-Schelter regular algebras. Artin-Schelter regular algebras was introduced by Artin and Schelter in [1]:

Definition 4.3 (AS-regular algebra). A connected graded $k$-algebra $A$ is called AS-regular of dimension $n$ if

1. $A$ has finite global dimension $n$, and
2. $A$ is Gorenstein, that is, $\text{Ext}^i_A(k, A) = 0$ for $i \neq n$ and $\text{Ext}^n_A(k, A) \simeq k$. 

2
In the literature, an AS-regular algebra is also called a twisted Calabi-Yau algebra, due to the following.

**Theorem 4.4** (Reyes-Rogalski-Zhang [30] Lemma 1.2). Suppose $A$ is as above. Then $A$ is AS-regular if and only if it is twisted Calabi-Yau; that is, $A$ satisfies the following two conditions:

1. $A$ is homologically smooth, that is, $A$, viewed as an $A^e$-module, has a bounded, finitely generated projective resolution;
2. there exists an integer $n$ and an algebra automorphism $\sigma$ of $A$ such that
\[
\text{Ext}_{A^e}^i(A, A^e) \cong \begin{cases} A_\sigma, & \text{if } i = n, \\ 0, & \text{otherwise} \end{cases}
\]
as $A^e$-modules.

In the above theorem, $A_\sigma$ is $A$ with the twisted $A$-bimodule structure given by
\[
a \cdot b \cdot c := ab\sigma(c).
\]
for any $a, b, c \in A$, and $\sigma$ is usually called the Nakayama automorphism of $A$. If $\sigma = \text{Id}$, then $A$ is called Calabi-Yau in the sense of Ginzburg [14].

In 2008, Brown and Zhang obtained a refinement of Van den Bergh’s noncommutative Poincaré duality:

**Theorem 4.5** ([2] Corollary 0.4). Suppose $A$ is an AS-regular algebra of dimension $n$. Then we have the following isomorphism
\[
\HH^\bullet(A) \cong \HH_{n-\bullet}(A, A_\sigma),
\]
where $\HH^\bullet(\cdot)$ and $\HH_{\bullet}(\cdot)$ are the Hochschild cohomology and homology respectively.

**Example 4.6.** Let $A = k\langle x_1, \cdots, x_n \rangle/(f)$, where
\[
f = (x_1, \cdots, x_n)g(x_1, \cdots, x_n)^T, g \in \text{GL}_n(k),
\]
and $(f)$ means the ideal generated by $f$. Then $A$ is an AS-regular algebra. Observe that $A$ is a graded algebra; for $x = \sum k_ix_i$, $k_i \in k$, let
\[
\sigma(x) = -(x_1, \cdots, x_n)g^T g^{-1}(k_1, \cdots, k_n)^T,
\]
and extend it to the whole $A$. The $\sigma$ thus defined is the Nakayama automorphism of $A$.

**Example 4.7** (The Quantum affine space). Let $Q = \left( \begin{array}{cccc} q_{11} & \cdots & q_{1n} \\ \vdots & \ddots & \vdots \\ q_{n1} & \cdots & q_{nn} \end{array} \right)$ be an $n \times n$ matrix over $k$ with $q_{ii} = 1$, $q_{ij}q_{ji} = 1$, for $1 \leq i, j \leq n$. Let $A = k\langle x_1, \cdots, x_n \rangle/(x_jx_i - q_{ij}x_ix_j)$. Then $A$ is an AS-regular algebra with the Nakayama automorphism $\sigma$ given by
\[
\sigma(x_i) = (\prod_{j=1}^n q_{ji})x_i, \quad i = 1, \cdots, n.
\]
4.3. Quantization of the modular vector fields. Now let $\nu$ be the modular vector field of a Poisson algebra $A$, and $\nu_{\hbar} = \nu + \nu_{1}\hbar + \cdots$ be the modular vector field with respect to $\pi_{\hbar}$. Since $\nu_{\hbar}$ is a Poisson cocycle, its image under Kontsevich’s $L_\infty$ map gives a Hochschild cocycle, denoted by $\sigma$, which is in fact $\exp(\nu_{\hbar})$; see Dolgushev [10, Theorem 2] for a proof.

Lemma 4.8 (Dolgushev). Let $A$ be as above. Let $\nu$ be the modular vector field of $A$. Then $(A_{\nu})_{\hbar} = (A_{\hbar})_{\sigma}$,

up to an automorphism of $A[\hbar]$ whose leading term is $\text{Id}$. In other words, $\sigma$ is the deformation quantization of $\nu$ by Kontsevich’s $L_\infty$-quasi-isomorphism $U$.

Proof. By the argument above, we only need to show that for any $a, m \in A$, they satisfy (15). In fact,

$$a \star m - m \star (a) = \hbar(a \star_{1} m - m \star_{1} a + m \cdot \nu(a)) = \hbar \{a, m\} + m \cdot \nu(a) = \hbar \{a, m\}_{\nu} \mod \hbar^2.$$  

The lemma now follows. □

Lemma 4.9. Let $(A = \mathbb{R}[x_{1}, \ldots, x_{n}], \pi)$ be a Poisson algebra and $\nu$ be the corresponding modular vector field. We have

$$HP_{\bullet}(A[\hbar], (A_{\nu})[\hbar]) \cong HH_{\bullet}(A_{\hbar}, (A_{\hbar})_{\sigma}).$$ (16)

Proof. By above lemma, $(A_{\nu})_{\hbar} = (A_{\hbar})_{\sigma}$. The lemma now follows Chemla’s result Theorem 4.2. □

Theorem 4.10. Suppose $A = \mathbb{R}[x_{1}, \ldots, x_{n}]$ is a Poisson algebra. Then the diagram

$$
\begin{array}{ccc}
HP^\bullet(A[\hbar]) & \cong & HP^\bullet(A[\hbar], (A_{\nu})[\hbar]) \\
\cong & & \cong \\
HH^\bullet(A_{\hbar}) & \cong & HH^\bullet(A_{\hbar}, (A_{\nu})_{\hbar})
\end{array}
$$ (17)

commutes.

Proof. Dolgushev showed that $A_{\hbar}$ satisfies the following

$$\text{Ext}^i_{A_{\hbar} \otimes \mathbb{R}[\hbar]} \Lambda^p A_{\hbar} \otimes \mathbb{R}[\hbar] A_{\hbar}^{op} \cong \begin{cases} (A_{\hbar})_{\sigma}, & i = n, \\ 0, & \text{otherwise,} \end{cases}$$

where $\sigma$ is the deformation quantization of $\nu$ as in the previous two lemmas (see Dolgushev [10, Proposition 2]). This means $A_{\hbar}$ is an AS-regular algebra over $\mathbb{R}[\hbar]$ of dimension $n$, which then implies the noncommutative Poincaré duality (see Theorem 4.5)

$$HH^\bullet(A_{\hbar}) \cong HH_{n-n}(A_{\hbar}, (A_{\hbar})_{\sigma}).$$

Combining (9), (14) and (16) we get the isomorphisms in (17).
Chemla proved in [7, Theorem 10] that for any Poisson $A$-bimodule $M$, there is a quasi-isomorphism of $L_\infty$-modules from the modules over $T_{\text{poly}}(A)$ to modules over $D_{\text{poly}}(A)$, which she denotes by $T_{\text{poly}}(M)$ and $D_{\text{poly}}(M)$ respectively. Such a quasi-isomorphism generalizes the $L_\infty$-quasi-isomorphism from $T_{\text{poly}}(A)$ to $D_{\text{poly}}(A)$ of Kontsevich. Then by a similar argument to that of Kontsevich she gets the above Theorem 4.2, which is more precisely the following commutative diagram

\[
\begin{array}{ccc}
HP^\bullet(A[h]) & \xrightarrow{\cong} & HP^\bullet(A[h], M[h]) \\
\downarrow \cong & & \downarrow \cong \\
HH^\bullet(A_h) & \xrightarrow{\cong} & HH^\bullet(A_h, M_h),
\end{array}
\]

where the horizontal curved arrows mean the Lie algebra actions. Restricting to the case where $M = A_\nu$, with the Poincaré duality taken into account we get the commutativity of (17).

4.3.1. Semi-simple Nakayama automorphism. Now we study the deformation quantization of semi-simple modular fields. First, observe that the following is straightforward.

Lemma 4.11. Suppose $A$ is a Poisson algebra. Let $A_h$ be its deformation quantization. If $\nu$ is semi-simple, then $\sigma = \exp(\hbar \nu)$ is the semi-simple Nakayama automorphism of $A_h$.

Kowalzig and Krahmer proved in [19, Theorem 1.5] that, for an AS-regular algebra with semi-simple Nakayama automorphism, its Hochschild cohomology has a Batalin-Vilkovisky algebra structure, whose Batalin-Vilkovisky operator generates the Gerstenhaber bracket on the cohomology. Thus in the light of the above lemma, combining this result with Theorem 4.10 we obtain the following.

Theorem 4.12. Suppose $A$ is a Poisson algebra. Let $A_h$ be its deformation quantization. If $A$ has semi-simple modular vector field, then we have an isomorphism

\[
HP^\bullet(A[h]) \cong HH^\bullet(A_h)
\]

of Batalin-Vilkovisky algebras.

5. Frobenius Poisson algebras

In [40], Zhu, Van Oystaeyen and Zhang introduced the notion of Frobenius Poisson algebras, that is, Poisson algebras with a non-degenerate pairing, and studied the structures on their (co)homology. In this subsection, we study these algebras with semi-simple modular vector fields, and their twisted Poincaré duality, Koszul duality and deformation quantization.

5.1. Modular symmetry and Poincaré duality. Let us start with the definition of Frobenius algebras.

Definition 5.1. A finite dimensional graded associative $k$-algebra $A$ is called Frobenius of dimension $n$ if it is equipped with a bilinear, non-degenerate pairing of degree $n$

\[
\langle -, - \rangle : A \otimes A \to k
\]
such that $\langle a \cdot b, c \rangle = \langle a, b \cdot c \rangle$, for all homogeneous $a, b, c \in A$.

Suppose $A$ is a Frobenius algebra, then the nondegeneracy of the pairing in the above definition is equivalent to saying that there is an isomorphism

$$\eta : A \rightarrow A^\ast, \quad a \mapsto (\cdot, a)$$

of left $A$-modules, but not necessarily an isomorphism of $A$-bimodules, where $A^\ast := \text{Hom}(A, k)$. We shall discuss this more in Example 5.3 below.

**Example 5.2.** Suppose $A^t = A(\xi_1, \cdots, \xi_n)$ is the exterior algebra; in what follows we view it as the graded symmetric algebra generated by $\xi_1, \cdots, \xi_n$ with each grading $|\xi_i| = -1$. There is a degree $n$ $A^t$-module isomorphism

$$\eta : A^t \rightarrow A^t, \quad \xi_i \cdots \xi_{ip} \mapsto \eta(\xi_i \cdots \xi_{ip})$$

where

$$\eta(\xi_i \cdots \xi_{ip}) := \sum_{s \in S_{p,n}} \langle \xi_i \cdots \xi_{ip}, \xi_{s_1} \cdots \xi_{s_p} \rangle \cdot \xi_{s_{p+1}} \cdots \xi_{s_n},$$

$A^t := (A^t)^\ast$, $\xi^t_i$'s are the linear duals of $\xi_i$'s, for $i = 1, \cdots, n$, and the sum runs over all $(p, n-p)$-shuffles $s$ of $(1, \cdots, n)$. Recall that a $(p, n-p)$-shuffle is a permutation $s$ of $(1, \cdots, n)$ such that $s_1 < \cdots < s_p, s_{p+1} < \cdots < s_n$. It is direct to see that $\eta$ is non-degenerate and hence gives a Frobenius algebra structure on $A^t$. We also write $\eta$ in the form $\xi^t_1 \cdots \xi^t_n$, and call it the volume form of $A^t$.

**Definition 5.3** (Zhu-Van Oystaeyen-Zhang [10]). A graded Poisson algebra $A$ is called a **Frobenius Poisson** if it is moreover a Frobenius algebra.

For a Frobenius Poisson algebra, say $A$, there is a differential calculus structure associated to it, which is different to the one given in Example 3.4. In fact, suppose $A$ is a Frobenius Poisson algebra. Then any $f \in \mathcal{X}^p(A)$ and $\alpha \in \mathcal{X}^q(A; A^\ast)$, let $f \cap \alpha \in \mathcal{X}^{p+q}(A; A^\ast)$ be given by

$$\langle f \cap \alpha \rangle(a_1, \cdots, a_{p+q}) := \sum_{s \in S_{p,q}} \text{sgn}(\sigma)f(a_{s_1}, \cdots, a_{s_p}) \cdot \alpha(a_{s_{p+1}}, \cdots, a_{s_{p+q}}),$$

(19)

where $\sigma$ runs over all $(p, q)$-shuffles of $(1, \cdots, p + q)$. Observing that

$$\mathcal{X}_A^\ast(A^\ast) = \text{Hom}_A(\Omega^\ast(A), A^\ast)$$

$$= \text{Hom}_A(\Omega^\ast(A), \text{Hom}_k(A, k))$$

$$= \text{Hom}_k(A \otimes_A \Omega^\ast(A), k)$$

$$= \text{Hom}_k(\Omega^\ast(A), k).$$

We dualize the de Rham differential $d$ on $\Omega^\ast(A)$ and obtain a differential $d^\ast$ on $\text{Hom}(\Omega^\ast(A), k)$, i.e., on $\mathcal{X}^\ast(A; A^\ast)$, which commutes with the Poisson coboundary (see [40] Theorem 4.10 for a proof). The following proposition is obtained by Zhu-Van Oystaeyen-Zhang (see [40] §3-4] for a complete proof).
**Proposition 5.4.** Let $A$ be a Frobenius Poisson algebra. Then

$$(\text{HP}^\bullet(A), \text{HP}^\bullet(A, A^\ast), \cup, \{-,-\}, \cap, d^\ast)$$

form a differential calculus, where $\cup$ and $\{-,-\}$ are as in the above example, and $\cap$ is $\iota$ given by (20) and $d^\ast$ is the dual de Rham differential given by (22).

In what follows we denote by $A^!$ a Frobenius Poisson algebra, and by $A^\$ its linear dual. From the nondegeneracy of the pairing we in fact get an isomorphism $\eta^! : A^! \to A^\$ which further induces an isomorphism of vector spaces

$$\iota(-) \eta^! : \mathcal{X}^\bullet_{A^!}(A^!) \to \mathcal{X}^\bullet_{A^\}(A^\)$$

given by

$$\iota\varphi \eta^! := \{(a_1, \cdots, a_p) \mapsto \eta^!(\varphi(a_1, \cdots, a_p))\}, \quad \text{for } \varphi \in \mathcal{X}^p(A^!), \ a_1, \cdots, a_p \in A^!.$$  

Again, $\iota(-) \eta^!$ gives the following diagram

$$\begin{array}{ccc}
\mathcal{X}^\bullet_{A^!}(A^!) & \xrightarrow{\iota(-) \eta^!} & \mathcal{X}^\bullet_{A^\}(A^!) \\
\downarrow \delta & & \downarrow \delta \\
\mathcal{X}^{n+1}_{A^!}(A^!) & \xrightarrow{\iota(-) \eta^!} & \mathcal{X}^{n+1}_{A^\}(A^!) \\
\end{array}$$

of vector spaces, which in general does not commute with the boundaries on each side, since $\eta^!$ is not a Poisson cocycle. To adjust this discrepancy, we do the same procedure as in the Poisson algebra case. Namely, let

$$\text{Div} : \mathcal{X}^\bullet_{A^!}(A^!) \to \mathcal{X}^{n-1}_{A^i}(A^i)$$

be such that the following diagram

$$\begin{array}{ccc}
\mathcal{X}^\bullet_{A^!}(A^!) & \xrightarrow{\iota(-) \eta^!} & \mathcal{X}^\bullet_{A^\}(A^!) \\
\downarrow \text{Div} & & \downarrow d^\ast \\
\mathcal{X}^{n-1}_{A^!}(A^!) & \xrightarrow{\iota(-) \eta^!} & \mathcal{X}^{n-1}_{A^\}(A^!) \\
\end{array}$$

commutes, where $d^\ast$ is the dual of the de Rham differential. Let $\nu^! = -\text{Div}(\pi^!)$, which is also called the modular vector field of $A^!$. Analogously to Lemma 2.5, for any $\varphi \in \mathcal{X}^p(A^!)$, we have

$$\partial(\varphi \cap \eta^!) + \nu^!(\varphi \cap \eta^!) = \delta(\varphi) \cap \eta^!.$$  

Combining (21) and (23), with the appropriate degree on the cohomology taken into accounted, yields the following.

**Theorem 5.5** ([28] §3.1). Let $A^!$ be a Poisson exterior algebra, and $\nu^!$ be the corresponding modular vector field. Then

$$\text{HP}^\bullet(A^!) \cong \text{HP}^{n-\eta}(A^!, A^\nu!),$$

(24)
Now let us move to the Frobenius Poisson algebra case. Let $A^!$ be a Frobenius Poisson algebra with semi-simple modular vector field $\nu^!$. The following three statements are completely parallel to Lemma 3.8–Theorem 3.10, and we leave their proofs to the interested reader.

**Lemma 5.6.** On the Poisson cochain complex $CP^\bullet(A^!, A^!)$, we have

$$\partial_{\nu^!} \circ d^* + d^* \circ \partial_{\nu^!} = \tilde{\nu}^!.$$  

**Corollary 5.7.** $(HP^\bullet(A^!), HP^\bullet(A^!, A^!) \cup \{-, -, \}, \iota, d^*)$ forms a differential calculus with duality.

**Theorem 5.8** (See also [37]). Suppose $A^!$ is a Frobenius Poisson algebra with semi-simple modular vector field $\nu^!$, then $HP^\bullet(A^!)$ has a Batalin-Vilkovisky algebra structure whose Batalin-Vilkovisky operator generates the Schouten bracket.

### 5.2. Koszul duality for Poisson algebras

From now on, we focus on quadratic Poisson algebras. As we mentioned before, Shoikhet showed that the Koszul dual of a quadratic Poisson polynomial algebra is again quadratic Poisson. In this section, we study the modular symmetry under Koszul duality, and the main result is Theorem 5.14.

**Definition 5.9.** Let $A = \mathbb{R}[x_1, \ldots, x_n]$ be the real polynomial algebra in $n$ variables. A Poisson structure on $A$, say $\pi$, is called quadratic if it is of the form

$$\pi = \sum_{i_1, i_2, j_1, j_2} c_{i_1 i_2 j_1 j_2}^{j_1 j_2} x_{i_1} x_{i_2} \frac{\partial}{\partial x_{j_1}} \wedge \frac{\partial}{\partial x_{j_2}}, \quad c_{i_1 i_2}^{j_1 j_2} \in \mathbb{R}. \quad (25)$$

**Definition 5.10.** If $A = \mathbb{R}[x_1, \ldots, x_n]$ is the polynomial algebra with a quadratic bivector $\pi = \sum_{i_1, i_2, j_1, j_2} c_{i_1 i_2}^{j_1 j_2} x_{i_1} x_{i_2} \frac{\partial}{\partial x_{j_1}} \wedge \frac{\partial}{\partial x_{j_2}}$, then its Koszul dual, denoted by $A^!$, is the graded symmetric algebra

$$A^! = \Lambda(\xi_1, \ldots, \xi_n), \quad |\xi_i| = -1, i = 1, \ldots, n$$

with the dual bivector

$$\pi^! = \sum_{i_1, i_2, j_1, j_2} c_{i_1 i_2}^{j_1 j_2} \xi_{i_1} \xi_{i_2} \frac{\partial}{\partial \xi_{j_1}} \frac{\partial}{\partial \xi_{j_2}}. \quad (26)$$

Under the correspondence

$$x_i \leftrightarrow \frac{\partial}{\partial \xi_i} \quad \text{and} \quad \frac{\partial}{\partial x_i} \leftrightarrow \xi_i \quad (27)$$

between the sets of polyvectors on $A$ and on $A^!$, it is direct to check that $\pi$ is Poisson if and only if $\pi^!$ is Poisson. We call $(A^!, \pi^!)$ the Koszul dual Poisson algebra of $(A, \pi)$.

**Proposition 5.11** (See also [38]). Let $(A, \pi)$ and $(A^!, \pi^!)$ be the quadratic Poisson algebras Koszul dual to each other as given in Definition 5.10. Then we have isomorphisms

$$HP^\bullet(A) \cong HP^\bullet(A^!) \quad \text{and} \quad HP_\bullet(A) \cong HP^{-\bullet}(A^!, A^!),$$

where $A^!$ is $(A^!)* = \text{Hom}(A^!, k)$. 
Proof. Since $A = \mathbb{R}[x_1, \cdots, x_n]$, we have
\[
\mathfrak{x}^*(A) = \Lambda(x_1, \cdots, x_n, \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})
\]  
(28)
and similarly,
\[
\mathfrak{x}^*(A^l) = \Lambda(\xi_1, \cdots, \xi_n, \frac{\partial}{\partial \xi_1}, \cdots, \frac{\partial}{\partial \xi_n}),
\]
(29)
where the gradings are given as follows:
\[|x_i| = 0, \quad |\frac{\partial}{\partial x_i}| = -1, \quad |\xi_i| = -1, \quad |\frac{\partial}{\partial \xi_i}| = 0, \quad i = 1, \cdots, n.\]
Under the correspondence (27) we obtain an isomorphism of chain complexes which gives the first isomorphism on the cohomology.

Koszul duality and modular symmetry. We now study the behavior of the modular vector field under Koszul duality.

Proposition 5.12. Suppose $A = (\mathbb{R}[x_1, \cdots, x_n], \pi)$ and $A^l = (\Lambda(\xi_1, \cdots, \xi_n), \pi^l)$ are Koszul dual Poisson algebras. Then under the correspondence (27) the modular vector field $\nu$ of $A$ corresponds to $\nu^l$ of $A^l$.

Proof. It is direct to check that the modular vector field
\[
\nu = -\text{Div}(\pi)
\]
and
\[
\nu^l = -\text{Div}(\pi^l)
\]
On the other hand, we have
\[
\nu = -\text{Div}(\pi)
\]
and
\[
\nu^l = -\text{Div}(\pi^l)
\]
Under the identification (27) these two modular derivations are isomorphic to each other. □

From the above computation of \( \nu \) we also get the following byproduct.

**Proposition 5.13.** Suppose \( A = k[x_1, \cdots, x_n] \) is a Poisson algebra with Poisson structure \( \pi \). Take the volume form to be \( \eta = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \). Then the modular vector field is semi-simple if and only if \( \pi \) is of the form

\[
\pi = \sum_{i,j} c_{ij} x_i \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_j}.
\]

Also as a corollary of Proposition 5.12, we obtain the following:

**Theorem 5.14.** Let \( A = \mathbb{R}[x_1, \cdots, x_n] \) be a quadratic Poisson algebra, and let \( A^! = \Lambda(\xi_1, \cdots, \xi_n) \) be its Koszul dual. Denote by \( \nu \) and \( \nu^! \) the modular vector fields of \( A \) and \( A^! \) respectively. Then the following diagram

\[
\begin{align*}
HP^\bullet(A, A_\nu) & \cong HP_{n-\bullet}(A, A_\nu) \\
HP^\bullet(A^!, A^!_\nu) & \cong HP^\bullet_{-n}(A^!, A^!_\nu)
\end{align*}
\]

commutes.

**Proof.** With the results in Proposition 5.11 and Theorems 2.6 and 5.5, the only thing that we need to prove is

\[
HP_\bullet(A, A_\nu) \cong HP^{-\bullet}(A^!, A^!_\nu).
\]

The proof of the second isomorphism in Proposition 5.11 shows that

\[
CP_\bullet(A, A_\nu) \cong CP^{-\bullet}(A^!, A^!_\nu)
\]

as chain complexes with respect to the Poisson boundary maps; now Proposition 5.12 says that the twistings on both sides of the above complexes are also identical. Taking the corresponding homology we get the commutative diagram (32), and the theorem follows. □

### 5.3. Deformation quantization of Frobenius Poisson algebras

We now show that the deformation quantizations of Frobenius Poisson algebras are Frobenius algebras.

In Definition 5.1 of a Frobenius algebra, since the pairing is non-degenerate, there exists an automorphism \( \sigma^! \) such that \( \langle ab, c \rangle = (-1)^{|a||b|+|c|} \langle a \sigma^!(c)b \rangle \). Such a \( \sigma^! \) is also called the *Nakayama automorphism* of \( A^! \). The non-degeneracy of the pairing given by is equivalent to saying that

\[
\eta^!: A^! \to A^!_{\sigma^!}\![\!-n], \ a \mapsto \langle \!-\!, a \rangle
\]

is an isomorphism of \( A^!\)-bimodules. In 2016, Lambre, Zhou and Zimmermann obtained the following “noncommutative Poincaré duality”: 

Theorem 5.15 ([21] Proposition 3.3). Let $A^!$ be a Frobenius algebra of degree $n$ with Nakayama automorphism $\sigma^!$. Then there is an isomorphism

$$\text{HH}^\bullet(A^!) \rightarrow \text{HH}^\bullet(-n)(A^!, A^{\sigma!}).$$

5.3.1. Deformation quantization. First we recall that for graded Poisson algebras over supermanifolds, Kontsevich’s deformation quantization remain valid (see Cattaneo and Felder [6, Theorem 4.6] for a proof).

Now by the same argument as in the polynomial case, $\nu^!$ can be deformation quantized via the Kontsevich map. Denote by $\sigma^!$ its deformation quantization; then we have (see Lemma 4.8)

$$(A^{\nu!})_h \cong (A^i_h)^{\sigma^i}.$$ (33)

This implies the following lemma.

Lemma 5.16. Let $A^!$ and $\sigma^!$ be as above. Then $\sigma^!$ is the Nakayama automorphism of $A^!$.

Proof. We have to show that for any $a, b \in A^!_h$,

$$\langle a, b \rangle = (-1)^{|a||b|} \langle \sigma^!(b), a \rangle.$$ (33)

This is equivalent to showing that $A^!_h \cong (A^i_h)^{\sigma^i}$ as $A_h$-bimodules.

In fact, $A^! \cong A^{\nu!}$ as Poisson $A$-modules, and therefore they have isomorphic deformation quantization. This implies that

$$A^!_h \cong (A^{\nu!})_h$$ as $A_h$-bimodules. Combining it with (33), we get the lemma. \qed

Similarly to Theorem 4.10, we have the following.

Theorem 5.17. Suppose $A^!$ is a Frobenius Poisson algebra. Then the diagram

$$\begin{align*}
\text{HP}^\bullet(A^! [\hbar]) & \cong \text{HP}^\bullet(-n)(A^! [\hbar], A^{\nu!} [\hbar]) \\
\text{HH}^\bullet(A^!_h) & \cong \text{HH}^\bullet(-n)(A^!_h, (A^{\nu!}_h)^{\sigma^i})
\end{align*}$$ (34)

commutes.

Proof. Observe that the left vertical isomorphism is Kontsevich’s isomorphism (14), the top horizontal isomorphism is given by (24), and the bottom horizontal isomorphism is the right vertical isomorphism of (38) with the Nakayama automorphism given by Lemma 5.16.

We now need to prove

$$\text{HP}^\bullet(-n)(A^! [\hbar], A^{\nu!} [\hbar]) \cong \text{HH}^\bullet(-n)(A^!_h, (A^i_h)^{\sigma^i}).$$ (35)

In fact this follows from combining (33) and Theorem 4.2. \qed
5.3.2. Semi-simple Nakayama automorphism. Analogously to Kowalzig and Krahmer [19], for a Frobenius algebra with semi-simple Nakayama automorphism, Lambre, Zhou and Zimmermann proved in [21, Theorem 4.1] that its Hochschild cohomology also admits a Batalin-Vilkovisky algebra structure, whose Batalin-Vilkovisky operator generates the Gerstenhaber bracket on the cohomology. Parallel to Theorem 4.12, we have the following.

**Theorem 5.18.** Suppose $A^1$ is a Frobenius Poisson algebra. Let $A^1_h$ be its deformation quantization. If $A^1$ has semi-simple modular symmetry, then we have an isomorphism

$$\text{HP}^*(A^1_h) \cong \text{HH}^*(A^1)$$

of Batalin-Vilkovisky algebras.

6. **Poincaré duality, Koszul duality and deformation quantization**

In this section we study the deformation quantization of quadratic Poisson algebras, which relates the theorems obtained in previous sections.

6.1. **Koszul duality of AS-regular algebras.** We start with the Koszul duality theory for associative algebras.

Let $V$ be a finite-dimensional (possibly graded) vector space over $k$. Denote by $TV$ the free algebra generated by $V$ over $k$; that is, $TV$ is the tensor algebra generated by $V$. Suppose $R$ is a subspace of $V \otimes V$, and let $(R)$ be the two-sided ideal generated by $R$ in $TV$, then the quotient algebra $A := TV/(R)$ is called a quadratic algebra. Let $A^l$ be the quadratic dual algebra of $A$; that is, $A^l = TV^*/(R^\perp)$, where $R^\perp = \{r^* \in V^* \otimes V^* | r^*(R) = 0\}$. Let $A^l$ be the linear dual of $A^l$, called the quadratic dual coalgebra of $A$. Choose a set of basis $\{e_i\}$ for $V$, and let $\{e^*_i\}$ be their duals in $V^*$. There is a natural chain complex associated to $A$, called the Koszul complex:

$$\cdots \xrightarrow{\delta} A \otimes A^1_{i+1} \xrightarrow{\delta} A \otimes A^1_i \xrightarrow{\delta} \cdots \xrightarrow{\delta} A \otimes A^1_0 \xrightarrow{\delta} k,$$

where for any $r \otimes f \in A \otimes A^l$, $\delta(r \otimes f) = \sum_i e_i r \otimes e^*_i f$.

**Definition 6.1** (Koszul algebra). A quadratic algebra $A = TV/(R)$ is called Koszul if the Koszul chain complex (37) is acyclic.

A typical example of Koszul algebras what we use throughout the paper is the polynomial algebra $A = k[x_1, \cdots, x_n]$, whose Koszul dual is the exterior algebra $A^l = \Lambda(\xi_1, \cdots, \xi_n)$.

We have the following Koszul duality for AS-regular and Frobenius algebras; see Smith [33, Proposition 5.10] and Van den Bergh [35, pp. 667] for a proof.

**Proposition 6.2.** Let $A$ be a Koszul algebra, and let $A^l$ be its Koszul dual algebra. Then $A$ is an AS-regular algebra if and only if $A^l$ is Frobenius. Under this correspondence, the Nakayama automorphism of $A$ is Koszul dual to the Nakayama automorphism of $A^l$.

Combining Theorems 4.5 and 5.15 and Proposition 6.2, the second author was able to prove the following.
Theorem 6.3 ([25] Lemma 5.8). Let $A$ be a Koszul AS-regular algebra. Let $A^!$ and $A^!$ be its Koszul dual algebra and coalgebra respectively. Then the Nakayama automorphism $\sigma$ of $A$ is mapped to the Nakayama automorphism $\sigma^!$ of $A^!$ under Koszul duality, and the following diagram

\[ \begin{array}{ccc}
\text{HH}^*(A) & \cong & \text{HH}^*_n(A, A_\sigma) \\
\cong & & \cong \\
\text{HH}^*(A^!) & \cong & \text{HH}^{*-n}(A^!, A^{!\sigma^!})
\end{array} \] (38)

commutes. Moreover, if the Nakayama on $A$ and hence on $A^!$ is semi-simple, then

\[ \text{HH}^*(A) \cong \text{HH}^*(A^!) \]

as Batalin-Vilkovisky algebras, whose Batalin-Vilkovisky operators generate the Gerstenhaber brackets on both sides.

6.2. Koszul duality and deformation quantization. One of the motivations of the current paper is the result of Shoikhet et al. on the Koszul duality between the deformation quantizations of quadratic Poisson polynomial algebras and their Koszul dual, which is stated as follows (see Shoikhet [32] Theorem 0.3 and Calaque et al. [5] Theorem 8.6 for a proof): Let $A = \mathbb{R}[x_1, \cdots, x_n]$ and $A^!$ its Koszul dual. Then Kontsevich’s deformation quantization of $A$ and $A^!$, denoted by $A_\hbar$ and $A^!_\hbar$ respectively, are also Koszul dual to each other as associative algebras over $\mathbb{R}[\hbar]$.

Notice that by Shoikhet [31], the Koszul duality theory remain valid if $\mathbb{R}$ is replaced by $\mathbb{R}[\hbar]$, and therefore, the Koszul duality between $A_\hbar$ and $A^!_\hbar$ over $\mathbb{R}[\hbar]$ in the above theorem makes sense. The following theorem is obtained in [8] Theorem 1.5: Let $A[\hbar]$ and $A^![[\hbar]]$ be Koszul dual Poisson algebras. Then we have the following commutative diagram of isomorphisms

\[ \begin{array}{ccc}
\text{HP}^*(A[\hbar]) & \cong & \text{HP}^*(A^![[\hbar]]) \\
\cong & & \cong \\
\text{HH}^*(A_\hbar) & \cong & \text{HH}^*(A^!_\hbar)
\end{array} \] (39)

For the twisted Poisson homology and the twisted Hochschild homology, we have the following.

Theorem 6.4. Let $A[\hbar]$ and $A^![[\hbar]]$ be Koszul dual Poisson algebras. Then we have the following commutative diagram of isomorphisms

\[ \begin{array}{ccc}
\text{HP}_*(A[\hbar], A_\nu[\hbar]) & \cong & \text{HP}_*^*(A^![[\hbar]], A^!_\nu[[\hbar]]) \\
\cong & & \cong \\
\text{HH}_*(A_\hbar, (A_\nu)_\hbar) & \cong & \text{HH}^*_*(A^!_\hbar, (A^!_\nu)_\hbar)
\end{array} \] (40)
Proof. The top horizontal isomorphism is the right vertical isomorphism of (32); the bottom horizontal isomorphism is the right vertical isomorphism of (38); the left vertical isomorphism is (16); and the right vertical isomorphism is (35). The commutativity follows from the Hochschild-Kostant-Rosenberg theorem.

The following theorem summarizes the above several results.

**Theorem 6.5.** Let \( A[\hbar] \) and \( A^!\hbar \) be Koszul dual Poisson algebras. Then the following diagram of isomorphisms

\[
\begin{array}{c}
\text{HP}^*(A[\hbar]) \cong \text{HP}^*-n(A^!\hbar, A^!\hbar, [\hbar]) \\
\text{HP}^*(A^!\hbar) \cong \text{HH}^*(A^!\hbar) \\
\text{HH}^*(A^!\hbar) \cong \text{HH}^*-n(A^!\hbar, A^!\hbar, [\sigma])
\end{array}
\]

(41)

commutes, where the horizontal arrows are the Poincaré duality, the vertical arrows are given by deformation quantization, and the slanted arrows are given by Koszul duality.

Proof. The top square of the diagram is given by (32), the bottom square is given by (38), the front square is given by (17), the back square is given by (34), and the left and the right squares are given by (39) and (40) respectively.

6.3. **Isomorphisms of Batalin-Vilkovisky algebras.** We continue to show that, for a quadratic Poisson algebra with semi-simple modular symmetry, the left side diagram in (41) is an commutative diagram of isomorphisms of Batalin-Vilkovisky algebras (see Theorem 6.8). It induces a commutative diagram of isomorphisms of gravity algebras on the corresponding negative cyclic homology (see Theorem 6.12).

**Lemma 6.6.** Let \( A \) be a quadratic Poisson algebra. Let \( A^! \) be its Koszul dual algebra. If the modular vector field \( \sigma \) is semi-simple, then so is its Koszul dual \( \sigma^! \). In this case, we have

\[
\text{HP}^*(A) \cong \text{HP}^*(A^!)
\]

as Batalin-Vilkovisky algebras.

Proof. The first half follows from Proposition 5.12. The Batalin-Vilkovisky algebra isomorphism follows from Theorem 5.14.

The following result is proved in [25, Theorem 1.1]:

**Lemma 6.7.** Let \( A \) be a Koszul AS-regular algebra. Let \( A^! \) be its Koszul dual algebra. If the Nakayama automorphism \( \sigma \) is semi-simple, then so is its Koszul dual \( \sigma^! \). In this case, we have

\[
\text{HH}^*(A) \cong \text{HH}^*(A^!)
\]

as Batalin-Vilkovisky algebras.
We now reach to the proof of the following two theorems, which supersede the results obtained in [8] [9] for unimodular Poisson algebras.

**Theorem 6.8.** Suppose \( A = \mathbb{R}[x_1, \cdots, x_n] \). For a quadratic Poisson structure on \( A[^\hbar] \) with semi-simple modular vector field, the following

\[
\begin{array}{ccc}
\text{HP}^\bullet(A[^\hbar]) & \cong & \text{HP}^\bullet(A[^\hbar]) \\
\cong & \cong & \\
\text{HH}^\bullet(A_\hbar) & \cong & \text{HH}^\bullet(A_\hbar)
\end{array}
\]

is a commutative diagram of isomorphisms of Batalin-Vilkovisky algebras.

**Proof.** Combine the left side diagram of (41) with the first halves of Theorems 4.12 and 5.18 and Lemmas 6.6 and 6.7. \( \square \)

6.4. **The gravity algebra structure.** In this last subsection, we briefly discuss the gravity algebra structure on the negative cyclic homology of Poisson algebras with semi-simple modular vector field.

The notion of gravity algebra was introduced by Getzler in [13]; it plays an important role in the study of equivariant topological conformal field theory. In [9], the first two authors of the current paper together with Eshmatov showed that the negative cyclic Poisson homology of a unimodular Poisson algebra has a gravity algebra structure. In what follows we generalize the result of [9] to the case of Poisson algebras whose modular vector field is semi-simple.

**Definition 6.9** (Getzler [13]). Suppose \( V \) is a (graded) vector space over \( k \). A **gravity algebra structure** on \( V \) consists of a sequence of (graded) skew symmetric operators (called the higher Lie brackets)

\[
\{x_1, \ldots, x_n\} : V^{\otimes n} \to V, n = 2, 3, \ldots
\]

such that

\[
\sum_{1 \leq i < j \leq n} (-1)^{\epsilon_{ij}} \{\{x_i, x_j\}, x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n, y_1, \ldots, y_m\} = \begin{cases} 
\{\{x_1, \ldots, x_n\}, y_1, \ldots, y_m\}, & \text{if } m > 0, \\
0, & \text{otherwise},
\end{cases}
\]

where \( \epsilon_{ij} = (|x_i| + 1)(|x_1| + \ldots + |x_{i-1}| + i - 1) + (|x_j| + 1)(|x_1| + \ldots + |x_{j-1}| + j - 1) - (|x_i| + 1)(|x_j| + 1). \)

Now suppose \((C_\bullet, b, B)\) is a mixed complex. Denote by \( \text{CC}^-_\bullet(C_\bullet) \) the negative cyclic complex of \( C_\bullet \). Then we have a short exact sequence

\[
0 \to u \cdot \text{CC}^-_\bullet(C_\bullet) \xrightarrow{\iota} \text{CC}^-_\bullet(C_\bullet) \xrightarrow{\pi} C_\bullet \to 0,
\]

where \( \iota : u \cdot \text{CC}^-_{\bullet+2}(C_\bullet) \to \text{CC}^-_\bullet(C_\bullet) \) is the embedding and

\( \pi : \text{CC}^-_\bullet(C_\bullet) \to C_\bullet \), \( \sum_i x_i \cdot u^i \mapsto x_0 \).
is called the negative cyclic Poisson homology. Suppose (C•, b, B) is a mixed complex. If HH•(C•) has a Batalin-Vilkovisky algebra structure such that B is the generator of the Gerstenhaber bracket, then the following sequence of maps

\[ \{−, ⋯, −\} : (\text{HC}_{−n}(C•)) \rightarrow \text{HC}_{−n}(C•) \]

\[ (x_1, \cdots, x_n) \mapsto (-1)^{\epsilon_n} \beta(\pi_s(x_1) \cdot \pi_s(x_2) \cdots \cdot \pi_s(x_n)), \quad n = 2, 3, \cdots \]

where \( \epsilon_n = (n-1)|x_1| + (n-2)|x_2| + \cdots + |x_{n-1}| \) and \( \cdot \) is the product on the Hochschild homology (coming from the Batalin-Vilkovisky algebra structure), gives on \( \text{HC}_{−n}(C•) \) a gravity algebra structure.

In what follows, we shall also study the cyclic cohomology of an associative and Poisson algebra. Suppose \((C•, \delta, B^*)\) is a mixed complex with degrees of \( \delta \) and \( B^* \) being 1 and \(-1\) respectively; in order to distinguish, we would call this type of mixed complex in what follows mixed cochain complex, and call the usual mixed complex, like \((C•, b, B)\) above, mixed chain complex. By our convention, the cyclic cohomology of a mixed cochain complex \((C•, \delta, B^*)\), denoted by \(\text{HC}^•(C•)\), is the cohomology of the negative cyclic complex of the mixed chain complex obtained from \((C•, \delta, B^*)\) by negating the gradings. Thus the cyclic cohomology is essentially the same as the negative cyclic homology.

Now suppose \( A \) is a Poisson algebra and respectively \( A^1 \) is a Frobenius Poisson algebra, both with semi-simple modular vector fields. In the previous section we have shown that \((\text{CP}^0(A, A_\nu), \partial_\nu, d)\) is a mixed chain complex and \((\text{CP}^•(A^1, A^1_\nu), \partial_\nu, d^*)\) is a mixed cochain complex.

**Definition 6.11.** Suppose \( A \) is a Poisson algebra with semi-simple modular vector field \( \nu \). The negative cyclic homology of the mixed complex

\[ (\text{CP}^0(A, A_\nu), \partial_\nu, d) \]

is called the negative cyclic Poisson homology of \( A \), and is denoted by \( \text{PC}^−(A) \). Similarly, suppose \( A^1 \) is a Frobenius Poisson algebra with semi-simple modular vector field \( \nu^l \). The cyclic cohomology of \((\text{CP}^•(A^1, A^1_\nu), \partial_\nu, d^*)\) is called the cyclic Poisson cohomology of \( A^1 \), and is denoted by \( \text{PC}^•(A^1) \).

**Theorem 6.12.** Suppose \( A = \mathbb{R}[x_1, \cdots, x_n] \). For a quadratic Poisson structure on \( A[\hbar] \) with semi-simple modular vector field, the following diagram

\[ \begin{array}{ccc}
\text{PC}^−(A[\hbar]) & \xrightarrow{\cong} & \text{PC}^•(A^1[\hbar]) \\
\cong & & \cong \\
\text{HC}^−(A_\hbar) & \xrightarrow{\cong} & \text{HC}^•(A^1_\hbar)
\end{array} \]
commutes, where $HC^{-\bullet}(A_{\hbar})$ and $HC^{\bullet}(A_{\hbar}^1)$ are the usual negative cyclic homology and the cyclic cohomology of $A_{\hbar}$ and $A_{\hbar}^1$ respectively. Moreover, after the degrees shifted down by $n$, the above is a commutative diagram of isomorphisms of gravity algebras.

**Proof.** By transporting the Batalin-Vilkovisky algebra structures in Theorem 6.8 to the right hand side of the diagram (41), the theorem then follows from Lemma 6.10. □

**Appendix A. Poincaré duality for Lie-Rinehart algebras**

The notion of Lie-Rinehart algebras was introduced by Huebschmann in [15]. Later in [16] he developed a general Poincaré duality theory for Lie-Rinehart algebras, which includes the Poincaré duality of Poisson algebras as a special case. It is expected that the twisted Poincaré duality in Theorem 1.1 can be deduced from his result. In this Appendix we give a proof of this statement.

During our study of [16] and particularly when chasing the literature, we found the paper of Lü, Wang and Zhuang [26], where Theorem 1.1 has been proved for Poisson Calabi-Yau affine varieties, and some comparisons with [16] have already been made.

The following Definitions A.1–A.5 are taken from Huebschmann [16].

**Definition A.1.** Let $R$ be a commutative ring. A **Lie-Rinehart algebra** is a pair $(A, L)$ where $L$ and $A$ are Lie and commutative algebras over $R$ respectively, and moreover, $L$ acts on $A$ from the left and is itself a left $A$-module with the following compatibility conditions:

$$[\alpha, u\beta] = \alpha(a)\beta + u[\alpha, \beta], \quad (u\alpha)(v) = u(\alpha(v)),$$

for all $u, v \in A$ and $\alpha, \beta \in L$. $L$ is called an $(R, A)$-Lie algebra.

**Definition A.2.** Suppose $M$ is an $A$-module $M$ and also a left $L$-module. $M$ is called a **left $(A, L)$-module** if

$$\alpha(um) = \alpha(u)m + u\alpha(m), \quad (u\alpha)(m) = u(\alpha(m)),$$

for all $u \in A, \alpha \in L, m \in M$.

An $A$-module $N$ which is also a right $L$-module is called a **right $(A, R)$-module** if

$$(un)\alpha = u(n\alpha) - \alpha(u)n, \quad n(u\alpha) = u(n\alpha) - \alpha(u)n,$$

for all $a \in A, \alpha \in L, n \in N$.

We refer the interested reader to [16, 2.1-2.5] for the induced Lie-Rinehart module structures on the tensor product and Hom space of Lie-Rinehart modules.

**Example A.3.** (See [15] Theorem 3.8] and thereafter) (1) Let $X$ be a smooth Poisson manifold. Denote by $\Omega^1(X)$ the $C^\infty(X)$-module of smooth 1-forms on $X$. Then $(C^\infty(X), \Omega^1(X))$ is a Lie-Rinehart algebra, where the Lie algebra on $\Omega^1(X)$ is given by

$$[adu, bdv] := a\{u, b\}dv + b\{a, v\}du + abd\{u, v\},$$

for all $adu, bdv \in \Omega^1(X)$. 
(2) Algebraically, suppose $A$ is a Poisson $R$-algebra with the Poisson bracket $\{−, −\}$. Let $D_A$ be the $A$-module of Kähler differentials of $A$. Then $(A, D_A)$ is a Lie-Rinehart algebra, where the Lie algebra on $D_A$ is the same as above.

**Definition A.4.** Suppose $(A, L)$ is a Lie-Rinehart algebra. Then its universal algebra is an associative $R$-algebra $U(A, L)$ together with a morphism $ι_A : A → U(A, L)$ of $R$-algebras and a morphism $ι_L : L → U(A, L)$ of Lie algebras over $R$ such that

\[
ι_A(u)ι_L(α) = ι_L(uα), \quad ι_L(α)ι_A(u) − ι_A(u)ι_L(α) = ι_A(α(u)),
\]

and moreover, $U(A, L)$ is universal among all triples $(B, φ_L, φ_A)$ having these properties.

From (44) and the universal property we see that a left (respectively right) $(A, L)$-module is automatically a left (respectively right) $U(A, L)$-module, and vice versa.

**Definition A.5** (Homology and cohomology of Lie-Rinehart algebras). Suppose $(A, L)$ is a Lie-Rinehart algebra. Let $M$ and $N$ be a left and right $(A, L)$-module respectively. Then the cohomology of $L$ with values in $M$ and the homology of $L$ with coefficients in $N$ are given by

\[
H^•_{LR}(L, M) := \text{Ext}^•_U(A, M), \quad H^•_{LR}(L, N) := \text{Tor}^•_U(N, A)
\]

respectively.

The following theorem is obtained in [16, Theorem 2.10 & Corollary 2.11]:

**Theorem A.6** (Huebschmann). Suppose $(A, L)$ is a Lie-Rinehart algebra such that $L$, as an $A$-module, is finitely generated and projective of constant rank $n$. Then

\[
H^•_{LR}(L, M) \cong H^•_{HR}(L, C_L ⊗_A M)
\]

for any left $(A, L)$-module $M$, where $C_L = \Lambda^n_A L^* = \text{Hom}_A(\Lambda^n_A L, A)$ is a right $(A, L)$-module, called the dualizing module.

The following theorem relates the Poisson homology and cohomology of a Poisson algebra with the homology and cohomology of the associated Lie-Rinehart algebra (see [15, §3] and [16, §7] for more details).

**Theorem A.7.** Suppose $A$ is a Poisson algebra over a commutative ring $R$. Then

\[
H^•_{LR}(A, M) \cong \text{HP}^•(A, M), \quad H^•_{LR}(A, N) \cong \text{HP}^•(A, N),
\]

for any left and right $(A, L)$-module $M$ and $N$.

The following is similar to the proofs of [26, Theorem 4.3 & Corollary 4.4]):

**Theorem A.8.** Suppose $A$ is the smooth functions on a smooth Poisson manifold, or the defining algebra of a Poisson Calabi-Yau affine variety. Then $C_L \cong A_ν$ as right Poisson $A$-modules, where $ν$ is the modular vector field, and therefore for any right Poisson $A$-module $M$, we have

\[
C_L ⊗_A M \cong M_ν
\]

as right Poisson $A$-modules.
Proof. Huebschmann showed in [16, §7] that with the assumption of the theorem, the corresponding Lie-Rinehart algebra \((A, \Omega^1(A))\) satisfies the condition of Theorem A.6 and hence the dualizing module exists, which is given by \(C_L = \mathfrak{X}^n(A)\), where \(n\) is the dimension of the manifold.

Choosing a volume form \(\eta\) on \(A\), we get an isomorphism

\[ \mathfrak{X}^n(A) \cong \mathfrak{A}, \phi \mapsto \iota_\phi \eta \]

of \(A\)-modules. By the same argument as in the proofs of Proposition 2.5 and Theorem 2.6 or simply by [26, Lemma 2.3], we get that \(\mathfrak{X}^n(A) \cong \mathfrak{A}_\nu\) as right Poisson \(A\)-modules. Therefore (see also the proof of [26, Corollary 4.4])

\[ C_L \otimes_A M \cong \mathfrak{X}^n(A) \otimes_A M \cong \mathfrak{A}_\nu \otimes_A M \cong M_\nu \]

as right Poisson \(A\)-modules. □

From the above theorem we deduce that for any right Poisson \(A\)-module \(M\), we have

\[ H_{\text{LR}}^*(A, C_L \otimes_A M) \cong H_{\text{LR}}^*(A, \mathfrak{A}_\nu) \cong H_{\text{P}}^*(A, M_\nu). \] (48)

Thus for \(A\) being the smooth functions on a smooth Poisson manifold, or the defining algebra of a Poisson Calabi-Yau affine variety, by the second isomorphism in (47) and (48), we see that (46) becomes

\[ H_{\text{P}}^*(A, M) \cong H_{\text{P}}^n_{\text{P}}(A, M_\nu). \]

In particular, if \(M = A\), then this isomorphism is exactly the twisted Poincaré duality in Theorem 1.1.

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