Asymptotics of Polynomials Orthogonal on a Cross with a Jacobi-type Weight

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Abstract. We investigate asymptotic behavior of polynomials $Q_n(z)$ satisfying non-Hermitian orthogonality relations

$$\int_{\Delta} s^k Q_n(s) \rho(s) ds = 0, \quad k \in \{0, \ldots, n-1\},$$

where $\Delta := [-a, a] \cup [-ib, ib]$, $a, b > 0$, and $\rho(s)$ is a Jacobi-type weight.

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1. Introduction

Let $a, b > 0$ be fixed. Set

$$\Delta := [-a, a] \cup [-ib, ib] \quad \text{and} \quad \Delta^\circ := \Delta \setminus \{0, a_1, a_2, a_3, a_4\},$$

(1.1)

where we put $a_1 = -a_3 = a$ and $a_2 = -a_4 = ib$. Denote by $\Delta_i$ (resp. $\Delta_i^\circ$), $i \in \{1, 2, 3, 4\}$, the closed (resp. open) segment joining the origin and $a_i$, which we orient towards the origin. In this work we are interested in strong asymptotics of polynomials $Q_n(z)$, $\deg(Q_n) \leq n$, satisfying orthogonality relations

$$\int_{\Delta} s^k Q_n(s) \rho(s) ds = 0, \quad k \in \{0, \ldots, n-1\},$$

(1.2)

where $\Delta$ inherits its orientation from the segments $\Delta_i$ and $\rho(s)$ is a certain weight function on $\Delta$. Orthogonality relations (1.2) are non-Hermitian as $s^k$ is not conjugated. Hence, there are no a priori reasons to assume that $\deg(Q_n) = n$. In what follows, we shall understand that $Q_n(z)$ stands for the monic polynomial of minimal degree satisfying (1.2). The weight functions we are interested in are holomorphic perturbations of the power functions. More precisely, we define the following nested sequence of classes of weights.

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Definition. Let $\ell$ be a positive integer or infinity. We shall say that a function $\rho(s)$ on $\Delta$ belongs to the class $\mathcal{W}_\ell$ if

\begin{enumerate}[(i)]
    \item $\rho_1(s) := \rho_1(\Delta \setminus \{a_i\})$ factors as a product $\rho_1(s) = \rho_1^*(s)(s-a_i)^{\alpha_i}$, where the function $\rho_1^*(z)$ is non-vanishing and holomorphic in some neighborhood of $\Delta_i$, $\alpha_i > -1$, and $(z-a_i)^{\alpha_i}$ is a branch holomorphic across $\Delta \setminus \{a_i\}$, $i \in \{1, 2, 3, 4\}$;
    \item the ratio $(\rho_1 \rho_3)(z)/(\rho_2 \rho_4)(z)$ is constant in some neighborhood of the origin (notice that each $\rho_i(s)$ extends holomorphically to a neighborhood of the origin by (i));
    \item it holds that $\rho_1(0) + \rho_2(0) + \rho_3(0) + \rho_4(0) = 0$;
    \item the quantities $\rho_1^{(l)}(0)/\rho_1(0)$, $0 \leq l < \ell$, do not depend on $i \in \{1, 2, 3, 4\}$.
\end{enumerate}

Observe that conditions (ii) and (iii) say that one of the functions $\rho_i(s)$ is fully determined by the other three. In particular, it must hold that

$$\rho_4(z) = -(\rho_1 + \rho_2 + \rho_3)(0)(\rho_2/\rho_1)(0)(\rho_1 \rho_3/\rho_2)(z).$$

Notice also that $\mathcal{W}_{\ell_1} \subset \mathcal{W}_{\ell_2}$ whenever $\ell_2 < \ell_1$ and that $\rho(s) \in \mathcal{W}_\infty$ if and only if there exists a function $F(z)$, holomorphic in some neighborhood of $\Delta \setminus \{a_1, a_2, a_3, a_4\}$, such that $\rho_i(s) = c_i F(\Delta \setminus \{a_i\})$ for some constants $c_i$ that add up to zero.

Holomorphy of the weights $\rho_i(s)$ allows one to deform $\Delta$ in (1.2) to any cross-like contour consisting of four arcs connecting the points $a_i$ to the origin (some central point if the weight add up to zero in a neighborhood of the origin). Hence, the following question arises: which contour do we expect to attract the zeros of the polynomials $Q_n(z)$ as $n \to \infty$? This fundamental question in the theory of non-Hermitian orthogonal polynomials was answered by Herbert Stahl in [16, 17, 18]. It turns out that the attracting contour is essentially characterized by having the smallest logarithmic capacity among all continua containing $\{a_1, a_2, a_3, a_4\}$. It is also known from the works [11, 15] that this contour must consist of the orthogonal critical trajectories of the quadratic differential

$$D(z)dz^2 = \frac{(z-b_1)(z-b_2)dz^2}{(z^2-a^2)(z^2+b^2)}$$

for some uniquely determined constants $b_1, b_2$. It can be readily verified that $\Delta$ is the desired contour and $b_1 = b_2 = 0$. In fact, the work of Stahl not only supplies us with the attracting contour, but also tell us that $\frac{1}{\pi i} \sqrt{D(s)} + ds$ is the limiting distribution of zeros of $Q_n(z)$, where the subscript $+$ stands for the trace on the positive side of $\Delta$ (according to the chosen orientation).

Strong asymptotics of the polynomials $Q_n(z)$ was considered as part of a study in [20] under much more restrictive assumption $\rho(s) = h(s)/w_+(s)$, where $h(z)$ is a holomorphic and non-vanishing function is some neighborhood of $\Delta$ and $w(z)$ is defined in (2.1) further below. It is also worth pointing out that if the points $\{a_1, a_2, a_3, a_4\}$ do not form a cross with two symmetries, then the points $b_1, b_2$ in (1.3) are distinct and the corresponding minimal capacity contour consists of five arcs: one joining $b_1$ and $b_2$, two connecting
be found in 

2. Statement of Results

The functions describing the asymptotics of the polynomials $Q_n(z)$ are constructed in three steps, carried out in Sections 2.2-2.4, and naturally defined on a Riemann surface corresponding to $\Delta$ that is introduced in Section 2.1. The main results of this work are stated in Sections 2.5 and 2.6. A more detailed description of the material in Sections 2.1–2.4 can be found in [21].
2.1. Riemann Surface

Let $\Delta = \cup_{i=1}^{4} \Delta_i$ be given by (1.1). Set
\[
w(z) := \sqrt{(z^2 - a^2)(z^2 + b^2)}, \quad z \in \mathbb{C} \setminus \Delta, \tag{2.1}\]
to be the branch normalized so that $w(z) = z^2 + O(z)$ as $z \to \infty$. Denote by $\mathcal{R}$ the Riemann surface of $w(z)$ realized as a two-sheeted ramified cover of $\mathbb{C}$ constructed in the following manner. Two copies of $\mathbb{C}$ are cut along each arc $\Delta_i$. These copies are glued together along the cuts in such a manner that the right (resp. left) side of the arc $\Delta_i$ belonging to the first copy, say $R(0)$, is joined with the left (resp. right) side of the same arc $\Delta_i$ only belonging to the second copy, $R(1)$. We denote by $\pi$ the canonical projection $\pi : \mathcal{R} \to \mathbb{C}$ and define $\Delta := \pi^{-1}(\Delta)$, $\Delta_i := \pi^{-1}(\Delta_i)$, $i \in \{1, 2, 3, 4\}$. Then $\Delta$ is a curve on $\mathcal{R}$ that intersects itself exactly twice (once at each point on top of the origin), see Figures 1 and 2. We orient $\Delta$ so that $\mathcal{R}(0)$ remains on the left when $\Delta$ is traversed in the positive direction. We shall denote by $z^{(k)}$, $k \in \{0, 1\}$, the point on $\mathcal{R}^{(k)}$ with the canonical projection $z$ and designate the symbol $^*$ to stand for the conformal involution that sends $z^{(k)}$ into $z^{(1-k)}$, $k \in \{0, 1\}$. We use bold lower case letters such as $z, t, s$ to indicate points on $\mathcal{R}$ with the canonical projections $z, t, s$. Since $\mathcal{R}$ has genus 1, any homology basis on $\mathcal{R}$ consists of only two cycles. In what follows, we choose cycle $\alpha$ (resp. $\beta$) to be involution-symmetric and such that $\pi(\alpha)$ (resp. $\pi(\beta)$) is a rectifiable Jordan arc joining $a_1$ and $a_2$ (resp. $a_4$ and $a_1$), that belongs to fourth (resp. first) quadrant and does not intersect $\Delta^* \cup \{0\}$, see Figure 1. We orient these cycles towards $a_1$ on $\mathcal{R}^{(0)}$ and therefore away from $a_1$ on $\mathcal{R}^{(1)}$, see Figure 2.
2.2. Geometric Term

The main goal of this subsection is to define the function $\Phi(z)$, see (2.6), that will be responsible for the rate of growth of the polynomials $Q_n(z)$ and is determined solely by the contour of orthogonality $\Delta$.

With a slight abuse of notation, let us set

$$w(z) := (-1)^k w(z), \quad z \in \mathcal{R}^{(k)} \setminus \Delta, \quad k \in \{0, 1\},$$

which we then extend by continuity to $\Delta$. Clearly, $w(z)$ is a meromorphic function on $\mathcal{R}$ with simple zeros at the ramification points of $\mathcal{R}$, double poles at $\infty^{(0)}$ and $\infty^{(1)}$, and otherwise non-vanishing and finite. Thus,

$$\Omega(z) := \left( \oint_{\alpha} \frac{ds}{w(s)} \right)^{-1} \frac{dz}{w(z)} \quad (2.2)$$

is the holomorphic differential on $\mathcal{R}$ normalized to have unit period on $\alpha$. In this case it was shown by Riemann that the constant

$$B := \oint_{\beta} \Omega \quad (2.3)$$

has positive purely imaginary part. Further, since $z/w(z)$ has simple poles at the ramification point of $\mathcal{R}$, simple zeros at $\infty^{(0)}$ and $\infty^{(1)}$, and behaves like $1/z$ around $\infty^{(0)}$, the differential

$$G(z) := \frac{dz}{w(z)}$$

is meromorphic on $\mathcal{R}$ having two simple poles at $\infty^{(1)}$ and $\infty^{(0)}$ with respective residues 1 and $-1$. $G(z)$ is also distinguished by having a purely
imaginary period on any cycle on $\mathcal{R}$. Indeed, it is enough to verify this claim on the cycles of any homology basis. To this end, define

$$\omega := -\frac{1}{2\pi i} \oint_{\beta} G \quad \text{and} \quad \tau := \frac{1}{2\pi i} \oint_{\alpha} G.$$  

(2.4)

By deforming $\alpha$ (resp. $\beta$) into $-\Delta_1 - \Delta_4$ (resp. $\Delta_1 + \Delta_2$) and using the symmetry $G(z^*) = -G(z)$, one gets that

$$\omega = \tau = \frac{1}{4\pi i} \oint_{\Gamma} \frac{z}{w(z)} = \frac{1}{2},$$  

(2.5)

where $\Gamma$ is any positively oriented rectifiable Jordan curve encircling $\Delta$, which does verify the claim about $G(z)$ having purely imaginary periods. Let

$$\Phi(z) := \exp \left\{ \int_{a_3}^{z} G \right\}, \quad z \in \mathcal{R}_{\alpha, \beta} \setminus \{ \infty^{(0)}, \infty^{(1)} \},$$

(2.6)

where $\mathcal{R}_{\alpha, \beta} := \mathcal{R} \setminus \{ \alpha, \beta \}$ and the path of integration lies entirely in $\mathcal{R}_{\alpha, \beta} \setminus \{ \infty^{(0)}, \infty^{(1)} \}$. The function $\Phi(z)$ is holomorphic and non-vanishing on $\mathcal{R}_{\alpha, \beta}$ except for a simple pole at $\infty^{(0)}$ and a simple zero at $\infty^{(1)}$. Furthermore, it possesses continuous traces on both sides of each cycle of the canonical basis that satisfy

$$\Phi_+(s) = -\Phi_-(s), \quad s \in \alpha \cup \beta,$$  

(2.7)

by (2.4)–(2.5). It is not a difficult computation to check that $\Phi(z)\Phi(z^*) \equiv 1$ and

$$|\Phi(z)| = \exp \left\{ (-1)^k g_\Delta(z; \infty) \right\}, \quad z \in \mathcal{R}^{(k)},$$

(2.8)

$k \in \{0, 1\}$, where $g_\Delta(z; \infty)$ is the Green function for $\mathbb{C} \setminus \Delta$ with pole at $\infty$. In fact, the above properties allow us to verify that

$$\Phi^2(z^{(k)}) = \frac{2}{a^2 + b^2} \left( z^2 + \frac{b^2 - a^2}{2} + (-1)^k w(z) \right),$$

(2.9)

$k \in \{0, 1\}$. In particular, this implies that the logarithmic capacity of $\Delta$ is equal to $\sqrt{a^2 + b^2}/2$ since

$$\Phi(z^{(0)}) = \frac{-2z}{\sqrt{a^2 + b^2}} + O(1) \quad \text{as} \quad z \to \infty$$

(2.10)

(the sign in (2.10) is determined by the fact that $\Phi(a_3) = 1$ and $\Phi(z)$ is non-vanishing on $\pi^{-1}((-\infty, -a))$). Observe also that a calculus level computation tells us that

$$\Phi(0) = \frac{\Phi(0^*)}{\Phi(0)} = \exp \left\{ i \arctan \left( \frac{a}{b} \right) \right\},$$

(2.11)

where the point $0$ and $0^*$ are defined as on Figure 1.

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1Here and in what follows we state jump relations understanding that they hold outside the points of self-intersection of the considered arcs.

2$g_\Delta(z; \infty)$ is equal to zero on $\Delta$, is positive and harmonic in $\mathbb{C} \setminus \Delta$, and satisfies $g(z; \infty) = \log |z| + O(1)$ as $z \to \infty$.
2.3. Szegő Function

It is known since the work of Szegő that the finer details of the asymptotics of $Q_n(z)$ are captured by the so-called Szegő function, which depends only on the weight of orthogonality. Below, we construct this function for $\rho(s) \in \mathcal{W}_1$.

Given $i \in \{1, 2, 3, 4\}$, fix $\log \rho_i(s)$ to be a branch continuous on $\Delta_i \setminus \{a_i\}$, selected so that

$$\nu := \frac{1}{2\pi i} \sum_{i=1}^{4} (-1)^i \log \rho_i(0) \quad \text{satisfies} \quad \text{Re}(\nu) \in \left(-\frac{1}{2}, \frac{1}{2}\right]. \quad (2.12)$$

Further, it can be readily verified that we can set

$$\log w_+(s) = \log |w_+(s)| + (-1)^i \frac{\pi i}{2}, \quad s \in \Delta_i, \quad (2.13)$$

where, as usual, $w_+(s)$ is the trace of (2.1) on the positive side of $\Delta^\circ_i$ according to the chosen orientation. We also let $\log(\rho_i w_+(s))$ to stand for $\log \rho_i(s) + \log w_+(s)$ with the just selected branches. Put

$$S_{\rho}(z) := \exp \left\{ -\frac{1}{4\pi i} \oint_{\Delta} \log(\rho w_+)(s) \Omega_{z,z^*}(s) \right\}, \quad (2.14)$$

where $\Omega_{z,z^*}(s)$ is the meromorphic differential with two simple poles at $z$ and $z^*$ with respective residues 1 and $-1$ normalized to have zero period on $\alpha$. When $z$ does not lie on top of the point at infinity, it can be readily verified that

$$\Omega_{z,z^*}(s) = \frac{w(z)}{s-z} \frac{ds}{w(s)} - \left( \int_{\alpha} \omega(t) \frac{dt}{t-z} w(t) \right) \Omega(s), \quad (2.15)$$

where $\Omega(s)$ is the holomorphic differential (2.2).

**Proposition 2.1.** Let $\rho(s) \in \mathcal{W}_1$ and $S_{\rho}(z)$ be given by (2.14). Define

$$c_{\rho} := \frac{1}{2\pi i} \oint_{\Delta} \log(\rho w_+) \Omega. \quad (2.16)$$

Then $S_{\rho}(z)$ is a holomorphic and non-vanishing function in $\mathfrak{H} \setminus \{\Delta \cup \alpha\}$ with continuous traces on $(\Delta \cup \alpha) \setminus \{a_1, a_2, a_3, a_4, 0, 0^*\}$ that satisfy

$$S_{\rho^+}(s) = S_{\rho^-}(s) \left\{ \begin{array}{ll}
\exp \{2\pi i c_{\rho}\}, & s \in \alpha, \\
1/(\rho w_+)(s), & s \in \Delta.
\end{array} \right. \quad (2.17)$$

It also holds that $S_{\rho}(z)S_{\rho^*}(z) \equiv 1$ and

$$|S_{\rho}(z(0))| \sim \left\{ \begin{array}{ll}
|z - a_i|^{-2(\alpha_i + 1)/4} & \text{as} \quad z \to a_i, \\
|z|^{(-1)^i \text{Re}(\nu)} & \text{as} \quad Q_j \ni z \to 0, \quad (2.18)
\end{array} \right.$$

for $i, j \in \{1, 2, 3, 4\}$, where $Q_j$ is the $j$-th quadrant and $\nu$ is given by (2.12).

Proposition 2.1 is proved in Section 5.

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3In what follows we write $|g_1(z)| \sim |g_2(z)|$ as $z \to z_0$ if there exists a constant $C > 1$ such that $C^{-1}|g_1(z)| \leq |g_2(z)| \leq C|g_1(z)|$ for all $z$ close to $z_0$. 
2.4. Theta Function

As it turns out, the product \((S_\rho \Phi^n)(z)\) is not sufficient to capture the strong asymptotics of the polynomials \(Q_n(z)\). What needs to be done now is to remove the jumps of this product from the cycles of the homology basis. This is done with the help of the functions \(T_k(z)\), \(k \in \{0, 1\}\), constructed further below in (2.22).

Let \(\text{Jac}(\mathfrak{M}) := \mathbb{C}/\{\mathbb{Z} + B\mathbb{Z}\}\) be the Jacobi variety of \(\mathfrak{M}\), where \(B\) is given by (2.3). We shall represent elements of \(\text{Jac}(\mathfrak{M})\) as equivalence classes \([s] = \{s + l + Bm : l, m \in \mathbb{Z}\}\), where \(s \in \mathbb{C}\). Since \(\mathfrak{M}\) has genus 1, Abel’s map

\[
z \in \mathfrak{M} \mapsto \left[ \int_{a_3}^z \Omega \right] \in \text{Jac}(\mathfrak{M})
\]

is a holomorphic bijection. Hence, given any \(s \in \mathbb{C}\), there exists a unique \(z_s \in \mathfrak{M}\) such that \(\left[ \int_{a_3}^{z_s} \Omega \right] = [s]\).

Denote by \(\theta(\zeta)\) the Riemann theta function associated to \(B\), i.e.,

\[
\theta(\zeta) := \sum_{n \in \mathbb{Z}} \exp \left\{ \pi iBn^2 + 2\pi in\zeta \right\}.
\]

As shown by Riemann, \(\theta(\zeta)\) is an entire, even function that satisfies

\[
\theta(\zeta + l + mB) = \theta(\zeta) \exp \{-\pi im^2B - 2\pi im\zeta\} \tag{2.19}
\]

for any integers \(l, m\). Moreover, its zeros are simple and \(\theta(\zeta) = 0\) if and only if \(\zeta = [(1 + B)/2]\). The constant \((1 + B)/2\), known as the Riemann constant, will appear often in our computations. So, we choose to abbreviate the representatives of its “half”-classes by

\[
K_+ := (1 + B)/4 \quad \text{and} \quad K_- := (1 - B)/4, \tag{2.20}
\]

i.e., \([2K_+] = [2K_-]\). The symmetries of \(\Omega(z)\) \((\Omega(-z) = -\Omega(z) = \Omega(z^*)\) yield that

\[
\int_{\infty}^0 \Omega = \frac{1}{2} \int_\delta \Omega = 2K_+ \quad \Rightarrow \quad \int_{a_3}^{\infty} = (-1)^kK_+, \tag{2.21}
\]

for \(k \in \{0, 1\}\), where \(\delta = \pi^{-1}\left((-\infty, -a] \cup [a, \infty)\right)\) is a cycle on \(\mathfrak{M}\) oriented from \((1)\) to \((0)\) (on Figure 2, \(\delta\) would be represented by the anti-diagonal), which is clearly homologous to \(\alpha + \beta\).

With \(c_\rho\) given by (2.16), define

\[
T_k(z) := \exp \left\{ \frac{\pi ik}{2} \int_{a_3}^z \Omega \right\} \frac{\theta \left( \int_{a_3}^z \Omega - c_\rho - (-1)^kK_+ \right)}{\theta \left( \int_{a_3}^z \Omega - K_+ \right)} \tag{2.22}
\]

for \(k \in \{0, 1\}\) and \(z \in \mathfrak{M}_{\alpha, \beta}\), where the path of integration lies entirely within \(\mathfrak{M}_{\alpha, \beta}\). Each \(T_k(z)\) is a meromorphic function that is finite and non-vanishing except for a simple pole at \(\infty^{(1)}\), see (2.21), and a simple zero at \(z_k := z_{c_\rho - (-1)^kK_+}\), where \(z_k \in \mathfrak{M}\) is uniquely characterized by

\[
\int_{a_3}^{z_k} \Omega = c_\rho - (-1)^kK_+ + l_k + m_kB, \tag{2.23}
\]
$k \in \{0, 1\}$, for some $l_0, m_0, l_1, m_1 \in \mathbb{Z}$. Furthermore, it follows from
the normalization in (2.2), the definition of $B$ in (2.3), and (2.19) that

$$
T_k^+(s) = T_k^-(s) \left\{ \begin{array}{ll}
\exp \{2\pi i(k/2 - c_\rho)\}, & s \in \alpha, \\
\exp \{\pi ik\}, & s \in \beta.
\end{array} \right.
$$  

(2.24)

Now we are ready to define the function that will be responsible for the
asymptotic behavior of the polynomials $Q_n(z)$. Given $\rho(s) \in W_1$, let $c_\rho$ be
defined by (2.16). Set

$$
\{0, 1\} \ni \nu(n) := n \mod 2, \quad n \in \mathbb{Z},
$$

to be the parity function. Then it follows from (2.7), (2.17), and (2.24) that
the function

$$
\Psi_n(z) := (\Phi^n S_\rho T_{\nu(n)})(z), \quad z \in \mathbb{R} \setminus \Delta,
$$

(2.25)
is meromorphic in $\mathbb{R} \setminus \Delta$ with a pole of order $n$ at $\infty(0)$, a zero of multiplicity
$n - 1$ at $\infty(1)$, a simple zero at $z_{\nu(n)}$, and otherwise non-vanishing and finite,
whose traces on $\Delta$ satisfy

$$
\Psi_{n+}(s) = \Psi_{n-}(s)/(\rho w_+)(s), \quad s \in \Delta,
$$

(2.26)
and whose behavior around the ramification points of $\mathbb{R}$ as well as $0*, 0$ is
governed by (2.18).

### 2.5. Asymptotics

In this section we formulate the main theorem on the behavior of the poly-
nomials $Q_n(z)$. As was alluded to in the introduction, we do not expect to be
able to handle all the possible indices $n$ as $Q_n(s)$ might have degree smaller
than $n$. One source of this degeneration already can be seen from (2.25) since
this function can have a pole of order $n - 1$ at $\infty(0)$ when $z_{\nu(n)} = \infty(0)$. In
fact, this is the only reason for the degeneration in the generic cases described
in [1]. However, this is no longer the case for the considered model.

To restrict the indices we need the following, unfortunately very tech-
nical, definition. Let us set

$$
\varsigma_{\nu} := \left\{ \begin{array}{ll}
1, & \text{Re}(\nu) > 0, \\
-1, & \text{Re}(\nu) < 0,
\end{array} \right. \quad \text{and} \quad \mathfrak{o} := \left\{ \begin{array}{ll}
0, & \text{Re}(\nu) > 0, \\
0^*, & \text{Re}(\nu) < 0.
\end{array} \right.
$$

(2.27)

We do not make any choice for $\varsigma_{\nu}$ and $\mathfrak{o}$ when $\text{Re}(\nu) = 0$. Given $\rho(s) \in W_1$
and the constant $c_\rho$ from (2.16), define

$$
A_{\rho, n} := \left\{ \begin{array}{ll}
\sigma_{\nu(n)} A'_{\rho, n} \Phi(z_{\nu(n)}) \Phi^{2(n-1)}(\mathfrak{o}), & \text{Re}(\nu) \neq 0, \\
0, & \text{Re}(\nu) = 0,
\end{array} \right.
$$

(2.28)

where $\sigma_k := (-1)^{l_k + m_k + k}$, $k \in \{0, 1\}$, see (2.23), and

$$
A'_{\rho, n} := A_{\rho} e^{\pi i \varsigma_{\nu}(c_{\nu} + 1/4)} \sqrt{\frac{a + b^2}{2}} \frac{\Gamma(1 - \varsigma_{\nu} \nu)}{\sqrt{2\pi}} \times
$$

$$
\lim_{z \to 0, \arg(z) = 5\pi/4} |z|^{2\nu} S_{\rho}^{2}(z^{(0)}) \varsigma_{\nu}(ab \frac{2n}{\rho})^{1/2 - \varsigma_{\nu} \nu},
$$
and
\[ A_\rho := e^{\pi i \rho_3(0)} \frac{(\rho_2 + \rho_3)(0)}{\rho_2(0)} \quad \text{or} \quad A_\rho := \frac{1}{(ab)^2} \frac{(\rho_3 + \rho_4)(0)}{(\rho_3 \rho_4)(0)} \]
depending on whether \( \Re(\nu) > 0 \) or \( \Re(\nu) < 0 \) (it follows from the last display in Section 5, devoted to the proof of Proposition 2.1, that the limit in the definition of the constant \( A'_{\rho,n} \) is indeed well defined).

Given the above constants \( A_{\rho,n} \) and \( \epsilon \in (0, 1/2) \), we define subsequences of allowable indices \( n \) for the weight \( \rho(s) \) by
\[
N_{\rho,\epsilon} := \left\{ n \in \mathbb{N} : z_{\iota(n)} \neq 0^{(0)} \text{ and } |1 - A_{\rho,n}| \geq \epsilon \right\}. \tag{2.29}
\]
The following proposition states that such sequences are non-empty.

**Proposition 2.2.** Let \( N_{\rho,\epsilon} \) be given by (2.29). If \( [c_\rho] = [0] \) or \( [c_\rho] = [(1 + B)/2] \), then it holds that
\[
N_{\rho,\epsilon} = N_{\rho} := \left\{ \begin{array}{ll}
2N & \text{when } [c_\rho] = [0], \\
N \setminus 2N & \text{when } [c_\rho] = [(1 + B)/2].
\end{array} \right. \tag{2.30}
\]
If \( [c_\rho] \neq [0] \) and \( [c_\rho] \neq [(1 + B)/2] \) while \( \Re(\nu) \in (-1/2, 1/2) \), it holds that
\[
N_{\rho,\epsilon} = N_{\rho} := N. \tag{2.31}
\]
If \( [c_\rho] \neq [0] \) and \( [c_\rho] \neq [(1 + B)/2] \), and \( \Re(\nu) = 1/2 \), then \( N_{\rho,\epsilon} \) is an infinite subsequence with gaps of size at most 2 (clearly, this is the only case when \( N_{\rho,\epsilon} \) might depend on \( \epsilon \)).

**Proof.** It readily follows from (2.23) and (2.21) that
\[
[c_\rho] = [k(1 + B)/2] \iff z_1 = \infty^{(k)} \iff z_0 = \infty^{(1-k)}
\]
for \( k \in \{0, 1\} \) (in which case \( \Phi(z_{\iota(n)}) = \Phi(\infty^{(1)}) = 0 = A_{\rho,n} \)). On the other hand, because Abel’s map is a bijection, we also get that \( |\pi(z_1)| < \infty \iff |\pi(z_0)| < \infty \). This proves (2.30). Observe that
\[
A_{\rho,n} = B_{\rho,\iota(n)} \Phi(o)^{2(n-1)} n^{-\nu-1/2}, \tag{2.32}
\]
where \( B_{\rho,\iota(n)} \) depends only on the parity of \( n \) and \( |\Phi(o)| = 1 \) by (2.11). Hence, \( A_{\rho,n} \to 0 \) as \( n \to \infty \) when \( \Re(\nu) \in (-1/2, 1/2) \), which proves (2.31). In the remaining situation,
\[
A_{\rho,n} = B_{\rho,\iota(n)} \exp\left\{ 2(n-1)i \arctan(a/b) + i\Im(\nu) \log n \right\}
\]
by (2.11). If \( |B_{\iota(n)}| \neq 1 \), then, in fact, \( N_{\rho,\epsilon} = N. \) Otherwise, we have that
\[
A_{\rho,n+2}/A_{\rho,n} = \exp\left\{ 2i \arctan(a/b) + i\Im(\nu) \log(1 + 2/n) \right\}.
\]
As \( \arctan(a/b) \in (0, \pi/2) \) and \( \log(1 + 2/n) = o(1) \), both constants \( A_{\rho,n+2} \) and \( A_{\rho,n} \) cannot be simultaneously close to 1. \( \square \)

When \( \Re(\nu) < 1/2 \), the sequence \( N_{\rho,\epsilon} = N_{\rho} \) is equal to the whole set of the natural numbers or consists of every other one. This is consistent with the explanation given at the beginning of the subsection and is supported by the examples in Sections 3.1 and 3.2 where two weights \( \rho(s) \) are provided for which \( Q_{2n}(z) = Q_{2n+1}(z) \). As mentioned before, this is a generic behavior.
observed in [1]. On the technical level this degeneration manifests itself as our inability to construct the “global parametrix”, see Section 6.3, since we are no longer able to properly renormalize \( Q_n(z) \) by \( \Psi_n(z^{(0)}) \) when \( z_{n(n)} = \infty^{(0)} \).

When \( \text{Re}(\nu) = 1/2 \), new phenomenon occurs. The sequence \( \mathbb{N}_{\rho,\ell} \) can have gaps of size 2 depending on the behavior of the constants \( A_n, \rho \). This suggests that there might be indices \( n \) such that \( Q_n(z) = Q_{n+1}(z) = Q_{n+2}(z) \). Such a possibility can in fact occur, see Section 3.3 for an example. On the technical level, the second condition in (2.29) appears in an attempt to match the behavior of \( Q_n(z) \) at the origin, that is, during the construction of the so-called “local parametrix”, see Sections 6.5 and 6.6, and manifests itself through the constants \( L_{n,i} \), see (2.38).

Recall that the weight \( p(s) \) defines two constants: \( \ell \), which says how well the restrictions of \( p(s) \) to different segments \( \Delta_i \) match each other at the origin, and \( \nu \), defined in (2.12). Our analysis does not allow us to handle all possible combinations of these constants. In what follows we assume that

\[
|\text{Re}(\nu)| \in \begin{cases} 
[0, \sqrt{7}/2 - 1) & \text{when } \ell = 1, \\
[0, 1/2) & \text{when } \ell = 2, \\
[0, 1/2] & \text{when } \ell > 3.
\end{cases} 
\tag{2.33}
\]

This technical condition appears in the rate of decay of the error, which we quantify by the following exponent:

\[
d_{\nu,\ell} := \begin{cases} 
\frac{2(1 + |\text{Re}(\nu)|)(\ell-2|\text{Re}(\nu)|)}{\ell^2 + 1 + 2|\text{Re}(\nu)|} & \ell \geq \frac{4|\text{Re}(\nu)|(1+|\text{Re}(\nu)|)}{1-2|\text{Re}(\nu)|}, \\
\frac{\ell(3-2|\text{Re}(\nu)|)-2|\text{Re}(\nu)||3+2|\text{Re}(\nu)|)}{2(\ell+3+2|\text{Re}(\nu)|)} & \text{otherwise},
\end{cases}
\tag{2.34}
\]

where we understand that \( d_{\nu,\infty} = 1/2 + |\text{Re}(\nu)| \). It is a straightforward computation to check that requiring positivity of the numerator of \( d_{\nu,\ell} \) in the second line of (2.34) produces restriction (2.33). Observe also that \( d_{1/2,\ell} = \frac{\ell - 2}{\ell + 4} \).

**Theorem 2.3.** Let \( p(s) \in \mathcal{W}_\ell \), where \( \ell \) is a positive integer or infinity. Define \( \nu \) by (2.12) and assume that (2.33) is satisfied. Let \( \Psi_n(z) \) be given by (2.25) and \( \mathbb{N}_{\rho,\varepsilon} \) be as in (2.29) for some \( \varepsilon \in (0, 1/2) \) fixed. Then it holds for all \( n \in \mathbb{N}_{\rho,\varepsilon} \) large enough that

\[
Q_n(z) = \gamma_n (1 + v_{n1}(z)) \Psi_n(z^{(0)}) + \gamma_n v_{n2}(z) \Psi_{n-1}(z^{(0)}) \tag{2.35}
\]

for \( z \in \mathbb{C} \setminus \Delta \), where \( \gamma_n := \lim_{z \to \infty} z^n \Psi_n^{-1}(z^{(0)}) \) is the normalizing constant,

\[
Q_n(s) = \gamma_n (1 + v_{n1}(s)) \left( \Psi_n^{(0)}(s) + \Psi_n^{(0)}(-s) \right) + \gamma_n v_{n2}(s) \left( \Psi_{n-1}^{(0)}(s) + \Psi_{n-1}^{(0)}(-s) \right) \tag{2.36}
\]

for \( s \in \Delta^\circ \), where \( \Psi_n^{(0)}(s) \) are the traces of \( \Psi_n(z^{(0)}) \) on the positive and negative sides of \( \Delta \). The functions \( v_{n1}(z) \) are such that

\[
v_{n1}(\infty) = 0 \text{ and } v_{n1}(z) = L_{n,i} z^{-1} + \mathcal{O}(n^{-d_{\nu,\ell}}) \tag{2.37}
\]
where \( O(\cdot) \) holds locally uniformly on \( \overline{\mathbb{C}} \setminus \Delta \) in (2.35) and on \( \Delta^\circ \) in (2.36), \( d_{\nu,\ell} \) was defined in (2.34), and \( L_{ni} \) are constants given by

\[
L_{ni} = (-1)^{(n)} \frac{A_{\rho,n}}{1 - A_{\rho,n}} \left( - \frac{\Phi T_{i(n)}}{T_{i(n-1)}} \right)^{i-1} (o) \frac{(T_0/T_1)(o)}{(T_0/T_1)'(o)}
\]

when \(|\pi(z_k)| < \infty, i \in \{1, 2\}\), where \( o \) was defined in (2.27) (when \(|\pi(z_k)| = \infty, the expressions for \( L_{ni} \) are even more cumbersome and therefore are omitted here).

Notice that the behavior of the polynomials \( Q_n(z) \) is qualitatively different for \( \text{Re}(\nu) < 1/2 \) and \( \text{Re}(\nu) = 1/2 \) as the first summand in (2.37) is decaying in the former case by (2.32), but does not decay in the latter.

Recall that the traces of \( \Phi(z) \) are unimodular on \( \Delta \), see (2.8). Since \( \Psi_n(z) = (S_nT_{i(n)})(z)\Phi^n(z) \), it is exactly the sum of the terms \((\Phi^{(0)}_+(s))^n\) and \((\Phi^{(0)}_-(s))^n\) that creates oscillations describing the zeros of \( Q_n(z) \). Of course, since the traces of \((S_nT_{i(n)})_{\pm}^{(0)}(s)\) are in general complex-valued, the zeros of \( Q_n(z) \) do not lie exactly on \( \Delta \). However, we do prove that (2.36) holds on compact subsets “close” to \( \Delta^\circ \), where \( \Psi_{i\pm}^{(0)}(s) \) are analytically continued from \( \Delta^\circ \) into the complex plane with the help of (2.26).

When \( \ell < \infty \), we cannot control the error functions \( v_{ni}(z) \) around the origin and therefore cannot describe the polynomials \( Q_n(z) \) there (however, we can extend (2.36) to hold on a sequence of compact subsets of \( \Delta^\circ \) that are allowed to approach the origin with a certain speed at the expense of worsening the rate of decay in the error estimates). When \( \ell = \infty \), we can provide an asymptotic formula for \( Q_n(z) \) around the origin, but due to its technical nature we placed it at the very end of the paper in Section 6.9.

Theorem 2.3, as well as Theorem 2.4 further below, is proved in Section 6 with the derivation of some technical identities relegated to Section 4.

### 2.6. Padé Approximation

For an integrable weight \( \rho(s) \) on \( \Delta \) define

\[
\hat{\rho}(z) := \frac{1}{2\pi i} \int_{\Delta} \frac{\rho(s)ds}{s - z}, \quad z \in \overline{\mathbb{C}} \setminus \Delta.
\]

In particular, it can be readily verified that the functions \( \sum_{i=1}^{4} C_i \log(z - a_i) \) and \( \prod_{i=1}^{4} (z - a_i)^{\alpha_i} \), where the constants \( C_i \) add up to zero and the exponents \(-1 < \alpha_i \notin \mathbb{Z}\) add up to an integer, possess branches holomorphic off \( \Delta \) that can be represented by (2.39) for certain weight functions in \( \mathcal{W}_\infty \) (the second function can be represented by (2.39) up to an addition of a polynomial).

Given \( \hat{\rho}(z) \), it follows from orthogonality relations (1.2) that

\[
R_n(z) := \frac{1}{2\pi i} \int_{\Delta} \frac{Q_n(s)\rho(s)ds}{s - z} = O(z^{-n-1}) \quad \text{as} \quad z \to \infty.
\]

Observe also that \( R_n(z) \) can be rewritten as

\[
R_n(z) = (Q_n\hat{\rho})(z) + \frac{1}{2\pi i} \int_{\Delta} \frac{Q_n(s) - Q_n(z)}{s - z}\rho(s)ds = (Q_n\hat{\rho})(z) - P_n(z),
\]
where \( P_n(z) \) is a polynomial of degree at most \( n - 1 \). The rational function \( (P_n/Q_n)(z) \) is called the \( n \)-th diagonal Padé approximant of \( \hat{\rho}(z) \).

**Theorem 2.4.** Let \( \hat{\rho}(z) \) be given by (2.39) and \( R_n(z) \) be defined by (2.40). In the setting of Theorem 2.3, it holds for all \( n \in \mathbb{N}_{\rho,\varepsilon} \) large enough that

\[
(wR_n)(z) = \gamma_n(1 + v_{n1}(z))\Psi_n(z^{(1)}) + \gamma_nv_{n2}(z)\Psi_{n-1}(z^{(1)})
\]

locally uniformly in \( \overline{C} \setminus \Delta \), where \( v_{n1}(z) \) are the same as in Theorem 2.3.

Theorem 2.4 has the following consequences for Padé approximation: it holds that

\[
\hat{\rho}(z) - \frac{P_n(z)}{Q_n(z)} = \frac{R_n(z)}{Q_n(z)} \times \frac{1}{w(z)(S^2\Phi^{2n-1})(z^{(0)})} \times \frac{(1 + v_{n1}(z))(\Phi T_{i(n)})(z^{(1)}) + v_{n2}(z)T_{i(n-1)}(z^{(1)})}{(1 + v_{n1}(z))T_{i(n)}(z^{(0)}) + v_{n2}(z)(T_{i(n-1)}/\Phi(z^{(0)})},
\]

where we used (2.25) and the fact that \( S_{\rho}(z)S_{\rho}(z^*) \equiv 1 \) and \( \Phi(z)\Phi(z^*) \equiv 1 \). It follows from (2.8) that the first fraction on the right-hand side of the equality above is geometrically small in \( \overline{C} \setminus \Delta \) with the zero of order \( 2n + 1 \) at infinity. However, if \( z_{i(n)} \in \mathcal{R}^{(0)} \setminus \Delta \), the second fraction, and hence the Padé approximant, will have a pole in the vicinity of \( z_{i(n)} \) by Rouche’s theorem, which will prevent the convergence around \( z_{i(n)} \). On the other hand, if \( z_{i(n)} \in \mathcal{R}^{(1)} \setminus \Delta \), then Rouche’s theorem yields that the Padé approximant has an additional interpolation point near \( z_{i(n)} \).

### 3. Examples

In this section, we illustrate Theorem 2.3 by three examples. In them, we shall not compute \( S_{\rho}(z) \) and \( c_{\rho} \) via their integral representations, (2.14) and (2.16), but rather construct a candidate \( \hat{S}_{\rho}(z) \) with the desired jump over \( \Delta \) and the singular behavior as in (2.18). This construction will also determine a candidate constant \( \hat{c}_{\rho} \). It is simple to argue that

\[
S_{\rho}(z) = \hat{S}_{\rho}(z)\exp\left\{2\pi im\int_{a_3}^{z} \Omega \right\}, \quad c_{\rho} = \hat{c}_{\rho} - mB,
\]

for some integer \( m \). Using \( \hat{c}_{\rho} \) in (2.22), we then construct \( \hat{T}_{i(n)}(z) \) for which it holds that

\[
T_{i(n)}(z) = \hat{T}_{i(n)}(z)\exp\left\{-2\pi im\int_{a_3}^{z} \Omega - \pi im^2B + 2\pi i(-1)^{(n)}K_+ \right\}
\]

with the same integer \( m \). This means that

\[
(S_{\rho}T_{i(n)})(z)/(S_{\rho}T_{i(n)})(\infty(0)) = (\hat{S}_{\rho}\hat{T}_{i(n)})(z)/(\hat{S}_{\rho}\hat{T}_{i(n)})(\infty(0))
\]

and therefore (2.35) and (2.41) remain valid with \( S_{\rho}(z) \), \( T_{i(n)}(z) \) replaced by \( \hat{S}_{\rho}(z), \hat{T}_{i(n)}(z) \). Furthermore, the value of \( A_{\rho,n} \) in (2.28) will not change either as the limit in the definition of \( A_{\rho,n}' \) will be augmented by \( e^{\pi im(1-B)} \), see (4.1),
that will be offset by the change in \( c_\rho \) and \( \sigma_k \) \((\bar{\sigma}_k = (-1)^m \sigma_k)\). Thus, with a slight abuse of notation, we shall keep on writing \( S_\rho(z) \), \( T_{i(n)}(z) \) below.

### 3.1. Chebyshëv-type case

Let \( 2\bar{\rho}(z) = 1/w(z) \), in which case it holds that

\[
\rho(s) = 1/w_+(s), \quad s \in \Delta,
\]

where \( \bar{\rho}(z) \) and \( w(z) \) were defined in (2.39) and (2.1), respectively, and the implication follows from the Plemelj-Sokhotski formulae and Privalov’s theorem. Using analytic continuations of \( w(z) \) one can easily see that \( \rho(s) \in W_\infty \) and \( \nu = 0 \). Since \((\rho w_+)(s) \equiv 1\), we get that \( S_\rho(z) \equiv 1 \) and necessarily \( c_\rho = 0 \). Thus, \( N_{\rho,\varepsilon} = 2N \) and \( z_0 = \infty^{(1)} \) \((z_1 = \infty^{(0)})\). Moreover, we get that \( T_0(z) \equiv 1 \) and \( T_1(z) = 1/\Phi(z) \), see (4.2). Hence, it follows from (2.9) and (2.35) that

\[
Q_{2n}(z) = \frac{1 + o(1)}{2^n} \left(z^2 + \frac{b^2 - a^2}{2} + w(z)\right)^n,
\]

where it holds that \( o(1) \) is geometrically small on closed subsets of \( \mathbb{C} \setminus \Delta \) (see [20] for the error rate in this case). To show that the above result is in a way best possible, assume that \( a = b = 1 \). Recall that the \( n \)-th monic Chebyshëv polynomial of the first kind is defined by

\[
2^n T_n(z) = \left(z + \sqrt{z^2 - 1}\right)^n + \left(z - \sqrt{z^2 - 1}\right)^n
\]

and is orthogonal to \( x^j \), \( j \in \{0, \ldots, n - 1\} \), on \((-1, 1)\) with respect to the weight \( 1/\sqrt{1-x^2} \). Hence,

\[
i \int_{\Delta} s^k T_n(s^2) \rho(s) ds = \left(\int_0^1 - \int_{-1}^0\right) \frac{x^k T_n(x^2) dx}{\sqrt{1-x^4}} - i^{k+1} \left(\int_0^1 - \int_{-1}^0\right) \frac{x^k T_n(-x^2) dx}{\sqrt{1-x^4}}.
\]

Clearly, the above expression is zero for all even \( k \). Assume now that \( k = 2j+1 \), \( j \in \{0, \ldots, n - 1\} \). Then we can continue the above chain of equalities by

\[
\int_0^1 x^j T_n(x) dx - (-1)^{j+1} \int_0^1 x^j T_n(-x) dx = \int_{-1}^1 \frac{x^j T_n(x) dx}{\sqrt{1-x^2}} = 0,
\]

where the last equality follows from the orthogonality properties of the Chebyshëv polynomials. Thus, it holds that

\[
Q_{2n+1}(z) = Q_{2n}(z) = T_n(z^2)
\]

in this case, which justifies the exclusion of odd indices from \( N_\rho = N_{\rho,\varepsilon} \) as for such indices polynomials can and do degenerate.
3.2. Legendre-type case

Let \( \tilde{\rho}(z) = \frac{1}{2\pi i} \left( \log(z^2 - 1) - \log(z^2 + 1) \right) \), in which case it holds that

\[
\rho(s) = (-1)^i, \quad s \in \Delta_i,
\]
i \( \in \{1, 2, 3, 4\} \), where the justification for the implication is the same as before. As in the previous case, it holds that \( \nu = 0 \). Let \( \sqrt{w}(z) \) be the branch holomorphic in \( \mathbb{C} \setminus \Delta \) such that \( \sqrt{w}(z) = z + \mathcal{O}(1) \) as \( z \to \infty \). Further, let

\[
\Phi_*(z) := \sqrt{\frac{2}{a^2 + b^2}} \left( z^2 + \frac{b^2 - a^2}{2} + w(z) \right)^{1/2},
\]
be the branch holomorphic in \( \mathbb{C} \setminus \Delta \) such that \( \Phi_*(z) = z + \mathcal{O}(1) \) as \( z \to \infty \). It easily follows from (2.7), (2.9), and (2.10) that \( \Phi_*(z) \) is an analytic continuation of \( -\Phi(z^{(0)}) \) across \( \pi(\alpha) \cup \pi(\beta) \). It is now straightforward to check that

\[
S_\rho(z^{(0)}) = e^{-\pi i/4} \Phi_*(z)/\sqrt{w}(z)
\]
and thus \( c_\rho = 0 \). Hence, as in the previous subsection, \( \mathbb{N}_{\rho, e} = 2\mathbb{N} \) and \( T_0(z) \equiv 1 \) while \( T_1(z) = 1/\Phi(z) \). Therefore, we again deduce from (2.9) and (2.35) that

\[
Q_{2n}(z) = \frac{1 + \mathcal{O}(n^{-1/2})}{2^{n+1/2}\sqrt{w}(z)} \left( z^2 + \frac{b^2 - a^2}{2} + w(z) \right)^{n+1/2},
\]
uniformly on closed subsets of \( \mathbb{C} \setminus \Delta \). Again, to show that the above result is best possible, assume that \( a = b = 1 \). Then we can check exactly as in the previous subsection that

\[
Q_{2n+1}(z) = Q_{2n}(z) = L_n(z^2),
\]
where \( L_n(x) \) is the \( n \)-th monic Legendre polynomial, that is, degree \( n \) polynomial orthogonal to \( x^j, \ j \in \{0, \ldots, n - 1\} \), on \((-1, 1)\) with respect to a constant weight.

3.3. Jacobi-1/4 case

Let \( \sqrt{2\tilde{\rho}}(z) = 1/\sqrt{w}(z) \), in which case it holds that

\[
\rho(s) = -i^{4-i}/|\sqrt{w}(s)|, \quad s \in \Delta_i, \quad i \in \{1, 2, 3, 4\},
\]
where \( \sqrt{w}(z) \) is the branch defined in the previous subsection. Observe that

\[
(\rho w_+)(s) = i^{i-1}|\sqrt{w}(s)|, \quad s \in \Delta_i,
\]
and that \( \nu = 1/2 \). In particular, the constant \( A_\rho \) appearing in the definition of \( A_{\rho, n} \) in (2.28) is equal to \( A_\rho = \sqrt{2e^{-\pi i/4}/ab} \).

To construct a Szegő function of \( \rho(s) \), let

\[
\Theta^2(z) := \frac{\theta(\int_{a_3}^z \Omega + K_-) \theta(\int_{a_3}^z \Omega - K_+)}{\theta(\int_{a_3}^z \Omega - K_-) \theta(\int_{a_3}^z \Omega + K_+)} , \quad z \in \mathfrak{R}_{\alpha, \beta},
\]
where the path of integration lies entirely in \( \mathfrak{R}_{\alpha, \beta} \). It follows from (2.21) and (4.1) further below that \( \Theta^2(z) \) is a meromorphic function in \( \mathfrak{R}_{\alpha, \beta} \) with two simple poles, namely, \( \infty^{(0)}, 0 \), and two simple zeros \( \infty^{(1)}, 0^* \). Moreover, \( \Theta^2(z) \) is continuous across \( \beta \) and satisfies \( \Theta^2_+(s) = \Theta^2(s)e^{-2\pi iB} \) on \( \alpha \) by (2.19) and
\(\Theta^2(z)\Theta^2(z^*) = 1\) by the symmetries of \(\theta(\zeta)\) and \(\Omega(z)\). Since each individual fraction in the definition of \(\Theta^2(z)\) is injective, we can define a branch \(\Theta(z)\) such that

\[
\Theta_+(s) = \Theta_-(s) \begin{cases} 
 e^{-\pi i b}, & s \in \alpha, \\
 -1, & s \in \Delta_3 \cup \pi^{-1}((-\infty, -a]),
\end{cases}
\]

and \(\Theta(z)\Theta(z^*) = 1\). Further, let \(w^{1/4}(z)\) be the branch holomorphic in \(\mathbb{C} \setminus (\Delta \cup (-\infty, a))\) that is positive for \(z > a\). Now, one can verify that \(c_\rho = -B/2\) and

\[
S_\rho(z^{(k)}) = \Theta(z^{(k)})w^{2k-1/4}(z), \quad k \in \{0, 1\}.
\]

Let us now compute \(A^*_\rho,n\) appearing in (2.28). Since \(\sqrt[w]{w(z)} \to e^{-3\pi i/4}\sqrt{ab}\) as \(Q_3 \ni z \to 0\), we get that

\[
\lim_{z \to 0, \text{arg}(z) = 5\pi/4} |z|S_\rho(z^{(0)}) = \frac{e^{-\pi i/2}}{\sqrt[4]{ab}} \lim_{Q_3 \ni z \to 0} z\Theta^2(z^{(0)}) = \frac{e^{\pi i B/2}}{\sqrt[4]{a+b^2}} \Phi(0),
\]

where the second equality follows from (4.1), (4.5), (4.9), and (4.10) further below. Therefore, it holds that \(A^*_\rho,n = \Phi(0)\). It is easy to see from (4.1) that \(z_0 = 0, l_0 = 0, m_0 = 1\), and \(z_1 = 0^*, l_1 = m_1 = 0\). Therefore, \(\sigma_i(n) = -1\) and the condition defining \(N_{\rho,\varepsilon}\) in Proposition 2.2 specializes to

\[
|1 + \exp\{2i(n - i(n))\arctan(a/b)\}| > \varepsilon
\]

by (2.11) and since \(\Phi(z_1)\Phi(z_0) = 1\), see (4.4) further below. As \(T_0(0) = 0\) and respectively \(L_{n1} = 0\), we then get that \(Q_n(z), n \in \mathbb{N}_{\rho,\varepsilon}\), is equal to

\[
\gamma_n(S_\rho\Phi^n)(z^{(0)}) \begin{cases} 
 (T_0(z^{(0)}) + O_{\varepsilon}(n^{-1})), & n \in 2\mathbb{N}, \\
 (T_1(z^{(0)}) + z^{-1}L_{n2}(T_0/\Phi)(z^{(0)}) + O_{\varepsilon}(n^{-1})), & n \not\in 2\mathbb{N},
\end{cases}
\]

uniformly on closed subsets of \(\bar{\mathbb{C}} \setminus \Delta\), where

\[
L_{n2} = \frac{-1}{(T_0/T_1)'(0)} \frac{\Phi^{2n-1}(0)}{1 + \Phi^{2n-1}(0)}
\]

for all odd \(n\). When \(a = b\), we further get that \(L_{n2} = -e^{\pi i/4}/[2(T_0/T_1)'(0)]\) for \(n \in \mathbb{N}_{\rho,\varepsilon}\) and

\[
N_{\rho,\varepsilon} = \{n = 4k, 4k + 1 : k \in \mathbb{N}\}.
\]

Assume further that \(a = b = 1\) and let \(P_{n,1}(x)\) be the \(n\)-th degree monic polynomial orthogonal on \([0, 1]\) to \(x^j, j \in \{0, \ldots, n-1\}\), with respect to the weight function \(x^{-3/4}(1-x)^{-1/4}\). Then

\[
\int_{\Delta} s^k P_{n,1}(s^4)\rho(s) ds = (1 + i^k) \int_{-1}^{1} y^k P_{n,1}(y^4) \frac{dy}{(1-y^4)^{1/4}},
\]

which is equal to zero for all \(k\) odd by symmetry and for all \(k = 4j + 2\) due to the factor \(1+i^k\). When \(k = 4j, j \in \{0, \ldots, n-1\}\), we can further continue
the above equality by
\[ 4 \int_0^1 y^4 P_{n,1}(y^4) \frac{dy}{(1-y^4)^{1/4}} = \int_0^1 x^j P_{n,1}(x) \frac{dx}{x^{3/4}(1-x)^{1/4}} = 0, \]
where the last equality now holds by the very choice of \( P_{n,1}(z) \). Hence, it holds that
\[ Q_{4n}(z) = P_{n,1}(z^4) \quad \text{and} \quad Q_{4n+1}(z) = Q_{4n+2}(z) = Q_{4n+3}(z) = z P_{n,2}(z^4), \]
where the second set of relations can be shown similarly with \( P_{n,2}(x) \) being the \( n \)-th degree monic polynomial orthogonal on \([0, 1]\) to \( x^j, j \in \{0, \ldots, n-1\} \), with respect to the weight function \( x^{1/4}(1-x)^{-1/4} \). That is, the restriction to the sequence of indices \( \{n = 4k, 4k+1 : k \in \mathbb{N}\} \) is not superfluous and the main term of the asymptotics of the polynomials does depend on the parity of \( n \).

4. Auxiliary Identities

In this section we state a number of identities, some of which we have already used and some of which we shall use later.

**Lemma 4.1.** Recall (2.20). It holds that
\[ \int_{\alpha_3}^0 \Omega = -K_- \quad \text{and} \quad \int_{\alpha_3}^{0^*} \Omega = K_-, \quad (4.1) \]
where the path of integration lies entirely in \( \Re \alpha, \beta \).

**Proof.** Exactly as in the case of (2.21), the symmetries of \( \Omega(z) \) imply that
\[ -\int_{\alpha_3}^{0} \Omega = \int_{\alpha_3}^{0^*} \Omega = \frac{1}{2} \int_{\Delta_3} \Omega = \frac{1}{4} \int_{\Delta_3 - \Delta_1} \Omega. \]
The claim now follows from the fact that \( \Delta_3 - \Delta_1 \) is homologous to \( \alpha - \beta \). \( \square \)

**Lemma 4.2.** It holds that
\[ \Phi(z) = \exp \left\{ -\pi i \int_{\alpha_3}^{z} \Omega \right\} \frac{\theta(\int_{\alpha_3}^{z} \Omega - K_+)\theta(\int_{\alpha_3}^{z} \Omega + K_+)}{\theta(\int_{\alpha_3}^{z} \Omega)}. \quad (4.2) \]

**Proof.** It follows from (2.21) and (2.19) that the right hand side of (4.2) is a meromorphic functions with a simple pole at \( \infty^{(0)} \), a simple zero at \( \infty^{(1)} \), and otherwise non-vanishing and finite that satisfies (2.7). As only holomorphic functions on \( \Re \) are constants, the normalization at \( \alpha_3 \) yields (4.2). \( \square \)

**Lemma 4.3.** Let \( l_0, l_1, m_0, m_1 \) be given by (2.23). Then it holds that
\[ \begin{cases} 
\Phi(z_0) = (-1)^{l_0+m_0} e^{-\pi i (c_\rho - K_+)} \frac{\theta(\int_{\alpha_3}^{z} \Omega - K_+)}{\theta(\int_{\alpha_3}^{z} \Omega + K_+)}, \\
\Phi(z_1) = (-1)^{l_1+m_1} e^{-\pi i (c_\rho + K_+)} \frac{\theta(\int_{\alpha_3}^{z} \Omega)}{\theta(\int_{\alpha_3}^{z} \Omega + K_+)}.
\end{cases} \quad (4.3) \]

In particular, when \( |\pi(z_k)| < \infty \), it holds that
\[ \Phi(z_0)\Phi(z_1) = -(-1)^{l_0-l_1+m_0-m_1}. \quad (4.4) \]
Moreover, we have that
\[
\Phi(0) = e^{\pi i K - \theta(1/2)/\theta(B/2)}.
\] (4.5)

\textbf{Proof.} Since \(-2K_+ = 2K_- - 1\), we get from (4.2) that
\[
\Phi(z_0) = e^{\pi i (K_+ - c_\rho - l_0 - m_0)} \frac{\theta(c_\rho + 2K_- + m_0 B)}{\theta(c_\rho + m_0 B)}.
\]
The first relation in (4.3) now follows from (2.19). Similarly, we have that
\[
\Phi(z_1) = e^{\pi i (-K_+ - c_\rho - l_1 - m_1 B)} \frac{\theta(c_\rho + m_1 B)}{\theta(c_\rho + 2K_+ + m_1 B)},
\]
which yields the second relation in (4.3), again by (2.19). To get (4.4), observe that
\[
\theta(c_\rho + 2K_-) = \theta(c_\rho + 2K_+ - B) = -e^{2\pi i c_\rho} \theta(c_\rho + 2K_+)
\]
by (2.19). Finally, (4.5) follows from (4.2) and (4.1). \hfill \Box

\textbf{Lemma 4.4.} Let
\[
X_n := \lim_{z \to \infty} z^{-2} \Psi_n(z^{(0)}) \Psi_{n-1}(z^{(1)}).
\] (4.6)

When \(|\pi(z_k)| < \infty\), it holds that
\[
X_n = \frac{4}{a^2 + b^2} \frac{\theta^2(c_\rho)}{\theta^2(0)} \frac{(-1)^{i(n)}}{\Phi^{2i(n)}(z_1)}.
\] (4.7)

\textbf{Proof.} Since \(\Phi(z)\Phi(z^*) \equiv 1\) and \(S_\rho(z)S_\rho(z^*) \equiv 1\), the desired limit is equal to
\[
\frac{4}{a^2 + b^2} T_{1(n)}(\infty^{(0)}) \lim_{z \to \infty} \Phi(z^{(1)}) T_{i(n-1)}(z^{(1)}),
\]
where we also used (2.10). Since \(-2K_+ = 2K_- - 1\), it follows from (2.22) and (2.21) that
\[
T_{i(n)}(\infty^{(0)}) = e^{\pi i n K_+} \frac{\theta(c_\rho + 2i(n) K_-)}{\theta(0)}.
\]

We further deduce from (2.22) and (4.2) that
\[
(\Phi T_{i(n-1)})(z) = \exp \left\{ -\pi i n \int_{a_3}^{z} \Omega \right\} \frac{\theta(\int_{a_3}^{z} \Omega - c_\rho + (-1)^{i(n) K_+})}{\theta(\int_{a_3}^{z} \Omega + K_+)}.
\]

Therefore, it follows from (2.21) that
\[
(\Phi T_{i(n-1)})(\infty^{(1)}) = e^{\pi i n K_+} \frac{\theta(c_\rho + 2i(n) K_+)}{\theta(0)}.
\]

Hence, we get from (4.3) that
\[
X_n = \frac{4}{a^2 + b^2} \frac{\theta^2(c_\rho)}{\theta^2(0)} \left( -1 \right)^{l_0 - l_1 + m_0 - m_1} \frac{\Phi(z_0)}{\Phi(z_1)}^{i(n)}.
\]

The claim of the lemma now follows from (4.4). \hfill \Box
Lemma 4.5. It holds that
\[ \frac{d}{d\zeta} \left( e^{\pi i \zeta} \frac{\theta(\zeta + K_+)}{\theta(\zeta - K_-)} \right) = i\pi \theta^2(0) e^{\pi i \zeta} \frac{\theta(\zeta - K_-) \theta(\zeta + K_+)}{\theta^2(\zeta - K_-)}. \]  
(4.8)

Proof. See [6, Eq. (20.7.25)] (observe that \( \theta(\zeta) = \theta_3(\pi \zeta | B) \) in the notation of [6, Chapter 20]). \( \square \)

Lemma 4.6. It holds that
\[ z = -\frac{\sqrt{a^2 + b^2}}{2} e^{-\pi i K_+} \frac{\theta^2(0)}{\theta(1/2) \theta(B/2) \theta(\int_{a_3}^z \Omega - K_-)} \theta(\int_{a_3}^z \Omega + K_-) \]  
(4.9)

Proof. It follows from (2.19), (2.21), and (4.1) that
\[ z = C \frac{\theta(\int_{a_3}^z \Omega - K_-) \theta(\int_{a_3}^z \Omega + K_-)}{\theta(\int_{a_3}^z \Omega - K_+) \theta(\int_{a_3}^z \Omega + K_+)} \]  
for some normalizing constant \( C \). It further follows from (2.10), (4.2), and (2.21) that
\[ -\frac{\sqrt{a^2 + b^2}}{2} = \lim_{z \to \infty} z \Phi^{-1}(z(0)) = C e^{\pi i K_+} \frac{\theta(1/2) \theta(B/2)}{\theta^2(0)}, \]  
which yields the desired result. \( \square \)

Lemma 4.7. It holds that
\[ e^{\pi i B/2} \frac{\theta^2(1/2) \theta^2(B/2)}{\theta^4(0)} = \frac{a^2 + b^2}{4ab}. \]  
(4.10)

Proof. To prove (4.10), evaluate (4.9) at \( a_3 \) to get
\[ \frac{\theta(1/2) \theta(B/2)}{\theta^2(0)} = \frac{\sqrt{a^2 + b^2}}{2a} e^{-\pi i K_+} \frac{\theta^2(K_-)}{\theta^2(K_+)}. \]  
Since \( \Delta_3 - \Delta_1 \) is homologous to \( \alpha - \beta \), one can easily deduce from Figure 1 that it also holds that
\[ \int_{a_3}^{a_2} \Omega = \left( \int_{a_3}^{a_1} \Omega + \int_{a_1}^{a_2} \right) = \frac{1}{2} \int_{\Delta_3 - \Delta_1 + \beta} \Omega = \frac{1}{2}, \]  
where the initial path of integration (except for \( a_2 \)) belongs to \( \Re_{\alpha, \beta} \). Thus, evaluating (4.9) at \( a_2 \) gives us
\[ \frac{\theta(1/2) \theta(B/2)}{\theta^2(0)} = -\frac{\sqrt{a^2 + b^2}}{2ib} e^{-\pi i K_+} \frac{\theta^2(K_+)}{\theta^2(K_-)}, \]  
where we used (2.19). Multiplying two expressions for \( \theta(1/2) \theta(B/2)/\theta^2(0) \) yields the desired result. \( \square \)

Lemma 4.8. It holds that
\[ \oint_{a} \frac{ds}{w(s)} = \frac{2\pi i}{\sqrt{a^2 + b^2}} e^{\pi i K_+} \theta(1/2) \theta(B/2). \]  
(4.11)
Proof. We can deduce from (4.2), (4.8), and the evenness of the theta function that

\[ \Phi'(z) = -i\pi \theta^2(0) \left( \int_{w(s)} \frac{ds}{\alpha w(s)} \right)^{-1} \frac{\Phi(z) \theta \left( \int_{a_3}^z \Omega + K_- \right) \theta \left( \int_{a_3}^z \Omega - K_- \right)}{w(z) \theta \left( \int_{a_3}^z \Omega + K_+ \right) \theta \left( \int_{a_3}^z \Omega - K_+ \right)}. \]

Since \( \Phi'(z) = z\Phi(z)/w(z) \) by (2.6), (4.11) follows from (4.9).

\[ \square \]

Lemma 4.9. Let

\[ Y_n := \left( T_{i(n)}' / T_{i(n-1)} / \Phi - T_{i(n)}(T_{i(n-1)}/\Phi') \right)(0). \quad (4.12) \]

When \( |\pi(z_k)| = \infty \), it holds that \( Y_n = 0 \), otherwise, we have that

\[ Y_n = (-1)^{l_0 + m_0 + i(n)} \frac{2e^{\pi ic\rho}}{a^2 + b^2} \frac{\Phi(z_0) \theta^2(c_\rho)}{\theta^2(0)}, \quad (4.13) \]

where the integers \( l_0, m_0 \) were defined in (2.23).

Proof. Since \( \Phi'(z) = z\Phi(z)/w(z) \) by (2.6), \( \Phi'(0) = 0 \). Therefore,

\[ Y_n = \left( T_{i(n-1)}/\Phi \right)(0) \left( T_{i(n)}/T_{i(n-1)} \right)'(0). \]

Assume that \( |\pi(z_k)| < \infty \). Then it follows from (2.22), (4.8), and (4.11) that

\[ \left( \frac{T_{i(n)}}{T_{i(n-1)}} \right)'(z) = -(-1)^{i(n)} \frac{\sqrt{a^2 + b^2}}{2w(z)} \frac{e^{-\pi i K_+} \theta^2(0)}{\theta(1/2) \theta(B/2)} \left( \frac{T_{i(n)}}{T_{i(n-1)}} \right)(z) \times \frac{\theta(\int_{a_3}^z \Omega - c_\rho + K_-) \theta(\int_{a_3}^z \Omega - c_\rho - K_-)}{\theta(\int_{a_3}^z \Omega - c_\rho + K_+) \theta(\int_{a_3}^z \Omega - c_\rho - K_+)} \times \theta(\int_{a_3}^z \Omega - c_\rho + K_+) \theta(\int_{a_3}^z \Omega - c_\rho - K_+). \]

We further deduce from (2.22), (4.1), and (4.5) that

\[ \left( T_{i(n-1)}T_{i(n)} \right)(0) = \frac{1}{\Phi(0)} \frac{\theta(c_\rho - B/2) \theta(c_\rho + 1/2)}{\theta(1/2) \theta(B/2)}. \]

Since \( w(0) = iab \), we therefore get from (4.1) that

\[ Y_n = \frac{\sqrt{a^2 + b^2}}{2ab} \frac{(-1)^{i(n)} \theta^4(0)}{\Phi^2(0)} \frac{\theta(c_\rho) \theta(c_\rho + 2K_-)}{\theta^2(1/2) \theta^2(B/2) \theta^2(0)}. \]

(4.13) now follows from (4.10) and the first formula in (4.3).

Let now \( z_0 = \infty^{(1)} \), in which case \( c_\rho = [0] \). Since \( \Phi(\infty^{(1)}) = 0 \), we get that \( Y_n = 0 \). Finally, let \( z_1 = \infty^{(1)} \). Then we have that \( c_\rho = -(-1)^k 2K_+ + l_k + m_k B \) and therefore

\[ \frac{T_1(z)}{T_0(z)} = \exp \left\{ \pi i \int_{a_3}^z \Omega \right\} \frac{\theta(\int_{a_3}^z \Omega + m_1 B + 3K_+)}{\theta(\int_{a_3}^z \Omega + (m_1 + 1) B - 3K_+)} = \exp \left\{ \pi i \int_{a_3}^z \Omega \right\} \frac{\theta(\int_{a_3}^z \Omega + (m_1 + 1) B - K_+)}{\theta(\int_{a_3}^z \Omega + m_1 B + K_+)} = e^{2\pi i (2m_1 + 1) K_+} \Phi(z) \]

by (2.19) and (4.2). As \( \Phi'(0) = 0 \), it also holds that \( Y_n = 0 \). \( \square \)
Lemma 4.10. Let
\[ Z_n := (T_{i(n)}' T_{i(n-1)}/\Phi - T_{i(n)}(T_{i(n-1)}/\Phi'))(0^*). \] (4.14)
When \(|\pi(z)\rho| \to \infty\), it holds that \( Z_n = 0 \), otherwise, we have that
\[ Z_n = (-1)^{l_0+m_0+n(n)} \frac{2e^{-\pi ic_0}}{\sqrt{a^2 + b^2}} \Phi(z_0) \frac{\theta^2(c_0)}{\Phi^2(0)}. \] (4.15)

**Proof.** The proof is the same as in the previous lemma. \( \square \)

Lemma 4.11. Let \( \sigma_0, \sigma_1 \) be as in (2.28). When \(|\pi(z)\rho| < \infty\), it holds that
\[ Y_nX^{-1} = \sigma_{i(n)}e^{\pi ic_0} \frac{\sqrt{a^2 + b^2}}{2} \Phi(z_{i(n)}) \] (4.16)
and
\[ Z_nX^{-1} = \sigma_{i(n)}e^{-\pi ic_0} \frac{\sqrt{a^2 + b^2}}{2} \Phi(z_{i(n)}) \] (4.17)
where \( X_n, Y_n, \) and \( Z_n \) are given by (4.6), (4.12), and (4.14), respectively.

**Proof.** The claims follow immediately from (4.7), (4.13), (4.15), and (4.4). \( \square \)

5. Proof of Proposition 2.1

It follows from (2.15) that \( \Omega_{z,z^*} = -\Omega_{z^*,z} \) for all \( z \in \mathfrak{R} \) such that \( \pi(z) \in \mathbb{C} \) and therefore \( S_\rho(z)S_\rho(z^*) \equiv 1 \) for such \( z \). Clearly, this relation extends to the points on top of infinity by continuity. It is also immediate from (2.14) and (2.15) that
\[ S_\rho(z^{(0)}) = \exp \left\{ - \sum_{i=1}^{4} \frac{w(z)}{2\pi i} \int_{\Delta_i} \frac{\log(\rho_i w_+(s))}{s - z} \frac{ds}{w_{|\Delta_i+}(s)} \right\} \times \]
\[ \times \exp \left\{ 2\pi i (wH)(z)c_0 \right\}, \] (5.1)
where, for emphasis, we write \( w_{|\Delta_i+}(s) \) for \( w_+(s) \) on \( s \in \Delta_i \) and
\[ H(z) := \frac{1}{2\pi i} \int_{\pi(\alpha)} \frac{dt}{(t-z)w(t)}. \] (5.2)

Relations (2.17) now easily follow from (5.1), (5.2), and Plemelj-Sokhotski formulae [10, equations (4.9)]. As for the behavior near \( a_i \), note that by [10, equation (8.8)], the function \( (wH)(z) \) is bounded as \( z \to a_i \). Furthermore, [10, equations (8.8) and (8.35)] yield that
\[ -\frac{w(z)}{2\pi i} \int_{\Delta_i} \frac{\log(\rho_i w_+(s))}{s - z} \frac{ds}{w_{|\Delta_i+}(s)} = -\frac{1}{2} \log(z - a_i)^{\alpha_i+1/2} + \mathcal{O}(1). \]
Since the above integral is the only one with the singular contribution around \( a_i \), the validity of the top line in (2.18) follows. As for the behavior near the origin, note that \( \lim_{Q_j \in z \to 0} w(z) = (-1)^{j-1}iab \), where, as before, \( Q_j \) stands
for the $j$-th quadrant. Recall that each segment $\Delta_i$ is oriented towards the origin, see Figure 1. Hence, it follows from [10, equation (8.2)] that
\[
-\frac{w(z)}{2\pi i} \int_{\Delta_i} \frac{\log(\rho_i w_+(s))}{s - z} \frac{ds}{w_{\Delta_i+}(s)} = -\frac{w(z)}{2\pi i} \frac{\log(\rho_i w_+(0))}{w|_{\Delta_i+}(0)} \log(z) + F_i(z)
\]
\[
= (-1)^{j+i} \frac{w(0)}{2\pi i} \log(\rho_i w_+(0)) \log(z) + F_i(z), \quad z \in Q_j,
\]
where $F_i(z)$ is a bounded function around the origin tending to a definite limit as $z \to 0$. Thus, summing over $i$ yields
\[
-\frac{w(z)}{2\pi i} \int_{\Delta} \frac{\log(\rho_i w_+(s))}{s - z} \frac{ds}{w_+(s)} = (-1)^j \nu \log(z) + \sum_{i=1}^{4} F_i(z), \quad z \in Q_j,
\]
where $\nu$ was defined in (2.12) and we used (2.13). Since $(wH)(z)$ is holomorphic around the origin, the second line in (2.18) follows.

6. Proofs of Theorems 2.3 and 2.4

6.1. Initial RH problem

Just as was first done by Fokas, Its, and Kitaev [8, 9], we connect the orthogonal polynomials $Q_n(z)$ to a $2 \times 2$ matrix Riemann-Hilbert problem. To this end, suppose that the index $n$ is such that
\[
\deg Q_n = n \quad \text{and} \quad R_{n-1}(z) \sim z^{-n} \quad \text{as} \quad z \to \infty, \quad (6.1)
\]
where $R_n(z)$ is given by (2.40). Let
\[
Y(z) := \begin{pmatrix} Q_n(z) & R_n(z) \\ k_{n-1}Q_{n-1}(z) & k_{n-1}R_{n-1}(z) \end{pmatrix}, \quad (6.2)
\]
where $k_{n-1}$ is a constant such that $k_{n-1}R_{n-1}(z) = z^{-n}(1+o(1))$ near infinity. Then $Y(z)$ solves the following Riemann-Hilbert problem (RHP-$Y$):
(a) $Y(z)$ is analytic in $\mathbb{C} \setminus \Delta$ and $\lim_{z \to \infty} Y(z)z^{-n\sigma_3} = I$ \(^4\).
(b) $Y(z)$ has continuous traces on $\Delta^o$ that satisfy
\[
Y_+(s) = Y_-(s) \begin{pmatrix} 1 & \rho(s) \\ 0 & 1 \end{pmatrix}, \quad s \in \Delta^o.
\]
(c) $Y(z)$ is bounded around the origin and
\[
Y(z) = \begin{cases} 
\mathcal{O} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \text{if } \alpha_i > 0, \\
\mathcal{O} \begin{pmatrix} 1 & \log |z - a_i| \\ 1 & \log |z - a_i| \end{pmatrix} & \text{if } \alpha_i = 0, \\
\mathcal{O} \begin{pmatrix} 1 & |z - a_i|^{\alpha_i} \\ 1 & |z - a_i|^{\alpha_i} \end{pmatrix} & \text{if } -1 < \alpha_i < 0,
\end{cases}
\]
as $z \to a_i$ for each $i \in \{1, 2, 3, 4\}$.

\(^4\)Hereafter, we set $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $I$ to be the identity matrix.
Indeed, property RHP-Y (a) is an immediate consequence of (6.1). The jump relations in RHP-Y (b) follow from (2.39), (2.40), and an application of the Plemelj-Sokhotski formulae. Behavior of the Cauchy integrals around the contours of integration, see [10, Section 8] and (2.40) yield RHP-Y (c) (to deduce boundedness around the origin one needs to utilize the third condition in the definition of the class $W_1$).

On the other hand, it also can be shown that if a solution of RHP-Y exists, then it must be of the form (6.2) with the diagonal entries satisfying (6.1) (see, for example, [1, Lemma 1]).

In what follows we prove solvability of RHP-Y for all $n \in \mathbb{N}_{\rho, \varepsilon}$ large enough via the matrix steepest descent method developed by Deift and Zhou [5].

6.2. Opening of the Lenses
Let $\delta_0 > 0$ be small enough so that all the functions $\rho_{i*}(z)$ are holomorphic in some neighborhood of $\{|z| \leq \delta_0\}$. Define $\tilde{\Delta}_i$ and $\tilde{\Delta}_o^i$ to be the closed and open segments connecting the origin and $\delta_0 e^{(2i-1)\pi i/4}$, $i \in \{1, 2, 3, 4\}$, that are oriented towards the origin. Further, let $\Gamma_i, \Gamma_{i+}$ be open smooth arcs that lie within the domain of holomorphy of $\rho_{i*}(z)$ and connect $a_i$ to $\delta_0 e^{(2i-1)\pi i/4}, \delta_0 e^{(2i-3)\pi i/4}$, respectively. We orient $\Gamma_i$ away from $a_i$ and assume that no open arcs $\Delta_i^o, \tilde{\Delta}_i^o, \Gamma_i, \Gamma_i$ intersect, see Figure 3. We denote by $\Omega_{i\pm}$ the domain partially bounded by $\Delta_i$ and $\Gamma_i$. Let

$$X(z) := Y(z) \begin{cases} 
\begin{pmatrix} 1 & 0 \\ \mp 1/\rho_i(z) & 1 \end{pmatrix}, & z \in \Omega_{i\pm}, \\
I, & z \not\in \overline{\Omega_{i+}} \cup \overline{\Omega_{i-}}. 
\end{cases}$$

(6.3)
Then $X(z)$ satisfies the following Riemann-Hilbert problem (RHP-$X$):
(a) $X(z)$ is analytic in $\mathbb{C} \setminus \cup_i (\Delta_i \cup \tilde{\Delta}_i \cup \Gamma_{i\pm})$ and $\lim_{z \to \infty} X(z) z^{-n\sigma_3} = I$;
(b) $X(z)$ has continuous traces on each $\Delta^\circ_i$, $\tilde{\Delta}^\circ_i$, and $\Gamma_{i\pm}$ that satisfy
\[
X_+(s) = X_-(s) \begin{cases}
1 & s \in \Gamma_{i+} \cup \Gamma_{i-}, \\
1/\rho_i(s) & s \in \Delta^\circ_i, \\
-1/\rho_i(s) & s \in \tilde{\Delta}^\circ_i,
\end{cases}
\]
where $i \in \{1, 2, 3, 4\}$ and $\rho_5 := \rho_1$.
(c) $X(z)$ is bounded around the origin and behaves like
\[
X(z) = \begin{cases}
O\left(\frac{1}{z}\right) & \text{if } \alpha_i > 0, \\
O\left(\frac{1}{\log |z - a_i|}\right) & \text{if } \alpha_i = 0, \\
O\left(\frac{1}{|z - a_i|^{\alpha_i}}\right) & \text{if } -1 < \alpha_i < 0,
\end{cases}
\]
as $z \to a_i$ from outside the lens while from inside the lens,
\[
X(z) = \begin{cases}
O\left(\frac{|z - a_i|^{-\alpha_i}}{|z - a_i|^{-\alpha_i}}\right) & \text{if } \alpha_i > 0, \\
O\left(\frac{1}{\log |z - a_i|}\right) & \text{if } \alpha_i = 0, \\
O\left(\frac{1}{|z - a_i|^{\alpha_i}}\right) & \text{if } -1 < \alpha_i < 0.
\end{cases}
\]

The following observation can be easily checked: RHP-$X$ is solvable if and only if RHP-$Y$ is solvable. When solutions of RHP-$X$ and RHP-$Y$ exist, they are unique and connected by (6.3).

6.3. Global Parametrix
Let $\Psi_n(z)$ be given by (2.25). For each $n \in \mathbb{N}_{\rho, \varepsilon}$, define
\[
N(z) := \gamma_n \begin{pmatrix}
\gamma_n & 0 \\
0 & \gamma_n^{*\!-1}
\end{pmatrix}
\begin{pmatrix}
\Psi_n(z^{(0)}) & \Psi_n(z^{(1)})/w(z) \\
\Psi_n(z^{(0)}) & \Psi_n(z^{(1)})/w(z)
\end{pmatrix},
\]
where the constants $\gamma_n$ and $\gamma_n^{*\!-1}$ are defined by the relations
\[
\lim_{z \to \infty} \gamma_n z^{-n} \Psi_n(z^{(0)}) = 1 \quad \text{and} \quad \lim_{z \to \infty} \gamma_n^{*\!-1} z^n \Psi_n(z^{(1)})/w(z) = 1.
\]
Such constants do exist by the very definition of the sequence $N_{\rho, \varepsilon}$ in (2.29) and the fact that the above normalization of $\Psi_n(z)$ is possible if and only if the above normalization of $\Psi_n(z)$ is possible, see the proof of Proposition 2.2.
The product $\gamma_n \gamma_{n-1}^*$ assumes only two necessarily finite and non-zero values depending on the parity of $n$ (when $|\pi(z_k)| < \infty$, it is equal to $X_n^{-1}$, see (4.6)). The matrix $N(z)$ solves the following Riemann-Hilbert problem (RHP-$N$):

(a) $N(z)$ is analytic in $\mathbb{C} \setminus \Delta$ and $\lim_{z \to \infty} N(z)z^{-n\sigma_3} = I$;

(b) $N(z)$ has continuous traces on $\Delta^o$ that satisfy

\[
N_+(s) = N_-(s) \begin{pmatrix} 0 & \rho(s) \\ -1/\rho(s) & 0 \end{pmatrix}, \quad s \in \Delta^o;
\]

(c) $N(z)$ satisfies

\[
N(i) = O \begin{bmatrix} |z - a_i|^{-(2\alpha_i + 1)/4} & |z - a_i|^{(2\alpha_i - 1)/4} \\ |z - a_i|^{-(2\alpha_i + 1)/4} & |z - a_i|^{(2\alpha_i - 1)/4} \end{bmatrix} \quad \text{as} \quad z \to a_i, \quad i \in \{1, 2, 3, 4\},
\]

\[
N(z) = O \begin{bmatrix} |z|^{-(1)^j \Re(\nu)} & |z|^{(-1)^{j+1} \Re(\nu)} \\ |z|^{-(1)^j \Re(\nu)} & |z|^{(-1)^{j+1} \Re(\nu)} \end{bmatrix} \quad \text{as} \quad z \to 0,
\]

where $j \in \{1, 2, 3, 4\}$ is the number of the quadrant from which $z \to 0$.

Indeed, RHP-$N$(a) holds by construction, while RHP-$N$(b,c) follow from (2.26) and (2.18), respectively (notice that the actual exponents in RHP-$N$(c) will change when the considered point happens to coincide with $z_{n(i)}$ or $z_{i(n-1)}$). Notice also that $\det(N(z)) \equiv 1$ since this is an entire function (it clearly has no jumps and it can have at most square root singularities at the points $a_i$) that converges to 1 at infinity.

For later calculations it will be convenient to set

\[
M^*(z) := \begin{pmatrix} (S_{\rho} T_{i(n)}) (z^{(0)}) & (S_{\rho} T_{i(n)}) (z^{(1)}) / w(z) \\ (S_{\rho} T_{i(n-1)}) \Phi(z^{(0)}) & (S_{\rho} T_{i(n-1)}) \Phi(z^{(1)}) / w(z) \end{pmatrix}, \quad (6.6)
\]

and $M(z) := (I + L_{\nu}/z) M^*(z)$, where $L_{\nu}$ is a certain constant matrix with zero trace and determinant defined further below in (6.26). Observe that $N(z) = CM^*(z) D(z)$, where

\[
C := \begin{pmatrix} \gamma_n & 0 \\ 0 & \gamma_{n-1}^* \end{pmatrix} \quad \text{and} \quad D(z) := \Phi^{n\sigma_3} (z^{(0)}). \quad (6.7)
\]

When $\Re(\nu) \in (-1/2, 1/2)$, it is possible to take $L_{\nu}$ to be the zero matrix, but this would worsen the error rates in (2.35) and (2.41). When $\Re(\nu) = 1/2$, our analysis necessitates introduction of $L_{\nu}$. Notice that neither the normalization of $M(z)$ at infinity nor its determinate depend on $L_{\nu}$. In fact, it holds that $\det(M(z)) = \det(M^*(z)) = (\gamma_n \gamma_{n-1}^*)^{-1}$.

### 6.4. Local Parametrix around $a_i$

Let $U_i$ be a disk around $a_i$ of small enough radius so that $\rho^*_i(z)$ is holomorphic around $\overline{U_i}, i \in \{1, 2, 3, 4\}$. In this section we construct solution of RHP-$X$ locally in each $U_i$. More precisely, we seeking a solution of the following local Riemann-Hilbert problem (RHP-$P_{a_i}$):
(a,b,c) $P_{a_i}(z)$ satisfies RHP-$X(a,b,c)$ within $U_i$;
(d) $P_{a_i}(s) = M(s)(I + O(1/n))D(s)$ uniformly for $s \in \partial U_i$.

We shall only construct a solution of RHP-$P_{a_1}$ as other constructions are almost identical.

6.4.1. Model Problem. Below, we always assume that the real line as well as its subintervals is oriented from left to right. Further, we set

$I_\pm := \{z : \arg(\zeta) = \pm 2\pi/3\}$,

where the rays $I_\pm$ are oriented towards the origin. Given $\alpha > -1$, let $\Psi_\alpha(\zeta)$ be a matrix-valued function such that

(a) $\Psi_\alpha(\zeta)$ is analytic in $\mathbb{C} \setminus (I_+ \cup I_- \cup (-\infty, 0])$;
(b) $\Psi_\alpha(\zeta)$ has continuous traces on $I_+ \cup I_- \cup (-\infty, 0)$ that satisfy

$\Psi_\alpha^+ = \Psi_\alpha^-$

\[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\quad \text{on} \quad (-\infty, 0),
\]

\[
\begin{pmatrix}
1 & 0 \\
e^{\pm \pi i\alpha} & 1
\end{pmatrix}
\quad \text{on} \quad I_\pm;
\]

(c) as $\zeta \to 0$ it holds that

$\Psi_\alpha(\zeta) = O\left(\frac{|\zeta|^{\alpha/2}}{|\zeta|^{\alpha/2}}\right) \quad \text{and} \quad \Psi_\alpha(\zeta) = O\left(\frac{\log|\zeta|}{\log|\zeta|}\right)$

when $\alpha < 0$ and $\alpha = 0$, respectively, and

$\Psi_\alpha(\zeta) = O\left(\frac{|\zeta|^{\alpha/2}}{|\zeta|^{\alpha/2}}\right) \quad \text{and} \quad \Psi_\alpha(\zeta) = O\left(\frac{|\zeta|^{-\alpha/2}}{|\zeta|^{-\alpha/2}}\right)$

when $\alpha > 0$, for $|\arg(\zeta)| < 2\pi/3$ and $2\pi/3 < |\arg(\zeta)| < \pi$, respectively;
(d) it holds uniformly in $\mathbb{C} \setminus (I_+ \cup I_- \cup (-\infty, 0])$ that

$\Psi_\alpha(\zeta) = S(\zeta) \left(I + O\left(\zeta^{-1/2}\right)\right) \exp\left\{2\zeta^{1/2}\sigma_3\right\},$

where $S(\zeta) := \frac{\zeta^{-\sigma_3/4}}{\sqrt{2}} \begin{pmatrix}
1 & i \\
i & 1
\end{pmatrix}$ and we take the principal branch of $\zeta^{1/4}$.

Explicit construction of this matrix can be found in [14] (it uses modified Bessel and Hankel functions). Observe that

$S_+(\zeta) = S_-(\zeta) \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}, \quad (6.8)$

since the principal branch of $\zeta^{1/4}$ satisfies $\zeta^{1/4}_+ = i\zeta^{1/4}_-$. Also notice that the matrix $\sigma_3\Psi_\alpha(\zeta)\sigma_3$ satisfies RHP-$\Psi_\alpha$ only with the reversed orientation of $(-\infty, 0]$ and $I_\pm$. 

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6.4.2. Conformal Map. Since $w(z)$ has a square root singularity at $a_1$ and satisfies $w_+(s) = -w_-(s)$, $s \in \Delta$, the function
\[
\zeta_{a_1}(z) := \left( \frac{1}{2} \int_{a_1}^{z} \frac{sd\rho}{w(s)} \right)^2, \quad z \in U_1,
\] 
(6.9)
is holomorphic in $U_1$ with a simple zero at $a_1$. Thus, the radius of $U_1$ can be made small enough so that $\zeta_{a_1}(z)$ is conformal on $U_1$. Observe that $sd\rho/w_\pm(s)$ is purely imaginary on $\Delta^*_1$ and therefore $\zeta_{a_1}(z)$ maps $\Delta_1 \cup U_1$ into the negative reals. It is also rather obvious that $\zeta_{a_1}(z)$ maps the interval $(a_1, \infty) \cap U_1$ into the positive reals. As we have had some freedom in choosing the arcs $\Gamma_{1,\pm}$, we shall choose them within $U_1$ so that $\Gamma_{1,-}$ is mapped into $I_+$ and $\Gamma_{1,+}$ is mapped into $I_-$. Notice that the orientation of the images of $\Delta_1, \Gamma_{1,+}, \Gamma_{1,-}$ under $\zeta_{a_1}(z)$ are opposite from the ones of $(-\infty, 0], I_-, I_+$.

In what follows, we understand that $\zeta_{a_1}^{1/2}(z)$ stands for the branch given by the expression in the parenthesis in (6.9).

6.4.3. Matrix $P_{a_1}$. According to the definition of the class $W_1$, it holds that
\[
\rho(z) = \rho^*_1(z)(a_1 - z)^{\alpha_1}, \quad z \in U_1,
\]
where $\rho_1^*(z)$ is non-vanishing and holomorphic in $U_1$ and $(a_1 - z)^{\alpha_1}$ is the branch holomorphic in $U_1 \setminus (a_1, \infty)$ and positive on $\Delta_1$. Define
\[
r_{a_1}(z) := \sqrt{\rho_1^*(z)(z - a_1)^{\alpha_1/2}}, \quad z \in U_1 \setminus \Delta_1,
\]
where $(z - a_1)^{\alpha_1/2}$ is the principal branch. It clearly holds that
\[
(z - a_1)^{\alpha_1} = e^{\pm \pi i \alpha_1}(a_1 - z)^{\alpha_1}, \quad z \in U_1^\pm,
\]
where $U_1^\pm := U_1 \cap \{\pm \text{Im}(z) > 0\}$. Then
\[
\begin{align*}
& r_{a_1,+}(s)r_{a_1,-}(s) = \rho(s), \quad s \in \Delta_1 \cap U_1, \\
& r_{a_1}^2(z) = \rho(z)e^{\pm \pi i \alpha_1}, \quad z \in U_1^\pm.
\end{align*}
\]
The above relations and RHP-$\Psi_{a_1}(a, b, c)$ imply that
\[
P_{a_1}(z) := E_{a_1}(z)\sigma_3\Psi_{a_1}(n^2\zeta_{a_1}(z))\sigma_3 r_{a_1}^{-\sigma_3}(z)
\]
(6.10)
satisfies RHP-$P_{a_1}(a, b, c)$ for any holomorphic matrix $E_{a_1}(z)$.

6.4.4. Matrix $E_{a_1}$. Now we choose $E_{a_1}(z)$ so that RHP-$P_{a_1}(a, b, c)$ is fulfilled. To this end, denote by $V_1, V_2, V_3$ the sectors within $U_1$ delimited by $\pi(\alpha) \cup \pi(\beta)$, $\pi(\beta) \cup \Delta_1$, and $\Delta_1 \cup \pi(\alpha)$, respectively, see Figure 1. Let $\gamma \subset \mathbb{C} \setminus \Delta$ be a path from $a_3$ to $a_1$ that does not intersect $\pi(\alpha), \pi(\beta)$. Further, let $\gamma := \pi^{-1}(\gamma)$ be a cycle oriented so that $\gamma^{(0)} := \gamma \cap \mathcal{R}^{(0)}$ proceeds from $a_3$ to $a_1$. Define
\[
K_{a_1}(z) := \begin{cases}
\exp\{ \int_{\gamma^{(0)}} G \} = \exp\{\pi i (\tau - \omega)\} = 1, & z \in V_1, \\
\exp\{ \int_{\gamma^{(0)} - \alpha} G \} = \exp\{-\pi i (\tau + \omega)\} = -1, & z \in V_2, \\
\exp\{ \int_{\gamma^{(0)} - \beta} G \} = \exp\{\pi i (\tau + \omega)\} = -1, & z \in V_3,
\end{cases}
\]
where we used the symmetry $G(z^*) = -G(z)$, the fact that $\gamma$ is homologous to $\alpha + \beta$, see Figure 2, and (2.4)–(2.5). Recalling the definition of $\Phi(z)$ in (2.6) (the path of integration must lie in $\mathcal{R}_{\alpha, \beta}$), one can see that

$$\Phi(z^{(0)}) = K_{a_1}(z) \exp \{2z_{a_1}^{1/2}(z)\}, \quad z \in V_1 \cup V_2 \cup V_3.$$ 

Clearly, $|K_{a_1}(z)| = 1$. It now follows from RHP-$\Psi_\alpha(d)$ that

$$P_{a_1}(s) = E_{a_1}(s)\sigma_3 S(n^2 \zeta_{a_1}(s)) \sigma_3 r_{a_1}^{-\sigma_3}(s) K_{a_1}^{-n\sigma_3}(s)(I + O(1/n)) D(s)$$

for $s \in \partial U_1$. Thus, if the matrix

$$E_{a_1}(z) := M(z)K_{a_1}^{-n\sigma_3}(z) r_{a_1}^{-\sigma_3}(z) \sigma_3 S^{-1}(n^2 \zeta_{a_1}(z)) \sigma_3$$

is holomorphic in $U_1$, RHP-$P_{a_1}(d)$ is clearly fulfilled. The fact that it has no jumps on $\Delta_1$, $\pi(\alpha), \pi(\beta)$ follows from RHP-$N(b)$, (6.8), (2.7), and the definition of $K_{a_1}(z)$. Thus, it is holomorphic in $U_1 \setminus \{a_1\}$. Since $|r_{a_1}(z)| \sim |z - a_1|^{a_1/2}$, $S^{-1}(n^2 \zeta_{a_1}(z)) \sim |z - a_1|^{\sigma_3/4}$, and $M(z)$ satisfies RHP-$N(c)$ around $a_1$, the desired claim follows.

### 6.5. Approximate Local Parametrix around the Origin

Let $0 < \delta \leq \delta_0$, see Section 6.2. We can assume that the closure of $U_\delta := \{|z| < \delta\}$ is disjoint from $\pi(\alpha), \pi(\beta)$. In this section we construct an approximate solution of RHP-$X$ in $U_\delta$ when $\ell < \infty$ and an exact solution of RHP-$X$ in $U_\delta$ when $\ell = \infty$.

To this end, let functions $b_i(z), i \in \{1, 2, 3, 4\}$, be defined in $\overline{U}_{\delta_0}$ by

$$b_1 := \frac{\rho_1 + \rho_2}{\rho_2}, \quad b_2 := -\frac{\rho_2 + \rho_3}{\rho_4}, \quad b_3 := -\frac{\rho_3 + \rho_4}{\rho_2}, \quad \text{and} \quad b_4 := \frac{\rho_1 + \rho_4}{\rho_4},$$

which are holomorphic and non-vanishing on $\overline{U}_{\delta}$. It follows from item (iv) in the definition of class $\mathcal{W}_l$ that

$$\frac{b_i(0)}{b_i(z)} - 1 = O(z^\ell) \quad \text{as} \quad z \to 0, \quad i \in \{1, 2, 3, 4\}. \quad (6.12)$$

Notice that $b_i(z) \equiv b_i(0)$ when $\ell = \infty$. Observe also that $b_1(0) = b_3(0)$ and $b_2(0) = b_4(0)$ by item (ii) in the definition of class $\mathcal{W}_l$. We are seeking a solution of the following local Riemann-Hilbert problem (RHP-$P_0$):

(a) $P_0(z)$ satisfies RHP-$X(a)$ within $U_\delta$;
(b) $P_0(z)$ satisfies RHP-$X(b)$ within $U_\delta$, where the jump matrix on each $\hat{\Delta}_i^\delta$ needs to be replaced by

$$\begin{pmatrix} 1 \\ b_i((s) \left( \frac{1}{\rho_i(s)} + \frac{1}{\rho_{i+1}(s)} \right) 0 \end{pmatrix};$$

(c) $P_0(s) = M(s)(I + O((n\delta^2)^{-1/2} - |\text{Re}(\nu)|)) D(s)$ uniformly for $s \in \partial U_\delta$ and $\delta \leq \delta_0$. 

6.5.1. Model Problem. A construction, similar the one below, has been introduced in [12], see also [5] and the book [7, Chapter 2], in the context of integrable systems. Unfortunately, the local problem is not stated in the form and generality we need in any of these references. Thus, for the convenience of the reader, we provide an explicit expression for the local parametrix.

Let \( s_1, s_2 \in \mathbb{C} \) be independent parameters and let \( \nu \in \mathbb{C} \), \( \text{Re}(\nu) \in (-\frac{1}{2}, \frac{1}{2}] \) be given by

\[
e^{-2\pi i \nu} := 1 - s_1 s_2	ag{6.13}
\]

(we slightly abuse the notation here as the parameter \( \nu \) of the reader, we provide an explicit expression for the local parametrix.

Let \( \Gamma(\nu) \) be given by

\[
\Gamma(1+\nu) \sqrt{2\pi}
\]

for \( \nu \in \mathbb{C} \), \( \text{Re}(\nu) \in (-\frac{1}{2}, \frac{1}{2}] \). Define constants \( d_1, d_2 \) by

\[
d_1 := -s_1 \Gamma(1+\nu) \sqrt{2\pi} \quad \text{and} \quad d_2 := -s_2 e^{\pi i \nu} \Gamma(1-\nu) \sqrt{2\pi},
\]

where \( \Gamma(z) \) is the standard Gamma function. It follows from the well-known Gamma function identities that

\[
d_1 d_2 = i\nu.	ag{6.15}
\]

Denote by \( D_\mu(\zeta) \) the parabolic cylinder function in Whittaker’s notations, see [6, Section 12.2]. It is an entire function with the asymptotic expansion

\[
D_\mu(\zeta) \sim e^{-\zeta^2/4} \zeta^\mu \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)} \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-2k)} \frac{1}{(2\zeta^2)^k}
\]

valid uniformly in each \( |\arg(\zeta)| \leq 3\pi/4 - \epsilon, \epsilon > 0 \), see [6, Equation (12.9.1)].

Let the matrix function \( \Psi_{s_1, s_2}(\zeta) \) be given by

\[
\begin{pmatrix}
D_\nu(2\zeta) & d_1 D_{-\nu-1}(-2i\zeta) \\
-d_2 D_{\nu-1}(2\zeta) & D_{-\nu}(-2i\zeta)
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & e^{-\pi i \nu/2}
\end{pmatrix}, \quad \arg(\zeta) \in (0, \frac{\pi}{2}),
\]

\[
\begin{pmatrix}
D_\nu(-2\zeta) & d_1 D_{-\nu-1}(-2i\zeta) \\
-d_2 D_{\nu-1}(-2\zeta) & D_{-\nu}(-2i\zeta)
\end{pmatrix}
\begin{pmatrix}
e^{\pi i \nu} & 0 \\
0 & e^{-\pi i \nu/2}
\end{pmatrix}, \quad \arg(\zeta) \in \left(\frac{\pi}{2}, \pi\right),
\]

\[
\begin{pmatrix}
D_\nu(-2\zeta) & -d_1 D_{-\nu-1}(2i\zeta) \\
-d_2 D_{\nu-1}(-2\zeta) & D_{-\nu}(2i\zeta)
\end{pmatrix}
\begin{pmatrix}
e^{-\pi i \nu} & 0 \\
0 & e^{\pi i \nu/2}
\end{pmatrix}, \quad \arg(\zeta) \in \left(-\frac{\pi}{2}, -\pi\right),
\]

\[
\begin{pmatrix}
D_\nu(2\zeta) & -d_1 D_{-\nu-1}(2i\zeta) \\
-d_2 D_{\nu-1}(2\zeta) & D_{-\nu}(2i\zeta)
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & e^{\pi i \nu/2}
\end{pmatrix}, \quad \arg(\zeta) \in \left(0, -\frac{\pi}{2}\right).
\]

Then, \( \Psi_{s_1, s_2}(\zeta) \) satisfies the following RH problem (RHP-\( \Psi_{s_1, s_2} \)):

(a) \( \Psi_{s_1, s_2}(\zeta) \) is analytic in \( \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R}) \);

(b) \( \Psi_{s_1, s_2}(\zeta) \) has continuous traces on \( \mathbb{R} \cup i\mathbb{R} \) outside of the origin that satisfy the jump relations shown in Figure 4;

(c) \( \Psi_{s_1, s_2}(\zeta) \) has the following asymptotic expansion as \( \zeta \to \infty \):

\[
\left(I + \frac{1}{2\zeta} \begin{pmatrix} 0 & id_1 \\ d_2 & 0 \end{pmatrix} + \frac{\nu(\nu-1)}{8\zeta^2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + O\left(\frac{1}{\zeta^3}\right)\right) (2\zeta)^{\nu\sigma_3} e^{-\zeta^2\sigma_3},
\]
which holds uniformly in $\mathbb{C}$.

Indeed, RHP-$\Psi_{s_1, s_2}(a)$ follows from the fact that $D_{\nu}(\zeta)$ is entire, while RHP-$\Psi_{s_1, s_2}(c)$ is a consequence of (6.16). The jump relations RHP-$\Psi_{s_1, s_2}(b)$ can be verified using the identities

\[
\Gamma(-\nu)\Gamma(1 + \nu) = -\frac{\pi}{\sin(\pi\nu)},
\]

and

\[
D_{\mu}(2\xi) = e^{-\mu\pi i}D_{\mu}(-2\xi) + \frac{\sqrt{2\pi}}{\Gamma(-\mu)}e^{-(\mu+1)i\pi/2}D_{-\mu-1}(2i\xi),
\]

suitably applied with parameter values $\mu = -\nu, \nu - 1$ and $\xi = \zeta, -\zeta, i\zeta$. For later, it will be important for us to make the following observation. Define

\[
d_{\nu} := \begin{cases} d_2, & \text{Re}(\nu) > 0, \\ 0, & \text{Re}(\nu) = 0, \\ id_1, & \text{Re}(\nu) < 0 \end{cases}
\]

and

\[
A_{\nu} := \begin{cases} (0 0), & \text{Re}(\nu) \geq 0, \\ (0 1), & \text{Re}(\nu) < 0, \end{cases}
\]

Recall that we set $\zeta_\nu = 1, 0, -1$ depending on whether Re$(\nu) > 0$, Re$(\nu) = 0$, or Re$(\nu) < 0$. Observe that

\[
(I - (2\zeta)^{-1}d_{\nu}A_{\nu})\Psi_{s_1, s_2}(\zeta)
\]

\[
= (2\zeta)^{i\sigma_3} \left( I + (2\zeta)^{-1-2\zeta_\nu}d_{-\nu}A_{-\nu} + O \left( \zeta^{-1-|\zeta_\nu|} \right) \right) e^{-\zeta^2\sigma_3}. \quad (6.18)
\]

6.5.2. Conformal Map. Let, as before, $Q_j$ stand for the $j$-th quadrant, $j \in \{1, 2, 3, 4\}$. Set

\[
\zeta_0(\zeta) := \left( (-1)^{j-1} \int_0^\zeta \frac{sds}{w(s)} \right)^{1/2}, \quad \zeta \in U_\delta \cap Q_j. \quad (6.19)
\]

Since $w(z)$ is bounded at 0 and satisfies $w_+(s) = -w_-(s)$, $s \in \Delta$, the branch of the square root can be chosen so that the function $\zeta_0(z)$ is in fact holomorphic in $U_\delta$ with a simple zero at the origin. Without loss of generality we can assume that $\delta$ is small enough for $\zeta_0(z)$ to be conformal on $\overline{U_\delta}$.

Since the integrand $(-1)^{j-1} sds/w(s)$ becomes negative purely imaginary on $\Delta_1 \cup \Delta_3$, the square root in (6.19) can be chosen so that arg $(\zeta_0(z)) = -\pi/4$, $z \in \Delta_3^\circ$. As we have had some freedom in selecting the arcs $\tilde{\Delta}_i$, we...
shall choose them so that $\tilde{\Delta}_3^o$ and $\tilde{\Delta}_1^o$ are mapped by $\zeta_0(z)$ into positive and negative reals, respectively, while $\tilde{\Delta}_4^o$ and $\tilde{\Delta}_2^o$ are mapped into positive and negative purely imaginary numbers.

6.5.3. Matrix $P_0$. Define the function $r(z) := r_j(z)$, $z \in Q_j$, where we let

$$
r_1 := e^{\pi i \nu} \sqrt{\rho_1}, \quad r_2 := e^{-\pi i \nu} \frac{\rho_2}{\sqrt{\rho_1}}, \quad r_3 := -ie^{-\pi i \nu} \frac{\rho_4}{\sqrt{\rho_1}}, \quad r_4 := -ie^{-\pi i \nu} \sqrt{\rho_1}
$$

for a fixed determination of $\sqrt{\rho_1(z)}$. Furthermore, let

$$
J(z) := \begin{cases} 
  e^{2\pi i \nu \sigma_3}, & \arg z \in \left(-\frac{\pi}{2}, 0\right), \\
  \begin{pmatrix} 0 & 1 \\
                 -1 & 0 \end{pmatrix} e^{2\pi i \nu \sigma_3}, & \arg z \in \left(0, \frac{\pi}{4}\right), \\
  \begin{pmatrix} 0 & 1 \\
                 -1 & 0 \end{pmatrix}, & \arg z \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, -\pi\right), \\
  I, & \arg z \in \left(\frac{\pi}{2}, \pi\right).
\end{cases}
$$

Finally, recalling (6.11), put

$$
s_1 := b_1(0) = b_3(0) \quad \text{and} \quad s_2 := b_2(0) = b_4(0).
$$

Notice that since $(\rho_1 + \rho_2 + \rho_3 + \rho_4)(0) = 0$, the parameters $s_1, s_2$ satisfy (6.13) with $\nu$ given by (2.12). Then

$$
P_0(z) := E_0(z) \Psi_{s_1,s_2}(n^{1/2}z_0(z)) \mathcal{J}^{-1}(z)r^{-\sigma_3}(z)
$$

satisfies RHP-$P_0(a,b)$ for any matrix $E_0(z)$ holomorphic in $U_\delta$. Indeed, RHP-$P_0(a)$ is an immediate consequence of RHP-$\Psi_{s_1,s_2}(a)$. It further follows from RHP-$\Psi_{s_1,s_2}(b)$ that the jumps of $P_0(z)$ are as on Figure 5. To verify RHP-$P_0(b)$, it remains

$$
\begin{pmatrix}
  1 & 0 \\
  s_2/r_2 & 1
\end{pmatrix}
\begin{pmatrix}
  0 & -r_1r_2 \\
  1/r_1r_2 & 0
\end{pmatrix}
\begin{pmatrix}
  1 & 0 \\
  -s_1e^{2\pi i \nu} / r_1^2 & 1
\end{pmatrix}
\begin{pmatrix}
  0 & r_1r_4 \\
  -1/r_1r_4 & 0
\end{pmatrix}
\begin{pmatrix}
  1 & 0 \\
  s_1/r_3 & 1
\end{pmatrix}
\begin{pmatrix}
  0 & -r_3r_4 \\
  1/r_3r_4 & 0
\end{pmatrix}
\begin{pmatrix}
  1 & 0 \\
  -s_2e^{-2\pi i \nu} / r_4^2 & 1
\end{pmatrix}
$$

Figure 5. The jump matrices of $P_0(z)$. 

to observe that
\[ r_1 r_4 = \rho_1, \quad -r_1 r_2 = \rho_2, \quad r_2 r_3 = e^{-2\pi i \nu} \rho_2 \rho_4 / \rho_1 = \rho_3, \quad -r_3 r_4 e^{2\pi i \nu} = \rho_4, \]
since \( e^{-2\pi i \nu} = (\rho_1 \rho_3) / (\rho_2 \rho_4) \), and that
\[
-e^{2\pi i \nu} \frac{s_1}{r_1} = \frac{b_1(0)}{\rho_1} = \frac{b_1(0)}{b_1} \left( \frac{1}{\rho_1} + \frac{1}{\rho_2} \right),
\]
\[
s_2 = -e^{2\pi i \nu} b_2(0) \frac{\rho_1}{\rho_2} = -b_2(0) \frac{\rho_4}{\rho_2 \rho_3} = \frac{b_2(0)}{b_2} \left( \frac{1}{\rho_2} + \frac{1}{\rho_3} \right),
\]
\[
s_3 = -e^{2\pi i \nu} b_3(0) \frac{\rho_1}{\rho_4} = -b_3(0) \frac{\rho_2}{\rho_3 \rho_4} = \frac{b_3(0)}{b_3} \left( \frac{1}{\rho_3} + \frac{1}{\rho_4} \right),
\]
\[
-e^{2\pi i \nu} \frac{s_2}{r_4} = \frac{b_4(0)}{\rho_1} = \frac{b_4(0)}{b_4} \left( \frac{1}{\rho_1} + \frac{1}{\rho_4} \right).
\]

Thus, it remains to choose \( E_0(z) \) so that RHP-\( P_0(c) \) is fulfilled.

### 6.5.4. Matrix \( E_0 \)

Let \( \gamma \) be the part of \( \Delta_3 \) that proceeds from \( a_3 \) to \( 0^* \), see Figures 1 and 2. Define
\[
K_0(z) := \left\{ \begin{array}{ll}
\exp \{- \int_\gamma G\} = \Phi (0), & z \in Q_1 \cup Q_3, \\
\exp \{ \int_\gamma G\} = \Phi (0^*), & z \in Q_2 \cup Q_4.
\end{array} \right.
\]
(2.11) immediately yields that \( |K_0(z)| \equiv 1 \). Define
\[
E_0^*(z) := M^*(z) r^{\sigma_3}(z) K_0^{-\sigma_3}(z) J(z) \zeta_0^{-\nu \sigma_3}(z),
\]
see (6.6). From RHP-\( N(b) \), the definition of \( J(z) \), and the fact that \( \zeta_0(z) \) maps \( \Delta_3^0 \) into the negative reals, it follows that \( E_0^*(z) \) is holomorphic in \( U_\delta \setminus \{0\} \). Furthermore, RHP-\( N(c) \) combined with the fact that \( \zeta_0(z) \) possesses a simple zero at \( z = 0 \) imply that \( E_0^*(z) \) is holomorphic in \( U_\delta \). Observe also that the moduli of the entries of \( E_0^*(z) \) depend only on the parity of \( n \).

Put for brevity \( \epsilon_{\nu,n} := (4n)^{\epsilon_{\nu} - 1/2} \), where, as before, \( \epsilon_\nu \) is equal to 1, 0, \(-1\) depending on whether \( \text{Re}(\nu) \) is positive, zero, or negative. Set
\[
L_\nu := \frac{d_\nu \epsilon_{\nu,n}}{\zeta_0(0) D_\nu} E_0^*(0) A_\nu E_0^{*-1}(0),
\]
where \( d_\nu, A_\nu \) were defined in (6.17) and we assume that
\[
0 \neq D_\nu := 1 - d_\nu \epsilon_{\nu,n}(\zeta_0'(0))^{-1} E_\nu
\]
with
\[
E_\nu := \left\{ \begin{array}{ll}
[E_0^{*-1}(0) E_0^{*\nu}(0)]_{12} & \text{if } \text{Re}(\nu) \geq 0, \\
[E_0^{*-1}(0) E_0^{*\nu}(0)]_{21} & \text{if } \text{Re}(\nu) < 0.
\end{array} \right.
\]
Notice that \( L_\nu \) is the zero matrix when \( \text{Re}(\nu) = 0 \) as \( d_\nu = 0 \) by (6.17). Let
\[
E_0(z) := (I + L_\nu / z) E_0^*(z) (4n)^{-\nu \sigma_3 / 2} (I - d_\nu (2n^{1/2} \zeta_0(z))^{-1} A_\nu).
\]
Let us show that thus defined matrix $E_0^*(z)$ is holomorphic at the origin. Indeed, it has at most double pole there. It is quite simple to see that the coefficient next to $z^{-2}$ is equal to

$$-d_{\nu}\epsilon_{\nu,n}(4n)^{-\nu/2}(\zeta_0'(0))^{-1}L_{\nu}E_0^*(0)A_{\nu},$$

which is equal to the zero matrix since $A_{\nu}^2$ is equal to the zero matrix. Using this observation we also get that the coefficient next to $z^{-1}$ is equal to

$$L_{\nu}E_0^*(0)(4n)^{-\nu/2} - d_{\nu}\epsilon_{\nu,n}(4n)^{-\nu/2}(\zeta_0'(0))^{-1}(E_0^*(0) + L_{\nu}E_0^*(0))A_{\nu},$$

which simplifies to

$$\frac{d_{\nu}\epsilon_{\nu,n}(4n)^{-\nu/2}}{\zeta_0'(0)D_n} \left( 1 - \frac{d_{\nu}\epsilon_{\nu,n}(4n)}{\zeta_0'(0)} E_{\nu} - D_n \right) E_0^*(0)A_{\nu},$$

that is equal to the zero matrix by the very definition of $D_n$.

Now, recalling the definition of $\Phi(z)$ in (2.6) and of $\zeta_0(z)$ in (6.19), one can see that

$$\exp\{-\zeta_0^2(z)\} = e^{-\int_\gamma G} \left\{ \begin{array}{ll} \Phi(z^{(1)}), & z \in \mathcal{Q}_1 \cup \mathcal{Q}_3, \\ \Phi(z^{(0)}), & z \in \mathcal{Q}_2 \cup \mathcal{Q}_4. \end{array} \right. \quad (6.29)$$

In particular, since $D(z) = \Phi^n\sigma_3(z^{(0)})$ and $\Phi(z^{(0)})\Phi(z^{(1)}) \equiv 1$, it follows from (6.24) that

$$\exp\{-n\zeta_0^2(z)\} J^{-1}(z) = J^{-1}(z)K^{-n\sigma_3}(z)D(z).$$

For brevity, let $H(z) := r^{-\sigma_3}(z)K^n\sigma_3(z)J(z)$. Then we get from (6.18) and the previous identity that

$$E_0(s)\Psi_{s_1,s_2}(n^{1/2}\zeta_0(s))J^{-1}(s)r^{-\sigma_3}(s) = M(s)H(s) \left( I + \mathcal{O}\left((n\zeta_0^2(s))^{-1/2-|\text{Re}(\nu)|}\right) \right) H^{-1}(s)D(s) = M(s) \left( I + \mathcal{O}\left((n\zeta_0^2(s))^{-1/2-|\text{Re}(\nu)|}\right) \right) D(s).$$

It remains to show that (6.27) holds for all $n \in \mathbb{N}_{\rho,\epsilon}$. It follows from (2.28) that it is enough to show that

$$A_{\rho,n} = d_{\nu}\epsilon_{\nu,n}(\zeta_0'(0))^{-1}E_{\nu}. \quad (6.30)$$

### 6.6. Existence of $L_{\nu}$

Assume that $\text{Re}(\nu) > 0$. It can be readily verified that

$$E_{\nu} = \gamma_n\gamma_{n-1}^*([E_0^*(0)]_{12}[E_0^*(0)]_{22} - [E_0^*(0)]_{22}[E_0^*(0)]_{12}),$$

where we used the fact that $\det(E_0^*(z)) = \det(M^*(z)) = (\gamma_n\gamma_{n-1})^{-1}$. Notice that $d_2 \neq 0$ by (6.15). Using (6.25), (6.21), and (6.24) gives us that $[E_0^*(z)]_{12}$
we obtain from \((6.34)\) that \(Y_n = 0\) in this case. Hence, \((6.30)\) does hold in this case.

Let now \(|\pi(z_k)| < \infty\) and therefore the first condition in the definition of \(N_{\rho, \varepsilon}\) is void. It follows from \((6.19)\) and \((2.1)\) as well as the fact that \(\zeta_0(z)\) maps \(\{\arg(z) = 5\pi/4\}\) into the positive reals that

\[
\frac{1}{\zeta_0'(0)} = e^{5\pi i/4} \sqrt{2ab}.  \tag{6.32}
\]

Since \(e^{-2\pi i\nu} = (\rho_1 \rho_3)(0)/(\rho_2 \rho_4)(0)\) by \((2.12)\), we get from \((6.20)\) that

\[
S^2(0) = - (\rho_3 \rho_4 / \rho_2)(0) (2ab)^{-\nu} \lim_{z \to 0, \arg(z) = 5\pi/4} \frac{|z|^{2\nu} S_\rho^2(z(0))}{2\pi}.  \tag{6.33}
\]

Observe also that

\[
d_2 = \frac{e^{\pi i\nu} (\rho_2 + \rho_3)(0)}{\rho_4(0)} \frac{\Gamma(1 - \nu)}{\sqrt{2\pi}}  \tag{6.34}
\]

by \((6.14), (6.22),\) and \((6.11)\). Then it follows from \((4.16)\) and the very definitions of \(A_{\rho, n}\) in \((2.28)\) that \((6.31)-(6.34)\) yield \((6.30)\). The proof of \((6.30)\) in the case \(\text{Re}(\nu) < 0\) is similar.

Since \(|\pi(z_k)| < \infty\), the quantities \(Y_n\) and \(Z_n\) in \((4.12)\) and \((4.14)\) are non-zero and equal to

\[
W'_{1(n)}(0) T^2_{1(n-1)}(0)^{-\nu}, \quad W_{1(n)}(z) = \frac{T_{1(n)}(z)}{T^2_{1(n-1)}(z)},
\]
where $o$ was defined in (2.27). Hence, it follows from (6.26), (6.30), (6.31), and a computation similar to the one carried out at the beginning of this subsection that

$$L_{\nu} = \frac{A_{\rho,n}}{1 - A_{\rho,n}} \frac{1}{W_{i(n)}'(o)} \begin{pmatrix} W_{i(n)}(o) & -\Phi(o)W_{i(n)}^2(o) \\ 1/\Phi(o) & -W_{i(n)}(o) \end{pmatrix}.$$ 

Moreover, since $W_1(z) = 1/W_0(z)$ we can rewrite the first row of $L_{\nu}$ as

$$(1 \ 0) L_{\nu} = (-1)^{(n)} \frac{A_{\rho,n}}{1 - A_{\rho,n}} \frac{W_0(o)}{W_0'(o)} \begin{pmatrix} 1 & -\Phi(o)W_{i(n)}(o) \end{pmatrix}. \quad (6.35)$$

6.7. Final Riemann-Hilbert Problem

In what follows, we assume that $\delta = \delta_n \leq \delta_0$ in Section 6.5 when $\ell < \infty$ and shall specify the exact dependence on $n$ later on in this section. When $\ell = \infty$, we simply take $\delta = \delta_0$. Set $U := \cup_{i=1}^4 U_{a_i}$ and define

$$\Sigma_n := (\partial U \cup \partial U_{\delta_n}) \cup \left( \cup_{i=1}^4 (\Gamma_{i-} \cup \Gamma_{i+} \cup \tilde{\Delta}_i) \setminus \overline{U} \right),$$

see Figure 6. We are looking for a solution of the following Riemann-Hilbert problem (RHP-$Z$):

(a) $Z(z)$ is analytic in $\overline{C \setminus \Sigma_n}$ and $\lim_{z \to \infty} Z(z) = I$;

(b) $Z(z)$ has continuous traces outside of non-smooth points of $\Sigma_n$ that satisfy

$$Z_+ = Z_- \begin{cases} P_{a_i}(MD)^{-1}, & \text{on } \partial U_{a_i}, \\ P_0(MD)^{-1}, & \text{on } \partial U_{\delta}, \\ MD \left( \begin{pmatrix} 1 & 0 \\ 1/\rho_i & 1 \end{pmatrix} (MD)^{-1} \right), & \text{on } (\Gamma_{i+}^0 \cup \Gamma_{i-}^0) \setminus \overline{U}, \end{cases}$$

FIGURE 6. Contour $\Sigma_n$ for RHP-$Z$ (dashed circle represents $\{|z| = \delta_0\}$).
and

\[
Z_+ = Z_- \begin{cases} 
\begin{align*}
MD \begin{pmatrix} 1 & 0 \\ \frac{\rho_i + \rho_{i+1}}{\rho_i \rho_{i+1}} & 1 \end{pmatrix} (MD)^{-1}, & \text{on } \tilde{\Gamma}_i \setminus U_{\delta_n}, \\
\rho_0 - \begin{pmatrix} 1 & 0 \\ \frac{\rho_i + \rho_{i+1}}{\rho_i \rho_{i+1}} & 1 \end{pmatrix} P_0^{-1}, & \text{on } \tilde{\Gamma}_i \cap U_{\delta_n}
\end{align*}
\end{cases}
\]

(notice that the second set of jumps is not present when \( \ell = \infty \) as \( \delta_n = \delta_0 \) and \( P_0(z) \) is the exact parametrix).

It follows from RHP-\( P_0 \) that the jump of \( Z \) on \( \partial U_{\delta_n} \) can be written as

\[
M(s)(I + \mathcal{O}(1/n))M^{-1}(s) = I + \mathcal{O}_{\varepsilon}(1/n)
\]

since the matrix \( M(z) \) is invertible (its determinant is equal to the reciprocal of \( \gamma_n \gamma_{n-1} \)), the matrix \( M^*(z) \) depends only on the parity of \( n \), see (6.6), and the matrix \( L_{\nu} \) has trace and determinant zero as well as bounded entries for all \( n \in \mathbb{N}_{\rho,\varepsilon} \) and each fixed \( \varepsilon > 0 \), see (6.26). Similarly, we get from RHP-\( P_0 \) (d) that the jump of \( Z \) on \( \partial U_{\delta_n} \) can be written as

\[
M(s)(I + \mathcal{O}(1/n))M^{-1}(s) = I + (I + L_{\nu}/s)\mathcal{O}\left((n\delta_n^2)^{-1/2-|\text{Re}(\nu)|}\right)(I - L_{\nu}/s),
\]

where \( \mathcal{O}(\cdot) \) does not depend on \( n \). Since \( L_{\nu} = \mathcal{O}_{\varepsilon}(n^{\text{Re}(\nu)-1/2}) \) by its very definition in (6.26), we get that the jump of \( Z \) on \( \partial U_{\delta_n} \) can further be written as

\[
I + \mathcal{O}_{\varepsilon}\left((n\delta_n^2)^{-1/2-|\text{Re}(\nu)|}\max\{1,n^{2|\text{Re}(\nu)|}/(n\delta_n^2)\}\right).
\]

One can easily check with the help of (6.4) and (6.6) that the jump of \( Z \) on \( (\Gamma_{i+}^0 \cup \Gamma_{i-}^0) \setminus \overline{U} \) is equal to

\[
I + \frac{\gamma_n \gamma_{n-1}}{(\omega^2 \rho_i)(s)}(I + L_{\nu}/s)\begin{pmatrix} (\Psi_n \Psi_{n-1})(s^{(1)}) & -\Psi_n^2(s^{(1)}) \\ -\Psi_n^2(s^{(1)}) & (\Psi_n \Psi_{n-1})(s^{(1)}) \end{pmatrix}(I - L_{\nu}/s) = I + \mathcal{O}_{\varepsilon}(e^{-cn})
\]

for some constant \( c > 0 \) by (2.25) and since the maximum of \( |\Phi(s^{(1)})| \) on \( \Gamma_{i\pm} \setminus U \) is less than 1. The estimate of the jump of \( Z \) on \( \tilde{\Delta}_i \setminus \overline{U}_{\delta_n} \) is analogous and yields

\[
I + \mathcal{O}_{\varepsilon}\left(e^{-cn\delta_n^2}\max\{1,n^{2|\text{Re}(\nu)|}/(n\delta_n^2)\}\right)
\]

for an adjusted constant \( c > 0 \), where the rate estimate follows from (6.29) as

\[
|\Phi(s^{(1)})| = \exp\{(-1)^i \text{Re}(\zeta_0^2(s))\} = \mathcal{O}(e^{-c\delta_n^2}), \quad s \in \tilde{\Delta}_i \setminus U_{\delta_n},
\]

since \( \zeta_0(z) \) is real on \( \tilde{\Delta}_1 \cup \tilde{\Delta}_3 \) and is purely imaginary on \( \tilde{\Delta}_2 \cup \tilde{\Delta}_4 \).
Finally, it holds on $\hat{\Delta}_2^2 \cap U_{\delta_n}$ that the jump of $Z$ is equal to

$$I + \left(1 - \frac{b_i(0)}{b_i(z)}\right) \frac{(\rho_i + \rho_{i+1})(s)}{(\rho_i \rho_{i+1})(s)} P_{0+}(s) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} P_{0+}^{-1}(s) =$$

$$I + O(\delta_n^\ell) E_0(s) \begin{pmatrix} [\Psi_+(s)]_{1j} & [\Psi_+(s)]_{2j} \\ [\Psi_+(s)]_{2j}^2 & -[\Psi_+(s)]_{1j} \end{pmatrix} E_0^{-1}(s)$$

by (6.12) and (6.23), where $j = 1$ for $s \in \hat{\Delta}_1 \cup \hat{\Delta}_3$ and $j = 2$ for $s \in \hat{\Delta}_2 \cup \hat{\Delta}_4$, and we set for brevity $\Psi(z) := \Psi_{n_1,n_2}(n^{1/\xi} \zeta_0(z))$ (observe also that $\det(\Psi(z)) \equiv 1$). It follows from the asymptotic expansion (6.16) that $D_\mu(x)$ is bounded for $x \geq 0$. Thus, we deduce from the definition of $\Psi(z)$ that the above jump matrix can be estimated as

$$I + O(\delta_n^\ell) E_0(s) O(1) E_0^{-1}(s) = I + O_\varepsilon \left(n^{\text{Re}(\nu)}|\delta_n^\ell|\delta_n^{2\ell}\right),$$

where the last equality follows from (6.25) and (6.28) as $E_0(z)$ is equal to a bounded matrix that depends only on $\varepsilon_{\nu,n}$ multiplied by $(4\nu)^{3/2}$ on the right.

When $\ell \geq 4|\text{Re}(\nu)|(1 + |\text{Re}(\nu)|)/(1 - 2|\text{Re}(\nu)|)$, choose

$$\delta_n = \delta_0 \exp \left\{- \frac{1}{2} \frac{1 + 4|\text{Re}(\nu)|}{1 + 2|\text{Re}(\nu)|} \ln n \right\}. \quad (6.36)$$

Then it holds that $n^{2|\text{Re}(\nu)|}/(n\delta_n^2) = O(1)$ and

$$n^{\text{Re}(\nu)}|\delta_n/\delta_0|^\ell = \left(n(\delta_n/\delta_0)^2\right)^{|\text{Re}(\nu)|-1/2} = n^{-d_{\nu,\ell}}$$

with $d_{\nu,\ell}$ defined in (2.34). Otherwise, take

$$\delta_n = \delta_0 \exp \left\{- \frac{1}{2} \frac{3}{1 + 2|\text{Re}(\nu)|} \ln n \right\}. \quad (6.36)$$

In this case $n^{2|\text{Re}(\nu)|}/(n\delta_n^2) \to \infty$ as $n \to \infty$ and

$$n^{\text{Re}(\nu)}|\delta_n/\delta_0|^\ell = n^{2|\text{Re}(\nu)|} \left(n(\delta_n/\delta_0)^2\right)^{-|\text{Re}(\nu)|-3/2} = n^{-d_{\nu,\ell}}.$$

Since $d_{\nu,\ell} < 1$, it holds that the jumps of $Z$ on $\Sigma_n$ are of order $I + O_\varepsilon(n^{-d_{\nu,\ell}})$, where $O_\varepsilon(\cdot)$ does not depend on $n$. Then, by arguing as in [4, Theorem 7.103 and Corollary 7.108] we obtain that the matrix $Z$ exists for all $n \in \mathbb{N}_{\rho,\varepsilon}$ large enough and that

$$\|Z_\pm - I\|_{2,\Sigma_n} = O_\varepsilon(n^{-d_{\nu,\ell}}).$$

Since the jumps of $Z$ on $\Sigma_n$ are restrictions of holomorphic matrix functions, the standard deformation of the contour technique and the above estimate yield that

$$Z = I + O_\varepsilon(n^{-d_{\nu,\ell}}) \quad \text{(6.37)}$$

locally uniformly in $\overline{\Sigma} \setminus \{0\}$. 

6.8. Proofs of Theorems 2.3 and 2.4

Given \( Z(z) \), a solution of RHP-\( Z \), \( P_{a_i}(z) \) and \( P_0(z) \), defined in (6.10) and (6.23), respectively, and \( C(MD)(z) \) from (6.6) and (6.7), it can be readily verified that

\[
X(z) := CZ(z) \begin{cases} P_{a_i}(z), & z \in U_i, \ i \in \{1,2,3,4\}, \\ P_0(z), & z \in U_\delta, \\ (MD)(z), & \text{otherwise}, \end{cases}
\]

solves RHP-\( X \). Given a closed set \( K \subset \mathbb{C} \setminus \Delta \), the contour \( \Sigma_n \) can always be adjusted so that \( K \) lies in the exterior domain of \( \Sigma_n \). Then it follows from (6.3) that \( Y(z) = X(z) \) on \( K \). Formulae (2.35) and (2.41) now follow immediately from (6.2), (6.3), (6.6), (6.7), and (2.25) since

\[
w^{i-1}(z)[(ZMD)(z)]_{1i} = (1 + v_{n1}(z))\Psi_n(z^{i-1}) + v_{n2}(z)\Psi_{n-1}(z^{i-1}),
\]

where \( 1 + v_{n1}(z) \), \( v_{n2}(z) \) are the first row entries of \( Z(z)(I + L_\nu/z) \). Estimates (2.37) are direct consequence of (6.26) and (6.37). Relations (2.38) follow from (6.35). Similarly, if \( K \) is a compact subset of \( \Delta^o \), the lens \( \Sigma_n \) can be arranged so that \( K \) does not intersect \( \overline{U} \cup \overline{U}_\delta \). As before, we get that

\[
[(ZMD)(z)]_{11} = \left((1 + v_{n1}(z))\Psi_n(z^{(0)}) + v_{n2}(z)\Psi_{n-1}(z^{(0)})\right) \pm (\rho_i w)^{-1}(z) \left((1 + v_{n1}(z))\Psi_n(z^{(1)}) + v_{n2}(z)\Psi_{n-1}(z^{(1)})\right)
\]

for \( z \in \Omega_{i\pm} \setminus (\overline{U} \cup \overline{U}_\delta) \). Formula (2.36) now follows by taking the trace of \( [(ZMD)(z)]_{11} \) on \( \Delta_{i\pm} \setminus (\overline{U} \cup \overline{U}_\delta) \) and using (2.26).

6.9. Behavior of \( Q_n(z) \) around the Origin when \( \ell = \infty \) and \( |\text{Re}(\nu)| < 1/2 \)

Assume that \( \ell = \infty \). In this case \( \delta = \delta_n = \delta_0 \) in (6.36) is independent of \( n \) and \( P_0(z) \) is the exact parametrix (that is, the second group of jumps in RHP-\( Z \))(b) is not present). Assume further that \( |\text{Re}(\nu)| < 1/2 \). The definition of the matrix function \( M(z) \) as \( (I + L_\nu/z) M^\ast(z) \) is absolutely necessary when \( |\text{Re}(\nu)| = 1/2 \), see (6.6), but can be simplified to \( M(z) = M^\ast(z) \) when \( |\text{Re}(\nu)| < 1/2 \). That is, we can take \( L_\nu \) to be the zero matrix. In this case the error rate in RHP-\( P_0(c) \) will become \( \mathcal{O}(n|\text{Re}(\nu)|^{-1/2}) \) and the matrix \( E_0(z) \) will simplify to

\[
E_0(z) = M(z)K_0^n{\sigma_3}(z)r^{\sigma_3}(z)J(z)(2\xi_n)^{-\nu\sigma_3}, \quad \xi_n := \sqrt{n}\zeta_0(z),
\]

see (6.25) and (6.28). Assume now that \( z \) is in the second quadrant, in which case \( J = I \). It then follows from (6.24) and (6.29) that \( K_0^n(z) = \Phi^n(z^{(0)})e^{\xi_n^2} \).

Thus, we get from (6.23) as well as (6.4) and (6.7) that

\[
P_0(z) = E_0(z)\Psi(\zeta)r_2^{-\sigma_3}(z), \quad E_0(z) = C^{-1}N(z)\left(r_2(z)e^{\xi_n^2/(2\xi_n)^\nu}\right)^{\sigma_3},
\]

where we write \( \Psi(\zeta) \) for \( \Psi_{s_1,s_2}(\zeta) \). Now, (6.2) and (6.3) yield that \( Q_n(z) = [X(z)]_{11} + \rho_3^{-1}(z)[X(z)]_{12} \) for \( z \in \Omega_{3+} \). Therefore, we get from (6.38) that

\[
\gamma_n^{-1}Q_n(s) = (1 \quad 0) Z(s) ([P_0(s)]_{1+} + \rho_3^{-1}(s)[P_0(s)]_{2+})
\]
for $s \in \Delta_3 \cap U_\delta$, where $[P_0(z)]_i$ stands for the $i$-th column of $P_0(z)$. It follows from the analyticity of $E_0(z)$ in $U_\delta$ that
\[
\gamma_n^{-1}Q_n(s) = (1, 0)Z(s) E_0(s) \left( r_2^{-1}(s)[\Psi(\xi_n)]_1 + r_3^{-1}(s)[\Psi(\xi_n)]_2 \right)
\]
since $r_2(s)/r_3(s) = \rho_3(s)$. Using the expression for $E_0(z)$ from above as well as (6.4) and (2.26) we get that
\[
\gamma_n^{-1}Q_n(s) = (1, 0)Z(s) \begin{pmatrix} \Psi^{(0)}_{n+}(s) & \Psi^{(0)}_{n-}(s) \\ \Psi^{(0)}_{n-1+}(s) & \Psi^{(0)}_{n-1-}(s) \end{pmatrix} \begin{pmatrix} (2\xi_n)^{-\nu}A_\rho(\xi_n) \\ (2i\xi_n)^{\nu}B_\rho(\xi_n) \end{pmatrix}
\]
(6.39)
for $s \in \Delta_3 \cap U_\delta$, where, since $\xi_0(s)$ has argument $-\pi/4$ for $s \in \Delta_3$, we set
\[
\begin{cases}
A_\rho(\zeta) := e^{\zeta^2}(D_{\nu}(2\zeta) + \alpha_\rho D_{-\nu-1}(2i\zeta)) \\
B_\rho(\zeta) := e^{-\zeta^2}(D_{-\nu}(2i\zeta) + \beta_\rho D_{\nu+1}(2\zeta))
\end{cases}
\]
with $\alpha_\rho := -e^{\pi i/2}d_1(r_2/r_3)(s) = d_1(\rho_4/\rho_2)(s)$, $\beta_\rho := -e^{-\pi i/2}d_2(\rho_2/\rho_4)(s)$ and $d_1, d_2$ given by (6.14), which are constants by the definition of $W_\infty$. Recall that $(1, 0)Z(s)$, the first row of $Z(s)$, behaves like $(1 + o(1) - o(1))$, where $o(1) = O(n^{\Re(\nu)-1/2})$, in the considered case. Therefore, by multiplying (6.39) out, we can get an asymptotic expression for $Q_n(s)$ around the origin on $\Delta_3$. Clearly, we can get similar expressions on the remaining arcs $\Delta_1, \Delta_2$ and $\Delta_4$.

A computation along these lines can be performed in the case $\Re(\nu) = 1/2$, but the resulting formula is even more involved than (6.39).

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