On the total mass of closed universes with a positive cosmological constant

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Abstract

The recently suggested notion of total mass density for closed universes is extended to closed universes with a positive cosmological constant. Assuming that the matter fields satisfy the dominant energy condition, it is shown that the cosmological constant provides a sharp lower bound for the total mass density, and that the total mass density takes this as its minimum value if and only if the spacetime is locally isometric with the de Sitter spacetime. This notion of total mass density is extensible to non-compact 3-spaces of homogeneity of Bianchi class A cosmological spacetimes.

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1. Introduction

In our previous paper [1], we suggested a notion of total mass $M$ (or rather the total mass density, depending on the normalization) of closed universes at any instant represented by a closed spacelike hypersurface $\Sigma$. This $M$ was defined by a non-negative expression, built from the integral of the 3-surface twistor operator and the energy–momentum tensor on $\Sigma$, which in the asymptotically flat/hyperboloidal case provided a lower bound for the ADM/Bondi–Sachs mass. It was shown in [1] that apart from a numerical coefficient, $M$ is just the first eigenvalue of the square of the Sen–Witten operator, Witten’s gauge condition admits a non-trivial solution if and only if $M = 0$, and that $M = 0$ holds on some (and hence any) Cauchy hypersurface $\Sigma$ if and only if the spacetime is flat with toroidal spatial topology: $\Sigma \approx S^1 \times S^1 \times S^1$. If we allow that the connection not to be determined completely by the metric, namely if locally flat but holonomically non-trivial spacetime configurations are allowed, then this theorem should be generalized. This generalization was done in [2]: $M = 0$ holds if and only if the spacetime is holonomically trivial. In [2], we discussed the properties of $M$ further (and changed its normalization such that its scale be that of the ADM/Bondi–Sachs mass). We showed that the multiplicity of each eigenvalue of the square of the Sen–Witten operator is always even. We also illustrated these ideas in the examples of the closed Bianchi I. and Friedman–Robertson–Walker (FRW) cosmological models.
In this paper, we extend the ideas above by allowing the presence of a positive cosmological constant in Einstein’s equations. Its potential significance is given by the phenomenological interpretation of the observed luminosity–redshift anomaly of distant type Ia supernovae as the indication of the strict positivity of the cosmological constant $\Lambda$ (see e.g. [3, 4]).

While the general idea behind the construction and the key technical (both geometric and functional analytic) results remain the same in the presence of a positive $\Lambda$, the cosmological constant provides a sharp, strictly positive lower bound for $M$. Thus, the minimal mass configurations will be different from that in the zero cosmological constant case above. The aim of this paper is to determine these configurations. We show that $M$ takes this sharp lower bound as its minimal value precisely when the spacetime is locally isometric with the de Sitter spacetime. In this case, $M$ turns out to be independent of the hypersurface $\Sigma$. We also indicate how this notion of total mass density can be extended to 3-spaces of homogeneity of Bianchi class A cosmological spacetimes even if these are not closed.

In the following section, we review the key results and formulae that we need in our analysis. In section 3, we determine the minimal mass configurations. In section 4, it is indicated how this notion of the total mass density can be defined for spatially non-compact Bianchi A cosmological models. In the appendices we show, using Aronszajn’s theorem, that the eigenspinors of the (square of the) Sen–Witten operators cannot vanish on any open subset of $\Sigma$, and we discuss the technical details on the geometry of the Bianchi cosmological spacetimes that we need in the proof of our main theorem. As a by-product, we show that the left-invariant frame fields always satisfy the special orthonormal frame gauge condition of Nester [5].

Here, we use the abstract index formalism, and our sign conventions are those of [6]. In particular, the signature of the spacetime metric is $(+, -, -, -)$, and the curvature tensor is defined by $-R^d_{abcd}X^aV^bW^d := V^c\nabla_a(W^d\nabla_bX^c) - W^d\nabla_c(V^b\nabla_aX^d) - [V,W]^c\nabla_aX^d$ for any vector fields $X^a$, $V^b$ and $W^c$. Thus, Einstein’s equations take the form of $G_{ab} := R_{ab} - \frac{1}{2}Rg_{ab} = -\kappa T_{ab} - \Lambda g_{ab}$, where $\kappa := 8\pi G$ and $G$ is Newton’s gravitational constant, and $\Lambda > 0$.

2. Preliminaries

In our investigations, the key geometric ingredient is the Reula–Tod (or SL(2, $\mathbb{C}$) spinor) form [7] of the Sen–Witten identity for any spinor field $\lambda^A$ on the spacelike hypersurface $\Sigma$:

$$D_a(f^{AB}t^B)D_{bb}\lambda^A - \tilde{\lambda}^A F_{AB}D_{bb}\lambda^B + 2\lambda^A(D_{AB}\tilde{\lambda}^B)(D_{BB}\lambda^B) = -t^{AA}h^A(D_{\lambda}^A)(D_{\tilde{\lambda}}^A) - \frac{1}{2}\kappa t^A G_{ab}\lambda^A t^B,$$

(2.1)

where $t^A$ is the future pointing unit timelike normal to $\Sigma$, $h_{ab} := g_{ab}P_a^EP_b^E$ is the induced (negative definite) metric, $D_a$ is the corresponding intrinsic Levi-Civita covariant derivative and $D_c := P_c^f\nabla_f$ is the derivative operator of the Sen connection. Here, $P_a^c := \delta_a^c - t^at_a$ is the $g_{ab}$-orthogonal projection to $\Sigma$. The key observation is that the algebraically irreducible decomposition of the unitary spinor form [8, 9] of the $D_c$-derivative of the spinor field $\lambda_A$ into its totally symmetric part and the trace(s),

$$D_{EF}\lambda_A = D_{(EF}\lambda_{AA)} + \frac{2\sqrt{2}}{3}t_F^Ep^{EC}_{EE}\delta_C\ldots\lambda^D,$$

(2.2)

is a $t_{AA}$-orthogonal decomposition, and hence it is an $L_2$-orthogonal decomposition also. Here, the $L_2$-scalar of two spinor fields, say $\lambda^A$ and $\mu^A$, is defined by $\langle \lambda^A, \mu^A \rangle := \int_{\Sigma} \sqrt{2t_{AA}}\lambda^A t^B d\Sigma$, and the corresponding norm will be denoted by $\| \cdot \|_{L_2}$. In (2.2), the totally symmetric part of the derivative defines the 3-surface twistor operator of Tod [10], while the
second term is proportional to the action of the Sen–Witten operator (i.e. the Dirac operator built from the Sen connection on \( \Sigma \)) on the spinor field.

Substituting this decomposition into identity (2.1), taking its integral on \( \Sigma \) and using the fact that \( \Sigma \) is closed (i.e. compact with no boundary), we obtain

\[
\frac{4\sqrt{2}}{3\kappa} \| D_{\mu} \lambda^A \|_{L_2}^2 = \frac{\sqrt{2}}{\kappa} \| D_{(AB)} \lambda^C \|_{L_2}^2 + \int_{\Sigma} \tau^a \left( T_{ab} + \frac{\Lambda}{\kappa} g_{ab} \right) \lambda^B \bar{\lambda}^B d\Sigma, \tag{2.3}
\]

where we have used Einstein’s equations. This equation will play a key role in what follows and we call it the basic norm identity.

Next, let us define

\[
\mathcal{M} := \inf \left\{ \frac{\sqrt{2}}{\kappa} \| D_{(AB)} \lambda^C \|_{L_2}^2 + \int_{\Sigma} \tau^a \left( T_{ab} + \frac{\Lambda}{\kappa} g_{ab} \right) \lambda^B \bar{\lambda}^B d\Sigma \right\}, \tag{2.4}
\]

where the infimum is taken on the set of the smooth spinor fields for which \( \| \lambda^A \|_{L_2}^2 = \sqrt{2} \).

The physical dimension of \( \mathcal{M} \) is mass density, and note that \( \mathcal{M} \) with this normalization is \( \sqrt{2} \) times of the \( \mathcal{M} \) introduced in [1]. (Although the physical dimension of \( \mathcal{M} \) with the normalization \( \| \lambda^A \|_{L_2}^2 = \sqrt{2} \text{ vol } (\Sigma) \) would be mass, it is more convenient to use the above normalization [2].)

By the basic norm identity and the definition of \( \mathcal{M} \) we have, for any spinor field \( \lambda^A \), that

\[
\frac{4\sqrt{2}}{3\kappa} \| D_{\mu} \lambda^A \|_{L_2}^2 \leq \left\{ \frac{\sqrt{2}}{\kappa} \| D_{(AB)} \lambda^C \|_{L_2}^2 + \int_{\Sigma} \tau^a \left( T_{ab} + \frac{\Lambda}{\kappa} g_{ab} \right) \lambda^B \bar{\lambda}^B d\Sigma \right\} \geq \frac{1}{\sqrt{2}} \| \mathcal{M} \| \lambda^A \|_{L_2}^2. \tag{2.5}
\]

Since \( \mathcal{M} \) was defined as the infimum of an expression on a set of certain smooth spinor fields, it is not a priori obvious that there is a smooth spinor field which saturates the inequality on the right. Nevertheless, one can in fact show that such a spinor field does exist [1]. We will call such a spinor field a minimizer spinor field. Thus, if \( \lambda^A \) is such a minimizer spinor field, then by (2.5) we have that

\[
2 D^{[A} D_{AB} \lambda^B = \frac{3}{4 \kappa} \mathcal{M} \lambda^A \lambda^A = 2 \| D_{\mu} \lambda^A \|_{L_2}^2 - \frac{3}{2 \kappa} \mathcal{M} \| \lambda^A \|_{L_2}^2 = 0. \tag{2.6}
\]

This implies that the minimizer spinor field is necessarily \( L_2 \)-orthogonal to the spinor field \( 2 D^{[A} D_{AB} \lambda^B = \frac{3}{4 \kappa} \mathcal{M} \lambda^A \), or that \( \frac{1}{\kappa} \mathcal{M} \) is an eigenvalue and the minimizer spinor field is a corresponding eigenspinor of the operator \( 2 D^{[A} D_{AB} \).

In fact, it has been proven in [1] that its smallest eigenvalue is just \( \alpha^2 = \frac{3}{4 \kappa} \mathcal{M} \), and the minimizer spinor field is a corresponding eigenspinor. Moreover, it was shown in [2] that the multiplicity of every eigenvalue is even. In particular, if \( \alpha \neq 0 \) and \( \lambda^A \) is an eigenspinor of \( 2 D^{[A} D_{AB} \) with the eigenvalue \( \alpha^2 \), then \( \mu^A := -i \frac{2 \kappa}{\alpha} D^{[A} \lambda^A \) is a linearly independent eigenspinor with the same eigenvalue. Clearly, all these general results hold true even in the presence of a positive cosmological constant, since the previous analysis can be repeated with the ‘effective energy–momentum tensor’ \( \tilde{T}_{ab} := T_{ab} + \frac{\Lambda}{\kappa} g_{ab} \), which also satisfies the dominant energy condition if \( T_{ab} \) does.

3. The minimal mass configurations

The (geometrical and physical) significance of \( \mathcal{M} \) in the case of the vanishing cosmological constant is shown by the result of [1, 2] that \( \mathcal{M} = 0 \) on some \( \Sigma \) is equivalent to the holonomic triviality of the spacetime with spatial topology \( \Sigma \approx S^1 \times S^1 \times S^1 \), provided the matter fields satisfy the dominant energy condition. Thus, \( \mathcal{M} \) satisfies the minimal requirement to be a measure of the ‘strength of the gravitational field’.
If the cosmological constant is positive, then the minimal mass configuration(s) will be different from that in the zero cosmological constant case. To see this, let $\lambda^A$ be any spinor field with $||\lambda^A||_{\Sigma_1}^2 = \sqrt{2}$, for which we have that

$$
\frac{\sqrt{2}}{\kappa} ||D(\Lambda^B C)\lambda^A||_{\Sigma_1}^2 + \int_E r^d \left(T_{ab} + \frac{\Lambda}{k} g_{ab}\right) \lambda^{aB} \overline{\lambda}^{B}\, d\Sigma \geq \frac{\Lambda}{k}.
$$

Then definition (2.4) yields that $\kappa \mathcal{M} \geq \Lambda$, i.e., the total mass density is bounded from below by the cosmological constant. This bound is sharp, as the following example shows.

Let $\tilde{\Sigma}_t$ be a $t =$ const spacelike hypersurface of the de Sitter spacetime with the line element

$$
ds^2 = dt^2 - a^2 \cosh^2 \left(\frac{t}{a}\right) \left(d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta \, d\phi^2)\right),
$$

where $t \in \mathbb{R}$, $(\chi, \theta, \phi) \in S^3$ and $a$ is a positive constant. Thus, the induced metric $h_{ab}$ on $\tilde{\Sigma}_t \approx S^3$ is that of the metric sphere with radius $a \cosh(\frac{t}{a})$, its (spatial) scalar curvature is $R = 6/(a^2 \cosh^2(\frac{t}{a}))$ and its extrinsic curvature is $\chi_{ab} = \frac{1}{a} \tanh(\frac{t}{a}) h_{ab}$. The spacetime is of constant curvature with scalar curvature $R = 12/a^2$. Considering this spacetime as a solution of the vacuum Einstein equations, for the cosmological constant we obtain that $R = 4A$. The first eigenvalue of $2D^{\Lambda^A} D_{\Lambda^B}$ on the $t =$ const spacelike hypersurfaces of the general $k = 1$ (closed) FRW spacetimes is $\alpha_1^2 = \frac{3}{2} R + \frac{3}{2} \chi^2$, and the two linearly independent eigenspinors are just the ones whose components in the globally defined spinor dyad associated with the $SU(2)$-left-invariant orthonormal frame are constant (see [1], and, for a more general analysis, appendix A.2.2). Substituting the specific expression for the spatial curvature scalar and the extrinsic curvature here, we obtain that $\kappa \mathcal{M} = \frac{2}{3} \alpha_1^2 = \frac{1}{a^2} = \Lambda$; i.e., the lower bound $\Lambda$ for $\kappa \mathcal{M}$ is, in fact, sharp.

However, the de Sitter spacetime with $S^3$ spatial topology is not the only minimal total mass density configuration. In fact, let us substitute $\tilde{\Sigma}_t$ in (3.1) by any homogeneous Riemannian 3-manifold $\Sigma_t$ which is locally isometric with $\tilde{\Sigma}_t$. It is known (see [14] theorem 3 in Note 4, pp 294–7) that any such $\Sigma_t$ is isometric with the quotient $\tilde{\Sigma}_t/G$, where $G$ is some discrete subgroup of the group $SU(2) \approx S^3 \approx \Sigma_t$. The projection $\pi : \tilde{\Sigma}_t \rightarrow \Sigma_t$ is clearly a local isometry, and hence $\Sigma_t$ inherits a globally defined left-invariant frame field. In addition, $\pi^*$ maps the extrinsic curvature of $\Sigma_t$ into that of $\tilde{\Sigma}_t$. Then the resulting spacetime is locally isometric with the de Sitter spacetime, but its global topology is $\mathbb{R} \times S^3/G$. For example, $S^3/G$ could be the three-dimensional real projective space $\mathbb{R}P^3 := S^3/\mathbb{Z}_2 \approx SO(3)$. Therefore, any spinor field on $\Sigma_t$ whose components in the spinor dyad associated with the left-invariant frame field are constant is an eigenspinor of $2D^{\Lambda^A} D_{\Lambda^B}$ with the same eigenvalue $\alpha_1^2 = \frac{1}{a^2}$ as in the de Sitter case (see the discussion following equations (A.10) and (A.16) of the appendix), yielding the same total mass density $\mathcal{M} = \Lambda/a$, too.

The next theorem gives the complete characterization of the minimal total mass density configurations in the presence of a positive cosmological constant.

**Theorem 3.1.** Let the cosmological constant $\Lambda$ be positive and the matter fields satisfy the dominant energy condition. Then $\kappa \mathcal{M} = \Lambda$ for some (and hence for any) $\Sigma$ if and only if $T_{ab} = 0$ and the spacetime is locally isometric with the de Sitter spacetime.

**Proof.** First we show that $\kappa \mathcal{M}$, calculated on any Cauchy surface of any locally de Sitter spacetime above, is $\Lambda$. We saw that $\alpha_1^2 = \frac{2}{3} \Lambda$ is an eigenvalue of $2D^{\Lambda^A} D_{\Lambda^B}$ on any maximally symmetric spacelike hypersurface $\Sigma_t$, and let $\lambda^A$ be a corresponding eigenspinor. As we noted above, its components in the spinor dyad associated with the left-invariant orthonormal frame field are constant. Since $\mu_A := -i \frac{2}{a^2} D_{A^A} \lambda^A$ is an independent eigenspinor of $2D^{\Lambda^A} D_{\Lambda^B}$ with
the same eigenvalue (see [2]), its components are also constant. Therefore, by equations (A.10) and (A.15) of section A.2.2, both $\lambda^A$ and $\mu^A$ satisfy the 3-surface twistor equation on $\Sigma_r$, and hence, by (2.2) and the eigenvalue equation, the equations

$$
\mathcal{D}_e \lambda^A = \frac{2}{3} p_{e}^{ABC} \mathcal{D}^e_{ABC} \lambda^B = -i \alpha \sqrt{2} p_{e}^{ABC} \bar{\mu}_A, \quad \mathcal{D}_e \mu^A = \frac{2}{3} p_{e}^{ABC} \mathcal{D}^e_{ABC} \mu^B = i \alpha \sqrt{2} p_{e}^{ABC} \lambda_A.
$$

(3.2)

too. In this way, we have a spinor field (or rather a pair of spinor fields) on every leaf $\Sigma_r$, and hence on the spacetime manifold itself up to an unspecified function of time as a factor of proportionality. To find out how to fix this factor, let us suppose that $\omega^A$ is any spinor field on the spacetime manifold which satisfies the 3-surface twistor equation on the leaves $\Sigma_r$, and decompose its covariant derivative $\nabla_e \omega^A$ using $\mathcal{D}_{(AB)e(C)} = 0$ in its equivalent form

$$
\mathcal{D}_e \omega^A = \frac{2}{3} p_{e}^{ABC} \mathcal{D}^e_{ABC} \omega^B. \quad \text{We obtain}
$$

$$
\nabla_e \omega^A = \frac{1}{2} \delta^A_{\bar{E}} \nabla_{EB} \omega^B = \frac{1}{3} (2 \delta^A_{\bar{E}} - \delta^A_{E}) (\frac{1}{2} t_B^B \nabla_f \omega^B - t_{EB}^B D_{EB} \omega^D).
$$

Since the vanishing of the left-hand side is just the 1-valence twistor equation on $M$ [6], we obtained that the spinor field $\omega^A$ satisfying the 3-surface twistor equation satisfies the 1-valence twistor equation if and only if $t^B \nabla_e \omega^A = \frac{2}{3} t_B^B \mathcal{D}_B e^D \omega^D$. Since the twistor equation would be the natural extension of the 3-surface twistor equation form the hypersurfaces to the spacetime, we require that the eigenspinors $\lambda^A$ and $\mu^A$ satisfy the ‘evolution equations’

$$
t^e (\nabla_e \lambda^A) = i \alpha \sqrt{2} \frac{2}{3} \bar{\mu}_A^B, \quad t^e (\nabla_e \mu^A) = -i \alpha \sqrt{2} \frac{2}{3} \lambda_A^B.
$$

(3.3)

These evolution equations preserve (3.2). In fact, by (3.2), (3.3) and the definition and the specific form of the curvature, we have that

$$
t^e \nabla_e \left( \mathcal{D}_e \lambda^A + i \alpha \sqrt{2} p_{e}^{ABC} \bar{\mu}_A^B \right) = -\frac{4}{9} \alpha^2 p_{e}^{ABC} \mathcal{D}^e_{ABC} \lambda^B - R^A_{BC} \epsilon^C \lambda^B
$$

$$
= \frac{1}{6} \left( \Lambda - \frac{4}{3} \alpha^2 \right) \lambda_A^B = 0,
$$

and, similarly, the time derivative of $\mathcal{D}_e \mu^A - i \alpha \sqrt{2} p_{e}^{ABC} \lambda_A$ is also vanishing. Thus the integrability conditions of the twistor equation on $M$ are satisfied. Therefore, by (3.3), there is a unique extension of the eigenspinor from e.g. $\Sigma_0$ to the whole locally de Sitter spacetime, which solves the 1-valence twistor equation

$$
\nabla^A (\lambda_A^B) = 0.
$$

1 Let $\Sigma$ be any Cauchy hypersurface of the spacetime. Since the spinor field $\lambda^A$ solves the 1-valence twistor equation on $M$, its restriction to $\Sigma$ solves the 3-surface twistor equation there. Since $T_{ab} = 0$, for this spinor field the expression between the curly brackets in (2.4) is just $\Lambda \lambda_A^B / (\sqrt{2} \lambda)$. Thus, after normalization, it is a minimizer spinor field, yielding that $\kappa = \Lambda$.

Conversely, let $\Sigma$ be a closed spacelike hypersurface, suppose that $\kappa = \Lambda$ on $\Sigma$, and let $\lambda^A$ be any eigenspinor of $2 \mathcal{D}^{ABC} \mathcal{D}_{ABC}$ with the eigenvalue $\alpha^2 = \frac{3}{4} \Lambda$. Let $\mu^A := -i \sqrt{2} \alpha \lambda_A^B$.

In fact, the pair $(\lambda^A, \mu^A)$ of spinor fields solves a pair of 1-valence twistor equations in which the secondary part of one solution is just the primary part of the other. It might be worth noting that this system of equations can be written in terms of Dirac spinors in the remarkably simple form

$$
\nabla_e \Psi^\alpha = \frac{i}{2} \gamma^\alpha \gamma^e \Psi^\beta,
$$

where $\Psi^\alpha = (\lambda^A, \bar{\mu})$ (as a column vector), $\alpha = A \oplus A^\prime$, $\beta = B \oplus B^\prime$ and Dirac’s ‘$\gamma$-matrices’ are given explicitly in the abstract index formalism by

$$
\gamma^e_{\alpha \beta} = \sqrt{2} \begin{pmatrix} 0 & \epsilon_{EBA}^\beta \delta^A_e \\ \epsilon_{EBA}^\beta \delta^A_e & 0 \end{pmatrix}
$$

(see e.g. [15, p 221]).
and hence $\mu^A$ is also an eigenspinor of $2D^{AA}D_{AB}$ with the same eigenvalue. Since $\kappa \mu = \Lambda$, by the dominant energy condition, it follows from (2.4) that $i^n T^B_{AB\hat{\lambda}^B} = i^n T^B_{AB\hat{\mu}^B} = 0$, and hence that $T_{ab} = f_{\hat{\lambda}^A}^{\hat{\lambda}^B} \hat{\lambda}_A^{\hat{\lambda}^B} = f_{\hat{\mu}^A}^{\hat{\mu}^B} \hat{\mu}_A^{\hat{\mu}^B}$ for some non-negative functions $f$ and $\hat{f}$ on $\Sigma$. These imply that if $\lambda^A$ and $\mu^A$ are not proportional with each other, then $f = 0$. (Note that the linear independence of $\lambda^A$ and $\mu^A$ means only that $\lambda^A \neq c \mu^A$ for any non-zero complex constant $c$. Thus, in principle, $\lambda^A$ and $\mu^A$ could be proportional to each other with some complex function as a factor of proportionality.)

Also, the condition $\kappa \mu = \Lambda$ yields from (2.4) that $D_{(AB\lambda C)} = D_{(AB\mu C)} = 0$. Substituting these into (2.2), we obtain (3.2). First, we show that $\lambda^A$ and $\mu^A$ are not proportional to each other. Thus, suppose, on the contrary, that there exists an open set $U \subset \Sigma$ and a smooth complex-valued function $F : U \rightarrow \mathbb{C}$, such that $D_{(AB\lambda)} = F\lambda^A$, i.e. $\mu^A = -i \sqrt{\frac{2}{n}} F\lambda^A$ on $U$. Then, on $U$, $\lambda_A^{\hat{\mu}} = 0$. Taking its derivative, using equation (3.2), our assumption of the proportionality of $\mu^A$ and $\lambda^A$ and the eigenvalue equation for $\lambda^A$, we find

$$0 = D_{\varepsilon}(\lambda_A^{\mu A}) = -(D_{\varepsilon} \lambda^A)\mu_A + (D_{\varepsilon} \mu^A)\lambda_A$$

$$= i\alpha \frac{\sqrt{2}}{3} P_{\varepsilon}^{A\mu} \mu_A^{\varepsilon} \lambda_{\varepsilon} A + i\alpha \frac{\sqrt{2}}{3} P_{\varepsilon}^{A\mu} \lambda_{\varepsilon} A = i\frac{\sqrt{2}}{3\alpha}(2FF^\prime + \alpha^2)P_{\varepsilon}^{A\mu} \lambda_A^{\varepsilon} A,$$

i.e. the projection $Z_{\varepsilon} := P_{\varepsilon}^{AA} \lambda_A^{\varepsilon} A$ would have to be vanishing on the open set $U$. Since $\Lambda^A := i\lambda_A^{\mu} \lambda^A$ is null and $\Sigma$ is spacelike, this would imply the vanishing of $\lambda^A$ on the open set $U$. However, Bär [11] showed that the zero-set of any eigenspinor of a Riemannian Dirac operator on an $n$-dimensional manifold is at most $(n-2)$ dimensional. Also, by Aronszajn’s theorem [12] if a function (or a set of functions) satisfying a second-order elliptic p.d.e. is (are) vanishing on an open set, then it is (they are) vanishing everywhere. These results indicate that $\mu^A$ and $\lambda^A$ cannot be proportional. In fact, in section A.1, we show that the eigenspinors of $2D^{AA}D_{AB}$ satisfy the conditions of Aronszajn’s theorem, and hence they cannot be vanishing on any open subset of $\Sigma$. Therefore, $\mu^A$ and $\lambda^A$ are not proportional to each other on any open subset of $\Sigma$, and hence, by the argumentation above, the energy–momentum tensor is vanishing on $\Sigma$.

Next, let us evaluate the integrability conditions of (3.2). The action of the commutator of two Sen derivative operators on any spinor field is

$$(D_{\varepsilon}D_{\delta} - D_{\delta}D_{\varepsilon})\lambda^A = -R_{\varepsilon\delta}^{\varepsilon\delta} P_{\varepsilon}^{\mu}\mu^B - 2\chi^{\varepsilon}_{[\varepsilon\delta]} D_{\varepsilon} \lambda^A,$$

where $\chi_{\varepsilon\delta}$ is the extrinsic curvature of $\Sigma$ in the spacetime (see e.g. [1]). Thus, it is a straightforward calculation to derive the integrability conditions of (3.2), for which we obtain

$$R_{\varepsilon\delta}^{\varepsilon\delta} P_{\varepsilon}^{\mu} \mu^B = -\frac{1}{2} \lambda^{\varepsilon} \delta_{[\varepsilon\delta]}^{[\varepsilon\delta]^{\mu}} \varepsilon, \varepsilon B^\varepsilon E^F \varepsilon C^\delta_{EF} \varepsilon C^\sigma \varepsilon D^\sigma$$

where we used the eigenvalue equation and the specific value $\alpha^2 = \frac{1}{2} \Lambda$ of the eigenvalue. However, the expression on the right is just the pullback to $\Sigma$ of the anti-self-dual part of the curvature of the constant positive curvature spacetime with scalar curvature $R = 4\Lambda$, i.e. of the de Sitter spacetime. Therefore, the anti-self-dual part of the curvature tensor of the spacetime has the structure $-R_{\varepsilon\delta}^{\varepsilon\delta} = \varepsilon_{\varepsilon\delta} \varepsilon C^\varepsilon \varepsilon D^\delta - \frac{1}{4} \delta_{[\varepsilon\delta]}^{[\varepsilon\delta]^{\mu}} \varepsilon, \varepsilon B^\varepsilon E^F \varepsilon C^\sigma \varepsilon D^\sigma$, where using Einstein’s equations we have already taken into account the vanishing of the trace-free part of the Ricci tensor. In terms of these quantities, the integrability condition (3.4) can be rewritten as

$$0 = -(R_{\varepsilon\delta} + \frac{1}{2} \Lambda \delta_{[\varepsilon\delta]}^{[\varepsilon\delta]^{\mu}} \varepsilon, \varepsilon B^\varepsilon E^F \varepsilon C^\sigma \varepsilon D^\sigma) P_{\varepsilon}^{\mu} \mu^B = \lambda^B \varepsilon_{\varepsilon\delta} \varepsilon C^\varepsilon \varepsilon D^\delta \varepsilon C^\sigma \varepsilon D^\sigma P_{\varepsilon}^{\mu}.$$

Now we show that the whole Weyl spinor is also vanishing.

For this, let us introduce the complex null vectors $M^a := \lambda^A \nu_A^B \lambda^B$ and $\bar{M}^a := \bar{\lambda}^A \nu_A^B \bar{\lambda}^B$, which are tangent to $\Sigma$. These vectors together with $Z^a := P_{\delta}^{\mu} \lambda^A \bar{\lambda}^B$ form a basis on
an open dense subset of \( \Sigma \) (namely on the subset where \( \lambda^A \) is non-zero). Clearly, these satisfy the orthogonality and normalization conditions \( \Omega_\mu M^\mu = M_\rho M^\rho = 0 \) and \( \Omega Z^\mu = M_\rho Z^\rho = M^\mu = \) holds. Then contracting (3.5), respectively, with \( M^\mu M^\mu, Z^\mu M^\mu \) and \( Z^\mu M^\mu \), we obtain the vanishing of \( \Psi_{\lambda BCD} = \Psi_{\lambda \lambda B C \lambda D} \). These imply that \( \Psi_{\lambda BCD} = \Psi_{\lambda \lambda B C \lambda D} \). Repeating this argument with the spinor field \( \mu^A \), we obtain that \( \Psi_{\lambda BCD} = \Psi_{\mu A \mu B \mu C \mu D} \). Since, however, the spinor fields \( \lambda^A \) and \( \mu^A \) are not proportional to each other on any open subset of \( \Sigma \), this yields that \( \Psi = \Psi = 0 \). Therefore, at the points of \( \Sigma \) the curvature tensor of the spacetime is that of the de Sitter spacetime.

Finally, let us foliate an open neighborhood of \( \Sigma \) in the spacetime by the spacelike hypersurfaces \( \Sigma \) obtained by Lie dragging \( \Sigma \) along its own unit timelike normal \( t^A \). Then, the Bianchi identities, written in their 3+1 form with respect to this foliation by Friedrich [13], yield that the spacetime curvature tensor is that of the de Sitter spacetime on the neighboring leaves of the foliation. Hence, the spacetime is locally isometric to the de Sitter spacetime with the cosmological constant \( \Lambda \) in which the typical Cauchy surface is homeomorphic to \( \Sigma \) (see [16]).

This result is slightly weaker than the analogous one in [1, 2] for the zero cosmological constant case, because there the topology of the 3-space has also been determined. In fact, there is a uniqueness for the topology of the holonomically trivial compact 3-spaces: they are necessarily tori (see [14], theorem 4.2 in chapter V, pp 211–21), while, as we already noted above, the topology of 3-spaces with constant positive curvature is far from being fixed.

In both the \( \Lambda = 0 \) and \( \Lambda > 0 \) cases, the first eigenvalue of \( 2D^{AB} D_{AB} \) in the minimal total mass density configurations is built only from the curvature of \( \mathcal{D} \), which has a uniform value in the 3-space. Thus, to give a more detailed characterization of the 3-spaces we should consider other eigenvalues. In the spatially flat, closed Bianchi I model, the first eigenvalue of the Riemannian Dirac operator is zero, and the three parameters specifying the ‘size’ of the flat 3-space are encoded in the higher eigenvalues (see [2]). Similarly, in the locally de Sitter case, the volume of the 3-space is expected to be recoverable from the higher eigenvalues. If this were indeed the case, then in terms of the curvature and the volume the topology of the 3-space could also be characterized, at least partly, by the eigenvalues of \( 2D^{AB} D_{AB} \). To see this, observe that the local isometry \( \pi : \Sigma \rightarrow \Sigma \) is a universal covering, because \( \Sigma \approx S^3 \) is simply connected. Since \( \pi \) is smooth and \( \Sigma \) is compact, it is a proper map, i.e. \( \pi^{-1}(K) \subset \Sigma \) is compact for any compact \( K \subset \Sigma \). Hence, \( \int_{\Sigma} \pi^* (\omega)_{abc} = \deg(\pi) \int_{\Sigma} \omega_{abc} \) holds for any 3-form \( \omega_{abc} \) on \( \Sigma \), where \( \deg(\pi) = \) the degree of \( \pi \) (see e.g. [17], p 275). Thus, in particular, \( \vol(\Sigma) = \deg(\pi) \vol(\Sigma) \). Actually, \( \deg(\pi) \) is positive because \( \pi \) is orientation preserving, and \( \deg(\pi) \) can be interpreted as the number of how many times \( \Sigma \) covers \( \Sigma \). Therefore, since the curvature determines the volume of \( \Sigma \), by \( \vol(\Sigma) / \vol(\Sigma) = \deg(\pi) \) the curvature and the volume of the 3-space \( \Sigma \) determine \( \deg(\pi) \), i.e. the number how many times \( S^3 \) covers the typical Cauchy surface \( \Sigma \).

4. Total mass density in Bianchi A models

In Bianchi A cosmological spacetimes by the spatial homogeneity, the eigenvalue equation for the Sen–Witten operator always has two independent eigenspinors whose components in the left-invariant frame are constant. In particular, on the \( t = \text{const} \) spacelike hypersurfaces of the \( k = 1 \) (closed) FRW spacetime these spinor fields are just the minimizer spinor fields.
of \( \mathfrak{H} \). (See the discussions following equations (A.12) and (A.16) of appendix A.2.) With this spinor field the integrand

\[
\frac{4}{\kappa} t^{\alpha A} t^{\beta B} t^{\gamma C} D_{(A \beta C)} D_{(B \gamma C)} + t^a \left( T_{ab} + \frac{A}{\kappa} \delta_{ab} \right) \lambda^B B^C
\]

in definition (2.4) of \( \mathfrak{H} \) is constant on the 3-spaces \( \Sigma_t \) of homogeneity. Thus, even though the integral of (4.1) does not exist on non-compact 3-spaces of homogeneity, this expression can be used to define the total mass density by fixing the scale of the spinor field by \( t_{AA} \lambda^A \lambda^A = 1 \) at any given point of \( \Sigma_t \). By equation (A.11) of section A.2 the 3-surface twistor term of (4.1) in general is non-trivial, and could be interpreted as the contribution of the gravitational ‘field’ to the total mass density of the matter+gravity system.

Appendix

A.1. An application of Aronszajn’s theorem to eigenspinors

In this appendix we prove the following statement:

**Theorem A.1.** If \( \lambda^A \) is an eigenspinor of \( 2D^{\lambda A} D_{\lambda B} \) on \( \Sigma_t \), then it cannot have any zero of infinite order in the \( 1 \)-mean (for the definition see below).

Let \( o \in \Sigma, r > 0 \) and let \( B(o, r) \) denote the set of points of \( \Sigma \) whose geodesic distance from \( o \) is less than \( r \). Following Aronszajn [12], we say that \( \lambda^A \) has a zero of order \( \omega \) at \( o \) in the \( p \)-mean if \( \int_{B(o, r)} |\lambda^A|^p d\Sigma = O(r^{\omega p+1}) \), where \( |\lambda^A|^2 := \sqrt{2} t_{AA} \lambda^A \bar{\lambda}^A \), the square of the pointwise norm of the spinor field. The zero is said to be of infinite order if it is of order \( \omega \) for all \( \omega > 0 \). Clearly, if \( \lambda^A \) is vanishing on an open subset \( O \subset \Sigma \), then any \( o \in O \) is a zero of infinite order. Hence, the theorem above guarantees that no eigenspinor of \( 2D^{\lambda A} D_{\lambda B} \) can be vanishing on any open subset of \( \Sigma \).

**Proof.** In [1], we derived a Lichnerowicz type identity for any spinor field (equation (2.11) of [1]), which, by the expression for the constraint parts of the spacetime Einstein tensor (equations (2.9) and (2.10) of [1]) and for the Sen derivative operator in terms of the intrinsic Levi-Civita derivative and the extrinsic curvature (equations (2.2) and (2.4) of [1]), takes the form

\[
2D^{\lambda A} D_{\lambda B} = D_{\gamma} D_{\lambda} \lambda^A + (D^{\lambda A} \chi)_{AB} \chi^B + \frac{1}{2} (\mathcal{R} + \chi^2) \lambda^A.
\]

(A.1)

Thus if \( \lambda^A \) is an eigenspinor of \( 2D^{\lambda A} D_{\lambda B} \) with the eigenvalue \( \alpha^2 \), then \( \lambda^A \) satisfies

\[
D_{\gamma} D_{\lambda} \lambda^A = (D^{\lambda A} \chi)_{AB} \chi^B + \frac{1}{2} (4\alpha^2 + \mathcal{R} + \chi^2) \lambda^A.
\]

Let \( U \subset \Sigma \) be an open set with compact closure which is the domain of the local coordinate system \( \{ e^\alpha \} \), and fix a normalized dual spin frame \( \{ e^\alpha_A, e^\beta_A \} \) on \( U \). Then the coordinate Laplacian of the components of the spinor field has the structure

\[
\lambda^A \lambda^B = (D_{\gamma} D_{\lambda} \lambda^A) e^\gamma_A + E^A_B \partial_{\gamma} \lambda^B + E^A_B \bar{\lambda}^B,
\]

where the coefficients \( E^A_B \) and \( E^A_B \) are built from the Christoffel symbols and the components of the connection 1-form on the spinor bundle in the spinor basis above. Taking into account (A.2) and writing its right-hand side as \( F^A_B \lambda^B \), we obtain

\[
|\lambda^A \lambda^B| \leq |E^A_B \lambda^B| + |E^A_B \bar{\lambda}^B| + |F^A_B \lambda^B|,
\]

(A.3)

where \( |.| \) means pointwise absolute value of spinor/tensor components. Next, recall that on any finite dimensional complex vector space, say \( \mathbb{C}^n \), any two norms \( \| x \|_p := (\sum_{k=1}^{n} |x^k|^p)^{1/p} \), \( x = \{ x^k \} \in \mathbb{C}^n \), are equivalent, where \( 1 \leq p < \infty \). In particular, \( \| x \|_1 \leq n \| x \|_2 \).
holds. Thus, if \( x, y, z \in \mathbb{C}^n \), then with the notation \( w := (x, y, z) \in \mathbb{C}^{3n} \) we have that \( \|x\|_1 + \|y\|_1 + \|z\|_1 = \|w\|_1 \leq 3n\|w\|_2 \). Applying this inequality to the pointwise norm on the right of (A.3), we obtain
\[
|\hat{R}^{\alpha\beta}_\gamma \partial_\alpha \partial_\beta \chi |^2 \leq 36(\|E^{\alpha\beta}_\gamma \partial_\alpha \lambda E^{\beta\gamma}_\delta \|^2 + \|E^{\alpha\beta}_\gamma \partial_\beta \lambda E^{\alpha\gamma}_\delta \|^2 + \|\delta E^{\alpha\beta}_\gamma \lambda E^{\alpha\beta}_\delta \|^2).
\]
Since the geometry is smooth and \( U \) is of compact closure, there exists a positive constant \( C \), such that for both \( A = 0, 1 \),
\[
|\hat{R}^{\alpha\beta}_\gamma \partial_\alpha \partial_\beta \chi |^2 \leq C\delta_{BG} (-h^{\alpha\beta} (\partial_\alpha \lambda E^{\beta\gamma}_\delta) + \lambda E^{\alpha\beta}_\delta E^{\alpha\beta}_\delta \).
\]
holds; here \( \delta_{BG} \) is the Kronecker delta. However, this is just the condition of Aronszajn’s theorem [12] (see remark 3 in Aronszajn’s paper), which guarantees the vanishing of \( \lambda^A \) on \( U \) if it has a zero of infinite order somewhere in \( U \). Finally, covering \( \Sigma \) by such coordinate domains \( U \) and applying the above result to the overlapping domains, we find that \( \lambda^A \) is vanishing on the whole of \( \Sigma \).

**A.2. The \( D \) and \( T \) operators in Bianchi 3-spaces.**

**A.2.1. The geometry of the Bianchi cosmological spacetimes.** Let the spacetime be a homogeneous Bianchi cosmological spacetime, foliated by the spacelike hypersurfaces \( \Sigma_t \) which are the transitivity surfaces of the isometry group (see e.g. [18]). Let \( \{e^i, \vartheta^i\}, i = 1, 2, 3 \), be a globally defined \( h_{ab} \)-orthonormal dual frame field on \( \Sigma_0 \), which is left invariant with respect to the (simply transitive) action of the isometry group \( G \) on \( \Sigma_0 \). Then the structure constants of \( G \) can be given in this basis, too, by \( c^i_{jk} := [e^j, e_k]^i \vartheta^a_i \), and \( D^a_i \vartheta^j_i = -\frac{1}{2} c^i_{jk} \vartheta^a_i e^k_j \) also holds. Following [18], we parametrize the structure constants as \( c^i_{jk} = M^{ji} \epsilon_{mjk} + \delta^i_{jk} \lambda \), where \( M^{ji} \) is a symmetric real matrix, \( M^{ji} \lambda_j = 0 \) holds and \( \delta_{mji} \) is the alternating Levi-Civita symbol. The group \( G \) is said to belong to Bianchi class \( A \) if \( \lambda = 0 \), otherwise it belongs to class \( B \). Extending the dual frame field \( \{e^i, \vartheta^i\} \) along the timelike geodesic normals to the other leaves \( \Sigma_t \), we obtain a globally defined frame field which provides a convenient ‘background’ to describe the dynamics, but, in general, it will not be \( h_{ab} \)-orthonormal on the leaves other than \( \Sigma_0 \). In this basis \( h_{ab} = h_{ij} \vartheta^a_i \vartheta^b_j \), where the components \( h_{ij} \) are functions only of the time coordinate \( t \).

Thus, let us write the vectors of an \( h_{ab} \)-orthonormal basis as \( E^a_i = e^i \Phi^a_i \), and similarly \( \vartheta^a_i = \vartheta^i_j \Phi^a_j \), where the matrices \( \Phi^a_i, \Phi^a_j \) depend only on \( t \), have positive determinant, satisfy \( \Phi^a_i \Phi^b_j = \delta^a_b \) and \( h_{ij} = \Phi^a_i \Phi^b_j \eta_{ab} \), and at \( t = 0 \) they reduce to the unit matrix \( \delta^i_j \). Clearly, \( \{E^a_i, \vartheta^a_i\} \) is also a left-invariant dual basis, but \( E^a_i = h^{ab} \vartheta^i_j \eta_{b1} \) holds, where boldface (i.e. frame name) index lowering and raising are defined by \( \eta_{ij} := -\delta_{ij} \) and its inverse. (Note, however, that \( h^{ab} \) denotes the inverse of the matrix \( h_{ij} \).) The structure ‘constants’ of this frame are \( C^a_{ji} := [E_j, E_k]^a_{ji} \), which depend on the time coordinate. It is a simple calculation to show that these structure constants determine completely the Ricci rotation coefficients of the spatial Levi-Civita connection in the frame \( \{E^a_i, \vartheta^a_i\} \):
\[
\gamma^a_{kj} := E^a_k \vartheta^b_j := \partial^a_b E^a_k (D_k E^b_j) = \frac{1}{2} (C^a_{kj} + \eta^{im} \epsilon_{mj} \eta_{ni} + \eta^{im} \epsilon_{mj} \eta_{nk}).
\]
Clearly, these are constant on each \( \Sigma_t \), and, with the parametrization of \( c^i_{jk} \) above, we have that
\[
\frac{1}{2} \gamma^a_{kj} \vartheta^b_j = -\frac{1}{\sqrt{|\lambda|}} M^{a0} \Phi^0_m \vartheta^m_k \eta_{kj} \eta_{ji} \frac{1}{\sqrt{|\lambda|}} \eta_{ji} M^{a0} \vartheta^m_j \eta_{nk} - \frac{1}{2} \vartheta^m_k \Phi^0_j e^0_{kl}.
\]
Equation (A.5) implies, in particular, that the frame field \( \{E^a_i\} \) satisfies Nester’s special orthonormal frame gauge condition on \( \Sigma_t \) [5]: Nester requires \( q := \gamma^0_k \delta_{ij} \) to be constant and \( q \delta^0_{ij} \) to be closed, where \( q_j := \gamma^0_j \). By \( q = M^{a0} \vartheta^a_j / |\lambda| \).
the first condition is clearly satisfied, and by \( q_I = A_I \Phi_I^1, M^0 A_I = 0 \) and \( D_\alpha \theta_I^\rho = -\frac{1}{2} C_{\rho \sigma \tau} \theta_I^\sigma \), the 1-form \( q_I \theta_I^\rho \) is, indeed, closed. Since the Ricci rotation coefficients are constants in the left-invariant frame, it is straightforward to compute the components of the curvature tensor and its scalar curvature. For the latter, we obtain

\[
\mathcal{R} = -\frac{1}{2} M^{kl} M^{mn} \eta_{kn} \eta_{lm} - 2 \eta_{lk} M^{km} \eta_{mn} M^{jl} \eta_{ij} + 2 M^{mn} \eta_{mn} M^{kl} \eta_{ij} - (M^{kl} \eta_{k})^2 \eta_{ij} \frac{2}{h_{ij}}
\]

which is constant on \( \Sigma_t \), but depends on the time coordinate \( t \) since \( h_{ij} \) does.

A.2.2. The \( D \) and \( T \) operators. If \( \{ \epsilon_A^i, \sigma^A_{k_1 k_2} \}, A = 0, 1 \), is a normalized spinor basis associated with the dual frame field \( \{ E_i^a, \theta_I^a \} \), then the spinor connection 1-forms of the intrinsic Levi-Civita covariant derivative on the spinor bundle are given by

\[
\gamma^A_{\alpha \beta} = -\frac{1}{2\sqrt{2}} \epsilon^{ij}_{k} \epsilon^{k}_{l} \sigma_{k_1 k_2}^{\alpha \beta},
\]

where \( \sigma_{k_1 k_2}^{\alpha \beta} \) are the standard \( SU(2) \) Pauli matrices (including the factor \( 1/\sqrt{2} \)). Then, by contracting it with \( E_i^a \sigma_{\alpha \beta}^{a} \), it is straightforward to compute its various irreducible parts. We obtain

\[
\gamma^{D}_{AB} = \frac{1}{2} A_I \Phi_I^1 \sigma_{\alpha \beta}^{A} - \frac{1}{4\sqrt{2} h} M^0 h_{ij} e_{ij}^{AB}, \quad \gamma^{A}_{\alpha \beta} = \frac{i}{2\sqrt{2}} \Delta^{A} \Phi_I^1 \sigma_{\alpha \beta}^{A} + \frac{i}{2\sqrt{2}} \Delta^{A} \Phi_I^1 \sigma_{\alpha \beta}^{A}
\]

i.e. the irreducible parts of the first are proportional to \( A_I \) and the trace of \( M^0 \) with respect to \( h_{ij} \), respectively, while the second to the \( h_4 \)-trace-free part of \( M^0 \).

By (A.8) the Riemannian Dirac and 3-surface operators, \( D_{AB} \lambda^B \) and \( T(\lambda)_{ABC} := D_{(AB} \lambda_{C)} \) on \( \Sigma_t \), respectively, take the form

\[
\epsilon^A_{\alpha \beta} D_{AB} \lambda^B = \sigma^A_{\alpha \beta} \left( E_i^a \partial_i \lambda^B + \frac{1}{2} A_I \Phi_I^1 \lambda^B \right) - \frac{i}{4\sqrt{2}} \frac{1}{h} M^0 h_{ij} \lambda_{ij},
\]

\[
T(\lambda)_{ABC} = E_i^a \partial_i \lambda_{(AC} \sigma^B_{\alpha \beta} - \frac{i}{2\sqrt{2}} \frac{1}{h} M^{mn} \Phi_m^1 \Phi_n^1 \sigma^B_{\alpha \beta} - \frac{i}{2\sqrt{2}} \frac{1}{h} M^{mn} \Phi_m^1 \Phi_n^1 \sigma^B_{\alpha \beta} \lambda_{ij}.
\]

(A.9) shows that spinor fields with constant components in the left-invariant frame field are eigenspinors of the Riemannian Dirac operator precisely when \( A_I = 0 \), i.e. when the group \( G \) belongs to Bianchi class \( A \), in which case the eigenvalue is \( \beta = -M^0 h_{ij} / (4 \sqrt{|h|}) \). The action of the Riemannian 3-surface twistor operator on such a spinor field is

\[
T(\lambda)_{ABC} = -\frac{i}{2\sqrt{2} \sqrt{|h|}} \left( \hat{M}^i_{ij} \Phi_i^1 \Phi_j^1 \sigma^B_{\alpha \beta} + \frac{1}{3} M^0 h_{ij} \sigma^B_{\alpha \beta} \right) \sigma^A_{\alpha \beta} \sigma^C_{\alpha \beta} \lambda_{ij}.
\]

where \( \hat{M}^i_{ij} := M^ij - \frac{1}{2} M^{kl} \eta_{kl} \eta_{ij} \) and \( \hat{M}^i_{ij} := h_{ij} - \frac{1}{2} h^{kl} \eta_{kl} \eta_{ij} \), the trace-free part of \( M^0 \) and \( h_{ij} \), respectively, with respect to \( \eta_{ij} \). This is vanishing if \( M^0 = 0 \) (Bianchi I models), or if \( \hat{M}^0 = 0 \) and the evolution of the spatial metric yields that \( h_{ij} = S^2 \eta_{ij} \) for some positive function \( S = S(t) \) (closed FRW spacetimes in Bianchi IX).

The action of the square of the Riemannian Dirac operator on the spinor field whose components \( \lambda^A \) in the left-invariant frame are constant

\[
\epsilon^A_{\alpha \beta} D_{AB} \lambda^B = \left( \left( q_I q_I \right)^{ij} + \frac{1}{2} q^2 \sigma^A_{ij} \right) \lambda^B - \frac{1}{2} \sqrt{2} \eta_{ij} \sigma^A_{ij} \lambda^B,
\]

where, for the sake of brevity, we used the quantities \( q_I \) and \( q_I \) introduced in connection with Nester’s gauge condition above. Therefore, such a spinor field can be an eigenspinor of \( 2 D_{AB} D_{BC} \) with the eigenvalue \( \beta^2 \) precisely when \( q_I q_I = 0 \), and the eigenvalue is \( \beta^2 = \frac{1}{2} q_I q_I + \frac{1}{16} q^2 \). In particular, if \( A_I = 0, M^0 = \frac{1}{3} M \eta_{ij} \) and \( h_{ij} = S^2 \eta_{ij} \) (e.g. for the
closed FRW spacetimes), then by (A.6) this yields that \( \beta^2 = \frac{1}{8} \mathcal{R} \), which is known to be just the smallest eigenvalue of \( 2D^{AB}D_{BC} \) on the metric 3-spheres.

If the symmetry group \( G \) belongs to Bianchi class B, then no eigenspinor of the Riemannian Dirac operator has constant components in the left-invariant frame field. Nevertheless, (A.9) motivates us to consider the weaker condition

\[
E^i_i (\partial_i \lambda^A) + \frac{1}{2} q_i \lambda^A = 0. \tag{A.13}
\]

It is a straightforward calculation to show that its integrability conditions are satisfied identically, and hence, this equation locally admits precisely two linearly independent solutions, which are specified completely by their own value at an arbitrary point of \( \Sigma_t \).

Let these two solutions be \( \chi^A_1 \) and \( \chi^A_2 \), and form their symplectic scalar product \( \omega := \lambda^A \mu^B \varepsilon_{AB} \).

Then (A.13) yields that \( q_i \theta^i_e \equiv i \partial_i \omega \), i.e., if (A.13) admits global solutions, then the combination \( q_i \theta^i_e \) of the left-invariant orthonormal 1-form basis would be exact. Then this global solution would be an eigenspinor of the Riemannian Dirac operator also with the eigenvalue \( \beta = -M^B j_B/(4 \sqrt{|T|}) \).

The unitary spinor form of the derivative operator of the Sen connection is well known to be \( D_{AB} \chi^C = D_{AB} \chi^C - \frac{1}{\sqrt{2}} \chi_{ABCDEF} \), where \( \chi_{ABCDEF} := 2 \epsilon^A \epsilon^B \chi_{CDE} \). Hence, if the components \( \chi_{ab} \) of the extrinsic curvature are defined by \( \chi_{ab} =: \chi \chi_{ab} \), which are also constant on \( \Sigma_t \), but in general depend on \( t \), then the Sen–Witten and the 3-surface twistor operators coincide. Since in general \( 2D^{AB} D_{AB} \chi^A = 2D^{AB} D_{AB} \chi^A + \chi^A \), in spatially homogeneous spacetimes, where \( \chi \) is spatially constant, we have for any eigenspinor \( \lambda^A \) of the square of the Riemannian Dirac operator with eigenvalue \( \beta^2 \) that

\[
2D^{AB} D_{AB} \chi^A = 2D^{AB} D_{AB} \chi^A + \frac{1}{2} \chi^2 \lambda^A = \left( \beta^2 + \frac{1}{2} \chi^2 \right) \lambda^A. \tag{A.16}
\]

Thus, \( \lambda^A \) will be an eigenspinor of the square of the Sen–Witten operator, too, with the eigenvalue \( \beta^2 + \frac{1}{2} \chi^2 \). In particular, in the closed FRW spacetimes the first eigenvalue of \( 2D^{AB} D_{AB} \) is \( \frac{1}{2} \mathcal{R} + \frac{1}{2} \chi^2 \), and the corresponding eigenspinors are just the ones whose components in the left-invariant frame field are constant.

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