ADJONIT-BASED SHAPE OPTIMIZATION FOR THE MITIGATION OF COASTAL EROSION VIA SMOOTHED PARTICLE HYDRODYNAMICS

Abstract

Adjoint-based shape optimization most often relies on Eulerian flow field formulations. However, since Lagrangian particle methods are the natural choice for solving sedimentation problems in oceanography, extensions to the Lagrangian framework are desirable. For the mitigation of coastal erosion, we perform shape optimization for fluid flows, that are described by Lagrangian shallow water equations and discretized via smoothed particle hydrodynamics. The obstacle’s shape is hereby optimized over an appropriate cost function to minimize the height of water waves along the shoreline based on shape calculus. Theoretical results will be numerically verified exploring different scenarios.

Keywords Shape Optimization · SPH · Coastal Erosion

1 Introduction

Coastal erosion describes the displacement of land caused by destructive sea waves, currents or tides. Major efforts have been made to mitigate these effects using groins, breakwaters and various other structures. Numerical investigations of useful shapes require accurate descriptions of propagating waves and sediment. Lagrangian particle movements appear to be the natural choice for solving sedimentation problems in oceanography, being generally free of diffusion. In contrast, inherent numerical diffusion in Eulerian methods can only affect the resolution of purely advective flows. It is therefore tempting to investigate shape optimization techniques for Lagrangian fluid flows. Typically, associated meshfree particle methods can be divided in two groups, the ones that approximate the weak form, e.g. the diffusive element [1], the element-free Galerkin [2] or the reproducing kernel method [3], such as the ones that approximate the strong form, e.g. the moving particle [4] or the vortex method [5]. In this chapter we investigate a particle flow that falls into the latter class - the smoothed particle hydrodynamics (SPH) [6]. Both groups share the property that shape optimization has been only rarely investigated. In general most approaches are based on the weak form, introduced in order to enable large deformations, e.g. for the element-free Galerkin method [7] or the reproducing kernel particle method [8]. Lately a one-way coupled, volume averaged transport model was used to circumvent the direct usage of the Lagrangian particle flow [10]. In particular for SPH fluids only boundary contributions via ghost particles and the direct differentiation method to obtain optimal fluid-structure interactions have been investigated [11]. However, this method comes with the drawback of solving an additional problem for each design parameter, which can become costly for highly resolved obstacles. Hence, adjoint-based shape optimization for SPH fluids and accordingly also the usage in mitigation of coastal erosion appears novel.

The paper is structured as follows: We derive adjoints for a general class of particle systems in Section 2 before Section 3 will build the foundation for discretizing shallow water equations (SWE) via SPH-fluids, restructuring rigid and outflow boundaries via signed-distance fields [12] for the use in optimization. Subsequently, Section 4 will derive shape derivatives based on the adjoints that have been developed for the general case. The technique boils down to derive shape derivatives for a finite element interpolator building up on discrete investigations [13] [14]. The results are numerically verified in simplified scenarios using a gradient-descent algorithm. In this setting, partial derivatives
The sensitivity of the objective function with regards to the control

Assume a particle system is solved iteratively using iteration \((1)\), then sequences

*Theorem 1.* The idea is related to \([16]\) for adjoint calculations of iteration equations. We first sum up the products of iterates and unknown multipliers together with the objective \((2)\) required for recursive adjoints are calculated via automatic differentiation and a deformation gradient is obtained from linear elasticity evaluations \([15]\).

2 Adjoint for Particle Systems

In this section adjoints for a system of \(d\)-dimensional particles consisting of position and velocities \(X_k = (x_k, u_k) \in \mathbb{R}^{dN} \times \mathbb{R}^{dN}\) for time steps \(k \in \{1, \ldots, n\}\) with constant particle mass \(m > 0\) are derived. The iteration laws are described by a symplectic Euler, i.e. for force functions \(F: \mathbb{R}^{dN} \times \mathbb{R}^{dN} \times \Psi \rightarrow \mathbb{R}^{dN}\) and to be determined control \(q \in \Psi\). We solve

\[
\begin{align*}
    u_k &= u_{k-1} + \frac{\Delta t}{m} F(u_{k-1}, x_{k-1}, q) \\
    x_k &= x_{k-1} + \Delta t u_k.
\end{align*}
\]  

\[(1)\]

The solution of the particle system constrains a time-dependent objective function \(J: \mathbb{R}^{ndN} \times \mathbb{R}^{ndN} \times \Psi \rightarrow \mathbb{R}\), i.e.

\[
J(u, x, q) = \sum_{k=1}^{n} J_k(u_k, x_k, q).
\]  

\[(2)\]

In this setting, we can derive the following theorem, consisting of recursively defined adjoints and an equation for the sensitivity calculation with respect to control.

*Theorem 1.* (Adjoint Particle System) Assume a particle system is solved iteratively using iteration \((1)\), then sequences \(\{\delta_k\}_{k=1}^{n}\) and \(\{\mu_k\}_{k=1}^{n}\) required in adjoint computations are obtained from backward recursion, i.e.

\[
\begin{align*}
    \delta_k &= \delta_{k+1} + \frac{\Delta t}{m} (F^{x_k})^T \mu_{k+1} + \frac{\Delta t^2}{m} (F^{x_k})^T \delta_{k+1} + (J^x_k)^T \\
    \mu_k &= \mu_{k+1} + \frac{\Delta t}{m} (F^{u_k})^T \mu_{k+1} + \frac{\Delta t}{m} \delta_{k+1} + \frac{\Delta t^2}{m} (F^{u_k})^T \delta_{k+1} + (J^u_k)^T,
\end{align*}
\]  

\[(3)\]

\[(4)\]

where the recursion starts at

\[
\mu_n = J^u_n \quad \delta_n = J^x_n.
\]  

\[(5)\]

The sensitivity of the objective function with regards to the control \(q\) are calculated as

\[
J^q = \sum_{k=1}^{n} J_k^q + \frac{\Delta t}{m} (F^q_{k-1})^T \mu_k + \frac{\Delta t^2}{m} (F^q_{k-1})^T \delta_k.
\]  

\[(6)\]

*Proof.* For simplicity we will rewrite the symplectic Euler by substitution of \(u_k\) in \((1)\)

\[
\begin{align*}
    u_k &= u_{k-1} + \frac{\Delta t}{m} F(u_{k-1}, x_{k-1}, q) \\
    x_k &= x_{k-1} + \Delta t \left( u_{k-1} + \frac{\Delta t}{m} F(u_{k-1}, x_{k-1}, q) \right).
\end{align*}
\]  

\[(7)\]

The idea is related to \([16]\) for adjoint calculations of iteration equations. We first sum up the products of iterates and unknown multipliers together with the objective \((2)\)

\[
0 = \sum_{k=1}^{n} \left[ u_k - u_{k-1} - \frac{\Delta t}{m} F(u_{k-1}, x_{k-1}, q) \right]^T \mu_k \\
+ \sum_{k=1}^{n} \left[ x_k - x_{k-1} + \Delta t \left( u_{k-1} - \frac{\Delta t}{m} F(u_{k-1}, x_{k-1}, q) \right) \right]^T \delta_k \\
+ J - \sum_{k=1}^{n} J_k(u_k, x_k, q).
\]  

\[(8)\]
In order to control a particle system, we only require the ability to calculate the matrix with partial derivatives.

Remark. In this paper we will be concerned with a total of \(N\) particles in two dimensions such that \(x_k, u_k \in \mathbb{R}^2\). The recursion (9) can be written as

\[
0 = \sum_{k=1}^{n-1} \left[ \mu_k - \mu_{k+1} - \frac{\Delta t}{m} (F^u_{k+1})^T \mu_{k+1} - \frac{\Delta t}{m} \delta_{k+1} - \frac{\Delta t^2}{m} (F^u_{k+1})^T \delta_{k+1} - J^u_k \right]^T u_k^q
\]

\[
+ \sum_{k=1}^{n-1} \left[ - \frac{\Delta t}{m} (F^x_{k+1})^T \mu_{k+1} + \delta_k - \delta_{k+1} - \frac{\Delta t^2}{m} (F^x_{k+1})^T \delta_{k+1} - J^x_k \right]^T x_k^q
\]

\[
+ \{ \mu_n - J^u_n \}^T u_n^q + \{ \delta_n - J^x_n \}^T x_n^q
\]

\[
+ J^q - \sum_{k=1}^{n} J_k^q - \frac{\Delta t}{m} (F^q_{k-1})^T \mu_k - \frac{\Delta t^2}{m} (F^q_{k-1})^T \delta_k.
\]

The recursion is finally obtained from the coefficients before the unknown partial derivatives, the final conditions follow from the third line. The sensitivity of the objective function with regards to the control follows from the last line.

Remark. In this paper we will be concerned with a total of \(N\) particles in two dimensions such that \(x_k, u_k \in \mathbb{R}^2\). The recursion (9) can be written as

\[
\mu_k = \left( I_{2N} + \frac{\Delta t}{m} F^u_{k+1} \right)^T (\mu_{k+1} + \Delta t \delta_{k+1}) + (J^u_k)^T
\]

\[
\delta_k = \delta_{k+1} + \left( \frac{\Delta t}{m} F^x_{k+1} \right)^T (\mu_{k+1} + \Delta t \delta_{k+1}) + (J^x_k)^T.
\]

In this form obtained adjoints equal the ones that have been derived earlier by different means [17][18][14].

Remark. In order to control a particle system, we only require the ability to calculate the matrix with partial derivatives of the forces. Further sensitivity calculations need to specify the particle system such as control. We will restrict to the aforementioned SPH fluids in the following section, where the shape of an obstacle is optimized. The sensitivity is obtained in Section 4.3 by evaluating the discrete shape derivative \(DJ(\Omega)[V]_l\) for domain \(\Omega\) in direction \(V_l\) via formula (6).

3 Smoothed Particle Hydrodynamics

Before being concerned with optimization of the SPH flows, Section 3.1 will discuss the basic idea, definitions and notations for this technique, while Section 3.2 is dealing with boundary interactions of SPH particles, that are crucial in shape sensitivity calculations.

3.1 Basics of SPH

Central for SPH-particles travelling on a domain \(\Omega \subset \mathbb{R}^d\) is the approximation of the delta distribution by the usage of kernels. Hence, a field value \(v : \Omega \to \mathbb{R}\) is written using the symbolical Dirac-delta identity and the Dirac-delta condition [6]

\[
v(x) = \int v(x') \delta(x - x') \, dx' = \lim_{h \to 0} \int v(x') W(x - x', h) \, dx',
\]

where \(W : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}\) is defined as a valid kernel if fulfilling properties of normalization, Dirac-delta limiting such as often positivity, symmetry and compactness of the support [6]. In turn this is intended to serve as limiting object in
the sense of
\[
\lim_{h \to 0} \int v(x') W(x - x', h) \, dx' = \lim_{h \to 0} \lim_{N \to \infty} \sum_{i=1}^{N} \frac{m_i v_i \rho_i}{\rho} W(x - x^i, h)
\]  \quad (13)
for constant particle density \( \rho^i \) and particle mass \( m_i \). Ultimately, the latter is approximated for fixed \( h > 0 \) and \( N \in \mathbb{N} \), forming the basis of SPH-techniques. In the literature various valid kernel functions are defined, we restrict ourselves here to the original one, hence define the Gaussian kernel as \[ W(x - x^i, h) = \sigma g \exp \left( -\frac{||x - x^i||^2}{h^2} \right) \]  \quad (14)
e.g. with two-dimensional normalization as
\[
\sigma g = \frac{1}{\pi h^2}.
\]  \quad (15)

Remark. The Gaussian kernel comes with the advantage of belonging to class \( C^\infty \), we will use this fact in Section 4. However, it lacks compact support, which results in a summation over all particles in the domain. To counter this, often cut-off kernels as the Cubic Spline \[6\] or Poly6 \[20\] kernel are used. In classical form these kernels are not differentiable at the cut-off, but are attractive from computational side, as the summation is only performed over particles in the limited support radius.

Remark. A limited number of publications have been dealing with the convergence of SPH methods, based on joint particle limits \( N \to \infty \) and smoothing limit \( h \to 0 \) as in (13). First attempts relied on spatial discretizations with time-continuous approximations \[21\] or at least the knowledge about exact particle trajectories \[22\]. In addition, convergence results have been presented for consecutive limits of discretization parameter and smoothing radius \[23\] or a selective choice of kernel functions \[24\].

3.2 Particle Boundary Interactions

Since we are ultimately interested in shape optimization of particle systems, we need to specify the particle boundary interaction. In classical SPH methods, this is typically done via boundary particles \[6\]. The obvious drawback are increased computational efforts, undesired boundary frictions such as the inability to model complex geometries \[12\]. All SPH boundary techniques have the common idea to approximate the second integral below, which arises naturally when being restricted to a bounded domain \( \Omega \subset D \subset \mathbb{R}^d \nabla \)
\[
\rho(x) = \lim_{h \to 0} \int_{\Omega} W(x - x', h) \rho(x') \, dx' + \lim_{h \to 0} \int_{D \setminus \Omega} W(x - x', h) \rho(x') \, dx'.
\]  \quad (16)
In the last decade various researchers addressed this problem e.g. by the usage of continuous boundary methods such as the boundary surface integral \[25\] or the signed distance field method \[12\]. This work will restrict to latter ideas and extends them for Lagrangian SWE with rigid and outflow boundary conditions. In \[12\] the boundary density portion \( \rho_D \) is approximated as
\[
\rho_D(x) = \lim_{h \to 0} \int_{D \setminus \Omega} \gamma(d_\Omega(x')) W(x - x', h) \, dx',
\]  \quad (17)
where a modification of the signed distance function
\[
d_\Omega(x) = \begin{cases} 
  d(x, \partial \Omega) & \text{if } x \in \Omega \\
  0 & \text{if } x \in \partial \Omega \\
  -d(x, \partial \Omega) & \text{if } x \in D \setminus \Omega
\end{cases}
\]  \quad (18)
is used with
\[
\gamma(d_\Omega) = \begin{cases} 
  \rho_0 (1 - \frac{d_\Omega}{h}) & \text{if } d_\Omega \leq h \\
  0 & \text{otherwise}
\end{cases}
\]  \quad (19)
for reference density \( \rho_0 \). In addition to rigid boundaries, we implement open-sea boundaries. Due to this reason, we extend the idea of buffered layers for the modelling of outflow conditions, as introduced in \[26\] for SPH-based computations, to mesh-based signed distance maps. The idea is to create a collecting channel, in which the particle
movement is decelerated. Suppose an additional layer as subdomain $\Omega_L \subset \mathbb{R}^d$ is introduced, that builds with $\Omega$ a conformal domain. Then as in (18), we require

$$d_{\Omega_L}(x) = \begin{cases} d(x, \partial \Omega_L) & \text{if } x \in (\Omega \cup D) \\ 0 & \text{if } x \in \partial \Omega \\ -d(x, \partial \Omega_L) & \text{if } x \in \Omega_L \setminus (\Omega \cup D). \end{cases} \quad (20)$$

Hence, outflow boundary conditions are created via subdomain-dependent modification of this very signed distance function, i.e.

$$\gamma^L(d_{\Omega_L}) = \begin{cases} \rho_0 d_{\Omega_L} & \text{if } x \in \Omega_L \\ 0 & \text{otherwise}. \end{cases} \quad (21)$$

In addition, we add a decelerating domain-dependent coefficient for the velocity in the particles iteration law of (1), i.e. for $\epsilon > 0$

$$\phi := \begin{cases} \phi_1 = 1 \text{ in } \Omega \cup D \\ \phi_2 = \epsilon \text{ in } \Omega_L. \end{cases} \quad (22)$$

The implementation in the discretized setting is done via interpolation for each particle and its position $x \in \Omega$ since this is not differentiable for $x \in \Omega_L \setminus (\Omega \cup D)$.

allowing cheap finite element interpolations, whenever a particle is in the proximity of the boundary.

For outflow boundaries we rely on an analogous $C^1$-counterpart of the min-function as

$$\min_\alpha(d_{\Omega_L}) = \begin{cases} \min(0, d_{\Omega_L}) & \text{for } d_{\Omega_L} \in \mathbb{R} \setminus [-\frac{1}{\alpha}, \frac{1}{\alpha}] \\ \frac{\alpha}{2} d_{\Omega_L}^2 + \frac{1}{2} d_{\Omega_L} - \frac{1}{4\alpha} & \text{otherwise} \end{cases} \quad (27)$$

with

$$\min'_\alpha(d_{\Omega_L}) = \begin{cases} 1 & \text{for } d_{\Omega_L} \in (-\infty, -\frac{1}{\alpha}) \\ -\frac{\alpha}{2} d_{\Omega_L} - \frac{1}{2} & \text{for } d_{\Omega_L} \in [-\frac{1}{\alpha}, \frac{1}{\alpha}] \\ 0 & \text{for } d_{\Omega_L} \in \left(\frac{1}{\alpha}, \infty\right). \end{cases} \quad (28)$$

The interpolation technique is then also used to compute the boundary forces by

$$F^H_{\text{height}}(x) = \nabla \rho_D(x) = \sum_{s=1}^S \nabla \gamma_s(x) \nabla N_s^p(x) + \sum_{s=1}^S \gamma^L_s(x) \nabla N_s^p(x). \quad (24)$$

Remark. The advantage in using this boundary representation lies in the possibility to precompute a solution field, allowing cheap finite element interpolations, whenever a particle is in the proximity of the boundary. Later on the sediment field $z : \Omega \to \mathbb{R}$ of (30) can be created in the same manner as the boundary density. In this setting, we can naturally identify domain boundaries via increased sediment elevations.

Remark. Rigid boundary contributions, as in (17), rely on the max-function, i.e.

$$\gamma(d_{\Omega_L}(x)) = \rho_0 \left( \max \left\{ 0, 1 - \frac{d_{\Omega_L}(x)}{h} \right\} \right),$$

since this is not differentiable for $x \in \Omega$ such that $d_{\Omega_L}(x) = h$, we will use a smoothed max-function $C^1(\Omega) \ni \max_\alpha : \Omega \to \mathbb{R}$ in the following section e.g. as [27]

$$\max_\alpha(d_{\Omega_L}) = \begin{cases} \max(0, 1 - \frac{d_{\Omega_L}}{h}) & \text{for } 1 - \frac{d_{\Omega_L}}{h} \in \mathbb{R} \setminus [-\frac{1}{\alpha}, \frac{1}{\alpha}] \\ \frac{\alpha}{4} (1 - \frac{d_{\Omega_L}}{h})^2 + \frac{1}{2} (1 - \frac{d_{\Omega_L}}{h}) + \frac{1}{4\alpha} & \text{otherwise} \end{cases} \quad (25)$$

for $\alpha > 0$ with derivative as

$$\max'_\alpha(d_{\Omega_L}) = \begin{cases} 0 & \text{for } 1 - \frac{d_{\Omega_L}}{h} \in (-\infty, -\frac{1}{\alpha}) \\ -\frac{\alpha}{2h} (1 - \frac{d_{\Omega_L}}{h}) - \frac{1}{2h} & \text{for } 1 - \frac{d_{\Omega_L}}{h} \in [-\frac{1}{\alpha}, \frac{1}{\alpha}] \\ -\frac{1}{h} & \text{for } 1 - \frac{d_{\Omega_L}}{h} \in \left(\frac{1}{\alpha}, \infty\right). \end{cases} \quad (26)$$

For outflow boundaries we rely on an analogous $C^1$-counterpart of the min-function as

$$\min_\alpha(d_{\Omega_L}) = \begin{cases} \min(0, d_{\Omega_L}) & \text{for } d_{\Omega_L} \in \mathbb{R} \setminus [-\frac{1}{\alpha}, \frac{1}{\alpha}] \\ \frac{\alpha}{2} d_{\Omega_L}^2 + \frac{1}{2} d_{\Omega_L} - \frac{1}{4\alpha} & \text{otherwise} \end{cases} \quad (27)$$

with

$$\min'_\alpha(d_{\Omega_L}) = \begin{cases} 1 & \text{for } d_{\Omega_L} \in (-\infty, -\frac{1}{\alpha}) \\ -\frac{\alpha}{2} d_{\Omega_L} + \frac{1}{2} & \text{for } d_{\Omega_L} \in [-\frac{1}{\alpha}, \frac{1}{\alpha}] \\ 0 & \text{for } d_{\Omega_L} \in \left(\frac{1}{\alpha}, \infty\right). \end{cases} \quad (28)
4 Adjoint-Based Shape Optimization for SPH Particles

The following section is devoted to the derivation of the shape derivative of an SPH-fluid with suitable boundary interaction for a model described in Section 4.1. For this, necessary definitions and notations are recalled in Section 4.2 and applied in Section 4.3. A numerical verification of results for two test cases follows in Section 4.4.

4.1 Model Formulation

Suppose we are given an open domain \( \Omega \subset \mathbb{R}^2 \), which is split into disjoint subdomains, consisting of a connected, interior domain \( \Omega_1 \) such that \( \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 := \Gamma := \partial \Omega_1 \), a simply connected obstacle domain \( D \) and an exterior domain \( \Omega_2 \cup \Omega_3 := \Omega \setminus \bar{\Omega}_1 \setminus D \), such that \( \bar{\Omega}_1 \cup \Omega_2 \cup \Omega_3 \cup D = \Omega \). We assume the variable, interior boundary \( \Gamma_3 \) and the fixed outer \( \Gamma_1 \cup \Gamma_2 \) of interior domain \( \Omega_1 \) to be at least Lipschitz. One simple example of such kind is visualized below in Figure 1.

![Illustrative Domain](image)

Figure 1: Illustrative Domain \( \Omega \) with Initial Circled Obstacle \( D \), Interior Domain \( \Omega_1 \), Exterior Domains \( \Omega_2, \Omega_3 \) and Boundaries \( \Gamma_1, \Gamma_2, \Gamma_3 \)

On this domain fluids will follow the laws of the Lagrangian viscous SWE, i.e. on \( \Omega \times (0, T) \) we have

\[
DH = -H \nabla \cdot \bar{\mathcal{Q}} \quad (29)
\]

\[
D\bar{\mathcal{Q}} = -g \nabla (H + z) + \mu \nabla^2 \bar{\mathcal{Q}}. \quad (30)
\]

subject to rigid boundary conditions on \( \Gamma_1, \Gamma_3 \times (0, T) \), open boundary conditions on \( \Gamma_2 \times (0, T) \) and suitable initial conditions on \( \Omega \times \{0\} \) for solution \( U = (H, \mathcal{Q}) = (H, H_u, H_v) \), with \( H \) being the water height and \( H_u, H_v \) the weighted horizontal and vertical discharge or velocity. The sediment is a scalar field \( z : \Omega \to \mathbb{R} \), \( g \) the gravitational acceleration and \( \mu > 0 \) the viscosity weight.

Our objective \( J : \Omega \to \mathbb{R} \) consists of three parts

\[
J(\Omega) = J_1(\Omega) + J_2(\Omega) + J_3(\Omega). \quad (31)
\]

We interpret \( \Gamma_1 \) as coastline and implement a tracking-type objective [28] for the rest height of the water, i.e.

\[
J_1(\Omega) = \int_0^T \int_{\Gamma_1} \frac{1}{2} (H - \bar{H})^2 \, ds \, dt. \quad (32)
\]

The objective is accompanied by a volume penalty as a second part

\[
J_2(\Omega) = \nu_1 \int_D 1 \, dx \quad (33)
\]

and a perimeter regularization

\[
J_3(\Omega) = \nu_2 \int_{\Gamma_3} 1 \, ds. \quad (34)
\]

The contribution of the penalty terms (33) and (34) is controlled by parameters \( \nu_1, \nu_2 > 0 \), which need to be defined a priori. Finally, with constraints of type (29)-(30) we obtain a PDE-constrained optimization problem, which is intended to be solved in a discretize-then-differentiate setting. The discretization is based on SPH as described in Section 3.1 with boundary interaction as in Section 3.2. From this we obtain a particle system, which provides us in two dimensions
with states via tuple \((x_k, v_k) \in \mathbb{R}^{2N} \times \mathbb{R}^{2N}\). Here we interpret \(\Omega\) as fluid and \(\Omega_2 \cup \Omega_3 \cup D\) as boundary domain for the particles in reference to Section 3.2. Based on this we define the discrete counterpart to objective (32), i.e.

\[
J_{1, h}(\Omega) = \sum_{k=1}^{n} J_{1, h, k}(x_k, v_k, \Omega)
\]

\[
= \sum_{(k,x) \in \{1, \ldots, n\} \times \Gamma_{1, h}} \frac{1}{2} \Delta t \left(\frac{\rho(x)}{\rho_0} - \bar{H}(x)\right)^2
\]

(35)

where \(x^j_k\) defines the position of particle \(j\) at time \(k\), that is obtained as solution to an SPH-discretization of the Lagrangian SWE. As it can be seen in (13) SPH methods allow for simplified density calculations, to utilize this benefit a density-water relation is used [29], i.e.

\[
H = \frac{\rho}{\rho_0}
\]

(36)

for \(\rho_0 > 0\). This definition gives only local conservation results, such that [29] and [30] hold only in the limit case of \(h \to 0\), which is shown in [29] for inviscid Lagrangian SWE. Furthermore, for constant particle number and mass the continuity equation (29) is automatically fulfilled by reinterpretation of the density as height of the waves [29]. In this setting (30) reduces to regular SPH calculations for incompressible Navier-Stokes equations [19] with pressure and viscosity corresponding forcing terms such as an additional term that arises from variations in the sediment.

Remark. By relying on (36) we circumvent the additional carry of information by the individual particles and instead rely on density-based calculation in (35).

In (35) and in what follows we are relying on a constant particle mass, in line with Section 2. Furthermore, we omit discretization indices for readability whenever it is clear from the context.

### 4.2 Basics of Shape Optimization

In this section we introduce methodologies commonly used in shape optimization, extensively elaborated in monographs [30] [31] [32]. We hereby mainly follow [13]. We start by introducing a family of mappings \(\{\phi_\epsilon\}_{\epsilon \in [0, \tau]}\) for \(\tau > 0\) that are used to map each current position \(x \in \Omega\) to another by \(\phi_\epsilon(x)\), where we choose the sufficiently smooth vector field \(\bar{V}\) as the direction for the so-called perturbation of identity

\[
x_\epsilon = \phi_\epsilon(x) = x + \epsilon \bar{V}(x).
\]

(37)

According to this methodology, we can map the whole domain \(\Omega\) to another \(\Omega_\epsilon\) such that

\[
\Omega_\epsilon = \{x_\epsilon | x + \epsilon \bar{V}(x), x \in \Omega\}.
\]

(38)

We define the Eulerian Derivative as

\[
DJ(\Omega)[\bar{V}] = \lim_{\epsilon \to 0^+} \frac{J(\Omega_\epsilon) - J(\Omega)}{\epsilon}.
\]

(39)

Commonly, this expression is called shape derivative of \(J\) at \(\Omega\) in direction \(\bar{V}\) and in this sense \(J\) shape differentiable at \(\Omega\) if for all directions \(\bar{V}\) the Eulerian derivative exists and the mapping \(\bar{V} \mapsto DJ(\Omega)[\bar{V}]\) is linear and continuous. In addition, we define the material derivative of some scalar function \(p : \Omega \to \mathbb{R}\) at \(x \in \Omega\) by the derivative of a composed function \(p_\epsilon \circ \phi_\epsilon : \Omega \to \mathbb{R}_{+} \to \mathbb{R}\) for \(p_\epsilon : \Omega_\epsilon \to \mathbb{R}\) as

\[
D_m p(x) := \lim_{\epsilon \to 0^+} p_\epsilon \circ \phi_\epsilon(x) \frac{p(x)}{\epsilon} = \frac{d}{d\epsilon}(p_\epsilon \circ \phi_\epsilon)(x)|_{\epsilon=0^+}
\]

(40)

and the corresponding shape derivative for a scalar \(p\) as

\[
Dp[\bar{V}] := D_m p - \bar{V} \cdot \nabla p.
\]

(41)

In Section 4.3 we will need the product rule, i.e. [13] \[m\]

\[
D_m(pq) = D_m pq + D_m q
\]

(42)
and the fact that material derivatives do not commute with spatial derivatives \[13\]

\[ D_m \nabla p = \nabla D_m p - \nabla (\vec{V}^T \nabla p). \]  

As mentioned in Section 4.1, we are relying on a discretize-then-differentiate approach. Hence, the perturbation of identity w.r.t. a single vertex perturbation \( \delta x_l \) for vertices \( l \in \{1, ..., L\} \) is defined as \[13\]

\[ x_\epsilon = \phi_\epsilon(x) = x + \epsilon \delta x_l N_1^l(x) \]

where \( N_1^l \) is a finite-element basis function of first order, i.e. a continuous piecewise-linear polynomial.

**Remark.** The ultimate position \( x_\epsilon \in \Omega_\epsilon \) can be calculated using summed vertex contributions, i.e.

\[ x_\epsilon = x + \epsilon \sum_{l=1}^{L} \delta x_l N_1^l(x). \]

The discretized perturbation of identity interpolates the deformation \( \epsilon \delta x_l \) of vertex \( l \in \{1, ..., L\} \) on the support of \( N_1^l \).

### 4.3 Adjoint-Based Shape Optimization for SPH Particles

The sensitivity with respect to domain deformations is obtained from Theorem [11] as

\[ DJ_1(\Omega)[\vec{V}_l] = \left. \frac{dJ_{1,h}(\Omega_\epsilon)}{d\epsilon} \right|_{\epsilon=0^+} = \left. \frac{d}{d\epsilon} \sum_{k=1}^{n} J_{1,h,k}(x_k, v_k, \Omega_\epsilon) \right|_{\epsilon=0^+} + \left. \frac{d}{d\epsilon} \sum_{k=1}^{n} (F_{k-1})^T \left( \frac{\Delta t}{m} \mu_k + \frac{\Delta \mu^2}{m} \delta_k \right) \right|_{\epsilon=0^+}, \]

where occurring functions are defined on the perturbed domain \( \Omega_\epsilon \subset \mathbb{R}^d \). In need of deriving the shape derivative, we first state the following lemma, following [13].

**Lemma 2.** For a finite element function \( g \), which is an element of a conformal approximation space of \( H^1(\Omega) \), i.e. \( g(x) = \sum_{s=1}^{S} g_s N_1^s(x) \) for finite element ansatz function \( N_1^s \), whose restriction on an Lagrangian element \( \kappa \) is a polynomial of order \( p \geq 1 \), the shape derivative is derived as

\[ Dg[\vec{V}_l] = \sum_{s=1}^{S} (Dg_s[\vec{V}_l] N_1^s - g_s \vec{V}_l \cdot \nabla N_1^s), \]

where \( N_1^s(x_\epsilon) = N_1^s(\tau_{\epsilon^{-1}}(x_\epsilon)) \) is moving along the deformation.

**Proof.** [13] Since \( N_1^s(x) \) is moving along the deformation, the material derivative vanishes

\[ D_m N_1^s = 0, \]

hence for the shape derivative it holds by (41)

\[ DN_1^s[\vec{V}_l] = -\vec{V}_l \cdot \nabla N_1^s. \]

Since \( g_s \) is spatially constant, we conclude according to [13]

\[ D_m g = \sum_{s=1}^{S} Dg_s[\vec{V}_l] N_1^s \]

and therefore obtain the shape derivative as

\[ Dg[\vec{V}_l] = D_m g - \vec{V}_l \nabla g = \sum_{s=1}^{S} \left( Dg_s[\vec{V}_l] N_1^s - g_s \vec{V}_l \cdot \nabla N_1^s \right) \]
Remark. For SPH boundary computations we recall the shape derivative of the signed distance function for \( x \notin \Sigma \)

\[
Dd_{\Omega}(x)[\vec{v}] = -\vec{v}(p_{\partial \Omega}(x)) \cdot \vec{n}(p_{\partial \Omega}(x)).
\]

(48)

with operator \( p_{\partial \Omega} \) that projects a point \( x \in \Omega \) onto its closest boundary, where \( \Sigma \) is referred to as the ridge. In this sense, \( Dd_{\Omega,s}[\vec{v}] \) is its nodal discretized counterpart w.r.t. the \( l^{th} \) vertex perturbation.

Since the boundary contribution is driven by surface gradient forces, we require the shape derivative of the gradient of a finite element function [24].

Lemma 3. For the gradient of a finite element function, i.e. \( \nabla g(x) = \sum_{s=1}^{S} g_s \nabla N_s^p(x) \) for finite element ansatz function \( N_s^p \), whose restriction on an Lagrangian element \( \kappa \) is a polynomial of order \( p \geq 2 \), the shape derivative is derived as

\[
D(\nabla g)[\vec{v}_l] = \sum_{s=1}^{S} Dg_s[\vec{v}_l] \nabla N_s^p - \left( \nabla \vec{v}_l \right)^T \left( \sum_{s=1}^{S} g_s \nabla N_s^p \right) \nabla g + \nabla \left( \sum_{s=1}^{S} g_s \nabla N_s^p \right) \vec{v}_l.
\]

(49)

Proof. The material derivative does not commute with the spatial derivative [43]

\[
D_m(\nabla g) = \nabla(D_mg) - \left( \nabla \vec{v}_l \right)^T \nabla g.
\]

(50)

which equals by the same argument as in proof to Lemma [2]

\[
D_m(\nabla g) = \nabla \left( \sum_{s=1}^{S} g_s \nabla [\vec{v}_l] N_s^p \right) - \left( \nabla \vec{v}_l \right)^T \left( \sum_{s=1}^{S} g_s \nabla N_s^p \right) \nabla g.
\]

(51)

Then we have for the shape derivative

\[
D(\nabla g)[\vec{v}_l] = D_m(\nabla g) - \nabla(\nabla g)\vec{v}_l
\]

\[=
\nabla \left( \sum_{s=1}^{S} Dg_s[\vec{v}_l] N_s^p \right) - \left( \nabla \vec{v}_l \right)^T \left( \sum_{s=1}^{S} g_s \nabla N_s^p \right) \nabla g + \nabla \left( \sum_{s=1}^{S} g_s \nabla N_s^p \right) \vec{v}_l.
\]

(52)

Theorem 4. (Shape Derivative) Assume \( \{x_k\}_{k=1}^{N} \) moves alongside the deformation, the shape derivative of objective \( J_1(\Omega) \) is then given by

\[
DJ_1(\Omega)[\vec{v}_l] = \frac{d}{dc} \sum_{k=1}^{n} J_{1,k}(x_k, u_k, \Omega) \bigg|_{c=0^+} + \frac{d}{dc} \sum_{k=1}^{n} (F_{k-1})^T \left( \frac{\Delta t}{m} \mu_k + \frac{\Delta t^2}{m} \delta_k \right) \bigg|_{c=0^+}
\]

\[
= \sum_{k=1}^{n} \sum_{i=1}^{N} \left[ \nabla \left( \sum_{s=1}^{S} D\gamma_s[\vec{v}_l] N_s^p(x_k^i) \right) - \left( \nabla \vec{v}_l(x_k^i) \right)^T \left( \sum_{s=1}^{S} \gamma_s \nabla N_s^p(x_k^i) \right) \nabla g + \nabla \left( \sum_{s=1}^{S} \gamma_s \nabla N_s^p(x_k^i) \right) \vec{v}_l(x_k^i) \right] \nabla g + \Delta t \mu_k + \Delta t^2 \delta_k.
\]

(53)

where \( D\gamma_s[\vec{v}_l] \) is the nodal discretization w.r.t. the \( l^{th} \) vertex perturbation of

\[
D\gamma(x)[\vec{v}] = \max_{s} \left( d \gamma_s(\vec{v}) \right)(-\vec{v}(p_{\partial \Omega}(x)) \cdot \vec{n}(p_{\partial \Omega}(x))).
\]

(54)

Proof. Since \( \Gamma_1 \) is fixed, the objective [35] is independent of mesh deformations. Hence, the shape derivative is zero, i.e.

\[
\frac{d}{dc} \sum_{k=1}^{n} J_{1,k}(x_k, u_k, \Omega) \bigg|_{c=0^+} = 0.
\]

We recall that for SPH-flows the fluid and boundary part are split up [16], i.e. for density of particle \( i \in \{1, ..., N\} \) in each time step \( k \in \{1, ..., n\} \)

\[
\rho(x_k^i) = \sum_{j=1}^{N} mW(x_k^i - x_j^i, h) + \sum_{s=1}^{S} mN_s(x_k^i) \gamma_s + \sum_{s=1}^{S} mN_s(x_k^i) \gamma_s^L.
\]
The first sum is only dependent on the position of the particles, hence the respective shape derivatives vanish due to adjoints and we have

\[ \frac{d}{d\epsilon} \sum_{j=1}^{N} mW(x^i_k - x^j_{-k}, h) \bigg|_{\epsilon=0^+} = 0. \]

The remainders follow from regarding terms w.r.t. the water gradient, Lemma 2 and 3 such as equation (53), by assuming that the particle position \( \{x^i_k\}_{i,k=1}^{N,n} \) moves alongside the deformation in each time step.

Remark. The assumption that \( \{x^i_k\}_{i,k=1}^{N,n} \) moves along the deformation drastically simplifies calculations, otherwise we are required to perform low-level computations similar to works in [14].

Remark. Using (44) and the fact that nodal values are spatially constant we can rewrite (53) as

\[ DJ^1(\Omega)[\vec{V}] = \frac{d}{d\epsilon} \sum_{k=1}^{n} (F^i_k)^T \left( \frac{\Delta t}{m} \mu_k + \frac{\Delta t^2}{m} \delta_k \right) \bigg|_{\epsilon=0^+} \]

\[ = \frac{1}{\Delta t} \sum_{k=1}^{n} \sum_{i=1}^{N} \left( \sum_{s=1}^{S} D\gamma_s[\vec{V}] \nabla N^p_s(x^i_k) \right) - \left( \sum_{s=1}^{S} \gamma_s \nabla N^p_s(x^i_k) \right) \sum_{k=1}^{n} \sum_{i=1}^{N} \left( \sum_{s=1}^{S} \gamma_s \nabla N^p_s(x^i_k) \right) \delta x^i_k \right) \left( \Delta t \mu_k + \Delta t^2 \delta_k \right). \]

(55)

Here we highlight, that the product of ansatz functions is zero, whenever, \( \{x^i_k\}_{i,k=1}^{N,n} \) is not in the support of shape functions \( N^p_s \) and \( N^1_s \).

Remark. Factoring out the respective \( \delta x^i_k \) in (55) we can calculate the remaining quantities for each mesh vertex perturbation. All quantities can be collected in a vector \( DJ^1(\Omega)[\vec{V}] \) of size \( dL \) to apply some mesh deformation strategy as discussed in [34, Section 6.3]. Linear elasticity calculations as in [35] can be enabled, by choosing a basis of the test function space, calculating all occurring quantities and split vectorial contributions, as it is common in vector-valued finite element methods.

Remark. For completeness shape derivatives of the penalty terms (33) and (34) are obtained as [31]

\[ DJ^2(\Omega)[\vec{V}] = \nu \int_D \nabla \cdot \vec{V} \ dx \]

(56)

\[ DJ^3(\Omega)[\vec{V}] = \nu \int_{\Gamma_3} \kappa(\vec{V}, \vec{n}) \ ds = \nu \int_{\Gamma_3} \nabla \cdot \vec{V} - (\frac{\partial \vec{V}}{\partial \vec{n}}, \vec{n}) \ ds. \]

(57)

The discretization of (56) and (57) and the subsequent usage in optimization routines follows naturally for standard finite element solvers [15].

4.4 Numerical Results

In this section we will first give details about the numerical implementation, before verifying results for a selected choice of test cases.
The SPH techniques from Section 3 to solve Lagrangian SWE with boundary contributions can be summarized in a pseudocode as

**Algorithm 1: 2D SPH for Lagrangian SWE with Boundary Contribution**

Initialize Particles carrying \((\vec{Q}_i^0, x_i^0)\) \(i = 1, \ldots, N\) with Constant Mass \(m > 0\) and Step-Size \(\Delta t > 0\)

foreach Time Step \(k \in \{1, \ldots, n\}\) do
  foreach Particle \(i\) do
    Find Neighbour Particles \(M \subset \{1, \ldots, N\}\)
  end
  foreach Particle \(i\) do
    Compute Fluid Density \(\rho^F_i\) [via (13)]
    Compute Boundary Density \(\rho^D_i\) [via (23)]
    Compute Water Height \(H_i\) [via (36)]
  end
  foreach Particle \(i\) do
    Calculate Forces:
    \[ F^i_k = F^\text{Height}_i + F^\text{Viscosity}_i + F^\text{Sediment}_i \]
  end
  foreach Particle \(i\) do
    Calculate States:
    \[ \vec{Q}^i_{k+1} = \vec{Q}^i_k + \Delta tF^i_k/m \]
    \[ x^i_{k+1} = x^i_k + \Delta t\vec{Q}^i_k \]
  end
end

**Remark.** The following remarks should guide through Algorithm [1]

(i) The neighbour search is only mentioned for completeness. For the Gaussian kernel [14], the search for neighbouring particles is trivially omitted.

(ii) Whenever quantities should be stored for subsequent computations the time step \(k \in \{1, \ldots, n\}\) is explicitly mentioned.

(iii) The change from Lagrangian SWE to incompressible Navier-Stokes equations is remarkable easy. Instead of computing water heights, a fluid pressure would be required as

\[
p^F_i = B_T \left( \left( \frac{\rho^F_i}{\rho_0} \right)^\xi - 1 \right) \tag{58}
\]

for \(B_T, \xi > 0\) e.g. using Tait’s law [36].

In all the following examples we will work with a simple mesh in line with definitions given in Section 4.3, as it can be seen in Figure 2. The solution to the signed distance function is based on the solution of the Eikonal equation with

\[
f(x) = 1, \quad q(x) = 0 \quad |\nabla w(x)| = f(x) \quad x \in \Omega \quad w(x) = q(x) \quad x \in \Gamma, \tag{59}
\]
where we have implemented a viscous and stabilized version to obtain \( w \in H^1(\Omega) \) for all \( v \in H^1(\Omega) \) i.e.

\[
\int_{\Omega} \sqrt{\nabla w \cdot \nabla w} \, dx - \int_{\Omega} fv \, dx + \int_{\Omega} \epsilon_{SDF} \nabla w \cdot \nabla v \, dx = 0, \tag{60}
\]

where \( \epsilon_{SDF} = \max h_\kappa \) is dependent on the element-diameter \( h_\kappa \) of mesh-cell \( \kappa \in T_h \). Build on this solution, modifications as in (25) are exercised downstream by prescribing respective nodal values. The recursive adjoints require the calculation of a matrix with partial derivatives w.r.t. the states. The partial derivatives matrix w.r.t. the positions is obtained by a \( 2N \times 2N \) matrix

\[
\frac{\partial F}{\partial x} = \begin{pmatrix}
\frac{\partial F^x}{\partial x^1} & \frac{\partial F^x}{\partial x^2} & \ldots & \frac{\partial F^x}{\partial x^N} \\
\frac{\partial F^x}{\partial y^1} & \frac{\partial F^x}{\partial y^2} & \ldots & \frac{\partial F^x}{\partial y^N} \\
\frac{\partial F^x}{\partial z^1} & \frac{\partial F^x}{\partial z^2} & \ldots & \frac{\partial F^x}{\partial z^N} \\
\vdots & \ddots & \ddots & \vdots \\
\frac{\partial F^x}{\partial x^1} & \frac{\partial F^x}{\partial x^2} & \ldots & \frac{\partial F^x}{\partial x^N}
\end{pmatrix} \tag{61}
\]

and the \( 2N \times 2N \) matrix of partial derivatives of the particle velocities as

\[
\frac{\partial F}{\partial u} = \begin{pmatrix}
\frac{\partial F^x}{\partial u^1} & \frac{\partial F^x}{\partial u^2} & \ldots & \frac{\partial F^x}{\partial u^N} \\
\frac{\partial F^y}{\partial u^1} & \frac{\partial F^y}{\partial u^2} & \ldots & \frac{\partial F^y}{\partial u^N} \\
\frac{\partial F^z}{\partial u^1} & \frac{\partial F^z}{\partial u^2} & \ldots & \frac{\partial F^z}{\partial u^N} \\
\vdots & \ddots & \ddots & \vdots \\
\frac{\partial F^x}{\partial u^1} & \frac{\partial F^x}{\partial u^2} & \ldots & \frac{\partial F^x}{\partial u^N}
\end{pmatrix}. \tag{62}
\]

with two-dimensional components \( x = (x, y) \) and \( u = (u, v) \). In the continuous setting, a deformation scheme can be obtained by regarding the Steklov-Poincaré metric [15], such that a solution of the linear elasticity equation \( \tilde{W} : \Omega \rightarrow \mathbb{R}^2 \) can be used, i.e.

\[
\int \sigma(\tilde{W}) : \epsilon(\tilde{V}) = DJ(\Omega)[\tilde{V}] \quad \forall \tilde{V} \in H^1_0(\Omega, \mathbb{R}^2)
\]

\[
\sigma = \lambda Tr(\epsilon(\tilde{W})) I_2 + 2\mu \epsilon(\tilde{W})
\]

\[
\epsilon(\tilde{W}) = \frac{1}{2}(\nabla \tilde{W} + \nabla \tilde{W}^T)
\]

\[
\epsilon(\tilde{V}) = \frac{1}{2}(\nabla \tilde{V} + \nabla \tilde{V}^T),
\]

where \( \sigma \) and \( \epsilon \) are called strain and stress tensor and \( \lambda \) and \( \mu \) are called Lamé parameters. We have chosen \( \lambda = 0 \) and \( \mu \) as the solution of the following Poisson problem

\[
- \Delta \mu = 0 \quad \text{in} \ \Omega \\
\mu = \mu_{\max} \quad \text{on} \ \Gamma_3 \\
\mu = \mu_{\min} \quad \text{on} \ \Gamma_1, \Gamma_2.
\]

\[
(64)
\]
In our discrete setting is the right hand-side in (63) manually evaluated using (55) and the associated remark. The full gradient-descent based shape optimization algorithm for SPH-flows is pseudocoded below.

**Algorithm 2:** Lagrangian SWE via SPH Shape Optimization Algorithm

| Initialization Mesh, Particles |
|--------------------------------|
| while $||D_J(\Omega_k)[\vec{V}]|| > \epsilon_{TOL}$ do |
| 1. Calculate Modified SDF $\gamma_k$ [via Viscous Eikonal Eq. (60)] |
| 2. Calculate States $x_k, u_k$ [via SPH Algorithm 1] |
| 3. Calculate Adjoint $\delta_k, \mu_k$ [via Section 2 & AD] |
| 4. Calculate Gradient $W_k$ [via (55-57) & Linear Elasticity (63)] |
| 5. Perform Linesearch for $\tilde{W}_k$ |
| 6. Calculate $\Omega_{k+1}$ [via $\tilde{W}_k$ and (44)] |
| end |

*Remark.* Calculation of partial derivatives in (61) and (62) is manually possible, however results in a tremendous calculative effort, prone to errors. For circumvention we use automatic differentiation (AD) via the Autograd library in either a forward or backward mode for the fluid portion, which is possible for code generated fully in Numpy. Here we highlight for boundary terms as defined in (23) the partial derivative matrix for positional state is diagonal and zero for the velocity state. We also point out that derivatives of boundary contributions are fully implemented in FEniCS [37].

The manual evaluation of (53) in Step 4 is relying on a constant evaluation of finite element ansatz functions, where the FEniCS build-in function evaluate_basis_derivatives_all is used.

**Example 1:**

In the first example we model the propagation of two particles, i.e. $N = 2$, with initial position $x_0^1 = (0.5, 0.5)$ and $x_0^2 = (0.63, 0.5)$, towards the shore by prescribing initial velocities as $u_0 = (0, -3)^2$. This test case is deliberately kept simple to analyse the procedure. In Figure 3 we have visualized the particle movements for time snapshots $t \in \{0, 0.6, 0.12, 0.24\}$ for smoothing radius $h = 0.04$, particle mass $m = 1$, smoothing factor $\alpha = 100$ and reference density $\rho_0 = 10$. The particle propagation is performed using time steps of size $\Delta t = 0.008$ with end time $T = 0.18$.

The red particle, with initial position $x_0^1$, travels towards the shore $\Gamma_1$ and effectively accounts for an increased objective in form of (35). In contrast, the blue particle contributes only marginally to the objective, starting from initial position $x_0^2$, being reflected from the obstacle and hence travelling back into the field and being collected in the outflow channel.

![Figure 3: Ex.1 Particle Propagation](image-url)
Both particles are within the interaction radius of the obstacle for a certain time frame, such that shape derivatives of boundary height forces are non-zero. However, as stated above, the objective contribution of blue ensures that adjoints \( \{ \delta^1_k \}_{k=1}^n \) and \( \{ \mu^1_k \}_{k=1}^n \) are vanishingly small. The initial deformation vector is hence predominantly activated in reds interaction region with the obstacle, as it can be observed in Figure 4 on the left. Relying on the shape optimization algorithm that we have presented before, for initial step-size \( \rho_{\text{step}} = 1e^{-4} \) and tolerance \( \epsilon_{TOL} = 1e^{-9} \), we are able to deform the obstacle setting \( \mu_{\text{min}} = 10 \) and \( \mu_{\text{max}} = 100 \) in the Poisson problem. The final mesh is able to decrease the objective up to a minimum, as it can be seen in Figure 4 in the mid and right part. We highlight, that even a small deformation is able to achieve these results in this simple setting.

![Deformation Field](image1)

![Optimized Mesh](image2)

![Objective](image3)

**Figure 4: Ex.1 Optimization Results**

**Example 2:**

In the second example we increase the number of particles, i.e. \( N = 180 \), with initial positions drawn from a normal distribution as \( x_0 = (0.7 + \mathcal{N}(0, 0.1), 0.6 + \mathcal{N}(0, 0.1))^N \). The particles once more travel towards the shore driven by initial velocity as \( u_0 = (0, -3)^N \). The remaining settings are as in the first example. We can observe the movement in Figure 5.

![Particles at t = 0](image4)

![Particles at t = 0.06](image5)

![Particles at t = 0.12](image6)

![Particles at t = 0.24](image7)

**Figure 5: Ex.2 Particle Propagation**

Once more relying on our shape optimization algorithm, we are able to deform the obstacle using resulting deformation fields, pictured in Figure 6 on the left for the first iteration. In this setting a larger deformation is necessary to effectively reduce the objective as we see in the final mesh Figure 6 in the middle and on the right. We would like to highlight that
the performance of the algorithm can be degrading by consecutive particle and boundary interactions, resulting from mesh deformations. Potentially this hinders us from obtaining improved results.

![Deformation Field](image1) ![Optimized Mesh](image2) ![Objective](image3)

Figure 6: Ex.2 Optimization Results

5 Conclusion

We have derived the discrete adjoint for a general symplectic Euler particle class. In addition, for an SPH-representative, based on boundary interactions by signed distance fields, the shape derivative was derived, that reduces to the shape derivative of a finite element interpolator. Results have been verified on a sample mesh inspired by practical applications to mitigate effects of coastal erosion. We point out that this can only serve as a first feasibility study, since the number of particles is still low and meshes are simplified. In future, comparisons between shape optimization results in an Eulerian, cf. to [38] and [39], and an Lagrangian framework, such as an Eulerian-Lagrangian coupling e.g. for a SWE-Exner model, appear of interest.

Acknowledgement

This work has been supported by the Deutsche Forschungsgemeinschaft within the Priority program SPP 1962 "Non-smooth and Complementarity-based Distributed Parameter Systems: Simulation and Hierarchical Optimization". The authors would like to thank Diaraf Seck (Université Cheikh Anta Diop, Dakar, Senegal) and Mame Gor Ngom (Université Cheikh Anta Diop, Dakar, Senegal) for helpful and interesting discussions within the project Shape Optimization Mitigating Coastal Erosion (SOMICE).

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