Chiral Dirac Fermions on the Lattice using Geometric Discretisation

Vivien de Beaucé and Samik Sen

School of Mathematics, University of Dublin, Dublin 2, Ireland.

Abstract

We propose a discretisation scheme based on the Dirac-Kähler formalism (DK) in which the algebraic relations between continuum operators \(\{\wedge, d, \star\}\) are captured by their discrete analogies, allowing the construction of the relevant projection operators necessary to prevent species doubling. We thus avoid the traditional form of species doubling as well as spectral doubling, which does not occur in the DK setting. Chirality is also captured, since we have \(\star\) from geometric discretisation. Some remarks regarding the gauging of the theory are made.
I. INTRODUCTION

The lattice provides a regularisation of continuum quantum chromodynamics which has played a central role in the study of non-perturbative effects such as quark confinement. However, problems arose which seem, at least at first sight, to be inherent to the lattice.

Putting fermions on the lattice naively leads to degeneracy, as can be seen from the dispersion relations obtained. When Wilson removed this degeneracy, it may have seemed that the loss of chirality was perhaps coincidental but subsequent work, based on topological arguments, found that this was not the case.

The argument, proposed by Nielsen and Ninomiya (NN), shows that there are, under reasonable assumptions such as exact conservation of discrete valued quantum numbers (lepton number $Q$) and locality, an equal number of species of left and right-handed neutrinos which means that you cannot have single fermions since they occur in pairs. Handedness is then taken into account by considering the homotopy class of maps from closed surfaces, encircling the degeneracy points $\omega_{\text{deg}}$ embedded in the Brillouin zone into the space of rays formed by the $N$ fundamental fermion components $\mathbb{CP}^{N-1}$. The degenerate fermionic states, $|\omega>$, live in this space and satisfy

$$H |\omega> = \omega_{\text{deg}} |\omega>.$$  

The Dirac-Kähler (DK) formulation, which is differential geometric in nature, is advantageous since it gives rise to a non-degenerate energy spectrum in the discrete formulation, which means that the NN theorem is not applicable. With this in mind a discrete DK theory can be expressed, using ideas from algebraic topology to provide a natural discrete analogue to differential geometry. Becher and Joos showed how doubling still arises in this theory after reduction, even though topology is captured and NN is not applicable, as the discrete operations used do not satisfy the desired properties.

The algebraic properties of the discrete analogies of the operators $\{\wedge, d, \star\}$ are used to describe spinors and the action of the Clifford algebra on them. The latter has a differential geometric analogue through the introduction of the Clifford product (CP), both in the continuum and on the lattice. Its action, denoted by $\lor$, is a combination of the operations in the triple $\{\wedge, d, \star\}$ and plays an important role in the construction.
We use the DK approach with geometric discretisation (GD)\cite{14} as our discrete differential geometry. GD uses discrete analogies to continuum objects and operators, capturing Stokes’ theorem and the Hodge decomposition theorem, in a similar fashion to Dodziuk\cite{15}. It also possesses a discrete Hodge star operator by using a subdivided space in which both the original and dual lattice are contained, which was previously not available. The discrete wedge enables us to impose the conditions needed to relate the Dirac-Kähler equations to the Dirac equation (DE)\cite{16}, which was the problem faced by Becher and Joos\cite{11}, and the discrete Hodge star allows us to deal with chirality as was discussed by Rabin\cite{17}.

To summarise:

• We use the DK approach and so do not suffer from spectral degeneracy.

• In our framework we use:
  – The GD wedge, which means that we can impose subsidiary conditions to avoid species doubling.
  – The GD Hodge star, which means that we have chirality.

In this work, we strengthen the relation between the lattice and the continuum in order to propose a more satisfactory lattice regularisation of the Chiral theory.

We begin with a brief review of the DK equation in section 2. We then look at how this is dealt with discretely in section 3; first, using standard methods where the wedge fails, and then using GD where it does not. We address some issues regarding the gauging of the free theory in section 4.

II. THE DIRAC-KÄHLER EQUATION

In the language of differential forms the Laplacian is given by

\[ \triangle = d\delta + \delta d. \]

Thus we note that

\[ -(d - \delta)^2 = \triangle, \]

from which we obtain the Dirac-Kähler equation (DKE)\cite{4, 5, 16}

\[ i(d - \delta + m)\psi = 0, \]
the solution of which has the following the functional form

$$\psi = 1 + f_\mu dx^\mu + \frac{1}{2!} f_{\mu\nu} dx^\mu \wedge dx^\nu + \frac{1}{3!} f_{\mu\nu\lambda} dx^\mu \wedge dx^\nu \wedge dx^\lambda + f_{0123} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3.$$  

In order to represent spinors in this language, we introduce the Clifford product, which acts on the space of inhomogeneous differential forms and satisfies the following relations:

$$1 \lor 1 = 1,$$

$$1 \lor dx^\mu = dx^\mu \lor 1 = dx^\mu,$$

$$dx^\mu \lor dx^\nu = g^{\mu\nu} \cdot 1 + dx^\mu \wedge dx^\nu,$$

where $g^{\mu\nu}$ is the Euclidean metric. Through the identification

$$dx^\mu \lor \mapsto \gamma^\mu,$$

which arises from representation theory, we relate the differential forms under the CP to the algebra of gamma matrices. It is now immediate that

$$\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu}.$$

The Dirac field then belongs to a 16 dimensional representation of the algebra of gamma matrices. Furthermore all the representations of the complex Clifford algebra of the 4-dimensional Euclidean space can be decomposed into 4-dimensional irreducible representations [20]; these being equivalent to those generated by the standard gamma matrices.

As we have introduced the Clifford product, an important observation due to Susskind is now in order, namely, that the DKE is invariant under the group $SU(4)$, referred to as a global flavour symmetry, which acts by right action of the Clifford product with a constant differential $U$. That is,

$$(d - \delta + m) (\Phi \lor U) = \{(d - \delta + m) \Phi\} \lor U = 0,$$

this being established through the identification $d - \delta \mapsto dx^\mu \lor \partial_\mu$ and use of the associativity of $\lor$ which is known to hold in the continuum.

We can thus decompose the 16D space of differential forms $\mathcal{D} = \{\Phi\}$ into 4-dimensional invariant subspaces

$$\mathcal{D} = \bigoplus_{b=1}^{4} \mathcal{D}^{(b)}.$$
on which the DKE implies the DE.

One can construct a collection of projection operators $P^{(b)}$ mapping forms onto the irreducible subspaces $D^{(b)}$, where

$$\Phi \lor P^{(b)} = \Phi,$$

if $\Phi \in D^{(b)}$. We also know that $\Phi \lor P^{(b)}$ is a solution of the DKE from the argument given above regarding the global $SU(4)$ symmetry. With this subsidiary condition (4), the DKE for fixed $b$ is equivalent to the DE.

From (1) we can see that $d - \delta = dx^\mu \lor \partial_\mu$. Considering

$$\Phi^{(b)} = \Phi \lor P^{(b)},$$

where $\Phi$ is a solution of the DKE, we get

$$(d - \delta + m) \Phi^{(b)} \mapsto (\gamma_\mu \partial_\mu + m) \Phi^{(b)},$$

leading to the Dirac equation.

We note that the DKE can also be expressed using the set of equations

$$i(d\omega^{(p-1)} - \delta\omega^{(p+1)}) = -m\omega^{(p)}, \quad p = \{0, 1, \ldots, 4\}.$$
III. THE DIRAC-KÄHLER EQUATION ON THE LATTICE

After having stressed the importance of the subsidiary conditions above, we now investigate their discrete counterpart \[32\]. We will first explain the nature of the problem by describing the usual method and then propose our alternative. Again, this will turn out to be linked to the algebraic relations of the discrete operators; particularly of the wedge $\wedge$.

At this point it is useful to note that the (continuum) subsidiary conditions can be given the form of a group property known as the reduction group denoted $\mathcal{R}$ generated by $\tau = id x^1 \wedge d x^2$ and $\epsilon = d x^1 \lor d x^2 \lor d x^3 \lor d x^4$ under $\lor$-multiplication.

On the lattice, we represent $p$-forms as $p$-cochains. Thus the Dirac fields are represented by inhomogeneous cochains. We now review the construction of the discrete analogue of the reduction group.

In that approach, where the cup product is taken as the discrete wedge, the operation is performed at a base point $x$. The left and right cochain arguments of the wedge must have the same base point, as illustrated by the above FIG. 1.

In the continuum $x \wedge y$ is equal to $y \wedge x$ up to sign \[33\] which is no longer the case on the lattice; unless we translate the right cochain in such a way that its base point coincides with the other, enabling us to wedge cochains which do not have the same base point.

In the continuum we have

$$d x^1 \lor (d x^1 \wedge d x^2) = 1 \wedge d x^2 = d x^2$$

because we remove any $d x^i$’s which two arguments have in common before wedging them, when taking their Clifford product.
So,

\[(dx \wedge dy) \lor (dx \wedge dy) = 1 \wedge 1 = I\]

whose discrete analogue is

\[(dx \wedge dy) \lor T(dx \wedge dy) = TeI,\]

where translation operation \(Te\) shifts simplices \(\sigma\) and \(\eta\) so that \(\sigma \wedge \eta\) and \(\eta \wedge \sigma\) have common support.

It is then found that \([11]\):

\[
\begin{align*}
\tau'^2 &= T_{-(e_1+e_2)}, \quad \epsilon'^2 = T_{-(e_1+e_2+e_3+e_4)}; \\
\tau \lor \epsilon &= \epsilon \lor \tau = Te_{e_1+e_2} T \epsilon,
\end{align*}
\]

thus supplementing the group \(R\) with translation elements. This means that the discrete reduction group does not close and it is not possible to impose the subsidiary conditions.

Note that the presence of translation operators in the discrete analogue of the reduction group is unavoidable in this formalism. One could consider the space resulting from quotienting translation elements, but this would lead to the lattice being identified as one unique point.

![FIG. 2: Discrete wedge in GD: the higher dimensional region, here a tetrahedron gives common support to the vertex at \(x\), the vertex at \(x+h\) and to the edge \([x, x+h]\).](image)

In geometric discretisation (GD) \([14]\) we map any \(p\)-cochains, \(\sigma^p\), to \(p\)-forms, \(\omega^p\), using the Whitney map \(W\) and wedge these, before mapping the result back to the lattice, using the de Rham map \(A\) given by

\[
A(\omega^p) = \sum_i \left( \int_{[\sigma_i^p]} \omega^p \right) [\sigma_i^p],
\]
where $AW = I$ which makes the Whitney map the right-inverse of $A$ and in the standard simplex coordinate system is given by

$$W([v_0 \ldots v_r]) = r! \sum_{i=0}^{r} (-1)^i \mu_i d\mu_0 \wedge \ldots \wedge d\mu_i \wedge \ldots \wedge d\mu_r,$$

(7)

and the functions $\{\mu_i, i \in [0, r]\}$ are the so called barycentric coordinate functions.

![FIG. 3: The standard simplex](image)

In the case of the triangle of FIG. 3, the barycenter coordinates are given by

$$\mu_i = \begin{cases} 
1 - x - y, & \text{for } i = 0, \\
x, & \text{for } i = 1, \\
y, & \text{for } i = 2.
\end{cases}$$

(8)

It is then immediate to check that

$$dW^K([v_0 \ldots v_r]) = W^K d^K [v_0 \ldots v_r],$$

(9)

displaying how the simplicial map $d^K$ is tied to its continuum analogue $d$. Starting with (8), and applying (7), one obtains the so called Whitney elements which form the basis list of differential forms captured by the cochain space under $W$.

We then define the discrete wedge of any two given simplices $\sigma$ and $\eta$ living in the complex $K$ as

$$\sigma \wedge^K \eta = A^K (W^K(\sigma) \wedge W^K(\eta)),$$

(10)

which satisfies $x \wedge^K y = \pm y \wedge^K x$ with no translation element required. Thus $\tau^2 = I$ and the reduction group closes. We note that the wedge product in our method is only associative.
up to a multiplicative factor which is a function of the degree of the cochains being wedged. Given three cochains $\sigma$, $\eta$, $\zeta$ in K, respectively of degrees $p$, $q$ and $r$, the rule is

$$\sigma \land^K (\eta \land^K \zeta) = \left(\frac{p+q+1}{r+q+1}\right) (\sigma \land^K \eta) \land^K \zeta. \quad (11)$$

Before we proceed to discuss the Dirac-Kähler equation in our framework, we would like to comment further on the simplicial operations defining GD by giving our discrete exterior derivative $d^K$ and its adjoint $\delta^K$. They are as follows:

$$d^K[v_0 \ldots v_r] = \sum_{k=0}^{N}[v_kv_0 \ldots v_r]; \quad (12)$$

while,

$$\delta^K[v_0 \ldots v_r] = \star^L d^L \star^K [v_0 \ldots v_r]$$
$$= \pm \partial_K [v_0 \ldots v_r]$$
$$= \sum_{k=0}^{N} (-1)^k [v_0 \ldots \hat{v}_k \ldots v_r]$$

where $N$ is the number of vertices. Note the role played by the dual complex $L$ in defining $\delta^K$.

Example: Consider FIG. 3 again and the zero-cochain

$$\sigma^0_K = A^K(f) = f([0])[0] + f([1])[1] + f([2])[2]$$

Applying $d^K$ we obtain

$$d^K \sigma^0 = (f([1]) - f([0]))[01] + (f([2]) - f([1]))[12] + (f([2]) - f([0]))[02].$$

Noting that for $[0] = (0,0)$ and $[1] = (h,0)$ we see that $f([1]) - f([0]) = f(x + h) - f(x)$. Also, since we started with a zero-cochain, $\delta \sigma^0_K = 0$. The emerging picture for the DK equation in the present framework is that one has a hyper-cubic complex with fields represented by inhomogeneous cochains with constant coefficients in front of simplices such as $[i]$, $[ij]$, $[ijkl]$ and so on. The fact that one has Whitney elements is of importance here in two related ways. First of all, they provide support to the associated form and also are used in the wedge formulae used below.
To compare this with the method described previously, we point out that the vertices at $x$ and at $x + h$ have overlapping support on the edge $[x, x + h]$ (FIG. 2). If one was to specify a base point to be $x$ or $x + h$, as is done in turn in FIG 1, one would still get the correct non-zero answer. It follows that no translation is necessary.

In the argument given above, we assumed that the inhomogeneous differential forms projected under the right action of $P^{(b)}$ remain solutions of the DKE. In the continuum, this step involved the associativity of the continuum Clifford product. As is apparent from (3) the associativity of the discrete Clifford is conditional on the discrete wedge being associative. In other methods such as [11] the discrete Clifford product is non-associative in general but does not affect the derivation of (3). In the present framework we point out that we are given the exact expression for the $P^{(b)}$ differentials in the discrete theory. Therefore, owing to the fact that the Clifford product gives rise to the algebra displayed in (11), we can assert that

$$\Phi' = \Phi \vee P^{(b)}$$

is also a solution of the DKE.

More explicitly, the $P^{(b)}$ are constant differentials, and our discrete wedge (10) is exact for constant differentials. Hence provided the lattice field $\Phi^K$ satisfies the discrete DK equation

$$(d^K - \delta^K + m)\Phi^K = 0,$$  \hspace{1cm} (14)

the equations

$$(d^K - \delta^K + m)(\Phi^K \vee^K P^{(b)}) = 0$$  \hspace{1cm} (15)

follow without requiring associativity of the discrete wedge. To make sense of this last equation in the lattice setting, we recall the formulae for the Clifford product does include the wedge but also contraction [35]. Given a two form for example which has support on a square $[0123]$ say with $[01]$ and $[32]$ opposite edges along the $x$-axis and $[03]$, $[12]$ parallel edges along the $y$-axis. Then contracting the two form represented by $[0123]$ with a $dx$ and using (see Becher and Joos [11])

$$P^{(b)} = \frac{1}{4}(1 + i\text{sign}(12)dx^1 \wedge dx^2) \vee (1 + \text{sign}(1234)\epsilon)$$

(16)
in (13), we have an analogous lattice operation

\[ e_x |_{K[0123]} = [03] + [12], \]

(17)

while \( W^K([03]) \) and \( W^K([12]) \) have respectively images given by \((1 - x)dy\) and \(xdy\) so the contraction operation at the simplicial level gives back the correct Whitney element of form degree one less. This is what is meant by capturing exactly the right multiplication with the constant differential \( P^{(b)} \). It is of interest to compare this approach with the discussion of Rabin [17] in which the importance of the discrete operators in the reduction is identified.

On the issue of chirality, our discrete Hodge star maps one space into its dual, that is there is a doubling of the space of cochains. As is customary in lattice theories we then deduce the relation:

\[ \{ \gamma_5, D \} = 0. \]

(18)

In the present DK formalism this is expressed (acting on fields in K, but one can equally swap K and L superscripts) as

\[ \{ \star, d - \delta \}^K \equiv \gamma^K(d^K - \delta^K) + (d^L - \delta^L)\gamma^K, \]

(19)

since \( \star \) plays the role of \( \gamma_5 \) and \( D = d - \delta \). Using \( \gamma^K \gamma^K = I \) and \( \delta^K = \gamma^K d^K \gamma^K \), which are both satisfied in GD, we find

\[
\star^K(d^K - \gamma^K d^L \gamma^K) + (d^L - \gamma^K d^K \gamma^K) = (\star^K d^K - \star^K \gamma^K d^K \gamma^K) + (d^L \gamma^K d^K - \gamma^K d^K \gamma^K) = 0.
\]

Note that \( \gamma_5 \) is identified with \( \star^K \) and \( \star^L \), up to some sign factor, since \( (\gamma_5)^2 = 1 \) where as \( \star^K \gamma^K = \pm 1 \). So we always have \( \star \star = 1 \) provided we insert the correct sign factor, leading to the desired result. Again, there are two spaces involved here, since \( \gamma^K D^K \) and \( D^K \gamma^K \) both map objects to their dual space, which is also the case in Ginsparg-Wilson [12] for example.

The presence of the two spaces, \( K \) and \( L \), is, we claim, not a problem in the discretised chiral fermion theory. In order to check this, we note that the subsidiary conditions carry through under application of \( \star \). Namely, if \( \omega \in P^{(b)}_K \) then \( \star \omega \in P^{(b)}_L \). Since the theory is well defined as we interchange \( K \) and \( L \), a space such as \( P^{(b)}_L \) is well-defined. That is, in a self-explanatory notation we can write an \( L \)-DKE as we were able to write a \( K \)-DKE in (14).
Given the inhomogeneous fields $\Phi_K$, represented by inhomogeneous cochains in the complex $K$, we can now consider the associated action functional $S_K$:

$$S_K = \langle \bar{\Phi}_K, (d - \delta + m)\Phi_K \rangle,$$

where (denoting complex conjugation by $c$)

$$\bar{\Phi}_K = \gamma^5 \Phi_K^c = \star^K \Phi_K^c,$$

and so belongs to the dual space $L$. It is worth pausing for a moment to appreciate the role played by $\star$ both as $\gamma^5$ and as providing the volume form for integration.

The inner product $\langle \cdot, \cdot \rangle: C^p(L) \times C^{n-p}(K) \rightarrow \mathbb{C} \quad [14]$ is given by

$$\langle \sigma^L, \eta^K \rangle = (\sigma^L, \star^K \eta^K) = \frac{(n+1)!}{p!(n-p)!} \int_M W^B(B\sigma^L) \wedge W^B(B\eta^K),$$

where $B$ is the subdivided space containing both $K$ and $L$ and $B\sigma^L$ is the projection of $\sigma^L$ onto $B$.

The action is of the form

$$S_K \equiv S \left( \Phi_K, \star^K \Phi_K' \right), \quad (20)$$

where there is a subtlety since the field $\star^K \Phi_K$ lives in $L$. Although there is a field in the original and dual complex, one is the image of the other. We can repeat the argument starting with the $L$ complex, and define $S_L(\Phi_L)$. This in contrast with methods such as $[11]$ in which the volume element is induced by right Clifford multiplication with $\epsilon$. Again, this is due to the close link between the lattice and differential geometry in our method.

Finally we look at the partition function for which the action is a direct sum of its components in the invariant spaces $\mathcal{D}^{(b)}$.

$$S = \sum_{b=1}^{4} S^{(b)}, \quad (21)$$

which according to the argument above can be readily rewritten as

$$S = \sum_{b=1}^{4} S^{(b)}_K \oplus S^{(b)}_L. \quad (22)$$

Furthermore, $S^{(b)}_K$ and $S^{(b)}_L$ are functionals respectively of the fields $\Phi_K$ and $\Phi_L$ and their associated representative in the dual spaces $L$ and $K$. It follows then that the counting of
the fields is correct, while we obtain the partition function squared given below. To see this, we introduce a matrix $M$ determined by taking

$$\Phi^T_K M K \Phi'_K = S_K(\Phi_K, \Phi'_K).$$  \hspace{1cm} (23)$$

The entries of $M^K$ are fixed, up to normalisation (volume factors), by

$$M^K_{ij} = S_K(\sigma_i, \sigma_j).$$

We can fix this by normalising the parameters in the general formula for the discretised fields,

$$\Phi_K = \sum_i \lambda_i [\sigma_i],$$

from Eq. (10). The space of parameters $\lambda_i$ determines the partition function measure:

$$Z = \int [d\Phi_K] [d\Phi'_K] [d\Phi_L] [d\Phi'_L] e^{\Phi^T_K M K \Phi' + \Phi^T_L M_L} = \int [d\Phi_K] [d\Phi'_K] e^{\Phi^T_K M_K} \int [d\Phi_L] [d\Phi'_L] e^{\Phi^T_L M_L}$$

(24)

which splits into two independent parts, one for fields defined on $K$ and the other for fields defined on $L$. So, there is no mixing between the two lattices $K$ and $L$.

**IV. REMARKS ON GAUGING THE FREE FERMIONS**

In the DK picture the global $SU(4)$ symmetry is linked to the specific representation of fermions using the CP, and it has been argued that one could attempt to make it into a local “flavour” gauge symmetry. Here however, one take the inhomogeneous differential forms, and add a gauge label to their coefficient turning a given form into a Lie algebra valued form. The central observation is that the group action on the gauge labels should leave the reduced spaces $P^{(b)}(\Omega)$ invariant. In the continuum language one would say that reduction (which is an action of $SU(4)$ on $\Omega$) commutes with gauging the free fermions.

In GD, some freedom is given to us in how one might gauge the fermions, let us choose minimal coupling. As is customary, introduce gauge fields $A_\mu(x)$ and consider the parallel transport operator from the vertex at $x$ to the vertex at $x + \hat{\mu}$:

$$U_\mu(x) = \exp[i e A_\mu(x)].$$  \hspace{1cm} (25)$$

The operator $U$ is then inserted in the fermion bilinear term of the free Lagrangian. The invariance of the Lagrangian under gauge transformation follows by the usual manipulation
of matrices and traces. In effect, we have coupled a zero-form (in the spinor representation of DK only) gauge field to an inhomogeneous form field. Furthermore, this guarantees that the $D^{(b)}$ are left invariant.

In the usual Dirac-Kähler methods, as described above in geometric language, one diagonalises the free Lagrangian leading to

$$L = a^d \sum_{r, \mu} \bar{\phi}(r) \frac{1}{2a} 2^\pi \left[ Tr \left[ (\Gamma^a)^\dagger \gamma_\mu \Gamma^{a+\hat{\mu}} \right] \phi(r + a\hat{\mu}) - Tr \left[ (\Gamma^a)^\dagger \gamma_\mu \Gamma^{a-\hat{\mu}} \phi(r - a\hat{\mu}) \right] \right]$$

(26)

by means of the transformation of Kawamoto and Smit [24] (introducing the $\Gamma$ matrices) followed by thinning which amounts to taking the trace leaving one component of $\phi$. At this point, one has the option of blocking the fields [25, 26]; this is done by introducing the so-called block coordinates and by means of a unitary transformation one obtains a Dirac $\otimes$ flavour representation. The gauging is then done by inserting link variables in the Lagrangian [26].

Of particular interest to us, is the framework of staggered fermions in which the gauging is done after the thinning. One obtains directly:

$$L^S = a^d \sum_{r, \mu} \bar{\phi}(r) \frac{1}{2a} 2^\pi Tr \left[ (\Gamma^a)^\dagger \gamma_\mu \Gamma^{a+\hat{\mu}} \right] \frac{1}{2a} [U_\mu(r) \phi(r + a\hat{\mu}) - U_\mu^\dagger(r - a\hat{\mu}) \phi(r - a\hat{\mu})].$$

(27)

The reduction described in this paper which leads to (20) can be interpreted as the analogue of diagonalising and thinning, we reduce the degrees of freedom to obtain one fermion with $2^{\frac{d}{2}}$ spinorial components described in any given invariant subspace $D^{(b)}$ by an action of the form [27]. However we can also be more general and consider

$$S_K = \langle \Phi^{(b)}_K, U \left[ (d^K - \delta^K + m) \Phi^{(b)}_K \right] \rangle,$$

(28)

where $U = \sum_l U_l$ is the displacement operator labeled by the degree of the cochain it acts on.

It is immediate that $U$ exponentiates a zero form with value in the representation of the Lie algebra. It seems at first that we are coupling $(n - p)$-form fields to $p$-fields. However it is not the case, as it should be physically. The cochains cannot be used as the basic fields in the theory, since the gauge invariance of the resulting extended objects is not well-defined, as
illustrated by the no-go theorem of Teitelboim. Thus, although we use inhomogeneous differential forms to represent the algebra of spinors, they exist at points within a cell. This argument indicates that one may not attempt to make the $SU(4)$ symmetry local in our method as opposed to the suggestion of [11]. Nevertheless, our action has the discrete rotational symmetry within a given cubic cell.

Let us look at the 2D example and consider a square as below.

![Diagram](image)

FIG. 4: Two types of couplings: two edges with a vertex in common and the entire square with one vertex.

The displacement $U$ in the group then corresponds to matching fields at a fixed point in each cell.

To summarise [36], in gauging the fermion field, we insert an operator to the right of the wedge in the action. In our language, the operator $d - \delta$ affects a displacement and so one should propagate the field by inserting the appropriate $U$ operator.

Finally, a short comment about fermion masses. It has been established by various authors [26, 28] that the introduction of masses in the theory, either by hand as specifying bare masses or those induced by renormalization counter terms will depend on the gauging procedure. As pointed out above, we do not block the fields in the present method and it has been shown [29] that for the usual Susskind fermions with parallel transport operators inserted in the Lagrangian, no counterterms arise from renormalization and so the bare masses specified by hand are the actual masses in the theory. This means that our mass term which is

$$m\bar{\Phi} \wedge \star \Phi,$$

would give rise to no counterterms. In the usual methods, the blocking of the fields is central as the residual doubling present in these methods [11, 30] is identified as providing the multiple flavours. Moreover, careful analysis of the various symmetries of the Lagrangian
indicate that the DK lattice theory and the staggered theory differ as regards the generation of masses. For instance, the shift invariance is present in the staggered formalism thus preventing the dynamical generation of masses while it is not in DK after gauging.

V. A CRUCIAL REMARK

We have made some general comments about gauging, which ought to be the crucial step; but it is essential to note that this theory can be related to the staggered theory in the standard way of Becher and Joos, where one would expect the problem of doubling to reappear in the form of species doubling. What is novel in our approach is that by keeping inhomogeneous cochain fields without mapping their coefficients to vertices as is customary, we open the possibility of having reduced Dirac-Kähler fermions regularized in a way that has fields not defined at vertices solely but also on links and plaquettes. This is a consequence of the close relation of our formulation to differential geometry which makes the shift operators of FIG. 1 superfluous. Yet the extended objects should have no intrinsic meaning physically and only represent a fermionic field $\psi$.

This state of affairs is clearly different from the type of theories covered by the Nielsen-Ninomiya theorem [37], yet the theory is discretised so we expect to be able to put it on a computer.

VI. CONCLUSION

We are thus able to work with chiral Dirac fermions without seemingly suffering from degeneracy. In summary:

- Spectral degeneracy does not occur in the Dirac-Kähler formalism,

- Species doubling is avoided using the GD wedge which allows us to impose the subsidiary conditions,

- Species doubling is also avoided since we have $\delta = *d* \text{ and } ** = I$; the DKE is overdetermined if this is not the case,

- Chirality is captured since we have a discrete Hodge star.
• Some gauging prescriptions are made for the resulting staggered fermion theory and we have alluded to a new approach which is more natural in the present context.

We have tried to keep this note as short as possible, avoiding purposely to cloud the argument with the notation used in GD. Yet we intend to write a more exhaustive note which will include more technical details of our method (GD). The interest for this problem was initiated partly by the paper of Rabin in which the role of the Hodge star is presented as crucial to the lattice formulation of Chiral fermions. We also hope to have conveyed the idea that a large class of discretisation problems are caused by discrepancies in the relations between the operators $\wedge$, $d$ and $\star$. This is precisely what we tried to remedy here, emphasising the importance of differential geometric ideas in a discretised theory.

Acknowledgments

We would like to thank James Sexton for drawing our attention to applications of geometric discretisation to fermions and introducing references [11, 17] to us. Thanks also to Denjoe O’Connor and Siddhartha Sen for stimulating discussions. We are grateful to our referees for bringing [26, 28] to our attention. V de B was supported by the HEA through the IITAC.

[1] K. Wilson, Phys. Rev. D10, 2445 (1974).
[2] L. H. Karsten and J. Smit, Nuclear Physics B183, 103-140 (1981).
[3] H. B. Nielsen and M. Ninomiya, Phys. Lett. B105 219 (1981).
[4] E. Kähler, Rendiconti de Matematica (3-4) 21 425 (1962).
   W. Graf, Ann. Inst. Henri Poincaré A29 85 (1978).
[5] S. I Kruglov, hep-th/0110251.
[6] C. Nash and S. Sen, Topology and Geometry for Physicists, Academic Press, London (1983).
[7] S. Lang, Differential and Riemannian Manifolds, Springer-Verlag (1995).
[8] H. Flanders, Differential Forms with Applications to the Physical Sciences, Dover Publications (1989).
[9] P. Becher, Phys. Lett. 104B, 221(1981).
[10] J. Kogut and L. Susskind, *Phys. Rev.* **D11** 395-408 (1975).

[11] P. Becher and H. Joos, *Z. Phys.* **C15**, 343 (1982).

[12] P. H. Ginsparg and K. G. Wilson, *Phys. Rev.* **D25**, 2649 (1982).

[13] M. Creutz, *Rev. Mod. Phys* **73**, 119 (2001).

[14] Samik Sen, S. Sen, J. C. Sexton and D. H. Adams, *Phys. Rev.* **E61**, 3174 (2000); hep-th/0001030.

D. Adams, *Phys. Rev. Lett.* **78**, 4155-4158, (1997).

D. Adams, hep-th/9612009.

[15] J. Dodziuk, *Finite-Difference approach to the Hodge Theory of Harmonic Forms*, American Journal of Mathematics, Vol. 98, No. 1, 79-104 (1976).

[16] M. B Green, A. Schwarz, E. Witten, *Superstring Theory*, CUP (1987).

[17] J. M. Rabin, *Nuclear Physics* **B201**, 315 (1982).

[18] J. Vaz, *Advances in Applied Clifford algebras*, **7** (1997) 37.

J. Vaz, hep-th/9706189.

[19] I. Kanamori, N. Kawamoto, hep-th/0305094.

[20] E. P. Wigner, *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra*, Expanded and improved edition p210 ff, Academic Press, New York, London (1959).

[21] D. Birmingham, M. Rakowski, *Phys. Letters* **B299**, 299 (1993).

[22] S. Albeverio and J. Schäfer, *J. Math. Phys.* **36**, 2157 (1995).

[23] Samik Sen, hep-th/0307166.

[24] N. Kawamoto and J. Smit, *Nuclear Physics* **B192**, 100 (1981).

[25] H. Kluberg-Stern, A. Morel, O. Napoly and B. Petersson *Nuclear Physics* **B220**, 447 (1983).

[26] G. T Bodwin, E. V Kovács, *Phys. Rev.* **D38**, 1206 (1988).

[27] C. Teitelboim, *Phys. Letters*. **167B**, 63 (1986).

[28] P. Mitra and P. Weisz, *Phys. Letters* **B126**, 355 (1983).

[29] H. Sharatchandra, H.J Thun and P. Weisz, *Nuclear Physics* **B192**, 205 (1981).

[30] M. Golterman and J. Smit *Nuclear Physics* **B245**, 61 (1984).

[31] The no-go theorem of Nielsen-Ninomiya can also be avoided using Ginsparg-Wilson methods; namely domain wall and overlap fermions.

[32] Other approaches of interest are given in [18, 19].

[33] The discrete wedges of Birmingham-Rakowski and Albeverio-Schäfer satisfy this prop-
The complex dual to simplices is not simplicial, but hexagonal, for example, for which we do not have a Whitney map. We thus favour the hypercubic lattice whose dual is also hypercubic, and for which a Whitney map has been proposed [23].

An approach for contraction and symmetries by one of the authors V de B using GD will appear in a later paper.

One of the present authors; V de B, intends to address in more detail the issue of gauging in subsequent work.

V de B would like to thank Noboru Kawamoto for some discussions of these issues at Lattice 2003.