Extremal Cylinder Configurations II: Configuration $O_6$

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ABSTRACT

We study the octahedral configuration $O_6$ [Kuperberg] of six equal cylinders touching the unit sphere. We show that the configuration $O_6$ is a local sharp maximum of the distance function. Thus, it is not unlockable and, moreover, rigid.

1. Introduction

In the present article, we continue to study critical configurations of six infinite nonintersecting right circular cylinders touching the unit sphere in $\mathbb{R}^3$. We call a cylinder configuration critical if for each small deformation $t$ that keeps the radii of the cylinders, either

(T1) some cylinders start to intersect, or else

(T2) the distances between all of them increase, but by no more than

$$\sim ||t||^2,$$

or stay zero, for some. The norm $||t||$ is defined by formula (3) below.

A critical configuration is called a locally maximal configuration if all its deformations are of the first type. Any other critical configuration is called a saddle configuration, and the deformations of type (T2) are then called the unlocking deformations. This definition of a critical configuration is close in spirit to the definition of a critical point of the Morse function, but is adapted to our case of the function being non-smooth minimax function, compare with Definition 4.6 in [Kusner et al. 18].

For example, let $C_6$ be the configuration of six non-intersecting cylinders of radius 1, parallel to the $z$ direction in $\mathbb{R}^3$ and touching the unit ball centered at the origin. One of the results of [Ogievetsky and Shlosman 19a] is that the configuration $C_6$ is a saddle point configuration: there is a deformation of $C_6$ along which the unit cylinders cease touching each other; thus, the configuration $C_6$ can be unlocked. We note here that the structure of the critical point $C_6$ is complicated; in particular, the distance function $D$ (the minimum of the distances between the cylinders) is not even continuous at $C_6$, and the limits $\lim_{m \to C_6} D(m)$, $m \in M^6$, depend on the direction from which the point $C_6$ is approached. Here $M^6$ is the relevant configuration space, see the precise definition in Section 2.1.

We have constructed in [Ogievetsky and Shlosman 19a] the deformation $C_{6,x}$ of the configuration $C_6$. Moving along $C_{6,x}$ the common radius of cylinders grow when $x$ decreases from 1 to 1/2 ($x=1$ corresponds to the initial configuration $C_6$). For $x=1/2$ we obtain the configuration $C_m$, see Figure 1, for which the radius reaches its maximum value $\frac{1}{8}(3 + \sqrt{33})$.

In [Ogievetsky and Shlosman 18] we have shown that the configuration $C_m$ is a sharp local maximum of the distance function.

In the present article, we study the configuration $O_6$, comprised of the following six radius one cylinders:

- two cylinders are parallel to the $Oz$ axis and touch the sphere $S^2$ at points $(\pm 1, 0, 0)$ on the $Ox$ axis;
- two cylinders are parallel to the $Ox$ axis and touch the sphere $S^2$ at points $(0, \pm 1, 0)$ on the $Oy$ axis;
- two cylinders are parallel to the $Oy$ axis and touch the sphere $S^2$ at points $(0, 0, \pm 1)$ on the $Oz$ axis.
The letter “O” in the name of the configuration refers, probably, to the fact that the points at which the cylinders touch the sphere form the vertices of the regular octahedron. In a forthcoming publication [Ogievetsky and Shlosman 19b] we will give an interpretation of the configuration $O_6$ which rather relates it to the regular tetrahedron and suggest a generalization for dual pairs of Platonic bodies.

The configuration $O_6$ is centrally symmetric. There is a freedom in the definition of the configuration $O_6$: two cylinders, touching the sphere $S^2$ at points $(\pm 1,0,0)$ are parallel to the $z$-axis. Instead one can start with the two cylinders, touching the sphere $S^2$ at points $(\pm 1,0,0)$ but parallel to the $y$-axis; then add the two cylinders, touching the sphere $S^2$ at points $(0,\pm 1,0)$ but parallel to the $z$-axis and the two cylinders, touching the sphere $S^2$ at points $(0,0,\pm 1)$ but parallel to the $x$-axis. However, this configuration is obtained from $O_6$ by the rotation around an arbitrary coordinate axis through the angle $\pi/2$ or $-\pi/2$.

The configuration $O_6$ of cylinders is shown on Figure 2 (the green unit ball is in the center).

To visualize configurations of cylinders it is convenient to replace each cylinder by its unique generator (a line parallel to the axis of the cylinder) touching the sphere $S^2$. We define the value of the distance function on a configuration to be the minimum of distances between these tangent lines. The configuration $O_6$ of tangent lines is shown on Figure 3.

About the configuration $O_6$ W. Kuperberg was asking [Kuperberg 90] whether it can be unlocked, i.e., whether one can deform it in such a way that all the distances between the cylinders become positive. Our result is that the configuration $O_6$ is a sharp local maximum of the distance function $D$, and therefore is not unlockable. Moreover, it is rigid, that is, any continuous deformation, which does not decrease the radii of cylinders, reduces to a global rotation in the three-dimensional space.

In the process of the proof we will, in particular, show that the 15-dimensional tangent space to $M^6 \mod SO(3)$ at $O_6$ contains a six-dimensional subspace along which the function $D(m)$ decays quadratically, while along any other tangent direction it decays linearly.

As for the configuration $C_m$ it turns out that it is sufficient to study the variations of distances up to the second order.

For the configuration $O_6$ we distinguish twelve distances between the cylinders which are not parallel. Let $\tilde{D}(m)$ be the minimum of these twelve distances. We prove that the configuration $O_6$ is a sharp local maximum already of the function $\tilde{D}$.

We first show that there are three convex (i.e., having nonnegative coefficients) linear dependencies $\lambda_a$, $a = 1, 2, 3$, between the differentials of the twelve distances. We thus have a six-dimensional linear subspace $E$ of the tangent space on which all 12 differentials vanish. It so happens that in our coordinates the groups of coordinates entering these linear combinations are disjoint.

For the configuration $C_m$ there is one convex linear dependency between the differentials, see [Ogievetsky and Shlosman 18] and the restriction of same linear
combination $Q$ of the second differentials on the subspace, on which the differentials vanish, is negatively defined. We have shown in [Ogievetsky and Shlosman 18] that these conditions are sufficient for the local maximality. In the present article, we prove a generalization (of the above result for the configuration $C_m$) which allows us to make a conclusion about the local maximality of the configuration $O_6$. In this modification the negativity of the form $Q$ is replaced by the nonexistence of nontrivial solutions of the system of three inequalities $Q_d > 0$, $a = 1, 2, 3$, where $Q_d = \lambda_a(Q_{11}, \ldots, Q_{12})$. If there existed a convex linear combination of the forms $Q_d > 0$, $a = 1, 2, 3$, with negatively defined restriction on $E$, we could simply refer to the assertion made in [Ogievetsky and Shlosman 18]. However, this is not the case, see Section 3.3.

Let $Z$ be the configuration space of six-unit cylinders touching a unit ball. At his mathoverflow page [Kuperberg] Kuperberg asks, among other questions:

- Is the space $Z$ connected?
- In particular, within $Z$, is a continuous transition between the configurations $C_6$ and $O_6$ possible?

In the process of our proof of the local maximality of the configuration $O_6$, we establish that the configuration $O_6$ is an isolated point in the space $Z$ mod $SO(3)$ which implies the negative answer to these questions, see Corollary 3, Section 2.2. So, a modified question arises:

- How many components does the space $Z$ have?

The configuration $C_m$, in contrast to the configuration $O_6$, is not mirror symmetric. We make several conjectures concerning the components of the space $Z$ mod $O(3)$ and mod $SO(3)$.

The article is organized as follows. In the next section we recall notation, concerning the manifold $M^6$, formulate the maximality result and discuss the connected components of the space $Z$. Section 3 contains the calculation of the necessary differentials. In Section 4 we establish analytic results needed for the proofs of the local maximality of the configuration $O_6$.

2. Preliminaries

A cylinder $\varsigma$ touching the unit sphere $S^2$ has a unique generator (a line parallel to the axis of the cylinder) $i(\varsigma)$ touching $S^2$. We will usually represent a configuration $\{\varsigma_1, \ldots, \varsigma_L\}$ of cylinders touching the unit sphere by the configuration $\{i(\varsigma_1), \ldots, i(\varsigma_L)\}$ of tangent to $S^2$ lines. The manifold of all such six-tuples will be denoted by $M^6$.

Let $\varsigma', \varsigma''$ be two equal cylinders of radius $r$ touching $S^2$, which also touch each other, while $\varsigma', \varsigma''$ are the corresponding tangents to $S^2$. If $d = d_{\varsigma', \varsigma''}$ is the distance between $\varsigma', \varsigma''$ then we have

$$r = \frac{d}{2 - d}.$$

Indeed, when the cylinders touch each other, we have the proportion:

$$\frac{d}{1} = \frac{2r}{1 + r}.$$  

The study of the manifold of six-tuples of cylinders of equal radii, some of which are touching, is equivalent to the study the manifold $M^6$ and the function $D$ on it, defined by

$$D(t_1, \ldots, t_6) = \min_{1 \leq i < j \leq 6} d_{t_i, t_j}.$$  

2.1. Configuration manifold

Here we collect the notation of [Ogievetsky and Shlosman 19a].

Let $S^2 \subset \mathbb{R}^3$ be the unit sphere, centered at the origin. For every $x \in S^2$ we denote by $TL_x$ the set of all (unoriented) tangent lines to $S^2$ at $x$. We denote by $M$ the manifold of tangent lines to $S^2$. We represent a point in $M$ by a pair $(x, \xi)$, where $\xi$ is a unit tangent vector to $S^2$ at $x$, though such a pair is not unique: the pair $(x, -\xi)$ is the same point in $M$.

We shall use the following coordinates on $M$. Let $x, y, z$ be the standard coordinate axes in $\mathbb{R}^3$. Let $R_x^2, R_y^2$ and $R_z^2$ be the counterclockwise rotations about these axes by an angle $\alpha$, viewed from the tips of axes. We call the point $N = (0, 0, 1)$ the North pole, and $S = (0, 0, -1)$ — the South pole. By meridians we mean geodesics on $S^2$ joining the North pole to the South pole. The meridian in the plane $xz$ with positive $x$ will be called Greenwich. The angle $\alpha$ will denote the latitude on $S^2$, $\phi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, and the angle $\alpha \in [0, 2\pi)$ — the longitude, so that Greenwich corresponds to $\alpha = 0$. Every point $x \in S^2$ can be written as $x = (\alpha, x_z)$.

Finally, for each $x \in S^2$, we denote by $R_x^2$ the rotation by the angle $\alpha$ about the axis joining $(0, 0, 0)$ to $x$, counterclockwise if viewed from its tip.

Let $u = (x', x')$, $v = (x'', x'')$ be two lines in $M$. We denote by $d_{uv}$ the distance between $u$ and $v$; clearly $d_{uv} = 0$ iff $u \cap v \neq \emptyset$. If the lines $u, v$ are not parallel then the square of $d_{uv}$ is given by the formula
\[ d_{uv} = \frac{\det^2 [\zeta', \xi', X' - X']} {1 - (\zeta', \xi')^2}, \]  
(2-1)

where \((*, *)\) is the scalar product.

We are studying the critical points of the function

\[ D(m) = \min_{1 \leq i < j \leq N} d_{u_i u_j}, \]
on the manifold \(M^N\) of \(N\)-tuples

\[ m = \{ u_1, ..., u_N : u_i \in M, \ i = 1, ..., N \}. \]  
(2-2)

The norm which we mentioned in the Introduction is defined by

\[ ||u - v|| = ||x' - x'|| + \min \{ ||\xi' - \xi'||, ||\zeta' + \xi'|| \}. \]

(2-3)

### 2.2. Configuration \(O_6\)

We denote by \(e_i\) the orthonormal basis in \(\mathbb{R}^3\),

\[ e_1 \equiv e_x = (1, 0, 0), \quad e_2 \equiv e_y = (0, 1, 0), \quad e_3 \equiv e_z = (0, 0, 1). \]

Let \(g\) be the rotation of order 3, which cyclically permutes the vectors \(e_p\),

\[ g : e_1 \mapsto e_2 \mapsto e_3 \mapsto e_1, \]

\(I\) the central reflection,

\[ Iv = -v, \ v \in \mathbb{R}^3, \]

and \(r\) the rotation around the axis \(Ox\) by the angle \(\pi\),

\[ r : e_1 \mapsto e_1, \ e_2 \mapsto -e_2, \ e_3 \mapsto -e_3. \]

The maps \(g, I\) and \(r\) generate the group \(A_4 \times C_2\) where \(A_4\) is the alternating group on four letters, generated by \(g\) and \(r\), and \(C_2\) is the cyclic group of order 2, generated by \(I\).

Let \(\ell_j^+\) be the line in the direction \(e_j\) touching the unit sphere at the point \(e_1\) and let \(\ell_2^+ = g \ell_2^+, \ell_3^+ = g^2 \ell_1^+\). The images of the lines \(\ell_j^+, j = 1, 2, 3,\) under the central reflection \(I\) will be denoted by \(\ell_j^-\),

\[ \ell_j^- = I \ell_j^+, \ j = 1, 2, 3. \]

The six lines \(\ell_j^+, \ell_j^-, \ j = 1, 2, 3,\) form the configuration \(O_6\). The symmetry group of the configuration \(O_6\) is \(A_4 \times C_2\).

Let \(O_6(t)\) be a deformation of the configuration \(O_6\). We have 15 pairwise distances between the lines in \(O_6(t)\). There are three distances \(d_{\ell_j^+(t), \ell_j^-(t)}, j = 1, 2, 3,\) which do not have a well-defined limit when \(t \to 0\) because the lines \(\ell_j^+\) and \(\ell_j^-\) are parallel. The remaining twelve distances do have a well-defined limit, equal to 1, when \(t \to 0\). We shall first study these twelve distances. More generally, let \(m \in M^6\) be a point in a small enough neighborhood of \(O_6\) and let \(\ell_j^+, \ell_j^-, \ j = 1, 2, 3\) be the positions of perturbed lines \(\ell_j^+, \ell_j^-\), \(j = 1, 2, 3,\) Let

\[ \tilde{D}(m) := \min_{1 \leq i < j < 3, x', x''} \left( d_{\ell_i^+, \ell_j^-} \right). \]

(2-4)

**Theorem 1.** The configuration \(O_6\) is a point of a sharp local maximum of the function \(\tilde{D}\): for any point \(m\) in a vicinity of \(O_6\) we have

\[ \tilde{D}(m) < 1 = \tilde{D}(O_6). \]

We have \(\tilde{D}(O_6) = D(O_6) = 1\) and \(D(m) \leq \tilde{D}(m)\). This implies that the configuration \(O_6\) is locally maximal.

**Corollary 2.** The configuration \(O_6\) is a point of a sharp local maximum of the function \(D\).

In the process of the proof of Theorem 1, we will see that there exists a six-dimensional subspace \(L_{\text{quad}}\) in the tangent space to \(M^6\) mod \(SO(3)\) at \(O_6\), such that for any \(l \in L_{\text{quad}}\), \(||l|| = 1\), we have

\[ -c_d ||l||^2 \leq \tilde{D}(O_6 + tl) - \tilde{D}(O_6) \leq -c_d ||l||^2 \]

for \(t\) small enough. Here \(c_d\) and \(c_u\) are some constants, \(0 < c_d \leq c_u < +\infty\) and \(O_6 + tl \in M^6\) mod \(SO(3)\) stands for the exponential map applied to the tangent vector \(tl\).

For the tangent vectors outside \(L_{\text{quad}}\) we have

\[ -c'_d(l)t \leq \tilde{D}(O_6 + tl) - \tilde{D}(O_6) \leq -c'_d(l)t, \]

where now \(c'_d(l)\) and \(c'_u(l)\) are some positive valued functions of \(l\), \(0 < c'_d(l) \leq c'_u(l) < +\infty\).

### 2.3. Connected components

In [Ogievetsky and Shlosman 18] we have shown that the configuration \(C_m\) is a local maximum. Together with Theorem 1 we arrive at the following conclusion.

**Corollary 3.** The configuration space \(Z\) of six unit cylinders touching a unit ball is not connected.

**Proof.** Theorem 1 implies that the configuration \(O_6\) is an isolated point in the space \(Z\) mod \(SO(3)\). Hence the configurations

\[ \gamma(\varphi) = C_6(\varphi, \delta(\varphi), x(\varphi)), \varphi \in \left[ 0; \arcsin \left( \sqrt{\frac{5}{4}} \right) \right], \]

constructed in [Ogievetsky and Shlosman 19a], belong to another component of \(Z\) at \(\varphi = \arcsin \left( \sqrt{\frac{5}{4}} \right)\) the function \(D(\gamma(\varphi))\) gets back its initial value 1, see the end of Section 5 in [Ogievetsky and Shlosman 19a].
Remark. In contrast to the configuration $O_6$, the configuration $C_m$ is not congruent to its mirror image. To show this, we need several definitions.

A triple $T$ of straight lines is said to be in a generic position if there is no plane parallel to all three lines. A triple $T$ in a generic position defines an orientation of the space $\mathbb{R}^3$, or, if an orientation of $\mathbb{R}^3$ is given, a sign $\sigma(T)$. This sign is defined as follows. There is a unique hyperboloid $\mathcal{H}(T)$ of one sheet, passing through the three straight lines of $T$. Let the equation of the hyperboloid $\mathcal{H}(T)$, in its principal axes, be $z^2 = ax^2 + by^2 - r^2$. The hyperboloid $\mathcal{H}(T)$ has two families of rulings and the three lines of $T$ belong to the same family. So, viewed from a remote point on the $z$-axis (that is, a point $(0,0,z)$ with $z$ big enough), we will see a picture of the three lines of $T$ isotopic to the one shown on Figure 4. The isotopy class of the picture will not change if we are looking from the $z$-axis to the right picture the sign is $+$, we will see a picture of the three lines of $T$ isotopic to the one shown on Figure 5. The isotopy class of the picture will not change if we are looking from the $z$-axis to the right picture the sign is $-$. Therefore, the configuration on Figure 1, the sign of both triples is negative. For the configuration on Figure 1, the sign of both triples is negative. For the configuration on Figure 1, the sign of both triples is negative. For the configuration on Figure 1, the sign of both triples is negative.

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The configuration $C_m$ has a three-fold axis of symmetry. The action of the cyclic group $C_3$ on the set of six cylinders of $C_m$ has two orbits of length three. These are the brown and the red triplets of cylinders on Figure 1. The sign of both triples is negative. For the configuration, obtained by a central symmetry from the configuration $C_m$, the sign of the both reflected triples is positive. Therefore, the configuration $C_m$ is not congruent to its mirror image. Let us for the moment call $C^+_m$ the configuration on Figure 1, and $C^-_m$ its mirror image.

Let $Z(\geq R)$, respectively $Z(> R)$, denote the configuration space, mod $SO(3)$, of 6 cylinders, of radius bigger or equal to $R$, respectively, bigger than $R$, touching the unit ball.

In the space $Z(\geq 1)$, the configurations $C^+_m$ and $C^-_m$ are in the same connected component: one can move from $C^+_m$ to $C^-_m$ and then return to $C^-_m$.

Conjecture. The configurations $C^+_m$ and $C^-_m$ belong to different connected components of the space $Z(> 1)$.

The following observation might be helpful in testing this conjecture. There are twenty different triples of tangent lines in the configuration $C^+_m$. Among them there are twelve positive triples and eight negative triples. Therefore, in a motion from $C^+_m$ to $C^-_m$ some triples have to pass a nongeneric position; the formulas for the distances between the cylinders slightly simplify when there is a nongeneric triple.

Also, one can ask the following natural questions. Let $D$ be a configuration of nonoverlapping cylinders of the same radius. Supply each cylinder with an orientation. Let $p$ be a path in the space $Z(\geq R)$ or $Z(> R)$ which starts and ends with the configuration $D$. In general, the path $p$ might permute the cylinders or change their orientation. The first question is — what is the group of permutations and orientation changes induced by all possible such paths?

We conjecture that for the configuration $C_m$ the only permutations of oriented cylinders which can be achieved by a motion in the space $Z(\geq 1)$ are the rigid rotations from the dihedral group $D_3$.

The group generated by permutations and orientation changes of six cylinders is the wreath product $S_6 \wr C_2$. It is interesting to know what is the maximal radius $R$ of cylinders for which the whole group $S_6 \wr C_2$ is realizable by paths in $Z(\geq R)$.

Let us say that a subgroup $\mathcal{H}$ of $S_6 \wr C_2$ is path-realizable if there exists a configuration $D$ of six non-overlapping cylinders in $Z(\geq R)$ for some $R$, such that the elements of $S_6 \wr C_2$, realizable by paths in $Z(\geq R)$, form the subgroup $\mathcal{H}$. Which subgroups of $S_6 \wr C_2$ are path-realizable? For example, for twelve spheres of radius slightly larger than one, touching the unit sphere, it is known that the subgroup $A_{12}$ of $S_{12}$ is path-realizable, see Appendix to Chapter 1 in [Conway
and Sloane 13]. What is the maximal radius $R$ for each path-realizable subgroup of $\mathbb{G}_6 \cap \mathbb{G}_2$? Does this maximal radius $R$ depend on the connected component, to which the configuration $D$ belongs, of the configuration $Z(\geq R)$? Of course, these questions can be asked about any number of cylinders, not necessarily six.

3. Criticality of $O_6$

We shall study the deformed configuration $O_6(t)$, formed by the tangent lines

$$\ell_j^\varepsilon(t) = R^{\varphi_j}_{\varepsilon_0^{j'}} R^{\varphi_j}_{\varepsilon_0^{j'}} R^{\varphi_j}_{\varepsilon_0^{j'}} \ell_j^\varepsilon, \quad j = 1, 2, 3, \varepsilon \in \{+, -\}$$

where we write $R^{\varphi_j}_{\varepsilon_0^{j'}}$ (respectively, $R^{\varphi_j}_{\varepsilon_0^{j'}}$ and $R^{\varphi_j}_{\varepsilon_0^{j'}}$) for $R_x$ (respectively, $R_x^+$ and $R_x^-$).

To fix the rotational symmetry we keep the tangent line $\ell_1^t$ at its place, that is, $a_1^t = 0, b_1^t = 0$ and $c_1^t = 0$. The decompositions of the functions $a_j^t(t), b_j^t(t)$ and $c_j^t(t)$ start with $t^2$ and we denote the leading term of the function by the same letter, say, $a_j^t(t) = a_j^t = t + o(t)$.

We shall be studying the variations of twelve distances appearing in the function $D$, defined by the formula (2–4).

3.1. First differentials

First we directly calculate, by using the formula (2–1), the deformations of the squares of distances in the first order. For brevity, we shall write $[d_{uv}^2]$ for the coefficient at $t^2$ in the function $d_{uv}^2(t)$, where $u, v \in O_6(t)$. Here is the result:

$$[d_{ij}^2, d_{ij}^2]_1 = -2b_2, \quad [d_{ii}^2, d_{ii}^2]_1 = 2b_2^2,$$

$$[d_{ii}^2, d_{ii}^2]_1 = 2a_2, \quad [d_{ii}^2, d_{ii}^2]_1 = -2a_2,$$

$$[d_{jj}^2, d_{jj}^2]_1 = 2(b_2^2 - a_2^2), \quad [d_{jj}^2, d_{jj}^2]_1 = 2(a_2^2 - b_2^2),$$

$$[d_{ij}^2, d_{ij}^2]_1 = 2(b_1^2 - a_2^2), \quad [d_{ij}^2, d_{ij}^2]_1 = 2(a_2^2 - b_1^2),$$

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$$[d_{ij}^2, d_{ij}^2]_1 = 2(a_2^2 - b_2^2), \quad [d_{ij}^2, d_{ij}^2]_1 = 2(b_2^2 - a_2^2).$$

(3–1)

The expressions in the first two lines are shorter because the tangent line $\ell_1^t$ does not move.

The above differentials are not independent. All the linear relations between them are linear combinations of the following three relations:

$$[d_{ij}^2, d_{ij}^2]_1 + [d_{ii}^2, d_{ii}^2]_1 + [d_{jj}^2, d_{jj}^2]_1 = 0,$$

$$[d_{ij}^2, d_{ij}^2]_1 + [d_{ii}^2, d_{ii}^2]_1 + [d_{jj}^2, d_{jj}^2]_1 = 0,$$

$$[d_{ij}^2, d_{ij}^2]_1 + [d_{ii}^2, d_{ii}^2]_1 + [d_{jj}^2, d_{jj}^2]_1 = 0.$$
This is however not the case, and we do not have a geometrical interpretation neither of each group of equalities separately nor of the curious fact that there are exactly three linear dependencies between the first differentials.

### 3.2. Second differentials

We now consider the same combinations (3–2)–(3–4) but for the coefficients in \( t^2 \). Clearly, these combinations will contain only six parameters: \( \omega \) and all \( c_j, j = 1, 2, 3, \epsilon \in \{+, -\} \), except \( c_+^+ \), which is fixed to be zero. Explicitly (this is again a direct calculation using Mathematica [Wolfram Research, Inc. 18]) these combinations read

\[
\begin{align*}
Y_1 &= \frac{1}{2} \left( \left[ d_{i_1}^2 \right]_{i_2} + \left[ d_{i_3}^2 \right]_{i_2} + \left[ d_{i_4}^2 \right]_{i_2} \right) \\
&= c_1^+ c_2^+ - (c_3^+)^2 - c_1^+ c_2^- + 2c_1^+ \omega - 2\omega^2. \\
Y_2 &= \frac{1}{2} \left( \left[ d_{i_1}^2 \right]_{i_2} + \left[ d_{i_3}^2 \right]_{i_2} + \left[ d_{i_4}^2 \right]_{i_2} \right) \\
&= c_1^- c_2^- - (c_3^-)^2 - c_1^- c_2^+ - (c_3^-)^2, \\
Y_3 &= \frac{1}{2} \left( \left[ d_{i_1}^2 \right]_{i_2} + \left[ d_{i_3}^2 \right]_{i_2} + \left[ d_{i_4}^2 \right]_{i_2} \right) \\
&= c_2^- c_3^- + c_2^+ c_3^+ - c_2^- c_3^+ - c_2^- c_3^+ - (c_3^-)^2 - (c_3^+)^2.
\end{align*}
\]

(3–8)

The distances will not decrease in the second order only if

\[
Y_1 \geq 0, \quad Y_2 \geq 0, \quad Y_3 \geq 0.
\]

We will show now that the system (3–11) has only zero solution. We rewrite it in the form

\[
\begin{align*}
\omega^2 + (\omega - c_1^-)^2 \leq c_1^+ (c_2^+ - c_2^-), \\
(c_3^-)^2 + (c_3^+)^2 \leq c_1^- (c_3^- - c_3^+), \\
(c_2^-)^2 + (c_2^+)^2 \leq c_2^- (c_3^- - c_3^+) - (c_2^-)^2 - (c_2^+)^2.
\end{align*}
\]

The left-hand sides are nonnegative. Taking the product of the inequalities (3–12) and (3–13), we find

\[
(c_1^-)^2 (c_2^+ - c_2^-) (c_3^- - c_3^+) \geq 0.
\]

Assume that \( c_1^- \neq 0 \). Then \((c_2^+ - c_2^-)(c_3^- - c_3^+) \geq 0\). But (3–14) implies that \((c_2^+ - c_2^-)(c_3^- - c_3^+) \leq 0\). Therefore \( (c_2^+ - c_2^-)(c_3^- - c_3^+) = 0 \). Now it follows from (3–14) that \( c_2^- = c_2^+ = 0 \). Then (3–12) implies that \( c_1^- = 0 \).

Thus, we have checked that \( c_1^- \) must be 0. Now (3–12) implies that

\[
\omega = 0,
\]

(3–15) implies that

\[
c_3^- = c_3^+ = 0
\]

(3–16) and then (3–14) implies that

\[
c_2^- = c_2^+ = 0.
\]

(3–17)

The above computations show that along any path with tangent vector in the six-dimensional subspace (3–5)–(3–7) our function decays as \( t^2 \).

Together, equalities (3–5)–(3–7) and (3–15)–(3–17) show that order \( t^1 \) coefficients of all functions \( a_j^i(t), b_j^i(t) \) and \( c_j^i(t) \) vanish.

This is not, however, the end of the story, since we do not have uniform estimates on the lengths of all the paths entering into our argument. In general, it is possible that a \( C^\infty \) function decays along any analytic path starting at the origin, yet it increases along a nonanalytic path as the following example shows.

**Example.** Let us draw two graphs on the plane \( \mathbb{R}^2 \), of functions \( f_1(x) = e^{-\frac{1}{2}x} \) and \( f_2(x) = e^{-\frac{2}{2}x} \) for \( x \geq 0 \). An example is provided by an arbitrary \( C^\infty \) function in a vicinity of origin in \( \mathbb{R}^2 \) which increases in the horn between the graphs of \( f_1 \) and \( f_2 \) and decreases otherwise: for any analytic path, starting at the origin, there exists a duration when the path does not enter the interior of the horn.

So, to complete the argument we use the Theorem 8, Section 4.

### 3.3. Three forms

A straightforward calculation shows that each of the forms \( Y_1, Y_2, Y_3 \) has the matrix rank three.

If there existed a negatively defined strictly convex combination of three forms \( Y_1, Y_2, Y_3 \) then we could directly refer to Theorem 2, Section 5 of [Ogievetsky and Shlosman 18] to finish the proof of Theorem 1. Besides, an existence of such combination would give an easier proof of the statement that the system \( Y_1 \geq 0, Y_2 \geq 0 \) and \( Y_3 \geq 0 \) admits only a trivial solution. However, such combination does not exist as we will now show.

**Proposition 4.** There is no positively defined convex linear combination of the three forms \( (-Y_1), (-Y_2) \) and \( (-Y_3) \).

**Proof.** Let \( \tilde{\omega} = \omega - \frac{1}{2}c_1^- \). We have

\[
Y_1 = \tilde{Y}_1 - 2\tilde{\omega}^2,
\]

where
\[ \tilde{Y}_1 = c_1^- c_2^+ - \frac{1}{2} (c_1^-)^2 - c_1^- c_2^- . \]

The variable \( \tilde{\omega} \) is not involved in the forms \( Y_2 \) and \( Y_3 \). Therefore, a convex combination of the forms \( (\tilde{Y}_1), (Y_2) \) and \( (Y_3) \) is positively defined on the subspace with coordinates \( \{\tilde{\omega}, c_1^-, c_2^+, c_2^-, c_3^+, c_4^-\} \) if and only if the same convex combination of the forms \( (\tilde{Y}_1), (Y_2) \) and \( (Y_3) \) is positively defined on the five-dimensional subspace with coordinates \( \{c_1^-, c_2^+, c_2^-, c_3^+, c_4^-\} \). Thus it is sufficient to consider only this five-dimensional space.

Assume that a combination

\[ Y := - (\tilde{Y}_1 + x Y_2 + \beta Y_3), \]

where \( x, \beta > 0 \), is positively defined; without loss of generality we fixed the coefficient of the form \( (\tilde{Y}_1) \) to be 1.

In the coordinates \( \{c_1^-, c_2^+, c_2^-, c_3^+, c_4^-\} \) the form \( Y \) has the following Gram matrix:

\[
\begin{pmatrix}
1 & -1 & 1 & -x & x \\
-1 & 2\beta & 0 & \beta & -\beta \\
1 & 0 & 2\beta & -\beta & \beta \\
-x & \beta & -\beta & 2x & 0 \\
-x & \beta & -\beta & 0 & 2x
\end{pmatrix}.
\]

The Sylvester criterion says that the positivity of the form \( Y \) is equivalent to the following system of inequalities

\[
2\beta - 1 > 0, 4\beta(\beta - 1) > 0, -4\beta(2x - 4x\beta + x^2\beta + \beta^2) > 0, -16x\beta(x - 3x\beta + x^2\beta + \beta^2) > 0.
\]

(3–18)

Taking into account that the coefficients \( x \) and \( \beta \) are positive, the system (3–18) reduces to the system

\[
\beta > 1, 2x - 4x\beta + x^2\beta + \beta^2 < 0, x - 3x\beta + x^2\beta + \beta^2 < 0,
\]

which is incompatible. Already the first and the third inequalities are not compatible. Indeed, consider the left hand side

\[
m := x - 3x\beta + x^2\beta + \beta^2
\]

of the third inequality as the polynomial in \( x \). The quadratic polynomial \( m \) can take a negative value only if its roots are real. However, the discriminant of the polynomial \( m \) is

\[
-(\beta - 1)^2 (4\beta - 1),
\]

which is negative for \( \beta > 1 \). \( \square \)

### 4. Sufficient condition

Let \( \{F_1(x), ..., F_m(x)\} \) be a family of functions \( U \to \mathbb{R} \), where \( U \subseteq \mathbb{R}^n \) is a neighborhood of the origin \( 0 \in \mathbb{R}^n \), such that \( F_u(0) = 0, u = 1, ..., m \). We assume that the number of functions is not greater than the number of variables, \( m \leq n \).

For the configurations of tangent lines in Theorem 1 the functions \( F_u \) are the differences between the squares of distances in the perturbed and non-perturbed configurations.

We are studying the function

\[
F(x) := \min \{F_1(x), ..., F_m(x)\}.
\]

In [Ogievetsky and Shlosman 18] we have proved the local maximality of the configuration \( C_m \). For the configuration \( C_m \) there is exactly one convex linear dependency between the differentials of the functions \( F_u(x), u = 1, ..., m \), at the origin. We have given in [Ogievetsky and Shlosman 18] a sufficient condition ensuring that the point \( 0 \in \mathbb{R}^n \) is a sharp local maximum of the function \( F(x) \).

As we have seen in Section 3, the space of linear dependencies between \( dF_u(0), u = 1, ..., m \), is three-dimensional and has a basis consisting of three convex dependencies. Moreover, in our coordinates the groups of coordinates entering these linear combinations are disjoint.

In this section we establish an analytic result, Theorem 8, needed to complete the proof of Theorem 1. Theorem 8 is a sufficient condition, applicable to the configuration \( O_0 \), which ensures that the point \( 0 \in \mathbb{R}^n \) is a sharp local maximum of the function \( F(x) \). Theorem 8 is a generalization of Theorem 2, Section 5 in [Ogievetsky and Shlosman 18].

**4.1. Notation**

We recall some notation from [Ogievetsky and Shlosman 18]. Till the end of the Section the summation over repeated indices is assumed.

We denote by \( l_{ij} \) and \( q_{ijk} \) the coefficients of the linear and quadratic parts of the function \( F_u(x), u = 1, ..., m \),

\[
F_u(x) = l_{ij}x^i + q_{ijk}x^j x^k + o(2),
\]

where \( o(2) \) stands for higher order terms.

Let \( \xi^j := dx^j, j = 1, ..., n \), be the coordinates, corresponding to the coordinate system \( x^1, ..., x^n \), in the tangent space to \( \mathbb{R}^n \) at the origin. We define the linear and quadratic forms \( l_u(\xi) := l_{ij} \xi^j \) and \( q_u(\xi) := q_{ijk} \xi^j \xi^k \) on the tangent space \( T_0 \mathbb{R}^n \).

Let \( E \) be the subspace in \( T_0 \mathbb{R}^n \) defined as the intersection of kernels of the linear forms \( l_u(\xi) \),

\[
E = \bigcap_{u=1}^m \ker l_u(\xi).
\]
Let $\mu = \{\mu^1, ..., \mu^m\}$ be a linear dependency between the linear parts of the functions $F_u(x), u = 1, ..., m$, that is,

$$\mu^i l_{ij} = 0 \text{ for all } j = 1, ..., n.$$  

We denote by $q[\mu]$ the corresponding quadratic form on the space $E$ defined by

$$q[\mu] = \mu^i a_{ijk} s^i s^j |E|.$$  

4.2. Positively defined families of quadratic forms

We shall say that a family \{\Omega_1, ..., \Omega_L\} of quadratic forms on a real vector space $\mathbb{V}$ is positively defined if the following condition holds

the system of inequalities $\Omega_u(x) \leq 0, u = 1, ..., L,$ admits only the trivial solution $x = 0.$

(4–1)

Also, we say that a family \{\Omega_1, ..., \Omega_L\} of quadratic forms on a space $\mathbb{V}$ is negatively defined if the family \{- \Omega_1, ..., - \Omega_L\} is positively defined.

The notion of a positively defined family of quadratic forms generalizes the notion of a positively defined quadratic form (it corresponds to $L = 1$).

Let $\Omega(x) := \max(\Omega_1(x), ..., \Omega_L(x)).$

The condition (4–1) is satisfied if and only if the constant

$$v := \min_{|x| = 1} (\Omega(x))$$

is positive, $v > 0.$ Because of the homogeneity we have

$$\Omega(x) \geq v|\|x\|^2 \text{ for any } x \in \mathbb{R}^n.$$  

So, we can reformulate the positivity of a family in the following form.

Definition 5. A family \{\Omega_1, ..., \Omega_L\} of quadratic forms is positively defined if there exists a positive constant $v > 0$ such that for any $x \in \mathbb{R}^n$

there exists $a_0 \in \{1, ..., L\}$ such that $\Omega_{a_0}(x) \geq v|\|x\|^2.$

(4–2)

We shall say that such family \{\Omega_1, ..., \Omega_L\} is $v$-positively defined.

As for $L = 1,$ the positivity of a family of quadratic forms is an open condition in the following sense.

Lemma 6. The condition (4–1) is stable under small perturbations of the forms of the family.

Proof. Assume that a family \{\Omega_1, ..., \Omega_L\} of quadratic forms is positively defined and let $v$ be a constant from Definition 5.

Let $\mathbb{P}_{\mu}, u = 1, ..., L,$ be an arbitrary family of quadratic forms. There exists a positive constant $w$ such that

$$|\mathbb{P}_{\mu}(x)| \leq w|\|x\|^2, u = 1, ..., L.$$  

Given $x \in \mathbb{R}^n,$ let $a_0$ be the index defined by (4–2).

For a positive $\epsilon$ we have

$$|\Omega_{a_0}(x) + \epsilon \mathbb{P}_{\mu}(x)| \geq |\Omega_{a_0}(x)| - \epsilon |\mathbb{P}_{\mu}(x)|$$

$$\geq (v - \epsilon w)|\|x\|^2,$$

therefore, the family \{\Omega_u + \epsilon \mathbb{P}_{\mu}, u = 1, ..., L\} satisfies the condition of Definition 5 for $\epsilon$ small enough.

4.3. Analytic theorem

The particularity of the situation analyzed in Sections 3.1 and 3.2 can be described as follows. The family of functions \{F_1(x), ..., F_m(x)\} splits into several subfamilies

$$\mathcal{F}_1 = \{F_{1,1}(x), ..., F_{1,m}(x)\}, ...,$$

$$\mathcal{F}_L = \{F_{L,1}(x), ..., F_{L,m}(x)\}$$

such that:

(A) In each subfamily $\mathcal{F}_a$ there is exactly one linear dependency $\lambda_a$ between the linear parts of the functions in the subfamily, and this dependency is strictly convex for each subfamily.

(B) The set of variables $x^1, ..., x^n$ is a union of disjoint sets $X_a, a = 1, ..., L,$ and a set $\mathcal{Y}$ with the following property: the linear parts of functions from the subfamily $\mathcal{F}_a$ depend only on the variables from the set $X_a$ for each $a = 1, ..., L.$

(C) The family of quadratic forms $q[\lambda_a], a = 1, ..., L$ is negatively defined on the subspace $E.$

We shall use the following notation for the variables from the subsets $X_a$ and $\mathcal{Y}$:

$$X_a = \{x^1_a, ..., x^n_a\}, a = 1, ..., L, \text{ and } \mathcal{Y} = \{y^1, ..., y^d\}.$$  

In particular, $d_1 + \cdots + d_L + d = n.$ The variables $\{y^1, ..., y^d\}$ do not enter the linear parts of functions $\{F_1(x), ..., F_m(x)\}.$

Lemma 7. The conditions (A) and (C) are invariant under an arbitrary analytic change of variables, preserving the origin, such that the linear parts transform inside each group $X_a$ of variables.
Here each of matrices $A_{ao}$, $a = 1, ..., L$, is nondegenerate.

Proof. This is a straightforward generalization of the proof of Lemma 3, Section 5, in [Ogievetsky and Shlosman 18].

We now formulate our analytic theorem.

**Theorem 8.** Under the conditions (A), (B) and (C), the origin is the strict local maximum of the function $F(x)$.

**Remarks 1.** Our particular case of the configuration $O_6$ corresponds to $L = 3$; we have the following subfamilies of functions

$$F_1 = \left\{ \frac{d_2}{d_1^2}, \frac{d_3}{d_1d_2}, \frac{d_4}{d_1d_2}, \frac{d_5}{d_1d_2} \right\},$$

$$F_2 = \left\{ \frac{d_2}{d_1^2}, \frac{d_3}{d_1d_2}, \frac{d_4}{d_1d_2}, \frac{d_5}{d_1d_2} \right\},$$

$$F_3 = \left\{ \frac{d_2}{d_1^2}, \frac{d_3}{d_1d_2}, \frac{d_4}{d_1d_2}, \frac{d_5}{d_1d_2} \right\}.$$

Each subfamily contains four functions.

The property (A) refers to formulas (3–2)–(3–4). For the property (B) see expressions (3–1).

The property (C) is the statement about three quadratic forms $Y_1, Y_2, Y_3$ defined by formulas (3–8)–(3–10); we have checked in Section 3.1 that $\min(Y_1, Y_2, Y_3)|_E < 0$ everywhere except the origin.

2. Similarly to the case of the configuration $C_m$, it follows from the proof that the function $F(x)$ decays quadratically at zero along any direction in $E$ and decays linearly along any direction outside $E$.

As for Theorem 2, Section 5, from [Ogievetsky and Shlosman 18], we need the assertion of Theorem 8 for a family of analytic functions $F_\alpha$; however, a careful analysis shows that the assertion of Theorem 8 holds for functions $F_\alpha$ of the class $C^3$ (more precisely functions which differ from their second order decomposition by a term decreasing faster than $t^3$).

In [Ogievetsky and Shlosman 18] we have given two different proofs of Theorem 2. Here we present an analog of the first proof in [Ogievetsky and Shlosman 18]. A proof of Theorem 8 generalizing the second proof of Theorem 2 in [Ogievetsky and Shlosman 18] can be given as well, but it looks less natural and more involved, so we have decided to omit it.

### 4.4. Proof

We proceed as in the first proof of Theorem 2 in [Ogievetsky and Shlosman 18]. Performing, if necessary, a suitable change of variables, satisfying the conditions of Lemma 7, we may assume that each subfamily $F_{ao}, a = 1, ..., L$, consists of linear functions, except for the first one,

$$F_{ao,1} = -\sum_{i=2}^{m_a} i^2 x_{ai} + q_a(x) + o(2)$$

where $i^2 > 0$ for $i = 2, ..., m_a$.

$$F_{ao,2} = x_{ai}^{2}, \quad F_{ao,3} = x_{ai}^{3}.$$ The set of variables is split into two disjoint parts,

$$V_1 := \{ x_{ai}^{2} \}_{i=2}^{m_a} \quad \text{and} \quad V_2 := \{ x_{ai}^{3} \}_{i=1}^{m_a} \cup \{ y \}.$$

We rename, for convenience, the variables from $V_2 : V_2 = \{ y^{1}, ..., y^{k} \}$. We identify the points of the tangent subspace $E \subset T_{0} \mathbb{R}^{n}$ (in a small enough neighborhood $U$ of the origin) with the plane defined by $x^{a} = 0, x^{a} \in V_1$, and coordinatize the space $E$ by the variables $y \in V_2$.

We need to prove that in the set $E^{\perp}$ defined by the system of inequalities $x^{a} \geq 0, x^{a} \in V_1$, the function $F^{(1)}(x) = \min_{1 \leq a \leq L}(F_{ao,1}(x))$ has a sharp local maximum at the origin. We make a substitution, allowed in $E^{\perp}$, $x^{a} = (z^{a})^{2}, x^{a} \in V_1$. Our functions have the form

$$F_{ao,1} = \Omega_{ao} + \psi_{ao}, \quad \Omega_{ao} = -\sum_{i=2}^{m_a} i^2 (z_{ai}^{a})^{2} + q_{a}(y) \quad \text{and} \quad \psi_{ao} = o(2).$$

The family $\{ q_{a} \}$ is negatively defined on $E$ hence the family $\{ \Omega_{ao} \}$ is negatively defined on the space $\mathbb{R}^{n}$.

Due to the order of smallness of functions $\psi_{ao}$ there exists a positive constant $v > 0$ such that for any $x \in \mathbb{R}^{n}$ there exists $\alpha_{ao}(x) \in \{ 1, ..., L \}$ for which

$$\Omega_{ao}(x) \leq -2v ||x||^{2}. $$

Therefore, for any $x \in U$ we have $F_{ao}(x) \leq F_{ao,1}(x) \leq -v ||x||^{2}$. □

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