Double Well Potential: Perturbation Theory,
Tunneling, WKB
(beyond instantons)
Alexander V Turbiner

Abstract

A simple approximate solution for the quantum-mechanical quartic oscillator $V = m^2x^2 + gx^4$ in the double-well regime $m^2 < 0$ at arbitrary $g \geq 0$ is presented. It is based on a combining of perturbation theory near true minima of the potential, semi-classical approximation at large distances and a description of tunneling under the barrier. It provides 9-10 significant digits in energies and gives for wavefunctions the relative deviation in real $x$-space less than $\lesssim 10^{-3}$.
Anharmonic oscillator and double-well potential

Needless to say that since the creation of quantum mechanics the one-dimensional quartic anharmonic oscillator
\[ \mathcal{H} = -\frac{d^2}{dx^2} + m^2 x^2 + g x^4, \] (1)
always attracted a lot of attention being among the most celebrated problem of quantum mechanics. The interest to these problems ranges from various branches of physics, from quantum field theory to chemistry and biology. It is especially true for the case when \( m^2 < 0 \) and the potential has two minima. This problem is known in literature as the double-well potential. It was studied in hundreds papers, appeared practically in all books on quantum mechanics. A special emphasis was made to a domain \( m^2 \to \infty \) - a domain where the barrier penetration is described by instantons (see e.g. [1, 2, 3]) that gives rise to the instanton physics.

The first detailed study of (1) carried out by Bender-Wu at 1969-1973 [4] revealed in this seemingly simple Hamiltonian the extremely rich analytic structure which looks intrinsic for any non-trivial eigenvalue problem of quantum mechanics and even though for quantum field theory. In fact, one of the most important unwritten conclusions was that in no way this problem can be solved exactly. The goal of the present talk is to give an approximate solution valid, actually, for any \( g > 0 \) and \( m^2 \). The solution is given in a form of the fairly simple expression for the ground state and the first excited state wavefunctions, which for any real \( x \) and for \( g \geq 0, \) real \( m^2 \) differs from the exact wavefunction for not more than a small number \( \delta \),
\[ \left| \frac{\Psi_{\text{approximate}} - \Psi_{\text{exact}}}{\Psi_{\text{approximate}}} \right| \leq \delta. \]
In our case the \( \delta \approx 10^{-6} \). Evidently, it implies that any quantity related with the first two eigenstates like expectation values can be calculated with accuracy \( \delta^2 \).

There are three basic analytic approaches to study the spectra in quantum mechanics: (i) perturbation theory, (ii) WKB method and (iii) instanton calculus [15]. Each approach has its domain of applicability and usually these domains do not overlap. We attempt to combine (incorporate, unify) all three approaches into one by making interpolation. The most convenient object to incorporate (i) and (ii) is the logarithmic derivative of the wavefunction. While the suitable object to incorporate the property (iii) is the wavefunction. A final form of the approximation depends on a few free parameters. Roughly speaking, their behavior as a function of \( g, m^2 \) is rather smooth and simple. They can be fixed variationally, although it is not very important: a small variation of the parameters does not lead to dramatic loss of accuracy.
As a first step to approach the problem let us remind the Symanzik rescaling for eigenvalues and eigenfunctions

\[
E(m^2, g) = g^{1/3}E\left(\frac{m^2}{g^{2/3}}, 2\right), \quad \Psi(x; m^2, g) = \Psi\left(xg^{1/6}; \frac{m^2}{g^{2/3}}, 2\right).
\]

It manifests that the original problem (1) is in fact one-parametric. The Hamiltonian (1) can be rewritten in the form

\[
\mathcal{H} = -\frac{d^2}{dx^2} + ax^2 + 2x^4, \quad x \in (-\infty, +\infty),
\]

where \(a \equiv \frac{m^2 g^2}{3} \). This is the form of the Hamiltonian we are going to study with the real parameter \(a\) varying from negative to positive values. The Schrödinger equation for (2) reads

\[
-\frac{d^2\Psi}{dx^2} + ax^2\Psi + 2x^4\Psi = E\Psi, \quad \int_{-\infty}^{+\infty} |\Psi|^2 dx < \infty.
\]

Eigenfunctions of (3) are sharply changing functions in \(x \in \mathbb{R}\) being characterized by a power-like behavior at \(|x| \to 0\) and an exponentially-decaying one at \(|x| \to \infty\). Following the oscillation (Sturm) theorem the \(n\)th eigenfunction has \(n\) simple (real) zeros. It seems natural to introduce the representation for eigenfunctions as follows

\[
\Psi(x) = p_n(x)e^{-\varphi(x)},
\]

where the phase \(\varphi(x)\) is a slow-changing smooth function and \(p_n(x)\) is a polynomial of \(n\)th degree with real coefficients which has \(n\) real roots \([5]\). Recently, it was obtained a remarkable result \([6]\): Any eigenfunction \(\Psi(x)\) of (2) for any real \(a\) is entire function and it has infinitely-many simple complex zeros all situated on imaginary axis symmetrically! It implies that the phase \(\varphi(x)\) has infinitely-many logarithmic branch points in complex \(x\)-plane and has no singularities at real \(x\).

After substitution of (4) into (3) we get the following equation

\[
y' - y^2 - \frac{p'_n - 2yp_n}{p_n} = E - ax^2 - 2x^4, \quad y = \varphi' = (\log \Psi(x))', \quad y(0) = 0.
\]

In order to define the problem (5) we impose two conditions that \(y\) has no simple poles at real \(x\) (i) and it grows at \(|x| \to \infty\) not faster than polynomial (ii). The condition (i) implies that the coefficients of the polynomial \(p_n(x)\) are those that the residues in the simple poles in the third term in l.h.s. of (5) vanish. The condition (ii) assures square-integrability of the wavefunction (4). It is evident that the polynomial \(p_n(x)\) has parity \(p = (-1)^n\), hence, can be written as \(p_n(x) = x^pP_{\frac{1}{2}}(x^2)\), and \(y(x)\) is odd, \(y(x) = y(-x)\) (for discussion, see \([7]\)).
From the analysis of (5) it is easy to find asymptotic behavior of the phase \[16\],
\[
\varphi = \frac{2^{1/2}}{3} x^2 |x| + \frac{a}{2^{3/2}} |x| + (n + 1) \log |x| + \frac{8E + a^2}{2^9/2} \frac{1}{|x|} + \left( \frac{a}{2} - 4A \right) \frac{1}{x^2} + \ldots \quad \text{at} \quad |x| \to \infty ,
\]
where \(A\) is sum of squared of nodes, while
\[
\varphi = \frac{E}{2} x^2 + \frac{E^2 - a}{12} x^4 + \frac{2E(E^2 - a) - 6}{90} x^6 + \ldots \quad \text{at} \quad |x| \to 0 ,
\]
(see \[8\]). It is important to note that the first two terms in (6) are defined by the equation (5) with omitted \(y'\) term, which is actually the Hamilton-Jacobi equation. Therefore, these two terms coincide with first two terms of the asymptotics of the classical action at \(|x| \to \infty\). The third term in (6) is also reproduced in the expansion of the classical action but with a wrong coefficient. The correct coefficient can be obtained if the first correction to the classical action is taken into account (quadratic fluctuations). The first three terms in (6) grow when \(|x|\) tends to \(\infty\). In fact, they characterize a singularity at \(|x| = \infty\). These terms do not depend on energy \(E\) and are found explicitly. For the single well potential \(a \geq 0\) the expansion (7) is nothing but the (divergent) perturbation theory series near the minimum of the potential. It is an expansion around the true vacuum.

Now let us construct a simplest function for phase (4) which interpolates small and large distance expansions, reproducing exactly the first three (growing) terms in (6). It has the form,
\[
\varphi_{\text{int}} = \frac{A + (D^2 + 3a)x^2 + 4x^4}{6(D^2 + 2x^2)^{1/2}} - \frac{n + 1}{2} \log (D^2 + 2x^2) .
\]
The corresponding wavefunction (4) for \(n \equiv (2k + p)\) excited state (for \(k = 0, 1, \ldots\) and parity \(p = 0, 1\)) is equal to
\[
\psi_{0}^{(k,p)} = \frac{x^p P_k(x^2)}{(D^2 + 2x^2)^{k+1/2}} \exp \left\{ - \frac{A + (D^2 + 3a)x^2 + 4x^4}{6(D^2 + 2x^2)^{1/2}} \right\} ,
\]
(cf. \[8\] at \(k = 0\)), where \(A, D\) are free parameters and \(P_k\) is a polynomial with real coefficients having positive roots only. There are two ways to find the polynomial \(P_k\): either imposing the orthogonality conditions to the functions with smaller quantum numbers \(k = 0, 1, \ldots (k - 1)\), or requiring the absence of simple poles in the third term in l.h.s. of (5). Surprisingly, in concrete calculations these two conditions lead to polynomials whose coefficients coincide with high accuracy. Each function \(\psi_{0}^{(k,p)}\) is characterized by two free parameters \(A, D\), which can be fixed if the function (9) is taken as variational trial function. If \(k = 0\) in (9) both ground (\(p = 0\)) and the first excited (\(p = 1\)) states occur, respectively. Concrete calculations of the variational energies of these states
with (9) taken as a trial function lead to unprecedented accuracy \(8\) (see below). Furthermore, if in the exponential in (9) in the term \(4x^4\) (this term governs the asymptotic behavior of the wave function at large distances where the wave function is exponentially small) the factor 4 is replaced by a parameter, \(Wx^4\), minimization of such a trial function leads to a value of \(W\) which is equal to 4 with accuracy \(10^{-4}\)!

In the case \(a < 0\) the vacuum at \(x = 0\) becomes the false vacuum and two classically degenerate vacua at \(x_\pm = \pm a^{1/2}/2\) appear. The expansion (7) becomes the expansion around false minimum (maximum), it is not relevant physically. The expansion around one or another true vacuum should be considered instead, as a relevant one. This expansion can be easily derived and we skip it.

In this case a new physical phenomenon of the quantum mechanical tunneling (barrier penetration) occurs. There is a probability to meet the particle under the barrier, near \(x \approx 0\). It decays exponentially when \(a \to -\infty\). This phenomenon is absent in (9). A prescription how to describe tunneling it is given in the celebrated Landau-Lifschitz book \(9\). It employs a linear superpositions of the wavefunctions centered at different minima which has positive parity for the ground state and similar one of negative parity for the first excited state. From the viewpoint of construction of the interpolation it is similar to making an interpolation between the expansion at one minimum and another one, and at \(|x| \to \infty\). Finally, such an interpolating function is a linear superposition of two off-centered functions (9), which can be written as

\[
\psi_0^{(k,+)} = q^+_k(x^2) \frac{\cosh \alpha_+ x}{(D^2_+ + 2x^2)^{k+\frac{1}{2}}} \exp \left\{ - \frac{A_+ + (D^2_+ + 3a)x^2 + 4x^4}{6(D^2_+ + 2x^2)^{1/2}} \right\},
\]

for the states of positive parity and

\[
\psi_0^{(k,-)} = q^-_k(x^2) \frac{\sinh \alpha_- x}{(D^2_- + 2x^2)^{k+1}} \exp \left\{ - \frac{A_- + (D^2_- + 3a)x^2 + 4x^4}{6(D^2_- + 2x^2)^{1/2}} \right\},
\]

for the states of negative parity. Parameter \(\alpha_+ (\alpha_-)\) ‘measures’ a displacement of the peaks of wave function from the origin, \(x = 0\). If \(\alpha_+ = 0\) the function (10) becomes (9). Here \((q^\pm_k)\) is a polynomial of degree \(k\) with real coefficients having positive roots only. At fixed \(k\) any function depends on three free parameters \(\alpha, A, D\).

In order to proceed further I need to remind two important and poorly known results: (i) a special form of perturbation theory in QM sometimes called ‘Logarithmic Perturbation Theory’ or ‘Non-linearization Method’, and (ii) a connection between variational calculation and perturbation theory.
(i) A special form of perturbation theory is a certain iterative procedure developed for solving the Riccati equation (5) instead of the Schrödinger equation. For finding the wave function it is a multiplicative perturbation theory unlike a standard additive Rayleigh-Schrödinger perturbation theory. Such a multiplicative perturbation theory was developed for the first time by Price [10] and then it was numerously rediscovered (for early history and discussion see [7] and references therein). In presentation we follow closely to [11]. For simplicity we will consider the eigenstates for which the nodes are absent (ground state) or nodal positions are known.

As a first step to develop the perturbation theory we make a choice of some square-integrable function \( \Psi_0 \) and calculate its logarithmic derivative
\[
y_0 = (\log \Psi_0)' = \frac{\Psi_0'}{\Psi_0}.
\]
(12)
It is clear that \( \Psi_0 \) is the exact eigenfunction of the Schrödinger operator with a potential
\[
V_0 = \frac{\Psi_0''}{\Psi_0} = y_0^2 - y_0',
\]
(13)
where without a loss of generality we put their eigenvalue equals to zero, \( E_0 = 0 \). It is nothing but a choice of the reference point for eigenvalues. Now we can construct a perturbation theory for Riccati equation taking \( \Psi_0 \) and \( y_0, V_0 \) as zero approximation, which characterizes the unperturbed problem. One can write the original potential \( V = m^2 x^2 + gx^4 \) as a sum,
\[
V = V_0 + (V - V_0) \equiv V_0 + V_1,
\]
(14)
thus, taking a deviation of the original potential from the potential of the zero approximation as a perturbation. We always can insert a formal parameter \( \lambda \) in front of \( V_1 \) and develop a perturbation theory in powers of \( \lambda \),
\[
E = \sum_{k=0}^{\infty} \lambda^k E_k, \quad y = \sum_{k=0}^{\infty} \lambda^k y_k,
\]
(15)
putting \( \lambda = 1 \) afterwards. Perhaps, it is worth emphasizing that in spite of the fact that we study iteratively the equation (5), in general, this perturbation series has nothing to do with a standard WKB expansion. By substituting (15) into (5) we arrive at the equations which defines iteratively the corrections
\[
y_k' - 2y_0y_k = E_k - Q_k,
\]
(16)
where
\[
Q_1 = V_1,
\]
\[
Q_k = - \sum_{i=1}^{k-1} y_i \cdot y_{k-i}, \quad k = 2, 3, \ldots.
\]
It is interesting that the operator in the l.h.s. of (16) does not depend on \( k \), while \( Q_k \) in the r.h.s. can be interpreted as a perturbation on the level \( k \). The solution of (16) can be found explicitly and is given by

\[
E_k = \frac{\int_{-\infty}^{\infty} Q_k \Psi_0^2 \, dx}{\int_{-\infty}^{\infty} \Psi_0^2 \, dx},
\]

\[
y_k = \Psi_0^{-2} \int_{-\infty}^{x} (E_k - Q_k) \Psi_0^2 \, dx'.
\]

It is easy to demonstrate that if the first correction \( y_1 \) is bounded,

\[
|y_1| \leq \text{Const},
\]

it provides a sufficient condition for this perturbation theory (15) to be convergent \([7]\). Note that this condition is very rough and very likely can be strengthened.

(ii) The first two terms in the expansion of energy (15) in the above-described perturbation theory admit an interpretation in the framework of the variational calculus \([5]\). Let us assume that our variational trial function \( \Psi_0(x) \) is normalized to 1. We can calculate the potential \( V_0 \) where \( \Psi_0(x) \) is the ground state eigenfunction and even put \( E_0 = 0 \) (see a discussion above). Formally, we construct the Hamiltonian \( H_0 = p^2 + V_0 \) for which \( H_0 \Psi_0(x) = 0 \). The variational energy is equal to

\[
E_{\text{var}} = \int \psi_0 H \psi_0 = \left[ \int \psi_0 H_0 \psi_0 + \int \psi_0 (H - H_0) \psi_0 \right]_{\substack{E_0 \quad \text{=} E_0 \quad \text{=} E_1 \quad \text{=} E_{\text{exact}}} \substack{V - V_0 \quad \text{=} E_1}} = E_0 + E_1 (V_1 = V - V_0) \geq E_{\text{exact}}.
\]

Of course, \( \Psi_0(x) \) could depend on free parameters. In this case both \( V_0 \) and \( V_1 \) depend on parameters as well. Minimization of \( E_{\text{var}} \) with respect to the parameters can be performed and the variational principle guarantees that \( E_{\text{var}} \) gives upper bound to the ground state energy. This simple interpretation \([20]\) reveals a fundamental difference between perturbation theory and variational calculus. Variational estimates can be obtained independently on the fact that the perturbation theory associated with trial function \( \Psi_0(x) \) is convergent or divergent. However, it seems natural to remove this difference by requiring a convergence of the perturbation series. In this case by calculating the next terms \( E_2, E_3, \ldots \) in (15) one can estimate the accuracy of variational calculation from one side and improve it iteratively from another side. An immediate criteria how to choose \( \Psi_0(x) \) in order to get a convergent perturbation theory is to have the perturbation potential \( V_1 \) to
be subordinate with respect to the non-vanishing potential of zero approximation $V_0$,

$$\left| \frac{V_1}{V_0} \right| < 1, \quad \text{for } |x| > R .$$

(21)

From this point of view any function (9), (10) or (11) taken as an entry leads to a convergent perturbation theory. An open question is how to estimate the radius of convergency.

The requirement (21) has a non-trivial physical implication: in order to guarantee a convergence of perturbation theory a domain where the wavefunction is exponentially small (classically-prohibited domain) should be reproduced as precise as possible. The same time a description of a domain where the wavefunction is of the order 1 is not important. It contradicts to a straightforward physics intuition and underlying idea of variational calculus which, in particular, requires a precise description of the domain where the wavefunction is of the order 1. Needless to say that namely the latter domain gives a dominant contribution to the integrals which define the energy in the variational calculations. Similar conclusion was presented in (12).

**Results.**

**Ground state**

$$a = 1$$

This is the case of a single-well potential (anharmonic oscillator). The variational parameters in (10) are

$$D = 4.33441 , \quad A = -9.23456 , \quad \alpha = 2.74573 .$$

Ground state energy is

$$E_{\text{var}} \equiv E_0 + E_1 = 1.607541302594 ,$$

while the first correction to it is

$$\Delta E_{\text{var}} \equiv E_2 = -1.2552 \times 10^{-10} .$$

Eventually,

$$\bar{E}_{\text{var}} = E_{\text{var}} + \Delta E_{\text{var}} = 1.607541302469 ,$$

(22)

where all 13 digits are correct, since the next correction $E_3 \sim 10^{-14}$. The rate of convergency seems extremely high, $\sim 10^{-4}$! The dependence the energy on $\alpha$ is very weak: the results are almost unchanged if $\alpha = 0$ (and (10) becomes (9)). It is the case for $a \geq 0$. 

FIG. 1: Logarithmic derivative $y_0 = \varphi_{int}'$ (see (8)) as function of $x$ for double-well potential (2) with $a = -1$

\begin{align*}
a = -1
\end{align*}

This is already the case of a double-well potential. The variational parameters in (10) are

\begin{align*}
D &= 4.059888 \ , \ A = -12.4816 \ , \ \alpha = 3.07041 .
\end{align*}

In comparison with the case $a = 1$ the parameters $D, \alpha$ are slightly changed unlike the parameter $A$ which decreased in $\sim 40\%$. Ground state energy is

\begin{align*}
E_{\text{var}} \ (\equiv E_0 + E_1) &= 1.029560832093 ,
\end{align*}

while the first correction to it is

\begin{align*}
\Delta E_{\text{var}} \ (\equiv E_2) &= -1.0382 \times 10^{-9} .
\end{align*}

Eventually,

\begin{align*}
\tilde{E}_{\text{var}} = E_{\text{var}} + \Delta E_{\text{var}} &= 1.029560831054 \ , \quad (23)
\end{align*}

where all 13 digits are correct, since the next correction $E_3 \sim 10^{-13}$. Similar to the case $a = 1$ the rate of convergency seems extremely high, $\sim 10^{-4}!$

On Fig.1 one can see the behavior of logarithmic derivative $y_0$ vs $x \geq 0$. It is a very smooth function. The first correction $|y_1|$ (see Fig.2) has very interesting behavior: it is of the order of $10^{-4}$ at $x \lesssim 1$ (in the domain which give a dominant contribution to the energy integral (20)), then it starts to grow and reaches the maximum $\sim 0.006$ at $x \sim 3.9$ in the domain which gives negligibly small contribution to the energy integral, $y_0(x = 3.9) \sim 15.2$ and $\Psi_0(x = 3.9) \sim 10^{-9}$. In this case $\delta \sim \max |y_1| \sim 0.006$ (see p.1). Notably, $y_1 \propto 1/x^2$ at $x \gg 1$. Similar behavior is demonstrated by the higher corrections $|y_n|, n = 2, 3$: they are very small at $x \lesssim 1$, its maximum is reached for $x > 1$ but its position is systematically reduced with $n$. 
Let us consider the Hamiltonian (1). The ground state function is symmetric w.r.t. \( x \to -x \). Hence, it has to have an extremum at \( x = 0 \). For any fixed \( g \) there exists a value \( m_{\text{crit}}^2 < 0 \) such that for \( m^2 > m_{\text{crit}}^2 \) this extremum is a maximum, otherwise a minimum. It is easy to find out that the critical point \( m_{\text{crit}}^2 \) corresponds to the vanishing ground state energy, \( E = 0 \) [17]. Using the function [10] it was calculated the critical value \( m_{\text{crit}}^2 = -2.2195970861 \) for \( g = 1 \). For the Hamiltonian [2]

\[
E(a_{\text{crit}} = -3.523390749) = 0 .
\]

It is quite interesting from physical point of view that for a family of double-well potentials with fixed \( g \) there exists a domain \( 0 > m^2 > (m^2)_{\text{crit}} \) where the ground-state eigenfunction has the maximum at the origin, which corresponds to the position of the unstable equilibrium similar to what takes place for the single-well case. It implies that the particle in such a potential with the ground state energy above the barrier, \( E > 0 \), somehow does not feel the existence of two minima. In this domain WKB consideration at \( x \sim 0 \) is not valid.

\[
a = -20
\]

This is the case of a double-well potential. The variational parameters in [10] are

\[
D = 6.765663 ,
A = -286.6456 ,
\alpha = 49.6136 .
\]

Ground state energy is

\[
E_{\text{var}} (\equiv E_0 + E_1) = -43.7793127 ,
\]

while the first correction to it is

\[
\Delta E_{\text{var}} (\equiv E_2) = -3.81 \times 10^{-6} .
\]
Eventually,

$$\tilde{E}_{\text{var}} = E_{\text{var}} + \Delta E_{\text{var}} = -43.7793165 \ ,$$

(24)

where all 9 digits are correct, since the next correction $E_3 \sim 10^{-8}$. Similar to the cases $a = \pm 1$ the rate of convergency seems still high, $\sim 10^{-2}$!

**First excited state**

$$a = -20$$

This is the case of a double-well potential. The variational parameters in (11) are

$$D = 5.584376 \ , \ A = -246.64375 \ , \ \alpha = 38.82768 \ .$$

The parameters seems quite close to those for $a = 20$ for the ground state. Ground state energy is

$$E_{\text{var}} (\equiv E_0 + E_1) = -43.77931637 \ ,$$

while the first correction to it is

$$\Delta E_{\text{var}} (\equiv E_2) = -9.3618 \times 10^{-8} \ .$$

Eventually,

$$\tilde{E}_{\text{var}} = E_{\text{var}} + \Delta E_{\text{var}} = -43.77931646 \ ,$$

(25)

where all 10 digits are correct, since the next correction $E_3 \sim 10^{-10}$. Similar to the cases $a = 20$ for the ground state the rate of convergency seems still high, $\sim 10^{-2}$!

**Energy Gap**

By definition the energy gap is

$$\Delta E = E_{\text{first excited state}} - E_{\text{ground state}} \ .$$

For the double-well potential (2) at $a \to -\infty$ it can be calculated and in one-instanton approximation it reads [9, 13, 14]

$$\Delta E = \frac{2^{11/4}}{\sqrt{\pi}} |a|^{5/4} e^{-\sqrt{2}|a|^{3/2} / 6} \left( 1 - \frac{71}{12} \frac{1}{\sqrt{2}|a|^{3/2}} - \frac{629}{288} \frac{1}{|a|^{3}} - \frac{269107}{10368} \frac{1}{2\sqrt{2}|a|^{9/2}} - \frac{2125346615}{497664} \frac{1}{4|a|^{16}} \cdots \right)$$

(26)

The first term in the expansion is a common knowledge. All other coefficients in the expansion are due to J Zinn-Justin, 1981-2005 (see [13] and references therein), they were obtained using the so called exact Bohr-Sommerfeld quantization condition. The coefficient $71/12$ was independently
calculated (and confirmed) in two-loop instanton calculation by E Shuryak (see [14] and references therein). It seems highly desirable to perform three-loop calculation to check the next coefficient in the expansion. It is an asymptotic expansion [13].

It is interesting to compare the energy gap \( \Delta E \) at \( a = -20 \) calculated in the (convergent) perturbation theory with (10) and (11) as zero approximations and with use of the (asymptotic) expansion (26). Subsequent expressions show how \( \Delta E \) evolves from pure variational results to ones with the first corrections \( E_2 \) taken into account and then to ones with the second corrections \( E_3 \) involved

\[
\Delta E_{\text{var}} = 1.03282 \times 10^{-7},
\]
\[
\Delta E^{(1)}_{\text{var}} = 1.06529 \times 10^{-7},
\]
\[
\Delta E^{(2)}_{\text{var}} = 1.06525 \times 10^{-7}.
\]  

In the last expression \( \Delta E^{(2)}_{\text{var}} \) all six significant digits are correct. Now how the energy gap \( \Delta E \) looks like as function of a number of corrections to one-instanton result included (see (26)) [18]:

- one – instanton = \( 1.12154 \times 10^{-7} \) (5.3\% deviation)
- one – instanton + 1st correction = \( 1.06908 \times 10^{-7} \) (0.36\% deviation)
- one – instanton + 1st and 2nd corrections = \( 1.06754 \times 10^{-7} \) (0.22\% deviation)
- one – instanton + four corrections = \( 1.06738 \times 10^{-7} \) (0.20\% deviation) \( (27) \)

The numbers in brackets are relative deviations from (27). The fact that a deviation stays almost the same after adding 2nd, 3rd and 4th corrections likely indicates the maximal accuracy based on a use of asymptotic expansion is reached and in any moment the result can blow up. A comparison of two values (27) and (28) shows that they do not agree in the 4th digit.

In a conclusion I have to say that a similar consideration based on interpolation of phase between small and large distances was done for sextic oscillator and the Zeeman effect on hydrogen. In both cases the exceptionally high accuracies were obtained.

Acknowledgments

Author thanks E. Shuryak, A.I. Vainshtein, J.C.L. Vieyra for valuable discussions. The work is
Anharmonic oscillator and double-well potential

supported in part by grants: CONACyT 58942-F, DGAPA IN115709-3.

[1] A.M. Polyakov, Nucl. Phys. B122, 429 (1977)
[2] S.R. Coleman, Subnucl. Ser. 15, 805 (1979)
[3] M.A. Shifman, “Instantons in Gauge Theories” (World Scientific, Singapore, 1994)
[4] C.M. Bender, T.T. Wu, Phys. Rev. 184, 1231 (1969); Phys. Rev. D 7, 1620 (1973)
[5] A.V. Turbiner, Soviet Phys. – ZhETF 79, 1719-1745 (1980);
    JETP 52, 868-876 (1980) (English Translation)
[6] A. Eremenko, A. Gabrielov, B. Shapiro, Ann. Inst. Fourier, Grenoble 58, 603-624 (2008)
[7] A.V. Turbiner, Usp. Fiz. Nauk. 144, 35-78 (1984),
    Sov. Phys. - Uspekhi 27, 668-694 (1984) (English Translation)
[8] A.V. Turbiner, Letters in Mathematical Physics 74, 169-180 (2005)
[9] L.D. Landau and E.M. Lifshitz, Quantum Mechanics, Pergamon Press (Oxford - New York -
    Toronto - Sydney - Paris - Frankfurt), 1977
[10] P.J. Price, Proc. Phys. Soc. London 67, 383 (1954)
[11] A.V. Turbiner, Soviet Phys. – Pisma ZhETF 30, 379-383 (1979).
    JETP Lett. 30, 352-355 (1979) (English Translation)
[12] R.P. Feynman, ‘Difficulties In Applying The Variational Principle To Quantum Field
    Theories’, in PROCEEDINGS of Int. Workshop on Variational Calculus in Quantum Field
    Theory, Wangerooge, West Germany, Sept. 1-4, 1987
    (World Scientific, Singapore, 1987) pp. 28-40
[13] J. Zinn-Justin and U.D. Jentschura, Annals Phys. 313, 269-325 (2004); quant-ph/0501137
    (updated, January 2005)
[14] C.E. Wöhler, E. Shuryak, Phys. Lett. B 333 (1994) 467–470
[15] It must be noted that, in fact, any one-dimensional Schroedinger equation can be solved
    numerically with any desirable accuracy. However, it is not true for multidimensional case.
[16] The phase is defined up to additive constant which is fixed following a normalization of the
    wavefunction. We will omit it.
[17] In classical case, it corresponds to the particle stopping on the top of the barrier.
[18] three and more instanton contributions are negligibly small for $a = -20$