Approximate counting with a floating-point counter

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Abstract

Memory becomes a limiting factor in contemporary applications, such as analyses of the Webgraph and molecular sequences, when many objects need to be counted simultaneously. Robert Morris [Communications of the ACM, 21:840–842, 1978] proposed a probabilistic technique for approximate counting that is extremely space-efficient. The basic idea is to increment a counter containing the value $X$ with probability $2^{-X}$. As a result, the counter contains an approximation of $\log n$ after $n$ probabilistic updates stored in $\log \log n$ bits. Here we revisit the original idea of Morris, and introduce a binary floating-point counter that uses a $d$-bit significand in conjunction with a binary exponent. The counter yields a simple formula for an unbiased estimation of $n$ with a standard deviation of about $0.6 \cdot n^{2^{-d/2}}$, and uses $d + \log \log n$ bits.

We analyze the floating-point counter’s performance in a general framework that applies to any probabilistic counter, and derive practical formulas to assess its accuracy.

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1 Introduction

An elementary information-theoretic argument shows that $\lceil \log_2(n + 1) \rceil$ bits are necessary to represent integers between 0 and $n$ ($\log$ denotes binary logarithm throughout the paper). Counting some interesting objects in a data set thus takes logarithmic space. Certain applications need to be more economical because they need to maintain many counters simultaneously while, say, tracking patterns in large data streams. Notable examples where memory becomes a limiting factor include analyses of the Webgraph [1] [3]. Numerous bioinformatics studies also require space-efficient solutions when searching for recurrent motifs in protein and DNA sequences. These frequent sequence motifs are associated with mobile, structural, regulatory or other functional elements, and have been studied since the first molecular sequences became available [6]. Some recent studies have concentrated on patterns involving long oligonucleotides, i.e., “words” of length 16–40 over the 4-letter DNA alphabet, revealing potentially novel regulatory features [5] [10], and general characteristics of copying processes in genome evolution [2] [11]. Hashtable-based indexing techniques [9] used in homology search and genome assembly procedures also rely on counting in order to identify repeating sequence patterns. In these applications, billions of counters need to be handled, making implementations difficult in mainstream computing environments. The need for many counters is aggravated by the fact that the counted features often have heavy-tailed frequency distributions [2, 3, 11], and there is thus no “typical” size for individual counters that could guide the memory allocation at the outset. As a numerical example, consider a study [2] of the 16-mer distribution in the human genome sequence, which has a length surpassing three billion. More than four billion ($4^{16}$) different words need to be counted, and the counter values span more than sixteen binary magnitudes even though the average 16-mer occurs only once or twice.

One way to greatly reduce memory usage is to relax the requirement of exact counts. Namely, approximate counting to $n$ is possible using $\log \log n + O(1)$ bits with probabilistic techniques [1] [8]. The idea of probabilistic counting was introduced by Morris [8]. In the simplest case, a counter is initialized as $X = 0$. The counter is incremented by one at the occurrence of an event with probability $2^{-X}$. The counter is meant to track the magnitude of the true number of events. More precisely, after $n$ events, the expected value of $2^X$ is exactly $(n + 1)$.

A generalization of the binary Morris counter is the so-called $q$-ary counter with some $r \geq 1$ and $q = 2^{1/r}$. In such a setup, the counter is incremented with probability $q^{-X}$. The actual event count is estimated as $f(X)$, using the transformation

$$f(x) = \frac{q^x - 1}{q - 1} = \frac{2^{x/r} - 1}{2^{1/r} - 1}.$$ 

The function $f$ yields an unbiased estimate, as $E f(X) = n$ after $n$ probabilistic updates. The accuracy of a probabilistic counting method is characterized by
the variance of the estimated count. For the $q$-ary counter,

$$\text{Var}(f(X)) = (q - 1) \frac{n(n + 1)}{2},$$

which is approximately $\frac{\ln 2}{2^r} n^2$ for large $n$ and $r$. The parameter $r$ governs the tradeoff between memory usage and accuracy. The counter stores $X$ (with $n = f(X)$) using $\lg r + \lg \lg n + o(1)$ bits; larger $r$ thus increases the accuracy at the expense of higher storage costs.

The main goal of this study is to introduce a novel algorithm for approximate counting. Our floating-point counter is defined with the aid of a design parameter $M = 2^d$, where $d$ is a nonnegative integer. As we discuss later, $M$ determines the tradeoff between memory usage and accuracy, analogously to parameter $r$ of the $q$-ary counter. The procedure relies on a uniform random bit generator $\text{RandomBit}()$. Algorithm FP-Increment below shows the incrementation procedure for a floating-point counter, initialized with $X = 0$. Notice that the first $M$ updates are deterministic.

| FP-Increment($X$) // returns new value of $X$ |
|---|
| 1 set $t \leftarrow \lfloor X/M \rfloor$ // bitwise right shift by $d$ positions |
| 2 while $t > 0$ do |
| 3 if RandomBit() = 1 then return $X$ |
| 4 set $t \leftarrow t - 1$ |
| 5 return $X + 1$ |

The counter value $X = 2^d \cdot t + u$, where $u$ denotes the lower $d$ bits, is used to estimate the actual count $f(X) = (M + u) \cdot 2^d - M$. The counter thus stores $X$ using $d + \lg \lg n + o(1)$ bits. The estimate’s standard deviation is $\frac{c}{\sqrt{M}} n$ where $c$ fluctuates between about 0.58 and 0.61 asymptotically (see Corollary 7 for a precise characterization). Notice that a $q$-ary counter with $r = M$ has asymptotically the same memory usage, and a standard deviation of about $\frac{0.59}{\sqrt{r}} n$ (see Eq. (1)). Our algorithm thus has similar memory usage and accuracy as $q$-ary counting. The floating-point counter is more advantageous in two aspects. First, the first $M$ updates are deterministic, i.e., small values are exactly represented with convenience. Second, the counter can be implemented with a few elementary integer and bitwise operations, whereas a $q$-ary counter works with irrational probabilities. The random updates in the floating-point counter occur with exact integer powers $2^{-i}$, and such random values can be generated using an average of 2 random bits. Specifically, the FP-Increment procedure uses an expected number of $2^d \cdot (t - \frac{1}{2^d - 1})$ calls to the random bit generator $\text{RandomBit}()$.

In contrast, a $q$-ary counter needs a uniform random number in the range $(0, 1)$ to produce a random event with probability $2^{-X/r}$.

The rest of the paper is organized as follows. In order to quantify the performance of floating-point counters, we found it fruitful to develop a general
analysis of probabilistic counting, which is of independent mathematical interest. Section 2 presents the main results about the accuracy of probabilistic counting methods. First, Theorem 1 shows that every probabilistic counting method has a unique unbiased estimator $f$ with $E(f(X)) = n$ after $n$ probabilistic updates. Second, Theorem 2 shows that the accuracy of any such method is computable directly from the counter value. Finally, Theorem 3 gives relatively simple upper and lower bounds on the asymptotic accuracy of the unbiased estimator. The proofs of the theorems are given in Section 3 which can be safely skipped on first reading. Section 4 presents floating-point counters in detail, and mathematically characterizes their utility by relying on the results of Section 2. Section 4 further illustrates the theoretical analyses with simulation experiments comparing $q$-ary and floating-point counters.

## 2 Probabilistic counting

For a formal discussion of probabilistic counting, consider the Markov chain formed by the successive counter values.

**Definition 1.** A counting chain is a Markov chain $(X_n: n = 0, 1, \ldots)$ with

\[ X_0 = 0; \quad \text{(2a)} \]
\[ P\{X_{n+1} = k + 1 \mid X_n = k\} = q_k \quad \text{(2b)} \]
\[ P\{X_{n+1} = k \mid X_n = k\} = 1 - q_k, \quad \text{(2c)} \]

where $0 < q_k \leq 1$ are the transition probabilities defining the counter.

It is a classic result associated with probabilities in pure-birth processes \[7\] that the $n$-step probabilities $p_n(k) = P\{X_n = k\}$ are computable by a simple recurrence (see Equations (8a–8b) later). In case of probabilistic counting, we want to infer $n$ from the value of $X_n$ alone through a computable function $f$. A given probabilistic counting method is defined by the transition probabilities and the function $f$. As we will see later (Theorem 1), the transition probabilities determine a unique function $f$ that gives an unbiased estimate of the update count $n$.

**Definition 2.** A function $f: \mathbb{N} \mapsto \mathbb{N}$ is an unbiased count estimator for a given counting chain if and only if $E(f(X_n)) = n$ holds for all $n = 0, 1, \ldots$.

In the upcoming discussions, we assume that the probabilistic counting method uses an unbiased count estimator $f$. The merit of a given method is gauged by its accuracy, as defined below.

**Definition 3.** The accuracy of the counter is the coefficient of variation

\[ A_n = \frac{\sqrt{\text{Var}(f(X_n))}}{E(f(X_n))} \]
The theorems below provide an analytical framework for evaluating probabilistic counters. Theorem 1 shows that the unbiased estimator is uniquely defined by a relatively simple expression involving the transition probabilities. Theorem 2 shows that the uncertainty of the estimate can be determined directly from the counter value. Theorem 3 gives a practical bound on the asymptotic accuracy of the counter.

**Theorem 1.** The function

\[ f(0) = 0 \]  
\[ f(k) = \frac{1}{q_0} + \frac{1}{q_1} + \ldots + \frac{1}{q_{k-1}}. \] \[ \{ k > 0 \} \]

uniquely defines the unbiased count estimator \( f \) for any given set of transition probabilities \((q_k: k = 0, 1, \ldots)\). Thus, for any given counting chain, we can determine efficiently an unbiased estimator.

Theorem 1 confirms the intuition that the transition probabilities must be exponentially decreasing in order to achieve storage on \( \lg \lg n + O(1) \) bits. Otherwise, with subexponential \( q_k^{-1} = 2^{o(k)} \), one would have \( f(k) = 2^{o(k)} \), leading to \( \lg n = o(k) \).

The next definition provides a computable function for quantifying the uncertainty of \( f(X) \).

**Definition 4.** The variance function for a given counting chain is defined by

\[ g(0) = 0 \]  
\[ g(k) = \frac{1 - q_0}{q_0^2} + \frac{1 - q_1}{q_1^2} + \ldots + \frac{1 - q_{k-1}}{q_{k-1}^2}. \] \[ \{ k > 0 \} \]

Theorem 2 below shows that the accuracy is computable directly from the counter value for any counting chain. The statement has a practical relevance (since count estimates can be coupled with the variance function’s value), and the variance function is used to evaluate the asymptotic accuracy of any counting chain (see Theorem 3).

**Theorem 2.** The variance function \( g \) of Definition 4 provides an unbiased estimate for the variance of \( f \) from Theorem 1. Specifically,

\[ \text{Var} f(X_n) = \mathbb{E} g(X_n) \]  
holds for all \( n \geq 0 \), where the moments refer to the space of \( n \)-step probabilities.

Theorem 3 is the last main result of this section. The statement relates the asymptotics of the variance function, the unbiased count estimator, and the counting chain’s accuracy.

**Theorem 3.** Let \( A_n \) be the accuracy of Definition 3, and let

\[ B_k = \frac{\sqrt{g(k)}}{f(k)}. \]  


Let \( \lim \inf_{k \to \infty} B_k = \mu \). Suppose that \( \lim \sup_{k \to \infty} B_k = \lambda < 1 \) (and, thus, \( \mu < 1 \)). Then

\[
\begin{align*}
\limsup_{n \to \infty} A_n & \leq \frac{\lambda}{\sqrt{1 - \lambda^2}} \quad (7a) \\
\liminf_{n \to \infty} A_n & \geq \frac{\mu}{\sqrt{1 - \mu^2}}. \quad (7b)
\end{align*}
\]

**Example**  Consider the case of a \( q \)-ary counter, where \( q_i = q^{-i} \) with some \( q > 1 \). Theorem 1 automatically gives the unbiased count estimator

\[ f(k) = \sum_{i=0}^{k-1} q_i^{-1} = \frac{q^k - 1}{q - 1}. \]

Theorem 2 yields the variance function

\[ g(k) = \sum_{i=0}^{k-1} (q_i^{-2} - q_i^{-1}) = \frac{q^{2k} - 1}{q^2 - 1} - \frac{q^k - 1}{q - 1}. \]

In order to use Theorem 3, observe that

\[ \lambda^2 = \lim_{k \to \infty} \frac{g(k)}{f^2(k)} = \frac{q - 1}{q + 1} < 1. \]

Therefore, we obtain the known result [4] that \( \lim_{n \to \infty} A_n^2 = \frac{\lambda^2}{1 - \lambda^2} = \frac{q - 1}{q + 1} \).

### 3 Proofs

In what follows, we use the shorthand notation

\[ p_n(k) = \mathbb{P}\{X_n = k\} \]

for the \( n \)-step probabilities. By [2], \( p_0(0) = 1 \), and the recurrences

\[
\begin{align*}
p_{n+1}(0) &= (1 - q_0)p_n(0) \quad (8a) \\
p_{n+1}(k) &= (1 - q_k)p_n(k) + q_{k-1}p_n(k - 1) \quad \{k > 0\} \quad (8b)
\end{align*}
\]

hold for all \( n \geq 0 \).

**Lemma 4.** *The unbiased estimator is unique.*

**Proof.** Since \( \mathbb{E}f(0) = 0 \) is imposed, and \( X_0 = 0 \) with certainty, \( f(0) = 0 \). For all \( n \), \( \mathbb{P}\{X_n > n\} = 0 \), so

\[ \mathbb{E}f(X_n) = \sum_{k=0}^{n} p_n(k) f(k) = n. \]
Thus, for all \( n > 0 \),
\[
f(n) = \frac{n - \sum_{k=0}^{n-1} p_n(k) f(k)}{p_n(n)} = \frac{n - \sum_{k=0}^{n-1} p_n(k) f(k)}{q_0 q_1 \cdots q_{n-1}},
\]
which shows that \( f(n) \) is uniquely determined by \( f(0), \ldots, f(n-1) \) and the \( n \)-step probabilities.

**Proof of Theorem 1.** Define the durations \( L_k(n) = \sum_{i=0}^{n-1} \{X_i = k\} \), i.e., the number of times \( X_i = k \) for \( i < n \). Define also \( L_k = \lim_{n \to \infty} L_k(n) = \sum_{i=0}^{\infty} \{X_i = k\} \). Clearly, \( \mathbb{E} L_k = 1/q_k \). By the linearity of expectations,
\[
\mathbb{E} L_k = \mathbb{E} L_k(n) + \mathbb{E} \sum_{i=n}^{\infty} \{X_i = k\}
= \mathbb{E} L_k(n) + \mathbb{E} \left[ \sum_{i=n}^{\infty} \{X_i = k\} \mid X_n \leq k \right] \mathbb{P}\{X_n \leq k\}
= \mathbb{E} L_k(n) + \mathbb{P}\{X_n \leq k\} \mathbb{E} L_k,
\]
where we used the memoryless property of the geometric distribution in the last step. Consequently,
\[
\mathbb{E} L_k(n) = \frac{\mathbb{P}\{X_n > k\}}{q_k}.
\]

Now,
\[
\mathbb{E} \sum_{k=0}^{\infty} L_k(n) = \sum_{k=0}^{\infty} \mathbb{P}\{X_n > k\} \frac{1}{q_k} = \sum_{k=0}^{n} p_n(k) \sum_{i=0}^{k-1} \frac{1}{q_i} = \sum_{k=0}^{n} p_n(k) f(k) = \mathbb{E} f(X_n).
\]

Since \( \sum_{k=0}^{\infty} L_k(n) = n \), we have \( \mathbb{E} f(X_n) = n \) for all \( n \). By Lemma 4, no other function \( f \) has the same property.

**Proof of Theorem 2.** By (8), for all \( n \geq 0 \),
\[
\mathbb{E} f^2(X_{n+1}) = \sum_{k=0}^{n+1} p_{n+1}(k) f^2(k)
= \sum_{k=0}^{n} (1 - q_k) p_n(k) f^2(k) + \sum_{k=1}^{n+1} q_k p_n(k) f^2(k)
= \mathbb{E} f^2(X_n) - \sum_{k=0}^{n} q_k p_n(k) f^2(k) + \sum_{k=1}^{n+1} q_k p_n(k - 1) (f(k - 1) + q_{k-1})^2
= \mathbb{E} f^2(X_n) + 2 \sum_{k=0}^{n} p_n(k) f(k) + \sum_{k=0}^{n} p_n(k) q_{k-1}^{-1}
= \mathbb{E} f^2(X_n) + 2n + \sum_{k=0}^{n} p_n(k) q_{k-1}^{-1}.
\]
Since \( \text{Var} f(X_n) = \mathbb{E}f^2(X_n) - \left( \mathbb{E}f(X_n) \right)^2 = \mathbb{E}f^2(X_n) - n^2 \),

\[
\text{Var} f(X_{n+1}) = \text{Var} f(X_n) + \sum_{k=0}^{n} p_n(k)q_k^{-1} - 1.
\] (10)

By (4) and (8),

\[
Eg(X_{n+1}) = \sum_{k=0}^{n+1} p_{n+1}(k)g(k)
\]

\[
= Eg(X_n) - \sum_{k=0}^{n} q_k p_n(k)g(k) + \sum_{k=1}^{n+1} q_{k-1} p_n(k-1) \left( g(k-1) + \frac{1 - q_{k-1}}{q_{k-1}} \right)
\]

\[
= Eg(X_n) + \sum_{k=0}^{n} p_n(k) \frac{1 - q_k}{q_k}.
\]

\[
= Eg(X_n) + \sum_{k=0}^{n} p_n(k)q_k^{-1} - 1.
\]

By (10), \( \text{Var} f(X_{n+1}) - \text{Var} f(X_n) = Eg(X_{n+1}) - Eg(X_n) \) holds for all \( n \geq 0 \).

Since \( \text{Var} f(X_0) = Eg(X_0) = 0 \), \( \text{Var} f(X_n) = Eg(X_n) \) holds for all \( n \).

Proof of Theorem 3: Define

\[
W_n = \frac{\text{Var} f(X_n)}{\mathbb{E}f^2(X_n)} = \frac{\sum_{k=0}^{\infty} p_n(k) \cdot g(k)}{\sum_{k=0}^{\infty} p_n(k) \cdot f^2(k)}
\]

Let \( \epsilon > 0 \) be an arbitrary threshold. By the definition of \( \lambda \), there exists \( K \) such that

\[
\frac{g(k)}{f^2(k)} < (1 + \epsilon)\lambda^2
\]

for all \( k > K \). Therefore,

\[
W_n = \frac{\sum_{k=0}^{K} p_n(k)g(k) + \sum_{k>K} p_n(k) \cdot g(k)}{\sum_{k=0}^{K} p_n(k)f^2(k) + \sum_{k>K} p_n(k)f^2(k)} < \frac{\sum_{k=0}^{K} p_n(k)g(k) + (1 + \epsilon)\lambda^2 \sum_{k>K} p_n(k)f^2(k)}{\sum_{k>K} p_n(k)f^2(k)}
\]

\[
= (1 + \epsilon)\lambda^2 + \frac{\sum_{k=0}^{K} p_n(k)g(k)}{\sum_{k>K} p_n(k)f^2(k)}.
\]

Since \( q_k > 0 \) for all \( k \), \( \lim_{n \to \infty} p_n(k) = 0 \) for all \( k \). Consequently, \( \lim_{n \to \infty} \sum_{k=0}^{K} p_n(k)g(k) = 0 \). As \( \lim_{n \to \infty} \sum_{k>K} p_n(k)f^2(k) = \infty \), there exists \( N \) such that

\[
W_n < (1 + 2\epsilon)\lambda^2 \quad \text{for all } n > N.
\] (11)
Since \( \text{Var} f(X_n) = \mathbb{E} f^2(X_n) - \mathbb{E}^2 f(X_n) \),
\[
W_n = \frac{\text{Var} f(X_n)}{\text{Var} f(x_n) + n^2}.
\]

By (11), \( \frac{\text{Var} f(X_n)}{n^2} \leq (1 + 2\epsilon)\lambda^2 \) for all \( n > N \). So,
\[
\frac{\text{Var} f(X_n)}{n^2} \leq \frac{(1 + 2\epsilon)\lambda^2}{1 - (1 + 2\epsilon)\lambda^2} = \frac{\lambda^2}{1 - \lambda^2 \left(1 + \frac{2\epsilon}{1 - (1 + 2\epsilon)\lambda^2}\right)}.
\]
Since \( \epsilon \) is arbitrarily small and \( \lambda^2 < 1 \),
\[
\limsup_{n \to \infty} \frac{\text{Var} f(X_n)}{n^2} \leq \frac{\lambda^2}{1 - \lambda^2}.
\]

The lower bound is proven analogously. Let \( \epsilon > 0 \) be an arbitrary threshold. Let \( K \) be such that \( \frac{q(k)}{f(k)} > (1 - \epsilon)\mu^2 \) for all \( k > K \). So,
\[
W_n > \frac{(1 - \epsilon)\mu^2 \sum_{k=0}^{K} p_n(k) f^2(k)}{\sum_{k=0}^{K} p_n(k) f^2(k) + \sum_{k>K} p_n(k) f^2(k)}.
\]
For \( n \) large enough, \( W_n > (1 - 2\epsilon)\mu^2 \) holds. Since \( \epsilon \) is arbitrarily small, and \( \mu^2 \leq \lambda^2 < 1 \),
\[
\liminf_{n \to \infty} \frac{\text{Var} f(X_n)}{n^2} \geq \frac{\mu^2}{1 - \mu^2}.
\]
\[
\square
\]

4 Floating-point counters

The counting chain for a floating-point counter is defined using a design parameter \( M = 2^d \) with some nonnegative integer \( d \):
\[
P\left\{ X_{n+1} = k + 1 \mid X_n = k \right\} = 2^{-\lceil k/M \rceil}; \quad (12a)
\]
\[
P\left\{ X_{n+1} = k \mid X_n = k \right\} = 1 - 2^{-\lceil k/M \rceil}. \quad (12b)
\]

Figure 1 illustrates the states of the floating-point counter. The counter’s designation becomes apparent from examining the binary representation of the counter value \( k \). Write \( k = Mt + u \) with
\[
t = \lceil k/M \rceil \quad u = k \mod M;
\]
i.e., \( u \) corresponds to the lower \( d \) bits of \( k \), and \( t \) corresponds to the remaining upper bits. Theorem 1 applies with \( q_k = 2^{-\lceil k/M \rceil} \), leading to the following Corollary.
Corollary 5. The unbiased estimator for \( k = Mt + u \) is

\[
    f(k) = f(t, u) = (M + u)2^t - M. \tag{13}
\]

In other words, \((t, u)\) is essentially a floating-point representation of the true count \(n\), where \(t\) is the exponent, and \(u\) is a \(d\)-bit significand without the hidden bit for the leading ‘1.’

Theorem 2 yields the following Corollary.

Corollary 6. The variance function for the floating-point counter is

\[
    g(k) = g(t, u) = \left(\frac{M}{3} + u\right)4^t - (M + u)2^t + \frac{2}{3}M. \tag{14}
\]

Combining Corollaries 5 and 6, we get the following bounds.

Corollary 7. The accuracy of the floating-point counter is asymptotically bounded as

\[
    \limsup_{n \to \infty} A_n \leq \sqrt[8]{\frac{3}{8M - 3}}
\]

\[
    \liminf_{n \to \infty} A_n \geq \sqrt[3]{\frac{1}{3M - 1}}
\]

Proof. By Equations (13) and (14), we have

\[
    \lim_{t \to \infty} \frac{g(t, u)}{f^2(t, u)} = \frac{M}{3} + u \quad \text{for} \quad (M + u)^2.
\]
Considering the extreme values at \( u = 0 \) and \( u = M/3 \), respectively:

\[
\mu^2 = \liminf_{k \to \infty} \frac{g(k)}{f^2(k)} = \frac{1}{3} M^{-1}; \quad \lambda^2 = \limsup_{k \to \infty} \frac{g(k)}{f^2(k)} = \frac{3}{8} M^{-1}. \quad (15)
\]

Plugging these limits into Theorem \text{} leads to the Corollary.

For large \( M = 2^d \), the bounds of Corollary \text{} become

\[
\limsup_{n \to \infty} A_n \lesssim 2^{-d/2} \sqrt{3/8} \approx 0.612 \cdot 2^{-d/2}
\]

\[
\liminf_{n \to \infty} A_n \gtrsim 2^{-d/2} \sqrt{1/3} \approx 0.577 \cdot 2^{-d/2}.
\]

The accuracy is thus comparable to the accuracy of a \( q \)-ary counter with \( q = 2^{2^{-d}} \), which is approximately \( 2^{-d/2} \sqrt{0.5 \cdot \ln 2} \approx 0.589 \cdot 2^{-d/2} \). The memory requirements of the two counters are equivalent: in order to count up to \( n = f(k) \), \( \lg k = d + \lg \lg n + o(1) \) bits are necessary.

Figures 2 and 3 compare the performance of the floating-point counters with equivalent base-\( q \) counters in simulation experiments. The equivalence is manifest on Figure 2 that illustrates the trajectories of the estimates by the different counters. Figure 3 plots statistics about the estimates across multiple experiments: the estimators are clearly unbiased, and the two counters display the same accuracy.
Figure 3: Distribution of the estimates for a floating-point counter (top) and a comparable q-ary counter (bottom). Each plot depicts the result of 1000 experiments, in which a floating-point counter with $d = 4$-bit mantissa, and a q-ary counter with $q = 2^{1/16}$ were run until $n = 100,000$. The dots in the middle follow the averages; the black segments depict the standard deviations (for each $\sigma$, they are of length $\sigma$ spaced at $\sigma$ from the average), and grey dots show outliers that differ by more than $\pm 2\sigma$ from the average. The shading highlights the asymptotic relative accuracy of the q-ary counter ($\approx 0.59 \cdot 2^{-d/2}$).
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