Microscopic theory of multi-stage Fermi surface reconstruction in heavy fermion systems with quartet multipolar local moments

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Recent experiments on Ce$_{3}$Pd$_{20}$(Si, Ge)$_{6}$ show novel quantum critical behaviors associated with two consecutive quantum phase transitions upon varying the external magnetic field. Interestingly, the derivative of the Hall conductivity shows a discontinuous jump at each phase transition, which was attributed to sequential Fermi surface reconstructions. Motivated by this discovery and previous theory work, we consider a microscopic model of itinerant electrons coupled to the local moments described by a quartet of ground states in a crystal-electric-field (CEF). Such a quartet arises due to two degenerate Kramers doublets of Ce$^{3+}$ ions in a cubic CEF and supports a large number of dipolar, quadrupolar, and octupolar moments. Specifically, we investigate emergent quantum phase transitions and criticality in a local effective model, the so-called Bose-Fermi Kondo model. This model describes the competition between the Kondo effect with the itinerant electrons and RKKY interaction for all of the 15 symmetry-allowed multipolar moments. Using renormalization group analyses, we demonstrate that a multitude of quantum phase transitions can occur depending on which multipolar moments participate in the Fermi surface formation and which other multipolar moments are decoupled via Kondo destruction. We provide a concrete example of two consecutive quantum phase transitions that involve the quadrupolar and dipolar/octupolar moments at two different stages. Our work provides an illuminating insight as to the importance of local symmetries in understanding multipolar Kondo lattice systems and an outlook for future directions.

I. INTRODUCTION

Experimental work on rare-earth metallic systems has shown a wide variety of quantum phases of matter and novel quantum phase transitions. In some systems, f-electrons give rise to multipolar moments, and these moments couple to itinerant conduction electrons: this situation is described by a multipolar Kondo lattice model. Since we have a large number of degrees of freedom and constraining crystal field symmetries, Kondo couplings become highly anisotropic in contrast to the conventional dipolar Kondo lattice model. Hence we can expect to find more diverse novel quantum phenomena such as the emergence of the new types of Kondo phases, RKKY mediated multipolar ordered phases [1–5], and novel quantum criticality from the competition between them [6–29]. Many of these phenomena are not yet well understood, and researchers face ongoing challenges to theoretically describe and experimentally detect these multipolar quantum phases [30–37]. For example, the Pr$^{3+}$ ion (4$f^2$ configuration) in Pr(Ti, V)$_2$Al$_{20}$ systems provides a non-Kramers doublet due to a cubic crystal electric field, supporting quadrupolar $\sim \{J_x^2 - J_y^2, 3J_x^2 - J_y^2\}$ and octupolar $\sim \{J_xJ_yJ_z\}$ moments [38–40]. This leads to a multipolar Kondo effect, anisotropic RKKY interactions responsible for multipole ordering, and unconventional superconducting behavior [16, 41–49]. An example of this multipolar Kondo effect even produces a novel multipolar non-Fermi liquid Kondo phase which is not classifiable into any multi-channel Kondo model [36, 50–52].

Another class of metallic systems which contain multipolar moments is Ce$_3$Pd$_{20}$(Si, Ge)$_6$. Here, the magnetically active Ce$^{3+}$ ions (4$f^1$ configuration) are surrounded by a cage of 16 Pd atoms, whose tetrahedral (Td) symmetry constrains the Ce$^{3+}$ ground state to be a fourfold degenerate quartet [53]. The states consist of two degenerate Kramers doublets, and these four states support a large number of the multipolar moments: 3 dipolar, 5 quadrupolar, and 7 octupolar moments (see Table I) [54].

Studies of Ce$_3$Pd$_{20}$Si$_6$ in particular show novel quantum critical behaviors corresponding to two consecutive field-induced quantum phase transitions [55, 56]. At zero magnetic field, the system exhibits coexisting antiferromagnetic ($\sim J_z$) and antiferroquadrupolar ($\sim 3J_x^2 - J_y^2$) order, and by increasing the external magnetic field along [001], the antiferromagnetic order disappears but the antiferroquadrupolar order is maintained. Upon increasing the field further, the system arrives at another phase that has not been clearly identified yet. Interestingly, the experiment observed that the derivative of the Hall conductivity, when it is extrapolated to zero temperature, jumps at both phase transitions [57–61]. This jump indicates sequential Fermi surface reconstruction [7, 58, 59, 62]. A recent study using a toy model provides a possible theoretical mechanism for this reconstruction by proposing an explanation via two-stage Kondo destruction [63]. While this illuminating study offers valuable insight, it relaxed microscopic details such as the explicit classification and identification of symmetry-allowed multipolar moments. Thus, understanding the microscopic origin of the two consecutive quantum phase transitions and the nature of the types of order in each phase remains elusive.

In this work, motivated by experiments on Ce$_3$Pd$_{20}$Si$_6$, we construct a multipolar Kondo lattice model of the 15
multipolar moments of the Ce\(^{3+}\) quartet (3 dipoles, 5 quadrupoles, 7 octupoles) coupled to \(p\)-wave conduction electrons, which may be regarded as molecular orbitals of the surrounding Pd\(_{16}\) cage. For simplicity, we consider a multipolar Bose-Fermi Kondo model which is the local approximation of the multipolar Kondo lattice model wherein the RKKY magnetic fluctuations are replaced by a dynamical bosonic bath [64–69]; the model now consists of the 15 multipolar moments coupled to both the fermionic (conduction electrons) and bosonic (RKKY) baths via the Fermi-Kondo and Bose-Kondo couplings, respectively. This model facilitates a study of the competition between the Kondo effect and magnetic ordering. Our goal is to determine the permissible types of magnetic order and Kondo destruction pathways purely on the basis of local symmetry. Specifically, we want to determine which local moments participate in the Fermi surface and which local moments are ordered and decouple from the conduction electrons in each part of the zero temperature phase diagram. We carry out a perturbative renormalization group (RG) analysis to two loop order by using an \(\epsilon\) expansion, with the density of states of the bosonic bath scaling as \(\sim |\omega|^{1-\gamma}\).

Our primary result is the identification of two consecutive quantum phase transitions. In the first transition, the system starts in a phase with coexisting quadrupolar \(\sim \{J_{2} - J_{z}^{2}, 3J_{2} - J_{z}^{2}\}\) and dipolar/octupolar \(\sim \{-J_{x,y,z} + \frac{2}{21}T_{x,y,z}^{\alpha}\}\) order (see Table II), and none of the local moments hybridize with the conduction bands. Then, the system transitions to a phase wherein the dipolar/octupolar moments are hybridized into the Fermi surface, causing its enlargement; the quadrupolar moments remain ordered and decoupled from the conduction bands. The second phase transition occurs as the quadrupolar moments are also hybridized into the Fermi surface, leading to a paramagnetic phase with an even larger Fermi surface. We emphasize that none of the ordered phases in our model correspond to spin or multipolar density waves, so the conduction electrons do not participate in the ordering. Experimental results on Ce\(_{2}\)Pd\(_{20}\)Si\(_{6}\) also show such a sequential Fermi surface reconstruction with a coexisting antiferromagnetic/antiferroquadrupolar order transitioning to a pure antiferroquadrupolar order transitioning to another ordered phase, with the Fermi surface enlarging at both stages as the external magnetic field strength is increased [55, 56]. This remarkable result demonstrates that, purely on the basis of local symmetry, profound conclusions can be drawn about the nature of magnetic ordering and Fermi surface reconstruction in the parent Kondo lattice problem.

The remainder of the paper is organized as follows. Because our paper contains many important results presented throughout the main text, a concise highlight of some key results is presented in Section II. Here, we expand on the aforementioned primary result, further explain physical interpretations, and introduce some other phases and phase transitions that our model exhibits. In Section III, we explain the microscopic origin of the quartet local moment and derive its couplings to the conduction electrons and bosonic bath. In Section IV, we perform the RG analysis and describe the stable phases and critical points of our models. Section V contains further comments on the observed phase transitions as well as a discussion of the multipolar susceptibilities at the different quantum critical points and their measurement. Finally in Section VI we discuss broad applications of our work and implications for other heavy fermion compounds.

### II. HIGHLIGHTS OF THE KEY RESULTS

In this section, we will expand on the consecutive quantum phase transitions mentioned in the introduction, as well as discuss some additional results of the model. The consecutive quantum phase transitions described in the introduction were discussed going from small Fermi surface to large Fermi surface to line up with the experimental procedure of starting at zero magnetic field and increasing it from there. However, we find it more intuitive to go in the reverse direction and start from a system with...
a large Fermi surface and tune the parameters such that the Fermi surface shrinks as local moments decouple from the conduction electrons. We will take this explanatory approach in the remainder of the paper. Let us briefly expand on the different types of phases we can observe in the model, which are described by stable fixed points of the RG flow. We ask the reader to refer to Fig. 1 while reading this section. First, phases labelled by $F_1$ are the Fermi-Kondo phases, which have some nonzero Fermi-Kondo couplings but all zero Bose-Kondo couplings (blue and indigo phases in Fig. 1). These phases have the largest Fermi surface, and no magnetic ordering. Second, we find one phase labelled $P$ which is a partially Kondo-destroyed phase (red phase in Fig. 1). Here, fewer of the Fermi-Kondo couplings are nonzero, but importantly there is also a nonzero Bose-Kondo coupling. This means that there is a medium sized Fermi surface, which coexists with magnetic ordering. Third, there are the fully Kondo destroyed phases, denoted by $B_1$ (green, yellow, orange phases in Fig. 1). Here, all of the Fermi-Kondo couplings are zero, but there are nonzero Bose-Kondo couplings and therefore correspond to magnetically ordered phases. In this final case, the local moments are completely decoupled from the conduction electrons and the Fermi surface is the smallest.

The model, which is derived in Section III, has two components: the Fermi-Kondo part and the Bose-Kondo part. In the Fermi-Kondo model, we uncover numerous Fermi-liquid and non-Fermi-liquid Fermi phases labelled by $F_i$, where the conduction electrons are actively coupled with the local multipolar moments; this signifies phases with a large Fermi surface. In Fig. 1, $F_1^*$ and $F_3$ are two examples of such fixed points. Let us revisit the paramagnetic phase mentioned in the introduction, $F_1^*$. This is a Kondo phase where all of the dipolar/octetual $\{-J_{x,y,z} \pm \frac{2}{\Pi} J_{x,y,z}^2\}$, quadrupolar $\{J_x^2 - J_y^2, 3J_z^2 - J^2\}$ moments, and pure octupolar moments $\{J_x J_y J_z\}$ are participating via Kondo hybridization in the formation of the Fermi surface. When passing through the critical point $C_{FP}$, at $P$ the pure octupolar moments decouple from the Fermi surface but do not undergo ordering. The quadrupolar moments also no longer participate in the Fermi surface, but they experience ordering, indicated by a flow to nonzero coupling between the $\{J_x^2 - J_y^2, 3J_z^2 - J^2\}$ moments and bosonic bath. The dipolar/octetual moments remain a part of the Fermi surface, so $P$ has a medium-sized Fermi surface. The quadrupolar ordering at $P$ is only possible because the parent phase $F_1^*$ had the quadrupolar moments hybridizing with conduction electrons and participating in the Fermi surface; one must start with a Kondo effect in order to destroy it. The second phase transition passes through the critical point $C_{BP}$ and destroys the dipolar/octetual moments as well, leading to the coexistence of dipolar/octetual ordering with the quadrupolar ordering inherited from $P$ (see Figs. 1,5). Now, at $B_{3M}$, the Fermi surface is made up purely of conduction electrons and the local moments are either ordered, as is the case for the quadrupolar and dipolar/octetual moments, or behave as spectators, as is the case for the pure octupolar moments.

Now that we have arrived at $B_{3M}$, we note from Fig. 1 that there are two other magnetically ordered phases possible ($B_1$ and $B_2$). These are the three magnetically ordered phases of the Bose-Kondo model, where the Kondo effect has been fully destroyed so the Fermi surface is at its smallest. The three phases are a mixed dipolar/octetual $\{-J_{x,y,z} \pm \frac{2}{\Pi} J_{x,y,z}^2\}$ ordered phase $B_1$, a pure octupolar $\{J_{x,y,z}^3\}$ ordered phase $B_2$, and finally the previously described phase $B_{3M}$ with coexistence of quadrupolar $\{J_x^2 - J_y^2, 3J_z^2 - J^2\}$ and mixed dipolar/octetual $\{-J_{x,y,z} \pm \frac{2}{\Pi} J_{x,y,z}^2\}$ orderings, (see Table I). We also find three critical points $C_{12,M1,M2}$ describing phase transitions between the magnetically ordered phases. These other magnetically ordered phases may be experimentally accessible by tuning other external parameters such as pressure.

In the full Bose-Fermi Kondo model, we not only find the $F_1^* \leftarrow C_{FP} \rightarrow P \leftarrow C_{BP} \rightarrow B_{3M}$ pathway of quantum phase transitions, but also quantum critical points between $B_{1,2}$ and Kondo phases, for example, $F_3$ (further examples are given in Sec. V A). At $F_3$, all of the 15 multipolar moments participate in the formation of a large Fermi surface. This means that, a priori, any magnetic order is possible since there are many Kondo effects available to be destroyed. The analysis reveals however that only $B_1$ (dipolar/octetual) and $B_2$ (pure octupolar) are stable descendants of $F_3$ (see Figs. 1,4). Since all of the Fermi-Kondo couplings completely vanish at multipolar ordered phases $B_{1,2}$, they have a small Fermi surface. These critical points indicate that our model contains the quantum phase transitions $F_3 \leftarrow C_{1,2} \rightarrow B_{1,2}$ associated with single Kondo destruction in addition to the two-stage Kondo destruction, depending on the microscopic details (which determine the initial conditions of the RG flow). The full schematic diagram is presented in Fig. 1. Further fixed points and transitions are listed in the main text.

### III. CONSTRUCTION OF THE MODELS

In this section, we begin with the description of the multipolar moments supported in the quartet ground state of the Ce$^{3+}$ ions in a cubic CEF. We then explain how to construct the conduction electron orbitals that couple to such multipolar moments. This leads to the Fermi-Kondo models, which can then be used to derive the most general bilinear RKKY interactions between multipolar moments in the parent lattice problem. Replacing one of the multipolar moment operators in the bilinear interactions by a dynamical bosonic field, we arrive at the appropriate Fermi-Bose Kondo problem. To construct the model, we consider the local symmetry at the Ce$^{3+}$ ion site in Ce$_2$Pd$_2$(Si, Ge)$_6$. For this family of materials specifically, there are two crystallographically
distinct sites for Ce ions: the 4a and 8c sites [55, 70, 71]. The magnetically active Ce ions occupy the 8c sites and are surrounded by a Pd$_{16}$ cage, which has a tetrahedral $T_d$ symmetry. The conduction electron wave functions are considered to be molecular orbitals centred on the Ce ion and constructed from electrons hopping on the Pd$_{16}$ cage. The resulting wave functions can be classified according to irreducible representations of $T_d$.

A. Multipolar moment

The degenerate ground states of an ion in a vacuum can be described by an effective higher-spin system through Hund’s rules. When the ion experiences a crystal electric field, the degenerate states then split. In Ce$_3$Pd$_{20}$(Si,Ge)$_6$, the Ce$^{3+}$ ion has a $4f^1$ configuration and resulting $J = 5/2$ moment. The surrounding $T_d$ crystal electric field yields a $Γ_8$ quartet ground state. The states of this quartet are listed in Appendix A. Since there are 4 degenerate ground states, numerous multipolar moments can be formed; in particular we have 3 dipolar, 5 quadrupolar, and 7 octupolar moments [54]. These moments are tabulated in Table I.

We classify the multipolar moments according to irreducible representations of $T_d$. All moments are parity even, the dipolar and octupolar moments are time-reversal odd, and the quadrupolar moments are time-reversal even. The $+/−$ subscripts on the $T_2$ irreps denote this time-reversal even/odd character. We do not include a $+/-$ subscript where there is no ambiguity of the time-reversal nature of the moment. Note that, despite the writing of these moments as generators of SU(4), the transformations of the operators under $T_d$ (and time-reversal) reflect the underlying multipolar properties of the operators.

In order to clean up the RG calculation, the multipolar operators are better expressed if we perform rescalings and linear combinations, as shown in Table II. Additionally, since there are 6 moments which transform as $T_1$, we give different labels to the different instances, $T_{1a}$ containing $S^1, S^2, S^3$, and $T_{1b}$ containing $S^{10}, S^{11}, S^{12}$ [54]. The linear combinations introduce mixed dipolar/octupolar (D&O) moments, and we label the quadrupolar and pure octupolar moments with the shorthands Q, O, respectively. The results of this rearrangement are enumerated in Table II. With this choice of linear combinations, these operators SU(4) also yield simpler structure constants which simplifies the RG analysis.

| Irrep. | Stevens | In terms of $J_x, J_y, J_z$ | Moment |
|--------|---------|-----------------------------|--------|
| $T_1$ | $J_x$   | $J_x$                        | D      |
| $T_1$ | $J_y$   | $J_y$                        | D      |
| $T_1$ | $J_z$   | $J_z$                        | D      |
| $E$   | $O_{22}$| $\sqrt{2}(J_x^2 - J_y^2)$   | Q      |
| $E$   | $O_{20}$| $\frac{1}{2}(3J_x^2 - J_y^2)$| Q      |
| $T_{2+}$ | $O_{yz}$| $\frac{\sqrt{3}}{2}J_yJ_z$ | Q      |
| $T_{2+}$ | $O_{zx}$| $\frac{\sqrt{3}}{2}J_xJ_y$ | Q      |
| $T_{2+}$ | $O_{xy}$| $\frac{\sqrt{3}}{2}J_xJ_y$ | Q      |
| $A_2$ | $T_{xyz}$| $\sqrt{15}J_xJ_yJ_z$       | O      |
| $T_1$ | $T_{α}^x$ | $\frac{1}{2}(2J_x^3 - J_yJ_z - J_y^2J_x)$ | O   |
| $T_1$ | $T_{α}^y$ | $\frac{1}{2}(2J_y^3 - J_xJ_z - J_x^2J_y)$ | O   |
| $T_1$ | $T_{α}^z$ | $\frac{1}{2}(2J_z^3 - J_xJ_y - J_x^2J_z)$ | O   |
| $T_{2−}$ | $T_{β}^x$ | $\frac{15}{2}(J_xJ_y^2 - J_y^2J_x)$ | O   |
| $T_{2−}$ | $T_{β}^y$ | $\frac{15}{2}(J_yJ_z^2 - J_z^2J_y)$ | O   |
| $T_{2−}$ | $T_{β}^z$ | $\frac{15}{2}(J_zJ_x^2 - J_x^2J_z)$ | O   |

TABLE I. Multipolar Moments, $J_x, J_y, J_z$ are $J = 5/2$ operators. The overline notation means full symmetrization. For example $\overline{AB} = AB + BA$, $\overline{AB}^2 = A^2B + ABA + BAA$, and $\overline{ABC} = ABC + ACB + BAC + BCA + CAB + CBA$. The irrep. column denotes irreducible representations of $T_d$, and the Stevens column contains Stevens operators. The $+$ subscripts on the $T_2$ moments denote the time reversal even/odd nature of the moments for quadrupole/octupoles. We do not include a $+/-$ label if there is no ambiguity. In the moment column, we indicate if the moment is dipolar (D), quadrupolar (Q), or octupolar (O).

B. Fermi-Kondo couplings

We construct a model of 3 degenerate bands of conduction electrons, made up of Wannier functions which lie in the $T_2$ irrep. of $T_d$. We may use $p$-wave \{$x, y, z$\} orbitals, or $d$-wave \{$yz, zx, xy$\} orbitals; the results are identical with either choice and we use $p$-wave in this work. The kinetic term for the conduction electrons is given by:

$$H_0^p = \sum_{knp} \xi_k c_{knp}^\dagger c_{knp},$$

where $\xi_k$ is the dispersion of the electrons, $n = x, y, z$ is the orbital index, and $\rho = \uparrow, \downarrow$ is the spin index. The electrons are then coupled to the SU(4) local moment, which leads to 15 coupling constants (this is unrelated to the fact that there are 15 multipolar moments). The couplings are four $T_{1a}$ irrep. couplings $K_{a(1,2,3,4)}$, four $T_{1b}$ irrep. couplings $K_{b(1,2,3,4)}$, two $E$ irrep. couplings $K_{E(1,2)}$, two $T_{2+}$ irrep. couplings $K_{2(α, β)+}$, one $A_2$ irrep. coupling,
In terms of Stevens notation, linear combinations and normalizations are chosen such that operators are quite lengthy and thus are relegated to Appendix D. 

In order to later perform the RG analysis, we write the multipolar spin operators in terms of Abrikosov pseudofermions. This is done by writing the operators as $S^i = \sum_{p=1}^{4} f_p^i \tilde{\Lambda}^i_{\rho r} f_r$, with $\tilde{\Lambda}^i$ being a $4 \times 4$ traceless Hermitian matrix which corresponds to the specific form of the multipolar operator. Not that the rescalings and linear combinations ensure that $\text{tr}(\Lambda^i \Lambda^j) = \delta_{ij}$. In order to restrict the Hilbert space of the impurity states to the original 4 states, we impose the single occupation condition $\sum_{p=1}^{4} f_p^i \tilde{\Lambda}^i_{\rho r} f_r = 1$. To ensure that the condition is satisfied, we introduce a chemical potential $\lambda$ for the pseudofermion, which is taken to $\lambda \to \infty$ at the end of the calculation [65].

### Table II. The multipolar moments used in the Hamiltonians, expressed in terms of $J = 5/2$ Stevens operators. The operators $S^i$, $i = 1, \ldots, 15$ are $4 \times 4$ traceless Hermitian matrices projected to acting on the quartet ground state. The linear combinations and normalizations are chosen such that $\text{tr}(\Lambda^i \Lambda^j) = \delta_{ij}$ where $\Lambda^i$ is the traceless $4 \times 4$ matrix defined by $\Lambda^i = \sum_{\rho, \tau = 1}^{4} (f_p^i \tilde{\Lambda}^i_{\rho r} f_r)$. We redefine and classify the six $T_1$ moments as $T_{1a}$ and $T_{1b}$ [54]. In the moment column, we indicate if the moment is a mixed dipolar/octupolar (D&O), quadrupolar (Q), or pure octupolar (O).

| Irrep. | Notation | In terms of Stevens | Moment |
|--------|----------|---------------------|--------|
| $T_{1a}$ | $S^1$ | $-\frac{1}{15} J_x + \frac{2}{90} T_x^\alpha$ | D&O |
| $T_{1a}$ | $S^2$ | $-\frac{1}{15} J_y + \frac{7}{90} T_y^\alpha$ | D&O |
| $T_{1a}$ | $S^3$ | $-\frac{1}{15} J_z + \frac{7}{90} T_z^\alpha$ | D&O |
| $E$ | $S^4$ | $\frac{1}{8} O_{2z}$ | Q |
| $E$ | $S^5$ | $\frac{1}{8} O_{20}$ | Q |
| $T_{2+}$ | $S^6$ | $\frac{1}{2} O_{yz}$ | Q |
| $T_{2+}$ | $S^7$ | $\frac{1}{2} O_{xz}$ | Q |
| $T_{2+}$ | $S^8$ | $\frac{1}{2} O_{xy}$ | Q |
| $A_2$ | $S^9$ | $\frac{1}{9\sqrt{5}} T_{yz}$ | O |
| $T_{1b}$ | $S^{10}$ | $-\frac{7}{15} J_x + \frac{2}{45} T_x^\alpha$ | D&O |
| $T_{1b}$ | $S^{11}$ | $-\frac{7}{15} J_y + \frac{2}{45} T_y^\alpha$ | D&O |
| $T_{1b}$ | $S^{12}$ | $-\frac{7}{15} J_z + \frac{2}{45} T_z^\alpha$ | D&O |
| $T_{2-}$ | $S^{13}$ | $\frac{1}{6\sqrt{5}} T_x^\beta$ | O |
| $T_{2-}$ | $S^{14}$ | $\frac{1}{6\sqrt{5}} T_y^\beta$ | O |
| $T_{2-}$ | $S^{15}$ | $\frac{1}{6\sqrt{5}} T_z^\beta$ | O |

TABLE III. Table of Fermi-Kondo couplings and bosonic bath couplings classified by irreducible representations. $K_i$ are the Kondo couplings and $g_i$ are the bosonic bath couplings. We have 15 Fermi-Kondo couplings and 6 Bose Kondo couplings. We also denote which multipolar moments the coupling constants correspond to.

| Irrep. | $K_i$ | $g_i$ | Moments |
|--------|-------|-------|---------|
| $T_{1a}$ | $K^{(1,2,3,4)}_g$ | $g_a$ | $S^1, S^{22}, S^3$ |
| $E$ | $K_{E(a,b)}$ | $g_E$ | $S^4, S^5$ |
| $T_{2+}$ | $K_{2(a,b)}^+$ | $g_{2+}$ | $S^6, S^7, S^8$ |
| $A_2$ | $K_A$ | $g_A$ | $S^9$ |
| $T_{1b}$ | $K_{(1,2,3,4)}^g$ | $g_b$ | $S^{10}, S^{11}, S^{12}$ |
| $T_{2-}$ | $K_{2(a,b)}^-$ | $g_{2-}$ | $S^{13}, S^{14}, S^{15}$ |

C. Bose-Kondo couplings

In the case of a lattice of multipolar moments, the spin bilinear RKKY interaction is induced by the conduction electrons. Capturing such an interaction in the Bose-Fermi Kondo model is done by introducing a bosonic bath, which can be thought of as a dynamical Weiss mean field. The details of deriving the symmetry-allowed bosonic couplings for the model are given in Appendix E. The result of the derivation is 6 couplings between the bosonic bath and local moment, which are $g_a, g_E, g_{2+}, g_A, g_b$, and $g_{2-}$ for local moments in the irreps $T_{1a}, E, T_{2+}, A_2, T_{1b}$, and $T_{2-}$, respectively. The coupling constants are summarized in Table III. The Hamiltonian for the kinetic part of the bosonic bath is shown here

$$H_B^0 = \sum_{i,k} \Omega_k \phi_k^i \phi_k^i,$$

where we assume that all flavors are degenerate. The index $i = 1, \ldots, 15$ runs over all bosonic baths, and $\Omega_k$ is the dispersion of the bosonic fields. In order to perform the RG analysis, we set up an $\epsilon$ expansion, where $\epsilon$ controls the sub-linearity of the spectral function of the bosonic bath:

$$\sum_k \left[ \delta(\omega - \Omega_k) - \delta(\omega + \Omega_k) \right] = \frac{N_{B}^{2}}{2} |\epsilon|^{1-\epsilon} \text{sgn}(\omega).$$

Because we assumed that all flavors of the bosonic bath were degenerate, they also all have the same $\epsilon$ controlling their densities of states.

To construct the full Fermi-Bose Kondo model, we simply add all the terms of the Fermi-Kondo contributions in Eqs. (1), (D1)-(D15) and Bose-Kondo contributions in Eqs. (2), (E1)-(E6). This yields a model with a grand total of $15 + 6 = 21$ coupling constants.
TABLE IV. A selection of stable fixed points for the Fermi-Kondo models. $F_\Delta$ stands for the stable fixed point with the scaling dimension $\Delta$, and $K^*_i$ stands for the fixed point value of $K_i$.

| Name | $(K_{a1}^*, K_{b1}^*, K_{a2}^*, K_{b2}^*, K_{a3}^*, K_{b3}^*, K_{a4}^*, K_{b4}^*, K_{E1}^*, K_{E2}^*, K_{2a1}^*, K_{2b1}^*, K_{a2}^*, K_{b2}^*)$ | $\Delta$ |
|------|-----------------------------------------------------------------|-----|
| $F_3$ | $\left(\pm \frac{\sqrt{5}}{3\sqrt{3}}, \frac{1}{3\sqrt{3}}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6\sqrt{3}}, -\frac{2}{3\sqrt{3}}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ | 3 |
| $F_2$ | $\left(\pm \frac{\sqrt{7}}{3\sqrt{3}}, -\frac{1}{3\sqrt{3}}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ | 2 |
| $F_1$ | $\left(\pm \frac{\sqrt{7}}{3\sqrt{3}}, -\frac{1}{3\sqrt{3}}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ | 1 |
| $F_1^{1/2}$ | $\left(\pm \frac{\sqrt{7}}{3\sqrt{3}}, -\frac{3}{18}, \frac{1}{3}, -\frac{1}{3}, 0, -\frac{2}{3\sqrt{3}}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ | 1/2 |

IV. RENORMALIZATION GROUP ANALYSIS

A. Dimensional regularisation and $\epsilon$ expansion

The renormalization group analysis is carried out by using dimensional regularization with minimal subtraction [65] in this work. In this scheme, the density of states of the conduction electrons and bosonic bath are parameterized by small factors $\epsilon$ and $\epsilon'$, respectively. This factor is already in place for the bosonic bath (see Eq. (3)), and we now introduce the $\epsilon'$ factor for the fermionic density of states:

$$\sum_k \delta(\omega - \xi_k) = N_0 |\omega|^{-\epsilon'}. \quad (4)$$

These $\epsilon, \epsilon'$ factors are used in the minimal subtraction of poles, and $\epsilon' \to 0$ at the end of the calculation. This necessitates defining a renormalized field $f$ and dimensionless coupling constants $g_i$ and $K_j$,

$$f^B = Z_f^{1/2} f, \quad (5)$$

$$g_i^B = g_i Z_f^{-1} Z_{g_i} \mu^{1/2}, \quad (6)$$

$$K_j^B = K_j Z_f^{-1} Z_{K_j} \mu^{\epsilon'}, \quad (7)$$

where $\mu$ is the renormalization energy scale, and $Z_f$, $Z_{g_i}$, and $Z_{K_j}$ are the renormalization constants for the pseudofermion $f$, bosonic couplings $g_i$ (here $i = a, b, 2 \pm E, A$), and fermionic couplings $K_j$ (here $j = a(1, 2, 3, 4), b(1, 2, 3, 4), 2(\alpha, \beta) \pm E(K_1, 2), A$). The superscript $B$ stands for the bare value which does not evolve under the RG flow. In addition, we absorb the density of states $N_i$ into the dimensionless couplings as $N_i K_j \to K_j$ and $N_i^2 g_i \to g_i$, respectively, in the following section. The details of the RG analysis and corresponding Feynman diagrams are enumerated in Appendix F.

B. Beta functions for Fermi-Kondo Hamiltonians

When we consider the Fermi-Kondo Hamiltonians only, the beta functions are quite lengthy and are given in Appendix G. From the beta functions, we find numerous stable fixed points, each of which corresponds to some low temperature phase of the high-energy Kondo models in Eqs. (D1)-(D15). We label the fixed points by $F_\Delta$, where $1 + \Delta$ is the scaling dimension of the leading irrelevant operator. Note that, in the Kondo problem, it is possible for an apparently stable fixed point of the RG flow to actually be unstable. This occurs when the strong coupling limit of the fixed point Hamiltonian is stable, or, equivalently, has a unique ground state. This indicates that the fixed point at intermediate coupling is in fact unstable, and should actually flow to strong coupling [72–74]. For the case of SU(N) $k$-channel Kondo problems, fixed points with unstable strong coupling limit correspond to non-Fermi liquid phases (for $k \geq N$), as shown through non-Abelian bosonization and conformal field theory approaches [75–80]. In this work, we classify the fixed points into Fermi liquid phases and non-Fermi liquid phases depending only on whether they have unique or degenerate ground states in the strong coupling limit because none of the points can exactly be mapped to an SU(N) $k$-channel Kondo model. A more rigorous classification may be provided in future works with nonperturbative techniques.

We find numerous stable fixed points from the Fermi-Kondo Hamiltonian, two of which we will discuss here. The remainder will be described in a future work [81]. A selection of fixed points are summarized in Table IV. However, in the present section, we focus on and present only the two fixed points $F_3$ and $F_3^1$ as representative examples. The descriptions of these two fixed point Hamiltonians are as follows:

(i) $F_3$: The fixed point Hamiltonian for $F_3$ corresponds to a fully SU(4) symmetric model as follows:

$$H_3 = \sum_{r=1}^{15} \sum_{\rho \tau} \psi^\dagger_{\rho r} A^i_{\rho r} \psi^i S^i, \quad (8)$$

Here, $\psi$ is a 4-component spinor. To bring the fixed point Hamiltonian to this form, we perform a change of basis for conduction electrons (for details, see Appendix H). $F_3$ has four degenerate ground states in the strong cou-
pling limit; thus the IR fixed point is valid. The lack of a unique strong coupling ground state is indicative of non-Fermi liquid behavior according to the criterion mentioned above, which should be contrasted with the ordinary single channel SU(4) symmetric model, which is a Fermi liquid with strong coupling singlet ground state [82, 83].

(ii) $F_1'$: The fixed point Hamiltonian for $F_1'$ corresponds to a 6 generator truncated SU(4) fixed point [84]. The fixed point Hamiltonian at $F_1'$ can be split as follows:

$$H_{F_1'} = \frac{1}{2} \sum_{\rho, \tau} \psi_{\rho}^\dagger (\sigma^0 \otimes \bar{\sigma})_{\rho\tau} \cdot (S^4, S^8, S^S) \psi_{\tau} + \sum_{\rho, \tau} c_1 \tilde{\psi}_{\rho \tau} \cdot (-S^{10} - S^{11} - S^{12}) c_2 \tilde{\psi}_{\tau}, \tag{9}$$

under the change of basis for the conduction electrons, where $\bar{\sigma} = (\sigma^+, \sigma^0, \sigma^-)$, $\tilde{\psi}$ and $c_1$ are 4 and 2 component spinors, respectively (see Appendix II for details). The multipolar moments in each line on the right hand side satisfy an SU(2) algebra, respectively, and the two SU(2) algebras are mutually commuting. This leads us to incorrectly consider the fixed point Hamiltonian as the sum of a previously suggested 6 generator truncated SU(4) fixed point [84]. Also, this fixed point Hamiltonian has two-fold degenerate ground states in the strong coupling limit. So, the IR fixed point is valid and likely shows non-Fermi liquid behavior.

C. Beta Functions for the Bose-Kondo Hamiltonians

The beta functions with only the bosonic bath coupling are given in Appendix I. Here, we find three distinct stable fixed points: $B_1$, $B_2$, and $B_M$. Their fixed point values are presented in Table V, and their properties are as follows:

(i) $B_1$: This fixed point has nonzero $g_b$ bosonic bath coupling, so it corresponds to a mixed dipolar and octupolar ordered fixed point with moments in the $T_1$ irrep. We can regard this as a dipolar ordered phase because the octupoles in $T_1$ cannot be practically distinguished from the dipoles; both the dipoles and octupoles transform the same way and therefore both couple linearly to an external magnetic field. $B_1$ only exists for $0 < \epsilon < 1/8$.

(ii) $B_2$: This point has nonzero $g_{2b}$ bosonic bath coupling, so it corresponds to an octupolar ordered fixed point with moments in the $T_{2-}$ irrep. $B_2$ only exits for $0 < \epsilon < 1/8$.

(iii) $B_M$: This point has nonzero $g_E$ and $g_b$ bosonic bath couplings, so it corresponds to a mixed multipolar ordered fixed point which has quadrupolar ordering and mixed dipolar/octetopolar ordering with $E$ and $T_1$ irrep. As with $B_1$, the octupolar component of the dipolar/octetopolar ordering cannot be practically separately detected, and may be regarded simply as dipolar order. $B_M$ exists for $0 < \epsilon < 1/4$, which is a wider range than the cases of $B_1$ and $B_2$. This means that $B_M$ is the only possible bosonic fixed point if $1/8 < \epsilon < 1/4$.

We also find critical points between the bosonic fixed points. The fixed point values of the critical points are presented in Table VI, and the RG flow diagrams between the stable bosonic fixed points and critical points are presented in Fig. 2. The scaling of the multipolar susceptibility at these critical points will be discussed in Section V C.

D. Beta Functions for the Fermi-Bose-Kondo Hamiltonians

The beta functions for the Bose-Fermi Kondo Hamiltonians are enumerated in Appendix J. The fermionic and bosonic fixed points presented in Sections IV B, IV C are all still stable under the full renormalization group flow. In addition to these fixed points, we also find critical points between the fermionic and bosonic stable fixed points. In particular, the critical points connect between $F_1$ and $B_1$, or $F_1$ and $B_2$. The non-zero fixed point values of some of the critical points for are listed in Table VII, and the numerical values are in Table XI.

Moreover, we find a (stable) partially Kondo-destroyed fixed point $P$. The only couplings which are nonzero at $P$
are \(K_{k_1}^1, K_{k_2}^2, K_{k_3}^3, K_{k_4}^4, g_{g_4}^2\) = \(\left(\frac{1}{3\sqrt{2}}, -\frac{1}{3}, -\frac{1}{3}, -\frac{2}{3\sqrt{3}}, e^2\right)\). Furthermore, we find critical points \(C_{FP}\) between \(F_1\) and \(P\), and \(C_{BP}\) between \(B_M\) and \(P\). The fixed point values are presented in Table VIII. The RG flow diagrams between them are presented in Fig. 3. This path of fixed points corresponds to the one described in the introduction as the primary result. We will discuss and explain the nature of the critical points of the full beta functions in Sec. V.

| Name | Nonzero Couplings | Between |
|------|-------------------|---------|
| \(C_1\) | \((k_i^1/s, g_{k_i^2}^a)\) | \(F_3\) and \(B_1\) |
| \(C_2\) | \((k_i^1/s, g_{k_i^2}^a)\) | \(F_3\) and \(B_2\) |
| \(C_3\) | \((k_i^{a(i)}, k_i^{b(i)}, g_2^a, g_2^a)\) | \(F_2\) and \(B_2\) |
| \(C_4\) | \((k_i^{a(i)}, k_i^{b(i)}, k_i^{A(k_2, b_2), g_2^a, g_2^a})\) | \(F_{1/2}\) and \(B_2\) |

TABLE VII. The critical points between fermionic and bosonic stable fixed points. \(C_i\) stands for the critical point between \(F_i\) and \(B_k\), and ‘Between’ means that the critical point exists between which \(F_j\) and \(B_k\). Here, we only show which fixed point values are non-zero because they have many non-zero values. \(a(i)\) and \(b(i)\) stand for \(a(1, 2, 3, 4)\) and \(b(1, 2, 3, 4)\). The actual values are presented in Appendix J.

V. QUANTUM PHASE TRANSITIONS

In the previous section, we demonstrated the existence of many stable fixed points and critical points between them. This suggests a diverse range of possible scenarios such as a direct quantum phase transition from the fermionic Kondo phases to bosonic phases, or two consecutive quantum phase transitions from the fermionic Kondo phase to bosonic phase through the partially Kondo-destroyed phase (Fig. 1). In this section, we will explain the nature of the different Kondo destruction critical points and their relation to the Fermi surface.

A. Pathways for Single Kondo Destruction

We start by discussing the direct Kondo destruction quantum phase transitions. As we mentioned in the previous section, we find 4 critical points which connect fermionic and bosonic fixed points, as listed in Table VII. Physical insight is gained about these transitions by identifying which constants are nonzero at each fixed point in a particular Kondo destruction path. In particular, we need to closely examine which irreps the coupling constants are associated with. We see that, for a given Kondo destruction path, if the Fermi-Kondo coupling for a given irrep, is nonzero, then it may (but is not guaranteed to) induce a nonzero bosonic bath coupling associated with the same irrep. For example, nonzero \(K_{a_4}^{i(i)}\) \((i = 1, 2, 3, 4)\) and \(K_{2(a, b)_2}^{i(i)}\) are able to generate \(g_4\) and \(g_2^a\), respectively.

From these considerations, it should be no surprise that we find quantum phase transitions from \(F_3\) to \(B_1\) or \(B_2\), passing through \(C_1\) or \(C_2\), respectively because \(F_3\) starts with every Fermi-Kondo coupling nonzero. Furthermore, we can also find quantum phase transitions from \(F_2\) or \(F_{1/2}\) to \(B_1\) passing through the critical points \(C_{4,4}\), respectively, because \(F_2\) and \(F_{1/2}\) have nonzero \(K_{a_4}^{i(i)}\) couplings, respectively. These results should be interpreted as the Kondo couplings, which are responsible for hybridization of particular local moments with the Fermi sea, being destroyed and replaced by the ordering of these local moments. A signature of Kondo destruction is the shrinking of the Fermi surface as the local moments order (Fig. 4). Shrinking of the Fermi surface can be detected by a jump in Hall constant \([12, 57–61, 85, 86]\). Note that,
Nonzero Couplings

\[ \Delta_c \]

Stability \( \Delta \) Between

| Name | Nonzero Couplings | Between |
|------|-------------------|---------|
| \( P \) | \( (K_{b1}, K_{b2}, K_{b3}, K_{b4}, g_E^2) = \left( \frac{1}{3\sqrt{3}}, -\frac{1}{3}, -\frac{2}{3\sqrt{3}}, \epsilon + \epsilon^2 \right) \) | Stable \( \frac{\epsilon}{2} \) |
| \( C_{FP} \) | \( (K_{b1}, K_{b2}, K_{b3}, K_{b4}, K_{E1}, K_{E2}, K_A, g_E^2) = \left( \frac{1}{3\sqrt{6}}, -\frac{1}{3}, -\frac{2}{3\sqrt{6}}, \frac{\epsilon}{2\sqrt{6}} + \frac{\sqrt{3}\epsilon^2}{16\sqrt{2}}, \frac{\epsilon}{2\sqrt{6}} + \frac{\epsilon^2}{8\sqrt{2}}, -\frac{\epsilon}{4\sqrt{3}}, \epsilon + \epsilon^2 \right) \) | Critical \( - F'_1 \) and \( P \) |
| \( C_{BP} \) | \( (K_{b1}, K_{b2}, K_{b3}, K_{b4}, g_E^2, g^2) = \left( \frac{\epsilon}{12\sqrt{7}6}, -\frac{\epsilon}{12\sqrt{7}6}, -\frac{\epsilon}{6\sqrt{7}6}, \epsilon + \epsilon^2, \frac{\epsilon}{2} + \frac{\epsilon^2}{8} \right) \) | Critical \( - B_M \) and \( P \) |

TABLE VIII. The nonzero fixed point values of the partially Kondo destroyed fixed point \( P \) and related critical points \( C_{FP} \) and \( C_{BP} \) in the full beta functions up to \( \epsilon^2 \) order. Here, \( P \) is the partially Kondo destroyed fixed point, and \( C_{FP} \) and \( C_{BP} \) are the critical points between \( F'_1 \) and \( P \), and \( P \) and \( B_M \), respectively.

![Diagram](image)

FIG. 3. The RG flow diagrams between \( F'_1 \) and \( P \), and \( B_M \) and \( P \) when \( \epsilon = 0.1 \). (a-b) The RG flow diagrams between \( F'_1 \) and \( P \); (b) zooms in on the quantum critical point of (a). The blue and red points stand for \( F'_1 \) and \( P \). The purple point is the critical point \( C_{FP} \) between \( F'_1 \) and \( P \). Here, \( s_1 = -0.484K_E + 1.484K_A \) and \( t_1 = 1 - 1.236K_E + 0.236K_A \) with the constraint \( g_E^2 = 1 - 1.675K_E + 0.675K_A \) where \( K_E = 3K_{E1} = 3K_{E2} \), \( K_A = -2\sqrt{3}K_A \), and \( g^2 = 8.873g^2 \), and we set \( (K_{b1}, K_{b2}, K_{b3}, K_{b4}) = \left( \frac{1}{3\sqrt{7}}, -\frac{1}{3}, -\frac{2}{3\sqrt{7}}, \frac{1}{3\sqrt{7}} \right) \), and other coupling constants as zero. (c-d) The RG flow diagrams between \( F'_1 \) and \( P \); (d) zooms in on the critical point of (c). The green and red points stand for \( B_M \) and \( P \). The olive point is the critical point \( C_{BP} \) between \( B_M \) and \( P \). Here, \( s_2 = 3\sqrt{6}K_{b1} = -3K_{b2} = -3K_{b3} = -2\sqrt{3}K_{b4} \), and \( t_2 = 18.944g_E^6 \), and we set \( g_E^2 = 0.113 \) and other coupling constants as zero.

since the bosonic fixed points \( B_1 \) and \( B_2 \) only exist for \( 0 < \epsilon < 1/8 \), these direct Kondo destruction paths are only possible for \( 0 < \epsilon < 1/8 \).

### B. Pathway for Two-Stage Kondo Destruction

Another possible scenario is two consecutive quantum phase transitions. In the full beta functions, not only do we find the critical points discussed in the previous section, but we also have a partially Kondo-destructed stable fixed point. The partially Kondo-destructed fixed point can be connected with \( F'_1 \) and \( B_M \) via the critical points \( C_{FP} \) and \( C_{BP} \), respectively. When we pass through \( C_{FP} \) from \( F'_1 \) to \( P \), the Kondo couplings \( K_{Ei} \) \( (i = 1, 2) \) and \( K_A \) flow to zero. Similar to the previous direct quantum phase transition, the nonzero \( K_{Ei} \) of \( F'_1 \) induces a nonzero \( g_E \) coupling, so we can reach \( P \). At \( P \), since we lose a part of the Kondo couplings, the Kondo effect is partially destroyed, but we still have nonzero Kondo couplings. This corresponds to the Fermi surface shrinking one time. Furthermore, the nonzero \( g_E \) at \( P \) signifies quadrupolar order with potential order parameters \( \{J_E^2 - J_E^2, 3J_E^2 - J_E^2\} \). Next, when we pass through \( C_{BP} \) between \( P \) and \( B_M \), the remaining Kondo couplings \( K_{bi} \) \( (i = 1, 2, 3, 4) \) flow to zero and induce a nonzero \( g_b \). This corresponds to a second shrinking of the Fermi surface, as well as the magnetic ordering of the dipolar/octet sequences in the \( T_b \) irrep. At \( B_M \), since all the Kondo couplings are zero, the Kondo effect is completely destroyed; the full picture of going from \( F'_1 \) to \( B_M \) is therefore two-stage Kondo destruction (Fig. 5). Since both \( g_E \) and \( g_b \) are nonzero at \( B_M \), we have coexistence between quadrupolar ordering and dipolar/octet ordering. Note that this may be regarded as coexistence between quadrupolar and dipolar/octet ordering because the octupoles in \( T_1 \) are not practically distinguishable from the dipoles. We also remind the reader that \( B_M \) exists for \( 0 < \epsilon < 1/4 \), which is a wider range than \( B_1 \) and \( B_2 \), which exist for \( 0 < \epsilon < 1/8 \). Therefore, if \( 0 < \epsilon < 1/8 \), we have the possibilities for either single or two-stage Kondo destruction phase transitions, but there is only the two-stage Kondo destruction phase transition if \( 1/8 < \epsilon < 1/4 \).
FIG. 4. A schematic diagram for direct quantum phase transition with single Kondo destruction. The indigo and yellow circles in the upper panel are the Kondo phase $F_3$ and magnetically ordered phases $B_{1,2}$, respectively, and the gray dashed circle indicates the critical points $C_{1,2}$ to $B_1$ or $B_2$. The arrows mean the direction of RG flow. $F_3$ has nonzero Kondo couplings (denoted by $K \neq 0$), so the multipolar moments hybridize with the conduction electrons and the Fermi surface is enlarged, and there is no multipolar ordering (denoted by $g = 0$). This large Fermi surface is denoted by $c + D\&O + Q + O$ and the indigo circle with ‘FS’ in the lower panel. Here, $c$, D\&O, Q, and O stand for the conduction electron, mixed dipolar/octupolar moments, quadrupolar moments, and octupolar moments, respectively. In $B_{1,2}$, the Kondo effect is fully destroyed (denoted by $K = 0$), so all the multipolar moments are decoupled from the conduction electrons. The Fermi surface becomes small, so its contents are just the conduction electrons ‘$c$’, indicated by the yellow small circle. Also at $B_{1,2}$, some multipolar moments form RKKY-mediated multipolar orderings (denoted by $g \neq 0$). These are mixed dipolar/octupolar ordering in $B_1$ or pure octupolar ordering in $B_2$, which is denoted by ‘$+D\&O$ or O’ in the last line. Other moments do not participate in the ordering.

C. Multipolar Susceptibilities

In addition to the qualitative signature of Fermi surface reconstruction, the multipolar susceptibility exponent can be used to quantitatively identify the fixed points. The multipolar susceptibility is defined by

$$\chi_i(\tau) = \langle T_\tau S_i(\tau) S_i(0) \rangle \sim \left( \frac{\tau_0}{|\tau|} \right)^{\gamma_i}, \quad (10)$$

where $\gamma_i$ is the multipolar susceptibility exponent, $i = a, E, 2+, A, b, 2-$ for each type of multipolar moment (two multipolar moments are of the same type if they are in the same irrep.), the $S_i$ operators are listed in Table II, $\tau$ is imaginary time, and $\tau \gg \tau_0$ with the cut-off $\tau_0 = 1/\Lambda \sim 1/\mu$. In Table II, $S_j$ operators in the same section give us the same multipolar susceptibility, i.e. $S_{1,2,3}$, $S_{4,5}$, $S_{6,7,8}$, $S_9$, $S_{10,11,12}$, and $S_{13,14,15}$ give us $\gamma_0$, $\gamma_E$, $\gamma_{2+}$, $\gamma_A$, $\gamma_b$, and $\gamma_{2-}$, respectively. From the beta functions, we can compute the multipolar susceptibility exponent. Depending on the fixed point values of the bosonic couplings, the multipolar susceptibility exponent is given by [65, 66, 87]:

$$\gamma_i = \begin{cases} \epsilon + \left[ \frac{2}{d} \frac{dg_i}{d\ln \mu} \right]_{f.p.} & = \epsilon, \quad g_i^* \neq 0, \\ \epsilon + 2 \left[ \frac{\partial}{\partial g_i} \frac{dg_i}{d\ln \mu} \right]_{f.p.} & , \quad g_i^* = 0, \end{cases} \quad (11)$$

FIG. 5. A schematic diagram for quantum phase transition with two-stage Kondo destruction. The blue, red, and green circles in the upper panel stand for the fermionic Kondo phase $F'_1$, partially Kondo-destroyed phase $P$, and magnetically ordered phase $B_M$, respectively. $F'_1$ has nonzero Kondo couplings ($K \neq 0, K' \neq 0$) and zero bosonic bath couplings ($g = 0, g' = 0$), so it corresponds to a Kondo phase. At $F'_1$, the conduction electrons ($c$) hybridize with mixed dipolar/octupolar moments (D\&O), quadrupolar moments (Q), and octupolar moments (O), so $F'_1$ has the largest Fermi surface; its contents and size are denoted by ‘$c + D\&O + Q + O$’ and the large blue circle with ‘FS’, respectively. After the phase transition from $F'_1$ to $P$, some of the Kondo couplings flow to zero ($K' = 0$ but $K \neq 0$). Concretely, this corresponds to the quadrupolar and pure octupolar moments getting decoupled from the conduction electrons, so the Kondo effect is partially destroyed. There is a remaining Kondo effect however because the Kondo couplings which represent the hybridization between the mixed dipolar/octupolar and conduction electrons is still nonzero (schematically $K$ in this figure). Due to this, $P$ has a smaller Fermi surface than $F'_1$. The contents and size of this Fermi surface are denoted by the $c + D\&O$ label and red medium-sized circle with ‘FS’. Furthermore, the decoupled quadrupolar moments become ordered, denoted by ‘$+Q$’ in the last line, which is due to $g' \neq 0$, which occurs as $K'$ flows to zero. After the phase transition from $P$ to $B_M$, all of the Kondo couplings become zero ($K = 0, K' = 0$). Then, all the multipolar moments are decoupled from the conduction electrons, so the Kondo effect is completely destroyed and $B_M$ has the smallest Fermi surface. This surface contains only conduction electrons $c$ and the small-sized green circle ‘FS’. Additionally, since $B_M$ has two nonzero bosonic bath couplings ($g \neq 0, g' \neq 0$), this means it supports coexistence of two types of ordering. Concretely, $B_M$ has both mixed dipolar/octupolar and quadrupolar moments. This is denoted by ‘$+Q + D\&O$’ in the last line.
where $\gamma_0$ becomes $\epsilon$ to all orders of $\epsilon$ by definition of the fixed point: $(dq/d\ln \mu)_{\text{f}p.} = 0$ when $q^*_f \neq 0$ [65, 66]. In the other case where $q^*_f = 0$, $\gamma_0$ is computed up to $\epsilon^2$ order. By assuming that the multipolar moments are primary fields with conformal dimension $\gamma = 2$, the finite temperature scaling of the multipolar susceptibility is given by [69, 87, 88]

$$\chi'_i(\omega, T) \sim \left\{ \begin{array}{ll}
T^{\gamma_i - 1} (1 + C_{\text{Re}1} (\frac{T}{\mu})^2), & |\omega/T| \ll 1, \\
\omega^{\gamma_i - 1}, & |\omega/T| \gg 1,
\end{array} \right.$$  

$$\chi''_{ii}(\omega, T) \sim \left\{ \begin{array}{ll}
T^{\gamma_i - 1} (\frac{T}{\mu}), & |\omega/T| \ll 1, \\
\omega^{\gamma_i - 1}, & |\omega/T| \gg 1,
\end{array} \right.$$  

where $\chi_i(\omega, T) = \chi_i'(\omega, T) + i\chi''_{ii}(\omega, T)$ and $C_{\text{Re}1}$ is a real constant. The temperature scaling of the multipolar susceptibility can be measured by observing the temperature dependence of elastic constants through ultrasound measurements in an external magnetic field [30, 32, 87, 89].

D. Ultrasound Measurement of Multipolar Susceptibility

The susceptibility exponents listed in the previous section include both dipolar and multipolar susceptibilities. Although the dipolar susceptibilities can be detected by conventional techniques, the purely multipolar susceptibilities require elastic measurements. One way to achieve this is through ultrasound experiments. The symmetry allowed free energy produces a linear coupling between strain and quadrupolar moments, which facilitates a relationship between elastic constants and quadrupolar susceptibilities. Furthermore, in the presence of an external magnetic field, a product of magnetic field and strain couples linearly to octupolar moments, adding octupolar susceptibility corrections to the elastic constants. The symmetry allowed elastic energy is then [30, 89]

$$F = \frac{C_{11}^0 - C_{12}^0}{2} (\epsilon_x^2 + \epsilon_y^2) + \frac{C_{44}^0}{2} (\epsilon_{yx}^2 + \epsilon_{yy}^2 + \epsilon_{zz}^2) + C_{11}^0 + 2C_{12}^0 \epsilon_B - s_E (\epsilon_x O_{22} + \epsilon_y O_{20})$$

$$- s_A T_{x y z} (h_x \epsilon_{y z} + h_y \epsilon_{x z} + h_z \epsilon_{x y})$$

$$- s_2 (\epsilon_{xy} O_{xy} + \epsilon_{yz} O_{yz} + \epsilon_{zx} O_{zx})$$

$$- s_{2\prime} (T_{x}^{\beta} h_x (\epsilon_{y y} - \epsilon_{z z}) + T_{y}^{\beta} h_y (\epsilon_{z z} - \epsilon_{x x}) + T_{z}^{\beta} h_z (\epsilon_{x x} - \epsilon_{y y})),$$

where $\epsilon'' = (2\epsilon_{x z} - \epsilon_{x x} - \epsilon_{y y})/\sqrt{6}$, $\epsilon' = (\epsilon_{x x} - \epsilon_{y y})/\sqrt{2}$, and $\epsilon_B = (\epsilon_{x z} + \epsilon_{y y} + \epsilon_{z z})/\sqrt{3}$. Under the assumption that we apply the magnetic field along the $z$-direction and that the field is sufficiently small such that the cubic symmetry is negligibly affected, the renormalized elastic constants are then given by second order perturbation theory as:

$$(C_{11} - C_{12}) = (C_{11}^0 - C_{12}^0) - (s_E^0) \chi_E - 2(s_z^0 h_z^2) \chi_{z-},$$  

$$C_{44} = C_{44}^0 - (s_z^2 \gamma_z^2) \chi_{z+} - (s_z^2 h_z^2) \chi'_{A}.$$  

where $h_z$ is the magnetic field along $z$-direction. Similar formulae may be obtained for magnetic fields applied in different directions. We see that the multipolar susceptibilities $\chi_E$ and $\chi_{z\pm}$ can both be measured without an external magnetic field. Once these are determined, the susceptibilities $\chi_A$ and $\chi_{z0}$ can then be found. In the case of the susceptibilities $\gamma_a$ and $\gamma_b$, they contain both dipolar and octupolar moments. However, these octupolar moments are not practically distinguishable from the dipolar moments due to being in the same irrep. as the dipolar moments. Thus, they couple to the magnetic field linearly, and can be measured by conventional magnetic susceptibility probes such as neutron scattering.

VI. DISCUSSIONS

In this work, we provided a detailed perturbative renormalization group analysis of the Fermi-Bose Kondo model describing a quartet of local states from Ce$_3$Pd$_{20}$ (Si, Ge)$_6$ [54] coupled to 3 bands of $p$-wave conduction electrons. We find numerous Fermi and non-Fermi liquid fixed points of the Fermi-Kondo impurity problem, which correspond to phases with local moments hybridizing with the conduction electrons and thereby enlarging the Fermi surface. We also find three magnetically ordered phases in the Bose-Kondo problem, wherein the Kondo effect is fully destroyed and local moments are ordered and decoupled from the Fermi surface. One of these phases is particularly relevant to recent experiments, and exhibits the coexistence of quadrupolar $\{J_x^2 - J_y^2, J_z^2 - J^2 \}$ and dipolar/octupolar $\{-J_{x,y,z} \frac{2}{27} T_{x,y,z}^a \}$ order. This is similar to the low temperature and zero magnetic field phase of Ce$_3$Pd$_{20}$Si$_8$, which has coexistence of antiferromagnetic ($J_z$) and antiferroquadrupolar ($3J_z^2 - J^2$) order [56, 62, 90].

Connected to this magnetically ordered phase is a partially Kondo destroyed phase wherein the dipolar/octupolar moments hybridize with and enlarge the Fermi surface, whereas the quadrupolar moments remain ordered and decoupled from the conduction electrons. This partially Kondo destroyed phase is potentially related to the quadrupolar ordered phase observed in the experiment at low temperatures for magnetic fields between 1T and 2T [55]. Our results show a further phase transition to a paramagnetic phase, where the quadrupolar moments also get hybridized with the Fermi surface and enlarge it a second time. Interestingly, all of these phases are also observed at zero magnetic field as a function of temperature. Experimentally, additional reconstruction of the Fermi surface is observed above 2T. However, this unidentified phase above 2T is not connected
The remaining moments may enter a (potentially multipolar) spin liquid phase [91–93], with interactions mediated by the RKKY coupling providing a mechanism for frustration. In the context of the Kondo impurity problem, experimental work has been done on cubic systems with the Ce diluted by La, providing a platform for Kondo impurity physics, with experiment results indicating non-Fermi liquid behavior. Thus the fixed points found in the Fermi-Kondo model may provide explanations for this behavior if studied with conformal field theory and non-Abelian bosonization techniques [50, 75–80, 94–97].

Although the construction of our model in the tetrahedral $T_d$ environment was inspired by work on Ce$_3$Pd$_{20}$Si$_6$, the results apply equally as well to other materials with a $T_d$ quartet in a cubic environment. This quartet can also arise in the presence of an octahedral $O_h$ crystal field, as is the case for the ground state of Ce$^{3+}$ in CeB$_6$ [54]. In fact, it is likely that such a rich phase diagram with possibility of both single and two-stage Kondo destruction is the case in any rare earth metallic system with a quartet of local moment states; this even applies to compounds with the Ce diluted by La [55, 56].

| F.P. | $\gamma_a$ | $\gamma_E$ | $\gamma_2^+$ | $\gamma_A$ | $\gamma_b$ | $\gamma_2^-$ |
|------|-----------|------------|-------------|------------|------------|-------------|
| $B_1$ | $\epsilon$ | $3\epsilon + 6\epsilon^2$ | $2\epsilon + 2\epsilon^2$ | $6\epsilon + 12\epsilon^2$ | $4\epsilon + 6\epsilon^2$ | $5\epsilon + 8\epsilon^2$ |
| $B_2$ | $5\epsilon + 8\epsilon^2$ | $3\epsilon + 6\epsilon^2$ | $2\epsilon + 2\epsilon^2$ | $6\epsilon + 12\epsilon^2$ | $4\epsilon + 6\epsilon^2$ | $\epsilon$ |
| $B_M$ | $2\epsilon$ | $\epsilon$ | $3\epsilon + 2\epsilon^2$ | $2\epsilon + 2\epsilon^2$ | $\epsilon$ | $2\epsilon$ |
| $P$ | $4 + \epsilon$ | $\epsilon$ | $4 + 2\epsilon + 2\epsilon^2$ | $2\epsilon + 2\epsilon^2$ | $\epsilon$ | $2\epsilon$ |
| $C_{0P}$ | $4 + \epsilon$ | $\epsilon$ | $4 + 2\epsilon + 6\epsilon^2$ | $2\epsilon + 3\epsilon^2$ | $4 + 4\epsilon$ | $4 + 4\epsilon$ |
| $C_{1P}$ | $2\epsilon$ | $\epsilon$ | $3\epsilon + 2\epsilon^2$ | $2\epsilon + 2\epsilon^2$ | $\epsilon$ | $2\epsilon$ |
| $C_{12}$ | $\epsilon$ | $\frac{3\epsilon}{2} + \frac{3\epsilon^2}{4}$ | $\epsilon$ | $\frac{3\epsilon}{2} + \frac{3\epsilon^2}{4}$ | $\frac{3\epsilon}{2} + \frac{3\epsilon^2}{4}$ | $\epsilon$ |
| $C_{M1}$ | $\epsilon$ | $\frac{5\epsilon}{3} + \frac{5\epsilon^2}{3}$ | $\epsilon$ | $\frac{5\epsilon}{3} + \frac{5\epsilon^2}{3}$ | $\epsilon$ | $\frac{5\epsilon}{3} + \frac{5\epsilon^2}{3}$ |
| $C_{M2}$ | $\frac{5\epsilon}{3} + \frac{5\epsilon^2}{3}$ | $\epsilon$ | $\frac{5\epsilon}{3} + \frac{5\epsilon^2}{3}$ | $2\epsilon + 2\epsilon^2$ | $\epsilon$ | $\frac{5\epsilon}{3} + \frac{5\epsilon^2}{3}$ |
| $C_1$ | $\epsilon$ | $3\epsilon + 5.295\epsilon^2$ | $2\epsilon + 1.762\epsilon^2$ | $6\epsilon + 9.491\epsilon^2$ | $4\epsilon + 4.638\epsilon^2$ | $5\epsilon + 6.077\epsilon^2$ |
| $C_2$ | $5\epsilon + 6.077\epsilon^2$ | $3\epsilon + 5.295\epsilon^2$ | $2\epsilon + 1.762\epsilon^2$ | $6\epsilon + 9.491\epsilon^2$ | $4\epsilon + 4.638\epsilon^2$ | $\epsilon$ |
| $C_3$ | $\epsilon$ | $3\epsilon + 2.51\epsilon^2$ | $2\epsilon + 0.77\epsilon^2$ | $6\epsilon + 2.51\epsilon^2$ | $\epsilon$ | $5\epsilon + 2\epsilon^2$ |
| $C_4$ | $\epsilon$ | $3\epsilon + 4.423\epsilon^2$ | $2\epsilon + 1.306\epsilon^2$ | $6\epsilon + 8.813\epsilon^2$ | $\epsilon$ | $5\epsilon + 5.582\epsilon^2$ |

TABLE IX. Multipolar susceptibilities at different fixed points. ‘F.P.’ stands for the fixed points. $\gamma_i$ stands for the multipolar susceptibility exponent for the multipolar moment in $i$-irrep. The exponents are calculated up to $\epsilon^2$ order. For the case when the bosonic bath coupling is nonzero at a particular fixed point, the exponent is exactly $\epsilon$ to all orders. We note that, when the leading contribution to the susceptibility exponent is not proportional to a power of $\epsilon$, then the perturbative estimation is not controlled by it. The leading nonperturbative contribution may therefore be different for these cases.
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Appendix A: \( \Gamma_8 \) Quartet wavefunctions

In a vacuum, a Ce\(^{3+} \) ion forms a spin-\( J = 5/2 \) system by Hund’s rules. In the presence of a tetrahedral crystal field, these 6 degenerate states are split, and in Ce\(_3\)Pd\(_20\)(Si, Ge)_6, the resulting ground state of the Ce\(^{3+} \) at the 8c site is a quartet spanned by the following four states [54]:

\[
|\Gamma_{8}^{(1)} \rangle = \sqrt{\frac{5}{6}} \left( |5/2 \rangle + |3/2 \rangle \right), \quad (A1)
\]
\[
|\Gamma_{8}^{(2)} \rangle = \sqrt{\frac{1}{3}} \left( |3/2 \rangle - |5/2 \rangle \right), \quad (A2)
\]
\[
|\Gamma_{8}^{(3)} \rangle = |1/2 \rangle, \quad (A3)
\]
\[
|\Gamma_{8}^{(4)} \rangle = | -1/2 \rangle. \quad (A4)
\]

To determine which multipolar moments are supported by these wave functions, we can compute the matrix elements of Stevens operators in the quartet \( \{ |\Gamma_{8}^{(1)} \rangle, |\Gamma_{8}^{(2)} \rangle, |\Gamma_{8}^{(3)} \rangle, |\Gamma_{8}^{(4)} \rangle \} \). More specifically, we compute

\[
\Lambda_{\rho \tau}^i = \langle \Gamma_{8}^{(\rho)} | \mathcal{O}_i | \Gamma_{8}^{(\tau)} \rangle,
\]

where \( \mathcal{O}_i \) is any (rescaled or linear combination of) Stevens operator (from “In terms of Stevens” column in Table II), and \( \Lambda_{\rho \tau}^i \) is some SU(4) generator. Note that the spin operators are then constructed as \( S^i = \sum_{\rho, \tau}^4 f^{1}_\rho \Lambda_{\rho \tau}^i f_\tau \).

Appendix B: Action of Tetrahedral Group

To derive the symmetry allowed Hamiltonian, we need to test all possible candidates and determine which remain invariant under action of the tetrahedral group \( T_d \), and under time-reversal \( \mathcal{T} \). The most efficient way to verify that a term is invariant under all elements of \( T_d \) is to pick two of its generators, which we select to be \( C_{31} \) and \( S_{12} \). \( C_{31} \) is a rotation by \( 2\pi/3 \) about the \((1,1,1)\) axis, and \( S_{12} \) is a rotation by \( \pi/2 \) about the \( z \)-axis followed by a mirror reflection across the \( xy \) plane. Both of these transformations map a tetrahedron to itself. Adding in time-reversal symmetry yields Table X. Checking all possible Hermitian Kondo terms respecting these symmetries yields Eqs. (D1)-(D15).

| Object | \( S_{12} \) | \( C_{31} \) | \( \mathcal{T} \) |
|--------|--------|--------|--------|
| \( x \) | \( -y \) | \( y \) | \( x \) |
| \( y \) | \( x \) | \( z \) | \( y \) |
| \( z \) | \( -z \) | \( x \) | \( z \) |
| \( \sigma^x \) | \( \sigma^y \) | \( \sigma^z \) | \( \sigma^x \) |
| \( \sigma^y \) | \( \sigma^y \) | \( \sigma^z \) | \( -\sigma^x \) |
| \( \sigma^z \) | \( \sigma^z \) | \( \sigma^x \) | \( -\sigma^z \) |
| \( J_x \) | \( J_y \) | \( J_x \) | \( -J_x \) |
| \( J_y \) | \( -J_x \) | \( J_z \) | \( -J_y \) |
| \( J_z \) | \( J_z \) | \( J_x \) | \( -J_z \) |

TABLE X. Symmetry transformations of orbitals, Pauli matrices, and multipolar moments, under two generators of the tetrahedral group as well as time-reversal.

Appendix C: SU(3) Gell-Mann matrices

When constructing the multipolar Kondo models for \( p \)-wave electrons coupled to the local moment, we have to consider all possible fermion bilinears of these 3 orbitals. For this, we use the generators of SU(3), normalized by \( \text{tr}(\lambda^i \lambda^j) = 2\delta_{ij} \). The \( 3 \times 3 \) Gell-Mann matrices are enumerated here:

\[
\lambda^0 = \sqrt{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (C1)
\]
\[
\lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (C2)
\]
\[
\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad (C3)
\]
\[
\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad (C4)
\]
\[ \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \]  \hspace{1cm} (C5)

**Appendix D: Fermi-Kondo Models**

The Kondo Hamiltonians coupling the conduction electrons to local moments described in Sec. III B are given by Eqs. (D1)-(D15). In these Hamiltonians, \( c^{\dagger} \) and \( c \) are conductin electron creation and annihilation operators, the subscript 0 on the conduction electron operators denotes the interaction taking place at the impurity site, which we choose to be the origin. The Latin indices sum over orbitals, \( m, n = x, y, z \), and the Greek indices sum over spins, \( \rho, \tau = \uparrow, \downarrow \). The \( \sigma \) are the standard Pauli matrices satisfying \( [\sigma^j, \sigma^k] = 2i\epsilon_{ijk}\sigma^i \), and the \( \lambda^i \) are the 3 x 3 Gell-Mann matrices, listed in Appendix C.

\[
H_{a1} = K_{a1} \sum_{mnp\tau} c^{\dagger}_{0mp} c_{0nt} \lambda_{mn}^0 \left( \sigma_{\rho\tau}^x S_1^1 + \sigma_{\rho\tau}^y S_2^2 + \sigma_{\rho\tau}^z S_3^3 \right), \tag{D1}
\]

\[
H_{b1} = K_{b1} \sum_{mnp\tau} c^{\dagger}_{0mp} c_{0nt} \lambda_{mn}^0 \left( \sigma_{\rho\tau}^x S_1^{10} + \sigma_{\rho\tau}^y S_1^{11} + \sigma_{\rho\tau}^z S_2^{12} \right), \tag{D2}
\]

\[
H_{a2} = K_{a2} \sum_{mnp\tau} c^{\dagger}_{0mp} c_{0nt} \sigma_{\rho\tau}^0 \left( \lambda_{mn}^7 S_1^1 - \lambda_{mn}^5 S_2^2 + \lambda_{mn}^2 S_3^3 \right), \tag{D3}
\]

\[
H_{b2} = K_{b2} \sum_{mnp\tau} c^{\dagger}_{0mp} c_{0nt} \sigma_{\rho\tau}^0 \left( \lambda_{mn}^7 S_1^{10} - \lambda_{mn}^5 S_1^{11} + \lambda_{mn}^2 S_2^{12} \right), \tag{D4}
\]

\[
H_{a3} = K_{a3} \sum_{mnp\tau} c^{\dagger}_{0mp} c_{0nt} \left[ (\sigma_{\rho\tau}^x \lambda_{mn}^1 + \sigma_{\rho\tau}^y \lambda_{mn}^4) S_3^1 + (\sigma_{\rho\tau}^x \lambda_{mn}^6 + \sigma_{\rho\tau}^y \lambda_{mn}^1) S_2^2 + (\sigma_{\rho\tau}^x \lambda_{mn}^4 + \sigma_{\rho\tau}^y \lambda_{mn}^6) S_3^3 \right], \tag{D5}
\]

\[
H_{b3} = K_{b3} \sum_{mnp\tau} c^{\dagger}_{0mp} c_{0nt} \left[ (\sigma_{\rho\tau}^x \lambda_{mn}^1 + \sigma_{\rho\tau}^y \lambda_{mn}^4) S_1^{10} + (\sigma_{\rho\tau}^x \lambda_{mn}^6 + \sigma_{\rho\tau}^y \lambda_{mn}^1) S_1^{11} + (\sigma_{\rho\tau}^x \lambda_{mn}^4 + \sigma_{\rho\tau}^y \lambda_{mn}^6) S_2^{12} \right], \tag{D6}
\]

\[
H_{a4} = K_{a4} \sum_{mnp\tau} c^{\dagger}_{0mp} c_{0nt} \left[ \sigma_{\rho\tau}^x \left( \frac{\sqrt{3}}{2} \lambda_{mn}^3 + \frac{1}{2} \lambda_{mn}^8 \right) S_1^1 + \sigma_{\rho\tau}^y \left( -\frac{\sqrt{3}}{2} \lambda_{mn}^3 + \frac{1}{2} \lambda_{mn}^8 \right) S_2^2 - \sigma_{\rho\tau}^z \lambda_{mn}^8 S_3^3 \right], \tag{D7}
\]

\[
H_{b4} = K_{b4} \sum_{mnp\tau} c^{\dagger}_{0mp} c_{0nt} \left[ \sigma_{\rho\tau}^x \left( \frac{\sqrt{3}}{2} \lambda_{mn}^3 + \frac{1}{2} \lambda_{mn}^8 \right) S_1^{10} + \sigma_{\rho\tau}^y \left( -\frac{\sqrt{3}}{2} \lambda_{mn}^3 + \frac{1}{2} \lambda_{mn}^8 \right) S_1^{11} - \sigma_{\rho\tau}^z \lambda_{mn}^8 S_2^{12} \right], \tag{D8}
\]

\[
H_{E1} = K_{E1} \sum_{mnp\tau} c^{\dagger}_{0mp} c_{0nt} \sigma_{\rho\tau}^0 \left( \lambda_{mn}^3 S_4^4 - \lambda_{mn}^8 S_5^5 \right), \tag{D9}
\]

\[
H_{E2} = K_{E2} \sum_{mnp\tau} c^{\dagger}_{0mp} c_{0nt} \left[ \sigma_{\rho\tau}^x \lambda_{mn}^7 S_4^4 + \sigma_{\rho\tau}^y \lambda_{mn}^5 S_4^5 \right] + \sigma_{\rho\tau}^z \lambda_{mn}^2 S_4^5, \tag{D10}
\]

\[
H_{2a+} = K_{2a+} \sum_{mnp\tau} c^{\dagger}_{0mp} c_{0nt} \sigma_{\rho\tau}^0 \left( \lambda_{mn}^6 S_4^6 + \lambda_{mn}^4 S_7^7 + \lambda_{mn}^1 S_8^8 \right), \tag{D11}
\]

\[
H_{2\beta+} = K_{2\beta+} \sum_{mnp\tau} c^{\dagger}_{0mp} c_{0nt} \left[ (\sigma_{\rho\tau}^x \lambda_{mn}^2 - \sigma_{\rho\tau}^y \lambda_{mn}^5) S_6^6 + (\sigma_{\rho\tau}^x \lambda_{mn}^7 + \sigma_{\rho\tau}^y \lambda_{mn}^2) S_7^7 - (\sigma_{\rho\tau}^x \lambda_{mn}^5 - \sigma_{\rho\tau}^y \lambda_{mn}^7) S_8^8 \right], \tag{D12}
\]

\[
H_A = K_A \sum_{mnp\tau} c^{\dagger}_{0mp} c_{0nt} \left( \sigma_{\rho\tau}^x \lambda_{mn}^6 + \sigma_{\rho\tau}^y \lambda_{mn}^4 + \sigma_{\rho\tau}^z \lambda_{mn}^1 \right) S_9^9, \tag{D13}
\]

\[
H_{2a-} = K_{2a-} \sum_{mnp\tau} c^{\dagger}_{0mp} c_{0nt} \left[ \sigma_{\rho\tau}^x \left( \frac{1}{2} \lambda_{mn}^3 - \frac{\sqrt{3}}{2} \lambda_{mn}^8 \right) S_{13}^{13} + \sigma_{\rho\tau}^y \left( \frac{1}{2} \lambda_{mn}^3 + \frac{\sqrt{3}}{2} \lambda_{mn}^8 \right) S_{14}^{14} - \sigma_{\rho\tau}^z \lambda_{mn}^3 S_{15}^{15} \right], \tag{D14}
\]

\[
H_{2\beta-} = K_{2\beta-} \sum_{mnp\tau} c^{\dagger}_{0mp} c_{0nt} \left[ (\sigma_{\rho\tau}^x \lambda_{mn}^7 - \sigma_{\rho\tau}^y \lambda_{mn}^2) S_{13}^{13} + (\sigma_{\rho\tau}^x \lambda_{mn}^6 - \sigma_{\rho\tau}^y \lambda_{mn}^5) S_{14}^{14} + (\sigma_{\rho\tau}^x \lambda_{mn}^4 - \sigma_{\rho\tau}^y \lambda_{mn}^6) S_{15}^{15} \right] . \tag{D15}
\]

**Appendix E: Bose Kondo Coupling**

The dynamical bosonic bath field representing local magnetic fluctuations is derived to respect the local symmetry. To do this, we construct the effective interaction between spins in the corresponding parent Kondo lattice. Starting with the Fermi-Kondo Hamiltonians in Eqs. (D1)-(D15), we can compute the effective interac-
FIG. 6. Effective Kondo Lattice RKKY Interaction; dotted lines refer to the spin operators and the solid lines are fermion propagators (not to be confused with dashed lines in other diagrams referring to pseudofermion propagators). The $S_i$ and $S_j$ are the (15 component) spin operators on different sites $i$ and $j$ in the parent Kondo lattice.

FIG. 7. Pseudofermion self-energy, both direct and counterterm contribution.

FIG. 8. Order $K$ corrections to the Fermi-Kondo vertex. There are only direct contributions at this order.

dition between two spins on different sites $i$ and $j$ by computing the diagram in Fig. 6.

By replacing one of the multipolar moment operators in the RKKY interaction by the dynamical bosonic field, we find the symmetry-allowed coupling between the local moment and bosonic bath. The allowed Bose-Kondo Hamiltonians are as follows:

\[ H^B_a = g_a(\phi_0^1, \phi_0^2, \phi_0^3) \cdot (S^1, S^2, S^3), \]  
(E1)

\[ H^B_E = g_E(\phi_0^4, \phi_0^5) \cdot (S^4, S^5), \]  
(E2)

\[ H^B_{z+} = g_{z+}(\phi_0^6, \phi_0^7, \phi_0^8) \cdot (S^6, S^7, S^8), \]  
(E3)

\[ H^B_A = g_A\phi_0^9 S^9, \]  
(E4)

\[ H^B_{z-} = g_{z-}(\phi_0^{10}, \phi_0^{11}, \phi_0^{12}) \cdot (S^{10}, S^{11}, S^{12}), \]  
(E5)

\[ H^B_{2-} = g_{2-}(\phi_0^{13}, \phi_0^{14}, \phi_0^{15}) \cdot (S^{13}, S^{14}, S^{15}). \]  
(E6)

Here, the subscript 0 denotes the Bose bath at the impurity site, whose components can be expanded as $\phi_0^i = \sum_k (\phi_0^i)^k$. We note that this is not the most general symmetry allowed Bose-Kondo Hamiltonian. It is in fact possible to have terms such as $(\phi_0^1, \phi_0^2, \phi_0^3) \cdot (S^{10}, S^{11}, S^{12})$. We ignore these mixing terms in our analysis [54, 62].

Appendix F: Details of the renormalization group method

From the bare kinetic Hamiltonian, Fermi-Kondo couplings, and bosonic bath couplings presented in the main text and appendix, we compute the vertex functions and self energy of the pseudofermion to 2-loop order. Along the way, we introduce counterterms order by order in order to remove the divergences from the loop integrals. The counterterms at order $n$ for the pseudofermion, Fermi-Kondo couplings, and Bose-Kondo couplings are denoted $(Z_f - 1)^{(n)}$, $(Z_k - 1)^{(n)}$, and $(Z_b - 1)^{(n)}$, respectively. The pseudofermion counterterm is denoted $\times$, and the Fermi-Kondo and Bose-Kondo counterterms are denoted $\otimes$; all are labelled in the appropriate Feynman diagrams. The corresponding Feynman diagrams for the pseudofermion self-energy are given in Fig. 7, the diagrams [100] for the Fermi-Kondo vertex corrections are given in Figs. 8-13, and the diagrams for the Bose-Kondo vertex corrections are given in Figs. 14-15. The details of the calculation of renormalization constants from the Feynman diagrams are explained for a simpler case in a wonderful reference [65]. In the Feynman diagrams of Figs. 7-15, solid, dashed, and squiggly lines refer to propagators of the conduction electrons (Eq. (F1)), pseudofermions (Eq. (F2)), and bosonic bath (Eq. (F3)), respectively, and their expressions are as follows:

\[ G_0^f(i\omega, k) = \frac{1}{i\omega - \xi_k}, \]  
(F1)

\[ G_0^f(i\omega, k) = \frac{1}{i\omega - \lambda}, \]  
(F2)

\[ G_0^\phi(i\omega, k) = \frac{2\Omega_k}{(i\omega - \Omega_k)(i\omega + \Omega_k)}. \]  
(F3)

For the explicit relation between the beta functions and renormalization constants, see Ref. [87].

Appendix G: Beta Functions for the Fermi-Kondo Model

The beta functions are very long so to save space we define a couple quantities which appear repeatedly:

\[ W_K = \sum a_i + 4b_1 + 2a_2 + 4b_2 + 2a_3 + 8b_3 + 8c_4 + 6d_4 + 10e_2, \]  
(G1)

\[ W_K' = \sum f_i + 4g_1 + 4h_2 + 4i_2 + 8j_3 + 8k_4 + 8l_5 + 8m_6 + 8n_7 + 8o_8 + 8p_9 + 8q_10 + 8r_11 + 8s_12 \]  
(G2)

\[ W_K'' = \sum t_i + 3u_2 + 3v_3 + 3w_4 + 3x_5 + 2y_6 + 3z_7, \]  
(G3)
FIG. 9. Order $K^2$ and $g^2$ corrections to the Fermi-Kondo vertex.

$$(ZK - 1)(1)$$

FIG. 10. Order $Kg^2$ direct corrections to the Fermi-Kondo vertex.

$$(Zf - 1)(2)$$

FIG. 11. Order $Kg^2$ counterterm corrections to the Fermi-Kondo vertex.

$$(Zf - 1)(2)$$

FIG. 12. Order $g^4$ direct corrections to the Fermi-Kondo vertex.

$$(ZK - 1)(2)$$

The beta functions for the Fermi-Kondo model are then given by Eqs. (G7)-(G21).

$$\frac{dK_{a1}}{d\ln \mu} = K_{a1}(W_{K_a} - 4K_{b1}^2) - \frac{4\sqrt{6}K_A K_{2\beta_+}}{3} - 2\sqrt{2}K_{E2} K_{2\beta_+} - \frac{2\sqrt{6}K_{a2} K_{b1}}{3} - 4K_{b1} K_{a2} K_{b2} - 8K_{b1} K_{a3} K_{b3}$$

$$- 4K_{b1} K_{a4} K_{b4} + \frac{2\sqrt{6}K_{a5} K_{b3}}{3} + 2\sqrt{2}K_{b3} K_{2\beta_-} + \frac{\sqrt{6}K_{a4} K_{b4}}{3} + \sqrt{6}K_{b4} K_{2\alpha_-}$$

$$+ 6K_{2\beta_-}$$

$$(G3)$$

$${W}_{K_{2+}} = 2K_{a1}^2 + 4K_{b1}^2 + 2K_{a2}^2 + 4K_{b2}^2 + 4K_{a3}^2 + 8K_{b3}^2$$

$$+ 2K_{a4}^2 + 4K_{b4}^2 + 4K_{E1}^2 + 6K_{E2}^2 + 4K_{2\alpha_+}$$

$$+ 8K_{2\beta_+} + 2K_{2\alpha_-} + 4K_{2\beta_-}$$

$$(G4)$$

$${W}_{A} = 6K_{a1}^2 + 6K_{a2}^2 + 12K_{a3}^2 + 6K_{a4}^2 + 4K_{E1}^2 + 6K_{E2}^2$$

$$+ 6K_{2\alpha_+} + 12K_{2\beta_+} + 6K_{A}^2 + K_{2\alpha_-} + 2K_{2\beta_-}$$

$$(G5)$$

$${W}_{2-} = 5K_{a1}^2 + 4K_{b1}^2 + 5K_{a2}^2 + 4K_{b2}^2 + 10K_{a3}^2 + 8K_{b3}^2$$

$$+ 5K_{a4}^2 + 4K_{b4}^2 + 2K_{E1}^2 + 3K_{E2}^2 + 2K_{2\alpha_+}$$

$$+ 4K_{2\beta_+} + 6K_{A}^2 + K_{2\alpha_-} + 2K_{2\beta_-}$$

$$(G6)$$
\[
\frac{dK_{b1}}{d\ln \mu} = K_{b1}(W_{Kb} - 4K_{a1}^2) - \frac{\sqrt{6}K_{a1}^2}{3} - 4K_{a1}K_{a2}K_{b2} - 8K_{a1}K_{a3}K_{b3} - 4K_{a1}K_{a4}K_{b4} + \frac{2\sqrt{6}K_{b1}^2}{3} + \frac{\sqrt{6}K_{a3}^2}{3}
- 2\sqrt{2}K_{a3}K_{2\beta} - \frac{2\sqrt{6}K_{b3}^2}{3} + \frac{\sqrt{6}K_{a4}^2}{3} - \frac{\sqrt{6}K_{2\alpha}^2}{3} + \frac{\sqrt{6}K_{2\beta}^2}{3} - \frac{2\sqrt{6}K_{2\beta+}^2}{3},
\]

\[\quad \quad (G8)\]

\[
\frac{dK_{a2}}{d\ln \mu} = K_{a2}(W_{Ka} - 4K_{b2}^2) + 2K_{A}K_{2\alpha} - 2K_{A}K_{2\beta} - 2K_{E1}K_{2\alpha} - \sqrt{3}K_{E2}K_{2\beta} - 4K_{a1}K_{b1}K_{b2} - K_{a2}K_{b2}
- 8K_{b2}K_{a3}K_{b3} - 4K_{b2}K_{a4}K_{b4} - K_{a3}K_{b3} - \sqrt{3}K_{a3}K_{b4} - \sqrt{3}K_{b3}K_{2\alpha} - \sqrt{3}K_{b3}K_{2\beta} - 3K_{b4}K_{2\beta},
\]

\[\quad \quad (G9)\]

\[
\frac{dK_{b2}}{d\ln \mu} = K_{b2}(W_{Kb} - 4K_{a2}^2) - 4K_{a1}K_{b1}K_{a2} - \frac{K_{b2}^2}{2} - 8K_{a2}K_{a3}K_{b3} - 4K_{a2}K_{a4}K_{b4} + K_{b2}^2 - \frac{K_{b3}^2}{2} - \sqrt{3}K_{b3}K_{a4}
+ \sqrt{3}K_{a3}K_{2\alpha} - \sqrt{3}K_{a3}K_{2\beta} + K_{b3}^2 + 2\sqrt{3}K_{b3}K_{b4} + 3K_{a4}K_{2\beta} + K_{2\alpha}K_{2\beta} + \frac{K_{2\beta}^2}{2} - K_{2\alpha}^2
- K_{2\beta+},
\]

\[\quad \quad (G10)\]

\[
\frac{dK_{a3}}{d\ln \mu} = K_{a3}(W_{Ka} - 8K_{b3}^2) - K_{a3}K_{2\alpha} - K_{A}K_{2\beta} - K_{E1}K_{2\beta} - \sqrt{3}K_{E2}K_{2\alpha} - \frac{2\sqrt{6}K_{b1}K_{b3}}{3} + \frac{\sqrt{6}K_{a1}K_{a3}^3}{3}
\]

\[\quad \quad (G11)\]

\[
\frac{dK_{b3}}{d\ln \mu} = K_{b3}(W_{Kb} - 8K_{a3}^2) - 4K_{a1}K_{b1}K_{a3} + \frac{\sqrt{6}K_{a1}K_{a3}^3}{3} + \frac{2\sqrt{6}K_{b1}K_{b3}}{3} - 2\sqrt{6}K_{b3}K_{a3} + \frac{K_{2\alpha}}{2} - 4K_{a2}K_{b2}K_{a3} - \frac{K_{a2}K_{a3}}{2},
\]

\[\quad \quad (G12)\]
\[
\begin{align*}
-Z_f & = \frac{\sqrt{3}K_{a2}K_{a1}}{2} - \frac{\sqrt{3}K_{a2}K_{a0-}}{2} + \frac{\sqrt{3}K_{a2}K_{2\beta-}}{2} + K_{b2}K_{b3} + \frac{\sqrt{3}K_{b2}K_{b4}}{2} - \frac{K_{a3}K_{a4}K_{b4}}{2} - \frac{\sqrt{3}K_{a3}K_{a4}}{3} \\
& + K_{b3}^2 - K_{b2}^2, \\
\frac{dK_{a4}}{d\ln \mu} & = K_{a4}(W_{ka} - 4K_{a4}^2) - \frac{2\sqrt{3}K_{a4}K_{2\beta-}}{3} - \frac{K_{a4}K_{b2}}{2} - \frac{K_{a3}K_{a4}K_{b4}}{3} - \frac{\sqrt{6}K_{a4}K_{b4}}{3} - \frac{\sqrt{6}K_{b1}K_{a4}}{3} - \frac{\sqrt{6}K_{b1}K_{2\alpha-}}{3} - \frac{\sqrt{6}K_{b1}K_{2\beta-}}{3} - \frac{\sqrt{6}K_{b1}K_{2\alpha+}}{3} - \frac{\sqrt{6}K_{b1}K_{2\beta+}}{3}, \\
\frac{dK_{b4}}{d\ln \mu} & = K_{b4}(W_{kb} - 4K_{b4}^2) - 4K_{a1}K_{b1}K_{a4} + \frac{\sqrt{6}K_{a4}K_{b4}}{3} + \frac{6K_{a4}K_{b4}}{3} - \frac{2\sqrt{6}K_{b1}K_{b4}}{3} - \frac{4K_{a2}K_{b2}K_{a4} - \sqrt{3}K_{a2}K_{a3}}{3} - 3K_{a2}K_{2\beta-} + 2\sqrt{3}K_{b2}K_{b3} - \frac{\sqrt{3}K_{a2}K_{2\beta-}}{3} - \frac{\sqrt{3}K_{b2}K_{b3}}{3} + 2K_{a3}K_{b3}K_{a4} + 2K_{a3}K_{2\beta-} + \frac{2\sqrt{3}K_{a4}K_{b3}}{3} + 2\frac{\sqrt{3}K_{a4}K_{b3}}{3} + 2\frac{\sqrt{3}K_{b3}^2}{3} + \frac{\sqrt{3}K_{b3}^2}{3} - 2\sqrt{3}K_{a4}K_{b4} + \frac{2\sqrt{3}K_{b4}^2}{3} + \frac{2\sqrt{3}K_{b4}^2}{3}, \\
\frac{dK_{E1}}{d\ln \mu} & = K_{E1}W_{KE} - 3K_{a}K_{E2} - 3K_{a2}K_{2\alpha-} - 3K_{a3}K_{2\beta-} + 3\sqrt{3}K_{a2}K_{2\beta+}, \\
\frac{dK_{E2}}{d\ln \mu} & = K_{E2}W_{KE} - 2K_{a}K_{E1} + 3K_{a}K_{E2} - 2\sqrt{3}K_{a}K_{2\alpha+} + 3\sqrt{3}K_{a}K_{2\beta+} - 3\sqrt{3}K_{a}K_{2\alpha-} - 3\sqrt{3}K_{a}K_{2\beta-} - K_{a4}K_{2\beta+} + 2K_{2\alpha-}K_{a2} - 2K_{2\alpha+}K_{2\beta-} + 2K_{2\alpha+}K_{2\beta+} + 2K_{2\beta+}K_{2\beta+}, \\
\frac{dK_{2\alpha-}}{d\ln \mu} & = K_{2\alpha-}W_{K2-} - 2K_{E1}K_{a2} - \sqrt{3}K_{E2}K_{a3} + 2K_{E2}K_{2\alpha-} - K_{E2}K_{2\beta-} - 2K_{b2}K_{2\alpha-} + 2K_{b3}K_{2\beta+} - 2\sqrt{3}K_{b4}K_{2\beta+}, \\
\frac{dK_{2\beta+}}{d\ln \mu} & = K_{2\beta+}W_{K2+} - K_{E1}K_{a3} - \sqrt{3}K_{E1}K_{2\beta-} - \sqrt{3}K_{E2}K_{a1} - \frac{\sqrt{3}K_{E2}K_{a2}}{2} - \frac{K_{E2}K_{a4}}{2} - \frac{K_{E2}K_{2\alpha+}}{2} + K_{E2}K_{2\beta-} - \frac{2\sqrt{6}K_{b1}K_{2\beta+}}{3} - 2K_{b2}K_{2\beta+} - 2K_{b3}K_{2\beta+} + \sqrt{3}K_{b4}K_{2\beta+} - \frac{2\sqrt{3}K_{b4}K_{2\beta+}}{3} - \frac{2\sqrt{3}K_{b4}K_{2\beta+}}{3}, \\
\frac{dK_{A}}{d\ln \mu} & = K_{A}W_{A} - 2K_{E1}K_{E2} + 4\sqrt{3}K_{b2}^2 - \frac{4\sqrt{6}K_{a1}K_{2\beta-}}{3} + 2K_{a2}K_{2\alpha-} - 2K_{a2}K_{2\beta-} - 2K_{a3}K_{2\alpha-} + 2K_{a3}K_{2\beta-} - 2\sqrt{3}K_{a4}K_{2\beta-},
\end{align*}
\]

FIG. 15. Order $g^4$ direct and counterterm corrections to the Bose-Kondo vertex.
The spin-orbit coupled angular momentum basis is given by

\[
\frac{dK_{2\alpha}}{d \ln \mu} = K_{2\alpha} - W_2 - 2K_A K_{a2} - 2K_A K_{a3} + 2K_{E2} K_{2\alpha} + K_{E2} K_{2\alpha} + \sqrt{6}K_{A1} K_{b4} - \sqrt{6}K_{b1} K_{A4} + \frac{\sqrt{6}K_{b1} K_{2\alpha}}{3}
\]

\[
\frac{dK_{2\beta}}{d \ln \mu} = K_{2\beta} - W_2 - \frac{2\sqrt{6}K_A K_{a1}}{3} - K_A K_{a2} + K_A K_{a3} - \frac{\sqrt{6}K_{A4} K_{b4}}{3} - \frac{3K_{2\alpha} K_{b3}}{2} + \frac{3K_{b2} K_{2\alpha}}{2} + \frac{K_{b2} K_{2\alpha}}{2} - \frac{K_{b2} K_{2\alpha}}{2} + \frac{K_{a3} K_{b4} - K_{b3} K_{a4} + K_{b3} K_{2\alpha}}{2} + \frac{\sqrt{6}K_{b1} K_{2\alpha}}{3}.
\]

(G20)

(G21)

The resulting fixed points of the Fermi-Kondo beta functions are already discussed in Section IV B. To simplify equations in the following appendices, we introduce the following for the beta functions in the Fermi-Kondo model:

\[
\beta^F(K_i) = \left[ \frac{dK_i}{d \ln \mu} \right] \bigg|_{F},
\]

where the superscript \( F \) on the left hand side and the subscript \( F \) on the right hand side stand for the Fermi-Kondo model only. In other words, the beta functions in Eqs. (G7)-(G21) are given by \( \beta^F(K_i) \).

### Appendix H: Fixed point Hamiltonians for \( F_3 \) and \( F'_3 \)

In this section, we will discuss how to obtain the fixed point Hamiltonians for \( F_3 \) and \( F'_3 \). The most important step is a change of basis for the conduction electrons. The original orbital basis (cubic harmonics) can be rewritten in terms of the angular momentum basis (spherical harmonics)\[51,\]

\[
|x\rangle = \frac{1}{\sqrt{2}} (-|1,1\rangle + |1,-1\rangle),
\]

(H1)

\[
|y\rangle = \frac{i}{\sqrt{2}} (|1,1\rangle + |1,-1\rangle),
\]

(H2)

\[
|z\rangle = |1,0\rangle.
\]

(H3)

By using the Clebsch-Gordan coefficients, the composite spin-orbit coupled angular momentum basis is given by

\[
|1, m\rangle \pm \frac{1}{2} = \sqrt{\frac{2 + m}{3}} |\frac{1}{2}, m \pm \frac{1}{2}\rangle \pm \sqrt{\frac{1 + m}{3}} |\frac{1}{2}, m \pm \frac{1}{2}\rangle.
\]

(H4)

Let \( U \) be the basis transformation from the cubic harmonic and spin basis to the total angular momentum basis. Then,

\[
(c_{x,\uparrow} c_{x,\downarrow} c_{y,\downarrow} c_{y,\uparrow} c_{z,\uparrow} c_{z,\downarrow})^T U^T = (c_{\uparrow} c_{\downarrow} c_{\uparrow} c_{\downarrow} c_{\downarrow} c_{\downarrow})^T,
\]

(H5)

where

\[
U = \left( \begin{array}{cccccc}
-\frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} & 0 & 0 & 0 \\
0 & -\frac{1}{\sqrt{6}} & 0 & \frac{i}{\sqrt{6}} & \sqrt{\frac{2}{3}} & 0 \\
0 & 0 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 \\
0 & -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \\
\end{array} \right).
\]

(H6)

The fixed point Hamiltonians are as follows:

(i) \( F_3 \): The fixed point Hamiltonian for \( F_3 \) corresponds to a fully SU(4) symmetric model as follows:

\[
H_3 = \sum_{i=1}^{15} \sum_{\rho \tau} (\psi^\dagger_0 \Lambda_{\rho \tau} \psi) S^i,
\]

(H7)

where \( \psi \) is the 4 component spinor, for example, defined by \( \psi = U_3 \tilde{c}_\downarrow \) with

\[
U_3 = \left( \begin{array}{cccc}
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{array} \right).
\]

(H8)

The form of \( U_3 \) may be modified slightly if a different sign structure of the fixed point value of \( F_3 \) is chosen.

(ii) \( F'_3 \): The fixed point Hamiltonian for \( F'_3 \) corresponds to a 6 generator truncated SU(4) fixed point \[84\] as follows:

\[
H_{F'_3} = \frac{1}{2} \sum_{\rho \tau} \psi_0^\dagger (\sigma^0 \otimes \tilde{\sigma})_{\rho \tau} \cdot (S^4, S^9, S^9) \tilde{\psi}_\tau
\]

\[
+ \sum_{\rho \tau} c_{\downarrow}^\dagger (\tilde{\sigma}_{\rho \tau} \cdot (-S^{10}, -S^{11}, -S^{12})) c_{\uparrow, \tau},
\]

(H9)

where \( \tilde{\sigma} = (\sigma^x, \sigma^y, \sigma^z) \), \( c_{\downarrow} \) is a 2-component spinor, and \( \tilde{\psi} \) is a 4-component spinor obtained by a further change
of basis, for example, \( \psi = U'_1 e_2 \).

\[
U'_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix},
\]

where the form of \( U'_1 \) may change slightly for a different chosen sign structure of the fixed point. Here, \( S^{4,9,5} \) and \(-S^{10,11,12}\) satisfy two mutually commuting SU(2) algebras.

### Appendix I: Beta Functions for the Bose-Kondo Model

To neatly express the beta functions for the Bose-Kondo model, we define some commonly occurring quantities in Eqs. (11)-(16).

\[
W_{g_a} = -\frac{g_a^2}{2} - \frac{g_b^2 g_a^2}{2} \frac{g_a^2 g_b^2}{2} - \frac{3g_a^2 g_b^2}{2} - \frac{g_b^4}{2} + \frac{5g_a^2 g_b^2}{4} - \frac{g_a^2}{4} - \frac{g_b^4}{4} - 2g_a^2 g_b^2 - \frac{3g_a^2 g_b^2}{4},
\]

\[
W_{g_E} = -\frac{g_E^2}{2} - \frac{g_a^2 g_E^2}{2} - \frac{3g_a^2 g_E^2}{2} - \frac{3g_a^2 g_E^2}{2} + \frac{g_a^2}{2} - \frac{3g_a^2 g_E^2}{4} - \frac{3g_a^2 g_E^2}{4} + \frac{g_a^2}{2} - \frac{3g_a^2 g_E^2}{4} - \frac{3g_a^2 g_E^2}{4}
\]

\[
W_{g_{2+}} = -\frac{g_a^2 g_E^2}{2} - \frac{g_a^2 g_E^2}{2} - \frac{3g_a^2 g_E^2}{2} + \frac{g_a^2}{2} - \frac{3g_a^2 g_E^2}{4} - \frac{3g_a^2 g_E^2}{4} + \frac{g_a^2}{2} - \frac{3g_a^2 g_E^2}{4} - \frac{3g_a^2 g_E^2}{4},
\]

\[
W_{g_{2-}} = -\frac{g_a^2 g_E^2}{2} - \frac{g_a^2 g_E^2}{2} - \frac{3g_a^2 g_E^2}{2} + \frac{g_a^2}{2} - \frac{3g_a^2 g_E^2}{4} - \frac{3g_a^2 g_E^2}{4} + \frac{g_a^2}{2} - \frac{3g_a^2 g_E^2}{4} - \frac{3g_a^2 g_E^2}{4}
\]

The beta functions for the Bose-Kondo model are then in Eqs.(17)-(12):

\[
\beta^B(g_i) = \left[ \frac{dg_i}{d\ln \mu} \right]_B,
\]

The fixed points of this RG flow are described in Section IV C, and the fixed point values are listed in Tables V, VI. In addition, we introduce the notation for the beta functions in the Bose-Kondo model as follows:

\[
\beta^B(g_i) = \left[ \frac{dg_i}{d\ln \mu} \right]_B,
\]

where the superscript \( B \) on the left hand side and the subscript \( B \) on the right hand side stand for the Bose-Kondo model. In other words, \( \beta^B(g_i) \) are the beta functions in Eqs.(17)-(12).
Appendix J: Beta Functions for the Bose-Fermi Kondo Model

Because the full beta functions for the Bose-Fermi Kondo model extend the previous results, we can express them in terms of the Fermi-Kondo only or Bose-Kondo only results, plus some contributions due to the mixing between fermionic and bosonic fields. Furthermore, these beta functions also contain the commonly-occurring expressions defined in Appendices G and I. Thus they are expressed succinctly as

\[
\begin{align*}
\frac{dK_{ai}}{d\ln \mu} &= \beta^F(K_{ai}) + K_{ai}W_{ga}, \quad i = 1, 2, 3, 4, \quad (J1) \\
\frac{dK_{bi}}{d\ln \mu} &= \beta^F(K_{bi}) + K_{bi}W_{gb}, \quad i = 1, 2, 3, 4, \quad (J2) \\
\frac{dK_{Ei}}{d\ln \mu} &= \beta^F(K_{Ei}) + K_{Ei}W_{gE}, \quad i = 1, 2, \quad (J3) \\
\frac{dK_{2i+}}{d\ln \mu} &= \beta^F(K_{2i+}) + K_{2i+}W_{g2i+}, \quad i = \alpha, \beta, \quad (J4) \\
\frac{dK_{A}}{d\ln \mu} &= \beta^F(K_{A}) + K_{A}W_{gA}, \quad (J5) \\
\frac{dK_{2i-}}{d\ln \mu} &= \beta^F(K_{2i-}) + K_{2i-}W_{g2i-}, \quad i = \alpha, \beta, \quad (J6)
\end{align*}
\]

where \( W_{ab} \equiv 4(K_{a1}K_{b1} + K_{a2}K_{b2} + 2K_{a3}K_{b3} + K_{a4}K_{b4}) \).

From the above full beta functions, we can find numerous stable fixed points and critical points as presented in the main text. The nonzero fixed point values for the full beta functions up to \( e^2 \) order are presented in Tables XI and VIII. In addition the critical points \( C_{1,2,3,4} \) between \( F_i \) and \( B_{1,2} \) are mentioned in the main text.

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TABLE XI. Nonzero fixed point values of the critical points between fermionic and bosonic stable fixed points up to $\epsilon^2$.

| Name | Nonzero fixed point values | Between |
|------|---------------------------|---------|
| $C_1$ | $(K_{a1}^*, K_{b1}^*, K_{b2}^*, K_{a2}^*, K_{b3}^*, K_{a3}^*, K_{b4}^*, K_{b5}^*, K_{b6}^*, K_{a6}^*, K_{b7}^*, K_{a7}^*, K_{b8}^*, K_{a8}^*, K_{b9}^*, K_{a9}^*, K_{b10}^*, K_{a10}^*, K_{b11}^*, K_{a11}^*, K_{b12}^*, K_{a12}^*, K_{b13}^*, K_{a13}^*, K_{b14}^*, K_{a14}^*, K_{b15}^*, K_{a15}^*, K_{b16}^*, K_{a16}^*, K_{b17}^*, K_{a17}^*)$ | $F_3$ and $B_1$ |
| $C_2$ | $(K_{a1}^*, K_{b1}^*, K_{b2}^*, K_{a2}^*, K_{b3}^*, K_{a3}^*, K_{b4}^*, K_{b5}^*, K_{a6}^*, K_{b7}^*, K_{a7}^*, K_{b8}^*, K_{a8}^*, K_{b9}^*, K_{a9}^*, K_{b10}^*, K_{a10}^*, K_{b11}^*, K_{a11}^*, K_{b12}^*, K_{a12}^*, K_{b13}^*, K_{a13}^*, K_{b14}^*, K_{a14}^*, K_{b15}^*, K_{a15}^*, K_{b16}^*, K_{a16}^*, K_{b17}^*, K_{a17}^*)$ | $F_1$ and $B_1$ |
| $C_3$ | $(K_{a1}^*, K_{b1}^*, K_{b2}^*, K_{a2}^*, K_{b3}^*, K_{a3}^*, K_{b4}^*, K_{b5}^*, K_{a6}^*, K_{b7}^*, K_{a7}^*, K_{b8}^*, K_{a8}^*, K_{b9}^*, K_{a9}^*, K_{b10}^*, K_{a10}^*, K_{b11}^*, K_{a11}^*, K_{b12}^*, K_{a12}^*, K_{b13}^*, K_{a13}^*, K_{b14}^*, K_{a14}^*, K_{b15}^*, K_{a15}^*, K_{b16}^*, K_{a16}^*, K_{b17}^*, K_{a17}^*)$ | $F_2$ and $B_1$ |
| $C_4$ | $(K_{a1}^*, K_{b1}^*, K_{b2}^*, K_{a2}^*, K_{b3}^*, K_{a3}^*, K_{b4}^*, K_{b5}^*, K_{a6}^*, K_{b7}^*, K_{a7}^*, K_{b8}^*, K_{a8}^*, K_{b9}^*, K_{a9}^*, K_{b10}^*, K_{a10}^*, K_{b11}^*, K_{a11}^*, K_{b12}^*, K_{a12}^*, K_{b13}^*, K_{a13}^*)$ | $F_{1/2}$ and $B_1$ |

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