SOME INEQUALITIES AND ASYMPTOTIC FORMULAS FOR EIGENVALUES ON RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we establish sharp inequalities for four kinds of classical eigenvalues on a bounded domain of a Riemannian manifold. We also establish asymptotic formulas for the eigenvalues of the buckling and clamped plate problems. In addition, we give a negative answer to the Payne conjecture for the one-dimensional case.

1. Introduction

Let \((M, g)\) be an \(n\)-dimensional oriented Riemannian manifold, and let \(\Omega \subset M\) be a bounded domain with \(C^2\)-smooth boundary \(\partial \Omega\). We consider the following classical eigenvalue problems:

\begin{align*}
(1.1) & \quad \left\{ 
\begin{array}{ll}
\Delta_g u + \mu u &= 0 & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} &= 0 & \text{on } \partial \Omega,
\end{array}
\right. \\
(1.2) & \quad \left\{ 
\begin{array}{ll}
\Delta_g u + \lambda u &= 0 & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega,
\end{array}
\right. \\
(1.3) & \quad \left\{ 
\begin{array}{ll}
\Delta_g^2 u - \Gamma^2 u &= 0 & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega,
\end{array}
\right. \\
(1.4) & \quad \left\{ 
\begin{array}{ll}
\Delta_g^2 u + \Lambda \Delta_g u &= 0 & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega,
\end{array}
\right.
\end{align*}

where \(\nu\) denotes the outward unit normal vector to \(\partial \Omega\), and \(\Delta_g\) is the Laplace-Beltrami operator that is given in local coordinates by the expression,

\[
\Delta_g = \frac{1}{\sqrt{|g|}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( \sqrt{|g|} g^{ij} \frac{\partial}{\partial x_j} \right).
\]
Here \( |g| := \det(g_{ij}) \) is the determinant of the metric tensor, and \( g^{ij} \) are the components of the inverse of the metric tensor \( g \).

(1.1) is the Neumann problem (see [7]); (1.2) is the Dirichlet problem (see [7] or [9]); (1.3) occurs in the treatment of the vibration problem for a clamped plate (see [9] and [34]), and (1.4) is the well-known buckling problem for a clamped plate (see [7], [9], [27], [28] or [34]). In each of these cases, the spectrum is discrete and we arrange the eigenvalues in non-decreasing order (repeated according to multiplicity)

\[
0 = \mu_1 < \mu_2 \leq \cdots \leq \mu_k \leq \cdots ;
0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots ;
\Gamma_1^2 \leq \Gamma_2^2 \leq \cdots \leq \Gamma_k^2 \leq \cdots ;
A_1 \leq A_2 \leq \cdots \leq A_k \leq \cdots .
\]

The corresponding eigenfunctions are expressed as \( v_1, v_2, v_3, \cdots ; u_1, u_2, u_3, \cdots ; U_1, U_2, U_3, \cdots ; \) and \( W_1, W_2, W_3, \cdots \).

For any bounded domain \( \Omega \) in \((M, g)\), the variational formulation of the Neumann and Dirichlet eigenvalue problems (in terms of Rayleigh quotients, cf. Sect.VI.1 of [9]) immediately implies the inequalities

\[
\mu_k \leq \lambda_k, \quad k = 1, 2, 3, \cdots .
\]

Moreover, Pólya [33] proved in 1952 that for \( \Omega \subset \mathbb{R}^2 \)

\[
\mu_2 < \lambda_1,
\]

answering a question of Kornhauser and Stakgold [20]. In the case that \( \Omega \) is a bounded convex domain \( \Omega \subset \mathbb{R}^2 \) with a piecewise \( C^2 \)-smooth boundary, Payne [27] showed that

\[
\mu_{k+2} < \lambda_k, \quad k = 1, 2, 3, \cdots .
\]

Levine and Weinberger [22] proved that

\[
\mu_{k+n} < \lambda_k, \quad k = 1, 2, 3, \cdots
\]

for smooth bounded convex domains \( \Omega \subset \mathbb{R}^n \) (cf. [34]), as well as

\[
\mu_{k+m} \leq \lambda_k, \quad k = 1, 2, 3, \cdots ; 1 \leq m \leq n
\]

for arbitrary bounded convex domains. In 1991, Friedlander [13] proved that

\[
\mu_k \leq \lambda_{k+1}, \quad k = 1, 2, 3, \cdots
\]

when \( \Omega \subset \mathbb{R}^n \) is a bounded domain with a \( C^1 \)-smooth boundary \( \partial \Omega \). We also refer to Mazzeo [23] for an extension to certain smooth manifolds, and to Ashbaugh and Levine [2] and Hsu and Wang [17] for the case of subdomains of the \( n \)-sphere \( \mathbb{S}^n \) with a smooth boundary and nonnegative mean curvature. Finally, in 2004 Filonov [12] proved strict inequality

\[
\mu_{k+1} < \lambda_k \quad (k = 1, 2, 3, \cdots).
\]

when \( \Omega \subset \mathbb{R}^n \) is a domain with finite volume, and with the embedding \( W^{1,2}(\Omega) \hookrightarrow L^2(\Omega) \) compact.

In regard to the vibration problem of a clamped plate, Pólya in [32] obtained that

\[
\lambda_k \leq \Gamma_k, \quad k = 1, 2, 3, \cdots
\]
for any bounded domain in $\mathbb{R}^2$. This result had actually been improved to be
\begin{equation}
\lambda_k < \Gamma_k, \quad k = 1, 2, 3, \ldots
\end{equation}
for bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary by Weinstein [42] (which is referred as Weinstein’s inequality, see [43]). In [2], Ashbaugh and Laugesen obtained the inequalities
\begin{equation}
\lambda_1^2 \leq \lambda_1 \lambda_2 \leq \Lambda_1 \lambda_1 \leq \Gamma_1 \leq \Lambda_1^2.
\end{equation}

Concerning the buckling problem of a clamped plate, in 1937 Weinstein [42] proved the following strictly inequality
\begin{equation}
\lambda_k < \Lambda_k, \quad k = 1, 2, 3, \ldots
\end{equation}
for bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary. Payne [27] in 1955 proved that for any bounded domain $\Omega \subset \mathbb{R}^2$ with $C^2$-smooth boundary
\begin{equation}
\lambda_2 \leq \Lambda_1,
\end{equation}
solving a conjecture of Weinstein. In [27], Payne further made the conjecture:
\begin{equation}
\lambda_{k+1} \leq \Lambda_k, \quad \text{for } k = 1, 2, 3, \ldots
\end{equation}
Note that this conjecture remains open in $\mathbb{R}^n$ ($n \geq 2$); however, in this paper we give a negative answer to the Payne conjecture for the one-dimensional case (see Example 4.1).

The first purpose of this paper is to prove:

**Theorem 1.1.** Let $(M, g)$ be an $n$-dimensional oriented Riemannian manifold ($n \geq 2$), and let $\Omega \subset M$ be a bounded domain with $C^2$-smooth boundary. Then
\begin{equation}
\mu_k < \lambda_k < \Gamma_k < \Lambda_k \quad \text{for } k = 1, 2, 3, \ldots,
\end{equation}
where $\mu_k$, $\lambda_k$, $\Gamma_k$ and $\Lambda_k$ are the $k$-th eigenvalues of the Neumann, Dirichlet, clamped plate and buckling problems for the domain $\Omega$, respectively.

We also show by some examples that in the Riemannian manifold setting, (1.16) are the best possible inequalities for these classical eigenvalue problems (see Remark 3.1).

H. Weyl in 1912, was the first to establish asymptotic formulas in $\mathbb{R}^n$ for the Dirichlet and Neumann eigenvalues (see [41] or [15]):
\begin{equation}
\lambda_k \sim (2\pi)^2 \left( \frac{k}{\omega_n |\Omega|} \right)^{2/n}, \quad k \to \infty,
\end{equation}
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\mu_k \sim (2\pi)^2 \left( \frac{k}{\omega_n |\Omega|} \right)^{2/n}, \quad k \to \infty,
\end{equation}
where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$, and $|\Omega|$ is the $n$-dimensional Lebesgue measure of $\Omega$ ($\sim$ means the ratio of the RHS to the LHS approaches 1 as $k \to \infty$). In the case of two-dimensional Euclidean space, Pleijel [31] in 1950 given the asymptotic formula for the eigenvalues of a clamped plate based on a Carleman’s method in [4] and [5]. In 1967, Mckean and Singer [24] generalized Weyl’s asymptotic formulas to a bounded domain of a Riemannian manifold by investigating asymptotic expansion of the trace of heat operator.
The second purpose of the paper is to establish the asymptotic formulas for the eigenvalues of the buckling and clamped plate problems on a Riemannian manifold. We have the following:

**Theorem 1.2.** Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold, and let \(\Omega \subset M\) be a bounded domain with \(C^2\)-smooth boundary. Then

\[
\Lambda_k \sim (2\pi)^2 \left( \frac{k}{\omega_n(\text{vol}(\Omega))} \right)^{2/n}, \quad \text{as } k \to +\infty,
\]

\[
\Gamma_k \sim (2\pi)^2 \left( \frac{k}{\omega_n(\text{vol}(\Omega))} \right)^{2/n}, \quad \text{as } k \to +\infty,
\]

where \(\text{vol}(\Omega)\) is the volume of \(\Omega\).

Weyl’s asymptotic formulas and Theorem 1.2 show that for general bounded domain with \(C^2\)-smooth boundary in a Riemannian manifold, the four kinds of classical quantities \(\mu_k, \lambda_k, \Gamma_k\) and \(\Lambda_k\) have the same asymptotic behavior as \(k \to +\infty\).

The proof of Theorem 1.1 uses a key result (Lemma 2.1), which generalizes the classical Holmgren’s uniqueness theorem (see [36]) to the Riemannian manifold, and a technique of [12] by which Filonov proved the inequalities (1.10). In order to prove Theorem 1.2, we first consider the case of the Euclidean space and then obtain the version in Riemannian manifold by applying metric expansion in normal coordinates system. The main method is to approximate \(\Omega\) by a union of subdomains whose boundary are piecewise smooth that has been suitably contracted. We thus get a lower estimate for the counting function of the buckling eigenvalues if these subdomains are open, disjoint and lie inside \(\Omega\). On the other hand, an upper estimate had been given in [24] (see also p.441 of [9]) by investigating the Neumann and Dirichlet eigenvalue problems. Thus the desired result is proved.

2. Some lemmas

The following several lemmas will be needed.

**Lemma 2.1.** Let \(\Omega\) be a bounded domain with \(C^2\)-smooth boundary in an \(n\)-dimensional Riemannian manifold \((M, g)\), and let \(0 \neq u \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)\) be a solution of (1.2). Then \(\frac{\partial u}{\partial n}|_{\partial \Omega}\) does not vanish identically on \(\partial \Omega\).

**Proof.** Let \(F(x, \xi)\) be a fundamental solution for the Helmholtz operator \(\triangle_g + \lambda\) on \(M\) (i.e., \(F(x, \xi)\) satisfying

\[
(\triangle_g + \lambda)F(x, \xi) = \delta_x(\xi),
\]

where \(\triangle_g\) denotes the Laplace operator taken with respect to the variables \(\xi\), and \(\delta_x(\xi)\) is the Dirac \(\delta\)-function. More precisely, \((\triangle_g + \lambda)F(x, \xi) = 0\) with respect to \(\xi \neq x\) for any fixed \(x\). For \(x \in M\), we choose the normal coordinates centered at \(x\). Since \(F(x, \xi)\) is singular at \(\xi = x\) we cut out from \(\Omega\) a geodesic ball \(B(x, \epsilon)\) contained in \(\Omega\) with center
where \( d\mu \) denotes the \((n-1)\)-dimensional volume element, and \( \frac{\partial}{\partial \nu} \) denotes the derivative in the direction of the outward unit normal vector \( \nu_\xi \) at \( \xi \). We now wish to evaluate the limits of the individual integrals in this formula for \( \epsilon \to 0 \). On \( \partial B(x, \epsilon) \), we have \( F(x, \xi) = F_0(\epsilon) + O(\epsilon) \) since we have used the normal coordinates. From proof of Theorem 9.4 of \([23]\), we know that for \( n \geq 2 \),

\[
F_1(z) = F_0(z)[1 + f(z)] + h(z) \quad \text{as} \quad |z| \to 0,
\]

where

\[
F_0(z) = \begin{cases} 
\frac{|z|^{2-n}}{n(2-n)\omega_n} & \text{for} \ n > 2, \\
\frac{1}{2\pi} \log |z| & \text{for} \ n = 2,
\end{cases}
\]

\( f(z) = O(|z|^2) \)

and

\[
h(z) = \begin{cases} 
\text{const} + O(|z|^2) & \text{for} \ n = 2, \\
0 & \text{for odd} \ n > 2, \\
\text{const} \times \log(\sqrt{\lambda}|z|/2)[1 + O(|z|^2)] & \text{for} \ n > 2;
\end{cases}
\]

here \( \omega_n \), as before, denotes the volume of the unit ball in \( \mathbb{R}^n \), and the \( O(|z|^2) \) terms are analytic functions of \( |z|^2 \).

Under the normal coordinates,

\[ \xi = q(\epsilon, \eta) = \exp_x \epsilon \eta, \]

\( \epsilon \geq 0, \eta \in \mathcal{S}_x = \{ \eta \in M_x \mid |\eta| = 1 \} \), about \( x \). As discussed in Section III.1 of \([17]\), the volume element \( dV_\eta \) is given by

\[
dV_\eta(q(\epsilon, \eta)) = \sqrt{|g(\epsilon, \eta)|} \, d\epsilon \, d\mu_x(\eta),
\]

where \( d\mu_x \) is the standard \((n-1)\)-measure on \( \mathcal{S}_x \); and the \((n-1)\)-dimensional volume element of \( \partial B(x, r) \) is given by

\[
dS_\eta(q(\epsilon, \eta)) = \sqrt{g(\epsilon, \eta)} d\mu_x(\eta).
\]
The discussion of Sections III.1 and XII.8 of [7] shows that

$$\lim_{\epsilon \to 0} \sqrt{|g(\epsilon, \eta)|} e^{n-1} = 1.$$ 

Since $u \in W^{2,2}(\Omega)$, by applying local regularity of elliptic equations (see, for example, Theorem A.2.1 of [10]) repeatedly, we get that $u \in W^{2,j}(\overline{B}(x, \epsilon))$ for all $j = 1, 2, 3, \cdots$, which implies $u \in C^\infty(B(x, \epsilon))$. It follows from (2.2) and (2.3) that for $\epsilon \to 0$,

$$\frac{1}{\epsilon} \left| \int_{\partial B(x, \epsilon)} F(x, \xi) \frac{\partial u(\xi)}{\partial \nu_\xi} dS_\xi(\xi) \right| \leq (n\omega_n e^{n-1} + o(\epsilon^{n-1})) |F_1(\epsilon) + O(\epsilon)| \sup_{B(x, \epsilon)} |\nabla u| \to 0.$$

Furthermore,

$$\int_{\partial B(x, \epsilon)} u(\xi) \frac{\partial F(x, \xi)}{\partial \nu_\xi} dS_\xi(\xi) = (\frac{\partial F_1(\epsilon)}{\partial \xi} + O(1)) \int_{\partial B(x, \epsilon)} u(\xi) dS_\xi(\xi)$$

$$= \left( n\omega_n e^{n-1} + b(\epsilon) + O(1) \right) \int_{\partial B(x, \epsilon)} u(\xi) dS_\xi(\xi) \to u(x),$$

where $b(\epsilon)$ satisfies $\lim_{\epsilon \to 0} n\omega_n e^{n-1} b(\epsilon) = 0$. Altogether, we get

$$u(x) = - \int_{\partial \Omega} F(x, \xi) \frac{\partial u(\xi)}{\partial \nu_\xi} dS_\xi(\xi).$$

Since $u$ does not vanish identically in $\Omega$, by the above formula we get that $\frac{\partial u}{\partial \nu_\xi}|_{\partial \Omega}$ does not vanish identically on $\partial \Omega$. □

**Remark 2.2.** (a) When $M$ is a bounded domain with $C^{2,\alpha}$-smooth boundary in a real analytic Riemannian manifold $(M, g)$, Lemma 2.1 can be immediately proved as follows. Suppose by contradiction that $u \in W^{1,2}_0(\Omega) \cap W^{2,2}(\Omega)$ satisfies

$$\begin{cases} \triangle_g u + \lambda u = 0 & \text{in } \Omega, \\ u = 0, \quad \frac{\partial u}{\partial \nu_m} = 0 & \text{on } \partial\Omega. \end{cases}$$

Since the elliptic operator $\triangle_g + \lambda$ has real analytic coefficients in local coordinates chart, it follows from Shauder’s estimate (see, for example, Theorem 6.15 of [15]) that $u \in C^{2,\alpha}(\Omega)$. Applying Holmgren’s uniqueness theorem (see Theorem 2 of p.42 in [30] or p.433 of [10]), we obtain $u \equiv 0$ in $\Omega$. This contradicts the assumption that $u$ does not vanish identically in $\Omega$.

(b) When $M = \mathbb{R}^n$, the proof of Lemma 2.1 is quite easy. Indeed, it follows form Rellich’s formula for the Dirichlet eigenvalue (see [57]) that

$$\lambda = \frac{\int_{\partial \Omega} \sum_{m=1}^n \left( \frac{\partial u}{\partial \nu_m} \right)^2 x_m \nu_m dS}{2 \int_{\Omega} u^2 dx},$$

where $\nu(x) = (\nu_1(x), \cdots, \nu_n(x))$ with $x \in \partial \Omega$. Since $\lambda \neq 0$, we get that $\frac{\partial u}{\partial \nu_m}|_{\partial \Omega}$ cannot vanish identically on $\partial \Omega$.

**Lemma 2.3.** Let $(M, g)$ be an $n$-dimensional Riemannian manifold ($n \geq 2$), and let $\Omega \subset M$ be a bounded domain with $C^2$-smooth boundary. Then, for any $\tau$ we have

$W^{2,2}_0(\Omega) \cap M_\tau = \{0\}$,

where $M_\tau = \{ u \in W^{1,2}_0(\Omega) \cap W^{2,2}(\Omega) | \triangle_g u + \tau u = 0 \text{ in } \Omega \}$. 
Proof. Let $v \in W_0^{2,2}(\Omega)$. It follows from Corollary 6.2.43 of [10] that
\[ \frac{\partial^j v}{\partial \nu^j} = 0 \text{ on } \partial \Omega, \text{ for } 0 \leq j < \frac{3}{2}. \]
Thus, for any $v \in W_0^{2,2}(\Omega) \cap M_\tau$, we have
\[
\begin{cases}
\Delta_g v + \tau v = 0 & \text{in } \Omega, \\
v = 0, \quad \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial \Omega.
\end{cases}
\]
By applying Lemma 2.1, we get $v \equiv 0$ in $\Omega$. \qed

Denote by $\sigma_N$ (respectively, $\sigma_D$, $\sigma_P$ and $\sigma_B$) the spectra of the Neumann (respectively, the Dirichlet, the clamped plate and the buckling) problem for a bounded domain in Riemannian manifold $(M, g)$. Let
\[
\begin{align*}
N^{(N)}(\tau) &= \# \{ \mu_k \in \sigma_N | \mu_k \leq \tau \}, \quad N^{(D)}(\tau) = \# \{ \lambda_k \in \sigma_D | \lambda_k \leq \tau \}, \\
N^{(P)}(\tau) &= \# \{ \Gamma_k^2 \in \sigma_P | \Gamma_k \leq \tau \}, \quad N^{(B)}(\tau) = \# \{ \Lambda_k \in \sigma_B | \Lambda_k \leq \tau \}
\end{align*}
\]
be the counting functions of $\sigma_N$, $\sigma_D$, $\sigma_P$ and $\sigma_B$, respectively. Each eigenvalue is counted as many times as its multiplicity.

Lemma 2.4. For any $\tau$ we have
\[
\begin{align*}
(2.6) \quad N^{(N)}(\tau) &= \max \{ \dim L | L \subset W^{1,2}(\Omega), \int_\Omega |\nabla_g u|^2 \, dv_g \leq \tau \int_\Omega |u|^2 \, dv_g, \ u \in L \}, \\
(2.7) \quad N^{(D)}(\tau) &= \max \{ \dim L | L \subset W_0^{1,2}(\Omega), \int_\Omega |\nabla_g u|^2 \, dv_g \leq \tau \int_\Omega |u|^2 \, dv_g, \ u \in L \}, \\
(2.8) \quad N^{(B)}(\tau) &= \max \{ \dim L | L \subset W_0^{2,2}(\Omega), \int_\Omega |\Delta_g u|^2 \, dv_g \leq \tau \int_\Omega |u|^2 \, dv_g, \ u \in L \}, \\
(2.9) \quad N^{(P)}(\tau) &= \max \{ \dim L | L \subset W_0^{2,2}(\Omega), \int_\Omega |\Delta_g u|^2 \, dv_g \leq \tau^2 \int_\Omega |u|^2 \, dv_g, \ u \in L \},
\end{align*}
\]
where $\nabla_g u$ is the gradient of $u$ which has the expression in local coordinates
\[ \nabla_g u = \sum_{i,j=1}^n (g_{ij})^{g} \frac{\partial u}{\partial x_i} \frac{\partial}{\partial x_j}. \]

Proof. (i) The argument proving (2.6) and (2.7) is completely analogous to the one used in the Euclidean space (see [14] or [11]). Actually, the formulas (2.6) and (2.7) are known as Glazman's variational principle.

(ii) For any fixed $\tau$, let $\Lambda_1, \cdots, \Lambda_k$ be all the buckling eigenvalues that are not greater than $\tau$. Then the corresponding buckling eigenfunctions $W_1, \cdots, W_k$ span a $k$-dimensional linear subspace $S_k$ of $W_0^{2,2}(\Omega)$ (Suppose by contradiction that $W_m = c_1W_1 + \cdots + c_{m-1}W_{m-1} + c_{m+1}W_{m+1} + \cdots + c_kW_k$ for some $m$, where $c_1, \cdots, c_{m-1}, c_{m+1}, \cdots, c_k$ are constants. Therefore, $\int_\Omega \nabla_g W_m \cdot (\nabla_g W_m - \sum_{i \neq m} c_i \nabla_g W_i) \, dv_g = 0$. Noticing that
\[ \int_\Omega \nabla_g W_i \cdot \nabla_g W_j \, dv_g = \begin{cases} 1 & \text{when } i = j, \\
0 & \text{when } i \neq j, \end{cases} \]
we obtain $\int_\Omega |\nabla_g W_m|^2 \, dv_g = 0$, so that $W_m$ is a constant in $\Omega$. In view of $W_m |_{\partial \Omega} = 0$ we get that $W_m \equiv 0$ in $\Omega$, which is a contradiction). It suffices to prove that the right-hand side of (2.8) is also $k$. If it is not this case, then there exists a $(k+1)$-dimensional linear subspace $L_{k+1}$ of $W_0^{2,2}(\Omega)$ such that
\[
(2.10) \quad \int_\Omega |\Delta_g u|^2 \, dv_g \leq \tau \int_\Omega |\nabla_g u|^2 \, dv_g \quad \text{for all } u \in L_{k+1}.
\]
Thus, $E \cap L_{k+1} \neq 0$ for any linear subspace $E$ of $W^{2,2}_0(\Omega)$ with $\text{codim}(E) = k$. It follows from this and the variational formula

$$\Lambda_{k+1} = \sup_{E \subset W^{2,2}_0(\Omega), \text{codim } E = k} \left( \inf_{w \in E} \frac{\int_{\Omega} |\Delta_g w|^2 dV_g}{\int_{\Omega} |\nabla_g w|^2 dV} \right)$$

that $\Lambda_{k+1} \leq \tau$, which is a contradiction. Therefore (2.8) holds.

(iii) The proof of (2.9) is similar to (ii). □

**Lemma 2.5.** Let $(M, g)$ be a Riemannian manifold, and let $\Omega \subset M$ be a bounded domain with $C^2$-smooth boundary. Suppose $\Omega_1, \cdots, \Omega_m$ are pairwise disjoint domains in $\Omega$, each of which has piecewise $C^2$-smooth boundary. Arrange all the buckling eigenvalues of $\Omega_1, \cdots, \Omega_m$ in an increasing sequence

$$\Lambda_1^* \leq \Lambda_2^* \leq \Lambda_3^* \leq \cdots \tag{2.11}$$

with each eigenvalue repeated according to its multiplicity, and let

$$\Lambda_1 \leq \Lambda_2 \leq \Lambda_3 \leq \cdots \tag{2.12}$$

be the buckling eigenvalues for $\Omega$. Then we have for all $k = 1, 2, 3, \cdots$

Proof. For $j = 1, 2, \cdots, k$, let $\psi_j : \Omega \rightarrow \mathbb{R}^n$ be a buckling eigenfunction of $\Lambda_j^*$ when restricted to the appropriate subdomain, and identically zero, otherwise. Obviously,

$$\int_{\Omega} \nabla_g \psi_i \cdot \nabla_g \psi_j dV_g = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Let $W_1, \cdots, W_{k-1}$ be the buckling eigenfunctions corresponding to eigenvalues $\Lambda_1, \cdots, \Lambda_{k-1}$, respectively, which satisfy

$$\int_{\Omega} (\nabla_g W_i \cdot \nabla_g W_j) dV_g = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Consider the functions $f$ of the form

$$f = \sum_{j=1}^{k} \beta_j \psi_j,$$

where $f$ satisfies

$$\sum_{j=1}^{k} \beta_j \int_{\Omega} (\nabla_g W_i \cdot \nabla_g \psi_j) dV_g = 0, \quad i = 1, 2, \cdots, k-1. \tag{2.13}$$

If we think of $\beta_1, \cdots, \beta_k$ as unknowns and $\int_{\Omega} (\nabla_g W_i \cdot \nabla_g \psi_j) dV_g$ as given coefficients, then system has more unknowns than equations and a nontrivial solution of (2.13) must exist. Applying Green’s formula and the definition of $\psi_j$, we have

$$\int_{\Omega} (\Delta_g \psi_i)(\Delta_g \psi_j) dV_g = \int_{\Omega} \psi_i(\Delta_g^2 \psi_j) dV_g = -\Lambda_j^* \int_{\Omega} \psi_i(\Delta_g \psi_j) dV_g$$

$$= \Lambda_j^* \int_{\Omega} (\nabla_g \psi_i \cdot \nabla_g \psi_j) dV_g = \begin{cases} \Lambda_j^* & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$
Hence
\[ \Lambda_k \int_{\Omega} |\nabla_g f|^2 dV_g \leq \int_{\Omega} |\triangle_g f|^2 dV_g = \sum_{j=1}^{k} \Lambda_j^2 \beta_j^2 \leq \Lambda_k^2 \int_{\Omega} |\nabla_g f|^2 dV_g, \]
which implies the desired result. \( \square \)

3. Inequalities of eigenvalues

Proof of Theorem 1.1. (i) We shall prove \( \mu_k < \lambda_k \) for all positive integer \( k \). The proof is analogous to [12]. For any fixed \( \tau \), it follows from (2.7) of Lemma 2.4 that there exists a subspace \( F \) of \( W^{1,2}_0(\Omega) \) such that \( \dim F = N^{(D)}(\tau) \) and
\[ \int_{\Omega} |\nabla_g u|^2 dV_g \leq \tau \int_{\Omega} |u|^2 dV_g, \quad u \in F. \]
Let \( u \in F \cap M_\tau \), where \( M_\tau = \{ v | \triangle_g v + \tau v = 0 \text{ in } \Omega, \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega \} \). Since \( \partial \Omega \in C^2 \), it follows from \( u \in M_\tau \) and the regularity of elliptic equations (see, for example, [1] or Theorem 8.12 of [13]) that \( u \in W^{2,2}(\Omega) \). From \( u \in W^{1,2}_0(\Omega) \), we get that \( u = 0 \) on \( \partial \Omega \), as mentioned earlier. This implies that \( u \) is also a Dirichlet eigenfunction with eigenvalue \( \tau \). By Lemma 2.1, we get that \( \frac{\partial u}{\partial \nu} \) cannot vanish identically in \( \Omega \), which contradicts the fact that \( \frac{\partial u}{\partial \nu} = 0 \) on \( \partial \Omega \). Thus \( F \oplus M_\tau \) is a direct sum and we denote it by \( G_\tau \). Let \( u + v \in G_\tau \subset W^{1,2}_0(\Omega) \), where \( u \in F \) and \( v \in M_\tau \). We have
\[
\int_{\Omega} |\nabla_g (u + v)|^2 dV_g = \int_{\Omega} (|\nabla_g u|^2 + |\nabla_g v|^2 + 2\nabla_g u \cdot \nabla_g v) dV_g
\leq \tau \int_{\Omega} |u + v|^2 dV_g,
\]
so that
\[ N^{(N)}(\tau) \geq \dim G_\tau = N^{(D)}(\tau) + \dim M_\tau, \]
Taking \( \tau = \lambda_k \), we have
\[ \# \{ \mu_j \in \sigma_N | \mu_j < \lambda_k \} = N^{(N)}(\lambda_k) - \dim M_{\lambda_k} \geq N^{(D)}(\lambda_k) = k. \]
That is, \( \mu_k < \lambda_k \).

(ii) It follows from (2.9) of Lemma 2.4 that there exists a subspace \( H_\tau \) of \( W^{2,2}_0(\Omega) \) such that \( \dim H_\tau = N^{(D)}(\tau) \) and
\[ \int_{\Omega} |\triangle_g w|^2 dV_g \leq \tau^2 \int_{\Omega} |w|^2 dV_g, \quad \forall w \in H_\tau. \]
Let \( K_\tau = \{ v | \triangle_g v + \tau v = 0 \text{ in } \Omega, \text{ and } v = 0 \text{ on } \partial \Omega \} \), and let \( u \in H_\tau \cap K_\tau \). Since \( u \in H_\tau \subset W^{2,2}_0(\Omega) \), we find by Corollary 6.2.43 of [10] that \( u = \frac{\partial u}{\partial \nu} = 0 \) on \( \partial \Omega \). Lemma 2.1 implies that \( u = 0 \) in \( \Omega \), therefore, the sum \( G_\tau := H_\tau \bigoplus K_\tau \) is direct. Let
$u = w + v \in G_\tau$, where $w \in H_\tau$, $v \in K_\tau$. It follows from Green’s formula and Schwarz’s inequality that for $w \neq 0$,
\[
\left( \frac{\int_\Omega |\nabla_g w|^2 dV_g}{\int_\Omega |w|^2 dV_g} \right)^2 \leq \frac{\int_\Omega |\Delta_g w|^2 dV_g}{\int_\Omega |w|^2 dV_g}.
\]
From this and the definition of $H_\tau$, we get
\[
\int_\Omega |\nabla_g w|^2 dV_g \leq \tau \int_\Omega |w|^2 dV_g.
\]
Note that
\[
\int_\Omega |\nabla_g v|^2 dV_g = \tau \int_\Omega |v|^2 dV_g, \text{ for } v \in K_\tau
\]
and
\[
\int_\Omega \nabla_g w \cdot \nabla_g v dV_g = -\int_\Omega w(\Delta_g v) dV_g = \tau \int_\Omega w v dV_g.
\]
Therefore, for any $u = w + v \in G_\tau \subset W^{2,2}_0(\Omega)$ we have
\[
\int_\Omega |\nabla_g (w + v)|^2 dV_g = \int_\Omega \left( |\nabla_g w|^2 + |\nabla_g v|^2 + 2 \nabla_g w \cdot \nabla_g v \right) dV_g 
\]
\[
\leq \tau \int_\Omega |w + v|^2 dV_g.
\]
For $0 = w \in H_\tau$, there is equality in the above inequality. It follows that
\[
N^{(D)}(\tau) \geq \dim G_\tau = N^{(P)}(\tau) + \dim K_\tau.
\]
By taking $\tau = \Gamma_k$, we obtain
\[
\# \{\lambda_j \in \sigma_D | \lambda_j < \Gamma_k\} = N^{(D)}(\Gamma_k) - \dim K_{\Gamma_k} \geq N^{(P)}(\Gamma_k) = k,
\]
Hence $\lambda_k < \Gamma_k$.

(iii) For fixed $\tau > 0$, (2.8) of Lemma 2.4 implies that there exists a subspace $L_\tau$ of $W^{2,2}_0(\Omega)$ such that $\dim L_\tau = N^{(B)}(\tau)$ and
\[
\int_\Omega |\Delta_g w|^2 dV_g \leq \tau \int_\Omega |\nabla_g w|^2 dV_g, \quad w \in L_\tau.
\]
Denote $J_\tau = \{v|\Delta_g^2 v - \Delta_g v = 0 \text{ in } \Omega, \text{ and } v = \frac{\partial w}{\partial n} = 0 \text{ on } \partial \Omega\}$. and put $G_\tau = L_\tau + J_\tau$. We shall prove $L_\tau \cap J_\tau = \{0\}$. Suppose that $0 \neq u \in L_\tau \cap J_\tau$. Then, in view of $u = 0$ on $\partial \Omega$ we get that $\nabla_g u$ and $\Delta_g u$ don’t vanish identically in $\Omega$. It follows from Green’s formula and Schwarz’s inequality that for any $u \in W^{2,2}_0(\Omega)$,
\[
(3.1) \quad \int_\Omega |\nabla_g u|^2 dV_g = \left| - \int_\Omega u(\Delta_g u) dV_g \right| \leq \left( \int_\Omega |u|^2 dV_g \right)^{1/2} \left( \int_\Omega |\Delta_g u|^2 dV_g \right)^{1/2},
\]
i.e.,
\[
(3.2) \quad \frac{\int_\Omega |\Delta_g u|^2 dV_g}{\int_\Omega |u|^2 dV_g} \leq \left( \frac{\int_\Omega |\Delta_g u|^2 dV_g}{\int_\Omega |\nabla_g u|^2 dV_g} \right)^2, \quad \forall u \in W^{2,2}_0(\Omega).
\]
Note that
\[
(3.3) \quad \frac{\int_\Omega |\Delta_g u|^2 dV_g}{\int_\Omega |\nabla_g u|^2 dV_g} \leq \tau, \quad \forall u \in L_\tau
\]
and
\begin{equation}
\tau^2 = \frac{\int_{\Omega} |\triangle_g u|^2 dV_g}{\int_{\Omega} |u|^2 dV_g}, \quad \forall u \in J_\tau.
\end{equation}

Therefore, Schwarz’s inequality in (3.1) is an equality, which implies there exists a constant \( \beta \) such that \( \triangle_g u + \beta u = 0 \) in \( \Omega \). Since \( u = 0 \) on \( \partial \Omega \), it follows that \( \beta > 0 \) and \( u \) is a Dirichlet eigenfunction. Thus, we find by \( \frac{\partial u}{\partial \nu} |_{\partial \Omega} = 0 \) and Lemma 2.1 that \( u \equiv 0 \) in \( \Omega \).

This shows that the sum \( L_\tau \oplus J_\tau \) is direct (we still denote the direct sum by \( G_\tau \)). For any \( u = w + v \in G_\tau \subset W^{2,2}_0(\Omega) \), where \( w \in L_\tau \), \( v \in J_\tau \), we have
\begin{equation}
\int_{\Omega} |\triangle_g (w + v)|^2 dV_g = \int_{\Omega} \left[ |\triangle_g w|^2 + |\triangle_g v|^2 + 2w(\triangle_g^2 v) \right] dV_g
\end{equation}
By (3.2)–(3.5), we arrive at
\begin{equation}
\int_{\Omega} |\triangle_g (w + v)|^2 dV_g \leq \tau^2 \int_{\Omega} |w + v|^2 dV_g.
\end{equation}
This implies that
\begin{equation}
N^{(P)}(\tau) \geq \dim G_\tau = N^{(B)}(\tau) + \dim J_\tau,
\end{equation}
i.e.,
\begin{equation}
N^{(P)}(\tau) - \dim J_\tau \geq N^{(B)}(\tau).
\end{equation}
Setting \( \tau = \Lambda_k \), we see that
\begin{equation}
\# \{ \Gamma^2_j \in \sigma_P | \Gamma_j \subset \Lambda_k \} = N^{(P)}(\Lambda_k) - \dim J_{\Lambda_k} \geq N^{(B)}(\Lambda_k) = k,
\end{equation}
that is, \( \Gamma_k \lesssim \Lambda_k \). □

Remark 3.1. (i) For a bounded domain of a Riemannian manifold, Mazzeo \cite{23} had showed that
\begin{equation}
\mu_k \leq \lambda_k, \quad k = 1, 2, 3, \ldots.
\end{equation}
(Actually, Mazzeo proved that inequalities \( \mu_k + 1 \leq \lambda_k \) when \( M \) is a Riemannian symmetric space of noncompact type). Therefore, the strict inequalities
\begin{equation}
\mu_k < \lambda_k, \quad k = 1, 2, 3, \ldots
\end{equation}
is an improvement of Mazzeo’s result in the general Riemannian manifold. The following example shows inequalities (3.7) cannot be improved such that \( \mu_{k+1} \leq \lambda_k \) holds for \( k = 1, 2, 3, \ldots \). In fact, for the spherical cap of radius \( \delta > \frac{\pi}{2} \) on the sphere of radius 1 in \( \mathbb{R}^n \), one has \( \mu_2(\Omega) > \lambda_1(\Omega) \) (see, Theorem 3 of p.44 in \cite{17}). This fact was also pointed out by Mazzeo in \cite{23}. Therefore our strict inequalities (3.7) are sharp.

(ii) The inequalities
\begin{equation}
\lambda_k \lesssim \Gamma_k, \quad \text{for } k = 1, 2, 3, \ldots
\end{equation}
are a generalization of Weinstein’s inequality to \( n \)-dimensional Riemannian manifold. Here our proof is completely different from that of \cite{42}. The inequalities (3.8) cannot be improved to be \( \lambda_{k+1} \leq \Gamma_k \) for \( k = 1, 2, 3, \ldots \). Indeed, let \( \Omega \) be the unit disk \( \{ x \in \mathbb{R}^n \mid \}

\end{equation}
$\mathbb{R}^2 | |x| < 1$. Denote by $J_m(r)$ the Bessel function of order $m$ and by $j_m^{(l)}$ its $l$-th positive zero. Then the Dirichlet eigenfunctions are

$$\phi_{m,l} = a_{m,l} J_m(\sqrt{\lambda_{m,l}} r) \begin{cases} \cos m\theta, & m = 0, 1, 2, \ldots; l = 1, 2, 3, \ldots, \\ \sin m\theta, & \end{cases}$$

and the corresponding eigenvalues are $\lambda_{m,l} = (j_m^{(l)})^2$. Thus $\lambda_1(\Omega) \approx (2.4048)^2$, $\lambda_2(\Omega) = \lambda_3(\Omega) \approx (3.832)^2$. It follows from p.231 of [34] that $\Gamma_1(\Omega) \approx (3.1962)^2$ (where $3.1962 \cdots$ is the first zero of $J_0(r)I_1(r) + J_1(r)I_0(r)$, $r > 0$, and $I_m(r)$ is the modified Bessel function of order $m$). This means that $\lambda_2(\Omega) > \Gamma_1(\Omega)$.

(iii) For $k = 2, 3, 4, \cdots$, our inequalities $\Gamma_k < \Lambda_k$ ($k = 2, 3, 4, \cdots$) are completely new even for the case $M = \mathbb{R}^n$. It is also sharp since it cannot be improved such that $\Gamma_{k+1} \leq \Lambda_k$ holds for $k = 1, 2, 3, \cdots$. In fact, let $\Omega = \{x \in \mathbb{R}^2 | |x| < 1\}$. Then we claim that $\Gamma_2(\Omega) > \Lambda_1(\Omega)$. Suppose by contradiction that $\Gamma_2(\Omega) \leq \Lambda_1(\Omega)$. It follows from Theorem 1.1 that $\lambda_2(\Omega) < \Gamma_2(\Omega)$. Thus we get $\lambda_2(\Omega) < \Lambda_1(\Omega)$. However, for the unit disk $\Omega$, it must be $\lambda_2(\Omega) = \Lambda_1(\Omega) \approx (3.832)^2$. This is a contradiction, and the claim is verified.

4. ASYMPTOTIC FORMULA FOR THE BUCKLING EIGENVALUES IN $\mathbb{R}^n$

First, we consider the one-dimensional buckling problem:

(4.1) \[ u'''(x) + \Lambda u''(x) = 0, \quad 0 \leq x \leq L, \]
(4.2) \[ u(0) = u(L) = 0, \quad u'(0) = u'(L) = 0. \]

It is easy to check that the general solution of (4.1) is

$$u(x) = C_1 + C_2 x + C_3 \cos \sqrt{\Lambda} x + C_4 \sin \sqrt{\Lambda} x.$$ 

The boundary conditions yield the following equations for the coefficients:

$$\begin{cases} C_1 = -C_3, & C_2 = -\sqrt{\Lambda} C_4, \\ C_1 + C_2 L + C_3 \cos \sqrt{\Lambda} L + C_4 \sin \sqrt{\Lambda} L = 0, \\ C_2 - \sqrt{\Lambda} C_3 \sin \sqrt{\Lambda} L + \sqrt{\Lambda} C_4 \cos \sqrt{\Lambda} L = 0. \end{cases}$$

In order that this system of equations has a nontrivial solution, $\sqrt{\Lambda}$ must satisfy

$$\sin \frac{\sqrt{\Lambda} L}{2} \left[ 2 \sin \frac{\sqrt{\Lambda} L}{2} - \sqrt{\Lambda} L \cos \frac{\sqrt{\Lambda} L}{2} \right] = 0.$$ 

From the equation $\sin \frac{\sqrt{\Lambda} L}{2} = 0$, we obtain that

$$\Lambda_{1,k}(L) = \left( \frac{2k\pi}{L} \right)^2, \quad k = 1, 2, 3, \cdots,$$

and the associated eigenfunctions are

$$u_{1,k}(L, x) = 1 - \cos \frac{2k\pi}{L} x, \quad k = 1, 2, 3, \cdots.$$
From the equation $2 \sin \frac{\sqrt{\Lambda L}}{2} - \sqrt{\Lambda L} \cos \frac{\sqrt{\Lambda L}}{2} = 0$, we get that

$$\tan \frac{\sqrt{\Lambda L}}{2} = \frac{\sqrt{\Lambda L}}{2}. \quad (4.3)$$

If we denote by $\{\sqrt{\Lambda_{2,k}(L)}|k = 1, 2, 3, \ldots\}$ all the positive roots of $(4.3)$, then

$$u_{2,k}(L, x) = 1 + \frac{\sqrt{\Lambda_{2,k}(L)} \sin(L \sqrt{\Lambda_{2,k}(L)})}{\cos(L \sqrt{\Lambda_{2,k}(L)}) - 1} x - \cos(L \sqrt{\Lambda_{2,k}(L)})$$

$$- \frac{\sin(L \sqrt{\Lambda_{2,k}(L)})}{\cos(L \sqrt{\Lambda_{2,k}(L)}) - 1} \sin(L \sqrt{\Lambda_{2,k}(L)}) x$$

is the buckling eigenfunction corresponding to the eigenvalue $\Lambda_{2,k}(L)$. By solving the system of equations

$$\begin{cases} y = x \\ y = \tan x \end{cases},$$

we find that $\frac{2\pi}{L} < \sqrt{\Lambda_{2,k}(L)} < \frac{(2k+1)\pi}{L}$ for all $k = 1, 2, 3, \ldots$.

**Example 4.1.** From the above argument, we obtain all the buckling eigenvalues for the interval $[0, L]$:

$$\Lambda_1 = \left(\frac{2\pi}{L}\right)^2, \quad \Lambda_2 = \Lambda_{2,1}(L), \quad \Lambda_3 = \left(\frac{4\pi}{L}\right)^2, \quad \Lambda_4 = \Lambda_{2,2}(L), \ldots, \ldots. \quad (4.4)$$

A simple calculation shows that the Dirichlet eigenvalues for the interval $[0, L]$ are

$$\lambda_1 = \left(\frac{\pi}{L}\right)^2, \quad \lambda_2 = \left(\frac{2\pi}{L}\right)^2, \quad \lambda_3 = \left(\frac{3\pi}{L}\right)^2, \quad \lambda_4 = \left(\frac{4\pi}{L}\right)^2, \ldots, \ldots, \quad (4.5)$$

and the corresponding Dirichlet eigenfunctions are $u_k(x) = \sin\left(\frac{k\pi x}{L}\right), \ k = 1, 2, 3, \ldots$. Recall that $\Lambda_{2,1}(L) < \left(\frac{2\pi}{L}\right)^2$, i.e., $\Lambda_2 < \Lambda_3$. This shows that the Payne conjecture is not true for the one-dimensional case.

**Theorem 4.2.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with $C^2$-smooth boundary. Then,

$$N^{(B)}(\tau) = (2\pi)^{-n} \omega_n |\Omega| \tau^{n/2} (1 + o(1)) \quad \text{as} \quad \tau \to +\infty. \quad (4.6)$$

**Proof.** We give the proof for $n = 2$ only, indicating at its conclusion how it can be augmented to yield the $n$-dimensional case.

(i) Let $Q = [0, a] \times [0, b]$ be a rectangle in $\mathbb{R}^2$. Let us consider the buckling eigenvalue problem on $Q$:

$$\begin{cases} \Delta^2 u + \Lambda \Delta u = 0, & \text{in} \ Q, \\
u = \frac{\partial u}{\partial \nu} = 0, & \text{on} \ \partial Q. \end{cases}$$

It is easy to check that the buckling eigenvalues are

$$\pi^2 \left(\frac{(2l)^2}{a^2} + \frac{(2m)^2}{b^2}\right), \quad \Lambda_{2,l}(a) + \Lambda_{2,m}(b), \quad \Lambda_{2,l}(a) + \Lambda_{2,m}(b),$$

$$\Lambda_{2,1}(a) + \Lambda_{2,1}(b), \quad \Lambda_{2,1}(a) + \Lambda_{2,1}(b).$$
If the number of the buckling eigenvalues for $\Omega$ no greater than a bound $N$ denoted by each lattice point $(2l, 2m)$, this increases only with $\max\{a, b\}$. As in p.431 of [9], a of these squares is contained in this sector of the ellipse; if we add more squares with the region formed by all the squares contains the sector of the ellipse. Thus, we get
\[
\int_{-\pi/4}^{\pi/4} \frac{1}{\sqrt{1 - \frac{2-a^2 b^2}{a^2} \cos^2 t}} dt \text{ if } a > b.
\]
Therefore, we have the asymptotic formula
\[
\lim_{\tau \to +\infty} \frac{N_1^B(\tau)}{\tau} = \frac{ab}{16\pi}.
\]
More precisely, we can write
\[ N_1(B)(\tau) = \frac{ab}{16\pi} \tau + \theta_1 c_1(\max\{a, b\}) \sqrt\tau, \]
where \( c_1 \) is a constant independent of \( \tau \), and \( -1 < \theta_1 < 1 \).

Next, we shall estimate \( N_2(B)(\tau) \). Recall that \( (\frac{2m\pi}{b})^2 < \Lambda_2, m(b) < (\frac{(2m+1)\pi}{b})^2 \) for all \( m = 1, 2, 3, \ldots \). Then, we have that
\[ N_2(B)(\tau) \geq \# \left\{ (l, m) \left| \frac{(2l)^2}{a^2} + \frac{(2m+1)^2}{b^2} \leq \frac{\tau}{\pi^2}, l = 1, 2, \ldots ; m = 1, 2, \ldots \right. \right\}. \]

Similar to the argument for \( N_1(B)(\tau) \), the ratio of the area of this sector to the number of lattice points \( \{(2l, 2(m+1))\} \) contained in it is arbitrarily close to 4 for sufficiently large \( \tau \). In fact, if a square with the side length 2 lying below and to the left of each lattice point is associated with it, then the region composed of these squares is contained in this sector of the ellipse. Thus, we get
\[ 4N_2(B)(\tau) + T(\tau) + 4R(\tau) \geq \tau \frac{ab}{4\pi}, \]
where \( R(\tau) \) is as before, and \( T(\tau) \) is the number of the unit squares (in this sector) whose bottom sides lie in the x-axis. Obviously, \( T(\tau) \leq a\sqrt{\tau} \). Therefore, we have the asymptotic relation
\[ \lim_{\tau \to \infty} \frac{N_2(B)(\tau)}{\tau} \geq \frac{ab}{16\pi}. \]

In other words, we can write
\[ N_2(B)(\tau) \geq \frac{ab}{16\pi} \tau + \theta_2 c_2(\max\{a, b\}) \sqrt{\tau}, \]
where \( c_2 \) is a constant independent of \( \tau \), and \( -1 < \theta_2 < 1 \). With the similar argument as before, we can get
\[ N_i(B)(\tau) \geq \frac{ab}{16\pi} \tau + \theta_i c_i(\max\{a, b\}) \sqrt{\tau}, \quad \text{for } i = 3, 4. \]

It follows from (4.8) — (4.10) that
\[ N(B)(\tau) \geq \frac{ab}{4\pi} \tau + \theta c(\max\{a, b\}) \sqrt{\tau}. \]

(ii) Let \( \Omega \) be a domain which may be decomposed into a finite number, say \( h \), of squares \( Q_1, Q_2, \ldots, Q_h \) (or \( n \)-dimensional cubes in the case of \( n \) independent variables) of side \( a \). Such domains will be called square-domains (or \( n \)-dimensional cube-domains). The area of \( \Omega \) is then \( |\Omega| = ha^2 \) (or its \( n \)-dimensional volume is \( |\Omega| = ha^n \)). We consider the buckling problem for the domain \( \Omega \). From the previous argument, we have
\[ N_i(Q_j)(\tau) \geq \frac{a_j^2}{4\pi} \tau + \theta_j c_j a_j \sqrt{\tau}, \quad j = 1, 2, \ldots, h, \]
where \( N_i(Q_j)(\tau) \) is the number of the buckling eigenvalues less than or equal to \( \tau \) for the subsquare \( Q_j \). It follows from Lemma 2.5 that for the square-domain, the \( k \)-th buckling eigenvalue \( \Lambda_k \) for the domain \( \Omega \) is at most equal to the \( k \)-th number \( \lambda^*_k \) in the sequence consisting of all the eigenvalues of the subdomains \( Q_j \) (ordered according to increasing magnitude and taken with their respective multiplicity). Therefore we have
\[ N(B)(\tau) \geq N(Q_1)(\tau) + \cdots + N(Q_h)(\tau), \]
where \( N^{(B)}(\tau) \) is the number of the buckling eigenvalues less than or equal to \( \tau \) for the domain \( \Omega \). Since the number \( N^{(B)}_{Q_j}(\tau) \) have the form given by inequalities (4.12), we get

\[
N^{(B)}(\tau) \geq \left\lfloor \frac{|\Omega|}{4\pi} \tau + \theta c |\partial \Omega| \sqrt{\tau},
\]

where \(-1 < \theta < 1\) and \( c \) is a constant independent of \( \tau \).

If there are \( n \) independent variables instead of two, the preceding discussion is still valid except for the expressions \( N^{(B)}_{Q_j}(\tau) \). It is easy to see that

\[
N^{(B)}_{Q_j}(\tau) \geq \frac{1}{2\pi^2} \omega_n a^n \tau^{n/2} \frac{n}{\pi^n} + \theta_j c_j a^{n-1} \tau^{n-1} \frac{n}{\pi^n}, \quad j = 1, 2, \ldots, h.
\]

where \( \omega_n \) is the volume of the unit ball in \( \mathbb{R}^n \). This implies that for an \( n \)-dimensional polyhedron of volume \( |\Omega| \) consisting of a finite number \( h \) of congruent cubes, we have

\[
N^{(B)}(\tau) \geq \left\lfloor \frac{\omega_n |\Omega| \tau^{n/2}}{(2\pi)^n} + \theta c |\partial \Omega| |\tau|^{n-1} \frac{n}{\pi^n},
\]

where \(-1 < \theta < 1\) and \( c \) is a constant independent of \( \tau \).

(iii) We now consider the buckling problem for arbitrary bounded domain with \( C^2 \)-smooth boundary. With the above argument, it is possible to obtain a lower bound for \( N^{(B)}(\tau) \).

Suppose the plane is partitioned into squares of side \( a \), inducing a decomposition of the domain \( \Omega \) into \( h \) squares \( Q_1, Q_2, \ldots, Q_h \) and \( r \) boundary domains \( G_1, G_2, \ldots, G_r \). It follows from Lemma 2.5 that

\[
N^{(B)}(\tau) \geq N^{(B)}_{Q_1}(\tau) + N^{(B)}_{Q_2}(\tau) + \cdots + N^{(B)}_{Q_h}(\tau);
\]

furthermore, by (4.12) we have

\[
N^{(B)}_{Q_j}(\tau) + \cdots + N^{(B)}_{Q_h}(\tau) \geq \frac{ha^2}{4\pi} \tau + \theta cha \sqrt{\tau} = \tau \left( \frac{ha^2}{4\pi} + \frac{\theta cha}{\sqrt{\tau}} \right),
\]

where, as before, \(-1 < \theta < 1\) and \( c \) is a constant independent of \( a \) and \( \tau \).

By applying (1.10) of Theorem 1.1 we obtain that

\[
N^{(N)}(\tau) \geq N^{(D)}(\tau) \geq N^{(P)}(\tau) \geq N^{(B)}(\tau), \quad \text{for any } \tau.
\]

On the other hand, for the same partition of the domain \( \Omega \), it follows from p.441 of (9) that

\[
N^{(D)}(\tau) \leq N^{(N)}_{Q_1}(\tau) + N^{(N)}_{Q_2}(\tau) + \cdots + N^{(N)}_{Q_h}(\tau) + N^{(N)}_{G_1}(\tau) + \cdots + N^{(N)}_{G_r}(\tau)
\]

\[
\leq \tau \left( \left( \frac{ha^2}{4\pi} + \theta_1 c_1 a^2 \right) + \left( \theta_2 c_2 h a + \theta_3 c_3 r a \right) \frac{1}{\sqrt{\tau}} \right),
\]

where \( N^{(N)}_{Q_j}(\tau) \) and \( N^{(N)}_{G_j}(\tau) \) are the numbers of the Neumann eigenvalues less than or equal to \( \tau \) for \( Q_j \) and \( G_j \), respectively. From (4.10) to (4.19), we obtain

\[
\tau \left( \left( \frac{ha^2}{4\pi} + \theta_1 c_1 a^2 \right) + \left( \theta_2 c_2 h a + \theta_3 c_3 r a \right) \frac{1}{\sqrt{\tau}} \right)
\]

\[
\geq N^{(D)}(\tau) \geq N^{(P)}(\tau) \geq N^{(B)}(\tau) \geq \tau \left( \frac{ha^2}{4\pi} + \frac{\theta cha}{\sqrt{\tau}} \right).
\]
Note that $ar < \tilde{c}$. Hence, for sufficiently small $a$, we see that $a^2r$ and $|ha^2 - |\Omega||$ are arbitrarily small. It follows from these inequalities that
\[
\lim_{\tau \to \infty} \frac{4\pi N^{(D)}(\tau)}{\tau|\Omega|} = \lim_{\tau \to \infty} \frac{4\pi N^{(P)}(\tau)}{\tau|\Omega|} = \lim_{\tau \to \infty} \frac{4\pi N^{(B)}(\tau)}{\tau|\Omega|} = 1
\]
since we may take a sufficiently small fixed $a$ (the quantity $a$ can be arbitrarily chosen) such that the factor of $\tau$ in (4.20) arbitrarily close to the value $|\Omega|/4\pi$ for sufficiently large $\tau$.

(iv) With the similar way as for the plane, we can get the desired result for the buckling eigenvalues in the $n$-dimensional case. \qed

5. Asymptotic formulas in Riemannian manifolds

Theorem 5.1. Let $(M, g)$ be an $n$-dimensional Riemannian manifold, and let $\Omega \subset M$ be a bounded domain with $C^2$-smooth boundary. Then,
\[
(5.1) \quad N^{(B)}(\tau) \sim (2\pi)^{-n} \omega_n (\text{vol}(\Omega)) \tau^{n/2} \quad \text{as} \quad \tau \to +\infty,
\]
\[
(5.2) \quad N^{(P)}(\tau) \sim (2\pi)^{-n} \omega_n (\text{vol}(\Omega)) \tau^{n/2} \quad \text{as} \quad \tau \to +\infty.
\]

Proof. For any $x_0 \in M$, we consider a geodesic, normal coordinates system at $x_0$. Under the normal coordinates one can expand the metric as follows:
\[
g_{ij} = \delta_{ij} - \frac{1}{3} \sum_{k,l=1}^{n} R_{ikjl} x^k x^l + O(|x|^3)
\]
and
\[
\sqrt{\det(g_{ij})} = 1 - \frac{1}{6} \sum_{i,j=1}^{n} R_{ij} x^i x^j + O(|x|^3),
\]
where $R_{ikjl}$ and $R_{ij}$ are, respectively, the components of the curvature tensor and Ricci tensor associated with $g$; this is accomplished by applying the exponential map to the tangent space at $0$ to obtain coordinates on a patch and then fixing things up outside (see [29], p.59 of [8] or Chapter 10 of [6]). We let $B_{x_0}(\varrho)$ be the ball on which this coordinates system is defined. We can choose $\varrho$ sufficiently small such that in $B_{x_0}(\varrho)$, the eigenvalues of $g_{ij}$ and $g^{ij}$ are between $(1 + \epsilon(\varrho))^{-1}$ and $(1 + \epsilon(\varrho))$, and furthermore
\[
dV_g = \sqrt{\det(g_{ij})} dx \quad \text{where} \quad (1 + \epsilon(\varrho))^{-1} < \sqrt{\det(g_{ij})} < (1 + \epsilon(\varrho)).
\]
Here $\epsilon(\varrho)$ is a positive function of variable $\varrho$, and $\epsilon(\varrho) \to 0$ as $\varrho \to 0$. Let $N$ be any compact sub-manifold in $M$ with
\[
(5.3) \quad Rc_g \geq -c \quad \text{on} \quad N,
\]
where $c$ is a positive constant. The classical Bochner-Lichnerowicz-Weitzenböck formula (see [21]) reads
\[
\int_N |\nabla_g u|^2 dV_g = \int_N |\Delta_g u|^2 dV_g - \int_N Rc_g(\nabla u, \nabla u) dV_g \quad \text{for any} \quad u \in C_0^\infty(N),
\]
where $|\nabla^2_g u|^2$ is defined in an invariant ways as

$$|\nabla^2_g u|^2 = \nabla^i \nabla^k u \nabla_i \nabla_k u = g^{pl} g^{kq} \left( \frac{\partial^2 u}{\partial x^k \partial x^l} - \Gamma^m_{kli} \frac{\partial u}{\partial x^m} \right) \left( \frac{\partial^2 u}{\partial x^p \partial x^q} - \Gamma^r_{pkl} \frac{\partial u}{\partial x^r} \right).$$

Together with (5.3), it implies that

$$\int_N |\nabla^2_g u|^2 dV_g \leq \int_N |\triangle_g u|^2 dV_g + c \int_N |\nabla_g u|^2 dV_g.$$  

Denote by $B_0(\varrho)$ the ball of $\mathbb{R}^n$ with the center 0 and radius $\varrho > 0$, and denote by $\triangle u$ and $\nabla u$ the usual the Laplacian and gradient of $u$ in $\mathbb{R}^n$. Passing in the coordinates system, we find by a similar way as in p.135 of [10] that

$$(\triangle_g u)^2 \leq (\triangle u)^2 + \tilde{\epsilon}(\varrho) |\nabla^2 u|^2 + \tilde{\epsilon}(\varrho) |\nabla u|^2,$$

for $u \in C^2_0(B_0(\varrho))$ where $\tilde{\epsilon}(\varrho) \to 0$ as $\varrho \to 0$, while by the Bochner-Lichnerowicz-Weitzenböck formula,

$$\int_{B_0(\varrho)} |\nabla^2 u|^2 dx = \int_{B_0(\varrho)} (\triangle u)^2 dx.$$  

Note that for any $u \in C^2_0(B_0(\varrho))$,

$$\int_{B_{2\varrho}(\varrho)} |\nabla_g u|^2 dV_g = \int_{B_0(\varrho)} \sum_{i,j=1}^n g^{ij} \sqrt{\det(g_{ij})} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx \geq \int_{B_0(\varrho)} (1 + \epsilon(\varrho))^{-2} |\nabla u|^2 dx.$$  

Thus, we have that for any $u \in C^2_0(B_0(\varrho))$,

$$\frac{\int_{B_{2\varrho}(\varrho)} (\triangle_g u)^2 dV_g}{\int_{B_0(\varrho)} |\nabla_g u|^2 dV_g} \leq (1 + \epsilon(\varrho))^2 (1 + \tilde{\epsilon}(\varrho)) \frac{\int_{B_0(\varrho)} (\triangle u)^2 dx}{\int_{B_0(\varrho)} |\nabla u|^2 dx} + (1 + \epsilon(\varrho))^2 \tilde{\epsilon}(\varrho).$$

We may always assume $\varrho$ is small enough such that $\lambda_1(B_0(\varrho)) > 1$, where $\lambda_1(B_0(\varrho))$ is the first Dirichlet eigenvalue for $B_0(\varrho)$. Since the geodesic open balls $\{B_{2\varrho}(\varrho) | x_0 \in M \}$ cover $\bar{\Omega}$, it follows from Lebesgue’s lemma (see, for example, Theorem 6.27 of [35]) that there exists a constant $\gamma > 0$ such that if any subdomain $G \subset \Omega$ satisfies $\text{diam}(G) < \gamma$, then $G$ must be contained in some $B_{\varrho}(\varrho)$. Let us part the domain $\Omega$ into $h$ subdomains $G_1, G_2, \cdots, G_h$ with piecewise $C^2$-smooth boundaries such that $\text{diam}(G_j) < \gamma, 1 \leq j \leq h$. It follows from Lemma 2.5 that the $k$-th buckling eigenvalue $\Lambda_k$ for the domain $\Omega$ is not greater than the $k$-th number $\Lambda_k^*$ in the sequence consisting of all the buckling eigenvalues of the subdomains $G_j$ (arranged according to increasing magnitude and taken with their respective multiplicity). Thus, we have

$$N^B(\tau) \geq N^{B}_{G_1}(\tau) + N^{B}_{G_2}(\tau) + \cdots + N^{B}_{G_h}(\tau),$$

where $N^B(\tau)$ and $N^{B}_{G_j}(\tau)$ are the numbers of the buckling eigenvalues less than or equal to $\tau$ for $\Omega$ and $G_j$, respectively. For each subdomain $G_j$, we take a point $p_j \in G_j$ such that $G_j' \subset B_0(\varrho)$, where $G_j' = \{x' \in \mathbb{R}^n | x' = \text{Exp}_{p_j}^{-1}x, x \in G_j\}$. Therefore, under normal coordinates at $p_j$, the inequality (5.4) holds for any $u \in W^{2,2}_0(G_j')$. This implies that

$$\Lambda_k(G_j) \leq (1 + \epsilon(\varrho))^2 (1 + \tilde{\epsilon}(\varrho)) \Lambda_k(G_j') + (1 + \epsilon(\varrho))^2 \tilde{\epsilon}(\varrho), \quad k = 1, 2, 3, \cdots.$$
By Theorem 1.1 and the Faber-Krahn inequality (see, for example, Theorem 2 of p.87 in [7]), we have
\[ \Lambda_k(G_j) \geq \Lambda_1(G_j) \geq \lambda_1(G_j) \geq \lambda_1(B_0(\varrho)) > 1, \quad 1 \leq j \leq h. \]
It follows from this and (5.6) that
\[ \Lambda_k(G_j) \leq (1 + \epsilon(\varrho))^2 (1 + 2\bar{\epsilon}(\varrho)) \Lambda_k(G_j'), \quad j = 1, 2, \cdots, h, \quad k = 1, 2, 3, \cdots, \]
so that
\[ N(B_0(G_j))(\tau) \geq N(B_0(G_j'))(\tau) \left( \frac{\tau}{(1 + \epsilon(\varrho))^2 (1 + 2\bar{\epsilon}(\varrho))} \right)^{n/2} (1 + o(1)) \quad \text{as } \tau \to \infty. \]
By Theorem 4.2, we have that
\[ N(B_0(G_j'))(\tau) = (2\pi)^{-n} \omega_n \frac{\tau}{(1 + \epsilon(\varrho))^2 (1 + 2\bar{\epsilon}(\varrho))} \frac{1}{(1 + o(1))}. \]
It follows from (5.5), (5.7) and (5.8) that, as \( \tau \to +\infty \),
\[ N(B_0(G_j))(\tau) \geq (2\pi)^{-n} \omega_n \frac{\tau}{(1 + \epsilon(\varrho))^2 (1 + 2\bar{\epsilon}(\varrho))} \frac{1}{(1 + o(1))}. \]
Recall that \( dV_g = \sqrt{\det(g_{ij})} \, dx \) with \((1 + \epsilon(\varrho))^{-1} < \sqrt{\det(g_{ij})} < (1 + \epsilon(\varrho))\). We have
\[ (1 + \epsilon(\varrho))^{-1}|G_j'| < \text{vol}(G_j) < (1 + \epsilon(\varrho))|G_j'|, \]
so that
\[ \sum_{j=1}^{h} |G_j'| \geq (1 + \epsilon(\varrho))^{-1} \sum_{j=1}^{h} (\text{vol}G_j) = (1 + \epsilon(\varrho))^{-1} (\text{vol}(\Omega)). \]
This implies that
\[ N(B_0(G_j))(\tau) \geq (2\pi)^{-n} \omega_n (1 + \epsilon(\varrho))^{-1} (\text{vol}(\Omega)) \times \frac{\tau}{(1 + \epsilon(\varrho))^2 (1 + 2\bar{\epsilon}(\varrho))} \frac{1}{(1 + o(1))} \quad \text{as } \tau \to \infty. \]
Hence
\[ \lim_{\tau \to \infty} \frac{N(B_0(G_j))(\tau)}{\tau^{n/2}} \geq (2\pi)^{-n} \omega_n (\text{vol}(\Omega)). \]
For, we may choose the quantity \( \varrho \) arbitrarily, and by taking a sufficiently small fixed \( \varrho \), make the factor of \( \tau^{n/2} \) in (5.9) arbitrarily close to \((2\pi)^{-n} \omega_n (\text{vol}(\Omega))\) for sufficiently large \( \tau \).
On the other hand, it follows from (6) of [24] that, for the bounded domain \( \Omega \) in a Riemannian manifold \((M, g)\),

\[
\sum_{k=1}^{\infty} e^{-t\mu_k} = (4\pi t)^{-n/2} \left[ \text{vol}(\Omega) + \frac{1}{4} \sqrt{4\pi t} \text{vol}(\partial\Omega) \right] + \frac{t}{3} \int_{\Omega} R \, dV_g - \frac{t}{6} \int_{\partial\Omega} J \, dS_g + o(t^{3/2}),
\]

where \( R \) is the scalar curvature at a point of \( M \), and \( J \) the mean curvature at a point of \( \partial\Omega \). From (5.11), we have

\[
\int_{0}^{\infty} e^{-t\tau} dN^{(N)}(\tau) = \sum_{k=1}^{\infty} e^{-t\mu_k} \sim (4\pi t)^{-n/2} \text{vol}(\Omega), \quad \text{as} \ t \to 0,
\]

where \( N^{(N)}(\tau) \) is the number of the Neumann eigenvalues less than or equal to \( \tau \) for \( \Omega \), and \( \int_{0}^{\infty} e^{-t\tau} dN^{(N)}(\tau) \) is the Riemann-Stieltjes integral on \([0, +\infty)\) (Note that \( \int_{0}^{\infty} e^{-t\tau} dN^{(N)}(\tau) \) means \( \lim_{\delta \to 0^+} \int_{\infty}^{\delta} e^{-t\tau} dN^{(N)}(\tau) \)). It follows from Proposition 3.2 of p.89 in [41] that

\[
N^{(N)}(\tau) \sim (2\pi)^{-n} \omega_n (\text{vol}(\Omega))^{\tau n/2}, \quad \text{as} \ \tau \to \infty,
\]

i.e.,

\[
\lim_{\tau \to \infty} \frac{N^{(N)}(\tau)}{\tau^{n/2}} = (2\pi)^{-n} \omega_n (\text{vol}(\Omega)),
\]

By (1.16) of Theorem 1.1, we have that

\[
N^{(N)}(\tau) \geq N^{(D)}(\tau) \geq N^{(P)}(\tau) \geq N^{(B)}(\tau), \quad \text{for any} \ \tau.
\]

It follows from (5.10), (5.12) and (5.13) that

\[
\lim_{\tau \to -\infty} \frac{N^{(B)}(\tau)}{\tau^{n/2}} = \lim_{\tau \to -\infty} \frac{N^{(P)}(\tau)}{\tau^{n/2}} = \lim_{\tau \to -\infty} \frac{N^{(D)}(\tau)}{\tau^{n/2}} = \lim_{\tau \to -\infty} \frac{N^{(N)}(\tau)}{\tau^{n/2}} = (2\pi)^{-n} \omega_n (\text{vol}(\Omega)). \quad \square
\]

**Remark 5.2.** (i) For the Dirichlet and Neumann eigenvalue problems, Seeley [38] and Pham [39] showed that if \( \Omega \subset \mathbb{R}^n \) is a bounded domain with \( C^\infty \)-smooth boundary, then the following sharp remainder estimate holds:

\[
N^{(D)}(\tau) = (2\pi)^{-n} \omega_n |\Omega| \tau^{n/2} (1 + O(\tau^{-\frac{1}{2}})), \quad \text{as} \ \tau \to \infty.
\]

In [38], Seeley used the method of hyperbolic equations which is the most precise of the known Tauberian methods. Seeley in [39] has generalized the above result to \( n \)-dimensional Riemannian manifolds.

(ii) For the Dirichlet and Neumann eigenvalues of a bounded domain \( \Omega \) in a smooth Riemannian manifold \( M \), Ivrii (see [18]) has established:

\[
N^{\pm}(\tau) = (2\pi)^{-n} \omega_n \cdot \text{vol}(\Omega) \cdot \tau^{n/2} \pm \frac{1}{4} (2\pi)^{-n+1} \omega_{n-1} \cdot \text{vol}(\partial\Omega) \cdot \tau^{(n-1)/2} + o(\tau^{(n-1)/2}), \quad \text{as} \ \tau \to +\infty,
\]

where \( \omega_n \) is the volume of the unit ball in \( \mathbb{R}^n \).
under an additional assumption (roughly, that the set of “multiply reflected periodic geodesics in \( \bar{\Omega} \) is of measure zero”), where \( N^+(\tau) \) and \( N^-(\tau) \) denote the counting functions of \( \sigma_N \) and \( \sigma_D \), respectively. Melrose [26] independently obtained the same asymptotic estimate [5,15] for Riemannian manifolds with concave boundary. However, Ivrii’s method is no longer valid for the buckling and the clamped eigenvalues.

**Proof of Theorem 1.2.** Taking \( \tau = \Lambda_k \) in Theorem 5.1, we immediately obtain the conclusion of the theorem. □

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