ALGEBRAIC APPROXIMATIONS OF COMPACT KÄHLER MANIFOLDS OF
ALGEBRAIC CODIMENSION 1

by

Hsueh-Yung Lin

Abstract. — For every compact Kähler manifold $X$ of algebraic dimension $a(X) = \dim X - 1$, we prove that $X$ has arbitrarily small deformations to some projective manifolds.

1 Introduction

1.1 The Kodaira problem

Let $X$ be a compact Kähler manifold. An algebraic approximation of $X$ is a deformation $\mathcal{X} \to \Delta$ of $X$ such that up to shrinking $\Delta$, the subset of $\Delta$ parameterizing projective manifolds is dense. The so-called Kodaira problem asks whether a compact Kähler manifold in general admits an algebraic approximation. For compact Kähler surfaces, the Kodaira problem has a positive answer [16, Theorem 16.1]; see also [28, 8, 12, 9, 19, 10] for other positive results. However starting from dimension 4, there exist compact Kähler manifolds which cannot deform to any projective manifold at all (even homotopically) [31]. Despite these works, the existence of algebraic approximations is still unknown for most compact Kähler manifolds.

Recall that the algebraic dimension $a(X)$ of a compact complex manifold $X$ is the transcendental degree (over $\mathbb{C}$) of its field of meromorphic functions. For compact Kähler manifolds, those satisfying $a(X) = \dim X$ are exactly the projective ones by Moishezon’s criterion [21]. Among non-projective compact Kähler manifolds, $a(X) = \dim X - 1$ is the highest possible algebraic dimension. The aim of this text is to prove that such manifolds have algebraic approximations.

Theorem 1.1. — Every compact Kähler manifold $X$ of algebraic dimension $a(X) = \dim X - 1$ admits an algebraic approximation.

An elliptic fibration is a holomorphic surjective map $f : X \to B$ whose general fiber is an elliptic curve. The algebraic reduction of a compact Kähler manifold $X$ with $a(X) = \dim X - 1$ is bimeromorphic to an elliptic fibration over a projective variety [30, Theorem 12.4]. Conversely, a compact Kähler manifold $X$ bimeromorphic to an elliptic fibration $X' \to B$ over a projective variety $B$ has algebraic dimension

$$a(X) \geq a(B) = \dim B = \dim X - 1.$$
Thus we have the following variant of Theorem 1.1, which we will prove instead in this text.

**Theorem 1.2 (Variant of Theorem 1.1).** — Let $X$ be a compact Kähler manifold. If $X$ is bimeromorphic to the total space of an elliptic fibration over a projective variety, then $X$ has an algebraic approximation.

In [10], it was already proven that every compact Kähler manifold $X$ as in Theorem 1.2 is bimeromorphic to an elliptic fibration admitting an algebraic approximation.

**Remark 1.3.** — Theorem 1.2 fails in general if the base of the elliptic fibration is not assumed to be projective: the product of an elliptic curve with Voisin’s example constructed in [31, Section 2] is a counterexample [31, Proposition 1].

The notion of algebraic approximations can be defined more generally for any compact complex manifold $X$: in the definition, we require instead that Moishezon manifolds (i.e. compact complex manifolds $X'$ with $a(X') = \dim X'$) are dense in the deformation $\mathcal{X} \to \Delta$ of $X$ (see Definition 2.4). Since small deformations of Kähler manifolds remain Kähler, this definition of algebraic approximation coincides with the one introduced at the beginning of the text for compact Kähler manifolds by Moishezon’s criterion. Recall that the Fujiki class $\mathcal{C}$ is the collection of compact complex varieties which are bimeromorphic to compact Kähler manifolds. As an immediate corollary of Theorem 1.1, we have the following more general statement.

**Corollary 1.4.** — Every compact complex variety $X$ in the Fujiki class $\mathcal{C}$ with rational singularities and algebraic dimension $a(X) = \dim X - 1$ admits an algebraic approximation.

**Proof.** — Let $\tilde{X} \to X$ be a bimeromorphic morphism from a compact Kähler manifold $\tilde{X}$. Since $X$ has at worst rational singularities, every deformation of $\tilde{X}$ induces a deformation of $\tilde{X} \to X$ [27, Theorem 2.1]. As $a(\tilde{X}) = \dim \tilde{X} - 1$, Corollary 1.4 follows from Theorem 1.1. □

**Remark 1.5.** — Small deformations of varieties in the Fujiki class $\mathcal{C}$ might no longer be in the Fujiki class $\mathcal{C}$ [6, 18] and in the definition of algebraic approximations, it is not required that nearby fibers of the deformation of $X$ remain in the Fujiki class $\mathcal{C}$.

In [20], we will apply Theorem 1.1 to prove the existence of algebraic approximations for compact Kähler threefolds, which was one of our motivations of this work. More precisely, note that since $\kappa(X) \leq a(X)$ where $\kappa(X)$ is the Kodaira dimension, Theorem 1.1 holds in particular for compact Kähler manifolds $X$ with $\kappa(X) = \dim X - 1$. We will apply Theorem 1.1 to cover the case of threefolds with $\kappa = 2$.

The rest of the introduction is devoted to an overview of the proof of Theorem 1.2.

### 1.2 Algebraic approximations and bimeromorphic models

Given a compact Kähler manifold $X$ as in Theorem 1.2, we will prove Theorem 1.2 by constructing a bimeromorphic model $X \to X'$ of $X$ together with an algebraic approximation of $X'$ which induces a deformation of $X \to X'$.

In general, based on Ancona, Tomassini, and Bingener’s work on formal modifications [4], we will observe that if $X'$ is a bimeromorphic model of a given compact complex variety $X$ and $Y \subset X'$ a subvariety containing the locus that are modified under the bimeromorphic map $X' \to X$, then a deformation $\Pi : \mathcal{X}' \to \Delta$...
\( \Delta \) of \( X' \) induces a deformation of \( X \to X' \) provided \( \Pi \) induces a trivial deformation of the completion \( \hat{Y} \) of \( X' \) along \( Y^{(1)} \). In particular if \( X \) is a compact Kähler manifold, then an algebraic approximation of \( X' \) preserving \( \hat{Y} \) induces an algebraic approximation of \( X \). The exact statement that we will prove and apply is Proposition 5.1, allowing us to transform the problem of finding an algebraic approximation of \( X \) into the problem of finding a bimeromorphic model \( X' \) of \( X \) together with an algebraic approximation \( \mathcal{X}' \to \Delta \) of \( X' \) preserving \( \hat{Y} \) for a certain subvariety \( Y \subset X' \).

Let us return to the case where \( X \) is a compact complex manifold as in Theorem 1.2. By Hironaka’s desingularization, \( X \) is bimeromorphic to the total space of an elliptic fibration \( f' : X' \to B \) over a projective manifold \( B \) such that the discriminant locus \( \Delta \subset B \) is a simple normal crossing divisor. We can find a finite Galois cover \( r : \hat{B} \to B \) such that the base change \( f^\prime : \hat{X}' = X' \times_B \hat{B} \to \hat{B} \) has local meromorphic sections at every point of \( \hat{B} \) and the local monodromies of \( H = (R^1 f^\prime_\ast \mathcal{Z})_{|\hat{B} \setminus \hat{\Delta}} \) around \( \hat{\Delta} = r^{-1}(\Delta) \) are unipotent. In order to simplify the discussion, we shall first assume that the finite cover \( \hat{B} \to B \) is trivial so that \( f' \) already satisfies the later properties. Tactically in the following discussion, these are the properties that we need in most of the arguments involved in the proof of Theorem 1.2.

As we mentioned above, we will prove Theorem 1.2 by constructing a bimeromorphic model \( X \to X \) of \( X \) together with an algebraic approximation of \( X \) preserving \( \hat{Y} \), where \( Y \subset X \) is a subvariety containing the locus that are modified under the bimeromorphic map \( X \to X \). Up to a finite Galois cover, the bimeromorphic model that we consider will be the total space of some particular elliptic fibration \( f : X \to B \) called tautological model, and the algebraic approximation of \( X \) will be realized by a subfamily of the tautological family associated to \( f \). The next paragraph is an informal discussion on how these notions are introduced. For the details, the reader is referred to Section 3.

### 1.3 Elliptic fibrations and tautological families

Let \( B \) be a compact complex manifold and \( \Delta \subset B \) a normal crossing divisor. To each local system \( \mathbf{H} \) of stalks \( \mathcal{Z}^2 \) over \( B^\ast := B \setminus \Delta \), we can associate a unique minimal Weierstraß fibration \( p : W \to B \), which is an elliptic fibration satisfying \((R^1 p_\ast \mathcal{Z})_{B^\ast} = \mathbf{H} \ [22, \text{Theorem } 2.5]\). A minimal Weierstraß fibration can be considered as a standard compactification of the Jacobian fibration \( f \to B^\ast \) associated to \( \mathbf{H} \).

Let \( \mathcal{E}(B, \Delta, \mathbf{H}) \) be the set of bimeromorphic classes of elliptic fibrations \( f' : X' \to B \) such that \( f' \) is smooth over \( B^\ast \) with \((R^1 f'_\ast \mathcal{Z})_{B^\ast} = \mathbf{H} \) and admits local meromorphic sections over every point of \( B \). Like in the smooth case where smooth elliptic fibrations are torsors under their Jacobian fibrations, elements of \( \mathcal{E}(B, \Delta, \mathbf{H}) \) can roughly be regarded as "meromorphic \( W \)-torsors" where \( p : W \to B \) is the minimal Weierstraß elliptic fibration associated to \( \mathbf{H} \). So if \( \mathcal{F}_{H/B} \) denotes the sheaf of abelian groups of germs of meromorphic sections of \( p \), then we have an injective map

\[
\eta : \mathcal{E}(B, \Delta, \mathbf{H}) \hookrightarrow H^1(B, \mathcal{F}_{H/B}).
\]

In general, not every element in \( H^1(B, \mathcal{F}_{H/B}) \) corresponds to (the bimeromorphic class of) an elliptic fibration. However, consider the subsheaf \( \mathcal{F}_{H/B}^W \subset \mathcal{F}_{H/B} \) of germs of local holomorphic sections of \( p : W \to B \)

---

1. Usually the completion of \( X \) along a subvariety \( Y \) is denoted by \( \hat{X} \). In this text, since we want to keep track of the subvariety along which the completion is done, we will often use the notation \( \hat{X} \) instead of \( X \).
whose image is contained in the smooth locus of \( p \), then each element \( \eta \in H^1(B, \mathcal{O}_W) \) corresponds to some elliptic fibration \( p^* : W^* \to B \), called the locally Weierstraß fibration twisted by \( \eta \). Torsion points of \( H^1(B, \mathcal{O}_W) \) correspond to elliptic fibrations that are projective. The sheaf \( \mathcal{O}_W \) lies in the short exact sequence

\[
0 \longrightarrow \mathcal{O}_W \longrightarrow \mathcal{H}_W \longrightarrow \exp \longrightarrow \mathcal{O}_W \longrightarrow 0.
\]

where \( j : B^* \to B \) is the inclusion and \( \mathcal{H}_W = R^1 p_* \mathcal{O}_W \). Each \( \eta \in H^1(B, \mathcal{O}_W) \) comes equipped with a family of elliptic fibrations

\[
\Pi : \mathcal{W} \to B \times V \to V := H^1(B, \mathcal{O}_W)
\]

parameterized by \( V \) that is tautological in the sense that the elliptic fibration parameterized by \( t \in V \) corresponds to \( \eta + \exp(t) \in H^1(B, \mathcal{O}_W) \).

More generally, let \( f' : X' \to B \) be an elliptic fibration such that \( X' \) is a compact Kähler manifold. Assume that the bimeromorphic class of \( f' \) lies in \( \mathcal{E}(B, \Delta, \mathcal{H}) \), then we can construct a bimeromorphic model \( f : X \to B \) of \( f' \) together with a family of elliptic fibrations

\[
\Pi : \mathcal{X} \to B \times V \to V
\]

which contains \( f \) as the central fiber and is tautological in the sense that the bimeromorphic class of the elliptic fibration parameterized by \( t \in V \) corresponds to \( \eta(f) + \exp'(t) \) where \( \exp' \) is the composition of

\[
\exp : H^1(B, \mathcal{O}_W) \to H^1(B, \mathcal{O}_W)
\]

with \( H^1(B, \mathcal{O}_W) \to H^1(B, \mathcal{O}_W) \) (see Proposition-Definition 3.16). We call \( f : X \to B \) a tautological model of \( f' \) and \( \Pi \) the tautological family associated to \( f \). Such a family has already been constructed in [10] and proven to be an algebraic approximation of \( f \). Up to some finite Galois cover, this is how the bimeromorphic version of Theorem 1.2 was proven in [10].

However, for such a bimeromorphic model \( f \), there may be no algebraic approximation of \( X \) contained in \( \Pi : \mathcal{W} \to V \) as a subfamily preserving the modification \( X \to X \) (for instance, this will be the case as long as the subvariety \( Y \subset X \) along which \( X \) is modified is non-algebraic). To prove Theorem 1.2, we need to refine our choice of bimeromorphic model. Let \( Y_{\text{min}} \subset X \) be the minimal subvariety which is modified under the induced bimeromorphic map \( X \to X \) and let \( \hat{Y} := f^{-1}(f(Y_{\text{min}})) \). Up to replacing \( f' \) with a more carefully constructed bimeromorphic model (see Step 2, 3, and 4 in the proof of Lemma 5.3), we may assume that the projection \( Y \to f(Y) \) has a multi-section. It is for the pair \((X, Y)\) that we will prove that the tautological family \( \Pi \) associated to \( f : X \to B \) contains a subfamily which is an algebraic approximation of \( X \) preserving the completion \( \hat{Y} \) of \( X \) along \( Y \). Eventually it can be reduced to the case where \( f \) is a locally Weierstraß fibration twisted by some \( \eta \in H^1(B, \mathcal{O}_W) \) and we will find algebraic approximations in the associated tautological family using Hodge theory, as we will discuss in the next paragraph.

1.4 The Hodge theory of Weierstraß fibrations

With the same notations introduced in 1.3, a new ingredient in this work allowing us to go further than [10] is the observation that \( H^1(B, j_*\mathcal{H}) \) can be endowed with a natural pure \( \mathbb{Z} \)-Hodge structure of weight 2 such that the map \( H^1(B, j_*\mathcal{H}) \to H^1(B, \mathcal{H}_W) \) induced by \( (1.1) \) is the projection to the \((0, 2)\)-part
As \( j: H \to \mathcal{L}_{H/B} \) is isomorphic to \( R^1p_*Z \to R^1p_*\mathcal{O}_W \) (Lemma 3.4) where \( p: W \to B \) is the minimal Weierstraß fibration associated to \( H \), when \( p \) is smooth this result is a particular case of [32, Theorem 2.9]. The Hodge-theoretic interpretation of \( H^1(B, j_*H) \to H^1(B, \mathcal{L}_{H/B}) \) allows us to show without difficulties that for every \( \eta \in H^1(B, \mathcal{J}_W_{H/B}) \), the subset of points \( t \in V = H^1(B, \mathcal{L}_{H/B}) \) such that \( \eta + \exp(t) \) is contained in the torsion part of \( H^1(B, \mathcal{J}_W_{H/B}) \) is dense in \( V \). This also gives an alternative way (in comparison to the proof of [10, Theorem 3.25]) closer to the spirit of [9, 19] to show that the tautological family \( \Pi \) is an algebraic approximation.

Recall that we want to prove that the tautological family \( \Pi : \mathcal{W} \to B \times V \to V \) associated to \( \eta \in H^1(B, \mathcal{J}_W_{H/B}) \) contains a subfamily which is an algebraic approximation of \( p^n : W^n \to B \) preserving the completion \( \bar{Y} \). First of all, it is easy to see that if \( t : Z = p^n(Y) \to B \) is the inclusion, then the subfamily of \( \Pi \) parameterized by

\[
V_{Z} := \ker \left( \iota^* \circ H^1(B, \mathcal{L}_{H/B}) \to H^1(Z, \iota^* \mathcal{L}_{H/B}) \right),
\]

is a deformation of \( p^n \) preserving \( Y \to Z \) (Lemma 3.2). However, we do not know whether \( \iota^* \) is the (0, 2)-part of a morphism of pure Hodge structures of weight 2, so cannot (use Hodge theory to) conclude that the subfamily parameterized by \( V_{Z} \) is an algebraic approximation. So instead of \( t : Z \to Z \) preserves the \( Y \), consider a log-desingularization \( t : Z \to Z \to B \) of the pair \((Z\setminus\Delta, Z\setminus\Delta \cap \Delta)\). By doing so, the pullback

\[
\iota^* : H^1(B, \mathcal{L}_{H/B}) \to H^1(Z, \iota^* \mathcal{L}_{H/B})
\]

will be the (0, 2)-part of the morphism of pure Hodge structures

\[
\iota^* : H^1(B, j_*H) \to H^1(Z, j_*Z_{H/B})
\]

of weight 2. We will then show that

\[
V_{Z} := \ker \left( \iota^* \circ H^1(B, \mathcal{L}_{H/B}) \to H^1(Z, \iota^* \mathcal{L}_{H/B}) \right) \subset V_{Z},
\]

so the subfamily of \( \Pi \) parameterized by \( V_{Z} \) also preserves \( Y \to Z \) (Proposition 4.11). Based on the assumption that \( Y \to Z \) has a multi-section, we will also show that this subfamily is an algebraic approximation of \( p^n \) (Proposition 4.6). The proofs of both Proposition 4.11 and Proposition 4.6 use Hodge theory in an essential way. Another property that we can prove by Hodge theory is that \( V_{Z} \) contains a lattice \( \Delta \) such that for every \( t \in V_{Z} \) and \( \lambda \in \Delta \), the elliptic fibrations parameterized by \( t \) and \( t + \lambda \) are isomorphic (Lemma 4.7). We summarize these results as follows for the sake of reference.

**Proposition 1.6 (Proposition 4.6 + Lemma 4.7 + Proposition 4.11 without the G-action)**

If \( Y \to Z \) has a multi-section, then the tautological family \( \Pi \) contains a subfamily

\[
\Pi' : \mathcal{W}_Z \to B \times V \to V
\]

parameterized by a linear subspace \( V_{Z} \subset H^1(B, \mathcal{L}_{H/B}) \) which is an algebraic approximation of \( p^n \) preserving \( Y \to Z \). Moreover, there exists a lattice \( \Delta \subset V_{Z} \) such that \( t \) and \( t + \lambda \) parameterize isomorphic elliptic fibrations for each \( t \in V_{Z} \) and \( \lambda \in \Delta \).
Remark 1.7. — The assumption in Proposition 1.6 that \( Y \to Z \) has a multi-section is also an obvious necessary condition for \( p^n \) to have an algebraic approximation preserving \( Y \to Z \). Indeed, if such an algebraic approximation exists, then \( Y \to Z \) is algebraic, thus admits a multi-section.

1.5 From subvarieties to the completions and the end of the proof

The geometric properties of the family \( \Pi' \) described in Proposition 1.6 allow us to apply Lemma 4.18 to deduce that \( \Pi' \) preserves in fact the completion \( \hat{Y} \to Z \) of \( p^n \) along \( Y \to Z \). The argument goes roughly as follows: It suffices to prove by induction that the induced deformation \( \mathcal{Y}_n \to Z_n \times V_2 \to V_2 \) of the \( n \)-th infinitesimal neighborhood \( Y_n \to Z_n \) of \( Y \to Z \) is trivial. Due to the flatness of \( q \), we will see that each fiber \( \mathcal{Y}_{n,t} \) of \( \mathcal{Y}_n \to V_2 \) is a square-zero extension of \( Y_{n-1} \) by a sheaf \( \mathcal{F} \) independent of \( t \). Thus we have a holomorphic map \( \phi : V_2 \to \text{Ext}^1(L_{Y_{n-1}}^* \mathcal{F}) \), which is constant if and only if \( \mathcal{Y}_n \to V_2 \) is a trivial deformation where \( \text{Ext}^1(L_{Y_{n-1}}^* \mathcal{F}) \) is the vector space parameterizing square-zero extension of \( Y_{n-1} \) by \( \mathcal{F} \). We will see that the last statement of Proposition 1.6 together with the induction hypothesis implies that \( \phi \) factorizes through \( V_2/\Lambda \), which is a complex torus. Therefore \( \phi \) is indeed constant.

This is an overview of the proof of Theorem 1.2 under the assumption that the Galois cover of \( r : \bar{B} \to B \) in 1.2 is trivial. If \( r \) is non-trivial with Galois group \( G \), we replace \( f' \) in 1.2 with \( f' : \bar{X}' = X' \times_B \bar{B} \to \bar{B} \) and run the whole argument sketched above for \( f' \) in the \( G \)-equivariant setting. The algebraic approximation of the chosen bimeromorphically model will be realized by the finite quotient of a subfamily of the tautological family.

Remark 1.8. — Instead of constructing an algebraic approximation of \( X \) preserving \( \hat{Y} \), one would have tried to find directly an algebraic approximation \( \mathcal{X}' \to \Lambda \) of \( X \) such that a neighborhood \( U \subset X \) of \( Y \) deforms trivially in \( \mathcal{X}' \to \Lambda \), which is the way we prove the existence of algebraic approximations of compact Kähler threefolds with \( \kappa \leq 1 \) in [20]. However for the elliptic fibrations that we consider in this text, it is difficult (and maybe impossible) to find such algebraic approximations in the tautological families. This is why we choose to verify the weaker property that the completion \( \hat{Y} \) is preserved along \( \mathcal{X}' \to \Delta \) and use Proposition 5.1 to conclude.

This text is organized as follows. We will first introduce some basic terminology including various definitions of deformations of complex spaces and maps in Section 2. In Section 3, we will recall and improve some results of the theory of elliptic fibrations developed in [22, 24, 23, 10]. It is also in Section 3 that we will construct the tautological families (Proposition-Definition 3.16). Section 4 is the Hodge-theoretic part of the text. We will prove Proposition 4.6, Proposition 4.11, and other related results therein. In Section 5, we will prove Proposition 5.1 and finally the main result Theorem 1.2.

2 Preliminaries

2.1 Notation and terminology

In this text, a fibration is a proper holomorphic surjective map \( f : X \to B \) between complex spaces. The fiber of \( f \) over \( b \in B \) will frequently be denoted by \( X_b \). A multi-section of \( f \) is a subvariety \( S \subset X \) such that \( f_{|S} \) is surjective and generically finite. A deformation of a complex space \( X \) is a flat surjective morphism
\[ \Pi : \mathcal{X} \to \Delta \] containing \( X \) as a fiber. Let \( f : X \to B \) be a holomorphic map. A deformation of \( f \) is a composition \( \Pi : \mathcal{X} \to B \times \Delta \overset{pr_2}{\to} \Delta \) such that \( \Pi \) is flat and \( q_{\Pi,x} : \mathcal{X}_o \to B \) equals \( f \) for some \( o \in \Delta \). Note that in our definition, a deformation of \( f \) always preserves the base \( B \).

Let \( G \) be a group and \( X \) a complex space endowed with a \( G \)-action. We say that a deformation \( \Pi : \mathcal{X} \to \Delta \) of \( X \) preserves the \( G \)-action if there exists a fiberwise \( G \)-action on \( \mathcal{X} \to \Delta \) extending the given \( G \)-action on \( X \). Similarly, let \( f : X \to B \) be a \( G \)-equivariant map. We say that a deformation \( \Pi : \mathcal{X} \to B \times \Delta \overset{pr_2}{\to} \Delta \) of \( f \) preserves the \( G \)-action if there exists a fiberwise \( G \)-action on \( \mathcal{X} \to \Delta \) extending the \( G \)-action on \( X \) such that \( q \) is \( G \)-equivariant.

**Definition 2.1.** —

1. Let \( \Pi : \mathcal{X} \to \Delta \) be a deformation of a complex variety \( X \) and let \( Y \subset X \) be an analytic subset. We say that \( \Pi \) preserves \( Y \) if there exists \( \mathcal{Y} \subset \mathcal{X} \) such that \( \mathcal{Y} \) is isomorphic to \( Y \times \Delta \) over \( \Delta \).
2. We say that \( \Pi \) preserves the completion \( \hat{Y} \) of \( X \) along \( Y \) if there exist \( \mathcal{Y} \subset \mathcal{X} \) and an isomorphism \( \mathcal{Y} \cong \hat{Y} \times \Delta \) over \( \Delta \) where \( \hat{Y} \) is the completion of \( \mathcal{X} \) along \( \mathcal{Y} \).
3. Let \( \Pi : \mathcal{X} \to B \times \Delta \to \Delta \) be a deformation of \( f : X \to B \) and let \( Z \) be an analytic subset of \( B \). We say that \( \Pi \) preserves \( Y := f^{-1}(Z) \to Z \) if \( q^{-1}(Z \times \Delta) \) is isomorphic to \( Y \times \Delta \) over \( B \times \Delta \). We define in a similar way deformations of \( f \) that preserve the completion \( \hat{Y} \to \hat{Z} \) of \( f \) along \( Y \to Z \).
4. Let \( G \) be a group acting on a complex variety \( X \) and \( Y \subset X \) a \( G \)-stable analytic subset. Let \( \Pi : \mathcal{X} \to \Delta \) be a deformation of \( X \) preserving the \( G \)-action. We say that \( \Pi \) preserves \( G \)-equivariantly \( Y \) if there exists a \( G \)-stable subvariety \( \mathcal{Y} \subset \mathcal{X} \) together with a \( G \)-equivariant isomorphism \( \mathcal{Y} \cong Y \times \Delta \) over \( \Delta \). The \( G \)-equivariant versions of ii) and iii) are defined similarly.

**Definition 2.2.** —

1. A deformation \( \Pi : \mathcal{X} \to B \times \Delta \to \Delta \) of \( f : X \to B \) is said to be locally trivial (over \( B \)) if there exists an open cover \( \{U_i\} \) of \( B \) such that \( q^{-1}(U_i \times \Delta) \cong f^{-1}(U_i) \times \Delta \) over \( U_i \times \Delta \).
2. Let \( G \) be a group and \( f : X \to B \) a \( G \)-equivariant map. We say that \( \Pi \) is \( G \)-equivariantly locally trivial if \( \Pi \) preserves the \( G \)-action and the isomorphisms \( q^{-1}(U_i \times \Delta) \cong f^{-1}(U_i) \times \Delta \) above are \( G \)-equivariant for some \( G \)-invariant open cover \( \{U_i\} \) of \( B \).

Obviously, the quotient of a \( G \)-equivariantly locally trivial deformation is a locally trivial deformation.

**Lemma 2.3** ([19, Lemma 2.2]). — If \( \Pi : \mathcal{X} \to B \times \Delta \to \Delta \) is a \( G \)-equivariantly locally trivial deformation of a \( G \)-equivariant fibration \( f : X \to B \) for some finite group \( G \), then the quotient \( \mathcal{X}/G \to (B/G) \times \Delta \to \Delta \) is a locally trivial deformation of \( X/G \to B/G \).

Now we come to the notion of algebraic approximations. Recall that a compact complex variety \( X \) is called Moishezon if its algebraic dimension \( a(X) \) equals \( \dim X \).

**Definition 2.4 (Algebraic approximation).** — Let \( X \) be a compact complex variety. An algebraic approximation of \( X \) is a deformation \( \pi : \mathcal{X} \to \Delta \) of \( X \) such that the subset of points in \( \Delta \) parameterizing Moishezon varieties is dense for the Euclidean topology.
2.2 Campana’s criterion

Let \( X \) be a complex variety. We say that \( X \) is algebraically connected if a general pair of points \( x, y \in X \) is contained in a connected (but not necessarily irreducible) and complete curve in \( X \). For a compact complex variety \( X \) in the Fujiki class \( \mathcal{C} \) (i.e. bimeromorphic to a compact Kähler manifold), Campana shows that \( X \) is Moishezon if and only if \( X \) is algebraically connected [7, Corollaire, p.212]. Since we will mainly deal with fibrations in curves over a projective base, here we state a variant of Campana’s criterion in this situation.

\textbf{Theorem 2.5 (Campana).} — Let \( f : X \to B \) be a fibration whose general fiber is a curve. Assume that \( X \) is in the Fujiki class \( \mathcal{C} \) and \( B \) is projective, then \( X \) is Moishezon if and only if \( f \) has a multi-section.

3 Locally Weierstraß fibrations and tautological families

The theory of elliptic fibrations had first been developed by Kodaira for fibrations over a curve [16], then by N. Nakayama in any dimension [22, 24, 23]. We will start with a concise summary of the part of the theory of elliptic fibrations developed in [22, 24, 23, 10] that we will use in this text. Then we will construct tautological models and the associated tautological families in 3.4 and 3.5.

3.1 Weierstraß fibrations

3.1.1 In the following we recall the definition of Weierstraß fibrations. These fibrations can be considered as standard compactifications of Jacobian fibrations associated to smooth elliptic fibrations: we will see that for every elliptic fibration \( f : X \to B \) (over a complex manifold \( B \)), there exists a unique minimal Weierstraß fibration \( p : W \to B \) whose restriction to \( p^{-1}(B^*) \) is isomorphic to the Jacobian fibration \( J \to B^* \) associated to the smooth part \( X^* \to B^* \) of \( f \).

\textbf{Definition 3.1.} — Let \( \mathcal{L} \) be a line bundle over a complex space \( B \) and let

\[ Z \in H^0(\mathcal{O}(1)), \quad X \in H^0(\mathcal{O}(1) \otimes p^* \mathcal{L}(-2)), \quad \text{and} \quad Y \in H^0(\mathcal{O}(1) \otimes p^* \mathcal{L}(-3)) \]

be the three coordinate sections of the projectivization \( \mathbf{P} := \mathbf{P}(\mathcal{O} \oplus \mathcal{L}^2 \oplus \mathcal{L}^3) \).

i) A \textit{Weierstraß fibration} is a fibration \( p : W := W(\mathcal{L}, a, \beta) \to B \) defined by the projection onto \( B \) of the hypersurface in \( \mathbf{P} \) defined by a nonzero section of the form

\[ Y^2Z - X^3 - aXZ^2 - \beta Z^3 \in H^0(\mathbf{P}, \mathcal{O}_\mathbf{P}(3) \otimes p^* \mathcal{L}(-6)) \]

for some sections \( a \in H^0(B, \mathcal{L}(-4)) \) and \( \beta \in H^0(B, \mathcal{L}(-6)) \) such that a general fiber of \( p \) is smooth.

ii) Without assuming that a general fiber of \( p : W \to B \) is smooth, \( p \) is called a \textit{singular Weierstraß} fibration.

iii) The section defined by \( \Sigma := \{X = Z = 0\} \subset W \) is called the \textit{zero-section} of \( p \).

iv) A Weierstraß fibration defined above is called \textit{minimal} if there is no prime divisor \( D \subset B \) such that \( \text{Div}(a) \geq 4D \) and \( \text{Div}(\beta) \geq 6D \).

v) A (singular) (minimal) \textit{locally Weierstraß fibration} is a fibration \( f : X \to B \) such that there exists an open cover \( \{U_i\} \) of \( B \) such that each \( f^{-1}(U_i) \to U_i \) is a (singular) (minimal) Weierstraß fibration.
Obviously, any base change $W \times_B Z \to Z$ of a (singular) Weierstraß fibration $W \to B$ is still a (singular) Weierstraß fibration. Locally a (singular) Weierstraß fibration $p : W \to B$ is a pullback of the standard Weierstraß fibration
\[ \{Y^2Z = X^3 + aXZ^2 + bZ^3\} \subset \mathbb{P}^2 \times \mathbb{C}^2 \]
parameterized by $(a, b) \in \mathbb{C}^2$. So $p$ is flat with irreducible and reduced fibers which are either elliptic curves, nodal cubic curves, or cuspidal cubic curves.

When $B$ is normal, the total space $W$ of a Weierstraß fibration is also normal [22, 1.2.(1)]. The notion of minimal Weierstraß fibrations first appears in [22] and the main interests of this notion reside in the properties that, under the assumption that $B$ is normal and the discriminant locus $\Delta = \text{Div}(4a^3 + 27b^2)$ is a normal crossing divisor, a Weierstraß fibration $p : W \to B$ is minimal if and only if $W$ has at worst rational singularities [22, Corollary 2.4]; also, when $B$ is smooth, each bimeromorphic class of Weierstraß fibrations contains a unique minimal Weierstraß fibration [22, Theorem 2.5].

3.1.2 Let $p : W \to B$ be a (singular) Weierstraß fibration and let $W^a \subset W$ denote the smooth locus of $p$. The fibration $W^a \to B$ can be considered as an analytic group variety over $B$ with connected fibers whose zero-section is $\Sigma$ (see Definition 3.1.iii)). The relative tangent space at $\Sigma$ is isomorphic to $\mathcal{L}$ [23, Lemma 5.1.1.(8)], so the exponential map induces a surjective morphism
\[ \exp : \mathcal{L} \longrightarrow \mathcal{J}^W \]  
(3.1)
where $\mathcal{J}^W$ is the sheaf of germs of holomorphic sections of $p$ whose image is contained in $W^a$. We also have
\[ \mathcal{L} \cong R^1p_*\mathcal{O}_W \]  
(3.2)
[22, 1.2.(6)] since fibers of $p$ are curves of arithmetic genus 1.

Let $B$ be a complex manifold and $f : B^* \to B^*$ a Jacobian fibration over a Zariski open $B^* \subset B$. By [22, Theorem 2.5], there exists a unique minimal Weierstraß fibration $p : W = W(\mathcal{L}, \alpha, \beta) \to B$ such that $p^{-1}(B^*)$ is isomorphic to $f$ over $B^*$. Since the local system $\mathcal{H} := (R^1p_*\mathcal{Z})_{B^*}$ is uniquely determined by $f : B^* \to B^*$ up to isomorphism, we also say that $p : W \to B$ is the minimal Weierstraß fibration associated to $\mathcal{H}$. If $f : X \to B$ is an elliptic fibration, then the minimal Weierstraß fibration associated to $f$ is defined to be the minimal Weierstraß fibration associated to $(R^1f_*\mathcal{Z})_{B^*}$, where $B^* \subset B$ is a Zariski open over which $f$ is smooth. If $G$ is a group acting on $B$ and on $\mathcal{H}$ in a compatible way, then by [22, Corollary 2.6], the $G$-action extends to a $G$-equivariant $G$-action on $W$ such that $p$ is $G$-equivariant.

In the case where $p : W \to B$ is the minimal Weierstraß fibration associated to $\mathcal{H}$, we use the notations
\[ \mathcal{L}_{H/B} := \mathcal{L} \quad \text{and} \quad \mathcal{J}^W_{H/B} := \mathcal{J}^W. \]

By [22, Proposition 2.10], the exponential map lies in the short exact sequence
\[ 0 \longrightarrow j_*\mathcal{H} \xrightarrow{q^*} \mathcal{L}_{H/B} \xrightarrow{\exp} \mathcal{J}^W_{H/B} \longrightarrow 0. \]  
(3.3)
where $j : B^* \hookrightarrow B$ is the inclusion. The restriction of (3.3) to $B^*$ is the short exact sequence
\[ 0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{E} := \mathcal{H}/\Gamma^1\mathcal{H} \longrightarrow \mathcal{J} \longrightarrow 0. \]  
(3.4)
where \( F^i, H \) is the first piece of the Hodge filtration of \( \mathcal{H} := H \otimes \mathcal{O}_B \), and \( \mathcal{J} \) is the sheaf of germs of holomorphic sections of \( J \to B^* \).

3.1.3 The action of the group variety \( W^g \) over \( B \) on \( W^g \) extends to a \( W^g \)-action on \( W \) [23, Lemma 5.1.1.(7)], so each Čech 1-cocycle \( \eta \) of the sheaf \( \mathcal{J}^W \) defines a (singular) locally Weierstraß fibration \( p^\eta : W^\eta \to B \) (cf. [10, Construction 3.14]). If \( \eta' \) is another 1-cocycle representing the same class in \( H^1(B, \mathcal{J}^W) \), then \( W^{\eta'} \cong W^\eta \) over \( B \). We say that \( p^\eta : W^\eta \to B \) is the locally Weierstraß fibration associated to \( \eta \) (or twisted by \( \eta \)).

More generally, let \( G \) be a finite group and \( p : W \to B \) a \( G \)-equivariant Weierstraß fibration such that the zero-section \( \Sigma \subset W \) is \( G \)-stable. The sheaf \( \mathcal{J}^W \) is endowed with a natural \( G \)-action and to each element \( \eta_G \in H^1_G(B, \mathcal{J}^W) \), we can associate a \( G \)-equivariant locally Weierstraß fibration \( p^\eta_G : W^g \to B \) [10, Section 3.E]. For later use, we shall briefly recall the construction.

Given an element \( \eta_G \in H^1_G(B, \mathcal{J}^W) \). Let \( \{ U_i \}_{i \in I} \) be a \( G \)-invariant good open cover of \( B \) and let \( G \) act on \( I \) such that \( p^{-1}(U_i) = U_{g_i} \). The element \( \eta_G \) can be represented by a 1-cocycle \( \{(\eta_{ij})_{i \in I}, (\eta_i^g)_{i \in I, g \in G} \} \) where \( \eta_{ij} \) is a 1-cocycle representing the image \( \eta \) of \( \eta_G \) in \( H^1(B, \mathcal{J}^W) \) and \( \eta_i^g \) is a local section of \( \mathcal{J}^W \) defined over \( U_i \). Let \( p^\eta : W^\eta \to B \) be the locally Weierstraß fibration twisted by \( \eta \). Fix biholomorphic maps

\[
\eta_i : W_i^\eta := (p^\eta)^{-1}(U_i) \to W_1 := p^{-1}(U_1)
\]

such that \( \eta_i \circ \eta_j^{-1} = \text{tr}(\eta_{ij}) \) where \( \text{tr}(\eta_{ij}) \) denotes the translation by the holomorphic section \( \eta_{ij} \). For each \( g \in G \), the automorphism \( \psi_g : W^\eta \to W^\eta \) that defines the \( G \)-action on \( p^\eta : W^\eta \to B \) associated to \( \eta_G \) is constructed by patching together the isomorphisms

\[
\psi_g := \eta_i^{-1} \circ \text{tr}(\eta_i^g) \circ g \circ \eta_{g_i} : W_i^\eta \to W_i^\eta.
\]

Conversely, given a \( G \)-equivariant minimal locally Weierstraß fibration \( p^\eta : W^\eta \to B \) twisted by \( \eta \in H^1(B, \mathcal{J}^W) \), at least when \( B \) is smooth we can also reconstruct the element \( \eta_G \in H^1_G(B, \mathcal{J}^W) \) starting with which \( p^\eta \) is constructed. First of all, the \( G \)-action on \( p^\eta \) induces a \( G \)-action on \( H := (R^1 p^*_i \mathbb{Z})_g \), where \( B^* \subset B \) is a Zariski open over which \( p^\eta \) is smooth. By [22, Corollary 2.6], the \( G \)-action on \( H \) extends to a \( G \)-action on \( W \) such that \( p \) is \( G \)-equivariant, and it is for this \( G \)-action we define the \( G \)-equivariant cohomology group \( H^1_G(B, \mathcal{J}^W) \). Now let \( \{ U_i \} \) be a good open cover of \( B \) and let \( \eta_i : W_i^\eta \to W_1 \) be biholomorphic maps such that \( \eta_i \circ \eta_j^{-1} = \text{tr}(\eta_{ij}) \) for some 1-cocycle \( \{ \eta_{ij} \} \) representing \( \eta \in H^1(B, \mathcal{J}^W) \). Let \( \psi_g : W^\eta \to W^\eta \) be the action of \( g \in G \) on \( W^\eta \). We define

\[
\text{tr}(\eta_i^g) := \eta_i \circ \psi_g \circ \eta_i^{-1} \circ \eta_{g_i}^{-1}.
\]

The element \( \eta_G \in H^1_G(B, \mathcal{J}^W) \) represented by the 1-cocycle \( \{(\eta_{ij}), (\eta_i^g) \} \) is the element we look for.

3.1.4 Let \( p : W \to B \) be a Weierstraß fibration over a compact complex manifold \( B \). For each \( \eta \in H^1(B, \mathcal{J}^W) \) there exists a family of elliptic fibrations

\[
\Pi : \mathcal{W} \to B \times V \to V := H^1(B, \mathcal{L})
\]

such that the elliptic fibration parameterized by \( t \in V \) corresponds to \( \eta + \exp(t) \in H^1(B, \mathcal{J}^W) \) where

\[
\exp : H^1(B, \mathcal{L}) \to H^1(B, \mathcal{J}^W)
\]
is the map induced by \( \exp : \mathcal{L} \to \mathcal{F}^W \). Indeed, let \( \text{pr}_1 : B \times V \to B \) be the first projection and let
\[
x \in V \otimes H^0(V, \mathcal{O}_V) \cong H^1(B \times V, \text{pr}_1^* \mathcal{L})
\]
be the element corresponding to the identity map \( \text{Id}_V \). Let
\[
\lambda_\eta := \text{pr}_1^* \eta + \tilde{\exp}(x) \in H^1(B \times V, \mathcal{F}^{W \times V})
\]
where
\[
\text{pr}_1^* : H^1(B, \mathcal{F}^W) \to H^1(B \times V, \mathcal{F}^{W \times V})
\]
is the map induced by pulling back sections \( \mathcal{F}^W \to \text{pr}_1^* \mathcal{F}^{W \times V} \) and
\[
\tilde{\exp} : H^1(B \times V, \text{pr}_1^* \mathcal{L}) \to H^1(B \times V, \mathcal{F}^{W \times V})
\]
the map induced by \( \text{pr}_1^* \mathcal{L} \to \mathcal{F}^{W \times V} \). Then the locally Weierstraß fibration \( q : \mathcal{W} \to B \times V \) twisted by \( \lambda_\eta \) will define such a family \( \Pi \). The family \( \Pi \) is called the tautological family associated to \( \eta \). Such a family can also be constructed when \( p : \mathcal{W} \to B \) is a singular Weierstraß fibration.

If, in addition, the fibration \( p \) is \( G \)-equivariant for some finite group \( G \) such that the zero section \( \Sigma \subset \mathcal{W} \) is preserved under the \( G \)-action and assume that \( \eta \) is the image of a class \( \eta_G \in H^1_G(B, \mathcal{F}^W) \), then \( \mathcal{W}^G := \Pi^{-1}(V^G) \) can be endowed with a \( G \)-action such that the restriction \( q_{|\mathcal{W}^G} \) is the \( G \)-equivariant locally Weierstraß fibration twisted by
\[
\lambda_{\eta_G} := \text{pr}_1^* \eta_G + \tilde{\exp}_G(x_G) \in H^1_G(B \times V^G, \mathcal{F}^{W \times V^G})
\]
where \( x_G \in V^G \otimes H^0(V^G, \mathcal{O}_{V^G}) \cong H^1_G(B \times V, \text{pr}_1^* \mathcal{L}) \) is the element corresponding to the identity map \( \text{Id}_{V^G} \) and
\[
\tilde{\exp}_G : H^1_G(B \times V, \text{pr}_1^* \mathcal{L}) \to H^1_G(B \times V, \mathcal{F}^{W \times V})
\]
is again the map induced by \( \exp : \text{pr}_1^* \mathcal{L} \to \mathcal{F}^{W \times V} \). So each point \( t \in V^G \) parameterizes the \( G \)-equivariant locally Weierstraß fibration twisted by \( \eta_G + \exp_G(t) \in H^1_G(B, \mathcal{F}^W) \) in the tautological family \( \Pi \).

For every \( G \)-stable analytic subset \( Z \subset B \), the next lemma gives a standard way to produce a subspace \( V^G_Z \) of \( V \) along which the deformation of \( p^n \) in the tautological family preserves \( G \)-equivariantly the fibration \( (p^n)^{-1}(Z) \to Z \) contained in \( p^n \).

**Lemma 3.2.** — Let \( Z \subset B \) be a \( G \)-stable subvariety and let
\[
V^G_Z = \ker \left( \iota^* : H^1(B, \mathcal{L}) \to H^1(Z, \iota^* \mathcal{L}) \right)^G
\]
where \( \iota : Z \hookrightarrow B \) is the inclusion. Then the subfamily of the tautological family associated to \( \eta_G \in H^1_G(B, \mathcal{F}^W) \) parameterized by \( V^G_Z \) preserves the \( G \)-action and \( G \)-equivariantly the fibration \( \mathcal{W}_Z^G := (p^n)^{-1}(Z) \to Z \).

**Proof.** — We already saw that the subfamily parameterized by \( V^G_Z \) preserves the \( G \)-action. Let \( \Psi : Z \times V^G_Z \hookrightarrow B \times V^G \) be the product of \( \iota \) with \( V^G_Z \hookrightarrow V \). The restriction
\[
\Psi := q^{-1}(Z \times V^G_Z) \to Z \times V^G_Z
\]
of \( \eta \) to \( \mathcal{Y} \) is the \( G \)-equivariant locally Weierstraß elliptic fibration twisted by \( \Psi^* \lambda_{H_0} \in H^1_G \left( \mathcal{Z} \times V^G_Z, \mathcal{J}^{W_x V^G_Z} \right) \) where \( W_Z := p^{-1}(Z) \) and

\[
\Psi^* : H^1_G \left( B \times V^G, \mathcal{J}^{W_x V^G} \right) \to H^1_G \left( \mathcal{Z} \times V^G_Z, \mathcal{J}^{W_x V^G_Z} \right)
\]

is the map induced by pulling back sections \( \mathcal{J}^{W_x V^G} \) to \( \Psi \). By definition of \( \xi_G \) and \( V^G_Z \), we have

\[
\Psi^* \xi_G = 0 \in H^1(\mathcal{Z}, \xi^G) \otimes \Omega^1(V^G_Z, \mathcal{O}_V^G).
\]

So \( \Psi^* \lambda_{H_0} = \operatorname{pr}_1^* \eta_G \) where \( \operatorname{pr}_1 : Z \times V^G_Z \to Z \) is the first projection and

\[
\operatorname{pr}_1^* : H^1_G(\mathcal{Z}, \mathcal{J}^{W_x}) \to H^1_G \left( \mathcal{Z} \times V^G_Z, \mathcal{J}^{W_x V^G_Z} \right)
\]

the map induced by \( \mathcal{J}^{W_x} \to \operatorname{pr}_1^*, \mathcal{J}^{W_x V^G_Z} \). It follows that \( \mathcal{Y} \to Z \times V^G_Z \) is \( G \)-equivariantly isomorphic to \( W_Z^1 \times V^G_Z \to Z \times V^G_Z \), which proves the first statement of Lemma 3.2.

Finally, when \( p : W \to B \) is the minimal Weierstraß fibration associated to \( H \), it follows from (3.3) that \( t \) and \( t' \in H^1(B, \xi^W) \) parameterize isomorphic elliptic fibrations (over \( B \)) in \( \Pi \) if and only if \( t - t' \) lies in the image of \( H^1(B, j, H) \).

3.1.5 Let \( p : W \to B \) be a Weierstraß fibration such that \( W \) is normal. As the codimension of \( W \setminus W^# \) in \( W \) is at least 2, for every \( m \in \mathbb{Z} \), the multiplication-by-\( m \) map \( W^# \to W^# \) extends to a meromorphic map \( m : W \to W \). Gluing these maps locally, we obtain for each \( \eta \in H^1(B, \xi^W) \) a map \( m : W^\eta \to W^{\eta m} \), which is generically finite if \( m \neq 0 \). An immediate consequence of the existence of \( m \) is the following cohomological criterion for the existence of a multi-section of \( p^\eta : W^\eta \to B \).

**Lemma 3.3.** Assume that \( W \) is normal. If \( \eta \in H^1(B, \xi^W) \) is torsion, then \( p^\eta : W^\eta \to B \) has a multi-section.

**Proof.** Assume that \( m \) is the order of \( \eta \), then \( m : W^\eta \to W \) defines a generically finite map onto \( W \). Thus \( m^{-1}(\Sigma) \) is a multi-section of \( W^\eta \) where we recall that \( \Sigma \subset W \) is the zero-section of \( p : W \to B \).

The reader is referred to Lemma 3.7 for a converse of Lemma 3.3.

3.2 Locally Weierstraß fibrations over a smooth variety with a normal crossing discriminant divisor

3.2.1 Let \( p : W \to B \) be a minimal Weierstraß fibration over a complex manifold and let \( \Delta \subset B \) be its discriminant locus. In this paragraph, we assume that \( \Delta \) is a normal crossing divisor. Under this assumption, \( W \), and more generally the total space of the fibration \( W^\eta \to B \) twisted by \( \eta \in H^1(B, \xi^W) \) have at worst rational singularities [22, Corollary 2.4]. In this situation, the map \( \varphi \) in (3.3) can be described as follows.

**Lemma 3.4 ([10, Lemma 3.15]).** Let \( p : W \to B \) be the minimal Weierstraß fibration associated to \( H \). Assume that \( B \) is smooth and \( \Delta \) is a normal crossing divisor. Then \( \varphi : j, H \to \mathcal{L}_{H/B} \) is isomorphic to \( R^1 p_* \mathcal{O} \) induced by \( Z \to \mathcal{O}_W \).

If the local monodromies of \( H \) around \( \Delta \) are unipotent, then the construction of the minimal Weierstraß fibration associated to \( H \) is functorial under pullback.
**Lemma 3.5 (23, Remark on p.549).** — Let $p : W := W(\mathcal{L}_W, \alpha, \beta) \to B$ be the minimal Weierstraß fibration associated to a local system $\mathcal{H}$. Assume that $B$ is smooth and the discriminant locus $\Delta$ of $p$ is a normal crossing divisor. Let $\psi : Z \to B$ be a holomorphic map from a complex manifold such that $\Delta' := \psi^{-1}(\Delta)$ is also a normal crossing divisor. Let $Z^* = Z \setminus \Delta'$ and $\mathcal{H}' = \psi_1^1(\mathcal{H})$. If the local monodromies of $\mathcal{H}$ around $\Delta$ are unipotent, then $\mathcal{L}_{W/Z} \simeq \psi_! \mathcal{L}_{W/B}$ and the base change $W_Z := W \times_B Z \to Z$ is the minimal Weierstraß fibration associated to $\mathcal{H}'$.

**Proof.** — This proof is pointed out to us by N. Nakayama. For simplicity, let $\mathcal{L} = \mathcal{L}_{W/B}$ and $\mathcal{L}' = \mathcal{L}_{W/Z}$. As the local monodromies of $\mathcal{H}$ around $\Delta$ are unipotent and $W$ has at worst rational singularities, by [17, Theorem 2.6] the sheaf $R^1 p_* \mathcal{O}_W$ is isomorphic to $\text{Gr}_0^1 \mathcal{H}'$, the zeroth graded piece of the Hodge filtration on the canonical extension $\mathcal{H}'$ of the VHS that $\mathcal{H}$ underlies. So $\mathcal{L} \simeq \text{Gr}_0^1 \mathcal{H}'$ by (3.2). Since the canonical extension is functorial, if $\mathcal{H}'$ denotes the canonical extension of the VHS that $\mathcal{H}'$ underlies, then

$$
\mathcal{L}' \simeq \text{Gr}_0^1 \mathcal{H}' \simeq \psi_! \text{Gr}_0^1 \mathcal{H} = \psi_! \mathcal{L}.
$$

Let $p' : W' := W(\mathcal{L}', \alpha', \beta') \to Z$ be the minimal Weierstraß fibration associated to $\mathcal{H}'$. Since $W_Z \to Z$ and $W' \to Z$ are both Weierstraß fibrations extending the Jacobian fibration associated to $\mathcal{H}'$, by [22, Lemma 1.4] there exists $\epsilon \in H^0(Z^*, \psi_! \mathcal{O} \otimes (\mathcal{L}')^\vee) \cong H^0(Z^*, \mathcal{O}_{Z'})$ such that $(\psi_* \alpha)_{Z'} = \epsilon \cdot \alpha'_{Z'}$ and $(\psi_* \beta)_{Z'} = \epsilon \cdot \beta'_{Z'}$. As $\epsilon^4$ is the restriction to $Z^*$ of the meromorphic function $(\psi_* \alpha)/\alpha'$, $\epsilon$ extends to a meromorphic function on $Z$ (still denoted by $\epsilon$). So $\psi_! \alpha = \epsilon^4 \alpha'$ and $\psi_! \beta = \epsilon^6 \beta'$. Since there is no prime divisor $D$ of $Z$ such that $\text{Div}(\alpha') \geq 4D$ (resp. $\text{Div}(\alpha) \geq 4D$), $\epsilon$ has no pole (resp. no zero). Therefore $W_Z \cong W'$ over $Z$, which proves Lemma 3.5.

### 3.2.2 Let $f : X := W^t \to B$ be the locally Weierstraß fibration twisted by $\eta \in H^1(B, \mathcal{J}_B^W)$. We still assume that $B$ is smooth and the discriminant locus $\Delta$ is a normal crossing divisor. In addition to (3.3), the sheaf $\mathcal{J}_B^W$ sits inside another short exact sequence

$$
0 \longrightarrow \mathcal{J}_B^W \longrightarrow R^1 f_* \mathcal{O}_X^\times \longrightarrow Z \longrightarrow 0.
$$

where $R^1 f_* \mathcal{O}_X^\times \to Z$ maps locally a line bundle to the degree of its restriction to a general fiber of $f$. The class $\eta \in H^1(B, \mathcal{J}_B^W)$ coincides with the one associated to the extension (3.5) [10, short exact sequence (9)].

Now let $f : X \to B$ be an elliptic fibration such that both $X$ and $B$ are smooth, the discriminant locus $\Delta \subset B$ is a normal crossing divisor, and $f$ has local meromorphic sections at every point of $B$. For such an elliptic fibration, there exists a short exact sequence similar to (3.5): Let $H := (R^1 f_* Z)|_{B, \Delta}$ and let $p : W \to B$ be the minimal Weierstraß fibration associated to $\mathcal{H}$. Regarding the zero-section $\Sigma \subset W$ as the neutral element, let $\mathcal{J}_B^W$ be the sheaf of abelian groups of germs of meromorphic sections of $p$. By [23, Lemma 5.4.8], there exists a short exact sequence

$$
0 \longrightarrow \mathcal{J}_B^W \longrightarrow R^1 f_* \mathcal{O}_X^\times / \mathcal{Y}_X \longrightarrow Z \longrightarrow 0
$$

where $\mathcal{Y}_X := \ker (R^1 f_* \mathcal{O}_X^\times \to j_* R^1 f_* \mathcal{O}_X^\times)$ and the third arrow comes from the map $R^1 f_* \mathcal{O}_X^\times \to Z$ defined by the degree of a line bundle restricted to a general fiber as in (3.5).

By [23, Proposition 5.5.1], there exists an inclusion

$$
\eta : \mathcal{E}(B, \Delta, \mathcal{H}) \hookrightarrow H^1(B, \mathcal{J}_B^W)
$$

where $\mathcal{E}(B, \Delta, \mathcal{H})$ is the sheaf of abelian groups of germs of meromorphic sections of $p$.
where \( \mathcal{E}(B, \Delta, \mathbf{H}) \) is the set of bimeromorphic classes of elliptic fibrations \( f : X \to B \) admitting local meromorphic sections at every point of \( B \) and whose restriction over \( B \setminus \Delta \) is bimeromorphic to a smooth fibration with \( (R^1f_!Z)_{B \setminus \Delta} = H \). The image \( \eta(f) \in H^1(B, \mathcal{F}_{H/B}) \) of \( f \) coincides with the element associated to the extension \((3.6)\). According to the above, there exists a map \( H^1(B, \mathcal{F}^W_{H/B}) \to \mathcal{E}(B, \Delta, \mathbf{H}) \) which associates \( \eta \in H^1(B, \mathcal{F}^W_{H/B}) \) to the bimeromorphic class of \( p^\eta : W^\eta \to B \) and the composition

\[
i^W : H^1(B, \mathcal{F}^W_{H/B}) \to \mathcal{E}(B, \Delta, \mathbf{H}) \hookrightarrow H^1(B, \mathcal{F}_{H/B})
\]
equals the map induced by the inclusion \( \mathcal{F}^W_{H/B} \subset \mathcal{F}_{H/B} \).

We can also generalize the above discussion to the \( G \)-equivariant setting \([10, \text{Section 3.E}]\). Let \( G \) be a finite group acting on \( B \) and on \( H \) in a compatible way. Let \( \mathcal{E}_G(B, \Delta, \mathbf{H}) \) denote the set of bimeromorphic classes of \( G \)-equivariant elliptic fibrations \( f \in \mathcal{E}(B, \Delta, \mathbf{H}) \) such that \( (R^1f_!Z)_{B \setminus \Delta} \) is \( G \)-equivariantly isomorphic to \( H \). \((2)\)

To each \( G \)-equivariant elliptic fibration \( f \in \mathcal{E}_G(B, \Delta, \mathbf{H}) \), we can associate an element \( \eta_G(f) \in H^1_G(B, \mathcal{F}_{H/B}) \) in an injective manner using a similar construction to the one sketched in \(3.1.3\). According to the above, there exists a map \( H^1_G(B, \mathcal{F}^W_{H/B}) \to \mathcal{E}_G(B, \Delta, \mathbf{H}) \) that associates \( \eta_G \in H^1_G(B, \mathcal{F}^W_{H/B}) \) to the bimeromorphic class of the \( G \)-equivariant elliptic fibration \( p^\eta : W^\eta \to B \) and the composition

\[
i^G_W : H^1_G(B, \mathcal{F}^W_{H/B}) \to \mathcal{E}_G(B, \Delta, \mathbf{H}) \hookrightarrow H^1_G(B, \mathcal{F}_{H/B})
\]
equals the map induced by the inclusion \( \mathcal{F}^W_{H/B} \subset \mathcal{F}_{H/B} \).

### 3.3 Kähler or projective elliptic fibrations

Let \( f : X \to B \) be an elliptic fibration over a complex manifold \( B \) with a normal crossing discriminant divisor. The aim of this paragraph is to recall further properties of \( f \) under the additional assumption that \( X \) is in the Fujiki class \( \mathcal{E} \).

Given a minimal \( \eta \)-twisted locally Weierstraß fibration \( p^\eta : W^\eta \to B \) over a compact Kähler manifold, we have the following cohomological characterization for the total space \( W^\eta \) to be bimeromorphically Kähler.

**Proposition 3.6** ([10, Theorem 3.20 and Proposition 3.23]). — Let \( p : W \to B \) be a minimal Weierstraß fibration over a compact Kähler manifold and assume that the discriminant locus is a normal crossing divisor. Let \( \eta \in H^1(B, \mathcal{F}^W_{H/B}) \). The following assertions are equivalent.

1) The total space \( W^\eta \) is in the Fujiki class \( \mathcal{E} \).

2) The "Chern class" \( c(\eta) \) of \( \eta \) is a torsion element, where \( c : H^1(B, \mathcal{F}^W_{H/B}) \to H^2(B, j_!\mathbf{H}) \) is defined to be the connecting morphism induced by \((3.3)\).

When \( W^\eta \) is in the Fujiki class \( \mathcal{E} \) and \( B \) is projective, we can prove the following converse of Lemma 3.3.

**Lemma 3.7.** — Let \( p : W \to B \) be a minimal Weierstraß fibration and \( f : W^\eta \to B \) the locally Weierstraß fibration twisted by \( \eta \in H^1(B, \mathcal{F}^W_{H/B}) \). Assume that \( W^\eta \) is in the Fujiki class \( \mathcal{E} \) and \( B \) is projective. If \( f \) has a multi-section, then \( \eta \) is a torsion element.

\(2\) We could have defined \( \mathcal{E}_G(B, \Delta, \mathbf{H}) \) to be a larger set by allowing the \( G \)-action on the total space of \( f \) to be only meromorphic (but still holomorphic over \( B \setminus \Delta \)). However in this text, we will only consider \( G \)-actions that are holomorphic.
Proof. — Since $W^i$ is in the Fujiki class $\mathcal{C}$, by Proposition 3.6 $c(\eta)$ is torsion. So up to replacing $\eta$ with a larger multiple, we may assume that $\eta = \exp(t)$ for some $t \in H^1(B, \mathcal{L}_{B/B})$. Since the multiplication by $m$ is generically finite, the elliptic fibration $f : W^{\exp(t)} \to B$ still has a multi-section. For any $m \in \mathbb{Z}_{>0}$, a multi-section of $f$ can be pulled back to a multi-section of $f' = f^{\exp(t/m)} : W^{\exp(t/m)} \to B$ under the generically finite map $m : W^{\exp(t/m)} \to W^{\exp(t)}$. Assume that $m$ is sufficiently large so that $W^{\exp(t/m)}$ is closed enough to $W$ in the tautological family associated to $p : W \to B$. As $B$ is projective, $W$ is also projective. So $W^{\exp(t/m)}$, being a small deformation of $W$ with at worst rational singularities (because $p : W \to B$ is minimal), is Kähler [25, Proposition 5]. Again as $B$ is projective, $W^{\exp(t/m)}$ is Moishezon by Theorem 2.5. It follows that $W^{\exp(t/m)}$ is projective [26, Theorem 1.6] and an ample line bundle on $W^{\exp(t/m)}$ gives rise to an element $s \in H^0(B, R^1f^\star \mathcal{O}_{W^{\exp(t/m)}})$ whose restriction to a general fiber of $f'$ has nonzero degree. So the image of $s$ in $H^0(B, \mathbb{Z})$ under the map induced by (3.5) is not zero. Therefore $\exp(t/m)$ is torsion, so $\eta$ is torsion as well. —

At the end of 3.2, we introduced the map $i^W : H^1(B, \mathcal{G}_{W/B}) \to H^1(B, \mathcal{F}_{W/B})$. This map is not surjective in general. However when $X$ is in the Fujiki class $\mathcal{C}$, the cokernel of $H^1(B, \mathcal{G}_{W/B}) \to H^1(B, \mathcal{F}_{W/B})$ is always torsion.

Lemma 3.8 ([10, Lemma 3.19]). — Let $f : X \to B$ be an elliptic fibration over a compact complex manifold $B$. Assume that the discriminant locus is a normal crossing divisor and $f$ has local meromorphic sections over every point of $B$. Let $W \to B$ be the minimal Weierstraß fibration associated to $f$. Assume that $X$ is in the Fujiki class $\mathcal{C}$, then there exists $m \in \mathbb{Z} \setminus \{0\}$ such that

$$m \cdot \eta(f) \in \text{Im}(i^W : H^1(B, \mathcal{G}_{W/B}) \to H^1(B, \mathcal{F}_{W/B})).$$

For $G$-equivariant elliptic fibrations $f : X \to B$, if the conclusion of Lemma 3.8 holds, then it also holds $G$-equivariantly:

Lemma 3.9 ([10, Lemma 3.24]). — Let $p : W \to B$ be a $G$-equivariant Weierstraß fibration for some finite group $G$ such that the zero-section $\Sigma \subset W$ is $G$-stable. Let $\eta_G \in H^1_G(B, \mathcal{F}_{W/B})$ and let $\eta$ denote its image in $H^1(B, \mathcal{F}_{W/B})$. Assume that there exist $m \in \mathbb{Z}_{>0}$ and $\eta' \in H^1(B, \mathcal{G}_{W/B})$ such that $m\eta = i^W(\eta')$, then up to replacing $m$ with a larger multiple of it, $m\eta_G$ can be lifted to an element $H^1_G(B, \mathcal{G}_{W/B})$.

3.4 Tautological models

Let $B$ be a complex manifold and $B^* \subset B$ a Zariski open such that $B \setminus B^*$ is a normal crossing divisor. Let $H$ be a local system of stalk $\mathbb{Z}$ over $B^*$ and let $G$ be a finite group acting on both $B$ and $H$ in a compatible way. We denote by $p : W \to B$ the minimal $G$-equivariant Weierstraß fibration associated to $H$. Given $\eta_G \in H^1_G(B, \mathcal{F}_{W/B})$ and assume that $m\eta_G$ can be lifted to an element $\eta_G' \in H^1_G(B, \mathcal{G}_{W/B})$ for some $m \in \mathbb{Z} \setminus \{0\}$. Let $\cup_i$ be a $G$-invariant good open cover of $B$ and $\{(\eta_i), (\eta_i^0)\}$ a 1-cocycle representing $\eta_G \in H^1_G(B, \mathcal{F}_{W/B})$. In this paragraph, we will construct a $G$-equivariant elliptic fibration

$$f : \mathcal{X} := \mathcal{X}(m; (\eta_i), (\eta_i^0); \eta_G') \to B$$

representing $\eta_G$ together with a $G$-equivariant finite holomorphic map

$$m : \mathcal{X} \to W^{\eta'}$$
over $B$, where $W^\eta \to B$ is the $G$-equivariant locally Weierstraß fibration associated to $\eta_G' \in H^1_G(B, \mathcal{F}_W^W)$. The map $\mu$ generalizes the multiplication-by-$m$ map defined before. We will call this $G$-equivariant elliptic fibration $f$ a tautological model associated to $\eta_G$. The reason we call such an elliptic fibration a tautological model is that in the next section, we will show that each tautological model $f : X \to B$ comes equipped with a tautological family of elliptic fibrations parameterized by $H^1(B, \mathcal{Z}_{H/B})$ generalizing the one defined before for twisted locally Weierstraß fibrations.

An argument similar to that of [23, Proposition 5.5.4] will allow us to construct $f : X \to B$. For each $g \in G$, let $\phi_g : W \to W$ be the biholomorphic map defining the $G$-action on $W$. Let $(\eta'_G, (\eta''_G))$ be a 1-cocycle representing $\eta'_G \in H^1_G(B, \mathcal{F}_W^W)$. We can also glue the $X$-fibers over $h_i \cap U_i$. For each $i$, let $h_i : X_i \to W_i$ be the finite map in the Stein factorization of $\text{tr}(\sigma) \circ \mu$. Then there exist bimeromorphic maps $h_i : X_i \to W_i$ over $U_i$ such that

\[
\begin{align*}
\eta'_G \circ \phi_g &\quad \eta''_G \circ \phi_g \\
\mu_i &\quad \mu_i \\
\text{tr}(\sigma) &\quad \text{tr}(\sigma) \\
\end{align*}
\]

are commutative where the notation $\text{tr}(\sigma)$ denotes the translation by the meromorphic section $\sigma$ and $\mu_i$ (resp. $\mu_i$) the restriction to $W_i$ (resp. $W_i$) of the multiplication-by-$m$ map $W \to W$. Let $\mu_i : X_i \to W_i$ be the finite map in the Stein factorization of $\text{tr}(\sigma) \circ \mu$. Then there exist bimeromorphic maps $h_i : X_i \to W_i$ over $U_i$ such that

\[
\begin{align*}
X_i &\quad X_i \\
\mu_i \quad \mu_i \\
\text{tr}(\sigma) \quad \text{tr}(\sigma) \\
W_i &\quad W_i \\
\end{align*}
\]

are commutative where $X_i := \mu_i^{-1}(W_i)$ and $X_i := \mu_i^{-1}(W_i)$. As $\mu_i$ and $\mu_j$ are finite and the varieties in (3.8) are normal, the maps $h_i$ and $h_j$ are biholomorphic. Thus we obtain an elliptic fibration $f : X \to B$ by gluing the $X_i \to U_i$ using the 1-cocycle $(h_i)$ of biholomorphic maps. The maps $h_i$ can also be glued to a biholomorphic map $\psi : X \to X$ using the cocycle condition of $\{(\eta'_G, (\eta''_G))\}$ and $g \mapsto \psi_g$ defines a $G$-action on $X$ such that $f : X \to B$ is $G$-equivariant. By construction, $f$ is a $G$-equivariant elliptic fibration representing $\eta_G$. We can also glue the $\mu_i$ and obtain a $G$-equivariant finite map $\mu : X \to W^\eta$, which generalizes the multiplication-by-$m$ maps defined previously.

**Definition 3.10.** Let $\eta_G \in H^1_G(B, \mathcal{F}_W^W)$ be an element such that $m\eta_G$ can be lifted to an element $\eta_G' \in H^1(B, \mathcal{F}_W^W)$ for some integer $m \neq 0$. The $G$-equivariant elliptic fibration

\[ f : X \to \mathcal{X}(m; (\eta'_G), (\eta''_G); \eta_G') \to B \]

representing $\eta_G$ constructed above is called a $(G$-equivariant) tautological model (associated to $\eta_G$).

If the $G$-action is trivial (so that $H^1_G(B, \mathcal{F}_W^W) = H^1(B, \mathcal{F}_W^W)$), we will use the notation $\mathcal{X}(m; (\eta'_G); (0); \eta''_G)$ instead of $\mathcal{X}(m; (\eta'_G), (0); \eta''_G)$.
Remark 3.11. — Up to isomorphism, the construction of \( f : X \to B \) and \( m \) depends on \( m \), the 1-cocycle \( \left( \eta_B, \eta^0_i \right) \) representing \( \eta_G \), and \( \eta^0_G \). But it is easy to see from the construction that \( f \) and \( m \) do not depend on the 1-cocycle \( \left( \eta^0_i, \eta^0_G \right) \) representing \( \eta^0_G \).

Remark 3.12. — Since the Stein factorization is functorial under flat pullbacks, the flat pullback of a tautological model is still a tautological model.

The next lemma shows that a bimeromorphic class of elliptic fibrations \( \left[ g \right] \in \mathcal{E}_G(B, \Delta, H) \) contains a tautological model if the total space of \( g \) is in the Fujiki class \( \mathcal{C} \).

Lemma-Definition 3.13. — Let \( g : X \to B \) be a \( G \)-equivariant elliptic fibration over a compact complex manifold for some finite group \( G \). Assume that \( X \) is in the Fujiki class \( \mathcal{C} \), \( g \) is smooth over the complement of a normal crossing divisor in \( B \), and \( g \) has local meromorphic sections over every point of \( B \). Then the element \( \eta_G(g) \in H^1_G(B, \mathcal{J}_{H/B}) \) associated to \( g \) has a tautological model, called a \( (G\text{-equivariant}) \) tautological model of \( g \).

Proof. — Let \( \eta \) be the image of \( \eta_G(g) \) in \( H^1(B, \mathcal{J}_{H/B}) \). Since \( X \) is in the Fujiki class \( \mathcal{C} \), by Lemma 3.8 some nonzero multiple of \( \eta \) can be lifted to an element in \( H^1(B, \mathcal{J}^W_{H/B}) \). So by Lemma 3.9, some nonzero multiple of \( \eta_G \) can be lifted to an element \( \eta_G' \in H^1_G(B, \mathcal{J}^W_{H/B}) \). Thus \( \eta_G(g) \) has a tautological model. \( \square \)

We finish this paragraph with some geometric properties of the tautological models.

Lemma 3.14. — The total space of \( f : X \to B \) is normal and the discriminant locus \( \Delta_f \subset B \) of \( f \) coincides with the discriminant locus \( \Delta \) of the minimal Weierstraß fibration \( p : W \to B \) associated to \( H \).

Proof. — As \( \mu_i : X_i \to W_i \) is the finite map in a Stein factorization, each \( X_i \) is normal. So \( X \) is normal.

Since the restriction of \( p \) to \( W \setminus p^{-1}(\Delta) \) is the minimal Weierstraß fibration associated to the restriction of \( f \) to \( X \setminus f^{-1}(\Delta_f) \), which is a smooth fibration, we have \( \Delta \subset \Delta_f \). For the other inclusion, since a meromorphic section of a smooth elliptic fibration over a smooth variety is holomorphic [24, Lemma 1.3.5], the bimeromorphic maps in (3.7) are biholomorphic outside of \( p^{-1}(\Delta) \). So \( X_i \to U_i \) is isomorphic to \( W_i \to U_i \) over \( U_i \setminus \Delta \). In particular \( X \to B \) is smooth over \( B \setminus \Delta \). \( \square \)

Lemma 3.15. — If \( f : X \to B \) is a tautological model associated to a \( G \)-equivariant elliptic fibration \( g : X \to B \) by virtue of Lemma-Definition 3.13, then there exists a \( G \)-equivariant bimeromorphic map \( X \to X \) over \( B \) which is biholomorphic over \( B \setminus \Delta_g \) where \( \Delta_g \) is the discriminant locus of \( g \).

Proof. — Since both \( f \) and \( g \) represent the same class \( \eta_G(g) \in H^1_G(B, \mathcal{J}_{H/B}) \), we have a \( G \)-equivariant bimeromorphic map \( X \to X \) over \( B \). Let \( p : W \to B \) be the minimal Weierstraß fibration associated to \( g \) and \( \Delta \) the discriminant locus of \( p \). Since \( \Delta \subset \Delta_g \), by Lemma 3.14 \( f \) is smooth over \( B \setminus \Delta_g \subset B \setminus \Delta = B \setminus \Delta_f \). As \( g \) is also smooth over \( B \setminus \Delta_g \), by [23, Lemma 5.3.3] \( X \to X \) is biholomorphic over \( B \setminus \Delta_g \). \( \square \)

3.5 The tautological family associated to a tautological model

We continue to use the notations introduced in the last paragraph. The goal of this paragraph is to construct a family of elliptic fibrations parameterized by \( V := H^1(B, \mathcal{J}_{H/B}) \) which contains \( f : X \to B \) as
the central fiber and which is tautological in the sense that the elliptic fibration parameterized by \( t \in V \) represents \( \eta + \exp'(t) \in H^1(B, \mathcal{F}_{\mathcal{H}/B}) \), where

\[
\exp' : H^1(B, \mathcal{L}_{\mathcal{H}/B}) \to H^1(B, \mathcal{F}_{\mathcal{H}/B})
\]

is the composition of \( \exp : H^1(B, \mathcal{L}_{\mathcal{H}/B}) \to H^1(B, \mathcal{F}_{\mathcal{H}/B}^W) \) with \( i^W : H^1(B, \mathcal{F}_{\mathcal{H}/B}^W) \to H^1(B, \mathcal{F}_{\mathcal{H}/B}) \). Moreover, if \( t \in V^G \), then \( t \) parameterizes the \( G \)-equivariant elliptic fibration representing \( \eta_G + \exp'_G(t) \in H^1_G(B, \mathcal{F}_{\mathcal{H}/B}) \), where

\[
\exp'_G : H^1(B, \mathcal{L}_{\mathcal{H}/B})^G \to H^1_G(B, \mathcal{F}_{\mathcal{H}/B})
\]

is defined similarly. When \( \dim B = 1 \), tautological families have already been constructed by Kodaira in [16]. In higher dimension, such families have also appeared in [23, Lemma 7.4.3] and [10, Lemma 4.1].

**Proposition-Definition 3.16.** — Let \( B \) be a compact complex manifold and \( B^* \subseteq B \) a normal crossing divisor. Let \( \mathcal{H} \) be a local system of stalk \( \mathcal{L}^2 \) over \( B^* \) and let \( G \) be a finite group acting on both \( B \) and \( \mathcal{H} \) in a compatible way. Let \( p : W \to B \) denote the \( G \)-equivariant minimal Weierstraß fibration associated to \( \mathcal{H} \) and \( \Delta \subseteq B \) the discriminant divisor of \( p \). Given \( \eta_G \in H^1_G(B, \mathcal{F}_{\mathcal{H}/B}) \) and assume that \( m \eta_G \) can be lifted to an element \( \eta'_G \in H^1_G(B, \mathcal{F}_{\mathcal{H}/B}^W) \) for some \( m \in \mathbb{Z} \setminus \{0\} \). Consider the tautological model \( f : \mathcal{X} = \mathcal{X} \left( m; (\eta_i), (\eta'_i); \eta'_G \right) \to B \) of \( \eta_G \) where \( \left\{(\eta_i), (\eta'_i)\right\} \) is a 1-cocycle representing \( \eta_G \). Then there exists a family of elliptic fibrations

\[
\Pi : \mathcal{X} \to B \times V \to V := H^1(B, \mathcal{L}_{\mathcal{H}/B})
\]

satisfying the following properties.

1. *(Tautologicality)* The central fiber of \( \Pi \) is \( \mathcal{X} \to B \) and the elliptic fibration parameterized by \( t \in V \) represents the element \( \eta + \exp'(t) \in H^1(B, \mathcal{F}_{\mathcal{H}/B}) \) where \( \eta \in H^1(B, \mathcal{F}_{\mathcal{H}/B}) \) is the image of \( \eta_G \).

2. *(\( G \)-equivariance)* There exists a \( G \)-action on \( \mathcal{X}^G := q^{-1}(B \times V) \) such that the subfamily

\[
\Pi^G : \mathcal{X}^G \to B \times V^G \to V^G
\]

of \( \Pi \) parameterized by \( V^G \) is \( G \)-equivariantly locally trivial over \( B \) and the \( G \)-equivariant elliptic fibration parameterized by \( t \in V^G \) corresponds to \( \eta_G + \exp'_G(t) \in H^1_G(B, \mathcal{F}_{\mathcal{H}/B}) \).

3. *(Multiplication-by-m)* Let \( \Pi' : W \to B \times V \to V \) be the tautological family associated to the image \( \eta' \in H^1(B, \mathcal{F}_{\mathcal{H}/B}^W) \) of \( \eta'_G \). There exists a map \( m : \mathcal{X} \to \mathcal{W} \) over \( B \times V \) whose restriction to each fiber over \( V \) is the multiplication-by-\( m \) defined in 3.4. Over \( V^G \), the map \( m \) is \( G \)-equivariant.

The family \( \Pi \) is called the tautological family associated to \( f : \mathcal{X} \to B \).

**Proof.** — First we define 1-cocycle classes \( \lambda, \lambda_G, \theta, \) and \( \theta_G \) that we will use to construct the family \( \Pi \) and the \( G \)-action over \( V^G \). Let

\[
\xi \in V \otimes H^0(V, \mathcal{O}_V) = H^1(B \times V, \mathcal{L}_{\mathfrak{pr}^1_{\mathcal{H}/B \times V}},)
\]

\[
\xi_G \in V^G \otimes H^0(V^G, \mathcal{O}_{V^G}) = H^1(B \times V^G, \mathcal{L}_{\mathfrak{pr}^1_{\mathcal{H}/B \times V^G}})
\]
be the elements corresponding to the identity maps $\Id_V$ and $\Id_{V^G}$. Let $\pr_1 : B \times V \to B$ be the first projection and define
\[
\lambda := \pr_1^* \eta + \overline{\exp'}(\xi) \in H^1(B \times V, \mathcal{F}_{pr_1^*}H/B \times V),
\]
\[
\lambda_G := \pr_1^* \eta_G + \overline{\exp_p'}(\xi_G) \in H^1_G(B \times V^G, \mathcal{F}_{pr_1^*}H/B \times V^G),
\]
\[
\theta := \pr_1^* \eta' + \exp(m\xi) \in H^1(B \times V, \mathcal{F}_{pr_1^*}H/B \times V),
\]
\[
\theta_G := \pr_1^* \eta_G' + \overline{\exp}(m\xi_G) \in H^1_G(B \times V^G, \mathcal{F}_{pr_1^*}H/B \times V^G),
\]
where $\pr_1 : H^1(B, \mathcal{F}_{H/B}) \to H^1(B \times V, \mathcal{F}_{pr_1^*}H/B \times V)$ in the definition of $\lambda$ is the map induced by $\mathcal{F}_{H/B} \to \pr_1^* \mathcal{F}_{pr_1^*}H/B \times V$ (the definitions of the other $\pr_1^*$ are similar), $\exp$ is the map induced by $\exp : \mathcal{F}_{pr_1^*}H/B \times V \to \mathcal{F}_{pr_1^*}H/B \times V$, and $\overline{\exp'}$ is the composition of $\overline{\exp}$ with $H^1(B \times V, \mathcal{F}_{pr_1^*}H/B \times V) \to H^1(B \times V, \mathcal{F}_{pr_1^*}H/B \times V)$ (similar for $\overline{\exp}$ and $\overline{\exp'}$).

We specify the Čech 1-cocycles representing $\lambda$ and $\lambda_G$ as follows. Let $\{U_i\}_{i \in I}$ be a $G$-invariant good open cover of $B$ and let $U_{ij} := U_i \cap U_j$. Let $\{(\eta_{ij}), (\eta_{ij}^G)\}$ be a 1-cocycle representing $\eta_G \in H^1_G(B, \mathcal{F}_{H/B})$ and let
\[
\eta_{ij} \in \mathcal{F}_{pr_1^*}H/B \times V(U_{ij} \times V) \quad \text{and} \quad \eta_{ij}^G \in \mathcal{F}_{pr_1^*}H/B \times V^G(U_{ij} \times V^G)
\]
be the pullbacks of $\eta_{ij}$ and $\eta_{ij}^G$ under $U_{ij} \times V \to U_{ij}$ and $U_{ij} \times V^G \to U_{ij}$ respectively. As $V$ and $V^G$ are contractible, $\{U_i \times V\}$ and $\{U_i \times V^G\}$ are good covers of $B \times V$ and $B \times V^G$. Fix a $G$-invariant 1-cocycle $(\xi_{ij})$ representing $\xi$. The 1-cocycles $\{\lambda_{ij}\}$ and $\{(\lambda_{ij}^G), (\lambda_{ij}^G)\}$ we choose to represent $\lambda$ and $\lambda_G$ are defined by
\[
\lambda_{ij} := \eta_{ij} + \exp(U_{ij} \times V)(\xi_{ij}) \in \mathcal{F}_{pr_1^*}H/B \times V(U_{ij} \times V),
\]
\[
\lambda_{ij}^G := \lambda_{ij} |_{U_i \times V^G} \in \mathcal{F}_{pr_1^*}H/B \times V^G(U_{ij} \times V^G),
\]
(3.9)
where $\exp : \mathcal{F}_{pr_1^*}H/B \times V \to \mathcal{F}_{pr_1^*}H/B \times V$ is the exponential map. We define the elliptic fibrations $q : \mathcal{X} \to B \times V$ and $q^G : \mathcal{X}^G \to B \times V^G$ to be the tautological models $X(m_1; (\lambda_{ij}), \theta) \to B \times V$ and $X(m; (\lambda_{ij}^G), (\lambda_{ij}^G), \theta_G) \to B \times V^G$. In order to prove Proposition-Definition 3.16 for $q$ and $q^G$, we need to look into the construction of these tautological models. This will also be useful when we prove Proposition 4.17 in Section 4.

Let $\{(\eta_{ij}^G), (\eta_{ij}^G)\}$ be a 1-cocycle representing $\eta_{ij}^G$ and let
\[
\tilde{\eta}_{ij}^G \in \mathcal{F}_{pr_1^*}H/B \times V(U_{ij} \times V) \quad \text{and} \quad \tilde{\eta}_{ij}^G \in \mathcal{F}_{pr_1^*}H/B \times V^G(U_{ij} \times V^G)
\]
be the pullbacks of $\eta_{ij}^G$ and $\eta_{ij}^G$ under $U_{ij} \times V \to U_{ij}$ and $U_{ij} \times V^G \to U_{ij}$ respectively. Define
\[
\theta := \tilde{\eta}_{ij}^G + \exp(U_{ij} \times V)(m\xi_{ij}) \in \mathcal{F}_{pr_1^*}H/B \times V(U_{ij} \times V),
\]
\[
\theta_G := \tilde{\eta}_{ij}^G |_{U_i \times V^G} \in \mathcal{F}_{pr_1^*}H/B \times V^G(U_{ij} \times V^G),
\]
(3.10)
so that $\theta$ and $\theta_G$ are represented by $[\theta_{ij}]$ and $\{(\theta_{ij}^G), (\theta_{ij}^G)\}$. Since $m\eta_{ij} = t^G(m\eta_{ij})$, up to refining $\{U_i\}$ there exist meromorphic sections $\sigma_i$ of $W_i := p^{-1}(U_i) \to U_i$ such that the 1-coboundary associated to $[\sigma_i]$ modifies the 1-cocycle $\{(m\eta_{ij}), (m\eta_{ij}^G)\}$ to $\{(\eta_{ij}^G), (\eta_{ij}^G)\}$. So by construction, the 1-coboundary associated to $[\sigma_i \times \Id_{V^G}]$ (resp. $[\sigma_i \times \Id_{V^G}]$) modifies the 1-cocyle $m\lambda_{ij}$ to $[\theta_{ij}]$ (resp. $\{(m\lambda_{ij}^G), (m\lambda_{ij}^G)\}$ to $\{(\theta_{ij}^G), (\theta_{ij}^G)\}$. We emphasize that
the 1-coboundaries modifying these 1-cocycles are of the form \( [\sigma_1 \times \text{Id}] \), which we will use to deduce that the bimeromorphic maps \( \mathcal{X}_i \times V \to W_i \times V \) induced by the Stein factorizations in the construction of the tautological model are of the form \( h_i \times \text{Id} : \mathcal{X}_i \times V \to W_i \times V \).

As in the construction of \( f : \mathcal{X} \to B \), if \( m_i \) and \( m_{ij} \) denote the restrictions of the multiplication-by-\( m \) map \( W \to W \) to \( W_i \) and \( W_{ij} := p^{-1}(U_{ij}) \), then we have commutative diagrams

\[
\begin{array}{ccc}
W_{ij} \times V & \xrightarrow{\text{tr}(\lambda_{ij})} & W_{ji} \times V \\
\downarrow_{m_{ij} \times \text{Id}} & & \downarrow_{m_{ji} \times \text{Id}} \\
W_{ij} \times V & \xrightarrow{\text{tr}(\sigma_{ij}) \times \text{Id}} & W_{ji} \times V
\end{array}
\quad \quad \quad
\begin{array}{ccc}
W_{ij} \times V & \xrightarrow{\text{tr}(\lambda_{ij})} & W_{ji} \times V \\
\downarrow_{\phi \times \text{Id}} & & \downarrow_{\phi \times \text{Id}} \\
W_{ij} \times V & \xrightarrow{\text{tr}(\sigma_{ij}) \times \text{Id}} & W_{ji} \times V
\end{array}
\]

where \( \phi : W \to W \) denotes the action of \( g \in G \) on \( W \) induced by the \( G \)-action on \( H \). Let \( \mu_i : \mathcal{X}_i \to W_i \) be the finite map in the Stein factorization of \( \text{tr}(\sigma_i) \circ \mu \), and let \( X_{ij} := \mu_i^{-1}(W_{ij}) \) for \( X_{ji} \). Then there exist bimeromorphic maps \( h_i : \mathcal{X}_i \to W_i \) over \( U_i \) such that

\[
\begin{array}{ccc}
X_{ij} \times V & \xrightarrow{H_{ij}} & X_{ji} \times V \\
\downarrow_{\mu_i \times \text{Id}} & & \downarrow_{\mu_j \times \text{Id}} \\
W_{ij} \times V & \xrightarrow{\text{tr}(\lambda_{ij})} & W_{ji} \times V
\end{array}
\quad \quad \quad
\begin{array}{ccc}
X_{ij} \times V & \xrightarrow{H_{ij}} & X_{ji} \times V \\
\downarrow_{\mu_i \times \text{Id}} & & \downarrow_{\mu_j \times \text{Id}} \\
W_{ij} \times V & \xrightarrow{\text{tr}(\lambda_{ij})} & W_{ji} \times V
\end{array}
\]

are commutative. As both \( X_i \) and \( W_i \) are normal and \( \mu_i \) is finite, the bimeromorphic maps \( H_{ij} \) and \( H_{ji} \) are biholomorphic. Thus by gluing the \( \mu_i : \mathcal{X}_i \times V \to W_i \times V \) together using the 1-cocycle of biholomorphic maps \( \{H_{ij}\} \), we obtain a tautological model \( q : \mathcal{X} \to B \times V \) of \( \lambda \), a locally Weierstraß fibration \( q' : \mathcal{W} \to B \times V \) twisted by \( \theta \), and a finite surjective map \( m : \mathcal{X} \to \mathcal{W} \) over \( B \times V \). Moreover, over \( B \times V \) the fibration \( q \) (resp. \( q' \)) is the \( G \)-equivariant tautological model \( \mathcal{X}^G := \mathcal{X}(m; (\lambda_{ij}^G, \lambda_{ji}^G); \theta_G) \to B \times V^G \) (resp. the \( G \)-equivariant locally Weierstraß fibration twisted by \( \theta_G \)) and the restriction of \( m \) to \( \mathcal{X}^G \) is \( G \)-equivariant. This is the construction of the family \( \Pi : \mathcal{X}^G \to B \times V \to V \) together with the \( G \)-action on \( \mathcal{X}^G \) and the map \( m : \mathcal{X} \to \mathcal{W} \).

Obviously \( \Pi^G \) is \( G \)-equivariantly locally trivial over \( B \). By construction, the central fiber of \( \Pi^G \) is the \( G \)-equivariant tautological model \( f \). As the elliptic fibration \( \mathcal{X}_i \to B \) parameterized by \( t \) in the family \( \Pi \) is the tautological model \( \mathcal{X}(m; (\lambda_{ij}^G(t), \lambda_{ji}^G(t)); \theta_G(t)) \to B \times V^G \) associated to \( \lambda_{ij}^G(t) = \eta + \exp(t) \), it represents the element \( \eta + \exp(t) \). Similarly, the elliptic fibration parameterized by \( t \) in \( V^G \) represents \( \eta_G + \exp_G(t) \). This proves Proposition-Definition 3.16. 

\[ \square \]

4 The Hodge theory of minimal Weierstraß fibrations

In this section, we will use Hodge theory to study minimal Weierstraß fibrations and derive some geometric consequences on the tautological families introduced at the end of Section 3. Consider the following hypotheses on an elliptic fibration \( f : X \to B \):
Hypotheses 4.1. —

i) The base $B$ is a projective manifold.

ii) The total space $X$ is in the Fujiki class $C$.

iii) The discriminant locus $\Delta \subset B$ is a normal crossing divisor.

iv) The local monodromies of $H := (R^1 f_* Z)|_{B \setminus \Delta}$ around $\Delta$ are unipotent.

The main result of this section that will be useful for proving Theorem 1.2 in Section 5 is the following.

Proposition 4.2. — Let $G$ be a finite group and $f : X \to B$ a $G$-equivariant tautological model satisfying Hypotheses 4.1. Let $Z \subset B$ be a $G$-stable subvariety. If $f^{-1}(Z \setminus \Delta) \to Z \setminus \Delta$ has a multi-section, then the tautological family associated to $f$ contains a subfamily

$$
\Pi' : Z' \to B \times V' \to V'
$$

parameterized by some linear subspace $V' \subset H^1(B, \mathcal{Z}_{H/B})$ such that $\Pi'$ is an algebraic approximation of $X$ which is $G$-equivariantly locally trivial over $B$ and preserves $G$-equivariantly the completion $f^{-1}(Z) \to Z$ of $f$ along $f^{-1}(Z) \to Z$.

We postpone the discussion on Hypotheses 4.1 to the beginning of 4.2. With the notations introduced in Section 3, we will first observe in 4.1 that $H^1(B, j_* \mathcal{H})$ carries a natural (polarized) pure Hodge structure of weight $2$ such that $H^1(B, j_* \mathcal{H}) \to H^1(B, \mathcal{Z}_{H/B})$ induced by (3.3) is the projection to its $(0, 2)$-part (see Lemma 4.4). Then we will apply Lemma 4.4 to prove two results using Hodge theory (Proposition 4.6 and Proposition 4.11) in 4.2 and 4.3. With the notations in Proposition 4.2, the first one is the density of the algebraic members in the subfamily of the tautological family parameterized by $V'$, and the second one states that $f^{-1}(Z) \to Z$ is preserved under the deformation of $f$ along $V'$, both assuming that $f$ is a $G$-equivariant locally Weierstrass fibration twisted by some element $\eta_C \in H^1_G(B, \mathcal{J}^W_{H/B})$. Based on these results, we will prove Proposition 4.2 in 4.4.

4.1 A pure Hodge structure on $H^1(B, j_* \mathcal{H})$

The following result about the degeneration of Leray spectral sequences holds for any projective fibration over a projective manifold whose fibers are irreducible curves.

Lemma 4.3. — Let $p : W \to B$ be a projective fibration over a projective manifold $B$ such that fibers of $p$ are irreducible curves. Then the Leray spectral sequences of $p$ computing $H^2(W, \mathbb{Q})$ and $H^2(W, \mathcal{O}_W)$ degenerate at $E_2$. In particular,

$$
H^1(B, R^1 p_* \mathbb{Q}) \cong \ker \left( H^2(W, \mathbb{Q})/p^* H^2(B, \mathbb{Q}) \to H^2(F, \mathbb{Q}) \cong \mathbb{Q} \right),
$$

where $i : F \hookrightarrow W$ is the inclusion of a fiber of $p$ in $W$ and

$$
H^1(B, R^1 p_* \mathcal{O}_W) \cong H^2(W, \mathcal{O}_W)/p^* H^2(B, \mathcal{O}_B).
$$

For the Leray spectral sequence computing $H^2(W, \mathbb{Q})$, without assuming that $B$ is smooth, we have

$$
E_1^{0,2} = E_2^{0,2} \cong H^0(B, R^2 p_* \mathbb{Q}) \cong \mathbb{Q}.
$$

curves, we have
\[ R \] and \( \Lambda \) the projection to its \((0,1)\)-part. The pullback
\[ \psi^* : H^1(B, \mathcal{O}_B) \to H^1(W, \mathcal{O}_W) \] is injective, because \( B \) and \( W \) are compact Kähler manifolds, the map \( p^* : E_2^{1,0} \to H^1(B, \mathcal{O}_B) \to H^1(W, \mathcal{O}_W) \) through which it factorizes is also injective, especially for \( j = 2 \) or 3. So \( E_2^{1,0} = E_2^{2,0} = H^2(B, \mathcal{O}_B) \) and \( E_1^{1,1} = E_2^{1,1} \Rightarrow H^1(B, R^2p_*Q) \). As fibers of \( p \) are irreducible curves, \( R^2p_*Q \) is a local system of stalk \( Q \). Since \( W \) is projective by assumption, the restriction of a very ample class \( [H] \in H^2(W, Q) \) to \( H^2(F, Q) \) is nonzero. So \( H^2(W, Q) \to H^0(B, R^2p_*Q) \) is nonzero. In particular, \( R^2p_*Q \) has a non-trivial section, so \( R^2p_*Q = Q \). It follows that \( H^2(W, Q) \to H^0(B, R^2p_*Q) \Rightarrow Q \), being nonzero, is surjective. Hence (4.3) holds and (4.1) follows easily.

For the Leray spectral sequence computing \( H^2(X, \mathcal{O}_X) \), by the same argument the pullback \( \psi^* : E_2^{1,0} = H^1(B, \mathcal{O}_B) \to H^1(W, \mathcal{O}_W) \) is injective, so \( E_2^{2,0} = E_2^{1,0} = E_1^{1,1} \). As \( R^2p_*\mathcal{O}_W = 0 \) because fibers of \( p \) are curves, we have \( E_2^{2,0} = E_2^{0,2} = 0 \), which proves the degeneration of the Leray spectral sequence and (4.2). □

Using Lemma 4.3, we can now define a Hodge structure on \( H^1(B, j_*\mathcal{H}) \).

**Lemma 4.4.** — Let \( p : W \to B \) be a minimal Weierstraß fibration over a projective manifold \( B \). Assume that the discriminant locus \( \Lambda \) is a normal crossing divisor. The morphism \( H^1(B, j_*\mathcal{H}) \otimes Q \to H^1(B, \mathcal{L}_{\mathcal{H}/B}) \) induced by \( \phi : j_*\mathcal{H} \to \mathcal{L}_{\mathcal{H}/B} \) introduced in (3.3) is isomorphic to

\[
\ker \left( H^2(W, \mathcal{O}_W)/p^*H^2(B, \mathcal{O}_B) \xrightarrow{\iota} H^2(F, \mathcal{O}_F) \Rightarrow Q \right) \to H^2(W, \mathcal{O}_W)/p^*H^2(B, \mathcal{O}_B)
\] (4.4)

induced by \( Q \leftarrow \mathcal{O}_W \) where \( \iota : F \leftarrow W \) is the inclusion of a fiber of \( p \) in \( W \). In particular, there is a polarized pure \( Z \)-Hodge structure of weight 2 on \( H^1(B, j_*\mathcal{H}) \) satisfying the Hodge symmetry and the morphism \( H^1(B, j_*\mathcal{H}) \to H^1(B, \mathcal{L}_{\mathcal{H}/B}) \) is the projection to the \((0,2)\)-part of the Hodge structure.

Finally, let \( \psi : Z \to B \) be a holomorphic map such that \( Z \) is a projective manifold and \( \psi^{-1}(\Lambda) \) is a normal crossing divisor. Assume that the Weierstraß fibration \( W \times_B Z \to Z \) is minimal, then the pullback

\[
\psi^* : H^1(B, j_*\mathcal{H}) \to H^1(Z, j_*j^*\mathcal{H})
\]
is a morphism of Hodge structures where \( j^* : Z^* := Z \setminus \psi^{-1}(\Lambda) \hookrightarrow Z \) is the open immersion and \( \mathcal{H}' := \psi^{-1}_{|Z^*}\mathcal{H} \).

**Proof.** — By Lemma 3.4, the morphism \( H^1(B, j_*\mathcal{H}) \to H^1(B, \mathcal{L}_{\mathcal{H}/B}) \) is isomorphic to

\[
H^1(B, R^1p_*\mathcal{Z}) \to H^1(B, R^1p_*\mathcal{O}_W)
\]
and by Lemma 4.3, \( H^1(B, R^1p_*\mathcal{H}) \to H^1(B, R^1p_*\mathcal{O}_W) \) is isomorphic to (4.4), which proves the first statement.

Since \( p : W \to B \) is a minimal Weierstraß fibration, \( W \) has at worst rational singularities. So the underlying Hodge structure on \( H^2(W, \mathcal{Z}) \) is pure of weight 2 satisfying the Hodge symmetry and \( H^2(W, \mathcal{Z}) \to H^2(F, \mathcal{O}_F) \) is the projection to its \((0,2)\)-part [2, Lemma 2.1 and Corollary 2.3.(1)]. As the \((0,2)\)-part of \( H^2(F, \mathcal{Q}) \) is trivial, (4.4) is the projection of a Hodge structure to its \((0,2)\)-part. This defines a polarized pure Hodge structure of weight 2 on \( H^1(B, j_*\mathcal{H}) \) satisfying the Hodge symmetry such that \( H^1(B, j_*\mathcal{H}) \to H^1(B, \mathcal{L}_{\mathcal{H}/B}) \) is the projection to its \((0,2)\)-part.

The pullback \( \psi^* \) in the last statement is defined to be the composition

\[
H^1(B, j_*\mathcal{H}) \xrightarrow{\psi^*} H^1(Z, \psi^{-1}(j_*\mathcal{H})) = H^1(Z, j_*j^*\mathcal{H})
\]
From the isomorphism
\[ H^1(B, j_! \mathcal{H}) \otimes \mathbb{Q} = \ker \left( H^2(W, \mathcal{Q})/p' H^2(B, \mathcal{Q}) \to H^2(F, \mathcal{Q}) = \mathbb{Q} \right) \]
and the similar one for \( H^1(Z, j_! \mathcal{H}') \otimes \mathbb{Q} \), it follows immediately that the pullback \( \psi^* : H^1(B, j_! \mathcal{H}) \to H^1(Z, j_! \mathcal{H}') \) is a morphism of Hodge structures.

**Remark 4.5.** — In particular, Lemma 4.4 implies that \( H^1(B, j_! \mathcal{H}) \otimes \mathbb{R} \to H^1(B, \mathcal{L}_{W/B}) \) is surjective. This gives an alternative proof of the equivalent statement that the tautological family of a minimal Weierstraß fibration over a compact Kähler manifold is an algebraic approximation [10, Theorem 3.25 and Remark 3.26].

### 4.2 Density of algebraic elliptic fibrations in the tautological family

In most of the statements we prove in the rest of Section 4 about elliptic fibrations, we will assume Hypotheses 4.1. Some of the hypotheses therein are considered for the following reasons. As we will apply Lemma 4.4 to define a Hodge structure on \( H^1(B, j_! \mathcal{H}) \), we need to assume that \( B \) is a projective manifold and the discriminant locus \( \Delta \) a normal crossing divisor. Since the Hodge structure on \( H^1(B, j_! \mathcal{H}) \) is defined using the minimal Weierstraß fibration associated to \( \mathcal{H} \) and we want this construction to be functorial under reasonable pullback, the corresponding pullback of the minimal Weierstraß fibration associated to \( \mathcal{H} \) need to be minimal. By Lemma 3.5, this can be achieved if we assume that the local monodromies of \( \mathcal{H} \) around \( \Delta \) are unipotent.

Our first application of Lemma 4.4 is the following density result.

**Proposition 4.6.** — Let \( f : X = W^\eta \to B \) be a \( G \)-equivariant locally Weierstraß fibration twisted by \( \eta \in H^1(B, \mathcal{J}_W^{\mathcal{H}/B}) \) satisfying Hypotheses 4.1 for some finite group \( G \). Let \( Z \subset B \) be a \( G \)-stable subvariety such that none of the irreducible components of \( Z \) is contained in \( \Delta \) and let \( \iota : \tilde{Z} \to Z \) be a \( G \)-equivariant log-desingularization of \((Z, Z \cap \Delta)\). Assume that \( \mathcal{Y} := f^{-1}(Z) \to Z \) has a multi-section, then the subfamily parameterized by
\[ V^G_Z := \ker \left( \iota^* : H^1(B, \mathcal{L}_{W/B}) \to H^1(\tilde{Z}, \iota^* \mathcal{L}_{W/B}) \right)^G \]
of the tautological family \( \mathcal{W} : V \to B \times V \to V \) associated to \( f \) is an algebraic approximation of \( X \).

While proving Proposition 4.6, we will also prove the following lemma along the way.

**Lemma 4.7.** — In the setting of Proposition 4.6, there exists a lattice \( \Lambda \subset V^G_Z \) such that \( \mathcal{W}_t \) is isomorphic to \( \mathcal{W}_{t+\Lambda} \) over \( B \) for all \( t \in V^G_Z \) and \( \lambda \in \Lambda \).

**Proof of Proposition 4.6 and Lemma 4.7.** — To prove Proposition 4.6, it suffices by Theorem 2.5 to prove that the subset of \( V^G_Z \) parameterizing elliptic fibrations admitting a multi-section in the tautological family \( \Pi \) is dense in \( V^G_Z \) (for the Euclidean topology). By Lemma 3.3, it suffices to show that \( V^G_Z \) contains a dense subset \( V_{\text{tors}} \) such that \( \eta + \exp(t) \in H^1(B, \mathcal{J}_W^{W/B}) \) is torsion for all \( t \in V_{\text{tors}} \).

We need the following two lemmas. Let \( j_\Delta : \tilde{Z}^* := \tilde{Z} \setminus \iota^{-1}(\Delta) \to \tilde{Z} \) be the open embedding and \( H_\Delta := (\iota^{-1} \mathcal{H})_{|\Delta^*} \). Let
\[ \phi : H^1(B, j_! \mathcal{H}) \otimes \mathbb{Q} \to H^1(B, \mathcal{L}_{W/B}) \]
denote the map induced by \( j : H \to \mathcal{L}_{W/\beta} \) introduced in (3.3) and let

\[ \iota' : H^1(B, j_* H)^G \to H^1(\hat{Z}, j_{\hat{Z}} H_{\hat{Z}}) \]

be the restriction to \( H^1(B, j_* H)^G \) of the pullback \( H^1(B, j_* H) \to H^1(\hat{Z}, j_{\hat{Z}} H_{\hat{Z}}) \).

**Lemma 4.8.** — Let \( K := \ker\left( \iota' : H^1(B, j_* H)^G \to H^1(\hat{Z}, j_{\hat{Z}} H_{\hat{Z}}) \right) \). Then \( \phi(K) \) is dense in \( V^G \).

In order to state the second lemma, recall that since \( X \) is in the Fujiki class \( \mathcal{C} \), \( c(\eta) \) is torsion by Proposition 3.6. So there exist \( m \in \mathbb{Z}_{>0} \) and \( \beta \in H^1(B, \mathcal{L}_{W/\beta}) \) such that \( \exp(\eta) = m\eta \). Since \( G \) is finite, up to replacing \( m \) with a larger multiple of it, we may assume that \( \beta \in H^1(B, \mathcal{L}_{W/\beta})^G \).

**Lemma 4.9.** — Up to replacing \( m \) with a larger multiple of it, there exists \( \alpha \in H^1(B, j_* H)^G \) such that \( \beta - \phi(\alpha) \in V^G \).

Let us admit Lemma 4.8 and 4.9 for the moment and finish the proof. As \( \phi(K) \) is dense in \( V^G \) by Lemma 4.8 and \( \beta - \phi(\alpha) \in V^G \) by Lemma 4.9, the subset \( V' := \beta - \phi(\alpha) - \phi(K) \) is dense in \( V^G \). If \( t \in V' \), then \( \beta - t = \text{Im}(\phi) \), so \( m\eta - \exp(t) = \exp(\beta - t) \in H^1(B, \mathcal{J}_{W/\beta}) \) is torsion. It follows that \( \eta + \exp(t) \in H^1(B, \mathcal{J}_{W/\beta}) \) is torsion whenever \( t \) is in the dense subset \( V_{\text{tors}} := \frac{1}{m} V' \) of \( V^G \), which proves Proposition 4.6. To prove Lemma 4.7, note that Lemma 4.8 implies that \( \phi(K) \) contains a lattice \( \Lambda \) of \( V^G \). As \( K \subset H^1(B, j_* H) \), we have \( \mathcal{W}_t \simeq \mathcal{W}_{t+\lambda} \) over \( B \) for all \( t \in V^G \) and \( \lambda \in \Lambda \).

Now we prove Lemma 4.8 and 4.9. Both proofs use Hodge theory in an essential way.

**Proof of Lemma 4.8 and 4.9.** — Since \( (\hat{Z}, \iota^{-1}(\Delta)) \) is a log-smooth projective pair and since the local monodromies of \( H \) around \( \Delta \) are assumed to be unipotent, by Lemma 3.5 \( \mathcal{L}_{H_{\hat{Z}}/\hat{Z}} = \iota' \mathcal{L}_{W/\beta} \) and the base change \( W_{\hat{Z}} := W \times_B \hat{Z} \to \hat{Z} \) by \( \iota : \hat{Z} \to B \) of the minimal Weierstraß fibration \( p : W \to B \) associated to \( H \) is the minimal Weierstraß fibration associated to \( H_{\hat{Z}} \). Also, since both \( B \) and \( \hat{Z} \) are projective manifolds and \( \Delta \) and \( \iota^{-1}(\Delta) \) are normal crossing divisors, by Lemma 4.4 \( K \) is a pure Hodge structure of weight 2 satisfying the Hodge symmetry and the restriction of \( \phi \) to \( K \) is the projection of \( K \) to its \((0,2)\)-part

\[ \ker\left( H^1(B, \mathcal{L}_{W/\beta})^G \to H^1(\hat{Z}, \mathcal{L}_{H_{\hat{Z}}/\hat{Z}}) \right) = \ker\left( H^1(B, \mathcal{J}_{W/\beta})^G \to H^1(\hat{Z}, \mathcal{J}_{H_{\hat{Z}}/\hat{Z}}) \right) = V^G. \]

Hence \( \phi(K) \) is dense in \( V^G \), which proves Lemma 4.8.

The short exact sequence (3.3) together with its pullback by \( \iota \) induces the following commutative diagram with exact rows.

\[
\begin{array}{cccc}
H^1(B, j_* H) & \xrightarrow{\phi} & H^1(B, \mathcal{L}_{W/\beta}) & \xrightarrow{\exp} & H^1(B, \mathcal{J}_{W/\beta}) \\
\downarrow \iota' & & \downarrow \iota' & & \downarrow \iota' \\
H^1(\hat{Z}, (j_{\hat{Z}})_* H_{\hat{Z}}) & \xrightarrow{\phi_{\hat{Z}}} & H^1(\hat{Z}, \mathcal{L}_{H_{\hat{Z}}/\hat{Z}}) & \xrightarrow{\exp} & H^1(\hat{Z}, \mathcal{J}_{H_{\hat{Z}}/\hat{Z}}) \\
\end{array}
\]

Here in this proof, we use \( \rho^2 \) to denote the pullback \( \iota' : H^1(B, \mathcal{L}_{W/\beta}) \to H^1(\hat{Z}, \mathcal{L}_{H_{\hat{Z}}/\hat{Z}}) \).

Let \( g : Y := X \times_B \hat{Z} \to \hat{Z} \), which is isomorphic to the locally Weierstraß fibration \( W_{\hat{Z}}^{\iota \eta} \to \hat{Z} \) twisted by \( \iota' \eta \in H^1(\hat{Z}, \mathcal{J}_{H_{\hat{Z}}/\hat{Z}}) \). Since \( g \) has a multi-section by assumption and \( Y \) is in the Fujiki class \( \mathcal{C} \), \( \iota' \eta \) is torsion by Lemma 3.7. Therefore up to replacing \( m \) with a larger multiple, \( \exp(\rho^2(\beta)) = 0 \), thus again up to replacing \( m \) with a larger multiple of it, there exists \( a_0 \in H^1(\hat{Z}, (j_{\hat{Z}})_* H_{\hat{Z}})^G \) such that \( \phi_{\hat{Z}}(a_0) = \rho^2(\beta) \).
Let $H := H^1(B, j^* \mathcal{H})$ and $H_2 := H^1(Z, (j_2)_* \mathcal{H}_2)$ for simplicity. As $H_2$ is a polarized Hodge structure, there exists a $\mathbb{Q}$-Hodge structure $H'_Q$ such that $H_2 \otimes \mathbb{Q} = r'H'_Q \oplus H'_Q$. As $\phi_2$ is the projection of the Hodge structure to its $(0, 2)$-part by Lemma 4.4, $\phi_2 : r'H''_Q \oplus H' Q \rightarrow \rho^2(H^{0,2}) \oplus H^{0,2}$ preserves the two factors in the direct sum where $H^{0,2}$ and $H''^{0,2}$ are the $(0, 2)$-parts of $H_2$ and $H'_Q$. So, since $\beta \in H^1(B, \mathcal{Z}_{H/B})^G = (H^{0,2})^G$ where the equality follows from Lemma 4.4, there exists $\alpha \in H^2_G$ such that $\phi_2(\rho(\alpha)) = \rho^2(\beta)$. As $\phi_2(\rho(\alpha)) = \rho^2(\phi(\alpha))$, we have $\rho^2(\beta - \phi(\alpha)) = 0$. Hence $\beta - \phi(\alpha) \in V^G_2$.

**Remark 4.10.** — In Lemma 4.7, the hypotheses that the total space $X$ is in the Fujiki class $\mathcal{C}$ and $Y \rightarrow Z$ has a multi-section are unnecessary.

### 4.3 A stability result

Let $f : W^n \rightarrow B$ be an $\eta$-twisted locally Weierstraß fibration satisfying Hypotheses 4.1 for some $\eta \in H^1(B, \mathcal{F}_H^W)$. Let $Z \subset B$ be a subvariety of $B$. In the last paragraph, we showed that if $f^{-1}(Z \setminus \Delta) \rightarrow Z \setminus \Delta =: Z_0$ has a multi-section, then the subfamily parameterized by $V_{Z_0} = \ker (\rho : H^1(B, \mathcal{L}_{H/B}) \rightarrow H^1(\hat{Z}_0, \mathcal{L}_{\hat{H}/\hat{B}}))$

of the tautological family associated to $f$ is an algebraic approximation of $f$ where $\hat{i}_0 : \hat{Z}_0 \rightarrow Z_0 \subset B$ is a projective log-desingularization of $(Z_0, Z_0 \cap \Delta)$. Our next application of Lemma 4.4 is to show that this subfamily preserves the fibration $f^{-1}(Z) \rightarrow Z$.

**Proposition 4.11.** — Let $\eta \in H^1(B, \mathcal{F}_H^W)$ and $f : W^n \rightarrow B$ be the associated locally Weierstraß fibration. Assume that $f$ satisfies Hypotheses 4.1. Let $Z \subset B$ be a subvariety of $B$ and $Z_0 := \overline{Z \setminus \Delta}$. Let $i_0 : Z_0 \rightarrow Z_0 \subset B$ be a projective log-desingularization of $(Z_0, Z_0 \cap \Delta)$. Then the subfamily of the tautological family associated to $f$ parameterized by $V_{Z_0} := \ker (\rho_0 : H^1(B, \mathcal{L}_{H/B}) \rightarrow H^1(Z_0, \mathcal{L}_{H/B}))$

preserves the fibration $f^{-1}(Z) \rightarrow Z$.

We first prove some auxiliary results before we start the proof of Proposition 4.11. Let $f : W \rightarrow B$ be a Weierstraß fibration over a projective manifold and $\psi : Z \rightarrow B$ a map from a projective manifold $Z$.

**Lemma 4.12.** — Assume that the image of $\psi : Z \rightarrow B$ is contained in the discriminant locus $\Delta$. Let $p : Y := W \times_B Z \rightarrow Z$ denote the base change fibration and $\tau : \hat{Y} \rightarrow Y$ a minimal desingularization of $Y$ such that $\hat{Y}$ is projective. Let $\rho := p \circ \tau$. Then the quotient $H^2(\hat{Y}, \mathcal{Z})/\rho^*H^2(Z, \mathcal{Z})$ is a pure Hodge structure of weight two concentrated in bi-degree $(1, 1)$.

**Proof.** — It suffices to show that $\rho^* : H^0(Z, \Omega^2_Z) \rightarrow H^0(\hat{Y}, \Omega^2_{\hat{Y}})$ is an isomorphism. The morphism $\rho^*$ is injective because $\rho$ is a surjective map between projective manifolds. All we need to prove is that $\dim H^2(\hat{Y}) \geq \dim H^2(\hat{Y})$. Since the zero-section $\Sigma \subset W$ of $f : W \rightarrow B$ is contained in the smooth locus of $f$, its pullback to $p : Y \rightarrow Z$ lies in the smooth part of $Y$, so $\rho$ has a holomorphic section $\sigma : Z \rightarrow \hat{Y}$. Consider the pushforward $\chi : (\rho)_* \Omega^2_{\hat{Y}} \rightarrow \rho_\sigma \Omega^2_Z = \rho_\sigma \Omega^2_Z$ of $\Omega^2_{\hat{Y}} \rightarrow \sigma_* \Omega^2_Z$ by $\rho$. Since $\rho$ is a $\mathbb{P}^1$-bundle over a dense Zariski open of $Z$, $\chi$ is generically an isomorphism. Moreover since $(\rho)_* \Omega^2_Z$ is torsion free, $\chi$ is injective. It follows that $\dim H^2(\hat{Y}) \geq \dim H^2(\hat{Y})$. □
Lemma 4.12 will be used to prove the following result.

**Lemma 4.13.** — In the setting of Lemma 4.12, assume that \( f : W \to B \) is a minimal Weierstrass fibration. If we define

\[
K_0 := \ker \left( \frac{H^2(W, \mathbb{Z})}{f^*H^2(B, \mathbb{Z})} \right),
\]

then \( K_0 \) is a pure sub-Hodge structure of \( \frac{H^2(W, \mathbb{Z})}{f^*H^2(B, \mathbb{Z})} \) of weight 2 whose \((0,2)\)-part coincides with that of \( \frac{H^2(W, \mathbb{Z})}{f^*H^2(B, \mathbb{Z})} \), which is \( H^2(W, \mathcal{O}_W) / f^*H^2(B, \mathcal{O}_B) \).

**Proof.** — Since \( p \) is assumed to be minimal, \( W \) has at worst rational singularities. It follows from [2, Lemma 2.1] that the underlying Hodge structure of \( \frac{H^2(W, \mathbb{Z})}{f^*H^2(B, \mathbb{Z})} \) is pure of weight 2, and so is \( K_0 \). Let \( H := H^2(W, \mathbb{C}) / f^*H^2(B, \mathbb{C}) \) and \( H' := H^2(Y, \mathbb{C}) / p^*H^2(Z, \mathbb{C}) \) for simplicity. The map \( \psi^p \) is a morphism of mixed Hodge structures and it suffices to show that \( \psi^*(\tau F^1H) = \psi^*(H) \) to prove the assertion about the \((0,2)\)-part of \( K_0 \).

To this end, let \( \alpha \in H \). Let \( \tilde{\psi} : H \to \text{Gr}^0_{W}H' \) be the composition of \( \psi^p \) with the projection \( H' \to \text{Gr}^0_{W}H' \). By Lemma 4.12, \( \text{Gr}^0_{W}H' \subset H^2(Y, \mathbb{C}) / p^*H^2(Z, \mathbb{C}) \) is a pure Hodge structure of weight two concentrated in bi-degree \((1,1)\), so \( \tilde{\psi}(H) = F^1\text{Gr}^0_{W}H' \cap \tilde{\psi}(H) = \tilde{\psi}(F^1H) \). Thus there exist \( \beta \in F^1H \) and \( \gamma \in W_{-1}H' \) such that \( \psi^*(\alpha) = \psi^*(\beta) + \gamma \). Since \( \text{Im}\psi^p \cap W_{-1}H' = \psi^p(W_{-1}H) = 0 \) as \( H \) is pure, we have \( \gamma = \psi^*(\alpha - \beta) = 0 \). Hence \( \psi^*(\alpha) = \psi^*(\beta) \in \psi^*(F^1H) \).

**Proof of Proposition 4.11.** — By Lemma 3.2, it suffices to prove that the restriction to \( V_{\tau_0} \) of the pullback \( \tau^* : H^1(B, \mathcal{L}_{H/B}) \to H^1(Z, \tau^*\mathcal{L}_{H/B}) \) by the inclusion \( \tau : Z \hookrightarrow B \) is zero.

For simplicity, we may assume that \( Z \) contains \( \Delta \). Let \( \tau_\Delta : \tilde{\Delta} \to \Delta \) be a desingularization of \( \Delta \) and let

\[
\tau := (\tau_0 \cup \tau_\Delta) : \tilde{Z} := Z_0 \cup \tilde{\Delta} \to Z.
\]

We denote by \( p : W \to B \) the minimal Weierstrass fibration associated to \( H = (R^1f_*\mathcal{L})_{|B/\Lambda} \) and let

\[
q : Y := W \times_B Z \to Z,
\]

\[
\tilde{q}_0 : \tilde{Y}_0 := W \times_B Z_0 \to Z_0,
\]

\[
\tilde{q}_\Delta : \tilde{D} := W \times_B \tilde{\Delta} \to \tilde{\Delta},
\]

\[
\tilde{q} : \tilde{Y} := W \times_B \tilde{Z} \to \tilde{Z}
\]

be various base changes of \( p : W \to B \). Let \( \tau := \tau \circ \tilde{q} \) and

\[
K := \ker \left( \tau^* : H^2(W, \mathbb{C}) / p^*H^2(B, \mathbb{C}) \to H^2(Y, \mathbb{C}) / \tilde{q}^*H^2(Z, \mathbb{C}) \right).
\]

We have a commutative diagram

\[
\begin{array}{cccccc}
K & \xrightarrow{\pi} & H^2(W, \mathbb{C}) & \xrightarrow{\alpha} & H^2(Y, \mathbb{C}) & \xrightarrow{\beta} & H^2(Y, \mathbb{C}) \\
& & \downarrow \psi & & \downarrow \gamma & & \downarrow \gamma \\
V_{\tau_0} & \xrightarrow{\tau} & H^1(B, \mathcal{L}_{H/B}) & \xrightarrow{\tau^*} & H^1(Z, \tau^*\mathcal{L}_{H/B}) & \xrightarrow{\tau^*} & H^1(\tilde{Z}, \tau^*\mathcal{L}_{H/B})
\end{array}
\]

where the arrows are defined as follows. The horizontal arrows (except the leftmost ones) are induced by pullbacks under various morphisms. The map \( H^2(Y, \mathbb{C}) / \tilde{q}^*H^2(Z, \mathbb{C}) \to H^1(\tilde{Z}, \tau^*\mathcal{L}_{H/B}) \) is defined to be the
composition
\[ H^2(Y, C)/q^*H^2(Z, C) \twoheadrightarrow H^2(Y, \mathcal{O}_Y)/q^*H^2(Z, \mathcal{O}_Z) \cong H^1(Z, \mathcal{O}_Z) \cong H^1(Z, \mathcal{O}_W) \cong H^1(Z, \tau^*\mathcal{L}_{H/B}), \]
where the first and the third isomorphisms result from Lemma 4.3 and Lemma 3.4 respectively, and the second from the base change theorem since \( Y \to Z \) is the base change of the flat fibration \( W \to B \). The other vertical arrows are defined similarly except for \( \pi \). As
\[ H^1(Z, \tau^*\mathcal{L}_{H/B}) = H^1(\tilde{\lambda}, \tau_0^*\mathcal{L}_{H/B}) \oplus H^1(\tilde{Z}_0, \tau_0^*\mathcal{L}_{H/B}) \]
where \( \tilde{\lambda} := \iota \circ \tau_\iota \) and the composition
\[ K \xrightarrow{q} H^2(W, C) \xrightarrow{p^*H^2(B, C)} H^1(B, \mathcal{L}_{H/B}) \xrightarrow{\gamma} H^1(Z, \tau^*\mathcal{L}_{H/B}) \xrightarrow{pr_2} H^1(\tilde{Z}_0, \tau_0^*\mathcal{L}_{H/B}) \]
is zero, the image of the composition \( K \to H^2(W, C)/p^*H^2(B, C) \to H^1(B, \mathcal{L}_{H/B}) \) is contained in \( V_{Z_0} \), which defines \( \pi : K \to V_{Z_0} \) in (4.5).

**Lemma 4.14.** — The map \( \pi : K \to V_{Z_0} \) is the projection of the (pure) Hodge structure \( K \) of weight 2 onto its (0,2)-part. In particular, \( \pi \) is surjective.

**Proof.** — First of all
\[ \frac{H^2(Y, C)}{q^*H^2(Z, C)} = \frac{H^2(\tilde{\lambda}, C)}{q_0^*H^2(\tilde{\lambda}, C)} \oplus \frac{H^2(\tilde{Z}_0, C)}{q_0^*H^2(\tilde{Z}_0, C)}, \]
so
\[ K = \ker \left( K_0 \to H^2(W, C)/p^*H^2(B, C) \xrightarrow{\gamma} H^2(\tilde{Z}_0, C)/q_0^*H^2(\tilde{Z}_0, C) \right) \]
where
\[ K_0 := \ker \left( \frac{H^2(W, C)}{p^*H^2(B, C)} \to H^2(\tilde{\lambda}, C)/q_0^*H^2(\tilde{\lambda}, C) \right). \]
By Lemma 4.13, since \( \Delta \subset \Delta \), \( K_0 \) is a pure Hodge structure of weight 2 whose (0,2)-part is isomorphic to that of \( H^2(W, C)/p^*H^2(B, C) \), which is \( H^2(W, \mathcal{O}_W)/p^*H^2(B, \mathcal{O}_B) \), and is further isomorphic to \( H^2(B, \mathcal{L}_{H/B}) \) by Lemma 4.4. By Lemma 3.5, \( q_0^* : \tilde{Z}_0 \to Z_0 \) is the minimal Weierstrass fibration associated to \( H' := \frac{1}{\tau_0^1 \tau_0 \tau_1} \mathcal{H} \) and \( \mathcal{L}_{H/Z_0} = \tau_0^* \mathcal{L}_{H/B} \). So again by Lemma 4.4, \( H^2(\tilde{Z}_0, C)/q_0^*H^2(\tilde{Z}_0, C) \) is a pure Hodge structure of weight 2 whose (0,2)-part is isomorphic to \( H^1(Z_0, \mathcal{L}_{H/Z_0}) = H^1(\tilde{Z}_0, \tau_0^*\mathcal{L}_{H/B}) \). Therefore \( V_{Z_0} = \ker \left( \frac{\tau_0^0 : H^1(B, \mathcal{L}_{H/B}) \to H^1(\tilde{Z}_0, \tau_0^*\mathcal{L}_{H/B}) \right) \) is the (0,2)-part of the pure Hodge structure \( K \) of weight 2. \( \square \)

**Lemma 4.15.** — In (4.5), the restriction of \( \gamma \) to \( \text{Im}(\beta) \) is injective.

**Proof.** — First of all let \( E_{\gamma,Y}^\bullet \) and \( E_{\gamma,\tilde{Y}}^\bullet \) be the Leray spectral sequences of \( q : Y \to Z \) and \( \tilde{q} : \tilde{Y} \to Z \) computing \( H^2(Y, Q) \) and \( H^2(\tilde{Y}, Q) \) respectively. By Lemma 4.3, we have \( E_{\gamma,Y}^{0,2} = H^0(Z, R^2q_*Q) = Q \) and \( E_{\gamma,\tilde{Y}}^{0,2} = H^0(\tilde{Z}, R^2\tilde{q}_*Q) = Q \).

Let \( \xi \in H^2(Y, Q) \) such that \( \tau_Y^*\xi \in \tilde{q}^*H^2(\tilde{Z}, Q) \) where \( \tau_Y^* : H^2(Y, Q) \to H^2(\tilde{Y}, Q) \) is the pullback by the projection \( \tau_Y : \tilde{Y} = Y \times _Z \tilde{Z} \to Y \). Since \( \tau_Y^*\xi \in \tilde{q}^*H^2(\tilde{Z}, Q) \), the projection of \( \tau_Y^*\xi \) in \( E_{\gamma,\tilde{Y}}^{0,2} \) is zero. As the pullback
\[ \tau^* : Q = H^0(Z, R^2q_*Q) \to H^0(\tilde{Z}, R^2\tilde{q}_*Q) = Q \]
is an isomorphism, it follows that the projection of \( \xi \) in \( E_{\gamma,Y}^{0,2} \) is zero. So \( \xi \in L^1H^2(Y, Q) \).
It remains to show that the projection of $\xi$ in $E^{1,1}_{\infty,Y}$ is zero, so that $\xi \in E^{2,0}_{\infty,Y} = q^*H^2(Z, Q)$. Note that since $\tau^*_Y \xi \in \hat{\partial}^*H^2(\bar{Z}, Q)$, the image of $\tau^*_Y \xi$ in $E^{1,1}_{\infty,Y}$ is zero. Thus it suffices to show that the pullback $E^{1,1}_{\infty,Y} \to E^{1,1}_{\infty,Y}$ is injective. Since $E^{1,1}_{\infty,Y} \to E^{1,1}_{\infty,Y}$ is contained in $E^{1,1}_{\infty,Y}$, it suffices to show that $H^1(Z, R^1\theta, Q)$ is injective. By the base change theorem [14, VII.2.6], $R^1\theta, Q \cong \tau^{-1}R^1\theta, Q$. So by the projection formula [14, VII.2.4] $H^1(Z, R^1\theta, Q)$ is in fact the $E^{2,0}$-term of the Leray spectral sequence of $\tau$ computing $H^1(Z, R^1\theta, Q)$. Hence $H^1(Z, R^1\theta, Q) \to H^1(\bar{Z}, R^1\bar{\theta}, Q)$ is injective.

Given a morphism $\phi : M_1 \to M_2$ of mixed Hodge structures, let $\tilde{\phi} : Gr^W_1M_1 \to Gr^W_1M_2$ denote the induced morphism on the 0-th graded pieces. Recall that by the strictness of the weight filtration, if $\phi : H \to M$ is a morphism of mixed Hodge structures such that $H$ is proper (in the sense that $W_0 = H$ and $W_{-1}H = 0$), then $\phi(H) \cong \tilde{\phi}(H)$. This is a property that will be repeatedly used in the next paragraph concluding the proof of Proposition 4.11.

Assume that $a(K) \neq 0$, then $a(K) \neq 0$ because $K$ is a pure Hodge structure. As $\bar{Y} \to Y$ is proper and surjective, $\tilde{\beta}$ is injective, so $\tilde{\beta}(a(K)) \neq 0$. Accordingly $\beta(a(K)) \neq 0$, so $\gamma(\beta(a(K))) \neq 0$ by Lemma 4.15, which contradicts the definition of $K$. Hence $a(K) = 0$. As $K \to V_{Z_0}$ is surjective, the map $V_{Z_0} \to H^1(Z, \tau^*\mathcal{L}_{H/B})$ is zero, which proves Proposition 4.11. □

Before we turn to the proof of Proposition 4.2, let us prove the following corollary of Proposition 4.6 and Proposition 4.11 as an aside.

**Corollary 4.16.** — Let $f : X \to B$ be a locally Weierstraß fibration twisted by $\eta \in H^1(B, \mathcal{F}_{H/B})$ satisfying Hypotheses 4.1 and let $Z \subset B$ be a subvariety. Assume that $Y := f^{-1}(Z)$ has a multi-section and $Y \to Z$ is not preserved along any direction in the tautological family associated to $\eta$. Then $X$ is already Moishezon.

**Proof.** — Since $Y \to Z$ is not preserved along any direction in the tautological family, the subspace $V_{Z_0}$ defined in Proposition 4.11 is zero. Now by Proposition 4.6, the subfamily of the tautological family associated to $f$ parameterized by $V_{Z_0}$ is an algebraic approximation of $f$. So $X$ is already Moishezon. □

### 4.4 Proof of Proposition 4.2

The following proposition generalizes Proposition 4.11, in which locally Weierstraß fibration is replaced by $G$-equivariant tautological model (see 3.4), and the fibration $f^{-1}(Z) \to Z$ is replaced by the formal completion $\hat{f}^{-1}(Z) \to \hat{Z}$.

**Proposition 4.17.** — Let $G$ be a finite group and $f : X \to B$ a $G$-equivariant tautological model associated to an element $\eta_G \in H^1_0(B, \mathcal{F}_{H/B})$ satisfying Hypotheses 4.1. Let $Z_0 \subset B$ be a $G$-stable subvariety of $B$ and $Z_0 := Z_{0, \Delta}$. Let $\tilde{t}_0 : Z_0 \to Z_0$ be a $G$-equivariant log-dsingularization of $(Z_0, Z_0 \cap \Delta)$ such that $Z_0$ is projective. Then the subfamily parameterized by $V_{Z_0}^G := \ker \left( \tilde{t}_0^* : H^1(B, \mathcal{L}_{H/B}) \to H^1(\hat{Z}_0, \tilde{t}_0^*\mathcal{L}_{H/B}) \right)^G$ of the tautological family $\Pi : \mathcal{Y} \to B \times V \to V := H^1(B, \mathcal{L}_{H/B})$ associated to $f$ preserves $G$-equivariantly the completion $\hat{f}^{-1}(Z) \to Z$ of $f$ along $f^{-1}(Z) \to Z$. 
Let us first prove some general results before we prove Proposition 4.17.

**Lemma 4.18.** — Let $\Pi : \mathcal{X} \rightarrow B \times V \rightarrow V$ be a deformation of a fibration $f : X \rightarrow B$ over a complex vector space $V$ and assume that $q$ is flat. Let $Z$ be a subvariety of $B$ such that $Y := f^{-1}(Z) \rightarrow Z$ is preserved by $\Pi$. Assume that there exists a lattice $\Lambda \subset V$ such that $\mathcal{X}$ is isomorphic to $\mathcal{X}_{t+\lambda}$ over $B$ for all $t \in V$ and $\lambda \in \Lambda$. Then the completion $\hat{Y} \rightarrow Z$ of $X \rightarrow B$ along $Y \rightarrow Z$ is also preserved by $\Pi$.

If moreover $f$ is $G$-equivariant, $Z$ is $G$-stable, and $\Pi$ preserves $G$-equivariantly $Y \rightarrow Z$ for some finite group $G$, then $\Pi$ preserves $G$-equivariantly $\hat{Y} \rightarrow \hat{Z}$ as well.

*Proof.* — For the first statement, it suffices to prove by induction on $n \in \mathbb{Z}_{\geq 0}$ that the $n$-th order infinitesimal neighborhood $g_n : Y_n \rightarrow Z_n$ of $Y \rightarrow Z$ is preserved by $\Pi$. The case where $n = 0$ is covered by the assumption. Assume that $g_{n-1} : Y_{n-1} \rightarrow Z_{n-1}$ is preserved by $\Pi$. Let $\mathcal{I}_Z \subset \mathcal{O}_B$, $\mathcal{I}_Y \subset \mathcal{O}_X$, $\mathcal{I}_\mathcal{X} \subset \mathcal{O}_\mathcal{X}$, and $\mathcal{I}_\mathcal{X} \subset \mathcal{O}_\mathcal{X}$ denote the ideal sheaves of $Z \subset B$, $Y \subset X$, $\mathcal{X} := q^{-1}(Z \times V) \subset \mathcal{X}$, and $\mathcal{X} \subset \mathcal{X}$ respectively for every $t \in V$. Since $q$ is flat, we have

$$q^* \pi_1^* \mathcal{I}_Z = \mathcal{I}$$

where $\pi_1 : B \times V \rightarrow B$ is the first projection. Similarly,

$$g_{n-1}^*(\mathcal{I}_Z|_{Z_{n-1}}) = \mathcal{I}|_{\mathcal{Y}|_{\mathcal{Y}_{n-1}}}.$$  

Let $\mathcal{Y}_i$ denote the $i$-th infinitesimal neighborhood of $\mathcal{X}_i$ in $\mathcal{X}$ for each $i \in \mathbb{Z}_{\geq 0}$, then we have $q^{-1}(Z_t \times V) = \mathcal{Y}_t$. Therefore as $Y_{n-1} \rightarrow Z_{n-1} \times V \rightarrow V$ is a family isomorphic to the constant family $Y_{n-1} \times V \rightarrow Z_{n-1} \times V \rightarrow V$ by the induction hypothesis, we have

$$\mathcal{I}|_{\mathcal{Y}_{n-1}} = (q^* \pi_1^* \mathcal{I}_Z)|_{\mathcal{Y}_{n-1}} \cong \pi^* g_{n-1}^*(\mathcal{I}_Z|_{Z_{n-1}}) = \pi^*(\mathcal{I}_Y|_{\mathcal{Y}_{n-1}}) \quad (4.6)$$

where $\pi : Y_{n-1} \times V \rightarrow Y_{n-1}$ is the first projection. Recall that for every morphism of complex spaces $T \rightarrow S$ and every sheaf $\mathcal{F}$ over $T$, there is a natural (e.g. functorial under pullback) one-to-one correspondence between the set of square-zero extensions of $T$ by $\mathcal{F}$ over $S$ with $\text{Ext}^1_{\mathcal{O}_T}(L^*_T/S, \mathcal{F})$ [11, Satz 3.16] where $L^*_T/S$ is the cotangent complex of $T$ over $S$. Let

$$\mathcal{F} := \left(\mathcal{F}_T/\mathcal{I}^{n+1}_T\right)|_{Y_{n-1}}.$$  

According to (4.6), $\mathcal{Y}_n$ is a square-zero extension of $\mathcal{Y}_{n-1} \cong Y_{n-1} \times V$ by $\pi^* \mathcal{F}$ over $Z_n \times V$; let

$$\xi \in \text{Ext}^1_{\mathcal{O}_{\mathcal{Y}_{n-1}}}(L^*_n|_{\mathcal{Y}_{n-1} \times V/Z_n \times V}, \pi^* \mathcal{F}) \cong \text{Ext}^1_{\mathcal{O}_{\mathcal{Y}_{n-1}}}(\pi^* L^*_n|_{\mathcal{Y}_{n-1}/Z_n}, \pi^* \mathcal{F}) \cong \text{Ext}^1_{\mathcal{O}_{\mathcal{Y}_{n-1}}}(L^*_n|_{\mathcal{Y}_{n-1}/Z_n}, \mathcal{F}) \otimes H^0(V, \mathcal{O}_V)$$

be the corresponding element, which we regard as a holomorphic map

$$\xi : V \rightarrow \text{Ext}^1_{\mathcal{O}_{\mathcal{Y}_{n-1}}}(L^*_n|_{\mathcal{Y}_{n-1}/Z_n}, \mathcal{F}).$$  

Again by (4.6), for every $t \in V$, the fiber $\mathcal{Y}_t$ of $\mathcal{Y}_n \rightarrow V$ over $t$ is a square-zero extension of $\mathcal{Y}_{n-1} \cong Y_{n-1}$ by $\left(\mathcal{Y}_n/\mathcal{I}^{n+1}_n\right)|_{\mathcal{Y}_{n-1}} \cong \mathcal{F}$ over $Z_n$ and by functoriality, this extension corresponds to the element $\xi(t) \in \text{Ext}^1_{\mathcal{O}_{\mathcal{Y}_{n-1}}}(L^*_n|_{\mathcal{Y}_{n-1}/Z_n}, \mathcal{F})$.

Since an isomorphism $\mathcal{X}_i \cong \mathcal{X}_{i+\lambda}$ over $B$ restricts to an isomorphism $\mathcal{Y}_i \cong \mathcal{Y}_{i+\lambda}$ over $Z$ for each $i \in \mathbb{Z}_{\geq 0}$, the fibers $\mathcal{Y}_{n-1}$ and $\mathcal{Y}_{n+\lambda}$ are isomorphic as square-zero extensions of $Y_{n-1}$ by $\mathcal{F}$ over $Z_n$. It follows that $\xi(t + \lambda) = \xi(t)$ for all $t \in V$ and $\lambda \in \Lambda$, so $\xi$ descends to a holomorphic map $V/\Lambda \rightarrow \text{Ext}^1_{\mathcal{O}_{\mathcal{Y}_{n-1}}}(L^*_n|_{\mathcal{Y}_{n-1}/Z_n}, \mathcal{F})$.  


As $V/\Lambda$ is a complex torus and $\text{Ext}^1_{\mathcal{O}_{\mathcal{X}_{\pi_{-1}}}}(L_{Y_{n-1}(Z_n)}^* B)$ a complex vector space, $\xi$ is constant whose image represents the square-zero extension $Y_n$ of $Y_{n-1}$ by $B$ over $Z_n$. It follows that as square-zero extensions, $Y_n = q^{-1}(Z_n \times V)$ is isomorphic to $Y_n \times V$ over $Z_n \times V$. So $g_n : Y_n \to Z_n$ is preserved by $\Pi$.

For the last statement, we prove again by induction on $n$ that the isomorphism $\mathcal{Y}_n \simeq Y_n \times V$ is in fact $G$-equivariant. The case $n = 0$ is covered by the assumption. Suppose that the statement is proven for $n - 1$. Since $(\mathcal{Y}_{n-1} \subset \mathcal{Y}_n)$ is isomorphic to $(Y_{n-1} \times V \subset Y_n \times V)$ the short exact sequences

$$0 \xrightarrow{} (\mathcal{Y}^n / \mathcal{Y}^{n+1})|_{\mathcal{Y}_n} \xrightarrow{} \mathcal{O}_{\mathcal{Y}_n} \xrightarrow{} \mathcal{O}_{\mathcal{Y}_n-1} \xrightarrow{} 0$$

(4.7)

$$0 \xrightarrow{} \pi^* (\mathcal{Y}^n / \mathcal{Y}^{n+1})|_{Y_n \times V} \xrightarrow{} \mathcal{O}_{Y_n \times V} \xrightarrow{} \mathcal{O}_{Y_{n-1} \times V} \xrightarrow{} 0.$$  \hspace{1cm} (4.8)

are isomorphic. As $\mathcal{Y} \subset Y$ and $\mathcal{Y} \subset \mathcal{X}$ are $G$-stable, their corresponding ideal sheaves $\mathcal{I}_Y \subset \mathcal{O}_X$ and $\mathcal{I} \subset \mathcal{O}_\mathcal{X}$ are also $G$-stable. So the extensions (4.7) and (4.8) are $G$-equivariant for the $G$-actions induced by the $G$-actions on $\mathcal{X}$ and $X \times V$. On the one hand, since the isomorphism $\mathcal{Y}_n \simeq Y_n \times V$ is $G$-equivariant by the induction hypothesis, the isomorphisms $\mathcal{O}_{\mathcal{Y}_n-1} \simeq \mathcal{O}_{Y_{n-1} \times V}$ and (4.6) are $G$-equivariant. So we can pullback the $G$-action on (4.8) to a $G$-action on (4.7), which might induce a different $G$-structure on $\mathcal{O}_{\mathcal{Y}_n}$. But on the other hand, since $G$ is a finite group, we have

$$\text{Ext}^1(\mathcal{O}_{\mathcal{Y}_n-1}, (\mathcal{Y}^n / \mathcal{Y}^{n+1})|_{\mathcal{Y}_n})^G = \text{Ext}^1(\mathcal{O}_{\mathcal{Y}_n-1}, (\mathcal{Y}^n / \mathcal{Y}^{n+1})|_{\mathcal{Y}_n}),$$

so the $G$-actions on $\mathcal{O}_{\mathcal{Y}_n-1}$ and on $(\mathcal{Y}^n / \mathcal{Y}^{n+1})|_{\mathcal{Y}_n}$ determine uniquely the $G$-action on $\mathcal{O}_{\mathcal{Y}_n}$. Therefore the two $G$-actions on (4.7) are in fact the same. In other words, $\mathcal{Y}_n \simeq Y_n \times V$ identifies the $G$-action on $\mathcal{Y}_n$ and the $G$-action on $Y_n \times V$.

We can apply Lemma 4.18 to the tautological families of twisted locally Weierstraß fibrations.

**Lemma 4.19.** — In the setting of Proposition 4.11, let $G$ be a finite group acting on $B$ and on $H$ in a compatible way. Assume that $f$ and $i_0 : Z_0 \to B$ are $G$-equivariant. Then the subfamily

$$\Pi^G_{Z_0} : \mathcal{Y}^G_{Z_0} \xrightarrow{q} B \times V^G_{Z_0} \to V^G_{Z_0}$$

of the tautological family associated to $f$ parameterized by $V^G_{Z_0}$ preserves $G$-equivariantly the completion $\hat{Y} \to Z$ of $f$ along $Y := f^{-1}(Z) \to Z$.

**Proof.** — By Lemma 4.7, there exists a lattice $\Lambda \subset V_{Z_0}$ such that $t$ and $t + \lambda$ parameterize isomorphic elliptic fibrations in $\Pi^G_{Z_0}$ for every $t \in V_{Z_0}$ and $\lambda \in \Lambda$. By Proposition 4.11, the family $\Pi^G_{Z_0}$ preserves $G$-equivariantly $Y \to Z$. As $q'$ is a locally Weierstrass fibration, $q'$ is flat. Applying Lemma 4.18 to the family $\Pi^G_{Z_0}$ and the fibration $Y \to Z$ contained in $f$ yields Lemma 4.19.

The next general result that we prove is the following.

**Lemma 4.20.** — Let $f : X \to B$ and $g : Y \to B$ be two fibrations where $X$, $Y$, and $B$ are completions of reduced complex spaces and assume that $Y$ has only finitely many irreducible components. Let $m : X \to Y$ be a finite surjective map over $B$. Let $\Pi : \mathcal{X} \xrightarrow{\partial} B \times \Delta \to \Delta$ and $\Pi^G : \mathcal{Y} \xrightarrow{\partial} B \times \Delta \to \Delta$ be deformations of $f$ and $g$ over a connected base $\Delta$. 
Suppose that \( m \) can be extended to a finite surjective map \( \mu : \mathcal{X} \to \mathcal{Y} \) over \( B \times \Delta \) and that for some open cover \( \{U_i\} \) of \( B \), the restriction \( \mu_i = \mu_{|\mathcal{X}_i} : \mathcal{X}_i \to \mathcal{Y}_i \) is isomorphic to \( m_i \times \text{Id} : X_i \times \Delta \to Y_i \times \Delta \) over \( B \times \Delta \) where \( \mathcal{X}_i := q^{-1}(U_i \times \Delta) \), \( \mathcal{Y}_i := q^{-1}(U_i \times \Delta) \), \( X_i := \mathcal{X}_i \cap X \), and \( Y_i := \mathcal{Y}_i \cap Y \). Furthermore, assume that the restriction of the isomorphism \( \mu_i \cong m_i \times \text{Id} \) to the central fiber \( X_i \to Y_i \) is the identity.

If \( \Pi' \) is a trivial deformation of \( g \), then \( \mu \) is isomorphic to \( m \times \text{Id} : X \times \Delta \to Y \times \Delta \) over \( B \times \Delta \). In particular, \( \mathcal{X} \to B \times \Delta \to \Delta \) is a trivial deformation. Moreover, let \( G \) be a group acting on \( X, Y, \) and \( B \) such that \( f, g, \) and \( m \) are \( G \)-equivariant and assume that \( \Pi, \Pi', \) and \( \mu \) preserve the \( G \)-action. Assume that \( \{U_i\} \) is \( G \)-invariant and the isomorphisms \( \mu_i \cong m_i \times \text{Id} \) are \( G \)-equivariant as well. Then \( \mu \) is \( G \)-equivariantly isomorphic to \( m \times \text{Id} \).

**Proof.** — It suffices to show that for every \( i \) and \( j \), the isomorphisms \( \phi_i : X_i \times \Delta \to \mathcal{X}_i \) and \( \phi_j : X_j \times \Delta \to \mathcal{X}_j \) induced by \( \mu_i \cong m_i \times \text{Id} \) agree on the intersection \( X_{ij} \times \Delta = (X_i \cap X_j) \times \Delta \). Let \( Y_{ij} = Y_i \cap Y_j \). Consider the map \( \Delta \to \text{Aut}(X_{ij}/Y_{ij}) \) which associates \( t \in \Delta \) to the unique automorphism \( \psi_t : X_{ij} \to X_{ij} \) satisfying \( \phi_i(x, t) = \phi_j(\psi_t(x, t)) \). As \( m : X \to Y \) is finite and \( X_i \) is reduced, \( \text{Aut}(X_{ij}/Y_{ij}) \) is finite by the following lemma.

**Lemma 4.21.** — Let \( f : S \to T \) be a finite morphism of formal complex spaces. Assume that \( S \) is a completion of a reduced complex space and \( T \) has only finitely many irreducible components, then \( \text{Aut}(S/T) \) is finite.

Accordingly, \( t \mapsto \psi_t \) is constant. Since the restriction of the isomorphism \( \mu_i \cong m_i \times \text{Id} \) to the central fiber \( X_i \to Y_i \) is the identity, we have \( \psi_{t_i} = \psi_{t_j} = \text{Id} \). Hence \( \phi_i \) and \( \phi_j \) agree on \( X_{ij} \times \Delta \). \( \square \)

**Proof of Lemma 4.21.** — Let \( S = \bigcup_{i,j} S_{ij} \) be the decomposition of \( S \) into its irreducible components such that \( f(S_{ij}) = f(S_{i,j'}) \) if and only if \( i = i' \). Since

\[
\text{Aut}(S/T) \subset \prod_i \text{Aut}(S_{ij}/f(S_{ij})),
\]

we can assume that \( T \) is irreducible and the image of each irreducible component of \( S \) is \( T \). By [3, (3.5)], the map \( (f : S \to T) \mapsto (f, \mathcal{O}_S) \) defines an equivalence of categories between the category of finite formal complex spaces over \( T \) and the category of coherent \( \mathcal{O}_T \)-algebras. As \( S \) is a completion of a reduced complex space, \( S \) is also reduced due to the existence of resolution of singularities. So for every \( t \in T \), \((f, \mathcal{O}_S)_t \) is a reduced \( \mathcal{O}_{T_t} \)-algebra of finite type. It follows that the automorphism group \( \text{Aut}_{\mathcal{O}_T}((f, \mathcal{O}_S)_t) \) of the \( \mathcal{O}_{T_t} \)-algebra \( (f, \mathcal{O}_S)_t \) is finite. Therefore, it suffices to show that the group homomorphism

\[
\Psi_t : \text{Aut}_{\mathcal{O}_T}(f, \mathcal{O}_S) \to \text{Aut}_{\mathcal{O}_{T_t}}((f, \mathcal{O}_S)_t)
\]

defined by the restriction is injective.

To this end, let \( g : S \to S \) be an automorphism over \( T \) such that \( \Psi_t(g) = \text{Id} \). The maps \( g \) and \( \text{Id}_S : S \to S \) give rise to two elements

\[
g, \text{Id}_S \in \text{Hom}_{\mathcal{O}_T}(f, \mathcal{O}_S, f, \mathcal{O}_S) = \text{Hom}_{\mathcal{O}_S}(f^* f, \mathcal{O}_S, \mathcal{O}_S)
\]

where \( f, \mathcal{O}_S \) is considered as a coherent sheaf over \( \mathcal{O}_T \) (instead of a coherent \( \mathcal{O}_T \)-algebra). Let

\[
\mathcal{I} := (g - \text{Id}_S)(f^* f, \mathcal{O}_S) \subset \mathcal{O}_S.
\]
As $\Psi_t(g) = \text{Id}$, there exists a neighborhood $U \subset T$ of $t$ such that $g_{f^{-1}(U)}$ is the identity. So $\mathcal{F}_{f^{-1}(U)} = 0$. For every irreducible component $S_i$ of $S$, since $f(S_i) = T$ by assumption, the intersection $S_i \cap f^{-1}(U)$ is a nonempty open subset of $S_i$. Because $\mathcal{F}$ is an ideal sheaf and $S_i$ is reduced, it follows from $\mathcal{F}|_{S_i \cap f^{-1}(U)} = 0$ that $\mathcal{F}|_{S_i} = 0$. So $\mathcal{F} = 0$, hence $g = \text{Id}_S$.

Proof of Proposition 4.17. — Let $Y := f^{-1}(Z)$. It suffices to prove that

$$\mathcal{D} := q^{-1}(\hat{Z} \times V^G_{Z_0}) \rightarrow \hat{Z} \times V^G_{Z_0}$$

is $G$-equivariantly isomorphic to the product $\hat{Y} \times V^G_{Z_0}$ of $\hat{Y} \rightarrow \hat{Z}$ with the identity $\text{Id} : V^G_{Z_0} \rightarrow V^G_{Z_0}$.

Assume that $\mathcal{X} = \mathcal{X}(m; (\eta_i), (\eta_i^G))$ where $\{\eta_i, (\eta_i^G)\}$ is a 1-cocycle representing $\eta_G$ and $m \in \mathbb{Z}\backslash\{0\}$ and $\eta_i^G \in H^1(B, \mathcal{F}_W)$ are such that $m\eta_i^G$ can be lifted to $\eta_i^G$. We will use the construction of the tautological family described in the proof of Proposition-Definition 3.16 and the notations therein.

Let $\Psi : \hat{Z} \times V^G_{Z_0} \hookrightarrow B \times V$ be the product of $\hat{Z} \hookrightarrow B$ with $V^G_{Z_0} \hookrightarrow V$ and let

$$\hat{\mathcal{X}}_{Z,i} \times V^G_{Z_0} \xrightarrow{\mu_i \times \text{Id}} \hat{\mathcal{X}}_{Z,i} \times V^G_{Z_0}$$

$$\hat{W}_{Z,i} \times V^G_{Z_0} \xrightarrow{\omega_i} \hat{W}_{Z,i} \times V^G_{Z_0}$$

be the pullback of the left side of (3.12) by $\Psi$. Let $\hat{X}_{Z,i} = \chi_{x_B} \hat{Z}$ and $\hat{W}_{Z,i} = W_i \times_B \hat{Z}$. Let

$$\hat{\mathcal{D}}' \xrightarrow{\hat{\mathcal{D}}} B \times V \rightarrow V$$

be the tautological family associated to the locally Weierstraß fibration $f' : W' := W'' \rightarrow B$ twisted by $\eta'$ where we recall that $\eta'$ is the image of $\eta_i^G$ in $H^1(B, \mathcal{F}_W)$. By Lemma 4.19, the complex space obtained by gluing the $\hat{W}_{Z,i} \times V^G_{Z_0}$ using $\Theta_{ij}$, which is $\hat{\mathcal{D}}' := q^{-1}(\hat{Z} \times V^G_{Z_0})$ by construction, is $G$-equivariantly isomorphic to $\hat{W}_{Z} \times V^G_{Z_0}$ over $\hat{Z} \times V^G_{Z_0}$, where $\hat{W}_{Z} \rightarrow \hat{Z}$ is the completion of $f' : W' \rightarrow B$ along $f^{-1}(Z) \rightarrow Z$. Since $\hat{X}_{Z} := \chi_{x_B} \hat{Z}$ is a completion of $\chi$ which is reduced and the finite map $m \times x_B : \hat{\mathcal{D}} \rightarrow \hat{\mathcal{D}}'$ extending $m \times x_B : \hat{X}_{Z} \rightarrow \hat{W}_{Z}$ is obtained by gluing the

$$\mu_i \times \text{Id} : \hat{X}_{Z,i} \times V^G_{Z_0} \rightarrow \hat{W}_{Z,i} \times V^G_{Z_0}$$

using (4.9), by Lemma 4.20 $\hat{\mathcal{D}} \rightarrow \hat{Z} \times V^G_{Z_0}$ is $G$-equivariantly isomorphic to $\hat{Y} \times V^G_{Z_0} \rightarrow \hat{Z} \times V^G_{Z_0}$. —

Remark 4.2. — In both Proposition 4.11 and Proposition 4.17, the hypothesis that the total space of $f$ is in the Fujiki class $\mathcal{C}$ is unnecessary.

Combining Proposition 4.6 and Proposition 4.17, now we can prove Proposition 4.2.

Proof of Proposition 4.2. — Let $Z_0 = \overline{Z \setminus \Delta}$ and $\iota_0 : Z_0 \rightarrow Z_0 \hookrightarrow B$ be a $G$-equivariant log-desingularization of $(Z_0, Z_0 \cap \Delta)$. Then the subfamily $\Pi'$ of the tautological family associated to $f$ parameterized by

$$V' = \ker \left( \iota_0 : H^1(B, \mathcal{L}_B) \rightarrow H^1(\hat{Z_0}, \mathcal{L}_{\hat{Z_0}}) \right)^G$$

is a deformation of $f$ which is $G$-equivariantly locally trivial over $B$ (Proposition-Definition 3.16.ii) and preserves $G$-equivariantly $f^{-1}(Z) \rightarrow \hat{Z}$ (Proposition 4.17).
By Proposition-Definition 3.16, there exists a multiplication-by-$m$ map $m: \mathcal{X} \to \mathcal{W}$ over $B \times V$ where $V := H^1(B, \mathcal{L}_{\mathcal{H}/B})$ and $\mathcal{X} \to B \times V \to V$ (resp. $\Pi: \mathcal{W} \to B \times V \to V$) denotes the tautological family associated to $f$ (resp. the minimal locally Weierstrass fibration $\varphi$ twisted by some element $\eta \in H^1(B, \mathcal{L}_{\mathcal{W}/B})$).

Since $f^{-1}(Z_0) \to Z_0$ has a multi-section, $(\varphi^{-1})^{-1}(Z_0) \to Z_0$ has a multi-section as well. By Proposition 4.6, the subfamily of $\Pi$ parameterized by $V'$ is an algebraic approximation of $W_0$. Since $m$ is finite and surjective, it follows that $\Pi'$ is an algebraic approximation of $\mathcal{X}$.

\section{Algebraic approximations of elliptic fibrations over a projective variety}

We will prove Theorem 1.2 in this section. The following proposition provides a general approach that we will follow to proving the existence of algebraic approximations.

**Proposition 5.1.** Let $X$ be a compact complex variety and $v: X \to X'$ a bimeromorphic map. Suppose that there exist a subvariety $Y' \subset X'$ and an algebraic approximation $\mathcal{X}' \to \Delta$ of $X'$ preserving the formal completion $\mathcal{X}'^{(3)}$ of $X'$ along $Y'$ (cf. Definition 2.1) such that $v|_{\mathcal{X}' \setminus Y'}$ is biholomorphic onto its image. Then $X$ has an algebraic approximation.

**Proof of Proposition 5.1.** Let

$$X \xleftarrow{\mu'} \hat{X} \xrightarrow{\nu'} X'$$

be a minimal resolution of $v$. Let $Y := X \setminus (v^{-1}(X' \setminus Y'))$, namely $Y$ is the subvariety of $X$ such that the restriction of $v$ to $X \setminus Y$ is biholomorphic onto $X \setminus Y'$. We have

$$E := \mu'^{-1}(Y) = v'^{-1}(Y').$$

If $\hat{E}$ (resp. $\hat{Y}$ and $\hat{Y}'$) denotes the completion of $\hat{X}$ (resp. $X$ and $X'$) along $E$ (resp. $Y$ and $Y'$), then $\hat{E} \to \hat{Y}$ and $\hat{E} \to \hat{Y}'$ are formal modifications. So their products $\hat{E} \times \Delta \to \hat{Y} \times \Delta$ and $\hat{E} \times \Delta \to \hat{Y}' \times \Delta$ with the identity $\iota: \Delta \to \Delta$ are also formal modifications. By assumption, there exists an algebraic approximation $\mathcal{Y} \subset \mathcal{X}' \xrightarrow{\pi} \Delta$ of the pair $(Y' \subset X')$ such that the completion $\mathcal{Y}'$ of $\mathcal{X}'$ along $\mathcal{Y}'$ is isomorphic to $\hat{Y}' \times \Delta$ over $\Delta$. Therefore by [4, Theorem 9.1], there exist a bimeromorphic morphism $\tilde{v}: \mathcal{X} \to \mathcal{Y}'$ and a subvariety $\mathcal{E} \subset \mathcal{X}$ mapping to $\mathcal{Y}'$ such that the restriction $\tilde{v}|_{\mathcal{X} \setminus \mathcal{E}}: \mathcal{X} \setminus \mathcal{E} \to \mathcal{X} \setminus \mathcal{Y}'$ is an isomorphism and the completion of $\tilde{v}$ along $\mathcal{E} \to \mathcal{Y}'$ is isomorphic to $\hat{E} \times \Delta \to \hat{Y}' \times \Delta$. Similarly by [4, Theorem 8.1], there exist a bimeromorphic morphism $\tilde{\mu}: \mathcal{X} \to \mathcal{Y}$ and a subvariety $\mathcal{Y} \subset \mathcal{X}$ such that the restriction $\tilde{\mu}|_{\mathcal{X} \setminus \mathcal{Y}}: \mathcal{X} \setminus \mathcal{Y} \to \mathcal{X} \setminus \mathcal{Y}$ is an isomorphism where $\mathcal{Y} := \tilde{\mu}(\mathcal{E})$ and furthermore, the completion of $\tilde{\mu}$ along $\mathcal{E} \to \mathcal{Y}$ is isomorphic to $\hat{E} \times \Delta \to \hat{Y} \times \Delta$. Let $\mathcal{A} := \mathcal{X} \setminus \mathcal{Y} \to \mathcal{A}$. As $\mathcal{E} \to \mathcal{Y}$, being isomorphic to $E \times \Delta \to Y \times \Delta$, is a morphism over $\Delta$, the map $\pi$ induces a map $\pi: \mathcal{X} \to \Delta$. To sum up, we have a diagram

$$\begin{array}{ccc}
\mathcal{X} & \xleftarrow{\tilde{\mu}} & \hat{E} \\
\downarrow{\pi} & & \downarrow{\tilde{v}} \\
\Delta & \xrightarrow{\pi'} & \mathcal{Y}'
\end{array}$$

of morphisms over $\Delta$ containing (5.1) as a fiber over the point $o \in \Delta$ parameterizing $X'$ in $\nu': \mathcal{X}' \to \Delta$.

3. Usually the completion of $X$ along a subvariety $Y$ is denoted by $\hat{X}$. In this text, since we want to keep track of the subvariety along which the completion is done, we will often use the notation $\tilde{Y}$ instead of $\hat{X}$. 

---
Now we show that \( \pi \) and \( \pi' \) are deformations of \( X \) and \( X' \) respectively. First we show that \( \pi \) and \( \pi' \) are flat. For lack of a reference, we shall first prove the following infinitesimal criterion of flatness.

**Lemma 5.2.** — Let \( \phi : S \rightarrow T \) be a morphism between complex spaces and \( \Sigma \subset S \) an analytic closed subset. Let \( \Sigma' \) be the completion of \( S \) along \( \Sigma \). Suppose that the induced map \( \tilde{\Sigma} \rightarrow T \) is flat, then \( \phi \) is flat at every point \( p \in \Sigma' \subset S \).

**Proof.** — Let \( p \in \Sigma \) and let \( l \subset \mathcal{O}_{S_p} \) be the ideal defining \( \Sigma \) at \( p \). Let \( \mathcal{O}_{S_p} \) be the \( l \)-adic completion of \( \mathcal{O}_{S_p} \). Since \( \mathcal{O}_{S_p} \) is Noetherian \([13, \text{p.44, Corollary}]\), \( \mathcal{O}_{S_p} \rightarrow \mathcal{O}_{S_p} \) is faithfully flat \([29, \text{Tag 00MC}]\). It follows that as \( \mathcal{O}_{T_{\phi(p)}} \rightarrow \mathcal{O}_{S_p} \) is flat by assumption, \( \mathcal{O}_{T_{\phi(p)}} \rightarrow \mathcal{O}_{S_p} \) is also flat \([29, \text{Tag 0584}]\). \( \square \)

On the one hand since \( \hat{\theta} \rightarrow \Delta \) is isomorphic to the projection \( \hat{E} \times \Delta \rightarrow \Delta \), which is flat, by Lemma 5.2 the composition \( \hat{\pi} : \hat{E} \rightarrow \hat{X} \rightarrow \Delta \) is flat at every point of \( \hat{\mathcal{E}} \subset \hat{E} \). On the other hand as the restriction \( \mu_{\hat{E}\setminus \hat{\mathcal{E}}} : \hat{E}\setminus \hat{\mathcal{E}} \rightarrow \hat{X}\setminus \hat{\mathcal{Y}} \) is an isomorphism and \( \hat{X}\setminus \hat{\mathcal{Y}} \rightarrow \Delta \) is flat, the restriction of \( \hat{\pi} \) to the Zariski open \( \hat{E}\setminus \hat{\mathcal{E}} \) is also flat. Hence \( \pi \) is flat everywhere. A similar argument shows that \( \pi' \) is flat as well.

Let \( \hat{X}_o \) (resp. \( \hat{X}_e \)) denote the central fiber of \( \hat{\pi} \) (resp. \( \pi \)). The restriction \( \hat{X}_o \rightarrow \hat{X}' \) of \( \hat{\pi} : \hat{E} \rightarrow \hat{X}' \rightarrow \hat{X} \to \hat{X}' \) to \( \hat{X}_o \) is a modification of \( \hat{X}' \), whose completion along the exceptional locus is isomorphic to \( \hat{E} \rightarrow \hat{Y}' \). So by the uniqueness of extensions of formal modifications \([4, \text{Theorem 9.1}]\), the restriction of \( \hat{\pi} \) to \( \hat{X}_o \) is isomorphic to \( \nu : \hat{X}_o \rightarrow \hat{X}' \). Thus \( \pi' : \hat{X}' \rightarrow \Delta \) is a deformation of \( \hat{X}' \). A similar argument (in which we apply \([4, \text{Theorem 8.1}]\) instead of \([4, \text{Theorem 9.1}]\)) shows that the restriction of \( \mu \) to \( \hat{X}_o \) is isomorphic to \( \mu : \hat{X}_o \rightarrow \hat{X} \). Thus \( \pi : \hat{X} \rightarrow \Delta \) is a deformation of \( X \). Finally since \( \pi' : \hat{X}' \rightarrow \Delta \) is an algebraic approximation of \( X' \), we conclude that \( \pi \) is an algebraic approximation of \( X \).

**Proof.** — Let \( X \) be a compact Kähler manifold bimeromorphic to the total space of an elliptic fibration over a projective variety. We will first construct bimeromorphic maps

\[
X \leftarrow^\mu X'/G \longrightarrow^\tau X/G
\]

such that \( \tau \) satisfies the hypothesis of Proposition 5.1.

**Lemma 5.3.** — There exist a compact complex manifold \( X' \), a finite group \( G \) acting on it, a \( G \)-equivariant tautological model \( f : X \rightarrow B \) satisfying Hypotheses 4.1, and bimeromorphic maps

\[
X \leftarrow^\mu X'/G \longrightarrow^\tau X/G
\]

such that \( \mu \) is holomorphic and \( \tau \) is the quotient of a \( G \)-equivariant bimeromorphic map \( \tau : X' \rightarrow X \) satisfying the following property: There exists a \( G \)-stable subvariety \( Z \subset B \) containing \( \Delta \) such that \( f^{-1}(Z \setminus \Delta) \rightarrow Z \setminus \Delta \) has a multi-section and \( \tau_{\Delta/\{0\}}^{-1} \) is biholomorphic onto its image.

**Proof.** — The proof of Lemma 5.3 consists of several steps.

**Step 1:** \( X \) is bimeromorphic to \( X_0/G \) where \( G \) is a finite group and \( X_0 \) is the total space of a \( G \)-equivariant tautological model \( f_0 : X_0 \rightarrow B_0 \) satisfying Hypotheses 4.1.

Let \( g : X' \rightarrow B_0 \) be an elliptic fibration over a projective variety \( B_0 \) such that \( X' \) is bimeromorphic to \( X \). Let \( \Delta_0 \subset B_0 \) be the discriminant locus of \( g \). Up to base changing \( g \) with a log-desingularization of \((B_0, \Delta_0)\), we can assume that \( B_0 \) is smooth and \( \Delta_0 \) a simple normal crossing (SNC) divisor. Since \( g \) is a smooth elliptic
fibration over a nonempty Zariski open of $B_0$, up to replacing $X'$ with a minimal Kähler desingularization of it, we can assume that $X'$ is a compact Kähler manifold and $g$ remains an elliptic fibration.

Since $X'$ is a compact Kähler manifold and $B_0$ is smooth, $g$ is locally projective [24, Theorem 3.3.3]. So there exists a Galois cover $r_0: B_0 \to \tilde{B}_0$ such that the base change $g: X' := \tilde{X}' \times_{\tilde{B}_0} B_0 \to B_0$ of $g$ is an elliptic fibration over a projective manifold $B_0$ whose discriminant locus $r_0^{-1}(\Delta_0)$ is an SNC divisor and $g$ has local meromorphic sections at every point of $B_0$ [10, Proposition 3.11]. Let $\Delta_0 \subset r_0^{-1}(\tilde{\Delta}_0) \subset B_0$ be the discriminant divisor of the minimal Weierstraß fibration associated to $g$. As $g$ is locally projective, the local monodromies of the local system $H_0 := (R^1g_*\mathcal{O}_X)_{|B_0\setminus\Delta_0}$ around $\Delta_0$ are quasi-unipotent. By Kawamata’s unipotent reduction [15, Corollary 18], we can further assume that the Galois cover $r_0: B_0 \to \tilde{B}_0$ we take is such that the local monodromies of $H_0$ around $\Delta_0$ are unipotent and $(B_0, \Delta_0)$ remains a log-smooth projective pair.

Since $g: X' \to B_0$ has local meromorphic sections at every point of $B_0$, we can apply Lemma-Definition 3.13 and obtain a $G$-equivariant tautological model $f_0: X_0 \to B_0$ of $g$ where $G := \text{Gal}(B_0/B_0)$. So $X_0/G$ is bimeromorphic to $X'/G$, which is further bimeromorphic to $X$. As the discriminant locus of $f_0: X_0 \to B_0$ is $\Delta_0$ by Lemma 3.14 and $(B_0, \Delta_0)$ is a log-smooth projective pair, $f_0$ satisfies i) and iii) in Hypotheses 4.1. Since $f_0$ is bimeromorphic to $g$ and $g$ satisfies ii) and iv) in Hypotheses 4.1, so does $f_0$. □

**Step 2:** There exist a sequence of blow-ups

$$X_n \to \cdots \to X_0$$

(5.3)

along $G$-stable subvarieties $C_i \subset X_i$ which do not dominate $B_0$ and a bimeromorphic morphism $\mu_n: X_n/G \to X$.

By Step 1, we have a bimeromorphic map $\phi: \tilde{X}_0 := X_0/G \to X$. Let

$$X_n \to \cdots \to X_0$$

(5.4)

be a sequence of blow-ups and $\phi': \tilde{X}_n \to X$ a (bimeromorphic) morphism resolving $\phi$. As $X_0$ is normal by Lemma 3.14, so is the finite quotient $\tilde{X}_0$. Thus the indeterminacy locus of $\phi'$ is of codimension at least 2. As $\dim \tilde{B}_0 = \dim \tilde{X}_0 - 1$, we may assume that the blow-up centers $\tilde{C}_i \subset \tilde{X}_i$ of (5.4) do not dominate $\tilde{B}_0$.

We construct by induction the blow-up sequence (5.3) together with $G$-equivariant morphisms $q_i: X_i \to X_i$ (where $X_i$ is endowed with the trivial $G$-action) such that the quotients $\tilde{q}_i: \tilde{X}_i/G \to \tilde{X}_i$ are bimeromorphic. For $i = 0$, we define $\tilde{q}_0: \tilde{X}_0 \to \tilde{X}_0$ to be the quotient map, so $\tilde{q}_0$ is the identity. Suppose that $X_i$ and $q_i$ are constructed and $q_i$ is known to be bimeromorphic. Let $C_i := q_i^{-1}(\tilde{C}_i)$. We define $X_{i+1} \to X_i$ to be the blow-up of $X_i$ along $C_i$. As $q_i$ is $G$-equivariant, $C_i$ is $G$-stable. Since $C_i$ does not dominate $\tilde{B}_0$, $C_i$ does not dominate $B_0$. The universal property of blowing up provides a $G$-equivariant map $\tilde{q}_{i+1}: \tilde{X}_{i+1} \to \tilde{X}_{i+1}$ and a commutative diagram

$$\begin{array}{ccc}
X_{i+1} & \longrightarrow & X_i \\
\downarrow q_{i+1} & & \downarrow \tilde{q}_i \\
\tilde{X}_{i+1} & \longrightarrow & \tilde{X}_i
\end{array}$$

(5.5)

where the horizontal arrows are the blow-ups along $C_i$ and $\tilde{C}_i$ respectively. As $\tilde{q}_i$ is bimeromorphic, $\tilde{q}_{i+1}$ is also bimeromorphic. It follows that the map $\mu_n$ defined by

$$\mu_n := \phi' \circ \tilde{q}_n: X_n/G \to X_n \to X$$
is bimeromorphic as well.

**Step 3:** We construct by induction starting from \( i = 0 \) a sequence of \( G \)-equivariant blow-ups

\[
\nu : B_\nu \xrightarrow{\nu_{i+1}} \cdots \xrightarrow{\nu_0} B_0
\]

and \( G \)-stable subvarieties \( Z_i \subset B_i \) containing the pre-images \( \Delta_i \subset B_i \) of \( \Delta_0 \subset B_0 \) and satisfying the following properties.

i) The induced \( G \)-equivariant meromorphic map \( f_i : X_i \dashrightarrow B_i \) is almost holomorphic and well-defined over \( B_i \setminus Z_i \), which is nonempty.

ii) Let \( f'_i : X'_i := X_0 \times_{B_0} B_i \rightarrow B_i \) be the base change of the fibration \( f_0 : X_0 \rightarrow B_0 \). Then \( f'_i \) is a smooth elliptic fibration over \( B_i \setminus Z_i \) and \( f^{-1}_i(Z_i \cap \Delta_i) \rightarrow Z_i \setminus \Delta_i \) has a multi-section.

iii) The map \( f_i \) is \( G \)-equivariantly biholomorphic to \( f'_i \) over \( B_i \setminus Z_i \).

For \( i = 0 \), we set \( Z_0 = \Delta_0 \) and the three properties above hold obviously. Assume that the blow-up \( \nu_{i+1} : B_i \rightarrow B_{i+1} \) and \( Z_i \) are constructed and satisfy the properties listed above. Let \( C'_i \) be the union of the irreducible components of the blow-up center \( C_i \subset X_i \) of \( X_{i+1} \rightarrow X_i \) that are of the form \( \overline{f^{-1}_i(\Sigma \setminus Z_i)} \) for some subvariety \( \Sigma \subset B_i \); we write \( C_i = C'_i \cup C''_i \) where \( C''_i \) is the union of the irreducible components of \( C_i \) complementary to \( C'_i \). We define \( \nu_i : B_{i+1} \rightarrow B_i \) to be the blow-up of \( B_i \) along \( f_i(C'_i) \) (4) and

\[
Z_{i+1} := \nu_i^{-1}(Z_i \cup f_i(C''_i)).
\]

Obviously, \( Z_{i+1} \) contains \( \Delta_{i+1} \). As \( C_i \) and \( Z_i \) are \( G \)-stable, \( C'_i \) and \( C''_i \) are \( G \)-stable as well. Thus the blow-up center \( f_i(C'_i) \) of \( \nu_i \) and \( Z_{i+1} \) are also \( G \)-stable.

As \( f_i \) is biholomorphic to \( f'_i \) over \( B_i \setminus Z_i \) and \( f'_i \) is a smooth elliptic fibration over \( B_i \setminus Z_i \), the map \( f_i : X_i \dashrightarrow B_i \) is flat over \( B_i \setminus (Z_i \cup f_i(C''_i)) \). Since blowing up commutes with flat base change, we have the following cartesian diagram

\[
\begin{array}{ccc}
X^*_{i+1} := f^{-1}_i \left( B_i \setminus \overline{f_i(C'_i)} \right) & \longrightarrow & f^{-1}_i \left( B_i \setminus (Z_i \cup f_i(C''_i)) \right) \\
\downarrow & & \downarrow f_i \\
B_{i+1} \setminus Z_{i+1} & \xrightarrow{\nu_i} & B_i \setminus (Z_i \cup f_i(C''_i))
\end{array}
\]  

(5.6)

where the upper horizontal arrow is the blow-up of the Zariski open \( f^{-1}_i \left( B_i \setminus (Z_i \cup f_i(C''_i)) \right) \subset X_i \) along

\[
f^{-1}_i \left( f_i(C'_i) \setminus (Z_i \cup f_i(C''_i)) \right) = C'_i \setminus f^{-1}_i(Z_i \cup f_i(C''_i)) = C'_i \setminus f^{-1}_i(Z_i \cup f_i(C''_i)) = C_i \setminus f^{-1}_i \left( B_i \setminus (Z_i \cup f_i(C''_i)) \right).
\]

(5.7)

Here, the first equality of (5.7) follows from the assumption that irreducible components of \( C'_i \) are of the form \( \overline{f^{-1}_i(\Sigma \setminus Z_i)} \). Recall that \( X_{i+1} \) is the blow-up of \( X_i \) along \( C_i \), so \( X^*_{i+1} \) is a Zariski open of \( X_{i+1} \) and the left vertical arrow is the restriction of \( f_{i+1} \) to \( X^*_{i+1} \). As \( X^*_{i+1} \rightarrow B_{i+1} \setminus Z_{i+1} \) is proper, \( f_{i+1} \) is almost holomorphic and well-defined over \( B_{i+1} \setminus Z_{i+1} \). As \( C_i \subset X_i \) does not dominate \( B_i \) by Step 2, we have \( B_{i+1} \setminus Z_{i+1} \neq \emptyset \), which shows i). Since (5.6) is cartesian and by the induction hypothesis, the right vertical arrow of (5.6) is \( G \)-equivariantly isomorphic to the base change

\[
X_0 \times_{B_0} \left( B_i \setminus (Z_i \cup f_i(C''_i)) \right) \rightarrow B_i \setminus (Z_i \cup f_i(C''_i))
\]

4. For any subvariety \( Y \subset X_i \), \( f_i(Y) \) is defined as \( \overline{f_i(\nu_i^{-1}(B_i \setminus Z_i))} \), which is well-defined since \( f_i \) is holomorphic over \( B_i \setminus Z_i \). In particular \( f_i(Y) \) is empty if and only if \( Y \subset f^{-1}_i(Z_i) \).
of \( f_0 : X_0 \to B_0 \) iii) holds.

As for ii), since \( f_0 \) is smooth over \( B_0 \setminus Z_0 \) and \( Z_{i+1} \) contains the pre-image of \( Z_0 \), the elliptic fibration \( f'_{i+1} \) is smooth over \( B_{i+1} \setminus Z_{i+1} \). Since \( Z_{i+1} = v_i^{-1}(Z_i \cup f_i(C'_i)) \) and \( f'_{i+1} \) is the base change of \( f'_i \) by \( v_i : B_{i+1} \to B_i \), it suffices to show that \( f_i^{-1}(Z_i \setminus \Delta) \to Z_i \setminus \Delta \) and \( f_i^{-1}(f_i(C'_i)) \to f_i(C'_i) \) have multi-sections. The former holds by the induction hypothesis. As for the latter, we have to show that for every irreducible component \( Y'' \) of \( C'_i \), if \( Z'' := f_i(Y'') \) (which we can assume to be nonempty, namely \( Z'' \neq Z_0 \)), then the fibration \( f_i^{-1}(Z'') \to Z'' \) has a multi-section. Over the dense Zariski open \( Z'' \setminus Z_i \) of \( Z'' \), the map \( f_i^{-1}(Z'') \to Z'' \) is isomorphic to the almost holomorphic map \( f_i^{-1}(Z'') := f_i^{-1}(Z'' \setminus Z_i) \to Z'' \). In particular, fibers of \( f_i^{-1}(Z'') \to Z'' \) over \( Z'' \setminus Z_i \) are elliptic curves. So if \( Y'' \) is not a multi-section of \( f_i^{-1}(Z'') \to Z'' \), then \( Y'' = f_i^{-1}(Z'' \setminus Z_i) \), which would be an irreducible component of \( C'_i \) and yield a contradiction. Accordingly, the image of \( Y'' \) in \( f_i^{-1}(Z'') \) is a multi-section of \( f_i^{-1}(Z'') \to Z'' \).

**Step 4:** Construction of the G-equivariant tautological model \( f : X \to B \).

Let \( \nu' : (B, \Delta) \to (B_n, \Delta_n) \) be a G-equivariant log-resolution of singularities [1] of the pair \((B_n, \Delta_n)\), where we recall that \( \Delta_n \subset B_n \) is the pre-image of \( \Delta_0 \subset B_0 \), the discriminant locus of \( f_0 : X_0 \to B_0 \). Let \( f : X \to B \) be the G-equivariant tautological model associated to the G-equivariant elliptic fibration \( X_n \times_{B_n} B = X_n \times_{B_n} B \to B \) by virtue of Lemma-Definition 3.13. Since \( \Delta = (\nu \circ \nu')^{-1}(\Delta_0) \) is a normal crossing divisor and the local monodromies of \( H_0 \) around \( \Delta_0 \) are unipotent, by Lemma 3.5 the pullback of the minimal Weierstraß fibration associated to \( H_0 \) by \( \nu \circ \nu' \) is still a minimal Weierstraß fibration. Therefore by Lemma 3.14, the discriminant locus of \( f \) is \( \Delta \). As \((B, \Delta)\) is log-smooth and projective, \( f \) satisfies i) and ii) in Hypotheses 4.1. Since \( f \) is the pullback of \( f_0 \) by \( \nu \circ \nu' \) and \( f_0 \) satisfies ii) and iii) in Hypotheses 4.1, so does \( f \).

**Step 5:** Construction of the G-stable subvariety \( Z \subset B \) and the end of the proof of Lemma 5.3.

Let \( Z := \nu'^{-1}(Z_n) \). As \( \Delta_n \subset Z_n \), we have \( \Delta \subset Z \). Since \( f_n^{-1}(Z_n \setminus \Delta_n) \to Z_n \setminus \Delta_n \) has a multi-section by Step 3 and \( f \) is bimeromorphic to \( X_n \times_{B_n} B \to B \) over \( B \setminus \Delta \), the fibration \( f^{-1}(Z \setminus \Delta) \to Z \setminus \Delta \) has a multi-section.

Let \( f'' : X'' \to B_n \) be a G-equivariant minimal resolution of the G-equivariant meromorphic map \( f_n : X_n \to B_n \) and let \( f'' : X'' := X'' \times_{B_n} B \to B \) be the base change of \( f'' \). As the resolution is minimal and \( X_n \to B_n \) is almost holomorphic and well-defined over \( B_n \setminus Z_n \), the map \( f'' \) is (G-equivariantly) biholomorphic to \( f_n \) over \( B_n \setminus Z_n \), which is further (G-equivariantly) biholomorphic to \( f'_i \) over \( B_i \setminus Z_i \) by Step 3. It follows that the induced bimeromorphic map \( \alpha : X'' \to X_0 \times_{B_0} B \) is biholomorphic over \( B \setminus Z \). In particular, \( f''^{-1}(B \setminus Z) \) is a smooth elliptic fibration because \( f_0 : X_0 \to B_0 \) is smooth over \( B_0 \setminus \Delta_0 \) and \( (\nu \circ \nu')^{-1}(\Delta_0) = \Delta \subset Z \). Therefore the singular locus of \( X'' \) is contained in \( f''^{-1}(Z) \), so if \( \mu' : X' \to X'' \) is a G-equivariant minimal desingularization of \( X'' \), then \( \mu' \) is biholomorphic over \( B \setminus Z \). By Lemma 3.15, there exists a G-equivariant bimeromorphic map \( \beta : X_0 \times_{B_0} B \to X \) over \( B \) that is biholomorphic over \( B \setminus \Delta \). Thus if we define the bimeromorphic map \( \tau \) in the statement of Lemma 5.3 to be the composition of the bimeromorphic maps

\[
\tau : X' \xrightarrow{\mu'} X'' \xrightarrow{\alpha} X_0 \times_{B_0} B \xrightarrow{\beta} X,
\]

then as \( \alpha, \beta, \) and \( \mu' \) are G-equivariant and biholomorphic over \( B \setminus Z \), so is \( \tau \). Hence \( \tau \) satisfies the desired property in Lemma 5.3.
Finally we define the holomorphic map $\mu$ in the statement of Lemma 5.3 to be the composition of the bimeromorphic morphisms

$$\mu : X'/G \xrightarrow{\mu'/G} X''/G \to X'''/G \to X_n/G \xrightarrow{\mu_1} X,$$

where the second arrow is the quotient by $G$ of the projection $X'' = X''_n \times_{B_n} B \to X''_n$ and the third arrow the quotient of the $G$-equivariant resolution $X''_n \to X_n$ of the map $X_n \to B_n$ introduced in Step 4. \hfill \Box

By Lemma 5.3 we have a diagram

$$X \leftarrow^\mu X'/G \xrightarrow{\tau} X/G,$$

where both arrows are bimeromorphic. Since $X'/G$ is a finite quotient of a complex manifold, it has at worst rational singularities [5, Corollaire]. As $X$ is smooth, we have $\mu_* O_{X'/G} = O_X$ and $R^1 \mu_* O_{X'/G} = 0$. It follows from [27, Theorem 2.1] that if $X'/G$ has an algebraic approximation, then it induces an algebraic approximation of $X$. Since $\hat{Z}$ is $G$-stable and $f$ is $G$-equivariant, $Y := f^{-1}(\hat{Z})$ is also $G$-stable. As $\tau^{-1}_{X/G}$ is biholomorphic onto its image, so is $\tau^{-1}_{(X/G)-(Y/G)}$ Thus by Proposition 5.1, it suffices to prove that there exists an algebraic approximation of $\hat{X}/G$ preserving the completion $\hat{Y}/\hat{G}$ of $X/G$ along $Y/G$.

Since $f : X \to B$ is a $G$-equivariant tautological model satisfying Hypotheses 4.1 and since $f^{-1}(\hat{Z} \setminus \Delta) \to \hat{Z} \setminus \Delta$ is a fibration over a $G$-stable subvariety of $B$ which has a multi-section by Lemma 5.3, according to Proposition 4.2 there exists an algebraic approximation

$$\Pi : \mathcal{X} \xrightarrow{\mathcal{Y}} B \times V' \to V'$$

of $f : \mathcal{X} \to B$ which is $G$-equivariantly locally trivial over $B$ and preserves $G$-equivariantly the completion $\hat{Y} \to \hat{Z}$ of $f$ along $Y := f^{-1}(\hat{Z}) \to \hat{Z}$. By Lemma 2.3, the quotient $\mathcal{X}/G \to V'$ of $\Pi$ is an algebraic approximation of $X/G$. As $\hat{Y}$ is $G$-equivariantly preserved by $\Pi$, we have a $G$-stable subvariety $\mathcal{Y} \subset \mathcal{X}$ along which the completion $\hat{\mathcal{Y}}$ of $\mathcal{X}$ is $G$-equivariantly isomorphic to $\hat{Y} \times V'$ over $V'$. So

$$\mathcal{Y}/G \simeq \hat{\mathcal{Y}}/G \simeq (\hat{Y}/G) \times V' \simeq \hat{Y}/\hat{G} \times V'$$

over $V'$, where $\hat{\mathcal{Y}}/G$ is the completion of $\mathcal{X}/G$ along $\mathcal{Y}/G$. It follows that the quotient $\mathcal{X}/G \to V'$ of $\mathcal{X} \to V'$ is an algebraic approximation of $X/G$ preserving $\hat{Y}/\hat{G}$, which finishes the proof of Theorem 1.2. \hfill \Box

Acknowledgement

The theory of elliptic fibrations developed by Nakayama plays a crucial role in this work, and I had a more thorough understanding of this subject when collaborating with Benoît Claudon and Andreas Höring. I am grateful to them for the e-mail correspondence related to this subject and their comments on the preliminary version of the text. I am also in debt to Noboru Nakayama for explaining to me several details of his work [23].
References

[1] Dan Abramovich and Jianhua Wang. Equivariant resolution of singularities in characteristic 0. *Math. Res. Lett.*, 4(2-3) :427–433, 1997. 37

[2] Benjamin Bakker and Christian Lehn. A global torelli theorem for singular symplectic varieties. *arXiv*:1612.07894, 2017. 22, 26

[3] Jürgen Bingener. Über formale komplexe Räume. *Manuscripta Math.*, 24(3) :253–293, 1978. 31

[4] Jürgen Bingener. On the existence of analytic contractions. *Invent. Math.*, 64(1) :25–67, 1981. 2, 33, 34

[5] Jean-François Boutot. Singularités rationnelles et quotients par les groupes réductifs. *Invent. Math.*, 88(1) :65–68, 1987. 38

[6] Frédéric Campana. The class $G$ is not stable by small deformations. *Math. Ann.*, 290(1) :19–30, 1991. 2

[7] Frédéric Campana. Coréduction algébrique d’un espace analytique faiblement kählérien compact. *Invent. Math.*, 63(2) :187–223, 1981. 8

[8] Junyan Cao. On the approximation of Kähler manifolds by algebraic varieties. *Math. Ann.*, 363(1-2) :393–422, 2015. 1

[9] Benoît Claudon. Smooth family of tori and linear Kähler groups. To appear in Annales de la Faculté des Sciences de Toulouse (*arXiv*:1604.0367v2), 2016. 1, 5

[10] Benoît Claudon, Andreas Höring, and Hsueh-Yung Lin. The fundamental group of compact Kähler threefolds. *arXiv*:1612.04224, 2018. 1, 2, 4, 5, 6, 8, 10, 12, 13, 14, 15, 18, 23, 35

[11] Hubert Flenner. Über Deformationen holomorpher Abbildungen. Habilitation thesis, Universität Osnabrück, 1979. 29

[12] Patrick Graf. Algebraic approximation of Kähler threefolds of Kodaira dimension zero. *Mathematische Annalen*, Aug 2017. 1

[13] Hans Grauert and Reinhold Remmert. *Coherent analytic sheaves*, volume 265 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984. 34

[14] Birger Iversen. *Cohomology of sheaves*. Berlin: Springer, 1986. 28

[15] Yujiro Kawamata. Characterization of abelian varieties. *Compositio Math.*, 43(2) :253–276, 1981. 35

[16] Kunihiko Kodaira. On compact analytic surfaces. II, III. *Ann. of Math. (2) 77 (1963), 563–626; ibid.*, 78 :1–40, 1963. 1, 8, 18

[17] János Kollár. Higher direct images of dualizing sheaves. II. *Ann. of Math. (2)*, 124(1) :171–202, 1986. 13

[18] Claude LeBrun and Yat Sun Poon. Twistor, Kähler manifolds, and bimeromorphic geometry. II. *J. Amer. Math. Soc.*, 5(2) :317–325, 1992. 2

[19] Hsueh-Yung Lin. Algebraic approximations of fibrations in abelian varieties over a curve. *arXiv*:1612.09271, 2016. 1, 2, 5, 7

[20] Hsueh-Yung Lin. Algebraic approximations of compact Kähler threefolds. *arXiv*:1710.01083, 2018. 2, 6

[21] B. G. Moisëzon. On $n$-dimensional compact complex manifolds having $n$ algebraically independent meromorphic functions. I. *Izv. Akad. Nauk SSSR Ser. Mat.*, 30 :133–174, 1966. 1

[22] Noboru Nakayama. On Weierstrass models. In *Algebraic geometry and commutative algebra, Vol. II*, pages 405–431. Kinokuniya, Tokyo, 1988. 3, 6, 8, 9, 10, 12, 13

[23] Noboru Nakayama. Global structure of an elliptic fibration. *Publ. Res. Inst. Math. Sci.*, 38(3) :451–649, 2002. 6, 8, 9, 10, 13, 16, 17, 18, 38

[24] Noboru Nakayama. Local structure of an elliptic fibration. In *Higher dimensional birational geometry (Kyoto, 1997)*, volume 35 of *Adv. Stud. Pure Math.*, pages 185–295. Math. Soc. Japan, Tokyo, 2002. 6, 8, 17, 35

[25] Yoshinori Namikawa. Extension of 2-forms and symplectic varieties. *J. Reine Angew. Math.*, 539 :123–147, 2001. 15

[26] Yoshinori Namikawa. Projectivity criterion of Moishezon spaces and density of projective symplectic varieties. *Internat. J. Math.*, 13(2) :125–135, 2002. 15

[27] Ziv Ran. Stability of certain holomorphic maps. *J. Differential Geom.*, 34(1) :37–47, 1991. 2, 38

[28] Florian Schrack. Algebraic approximation of Kähler threefolds. *Math. Nachr.*, 285(11-12) :1486–1499, 2012. 1

[29] The Stacks Project Authors. *Stacks Project*. *http://stacks.math.columbia.edu*, 2018. 34

[30] Kenji Ueno. *Classification theory of algebraic varieties and compact complex spaces*. Lecture Notes in Mathematics, Vol. 439. Springer-Verlag, Berlin-New York, 1975. Notes written in collaboration with P. Cherenack. 1
[31] Claire Voisin. On the homotopy types of compact Kähler and complex projective manifolds. *Invent. Math.*, 157:329–343, 2004.

[32] Steven Zucker. Hodge theory with degenerating coefficients. \(L_2\) cohomology in the Poincaré metric. *Ann. of Math. (2)*, 109(3):415–476, 1979.

HSHUEH-YUNG LIN, Mathematisches Institut der Universität Bonn, Endenicher Allee 60, Office 303, 53115 Bonn, Germany.

E-mail: linhsueh@math.uni-bonn.de