GEOMETRIC CHARACTERIZATION OF FLAT MODULES

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Abstract. Let $R$ be a commutative ring. Roughly speaking, we prove that an $R$-module $M$ is flat iff it is a direct limit of affine algebraic varieties of $R$-modules, and $M$ is a flat Mittag-Leffler module iff it is the union of its affine algebraic subvarieties of $R$-modules.

1. Introduction

Let $R$ be a commutative (associative with unit) ring. All the functors considered in this paper are covariant functors from the category of commutative $R$-algebras to the category of sets (groups, rings, etc.).

$R$-algebraic varieties or $R$-schemes are not mere sets of points, but they are determined by their functors of points. Given an $R$-scheme $X$ the functor of points of $X$, $X^*$ is defined by

$$X^*(S) := \text{Hom}_{\text{Spec } S}(\text{Spec } S, X \times_{\text{Spec } R} \text{Spec } S),$$

for any commutative $R$-algebra $S$. If $X = \text{Spec } R[x_1, \ldots, x_n]/(p_1, \ldots, p_m)$ then

$$X^*(S) = \text{Hom}_{R-\text{sch}}(\text{Spec } S, X) = \text{Hom}_{R-\text{alg}}(R[x_1, \ldots, x_n]/(p_1, \ldots, p_m), S)$$

$$= \{s \in S^n : p_1(s) = \cdots = p_m(s) = 0 \}.$$

Let $R$ be the functor defined by $R(S) := S$ for any commutative $R$-algebra $S$. Let $M$ be an $R$-module. Consider the functor of $R$-modules, $\mathcal{M}$, defined by $\mathcal{M}(S) := M \otimes_R S$. $\mathcal{M}$ is said to be the quasi-coherent functor of $R$-modules associated with $M$. It is easy to prove that the category of $R$-modules is equivalent to the category of functors of quasi-coherent modules. Consider the dual functor $\mathcal{M}^* := \text{Hom}_{\mathcal{M}}(\mathcal{M}, R)$ defined by $\mathcal{M}^*(S) := \text{Hom}_S(M \otimes_R S, S)$. $\mathcal{M}^*$ is called an $R$-module scheme, because it is isomorphic to the functor of points of the affine $R$-scheme $\text{Spec}(S'M)$. A surprising result states that $\mathcal{M} = \mathcal{M}^{**}$ (see 2.11). This result has many applications in Algebraic Geometry (see 3), for example the Cartier duality of commutative affine groups and commutative formal groups.

In [2], we proved that an $R$-module $M$ is a projective module of finite type iff $M$ is a module scheme. In Corollary 3.3 we will prove that $M$ is a projective module of finite type iff $M$ is equal to the functor of points of an affine $R$-scheme. The Govorov-Lazard Theorem states that an $R$-module $M$ is flat iff $M$ is isomorphic to a direct limit of (functors of points of) affine $n$-spaces, $A^n_R$ (with the natural structure of $R$-modules). In this paper we prove that $M$ is a flat $R$-module iff $M$ is a direct limit of (functors of points of) affine algebraic varieties of modules (see 4.7). We also prove that $M$ is a flat $R$-module iff $M$ is a direct limit of module schemes.

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Given a functor of $\mathcal{R}$-modules $\mathbb{M}$, we will denote by $M(\mathcal{R})$ the quasi-coherent $\mathcal{R}$-module associated with the $R$-module $M(R)$. Now, let $R$ be a Noetherian ring. We prove that an $R$-module $M$ is flat iff
\[ M = \lim_{\rightarrow \, i \in I} M_i(\mathcal{R})^*, \]
where $\{M_i\}_{i \in I}$ is the set of the submodules of finite type of $M$. In [4.10] we generalize this result to characterize flat quasi-coherent sheaves on Noetherian schemes. This theorem can be considered as the generalization of the Govorov-Lazard Theorem to Noetherian schemes.

Finally, we prove that $M$ is a flat Mittag-Leffler module iff $M$ is the direct limit of its submodule schemes, $M = \lim_{\rightarrow \, i} N_i^*$, where $\{N_i^*\}$ is the set of submodule schemes of $M$.

2. Preliminaries

Let $\mathcal{R}$ be the functor of rings defined by $\mathcal{R}(S) := S$, for any commutative $R$-algebra $S$. A functor of Abelian groups $\mathbb{M}$ is said to be a functor of (left) $R$-modules if we have a morphism of functors of sets, $\mathcal{R} \times \mathbb{M} \to \mathbb{M}$, so that $\mathbb{M}(S)$ is a (left) $S$-module, for every commutative $R$-algebra $S$.

Let $\mathbb{M}$ and $\mathbb{M}'$ be functors of $\mathcal{R}$-modules. A morphism of functors of $\mathcal{R}$-modules $f: \mathbb{M} \to \mathbb{M}'$ is a morphism of functors such that the morphisms $f_S: \mathbb{M}(S) \to \mathbb{M}'(S)$ defined by $f$ are morphisms of $S$-modules, for any commutative $R$-algebra $S$. We will denote by $\text{Hom}_\mathcal{R}(\mathbb{M}, \mathbb{M}')$ the family of all morphisms of $\mathcal{R}$-modules from $\mathbb{M}$ to $\mathbb{M}'$.

Let $S$ be a commutative $R$-algebra and $M_S$ the functor $\mathbb{M}$ restricted to the category of commutative $S$-algebras. We will denote by $\prod_{\text{Hom}_\mathcal{R}(\mathbb{M}, \mathbb{M}')} S := \text{Hom}_S(M_S, M'_S)$.

Obviously,
\[ (\text{Hom}_\mathcal{R}(\mathbb{M}, \mathbb{M}'))|_S = \text{Hom}_S(M_S, M'_S). \]

Notation 2.1. We will denote $M^* = \text{Hom}_\mathcal{R}(\mathbb{M}, \mathcal{R})$.

Notation 2.2. For simplicity, given a (covariant) functor (from the category of commutative $R$-algebras to the category of sets), $\mathcal{X}$, we will sometimes use $x \in \mathcal{X}$ to denote $x \in \mathcal{X}(S)$. Given $x \in \mathcal{X}(S)$ and a morphism of commutative $R$-algebras $S \to S'$, we will still denote by $x$ its image by the morphism $\mathcal{X}(S) \to \mathcal{X}(S')$.

Remark 2.3. Tensor products, direct limits, inverse limits, etc., of functors of $\mathcal{R}$-modules and kernels, cokernels, images, etc., of morphisms of functors of $\mathcal{R}$-modules are regarded in the category of functors of $\mathcal{R}$-modules.

We have that
\[
(M \otimes_R M')(S) = M(S) \otimes_S M'(S), \quad (\text{Ker } f)(S) = \text{Ker } f_S, \quad (\text{Coker } f)(S) = \text{Coker } f_S,
\]
\[
(\text{Im } f)(S) = \text{Im } f_S, \quad (\frac{\lim}{\rightarrow \, i \in I} M_i)(S) = \lim_{\rightarrow \, i \in I} (M_i(S)), \quad (\frac{\lim}{\rightarrow \, j \in J} M_j')(S) = \lim_{\rightarrow \, j \in J} (M_j'(S)).
\]
(where $I$ is an upward directed set and $J$ a downward directed set).

\[\text{In this paper, we will only consider well-defined functors } \prod_{\text{Hom}_\mathcal{R}(\mathbb{M}, \mathbb{M}')} S, \text{ that is to say, functors such that } \text{Hom}_S(M_S, M'_S) \text{ is a set, for any } S.\]
Proposition 2.4. Let \( M \) and \( N \) be two functors of \( R \)-modules. Then,

\[
\text{Hom}_R(M, N^*) = \text{Hom}_R(N, M^*)
\]

\((f \in \text{Hom}_R(M, N^*)) \text{ is mapped to } \tilde{f}, \text{ which is defined as follows: } \tilde{f}(n)(m) := f(m)(n), \text{ for any } m \in M \text{ and } n \in N\).

Proof. \( \text{Hom}_R(M, N^*) = \text{Hom}_R(M \otimes_R N, R) = \text{Hom}_R(N, M^*) \). \( \square \)

Definition 2.5. Given an \( R \)-module \( M \) (resp. \( N, V, \) etc.), \( M \) (resp. \( N, V, \) etc.) will denote the functor of \( R \)-modules defined by \( \mathcal{M}(S) := M \otimes_R S \) (resp. \( \mathcal{N}(S) := N \otimes_R S, \) etc.). \( M \) will be called a quasi-coherent \( R \)-module (associated with \( M \)). If \( M \) is a finitely generated \( R \)-module we will say that \( M \) is a coherent \( R \)-module. Finally, \( M^* \) will be called a module scheme.

Proposition 2.6. [1, 1.3] For every functor of \( R \)-modules \( M \) and every \( R \)-module \( M \), it is satisfied that

\[
\text{Hom}_R(M, M) = \text{Hom}_R(M, M(R)), \ f \mapsto f_R.
\]

The functors \( M \leadsto \mathcal{M}, \mathcal{M} \leadsto \mathcal{M}(R) = M \) establish an equivalence between the category of \( R \)-modules and the category of quasi-coherent \( R \)-modules ([1, 1.12]). In particular, \( \text{Hom}_R(M, M') = \text{Hom}_R(M, M') \). For any pair of \( R \)-modules \( M \) and \( N \), the quasi-coherent module associated with \( M \otimes_R N \) is \( \mathcal{M} \otimes_R \mathcal{N} \). \( \mathcal{M}|_S \) is the quasi-coherent \( S \)-module associated with \( M \otimes_R S \).

Notations 2.7. Given an \( R \)-module \( N \), let \( R \oplus N \) be the commutative \( R \)-algebra defined by

\[(r, n) \cdot (r', n') := (rr', r'n + r'n), \ \forall r, r' \in R, \text{ and } \forall n, n' \in N.\]

Consider the morphism of \( R \)-algebras \( \pi_1 : R \oplus N \to R, \pi_1(r, n) := r \). Given a functor of \( R \)-modules \( F \), define \( F(N) := \text{Ker}([\pi_1]: F(R \oplus N) \to F(R)) \). For example, \( \mathcal{M}^*(N) = \text{Hom}_R(M, N) \) and \( \mathcal{M}(N) = M \otimes_R N \).

Let \( S \) be a commutative \( R \)-algebra and let \( S[\epsilon] := S \oplus S \cdot \epsilon \) be the commutative \( S \)-algebra, where \( \epsilon^2 = 0 \). Let \( S[\epsilon, \epsilon'] := S \oplus S \cdot \epsilon \oplus S \cdot \epsilon' \) be the commutative \( R \)-algebra, where \( \epsilon^2 = \epsilon \cdot \epsilon' = \epsilon' \epsilon = 0 \).

Lemma 2.8. Let \( F \) be a functor of \( R \)-modules. Assume the natural morphism of \( S[\epsilon, \epsilon'] \)-modules

\[
F(S) \oplus F(S) \cdot \epsilon \oplus F(S) \cdot \epsilon' \to F(S[\epsilon, \epsilon'])
\]

is an isomorphism for any commutative \( R \)-algebra \( S \). Then,

\[
\text{Hom}_R(N^*, F) = F(N).
\]

Proof. It holds that \( F(S) \oplus F(S) \cdot \epsilon = F(S[\epsilon]): \) the natural inclusion \( S[\epsilon] \to S[\epsilon, \epsilon'] \) has an obvious retraction, then the morphism \( F(S[\epsilon]) \to F(S[\epsilon, \epsilon']) \) is injective and \( F(S[\epsilon]) \cap \epsilon' \cdot F(S[\epsilon, \epsilon']) = 0 \). Then, the natural morphism of \( S[\epsilon] \)-modules \( F(S) \oplus F(S) \cdot \epsilon \to F(S[\epsilon]) \) is an isomorphism.

Let \( \phi : N^* \to F \) be a morphism of \( R \)-modules. Let \( I \in N^*(R \oplus N) \) be defined by \( I(n) := (0, n), \ I \in N^*(N) \), then \( \phi_{R \oplus N}(I) \in F(N) \). Given \( w \in N^*(S) \), consider...
the morphism $\tilde{w}: R \oplus N \to S[\epsilon]$, $\tilde{w}(r, n) := r + w(n) \cdot \epsilon$. We have the commutative diagram

$$
\begin{array}{ccc}
\mathcal{N}^*(S[\epsilon]) & \xrightarrow{\mathcal{F}(S[\epsilon])} & \mathcal{F}(S) \oplus \mathcal{F}(S) \cdot \epsilon \\
\downarrow & & \downarrow \\
\mathcal{N}^*(R \oplus N) & \xrightarrow{\mathcal{F}(\tilde{w})} & \mathcal{F}(R \oplus N)
\end{array}
\quad \mathcal{N}^*(R) = \mathcal{F}(R) \oplus \mathcal{F}(R) \cdot \epsilon
$$

Then, $\phi$ is determined by $\phi \circ \mathcal{N}(I) \in \mathcal{F}(N)$. Given $f \in \mathcal{F}(N)$ let us define $\phi$. Given $w \in \mathcal{N}^*(S)$, consider $\tilde{w}: R \oplus N \to S[\epsilon]$, $\tilde{w}(r, n) := r + w(n) \cdot \epsilon$ and the composite morphism $\mathcal{F}(R \oplus N) \to \mathcal{F}(S[\epsilon]) = \mathcal{F}(S) \oplus \mathcal{F}(S) \cdot \epsilon$, $f \mapsto \mathcal{F}(\tilde{w})(f) = 0 + f_w \cdot \epsilon$. Then, $\phi: \mathcal{N}^* \to \mathcal{F}$ is defined by $\phi(w) := f_w$. Let us only check that $\phi(w + w') = \phi(w) + \phi(w')$: Let $\pi_i: S[\epsilon, \epsilon'] \to S[\epsilon]$ for $i = 1, 2, 3$, be defined by $\pi_1(\epsilon) = \epsilon$ and $\pi_2(\epsilon') = 0$ and $\pi_2(\epsilon') = \epsilon$, and finally $\pi_3(\epsilon) = \epsilon$ and $\pi_3(\epsilon') = \epsilon$. Consider the morphism $v: R \oplus N \to S[\epsilon, \epsilon']$, $v(r, n) := r + w(n) \cdot \epsilon + w'(n) \cdot \epsilon'$. Then, $\mathcal{F}(v)(I) = f_w \cdot \epsilon + f_{w'} \cdot \epsilon'$, because $\mathcal{F}(\pi_1) \circ \mathcal{F}(v) = \mathcal{F}(\pi_1 \circ v) = \mathcal{F}(\tilde{v})$ and $\mathcal{F}(\pi_2) \circ \mathcal{F}(v) = \mathcal{F}(\pi_2 \circ v) = \mathcal{F}(\tilde{v})$. Finally, $(f_w + f_{w'}) \cdot \epsilon = \mathcal{F}(\pi_3)(\mathcal{F}(v)(I)) = \mathcal{F}(\tilde{v})(I) = f_w + f_{w'} \cdot \epsilon$, then $f_w + f_{w'} = f_{w + w'}$ and $\phi(w + w') = \phi(w) + \phi(w') = \phi(w + w')$. \\n
\[\square\]

**Proposition 2.9.** It holds that $\text{Hom}_R(\mathcal{N}^*, \text{lim}_i \mathcal{M}_i^*) = \text{lim}_i \text{Hom}_R(\mathcal{N}^*, \mathcal{M}_i^*)$.

**Proof.** By Lemma 2.8, $\text{Hom}_R(\mathcal{N}^*, \text{lim}_i \mathcal{M}_i^*) = (\text{lim}_i \mathcal{M}_i^*) (\mathcal{N}) = \text{lim}_i \text{Hom}_R(\mathcal{M}_i, \mathcal{N}) = \text{lim}_i \text{Hom}_R(\mathcal{N}^*, \mathcal{M}_i^*)$. \\n
\[\square\]

**Corollary 2.10.** [\cite{1, 8}] Let $M$ and $M'$ be $R$-modules. Then,

$$\text{Hom}_R(\mathcal{M}^*, \mathcal{M}') = \mathcal{M} \otimes_R \mathcal{M}'$$

**Proof.** $\text{Hom}_R(\mathcal{M}^*, \mathcal{M}') \cong \mathcal{M}'(M) = M \otimes_R \mathcal{M}'$. \\n
\[\square\]

If we make $\mathcal{M}' = R$ in the previous corollary, we obtain the following theorem.

**Theorem 2.11.** [\cite{3, II, §1, 2, 5}] \[\cite{1, 10}\] Let $M$ be an $R$-module. Then

$$\mathcal{M}^{**} = \mathcal{M}.$$  

**Definition 2.12.** Let $\mathcal{M}$ be a functor of $R$-modules. We will say that $\mathcal{M}^*$ is a dual functor. We will say that a functor of $R$-modules $\mathcal{M}$ is reflexive if $\mathcal{M} = \mathcal{M}^{**}$.

**Example 2.13.** Quasi-coherent modules are reflexive functors of $R$-modules.

**Notation 2.14.** Let $\mathcal{M}$ be an $R$-module. We will denote $\mathcal{M}(R)$ the quasi-coherent module associated with the $R$-module $\mathcal{M}(R)$, that is, $\mathcal{M}(R)(S) := \mathcal{M}(R) \otimes_R S$.

There exists a natural morphism $\mathcal{M}(R) \to \mathcal{M}$, $m \otimes s \mapsto s \cdot m$. Observe that

$$\text{Hom}_R(\mathcal{N}, \mathcal{M}) = \text{Hom}_R(\mathcal{N}, \mathcal{M}(R)) = \text{Hom}_R(\mathcal{N}, \mathcal{M}(R))$$

for any quasi-coherent module $\mathcal{N}$.

**Notation 2.15.** Let $i: R \to S$ be a ring homomorphism between commutative rings. Given a functor of $R$-modules, $\mathcal{M}$, let $i^* \mathcal{M}$ be the functor of $S$-modules defined by $(i^* \mathcal{M})(S') := \mathcal{M}(S')$. Given a functor of $S$-modules, $\mathcal{M}'$, let $i_* \mathcal{M}'$ be the functor of $R$-modules defined by $(i_* \mathcal{M}')(R') := \mathcal{M}(S \otimes_R R')$. 

3. Affine schemes of modules and module schemes

Let $M$ be an $R$-module. There exists an $R$-module $N$ such that $M = N^*$ iff $M$ is a projective $R$-module of finite type (see [2]). Observe that $N = M^*$. In other words, $M = M^*(R)^*$ iff $M$ is a projective $R$-module of finite type.

It is easy to prove that $M = M^*(R)^*$ iff $M \otimes_R N = \text{Hom}_R(M^*, N)$, for any $R$-module $N$.

**Definition 3.1.** A functor of $R$-modules is said to be an affine scheme of $R$-modules if it is isomorphic to the functor of points of an affine scheme.

Recall that the functor of points of Spec $A$, $(\text{Spec } A)^*$ is defined by

$$(\text{Spec } A)(S) := \text{Hom}_{R-\text{alg}}(A, S)$$

for any commutative $R$-algebra $S$.

**Example 3.2.** Module schemes are affine schemes of $R$-modules.

**Theorem 3.3.** Let $\mathcal{M}$ be an affine scheme of $R$-modules. Then, $\mathcal{M}^* = \mathcal{M}^*(R)$.

**Proof.** $\Rightarrow$ Let $\mathcal{M}$ be the functor of points of Spec $A$, i.e., $\mathcal{M}(S) = \text{Hom}_{R-\text{alg}}(A, S)$.

1. $G_m = \text{Spec } R[x, 1/x]$ (in fact $R$) acts on $\mathcal{M}$. Then, $G_m = \text{Spec } R[x, 1/x]$ (in fact, $R$) acts on $A$ and $A = \bigoplus_{n \in \mathbb{N}} A_n$, where $\lambda \cdot a_n = \lambda^n \cdot a_n$, for any $\lambda \in G_m$ and $a_n \in A_n$.

2. Let $\Delta: A \rightarrow A \otimes_R A$ be the coproduct morphism (which is obtained from the addition morphism $\mathbb{M} \times \mathbb{M} \rightarrow \mathbb{M}$). Then, $\Delta(A_n) \subseteq \bigoplus_{i=0}^n A_i \otimes A_{n-i}$, because $G_m$ acts linearly on $\mathbb{M}$.

3. $A_0 = R$, because Spec $A_0 = \mathcal{M}^G_m = \{0\}$, where $\mathcal{M}^G_m(S) := \{m' \in \mathcal{M}(S): x \cdot m' = m', \text{ for any } x \in G_m\}$.

4. $\Delta(w) = w \otimes 1 + 1 \otimes w$, for any $w \in A_1$: $\Delta(w) = w' \otimes 1 + 1 \otimes w'$ for some $w' \in A$, because $\mathcal{M}$ is commutative (and $A_0 = R$). Let $0: A \rightarrow A_0$ the morphism induced by the obvious morphism $\{0\} \rightarrow \mathcal{M}$. The composite morphism

$$A \xrightarrow{\Delta} A \otimes_R A \xrightarrow{1 \otimes 0} A \otimes_R A_0 = A$$

is the identity morphism, then $w = (1 \otimes 0)(\Delta(w)) = (1 \otimes 0)(w' \otimes 1 + 1 \otimes w') = w' + 0 = w'$.

5. $A_1 = \mathcal{M}^*(R)$:

$$\mathcal{M}^*(R) = \text{Hom}_R(\mathcal{M}, R) = \text{Hom}_{R-\text{alg}}(\mathcal{M}, R)^G_m = \text{Hom}_{R-\text{alg}}(R[x], A)^G_m = A_1$$

6. By change of rings, $R \rightarrow S$, we have that Spec$(A \otimes_R S) = (\text{Spec } A)|_S = \mathcal{M}|_S$. By 5., $A_1 \otimes_R S = \text{Hom}_S(\mathcal{M}|_S, S) = \mathcal{M}^*(S)$. Hence, $A_1 = \mathcal{M}^*$. \qed

Assume $R$ is a ring of characteristic 2. Let $\mathcal{M} := R$ be the obvious functor of Abelian groups. Consider $\mathcal{M}$ as a functor of $R$-modules as follows:

$$\lambda \cdot m := \lambda^2 \cdot m, \forall \lambda \in R \text{ and } m \in \mathcal{M}$$

$\mathcal{M}$ is an affine scheme of $R$-modules, but $\mathcal{M}$ is not a module scheme because $\mathcal{M}^*(R) = 0$.

**Corollary 3.4.** A reflexive functor of $R$-modules is an affine scheme of $R$-modules iff it is a module scheme. A quasi-coherent module $\mathcal{M}$ is an affine scheme of $R$-modules iff $M$ is a projective module of finite type.
Proof. Let $M$ be a reflexive affine scheme of $R$-modules. Then, $M = M^{**}$ and $M$ is a module scheme.

Now, if $M$ is an affine scheme of $R$-modules, then $M$ is a module scheme. $M$ is a projective module of finite type by [2].

If $M$ is a projective module of finite type, then $M$ is a module scheme by [2]. In particular, $M$ is an affine scheme of $R$-modules. □

Proposition 3.5. Let $N$ be a functor of $R$-modules and let $M$ be an $R$-module. Then,

$$\text{Hom}_R(N, M^*) = \text{Hom}_R(N^*(R)^*, M^*).$$

Proof. $\text{Hom}_R(N, M^*) = \text{Hom}_R(M, N^*) = \text{Hom}_R(M, N^*(R)) = \text{Hom}_R(M, N^*(R)^*) = \text{Hom}_R(N^*(R)^*, M^*)$. □

Proposition 3.6. Let $M$ be an affine scheme of $R$-modules, and $N$ a dual functor of $R$-modules. Then,

$$\text{Hom}_R(M, N) = \text{Hom}_R(M^*(R)^*, N).$$

Let $N'$ be a functor of $R$-modules. A morphism $f: N' \to N$ factors through a morphism from $N'$ to an affine scheme of $R$-modules iff $f$ factors through the morphism $N' \to N^*(R)^*$. Proof. Let us write $N = N^*$. Then,

$$\text{Hom}_R(M, N) = \text{Hom}_R(M^*(R)^*, N).$$

Now, it is easy to prove the last statement by Proposition 3.5. □

4. Characterization of flat modules

Lemma 4.1. Let $N$ and $M$ be $R$-modules. Given a morphism $f: N^* \to M$ of functors of $R$-modules, there exist a free module of finite type $L$ and morphisms $g: N^* \to L$ and $h: L \to M$, such that the diagram

$$\begin{array}{ccc}
N^* & \xrightarrow{f} & M \\
g \downarrow & & \downarrow h \\
L & \xrightarrow{g} & M
\end{array}$$

is commutative.

Proof. We have $f = \sum_{i=1}^r n_i \otimes m_i \in N \otimes M$ homomorphism $N^*, M$. Let $L = R^r$, $\{e_i\}$ the standard basis of $L$, $h(e_i) := m_i$ and $g := \sum_{i=1}^r n_i \otimes e_i \in N \otimes L$ homomorphism $N^*, L$. □

Note 4.2. In particular, every morphism $f: N^* \to M$ factors through the coherent module associated with a submodule of finite type of $M$ (for example, $\text{Im} h_R$).

Lemma 4.3. Let $N$ be a finitely presented $R$-module and let $M$ be a flat $R$-module. Every morphism $N \to M$ uniquely factors through $N \to N^*(R)^*$, that is,

$$\text{Hom}_R(N^*(R)^*, M) \cong \text{Hom}_R(N, M).$$
Proof. Let us recall that $N^* \otimes M = \text{Hom}_R(N, M)$:

If $N = R^n$ then it is obvious. $\text{Hom}_R(-, M)$ and $(-)^* \otimes_R M$ are contravariant left exact functors. Finally, $N$ is equal to the cokernel of a morphism between finitely generated free modules.

□

**Proposition 4.4.** [4, 6.6] Let $M$ be a flat $R$-module, $N$ a finitely presented $R$-module and $f : N \to M$ a morphisms of $R$-modules. Then, there exist a free module of finite type $L$ and morphisms $g : N \to L$, $h : L \to M$ such that the diagram

$$
\begin{array}{ccc}
N & \xrightarrow{f} & M \\
\downarrow{g} & & \downarrow{h} \\
L & & \\
\end{array}
$$

is commutative.

Proof. By Lemma 4.3 and Lemma 4.1, we have a commutative diagram

$$
\begin{array}{ccc}
N & \xrightarrow{f} & M \\
\downarrow{g} & & \downarrow{h} \\
N^*(R)^* & & L \\
\end{array}
$$

Any module is a direct limit of finitely presented modules. The Govorov-Lazard states that any flat module is a direct limit of free modules of finite type (see [4, A6.6]). The proof of this theorem is based on Proposition 4.4.

**Lemma 4.5.** If $M = \lim_{\leftarrow i} N_i^*$, then $M$ is flat and $M \otimes_R N = \lim_{\leftarrow i} \text{Hom}_R(N_i, N)$ for any $R$-module $N$.

Proof. $M \otimes_R S = \lim_{\leftarrow i} \text{Hom}_R(N_i, S)$ for any commutative $R$-algebra $S$. Given an $R$-module $N$, consider the $R$-algebra $S = R \oplus N$, where $(r, n) \cdot (r', n') := (rr', rn' + r'n)$. Then, $M \otimes_R (R \oplus N) = \lim_{\leftarrow i} \text{Hom}_R(N_i, R \oplus N)$. Hence, $M \otimes_R N = \lim_{\leftarrow i} \text{Hom}_R(N_i, N)$. Therefore, the functor $M \otimes -$ is left exact because the functor $\lim_{\leftarrow i} \text{Hom}_R(N_i, -)$ is left exact. Then, $M$ is a flat $R$-module.

□

**Theorem 4.6.** $\mathcal{M}$ is a direct limit of module schemes iff $M$ is a flat $R$-module.

Proof. $\Leftarrow$) $M$ is a direct limit of free modules of finite type, $M = \lim_{\leftarrow i} L_i$, by the Govorov-Lazard Theorem. Then, $\mathcal{M} = \lim_{\leftarrow i} (L_i^*)^*$ is a direct limit of module schemes.

$\Rightarrow$) This implication is a consequence of Lemma 4.5.
Theorem 4.7. $\mathcal{M}$ is a direct limit of affine schemes of $\mathcal{R}$-modules iff $\mathcal{M}$ is a flat $\mathcal{R}$-module.

Proof. $\Rightarrow$ Write $\mathcal{M} = \varprojlim_i \mathcal{N}_i$, where every $\mathcal{N}_i$ is an affine scheme of $\mathcal{R}$-modules.

Each morphism $\mathcal{N}_i \to \mathcal{M}$ uniquely factors through $\mathcal{N}_i^*(\mathcal{R})^*$, by Proposition 3.6. Then, we have the morphisms

$$\mathcal{M} = \varprojlim_i \mathcal{N}_i \to \varprojlim_i \mathcal{N}_i^*(\mathcal{R})^* \to \mathcal{M}$$

Hence, $\mathcal{M}$ is a direct summand of $\varprojlim_i \mathcal{N}_i^*(\mathcal{R})^*$. Let $\mathcal{N}$ be an $\mathcal{R}$-module. Consider the morphisms

$$M \otimes_\mathcal{R} N \overset{\text{4.3}}{\to} \text{Hom}_\mathcal{R}(\mathcal{N}^*, M) \hookrightarrow \text{Hom}_\mathcal{R}(\mathcal{N}^*, \varprojlim_i \mathcal{N}_i^*(\mathcal{R})^*) \overset{\text{4.10}}{\to} \varprojlim_i \text{Hom}_\mathcal{R}(\mathcal{N}_i^*(\mathcal{R}), N) = \lim_i \text{Hom}_\mathcal{R}(\mathcal{N}_i^*(\mathcal{R}), N).$$

Then, the functor $M \otimes_\mathcal{R} -$ from the category of $\mathcal{R}$-modules is left exact, and $\mathcal{M}$ is a flat $\mathcal{R}$-module.

$\Leftarrow$ $\mathcal{N}^n$ is an $\mathcal{R}$-module scheme. $\mathcal{M}$ is a direct limit of affine schemes of $\mathcal{R}$-modules by the Govorov-Lazard Theorem. \hfill $\square$

Theorem 4.8. Let $\mathcal{R}$ be a Noetherian ring. Let $\mathcal{M}$ be an $\mathcal{R}$-module and $\{M_i\}$ the set of the submodules of finite type of $\mathcal{M}$. $\mathcal{M}$ is a flat $\mathcal{R}$-module iff

$$\mathcal{M} = \varprojlim_i M_i^*(\mathcal{R}).$$

Proof. $\Rightarrow$ Every morphism $M_i \to \mathcal{M}$ uniquely factors through $M_i \to M_i^*(\mathcal{R})^*$, by 4.3. Every morphism $\mathcal{Y}^* \to \mathcal{M}$ factors through $M_j \to \mathcal{M}$, for some $j$, by 4.2. Then,

$$\mathcal{M} = \varprojlim_i M_i = \varprojlim_i M_i^*(\mathcal{R}).$$

$\Leftarrow$ It is a consequence of Theorem 4.7. \hfill $\square$

Corollary 4.9. Let $\mathcal{R}$ be a Noetherian ring. Let $\mathcal{M}$ be an $\mathcal{R}$-module and $\{M_i\}$ the set of submodules of $\mathcal{M}$ of finite type. $\mathcal{M}$ is a flat $\mathcal{R}$-module iff

$$\mathcal{M} \otimes_\mathcal{R} N = \varprojlim_i \text{Hom}_\mathcal{R}(M_i^*, N),$$

for every $\mathcal{R}$-module $N$.

Proof. $\Rightarrow$ It is a consequence of 4.8 and 4.5.

$\Leftarrow$ $\mathcal{M} \otimes_\mathcal{R} -$ is a left exact functor. \hfill $\square$

Let $X$ be a Noetherian scheme and $\mathcal{F}$ a quasi-coherent sheaf on $X$ (in the standard sense, see [6, II 5.]). Let us consider $\mathcal{F}$ as a functor from the category of quasicompact schemes over $X$ to the category of Abelian groups as follows: $\mathcal{F}(Y) := (f^*\mathcal{F})(Y)$, for any quasicompact scheme $Y$ over $X$, $f: Y \to X$. Given a coherent sheaf $\mathcal{G}$, we will say that $\mathcal{G}^* := \text{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{O}_X)$ is a module scheme. $\mathcal{F}$ and $\mathcal{G}^*$ are sheaves in the Zariski topos.
Theorem 4.10. Let $X$ be a Noetherian scheme and $\mathcal{F}$ a quasi-coherent module on $X$. $\mathcal{F}$ is a flat $O_X$-module iff $\check{\mathcal{F}}$ is a direct limit of module schemes.

Proof. $\Rightarrow$) $\mathcal{F}$ is equal to the direct limit of its coherent subsheaves, $\mathcal{F} = \varinjlim \mathcal{F}_i$, and given a coherent subsheaf $\mathcal{G}$ of $\mathcal{F}|_U$, there exists a coherent subsheaf $\mathcal{H} \subseteq \mathcal{F}$ such that $\mathcal{H}|_U = \mathcal{G}$ (see [4, Ex. II 5.15]). Consider the natural morphism $\check{\mathcal{F}}_i \to \check{\mathcal{F}}_i^*$, then the morphism
\[ \text{Hom}_{O_X}(\check{\mathcal{F}}_i^*, \check{\mathcal{F}}) \to \text{Hom}_{O_X}(\check{\mathcal{F}}_i^*, \check{\mathcal{F}}) \]
is a bijection, by Lemma 4.3. Finally, the morphism
\[ \varinjlim \check{\mathcal{F}}_i^* \to \check{\mathcal{F}} \]
is an isomorphism by Theorem 4.8.

$\Leftarrow$) Assume $\check{\mathcal{F}} = \varinjlim \mathcal{G}_i^*$. Let us restrict this equation to the open set $U = \text{Spec } R \subseteq X$. Let $M := \mathcal{F}(U)$ and $N_i := \mathcal{G}_i(U)$. Then, $M = \varprojlim N_i^*$. By 4.0 $M$ is a flat $R$-module. Then, $\mathcal{F}$ is flat.

5. Characterization of Mittag-Leffler modules

Definition 5.1. [9, Chap. 2 Def. 3.] An $R$-module $M$ is said to be a Mittag-Leffler module if $M$ is the direct limit of free finite $R$-modules $\{L_i^*\}$ such that the inverse system $\{L_i^*\}$ satisfies the usual Mittag-Leffler condition (that is, for each $i$ there exists $j \geq i$ such that for $k \geq j$ we have $\text{Im}(L_k^* \to L_i^*) = \text{Im}(L_j^* \to L_i^*)$).

Mittag-Leffler modules are flat modules because any direct limit of free modules is flat.

Proposition 5.2. Let $M$ be an $R$-module. Let $f: M^* \to N$ be a morphism of $R$-modules.

1. The dual morphism $f^*: N^* \to M$ is a monomorphism iff there is not a submodule $N' \subseteq N$ such that $f$ factors through $N' \to N$.

2. Consider the set of all the pairs $(N', g)$ such that $N' \subseteq N$ is a submodule and $g: M^* \to N'$ is a morphism such that the composite morphism $M^* \to N' \to N$ is $f$. The pair $(N', g)$ is minimal iff $g^*: N'^* \to M$ is a monomorphism.

Proof. 1. Let $0 \neq w \in \text{Ker } f^*(S) \subseteq \text{Hom}_R(N, S)$, that is, we have the commutative diagram

\[
\begin{array}{ccc}
M^* & \xrightarrow{f} & N \\
\downarrow{0} & \downarrow{w} & \\
S & \xrightarrow{w} & S
\end{array}
\]

Then $f$ factors through the quasi-coherent module associated to $\text{Ker } w$.

Reciprocally, if there exists a submodule $N' \subseteq N$ such that $f$ factors through $N' \to N$, let $S = R \oplus N/N'$ and let $w: N \to S$ be defined by $w(n) := (0, \bar{n})$, then $0 \neq w \in \text{Ker } f^*(S)$. 

2. It is an immediate consequence of 1. \[\square\]

**Proposition 5.3.** Let \(M\) be a flat \(R\)-module, \(N\) an \(R\)-module and \(f : M^* \to N\) a morphism of \(R\)-modules. Then,

1. If \(N' \subseteq N\) is a submodule such that \(f\) factors through the associated morphism \(N' \to N\), then \(f\) factors in a unique way.
2. The set of the submodules \(N'\) of \(N\) such that \(f\) factors through \(N' \to N\) is a downward directed set.

**Proof.**
1. The map \(\text{Hom}_R(M^*, N') = M \otimes_R N' \to M \otimes_R N = \text{Hom}_R(M^*, N)\) is injective, because \(M\) is a flat \(R\)-module.
2. Let \(N_1\) and \(N_2\) be two submodules of \(N\). Consider the morphisms \(N_1 \cap N_2 \to N_1 \oplus N_2, n \mapsto (n, n)\) and \(N_1 \oplus N_2 \to N, (n_1, n_2) \mapsto n_1 - n_2\). The sequence of morphisms
\[
0 \to N_1 \cap N_2 \to N_1 \oplus N_2 \to N
\]
is exact. Let \(\widetilde{N_1 \cap N_2}\) be the quasi-coherent module associated with \(N_1 \cap N_2\). Taking \(\text{Hom}_R(M^*, -)\) we obtain the exact sequence
\[
0 \longrightarrow M \otimes_R (N_1 \cap N_2) \longrightarrow M \otimes_R N_1 \oplus M \otimes_R N_2 \longrightarrow M \otimes_R N
\]
\[
\text{Hom}_R(M^*, N_1 \cap N_2) \longrightarrow \text{Hom}_R(M^*, N_1 \oplus N_2) \longrightarrow \text{Hom}_R(M^*, N)
\]
Then, \(f\) factors through \(N_1 \to N\) and \(N_2 \to N\) iff \(f\) factors through \(\widetilde{N_1 \cap N_2} \to N\).
\[\square\]

**Lemma 5.4.** Let \(\{N_i^*\}_i\) be a direct system of module schemes and assume \(\lim_i N_i^*\) is a reflexive \(R\)-module, then
\[
\text{Hom}_R(\lim_i N_i, N) = \lim_i \text{Hom}_R(N_i, N).
\]

**Proof.** Observe that \((\lim_i N_i)^* = (\lim_i N_i^*)^{**} = \lim_i N_i^*\). Now, \(\text{Hom}_R(\lim_i N_i, N) \xrightarrow{2.3} \lim_i \text{Hom}_R(N_i^*, N) \xrightarrow{2.3} \lim_i \text{Hom}_R(N_i, N)\).
\[\square\]

**Theorem 5.5.** Let \(M\) be an \(R\)-module. The following statements are equivalent

1. \(M\) is a Mittag-Leffler module.
2. \(M\) is equal to the direct limit of its submodule schemes.
3. Any morphism \(N^* \to M\) factors through a submodule scheme of \(M\).
4. For every \(R\)-module \(N\) and every morphism \(f : M^* \to N\), there exists a minimal submodule \(P \subseteq N\) such that \(f\) factors through \(P \to N\).

**Proof.**
1. \(\Rightarrow\) 2. Write \(M = \lim_i L_i\), where the inverse system \(\{L_i^*\}\) satisfies the Mittag-Leffler condition. Obviously, \(M\) is a flat \(R\)-module and \(M^* = \lim_i L_i^*\).
Let \(N_i := \text{Im}(L_k^* \to L_i^*)\), for \(k \gg i\). Obviously, the morphisms \(N_i \to N_j\) are epimorphisms. The morphism \(L_k^* \to N_i\) is an epimorphism for every \(k \gg i\), then
the morphism $N^*_i \to \mathcal{L}_k$ is a monomorphism for every $k >> i$. Taking direct limits we have morphisms
\[ \mathcal{M} \to \lim_{i \to} N^*_i \to \mathcal{M} \]
Then, $\mathcal{M} = \lim_{i \to} N^*_i$. Finally, any morphism $N^* \to \mathcal{M}$ factors through some $N^*_i$.

Proof. Observe that $\text{Hom}_R \mathcal{M} \to \mathcal{N}$ is a minimal submodule of $\mathcal{N}$ since it is a consequence of the equivalence of (1) and (3), in Theorem 5.5.

2. $\Rightarrow$ 3. It is a consequence of 2.9.

3. $\Rightarrow$ 4. Let $f: \mathcal{M} \to \mathcal{N}$ a morphisms of $\mathcal{R}$-modules. Consider the dual morphism $f^*: \mathcal{N}^* \to \mathcal{M}$. Let $i: \mathcal{N}^*_i \to \mathcal{M}$ a submodule scheme and $g: \mathcal{N}^* \to N^*$ a morphism of $\mathcal{R}$-modules such that $f^* = i \circ g$. Let $g^*: \mathcal{N}^* \to \mathcal{N}$ the dual morphism of $g$, and $P = \text{Im} g^*_R$. Consider the sequence of morphisms
\[ N^* \to P^* \to N^* \to \mathcal{M} \]
$P$ is a minimal submodule of $N$ such that $f$ factors through $P \to \mathcal{N}$, by Proposition 5.2.

4. $\Rightarrow$ 1. $M$ is a direct limit of finite free modules, $M = \lim_{i \to} L_i$, by the Govorov-Lazard Theorem. Consider the canonical morphisms $f_i: \mathcal{M}^* \to L_i^*$. Let $N_i$ be a minimal submodule of $L_i^*$ such that $f_i$ factors through a morphism $g_i: \mathcal{M}^* \to N_i$. Again, by 5.3, $g_i : L_i^* \to \mathcal{N}_i$. The morphism $g$ is an epimorphism, and for $k \geq j$ the composite morphism $L_k^* \to L_j^* \to N_i$ is an epimorphism. Then, $\text{Im}(L_k^* \to L_i^*) = N$ for any $k \geq j$.
\[ \square \]

Let us prove some well-known results on Mittag-Leffler modules (see [9] and [5]).

**Corollary 5.6.** Let $M$ be an $\mathcal{R}$-module. $M$ is a Mittag-Leffler module if and only if for every $\mathcal{R}$-module $N$ and every $x \in M \otimes_R N$, there exists a smallest submodule $P$ of $N$ such that $x \in M \otimes_R P$.

Proof. Observe that $\text{Hom}_R(\mathcal{M}^*,N) = M \otimes_R N$. Then, this corollary is an immediate consequence of the equivalence of (1) and (3), in Theorem 5.5.
\[ \square \]

**Corollary 5.7.** Let $M$ be a flat $\mathcal{R}$-module. $M$ is a Mittag-Leffler module if the natural morphism $M \otimes_R \prod_{i \in I} Q_i \to \prod_{i \in I}(M \otimes_R Q_i)$ is injective, for every set of $\mathcal{R}$-modules $\{Q_i\}_{i \in I}$.

Proof. $\Rightarrow$) Let $\{\mathcal{N}^*_j\}$ be the set of submodule schemes of $\mathcal{M}$. Then, $\mathcal{M} = \lim_{j \to} \mathcal{N}^*_j$.

Recall Notations 2.7. Obviously,
\[ M \otimes_R N = \mathcal{M}(N) = (\lim_{j \to} \mathcal{N}^*_j)(N) = (\lim_{j \to} \mathcal{N}^*_j)(N) = \lim_{j \to} \text{Hom}_R(N_j,N), \]
for any $\mathcal{R}$-module $N$, and the morphisms, $\text{Hom}_R(N_j,N) = \mathcal{N}^*_j(N) \to \mathcal{N}^*_k(N) = \text{Hom}_R(N_k,N)$ are injective, for every $k > j$. Then,
\[ M \otimes_R \prod_{i \in I} Q_i = \lim_{j \to} \text{Hom}_R(N_j, \prod_{i \in I} Q_i) = \lim_{j \to} \prod_{i \in I} \text{Hom}_R(N_j, Q_i) \]
\[ \to \lim_{j \to} \prod_{i \in I} \text{Hom}_R(N_j, Q_i) = \prod_{i \in I}(M \otimes_R Q_i) \]
LET $f: \mathcal{M}^* \to \mathcal{N}$ be a morphism of $\mathcal{R}$-modules. Let $\{N_i\}$ be the set of submodules of $N_i \subseteq N$, such that the morphism $f$ factors through $N_i \to \mathcal{N}$. Let $N' = \cap_i N_i$ and $N'_i = N/N_i$. We have to prove that $f$ factors through $N' \to \mathcal{N}$, by the equivalence of (1) and (4) in Theorem 5.5. The obvious sequence of morphisms of modules

$$0 \to N' \to N \to \prod_i N'_i$$

is exact. Then, the sequence of morphisms of modules

$$0 \to M \otimes_R N' \to M \otimes_R N \to M \otimes_R \prod_i N'_i$$

is exact. The natural morphism $M \otimes_R \prod_{i \in I} N'_i \to \prod_{i \in I} (M \otimes_R N'_i)$ is injective. Then, the sequence of morphisms of modules

$$0 \to \text{Hom}_R(\mathcal{M}^*, N') \to \text{Hom}_R(\mathcal{M}^*, \mathcal{N}) \to \prod_i \text{Hom}_R(\mathcal{M}^*, N'_i)$$

is exact, and $f$ factors through $N' \to \mathcal{N}$.

\[\square\]

An immediate consequence of Corollary 5.7 is the following corollary.

**Corollary 5.8.** Let $f: M_1 \to M_2$ be a universally injective morphism of $\mathcal{R}$-modules. If $M_2$ is a Mittag-Leffler module then $M_1$ is a Mittag-Leffler module. If $M_1$ and $\text{Coker} \ f$ are Mittag-Leffler modules then $M \text{ is a Mittag-Leffler module}.

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