An exact algorithm for the static pricing problem under discrete mixed logit demand

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Price differentiation is a common strategy in many markets. In this paper, we study a static multiproduct price optimization problem with demand given by a discrete mixed multinomial logit model. By considering a mixed logit model that includes customer specific variables and parameters in the utility specification, our pricing problem reflects well the discrete choice models used in practice. To solve this pricing problem, we design an efficient iterative optimization algorithm that asymptotically converges to the optimal solution. To this end, a linear optimization (LO) problem is formulated, based on the trust-region approach, to find a “good” feasible solution and approximate the problem from below. A convex optimization problem is designed using a convexification technique to approximate the optimization problem from above. Then, using a branching method we tighten the optimality gap. The effectiveness of our algorithm is illustrated on several cases, and compared against solvers and existing state-of-the-art methods in the literature.

Key words: pricing, static multi-product, mixed logit model, nonlinear optimization

1. Introduction
Offering different products at different prices to different customers is a common practice in many markets, including transportation, retail, and entertainment, among others. Classic examples include discount prices for seniors and children, business-, first-, and economy-class flight tickets, or first- and second-class railway tickets. With product and price differentiation, operators and retailers are able to get higher revenues by adapting their prices based on the price sensitivity of their customers. Basically, higher fares are offered to the ones who are willing to pay more.

Inferring customers’ willingness to pay (WTP) is a long-standing practice in applied economics (Hensher et al. 2005). Discrete-choice modeling (DCM) has established itself as an important and widely-used methodology for extracting valuations such as willingness to pay (Hess et al. 2018). Researchers have used these disaggregate demand models for more
than 40 years, from the pioneering work of McFadden and Zarembka (1974) to more recent studies on WTP for self-driving vehicles (Daziano et al. 2017) or willingness to travel with green modes in the context of shared mobility (Li and Kamargianni 2019).

Formulating pricing policies based on such disaggregate demand representations allows to better account for the heterogeneity of the population of interest, where different customers have different tastes and preferences. Even more importantly, it better reflects the supply-demand interactions by capturing the tradeoff between the operator’s objective of maximizing the expected revenue and the customer objective of maximizing the expected utility (Sumida et al. 2019).

Despite a more comprehensive representation, including discrete choice models within pricing problems increases the computational complexity because the choice probabilities are nonlinear. As a result, the expected revenue is highly nonlinear in the prices of the products, and customary used nonlinear algorithms may get terminated at a local optimum.

Due to the importance of the problem, the operations research and management science communities put remarkable efforts into analyzing it. Hanson and Martin (1996) pioneer this research by showing that the expected revenue function is not concave in prices, even for the simple multinomial logit (MNL) model. Subsequent authors have demonstrated that, under uniform price sensitivities across all products, the expected revenue function is concave in the choice probability vector (Song and Xue 2007, Dong et al. 2009, Zhang and Lu 2013). Li and Huh (2011) show that this concavity result also holds under asymmetric price-sensitivities, not only for the MNL model, but also for the nested logit (NL) model that generalizes the MNL model by grouping product alternatives into different nests based on their degree of substitution (McFadden 1977).

Parallel to these works, several authors have shown that under restrictive conditions on the degree of asymmetry in the price sensitivity parameters, unique price solutions exist for some logit models. This has been shown for the MNL model (e.g., Aydin and Ryan (2000), Hopp and Xu (2005), Maddah and Bish (2007), Aydin and Porteus (2008), Akçay et al. (2010)), the NL model (e.g., Aydin and Ryan (2000), Hopp and Xu (2005), Maddah and Bish (2007), Aydin and Porteus (2008), Akçay et al. (2010), Gallego and Wang (2014), Huh and Li (2015)), the paired combinatorial logit (PCL) model (Li and Webster 2017) and lately generalized to any generalized extreme value (GEV) model (Zhang et al. 2018). In this stream of research, a first-order condition is generally used to find optimal prices. It is worth noting that in some
of these studies and additional recent ones, (1) pricing decisions are optimized jointly with other decisions such as assortment or scheduling decisions (e.g., Du et al. (2016), Jalali et al. (2019), Bertsimas et al. (2020)), (2) decisions of multiple firms (or players) are studied, mainly from a non-cooperative game theory perspective (e.g., Li and Huh (2011), Aksoy-Pierson et al. (2013), Gallego and Wang (2014), Bortolomiol et al. (2021)), but also more recently, from a cooperative game theory perspective (Schlicher and Lurkin 2022).

Only a few papers (Gilbert et al. (2014), Li et al. (2019), and van de Geer and den Boer (2022)) consider pricing problems under a mixed logit (ML) model, a choice model that better accommodates customer heterogeneity by allowing some parameters to vary across customers. As shown by McFadden and Train (2000), under mild regularity conditions, the ML model can approximate choice probabilities of any discrete choice model derived from random utility maximization (RUM) assumption, making it a popular choice model. However, as explained by Li et al. (2019), the expected revenue function under the mixed logit model is not well-behaved, and the concavity property with respect to the choice probabilities breaks down, even for entirely symmetric price sensitivities across products and segments. Accordingly, the theoretical results as well as the solution methods developed for other logit models do not apply to the pricing problem with demand characterized by a mixed logit model.

In Gilbert et al. (2014), the authors consider a ML demand model within a revenue-maximizing network pricing problem whose objective is to improve the performance of a congested network through the selection of appropriate tolls. The price sensitivity parameter is distributed across the population according to a continuous random variable. The authors rely on a tractable approximation of the ML pricing problem. Their approach consists of two phases. They first solve optimally a mixed integer program that approximates the original problem by assuming a simpler distribution for the price sensitivity parameter. The optimal solution of this program is then considered as the starting solution of an ascent algorithm that is used to solve a differentiable optimization problem that better approximates the original ML pricing problem.

In Li et al. (2019), the authors assume that the market is divided into a finite number of market segments, with product demand in each segment governed by the multinomial logit model. The problem under investigation is therefore a price optimization problem with demand given by a discrete ML model. To solve this problem, the authors propose
two concave maximization problems that work as lower and upper bounds for the objective value of the revenue function, under some conditions. Then, they propose an algorithm that converges to a local optimum.

In van de Geer and den Boer (2022), the authors also assume that the market is divided into a finite number of market segments and that customers’ intrinsic product valuations are both product and segment dependent. However, the customers’ price sensitivity parameters are product dependent only, and thus identical for all customer segments. This assumption was critical for them to develop a scalable algorithm that quickly converges to an optimal price of products.

Our pricing problem also considers the discrete setting of the ML model and therefore includes customer specific variables in the utility specifications. However, unlike former contributions that either consider local optimality (Hanson and Martin 1996) or impose restrictive conditions on price sensitivity parameters (Li et al. (2019) and van de Geer and den Boer (2022)) to have global optimality, our pricing problem can handle customers’ price sensitivity parameters that differ among products and customer segments. Considering that customers have different willingness to pay (WTP) is more realistic, but it comes at the price of additional computational complexity.

To solve our pricing problem, we follow an approach similar to van de Geer and den Boer (2022) by designing an exact iterative optimization algorithm that asymptotically converges to a global optimal solution. More specifically, we develop a method to find “good” solutions, design an approach to check the quality of the obtained solution, and branch to make sure the solution is optimal. To this end, a linear optimization (LO) problem is formulated, based on the trust-region approach, to find a “good” feasible solution and approximate the problem from below. A convex optimization problem is designed using a convexification technique as well as the McCormick relaxation (McCormick 1976) to approximate the optimization problem from above. Then, we develop a branching method to tighten the optimality gap and show that the algorithm converges to the optimal solution asymptotically. The effectiveness of this algorithm is illustrated on several instances, including a parking services pricing case for which the demand model comes from a published, non-trivial parking choice model.

The remaining sections are organized as follows. Section 2 further defines the problem under consideration. Section 3 presents our global algorithm, while Section 4 shows the results of our numerical experiments. The final section concludes our paper.
2. Problem description

In this paper, we are interested in solving a static multi-product pricing problem under a discrete ML model. Static pricing involves the simultaneous pricing of multiple products, where a fixed price is set for each product (Soon 2011). In our setting, we assume that a single seller must decide at what price to offer each product from a finite set of alternatives (also known as product assortment). On the demand side, we assume that customers choose among the products according to a consumer choice model. The demand for each product is thus the result of the customer purchase choice of \( N \) customers. The purchase choice is captured by a discrete choice model, that predicts the customer choice from a finite set of discrete alternatives (Ben-Akiva and Bierlaire 2003).

Let \( N \) represent the set of \( N \) customers and let \( C \) indicate the set of \( C \) available alternatives, among which \( I \) alternatives are offered by the seller. We denote by \( I \) the offered alternatives. So, we assume \( I \subseteq C \), and \( C \setminus I \) is the set of other alternatives not offered by the seller, such as competitive products or the standard non-purchase alternative. For each customer \( n \in N \) and alternative \( i \in C \), the utility function \( U_{in} \) is a function of the socio-economic characteristics of the customers and/or the attributes of the alternatives. According to Random Utility Maximization (RUM) theory (Manski 1977), \( U_{in} \) can be decomposed into a systematic component \( V_{in}(\beta) \), which includes all attributes observed by the decision maker, \( \beta \), and a random term \( \varepsilon_{in} \), which captures the uncertainties caused by unobserved attributes and unobserved taste variations:

\[
U_{in} = V_{in}(\beta) + \varepsilon_{in} = \beta_{in}^p p_i + q_{in}(\beta^q) + \varepsilon_{in},
\]  

(1)

where \( p_i \) is the price of the alternative \( i \), \( \beta_{in}^p \) is the willingness to pay of customer \( n \) for alternative \( i \), and \( q_{in}(\beta^q) \) is the exogenous part of the utility, obtained by adding all observed product attributes other than price, weighted based on customers’ preferences. Note that for the alternatives that are not offered by the seller, the price is assumed to be fixed and given, i.e., \( p_i = \bar{p}_i, \forall i \in C \setminus I \).

The resulting discrete choice model is, therefore, naturally probabilistic. The probability that customer \( n \) chooses alternative \( i \) is defined as

\[
P_{in} = \Pr \left[ V_{in}(\beta) + \varepsilon_{in} = \max_{j \in I} \{ V_{jn}(\beta) + \varepsilon_{jn} \} \right].
\]
The optimal expected revenues of the seller, obtained from the sales of all product within the set \( \mathcal{I} \), is then given by:

\[
\max_{p \in \mathbb{R}^C} \sum_{i \in \mathcal{I}} \sum_{n \in \mathcal{N}} p_i P_{in},
\]

s.t. \( P_{in} = \Pr [V_{in}(\beta) + \varepsilon_{in} \geq V_{jn}(\beta) + \varepsilon_{jn}, \forall j \in \mathcal{C}], \forall i \in \mathcal{C}, \forall n \in \mathcal{N}, \)

\[
V_{in}(\beta) = \beta_{in} p_i + q_{in}(\beta'), \quad \forall i \in \mathcal{C}, \forall n \in \mathcal{N},
\]

\[
p_i = \bar{p}_i, \quad \forall i \in \mathcal{C} \setminus \mathcal{I},
\]

\[
0 \leq p_i \leq p_i^u, \quad \forall i \in \mathcal{I},
\]

where \( p^u \in \mathbb{R}^I \) is a vector containing upper bounds on the prices of products of the seller and \( \bar{p}_i \) is the given price of the exogenous alternative \( i \in \mathcal{C} \setminus \mathcal{I} \). The most commonly used discrete choice models, the multinomial logit (MNL) model, is built upon the assumption of independent and identically extreme value distributed error terms (Manski 1977), that is \( \varepsilon_{in} \overset{i.i.d.}{\sim} EV(0, 1) \). Under this assumption, the probability for customer \( n \) to select choice alternative \( i \) is given by

\[
P_{in} = \frac{e^{V_{in}(\beta)}}{\sum_{j \in \mathcal{C}} e^{V_{jn}(\beta)}}. \tag{3}
\]

Mixed logit probabilities are the weighted sum of these standard logit probabilities over a density of parameters (Train 2003). The choice probabilities can then be expressed as

\[
P_{in} = \int \frac{e^{V_{in}(\beta)}}{\sum_{j \in \mathcal{C}} e^{V_{jn}(\beta)}} d\nu_{\beta}, \tag{4}
\]

where \( \nu_{\beta} \) is a multivariate probability measure.

In this paper, we assume that the probability measure \( \nu_{\beta} \) is discrete and can be dependent on customers. In other words, \( \beta \) can take only \( L \) distinct values \( b_1, b_2, ..., b_L \), probability of which differs per customer. So, we have the following logit choice probability:

\[
P_{in} = \sum_{\ell=1}^{L} w_{in} \frac{e^{V_{in}(b_\ell)}}{\sum_{j \in \mathcal{C}} e^{V_{jn}(b_\ell)}}, \tag{5}
\]

where \( w_{in} \) is the probability that \( \beta^p = b_\ell^p \) for customer \( n \).

We are thus interested in solving the following nonlinear maximization problem:
\[
\begin{align*}
\max_{p \in \mathbb{R}^C} & \quad \sum_{i \in I} \sum_{n \in \mathcal{N}} p_i P_{in}, \\
\text{s.t.} & \quad P_{in} = \sum_{\ell = 1}^L w_{in} \frac{e^{V_{in}(b_{\ell})}}{\sum_{j \in C} e^{V_{jn}(b_{\ell})}}, \quad \forall i \in \mathcal{C}, \forall n \in \mathcal{N}, \\
V_{in}(\beta) & = \beta^p_{in} p_i + q_{in}(\beta^q), \quad \forall i \in \mathcal{C}, \forall n \in \mathcal{N}, \\
p_i = \bar{p}_i, \quad & \forall i \in \mathcal{C} \setminus \mathcal{I}, \\
0 & \leq p_i \leq p^u, \quad \forall i \in \mathcal{I},
\end{align*}
\]

(6)

It is worth noting that in Li et al. (2019) and van de Geer and den Boer (2022), the closest related works, the probability measure \(\nu_{\beta}\) is also assumed to be discrete. In Li et al. (2019), there is no customer specific variables included in their utility specifications. As a result, the choice probability is the same for all customers (i.e., \(P_{in} = P_i, \forall n \in \mathcal{N}\)) and the objective function becomes \(\max_{p \in \mathbb{R}^I} \sum_{i \in I} P_i P_i\). In van de Geer and den Boer (2022), only the exogenous part of the utility is linked to the customers. So, unlike van de Geer and den Boer (2022), our pricing problem handles customer heterogeneity also in terms of price sensitivity parameters.

3. Methodology

In this section, we introduce a new efficient optimization algorithm for solving Problem (6). The proposed algorithm is a global optimizer, meaning that it asymptotically converges to the optimal solution. This is done by designing a method to find a “good” feasible solution, which provide a lower bound, as well as a method to check the quality of the obtained solution, which provides an upper bound.

Let us reformulate the optimization problem (6) to

\[
\begin{align*}
\text{opt} = & \max_{p \in \mathbb{R}^C} f(p) \\
\text{s.t.} & \quad Ap \geq b, \quad p \geq 0,
\end{align*}
\]

(7)

where \(f : \mathbb{R}^I \to \mathbb{R}\) is \(\sum_{i \in I} \sum_{n \in \mathcal{N}} \sum_{\ell = 1}^L \frac{w_{in} P_i}{f_{in}(p)}\), \(f_{in} : \mathbb{R}^I \to \mathbb{R}_+\) is \(\sum_{j \in \mathcal{C}} e^{V_{jn}(b_{\ell}) - V_{in}(b_{\ell})}\), for any \(i \in \mathcal{I}, n \in \mathcal{N}, \) and \(\ell = 1, ..., L\), \(\mathbb{R}_+\) is the set of positive real numbers, and the feasible region is a polytope defined by the matrix \(A \in \mathbb{R}^{m \times C}\) and the vector \(b \in \mathbb{R}^m\).

3.1. Designing a method to construct lower bounds

To construct the lower bounds, we use a trust-region method (Conn et al. 2000), where solutions are obtained iteratively in the neighborhood of the previous feasible solution. A
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A typical way of finding a better solution is by approximating the objective function with a quadratic function and solving the following optimization problem in the \( k \)th iteration:

\[
\max_{p \in \mathbb{R}^C} \frac{1}{2} p^T H_k p + g_k^T p \\
\text{s.t. } \|p - p^k\|_2 \leq r_k \\
Ap \geq b, \quad p \geq 0,
\]

where \( H_k \in \mathbb{R}^{C \times C} \) is the Hessian matrix and \( g_k \in \mathbb{R}^C \) is the gradient vector of the objective function at the feasible solution \( p^k \) obtained in the \((k - 1)\)st iteration, \( r_k \) is the radius of the neighborhood, and where \( \| \cdot \|_2 \) is the Euclidean norm. The issue is that the objective function of (7) is neither convex nor concave, and hence (8) might be a nonconcave quadratic optimization problem, known to belong to the class of NP-hard problems (Pardalos and Vavasis 1991). To avoid this issue, we use the linear approximation of the objective function in each iteration and use the following optimization problem:

\[
\max_{p \in \mathbb{R}^C} g_k^T p \\
\text{s.t. } \|p - p^k\|_1 \leq r_k \\
Ap \geq b, \quad p \geq 0,
\]

where \( \| \cdot \|_1 \) is the \( \ell_1 \)-norm.

Algorithm 1 provides the steps taken to find a “good” feasible solution using (9). As one can see, (9) is a linear optimization problem; hence optimal solutions are in the boundary points of its feasible region. So, the algorithm starts with searching for a good solution in the boundary of the neighborhood of the initial solution with a radius 1. It continues the search unless it does not reach a point with improvement in the objective function. Then, the radius of the neighborhood gets decreased with the hope of finding a better solution (in the numerical results, we set the decreasing scale to 10; i.e., we multiply the radius with 0.1). The algorithm gets terminated when the improvement in the last two iterations is less than a given tolerance error \( \theta \), hence a local optimum. We emphasize that we chose the trust region approach as it is known to have an extremely fast convergence rate to a local optimum (Higham 1999). However, any other method to efficiently find a “good” solution works.
Algorithm 1 Steps to obtain a “good” feasible solution using (9)

1: Input: $A$, $b$, $r^0$, $\theta$, and the gradient of $f(p)$
2: select a random feasible solution $p^0$
3: $f^1 := +\infty$, $r^0 := 1$, $k = 0$
4: while $|f^1 - f(p^k)| > \theta$, do
5: find $p^{k+1}$ by solving (9) with radius $r^0$
6: $\bar{p}^0 \leftarrow p^k$, $\bar{p}^1 \leftarrow p^{k+1}$, $f^0 \leftarrow f(\bar{p}^0)$, $f^1 \leftarrow f(\bar{p}^1)$
7: while $f^1 > f^0$ do
8: $\bar{p}^0 \leftarrow \bar{p}^1$, $f^0 \leftarrow f^1$
9: find $p^1$ by solving (9) with initial point $\bar{p}^0$ and radius $r^0$
10: $f^1 \leftarrow f(\bar{p}^1)$, $r^0 \leftarrow 1$
11: $r^0 \leftarrow \frac{r^0}{10}$
12: $p^{k+1} \leftarrow \bar{p}^1$, increase $k$ by 1
13: return $p^k$

3.2. Designing a method to construct upper bounds

In this section, we explore the properties of the optimization problem (7) and use them to develop an overestimator to construct an upper bound on the objective value of the problem. To this end, we first reformulate (7) as the following optimization problem:

$$
\max_{p \in \mathbb{R}^C} \sum_{i \in \mathcal{I}} \sum_{n \in \mathcal{N}} \sum_{\ell = 1}^{L} w_{\ell n} p_{i} \tau_{i n \ell} \\
\text{s.t. } f_{in\ell}(p) \tau_{i n \ell} \leq 1, \quad \forall i \in \mathcal{I}, n \in \mathcal{N}, \ell = 1, \ldots, L, \quad (10)
$$

$$
A p \geq b, \\
\tau_{i n \ell}, p_{i} \geq 0, \quad \forall i \in \mathcal{I}, n \in \mathcal{N}, \ell = 1, \ldots, L.
$$

It is clear that (7) and (10) are equivalent as $f_{in\ell}(p)$ is a positive function when $p \geq 0$, for any $i \in \mathcal{I}$, $n \in \mathcal{N}$, and $\ell = 1, \ldots, L$. Inspired by Zhen et al. (2021) and the fact that $\frac{1}{f_{in\ell}(p)}$ and hence $\tau_{in\ell}$ are positive, we can introduce a new variable $W_{ijn\ell} = p_{j} \tau_{i n \ell}$ and rewrite (10) as

$$
\max_{p \in \mathbb{R}^C} \sum_{i \in \mathcal{I}} \sum_{n \in \mathcal{N}} \sum_{\ell = 1}^{L} w_{\ell n} W_{i n \ell} \\
W_{ijn\ell} \geq 1, \quad \forall i \in \mathcal{I}, n \in \mathcal{N}, \ell = 1, \ldots, L, 
$$

$$
A p \geq b, \\
\tau_{i n \ell}, p_{i} \geq 0, \quad \forall i \in \mathcal{I}, n \in \mathcal{N}, \ell = 1, \ldots, L.
$$
\begin{align}
\text{s.t. } & f_{i\ell}(\frac{W_{i\ell}}{\tau_{i\ell}})\tau_{i\ell} \leq 1, & \forall i \in \mathcal{I}, n \in \mathcal{N}, \ell = 1, \ldots, L, & (11b) \\
& Ap \geq b, & & (11c) \\
& W_{ij\ell} = p_j \tau_{i\ell} & \forall i \in \mathcal{I}, j \in \mathcal{C}, n \in \mathcal{N}, \ell = 1, \ldots, L, & (11d) \\
& W_{ij\ell}, p_j \geq 0, & \forall i \in \mathcal{I}, j \in \mathcal{C}, n \in \mathcal{N}, \ell = 1, \ldots, L, & (11e) \\
& \tau_{i\ell} > 0 & \forall i \in \mathcal{I}, n \in \mathcal{N}, \ell = 1, \ldots, L, & (11f)
\end{align}

where \(W_{i\ell} \in \mathbb{R}^C\) is the vector containing \(W_{ijn\ell}\) for \(j \in \mathcal{C}\).

Problem (11) belongs to the class of biconvex optimization problems, as it contains functions that are convex in \(p\), and \((W, \tau)\) but not in \((p, W, \tau)\). To better see this, let us fix \(i \in \mathcal{I}, j \in \mathcal{C}, n \in \mathcal{N}, \ell = 1, \ldots, L\). The function \(f_{i\ell}(p)\) is convex as it is the summation of exponential functions with linear exponents in \(p\). As shown in Section 3.2.6 of Boyd and Vandenberghe (2004), \(f_{i\ell}(\frac{W_{i\ell}}{\tau_{i\ell}})\tau_{i\ell}\) is the perspective map of the convex function \(f_{i\ell}(W_{i\ell})\), which is convex in \((W_{i\ell}, \tau_{i\ell})\). Finally, the term \(p_j \tau_{i\ell}\) appearing in (11d) is bilinear. Therefore, (11) is a biconvex optimization problem.

The main challenge in solving (11) is on dealing with the bilinear terms in constraint (11d). In this section, we use McCormick relaxation (McCormick 1976) to construct a convex optimization problem that approximates (11) from above. More specifically, we obtain an upper bound by solving the following optimization problem:

\begin{align}
\max_{\{p,C, W \}} & \sum_{i \in \mathcal{I}} \sum_{n \in \mathcal{N}} \sum_{\ell = 1}^{L} \sum_{n \in \mathcal{N}} \sum_{\ell = 1}^{L} w_{i\ell \ell} W_{i\ell} \\
\text{s.t. } & Ap \geq b, & & (12a) \\
& f_{i\ell}(\frac{W_{i\ell}}{\tau_{i\ell}})\tau_{i\ell} \leq 1, & \forall i \in \mathcal{I}, n \in \mathcal{N}, \ell = 1, \ldots, L, & (12c) \\
& AW_{i\ell} \geq b_{i\ell}, & \forall i \in \mathcal{I}, & (12d) \\
& LB_{i\ell} (Ap - b) \leq AW_{i\ell} - b_{i\ell}, & \forall n \in \mathcal{N}, \ell = 1, \ldots, L, & (12e) \\
& AW_{i\ell} - b_{i\ell} \leq UB_{i\ell} (Ap - b), & \forall i \in \mathcal{I}, & (12f) \\
& W_{ij\ell} \geq LB_{i\ell} p_j + \tau_{i\ell} LB_{p_j} - LB_{i\ell} LB_{p_j}, & \forall n \in \mathcal{N}, \ell = 1, \ldots, L, & (12g) \\
& W_{ij\ell} \geq UB_{i\ell} p_j + \tau_{i\ell} UB_{p_j} - UB_{i\ell} UB_{p_j}, & \forall n \in \mathcal{N}, \ell = 1, \ldots, L, & (12h)
\end{align}
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\begin{align}
W_{ijn\ell} & \leq UB_{\tau_{int}} p_j + \tau_{int} LB_{p_j} - UB_{\tau_{int}} LB_{p_j}, & \forall i, j \in \mathcal{I}, \forall n \in \mathcal{N}, \ell = 1, \ldots, L, \quad (12i) \\
W_{ijn\ell} & \geq LB_{\tau_{int}} p_j + \tau_{int} UB_{p_j} - LB_{\tau_{int}} UB_{p_j}, & \forall i, j \in \mathcal{I}, \forall n \in \mathcal{N}, \ell = 1, \ldots, L, \quad (12j) \\
LB_{\tau} & \leq \tau \leq UB_{\tau}, \quad (12k) \\
LB_{p} & \leq p \leq UB_{p}, \quad (12l)
\end{align}

where $LB_{p}, UB_{p} \in \mathbb{R}^{C}$ are the vectors containing component-wise lower and upper bounds of $p$, and $LB_{\tau}, UB_{\tau} \in \mathbb{R}^{I \times N \times L}$ contain the positive component-wise lower and upper bounds of $\tau$, respectively. Problem (12) is constructed by convexification of the biconvex optimization problem equivalent to (11) including some redundant constraints. Constraint (12d) linearizes the redundant constraint $(Ap - b)\tau_{int} \geq 0$, for $i \in \mathcal{I}, n \in \mathcal{N}, \ell = 1, \ldots, L$. Constraints (12e) and (12f) are the constraints proposed by Zhen et al. (2022) to tighten the linear relaxation. Constraints (12g), (12h), (12i), and (12j) are obtained by using McCormick relaxation (McCormick 1976). Therefore, the objective value of (12) is an upper bound on the objective value of (11) and hence (10).

**Remark 1.** To construct (12), we need to compute $LB_{\tau}$ and $UB_{\tau}$. Since in the optimal solution $(\tau^*, p^*)$ of (10), we have $\tau_{int}^* = \max_{p \in \mathbb{R}^{C}} \frac{1}{f_{int}(p)}$, we can compute $\frac{1}{LB_{\tau_{int}}}$ by solving

\begin{align}
\max_{p \in \mathbb{R}^{C}} & \quad \frac{1}{f_{int}(p)} \\
\text{s.t.} & \quad Ap \geq b, p \geq 0.
\end{align}

Since $f_{int}(p)$ is a convex function, the above optimization problem is known to be \mathcal{NP}-hard (even checking local optimality for a quadratic objective function is \mathcal{NP}-hard (Pardalos and Schnitger 1988)). However, we are not looking for an exact optimal value but for an upper bound. By definition, we have $f_{int}(p) = \sum_{j \in \mathcal{C}} e^{V_{jn}(b_{\ell})-V_{in}(b_{\ell})}$. So, we know

\begin{align}
\max_{p \in \mathbb{R}^{C}} f_{int}(p) & \leq \sum_{j \in \mathcal{C}} \max_{p \in \mathbb{R}^{C}} e^{V_{jn}(b_{\ell})-V_{in}(b_{\ell})} = \left\{ \max_{p \in \mathbb{R}^{C}} V_{jn}(b_{\ell}) - V_{in}(b_{\ell}) \right\} \sum_{j \in \mathcal{C}} e^{s.t. Ap \geq b, p \geq 0,}
\end{align}

so, by solving some linear optimization problems, we can obtain a lower bound on $\tau$.

Also, to compute $\frac{1}{UB_{\tau_{int}}}$, we solve

\begin{align}
\min_{p \in \mathbb{R}^{C}} & \quad f_{int}(p) \\
\text{s.t.} & \quad Ap \geq b, p \geq 0,
\end{align}
which is a convex optimization problem; hence can be solved efficiently. □

Hitherto, we have provided a method to obtain a “good” feasible solution (Section 3.1) and an optimization problem to provide an upper bound on the objective value of (7) (Section 3.2). We emphasize that the $p$-part of the solution obtained from (12) can also be used to find a lower bound on (10). So, we also check the quality of such solutions.

In the next section, we show how a branching technique can tighten the gap between the lower and upper bounds.

### 3.3. Branching method

In this section, we use a typical branching method in continuous optimization. In this method, we first choose the branching variable $i$ and then split its feasible interval into two intervals (Misener and Floudas 2014, Floudas et al. 2005, Akrotirianakis and Floudas 2004). The index $i$ is usually the dimension where the feasible region has its largest length. Mathematically speaking,

$$i \in \arg \max \{ UB_{p_j} - LB_{p_j} : j \in C \}.$$ 

Then, the partitions are

$$S_1 = \left\{ p \in \mathbb{R}^C : LB_{p_j} \leq p_i \leq \frac{UB_{p_j} + LB_{p_j}}{2} \right\} \cap S, \quad S_2 = \left\{ p \in \mathbb{R}^C : \frac{UB_{p_j} + LB_{p_j}}{2} \leq p_i \leq LB_{p_j} \right\} \cap S.$$

As one can notice, such branching methods result in binary trees, as in each iteration, we only have two branches. Furthermore, in each iteration, the volume of the feasible region is getting smaller. Therefore, we can have the following convergence result.

**Theorem 1.** Let us denote by $S$ the feasible region of (7), and its partitions $S^m_k$, $k = 1, \ldots, K^m$, in the $m$th iteration. Also, let us denote by $B_t(p)$ a hyperball with the center $p$ and radius $t$. Let $\text{opt}^m$ be the upper bound obtained in the $m$th iteration. Set

$$t^m := \max_{k=1, \ldots, K^m} \left\{ \min \{ t : S^m_k \subseteq B_t(p), \text{ for some } p \in S^m_k \} \right\}.$$ 

In other words, $t^m$ is the maximum radius of the smallest hyperball among those covering the partitions $S^m_k$. If $t_m \to 0$ as $m$ tends to $+\infty$, then $\text{opt}^m \searrow \text{opt}$, meaning the sequence of upper bounds asymptotically converges to the optimal value of (7).
For each \( \ell \) is able to provide the user a collection of optimal solutions, which is important in pricing problems.

Proof. Without loss of generality, we assume that \( b_\ell \) can be written as \([b^p_\ell, b^q_\ell]\), where \( b^p_\ell, b^q_\ell \in \mathbb{R}^{C \times N} \), for any \( \ell = 1, ..., L \). Let us set \( M_{n\ell} = \max_{i,j \in C} \left\| \begin{bmatrix} b^p_{ijn\ell} \\ -b^t_{ijn\ell} \end{bmatrix} \right\|_2 \), for any \( n \in N \) and \( \ell = 1, ..., L \). Let us assume that \( p \in \mathcal{B}_t(p^0) \), for a given feasible \( p^0 \) and \( t > 0 \). We see from Remark 1 that

\[
\sum_{j \in C} \epsilon \min_{p \in \mathcal{B}_t(p^0)} \{ V_{jn}(b_\ell) - V_{in}(b_\ell) \} \leq \frac{1}{\tau_{int}} \sum_{j \in C} \epsilon \max_{p \in \mathcal{B}_t(p^0)} \{ V_{jn}(b_\ell) - V_{in}(b_\ell) \}.
\]

For each \( j \in C \), we know

\[
\max_{p \in \mathcal{B}_t(p^0)} \{ V_{jn}(b_\ell) - V_{in}(b_\ell) \} = q_{jn}(b^q_\ell) - q_{jn}(b^p_\ell) + \max_{p \in \mathcal{B}_t(p^0)} \left[ \begin{bmatrix} b^p_{jn\ell} \\ -b^t_{jn\ell} \end{bmatrix} \right]^T \left[ \begin{bmatrix} p_j \\ p_i \end{bmatrix} \right]
\]

\[
= q_{jn}(b^q_\ell) - q_{jn}(b^p_\ell) + \left[ \begin{bmatrix} b^p_{jn\ell} \\ -b^t_{jn\ell} \end{bmatrix} \right]^T \left[ \begin{bmatrix} p^0_j \\ p^0_i \end{bmatrix} \right] + t \left\| \begin{bmatrix} b^p_{jn\ell} \\ -b^t_{jn\ell} \end{bmatrix} \right\|_2
\]

\[
\leq q_{jn}(b^q_\ell) - q_{jn}(b^p_\ell) + \left[ \begin{bmatrix} b^p_{jn\ell} \\ -b^t_{jn\ell} \end{bmatrix} \right]^T \left[ \begin{bmatrix} p^0_j \\ p^0_i \end{bmatrix} \right] + t M_{n\ell},
\]

and, analogously,

\[
\min_{p \in \mathcal{B}_t(p^0)} \{ V_{jn}(b_\ell) - V_{in}(b_\ell) \} \geq q_{jn}(b^q_\ell) - q_{jn}(b^p_\ell) + \left[ \begin{bmatrix} b^p_{jn\ell} \\ -b^t_{jn\ell} \end{bmatrix} \right]^T \left[ \begin{bmatrix} p^0_j \\ p^0_i \end{bmatrix} \right] - t M_{n\ell}.
\]

So, we have

\[
\frac{e^{-t M_{n\ell}}}{f_{int}(p^0)} \leq \tau_{int} \leq \frac{e^{t M_{n\ell}}}{f_{int}(p^0)}.
\]

Therefore, for given \( i \in I \), \( n \in N \), and \( \ell = 1, ..., L \), when \( t \to 0 \), the differences of the lower and upper bounds of \( p_i \) and \( \tau_{int} \), tends to zero. Furthermore, \( opt^m \) is a non-increasing sequence. Hence, if \( t_m \to 0 \) as \( m \) tends to \( +\infty \), then \( opt^m \searrow opt \) (McCormick 1976).

The main assumption in Theorem 1 is that the radius of the minimum ball covering the partitions goes to 0. This assumption is satisfied by the partitioning method used in the algorithm. Therefore, the theorem guarantees that the algorithm achieves an \( \epsilon \)–approximation of the optimal value of (6) within a finite number of steps for any given \( \epsilon > 0 \). With this assumption, we can obtain multiple optimal solutions from the trust–region approach as well as the \( p \)–part of the solution of (12). So, the proposed algorithm is able to provide the user a collection of optimal solutions, which is important in pricing problems.

In the next section, we show how the algorithm efficiently works on several instances.
4. Numerical experiments

In this section, we discuss the effectiveness of our method in solving static pricing problems under a discrete mixed logit model. We refer to our method as CoBiT, as it is based on Covexification of a bi-convex optimization and trust-region algorithm. To have a better understanding of how CoBiT works, we start by presenting its logic on an illustrative example in Section 4.1. We then analyze CoBiT from an algorithmic perspective in Section 4.2 and Section 4.3 by comparing its performance to state-of-the-art nonlinear optimizers and the two closest algorithms available in the literature: the algorithm 2 of Li et al. (2019) and the algorithm of van de Geer and den Boer (2022). Finally, Section 4.4 investigates CoBiT from a problem definition perspective by comparing the results obtained with pricing problems that cannot handle price sensitivity parameters that differ among products and customer segments. To do so, we use an additional case study, inspired by a real parking choice model.

The numerical results of this work were carried out on a Laptop featuring 4 processors 2.60 GHz and 8.00 GB RAM running Julia 1.8.0 (Bezanson et al. 2017). We use JuMP 1.7.0 (Dunning et al. 2017) to pass Linear Optimization problems to Gurobi 9.5.1. We solve (12) by reformulating it in a conic form and using Mosek Optimization Tool 10.0. We compare the performance of our method with SCIP 7.0 (Gamrath et al. 2020), Couenne (Belotti 2009), BARON 21.1.13 (Sahinidis 1996), and ANTIGONE 37.1 (Misener and Floudas 2014). We have passed the problems to SCIP, BARON, and ANTIGONE using GAMS modeling language version 37.1.0. For Couenne, we use JuMP to pass the problem to the solver.

4.1. Illustration of CoBiT

To solve the optimization problem, CoBiT starts by partitioning the feasible region into two parts. The trust-region algorithm, Algorithm 1, is employed to obtain new local solutions in each of the rectangles (Figure 1a).

Then, CoBiT further partitions each of the rectangles (Figure 1b). After using the trust region in the top left square, CoBiT finds the same local solution as in the previous iteration. In the top right square, CoBiT finds a new solution (red square in the figure). For both two bottom squares, CoBiT detects that no effort is needed there as the upper bounds on these regions are, at most, as high as the objective value of the best-obtained solution.
So, in the next iteration, the top squares are further partitioned, and local solutions are obtained (blue dots in Figure 1c).

CoBiT continues the procedure until either the time limit is reached or the objective value of the best obtained feasible solution does not deviate by more than $10^{-5}$ from the upper bound obtained by the linearization.

![Figure 1](image)

**Figure 1** An illustration of first three iterations of CoBiT

4.2. CoBiT versus Li et al. (2019)

In this section, we apply CoBiT to the Intel Corporation case presented in Li et al. (2019).

4.2.1. The Intel Corporation case The authors assume that the Intel market is divided into a finite number of segments and that a multinomial logit model characterizes the product demand in each segment. Unlike us, the authors do not allow any customer specific variables in the utility functions, and their pricing optimization problem is therefore given by:

$$\max_{p \in \mathbb{R}^I} \sum_{i \in I} p_i P_i,$$

subject to

$$P_i = \sum_{k=1}^{K} w_k \sum_{j \in I} e^{q_{ik} + \beta_{ik} p_i} e^{q_{jk} + \beta_{jk} p_j}, \quad \forall i \in I. \tag{15}$$

The authors apply this model to Intel’s microprocessor stock-keeping units (SKUs) used in computer servers. Intel customers are categorized into $K = 7$ segments, corresponding to the seven groupings used by Intel’s sales division. The weights $w_k$, computed based on historical purchasing volumes, are provided in Table 1.

Sales data of the first three generations of products (13 SKUs) were used to parameterize the demand model, while the fourth generation of products, representing 3 SKUs ($I = \{1, 2, 3\}$), were used to test the demand model. The segment-specific coefficients are
Table 1 \( w_k \) values from Li et al. (2019)

| \( k \) | 1   | 2   | 3   | 4   | 5   | 6   | 7   |
|--------|-----|-----|-----|-----|-----|-----|-----|
| \( w_k \) | 0.0753 | 0.1126 | 0.1285 | 0.1180 | 0.0859 | 0.2842 | 0.1953 |

providing in Tables 2 and 3. We refer to Li et al. (2019), the original work, for more details on data fitting and parameterization. Prices were then optimized for the three SKUs of this fourth generation of products.

Table 2 \( q_{ik} \) values from Li et al. (2019)

| \( i \) | 1   | 2   | 3   | 4   | 5   | 6   | 7   |
|--------|-----|-----|-----|-----|-----|-----|-----|
| \( q_{ik} \) | -1.0334 | 3.2480 | -0.9336 | 1.7094 | 0.4187 | -0.8904 | -0.9804 |

Table 3 \( \beta_{ik} \) values from Li et al. (2019)

| \( i \) | 1   | 2   | 3   | 4   | 5   | 6   | 7   |
|--------|-----|-----|-----|-----|-----|-----|-----|
| \( \beta_{ik} \) | -0.00416 | -0.01840 | -0.00525 | -0.01165 | -0.01015 | -0.00325 | -0.00331 |

4.2.2. Numerical results Using the provided data, Algorithm 2 in Li et al. (2019) finds the local solution \([608.2695, 365.079, 1,209.09]\) within 0.079 seconds with the objective value of 362.3389. CoBiT can find this solution and prove optimality in 105.78 seconds. As explained in Li et al. (2019), their Algorithm 2 performs very well “when the degree of segment asymmetry is sufficiently small, because then the total profit is concave”, which is the case for this instance.

To see the effect of this algorithm in other cases and compare it fairly with CoBiT, we generate 10 instances randomly. More specifically, we choose \( q_{ik} \) from a uniform distribution over \([-5,5]\) and \( \beta_{ik} \) from a uniform distribution over \([-5,-0.025]\). Table 4 provides the result of applying CoBiT and Algorithm 2 in Li et al. (2019) on these 10 instances.
| Instance | Method             | Expected revenue | Solution time | Price  \\  | Price  \\  | Price  \\  |
|----------|-------------------|------------------|--------------|----------|----------|----------|
|          |                   |                  |              | $p_1$    | $p_2$    | $p_3$    |
| 1        | CoBiT             | 1.40             | 672.88       | 4.19     | 7.81     | 3.13     |
|          | Algorithm 2 (Li et al. 2019) | 1.40 | 2.32 | 4.23 | 8.04 | 2.92 |
| 2        | CoBiT             | 1.41             | 71.67        | 5.7      | 13.50    | 10.50    |
|          | Algorithm 2 (Li et al. 2019) | 1.37 | 5.77 | 5.71 | 10.46 | 4.61 |
| 3        | CoBiT             | 4.46             | 3.94         | 68.60    | 2.50     | 17.40    |
|          | Algorithm 2 (Li et al. 2019) | 0.554 | 3.11 | 1.0 | 1.36 | 0.85 |
| 4        | CoBiT             | 0.65             | 313.36       | 1.38     | 3.00     | 1.39     |
|          | Algorithm 2 (Li et al. 2019) | 0.65 | 1.95 | 1.38 | 3.11 | 1.39 |
| 5        | CoBiT             | 0.80             | 267.89       | 1.25     | 3.63     | 1.18     |
|          | Algorithm 2 (Li et al. 2019) | 0.80 | 2.57 | 0.32 | 0.91 | 0.29 |
| 6        | CoBiT             | 1.24             | 91.17        | 2.19     | 6.13     | 2.04     |
|          | Algorithm 2 (Li et al. 2019) | 1.24 | 1.59 | 2.19 | 6.16 | 2.04 |
| 7        | CoBiT             | 2.20             | 74.92        | 2.25     | 0.92     | 15.08    |
|          | Algorithm 2 (Li et al. 2019) | 2.20 | 1.75 | 2.26 | 0.92 | 15.07 |
| 8        | CoBiT             | 0.69             | 153.51       | 0.86     | 3.98     | 0.94     |
|          | Algorithm 2 (Li et al. 2019) | 0.69 | 1.64 | 0.84 | 4.33 | 0.94 |
| 9        | CoBiT             | 3.66             | 19.99        | 65.73    | 1.14     | 16.32    |
|          | Algorithm 2 (Li et al. 2019) | 3.66 | 1.77 | 65.57 | 1.13 | 16.29 |
| 10       | CoBiT             | 0.91             | 40.19        | 9.00     | 13.50    | 7.20     |
|          | Algorithm 2 (Li et al. 2019) | 0.72 | 2.12 | 1.29 | 2.16 | 1.54 |

Table 4: Comparison of the solution obtained by the local algorithm in Algorithm 2 (Li et al. 2019) and global solver CoBiT. The time inside parentheses is the time took for CoBiT to find an optimal solution.

As one can see, for 7 out of 10 randomly generated instances, Algorithm 2 in Li et al. (2019) finds an optimal solution of the problem. However, in the other 3 instances, the optimality gap can go up to 80%. From the comparison of the optimal prices in these 3 instances, we see the price vector reported by Algorithm 2 is far from the optimal prices. This shows that the local solutions may not necessarily be close to the optimal prices.

Furthermore, we can see that in 5 instances (2, 3, 7, 9, and 10), the solution time of CoBiT is much smaller than the others. The main reason behind it is the shape of the
expected revenue function. If the function is rather flat, then the upper bounds obtained by solving Problem (12) may not help much in deciding not to branch some nodes. In other words, when the expected revenue function is flat, then we need to have partitions that are really small to have a good convex approximation. However, in the 5 instances where CoBiT is fast, the expected revenue function is not very flat, and therefore, the number of nodes that are explored in each iteration of CoBiT is reasonable, which results in a more reasonable solution time.

4.3. CoBiT versus van de Geer and den Boer (2022)

In this section, we compare CoBiT with the algorithm developed by van de Geer and den Boer (2022). We follow the same steps in generating random instances. More specifically, we consider $I, N \in \{3, 4, 5\}$. We consider $C = I \cup \{0\}$, where 0 refers to the non-purchase alternative. Given the number of alternatives $I$ and the number of customers $N$, we randomly generate three instances from different seeds. For each instance, $q_{in} \sim U(-7.0, 7.0)$ and $\beta_i^p \sim U(-0.01, -0.001)$, where $U(a, b)$ denotes a uniform distribution in the range $(a, b)$. We have put a time limit of 7200 seconds on both methods. Regarding the quality, we set the relative error to be $\epsilon = 0.00001$ in both algorithms.

The results obtained by applying the algorithm proposed by Geer and Branch van de Geer and den Boer (2022), referred to as GB, as well as applying CoBiT, are summarized in Table 5. The table shows a comparison of the performance of the two algorithms in different instances. The ‘Instance’ rows provide the name of the instance, which is composed of three numbers. The first two numbers represent $I$ and $N$, respectively, and the last number indicates the index of the random instance in this class of instances, which can take values in the set \{0, 1, 2\}. To compare the solutions, we report the time (in seconds) taken by the algorithm to solve the instance. In case the time limit of 7200 seconds is reached, we report the optimality gap, calculated by $\frac{U - L}{L}$, where $U$ and $L$ denote the upper and lower bounds, respectively.

From this table, interesting observations can be made: (i) GB reports negative optimality gaps in 4 instances, meaning the upper bounds are lower than the lower bounds for these instances. Such errors occur in branching techniques due to numerical rounding. Therefore, we can safely assume that GB solves these instances to optimality. (ii) CoBiT can solve all the instances to optimality within the given time limit, but GB can only solve 16 out
| Instance | 330 | 331 | 332 | 430 | 431 |
|----------|-----|-----|-----|-----|-----|
| Method   | CoBiT | GB | CoBiT | GB | CoBiT | GB | CoBiT | GB |
| Time (S) | 32.78 | 7200 | 9.49 | 613.57 | 17.05 | 249.68 | 35.59 | 62.25 | 16.3 |
| Gap (%)  | 0 | 1.37 | 0 | 0 | 0 | 0 | 0 | 0 | -0.03 |

| Instance | 432 | 530 | 531 | 532 | 340 |
|----------|-----|-----|-----|-----|-----|
| Method   | CoBiT | GB | CoBiT | GB | CoBiT | GB | CoBiT | GB |
| Time (S) | 74.08 | 89.49 | 268.90 | 470.47 | 526.73 | 520.81 | 107.00 | 183.39 | 5.47 |
| Gap (%)  | 0 | 0 | 0 | 0 | -0.01 | 0 | 0.04 |

| Instance | 341 | 342 | 440 | 441 | 442 |
|----------|-----|-----|-----|-----|-----|
| Method   | CoBiT | GB | CoBiT | GB | CoBiT | GB | CoBiT | GB |
| Time (S) | 35.6 | 169.34 | 5.06 | 7200 | 13.68 | 6876.57 | 4.23 | 217.15 | 38.29 |
| Gap (%)  | 0 | -0.04 | 0 | 0.04 | 0 | 0 | 0 | 0 | -0.02 |

| Instance | 540 | 541 | 542 | 350 | 351 |
|----------|-----|-----|-----|-----|-----|
| Method   | CoBiT | GB | CoBiT | GB | CoBiT | GB | CoBiT | GB |
| Time (S) | 1020.46 | 2862.82 | 325.27 | 3303.98 | 371.05 | 1101.4 | 8.58 | 7200 | 15.58 |
| Gap (%)  | 0 | 0 | 0 | 0 | 1.81 | 0 | 14.18 |

| Instance | 352 | 450 | 451 | 452 | 550 |
|----------|-----|-----|-----|-----|-----|
| Method   | CoBiT | GB | CoBiT | GB | CoBiT | GB | CoBiT | GB |
| Time (S) | 66.00 | 7200 | 80.87 | 7200 | 25.06 | 7200 | 93.06 | 7200 | 1612.18 |
| Gap (%)  | 0 | 2.4 | 0 | 3.13 | 0 | 0 | 6.56 | 0 | 3.7 |

| Instance | 551 | 552 |
|----------|-----|-----|
| Method   | CoBiT | GB | CoBiT | GB |
| Time (S) | 77.88 | 7200 | 15.79 | 7200 |
| Gap (%)  | 0 | 2.46 | 0 | 7.81 |

Table 5  Comparison between the method proposed by van de Geer and den Boer (2022), denoted by GB, and CoBiT.
of 27 instances. Also, among the instances solved by both algorithms, CoBiT has a significantly lower time than GB, except the instance 531. (iii) The performance of GB varies significantly across different classes of instances. While GB may appear to perform well on instances with $N = 3$ based on the experiments conducted by van de Geer and den Boer (2022), it is important to note that the complexity of GB is $O(\epsilon^{-N I^{5.5+3N}})$. The authors of the study used $\epsilon = 0.01$ in their numerical experiments, while we consider $\epsilon = 0.00001$ to ensure that the obtained solutions are indeed optimal. (iv) As expected, CoBiT is quite sensitive to an increase in $I$ rather than an increase in $N$. We see that for $I = 3$, 4, and 5, the instances are solved in less than 67, 95, and 1615 seconds, respectively, while changes in $N$ do not necessarily change the solution time.

4.4. CoBiT versus restrictive pricing problems

In this section, we investigate CoBiT from a problem definition perspective by comparing pricing problems that differ in terms of the attributes and price sensitivity parameters included in the utility functions.

4.4.1. The parking choice model

The selection of this case study is motivated by the availability of a published, non-trivial, disaggregate parking choice model by Ibeas et al. (2014), which we can use to characterize the demand. Furthermore, this case study has been recently used by Paneque et al. (2018) to demonstrate how to integrate advanced discrete choice models in pricing problems using a mixed integer linear programming (MILP) formulation. The parking choice consists of three services: (1) paid on-street parking (PSP), (2) paid parking in an underground car park (PUP), and (3) free on-street parking (FSP). The latter does not provide any revenue to the operator. Table 6 shows all explanatory variables used in the utility functions of the logit model. These are features related to the age of the vehicle, the income of customers, the type of trip, the access time to the destination from the parking, and information on whether the customer is a resident or not.

Following the logit model proposed by Ibeas et al. (2014), we build the following three utility specifications:

\[
V_{FSP,n} = \beta_{FSP,n} p_{FSP} + q_{FSP,n} = q_{FSP,n}
\]

\[
V_{PSP,n} = \beta_{PSP,n} p_{PSP} + q_{PSP,n},
\]

\[
V_{PUP,n} = \beta_{PUP,n} p_{PUP} + q_{PUP,n}.
\]
| Features      | Definition                                                                 |
|---------------|---------------------------------------------------------------------------|
| ASC<sub>PSP</sub> | Alternative specific constant for the PSP alternative.                   |
| ASC<sub>PUP</sub> | Alternative specific constant for the PUP alternative.                   |
| AT<sub>FSP</sub>  | Access time to the free on-street parking.                               |
| AT<sub>PSP</sub>  | Access time to the paid on-street parking.                               |
| AT<sub>PUP</sub>  | Access time to the paid underground parking.                             |
| TD<sub>FSP</sub>  | Access time to the destination from the free on-street parking.          |
| TD<sub>PSP</sub>  | Access time to the destination from the paid on-street parking.          |
| TD<sub>PUP</sub>  | Access time to the destination from the paid underground parking.        |
| Origin        | Dummy parameter that is 1 if the origin of the trip is internal to the town. |
| p<sub>PSP</sub>  | Fee for the paid on-street parking.                                      |
| p<sub>PUP</sub>  | Fee for the paid underground parking.                                    |
| LowInc        | Dummy parameter that is 1 if the income of the customer is below 1200€/month. |
| Residence     | Dummy parameter that is 1 if the customer is a resident.                 |
| AgeVeh<sub>≤3</sub> | Dummy parameter that is 1 if the age of the vehicle is lower than 3 years. |

Table 6 Features used in the parking choice model.

The utility specification of the free on-street parking only contains the exogenous part \( q_{FSP,n} \) since there is no fee to pay for that option (\( p_{FSP} = 0 \)). The price sensitivities parameters \( \beta_{PSP,n} \) and \( \beta_{PUP,n} \) are then further expressed as:

\[
\begin{align*}
\beta_{PSP,n} &= \beta_{FEE} + \beta_{FEE_{PSP}(LowInc)} \times LowInc_n + \beta_{FEE_{PSP}(Resident)} \times Residence_n, \\
\beta_{PUP,n} &= \beta_{FEE} + \beta_{FEE_{PUP}(LowInc)} \times LowInc_n + \beta_{FEE_{PUP}(Resident)} \times Residence_n.
\end{align*}
\]

The exogenous parts of utilities are modeled as:

\[
\begin{align*}
q_{FSP,n} &= \beta_{AT} \times AT_{FSP} + \beta_{TD} \times TD_{FSP} + \beta_{Origin} \times Origin_n, \\
q_{PSP,n} &= ASC_{PSP} + \beta_{AT} \times AT_{PSP} + \beta_{TD} \times TD_{PSP}, \\
q_{PUP,n} &= ASC_{PUP} + \beta_{AT} \times AT_{PUP} + \beta_{TD} \times TD_{PUP} + \beta_{AgeVeh\leq3} \times AgeVeh\leq3_n.
\end{align*}
\]

The values of coefficient parameters used in Ibeas et al. (2014) are depicted in Table 7. Note that in Ibeas et al. (2014), \( \beta_{AT} \) and \( \beta_{FEE} \) are assumed to be normally distributed and correlated, with \( \text{cov}(\beta_{AT}; \beta_{FEE}) = -12.8 \).
The pricing problem is to determine the optimal prices (or parking fees) of the two paid parking services, i.e., $p_{PSP}$ and $p_{PUP}$, so that the revenue of the operator is maximized. Since the purpose is to show the practicality of CoBiT, we consider an unlimited capacity for the parking services. In the pricing problem, $p_{PSP}$ and $p_{PUP}$ are the only endogenous variables, and all others are exogenous demand variables for which values are given.

**4.4.2. Numerical results** As mentioned, CoBiT is capable of solving pricing problems under discrete mixed logit models. Unlike existing contributions, CoBiT is capable to handle heterogeneous price sensitivity parameters in the utility function, better reflecting the demand models that have been used in the DCM literature. In this section, we perform experiments with two main goals in mind:

**G1** First, we want to show that CoBiT outperforms the global optimizers SCIP 7.0, BARON 21.1.13, Couenne, and ANTIGONE 37.1, both in time and optimality gap. To do so, we assume an MNL model, with $\beta_{AT}$ and $\beta_{FEE}$ fixed to their mean values (i.e., $\beta_{AT} = -0.788$ and $\beta_{FEE} = -32.3$) and we use instances with 10 and 50 customers. It is worth noting that under these assumptions, the price sensitivity parameters ($\beta^p_{PSP,n}$ and $\beta^p_{PUP,n}$) are still both product and customers dependent (see Equations (16)-(17)), which means that state-of-the-art methods cannot be used to solve these problems.
Second, we show the consequences, mainly in terms of lost revenues, that would arise from optimizing under simplified assumptions regarding the price sensitivity parameters, either by assuming a single value instead of a distribution or by neglecting that the price sensitivity parameters can be both product and customers dependent.

| Number of customers | Method  | Best solution | Upper bound | Opt. Gap | $P^*_{PSP}$ | $P^*_{PUP}$ | Time (Seconds) |
|---------------------|---------|---------------|-------------|----------|------------|------------|----------------|
| 10                  | CoBiT   | 6.36          | 6.36        | 0.00%    | 0.7036     | 0.7137     | 514.45         |
| 10                  | SCIP    | 0.0           | $3 \times 10^{19}$ | 10^{21}% | 0          | 0          | 7,200          |
| 10                  | BARON   | 6.01          | 6.47        | 7.65%    | 0.6117     | 0.6202     | 7,200          |
| 10                  | Couenne | 6.36          | 774.91      | 1.2 $\times 10^{10}$% | 0.7025     | 0.7176     | 7,200          |
| 10                  | ANTIGONE| 6.35*         | -*          | 0.00%    | 0.6551     | 0.6638     | 588            |
| 50                  | CoBiT   | 31.93         | 31.93       | 0.00%    | 0.7142     | 0.7210     | 780.02         |
| 50                  | SCIP    | 31.74         | $10^{21}$   | 10^{21}% | 0.6531     | 0.6642     | 7,200          |
| 50                  | BARON   | 30.51         | 587,986     | 1.9 $\times 10^{6}$% | 0.6178     | 0.6291     | 7,200          |
| 50                  | Couenne | 31.93         | 476,747.02  | 1.5 $\times 10^{6}$% | 0.7142     | 0.7210     | 7,200          |
| 50                  | ANTIGONE| 31.74         | 10,000      | 3.1 $\times 10^{4}$% | 0.6531     | 0.6642     | 7,200          |

Table 8 Information obtained on solving (15) with degenerate mixing probability measures with $N = 10$ and $N = 50$ customers.

*:Looking at the log of ANTIGONE, it seems that there is a bug in the solver. After 525 Seconds, the solver reports a solution whose objective value is 7.449, and after 588 Seconds ANTIGONE proves optimality of that solution. We have reported this bug.

Table 8 provides the comparison between CoBiT and the other solvers. Since the problem contains a fraction and exponential functions, its solution is quite sensitive to the errors. That made the solution obtained by ANTIGONE unreliable. As one can see, CoBiT is the only solver that can solve the problem to optimality. Couenne solver can find the optimal solution within two hours but it has difficulty proving optimality of it. On the other hand, BARON is the solver with second best optimality gap. One of the main reasons why solvers are unable to solve these instances is the way they construct an upper bound. Most of the solvers use a technique called term-based underestimates, where they introduce new variables to represent each nonlinear terms (in our instances, the nonlinear terms are the fraction and the exponential functions). Using this technique and then convexification of the problem results in loose relaxations (convexification of each term comes with some...
gap; hence putting all the relaxations into one problem aggregate the error). However, in CoBiT, we only consider the summation of the exponential functions as one nonlinear term and the fraction as the other one. Therefore, our convexification is tighter.

To have a better understanding on how CoBiT converges to the optimal solution, we illustrate how the optimality gap is reduced over time. In all instances, CoBiT finds optimal solutions in the first few iterations and attempts to close the optimality gap in the later ones. As expected, CoBiT converges faster for the instance with \( N = 10 \) customer classes, because in this instance each iteration can be solved much faster.

Next to the optimality gap, it is also interesting to see the number of nodes generated in each iteration of CoBiT. Figure 3 provides this information. As one can see, the behavior of CoBiT in both instances is similar. More specifically, we see that the number of nodes that are explored has its peak around the 15th iteration, and then it decreases. Moreover, an interesting observation is on the solid blue curve (corresponding to 50 customers), which is below the solid red curve (corresponding to 10 customers). The reason is the shape of the expected revenue function. As we can see in Appendix A, the expected revenue function is rather flat when considering the instance with 10 customers compared to the one with
50 customers. Therefore, CoBiT needs small partitions to make sure where the optimal solution is in the instance with 10 customers.

G2. In this part, we focus on evaluating the impact of simplified assumptions on the quality of the solution. One possible way to simplify the pricing model is by assuming that the price sensitivity is customer-independent (as suggested in van de Geer and den Boer (2022)). Mathematically speaking, this simplification reduces the dimensionality of $\beta^p$ from $\mathbb{R}^{C \times N}$ to $\mathbb{R}^C$. To investigate the effect of this simplification, we restrict ourselves to three columns of $\beta^p \in \mathbb{R}^{C \times N}$, which we refer to by Classes 1, 2, and 3.

To assess the impact of this simplification, we evaluate the optimal prices obtained by restricting ourselves to these three classes on the revenue function of the parking case study (where $\beta^p \in \mathbb{R}^{C \times N}$). Table 9 displays the results obtained from this analysis. As the table illustrates, the optimal prices obtained are not optimal in all cases and can be far from optimal in some cases. Thus, neglecting customer-specific price sensitivity in modeling the problem can result in solutions that may not even be locally optimal, indicating the importance of considering customer-specific price sensitivity in the problem’s modeling.

| Customers’ price sensitivity | Evaluation for problem with 10 classes | Gap(%) |
|-----------------------------|---------------------------------------|--------|
| Class 1                     | $0.62$ | $0.62$ | $6.20$ | $2.60$ |
| Class 2                     | $0.77$ | $0.77$ | $5.38$ | $15.43$ |
| Class 3                     | $0.82$ | $0.82$ | $3.73$ | $41.40$ |

Table 9 The result of ignoring the customers’ sensitivity by only considering one customer class to calculate $\beta^p$.

As mentioned, the original distributions of $\beta_{AT}$ and $\beta_{FEF}$ are continuous, while the above-mentioned results are achieved by considering a single value (mean value) of these parameters. Thanks to CoBiT, we are able to integrate a discrete mixed logit model within our pricing problem, i.e., to discretize the distribution and approximate the integral.

Let us consider the situation with $N = 10$ customer classes. We limit the support set of $[\beta_{AT} \atop \beta_{FEF}]$ to its 0.99 confidence set, i.e., $[-3.6, 1.94] \times [-68.52, 3.92]$. To discretize the distribution and approximate the integral, we break the length and the width of the confidence set into $n$ parts with the same length; hence $n^2$ break points. We check the solution obtained by
this approximation when $n^2$ varies in \{9, 16, 25, 49, 64, 100, 121, 144, 169, 400, 900\}. As mentioned before, the performance of the available solvers depends on the number of nonlinear terms in the optimization problem. Discretization of the continuous distribution increases the number of nonlinear terms dramatically, hence negatively affecting the performance of the solvers. Since we have seen that the available solvers cannot tackle the simple MNL model, we only apply CoBiT for the discrete cases.

Table 10 shows how the optimal value and optimal solutions change when the number of breakpoints increases. In the fifth column, we also report the value of the objective function of the continuous mixed logit model for the obtained optimal prices.

| Num. break points | Opt. value | $p_{PSP}$ | $p_{PU}$ | Expected revenue of continuous ML | Time (Seconds) |
|-------------------|------------|-----------|----------|----------------------------------|----------------|
| 1 (MNL case)      | 6.36       | 0.70      | 0.71     | 4.43                             | 514.45         |
| 9                 | 6.80       | 0.53      | 0.75     | 5.05                             | 62.16          |
| 16                | 5.32       | 0.49      | 0.64     | 5.07                             | 184.42         |
| 25                | 5.11       | 0.56      | 0.70     | 5.00                             | 225.81         |
| 49                | 5.09       | 0.51      | 0.66     | 5.09                             | 323.58         |
| 64                | 5.09       | 0.50      | 0.66     | 5.08                             | 362.45         |
| 100               | 5.06       | 0.50      | 0.66     | 5.08                             | 278.66         |
| 121               | 5.06       | 0.50      | 0.67     | 5.08                             | 53.52          |
| 144               | 5.07       | 0.50      | 0.66     | 5.08                             | 407.09         |
| 169               | 5.07       | 0.50      | 0.66     | 5.08                             | 84.13          |
| 400               | 5.08       | 0.50      | 0.67     | 5.08                             | 989.66         |
| 900               | 5.08       | 0.50      | 0.67     | 5.08                             | 2,912.58       |

Table 10  Sensitivity of the optimal prices to the number of break points.

We see that the sequence of solutions converges as we increase the number of breakpoints. Doing so, we better approximate the continuous mixed logit model, which is the best choice model reported in Ibeas et al. (2014). Accordingly, our pricing model better reflects the heterogeneity of the parking users’ behaviors. While we expect this additional complexity in the demand model to come with an increased computational time, we can see that this is not always true. In the end, it depends, as mentioned already, on the resulting shape of the expected revenue function.
In Table 11, we show the revenue, as well as the markets shares obtained while using the optimal prices of the simple MNL, as well as several discrete mixed logit models, into the revenue maximization objective function of the continuous mixed logit demand.

| Model         | N | \(p^*_{\text{FSP}}\) | \(p^*_{\text{PSP}}\) | \(p^*_{\text{PUP}}\) | Expected revenue | Market shares (%) |
|---------------|---|----------------------|----------------------|----------------------|------------------|------------------|
| Simple MNL    | 10| 0.70                 | 0.71                 | 4.43                 | 36.35            | 1.41         60.66 |
| Discrete ML(9)| 10| 0.53                 | 0.75                 | 5.05                 | 14.20            | 57.70    26.53  |
| Discrete ML(16)|10 | 0.49                | 0.64                 | 5.07                 | 8.00             | 48.75    41.68  |
| Discrete ML(25)|10 | 0.51                | 0.66                 | 5.09                 | 10.52            | 47.50    40.41  |
| Discrete ML(900)|10| 0.50                | 0.67                 | 5.08                 | 8.97             | 52.63    36.83  |

Table 11  Expected revenue and market shares associated with the optimal prices of different logit models.

On the parking choice instances of 10 customers, we see that assuming a simple MNL model leads to a significant drop in revenue (4.43 instead of 5.09). Even more interesting is to see that these assumptions regarding the demand also have an important impact on the market shares associated with the different parking options. We see that assuming a simple MNL model in our case would cause a significant shift in the market shares distribution. These are an important consideration to take into account when deciding about pricing. Naturally, the magnitude of the revenue loss and market share shifts will depend on the realism of the assumptions regarding the demand model. With this work, we do not claim that a discrete mixed logit model should be used in all cases, but we offer an algorithm able to include this complex choice model if the analyst considers that this is the right choice model to use. Ultimately, the decision should be based on the trade-off between the demand model realism on one side, and the complexity of the resulting pricing model on the other side, and this should be assessed on a case-by-case basis by the analyst.

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6. Conclusions

Pricing problems under disaggregate demand assumptions is a challenging but relevant area of research due to its numerous applications. In this paper, we explored a static multi-product pricing problem under a discrete mixed logit model that can accommodate heterogeneous price sensitivity. We designed an efficient iterative optimization algorithm that asymptotically converges to the optimal solution. We used linear optimization problems designed based on a trust-region approach to approximate the problem from below and therefore find a “good” feasible solution. We then used convex approximations as well as McCormick relaxation to obtain an upper bound on the optimal value of the nonlinear optimization problem. Thanks to the branching method, we then tightened the optimality gap and proved the asymptotic convergence of our algorithm.

The effectiveness of this general algorithm was demonstrated on several case studies. Benchmarks against solvers and existing contributions in the literature were performed. Our results showed that our algorithm could find optimal prices, even for a higher degree of segment asymmetry. Furthermore, we see that the considered static multi-product pricing problems under a discrete mixed logit model can be considered as the hard instances for global optimization solvers and can be used as the test instances.

The computational complexity of the proposed algorithm is an important issue that needs further investigation. A formal computational complexity will show how the computation of our algorithm is linked to the number of customers as well as the number of alternatives. Furthermore, this paper focused on the static pricing problem under discrete mixed logit demand. However, the proposed algorithm could be extended to other types of demand models and pricing problems. Exploring the generalization of the algorithm to other settings is an interesting direction for future research. Finally, the partitioning method used in the proposed algorithm is a key component for ensuring the convergence of the algorithm. Investigating alternative partitioning methods and comparing their performance could be an interesting avenue for future research.

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A. Illustration of the objective functions of the continuous mixed logit model for the case study

Figure 4 Illustration of the objective function of (15) for the parking choice model with $N = 10$ and $N = 50$ customers.
This figure "universe.jpg" is available in "jpg" format from:

http://arxiv.org/ps/2005.07482v5