Rays, modes, wavefield structure and wavefield stability

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Sound propagation is considered in range-independent environments and environments consisting of a range-independent background on which a weak range-dependent perturbation is superimposed. Recent work on propagation of both types of environment, involving both ray- and mode-based wavefield descriptions, have focused on the importance of $\alpha$, a ray-based “stability parameter,” and $\beta$, a mode-based “waveguide invariant.” It is shown that, when $\beta$ is evaluated using asymptotic mode theory, $\beta = \alpha$. Using both ray and mode concepts, known results relating to the manner by which $\alpha$ (or $\beta$) controls both the unperturbed wavefield structure and the stability of the perturbed wavefield are briefly reviewed.

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I. INTRODUCTION

In recent work on sound propagation in the ray limit in weakly range-dependent ocean environments it has been shown that both ray amplitude and phase (travel time) distributions are largely controlled by a property—described by the stability parameter $\alpha$—of the background sound speed profile. Other investigators, making use of a modal description of the wavefield in both range-independent environments and range-independent environments with weak perturbations superimposed have shown that many wavefield properties are controlled by a property—the waveguide invariant $\beta$—which is defined using mode-based quantities in the background environment. In this letter it is shown that when $\beta$ is evaluated using asymptotic mode-theoretical results, $\beta = \alpha$.

In the two sections that follow $\alpha$ and $\beta$, respectively, are defined and briefly discussed. In the final section those properties of wavefields that are known to be controlled by $\alpha$ or $\beta$ are briefly reviewed. To keep our presentation brief we show only equations required to derive the result $\beta \sim \alpha$ or those that provide insight into wavefield properties discussed in the final section.

II. RAYS: $\alpha$

We consider underwater acoustic wavefields in three space dimensions that are excited by a point source. We assume that the environment consists of a depth-dependent background, with a weak perturbation superimposed, so the sound speed is written as $c(z,r) = C(z) + \delta c(z,r)$. Here $-z$ is depth and the range $r$ is the horizontal distance from the source. When $\delta c \neq 0$ it is assumed that azimuthal coupling of the wavefield is negligible, so that one need only consider propagation in the $(r,z)$ plane. To make our discussion concrete we will assume that $C(z)$ has a single minimum, but this assumption is not necessary.

It is well known (cf. e.g. Refs. 1,2,3,4,5,6) that substitution of the geometric ansatz
\[ \bar{u}(z,r,\sigma) = a(z,r)e^{i\sigma T(z,r)}, \]
where $\bar{u}(z,r,\sigma)$ is the Fourier transform of the pressure $u(z,r,t)$ and $\sigma$ is the acoustic frequency, into the Helmholtz equation and collecting terms in descending powers of $\sigma$ yields the eikonal and transport equations.

The eikonal equation can be solved for the travel time $T$ by integrating the ray equations,
\[ \frac{dp_z}{dr} = -\frac{\partial H}{\partial z}, \quad \frac{dz}{dr} = \frac{\partial H}{\partial p_z}, \quad \frac{dT}{dr} = p_z \frac{dz}{dr} - H \]
where
\[ H(p_z, z, r) = -\sqrt{c^2(z,r) - p_z^2}. \]

It follows from these equations and the relationship $dz/dr = \tan \varphi$, where $\varphi$ is ray angle with respect to the horizontal, that $c p_z = \sin \varphi$, so the vertical slowness $p_z$ can be thought of as a scaled angle variable. For a point source it is convenient to label rays by their $p_z$ value at $(z,r) = (z_0,0)$, $p_{z,0}$. The Hamiltonian $H = -p_r$, where $p_z$ and $p_r$ are the vertical and horizontal components of the slowness vector $p$ with $||p|| = c^{-1}(z,r)$. In a range-independent environment $c = C(z)$, $p_r$ is constant following a ray. The transport equation can be reduced to a statement of the constancy of energy flux in ray tubes; its solution, accounting for azimuthal spreading, can be written
\[ a^2 = a_0^2 \frac{r^2}{r} \left| H(\partial z/\partial p_{z,0}) \right|^{-1}. \]
Here $a_0^2$ is the value of $a^2$ at the small distance (1 m by convention) $r_0$ from the source. The partial derivative in $H$ is evaluated keeping $r$ fixed: $z(p_{z,0}, r)$ is ray depth. In a range-independent environment an alternative form of $a^2$ is
\[ a^2 = a_0^2 \frac{r^2}{r} \left| \frac{p_r(p_z,0)}{|p_z(\partial R/\partial p_r)|} \right|, \]
where $R(p_r, z)$ is the range of a ray, and ray depth is held constant in the partial derivative.

For the range-independent problem $z(r)$ and $p_z(r)$ (following rays) are periodic functions. This periodic motion is most naturally described using action–angle variables $(I, \vartheta)$. The transformed ray equations \cite{2,3} for details of the transformation) are

\[
\begin{align*}
\frac{dI}{dr} & = -\frac{\partial H}{\partial \vartheta} = 0, \\
\frac{d\vartheta}{dr} & = \frac{\partial H}{\partial I} = \omega(I), \\
\frac{dT}{dr} & = I \omega(I) - \bar{H}(I)
\end{align*}
\]

where $\bar{H}(I)$ is the transformed Hamiltonian. The action

\[
I = \frac{1}{2\pi} \int_{z_{-}}^{z_{+}} dz \, p_z(z) = \frac{1}{\pi} \int_{z_{-}}^{z_{+}} dz \sqrt{C^{-2}(z) - p_r^2},
\]

where $z_{\pm}$ correspond to the ray upper (+) and lower (−) turning depths where $C^{-1}(z_-) = C^{-1}(z_+) = p_r$. The angle variable $\vartheta$ increases by $2\pi$ each time a ray completes a cycle, and $\omega(I) = 2\pi/R_\ell(I)$ where $R_\ell$ is the range of a ray cycle (double loop),

\[
R_\ell(p_r) = -2\pi \frac{dI}{dp_r} = 2p_r \int_{z_{-}}^{z_{+}} dz \sqrt{C^{-2}(z) - p_r^2},
\]

where $I(p_r)$ is defined in \ref{10}. The action–angle form of the ray equations can trivially be integrated:

\[
\begin{align*}
\vartheta(r) & = \vartheta_0 + \omega(I)r \text{ mod } 2\pi, \\
I(r) & = I_0 \\
T(r) & = [I \omega(I) - \bar{H}(I)] r.
\end{align*}
\]

[More correctly, a term $d(G - I \vartheta)/dr$ should be added to the r.h.s. of \ref{10} where $G$ is the generating function of the canonical transformation $(p_z, z) \to (I, \vartheta)$. The corresponding endpoint corrections to \ref{12} is both numerically insignificant and not relevant to the discussion that follows.]

The stability parameter $\alpha(I)$, whose relationship to wavefield properties will be reviewed in the final section, is defined as

\[
\alpha(I) = \frac{I}{\omega(I)} \frac{d\omega}{dI}.
\]

It follows from the relationship $\omega(I) = 2\pi/R_\ell(I)$ that $\alpha(I)$ can be expressed in the form

\[
\alpha(p_r) = \frac{2\pi I(p_r)}{R_\ell(I)} \frac{dR_\ell}{dp_r}.
\]

III. MODES: $\beta$

In a stratified environment $c = C(z)$ in three space dimensions the modal decomposition of the wavefield has the form

\[
\bar{u}(z, r, \sigma) = \frac{i}{4} \sum_{m=0}^{\infty} \phi_m(z_0, \sigma) \phi_m(z, \sigma) H_0^{(1)}(\sigma \rho_m r).
\]

Here $H_0^{(1)}$ is the zeroth-order Hankel function of the first kind and the normal modes $\phi_m(z; \sigma)$ satisfy

\[
\frac{d^2 \phi_m}{dz^2} + \frac{\sigma^2}{C^2(z)} \phi_m = 0
\]

together with a pair of boundary conditions. Here $(\sigma \rho_m)^2$ is a separation constant. For sound speed profiles with a single minimum it is well known (cf. e.g. Ref. \ref{9} for the outline of a uniform asymptotic derivation) that an asymptotic analysis of \ref{17} for modes with turning depths within the water column reveals that each $\phi_m$ is associated with a discrete value of the action $I$,

\[
\sigma I(p_r) = m + \frac{1}{2}, \quad m = 0, 1, 2, \cdots,
\]

where $I(p_r)$ is defined in \ref{10}. We now introduce, following the argument given in Ref. \ref{9} the important concepts of modal group and phase slowness. Owing to the orthogonality of the modes, the quantity $\bar{u}_m(r, \sigma) = \int dz \, \bar{u}(z, r, \sigma) \phi_m(z, \sigma)/\int dz \, \phi_m^2(z, \sigma)$ isolates the contribution to the wavefield from the mode with frequency $\sigma$ and mode number $m$. The inverse Fourier transformation of $\bar{u}_m(r, \sigma)$, weighted by $s(\sigma)$, the Fourier transformation of the source time history $s(t)$, is denoted $u_m(r, t)$. If $s(\sigma)$ has a narrow bandwidth, centered at $\sigma_0$, then a Taylor series expansion of $k_{rm} = \sigma p_{rm}$, with $p_{rm} = p_r(\sigma)$ via Eq. \ref{18}, about $\sigma_0$ yields the result

\[
u_m(r, t) = e^{ik_{rm}(\sigma_0 r - \sigma_0 t)} \psi_m(r, t)
\]

where the envelope function $\psi_m(r, t)$ travels at the group speed, $(\partial k_{rm}/\partial \sigma)^{-1}$, evaluated at the center frequency and mode number $m$. The group slowness is defined as

\[
S_g = \frac{\partial k_{rm}}{\partial \sigma}.
\]

Note that \ref{19} represents a slowly varying dispersive wavetrain whose envelope moves at the group slowness, but within which surfaces of constant phase move at the phase slowness $k_r/\sigma = p_r$.

Consistent with the asymptotic analysis presented here

\[
S_g(p_r) = \frac{\ell(t)}{R_\ell(p_r)}
\]

(cf. e.g. Ref. \ref{13} or \ref{9}, the latter reference also includes, with additional references, the exact expression for $S_g$). Here $R_\ell(p_r)$ is given in \ref{10} and $T_\ell(p_r)$ is the corresponding expression for the single-cycle travel time,

\[
T_\ell(p_r) = 2\pi I(p_r) + p_r R_\ell(p_r) = 2 \int_{z_{-}}^{z_{+}} dz \sqrt{C^{-2}(z)} - p_r^2.
\]
Note that although, according to (21), $S_p$ depends only on the turning depths of a mode, this dictates via the quantization condition that $S_p$ is in general a function of both frequency $\sigma$ and mode number $m$.

The waveguide invariant $\beta$, whose relationship to wavefield properties will be reviewed in the next section, is defined as

$$\beta = -\frac{\partial S_\beta}{\partial p_r}$$

where $p_r$ is the modal phase slowness (often written as $S_p$). Like $S_p$, in general $\beta$ is a function of both $\sigma$ and $m$. It follows from Eqs. (13) (21) (28) that $\beta$ can be expressed as

$$\beta(p_r) = 2\pi \frac{I(p_r) dR_\ell}{R_\ell^2(I) \partial p_r} = \alpha(p_r).$$

This is the main result of this letter. It should be emphasized that our modal analysis is based on asymptotic results, so that we have only demonstrated the asymptotic equivalence of $\beta$ and $\alpha$.

IV. WAVEFIELD STRUCTURE AND STABILITY

In this section we briefly review those properties of wavefield structure and wavefield stability (to a small range-dependent perturbation) that are known to be controlled by $\alpha$ or $\beta$. Fig. 1 shows wavefield intensity $|u(z, r, t)|^2$ in a range-independent sound channel in the depth–time plane at a fixed range, $r = 500$ km, for waves excited by a transient compact source with center frequency $f_0 = 75$ Hz and bandwidth $\Delta f \approx 30$ Hz where $\sigma = 2\pi f$. This plot was produced by solving the Thomson–Chapman

parabolic equation. Use of a parabolic approximation to the Helmholtz equation introduces some minor distortion of the wavefield, but the connections described here between $\alpha$, $\beta$, wavefield structure and wavefield stability hold whether a parabolic approximation is introduced or not.

Perhaps the simplest interpretation of $\alpha$ is that it is a measure of the rate at which small elements of the extended phase space $(z, p, T)$ are deformed by the background flow (with $dz/dr$ treated as the $z$-coordinate of a fluid motion, etc.) by shearing motion. Although this property does not correspond directly to any observable wavefield feature, this property helps to understand other observable wavefield features. A very simple observable wavefield property controlled by $\alpha$ is travel time dispersion. It follows from the third of Eqs. (5) that $dT/d\ell = I(\omega/\partial\ell)$ is a ray label that increases monotonically with increasing axial ray angle, so this equation describes the rate of change of ray travel time with increasing axial ray angle. This is illustrated in the ray simulation in Fig. 1 where $\alpha < 0$ except in small angular bands corresponding to steep rays; note that this ray property is also clearly visible in the corresponding finite frequency wavefield. Note also that zeros of $\alpha$ correspond approximately to cusps in the $(z, T)$ plane. At such cusps geometric amplitudes diverge. This can be seen from Eq. (5). In that expression $R$ is the total range of a ray, which can be written as $nR_\ell$ ($n$ complete ray loops) plus end-segment corrections which depend on source and receiver depths and ray inclination (positive or negative) at the source and receiver. At long range the dominant contribution to $R$ is $nR_\ell$. Then it follows from (5) and (15) that at caustics either $p_r = 0$ or $\alpha = 0$.

[We emphasize that is true only in the large $r$ asymptotic limit. At short range $(\partial R/\partial p_r)_z$ has complicated structure, usually including a singularity when $p_r = 0$, so that application of (5) generally requires great care near such points.] The same argument reveals that at long range geometric amplitudes are inversely proportional to $|\alpha|$; this property is also seen (away from caustics and interference fringes) in the finite frequency wavefield shown in Fig. 1.

In the presence of a weak range-dependent perturbation $\alpha$ has been shown to control both ray stability quantified by Lyapunov exponents, for instance, and several measures of travel time spread. The latter property is linked to the travel time dispersion property in the background environment noted above. Also, in the presence of a weak perturbation, $\alpha$ plays a critical role in ideas relating to ray dynamics. References 1, 2, 3, 4 show that unperturbed rays for which $|\alpha|$ is large are generally more sensitive to environmental perturbations...
than those for which \(|\alpha|\) is small.

In a range-independent environment \(\beta\) controls modal group delay dispersion: modal group delays satisfy \(T_g = S_g p_r\), where \(T_g = T_g(m, \sigma)\) and \(S_g = S_g(m, \sigma)\), so \((\partial T_g/\partial p_r)\) is \(-\beta\). The property that is most commonly used\(^{22}\) to motivate the introduction of the waveguide invariant \(\beta\) is that, following a surface of constant wavefield intensity, a relationship \(\delta r/r = \beta \delta \sigma/\sigma\) must be satisfied. This leads naturally to the defining relationship \(\delta r/r = \beta \delta \sigma/\sigma\). (Note that in some references, including the original work by Chuprov\(^{19}\), \(\delta S_g/\partial p_r\), is defined to be \(1/\beta\) rather than \(\beta\)). This defining relationship\(^{23}\) implies a power-law dependence of the difference in eigenvalues of the Helmholtz equation on the acoustic frequency, \(k_{\sigma m} - k_{\tau m} = a_{mn} \sigma^{-\beta}\) where \(m\) and \(n\) are mode numbers and \(a_{mn}\) is a constant. More generally Grachev\(^{24}\) argued that \(k_{\sigma m} - k_{\tau m} = b_{mn} \eta^{-\beta/\gamma} \sigma^{-\beta}\) where \(\eta\) is a parameter, such as water depth, describing the environment and \(\gamma\) is a constant. Consistent with this dependence is the relationship \(\delta r/r = \beta \delta \sigma/\sigma + (\beta/\gamma) \delta \eta/\eta\). It is important to note (cf. also Ref. \(^{14}\)) that because \(\beta = \beta(\sigma, m)\), when modes corresponding to different values of \(\beta\) are not temporally resolved\(^{24}\), application of the above results may be difficult. Much work relating to \(\beta\) has focused on the homogeneous, constant depth waveguide for which \(\beta\) is independent of \(\sigma\) and \(m\) for modes corresponding to small values of axial ray angle; for that problem \(\beta\) can be approximated as a constant.

In an environment consisting of a range-independent background on which a range-dependent perturbation is superimposed, \(\beta\) controls the spread of modal group delays owing to mode coupling. This is seen by noting that the total group delay of modal energy that has been scattered among mode numbers \(m_i, i = 1, \cdots, N\), is \(T_g = \sum S_g(m_i, \sigma) r_i\), where \(r = \sum_i r_i\) is the range. It follows that

\[
T_g = S_g(\bar{m}, \sigma)r + \frac{\partial S_g(\bar{m}, \sigma)}{\partial p_r} \sum_i \frac{1}{\bar{I}} (I_i - \bar{I}) r_i
\]

where \(\bar{I} = (\bar{m} + \frac{1}{2})/\sigma = I(\bar{p}_r)\) and \(\bar{m}\) is a suitably chosen mode number (e.g. the mode number at \(r = 0\) or the average mode number). Also in range-dependent environments, the relationship \(\delta r/r = \beta \delta \sigma/\sigma + (\beta/\gamma) \delta \eta/\eta\) has been used in the interpretation of measurements in shallow water waveguides\(^{20,21,22,23}\) and time reversal applications\(^{14,17,18,19}\). Note that large \(|\beta|\) is associated with a high degree of wavefield sensitivity, e.g. high sensitivity of modal group delays to an environmental perturbation, or high sensitivity of the location of a time reversal focus to an environmental perturbation \(\delta \eta\).

In retrospect it is not surprising that there should be a simple connection between \(\alpha\) and \(\beta\) inasmuch as, whether one adopts a ray or modal wavefield description, the object of study, the wavefield, is the same. The result \(\beta \sim \alpha\) is an aspect of ray–mode duality (cf. e.g. Ref. \(^{14}\)). We expect that there are many wavefield properties that we have overlooked that are controlled by this parameter. We are unaware of any parameter that characterizes an acoustic environment whose important rivals that of \(\alpha\) (or \(\beta\)).

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