On the gauge transformation of scalar induced gravitational waves

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Abstract

The gauge dependence of the scalar induced gravitational waves (SIGWs) generated at the second order imposes a challenge to the discussion of the secondary gravitational waves generated by scalar perturbations. We provide a general formula that is valid in any gauge for the calculation of SIGWs and the relationship for SIGWs calculated in various gauges under the coordinate transformation. The formula relating SIGWs in the Newtonian gauge to other gauges is used to calculate SIGWs in six different gauges. We find that the Newtonian gauge, the uniform curvature gauge, the synchronous gauge and the uniform expansion gauge yield the same result for the energy density of SIGWs. We also identify and eliminate the pure gauge modes existed in the synchronous gauge. In the total matter gauge and the comoving orthogonal gauge, the energy density of SIGWs increases as $\eta^2$. While in the uniform density gauge, the energy density of SIGWs increases as $\eta^6$.

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I. INTRODUCTION

The detection of gravitational waves (GWs) by LIGO collaboration and Virgo collaboration opens up a new avenue for probing the property of gravity in the strong field and nonlinear regions [1–7]. Although the primordial GWs may be too small to be detected even by the third generation ground based and the spaced based GW detectors, the scalar induced GWs (SIGWs) generated at the second order can be large enough because the amplitude of the primordial scalar power spectrum is constrained by the cosmic microwave background anisotropy measurements to be $\sim 10^{-9}$ only at large scales and it can be as large as $\sim 0.01$ at small scales [8]. The SIGWs [9–36] produced by the large scalar perturbations during radiation dominated era can be much larger than the primordial GWs and they have peak frequency at nanohertz or millihertz which can be detected by the space based GW observatory like Laser Interferometer Space Antenna (LISA) [37, 38], TianQin [39] and TaiJi [40], and the Pulsar Timing Array (PTA) [41–44] including the Square Kilometer Array (SKA) [45] in the future. In the literature, SIGWs are also called secondary GWs. Because in addition to the first order scalar perturbation as the source of second order tensor perturbation, there are other sources coming from the first order vector and tensor perturbations such as the scalar-vector, scalar-tensor, vector-vector, vector-tensor and tensor-tensor combinations [46]. In order to distinguish GWs produced by these sources, we use SIGWs in this paper.

Unlike the primordial GWs which are the first order tensor perturbations and gauge invariant, SIGWs sourced by the first order scalar perturbations due to the non-linearity of Einstein’s equation are gauge dependent [47–52]. Apart from the issue of the choice of physical gauge and the true observable measured by GW detectors, the gauge dependence requires us to calculate SIGWs in each gauge. To avoid this problem, in this paper we discuss the relationship for SIGWs in various gauges under the coordinate transformation. Although it was discussed in the literature, but the formula was derived without including the contribution of the first order perturbation $E$ [31, 46, 50]. For this reason, the results of SIGWs obtained in the synchronous gauge cannot be derived from other gauges by the coordinate transformation although their results on the energy density agree with each other [31, 51, 53]. We provide a general formula that is valid in any gauge for the calculation of SIGWs and the relationship for SIGWs calculated in various gauges under the coordinate transformation.
The paper is organized as follows. In the next section, we review the basic formulae to calculate SIGWs and discuss the gauge transformation. We use the mathematica package xPand \[54\] to derive some equations. We also provide the prescription to obtain the expressions in other gauges from the result in the Newtonian gauge by using the gauge transformation of the second order tensor perturbation. In Sec. III, we use the prescription presented in Sec. II to derive the kernels in the uniform curvature gauge, the synchronous gauge, the uniform expansion gauge, the total matter gauge, the comoving orthogonal gauge and the uniform density gauge. The late time behaviors of SIGWs in these gauges are then analyzed. We also calculate the kernels directly in the uniform curvature gauge, the synchronous gauge and the total matter gauge, and show that they agree with those obtained from the coordinate transformations. The pure gauge modes in the synchronous gauge are identified. We conclude the paper in Sec. IV. The discussion on the perturbations is presented in the appendix.

II. BASICS OF SIGWS AND GAUGE TRANSFORMATIONS

We consider the following perturbed metric

$$ds^2 = a^2 \left[ -(1 + 2\phi)d\eta^2 + 2B_i dx^i d\eta + \left( (1 - 2\psi)\delta_{ij} + 2E_{ij} + \frac{1}{2}h_{ij}^{TT} \right) dx^i dx^j \right],$$

(1)

where the scalar perturbations $\phi$, $\psi$, $B$ and $E$ are of first order, but the transverse traceless (TT) part $h_{ij}^{TT}$ is of second order for the purpose of calculating the scalar induced tensor perturbations - $h_{ii}^{TT} = 0$ and $\partial_i h_{ij}^{TT} = 0$ while the first-order vector and tensor perturbations are not considered.

Perturbing Einstein’s equation $G_{\mu\nu} = 8\pi GT_{\mu\nu}$ to the second order, we get the contribution of the first order scalar perturbations to the second order tensor perturbation as

$$h_{ij}^{TT''} + 2\mathcal{H}h_{ij}^{TT'} - \nabla^2 h_{ij}^{TT} = 4T_{ij}^{lm}s_{lm},$$

(2)

where $\mathcal{H} = a'/a$, the prime denotes the derivative with respect to the conformal time $\eta$, and the projection tensor $T_{ij}^{lm}$ extracting the transverse, trace-free part of a tensor will be
discussed explicitly below. The source $s_{ij}$ is

\begin{equation}
\begin{aligned}
-s_{ij} &= \psi_i \psi_j + \phi_i \phi_j - \sigma_{ij} \left( \phi' + \psi' - \nabla^2 \sigma \right) + \left( \psi_i' \sigma_j + \psi_j' \sigma_i \right) - \sigma_{ik} \sigma_{jk} + 2 \psi_{ij} (\phi + \psi) \\
&- 8\pi G a^2 (\rho_0 + P_0) \delta V_i \delta V_j - 2 \psi_{ij} \nabla^2 E + 2 E_{ij} \left( \psi'' + 2 H \psi' - \nabla^2 \psi \right) - E'_{ik} E_{jk} \\
&+ E_{ijkl} E_{jkl} + 2 \left( \psi_{ijk} E_{ik} + \psi_{ijk} E_{jk} \right) - 2 H (\psi_i E_{ij}' + \psi_j E_{ij}') - (\psi_i' E_{ij} + \psi_j' E_{ij}') \\
&- (\psi_i E_{ij}'' + \psi_j E_{ij}'') + 2 E_{ij} \psi' + E_{ij} \left( E'' + 2 H E' - \nabla^2 E \right)_{ik},
\end{aligned}
\end{equation}

where $\sigma = E' - B$ is the shear potential, the anisotropic stress tensor $\Pi_{ij}$ is assumed to be zero, $\delta V$ is the scalar part of the velocity perturbation of the fluid, $\rho_0$ and $P_0$ are the background values of energy density and pressure of the fluid. The detailed discussion of these variables is presented in appendix A. In gauges with $E = 0$, the above Eq. (3) reduces to the results given in [31, 46, 50] with vanishing anisotropic stress. In general, we need to use Eq. (3) instead. In particular, we should include all the terms involving $E$ in the synchronous gauge.

For GWs propagating along the direction $k$, we introduce the orthonormal bases $e$ and $\bar{e}$ with $k \cdot e = k \cdot \bar{e} = e \cdot \bar{e} = 0$ and $|e| = |\bar{e}| = 1$, then the plus and cross polarization tensors are expressed as

\begin{equation}
\begin{aligned}
e^+_{ij} &= \frac{1}{\sqrt{2}} (e_i e_j - e_j e_i), \\
e^x_{ij} &= \frac{1}{\sqrt{2}} (e_i \bar{e}_j + e_j \bar{e}_i).
\end{aligned}
\end{equation}

The polarization tensors (4) are transverse and traceless because $k_i e^+_{ij} = k_i e^x_{ij} = 0$ and $e^+_{ii} = e^x_{ii} = 0$, and they can be used to expand $h^\text{TT}_{ij}$,

\begin{equation}
h^\text{TT}_{ij}(x, \eta) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{ik \cdot x} \left[ h^+_{k}(\eta) e^+_{ij} + h^x_{k}(\eta) e^x_{ij} \right].
\end{equation}

The projection tensor is

\begin{equation}
T^l_{ij} s_{lm} = \int \frac{d^3k}{(2\pi)^{3/2}} e^{ik \cdot x} [e^+_{ij} e^{+lm} + e^x_{ij} e^{xlm}] s_{lm}(k, \eta),
\end{equation}

where $s_{ij}(k, \eta)$ is the Fourier transformation of $s_{ij}(x, \eta)$.

Assuming equal contributions from the two polarizations, we can use one polarization to calculate its energy density and obtain the total energy density by doubling it. Working in Fourier space, the solution to Eq. (2) for the plus polarization $e^+_{ij}$ is

\begin{equation}
h^+_{k}(\eta) = 4 \int \frac{d^3p}{(2\pi)^{3/2}} e^{+ij} p_i p_j \zeta(p) \zeta(k - p) \frac{1}{k^2} I(u, v, x),
\end{equation}

4
where \( x = k\eta \), \( u = p/k \), \( v = |k - p|/k \), \( \zeta(k) = \psi + \mathcal{H}\delta\rho/\rho_0 \) is the primordial curvature perturbation, \( I(u, v, x) \) is given by [13, 16, 17, 24]

\[
I(u, v, x) = \int_0^x d\tilde{x} \frac{a(\tilde{\eta})}{a(\eta)} kG_k(\eta, \tilde{\eta}) f(u, v, \tilde{x}),
\]

(8)

the Green’s function \( G_k(\tilde{\eta}, \eta) \) to Eq. (2) is,

\[
G_k(\tilde{\eta}, \eta) = \frac{\sin(x - \tilde{x})}{k},
\]

(9)

\( f(u, v, x) \) is symmetric about \( u \) and \( v \) and it is related with the source \( S^+_k = e^{+ij}s_{ij}(k, \eta) \) as

\[
S^+_k(\eta) = \int \frac{d^3p}{(2\pi)^3} \zeta(p)\zeta(k - p)e^{+ij}p_i p_j f(u, v, x).
\]

(10)

In the above derivation, we assume that the production of induced GWs begins long before the horizon reentry during radiation domination. In this paper, we consider the production of SIGWs in the radiation dominated era only. During radiation domination, \( \mathcal{H} \sim \eta^{-1} \), the power spectrum of SIGWs is given by

\[
\mathcal{P}_h(k, x) = 4 \int_0^\infty du \int_{|1 - u|}^{1+u} dv \left[ \frac{4u^2 - (1 + u^2 - v^2)}{4uv} \right]^2 I^2(u, v, x) \mathcal{P}_\zeta(uk)\mathcal{P}_\zeta(vk),
\]

(11)

and the fractional energy density of SIGWs is

\[
\Omega_{GW} = \frac{1}{24} \left( \frac{k}{\mathcal{H}} \right)^2 \frac{\mathcal{P}_h(k, x)}{\mathcal{P}_\zeta(k)},
\]

(12)

where \( \mathcal{P}_\zeta \) is the primordial scalar power spectrum and \( \mathcal{P}_h \) is given by

\[
\langle h^s_{k_1}(\eta)h^s_{k_2}(\eta) \rangle = \frac{2\pi^2}{k_1^3} \delta_{s_1s_2}\delta^3(k_1 + k_2)\mathcal{P}_h(k_1, \eta), \quad s_i = +, \times.
\]

(13)

To separate the time evolution, we introduce the transfer function \( T \) by defining \( \phi(k, \eta) = \phi(k, 0)T(\eta) \). In the Newtonian gauge (also referred as Poisson gauge and zero-shear gauge), \( B = E = 0 \), we have \( \phi_N = \psi_N = \Phi = \Psi \) if the anisotropic stress vanishes, here \( \Phi \) and \( \Psi \) are the Bardeen’s potentials defined in (A17) and (A18), and during radiation domination \( \Phi = \Psi = 2\zeta/3 \) on superhorizon scales. Therefore, in the Newtonian gauge, \( \phi_N(k, 0) = 2\zeta(k)/3 \), the transfer function \( T_N(x) \) is

\[
T_N(x) = \frac{9}{x^2} \left( \frac{\sin(x/\sqrt{3})}{x/\sqrt{3}} - \cos(x/\sqrt{3}) \right),
\]

(14)
and we have

\[ f_N(u, v, x) = 2T_N(vx)T_N(ux) + [T_N(vx) + vxT'_N(vx)][T_N(ux) + uxT'_N(ux)]. \]  

The explicit expression for \( I_N(u, v, x) \) is \([16, 24]\)

\[ I_N(u, v, x) = \frac{3}{4u^2v^3} \left( -\frac{4}{x^3} \left( (u^2 + v^2 - 3)uvx^3 \sin x - 6uvx^2 \cos \frac{ux}{\sqrt{3}} \cos \frac{vx}{\sqrt{3}} \right) + 6\sqrt{3}ux \cos \frac{ux}{\sqrt{3}} \sin \frac{vx}{\sqrt{3}} + 6\sqrt{3}vx \sin \frac{ux}{\sqrt{3}} \cos \frac{vx}{\sqrt{3}} \right) - 3(6 + (u^2 + v^2 - 3)x^2) \sin \left( \frac{ux}{\sqrt{3}} \sin \frac{vx}{\sqrt{3}} \right) + (u^2 + v^2 - 3)^2 \]

\[ \times \left[ \left( \text{Ci} \left[ \left( 1 + \frac{u - v}{\sqrt{3}} \right) x \right] + \text{Ci} \left[ \left( 1 + \frac{v - u}{\sqrt{3}} \right) x \right] - \text{Ci} \left[ \left( 1 + \frac{u + v}{\sqrt{3}} \right) x \right] \right) - \text{Ci} \left[ \left( 1 - \frac{u + v}{\sqrt{3}} \right) x \right] + \ln \left[ \frac{3 - (u + v)^2}{3 - (u - v)^2} \right] \sin x + \left( -\text{Si} \left[ \left( 1 + \frac{u - v}{\sqrt{3}} \right) x \right] \right) \right] \]

\[ - \left[ \left( 1 + \frac{u - v}{\sqrt{3}} \right) x \right] + \ln \left[ \frac{3 - (u + v)^2}{3 - (u - v)^2} \right] \sin x + \left( -\text{Si} \left[ \left( 1 + \frac{u + v}{\sqrt{3}} \right) x \right] \right) \cos x \right). \]  

The subscript “\( N \)” indicates that they are evaluated in the Newtonian gauge. The evolution of \( I_N^2(u, v, x) \) with \( u = v = 1 \) is shown in Fig. 1. Note that \( I_N(u, v, x \to \infty) \propto x^{-1} \), and \( \Omega_{GW}(k, x \to \infty) \) is a constant. This implies that SIGWs behave like free radiation deep within horizon.

Now we discuss the gauge transformation. The infinitesimal coordinate transformation is \( x^\mu \to x^\mu + \epsilon^\mu \) with \( \epsilon^\mu = [\alpha, \delta^{ij}\partial_j\beta] \). For the discussion of SIGWs, we do not consider the vector degrees of freedom for the coordinate transformation, and the scalars \( \alpha \) and \( \beta \) are of first order. Since the gauge transformation of tensor modes does not depend on the coordinate transformation of the same order, therefore we do not need to consider the second order coordinate transformation. For the second order tensor perturbation, we have \([48, 49]\)

\[ h_{ij}^{TT} \to h_{ij}^{TT} + \chi_{ij}^{TT}, \]  

where

\[ \chi_{ij}^{TT}(x, \eta) = T_{ij}^{lm} \chi_{lm} = \int \frac{d^3k}{(2\pi)^{3/2}} e^{ik\cdot x} \left[ \chi_k^+(\eta)e_{ij}^+ + \chi_k^-(\eta)e_{ij}^- \right], \]  

(18)
\[ \chi^+_k(\eta) = - \int \frac{d^3p}{(2\pi)^{3/2}} e^{+ij}p_ip_j (4\alpha(p)\sigma(k - p) + 8\dot{H}\alpha(p)[E(k - p) + \beta(k - p)] + p \cdot (k - p)\beta(p)[4E(k - p) + 2\beta(k - p)] - 8\psi(p)\beta(k - p) + 2\alpha(p)\alpha(k - p)) , \]

\[ = 4 \int \frac{d^3p}{(2\pi)^{3/2}} e^{+ij}p_ip_j \zeta(p)\zeta(k - p) \frac{1}{k^2} I_\chi(u, v, x), \tag{19} \]

\[ I_\chi(u, v, x) = - \frac{1}{9uv} \left[ 2T_\alpha(ux)T_\sigma(vx) + 2T_\alpha(vx)T_\sigma(ux) + 2T_\alpha(ux)T_\alpha(vx) \right. \]
\[ - 4 \left( \frac{u}{v} T_\psi(ux)T_\beta(vx) + \frac{v}{u} T_\psi(vx)T_\beta(ux) \right) \]
\[ + \frac{1 - u^2 - v^2}{uv} \left[ T_\beta(ux)T_E(vx) + T_\beta(vx)T_E(ux) + T_\beta(ux)T_\beta(vx) \right] \tag{20} \]
\[ + \frac{4}{k} \left( \frac{1}{v} T_\alpha(ux)T_E(vx) + \frac{1}{u} T_E(ux)T_\alpha(vx) \right. \]
\[ + \frac{1}{v} T_\alpha(ux)T_\beta(vx) + \frac{1}{u} T_\beta(ux)T_\alpha(vx) \right] , \]

and the transfer functions \( T(x) \) are

\[ \alpha(k, x) = \frac{2}{3} \zeta(k) \frac{1}{k} T_\alpha(x) , \tag{21} \]
\[ \beta(k, x) = \frac{2}{3} \zeta(k) \frac{1}{k^2} T_\beta(x) , \tag{22} \]
\[ \sigma(k, x) = \frac{2}{3} \zeta(k) \frac{1}{k} T_\sigma(x) , \tag{23} \]
\[ E(k, x) = \frac{2}{3} \zeta(k) \frac{1}{k^2} T_E(x) , \tag{24} \]
\[ B(k, x) = \frac{2}{3} \zeta(k) \frac{1}{k} T_B(x) , \tag{25} \]
\[ \psi(k, x) = \frac{2}{3} \zeta(k) T_\psi(x) , \tag{26} \]
\[ \phi(k, x) = \frac{2}{3} \zeta(k) T_\phi(x) . \tag{27} \]

We have symmetrized \( I_\chi(u, v, x) \) under \( u \leftrightarrow v \). Note that the first order scalar coordinate transformation appears in the transformed second-order tensor perturbations. With the gauge transformation (17) and the result for SIGWs in the Newtonian gauge, it is straightforward to derive the semi-analytic expression for SIGWs in other gauges without performing the detailed calculation in that gauge.

Combining Eqs. (5), (7), (17), (18) and (19), we get the following gauge transformation

\[ h^+_k \rightarrow h^+_k + \chi^+_k = 4 \int \frac{d^3p}{(2\pi)^{3/2}} e^{+ij}k_ip_ip_j \zeta(p)\zeta(k - p) \frac{1}{k^2} \left[ I(u, v, x) + I_\chi(u, v, x) \right] . \tag{28} \]
This gauge transformation (28) is the main result of our paper. It shows how the solution or the power spectrum of SIGWs transforms under the gauge transformation. For example, with the solution in the Newtonian gauge [16, 24], we can obtain the solution in any other gauge by replacing the Newtonian gauge kernel $I_N(u,v,x)$ in Eq. (11) according to the following rule

$$I_N(u,v,x) \rightarrow I_N(u,v,x) + I_\chi(u,v,x),$$

where

$$I_\chi(u,v,x) = -\frac{1}{9uv} \left( -4 \left[ \frac{u}{v} T_N(ux)T_\beta(vx) + \frac{v}{u} T_N(vx)T_\beta(ux) \right] + 2T_\alpha(ux)T_\alpha(vx) ight) + \frac{4\mathcal{H}}{k} \left( \frac{1}{v} T_\alpha(ux)T_\beta(vx) + \frac{1}{u} T_\beta(ux)T_\alpha(vx) \right) + \frac{1}{uv} \left[ T_\beta(ux)T_\beta(vx) \right] - \frac{1}{9uv} \left( -4 \left[ \frac{u}{v} T_N(ux)T_\beta(vx) + \frac{v}{u} T_N(vx)T_\beta(ux) \right] + 2T_\alpha(ux)T_\alpha(vx) \right),$$

which is obtained by substituting the transfer functions $T_\sigma = T_E = 0$ and $T_\psi = T_N$ in the Newtonian gauge into Eq. (20), and the coordinate transformations from the Newtonian gauge to the other gauge give the transfer functions $T_\alpha$ and $T_\beta$.

III. THE KERNEL IN VARIOUS GAUGES

In this section, we derive the analytic expressions for the kernel $I$ in six other gauges by the coordinate transformation from the Newtonian gauge to any other gauge. To show the effectiveness of the method by the coordinate transformation, we also calculate the kernels directly in the uniform curvature gauge, the synchronous gauge and the comoving gauge (total matter gauge), and show that they agree with the expressions obtained from the coordinate transformations. In some gauges, the residual gauge transformations are used to eliminate the pure gauge modes. We then discuss the late time limit of $\Omega_{GW}$ in those gauges.

A. Uniform curvature gauge

First, let us examine the nature of the uniform curvature gauge, also called the flat gauge where the curvature perturbation vanishes. Moreover, $\psi = E = 0$ and the transfer functions
are
\begin{align}
T_\phi(x) &= \frac{3\sin(x/\sqrt{3})}{x/\sqrt{3}}, \\
T_B(x) &= -\frac{9}{x}\left[\sin(x/\sqrt{3}) - \cos(x/\sqrt{3})\right],
\end{align}
where we assume the same initial condition as that in the Newtonian gauge for the gauge-invariant perturbation $\zeta$ that it is conserved well outside the horizon. Combining Eqs. (3) and (10), we obtain
\begin{equation}
f_{UC}(u, v, x) = \frac{2}{9} \left[ \frac{v}{u}T_B(ux)T_\phi(vx) + \frac{u}{v}T_B(vx)T_\phi(ux) - \frac{1}{uv}T_B(ux)T_B(vx) \right]
= \frac{6(u^2 + v^2 - 3)}{u^3 v^3 x^4} \left( u x \cos \left( \frac{ux}{\sqrt{3}} \right) - \sqrt{3} \sin \left( \frac{ux}{\sqrt{3}} \right) \right)
\times \left( v x \cos \left( \frac{vx}{\sqrt{3}} \right) - \sqrt{3} \sin \left( \frac{vx}{\sqrt{3}} \right) \right).
\end{equation}
Substituting Eq. (33) into Eq. (8), we obtain
\begin{equation}
I_{UC}(u, v, x) = \frac{3}{4uv^3x^4} \left[ -24 \left( -ux \cos \frac{ux}{\sqrt{3}} + \sqrt{3} \sin \frac{ux}{\sqrt{3}} \right) \left( -vx \cos \frac{vx}{\sqrt{3}} + \sqrt{3} \sin \frac{vx}{\sqrt{3}} \right)
- 4 \left( uv(u^2 + v^2 - 3)x^3 \sin x + 6ux \cos \frac{ux}{\sqrt{3}} \left( -vx \cos \frac{vx}{\sqrt{3}} + \sqrt{3} \sin \frac{vx}{\sqrt{3}} \right) \right)
- 3 \sin \frac{ux}{\sqrt{3}} \left( -2\sqrt{3}vx \cos \frac{vx}{\sqrt{3}} + (6 + (u^2 + v^2 - 3)x^2) \sin \frac{vx}{\sqrt{3}} \right) \right)
+ (u^2 + v^2 - 3)^2x^3 \left( \sin x \left( \text{Ci} \left[ \left( 1 + \frac{u - v}{\sqrt{3}} \right) x \right] + \text{Ci} \left[ \left( 1 + \frac{v - u}{\sqrt{3}} \right) x \right] \right) - \text{Ci} \left[ \left( 1 + \frac{u + v}{\sqrt{3}} \right) x \right] - \text{Ci} \left[ \left( 1 - \frac{u + v}{\sqrt{3}} \right) x \right] + \ln \left[ \frac{3 - (u + v)^2}{3 - (u - v)^2} \right] \right)
+ \cos x \left( -\text{Si} \left[ \left( 1 + \frac{u - v}{\sqrt{3}} \right) x \right] - \text{Si} \left[ \left( 1 + \frac{v - u}{\sqrt{3}} \right) x \right] + \text{Si} \left[ \left( 1 - \frac{u + v}{\sqrt{3}} \right) x \right] + \text{Si} \left[ \left( 1 + \frac{v + u}{\sqrt{3}} \right) x \right] \right)
\right]
\end{equation}
To compare the result (34) with that from the Newtonian gauge by the gauge transformation, we need the coordinate transformation from the Newtonian gauge to the uniform curvature gauge
\begin{align}
\alpha &= \frac{\phi_N}{\mathcal{H}} = \frac{2}{3} \zeta(k) \frac{1}{k}T_\alpha(x), \\
\beta &= 0,
\end{align}
where $T_\alpha = xT_N$.

Substituting these results into Eq. (30), we get

$$I_\chi(u, v, x) = -\frac{18}{w^3v^3x^4} \left[ uvx^2 \cos \left( \frac{ux}{\sqrt{3}} \right) \cos \left( \frac{vx}{\sqrt{3}} \right) + 3 \sin \left( \frac{ux}{\sqrt{3}} \right) \sin \left( \frac{vx}{\sqrt{3}} \right) ight]$$

$$-\sqrt{3}uvx^2 \sin \left( \frac{ux}{\sqrt{3}} \right) \cos \left( \frac{vx}{\sqrt{3}} \right) - \sqrt{3}uvx^2 \sin \left( \frac{ux}{\sqrt{3}} \right) \sin \left( \frac{vx}{\sqrt{3}} \right).$$

(37)

Therefore, we confirm that $I_{UC} = I_N + I_\chi$. The evolution of $I_{UC}^2(u, v, x)$ with $u = v = 1$ is shown in Fig. 1. Since $I_\chi(u, v, x)$ decays as $x^{-2}$ as $x \to \infty$, while $I_N(u, v, x)$ decays as $x^{-1}$, so the late time result in the uniform curvature gauge is the same as that obtained in the Newtonian gauge, i.e., $I_{UC}(u, v, x \to \infty) = I_N(u, v, x \to \infty)$. This agrees with the conclusion in Ref. [51, 52]. Notice that during the late time, the fractional energy density $\Omega_{GW} \propto x^2I_{UC}^2$. Thus $\Omega_{GW}$ becomes constant at late time in the uniform curvature gauge.

B. Synchronous gauge

In the synchronous gauge, $\phi = B = 0$, the equation for the transfer function $T_E$ is

$$x^3T_{E'''} + 5x^2T_E'' + \left( 2 + \frac{x^2}{3} \right) xT_E' - \left( 2 - \frac{x^2}{3} \right) T_E' = 0,$$

(38)

and the general solution is

$$T_E(x) = C_1 + C_2 \left( \text{Ci}(x/\sqrt{3}) - \frac{\sin(x/\sqrt{3})}{x/\sqrt{3}} \right) + C_3 \ln(x/\sqrt{3}) + C_4 \left( \text{Si}(x/\sqrt{3}) + \frac{\cos(x/\sqrt{3})}{x/\sqrt{3}} \right).$$

(39)
where \( C_i \) are integration constants. Note that there are two gauge modes in Eq. (39) because of the residual gauge freedom in the synchronous gauge [55–58]. To identify these two gauge modes, during the radiation domination we take the residual gauge transformation [55, 56]

\[
\begin{align*}
\alpha &= -\frac{C_5}{x}, \\
\beta &= -C_5 \ln x + C_6.
\end{align*}
\]

(40)

From the transformation (A11), we see that the constant \( C_6 \) term in \( \beta \) contributes to the integration constant \( C_1 \) in Eq. (39) and the \( C_5 \) term in \( \beta \) contributes to the \( \ln(x) \) term in Eq. (39). Therefore, \( C_1 \) and \( C_3 \) terms in Eq. (39) are just pure gauge modes. Now we determine the remaining integration constants from the initial condition. At the initial time \( x = 0 \), \( T_E(0) = 0 \), so we get \( C_4 = 0 \), \( C_1 = -(1 - \gamma_E)C_2 \) and \( C_3 = -C_2 \), where the Euler gamma constant \( \gamma_E \approx 0.577216 \). Since \( x \to 0 \), \( \text{Ci}(x/\sqrt{3}) - \ln(x/\sqrt{3}) - \gamma_E \to 0 \), so we need to add \( C_1 \) and \( C_3 \) terms to eliminate these gauge modes in \( \text{Ci}(x) \) when \( x \to 0 \). Finally we use the initial condition of the gauge invariant Bardeen potential to fix the constant \( C_2 \). The gauge invariant Bardeen potential in synchronous gauge is

\[
\Phi = -\mathcal{H}E' - E'',
\]

(41)

so the transfer function \( T_\Phi \) is

\[
T_\Phi = \frac{C_2}{x^2} \left( \frac{\sin(x/\sqrt{3})}{x/\sqrt{3}} - \cos(x/\sqrt{3}) \right).
\]

(42)

From the initial condition \( T_\Phi(0) = 1 \), we get \( C_2 = 9 \), so \( T_\Phi = T_N \) as expected because it is a gauge invariant variable. Therefore, the transfer functions \( T_E \) and \( T_\psi \) are

\[
\begin{align*}
T_E(x) &= 9 \left[ C + \text{Ci}(x/\sqrt{3}) - \ln(x/\sqrt{3}) - \frac{\sin(x/\sqrt{3})}{x/\sqrt{3}} \right], \\
T_\psi(x) &= \frac{9}{x^2} \left( 1 - \cos(x/\sqrt{3}) \right),
\end{align*}
\]

(43)

where \( C = 1 - \gamma_E \). Recall that at late time \( x \gg 1 \), \( C \) and \( \ln(x/\sqrt{3}) \) terms are gauge modes and they are physical only at \( x \ll 1 \). Therefore, at late time \( x \gg 1 \), the transfer function is

\[
T_E(x) \approx 9 \left[ \text{Ci}(x/\sqrt{3}) - \frac{\sin(x/\sqrt{3})}{x/\sqrt{3}} \right].
\]

(44)
Combining Eqs. (3) and (10), we get

\[ f_{\text{syn}} = - [T_\psi(u x)T_\psi(v x)] - \frac{1}{2} \left[ \frac{v}{u} T'_E(u x) (T'_\psi(v x) + T'_E(v x)) + \frac{u}{v} T'_E(v x) (T'_\psi(u x) + T'_E(u x)) \right] - \frac{1}{2uv} T'_E(v x) \]

\[ + 2T_\psi(u x)T_\psi(v x) + x^2 uv T'_\psi(v x) + [T_\psi(u x)T_\psi(v x) + T_\psi(v x)T_\psi(u x)] \]

\[ + \left( \frac{1}{u^2} T_E(u x) \right) \left[ u^2 T'_\psi(v x) + \frac{2v}{x} T'_\psi(v x) + v^2 T_\psi(v x) \right] \]

\[ + \frac{1}{v^2} T_E(v x) \left[ u^2 T'_\psi(u x) + \frac{2u}{x} T'_\psi(u x) + u^2 T_\psi(u x) \right] - \frac{1 - u^2 - v^2}{2uv} T'_E(u x) T'_E(v x) \]

\[ - \left( \frac{1 - u^2 - v^2}{2uv} \right)^2 T_E(u x) T_E(v x) + (1 - u^2 - v^2) \left[ \frac{1}{v^2} T_\psi(u x) T_E(v x) \right] \]

\[ + \frac{1}{u^2} T_\psi(v x) T_E(u x) \] + 2 \left[ \frac{1}{vu} T_\psi(u x) T'_E(v x) + \frac{1}{ux} T_\psi(v x) T'_E(u x) \right] + \frac{u}{v} T'_\psi(u x) T'_E(v x) \]

\[ + \frac{v}{u} T'_\psi(v x) T'_E(u x) + [T_\psi(u x) T'_E(v x) + T_\psi(v x) T'_E(u x)] + \left[ \frac{v}{u} T'_E(u x) T'_\psi(v x) \right] \]

\[ + \frac{u}{v} T'_E(v x) T'_\psi(u x) - \frac{1 - u^2 - v^2}{4} \left( \frac{1}{u^2} T_E(u x) \right) \left[ T'_E(v x) + \frac{2}{ux} T'_E(v x) + T_E(v x) \right] \]

\[ + \frac{1}{v^2} T_E(v x) \left[ T'_E(u x) + \frac{2}{ux} T'_E(u x) + T_E(u x) \right] \].

(45)

Since \( I(u, v, x) \) depends on \( f(u, v, x) \) linearly in Eq. (8), we split \( f_{\text{syn}}(u, v, x) \) as \( f_{\text{syn}} = f_N + \Delta f \) with \( f_N \) given by (15). Substituting the result (45) into Eq. (8), we get \( I_{\text{syn}} = I_N + \Delta I_{\text{syn}}(u, v, x) \) and

\[ \Delta I_{\text{syn}}(u, v, x) = \int_0^x d\tilde{x} \tilde{k} G_k(x ; \tilde{x}) [f_{\text{syn}}(u, v, \tilde{x}) - f_N(u, v, \tilde{x})] \]

\[ = - \frac{9}{w^2 u^2 v^2 x^2} \left( 1 - u^2 - v^2 \right) x^2 \left[ \text{Ci} \left( \frac{uv}{\sqrt{3}} \right) + C - \ln \frac{ux}{\sqrt{3}} - \frac{\sin(ux/\sqrt{3})}{ux/\sqrt{3}} \right] \]

\[ \times \left[ \text{Ci} \left( \frac{vx}{\sqrt{3}} \right) + C - \ln \frac{vx}{\sqrt{3}} - \frac{\sin(vx/\sqrt{3})}{vx/\sqrt{3}} \right] \]

\[ + 2 \left[ \frac{\sin(ux/\sqrt{3})}{ux/\sqrt{3}} - 1 \right] \left[ \frac{\sin(vx/\sqrt{3})}{vx/\sqrt{3}} - 1 \right] \]

\[ + 4 \left[ -\text{Ci} \left( \frac{ux}{\sqrt{3}} \right) - C + \ln \frac{ux}{\sqrt{3}} + \frac{\sin(ux/\sqrt{3})}{ux/\sqrt{3}} \right] \left[ 1 - \cos \frac{ux}{\sqrt{3}} \right] \]

\[ + 4 \left[ -\text{Ci} \left( \frac{vx}{\sqrt{3}} \right) - C + \ln \frac{vx}{\sqrt{3}} + \frac{\sin(vx/\sqrt{3})}{vx/\sqrt{3}} \right] \left[ 1 - \cos \frac{vx}{\sqrt{3}} \right] \].

(46)
We expect the result (46) to equal \( I_\chi(u,v,x) \) by the coordinate transformation from the Newtonian gauge to the synchronous gauge. As emphasized above, in the synchronous gauge, we should include the contribution from \( E \) in Eq. (3). If the contribution from \( E \) is not included the \( I_{\text{Syn}} \) cannot be obtained from \( I_N \) by the coordinate transformation from the Newtonian gauge to the synchronous gauge [31, 51, 53]. To confirm our result (46), we now discuss the gauge transforms. The coordinate transformation from the Newtonian gauge to the synchronous gauge is

\[
\begin{align*}
\alpha(k,x) &= \frac{2}{3} \zeta(k) \frac{1}{k} T_\alpha(x), \\
\beta(k,x) &= \frac{2}{3} \zeta(k) \frac{1}{k^2} T_\beta(x),
\end{align*}
\]

(47)

where the transfer functions are

\[
\begin{align*}
T_\alpha(x) &= \frac{9}{x} \left[ \sin\left(\frac{x}{\sqrt{3}}\right) \right], \\
T_\beta(x) &= 9 \left[ C + \text{Ci} \left( \frac{x}{\sqrt{3}} \right) - \ln \left( \frac{x}{\sqrt{3}} \right) - \frac{\sin(x/\sqrt{3})}{x/\sqrt{3}} \right].
\end{align*}
\]

(48)

Substituting Eq. (48) into Eq. (30), we find \( I_{\chi}^{\text{Syn}}(u,v,x) = \Delta I_{\text{Syn}}(u,v,x) \) and confirm that \( I_{\text{Syn}}(u,v,x) = I_N(u,v,x) + I_\chi(u,v,x) \). The evolution of \( I_{\text{Syn}}^2(u,v,x) \) with \( u = v = 1 \) is shown in Fig. 1. As discussed above, at late-time with \( x \gg 1 \) the growing mode \( \ln(u x/\sqrt{3}) \ln(v x/\sqrt{3}) \) in \( I_\chi(u,v,x) \) is a gauge mode. Dropping the gauges terms, we find that the contribution of \( E \) is negligible. Therefore, at late-time \( I_{\text{Syn}} = I_N \) and the result on the energy density of SIGWs is the same in both the Newtonian gauge and the synchronous gauge, as found in [31, 51, 53]. Although at late times the contribution of \( E \) is negligible and it does not affect the obtained energy density of SIGWs, we still need to include \( E \) in the calculation so that the covariance of \( h_{ij} \) is guaranteed and the relation between \( I_{\text{Syn}} \) and \( I_N \) under the coordinate transformation is retained. In the synchronous gauge the subtlety arises in determining when to eliminate the gauge modes in \( C \text{i}(x/\sqrt{3}) - \ln(x/\sqrt{3}) + C \). Accordingly, this gauge is not a particularly good choice for calculating the production of SIGWs.
C. Comoving gauge (Total matter gauge)

In this subsection, the comoving gauge (also referred as the total matter gauge [49]), $\delta V = E = 0$, and the transfer functions are

$$T_\psi(x) = \frac{3}{2} \frac{\sin(x/\sqrt{3})}{x/\sqrt{3}},$$

$$T_B(x) = \frac{3}{2x^2} \left[ 6x \cos(x/\sqrt{3}) + \sqrt{3}(x^2 - 6) \sin(x/\sqrt{3}) \right], \quad (49)$$

$$T_\phi(x) = \frac{3}{2} \left[ \frac{\sin(x/\sqrt{3})}{x/\sqrt{3}} - \cos(x/\sqrt{3}) \right].$$

As expected, on the superhorizon scales, $\psi(k, 0) = \zeta(k)$, so $T_\psi(0) = 3/2$ and at late time with $x \gg 1$, the perturbation $\psi$ decays as [52]

$$\psi(k, \eta) = \psi(k, 0) \frac{\sin(x/\sqrt{3})}{x/\sqrt{3}}. \quad (50)$$

However, the perturbations $B$ and $\phi$ do not decay at late time with $x \gg 1$ [52], they will induce SIGWs continuously. Combining Eqs. (3), (10) and (49), we have [52]

$$f_{TM}(u, v, x) = \frac{1}{2u^3v^3x^4} \left[ 2ux \cos \frac{ux}{\sqrt{3}} \left( -vx(18 - 12v^2 + u^2(v^2x^2 - 12)) \cos \frac{vx}{\sqrt{3}} ight. 
+ \sqrt{3}(18 + v^4x^2 - 3v^2(-4 + x^2) + 2u^2(v^2x^2 - 6)) \sin \frac{vx}{\sqrt{3}} 
+ \sin \frac{ux}{\sqrt{3}} \left( 2\sqrt{3}vx(18 - 12v^2 + u^4x^2 + u^2(-12 + (2v^2 - 3)x^2)) \cos \frac{vx}{\sqrt{3}} 
+ (u^4x^2(v^2x^2 - 6) + u^2(72 - 6(2v^2 - 3)x^2 + v^2(v^2 - 3)x^4) 
- 6(18 + v^4x^2 - 3v^2(-4 + x^2))) \sin \frac{vx}{\sqrt{3}} \right) \right]. \quad (51)$$
Substituting Eq. (51) into Eq. (8), we get

\[
I_{TM}(u, v, x) = \frac{1}{4u^3v^3} \left( -2 \left( 6ux \cos \frac{ux}{\sqrt{3}} + \sqrt{3}(u^2x^2 - 6) \sin \frac{ux}{\sqrt{3}} \right) \right. \\
\times \left( 6vx \cos \frac{vx}{\sqrt{3}} + \sqrt{3}(v^2x^2 - 6) \sin \frac{vx}{\sqrt{3}} \right) \\
- 12 \left[ uv(u^2 + v^2 - 3)x^3 \sin x + 6ux \cos \frac{ux}{\sqrt{3}} \left( -vx \cos \frac{vx}{\sqrt{3}} + \sqrt{3} \sin \frac{vx}{\sqrt{3}} \right) \right. \\
- 3 \sin \frac{ux}{\sqrt{3}} \left( -2\sqrt{3}vx \cos \frac{vx}{\sqrt{3}} + [6 + (u^2 + v^2 - 3)x^2] \sin \frac{vx}{\sqrt{3}} \right) \\
+ 3(u^2 + v^2 - 3)x^3 \left[ \sin x \left( \text{Ci} \left[ \left( 1 + \frac{u - v}{\sqrt{3}} \right) x \right] + \text{Ci} \left[ \left( 1 + \frac{v - u}{\sqrt{3}} \right) x \right] \right) \\
- \text{Ci} \left[ \left( 1 + \frac{u + v}{\sqrt{3}} \right) x \right] - \text{Ci} \left[ \left( 1 - \frac{u + v}{\sqrt{3}} \right) x \right] + \ln \left[ \frac{3 - (u + v)^2}{3 - (u - v)^2} \right] \right] \\
\left. \right) \cdot \left( -3 \sin \frac{vx}{\sqrt{3}} \left( -2\sqrt{3}ux \cos \frac{ux}{\sqrt{3}} + \sqrt{3} \sin \frac{ux}{\sqrt{3}} \right) \right) \\
\left. \right) \cdot \left( \text{Si} \left[ \left( 1 + \frac{u - v}{\sqrt{3}} \right) x \right] - \text{Si} \left[ \left( 1 + \frac{u - v}{\sqrt{3}} \right) x \right] \right) \right).
\]

At the late time, \(I_{TM}(u, v, x \to \infty)\) approaches to a constant.

From the Newtonian gauge to the comoving gauge, the transfer functions for the coordinate transformation are

\[
\alpha = \frac{\mathcal{H}\phi_N + \phi_N'}{\mathcal{H}' - \mathcal{H}^2} = \frac{2}{3} \zeta(k) \frac{1}{k} T_{\alpha}(x),
\]

\[
\beta = 0,
\]

where

\[
T_{\alpha}(x) = -\frac{1}{2}(xT_N(x) + x^2T_N'(x)) \\
= -\frac{3}{2x^2} \left[ 6x \cos(x/\sqrt{3}) + \sqrt{3}(x^2 - 6) \sin(x/\sqrt{3}) \right].
\]

Substituting Eq. (55) into Eq. (30), we get

\[
I_\chi(u, v, x) = -\frac{3}{2u^3v^3x^4} \left[ u^2x^2 - 6 \right] \sin \left( \frac{ux}{\sqrt{3}} \right) \left( v^2x^2 - 6 \right) \sin \left( \frac{vx}{\sqrt{3}} \right) + 2\sqrt{3}ux \cos \left( \frac{vx}{\sqrt{3}} \right) \\
+ 2ux \cos \left( \frac{ux}{\sqrt{3}} \right) \left( \sqrt{3}(v^2x^2 - 6) \sin \left( \frac{vx}{\sqrt{3}} \right) + 6vx \cos \left( \frac{vx}{\sqrt{3}} \right) \right).
\]
Again we confirm that \( I_{TM}(u, v, x) = I_N(u, v, x) + I_\chi(u, v, x) \). The evolution of \( I_{TM}^2(u, v, x) \) with \( u = v = 1 \) is shown in Fig. 1. It is obvious that at late time, the constant term in \( I_\chi(u, v, x) \) dominates over \( I_N \), so \( I_{TM}(u, v, x \to \infty) \) approaches a constant. In this gauge, the perturbations \( B \) and \( \phi \) do not decay as \( x \to \infty \) and they induce SIGWs continuously, so \( \Omega_{GW} \) for SIGWs grows as \( x^2 \). This result agrees with Ref. [52].

D. Comoving orthogonal gauge

Let us now focus on comoving orthogonal gauge, \( \delta V = B = 0 \) [49]. In this gauge, there remains a residual coordinate transformation with \( \beta = \mathcal{C} \) which corresponds to the arbitrary choice of the origin of the spatial coordinates. The variable \( \alpha \) for the time coordinate transformation from the Newtonian gauge to this gauge is the same as that from the Newtonian gauge to the total matter gauge. The variable \( \beta \) for the spatial coordinate transformation from the Newtonian gauge to this gauge is

\[
\beta(k, \eta) = \frac{2}{3} \zeta(k) \frac{1}{k^2} T_\beta(x),
\]

(57)

where

\[
T_\beta(x) = -9 \left[ \cos \left( \frac{x}{\sqrt{3}} \right) - \frac{2\sqrt{3}}{x} \sin \left( \frac{x}{\sqrt{3}} \right) + \mathcal{C} \right].
\]

(58)

We may choose \( \mathcal{C} = 1 \) so that \( T_\beta(x = 0) = 0 \). At late time, \( x \gg 1 \), the last constant \( \mathcal{C} \) term is a pure gauge mode, both \( T_\alpha \) and \( T_\beta \) do not decay. Substituting Eq. (58) into Eq. (30), we get

\[
I_\chi(u, v, x) = \frac{3}{4u^3v^3x^4} \left[ 3\mathcal{C}^2uv(2v^2 - u^2 - 1)x^4 - 2\sqrt{3}\mathcal{C}v(5u^2 + 3v^2 - 3)x^3 \sin \frac{ux}{\sqrt{3}} \\
- 2\sqrt{3}uv(3u^2 + 5v^2 - 3)x^3 \sin \frac{vx}{\sqrt{3}} \\
- 2[36 - 18(2u^2 + 2v^2 - 1)x^2 + u^2v^2x^4] \sin \frac{ux}{\sqrt{3}} \sin \frac{vx}{\sqrt{3}} \\
+ 3uvx^2[-8 + (u^2 + v^2 - 1)x^2] \cos \frac{ux}{\sqrt{3}} \cos \frac{vx}{\sqrt{3}} \\
+ vx \cos \frac{vx}{\sqrt{3}} \left( 3\mathcal{C}u(3u^2 + v^2 - 1)x^3 - 2\sqrt{3}[12 + (7u^2 + 3v^2 - 3)x^2] \sin \frac{ux}{\sqrt{3}} \right) \\
+ ux \cos \frac{ux}{\sqrt{3}} \left( 3\mathcal{C}v(u^2 + v^2 - 1)x^3 - 2\sqrt{3}[12 + (3u^2 + 7v^2 - 3)x^2] \sin \frac{vx}{\sqrt{3}} \right) \right],
\]

(59)
and the analytic expression for the kernel $I_{CO}$ in comoving orthogonal gauge is $I_{CO} = I_{N} + I_{χ}$. The evolution of $I_{CO}^{2}(u, v, x)$ with $u = v = 1$ is shown in Fig. 1. At late time, even after dropping the gauge mode, $I_{χ}$ still approaches to a constant because $φ$ and $E$ do not decay, so $Ω_{GW}$ for SIGWs grows as $x^{2}$.

E. Uniform density gauge

The uniform density gauge is defined as $δρ = E = 0$. The coordinate transformation from the Newtonian gauge to this gauge is

$$\alpha = -\frac{δρ_{N}}{ρ^{′}_{0}},$$
$$\beta = 0.$$  \hspace{1cm} (60)

Using the first order perturbation equation and the background equation, we obtain

$$\alpha = \frac{Hφ_{N} + φ^{′}_{N} + \frac{k^{2}φ_{N}}{3H(H^{′} - H^{2})}}{H^{′} - H^{2}}$$
$$= \frac{2}{3}ζ(k)\frac{1}{k}T_{α}(x),$$  \hspace{1cm} (61)

where

$$T_{α}(x) = \frac{3}{2x^{2}} \left[ x(x^{2} - 6) \cos(x/\sqrt{3}) - 2\sqrt{3}(x^{2} - 3) \sin(x/\sqrt{3}) \right].$$  \hspace{1cm} (62)

At late time, the first term in Eq. (62) grows, we may wonder whether it violates the condition of infinitesimal coordinate transformation. To check this, we consider the dimensionless variable $α/η$,

$$\frac{α}{η} \rightarrow ζ(k) \cos(x/\sqrt{3}), \hspace{0.5cm} x \rightarrow ∞.$$  \hspace{1cm} (63)

Therefore, the condition of infinitesimal coordinate transformation is still satisfied at late time. Substituting Eq. (62) into Eq. (30), we get

$$I_{χ}(u, v, x) = \frac{1}{4u^{2}v^{2}x^{4}} \left[ 2ux(u^{2}x^{2} - 6) \cos(\frac{ux}{\sqrt{3}}) \left( vx(v^{2}x^{2} - 6) \cos(\frac{vx}{\sqrt{3}}) - 2\sqrt{3}(v^{2}x^{2} - 3) \sin(\frac{vx}{\sqrt{3}}) - 4(u^{2}x^{2} - 3) \sin(\frac{ux}{\sqrt{3}}) \right) - 6(v^{2}x^{2} - 3) \sin(\frac{ux}{\sqrt{3}}) \right] \times \left( \sqrt{3}vx(v^{2}x^{2} - 6) \cos(\frac{vx}{\sqrt{3}}) - 6(v^{2}x^{2} - 3) \sin(\frac{vx}{\sqrt{3}}) \right),$$  \hspace{1cm} (64)

and the analytic expression for the kernel in the uniform density gauge $I_{UD} = I_{N} + I_{χ}$. The evolution of $I_{UD}^{2}(u, v, x)$ with $u = v = 1$ is shown in Fig. 1. At late time, $I_{χ}$ grows as $x^{2}$,
so $\Omega_{\text{GW}}^{\text{UD}} \sim x^6$. In this gauge, $\phi$ and $\psi$ do not decay, and the variable $B = -\alpha$ even grows with $x$ as $x \to \infty$. If we assume $\zeta(k) \sim 0.01$, then $kB$ approaches order 1 when $T_B$ is about 100 at $x \sim 100$ as shown in Fig. 2, so the linear perturbation breaks down and the above calculation cannot be applied. Thus, we need to be careful about the calculation of $\Omega_{\text{GW}}$ in this gauge.

**F. Uniform expansion gauge**

Let us finally consider the uniform expansion gauge, $3(\mathcal{H}\phi + \psi') + k^2\sigma = 0$ and $E = 0$ [50]. From the Newtonian gauge to this gauge, the coordinate transformation is

$$\alpha = \frac{2}{3} \zeta(k) \frac{1}{k} T_\alpha(x),$$

$$\beta = 0,$$

where

$$T_\alpha(x) = -\frac{3}{6 + x^2} \left( xT_N(x) + x^2T'_N(x) \right)$$

$$= -\frac{9}{x^2(6 + x^2)} \left[ 6x \cos(x/\sqrt{3}) + \sqrt{3}(x^2 - 6) \sin(x/\sqrt{3}) \right].$$
Substituting Eq. (66) into Eq. (30), we get
\[ I_\chi(u,v,x) = \frac{54}{u^3v^3x^4(u^2x^2 + 6)(v^2x^2 + 6)} \left[ u^2v^2x^4 \sin \left( \frac{ux}{\sqrt{3}} \right) \sin \left( \frac{vx}{\sqrt{3}} \right) 
+ 2\sqrt{3}u^2vx^3 \sin \left( \frac{ux}{\sqrt{3}} \right) \cos \left( \frac{vx}{\sqrt{3}} \right) - 6u^2x^2 \sin \left( \frac{ux}{\sqrt{3}} \right) \sin \left( \frac{vx}{\sqrt{3}} \right) 
+ 2\sqrt{3}uv^2x^3 \cos \left( \frac{ux}{\sqrt{3}} \right) \sin \left( \frac{vx}{\sqrt{3}} \right) - 6v^2x^2 \sin \left( \frac{ux}{\sqrt{3}} \right) \sin \left( \frac{vx}{\sqrt{3}} \right) 
+ 12uvx^2 \cos \left( \frac{ux}{\sqrt{3}} \right) \cos \left( \frac{vx}{\sqrt{3}} \right) + 36 \sin \left( \frac{ux}{\sqrt{3}} \right) \sin \left( \frac{vx}{\sqrt{3}} \right) 
- 12\sqrt{3}vx \sin \left( \frac{ux}{\sqrt{3}} \right) \cos \left( \frac{vx}{\sqrt{3}} \right) - 12\sqrt{3}ux \cos \left( \frac{ux}{\sqrt{3}} \right) \sin \left( \frac{vx}{\sqrt{3}} \right) \right] \] (67)

So the analytic expression for the kernel \( I_{UE} \) in the uniform expansion gauge is \( I_{UE}(u,v,x) = I_N + I_\chi \), and it decays as \( x^{-4} \) when \( x \to \infty \), indicating that \( \Omega_{GW}^{UE} \) approaches \( \Omega_{GW}^N \) at late time. The evolution of \( I_{UE}^2(u,v,x) \) with \( u = v = 1 \) is shown in Fig. 1.

IV. CONCLUSION

In this paper, we have derived the general formula valid in any gauge for the calculation of SIGWs. In particular, we have provided the prescription to use the result in the Newtonian gauge to obtain SIGWs in several other gauges by the coordinate transformation from the Newtonian gauge to the other gauges, and also provide the general expression for the kernel function \( I_\chi(u,v,x) \). Besides, we directly derive the kernel functions in the uniform curvature gauge, the synchronous gauge and the total matter gauge, and confirm that they are the same as those obtained by the coordinate transformation from the Newtonian gauge to any other gauge. With the general kernel function \( I_\chi(u,v,x) \) and the result of \( \Omega_{GW} \) in the Newtonian gauge, we derive the results of \( \Omega_{GW} \) in the comoving orthogonal gauge, the uniform density gauge and the uniform expansion gauge by the coordinate transformation form the Newtonian gauge to these gauges. The Newtonian gauge, the uniform curvature gauge, the synchronous gauge and the uniform expansion gauge have the same result on \( \Omega_{GW} \).

We have also identified the two gauge modes in the synchronous gauge, which lead to the growing of the kernel function, and we find that \( \Omega_{GW} \) in the synchronous gauge is the same as that in the Newtonian gauge after eliminating the gauge modes. Although the contribution of the perturbation \( E \) is negligible after eliminating the gauge modes at late time, and \( E \)
does not affect the final result on the energy density of SIGWs, its contribution cannot be neglected. Otherwise the relationship between the results for $\Omega_{GW}$ in the synchronous gauge and the other gauges under the gauge transformation is not satisfied. Since the constant term $C$ and the $\ln(x)$ term are gauge modes only at late time with $x \gg 1$, they are physical modes at early time with $x \ll 1$, thus determining when to drop these terms is quite subtle. Thus, in our view the synchronous gauge is not a good choice for the calculation of the production of SIGWs.

Finally, in the total matter gauge and the comoving orthogonal gauge, the perturbation $\phi$ does not decay at late time, so as $x \gg 1$ the energy density of SIGWs in both gauges increases as $x^2$. In the uniform density gauge, the perturbation $B$ grows as $x$ at late time, so as $x \gg 1$ the energy density of SIGWs in this gauge increase as $x^6$. Of course, we need to be careful about this result because the perturbation theory breaks down due to the growth of $B$. The reason for the gauge dependence of the energy density of SIGWs and the issue of observable need to be further studied.

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Appendix A: The perturbation and gauge transformation

The energy-momentum tensor of a perfect fluid is

$$T_{\mu\nu} = (\rho + P)U_{\mu}U_{\nu} + Pg_{\mu\nu} + \Pi_{\mu\nu}, \quad (A1)$$

where the background anisotropic stress $\Pi_{0\mu\nu}$ is assume to be zero. The first order perturbations of the velocity $U_{\mu}$, the energy density, the pressure and the anisotropic stress are $\delta U_{\mu}$, $\delta \rho$, $\delta P$ and $\delta \Pi_{ij}$, respectively. The first order velocity perturbation $\delta U_{\mu}$ is decomposed via $\delta U_{\mu} = a[\delta V_0, \delta V_i + \delta V_i]$ with $\delta V_{i,i} = 0$.

For flat Friedmann-Robertson-Walker space-time, the background cosmological equations
implies that

\[ \mathcal{H}^2 = \frac{8\pi G}{3} a^2 \rho_0, \]

\[ \mathcal{H}' = - \frac{4\pi G}{3} a^2 (\rho_0 + 3P_0). \]  

(A2)

For the discussion of the perturbed equation, we also write the above Friedmann equations as

\[ \rho_0 + P_0 = \frac{\mathcal{H}^2 - \mathcal{H}'}{4\pi Ga^2}, \]

\[ P_0 = -\frac{\mathcal{H}^2 + 2\mathcal{H}'}{8\pi Ga^2}. \]  

(A3)

In the absence of the anisotropic stress, the first order perturbed cosmological equations are

\[ 3\mathcal{H}(\psi' + \mathcal{H}\phi) - \nabla^2(\psi + \mathcal{H}\sigma) = -4\pi Ga^2 \delta \rho, \]  

(A4)

\[ \psi' + \mathcal{H}\phi = -4\pi Ga^2 (\rho_0 + P_0) \delta V, \]  

(A5)

\[ \sigma' + 2\mathcal{H}\sigma + \psi - \phi = 0, \]  

(A6)

\[ \psi'' + 2\mathcal{H}\psi' + \mathcal{H}\phi' + (2\mathcal{H}' + \mathcal{H}^2)\phi = 4\pi Ga^2 \delta P. \]  

(A7)

Under the infinitesimal coordinate transformation \( x^\mu \rightarrow \tilde{x}^\mu = x^\mu + \epsilon^\mu(x) \) with \( \epsilon^\mu = [\alpha, \delta^{ij}\partial_j \beta] \), the scalar parts of the perturbations transform as

\[ \tilde{\phi} = \phi + \mathcal{H} \alpha + \alpha', \]  

(A8)

\[ \tilde{\psi} = \psi - \mathcal{H} \alpha, \]  

(A9)

\[ \tilde{B} = B - \alpha + \beta', \]  

(A10)

\[ \tilde{E} = E + \beta, \]  

(A11)

\[ \tilde{\sigma} = \sigma + \alpha, \]  

(A12)

\[ \delta \tilde{\rho} = \delta \rho + \rho_0' \alpha, \]  

(A13)

\[ \delta \tilde{P} = \delta P + P_0' \alpha, \]  

(A14)

\[ \delta \tilde{V} = \delta V - \alpha, \]  

(A15)

\[ \delta \tilde{\Pi} = \delta \Pi, \]  

(A16)

where \( \Pi \) is the scalar part of the anisotropic stress. Using the above gauge transformation, we obtain two gauge invariant Bardeen potentials [59]

\[ \Phi = \phi - \mathcal{H} \sigma - \sigma', \]  

(A17)

\[ \Psi = \psi + \mathcal{H} \sigma. \]  

(A18)
For the SIGWs, under the infinitesimal coordinate transformation, we have \( h_{ij}^{\text{TT}} \rightarrow h_{ij}^{\text{TT}} + \chi_{ij}^{\text{TT}} \), and

\[
\chi_{ij} = 2 \left[ \left( \mathcal{H}^2 + \frac{a^n}{a} \right) \alpha^2 + \mathcal{H} \left( \alpha \alpha' + \alpha_{,\kappa} \epsilon^\kappa \right) \right] \delta_{ij} \\
+ 4 \left[ \alpha \left( C_{ij}' + 2 \mathcal{H} C_{ij} \right) + C_{ij,k} \epsilon^k + C_{ik} \epsilon_{j}^j + C_{jk} \epsilon_{i}^i \right] \\
+ 2 \left( B_i \alpha_j + B_j \alpha_i \right) + 4 \mathcal{H} \alpha \left( \epsilon_{i,j} + \epsilon_{j,i} \right) - 2 \alpha_{,i} \alpha_{,j} + 2 \epsilon_{k,i} \epsilon_{j}^j + \alpha \left( \epsilon_{i,j} + \epsilon_{j,i} \right) \\
+ \left( \epsilon_{i,j,k} + \epsilon_{j,i,k} \right) \epsilon^k + \epsilon_{i,k} \epsilon_{j}^j + \epsilon_{j,k} \epsilon_{i}^i + \epsilon_{i}^i \alpha_{,j} + \epsilon_{j}^j \alpha_{,i},
\]

where \( C_{ij} = -\psi \delta_{ij} + E_{,ij} \).

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