Space-time wave packets localized in all dimensions

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Optical wave packets that are localized in space and time, but nevertheless overcome diffraction and travel rigidly in free space, are a long sought-after field structure with applications ranging from microscopy and remote sensing, to nonlinear and quantum optics. However, synthesizing such wave packets requires introducing non-differentiable angular dispersion with high spectral precision in two transverse dimensions, a capability that has eluded optics to date. Here, we describe an experimental strategy capable of sculpting the spatio-temporal spectrum of a generic pulsed beam by introducing arbitrary radial chirp via two-dimensional conformal coordinate transformations of the spectrally resolved field. This procedure yields propagation-invariant ‘space-time’ wave packets localized in all dimensions, with tunable group velocity in the range from 0.7c to 1.8c in free space, and endowed with prescribed orbital angular momentum. By providing unprecedented flexibility in sculpting the three-dimensional structure of pulsed optical fields, our experimental strategy promises to be a versatile platform for the emerging enterprise of space-time optics.

Creating spatio-temporally localized optical wave packets that overcome diffraction and propagate rigidly in free space has been a long-standing yet elusive goal in optics. Such wave packets can have applications ranging from remote optical sensing and biological imaging, to nonlinear and quantum optics. To date, this challenge has been addressed via nonlinear optical effects that sustain solitons1, wave-guiding structures2, or by exploiting particularly shaped waveforms such as Bessel-Airy wave packets in linear dispersive media3. Propagation invariance in a linear nondispersive medium necessitates inculcating a precise spatio-temporal spectral structure into the field by introducing angular dispersion (AD)4,5; i.e., associating each wavelength with a single propagation direction6,7. Examples of such wave packets date back to Brittingham’s focus-wave mode8, X-waves9,10, and more recently the general class of ‘space-time’ (ST) wave packets11–19. The challenge of producing the AD necessary for propagation-invariant wave packets localized in all dimensions (referred to hereon as 3D ST wave packets) is twofold. First, the AD must be inculcated in two transverse dimensions rather than in one as typically realized via gratings or prisms1. Second, non-differentiable AD is required20; i.e., it is necessary that the derivative of the wavelength-dependent propagation angle not be defined at some wavelength21,22 – a field configuration that cannot be directly produced with conventional optical components. Consequently, with the exception of X-waves that are AD-free, no propagation-invariant optical wave packets that are localized in all dimensions have been observed in free space2.

The challenge of introducing arbitrary AD into a generic pulsed beam along one transverse dimension has been recently addressed by constructing a universal AD synthesizer23. This experimental strategy has enabled the realization of ST wave packets in the form of light sheets16 (referred to hereon as 2D ST wave packets), which exhibit a broad host of sought-after effects, such as long-distance propagation invariance17, tunable group velocities18,19,25–29, anomalous refraction at

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planar interfaces\textsuperscript{39}, and the space-time Talbot effect\textsuperscript{31}. Although this arrangement produces non-differentiable AD with high spectral resolution, these features cannot be extended to both transverse dimensions. Crucially, the centerpiece of this configuration is a spatial light modulator that modifies the temporal spectrum along one dimension, leaving only one dimension to manipulate the field spatially – a limitation that is shared by other recently investigated spatio-temporal field structures\textsuperscript{32–38}. Therefore, the fundamental challenge of producing non-differentiable AD encompassing both transverse dimensions remains outstanding.

Here, we demonstrate a spatio-temporal modulation strategy that efficiently produces arbitrary yet precise AD in two transverse dimensions, and thus yields ST wave packets localized in all dimensions – while preserving all the key attributes of its reduced-dimension counterpart. This modulation scheme is implemented in three stages. In the first stage, the spectrum of a generic plane-wave pulse is spatially resolved along one dimension after a double-pass through a volume chirped Bragg grating. In the second stage, a spectral transformation ‘reshuffles’ the wavelengths into a prescribed sequence. In the third stage, a log-polar-to-Cartesian conformal coordinate transformation converts the spatial locus of each wavelength from a line into a circle. A lens finally converts the spatially resolved wave front into a 3D ST wave packet localized in all three dimensions. Utilizing this approach, we produce 3D ST wave packets with ∼30 pm transverse beam width and ∼6 ps pulse width that propagate for over 50 mm. Moreover, by modulating the spatio-temporal spectral structure, we realize group velocities extending from the subluminal to the superluminal regimes over the range from 0.7c to 1.8c (c is the speed of light in vacuum). Furthermore, by providing access to both transverse dimensions in a ST wave packet, new degrees of freedom of the optical field can be accessed, such as orbital angular momentum (OAM)\textsuperscript{40–44}. Specifically, by encoding a helical phase structure in the spatio-temporal spectrum, we demonstrate propagation-invariant pulsed OAM wave packets with controllable group velocity in free space, which we refer to as ST-OAM wave packets. In addition to the propagation-invariance and arbitrary group velocities of ST-OAM wave packets, their underlying spatio-temporal structure may lead to variations of some of the recently uncovered behaviors of conventional OAM pulses, such as the trade-off between the topological charge and pulse duration\textsuperscript{40,41,46}. Such 3D ST wave packets that are fully localized in all dimensions have potential uses in areas such as free-space optical communications, imaging, and nonlinear optics.

Results

Theory of 3D space-time wave packets

A useful conceptual tool for understanding the characteristics of ST wave packets and the requirements for their synthesis is to visualize their spectral support domain on the surface of the light cone. The light-cone is the geometric representation of the free-space dispersion relationship $\kappa_x^2 + \kappa_y^2 + \kappa_z^2 = \omega^2/c^2$, where $\omega$ is the temporal frequency, $c$ is the speed of light in vacuum, $(\kappa_x, \kappa_y, \kappa_z)$ are the components of the wave vector in the Cartesian coordinate system ($x, y, z$) and $x$ and $y$ are the transverse coordinate, and $z$ is the axial coordinate. Although this relationship corresponds to the surface of a four-dimensional hypercone, a useful representation follows from initially restricting our attention to azimuthally symmetric fields in which $\kappa_x$ and $\kappa_y$ are combined into a radial wave number $k_r = \sqrt{k_x^2 + k_y^2}$, so that the light-cone can be then visualized in $(k_r, k_z, \phi)$-space (Fig. 1). The spectral support domain for 3D ST wave packets is restricted to the conic section at the intersection of the light-cone with a spectral plane that is parallel to the $k_z$ axis and makes an angle $\theta$ (the spectral tilt angle) with the $k_r$ axis, which is given by the equation $\Omega = (k_r - k_0)\tan \theta$; here $\Omega = \omega - \omega_0$, $\omega_0$ is a carrier frequency, and $k_0 = \omega_0/c$. It can be readily shown that such a construction in the narrowband paraxial regime results in a propagation-invariant 3D ST wave packet $E(r, z, t) = e^{i(k_zz - \omega_0t)}\psi(k_r, k_z, t)$, where the slowly varying envelope $\psi(k_r, k_z, t)$ travels rigidly at a group velocity $\vec{v} = \nabla \theta$, $\psi(k_r, k_z, t) = \psi(r, 0, t - z/\vec{v})$, where $\psi(r, 0, t) = \int dk_r \hat{k}_r \hat{\psi}(k_r) e^{i(k_r r - \omega t - \sigma)}$, and $\hat{\psi}(k_r)$ is the spectrum. Here $k_r$ and $\Omega$ are no longer independent variables, but are instead related via the paraxial spectral trajectory on the light-cone (Supplementary Note 1). Although this spectral trajectory is a conic section whose kind is determined by the spectral tilt angle $\theta$, it can nevertheless be approximated in the narrowband paraxial regime by a parabola in the vicinity of $k_r = 0$:

\[
\frac{\Omega}{\omega_0} = \frac{k_r^2}{2k_0^2(1 - n)}.
\]

where $n = \cot \theta$ is the wave-packet group index in free space. By setting $k_r = k_0 \sin(\phi \omega)$, where $\phi \omega$ is the propagation angle for $\omega$ as shown in Fig. 1a, we have $\omega \phi = q \Omega \phi$, which is not differentiable at $\Omega = 0$\textsuperscript{33}; here $n = 1 - q^2\sigma^2$, $\sigma = 1$ in the superluminal regime, and $\sigma = 1$ in the subluminal regime. In other words, non-differentiable AD is required to produce a propagation-invariant ST wave packet. This result is similar to that for ST light-sheets\textsuperscript{8} except that the transverse coordinate $x$ is now replaced with the radial coordinate $r$.

The representation in Fig. 1 is particularly useful in identifying a path towards synthesizing 3D ST wave packets. When $45^\circ < \theta < 90^\circ$, the ST wave packet is superluminal $\vec{v} > c$, $\Omega$ is positive, and $\omega_0$ is the minimum allowable frequency in the spectrum. When viewed in $(k_r, k_z, \phi)$-space, the wavelengths are arranged in concentric circles, with long wavelengths (low frequencies) at the center, and shorter wavelengths (higher frequencies) extending outward. On the other hand, when $0^\circ < \theta < 45^\circ$, the ST wave packet is subluminal $\vec{v} < c$, $\Omega$ is negative, and $\omega_0$ is the maximum allowable frequency in the spectrum. The wavelengths are again arranged in concentric circles in $(k_r, k_z, \phi)$-space – but in the opposite order: short wavelengths are close to the center and longer wavelengths extend outward. For both subluminal and superluminal 3D ST wave packets, each $\omega$ is associated with a single radial spatial frequency $k_r(\omega)$, and is related to it via the relationship in Eq. (1). This representation indicates the need for arranging the wavelengths in concentric circles with square-root radial chirp, and then converting the spatial spectrum into physical space via a spherical lens. Moreover, adding a spectral phase factor $e^{\imath \psi}$, where $\ell$ is an integer and $\chi$ is the azimuthal angle in spectral space, produces OAM in physical space (Supplementary Note 1B).

Closed-form expressions can be obtained for 3D ST wave packets by applying Lorentz boosts to an appropriate initial field\textsuperscript{26–32}. For example, starting with a monochromatic beam $E(r, z, t)$, a subluminal 3D ST wave packet at a group velocity $\vec{v}$ is obtained by the Lorentz boost $E(r, z, t) = E_0\left(\frac{z - vt}{\sqrt{1 - \beta^2}}, \frac{r - vt}{\sqrt{1 - \beta^2}}\right)$, where $\beta = \frac{v}{c}$ is the Lorentz factor. On the other hand, closed-form expressions for superluminal 3D ST wave packets can be obtained by applying a Lorentz boost to the ‘needle beam’ in\textsuperscript{26}. The time-averaged intensity is $I(r, \varphi, z) = 2\pi n k_f^2(1 - n) \int dk_r k_r^2 |\hat{\psi}(k_r)|^2 f^2(\hat{\psi}(k_r))$, which is independent of $\varphi$ even if the field is endowed with OAM. In the case of 2D ST light-sheets, the time-averaged intensity separates into a sum of a constant background pedestal and a spatially localized feature at the center\textsuperscript{26}. A similar decomposition is not possible for 3D ST wave packets. However, using the asymptotic form for Bessel functions that is valid far from $r = 0$, we have:

\[
I(r) = \frac{2\pi n k_f^2(1 - n)}{\pi r} \int dk_r \sqrt{k_r^2 + \ell^2} |\hat{\psi}(k_r)|^2 + \frac{2\pi n k_f^2(1 - n)(-1)^\ell}{\pi r} \int dk_r \sqrt{k_r^2 + \ell^2} |\hat{\psi}(k_r)|^2 \sin(2k_r r).
\]
where the first term is a pedestal decaying at a rate of $r$, and the second term tends to be localized closer to the beam center. In the vicinity of $r=0$, the two terms merge and cannot be separated. The spatio-temporal intensity profile of such a 3D ST wave packet is depicted in Fig. 1c, two conic field structures emanate from the wave-packet center, such that the profile is X-shaped in any (meridional plane containing the optical axis, and the intensity profile is circularly symmetric in any transverse plane.

**Fig. 1** | Visualization of the spectral support domain for 3D ST wave packets on the surface of the free-space light-cone. 

- **a** The spectral support domain for a superluminal 3D ST wave packet at the intersection of the light-cone $k_r^2 + k_z^2 = \left(\frac{\omega}{c}\right)^2$ with a spectral plane that is parallel to the $k_r$-axis and makes an angle $\theta > 45^\circ$ with the $k_z$-axis. The conic section at the intersection is a hyperbola. In $(k_x, k_y, \omega/c)$-space the spectrum is one half of a two-sheet hyperboloid (an elliptic hyperboloid).

- **b** Same as **a** for a subluminal ST wave packet with $\theta < 45^\circ$, where the spectral support domain on the light-cone in $(k_x, k_y, \omega/c)$-space is an ellipse. In $(k_x, k_y, \omega/c)$-space, the spectrum is an ellipsoid of revolution (a spheroid, which may be prolate or oblate according to the value of $\theta$). 

- **c** Plot of the spatio-temporal intensity profile $I(x, y, z=0; t)$ at a fixed axial plane $z=0$, the intensity profile in a meridional plane $I(x, 0, z=0; t)$, and the transverse profiles at the wave-packet center $I(x, y, 0; 0)$ and off-center $I(x, y, 0; t>0)$.
Synthesizing ST wave packets localized in all dimensions

Central to converting a generic pulsed beam into a ST wave packet localized in all dimensions is the construction of an optical scheme that can associate each wavelength $\lambda$ with a particular azimuthally symmetric spatial frequency $k(\lambda)$ and arrange the wavelengths in concentric circles with the order prescribed in Eq. (1) (Fig. 2a). This system realizes two functionalities, producing a particular wavelength sequence, and changing the coordinate system, which are implemented in succession via the three-stage strategy outlined in Fig. 2b. In the first stage, the spectrum of a plane-wave pulse is resolved along one spatial dimension. At this point, the field is endowed with linear spatial chirp and the wavelengths are arranged in a fixed sequence. The second stage rearranges the wavelengths in a new prescribed sequence. This spectral transformation is tunable; that is, a wide range of spectral structures can be obtained from a fixed input. In the third stage, a 2D conformal coordinate transformation converts vertical lines into circles, thereby realizing the targeted spatio-temporal spectra $S_1$, $S_2$, and $S_3$.

For the sake of benchmarking, we also synthesized pulsed Bessel beams with separable spatio-temporal spectrum by circumventing the spectral analysis and 1D spectral transformation, and sending the input laser pulses directly to the 2D coordinate transformation. To match the temporal bandwidth of the pulsed Bessel beams to that of the 3D ST wave packets, we spectrally filter $\Delta \lambda = 0.3$ nm from the input spectrum via a planar Fabry-Perot cavity.

Characterizing 3D ST wave packets

To verify the structure of the synthesized 3D ST wave packet, we characterize the field in four distinct domains: (1) the spatio-temporal spectrum to verify the square-root radial chirp (Fig. 4); (2) the time-averaged intensity to confirm diffraction-free propagation along $z$
the measured spatial chirp engendered by the 2D coordinate transformation, and combining with the spectrum by scanning a single-mode spectral-domain characterization $k_r$ phase of the ST-OAM wave packets (Fig. 7).

scanning a CCD camera along the propagation axis position a narrow spectrum (domain, as shown in Fig. 4a. By calibrating the conversion averaged intensity pro

superluminal and subluminal cases (Fig. 4c). The measurement is repeated for superluminal

Bessel beam whose spatio-temporal spectrum is separable, where the spatial bandwidth is $\Delta k_r = 0.02 \text{ rad/\mu m}$ and is centered at $k_r = 0.06 \text{ rad/\mu m}$ (Fig. 5a). Here, the full temporal bandwidth $\Delta \lambda$ is associated with each spatial frequency $k_r$. The finite spatial bandwidth $\Delta k_r$ renders the propagation distance finite$, and we observe a Bessel beam comprising a main lobe of width $\Delta r = 30 \text{ mm (FWHM)}$ accompanied by several side lobes, which propagates for a distance $L_{\text{max}} = 50 \text{ mm}$. For comparison, the Rayleigh range of a Gaussian beam with a similar size and central wavelength is $z_R = 1 \text{ mm}$. By further increasing $\Delta k_r$ to $0.07 \text{ rad/\mu m}$ while remaining centered at $k_r = 0.06 \text{ rad/\mu m}$ as shown in Fig. 5b, the axial propagation distance is reduced proportionately to $L_{\text{max}} = 15 \text{ mm}$, and the side lobes are diminished.

Now, rather than the separable spatio-temporal spectra for pulsed Bessel beams (Fig. 5a, b), we utilize the structured spatio-temporal spectra associated with 3D ST wave packets in which each $k_r$ is associated with a single $\lambda$ (Fig. 4e), whose spatial bandwidths are all $\Delta k_r = 0.07 \text{ rad/\mu m}$ centered at $k_r = 0.06 \text{ rad/\mu m}$, similarly to the pulsed Bessel beam in Fig. 5b. Despite the large spatial bandwidth, the one-to-one correspondence between $k_r$ and $\lambda$ curtails the effect of diffraction, leading to an increase in the propagation distance (Fig. 5c–e). The subluminal 3D ST wave packet ($\Delta k = 0.83c$) in Fig. 5e propagates for $L_{\text{max}} = 60 \text{ mm}$, which is a $4 \times$ improvement compared with the separable Bessel beam and a $60 \times$ improvement compared with a Gaussian beam of the same spatial bandwidth. We observe a similar behavior for a superluminal 3D ST wave packet ($\Delta k = 1.37c$) in Fig. 5d, and a superluminal ST-OAM wave packet ($\Delta k = 1.16c$) with $\ell = 1$ in Fig. 5e.

Reconstructing the spatio-temporal profile and measuring the group velocity. The spatio-temporal intensity profile $|E(x, y, z; t)|^2$ of the 3D ST wave packet is reconstructed by placing the
synthesizer (Fig. 3) in one arm of a Mach-Zehnder interferometer, while the initial 100-fs plane-wave pulses from the laser traverse an optical delay line $\tau$ in the reference arm (Fig. 6a). By scanning $\tau$ we reconstruct the spatio-temporal intensity profile in a meridional plane from the visibility of spatially-resolved interference fringes recorded by a CCD camera showing an annular structure. The reconstructed spatio-temporal intensity profile $I(0, y, z; t)$ of the 3D ST wave packets corresponding to those in Fig. 5c–e are plotted in Fig. 6b–d at multiple axial planes, which reveal clearly the expected X-shaped profile that remains invariant over the propagation distance $L_{\text{max}}$. In all cases, the on-axis pulse width, taken as the FWHM of $I(0, 0, 0; t)$, is $\Delta t = 6$ ps. The spatio-temporal intensity profile of the superluminal ST-OAM wave packet with $\ell = 1$ in Fig. 6d reveals a similar X-shaped profile, but with a central null instead of a peak, as expected from the helical phase structure associated with the OAM mode.

A subtle distinction emerges between the subluminal and superluminal wave packets regarding the axial evolution of their spatio-temporal spectral structure of 3D ST wave packets. Measurements for a superluminal wave packet ($\nu = 1.37c$) are plotted in the left column, and those for its subluminal counterpart ($\nu = 0.83c$) are plotted in the column on the right. a Measured spatial spectrum $|\tilde{\psi}(k_x, k_y)|^2$ by a wavelength-insensitive camera showing an annular structure. b Measured temporal spectra at selected radial positions revealing the radial chirp and the spectral uncertainty. c Measured radial chirp by plotting the central wavelength $\lambda_c$ of the spectrum with radial spatial frequency $k_r$. Error bars in c represent the spectral resolution of the optical spectrum analyzer (OSA; Advantest AQ6317B) we made use to perform spectral measurements.

Fig. 4 | The spatio-temporal spectral structure of 3D ST wave packets. Measurements for a superluminal wave packet ($\nu = 1.37c$) are plotted in the left column, and those for its subluminal counterpart ($\nu = 0.83c$) are plotted in the column on the right. a Measured spatial spectrum $|\tilde{\psi}(k_x, k_y)|^2$ by a wavelength-insensitive camera showing an annular structure. b Measured temporal spectra at selected radial positions revealing the radial chirp and the spectral uncertainty. c Measured radial chirp by plotting the central wavelength $\lambda_c$ of the spectrum with radial spatial frequency $k_r$. Error bars in c represent the spectral resolution of the optical spectrum analyzer (OSA; Advantest AQ6317B) we made use to perform spectral measurements.
It can be shown that in presence of finite spectral uncertainty $\delta \lambda$, the realized ST wave packet can be separated into the product of an ideal ST wave packet traveling indefinitely at $c$ and a long ‘pilot envelope’ traveling at $c$. The finite propagation distance $L_{\text{max}}$ is then a consequence of temporal walk-off between the ST wave packet and the pilot envelope. For subluminal ST wave packets, this results initially in a ‘clipping’ of the leading edge of the wave packet in Fig. 6b at $z = 20$ mm, and ultimately a clipping of the trailing edge of the ST wave packet as the faster pilot envelope catches up with it (Fig. 6b at $z = 40$ mm). The opposite behavior occurs for the superluminal ST wave packet in Fig. 6c, d.

This experimental methodology also enables us to estimate the group velocity $e_v$. After displacing the CCD camera until the interference fringes are lost due to the mismatch between $\bar{v} = c \tan \theta$ for the ST wave packets and the reference pulses traveling at $\bar{v} = c$, we restore the interference by inserting a delay $\Delta t$ (Fig. 6e), which allows us to estimate $\bar{v}$ for the 3D ST wave packet. By tuning $B$, we record a broad span of group velocities in the range from $\bar{v} = 0.7c$ to $\bar{v} = 1.8c$ in free space (Fig. 6f). The continuous tunability of the group velocity of 3D ST wave packets over the subluminal and superluminal ranges allows them to be exploited in applications previously proposed for ST light-sheets, such as for constructing in-line optical delay lines for all-optical communications, whereby the localization of 3D ST wave packets in both transverse dimensions can provide a significant advantage with regards to efficiently coupling into optical fibers.

Field amplitude and phase measurements. Lastly, we modify the measurement system in Fig. 6a by adding a small relative angle between the propagation directions of the 3D ST wave packets and the reference pulses, and make use of off-axis digital holography to reconstruct the amplitude $|\psi(x, y, z; \tau)|$ and phase $\phi(x, y, z; \tau)$ of their complex field envelope $\psi(x, y, z; \tau) = |\psi(x, y, z; \tau)| e^{i \phi(x, y, z; \tau)}$ (Supplementary Note 3D). We

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Fig. 5 | Measured transverse and axial time-averaged intensity for separable pulsed Bessel beams and 3D ST wave packets. In the first column, we illustrate the spatio-temporal structure; in the second, we plot the measured transverse intensity $I(x, y, z)$ at $z = 30$ mm, in addition to sections through $x = 0$ and $y = 0$ (white curves); and, in the third, we plot the measured intensity in a meridional plane $I(0, y, z)$. The white curve at the bottom of the panels in the last column is the on-axis intensity $I(0, 0, z)$, except in e where we use $y = 30 \mu m$. For all cases, $\Delta l = 0.3$ nm. a A separable pulsed Bessel beam with $\Delta k_r = 0.02$ rad/$\mu m$. b A pulsed Bessel beam with $\Delta k_r = 0.07$ rad/$\mu m$. c–e In all cases $\Delta k_r = 0.07$ rad/$\mu m$ as in b. c A subluminal $(\bar{v} = 0.83c)$ 3D ST wave packet; d a superluminal $(\bar{v} = 1.37c)$ 3D ST wave packet; and e a superluminal $(\bar{v} = 1.16c)$ 3D ST-OAM wave packet with $\ell = 1$ (the inset in the first column is the associated transverse spectral phase distribution). The dotted vertical white lines in the third column in c–e identify the axial planes for the time-resolved measurements in Fig. 6.

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Δλ = 0.3 nm. a A separable pulsed Bessel beam with Δk_r = 0.02 rad/μm. b A pulsed Bessel beam with Δk_r = 0.07 rad/μm. c–e In all cases Δk_r = 0.07 rad/μm as in b. c A subluminal (v_bar = 0.83c) 3D ST wave packet; d a superluminal (v_bar = 1.37c) 3D ST wave packet; and e a superluminal (v_bar = 1.16c) 3D ST-OAM wave packet with ℓ = 1 (the inset in the first column is the associated transverse spectral phase distribution). The dotted vertical white lines in the third column in c–e identify the axial planes for the time-resolved measurements in Fig. 6.
Fig. 6 | Reconstructing the spatio-temporal intensity profile and estimating the group velocity for 3D ST wave packets. 

**a** Schematic of the interferometric configuration for reconstructing \( I(x, y, z; t) \) and estimating \( \vec{v} \).

**b** Measured \( I(0, y, z; \tau) \) at \( z = 20, 30, \) and 40 mm for a subluminal \( (\vec{v} = 0.83c) \) wave packet; **c** for a superluminal \( (\vec{v} = 1.37c) \) wave packet; and **d** for a superluminal \( (\vec{v} = 1.16c) \) wave packet endowed with the OAM mode \( \ell = 1 \). We also plot the section \( y = 0 \) through the intensity profile (white curve at the bottom of each panel), except in **d** where we use \( y = 30 \mu m \). **e** Measured group delay \( \Delta t \) at different axial planes for subluminal and superluminal 3D ST wave packets. The straight lines are theoretical expectations and the symbols are data points. **f** Plot of the estimated group velocity \( \vec{v} \) with the 1D spectral transformation parameter \( B \). The curve is the theoretical expectation \( \vec{v} = c/n \), with \( n = 1 - \frac{\Delta t}{\Delta z} (B \text{ in mm}) \). Error bars correspond to the uncertainty in the measurement of \( \vec{v} \) due to the finite pulse width of 3D ST wave packets; see Supplementary Note 3C.
reconstruct the complex field at a fixed axial plane \( z = 30 \) mm for the time delays: \( \tau = -5 \), 0, and \( 5 \) ps (Fig. 7). First, we plot the results for \( |\psi(x, y, z; \tau)|^2 \) and phase \( \phi(x, y, z; \tau) \) for a superluminal 3D ST wave packet (\( \Omega = 1.1c \)) with no OAM (\( \ell = 0 \)). At the pulse center \( \tau = 0 \), the field is localized on the optical axis, whereas at \( \tau = \pm 5 \) ps the field spreads away from the center (Fig. 7a). For \( \tau = 0 \) we find a spherical transverse phase distribution that is almost flat at \( \tau = 0 \), similar to what one finds during the axial evolution of a Gaussian beam in space through a focal plane. After adding the OAM mode \( \ell = 1 \) to the field structure, a similar overall behavior is observed for the superluminal ST-OAM wave packet except for two significant features. First, a dip is observed on axis in Fig. 7b, in lieu of the central peak in Fig. 7a, as a result of the phase overall behavior is observed for the superluminal ST-OAM wave packet two 3D ST wave packets in Fig. 7a, b. We increase the propagation length to the kilometer range 24. The spectral uncertainty reducing the complexity of the synthesis system, potentially without 24. The 2D transformation used to construct the 3D STWP can be implemented by making use of diffractive optics 25,27,23, or refractive optics 54. We exploited both types of phase plates in our experiments to imprint the desired phase profiles: diamond-edged refractive phase plates 24 and analog diffractive phase plates 54.

**Discussion**

We have demonstrated a general procedure for spatio-temporal spectral modulation of pulsed optical fields that is capable of synthesizing 3D ST wave packets localized in all dimensions. At the heart of our experimental methodology lies the ability to sculpt the angular dispersion of a generic optical pulse in two transverse dimensions. Crucially, this approach produces the non-differentiable angular-dispersion necessary for propagation invariance 54. Because such a capability has proven elusive to date, AD-free X-waves have been the sole class of 3D propagation-invariant wave packets conclusively produced in free space. Unfortunately, X-waves can exhibit only minuscule changes in the group velocity with respect to \( c \) (typically \( \Delta v \sim 0.001c \)) in the paraxial regime, and only superluminal group velocities are supported. Furthermore, ultrashort pulses of width \(< 20 \) fs are required to observe a clear X-shaped profile 54, and OAM-carrying X-waves have not been realized to date. Even more stringent requirements are necessary for producing focus-wave modes, and consequently they have not been synthesized in three dimensions to date. By realizing instead propagation-invariant 3D ST wave packets, an unprecedented tunable span of group velocities has been realized, clear X-shaped profiles are observed with pulse widths in the picosecond regime, and they outperformed spectrally separable puls Bessel beams of the same spatial bandwidth with respect to their propagation distance and transverse side-lobe structure. In addition, we demonstrated propagation-invariant ST-OAM wave packets with tunable group velocity in free space. Further optimization of the experimental layout is possible. We made use of four phase patterns to produce the target spatio-temporal spectral structure. It is conceivable that this spectral modulation scheme can be performed with only three phase patterns, or perhaps even fewer. Excitingly, a new theoretical proposal suggests that a single non-local nanophotonic structure can produce 3D ST wave packets through a process of spatio-temporal spectral filtering 54. This theoretical proposal indicates the role nanophotonics is poised to play in reducing the complexity of the synthesis system, potentially without recourse to filtering strategies.

Finally, efforts in the near future will be directed to reducing the spectral uncertainty \( \delta \ell \) and concomitantly approaching \( \theta \sim 45^\circ \) to increase the propagation length to the kilometer range 55. The experimental procedure presented here can in principle be extended to the synthesis of other exotic variants of ST wave packet, such as abruptly focusing needle pulses 56 among other possibilities 57,56. With access to 3D ST wave packets, previous work on guided ST modes in planar wave-guides 54 can be extended to conventional single-mode and multi-mode waveguides 62, and potentially to optical fibers 53-57. Moreover, the localization in both transverse dimensions provided by 3D ST wave packets opens new avenues for nonlinear optics by increasing the intensity with respect to 2D ST wave packets, for introducing topological features such as spin texture in momentum space 58, and for the exploration of spatio-temporal vortices and polarization singularities 67. Our findings point therefore to profound new opportunities provided by the emerging field of space-time optics 60.

**Methods**

The 2D transformation used to construct the 3D STWP can be implemented by making use of diffractive optics 25,27,23, or refractive optics 54.

The diffractive phase plates were fabricated in fused silica using Clemson University facilities. The fabrication process is outlined in 54, which involves writing a binary phase grating on a stepper mask with an electron-beam and subsequently transferring this analog mask into a fused silica substrate with projection lithography. The phase grating period is designed to be larger than the cutoff period of the projection stepper for higher diffraction orders, so only the zeroth-order diffracted light from the stepper can be transmitted. The transmission coefficient of the stepper light is then a function of the duty cycle of the electron-beam patterned binary phase grating. The spatial intensity distribution of light in the wafer plane can be controlled with a spatial duty cycle function, which then exposes the Hine resist with a spatially varying analog intensity profile. This allows fabrication of analog diffractive optics with a single exposure from the stepper rather than binary 2 diffractive optics, resulting in excellent spatial and temporal resolution.

**Refractive phase plates**

The refractive optical elements used in our experiments are similar to those outlined by Lavery et al. in 54, in which the transformation parameters are \( C = 4.77 \) mm, \( D = 12 \pm 1 \) mm, and \( d = 310 \) mm. Each phase plate is made of the polymer PMMA (Poly methyl methacrylate) with accurately manufactured height profiles \( Z(x, y) \) and \( Z(x, y) \) to imprint the required phase profiles. The phase encountered by light at a wavelength \( \lambda \) traversing a height \( Z \) of refractive index \( n \) with respect to the phase encountered over the same distance in vacuum – is given by \( \Phi = 2\pi(n-1)Z/\lambda \). Thus, the height profile of the first element is \( Z(x, y) = \frac{\delta L}{\delta x} \Phi(x, y) \) (Supplementary Fig. 14a) and that of the profile of the second element is \( Z(x, y) = \frac{\delta L}{\delta x} \Phi(x, y) \) (Supplementary Fig. 14b). Note that each surface height is wavelength-independent, and dispersion effects in the material manifest themselves as a change in the focal length \( d \) of the integrated lens for different wavelengths. Hence, in the experiment the system can be tuned to a specific wavelength by changing the distance between the two elements.

The elements were diamond-machined using a Natotech, 3-axis \((X, Z, C)\) ultra precision lathe (UPL) in combination with a Nanotech NFTS6000 fast tool servo (FTS) system. The machined PMMA surfaces had a radius of 5.64 mm, angular spacing \( 1^\circ \), radial spacing of 5 \( \mu \)m, a spindle speed of 500 RPM, a roughing feed rate 5 mm/minute with a finishing feed rate of 1 mm/minute with a cut depth of 20 \( \mu \)m, and a finishing feed rate of 1 mm/minute with a cut depth of 20 \( \mu \)m. The total sag height difference for each part was relatively small (<15 \( \mu \)m for surface 1 and <144 \( \mu \)m for surface 2). The transmission efficiency of the combination of the elements is >85%.
Fig. 7 | Measured complex-field amplitude and phase profiles for 3D ST wave packets with and without OAM. a Measured amplitude $\psi(x, y)$ (first row) and phase $\phi(x, y)$ (second row) at a fixed axial plane $z = 30$ mm (see Figs. 5 and 6) at delays $\tau = -5$ ps, $\tau = 0$ corresponding to the wave-packet center, and $\tau = 5$ ps for a superluminal 3D ST wave packet with $\ell = 0$ and $a \ell = 1$. c Iso-amplitude contour $I = 0.6 I_{\text{max}}$ for the 3D ST wave packet from (a) and ST-OAM from (b). d Same as c but for the iso-amplitude contour $I = 0.15 I_{\text{max}}$. 

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in high-efficiency optics. The transmission efficiency of the combination of the two faces is ≈92%.

The design parameters for the analog diffractive phase plates are chosen as follows: \( D = \frac{2}{\pi} = 2.2 \text{ mm}, C = 6 \text{ mm}, \lambda_n = 798 \text{ nm}, \) and \( d_s = 225 \text{ mm}. \) These design parameters were optimized so the paraxial approximation remains valid over the desired transformation range of 5 mm.

### Data availability

The data that support the plots within this paper and other findings of this study are available from the corresponding author upon reasonable request.

### Code availability

The software code used for data acquisition and data analysis are available from the corresponding author upon reasonable request.

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