Local Stable Manifold for the Bidirectional Discrete-Time Dynamics

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Abstract. We show the existence of a local stable manifold for a bidirectional discrete-time nondiffeomorphic nonlinear Hamiltonian dynamics. This is the case where zero is a closed loop eigenvalue and therefore the Hamiltonian matrix is not invertible. In addition, we show the eigenstructure and the symplectic properties of the mixed direction nonlinear Hamiltonian dynamics. We extend the Local Stable Manifold Theorem for the nonlinear discrete-time Hamiltonian map with a hyperbolic fixed point. As a consequence, we show the existence of a local solution to the Dynamic Programming Equations, the equations corresponding to the discrete-time optimal control problem.

Keywords: Stable Manifold, Dynamic Programming Equations, Optimal Control, Discrete-Time Hamiltonian Dynamics

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1. Introduction

One application of the local stable manifold theorem in optimal control problems is that the stable manifold of the associated Hamiltonian dynamics describes the graph of the optimal cost. The optimal cost and the optimal control satisfy the Dynamic Programming Equations (DPE), which are obtained from the optimal control problem of minimizing a discrete-time, nonlinear cost subject to a nonlinear discrete-time dynamics through the dynamic programming technique. In proving the existence of the solutions of the DPE, we will use the Pontryagin Maximum Principle (PMP) which gives the bidirectional nonlinear Hamiltonian dynamics and the condition for a control to be optimal satisfied by the optimal state and costate trajectories.

In the case in which the Hamiltonian matrix of the dynamics is invertible, the nonlinear Hamiltonian dynamics can be rewritten as a dynamics with both state and costate dynamics propagating in the direction where time approaches infinity. Assuming the invertibility of the Hamiltonian matrix is to exclude zero as a closed loop eigenvalue. The formal solutions to the DPE have been worked out using Al’brecht’s method ([1]) in [24] and are valid for all closed loop eigenvalues lying inside the unit circle. To our knowledge the local stable manifold theorem has only been proven for invertible maps, see ([11], [12]). Our main result is the extension of the local stable manifold theorem to the bidirectional, discrete-time, nondiffeomorphic, nonlinear Hamiltonian dynamics. This generalizes the proof of the existence of the local solutions to the DPE found in [24].

For invertible maps, Hartmann [12] had shown the existence of a stable manifold by the method of successive approximations on the implicit functional equation. Another method, developed by Kelley [14], is the technique of using the Contraction Mapping Theorem on a complete space. There is another method by Irwin [13] based on an application of the inverse function theorem on a Banach space of sequences. After a two-step process of diagonalizing the bidirectional discrete-time Hamiltonian dynamics, we apply the technique of Kelley [14] on a complete space of Lipschitz functions endowed with the supremum norm.

The paper is organized as follows. In the next section we introduce the bidirectional discrete-time Hamiltonian dynamics from the Pontryagin Maximum Principle that is associated with the optimal control problem. In Section 3 we give some discussion of Gronwall’s inequalities in the discrete-time case, which will then be used in the proof. In Section 4 we state and prove the local stable manifold theorem. In Section 5 we discuss the eigenstructure and the symplectic properties of the dynamics. Finally, in Section 6 we show how the Theorem along with these properties give the local solvability of the Dynamic Programming Equations.

2. Nonlinear Dynamics

We formulate a discrete in time infinite horizon optimal control problem of minimizing the cost functional,

$$\min_{u} \sum_{k=0}^{\infty} l(x_k, u_k)$$

subject to the dynamics

$$x^+ = f(x, u)$$
$$x(0) = x_0$$
where the state vector $x \in \mathbb{R}^n$, the control $u \in \mathbb{R}^m$, and
\begin{align}
(2.1) \quad f(x, u) &= Ax + Bu + f^{[2]}(x, u) + f^{[3]}(x, u) + \ldots \\
(2.2) \quad l(x, u) &= \frac{1}{2}x'Qx + x'Su + \frac{1}{2}u'Ru + l^{[3]}(x, u) + \ldots 
\end{align}
where $f^{[m]}(x, u)$ and $l^{[m]}(x, u)$ as homogeneous polynomials in $x$ and $u$ of degree $m$.
We let $x^+ = x_{k+1}$ and $x = x_k$.

Associated with optimal control problem formulation there is a nonlinear Hamiltonian,
\begin{equation}
H(x, u, \lambda^+) = (\lambda^+)'f(x, u) + l(x, u) \tag{2.3}
\end{equation}
where $\lambda^+ = \lambda_{k+1}$ and the functions $f$ and $g$ are given by equations (2.1) and (2.2).

The Pontryagin Maximum Principle (PMP) states the following:
\begin{equation}
\text{Theorem 2.1. If } x_k \text{ and } u_k \text{ are optimal for } k \in 0, 1, 2, \ldots, \text{ then there exists } \lambda_k \neq 0 \\
\text{for } k \in 0, 1, 2, \ldots \text{ such that }
\end{equation}
\begin{align}
(2.4) \quad x^+ &= \frac{\partial H}{\partial \lambda^+}(x, u, \lambda^+) \\
(2.5) \quad \lambda &= \frac{\partial H}{\partial x}(x, u, \lambda^+)
\end{align}
and
\begin{equation}
(2.6) \quad u^* = \arg\min_u H(x, u, \lambda^+).
\end{equation}

Thus, the minimizer of the nonlinear Hamiltonian evaluated at the optimal $x$ and $\lambda^+$ is the optimal control $u^*$ amongst all admissible controls $u$. Note that $u^*(x, \lambda^+) \in C^{r-1}$ since $f \in C^{r-1}$ and $l \in C^r$ in (2.3). We assume that $H$ is convex in $u$ to guarantee a unique optimal control. With the Hamiltonian (2.3), the equations (2.4) and (2.5) are the following
\begin{equation}
\begin{bmatrix}
  x^+ \\
  \lambda
\end{bmatrix} = \begin{bmatrix}
  A - BR^{-1}S' & -BR^{-1}B' \\
  Q - SR^{-1}S' & A' - SR^{-1}B'
\end{bmatrix} \begin{bmatrix}
  x \\
  \lambda^+
\end{bmatrix} + \begin{bmatrix}
  F(x, \lambda^+) \\
  G(x, \lambda^+)
\end{bmatrix} \tag{2.7}
\end{equation}
where $x, \lambda \in \mathbb{R}^n$ and $F(x, \lambda^+)$ and $G(x, \lambda^+)$ contain the nonlinear terms. Observe the opposing directions of the propagation of the state and costate dynamics in (2.7).

If we linearize the $2n$ dimensional difference equations (2.7) around zero, we obtain exactly the linear Hamiltonian system,
\begin{equation}
\begin{bmatrix}
  x^+ \\
  \lambda
\end{bmatrix} = \mathbb{H} \begin{bmatrix}
  x \\
  \lambda^+
\end{bmatrix}, \tag{2.8}
\end{equation}
where
\begin{equation}
\mathbb{H} = \begin{bmatrix}
  A - BR^{-1}S' & -BR^{-1}B' \\
  Q - SR^{-1}S' & A' - SR^{-1}B'
\end{bmatrix}
\end{equation}
is the associated Hamiltonian matrix and the corresponding Hamiltonian is
\begin{equation}
H(x, \lambda^+, u) = \lambda^+'(Ax + Bu) + \frac{1}{2}x'Qx + x'Su + \frac{1}{2}u'Ru. \tag{2.9}
\end{equation}

In fact, the stable subspace of $\mathbb{H}$ is described by
\begin{equation}
\lambda = Px
\end{equation}
where the nonegative definite $P$ satisfies the discrete-time algebraic Riccati equation (DTARE),

\begin{equation}
\label{eq:dtare}
 P - A'PA + (A'PB + S)(B'PB + R)^{-1}(A'PB + S)' - Q = 0.
\end{equation}

Moreover, the stable linear subspace is

$$E_s = \begin{bmatrix} I \\ P \end{bmatrix}$$

where $E_s$ is spanned by the $n$ stable eigenvalue of the Hamiltonian matrix lying inside the unit circle. A detailed proof is found in [24]. Since the linear part of the nonlinear bidirectional dynamics is the linear Hamiltonian system, we have that the linear part of the local stable manifold is $\lambda = Px$.

### 3. Discrete-Time Version of Gronwall’s Inequalities

We first discuss various results based on the discrete-time version of Gronwall’s inequalities as these following lemmas will be useful in the proof of the local existence of a stable manifold.

**Lemma 3.1.** Suppose the sequence of scalars $\{u_j\}_{j=0}^\infty$ satisfies the difference inequality

\begin{equation}
\label{eq:discrete_ineq1}
 u_{k+1} \leq \delta u_k + L
\end{equation}

where $\delta, L \geq 0$, then

$$u_k \leq \delta^k u_0 + L \sum_{j=0}^{k-1} \delta^{k-1-j}.$$

The proof of the lemma above clearly follows from summing equation \eqref{eq:discrete_ineq1} from 1 to $k$.

**Lemma 3.2.** Suppose $\{\xi_j\}_{j=0}^\infty$ is a sequence that satisfies

$$|\xi_k| \leq C_1 \sum_{j=0}^{k-1} |\xi_j| + C_2$$

with constants $C_1, C_2 \geq 0$. Then

$$|\xi_k| \leq C_2 \sum_{j=1}^{k} (1 + C_1)^j.$$

**Proof:** Let $s_k = \sum_{j=0}^{k-1} |\xi_j|$. Then, the sequence $\{s_j\}_{j=0}^\infty$ satisfies

$$s_{k+1} \leq (1 + C_1)s_k + C_2$$

where $C_1, C_2 \geq 0$. By Lemma 3.2,

$$|s_k| \leq (1 + C_1)^k |s_0| + C_2 \sum_{j=0}^{k-1} (1 + C_1)^{k-1-j}.$$
It follows that

\[ |\xi_k| \leq (1 + C_1)|s_k| + C_2 \]
\[ \leq (1 + C_1)\left[(1 + C_1)^k|s_0| + C_2 \sum_{j=0}^{k-1} (1 + C_1)^{k-1-j}\right] \]
\[ \leq C_2 \sum_{j=0}^{k-1} (1 + C_1)^{k-j} \]
\[ \leq C_2 \sum_{j=1}^{k} (1 + C_1)^j \]

since \(|s_0| = 0.\]

4. Local Stable Manifold Theorem for the Bidirectional Discrete-Time Dynamics

In this section we prove the existence of a local stable manifold

\[ \lambda = \phi(x) \]

for the Hamiltonian dynamics,

\[ \begin{bmatrix} x^+ \\ \lambda \end{bmatrix} = \begin{bmatrix} A & -BR\lambda^{-1}B' \\ Q & A' \end{bmatrix} \begin{bmatrix} x \\ \lambda^+ \end{bmatrix} + \begin{bmatrix} F(x, \lambda^+) \\ G(x, \lambda^+) \end{bmatrix} \]

where \(x, \lambda \in \mathbb{R}^n\) and zero is an eigenvalue of \(A\). The nonlinear terms, \(F\) and \(G\), are \(C^r\) functions for \(r \geq 1\) such that

\[ F(0, 0) = 0, \quad G(0, 0) = 0 \]
\[ \frac{\partial F}{\partial (x, \lambda)}(0, 0) = 0, \quad \frac{\partial G}{\partial (x, \lambda)}(0, 0) = 0. \]

Since we are locally proving the existence of the stable manifold, we just need the local behavior of the dynamics; so first we talk about cut-off functions. The proof also requires the discussion on the stability of the nonlinear state dynamics. So the following subsection deals the local asymptotic stability of the state dynamics. Then, we describe the diagonalization of the bidirectional Hamilton system. Finally, we show the existence of \(\lambda = \phi(x)\).

4.1. Cut-off Functions. First, we introduce a \(C^\infty\) cut-off function \(\rho(y) : \mathbb{R}^n \rightarrow [0, 1]\) such that

\[ \rho(y) = \begin{cases} 1, & \text{if } 0 \leq |y| \leq 1 \\ 0, & \text{if } |y| > 2 \end{cases} \]

and \(0 \leq \rho(y) \leq 1\) otherwise. Then we define the functions

\[ F(x, \lambda^+; \epsilon) := F(x \rho\left(\frac{x}{\epsilon}\right), \lambda^+ \rho\left(\frac{\lambda^+}{\epsilon}\right)) \]
\[ G(x, \lambda^+; \epsilon) := G(x \rho\left(\frac{x}{\epsilon}\right), \lambda^+ \rho\left(\frac{\lambda^+}{\epsilon}\right)) \]
for \( x, \lambda^+ \in \mathbb{R}^n \). Since the \( F(x, \lambda^+) \) and \( G(x, \lambda^+) \) agree with \( F(x, \lambda^+; \epsilon) \) and \( G(x, \lambda^+; \epsilon) \), respectively, for \( |x|, |\lambda^+| \leq \epsilon \), it suffices to prove the existence of a stable manifold for some \( \epsilon > 0 \).

Now, we show that there exists \( N_1 > 0 \) and \( N_2 > 0 \) such that

\[
|F(x, \lambda; \epsilon) - F(\bar{x}, \tilde{\lambda}; \epsilon)| \leq N_1 \epsilon |x - \bar{x}| + |\lambda - \tilde{\lambda}|
\]

(4.15)

\[
|G(x, \lambda; \epsilon) - G(\bar{x}, \tilde{\lambda}; \epsilon)| \leq N_1 \epsilon |x - \bar{x}| + |\lambda - \tilde{\lambda}|
\]

(4.16)

and

\[
\left| \frac{\partial F}{\partial(x, \lambda)}(x, \lambda; \epsilon) - \frac{\partial F}{\partial(x, \lambda)}(\bar{x}, \tilde{\lambda}; \epsilon) \right| \leq N_2 |x - \bar{x}| + |\lambda - \tilde{\lambda}|
\]

(4.17)

\[
\left| \frac{\partial G}{\partial(x, \lambda)}(x, \lambda; \epsilon) - \frac{\partial G}{\partial(x, \lambda)}(\bar{x}, \tilde{\lambda}; \epsilon) \right| \leq N_2 |x - \bar{x}| + |\lambda - \tilde{\lambda}|
\]

(4.18)

Since \( \rho(y) \) and its partial derivatives are continuous functions with compact support there exists \( M > 0 \) such that

\[
\left| \frac{\partial \rho}{\partial y}(y) \right| \leq M
\]

\[
\left| \frac{\partial^2 \rho}{\partial y^2}(y) \right| \leq M
\]

for all \( \lambda \in \mathbb{R}^n \). We also choose \( M > 0 \) large enough that

\[
\left| \frac{\partial F}{\partial x}(x, \lambda) \right| \leq M |x|
\]

(4.19)

\[
\left| \frac{\partial F}{\partial \lambda}(x, \lambda) \right| \leq M |\lambda|
\]

(4.20)

\[
\left| \frac{\partial^2 F}{\partial x \partial \lambda}(x, \lambda) \right| \leq M, \ i, j = 1, 2
\]

(4.21)

because of the condition (4.13) for \( |x|, |\lambda| < 1 \). By the Mean Value Theorem,

\[
|F(x, \lambda; \epsilon) - F(\bar{x}, \tilde{\lambda}; \epsilon)| \leq |F(x, \lambda; \epsilon) - F(\bar{x}, \tilde{\lambda}; \epsilon) + F(\bar{x}, \tilde{\lambda}; \epsilon) - F(\bar{x}, \tilde{\lambda}; \epsilon)|
\]

\[
\leq |F(x, \lambda; \epsilon) - F(\bar{x}, \tilde{\lambda}; \epsilon)| + |F(\bar{x}, \tilde{\lambda}; \epsilon) - F(\bar{x}, \tilde{\lambda}; \epsilon)|
\]

\[
\leq \left| \frac{\partial F}{\partial x}(\xi_1, \lambda; \epsilon) \right| |x - \bar{x}| + \left| \frac{\partial F}{\partial \lambda}(\xi_2, \lambda; \epsilon) \right| |\lambda - \tilde{\lambda}|
\]

where \( \xi_1 \) is between \( x \) and \( \bar{x} \) and \( \xi_2 \) is between \( \lambda \) and \( \tilde{\lambda} \). Similarly,

\[
\left| \frac{\partial F}{\partial x}(x, \lambda; \epsilon) - \frac{\partial F}{\partial x}(\bar{x}, \tilde{\lambda}; \epsilon) \right| \leq \left| \frac{\partial^2 F}{\partial x^2}(\xi_1, \lambda; \epsilon) \right| |x - \bar{x}| + \left| \frac{\partial^2 F}{\partial x \partial \lambda}(\xi_2, \lambda; \epsilon) \right| |\lambda - \tilde{\lambda}|
\]

\[
\left| \frac{\partial F}{\partial \lambda}(x, \lambda; \epsilon) - \frac{\partial F}{\partial \lambda}(\bar{x}, \tilde{\lambda}; \epsilon) \right| \leq \left| \frac{\partial^2 F}{\partial \lambda \partial x}(\xi_1, \lambda; \epsilon) \right| |x - \bar{x}| + \left| \frac{\partial^2 F}{\partial \lambda^2}(\xi_2, \lambda; \epsilon) \right| |\lambda - \tilde{\lambda}|.
\]

Next we estimate for \( 0 \leq \epsilon < \frac{1}{2} \)

\[
\left| \frac{\partial F}{\partial x}(\xi_1, \xi_2; \epsilon) \right| = \left| \frac{\partial F}{\partial x}(\rho\left(\frac{\xi_1}{\epsilon}\right)\xi_1, \rho\left(\frac{\xi_2}{\epsilon}\right)\xi_2) \right| \left| \frac{\partial \rho}{\partial y}(\rho\left(\frac{\xi_1}{\epsilon}\right)\xi_1, \rho\left(\frac{\xi_2}{\epsilon}\right)\xi_2) \right| \left| \frac{\partial \rho}{\partial y}(\rho\left(\frac{\xi_1}{\epsilon}\right)\xi_1, \rho\left(\frac{\xi_2}{\epsilon}\right)\xi_2) \right| + \rho\left(\frac{\xi_1}{\epsilon}\right)\xi_1 \right|
\]

\[
\leq M|\rho(\xi_1)| ||\xi_1|| \left( \left| \frac{\partial \rho}{\partial y}(\rho\left(\frac{\xi_1}{\epsilon}\right)\xi_1, \rho\left(\frac{\xi_2}{\epsilon}\right)\xi_2) \right| + |\rho(\xi_1)| \right)
\]

\[
\leq M[|M| + 1] \epsilon
\]
and
\[
\left| \frac{\partial^2 F}{\partial x^2}(\xi_1, \xi_2; \epsilon) \right| \leq \left| \frac{\partial^2 F}{\partial x^2}(\rho(\frac{\xi_1}{\epsilon})\xi_1, \frac{\xi_2}{\epsilon})\xi_2 \right| + \left| \frac{\partial F}{\partial x}(\rho(\frac{\xi_1}{\epsilon})\xi_1, \frac{\xi_2}{\epsilon})\xi_2 \right| + \left| \frac{\partial^2 \rho}{\partial y^2}(\frac{\xi_1}{\epsilon})\xi_1, \frac{\xi_2}{\epsilon})\xi_2 \right| \leq M(1 + 2M) + 8M^2
\]
for $|\xi_1|, |\xi_2| \leq \epsilon$.

Let
\[
N_1 = M^2[M + 1], \quad N_2 = M(1 + 2M) + 8M^2.
\]
It follows that
\[
(4.22) \quad \left| \frac{\partial F}{\partial (x, \lambda)}(\xi_1, \xi_2; \epsilon) \right| \leq N_1 \epsilon
\]
\[
(4.23) \quad \left| \frac{\partial^2 F}{\partial x^2}(\xi_1, \xi_2; \epsilon) \right| \leq N_2, \quad i, j = 1, 2
\]
for $|\xi_1|, |\xi_2| < \epsilon$. The inequalities above also hold for $G$ as well.

Thus,
\[
(4.24) \quad |F(x, \lambda; \epsilon) - F(\tilde{x}, \tilde{\lambda}; \epsilon)| \leq N_1 \epsilon \left[ |x - \tilde{x}| + |\lambda - \tilde{\lambda}| \right]
\]
\[
(4.25) \quad |G(x, \lambda; \epsilon) - G(\tilde{x}, \tilde{\lambda}; \epsilon)| \leq N_1 \epsilon \left[ |x - \tilde{x}| + |\lambda - \tilde{\lambda}| \right]
\]
and
\[
(4.26) \quad \left| \frac{\partial F}{\partial (x, \lambda)}(x, \lambda; \epsilon) - \frac{\partial F}{\partial (x, \lambda)}(\tilde{x}, \tilde{\lambda}; \epsilon) \right| \leq N_2 \left[ |x - \tilde{x}| + |\lambda - \tilde{\lambda}| \right]
\]
\[
(4.27) \quad \left| \frac{\partial G}{\partial (x, \lambda)}(x, \lambda; \epsilon) - \frac{\partial G}{\partial (x, \lambda)}(\tilde{x}, \tilde{\lambda}; \epsilon) \right| \leq N_2 \left[ |x - \tilde{x}| + |\lambda - \tilde{\lambda}| \right].
\]

Henceforth we suppress $\epsilon$ and write $F(x, \lambda), G(x, \lambda)$ for $F(x, \lambda; \epsilon), G(x, \lambda; \epsilon)$.

4.2. Stability of the Nonlinear Dynamics. From Section 2, we mentioned that the linear term of the stable manifold for the nonlinear bidirectional Hamiltonian dynamics is $Px$. Then, the local stable manifold is of the form
\[
\lambda = \phi(x) = Px + \psi(x)
\]
where $\psi(x)$ contains all the nonlinear terms.

Suppose we substitute into the state dynamics in (4.12), then the nonlinear state dynamics becomes
\[
(I + BR^{-1}B'P)x^+ = Ax - BR^{-1}B'\psi(x^+) + F(x, Px^+ + \psi(x^+)).
\]
By the Matrix Inversion Lemma (22), we have that
\[
(I + BR^{-1}B'P)^{-1} = (I - B(B'PB + R)^{-1}B'P).
\]
Then, it follows that
\[
(4.29) \quad x^+ = (A + BK)x + f_\psi(x, x^+)
\]
\[
x(0) = x_0
\]
where \( K = -(B'PB + R)^{-1}B'P \) and \( f_{\psi}(x, x^+) = (I + BR^{-1}B')^{-1}(F(x, Px^+ + \psi(x^+)) - BR^{-1}B'\psi(x^+)) \).

The implicit equation above can be solved. Let \( F : \mathcal{N}(0) \subset \mathbb{R}^{2n} \rightarrow \mathbb{R}^n \) such that
\[
F(x, x^+) = x^+ - (A + BK)x - f_{\psi}(x, x^+) = 0
\]
for \( x, x^+ \in \mathcal{N}(0) \) where \( \mathcal{N}(0) \) is an open neighborhood of radius \( \epsilon \) around 0. Then, for \( 0 \in \mathcal{N}(0) \) the Jacobian
\[
\frac{\partial F}{\partial x^+}(0) = I - \frac{\partial f_{\psi}}{\partial x^+}(0)
\]
\[
= I - (I + BR^{-1}B')^{-1}
\left( \frac{\partial F}{\partial \lambda^+}(0, \phi(0)) \frac{\partial \psi}{\partial x^+}(0) - BR^{-1}B' \frac{\partial \psi}{\partial x^+}(0) \right)
\]
\[
= I,
\]
because of the condition (4.13) and \( \psi(x^+) \) only contains nonlinear terms. Then, by the Implicit Function Theorem there exists \( F(x) \) such that
\[
x^+ = F(x)
\]
is equivalent to the earlier state dynamics (4.29). Moreover, the linear term of \( F(x) \) is \( (A + BK)x \), i.e.;
\[
F(x) = (A + BK)x + F_{\psi}(x)
\]

(4.31)

because
\[
\frac{\partial F}{\partial x}(0) = -\left( \frac{\partial F}{\partial x^+}(0) \right)^{-1} \frac{\partial F}{\partial x}(0)
\]
\[
= -I(-A + BK)
\]
\[
= A + BK.
\]

It follows that \( F_{\psi}(x) \) contains only the nonlinear terms and thus,
\[
F_{\psi}(0) = 0
\]

(4.32) and
\[
\frac{\partial F_{\psi}}{\partial x_i}(0) = 0 \quad i = 1, \ldots, n.
\]

The linear part of (4.29) is
\[
x^+ = (A + BK)x.
\]

(4.33)

Since the eigenvalues of \( (A + BK) \) lie strictly inside the unit circle, the term \( (A + BK)^k x_0 \rightarrow 0 \) as \( k \rightarrow \infty \). Thus, the system (4.33) is asymptotically stable. Also, it implies that there exists a unique positive definite \( P \) that satisfies the Lyapunov equation
\[
(A + BK)'P(A + BK) - P = -I.
\]

Now we show the stability of the nonlinear dynamics
\[
x_k = (A + BK)x + F_{\psi}(x)
\]
\[
x(0) = 0.
\]

We must prove that
\[
\lim_{x \rightarrow 0} \frac{|F_{\psi}(x)|}{|x|} = 0;
\]
i.e., given any \( \varepsilon > 0 \) and any \( \psi(x) \) satisfying the conditions

\[
\begin{align*}
\psi(0) &= 0 \\
|\psi(x) - \psi(\bar{x})| &\leq l(\varepsilon)|x - \bar{x}|
\end{align*}
\]

where \( l(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \), there exists \( \delta > 0 \) such that

\[
\frac{|F_\psi(x)|}{|x|} < \varepsilon \text{ whenever } |x| < \delta.
\]

We define \( \psi(x) \) to be the nonlinear term of the stable manifold in (4.36). The conditions (4.34) and (4.35) will be necessary for the proof of the local stable manifold theorem.

Recall that

\[
x^+ = \mathcal{F}(x) = (A + BK)x + F_\psi(x).
\]

Then,

\[
0 = \mathcal{F}(x, x^+) = x^+ - (A + BK)x - f_\psi(x, x^+)
\]

\[
= (A + BK)x + F_\psi(x) - (A + BK)x - f_\psi(x, (A + BK)x + F_\psi(x))
\]

\[
= F_\psi(x) - f_\psi(x, (A + BK)x + F_\psi(x)).
\]

It follows that

\[
F_\psi(x) = (I + BR^{-1}B^P)^{-1} \left[ F(x, P((A + BK)x + F_\psi(x)) + \psi((A + BK)x + F_\psi(x))) \right]
\]

(4.36) \( -BR^{-1}B^P \psi((A + BK)x + F_\psi(x)) \).\]

Let \( B_1 = \|(I + BR^{-1}B^P)^{-1}\|, B_2 = \|BR^{-1}B^P\|, P = \|P\| \) and \( \alpha = \max_i |\lambda_i| \) where \( \lambda_i \in \sigma(A + BK) \) and \( |\lambda_i| < 1 \). We have the following from (4.36),

\[
|F_\psi(x)| \leq B_1 N_1 \varepsilon \left[ |x| + |P(A + BK)x + PF_\psi(x) + \psi((A + BK)x + F_\psi(x))| \right]
\]

\[
+ B_1 B_2 |\psi((A + BK)x + F_\psi(x))|.
\]

because of (4.24). Using the Lipschitz condition (4.35) for \( \psi(x) \),

\[
|F_\psi(x)| \leq \begin{cases} B_1 N_1 \varepsilon & |x| + |P(A + BK)x + PF_\psi(x) + \psi((A + BK)x + F_\psi(x))| \leq \frac{|F_\psi(x)|}{|x|} \\
B_1 N_1 \varepsilon \||x| + |P(A + BK)x + PF_\psi(x) + \psi((A + BK)x + F_\psi(x))| \leq \frac{|F_\psi(x)|}{|x|}\end{cases}
\]

Solving for \( |F_\psi(x)| \),

\[
|F_\psi(x)| \leq \frac{B_1 N_1 \varepsilon + \alpha (B_1 N_1 \varepsilon)|P| + B_1 N_1 \varepsilon \lambda_1 l(\varepsilon) + B_1 B_2 l(\varepsilon)}{1 - (B_1 N_1 \varepsilon + B_1 N_1 \varepsilon \lambda_1 l(\varepsilon) + B_1 B_2 l(\varepsilon))} |x|.
\]

Let \( \delta = \frac{1 - (B_1 N_1 \varepsilon + B_1 N_1 \varepsilon \lambda_1 l(\varepsilon) + B_1 B_2 l(\varepsilon))}{B_1 N_1 \varepsilon + \alpha (B_1 N_1 \varepsilon)|P| + B_1 N_1 \varepsilon \lambda_1 l(\varepsilon) + B_1 B_2 l(\varepsilon)} \varepsilon \). For some \( \varepsilon > 0 \) and \( \varepsilon > 0 \), we have that \( \delta > 0 \). Then,

\[
|F_\psi(x)| \leq \begin{cases} \frac{B_1 N_1 \varepsilon + \alpha (B_1 N_1 \varepsilon)|P| + B_1 N_1 \varepsilon \lambda_1 l(\varepsilon) + B_1 B_2 l(\varepsilon)}{1 - (B_1 N_1 \varepsilon + B_1 N_1 \varepsilon \lambda_1 l(\varepsilon) + B_1 B_2 l(\varepsilon))} |x| \\
\frac{B_1 N_1 \varepsilon + \alpha (B_1 N_1 \varepsilon)|P| + B_1 N_1 \varepsilon \lambda_1 l(\varepsilon) + B_1 B_2 l(\varepsilon)}{1 - (B_1 N_1 \varepsilon + B_1 N_1 \varepsilon \lambda_1 l(\varepsilon) + B_1 B_2 l(\varepsilon))} \delta \end{cases}
\]

\[
\leq \varepsilon.
\]

Thus,

\[
F_\psi(x) = o(|x|).
\]
Now, we use the Lyapunov argument. Let $v(x) = x'Px$. Then,

$$
\Delta v(x) = v(x^+) - v(x) \\
= x^+'Px^+ - x'Px \\
= [(A + BK)x - F_\psi(x)]'P[(A + BK)x - F_\psi(x)] - x'Px \\
= x'((A + BK)'P(A + BK) - P)x + 2x'(A + BK)'PF_\psi(x) \\
= -|x|^2 + 2x'(A + BK)'PF_\psi(x).
$$

since

$$
|F_\psi(x)| \leq \frac{1}{3p}|x|
$$

and

$$
|2x'(A + BK)'PF_\psi(x)| \leq \frac{2}{3}|x|^2
$$

for some $p > 0$. Thus,

$$
\Delta v(x) = -\frac{|x|^2}{3} < 0.
$$

Therefore, the nonlinear dynamics is locally asymptotically stable uniform for all $\psi \in \mathbb{X}$.

4.3. Diagonalization of the Hamiltonian Matrix. Recall the bidirectional nonlinear dynamics in (4.12) and the condition (4.13).

By substituting (4.37)

$$
\lambda = Px + \psi(x)
$$

into the state dynamics above (4.12), we get

$$
x^+ = (A + BK)x + f_\psi(x, x^+)
$$

where $f_\psi(x, x^+) = (I + BR^{-1}B'P)^{-1}(F(x, Px^+ + \psi(x^+)) - BR^{-1}B'\psi(x^+))$.

As we substitute (4.37) and (4.38) into the costate dynamics in (4.12), we also add

$$
0 = (BK)'\lambda^+ - (BK)'\lambda^+. 
$$

Then, the costate dynamics becomes

$$
\lambda = (A + BK)'\lambda^+ + Qx + g_\psi(x, x^+).
$$

where $\bar{Q} = Q - K'B'B'(A + BK)$ and $g_\psi(x, x^+) = G(x, Px^+ + \psi(x^+)) + K'B'(\psi(x^+) - Pf_\psi(x, x^+))$.

Thus, the substitution of

$$
\lambda = Px + \psi(x)
$$

into the dynamics (4.12) results in a new nonlinear dynamics

$$
\begin{bmatrix}
x^+ \\
\lambda
\end{bmatrix} = 
\begin{bmatrix}
A + BK & 0 \\
\bar{Q} & (A + BK)' \\
\end{bmatrix}
\begin{bmatrix}
x \\
\lambda^+
\end{bmatrix} +
\begin{bmatrix}
f_\psi(x, x^+) \\
g_\psi(x, x^+)
\end{bmatrix}
$$

where

$$
f_\psi(x, x^+) = (I + BR^{-1}B'P)^{-1}(F(x, \psi(x^+)) - BR^{-1}B'\psi(x^+))
$$

and

$$
g_\psi(x, x^+) = G(x, \psi(x^+)) + K'B'(\psi(x^+) - Pf_\psi(x, x^+)).$$
The nonlinear terms \( f_\psi \) and \( g_\psi \) are \( C^k \) functions for \( k \geq 1 \) such that
\[
\begin{align*}
f_\psi(0, 0) &= 0, \quad g_\psi(0, 0) = 0 \\
\frac{\partial f_\psi}{\partial (x, x^+)}(0, 0) &= 0, \quad \frac{\partial g_\psi}{\partial (x, x^+)}(0, 0) = 0.
\end{align*}
\]
because of (4.40), \( \psi(x) \) only contains nonlinear terms and
\[
\frac{\partial \psi}{\partial x}(0) = 0.
\]
Now we introduce the \( z \) coordinate by the transformation
\[
\lambda = z + Sx
\]
for some matrix \( S \) to block diagonalize the block lower triangular Hamiltonian matrix in (4.39). By substitution, the system (4.39) becomes
\[
\begin{align*}
x^+ &= (A + BK)x + f_\psi(x, x^+) \\
z &= (A + BK)'z^+ + (A + BK)'S(A + BK)x - Sx + \bar{Q}x + h_\psi(x, x^+)
\end{align*}
\]
where
\[
h_\psi(x, x^+) = (A + BK)'Sf_\psi(x, x^+) + g_\psi(x, x^+).
\]
From the \( z \) dynamics above observe that the terms
\[
(A + BK)'S(A + BK)x - Sx + \bar{Q}x = 0
\]
where \( \bar{Q} = Q - K'BP(A + BK) \). Indeed,
\[
(4.43) - S + A'S(A + BK) + K'B'S(A + BK) = -Q + K'B'P(A + BK).
\]
We know that
\[
(4.44) - S + A'S(A + BK) = -Q,
\]
is the discrete-time algebraic Riccati equation (DTARE). Subtracting (4.44) from (4.43), we have
\[
K'B'S(A + BK) = K'B'P(A + BK).
\]
Thus,
\[
(4.45) S = P
\]
and \( S \) satisfies the DTARE. Therefore, we have a diagonalized system
\[
\begin{align*}
\begin{bmatrix} x^+ \\ z \end{bmatrix} &= \begin{bmatrix} A + BK & 0 \\ 0 & (A + BK)' \end{bmatrix} \begin{bmatrix} x \\ z^+ \end{bmatrix} + \begin{bmatrix} f_\psi(x, x^+) \\ h_\psi(x, x^+) \end{bmatrix}
\end{align*}
\]
where
\[
f_\psi(x, x^+) = (I + BR^{-1}B'P)^{-1}(F(x, \psi(x^+)) - BR^{-1}B'\psi(x^+))
\]
and
\[
h_\psi(x, x^+) = (A + BK)'Pf_\psi(x, x^+) + g_\psi(x, x^+).
\]
The nonlinear terms \( f_\psi \) and \( h_\psi \) are \( C^r \) functions for \( r \geq 1 \) such that
\[
\begin{align*}
f_\psi(0, 0) &= 0, \quad h_\psi(0, 0) = 0 \\
\frac{\partial f_\psi}{\partial (x, x^+)}(0, 0) &= 0, \quad \frac{\partial h_\psi}{\partial (x, x^+)}(0, 0) = 0.
\end{align*}
\]
4.4. **The Local Stable Manifold Theorem.** Given the original dynamics \( \mathbf{4.12} \), we look for the local stable manifold described by \( \lambda = \phi(x) \). Since the linear term of the stable manifold for the system \( \mathbf{4.12} \) is \( \mathbf{P}x \) where \( \mathbf{P} \) is the solution to DTARE \( \mathbf{2.40} \), we assume the local stable manifold is of the form

\[
\lambda = \mathbf{P}x + \psi(x)
\]

where \( \psi(x) \) only contains the nonlinear term of \( \phi(x) \). In the two-step process of diagonalization of the system \( \mathbf{4.12} \), we introduce the \( z \) coordinate through the transformation

\[
\lambda = z + \mathbf{S}x.
\]

Since \( \mathbf{S} = \mathbf{P} \), it must be that

\[
z = \lambda - \mathbf{P}x = \mathbf{P}x + \psi(x) - \mathbf{P}x = \psi(x).
\]

Then, it suffices to prove existence of the local stable manifold \( z = \psi(x) \) for the diagonalized system \( \mathbf{4.46} \). In order to show the existence of \( z = \psi(x) \), we use the Contraction Mapping Principle (CMP). To invoke the CMP, we will need a map \( T : \mathbb{X} \rightarrow \mathbb{X} \) that is a contraction on a complete metric space \( \mathbb{X} \).

**Theorem 4.1.** Given the dynamics in \( \mathbf{4.46} \) with the nonlinear terms \( f_\psi \) and \( h_\psi \) that are \( \mathcal{C}^r \) functions satisfying the conditions \( \mathbf{4.47} \) and \( \mathbf{4.48} \) and a hyperbolic fixed point \( 0 \in \mathbb{R}^{2n} \), there exists a local stable manifold

\[
z = \psi(x)
\]

around the fixed point \( 0 \) where \( \psi \) is a \( \mathcal{C}^r \) function.

**Proof:**

First notice that \( f_\psi \) and \( g_\psi \) are cut-off functions. It follows that \( h_\psi \) is also a cut-off function. It suffices to prove the theorem for some \( \epsilon > 0 \) since the cut-off functions \( f_\psi(x, x^+; \epsilon) \) and \( h_\psi(x, x^+; \epsilon) \) agree with \( f_\psi(x, x^+) \) and \( h_\psi(x, x^+) \) for \( |x|, |x^+| \leq \epsilon \).

By \( \mathbf{4.24} - \mathbf{4.27} \), \( \mathbf{4.34} - \mathbf{4.35} \), and \( \mathbf{4.44} - \mathbf{4.48} \), there exists \( N_1, N_2 > 0 \) such that

\[
|f_\psi(x, y; \epsilon) - f_\psi(\tilde{x}, \tilde{y}; \epsilon)| \leq N_1 \epsilon |x - \tilde{x}| + |y - \tilde{y}|
\]

and

\[
|h_\psi(x, y; \epsilon) - h_\psi(\tilde{x}, \tilde{y}; \epsilon)| \leq N_1 \epsilon |x - \tilde{x}| + |y - \tilde{y}|
\]

Moreover, from the bound \( \mathbf{4.22} \) we know the following

\[
|\partial f_\psi(x, y; \epsilon)| \leq \tilde{N} \epsilon
\]

and

\[
|\partial h_\psi(x, y; \epsilon)| \leq \tilde{N} \epsilon.
\]

for \( \tilde{N} > 0 \) and \( |\xi_1|, |\xi_2| < \epsilon \).
Henceforth we suppress \( \epsilon \) and write \( f_\psi(x, y), h_\psi(x, y) \) for \( f_\psi(x, y; \epsilon), h_\psi(x, y; \epsilon) \).

Let \( l(\epsilon) \) with \( l(0) = 0 \) and \( \psi \in X \subset C^0(|x| \leq \epsilon) \) where \( X \) is space of \( \psi : \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that

\[
(4.57) \quad \psi(0) = 0
\]

\[
(4.58) \quad |\psi(x) - \psi(x)| \leq l(\epsilon)|x - x|.
\]

for \( x, \bar{x} \in \bar{B}_\epsilon(0) \subset \mathbb{R}^n \) where \( \bar{B}_\epsilon(0) \) is closed ball around \( 0 \) with radius \( \epsilon \). We define

\[
(4.59) \quad \|\psi\| = \sup_{|x| \leq \epsilon} \frac{|\psi(x)|}{x}.
\]

To show \( X \) is a complete metric space it suffices to show that \( X \) is closed since \( X \subset C^0(|x| \leq \epsilon) \). We take a sequence \( \{\psi_n\} \in X \) such that \( \psi_n \rightarrow \psi \) in \( C^0 \) norm. For large \( N > n \), \( |\psi_n(x) - \psi(x)| \leq \frac{\epsilon}{2} \) for all \( x \in \bar{B}_\epsilon(0) \). Then,

\[
|\psi(x) - \psi(x)| \leq |\psi(x) - \psi_n(x)| + |\psi_n(x) - \psi_n(x)| + |\psi_n(x) - \psi(x)|
\]

\[
\leq \frac{\epsilon}{2} + l(\epsilon)|x - x| + \frac{\epsilon}{2}.
\]

By letting \( \epsilon \rightarrow 0 \), we have that \( |\psi(x) - \psi(x)| \leq l(\epsilon)|x - x| \). Thus, \( \psi \) is a Lipschitz function. Similarly, the condition \( 4.57 \) is easily satisfied. It follows that

\[
|\psi(0) - 0| \leq |\psi(0) - \psi_n(0)| + |\psi_n(0) - 0| \leq \epsilon.
\]

Thus, \( \psi \in X \). Hence \( X \) is closed. Moreover, \( X \) is a complete metric space with the norm defined on \( 4.59 \).

Solving the \( z \) dynamics in \( 4.46 \) via the variation of constants formula, we have

\[
(4.60) \quad z_j = (A' + K'B')^{k-j} z_k + \sum_{l=j}^{k-1} (A' + K'B')^{l-j} h_\psi(x_l, x_{l+1})
\]

for \( j < k \). Let \( j = 0 \) and \( k = \infty \), then \( 4.60 \) changes to

\[
(4.61) \quad z_0 = \sum_{l=0}^{\infty} (A' + K'B')^l h_\psi(x_l, x_{l+1})
\]

We define a mapping \( T : X \rightarrow X \) by

\[
(T\psi)(x_0) = \sum_{l=0}^{\infty} (A' + K'B')^l h_\psi(x_l, x_{l+1}).
\]

where \( x_l, x_{l+1} \) satisfy

\[
x^+ = A + BKx + f_\psi(x, x^+).
\]

From this fixed point equation, we look for the solution

\[
(T\psi)(x_0) = \psi(x_0).
\]

We must show \( T\psi \in X \) and prove \( T \) is a contraction on \( X \).

Suppose \( x_0 = 0 \) is the initial condition. Clearly from the equation \( 4.31 \) - \( 4.32 \), \( x_k = 0 \) for all \( k \). Together with \( x_k = 0 \) for all \( k \) and the condition \( 4.18 \) \( h_\psi(0, 0) = 0 \), we have

\[
(4.63) \quad T\psi(0) = 0.
\]

Hence \( T\psi \) satisfies the condition \( 4.57 \).

We now prove the Lipschitz condition \( 4.58 \) for \( T\psi \).
For \( \psi \in X \) and the initial conditions \( x_0, \bar{x}_0 \in \mathbb{R}^n \), we denote \( x_k = x(k, x_0, \psi) \) to be the solution of the state dynamics.

\[
x^+ = (A + BK)x + f_\psi(x, x^+)
\]

\[
x(0) = x_0.
\]

Similarly, for \( \psi \in X \) and the initial conditions \( \bar{x}_0 \in \mathbb{R}^n \), let \( x_k = x(k, \bar{x}_0, \psi) \) the solution of

\[
x^+ = (A + BK)x + f_\psi(x, x^+)
\]

\[
x(0) = \bar{x}_0.
\]

Recall \( \alpha = \max_j |\lambda_j| \) where \( \lambda_j \in \sigma(A + BK) \) and \( |\lambda_j| < 1 \). Using the estimate (4.51), at one-time step

\[
|x_{k+1} - \bar{x}_{k+1}| \leq \alpha |x_k - \bar{x}_k| + \bar{N}_1 \left[ |x_k - \bar{x}_k| + |x_{k+1} - \bar{x}_{k+1}| \right].
\]

For \( 1 - \bar{N}_1 \epsilon > 0 \),

\[
|x_{k+1} - \bar{x}_{k+1}| \leq \frac{\alpha + \bar{N}_1 \epsilon}{1 - \bar{N}_1 \epsilon} |x_k - \bar{x}_k|
\]

and recursively,

\[
|x_k - \bar{x}_k| \leq \left( \frac{\alpha + \bar{N}_1 \epsilon}{1 - \bar{N}_1 \epsilon} \right)^k |x_0 - \bar{x}_0|
\]

As long as

\[
\epsilon < \frac{1 - \alpha}{2\bar{N}_1},
\]

then

\[
\frac{\alpha + \bar{N}_1 \epsilon}{1 - \bar{N}_1 \epsilon} < 1.
\]

Thus,

\[(4.64)\]

\[
|x_k - \bar{x}_k| \leq |x_0 - \bar{x}_0|.
\]

Using the bounds (4.62) and (4.64),

\[
|T_\psi(x_0) - T_\psi(\bar{x}_0)| \leq \sum_{l=0}^{\infty} \left| \sum_{l=0}^{\infty} (A' + K'B')^l (h_\psi(x_l, x_{l+1}) - h_\psi(\bar{x}_l, \bar{x}_{l+1})) \right|
\]

\[
\leq \sum_{l=0}^{\infty} \alpha^l |h_\psi(x_l, x_{l+1}) - h_\psi(\bar{x}_l, \bar{x}_{l+1})|
\]

\[
\leq \sum_{l=0}^{\infty} \alpha^l \bar{N}_1 \epsilon \left[ |x_l - \bar{x}_l| + |x_{l+1} - \bar{x}_{l+1}| \right]
\]

\[
\leq \sum_{l=0}^{\infty} \alpha^l 2\bar{N}_1 \epsilon |x_0 - \bar{x}_0|
\]

\[
\leq \frac{2\bar{N}_1 \epsilon}{1 - \alpha} |x_0 - \bar{x}_0|.
\]

Let

\[
l(\epsilon) = \frac{2\bar{N}_1 \epsilon}{1 - \alpha}.
\]
Notice that $l(\epsilon) \to 0$ as $\epsilon \to 0$. Thus,
\[ |T\psi(x_0) - T\psi(\tilde{x}_0)| \leq l(\epsilon)|x_0 - \tilde{x}_0| \]
and so $(T\psi)$ satisfies the condition (4.58) for $\epsilon > 0$ sufficiently small. Hence $T$ maps from $\mathcal{X} \to \mathcal{X}$.

Next we show $T$ is a contraction on $\mathcal{X}$.

We express the solutions to the state dynamics
\[ x_k = x(k, x_0, \psi) \]
and
\[ \bar{x}_k = \bar{x}(k, \bar{x}_0, \psi) \]
in the implicit form,
\[ x_k = (A + BK)^k x_0 + \sum_{j=0}^{k-1} (A + BK)^{k-1-j} f_\psi(x_j, x_{j+1}) \]
and
\[ \bar{x}_k = (A + BK)^k \bar{x}_0 + \sum_{j=0}^{k-1} (A + BK)^{k-1-j} f_\psi(\bar{x}_j, \bar{x}_{j+1}), \]
respectively.

We now denote $x_j = x(j, x_0, \psi)$ and $\bar{x}_j = \bar{x}(j, x_0, \bar{\psi})$ be the solutions to the state dynamics and satisfy the implicit form equations
\[ x_k = (A + BK)^k x_0 + \sum_{j=0}^{k-1} (A + BK)^{k-1-j} f_\psi(x_j, x_{j+1}) \]
and
\[ \bar{x}_k = (A + BK)^k \bar{x}_0 + \sum_{j=0}^{k-1} (A + BK)^{k-1-j} f_\psi(\bar{x}_j, \bar{x}_{j+1}), \]
respectively.

The estimates (4.51) - (4.52) with the trajectories $x(j, x_0, \psi)$ and $x(j, x_0, \bar{\psi})$ becomes
\begin{align*}
|f_\psi(x, y) - f_\psi(\bar{x}, \bar{y})| &\leq r_1(\epsilon)|y - \bar{y}| + r_2(\epsilon)\|\psi - \bar{\psi}\| + r_3(\epsilon)|x - \bar{x}| \\
|h_\psi(x, y) - h_\psi(\bar{x}, \bar{y})| &\leq r_1(\epsilon)|y - \bar{y}| + r_2(\epsilon)\|\psi - \bar{\psi}\| + r_3(\epsilon)|x - \bar{x}|
\end{align*}
where
\begin{align*}
r_1(\epsilon) &= n_{1,1}l(\epsilon) + n_{1,2}\epsilon + n_{1,3}l(\epsilon)\epsilon \\
r_2(\epsilon) &= n_{2,1}\epsilon + n_{2,2}\epsilon^2 \\
r_3(\epsilon) &= n_3\epsilon
\end{align*}
and $n_{i,j}$ are positive constants. Observe that $r_i(\epsilon) \to 0$ as $\epsilon \to 0$.

At one-time step,
\[ |x_{k+1} - \bar{x}_{k+1}| \leq m_2(\epsilon)|x_k - \bar{x}_k| + m_3(\epsilon)\|\psi - \bar{\psi}\| \]
where
\[ m_2(\epsilon) = \frac{\alpha + r_3(\epsilon)}{1 - r_1(\epsilon)} \]
and
\[ m_3(\epsilon) = \frac{r_2(\epsilon)}{1 - r_1(\epsilon)}. \]
By invoking Gronwall’s inequality (3.1) and assuming that for some small $\epsilon > 0$

$$m_2(\epsilon) < 1,$$

then

$$|x_k - \bar{x}_k| \leq m_3(\epsilon) \left[ \sum_{j=0}^{k-1} m_2(\epsilon)^{k-1-j} \right] \|\psi - \tilde{\psi}\|$$

$$\leq m_3(\epsilon) \frac{1 - m_2(\epsilon)^k}{1 - m_2(\epsilon)} \|\psi - \tilde{\psi}\|$$

(4.67)

With the bounds (4.66) and (4.67), we get

$$|T\psi(x_0) - T\tilde{\psi}(x_0)| \leq \sum_{l=0}^{\infty} \left( A' + K'B' \right)^l \left( h(\psi(x_l, x_{l+1})) - h(\tilde{\psi}(\bar{x}_l, \bar{x}_{l+1})) \right)$$

$$\leq \sum_{l=0}^{\infty} \alpha^l \left| h(\psi(x_l, x_{l+1})) - h(\tilde{\psi}(\bar{x}_l, \bar{x}_{l+1})) \right|$$

$$\leq \sum_{l=0}^{\infty} \alpha^l \left[ r_1(\epsilon) |x_{l+1} - \bar{x}_{l+1}| + r_2(\epsilon) \|\psi - \tilde{\psi}\| + r_3(\epsilon) |x_l - \bar{x}_l| \right]$$

$$\leq \sum_{l=0}^{\infty} \alpha^l \left[ 2(\alpha^l) + r_3(\epsilon) \frac{m_3(\epsilon)}{1 - m_2(\epsilon)} \|\psi - \tilde{\psi}\| + r_2(\epsilon) \|\psi - \tilde{\psi}\| \right].$$

It follows that

$$|T\psi(x_0) - T\tilde{\psi}(x_0)| \leq c(\epsilon) \|\psi - \tilde{\psi}\|$$

where

$$c(\epsilon) = 2m_3(\epsilon)(r_1(\epsilon) + r_3(\epsilon)) \frac{r_2(\epsilon)}{(1 - m_2(\epsilon))(1 - \alpha)}.$$ 

Notice that $c(\epsilon) \to 0$ as $\epsilon \to 0$. Thus, for sufficiently small $\epsilon > 0$

$$c(\epsilon) < 1.$$

Hence, $T$ is a contraction on $X$ for $\epsilon$ sufficiently small. Therefore, there exists a unique

$$\psi \in X$$

such that

$$\psi = T\psi.$$

Let $\bar{x}, \bar{z}$ satisfy

$$x^+ = A + BKx + f_\theta(x, x^+)$$

and

$$z^+ = (A + BK)'z + g_\theta(x, x^+),$$

respectively. Choose $\bar{z} = \psi(\bar{x})$. Using the definition of the contraction $T$ in 4.62, we have

$$\bar{z}_0 = (T\psi)(\bar{x}_0)$$

$$= \sum_{l=0}^{\infty} (A' + K'B')^l h(\psi(\bar{x}_l, x_{l+1})).$$

It follows that $(\bar{x}, \bar{z})$ is a solution to the difference equation (4.46) and (4.50) defines a $C^1$ invariant manifold. Thus, (4.11) is the graph of a $C^1$ invariant manifold. As
described in \[19\], one can show straightforwardly that \(4.11\) defines \(C^r\) invariant manifold given the data are \(C^r\) smooth.

5. Some Properties

5.1. Eigenstructure. Recall from Section 1 the bidirectional linear Hamiltonian dynamics

\[
\begin{bmatrix}
  x^+ \\
  \lambda
\end{bmatrix}
= \mathbb{H}
\begin{bmatrix}
  x \\
  \lambda^+
\end{bmatrix}
\]

where

\[
\mathbb{H} = \begin{bmatrix}
  A & \quad -BR^{-1}B' \\
  Q & \quad A'
\end{bmatrix}.
\]

**Definition 5.1.** Suppose

\[
\mathbb{H}
\begin{bmatrix}
  \delta x \\
  \mu \delta \lambda
\end{bmatrix}
= \begin{bmatrix}
  \mu \delta x \\
  \delta \lambda
\end{bmatrix}.
\]

Then we call \(\mu\) the eigenvalue of the dynamics \(5.68\).

We would like to show that the linear bidirectional Hamiltonian matrix \(\mathbb{H}\) is hyperbolic; i.e., the eigenvalue of the dynamics \(5.68\) lies strictly inside and outside the unit circle.

**Theorem 5.2.** If \(\mu\) is an eigenvalue of the dynamics \(5.68\) satisfying the relation \(5.69\), then \(1/\mu\) is also an eigenvalue of the dynamics \(5.68\).

**Proof:** First, we decompose \(\mathbb{H}\) into

\[
\begin{bmatrix}
  0 & I \\
  I & 0
\end{bmatrix}
\begin{bmatrix}
  Q & A'
  A & -BR^{-1}B'
\end{bmatrix}
= \mathbb{H}.
\]

Let’s call

\[
\mathbb{S} = \begin{bmatrix}
  Q & A'
  A & -BR^{-1}B'
\end{bmatrix},
\]

\[
\mathbb{I}_\mu = \begin{bmatrix}
  I & 0 \\
  0 & \mu I
\end{bmatrix},
\]

\[
\mathbb{I}^\mu = \begin{bmatrix}
  \mu I & 0 \\
  0 & I
\end{bmatrix}
\]

and notice that \(\mathbb{S}\) is symmetric.

We rewrite \(5.69\) to

\[
\mathbb{H} \mathbb{I}_\mu \begin{bmatrix}
  \delta x \\
  \delta \lambda
\end{bmatrix}
= \mathbb{I}^\mu \begin{bmatrix}
  \delta x \\
  \delta \lambda
\end{bmatrix}.
\]

It follows that

\[
\mathbb{I}^\dagger \mathbb{H} \mathbb{I}_\mu \begin{bmatrix}
  \delta x \\
  \delta \lambda
\end{bmatrix}
= \begin{bmatrix}
  \delta x \\
  \delta \lambda
\end{bmatrix}.
\]

We denote

\[
\mathbb{H}_\mu = \mathbb{I}^\dagger \mathbb{H} \mathbb{I}_\mu.
\]

Observe that 1 is an eigenvalue of \(\mathbb{H}_\mu\) from \(5.70\). Then,

\[
det [I - \mathbb{H}_\mu] = 0 \implies det [I - \mathbb{H}'_\mu] = 0
\]

where

\[
\mathbb{H}'_\mu = \mathbb{I}_\mu \mathbb{S} \begin{bmatrix}
  0 & I \\
  I & 0
\end{bmatrix} \mathbb{I}^\dagger.
\]
Again, 1 is an eigenvalue of $H'_\mu$; i.e.,
\[
\mathbb{H}'_{\mu} \left( \frac{\tilde{\delta}x}{\delta \lambda} \right) = \left( \frac{\tilde{\delta}x}{\delta \lambda} \right).
\]
By multiplying $I_{\frac{1}{\mu}}$ to equation above, we have
\[
S \left( \begin{array}{cc} 0 & I \\ I & 0 \end{array} \right) \left( \frac{\tilde{\delta}x}{\delta \lambda} \right) = \left( \frac{\tilde{\delta}x}{\delta \lambda} \right) \iff S \left( \begin{array}{cc} 0 & I \\ \frac{1}{\mu} \delta \lambda & 0 \end{array} \right) = \left( \frac{\tilde{\delta}x}{\delta \lambda} \right).
\]
Thus,
\[
H \left( \frac{\tilde{\delta}x}{\delta \lambda} \right) = \left( \frac{\tilde{\delta}x}{\delta \lambda} \right).
\]
Hence, $\frac{1}{\mu}$ is an eigenvalue of (5.68).

Note that the dynamics (5.68) admits the infinite eigenvalues, 0 and $\infty$, due to the singularity of $A$.

5.2. Symplectic Form. The nonlinear dynamics tangent to (5.12) as derived explicitly using perturbation technique in citeNa02 is
\[
\frac{\delta x^+}{\delta \lambda} = \begin{bmatrix} H_{\lambda+x} & H_{\lambda+\lambda^+} \\ H_{xx} & H_{\lambda+x} \end{bmatrix} \frac{\delta x}{\delta \lambda^+}
\]
where $H_{\lambda+x}$, $H_{xx}$, and $H_{\lambda+\lambda^+}$ are the partial derivatives of the Hamiltonian defined in (2.3) and $(\delta x, \delta \lambda)$ are tangent vectors in $T_{(x,\lambda)}M$ for $M = \{(x, \lambda) | x \in \mathbb{R}^n, \lambda \in \mathbb{R}^n\}$. The nondegenerate and bilinear symplectic two-form $\Omega : T_{(x,\lambda)}M \times T_{(x,\lambda)}M \mapsto \mathbb{R}$ is
\[
\Omega(v, w) = v' J w
\]
with
\[
\Omega(v, w) = -\Omega(w, v)
\]
where the symplectic matrix,
\[
J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},
\]
and
\[
v = \left[ \begin{array}{c} \delta x \\ \delta \lambda \end{array} \right], \quad w = \left[ \begin{array}{c} \tilde{\delta}x \\ \tilde{\delta} \lambda \end{array} \right] \in T_{(x,\lambda)}M.
\]

We would like to show that under the tangent dynamics (5.11), the two-form $\Omega$ is invariant; i.e.,
\[
\Omega(v, w) = \Omega(v^+, w^+).
\]
Then,
\[
\Omega(v, w) = v' J w = -\delta \lambda' \tilde{\delta} x + \delta x' \tilde{\delta} \lambda.
\]
By substituting the tangent dynamics of $\delta x$ in (5.71) into the above equation, we have
\[
\Omega(v, w) = -(H_{xx} \delta x + H_{\lambda+x} \delta \lambda^+) \tilde{\delta} x + \delta x' \left( H_{xx} \delta x + H_{\lambda+x} \tilde{\delta} \lambda^+ \right) = -\delta \lambda^+ H_{\lambda+x} \delta x + \delta x H_{\lambda+x} \delta \lambda^+.
\]
(5.73)
Similarly,
\[ \Omega(v^+, w^+) = v^+ J w^+ \]
\[ = - (\delta \lambda^+)' \tilde{\delta} x^+ + (\delta x^+)' \tilde{\delta} \lambda^+ \]
\[ = - (\delta \lambda^+)' (H_{\lambda^+ + \delta x} + H_{\lambda} + \delta \lambda^+) + (H_{\lambda^+ + \delta x} \delta x + H_{\lambda^+ + \delta \lambda^+}' \delta \lambda^+) \]
\[ = - \delta \lambda^+ H_{\lambda^+ + \delta x} \delta x + \delta x H_{\lambda^+ + \delta \lambda^+}. \]

(5.74)

Since (5.74) and (5.73) are equal, then
\[ \Omega(v, w) = \Omega(v^+, w^+). \]

Thus, for any two tangent vectors satisfying the dynamics (5.71), the value of \( \Omega \) does not change.

6. **Lagrangian Submanifold**

The two-form calculated from the last section is
\[ \Omega(v, w) = - \delta \lambda^+ H_{\lambda^+ + \delta x} \delta x + \delta x H_{\lambda^+ + \delta \lambda^+}. \]

We calculate the state dynamics tangent to (4.46) around the trajectories \((x_j, x_{j+1})\) as

(6.75)
\[ \delta x_{j+1} = \left( (A + BK) + \frac{\partial f}{\partial x_j}(x_j, x_{j+1}) \right) \delta x_j \]
\[ + \frac{\partial f}{\partial x_{j+1}}(x_j, x_{j+1}) \delta x_{j+1} \]

By the Inverse Function Theorem, we can choose \( \epsilon > 0 \) small enough so that
\[ I - \frac{\partial f_{\psi}}{\partial x_{k+1}}(0, 0) = I. \]

It follows that (6.75) is equivalent to
\[ \delta x_{k+1} = \left( I - \frac{\partial f_{\psi}}{\partial x_{k+1}}(x_k, x_{k+1}) \right)^{-1} \left( (A + BK) + \frac{\partial f_{\psi}}{\partial x_k}(x_k, x_{k+1}) \right) \delta x_k \]
and
\[ \delta x_{k+1} = \prod_{i=0}^{k} \left( I - \frac{\partial f_{\psi}}{\partial x_{i+1}}(x_i, x_{i+1}) \right)^{-1} \left( (A + BK) + \frac{\partial f_{\psi}}{\partial x_i}(x_i, x_{i+1}) \right) \delta x_0. \]

for \( |x_k|, |x_{k+1}| < \epsilon \). As we let \( k \to \infty \), we have \( x_k \to 0, \frac{\partial f_{\psi}}{\partial (x, x^+)}(0) \to 0 \). It follows that from (6.76)
\[ \delta x_{k+1} \to (A + BK)^k \delta x_0 \to 0. \]

Thus,
\[ \Omega(v, w) = - \delta \lambda^+ H_{\lambda^+ + \delta x} \delta x + \delta x' H_{\lambda^+ + \delta \lambda^+} \to 0 \]
since \( \delta x_k \to 0 \) as \( k \to \infty \) for \( v, w \) restricted to the tangent dynamics. Hence, the local stable manifold is a Lagrangian submanifold.
Denote $W_s$ as the local stable manifold described by the graph $\lambda = \phi(x)$. The basis for $T_{(x,\phi(x))}W_s$ is of the form

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \\ \phi_1 \\ \vdots \\ \phi_n \end{pmatrix} \quad \text{where } i = 1, \ldots, n.$$ 

Then, the two-form

$$\Omega(v^i, w^j) = (v^i)' J w^j$$

where

$$v^i = \frac{\partial}{\partial x^i} \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ \phi_1 \\ \vdots \\ \phi_n \end{pmatrix}, \quad w^j = \frac{\partial}{\partial x^j} \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ \phi_1 \\ \vdots \\ \phi_n \end{pmatrix} \in T_{(x,\phi(x))}W_s$$

is equivalent to

$$\frac{\partial \phi_i}{\partial x^j} - \frac{\partial \phi_j}{\partial x^i} = 0 \quad \text{for } i, j = 1, \ldots, n.$$

The equation above implies that $\phi(x)$ is closed. Then, by Stokes’ Theorem there exists $\pi \in C^r(\mathbb{R}^n)$ such that

$$(6.77) \quad \phi(x) = \frac{\partial \pi}{\partial x}(x) \text{ where } \phi(0) = 0$$

locally on some neighborhood of 0. Thus, there exists $\pi \in C^r$ such that

$$(6.78) \quad \lambda = \frac{\partial \pi}{\partial x}(x).$$

Hence, the local stable manifold $\lambda$ in (6.78) is the gradient of the optimal cost for the bidirectional Hamiltonian dynamics $\mathbf{23}$.

**7. Application to Dynamic Programming Equations**

Recall from Section 2 the formulation of a discrete in time infinite horizon optimal control problem of minimizing the cost functional,

$$\min_u \sum_{k=0}^{\infty} l(x_k, u_k)$$

subject to the dynamics

$$x^+ = f(x, u)$$

$$x(0) = x_0$$

We assume $l(x, u)$ is convex in $x$ and $u$ so that

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \geq 0$$
and let $R > 0$. In addition, it is assumed that the pair $(A,B)$ is stabilizable and the pair $(A,Q^{1/2})$ is detectable.

Given that $x(0) = x_0$ the optimal value function $\pi(x)$ is defined by

$$\pi(x_0) = \min_u \sum_{k=0}^{\infty} l(x_k, u_k).$$

This value function satisfies a functional equation, called the dynamic programming equation. The optimal feedback $\kappa(x)$ is constructed from the dynamic programming equation. We state the optimality principle:

**Theorem 7.1. Discrete-Time Optimality Principle:**

$$\pi(x) = \min_u \{\pi(f(x,u)) + l(x,u)\}$$

**Proof:** We have that

$$\pi(x_0) = \min_u \{\sum_{k=0}^{\infty} l(x_k, u_k)\}$$

$$= \min_u \{l(x_0, u_0) + \sum_{k=1}^{\infty} l(x_k, u_k)\}$$

$$= \min_u \{l(x_0, u_0) + \pi(x_1)\}$$

Generalizing the optimality principle at the $k^{th}$-step, we have

$$\pi(x) = \min_u \{\pi(x^+ + l(x,u))\}.$$

The optimality equation (7.79) is the first equation of the dynamic programming equations. An optimal policy $u^* = \kappa(x)$ must satisfy

$$\pi(x) - \pi(f(x,u^*)) - l(x,u^*) = 0$$

if we assume convexity of the LHS of (7.80). We can find $u^*$ through

$$\frac{\partial \pi(x) - \pi(f(x,u)) - l(x,u)}{\partial u} = 0$$

which by the chain rule becomes

$$\frac{\partial \pi}{\partial x}(f(x,u)) \frac{\partial f}{\partial u}(x,u) + \frac{\partial l}{\partial u}(x,u) = 0$$

Thus, $\pi(x)$ and $\kappa(x)$ satisfy these equations, the Dynamic Programming Equations (DPE):

$$(7.81) \quad \pi(x) - \pi(f(x,u)) - l(x,u) = 0$$

$$(7.82) \quad \frac{\partial \pi}{\partial x}(f(x,u)) \frac{\partial f}{\partial u}(x,u) + \frac{\partial l}{\partial u}(x,u) = 0$$

To show the existence of the local solutions, $\pi(x)$ and $\kappa(x)$, we use the Pontryagin Maximum Principle (Theorem 2.1). From the condition (2.6) and the consequence of Local Stable Manifold Theorem where $\lambda$ is function of $x$; i.e., $\lambda = \frac{\partial \pi}{\partial x}(x)$, we have

$$u^* = \kappa(x) = \arg \min_v H(x, \frac{\partial \pi}{\partial x}(x), v).$$
The above equation is equivalent to
$$\frac{\partial H}{\partial u}(x, \frac{\partial \pi}{\partial x}(x), \kappa(x)) = 0$$
which is essentially (7.82). Thus, $\frac{\partial \pi}{\partial x}(x)$ and $\kappa(x)$ solve (7.82). Since $\lambda = \frac{\partial H}{\partial x}$ from the PMP and $\lambda = \frac{\partial \pi}{\partial x}$, it follows that

$$(\text{7.83}) \quad \frac{\partial \pi}{\partial x} = \frac{\partial \pi}{\partial x}(x) \frac{\partial f}{\partial x}(x, u^\ast) + \frac{\partial l}{\partial x}(x, u^\ast)$$

Integrating (7.83) w.r.t. $x$, we get
$$\pi(x) - \pi(f(x, \kappa(x))) - l(x, \kappa(x)) = 0$$
which is (7.81). Therefore, $\pi$ and $\kappa$ solve the DPE (7.81, 7.82). Furthermore $\pi \in C^r$ and $\kappa \in C^{r-1}$ since $l \in C^r$ and $f \in C^{r-1}$ in the Hamiltonian.

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REFERENCES

[1] E. G. Al’brecht. *On the optimal stabilization of nonlinear systems*, PMM-J. Appl. Math. Mech., 25:1254-1266, 1961.
[2] B. D. O. Anderson and J. B. Moore. *Optimal Control, Linear Quadratic Methods*, Prentice Hall, Englewood Cliffs, NJ, 1990.
[3] P. J. Antsaklis and A. N. Michel. *Linear Systems*, McGraw-Hill, New York, 1997.
[4] M. Bardi and I. Capuzzo-Dolcetta. *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*, Birkhäuser, Boston, 1997.
[5] A. Carlson, A. B. Haurie, and A. Leizarowitz. *Infinite Horizon Optimal Control: Deterministic and Stochastic Systems*, Springer-Verlag, Berlin 1991.
[6] J. Carr. *Applications of Centre Manifold Theory*, Springer-Verlag, New York, 1981.
[7] C. Chen. *Linear System Theory and Design*, Oxford Univ. Press, New York, 1999.
[8] C. K. Chai and G. Chen. *Linear Systems and Optimal Control*, Springer-Verlag, Berlin, Heidelberg, 1989.
[9] L. C. Evans. *Partial Differential Equations*. American Mathematical Society. Providence, 1998.
[10] W. H. Fleming and H. M. Soner. *Controlled Markov Processes and Viscosity Solutions*. Springer-Verlag, New York, 1992.
[11] J. Guckenheimer and P. Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Springer-Verlag, New York, 1986.
[12] P. Hartman. *Ordinary Differential Equations*. Birkhauser, Boston, 1982.
[13] M. C. Irwin. *On the stable manifold theorem*. Bull. London Math. Soc., 2, 196-198.
[14] A. Kelley. *The Stable, Center-Stable, Center, Center-Unstable, Unstable Manifolds*. Journal of Differential Equations, 3, 546-570, 1967.
[15] A. J. Krener. *The construction of optimal linear and nonlinear regulators*, in A. Isidori and T.J. Tarn, editors, *Systems, Models and Feedback: Theory and Applications*, Birkhauser, Boston, 1992, 301–322.
[16] A. J. Krener. *Optimal model matching controllers for linear and nonlinear systems*, in M. Fliess, editor, *Nonlinear Control System Design 1992*, Pergamon Press, Oxford, 1993, 209–214.
[17] A. J. Krener. Necessary and sufficient conditions for nonlinear worst case (H-infinity) control and estimation, summary and electronic publication, J. Mathematical Systems, Estimation, and Control, 4:485-488, 1994, full manuscript in J. Mathematical Systems, Estimation, and Control, 7:81-106, 1997.
[18] A. J. Krener. The existence of optimal regulators, Proc. of 1998 CDC, Tampa, FL, 3081–3086.
[19] A. J. Krener. The local solvability of a Hamilton-Jacobi-Bellman PDE around a nonhyperbolic critical point, SIAM J. Control Optimization, 39:1461-1484, 2001.
[20] A. J. Krener and C. L. Navasca. Solution of Hamilton Jacobi Bellman Equations, Proceedings of the IEEE Conference on Decision and Control, Sydney, December 2000.
[21] H. J. Kushner and P. G. Dupuis. Numerical Methods for Stochastic Control Problems in Continuous Time, Springer-Verlag, New York, 1992.
[22] F. L. Lewis and Vassilis L. Syrmos. Optimal Control, Wiley and Sons, Inc, New York, 1995.
[23] D. L. Lukes. Optimal regulation of nonlinear dynamical systems, SIAM J. Contr., 7:75–100, 1969.
[24] C. Navasca. Local Solutions of the Dynamic Programming Equations and the Hamilton-Jacobi-Bellman PDE, PhD Dissertation, University of California, Davis, September 2002.
[25] S. Wiggins. Normally Hyperbolic Invariant Manifolds in Dynamical Systems. Springer-Verlag, 1994.