Exact Solution of selfconsistent Vlasov equation

K. M. Orzewski

M PG - AG "Theoretical M any-Body Physics", Universität Rostock, 0 -18055 Rostock, Germany

An analytical solution of the selfconsistent Vlasov equation is presented. The time evolution is entirely determined by the initial distribution function. The largest Lyapunov exponent is calculated analytically. For special parameters of the model potential positive Lyapunov exponent is possible. This model may serve as a check for numerical codes solving selfconsistent Vlasov equations. The here presented method is also applicable for any system with analytical solution of the Hamiltonian equation for the form factor of the potential.

The selfconsistent Vlasov equation is one of the most frequently used equations for the time dependent description of any-particle system. Especially in nuclear physics this equation has been employed to describe multifragmentation phenomena and collective oscillations. The numerical demands are appreciable to solve this equation in six phase space dimensions. It is apparently not widely known that there exists an analytical solvable model from which the effects of selfconsistency can be studied. Here such a model is presented which shows that selfconsistency can lead to positive Lyapunov exponents. The explicit analytical solution provides a tool for checking numerical codes.

The model single-particle Hamiltonian reads

\[ H = \frac{p^2}{2m} + V(\vec{r};t) \]  

where \( V(\vec{r};t) \) is the mean field potential associated with the separable multipole-multipole force \( V_{1234} = g_{1234} \) resulting in

\[ V(\vec{r};t) = g(\vec{r})Q(t). \]  

This model has been employed e.g. in [1] for numerical study of intrinsic chaos. We show in the following that this is indeed a consequence of selfconsistency. The selfconsistent solution requires

\[ Q(t) = \frac{2}{\text{det} \mathbf{g}(\vec{r})} \]  

where the one-particle distribution function obeys the quasiclassical Vlasov equation

\[ \partial_t f + \frac{p}{m} \partial_{\vec{r}} f - \frac{\vec{r}}{m} \nabla f = 0 \]  

I. Method of Solution

A. Nonselfconsistent solution

First, we solve the di erential equation (4) in nonselfconsistent manner. It means we consider the time dependence of \( V \) due to selfconsistency as an external time dependence. Then the di erential equation is a linear partial one and can be solved easily.

We solve this equation by examining the differential equations for the equipotential lines. This can be found by rewriting (4) in the form of a second-dimensional gradient

\[ \left( \frac{p_i}{m} \right) \partial_{\vec{r}} f = 0 \]  

where \( \partial_{\vec{r}} f = (\partial_{\vec{r}}, \partial_{\vec{p}}) \). Because of the fact that any gradient is perpendicular to the hypersphere we can see that any curve in this hyperplane, which can be characterized by a parametric representation in the way \( (s; \vec{r}; t; \vec{p}(s)) \), obeys the relation

\[ \vec{r}(t) = \vec{r}(s) + \vec{p}(s) t; \quad i = 1; \ldots; n; \quad t(s) = \frac{p_i}{m}. \]  

From this we can read off the di erential equations for the hyperplane. Usually one eliminates the parameter \( s \) choosing the time \( t \) as a parameter. The result is the well known Hamilton equations

\[ \partial_{s_i} = \partial_{\vec{r}} f (\vec{r}; s) \]  

In the case we can solve these equations we would obtain a sixdimensional parametric solution of the Vlasov equation \( c_i(\vec{s}; t; \vec{p}) \). Here the \( c_i \) are the integration constants of (4). The general solution of the differential equation (4) is given as a function of these \( c_i \). This function itself is determined by the initial distribution \( f_0(\vec{p}_i; t=0) \). We reformulate the latter distribution therefore as a function of the \( c_i(\vec{s}_0; t=0) \)

\[ f_0(\vec{p}_0; t=0) = f_0(c_i(\vec{p}_0; t=0)) \]  

which represents a variable substitution from \( (\vec{p}_0; t=0) \) coordinates into the new set of variables \( (c) \). Therefore the initial distribution \( f_0(\vec{p}_0; t=0) \) is changed into \( f(c) \). The general solution of the Vlasov equation at any time can then be represented by

\[ f(\vec{s}; t) = f_0(c_i(\vec{p}; t)). \]
W e like to point out that instead of choosing the time as a parameter we have also the possibility to eliminate it by any variable \( p_i \) or \( r_i \). This is especially helpful for other models because then the energy appears as an explicit integral of motion.

**B. Selfconsistency**

Provided we know the nonselfconsistent solution \([3]\) of the Vlasov equation \([4]\), we can easily build in selfconsistency by employing \([3]\). Introducing \([3]\) into \([3]\) we obtain

\[
Z = \frac{\partial}{\partial \mathbf{p}_0} \frac{\partial}{\partial \mathbf{r}_0} \left( g(\mathbf{r}_0, \mathbf{p}_0) f_0(\mathbf{p}_0, \mathbf{r}_0) \right) = 0
\]

which produces a complicated equation for \( Q \) and moments of the initial distribution. If this can be solved, eq. \([3]\) is the selfconsistent solution when \( Q(\mathbf{t}) \) is introduced.

**II. ANALYTICAL MODEL**

Here we like to demonstrate the application of the method by an exactly solvable model. We choose a form factor of the form

\[
g(\mathbf{r}) = a_x x + a_y y + a_z z
\]

For such a model system we can solve the Hamiltonian equations exactly. This is performed by differentiating the second equation of \([3]\) and inserting the first one

\[
\frac{\partial^2 c_i}{\partial t^2} = \frac{a_i}{m} Q(\mathbf{t}); \quad i = x, y, z
\]

This is easily solved as

\[
r_i = \frac{a_i}{m} Q_2(\mathbf{t}) + c_i^0 t + c_i^1
\]

with

\[
Z^x = \int d\mathbf{c}_0 \int d\mathbf{m}_0 Q(\mathbf{t}^0)
\]

\[
Q_2(\mathbf{t}) = \int dt^0 \frac{d}{dt^0} Q(\mathbf{t}^0)
\]

\[
Q_1(\mathbf{t}) = \int dt^0 \frac{d}{dt^0} Q(\mathbf{t}^0)
\]

Rearranging now for \( c_i^0(\mathbf{p}_i; \mathbf{r}_i) \) we obtain

\[
c_i^0(\mathbf{p}_i; \mathbf{r}_i) = \frac{P_i}{m} + \frac{a_i}{m} Q_1(\mathbf{t})
\]

\[
c_i^2(\mathbf{p}_i; \mathbf{r}_i) = r_i + \frac{a_i}{m} Q_2(\mathbf{t})
\]

\[
(\frac{P_i}{m} + \frac{a_i}{m} Q_1(\mathbf{t}) \mathbf{r}_i)
\]

such that the general solution of the Vlasov equation is any function of these \( c_i \). Taking the initial distribution as \( f_0(\mathbf{p}_0) \) into account corresponding to \([3]\) we see that the general solution can be represented as

\[
f(\mathbf{p}_i; \mathbf{r}_i; t) = f_0(\mathbf{p}_i) + a_i Q_1(\mathbf{t})
\]

\[
r_i + \frac{a_i}{m} Q_2(\mathbf{t})
\]

\[
(\frac{P_i}{m} + \frac{a_i}{m} Q_1(\mathbf{t})) t
\]

\[
(16)
\]

\[
Q(\mathbf{t}) = \frac{\partial}{\partial \mathbf{p}_0} \frac{\partial}{\partial \mathbf{r}_0} \left( f(\mathbf{p}_0, \mathbf{r}_0) \right) = 0
\]

\[
Q_0(t) = a_i < r_i > + \frac{a_i}{m} Q_1(t)
\]

\[
Q_0(0) = a_i < r_i > + \frac{a_i}{m} Q_1(0)
\]

\[
= a_i < r_i > + \frac{a_i}{m} Q_1(0)
\]

\[
(17)
\]

\[
Q(\mathbf{t}) = \int d\mathbf{p} d\mathbf{r} \mathbf{f}(\mathbf{p}, \mathbf{r}) = \frac{1}{2}\hbar^2 a_i Q_2(\mathbf{t}) + \frac{a_i}{m} Q_1(\mathbf{t})
\]

\[
(18)
\]

Double occurring indices \( i \) are summed over. Here we have introduced \( < a > = \frac{\partial}{\partial \mathbf{r}_0} \mathbf{f}(\mathbf{p}_0, \mathbf{r}_0) \) and the density \( n < r > \) of the initial distribution \( f_0 \). This selfconsistent condition \([3]\) is solved by rewriting it as a differential equation

\[
Q(\mathbf{t}) = \frac{2}{m} Q(\mathbf{t})
\]

\[
Q(0) = a_i < r_i > + \frac{a_i}{m} Q_1(0)
\]

\[
Q(0) = a_i < r_i > + \frac{a_i}{m} Q_1(0)
\]

\[
= a_i < r_i > + \frac{a_i}{m} Q_1(0)
\]

\[
(19)
\]

where the averaging \( < > \) is performed about the initial distribution. The solution reads then

\[
Q(\mathbf{t}) = < r > \cosh t + < \frac{P_i}{m} > \frac{1}{2} \sinh t
\]

\[
(20)
\]

from which one nds \( Q_1, Q_2 \) via \([13]\) and the selfconsistent solution \([3]\) follows. The selfconsistent solution is entirely determined by the initial distribution function. The further evolution is then given according to this explicit time dependence.

We see that we obtain in the case of \( < > > 0 \), which means effective repulsive force, an oscillatory solutions. There is no chaotic behaviour.
The interesting solution is given by $q < 0$. There we have an exponentially decreasing $q = \frac{n a_i^2}{m}$ and increasing mode $q = \frac{n a_i^2}{m}$. The latter one defines indeed the largest Lyapunov exponent which is found to be positive here. This can be seen as follows.

The mean momentum and position in any direction at a time $t$ takes the form

$$< p_i >_t = < \frac{p}{m} >_0 + \frac{a_i n}{m} Q_1(t)$$

$$< r_i >_t = < r_i >_0 + \frac{p_i}{m} t + \frac{a_i n}{m} Q_2(t)$$

Then we can calculate easily the mean phase-space distance to an initial point via $d = r^2 + \frac{p^2}{m}$ from which we deduce the largest Lyapunov exponent as

$$\lim_{t \to \infty} \frac{1}{t} \ln \left( \frac{r}{\frac{n a_i^2}{m}} \right) < p^2_t + < r^2_t > = 0$$

With this expression we have presented a model which can be exactly solved within selfconsistent Vlasov equation and shows explicitly that positive Lyapunov exponents are created by selfconsistency.

III. SUMMARY

A method is presented to solve the selfconsistent Vlasov equation. The following recipe is proposed which is applicable for a model mean field $V = g(r)Q(t)$ if the Hamilton equations for this form factor $g(r)$ are integrable.

1. Solution of the differential equations for equipotential lines as a function of the nonselfconsistent (time dependent) potential. The solution is a parametric representation of the general solution.

2. The initial distribution has to be expressed into this parameters. Then the time evolution is entirely determined by this parametric form of the initial distribution replacing the parameters by their time dependent form as derived in 1.

3. The selfconsistent condition leads now to a generally highly involved equation for the selfconsistent potential. This equation is derived using the nonselfconsistent solution of 2, which is a function of the potential itself.

4. Reintroducing this selfconsistent potential into the solution 2 the time evolution of the selfconsistent Vlasov equation is determined completely by the initial distribution.

For an explicitly solvable model the method is demonstrated. The largest Lyapunov exponent is calculated analytically and the conditions are investigated for the occurrence of positive Lyapunov exponent. It is found that positive Lyapunov exponents are generated under certain potential parameters by selfconsistency. The model may serve as a useful check for numerical codes.

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[1] W. Bauer, D. McGrew, V. Zelevinsky, and P. Schuck, Phys. Rev. Lett. 72, 3771 (1994).