Remark on Bi-Ideals and Quasi-Ideals of Variants of Regular Rings

1Samruam Baupradist and 2,3Ronnason Chinram

1Department of Mathematics, Chulalongkorn University, Bangkok 10330, Thailand
2Department of Mathematics and Statistics, Prince of Songkla University, Hat Yai, Songkhla 90110, Thailand
3Centre of Excellence in Mathematics, CHE, Si Ayuthaya Road, Bangkok 10400, Thailand

Abstract: Problem statement: Every quasi-ideal of a ring is a bi-ideal. In general, a bi-ideal of a ring need not be a quasi-ideal. Every bi-ideal of regular rings is a quasi-ideal, so bi-ideals and quasi-ideals of regular rings coincide. It is known that variants of a regular ring need not be regular. The aim of this study is to study bi-ideals and quasi-ideals of variants of regular rings. Approach: The technique of the proof of main theorem use the properties of regular rings and bi-ideals. Results: It shows that every bi-ideal of variants of regular rings is a quasi-ideal. Conclusion: Although the variant of regular rings need not be regular but every bi-ideal of variants of regular rings is a quasi-ideal.

Key words: Bi-ideals, quasi-ideals, variants, regular rings, BQ-rings

INTRODUCTION

The notion of quasi-ideals in rings was introduced by (Steinfeld, 1953) while the notion of bi-ideals in rings was introduced much later. It was actually introduced (Lajos and Sza'sz, 1971).

For nonempty subsets A, B of a ring R, AB denotes the set of all finite sums of the form \( \sum_{a_i, b_i \in A, B} a_i b_i \). A subring Q of a ring R is called a quasi-ideal of R if \( RQ \cap QR \subseteq Q \) and a bi-ideal of R is a subring B of R such that \( BRB \subseteq B \). Every quasi-ideal of R is a bi-ideal. In general, bi-ideals of rings need not be quasi-ideals. See the following example. Consider the ring \( (SU_4(\mathbb{R}), +, \cdot) \) of all strictly upper triangular 4x4 matrices over the field \( \mathbb{R} \) of real numbers under the usual addition and multiplication of matrices.

Let \( B = \begin{bmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \) \( x \in \mathbb{R} \).

Then B is a zero subring of \( (SU_4(\mathbb{R}), +, \cdot) \). Moreover, \( BSU_4(\mathbb{R})B = \{0\} \). Thus B is a bi-ideal of \( (SU_4(\mathbb{R}), +, \cdot) \).

But

\[
\begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} 
= \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} 
\notin (SU_4(\mathbb{R})B \cap BSU_4(\mathbb{R})) \setminus B.
\]

So B is not a quasi-ideal of \( (SU_4(\mathbb{R}), +, \cdot) \).

MATERIALS AND METHODS

An element a of a ring R is called regular if there exists x in S such that \( a = axa \). A ring R is called regular if every element in R is regular. The following known result shows a sufficient condition for a bi-ideal of a ring to be a quasi-ideal.

Theorem 1: If B is a bi-ideal of a ring R such that every element of B is regular in R, then B is a quasi-ideal of R. In particular, if R is a regular ring, then every bi-ideal of R is a quasi-ideal.

Let R be a ring and \( a \in R \). A new product \( o \) defined on R by \( x \circ y = xay \) for all \( x, y \in R \). Then \( (R, +, o) \) is a...
ring. We usually write \((R, +, a)\) rather than \((R, +, o)\) to make the element \(a\) explicit. The ring \((R, +, a)\) is called a variant of \(R\) with respect to \(a\). It is well-known that the variant of regular rings need not be regular ring (see (Kemprasit, 2002) and (Chinram, 2009).

Our aim is to prove that every bi-ideal of variants of regular rings is a quasi-ideal. In fact, the technique of the proof of Theorem 1 is helpful for our work. However, our proof is more complicated.

**RESULTS**

The following theorem is our main result.

**Theorem 2:** Let \(R\) be a regular ring and \(a \in R\). Then every bi-ideal of the ring \((R, +, a)\) is a quasi-ideal.

**Proof:** Let \(B\) be a bi-ideal of a ring \((R, +, a)\). Then \(Ba Ra B \subseteq B\). To show that \(Ra B \cap Ba R \subseteq B\), let \(x\) be an element of \(Ra B \cap Ba R\).

Then:

\[
x \in Ra B\tag{1}
\]

and

\[
x = b_1 a r_n + b_2 a r_{n-1} + \ldots + b_n a r_1 \tag{2}
\]

for some \(b_1, b_2, \ldots, b_n \in B\) and \(r_1, r_2, \ldots, r_n \in R\).

Since each \(b_i a \in R\) and \((R, +, \cdot)\) is a regular ring, there exists \(s_i \in R\) such that \(b_i a = b_i a s_i b_i a\). By (2), we have:

\[
x = b_1 a s_1 b_1 a r_1 + b_2 a s_2 b_2 a r_2 + \ldots + b_n a s_n b_n a r_1 \tag{3}
\]

and

\[
b_1 a s_1 b_1 a r_1 = b_1 a s_1 (x - b_2 a r_2 - \ldots - b_n a r_n) = b_1 a s_1 x - b_1 a s_1 b_2 a r_2 - \ldots - b_1 a s_1 b_n a r_n. \tag{4}
\]

It then follows from (3) and (4) that:

\[
x = b_1 a s_1 x + (b_2 a s_2 b_2 a - b_1 a s_1 b_2 a) a r_2 + \ldots + (b_n a s_n b_n a - b_1 a s_1 b_n a) a r_n. \tag{5}
\]

But from (1) and (2):

\[
b_1 a s_1 x \in Bas_1 Ra B \subseteq Ba Ra B
\]

and for \(i \in \{2, 3, \ldots, n\},\)

\[
b_i a s_i b_i a - b_i a s_1 b_i a \in Bas_i B - Bas_1 B \subseteq Ba R.
\]

So:

\[
x = b_1 + b_2 a r_2 + \ldots + b_n a r_n \tag{5}
\]

for some \(b_1 \in Ba Ra B\) and \(b_2, \ldots, b_n \in Ba R\).

Since for \(i \in \{2, 3, \ldots, n\}, b_i a \in R\), we have that for each \(i \in \{2, 3, \ldots, n\}, b_i a = b_i a s_i b_i a\) for some \(s_i \in R\). Thus from (5),

\[
x = b_1 + b_2 a s_2 b_2 a r_2 + \ldots + b_n a s_n b_n a r_n \tag{6}
\]

and

\[
b_2 a s_2 b_2 a r_2 = b_2 a s_2 (x - b_3 a r_3 - \ldots - b_n a r_n) = b_2 a s_2 x - b_2 a s_2 b_3 a r_3 - \ldots - b_2 a s_2 b_n a r_n. \tag{7}
\]

We then deduce from (6) and (7) that:

\[
x = b_1 + b_2 a s_2 b_2 a r_2 + b_2 a s_3 b_3 a r_3 + \ldots + b_n a s_n b_n a r_n + \ldots + (b_n a s_n b_n a - b_2 a s_2 b_2 a) a r_n
\]

But from (1) and (5):

\[
b_1 \in Ba Ra B,
\]

\[
b_2 a s_2 a r_2 \in Ba Ra B \subseteq Ba Ra B,
\]

\[
b_2 a s_2 b_2 a \in Ba Ra B \subseteq Ba Ra B
\]

and for \(i \in \{3, \ldots, n\},\)

\[
b_2 a s_2 b_2 a = b_2 a s_2 a r_2 + (b_2 a s_3 b_3 a - b_2 a s_2 b_2 a) a r_2 + \ldots + (b_n a s_n b_n a - b_2 a s_2 b_2 a) a r_n \in Ba R a s R a B R a B \subseteq Ba R a s R a B R a B
\]

Thus \(b_2 a s_2 b_3 a - b_2 a s_2 b_2 a \in Ba R\), so we have:

\[
x = b_2 + b_3 a r_3 + \ldots + b_n a r_n
\]

for some \(b_2 \in Ba Ra B\) and \(b_3, \ldots, b_n \in Ba R\).

Continuing in this fashion, we obtain the \(n\)-th step that:

\[
b_2 a s_2 b_2 a = b_2 a s_2 a r_2 + b_2 a s_3 b_3 a r_3 + \ldots + b_n a s_n b_n a r_n
\]

and for \(i \in \{3, \ldots, n\},\)

\[
b_i a s_i b_i a - b_i a s_1 b_i a \in Bas_i B - Bas_1 B \subseteq Ba R.
\]
\[ x = b_{n-1} + b_m a r_n \quad (8) \]

for some \( b_{n-1} \in \text{BaRaB} \) and \( b_m \in \text{BaR}. \)

Let \( s_m \in R \) be such that \( b_m a = b_m a s_m b_m a. \) Then from (8):

\[ x = b_{n-1} + b_m a s_m b_m a r_n \quad (9) \]

and

\[ b_m a s_m b_m a s_m b_m a r_n = b_m a s_m (x - b_{n-1}) \]
\[ = b_m a s_m x - b_m a s_m b_{n-1}. \quad (10) \]

Thus we obtain from (9) and (10) that:

\[ x = b_{n-1} + b_m a s_m x - b_m a s_m b_{n-1}. \]

But since by (1) and (8):

\[ b_{n-1} \in \text{BaRaB}, \]
\[ b_m a s_m x \in \text{BaRaB} \subseteq \text{BaRaB} \]
\[ b_m a s_m b_{n-1} \in \text{BaRaB} \subseteq \text{BaRaB}, \]

it follows that \( x \in \text{BaRaB} \) which implies that \( x \in B. \)

This proves that \( \text{RaB} \cap \text{BaR} \subseteq B, \) so \( B \) is a quasi-ideal of the ring \((R, +, a).\)

Hence the theorem is proved.

DISCUSSION

It is known that every bi-ideal of regular rings is a quasi-ideal. However, although the variant of regular rings need not be a regular ring but every bi-ideal of variants of regular rings is a quasi-ideal.

CONCLUSION

Every bi-ideal of variants of regular rings is a quasi-ideal, so bi-ideals and quasi-ideals of variants of regular rings coincide.

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