Homogenization of a thermo-poro-elastic medium with two-components and interfacial hydraulic/thermal exchange barrier

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Abstract

The paper addresses the homogenization of a micro-model of poroelasticity coupled with thermal effects for two-constituent media and with imperfect interfacial contact. The homogenized model is obtained by means of the two-scale convergence technique. It is shown that the macro-thermo-poro-elasticity model with double porosity/diffusivity contains in particular some extra terms accounting for the micro-heterogeneities and imperfect contact at the local scale. Finally, a corrector result is given under more regularity assumptions on the data.

1 Introduction

Materials like sedimentary rocks and living tissues are generally considered as porous, compressible and elastic. Generally speaking, these porous materials allow for transport mass of some substance through their open channels such as liquid or gas. The presence of a fluid in such diffusive porous materials affects its mechanical responses. Their elasticity properties are then clearly highlighted by the compression resulting from the fluid pressure since any release of fluid storage causes shrinkage of the pore volume. This approach, called poroelasticity, accounts for the coupling of the pore pressure field with the stresses in the skeleton. It is first derived by K. Von Terzaghi in the one-dimensional setting and later generalized by M. Biot to the three-dimensional case. It combines the Hookean classic elasticity theory for the mechanical response of the solid with the Darcy flow diffusion model for the fluid transport within the pores. However, coupling thermal effects with poromechanical processes is of great importance in real-world applications such as geomechanics, civil engineering, biophysics. For instance, cold water injection into a hot hydrocarbon reservoir causes changes in porosity and permeability. These heat transfer processes yield soil deformations of the geothermal well, see M. C. Suárez-Arriaga. An other example is given by the effects of the temperature on concrete

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(buildings, bridges,...) which highly influence its strength development and its durability. It is well known that high temperature causes cracking and strength loss within the concrete mass, see for instance, W. Khaliq and V. Kodur [32], J. Shi & al. [42], K. Y. Shin & al. [43], G. Weidenfeld & al. [51], W. Wang & al [52]. In bio-engineering, there are many important applications of thermal diffusion processes in biological tissues such as thermoregulation, thermotherapy, radiotherapy, ... In some situations tissues are often assumed poroelastic materials such as brain tissues [19], bone [24], skin [29-31], living organs [34, 39] and tumors [8, 14].

The thermoporoelasticity theory aims to describe the thermal/fluid flows and elastic behavior of heat conductive, porous and elastic media, see for instance R.W. Zimmerman [55], O. Coussy [23], A. H. Cheng [18]. A mathematical model for such conductive and porous materials is a set of governing equations which consists of the momentum balance, the conservation of mass and the energy balance equations. The purpose is to predict deformations, fluid pressure and the temperature under different internal, external forces and thermal sources. It is the classical Biot’s theory of consolidation processes coupled to thermal stresses.

Generally, mathematical models of flows in porous elastic media assume that the domain consists of a system of single network where the pores have the same size. However, in many natural situations such as aggregated soils or fissured rocks, materials exhibit two (or more) dominant pore scales. Mathematical modeling of multiple porosity media is the subject of great research activity since the pioneering work of Barenblatt et al. [11]. This concept has been applied to many areas of engineering, see for instance Bai & al. [9], Berryman and Wang [13], Cowin [24], Khalil and Selvadurai [31], Straughan [45] and T. D. Tran Ngoc & al. [48], Wilson and Aifantis [53]. In this context, it is assumed that there exist two porous structures: one is related to macro-porosity connected to the pores of the material and the other to micro-porosity connected to fissures of the skeleton. This causes different pressure fields in the micropores and macropores. Furthermore, the main temperature effect on any kind of media (solid, liquid or gas) is to induce the phenomenon of thermal expansion that is an increase or a shrinkage in volume. Furthermore, the temperature is also considered different in each phase. The concept of thermoelasticity with at least two (or multiple) temperatures was first initiated by Chen, Gurtin and Williams see [17] and further developed by many researchers see for instance Masters & al. [35], D. Iešan [30], H.M. Youssef [50].

The goal of this paper is to derive rigorously, by means of the two-scale convergence technique, a new fully coupled model of thermoporoelasticity for biphasic media. In particular, the work contains some original and essential advances in the study of homogenization problems applied to poroelasticity. Notice that there are many works devoted to the homogenization in poroelasticity and in thermoporoelasticity. We refer the reader for instance to [2, 3, 4, 21, 26, 34]. The outline of the paper is divided into 4 main sections: Firstly, a micro-model of poroelasticity coupled with thermal effects for two-constituent media is given in section 2. It is taken into account that contact between these constituents is
imperfect so that fluid and heat flows through the interface is proportional to the jump of the pressure and temperature field, respectively. Then in section 3 the weak formulation of the microscale problem and the main results are given. In section 4 the homogenization of the micro-model is given with the help of the two-scale convergence technique. The obtained macro-model for thermo-poroelasticity with double porosity/diffusivity is, at my knowledge, new in the literature. It is mainly shown that micro-heterogeneities and imperfect contact at the local scale lead to a Biot/Thermal matrices and zeroth order term at the macroscale, giving rise to absorption/diffusion term. Finally, in section 5 a corrector result is given under more regularity assumptions on the data.

2 Derivation of the micro-model

In this section we derive the set of thermo-hydro-mechanical equations for poroelastic materials, for more details see for e.g. O. Coussy [23].

2.1 Linear thermoporoelasticity equations

Let $\Omega$ be a bounded and smooth domain in $\mathbb{R}^3$ occupying during the time interval $[0,T], T > 0$ a saturated and poroelastic body which is subjected to a given body force per unit volume $f_0 \quad [\text{N}=\text{Kg.m}^{-2} \cdot \text{s}^{-2}]$, to a source/sink term $g_0$ $[\text{Kg.m}^{-3} \cdot \text{s}^{-1}]$ and to a heat source $h_0$ per unit time $[\text{Kg.m}^{-1} \cdot \text{s}^{-3}]$. Let $u \quad [\text{m}]$ denote its displacement, $p \quad [\text{Pa}=\text{Kg.m}^{-1} \cdot \text{s}^{-2}]$ its pressure, $\theta \quad [\text{K}]$ its temperature and $\sigma \quad [\text{Pa}]$ its Cauchy stress tensor. Throughout this paper the volumetric density $\rho$ is taken for simplicity to be a positive constant $[\text{Kg.m}^{-3}]$. Now, we introduce the Helmholtz free energy:

$$A := W - S\theta$$  \hspace{1cm} (2.1)

where $W \quad [\text{J.m}^{-3}, \quad J=\text{Kg.m}^2 \cdot \text{s}^{-2}]$ is the internal energy per unit of volume related to the strain work density/porosity and $S \quad [\text{J.m}^{-3} \cdot \text{K}^{-1}]$ is the entropy per unit of volume. The free energy $A$ is the basic quantity to define the material. In the theory of poromechanics and under the infinitesimal transformation, we assume that the internal energy is a function of the state quantities $e$ and $\phi$:

$$W = W(e, \phi)$$

where

$$e_{ij}(u) = \frac{1}{2} \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right), \quad u = (u_i), \quad 1 \leq i, j \leq 3$$

is the (linearized) strain tensor [dimensionless] and $\phi$ [dimensionless] is the porosity. Assuming an isentropic process, that is $dS = 0$, we get from (2.1) that

$$dA = dW - Sd\theta = \sigma d\varepsilon + pd\phi - Sd\theta$$ \hspace{1cm} (2.2)

where

$$\sigma_{ij} := \frac{\partial W}{\partial e_{ij}}, \quad p := \frac{\partial W}{\partial \phi}.$$
Let us introduce the Legendre transform of the free energy $A$:

$$
\eta := A - \phi p.
$$

(2.3)

It follows then from (2.2) and (2.3) that

$$
d\eta = \sigma d e - \phi d p - S d \theta.
$$

(2.4)

Expanding the free energy $\eta$ in a Taylor series around a reference state $\eta_0$ which corresponds to a free strain $e_0 = 0$, a reference pressure $p_0$, a reference temperature $\theta_0$ and neglecting all terms up to the second order we obtain

\[
\eta = \eta(e, p, \theta) = \eta_0 + \left. \frac{\partial \eta}{\partial e_{ij}} \right|_0 e_{ij} + \left. \frac{\partial \eta}{\partial p} \right|_0 (p - p_0) + \left. \frac{\partial \eta}{\partial \theta} \right|_0 (\theta - \theta_0) + \\
\frac{\partial^2 \eta}{\partial e_{ij} \partial p} \left|_0 (p - p_0) \right. e_{ij} + \left. \frac{\partial^2 \eta}{\partial e_{ij} \partial \theta} \right|_0 (\theta - \theta_0) + \left. \frac{\partial^2 \eta}{\partial p^2} \right|_0 (p - p_0) (\theta - \theta_0) + \\
+ \frac{1}{2} \left. \frac{\partial^2 \eta}{\partial e_{ij} \partial e_{kl}} \right|_0 e_{kh} e_{ij} + \frac{1}{2} \left. \frac{\partial^2 \eta}{\partial e_{ij} \partial \theta} \right|_0 (p - p_0)^2 + \frac{1}{2} \left. \frac{\partial^2 \eta}{\partial \theta^2} \right|_0 (\theta - \theta_0)^2
\]

(2.5)

where (and in the sequel) summation over repeated indices is used. Assuming without no loss of generality that the energy function $\eta$ presents an equilibrium point at $\eta_0 = 0$, that is

\[
\left. \frac{\partial \eta}{\partial e_{ij}} \right|_0 = \left. \frac{\partial \eta}{\partial p} \right|_0 = \left. \frac{\partial \eta}{\partial \theta} \right|_0 = 0,
\]

equation (2.5) reduces then to

\[
\eta = \frac{1}{2} e_{ij} \left. \frac{\partial^2 \eta}{\partial e_{ij} \partial e_{kl}} \right|_0 e_{kh} + e_{ij} \left. \frac{\partial^2 \eta}{\partial e_{ij} \partial \theta} \right|_0 (p - p_0) + \\
e_{ij} \left. \frac{\partial^2 \eta}{\partial e_{ij} \partial \theta} \right|_0 (\theta - \theta_0) + \left. \frac{\partial^2 \eta}{\partial p^2} \right|_0 (p - p_0) (\theta - \theta_0) + \\
+ \frac{1}{2} \left. \frac{\partial^2 \eta}{\partial e_{ij} \partial e_{kl}} \right|_0 (p - p_0)^2 + \frac{1}{2} \left. \frac{\partial^2 \eta}{\partial \theta^2} \right|_0 (\theta - \theta_0)^2
\]

which can be rewritten in a more simplified form:

\[
\eta = \frac{1}{2} (e : A e) - (B : e) (p - p_0) - (D : e) (\theta - \theta_0) + \\
+ \alpha (p - p_0) (\theta - \theta_0) - \frac{1}{2N} (p - p_0)^2 - \frac{\nu}{2N} (\theta - \theta_0)^2
\]

(2.6)

where

\[
a_{ijkh} = \left. \frac{\partial^2 \eta}{\partial e_{ij} \partial e_{kh}} \right|_0 , b_{ij} = - \left. \frac{\partial^2 \eta}{\partial e_{ij} \partial \theta} \right|_0 , d_{ij} = - \left. \frac{\partial^2 \eta}{\partial e_{ij} \partial \theta} \right|_0 \\
\alpha = - \left. \frac{\partial^2 \eta}{\partial \theta^2} \right|_0 , N = - \left. \frac{\partial^2 \eta}{\partial p^2} \right|_0 , \nu = - \theta_0 \left. \frac{\partial^2 \eta}{\partial \theta^2} \right|_0
\]

(2.7)

(2.8)
and
\[ E : F = e_{ij} f_{ij} = \sum_{i,j=1}^{3} e_{ij} f_{ij}, \quad E = (e_{ij})_{1 \leq i,j \leq 3}, \quad F = (f_{ij})_{1 \leq i,j \leq 3}. \]

In (2.7), \( A = (a_{ijkh}) \) [Kg.m\(^{-1}\).s\(^{-2}\)] is the (fourth-rank) elasticity stiffness tensor, \( B = (b_{ij}) \) [dimensionless] the symmetric stress-pressure tensor, expressing the change in porosity to the strain variation when pressure and temperature are kept constant and \( D \) is the thermal dilation (symmetric) tensor related to the solid deformation by the following expression:
\[ D = (d_{ij}) , \quad d_{ij} = -\frac{\partial^2 \eta}{\partial e_{ij} \partial e_{kk}} \bigg|_0 = a_{ijkh} \gamma_{kh} \]
where
\[ \gamma_{kh} = -\frac{\partial e_{kh}}{\partial \theta} \quad [K^{-1}] \]

In (2.8), \( \alpha \) [K\(^{-1}\)] expresses the volumetric thermal dilation coefficient with respect to the pore pressure. Furthermore \( N \) [Kg.m\(^{-1}\).s\(^{-2}\)] is the inverse of the compressibility coefficient so it refers to a modulus relating the pressure \( p \) linearly to the porosity variation \( \phi \) when the volumetric dilation is kept zero. Finally \( v \) [Kg.m\(^{-1}\).s\(^{-2}\).K\(^{-1}\)] is the volumetric heat capacity. From (2.4) and (2.6) we deduce the constitutive equations:
\[ \sigma = \partial_e \eta = A e - (p - p_0) B - (\theta - \theta_0) D, \quad (2.9) \]
\[ \phi = -\partial_p \eta = B : e + \frac{1}{N} (p - p_0) + \alpha (\theta - \theta_0), \quad (2.10) \]
\[ S = -\partial_\theta \eta = D : e + \alpha (p - p_0) + \frac{v}{\theta_0} (\theta - \theta_0). \quad (2.11) \]

Equation (2.9) is well-known in the literature as the Duhamel–Neumann relation.

In the framework of consolidation assumption, the inertia effects are neglected, that is \( \rho \partial_t^2 u \simeq 0 \). In this case, the conservation of linear momentum equation reads in its differential form as
\[ \text{div} \sigma + f_0 = 0. \quad (2.12) \]

Notice that
\[ \sigma^*_ij (u) = \sigma_{ij} (u) - (b_{ij} (p - p_0) + d_{ij} (\theta - \theta_0)) = a_{ijkh} e_{kh} (u) \]
is the effective stress tensor and equation (2.12) becomes then
\[ -\text{div} (A e (u)) + B \nabla p + D \nabla \theta = f_0. \quad (2.13) \]

In the case of homogeneous and isotropic materials, the phenomenological tensors \( A, B \) and \( D \) take the simplified forms:
\[ a_{ijkh} = \lambda \delta_{ij} \delta_{kh} + \mu (\delta_{ik} \delta_{jh} + \delta_{ih} \delta_{jk}), \quad b_{ij} = \beta \delta_{ij}, \quad (2.14) \]
\[ d_{ij} = (3 \lambda + 2 \mu) \hat{\gamma} \delta_{ij}, \quad \gamma_{kh} = \hat{\gamma} \delta_{kh} \quad (2.15) \]
where \((\delta_{ij})\) is the Kronecker symbol, \(\beta\) the Biot-Willis coefficient \[15\], \(\hat{\gamma}\) \([K^{-1}]\) is well-known as the thermal expansion coefficient and \(\lambda\) \([Pa]\), \(\mu\) \([Pa]\) are the Lamé’s constants which are related to the Young’s modulus \(E\) \([Pa]\) and the Poisson’s ratio \(\nu\) (dimensionless) through the expression:

\[
\lambda = E\frac{\nu}{(1-2\nu)(1+\nu)}, \quad \mu = E\frac{\nu}{2(1+\nu)}.
\]

The effective stress is then related to \(e\) through the linear Hooke’s constitutive law:

\[
\sigma_{ij}^* = 2\mu e_{ij} + \lambda e_{kk}\delta_{ij}.
\]

The Biot-Willis coefficient \(\beta\) expresses, at constant fluid pressure, the ratio of the volume of fluid squeezed out of a solid to total volume change for elastic deformation. The coefficient \(\hat{\gamma}\) refers to the fractional change in volume per degree of temperature change. The coefficient \(2\lambda + 3\mu\) is the bulk modulus which relates the volumetric dilation \(e_{ii}\) to \(\sigma\) at a constant pore pressure.

If the porous solid is saturated by a compressible fluid whose mass density \(\rho_0\) assumed to be constant, then the equation of continuity for the flow fluid is given by

\[
\partial_t \phi + \nabla \cdot \mathbf{v} = g_0
\]

where \(\mathbf{v}\) \([m\cdot s^{-1}]\) is the velocity of the total amount of the fluid content. For laminar flow, the fluid flux is related to the fluid velocity \(\mathbf{v}\) through the Darcy’s law (neglecting the gravity force \(g\)):

\[
\mathbf{v} = -\frac{k_f}{\mu_d} \nabla p
\]

where \(k_f\) is the intrinsic permeability coefficient \([m^2]\) and \(\mu_d\) \([Pa\cdot s]\) is the dynamic viscosity. The negative sign in the Darcy law is needed because fluids flow from high pressure to low one, opposite to the direction of the pressure gradient. Using (2.10), (2.16) and (2.17) we get

\[
\partial_t \left(\frac{1}{N}p + B : e(\mathbf{u}) + a\theta\right) - \nabla \cdot \left(\frac{k_f}{\mu_d} \nabla p\right) = g_0
\]

Similarly, the classical Fourier’s law relates heat flux vector \(\mathbf{q}\) \([Kg\cdot s^{-3}]\) with temperature gradient \(\nabla \theta\) by the equation

\[
\mathbf{q} = -\lambda_0 \nabla \theta
\]

where \(\lambda_0\) \([kg\cdot m\cdot s^{-3}\cdot K^{-1}]\) is the thermal conductivity. Using the first and second laws of thermodynamics \[28\], the energy balance equation reads as follows:

\[
\partial_t \mathcal{W} = \sigma : e(\partial_t \mathbf{u}) + p\partial_t \phi - \nabla \cdot \mathbf{q} + h_0
\]

where the left hand side represents the instantaneous change in storage energy in the body, the first term of the right hand side is heat entering the material
and the last term stands for energy generation in the body. From (2.4), differentiating η with respect to t and taking into account the fact that \( S = -\partial_\theta \eta \), it follows that
\[
\partial_t \eta = \sigma : e (\partial_t u) - \phi \partial_t p - S \partial_\theta \theta.
\] (2.20)

But, using (2.1) and (2.3), we also have
\[
\partial_t \eta = \partial_t W - \phi \partial_t p - p \partial_t \phi - S \partial_\theta \theta - \theta \partial_t S.
\] (2.21)

Equating (2.20) and (2.21), equation (2.19) is rewritten as
\[
\theta \partial_t S = -\text{div} q + h_0.
\] (2.22)

Assuming small variations of temperature so that one can replace θ by \( \theta_0 \), whenever required, and inserting (2.11) into (2.22) yield to the following equation:
\[
\theta_0 \partial_t \left\{ D : e (u) + \alpha p + \frac{\mu}{\theta_0} \right\} = \text{div} (\lambda_0 \nabla \theta) + h_0.
\] (2.23)

Equations (2.13), (2.18) and (2.23) are then the fundamental equations of linear thermoporoelasticity. Denoting
\[
\gamma = (3\lambda + 2\mu) \hat{\gamma}, \quad \phi_0 = \frac{1}{N}, \quad \kappa = \mu^{-1} \kappa_f, \\
g = \rho^{-1} g_0, \quad \hat{\lambda} = \theta_0^{-1} \lambda_0, \quad c = \theta_0^{-1} c, \quad h = \theta_0^{-1} h_0
\]
and taking into account (2.14), (2.15), Equations (2.13), (2.18) and (2.23) are reduced to the following system:
\[
\begin{align*}
- (\lambda + \mu) \nabla (\text{div} u) - \mu \Delta u + \beta \nabla p + \gamma \nabla \theta &= f_0, \\
\partial_t (\phi_0 p + \beta \text{div} u + \alpha \theta) - \kappa \Delta p &= g, \\
\partial_t (c \theta + \gamma \text{div} u + \alpha p) - \hat{\lambda} \Delta \theta &= h.
\end{align*}
\] (2.24)

**Remark 2.1** Note that if we let γ and α to be negligible then the system (2.24) decouples to the classical Biot system [14]:
\[
\begin{align*}
- \text{div} \sigma &= - (\lambda + \mu) \nabla \text{div} (u) - \mu \Delta u + \beta \nabla p = f_0, \\
\partial_t (\phi p + \beta \text{div} u) - \kappa \Delta p &= g,
\end{align*}
\]

**Remark 2.1** On the other hand, neglecting β and α, the system (2.24) decouples to the thermoelasticity system:
\[
\begin{align*}
- \text{div} \sigma &= - (\lambda + \mu) \nabla \text{div} (u) - \mu \Delta u + \gamma \nabla \theta = f_0, \\
\partial_t (c \theta + \gamma \text{div} u) - \hat{\lambda} \Delta \theta &= h
\end{align*}
\] with the single pressure diffusion equation
\[
\phi \partial_t p - \kappa \Delta p = g.
\]
2.2 A mathematical model for two-components materials

In this subsection we shall derive from (2.24) the mathematical model of thermo-poroelasticity for isotropic materials made of two constituents, namely matrix and inclusions and which are in imperfect interfacial contact. The geometrical setting is described as follows: Let $\Omega$ be a bounded and a smooth domain of $\mathbb{R}^3$. We assume that $\Omega$ is divided into two open sets $\Omega_1$ and $\Omega_2$ such that $\Omega = \Omega_1 \cup \Omega_2$ and $\partial \Omega_2 \cap \partial \Omega = \emptyset$. The medium occupying $\Omega$ is composed of two poro-elastic solids $\Omega_1$ and $\Omega_2$ separated by the interface $\Sigma := \partial \Omega_2$. According to (2.24), the local description is given by the following system: for each phase $m = 1, 2$ corresponding to the material $\Omega_m$

$$
\begin{align*}
-\text{div} \sigma_m &= f_m, \\
\sigma_m &= - (\lambda_m + \mu_m) \Delta u_m - \mu_m \Delta u_m + \beta_m \nabla p_m + \gamma_m \nabla \theta_m, \\
\partial_t (\phi_m p_m + \beta_m \nabla u_m + \alpha_m \theta_m) - \kappa_m \Delta p_m &= g_m, \\
\partial_t (c_m \theta_m + \gamma_m \nabla u_m + \alpha_m p_m) - \lambda_m \Delta \theta_m &= h_m.
\end{align*}
$$

(2.25)

For piecewise homogeneous media, the system (2.25) is complemented by interface, boundary and initial conditions. They read as follows: on the interface $\Sigma$

$$
\begin{align*}
\mathbf{u}_1 &= \mathbf{u}_2, \\
\sigma_1 \cdot \mathbf{n} &= \sigma_2 \cdot \mathbf{n}, \\
\kappa_1 \nabla p_1 \cdot \mathbf{n} &= \kappa_2 \nabla p_2 \cdot \mathbf{n}, \\
\hat{\lambda}_1 \nabla \theta_1 \cdot \mathbf{n} &= \hat{\lambda}_2 \nabla \theta_2 \cdot \mathbf{n}, \\
\kappa_1 \nabla p_1 \cdot \mathbf{n} &= - \zeta (p_1 - p_2), \\
\hat{\lambda}_1 \nabla \theta_1 \cdot \mathbf{n} &= - \omega (\theta_1 - \theta_2).
\end{align*}
$$

(2.26)

where $\mathbf{n}$ is the unit normal vector on $\Sigma$ pointing outwards to $\Omega_2$. In (2.26c) $\zeta$ [kg$^{-1}$m$^2$s] is the interfacial hydraulic permeability and in (2.26d) $\omega$ [kg.s$^{-3}$K$^{-1}$] is the interface thermal conductance. Conditions (2.26a)-(2.26d) are the continuity of the displacements, of the normal stresses and of the normal fluxes (hydraulic and thermal). It is assumed that the hydraulic/thermal contact between these two materials is imperfect, so that the fluxes are related to the jump of pressures and temperatures, see (2.26e) and (2.26f). For example the case $\zeta = \infty$ corresponds to a perfect hydraulic contact, so that $p_1 = p_2$ across the interface and the pressure is continuous. If $\zeta = 0$ there is no hydraulic contact across the interface, yielding no motion of the fluid relative to the solid, that is, $\kappa_1 \nabla p_1 \cdot \mathbf{n} = \kappa_2 \nabla p_2 \cdot \mathbf{n} = 0$. In the literature (2.26c) is known as the Deresiewicz-Skalak condition [25] and (2.26d) as the Newton’s cooling law [16].

On the exterior boundary $\Gamma$ we assume the following homogeneous Dirichlet boundary conditions:

$$
\mathbf{u}_1 = 0, \ p_1 = 0, \ \theta_1 = 0.
$$

(2.27)
Finally the initial conditions are given as follows:

\[
\begin{align*}
\mathbf{u}_m(0, x) &= 0, \quad p_m(0, x) = 0, \quad \theta_m(0, x) = 0, \quad x \in \Omega_m. 
\end{align*}
\]  

In summary, the system \((2.25) - (2.28)\) is the complete set of equations for thermoporoelastic media with two-components.

In this work we shall study a model consisting of "very" small inclusions embedded in a matrix, so we have to introduce a small and dimensionless parameter expressing the ratio between the local scale of the inclusions and the macroscopic scale of the matrix. This will be done in the next subsection.

2.3 Scaling

We consider a poroelastic composite of dimension \(\mathcal{O}(L^3)\) where \(L\) [m] is the characteristic length of the medium at the macroscopic scale. We assume that the composite has a periodic structure with period \(Y\) with dimension \(\mathcal{O}(\ell^3)\) where \(\ell\) [m] is the microscopic characteristic length. The fundamental assumption in the periodic homogenization theory \([12, 41]\) is that these scales are separated which in this case can be read as follows:

\[
\varepsilon := \frac{\ell}{L} \ll 1.
\]

We assume that the stiffness tensors \(A_1\) and \(A_2\), permeabilities \(\kappa_1\) and \(\kappa_2\), thermal conductivities \(\lambda_1\) and \(\lambda_2\) are of the same order of magnitude. Pressures \(p_1\), \(p_2\) and temperatures \(\theta_1\), \(\theta_2\) are also considered of the same order. More precisely, we assume that

\[
\begin{align*}
|A_1| &= |A_2| = \mathcal{O}(\varepsilon^0), \quad |\kappa_1| = |\kappa_2| = \mathcal{O}(\varepsilon^0), \quad |\lambda_1| = |\lambda_2| = \mathcal{O}(\varepsilon^0), \\
p_1 &= p_2 = \mathcal{O}(\varepsilon^0), \quad \theta_1 = \theta_2 = \mathcal{O}(\varepsilon^0).
\end{align*}
\]

Equations \((2.26e)\) and \((2.26f)\) give at the microscale two dimensionless (Biot) numbers:

\[
B_{i1} = \frac{|\varsigma(p_1 - p_2)|}{|\kappa_1 \nabla p_1|} = \frac{\varsigma l}{\kappa_1}, \quad B_{i2} = \frac{|\omega(\theta_1 - \theta_2)|}{|\lambda_1 \nabla \theta_1|} = \frac{\omega l}{\lambda_1}.
\]

Since the area of \(\Sigma^c\) is of order \(\varepsilon^{-1}\), a convenient scaling of those Biot numbers is \(B_{i1} = B_{i2} = \mathcal{O}(\varepsilon)\), see \([6, 37]\).

Remark 2.2 Note that other assumptions could be studied as well. One also could consider the following case: \(|A_1| = |A_2| = |\kappa_1| = |\lambda_1| = \mathcal{O}(\varepsilon^0), \quad p_1 = p_2 = \theta_1 = \theta_2 = \mathcal{O}(\varepsilon^0)\) and \(|\kappa_2| = |\lambda_2| = \mathcal{O}(\varepsilon^2)\). See for instance \([5, 7]\) and for more general situations we refer the reader to \([40]\).
2.4 Problem statement

In this subsection a micro-model for a thermoporoelastic medium with two-components and with interfacial hydraulic/thermal exchange barrier is presented. As before, we consider Ω a bounded domain in \( \mathbb{R}^3 \) with a smooth boundary \( \Gamma \). The region \( \Omega \) represents a part of a medium made of two constituents: the matrix and the inclusions, separated by a thin and periodic layer so that the hydraulic/thermal flux are proportional to the jump of the pressure/temperature field. To describe the periodicity of the medium, we consider \( Y := ]0, 1[^3 \) as the generic cell of periodicity divided as \( \bar{Y} := Y_1 \cup Y_2 \cup \Sigma \) where \( Y_1, Y_2 \) are two connected, open and disjoint subsets of \( Y \) and \( \Sigma := \partial Y_1 \cap \partial Y_2 \) is a smooth surface that separates them. We assume that \( \bar{Y} \) denote the \( Y \)-periodic characteristic function of \( Y_m \) \( (m = 1, 2) \). Let \( \varepsilon > 0 \) be a sufficiently small parameter and set

\[
\Omega^\varepsilon_m := \{ x \in \Omega : \chi_m(\frac{x}{\varepsilon}) = 1 \}, \quad \Sigma^\varepsilon := \bar{\Omega}_1^m \cap \bar{\Omega}_2^m.
\]

We assume that \( \bar{\Omega}_2^m \subset \Omega \). The space-time regions are denoted by

\[
Q^\varepsilon := (0, T) \times \Omega, \quad \Gamma_T := (0, T) \times \Gamma, \quad \Sigma_T := (0, T) \times \Sigma, \quad Q^\varepsilon_m := (0, T) \times \Omega^\varepsilon_m, \quad \Sigma_T^\varepsilon := (0, T) \times \Sigma^\varepsilon.
\]

In view of (2.25), the thermoporoelasticity system is given in each phase \( Q^\varepsilon_m \) by

\[
-\text{div} \sigma^\varepsilon_m = f_m, \quad \sigma^\varepsilon_m = (\sigma^\varepsilon_{m,ij})_{ij},
\]

\[
\partial_t (\phi_m \rho^\varepsilon_m + \beta_m \text{div} \bm{u}^\varepsilon_m + \alpha_m \theta^\varepsilon_m) - \kappa_m \Delta \rho^\varepsilon_m = g_m, \quad \partial_t (\epsilon_m \theta^\varepsilon_m + \gamma_m \text{div} \bm{u}^\varepsilon_m + \alpha_m \rho^\varepsilon_m) - \lambda_m \Delta \theta^\varepsilon_m = h_m
\]

where

\[
\sigma^\varepsilon_{m,ij} (\bm{u}^\varepsilon) = a_{m,ijkl} \epsilon_{kl} (\bm{u}^\varepsilon_m) - (\beta_m \epsilon_{m,ij} + \gamma_m \theta^\varepsilon_m) \delta_{ij}
\]

with \( (a_{m,ijkl})_{1 \leq i,j,k,l \leq 3} \) the elasticity tensor stiffness satisfying the Hooke’s law for isotropic materials:

\[
a_{m,ijkl} = \lambda_m \delta_{ij} \delta_{kl} + \mu_m (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \quad m = 1, 2, \quad 1 \leq i,j,k,l \leq 3.
\]

As in (2.26a)-(2.26b), the transmission conditions on the interface \( \Sigma_T^\varepsilon \) are as follows:

\[
\bm{u}^\varepsilon_1 = \bm{u}^\varepsilon_2,
\]

\[
\sigma^\varepsilon_1 \cdot \bm{n}^\varepsilon_1 = \sigma^\varepsilon_2 \cdot \bm{n}^\varepsilon_2,
\]

\[
\kappa_1 \nabla \rho^\varepsilon_1 \cdot \bm{n}^\varepsilon = \kappa_2 \nabla \rho^\varepsilon_2 \cdot \bm{n}^\varepsilon,
\]

\[
\hat{\lambda}_1 \nabla \theta^\varepsilon_1 \cdot \bm{n}^\varepsilon = \hat{\lambda}_2 \nabla \theta^\varepsilon_2 \cdot \bm{n}^\varepsilon,
\]

\[
\kappa_1 \nabla \rho^\varepsilon_1 \cdot \bm{n}^\varepsilon = -c^\varepsilon (p^\varepsilon_1 - p^\varepsilon_2),
\]

\[
\hat{\lambda}_1 \nabla \theta^\varepsilon_1 \cdot \bm{n}^\varepsilon = -\omega^\varepsilon (\theta^\varepsilon_1 - \theta^\varepsilon_2)
\]
where $\mathbf{n}^\varepsilon$ is the unit normal of $\Sigma^\varepsilon$ pointing outwards of $\Omega_2^\varepsilon$. Furthermore, boundary conditions on $\Gamma_T$ read as:

$$u_1^\varepsilon = 0, \quad \theta_1^\varepsilon = p_1^\varepsilon = 0. \quad (2.34)$$

Observe that no boundary conditions on $\Gamma$ are required for the phase 2 since the inclusions $\Omega_2^\varepsilon$ are strictly embedded in $\Omega$. Finally, the initial conditions give

$$u_m^\varepsilon (x, 0) = 0, \quad \theta_m^\varepsilon (x, 0) = p_m^\varepsilon (x, 0) = 0 \text{ in } \Omega_m^\varepsilon. \quad (2.35)$$

We assume that the elastic moduli $a_m, i, j, k, l$, the Biot-Willis parameter $\beta_m$, the thermal dilation coefficients $\gamma_m$, and the compressibility $\phi_m$, the diffusivity coefficients $\kappa_m, \lambda_m$, and the heat capacity $c_m$ are positive constants. We also assume that the body force $f_m$ is in $L^2 (\Omega)^3$, the sink source $g_m$ and the heat source $h_m$ are in $L^2 (\Omega)$. Furthermore, the interface hydraulic permeability and the interface thermal conductance are such that $\varsigma^\varepsilon (x) = \varepsilon \varsigma \left( \frac{x}{\varepsilon} \right)$ and $\omega^\varepsilon (x) = \varepsilon \omega \left( \frac{x}{\varepsilon} \right)$ (see Sec. 2.3) where $\varsigma$ and $\omega$ are continuous on $\mathbb{R}^3$, $Y$-periodic and bounded from below: $\exists C > 0$ such that for all $y \in \mathbb{R}^3$

$$\varsigma (y) \geq C, \quad \omega (y) \geq C.$$

In what follows, $C$ will denote a positive constant independent of $\varepsilon$.

### 3 Statement of the main results

We first introduce some notations: if $E$ is a Banach space then for $p = 2, \infty$, $L_p^p (E)$ denotes the Bochner space $L^p (0, T; E)$ of (class of ) functions $u : t \mapsto u(t)$ defined a.e. on $(0, T)$ with values in $E$ such that $\|u(t)\|_{L_p^p (E)} := \int_0^T \|u(t)\|_E^p \, dt$ is finite (dt denotes the Lebesgue measure on the interval $(0, T)$). Let $L^2_{\text{loc}} (Y)$ (resp. $L^2_{\text{loc}} (Y_{\text{m}})$) be the space of (class of) functions belonging to $L^2_{\text{loc}} (\mathbb{R}^3)$ (resp. $L^2_{\text{loc}} (Z_m)$) which are $Y$-periodic, where $Z_m = \bigcup_{k \in \mathbb{Z}^3} \left( Y_m + \tilde{k} \right)$. Let $H^1 (Y)$ (resp. $H^1 (Y_{\text{m}})$) to be the space of those functions together with their derivatives belonging to $L^2_{\text{loc}} (Y)$ (resp. $L^2_{\text{loc}} (Y_{\text{m}})$) having the same trace on the opposite faces of $\partial Y$ (resp. $\partial Y_m \cap \partial Y$). Let

$$V := H^1_0 (\Omega)^3, \quad H^\varepsilon := L^2 (\Omega_1^\varepsilon) \times L^2 (\Omega_2^\varepsilon),$$

$$H^1_0 (\Omega_1^\varepsilon) := \{ q \in H^1 (\Omega_1^\varepsilon) : q|_{\Gamma} = 0 \},$$

$$V^\varepsilon := V_1^\varepsilon \times V_2^\varepsilon = H^1_0 (\Omega_1^\varepsilon) \times H^1 (\Omega_2^\varepsilon).$$

The space $V^\varepsilon$ is equipped with the inner product:

$$(q, \psi)_{V^\varepsilon} := \int_{\Omega_1^\varepsilon} \nabla q_1 \nabla \psi_1 \, dx + \int_{\Omega_2^\varepsilon} \nabla q_2 \nabla \psi_2 \, dx + \varepsilon \int_{\Sigma^\varepsilon} (q_1 - q_2) (\psi_1 - \psi_2) \, ds^\varepsilon (x), \quad q = (q_1, q_2), \quad \psi = (\psi_1, \psi_2) \in V^\varepsilon$$
where $dx$ and $ds^\varepsilon(x)$ stands for the Lebesgue measure in $\mathbb{R}^3$ and the surfacic measure on $\Sigma^\varepsilon$, respectively. Let us denote for a.e. $t \in (0, T)$

$$
\mathbf{u}^\varepsilon(t, x) = \begin{cases} 
\mathbf{u}_1^\varepsilon(t, x), & x \in \Omega_1^\varepsilon, \\
\mathbf{u}_2^\varepsilon(t, x), & x \in \Omega_2^\varepsilon,
\end{cases} 
$$

$$
p^\varepsilon(t, x) = (p_1^\varepsilon(t, x), p_2^\varepsilon(t, x)), \quad \theta^\varepsilon(t, x) = (\theta_1^\varepsilon(t, x), \theta_2^\varepsilon(t, x))
$$

and let us define

$$
\mathbf{A}^\varepsilon(x) := \chi_1(\frac{x}{\varepsilon})\mathbf{A}_1 + \chi_2(\frac{x}{\varepsilon})\mathbf{A}_2, \\
\mathbf{f}^\varepsilon(x) := \chi_1(\frac{x}{\varepsilon})\mathbf{f}_1(x) + \chi_2(\frac{x}{\varepsilon})\mathbf{f}_2(x), \\
g^\varepsilon(x) := \chi_1(\frac{x}{\varepsilon})g_1(x) + \chi_2(\frac{x}{\varepsilon})g_2(x), \\
h^\varepsilon(x) := \chi_1(\frac{x}{\varepsilon})h_1(x) + \chi_2(\frac{x}{\varepsilon})h_2(x)
$$

where $\mathbf{A}_m := (a_{m,ijkl})_{1 \leq i,j,k,l \leq 3}$. Now we are in position to give the weak formulation.

**Definition 3.1** A weak solution of the micro-model (2.29)-(2.35) is a triple $(\mathbf{u}^\varepsilon, p^\varepsilon, \theta^\varepsilon) \in L^\infty_T(\mathbb{R}^3) \times L^2_T(V^\varepsilon)^2$ such that $p^\varepsilon, \theta^\varepsilon \in L^2_T(H^\varepsilon)$ and for $m = 1, 2$

$$
\partial_t (\phi_m p_m^\varepsilon + \beta_m \text{div} \mathbf{u}_m^\varepsilon + \alpha_m \theta_m^\varepsilon) \in L^2_T(V_m^\varepsilon^*), \\
\partial_t (c_m \theta_m^\varepsilon + \gamma_m \text{div} \mathbf{u}_m^\varepsilon + \alpha_m p_m^\varepsilon) \in L^2_T(V_m^\varepsilon^*)
$$

and for all $\mathbf{v} \in \mathbb{V}$, $(q_1, q_2) \in V^\varepsilon$, we have the three following coupled systems: for a.e. $t \in (0, T)$,

$$
\begin{cases}
\int_{\Omega} \mathbf{A}^\varepsilon e(\mathbf{u}^\varepsilon) e(\mathbf{v}) \, dx + \int_{\Omega_1} (\beta_1 \nabla p_1^\varepsilon + \gamma_1 \nabla \theta_1^\varepsilon) \cdot \mathbf{v} \, dx \\
+ \int_{\Omega_2} (\beta_2 \nabla p_2^\varepsilon + \gamma_2 \nabla \theta_2^\varepsilon) \cdot \mathbf{v} \, dx = \int_{\Omega} \mathbf{f}^\varepsilon \cdot \mathbf{v} \, dx, \\
\langle \partial_t (\phi_1 p_1^\varepsilon + \beta_1 \text{div} \mathbf{u}_1^\varepsilon + \alpha_1 \theta_1^\varepsilon), q_1 \rangle_{V_1^\varepsilon^*, V_1^\varepsilon} \\
+ \langle \partial_t (\phi_2 p_2^\varepsilon + \beta_2 \text{div} \mathbf{u}_2^\varepsilon + \alpha_2 \theta_2^\varepsilon), q_2 \rangle_{V_2^\varepsilon^*, V_2^\varepsilon} \\
+ \int_{\Omega_1} \gamma_1 \nabla p_1^\varepsilon \cdot \nabla q_1 \, dx + \int_{\Omega_2} \gamma_2 \nabla p_2^\varepsilon \cdot \nabla q_2 \, dx \\
+ \int_{\Sigma} \mathcal{S}^\varepsilon (p_1^\varepsilon - p_2^\varepsilon) (q_1 - q_2) \, ds^\varepsilon(x) = \int_{\Omega} g^\varepsilon q^\varepsilon \, dx
\end{cases}
$$

and

$$
\begin{cases}
\langle \partial_t (c_1 \theta_1^\varepsilon + \gamma_1 \text{div} \mathbf{u}_1^\varepsilon + \alpha_1 p_1^\varepsilon), q_1 \rangle_{V_1^\varepsilon^*, V_1^\varepsilon} \\
\langle \partial_t (c_2 \theta_2^\varepsilon + \gamma_2 \text{div} \mathbf{u}_2^\varepsilon + \alpha_2 p_2^\varepsilon), q_2 \rangle_{V_2^\varepsilon^*, V_2^\varepsilon} \\
+ \int_{\Omega_1} \lambda_1 \nabla \theta_1^\varepsilon \cdot \nabla q_1 \, dx + \int_{\Omega_2} \lambda_2 \nabla \theta_2^\varepsilon \cdot \nabla q_2 \, dx \\
+ \int_{\Sigma} \omega^\varepsilon (\theta_1^\varepsilon - \theta_2^\varepsilon) (q_1 - q_2) \, ds^\varepsilon(x) = \int_{\Omega} h^\varepsilon q^\varepsilon \, dx
\end{cases}
$$

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with the initial conditions:
\[
\begin{align*}
\mathbf{u}(0, \cdot) &= \mathbf{0}, \\
\rho_1^1(0, x) &= \theta_1^1(0, x) = 0, \quad x \in \Omega_1^1, \\
\rho_2^2(0, x) &= \theta_2^2(0, x) = 0, \quad x \in \Omega_2^2.
\end{align*}
\]  
(3.5)

where we have denoted
\[
q^\varepsilon(x) = \begin{cases} 
q_1(x), & x \in \Omega_1^1, \\
q_2(x), & x \in \Omega_2^2.
\end{cases}
\]

Existence and uniqueness results for the system (3.2)-(3.5) can be performed by using the Galerkin technique or the semi-group method. For more details, we refer the reader for instance to Showalter and Momken [44]. Hence we give without proof the following result.

**Theorem 3.1** There exists a unique \((u^\varepsilon, p^\varepsilon, \theta^\varepsilon) \in L^\infty_T(V) \times L^2_T(V^\varepsilon)^2\) with \((p^\varepsilon, \theta^\varepsilon) \in L^\infty_T(H^1)^2\) solution of the weak system (2.29), (2.30) such that
\[
\begin{align*}
\|u^\varepsilon\|_{L^\infty_T(V)} &\leq C, \\
\|p^\varepsilon\|_{L^2_T(V^\varepsilon)} + \|p^\varepsilon\|_{L^\infty_T(H^1)} &\leq C, \\
\|\theta^\varepsilon\|_{L^2_T(V^\varepsilon)} + \|\theta^\varepsilon\|_{L^\infty_T(H^1)} &\leq C.
\end{align*}
\]  
(3.6)

The key ingredient of our convergence results are the uniform estimates (3.6) and the use of the two-scale convergence technique, see Sect. 4.1 below. In order to give our main result, we introduce some notations related to the local scale models: let \(d^{kl} = (y_k \delta_{il})_{1 \leq i, l \leq 3}\) and \(w^{ij} \in (H^1_\#(Y)/\mathbb{R})^3\) denote the solution to the microscopic system:
\[
\begin{align*}
-\text{div}_y \left[ A \left( e_y \left( w^{ij} + d^{ij} \right) \right) \right] &= 0 \quad \text{a.e. in } Y, \\
\left[ w^{ij} \right] &= 0 \quad \text{a.e. on } \Sigma, \\
y \mapsto w^{ij}(x,y) &\text{ is } Y - \text{ periodic}
\end{align*}
\]  
(3.7)

where
\[
A(y) := \begin{cases} 
A_1, & y \in Y_1, \\
A_2, & y \in Y_2.
\end{cases}
\]

and \([\cdot]\) denotes the jump on \(\Sigma\). Let us define the "micro-pressure" \(\pi_m^{ij} \in H^1_{\#}(Y_m)/\mathbb{R}\) to be the solution of
\[
\begin{align*}
-\text{div} \left( \kappa_m \left( e^i + \nabla_y \pi_m^{ij} (y) \right) \right) &= 0 \quad \text{in } \Omega \times Y_m, \\
\left[ \kappa_m \left( e^i + \nabla_y \pi_m^{ij} (y) \right) \right] \cdot \nu(y) &= 0 \quad \text{on } \Omega \times \Sigma, \\
y \mapsto \pi_m^{ij}(y) &\text{ is } Y - \text{ periodic.}
\end{align*}
\]  
(3.8)

Here \(e^i\) is the \(i^{th}\) vector of the standard basis of \(\mathbb{R}^3\). Likewise, let the "micro-temperature" \(\vartheta_m^{ij} \in H^1(Y_m)/\mathbb{R}\) to be the solution of
\[
\begin{align*}
-\text{div} \left( \lambda_m(y) \left( e^i + \nabla_y \theta_m^{ij} \right) \right) &= 0 \quad \text{in } \Omega \times Y_m, \\
\left[ \lambda_m(y) \left( e^i + \nabla_y \theta_m^{ij} \right) \right] \cdot \nu(y) &= 0 \quad \text{on } \Omega \times \Sigma, \\
y \mapsto \theta_m^{ij}(y) &\text{ is } Y - \text{ periodic.}
\end{align*}
\]  
(3.9)
Remark 3.1 It is worthwhile noticing that these three boundary value problems (3.7), (3.8) and (3.9) are well-posed in the sense that each problem admits a unique weak solution (see for instance A. Bensoussan & al. [12]). Let us denote

\[ A_{ijkl} := \int_Y A e^y (w_t^{ij} + d^{ij}) e_y (w_k^{kl} + d^{kl}) \, dx, \]

\[ A_{ijkh} := 3 \sum_{r,s=1} a_{ijrs} (y) (\delta_{ir} \delta_{js} + e_{rs,y}(w^{kh})(y)) dy. \]

(3.10)

In other words

\[ A_{ijkh} = \sum_{r,s=1} \int_Y a_{ijrs} (y) (\delta_{ir} \delta_{js} + e_{rs,y}(w^{kh})(y)) dy. \]

(3.11)

Put

\[ B_m := (B_{m,ij})_{1 \leq i,j \leq 3}, D_m := (D_{m,ij})_{1 \leq i,j \leq 3}, C_m := (C_{m,ij})_{1 \leq i,j \leq 3}, K_m := (K_{m,ij})_{1 \leq i,j \leq 3}, L_m := (L_{m,ij})_{1 \leq i,j \leq 3}, \]

\[ f^* := |Y_1| f_1 + |Y_2| f_2, \quad g^*_m := |Y_m| g_m, \quad h^*_m := |Y_m| h_m, \]

\[ c^*_m := |Y_m| c_m, \quad \phi^*_m := |Y_m| \phi_m, \quad \gamma^*_m := |Y_m| \gamma_m, \]

\[ \omega^* := \int_\Sigma \omega (y) ds (y), \quad \zeta^* := \int_\Sigma \zeta (y) ds (y), \quad \alpha^*_m := |Y_m| \alpha_m \]

(3.16)

and

\[ \kappa_m := \int_Y \kappa (\delta^j_i + \text{div} (w^{ij})), \quad \lambda_m := \int_Y \lambda (\delta^j_i + \text{div} (w^{ij})). \]

(3.15)

Theorem 3.2 Let \((u^\varepsilon, p^\varepsilon, \theta^\varepsilon) \in L^\infty_t (V^*) \times L^2_t (V^2) \) be the weak solution of (2.29)-(2.35). Then, up to a subsequence, there exist \( u \in L^\infty_t (V), \) \( p_1, \theta_1 \in \)
functions. Y

and later developed by G. Allaire [5]. This technique is intended to handle homogenization problems involving periodic microstructures. Hereafter, we re-

The two-scale convergence method was first introduced by G. Nguetseng [38] and later developed by G. Allaire [5]. This technique is intended to handle homogenization problems involving periodic microstructures. Hereafter, we recall its definition and its main results. For more details, we refer the reader to [5] 33 [38].

We denote $C^\infty_\#(Y)$ to be the space of all continuous functions in $\mathbb{R}^3$ which are $Y$-periodic. Let $C^\infty_\#(Y')$ denote the subspace of $C^\infty_\#(Y)$ of infinitely differentiable functions.
Lemma 4.1 Let \( q \in L^2(\Omega; C_\#(Y)) \). Then \( q(x,x/\varepsilon) \in L^2(\Omega) \) and
\[
\int_{\Omega} \left| q(x, x/\varepsilon) \right|^2 \, dx \leq \int_{\Omega} \sup_{y \in Y} |q(x,y)|^2 \, dx, \\
\lim_{\varepsilon \to 0} \int_{\Omega} \left| q(x, x/\varepsilon) \right|^2 \, dx = \lim_{\varepsilon \to 0} \int_{\Omega \times Y} q(x,y)^2 \, dx \, dy.
\]
Such a function will be called in the sequel an admissible test function.

Definition 4.1 A sequence \((v^\varepsilon)\) in \( L^2(\Omega) \) two-scale converges to \( v \in L^2(\Omega \times Y) \) and we write \( v^\varepsilon \rightharpoonup^2 v \) if, for any \( q \in L^2(\Omega; C_\#(Y)) \),
\[
\lim_{\varepsilon \to 0} \int_{\Omega} v^\varepsilon(x)q(x,x/\varepsilon) \, dx = \int_{\Omega \times Y} v(x,y)q(x,y) \, dx \, dy.
\]

Theorem 4.1 Let \((v^\varepsilon)\) be a sequence of functions in \( L^2(\Omega) \) which is uniformly bounded. Then, there exist \( v \in L^2(\Omega \times Y) \) and a subsequence of \((v^\varepsilon)\) which two-scale converges to \( v \).

Remark 4.1 Thanks to Theorem 4.1, it is easy to see that for all \( q \in L^2(\Omega; C_\#(Y)) \)
\[
\lim_{\varepsilon \to 0} \int_{\Omega} \varepsilon \int_{\Sigma_{\varepsilon}} f_m(x)q(x, x/\varepsilon) \, dx = \int_{\Omega \times Y} f_m(x)q(x,y) \, dx \, dy.
\]

since
\[
\int_{\Omega_{\varepsilon_m}} f_m(x)q(x, x/\varepsilon) \, dx = \int_{\Omega} f_m(x)\chi_m \left( \frac{x}{\varepsilon} \right) q(x, x/\varepsilon) \, dx
\]
and \( x \mapsto \chi_m \left( \frac{x}{\varepsilon} \right) q(x, x/\varepsilon) \) is an admissible test function.

Theorem 4.2 Let \((v^\varepsilon)\) be a uniformly bounded sequence in \( H^1(\Omega) \) (resp. \( H^1_0(\Omega) \)). Then there exist \( v \in H^1(\Omega) \) (resp. \( H^1_0(\Omega) \)) and \( \hat{v} \in L^2(\Omega; H^1_\#(Y)/\mathbb{R}) \) such that, up to a subsequence,
\[
v^\varepsilon \rightharpoonup^2 v; \quad \nabla v^\varepsilon \rightharpoonup^2 \nabla v + \nabla_y \hat{v}.
\]

In (4.1) and in the sequel the subscript \( y \) on a differential operator as in \( \nabla_y \) indicates that the differentiation acts only on \( y \).

Theorem 4.3 operators act only on those variables

We now extend the notion of two-scale convergence to periodic surfaces [6, 37].

Definition 4.2 A sequence \((w^\varepsilon)\) in \( L^2(\Sigma^\varepsilon) \) two-scale converges to \( w_0(x,y) \in L^2(\Omega \times \Sigma) \) if for any \( q \in D \left( \Omega; C_\#(\Sigma) \right) \)
\[
\lim_{\varepsilon \to 0} \int_{\Sigma^\varepsilon} w^\varepsilon(x)q(x, x/\varepsilon) \, ds^\varepsilon(x) = \int_{\Omega \times \Sigma} w_0(x,y)q(x,y) \, dx \, ds(y).
\]
We state the following compactness result:

**Theorem 4.4** Let \((w^\varepsilon)\) be a sequence in \(L^2(\Sigma^\varepsilon)\) such that

\[
\sqrt{\varepsilon} \int_{\Sigma^\varepsilon} |w^\varepsilon(x)|^2 \, ds^\varepsilon(x) \leq C.
\]

Then, up to a subsequence, there exists \(w_0(x,y) \in L^2(\Omega \times \Sigma)\) such that \((w^\varepsilon)\) two-scale converges in the sense of Definition 4.2 to \(w_0(x,y) \in L^2(\Omega \times \Sigma)\).

**Corollary 4.1** Let \(v(y) \in L^2_\#(\Sigma)\). Then for any \(q \in H^1(\Omega)\)

\[
\lim_{\varepsilon \to 0} \int_{\Sigma^\varepsilon} \varepsilon v \left( \frac{x}{\varepsilon} \right) q(x) \, ds^\varepsilon(x) = \int_{\Omega \times \Sigma} v(y) q(x) \, dx \, ds(y).
\]

**Theorem 4.5** Let \((w^\varepsilon)\) be a sequence of functions in \(H^1(\Omega)\) such that

\[
\|w^\varepsilon\|_{L^2(\Omega)} + \varepsilon \|
abla w^\varepsilon\|_{L^2(\Omega)^N} \leq C.
\]

Then, there exist a subsequence of \((w^\varepsilon)\), still denoted by \((w^\varepsilon)\), and \(w_0(x,y) \in L^2 \left( \Omega; H^1_\#(Y) \right)\) such that \(w^\varepsilon \rightharpoonup w_0\) and \(\varepsilon \nabla w^\varepsilon \rightharpoonup \nabla_y w_0\) and for every \(q \in \mathcal{D} \left( \Omega; C_\#(Y) \right)\) we have

\[
\lim_{\varepsilon \to 0} \varepsilon \int_{\Sigma^\varepsilon} w^\varepsilon(x) q(x) \, ds^\varepsilon(x) = \int_{\Omega \times \Sigma} w_0(x,y) q(x,y) \, dx \, ds(y).
\]

**Remark 4.2** Notice that two-scale convergence can also handle problems involving a parameter without affecting the results stated above. Therefore, we shall use this technique to study the homogenization of our model which involves the time parameter \(t\). For more details, see [20].

### 4.2 Homogenization process

In this subsection, we shall prove Theorem 4.2. The proof is divided into Lemmata 4.2-4.8.

**Lemma 4.2** There exists a subsequence of \((u^\varepsilon, \theta^\varepsilon, p^\varepsilon)\), still denoted \((u^\varepsilon, \theta^\varepsilon, p^\varepsilon)\), and there exist

\[
u \in L^\infty(\Omega)^3, \quad \hat{u} \in L^\infty(\Omega)$$(\mathbb{H}^1(Y)/\mathbb{R})^3)$, \quad p_m, \theta_m \in L^\infty(\Omega)
\]

and

\[
\hat{p}_m, \hat{\theta}_m \in L^2(Q; \mathbb{H}^1(Y_m)/\mathbb{R})
\]

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such that, for a.e. $t \in (0,T)$

$$u^\varepsilon(t, x) \xrightarrow{2-*} u(t, x),$$

(4.2)

$$\chi_m^\varepsilon (x) p_m^\varepsilon(t, x) \xrightarrow{2-*} \chi_m(y) p_m(t, x),$$

(4.3)

$$\chi_m^\varepsilon (x) \theta_m^\varepsilon(t, x) \xrightarrow{2-*} \chi_m(y) \theta_m(t, x),$$

(4.4)

$$\nabla u^\varepsilon(t, x) \xrightarrow{2-*} \nabla u(t, x) + \nabla_y \hat{u}(t, x, y),$$

(4.5)

Moreover, for any $\psi \in D(Q; C_\#(Y))$ we have

$$\lim_{\varepsilon \to 0} \int_{\Omega_2} \zeta^\varepsilon(x) (p_1^\varepsilon(t, x) - p_2^\varepsilon(t, x)) \psi^\varepsilon(t, x) \, ds^\varepsilon(x) \, dt = \int_{Q \times \Sigma} \zeta(y) (p_1(t, x) - p_2(t, x)) \psi(t, x, y) \, dt \, dz \, ds(y),$$

(4.8)

$$\lim_{\varepsilon \to 0} \int_{\Omega_2} \omega^\varepsilon(x) (\theta_1^\varepsilon(t, x) - \theta_2^\varepsilon(t, x)) \psi^\varepsilon(t, x) \, ds^\varepsilon(x) \, dt = \int_{Q \times \Sigma} \omega(y) (\theta_1(t, x) - \theta_2(t, x)) \psi(t, x, y) \, dt \, dz \, ds(y),$$

(4.9)

where we have denoted $\psi^\varepsilon(t, x) = \psi(x, t, x/\varepsilon)$.

**Proof.** The two-scale limits (4.2)-(4.3) are a straightforward application of the a priori estimates (3.6) and the compactness Theorems 4.1, 4.2, 4.4 and 4.5. □

**Lemma 4.3** The corrector displacement $\hat{u}$ can be written as:

$$\hat{u}(t, x, y) = w^{ij}(y) e_{ij}(u)(t, x) \quad \text{for a.e.} \quad (t, x, y) \in Q \times Y$$

(4.10)

where $w^{ij} \in (H^1_\#(Y)/\mathbb{R})^3$ is the solution to the microscopic system (4.7).

**Proof.** We choose adequate test functions in (3.2): Let

$$\nu(x) := \nu^\varepsilon(x) = \varepsilon \hat{\nu}(x, \frac{x}{\varepsilon})$$

where $\hat{\nu} \in D(\Omega; C_\infty^\#(Y))^3$. Then, we have for a.e. $t \in (0,T)$

$$\int_\Omega A \left( \frac{x}{\varepsilon} \right) e(u^\varepsilon)(t, x) \left\{ \varepsilon e_x(\hat{\nu})(x, \frac{x}{\varepsilon}) + e_y(\hat{\nu})(x, \frac{x}{\varepsilon}) \right\} \, dx$$

$$+ \varepsilon \int_{\Omega_1^1} (\beta_1 \nabla p_1^\varepsilon(t, x) + \gamma_1 \nabla \theta_1^\varepsilon(t, x)) \hat{\nu}(x, \frac{x}{\varepsilon}) \, dx$$

$$+ \varepsilon \int_{\Omega_2^2} (\beta_2 \nabla p_2^\varepsilon(t, x) + \gamma_2 \nabla \theta_2^\varepsilon(t, x)) \hat{\nu}(x, \frac{x}{\varepsilon}) \, dx$$

(4.11)

$$= \varepsilon \int_{\Omega_1^1} f_1(x) \hat{\nu}(x, \frac{x}{\varepsilon}) \, dx + \varepsilon \int_{\Omega_2^2} f_2(x) \hat{\nu}(x, \frac{x}{\varepsilon}) \, dx.$$
In view of (4.5), we pass to the limit in the first term of the l.h.s. of (4.11) to get for a.e. $t \in (0, T)$

$$
\lim_{\varepsilon \to 0} \int_{\Omega} A^\varepsilon (x) e(u^\varepsilon) (t, x) \left\{ \varepsilon e_x (\hat{v}) (x, \frac{x}{\varepsilon}) + e_y (\hat{v}) (x, \frac{x}{\varepsilon}) \right\} \, dx
$$

$$
= \int_{\Omega \times Y} A (y) (e(u) (t, x) + e_y (\hat{u}) (t, x, y)) e_y (\hat{v}) (x, y) \, dx \, dy. \quad (4.12)
$$

Next, since

$$
\left| \varepsilon \int_{\Omega_m} \beta_m \nabla p^\varepsilon_m (t, x) \hat{v}(x, \frac{x}{\varepsilon}) \, dx \right| \leq \varepsilon \beta_m \left\| \hat{v}(x, \frac{x}{\varepsilon}) \right\|_{L^2(\Omega)} \left\| \nabla p^\varepsilon_m \right\|_{L^2(\Omega_m)},
$$

$$
\left| \varepsilon \int_{\Omega_m} \gamma_m \nabla \theta^\varepsilon_m (t, x) \hat{v}(x, \frac{x}{\varepsilon}) \, dx \right| \leq \varepsilon \gamma_m \left\| \hat{v}(x, \frac{x}{\varepsilon}) \right\|_{L^2(\Omega)} \left\| \nabla \theta^\varepsilon_m \right\|_{L^2(\Omega_m)}
$$

and taking into account (3.6) together with Lemma 4.1, we see that for a.e. $t \in (0, T)$

$$
\lim_{\varepsilon \to 0} \varepsilon \int_{\Omega_m} (\beta_m \nabla p^\varepsilon_m (t, x) + \gamma_m \nabla \theta^\varepsilon_m (t, x)) \hat{v}(x, \frac{x}{\varepsilon}) \, dx = 0.
$$

In the same way, letting $\varepsilon \to 0$ in the r.h.s. of (4.11) we obtain:

$$
\lim_{\varepsilon \to 0} \varepsilon \left( \int_{\Omega_1} f_1 (x) \hat{v}(x, \frac{x}{\varepsilon}) \, dx + \int_{\Omega_2} f_2 (x) \hat{v}(x, \frac{x}{\varepsilon}) \, dx \right) = 0. \quad (4.13)
$$

Collecting (4.12), (4.13) and passing to the limit in (4.11) yield for a.e. $t \in (0, T)$

$$
\int_{\Omega \times Y} A (y) (e(u) (t, x) + e_y (\hat{u}) (t, x, y)) e_y (\hat{v}) (x, y) \, dx = 0
$$

which gives after an integration by parts the following boundary value problem:

$$
\begin{cases}
-\text{div}_y \{ A (y) (e(u) (t, x) + e_y (\hat{u}) (t, x, y)) \} = 0 & \text{a.e. in } Q \times Y, \\
y \mapsto \hat{u}(t, x, y) = 0 & \text{is } Y \text{-periodic.}
\end{cases} \quad (4.14)
$$

According to the linearity of the system (4.14), we see that $\hat{u}$ can be written in terms of $u$ through the following scale separation expression:

$$
\hat{u}(t, x, y) = e_{ij} (u(t, x)) w^{ij} (y), \quad \text{a.e. (} t, x, y \text{) } Q \times Y
$$

where $w^{ij} \in (H^1_{\text{Div}} (Y) / \mathbb{R})^3$ is the solution to the microscopic system defined by (3.7), see for instance [12, page 15]. Hence (4.10) is proved.

**Lemma 4.4** The corrector pressure $\hat{p}_m$ satisfies

$$
\hat{p}_m (t, x, y) = \pi^i_m (y) \frac{\partial p_m}{\partial x_i} (t, x), \quad \text{a.e. (} t, x, y \text{) } Q \times Y_m \quad (4.15)
$$

where $\pi^i_m (y)$ $i = 1, 2, 3, m = 1, 2$ are defined by (3.8).
Proof. Let \( \hat{q}_m \in \mathcal{D}(Q; C_\#_\infty(Y)) \). Taking
\[
q_m(t, x) := \hat{q}_m(t, x, \frac{x}{\varepsilon})
\]
in (3.3) and integrating with respect to \( t \in (0, T) \), we get
\[
\langle \partial_t(\phi_1 p_1^\varepsilon + \beta_1 \text{div} u^\varepsilon + \alpha_1 \theta_1^\varepsilon), q_1^\varepsilon \rangle_{V_1^\varepsilon, V_1^\varepsilon}
+
\langle \partial_t(\phi_2 p_2^\varepsilon + \beta_2 \text{div} u^\varepsilon + \alpha_2 \theta_2^\varepsilon), q_2^\varepsilon \rangle_{V_2^\varepsilon, V_2^\varepsilon}
+
\int_{Q_1} \kappa_1 \nabla p_1^\varepsilon(t, x) \left( \varepsilon \nabla_x \hat{q}_1(t, x, \frac{x}{\varepsilon}) + \nabla_y \hat{q}_1(t, x, \frac{x}{\varepsilon}) \right) dt dx
+
\int_{Q_2} \kappa_2 \nabla p_2^\varepsilon(t, x) \left( \varepsilon \nabla_x \hat{q}_2(t, x, \frac{x}{\varepsilon}) + \nabla_y \hat{q}_2(t, x, \frac{x}{\varepsilon}) \right) dt dx
+ \varepsilon R_\varepsilon = 0
\]
where
\[
R_\varepsilon = \int_{Q_1} \kappa_1 \nabla p_1^\varepsilon \left( \frac{x}{\varepsilon} \right) \left( p_1^\varepsilon - p_2^\varepsilon \right) \left( \hat{q}_1^\varepsilon - \hat{q}_2^\varepsilon \right) dt ds^\varepsilon(x)
\]
and \( \hat{q}_m^\varepsilon(t, x) := \hat{q}_m(t, t, x/\varepsilon) \). From (4.3) it is easy to see that
\[
\left| \int_{Q_1} \varepsilon \kappa \left( \frac{x}{\varepsilon} \right) \left( p_1^\varepsilon - p_2^\varepsilon \right) \left( \hat{q}_1^\varepsilon - \hat{q}_2^\varepsilon \right) dt ds^\varepsilon(x) \right| \leq C. \tag{4.18}
\]
Furthermore, thanks to Lemma 4.4 we have
\[
\lim_{\varepsilon \to 0} \left( \int_{Q_1} g_1 \hat{q}_1^\varepsilon dt dx + \int_{Q_2} g_2 \hat{q}_2^\varepsilon dt dx \right) =
\int_{Q \times Y_1} g_1 \hat{q}_1 dt dx dy + \int_{Q \times Y_2} g_2 \hat{q}_2 dt dx dy. \tag{4.19}
\]
By virtue of the uniform estimates (3.3), the sequences \( \{p_m^\varepsilon\}_\varepsilon \), \( \{\text{div} u^\varepsilon\}_\varepsilon \) and \( \{\theta_m^\varepsilon\}_\varepsilon \) are uniformly bounded in \( L^2(Q_m^\varepsilon) \) so that
\[
\left| \int_{Q_m^\varepsilon} \left( \phi_m^\varepsilon p_m^\varepsilon + \beta_m \text{div} u^\varepsilon + \alpha_m \theta_m^\varepsilon \right) \partial_t \hat{q}_m^\varepsilon dt dx \right| \leq C. \tag{4.20}
\]
Taking into account (4.18) - (4.20) we get from (4.17) that
\[
\lim_{\varepsilon \to 0} \varepsilon R_\varepsilon = 0.
\]

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On the other hand passing to the limit in (4.16) and taking into account (4.6) we are led to

$$\int_{Q \times Y} \chi_1(y) \kappa_1(\nabla p_1(t,x) + \nabla_y \hat{p}_1(t,x)) \nabla_y \hat{q}_1(t,x,y) dt dx = 0,$$

so that an integration by parts yields:

$$\begin{cases}
- \text{div} \left( \kappa_1(\nabla p_1(t,x) + \nabla_y \hat{p}_1(t,x,y)) \right) = 0 \text{ a.e. in } Q \times Y_1, \\
- \text{div} \left( \kappa_2(\nabla p_2(t,x) + \nabla_y \hat{p}_2(t,x,y)) \right) = 0 \text{ a.e. in } Q \times Y_2, \\
[\kappa_m(\nabla p_m(t,x) + \nabla_y \hat{p}_m(t,x,y)) \cdot \nu(y)] \text{ a.e. on } Q \times \Sigma,
\end{cases}$$

(4.21)

As in Lemma 4.3 and thanks to the linearity of the system (4.21) we can write that

$$\hat{p}_m(t,x,y) = \pi^i_m(y) \frac{\partial p_m(t,x)}{\partial x_i} \text{ a.e. } (t,x,y) \in Q \times Y_m,$$

where $\pi^i_m(y) \in \pi_{m,1} \in H^1(Y_m) \cap \mathbb{R}$ is the solution of (3.8). The Lemma is then proved.

Lemma 4.5 The corrector temperature $\hat{\theta}_m$ is related to the homogenized temperature $\theta_m$ via the linear relation:

$$\hat{\theta}_m(t,x,y) = \vartheta^i_m(y) \frac{\partial \theta_m(t,x)}{\partial x_i} + C \text{ a.e. } (t,x,y) \in Q \times Y_m$$

(4.22)

where, for $i = 1, 2, 3$, the ”micro-temperature” $\vartheta^i_m \in H^1(Y_m) \cap \mathbb{R}$ is the solution of (3.9).

The proof of this Lemma follows the same lines as that of Lemma 4.4 and therefore will not be given.

Lemma 4.6 The macroscopic balance equation reads as follows:

$$\begin{cases}
- \text{div} \left( \sigma^\theta(u) \right) + B_1 \nabla p_1 + B_2 \nabla p_2 + D_1 \nabla \theta_1 + D_2 \nabla \theta_2 = f \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega
\end{cases}$$

(4.23)

where $\sigma$, $B_m$, $D_m$ and $f$ are defined by (3.11), (3.12), (3.13) and (3.16) respectively.

Proof. The convergence results obtained in Lemma 4.2 allow us to derive the macroscopic equations (3.17). To do so, we first determine the limiting equations.
Likewise, by using (4.15) and (4.22), there holds (3.10), (3.11) and (4.10). We have

Let us rewrite the first integral term in the l.h.s. of (4.24) with the help of (4.5)-(4.7) and Remark 4.1 that for a.e. $t \in (0, T)$

$$\int_{\Omega \times Y} A|e(u) + e_y(\hat{u})|e(v) dx dy + \beta_1 \int_{\Omega \times Y_1} (\nabla p_1 + \nabla_y \hat{p}_1) v dx dy$$

$$+ \beta_2 \int_{\Omega \times Y_2} (\nabla p_2 + \nabla_y \hat{p}_2) v dx dy + \gamma_1 \int_{\Omega \times Y_1} \left( \nabla \theta_1 + \nabla_y \hat{\theta}_1 \right) v dx dy$$

$$+ \gamma_2 \int_{\Omega \times Y_2} \left( \nabla \theta_2 + \nabla_y \hat{\theta}_2 \right) v dx dy$$

$$= \int_{\Omega \times Y_1} f_1 v dx dy + \int_{\Omega \times Y_2} f_2 v dx dy. \quad (4.24)$$

Let us rewrite the first integral term in the l.h.s. of (4.24) with the help of (3.10), (3.11) and (4.10). We have

$$\int_{\Omega \times Y} A_{ijkl} [e(u) + e_y(\hat{u})] e_i(v) dx dy = \int_\Omega \sigma_{ij}^{\text{hom}}(u) e_i(v) dx. \quad (4.25)$$

Likewise, by using (4.16) and (4.22), there holds

$$\beta_1 \int_{\Omega \times Y_1} (\nabla p_1 + \nabla_y \hat{p}_1) v dx dy + \beta_2 \int_{\Omega \times Y_2} (\nabla p_2 + \nabla_y \hat{p}_2) v dx dy$$

$$= \beta_1 \int_{\Omega \times Y} \chi_1 \left( \delta_{ik} + \frac{\partial \pi_1^{ij}}{\partial y_k} \right) \frac{\partial p_1}{\partial x_i} v_k dx$$

$$+ \beta_2 \int_{\Omega \times Y} \chi_2 \left( \delta_{ik} + \frac{\partial \pi_2^{ij}}{\partial y_k} \right) \frac{\partial p_2}{\partial x_i} v_k dx \quad (4.26)$$

where $v_k$ is the $k$th component of $v$. After simple algebraic calculations, (4.26) becomes then

$$\beta_1 \int_{\Omega \times Y_1} (\nabla p_1 + \nabla_y \hat{p}_1) v dx dy + \beta_2 \int_{\Omega \times Y_2} (\nabla p_2 + \nabla_y \hat{p}_2) v dx dy$$

$$= \int_\Omega B_1 \nabla p_1(x) v dx + \int_\Omega B_2 \nabla p_2 v dx. \quad (4.27)$$

In the same way, one can show that

$$\gamma_1 \int_{\Omega \times Y_1} \left( \nabla \theta_1 + \nabla_y \hat{\theta}_1 \right) v dx dy + \gamma_2 \int_{\Omega \times Y_2} \left( \nabla \theta_2 + \nabla_y \hat{\theta}_2 \right) v dx dy$$

$$= \int_\Omega D_1 \nabla \theta_1 v dx + \int_\Omega D_2 \nabla \theta_2 v dx. \quad (4.28)$$

Finally, inserting (3.10), (4.25), (4.27) and (4.28) into (4.24) and using the fact that $D(\Omega)^3$ is dense in $H_0^1(\Omega)^3$, we deduce the homogenized balance formulation:

$$\int_\Omega A_{ijkl} e_{kh}(u) e_i(v) dx + \int_\Omega B_1 \nabla p_1 v dx + \int_\Omega B_2 \nabla p_2 v dx$$

$$+ \int_\Omega D_1 \nabla \theta_1 v dx + \int_\Omega D_2 \nabla \theta_2 v dx = \int_\Omega f v dx \quad (4.29)$$

which by an integration by parts yields (4.23). The Lemma is then proved.
Lemma 4.7 The macroscopic mass conservation equation is given by:

\[
\begin{align*}
\partial_t (\phi^*_1 p_1 + \beta_1 C_1 : e (u) + \alpha_1^* \theta_1) - \text{div} (K_1 \nabla p_1) + \zeta^* (p_1 - p_2) &= g^*_1, \text{ in } Q, \\
\partial_t (\phi^*_2 p_2 + \beta_2 C_2 : e (u) + \alpha_2^* \theta_2) - \text{div} (K_2 \nabla p_2) + \zeta^* (p_2 - p_1) &= g^*_2, \text{ in } Q, \\
p_1 = 0, &\text{ on } \Gamma_T, \\
K_2 \nabla p_2 \cdot \nu = 0 &\text{ on } \Gamma_T, \\
p_1 (0, \cdot) = p_2 (0, \cdot) &= 0 \text{ in } \Omega. 
\end{align*}
\]  

(4.30)

where \(C_m, K_m\) and \((\phi^*_m, \alpha^*_m, \zeta^*, g^*_m)\) are given in (3.14), (3.15) and (3.16) respectively.

Proof. Let \(q_m (t, x) \in D((0, T) \times \Omega)\). Integration by parts in (3.3) with respect to the time variable \(t \in (0, T)\) yields:

\[
\begin{align*}
\int_{Q^1} (\phi^*_1 p_1 (t, x) + \beta_1 \text{div} u^* (t, x) + \alpha_1^* \theta_1 (t, x)) \partial_t q_1 (t, x) \, dt \, dx &+ \int_{Q^1} (\phi^*_2 p_2 (t, x) + \beta_2 \text{div} u^* (t, x) + \alpha_2^* \theta_2 (t, x)) \partial_t q_2 (t, x) \, dt \, dx \\
&+ \int_{Q^1} \kappa_1 \nabla \phi^*_1 (t, x) \nabla q_1 (t, x) \, dt \, dx + \int_{Q^2} \kappa_2 \nabla p_2 (t, x) \nabla q_2 (t, x) \, dt \, dx \\
&+ \int_{Q_2} \zeta^* (x) (p^*_1 (t, x) - p^*_2 (t, x)) (q_1 (t, x) - q_2 (t, x)) \, ds^* (x) \, dt \\
&= \int_{Q^1} g_1 (x) q_1 (t, x) \, dt \, dx + \int_{Q^2} g_2 (x) q_2 (t, x) \, dt \, dx.
\end{align*}
\]

Using (4.31) and (4.41) we obtain:

\[
\begin{align*}
\lim_{\varepsilon \to 0} \int_{Q^m} (\phi_m p^*_m (t, x) + \alpha_m \theta^*_m (t, x)) \partial_t q_m (t, x) \, dt \, dx \\
&= \int_{Q \times Y_m} (\phi_m p_m (t, x) + \alpha_m \theta_m (t, x)) \partial_t q_m (t, x) \, dt \, dx \, dy. 
\end{align*}
\]

(4.32)

Furthermore, thanks to (4.3) and (4.6) we see that

\[
\lim_{\varepsilon \to 0} \int_{Q^m} \beta_m \text{div} u^* (x) \partial_t q_m (t, x) \, dt \, dx \\
= \int_{Q \times Y_m} \beta_m (\text{div} u (t, x) + \text{div} \hat{u} (t, x, y)) \partial_t q_m (t, x) \, dt \, dx \, dy
\]

and

\[
\lim_{\varepsilon \to 0} \int_{Q^m} \kappa_m \nabla p^*_m (t, x) \nabla q_m (t, x) \, dt \, dx \\
= \int_{Q \times Y_m} \left( \kappa_m \left( \nabla p_m (t, x) + \nabla \hat{p}_m (t, x, y) \right) \right) \nabla q_m (t, x) \, dt \, dx \, dy.
\]

(4.34)
Now, using (4.10) we observe that
\[
\int_{Q \times Y_m} \beta_m (\text{div} \, u (t, x) + \text{div} \, \hat{u} (t, x, y)) \, \partial_t q_m (t, x) \, dt \, dx \, dy \\
= \int_{Q} \beta_m c_m : e (u) (t, x) \, \partial_t q_m (t, x) \, dt.
\]
In the same way, using (4.15) we find that
\[
\int_{Q \times Y_m} (\kappa_m \{ \nabla p_m (t, x) + \nabla \hat{p}_m (t, x, y) \}) \nabla q_m (t, x) \, dx \, dy \, dt \\
= \int_{Q} K_m \nabla p_m (x, t) \nabla q_m (x, t) \, dt \, dx.
\]
Now, taking into account (4.3) we have
\[
\lim_{\varepsilon \to 0} \int_{Q \times Y} \frac{c}{\varepsilon} \left( \frac{p_1 (x, t)}{2} - p_2 (x) \right) (q_1 (x, t) - q_2 (x, t)) \, dx \, dt \\
= \int_{Q \times Y_m} (\kappa_m \{ \nabla p_m (t, x) + \nabla \hat{p}_m (t, x, y) \}) \nabla q_m (t, x) \, dx \, dy \, dt \\
= \int_{Q} K_m \nabla p_m (x, t) \nabla q_m (x, t) \, dt \, dx.
\]
Using Remark 4.1 we infer that
\[
\lim_{\varepsilon \to 0} \left( \int_{Q} g_1 (x) q_1 (t, x) \, dt \, dx + \int_{Q} g_2 (x) q_2 (t, x) \, dt \, dx \right) \\
= \int_{Q} \hat{g}_1 q_1 (t, x) \, dt \, dx + \int_{Q} \hat{g}_2 q_2 (t, x) \, dt \, dx.
\]
Finally, having in mind (4.32)-(4.38), we can now pass to the limit in (4.31) to get
\[
\int_{Q} \{ \phi^*_1 p_1 (t, x) + \beta_1 C_1 : e (u) (t, x) + \alpha^*_1 \theta_1 (t, x) \} \, \partial_t q_1 (t, x) \, dt \, dx \\
+ \int_{Q} \{ \phi^*_2 p_2 (t, x) + \beta_2 C_2 : e (u) (t, x) + \alpha^*_2 \theta_2 (t, x) \} \, \partial_t q_2 (t, x) \, dt \, dx \\
+ \int_{Q} \kappa_1 \nabla p_1 (t, x) \nabla q_1 (t, x) \, dt \, dx + \int_{Q} \kappa_2 \nabla p_2 (t, x) \nabla q_2 (t, x) \, dt \, dx \\
+ \int_{Q} \zeta^* (p_1 (t, x) - p_2 (t, x)) (q_1 (t, x) - q_2 (t, x)) \, dt \, dx \\
= \int_{Q} g_1 (x) q_1 (t, x) \, dt \, dx + \int_{Q} g_2 (x) q_2 (t, x) \, dt \, dx
\]
which by integration by parts yields (4.39). This gives the desired result. ■

Likewise, as in the proof of Lemma 4.7 one can analogously show the following result:

**Lemma 4.8** The macroscopic heat equation is given by:
\[
\begin{align*}
\partial_t \{ c^*_1 \theta_1 + \gamma_1 C_1 : e (u) + \alpha^*_1 p_1 \} - \text{div}(L_1 \nabla \theta_1) + \omega^* (\theta_1 - \theta_2) &= h^*_1, \quad \text{in} \, Q, \\
\partial_t \{ c^*_2 \theta_2 + \gamma_2 C_2 : e (u) + \alpha^*_2 p_2 \} - \text{div}(L_2 \nabla \theta_2) + \omega^* (\theta_2 - \theta_1) &= h^*_2, \quad \text{in} \, Q, \\
\theta_1 &= 0, \quad \text{on} \, \Gamma_T, \\
L_2 \nabla \theta_2 \cdot \nu &= 0 \quad \text{on} \, \Gamma_T, \\
\theta_1 (0, \cdot) &= \theta_2 (0, \cdot) = 0 \quad \text{in} \, \Omega
\end{align*}
\]
where $L_m$ and $(c^*_m, \omega^*, h^*_m)$ are given in (3.15) and (3.10) respectively.

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Proof of Theorem 5.2 Collecting all the results given in Lemmata 4.6, 4.7, and 4.8 we arrive at the homogenized system \((3.17)\). □

5 A corrector result

Let \(\hat{u}, \hat{p}_m\) and \(\hat{\theta}_m\) to be the corrector terms of \(u^\varepsilon, p^\varepsilon_m\) and \(\theta^\varepsilon_m\) respectively. They are given by (4.3), (4.9) and (4.7) respectively. Now we first give the following corrector result.

Theorem 5.1 Assume that \(A \in L^\infty(\Lambda)\). Then \(e_y(\hat{u})\) is an admissible test function and the sequence \([e(u^\varepsilon)(x) - e(u)(x) - e_y(\hat{u}) (x, \frac{\varepsilon}{2})]\) converges strongly to 0 in \(L^2(\Omega)^{3 \times 3}\). We also have

\[
\lim_{\varepsilon \to 0} \| \chi_m \left( \frac{x}{\varepsilon} \right) \left\{ \nabla p^\varepsilon_m (t, x) - \nabla p_m (t, x) - \nabla y \hat{p}_m (t, x, \frac{\varepsilon}{2}) \right\} \|_{0, Q_T} = 0,
\]

\[
\lim_{\varepsilon \to 0} \| \chi_m \left( \frac{x}{\varepsilon} \right) \left\{ \nabla \theta^\varepsilon_m (t, x) - \nabla \theta_m (t, x) - \nabla y \hat{\theta}_m (t, x, \frac{\varepsilon}{2}) \right\} \|_{0, Q_T} = 0
\]

and

\[
\lim_{\varepsilon} \left( \int_{\Omega} A^\varepsilon e(u^\varepsilon)e(u^\varepsilon) dx + \sum_m \int_{\Omega_m} (\alpha_m \nabla p^\varepsilon_m + \gamma_m \nabla \theta^\varepsilon_m) u^\varepsilon dx \right)
\]

\[
= \int_{\Omega} A e(u) e(u) dx + \sum_m \int_{\Omega} (\xi_m \nabla \theta_m + \eta_m \nabla p_m) u^0 dx.
\]

To prove Theorem 5.1 we begin by establishing an integral identity.

Proposition 5.1 We have

\[
- \int_{\Omega_m} \phi^*_1(p_1)^2 (T, x) dx - \int_{Q \times Y_1} [\partial_t(\beta_1 (\text{div} u + \text{div} \hat{u}) + \alpha_1 \beta_1)] p_1 dt dx dy
\]

\[
- \int_{\Omega_m} \phi^*_2(p_2)^2 (T, x) dx - \int_{Q \times Y_2} [\partial_t(\beta_2 (\text{div} u + \text{div} \hat{u}) + \alpha_2 \beta_2)] p_2 dt dx dy
\]

\[
+ \int_{Q \times Y_1} K_1 \nabla p_1 \nabla p_1 dt dx dy + \int_{Q \times Y_2} K_2 \nabla p_2 \nabla p_2 dt dx dy
\]

\[
+ \int_{Q} \omega^*(p_1 - p_2)^2 dt d x = \int_{\Omega} g_1^* p_1 dx + \int_{\Omega} g_2^* p_2 dx \tag{5.1}
\]

and

\[
- \int_{\Omega_m} c^*_1(\theta_1)^2 (T, x) dx dy - \int_{Q \times Y_m} [\partial_t(\gamma_1 (\text{div} u + \text{div} \hat{u}) + \alpha_1 \gamma_1)] \theta_1 dt dx dy
\]

\[
- \int_{\Omega_m} c^*_2(\theta_2)^2 (T, x) dx dy - \int_{Q \times Y_m} [\partial_t(\gamma_2 (\text{div} u + \text{div} \hat{u}) + \alpha_2 \gamma_2)] \theta_2 dt dx dy
\]

\[
+ \int_{Q \times Y_1} L_1 \nabla \theta_1 \nabla \theta_1 dt dx dy + \int_{Q \times Y_2} L_2 \nabla \theta_2 \nabla \theta_2 dt dx dy + \int_Q \omega^*(\theta_1 - \theta_2)^2 dt dx = \int_{\Omega} h_1^* \theta_1 dx + \int_{\Omega} h_2^* \theta_2 dx \tag{5.2}
\]
Proof. Take \( q_m \in L^2 (0, T; H^1 (\Omega^m)) \), \( m = 1, 2 \) in (3.3) and integrate by parts to yield:

\[
\begin{align*}
- \int_{Q_1} (\phi_1 p_1 (t, x) + \beta_1 \text{div} u^+ (T, x) + \alpha_1 \theta_1 (t, x)) \partial_t q_1 (t, x) \, dt \, dx \\
- \int_{Q_2} (\phi_2 p_2 (t, x) + \beta_2 \text{div} u^+ (t, x) + \alpha_2 \theta_2 (t, x)) \partial_t q_2 (t, x) \, dt \, dx \\
+ \int_{\Omega_1} (\phi_1 p_1 (T, x) + \beta_1 \text{div} u^+ (T, x) + \alpha_1 \theta_1 (T, x)) q_1 (T, x) \, dx \\
+ \int_{\Omega_2} (\phi_2 p_2 (T, x) + \beta_2 \text{div} u^+ (T, x) + \alpha_2 \theta_2 (T, x)) q_2 (T, x) \, dt \, dx \\
- \int_{\Omega_1} (\phi_1 p_1 (0, x) + \beta_1 \text{div} u^+ (0, x) + \alpha_1 \theta_1 (0, x)) q_1 (0, x) \, dx \\
- \int_{\Omega_2} (\phi_2 p_2 (0, x) + \beta_2 \text{div} u^+ (0, x) + \alpha_2 \theta_2 (0, x)) q_2 (0, x) \, dx \\
\int_{Q_1} \kappa_1 \nabla p_1 (t, x) \nabla q_1 (t, x) \, dt \, dx + \int_{Q_2} \kappa_2 \nabla p_2 (t, x) \nabla q_2 (t, x) \, dt \, dx \\
+ \int_{\Sigma_T} \kappa^c (x) (p_1 (t, x) - p_2 (t, x)) (q_1 (t, x) - q_2 (t, x)) \, ds^c (x) \, dt \\
= \int_{Q_1} q_1 (t, x) \, dt \, dx + \int_{Q_2} q_2 (x) q_2 (t, x) \, dt \, dx.
\end{align*}
\]

Passing to the limit in the last identity we get

\[
\begin{align*}
- \int_{Q_1} (\phi_1 p_1 (t, x) + \beta_1 \text{div} u (t, x) + \text{div}_y u_1 (t, x, y)) + \alpha_1 \theta_1 (t, x)) \partial_t q_1 (t, x) \, dt \, dx \, dy \\
- \int_{Q_2} (\phi_2 p_2 (t, x) + \beta_2 \text{div} u (t, x) + \text{div}_y u_1 (t, x, y)) + \alpha_2 \theta_2 (t, x)) \partial_t q_2 (t, x) \, dt \, dx \, dy \\
+ \int_{\Omega_1 \times Y_1} (\phi_1 p_1 (T, x) + \beta_1 \text{div} u (T, x) + \text{div}_y u_1 (T, x, y)) + \alpha_1 \theta_1 (T, x)) q_1 (T, x) \, dt \, dx \, dy \\
+ \int_{\Omega_2 \times Y_2} (\phi_2 p_2 (T, x) + \beta_2 \text{div} u (T, x) + \text{div}_y u_1 (T, x, y)) + \alpha_2 \theta_2 (T, x)) q_2 (T, x) \, dx \, dy \\
- \int_{\Omega_1 \times Y_1} (\phi_1 p_1 (0, x) + \beta_1 \text{div} u (0, x) + \text{div}_y u_1 (0, x, y)) + \alpha_1 \theta_1 (0, x)) q_1 (0, x) \, dx \, dy \\
- \int_{\Omega_2 \times Y_2} (\phi_2 p_2 (0, x) + \beta_2 \text{div} u (0, x) + \text{div}_y u_1 (0, x, y)) + \alpha_2 \theta_2 (0, x)) q_2 (0, x) \, dx \, dy \\
+ \int_{Q_1} \kappa_1 (\nabla p_1 (t, x) + \nabla_y \tilde{p}_1 (t, y)) \nabla q_1 (t, x) \, dt \, dx \, dy \\
+ \int_{Q_2} \kappa_2 (\nabla p_2 (t, x) + \nabla_y \tilde{p}_2 (t, y)) \nabla q_2 (t, x) \, dt \, dx \, dy \\
+ \int_{\Sigma_T} \kappa (y) (p_1 (t, x) - p_2 (t, x)) (q_1 (t, x) - q_2 (t, x)) \, dt \, dx \, dy \, ds (y) \\
= \int_{Q_1 \times Y} g_1 (x) q_1 (t, x) \, dx \, dy \, dt + \int_{Q_2 \times Y} g_2 (x) q_2 (t, x) \, dt \, dx \, dy.
\end{align*}
\]

Now, taking any sequence \( (q_m^n) \) converging to \( p_m \) in the last identity and
Finally, thanks to the coercivity of $\mathbf{A}$, Proposition 5.2 goes along the same lines as that of \[5.1\].

**Proposition 5.2** $e_y(\tilde{u})$ is an admissible test function and the sequence $[e(u^\varepsilon)(x) - e(u)(x) - e_y(\tilde{u})(x, \frac{x}{\varepsilon})]$ converges strongly to 0 in $L^2(\Omega)^{3\times 3}$.

**Proof.** We argue as in \[5.7\] - \(4.10\) and standard results on regularity of elliptic equations \[27\], $e_y(\tilde{u})$ is an admissible test function. Applying (3.2) yields for a.e. $t \in (0, T)$

$$
\int_{\Omega} \left\{ \mathbf{A} \left( \frac{x}{\varepsilon} \right) \left[ e(u^\varepsilon)(x) - e(u)(x) - e_y(\tilde{u}) \left( x, \frac{x}{\varepsilon} \right) \right] \right\} \, dx =
$$

$$
- \int_{\Omega} (\mathbf{A} + \varepsilon \mathbf{A}) e(u^\varepsilon)[e(u) + e_y(\tilde{u}) \left( x, \frac{x}{\varepsilon} \right)] \, dx
$$

$$
+ \int_{\Omega} \mathbf{A} e(u) + e_y(\tilde{u}) \left( x, \frac{x}{\varepsilon} \right) \left[ e(u) + e_y(\tilde{u}) \left( x, \frac{x}{\varepsilon} \right) \right] \, dx
$$

$$
\int_{\Omega} \mathbf{F}_1 u^\varepsilon \, dx - \int_{\Omega} \left( \beta_1 p_m^\varepsilon + \gamma_1 \theta_m^\varepsilon \right) \text{div} u^\varepsilon \, dx
$$

$$
\int_{\Omega} \mathbf{F}_2 u^\varepsilon \, dx - \int_{\Omega} \left( \beta_2 p_m^\varepsilon + \gamma_2 \theta_m^\varepsilon \right) \text{div} u^\varepsilon \, dx. \quad (5.3)
$$

Using the strong convergences in $L^2(\Omega)$ of $u^\varepsilon$, $p_m^\varepsilon$, and $\theta_m^\varepsilon$ to $u$, $p_m$ and $\theta_m$ respectively and the weak convergence of $\chi_m \text{div} u^\varepsilon$ to $\int_{\Omega} (\text{div} u + \text{div} y \tilde{u})$ and taking the limit in the last term of the r.h.s. of \(5.3\) we obtain

$$
\lim_{\varepsilon \to 0} \int_{\Omega} \mathbf{F}_m u^\varepsilon \, dx - \int_{\Omega} \left( \beta_m p_m^\varepsilon + \gamma_m \theta_m^\varepsilon \right) \text{div} u^\varepsilon \, dx
$$

$$
= \int_{\Omega} \mathbf{F}_m u - \int_{\Omega} \left( \beta_m p_m + \gamma_m \theta_m \right) \int_{\Omega} (\text{div} u + \text{div} y \tilde{u}). \quad (5.4)
$$

Now, as $(x, y) \mapsto e_y(\tilde{u})(x, y)$ is an admissible test function, the first two terms of the right hand side of \(5.3\) converges to

$$
- \int_{\Omega \times Y} (a + t^a) [e(u) + e_y(\tilde{u})(x, y)] [e(u) + e_y(\tilde{u})(x, y)] \, dx dy
$$

$$
+ \int_{\Omega \times Y} a[e(u) + e_y(\tilde{u})(x, y)] [e(u) + e_y(\tilde{u})(x, y)] \, dx dy
$$

$$
= - \int_{\Omega \times Y} a[e(u) + e_y(\tilde{u})(x, y)] [e(u) + e_y(\tilde{u})(x, y)] \, dx dy. \quad (5.5)
$$

Finally, thanks to the coercivity of $\mathbf{A}$, \[4.21\] (with $\tilde{\mathbf{v}} = \tilde{\mathbf{u}}$) and \[5.4\] - \(5.5\) we
find that
\[ \alpha_0 \left\| e(u^\varepsilon)(x) - e(u^\varepsilon)(x) - e_y(\tilde{u}^\varepsilon)(x, \tilde{x}) \right\|_{L^2(\Omega)^3} \leq \]
\[ \int_\Omega A(\tilde{x}) \left[ e(u^\varepsilon)(x) - e(u^\varepsilon)(x) - e_y(\tilde{u}^\varepsilon)(x, \tilde{x}) \right] \left[ e(y^\varepsilon)(x, \tilde{x}) \right] \]
\[ \sum_m \left\{ \int_\Omega F_m u - \int_\Omega (\beta_m p_m + \gamma_m \theta_m) \int_{Y_m} (\text{div} u + \text{div} \tilde{u}) \right\}
\[ - \int_\Omega A[e(u) + e_y(\tilde{u})(x, y)]\left[ e(u) + e_y(\tilde{u})(x, y) \right] = 0 \]

where \( \alpha_0 = \min_{y \in Y} A(y) \). Hence the proposition is proved.

We now establish some corrector results on the mass conservation equation. Let us first give some technical results.

**Lemma 5.1 (G. Allaire and F. Murat [7, Lemma A.4])** There exists a constant \( C > 0 \) such that for all \( q_1 \in H^1_0(\Omega) \cap H^1(\Omega_\varepsilon) \) we have
\[ \left\| q_1 \right\|_{0, \Omega_\varepsilon} \leq C \left\{ \left\| \nabla q_1 \right\|_{0, \Omega_1} \right\}. \] (5.6)

**Lemma 5.2** There exists a constant \( C > 0 \) such that for all \( q \in H^1(\Omega_\varepsilon) \) we have
\[ \varepsilon \left\| q \right\|_{0, \Sigma^\varepsilon}^2 \leq C \left( \varepsilon^2 \left\| \nabla q \right\|_{0, \Omega}^2 + \left\| q \right\|^2_{0, \Omega_1} \right)^2, \] (5.7)
and
\[ \varepsilon \left\| q \right\|_{0, \Sigma^\varepsilon}^2 \leq C \left( \left\| \nabla q \right\|_{0, \Omega}^2 \right)^2. \] (5.8)

**Proof.** We argue as in [22]. Using the trace theorem on \( Y_1 \) (see for e.g. R. A. Adams and J. F. Fournier [1]), we know that there exists a constant \( C(Y_1) > 0 \) such that for every \( \psi \in H^1(Y_1) \)
\[ \int_\Sigma |\psi|^2 \, d\sigma \leq C \left( \int_{Y_1} |\nabla \psi|^2 \, dy + \int_{Y_1} |\psi|^2 \, dy \right). \]
Then, using the change of variables \( y := x/\varepsilon \) we have for every \( q \in H^1(Y_1^\varepsilon) \)
\[ \varepsilon \int_{\Sigma^\varepsilon^\varepsilon} |q|^2 \, d\sigma^\varepsilon \leq C \left( \varepsilon^2 \int_{Y_1^\varepsilon^\varepsilon} |\nabla q|^2 \, dx + \int_{Y_1^\varepsilon^\varepsilon} |q|^2 \, dx \right) \] (5.9)
where \( Y_1^\varepsilon^\varepsilon = \varepsilon(k + \Omega_\varepsilon) \) and \( \Sigma^\varepsilon^\varepsilon = \varepsilon(k + \Sigma) \quad k \in \mathbb{Z}^3 \). We mention that \( C \) appearing in the inequality (5.9) is independent of \( k \in \mathbb{Z}^3 \). Summing up these inequalities (5.9) over all \( Y_1^\varepsilon^k \) contained in \( \Omega \), we get (5.3). To obtain (5.8), it suffices to write (5.7) for sufficiently small \( \varepsilon, \varepsilon \ll 1 \) and use the Friedrich inequality (5.6). ■
Lemma 5.3 We have

\[
\lim_{\varepsilon \to 0} \int_{\Sigma^\varepsilon_T} \varepsilon \zeta (p_{m}^\varepsilon - p_m)^2 ds^\varepsilon (x) = 0, \quad (5.10)
\]

\[
\lim_{\varepsilon \to 0} \int_{\Sigma^\varepsilon_T} \varepsilon \zeta (p_1 - p_2)^2 ds^\varepsilon (x) = \int_Q \tilde{\zeta} (p_1 - p_2)^2 dx. \quad (5.11)
\]

\[
\lim_{\varepsilon \to 0} \int_{\Sigma^\varepsilon_T} \varepsilon \zeta (p_1^\varepsilon - p_2^\varepsilon)^2 ds^\varepsilon (x) = \int_Q \tilde{\zeta} (p_1 - p_2)^2 dx. \quad (5.12)
\]

**Proof.** Using (5.9) yields

\[
\varepsilon \left\| (p_{m}^\varepsilon - p_m) \right\|_{L^2(\Sigma^\varepsilon_T)}^2 \leq C \left( \varepsilon^2 \left\| \nabla (p_{m}^\varepsilon - p_m) \right\|_{L^2(\Omega)}^2 + \left\| (p_{m}^\varepsilon - p_m) \right\|_{L^2(\Omega)}^2 \right). \quad (5.13)
\]

We know that \( \left\| \nabla (p_{m}^\varepsilon - p_m) \right\|_{L^2(\Omega)} \) is uniformly bounded with respect to \( \varepsilon \) and thanks to Rellich’s Theorem, \( \chi_m p_{m}^\varepsilon \) converges strongly to \( \chi_m p_m \) in \( L^2(\Omega) \). So, by passing to the limit in (5.13) we easily arrive at (5.10). The convergence in (5.11) is a direct consequence of Corollary 4.1. To complete the proof it remains to show (5.12). Let us first observe that, according to Cauchy-Schwarz inequality and (5.10), we have

\[
\left| \int_{\Sigma^\varepsilon_T} \varepsilon \zeta (p_{m}^\varepsilon - p_m) (p_1 - p_2) ds^\varepsilon (x) \right| \leq C \sqrt{\varepsilon} \left\| (p_{m}^\varepsilon - p_m) \right\|_{L^2(\Sigma^\varepsilon_T)} \left\| p_1 - p_2 \right\|_{L^2(\Sigma^\varepsilon_T)} \to 0.
\]

Hence, from (5.10), (5.11) and (5.14) we obtain

\[
\int_0^T \int_{\Sigma^\varepsilon_T} \varepsilon \zeta (p_1^\varepsilon - p_2^\varepsilon)^2 ds^\varepsilon (x) dt = \int_0^T \int_{\Sigma^\varepsilon_T} \varepsilon \zeta (p_1^\varepsilon - p_1)^2 ds^\varepsilon (x) dt
\]

\[
+ \int_{\Sigma^\varepsilon_T} \varepsilon \zeta (p_2^\varepsilon - p_2)^2 ds^\varepsilon (x) dt + 2 \int_{\Sigma^\varepsilon_T} \varepsilon \zeta (p_1^\varepsilon - p_1) (p_1 - p_2) ds^\varepsilon (x) dt
\]

\[
+ 2 \int_{\Sigma^\varepsilon_T} \varepsilon \zeta (p_2^\varepsilon - p_2) (p_1 - p_2) ds^\varepsilon (x) dt
\]

\[
+ \int_{\Sigma^\varepsilon_T} \varepsilon \zeta (p_1 - p_2)^2 ds^\varepsilon (x) dt
\]

\[
\to \int_Q \tilde{\zeta} (p_1 - p_2)^2 dx.
\]

Hence (5.12). \( \blacksquare \)

**Proposition 5.3** We have

\[
\lim_{\varepsilon \to 0} \left\| \chi_m \left( \frac{x}{\varepsilon} \right) \left\{ \nabla p_{m}^\varepsilon (t,x) - \nabla p_m (t,x) - \nabla_y \hat{p}_m \left( t, x, \frac{x}{\varepsilon} \right) \right\} \right\|_{L^2(0,T)} = 0.
\]
Proof. As in the proof of Proposition 5.2 we first write that

\[
\begin{align*}
\int_{Q_1^{\varepsilon}} K_1 \left( \frac{x}{\varepsilon} \right) & \left| \nabla p_1^\varepsilon (t, x) - \nabla p_1 (t, x) - \nabla_y \hat{p}_1 \left( t, x, \frac{x}{\varepsilon} \right) \right|^2 \, dt \, dx \\
+ \int_{Q_2^{\varepsilon}} K_2 \left( \frac{x}{\varepsilon} \right) & \left| \nabla p_2^\varepsilon (t, x) - \nabla p_2 (t, x) - \nabla_y \hat{p}_2 \left( t, x, \frac{x}{\varepsilon} \right) \right|^2 \, dt \, dx \\
= & \int_{Q_1^{\varepsilon}} K_1 \left( \frac{x}{\varepsilon} \right) \left| \nabla p_1^\varepsilon (t, x) \right|^2 \, dt \, dx + \int_{Q_1^{\varepsilon}} K_m \left( \frac{x}{\varepsilon} \right) \left| \nabla p_m^\varepsilon (t, x) \right|^2 \, dt \, dx \\
- & 2 \int_{Q_1^{\varepsilon}} K_1 \left( \frac{x}{\varepsilon} \right) \nabla p_1^\varepsilon (t, x) \left( \nabla p_1 (t, x) + \nabla_y \hat{p}_1 \left( t, x, \frac{x}{\varepsilon} \right) \right) \, dt \, dx \\
- & 2 \int_{Q_2^{\varepsilon}} K_2 \left( \frac{x}{\varepsilon} \right) \nabla p_2^\varepsilon (t, x) \left( \nabla p_2 (t, x) + \nabla_y \hat{p}_2 \left( t, x, \frac{x}{\varepsilon} \right) \right) \, dt \, dx \\
\int_{Q_1^{\varepsilon}} & K_1 \left( \frac{x}{\varepsilon} \right) \left| \nabla p_1 (t, x) + \nabla_y \hat{p}_1 \left( t, x, \frac{x}{\varepsilon} \right) \right|^2 \, dt \, dx \\
+ & \int_{Q_2^{\varepsilon}} K_2 \left( \frac{x}{\varepsilon} \right) \left| \nabla p_2 (t, x) + \nabla_y \hat{p}_2 \left( t, x, \frac{x}{\varepsilon} \right) \right|^2 \, dt \, dx.
\end{align*}
\]

(5.15)

It is easy to show that the last two terms of the right hand side of (5.15) converge to

\[
-2 \int_{Q_m^{\varepsilon}} K_m \left( \frac{x}{\varepsilon} \right) \nabla p_m^\varepsilon (t, x) \left( \nabla p_m (t, x) + \nabla_y \hat{p}_m \left( t, x, \frac{x}{\varepsilon} \right) \right) + \\
\int_{Q_m^{\varepsilon}} K_m \left( \frac{x}{\varepsilon} \right) \left| \nabla p_m (t, x) + \nabla_y \hat{p}_m \left( t, x, \frac{x}{\varepsilon} \right) \right|^2 \\
\longrightarrow - \int_{Q \times Y_m^{\varepsilon}} K_m (y) \left| \nabla p_m (t, x) + \nabla_y \hat{p}_m (t, x, y) \right|^2, \ m = 1, 2. \tag{5.16}
\]

Now by (3.3) one can see that

\[
\begin{align*}
\int_{Q_m^{\varepsilon}} K_1 \left( \frac{x}{\varepsilon} \right) \nabla p_1^\varepsilon \nabla p_1^\varepsilon \, dt \, dx + \int_{Q_m^{\varepsilon}} K_2 \left( \frac{x}{\varepsilon} \right) \nabla p_2^\varepsilon \nabla p_2^\varepsilon \, dt \, dx \\
- \int_{Q_1^{\varepsilon}} \partial_t (\phi_1 p_1^\varepsilon + \beta_1 \text{div} u_1^\varepsilon + \gamma_1 \theta_1^\varepsilon) p_1^\varepsilon \, dt \, dx \\
- \int_{Q_m^{\varepsilon}} \partial_t (\phi_2 p_2^\varepsilon + \beta_2 \text{div} u_2^\varepsilon + \gamma_2 \theta_2^\varepsilon) p_2^\varepsilon \, dt \, dx \\
+ \int_{\Sigma_T} \varepsilon (p_1^\varepsilon - p_2^\varepsilon)^2 \, ds \, (x) \, dt = \int_{Q_m^{\varepsilon}} G_m p_m^\varepsilon \, dt \, dx. \tag{5.17}
\end{align*}
\]

Taking the limit in the r.h.s. of (5.17) gives

\[
\lim_{\varepsilon \to 0} \int_{Q_m^{\varepsilon}} G_m p_m^\varepsilon \, dt \, dx = |Y_m| \int_Q G_m p_m \, dt \, dx, \ m = 1, 2. \tag{5.18}
\]
Using Lemma 5.3 and more precisely (5.12) and passing in the last term of the l.h.s. of (5.17) yields

\[
\lim_{\varepsilon} \varepsilon \int_{\Sigma_1} \int_{Q_2} \nabla \phi_m (p_{m}^\varepsilon)^2 \, dx \, dt = \int_{Q_2} \phi_m (p_{m})^2 (T, x) \, dx.
\]

Regarding the second term of the right hand side of (5.17) we proceed as follows: Firstly, integrating by parts with respect to the time variable \( t \) we have for \( m = 1, 2 \)

\[
\int_{Q_2^m} \partial_t (\phi_m p_{m}^\varepsilon) \, dx \, dt = \int_{Q_2^m} \phi_m (p_{m}^\varepsilon)^2 (T, x) \, dx \rightarrow \int_{Q \times Y_m} \phi_m (p_{m})^2 (T, x) \, dx. \tag{5.19}
\]

Secondly, since \( \chi_{m} \partial_t (\text{div} \mathbf{u}_m) \) converges weakly to \( \chi_{m} \{ \partial_t (\text{div} \mathbf{u}_m + \text{div}_y \mathbf{\hat{u}}_m) \} \) and \( \chi_{m} p_{m}^\varepsilon \) converges strongly to \( \chi_{m} p_{m} \) it follows that

\[
\lim_{\varepsilon} \int_{Q_2^m} \beta_{m} \partial_t (\phi_m p_{m}^\varepsilon) \, dx \, dt = \int_{Q \times Y_m} \beta_{m} \{ \partial_t (\text{div} \mathbf{u}_m + \text{div}_y \mathbf{\hat{u}}_m) \} p_{m} \, dx. \tag{5.20}
\]

Furthermore, as \( \chi_{m} \partial_t \theta_{m}^\varepsilon \) converges strongly to \( \chi_{m} \partial_t \theta_{m} \), we see that

\[
\lim_{\varepsilon} \int_{Q_2^m} \gamma_{m} \partial_t \theta_{m}^\varepsilon \, dx \, dt = \int_{Q \times Y_m} \gamma_{m} \partial_t \theta_{m} \, dx. \tag{5.21}
\]

Using the integral identity (5.1) and (5.18)-(5.21) we pass to the limit in (5.17) to get

\[
\lim_{\varepsilon} \left\{ \int_{Q_2^m} \mathcal{K}_1 \left( \frac{\varepsilon}{\varepsilon} \phi_m (p_{m}^\varepsilon)^2 \, dx \, dt + \int_{Q_2^m} \mathcal{K}_2 \left( \frac{\varepsilon}{\varepsilon} \phi_m (p_{m}^\varepsilon)^2 \, dx \, dt \right) \right. \right. \nonumber
\]

\[
= |Y_1| \int_{Q} G_1 p \, dt \, dx + |Y_2| \int_{Q} G_2 p \, dt \, dx \nonumber
\]

\[
+ \int_{\Omega \times Y_1} \phi_1 (p_1)^2 (T, x) \, dx + \int_{\Omega \times Y_2} \phi_2 (p_2)^2 (T, x) \, dx + \int_{Q \times Y_2} \partial_t (\beta_1 (\text{div} \mathbf{u}_1 + \text{div}_y \mathbf{\hat{u}}_1 + \gamma_1 \theta_1)) p_1 \, dx \nonumber
\]

\[
+ \int_{Q \times Y_2} \partial_t (\beta_2 (\text{div} \mathbf{u}_2 + \text{div}_y \mathbf{\hat{u}}_2 + \gamma_2 \theta_2)) p_2 \, dx \nonumber
\]

\[
- \int_{Q \times \Sigma} \varepsilon (p_1 - p_2)^2 \, dt \, dx \, ds \left( y \right). \tag{5.22}
\]
Next, using (4.30), the convergence in (5.22) can be reduced to
\[
\lim_{\varepsilon \to 0} \left\{ \int_{Q_1^\varepsilon} \mathcal{K}_1 \left( \frac{x}{\varepsilon} \right) \nabla p_1^\varepsilon \nabla p_1^\varepsilon \, dt \, dx + \int_{Q_2^\varepsilon} \mathcal{K}_2 \left( \frac{x}{\varepsilon} \right) \nabla p_2^\varepsilon \nabla p_2^\varepsilon \, dt \, dx \right\} + \\
= \int_{Q \times Y_1} \mathcal{K}_1 (y) (\nabla p_1 + \nabla y \hat{p}_1)(\nabla p_1 + \nabla y \hat{p}_1) \, dt \, dy + \\
\int_{Q \times Y_2} \mathcal{K}_2 (y) (\nabla p_2 + \nabla y \hat{p}_2)(\nabla p_2 + \nabla y \hat{p}_2) \, dt \, dy.
\] (5.23)

Finally, collecting all the limits (5.16) and (5.23), Equation (5.15) becomes
\[
\lim_{\varepsilon \to 0} \sum_{m=1,2} \int_{Q_m^\varepsilon} \mathcal{K}_m \left| \nabla p_m^\varepsilon (t,x) - \nabla p_m (t,x) - \nabla y \hat{p}_m \left( t, x, \frac{x}{\varepsilon} \right) \right|^2 \leq \int_{Q \times Y} a_{\text{hom}} (u) e(u) \, dx + \sum_{m=1,2} \int_{Q_m} (A_m \nabla \theta_m + B_m \nabla p_m) \, u \, dx.
\] (5.24)

Proof. Taking \( v = u^\varepsilon \) in (5.2) gives
\[
\int_{\Omega} a \left( \frac{x}{\varepsilon} \right) e(u^\varepsilon) \, dx + \sum_{m=1,2} \int_{\Omega_m^\varepsilon} (\beta_m \nabla p_m^\varepsilon + \gamma_m \nabla \theta_m^\varepsilon) \, u^\varepsilon \, dx \\
- \int_{\Omega} Fu^\varepsilon = 0
\]
which, by taking into account (4.29), tends to
\[
\int_{\Omega} a_{\text{hom}} e(u) \, dx + \sum_{m=1,2} \int_{\Omega_m} (A_m \nabla \theta_m + B_m \nabla p_m) \, u \, dx \\
- \int_{\Omega} Fu \, dx = 0.
\]
Hence (5.24).
6 Conclusion

In this paper we derived by a homogenization technique a more general model of thermoporoelasticity with double porosity and two temperatures. More precisely, we studied a micro-model of fluid and thermal flows in two component poroelastic media consisting of matrix and inclusions with the same order of permeabilities and conductivities, separated by a periodic and thin layer which forms an exchange fluid/thermal barrier. In particular, we have shown that the Biot-Willis and thermal expansion parameters are in that case matrices and no longer scalars, see for instance \cite{2, 3}. Let us mention that the result of the paper remains valid if one considers non homogeneous initial and/or Dirichlet conditions. An interesting problem is to investigate the limiting behavior of such media when the flow potential in the inclusions is rescaled by $\varepsilon^2$. This occurs especially when the flow in the inclusions presents very high frequency spatial variations due to a relatively very low permeability, see Remark \cite{2, 3}.

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