On Doctrines and Cartesian Bicategories

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Abstract  
We study the relationship between cartesian bicategories and a specialisation of Lawvere’s hyperdoctrines, namely elementary existential doctrines. Both provide different ways of abstracting the structural properties of logical systems: the former in algebraic terms based on a string diagrammatic calculus, the latter in universal terms using the fundamental notion of adjoint functor. We prove that these two approaches are related by an adjunction, which can be strengthened to an equivalence by imposing further constraints on doctrines.

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1 Introduction

In [20, 21, 22] Lawvere introduced the notion of hyperdoctrine in an effort to capture the universal content of logical theories, and first-order logic in particular. Here, by universal, we intend by means of universal properties in category theory. The starting point is the notion of Lawvere theory [19], the universal way of capturing the notion of algebraic theory – where the universal property is that of cartesian categories, namely categories with finite products. In terms of logical content, Lawvere theories provide the notion of term. Now, a hyperdoctrine is a certain contravariant functor \( P \) from the Lawvere theory of terms to a posetal 2-category, e.g. lattices or Heyting algebras. The basic, high-level idea is that the functor takes us from terms to formulas; more precisely, the objects of the Lawvere theories, which can be thought of as variable contexts, are taken to the Lindenbaum-Tarski algebra of formulas over these contexts. In this way, the concept of quantifier can be captured by means of a universal property – the existence of left-adjoints (existential quantification) and right-adjoints (universal quantification) to the image along \( P \) of the projections.

In recent years, there has been a large number of contributions that use string diagrams in order to model computational phenomena of different kinds. Typically, the languages come with an equational theory, which can be used to reason about systems via diagrammatic reasoning. Interestingly, the same algebraic structures seem to appear in many different contexts, e.g. commutative monoids and comonoids,
Frobenius algebras, Hopf algebras, etc. These applications, while using the language and tools of (monoidal) category theory, are of rather different nature than the more established "universal" approaches, such as Lawvere theories and hyperdoctrines sketched in the previous paragraph. A bridge between the universal and algebraic worlds is given by a theorem of Fox [14] that characterises cartesian categories as symmetric monoidal categories where each object is equipped with a well-behaved commutative comonoid structure. This means that any Lawvere theory can be seen concretely as a string diagrammatic language (see e.g., [5]).

More recently, the notion of discrete cartesian restriction category was characterised in a similar way [10], with partial Frobenius algebras taking the place of commutative comonoids. This raises a natural question: can we capture the universal content of logical theories algebraically in a similar way? In other words, what are the “Fox theorems” for logic?

In this paper we turn our attention to the regular fragment of first-order logic with equality: formulas are built up from terms and the equality relation using the existential quantifier and conjunction. There has been much work on categorifying this fragment, notably the significant corpus of work on allegories [15]. More relevant to our story, we focus on the contrast between universal and algebraic approaches. A universal treatment, the notion of elementary existential doctrine, was introduced in [25]. The basic setup is the same as for Lawvere’s hyperdoctrines, but one asks only for the left adjoints, which, as we have previously mentioned, are the universal explanation for existential quantifiers. On the algebraic side, the concept that stands out is that of Carboni and Walters’ cartesian bicategories (of relations) [8], which are symmetric monoidal categories where objects are equipped with a special Frobenius algebra and a lax-natural commutative comonoid structures. While Carboni and Walters emphasised the relational algebraic aspects, they were certainly aware of the logical connections. In fact, some recent works [3, 13, 28] exploited various ramifications of the correspondence between cartesian bicategories and regular logic.

Our goal for this paper is a “Fox theorem” for the regular fragment, connecting the universal and the algebraic approaches. Our starting observation is that, given a cartesian bicategory \((\mathcal{B}, \otimes, I)\), one obtains an elementary existential doctrine by restricting the hom functor \(\text{Hom}_\mathcal{B}(\mathcal{B})\) to the (cartesian) category of maps of \(\mathcal{B}\). In Remark 2.5 of [26], it is mentioned that the other direction is also possible: given an elementary existential doctrine one can construct a cartesian bicategory. We explore the ramifications of this remark in detail. We show that these two translations are functorial and, actually, that they form an adjunction. More precisely, it turns out that the category of cartesian bicategories is a reflective subcategory of the category of doctrines.

The adjunction, however, is not an equivalence. We prove this with a counterexample that captures the crux of the matter: there are doctrines \(P: \mathcal{C}^{\text{op}} \to \text{InfSL}\) where the indexing categories of terms \(\mathcal{C}\) are not tailored to the represented logics. In doctrines-as-logical-theories, roughly speaking, equality can come from two places: implicitly, from the indexing term category, and explicitly, via logical equivalence. Doctrines, therefore, have an additional degree of intensionality: doctrines that “substantially represent” the same logic may have distinct index categories and thus not be isomorphic. This issue does not arise in cartesian bicategories where the role of \(\mathcal{C}\) is played by the subcategory of maps: maps are arrows satisfying certain properties, rather than given a priori in a fixed index category.

We conclude by observing that, by adding further constraints to the notion of elementary existential doctrine, namely comprehensive diagonals and the Rule of Unique Choice from [26], it is possible to exclude such problematic doctrines. By doing so, we restrict the adjunction to an equivalence, thus obtaining a satisfactory “Fox theorem”.

\textbf{Notation.} Given \(f: X \to Y\) and \(g: Y \to Z\) morphisms in some category, we denote their
composite as \( f \circ g \). We write \( Pf \) for the action of a doctrine \( P \) on a morphism \( f \).

## 2 Cartesian Categories

The starting point of our exposition is the definition of cartesian category that, thanks to the results of Fox [14], can be given in the following form, which is particularly convenient for the purposes of this paper.

**Definition 1.** A cartesian category is a symmetric monoidal category \((C, \otimes, I)\) where every object \( X \in C \) is equipped with morphisms

\[
\begin{align*}
\Delta_X &: X \to X \otimes X \\
\varepsilon_X &: X \to I
\end{align*}
\]

1. \( \Delta_X \) and \( \varepsilon_X \) form a cocommutative comonoid, that is they satisfy

\[
\begin{align*}
\Delta_X \circ \Delta_X &= \Delta_X \\
\varepsilon_X \circ \varepsilon_X &= \varepsilon_X \\
\Delta_X \circ \varepsilon_X &= \varepsilon_X \\
\varepsilon_X \circ \Delta_X &= \varepsilon_X
\end{align*}
\]

2. Each morphism \( f : X \to Y \) is a comonoid homomorphism, that is

\[
\begin{align*}
\Delta_Y \circ f &= f \circ \Delta_X \\
\varepsilon_Y \circ f &= \varepsilon_X
\end{align*}
\]

3. The choice of comonoid on every object is coherent with the monoidal structure in the sense that

\[
\begin{align*}
\Delta_X \otimes \varepsilon_Y &= \Delta_X \\
\varepsilon_X \otimes \varepsilon_Y &= \varepsilon_X \\
\varepsilon_Y \otimes \varepsilon_Y &= \varepsilon_Y \\
\Delta_Y \otimes \Delta_X &= \Delta_Y \otimes \Delta_X
\end{align*}
\]

Indeed, given a category \( C \) with finite products, one can construct a monoidal category as in the definition above, by taking as monoidal product \( \otimes \) the categorical product and as its unit \( I \) the terminal object; for every object \( X \)

\[
\begin{align*}
\Delta_X &: X \to X \otimes X \\
\varepsilon_X &: X \to I
\end{align*}
\]

\[
\begin{align*}
\Delta_X \circ \Delta_X &= \Delta_X \\
\varepsilon_X \circ \varepsilon_X &= \varepsilon_X \\
\Delta_X \circ \varepsilon_X &= \varepsilon_X \\
\varepsilon_X \circ \Delta_X &= \varepsilon_X
\end{align*}
\]

\[
\begin{align*}
\Delta_X \otimes \varepsilon_Y &= \Delta_X \\
\varepsilon_X \otimes \varepsilon_Y &= \varepsilon_X \\
\varepsilon_Y \otimes \varepsilon_Y &= \varepsilon_Y \\
\Delta_Y \otimes \Delta_X &= \Delta_Y \otimes \Delta_X
\end{align*}
\]

\[
\begin{align*}
\Delta_X \circ \Delta_X &= \Delta_X \\
\varepsilon_X \circ \varepsilon_X &= \varepsilon_X \\
\Delta_X \circ \varepsilon_X &= \varepsilon_X \\
\varepsilon_X \circ \Delta_X &= \varepsilon_X
\end{align*}
\]

\[
\begin{align*}
\Delta_X \otimes \varepsilon_Y &= \Delta_X \\
\varepsilon_X \otimes \varepsilon_Y &= \varepsilon_X \\
\varepsilon_Y \otimes \varepsilon_Y &= \varepsilon_Y \\
\Delta_Y \otimes \Delta_X &= \Delta_Y \otimes \Delta_X
\end{align*}
\]

Conversely, given a symmetric monoidal category \((C, \otimes, I)\) as in Definition 1, \( \otimes \) forms a categorical product where projections \( \pi_X : X \times Y \to X \) are given as \( \text{id}_X \otimes \Delta_Y \) and the pairing \( \langle f, g \rangle : X \to Y \otimes Z \) as \( \Delta_X ; (f \otimes g) \) for all arrows \( f : X \to Y \) and \( g : X \to Z \).

Hereafter, we will use \( \times \) to denote both the cartesian product of sets and the categorical product in an arbitrary cartesian category. It is worth to remark that while Set (sets and functions) and Rel (sets and relations) are both cartesian categories, the categorical product in Set is indeed the cartesian product, while in Rel it is actually the disjoint union.

**Example 2.** Another cartesian category that will play an important role is \( L_\Sigma \), the Lawvere theory [19] generated by a cartesian signature \( \Sigma \) (a set of symbols \( f \) equipped with some arity \( \text{ar}(f) \in \mathbb{N} \)). In \( L_\Sigma \), objects are natural numbers and arrows are tuples of terms over a countable set of variables \( V = \{ x_1, x_2, \ldots \} \). More precisely, arrows from \( n \) to \( m \) are tuples \( \langle t_1, \ldots, t_m \rangle \) of sort \( (n, m) \) as defined by the inference rules in the first line of Figure 1. It is easy to check that \( \langle t_1, \ldots, t_m \rangle \) has sort \( (n, m) \) if each term \( t_i \) has variables in \( \{ x_1, \ldots, x_m \} \).

Composition is defined by (simultaneous) substitution: the composition of \( \langle t_1, \ldots, t_m \rangle : (n, m) \) with \( \langle s_1, \ldots, s_l \rangle : (m, l) \) is the tuple \( \langle u_1, \ldots, u_l \rangle : (n, l) \) where \( u_i = s_i[t_1, \ldots, t_m/x_1, \ldots, x_m] \) for all \( i = 1, \ldots, l \). One can readily check that \( L_\Sigma \) is a SMC having \((\mathbb{N}, +, 0)\) as the monoid of objects,
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\[ \frac{i \leq n}{x_i : (n, 1)} \quad \frac{f \in \Sigma \text{ ar}(f) = m \quad (t_1, \ldots, t_m) : (n, m)}{f/t_1, \ldots, t_m : (n, 1)} \quad \frac{t_1 : (n, 1) \quad (t_2, \ldots, t_m) : (n, m)}{(\ldots) \quad (\ldots)} \quad \frac{\{1, \ldots, (n, 0)\}}{[0, 0]} \]

**Figure 1** Sort inference rules

i.e., it is a prop (product and permutation category, see [18, 23]). Identities \( \text{id}_n \) and symmetries \( \sigma_{n,m} \) are defined as expected; \( n \mapsto \ldots n \mapsto n + n \) is the tuple \( \langle x_1, \ldots, x_n, x_1, \ldots, x_n \rangle \) thus acting as a duplicator of variables; \( n \mapsto 0 \) is the empty tuple \( \langle \rangle \), acting as a discharger.

**Definition 3.** A morphism of cartesian categories is a strict monoidal functor preserving the chosen comonoid structures.

**Example 4.** Let \( \Sigma_1 \) be the cartesian signature consisting of a single symbol \( f \) with arity 1, and \( \Sigma_2 \) be the signature with two symbols, \( g_1 \) and \( g_2 \), both of arity 1. Consider the corresponding Lawvere theories \( L_{\Sigma_1} \) and \( L_{\Sigma_2} \). The assignment \( f \mapsto g_1 \) induces a morphism of cartesian categories, hereafter denoted by \( F_1 : L_{\Sigma_1} \to L_{\Sigma_2} \). Similarly, let \( F_2 : L_{\Sigma_1} \to L_{\Sigma_2} \) denote the morphism of cartesian categories where \( f \) is mapped to \( g_2 \). Finally, there is a unique morphism of cartesian categories \( Q : L_{\Sigma_2} \to L_{\Sigma_1} \) mapping \( g_1 \) and \( g_2 \) to \( f \).

3 Elementary Existential Doctrines

Recall that an inf-semilattice is a partially ordered set with all finite infima, including a top element \( \top \). We denote the category of inf-semilattices and inf-preserving functions by \( \text{InfSL} \).

The following definition is taken almost verbatim from [26]. The difference is that there the base category \( C \) only needs binary products, whereas we also require a terminal object.

**Definition 5.** Let \( C \) be a cartesian category. An elementary existential doctrine is given by a functor \( P : C^{op} \to \text{InfSL} \) that is:

- Elementary, namely for every object \( A \) in \( C \) there is an element \( \delta_A \in P(A \times A) \) such that for every map \( e = \text{id}_X \times \Delta_A : X \to X \times A \times A \), the function \( P_e : P(X \times A \times A) \to P(X \times A) \) has a left adjoint \( \exists_e : P(X \times A) \to P(X \times A \times A) \) defined by the assignment

\[ \exists_e(\alpha) = P_{(\pi_1, \pi_2)} : X \times A \times A \to X \times A(\alpha) \land P_{(\pi_2, \pi_2)} : X \times A \times A \to X \times A(\delta_A). \]

- Existential, namely for every \( A_1, A_2 \in C \) and projection \( \pi_i : A_1 \times A_2 \to A_i \) with \( i \in \{1, 2\} \), the function \( P_\pi : P(A_i) \to P(A_1 \times A_2) \) has a left adjoint \( \exists_\pi \) that satisfies

  - the Beck-Chevalley condition: for any projection \( \pi : X \times A \to A \) and any pullback

\[ \begin{array}{ccc} X' & \xrightarrow{\pi'} & A' \\ f' \downarrow & & \downarrow f \\ X \times A & \xrightarrow{\pi} & A \end{array} \]

  it holds that \( \exists_{\pi'}(P_f(\beta)) = P_f(\exists_\pi(\beta)) \) for any \( \beta \in P(X \times A) \).

- Frobenius reciprocity: for any projection \( \pi : X \times A \to A \), \( \alpha \in P(A) \) and \( \beta \in P(X) \), it holds that \( \exists_\pi(P_\pi(\alpha) \land \beta) = \alpha \land \exists_\pi(\beta) \).
The regular fragment of first order logic consists of formulas built from conjunction \( \land \) and existential quantification \( \exists \). In a cartesian category, the diagram below is a pullback of \( f: A' \to A \) with a projection \( \pi: X \times A \to A \).

\[
\begin{array}{ccc}
X \times A' & \xrightarrow{\pi_2} & A' \\
\downarrow \alpha & & \downarrow f \\
X \times A & \xrightarrow{n} & A
\end{array}
\]

Therefore, given a functor \( \mathcal{C}^{op} \to \text{InfSL} \), to check that \( \mathcal{P} \) satisfies the Beck-Chevalley condition it suffices to show that (2) holds for \( X' = X \times A' \), \( \pi' = \pi_2 \) and \( f' = \text{id}_X \times f \).

The contravariant powerset \( \mathcal{P} \), while often seen as an endofunctor on \( \text{Set} \), can also be seen as a functor \( \mathcal{P}: \text{Set}^{op} \to \text{InfSL} \). It is the classical example of an elementary existential doctrine.

Given a projection \( \pi: X \times A \to A \), \( \mathcal{P}_\pi \) and its left adjoint \( \exists_\pi \) are as follows:

\[
\begin{array}{ccc}
\mathcal{P}(A) & \xrightarrow{P_\pi} & \mathcal{P}(X \times A) \\
\mathcal{P}(X \times A) & \xrightarrow{\exists_\pi} & \mathcal{P}(A)
\end{array}
\]

\[
\begin{array}{c}
B \longleftarrow \{(x, b) \in X \times A \mid b \in B\} \\
S \longleftarrow \{a \in A \mid \exists x \in X. (x, a) \in S\}
\end{array}
\]

For every set \( A \), \( \delta_A \) is fixed to be \( \{(a, a) \mid a \in A\} \in \mathcal{P}(A \times A) \). With this information, it is easy to check that \( \mathcal{P}: \text{Set}^{op} \to \text{InfSL} \) satisfies the conditions in Definition 5. For the convenience of the reader, we work out the details in Appendix A.

**Example 8.** Let \( \Sigma \) and \( \mathcal{P} \) be signatures of function and predicate symbols respectively. The regular fragment of first order logic consists of formulas built from conjunction \( \land \) true \( \top \), existential quantification \( \exists \), equality \( t_1 = t_2 \) and atoms \( P(t_1, \ldots, t_m) \) where \( P \in \mathcal{P} \) and \( t_i \) are terms over \( \Sigma \). Formulas are sorted according to the rules in Figure 1; \( \phi \) has sort \( (n, 0) \) if the free variables of \( \phi \) are in \( \{x_1, \ldots, x_n\} \).

The indexed Lindenbaum-Tarski algebra functor \( LT: \mathcal{L}_\Sigma^{op} \to \text{InfSL} \) assigns to each \( n \in \mathbb{N} \) the set of formulas of sort \((n, 0)\) modulo logical equivalence (defined in the usual way). These form a semilattice with top given by \( \top \) and meet by \( \land \), where \( \phi = \psi \) if and only if \( \psi \) is a logical consequence of \( \phi \). To the arrow \( \langle t_1, \ldots, t_m \rangle: n \to m \), \( LT \) assigns the substitution mapping each \( \phi: (m, 0) \) to \( \phi[t_1/\ldots/t_m/x_1, \ldots, x_m]: (n, 0) \). In particular, for the projection \( \pi: n + m \to n \) (that is \( \langle x_1, \ldots, x_n; x_{n+1}, \ldots, x_{n+m} \rangle \) in \( n \to m \)), \( LT \) maps a formula of sort \((n, 0)\) to the same formula but with sort \((n + m, 0)\). Its left adjoint \( \exists_\pi \) maps a formula \( \phi: (n+m, 0) \) to the formula \( \exists x_{n+1}, \ldots, \exists x_{m+n}. \phi: (n, 0) \). The Beck-Chevalley condition asserts that “substitution commutes with quantification”: if \( \phi \) is a formula with at most \( x_{n+1} \) free variables and \( t \) is a term that does not contain \( x_{n+1} \), then \( \exists x_{n+1}. (\phi[t/x_{n+1}]) = (\exists x_{n+1}. \phi)[t/x_{n+1}] \).

Frobenius reciprocity states that \( \exists x_i. (\phi \land \psi) = \phi \land (\exists x_i. \psi) \) if \( x_i \) is not a free variable of \( \phi \).

For all \( n \in \mathbb{N} \), \( \delta_n \) is the formula \( (x_1 = x_{n+1}) \land (x_2 = x_{n+2}) \land \cdots \land (x_n = x_{n+n}) \).

---

1. The second projection \( \pi: n + m \to m \) requires the reindexing of variables. We do not discuss this case in order to keep the presentation simpler.
Definition 9 (Cf. [26]). The category $EED$ consists of the following data.

- Objects are elementary existential doctrines $P : C^{\text{op}} \to \text{InfSL}$.
- Morphisms from $P : C^{\text{op}} \to \text{InfSL}$ to $R : D^{\text{op}} \to \text{InfSL}$ are pairs $(F, b)$, where $F : C \to D$ is a strict cartesian functor while $b : P \to R \circ F^{\text{op}}$ is a natural transformation that preserves equalities and existential quantifiers, that is $b_{A \times A}(\delta^P_A) = \delta^R_{F(A)}$ for all $A$ in $C$ and, for any projection $\pi : X \times A \to A$ in $C$, the following diagram commutes.

\[ \begin{array}{ccc}
P(X \times A) & \xrightarrow{\exists^P_\pi} & P(A) \\
\downarrow{b_{X \times A}} & & \downarrow{b_A} \\
RF(X \times A) & \xrightarrow{\exists^R_{F(\pi)}} & RF(A)
\end{array} \]

Composition of $(F, b) : P \to R$ as above with $(G, c) : R \to S$ is given by $(GF, cF \circ b)$.

Remark 10. In [26], morphisms of elementary existential doctrines $P \to R$ are pairs $(F, b)$ where $F : C \to D$ is a functor between the base categories that preserves binary products merely up to isomorphism, so $F(X) \xleftarrow{F(\pi_1)} F(X \times Y) \xrightarrow{F(\pi_2)} F(Y)$ is a product diagram of $F(X)$ and $F(Y)$ in $D$ but it might not coincide with the chosen product of $D$. For this reason, preservation of equality for $b$ in [26] means that $b_{A \times A}(\delta^P_A) = R(\delta^R_{F(A)})(\delta^R_{F(A)})$.

Remark 11. Let $P : D^{\text{op}} \to \text{InfSL}$ be an elementary existential doctrine and $F : C \to D$ a strict cartesian functor. Then $P \circ F^{\text{op}} : C^{\text{op}} \to \text{InfSL}$ is again an elementary existential doctrine, where $\delta^P_{F(A)} = \delta^R_{F(A)}$ for all $A$ in $B$ and, for very $\pi_i : X_1 \times X_2 \to X_i$, $\exists^{P \circ F^{\text{op}}}_{\pi_i} = \exists^P_{\pi_i}$.

4 Cartesian Bicategories

In this section we recall from [8] definitions and properties of cartesian bicategories.

Definition 12. A cartesian bicategory is a symmetric monoidal category $(B, \otimes, I)$ enriched over the category of posets (that is every hom-set is a partial order and both composition and $\otimes$ are monotonous operations) where every object $X \in B$ is equipped with morphisms

\[ \begin{array}{c}
\xrightarrow{\Delta} : X \to X \otimes X \\
\xrightarrow{\iota} : X \to I
\end{array} \]

such that

1. $\xrightarrow{\Delta}$ and $\xrightarrow{\iota}$ form a cocommutative comonoid, as in Definition 1.1
2. $\xrightarrow{\Delta}$ and $\xrightarrow{\iota}$ have right adjoints $\xleftarrow{\Delta}$ and $\xleftarrow{\iota}$ respectively, that is

\[ \begin{array}{c}
\xleftarrow{\Delta} \leq \xrightarrow{\Delta} \xrightarrow{\iota} \leq \xrightarrow{\iota} \\
\xleftarrow{\Delta} \leq \xleftarrow{\iota} \leq \xleftarrow{\Delta}
\end{array} \]

3. The Frobenius law holds, that is

\[ \begin{array}{c}
\xrightarrow{\Delta} \xleftarrow{\Delta} \xrightarrow{\iota} \xrightarrow{\Delta} \xrightarrow{\iota} \xrightarrow{\Delta} = \xrightarrow{\iota} \xleftarrow{\iota} \xleftarrow{\Delta} \xleftarrow{\Delta} \xleftarrow{\iota} \xrightarrow{\iota} \xrightarrow{\Delta} = \xrightarrow{\iota} \xleftarrow{\Delta}
\end{array} \]
4. Each morphism \( R : X \to Y \) is a lax comonoid homomorphism, that is

\[
\Delta_Y [R] \leq [R] \quad \Delta_X [R] \leq [R]
\]

5. The choice of comonoid is coherent with the monoidal structure as in Definition 1.3.

The archetypal example of a cartesian bicategory is the category of sets and relations, with cartesian product of sets as monoidal product and \( 1 = \{ \bullet \} \) as unit \( I \). To be precise, \( \mathbf{Rel} \) has sets as objects and relations \( R \subseteq X \times Y \) as arrows \( X \to Y \). Composition and monoidal product are defined as expected:

\[
R \circ S = \{ (x, z) \mid \exists y \text{ s.t. } (x, y) \in R \text{ and } (y, z) \in S \}
\]

and

\[
R \otimes S = \{ ((x_1, x_2), (y_1, y_2)) \mid (x_1, y_1) \in R \text{ and } (x_2, y_2) \in S \}.
\]

For each set \( X \), the comonoid structure is given by the diagonal function \( \Delta_X : X \to X \times X \) and the unique function \( !_X : X \to 1 \), considered as relations, that is \( \Delta_X = \{(x, (x, x)) \mid x \in X\} \) and \( !_X = \{(x, \bullet) \mid x \in X\} \). Their right adjoints are given by their opposite relations: 

\[
\Delta_X^\ast = \{(x, x) \mid x \in X\} \quad \text{and} \quad !_X^\ast = \{\bullet \mid x \in X\}.
\]

Following the analogy with \( \mathbf{Rel} \), we will often call ‘relations’ arbitrary morphisms of a cartesian bicategory.

One of the fundamental properties of cartesian bicategories that follows from the existence of right adjoints (Property 2) is that every local poset \( \text{Hom}_B(X, Y) \) allows to take the intersection of relations and has a top element.

\[\blacktriangleright\text{Lemma 13. Let } B \text{ be a cartesian bicategory and } X, Y \in B. \text{ The poset } \text{Hom}_B(X, Y) \text{ has a top element given by } \Delta_X \quad \text{and the meet of relations } R, S : X \to Y \text{ is given by } R \wedge S \]
Lemma 15. A morphism \( \overline{f} \) is a map if and only if it has a right adjoint – a morphism \( R \) such that \( \overline{f} \leq \overline{R \circ f} \) and \( \overline{R \circ f} \leq \overline{f} \).

As expected, maps in \( \text{Rel} \) are precisely functions. Thus, \( \text{Map}(\text{Rel}) \) is exactly \( \text{Set} \). For maps it makes sense to imagine a flow of information from left to right. We will therefore draw \( \overline{f} \) to denote a map \( f \). Note that we use lower-case letters for maps and upper-case for arbitrary morphisms. Since \( X \) and \( X \) are maps by Lemma 15 and since by definition, every map respects them, then \( \text{Map}(\mathcal{B}) \), which inherits the monoidal product from \( \mathcal{B} \), has the structure of a cartesian category (Definition 1).

Lemma 16. For a cartesian bicategory \( \mathcal{B} \), the monoidal product \( \odot \) is a product on \( \text{Map}(\mathcal{B}) \), and the monoidal unit \( I \) is terminal. In other words, \( (\text{Map}(\mathcal{B}), \odot, I) \) is a cartesian category.

Definition 17. A morphism of cartesian bicategories is a strict monoidal functor that preserves the ordering, the chosen monoid and the comonoid structures. Cartesian bicategories and their morphisms form a category \( \text{CBC} \).

Proposition 18. Let \( F : \mathcal{A} \to \mathcal{B} \) be a morphism of cartesian bicategories. Then restricting its domain to \( \text{Map}(\mathcal{A}) \) yields a strict cartesian functor \( F|_{\text{Map}(\mathcal{A})} : \text{Map}(\mathcal{A}) \to \text{Map}(\mathcal{B}) \).

The remaining sections focus on the relationship between cartesian bicategories and elementary existential doctrines. First, a little taste of the similarity between them.

Example 19. In Example 8, we outlined the Lindenbaum-Tarski doctrine for the regular fragment of first order logic. We now introduce a cartesian bicategory, denoted by \( \text{CB}_{\Sigma, P} \), that provides a string diagrammatic calculus for this fragment. Like Lawvere theories, \( \text{CB}_{\Sigma, P} \) has \( (\mathbb{N}, +, 0) \) as monoid of objects. Arrows are (equivalence classes of) string diagrams \( \Sigma \) generated according to the rules in Figure 2 (top). For all \( n,m \in \mathbb{N} \), identities \( \overline{n} : n \to n \), symmetries \( \overline{n+m} : n+m \to m+n \), duplicators \( \overline{n} \odot \overline{n} : n \to n+n \), dischargers \( \overline{n} : n \to 0 \) and their adjoints can be constructed from the basic diagrams in the first row. Observe that function symbols \( f \) with arity \( n \) are depicted \( \overline{f} \) with coarity 1, while predicate symbols \( P \in P \) with arity \( n \) as \( \overline{n} \odot \overline{P} \) with coarity 0.
We need to show that Theorem 20.

The functor \( \mathcal{R}(\mathcal{B}) \) in (6) is an elementary existential doctrine.

Proof. We need to show that \( \mathcal{R}(\mathcal{B}) \) is elementary and existential.
1. First we prove that $\mathcal{R}(\mathbb{B})$ is elementary. We fix $\delta_A = \bullet \in \text{Hom}(A \otimes A, I)$. The string diagram for $e = id_X \times \Delta_A : X \times A \to X \times A \times A$ is $\xymatrix{ X \ar@/^/[r]^R & A \ar@/^/[l]^X \ar@/^/[r]^R & X}$. The function $\mathcal{R}(\mathbb{B})_e$ maps $\xymatrix{ A \ar[r]^R & A }$ to $\xymatrix{ A \ar@/^/[r]^R & A \ar@/^/[l]^A }$. The function $\exists_e : \mathcal{R}(\mathbb{B})(A \times X) \to \mathcal{R}(\mathbb{B})(A \times X \times X)$ defined in $\xymatrix{ }$ maps every $R \in \text{Hom}(X \otimes A, I)$ to $\xymatrix{ X \ar@/^/[r]^R & A \ar@/^/[l]^X }$. Now $\exists_e$ is left-adjoint to $\mathcal{R}(\mathbb{B})_e$ because $\xymatrix{ }$ is left-adjoint to $\xymatrix{ }$.

2. To show that $\mathcal{R}(\mathbb{B})$ is existential, let $\pi : X \times A \to A$ be a projection. Then $\exists_\pi$ maps $\xymatrix{ A \ar[r]^R & X \ar@/^/[l]^A \ar@/^/[r]^R & A }$. This is left-adjoint to $\mathcal{R}(\mathbb{B})_\pi$ since $\xymatrix{ }$ is left-adjoint to $\xymatrix{ }$.
   - For the Beck-Chevalley condition, consider the diagram in Remark 7. We need to show that $\exists_\pi((\mathcal{R}(\mathbb{B}))_{d_X \times f}(\beta)) = \mathcal{R}(\mathbb{B})_{f}(\exists_\pi(\beta))$. Translated to diagrams,
     $$\xymatrix{ D_A \ar@/^/[r]^\pi & X \ar@/^/[l]^A \ar@/^/[r]^R & A }$$
   which holds trivially.
   - For Frobenius reciprocity, take a projection $\pi : X \times A \to A$, $\alpha \in \mathcal{R}(\mathbb{B})(A)$ and $\beta \in \mathcal{R}(\mathbb{B})(X \times A)$. We need that $\exists_\pi(\mathcal{R}(\mathbb{B})_\pi(\alpha) \land \beta) = \alpha \land \exists_\pi(\beta)$, which translates to
     $$\xymatrix{ X \ar@/^/[r]^\pi & A \ar@/^/[l]^\alpha \ar@/^/[r]^R & A \ar@/^/[l]^\beta }$$
   This holds by naturality of the symmetry and since $\xymatrix{ }$ is the counit of $\xymatrix{ }$. ▽

By applying $\mathcal{R}$ to the cartesian bicategory $\text{Rel}$, one obtains $\mathcal{P} : \text{Set}^{\text{op}} \to \text{InfSL}$, in the sense that $\mathcal{P} \cong \mathcal{R}(\text{Rel})$ in EED. The isomorphism is the pair $(\Gamma, b)$, where $\Gamma : \text{Set} \to \text{Map}(\text{Rel})$ is the strict cartesian functor computing the graph of a function and $b : \mathcal{P} \to \mathcal{R}(\text{Rel}) \circ \Gamma^{\text{op}}$ is the natural transformation defined for all sets $A$ and $S \in \mathcal{P}(A)$ as $b_A(S) = \{(s, \bullet) \mid s \in S\}$.

**Example 21.** Consider $\text{CB}_{\Sigma, P}$ of Example 19 then $\mathcal{R}(\text{CB}_{\Sigma, P})$ is isomorphic to the doctrine $LT : L^{\text{op}}_{\Sigma} \to \text{InfSL}$ of Example 8. Here is why: the first two rows in Figure 9 define a cartesian isomorphism $F_{1[-1]}^{\text{op}} : L_{\Sigma} \to \text{Map}($CB$_{\Sigma, P})$. For all $n \in \mathbb{N}$, we define $b_n : LT(n) \to \text{Hom}(F_{1[-1]}^{\text{op}}(n), 0) = \text{Hom}(n, 0)$ as the function mapping $\phi : (n, 0)$ to $[\phi] : n \rightarrow 0$. This give rise to a natural isomorphism $b : LT \to \mathcal{R}($CB$_{\Sigma, P}) \circ F_{1^{\text{op}}}^{\text{op}}$. To show that $b$ preserves equalities and existential quantifiers (Definition 6) it suffices to note that $[x_1 = x_2] = \xymatrix{ }$ and $[\exists x_{n+1}. \phi] = \xymatrix{ }$. The pair $(F_{1[-1]}^{\text{op}}, b)$ witnesses the isomorphism of $LT$ and $\mathcal{R}($CB$_{\Sigma, P})$.

**Proposition 22.** Assigning doctrines to cartesian bicategories as in (6) extends to a functor $\mathcal{R} : \text{CBC} \to \text{EED}$. For a morphism of cartesian bicategories $F : \mathbb{A} \to \mathbb{B}$ in CBC, $\mathcal{R}(F) = (F|_{\text{Map}(\mathbb{A})}, b^F)$ and $b^F : \mathcal{R}(\mathbb{A}) \to \mathcal{R}(\mathbb{B}) \circ (F|_{\text{Map}(\mathbb{A})})^{\text{op}}$ is defined as

$$b^F_A(U) = F(U) \in \text{Hom}_{\mathbb{B}}(F(A), I) \text{ for all } A \in \mathbb{A} \text{ and } U \in \text{Hom}_\mathbb{A}(A, I).$$  \hfill (7)
Proof. Let $F : \mathcal{A} \to \mathcal{B}$ be a morphism in CBC. By Proposition 18 we have that $F^t_{\text{Map}(\mathcal{A})}$ is a strict cartesian functor. Regarding $b^F$, its naturality is ensured by (in fact, equivalent to) the functoriality of $F$, while the preservation of equalities and existential quantifiers of $\mathcal{R}(\mathcal{A})$ follows from the fact that $F$ preserves the structure of cartesian bicategory of $\mathcal{A}$, which is used to define the structure of elementary existential doctrine of $\mathcal{R}(\mathcal{A})$. Therefore $\mathcal{R}(F)$ is indeed a morphism in EED. Preservation of compositions and identities is straightforward. ▶

6 From Doctrines to Cartesian Bicategories

Given an elementary existential doctrine $P : \mathcal{C}^{op} \to \text{InfSL}$, we can form a category $\mathcal{A}_P$ whose objects are the same as $\mathcal{C}$ and whose morphisms are given by the elements of $P(X \times Y)$, intuitively seen as relations, as observed in [20]. Inspired by the calculus of ordinary relations on sets, using the structure of $P$ we can define a notion of composition and tensor product of these inner relations, and endow each object with a comonoid structure that makes $\mathcal{A}_P$ a cartesian bicategory. Here we recall the essential definitions, while the proof of the fact that $\mathcal{A}_P$ actually satisfies Definition 12 is rather laborious and therefore omitted here: the interested reader can find it in all details in Appendix B.

Since $\text{Hom}_{\mathcal{A}_P}(X, Y) = P(X \times Y)$, we get that $\mathcal{A}_P$ is poset-enriched. Given that $P$ is elementary, the obvious candidate for identity on $X$ is $\delta_X \in \text{Hom}_{\mathcal{A}_P}(X, X)$. Composition works as follows: let $f \in \text{Hom}_{\mathcal{A}_P}(X, Y)$ and $g \in \text{Hom}_{\mathcal{A}_P}(Y, Z)$. Consider the projections above. Then the composite $f : g$ is defined as

$$f : g = \exists_{\pi_Y} \left( P_{\pi_Z}(f) \land P_{\pi_X}(g) \right).$$

The monoidal structure of $\mathcal{A}_P$ is very straightforward: on objects, the monoidal product $\otimes$ is given by the cartesian product in $\mathcal{C}$. On morphisms, for $f \in \text{Hom}_{\mathcal{A}_P}(A, B) = P(A \times B)$ and $g \in \text{Hom}_{\mathcal{A}_P}(C, D) = P(C \times D)$, consider the projections

$$A \times C \times B \times D \xrightarrow{(\pi_1, \pi_3)} A \times B \quad \text{and} \quad \xrightarrow{(\pi_2, \pi_4)} C \times D$$

Then let

$$f \otimes g = P_{(\pi_1, \pi_3)}(f) \land P_{(\pi_2, \pi_4)}(g).$$

This makes $\mathcal{A}_P$ a monoidal, poset-enriched category. The rest of the structure of cartesian bicategory is inherited from $\mathcal{C}$ by means of a crucial tool: the graph functor of $P$, hereafter denoted by $\Gamma_P : \mathcal{C} \to \mathcal{A}_P$. It is the identity on objects and sends arrows $f : X \to Y$ in $\mathcal{C}$ to

$$\Gamma_P(f) = P_{f \times id_Y} (\delta_Y) \in P(X \times Y) = \text{Hom}_{\mathcal{A}_P}(X, Y).$$

\textbf{Proposition 23.} Let $P : \mathcal{C}^{op} \to \text{InfSL}$ be an elementary, existential doctrine. Then:

- $\Gamma_P$ is strict monoidal,
- $\Gamma_P(f)$ has a right adjoint, namely $P_{id_Y \times f}(\delta_Y)$, for every $f : X \to Y$ in $\mathcal{C}$. Therefore, by Lemma 15, it corestricts to a strict cartesian functor $\Gamma_P : \mathcal{C} \to \text{Map}(\mathcal{A}_P).$
Consider for instance the powerset doctrine \( \mathcal{P} : \text{Set}^{\text{op}} \to \text{InfSL} \): the elements of \( \mathcal{P}(X \times Y) \) are precisely the relations from \( X \) to \( Y \), while composition and monoidal product in \( \mathcal{A}_P \) coincide with the usual composition and product of relations, see § 4. In other words, \( \mathcal{A}_P = \text{Rel} \). The functor \( \Gamma_P \) calculates graphs of functions, which are exactly the maps in \( \text{Rel} \).

**Example 24.** Recall the doctrine \( LT : \Gamma \to \text{InfSL} \) and the cartesian bicategory \( \text{CBC}_{\Gamma, \text{InfSL}} \) from Examples 8 and 19. The functor \( \Gamma_{LT} : \Gamma \to \text{Map}(\mathcal{A}_{LT}) \) is inductively defined as the unique cartesian functor mapping each \( f \in \Gamma \) with arity \( \text{ar}(f) = n \) to the formula \( f(x_1, \ldots, x_n) = x_{n+1} : (n + 1, 0) \).

Now, since \( \mathcal{C} \) is a cartesian category, every object is canonically equipped with a natural and coherent comonoid structure. We can use the strict monoidal functor \( \Gamma_P : \mathcal{C} \to \mathcal{A}_P \) to transport this comonoid to \( \mathcal{A}_P \), and by Proposition 23 we have that copying and discarding in \( \mathcal{A}_P \) both have right adjoints. Finally, one can prove that every morphism in \( \mathcal{A}_P \) is a lax-comonoid homomorphism and that the Frobenius law holds.

**Theorem 25.** Let \( P : \mathcal{C}^{\text{op}} \to \text{InfSL} \) be an elementary existential doctrine. Then \( \mathcal{A}_P \) is a cartesian bicategory. Moreover, the assignment \( P \mapsto \mathcal{A}_P \) extends to a functor \( \mathcal{L} : \text{EED} \to \text{CBC} \) as follows: for \( P \) as above, \( R : \mathcal{D}^{\text{op}} \to \text{InfSL} \) and \( (F, b) : P \to R \) in \( \text{EED} \),

\[
\begin{array}{ccc}
\mathcal{A}_P & \xrightarrow{\mathcal{L}(F, b)} & \mathcal{A}_R \\
X & \xleftarrow{\Gamma(P)} & FX \\
(\mathcal{L}(\eta_P), \epsilon_R) & \xrightarrow{(\epsilon_P, \eta_R)} & (\mathcal{L}P, \mathcal{R}P) \\
Y & \xleftarrow{\Gamma(P)} & FY \\
\end{array}
\]

**7 An Adjunction**

We saw in Sections 5 and 6 that using \( \mathcal{L} \) and \( \mathcal{R} \) one can pass, in a functorial way, between the worlds of cartesian bicategories and elementary existential doctrines. Here we show that, in fact, they define an adjunction \( \mathcal{L} \dashv \mathcal{R} \). For this we need natural transformations

\[
\eta : \text{id}_{\text{EED}} \to \mathcal{R}\mathcal{L}, \quad \epsilon : \mathcal{L}\mathcal{R} \to \text{id}_{\text{CBC}}
\]

that make the following triangles commute for every \( P : \mathcal{C}^{\text{op}} \to \text{InfSL} \) in \( \text{EED} \) and \( B \) in \( \text{CBC} \).

\[
\begin{array}{ccc}
\mathcal{L}(P) & \xrightarrow{\mathcal{L}(\eta_P)} & \mathcal{L}\mathcal{R}\mathcal{L}(P) \\
\mathcal{L}(P) & \xleftarrow{\epsilon_P} & \mathcal{R}\mathcal{L}\mathcal{R}(P) \\
\mathcal{R}(B) & \xrightarrow{\eta_B} & \mathcal{R}\mathcal{L}\mathcal{R}(B) \\
\mathcal{R}(B) & \xleftarrow{\epsilon_B} & \text{id}_{\mathcal{R}(B)} \\
\end{array}
\]

(8)

Let us start with \( \epsilon \). Recall that \( \mathcal{R}(B) = \text{Hom}_\mathcal{B}(\cdot, I) : \text{Map}(\mathcal{B})^{\text{op}} \to \text{InfSL} \) and that, for \( f : X \to Y \) in \( \text{Map}(\mathcal{B}) \) and \( U \in \text{Hom}_\mathcal{B}(Y, I) \), \( \mathcal{R}(B)f(U) \) is equal to \( U \circ f : X \to I \). By definition then, \( \mathcal{L}\mathcal{R}(\mathcal{B}) = \mathcal{A}_{\mathcal{R}(\mathcal{B})} \) has the same objects of \( \mathcal{B} \) while a morphism \( X \to Y \) in \( \mathcal{L}\mathcal{R}(\mathcal{B}) \) is an element of \( \text{Hom}_\mathcal{B}(X \times Y, I) \). Hence, \( \epsilon_B \) has to take a \( \frac{x}{y} \) in \( \mathcal{B} \) and produce a morphism \( \epsilon_B : X \to Y \) in \( \mathcal{B} \). We can do so by “bending” the \( Y \) string, defining:

\[
\epsilon_B \left( \frac{x}{y} \right) = \frac{x}{y} 
\]

(9)

In fact, by the snake equation in [5], \( \epsilon_B \) has an inverse: define, for \( S : X \to Y \) in \( \mathcal{B} \),

\[
\epsilon_B^{-1} \left( \frac{x}{y} \right) = \frac{x}{y} 
\]
and it is immediate to see that \( \varepsilon_S^{-1} \varepsilon_B(R) = R \) and \( \varepsilon_S \varepsilon_B^{-1}(S) = S \).

**Example 26.** Recall from Example 21 that \( \mathcal{R}(\mathcal{CB}_{\Sigma,P}) \cong \mathcal{LT} \). Applying \( \mathcal{L} \) to both, one gets \( \mathcal{L} \mathcal{R}(\mathcal{CB}_{\Sigma,P}) \cong \mathcal{L}(\mathcal{LT}) = \mathcal{A}_{\mathcal{LT}} \), hence using \( \varepsilon \) we have that \( \mathcal{CB}_{\Sigma,P} \cong \mathcal{A}_{\mathcal{LT}} \). In particular, given a formula \( \phi : (n + m, 0) \), one obtains a morphism \( n \to m \) in \( \mathcal{CB}_{\Sigma,P} \) by first translating \( \phi \) to the string diagram \( \{\phi\} : n + m \to 0 \) (which is a morphism in \( \mathcal{L} \mathcal{R}(\mathcal{CB}_{\Sigma,P}) \) of type \( n \to m \)), and then bending the last \( m \) inputs using \( \varepsilon \) as illustrated in (9).

**Remark 27.** \( \varepsilon_S^{-1} \) coincides with \( \Gamma_{\mathcal{R}(\mathcal{B})} \) on \( \text{Map}(\mathcal{B}) \), since \( \varepsilon_S^{-1}(f) = \delta_{\mathcal{Y}} Y(f \times \text{id}_Y) = \Gamma_{\mathcal{R}(\mathcal{B})}(f) \) when \( f : X \to Y \) is a map. In other words, \( \varepsilon_S^{-1} \) is an extension of \( \Gamma_P \) to the whole of \( \mathcal{B} \).

Regarding \( \eta \), we have \( \mathcal{L}(P) = \mathcal{A}_P \), whose objects are the objects of \( \mathcal{C} \), and hom-sets are \( \mathcal{A}_P(X,Y) = P(X \times Y) \). This means that

\[
\mathcal{R} \mathcal{L}(P) = \text{Hom}_{\mathcal{A}_P}((-I) = P(- \times I) : \text{Map}(\mathcal{A}_P)^{\text{op}} \to \text{InfSL}.
\]

To give a morphism \( \eta_P : P \to \mathcal{R} \mathcal{L}(P) \) in EED means therefore to give a functor \( F : \mathcal{C} \to \text{Map}(\mathcal{A}_P) \) and a natural transformation \( b : P \to P(F(-) \times I) \) satisfying certain conditions. We have a natural candidate for \( F \): we proved in Proposition 22 that \( \Gamma_P \) is a functor whose image, in fact, is included in \( \text{Map}(\mathcal{A}_P) \) and, moreover, it is cartesian. Being the identity on objects, the natural transformation part of the definition of \( \eta_P \) must have components \( b_X : P(X) \to P(X \times I) \): it is clear then that the natural transformation

\[
P_P = \left( P_{\rho_X} : P(X) \to P(X \times I) \right)_{X \in \mathcal{C}},
\]

obtained by whiskering \( P \) with the right unitor \( \rho_X : X \times I \to X \) of \( \mathcal{C} \), is a sensible choice for \( b \). In short, we define

\[
\eta = \{(\Gamma_P, P_P) : P \to \mathcal{R} \mathcal{L}(P)\}_{P \in \text{EED}}.
\]

The interested reader can find a proof of the fact that \( \eta \) and \( \varepsilon \) are well-defined natural transformations that satisfy the triangular equalities (8) in Appendix C.

**Theorem 28.** The functors \( \mathcal{L} \) and \( \mathcal{R} \) form an adjunction

\[
\begin{array}{c}
\text{EED} \\
\downarrow \mathcal{L} \\
\mathcal{CBC}
\end{array} \\
\downarrow \mathcal{R}
\begin{array}{c}
\mathcal{L} \\
\downarrow \mathcal{R}
\end{array}
\]

whose unit is \( \eta \) (10) and whose counit, which is a natural isomorphism, is \( \varepsilon \) (9).

Since the counit \( \varepsilon \) of the adjunction \( \mathcal{L} \dashv \mathcal{R} \) is actually a natural isomorphism, the functor \( \mathcal{R} : \mathcal{CBC} \to \text{EED} \) is full and faithful. It turns out, however, that the adjunction \( \mathcal{L} \dashv \mathcal{R} \) is not an equivalence, because \( \mathcal{L} \) is not faithful. The following example shows why.

**Example 29.** Let \( \Sigma_1 \) and \( \Sigma_2 \) be the signatures in Example 4 and \( \mathbb{P} \) be some signature of predicate symbols. Let \( \mathcal{L}T_1 : \mathcal{L}_{\Sigma_1}^{\mathbb{P}} \to \text{InfSL} \) be the indexed Lindenbaum-Tarski algebras for \( \Sigma_1 \) defined as in Example 8. Recall from Example 4 the strict cartesian functor \( Q : \mathcal{L}_{\Sigma_2} \to \mathcal{L}_{\Sigma_1} \), mapping \( g_1, g_2 \in \Sigma_2 \) to \( f \in \Sigma_1 \), and observe that

\[
\mathcal{L}T_1^r = \mathcal{L}T_1 \circ Q^{\mathbb{P}} : \mathcal{L}_{\Sigma_2}^{\mathbb{P}} \to \text{InfSL}
\]

is an elementary, existential doctrine by Remark 11. Its behaviour is somewhat peculiar: it maps \( n \in \mathbb{N} \) to the set of formulas \( \phi : (n, 0) \) built from \( \Sigma_1 \) and \( \mathbb{P} \) and to any tuple
of terms \(\langle t_1, \ldots, t_m \rangle : n \to m\) in \(\Sigma_2\) assigns the function mapping a formula \(\phi : (m, 0)\) to 
\[\phi(Q(t_1), \ldots, Q(t_m)) / x_1, \ldots, x_m\]. Observe that each \(Q(t_i)\) is again a term in \(\Sigma_1\): the symbols \(g_1\) and \(g_2\) are both translated to \(f\). Somehow \(LT^\alpha_1\) behaves like \(LT^\alpha_1\) but they are different doctrines since their index categories, \(L_{\Sigma_2}\) and \(L_{\Sigma_1}\), are not isomorphic. However, when transforming them into cartesian bicategories via \(L\), one obtains that \(L(LT^\alpha_1) = L(LT^\alpha_1)\): objects are natural numbers and morphisms \(n \to m\) are the elements of the set

\[LT^\alpha_1(n + m) = LT^\alpha_1(Q(n + m)) = LT^\alpha_1(Q(n) + Q(m)) = LT^\alpha_1(n + m).\]

To formally show that \(L\) is not faithful we now define two morphisms in \(EED\), both from \(LT^\alpha_1\) to \(LT^\alpha_1\), and show that their image along \(L\) is the same. Consider the strict cartesian functors \(F_1, F_2 : L_{\Sigma_1} \to L_{\Sigma_2}\) from Example \(\square\) and observe that \(QF_i = \id_{k_{EED}}\) for \(i = 1, 2\). We are in the following situation:

\[
\begin{align*}
\text{id} & \downarrow \quad \downarrow \text{id}^\op \\
L_{\Sigma_1}^\op & \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow L_{\Sigma_2}^\op \\
\Gamma^\op & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow \text{InfSL} \\
L_{\Sigma_1} & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow L_{\Sigma_2} \\
F_1^\op & \downarrow Q^\op \quad \downarrow \quad \downarrow F_2^\op \\
\Gamma^\op & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow \text{InfSL} \\
L_{\Sigma_1} & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow L_{\Sigma_2} \\
\end{align*}
\]

which means that \(LT_1^\alpha \circ F_i^\op = LT_1^\alpha \circ Q^\op \circ F_i^\op = LT_1^\alpha\), thus \((F_1, \id_{LT_1^\alpha})\) and \((F_2, \id_{LT_1^\alpha})\) are distinct morphisms in \(EED\) from \(LT_1^\alpha\) to \(LT_1^\alpha\). Since \(L(LT^\alpha_1) = L(LT^\alpha_1)\), it follows from the definition of \(L\) that \(L(F_1, \id) = \id_{\text{InfSL}} = L(F_2, \id)\).

### 8 An Equivalence

In the previous section we identified a doctrine with peculiar behaviour, and used it to show that \(\square\) is not an equivalence. Here we characterise the additional constraints on doctrines that are needed for an equivalence.

To make the adjunction \(\square\) an equivalence, we need its unit \(\eta : \id_{\text{EED}} \to \text{RL}\) to be a natural isomorphism. This would mean that

\[\eta_P = (\Gamma_P, \rho_P) : P \to \text{RL}(P)\]

ought to be invertible in \(\text{EED}\) for any elementary existential doctrine \(P : \text{C}^{\op} \to \text{InfSL}\). Since

\[\text{RL}(P) = \text{Hom}_{\text{InfSL}}(-, I) : \text{Map}(\text{InfSL})^{\op} \to \text{InfSL},\]

by definition \(\eta_P\) has an inverse in \(\text{EED}\) if and only if the functor \(\Gamma_P : \text{C} \to \text{Map}(\text{InfSL})\) is an isomorphism of cartesian categories, in which case \((\Gamma_P^1, \rho_P^{-1})\) would be the inverse of \(\eta_P\).

But \(\Gamma_P\) is the identity on objects: to be an isomorphism, full and faithful suffices. That is, the maps of \(\text{InfSL}\) must bijectively correspond with the arrows of the base category \(\text{C}\) of \(P\). This is not necessarily the case, as arrows of \(\text{C}\) are not involved in the construction of \(\text{InfSL}\).

The technical tools that allow us to bridge the gap between morphisms in \(\text{C}\) and maps in \(\text{InfSL}\) are provided by \(\square\): the next two definitions are equivalent to faithfulness and, respectively, fullness of \(\Gamma_P\).

**Definition 30.** Let \(P : \text{C}^{\op} \to \text{InfSL}\) be an elementary existential doctrine and \(\alpha \in P(A)\). A comprehension of \(\alpha\) is an arrow \(\langle \alpha \rangle : X \to A\) in \(\text{C}\) such that \(\top_PX \leq P(\langle \alpha \rangle)\) and such that for every \(h : Z \to A\) for which \(\top_PZ \leq P(h)\), there is a unique arrow \(h' : Z \to X\) such that \(h = \langle \alpha \rangle \circ h'\). \(P\) has comprehensive diagonals if every diagonal arrow \(\Delta_A : A \to A \times A\) is the comprehension of \(\delta_A^P\).
there exists an arrow

We gave an exhaustive analysis of the relationship between two different categorifications of

denoted by

Yet there exists no

This direction has already started to be explored, from diverse perspectives, in [7, 6, 17].

Example 31. The doctrine \( LT^1_1 \colon L^0_{\Sigma_2} \to \text{InfSL} \) from Example 26 does not have comprehensive diagonal. Indeed \( \Delta_1 \) is not the comprehension of \( \delta_1 \); take as \( h \) the arrow \( \langle g_1, g_2 \rangle \colon 1 \to 2 \) and observe that \( LT(1, g_1, g_2) \delta_1) = \langle f(x_1) = f(x_1) \rangle : (1, 0) \) and \( \top : (1, 0) \leq \langle f(x_1) = f(x_1) \rangle : (1, 0) \). Yet there exists no \( h' : 1 \to 1 \) in \( L_{\Sigma_2} \) such that \( \langle g_1, g_2 \rangle = \Delta_1 \circ h' \).

Definition 32. Let \( P \colon C^{op} \to \text{InfSL} \) be an elementary, existential doctrine. We say that \( P \) satisfies the Rule of Unique Choice (RUC) if for every \( R \in P(X \times Y) \) which is a map in \( \mathcal{P} \) there exists an arrow \( f : X \to Y \) such that \( \top_{FX} \leq P_{(id_X, f)}(R) \).

Example 33. All the doctrines considered so far satisfy RUC. For an example of a doctrine that does not satisfy it consider the composition of \( F_1^{op} \colon L^0_{\Sigma_1} \to L^0_{\Sigma_2} \) (Example 4) and \( LT_2 \colon L^0_{\Sigma_2} \to \text{InfSL} \) (Example 5) that, by Remark 11 is an elementary existential doctrine. This doctrine maps \( n \) to the set of formulas \( \phi \colon (n, 0) \) where terms are built from \( g_1 \) and \( g_2 \), but the index category \( L_{\Sigma_2} \) contains terms built from \( f \) that is translated to \( g_1 \) by \( F_1 \). Now, the formula \( \phi \equiv \langle g_2(x_1) = x_2 \rangle : (2, 0) \) belongs to \( LT_2 \circ F_1^{op}(1 + 1) \) and gives rise to a map in \( \mathcal{P} \), but there is no arrow \( t : 1 \to 1 \) in \( L_{\Sigma_1} \) such that \( \top \leq LT_2(id, F_1(t))(\phi) \).

We denote by \( \text{EED} \) the full sub-category of \( \text{EED} \) consisting only of those elementary existential doctrines with comprehensive diagonals and satisfying the Rule of Unique Choice. Conveniently, it turns out that the image of \( R \) is already contained in \( \text{EED} \).

Proposition 34. Let \( \mathcal{B} \) be a cartesian bicategory. Then \( \mathcal{R}(\mathcal{B}) \) is in \( \text{EED} \).

Proof. It is enough to show that \( \Gamma_{\mathcal{R}(\mathcal{B})} : \text{Map}(\mathcal{B}) \to \text{Map}(\mathcal{R}(\mathcal{B})) = \text{Map}(L\mathcal{R}(\mathcal{B})) \) is full and faithful. As noticed in Remark 27, \( \Gamma_{\mathcal{R}(\mathcal{B})} = \varepsilon_{\mathcal{B}}^{-1} |_{\text{Map}(\mathcal{B})} \). Since \( \varepsilon_{\mathcal{B}}^{-1} \) is an isomorphism in \( \text{CBC} \), it is faithful, therefore its restriction \( \Gamma_{\mathcal{R}(\mathcal{B})} \) is as well. Moreover, \( \varepsilon_{\mathcal{B}}^{-1} \) is full: if \( R : X \to Y \) in \( L\mathcal{R}(\mathcal{B}) \) is a map, then there exists \( f : X \to Y \) in \( \mathcal{B} \) such that \( \varepsilon_{\mathcal{B}}^{-1}(f) = R \). In fact, \( f = \varepsilon_{\mathcal{B}}(R) \) and since \( \varepsilon_{\mathcal{B}} \) is a morphism in \( \text{CBC} \), by Proposition 18 we have that \( f \) is a map in \( \mathcal{B} \). Therefore, \( \Gamma_{\mathcal{R}(\mathcal{B})} \) is full.

Theorem 35. The categories \( \text{EED} \) and \( \text{CBC} \) are equivalent via adjunction \( L \to \text{CBC} \) and \( R \colon \text{CBC} \to \text{EED} \) which are moreover adjoint (Theorem 28).

9 Conclusion

We gave an exhaustive analysis of the relationship between two different categorifications of regular logic: the universal approach of elementary existential doctrines, and the algebraic approach of cartesian bicategories. We showed that cartesian bicategories give rise to elementary existential doctrines and, expanding a remark in [26], that also the other direction is possible. We proved that this correspondence is functorial, in the sense that we have a pair of functors \( L \colon \text{EED} \to \text{CBC} \) and \( R \colon \text{CBC} \to \text{EED} \) which are moreover adjoint (Theorem 28).

This adjunction can be strengthened to an equivalence provided that we refine the notion of doctrine, excluding some problematic examples (e.g. Example 29). These cases lay outside the image of \( R \) and thus this restriction does not affect cartesian bicategories (Theorem 35).

We hope that understanding the relationship between \( \text{CBC} \) and \( \text{EED} \) may provide some insights on the nature of the additional algebraic structure needed for cartesian bicategories to capture full first order logic, which one can do on the \( \text{EED} \), universal side by considering Lawvere’s original hyperdoctrines. It is probable that the end result will be closely related to Peirce’s existential graphs [30], a 19th century proto-string-diagramatic logical syntax. This direction has already started to be explored, from diverse perspectives, in [7] [6] [17].
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A Appendix to §3

In the following, we will write $\Delta_A$ or even $A \to AA$ for the morphism $(\text{id}_A, \text{id}_A)$ in a cartesian category. Also, projections $\pi_i: A_1 \times \cdots \times A_n \to A_i$ or pairings of them $(\pi_i, \pi_j): A_1 \times \cdots \times A_n \to A_i \times A_j$ will simply be written as $A_1 \ldots A_n \to A_i$, or $A_1 \ldots A_n \to A_iA_j$ respectively. For example, $\pi_2: A \times B \times C \to B$ will be denoted as $ABC \to B$, and $(\pi_1, \pi_3): A \times B \times C \to A \times C$ as $ABC \to AC$. In particular, the symmetry in a cartesian category $\sigma_{A,B} = (\pi_2, \pi_1): A \times B \to B \times A$ will be written as $AB \to BA$. 
Details on the Elementary Existential Doctrine \( \mathcal{P} : \text{Set}^{\text{op}} \to \text{InfSL} \)

The fact that \( \mathcal{P} : \text{Set}^{\text{op}} \to \text{InfSL} \) is an elementary existential doctrine is well-known and the computations to check it are rather easy. Anyway, we think that it might be useful for some readers to report such computations hereafter.

It is convenient to start with (3), even if this property is entailed by Definition 5 (see Remark 4). Recall that, for \( f : Y \to X \) in Set, \( \mathcal{P}_f : \mathcal{P}(X) \to \mathcal{P}(Y) \) is defined as \( \mathcal{P}_f(Z) = \{ y \in Y \mid f(y) \in Z \} \), in particular for a set \( A \),

\[
\begin{align*}
\mathcal{P}(A \times A) & \xrightarrow{\mathcal{P}_{\Delta_A}} \mathcal{P}(A) \\
Q & \longleftarrow \{ a \in A \mid (a, a) \in Q \}
\end{align*}
\]

Then the function \( \exists_{\Delta_A} : \mathcal{P}(A) \to \mathcal{P}(A \times A) \) defined as in (3), namely for all \( B \subseteq A \)

\[
\exists_{\Delta_A}(B) = \mathcal{P}_{\pi_1}(B) \land \delta_A
\]

\[
= \{ (x, b) \mid b \in B \} \cap \{ (a, a) \mid a \in A \}
\]

\[
= \{ (b, b) \mid b \in B \}
\]

is in fact the left adjoint of \( \mathcal{P}_{\Delta_A} \): indeed for \( B \in \mathcal{P}(A) \) and \( Q \in \mathcal{P}(A \times A) \)

\[
\exists_{\Delta_A}(B) \subseteq Q \iff \forall b \in B. (b, b) \in Q \iff B \subseteq \mathcal{P}_{\Delta_A}(Q).
\]

Similarly, for \( e : X \times A \to X \times A \times A \) given by \( e(x, a) = (x, a, a) \), the function \( \exists_e : \mathcal{P}(X \times A) \to \mathcal{P}(X \times A \times A) \) defined as in (1):

\[
\exists_e(S) = \mathcal{P}_{(\pi_1, \pi_2)}(S) \land \mathcal{P}_{(\pi_2, \pi_3)}(\delta_A)
\]

\[
= \{ (x, a, b) \mid \{ (x, a) \} \subseteq S \} \cap \{ (x, a, b) \mid (a, b) \in \delta_A \}
\]

\[
= \{ (x, a, a) \mid \{ (x, a) \} \subseteq S \}
\]

is easily seen to be left adjoint of \( \mathcal{P}_e : \mathcal{P}(X \times A \times A) \to \mathcal{P}(X \times A) \). Moreover, for \( \pi : X \times A \to A \), \( S \subseteq X \times A \) and \( B \subseteq A \),

\[
\exists_{\pi}(S) \subseteq B \iff \forall a \in A. (\exists x \in X. (x, a) \in S \implies a \in B)
\]

\[
\iff \forall (x, a) \in S. a \in B
\]

\[
\iff S \subseteq \mathcal{P}_{\pi}(B)
\]

so \( \exists_{\pi} \) is left adjoint to \( \mathcal{P}_{\pi} \).

Regarding the Beck-Chevalley condition: if \( f : A' \to A \) and \( \pi : X \times A \to A \), then the pullback of \( f \) against \( \pi \) is:

\[
\begin{array}{ccc}
X \times A' & \xrightarrow{\pi'} & A' \\
\downarrow{id_X \times f} & & \downarrow{f} \\
X \times A & \xrightarrow{\pi} & A
\end{array}
\]

Take \( S \in \mathcal{P}(X \times A) \). Then \( \mathcal{P}_{id_X \times f}(S) = \{ (x, a') \mid (x, f(a')) \in S \} \), therefore

\[
\exists_{\pi'}(\mathcal{P}_{id_X \times f}(S)) = \{ a' \in A' \mid \exists x \in X. (x, a') \in \mathcal{P}_{id_X \times f}(S) \}
\]

\[
= \{ a' \in A' \mid \exists x \in X. (x, f(a')) \in S \}
\]
while
\[ P_f(\exists_x(S)) = \{ a' \in A' \mid f(a') \in P_f(\exists_x(S)) \} \]
\[ = \{ a' \in A' \mid \exists x \in X. (x, f(a')) \in S \} \]

Hence the Beck-Chevalley condition is satisfied. Finally, for \( \pi : X \times A \to A, B \in \mathcal{P}(A) \) and \( S \in \mathcal{P}(X \times A) \):
\[
\exists_\pi(\mathcal{P}_\pi(B) \land S) = \exists_\pi(\{(x, b) \in X \times A \mid b \in B\} \cap S)
\[
= \exists_\pi(\{(x, b) \in S \mid b \in B\})
\[
= \{ b \in B \mid \exists x \in X. (x, b) \in S \}
\[
= B \cap \{ a \in A \mid \exists x \in X. (x, a) \in S \}
\[
= B \land \exists_\pi(S)
\]
therefore also the Frobenius reciprocity is satisfied.

**Additional General Facts about Elementary Existential Doctrines**

We report a few well known, simple results that are going to be useful later on.

**Lemma 36.** Let \( P \) be an existential, elementary doctrine. Then \( \delta_A = \exists_\Delta_A(\top) \). Moreover,
\[ \delta_{A \times B} = P_{ABAB \to AA}(\delta_A) \land P_{ABAB \to BB}(\delta_B) \]
for all objects \( A \) and \( B \).

**Proof.** By \((3)\), we have
\[ \exists_\Delta(\top) = P_{\pi_1}(\top) \land \delta_A = \top \land \delta_A = \delta_A. \]

As for the second part of the statement: since the projection \( ABAB \to AA \) coincides with the composite \( ABAB \to AAB \to AA \) in \( \mathcal{C} \), by functoriality of \( P \) and the fact that \( P_f \) preserves meets for any morphism \( f \) in \( \mathcal{C} \) we have
\[
P_{ABAB \to AA}(\delta_A) \land P_{ABAB \to BB}(\delta_B)
\[
= P_{ABAB \to AAB} \left( P_{AAB \to AA}(\delta_A) \land P_{AAB \to BB}(\delta_B) \right)
\[
= P_{ABAB \to AAB} \left( \exists_\epsilon( P_{AAB \to AA}(\delta_A)) \right)
\]
where \( \epsilon = \text{id}_A \times \text{id}_A \times \Delta_B \), due to \((1)\). Notice that the morphism \( \text{id}_A \times \sigma_{B,A} \times \text{id}_B \) is an isomorphism, therefore \( P_{ABAB \to AAB} \) is the left adjoint of its inverse. Thus we write \( P_{ABAB \to AAB} = \exists_{AAB \to ABAB} \) and we have:
\[
\exists_{AAB \to ABAB} \exists_\epsilon( P_{AAB \to AA}(\delta_A) \land P_{AAB \to AB}(\top) )
\[
= \exists_{AAB \to ABAB} \exists_\epsilon( \exists_\top )
\]
where \( \epsilon' = \Delta_A \times \text{id}_B \). Since left adjoints compose and \( \epsilon' : (\text{id}_A \times \sigma \times \text{id}_B) = \Delta_{A \times B} \), we obtain
\[
P_{ABAB \to AA}(\delta_A) \land P_{ABAB \to BB}(\delta_B) = \exists_{A \times B}(\top)
\[
= \delta_{A \times B}. \]

\[ \blacksquare \]
Proposition 37. Let $P : \mathcal{C}^{\text{op}} \to \text{Poset}$ be any functor. Let

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{k} & & \downarrow{g} \\
C & \xrightarrow{h} & D
\end{array}
\]

be a commutative diagram such that $P_f$ has a left-adjoint $\exists_f$ and likewise for $k$. Then

\[
\exists_f(P_h(\gamma)) \leq P_g(\exists_k(\gamma))
\]

for any $\gamma \in P(C)$.

Proof. Let $\gamma \in P(C)$. Then $\exists_f(P_h(\gamma)) \leq P_g(\exists_k(\gamma))$ if and only if

\[
P_h(\gamma) \leq P_f(P_g(\exists_k(\gamma)))
\]

which is always true because $\gamma \leq P_k(\exists_k(\gamma))$ and $P_h$ is monotone.

Combining Lemma 36 and Proposition 37 we obtain the following simple result.

Corollary 38. Let $P : \mathcal{C}^{\text{op}} \to \text{InfSL}$ be an existential elementary doctrine and $I$ be a terminal object in $\mathcal{C}$. Then for all $X$

\[
P_I : X \to I \times I(\delta_I) = \top
\]

Proof. For $f : X \to Y$, we define $f^{\text{op}} = P_{\sigma_X,Y}(f)$, where $\sigma_X,Y : Y \times X \to X \times Y$ is the symmetry of $\mathcal{C}$.

Consider also $g : Y \to Z$. Then

\[
(f : g)^{\text{op}} = P_{\sigma_X,Y}(\exists_{XYZ \to YZ} (P_{XYZ \to XY} (f) \land P_{XYZ \to YZ} (g)))
\]

while

\[
g^{\text{op}} : f^{\text{op}} = \exists_{YX \to ZX} (P_{ZYX \to YZ} (P_{\sigma_Z,Y}(g)) \land P_{ZYX \to YX} (P_{\sigma_Y,X}(f)))
\]

which simplifies to

\[
g^{\text{op}} : f^{\text{op}} = \exists_{YX \to ZX} (P_{ZYX \to YZ} (P_{\sigma_Y,X}(g) \land P_{ZYX \to YX} (P_{\sigma_Y,X}(f))))
\]

Proof of Proposition 23

Proposition 39. The symmetry $\sigma_{A,B} : A \times B \to B \times A$ of the cartesian product on $\mathcal{C}$ induces a bijection

\[
\Hom_{\mathcal{A}_P}(X,Y) \xrightarrow{\bullet^{\text{op}}} \Hom_{\mathcal{A}_P}(Y,X)
\]

This is contravariant with respect to composition in $\mathcal{A}_P$, meaning that

\[
g^{\text{op}} : f^{\text{op}} = (f : g)^{\text{op}} \quad \text{and} \quad \delta_X^{\text{op}} = \delta_X.
\]

Proof. For $f : X \to Y$, we define $f^{\text{op}} = P_{\sigma_Y,X}(f)$, where $\sigma_Y,X : Y \times X \to X \times Y$ is the symmetry of $\mathcal{C}$.

Consider also $g : Y \to Z$. Then

\[
(f : g)^{\text{op}} = P_{\sigma_X,Y}(\exists_{XYZ \to YZ} (P_{XYZ \to XY} (f) \land P_{XYZ \to YZ} (g)))
\]

while

\[
g^{\text{op}} : f^{\text{op}} = \exists_{YX \to ZX} (P_{ZYX \to YZ} (P_{\sigma_Z,Y}(g)) \land P_{ZYX \to YX} (P_{\sigma_Y,X}(f)))
\]

This simplifies to

\[
g^{\text{op}} : f^{\text{op}} = \exists_{YX \to ZX} (P_{ZYX \to YZ} (P_{\sigma_Y,X}(g) \land P_{ZYX \to YX} (P_{\sigma_Y,X}(f))))
\]
which is equal to \((f : g)^{op}\) thanks to the Beck-Chevalley condition applied to the pullback:

\[
\begin{align*}
Z \times Y \times X &\xrightarrow{(\pi_3, \pi_2, \pi_1)} Z \times X \\
&\xrightarrow{(\pi_3, \pi_2, \pi_1)} Z \times X \\
X \times Y \times Z &\xrightarrow{(\pi_3, \pi_2, \pi_1)} X \times Z
\end{align*}
\]

\[\langle \pi_1, \pi_3 \rangle \]

\[\langle \pi_3, \pi_2, \pi_1 \rangle \]

\[\sigma_{Z, X} \]

Lemma 40. Let \(P : \mathcal{C}^{op} \to \text{InfSL}\) be an elementary existential doctrine. There is a functor \(\Gamma_P : \mathcal{C} \to \mathcal{A}_P\), called the graph functor of \(P\), defined to be the identity on objects and on morphisms defined by sending \(f : X \to Y\) to

\[
\Gamma_P(f) = P_{f \times \text{id}_Y}(\delta_Y) \in P(X \times Y) = \text{Hom}_{\mathcal{A}_P}(X, Y).
\]

Proof. Firstly \(\Gamma_P\) preserves identities because \(\Gamma_P(\text{id}_X) = P_{\text{id}_X \times \text{id}_X}(\delta_X) = \delta_X\). Therefore it remains to show that \(\Gamma_P\) preserves composition. Let \(f : X \to Y\) and \(g : Y \to Z\) be morphisms in \(\mathcal{C}\). Consider the projections

\[
\begin{array}{ccc}
X \times Y \times Z & \xrightarrow{\pi_Z} & X \times Z \\
\downarrow \pi_Y & & \downarrow \pi_X \\
X \times Y & & Y \times Z
\end{array}
\]

Then

\[
\Gamma_P(f) \cdot \Gamma_P(g) = \exists_{\pi_Y} (P_{\pi_Z}(P_{f \times \text{id}_Y}(\delta_Y)) \land P_{\pi_X}(P_{g \times \text{id}_Z}(\delta_Z))).
\]

Now the diagram

\[
\begin{array}{ccc}
X \times Y \times Z & \xrightarrow{f \times \text{id}_Y \times \text{id}_Z} & Y \times Y \times Z \\
\downarrow \pi_Z & & \downarrow \pi \\
X \times Y & \xrightarrow{f \times \text{id}_Y} & Y \times Y
\end{array}
\]

commutes and therefore

\[
P_{\pi_Z}(P_{f \times \text{id}_Y}(\delta_Y)) = P_{f \times \text{id}_Y \times \text{id}_Z}(P_{\pi}(\delta_Y))
\]

and likewise

\[
X \times Y \times Z \xrightarrow{\pi_X \times \text{id}_Y \times \text{id}_Z} Y \times Y \times Z
\]

commutes, where \(\pi'\) projects onto the second and third component. Therefore we also have

\[
P_{\pi_X}(P_{g \times \text{id}_Z}(\delta_Z)) = P_{f \times \text{id}_Y \times \text{id}_Z}(P_{\pi'}(P_{g \times \text{id}_Z}(\delta_Y))).
\]

This gives us

\[
\Gamma_P(f) \cdot \Gamma_P(g) = \exists_{\pi_Y} (P_{f \times \text{id}_Y \times \text{id}_Z}(P_{\pi}(\delta_Y) \land P_{\pi'}(P_{g \times \text{id}_Z}(\delta_Y)))).
\]
Let $e = \Delta_Y \times \text{id}_Z : Y \times Z \to Y \times Y \times Z$, then by [1], we have

$$
\Gamma_P(f) \cdot \Gamma_P(g) = \exists_{\pi_Y}(P_{f \times \text{id}_Y \times \text{id}_Z}(\exists_{\pi_Y}(P_{g \times \text{id}_Z}(\delta_Z)))).
$$

Using the Beck-Chevalley condition with the square

$$
\begin{array}{ccc}
X \times Y \times Z & \xrightarrow{\pi_Y} & X \times Z \\
\downarrow f \times \text{id}_Y \times \text{id}_Z & & \downarrow f \times \text{id}_Z \\
Y \times Y \times Z & \xrightarrow{\rho} & Y \times Z
\end{array}
$$

where $\rho$ projects onto the first and third component, this becomes

$$
\Gamma_P(f) \cdot \Gamma_P(g) = P_{f \times \text{id}_Y}(\exists_\rho(\exists_{\pi_Y}(P_{g \times \text{id}_Z}(\delta_Z))))
$$

Now since $e : \rho = \text{id}$ and by uniqueness of adjoints, this becomes

$$
\Gamma_P(f) \cdot \Gamma_P(g) = P_{f \times \text{id}_Y}(\exists_\rho(\exists_{\pi_Y}(P_{g \times \text{id}_Z}(\delta_Z)))) = \Gamma_P(f \cdot g).
$$

**Remark 41.** It follow immediately from the definitions of $\Gamma_P$ and of tensor product in $\mathcal{A}_P$ that for any $f, g \in \mathcal{C}$, $\Gamma_P(f \times g) = \Gamma_P(f) \otimes \Gamma_P(g)$.

**Lemma 42.** Let $P : \mathcal{C}^{\text{op}} \to \text{InfSL}$ be an elementary, existential doctrine, $\Gamma_P : \mathcal{C} \to \mathcal{A}_P$ the functor from Lemma [41]. For any morphism $f : X \to Y$ in $\mathcal{C}$, $\Gamma_P(f)^{\text{op}}$ is right-adjoint to $\Gamma_P(f)$, that is

$$
\delta_X \leq \Gamma_P(f) \cdot \Gamma_P(f)^{\text{op}} \quad \text{and} \quad \Gamma_P(f)^{\text{op}} \cdot \Gamma_P(f) \leq \delta_Y.
$$

**Proof.** It is easy to check that $\Gamma_P(f)^{\text{op}} = P_{\text{id} \times f}(\delta_Y)$.

We start by proving $\delta_X \leq \Gamma_P(f) \cdot \Gamma_P(f)^{\text{op}}$. Let

$$
\begin{array}{ccc}
X \times Y \times X & \xrightarrow{\pi_{1,2}} & X \times Y \\
\downarrow \pi_{1,3} & & \downarrow \pi_{2,3} \\
X \times X & \xrightarrow{\pi_{1,3}} & Y \times X
\end{array}
$$

be the projections. Then

$$
\Gamma_P(f) \cdot \Gamma_P(f)^{\text{op}} = \exists_{\pi_{1,3}}(P_{\pi_{1,3}}(P_{f \times \text{id}_Y}(\delta_Y)) \wedge P_{\pi_{2,3}}(P_{\text{id}_Y \times f}(\delta_Y)))
$$

$$
= \exists_{\pi_{1,3}}(P_{f \times \text{id}_Y \times f}(P_{\pi_{1,3}}(\delta_Y) \wedge P_{\pi_{2,3}}(\delta_Y)))
$$

$$
= \exists_{\pi_{1,3}}(P_{f \times \text{id}_Y \times f}(\exists_{\pi_{1,3}}(\delta_Y)))
$$

where the last step uses [1] with $e = \Delta_Y \times \text{id}_Y$. Applying Beck-Chevalley using the pullback

$$
\begin{array}{ccc}
X \times Y \times X & \xrightarrow{\pi_{1,3}} & X \times X \\
\downarrow f \times \text{id}_f & & \downarrow f \times f \\
Y \times Y \times Y & \xrightarrow{\pi_{1,3}} & Y \times Y
\end{array}
$$

yields

$$
\Gamma_P(f) \cdot \Gamma_P(f)^{\text{op}} = P_{f \times f}(\exists_{\pi_{1,3}}(\exists_{\pi_{1,3}}(\delta_Y)) = P_{f \times f}(\delta_Y)
$$

where $\exists_{\pi_{1,3}} \circ \exists_{\pi_{1,3}} = \text{id}$. Now applying Lemma [46] and Proposition [47] yields

$$
\Gamma_P(f) \cdot \Gamma_P(f)^{\text{op}} = P_{f \times f}(\exists_{\Delta}(\top)) \geq \exists_{\Delta}(P_{f}(\top)) = \exists_{\Delta}(\top) = \delta_Y.
$$
It remains to prove that $\Gamma P(f)^{op} \subseteq \delta Y$. Let

$$\begin{array}{ccc}
Y \times X \times Y & \xrightarrow{\pi_{1,3}} & Y \\
\downarrow \pi_{1,3} & & \downarrow \pi_{2,3} \\
Y \times Y & \xrightarrow{\pi_{1,3}} & X \times Y
\end{array}$$

be the projections. Then

$$\begin{align*}
\Gamma P(f)^{op} \subseteq \Gamma P(f) &= \exists_{\pi_{1,3}} (P_{\pi_{1,2}}(P_{\text{id}_Y \times f}(\delta Y)) \land P_{\pi_{2,3}}(P_{f \times \text{id}_Y}(\delta Y))) \\
&= \exists_{\pi_{1,3}} P_{\text{id}_Y \times f \times \text{id}_Y}(P_{\pi_{1,2}}(\delta Y)) \land P_{\pi_{2,3}}(\delta Y) \\
&= \exists_{\pi_{1,3}} P_{\text{id}_Y \times f \times \text{id}_Y}(\exists_e(\delta Y))
\end{align*}$$

where the last step uses (1) with $e = \Delta_Y \times \text{id}_Y$. Using Proposition 37 with the commutative square

$$\begin{array}{ccc}
Y \times X \times Y & \xrightarrow{\pi_{1,3}} & Y \times Y \\
\downarrow \pi_{1,3} \times \text{id} & & \downarrow \text{id} \\
Y' \times Y \times Y & \xrightarrow{\pi_{1,3}} & Y \times Y
\end{array}$$

we get

$$\Gamma P(f)^{op} \subseteq \exists_{\pi_{1,3}} (\exists_e(\delta Y)) = \delta Y.$$

\[\Box\]

**Proof of Theorem 25**

Here we prove that $\mathcal{A}_P$ is a cartesian bicategory (Theorem 45) and that the assignment $P \mapsto \mathcal{A}_P$ extends to a functor $L : \text{EED} \to \text{CBC}$ (Proposition 46).

**Theorem 43.** Let $P : \mathbb{C}^{op} \to \text{InfSL}$ be an existential, elementary doctrine. Then $\mathcal{A}_P$ is a poset-enriched category.

**Proof.** That $\delta_X$ serves as the identity of composition follows easily from (1): Let $f \in \text{Hom}_{\mathcal{A}_P}(X,Y) = P(X \times Y)$ and

$$\begin{array}{ccc}
X \times Y \times Y & \xrightarrow{\pi_{1,3}} & X \times Y \\
\downarrow \pi_{1,3} & & \downarrow \pi_{2,3} \\
X \times Y & \xrightarrow{\pi_{1,3}} & Y \times Y
\end{array}$$

the projections. Then

$$f : \delta_Y = \exists_{\pi_{1,3}} (P_{\pi_{1,2}}(f) \land P_{\pi_{2,3}}(\delta Y)).$$

By (1), we have $P_{\pi_{1,2}}(f) = P_{\pi_{2,3}}(\delta Y) = \exists_e(\delta Y)$ where $e = \text{id}_X \times \Delta_Y : X \times Y \to X \times Y \times Y$.

Therefore, $f : \delta_Y = \exists_{\pi_{1,3}} (\exists_e(\delta Y))$. Since left-adjoints compose, we have that $\exists_{\pi_{1,3}} \circ \exists_e$ is left-adjoint to $P_{\pi_{1,2} \circ e} = P\text{id} = \text{id}$. By uniqueness of adjoints, also $\exists_{\pi_{1,3}} \circ \exists_e$ is the identity and hence

$$f : \delta_Y = \exists_{\pi_{1,3}} (\exists_e(\delta Y)) = f.$$

That also $\delta_X : f = f$ follows by Proposition 50.
To see that composition is associative, let \( f : A \rightarrow B, g : B \rightarrow C, h : C \rightarrow D \). We have

\[
f \circ g = \exists_{ABC \rightarrow AC}(P_{ABC \rightarrow AB}(f) \land P_{ABC \rightarrow BC}(g)).
\]

Let us give the inner expression a name, and let \( \alpha = P_{ABC \rightarrow AB}(f) \land P_{ABC \rightarrow BC}(g) \) so that

\[
f \circ g = \exists_{ABC \rightarrow AC}(\alpha).
\]

Then

\[
(f \circ g) \circ h = \exists_{ACD \rightarrow AD}(P_{ACD \rightarrow AC}(f \circ g) \land P_{ACD \rightarrow CD}(h)) = \exists_{ACD \rightarrow AD}(P_{ACD \rightarrow AC}(\exists_{ABC \rightarrow AC}(\alpha) \land P_{ACD \rightarrow CD}(h))).
\]

Now we can use the Beck-Chevalley condition with the following square of projections:

\[
\begin{array}{ccc}
A \times B \times C \times D & \longrightarrow & A \times C \times D \\
\downarrow & & \downarrow \\
A \times B \times C & \longrightarrow & A \times C
\end{array}
\]

to infer that

\[
P_{ACD \rightarrow AC}(\exists_{ABC \rightarrow AC}(\alpha)) = \exists_{ABCD \rightarrow ACD}(P_{ABCD \rightarrow ABC}(\alpha))
\]

and therefore

\[
(f \circ g) \circ h = \exists_{ACD \rightarrow AD}(\exists_{ABCD \rightarrow ACD}(P_{ABCD \rightarrow ABC}(\alpha) \land P_{ABCD \rightarrow CD}(h))).
\]

Using Frobenius reciprocity to get everything into the existential quantifier we get

\[
(f \circ g) \circ h = \exists_{ACD \rightarrow AD}(\exists_{ABCD \rightarrow ACD}(P_{ABCD \rightarrow ABC}(\alpha) \land P_{ABCD \rightarrow CD}(h))
\]

and therefore by uniqueness of adjoints also

\[
(f \circ g) \circ h = \exists_{ABCD \rightarrow AD}(P_{ABCD \rightarrow ABC}(\alpha) \land P_{ABCD \rightarrow CD}(h)).
\]

Now using the definition of \( \alpha \) and the fact that any \( P_\pi \) preserves meet, this expands into

\[
(f \circ g) \circ h = \exists_{ABCD \rightarrow AD}(P_{ABCD \rightarrow AB}(f) \land P_{ABCD \rightarrow BC}(g) \land P_{ABCD \rightarrow CD}(h)).
\]

Completely analogously one can prove that also

\[
f \circ (g \circ h) = \exists_{ABCD \rightarrow AD}(P_{ABCD \rightarrow AB}(f) \land P_{ABCD \rightarrow BC}(g) \land P_{ABCD \rightarrow CD}(h))
\]

thus composition is therefore associative.

Finally, composition is a monotonous operation because \( P_\pi \) and \( \exists_\pi \) are monotonous functions, thus if \( f_1 \leq f_2 \) then \( f_1; g \leq f_2; g \): therefore the category \( \mathcal{A}_P \) is poset-enriched. ◀

**Proposition 44.** Let \( P : \mathcal{C}^{\text{op}} \rightarrow \text{InfSL} \) be an existential, elementary doctrine. Then \( \mathcal{A}_P \) is a symmetric monoidal category.
We notice that the projection $\otimes$ is a functor. Let $f_1 : A \to B$, $g_1 : C \to D$, $f_2 : B \to E$, $g_2 : D \to F$. We need to show that

$$(f_1 \otimes g_1) ; (f_2 \otimes g_2) = (f_1 ; f_2) \otimes (g_1 ; g_2).$$

Starting from the left-hand side:

$$(P_{ACBDE} \to AB(f_1) \land P_{ACBDE} \to CD(g_1)) ; (P_{BDEF} \to BE(f_2) \land P_{BDEF} \to DF(g_2))$$

$$= \exists_{ACBDEF} \to ACF \left( P_{ACBDEF} \to ABD \left( P_{ACBDE} \to AB(f_1) \land P_{ACBDE} \to CD(g_1) \right) \right)$$

$$\land P_{ACBDEF} \to BDF \left( P_{BDEF} \to BE(f_2) \land P_{BDEF} \to DF(g_2) \right)$$

$$= \exists_{ACBDEF} \to ACF \left( P_{ACBDEF} \to ABE \left( P_{ABE} \to AB(f_1) \land P_{ABE} \to BE(f_2) \right) \right)$$

$$\land P_{ACBDEF} \to CDF \left( P_{CDF} \to CD(g_1) \land P_{CDF} \to DF(g_2) \right) \right).$$

Let us call, for the sake of brevity,

$$\alpha = P_{ABE} \to AB(f_1) \land P_{ABE} \to BE(f_2)$$

and

$$\beta = P_{CDF} \to CD(g_1) \land P_{CDF} \to DF(g_2).$$

Then we proved that

$$(f_1 \otimes g_1) ; (f_2 \otimes g_2) = \exists_{ACBDEF} \to ACF \left( P_{ACBDEF} \to ABE(\alpha) \land P_{ACBDEF} \to CDF(\beta) \right).$$

We notice that the projection $ACBDEF \to ACF$, over which we compute the existential quantifier, can be performed in two steps: first we forget $D$, then $B$. Similarly, the projection $ACBDEF \to ABE$, whose image along $P$ we calculate in $\alpha$, can be decomposed in two steps. We use this to put ourselves in the position of applying the Frobenius reciprocity property of $P$:

$$\exists_{ACBDEF} \to ACF \left( P_{ACBDEF} \to ABE(\alpha) \land P_{ACBDEF} \to CDF(\beta) \right)$$

$$= \exists_{ACBDEF} \to ACF \left( P_{ACBDEF} \to ABE(\alpha) \right)$$

$$\land P_{ACBDEF} \to CDF(\beta) \right)$$

$$= \exists_{ACBDEF} \to ACF \left( P_{ACBDEF} \to ABE(\alpha) \right)$$

$$\land \exists_{ACBDEF} \to ACF \left( P_{ACBDEF} \to CDF(\beta) \right) \right).$$

Now we use the Beck-Chevalley condition on the following pullback, consisting only of projections:

$$\begin{array}{ccc}
ACBDEF & \to & ACBEF \\
\downarrow & & \downarrow \\
CDF & \to & CF
\end{array}$$

resulting in exchanging the order of $\exists$ and $P$, obtaining:

$$\exists_{ACBDEF} \to ACF \left( P_{ACBDEF} \to CF(\exists_{CDF} \to CF(\beta)) \land P_{ACBDEF} \to ABE(\alpha) \right).$$
By writing the projection $ACBEF \rightarrow CF$ as a the composite

$$ACBEF \rightarrow ACEF \rightarrow CF,$$

one can again use the Frobenius reciprocity and then the Beck-Chevalley condition to finally obtain

$$P_{ACEF \rightarrow CF}(\exists_{CDF \rightarrow CF}(\beta)) \land P_{ACEF \rightarrow AE}(\exists_{ABE \rightarrow AE}(\alpha))$$

which is equal, by definition, to $(f_1 : f_2) \otimes (g_1 : g_2)$, as required. Thus $\otimes$ preserves compositions; preservation of identities, which is tantamount to

$$P_{ABAB \rightarrow AA}(\delta_A) \land P_{ABAB \rightarrow BB}(\delta_B) = \delta_{A \times B},$$

follows from Lemma 36. Therefore $\otimes$ is a functor.

Next, we show that $\otimes$ is coherently associative, unitary and commutative. We have already noticed in Remark 41 that the functor $\Gamma_P$ from Lemma 40 transports the product functor of $\mathbb{C}$ into the tensor functor $\otimes$ of $\mathcal{A}_P$: this suggests to transfer the whole cartesian structure of $\mathbb{C}$ into $\mathcal{A}_P$, thus the associator, the unitors and the symmetry for $\otimes$ are defined as the image of their correspondent isomorphisms in $\mathbb{C}$. This automatically ensures that they are all natural isomorphisms (being whiskerings of a functor with natural isomorphisms) coherent with $\otimes$.

Theorem 45. Let $P: \mathbb{C}^op \rightarrow \infsl$ be an elementary existential doctrine. Then $\mathcal{A}_P$ is a cartesian bicategory.

Proof. By Theorem 43 and Proposition 44, $\mathcal{A}_P$ is a poset-enriched, symmetric monoidal category. Since $\mathbb{C}$ is a cartesian category, every object is canonically equipped with a comonoid given by copying and discarding. We can use the functor $\Gamma_P: \mathbb{C} \rightarrow \mathcal{A}_P$ to transport this comonoid to $\mathcal{A}_P$ and Lemma 42 shows that copying and discarding both have right-adjoints. Since the comonoids in $\mathbb{C}$ are coherent with the monoidal structure, the same is true in $\mathcal{A}_P$ as the structure is transported through the strict monoidal functor $\Gamma_P$. It therefore remains to prove that every morphism is a lax-comonoid homomorphism and that the Frobenius law holds. To see that every morphism is a lax-comonoid homomorphism, we need to prove that

\[
\begin{array}{ccc}
\begin{array}{ccc}
\circ & \circ & \circ \\
\longrightarrow & \longrightarrow & \longrightarrow \\
x & y & z
\end{array}
\end{array} 
\leq 
\begin{array}{ccc}
\begin{array}{ccc}
\circ & \circ & \circ \\
\longrightarrow & \longrightarrow & \longrightarrow \\
x & z & y
\end{array}
\end{array}
\quad \text{and} \quad 
\begin{array}{ccc}
\begin{array}{ccc}
\circ & \circ & \circ \\
\longrightarrow & \longrightarrow & \longrightarrow \\
y & x & z
\end{array}
\end{array} 
\leq 
\begin{array}{ccc}
\begin{array}{ccc}
\circ & \circ & \circ \\
\longrightarrow & \longrightarrow & \longrightarrow \\
z & y & x
\end{array}
\end{array}
\]  

(12)

for any morphism $R \in \text{Hom}_{\mathcal{A}_P}(X, Y) = P(X \times Y)$. The latter is very easy to see because the right-hand side is the top element in $P(X \times I)$, see Corollary 38. So we focus on the former inequality.

We start with the left-hand side. In order to distinguish among the three occurrences of $Y$ in the composite of $R$ with the diagonal in $\mathcal{A}_P$, we will write $Y_1, Y_2$ and $Y_3$, even though they all represent the same object $Y$ of $\mathcal{A}_P$. We have:

$$\exists_{XY_1, Y_2, Y_3 \rightarrow XY_1 Y_2 Y_3} \left( P_{XY_1 Y_2 Y_3 \rightarrow XY_1 Y_2 Y_3}(\alpha_{XY_1 Y_2}) \land P_{XY_1 Y_2 Y_3 \rightarrow XY_1 Y_2 Y_3}(\delta_{XY_2 Y_3}) \right)$$

and since $\delta_{Y_2} = P_{(\pi_1, \pi_3)}(\delta_Y) \land P_{(\pi_2, \pi_4)}(\delta_Y)$ by Lemma 36 we get

$$\exists_{XY_1 Y_2, Y_2 \rightarrow XY_2 Y_3} \left( P_{XY_2 Y_3 \rightarrow XY_2 Y_3}(\alpha_{XY_2}) \land P_{XY_2 Y_3 \rightarrow XY_2 Y_3}(\delta_{XY_3}) \right)$$

$$= \exists_{XY_1, Y_2 Y_3 \rightarrow XY_2 Y_3} \left( P_{XY_1 Y_2 Y_3 \rightarrow XY_1 Y_2 Y_3}(\alpha_{XY_1}) \land P_{XY_1 Y_2 Y_3 \rightarrow XY_1 Y_2 Y_3}(\delta_{XY_2}) \right) \land P_{XY_1 Y_2 Y_3 \rightarrow XY_1 Y_2 Y_3}(\delta_Y).$$
Using \( \iota \), the symmetry in \( C \) and \( \iota \) again, we obtain:
\[
\exists_{XY\,Y^2 \rightarrow XY\,Y^2} \left( P_{XY\,Y^2 \rightarrow XY\,Y^2} = P_{XY\,Y^2 \rightarrow XY\,Y^2} = P_{XY\,Y^2 \rightarrow XY\,Y^2} \delta Y \right)
\]
\[
\lambda_{XY\,Y^2 \rightarrow XY\,Y^2} \left[ P_{XY\,Y^2 \rightarrow XY\,Y^2} \left( P_{XY\,Y^2 \rightarrow XY\,Y^2} \delta Y \right) \right]
\]
\[
\exists_{XY\,Y^2 \rightarrow XY\,Y^2} \left( P_{XY\,Y^2 \rightarrow XY\,Y^2} \left( \exists_{XY\,Y^2 \rightarrow XY\,Y^2} \left( P_{XY\,Y^2 \rightarrow XY\,Y^2} \delta Y \right) \right) \right)
\]
Since \( P_{XY\,Y^2} \) and \( P_{XY\,Y^2} \) are isomorphisms, they are left adjoints (to their inverses): by composing all the left adjoints in the above expression we obtain
\[
\exists_{XY\,Y^2 \rightarrow XY\,Y^2} \delta Y
\]
The right-hand side, instead, is
\[
\exists_{XY\,Y^2 \rightarrow XY\,Y^2} \left( P_{XY\,Y^2 \rightarrow XY\,Y^2} \delta X \right)
\]
\[
= \exists_{XY\,Y^2 \rightarrow XY\,Y^2} \left( P_{XY\,Y^2 \rightarrow XY\,Y^2} \delta X \right)
\]
Using that
\[
P_{\Delta X \times Y^2} \left( \exists_{\Delta_X \times Y^2} \left( R \otimes R \right) \right)
\]
we get
\[
P_{\Delta_X \times Y^2} \left( \exists_{\Delta_X \times Y^2} \left( R \otimes R \right) \right)
\]
Using that
\[
R \otimes R = P_{\sigma_1} \left( R \otimes R \right)
\]
we have
\[
P_{\Delta_X \times Y^2} \left( \exists_{\Delta_X \times Y^2} \left( R \otimes R \right) \right)
\]
Since we have established the right-hand side as a meet, it suffices to prove that
\[
\exists_{XY\,Y^2 \rightarrow XY\,Y^2} \delta Y \leq P_{XY\,Y^2} \left( \exists_{XY\,Y^2 \rightarrow XY\,Y^2} \delta Y \right)
\]
for \( i \in \{1,2\} \) and since
\[
P_{XY\,Y^2} \left( \exists_{XY\,Y^2} \left( R \right) \right) = R
\]
this follows by virtue of \( \exists_{XY\,Y^2} \) being a left-adjoint.
Lastly, we need to check the Frobenius law. For that it suffices to prove that
\[
\begin{array}{c}
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\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
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\end{array}
\end{array}
\end{array}
\end{array}
which are elements of $P(A^4)$. In the following we will calculate in detail the left-hand side of the equation above, where there are seven occurrences of the same object $A$. For notational convenience, we want to distinguish them, like this:

\[
\begin{array}{c}
\Delta_1 \\
\Delta_2 \\
\Delta_3
\end{array}
\]

To name a projection out of a product of copies of $A$, we will use the subscripts to specify which factors we are projecting onto: for example, $(\pi_1, \pi_2, \pi_3): A^7 \to A_1 \times A_2 \times A_3$ will simply be denoted as $1234567 \to 123$.

By simply unravelling the definition of composition and tensor, and using the fact that $\delta_{A \times A} = \delta_{A} \otimes \delta_{A}$ (Lemma 36), it is easy to see that the left-hand side is equal to:

\[
\exists_{1234567 \to 1267} (P_{1234567 \to 123} \land P_{1234567 \to 14} \land P_{1234567 \to 47} (\delta_A) \\
\land P_{1234567 \to 57} (\delta_A) \land P_{1234567 \to 25} (\delta_A) \land P_{1234567 \to 36} (\delta_A))
\]

which can be rewritten, using [1], as

\[
\exists_{1234567 \to 1267} (\exists_{A \times \Delta_{A}} (\delta_A)) \land P_{1234567 \to 147} (\exists_{A \times \Delta_{A}} (\delta_A)) \\
\land P_{1234567 \to 257} (\exists_{A \times \Delta_{A}} (\delta_A))
\]

The projection $1234567 \to 1267$ of the existential quantifier can be decomposed in three projections which forget only one instance of $A$ at a time. Using this and rearranging the expression above, we can put ourselves in the position of being able to use the Frobenius reciprocity as follows: the above is equal to

\[
\exists_{12567 \to 1267} \exists_{124567 \to 12567} (P_{124567 \to 147} (\exists_{A \times \Delta_{A}} (\delta_A)) \\
\land P_{124567 \to 257} (\exists_{A \times \Delta_{A}} (\delta_A)))
\]

By the Beck-Chevalley condition applied to the following pullback, made just of projections:

\[
\begin{array}{c}
1234567 \\
136 \\
16
\end{array}
\]

we have that

\[
\exists_{1234567 \to 124567} (P_{1234567 \to 136} (\exists_{A \times \Delta_{A}} (\delta_A))) = P_{124567 \to 16} (\exists_{136 \to 16} (\exists_{A \times \Delta_{A}} (\delta_A))) \\
= P_{124567 \to 16} (\delta_A).
\]
Using this fact and a similar strategy as before, we can reduce the left-hand side to
\[ P_{1267} \rightarrow 16(\delta_A) \land P_{1267} \rightarrow 17(\delta_A) \land P_{1267} \rightarrow 27(\delta_A). \]

We can use symmetries in \( C \) to arrange the above expression so that we can use (1) in the following:
\[
P_{1267} \rightarrow 16(\delta_A) \land P_{1267} \rightarrow 17(\delta_A) \land P_{1267} \rightarrow 27(\delta_A)
= P_{1267} \rightarrow 2716 \left[ P_{2716} \rightarrow 271 \left( P_{271} \rightarrow 27(\delta_A) \land P_{271} \rightarrow 71(\delta_A(\Pi)) \right) \land P_{2716} \rightarrow 16(\delta_A) \right]
= P_{1267} \rightarrow 2716 \left[ \exists_{A \times \Delta_A} \left( \exists_{\Delta_A} (\exists_{\Delta_A} (\top)) \right) \right]
\]

Now, \( P_{1267} \rightarrow 2716 \) is the left-adjoint of its inverse, which composed to the other existential quantifiers yields simply
\[ \exists_{[\text{id}_A, \text{id}_A, \text{id}_A, \text{id}_A]} (\top). \]

Analogously, one can prove that the right-hand side of the Frobenius equation corresponds to the same expression and therefore
\[
\begin{aligned}
&\xymatrix{L \ar@{=}[rr] & & L} \\
\end{aligned}
\]

\begin{proposition}
The assignment \( P \mapsto \mathcal{A}_P \) extends to a functor \( \mathcal{L} : \text{EED} \rightarrow \text{CBS} \) as follows.

- On objects: for \( P : \mathcal{C}^{\text{op}} \rightarrow \text{InfSL} \), \( \mathcal{L}(P) = \mathcal{A}_P \).
- On morphisms: for \( P \) as above, \( R : \mathcal{D}^{\text{op}} \rightarrow \text{InfSL} \) and \( (F, b) : P \rightarrow R \) in \( \text{EED} \),
\[
\begin{array}{c}
\mathcal{A}_P \xrightarrow{\mathcal{L}(F,b)} \mathcal{A}_R \\
X \xrightarrow{\beta_{X,Y}} FX \\
P(X \times Y) \xrightarrow{\beta_{X,Y}} b_{X \times Y}(r) \in B(R(X) \times R(Y)) \\
Y \xrightarrow{\beta_{X,Y}} FY
\end{array}
\]
\end{proposition}

\begin{proof}
First we check that \( \mathcal{L}(F,b) \) is a morphism of cartesian bicategories. Let \( X \) be an object of \( C \): the identity morphism of \( X \) in \( \mathcal{A}_P \) is \( \delta_X^P \) and
\[
\mathcal{L}(F,b)(\delta_X^P) = b_{X \times X}(\delta_X^P) = \delta_{F(X)}^R = \text{id}_{\mathcal{A}_{F(X)}}
\]
thus \( \mathcal{L}(F,b) \) preserves identities. Let now \( r \in P(X \times Y) \) and \( s \in P(Y \times Z) \). Then
\[
\begin{aligned}
\mathcal{L}(F,b)(s \circ_{\mathcal{A}_P} \tau) &= \mathcal{L}(F,b)(\exists_{X \times Y \times Z} (P_{X \times Y \times Z}(r) \land P_{X \times Y \times Z}(s))) \\
&= b_{X \times Z} (\exists_{X \times Y \times Z} (P_{X \times Y \times Z}(r) \land P_{X \times Y \times Z}(s))) \\
&= \exists_{F(X \times Y) \rightarrow F(X \times Z)} (b_{X \times Y \times Z} (P_{X \times Y \times Z}(r) \land P_{X \times Y \times Z}(s))) \\
&= \exists_{F(X \times Y) \rightarrow F(X \times Z)} (b_{X \times Y \times Z} (P_{X \times Y \times Z}(r) \land P_{X \times Y \times Z}(s))) \\
&= \exists_{F(X \times Y) \rightarrow F(X \times Z)} (R_{F(X \times Y) \rightarrow F(Y)} (b_X X \times Y(r) \land R_{F(Y) \times Z} (b_{Y \times Z}(s))) \\
&= b_{X \times Y}(s) \circ_{\mathcal{A}_R} b_{X \times Y}(r) \\
&= \mathcal{L}(F,b)(s) \circ_{\mathcal{A}_R} \mathcal{L}(F,b)(r)
\end{aligned}
\]
\end{proof}
where we used the fact that $b$ preserves existential quantifiers (third equation) and infima (fourth equation) and that $b$ is natural (fifth equation). Therefore $\mathcal{L}(F, b)$ is a functor, we need to check that it preserves all the structure of cartesian bicategory of $\mathcal{A}_P$.

Preservation of the monoidal product of $\mathcal{A}_P$ is done in an analogous way to the proof above of preservation of compositions, where commutativity of $b$ with existential quantifiers is not needed. Regarding the symmetry, recall that $\sigma_{X,Y}^\mathcal{A} = \Gamma_P(\sigma_{X,Y}^\mathcal{C})$, hence:

$$
\mathcal{L}(F, b)(\sigma_{X,Y}^\mathcal{A}) = b_{X \times Y \times Y \times X}(\sigma_{X,Y}^\mathcal{A})
= b_{X \times Y \times Y \times X}(P_{\sigma_{X,Y}^\mathcal{C}, \text{id}_Y, \text{id}_X}(\delta_Y^\mathcal{C}))
= R_F(\sigma_{X,Y}^\mathcal{C})(b_{Y \times X \times Y \times X}(\delta_Y^\mathcal{C}))
$$

by naturality of $b$. Since $F$ preserves the cartesian structure of $\mathcal{C}$, we have that $F(\sigma_{X,Y}^\mathcal{A} \times \text{id}_Y \times \text{id}_X) = \sigma_{FX,FY}^\mathcal{D} \times \text{id}_{FY \times FX}$; moreover, $b_{Y \times X \times Y \times X}(\delta_Y^\mathcal{C}) = \delta_Y^\mathcal{R}$, therefore

$$
\mathcal{L}(F, b)(\sigma_{X,Y}^\mathcal{A}) = R_{\sigma_{FX,FY}^\mathcal{D}, \text{id}_{FX}, \text{id}_Y}(\delta_Y^\mathcal{R}) = \sigma_{FX,FY}^\mathcal{A}.
$$

Analogously one shows that $\mathcal{L}(F, b)$ preserves the comonoid structure of any object $X$ in $\mathcal{A}_P$, since it is defined as the image along $\Gamma_P$ of copying and discarding in $\mathcal{C}$.

Finally, monotonicity of $b$ immediately implies that $\mathcal{L}(F, b)$ preserves the partial order of $P(X \times Y)$ for every $X, Y$ in $\mathcal{A}_P$. This proves that $\mathcal{L}(F, b)$ is indeed a morphism of cartesian bicategories.

It is left to prove that $\mathcal{L}$ is functorial. It is immediate to see that for $P: \mathcal{C}^\text{op} \to \text{InfSL}$, $\mathcal{L}(\text{id}_\mathcal{C}, \text{id}_P) = \text{id}_{\mathcal{A}_P}$. As per preservation of compositions in EED: let $R: \mathcal{D}^\text{op} \to \text{InfSL}$ and $S: \mathcal{E}^\text{op} \to \text{InfSL}$ be elementary existential doctrines, and let $(F, b): P \to R$ and $(G, c): R \to S$ in EED. We have that $(G, c) \circ (F, b) = (GF, cF \circ b)$, therefore

$$
\begin{array}{ccc}
\mathcal{A}_P & \xrightarrow{\mathcal{L}((G, c) \circ (F, b))} & \mathcal{A}_S \\
X \downarrow \mathcal{A}_P & \xrightarrow{GF} & FX \\
P(X \times Y) \downarrow \mathcal{A}_P & \xrightarrow{b_{X \times Y}(r)} & FY
\end{array}
$$

while

$$
\begin{array}{ccc}
\mathcal{A}_P & \xrightarrow{\mathcal{L}(F,b)} & \mathcal{A}_R \\
X \downarrow \mathcal{A}_P & \xrightarrow{FX} & GFX \\
P(X \times Y) \downarrow \mathcal{A}_P & \xrightarrow{b_{X \times Y}(r)} & GFY
\end{array}
$$

The two functors indeed coincide. \hfill ◀

## Appendix to § 7

We first show that $\eta(10)$ is a natural transformation in Lemma 47 and that $\varepsilon(9)$ is a natural isomorphism in Lemma 48, and then that they satisfy the triangular equalities required by the definition of adjunction in Theorem 49.
Proof of Theorem 28

Lemma 47. There is a natural transformation
\[
\eta: \text{id}_{\text{EED}} \to \mathcal{R} \mathcal{L}
\]
whose \(P\)-th component, for \(P: \mathbb{C}^{\text{op}} \to \text{InfSL} \) an elementary existential doctrine, is \(\eta_P = (\Gamma_P, P\rho)\), where \((P\rho)_X = P_{\rho_X}: X \times I \to X\) the right unitor in \(\mathbb{C}\) (and \(I\) the terminal object).

Proof. Let us fix \(P: \mathbb{C}^{\text{op}} \to \text{InfSL}\) an elementary existential doctrine. Then \(\mathcal{L}(P) = \mathcal{A}_P\), whose objects are the objects of \(\mathbb{C}\), and hom-sets are \(\mathcal{A}_P(X, Y) = P(X \times Y)\). This means that
\[
\mathcal{R} \mathcal{L}(P) = \text{Hom}_{\mathcal{A}_P}(-, I) = P(- \times I): \text{Map}(\mathcal{A}_P) \to \text{InfSL}.
\]

To give a morphism \(\eta_P: P \to \mathcal{R} \mathcal{L}(P)\) in EED means therefore to give a functor \(F: \mathbb{C} \to \text{Map}(\mathcal{A}_P)\) and a natural transformation \(b: P \to P(F(-) \times I)\) satisfying certain conditions. We have proved in Lemma 40, Remark 41 and Lemma 42 that \(\Gamma_P\) is a functor whose image, in fact, is included in \(\text{Map}(\mathcal{A}_P)\) and, moreover, it is cartesian. Being the identity on objects, the natural transformation part of the definition of \(\eta_P\) must have components \(b_X: P(X) \to P(X \times I)\), therefore \(P\rho\) type-checks. That \(P\rho\) is natural is obvious, being the whiskering of a functor with a natural transformation; we need to check that it preserves the equalities and existential quantifiers of \(P\).

Regarding equalities, we have to show that \(P_{\rho_{A \times A}}(\delta^P_A) = \delta^{\mathcal{R} \mathcal{L}(P)}_A\). We have that
\[
\delta^{\mathcal{R} \mathcal{L}(P)}_A = \delta^{\mathcal{R} \mathcal{L}(P)}_{A^2 \times A \times I} \to A^2 \times I
\]
which is the composite, in \(\mathcal{A}_P\), of \(\Gamma_P(\Delta_A)^{\text{op}}\) and \(\Gamma_P(!: A \to I)\). Since \(\Gamma_P(!: A \to I) = \top_{P(A \times I)}\), we have:
\[
\delta^{\mathcal{R} \mathcal{L}(P)}_A = \exists_{A^2 \times A \times I \to A^2 \times I} \left( P_{A^2 \times A \times I \to A^2 \times I} \left( P_{\text{id}_{A^2} \times \Delta_A} (P_{\rho_{A^2}} (\delta^P_A)) \right) \right).
\]

Using the Beck-Chevalley condition of \(P\) on the following pullback:
\[
\begin{array}{ccc}
A^2 \times A \times I & \xrightarrow{(\pi_1, \pi_3)} & A^2 \times I \\
\rho_{A^2 \times A} & & \downarrow \rho_{A^2} \\
A^2 \times A & \xrightarrow{\pi_1} & A^2
\end{array}
\]
and recalling that \(\delta^P_{A \times A} = P_{(\pi_1, \pi_3)}(\delta^P_A) \land P_{(\pi_2, \pi_4)}(\delta^P_A)\), we obtain:
\[
\delta^{\mathcal{R} \mathcal{L}(P)}_A = P_{\rho_{A^2}} \left( \exists_{A^2 \times A \to A^2 \times A^2} \left( P_{\text{id}_{A^2} \times \Delta_A} (P_{\rho_{A^2}} (\delta^P_A)) \right) \right) \\
= P_{\rho_{A^2}} \left( \exists_{A^2 \times A \to A^2 \times A^2} \left( P_{(\pi_1, \pi_3)}: A^2 \to A^2 \left( \delta^P_A \right) \right) \right) \\
= P_{\rho_{A^2}} \left( \exists_{A^2 \times A \to A^2} \left( \exists_{A^2 \times \Delta_A} (\delta^P_A) \right) \right) \\
= P_{\rho_{A^2}} (\delta^P_A).
\]

Regarding the preservation of the existential quantifiers of \(P\), we need to check that the square (13) commutes, which means that, for \(r \in P(X \times A)\) and \(\pi: X \times A \to A\), we have to prove that
\[
P_{\rho_A}(\exists^P_{\pi}(r)) = \exists^{\mathcal{R} \mathcal{L}(P)}_{\pi(r)} (P_{\rho_{X \times A}}(r))
\]
On Doctrines and Cartesian Bicategories

(Where we will continue to write $\Gamma$ instead of $\Gamma_P$ since there is no possibility of confusion).

First we compute $\exists P_{\Gamma(z)} : \mathcal{S}_P(X \times A, I) \to \mathcal{S}_P(A, I)$; by definition, it maps $u : X \times A \to I$ morphism in $\mathcal{S}_P$ to the following composite in $\mathcal{S}_P$:

$$A \xrightarrow{\lambda_A^{-1}} I \times A \xrightarrow{f} X \times A \xrightarrow{u} I$$

where $f = \bullet \times \text{id}_A = \Gamma(\bullet \times \text{id}_A)^{op} \times \delta_A^P$ (product computed in $\mathcal{S}_P$) and $\lambda_A^{-1} = \Gamma(\lambda_A^{-1})$ (the first $\lambda$ being the unitor in $\mathcal{S}_P$, the second in $\mathcal{C}$). We begin by calculating the composition in $\mathcal{S}_P$ of $\Gamma(\lambda_A^{-1})$ with $f$. Since $\Gamma(\bullet \times \text{id}_A)^{op} = P_{\text{id}_I \times !_X}$ and $\text{id}_I \times !_X = !_I \times X : I \times X \to I \times I$, we have, by Corollary 38 that $\Gamma(\bullet \times \text{id}_A)^{op} = \top_{P(I \times X)}$. Therefore, if we annotate with subscripts different copies of the same object $A$ to indicate the various projections in a concise way:

$$f \circ \lambda_A^{-1} = \exists A_1A_2AXA_3 \to A_1A_3 \left( P_{A_1A_2AXA_3 \to A_1A_3} (P_{A_1A_2X \to A_1A_3} (\delta_{I \times A})) \right.$$

$$\land P_{A_1A_2AXA_3 \to A_2A_3} (\delta_A) \right).$$

Since, moreover,

$$\delta_{I \times A} = P_{I_AIA \to IA} (\delta_I) \land P_{I_AIA \to AA} (\delta_A) = \top \land P_{I_AIA \to AA} (\delta_A) = P_{I_AIA \to AA} (\delta_A),$$

we have

$$f \circ \lambda_A^{-1} = \exists A_1A_2AXA_3 \to A_1XA_3 \left( P_{A_1A_2AXA_3 \to A_1XA_3} (\delta_A) \land P_{A_1A_2XA \to A_2A_3} (\delta_A) \right)$$

$$= \exists A_1A_2AXA_3 \to A_1XA_3 (P_{A_1A_2AXA_3 \to A_1X} (P_{A_1A_2XA \to A_2} (\delta_A))$$

$$\land P_{A_1A_2AXA_3 \to A_2A_3} (\delta_A) \right)$$

$$= \exists A_1A_2AXA_3 \to A_1XA_3 (\exists A_1A_2X \to A_1A_2XA \left( P_{A_1A_2X \to A_1A_2} (\delta_A) \right)$$

$$= \exists A_1A_2X \to A_1A_2 \left( P_{A_1A_2X \to A_1A_2} (\delta_A) \right)$$

$$\land P_{A_1A_2X \to A_2A_3} (\delta_A) \right).$$

Composing the above with $P_{\rho_{X \times A}}(r)$, we obtain the right-hand side of (13), as follows:

$$\exists P_{\Gamma(z)} (P_{\rho_{X \times A}}(r)) = \exists A_{XAI} \to A_I \left( P_{A_{XAI} \to AA} (P_{XAI \to AA} (\delta_A)) \right.$$

$$\land P_{A_{XAI} \to XA} (P_{\rho_{X \times A}}(r)) \right)$$

$$= \exists A_{XAI} \to A_I (P_{A_{XAI} \to AA} (\delta_A) \land P_{A_{XAI} \to XA} (P_{\rho_{X \times A}}(r))$$

$$= \exists A_{XAI} \to A_I \left( P_{A_{XAI} \to XA} (P_{\rho_{X \times A}}(r)) \right.$$

$$\land P_{A_{XAI} \to AA} (\delta_A) \right)$$

$$= \exists A_{XAI} \to A_I \left( \exists XAI \to AXAI (P_{XAI \to XA} (r)) \right.$$

$$\right)$$

Therefore equation 13 reduces to

$$P_{\rho_A}(\exists P(r)) = \exists P_{XAI} \to A_I (P_{\rho_{X \times A}}(r))$$
which holds because the following square is a pullback in $\mathbb{C}$:

$$
\begin{array}{ccc}
X \times A \times I & \xrightarrow{\langle \pi_2, \pi_3 \rangle} & A \times I \\
\rho_{X \times A} \downarrow & & \downarrow \rho_A \\
X \times A & \xrightarrow{\pi} & A
\end{array}
$$

therefore we can use the Beck-Chevalley condition of $P$.

Finally, we prove that $\eta$ is a natural transformation. Let $R: \mathbb{D}^{\text{op}} \to \text{InfSL}$ and $(F, c): P \to R$ be a morphism in $\text{EED}$. Then

$$\mathcal{RL}(F, c) = \{\mathcal{L}(F, c)|_{\text{Map}(\mathscr{A}_P)}, b^\mathcal{L}(F, c)\}$$

where recall that

$$
\begin{array}{ccc}
\mathscr{A}_P & \xrightarrow{\mathcal{L}(F, c)} & \mathscr{A}_R \\
P(X \times Y) \xrightarrow{\delta_R} & & \xrightarrow{c_{X \times Y}(r)} \\
\mathcal{L}(F, c) & \xrightarrow{\delta_{\mathcal{L}(F, c)}} & R(FX \times I) \\
Y & \xrightarrow{\mathcal{L}(F, c)(u)} & c_{X \times I}(u)
\end{array}
$$

We then need to prove that

$$
\begin{array}{ccc}
P & \xrightarrow{\eta_P} & \mathcal{RL}(P) \\
(F, c) \downarrow & & \downarrow \mathcal{RL}(F, c) \\
R & \xrightarrow{\eta_R} & \mathcal{RL}(R)
\end{array}
$$

commutes in $\text{EED}$. In the first component, we are asking for the following square of functors to commutate:

$$
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{\Gamma_P} & \text{Map}(\mathscr{A}_P) \\
\mathbb{D} & \xrightarrow{\Gamma_R} & \text{Map}(\mathscr{A}_R) \\
F & \downarrow \mathcal{L}(F, c)|_{\text{Map}(\mathscr{A}_P)} & \downarrow \mathcal{L}(F, c)|_{\text{Map}(\mathscr{A}_R)}
\end{array}
$$

The upper and lower legs send a morphism $f: X \to Y$ in $\mathbb{C}$ to

$$
c_{X \times Y}(P_{f \times \text{id}_Y}(\delta^P_Y)) \quad \text{and} \quad R_{F_{f \times \text{id}_Y}}(\delta^{R}_{FYX})
$$

respectively, and those expressions are the same map in $\mathscr{A}_R$ due to the naturality condition of $c: P \to RF^{\text{op}}$ applied to the morphism $f \times \text{id}_Y: X \times Y \to Y \times Y$, which states that

$$
\begin{array}{ccc}
P(X \times Y) & \xrightarrow{c_{X \times Y}} & RF(X \times Y) \\
P_{f \times \text{id}_Y} \downarrow & & \downarrow RF_{f \times \text{id}_Y} = RF_{F_{f \times \text{id}_Y}} \\
P(Y \times Y) & \xrightarrow{c_{Y \times Y}} & RF(Y \times Y)
\end{array}
$$

commutes, hence, given that $\delta^P_Y \in P(Y \times Y)$, we have

$$
c_{X \times Y}(P_{f \times \text{id}_Y}(\delta^P_Y)) = RF_{F_{f \times \text{id}_Y}}(c_{Y \times Y}(\delta^P_Y)) = RF_{F_{f \times \text{id}_Y}}(\delta^{R}_{FYX}).$$
In the second component, (14) becomes
\[
P \xrightarrow{Pρ} P(− × I) \quad \xrightarrow{c} \quad R \xrightarrow{Rρ} R(− × I)
\]
which, on the \(X\)-th component, is a square in \(\text{InfSL}\):
\[
P(X) \xrightarrow{P(ρX)} P(X × I) \quad \xrightarrow{cX} \quad RF(X) \xrightarrow{RF(ρFX)} R(FX × I)
\]
(Notice that \(ρFX = F(ρX)\) because \(F\) is strict cartesian.) This square does indeed commute due to the naturality of \(c\) applied to the morphism \(ρX : X \to X × I\) in \(C^\text{op}\).

It is convenient to analyse the category \(LR(\mathcal{B})\) for a cartesian bicategory \(\mathcal{B}\). Recall that \(R(\mathcal{B}) = \text{Hom}_\mathcal{B}(−, I)\) and that, for \(f : X \to Y\) in \(\text{Map}(\mathcal{B})\) and \(u : Y \to I\) morphism in \(\mathcal{B}\), \(\mathcal{R}(\mathcal{B})f(u) = u \circ f : X \to I\). By definition then, \(LR(\mathcal{B}) = \mathcal{R}(\mathcal{B})\) has the same objects of \(\mathcal{B}\) while a morphism \(X \to Y\) in \(LR(\mathcal{B})\) is an element of \(\mathcal{B}(X × Y, I)\).

- **Composition.** If \(R : X × Y \to I\) and \(S : Y × Z \to I\) are morphisms of \(\mathcal{B}\), then
  \[
  S \circ_L R = \exists_{LR(\mathcal{B})} XYZ \to XZ \left( \mathcal{R}(\mathcal{B})_{XYZ} \to XY(R) \land \mathcal{R}(\mathcal{B})_{XYZ} \to YZ(S) \right)
  \]
  \[
  = \exists_{LR(\mathcal{B})} XYZ \to XZ \left( \begin{array}{c}
  x \\
  y \\
  z
  \end{array} \right) \begin{array}{c}
  R \\
  S
  \end{array}
  \]
  \[
  = \begin{array}{c}
  X \\
  Y \\
  Z
  \end{array} \begin{array}{c}
  R \\
  S
  \end{array}
  \]
  (where the above projections appearing in the subscripts are projections in \(\text{Map}(\mathcal{B})\), which are given by \(\left\downarrow \right\) and the unitors).

- **Monoidal product.** Take \(R : X × X' \to I\) and \(S : Y × Y' \to I\) in \(\mathcal{B}\), which means that \(R : X \to X'\) and \(S : Y \to Y'\) in \(LR(\mathcal{B})\). Then
  \[
  R ⊗ S = \mathcal{R}(\mathcal{B})_{XX'Y'→XY}(R) \land \mathcal{R}(\mathcal{B})_{XX'Y'→YY'}(S)
  \]
  \[
  = \begin{array}{c}
  X \\
  X'
  \end{array} \begin{array}{c}
  R \\
  S
  \end{array}
  \]
  \[
  = \begin{array}{c}
  X \\
  X'
  \end{array} \begin{array}{c}
  R \\
  S
  \end{array}
  \]
  (15)
Lemma 48. There is a natural isomorphism

\[ \epsilon: \mathcal{LR} \rightarrow \text{id}_{\mathcal{CBC}} \]

whose \( \mathbb{B} \)-th component, for \( \mathbb{B} \) a cartesian bicategory, sends \( R \in \mathcal{LR}(\mathbb{B})(X, Y) = \mathbb{B}(X \times Y, I) \) to the composite

\[
X \xrightarrow{\epsilon_X^1} X \times I \xrightarrow{\hat{R}} I \times Y \xrightarrow{\lambda_Y} Y
\]

where \( \rho \) and \( \lambda \) are, respectively, the right and left unitors in \( \mathbb{B} \) and \( \hat{R} \) is:

\[
\begin{array}{ccc}
X & \xrightarrow{R} & Y \\
\downarrow & & \downarrow \\
X & & Y \\
\end{array}
\]

The inverse \( \epsilon^{-1}_B \) of \( \epsilon_B \) sends \( S: X \rightarrow Y \) in \( \mathbb{B} \) to

\[
\begin{array}{ccc}
X & \xrightarrow{S} & Y \\
\downarrow & & \downarrow \\
X & & Y \\
\end{array}
\]

Proof. The plan of the proof is as follows: first we show that \( \epsilon_B \) and \( \epsilon^{-1}_B \) are inverse of each other as functions on hom-sets, in the sense that for any \( R: X \times Y \rightarrow I \) and \( S: X \rightarrow Y \) in \( \mathbb{B} \),

\[
\epsilon_B^{-1}\epsilon_B(R) = R \quad \text{and} \quad \epsilon_B\epsilon^{-1}_B(S) = S.
\]

Next we observe that \( \epsilon^{-1}_B \) and \( \Gamma_R(\mathbb{B}) \) coincide on \( \text{Map}(\mathbb{B}) \).

This immediately implies that \( \epsilon^{-1}_B \) preserves symmetries and the comonoid structure on every object of \( \mathbb{B} \), while the Snake Lemma will imply that the same holds for \( \epsilon_B \). Then we prove that both \( \epsilon_B \) and its inverse are strict monoidal functors, thus morphisms in \( \mathcal{CBC} \).

Finally, we show naturality of \( \epsilon \).

First, from the snake equations (5) it immediately follows that for any \( R: X \times Y \rightarrow I \) and \( S: X \rightarrow Y \) in \( \mathbb{B} \),

\[
\epsilon_B^{-1}\epsilon_B(s \circ r) = \epsilon_B(s) \circ \epsilon_B(r).
\]

The same can be said for the comultiplication and the counit of every object \( X \) in \( \mathbb{B} \). Applying \( \epsilon_B \) to both sides of the above equation, one can see that the same holds for \( \epsilon_B \), therefore \( \epsilon^{-1}_B \) and \( \epsilon_B \) preserve symmetries and the comonoid structure of every object of \( \mathbb{B} \) and \( \mathcal{LR}(\mathbb{B}) \) respectively. Preservation of the ordering of the hom-sets is guaranteed by the fact that tensor and composition in \( \mathbb{B} \) are monotone operations.

We proceed by showing that \( \epsilon_B \) and \( \epsilon^{-1}_B \) are strict monoidal functors.

Given \( R: X \times Y \rightarrow I \) and \( S: Y \times Z \rightarrow I \) in \( \mathbb{B} \), from (15) we get that

\[
\epsilon_B(s \circ r) = \epsilon_B(s) \circ \epsilon_B(r).
\]
Preservation of the identity is a direct consequence of the snake equations (5). Therefore \( \varepsilon_B \) is a functor: we need to show that it is strict monoidal. Take now \( R: X \otimes X' \to I \) and \( S: Y \otimes Y' \to I \); by (15), we have
\[
\varepsilon_B(R \otimes S) = \varepsilon_B(f) \otimes \varepsilon_B(g)
\]
where the middle equality is due to the naturality of the unitors and the symmetry of \( B \), as well as the coherence for symmetric monoidal categories.

As per \( \varepsilon_B^{-1} \), from (15) and the snake equations (5) we get
\[
\varepsilon_B^{-1}(S) \circ \varepsilon_B^{-1}(R) = \varepsilon_B^{-1}(S \circ R)
\]
therefore \( \varepsilon_B^{-1} \) preserves composition; preservation of the identity is immediate from the definition of \( \varepsilon_B \). Regarding the monoidal product, from (16) and naturality of the symmetry of \( B \) we have:
\[
\varepsilon_B^{-1}(R \otimes S) = \varepsilon_B^{-1}(R) \otimes \varepsilon_B^{-1}(S).
\]
This proves that \( \varepsilon_B \) and \( \varepsilon_B^{-1} \) are inverse morphisms in \( \text{CBC} \).

All that is left to do is to prove the naturality of \( \varepsilon \). To that end, let \( F: \mathcal{A} \to \mathcal{B} \) be a morphism in \( \text{CBC} \): we show that the following square
\[
\begin{array}{ccc}
\mathcal{LR}(\mathcal{A}) & \overset{\varepsilon_B}{\longrightarrow} & \mathcal{A} \\
\mathcal{LR}(F) \downarrow & & \downarrow F \\
\mathcal{LR}(\mathcal{B}) & \overset{\varepsilon_B}{\longrightarrow} & \mathcal{B}
\end{array}
\]
commutes. We have that \( \mathcal{R}(F) = (F|_{\text{Map}(\mathcal{A})}, b^F) \) where for \( u: A \to I \) in \( \mathcal{A} \), \( b^F_A(u) = F(u) \). Therefore \( \mathcal{LR}(F): \mathcal{A}_{\mathcal{R}(\mathcal{A})} \to \mathcal{A}_{\mathcal{R}(\mathcal{B})} \) is the functor that maps \( R \in \mathcal{A}(X \times Y, I) \) into \( F(R) \in \mathcal{B}(FX \times FY, I) \). Hence the square above acts on \( R: X \times Y \to I \) in \( \mathcal{A} \) as follows:
\[
\begin{array}{ccc}
\mathcal{LR}(x) \otimes \mathcal{LR}(y) & \overset{\varepsilon_B}{\longrightarrow} & \mathcal{A} \\
\mathcal{LR}(F) \downarrow & & \downarrow F \\
\mathcal{LR}(F) \otimes \mathcal{LR}(F) & \overset{\varepsilon_B}{\longrightarrow} & \mathcal{B}
\end{array}
\]
where the right vertical arrow is correct because \( F \) preserves all the structure of cartesian bicategory. Thus \( \varepsilon \) is natural.
\textbf{Theorem 49.} $\mathcal{L}$ is left adjoint to $\mathcal{R}$.

\textbf{Proof.} All that is left to do is to show that the triangular equalities are satisfied. The first one requires that, for $P : \mathcal{C}^{\text{op}} \to \mathsf{InfSL}$ an elementary existential doctrine, the following commutes:

\[
\begin{array}{c}
\mathcal{L}(P) \\
\mathcal{L}(P) \xrightarrow{\eta_P} \\
\mathcal{L}(P) \xrightarrow{\epsilon_{\mathcal{L}(P)} \circ \mathcal{L}(\eta_P)} \\
\mathcal{L}(P) \xrightarrow{id_{\mathcal{L}(P)}} \\
\mathcal{L}(P)
\end{array}
\]

(18)

Recall that $\mathcal{L} \mathcal{R} \mathcal{L}(P) = \mathcal{R} \mathcal{L}(P)$, therefore its objects are those of the domain category of the doctrine

$\mathcal{R} \mathcal{L}(P) = \mathcal{R}(\mathcal{A}_P) = \text{Hom}_{\mathcal{A}_P}(-, I) : \text{Map}(\mathcal{A}_P)^{\text{op}} \to \mathsf{InfSL}$

while its homsets are

$\mathcal{L} \mathcal{R} \mathcal{L}(P)(X, Y) = \mathcal{L} \mathcal{R}(P)(X \times Y) = \text{Hom}_{\mathcal{A}_P}(X \times Y, I) = P(X \times Y \times I)$.

By definition of $\mathcal{L}$ and $\epsilon$, we have

$\mathcal{A}_P \xrightarrow{\mathcal{L}(\eta_P)} \mathcal{A}_P \mathcal{R} \mathcal{L}(P) \xrightarrow{\mathcal{L}(\epsilon_P)} \mathcal{A}_P$

$X \quad \xrightarrow{P(X \times Y) \supseteq R} \quad X \quad \xrightarrow{P_{X \times Y} \delta_1 \in P(X \times Y \times I) \supseteq Q} \quad X \quad \xrightarrow{\Gamma_P(\rho_X^{-1}) \circ \mathcal{L}(\eta_P) \circ \mathcal{L}(\epsilon_P) \circ \mathcal{L}(\delta_1)} \quad X$

where $Q$ is given as in (17). Therefore $\mathcal{L}(\epsilon_P) \circ \mathcal{L}(\eta_P)$ is the identity on objects and its action on $R \in P(X \times Y)$ is

$\Gamma_P \left( \begin{array}{c}
X \\
P_{X \times Y}(R) \\
\mathcal{L}(\epsilon_P) \circ \mathcal{L}(\eta_P)
\end{array} \right) = : (P_{X \times Y}(R) \circ \delta_1) ; \Gamma_P(\lambda_Y)$

(19)

where the above composite is performed in $\mathcal{A}_P$.

In the following calculations, we will have to refer to various instances of the two objects $X$ and $Y$, and we will do so as annotated in the following string diagram in $\mathcal{A}_P$, showing $\mathcal{L}(\epsilon_P) \circ \mathcal{L}(\eta_P)(R)$:

\[
\begin{array}{c}
\alpha \\
X \\
P_{X \times Y}(R) \\
\mathcal{L}(\epsilon_P) \circ \mathcal{L}(\eta_P) \\
Y \quad Y \\
Y \quad Y
\end{array}
\]

In this sense, the first morphism in the chain goes from $X_1$ to $X_2 \otimes Y_1 \otimes Y_2$ (as a morphism in $\mathcal{A}_P$), the middle one goes from $X_2 \otimes Y_1 \otimes Y_2$ to $I \otimes Y_3$, whereas $\Gamma_P(\lambda_Y)$ goes from $I \otimes Y_3$ to $Y_4$. (Again, the use of subscripts is only a notational convenience to clarify which projections we are taking into consideration in the following computations; as objects of $\mathcal{C}$, we have $X = X_1 = X_2$ and $Y = Y_1 = Y_2 = Y_3 = Y_4$.)

We begin by computing the middle morphism:

\[
P_{X \times Y}(R) \otimes \delta_1 = P_{X, Y, Y_5 \rightarrow X_2, Y_1}(P_{X \times Y}(R)) \land P_{X, Y_2, Y_5 \rightarrow Y_2, Y_3}(\delta_1) = P_{X, Y_2, Y_5 \rightarrow X_2, Y_1}(R) \land P_{X, Y_2, Y_5 \rightarrow Y_2, Y_3}(\delta_1)
\]
therefore we have that $(P_{\mu \times \nu} (R) \otimes \delta_Y) ; \Gamma P(\lambda_Y)$ is equal to:

$$
\exists X_1Y_1Y_2Y_3Y_4 \rightarrow X_2Y_1Y_2Y_3 \left[ P_{X_1Y_1Y_2Y_3Y_4 \rightarrow X_2Y_1Y_2Y_3} (P_{X_1Y_1Y_2Y_3Y_4 \rightarrow X_2Y_1Y_2Y_3} (R) \wedge P_{X_1Y_1Y_2Y_3Y_4 \rightarrow X_2Y_1Y_2Y_3} (\delta_Y)) \right. \\
\left. \wedge P_{X_1Y_1Y_2Y_3Y_4 \rightarrow X_2Y_1Y_2Y_3} (P_{\lambda_Y \times id_Y} (\delta_Y)) \right] \\
= \exists X_2Y_1Y_2Y_3Y_4 \rightarrow X_2Y_1Y_2Y_3 \left[ P_{X_2Y_1Y_2Y_3Y_4 \rightarrow X_2Y_1Y_2Y_3} (P_{X_2Y_1Y_2Y_3Y_4 \rightarrow X_2Y_1Y_2Y_3} (R) \wedge P_{X_2Y_1Y_2Y_3Y_4 \rightarrow X_2Y_1Y_2Y_3} (\delta_Y)) \right] \\
\wedge \exists X_2Y_1Y_2Y_3Y_4 \rightarrow X_2Y_1Y_2Y_3 \left[ P_{X_2Y_1Y_2Y_3Y_4 \rightarrow X_2Y_1Y_2Y_3} (P_{X_2Y_1Y_2Y_3Y_4 \rightarrow X_2Y_1Y_2Y_3} (R)) \wedge P_{X_2Y_1Y_2Y_3Y_4 \rightarrow X_2Y_1Y_2Y_3} (\delta_Y) \right] \\
= \exists X_2Y_1Y_2Y_3Y_4 \rightarrow X_2Y_1Y_2Y_3 \left[ P_{X_2Y_1Y_2Y_3Y_4 \rightarrow X_2Y_1Y_2Y_3} (P_{X_2Y_1Y_2Y_3Y_4 \rightarrow X_2Y_1Y_2Y_3} (R)) \wedge P_{X_2Y_1Y_2Y_3Y_4 \rightarrow X_2Y_1Y_2Y_3} (\delta_Y) \right] \\
\wedge \exists X_2Y_1Y_2Y_3Y_4 \rightarrow X_2Y_1Y_2Y_3 \left[ P_{X_2Y_1Y_2Y_3Y_4 \rightarrow X_2Y_1Y_2Y_3} (P_{X_2Y_1Y_2Y_3Y_4 \rightarrow X_2Y_1Y_2Y_3} (R)) \wedge P_{X_2Y_1Y_2Y_3Y_4 \rightarrow X_2Y_1Y_2Y_3} (\delta_Y) \right] \\
= P_{X_2Y_1Y_2Y_3Y_4 \rightarrow X_2Y_1Y_2Y_3} (R) \\
\wedge \exists X_2Y_1Y_2Y_3Y_4 \rightarrow X_2Y_1Y_2Y_3 \left[ P_{X_2Y_1Y_2Y_3Y_4 \rightarrow X_2Y_1Y_2Y_3} (P_{X_2Y_1Y_2Y_3Y_4 \rightarrow X_2Y_1Y_2Y_3} (R)) \wedge P_{X_2Y_1Y_2Y_3Y_4 \rightarrow X_2Y_1Y_2Y_3} (\delta_Y) \right] \\
= P_{X_2Y_1Y_2Y_3Y_4 \rightarrow X_2Y_1Y_2Y_3} (R) \wedge \exists X_2Y_1Y_2Y_3Y_4 \rightarrow X_2Y_1Y_2Y_3 \left[ P_{X_2Y_1Y_2Y_3Y_4 \rightarrow X_2Y_1Y_2Y_3} (P_{X_2Y_1Y_2Y_3Y_4 \rightarrow X_2Y_1Y_2Y_3} (R)) \wedge P_{X_2Y_1Y_2Y_3Y_4 \rightarrow X_2Y_1Y_2Y_3} (\delta_Y) \right] \\
= P_{X_2Y_1Y_2Y_3Y_4 \rightarrow X_2Y_1Y_2Y_3} (R) \wedge P_{X_2Y_1Y_2Y_3Y_4 \rightarrow X_2Y_1Y_2Y_3} (\delta_Y). \\
$$

The last equation above is due to the fact that the morphism $e = X_2Y_1Y_2Y_3 \rightarrow X_2Y_1Y_2Y_3$ is an isomorphism and therefore the left adjoint of $P_e$, which is $\exists e$, is actually $P_{e^{-1}}$. Hence, the composite $\left[ {19} \right]$ is equal to:

$$
\exists X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3 \\
\left. \exists X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3 \left[ P_{X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3} (\delta_X) \wedge P_{X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3} (R) \right. \\
\left. \wedge P_{X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3} (\delta_Y) \wedge P_{X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3} (\delta_Y) \right] \right. \\
= \exists X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3 \\
\left. \exists X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3 \left[ P_{X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3} (P_{X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3} (R)) \wedge P_{X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3} (\delta_Y) \right. \\
\left. \wedge P_{X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3} (\delta_Y) \right] \right. \\
= \exists X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3 \\
\left. \exists X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3 \left[ P_{X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3} (P_{X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3} (R)) \wedge P_{X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3} (\delta_Y) \right. \\
\left. \wedge P_{X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3} (\delta_Y) \right] \right. \\
\left. \wedge \exists X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3 \left[ P_{X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3} (P_{X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3} (R)) \wedge P_{X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3} (\delta_Y) \right. \\
\left. \wedge P_{X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3} (\delta_Y) \right] \right. \\
= \exists X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3 \\
\left. \exists X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3 \left[ P_{X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3} (P_{X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3} (R)) \wedge P_{X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3} (\delta_Y) \right. \\
\left. \wedge P_{X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3} (\delta_Y) \right] \right. \\
\left. \wedge \exists X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3 \left[ P_{X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3} (P_{X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3} (R)) \wedge P_{X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3} (\delta_Y) \right. \\
\left. \wedge P_{X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3} (\delta_Y) \right] \right. \\
= P_{X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3} (R) \wedge P_{X_1Y_1Y_2Y_3Y_4 \rightarrow X_1Y_1Y_2Y_3} (\delta_Y). \\
$$
Thus triangle (20) commutes.

The second triangular equality requires that, for \( \mathcal{B} \) a cartesian bicategory, the following diagram commute:

\[
\begin{array}{ccc}
\mathcal{R}(\mathcal{B}) & \xrightarrow{\eta_{\mathcal{R}(\mathcal{B})}} & \mathcal{L}\mathcal{R}(\mathcal{B}) \\
\downarrow{\text{id}_{\mathcal{R}(\mathcal{B})}} & & \downarrow{\mathcal{R}(\varepsilon_{\mathcal{B}})} \\
\mathcal{R}(\mathcal{B}) & & \\
\end{array}
\]

(20)

Recall that \( \eta_{\mathcal{R}(\mathcal{B})} = (\Gamma_{\mathcal{R}(\mathcal{B})}, \mathcal{R}(\mathcal{B})\rho) \), where \( \Gamma_{\mathcal{R}(\mathcal{B})} : \text{Map}(\mathcal{B}) \to \mathcal{L}\mathcal{R}(\mathcal{B}) \) coincides with the restriction of \( \varepsilon_{\mathcal{B}}^{-1} \) (see (17)) to \( \text{Map}(\mathcal{B}) \) as shown in the proof of Lemma 48 while

\[\mathcal{R}(\mathcal{B})\rho_X = \text{Hom}_\mathcal{B}(-, I)(\rho_X) = - \circ \rho_X : \text{Hom}_\mathcal{B}(X, I) \to \text{Hom}_\mathcal{B}(X \otimes I, I)\]
as \( \rho_X : X \otimes I \to I \) is the right unitor of \( \mathcal{B} \).

Now,

\[\mathcal{R}\mathcal{L}\mathcal{R}(\mathcal{B}) = \text{Hom}_{\mathcal{L}\mathcal{R}(\mathcal{B})}(-, I) = \text{Hom}_\mathcal{B}(- \otimes I, I) : \text{Map}(\mathcal{L}\mathcal{R}(\mathcal{B})) \to \text{InfSL}\]

and

\[\mathcal{R}(\varepsilon_{\mathcal{B}}) = (\varepsilon_{\mathcal{B}}|_{\text{Map}(\mathcal{L}\mathcal{R}(\mathcal{B}))}, b^{(2)\mathcal{B}})\]

where by (7) we have:

\[\text{Hom}_\mathcal{B}(X \otimes I, I) \xrightarrow{b^{(2)\mathcal{B}}_R} \text{Hom}_\mathcal{B}(X, I)\]

\[R \xrightarrow{\varepsilon_{\mathcal{B}}(R)} \varepsilon_{\mathcal{B}}(R)\]

and it is easy to see that, for \( R : X \otimes I \to I \) in \( \mathcal{B} \), \( \varepsilon_{\mathcal{B}}(R) = R \circ \rho_X^{-1} \).

Composition in EED is component-wise: on the first component of \( \eta_{\mathcal{R}(\mathcal{B})} : \mathcal{R}(\mathcal{B}) \) we have

\[\text{Map}(\mathcal{B}) \xrightarrow{\Gamma_{\mathcal{R}(\mathcal{B})}} \text{Map}(\mathcal{L}\mathcal{R}(\mathcal{B})) \xrightarrow{\varepsilon_{\mathcal{B}}^{-1}|_{\text{Map}(\mathcal{L}\mathcal{R}(\mathcal{B}))}} \text{Map}(\mathcal{B})\]

\[f \xrightarrow{\varepsilon_{\mathcal{B}}^{-1}(f)} \varepsilon_{\mathcal{B}}^{-1}(f) = f\]

while on the second component we have

\[\text{Hom}_\mathcal{B}(X, I) \xrightarrow{(\mathcal{R}(\mathcal{B})\rho)_X} \text{Hom}_\mathcal{B}(X \otimes I, I) \xrightarrow{b^{(2)\mathcal{B}}_R} \text{Hom}_\mathcal{B}(X, I)\]

\[R \xrightarrow{R \circ \rho_X} R \circ \rho_X \circ \rho_X^{-1} = R\]

Thus triangle (20) commutes.
D  Appendix to §8

We first characterise the maps of $\mathcal{A}_P$ in Proposition 50, which we need to prove that the Rule of Unique Choice is equivalent to the fullness of the graph functor in Proposition 51. The fact that a doctrine $P$ has comprehensive diagonals if and only if $\Gamma_P$ is faithful is stated in [24, Theorem 3.2]. Finally, we give a proof of Theorem 35.

Proposition 50 (Cf. [24]). Let $P : \mathbb{C}^{\text{op}} \to \text{InfSL}$ be an elementary existential doctrine, $R \in P(X \times Y)$. Then $R$ is a map in $\mathcal{A}_P$ if and only if

\begin{align}
\exists_{\pi_1} : X \to X(R) &= \top_{PX} \\
\Pi_{(\pi_1, \pi_2)} : XYY \to XY(R) \land \Pi_{(\pi_1, \pi_3)} : XYY \to XY(R) &\leq \Pi_{(\pi_2, \pi_3)} : XYY \to YY(\delta_Y).
\end{align}

Proof. Equation (21) stems from calculating the two sides of

\[ \frac{\pi_1}{\delta_Y} \]

while (22) is equivalent to

\[ \Pi_{XY_1Y_2 \to XY_1} (R) \land \Pi_{XY_1Y_2 \to Y_1Y_2} (\delta_Y) = \Pi_{XY_1Y_2 \to XY_1} (R) \land \Pi_{XY_1Y_2 \to XY_2} (R) \]

which is the result of computing the two sides of the equation

\[ \left[ \begin{array}{c}
\pi_1 \\
\delta_Y
\end{array} \right] \]

in the category $\mathcal{A}_P$. ▶

Proposition 51. An elementary existential doctrine $P$ satisfies the Rule of Unique Choice if and only if its graph functor $\Gamma_P$ is full.

Proof. We show that for $P : \mathbb{C}^{\text{op}} \to \text{InfSL}$, $R \in P(X \times Y)$ map in $\mathcal{A}_P$ and $h : X \to Y$ in $\mathbb{C}$:

\[ \top_{PX} \leq \Pi_{(\text{id}_X, h)} (R) \iff R = \Pi_{h \times \text{id}_Y} (\delta_Y). \]

First, if $R = \Pi_{h \times \text{id}_Y} (\delta_Y)$, then

\[ \Pi_{(\text{id}_X, h)} = \Pi{(\text{id}_X, h)} \Pi_{h \times \text{id}_Y} (\delta_Y) = \Pi_{h,h} (\delta_Y) = \Pi_{h} \Delta_Y (\delta_Y) = \Pi_{h}(\top_{PY}) = \top_{PX} \]

because $\delta_Y \equiv \Delta_Y (\top_{PY})$ and, since $\Delta_Y$ is left adjoint to $\Delta_Y$, $\top_{PY} \leq \Delta_Y (\top_{PY})$. Vice versa, suppose that $\top_{PX} \leq \Pi_{(\text{id}_X, h)} (R)$. Then $\Pi_{XY \to X} (\top_{PX}) = \Pi_{XY \to X} \Pi_{(\text{id}_X, h)} (R)$.

Now, since the following two diagrams commute:

\begin{align*}
X \times Y &\xrightarrow{\pi_1} X & A \times B &\xrightarrow{h \times \text{id}_Y} X \times Y \\
X \times Y \times Y &\xrightarrow{(\pi_1, \pi_2)} X \times Y & X \times Y \times Y &\xrightarrow{(\pi_2, \pi_3)} X \times Y
\end{align*}

we have:

\[ R = R \land \top_{P(XY)} = R \land P_{XY \to X} \Pi_{(\text{id}_X, h)} (R) = \Pi_{(\text{id}_X, h) \times \text{id}_Y} (\Pi_{XY_1Y_2 \to XY_1} (R) \land \Pi_{XY_1Y_2 \to XY_2} (R)) \leq \Pi_{(\text{id}_X, h) \times \text{id}_Y} (\Pi_{XY_1Y_2 \to Y_1Y_2} (\delta_Y)) = \Pi_{h \times \text{id}_Y} (\delta_Y) \]
where the inequality above is due to (22). So we proved that $R \leq P_{h \times \text{id}_Y}(\delta_Y)$, while for the other inequality:

\[
P_{h \times \text{id}_Y}(\delta_Y) = P_{h \times \text{id}_Y}(\delta_Y) \land \top_{P(XY)}
\]

\[
= P_{h \times \text{id}_Y}(\delta_Y) \land P_{XY \to X}P_{(\text{id}_X, h)}(R)
\]

\[
= P_{(\text{id}_X, h) \times \text{id}_Y}(\exists_{\text{id}_X \times \delta_Y}(R))
\]

because of (1). Since $R = P_{\text{id}_X \times \delta_Y}P_{XY_1Y_2 \to XY_2}(R)$, by adjunction we have $\exists_{\text{id}_X \times \delta_Y}(R) \leq P_{XY_1Y_2 \to XY_2}(R)$, hence we get

\[
P_{h \times \text{id}_Y}(\delta_Y) \leq P_{(\text{id}_X, h) \times \text{id}_Y}P_{XY_1Y_2 \to XY_2}(R) = R.
\]

\[\blacktriangleright\]

**Proof of Theorem 35**

By restricting $L$ and corestricting $R$ to $\text{EED}$, one obtains two functors $\overline{L}$ and $\overline{R}$ and we can define two natural transformations $\overline{\eta}$ and $\overline{\varepsilon}$ whose individual components are the same as $\eta$ and $\varepsilon$ respectively. This means that $\overline{\eta}$ and $\overline{\varepsilon}$ are still natural transformations satisfying the triangular equalities (8), therefore $L \perp R$, with $\overline{\varepsilon}$ being a natural isomorphism.

Let $P : \mathcal{C} \to \text{InfSL}$ be in $\text{EED}$. Then $\Gamma_P$ is faithful because of [23, Theorem 3.2] and full because of Proposition 51. Hence $\overline{\eta}_P = (\Gamma_P, P\rho)$ has an inverse in $\text{EED}$, namely $(\Gamma_P^{-1}, P\rho^{-1})$, where $\Gamma_P^{-1}$ is the identity on objects and sends every map $R : X \to Y$ in $R\mathcal{L}(P)$ into the unique morphism $f : X \to Y$ of $\mathcal{C}$ such that $\Gamma_P(f) = R$. Therefore also $\overline{\eta}$ is a natural isomorphism, hence $(\overline{L}, \overline{R}, \overline{\eta}, \overline{\varepsilon})$ is an adjoint equivalence.  

\[\blacktriangleright\]