SEMILINEAR BOUNDARY PROBLEMS OF
COMPOSITION TYPE IN $L_p$-RELATED SPACES

JOHN JOHSEN
Institute of Mathematical Sciences, Mathematics Department;
Universitetsparken 5, DK-2100 Copenhagen O; Denmark.
E-mail: jjohnsen@math.ku.dk

THOMAS RUNST
Mathematical Institute, Friedrich–Schiller–Universität Jena;
Ernst–Abbe–Platz 1–4, D-07743 Jena; Germany.
E-mail: runst@minet.uni-jena.de

1. Introduction

We address the $L_p$-theory of semi-linear boundary problems of the form:

$$Au(x) + g(u(x)) = f(x) \quad \text{in} \quad \Omega,$$

$$Tu(x) = 0 \quad \text{on} \quad \Gamma := \partial \Omega.$$  \hspace{1cm} (1.1)

Here $\{A, T\}$ defines a linear elliptic problem (specified below), $g(t) \in C^\infty_b(\mathbb{R})$, and we seek solutions $u(x)$ with $s$ derivatives in $L_p(\Omega)$, roughly speaking.

The purpose is to study effects caused by the non-linearity $g(u)$, when one wants a maximal range of both $s$ and $p$. As a main result we describe and determine in Theorem 2.1 ff. below a certain borderline occurring for $s \in ]1, \frac{n}{p}\]$. To our knowledge neither the borderline nor the range $]1, \frac{n}{p}\]$ has been treated before.

$^1$partly supported by the Danish Natural Sciences Research Council, grant no. 11–1221–1 and no. 11–9030
$^2$partly supported by Deutsche Forschungsgemeinschaft, grant Tr 374/1-1.

Appeared in Communications in partial differential equations, 22 (1997), no.7--8, 1283--1324.
Moreover, for each $n \geq 6$ and fixed $p$ in $[1, \frac{n}{3+\sqrt{8}}]$ the $H^s_p$-theory is split into two parts by the borderline (loosely speaking $0 < s \lesssim 3$ and $s \gtrsim \frac{n}{p}$). In particular this is so for the $H^s$-theory when $n \geq 12$.

These phenomena actually occur in any dimension when $p$ is taken arbitrarily in $[0, \infty]$. Thus it is advantageous for the full understanding of (1.1) to use spaces with $p < 1$, and this we do in the framework of the Besov and Triebel–Lizorkin spaces, $B^s_{p,q}$ and $F^s_{p,q}$.

In this context we treat the existence and regularity of solutions, with Landesman–Lazer conditions for the self-adjoint case.

Our methods combine two general investigations in $B^s_{p,q}$ and $F^s_{p,q}$ spaces: (i) Boutet de Monvel’s pseudo-differential calculus of linear boundary problems, which gives the framework for $\{A,T\}$, with [Job96] by the first author as source (extending works of Grubb and Franke [Gru90, Fra86]); and (ii) estimates of composition operators $u \mapsto g(u)$ in works of Sickel and the second author [Run86, RS96, Sic89].

The borderline phenomena occur although we assume that $g(t)$ is real-valued with bounded derivatives of any order, i.e.

$$g(t) \in C^\infty_b(\mathbb{R}).$$

(1.2)

Such non-linearities constitute only a narrow class, but on one hand new insight can be obtained even for these, and on the other hand our methods do not allow us to go further since a full set of composition estimates have not yet been established for wider classes.

As motivated above we treat solutions $u(x)$ in the Besov and Triebel–Lizorkin spaces, $B^s_{p,q}$ and $F^s_{p,q}$, with $s \in \mathbb{R}$ and $p$ and $q$ in $[0, \infty]$; throughout with $p < \infty$ for $F^s_{p,q}$, however. Both $u(x)$ and $f(x)$ are assumed real-valued.

Recall that e.g. Hölder–Zygmund spaces $C^s = B^s_{\infty,\infty}$ ($s > 0$), Sobolev–Slobodetskii spaces $W^s_p = B^s_{p,p}$ ($s \in \mathbb{R} \setminus \mathbb{N}$, $1 < p < \infty$), Bessel potential spaces $H^s_p = F^s_{p,2}$ ($s \in \mathbb{R}$, $1 < p < \infty$) and local Hardy spaces $h_p = F^0_{p,2}$ ($0 < p < \infty$), cf. [Tri83, Tri92], so that these are covered by our treatment.
In (1.1), $\Omega \subset \mathbb{R}^n$ is a bounded open set with $C^\infty$-smooth boundary $\Gamma$ for $n \geq 1$. $A = \sum_{|\alpha| \leq 2} a_\alpha(x)D^\alpha$ is an elliptic operator and the trace operator $T = S_0 \gamma_0 + S_1 \gamma_1$, where $\gamma_0 u = u|_\Gamma$ is restriction to the boundary while $\gamma_1 u = \gamma_0(\vec{n} \cdot \text{grad}(u))$ for a unit outward normal vectorfield, $\vec{n}$, near $\Gamma$. For simplicity $A$ is taken of order 2 and the boundary condition is homogeneous, so we only need to treat $A_T$, the $T$-realisation of $A$; for this reason $T$ is assumed to be right invertible (e.g. $T$ could be normal). Moreover, $A$ and $T$ have coefficients in $C^\infty(\Omega)$, and the $S_j$ are differential operators in $\Gamma$ of order $d - j$ for some $d < 2$. The class of $T$ is denoted by $r$; by definition $r = 1$ here if $S_1 \equiv 0$, and else $r = 2$.

Finally, $\{A, T\}$ is assumed elliptic in the Boutet de Monvel calculus [BdM71]; see (4.6)–(4.7) below.

**Review.** Under the assumptions above we deduce three consequences for the non-linear problem (1.1):

(i) (Theorem 2.1) For $(s, p, q)$ belonging to a domain $\mathbb{D}(A_T + g(\cdot))$, specified below, the condition $Tv = 0$ makes sense and $v \mapsto g(v)$ has order strictly less than 2 when $v(x)$ in $B^{s}_{p,q}$ or $F^{s}_{p,q}$.

In particular $g(\cdot)$ is better behaved than $A_T$ on $B^{s}_{p,q}$ and $F^{s}_{p,q}$ whenever $(s, p, q) \in \mathbb{D}(A_T + g(\cdot))$. Because the range $1 < s < \frac{n}{p}$ is included, the transformation $(s, p, q) \mapsto (s, \frac{p}{p}, \frac{q}{q})$ will for $n \geq 2$ take $\mathbb{D}(A_T + g(\cdot))$ into a non-convex subset of $\mathbb{R}^3$.

(ii) (Theorem 2.2) Given a solution $u(x)$ in $B^{s_1}_{p_1,q_1}$ for data $f(x)$ in $B^{s_0 - 2}_{p_0,q_0}$, where $(s_j, p_j, q_j) \in \mathbb{D}(A_T + g(\cdot))$ for both $j = 0$ and 1, then $u(x)$ also belongs to $B^{s_0}_{p_0,q_0}$, as in the linear case, and similarly in the $F$-case.

Using that $A_T$ has a parametrix in the pseudo-differential calculus, this follows from a bootstrap argument with varying integral exponents; even for $p_0 = p_1$ the $p$’s cannot in general for $n \geq 4$ be kept fixed because $\mathbb{D}(A_T + g(\cdot))$ is not convex.
(iii) (Theorem 2.3.) For \((s, p, q)\) in \(\mathbb{D}(A_T + g(\cdot))\) and \(f(x)\) in \(B^{s-2}_{p,q}\) there exists a solution \(u(x)\) in \(B^s_{p,q}\), and similarly for the \(F^s_{p,q}\) scale. This is proved by means of the Leray–Schauder theorem when \(A_T\) is invertible, as well as when \(A_T\) is self-adjoint and \(f(x)\) satisfies generalised Landesman–Lazer conditions, cf. [RL95].

The proof is standard for \(s < 2\), for then the embedding, say, \(L_\infty \hookrightarrow B^{s-2}_{p,q}\) shows that \(\|g(u)\|_{B^{s-2}_{p,q}}\) is estimated independently of \(u\) by \(\|g\|_{L_\infty}\). For larger \(s\) such a procedure seems impossible, but we consider \(f(x)\) as an element of some \(X \supset L_\infty\) to which the result for \(s < 2\) applies; the inverse regularity result in (ii) yields that the found solution belongs to \(B^s_{p,q}\) or \(F^s_{p,q}\) as required.

Throughout the set \(\mathbb{D}(A_T + g(\cdot))\) is termed the parameter domain of the operator \(A_T + g(\cdot)\), cf. Figure 1. In addition to (i) above, for \(T\) of class 2 we characterise the largest possible parameter domain (except for the borderline cases, which are undiscussed here).

**Example 1.1** (General data). When \(\Omega\) is connected in \(\mathbb{R}^n\) for \(n \geq 2\) and \(0 \in \Omega\), we get the following:

(a) For \(r = 1\), take any \(A_T + g(\cdot)\), say \(-\Delta_{\gamma_0} u + (1 + u^2)^{-1}\). With \(x = (x', x_n)\), let \(f\) be the restriction to \(\Omega\) of one the distributions

\[1(x') \otimes \text{pv}(\frac{1}{x_n}), \quad 1(x') \otimes \delta_0(x_n);\]

then \(f\) is in \(B^{\frac{1}{p}-1}_{p,\infty}(\overline{\Omega})\) for \(p \in ]1, \infty]\), cf. Example 2.9. By Theorem 2.3 there is, whenever \(1 < p \leq \infty\), a solution \(v_0(x)\) lying in \(B^{\frac{1}{p}+1}_{p,\infty}(\overline{\Omega})\).

(b) \(r = 2\). When \(A_T = -\Delta_{\gamma_1}\) and \(g(t) = \frac{\pi}{2} + \arctan t\), then,

\[f(x) = \chi(x)|x|^\alpha \in B^{\frac{n}{p}+\alpha}_{p,\infty} \quad \text{for each} \quad p \in ]0, \infty],\]

when \(-n < \alpha < 0\) and \(\chi\) is a cut-off function with \(\chi(0) = 1\), cf. [RS96]. Here each \(\alpha \geq -2\) yields \(\frac{n}{p} + 2 + \alpha > \frac{n}{p}\) and hence \((\frac{n}{p} + 2 + \alpha, p, \infty) \in \mathbb{D}(-\Delta_{\gamma_1} + g(\cdot))\) if \(p\) satisfies \(\frac{n-1}{p} + 1 + \alpha > 0\), and for \(\chi\) such that \(\int_\Omega f < \pi\)
there is then a solution $v_1(x)$ in $B^{\frac{\alpha+2}{p}}_{p,\infty}(\Omega)$ according to Theorem 2.3 (Even $-n < \alpha < -2$ may be treated for $p$ in a smaller interval.)

However, when $-2 < \alpha \leq -1$ the function $f$ in (1.4) is not in $B^t_{\infty,\infty}$ for $t > -1$, so the existence of $v_1$ is not provided by [FR88, RR96].

**Example 1.2** (Optimal regularity). By Theorem 2.2 each $v_0$ in (a) of Example 1.1 also belongs to $B^{r+1}_{r,\infty}(\Omega)$ for every $r \in [1, \infty]$.

That $v_1$ exists in $H^2$ is known for $-2 < \alpha \leq -1$ when $n > -2\alpha$, for $f$ is in $L_2$ in such dimensions. However, that $v_1$ is in $B^{\frac{\alpha+2}{p}}_{p,\infty}$ is a stronger fact provided by Example 1.1. For $n \geq 6$ this even holds for the classical range $p \in [1, \frac{n}{3+\sqrt{8}}]$, so in particular, for $\alpha = -2$ and $n = 12$ we conclude that $v_1$ belongs to $H^{6-\varepsilon}$ for $\varepsilon > 0$.

The typical difficulties caused by the boundary of the parameter domain $\mathbb{D}(-\Delta_{\gamma_1} + g(\cdot))$ are illustrated in Figure 1 below; especially the dotted line indicates that one cannot just ‘go upwards’ to obtain, say, $v_1 \in H^{6-\varepsilon}$.

**Other works.** There are numerous articles on semi-linear problems, so we shall only compare results for the one specified in (1.1) ff., and thus leave out the more liberal assumptions found on e.g. $g$ in many papers.

Solutions for $s = 1$ or 2 and $p = 2$ have been treated by e.g. Landesman and Lazer [LL70], Ambrosetti and Mancini [AM78], Brézis and Nirenberg [BN78] and Robinson and Landesman [RL95], and for $p > 1$ by Amann, Ambrosetti and Mancini [AAM78] and Nečas [Nec83] whereas the $B^s_{p,q}$ and $F^s_{p,q}$ have been dealt with for $s > \frac{n}{p}$ in works of Franke, Runst and Robinson [FR88, RR96].

Spaces with $1 < s < \frac{n}{p}$ have not been treated systematically for (1.1) before, so the non-convexity and the borderline of $\mathbb{D}(A_T + g(\cdot))$ in this region should be novelties, together with its maximality when $T$ has class 2.

The crucial inverse regularity properties of $A_T + g(\cdot)$ in (ii) above do not as far as we know have any forerunners, not even under further assumptions on
the \((s, p, q)\)'s or on \(g(t)\). However, the simpler property that \(u(x)\) is in \(C^\infty\) when \(f(x)\) is so (hypoellipticity) was obtained in \(\text{[AAM78 AM78 BN78]}\).

In contrast to this the solvability of \(\text{[1.1]}\) has been treated extensively with some of the original applications of the Leray–Schauder theorem containing the case \(A_T = \Delta_{\gamma_0}\) \(\text{[LS34]}\). In general, when \(A_T\) is invertible, it was assumed in \(\text{[FR88 RR96]}\) that the data \(f\) given in \(B_{p,q}^{s-2}\) or \(F_{p,q}^{s-2}\) for \(s > \frac{n}{p}\) should also belong to \(B_{t,\infty}^{1}\) for some \(t > -1\) when \(T\) has class \(r = 2\). For \(p < \infty\) and \(s < \frac{n}{p} + 1\) this is a serious restriction, which is removed in our work.

For \(A_T\) self-adjoint, the Landesman–Lazer conditions appeared in \(\text{[LL70]}\) and was further investigated by Hess, Fučík and the abovementioned \(\text{[Hes74 AAM78 AM78 BN78 FH78]}\). Extensions to slowly decaying \(g\) was given in \(\text{[FK77 Hes77 Nec83]}\), and more general versions in \(\text{[RL95]}\); see \(\text{[RL95]}\) for more references and a survey on the development of solvability conditions, and in general also \(\text{[Run90 RR96]}\).

Here the generalised Landesman–Lazer conditions of \(\text{[RL95 RR96]}\) are extended to the \(B_{p,q}^s\) and \(F_{p,q}^s\) with \((s, p, q)\) running in the full \(\mathbb{D}(A_T + g(\cdot))\), including the range \(1 < s < \frac{n}{p}\); various other improvements in this extension are collected in Remarks 2.4–2.6 below.

**Contents.** 2. Main results and notation, 3. Composition estimates, 4. Proof of the regularity theorem, 5. The existence results and 6. Final remarks.

### 2. Main Results and Notation

In general the notation and the spaces are described in Sections 2.1–2.2 below, so we proceed to present the results.

For convenience, we shall first of all let \(E_{p,q}^s\) stand for a space which can be either \(B_{p,q}^s(\Omega)\) or \(F_{p,q}^s(\Omega)\). Hereby we avoid repetition when properties in the \(B_{p,q}^s\) spaces carry over verbatim to the \(F_{p,q}^s\) spaces (but \(p < \infty\) must be understood in the \(F_{p,q}^s\) case, of course).
Secondly, $A_T$ will denote the $T$-realisation of $A$. That is, for

$$
s > r + \max(\frac{1}{p} - 1, \frac{2}{p} - n),
$$

(2.1)

where $r = 1$ or $r = 2$ denotes the class of $T$, the operator $A_T$ acts like $A$ in the distribution sense and it is defined for those $u \in E^s_{p,q}$ that satisfy the boundary condition; hence

$$
A_T u = Au = \sum_{|\alpha| \leq 2} a_\alpha D^\alpha u,
$$

(2.2)

$$
D(A_T) = \{ u \in E^s_{p,q} \mid Tu = 0 \} =: E^{s}_{p,q;T}.
$$

(2.3)

For $(s, p, q) = (2, 2, 2)$ this is just the usual $H^2$-realisation (in $L^2$), cf. [Gru86, Def. 1.4.1].

Thirdly, the problem is then given by the operator equation

$$
A_T u + g(u) = f \quad \text{in} \quad E^{s-2}_{p,q},
$$

(2.4)

with $u(x)$ sought in $E^s_{p,q;T}$ for a parameter $(s, p, q)$ satisfying (2.1).

In our treatment of (2.4) we build on results for the solution operator for $A_T$ derived in Section 4.1.2 below from [Joh96], where the Boutet de Monvel calculus of pseudo-differential boundary operators is extended to the $B^s_{p,q}$ and $F^s_{p,q}$ spaces. See also [Joh93, Ch. 4] for this.

Another basic ingredient is the results for composition (or Nemytskii) operators $u(x) \mapsto g(u(x))$, written $g(\cdot)$ for short, that have been derived in [Sic89] and [Run86]; see also [Run85]. For an overview concerning the Bessel potential spaces see [Sic92], and for more results [RS96].

Once the function $g(t)$ is given, it is natural to ask for the parameters $(s, p, q)$ such that $T$ and $g(u)$ both make sense on $E^s_{p,q}$ and such that $g(\cdot)$ respects the continuity properties of $A$ on $E^s_{p,q}$; i.e. we could introduce

$$
\mathbb{D} = \{ (s, p, q) \mid T \text{ and } g(\cdot) \text{ are bounded from } E^s_{p,q},
$$

$$
\exists \varepsilon > 0 : g(E^s_{p,q}) \subset E^{s-2+\varepsilon}_{p,q} \},
$$

(2.5)
which would provide a domain of parameters for the non-linear operator $A_T + g(\cdot)$ in the sense that it goes from $E^s_{p,q,T} \subset E^s_{p,q}$ to $E^{s-2}_{p,q}$ for each $(s,p,q) \in \mathcal{D}$—through $\epsilon$, even with a good control over $g(\cdot)$.

However, our results only allow us to treat a slightly smaller set denoted $\mathbb{D}(A_T + g(\cdot))$ and characterised in the following:

**Theorem 2.1.** Let $(s,p,q)$ be an admissible parameter for which the following conditions are fulfilled:

(i) $s > r + \max\left(\frac{1}{p} - 1, \frac{p}{p} - n\right)$,

(ii) $s > \begin{cases} 0 & \text{for } 1 \leq p < \infty, \\ \frac{1}{p} + \max\left(-n, -\frac{n}{1-p}\right) & \text{for } p < 1; \end{cases}$

(iii) $s > \frac{1}{2}\left(\frac{n}{p} + 3 + \sqrt{\left(\frac{n}{p} - 3\right)^2 - 8}\right)$ or $s < \frac{1}{2}\left(\frac{n}{p} + 3 - \sqrt{\left(\frac{n}{p} - 3\right)^2 - 8}\right)$, if $\frac{n}{p} \geq 3 + \sqrt{8}$.

Then (i) and (ii)–(iii), respectively, assure that

\[ T: B^s_{p,q}(\Omega) \to B^{s-d-\frac{1}{p}}_{p,q}(\Gamma), \quad T: F^s_{p,q}(\Omega) \to B^{s-d-\frac{1}{p}}_{p,p}(\Gamma), \quad (2.6) \]

\[ g(\cdot): E^s_{p,q} \to F^s_{p,q} \quad (2.7) \]

are bounded for some $\sigma > s - 2$.

Moreover, in the $F^s_{p,q}$ case, (ii) alone implies that (2.7) holds for $q = \infty$ and $\sigma$ equal, for any $\varepsilon > 0$, to

\[ \sigma(s,p) = \begin{cases} s & \text{for } s > \frac{n}{p} \text{ or } 0 < s < 1, \\ s - \varepsilon & \text{for } s = \frac{n}{p} \text{ or } s = 1, \\ \frac{n}{p} - \varepsilon & \text{otherwise}. \end{cases} \quad (2.8) \]

For $E^s_{p,q}$ with $q \in [0, \infty]$ it is possible to take $\sigma = \sigma(s,p) - \varepsilon$, for any $\varepsilon > 0$.

When (i)–(iii) hold, we say that $(s,p,q)$ belongs to $\mathbb{D}(A_T + g(\cdot))$.

This theorem gives sufficient conditions for $g(\cdot)$ to be of a lower order than $A_T$, so it may be termed the Direct Regularity Theorem for (1.1).
In comparison with (2.5), we have excluded borderline cases with equality in (i) and values of $s$ between $\frac{n}{p} - n$ and $\frac{n}{p} - \frac{p}{1-p}$. The latter restriction is felt in a small set of $(s, p, q)$'s, for in (ii) it only applies for $p < 1$ and in this region $s > r + \frac{n}{p} - n$ is stronger to begin with (since $r = 1$ or 2) and afterwards the second requirement in (iii) quickly takes over, cf. Figure 1. The first part of (iii) is stronger than $s > \frac{n}{p} - \frac{p}{1-p}$, hence stronger than (ii).

Exceptions for $n = 1, 2, 3$ or $r = 2$ are given in Remarks 3.2–3.5 below.

It is expected, but not proved, that the function $\sigma(s, p)$ in (2.8) may be used in (2.7) also for $q < \infty$, and even then also in the Besov case.

Nevertheless the function $\sigma(s, p)$ gives the right understanding of the conditions (ii)–(iii) (the sum-exponents are less important because $E^s_{p,q} \hookrightarrow E^s_{p,r}$ for $q \leq r$). On the one hand, (ii) gives either $s > (\frac{n}{p} - n)_+$, so that $E^s_{p,q} \subset L^{1\text{loc}}_1$ and hence $g(\cdot)$ makes sense, or $s > \frac{n}{p} - \frac{p}{1-p}$, which may be seen to yield $E^{\sigma(s,p)}_{p,q} \subset L^{1\text{loc}}_1$. Perhaps the latter condition is only proof-technical; it is used to make sense of products $u \ldots u$ when estimating $g(u)$.

On the other hand, asking for the identity

$$\sigma(s, p) = s - 2,$$

or for the level curve for the value 2 of the loss-of-smoothness function $s - \sigma(s, p)$, one finds

$$(2s - \frac{n}{p} - 3)^2 = \left(\frac{n}{p} - 3\right)^2 - 8,$$

which leads to (iii) with $=$ instead of the inequalities for $s$.

In other words: condition (iii), or (2.10), determines a borderline to a region of spaces where the loss of smoothness equals or exceeds 2. Generally speaking this is correct, for if (iii) is violated by $E^s_{p,q}$ then $u \mapsto \sin(u)$, for example, cannot map into $E^{s-2+\varepsilon}_{p,q}$ for any $\varepsilon > 0$; cf. Remark 6.1 below.

The identity in (2.10) describes a hyperbola in the $(\frac{n}{p}, s)$-halfplane, that lies entirely in the area with $1 < s < \frac{n}{p}$. Hence (iii) is relevant only for the consideration of unbounded solutions in (2.4).
To present an overview, the spaces $E^{s}_{p,q,T}$ for which the perturbation $g(u)$ is studied in the present article are illustrated in Figure 1 (for simplicity only for $r = 1$). The sum-exponent $q$ is not represented in the diagram, but because of the sharp inequalities in Theorem 2.1 and the existence of simple embeddings, $q$ does not have any influence.

The lines with $s = 3$ and $s = \frac{n}{p}$ are the asymptotes of the hyperbola, and for all points on this level curve,

$$\frac{n}{p} \geq 3 + \sqrt{8}. \quad (2.11)$$
The interest of this is that for \( n \geq 12 \) even the theory within the classical \( H^s \) Sobolev spaces is affected by (iii) in Theorem 2.1. Actually \( s \) should be taken outside of an interval of length \( \sqrt{(\frac{n}{2} - 3)^2 - 8} \), which is at least 1 and \( O(\frac{n}{2}) \) for \( n \to \infty \). Moreover, for each \( n \geq 6 \) there are \( p > 1 \) fulfilling (2.11), so restrictions occur also in the \( W^s_p \) and \( H^s_p \) spaces for such dimensions.

In addition to the general pattern described above, see Section 3.3 below for the atypical cases with \( n = 1, n = 2 \) or \( r = 2 \).

At the moment it is not clear whether the condition \( s > \frac{n}{p} - \frac{p}{1-p} \) is necessary or not, but in any case it won’t change the fact that the sets \( \mathbb{D}(A_T + g(\cdot)) \) are non-convex, because already for \( g(u) = \sin(u) \) the condition (iii) is best possible. We believe that the specific form of the \( \mathbb{D} \)'s and in particular the non-convexity constitutes a novelty.

Because \( \sigma > s - 2 \) is possible in \( \mathbb{D}(A_T + g(\cdot)) \), the non-linear operator \( g(\cdot) \) also respects the inverse regularity properties of \( A_T \) on every \( E^{s,p,q;T} \) with parameter in \( \mathbb{D}(A_T + g(\cdot)) \):

**Theorem 2.2.** Let \( u(x) \) in \( E^{s_1,p_1,q_1;T} \) solve

\[
A_T u + g(u) = f
\]

(2.12)

for data \( f(x) \) in \( E^{s_0,-2,p_0,q_0;T} \) and suppose that

\[
(s_1,p_1,q_1),(s_0,p_0,q_0) \in \mathbb{D}(A_T + g(\cdot)).
\]

Then the solution \( u(x) \) also belongs to the space \( E^{s_0,p_0,q_0;T} \).

To prove this we use Theorem 2.1 for \( g(u) \) and results for the Boutet de Monvel calculus in [Joh96] for \( A_T \). These tools are combined into a bootstrap argument, but one has to ‘go around the corner’ inside \( \mathbb{D}(A_T + g(\cdot)) \), because of the non-convexity; cf. Figure 2 below.

It is interesting to observe that the set \( \mathbb{D}(A_T + g(\cdot)) \)—in contrast to Theorem 2.1—is non-optimal with respect to \((s_0,p_0,q_0)\), cf. Remark 6.5.
Concerning the solvability of the problem in (2.4) it is noted that the Fredholm properties of $A_T$ depend neither on the parameter $(s, p, q)$ nor on whether the $B_{p,q}^s$ or the $E_{p,q}^s$ spaces are considered.

That is to say, because of the ellipticity and the right-invertibility of $T$, there exists two finite dimensional subspaces $\ker A_T$ and $N$ of $C^\infty(\Omega)$ such that when $s > r + \max(\frac{1}{p} - 1, \frac{n}{p} - n)$ the following holds:

$$\ker A_T = \{ u \in E_{p,q}^s \mid A_T u = 0 \}, \quad (2.14)$$

$$E_{p,q}^{s-2} = N \oplus A_T(E_{p,q}^s); \quad (2.15)$$

and $A_T(E_{p,q}^s)$ is closed. This is a consequence of [Joh96, Thm. 1.3]; see Section 4 below for details. In particular $A_T$ is bijective for all admissible parameters $(s, p, q)$ if (and only if) it is so for one.

Among the conditions that assure solvability of (2.4) we consider:

(I) $A_T$ is invertible.

(II) For each bounded sequence $(v_k)$ in $L_{t-0}$, $\frac{1}{t} = (\frac{1}{p} - \frac{s}{n})_+$, and each $L_\infty$-convergent sequence $(w_k)$ in $\ker A_T$ with $\|w_k\|_{L_\infty} = 1$,

$$\int_\Omega g(v_k + t_k w_k)w_k \, dx - \langle f, w_k \rangle \geq 0 \quad (2.16)$$

holds for some $k \in \mathbb{N}$ when $t_k \to \infty$ for $k \to \infty$.

(III) Under the hypothesis of (II),

$$\int_\Omega g(v_k + t_k w_k)w_k \, dx - \langle f, w_k \rangle \leq 0 \quad (2.17)$$

holds for some $k \in \mathbb{N}$ when $t_k \to \infty$ for $k \to \infty$.

It should be understood that $L_{t-0}$ means $L_t$, except when $B_{p,q}^t$ is considered for $q > t$ where $t - 0$ denotes any $t' < t$. This ensures $E_{p,q}^s \hookrightarrow L_{t-0}$ in any case, cf. (2.28)–(2.31).

Both (II) and (III) are posed for each $f$ in $E_{p,q}^{s-2}$ with $(s, p, q)$ in $\mathbb{D}(A_T + g(\cdot))$; since the requirements are void if $A_T$ is injective, (I) implies both of them. When $g(t)$ is odd, (II) $\iff$ (III) holds, reflecting that $A_T + g(\cdot)$ then sends $u$ to $f$ if and only if $-u$ is mapped to $-f$. If $g$ is even, then (II)
holds for $f$ precisely when $-f$ satisfies (III) for $-g$ (and $A_T u + g(u) = f$ if and only if $A_T - g(\cdot)$ maps $-u$ to $-f$, then).

**Theorem 2.3.** Let $(s, p, q)$ fulfil (i)–(iii) in Theorem 2.1, let $f(x)$ be given in $E_{p,q}^{s-2}$, and let $A_T$ satisfy (I), or let $A_T$ be self-adjoint and $f(x)$ have one of the properties in (II) or (III) above. Then the equation

$$A_T u + g(u) = f$$

(2.18)

has at least one solution $u(x)$ belonging to $E_{p,q;T}^s$.

This generalises the $L_2$-versions of (III) of Robinson and Landesman [RL95] and the $B_{p,q}^s$- and $F_{p,q}^s$-version of (II) in [RR96] to the case with $(s, p, q)$ in the full parameter domain $\mathcal{D}(A_T + (\cdot))$ as defined here. See Remarks 2.4–2.6 below for specific comparisons.

Simple cases of Theorem 2.3 are given in Examples 1.1–1.2 above. In addition, note that we can have, say, $-\Delta_{\gamma_0} - \lambda$ where $\lambda$ is any eigenvalue.

One-dimensional examples may be found in e.g. [RL95]; they also elucidate the connection to other and earlier conditions, mainly formulated in terms of $g(t)$’s properties and without reference to sequences. For the $B_{p,q}^s$ and $F_{p,q}^s$ conditions there is a similar treatment in [RR96]. Drawing on this, we do not give further examples on (II) and (III).

Concerning the proof we use when $s < 2$ that $L_\infty(\Omega) \hookrightarrow E_{p,q}^{s-2}$ to obtain Theorem 2.3 from the Leray–Schauder theorem. The remaining cases are reduced to this by a crucial application of Theorem 2.2, cf. Section 5.

**Remark 2.4.** In (II) and (III) it suffices when $s < 2$ and $1 < p \leq \infty$ to consider sequences $(v_k)$ that are merely bounded in $E_{p,q;T}^s$ itself. Our proof gives this directly, but the $L_{t=0}$-condition is convenient to state.

**Remark 2.5.** Formally the requirements in (II) and (III) are weaker than those in e.g. [RL95] in the sense that the inequalities should hold for one $k$ in $\mathbb{N}$, and not for all $k$ eventually. However, it is easy to infer that this must be the case when (II) or (III) holds.
Seemingly (II) and (III) have not been considered simultaneously before.

Remark 2.6. Extension to $B_{p,q}^s$ and $F_{p,q}^s$ of the conditions in [RL95] has been done by Robinson and Runst [RR96], but only for $s > \frac{n}{p}$. Conditions (II) and (III) are also more general in other respects. Most importantly, we have removed the additional assumption that $f \in B_{t,\infty}^t$ for $t > -1$ when $T$ has class 2. Secondly, (II) and (III) may by Remark 2.4 in some cases refer to the $E_{p,q}^s$-norms (implying their $L_\infty$-conditions when $s > \frac{n}{p}$); thirdly $(v_k)$ is assumed bounded, so that it is unnecessary to consider the case when their norms tend slower to infinity than $(t_k)$.

2.1. Notation. For real numbers $a$ the convention $a_\pm = \max(0, \pm a)$ is used. When $A \subset \mathbb{R}^n$ is open, $L_p(A)$ denotes the classes of functions whose $p^{th}$ power is integrable for $0 < p < \infty$, while $p = \infty$ gives the essentially bounded ones; $L_{1,\text{loc}}^1(A)$ stands for the locally integrable functions.

When $\Omega \subset \mathbb{R}^n$ is open, $C^\infty(\Omega)$ denotes the infinitely differentiable functions; $C^\infty_0(\mathbb{R}^n)$ the subspace of $C^\infty(\mathbb{R}^n)$ for which derivatives of any order are bounded. $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of rapidly decreasing functions; $\mathcal{S}^\prime(\mathbb{R}^n)$ its dual of tempered distributions. The Fourier transformation $\mathcal{F}$ is extended to $\mathcal{S}^\prime$ by duality. The Sobolev–Slobodetskii spaces $W_p^s$ are defined by derivatives and differences thereof for $s > 0$ and $1 < p < \infty$; the Bessel potential spaces $H_p^s = \mathcal{F}^{-1} (1 + |\xi|^2)^{-s/2} \mathcal{F}(L_p)$ for $s \in \mathbb{R}$, $1 < p < \infty$. Besov and Triebel–Lizorkin spaces are written $B_{p,q}^s(\mathbb{R}^n)$ and $F_{p,q}^s(\mathbb{R}^n)$ with $s \in \mathbb{R}$ while $p, q \in [0, \infty]$, except that $p < \infty$ is required for $F_{p,q}^s$.

The subspaces of real-valued elements are all denoted by the same symbols as the complex ones, for throughout we only consider the former versions.

For open sets $\Omega \subset \mathbb{R}^n$ the corresponding spaces are defined by restriction, that is $B_{p,q}^s(\Omega) = r_\Omega B_{p,q}^s(\mathbb{R}^n)$ etc. Hereby $r_\Omega$ is the transpose of $e_\Omega$, the extension by 0 outside of $\Omega$. Spaces over $\Omega$ are given the infimum (quasi-) norm. Similarly for $C^\infty(\Omega)$. For the testfunction space $C^\infty_0(\Omega)$ the dual is
written $\mathcal{D}'(\Omega)$, and $\langle u, \varphi \rangle = u(\varphi)$ for $u \in \mathcal{D}'$ and $\varphi \in C_0^\infty$. The spaces over $\Gamma = \partial \Omega$ are defined by means of local coordinates.

2.2. The spaces. In the following $\mathbb{R}^n$ is suppressed as the underlying set.

First a partition of unity, $1 = \sum_{j=0}^{\infty} \Phi_j$, is constructed: From $\Psi \in C^\infty(\mathbb{R})$, such that $\Psi(t) = 1$ for $0 \leq t \leq \frac{11}{10}$ and $\Psi(t) = 0$ for $\frac{13}{10} \leq t$, the functions $\Psi_j(\xi) = \Psi(2^{-j}|\xi|)$, with $\Psi_j \equiv 0$ for $j < 0$, are used to define $\Phi_j(\xi) = \Psi_j(\xi) - \Psi_{j-1}(\xi)$, for $j \in \mathbb{Z}$.

Secondly there is then a decomposition, with (weak) convergence in $S'$,

$$u = \sum_{j=0}^{\infty} F^{-1}(\Phi_j Fu), \quad \text{for every } u \in S'. \quad (2.20)$$

Now the Besov space $B^s_{p,q}(\mathbb{R}^n)$ and the Triebel–Lizorkin space $F^s_{p,q}(\mathbb{R}^n)$ with smoothness index $s \in \mathbb{R}$, integral-exponent $p \in ]0, \infty]$ and sum-exponent $q \in ]0, \infty]$ is defined as

$$B^s_{p,q} = \{ u \in S' \mid \| 2^{sj} \| F^{-1}(\Phi_j Fu) \|_{L_p} \| \|_{j=0}^{\infty} \| \ell_q \| < \infty \}, \quad (2.21)$$

$$F^s_{p,q} = \{ u \in S' \mid \| 2^{sj} \| F^{-1}(\Phi_j Fu) \|_{j=0}^{\infty} \| \ell_q \| \| (\cdot) \|_{L_p} \| < \infty \}, \quad (2.22)$$

respectively. For the history of these spaces we refer to Triebel’s books [Tri83, Tri92]. Identifications with other spaces are found in Section 1.

In the rest of this subsection the explicit mention of the restriction $p < \infty$ concerning the Triebel–Lizorkin spaces is omitted. E.g., (2.23) below should be read with $p \in ]0, \infty]$ in the $B^s_{p,q}$ part and with $p \in ]0, \infty]$ in the $F^s_{p,q}$ part.

The $B^s_{p,q}$ and $F^s_{p,q}$ are complete, for $p$ and $q \geq 1$ Banach spaces, and $S \hookrightarrow E^s_{p,q} \hookrightarrow S'$ are continuous. Moreover, $S$ is dense in $E^s_{p,q}$ when both $p$ and $q$ are finite, and $C^\infty$ is so in $B^s_{p,q}$ for $q < \infty$.

The definitions imply that $B^s_{p,p} = F^s_{p,p}$, and they give the existence of simple embeddings for $s \in \mathbb{R}$, $p \in ]0, \infty]$ and $o$ and $q \in ]0, \infty]$,

$$E^s_{p,q} \hookrightarrow E^s_{p,o} \quad \text{when } q \leq o, \quad E^s_{p,q} \hookrightarrow E^{s-\varepsilon}_{p,o}, \quad \varepsilon > 0, \quad (2.23)$$

$$B^s_{p,\min(p,q)} \hookrightarrow F^s_{p,q} \hookrightarrow B^s_{p,\max(p,q)}. \quad (2.24)$$
There are Sobolev embeddings if \( s - \frac{n}{p} \geq t - \frac{n}{r} \) and \( r > p \), more specifically

\[
B_{p,q}^s \hookrightarrow B_{r,o}^t, \quad \text{provided } q \leq o \text{ when } s - \frac{n}{p} = t - \frac{n}{r}, \tag{2.25}
\]

\[
F_{p,q}^s \hookrightarrow F_{r,o}^t, \quad \text{for any } o \text{ and } q \in [0, \infty]. \tag{2.26}
\]

Furthermore, Sobolev embeddings also exist between the two scales, in fact under the assumptions \( \infty \geq p_1 > p > p_0 > 0 \) and \( s_0 - \frac{n}{p_0} = s - \frac{n}{p} = s_1 - \frac{n}{p_1} \),

\[
B_{p_0,q_0}^{s_0} \hookrightarrow F_{p,q}^{s} \hookrightarrow B_{p_1,q_1}^{s_1}, \quad \text{for } q_0 \leq p \text{ and } p \leq q_1. \tag{2.27}
\]

When \( C_b \) denotes the bounded uniformly continuous functions on \( \mathbb{R}^n \), then

\[
B_{p,q}^s \hookrightarrow B_{\infty,1}^0 \hookrightarrow C_b \hookrightarrow L_\infty \hookrightarrow B_{\infty,\infty}^0, \tag{2.28}
\]

whereas

\[
F_{p,q}^s \hookrightarrow B_{\infty,1}^0 \hookrightarrow C_b \hookrightarrow L_\infty, \tag{2.29}
\]

for \( s = 0 \) this is provided that \( q \leq 1 \) for \( p = 1 \) and that \( q \leq 2 \) for \( p > 1 \).

Correspondingly

\[
B_{p,q}^s \hookrightarrow \bigcap \{ L_r \mid p \leq r < t \}, \tag{2.30}
\]

where \( r = t \) can be included in general when \( q \leq t \). For \( s = 0 \) one has

\[
B_{p,q}^s \hookrightarrow L_p \quad \text{for } q \leq \min(2,p) \text{ and } p \geq 1. \tag{2.31}
\]

For an open set \( \Omega \subset \mathbb{R}^n \) the space \( E_{p,q}^s(\Omega) \) is defined by restriction,

\[
E_{p,q}^s(\Omega) = r_{\Omega} E_{p,q}^s = \{ u \in \mathcal{D}'(\Omega) \mid \exists v \in E_{p,q}^s; r_{\Omega} v = u \} \tag{2.32}
\]

\[
\| u \|_{E_{p,q}^s(\Omega)} = \inf\{ \| v \|_{E_{p,q}^s} \mid r_{\Omega} v = u \}. \tag{2.33}
\]

By the definitions all the embeddings in (2.23) – (2.31) carry over to the scales over \( \Omega \). When \( \infty \geq p \geq r > 0 \) the inclusion \( L_p(\Omega) \hookrightarrow L_r(\Omega) \) gives

\[
B_{p,q}^s(\Omega) \hookrightarrow B_{r,o}^{s_1}(\Omega), \quad F_{p,q}^s(\Omega) \hookrightarrow F_{r,o}^{s_1}(\Omega), \tag{2.34}
\]

for \( \Omega \), say smooth and bounded; cf. [Joh95a] for a proof (in full generality).
Proposition 2.7. For $s < 0$ and $p, q \in [0, \infty]$ there exists $c < \infty$ such that
\[
\| u \otimes v |B_{p,q}^s(\mathbb{R}^{n+m})\| \leq c \| u |B_{p,q}^s(\mathbb{R}^n)\| \| v |L_p(\mathbb{R}^m)\|, \tag{2.35}
\]
\[
\| u \otimes v |B_{p,q}^{s+t}(\mathbb{R}^{n+m})\| \leq c \| u |B_{p,0}^s(\mathbb{R}^n)\| \| v |B_{p,q}^t(\mathbb{R}^m)\|, \tag{2.36}
\]
when $p > 1$ in (2.35) and $t < 0$ and $\frac{1}{q} = \frac{1}{q_0} + \frac{1}{q_1}$ in (2.36), respectively.

Proof. Using Littlewood–Paley decompositions, this may be proved in the same manner as [Joh96, Prop. 2.5] (where $v = \delta_0$ was treated). \qed

Example 2.8. Precisely when $1 < p \leq \infty$ does
\[
\text{pv}(\frac{1}{x}) \in B_{p,\infty}^{\frac{1}{p}-1}(\mathbb{R}). \tag{2.37}
\]
Indeed, since $\text{pv}(\frac{1}{x}) = i\mathcal{F}H - i\pi\delta_0$, where $H$ is the Heaviside function it suffices to consider $i\mathcal{F}H$. Since $H$ is homogeneous of degree 0, $\mathcal{F}H$ is in $B_{p,q}^{\frac{1}{p}-1}$ if and only if $\mathcal{F}^{-1}(\Phi_0 H(\cdot))$ is in $L_p$. But since
\[
-x \mathcal{F}^{-1}(\Phi_0 H(\cdot)) - \frac{i}{2\pi} = \mathcal{F}^{-1}(H(\cdot)D_x \Phi_0) \in L_\infty(\mathbb{R}), \tag{2.38}
\]
and $\mathcal{F}^{-1}(\Phi_0 H(\cdot))$ is in $C_b(\mathbb{R})$, it is in $L_p$ for $1 < p \leq \infty$.

Example 2.9. By the proposition and Example 2.8 with $x = (x', x_n)$ in $\mathbb{R}^n$ for $n \geq 2$, one has for $1 < p \leq \infty$
\[
r_\Omega(1(x') \otimes \text{pv} \frac{1}{x_n}) \in B_{p,\infty}^{\frac{1}{p}-1}(\Omega), \tag{2.39}
\]
for tensoring instead with $1_B$, the characteristic function of a bounded set with $\Omega \subset B \times \mathbb{R}$, which is in $L_p(\mathbb{R}^{n-1})$, yields the same restriction to $\Omega$.

3. Composition Estimates

Here we prove Theorem 2.1 and substantiate the remarks made after it.

3.1. Proof of Theorem 2.1. That $T$ is bounded as in (2.10) when (i) holds is well known. Concerning the standard traces $\gamma_0$ and $\gamma_1$ one can consult [Tri83, Thm. 3.3.3], and in general this is combined with the fact that $S_0$ and $S_1$ has order $d$ and $d - 1$, respectively, in both $B_{p,q}^s(\Gamma)$ and $F_{p,q}^s(\Gamma)$. 
Secondly, it suffices to show (2.8) for \( g(\cdot) \), for the fact in (2.7) that \( E_p^s \) is sent into \( E_{p,q}^\sigma \) for some \( \sigma > s - 2 \) is a consequence of this. Indeed, given the property in (2.8) it follows at once that (2.7) holds if \( s > \frac{n}{p} \) or if \( 0 < s < 1 \) does so: for any \( \varepsilon > 0 \) one can take \( \sigma = s - \varepsilon \) and use embeddings, e.g.

\[
B_{p,q}^s(\Omega) \hookrightarrow F_{p,\infty}^{s-\epsilon}(\Omega) 
\xrightarrow{g(\cdot)} F_{p,\infty}^{s-\epsilon}(\Omega) 
\hookrightarrow B_{p,q}^{s-\epsilon}(\Omega)
\]  

(3.1)

when \( k \) is so big that \( s - \frac{k}{p} > \frac{n}{p} \) and \( s - \frac{k}{p} > \max(0, \frac{n}{p} - n, \frac{n}{p} - \frac{n}{1-p}) \). For \( s = 1 \), or in the \( F \)-case even for \( s = \frac{n}{p} \), a similar argument applies.

For \( 1 < s < \frac{n}{p} \) we consider for \( p \) fixed \( s - \sigma(s,p) \), that is

\[
d(s) = s - \frac{\frac{n}{p}}{p - s + 1} = \frac{(s-1)(\frac{n}{p} - s)}{n - s + 1},
\]

(3.2)

which measures the loss of smoothness under \( g(\cdot) \). (There exists for \( \varepsilon > 0 \) a \( u_\varepsilon \in E_p^s \) such that \( g(u_\varepsilon) \notin E_{p,q}^{\sigma(s,p)+\varepsilon} \), cf. Remark 6.1.) Since

\[
d(s) = 2 \iff s^2 - (\frac{n}{p} + 3)s + 3\frac{n}{p} + 2 = 0,
\]

(3.3)

where the discriminant \( D = (\frac{n}{p} - 3)^2 - 8 \), it is found that \( d(s) < 2 \) holds

\[
\begin{align*}
\text{if} & \quad s > \frac{1}{2}(\frac{n}{p} + 3 + \sqrt{(\frac{n}{p} - 3)^2 - 8}) \\
\text{or if} & \quad s < \frac{1}{2}(\frac{n}{p} + 3 - \sqrt{(\frac{n}{p} - 3)^2 - 8});
\end{align*}
\]

(3.4)\( (3.5) \)

this is condition (iii) in the theorem, for \( D \geq 0 \) holds when \( \frac{n}{p} \geq 3 + \sqrt{8} \). Observe that \((\sqrt{\frac{n}{p}} - 1)^2 = \max\{d(s) \mid 1 < s < \frac{n}{p}\} \), and that this equals 2 for \( \frac{n}{p} = 3 + \sqrt{8} \) since \( D = 0 \) then. If \( \frac{n}{p} < 3 + \sqrt{8} \), then \((\sqrt{\frac{n}{p}} - 1)^2 < 2 \).

For a given \((s,p,q)\) with \( 1 < s < \frac{n}{p} \) and (iii) satisfied we can now take \( \varepsilon > 0 \) so that \( \sigma(s,p) - \varepsilon > s - 2 \) and obtain

\[
F_{p,q}^s(\Omega) \xrightarrow{g(\cdot)} F_{p,\infty}^{\sigma(s,p)}(\Omega) \hookrightarrow F_{p,q}^{\sigma(s,p)-\varepsilon}(\Omega),
\]

(3.6)

which gives (2.7) in this case. Moreover, the fact that (ii),(iii) and \( 1 < s < \frac{n}{p} \) specify an open set of parameters \((s,p,q)\) together with the continuity of \( \sigma(\cdot,p) \) gives an \( \eta > 0 \) such that \( \sigma(s-\eta,p) > s - 2 \), and then

\[
B_{p,q}^s(\Omega) \hookrightarrow F_{p,\infty}^{s-\eta}(\Omega) \xrightarrow{g(\cdot)} F_{p,\infty}^{\sigma(s-\eta,p)}(\Omega) \hookrightarrow B_{p,q}^\sigma(\Omega)
\]

(3.7)

holds for any \( \sigma < \sigma(s-\eta,p) \).
Finally, when \( s = \frac{n}{p} \) in the \( B \)-case an argument similar to (3.7), but with \( \sigma(s - \eta, p) > s - \epsilon \), works because \( \lim_{s \to \frac{n}{p}} \sigma(s, p) = \frac{n}{p} \). The statement on \( \bar{\sigma} \) follows analogously if the effects of (iii) are disregarded, for in (3.4) ff. any \( \epsilon > 0 \) and in (3.7) ff. any \( \sigma < \sigma(s, p) \) may be obtained. Similarly \( \sigma = \sigma(s, p) - \epsilon \) is always possible.

It remains to show (2.8). Here we draw on the literature, where \( \Omega = \mathbb{R}^n \) has been considered by many. On \( \mathbb{R}^n \) the condition \( g(0) = 0 \) is posed in order to have \( g(0) \in L_p \) also for \( p < \infty \), so strictly speaking we should replace \( g(\cdot) \) by \( g(\cdot) - g(0) \); this is harmless because \( g(0) \) belongs to \( \cap_{s, p, q} B^{s, p, q}_{s, p, q}(\mathbb{R}) \).

Once boundedness has been established on \( \mathbb{R}^n \) through an inequality like
\[
\| g(u) F^{s, p}_{p, \infty} \| \leq c \| u \| F^{s, p}_{p, \infty} \|(1 + \| u \| F^{s, p}_{p, \infty})^{\mu - 1}) \tag{3.8}
\]
this carries over to \( \Omega \) by restriction: if \( r \Omega u = v \) for \( v \in F^{s, p}_{p, \infty}(\mathbb{R}^n) \), then \( g(v) \in F^{s, p}_{p, \infty}(\mathbb{R}^n) \) restricts to \( g(u) \). Thus it suffices to consider \( \Omega = \mathbb{R}^n \).

For \( s > \frac{n}{p} \) it was shown in [Run86] that for every real-valued \( u \in F^{s, p}_{p, q}(\mathbb{R}^n) \),
\[
\| g(u) F^{s, p}_{p, q} \| \leq c \| u \| F^{s, p}_{p, q} \|(1 + \| u \| F^{s, p}_{p, q})^{\mu - 1}) \tag{3.9}
\]
when \( \mu > \max(1, s) \), cf. Theorem 5.4.2 there. Here the general assumption that \( g^{(j)} \in L_\infty(\mathbb{R}) \) for every \( j \in \mathbb{N}_0 \) is used to obtain \( c \) independent of \( u \).

When \((\frac{p}{n} - n)_+ < s < 1 \) the estimate in (3.8) is, with \( \sigma(s, p) = s \) and \( \mu = 1 \), a well-known easy consequence of the characterisation of \( F^{s, p}_{p, q} \) by first order differences, cf. [Tri92, Thm. 3.5.3] and the estimate
\[
|g(u(x + h)) - g(u(x))| \leq \| g' \| L_\infty \cdot |u(x + h) - u(x)|. \tag{3.10}
\]

The cases with \( 1 < s < \frac{n}{p} \) are covered by [Sic89, Lemma 3], even with a sharper result in Theorem 1 there when \( s > 1 + (\frac{p}{n} - n)_+ \). In fact this lemma yields (3.8) for \( \sigma(s, p) = \frac{p}{p - s + 1} \) and \( \mu = \sigma(s, p) \), provided that \( 1 < s < \frac{n}{p} \) and \( \sigma(s, p) > (\frac{p}{n} - n)_+ \) hold. By definition \( \sigma(s, p) > 1 \) for \( s > 1 \), so this is trivially true for \( 1 \leq p < \infty \); for \( p \leq 1 \) the assumption \( s < \frac{n}{p} \) gives that
\[
\sigma(s, p) > \frac{n}{p} - n \iff s > \frac{(\frac{n}{p})^2 - n \frac{n}{p} - n}{\frac{n}{p} - n} \iff s > \frac{n}{p} - \frac{p}{1-p} \tag{3.11}
\]
so the second line of (ii) is found from the requirement $\sigma(s,p) > (\frac{n}{p} - n)_+$. Finally, for $s = 1$ we reduce to the case with $s < 1$ by an arbitrarily small loss of smoothness; for $s = \frac{n}{p}$ a reduction to $1 < s < \frac{n}{p}$ works because $\lim_{s \to \frac{n}{p}} \sigma(s,p) = \frac{n}{p} = s$. The proof of Theorem 2.1 is complete.

We include a few observations on the curve determined by (3.3) for $\frac{n}{p} > 0$. For the auxiliary function $h_1(t) = \frac{1}{2}(t + 3 + \sqrt{(t - 3)^2 - 8})$,

$$h_1(t) - t = \frac{1}{2}(t - 3)(\sqrt{1 - 8(t - 3)^2} - 1)$$

$$= -2(t - 3)^{-1} + O((t - 3)^{-3}) \to 0_+ \quad \text{for } t \to \infty,$$  \hspace{1cm} (3.12)

whereas $h_2(t) = \frac{1}{2}(t + 3 - \sqrt{(t - 3)^2 - 8})$ satisfies

$$h_2(t) - 3 = \frac{1}{2}(t - 3)(1 - \sqrt{1 - 8(t - 3)^2}) \to 0_+ \quad \text{for } t \to \infty.$$  \hspace{1cm} (3.13)

Thus $s = \frac{n}{p}$ and $s = 3$ are the asymptotes as claimed. The curve itself is a branch of a hyperbola since the equation in (3.3) may be written

$$0 = (s - 3)^2 - (\frac{n}{p} - 3)(s - 3) + 2$$

$$= \begin{pmatrix} \frac{n}{p} - 3 & s - 3 \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} \frac{n}{p} - 3 \\ s - 3 \end{pmatrix} + 2,$$  \hspace{1cm} (3.14)

where the matrix is symmetric and indefinite as the determinant is $-\frac{1}{4}$.

3.2. A lemma on continuity. The boundedness obtained for $g(\cdot)$ above means that every bounded set of $E_{p,q}^s$ is mapped into a bounded set in $E_{p,q}^\sigma$. Although $g(\cdot)$ is non-linear, this boundedness does imply a norm continuity if one can afford to loose a little smoothness.

For the reader’s convenience we include the next lemma, which is used in Section 5 below; it extends [Sic92, 3.1] and simplifies [RS96, Lem. 5.5.2]:

**Lemma 3.1.** When $\Omega$ is as above, and $g \in C^\infty(\mathbb{R})$ with $g' \in L_\infty(\mathbb{R})$, then boundedness, for some $s > (\frac{n}{p} - n)_+, 0 < p \leq \infty$ and some $\sigma \in \mathbb{R}$, of

$$g(\cdot) : E_{p,q}^s \to E_{p,q}^\sigma$$  \hspace{1cm} (3.15)

implies norm continuity of

$$g(\cdot) : E_{p,q}^s \to E_{p,q}^{\sigma - \varepsilon} \quad \text{for each } \varepsilon > 0.$$  \hspace{1cm} (3.16)
Proof. In the Besov case one has, when \( t < \min(0, \sigma - \varepsilon) \), that
\[
B_{p,q}^\sigma(\mathbb{T}) = (B_{p,q}^\sigma(\mathbb{T}), B_{p,q}^t(\mathbb{T}))_{\theta,q}
\] (3.17)
for some \( \theta \in ]0,1[ \), cf. [Tri83, Thm. 3.3.6]. When \( r = \max(1, r) \)
\[
\| g(u) - g(v) \|_{B_{p,q}^{\sigma-\varepsilon}} \leq c \| g(u) - g(v) \|_{L_r}^{1-\theta} \| g(u) - g(v) \|_{B_{p,q}^{\sigma}}^\theta,
\] (3.18)
since \( L_r(\Omega) \hookrightarrow B_{p,q}^t(\mathbb{T}) \) then. In \( L_r \) an estimate like (3.10) is applicable,
and thereafter \( B_{p,q}^s \hookrightarrow L_r \) may be used (for \( p < 1 \) this embedding is based
on the assumption \( s > \frac{n}{p} - n \)). Thus the first factor on the right hand side
tends to 0 for \( u \to v \) in \( B_{p,q}^s \) while the second remains bounded by (3.15).

In the \( F_{p,q}^s \) case, \( g(\cdot): F_{p,q}^s \to B_{p,\infty}^\sigma(\mathbb{T}) \) is bounded, so analogously
\[
g(\cdot): F_{p,q}^s(\mathbb{T}) \to B_{p,q}^{\sigma-\eta}(\mathbb{T})
\] (3.19)
is continuous for any \( \eta > 0 \). Then (3.16) follows. \( \square \)

3.3. Interrelations between conditions (i), (ii) and (iii).

Remark 3.2. In the definition of \( D(A_T + g(\cdot)) \) the condition:
\[
s > \frac{n}{p} - \frac{p}{1-p} \quad \text{for} \quad 0 < p < 1
\] (3.20)
in (ii) of Theorem 2.1 is always redundant when \( T \) has class \( r = 2 \).

Indeed, since one has
\[
\frac{n}{p} - \frac{p}{1-p} \leq \frac{n}{p} - n + 2 \iff p(n-1) \geq n - 2
\] (3.21)
it is clear that when \( (s, p, q) \) satisfies (i) for \( r = 2 \), then (3.20) holds if either
\( n = 1, n = 2 \) or if \( \frac{n-2}{n-1} \leq p < 1 \) when \( n \geq 3 \).

Therefore, when (i) and (iii) hold for \( r = 2 \), then it suffices to verify for
\( n \geq 3 \) and \( 0 < p \leq \frac{n-2}{n-1} \) that the first inequality in (iii) poses a stronger
condition than (3.20). This follows from Remark 3.5.

Remark 3.3. For \( n = 1 \) condition (i) in Theorem 2.1 amounts to
\[
s > \frac{n}{p},
\] (3.22)
since \( r \geq 1 \). Therefore any \( E_{p,q}^s \) in \( D(A_T + g(\cdot)) \) satisfies \( E_{p,q}^s \hookrightarrow C(\mathbb{T}) \), and
both (ii) and (iii) hold when (i) does so.
Hence Figure 1 is misleading for \( n = 1 \), and in fact
\[
\mathbb{D}(A_T + g(\cdot)) = \{ (s,p,q) \mid s > \frac{1}{p} - 1 + r \}, \tag{3.23}
\]
which in contrast to the general case (for \( n \geq 2 \)) is convex.

**Remark 3.4.** Also \( n = 2 \) gives an exception from the overview after Theorem 2.1.

In this case \( \mathbb{D}(A_T + g(\cdot)) \) is still not convex for \( r = 1 \), but (ii) implies (iii), so that the curved boundary is given by \( s = \frac{p}{p} - \frac{p}{1-p} \). See Remark 3.5 below for the details.

Moreover, for \( n = 2 = r \) it follows from Remark 3.2 that even (ii) is redundant, cf. (3.21), and hence
\[
\mathbb{D}(A_T + g(\cdot)) = \{ (s,p,q) \mid s > \max(\frac{1}{p} + 1, \frac{2}{p}) \}. \tag{3.24}
\]
Evidently this is convex, so also this case deviates from the general pattern.

**Remark 3.5.** Among the requirements in Theorem 2.1 the condition
\[
(iii)' \quad s > \frac{1}{2}(\frac{n}{p} + 3 + \sqrt{(\frac{n}{p} - 3)^2 - 8})
\]
turns out to be almost always stronger than
\[
(ii)' \quad s > \frac{n}{p} - \frac{n}{p - n}
\]
when they both apply, that is for \( \frac{n}{p} \in ]\max(n, 3 + \sqrt{8}), \infty[ \) and \( n \geq 2 \). The exceptions are for \( n = 3 \) in which case \( (ii)' \implies (iii)' \) in the narrow interval with \( 3 + \sqrt{8} \leq \frac{p}{n} < 6 \) and in general for \( n = 2 \).

Observe first that (ii)' and (iii)' are redundant for \( n = 1 \) by Remark 3.3.

To analyse when (iii)' \( \implies (ii)' \) for \( n \geq 2 \), consider
\[
t - 3 - \frac{2n}{t-n} \leq \sqrt{(t-3)^2 - 8} \tag{3.25}
\]
when \( t > n \) and \( t \geq 3 + \sqrt{8} \) as well as \( n = 2, 3, \ldots \). Notice that the left hand side equals \((t - n)^{-1}(t^2 - (n + 3)t + n)\) and is negative when
\[
t^2 - (n + 3)t + n < 0; \tag{3.26}
\]
the discriminant \( n^2 + 2n + 9 \) is \( > 0 \). Thus (3.25) always holds for \( t \in [\alpha_-(n), \alpha_+(n)] \) when \( 2\alpha_\pm(n) = n + 3 \pm \sqrt{n^2 + 2n + 9} \). Here \( \alpha_+(n) > n \) and \( \alpha_-(n) < \min(n, 3 + \sqrt{8}) \).

For \( t \geq \max(\alpha_+(n), 3 + \sqrt{8}) \) it is found by taking squares that
\[
(3.25) \iff \frac{4n^2}{(t-n)^2} - 2(t-3)\frac{2n}{t-n} \leq -8 
\iff 0 \leq (n-2)t^2 - n(n-1)t.
\]

The last inequality is false for \( n = 2 \), and since \( \alpha_+(2) < 3 + \sqrt{8} \) it is proved that \( (ii)' \implies (iii)' \) for \( n = 2 \).

Since \( t = 0 \) and \( t = n(n-1)/(n-2) \) are the roots of the polynomial \( (n-2)t^2 - n(n-1)t \), the implication \( (iii)' \implies (ii)' \) holds for all \( t \leq \max(\alpha_+(n), 3 + \sqrt{8}) \) precisely when
\[
\frac{n(n-1)}{n-2} \geq \max(\alpha_+(n), 3 + \sqrt{8})
\]
does so. A straightforward calculation shows that
\[
\frac{n(n-1)}{n-2} < \alpha_+(n) \iff n \geq 4,
\]
so (3.28) holds for all \( n \geq 4 \). In addition \( \alpha_+(3) = 3 + \sqrt{6} \) while \( \frac{n(n-1)}{n-2} \big|_{n=3} = 6 \), so by (3.27) the inequality (3.25) holds for \( t \in [6, \infty[ \) when \( n = 3 \).

Altogether this shows that, except for \( n = 2 \) and a small interval for \( n = 3 \), the condition \( s > \frac{n}{p} - \frac{p}{1-p} \), that is \( \mathcal{O}(\frac{n}{p}) \), only interferes with the second requirement in (iii). In other words, when \( n \geq 3 \) the domains \( \mathbb{D}(AT + g(\cdot)) \) are for \( \frac{n}{p} \geq 6 \) only defined by the stronger condition (iii)'.

4. PROOF OF THE INVERSE REGULARITY THEOREM

Before the regularity properties of Theorem 2.2 are proved in Section 4.2 below, we review the prerequisites on elliptic problems in Besov and Triebel–Lizorkin spaces for a better reading.

4.1. The Boutet de Monvel calculus. There are two sources for elliptic theory in the full \( B_{p,q}^s \) and \( F_{p,q}^s \) scales; the Agmon–Douglis–Nirenberg theory has been extended in [FR95], but this is not quite adequate here,
cf. Remark 4.3. Instead we use the pseudo-differential boundary operator calculus, which was generalised to these spaces in [Joh96] and [Joh93, Ch. 4].

As a general introduction to the calculus there is [Gru91] and the introduction and Section 1.1 in [Gru86].

4.1.1. Green Operators. In a systematic approach to boundary problems, the basic ingredient to study is a matrix operator

\[
A = \begin{pmatrix}
P_{\Omega} + G & K \\ T & S \\
\end{pmatrix} : \begin{array}{c}
\bigoplus C^\infty(\overline{\Omega})^N \\
\bigoplus C^\infty(\Gamma)^M \\
\end{array} \to \begin{array}{c}
\bigoplus C^\infty(\overline{\Omega})^{N'} \\
\bigoplus C^\infty(\Gamma)^{M'} \\
\end{array} \tag{4.1}
\]

where \(P_{\Omega} := r_{\Omega} Pe_{\Omega}\) is the truncation to \(\Omega\) of a pseudo-differential operator on \(\mathbb{R}^n\), \(K\) is a Poisson operator, \(T\) is a trace operator, \(S\) is a pseudo-differential operator in \(\Gamma\) whilst \(G\) is a singular Green operator.

As examples of (4.1), or of the so-called Green operators, one can take

\[
\begin{pmatrix}
-\Delta \\
\gamma_0
\end{pmatrix}, \quad \begin{pmatrix}
-\Delta \\
\gamma_1
\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}
A \\
T
\end{pmatrix}, \tag{4.2}
\]

whereby \(M = 0\) since they are column matrices, or their parametrices

\[
(R_D \ K_D), \quad (R_N \ K_N) \quad \text{resp.} \quad (R \ K) \tag{4.3}
\]

(when \(\frac{A}{T}\) is elliptic); hereby \(M' = 0\) because of the row-form.

For realisations like \(A_T\) considered above a variety of results follow easily from a study of \(\frac{A}{T}\), so we focus on the latter operator to begin with.

To get a good calculus of Green operators like \(A\) above, Boutet de Monvel [BdM71] introduced first of all the requirement that \(P\) should have the transmission property at \(\Gamma \subset \mathbb{R}^n\). That is to say, for \(N = N' = 1\), \(P_{\Omega}\) should map \(C^\infty(\overline{\Omega})\) into itself — when \(P\) merely belongs to the Hörmander class \(S^d_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)\), then \(P_{\Omega}(C^\infty(\overline{\Omega})) \subset H^{-d}(\overline{\Omega}) \cap C^\infty(\Omega)\) (since the singular support of \(P(\epsilon_{\Omega} \varphi)\), for \(\varphi \in C^\infty(\overline{\Omega})\), as a subset of \(\Gamma\), is not felt after application of \(r_{\Omega}\); thus the transmission property rules out blow-up at \(\Gamma\).

Secondly, the notion of singular Green operators \(G\) was introduced in order to encompass solution operators; e.g., when the inverse of \(\begin{pmatrix}
-\Delta \\
\gamma_0
\end{pmatrix}\) is denoted \(\begin{pmatrix} R_D \\ K_D \end{pmatrix}\), then \(R_D\) is not a truncated pseudo-differential operator.
In fact, \( R_D = \text{OP}(|\xi|^{-2})_\Omega + G_D \), where the compensating term \( G_D \) is a singular Green operator equal to \(-K_D G_0\text{OP}(|\xi|^{-2})_\Omega\).

For the precise symbol classes of \( P_\Omega, G, K, T \) and \( S \), with the uniformly estimated class \( S_{1,0}^d(\mathbb{R}^n \times \mathbb{R}^n) \) as the basis, the reader is referred to [GK93]. A discussion of the transmission property is found in a work of Grubb and Hörmander [GH91]; let us also mention [Gru91], [Joh96, Sect. 3.2] and Section 1.2 in the second edition of [Gru86].

We proceed to state relevant properties of \( \mathcal{A} \). Further details and proofs are given in [Joh96]. Specialising to \( \mathcal{A} = (\frac{\partial}{\partial t}) \) with \( A \) and \( T \) as in Section II \( P_\Omega = A \) is of order 2, \( G = 0 \) and \( (K \text{ and } S \text{ being redundant, i.e. } M = 0) \) \( T \) is of order \( d \) and class \( r = 1 \) or 2. Then

\[
\mathcal{A}: B^{s}_{p,q}(\overline{\Omega}) \to B^{s-2}_{p,q}(\overline{\Omega}) \oplus B^{s-d-\frac{1}{p}}_{p,p}(\Gamma) \quad (4.4)
\]

\[
\mathcal{A}: F^{s}_{p,q}(\overline{\Omega}) \to F^{s-2}_{p,q}(\overline{\Omega}) \oplus B^{s-d-\frac{1}{p}}_{p,p}(\Gamma) \quad (4.5)
\]

are bounded when \( s > r + \max(\frac{1}{p} - 1, \frac{n}{p} - n) \).

The assumed ellipticity of \( \mathcal{A} \) in the sense of the calculus amounts to

(I) \( \mathcal{A} \)'s principal symbol, \( a^0(x,\xi) = \sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha \), is non-zero,

\[
a^0(x,\xi) \neq 0, \quad \text{for } x \in \Omega \text{ and } |\xi| \geq 1; \quad (4.6)
\]

(II) the principal boundary symbol operator \( a^0(D_n) = a^0(x',0,\xi',D_n) \),

\[
a^0(D_n): S(\mathbb{R}^+) \to \mathcal{S}(\mathbb{R}^+) \oplus \mathbb{C} \quad (4.7)
\]

is a bijection for each \( x \in \Omega \) and \( |\xi'| \geq 1 \).

Here \( a^0(D_n) \) is defined from the principal part of \((\frac{\partial}{\partial n})\) by means of local coordinates in which \( \Gamma \) is a subset of \( \{x_n = 0\} \); there \( x_n \) is set equal to 0 and \( D_j \) is replaced by \( \xi_j \) when \( j < n \).

The ellipticity assures the existence of a parametrix \( \tilde{\mathcal{A}} \), that is, another Green operator in the calculus such that

\[
\tilde{\mathcal{A}} = 1 - \mathcal{R}, \quad \mathcal{A}\tilde{\mathcal{A}} = 1 - \mathcal{R}' \quad (4.8)
\]
for negligible operators $R$ and $R'$; i.e. Green operators of order $-\infty$. Although $A$ is purely differential, $\tilde{A}$ has the form $(R \ K)$ where $R = P_\Omega + G$ for a truly pseudo-differential operator $P$ with transmission property at $\Gamma$ and a non-trivial singular Green operator. The orders of $R$ and $K$ are $-2$ and $-d$, respectively, while $R$ may be taken of class $r - 2$ (best possible), cf. [Gru90, Thm. 5.4]. Hence, by (4.4)–(4.5),

\[ \tilde{A} : B^{s-2}_{p,q}(\Omega) \oplus B^{s-d-\frac{1}{2}}_{p,q}(\Gamma) \to B^s_{p,q}(\Omega) \]  

(4.9)

\[ \tilde{A} : F^{s-2}_{p,q}(\Omega) \oplus B^{s-d-\frac{1}{2}}_{p,p}(\Gamma) \to F^s_{p,q}(\Omega) \]  

(4.10)

are bounded for $s > r + \max(\frac{1}{p} - 1, \frac{n}{p} - n)$.

Using $\tilde{A}$ it may be shown that there exist two finite-dimensional subspaces

\[ \ker A \subset C^\infty(\Omega) \quad \mathcal{N} \subset C^\infty(\Omega) \oplus C^\infty(\Gamma), \]  

(4.11)

(and that $A(E^s_{p,q})$ is closed) such that whenever $s > r + \max(\frac{1}{p} - 1, \frac{n}{p} - n)$,

\[ \ker A = \{ u \in E^s_{p,q} \mid Au = 0 \}, \]  

(4.12)

\[ A(B^s_{p,q}) \oplus N = B^{s-2}_{p,q}(\Omega) \oplus B^{s-d-\frac{1}{2}}_{p,q}(\Gamma), \]  

(4.13)

\[ A(F^s_{p,q}) \oplus N = F^{s-2}_{p,q}(\Omega) \oplus B^{s-d-\frac{1}{2}}_{p,p}(\Gamma). \]

In other words, the kernel of $A$ is $(s, p, q)$-independent and the range complement may be picked with this property.

4.1.2. Realisations. For $A_T$ in (2.2)–(2.3) the subspaces $B^s_{p,q;T}$ and $F^s_{p,q;T}$ defined by $Tu = 0$ make sense for $s > r + \max(\frac{1}{p} - 1, \frac{n}{p} - n)$, and

\[ A_T : E^s_{p,q;T} \to E^{s-2}_{p,q} \]  

(4.14)

is bounded for such $(s, p, q)$, by (4.4) and (4.5).

Ellipticity of $A_T$ means that $(\frac{1}{p})$ is elliptic, i.e. that (I) and (II) are satisfied. In the elliptic case even $A_T$ has a parametrix, say $R_0$; it is of the form $(A_0)_\Omega + G_0$, where $A_0$ is a parametrix of $A$ on $\mathbb{R}^n$ and $G_0$ is a singular Green operator, both of order $-2$ and $(A_0)_\Omega + G_0$ of class $r - 2$, so

\[ R_0 : E^{s-2}_{p,q} \to E^{s}_{p,q} \]  

(4.15)
is bounded whenever \( s > r + \max\left(\frac{1}{p} - 1, \frac{n}{p} - n\right) \) by the general result in (4.4)–(4.5). More importantly, \( R_0 \) can be taken so that

- \( R_0 \) maps \( E^{s-2}_{p,q} \) into \( D(A_T) = E^s_{p,q,T} \);
- both \( R_0 A_T - I \) and \( A_T R_0 - I \) have finite-dimensional ranges in \( C^\infty(\Omega) \).

This follows as in [Gru86, Prop. 1.4.2]; when \( r \neq 2 \) or \( d \neq 2 \) one can modify the order and class reduction in (1.4.14) there, as in [Gru90, (5.32)].

For the Fredholm properties of \( A_T \) one has obviously that \( \ker A_T = \ker \mathcal{A} \), but it is a point to show that \( A_T(E^s_{p,q,T}) \) is complemented also for \( p, q < 1 \) in which case \( E^s_{p,q} \) is not locally convex. However, when \( T \) has a Poisson operator \( K \) as a right inverse, i.e. \( TK = I \), then

\[
\Phi = \begin{pmatrix} I & -AK \end{pmatrix},
\]

may be used in a way similar to the proof of [Gru86 4.3.1] to get

**Lemma 4.1.** 1° When \((s, p, q)\) is admissible and \( W \) is a range complement of \( \mathcal{A} \), then \( A_T(E^s_{p,q,T}) \) is closed while \( \dim \Phi(W) = \dim W \) and \( E^{s-2}_{p,q} = A_T(E^s_{p,q,T}) \oplus \Phi(W) \).

2° A subspace \( N \subset C^\infty \) is a range complement of \( A_T \) for some \((s, p, q)\) if and only if it is so for every \((s, p, q)\) admissible for \( A_T \).

**Proof.** As in [Gru86 4.3.1], \( \Phi \) is seen to be injective on \( W \), hence \( \dim \Phi(W) = \dim W \), and \( \Phi(W) \) to be linearly independent of \( R(A_T) := A_T(E^s_{p,q,T}) \).

Then, using the quotient \( Q \) onto \( E^{s-2}_{p,q}/R(A_T) \), \( \dim \Phi(W) \leq \dim Q(E^{s-2}_{p,q}) \) follows. But a finite dimensional \( U \subset Q(E^{s-2}_{p,q}) \) equals \( QV \) for some \( V \) linearly independent of \( R(A_T) \) and with \( \dim U = \dim V \leq \dim W \) (since \( V \times \{0\} \) is linearly independent of \( \mathcal{A}(E^s_{p,q}) \)). Altogether \( \dim Q(E^{s-2}_{p,q}) = \dim \Phi(W) < \infty \), so \( R(A_T) \) is closed by [Hörs5 19.1.1] (carried over to \( E^s_{p,q} \) by [Rud73 1.41(d)+2.12(b)]) and complemented by \( \Phi(W) \).
Since $\mathcal{N} = \Phi(\mathcal{N} \times \{0\})$, $W = \mathcal{N} \times \{0\}$ is possible for dimensional reasons. By Theorem 1.3 or 5.2 of [Joh96], $W$ is a range complement for every $(s, p, q)$; by $1^c$, so is $\mathcal{N}$.

Existence of such a $K$ is assured when $T$ is normal; see Proposition 1.6.5, Definition 1.4.3 and Remark 1.4.4 in [Gru86]. For $d = 0$ normality means that $T = S_0 \gamma_0$, where $S_0(x)$ is a function without roots on $\Gamma$; when $d = 1$, $T$ is normal when $S_1(x)$ is such a zero-free function.

Finally, one can in this case project onto the kernel and range of $A_T$.

**Proposition 4.2.** Let $A_T$ be an elliptic realisation of $A$ as described above, with a right inverse of $T$ (or $T$ normal).

For each $C^\infty$ range complement $\mathcal{N}$ and each $s > r + \max(\frac{1}{p} - 1, \frac{n}{p} - n)$ there is a continuous idempotent

$$Q : E_{p,q}^{s-2} \to E_{p,q}^{s-2},$$

projecting onto $\mathcal{N}$ along $A_T(E_{p,q}^s T)$.

(4.17)

When $\{w_1, \ldots, w_m\}$ is an $L_2$-orthonormal basis for $\ker A_T$,

$$Pu = \sum_{j=1}^m \langle u, w_j \rangle w_j \text{ is bounded } P : E_{p,q}^s \to E_{p,q}^s$$

(4.18)

and projects onto $\ker A_T$ whenever $s > r + \max(\frac{1}{p} - 1, \frac{n}{p} - n)$.

Furthermore, when $A_T$ is self-adjoint in $L_2(\Omega)$, one can take $\mathcal{N} = \ker A_T$ for every $(s, p, q)$ as above and then (4.18) holds even on $E_{p,q}^{s-2}$.

**Proof.** When (2.15) holds [Rud73, Thm. 5.16] gives the existence and continuity of $Q$. This does not just carry over to $\ker A_T$, for application of, say, [Rud73, Lem. 4.21] requires local convexity.

However, the given $P$ is defined for $u \in E_{p,q}^s$ when $s > r + \max(\frac{1}{p} - 1, \frac{n}{p} - n)$, for since $r \in \{1, 2\}$ we have $s > 0$ so that $E_{p,q}^s \hookrightarrow L_1(\Omega)$,

$$\langle u, w_j \rangle = \int_{\Omega} u w_j$$

is defined and

$$|\langle u, w_j \rangle| \leq \|u\|_{L_1} \|w_j\|_{L_\infty} \leq c \|u\|_{E_{p,q}^s} \|w_j\|_{L_\infty};$$

(4.19)

continuity of $P$ follows. By construction $P^2 = P$ and $\ker A_T = P(E_{p,q}^s)$. 

When $A_T = A_T^*$ in $L_2$, then $\ker A_T$ is a range complement in $E^{s-2}_{p,q}$ by the lemma. Consider first $r = 2$. Then the inequality for $s$ implies that $E^{s-2}_{p,q}$ is contained in the dual of some $E^{s_2}_{p_2,q_2} \supset \ker A_T$, and analogously to the above $P$ is a continuous projection in $E^{s-2}_{p,q}$ onto $\ker A_T$.

For $r = 1$ elements of e.g. $H^{-1}$ may occur in (4.18). However, $w \in \ker A_T$ implies $\gamma_0 w = 0$: evidently $T w = 0$ where $T = S_0 \gamma_0$ and $S_0(x)$ is a function on $\Gamma$ (being a differential operator of order 0 by assumption), and $S_0(x)$ cannot have any zeroes because $S_0 \gamma_0$ has a right inverse. Thus $\gamma_0 w = 0$.

So when $s > 1 + \max(\frac{1}{p} - 1, \frac{n}{p} - n)$, the space $E^{s-2}_{p,q}$ is embedded into some $E^{s_1-2}_{p_1,q_1}$ with $s_1 > 1 + \max(\frac{1}{p_1} - 1, \frac{n}{p_1} - n)$ and $p_1, q_1 \in [1, \infty]$. The latter is dual to $E^{s_2}_{p_2,q_2;0} = \{ u \in E^{s_2}_{p_2,q_2} \mid \gamma_0 u = 0 \}$ when $s_1 - 2 + s_2 = 0$, $\frac{1}{p_1} + \frac{1}{p_2} = 1$ and $\frac{1}{q_1} + \frac{1}{q_2} = 1$, and since $\ker A_T \subset E^{s_2}_{p_2,q_2;0}$, $P$ in (4.18) is defined on $E^{s_1-2}_{p_1,q_1}$, hence on $E^{s-2}_{p,q}$. Again $P$ is bounded and idempotent. □

### 4.2. Proof of Theorem 2.2

We now turn to one of the main subjects in this article: the Inverse Regularity Theorem for the problem in (1.1). For the proof the bootstrap method in [Joh93, Joh95b, Joh] is extended to overcome the difficulties caused by the non-convexity of $\mathcal{D}(A_T + g(\cdot))$.

Basically the non-linear estimates and the elliptic theory is used as follows: suppose $u(x)$ in $E^{s_1}_{p_1,q_1;T}$ is a solution of

$$A_T u + g(u) = f$$

(4.20) for $f(x)$ in $E^{s_0-2}_{p_0,q_0}$ with both $(s_0, p_0, q_0)$ and $(s_1, p_1, q_1)$ in $\mathcal{D}(A_T + g(\cdot))$. Then $R_0$, the parametrix of $A_T$ introduced in (4.15) ff., is bounded

$$R_0: E^{s_0-2}_{p_0,q_0} \rightarrow E^{s_0}_{p_0,q_0;T}$$

(4.21) because $(s_0, p_0, q_0) \in \mathcal{D}(A_T + g(\cdot))$. Thus $R_0$ can be applied to the right hand side of (4.20), hence to the left hand side. By Theorem 2.1 and (4.14), both $A_T u$ and $g(u)$ are in $E^{s_1-2}_{p_1,q_1}$, and so $R_0$ acts linearly on the left hand side of (4.20). After a rearrangement, cf. Remark 4.3 below, we get

$$u = R_0 f - R_0 g(u) + R u$$

(4.22)
where $\mathcal{R} := R_0 A_T - I$ is an operator with range in $C^\infty(\Omega)$.

Since $R_0 g(u) \in E_{p_0,q_0}^{s_0+2}$ for some $\sigma_1 > s_1 - 2$ by Theorem 2.1, one may now search for $E_{p_2,q_2;T}^{s_2}$ large enough to contain $E_{p_0,q_0;T}^{s_0} + E_{p_1,q_1;T}^{\sigma_1+2}$, and thus

$$R_0 f - R_0 g(u) + \mathcal{R} u \in E_{p_2,q_2;T}^{s_2}.$$  \hfill (4.23)

Then $u \in E_{p_2,q_2;T}^{s_2}$, and this fact is used to get a new knowledge about $R_0 g(u)$ and then for $u$ itself. Thus we seek spaces $E_{p_1,q_1;T}^{s_1}, E_{p_2,q_2;T}^{s_2}$, ... containing $u(x)$, and the task is to obtain $E_{p_j,q_j;T}^{s_j} \hookrightarrow E_{p_0,q_0}^{s_0}$ for some $j$.

Obviously it is irrelevant for the application of $g(\cdot)$ whether we consider $u$ in the subspace $E_{p_j,q_j;T}^{s_j}$ or not, so for simplicity we use the full space $E_{p_j,q_j;T}^{s_j}$.

Furthermore we shall first treat the case where $E_{p_0,q_0}^{s_0-2} = F_{p_0,\infty}^{s_0-2}(\Omega)$ and $E_{p_1,q_1;T}^{s_1} = F_{p_1,\infty;T}$; the other cases follow from this at the end. This allows us to work with the function $\sigma(s,p)$ from (2.8), or more relevantly

$$\delta(s,p) := \sigma(s,p) - (s - 2),$$  \hfill (4.24)

which measures the deviation of $g(\cdot)$’s order from that of $A_T$. Thus $\sigma_1 + 2$ above (4.23) should be replaced by $s_1 + \delta(s_1,p_1)$, but for convenience we let $\delta_j = \delta(s_j,p_j)$ in the following.

4.2.1. The Worst Case. The sets corresponding to $\mathcal{D}(A_T + g(\cdot))$ in [Joh93, Joh95b, Joh] are all convex, so to begin with we first consider the case when

$$\left(\frac{p_0}{p_1}, s_0\right) \text{ and } \left(\frac{p_0}{p_1}, s_1 + \delta_1\right)$$  \hfill (4.25)

cannot be connected by a straight line in the $(\frac{p}{p_1}, s)$-plane. The worst case is when this is caused by the hyperbola defined by condition (iii) in Theorem 2.1. (The other possibility stems from the condition $s > \frac{p}{p_1} - \frac{p}{p_0}$.)

If $s_1 + \delta_1 > s_0$ we note that also $s_1 + \delta_1 - \frac{n}{p_1} > s_0 - \frac{n}{p_0}$ (otherwise there would be a connecting straight line), and therefore $E_{p_1,q_1}^{s_1+\delta_1}$ is embedded into $E_{p_0,q_0}^{s_0}$. Thus $(s_2,p_2,q_2) = (s_0,p_0,q_0)$ is possible and the conclusion (reached above) that $u \in E_{p_2,q_2}^{s_2}$ is already the desired one.
Figure 2. An example of the worst case procedure. Spaces containing \( u(x) \) and the non-linear term \( R_0 g(u) \) are indicated by \( \times \) and \( \circ \), respectively; arrows stand for embeddings, while dotted lines indicate new information on \( R_0 g(u) \).

For the case \( s_0 > s_1 + \delta_1 \) we explain our procedure in the following; Figure 2 illustrates the strategy. Observe first that for \( k = 1 \) the inequality

\[
s_0 - \frac{n}{p_0} \geq s_k + \delta_k - \frac{n}{p_k}
\]

may be either true or false. If it is false, the point \( \left( \frac{n}{p_k}, s_k + \delta_k \right) \) lies above the line of slope 1 through \( \left( \frac{n}{p_0}, s_0 \right) \), hence these points can be connected by a straight line; this situation is treated further below in Subsection 4.2.2 (and also illustrated in Figure 2). We proceed to show that (4.26) is false eventually for a certain choice of the parameters \( (s_j, p_j, q_j) \) for \( j \geq 2 \).

Suppose therefore that for some \( j \in \mathbb{N} \) we have shown that \( u \) is in a space \( E_{p_j, q_j}^{s_j} \) fulfilling the inequality in (4.26) and \( \delta_j > 0 \). There are three
possibilities for the definition of \((s_{j+1}, p_{j+1}, q_{j+1})\), cf. 1°–3° below that apply in the given order (possibly 1° or even 1° and 2° is redundant).

1° First we consider the case where

\[
\begin{align*}
(1) & \quad \frac{n}{p_j} - \delta_j \geq 0 \\
(2) & \quad \frac{n}{p_j} > \min(\frac{n}{p_0}, 3 + \sqrt{8})
\end{align*}
\]  

(4.27)  

(4.28)

both hold. Then we take a Sobolev embedding

\[
E^{s_j+\delta_j}_{p_j,q_j} \hookrightarrow E^{s_j}_{p_{j+1},q_j}
\]

with \(\frac{n}{p_j+1} = \frac{n}{p_j} - \delta_j\); this is possible since the inequalities \(\infty \geq p_{j+1} > p_j\) follow from (I) and \(\delta_j > 0\). Moreover we let

\[
(s_{j+1}, p_{j+1}, q_{j+1}) = (s_j, p_{j+1}, q_j)
\]

(4.30)

and it is seen that \(s_j = s_1\) and \(q_j = q_1\) result from (4.30) for all \(j\). By the definition of \((s_{j+1}, p_{j+1}, q_{j+1})\), and since \((4.26)\) for \(k = j\) and \(s_0 > s_j\) are assumed to hold, it is clear that we have \(E^{s_0}_{p_0,q_0} \hookrightarrow E^{s_{j+1}}_{p_{j+1},q_{j+1}}\), and hence

\[
u = R_0 f - R_0 g(u) + \mathcal{R} u \in E^{s_{j+1}}_{p_{j+1},q_{j+1}}.
\]

(4.31)

For this space containing \(u\) we find

\[
s_{j+1} + \delta_{j+1} - \frac{n}{p_{j+1}} = s_j + \delta_j - \frac{n}{p_j} + \delta_{j+1} > s_j + \delta_j - \frac{n}{p_j},
\]

(4.32)

because by Theorem 2.1 \(\delta(s_1, \cdot)\) is a non-decreasing function of \(p\), so that the gain \(\delta_{j+1}\) in (4.32) is bounded from below by the amount \(\delta_1 > 0\); in addition \(\delta_j \in [\delta_1, 2]\) since \(\sigma(s, p) \leq s\). After finitely many steps either (I) or (II) is false (because \(\frac{n}{p_j}\) is decreasing with \(j\)), in which case we proceed by 2° and 3°, or (4.26) itself is false.

2° When (I) is false but (II) is true, \(\frac{n}{p_0} \leq 3 + \sqrt{8}\) (otherwise \(3 + \sqrt{8} < \frac{n}{p_j}\) and since \(\delta_j\) is at most 2, then (I) would be true). Now a Sobolev embedding as above is impossible since (I) is false, but we take a ‘shorter’ one into \(E^{s_j+\delta_j}_{\infty,q_j}\) and let this have parameter \((s_{j+1}, p_{j+1}, q_{j+1})\). That \(\frac{n}{p_{j+1}} = 0\) gives \(\delta_{j+1} = 2\), so

\[
s_{j+1} + \delta_{j+1} - \frac{n}{p_{j+1}} = s_j + \delta_j - \frac{n}{p_j} + 2 \geq s_j + \delta_j - \frac{n}{p_j} + \delta_1,
\]

(4.33)
and the gain is at least $\delta_1$. This construction is at most used once, for either it makes (4.26) false or it brings one to the third case (since $\frac{n}{p_j+1} = 0$).

3° When (II) is false we observe first that $\delta(s, p_j) > 0$ for all $s > 0$ if $\frac{m}{p_j} < 3 + \sqrt{8}$. Indeed, as noted after (3.5), $\max d(s) = (\sqrt{\frac{m}{p_j}} - 1)^2$ and

$$
(\sqrt{\frac{m}{p_j}} - 1)^2 = 2 \iff \sqrt{\frac{m}{p_j}} = 1 + \sqrt{2} \iff \frac{m}{p_j} = 3 + \sqrt{8}, \tag{4.34}
$$

so if $\frac{m}{p_j} < 3 + \sqrt{8}$ we have $d(s) < 2$ for all $s \in [1;\frac{m}{p_j}[,$ and hence $\delta(s, p_j) \geq 2 - \max d(\cdot)) =: \alpha > 0$ (regardless of whether $1 < s < \frac{m}{p_j}$ or not).

Now if $\frac{m}{p_j} = 3 + \sqrt{8}$ there is the freedom to make a single Sobolev embedding of $E^{s_j, q_j}_{p_j, q_j}$ (thereby defining $(s_j+1, p_j+1, q_j+1)$ without any gain), so we can assume that

$$
\frac{m}{p_j} < 3 + \sqrt{8}, \tag{4.35}
$$

whenever (II) is false. Then $\delta(s, p_j) > 0$ for all $s > 0$ as noted first.

Now we simply go upwards, that means we let

$$(s_{j+1}, p_{j+1}, q_{j+1}) = (s_j + \delta_j, p_j, q_j). \tag{4.36}$$

Because (4.26) holds for $k = j$, there is an embedding $E^{s_0, q_0}_{p_0, q_0} \rightarrow E^{s_j+\delta_j}_{p_j, q_j}$ since also $p_j \geq p_0$ holds by the negation of (II). Again $u \in E^{s_j+1}_{p_j+1, q_j+1}$, only this time with a gain $s_{j+1} + \delta_{j+1} - (s_j + \delta_j) = \delta_{j+1}$. Since $\frac{m}{p_k} = \frac{m}{p_j}$ for all $k > j$ in this procedure, (II) remains false; and we have $\delta_k \geq \alpha > 0$ for all $k$, so (4.26) is violated in a finite number of steps.

Consequently, when the $(s_j, p_j, q_j)$ are defined as above, then for a finite $k$ the function $u(x)$ belongs to some $E^{s_k}_{p_k, q_k}$ for which (4.26) false. Moreover, $(s_k, p_k, q_k) \in D(A_T + g(\cdot))$, for it is clear (but tedious to prove) that this set is stable under 1°, 2° and 3° above.

However, this means that the considered case has been reduced to one of those treated in the next subsection.

4.2.2. The Main Argument. We return to a sketch of the full proof, which eventually would go through the same cases as those considered in [Joh];
there a proper exposition for problems of product-type is given. \cite{Joh95b} gives a concise presentation of the ideas, which originated in \cite{Joh93}.

First of all, if \( R_0 f + Ru \in E_{p_0,q_0}^{s_0} \) and \( R_0 g(u) \in E_{p_1,q_1}^{s_1+\delta_1} \) in (4.22) with

\[
s_1 + \delta_1 \geq s_0 \quad \text{and} \quad s_1 + \delta_1 - \frac{n}{p_1} \geq s_0 - \frac{n}{p_0},
\]

then there is actually an embedding \( E_{p_1,q_1}^{s_1+\delta_1} \hookrightarrow E_{p_0,q_0}^{s_0} \), so from (4.22) it follows that \( u \in E_{p_0,q_0}^{s_0} \) (as also used in the beginning of Subsection 4.2.1).

Secondly, there is the case with

\[
s_1 + \delta_1 < s_0 \quad \text{and} \quad s_1 + \delta_1 - \frac{n}{p_1} \geq s_0 - \frac{n}{p_0},
\]

(This, and (4.37), is the one that the worst case was reduced to in Subsection 4.2.1 above.) The spaces \( E_{p_j,q_j}^{s_j} \) considered for this case in \cite{Joh} all have \( (\frac{n}{p_j}, s_j) \) lying on or above each of the two lines \( s = s_1 + \delta_1 \) and \( s = \frac{n}{p} + s_0 - \frac{n}{p_0} \), so it is geometrically clear that all these \( (s_j, p_j, q_j) \) belong to \( \mathbb{D}(A_T + g(\cdot)) \). See also Figure 2 after the first horizontal arrow. Hence, by \cite{Joh}, we obtain \( u \in E_{p_0,q_0}^{s_0} \).

Thirdly, when

\[
s_1 + \delta_1 \geq s_0 \quad \text{and} \quad s_1 + \delta_1 - \frac{n}{p_1} < s_0 - \frac{n}{p_0},
\]

already \( (s_2, p_2, q_2) \) defined as in (4.26) may be outside of \( \mathbb{D}(A_T + g(\cdot)) \) because the condition \( s_2 > r + \frac{n}{p_2} - 1 \) may be violated.

However, it is a main point of \cite{Joh93, Joh95b, Joh} that such problems can be overcome if \( \delta(s, p) \) satisfies additional conditions, and these can be verified in our case. (Phrased briefly, \( R_0 g(\cdot) \) should be defined on \( E_{p_2,q_2}^{s_2} \) when the problem occurs for \( (s_2, p_2, q_2) \), then \( p_2 > 1 \). For \( r = 1 \), \( R_0 g(\cdot) \) makes sense on \( E_{p,q}^{s_2} \) as soon as \( s > 0 \) and \( p > 1 \), for \( g(\cdot) \) has order 0 on \( L_p \), where \( R_0 \) is defined; and if \( r = 2 \), then \( s_2 > 1 \), and \( g(E_{p_2,q_2}^{s_2}) \subset H_{p_2}^1 \).

Non-convexity problems do not occur either.

Finally, when the spaces are such that

\[
s_1 + \delta_1 < s_0 \quad \text{and} \quad s_1 + \delta_1 - \frac{n}{p_1} < s_0 - \frac{n}{p_0}.
\]
the procedure in \([\text{Joh}]\) is just to go upwards as in (4.36). Evidently this may be inappropriate here if \(n_p \geq 3 + \sqrt{8}\), as one will hit the bulge defined by condition (iii) in Theorem 2.1.

However, as described in the worst case analysis in 4.2.1, it is possible first to move left of \(n_p = 3 + \sqrt{8}\) (1°), if necessary make sure that \(n_p < n_p^0\) too (2°), and then move upwards until a reduction to (4.37) or (4.38) is achieved (3°, with an intermediate step if some \(n_p^j\) equals \(3 + \sqrt{8}\)).

In general the strategy of \([\text{Joh}]\) in this case is to move upward s until (4.40) is not valid any longer (with \(s_j\) and \(p_j\) replacing \(s_1\) and \(p_1\)), thus obtaining a reduction to the cases in (4.37), (4.38), and (4.39). The procedure in Subsection 4.2.1 serves the same purpose, so the argument of \([\text{Joh}]\) may be applied the rest of the way to get \(u \in E_{s_0}^{p_0, q_0}\) also in this situation.

Finally, note that \(D(A_T + g(\cdot))\) is an open set defined by sharp inequalities, so we can weaken the assumption on \(u(x)\) slightly to begin with. Thus it is not a restriction to assume \(E_{s_1}^{p_1, q_1, T} = E_{p_1, \infty, T}^{s_1}\).

Since \(f \in F_{p_0, \infty} = E_{p_0, \infty, T}^{s_0, -2, -\varepsilon}(\Omega)\) and \((s_0 - \varepsilon, p_0, \infty) \in D(A_T + g(\cdot))\) for \(\varepsilon > 0\) small enough, \(u \in E_{p_0, \infty, T}^{s_0, -2, -\varepsilon}(\Omega)\) according to the proof given above. So by (4.23) and the fact that \(\sigma > s - 2\) is possible near \((s_0, p_0, q_0)\), we get \(u \in E_{p_0, q_0, T}^{s_0}\).

Altogether this completes the proof of Theorem 2.2.

**Remark 4.3.** Although the basic formula (4.22) is not surprising, it has to be derived in the indicated way, for if one rearranges before the application of \(R_0\), then \(R_0\) may be undefined on \(E_{p_0, q_0}^{s_0, -2} + E_{p_1, q_1}^{s_1, -2}\) (that contains \(f - g(u)\)). Moreover, in such cases the usual regularity statements for elliptic problems cannot be used, so then it is necessary to utilise the parametrix \(R_0\).

5. **The Existence Results**

From the Leray–Schauder theorem we now deduce that solutions exist as described in Theorem 2.3.
It suffices to treat the case where the data space has the form

\[ E_{p_1,q_1}^{s_1-2} \] for some \( s_1 < 2 \) and \( p_1, q_1 \in [1, \infty] \).

(5.1)

To see this, we may for the actual data space \( E_{p,q}^{s-2} \) use a Sobolev embedding

\[ E_{p,q}^{s-2} \hookrightarrow E_{p_1,q_1}^{s_1-2} \quad \text{for} \quad s - \frac{n}{p} = s_1 - \frac{n}{p_1}, \quad q = q_1 \]

(5.2)

when \( s - \frac{n}{p} < 2 \) (since \( s_1 - 2 - \frac{n}{p_1} < 0 \) in (5.1)); for \( s - \frac{n}{p} \geq 2 \) one can take

\[ E_{p,q}^{s-2} \hookrightarrow E_{p_1,q_1}^{-1/2} =: E_{p_1,q_1}^{s_1-2}. \]

(5.3)

For the corresponding solution spaces the inclusion \( E_{p,q,T}^s \subset L_{t_0} \) for \( t^{-1} = (\frac{1}{p} - \frac{s}{n})_+ \) carries over to \( E_{p_1,q_1,T}^{s_1} \) for the same \( t \); that is, both (II) and (III) are invariant under the reduction.

So when (5.1) is covered, there is to any \( f \in E_{p,q}^{s-2} \subset E_{p_1,q_1}^{s_1-2} \) a solution \( u \in E_{p_1,q_1,T}^{s_1} \); for it is easy to see that \( (s_1,p_1,q_1) \) is or may be taken in \( \mathcal{D}(AT + g(\cdot)) \) (as for (i), \( s_1 \) should be taken in the gap between the lines \( s = r + \frac{1}{p} - 1 \) and \( s = r \) (then \( p_1 > 1 \) follows since \( s - \frac{n}{p} > r - n \) by (i));

(i) implies (ii), and (iii) is redundant for \( s < 3 \).

But then, from the assumption \( (s,p,q) \in \mathcal{D}(AT + g(\cdot)) \), we infer from Theorem 2.2 that \( u \) belongs to \( E_{p,q,T}^s \).

So consider some \( (s,p,q) \) in \( \mathcal{D}(AT + g(\cdot)) \) with \( s < 2 \) and \( 1 < p, q \leq \infty \).

When \( AT = A_T^* \) in \( L_2 \), the space \( \ker AT \) with \( Q = P \) may be used as a range complement for \( AT \) for every \( (s,p,q) \) according to Proposition 4.2.

Moreover, with \( Q^c = I - Q \) it is clear that \( Q^c(E_{p,q,T}^s) \subset E_{p,q,T}^s \), and since \( AT \) by restriction is a bijection from \( Q^c(E_{p,q,T}^s) \) to \( Q^c(E_{p,q,T}^{s-2}) \), there is an inverse \( B \) of this, that is

\[ B : Q^c(E_{p,q,T}^{s-2}) \to Q^c(E_{p,q,T}^s), \]

(5.4)

\[ BA = 1 \quad \text{on} \quad Q^c(E_{p,q,T}^s), \quad AB = 1 \quad \text{on} \quad Q^c(E_{p,q,T}^{s-2}). \]

(5.5)

These facts apply formally equally well to the case when \( AT \) is invertible.
Obviously $A_T u + g(u) = f$ is equivalent to the system
\begin{align*}
v &= \lambda B Q^c (f - g(v + w)) \\
w &= \lambda w + \lambda Q (f - g(v + w))
\end{align*}
when $\lambda = 1$, $v = Q^c u$ and $w = Q u$. Here the transformation
\[ (v, w) \mapsto (B Q^c (f - g(v + w)), w + Q (f - g(v + w))) \]
is continuous on $Q^c (E_{p,q}^s \times \ker A_T)$ by Lemma 3.1 and maps bounded sets to compact ones because $g(\cdot)$ does so from $E_{p,q}^s$ to $E_{p,q}^{s-2}$. So by the Leray–Schauder theorem (5.6) is solvable for $\lambda = 1$, if there exist $c_1$ and $c_2$ in $[0, \infty[$ such that for every $\lambda \in [0, 1]$, any solution satisfies
\begin{equation}
\| v \|_{E_{p,q}^s} < c_1, \quad \| w \|_{E_{p,q}^s} < c_2.
\end{equation}
Assuming a solution of (5.6) does not exist for $\lambda = 1$, then $L_\infty \hookrightarrow E_{p,q}^{s-2}$ (which holds by (2.33) since $s < 2$) and (5.6) gives
\begin{equation}
\| v \|_{E_{p,q}^s} \leq c (\| f \|_{E_{p,q}^{s-2}} + \| g \|_{L_\infty}) =: c_1;
\end{equation}
hence $c_2$ does not exist. Thus there is for each $N \in \mathbb{N}$ a solution $(v_N, w_N)$ of (5.6) for some $\lambda_N \in [0, 1]$ such that
\begin{equation}
\| v_N \|_{E_{p,q}^s} < c_1 \quad \text{and} \quad \| w_N \|_{E_{p,q}^s} \geq N.
\end{equation}
Passing to a subsequence if necessary, a sequence of solutions $(v_k, t_k w_k)$ to (5.6) is found such that $\| v_k \|_{E_{p,q}^s} < c_1$ and
\begin{equation}
\| w_k \|_{L_\infty} = 1, \quad t_k \to \infty \quad \text{for} \ k \to \infty
\end{equation}
Here it is used that all norms on $\ker A_T$ are equivalent. Furthermore, we can assume that for some $w_0 \in \ker A_T$,
\begin{equation}
w_k \to w_0 \quad \text{in} \ L_\infty(\Omega);
\end{equation}
indeed, by (5.11) a subsequence converges $w^*$ in $L_\infty$ and, because $\ker A_T$ is finite dimensional, also uniformly with limit $w_0$ in $\ker A_T$.

By (5.11), $A_T$ is not invertible. Moreover, $\langle Q f, w_k \rangle = \langle f, Q w_k \rangle$ because $f$ may be approximated from $C^\infty(\Omega)$ and because $Q$ is $L_2$-selfadjoint. With
\( W_k := t_k w_k \), then the fact that \((v_k, W_k)\) is a solution of (5.6) gives
\[
\int_\Omega W_k^2 dx = \lambda_k \int_\Omega W_k^2 dx - \lambda_k \int_\Omega Q(g(v_k + W_k) - f)W_k dx \tag{5.13}
\]
or equivalently
\[
\int_\Omega g(v_k + W_k)w_k dx - \langle f, w_k \rangle = \frac{\lambda_k - 1}{\lambda_k t_k} \| W_k \|_{L_2}^2 \tag{5.14}
\]
Because \( \lambda_k \in [0, 1] \), the right hand side is strictly negative, so since \((v_k)\) is bounded in \( L_{t-\alpha} \) and \( k \) is arbitrary, (II) does not hold.

Replacing \( \lambda Q \) by \(-\lambda Q\) in (5.6) yields (5.14) with \( 1 - \lambda_k \) instead of \( \lambda_k - 1 \); hence (III) does not hold either. The proof is complete.

6. Final Remarks

Remark 6.1. As mentioned in Section 2, the function \( \sigma(s, p) \) is conjectured to give the best possible smoothness index of \( E_{p,q}^\sigma \), the codomain of \( g(\cdot) \) applied to \( E_{s,p,q}^s \), even for any \( p, q \in [0, \infty] \) and any \( s > \max(0, \frac{n}{p} - n) \).

On the one hand, for \( 1 < s < \frac{n}{p} \), this is known to be correct if e.g. \( g(t) = \sin t \), for then when \( \varepsilon > 0 \) there exists \( u_\varepsilon \in E_{s,p,q}^s \) with \( g(u_\varepsilon) \notin E_{p,q}^{\sigma(s,p) + \varepsilon} \). For this we refer to [Sic89] and the more extensive treatment in [RS96].

On the other hand \( g \) need not be periodic, cf. the classes introduced in [RS96]; there isn’t complete freedom since \( g(t) = ct \) evidently acts on \( E_{s,p,q}^s \).

However, for a subrange of \( 1 < s < \frac{n}{p} \), only this \( g(t) \) has that property, as proved by Dahlberg [Dah79] for the \( W_p^s \), and this function moreover falls outside \( C_0^\infty(\mathbb{R}) \), in which we seek \( g(t) \) in the present article. Thus it requires further knowledge on \( g(t) \) to have another boundary for the parameter domain \( \mathcal{D}(AT + g(\cdot)) \) than the hyperbola in (iii) of Theorem 2.1.

Remark 6.2 (Quasi-Banach spaces). Our existence results are all based on the Leray–Schauder theorem, although the spaces are merely quasi-Banach when \( p < 1 \) or \( q < 1 \); but the theorem was applied for \( p, q > 1 \), for in ff. we reduced to this case by means of the regularity result in Theorem 2.2. However, the mapping degree has been extended to the full Besov and Triebel–Lizorkin scales (although this was not used here), cf. [FR87].
Theorems 2.2 and 2.3 are based on the linear elliptic theory in \cite{Joh96}, where the Fredholm properties for \( p \) and \( q \in ]0,1[ \) are obtained from a reduction, this time by embeddings, to the Banach cases with \( p, q > 1 \); cf. \cite{Joh96} Rem. 5.1. In addition one can extend the Fredholm concept to quasi-Banach spaces with separating duals as in \cite{FR95}.

**Remark 6.3 (Continuity vs. boundedness).** In the definition of \( \mathbb{D}(A_T + g(\cdot)) \) it suffices to require \( g(\cdot) \) bounded \( E_{p,q}^s \to E_{p,q}^{s-2} \), for this is the only relevant property for whether \( A_T \) or \( g(\cdot) \) is the dominant operator. Hence continuity of \( g(\cdot) \) is not needed in Theorem 2.2 whereas it is for Theorem 2.3 in which case it is provided by Lemma 3.1 at once.

**Remark 6.4.** The present pseudo-differential approach to the inverse regularity properties has predecessors for simpler problems of product-type, primarily the stationary Navier–Stokes equations with various boundary conditions, cf. \cite{Joh93, Joh95a, Joh}. Comparisons with the present problem are made in the beginning of Section 4.2 and Subsections 4.2.1 and 4.2.2.

**Remark 6.5 (Data beyond the borderline).** In Theorem 2.2 the conclusion can be obtained even for \( f(x) \) in some \( E_{p_0,q_0}^{s_0-2} \) outside of \( \mathbb{D}(A_T + g(\cdot)) \), at least when \( (s_1,p_1,q_1) \in \mathbb{D}(A_T + g(\cdot)) \) with \( s_1 > 1 \). More precisely, a range of \( (s_0,p_0,q_0) \) violating (iii) in Theorem 2.1 can then be treated. E.g. if \( s_0 < \sigma(s_1,p_1) + 2 \) this is trivial since \( E_{p_1,q_1}^{\sigma(s_1,p_1)+2} \to E_{p_0,q_0}^{s_0} \) in (4.23) then.

More generally one could ask for \( s_0 > \sigma(s_1,p_1) + 2 \) with \( (s_0,p_0,q_0) \) outside of \( \mathbb{D}(A_T + g(\cdot)) \). We have an argument based on interpolation and composition estimates with fixed \( s \) and variable \( p \) that yields \( u \in E_{p_0,q_0}^{s_0} \) provided \( (s_0,p_0,q_0) \) is close to \( \mathbb{D}(A_T + g(\cdot)) \) — but we omit the details here.

However, this emphasises that direct regularity properties like those in Theorem 2.1 and inverse regularity properties, of which there are some in Theorem 2.2, should be analysed separately, since for non-linear problems these notions allow different sets of parameters \( (s,p,q) \) to be considered.
Acknowledgements

This work was done partly during the first author’s stay at the Friedrich–Schiller University of Jena, and J. Johnsen is grateful for the warm hospitality he enjoyed at the Mathematics Department there. In addition we thank W. Sickel and S. I. Pohožaev for discussions on the subject.

References

[AAM78] H. Amann, A. Ambrosetti, and G. Mancini, Elliptic equations with noninvertible Fredholm linear part and bounded nonlinearities, Math. Z. 158 (1978), 179–194.

[AM78] A. Ambrosetti and G. Mancini, Existence and multiplicity results for nonlinear elliptic problems with linear part at resonance, J. Differential Equations 28 (1978), 220–245.

[BdM71] L. Boutet de Monvel, Boundary problems for pseudo-differential operators, Acta Math. 126 (1971), 11–51.

[BN78] H. Brézis and L. Nirenberg, Characterizations of the range of some nonlinear operators and applications to boundary value problems, Ann. Scuola Norm. Sup. Pisa 5 (1978), 225–326.

[Dah79] B. E. J. Dahlberg, A note on Sobolev spaces, Proc. Symp. Pure Math., vol. 35, Part I, Amer. Math. Soc., 1979, pp. 183–185.

[FR87] J. Franke and T. Runst, On the admissibility of function spaces of type $B^{s}_{p,q}$ and $F^{s}_{p,q}$, and boundary value problems for non-linear partial differential equations, Anal. Math. 13 (1987), 3–27.

[FR88] J. Franke and T. Runst, Non-linear perturbations of linear non-invertible boundary value problems in function spaces of type $B^{s}_{p,q}$ and $F^{s}_{p,q}$, Czechoslovak Math. J. 38 (1988), 623–641.

[FR95] J. Franke and T. Runst, Regular elliptic boundary value problems in Besov–Triebel–Lizorkin spaces, Math. Nachr. 174 (1995), 113–149.

[Fra86] J. Franke, Elliptische Randwertprobleme in Besov–Triebel–Lizorkin–Raümen, 1986, Dissertation, Friedrich–Schiller–Universität, Jena.

[GH91] G. Grubb and L. Hörmander, The transmission property, Math. Scand. 67 (1991), 273–289.

[GK93] G. Grubb and N. J. Kokholm, A global calculus of parameter-dependent pseudodifferential boundary problems in $L_p$ Sobolev spaces, Acta Math. 171 (1993), 165–229.

[Gru86] G. Grubb, Functional calculus of pseudo-differential boundary problems, Progress in Mathematics, vol. 65, Birkhäuser, Boston, 1986.

[Gru90] G. Grubb, Pseudo-differential boundary problems in $L_p$-spaces, Comm. Part. Diff. Equations 15 (1990), 289–340.

[Gru91] G. Grubb, Parabolic pseudo-differential boundary problems and applications, Microlocal analysis and applications, Montecatini Terme, Italy, July 3–11, 1989 (Berlin) (L. Cattabriga and L. Rodino, eds.), Lecture Notes in Mathematics, vol. 1495, Springer, 1991.
