DONALDSON-UHLENBECK-YAU THEOREM OVER NORMAL VARIETIES WITH MULTI-KÄHLER METRICS

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Abstract. A singular Donaldson-Uhlenbeck-Yau theorem over a normal complex variety with multi-Kähler metrics is proven. Furthermore, it is shown that the Hermitian-Einstein metric gives a lower bound for the discriminants of any resolutions. In particular, this gives a Gieseker-Bogomolov inequality over normal varieties and a characterization of the equality using projectively flat connections. Special cases include normal Kähler varieties and projective normal varieties with multipolarizations.

1. Introduction

The celebrated Donaldson-Uhlenbeck-Yau theorem ([12] [27]) confirms the existence of Hermitian-Einstein metrics on stable holomorphic vector bundles over compact Kähler manifolds. This gives the Gieseker-Bogomolov inequality for stable bundles over Kähler manifolds together with a characterization of the equality using projectively flat metrics. There are also further important developments focusing on different aspects: stable Higgs bundles over Kähler manifolds by Simpson ([23]); stable bundles over compact complex manifold with Gauduchon metrics by Li and Yau ([18] [20]); stable reflexive sheaves over Kähler manifolds by Bando and Siu ([1]).

Recently it was observed in [6] that a new class of Gauduchon metrics can be given by the so-called balanced metrics of Hodge-Riemann type, of which the multipolarizations provide a natural class of examples. Namely, given $(n-1)$ Kähler forms $\omega_1 \cdots, \omega_{n-1}$, $\omega_1 \wedge \cdots \wedge \omega_{n-1}$ defines a balanced metric. More importantly, the Hodge-Riemann property still holds for $\omega_1 \wedge \cdots \omega_{n-2}$ (see [25]) which gives the Gieseker-Bogomolov inequality in the multipolarization setting (see Section 2.2). The latter had been known when $\omega_i$ are all Hodge metrics which follows from the restriction theorem ([17] [16]).

For a balanced metric coming from the multipolarization above, $[\omega_1 \wedge \cdots \wedge \omega_{n-1}]$ lies in the interior of the so-called cone of movable curves of compact complex manifolds ([2]). Also, the notion of stable sheaves defined via multipolarizations ([14]) already plays an important role in compactifying the moduli space of semistable sheaves over projective manifolds in higher dimensions. More generally the slope stability via movable class has been

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defined and studied on normal varieties, and it is a very important and useful concept in birational geometry (13 4).

We are led to ask the natural question about whether a version of Donaldson-Uhlenbeck-Yau theorem and Gieseker-Bogomolov inequality still exist in this situation. Suppose \( X \) is a normal variety of dimension \( n \) endowed with \((n-1)\) Kähler forms \( \omega_1, \cdots, \omega_{n-1} \) (see Section 2.1 for definitions). Through the equation \( \omega^{n-1} = \omega_1 \wedge \cdots \omega_{n-1} \), this defines a balanced metric \( \omega \). A stable reflexive sheaf can be defined by passing to a particular resolution \( p : \tilde{X} \to X \) so that \( \tilde{E} := (p^*\mathcal{E})^{**} \) is stable with respect to \( p^*\omega_1 \wedge \cdots p^*\omega_{n-1} \). The stability is independent of the resolutions since everything is smooth in codimension two, it can be computed by using any resolutions (see Section 2.3).

**Theorem 1.1.** Suppose \( \mathcal{E} \) is a stable reflexive sheaf over a normal variety \((X, \omega_1 \wedge \cdots \omega_{n-1})\). Then there exists an admissible Hermitian-Einstein metric on \( \mathcal{E} \). Furthermore, for any global section \( s \) of \( \mathcal{E} \), \( \log^+ |s|^2 \in W^{1,2} \cap L^\infty \). In particular, such a metric is unique up to scaling.

As a direct corollary, this gives

**Corollary 1.2.** Suppose \( \mathcal{E}_1, \mathcal{E}_2 \) are stable reflexive sheaves over \((X, \omega_1 \wedge \cdots \omega_{n-1})\), then \((\text{Sym}^b \mathcal{E}_1)^{**}, (\wedge^k \mathcal{E}_2)^{**} \) and \((\mathcal{E}_1 \otimes \mathcal{E}_2)^{**} \) are all polystable.

The Hodge-Riemann property ensures that the Hermitian-Einstein metric \( H \) gives an analytic Gieseker-Bogomolov inequality. Unless the variety is smooth in codimension two, it is in general expected that it does not compute any corresponding algebraic quantities. However, for the algebraic side, fix any \( 1 \leq i_1 < \cdots i_{n-2} \leq n-2 \), we can still define the discriminant as

\[
(2rc_2(\mathcal{E}) - (r-1)c_1(\mathcal{E})^2)[\omega_{i_1}] \cdots [\omega_{i_{n-2}}] = \inf_p (2rc_2(\tilde{\mathcal{E}}) - (r-1)c_1(\tilde{\mathcal{E}})^2)[p^*\omega_{i_1}] \cdots [p^*\omega_{i_{n-2}}]
\]

which is intrinsically associated to \( \mathcal{E} \); when \( X \) is smooth in codimension two, it can be computed by using any resolutions (see Section 2.3). It turns out the Hermitian-Einstein metric does give a lower bound for the discriminant, thus a version of Gieseker-Bogomolov inequality over normal varieties without requiring the variety being smooth in codimension two.

**Corollary 1.3.** Suppose \( \mathcal{E} \) is a stable reflexive sheaf over \((X, \omega_1 \wedge \cdots \omega_{n-1})\) and let \( H \) be the admissible Hermitian-Einstein metric as above. Then

\[
(2rc_2(\mathcal{E}) - (r-1)c_1(\mathcal{E})^2)[\omega_{i_1}] \cdots [\omega_{i_{n-2}}] \geq \int_X (2rc_2(H) - (r-1)c_1(H)^2) \wedge \omega_{i_1} \wedge \cdots \omega_{i_{n-2}}
\]

for any \( 1 \leq i_1 < \cdots i_{n-2} \leq n-1 \). In particular,

\[
(2rc_2(\mathcal{E}) - (r-1)c_1(\mathcal{E})^2)[\omega_{i_1}] \cdots [\omega_{i_{n-2}}] \geq 0
\]

where the equality holds if and only if \( \mathcal{E} \) is projectively flat.
Connections with recent work. In the Kähler variety case, i.e. \( \omega = \omega_1 = \cdots = \omega_{n-1} \), there has been some recent progress on the singular Donaldson-Uhlenbeck-Yau theorem. In the projective case ([7]), if \( \omega \) is the restriction of a Hodge metric from the ambient smooth variety, this has been proved by assuming the base is smooth in codimension two; a singular Donaldson-Uhlenbeck-Yau theorem has also been shown for stable sheaves over projective normal varieties which come from limits of stable sheaves over smooth projective varieties. Assuming a uniform Sobolev constant control for the resolution with perturbed Kähler metrics, the argument in [7] can be used to prove slightly more general results for normal projective varieties and subvarieties of Kähler manifolds smooth in codimension two. However, the control of the Sobolev constants for the perturbed Kähler metrics does not seem to follow from any known results unlike the smooth case in [1]. On the other hand, to use the heat flow to prove the singular Donaldson-Uhlenbeck-Yau theorem, the control of the Sobolev constants is very crucial. It is also the goal of this paper to remove all those restrictions over singular base and prove it in a more general natural context. When the base is smooth in codimension two, the Gieseker-Bogomolov inequality is known in the case of projective normal varieties, or more generally Kähler normal varieties, and the equality has been characterized with various characteristic classes vanishing conditions ([29, 9, 15, 13]).

Sketch of the proof. The strategy for the proof is pretty standard and has been used in [7]. It is done by passing to a resolution of singularities and studying the corresponding gauge theoretical limits, which goes back to [11, 27, 1]. The subtlety lies in how to take care of various technical difficulties in this process with new ideas. Start with any resolution \( p : \hat{X} \to X \) so that \( \hat{E} := (p^{*}E)^{**} \) is locally free. Fix any Kähler metric \( \theta \) on \( \hat{X} \). Then one can show \( \hat{E} \) is stable with respect to \( (p^{*}\omega_1 + i^{-1}\theta) \wedge \cdots \wedge (p^{*}\omega_{n-1} + i^{-1}\theta) \) for any \( i \gg 1 \), which essentially follows from the boundedness results in [26]. Thus the Donaldson-Uhlenbeck-Yau theorem for multipolarizations ([6]) gives a family of Hermitian-Einstein metrics \( H_i \) with respect to the perturbed metrics. Let \( \hat{E} \) be the underlying smooth bundle for \( \hat{E} \). By normalizing gauge, we get a sequence of Hermitian-Yang-Mills connections on \( \hat{E} \) with a unitary metric \( H \). By known gauge theoretical results ([8]), after passing to a subsequence, up to gauge transforms, \( A_i \) converges to a limiting Hermitian-Yang-Mills connection \( A_{\infty} \) over \( X \setminus Z \) where \( Z \) is some codimension two subvariety of \( X \). The proof is done in the following steps

1. there exists a nontrivial map \( \Phi : E_\infty \to E \); for any global section \( s \) of \( E_\infty \), \( \log^+|s|^2 \in W^{1,2} \cap L^\infty \).
2. prove the statement for rank 1 case by showing \( \Phi \) is an isomorphism and \( A_{\infty} \) computes \( \mu(E) \).
3. prove \( \Phi \) is an isomorphism for general rank.
For (1), we crucially use the fact that the singular set $Z$ has codimension at least 2. Fix a smooth metric $H'$ on $E$. The idea is to take limit of the sections given by the identity map $\text{id} \in \text{Hom}(E, E)$. We can take a precompact exhaustion $X_\epsilon$ of $X \setminus Z$ where $X \setminus X_\epsilon \to Z$ as $\epsilon \to 0$. Normalize the identity map to have $L^2$ norm equal to one over a fixed region $X_\epsilon$ with respect to the metric $H_\epsilon^* \otimes H'$ on $\text{Hom}(E, E)$ and the metric on $X$ given by $\omega_1 \wedge \cdots \wedge \omega_{n-1}$. This gives us a sequence of holomorphic sections. By standard elliptic theory, passing to a subsequence, one can take a limit of this sequence. But the problem is that we are working with noncompact base, thus the limit might be trivial. However, by using the fact that $Z$ has codimension two, we can prove a useful property: for any $z \in X_\epsilon^{\frac{1}{2}}$, there exists a holomorphic curve $D \subset X^{\text{reg}}$ passing $z$ and $\partial D \subset X_\epsilon$. This enables us to restrict our sections to $D$, and apply maximum principle to get control over $X_\epsilon^{\frac{1}{2}}$. In particular, we get a nontrivial limit over $X_\epsilon$, which by induction implies the existence of a nontrivial limit over $X^{\text{reg}}$. For the regularity statement about sections $s$ of $E_\infty$, one can first show $\log^+ |s|^2 \in W^{1,2}_{\text{loc}}$ by an adaption of Bando and Siu’s argument ([1, Theorem 2]) using one dimensional slices instead of two dimensional slices. Then it follows from [19] and [22] that there exists a global Sobolev inequality for $W^{1,2}(X^{\text{reg}})$ functions, thus one can apply Moser iteration to get the $L^\infty$ bound (see Section 2.4).

For (2), if we assume $\text{rank } E = 1$, then the Remmert-Stein extension theorem implies $E_\infty$ can be extended to be a reflexive sheaf over $X$ since we have a nontrivial map $\Phi : E_\infty \to E$. If we can show $\mu(E_\infty) = \mu(E)$ which will be a very crucial fact needed in (3) as well, the map has to be an isomorphism. This relies on a key observation that such a Chern-Weil formula still exists by using the fact that $A_\infty$ comes from the limit of smooth ones on the resolutions and the fact that $Z$ is a complex subvariety of codimension at least two.

For (3), by Siu’s theorem ([21]), one can show actually the saturation $G$ of the image of $\Phi$ defines a coherent analytic subsheaf of $E$. If $\text{rank } G < \text{rank } E$, then $\mu(G) < \mu(E)$ since $E$ is stable. By applying the Weizenböck formula to sections of $\text{Hom}(\wedge^{\text{rank } E_\infty} G), \text{det } G)$, this will give a contradiction. Here we need the crucial fact that $\text{det } G$ admits a Hermitian-Einstein metric which does compute the slope of $\text{det } G$ by step (2). In particular, $\Phi$ has full rank. Now by (2), we know $\text{det } (E_\infty) \cong \text{det } (E)$ which will force $\Phi$ to be an isomorphism. The conclusion follows.

Now we discuss the Bogomolov-Gieseker inequality. We first deal with $p : \hat{X} \to X$ so that $\hat{E} = (p^* \mathcal{E})^{**}$ is locally free. On the resolution, the quantities in the Bogomolov-Gieseker inequality can be directly computed by the perturbed Hermitian-Einstein metrics. Now the limit of this equation will give us what we need. This is due to the fact that by the Hodge-Riemann property for multipolarizations, the integrand given by the Hermitian-Einstein metrics on the resolution defines a sequence of Radon measures with uniformly bounded mass on $\hat{X}$, thus the inequality follows from Fatou’s lemma.
For general resolutions \( p : \hat{X} \to X \), this follows from the same argument by our main theorem applied to stable reflexive sheaves over \( \hat{X} \) and that the Chern-Weil formula still holds for such with admissible Hermitian-Einstein metrics.

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## 2. Preliminary results

### 2.1. Varieties with muti-Kähler metrics.

In this section, we will recall the notion of Kähler metrics on normal varieties from [28].

Let \( X \) be a normal variety. We will always write \( X = X^s \cup X^{\text{reg}} \) where \( X^s \) denotes the singular part of \( X \) having codimension at least two; \( X^{\text{reg}} \) denotes the smooth part of \( X \). A local function \( f \) on \( X \) is called to be smooth if for any local embedding of \( X \), i.e. \( X \cap U \to U \subset \mathbb{C}^N \), it can be extended to be a smooth function. Now a local smooth strongly plurisubharmonic function on \( X \) means a smooth function which can be extended to be a smooth strongly plurisubharmonic function for some embedding.

**Definition 2.1 (Definition-Lemma).** A Kähler metric \( \omega \) on \( X \) is defined by a cover \( \{(U_i, \rho_i)\} \), where \( U_i \subset \mathbb{C}^N \) is open subset, \( \rho_i \) is a smooth strongly plurisubharmonic function on \( U_i \), i.e. in a neighborhood of \( U_i \) in \( \mathbb{C}^N \), and \( \omega|_{X^{\text{reg}} \cap U_i} = \sqrt{-1} \partial \bar{\partial} \rho_i|_{X^{\text{reg}} \cap U_i} \). This gives a Čech class \( [\omega] \in H^2(X, \mathbb{R}) \), which is usually referred as a Kähler class on \( X \).

Now take any smooth resolution of \( p : \hat{X} \to X \). As a consequence of the definition, we have the following properties for Kähler metrics on varieties.

1. For each \( i \), \( p^*\omega_i \) defines a smooth de-Rham class in \( \hat{X} \) which follows from that we can pull back the smooth plurisubharmonic functions to define the pull-back of the corresponding Kähler class;
2. \( p^*[\omega_1] \wedge \cdots p^*[\omega_{n-1}]|_{p^{-1}(Z)} = 0 \) if \( Z \) has codimension at least two;
3. \( p^*[\omega_n] \wedge \cdots p^*[\omega_{n-2}]|_{p^{-1}(Z)} = 0 \) if \( Z \) has codimension at least three.

We need the following well-known fact, for which we include a short explanation

**Lemma 2.2.** Let \( Z \subset X \) be a subvariety containing \( X^s \). Suppose \( a \in H^*(\hat{X}, \mathbb{R}) \) with \( a|_{p^{-1}(Z)} = 0 \). Denote by \( [\Omega] \) the corresponding de-Rham class for \( a \), then

\[
\Omega = \Omega^0 + d\Phi
\]

where \( \Omega^0 \) is a smooth closed form with compact support in \( \hat{X} \setminus p^{-1}(Z) \).

**Proof.** Indeed, choose an open neighborhood \( U \) of \( p^{-1}(Z) \) which deformation retracts onto \( p^{-1}(Z) \), then \( H^*(U, \mathbb{R}) \cong H^*(p^{-1}(Z), \mathbb{R}) \) through the restriction map, which can be seen by using the singular Cohomology. In
particular, we know $a|_U$ is also trivial, thus the corresponding de-Rham cohomology class $[\Omega|_U]$ is also trivial, i.e. $\Omega|_U = d\Phi'$ over $U$. Choose a cut-off function $\rho$ which is 1 near $p^{-1}(Z)$. Then $\Phi = \rho \Phi'$ does the job. □

Given this, if $Z$ has codimension at least two, we can always write
\begin{equation}
p^*\omega_1 \wedge \cdots \wedge p^*\omega_{n-1} = \Omega^0_{n-1} + d\Phi_{n-1}
\end{equation}
where $\Omega^0_{n-1}$ is compact supported in $\tilde{X} \setminus p^{-1}(Z)$ and if $Z$ has codimension at least three then
\begin{equation}
p^*\omega_{i_1} \wedge \cdots \wedge p^*\omega_{n-2} = \Omega^0_{n-2} + d\Phi_{n-2}
\end{equation}
where $\Omega^0_{n-2}$ is compact supported in $\tilde{X} \setminus p^{-1}(Z)$.

We will also need the following observation

**Proposition 2.3.** Given two Kähler metrics $\omega_1, \omega_2$, there exists a constant $C > 0$ so that
\[ C^{-1} \omega_1 \leq \omega_2 \leq C \omega_1. \]

**Proof.** It suffices to build such a bound near each point $z \in X^s$. Cover $X$ with open sets $(U_i, \rho_i)$ where $U_i \subset \mathbb{C}^N$ and $\rho_i$ is a smooth strongly PSH function on $U_i$ and $\omega_1 = i\partial \bar{\partial} \rho_i$. Assume $z \in U_i$ for some fixed $i$. By assumption, we can always extend the local defining function $\rho_i'$ for $\omega_2$ to be a smooth function $U_i$. Thus we know
\[ \sqrt{-1} \partial \bar{\partial} \rho_i' \leq C_i \sqrt{-1} \partial \bar{\partial} \rho_i. \]
By compactness, we can cover $X$ with finitely many such open sets and choose $C$ to be the largest $C_i$. This gives $\omega_2 \leq C \omega_1$. The other inequality follows from symmetry where we might need to change $C$ a little. □

**Corollary 2.4.** Given $(n-1)$ Kähler metrics $\omega_{i_1}, \cdots, \omega_{i_{n-1}}$ on $X$, then for some constant $C > 0$
\[ C^{-1} \omega_{n-1} \leq \omega_1 \wedge \cdots \wedge \omega_{n-1} \leq C \omega_{i_{n-1}}. \]

2.2. Donaldson-Uhlenbeck-Yau theorem for balanced metrics of Hodge-Riemann type. In this section, we recall some results from [6]. Let $\mathcal{F}$ be a holomorphic vector bundle over a smooth compact complex manifold $Y$ with $n-1$ Kähler forms $\omega_1, \cdots, \omega_{n-1}$. Then by [25], we know $\omega_1, \cdots, \omega_{n-1}$ define a balanced metric $\omega$ through the following
\begin{equation}
\omega^{n-1} = \omega_1 \wedge \cdots \wedge \omega_{n-1}
\end{equation}
i.e. $\omega$ is a Hermitian metric with $d\omega^{n-1} = 0$. The Donaldson-Uhlenbeck-Yau theorem over complex manifolds with Gauduchon metrics gives ([18])

**Theorem 2.5.** Suppose $\mathcal{F}$ is stable over $(X, \omega_1 \wedge \cdots \omega_{n-1})$. There exists a Hermitian-Einstein metric $H$ on $\mathcal{F}$, i.e.
\[ \sqrt{-1} \Lambda_\omega F_H = \lambda \text{id} \]
where $F_H$ is the Chern curvature of the metric $H$. 


The Hermitian-Einstein equation implies
\[
\left( \sqrt{-1} \frac{1}{2\pi} F_H - \frac{1}{r} \text{tr} \left( \sqrt{-1} \frac{1}{2\pi} F_H \right) \text{id} \right) \wedge \omega_1 \wedge \cdots \wedge \omega_{n-1} = 0
\]
where \( r = \text{rank}(F) \). By Timorin’s results which generalize the classical Hodge-Riemann property for one Kähler form to multiple Kähler forms (see [25]), we know for any \( 1 \leq i_1 < \cdots < i_{n-2} \leq n-1 \) the following holds pointwisely
\[
-\text{tr} \left( \left( \sqrt{-1} \frac{1}{2\pi} F_H - \frac{1}{r} \text{tr} \left( \sqrt{-1} \frac{1}{2\pi} F_H \right) \text{id} \right) \wedge \left( \sqrt{-1} \frac{1}{2\pi} F_H - \frac{1}{r} \text{tr} \left( \sqrt{-1} \frac{1}{2\pi} F_H \right) \text{id} \right) \right) \wedge [\omega_{i_1}] \wedge \cdots \wedge [\omega_{i_{n-2}}] \geq 0,
\]
where the equality holds if and only if
\[
\sqrt{-1} \frac{1}{2\pi} F_H = \frac{1}{r} \text{tr} \left( \sqrt{-1} \frac{1}{2\pi} F_H \right) \text{id}.
\]
As a corollary, this generalizes the classical Gieseker-Bogomolov inequality to the multi-polarization setting

**Corollary 2.6** (Gieseker-Bogomolov inequality). Suppose \( F \) is stable over \((X, \omega_1 \wedge \cdots \omega_{n-1})\). Then for any \( 1 \leq i_1 < \cdots < i_{n-2} \leq n-1 \)
\[
(2rc_2(F) - (r-1)c_1(F))^2 \cdot [\omega_{i_1}] \cdots [\omega_{i_{n-2}}] \geq 0
\]
where the equality holds if and only if \( F \) is projectively flat.

### 2.3. Stability and admissible Hermitian-Einstein metric.
In this section, we will recall known definitions regarding stable reflexive sheaves in the case of normal Kähler varieties.

We fix \((X, \omega_1, \cdots, \omega_{n-1})\) to be a normal Kähler variety with \( n-1 \) Kähler metrics. Denote \( \omega \) to be the balanced metric defined by
\[
\omega^{n-1} = \omega_1 \wedge \cdots \omega_{n-1}.
\]
Let \( E \) be a reflexive sheaf over \( X \). We will recall the notion of Chern classes we want to deal with. We fix \( p : \hat{X} \to X \) to be any resolution and denote
\[
\hat{E} := (p^*E)^{**}.
\]
Then the Chern numbers we need can be defined as
\[
\text{deg}(E) = c_1(\hat{E})p^*[\omega_1] \cdots p^*[\omega_{n-1}]
\]
and for any \( 1 \leq i_1 < \cdots < i_{n-2} \leq n-1 \), the discriminant is defined as
\[
(2rc_2(E) - (r-1)c_1(E))^2 \cdot [\omega_{i_1}] \cdots [\omega_{i_{n-2}}]
\]
\[
= \inf_p \left( 2rc_2(\hat{E}) - (r-1)c_1(\hat{E})^2 \right) p^*[\omega_{i_1}] \cdots p^*[\omega_{i_{n-2}}]
\]
Define the slope of \( E \) as
\[
\mu(E) := \frac{\text{deg} E}{\text{rank} E}.
\]
Given a connection \( A \) on \( E \) defined away from the singular set, if everything is smooth, then the Chern numbers above can be computed using
Chern-Weil theory. In the following, we will still denote $c_i(A)$ as the forms corresponding to $c_i(E)$, and $c_i(H)$ for if $A$ is the Chern connection given by $H$.

As a direct corollary of Equation (2.1) and (2.2), we have

**Proposition 2.7.** $\deg(E)$ is independent of the choice of the resolutions $p$; if $X$ is smooth in codimension two, the discriminants are independent of the resolutions.

In particular, the following slope stability is well-defined

**Definition 2.8.** $E$ is called slope stable (resp. semistable) if for any $F \subset E$, $\mu(F) < \mu(E)$ (resp. $\mu(F) \leq \mu(E)$). $E$ is called polystable if it is a direct sum of stable ones.

The following fact will be used in later sections which we include a proof for completeness

**Lemma 2.9.** Suppose $E_1$ and $E_2$ are two stable reflexive sheaves with the same slope. Then any nontrivial map between $E_1$ and $E_2$ is an isomorphism. In particular, a stable reflexive sheaf is simple.

**Proof.** Suppose $\phi$ is such a map. By stability, we must have $\text{Ker} \phi = 0$, thus $\phi$ is injective. Then we have

$$0 \rightarrow \det(E_1) \xrightarrow{\det \phi} \det(E_2) \rightarrow \tau_D \rightarrow 0$$

for some $\tau_D$ which is a torsion sheaf supported on a subvariety $D \subset X$. By definition, we have

$$\deg(E_2) - \deg(E_1) = c_1(\tau_D).[\omega_1] \cdots [\omega_{n-1}]$$

Now any pure codimension one component of $D$ will contribute strictly positively to the equality above and give a contradiction, thus $\text{codim}_CD \geq 2$. The conclusion follows. \qed

2.4. **Admissible Hermitian-Einstein metrics and regularity results.**

Following [1], we use the following notion of admissible Hermitian-Einstein metric

**Definition 2.10.** An admissible Hermitian-Einstein metric is defined as a smooth Hermitian metric $H$ on a holomorphic bundle $F$ over $X \setminus Z$ where $Z$ is some codimension two subvariety of $X$ with $X_s \subset Z$, and the metric $H$ satisfies the following

- $\int_{X \setminus Z} |F_H|^2 < \infty$;
- $\sqrt{-1}\Lambda_\omega F_H = \lambda \text{id}$ where $\lambda$ is usually referred as the Einstein constant where $F_H$ denotes the curvature. The associated Chern connection $A$ is usually referred as an (admissible) Hermitian-Yang-Mills connection.

Now an adaption of the slicing argument in [1] using one dimensional slices instead of two dimensional slices gives the following regularity result. We include the proof for completeness here.
Proposition 2.11. For any local section $s$ of $F$, $\log^+ |s|^2 \in W^{1,2}_{\text{loc}}$.

Proof. We will use $Q_1 \lesssim (\geq) Q_2$ to denote $Q_1 \leq (\geq) C Q_2$ for some constant $C$. By Corollary 2.4, we can assume the metric is induced from the flat metric on $\mathbb{C}^N$. The statement is local. Fix $z \in X^s$. We can assume $z \in X \subset U \subset \mathbb{C}^N$ with $\omega$ being the restriction of the standard flat metric. By shrinking $U$, we can assume $U = B^t \times B^t$ and $z = 0$

$$p : X \cap U \to B^t$$

and for generic $t \in B^t \cap p(X \cap U)$, $p^{-1}(t)$ is a smooth curve while $p^{-1}(0)$ is a smooth curve with a point singularity at the origin. Let $u_t = \log^+ |s|^2|_{p^{-1}(t)}$. We have

$$\Delta_t u_t \gtrsim -|F_H|.$$ 

Let $\chi$ be a cut-off function on $B^t$ supported near 0. Then by doing integration by parts and applying Cauchy-Schwartz inequality, we have

$$\int_{p^{-1}(t)} |\nabla' (\chi u_t)|^2 \lesssim \delta \int_{p^{-1}(t)} (\chi u_t)^2 + \delta^{-1} \int_{p^{-1}(t)} \chi^2 |F_H|^2 + \int_{p^{-1}(t)} |\nabla' \chi|^2 u_t^2.$$ 

for any fixed $0 < \delta << 1$. With the induced metric on $p^{-1}(t)$, by [22, Theorem 18.6], there exists a Poincaré inequality for $p^{-1}(t)$ and the Sobolev constant is independent of $t$ since $p^{-1}(t)$ is a minimal submanifold of $U$. So we have

$$\int_{p^{-1}(t)} |\nabla' (\chi u_t)|^2 \lesssim \delta^{-1} \int_{p^{-1}(t)} \chi^2 |F_H|^2 + \int_{p^{-1}(t)} |\nabla' \chi|^2 u_t^2$$

which further implies

$$\int_{p^{-1}(K)} |\nabla' (\chi u_t)|^2 \lesssim \delta^{-1} \int_{p^{-1}(K)} \chi^2 |F_H|^2 + \int_{p^{-1}(K)} |\nabla' \chi|^2 u_t^2$$

for some compact set $0 \in K \subset p(X \cap U)$. Here $\nabla'$ denotes the derivative in the fiber direction. Now choose $(n-1)$ more such projections and add them together. This gives the bound we need. 

□

Corollary 2.12. For any global section $s$ of $F$, $\log^+ |s|^2 \in L^\infty$.

Proof. By [22], we know $X$ admits a Sobolev inequality for compact supported functions over $X^{\text{reg}}$ if $X$ is endowed with a fixed Kähler metric i.e.

$$\|f\|_{L^{2n/2}} \leq C(\|f\|_{L^2(X)} + \|\nabla f\|_{L^2(X)}).$$

for any $f \in C^1_c(X^{\text{reg}})$. More precisely, locally we can assume the Kähler metric is induced from the flat metric on $\mathbb{C}^N$. In particular, $X$ is locally a minimal submanifold of $X$, thus by [22, Theorem 18.6], there exists a Sobolev inequality for $X$. Now the approximation result in [19, Section 4] implies a Sobolev inequality for functions in $W^{1,2}(X^{\text{reg}})$. Note as pointed out in [19], this does not require the the variety to be projective. By Corollary 2.4, we
also have a Sobolev inequality for \((X, \omega_1 \wedge \cdots \omega_{n-1})\). Given this, we can apply the Moser iteration to \(\log^+ |s|^2\) which satisfies
\[
\Delta \log^+ |s|^2 \geq \lambda_\infty
\]
where \(\lambda_\infty\) is the Einstein constant of \(A_\infty\), thus \(\log^+ |s|^2 \in L^\infty\). This concludes the proof.

\[\square\]

**Corollary 2.13.** The admissible Hermitian-Yang-Mills connection on \(F\) is unique if it exists. Furthermore, if \(F\) can be extended to be a stable reflexive sheaf over \(X\), the Hermitian-Einstein metric is unique up to scaling.

**Proof.** Otherwise, suppose we have two such metrics \(H_1\) and \(H_2\) with Einstein constants \(\lambda_1 \geq \lambda_2\). Consider the identity map \(id \in \text{Hom}(F, F)\) with the endowed metric \(H_1^* \otimes H_2\). Then straightforward computation shows
\[
\Delta |id|^2 = 2|\nabla id|^2 - (\lambda_2 - \lambda_1) |id|^2.
\]
Since \(id \in L^\infty\) by Corollary 2.12, we can do integration by parts to conclude \(\nabla id = 0\), i.e. \(id\) is parallel and \(\lambda_2 = \lambda_1\). The uniqueness follows. If \(F\) comes from a stable reflexive sheaf over \(X\), write \(H_1(., .) = H_2(g, .)\) where \(g\) is Hermitian with respect to \(H_2\). The fact that \(id\) is parallel implies \(g\) is holomorphic, thus a multiple of the identity map by Lemma 2.9. \(\square\)

### 2.5. Hermitian-Einstein metrics using perturbations.

The following has been observed in the Kähler case (see [9, 29]) by using the boundedness result in [26]. Similar argument also works in our setting. Take a resolution \(p : \hat{X} \to X\) and let \(\hat{\mathcal{E}} := (p^* \mathcal{E})^{**}\).

**Proposition 2.14.** Suppose \(\mathcal{E}\) is a stable reflexive sheaf over \((X, \omega_1 \wedge \cdots \omega_{n-1})\), then \(\hat{\mathcal{E}}\) is stable over \((\hat{X}, (p^* \omega_1 + i^{-1} \theta) \wedge \cdots (p^* \omega_{n-1} + i^{-1} \theta))\) for \(i \gg 1\).

**Proof.** By passing to a further resolution ([21, Theorem 3.5]), it suffices to prove it when \(\hat{\mathcal{E}}\) is locally free. By definition, \(\hat{\mathcal{E}}\) is stable with respect to \((p^* \omega_1 \wedge \cdots p^* \omega_{n-1})\). Otherwise, the push-forward of the destabilizing subsheaf will destabilize \(\mathcal{E}\) as well. Now we argue by contradiction for the main statement. By passing to a subsequence, we have a sequence of quotient maps \(q_i : \hat{\mathcal{E}} \to \mathcal{F}_i\) and \(\mu_i(\mathcal{F}_i) \leq \mu_i(\hat{\mathcal{E}})\) where \(\mu_i\) denotes the slope of a sheaf with respect to the metric \((p^* \omega_1 + i^{-1} \theta) \wedge \cdots (p^* \omega_{n-1} + i^{-1} \theta)\). By choosing a metric on \(\hat{\mathcal{E}}\), an easy computation shows
\[
c_i(\mathcal{F}_i)[\theta][p^* \omega_{k_1}] \cdots [p^* \omega_{k_{l-1}}] \geq C
\]
for some \(C\) independent of \(l, k_1, \cdots k_{l-1}\). In particular, this implies
\[
\mu(\mathcal{F}_i) + O\left(\frac{1}{l}\right) \leq \mu_i(\hat{\mathcal{E}}).
\]
By the boundedness result in [20, Corollary 6.3] applied to the degree defined by \(\omega_1 \wedge \cdots \omega_{n-1}\), we can assume \(p_*q_i : \mathcal{E} \to \text{Im}(p_*q_i)\) lies in the same component of the corresponding Douady space and has a limit over \(X\). Thus we get a quotient map \(q_\infty : \mathcal{E} \to \mathcal{F}_\infty\) where \(\mu(\mathcal{F}_\infty) \leq \mu(\mathcal{E})\). This contradicts the stability of \(\mathcal{E}\). \(\square\)
Now we will fix a resolution \( p : \hat{X} \to X \) so that \( \hat{E} := (p^* E)^* \) is locally free. From the above, for any \( i > 1 \), by the Donaldson-Uhlenbeck-Yau theorem for multipolarizations (see Theorem 2.5), there exists a Hermitian-Einstein metric \( H_i \) on \( \hat{E} \) over \( \hat{X}, (p^* \omega_1 + i^{-1} \theta) \wedge \cdots (p^* \omega_{n-1} + i^{-1} \theta) \). In particular,

\[
(\sqrt{-1} F_{H_i} - \frac{\text{tr}(\sqrt{-1} F_{H_i})}{\text{rank} \ E}) \wedge (p^* \omega_1 + i^{-1} \theta) \wedge \cdots (p^* \omega_{n-1} + i^{-1} \theta) = 0
\]
i.e.

\[
\sqrt{-1} F_{H_i} - \frac{\text{tr}(\sqrt{-1} F_{H_i})}{\text{rank} \ E} \text{id}
\]
is primitive with respect to \( (p^* \omega_{k_1} + i^{-1} \theta) \wedge \cdots (p^* \omega_{k_{n-2}} + i^{-1} \theta) \) for any \( 1 \leq k_1 < \cdots < k_{n-2} \leq n - 1 \). Now the Hodge-Riemann property applied to \( (p^* \omega_{k_1} + i^{-1} \theta) \wedge \cdots (p^* \omega_{k_{n-2}} + i^{-1} \theta) \) (see Equation (2.4)) implies the following Gieseker-Bogomolov inequality

\[
(2.9) \quad \int_{\hat{X}} (2rc_2(A_i) - (r-1)c_1^2(A_i)) \wedge (p^* \omega_{i_1} + i^{-1} \theta) \wedge \cdots (p^* \omega_{i_{n-2}} + i^{-1} \theta) \geq 0
\]

where the inequality holds if and only if \( A_i \) is projectively flat.

Let \( \hat{E} \) be the underlying smooth bundle of \( \hat{E} \). By doing complex transformations ([27, Section 5]), we have a sequence of Hermitian-Yang-Mills connections \( A_i \) on a fixed unitary bundle \( (\hat{E}, H) \). Then we can take a Uhlenbeck limit using gauge theory (see [8]). Namely, by passing to a subsequence, up to gauge transforms, we can assume \( A_i \) converges locally smoothly to a limiting connection \( A_\infty \) defined on \( E|_{X \setminus \Sigma} \) where \( \Sigma \) is the so-called bubbling set defined as

\[
(2.10) \quad \Sigma = \{ z \in X^{\text{reg}} : \lim_{r \to 0^+} \lim_{i \to \infty} \inf r^{4-2n} \int_{B_r(z)} |F_{A_i}|^2 \ d\text{vol}_i > 0 \}.
\]

As [7, Lemma 2.15], we have

**Lemma 2.15.** \( \Sigma \) can be extended to be a subvariety of \( X \) of codimension at least two.

We need the following key observation

**Proposition 2.16.** Assume \( E_\infty \) can be extended to be a reflexive sheaf over \( X \). Then \( \mu(E_\infty) = \mu(E) \). In particular, the admissible Hermitian-Einstein metric computes the slope of \( E_\infty \).

**Proof.** Denote \( \hat{E}_\infty = (p^* E_\infty)^* \). Write \( p^*(\omega_1 \wedge \cdots \omega_{n-1}) = \Omega^0_{n-1} + d\Phi_{n-1} \) where \( \Omega^0_{n-1} \) is compact supported in \( \hat{X} \setminus p^{-1}(X^s \cup \Sigma) \) which is possible because \( X^s \cup \Sigma \) is a subvariety of codimension at least two (see Equation...
We have

\[
\mu(\mathcal{E}_\infty) = \frac{\int_X c_1(\hat{\mathcal{E}}_\infty) \wedge p^*(\omega_1 \wedge \cdots \omega_{n-1})}{\text{rank } \hat{\mathcal{E}}_\infty}
\]

\[
= \frac{\int_X c_1(\hat{\mathcal{E}}_\infty) \wedge (\Omega^0_{n-1} + d\Phi_{n-1})}{\text{rank } \hat{\mathcal{E}}_\infty}
\]

\[
= \frac{\int_X c_1(\hat{\mathcal{E}}_\infty) \wedge \Omega^0_{n-1}}{\text{rank } \hat{\mathcal{E}}_\infty}
\]

\[
= \frac{\int_X c_1(A_\infty) \wedge \Omega^0_{n-1}}{\text{rank } \hat{\mathcal{E}}_\infty}
\]

\[
= \lim_i \frac{\int_X c_1(A_i) \wedge \Omega^0_{n-1}}{\text{rank } \hat{\mathcal{E}}_\infty}
\]

\[
= \lim_i \frac{\int_X c_1(A_i) \wedge p^*(\omega_1 \wedge \cdots \omega_{n-1})}{\text{rank } \hat{\mathcal{E}}_\infty}
\]

\[
= \lim_i \frac{\int_X c_1(A_i) \wedge (p^*(\omega_1) + i^{-1}\theta) \wedge \cdots (p^*(\omega_{n-1}) + i^{-1}\theta)}{\text{rank } \hat{\mathcal{E}}_\infty}
\]

\[
= \lim_i \mu_i(\hat{\mathcal{E}})
\]

Combined with that the Einstein constant of $A_i$ converges to the Einstein constant of $A_\infty$, we know the admissible Hermitian-Einstein metric computes the slope of $\mathcal{E}_\infty$. \qed

### 2.6. Nontrivial limits of holomorphic sections away from the singular set

In this section, we will prove a technical result needed later. By Corollary 2.4, we can assume $X$ is endowed with a Kähler metric.

The convergence of the Hermitian-Yang-Mills connections in the previous sections can be now described by saying we have a sequence of connections, up to gauge transforms, converging locally smoothly over $X\setminus Z$ where $Z = X^s \cup \Sigma$ is a codimension at least two subvariety of $X$ endowed with a fixed Kähler metric. Here we fix the original metric on $X$ given by $\omega_1 \wedge \cdots \omega_{n-1}$ to look at the convergence. Namely, $A_i$ is a sequence of unitary connections on a unitary bundle $(\hat{\mathcal{E}}, H)$ satisfying

- $F^{0,2} = 0$;
- up to gauge transforms, $A_i$ converges to $A_\infty$ locally smoothly over $X\setminus Z$.

For any $\epsilon > 0$, define $Z^\epsilon$ to be a closed $\epsilon$-neighborhood of $Z$ in $X$. Furthermore, $Z^\epsilon \subset \text{Interior}(Z^{\epsilon'})$ if $\epsilon < \epsilon'$ and $Z^\epsilon$ converges to $Z$ as $\epsilon \to 0$. Suppose we have a sequence of holomorphic sections $\{s_i \in H^0(X\setminus Z, E_i)\}$ with $\|s_i\|_{L^2(X_i)} = 1$ where $X_i = X\setminus Z^\epsilon$. Standard elliptic theory for holomorphic sections guarantees the existence of a limit over $X^\epsilon$. The goal is to show the limit is actually nontrivial. See also [3, 10, 5] for dealing with similar problems in different settings.
2.6.1. Curve property.

**Definition 2.17.** A closed subset $S \subset X$ admits a good cover if $X$ can be covered by finitely many open sets $U_k \subset \mathbb{C}^N$ where $\omega_{|U_k \cap X} = \sqrt{-1} \partial \overline{\partial} \rho|_{X \cap U_k}$ and $\rho$ is a strongly smooth pluri-subharmonic function on $\overline{U_k}$ such that

- $U_k = B^l_{\delta_2} \times B^l_{\delta_3}$ for some $\delta_2, \delta_3 > 0$, where $B^l_{\delta_i}$ denotes the ball ${|z| < \delta_i^l}$ in $\mathbb{C}^l$ and $B^l_{\delta_i}$ denote the ball ${|z| < \delta_i^l}$ in $\mathbb{C}^l$. Here $l + l' = N$.
- $\overline{U_k} \cap S \subset V_k = B^l_{\delta_1} \times B^l_{\delta_3}$ for some $0 < \delta_1^l < \delta_2^l$ where $\overline{U_k} = B^l_{\delta_2} \times B^l_{\delta_3}$.
- for any $z \in B^l_{\delta_1} \cap \rho_k(X \cap \overline{U_k})$, $\rho_k^{-1}(z) \cap X$ is a smooth surface away from finitely many point. Here $\rho_k : B^l_{\delta_1} \times B^l_{\delta_3} \to B^l_{\delta_1} \times B^l_{\delta_3}$ denotes the natural projection.

**Lemma 2.18.** $Z \subset X$ admits a good cover.

**Proof.** For any $z \in X$, since $Z$ is a codimension 2 complex subvariety of $X$, locally we can cover $Z$ with an open set $U$ in $\mathbb{C}^N$ by assuming $z = 0$ so that for some orthogonal projection $\rho : U \subset \mathbb{C}^N = \mathbb{C}^l \times \mathbb{C}^{l'} \to \mathbb{C}^l$, $\rho^{-1}(y) \cap X$ is a smooth surface away from $\rho^{-1}(y) \cap X \cap B^l_{\delta_2}$ which consists of finitely many points for any $y \in \rho(X \cap U)$ for some $\delta_2 > 0$. Then near $z$, we can easily construct a neighborhood $U_p = B^l_{\delta_2} \times B^l_{\delta_3}$ for some $0 < \delta_1^l < \delta_2^l$ and $\overline{U_p} \cap \Sigma \subset V_p$ where $V_p = B^l_{\delta_1} \times B^l_{\delta_3}$ for some $0 < \delta_1 < \delta_2$. Now we get such an open cover of $X$ given by $\cup_p U_p$. Since $X$ is compact, we can take a finite subcover $\cup U_\alpha$ which gives a good cover for $Z \subset X$. □

**Corollary 2.19.** For $0 < \epsilon < 1$, $Z^\epsilon \subset X$ admits a good cover.

**Proof.** Otherwise, suppose the statement is false for a sequence $\epsilon_i \to 0^+$. By definition, $Z^\epsilon_i$ converges to $Z$. Let $\cup U_k$ be a good cover for $Z$. We want to show it is a good cover for $Z^\epsilon_i$ for $i$ large, thus get a contradiction. We only need to verify that $\overline{U_k} \cap Z^\epsilon_i \subset V_k$ for $i$ large. Otherwise, by passing to a subsequence and using the finiteness of $\{U_k\}_k$, we can assume for some fixed $k$, there always exists $z_i \in \overline{(U_k \cap Z^\epsilon_i)} \setminus V_k$ for each $i$ and $z_i$ converges to $z \in \overline{U_k} \cap Z$. Then $z \in V_k$ and thus $z_i \in V_k$ for $i$ large. Contradiction. □

**Lemma 2.20.** For any $\epsilon > 0$, let $\cup U_k$ be a good cover of $Z^\epsilon \subset X$. Then there exists a constant $C = C(\epsilon) > 0$ so that for any $z \in X \setminus Z^\epsilon$, there exists a curve $D_z \subset U_k \cap X$ for some $k$ such that $\partial D_z \subset X \setminus Z^\epsilon$, $d(\partial D_z, \partial(X \setminus Z^\epsilon)) \geq C$ and $d(D_z, Z) \geq C$. Here $d$ denotes the distance between two sets.

**Proof.** For each $k$, $\overline{U_k} \cap Z \subset \overline{U_k} \cap Z^\epsilon \subset \overline{U_k} \cap Z^\epsilon \subset V_k$ where $\overline{U_k} = B^l_{\delta_2} \times B^l_{\delta_3}$ and $V_k = B^l_{\delta_1} \times B^l_{\delta_3}$. Consider the projection $\rho_k : \overline{U_k} \rightarrow B^l_{\delta_1}$. By assumption, we have

$$(B^l_{\delta_2} \times B^l_{\delta_3}) \cap Z = \overline{U_k} \cap Z \subset B^l_{\delta_1} \times B^l_{\delta_3}$$
which implies \( \rho_k^{-1}(y) \cap Z \cap B_{\delta_k}^1 \) is a compact complex analytic subvariety of \( B_{\delta_k}^1 \), thus consists of finitely many points for any \( y \in \rho_k(Z) \). For any \( z \in X \setminus \mathbb{Z}^2 \), suppose \( z \in U_k \), then \( \rho_k^{-1}(\rho_k(z)) \cap Z \cap B_{\delta_k}^1 \) consists of finitely many points which lie in \( B_{\delta_k}^1 \). As a result, one can easily find a curve \( D_z \subset U_k \) containing \( z \) such that \( D_z \cap Z = \emptyset \) and \( \partial D_z \subset (U_k \setminus V_k) \cap X \subset U_k \cap X \setminus \mathbb{Z}^\epsilon \). By perturbing the curve \( D_z \), we can find an open neighborhood \( V_z \) of \( z \) so that for each \( z' \in V_z \) there exists such a curve \( D_{z'} \) so that \( D_{z'} \cap Z = \emptyset \) and \( \partial D_{z'} \subset (U_k \setminus V_k) \cap X \subset U_k \cap X \setminus \mathbb{Z}^\epsilon \). Furthermore, \( \inf_{z' \in V_z} d(D_{z'}, Z) > 0 \) and \( \inf_{z' \in V_z} d(\partial D_{z'}, \partial(X \setminus \mathbb{Z}^\epsilon)) > 0 \). As a result, we get an open cover \( \bigcup_{z \in X \setminus \mathbb{Z}^2} V_z \) of \( X \setminus \mathbb{Z}^\epsilon \). Since \( X \setminus \mathbb{Z}^2 \) is compact, we can find a finite subcover \( \bigcup_{z_i} V_{z_i} \). Let \( C(\epsilon) = \min \{ \inf_{z \in V_{z_i}} d(D_z, Z), \inf_{z \in V_{z_i}} d(\partial D_z, \partial(X \setminus \mathbb{Z}^\epsilon)) \} \). This finishes the proof.

2.6.2. Nontrivial limits.

**Proposition 2.21.** For any fixed \( 0 < \epsilon << 1 \), \( s_i \) converges to a nontrivial holomorphic section \( s_\infty \in H^0(X \setminus \mathbb{Z}, E_\infty) \).

**Proof.** We consider any \( 0 < \epsilon << 1 \) so that \( X^\epsilon = X \setminus \mathbb{Z}^\epsilon \) satisfies the conditions needed in Lemma 2.20. By induction, it suffices to show that

\[
\| s_i \|_{L^\infty(X^\epsilon)} \leq C := C(\epsilon, \| s_i \|_{L^2(X^\epsilon)})
\]

which implies the strong convergence of \( s_i \) over \( X^\epsilon \) for any \( 0 < \epsilon << 1 \). Namely, for any point \( z \in X^\epsilon \), we want to prove

\[
|s_i(z)| \leq C.
\]

Let \( D \) be the holomorphic curve obtained in Lemma 2.20. Let \( t = s_i|D \), we know

\[
\Delta_D \log(|t|^2 + 1) \geq -|F_{A_\infty}|D| \geq -C_\epsilon.
\]

Here since the disk \( D \) has a definite distance of \( Z \) by assumption which might depend on \( \epsilon \), we have \( |F_{A_\infty}|D| \leq C_\epsilon \) for some \( C_\epsilon \). Now near the point \( z \), \( X \cap U \subset U_k \subset \mathbb{C}^n \), \( \omega = i\partial \overline{\partial} \rho_k|X \cap U \) for some smooth strongly plurisubharmonic function on \( U_k \). By definition, \( \Delta_D \rho_k = 1 \) thus we have

\[
\Delta_D (\log(|t|^2 + 1) + C_\epsilon \rho_k) \geq 0
\]

which by Maximum principle implies

\[
\log(|t|^2 + 1) + C_\epsilon \rho_k \leq C_\epsilon \sup_{\partial D} \rho_k + \sup_{\partial D} \log(|t|^2 + 1) \leq C(\epsilon, \| s_i \|_{L^2(X^\epsilon)}).
\]

For the last inequality, the bound of the first term is trivial, and we only explain the second one. The interior estimate over \( X^\epsilon \) for holomorphic sections implies \( s_i \) is uniformly bounded within any precompact subset of \( X^\epsilon \). In particular, it is uniformly bounded over \( \partial D \) which lies in a fixed precompact subset of \( X^\epsilon \), since \( \partial D \subset X^\epsilon \) has a definite distance to \( \partial X^\epsilon \). This finishes the proof.

\[ \square \]
3. Proof of main results

3.1. Singular Donaldson-Uhlenbeck-Yau theorem. We first prove the rank 1 case. Recall we have a limiting Hermitian-Yang-Mills connection $A_\infty$ coming from the perturbed Hermitian-Yang-Mills connections and $A_\infty$ defines a holomorphic vector bundle $E_\infty$ over $X \setminus Z$ where $Z = X_s \cup \Sigma$.

By Proposition [2.21] we can conclude the existence of a nontrivial map $\Phi : E_\infty \to E$ coming from the normalized identity map $id : E \to E$. More precisely, to get the limit, we consider $\text{Hom}(E, E)$ with the metrics $H_i \otimes H'$ where $H_i$ are the sequence of Hermitian-Einstein metrics and $H'$ is any fixed smooth metric on $E$ away from $X_s$. Now we apply Proposition [2.21] to the sequence of holomorphic sections given by the identity map and get $\Phi : E_\infty \to E$.

**Lemma 3.1.** Given a rank 1 reflexive sheaf $E$ over $X$, there exists an admissible Hermitian-Einstein metric $H$ on $E$ with

$$\mu(E) = \int_X c_1(H) \wedge \omega_1 \wedge \cdots \wedge \omega_{n-1} \over \text{rank } E.$$

**Proof.** Since there exists a nontrivial map between $E$ and $E_\infty$ over $X^{\text{reg}}$, $E_\infty$ can be extended to be a reflexive rank 1 sheaf over $X$. Indeed, by the Remmert-Stein extension theorem, $(E \otimes E_\infty)^{**}$ can be extended to be a rank 1 reflexive sheaf from which the extension of $E_\infty$ follows trivially. It has to be an isomorphism since $E$ and $E_\infty$ have the same slope by Proposition [2.16] and Lemma [2.9].

In particular, for general rank, this gives

**Corollary 3.2.** The induced admissible Hermitian-Yang-Mills connection on $\text{det}(E_\infty)$ defines a sheaf isomorphic to $\text{det}(E)$. Furthermore, $A_\infty$ computes the slope of $\text{det}(E)$.

The proof can be concluded by showing

**Proposition 3.3.** $\Phi : E_\infty \to E$ is an isomorphism.

**Proof.** Let $G$ denote the saturated subsheaf of $E$ given by the image of $\Phi$ in $E$. By Siu’s theorem ([24, Page 441]), $G$ can be extended to be a reflexive sheaf over $X$. Indeed, it suffices to prove it locally. We can assume $G$ is a saturated subsheaf of $O^{\oplus N}$ over some open subset $U$. Then $G \subset O^{\oplus N}$ defines a map $f$ from the base to $Gr(N, N - \text{rank } G)$ away from some codimension two subvariety $Z'$. Since $U$ is normal, Siu’s theorem says $f$ can be extended to be a meromorphic map $\bar{f}$. The pull-back of the universal sheaf over the Grassmannian through $\bar{f}$ gives the extension. Assume rank $G < \text{rank } E$. Since $E$ is stable, $\mu(G) < \mu(E)$. By considering $\mathcal{F} := \text{Hom}(\wedge^{\text{rank } G} E_\infty, \text{det } G)$ we can reduce to the following setting by Corollary [3.1] a sheaf $\mathcal{F}$ defined over $X^{\text{reg}}$ admits an admissible Hermitian-Einstein metric with negative...
Einstein constant and a nonzero global section \( s := \wedge^{\text{rank} \Phi} \). By Corollary 2.12, we know \( s \in L^\infty \). Then we can use
\[
\Delta |s|^2 = 2|\nabla s|^2 - (\mu(\mathcal{G}) - \mu(\mathcal{E}))|s|^2.
\]
By doing a cut-off argument, we know \( \int_X \Delta |s|^2 = 0 \). This is a contradiction since the integral of the right hand side is strictly positive. In particular, \( \Phi \) has full rank at a generic point. By Corollary 3.2, \( \det \Phi \) is now here vanishing, thus \( \Phi \) is an isomorphism. \( \square \)

As a direct corollary, we have the following

**Corollary 3.4.** If a reflexive sheaf \( \mathcal{F} \) over \((X, \omega_1 \wedge \cdots \omega_{n-1})\) admits an admissible Hermitian-Einstein metric \( H \) so that \( c_1(H) \) computes \( \deg(\mathcal{F}) \), it is polystable.

In particular, Corollary 1.2 follows directly from this.

### 3.2. Gieseker-Bogomolov inequality

We separate the discussions into two cases.

#### 3.2.1. Locally free resolutions

Fix any resolution \( p : \hat{X} \to X \) so that \( \hat{E} = (p^*\mathcal{E})^{**} \) is locally free. Then
\[
\int_{\hat{X}} (2rc_2(\hat{E}) - (r - 1)c_1(\hat{E})^2) \wedge p^*\omega_{1} \wedge \cdots p^*\omega_{n-2}
\]
\[
= \lim_i \int_{\hat{X}} (2rc_2(\hat{E}) - (r - 1)c_1(\hat{E})^2) \wedge p^*\omega_{1} + i^{-1}\theta \wedge \cdots p^*\omega_{n-2} + i^{-1}\theta)
\]
\[
= \lim_i \int_{\hat{X}} (2rc_2(A_i) - (r - 1)c_1(A_i)^2) \wedge (p^*\omega_{1} + i^{-1}\theta) \wedge \cdots (p^*\omega_{n-2} + i^{-1}\theta)
\]
\[
\geq \int_{\hat{X}} (2rc_2(A_\infty) - (r - 1)c_1(A_\infty)^2) \wedge \omega_{1} \wedge \cdots \omega_{n-2}.
\]

The first equality is trivial while the second follows from the Chern-Weil theory in the smooth case. For the last inequality, it follows from the curvature of the induced connection on \( \text{Hom}(\hat{E}, \hat{E}) \) being primitive (see Equation 2.4), thus
\[
(2rc_2(A_i) - (r - 1)c_1(A_i)^2) \wedge (p^*\omega_{1} + i^{-1}\theta) \wedge \cdots (p^*\omega_{n-2} + i^{-1}\theta)
\]
defines a sequence of Radon measures on \( \hat{X} \) (see Equation 2.4). Furthermore, it has uniformly bounded mass. Then the inequality follows from Fatou’s lemma. Since the limiting Hermitian-Yang-Mills connection is independent of the resolutions by Corollary 2.13, the conclusion follows.

#### 3.2.2. General case

Fix a general resolutions \( p : \hat{X} \to X \). Our main theorem implies there exists a family of admissible Hermitian-Einstein metrics on \( (p^*\mathcal{E})^{**} \) on \( \hat{X} \) with respect to the perturbed metrics. Since over \( \hat{X} \), the quantities \( (2rc_2(A_i) - (r - 1)c_1(A_i)^2) \) given by the admissible Hermitian-Yang-Mills connections defines a sequence of closed currents (see \[\text{[8, Proposition 46]}\]), we know the quantities in the Bogomolov-Gieseker inequality can still
be computed by the admissible Hermitian-Yang-Mills connections on \((p^*\mathcal{E})^{**}\) over \(\tilde{X}\). Furthermore, exactly the same argument as the smooth case implies the family of admissible Hermitian-Yang-Mills connections converges to the admissible Hermitian-Yang-Mills connections on \(\mathcal{E}\) we obtained. So the same argument as above gives the statement about general resolutions.

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