THE TAU CONSTANT AND THE EDGE CONNECTIVITY OF A METRIZED GRAPH

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ABSTRACT. The tau constant is an important invariant of a metrized graph, and it has applications in arithmetic properties of curves. We show how the tau constant of a metrized graph changes under successive edge contractions and deletions. We discover identities which we call “contraction”, “deletion”, and “contraction-deletion” identities on a metrized graph. By establishing a lower bound for the tau constant in terms of the edge connectivity, we prove that Baker and Rumely’s lower bound conjecture on the tau constant holds for metrized graphs with edge connectivity 5 or more. We show that proving this conjecture for 3-regular graphs is enough to prove it for all graphs.

1. Introduction

Metrized graphs, which are graphs equipped with a distance function on their edges, appear in many places in arithmetic geometry. Metrized graphs and their invariants are studied in the articles [Zh1], [Zh2], [C1], [C2], [C3], [C5], and [C6]. Metrized graphs which arise as dual graphs of special fibers of curves, and Arakelov Green’s functions $g_{\mu}(x, y)$ on metrized graphs, play an important role in both articles [CR] and [Zh1] to study arithmetic properties of curves. T. Chinburg and R. Rumely [CR] introduced a canonical measure $\mu_{\text{can}}$ of total mass 1 on a metrized graph $\Gamma$. The diagonal values $g_{\mu_{\text{can}}}(x, x)$ are constant on $\Gamma$. M. Baker and Rumely called this constant “the tau constant” of a metrized graph $\Gamma$, and denoted it by $\tau(\Gamma)$. They posed the following conjecture concerning lower bound of $\tau(\Gamma)$.

Conjecture 1.1. [BR] There is a universal constant $C > 0$ such that for all metrized graphs $\Gamma$, $\tau(\Gamma) \geq C \cdot \ell(\Gamma)$ where $\ell(\Gamma)$ is the total length of $\Gamma$.

We call Conjecture 1.1 Baker and Rumely’s lower bound conjecture.

In summer 2003 at UGA, an REU group lead by Baker and Rumely studied properties of the tau constant and the lower bound conjecture. Baker and Rumely [BR] introduced a measure valued Laplacian operator $\Delta$ which extends Laplacian operators studied earlier in the articles [CR] and [Zh1]. This Laplacian operator combines the “discrete” Laplacian on a finite graph and the “continuous” Laplacian $-f''(x)dx$ on $\mathbb{R}$. Baker and Rumely [BR] studied harmonic analysis on metrized graphs. In terms of spectral theory, the tau constant is the trace of the inverse operator of $\Delta$ when $\Gamma$ has total length 1.

In this paper, we prove that Conjecture 1.1 holds with $C = \frac{1}{108}$ for a graph $\Gamma$ with edge connectivity at least 6, and with $C = \frac{1}{300}$ for a graph $\Gamma$ with edge connectivity 5. The proof involves establishing a set of identities, which we call “contraction”, “deletion”, and
“contraction-deletion” identities on a metrized graph. By using these identities, we show how the tau constant changes after successive edge deletions and contractions. In particular, when we consider successive edge contractions until we are left with only two vertices, we use our previous results \cite{C2} about the tau constant to obtain a set of inequalities between the terms adding up to the tau constant. In this way, we transform the tau lower bound problem into a linear optimization problem. Finally, we obtain a lower bound to the tau constant $\tau(\Gamma)$ of a metrized graph $\Gamma$ in terms of the edge connectivity and $\ell(\Gamma)$ when the edge connectivity is at least 5. The results here not only extend those obtained in \cite[Sections 3.6, 3.7, 3.9, 3.10 and 3.12]{C1} but we obtained in a more coherent and systematic manner.

Applications of these results to arithmetic of curves and specifically to Bogomolov Conjecture can be found in the article \cite{C5}.

Note that there is a 1-to-1 correspondence between equivalence classes of finite connected weighted graphs, metrized graphs, and resistive electric circuits. If an edge $e_i$ of a metrized graph has length $L_i$, then the resistance of $e_i$ is $L_i$ in the corresponding resistive electric circuit, and the weight of $e_i$ is $\frac{1}{L_i}$ in the corresponding weighted graph. Therefore, the identities we show in this paper have equivalent forms for weighted graphs.

### 2. Metrized Graphs and Their Tau Constants

In this section, we recall a few facts about metrized graphs, the canonical measure on a metrized graph $\Gamma$, and the tau constant of $\Gamma$.

A metrized graph $\Gamma$ is a finite connected graph equipped with a distinguished parametrization of each of its edges. A metrized graph $\Gamma$ can have multiple edges and self-loops. For any given $p \in \Gamma$, the number of directions emanating from $p$ will be called the valence of $p$, and will be denoted by $v(p)$. By definition, there can be only finitely many $p \in \Gamma$ with $v(p) \neq 2$.

For a metrized graph $\Gamma$, we will denote a vertex set for $\Gamma$ by $V(\Gamma)$. We require that $V(\Gamma)$ be finite and non-empty and that $p \in V(\Gamma)$ for each $p \in \Gamma$ with $v(p) \neq 2$. For a given metrized graph $\Gamma$, it is possible to enlarge the vertex set $V(\Gamma)$ by considering additional valence 2 points as vertices.

For a given metrized graph $\Gamma$ with vertex set $V(\Gamma)$, the set of edges of $\Gamma$ is the set of closed line segments with end points in $V(\Gamma)$. We will denote the set of edges of $\Gamma$ by $E(\Gamma)$. However, if $e_i$ is an edge, by $\Gamma - e_i$ we mean the graph obtained by deleting the interior of $e_i$.

Let $v := \#(V(\Gamma))$ and $e := \#(E(\Gamma))$. We define the genus $\Gamma$ to be the first Betti number $g := e - v + 1$ of the graph $\Gamma$. Note that the genus is a topological invariant of $\Gamma$. In particular, it is independent of the choice of the vertex set $V(\Gamma)$. Since $\Gamma$ is connected, $g(\Gamma)$ coincides with the cyclotomic number of $\Gamma$ in combinatorial graph theory. We will simply use $g$ to show $g(\Gamma)$ when there is no danger of confusion.

We denote the length of an edge $e_i \in E(\Gamma)$ by $L_i$. The total length of $\Gamma$, which will be denoted by $\ell(\Gamma)$, is given by $\ell(\Gamma) = \sum_{i=1}^{e} L_i$.

Let $\Gamma$ be a metrized graph. If we scale each edge in $\Gamma$ by multiplying its length by $\frac{1}{\ell(\Gamma)}$, we obtain a new graph which is called normalization of $\Gamma$ and denoted by $\Gamma^N$. Note that $\Gamma$ and $\Gamma^N$ have the same topology, and $\ell(\Gamma^N) = 1$. If $\Gamma = \Gamma^N$, we call $\Gamma$ be a normalized graph.

A metrized graph $\Gamma$ is called $n$-regular if it has a vertex set $V(\Gamma)$ such that $v(p) = n$ for all vertices $p \in V(\Gamma)$.
We will denote the minimum of the valences of vertices in \( V(\Gamma) \) by \( \delta(\Gamma) \). The minimum number of edges whose deletion disconnects \( \Gamma \) is called the “edge connectivity” of \( \Gamma \) and denoted by \( \Lambda(\Gamma) \). The minimum number of vertices whose deletion disconnects \( \Gamma \) is called the “vertex connectivity” of \( \Gamma \) and denoted by \( \kappa(\Gamma) \).

In the article [CR], a kernel \( j_z(x, y) \) giving a fundamental solution of the Laplacian is defined and studied as a function of \( x, y, z \in \Gamma \). For fixed \( z \) and \( y \) it has the following physical interpretation: When \( \Gamma \) is viewed as a resistive electric circuit with terminals at \( z \) and \( y \), with the resistance in each edge given by its length, then \( j_z(x, y) \) is the voltage difference between \( x \) and \( z \), when unit current enters at \( y \) and exits at \( z \) (with reference voltage 0 at \( z \)).

For any \( x, y, z \) in \( \Gamma \), the voltage function \( j_z(y, z) \) on \( \Gamma \) is a symmetric function in \( y \) and \( z \), and it satisfies \( j_z(x, z) = 0 \) and \( j_z(y, y) = r(x, y) \), where \( r(x, y) \) is the resistance function on \( \Gamma \). For each vertex set \( V(\Gamma) \), \( j_z(x, y) \) is continuous on \( \Gamma \) as a function of 3 variables. As the physical interpretation suggests, \( j_z(x, y) \geq 0 \) for all \( x, y, z \) in \( \Gamma \). For proofs of these facts, see the articles [CR], [BR] sec 1.5 and sec 6, and [Zh1], Appendix]. The voltage function \( j_z(x, y) \) and the resistance function \( r(x, y) \) on a metrized graph were also studied in the articles [BF] and [C2].

For any real-valued, signed Borel measure \( \mu \) on \( \Gamma \) with \( \mu(\Gamma) = 1 \) and \( |\mu|(\Gamma) < \infty \), define the function \( j_\mu(x, y) = \int_{\Gamma} j_z(x, y) d\mu(z) \). Clearly \( j_\mu(x, y) \) is symmetric, and is jointly continuous in \( x \) and \( y \). Chinburg and Rumely [CR] discovered that there is a unique measure \( \mu = \mu_{\text{can}} \) with above properties such that \( j_{\mu}(x, x) \) is constant on \( \Gamma \). The measure \( \mu_{\text{can}} \) is called the canonical measure. Baker and Rumely [BR] called the constant \( \frac{1}{2} j_{\mu}(x, x) \) the tau constant of \( \Gamma \) and denoted it by \( \tau(\Gamma) \).

The following lemma gives another description of the tau constant. In particular, it implies that the tau constant is positive.

**Lemma 2.1.** [BR] Lemma 14.4] For any fixed \( y \) in \( \Gamma \), \( \tau(\Gamma) = \frac{1}{4} \int_{\Gamma} (\frac{\partial}{\partial x} r(x, y))^2 dx. \)

We will use the following results frequently in later sections.

**Lemma 2.2.** [BR] pg. 37, [C2] Corollaries 2.17 and 2.22 If \( \Gamma \) is a tree, i.e. a graph without cycles, then \( \tau(\Gamma) = \frac{\ell(\Gamma)}{4}. \) If \( \Gamma \) is a circle graph, then \( \tau(\Gamma) = \frac{\ell(\Gamma)}{12}. \)

**Remark 2.3.** Whenever a graph \( \Gamma \) has vertex \( p \) such that removing \( p \) disconnects \( \Gamma \), i.e. \( p \) is a cut-vertex of \( \Gamma \), then \( \Gamma = \Gamma_1 \cup \Gamma_2 \) for subgraphs \( \Gamma_1 \) and \( \Gamma_2 \) with \( \Gamma_1 \cap \Gamma_2 = \{p\} \). In this case, we have \( \tau(\Gamma_1 \cup \Gamma_2) = \tau(\Gamma_1) + \tau(\Gamma_2) \), which we call the additive property of the tau constant (see also [C2] pg. 11). It was initially noted in [REU].

Therefore, proving Conjecture [1] for graphs with vertex connectivity \( \kappa(\Gamma) \geq 2 \) yields it for all graphs.

**Remark 2.4.** [BR] If we multiply all lengths on \( \Gamma \) by a positive constant \( c \), we obtain a graph \( \Gamma' \) of total length \( c \cdot \ell(\Gamma) \). Then \( \tau(\Gamma') = c \cdot \tau(\Gamma) \). This will be called the scale-independence of the tau constant. By this property, to prove Conjecture [1] it is enough to consider metrized graphs with total length 1.

**Remark 2.5.** Let \( \Gamma \) be any metrized graph with resistance function \( r(x, y) \). The tau constant \( \tau(\Gamma) \) is independent of the vertex set \( V(\Gamma) \) chosen. In particular, enlarging \( V(\Gamma) \) by including points \( p \in \Gamma \) with \( v(p) = 2 \) does not change \( \tau(\Gamma) \). Thus, \( \tau(\Gamma) \) depends only on the topology
and the edge length distribution of the metrized graph $\Gamma$. This will be called the valence property of the tau constant.

We will denote by $R_i(\Gamma)$, or by $R_i$ if there is no danger of confusion, the resistance between the end points of an edge $e_i$ of a graph $\Gamma$ when the interior of the edge $e_i$ is deleted from $\Gamma$. We will use the following notation in the rest of this paper:

$$z(\Gamma) = \sum_{e_i \in E(\Gamma)} \frac{L_i^2}{L_i + R_i}, \quad r(\Gamma) = \sum_{e_i \in E(\Gamma)} \frac{L_iR_i}{L_i + R_i}.$$  

Note that $\ell(\Gamma) = z(\Gamma) + r(\Gamma)$.

Chinburg and Rumely [CR, page 26] showed that the end points of the edge $e_i$ of a graph $\Gamma$ when the interior of the edge $e_i$ is deleted from $\Gamma$. This makes $\Gamma$ have end points $p$ and $q$, which are fairly difficult to understand. An important step in this direction is provided by Proposition 3.2 and Theorem 3.3, which depend on Theorem 2.6.

Properties of $A_{p,q,\Gamma}$ were studied in the article [C2, Sections 4 and 8]. For any $p, q \in \Gamma$, $0 \leq A_{p,q,\Gamma} \leq r(p,q)(r_T(p) - \frac{12}{p,q})$, where $r_T(p) = \max\{r(p,x) | x \in \Gamma\}$ and $r(x,y)$ is the resistance function in $\Gamma$. Here, the upper bound follows by combining [C2, Theorem 4.3 part (vi)] and [C2, Corollary 2.19].

We call an edge $e_i \in E(\Gamma)$ a bridge if $\Gamma - e_i$ is disconnected. If $\Gamma - e_i$ is connected for every $e_i \in E(\Gamma)$, we call $\Gamma$ a bridgeless graph.

Theorem 2.6. [C2, Theorem 5.7] Let $\Gamma$ be a bridgeless graph. Suppose that $p_i, q_i$ are the end points of the edge $e_i$, for each $i = 1, 2, \ldots, e$. Then,

$$\tau(\Gamma) = \frac{\ell(\Gamma)}{12} - \sum_{i=1}^{e} \frac{L_iA_{p_i,q_i,\Gamma-e_i}}{(L_i + R_i)^2}.$$  

Theorem 2.7. [C2, Theorem 2.21] For any $p, q \in \Gamma$, $\tau(\Gamma) = \frac{1}{4} \int_{\Gamma} (\frac{d}{dx}j_x(p,q))^2 dx + \frac{1}{4} r(p,q)$.  

3. Edge contractions and deletions

Let $\Gamma_i$ be the graph obtained by contracting the $i$-th edge $e_i, i \in \{1, 2, \ldots, e\}$, of a given graph $\Gamma$ to its end points. If $e_i \in \Gamma$ has end points $p_i$ and $q_i$, then in $\Gamma_i$, these points become identical, i.e., $p_i = q_i$. Let $\tilde{\Gamma}_i$ be the graph obtained by identifying the end points of the $i$-th edge $e_i \in E(\Gamma)$. This makes $e_i$ into a loop in $\tilde{\Gamma}_i$. Note that $\tau(\tilde{\Gamma}_i) = \tau(\Gamma_i) - \frac{L_i}{12}$ by the additive property of the tau constant and Lemma 6.2.

The following lemma sheds light on how the tau constant changes by contraction of an edge:

Lemma 3.1. [C2, Lemma 6.2] Let $e_i \in E(\Gamma)$ be such that $\Gamma - e_i$ is connected. Then we have

$$\tau(\Gamma) = \tau(\Gamma_i) + \frac{L_i}{12} - \frac{L_iA_{p_i,q_i,\Gamma-e_i}}{R_i(L_i + R_i)}.$$  

Note that Lemma 3.1 involves terms containing $A_{p_i,q_i,\Gamma-e_i}$, which are fairly difficult to understand. One wants to understand the effect of edge contraction in a better way. An important step in this direction is provided by Proposition 3.2 and Theorem 3.3, which depend on Theorem 2.6.
Proposition 3.2. Let $\Gamma$ be a bridgeless graph with $v = \#(V(\Gamma)) \geq 3$. Then,
\[
\tau(\Gamma) = \frac{1}{v-2} \sum_{i=1}^{e} \frac{R_i}{L_i + R_i} \tau(\tilde{\Gamma}_i) - \frac{z(\Gamma)}{12(v-2)}, \quad \tau(\Gamma) = \frac{1}{v-2} \sum_{i=1}^{e} \frac{R_i}{L_i + R_i} \tau(\tilde{\Gamma}_i) - \frac{\ell(\Gamma)}{12(v-2)}.
\]

Proof. Multiply both sides of the equation in Lemma 3.1 by $\frac{R_i}{L_i + R_i}$, sum over all edges of $\Gamma$, and use the fact that $\sum_{i=1}^{e} \frac{R_i}{L_i + R_i} = v - 1$ (see Equation (2)) to obtain
\[
(v-1)\tau(\Gamma) = \sum_{i=1}^{e} \frac{R_i}{L_i + R_i} \tau(\tilde{\Gamma}_i) + \sum_{i=1}^{e} \frac{R_i}{L_i + R_i} \tau(\tilde{\Gamma}_i) - \sum_{i=1}^{e} \frac{L_i A_{p_i q_i, v - e_i}}{(L_i + R_i)^2}.
\]
Recall that $z(\Gamma)$ and $r(\Gamma)$ are defined in Equation (1). We obtain the first formula by using Theorem 2.6. Then the second formula follows from the fact that $\tau(\tilde{\Gamma}_i) = \tau(\tilde{\Gamma}_i) - L_i$.

In the proof of Proposition 3.2, we used the fact that $\Gamma$ is bridgeless when we worked with terms $A_{p_i q_i, v - e_i}$. We will now extend the result of Proposition 3.2 to any connected graph $\Gamma$. For an edge $e_i$, which is a bridge in $\Gamma$, the end points $p_i$ and $q_i$ become disconnected in $\Gamma - e_i$, and so $R_i = \infty$. In such cases, if we use the limiting values of the corresponding terms, it is possible to extend Proposition 3.2 to a metrized graph with bridges. More precisely, note that
\[
\tau(\Gamma) = \frac{1}{v-2} \sum_{i=1}^{e} \left[ \lim_{t \to R_i} \frac{t}{L_i + t} \right] \tau(\tilde{\Gamma}_i) - \frac{1}{12(v-2)} \sum_{i=1}^{e} \left[ \lim_{t \to R_i} \frac{L_i^2}{L_i + t} \right].
\]
In short, we set $\frac{R_i}{L_i + R_i} := 1$ and $\frac{L_i}{L_i + R_i} := 0$ whenever $R_i = \infty$.

Theorem 3.3. Let $\Gamma$ be a metrized graph with $v = \#(V(\Gamma)) \geq 3$. Then we have
\[
\tau(\Gamma) = \frac{1}{v-2} \sum_{i=1}^{e} \frac{R_i}{L_i + R_i} \tau(\tilde{\Gamma}_i) - \frac{z(\Gamma)}{12(v-2)}, \quad \tau(\Gamma) = \frac{1}{v-2} \sum_{i=1}^{e} \frac{R_i}{L_i + R_i} \tau(\tilde{\Gamma}_i) - \frac{\ell(\Gamma)}{12(v-2)}.
\]

Proof. We already dealt with the case in which $\Gamma$ is bridgeless. Suppose that $\Gamma$ has bridges. Let $B = \{e_{i_1}, e_{i_2}, \ldots, e_{i_k}\}$ be the set of all bridges in $\Gamma$, for some positive integer $k$. Let $\gamma$ be the graph obtained from $\Gamma$ by contracting all of its bridges to their end points. Thus, an edge $e_i$ belongs to $E(\gamma)$ iff $e_i \notin B$. By the additive property of $\tau(\Gamma)$ (i.e., by Remark 2.3) and Lemma 2.2,
\[
(3) \quad \tau(\Gamma) = \tau(\gamma) + \frac{1}{4} \sum_{e_j \in B} L_j.
\]
Clearly, $\gamma$ is connected and bridgeless with $v - k$ vertices. Note that if $e_i \in B$, then
\[
(4) \quad \tau(\Gamma_i) = \tau(\gamma_i) + \frac{1}{4} \sum_{e_j \in B} L_j - \frac{L_i}{4},
\]
and if $e_i \notin B$, then
\[
(5) \quad \tau(\Gamma_i) = \tau(\gamma_i) + \frac{1}{4} \sum_{e_j \in B} L_j.
\]
In either case,

\( z(\Gamma) = z(\gamma) \), and

\[
(6) \quad z(\Gamma) = z(\gamma), \quad \text{and} \quad \sum_{e_i \in E(\Gamma) - B} \frac{R_i(\Gamma)}{L_i + R_i(\Gamma)} = \sum_{e_i \in E(\gamma)} \frac{R_i(\gamma)}{L_i + R_i(\gamma)} = v - k - 1.
\]

Since \( \gamma \) is bridgeless, we can apply Proposition 3.2 to obtain

\[
(7) \quad \tau(\gamma) = \frac{1}{v - k - 2} \sum_{e_i \in E(\gamma)} \frac{R_i}{L_i + R_i} \tau(\gamma_i) - \frac{z(\gamma)}{12(v - k - 2)}.
\]

Then by Equations (4) and (5)

\[
\sum_{e_i \in E(\Gamma)} \frac{R_i}{L_i + R_i} \tau(\Gamma_i) = \sum_{i \in B} \frac{R_i}{L_i + R_i} \left[ \tau(\gamma_i) + \frac{1}{4} \sum_{j \in B} L_j \right] + \sum_{i \in B} \frac{R_i}{L_i + R_i} \left[ \tau(\gamma) + \frac{1}{4} \sum_{j \in B} L_j - \frac{R_i}{4} \right],
\]

\[
= \sum_{i \in B} \frac{R_i}{L_i + R_i} \tau(\gamma_i) + \left( \frac{1}{4} \sum_{j \in B} L_j \right) \left( \sum_{i \in B} \frac{R_i}{L_i + R_i} \right) + k \tau(\gamma) + \frac{k - 1}{4} \sum_{j \in B} L_j
\]

\[
= (v - 2)\tau(\gamma) + \frac{z(\gamma)}{12} + \frac{v - 2}{4} \sum_{j \in B} L_j, \quad \text{by Equations (7) and (6)}.
\]

This is equivalent to the first formula we wanted to show. By using the fact that \( \tau(\tilde{\gamma}_i) = \tau(\gamma_i) + \frac{L_i}{12} \), for all \( e_i \in E(\Gamma) \), along with the first formula, we obtain the second formula. \( \square \)

The following lemma shows how the tau constant changes by deletion of an edge when the remaining graph is connected.

**Lemma 3.4.** \([C2] \) Corollary 5.3 \] Suppose that \( \Gamma \) is a graph such that \( \Gamma - e_i \), for some edge \( e_i \in E(\Gamma) \) with length \( L_i \) and end points \( p_i \) and \( q_i \), is connected. Then we have

\[
\tau(\Gamma) = \tau(\Gamma - e_i) + \frac{L_i}{12} - \frac{R_i}{6} + \frac{A_{p_i,q_i,\Gamma-e_i}}{L_i + R_i}.
\]

We can combine Lemma 3.1 and Lemma 3.4 to obtain the following Lemma:

**Lemma 3.5.** Suppose that \( \Gamma \) is a graph such that \( \Gamma - e_i \), for some edge \( e_i \in E(\Gamma) \) with length \( L_i \) is connected. Then we have

\[
\tau(\Gamma) = \frac{L_i}{L_i + R_i} \tau(\Gamma - e_i) + \frac{R_i}{L_i + R_i} \tau(\Gamma_i) + \frac{L_i^2 - L_i R_i}{12(L_i + R_i)}.
\]

**Proof.** Multiply the formula in Lemma 3.1 by \( \frac{R_i}{L_i + R_i} \) and the formula in Lemma 3.4 by \( \frac{L_i}{L_i + R_i} \). Then add the results.

To show the effect of edge deletion on the tau constant without using any terms with \( A_{p_i,q_i,\Gamma-e_i} \), we have the following theorem:

**Theorem 3.6.** Let \( \Gamma \) be a bridgeless graph with edges \( \{e_1, e_2, \ldots, e_g\} \). Then,

\[
\tau(\Gamma) = \frac{1}{g + 1} \sum_{i=1}^g \frac{L_i}{L_i + R_i} \tau(\Gamma - e_i) + \frac{\ell(\Gamma)}{6(g + 1)} - \frac{r(\Gamma)}{4(g + 1)}.
\]
Proof. Multiply both sides of the equation given in Lemma 3.4 by \( \frac{L_i}{L_i + R_i} \), sum over all edges of \( \Gamma \), and use the fact that \( \sum_{i=1}^{e} \frac{L_i}{L_i + R_i} = g \) (see Equation (2)) to obtain

\[
g \cdot \tau(\Gamma) = \sum_{i=1}^{e} \frac{L_i}{L_i + R_i} \tau(\Gamma - e_i) + \frac{z(\Gamma)}{12} - \frac{r(\Gamma)}{6} + \sum_{i=1}^{e} \frac{L_i A_{p_i,q_i,\Gamma-e_i}}{(L_i + R_i)^2}.
\]

Finally, we use Theorem 2.6 and the fact that \( z(\Gamma) + r(\Gamma) = \ell(\Gamma) \) to complete the proof. □

As a corollary, we obtain a lower bound to the tau constant in terms of the genus \( g \).

Corollary 3.7. Let \( \Gamma \) be a bridgeless metrized graph. Let edge \( e_i \) have end points \( p_i \) and \( q_i \). For the voltage function \( j_x^i(p_i,q_i) \) on \( \Gamma - e_i \), we have

\[
\tau(\Gamma) = \frac{1}{4(g+1)} \sum_{e_i \in E(\Gamma)} \frac{L_i}{L_i + R_i} \int_{\Gamma - e_i} \left( \frac{d}{dx} j_x^i(p_i,q_i) \right)^2 dx + \frac{\ell(\Gamma)}{6(g+1)}.
\]

In particular, \( \tau(\Gamma) \geq \frac{\ell(\Gamma)}{6(g+1)} \).

Proof. Applying Theorem 2.7 to \( \Gamma - e_i \) gives \( \tau(\Gamma - e_i) = \frac{1}{4} \int_{\Gamma - e_i} \left( \frac{d}{dx} j_x^i(p_i,q_i) \right)^2 dx + \frac{R_i}{4} \) for any edge \( e_i \). Thus, we obtain what we want by substituting this into Theorem 3.6. □

Using the results in this section, we attempted to apply induction arguments to prove Conjecture 1.1 without a satisfactory outcome. We believe that improving the results on \( A_{p_i,q_i,\Gamma-e_i} \) will make the induction arguments applicable.

4. Contraction, deletion and contraction-deletion identities

In this section, we will prove a number of identities, which we call “contraction identities”, “deletion identities” and “contraction-deletion” identities. These identities are interesting in their own right. One way to relate these identities to the tau constant can be explained as follows:

We know the exact values of the tau constant when the metrized graph is a tree or circle (see Lemma 2.2). If a metrized graph has vertex connectivity 1 or 2, we can express its tau constant in terms of the tau constants of its subgraphs (see Remark 2.3 and [C2, Theorems 5.1 and 8.1]). After a sequence of edge deletions and contractions we can pass to these type of graphs from an arbitrary metrized graph. In the previous section, we gave formulas expressing \( \tau(\Gamma) \) in terms of \( \tau(\Gamma - e_i) \)'s or \( \tau(\Gamma_i) \)'s by considering all edge deletions or contractions of depth 1 (see Theorem 3.3 and Theorem 3.6). One wonders if it is possible to generalize these two theorems with further depths of edge deletions and contractions. The solution is given by the identities shown in this section. The identities of this section have crucial roles in generalizing the results of the previous section, as in the following section where we deal with the successive edge contractions.

Some of these “contraction identities”, “deletion identities” and “contraction-deletion” identities were proven in [C1, Sections 3.6 and 3.7] using different methods. Our approach in this paper is to utilize Euler’s formula for homogeneous functions as in the proof of Theorem 2.6 in [C2].

Let \( \Gamma \) be a graph with edges \( E(\Gamma) = \{e_1, e_2, \ldots, e_e\} \), and let \( \Gamma - e_i \) be the graph obtained by deleting the \( i \)-th edge \( e_i \in E(\Gamma) \). As before \( L_i \) is the length of edge \( e_i \). Let \( \Gamma^{DA} \) be the
Given a graph $\Gamma$, we will compare $\tau$-constants of the following graphs: $\Gamma$, $\Gamma - e_i$, $\Gamma^{DA}$, $(\Gamma - e_i)^{DA}$, $\Gamma^{DA} - e_{i,j}$ and $\Gamma^{DA} - \{e_{i,1}, e_{i,2}\}$. It will turn out that by doing so, we will obtain non-trivial identities for $\tau(\Gamma)$, $z(\Gamma)$ and $r(\Gamma)$.

The graphs in Figure 1 illustrate what we will do. Graph I shows $\Gamma$ with an edge $e_i \in E(\Gamma)$ labeled by $i$. II shows $\Gamma - e_i$, III shows $(\Gamma - e_i)^{DA}$, IV shows $\Gamma^{DA}$ with edges $e_{i,1}$ and $e_{i,2}$ labeled by $i$ and $ii$, V shows $\Gamma^{DA} - e_{i,2}$ and VI shows $\Gamma^{DA} - \{e_{i,1}, e_{i,2}\}$.

Note that $(\Gamma - e_i)^{DA}$ and $\Gamma^{DA} - \{e_{i,1}, e_{i,2}\}$ are the same graphs. In Figure I they are the graphs in III and VI.

Let $E(\Gamma) = \{e_1, e_2, \ldots, e_e\}$, and let $L_i$ be the length of $e_i$. Then $E(\Gamma^{DA}) = \{e_{i,1}, e_{i,2}, e_{2,1}, e_{2,2}, \ldots, e_{e,1}, e_{e,2}\}$. We write $\Gamma^{d_i} := \Gamma^{DA} - \{e_{i,1}, e_{i,2}\}$ to simplify the notation.

**Theorem 4.1.** Let $\Gamma$ be a bridgeless metrized graph. Given an edge $e_i \in E(\Gamma)$ with end points $p_i$ and $q_i$,

$$\tau(\Gamma^{DA}) = \tau((\Gamma - e_i)^{DA}) + \frac{2L_i^2 - R_i^2}{24(L_i + R_i)} + \frac{4}{L_i + R_i}A_{p_i,q_i,\Gamma^{d_i}}.$$  

**Proof.** By applying Lemma 3.4 to $\Gamma^{DA}$ for the edge $e_{i,1} \in E(\Gamma^{DA})$ and using $R_{i,1}(\Gamma^{DA}) = \frac{L_i R_i}{2L_i + R_i}$ from [C2, Lemma 3.10 with $n = 2$], we get

$$\tau(\Gamma^{DA}) = \tau((\Gamma^{DA} - e_{i,1}) + \frac{L_i}{24} - \frac{1}{12} \frac{L_i R_i}{2L_i + R_i} + \frac{A_{p_i,q_i,\Gamma^{DA} - e_{i,1}}}{L_i^2 + \frac{L_i R_i}{2L_i + R_i}}.$$  

By applying [C2, Lemma 8.6] to $A_{p_i,q_i,\Gamma^{DA} - e_{i,1}}$ with edge $e_{i,2}$, we obtain

$$A_{p_i,q_i,\Gamma^{DA} - e_{i,1}} = \frac{4L_i^2 A_{p_i,q_i,\Gamma^{d_i}}}{(2L_i + R_i)^2} + \frac{1}{24} \left( \frac{L_i R_i}{2L_i + R_i} \right)^2.$$  

### Figure 1

Graphs used to obtain the deletion identities.

[Graphs shown with labels and arrows indicating the deletion process and the resulting graphs.]
Next, applying Lemma 3.4 to $\Gamma^\text{DA} - e_{i,1}$ with respect to the edge $e_{i,2}$ and using $R_{i,2}(\Gamma^\text{DA} - e_{i,1}) = \frac{R_i}{4}$ gives

\begin{equation}
\tau(\Gamma^\text{DA} - e_{i,1}) = \tau(\Gamma^d_i) + \frac{L_i}{24} - \frac{R_i}{24} + \frac{4A_{p_i,q_i,\Gamma^d_i}}{2L_i + R_i}
\end{equation}

We also note that $\Gamma^d_i = (\Gamma - e_i)^{\text{DA}}$.

Substituting Equations (9) and (10) into Equation (8) gives the result. □

**Notation.** Let $\Gamma$ be a bridgeless metrized graph. Then for any $e_i \in E(\Gamma)$, we set

\[ K_i(\Gamma) := \sum_{e_j \in E(\Gamma) \setminus e_i} \frac{L_j^2}{L_j + R_j} - \sum_{e_j \in E(\Gamma-e_i)} \frac{L_j^2}{L_j + R_j(\Gamma-e_i)} \]

**Remark 4.2.** Let $\Gamma$ be a metrized graph and let $e_i \in E(\Gamma)$. For every $j \neq i$ and $j \in \{1,2,\ldots,e\}$, $R_j(\Gamma-e_i) \geq R_j$ by Rayleigh’s Cutting law, which states that cutting branches can only increase the effective resistance between any two points in a circuit (See DS for more information). Therefore, $\frac{L_j^2}{L_j + R_j(\Gamma-e_i)} \leq \frac{L_j^2}{L_j + R_j}$. Hence, $K_i(\Gamma) \geq 0$.

**Theorem 4.3.** Let $\Gamma$ be a bridgeless metrized graph. For any edge $e_i \in E(\Gamma)$ with end points $p_i$ and $q_i$,

\[ \frac{A_{p_i,q_i,\Gamma-e_i}}{L_i + R_i} = \frac{16A_{p_i,q_i,\Gamma^d_i}}{L_i + R_i} - \frac{K_i(\Gamma)}{6} \]

**Proof.** Note that $\ell(\Gamma^\text{DA}) = \ell(\Gamma)$. Applying [C2, Corollary 3.5] to $\Gamma^\text{DA}$, we obtain

\begin{equation}
\tau(\Gamma^\text{DA}) = \frac{\ell(\Gamma)}{48} + \frac{\tau(\Gamma)}{4} + \frac{z(\Gamma)}{24}
\end{equation}

Applying [C2, Corollary 3.5] to $(\Gamma - e_i)^{\text{DA}}$, we obtain

\begin{equation}
\tau((\Gamma - e_i)^{\text{DA}}) = \frac{\ell(\Gamma - e_i)}{48} + \frac{\tau(\Gamma - e_i)}{4} + \frac{z(\Gamma - e_i)}{24}
\end{equation}

Substituting Equation (11) and Equation (12) into Theorem 4.1 and recalling that $\ell(\Gamma-e_i) = \ell(\Gamma) - L_i$ gives

\begin{equation}
\tau(\Gamma) = \tau(\Gamma - e_i) + \frac{L_i}{12} - \frac{R_i}{6} - \frac{K_i(\Gamma)}{6} + \frac{16A_{p_i,q_i,\Gamma^d_i}}{L_i + R_i}
\end{equation}

Comparing Equation (13) with Lemma 3.4 gives the result. □

Let $p$, $q$ be any two points in $\Gamma$, and let $e_0$ be a line segment of length $L$. By identifying the end points of $e_0$ with $p$ and $q$ of $\Gamma$ we obtain a new graph which we denote by $\Gamma_{(p,q)}$. Then $\ell(\Gamma_{(p,q)}) = \ell(\Gamma) + L$. Also, by identifying $p$ and $q$ with each other in $\Gamma$ we obtain a graph which we denote by $\Gamma_{pq}$. Then $\ell(\Gamma_{pq}) = \ell(\Gamma)$. If $p$ and $q$ are end points of an edge $e_i \in \Gamma$, then $\Gamma_{pq} = \tilde{\Gamma}_i$.

**Lemma 4.4.** [C2] Corollaries 7.1 and 7.2 Let $\Gamma$ be a metrized graph with resistance function $r(x,y)$. For $p$, $q$, $\Gamma_{(p,q)}$, and $\Gamma_{pq}$ as given above,

\[ \tau(\Gamma_{(p,q)}) = \tau(\Gamma) + \frac{L}{12} - \frac{r(p,q)}{6} + \frac{A_{p,q,\Gamma}}{L + r(p,q)}, \quad \tau(\Gamma_{pq}) = \tau(\Gamma) - \frac{r(p,q)}{6} + \frac{A_{p,q,\Gamma}}{r(p,q)}. \]
The following corollary is the initial step towards the contraction-deletion identities (Theorems 4.10 and 4.6).

**Corollary 4.5.** Let \( \Gamma \) be a metrized graph with resistance function \( r(x, y) \), and let \( p, q, e_0 \) and \( \Gamma_{(p,q)} \) be as above. Corresponding to the edge \( e_0 \), suppose that we have the pair of edges \( e_{0,1} \) and \( e_{0,2} \) in \( E((\Gamma_{(p,q)})^{DA}) \). Then we have

\[
\frac{A_{p,q,\Gamma}}{L + r(p, q)} = \frac{16A_{p,q,\Gamma^{DA}}}{L + r(p, q)} - \frac{1}{6} \left( \sum_{e_j \in E(\Gamma_{(p,q)}) \setminus e_0} \frac{L_j^2}{L_j + R_j(\Gamma(p,q))} - \sum_{e_i \in E(\Gamma)} \frac{L_i^2}{L_i + R_i} \right).
\]

Proof. Theorem 4.3 applied to \( \Gamma_{(p,q)} \) with edge \( e_0 \) gives

\[
\frac{A_{p,q,\Gamma^{DA}} - e_0}{L + r(p, q)} = \frac{16A_{p,q,\Gamma^{DA}} - e_0}{L + r(p, q)} - \frac{1}{6} \left( \sum_{e_j \in E(\Gamma_{(p,q)}) \setminus e_0} \frac{L_j^2}{L_j + R_j(\Gamma(p,q))} - \sum_{e_i \in E(\Gamma(p,q) - e_0)} \frac{L_i^2}{L_i + R_i} \right).
\]

On the other hand, we have \( \Gamma_{(p,q)} - e_0 = \Gamma_{(p,q)}' - e_0' \). Let \( \Gamma = t_1 \cdot r(p, q) \) and \( \Gamma' = t_2 \cdot r(p, q) \) for some positive real numbers \( t_1 \) and \( t_2 \). By applying Corollary 4.5 to \( \Gamma_{(p,q)} \) and \( \Gamma_{(p,q)}' \), we obtain

\[
(1) \quad (1 + t_1)(z(\Gamma_{(p,q)}) - \frac{L^2}{L + r(p, q)} - z(\Gamma)) = (1 + t_2)(z(\Gamma_{(p,q)}') - \frac{(L')^2}{L' + r(p, q)} - z(\Gamma)).
\]

As \( t_2 \to 0 \), we have \( L' \to 0 \) and \( \Gamma_{(p,q)}' \to \Gamma_{pq} \), and so \( z(\Gamma_{(p,q)}') \to z(\Gamma_{pq}) \). We substitute \( t_1 = \frac{L}{r(p, q)} \) into Equation (14). Then we obtain the following relation as \( t_2 \to 0 \):

\[
(15) \quad z(\Gamma_{(p,q)}) = \frac{L^2}{L + r(p, q)} + \frac{L}{L + r(p, q)} z(\Gamma) + \frac{r(p, q)}{L + r(p, q)} z(\Gamma_{pq}).
\]

We use Equation (15) to obtain the following Theorem:

**Theorem 4.6.** Let \( \Gamma \) be a metrized graph. For each edge \( e_i \in E(\Gamma) \) such that \( \Gamma - e_i \) is connected, we have

\[
z(\Gamma) = \frac{L_i^2}{L_i + R_i} + \frac{L_i}{L_i + R_i} z(\Gamma - e_i) + \frac{R_i}{L_i + R_i} z(\Gamma_i).
\]

Proof. In Equation (15), replace \( \Gamma_{(p,q)} \) by \( \Gamma \), \( L \) by \( L_i \), \( \Gamma \) by \( \Gamma - e_i \). This gives what we wanted to show.

We call the identity in Theorem 4.6 the contraction-deletion identity for \( z(\Gamma) \).

If \( e_i \) is a bridge (i.e., \( R_i = \infty \)), \( z(\Gamma) = z(\Gamma_i) \), which can also be seen from Theorem 4.6 as \( R_i \to \infty \).

Moreover, for any metrized graph \( \Gamma \) and for each edge \( e_i \in E(\Gamma) \) such that \( \Gamma - e_i \) is connected, we obtain the expression below for \( K_i(\Gamma) \) by using its definition and Theorem 4.6

\[
K_i(\Gamma) = \frac{R_i}{L_i + R_i} \left( z(\Gamma_i) - z(\Gamma - e_i) \right).
\]
A function $f : \mathbb{R}^n \to \mathbb{R}$ is called homogeneous of degree $k$ if $f(\lambda x_1, \lambda x_2, \cdots, \lambda x_n) = \lambda^k f(x_1, x_2, \cdots, x_n)$ for $\lambda > 0$. A continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ which is homogeneous of degree $k$ has the following property:

$$k \cdot f = \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}.$$  \hfill (17)

Equation (17) is called Euler’s formula.

For a given metrized graph $\Gamma$ with $\#(E(\Gamma)) = e$, let $\{L_1, L_2, \cdots, L_e\}$ be the edge lengths. Then $z : \mathbb{R}_{>0}^e \to \mathbb{R}$ given by $z(L_1, L_2, \cdots, L_e) = z(\Gamma)$ is a continuously differentiable homogeneous function of degree 1, when we consider all possible length distributions without changing the topology of the graph $\Gamma$.

**Lemma 4.7.** Let $\Gamma$ be a metrized graph, and let $e_i \in E(\Gamma)$ be of length $L_i$ such that $\Gamma - e_i$ is connected. Then we have

$$\frac{\partial z(\Gamma)}{\partial L_i} = \frac{L_i(L_i + 2R_i)}{(L_i + R_i)^2} + \frac{R_i}{(L_i + R_i)^2} z(\Gamma - e_i) - \frac{R_i}{(L_i + R_i)^2} z(\overline{\Gamma}_i).$$

**Proof.** Note that $z(\overline{\Gamma}_i)$, $z(\Gamma - e_i)$ and $R_i$ are independent of $L_i$. Thus, taking the partial derivatives of the both sides of the identity in Theorem 4.6 with respect to $L_i$ gives the result. \hfill \square

**Theorem 4.8.** Let $\Gamma$ be a bridgeless metrized graph. Then we have

$$\sum_{e_i \in E(\Gamma)} \frac{L_i K_i(\Gamma)}{L_i + R_i} = \sum_{e_i \in E(\Gamma)} \frac{L_i R_i}{(L_i + R_i)^2} \left( z(\overline{\Gamma}_i) - z(\Gamma - e_i) \right) = \sum_{e_i \in E(\Gamma)} \frac{L_i^2 R_i}{(L_i + R_i)^2}.$$  \hfill (18)

**Proof.** The first equality follows from Equation (16). By Euler’s formula, $z(\Gamma) = \sum_{e_i \in E(\Gamma)} L_i \cdot \frac{\partial z(\Gamma)}{\partial L_i}$. Then the second equality follows from Lemma 4.7. \hfill \square

For a given metrized graph $\Gamma$ with $\#(E(\Gamma)) = e$, let $\{L_1, L_2, \cdots, L_e\}$ be the edge lengths. Then $r : \mathbb{R}_{>0}^e \to \mathbb{R}$ given by $r(L_1, L_2, \cdots, L_e) = r(\Gamma)$ is a continuously differentiable homogeneous function of degree 1, when we consider all possible length distributions without changing the topology of the graph $\Gamma$.

**Lemma 4.9.** Let $\Gamma$ be a metrized graph, and let $e_i \in E(\Gamma)$ be of length $L_i$ such that $\Gamma - e_i$ is connected. Then we have

$$\frac{\partial r(\Gamma)}{\partial L_i} = \frac{R_i^2}{(L_i + R_i)^2} + \frac{R_i}{(L_i + R_i)^2} r(\Gamma - e_i) - \frac{R_i}{(L_i + R_i)^2} r(\overline{\Gamma}_i).$$

**Proof.** Since $\ell(\Gamma) = z(\Gamma) + r(\Gamma)$ for any graph, and $\ell(\Gamma - e_i) = \ell(\overline{\Gamma}_i) = \ell(\Gamma) - L_i$, Theorem 4.6 is equivalent to

$$r(\Gamma) = \frac{L_i R_i}{L_i + R_i} + \frac{L_i}{L_i + R_i} r(\Gamma - e_i) + \frac{R_i}{L_i + R_i} r(\overline{\Gamma}_i).$$

(18)

Note that $r(\overline{\Gamma}_i)$, $r(\Gamma - e_i)$ and $R_i$ are independent of $L_i$. Thus, taking the partial derivatives of the both sides of Equation (18) with respect to $L_i$ gives the result. \hfill \square
We call the following identities the contraction-deletion identities for (20)

\[
\text{transformed into a } Y\text{-shaped graph with the same resistances between } p_i, q_i, \text{ and } p \text{ as in } \Gamma - e_i \text{ (see the articles \cite{CR} and \cite{C2} Section 2). The resulting graph is shown by the first graph in Figure 2.}
\]

From now on, we will use the following notation: \( R_{a_i,p} := \hat{j}_{p_i}(p, q_i), R_{b_i,p} := \hat{j}_{q_i}(p_i, p), R_{c_i,p} := \hat{j}_{p}(p_i, q_i) \). Let \( R_i \) be the resistance between \( p_i \) and \( q_i \) in \( \Gamma - e_i \). Note that \( R_{a_i,p} + R_{b_i,p} = R_i \) for each \( p \in \Gamma \).

If \( \Gamma - e_i \) is not connected, we set \( R_{b_i,p} = R_i = \infty \) and \( R_{a_i,p} = 0 \) if \( p \) belongs to the component of \( \Gamma - e_i \) containing \( p_i \), and we set \( R_{a_i,p} = R_i = \infty \) and \( R_{b_i,p} = 0 \) if \( p \) belongs to the component of \( \Gamma - e_i \) containing \( q_i \).

In the rest of the paper, for any metrized graph \( \Gamma \) and a fixed vertex \( p \in V(\Gamma) \) we will use the following notation:

\[
y(\Gamma) = \frac{1}{4} \sum_{e_i \in E(\Gamma)} \frac{L_i R_i^2}{(L_i + R_i)^2} + \frac{3}{4} \sum_{e_i \in E(\Gamma)} \frac{L_i (R_{a_i,p} - R_{b_i,p})^2}{(L_i + R_i)^2},
\]

\[
x(\Gamma) = \sum_{e_i \in E(\Gamma)} \frac{L_i R_i^2}{(L_i + R_i)^2} + \frac{3}{4} \sum_{e_i \in E(\Gamma)} \frac{L_i R_i^2}{(L_i + R_i)^2} - \frac{3}{4} \sum_{e_i \in E(\Gamma)} \frac{L_i (R_{a_i,p} - R_{b_i,p})^2}{(L_i + R_i)^2}.
\]

If \( \Gamma - e_i \) is not connected for an edge \( e_i \), i.e. \( R_i \) is infinite (and \( (R_{a_i,p} - R_{b_i,p})^2 = R_i^2 \)), the summands should be considered as their corresponding limits as \( R_i \to \infty \).

It follows from \cite{C2} Proposition 2.9 that

\[
\tau(\Gamma) = \frac{\ell(\Gamma)}{12} - \frac{x(\Gamma)}{6} + \frac{y(\Gamma)}{6}.
\]

It is easy to see that

\[
r(\Gamma) = x(\Gamma) + y(\Gamma), \quad \text{and so } \ell(\Gamma) = x(\Gamma) + y(\Gamma) + z(\Gamma).
\]

We call the following identities the contraction-deletion identities for \( x(\Gamma) \) and \( y(\Gamma) \).
Theorem 4.10. Let $\Gamma$ be a metrized graph with an edge $e_i \in E(\Gamma)$ such that $\Gamma - e_i$ is connected. Then we have

\[ x(\Gamma) = \frac{L_i R_i}{L_i + R_i} x(\Gamma - e_i) + \frac{R_i}{L_i + R_i} x(\Gamma_i), \]

\[ y(\Gamma) = \frac{L_i}{L_i + R_i} y(\Gamma - e_i) + \frac{R_i}{L_i + R_i} y(\Gamma_i). \]

Proof. By Equations (18) and (20),

\[ x(\Gamma) + y(\Gamma) = \frac{L_i R_i}{L_i + R_i} + \frac{L_i}{L_i + R_i} (x(\Gamma - e_i) + y(\Gamma - e_i)) + \frac{R_i}{L_i + R_i} (x(\Gamma_i) + y(\Gamma_i)). \]

On the other hand, by Lemma 3.5 and Equation (19) applied to each of $\Gamma, \Gamma - e_i$ and $\Gamma_i$ we have

\[ x(\Gamma) - y(\Gamma) = \frac{L_i R_i}{L_i + R_i} + \frac{L_i}{L_i + R_i} (x(\Gamma - e_i) - y(\Gamma - e_i)) + \frac{R_i}{L_i + R_i} (x(\Gamma_i) - y(\Gamma_i)). \]

Hence, the result follows from Equation (21) and Equation (22). \qed

Lemma 4.11. Let $\Gamma$ be a metrized graph with an edge $e_i \in E(\Gamma)$ such that $\Gamma - e_i$ is connected. Let $p_i$ and $q_i$ be end points of $e_i$. Then we have

\[ x(\Gamma) - y(\Gamma) = x(\Gamma_i) - y(\Gamma_i) + 6 \frac{L_i A_{p_i,q_i,\Gamma - e_i}}{R_i (L_i + R_i)}. \]

Proof. It follows from Lemma 3.1 and Lemma 3.4 that

\[ \tau(\Gamma_i) = \tau(\Gamma - e_i) - \frac{R_i}{6} + \frac{A_{p_i,q_i,\Gamma - e_i}}{R_i}. \]

From Equation (23) and Equation (19) applied to both $\Gamma_i$ and $\Gamma - e_i$, we get

\[ x(\Gamma_i) - y(\Gamma_i) = x(\Gamma - e_i) - y(\Gamma - e_i) + R_i - 6 \frac{A_{p_i,q_i,\Gamma - e_i}}{R_i}. \]

Therefore, we obtain the result by solving Equation (24) for $x(\Gamma - e_i) - y(\Gamma - e_i)$ and substituting into Equation (22). \qed

For a given metrized graph $\Gamma$ with $\#(E(\Gamma)) = e$, let $\{L_1, L_2, \cdots, L_e\}$ be the edge lengths. Both of the functions $x : \mathbb{R}^e_{>0} \to \mathbb{R}$ given by $x(L_1, L_2, \cdots, L_e) = x(\Gamma)$ and $y : \mathbb{R}^e_{>0} \to \mathbb{R}$ given by $y(L_1, L_2, \cdots, L_e) = y(\Gamma)$ are continuously differentiable homogeneous functions of degree 1, when we consider all possible length distributions without changing the topology of $\Gamma$.

Theorem 4.12. Let $\Gamma$ be a bridgeless metrized graph. Then we have

\[ x(\Gamma) = \sum_{e_i \in E(\Gamma)} \frac{L_i R_i^2}{(L_i + R_i)^2} + \sum_{e_i \in E(\Gamma)} \frac{L_i R_i}{(L_i + R_i)^2} (x(\Gamma - e_i) - x(\Gamma_i)), \]

\[ y(\Gamma) = \sum_{e_i \in E(\Gamma)} \frac{L_i R_i}{(L_i + R_i)^2} (y(\Gamma - e_i) - y(\Gamma_i)). \]
Corollary 4.14. Let \( \sum \) with respect to \( L \)

Proof. By Theorem 4.13, we have

\[
\frac{\partial x(\Gamma)}{\partial L_i} = \frac{R_i^2}{(L_i + R_i)^2} + \frac{R_i}{(L_i + R_i)^2} x(\Gamma - e_i) - \frac{R_i}{(L_i + R_i)^2} x(\Gamma^i),
\]

\[
\frac{\partial y(\Gamma)}{\partial L_i} = \frac{R_i}{(L_i + R_i)^2} y(\Gamma - e_i) - \frac{R_i}{(L_i + R_i)^2} y(\Gamma^i).
\]

Therefore, by applying Euler’s formula we obtain the equalities we wanted.

We call the following identities the contraction identities for \( x(\Gamma) \) and \( y(\Gamma) \).

Theorem 4.13. Let \( \Gamma \) be a bridgeless metrized graph with \( v = \#(V(\Gamma)) \geq 2 \). Then we have

\[
(v - 2)x(\Gamma) = \sum_{e_i \in E(\Gamma)} \frac{R_i}{L_i + R_i} x(\Gamma^i), \quad (v - 2)y(\Gamma) = \sum_{e_i \in E(\Gamma)} \frac{R_i}{L_i + R_i} y(\Gamma^i),
\]

Proof. Multiplying both sides of the equalities in Theorem 4.10 by \( \frac{R_i}{L_i + R_i} \), and using the fact that \( \sum_{e_i \in E(\Gamma)} \frac{R_i}{L_i + R_i} = v - 1 \) (see Equation (2)), we obtain

\[
(v - 1)x(\Gamma) = \sum_{e_i \in E(\Gamma)} \frac{L_i R_i}{(L_i + R_i)^2} + \sum_{e_i \in E(\Gamma)} \frac{L_i R_i}{(L_i + R_i)^2} x(\Gamma - e_i) + \sum_{e_i \in E(\Gamma)} \frac{R_i^2}{(L_i + R_i)^2} x(\Gamma^i),
\]

\[
(v - 1)y(\Gamma) = \sum_{e_i \in E(\Gamma)} \frac{L_i R_i}{(L_i + R_i)^2} y(\Gamma - e_i) + \sum_{e_i \in E(\Gamma)} \frac{R_i^2}{(L_i + R_i)^2} y(\Gamma^i).
\]

Thus, the result follows from Equation (26) and Theorem 4.12.

We call the first identity in the corollary below the contraction identity for \( z(\Gamma) \).

Corollary 4.14. Let \( \Gamma \) be a bridgeless metrized graph with \( v = \#(V(\Gamma)) \geq 2 \). Then we have

\[
(v - 1)z(\Gamma) = \sum_{e_i \in E(\Gamma)} \frac{R_i}{L_i + R_i} z(\Gamma^i), \quad (v - 2)r(\Gamma) = \sum_{e_i \in E(\Gamma)} \frac{R_i}{L_i + R_i} r(\Gamma^i).
\]

Proof. The second equality follows by adding the expressions in Theorem 4.13 and using the fact that \( \ell(\Gamma) = z(\Gamma) + r(\Gamma) \). Using the second equality along with the facts that \( z(\Gamma^i) = \ell(\Gamma) - L_i - r(\Gamma^i) \) and \( \sum_{e_i \in E(\Gamma)} \frac{R_i}{L_i + R_i} = v - 1 \) (see Equation (2)), we obtain the first equality.

Corollary 4.15. Let \( \Gamma \) be a bridgeless metrized graph with \( v = \#(V(\Gamma)) \geq 3 \). Then we have

\[
\tau(\Gamma) = \frac{\ell(\Gamma)}{12} - \frac{1}{6(v-2)} \sum_{e_i \in E(\Gamma)} \frac{R_i}{L_i + R_i} (x(\Gamma^i) - y(\Gamma^i)).
\]

Proof. By Theorem 4.13, we have

\[
(v - 2)(x(\Gamma) - y(\Gamma)) = \sum_{e_i \in E(\Gamma)} \frac{R_i}{L_i + R_i} (x(\Gamma^i) - y(\Gamma^i)).
\]

Thus, the result follows from Equation (19).
We call the identities in Theorem 4.16 and Corollary 4.17 the deletion identities.

**Theorem 4.16.** Let $\Gamma$ be a bridgeless metrized graph. Then we have

$$g \cdot x(\Gamma) = y(\Gamma) + \sum_{e_i \in E(\Gamma)} \frac{L_i}{L_i + R_i} x(\Gamma - e_i), \quad (g + 1)y(\Gamma) = \sum_{e_i \in E(\Gamma)} \frac{L_i}{L_i + R_i} y(\Gamma - e_i).$$

**Proof.** Multiplying both sides of the equalities in Theorem 4.10 by $\frac{L_i}{L_i + R_i}$, and using the fact that $\sum_{e_i \in E(\Gamma)} \frac{L_i}{L_i + R_i} = g$ (see Equation (2)) we obtain

$$g \cdot x(\Gamma) = \sum_{e_i \in E(\Gamma)} \frac{L_i^2 R_i}{(L_i + R_i)^2} x(\Gamma - e_i) + \sum_{e_i \in E(\Gamma)} \frac{L_i R_i}{(L_i + R_i)^2} x(\Gamma_i),$$

$$g \cdot y(\Gamma) = \sum_{e_i \in E(\Gamma)} \frac{L_i^2 R_i}{(L_i + R_i)^2} y(\Gamma - e_i) + \sum_{e_i \in E(\Gamma)} \frac{L_i R_i}{(L_i + R_i)^2} y(\Gamma_i).$$

(28)

The first equality is obtained by adding the first equalities in Theorem 4.12 and Equation (28) and using the fact that $r(\Gamma) = x(\Gamma) + y(\Gamma)$.

Similarly, the second equality is obtained by adding the second equalities in Theorem 4.12 and Equation (28).

**Corollary 4.17.** Let $\Gamma$ be a bridgeless metrized graph. Then we have

$$(g - 1)z(\Gamma) = \sum_{e_i \in E(\Gamma)} \frac{L_i}{L_i + R_i} z(\Gamma - e_i), \quad g \cdot r(\Gamma) = \sum_{e_i \in E(\Gamma)} \frac{L_i}{L_i + R_i} r(\Gamma - e_i).$$

**Proof.** Adding the identities in Theorem 4.16 and using the fact that $r(\Gamma) = x(\Gamma) + y(\Gamma)$ give the second formula.

Then the first formula is obtained by using the second formula, Equation (2) and the fact that $\ell(\Gamma) = z(\Gamma) + r(\Gamma)$.

**Corollary 4.18.** Let $\Gamma$ be a bridgeless metrized graph. Then we have

$$\tau(\Gamma) = \frac{\ell(\Gamma)}{12} - \frac{1}{6(g + 1)} \sum_{e_i \in E(\Gamma)} \frac{L_i}{L_i + R_i} \left( x(\Gamma - e_i) - y(\Gamma - e_i) \right)$$

$$- \frac{1}{6(g + 1)} \sum_{e_i \in E(\Gamma)} \frac{L_i}{L_i + R_i} r(\Gamma - e_i).$$

**Proof.** By Theorem 4.16 and the fact that $r(\Gamma) = x(\Gamma) + y(\Gamma)$, we have

$$(g + 1) \cdot (x(\Gamma) - y(\Gamma)) = r(\Gamma) + \sum_{e_i \in E(\Gamma)} \frac{L_i}{L_i + R_i} (x(\Gamma - e_i) - y(\Gamma - e_i)).$$

(29)

Thus, the result follows from Equation (19) and the second identity in Corollary 4.17.

In this section, we proved the following identities among other things:
By Theorem 4.10 and Theorem 4.16 the contraction-deletion identities for a metrized graph \( \Gamma \) and for an edge \( e_i \in E(\Gamma) \) with connected \( \Gamma - e_i \) are

\[
x(\Gamma) = \frac{L_iR_i}{L_i + R_i} + \frac{L_i}{L_i + R_i}x(\Gamma - e_i) + \frac{R_i}{L_i + R_i}x(\Gamma_i),
\]

\[
y(\Gamma) = \frac{L_i}{L_i + R_i}y(\Gamma - e_i) + \frac{R_i}{L_i + R_i}y(\Gamma_i),
\]

\[
z(\Gamma) = \frac{L_i^2}{L_i + R_i} + \frac{L_i}{L_i + R_i}z(\Gamma - e_i) + \frac{R_i}{L_i + R_i}z(\Gamma_i).
\]

By Theorem 4.13 and Corollary 4.14 the contraction identities for a bridgeless metrized graph with \( v = \#(V(\Gamma)) \geq 2 \) are

\[
(v - 2)x(\Gamma) = \sum_{e_i \in E(\Gamma)} \frac{R_i}{L_i + R_i}x(\Gamma_i), \quad (v - 2)y(\Gamma) = \sum_{e_i \in E(\Gamma)} \frac{R_i}{L_i + R_i}y(\Gamma_i),
\]

\[
(v - 1)z(\Gamma) = \sum_{e_i \in E(\Gamma)} \frac{R_i}{L_i + R_i}z(\Gamma_i), \quad (v - 2)r(\Gamma) = \sum_{e_i \in E(\Gamma)} \frac{R_i}{L_i + R_i}r(\Gamma_i).
\]

By Theorem 4.16 and Corollary 4.17 the deletion identities for a bridgeless \( \Gamma \) are

\[
g \cdot x(\Gamma) = y(\Gamma) + \sum_{e_i \in E(\Gamma)} \frac{L_i}{L_i + R_i}x(\Gamma - e_i), \quad (g + 1)y(\Gamma) = \sum_{e_i \in E(\Gamma)} \frac{L_i}{L_i + R_i}y(\Gamma - e_i),
\]

\[
(g - 1)z(\Gamma) = \sum_{e_i \in E(\Gamma)} \frac{L_i}{L_i + R_i}z(\Gamma - e_i), \quad g \cdot r(\Gamma) = \sum_{e_i \in E(\Gamma)} \frac{L_i}{L_i + R_i}r(\Gamma - e_i).
\]

Also, for a bridgeless \( \Gamma \) the following identity of Theorem 4.18 deserves attention:

\[
\sum_{e_i \in E(\Gamma)} \frac{L_iK_i(\Gamma)}{L_i + R_i} = \sum_{e_i \in E(\Gamma)} \frac{L_iR_i}{(L_i + R_i)^2} (z(\Gamma_i) - z(\Gamma - e_i)) = \sum_{e_i \in E(\Gamma)} \frac{L_i^2R_i}{(L_i + R_i)^2}.
\]

5. Successive edge contraction

In this section, we will successively contract edges in \( E(\Gamma) \) for any metrized graph \( \Gamma \). The contraction identities developed in the previous section will enable us to generalize the results of \( \mathbb{F} \) and some of the results of \( \mathbb{G} \). The results of this section will help us to understand the effects of topological properties of \( \Gamma \), such as the edge connectivity, on \( \tau(\Gamma) \).

Let \( \Gamma \) be a metrized graph and let \( \Gamma_i \) be the metrized graph obtained by contracting \( i \)-th edge \( e_i \in E(\Gamma) \) to its end points. Similarly, for any integer \( k \geq 2 \), let \( \Gamma_{i_1,i_2,\ldots,i_k} \) be the metrized graph obtained by contracting \( i_k \)-th edge \( e_{i_k} \in E(\Gamma_{i_1,i_2,\ldots,i_{k-1}}) \) to its end points. Note that \( E(\Gamma_{i_1,i_2,\ldots,i_k}) = E(\Gamma) - \{e_{i_1}, e_{i_2}, \ldots, e_{i_k}\} \) for any \( k \). Let \( \Gamma_{i_0} := \Gamma \).

Let \( e_{i_k} \in E(\Gamma) \) be an edge of index \( i_k \). Recall that we denote the resistance between the end points of \( e_{i_k} \) in \( \Gamma - e_{i_k} \) by \( R_{i_k} \) and that we use \( L_{i_k} \) to denote the length of \( e_{i_k} \).

Now, we generalize Equation (27) as follows:
**Lemma 5.1.** Let $\Gamma$ be a bridgeless metrized graph with $(k + 2) \leq v = \#(V(\Gamma))$ for some integer $k \geq 1$. Then

$$
\frac{(v - 2)!}{(v - k - 2)!}(x(\Gamma) - y(\Gamma)) = \sum_{e_{i_1} \in E(\Gamma)} \frac{R_{i_1}}{L_{i_1} + R_{i_1}} \sum_{e_{i_2} \in \mathcal{E}(\Gamma_{i_1})} \frac{R_{i_2}}{L_{i_2} + R_{i_2}} \cdots \sum_{e_{i_k} \in \mathcal{E}(\Gamma_{i_1, \ldots, i_{k-1}})} \frac{R_{i_k}}{L_{i_k} + R_{i_k}}(x(\Gamma_{i_1, \ldots, i_k}) - y(\Gamma_{i_1, \ldots, i_k})).
$$

*Proof.* Note that if an edge of a bridgeless graph is contracted the resulting graph will be also bridgeless. If an edge $e_{i_j}$ is a self loop, then $\frac{R_{i_j}}{L_{i_j} + R_{i_j}} = 0$. Thus, contraction of self loops does not contribute to sums in contraction identities. Hence, we can inductively apply Equation (27) to obtain the result. \(\square\)

**Remark 5.2.** After contracting edges in a graph $\Gamma$, multiple edges or self-loops may appear. However, this does not cause any problem for contraction identities.

We can generalize Corollary 4.15 as follows:

**Theorem 5.3.** Let $\Gamma$ be a bridgeless metrized graph with $(k + 2) \leq v = \#(V(\Gamma))$ for some integer $k \geq 1$. Then we have

$$
\tau(\Gamma) = \ell(\Gamma) - \frac{(v - k - 2)!}{6(v - 2)!} \sum_{e_{i_1} \in E(\Gamma)} \frac{R_{i_1}}{L_{i_1} + R_{i_1}} \sum_{e_{i_2} \in \mathcal{E}(\Gamma_{i_1})} \frac{R_{i_2}}{L_{i_2} + R_{i_2}} \cdots \sum_{e_{i_k} \in \mathcal{E}(\Gamma_{i_1, \ldots, i_{k-1}})} \frac{R_{i_k}}{L_{i_k} + R_{i_k}}(x(\Gamma_{i_1, \ldots, i_k}) - y(\Gamma_{i_1, \ldots, i_k})).
$$

*Proof.* We can use Lemma 5.1 and Equation (19) to obtain the result. \(\square\)

Here is another formula for $r(\Gamma)$:

**Proposition 5.4.** Let $\Gamma$ be a bridgeless graph with $3 \leq v = \#(V(\Gamma))$. Then for any $k$ with $k + 2 \leq v$,

$$
r(\Gamma) = \frac{k(v - 2)!}{(v - k - 1)!}\sum_{e_{i_1} \in E(\Gamma)} \frac{R_{i_1}}{L_{i_1} + R_{i_1}} \cdots \sum_{e_{i_k} \in \mathcal{E}(\Gamma_{i_1, \ldots, i_{k-1}})} \frac{R_{i_k}}{L_{i_k} + R_{i_k}} \sum_{i=1}^{k} L_{i_i}.
$$

*Proof.* By applying the second part of Corollary 4.14 successively, we obtain

$$
r(\Gamma) = \sum_{e_{i_1} \in E(\Gamma)} \frac{R_{i_1}}{L_{i_1} + R_{i_1}} \cdots \sum_{e_{i_k} \in \mathcal{E}(\Gamma_{i_1, \ldots, i_{k-1}})} \frac{R_{i_k}}{L_{i_k} + R_{i_k}} r(\Gamma_{i_1, \ldots, i_k}).
$$

Now, we can use induction on $k$ to show the identity in the proposition. When $k = 1$, the result holds trivially by the definition of $r(\Gamma)$. Suppose the result is true for $k = n$ where
\( n + 3 \leq v \). Let \( A \) be the right hand side of Equation (54) for \( k = n + 1 \). By splitting the sum \( \sum_{t=1}^{n+1} L_{it} = \left( \sum_{t=1}^{n} L_{it} \right) + L_{n+1} \) we have
\[
A = \sum_{e_{i1} \in E(\Gamma)} \frac{R_{i1}}{L_{i1} + R_{i1}} \cdots \sum_{e_{in} \in E(\Gamma)} \frac{R_{in}}{L_{in} + R_{in}} \sum_{t=1}^{n} L_{it} \sum_{e_{in+1} \in E(\Gamma)} \frac{R_{in+1}}{L_{in+1} + R_{in+1}}
+ \sum_{e_{i1} \in E(\Gamma)} \frac{R_{i1}}{L_{i1} + R_{i1}} \cdots \sum_{e_{in} \in E(\Gamma)} \frac{R_{in}}{L_{in} + R_{in}} \sum_{t=1}^{n} L_{it} \sum_{e_{in+1} \in E(\Gamma)} \frac{R_{in+1}}{L_{in+1} + R_{in+1}}
= (v - n - 1) \sum_{e_{i1} \in E(\Gamma)} \frac{R_{i1}}{L_{i1} + R_{i1}} \cdots \sum_{e_{in} \in E(\Gamma)} \frac{R_{in}}{L_{in} + R_{in}} \sum_{t=1}^{n} L_{it}\]
by Equation (32) applied to \( \Gamma_{i1,\ldots,i_{n-1}} \), and by Equation (35),
\[
= \frac{(n + 1)(v - 2)!}{(v - n - 2)!} r(\Gamma), \quad \text{by the induction assumption.}
\]
Hence the result follows.

Note that Equation (35) generalizes the second equation in Corollary 4.14.

**Corollary 5.5.** Let \( \Gamma \) be a bridgeless graph with \( 3 \leq v = \#(V(\Gamma)) \). Then
\[
(v - 2)(v - 2)! r(\Gamma) = \sum_{e_{i1} \in E(\Gamma)} \frac{R_{i1}}{L_{i1} + R_{i1}} \cdots \sum_{e_{in} \in E(\Gamma)} \frac{R_{in}}{L_{in} + R_{in}} \sum_{t=1}^{n} L_{it}.
\]

**Proof.** The result follows from Proposition 5.4 with \( k = v - 2 \).

**Corollary 5.6.** Let \( \Gamma \) be a bridgeless graph with \( 3 \leq v = \#(V(\Gamma)) \) and \( e \) edges. For any \( k \in \{1, 2, \ldots, v - 2\} \), let \( A_k = \{ \sum_{t=1}^{k} L_{it} | \{i_1, \ldots, i_k\} \subseteq \{1, 2, \ldots, e\} \} \). Let \( C_k = \max(A_k) \) and \( c_k = \min(A_k) \). Then we have
\[
\frac{(v - 1)}{k} c_k \leq r(\Gamma) \leq \frac{(v - 1)}{k} C_k, \quad \text{and in particular,} \quad \frac{v - 1}{v - 2} c_{v-2} \leq r(\Gamma) \leq \frac{v - 1}{v - 2} C_{v-2}.
\]

**Proof.** The result follows from Proposition 5.4 and Equation (32).

Note also that successive application of the first part of Corollary 4.14 gives
\[
\frac{(v - 1)!}{(v - k - 1)!} z(\Gamma) = \sum_{e_{i1} \in E(\Gamma)} \frac{R_{i1}}{L_{i1} + R_{i1}} \cdots \sum_{e_{ik} \in E(\Gamma)} \frac{R_{ik}}{L_{ik} + R_{ik}} z(\Gamma_{i1,\ldots,i_k}).
\]

The following theorem generalizes Theorem 3.3.

**Theorem 5.7.** Let \( \Gamma \) be a bridgeless metrized graph with \( (k + 2) \leq v = \#(V(\Gamma)) \) for some integer \( k \geq 1 \). Then
\[
\tau(\Gamma) = \frac{(v - k - 2)!}{(v - 2)!} \sum_{e_{i1} \in E(\Gamma)} \frac{R_{i1}}{L_{i1} + R_{i1}} \cdots \sum_{e_{ik} \in E(\Gamma)} \frac{R_{ik}}{L_{ik} + R_{ik}} \tau(\Gamma_{i1,\ldots,i_k}) - \frac{k \cdot z(\Gamma)}{12(v - k - 1)}.
\]
Proof. First, we note that \( \tau(\Gamma_{1,\ldots,i_k}) = \frac{\ell(\Gamma_{1,\ldots,i_k})}{12} = \frac{x(\Gamma_{1,\ldots,i_k}) - y(\Gamma_{1,\ldots,i_k})}{6} \) by Equation (19), and \( \ell(\Gamma_{i_1,\ldots,i_k}) = \ell(\Gamma) - \sum_{t=1}^k L_{it} \). Then the result follows by applying Theorem 5.3 and Proposition 5.4.

Corollary 5.8. Let \( \Gamma \) be a bridgeless metrized graph with \( 3 \leq v = \#(V(\Gamma)) \). Then

\[
\tau(\Gamma) = \frac{1}{(v-2)!} \sum_{e_{i_1} \in E(\Gamma)} \frac{R_{i_1}}{L_{i_1} + R_{i_1}} \cdots \sum_{e_{i_v} \in E(\Gamma)} \frac{R_{i_v}}{L_{i_v} + R_{i_v}} - \frac{v-2}{12} z(\Gamma).
\]

Proof. The result follows from Theorem 5.7 with \( k = v - 2 \).

Recall that Theorem 3.3 is valid for graphs with more than 2 vertices. If an edge \( e_{i_k} \) is not a self loop in \( \Gamma_{i_1,i_2,\ldots,i_{k-1}} \), then \( \#(V(\Gamma_{i_1,i_2,\ldots,i_{k-1}})) = \#(V(\Gamma_i)) - 1 \). We call \( \Gamma_{i_1,i_2,\ldots,i_{k-1}} \) an admissible contraction of \( \Gamma \), if it is obtained from \( \Gamma \) by contracting edges with distinct end points, i.e., if we have \( \#(V(\Gamma_{i_1,i_2,\ldots,i_{k-1}})) = 2 \). Note that such graphs are the only ones that contribute to the sum in Corollary 5.8.

Let \( \Gamma_{i_1,i_2,\ldots,i_{k-1}} \) be an admissible contraction of \( \Gamma \), and let \( V(\Gamma_{i_1,i_2,\ldots,i_{k-1}}) = \{p,q\} \). The graph \( \Gamma_{i_1,i_2,\ldots,i_{k-1}} \) has \( n(i_1,\ldots,i_{k-1}) \) multiple edges between the vertices \( p \) and \( q \), and self-loops at \( p \) or \( q \). Figure 3 illustrates \( \Gamma_{i_1,i_2,\ldots,i_{k-1}} \). Let \( n' := n(i_1,\ldots,i_{k-1}) \) be the number of multiple edges in \( \Gamma_{i_1,i_2,\ldots,i_{k-1}} \), and let \( B' := \{e_{i_1},e_{i_2},\ldots,e_{i_{k-1}}\} \) be the set of multiple edges in \( \Gamma_{i_1,i_2,\ldots,i_{k-1}} \). For the resistance function \( r'(x,y) \) in \( \Gamma_{i_1,i_2,\ldots,i_{k-1}} \), we have \( r'(p,q) = \frac{1}{\sum_{t=1}^{n'} \frac{L_{it}}{L_{it} + R_{it}}} \) by circuit theory.

Therefore,

**Proposition 5.9.** Let \( \Gamma \) be a bridgeless metrized graph. Using the notation above, for each admissible contraction \( \Gamma_{i_1,i_2,\ldots,i_{k-1}} \) of \( \Gamma \), we have

\[
x(\Gamma_{i_1,i_2,\ldots,i_{k-1}}) = (n' - 1) \cdot r'(p,q), \quad y(\Gamma_{i_1,i_2,\ldots,i_{k-1}}) = r'(p,q).
\]

Proof. First note that \( r'(p,q) = \frac{L_{it}R_{it}}{L_{it} + R_{it}} \) for each \( e_{it} \in B' \), and \( R_{it} = 0 \) if \( e_{it} \notin B' \).

\[
(37) \quad r(\Gamma_{i_1,i_2,\ldots,i_{k-1}}) = \sum_{e_{it} \in B'} \frac{L_{it}R_{it}}{L_{it} + R_{it}} = n' \cdot r'(p,q).
\]

Moreover, \( (R_{i_t,p} - R_{i_t,p})^2 = R_{i_t}^2 \) for each \( e_{it} \in E(\Gamma_{i_1,i_2,\ldots,i_{k-1}}) \). Thus, by definition \( x(\Gamma_{i_1,i_2,\ldots,i_{k-1}}) = \sum_{e_{i_1} \in E(\Gamma_{i_1,i_2,\ldots,i_{k-1}})} \frac{L_{i_1}R_{i_1}}{(L_{i_1} + R_{i_1})^2} = \sum_{e_{it} \in B'} \frac{L_{it}R_{it}}{(L_{it} + R_{it})^2} = r'(p,q) \sum_{e_{it} \in B'} \frac{L_{it}}{L_{it} + R_{it}} = r'(p,q)(n' - 1) \), where the last equality follows from Equation (2). This proves the first equality, and the second equality follows from the first equality and Equation (37).

Here is another formula for the tau constant:
Proof. We have \( x(\Gamma_{i_1, \ldots, i_{v-2}}) - y(\Gamma_{i_1, \ldots, i_{v-2}}) = (n' - 2) \cdot r'(p, q) \), by Proposition 5.9. Therefore, we obtain what we want by using Theorem 5.3 with \( k = v - 2 \). \( \square \)

6. Edge connectivity and the tau constant

In this section, we will prove that Conjecture 1.1 holds with \( C = \frac{1}{108} \) for any graph \( \Gamma \) with edge connectivity more than or equal to 6, and we will give a lower bound to the tau constant in terms of edge connectivity.

Let \( \Gamma \) be a bridgeless metrized graph, and let \( \Gamma_{i_1, \ldots, i_{v-2}}, n', p, q, r'(p, q) \) and \( B' \) be as in [5]. Recall that \( n' := n(i_1, \ldots, i_{v-2}) \) is the number of multiple edges in \( \Gamma_{i_1, \ldots, i_{v-2}} \) and that \( B' := \{ e_1, e_2, \ldots, e_{n'} \} \) is the set of multiple edges in \( \Gamma_{i_1, \ldots, i_{v-2}} \). We will show that a lower bound for

\[
N(\Gamma) := \min\{ n' \mid \{ i_1, \ldots, i_{v-2} \} \subset \{ 1, 2, \ldots, e \} \}
\]
gives a lower bound for \( \tau(\Gamma) \). We will make some observations about \( N(\Gamma) \) after recalling some basic definitions from graph theory.

We recall the following inequality between the edge connectivity \( \Lambda(\Gamma) \), vertex connectivity \( \kappa(\Gamma) \), and the minimum degree of the valences \( \delta(\Gamma) \).

**Remark 6.1.** For a graph \( \Gamma \), we have \( \kappa(\Gamma) \leq \Lambda(\Gamma) \leq \delta(\Gamma) \) by basic graph theory [BB1, pg. 3].

Recall that a metrized graph is connected by definition.

**Lemma 6.2.** Let \( \Gamma \) be a graph. Then \( N(\Gamma) = \Lambda(\Gamma) \).

**Proof.** If \( V(\Gamma) = 2 \), then \( \Gamma \) is a banana graph with possibly self-loops. Then \( N(\Gamma) = \Lambda(\Gamma) \) clearly.

Note that when we contract an edge of a graph \( \Gamma \) with \( V(\Gamma) \geq 3 \), the edge connectivity either does not change or increases. Therefore, \( \Lambda(\Gamma_{i_1, \ldots, i_{v-2}}) \geq \Lambda(\Gamma) \) for the contraction of any edges \( e_{i_1}, \ldots, e_{i_{v-2}} \). Since \( n' = \Lambda(\Gamma_{i_1, \ldots, i_{v-2}}) \geq \Lambda(\Gamma) \), we have \( N(\Gamma) \geq \Lambda(\Gamma) \).

Let \( k = \Lambda(\Gamma) \), and let \( e_1, e_2, \ldots, e_k \) be edges such that \( \Gamma - \{ e_1, e_2, \ldots, e_k \} \) is disconnected but \( \Gamma - (\{ e_1, e_2, \ldots, e_k \} - e_j) \) is connected for each \( e_j \) where \( 1 \leq j \leq k \). Also, let \( p \) and \( q \) be the end points of the edge \( e_k \). Note that \( e_k \) is a bridge in \( \Gamma - \{ e_1, e_2, \ldots, e_{k-1} \} \). That is, \( \Gamma - \{ e_1, e_2, \ldots, e_{k-1} \} = \beta \cup e_k \cup \gamma \) for some graphs \( \beta \) and \( \gamma \) with \( \beta \cap e_k = \{ p \} \) and \( \gamma \cap e_k = \{ q \} \). Contract edges in \( E(\beta) \), say \( e_{i_1}, e_{i_2}, \ldots, e_{i_s} \), until \( \beta \) has 1 vertex. Similarly, contract edges in \( E(\gamma) \), say \( e_{l_1}, e_{l_2}, \ldots, e_{l_t} \), until \( \gamma \) has 1 vertex. Then, \( s + t = v - 2 \) and \( n(i_1, i_2, \ldots, i_s, l_1, l_2, \ldots, l_t) = k \) for the contraction graph \( \Gamma_{i_1, i_2, \ldots, i_s, l_1, l_2, \ldots, l_t} \). Thus, \( N(\Gamma) \leq \Lambda(\Gamma) \).

Hence, the result follows. \( \square \)

We will need the following computation before we relate the edge connectivity \( \Lambda(\Gamma) \) to \( \tau(\Gamma) \).
Corollary 6.3. Let $\Gamma$ be a bridgeless metrized graph with genus $g$. Then for any admissible contraction $\Gamma_{i_1,\ldots,i_{v-2}}$ of $\Gamma$ we have
\[ g \cdot y(\Gamma_{i_1,\ldots,i_{v-2}}) \geq x(\Gamma_{i_1,\ldots,i_{v-2}}) \geq (\Lambda(\Gamma) - 1) \cdot y(\Gamma_{i_1,\ldots,i_{v-2}}). \]

Proof. Since $\Gamma_{i_1,\ldots,i_{v-2}}$ has $e - (v - 2) = g + 1$ edges, $g + 1 \geq \max\{n'|\{i_1,\ldots,i_{v-2}\} \subset \{1,2,\ldots,e\}\}$. Then the first inequality follows from Proposition 5.9. The second inequality follows from Lemma 6.2 and Proposition 5.9.

When $k = v - 2$, Equation (36) becomes
\[ (v - 1)! z(\Gamma) = \sum_{e_{i_1} \in E(\Gamma)} \frac{R_{i_1}}{L_{i_1} + R_{i_1}} \cdots \sum_{e_{i_{v-2}} \in E(\Gamma_{1,\ldots,i_{v-3}})} \frac{R_{i_{v-2}}}{L_{i_{v-2}} + R_{i_{v-2}}} \sum_{e_{i_{v-1}} \in E(\Gamma_{i_1,\ldots,i_{v-2}})} \frac{L_{i_{v-1}}^2}{L_{i_{v-1}} + R_{i_{v-1}}}. \]

Lemma 6.4. For each admissible contraction $\Gamma_{i_1,\ldots,i_{v-2}}$ of $\Gamma$ as above we have
\[ \sum_{e_t \in B'} \frac{L_t^2}{L_t + R_t} \geq n' \cdot (n' - 1) r'(p,q). \]

Proof. We have $\frac{1}{n'} \sum_{e_t \in B'} L_t \geq \sum_{e_t \in B'} \frac{L_t^2}{L_t + R_t}$ by Arithmetic-Harmonic Mean inequality. On the other hand, $\sum_{e_t \in B'} L_t = \sum_{e_t \in B'} L_t^2 + \sum_{e_t \in B'} L_t R_t$, and $r'(p,q) = \frac{1}{\sum_{t=1}^{n'} \frac{1}{L_t}}$. Thus the result follows from Equation (37).

Lemma 6.5. Let $\Gamma$ be a bridgeless metrized graph. Then we have $z(\Gamma) \geq \frac{\Lambda(\Gamma)}{v-1} x(\Gamma)$.

Proof. If we apply the first part of Theorem 4.13 successively, we derive the following expression:
\[ x(\Gamma) = \frac{1}{(v-2)!} \sum_{e_{i_1} \in E(\Gamma)} \frac{R_{i_1}}{L_{i_1} + R_{i_1}} \cdots \sum_{e_{i_{v-2}} \in E(\Gamma_{1,\ldots,i_{v-3}})} \frac{R_{i_{v-2}}}{L_{i_{v-2}} + R_{i_{v-2}}} x(\Gamma_{i_1,\ldots,i_{v-2}}). \]

On the other hand, for each admissible contraction $\Gamma_{i_1,\ldots,i_{v-2}}$ of $\Gamma$, $x(\Gamma_{i_1,\ldots,i_{v-2}}) = (n' - 1) r'(p,q)$ by Proposition 5.9. Then the result follows from Equation (38), Lemma 6.2 Lemma 6.4 and Equation (39).

Set
\[ w(\Gamma) := \frac{1}{(v-2)!} \sum_{e_{i_1} \in E(\Gamma)} \frac{R_{i_1}}{L_{i_1} + R_{i_1}} \cdots \sum_{e_{i_{v-2}} \in E(\Gamma_{1,\ldots,i_{v-3}})} \frac{R_{i_{v-2}}}{L_{i_{v-2}} + R_{i_{v-2}}} \sum_{e_{i_{v-1}} \in E(\Gamma_{i_1,\ldots,i_{v-2}})} \frac{L_{i_{v-1}}^3}{(L_{i_{v-1}} + R_{i_{v-1}})^2}. \]

Then we have

Lemma 6.6. Let $\Gamma$ be a bridgeless metrized graph. Then $(v - 1) z(\Gamma) = w(\Gamma) + x(\Gamma)$.

Proof. The result follows from Equations (38) and (39).

Theorem 6.7. [C2] Theorem 2.26] Let $\Gamma$ be a normalized metrized graph. Then
\[ \sum_{e_t \in E(\Gamma)} \frac{L_t R_t^2}{(L_t + R_t)^2} \geq \left( \sum_{e_t \in E(\Gamma)} \frac{L_t R_t}{L_t + R_t} \right)^2. \]
Lemma 6.8. [C2] Lemma 2.12] Let \( \Gamma \) be a metrized graph and \( p \in V(\Gamma) \). Then if \( e_i \sim p \) indicates that edge \( e_i \) is incident to vertex \( p \)

\[
\sum_{e_i \in E(\Gamma)} \frac{L_i(R_{a_i,p} - R_{b_i,p})^2}{(L_i + R_i)^2} = \frac{2}{v} \sum_{e_i \in E(\Gamma)} \frac{L_iR_i^2}{(L_i + R_i)^2} + \frac{1}{v} \sum_{p \in V(\Gamma)} \left( \sum_{e_i \neq p, e_i \in E(\Gamma)} \frac{L_i(R_{a_i,p} - R_{b_i,p})^2}{(L_i + R_i)^2} \right).
\]

We have the following relations between \( x(\Gamma) \) and \( y(\Gamma) \):

Theorem 6.9. Let \( \Gamma \) be a normalized bridgeless metrized graph with \( \#(V(\Gamma)) = v \), and let \( x = x(\Gamma), y = y(\Gamma) \). Then we have

1. \( \tau(\Gamma) = \frac{1}{12} - \frac{x}{6} + \frac{y}{6} \),
2. \( 1 \geq \frac{\Lambda(\Gamma) + v - 1}{v} \cdot x + y, \quad x \geq 0, \quad \text{and} \quad y \geq 0 \),
3. \( y \geq \frac{v + 6}{4v} (x + y)^2 \),
4. \( g \cdot y \geq x \geq (\Lambda(\Gamma) - 1) y \).

Proof. Since \( \Gamma \) is normalized, \( \ell(\Gamma) = 1 \). Thus, part (1) follows from Equation (19). Part (2) follows from Lemma 6.5 and Equation (20). By Lemma 6.8 and the definition of \( y \), we have

\[
(40) \quad y \geq \frac{v + 6}{4v} \sum_{e_i \in E(\Gamma)} \frac{L_iR_i^2}{(L_i + R_i)^2}.
\]

Thus, part (3) follows from Equation (40) and Theorem 6.4.

We have \( g \cdot y(\Gamma_{i_1,\ldots,i_{v-2}}) \geq x(\Gamma_{i_1,\ldots,i_{v-2}}) \geq (\Lambda(\Gamma) - 1) \cdot y(\Gamma_{i_1,\ldots,i_{v-2}}) \) by Corollary 6.3. We inductively apply Theorem 4.13 to obtain

\[
(v - 2)! y(\Gamma) = \sum_{e_{i_1} \in E(\Gamma)} \frac{R_{i_1}}{L_{i_1} + R_{i_1}} \sum_{e_{i_2} \in E(\Gamma)} \frac{R_{i_2}}{L_{i_2} + R_{i_2}} \cdots \sum_{e_{i_{v-2}} \in E(\Gamma)} \frac{R_{i_{v-2}}}{L_{i_{v-2}} + R_{i_{v-2}}} y(\Gamma_{i_1,\ldots,i_{v-2}}).
\]

Thus, using Equation (39) we have part (4).

Now, we can state the main result of this paper:

Theorem 6.10. Let \( \Gamma \) be a metrized graph with \( v \) vertices. Then we have

1. \( \tau(\Gamma) \geq \ell(\Gamma) \left( \frac{1}{12} - \frac{1}{\Lambda(\Gamma)} \right)^2 + \frac{4(\Lambda(\Gamma) - 2)}{(v + 6)(\Lambda(\Gamma))^2} \), if \( \Lambda(\Gamma) \geq 4 \). In particular, \( \tau(\Gamma) \geq \frac{\ell(\Gamma)}{108} \) if \( \Lambda(\Gamma) \geq 6 \), and \( \tau(\Gamma) \geq \frac{\ell(\Gamma)}{300} \) if \( \Lambda(\Gamma) = 5 \).
2. \( \tau(\Gamma) \geq \frac{\ell(\Gamma)}{2(v + 6)} \). In particular, \( \tau(\Gamma) \geq \frac{\ell(\Gamma)}{108} \) if \( v \leq 48 \).

Proof. If an edge \( e_i \in E(\Gamma) \) is a bridge of length \( L_i \), then it contributes to \( \tau(\Gamma) \) by \( \frac{L_i}{4} \) (see [C2], Corollaries 2.22 and 2.23 for more information). Therefore, we can assume that \( \Gamma \) is bridgeless. On the other hand, by using the scale-independence of the tau constant (see Remark 2.4), we can assume that \( \Gamma \) is normalized.

Now, we look for \( x \) and \( y \) values that satisfy the inequalities in parts (2), (3), and (4) of Theorem 6.9 and minimize \( \frac{1}{12} - \frac{x}{6} + \frac{y}{6} \).

Whenever \( \Lambda(\Gamma) \geq 4 \), by elementary calculus, we see that the line \( x = (\Lambda(\Gamma) - 1) y \) and the parabola \( y = \frac{v + 6}{4v} (x + y)^2 \) intersect at the point with coordinates \( x = \frac{4v(\Lambda(\Gamma) - 1)}{(v + 6)(\Lambda(\Gamma))^2} \) and
Figure 4. The lower bound for $\tau(\Gamma)$ when $\Lambda(\Gamma) \geq 6$ and $v \to \infty$. $A = \left(\frac{5}{7}, \frac{1}{7}\right)$, $B = \left(\frac{3}{4}, \frac{1}{4}\right)$, $C = \left(\frac{13}{18}, \frac{2}{18}\right)$

$y = \frac{4v}{(v+6)\Lambda(\Gamma)^2}$, and that these give a lower bound to $\frac{1}{12} - \frac{x}{6} + \frac{y}{6}$. This proves the first inequality in part (1).

Again by elementary calculus, we see that the line $\frac{1}{12} - \frac{x}{6} + \frac{y}{6} = c$ is tangential to the parabola $y = \frac{v+6}{4v}(x+y)^2$ at the point with coordinates $x = \frac{3v}{4(v+6)}$ and $y = \frac{v}{4(v+6)}$, and that these give a lower bound to $\frac{1}{12} - \frac{x}{6} + \frac{y}{6}$. This proves the first inequality in part (2).

The remaining parts are immediate from what we have shown.

\begin{proof}
Since $L_i = \frac{1}{e}$ for each edge, $x(\Gamma) + y(\Gamma) = \frac{v-1}{e}$ by Equation (2) and Equation (20). Therefore, parts (3) and (4) of Theorem 6.9 are equivalent to $\frac{v-1}{\Lambda(\Gamma)}e \geq \frac{v+6}{4v}(\frac{v-1}{e})^2$, and Equation (19) is equivalent to $\tau(\Gamma) = \frac{1}{12} - \frac{v-1}{6e} + \frac{y}{3}$. These give the first two inequalities. The final two inequalities follow from the fact that $e = \frac{4v}{\tau}$ when $\Gamma$ is $n$-regular.
\end{proof}

7. Cubic graphs

In this section, we will show that Conjecture 1.1 holds for all metrized graphs if it holds for cubic metrized graphs. We call a 3-regular metrized graph a “cubic metrized graph” or “cubic graph” for short. We will consider the metrized graphs with $\kappa(\Gamma) \geq 2$ where $\kappa(\Gamma)$ is the vertex connectivity. By Remark 2.3, this would be enough to prove Conjecture 1.1.

We will use the following notation and graph constructions.
Suppose $\Gamma$ is a normalized metrized graph, i.e., $\ell(\Gamma) = 1$, and $p \in V(\Gamma)$ is a vertex with valence $n \geq 4$. We want to transform $\Gamma$ into another normalized metrized graph, $\Gamma_{p,(n-3)}$, by adding new edges and new vertices of valence 3 to $\Gamma$ in such a way that the valence of the vertex $p$ becomes 3 in $\Gamma_{p,(n-3)}$. In $\Gamma_{p,(n-3)}$, we add $n-3$ new vertices $p^1, p^2, \ldots, p^{n-3}$ and $n-3$ new edges $e_{p,1}, e_{p,2}, \ldots, e_{p,(n-3)}$ with pairs of end points $\{p^1, p^2\}, \{p^2, p^3\}, \ldots, \{p^{n-3}, p\}$, respectively. Figure 5 shows the details of the transformation. The first graph in Figure 5 shows $\Gamma$.

Suppose the edges with end point $p$ are given in a specified order. We disconnect the first and the second edges from $p$. Then we reconnect them to $p$ via adding edge $e_{p,1}$, with end points $\{p^1, p\}$ and of length $\varepsilon_{p,1}$, so that the new vertex $p^1$ becomes the end point of the first edge and the new edge $e_{p,1}$. We denote this graph by $\Gamma_{p,1}$. Note that $\ell(\Gamma_{p,1}) = \ell(\Gamma) + \varepsilon_{p,1} = 1 + \varepsilon_{p,1}$ and if we contract the new edge $e_{p,1}$, we obtain $\Gamma$. Also, the valence of $p$ in $\Gamma_{p,1}$ is $n - 1$. Then we obtain $\Gamma_{p,1}$ by normalizing $\Gamma_{p,1}$. $\Gamma_{p,1}$ is the second graph in Figure 5. Note that the graphs $\Gamma_{p,1}$ and $\Gamma_{p,2}$ have the same shape, i.e. the same topology. At the next step, we disconnect $e_{p,1}$ and the third edge with vertex $p$ from $p$, then we reconnect them via adding the edge $e_{p,2}$, with end points $\{p^2, p\}$ and of length $\varepsilon_{p,2}$, so that the new vertex $p^2$ becomes the end point of third edge, $e_{p,2}$. We denote this graph by $\Gamma_{p,2}$. Note that the valence of $p$ in $\Gamma_{p,2}$ is $n - 2$. Then by normalizing $\Gamma_{p,2}$ we obtain $\Gamma_{p,2}$ which is shown by the third graph in Figure 5. We continue this process until the valence of $p$ becomes 3, i.e., until we obtain the graphs $\Gamma_{p,(n-3)}$ and $\Gamma_{p,(n-3)}$.

Note that $\varepsilon_{p,k} > 0$ for each $k = 1, 2, \ldots, n - 3$. Since $\kappa(\Gamma) \geq 2$, $\Gamma_{p,k} - e_{p,k}$ is connected for each $k = 1, 2, \ldots, n - 3$. Let $\Gamma_{p,0} := \Gamma$.

Lemma 7.1. Let $k \in \{0, 1, \ldots, n - 4\}$ and let $\Gamma_{p,k+1}^N, \Gamma_{p,k}^N, p$ and $\varepsilon_{p,k+1}$ be as above. Then

$$
\tau(\Gamma_{p,k+1}^N) \leq \tau(\Gamma_{p,k}^N) + \frac{\varepsilon_{p,k+1}}{1 + \varepsilon_{p,k+1}} \cdot \left(1 - \frac{1}{12} - \tau(\Gamma_{p,k}^N)\right).
$$

Proof. Let $e_{p,k+1}, p^k, p^{k+1}, \Gamma_{p,k}, \Gamma_{p,k+1}^N, \Gamma_{p,k+1}^N, \varepsilon_{p,k+1}$ be as above.

Note that we can obtain $\Gamma_{p,k+1}^N$ from $\Gamma_{p,k+1}$ by contracting the edge $e_{p,k+1}$ to its end points. Since $\Gamma_{p,k} - e_{p,k}$ is connected, we can apply Lemma 3.1. This gives

$$
(41) \quad \tau(\Gamma_{p,k+1}) = \tau(\Gamma_{p,k}^N) + \frac{\varepsilon_{p,k+1}}{12} - \frac{\varepsilon_{p,k+1}A_k}{R_{k+1}(\varepsilon_{p,k+1} + R_{k+1})},
$$

where $A_k := A_{p,k}^{p,k+1} - e_{p,k+1}$ and $R_{k+1}$ is the resistance, in $\Gamma_{p,k+1} - e_{p,k+1}$, between $p^k$ and $p^{k+1}$.
Such that

\[ \tau(\Gamma_{p,k+1}) = (1 + \varepsilon_{p,k+1}) \cdot \tau(\Gamma_{p,k+1}^N). \]

Substituting Equation (42) into Equation (41) gives

\[ \tau(\Gamma_{p,k+1}^N) = \frac{\tau(\Gamma_{p,k+1}^N) + \varepsilon_{p,k+1}}{1 + \varepsilon_{p,k+1}} \cdot \left( \frac{1}{12} - \frac{A_k}{R_{k+1}(\varepsilon_{p,k+1} + R_{k+1})} \right) \]

\[ \leq \tau(\Gamma_{p,k}^N) + \frac{\varepsilon_{p,k+1}}{1 + \varepsilon_{p,k+1}} \cdot \left( \frac{1}{12} - \tau(\Gamma_{p,k}^N) \right), \]

since \( A_k \geq 0, R_{k+1} > 0 \) and \( \varepsilon_{p,k+1} > 0 \). This proves the result. \( \square \)

**Theorem 7.2.** If there exists a positive constant \( C \) such that \( \tau(\beta) \geq C \) for any normalized cubic graph \( \beta \), then \( \tau(\Gamma) \geq C \) for any normalized graph \( \Gamma \).

**Proof.** Let \( \Gamma \) be an arbitrary normalized metrized graph. By the additive property of the tau constant (Remark 2.3) we can assume that \( \Gamma \) has no cut vertices. If \( \Gamma \) is a loop, then \( \tau(\Gamma) = \frac{1}{12} \). Thus, we can assume that \( \Gamma \) has a vertices with valence at least 3. After removing all vertices of valence 2 from \( V(\Gamma) \), we can assume that all vertices have valence at least 3. Suppose \( \Gamma \) is not a cubic graph. Then by basic graph theory \( e > \frac{4}{3}v \), where \( e = #(E(\Gamma)) \) and \( v = #(V(\Gamma)) \). Let \( \varepsilon_0 := \frac{\varepsilon}{2e - 3v} \), for some arbitrary \( \varepsilon > 0 \).

Since \( \Gamma \) is not cubic, there exists a vertex \( p \in V(\Gamma) \) with \( v(p) \geq 4 \). We construct the graphs \( \Gamma_{p,k+1} \) and \( \Gamma_{p,k+1}^N \) for each \( k = 0, 1, \ldots, v(p) - 4 \) as mentioned at the beginning of this section. In these constructions, for each \( k \) we take

\[ \varepsilon_{p,k+1} = \begin{cases} \frac{\varepsilon_0}{12 - \tau(\Gamma_{p,k}^N)} & \text{if } \frac{1}{12} \neq \tau(\Gamma_{p,k}^N) \\ \text{a positive number} & \text{otherwise.} \end{cases} \]

Note that \( \frac{1}{12} \geq \tau(\Gamma_{p,k}^N) \) by [C2 Corollary 5.8]. Then in both cases we have

\[ \tau(\Gamma_{p,k+1}^N) \leq \tau(\Gamma_{p,k}^N) + \varepsilon_0. \]

By considering Equation (44) for each \( k = 0, 1, \ldots, v(p) - 4 \), we obtain

\[ \tau(\Gamma_{p,v(p)-3}^N) \leq \tau(\Gamma) + (v(p) - 3) \cdot \varepsilon_0. \]

By following the same procedure for each \( p \in V(\Gamma) \) with \( v(p) \geq 4 \), we obtain a normalized cubic graph \( \beta \) such that

\[ \tau(\beta) \leq \tau(\Gamma) + \sum_{p \in V(\Gamma)} (v(p) - 3) \cdot \varepsilon_0 = \tau(\Gamma) + (2e - 3v) \cdot \varepsilon_0 = \tau(\Gamma) + \varepsilon. \]

Thus \( \tau(\Gamma) \geq C - \varepsilon \). Since \( \varepsilon > 0 \) is arbitrary, \( \tau(\Gamma) \geq C \). \( \square \)

**Remark 7.3.** Theorem 7.2 shows that to prove Conjecture 7.1, it is enough to establish it for cubic graphs.

**Theorem 7.4.** Let \( \Gamma \) be a metrized graph with \( \Lambda(\Gamma) = 2 \). Then there exists a metrized graph \( \beta \) such that \( \tau(\Gamma) = \tau(\beta) \), \( \Lambda(\beta) \geq 3 \), \( #(E(\Gamma)) \geq #(E(\beta)) \), and \( g(\Gamma) = g(\beta) \).
Figure 6. All of these graphs have equal tau constant. The last graph has edge connectivity 3, and the others have edge connectivity 2. The length of the extended edges are shown, the other edges have length 1.

Proof. Since $\Lambda(\Gamma) = 2$, there is an edge $e_i \in E(\Gamma)$ such that $\Lambda(\Gamma - e_i) = 1$, and let $L_i$ be the length of $e_i$. Let $C(e_i) = \{e_{i_1}, e_{i_2}, \ldots, e_{i_s}\}$ be the set of bridges in $\Gamma - e_i$, and let $L_{i_j}$ be the length of $e_{i_j}$ for each $1 \leq j \leq s$. Let $\gamma$ be the metrized graph obtained from $\Gamma$ by contracting all of the edges in $C(e_i)$ to their end points, and by extending the length $L_i$ of the edge $e_i$ to $L_i + \sum_{j=1}^{s} L_{i_j}$. We have $\ell(\Gamma) = \ell(\gamma)$, and $\tau(\Gamma - e_i) = \tau(\gamma - e_i) + \frac{1}{4} \sum_{j=1}^{s} L_{i_j}$ by additive property of the tau constant (see Remark 2.3), $R_i(\Gamma) = R_i(\gamma) + \sum_{j=1}^{s} L_{i_j}$ by elementary circuit reductions, and $L_i(\gamma) = L_i + \sum_{j=1}^{s} L_{i_j}$ by our construction. Moreover, $A_{p,q,\Gamma - e_i} = A_{p,q,\gamma - e_i}$ by the additive property of $A_{p,q,\Gamma}$ (see [C2, Proposition 4.6]) and by [C2, Proposition 4.5]. By our construction, $\#(E(\Gamma)) \geq \#(E(\gamma))$, and $g(\Gamma) = g(\gamma)$. If we apply Lemma 3.4 to $\tau(\Gamma)$ and $\tau(\gamma)$ and use the equalities we derived, we see that $\tau(\Gamma) = \tau(\gamma)$.

Note that $\Lambda(\gamma - e_i) \geq 2$. If $\Lambda(\gamma) = 2$, we apply the same process to $\gamma$. We can repeat this process until we obtain a graph $\beta$ with the properties we wanted. Figure 6 shows an example in which this process applied four times. □

Remark 7.5. One of the implications of Theorem 7.4 is that if Conjecture 1.1 holds for metrized graphs with edge connectivity at least 3, then it holds for all metrized graphs.

We show in [C3] that $\tau(\Gamma)$ can be computed by using the discrete Laplacian of $\Gamma$ and its pseudo inverse. In [C6], we construct families of metrized graphs with the tau constants between $\frac{\ell(\Gamma)}{107}$ and $\frac{\ell(\Gamma)}{108}$, and the computations suggest that we can have sequences of metrized graphs with the tau constants approaching (but not equal) to $\frac{\ell(\Gamma)}{108}$.

Based on our theoretical and computational investigations, we refine Conjecture 1.1 as follows:

Conjecture 7.6. For all metrized graphs $\Gamma$, $\tau(\Gamma) > \frac{\ell(\Gamma)}{108}$.

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