Ideal Bose gas in fractal dimensions and superfluid $^4$He in porous media

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Abstract

Physical properties of ideal Bose gas with the fractal dimensionality between $D = 2$ and $D = 3$ are theoretically investigated. Calculation shows that the characteristic features of the specific heat and the superfluid density of ideal Bose gas in fractal dimensions are strikingly similar to those of superfluid Helium-4 in porous media. This result indicates that the geometrical factor is dominant over mutual interactions in determining physical properties of Helium-4 in porous media.

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I. INTRODUCTION

Superfluid Helium-4 is a typical boson fluid to occur in nature and Bose condensation is believed to be the fundamental reason for such a phenomenon. Recently, superfluidity of liquid helium-4 in highly connected porous structures (vycor glass, glass plate, xerogel, aerogel, graphite, fine powders, steel, German silver, plastic films, etc.) has been observed and studied intensively.\textsuperscript{1-8} However, so far no satisfactory explanation of experimental observations on these materials has been achieved.\textsuperscript{9-17} Since porous media can be treated as solids with fractal or non-integer dimensions,\textsuperscript{15-21} we believe that it is imperative to study the extent of the dimensionality contribution to the superfluidity in order to understand the experimental results.

In this paper, we examine the physical properties of ideal Bose gas with fractal dimensions between $D = 2$ (thin film limit) and $D = 3$ (the bulk limit). The results will be compared with experimental results obtained from liquid Helium-4 in porous media. Surprisingly, we find that most of salient features of experimental observations on liquid helium-4 in porous solids can be explained using the theoretical results from the ideal gas model in fractal dimensions. This indicates that the dimensionality contribution, not the mutual interactions, is the dominant factor for superfluidity in porous media.

II. IDEAL BOSE GAS IN NON-INTEGER DIMENSIONS

The ideal Bose gas system at integer dimensions was studied long time ago.\textsuperscript{22} However, study at fractal dimensions was carried out only quite recently.\textsuperscript{21,23} The density of states which is essential to calculate thermodynamic properties of the Bose gas in $D$-dimension, where $D$ is any real number, is given by\textsuperscript{18}

$$\rho_D(E) = a_D E^{\frac{D}{2} - 1}, \quad (1)$$

where $a_D$ is the $D$-dimensional coefficient which is known as
\[ a_D = \frac{V(D)}{\Gamma\left(\frac{D}{2}\right)} \left(\frac{m}{2\pi\hbar^2}\right)^{\frac{D}{2}}. \]  

Here, \( \Gamma \) is the Gamma function, \( m \) the mass, and \( V(D) \) the \( D \)-dimensional measure or volume.

In order to obtain physical quantities of the \( D \)-dimensional Bose gas, it is necessary to obtain the grand partition function in \( D \)-dimension,

\[
\ln Q(z, v, t) = -\ln(1 - z) - \int_0^\infty \ln(1 - ze^{-\beta E}) \rho_D(E)dE \\
= -\ln(1 - z) + \frac{V}{\lambda_D} g_{D/2+1}(z)
\]

where \( z \) is the thermodynamic fugacity defined by \( z = e^{\beta \mu} \), \( E(p) = p^2/2m \), and \( \lambda(T) \) is the thermal wavelength defined by \( \sqrt{2\pi\hbar^2/mk_BT} \). With the coefficient \( a_D \), we can readily calculate the grand partition function in \( D \)-dimension. The \( g_s(z) \) is the Bose gas function defined by

\[ g_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}. \]  

The coefficients \( s \) and \( z \) are restricted to \( s > 0 \) and \( 0 \leq z \leq 1 \) regions. The Bose gas function can be also extended to non-integer dimensions and has an integral expression,

\[
g_s(z) = -\frac{1}{\Gamma(s - 1)} \int_0^\infty dx x^{s-2} \ln(1 - z e^{-x}).
\]

Note that this representation is valid only when \( s > 1 \).

Using the above expressions, we obtain the average number of particles in \( D \)-dimension,

\[
N = z \frac{\partial}{\partial z} \ln Q(z, v, t) \\
= \sum_{p\neq0} <n_p> + <n_0> \\
= \frac{V}{\lambda_D} g_{D/2}(z) + \frac{z}{1 - z}.
\]  

We use the above equations to calculate thermodynamic properties in fractal dimensions in the following.
III. PHYSICAL PROPERTIES OF IDEAL BOSE GAS IN FRACTAL DIMENSIONS AND APPLICATION TO LIQUID HELIUM-4 IN POROUS MEDIA

Before we go into detailed calculations, we briefly summarize experimental findings on superfluid Helium-4 in porous media, since we are interested in whether the dimensionality considered above plays any role in real situations.

It is known that physical properties of superfluid Helium-4 in porous media show quite puzzling behaviors. Upon reducing the lab variables of porous media such as “pore filling,” “thickness,” “number of layers,” or “coverage,” the capacity curves show the following generic behaviors independent of the porous media or substrates: (i) The critical temperature for superfluidity onset shifts downwards almost linearly. (ii) The bulk-like sharp cusp disappears and the peak gets smaller. Furthermore, there appears a systematic crossing at low temperatures. (iii) The shape of the curve gets rounder and, eventually, becomes flat. These experimental findings are summarized in FIG. 1. These figures are reproduced from references [1-4], so that comparison with the theory can be made transparent. Here, we observe that the above experimental findings appear independent of detailed nature of porous media and also of interactions between particles. This fact strongly suggests that, at least, the qualitative nature of the above behavior may originate from geometric factors. (iv) Another interesting physical property is the superfluid density shown in FIG. 2. The sample E has smaller density, which implies more connectivity of porous structure than the sample F in FIG. 2(a). It is shown that the curves from different media do not cross one another. This again indicates that the superfluid density is strongly dependent on the porous structure.

Numerous theories have been suggested to explain the above experimental observations. However, so far no completely successful theory has emerged. For example, the KT model of the vortex mediated transition in two-dimensional space gives an excellent explanation of question (i) for very thin films of one or two layers but not for other
samples.

We now calculate the physical properties in fractal dimensions and compare the results with the experimental findings summarized above. Fractal dimensionality measures disorderness in terms of the connectivity of the system.

(i) The critical temperature, $T_c$, for the superfluidity of the ideal Bose gas in fractal space can readily obtained from Eq. (6) to be given by\textsuperscript{23}

$$k_B T_c = \frac{2\pi h^2}{m} \frac{1}{[v g_{D/2}(1)]^{\frac{1}{D}}}.$$ (7)

The almost linear behavior of $T_c$ as a function of dimensionality is plotted in FIG. 3. Necessary parameters are taken from reference 25. The formula can be simplified when $D$ approaches to 2 to be given by\textsuperscript{26}

$$T_c \sim \left| \frac{D}{2} - 1 \right|.$$ (8)

The results are surprisingly in good agreement with the experimental findings shown in FIG. 1. Even though we completely neglected the interactions between Bose particles, the $D \rightarrow 2$ limit has the same form with the KT theory prediction.\textsuperscript{12} Note that the KT theory cannot account the experimental results for thick films, but the present $D \rightarrow 3$ limit agrees with the correct bulk value.

(ii) The specific heat of the ideal Bose gas in fractal space can readily obtained, too, from the grand partition function $Q$ in Eq. (3). (a) when $T \leq T_c$

$$\frac{C_V(T)}{Nk_B} = \frac{D}{2} \left( \frac{D}{2} + 1 \right) \frac{v}{\lambda^D g_{D/2+1}(1)},$$ (9)

and, (b) when $T > T_c$

$$\frac{C_V(T)}{Nk_B} = \frac{D}{2} \left( \frac{D}{2} + 1 \right) \frac{v}{\lambda^D g_{D/2+1}(z)} - \left( \frac{D}{2} \right)^2 \frac{g_{D/2}(z)}{g_{D/2-1}(z)}.$$ (10)

The $C_V(T)$ curves for several values of fractal dimensions are plotted as functions of temperatures in FIG. 4. We observe that the cusp disappears when the dimension is less than 3, and the peak height becomes smaller with decreasing dimensionality. Also, FIG. 4 shows
that there exists a systematic crossover at low temperature regions. These results are in excellent agreement with the experiments shown in FIG. 1.

(iii) In FIG. 4, we observe that the peaks of the specific heat curves become less and less prominent and, eventually, the curves become flat with decreasing dimensionality. We show that this behavior originates from a hidden hierarchy in the superfluidity transition with fractal dimensions. The first temperature derivative of the specific heat can be readily obtained from Eqs. (9) and (10). (a) when \( T \leq T_c \)

\[
\left( \frac{\partial}{\partial T} \right)_V \frac{C_V(T)}{Nk_B} = \left( \frac{D}{2} \right)^2 \left( \frac{D}{2} + 1 \right) \frac{v g_{D/2+1}(1)}{\lambda^B T},
\]

(11)

(b) when \( T > T_c \)

\[
\left( \frac{\partial}{\partial T} \right)_V \frac{C_V(T)}{Nk_B} = - \left( \frac{D}{2} \right)^2 \frac{1}{T} \frac{g_{D/2}(z)}{g_{D/2-1}(z)} \left\{ 1 - \left( \frac{D}{2} + 1 \right) \frac{g_{D/2+1}(z) g_{D/2-1}(z)}{[g_{D/2}(z)]^2} + \frac{D}{2} \frac{g_{D/2}(z) g_{D/2-2}(z)}{[g_{D/2-1}(z)]^2} \right\}.
\]

(12)

The first derivative of \( C_V \) is plotted in FIG. 5 in arbitrary unit. The curves again show the shift of \( T_c \) with decreasing dimensionality. Also, it is shown that the discontinuity of \( (\partial C_V(T)/\partial T)_V \) disappears with decreasing dimensionality. In order to investigate this behavior more closely, we studied the relation between the continuity of higher derivatives of \( C_V \) and the fractional dimensionality. Taking higher derivatives on \( C_V \) and considering the behavior at \( T_c \), we obtain the following relation:

\[
\lim_{T \to T_c} \left[ \left( \frac{\partial}{\partial T} \right)_V^n C_V^-(T) - \left( \frac{\partial}{\partial T} \right)_V^n C_V^+(T) \right] = \lim_{\eta \to 0} \sum_{j=1}^{n} a_{nj} \eta^{2-((j+1)D/2)},
\]

(13)

where the coefficients \( a_{nj} \) are finite constants, and \( C_V^- \) and \( C_V^+ \) are the specific heats below and above \( T_c \). This formula can also be proved by mathematical induction method (see APPENDIX). The hierarchy of the superfluidity transition with the fractal dimensionality obtained from the above formula is summarized at TABLE 1. This table explains the physical origin of the roundness tendency of the specific heat curves with decreasing the dimensionality.
(iv) In real system the superfluid fraction is different from the condensate fraction, but the structure of the condensate fraction may shed some clue for the superfluid fraction. The condensate fraction in $D$-dimension can readily be obtained from Eqs. (6) and (7) to be given by

$$\frac{n_0}{n} = 1 - \left[ \frac{T}{T_c(D)} \right]^\frac{D}{2}$$

The curves are shown for several fractal dimensions in FIG. 6. It is shown that the curves with higher dimensions have higher $T_c$ and the curves do not cross each other. The FIG. 6 gives the information that the sample E has a higher fractal dimension than the sample F in FIG. 2(a). Also, the aerogel has the highest fractal dimension, the xerogel is the next, and the vycor has the lowest one in FIG. 2(b). Of course, they are all less than the bulk value of $D = 3$. The qualitative features of the theoretical curves are in excellent agreement with the experimental results in FIG. 2. Even the tail structure at $\rho_0 \to 0$ is reproduced.

IV. DISCUSSIONS

Physical properties of the ideal Bose gas in fractal dimensions are studied theoretically. The results are compared with the experimental findings from the superfluid Helium-4 in porous media. It is found that the main characteristics of the experimental results are in excellent agreement with the theoretical results obtained from the simple ideal Bose gas in fractal dimensions. Such a good agreement may not be totally unexpected. Careful examination of the experimental results reveals that the salient features are independent of materials and mutual interactions, thus suggesting that the dominant contributions should come from geometrical factors. Another important point of the theoretical study is the existence of a hierarchy of the superfluidity transitions with changing fractal dimensionality. The peak of heat capacity is not necessary for the superfluid transition in porous media. It will be interesting to study how the such physical properties modified when interactions are included in the calculations.
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APPENDIX A:

Here, we prove Eq. (13) using the mathematical induction method. First, we give some useful relations of the Bose gas function which are needed for the derivation:

\[ g_s(z) \sim z : z \to 0^+, \]
\[ \sim \Gamma(1-s)(-\ln z)^{s-1} + \zeta(s) : z \to 1^-, \quad (A1) \]

where \( \zeta \) is the Riemann-zeta function.

\[ \left( \frac{\partial z}{\partial T} \right)_V = -\frac{D}{2} \frac{z}{T} \frac{g_D/2(z)}{g_D/2-1(z)}. \quad (A2) \]

We can use above relations for \( (\partial \eta/\partial T)_V \), too. We put \( z = e^{-\eta} \), then \( z \to 1 \) as \( \eta \to 0 \).

We introduce a differential operator defined by

\[ \Delta^n(T) \equiv \left( \frac{\partial}{\partial T} \right)^n_v C_V^{-}(T) - \left( \frac{\partial}{\partial T} \right)^n_v C_V^{+}(T) \quad (A3) \]

where \( C_V^{-} \) and \( C_V^{+} \) are the specific heat below and above \( T_c \). For convenience, we drop the limit notation of \( \lim_{\eta \to 0} \) \( (\text{or } T \to T_c) \)’ during the proof.

(i) When \( n = 1 \),

\[ \Delta^1(T_c) = a_{11} \eta^{3-D}. \quad (A4) \]

This is clearly true from Eq. (13) and the known result for \( D = 3 \).

(ii) We assume that Eq. (13) is true for any positive integer \( k \). Then

\[ \Delta^k(T_c) = \sum_{i=1}^{k} a_{ki} \eta^{i+2-\frac{i+1}{2}D}. \quad (A5) \]
Using Eqs. (A1) and (A2), we obtain

\[ \Delta^{k+1}(T_c) = \sum_{i=1}^{k} a_{ki} (i + 2 - \frac{i + 1}{2} D) \eta^{i + 1 - \frac{i + 1}{2} D} \left( \frac{\partial \eta}{\partial T} \right)_V \]

\[ = \sum_{i=1}^{k} a_{ki} (i + 2 - \frac{i + 1}{2} D) \frac{D \zeta(\frac{D}{2})}{2 T_c \Gamma(2 - \frac{D}{2})} \eta^{i + 3 - \frac{i + 2}{2} D} \]

\[ = \sum_{j=2}^{k+1} a_{k+1,j-1} (j + 1 - \frac{j}{2} D) \frac{D \zeta(\frac{D}{2})}{2 T_c \Gamma(2 - \frac{D}{2})} \eta^{j + 2 - \frac{j+1}{2} D} \]

\[ = \sum_{j=2}^{k+1} a_{k+1,j} \eta^{j + 2 - \frac{j+1}{2} D}. \tag{A6} \]

The \( a_{ki} \) satisfies the recurrence relation

\[ a_{k+1,j} = \frac{(j + 1 - \frac{j}{2} D) D \zeta(\frac{D}{2})}{2 T_c \Gamma(2 - \frac{D}{2})} a_{k+1,j-1} \tag{A7} \]

where \( j = 2, 3, 4, ..., k+1 \).

Therefore, (i) and (ii) enables us, for any positive integer \( n \), to write

\[ \Delta^n(T_c) = \lim_{\eta \to 0} \sum_{j=1}^{n} a_{nj} \eta^{j + 2 - \frac{j+1}{2} D}, \tag{A8} \]

which completes the proof.
REFERENCES

1 H.P.R. Frederikse, Physica, 15, 860 (1949).

2 D.F. Brewer, J. Low Temp. Phys. 3, 205 (1970).

3 M. Bretz, Phys. Rev. Lett. 31, 1447 (1973).

4 D. Finotello, K.A. Gillis, A. Wong, and M.H.W. Chan, Phys. Rev. Lett. 61 (1988).

5 L.M. Steele and D. Finotello, J. Low Temp. Phys. 89, 645 (1992).

6 J.D. Reppy, Phase Transitions in Surface Films, edited by J.G. Dash and J. Ruvalds (Plenum, New York, 1980) p. 233.

7 J.D. Reppy, J. Low Temp. Phys. 87, 205 (1992).

8 G.K.S. Wong, P.A. Crowell, H.A. Cho, and J. D. Reppy, Phys. Rev. B. 48, 3858 (1993).

9 R.H. Bowley and S. Giorgini, J. Low Temp. Phys. 93, 987 (1993).

10 A. Khurana, Physics Today, 42, 21 (1989).

11 D.F. Goble and L.E.H. Trainor, Can. J. Phys. 44, 27 (1966).

12 J.M. Kosterlitz and D.J. Thouless, J. Phys. C 6, 1181 (1973).

13 P.A. Crowell, F.W. Van Keuls, J.D. Reppy, Phys. Rev. B. 55, 12620 (1997).

14 J.A. Phillips, D. Ross, P. Taborek, and J.E. Rutledge, Phys. Rev. B. 58, 3361 (1998).

15 F.M. Gaspirini and S. Mhlanga, Phys. Rev. B. 33, 5066 (1986).

16 G.A. Williams, Phys. Rev. Lett. 64, 978 (1990).

17 M. Saarela, B.E. Clements, E. Krotscheck, and F.V. Kusmartsev, J. Low Temp. Phys. 93, 971 (1993).

18 P. Pfeifer and M. Obert, The Fractal Approach to Heterogeneous Chemistry, edited by D. Avnir (Wiely, Chichester, 1989) p. 11.
19 P. Pfeifer, Y.J. Wu, M.W. Cole, and J. Krim, Phys. Rev. Lett. 62, 1977 (1989).

20 P. Pfeifer and M.W. Cole, New J. Chem. 14, 221 (1990).

21 P. Pfeifer, Chemistry and Physics of Solid Surfaces, edited by R. Vanselow and R. Howe (Springer-Verlag, Berlin, 1988) Vol. VII, p. 283.

22 R.M. Ziff, G.E. Uhlenbeck, and M. Kac, Phys. Rep. 32, 169 (North-Holland, Amsterdam, 1977).

23 S.-H. Kim, Int. J. Bif. Cha. 7, 1053 (1997).

24 J.E. Robinson, Phys. Rev. 83, 678 (1951).

25 International Critical Tables of Numerical Data (McGraw-Hill, New York, 1933) I-102.

26 I.S. Gradshteyn and I.M. Ryzhik, Tables of Integrals, series and products (Academic, New York, 1980) p. 946.
FIGURES

FIG. 1. Heat capacity measurements for helium-4 in various porous media. (a) Jewler’s rouge (powder) [1], (b) Vycor glass [2], (c) Grafoil [3]. (d) Xerogel [4]. Some data have been deleted for clarity. Arrows indicate temperatures below which the film relaxation times go to zero.

FIG. 2. Superfluid density of Helium-4 in several porous media. The sample E (0.133 g/cm$^3$ DESY aerogel) and F (0.200 g/cm$^3$ Air glass aerogel) are aerogels of two different porous structures. The solid lines represent superfluid fraction in bulk helium. [8,6]

FIG. 3. The critical temperatures of the ideal Bose gas system between $D = 2$ and $D = 3$. The slope is almost linear as observed in the experiments.

FIG. 4. Specific heat functions of the ideal Bose gas system between 2 and 3 dimensions. The unit of temperature is $T_o$ where $T_o \simeq 5.42K \left[ = g_{D/2}(1) \tilde{\pi} T_c(D) \right]$. 

FIG. 5. Plot of the first derivative of the specific heat functions of the ideal Bose gas between $D = 2$ and $D = 3$. There are no cusps when $D < 3$. The unit of y-axis is arbitrary.

FIG. 6. The condensate fraction in $D$-dimensional space. $D = 2.6, 2.7, 2.8, 2.9, 3.0$ from left to right.
TABLES

TABLE I. The hierarchy of the superfluidity transition between $D = 2$ and $D = 3$. The symbol ‘c’ stands for being continuous at $T_c$, and ‘d’ for being discontinuous at $T_c$. The Class stands for the class of function.

| Dimension | $C_V$ | $(\frac{\partial}{\partial T})C_V$ | $(\frac{\partial}{\partial T})^2C_V$ | $(\frac{\partial}{\partial T})^3C_V$ | $(\frac{\partial}{\partial T})^4C_V$ | ⋯ | Class |
|-----------|-------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|---|-------|
| $D = 3$   | c     | d                                |                                   |                                   |                                   |   | $C^0$ |
| $\frac{8}{3} \leq D < \frac{6}{2}$ | c     | c                                | d                                |                                   |                                   |   | $C^1$ |
| $\frac{10}{4} \leq D < \frac{8}{3}$ | c     | c                                | c                                | d                                |                                   |   | $C^2$ |
| ⋮         |       |                                   |                                   |                                   |                                   |   | ⋮     |
| $\frac{2(j+2)}{j+1} \leq D < \frac{2(j+1)}{j}$ | c     | c                                | c                                | c                                | ⋯ (d)                            |   | $C^{j-1}$ |
| ⋮         |       |                                   |                                   |                                   |                                   |   | ⋮     |
| $D = 2$   | c     | c                                | c                                | c                                | c                                | ⋯ | $C^\infty$ |

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(a) Vycor Glass

- Onset of Superflow
- $T_\lambda$

(b) Jouler’s Rouge
- 3-4 layers
- 5-6 layers
- 7-9 layers
- 9-12 layers

(c) Grafoil Films
- $N$
- 11.20
- 9.70
- 8.64
- 7.80
- 6.77
- 5.69
- 5.25
- 3.62

(d) Xerogel
- 34.82
- 36.18
- 39.13
- 41.07 (in $\mu$moles/m$^2$)
