A discrete time peakons lattice

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Abstract. A discretization of the peakons lattice is introduced, belonging to
the same hierarchy as the continuous–time system. The construction examplifies
the general scheme for integrable discretization of systems on Lie algebras with
r–matrix Poisson brackets. An initial value problem for the difference equations is
solved in terms of a factorization problem in a group. Interpolating Hamiltonian
flow is found. A variational (Lagrangian) formulation is also given.
1 Introduction

The subject of integrable symplectic maps received in the recent years a considerable attention. Given an integrable system of ordinary differential equations with such attributes as Lax pair, \( r \)-matrix and so on, one would like to construct its difference approximation, desirably also with a (discrete–time analog of) Lax pair, \( r \)-matrix etc. Recent years brought us several successful examples of such a construction [1–9].

Recently there appeared for the first time examples [6], [7] where the Lax matrix of the discrete–time approximation coincides with the Lax matrix of the continuous–time system, so that the discrete–time system belongs to the same integrable hierarchy as the underlying continuous–time one (systems of Calogero–Moser type). This led us to the formulation of a general recipe for producing discretizations of this type, which was applied to the two favorite examples of finite dimensional integrable systems of the classical mechanics – the Toda lattice and the relativistic Toda lattice [8], [9].

In the present Letter we want to describe an application of this scheme to another remarkable system, discovered recently by Camassa and Holm [10]. Such an application was made possible thanks to an \( r \)-matrix interpretation given to this system by Ragnisco and Bruschi [11].

2 Continuous–time peakons lattice

The peakons lattice has appeared in [10] as a system describing positions and velocities of ”peaked solitons” for a certain integrable shallow water equation. The Hamiltonian function of the peakons lattice reads:

\[
H(x, p) = \frac{1}{2} \sum_{j,k=1}^{N} p_j p_k \exp(-|x_k - x_j|).
\]

It can be proved [10] that, generally speaking, the peakons retain their ordering, so that one can assume \( x_1 > x_2 > \ldots > x_N \), and the Hamiltonian function takes the form

\[
H(x, p) = \frac{1}{2} \sum_{k=1}^{N} p_k^2 + \sum_{1\leq j<k\leq N} p_j p_k \exp(x_k - x_j). \tag{2.1}
\]
The canonical equations of motion read:

\[ \dot{x}_k = \sum_{j<k} p_j \exp(x_k - x_j) + p_k + \sum_{j>k} p_j \exp(x_j - x_k), \quad (2.2) \]

\[ \dot{p}_k = -\sum_{j<k} p_j \exp(x_k - x_j) + \sum_{j>k} p_j \exp(x_j - x_k). \quad (2.3) \]

One can consider (2.2) as a linear system for the impulses \( p_k \). Remarkably enough, the matrix of this linear system has a tridiagonal inverse, so that \( p_k \) depends only on \( x_j, \dot{x}_j \) with \( j = k - 1, k, k + 1 \):

\[ p_k = \frac{1 - \exp(2(x_{k+1} - x_{k-1}))}{[1 - \exp(2(x_{k+1} - x_k))][1 - \exp(2(x_k - x_{k-1}))]} \]

\[ -\frac{\exp(x_{k+1} - x_k)}{1 - \exp(2(x_{k+1} - x_k))} - \frac{\exp(x_k - x_{k-1})}{1 - \exp(2(x_k - x_{k-1}))}. \quad (2.4) \]

For \( k = 1 \) and \( k = N \) little modifications are necessary, which can be taken into account by imposing the boundary conditions

\[ x_0 = \infty, \quad x_{N+1} = -\infty. \quad (2.5) \]

The formula (2.4) implies that the Lagrangian function for the peakons lattice is local (with only nearest-neighbour contributions):

\[ L(x, \dot{x}) = \frac{1}{2} \sum_{k=1}^{N} \dot{x}_k^2 \frac{1 - \exp(2(x_{k+1} - x_{k-1}))}{[1 - \exp(2(x_{k+1} - x_k))][1 - \exp(2(x_k - x_{k-1}))]} \]

\[ -\sum_{k=1}^{N-1} \dot{x}_k \dot{x}_{k+1} \frac{\exp(x_{k+1} - x_k)}{1 - \exp(2(x_{k+1} - x_k))}, \quad (2.6) \]

The Lax pair formulation with the matrices from \( gl(N) \) for the system (2.2)–(2.3) was also given in [10]:

\[ \dot{T} = [T, A] \quad (2.7) \]

with the Lax matrix

\[ T(x, p) = \sum_{k=1}^{N} p_k E_{kk} + \sum_{j<k} \sqrt{p_k p_j} \exp \left( \frac{1}{2} (x_k - x_j) \right) (E_{kj} + E_{jk}), \quad (2.8) \]
and
\[ A(x, p) = \frac{1}{2} \sum_{j<k} \sqrt{p_k p_j} \exp \left( \frac{1}{2} (x_k - x_j) \right) (E_{kj} - E_{jk}). \quad (2.9) \]

An \( r \)-matrix interpretation of this result was given in [11]. To formulate it, let \( \langle \cdot, \cdot \rangle \) denote the standard scalar product in \( gl(N) \): \( \langle X, Y \rangle = \text{tr}(XY) \), and let the gradient \( \nabla \phi(T) \) of a smooth function \( \phi \) on \( gl(N) \) be defined by
\[ \langle \nabla \phi(T), X \rangle = \left. \frac{d}{d \varepsilon} \phi(T + \varepsilon X) \right|_{\varepsilon=0} \quad \forall X \in gl(N). \]

Let \( R(X) \) be a linear operator on \( gl(N) \), defined as
strictly lower triangular part \( (X) - \) strictly upper triangular part \( (X) \),

This operator satisfies the classical modified Yang–Baxter equation, so that the following Poisson bracket on \( gl(N) \) is defined:
\[ \{ \phi, \psi \} = \frac{1}{2} \langle T, [R(\nabla \phi(T)), \nabla \psi(T)] + [\nabla \phi(T), R(\nabla \psi(T))] \rangle. \]

Now one of the main results of [11] states that the set of the matrices \( T(x, p) \) from (2.8) forms a Poisson submanifold in \( gl(N) \) equipped with the above Poisson bracket. Hamiltonian equations generated in this bracket by a conjugation invariant Hamiltonian function \( \phi(T) \) read:
\[ \dot{T} = \left[ T, \frac{1}{2} R(\nabla \phi(T)) \right] = \left[ T, \pi_+ (\nabla \phi(T)) \right], \quad (2.10) \]
where
\[ \pi_+ = \frac{1}{2} (R + I), \quad \pi_- = \frac{1}{2} (R - I), \quad \text{so that} \quad \pi_+ - \pi_- = I. \]

This explains the formulas (2.7)–(2.9), since the Hamiltonian \( H(x, p) \) of the peakons lattice is nothing else then \( \phi(T) = \frac{1}{2} \text{tr}(T^2) \), so that \( \nabla \phi(T) = T \).

## 3 Discrete–time peakons lattice

3 Discrete–time peakons lattice

A general recipe for obtaining integrable time discretizations for equations of the form (2.10) was given in [8],[9]. Discrete time systems obtained by this
recipe share with the underlying continuous–time ones the Lax matrix, and hence the integrals of motion and the integrability property. The formula reads:

\[
T(t + h) = \Pi^{-1}_+ \left( I + h \nabla \varphi(T(t)) \right) T(t) \Pi^{-1}_+ \left( I + h \nabla \varphi(T(t)) \right) \tag{3.1}
\]

Here the factorization \( Y = \Pi_+(Y)\Pi^{-1}_-(Y) \) in the group \( GL(N) \) corresponds canonically to the additive decomposition \( X = \pi_+(X) - \pi_-(X) \) in its Lie algebra \( gl(N) \), and in our case is characterized by the following conditions: \( \Pi_+(Y) \) (\( \Pi_-(Y) \)) is a nondegenerate lower triangular (resp. upper triangular) matrix, and the diagonals of these matrices are mutually inverse.

The aim of the present Letter is to apply this recipe to the peakons lattice, i.e. to put in the equation (3.1) the Lax matrix (2.8) and \( \nabla \varphi(T) = T \).

**Theorem 1.** The equation

\[
T(t + h) = \Pi^{-1}_+ \left( I + hT(t) \right) T(t) \Pi_+ \left( I + hT(t) \right) \tag{3.2}
\]

serves as a Lax representation for the following map\(^1\) in the space \( \mathbb{R}^{2N} \{ x, p \} \):

\[
\exp(\bar{x}_k - x_k) = \frac{1}{\beta_k} + h p_k + h \sum_{j<k} p_j \exp(x_j - x_k), \tag{3.3}
\]

\[
\frac{\bar{p}_k}{p_k} = \beta_k + \frac{h\beta_k^2}{1 + hp_k \beta_k} \sum_{j>k} p_j \exp(x_j - x_k). \tag{3.4}
\]

Here the quantities \( \beta_k \) are defined recurrently by the relation

\[
\beta_{k+1} = 1 - \exp(x_{k+1} - x_k) + \frac{\exp(x_{k+1} - x_k)}{hp_k + \frac{1}{\beta_k}}, \tag{3.5}
\]

with an initial condition

\[
\beta_1 = 1.
\]

**Remark 1.** A simple induction shows that the finite continued fractions \( \beta_k \) have by \( h \to 0 \) the following asymptotics:

\[
\beta_k = 1 - h \sum_{j<k} p_j \exp(x_k - x_j) + O(h^2). \tag{3.6}
\]

\(^1\)We use the tilde to denote the time \( h \) shift, so that \( \bar{x}_k = x_k(t + h) \), if \( x_k = x_k(t) \).
Hence the discrete time equations (3.3), (3.4) in fact approximate the continuous time ones (2.2), (2.3).

**Remark 2.** Due to the general theory (cf. [8], [9]), the initial value problem for the discrete time peakons lattice can be solved in terms of the Cholesky factorization of the matrix $(I + hT(0))^n$, and the interpolating Hamiltonian for this system is given by $\text{tr}(\Phi(T))$, where

$$\Phi(\xi) = h^{-1} \int_0^\xi \log(1 + h\eta)d\eta = \frac{1}{2}\xi^2 + O(h).$$

**Proof of the Theorem 1.** Due to the symmetry of the matrix $T$, the $\Pi_+\Pi_-^{-1}$ factorization for $I + hT$ takes the form of the so-called Cholesky factorization:

$$I + hT = LL^T,$$

where

$$L = \Pi_+(I + hT) = \sum_{k \geq j} l_{kj}E_{kj}$$

is the (uniquely defined) lower triangular matrix with positive diagonal entries. With the help of (3.7), (3.8) the equation (3.2) can be rewritten as

$$I + h\tilde{T} = L^TL.$$

It turns out that the special structure of the matrix $T$ (2.8) allows for an almost explicit computation of the factor $L$.

**Lemma.**

i) For $k > j$

$$l_{kj} = \sqrt{\frac{p_k}{p_j}} \exp\left(\frac{1}{2}(x_k - x_j)\right) \left(l_{jj} - \frac{1}{l_{jj}}\right);$$

ii)

$$l_{k+1,k+1}^2 = 1 + hp_{k+1} - \exp(x_{k+1} - x_k)\frac{p_{k+1}}{p_k} \left(\frac{1}{l_{kk}^2} - 1 + hp_k\right).$$

The proof of this lemma is relegated to the end of this section. The Theorem 1 simply follows from this lemma. Indeed, setting in (3.11)

$$l_{kk}^2 = 1 + hp_k\beta_k,$$

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we immediately arrive at the recurrent equation (3.3) for $\beta_k$. The following relation, equivalent to (3.12), appears to be also useful:

$$
\left( l_{kk} - \frac{1}{l_{kk}} \right)^2 = \frac{h^2 p_k^2 \beta_k^2}{1 + hp_k \beta_k}.
$$

(3.13)

Substituting now (2.8) into (3.9), we get:

$$
1 + h\tilde{p}_k = \sum_{i \geq k} l_{ik}^2; \quad h\sqrt{\tilde{p}_k \tilde{p}_j} \exp \left( \frac{1}{2}(\bar{x}_k - \bar{x}_j) \right) = \sum_{i \geq k} l_{ik} l_{ij} \quad \text{for } k > j.
$$

(3.14)

With the help of (3.10) the first equation in (3.14) may be re-written as

$$
h\tilde{p}_k = l_{kk}^2 - 1 + \left( l_{kk} - \frac{1}{l_{kk}} \right)^2 \sum_{i > k} p_i \exp(x_i - x_k).
$$

(3.15)

Now (3.4) follows immediately, upon application of (3.12), (3.13).

The right-hand side of the second equation in (3.14) may be brought with the help of (3.10) into the form

$$
\sqrt{\frac{p_k}{p_j}} \exp \left( \frac{1}{2}(x_k - x_j) \right) \left( l_{jj} - \frac{1}{l_{jj}} \right) \left[ l_{kk} + \left( l_{kk} - \frac{1}{l_{kk}} \right) \sum_{i > k} p_i \exp(x_i - x_k) \right].
$$

Because of (3.15) this is equal to

$$
\sqrt{\frac{p_k}{p_j}} \exp \left( \frac{1}{2}(x_k - x_j) \right) \frac{l_{jj} - \frac{1}{l_{jj}}}{l_{kk} - \frac{1}{l_{kk}}} h\tilde{p}_k.
$$

Hence we arrive at the conclusion that the quantity

$$
\frac{\exp(\bar{x}_k - x_k)}{\tilde{p}_k p_k} \left( l_{kk} - \frac{1}{l_{kk}} \right)^2
$$

does not depend on $k$. Setting it equal to $h^2$, we get upon use of (3.13):

$$
\exp(\bar{x}_k - x_k) = \frac{\tilde{p}_k}{p_k} \frac{1 + hp_k \beta_k}{\beta_k^2}.
$$

(3.16)
This, together with (3.4) is equivalent to (3.3). The Theorem 1 is proved.

**Proof of the lemma.** The statement i) may be proved by induction in \( j \). Suppose that it holds for \( l_{ki} \) with \( k > i \) for all \( i < j \). From the definition of the matrix \( L \) (3.7) we have:

\[
\sum_{i \leq j} l_{ji}^2 = 1 + hp_j; \quad \sum_{i \leq j} l_{ki}l_{ji} = h\sqrt{p_kp_j} \exp \left( \frac{1}{2} (x_k - x_j) \right) \text{ for } k > j. \quad (3.17)
\]

Using the base of induction, i.e. the expressions (3.10) for \( l_{ki}, l_{ji} \) with \( i < j \), we deduce from the first equation in (3.17):

\[
l_{jj}^2 = 1 + p_j \left[ h - \sum_{i < j} \frac{\exp(x_j - x_i)}{p_i} \left( l_{ii} - \frac{1}{l_{ii}} \right)^2 \right]. \quad (3.18)
\]

and from the second one:

\[
l_{kj}l_{jj} = \sqrt{p_kp_j} \exp \left( \frac{1}{2} (x_k - x_j) \right) \left[ h - \sum_{i < j} \frac{\exp(x_j - x_i)}{p_i} \left( l_{ii} - \frac{1}{l_{ii}} \right)^2 \right].
\]

Comparing the right-hand side of the last equation with (3.18), we obtain:

\[
l_{kj} = \frac{1}{l_{jj}} \sqrt{p_kp_j} \exp \left( \frac{1}{2} (x_k - x_j) \right) \left( \frac{l_{jj}^2 - 1}{p_j} \right).
\]

This is identical with (3.10), which is hence proved for all \( k > j \).

The statement ii) follows now from the formula (3.18):

\[
l_{k+1,k+1}^2 - 1 - hp_{k+1} = -p_{k+1} \sum_{i \leq k} \frac{\exp(x_{k+1} - x_i)}{p_i} \left( l_{ii} - \frac{1}{l_{ii}} \right)^2 =
\]

\[
= -p_{k+1} \exp(x_{k+1} - x_k) \left[ \frac{1}{p_k} \left( l_{kk} - \frac{1}{l_{kk}} \right)^2 + \sum_{i < k} \frac{\exp(x_k - x_i)}{p_i} \left( l_{ii} - \frac{1}{l_{ii}} \right)^2 \right] =
\]

(using again (3.18) to express the last sum)

\[
= -\frac{p_{k+1}}{p_k} \exp(x_{k+1} - x_k) \left[ \left( l_{kk} - \frac{1}{l_{kk}} \right)^2 - (l_{kk}^2 - 1 - hp_k) \right],
\]

and (3.11) follows. The lemma is proved.
4 Lagrangian formulation

Recall that in the continuous time case, considering the evolution equation (2.2) as a linear system for $p_k$, we got a "tridiagonal" expression (2.4). In the discrete time case, the corresponding evolution equation (3.3) is highly nonlinear, due to the continued fraction expression (3.5) for $\beta_k$. Nevertheless, it turns out to be possible to obtain "tridiagonal" expressions for the impulses $p_k$ in the discrete time case, also.

In order to formulate the corresponding result, it will be convenient to introduce following short-hand notations:

$$A_k = \frac{1}{2} \left( \exp(-x_{k+1}) - \exp(-x_k) \right), \quad B_k = \frac{1}{2} \left( \exp(\bar{x}_k) - \exp(\bar{x}_{k+1}) \right). \quad (4.1)$$

**Theorem 2.** There hold following relations:

$$hp_k = \frac{1}{2} \left( \exp(\bar{x}_{k-1} - x_k) - \exp(\bar{x}_{k+1} - x_k) \right)$$

$$+ \exp(-x_k)B_k \sqrt{1 + \frac{1}{A_k B_k}} - \exp(-x_k)B_{k-1} \sqrt{1 + \frac{1}{A_{k-1} B_{k-1}}}; \quad (4.2)$$

$$h\bar{p}_k = \frac{1}{2} \left( \exp(\bar{x}_k - x_{k+1}) - \exp(\bar{x}_k - x_{k-1}) \right)$$

$$- \exp(\bar{x}_k)A_k \sqrt{1 + \frac{1}{A_k B_k}} + \exp(\bar{x}_k)A_{k-1} \sqrt{1 + \frac{1}{A_{k-1} B_{k-1}}}. \quad (4.3)$$

This Theorem will be proved at the end of this section. Several remarks are now in turn.

**Remark 3.** For $k = 1$ and $k = N$ minor correction in the formulas (4.2), (4.3) are necessary, due to the boundary conditions (2.5). The corresponding formulas read:

$$hp_1 = -1 + \frac{1}{2} \left( \exp(\bar{x}_1 - x_1) - \exp(\bar{x}_2 - x_1) \right) + \exp(-x_1)B_1 \sqrt{1 + \frac{1}{A_1 B_1}};$$

$$h\bar{p}_1 = \frac{1}{2} \left( \exp(\bar{x}_1 - x_2) + \exp(\bar{x}_1 - x_1) \right) - \exp(\bar{x}_1)A_1 \sqrt{1 + \frac{1}{A_1 B_1}};$$
Remark 4. The expressions (4.2), (4.3) serve as finite–difference approximations to (2.4). However, as opposed to the analogous statements in [8], [9], this is not immediately obvious, but rather requires for some (straightforward) calculations.

Now we are in a position to give a Lagrangian formulation of the discrete time peakons lattice, i.e. to represent it in the form
\[
\partial \left( \Lambda(x(t+h), x(t)) + \Lambda(x(t), x(t-h)) \right)/\partial x_k(t) = 0. \tag{4.4}
\]
The key point is the representation of the momenta in terms of the Lagrangian function [12]:
\[
p_k = -\partial \Lambda(\bar{x}, x)/\partial x_k, \quad \bar{p}_k = \partial \Lambda(\bar{x}, x)/\partial \bar{x}_k. \tag{4.5}
\]
Identifying these expressions with (4.2), (4.3), we see that the following statement holds.

**Theorem 3.** The discrete time peakons lattice is a Lagrangian system (4.4) with the Lagrangian function
\[
\Lambda(\bar{x}, x) = x_1 + \frac{1}{2} \exp(\bar{x}_1 - x_1) + \frac{1}{2} \exp(\bar{x}_N - x_N) - \bar{x}_N \\
+ \frac{1}{2} \sum_{k=1}^{N-1} \left( \exp(\bar{x}_k - x_{k+1}) - \exp(\bar{x}_{k+1} - x_k) \right) - \sum_{k=1}^{N-1} \Psi(A_k B_k), \tag{4.6}
\]
where
\[
\Psi(\xi) = 2 \int_0^\xi \sqrt{1 + \frac{1}{\eta^2}} d\eta.
\]

**Remark 5.** The function (4.6) serves as a rather non–trivial approximation to (2.6). This can be seen after some straightforward, though tedious calculations.
Proof of the Theorem 2. Subtract from the equation (3.3) the analogous equation for the subscript $k + 1$ multiplied by $\exp(x_{k+1} - x_k)$:

$$\left(\frac{1}{\beta_k} + hp_k\right) - \frac{1}{\beta_{k+1}} \exp(x_{k+1} - x_k) = \exp(\bar{x}_k - x_k) - \exp(\bar{x}_{k+1} - x_k).$$

Using the recurrence (3.5) to express $\beta_{k+1}$, we end up after some simple calculations with the following quadratic equation:

$$\left(\frac{1}{\beta_k} + hp_k\right)^2 - 2 \exp(-x_k)B_k \left(\frac{1}{\beta_k} + hp_k\right) - \exp(-2x_k)\frac{B_k}{A_k} = 0.$$ 

Its (positive) solution:

$$\frac{1}{\beta_k} + hp_k = \exp(-x_k)B_k \left(\sqrt{1 + \frac{1}{A_kB_k}} + 1\right). \quad (4.7)$$

In order to exclude $\beta_k$ from this expression, we use (3.5) in the form

$$\beta_k = \exp(x_k) \left(2A_{k-1} + \frac{\exp(-x_{k-1})}{\frac{1}{\beta_{k-1}} + hp_{k-1}}\right).$$

Using in the right–hand side (4.7), we get

$$\beta_k = \exp(x_k)A_{k-1} \left(\sqrt{1 + \frac{1}{A_{k-1}B_{k-1}}} + 1\right), \quad (4.8)$$

or

$$\frac{1}{\beta_k} = \exp(-x_k)B_{k-1} \left(\sqrt{1 + \frac{1}{A_{k-1}B_{k-1}}} - 1\right). \quad (4.9)$$

Now (4.2) follows directly from (4.7) and (4.9).

Finally, to prove (4.3) we use (3.16), re–written in the form

$$h\tilde{p}_k = hp_k \exp(\bar{x}_k + x_k) \frac{\beta_k \exp(-x_k)}{\frac{1}{\beta_k} + hp_k} \exp(x_k).$$

Putting (4.2), (4.7), (4.8) into this formula, we arrive after some manipulations at (4.3). The Theorem 2 is proved.
5 Conclusion

A new application of a general scheme for producing integrable discretizations for integrable Hamiltonian flows is described in the present Letter. Advantages of this approach are rather obvious: it is, in principle, applicable in a standartized way to every system admitting an $r$–matrix formulation, at least with a constant $r$–matrix satisfying the modified Yang–Baxter equation. We shall demonstrate elsewhere that the discrete time systems from [6], [7] with dynamical $r$–matrices may be also included into this framework. We hope also to report on numerous other applications of this approach in the future.

The drawback of this scheme is also obvious to any expert in this field. Namely, some of the most beautiful discretizations do not live on the same $r$–matrix orbits as their continuous time counterparts [1], [3], [4], [5], and there seems to exist no way of \textit{a priori} identifying the correct orbit for nice discretizations. However, we hope that continuing to collect examples will someday bring some light to this intriguing problem.

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