TRACE OF POWERS OF REPRESENTATIONS OF FINITE QUANTUM GROUPS

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Abstract. In this paper we study (asymptotic) properties of the ∗-distribution of irreducible characters of finite quantum groups. We proceed in two steps, first examining the representation theory to determine irreducible representations and their powers, then we study the ∗-distribution of their trace with respect to the Haar measure. For the Sekine family we look at the asymptotic distribution (as the dimension of the algebra goes to infinity).

Introduction

In [4] and then in [3], Diaconis, Shahshahani and Evans show that the traces of powers of a random unitary (respectively orthogonal) matrix behave asymptotically like independent complex (resp. real) Gaussian random variables. Later, Banica, Curran and Speicher investigate the case of easy quantum groups in [1, 2], and obtain similar results in the context of free probability.

Here we will look at finite quantum groups. They were introduced in the sixties as examples of Hopf-von Neumann algebras to recover symmetry in duality for non abelian locally compact groups. The eight-dimensional Kac-Paljutkin quantum group KP is the smallest non-trivial example, in the sense it is neither commutative nor cocommutative. In 1996, Sekine defines a new family of finite quantum groups [8], of dimension $2n^2$ for all $n \geq 2$.

To extend the study in this framework, let us recall the definition of a representation in the context of compact quantum group:

Definition 1. Let $G = (C(G), \Delta)$ be a compact quantum group. A corepresentation of the algebra $C(G)$, also called a representation of the quantum group $G$, is an element $u$ of $B(H) \otimes C(G)$, for some Hilbert space $H$, such that $(id_H \otimes \Delta)u = u_{12}u_{13}$ in $B(H) \otimes C(G) \otimes C(G)$.

Note that irreducible representations of compact quantum groups are finite dimensional, i.e. the Hilbert space is finite dimensional. Thus, we can see the representations as matrices with coefficients in the algebra of the quantum group satisfying $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$. Up to equivalence, we can consider only unitary finite dimensional representations. With these objects, we will use the following notation:
Definition 2. Let $\mathcal{M}$ be a matrix whose coefficients $M_{i,j}$ are elements of an algebra $\mathcal{A}$. The trace of $\mathcal{M}$, denoted $\chi(\mathcal{M})$, is the sum of all its diagonal elements, it means $\chi(\mathcal{M}) = \sum_i M_{ii}$ in $\mathcal{A}$.

This work is separated into two parts. The first section is devoted to the study of the Kac-Palyutkin quantum group $\mathbb{KP}$. After recalling the definition, we compute group-like elements and a fundamental representation. Finally, we determine the $\ast$-distribution of the trace of its powers in Theorem 1 and independence in Theorem 2.

In the second section, we work with the family of Sekine quantum groups. After recalling the definition, we give the representation theory. We also study the character space and a commutative subalgebra. Finally, we determine the asymptotic $\ast$-distribution of the trace of powers of two-dimensional representations in Theorems 6 and 7.

1. Kac-Palyutkin finite quantum group $\mathbb{KP}$

1.1. Definition. We will follow the definition of [5], but for convenience of the reader, we recall here the notations.

Consider the multi-matrix algebra $\mathcal{A} = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{M}_2(\mathbb{C})$ together with usual multiplication and involution. This is an eight-dimensional algebra, with the canonical basis

\[
e_1 = 1 \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad E_{11} = 0 \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

\[
e_2 = 0 \oplus 1 \oplus 0 \oplus 0 \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad E_{12} = 0 \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]

\[
e_3 = 0 \oplus 0 \oplus 1 \oplus 0 \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad E_{21} = 0 \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\]

\[
e_4 = 0 \oplus 0 \oplus 0 \oplus 1 \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad E_{22} = 0 \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

The unit is naturally $1 = 1 \oplus 1 \oplus 1 \oplus 1 \oplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = e_1 + e_2 + e_3 + e_4 + E_{11} + E_{22}$.

The following defines the coproduct:

\[
\Delta(e_1) = e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3 + e_4 \otimes e_4
\]
\[
+ \frac{1}{2} E_{11} \otimes E_{11} + \frac{1}{2} E_{12} \otimes E_{12} + \frac{1}{2} E_{21} \otimes E_{21} + \frac{1}{2} E_{22} \otimes E_{22}
\]

\[
\Delta(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 + e_3 \otimes e_4 + e_4 \otimes e_3
\]
\[
+ \frac{1}{2} E_{11} \otimes E_{22} + \frac{1}{2} E_{22} \otimes E_{11} - \frac{i}{2} E_{12} \otimes E_{21} + \frac{i}{2} E_{21} \otimes E_{12}
\]
\[ \Delta(e_3) = e_1 \otimes e_3 + e_3 \otimes e_1 + e_2 \otimes e_4 + e_4 \otimes e_2 + \frac{1}{2} E_{11} \otimes E_{22} + \frac{1}{2} E_{22} \otimes E_{11} + \frac{1}{2} E_{12} \otimes E_{21} - \frac{1}{2} E_{21} \otimes E_{12} \]
\[ \Delta(e_4) = e_1 \otimes e_4 + e_4 \otimes e_1 + e_2 \otimes e_3 + e_3 \otimes e_2 + \frac{1}{2} E_{11} \otimes E_{11} + \frac{1}{2} E_{22} \otimes E_{22} - \frac{1}{2} E_{12} \otimes E_{12} - \frac{1}{2} E_{21} \otimes E_{21} \]
\[ \Delta(E_{11}) = e_1 \otimes E_{11} + E_{11} \otimes e_1 + e_2 \otimes E_{22} + E_{22} \otimes e_2 + e_3 \otimes E_{22} + E_{22} \otimes e_3 + e_4 \otimes E_{11} + E_{11} \otimes e_4 \]
\[ \Delta(E_{12}) = e_1 \otimes E_{12} + E_{12} \otimes e_1 + \varepsilon e_2 \otimes E_{21} - iE_{21} \otimes e_2 - \varepsilon E_{21} \otimes e_3 - e_4 \otimes E_{12} - E_{12} \otimes e_4 \]
\[ \Delta(E_{21}) = e_1 \otimes E_{21} + E_{21} \otimes e_1 - \varepsilon e_2 \otimes E_{12} + iE_{12} \otimes e_2 + \varepsilon E_{12} \otimes e_3 - e_4 \otimes E_{21} - E_{21} \otimes e_4 \]
\[ \Delta(E_{22}) = e_1 \otimes E_{22} + E_{22} \otimes e_1 + e_2 \otimes E_{11} + E_{11} \otimes e_2 + e_3 \otimes E_{11} + E_{11} \otimes e_3 + e_4 \otimes E_{22} + E_{22} \otimes e_4 \]

the counit is given by \( \varepsilon \left( x_1 \oplus x_2 \oplus x_3 \oplus x_4 \oplus \begin{pmatrix} c_{11} \\ c_{21} \\ c_{22} \end{pmatrix} \right) = x_1 \) and the antipode is the transpose map, i.e. \( S(e_i) = e_i \) and \( S(E_{ij}) = E_{ji} \). This defines a finite quantum group, denoted by \( \mathbb{K} \mathbb{P} = (A, \Delta) \). We shall also need its Haar weight, denoted by \( \int_{\mathbb{K} \mathbb{P}} \):
\[ \int_{\mathbb{K} \mathbb{P}} \left( x_1 \oplus x_2 \oplus x_3 \oplus x_4 \oplus \begin{pmatrix} c_{11} \\ c_{21} \\ c_{22} \end{pmatrix} \right) = \frac{1}{8} (x_1 + x_2 + x_3 + x_4 + 2(c_{11} + c_{22})) \).

1.2. The group of group-like elements. Group-like elements correspond to one dimensional representations of \( \mathbb{K} \mathbb{P} \). They are non-zero elements of \( A \) such that \( \Delta(g) = g \otimes g \) and \( \varepsilon(g) = 1 \).

**Proposition 1.** The family of group-like elements of \( \mathbb{K} \mathbb{P} \) is a group isomorphic to \( \mathbb{Z}_2 \times \mathbb{Z}_2 \):
\[ \left\{ 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus \begin{pmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}, 1 \oplus -1 \oplus -1 \oplus 1 \oplus 1 \oplus \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ -1 \end{pmatrix}, 1 \oplus -1 \oplus -1 \oplus 1 \oplus 1 \oplus \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} . \]

**Proof.** This is done by direct computation, using the conditions \( \varepsilon(g) = 1 \) and \( \Delta(g) = g \otimes g \). \( \square \)

**Remark 1.** Note that in [6], Izumi and Kosaki give group-like elements which look a little bit different from ours. The fact is that they do not use the same basis: in their notation, \( z(a), z(b) \) and \( z(c) \) play respectively the role of our \( e_4, e_2 \) and \( e_3 \).
1.3. Matrix elements and fundamental representation. Let us look at representations of $\mathbb{K}^P$ with dimension at least 2. We will determine matrix elements of representation of dimension 2, i.e. elements $X_{(11)}$, $X_{(12)}$, $X_{(21)}$, $X_{(22)}$ of $\mathcal{A}$, such that $\Delta(X_{(ij)}) = X_{(11)} \otimes X_{(1j)} + X_{(12)} \otimes X_{(2j)}$ and $\varepsilon(X_{(ij)}) = \delta_{ij}$ for all $i, j \in \{1, 2\}$. These are matrix elements of two-dimensional representations of $\mathbb{K}^P$.

**Proposition 2.** For all $a \in \{-1, 1\}$ and all $\lambda \in \mathbb{T}$, let us fix

$$X_{a,\lambda} = \begin{pmatrix} X_{(11)} & X_{(12)} \\ X_{(21)} & X_{(22)} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \oplus a \oplus -a \oplus -1 \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & 0 \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} 0 & \lambda \\ ia\lambda & 0 \end{pmatrix} \\ 0 \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} 0 & \lambda \\ -ia\lambda & 0 \end{pmatrix} & 1 \oplus -a \oplus a \oplus -1 \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}.$$

Then $X_{a,\lambda}$ is a fundamental representation of $\mathbb{K}^P$, it means that its coefficients generate the algebra $\mathcal{A}$.

**Remark 2.** Remark [1] about [4], holds again. Moreover, since all $X_{a,\lambda}$ are unitary equivalent, Izumi and Kosaki fix $a = -1$ and $\lambda = e^{i\pi}$.

**Proof.** First of all, let us check that $X_{(11)}$ is a matrix element of a two-dimensional representation of $\mathbb{K}^P$. The computation for $X_{(12)}$, $X_{(21)}$ and $X_{(22)}$ are similar. We have on the first hand

$$\Delta(X_{(11)}) = e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3 + e_4 \otimes e_4 - (e_1 \otimes e_4 + e_4 \otimes e_1 + e_2 \otimes e_3 + e_3 \otimes e_2)$$

$$+ a (e_1 \otimes e_2 + e_2 \otimes e_1 + e_3 \otimes e_4 + e_4 \otimes e_3)$$

$$- a (e_1 \otimes e_3 + e_3 \otimes e_1 + e_2 \otimes e_4 + e_4 \otimes e_2)$$

$$+ E_{12} \otimes E_{12} + E_{21} \otimes E_{21} + iaE_{21} \otimes E_{12} - iaE_{12} \otimes E_{21}$$

and, on the other hand

$$X_{(11)} \otimes X_{(11)} + X_{(12)} \otimes X_{(21)} = e_1 \otimes e_1 + ae_1 \otimes e_2 - ae_1 \otimes e_3 - e_1 \otimes e_4 + a(e_2 \otimes e_1 + ae_2 \otimes e_2 - ae_2 \otimes e_3 - e_2 \otimes e_4)$$

$$- a(e_3 \otimes e_1 + ae_3 \otimes e_2 - ae_3 \otimes e_3 - e_3 \otimes e_4) - (e_4 \otimes e_1 + ae_4 \otimes e_2 - ae_4 \otimes e_3 - e_4 \otimes e_4)$$

$$+ |\lambda|^2 E_{12} \otimes E_{12} - ia|\lambda|^2 E_{12} \otimes E_{21}$$

$$+ ia|\lambda|^2 E_{21} \otimes E_{12} + a^2 |\lambda|^2 E_{21} \otimes E_{21}.$$

Hence, $\Delta(X_{(11)}) = X_{(11)} \otimes X_{(11)} + X_{(12)} \otimes X_{(21)}$.

Moreover, we can show that $X_{a,\lambda}$ and $\overline{X}_{a,\lambda}$ are unitary matrices and that $\int_{\mathbb{K}^P} \chi(X_{a,\lambda})^* \chi(X_{a,\lambda}) = 1$. Hence $X_{a,\lambda}$ defines a unitary irreducible representation of $\mathbb{K}^P$. 


Finally, the family \( \{ X_{(11)}, X_{(12)}, X_{(21)}, X_{(22)} \} \) generates \( \mathcal{A} \), since

\[
\begin{align*}
e_1 &= \frac{1}{4} \left( X_{(11)}^2 + X_{(11)}X_{(22)} \right) + \frac{1}{4} \left( X_{(11)} + X_{(22)} \right) \\
e_4 &= \frac{1}{4} \left( X_{(11)}^2 + X_{(11)}X_{(22)} \right) - \frac{1}{4} \left( X_{(11)} + X_{(22)} \right) \\
e_2 &= \frac{1}{4} \left( X_{(11)}^2 - X_{(11)}X_{(22)} \right) + \frac{a}{4} \left( X_{(11)} - X_{(22)} \right) \\
e_3 &= \frac{1}{4} \left( X_{(11)}^2 - X_{(11)}X_{(22)} \right) - \frac{a}{4} \left( X_{(11)} - X_{(22)} \right)
\end{align*}
\]

\[
\begin{align*}
E_{11} &= \frac{1}{2} \left( X_{(12)}X_{(12)}^* + iaX_{(12)}X_{(21)} \right) \\
E_{22} &= \frac{1}{2} \left( X_{(12)}X_{(12)}^* - iaX_{(12)}X_{(21)} \right) \\
E_{12} &= \frac{1}{2} \left( \bar{\lambda}X_{(12)} + \lambda X_{(21)} \right) \\
E_{21} &= -\frac{ia}{2} \left( \bar{\lambda}X_{(12)} - \lambda X_{(21)} \right).
\end{align*}
\]

\[\square\]

1.4. Powers of fundamental representations.

**Lemma 1.** For all non negative integers \( n \), \( (X_{a,\lambda})^{2n} \) is a diagonal matrix,

\[
(X_{a,\lambda})^{2n} = \begin{pmatrix}
1 \oplus 1 \oplus 1 \oplus 1 \oplus \left( (-ia)^n 0 0 0 \right) & 0 \oplus 0 \oplus 0 \oplus 0 \oplus \left( 0 0 0 \right) \\
0 \oplus 0 \oplus 0 \oplus 0 \oplus \left( 0 0 0 \right) & 1 \oplus 1 \oplus 1 \oplus 1 \oplus \left( (ia)^n 0 0 \right)
\end{pmatrix}
\]

*Proof.* By direct computation, we obtain that

\[
(X_{a,\lambda})^2 = \begin{pmatrix}
1 \oplus 1 \oplus 1 \oplus 1 \oplus \left( -ia 0 0 \right) & 0 \oplus 0 \oplus 0 \oplus 0 \oplus \left( 0 0 0 \right) \\
0 \oplus 0 \oplus 0 \oplus 0 \oplus \left( 0 0 0 \right) & 1 \oplus 1 \oplus 1 \oplus 1 \oplus \left( ia 0 0 \right)
\end{pmatrix}
\]

and the result follows.

\[\square\]

**Lemma 2.** For all non negative integers \( n \), \( \chi \left( (X_{a,\lambda})^{2n+1} \right) = \chi \left( X_{a,\lambda} \right) \).

*Proof.* The result comes from Lemma [1] and the classical formula

\[
(X_{a,\lambda})^{2n+1} = (X_{a,\lambda})^{2n} X_{a,\lambda}.
\]

\[\square\]

**Theorem 1.** Let \( k \) be a non negative integer. Let us denote by \( \mu_0, \mu_1, \mu_2 \) and \( \mu_4 \) the following distributions:

\[
\mu_0 = \delta_2, \quad \mu_1 = \frac{1}{8} \delta_{-2} + \frac{3}{4} \delta_0 + \frac{1}{8} \delta_2, \quad \mu_2 = \frac{1}{2} \delta_0 + \frac{1}{2} \delta_2 \quad \text{and} \quad \mu_4 = \frac{1}{2} \delta_{-2} + \frac{1}{2} \delta_2.
\]
Then for all \(a \in \{-1, 1\}\) and \(\lambda \in \mathbb{T}\), \(\chi((\mathcal{X}_{a,\lambda})^k)\) is self-adjoint and admits \(\mu_\kappa\) as \(*\)-distribution, with

\[
\kappa = \begin{cases} 
1 & \text{if } k \equiv 1[2] \\
2 & \text{if } k \equiv 2[4] \\
4 & \text{if } k \equiv 4[8] \\
0 & \text{if } k \equiv 0[8] 
\end{cases}
\]

Moreover, we have

\[
\chi((\mathcal{X}_{a,\lambda})^k) = \begin{cases} 
2 \oplus 0 \oplus 0 \oplus -2 \oplus 0_2 & , \text{if } k \equiv 1[2] \\
2 \oplus 2 \oplus 2 \oplus 2 \oplus 0_2 & , \text{if } k \equiv 2[4] \\
2 \oplus 2 \oplus 2 \oplus 2 \oplus -2I_2 & , \text{if } k \equiv 4[8] \\
2 \oplus 2 \oplus 2 \oplus 2 \oplus 2I_2 & , \text{if } k \equiv 0[8] 
\end{cases}
\]

Proof. Assume that \(k\) is odd, then \(\chi((\mathcal{X}_{a,\lambda})^k) = \chi(\mathcal{X}_{a,\lambda})\), by Lemma 2, and, by definition, we have

\[
\chi(\mathcal{X}_{a,\lambda}) = \mathcal{X}_{(11)} + \mathcal{X}_{(22)} = 2 \oplus 0 \oplus 0 \oplus -2 \oplus 0 \\
\]

so, \(\chi(\mathcal{X}_{a,\lambda})^* = \chi(\mathcal{X}_{a,\lambda})\), and for all non negative integer \(n\), we obtain that

\[
\int_{KP} (\chi(\mathcal{X}_{a,\lambda}))^n = \frac{1}{8}(2^n + 0^n + 0^n + (-2)^n + 2(0^n + 0^n)) \\
= \frac{(-2)^n}{8} + \frac{3}{4} \times 0^n + \frac{2^n}{8} = E[Z_1]
\]

where \(Z_1\) is a \(\mu_1\)-distributed random variable.

Now, assume that \(k\) is even. Then

\[
\chi((\mathcal{X}_{a,\lambda})^k) = 2 \oplus 2 \oplus 2 \oplus 2 \oplus \begin{pmatrix} (ia)^{\frac{k}{2}} + (-ia)^{\frac{k}{2}} & 0 \\ 0 & (ia)^{\frac{k}{2}} + (-ia)^{\frac{k}{2}} \end{pmatrix}
\]

is self-adjoint and

\[
\int_{KP} \left(\chi((\mathcal{X}_{a,\lambda})^k)\right)^n = \frac{1}{2} \left(2^n + \left(ia^{\frac{k}{2}} + (-ia)^{\frac{k}{2}}\right)^n\right).
\]

Let us note that \((ia)^2 = -1\), and \((ia)^4 = 1\). Hence, we obtain that the distribution of \(\chi((\mathcal{X}_{a,\lambda})^k)\) equals \(\mu_2\) if \(k = 4p + 2\), \(\mu_4\) if \(k = 8p + 4\) and \(\mu_0\) if 8 divides \(k\). \(\square\)

Remark 3. Like in the classical case, we can express traces of powers of the fundamental representation as linear combinations of characters, it means one-dimensional representations given in Proposition 1 and \(\chi(\mathcal{X}_{a,\lambda})\). Let us note that this is not true in general for quantum groups. Here we have

\[
2 \oplus 2 \oplus 2 \oplus 2 \oplus I_2 = 2 \oplus 1, 2 \oplus 2 \oplus 2 \oplus 2 \oplus -2I_2 = 2 \oplus 1 \oplus 1 \oplus 1 \oplus -I_2 \\
2 \oplus 2 \oplus 2 \oplus 2 \oplus 0_2 = 2 \oplus 1 \oplus 1 \oplus 1 \oplus -I_2 + 1).
\]
Theorem 2. For $i \in \{0, 1, 2, 4\}$, let $Z_i$ be a $\mu_i$-distributed random variable such that $Z_0$ and $Z_1$ are independent from all the others.

Then, for all $a \in \{-1, 1\}$, $\lambda \in \mathbb{T}$, and $(k_1, \ldots, k_r) \in \mathbb{N}^r$,

$$\int_{\mathbb{K}^p} \chi((X_{a,\lambda})^k_1) \ldots \chi((X_{a,\lambda})^k_r) = \mathbb{E}[Z_{m_1} \ldots Z_{m_r}]$$

$$= 2^{\#\{i, k_i = 0\}} \mathbb{E}[Z_{2}^{\#\{i, k_i = 1\}}] \mathbb{E}[Z_{2}^{\#\{i, k_i = 2\}}] Z_{4}^{\#\{i, k_i = 4\}}$$

with $m_i = \begin{cases} 1 & \text{if } k_i \equiv 1[2] \\ 2 & \text{if } k_i \equiv 2[4] \\ 4 & \text{if } k_i \equiv 4[8] \\ 0 & \text{if } k_i \equiv 0[8] \end{cases}$

Proof. Let us note that the $\chi((X_{a,\lambda})^i)$'s commute. Hence

$$\int_{\mathbb{K}^p} \prod_{i=1}^r \chi((X_{a,\lambda})^{k_i}) = \int_{\mathbb{K}^p} 2^r \oplus \alpha(k) \oplus \alpha(k) \oplus (-1)^{\#\{i, k_i = 1\}} 2^r \oplus \beta(k)$$

$$= \frac{1}{8} \left( 2^r \left( 1 + (-1)^{\#\{i, k_i = 1\}} \right) + 2\alpha(k) + 2\text{Tr}(\beta(k)) \right)$$

where $\alpha(k)$ is 0 if there exists $i$ such that $k_i = 1$ and $2^r$ otherwise, and $\beta(k)$ is the matrix null if there exists $i$ such that $k_i \in \{1, 2\}$ and $(-1)^{\#\{i, k_i = 4\}} 2^r I_2$ otherwise. So, we have

$$\int_{\mathbb{K}^p} \prod_{i=1}^r \chi((X_{a,\lambda})^{k_i})$$

$$= \begin{cases} 0 & \text{if } 2 \nmid \#\{i, k_i = 1\} \\ 2^{r-2} & \text{if } \#\{i, k_i = 1\} \in 2(\mathbb{N} \setminus \{0\}) \\ 2^{r-1} & \text{if } \#\{i, k_i = 1\} = 0, \#\{i, k_i = 2\} \geq 1 \\ 2^{r-1}(1 + (-1)^{\#\{i, k_i = 4\}}) & \text{otherwise} \end{cases}$$

Clearly, $\chi((X_{a,\lambda})^0)$ is independent from the other $\chi((X_{a,\lambda})^i)$'s. To study the independence of $\chi(X_{a,\lambda})$, let us look at classical cumulants. Let $\{p_1, p_2, \ldots, p_r\}$ be a subset of $\{1, 2, 4\}$, and $\kappa(p_1, \ldots, p_r)$ be the joint cumulant of $\chi((X_{a,\lambda})^{p_1}) \ldots \chi((X_{a,\lambda})^{p_r})$. By definition, we have

$$\kappa(p_1, \ldots, p_r) = \sum_{\pi \in P_r} (|\pi| - 1)! (-1)^{|\pi|-1} \prod_{B \in \pi} \int_{\mathbb{K}^p} \prod_{k \in B} \chi((X_{a,\lambda})^{p_k})$$

hence, if 1 is in $\{p_1, p_2, \ldots, p_r\}$ the cumulant is 0, it means that $\chi(X_{a,\lambda})$ is independent from the others. Moreover,

$$\kappa(2, 4) = \int_{\mathbb{K}^p} \chi((X_{a,\lambda})^2) \chi((X_{a,\lambda})^4) - \int_{\mathbb{K}^p} \chi((X_{a,\lambda})^2) \int_{\mathbb{K}^p} \chi((X_{a,\lambda})^4) = 2$$

thus $\chi((X_{a,\lambda})^2)$ and $\chi((X_{a,\lambda})^4)$ are not independent. \qed
2. The Sekine finite quantum groups $\mathbb{KP}_n$

2.1. Definition. We will follow the definition of [7], but for convenience of the reader, we recall here the notations.

Consider the multi-matrix algebra $A_n = \bigoplus_{i,j \in \mathbb{Z}} \mathbb{C} e_{(i,j)} \oplus M_n(\mathbb{C})$ together with usual multiplication and involution. This is a $2n^2$-dimensional algebra, with basis $\{ e_{(i,j)} \}_{i,j \in \mathbb{Z}^n} \cup \{ E_{i,j} \}_{1 \leq i,j \leq n}$. The unit is naturally

$$1 = \sum_{i,j \in \mathbb{Z}_n} e_{(i,j)} + \sum_{i=1}^n E_{i,i}$$

The following defines the coproduct:

$$\Delta(e_{(i,j)}) = \sum_{k,l \in \mathbb{Z}_n} e_{(k,l)} \otimes e_{(-k,j-l)} + \frac{1}{n} \sum_{k,l \in \mathbb{Z}_n} \eta^{i(k-l)} E_{k,l} \otimes E_{k+j,l+j}$$

$$\Delta(E_{i,j}) = \sum_{k,l \in \mathbb{Z}_n} e_{(-k,-l)} \otimes \eta^{i(k-j)} E_{i-l,j-l} + \sum_{k,l \in \mathbb{Z}_n} \eta^{i(j-l)} E_{i-l,j-l} \otimes e_{(k,l)}$$

with $\eta = e^{\frac{2\pi i}{n}}$ a primitive $n$th root of unity, the counit is given by

$$\varepsilon \left( \sum_{i,j \in \mathbb{Z}_n} x_{(i,j)} e_{(i,j)} + \sum_{1 \leq i,j \leq n} X_{i,j} E_{i,j} \right) = x_{(0,0)}$$

and the antipode satisfies $S(e_{(i,j)}) = e_{(-i,-j)}$ and $S(E_{i,j}) = E_{j,i}$.

This defines a finite quantum group, denoted by $\mathbb{KP}_n = (A_n, \Delta)$, called Sekine quantum group. We shall also need its Haar state, denoted by $\int_{\mathbb{KP}_n}$ and given by the following formula:

$$\int_{\mathbb{KP}_n} \left( \sum_{i,j \in \mathbb{Z}_n} x_{(i,j)} e_{(i,j)} + \sum_{1 \leq i,j \leq n} X_{i,j} E_{i,j} \right) = \frac{1}{2n^2} \left( \sum_{i,j \in \mathbb{Z}_n} x_{(i,j)} + n \sum_{i=1}^n X_{i,i} \right).$$

Remark 4. As noted in [7], with this definition, $\mathbb{KP}_2$ is cocommutative and equal to the virtual object $\tilde{\mathbb{D}}_4$, i.e. $\mathcal{A}_2 \simeq \mathbb{C}\tilde{\mathbb{D}}_4$.

2.2. Representation theory.

2.2.1. Case $n$ odd.

**Theorem 3** ([7]). If $n$ is odd, the finite quantum group $\mathbb{KP}_n$ admits $2n$ one-dimensional non equivalent unitary representations,

$$\forall l \in \{0,1,\ldots,n-1\}, \quad \rho^+_l = \sum_{i,j \in \mathbb{Z}_n} \eta^l e_{(i,j)} \pm \sum_{i=1}^n E_{i,i+l}.$$

It also admits $\frac{n(n-1)}{2}$ non equivalent unitary two-dimensional irreducible representations, indexed by $u \in \{0,1,\ldots,n-1\}$ and $v \in \{1,2,\ldots,\frac{n-1}{2}\}$,
given by their matrix-coefficients:

\[ X_{11}^{u,v} = \sum_{i,j \in \mathbb{Z}_n} \eta^{iu+jv} e_{(i,j)} \quad X_{12}^{u,v} = \sum_{i=1}^{n} \eta^{-iv} E_{i,i+u} \]

\[ X_{21}^{u,v} = \sum_{i=1}^{n} \eta^{iv} E_{i,i+u} \quad X_{22}^{u,v} = \sum_{i,j \in \mathbb{Z}_n} \eta^{iu-jv} e_{(i,j)} \]

### 2.2.2. Case \( n \) even.

**Theorem 4.** If \( n \) is even, the finite quantum group \( \mathbb{KP}_n \) admits 4\( n \) one-dimensional non equivalent unitary representations, \( \forall l \in \{0, 1, \ldots, n-1\} \),

\[ \rho^\pm_l = \sum_{i,j \in \mathbb{Z}_n} \eta^l e_{(i,j)} \pm \sum_{i=1}^{n} E_{i,i+l} \]

\[ \sigma^\pm_l = \sum_{i,j \in \mathbb{Z}_n} (-1)^j \eta^l e_{(i,j)} \pm \sum_{i=1}^{n} (-1)^i E_{i,i+l} \]

It also admits \( \frac{n(n-2)}{2} \) non equivalent unitary two-dimensional irreducible representations, indexed by \( u \in \{0, 1, \ldots, n-1\} \) and \( v \in \{1, 2, \ldots, \frac{n}{2} - 1\} \), given by their matrix-coefficients:

\[ X_{11}^{u,v} = \sum_{i,j \in \mathbb{Z}_n} \eta^{iu+jv} e_{(i,j)} \quad X_{12}^{u,v} = \sum_{i=1}^{n} \eta^{-iv} E_{i,i+u} \]

\[ X_{21}^{u,v} = \sum_{i=1}^{n} \eta^{iv} E_{i,i+u} \quad X_{22}^{u,v} = \sum_{i,j \in \mathbb{Z}_n} \eta^{iu-jv} e_{(i,j)} \]

**Proof.** The computations done for \( \rho^\pm_l \) and \( X^{u,v} \) in the odd case in [7] are still valid.

It is clear that \( \varepsilon(\sigma^\pm_l) = 1 \). By Proposition 3.1.7 in [9], it remains to prove that \( \sigma^\pm_l \) are group-like elements. The same steps as for \( \rho^\pm_l \) give

\[ \Delta(\sigma^\pm_l) = \sum_{i,j,s,t \in \mathbb{Z}_n} (-1)^j \eta^l e_{(s,t)} \otimes e_{(i-s,j-t)} \]

\[ + \sum_{m=1 \atop j \in \mathbb{Z}_n}^{n} (-1)^{j} E_{m,m+l} \otimes E_{m+j,m+j+l} \]

\[ \pm \sum_{m=1 \atop s,t \in \mathbb{Z}_n}^{n} (-1)^m \eta^l e_{(s,t)} \otimes E_{m+t,m+t+l} \]

\[ \pm \sum_{m=1 \atop s,t \in \mathbb{Z}_n}^{n} (-1)^m \eta^l E_{m-t,m-t+l} \otimes e_{(s,t)} . \]
On the other hand, we have
\[ \sigma_i^\pm \otimes \sigma_i^\pm = \sum_{i,j,s,t \in \mathbb{Z}_n} (-1)^{j+t} \eta^{(i+s)t} e_{(i,j)} \otimes e_{(s,t)} \]
\[ + \sum_{s,t=1}^n (-1)^{s+t} E_{s,s+t} \otimes E_{t,t+s} \]
\[ \pm \sum_{m=1}^n (-1)^{t+m} \eta^{sl} e_{(s,t)} \otimes E_{m,m+l} \]
\[ \pm \sum_{m=1}^n (-1)^{m+t} \eta^{sl} E_{m,m+l} \otimes e_{(s,t)} \]
which is the same, up to re-indexation \((s + i \to i, t + j \to j, i \to s\) and \(j \to t\) in the first term, \(t - s \to j\) in the second one, \(m - t \to m\) in the third one
and \(m + t \to m\) in the last one). \(\square\)

**Remark 5.** This is another way to see that the Kac-Palyutkin \(\mathbb{K}P\) finite quantum group is different from \(\mathbb{K}P_2\), since they do not have the same representation theory. The quantum group \(\mathbb{K}P\) admits four one-dimensional representations and one two-dimensional irreducible representation, whereas the Sekine quantum group \(\mathbb{K}P_2\) admits eight one-dimensional representations and no two-dimensional irreducible representation.

### 2.3. Characters.

**Lemma 3.** The even powers of \(X^{u,v}\) are diagonal matrices.

**Proof.** By definition and orthogonality of the \(e_{(i,j)}\)'s and the \(E_{i,j}\)'s, we have
\[ \left( (X^{u,v})^2 \right)_{12} = X_{11}^{u,v} X_{12}^{u,v} + X_{12}^{u,v} X_{22}^{u,v} = 0_{\mathbb{K}P_n} \]
\[ \left( (X^{u,v})^2 \right)_{21} = X_{21}^{u,v} X_{11}^{u,v} + X_{22}^{u,v} X_{21}^{u,v} = 0_{\mathbb{K}P_n} \]
so \((X^{u,v})^2\) is a diagonal matrix and therefore \((X^{u,v})^{2k} = \left( (X^{u,v})^2 \right)^k\) is also diagonal. \(\square\)

In the following, we (incorrectly) call \(k\)th character associated to \(X^{u,v}\) the trace of the \(k\)th power of this representation.

**Proposition 3.** The characters associated to \(X^{u,v}\) are
\[
\chi \left( (X^{u,v})^k \right) = 2 \sum_{s,t \in \mathbb{Z}_n} \eta^{ksu} \cos \left( \frac{2kvt\pi}{n} \right) e_{(s,t)} \\
+ \mathbf{1}_{2\mathbb{Z}}(k) 2 \cos \left( \frac{kvt\pi}{n} \right) \sum_{r=1}^n E_{r,r+k}.
\]
Proof. By definition, we have

\[
\left( (X_{u,v})^2 \right)_{11} = (X_{11}^{u,v})^2 + X_{12}^{u,v} X_{21}^{u,v}
\]

\[
= \sum_{i,j,k,l \in \mathbb{Z}_n} \eta^{iu+ju} e_{(i,j)} \eta^{ku+lv} e_{(k,l)} + \sum_{i,j=1}^n \eta^{-iv} E_{i,i+u} \eta^{jv} E_{j,j+u}
\]

\[
= \sum_{s,t \in \mathbb{Z}_n} \eta^{2su+2tv} e_{(s,t)} + \sum_{r=1}^n \eta^r E_{r,r+2u}
\]

and by similar calculations, and by Lemma 3 we have

\[
\left( (X_{u,v})^2 \right)_{11} = \sum_{s,t \in \mathbb{Z}_n} \eta^{2su+2tv} e_{(s,t)} + \sum_{r=1}^n \eta^r E_{r,r+2u}
\]

which leads to the result for even powers.

Let \( k = 2p + 1 \), then \((X_{u,v})^{2p+1} = (X_{u,v})^{2p} X_{u,v}\), hence we obtain

\[
\left( (X_{u,v})^k \right)_{11} = \left( (X_{u,v})^{2p} \right)_{11} (X_{u,v})_{11} + \left( (X_{u,v})^{2p} \right)_{12} (X_{u,v})_{21}
\]

\[
= \left( \sum_{i,j \in \mathbb{Z}_n} \eta^{2piu+2pju} e_{(i,j)} + \sum_{i=1}^n \eta^{pu} E_{i,i+2pu} \right) \sum_{k,l \in \mathbb{Z}_n} \eta^{ku+lv} e_{(k,l)}
\]

\[
= \sum_{s,t \in \mathbb{Z}_n} \eta^{ksu+ktv} e_{(s,t)}
\]

and similarly \((X_{u,v})^{k}_{22} = \sum_{s,t \in \mathbb{Z}_n} \eta^{ksu-ktv} e_{(s,t)}\) which leads to the result for odd powers. \( \square \)

2.4. Character spaces. Let us look more precisely at relations between the characters before to study their asymptotic distributions.

2.4.1. Algebra of characters.

Proposition 4. The algebra of characters, generated by all one-dimensional representations and all the \( \chi(X_{u,v}) \), contains all \( \chi((X_{u,v})^k) \).

Proof. If \( n \) does not divide \( kv \), consider \( w \) the absolute value of \( kv \) (mod \( n \)). If \( w \) belongs to \( \{1, \ldots, \lfloor \frac{n-1}{2} \rfloor \} \), then

\[
\chi((X_{u,v})^k) = \chi(X_{ku,w}) + \mathbb{1}_{2\mathbb{Z}}(k) \cos \left( \frac{kv^2 \pi}{n} \right) (\rho^+_{ku} - \rho^-_{ku})
\]

otherwise, \( n \) is even and \( w = \frac{n}{2} \), and

\[
\chi((X_{u,v})^k) = \sigma^+_{ku} + \sigma^-_{ku} + \mathbb{1}_{2\mathbb{Z}}(k) \mathbb{1}_{2\mathbb{Z}}(v)(-1)^{\frac{kw}{2}} (\rho^+_{ku} - \rho^-_{ku})
\].
If \( kv = an \) then, for \( s \) the sign of \((-1)^{av}\),
\[
\chi\left( (X^{u,v})^k \right) = \begin{cases} 
\rho^{s} + \rho^{-s} & \text{if } k \text{ is odd} \\
2\rho^{s} & \text{if } k \text{ is even}
\end{cases}
\]
\(\square\)

Remark 6. This proposition gives us another reason to call character the \( \chi\left( (X^{u,v})^k \right) \) since, for the classical groups, every linear combination \( \chi \), with coefficients in \( \mathbb{Z} \), of characters such that \( \chi(e) > 0 \) is again a character. Let us note once again that this is not true in general for quantum groups.

Proposition 5. The algebra of characters is commutative if \( n \) is odd.

Proof. We have for \( s \neq s' \) or \( t \neq t' \), \( e(s,t)e(s',t') = 0_{\mathbb{CP}_n} \), \( e^2(s,t) = e(s,t) \) and \( e(s,t)E_{i,j}e(s,t) = 0_{\mathbb{CP}_n} \). On the other hand, we also have
\[
\left( \sum_{m=1}^{n} E_{m,m+a} \right) \left( \sum_{\mu=1}^{n} E_{\mu,\mu+b} \right) = \sum_{r=1}^{n} E_{r,r+a+b} = \left( \sum_{\mu=1}^{n} E_{\mu,\mu+b} \right) \left( \sum_{m=1}^{n} E_{m,m+a} \right)
\]
which leads to the commutativity of the algebra. \(\square\)

Remark 7. If \( n \) is even, the subalgebra generated by the \( \rho^l \)'s and the \( \chi(X^{u,v}) \) is also commutative. But the \( \sigma^l \)'s do not commute with the \( \rho^l \)'s, since
\[
\left( \sum_{m=1}^{n} (-1)^m E_{m,m+a} \right) \left( \sum_{\mu=1}^{n} E_{\mu,\mu+b} \right) = \sum_{r=1}^{n} (-1)^r E_{r,r+a+b} \\
\neq \left( \sum_{\mu=1}^{n} E_{\mu,\mu+b} \right) \left( \sum_{m=1}^{n} (-1)^m E_{m,m+a} \right) = \sum_{r=1}^{n} (-1)^r E_{r,r+b+a}
\]
when \( b \) is odd.

2.4.2. A commutative (sub)algebra. For all \( n \), the algebra generated by all the \( \chi(X^{u,v}) \)'s and all the \( \rho^l \)'s is a classical commutative algebra. Thus, by the spectral theorem, it is \( \ast \)-isomorphic to a subalgebra of some \( L^{\infty}(\Omega_n, \mu_n) \), for a compact space \( \Omega_n \) and a probability distribution \( \mu_n \). It means that we can see these characters as classical random variables on the classical probability space \( (\Omega_n, \mu_n) \).

By the Gelfand-Naimark Theorem, \( \Omega_n \) is the spectrum of the algebra, it means the set of all its characters, which are all the non zero \( \ast \)-multiplicative linear forms. To determine this space and the measure \( \mu_n \), we need to investigate deeper the structure of the \( \ast \)-algebra
\[
\mathcal{C}_n = \ast-\text{alg} \left\{ \rho^l, \chi(X^{u,v}) \right\}, 0 \leq l, u \leq n - 1, 1 \leq v \leq \left\lfloor \frac{n-1}{2} \right\rfloor
\]
Lemma 4. For all integers \( k \) and \( l \) between 0 and \( n - 1 \), \( \rho_k^+ \rho_l^+ = \rho_{k+l}^+ \), \( \rho_k^- \rho_l^- = \rho_{k+l}^- \) and \( \rho_k^+ \rho_l^- = \rho_k^- \rho_l^+ \), where the sum \( k + l \) is taken in \( \mathbb{Z}_n \).

We also have that, for all integer \( v \) between 1 and \( \lfloor \frac{n-1}{2} \rfloor \), and for all integer \( n \) between 0 and \( n - 1 \), there exist integers \( a_i \in \mathbb{Z} \) such that

\[
\chi \left( x^{u,v} \right) = \rho_u^+ \left( \chi \left( x^{0,1} \right) \right)^v + a_{v-2} \left( \chi \left( x^{0,1} \right) \right)^{v-2} + \ldots + a_1 \chi \left( x^{0,1} \right)
\]

if \( v \) is odd, or, if \( v \) is even

\[
\chi \left( x^{u,v} \right) = \rho_u^+ \left( \chi \left( x^{0,1} \right) \right)^v + a_{v-2} \left( \chi \left( x^{0,1} \right) \right)^{v-2} + \ldots + a_0 \left( \rho_0^+ + \rho_0^- \right)
\]

Proof. The first assertion follows from the definition of the componentwise multiplication.

For the second part of the lemma, by the same way, we easily see that \( \rho_u^+ \chi \left( x^{0,v} \right) = \chi \left( x^{u,v} \right) \). Therefore, we only need to consider the \( \chi \left( x^{0,v} \right) \).

Let us note that the Tchebychev polynomials \( T_n \) satisfy, for all real \( \theta \), \( T_n(\cos(\theta)) = \cos(n\theta) \), so we have

\[
\chi \left( x^{0,v} \right) = 2 \sum_{s,t \in \mathbb{Z}_n} \cos \left( \frac{2\pi tv}{n} \right) e_{(s,t)}
\]

\[
= 2 \sum_{s,t \in \mathbb{Z}_n} T_v \left( \cos \left( \frac{2\pi t}{n} \right) \right) e_{(s,t)}
\]

\[
= 2 \tilde{T}_v \left( \chi \left( x^{0,1} \right) \right)
\]

thanks to the componentwise multiplication, where the constant term in \( \tilde{T}_v \left( \chi \left( x^{0,1} \right) \right) \) is 0 or \( \left( \rho_0^+ + \rho_0^- \right) = \sum_{s,t \in \mathbb{Z}_n} e_{(s,t)} \) but not \( 1_{\mathbb{KP}_n} \), what we should have if we substitute \( \chi \left( x^{0,1} \right) \) in \( T_v(x) \).

Moreover, the Tchebychev polynomial \( T_v \) has degree \( v \) with leading coefficient \( 2^{v-1} \), and all its coefficients are integers. It is symmetric if \( v \) is even, or antisymmetric if \( v \) is odd.

This Lemma means that

\[
C_n = \ast{-alg} \left\{ \rho_1^+, \rho_1^-, \chi \left( x^{0,1} \right) \right\}
\]

Remark 8. Since, when \( n \) is odd, we have \( \left( \rho_1^- \right)^{n+1} = \rho_1^+ \), the corresponding \( C_n \) is generated (as an algebra) by \( \{ \rho_1^-, \chi \left( x^{0,1} \right) \} \).

Thus, by the properties of the characters, they are defined by their values on \( \rho_1^+, \rho_1^- \) and \( \chi \left( x^{0,1} \right) \). Moreover, \( \sigma(a) = \{ \omega(a), \omega \in \Omega \} \), so \( \Omega_n \) is fixed by the spectra of the three elements and some relations. Let us note that \( C_n \) is a subalgebra of \( \mathbb{KP}_n \) containing the unit. Hence, the spectrum with respect to \( C_n \) is the spectrum with respect to \( \mathbb{KP}_n \). Direct calculations show that
Lemma 5. 

\[ \sigma(\chi(X^0,1)) = \begin{cases} 2 \cos\left(\frac{2t\pi}{n}\right), & t \in \mathbb{Z}_n \end{cases} \cup \{0\} \]

\[ \sigma(\rho_1^+) = \{\eta^s, s \in \mathbb{Z}_n\} \]

\[ \sigma(\rho_1^-) = \{\eta^s, s \in \mathbb{Z}_n\} \cup \{-\eta^s, s \in \mathbb{Z}_n\} \]

To determine \( \Omega_n \), let us note that 

\[ (\rho_1^+)^2 = (\rho_1^-)^2 = \rho_1^2 \]

and 

\[ \rho_1^+ \chi(X^0,1) = \rho_1^- \chi(X^0,1) = \chi(X^{1,1}) \]

which leads to the relations 

\[ \forall \omega \in \Omega_n, \ \omega(\rho_1^+) = \pm \omega(\rho_1^-) \]

\[ \forall \omega \in \Omega_n, \ \omega(\rho_1^+) = -\omega(\rho_1^-) \Rightarrow \omega(\chi(X^{1,1})) = 0 \] .

Finally, we get the following result.

Theorem 5. For all \( n \), \( C_n \), equipped with the Haar state, can be viewed as an algebra of random variables on the probability space 

\[ \Omega_n = \{\eta^s, s \in \mathbb{Z}_n\} \times \{1\} \times \left( \begin{cases} 2 \cos\left(\frac{2t\pi}{n}\right), & t \in \mathbb{Z}_n \end{cases} \cup \{0\} \right) \]

\[ \cup \{-\eta^s, s \in \mathbb{Z}_n\} \times \{-1\} \times \{0\} \]

endowed with the measure 

\[ \mu_n = \left( 1_{4\mathbb{Z}}(n) \right) \frac{2n(p+1) + 1 - 1_{4\mathbb{Z}}(n)}{2n(p+2)} \sum_{\omega=(a,b,c)\in\Omega_n} \delta_{\omega} + \frac{1}{2n} \sum_{\omega=(a,b,c)\in\Omega_n} \delta_{\omega} \]

where \( p = \left\lfloor \frac{n}{2} \right\rfloor \), thanks to the Gelfand transform, 

\[ \mathcal{F}: C_n \to L^\infty(\Omega_n, \mu_n) \]

\[ x \mapsto (\hat{x}: \omega \mapsto \omega(x)) \]

given by the following formula 

\[ \hat{\rho}_1^+(a,b,c) = a, \ \hat{\rho}_1^+(a,b,c) = ab, \ \hat{\chi(X^{0,1})}(a,b,c) = c \] .

2.5. Asymptotic laws.

Definition 3. We say that a complex random variable \( Z \) is \( C\text{-arcsine}(\alpha) \) distributed, if it admits 

\[ z \mapsto \frac{1}{\pi \sqrt{\alpha^2 - |z|^2}} \]

as density function. Let us denote by \( \mu_{C\text{-arc}(\alpha)} \) the corresponding distribution.

By direct calculations, we can prove the following result.

Lemma 6. If \( Z \) is a \( C\text{-arcsine}(2) \) random variable, for all \( k \) and \( l \), we have 

\[ \mathbb{E}[Z^k \bar{Z}^l] = \begin{cases} \binom{2k}{k} & \text{if } k = l \\
0 & \text{otherwise} \end{cases} \] .
Theorem 6. For all $u,v \geq 1$, $\chi(\mathcal{X}^{u,v})$ is asymptotically (when $n \to +\infty$) a $\frac{1}{2}\delta_0 + \frac{1}{2}\mu_{\text{arc}(2)}$-distributed random variable, and $\chi(\mathcal{X}^{0,v})$ admits asymptotically (when $n \to +\infty$) the $\star$-distribution $\frac{1}{2}\delta_0 + \frac{1}{2}\mu_{\text{arc}(-2,2)}$, where $\mu_{\text{arc}(-2,2)}$ represents the classical arcsine distribution on the open interval $(-2,2)$.

Moreover, the same holds for all $\chi\left((\mathcal{X}^{u,v})^k\right)$, with $k > 1$ odd.

Proof. We only do the computation for $\chi(\mathcal{X}^{u,v})$. The case $\chi\left((\mathcal{X}^{u,v})^k\right)$, with $k > 1$ odd, is similar.

Let us remember that, by Proposition 3,

$$\chi(\mathcal{X}^{u,v}) = 2 \sum_{s,t \in \mathbb{Z}_n} \eta^{su} \cos\left(\frac{2tv\pi}{n}\right) e_{(s,t)}.$$ 

Therefore, $\chi(\mathcal{X}^{u,v})$ commutes with $\chi(\mathcal{X}^{u,v})^\star$, and for all $m \geq 1$

$$(\chi(\mathcal{X}^{u,v}))^m = 2^m \sum_{s,t \in \mathbb{Z}_n} \eta^{msu} \cos\left(\frac{2tv\pi}{n}\right)^m e_{(s,t)}$$

so we have

$$\int_{\mathbb{K}_n} (\chi(\mathcal{X}^{u,v}))^{r_1} (\chi(\mathcal{X}^{u,v}))^{r_\star} = \frac{2^{r_1+r_\star}}{2n^2} \sum_{s,t \in \mathbb{Z}_n} \eta^{(r_1-r_\star)su} \cos\left(\frac{2tv\pi}{n}\right)^{r_1+r_\star}$$

$$= \frac{1}{2n^2} \left(\sum_{s \in \mathbb{Z}_n} \eta^{(r_1-r_\star)su}\right) \left(\sum_{l=0}^{r_1+r_\star} \left(\frac{r_1+r_\star}{l}\right) \sum_{t \in \mathbb{Z}_n} \eta^{(2l-(r_1+r_\star))tv}\right)$$

$$= \frac{1}{2} \mathbb{1}_{n\mathbb{Z}}((r_1 - r_\star)u) \sum_{l=0}^{r_1+r_\star} \left(\frac{r_1+r_\star}{l}\right) \mathbb{1}_{n\mathbb{Z}}((2l - (r_1 + r_\star))v).$$

For all $u \geq 1$, $r_1 \neq r_\star$, $n \geq u(r_1 - r_\star) + 1$, we have $0 < u(r_1 - r_\star) < n$, so

$$\int_{\mathbb{K}_n} (\chi(\mathcal{X}^{u,v}))^{r_1} (\chi(\mathcal{X}^{u,v}))^{r_\star} = 0.$$ 

Otherwise, if $r_1 = r_\star = r \geq 1$, for $n$ great enough ($n \geq 2(r_\nu + 1)$), we obtain

$$\int_{\mathbb{K}_n} (\chi(\mathcal{X}^{u,v}))^{r} (\chi(\mathcal{X}^{u,v}))^{r} = \frac{1}{2} \left(\frac{2r}{r}\right).$$

Hence, if $u \neq 0$, $\chi(\mathcal{X}^{u,v})$ is asymptotically a $\frac{1}{2}\delta_0 + \frac{1}{2}\mu_{\text{arc}(2)}$-distributed random variable.

If $u = 0$, let us note that $\chi(\mathcal{X}^{0,v})$ is selfadjoint, and

$$\int_{\mathbb{K}_n} (\chi(\mathcal{X}^{u,v}))^m = \frac{1}{2} \sum_{l=0}^{m} \binom{m}{l} \mathbb{1}_{n\mathbb{Z}}((2l - m)v).$$
Hence, for $n$ great enough, if $m$ is odd, the moment vanishes, and if $m = 2p$, it is $\frac{1}{2}\binom{m}{p}$, which corresponds to the distribution $\frac{1}{2}\delta_0 + \frac{1}{2}\mu_{arc(-2,2)}$. □

**Theorem 7.** For all integers $u, v \geq 1$ and any even $k$, $\chi\left((X_{u,v})^k\right)$ is asymptotically (when $n \to +\infty$) a $\frac{1}{2}U(2 \mathbb{T}) + \frac{1}{2}\mu_{arc(2)}$-distributed random variable, and $\chi\left((X_{0,v})^k\right)$ admits asymptotically (when $n \to +\infty$) $\frac{1}{2}\delta_1 + \frac{1}{2}\mu_{arc(-2,2)}$ as $\ast$-distribution.

**Proof.** The only difference with the characters in the preceding theorem is the part "$+2 \cos\left(\frac{kv\pi}{n}\right) \sum_{r=1}^{n} E_{r,r+ku}\" in the even characters.

By the properties of the multiplication in $\mathbb{KP}_n$ we only need to study it, and show that its moments are asymptotically one half of those for a random variable uniformly distributed on $2\mathbb{T}$.

Let us denote this normal matrix by $M_{k,u,v,n}$. Then, for all $m \geq 1$

$$
\begin{align*}
M_{k,u,v,n}^m &= 2^m \cos\left(\frac{kv\pi}{n}\right) \sum_{r=1}^{n} E_{r,r+mk} \\
(M_{k,u,v,n}^*)^m &= 2^m \cos\left(\frac{kv\pi}{n}\right) \sum_{r=1}^{n} E_{r,r-mk}.
\end{align*}
$$

Hence, we have

$$
\int_{\mathbb{KP}_n} M_{k,u,v,n}^\alpha (M_{k,u,v,n}^*)^\beta = \frac{1}{2n^2} \cos\left(\frac{kv\pi}{n}\right) \int_{\mathbb{KP}_n} \sum_{r=1}^{n} E_{r,r+ku(\alpha-\beta)} = \frac{2^{\alpha+\beta}}{2n} \cos\left(\frac{kv\pi}{n}\right) \sum_{r=1}^{n} \mathbb{1}_{n\mathbb{Z}}(ku(\alpha - \beta)).
$$

For $u = 0$, we get $\frac{1}{2} \left(2 \cos\left(\frac{kv\pi}{n}\right)\right)^{\alpha+\beta}$ which goes to $2^{\alpha+\beta}$.

For $u \geq 1$ and $n$ great enough, this moment goes to one half of the moment of $U(2 \mathbb{T})$: zero if $\alpha$ is different from $\beta$, $2^{2\alpha}$ otherwise, which completes the proof of the theorem. □

**Remark 9.** Let us note that $\sum_{r=1}^{n} E_{r,r+ku}$ is a permutation matrix, whose eigenvalues are the $n$th roots of unity. Therefore the eigenvalues of the matrix $2 \cos\left(\frac{kv\pi}{n}\right) \sum_{r=1}^{n} E_{r,r+ku}$ are the $n$th roots of unity multiplied by $2 \cos\left(\frac{kv\pi}{n}\right)$.

So when $n$ goes to infinity, the eigenvalues come out uniformly on $2\mathbb{T}$ if $u$ is not zero, which corresponds to the result of the Theorem [\(\square\).
Remark 10. If we let \( n \) go to \( \infty \) in Theorem\(^5\) we see that, \((\Omega_n, \mu_n)\) converges to \((\Omega, \mu)\) where

\[
\Omega = \mathbb{T} \times \{1\} \times [-2; 2] \sqcup \mathbb{T} \times \{-1\} \times \{0\}, \quad \mu = \frac{1}{2} \mu_{\text{C-arc}(2)} + \frac{1}{2} \mathcal{U}(\mathbb{T}).
\]

And, we can check easily that

\[
\hat{\rho}_1^{e}, \hat{\rho}_1^{f} \sim \mathcal{U}(\mathbb{T}) \quad \text{and} \quad \chi(\overline{X}^{0,1}) \sim \frac{1}{2} \mu_{\text{arc}(-2,2)} + \frac{1}{2} \delta_0.
\]

Remark 11. The same type of computations for the dihedral group \( D_{2n} \) gives similar results. The dihedral group \( D_{2n} \) admits indeed \( \lfloor \frac{n-1}{2} \rfloor \) two-dimensional non equivalent unitary irreducible representations \( \sigma_k \) given by

\[
s \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad t \mapsto \begin{pmatrix} \eta^k & 0 \\ 0 & \eta^{-k} \end{pmatrix}
\]

where \( s \) and \( t \) generate \( D_{2n} \) with \( s \) of order 2 and \( t \) of order \( n \). Let us denote by \( \chi_{k,l} \) the class function \( \text{Tr}(\sigma_k(\cdot)^l) \).

Then by the moments method, the asymptotic law of \( \chi_{k,l} \) is

\[
\begin{cases} 
\frac{1}{2} \left( \mu_{\text{arc}(1,1)} + \delta_2 \right) & \text{if } l \text{ is even and positive}, \\
\frac{1}{2} \left( \mu_{\text{arc}(1,1)} + \delta_0 \right) & \text{if } l \text{ is odd or } l = 0.
\end{cases}
\]

2.6. Asymptotic pairwise independence. Since we only consider \( \chi(X^{a,b}) \) or its adjoint, we work in a commutative setting, and we can use classical cumulants \( \kappa \). Let \( b, d, k, l \), be natural integers, \( a \) and \( c \) be non negative integers and \( e, f \) be in \( \{1, *\} \). Then direct calculation leads to

\[
\lim_{n \to +\infty} \kappa\left( \chi\left(\left(X^{a,b}\right)^{e}\right), \chi\left(\left(X^{c,d}\right)^{f}\right)\right) = \delta_{k,a,l} \delta_{e,f} \delta_{\{1,c\}, \{1,c\}} \left( \delta_{k,b,l} + 2 \mathbb{1}_{(2\mathbb{Z})^2}(k,l) \right)
\]

which proves

Proposition 6. Let \( a, b, c, d, e, f, k \) and \( l \) be as above. The followings are equivalent

1. \( \chi\left(\left(X^{a,b}\right)^{e}\right) \) and \( \chi\left(\left(X^{c,d}\right)^{f}\right) \) are asymptotically independent
2. at least one of the following conditions holds
   - \( ka \neq lc \)
   - \( e = f \)
   - \( k \) or \( l \) is odd, and \( kb \neq ld \)

2.7. Dual groups. We can define a notion of duality for a finite quantum group \( G = (C(G), \Delta) \). This dual, denoted \( \hat{G} = (C(\hat{G}), \hat{\Delta}) \), is again a finite quantum group. The algebra \( C(\hat{G}) \) is the set of all linear forms defined on \( C(G) \). It is also isomorphic to the direct sum of the non-equivalent unitary irreducible corepresentations of \( C(G) \), thanks to the Fourier transform. All the structures are defined by duality.
In particular, for the case of the Sekine family, let us denote by $e^{(i,j)}$ and $E^{i,j}$ the elements of the dual basis. Then, we have the coproduct, dual of the multiplication on $A_n$, given by

$$\hat{\Delta}(e^{(i,j)}) = e^{(i,j)} \otimes e^{(i,j)} \quad \text{and} \quad \hat{\Delta}(E^{i,j}) = \sum_{k=1}^{n} E^{i,k} \otimes E^{k,j}$$

and the Haar state, dual of the counit on $A_n$,

$$\int_{K \mathbb{P}^1_n} \left( \sum_{i,j \in \mathbb{Z}_n} x_{(i,j)} e^{(i,j)} + \sum_{1 \leq i,j \leq n} X_{i,j} E^{i,j} \right) = x_{(0,0)}.$$ 

We clearly obtain that the unitary irreducible representations of $\hat{K \mathbb{P}^1}_n$ are the $n^2$ $e^{(i,j)}$'s and the $n$-dimensional representation $\hat{X}$, given by its matrix-elements $\hat{X}_{i,j} = E^{i,j}$. Let us note that $\hat{X}$ is a fundamental representation of $\hat{K \mathbb{P}^1}_n$. Moreover, these representations are non-equivalent, so there is no other non-equivalent unitary irreducible representation.

The characters, $e^{(i,j)}$, $i,j \in \mathbb{Z}_n$, and $\chi(\hat{X}) = \sum_{i=1}^{n} E^{i,i}$, generate a commutative algebra, for the product in $C(\hat{K \mathbb{P}^1}_n)$. This central algebra is the linear span of these characters and may not contain all the traces of powers of $\hat{X}$, given for all positive integer $k$ by

$$\chi(\hat{X}^k) = \begin{cases} 
\sum_{s,t \in \mathbb{Z}_n, s \equiv t \mod n} E^{s,s+t} & \text{if } k \text{ is odd} \\
\sum_{s,t \in \mathbb{Z}_n, s \equiv t \mod n} \eta^{-stE(s,t/2)} & \text{if } k \text{ is even}
\end{cases}.$$ 

Note that $\chi(\hat{X}^k)$ equals $\chi(\hat{X})$ when $k$ is odd and $gcd(k,n) = 1$. If $gcd(k,n)$ is greater than 1 for some odd $k$, then $\chi(\hat{X}^k)$ is not in the character space.

However, these traces are all self-adjoint, and commute. Using the corepresentations basis, we can get that, for all positive integers $k_1, k_2, \ldots, k_r$

$$\int_{K \mathbb{P}^1_n} \chi(\hat{X}^{k_1}) \cdots \chi(\hat{X}^{k_r}) = n^{r-1} \frac{1}{2} \sum_{i=1}^{n} \frac{1}{2} \sum_{s \in \mathbb{Z}_n, \sum_{i=1}^{r} k_i t \mod n} \# \{ t \in \mathbb{Z}_n, \forall 1 \leq i \leq r, n \mid k_it \}.$$ 

In particular, by the moments method, the normalized character $\frac{1}{n} \chi(\hat{X})$ admits the $\ast$-distribution $\frac{n}{2n} (\delta_{-1} + \delta_1) + (1 - \frac{1}{n^2}) \delta_0$. Moreover, let us note that $\# \{ t \in \mathbb{Z}_n, \forall 1 \leq i \leq r, n \mid k_it \} = gcd(n, k_1, \ldots, k_r)$. Thus, for all positive integer $k$, the normalized trace $\frac{1}{n} \chi(\hat{X}^k)$ admits the $\ast$-distribution $\frac{gcd(k,n)}{2n} (\delta_{-1} + \delta_1) + (1 - \frac{gcd(k,n)}{n}) \delta_0$ if $k$ is odd, and $\frac{gcd(k,n)}{n} \delta_1 + (1 - \frac{gcd(k,n)}{n^2}) \delta_0$ if $k$ is even.

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