Restless Bandits with Constrained Arms: Applications in Social and Information Networks

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Abstract

We study a problem of information gathering in a social network with dynamically available sources and time varying quality of information. We formulate this problem as a restless multi-armed bandit (RMAB). In this problem, information quality of a source corresponds to the state of an arm in RMAB. The decision making agent does not know the quality of information from sources a priori. But the agent maintains a belief about the quality of information from each source. This is a problem of RMAB with partially observable states. The objective of the agent is to gather relevant information efficiently from sources by contacting them. We formulate this as an infinite horizon discounted reward problem, where reward depends on quality of information. We study Whittle’s index policy which determines the sequence of play of arms that maximizes long term cumulative reward. We illustrate the performance of index policy, myopic policy and compare with uniform random policy through numerical simulation.

I. INTRODUCTION

Suppose there is an agent in a social or information network. The agent has connections to $N$ neighbors which are its information sources. The agent needs information for its use and it gathers this information through its sources at regular intervals. In each interval, the agent can contact only $M < N$ of its sources for information. The information provided by a source may be either relevant (1) or non-relevant (0) to the agent. So, there are two states, say $\{0, 1\}$, which corresponds to the information quality. However, the agent does not know a priori whether a certain source has relevant information. Relevance of the information received becomes apparent to the agent at end of the interval after processing it. The assumption that information quality is binary comes from the following consideration. Agents who want to form informed opinion on a given matter will first decide to access relevant material from a source which they believe to be accurate. Also, in some scenarios relevant and irrelevant states may be interpreted as truth or falsehood. The agent however knows that the information quality of a source varies in a Markovian manner and also knows its Markov matrix. Assume that the sources are self interested entities. Their current relevance/truthfulness depends on their previous state. The reward from relevant information is high and non-relevant information is low.

Further, in a given interval each source may or may not be available. However, an unavailable source may be leveraged through an additional cost. Hence the immediate reward that such information gives may be lower. This is a situation where the choice of sources might affect their future availability and information quality of the sources. The agent here needs a policy to choose which sources it must contact in each interval along the time line so that its cumulative reward is maximized.

A dynamic information sourcing problem such as above is sequential decision problem where a current decision impacts future rewards. Such sequential decision problems can be modeled using restless multi-armed bandits (RMAB) (see [1], [2]). An RMAB is an agent with $N$ arms and each arm can be in one of finite states. The states of arms evolve along Markov chains whose transition probabilities are known to the agent. The agent want to find a policy to make these arm choices in each slot to maximize its long term cumulative reward.

We now briefly review some related work. Information gathering in social networks has been a topic of interest in context of business and management decisions. [3], [4] study the impact of parameters such as perception, cost, timely access, etc., on choice of sources for information seeking in a large organization. An empirical study on journalists’ use of social media for sourcing information has been conducted by [5]. Further, automated monitoring of social media to build geo-spatial awareness during disasters is attempted in [6]. In general, in information intensive applications such as governance, public relations, journalism etc., both manual and automated information gathering happen in a sequential manner. Sequential decision making models in information networks have been studied under different scenarios in [7], [8].

In seminal paper [2], author introduced RMAB problem and proposed a heuristic index based policy; such index policy is now referred to as Whittle index policy. RMAB assumes that the model of system state variations is known. The Whittle index based policies also studied for opportunistic communication systems in [9], [10], where authors studied partially observable model. The Whittle index policies are popular due to they are asymptotically optimal. In [9], index policy is shown to be

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optimal for certain model parameter. In [10], a hidden Markov RMAB is studied which generalizes the work of [9]. The myopic policy was also investigated for restless bandits in [11] and it is shown to be optimal under certain model assumptions.

All the above models of RMAB assume that each arm is always available to play, and the decision in each slot is whether to play or not play an arm. However, availability of arms can be dynamic. Multi-armed bandit problems with dynamic availability constraints have been studied for machine-repair problem in [12], where, if the machine breaks down, then it will be available in next time slot with some probability after getting repaired. This model assumes that the state is observable and the authors analyze index-type policy for rested bandits.

In this paper, we formulate the problem of an agent gathering information from neighbors in a social network as RMAB with constrained arms (each arm being a dynamically available source). We further study Whittle index policy and myopic policy. We show that, for a single-armed bandit with constrained arm, the optimal policy is of a threshold-type, and the arm is indexable. We next devise the algorithm to compute Whittle index for each arm and this algorithm is based on two timescales stochastic approximations scheme. We finally illustrate performance of this scheme via numerical example.

We next describe the system model.

II. Preliminaries and Model Description

There is an agent in a social or information network. The agent has connections to $N$ neighbors which are its information sources. The agent can contact only ($M < N$) neighbors for information. The system is assumed to be time slotted and it is indexed by $t$. The quality of information available at the source represented by a Markov chain with state space $\{0, 1\}$. Let $X_n(t)$ denote the state corresponding to information quality of source $n$ at beginning of time slot $t$, $X_n(t) \in \{0, 1\}$. We suppose that each source has dynamic availability i.e. in a given slot it may or may not be available. When a source is not available, it may be leveraged to provide information by incurring an additional cost. Let $Y_n(t) \in \{0, 1\}$ represent the availability of the source $n$ in time slot $t$. Since the agent contacts $M$ sources out of $N$ in each time slot to gather information, we define $A_n(t) \in \{0, 1\}$ as the action in slot $t$, where $A_n(t) = 1$ if source $n$ is contacted in slot $t$, and $A_n(t) = 0$ otherwise.

We can have $A_n(t) = 1$ in both available and unavailable scenarios. The state of arm $n$, i.e., $X_n(t)$ changes at the end of time slot $t$ according to transition probabilities that depend on $A_n(t), Y_n(t)$ and it is defined as follows.

$$P^n_{ij}(y, a) := \Pr\{X_n(t + 1) = j \mid X_n(t) = i, Y_n(t) = y, A_n(t) = a\}.$$ 

If source $n$ is contacted in slot $t$, then quality of information from source $n$ is known exactly at the end of slot, i.e., state of source is known exactly. Also, the agent makes a binary observation $Z^n_n(t)$ about source $n$ when that is contacted. Hence, we define $Z^n_n(t) := 1$ if information from source $n$ is relevant, and $Z^n_n(t) := 0$ otherwise. Let $\rho_n(i, y)$ be the probability of $Z^n_n(t) = 1$ given that $X_n(t) = i, Y_n(t) = y$ and $A_n(t) = 1$.

$$\rho_n(i, y) := \Pr\{Z^n_n(t) = 1 \mid X_n(t) = i, Y_n(t) = y, A_n(t) = 1\}.$$ 

We assume that $\rho_n(0, y) = 0$ and $\rho_n(1, y) = 1$ for all $y \in \{0, 1\}$. When source $n$ is not used, the agent do not know the quality of information, hence state of source $n$ is unobservable. Hence, the agent maintains a belief $\pi_n(t)$ about the state of source $n$. Here, belief is the probability that the source is in state 0 given all past availability, actions, observations and given as

$$\pi_n(t) = \Pr\{X_n(t) = 0 \mid (Y_n(s) = y_s, A_n(s), Z^n_n(s))_{s=1}^{t-1}\}.$$ 

We now define the reward as measure of the quality of information from different sources. When the agent uses source $n$, it obtains reward from the information it receives. This reward depends on current state of that source and availability of that source. Let $R^n_n(i, y)$ be the reward obtained from using source $n$ given that $X_n(t) = i, Y_n(t) = y, A_n(t) = a$, and it is as follows.

$$R^n_n(i, 1) = r_{n,i}, \quad R^n_n(i, 0) = 0_{n,i},$$ 

$$R^n_n(0, 1) = 0, \quad R^n_n(0, 0) = 0.$$ 

We further assume that $r_{n,0} = 0_{n,0} = 0$, no reward from source $n$ if it has $X_n(t) = 0$. Also, we suppose $r_{n,1} > 0_{n,1}$, for all $n$. This implies that an unavailable source may be leveraged through an additional cost. Hence, the immediate reward is lower than when source is available. However, agent knows that the availability of sources is dynamic. This dynamic availability of each source $n$ is modeled stochastically as probability of availability $\theta^n_n = \Pr\{Y_n(t + 1) = 1 \mid A_n(t) = a\}$. Thus availability of a source varies according to Bernoulli distribution with parameter $\theta^n_n$. This is known to the agent. Let $H_t$ denote the history up to time $t$.

$$H_t := (Y_n(s) = y_s, A_n(s), Z^n_n(s))_{1 \leq n \leq N; 1 \leq s < t}.$$
We can describe the state of source $n$ at time $t$ by $S_n(t) = (\pi_n(t), Y_n(t)) \in [0,1] \times \{0,1\}$. $(S_1(t), \ldots S_N(t))$ is the state information of all the sources at the beginning of time slot $t$. The expected reward from using source $n$ at time $t$ given that $Y_n(t) = y$ is

$$\tilde{R}_n^1(\pi_n(t), y) = \pi_n(t)R_n^1(0,y) + (1-\pi_n(t))R_n^1(1,y).$$

In each slot, agent uses exactly $M$ sources. Let $\phi(t)$ is the policy of agent such that $\phi(t) : H_t \rightarrow \{1, \ldots, N\}$ maps the history to $M$ sources at each slot $t$. Let

$$A_n^\phi(t) = \begin{cases} 1 & \text{if } n \in \phi(t), \\ 0 & \text{if } n \notin \phi(t), \end{cases}$$

and $\sum_{n=1}^N A_n^\phi(t) = M$.

We are now ready to define the infinite horizon discounted reward under policy $\phi$ for initial state information $(\pi, y)$, $\pi = (\pi_1(1), \ldots, \pi_N(1))$ and $y = (y_1(1), \ldots, y_N(1))$. It is given by

$$V_\phi(\pi, y) = E^\phi \left( \sum_{t=1}^\infty \beta^{t-1} \left[ \sum_{n=1}^N A_n^\phi(t)\tilde{R}_n^1(\pi_n(t), Y_n(t)) \right] \right).$$

Here, $\beta$ is discount parameter, $0 < \beta < 1$. Then

$$\phi^* = \arg \max_\phi V_\phi(\pi, y)$$

s.t. $\sum_{n=1}^N A_n^\phi(t) = M, \pi \in [0,1]^N, y \in \{0,1\}^N$.  

The optimization problem (1) is a restless multi-arm bandit problem with availability constraints. Here, each source will correspond to an arm. The state of information quality of source $n$ and its availability represent the state $S_n(t) = (\pi_n(t), Y_n(t))$ of an arm $n$. This is a generalized version of restless multi-arm bandits with partially observable states and availability constraints. This problem is known to be PSPACE-hard, [13]. In this paper we consider index based policies. In such index policies, the dimensionality of the problem is reduced by calculating the index for each arm separately. The $M$ arms with highest indices are played at each time slot. That is, the agent uses $M$ sources with highest indices. To use index policies, one requires to study relaxed version of optimization problem (1), where a subsidy $w$ is introduced for not playing arm (not using source by agent), see [1, 2]. We first analyze agent with a single-armed bandit (a single source scenario) in next section.

### III. A Single-armed Bandit Problem

For notational convenience, we will drop the subscript $n$. We use the terms arm and source interchangeably. In view of subsidy $w$, we can rewrite optimization problem (1) for a single-armed bandit as follows.

$$\phi^* = \arg \max_\phi V_\phi(\pi, y)$$

for initial belief $\pi \in [0,1]$ and availability $y \in \{0,1\}$. Here, action $A(t)$ under policy $\phi$ is

$$A^\phi(t) = \begin{cases} 1 & \text{if } \phi(t) = 1, \\ 0 & \text{if } \phi(t) = 0. \end{cases}$$

We further simplify the model and assume that $P_{\mu_0}(y,a) = \mu_0$ and $P_{\mu_1}(y,a) = \mu_1$ for $a, y \in \{0,1\}$ [1] Recall that $\pi(t) = \Pr(X(t) = 0|H_t)$ and using the Bayes rule, we update the belief $\pi(t+1)$ in following manner.

$$\pi(t+1) = \begin{cases} \mu_1 & \text{if } A(t) = 1, Y(t) = y, \text{ and } Z^\mu(t) = 1, \\ \mu_0 & \text{if } A(t) = 1, Y(t) = y, \text{ and } Z^\mu(t) = 0, \\ \Gamma(\pi(t)) & \text{if } A(t) = 0, \text{ and } Y(t) = y, \end{cases}$$

for $y \in \{0,1\}$. Here, $\Gamma(\pi(t)) = \pi(t)\mu_0 + (1-\pi(t))\mu_1$. If the agent uses the source in slot $t$, and it observes that the information is relevant, i.e., $A(t) = 1$, and $Z^\mu(t) = 1$ for any $y \in \{0,1\}$, then the state is known exactly and $X(t) = 1$, thus belief $\pi(t+1) = \mu_1$. Whereas if agent uses the source, $A(t) = 1$ but $Z^\mu(t) = 0$ then the state is known exactly and $X(t) = 0$, thus belief $\pi(t+1) = \mu_0$. If the source is not used, state is not observed but belief is updated.

1In general, Markov model for source availability and unavailability could be different.
Similarly we can show that of Chapter µ following dynamic program A. Structural Results likely provide relevant information in future also. In this, one can obtain the optimum value function by solving dynamic program. We first define the value function under initial action $A_1$ and availability $Y_1$. 

$$V_T := \text{value function under } A_1 = 1, Y_1 = 1,$$
$$\bar{V}_T := \text{value function under } A_1 = 1, Y_1 = 0,$$
$$V_{NT} := \text{value function under } A_1 = 0, Y_1 = 1,$$
$$\bar{V}_{NT} := \text{value function under } A_1 = 0, Y_1 = 0.$$

We can write the following.

$$V_T(\pi) = \rho(\pi) + \beta[(1 - \pi)\{\theta^1 V(\mu_1) + (1 - \theta^1)\bar{V}(\mu_1)\} + \pi\{\theta^1 V(\mu_0) + (1 - \theta^1)\bar{V}(\mu_0)\}]$$

$$V_{NT}(\pi) = \beta[\theta^0 V(\Gamma(\pi)) + (1 - \theta^0)\bar{V}(\Gamma(\pi))],$$

$$\bar{V}_T(\pi) = \xi(\pi) + \beta[(1 - \pi)\{\theta^1 V(\mu_1) + (1 - \theta^1)\bar{V}(\mu_1)\} + \pi\{\theta^1 V(\mu_0) + (1 - \theta^1)\bar{V}(\mu_0)\}]$$

$$\bar{V}_{NT}(\pi) = \beta[\theta^0 V(\Gamma(\pi)) + (1 - \theta^0)\bar{V}(\Gamma(\pi))].$$

Here $r(\pi) = (1 - \pi)\gamma_1, \xi(\pi) = (1 - \pi)\gamma_{1,1}$. The optimal value function $V(\pi, y)$ and $\bar{V}(\pi, y)$, is determined by solving the following dynamic program

$$V(\pi) = \max\{V_T(\pi), V_{NT}(\pi)\}; \quad \bar{V}(\pi) = \max\{\bar{V}_T(\pi), \bar{V}_{NT}(\pi)\}.$$

### A. Structural Results

We now derive structural results for value functions, convexity of value functions and a threshold type policy. We will derive all result for $\mu_0 > \mu_1$. This means that source is positively correlated; a source that provides relevant information will more likely provide relevant information in future also.

**Lemma 1:**

1) For fixed $w$, $V(\pi), V_T(\pi), V_{NT}(\pi), \bar{V}(\pi), \bar{V}_T(\pi)$ and $\bar{V}_{NT}(\pi)$ are convex functions of $\pi$.
2) For a fixed $\pi$, $V(\pi), V_T(\pi), V_{NT}(\pi), \bar{V}(\pi), \bar{V}_T(\pi)$ and $\bar{V}_{NT}(\pi)$ are non decreasing and convex in $w$.
3) For fixed subsidy $w$, $\beta$, and $\mu_0 > \mu_1$, the value functions $V(\pi), V_T(\pi)$ and $V_{NT}(\pi)$ are decreasing in $\pi$. Also, $\bar{V}(\pi), \bar{V}_T(\pi)$ and $\bar{V}_{NT}(\pi)$ are decreasing in $\pi$.
4) $(V_T(\pi) - V_{NT}(\pi))$ and $(\bar{V}_T(\pi) - \bar{V}_{NT}(\pi))$ are decreasing in $\pi$.

**Proof:** The proof of (1) and (2) is similar to the proof of Lemma 2 in [10].

3) The proof is done by induction technique. Assume that $V_n(\pi)$ and $\bar{V}_n(\pi)$ are non-increasing in $\pi$. Let $\pi' \geq \pi$ and playing the arm is optimal. Then,

$$V_{n+1}(\pi) = \rho(\pi) + \beta[(1 - \pi)\{\theta^1 V_n(\mu_1) + (1 - \theta^1)\bar{V}_n(\mu_1)\} + \pi\{\theta^1 V_n(\mu_0) + (1 - \theta^1)\bar{V}_n(\mu_0)\}]$$

Here, $\rho(\pi)$ is decreasing in $\pi$, i.e. $\rho(\pi') < \rho(\pi)$ for $\pi' > \pi$. Hence,

$$V_{n+1}(\pi) \geq \rho(\pi') + \beta[(1 - \pi)\{\theta^1 V_n(\mu_1) + (1 - \theta^1)\bar{V}_n(\mu_1)\} + \pi\{\theta^1 V_n(\mu_0) + (1 - \theta^1)\bar{V}_n(\mu_0)\}]$$

From our assumption $\mu_0 > \mu_1$, we get stochastic ordering on observation probability, i.e., $[1 - \pi, \pi]^T \leq_s [1 - \pi', \pi']^T$, and $V_n(\pi), \bar{V}(\pi)$ are decreasing in $\pi$. Then we have

$$V_{n+1}(\pi) \geq \rho(\pi') + \beta[(1 - \pi')\{\theta^1 V_n(\mu_1) + (1 - \theta^1)\bar{V}_n(\mu_1)\} + \pi'(\theta^1 V_n(\mu_0) + (1 - \theta^1)\bar{V}_n(\mu_0))]$$

$$V_{n+1}(\pi) \geq V_{n+1}(\pi').$$

Similarly we can show that $\bar{V}_{n+1}(\pi) \geq \bar{V}_{n+1}(\pi')$. This is true for every $n \geq 1$. From Chapter 7 of [14] and Proposition 2.1 of Chapter 2 of [14], $V_n(\pi) \to V(\pi)$, uniformly and similarly $\bar{V}_n(\pi) \to \bar{V}(\pi)$. Hence $V(\pi) \geq V(\pi')$ and $\bar{V}(\pi) \geq \bar{V}(\pi')$ for $\pi' \geq \pi$. 

Next we prove, $V_T(\pi)$ and $V_{NT}(\pi)$ is non increasing in $\pi$.

\[
V_T(\pi) = \rho(\pi) + \beta(1 - \rho(\pi))\{\theta V(\mu_1) + (1 - \theta)\tilde{V}(\mu_1)\} \\
V_{NT}(\pi) = w + \beta\theta V(\Gamma(\pi)) + (1 - \theta)\tilde{V}(\Gamma(\pi))
\]  
(2)

For $\pi_1 > \pi_2$,

\[
V_T(\pi_1) - V_T(\pi_2) = (\pi_1 - \pi_2)\beta\theta V(\mu_0) - V(\mu_1)) + (\pi_1 - \pi_2)\beta(1 - \theta)(\tilde{V}(\mu_0) - \tilde{V}(\mu_1))
\]

Using above result, $V_T(\pi)$ is non increasing in $\pi$. Similarly, $V_{NT}(\pi)$ is non increasing in $\pi$.

4) Let $D(\pi) = V_T(\pi) - V_{NT}(\pi)$ and $D(\pi)$ is decreasing in $\pi$, i.e $D(\pi) < D(\pi')$ for $\pi > \pi'$. We need to show

\[
V_T(\pi) - V_{NT}(\pi) < V_T(\pi') - V_{NT}(\pi').
\]

Rearranging

\[
V_T(\pi) - V_T(\pi') < V_{NT}(\pi) - V_{NT}(\pi').
\]

Now, the right hand side of the (5),

\[
V_{NT}(\pi) - V_{NT}(\pi') = \beta\theta V(\Gamma(\pi)) - V(\Gamma(\pi')) + \beta(1 - \theta)\{\tilde{V}(\Gamma(\pi)) - \tilde{V}(\Gamma(\pi'))\}
\]

The left hand side of the (5),

\[
V_T(\pi) - V_T(\pi') = (\rho(\pi) - \rho(\pi')) + \beta(\pi - \pi')\theta V(\mu_0) - V(\mu_1)) + V(\pi - \pi')\{1 - \theta\}(\tilde{V}(\mu_0) - \tilde{V}(\mu_1))
\]

Note that $\rho(\pi) - \rho(\pi') = r_1(\pi - \pi') < 0$ because $\pi > \pi'$. Also, from the above expressions of difference in value functions, we can easily see that that for $\theta^0 = \theta^1$, Eqn (5) is true.

Even for $\theta^0 \neq \theta^1$, Eqn (5) is true because $\rho(\pi) - \rho(\pi') < 0$ and $0 < \pi - \pi' < 1$. Hence, $V_T(\pi) - V_T(\pi') < V_{NT}(\pi) - V_{NT}(\pi')$. Similar steps follow for $(\tilde{V}_T(\pi) - \tilde{V}_{NT}(\pi))$.

\[
\Box
\]

**Remark 1:** The above proofs have been done assuming that the availability probability is independent of state and action. However, a similar argument can be made for the dependent case $\theta^0(\pi, y)$ by imposing following conditions. $\theta^0(\pi, 1) > \theta^1(\pi, 0)$, and $\theta^0(\pi, y) > \theta^1(\pi', y)$, for $\pi' > \pi$.

We now define the threshold type policy and later we prove that the optimal policy is threshold type.

**Definition 1:** (Threshold type policy) A policy is said to be threshold type if one of the following is true.

1) The optimal action is to play the arm for all $\pi$.
2) The optimal action is to not play the arm for all $\pi$.
3) There exists a threshold $\pi^*$ such that for all $\pi \leq \pi^*$ the optimal action is to play the arm and not to play otherwise.

**Theorem 1:** For fixed $w$ and $\beta$, 

1) The optimal policy is threshold type when arm is available, i.e., $\exists \pi^* \in [0, 1]$ such that, $V_T(\pi) \geq V_{NT}(\pi)$ for $\pi \leq \pi_{th}$ and $V_T(\pi) < V_{NT}(\pi)$ for $\pi > \pi_{th}$.
2) The optimal policy is threshold type when arm is unavailable, i.e., $\exists \bar{\pi} \in [0, 1]$ such that, $\tilde{V}_T(\pi) \geq \tilde{V}_{NT}(\pi)$ for $\pi \leq \bar{\pi}_{th}$ and $\tilde{V}_T(\pi) < \tilde{V}_{NT}(\pi)$ for $\pi > \bar{\pi}_{th}$.

**Proof:** From Lemma 1, the value functions $V(\pi)$, $\tilde{V}(\pi)$ are convex in $\pi$. Further, from Lemma 5 and Lemma 6 we know that $V_T(\pi) - V_{NT}(\pi)$ and $\tilde{V}_T(\pi) - \tilde{V}_{NT}(\pi)$ is decreasing with $\pi$. This implies that there exists $\pi_{th}$, $\bar{\pi}_{th} \in [0, 1]$ such that following is true

1) Either $V(\pi) = V_T(\pi)$ for all $\pi \in [0, 1]$ or $V(\pi) = V_{NT}(\pi)$ for all $\pi \in [0, 1]$ or

\[
V(\pi) = \begin{cases} 
V_T(\pi) & \text{for } \pi \leq \pi_{th}, \\
V_{NT}(\pi) & \text{for } \pi > \pi_{th}.
\end{cases}
\]

2) Either $\tilde{V}(\pi) = \tilde{V}_T(\pi)$ for all $\pi \in [0, 1]$ or $\tilde{V}(\pi) = \tilde{V}_{NT}(\pi)$ for all $\pi \in [0, 1]$ or there exists $\bar{\pi}_{th}$ such that

\[
\tilde{V}(\pi) = \begin{cases} 
\tilde{V}_T(\pi) & \text{for } \pi \leq \bar{\pi}_{th}, \\
\tilde{V}_{NT}(\pi) & \text{for } \pi > \bar{\pi}_{th}.
\end{cases}
\]

Thus, the claims follows

\[
\Box
\]
B. Indexability and Whittle index computation

Recall that our interest is to seek the index type policy. We use the threshold policy result to show indexability and later provide an index computation algorithm.

We now define indexability and index, it is motivated from \([2]\), \([12]\). Let \(G(w)\) be the subset of state vector \(S\) in which it is optimal to not play the arm with subsidy \(w\), it is given as follows.

\[
G(w) := \{ (\pi, y) \in [0, 1] \times \{0, 1\} : V_T(\pi, w) \leq V_{NT}(\pi, w), \tilde{V}_T(\pi, w) \leq \tilde{V}_{NT}(\pi, w) \}. \tag{6}
\]

For clarity, we have explicitly mentioned dependence of value function on \(w\). Using set \(G(w)\), indexability and index are defined as follows.

**Definition 2:** An arm is indexable if \(G(w)\) is increasing in subsidy \(w\), i.e.,

\[
w_2 \leq w_1 \Rightarrow G(w_2) \subseteq G(w_1).
\]

**Definition 3:** The index of an indexable arm is defined as

\[
w(\pi, y) := \inf \{ w \in \mathbb{R} : (\pi, y) \in G(w), \forall (\pi, y) \in S \}. \tag{7}
\]

**Remark 2:**

1) Note that we can rewrite the definition of set \(G(w)\) in following way.

\[
G(w) = \{ [\pi_{th}, 1] \times \{1\}, [\bar{\pi}_{th}, 1] \times \{0\} \},
\]

where \(\pi_{th} := \min \{ \pi \in [0, 1] : V_T(\pi, w) \leq V_{NT}(\pi, w) \}\) and \(\bar{\pi}_{th} := \min \{ \pi \in [0, 1] : \tilde{V}_T(\pi, w) \leq \tilde{V}_{NT}(\pi, w) \}\). If the optimal policy is of threshold type, then \(\pi_{th}\) and \(\bar{\pi}_{th}\) are singleton.

2) Here, the definition of indexability and index is motivated from work of \([2]\) on restless bandits. In standard restless bandits, arms are assumed to be always available and \(y = 0\) is not feasible option.

3) When \(\theta^2 = 0\) or \(\theta^2 = 1\) for all \(a \in \{0, 1\}\), our definitions of indexability and index are still valid.

To claim indexability, we will require to show that \(\pi_{th}(w)\) and \(\bar{\pi}_{th}(w)\) are non-increasing in \(w\). Now, we use the following lemma from \([10]\).

**Lemma 2:** If

\[
\frac{\partial V_T(\pi, w)}{\partial w} \bigg|_{\pi = \pi_{th}(w)} < \frac{\partial V_{NT}(\pi, w)}{\partial w} \bigg|_{\pi = \pi_{th}(w)},
\]

\[
\frac{\partial \tilde{V}_T(\pi, w)}{\partial w} \bigg|_{\pi = \bar{\pi}_{th}(w)} < \frac{\partial \tilde{V}_{NT}(\pi, w)}{\partial w} \bigg|_{\pi = \bar{\pi}_{th}(w)},
\]

then \(\pi_{th}(w)\) and \(\bar{\pi}_{th}(w)\) are monotonically decreasing functions of \(w\).

Now, using Lemma 2 and Definition 2 we can show that single-armed restless bandit is indexable.

**Theorem 2:** If \(\mu_0 > \mu_1\) and \(\beta < 1/3\), then a single-armed restless bandit is indexable.

**Proof:** The following inequalities obtain using induction technique,

\[
\left| \frac{\partial V(\pi, w)}{\partial w} \right|, \left| \frac{\partial V_T(\pi, w)}{\partial w} \right|, \left| \frac{\partial V_{NT}(\pi, w)}{\partial w} \right| \leq \frac{1}{1 - \beta}
\]

and

\[
\left| \frac{\partial \tilde{V}(\pi, w)}{\partial w} \right|, \left| \frac{\partial \tilde{V}_T(\pi, w)}{\partial w} \right|, \left| \frac{\partial \tilde{V}_{NT}(\pi, w)}{\partial w} \right| \leq \frac{1}{1 - \beta}
\]

Also,

\[
\frac{\partial V_T(\pi, w)}{\partial w} = \beta \left[ (1 - \pi) \theta \frac{\partial V(\mu_1, w)}{\partial w} + (1 - \theta^1) \frac{\partial V(\mu_1, w)}{\partial w} \right] + \pi \left[ \theta \frac{\partial V(\mu_0, w)}{\partial w} + (1 - \theta^1) \frac{\partial V(\mu_0, w)}{\partial w} \right]
\]

and

\[
\frac{\partial V_{NT}(\pi, w)}{\partial w} = 1 + \beta \left[ \theta \frac{\partial V(\Gamma_1(\pi), w)}{\partial w} + (1 - \theta^1) \frac{\partial V(\Gamma_1(\pi), w)}{\partial w} \right].
\]
Now from Lemma 2, we require the difference $\frac{\partial V_{NT}(\pi, w)}{\partial w} - \frac{\partial V_{T}(\pi, w)}{\partial w}$ to be non-negative at $\pi_{th}(w)$ and $\bar{\pi}_{th}(w)$. This reduces to following expression.

$$
\left[(1 - \pi)\theta \frac{\partial V(\mu_1, w)}{\partial w} + (1 - \theta^1)\frac{\partial V(\mu_1, w)}{\partial w}\right] + \pi \left[\theta \frac{\partial V(\mu_0, w)}{\partial w} + (1 - \theta^1)\frac{\partial V(\mu_0, w)}{\partial w}\right] - \left[\theta^0 \frac{\partial V(\Gamma_1(\pi), w)}{\partial w} + (1 - \theta^0)\frac{\partial V(\Gamma_1(\pi), w)}{\partial w}\right] < \frac{1}{\beta}.
$$

(8)

We can provide upper bound on LHS of above expression and it is upper bounded by $2/(1 - \beta)$. If $\beta < 1/3$, Eqn. 8 is satisfied. $\pi_{th}(w)$ is decreasing in $w$. Similarly $\bar{\pi}_{th}(w)$ is decreasing in $w$. And the claim follows. ■

Proof of indexability for $0 < \theta^a < 1$ requires assumption on $\beta$. Whereas for $\theta^a = 1$ or 0, indexability do not need assumption on $\beta$, because the value function expression can be easily derived and then differentiating w.r.t. subsidy $w$, we can get required, such result is studied in [9] Theorem 1. We now use Definition 3 and restate the Whittle index definition as follows.

Definition 4 (Whittle’s index): For a given belief $\pi \in [0, 1]$ and availability $y \in \{0, 1\}$, Whittle index $w(\pi, y)$ is the minimum subsidy $w$ for which not playing the arm is the optimal action.

$$
w(\pi, 1) = \inf\{w \in \mathbb{R} : V_{NT}(\pi) = V_T(\pi)\},
$$

$$
w(\pi, 0) = \inf\{w \in \mathbb{R} : V_{NT}(\pi) = \bar{V}_T(\pi)\}.
$$

(9)

When $\theta^a = 0, 1$ for all $a \in \{0, 1\}$, the expression for index can be computed and this is given in [9, Section IV]. But for $\theta^a \in (0, 1)$, it is very difficult to obtain closed form expression for value functions because there is coupling between action value functions.

Hence, we study a numerical scheme for Whittle index computation. This scheme uses the threshold result of value functions and two-timescales stochastic approximations. In two-timescales stochastic approximations, we update $w_t$ at slower timescales or natural timescales, and the value functions are updated using value iteration algorithm at faster timescales. This scheme here is inspired from stochastic approximation algorithms, see [15], [16].

In this scheme for fixed $w, y = 1$ and a threshold $\pi$, we know that $V_T(\pi, w) = V_{NT}(\pi, w)$. Using value iteration algorithm, we compute $V_T(\pi, w)$ and $V_{NT, w}(\pi, w)$ on faster time scales until difference $|V_T(\pi, w) - V_{NT, w}(\pi, w)|$ becomes smaller than tolerance $h$. To compute the index $w(\pi, 1)$, our algorithm starts with initial subsidy $w_0$ and it is updated iteratively at slower timescales according to following expression.

$$
w_{t+1} = w_t + \alpha (V_T(\pi, w_t) - V_{NT}(\pi, w_t)).
$$

These computations are performed till difference $|V_T(\pi, w_t) - V_{NT}(\pi, w_t)|$ is smaller than tolerance $h$.

Using similar procedure mentioned above, we update $w_t$ with slower timescales and run value iteration for $\bar{V}_T(\pi, w_t)$ and $\bar{V}_{NT}(\pi, w_t)$ on faster timescales when $\pi$ is threshold and $y = 0$. Hence this is used to compute the index $w(\pi, 0)$. The details are given in Algorithm 1. The convergence of two timescales stochastic approximation algorithm is presented in [15] Chapter 6.

Algorithm 1: Algorithm that computes Whittle index for the single arm

Input: Reward values $r_1, r_2$; Initial subsidy $w_0$, tolerance $h$, step size $\alpha$.
Output: Whittle index, $w(\pi, y)$
if $(y=1)$ then
  $w_t \leftarrow w_0$
  while $|V_T(\pi, w_t) - V_{NT}(\pi, w_t)| > h$ do
    $w_{t+1} = w_t + \alpha (V_T(\pi, w_t) - V_{NT}(\pi, w_t))$;
    $t = t + 1$;
    compute $V_T(\pi, w_t), V_{NT}(\pi, w_t)$;
  end
else
  $w_t \leftarrow w_0$
  while $|\bar{V}_T(\pi, w_t) - \bar{V}_{NT}(\pi, w_t)| > h$ do
    $w_{t+1} = w_t + \alpha (\bar{V}_T(\pi, w_t) - \bar{V}_{NT}(\pi, w_t))$;
    $t = t + 1$;
    compute $\bar{V}_T(\pi, w_t), \bar{V}_{NT}(\pi, w_t)$;
  end
end
return $w(\pi, y) = w_t$
We model an individual element in a larger framework for studying information acquisition and dissemination in social networks. For example, we may consider the impact of a set of compromised or fake news sources over the decisions of various agents across a network.

We formulated problem of information gathering in a social network with dynamic availability of sources and time varying information quality using RMAB model. We studied the Whittle's index policy. Also, the performance of this policy was always available are chosen. Future rewards and availability of sources through the action value functions. Interestingly, sources more frequently even though they have lesser rewards. This behavior of Whittle’s index policy is because it considers insight, we also plot source choice fraction as function of number of arms in Fig. 1-b. and which is the probability that an source/arm is chosen in a slot. Here, myopic policy contacts sources according to their expected immediate rewards, 3) uniform random policy (UR)—chooses randomly with uniform distribution.

At each decision making instant the agent chooses to contact M of N sources, where we use M = 3. Our parameter set represents the scenario where availability of sources is independent of agent’s decision but the perception of their usefulness depends on it. Further, the sources in our examples tend to maintain their information quality (relevant or irrelevant). In Fig. 1-a we plot the discounted cumulative reward as a function of time slots. It can be seen that the discounted cumulative reward under Whittle index policy (WI) is comparable with that of the myopic policy (MP). We also observe that performance of WI policy yields higher discounted cumulative reward compared to that of myopic policy and uniform random policy. To gain insight, we also plot source choice fraction as function of number of arms in Fig. 1-b. and which is the probability that an source/arm is chosen in a slot. Here, myopic policy contacts sources {1, 5, 14} most frequently compared to other sources and this is because sources {1, 5} are always available and 14 has high reward. Whereas WI policy contacts from broader set of sources more frequently even though they have lesser rewards. This behavior of Whittle’s index policy is because it considers future rewards and availability of sources through the action value functions. Interestingly, sources {9, 10, 15} that are not always available are chosen.

We formulated problem of information gathering in a social network with dynamic availability of sources and time varying information quality using RMAB model. We studied the Whittle’s index policy. Also, the performance of this policy was illustrated for moderate sized scenarios.

In this work we considered the decision model of a single agent in a social or information network. This can be used to model an individual element in a larger framework for studying information acquisition and dissemination in social networks. For example, one may consider the impact of a set of compromised or fake news sources over the decisions of various agents across a network.

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