STANDING WAVES FOR SCHRÖDINGER-POISSON SYSTEM WITH GENERAL NONLINEARITY

ZHİ CHEN, XIANJIUA TANG AND NİNG ZHANG
School of Mathematics and Statistics, Central South University
Changsha, 410083 Hunan, China

JIAN ZHANG
School of Mathematics and Statistics, Hunan University of Technology and Business
Changsha, 410205 Hunan, China

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Abstract. In this paper we consider the following Schrödinger-Poisson system with general nonlinearity
\[
\begin{align*}
-\varepsilon^2 \Delta u + V(x)u + \psi u &= f(u), \quad x \in \mathbb{R}^3, \\
-\varepsilon^2 \Delta \psi &= u^2, \quad u > 0, \\n  u &\in H^1(\mathbb{R}^3),
\end{align*}
\]
where \( \varepsilon > 0 \) is a small positive parameter. Under a local condition imposed on the potential \( V \) and general conditions on \( f \), we construct a family of positive semiclassical solutions. Moreover, the concentration phenomena around local minimum of \( V \) and exponential decay of semiclassical solutions are also explored. We do not need the monotonicity of the function \( u \to f(u)u^3 \), and our results include the case \( f(u) = |u|^{p-2}u \) for \( 3 < p < 6 \). Since without more global information on the potential, in the proofs we apply variational methods, penalization techniques and some analytical techniques.

1. Introduction and main results. In this paper, we will consider the following nonlinear Schrödinger-Poisson system
\[
\begin{align*}
-\varepsilon^2 \Delta u + V(x)u + \psi u &= f(u), \quad x \in \mathbb{R}^3, \\
-\varepsilon^2 \Delta \psi &= u^2, \quad u > 0, \\n  u &\in H^1(\mathbb{R}^3),
\end{align*}
\]
where \( \varepsilon > 0 \) is a small positive parameter, \( f \) is a continuous, superlinear and sub-critical nonlinearity. We are interested in the existence and concentration behavior of semiclassical solutions of system (1.1) when \( \varepsilon \to 0 \).

The first equation of (1.1) is a nonlinear Schrödinger equation in which the potential \( \psi \) satisfies a nonlinear Poisson equation. For this reason, system (1.1) is called Schrödinger-Poisson system. It is well known that the Schrödinger-Poisson system (1.1) has some strong physical meaning. For instance, in Abelian Gauge Theories, system (1.1) provides a model to describe the interaction of a nonlinear...
Schrödinger field with the electromagnetic field. In quantum electrodynamics, system (1.1) describes the interaction between a charge particle interacting with the electromagnetic field. Moreover, system (1.1) also appears in semiconductor theory, nonlinear optics and plasma physics (see [4, 23, 25]). For more details in the physical backgrounds, we refer the readers to [3, 4].

In recent years, there has been increasing attention to Schrödinger-Poisson system (1.1) or similar problem on obtaining existence of positive solutions, ground state solutions, multiple solutions and semiclassical states by using variational methods. For instance, for the unperturbed case, i.e., \( \varepsilon = 1 \), see the papers [1, 2, 3, 4, 7, 8, 9, 10, 11, 30, 33, 34, 35, 37, 42, 44, 45] and the reference therein.

For \( \varepsilon > 0 \) small, problem (1.1) is called perturbed problem. In this case the solutions of (1.1) are referred to as semiclassical solutions, which describes the transition from quantum mechanics to classical mechanics when the parameter \( \varepsilon \) goes to zero, and possess an important physical interest. In this framework, from a mathematical viewpoint, one is interested not only in existence of semiclassical solutions but also in their asymptotic behavior as \( \varepsilon \to 0 \). Typically, solutions tend to concentrate around critical points of the potentials functions: such solutions are called spikes. In the present paper we only care about the perturbed problem, so in the following we shall recall some previous results for this case. In [18], Ianni and Vaira considered the following system

\[
\begin{align*}
-\varepsilon^2 \Delta u + A(x)u + \psi u &= f(u), \quad x \in \mathbb{R}^3, \\
-\Delta \psi &= u^2, \quad x \in \mathbb{R}^3.
\end{align*}
\] (1.2)

Under some subcritical conditions on \( f \), the authors proved that (1.2) has a single-bump solution near critical points of the potential \( A \). Later, by applying a standard Lyapunov-Schmidt reduction methods, Ruiz and Vaira [32] studied the existence of multi-bump solutions, whose bumps concentrated around a local minimum of the potential \( A \). For other related results, we refer the readers to [12, 19, 20, 31] for concentration on spheres.

Observe that, the singular perturbation problem (1.1) has two perturbation parameters in the system, and is called the double parameters’ perturbation problem. Here we would like to bring attention to the readers that there is a big difference between system (1.1) and (1.2). Indeed, when the parameter \( \varepsilon \) is small, the limit problem for the latter is the classical nonlinear Schrödinger equation while the former is still the Schrödinger-Poisson system. Recently, problem (1.1) was also considered in [15, 16, 17, 39, 40, 43]. More specifically, under the subcritical and Ambrosetti-Rabinowitz type growth condition, He [16] studied the existence of positive solutions by using minimax theorems, and proved that these solutions concentrate around the global minimum of the potential \( V \) in the semiclassical limit. Moreover, a multiplicity result depending on the topology property of the potential is obtained by the Ljusternik-Schnirelmann theory. Here the author assumed that

\[
\liminf_{|x| \to \infty} V(x) \geq \inf_{x \in \mathbb{R}^3} V(x) > 0 \quad \text{(global condition)}
\] (1.3)

and \( f \) is a \( C^1 \) function such that

\[
\begin{align*}
\frac{f(s)}{s^3} &\text{ is increasing on } (0, +\infty), \quad 0 < \mu F(s) = \mu \int_0^s f(t)dt \leq sf(s), \mu > 4 \\
f'(s)s^2 - 3f(s)s &\geq cs^\sigma, \sigma \in (4, 6), c > 0 \quad \text{and } f(s) = o(s^3) \text{ as } s \to 0.
\end{align*}
\] (1.4)
Subsequently, this result has been extended to the critical growth case by He and Zou [16]. Also under the global condition on the potential functions, Wang et al. [39] considered the following system with competing potentials

\[
\begin{align*}
-\varepsilon^2 \Delta u + a(x)u + \psi u &= b(x)f(u), \quad x \in \mathbb{R}^3, \\
-\varepsilon^2 \Delta \psi &= u^2, \quad x \in \mathbb{R}^3.
\end{align*}
\]

The authors proved the existence of semi-classical ground state solutions by the Nehari manifold method under the following conditions

\[
f \in C(\mathbb{R}^3), \quad f(s)s > 0, \quad \text{and} \quad f(s) = o(s^3) \quad \text{as} \quad s \to 0
\]

\[
\frac{f(s)}{s^3} \quad \text{is strictly increasing on} \quad (0, +\infty), \quad \frac{F(s)}{s^4} \to \infty \quad \text{as} \quad s \to \infty.
\]

And moreover, some new concentration phenomenons of semi-classical solutions on the minimum points of \(a(x)\) and the maximum points of \(b(x)\) are also investigated.

For the critical case, we refer readers to [40] for details.

It is worth pointing out that the global condition (1.3) used in [15, 16, 39, 40] plays a crucial role in proving the existence of positive solutions. Indeed, the key point is that the property of the potential \(V\) can help us to restore the necessary compactness by comparing energy levels of original problem and limit problem. An interesting question is whether one can find solutions which concentrating around local minima of the potential. Very recently, it seems that the only work concerning the existence and concentration behavior of solutions is due to He and Li [17] for the local potential condition. In [17], the authors considered the critical growth model

\[
f(u) = \lambda |u|^{p-2}u + |u|^4u, \quad \lambda > 0 \quad \text{and} \quad 3 < p \leq 4,
\]

and assumed that the following local condition first introduced by del Pino and Felmer in [27]:

\[
(V_1) \quad V(x) \in C(\mathbb{R}^3, \mathbb{R}) \quad \text{and} \quad \inf_{x \in \mathbb{R}^3} V(x) = \alpha_0 > 0;
\]

\[
(V_2) \quad \text{there exists a bounded domain} \quad \Lambda \quad \text{such that}
\]

\[
V_0 := \inf_{\Lambda} V < \min_{\partial \Lambda} V.
\]

Using a version of quantitative deformation lemma due to Figueiredo, Ikoma and Santos Junior [13], they constructed a special bounded Palais-Smale sequence and recovered the compactness by using a penalization method which was introduced in [6].

Motivated by the above facts, in the present paper we focus on the existence and concentration behavior of semiclassical solutions of system (1.1) under the local potential and general subcritical nonlinearity. To the best of our knowledge, it seems that such a problem was not considered in literature before. More precisely, our purpose in this paper is twofold. First, we construct a family of positive semiclassical solutions for (1.1) with some properties, such as concentration, exponent decay, etc. Second, we treat more general subcritical nonlinearity \(f\) that is weaker than the previous works [15, 17, 39].

In order to state our results, we make the following assumptions on the nonlinearity \(f\):

\[
(f_1) \quad f \in C(\mathbb{R}, \mathbb{R}), \quad \text{and} \quad \text{there exist} \quad C > 0 \quad \text{and} \quad p \in (2, 6) \quad \text{such that}
\]

\[
|f(s)| \leq C (1 + |s|^{p-1}), \quad \forall \quad s \in \mathbb{R};
\]

\[
(f_2) \quad f(s) = o(s) \quad \text{as} \quad s \to 0;
\]

\[
(f_3) \quad F(s) \geq 0 \quad \text{and} \quad \frac{F(s)}{s^3} \to +\infty \quad \text{as} \quad s \to \infty;
\]

\[
(f_4) \quad \text{the function} \quad [2f(s)s - 3F(s)]/s^3 \quad \text{is nondecreasing on} \quad (0, +\infty).
\]
In order to describe some concentration phenomena of semiclassical solution, we define the concentration set by
\[ M := \{ x \in \Lambda : V(x) = V_0 \} \]. Moreover, without loss of generality, we may assume that 0 \in M. We are now in a position to state the main result of this paper.

**Theorem 1.1.** Suppose that \((V_1)-(V_2)\) and \((f_1)-(f_4)\) hold. Then there exists \(\varepsilon^* > 0\) such that for each \(\varepsilon \in (0, \varepsilon^*)\), system (1.1) possesses a positive solution \(u_\varepsilon \in H^1(\mathbb{R}^3)\) such that

(i) there exists a maximum point \(x_\varepsilon\) of \(u_\varepsilon\) such that
\[
\lim_{\varepsilon \to 0} \text{dist}(x_\varepsilon, M) = 0.
\]
Moreover, \(u_\varepsilon(\varepsilon x + x_\varepsilon)\) converges (up to a subsequence) locally uniformly to a positive ground state solution of limiting system
\[
\begin{cases}
-\Delta u + V_0 u + \psi u = f(u), & x \in \mathbb{R}^3, \\
-\Delta \psi = u^2, & u > 0, u \in H^1(\mathbb{R}^3).
\end{cases}
\]

(ii) there exist constants \(C_1, C_2 > 0\) such that
\[
u_\varepsilon(x) \leq C_1 \exp\left(-\frac{C_2}{\varepsilon} |x - x_\varepsilon|\right),
\]
where \(C_1, C_2\) are independent of \(\varepsilon\).

**Remark 1.2.** It is easy to see that \((f_4)\) implies that \((f'_4)\) holds
\[
(f'_4) \quad \left[\frac{2f(\tau)\tau - 3F(\tau)}{|\tau|^3} - \frac{2(f(\tau^2)\tau^2 - 3F(\tau^2))}{(\tau^2)^3}\right] \text{sign}(1 - t) \geq 0, \quad \forall t > 0, \tau \neq 0.
\]
Moreover, \((f_2), (f_3)\) and \((f_4)\) implies that \((f_5)\) holds.
\[(f_5) \quad f(s)s \geq 3F(s) \geq 0.\]

The proof will be given in Lemma 3.1. These relation will be used later in proving the existence of positive ground state solution of limiting problem.

**Remark 1.3.** Compared with [15] and [39], on the one hand, our condition \((V_2)\) is rather weak, without restriction on the global behavior of \(V\) is required other than (1.3), and the behavior of \(V\) outside \(\Lambda\) is irrelevant in this paper. On the other hand, the conditions we assumed for the nonlinearity \(f\) can not satisfy the classical Ambrosetti-Rabinowitz condition in (1.4) and the 4-superlinear condition in (1.5) and the monotonicity condition of the function \(f(s)/s^3\). Instead, \(f\) only satisfies the superlinear condition at origin and the general 3-superlinear condition at infinity, and a weaker monotonicity condition in this paper. Clearly, ours conditions are weaker than the ones in [15, 39]. Besides, in [16] the nonlinearity \(f\) is only a special power nonlinear model with critical term. But our conditions seem more general and are subcritical growth. In particular, our results include the case \(f(u) = |u|^{p-2}u\) for \(3 < p < 6\). From these reasons, our results extend and complement related results [15, 16, 39].

The motivation of the present paper mainly comes from the results of semiclassical Schrödinger equations
\[
-\varepsilon^2 \Delta u + V(x)u = f(u), \quad x \in \mathbb{R}^N.
\]
Many mathematicians studied the existence, concentration and multiplicity of solutions for (1.6). In [29], Rabinowitz used the Mountain-Pass theorem to show that
(1.6) possesses a positive ground state solution for \( \varepsilon > 0 \) small under the global condition (1.3). After that, the authors in [38] showed that the positive solution obtained in [29] concentrates at global minimum points of \( V \). Under the local conditions \((V_1)-(V_2)\), del Pino and Felmer [27] introduced a penalization approach, so-called local mountain pass, and managed to handle the case of a, possibly degenerate, local minimum of \( V \). They showed the existence of a single spike solution concentrating around minimizer of \( V \) in \( \Lambda \). Based on a new singular perturbation, essentially a penalization method, the localized bound state solutions concentrating at an isolated component of the local minimum of \( V \) under the almost optimal conditions on \( f \) (such as Berestycki-Lions condition) were also constructed in papers [5] and [6].

From the commentaries above, it is quite natural to ask that: can we obtain some similar results for the Schrödinger-Poisson system (1.1) as in Schrödinger equation (1.6)? In the present paper, we shall give some answers for this system. However, compared with the Schrödinger equation (1.6), the Schrödinger-Poisson system (1.1) becomes more complicated since there exists a competing effect of the non-local term with the nonlinear term. Hence our problem poses more challenges in the calculus of variation. In addition, there are also some additional difficulties for system (1.1). (i) The nonlinearity \( f \) does not satisfy Ambrosetti-Rabinowitz condition and the fact that the function \( f(s)/s^3 \) is not increasing on \((0,\infty)\), these features prevent us from obtaining a bounded Palais-Smale sequence ((PS) sequence in short) and using the usual Nehari manifold method. (ii) The unboundedness of the domain \( \mathbb{R}^3 \) leads to the lack of compactness. As we shall see later, the above two aspects prevent us from using variational method in a standard way.

Our argument is based on variational method, which can be outlined as follows. The solutions are obtained as critical points of the energy functional associated to system (1.1). Moreover, the above mentioned difficulties we should overcome. Since the monotonicity of the function \( f(s)/s^3 \) is not required, which gives rise to the Nehari manifold caused in this paper. On the one hand, we need to use a penalization method introduced by Byeon and Wang [6], which helps us to overcome the obstacle caused by the non-compactness due to the unboundedness of the domain and the lack of Ambrosetti-Rabinowitz condition. Proceeding by some arguments and techniques, then the existence of ground state solution follows.

On the other hand, inspired by [13, 17], use a version of quantitative deformation lemma due to Figueiredo, Ikoma and Junior [13] to construct a special bounded (PS) sequence in a neighborhood of the compact set.

More precisely, for the proof of our results, some arguments are in order. Firstly, in order to get concentrated solutions to system (1.1) we need to consider the existence of ground state solutions of the associated “limiting problem” of system (1.1):

\[
\begin{cases}
-\Delta u + au + \psi u = f(u), & x \in \mathbb{R}^3, \\
-\Delta \psi = u^2, & u > 0, u \in H^1(\mathbb{R}^3), \\
a > 0,
\end{cases}
\tag{1.7}
\]

with the corresponding energy functional

\[ I_a(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + au^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} \psi u^2 dx - \int_{\mathbb{R}^3} F(u) dx, \ u \in H^1(\mathbb{R}^3). \]

Denoting \( S_a \) the set of ground state solution \( U \) of (1.7) satisfying \( U(0) = \max_{x \in \mathbb{R}^3} U(x) \), we will show that \( S_a \) is compact in \( H^1(\mathbb{R}^3) \). To conquer the difficulties of
certifying the boundedness of (PS) sequence for the limiting equation, we consider an auxiliary functional
\[ G_a(u) = 2\langle I'_a(u), u \rangle - P_a(u), \]
where \( P_a(u) = 0 \) is the Pohozaev identity of (1.7). By applying the ideas employed by Jeanjean [21] and Tang and Chen [35, 36], we find a (PS) sequence \( \{u_n\} \subset H^1(\mathbb{R}^3) \) of \( I_a \) at level \( c \) such that \( G_a(u_n) \to 0 \), where \( c > 0 \) is mountain pass level defined later by (3.10).

To investigate (1.1), we will focus on the following equivalent system by making the change of variable \( \varepsilon y = x \)
\[
\begin{cases}
-\Delta u + V(\varepsilon x)u + \psi u = f(u), \ x \in \mathbb{R}^3, \\
-\Delta \psi = u^2, \ u > 0, u \in H^1(\mathbb{R}^3),
\end{cases}
\] (1.8)

with the energy functional
\[ I_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + V(\varepsilon x)u^2dx + \frac{1}{4} \int_{\mathbb{R}^3} \psi u^2 dx - \int_{\mathbb{R}^3} F(u) dx, \ u \in H_\varepsilon, \]
where
\[ H_\varepsilon := \left\{ u \in H^1(\mathbb{R}^3) | \int_{\mathbb{R}^3} V(\varepsilon x)u^2 dx < \infty \right\} \]
edowed with the norm
\[ ||u||_{H_\varepsilon} = \left( \int_{\mathbb{R}^3} (|\nabla u|^2 + V(\varepsilon x)u^2) dx \right)^{\frac{1}{2}}. \]

Different to [15] and [39], where the potential \( V \) has the global condition (1.3) and the nonlinear term \( f \) satisfies the Ambrosetti-Rabinowitz condition or the general 4-superlinear condition, in this case the mountain geometry of the functional and boundedness and compactness of (PS) sequences can be obtained easily. Here in the present paper, all conditions we assumed are weaker than the ones in [15] and [39], namely, the potential condition \( (V_2) \) is local and the nonlinear term \( f \) satisfies the general 3-superlinear condition and a weaker monotonicity condition. Then some arguments used in [15] and [39] are no longer valid, and some new methods and techniques need to be introduced in this paper. All we do is to build a modification of the energy functional associated to (1.1). In such a way, the functional is proved to satisfy the so-called (PS) condition. The modification of the energy functional corresponds to a penalization technique “outside \( \Lambda \)”, that is why no other global condition are required for the potential \( V \). Following [6], we set \( J_\varepsilon : H_\varepsilon \to \mathbb{R} \) be given by
\[ J_\varepsilon(v) = I_\varepsilon(v) + Q_\varepsilon(v), \]
where
\[ Q_\varepsilon(v) = \left( \int_{\mathbb{R}^3} \chi_\varepsilon v^2 dx - 1 \right)^2 \]
and
\[ \chi_\varepsilon(x) = \begin{cases} 0, & \text{if } x \in \Lambda / \varepsilon, \\ \varepsilon^{-1}, & \text{if } x \notin \Lambda / \varepsilon. \end{cases} \]

It will be shown that the functional \( Q_\varepsilon \) acts as a penalization to force the concentration phenomena to occur inside \( \Lambda \). It is standard to see that \( J_\varepsilon \in C^1(H_\varepsilon, \mathbb{R}) \). Clearly a critical point of \( I_\varepsilon \) corresponds to a solution of system (1.8). To find solutions of system (1.8) which concentrate in \( \Lambda \) as \( \varepsilon \to 0 \), we shall search critical points of the modified functional \( J_\varepsilon \) for which \( Q_\varepsilon \) is zero. To do this, we need to establish a delicate \( L^\infty \)-estimation for the critical point \( v_\varepsilon \) of \( J_\varepsilon \), and moreover show the critical point \( v_\varepsilon \) of \( J_\varepsilon \) is indeed a solution to the original problem.
Theorem 1.1. For any Lemma 2.1. The function ψ has the following properties, the proof can be found in [7] and [30].

Throughout the sequel, we denote the norms of usual Lebesgue space \( L^p(\mathbb{R}^3) \) by

\[
\|u\|_p^p = \int_{\mathbb{R}^3} |u|^p dx, \quad \text{for } 1 \leq p < \infty,
\]

and \( C_i \) and \( C \) denotes different positive constant in different places.

2. Preliminaries. In this section, we recall that by the Lax-Milgram theorem, for each \( u \in H^1(\mathbb{R}^3) \), there exists a unique \( \psi_u \in D^{1,2}(\mathbb{R}^3) \) such that \(-\Delta \psi_u = u^2\). Moreover, \( \psi_u \) can be expressed as

\[
\psi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy.
\]

The function \( \psi_u \) has the following properties, the proof can be found in [7] and [30].

Lemma 2.1. For any \( u \in H^1(\mathbb{R}^3) \), we get

(i) \( \|\psi_u\|_{D^{1,2}(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} \psi_u u^2 dx \leq C \|u\|_2^2 \leq C \|u\|_{H^1(\mathbb{R}^3)}^4 \);  
(ii) \( \psi_u \geq 0 \);  
(iii) If \( u_n \rightharpoonup u \) in \( H^1(\mathbb{R}^3) \), then \( \psi_{u_n} \rightharpoonup \psi_u \) in \( D^{1,2}(\mathbb{R}^3) \) and

\[
\int_{\mathbb{R}^3} \psi_{u_n} u_n^2 dx \leq \liminf_{n \to \infty} \int_{\mathbb{R}^3} \psi_{u_n} u^2 dx;
\]

(iv) If \( y \in \mathbb{R}^3 \) and \( \tilde{u} = u(x+y) \), then \( \psi_{u_n}(x) = \psi_u(x+y) \) and

\[
\int_{\mathbb{R}^3} \psi_{u_n} \tilde{u}^2 dx = \int_{\mathbb{R}^3} \psi_u u^2 dx.
\]

Define \( N : H^1(\mathbb{R}^3) \to \mathbb{R} \) by

\[
N(u) = \int_{\mathbb{R}^3} \psi_u u^2 dx.
\]

Lemma 2.2. Let \( u_n \rightharpoonup u \) in \( H^1(\mathbb{R}^3) \) and \( u_n \to u \) a.e. in \( \mathbb{R}^3 \), then, as \( n \to \infty \),

(i) \( N(u_n - u) = N(u_n) - N(u) + o(1) \);  
(ii) \( N'(u_n - u) = N'(u_n) - N'(u) + o(1) \) in \( H^1(\mathbb{R}^3) \) and \( N' : H^1(\mathbb{R}^3) \to H^{-1}(\mathbb{R}^3) \) is weakly sequentially continuous;  
(iii) \( N''(u_n - u) = N''(u_n) - N''(u) + o(1) \) in \( L(H^1(\mathbb{R}^3), H^{-1}(\mathbb{R}^3)) \) and \( N''(u) \in L(H^1(\mathbb{R}^3), H^{-1}(\mathbb{R}^3)) \) is compact for any \( u \in H^1(\mathbb{R}^3) \).

3. The limiting problem. We will make use of the limiting problem for proving our main result. To this end, we first discuss in this section the existence of the positive ground state solutions of the limiting problem. Consider the limiting system of (1.1) as follows

\[
\begin{cases}
-\Delta u + au + \psi u = f(u), \quad x \in \mathbb{R}^3, \\
-\Delta \psi = u^2, \quad u > 0, \quad u \in H^1(\mathbb{R}^3),
\end{cases}
\]

where \( a > 0 \). Since we need to seek a positive ground state solution, we restrict the nonlinearity \( f(s) = 0 \) if \( s \leq 0 \). In view of [28], if \( u \in H^1(\mathbb{R}^3) \) is a weak solution to problem (1.7), then we have the following Pohozaev’s identity:
\[ P_a(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + 3au^2 \, dx + \frac{5}{4} \int_{\mathbb{R}^3} \psi_u u^2 \, dx - 3 \int_{\mathbb{R}^3} F(u) \, dx = 0. \] (3.1)

As in [30], we introduce the following Nehari-Pohozaev manifold
\[ M_a = \{ u \in H^1(\mathbb{R}^3) \setminus \{0\} \mid G_a(u) = 0 \}, \]
where
\[ G_a(u) = 2 \langle I_a'(u), u \rangle - P_a(u) = \frac{1}{2} \int_{\mathbb{R}^3} 3|\nabla u|^2 + au^2 \, dx + \frac{3}{4} \int_{\mathbb{R}^3} \psi_u u^2 \, dx - \int_{\mathbb{R}^3} 2f(u)u - 3F(u) \, dx. \]

**Lemma 3.1.** Assume that (f_1) and (f_4) hold, then
\[ \frac{2(1-t^3)}{3} f(\tau) \tau + (t^3 - 2) F(\tau) + \frac{1}{t^3} F(t^2 \tau) \geq 0, \quad \forall \ t > 0, \ \tau \in \mathbb{R}. \] (3.2)

**Proof.** It is obvious that (3.2) holds for \( \tau \leq 0 \). For \( \tau > 0 \), let
\[ g(t) = \frac{2(1-t^3)}{3} f(\tau) \tau + (t^3 - 2) F(\tau) + \frac{1}{t^3} F(t^2 \tau) \geq 0, \quad \forall \ t > 0, \ \tau \in \mathbb{R}. \] (3.3)

Then from (f_4) and Remark 1.2, one has
\[ g'(t) = t^2 |\tau|^3 \left\{ \frac{2f(t^2 \tau) t^2 \tau - 3F(t^2 \tau)}{t^6 |\tau|^3} - \frac{2f(\tau) \tau - 3F(\tau)}{|\tau|^3} \right\} \leq 0, \quad \text{if} \quad 0 < t < 1, \]
\[ g'(t) \geq 0, \quad \text{if} \quad t \geq 1. \] (3.4)

It follows that \( g(t) \geq g(1) = 0 \) for \( t > 0 \). Then we complete the proof. Moreover, let \( t \to 0 \) in (3.2), so we complete the proof of (f_5) in Remark 1.2.

**Lemma 3.2.** Assume that (f_1), (f_2) and (f_4) hold. Then for any \( u \in H^1(\mathbb{R}^3) \) and \( t \geq 0 \), we have
\[ I_a(u) \geq I_a(t^2 u(tx)) + \frac{1-t^3}{3} G_a(u) + \left( \frac{a(1-t)^2 (2 + t)}{6} \right) \| u \|_{L^2}^2. \] (3.5)

**Proof.** Note that
\[ I_a(t^2 u(tx)) = \frac{1}{2} \int_{\mathbb{R}^3} t^3 |\nabla u|^2 + au^2 \, dx + \frac{t^3}{4} \int_{\mathbb{R}^3} \psi_u u^2 \, dx - \frac{1}{t^3} \int_{\mathbb{R}^3} F(t^2 u) \, dx. \] (3.6)

It follows from Lemma 3.1 that
\[ I_a(u) - I_a(t^2 u(tx)) = \frac{1}{2} \int_{\mathbb{R}^3} \left[ (1-t^3)|\nabla u|^2 + a(1-t)u^2 \right] \, dx + \frac{1-t^3}{4} \int_{\mathbb{R}^3} \psi_u u^2 \, dx + \frac{1-t^3}{3} G_a(u) + \frac{a(1-t)^2 (2 + t)}{6} \int_{\mathbb{R}^3} u^2 \, dx \]
\[ + \int_{\mathbb{R}^3} \left[ \frac{1}{t^3} F(t^2 u) - F(u) \right] \, dx \]
\[ = \frac{1-t^3}{3} G_a(u) + \frac{a(1-t)^2 (2 + t)}{6} \int_{\mathbb{R}^3} u^2 \, dx \]
\[ + \int_{\mathbb{R}^3} \left[ \frac{2(1-t^3)}{3} f(u)u + (t^3 - 2) F(u) + \frac{1}{t^3} F(t^2 u) \right] \, dx \]
\[ \geq \frac{1-t^3}{3} G_a(u) + \frac{a(1-t)^2 (2 + t)}{6} \int_{\mathbb{R}^3} u^2 \, dx, \]
which implies (3.5) holds.
From Lemma 3.2, we get the following corollary immediately.

**Corollary 3.3.** Assume that \(f_1\), \(f_2\) and \(f_4\) hold. Then for \(u \in M_a\)
\[
I_a(u) = \max_{t \geq 0} I_a(t^2u(tx)).
\] (3.7)

**Lemma 3.4.** Assume that \(f_1\), \(f_2\), \(f_3\) and \(f_4\) hold. Then for any \(u \in H^1(\mathbb{R}^3)\setminus\{0\}\), there exists a unique \(t(u) > 0\) such that \(t(u)^2u(t(u)x) \in M_a\).

**Proof.** For a fixed \(u \in H^1(\mathbb{R}^3)\setminus\{0\}\), define a function \(\xi(t) := I_a(t^2u(tx))\) on \([0, +\infty)\). Clearly, we have
\[
\xi'(t) = 0 \Leftrightarrow \frac{1}{2} \int_{\mathbb{R}^3} 3t^3 |\nabla u|^2 + au^2 dx + \frac{3}{4} \int_{\mathbb{R}^3} \psi u^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} f(t^2u)u - \frac{3}{t^3} \int_{\mathbb{R}^3} F(t^2u) dx \Leftrightarrow t^2u(tx) \in M_a.
\]

Using \((f_1)-(f_3)\), one can easily check that \(\xi(0) = 0, \xi(t) > 0\) for \(t > 0\) small enough and \(\xi(t) < 0\) for \(t\) large. Thus \(\max_0^t \xi(t)\) is achieved at some \(t_0 = t(u)\). So we have \(\xi'(t_0) = 0\) and \(t_0^2u(t_0x) \in M_a\). We also need to assert that \(t(u)\) is unique for any \(u \in H^1(\mathbb{R}^3)\setminus\{0\}\). Arguing by contradiction, for given \(u \in H^1(\mathbb{R}^3)\setminus\{0\}\), we suppose \(t_1, t_2 > 0\) such that \(t_1^2u(t_1x), t_2^2u(t_2x) \in M_a\). It follows from (3.5) that
\[
I_a(t_1^2u(t_1x)) \geq I_a(t_2^2u(t_2x)) + \frac{t_3^2 - t_2^2}{3t_1^2} G_a(t_1^2u(t_1x)) + \frac{a(t_1 - t_2)^2(2t_1 + t_2)}{6t_1^2} \|u\|_2^2
\]
and
\[
I_a(t_2^2u(t_2x)) \geq I_a(t_1^2u(t_1x)) + \frac{t_3^2 - t_2^2}{3t_2^2} G_a(t_2^2u(t_2x)) + \frac{a(t_2 - t_1)^2(2t_2 + t_1)}{6t_2^2} \|u\|_2^2
\]
which yield \(t_1 = t_2\). Thus we complete the proof.

**Lemma 3.5.** Assume that \(f_1\), \(f_2\), \(f_3\) hold. Then \(I_a\) possesses the Mountain-Pass geometry.

**Proof.** By \((f_1), (f_2)\) and Sobolev’s imbedding inequality, we get
\[
\int_{\mathbb{R}^3} F(u) dx \leq \lambda \|u\|^2_{H^1(\mathbb{R}^3)} + C \|u\|^p_{H^1(\mathbb{R}^3)}.
\] (3.8)

Then there exist \(\rho, \delta > 0\) small such that
\[
I_a(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + au^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} \psi u^2 dx - \int_{\mathbb{R}^3} F(u) dx \\
\geq \min \left\{ \frac{1}{2}, \frac{a}{2} \right\} \|u\|^2_{H^1(\mathbb{R}^3)} - \lambda \|u\|^2_{H^1(\mathbb{R}^3)} + C \|u\|^p_{H^1(\mathbb{R}^3)} \\
\geq \delta > 0
\]
for \( \|u\|_{H^1(\mathbb{R}^3)} = \rho > 0 \). Fix \( u \in H^1(\mathbb{R}^3) \setminus \{0\} \), by Lemma 2.1, \((f_1)\) and \((f_3)\) we obtain
\[
I_a(t^2u(tx)) = \frac{1}{2} \int_{\mathbb{R}^3} t^3|\nabla u|^2 + atu^2dx + \frac{t^3}{4} \int_{\mathbb{R}^3} \psi u^2dx - \frac{1}{t^3} \int_{\mathbb{R}^3} F(t^2u)dx < 0
\]
for \( t > 0 \) large. Then \( \exists \ t_0 > 0 \), set \( u_0 = t_0^2u(t_0x) \), \( I_a(u_0) < 0 \). Thus we complete the proof. \( \square \)

Therefore, we can define the Mountain-Pass level of \( I_a \):
\[
c_a := \inf_{\gamma \in \mathcal{T}_a} \max_{t \in [0,1]} I_a(\gamma(t)),
\]
\[
\mathcal{T}_a := \{ \gamma \in C([0,1],H^1(\mathbb{R}^3)) : \gamma(0) = 0 \text{ and } I_a(\gamma(1)) < 0 \}. \tag{3.9}
\]
Next, we will construct the relation of Mountain-Pass level and Nehari-Phozaev manifold of \( I_a \). This idea back to Jeanjean [21].

**Lemma 3.6.** There exists a sequences \( \{u_n\}_{n=1}^{\infty} \) in \( H^1(\mathbb{R}^3) \) such that, as \( n \to \infty \),
\[
I_a(u_n) \to c_a > 0, \quad I_a'(u_n) \to 0, \quad \text{and } G_a(u_n) \to 0. \tag{3.10}
\]

**Lemma 3.7.** Assume that \((f_1)\)-(\(f_4)\) hold, every sequence \( \{u_n\} \) satisfying (3.10) is bounded in \( H^1(\mathbb{R}^3) \).

**Proof.** It follows from (3.10) and \((f_5)\), one has
\[
c_a + o(1) = I_a(u_n) \geq I_a(t_n^2u_n(t_nx) + \frac{1 - t_n^3}{3} G_a(u_n)
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^3} t_n^3|\nabla u_n|^2 + at_n u_n^2dx + \frac{t_n^3}{4} \int_{\mathbb{R}^3} \psi u_n^2dx
\]
\[
- \frac{1}{t_n^3} \int_{\mathbb{R}^3} F(t_n^2u_n)dx + \frac{1 - t_n^3}{3} G_a(u_n)
\]
\[
\geq \frac{t_n^3}{2} \|\nabla u_n\|^2 - C t_n \|u_n\|^2 - \frac{S^3}{216c_a} t_n^2 \|u_n\|^6 + \frac{1}{3} - \frac{t_n}{3} G_a(u_n)
\]
\[
\geq \frac{t_n^3}{2} \|\nabla u_n\|^2 - C t_n \|u_n\|^2 - \frac{1}{216c_a} t_n^2 \|\nabla u_n\|^6 + \frac{1 - t_n^3}{3} G_a(u_n)
\]
\[
= 2c_a - \frac{(6c_a)^\frac{1}{2} C}{\|\nabla u_n\|^2} \|u_n\|^2 + \frac{1 - t_n^3}{3} G_a(u_n)
\]
\[
= 2c_a + o(1)
\]
This contradiction implies that \( \{\|\nabla u_n\|_2\} \) is also bounded, so \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^3) \). \( \square \)
Lemma 3.8. There exist $C > 0$, $r > 0$ and $\xi_n \in \mathbb{R}^3$ such that
\[
\int_{B_r(\xi_n)} u_n^2 dx \geq C,
\]
where $\{u_n\}$ is given in (3.10).

Proof. Suppose by contradiction that the lemma does not hold, then by Lion’s concentration compactness principle [24], for $2 < p < 6$,
\[
\int_{\mathbb{R}^3} |u_n|^p dx \to 0 \quad \text{and} \quad \int_{\mathbb{R}^3} \psi u_n u_n^2 dx \to 0, \ n \to \infty.
\]
Using (f1), (f2) and $(I'_a(u_n), u_n) = o(1)$, we get
\[
\int_{\mathbb{R}^3} |\nabla u_n|^2 + au_n^2 dx = o(1).
\]  
(3.11)

By (3.11), we have $I_a(u_n) \to c_a = 0$, which contradicts to $c_a > 0$. Therefore we complete the proof. \hfill \Box

Lemma 3.9. Suppose that (f1)-(f4) hold. One has
\[
c_a = \inf_{a \in H^1(\mathbb{R}^3) \setminus \{0\}} \max_{t > 0} I_a(t^2u(tx)) = \inf_{a \in M_a} I_a(u) > 0.
\]

Proof. By Corollary 3.3 and Lemma 3.4, it is easy to check that
\[
\inf_{a \in H^1(\mathbb{R}^3) \setminus \{0\}} \max_{t > 0} I_a(t^2u(tx)) = \inf_{a \in M_a} I_a(u).
\]

For $u \in M_a$ and $t \in [0, 1]$, we have $\gamma(t) := (tT_0)^2 u(tT_0x) \in T_a$, where $I_a(T_0^2 u(T_0x)) < 0$. Using Corollary 3.3, we obtain $c_a \leq \inf_{a \in M_a} I_a(u)$. On the other hand, The manifold $M_a$ separates $H^1(\mathbb{R}^3)$ into two components. $G_a(u) > 0$ when $||u||_{H^1(\mathbb{R}^3)}$ is enough small. Since
\[
0 > 3I_a(\gamma(1)) = G_a(\gamma(1)) + a \int_{\mathbb{R}^3} (\gamma(1))^2 dx + \int_{\mathbb{R}^3} 2f(\gamma(1))\gamma(1) - 6F(\gamma(1)) dx,
\]
we get $G_a(\gamma(1)) < 0$ by (f5). Hence $\gamma$ cross $M_a$, one has
\[
c_a \geq \inf_{a \in M_a} I_a(u).
\]

Thus we complete the proof. \hfill \Box

Proposition 3.10. Suppose that(f1)-(f4) hold. Then problem (1.7) has a positive ground state solution $\tilde{u} \in H^1(\mathbb{R}^3)$.

Proof. Let $\{u_n\}$ be the sequence given in (3.10) and $c_a$ be the Mountain-Pass value for $I_a$. Set $\tilde{u}_n(x) = u_n(x + \xi_n)$, where $\{\xi_n\}$ is the sequence given by in Lemma 3.8. Up to a subsequence, we have
\[
\tilde{u}_n \rightharpoonup \tilde{u} \quad \text{weakly in } H^1(\mathbb{R}^3),
\]
\[
\tilde{u}_n \rightarrow \tilde{u} \quad \text{strongly in } L^p_{loc}(\mathbb{R}^3) \quad \text{for } 1 \leq p < 6,
\]
\[
\tilde{u}_n \rightarrow \tilde{u} \quad \text{a.e. in } \mathbb{R}^3,
\]
(3.12)

and $\tilde{u}$ is nontrivial. Moreover, $\tilde{u}$ satisfies
\[
- \Delta u + au + \psi u = f(u), \ x \in \mathbb{R}^3
\]  
(3.13)
and \( G_a(\tilde{u}) = 0 \). By \((f_5)\), Lemma 3.9 and Fatou’s Lemma, one has
\[
c_a \leq I_a(\tilde{u}) = I_a(\tilde{u}) - \frac{1}{3} G_a(\tilde{u}) = \frac{a}{3} \int_{\mathbb{R}^3} \tilde{u}^2 dx + \int_{\mathbb{R}^3} \frac{2}{3} f(\tilde{u}) \tilde{u} - 2F(\tilde{u}) dx
\]
\[
\leq \liminf_{n \to \infty} \left[ \frac{a}{3} \int_{\mathbb{R}^3} \tilde{u}_n^2 dx + \int_{\mathbb{R}^3} \frac{2}{3} f(\tilde{u}_n) \tilde{u}_n - 2F(\tilde{u}_n) dx \right]
\]
\[
= \liminf_{n \to \infty} \left[ I_a(\tilde{u}_n) - \frac{1}{3} G_a(\tilde{u}_n) \right]
\]
\[
= \liminf_{n \to \infty} \left[ I_a(u_n) - \frac{1}{3} G_a(u_n) \right] = c_a.
\]

Hence, \( I_a(\tilde{u}) = c_a \) and \( I_a'(\tilde{u}) = 0 \). Moreover, by the standard elliptic estimate and strong maximum principle (see [14]), \( \tilde{u} > 0 \) for all \( x \in \mathbb{R}^3 \), so \( \tilde{u} \) is a positive ground state solution of (1.7). Thus, we complete the proof. \( \square \)

Let \( S_a \) the set of ground state solutions \( U \) of (1.7) satisfying \( U(0) = \max_{x \in \mathbb{R}^3} U(x) \). Then, we get the following compactness of \( S_a \).

**Proposition 3.11.** Suppose that \((f_1)-(f_4)\) hold. \( S_a \) is compact in \( H^1(\mathbb{R}^3) \).

**Proof.** For any \( U \in S_a \), similar to the proof of Lemma 3.7, we can get \( S_a \) is bounded in \( H^1(\mathbb{R}^3) \). For any sequence \( \{U_k\} \subset S_a \), up to a subsequence, we may assume that there is \( U_0 \in H^1(\mathbb{R}^3) \) such that
\[
U_k \rightarrow U_0 \text{ in } H^1(\mathbb{R}^3)
\]
and \( U_0 \) satisfies
\[
-\Delta U_0 + aU_0 + \psi U_0 = f(U_0), \ x \in \mathbb{R}^3, \ U_0 \geq 0.
\]

Next, we will show that \( U_0 \) is nontrivial. Using Brezis-Kato’s type arguments, we can get
\[
\|U_k\|_{L^p_{\text{loc}}(\mathbb{R}^3)} \leq C \text{ for } 1 < p < \infty.
\]
By the classical Calderón-Zegmund \( L^p \) regularity estimates (see Theorem 9.9, [14]) and Morry-Sobolev embedding theorem, we can get that
\[
\|U_k\|_{C^{1,\alpha}_{\text{loc}}(\mathbb{R}^3)} \leq C \text{ for some } \alpha \in (0,1).
\]
Then using Schauder’s estimate, we have
\[
\|U_k\|_{C^{2,\alpha}_{\text{loc}}(\mathbb{R}^3)} \leq C.
\]
By the Arzela-Ascoli’s Theorem, we get
\[
U_k(0) \rightarrow U_0(0) \text{ as } k \rightarrow \infty.
\]
Since \( \Delta U_k(0) \leq 0 \), from (1.7), we can check that there exists \( b > 0 \) such that \( U_k(0) \geq b > 0 \), then \( U_0(0) \geq b > 0 \), so \( U_0 \) is nontrivial. Since
\[
c_a \leq I_a(U_0) - \frac{1}{3} G_a(U_0)
\]
\[
= \left( \frac{a}{2} - \frac{a}{6} \right) \int_{\mathbb{R}^3} U_0^2 dx - 2 \int_{\mathbb{R}^3} F(U_0) dx + \frac{2}{3} \int_{\mathbb{R}^3} f(U_0) U_0 dx
\]
\[
\leq \liminf_{n \to \infty} \left[ \left( \frac{a}{2} - \frac{a}{6} \right) \int_{\mathbb{R}^3} U_k^2 dx - 2 \int_{\mathbb{R}^3} F(U_k) dx + \frac{2}{3} \int_{\mathbb{R}^3} f(U_k) U_k dx \right]
\]
\[
= \liminf_{n \to \infty} \left[ I_a(U_k) - \frac{1}{3} G_a(U_k) \right] = c_a,
\]
which means that $I_a(U_0) = c_a$ and
\[ U_k \to U_0 \text{ in } L^2(\mathbb{R}^3). \tag{3.14} \]
Similar to the proof of Lemma 3.7, we can deduced \{$\|U_k\|_{H^1(\mathbb{R}^3)}$\} is bounded. Using the interpolation inequality, we get
\[ U_k \to U_0 \text{ in } L^p(\mathbb{R}^3), \quad \forall \ p \in [2, 6]. \]
On the other hand,
\[ \langle J_a'(U_k) - J_a'(U_0), U_k - U_0 \rangle = 0. \tag{3.15} \]
We can deduce that
\[
\|U_k - U_0\|_{H^1(\mathbb{R}^3)}^2 + \alpha \|\nabla(U_k - U_0)\|_2^2 = \int_{\mathbb{R}^3} \psi_{U_0} U_0 (U_k - U_0) dx - \int_{\mathbb{R}^3} \psi_{U_k} U_k (U_k - U_0) dx + \int_{\mathbb{R}^3} f(U_k)(U_k - U_0) dx - \int_{\mathbb{R}^3} f(U_0)(U_k - U) dx.
\]
Using Lemma 2.1, Lemma 2.2 and (3.15), it is standard to prove that
\[ \|U_k - U_0\|_{H^1(\mathbb{R}^3)} \to 0 \text{ as } k \to \infty. \tag{3.16} \]
Next, we claim that $\limsup_{k \to \infty} U_k(x) = 0$ uniformly for all $k$. To the contrary, we assume that there exist $\delta > 0$ and \{ $x_k$ \} $\subset \mathbb{R}^3$ with $|x_k| \to \infty$ such that $U_k(x_k) \geq \delta > 0$. Using bootstrap argument, we have
\[ U_k(x_k) \leq C\|U_k\|_{H^1(B_1(x_k))}. \]
Combining with (3.16), we obtain
\[ \delta \leq \lim_{k \to \infty} U_k(x_k) \leq C \lim_{k \to \infty} \|U_k(x_k)\|_{H^1(B_1(x_k))} = C \lim_{|x| \to \infty} \|U_0(x)\|_{H^1(B_1(x))} = 0. \]
So we prove the claim. Then by the comparison principle and elliptic estimates, we conclude exist two positive constants $C_3$ and $C_4$ such that
\[ |U_k(x)| \leq C_3 \exp(-C_4|x|). \]
This completes the proof that $S_a$ is compact in $H^1(\mathbb{R}^3)$. \hfill \Box

4. Proof of the Theorem 1.1. For the proof of our results, we do not handle system (1.1) directly, but instead we handle an equivalent system to (1.1). For this purpose, making the change of variable $\varepsilon y = x$, then (1.1) can be rewritten as
\[
\begin{cases}
-\Delta u + V(\varepsilon x)u + \psi u = f(u), \quad x \in \mathbb{R}^3, \\
-\Delta \psi = u^2, \quad x \in \mathbb{R}^3, \quad u > 0, \ u \in H^1(\mathbb{R}^3).
\end{cases}
\]
It is standard to show that $J_{\varepsilon} \in C^1(H_{\varepsilon}, \mathbb{R})$. To find solutions of (1.8) which concentrate around the local minimum of $V$ in $\Lambda$ as $\varepsilon \to 0$, we shall search critical points of $J_{\varepsilon}$ for which $Q_{\varepsilon}$ is zero.

Let $c_{V_0} = I_{V_0}(w)$ for $w \in S_{V_0}$ and $10\delta = \text{dist}(\mathcal{M}, \mathbb{R}^3 \setminus \Lambda)$, we fix a $\beta \in (0, \delta)$ and a cut-off function $\varphi \in C_0^\infty(\mathbb{R}^3)$ such that $0 \leq \varphi \leq 1$, $\varphi(x) = 1$ for $|x| \leq \beta$, $\varphi(x) = 0$ for $|x| \geq 2\beta$ and $|\nabla \varphi| \leq \frac{\beta}{\varepsilon}$. We will find a solution of (1.8) near the set
\[ X_\varepsilon := \left\{ \varphi(\varepsilon x - x')w(x - \frac{x'}{\varepsilon}) : x' \in \mathcal{M}^\beta, \ w \in S_{V_0} \right\} \]
for sufficiently small $\varepsilon > 0$, where we use the notation $\mathcal{M}_\beta := \{ y \in \mathbb{R}^3 : \inf_{x \in \mathcal{M}} |y - z| \leq \beta \}$. For $A \subset H_{\varepsilon}$, we define $A^a := \{ u \in H_{\varepsilon} : \inf_{v \in A} \|u - v\|_{H_{\varepsilon}} \leq a \}$. 
Fixed a $U^* \in S_{V_0}$, we also define
$$W_{\varepsilon,t} := t^2 \varphi(\varepsilon x)U^*(tx).$$
We will prove that $J_\varepsilon$ possesses the mountain pass geometry. On the one hand, by $(f_1)$, $(f_2)$ and $(f_3)$ we have
$$I_{V_0}(t^2U^*(tx)) = \frac{1}{2} \int_{\mathbb{R}^3} t^3|\nabla U^*(x)|^2 + V_0(tU^*(x))^2dx$$
$$+ \frac{t^3}{4} \int_{\mathbb{R}^3} \psi(U^*(x))(U^*(x))^2dx - \frac{1}{t^3} \int_{\mathbb{R}^3} F(t^2U^*(x))dx \to -\infty,$$
then there exists $t_0 > 0$ such that $I_{V_0}(t_0^2U^*(t_0x)) < -3$.
We can easily check that $Q_\varepsilon(W_{\varepsilon,t_0}) = 0$. Using the Lebesgue’s dominated convergence theorem and Lemma 2.2-(i), then
$$J_\varepsilon(W_{\varepsilon,t_0})$$
$$= L(V_{\varepsilon,t_0})$$
$$= \frac{1}{2} \int_{\mathbb{R}^3} |\nabla V_{\varepsilon,t_0}|^2 + V(\varepsilon x)V_{\varepsilon,t_0}dx$$
$$+ \frac{1}{4} \int_{\mathbb{R}^3} \psi_{\varepsilon,t_0}V_{\varepsilon,t_0}dx - \int_{\mathbb{R}^3} F(V_{\varepsilon,t_0})dx$$
$$= \frac{1}{2} t_0^3 \int_{\mathbb{R}^3} |\varphi(\frac{x}{t_0})U^*(x) + \varphi(\frac{x}{t_0})\nabla U^*|^2 d\tilde{x}$$
$$+ \frac{1}{2} t_0^3 \int_{\mathbb{R}^3} V(\frac{x}{t_0})\varphi^2(\frac{x}{t_0})(U^*(x))^2d\tilde{x}$$
$$+ \frac{1}{4} t_0^3 \int_{\mathbb{R}^3} \psi_{\varphi(\frac{x}{t_0})U^*(x)}\varphi^2(\frac{x}{t_0})U^*(x)dx - \frac{1}{t_0^3} \int_{\mathbb{R}^3} F(t_0^2\varphi(\frac{x}{t_0})U^*(x))dx$$
$$= I_{V_0}(t_0^2U^*(t_0x)) + o(1) < -2$$
for $\varepsilon > 0$ small. Using $(f_1)$, $(f_2)$ and Sobolev’s imbedding theorem, one has
$$J_\varepsilon(u) \geq L(u)$$
$$= \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + V(\varepsilon u)u^2dx + \frac{1}{4} \int_{\mathbb{R}^3} \psi_xu^2dx - \int_{\mathbb{R}^3} F(u)dx$$
$$\geq \frac{1}{2} \|u\|_{H_\varepsilon}^2 - \lambda \|u\|_{H_\varepsilon}^2 - C\|u\|_{H_\varepsilon}^p > 0$$
for $\|u\|_{H_\varepsilon}$ small since $p > 2$.
Hence, we define the Mountain-Pass level value of $J_\varepsilon$ by
$$c_\varepsilon := \inf_{\gamma \in T_\varepsilon} \max_{s \in [0,1]} J_\varepsilon(\gamma(s)),$$
where
$$T_\varepsilon := \{ \gamma \in C([0,1], H_\varepsilon) : \gamma(0) = 0, \gamma(1) = W_{\varepsilon,t_0} \}.$$

**Lemma 4.1.**
$$\limsup_{\varepsilon \to 0} c_\varepsilon \leq c_{V_0}.$$

**Proof.** Define
$$W_{\varepsilon,0} = \lim_{t \to 0} W_{\varepsilon,t}$$
in $H_\varepsilon$, then $W_{\varepsilon,0} = 0$. Setting $\gamma(s) := W_{\varepsilon,s t_0}(0 \leq s \leq 1)$, we have $\gamma(s) \in T_\varepsilon$, so there exists an $\varepsilon_n \to W(\varepsilon s t_0)$.

Thus, for

$$c_\varepsilon \leq \max_{s \in [0,1]} J_\varepsilon(\gamma(s)) = \max_{t \in [0,t_0]} J_\varepsilon(W_{\varepsilon,t}).$$

So we only need to show that

$$\limsup_{\varepsilon \to 0} \max_{t \in [0,t_0]} J_\varepsilon(W_{\varepsilon,t}) \leq C_{V_0}.$$ 

Indeed, similar to (4.1), one has

$$\max_{t \in [0,t_0]} J_\varepsilon(W_{\varepsilon,t}) = \max_{t \in [0,t_0]} I_{V_0}(t^2 U^*(tx)) + o(1) \leq \max_{t \in [0,\infty]} I_{V_0}(t^2 U^*(tx)) + o(1) = I_{V_0}(U^*(x)) + o(1) = c_{V_0} + o(1).$$

Lemma 4.2.

$$\liminf_{\varepsilon \to 0} c_\varepsilon \geq c_{V_0}.$$ 

Proof. Assuming the contrary that $\liminf_{\varepsilon \to 0} c_\varepsilon < c_{V_0}$, then, there exist $\delta_0 > 0$, $\varepsilon_n \to 0$ and $\gamma_n \in T_{\varepsilon_n}$ satisfying $J_{\varepsilon_n}(\gamma_n(s)) < c_{V_0} - \delta_0$ for $s \in [0,1]$. Fixed an $\varepsilon_n$ such that

$$\frac{1}{2}V_0\varepsilon_n(1 + (1 + c_{V_0})^\frac{1}{2}) < \min\{\delta_0, 1\}. \quad (4.2)$$

Since $I_{\varepsilon_n}(\gamma_n(0)) = 0$ and

$$I_{\varepsilon_n}(\gamma_n(1)) \leq J_{\varepsilon_n}(\gamma_n(1)) = J_{\varepsilon_n}(W_{\varepsilon_n,t_0}) < -2,$$

so there exists an $s_n \in (0, 1]$ such that $I_{\varepsilon_n}(\gamma_n(s)) \geq -1$ for $s \in [0, s_n]$ and $I_{\varepsilon_n}(\gamma_n(s_n)) = -1$. Then, for any $s \in [0, s_n]$,

$$Q_{\varepsilon_n}(\gamma_n(s)) = J_{\varepsilon_n}(\gamma_n(s)) - I_{\varepsilon_n}(\gamma_n(s)) \leq 1 + c_{V_0} - \delta_0,$$

this implies that

$$\int_{\mathbb{R}^3 \setminus (A/\varepsilon_n)} \gamma_n^2(s) dx \leq \varepsilon_n(1 + (1 + c_{V_0})^\frac{1}{2}) \text{ for } s \in [0, s_n].$$

Thus, for $s \in [0, s_n]$,

$$I_{\varepsilon_n}(\gamma_n(s)) = I_{V_0}(\gamma_n(s)) + \frac{1}{2} \int_{\mathbb{R}^3} (V(\varepsilon_n x) - V_0)\gamma_n^2(s) dx$$

$$\geq I_{V_0}(\gamma_n(s)) + \frac{1}{2} \int_{\mathbb{R}^3 \setminus (A/\varepsilon_n)} (V(\varepsilon_n x) - V_0)\gamma_n^2(s) dx$$

$$\geq I_{V_0}(\gamma_n(s)) - \frac{1}{2} V_0\varepsilon_n(1 + (1 + c_{V_0})^\frac{1}{2}),$$

then

$$I_{V_0}(\gamma_n(s)) \leq I_{\varepsilon_n}(\gamma_n(s)) + \frac{1}{2} V_0\varepsilon_n(1 + (1 + c_{V_0})^\frac{1}{2})$$

$$= -1 + \frac{1}{2} V_0\varepsilon_n(1 + (1 + c_{V_0})^\frac{1}{2}) < 0,$$

and using (3.9), we obtain that

$$\max_{s \in [0, s_n]} I_{V_0}(\gamma_n(s)) \geq c_{V_0}.$$
So, we get that
\[ c_{\nu_0} - \delta_0 \geq \max_{s \in [0,1]} J_{\varepsilon_n}(\gamma_n(s)) \geq \max_{s \in [0,1]} I_{\varepsilon_n}(\gamma_n(s)) \]
\[ \geq \max_{s \in [0,s_n]} I_{\varepsilon_n}(\gamma_n(s)) \]
\[ \geq \max_{s \in [0,s_n]} I_{\nu_0}(\gamma_n(s)) - \frac{1}{2} V_0 \varepsilon_n(1 + (1+c_{\nu_0})^\frac{1}{2}), \]
we deduce that \( 0 < \delta_0 \leq \frac{1}{2} V_0 \varepsilon_n(1 + (1+c_{\nu_0})^\frac{1}{2}) \), which contradicts to (4.2).

Lemma 4.1 and Lemma 4.2 imply that
\[ \lim_{\varepsilon \to 0} \max_{s \in [0,1]} J_{\varepsilon}(\gamma_{\varepsilon}(s)) - c_{\varepsilon} = 0, \]
where \( \gamma_{\varepsilon}(s) = W_{\varepsilon,s,0} \) for \( s \in [0,1] \). Define \( \tilde{c}_{\varepsilon} := \max_{s \in [0,1]} J_{\varepsilon}(\gamma_{\varepsilon}(s)) \), it is easy to see that
\[ c_{\varepsilon} \leq \tilde{c}_{\varepsilon} \text{ and } \lim_{\varepsilon \to 0} c_{\varepsilon} = \lim_{\varepsilon \to 0} \tilde{c}_{\varepsilon} = c_{\nu_0}. \] (4.3)

In order to introduce the following lemma, we need some notations. \( H_0^1(B_R(0)) \) as a subspace of \( H_\varepsilon \), then \( \| \cdot \|_{H_\varepsilon} \) is equivalent to the standard norm of \( H_0^1(B_R(0)) \) for each \( R > 0, \varepsilon > 0 \). For each \( L \in (H_0^1(B_R(0)))^{-1} \), we define
\[ \| L \|_{*,\varepsilon,R} := \sup \{ L u : u \in H_0^1(B_R(0)), \| u \|_{H_\varepsilon} \leq 1 \}. \]
We note that \( \| \cdot \|_{*,\varepsilon,R} \) is equivalent to the standard norm of \( (H_0^1(B_R(0)))^{-1} \). Set
\[ J_{\varepsilon}^{\alpha} := \{ u \in H_\varepsilon : J_{\varepsilon}(u) \leq \alpha \} \]
and fix a \( R_0 > 0 \) such that \( B_{R_0}(0) \supset \Lambda \).

The next lemma plays a key role in proving the Theorem 1.1.

**Lemma 4.3.** (i) There exists a \( d_0 > 0 \) such that for any \( \{ \varepsilon_i \}_{i=1}^\infty \), \( \{ R_{\varepsilon_i} \}, \{ u_{\varepsilon_i} \} \) with
\[ \left\{ \begin{array}{l} \lim_{i \to \infty} \varepsilon_i = 0, \quad R_{\varepsilon_i} \geq R_0/\varepsilon_i, \quad u_{\varepsilon_i} \in X_{\varepsilon_i}^{d_0} \cap H_0^1(B_{R_{\varepsilon_i}}(0)), \\ \lim_{i \to \infty} J_{\varepsilon_i}(u_{\varepsilon_i}) \leq c_{\nu_0} \text{ and } \lim_{i \to \infty} \| J'_{\varepsilon_i}(u_{\varepsilon_i}) \|_{*,\varepsilon_i,R_{\varepsilon_i}} = 0, \end{array} \right. \] (4.4)
there exists, up to a subsequence, \( \{ y_i \}_{i=1}^\infty \subseteq \mathbb{R}^3 \), \( x_0 \in \mathcal{M}, U \in S_{\nu_0} \) such that
\[ \lim_{i \to \infty} \varepsilon_i y_i - x_0 = 0 \text{ and } \lim_{i \to \infty} \| u_{\varepsilon_i} - \varphi(\varepsilon_i x - \varepsilon_i y_i) U(x - y_i) \|_{H_{\varepsilon_i}} = 0. \]

(ii) If we drop \( \{ R_{\varepsilon_i} \} \) and replace (4.4) by
\[ \left\{ \begin{array}{l} \lim_{i \to \infty} \varepsilon_i = 0, \quad u_{\varepsilon_i} \in X_{\varepsilon_i}^{d_0}, \quad \lim_{i \to \infty} J_{\varepsilon_i}(u_{\varepsilon_i}) \leq c_{\nu_0} \text{ and } \lim_{i \to \infty} \| J'_{\varepsilon_i}(u_{\varepsilon_i}) \|_{(H_{\varepsilon_i})^{-1}} = 0, \end{array} \right. \] (4.5)
then the same conclusion holds.

**Proof.** We only prove (i). The proof of (ii) is similar. For notational brevity, we write \( \varepsilon \) for \( \varepsilon_i \), and still use \( \varepsilon \) after taking a subsequence. By the definition of \( X_{\varepsilon}^{d_0} \), there exist \( \{ U_\varepsilon \} \subseteq S_{\nu_0} \) and \( \{ x_\varepsilon \} \subseteq \mathcal{M}^d \) such that
\[ \| u_\varepsilon - \varphi(\varepsilon x - x_\varepsilon) U_\varepsilon(x - \frac{x_\varepsilon}{\varepsilon}) \|_{H_{\varepsilon}} \leq \frac{3}{2} d_0. \]
Since \( S_{\nu_0} \) and \( \mathcal{M}^d \) are compact, so there exists \( U_0 \in S_{\nu_0}, x_0 \in \mathcal{M}^3 \) such that \( U_\varepsilon \to U_0 \) in \( H^1(\mathbb{R}^3) \). Thus, for \( \varepsilon > 0 \) small,
\[ \| u_\varepsilon - \varphi(\varepsilon x - x_0) U_0(x - \frac{x_0}{\varepsilon}) \|_{H_{\varepsilon}} \leq 2 d_0, \] (4.6)
Claim 1. We claim that

$$\limsup_{\varepsilon \to 0} \int_{B_1(y)} |u_\varepsilon|^2 \, dx = 0,$$

where $A_\varepsilon = B_{\frac{3\varepsilon}{2}}(\frac{x_0}{\varepsilon}) \setminus B_{\frac{\varepsilon}{2}}(\frac{x_0}{\varepsilon})$.

Indeed, assuming the contrary, there exists $r > 0$ such that

$$\limsup_{\varepsilon \to 0} \int_{B_1(y)} |u_\varepsilon|^2 \, dx = 2r > 0,$$

then there exists $y_\varepsilon \in A_\varepsilon$ such that for $\varepsilon > 0$ small,

$$\int_{B_1(y_\varepsilon)} |u_\varepsilon|^2 \, dx \geq r > 0.$$

Note that $y_\varepsilon \in A_\varepsilon$, there exists $x^* \in \mathcal{M}^{1,3} \subset \Lambda$ such that $\varepsilon y_\varepsilon \to x^*$ as $\varepsilon \to 0$. Set $v_\varepsilon(x) := u_\varepsilon(x + y_\varepsilon)$, then, for $\varepsilon > 0$ small,

$$\int_{B_1(y_\varepsilon)} |v_\varepsilon|^2 \, dx \geq r > 0,$$

up to a subsequence, $v_\varepsilon \to v$ in $H^1(\mathbb{R}^3)$ and $v$ satisfies

$$-\Delta v + V(x^*)v + \psi_\varepsilon v = f(v), \quad x \in \mathbb{R}^3, \quad v \geq 0.$$

It is easy to see $v \neq 0$, then

$$c_{V(x^*)} \leq I_{V(x^*)}(v) - \frac{1}{3} G_{V(x^*)}(v)$$

$$= \frac{1}{3} V(x^*) \int_{\mathbb{R}^3} v^2 \, dx + \frac{2}{3} \int_{\mathbb{R}^3} f(v)v \, dx - 2 \int_{\mathbb{R}^3} F(v) \, dx,$$

one has

$$\|V\|_{L^\infty(\Lambda)} \int_{\mathbb{R}^3} v^2 \, dx + 2 \int_{\mathbb{R}^3} f(v)v \, dx - 6 \int_{\mathbb{R}^3} F(v) \, dx$$

$$\geq V(x^*) \int_{\mathbb{R}^3} v^2 \, dx + 2 \int_{\mathbb{R}^3} f(v)v \, dx - 6 \int_{\mathbb{R}^3} F(v) \, dx$$

$$\geq 3c_{V(x^*)} \geq 3c_{V_0}.$$

Therefore, for sufficiently large $R$,

$$\liminf_{\varepsilon \to 0} \left[ \|V\|_{L^\infty(\Lambda)} \int_{B_R(y_\varepsilon)} u_\varepsilon^2 \, dx + 2 \int_{B_R(y_\varepsilon)} f(u_\varepsilon)u_\varepsilon \, dx - 6 \int_{B_R(y_\varepsilon)} F(u_\varepsilon) \, dx \right]$$

$$= \liminf_{\varepsilon \to 0} \left[ \|V\|_{L^\infty(\Lambda)} \int_{B_R(0)} v_\varepsilon^2 \, dx + 2 \int_{B_R(0)} f(v_\varepsilon)v_\varepsilon \, dx - 6 \int_{B_R(0)} F(v_\varepsilon) \, dx \right]$$

$$\geq \left[ \|V\|_{L^\infty(\Lambda)} \int_{B_R(0)} v^2 \, dx + 2 \int_{B_R(0)} f(v)v \, dx - 6 \int_{B_R(0)} F(v) \, dx \right]$$

$$\geq \frac{1}{2} \left[ \|V\|_{L^\infty(\Lambda)} \int_{\mathbb{R}^3} v^2 \, dx + 2 \int_{\mathbb{R}^3} f(v)v \, dx - 6 \int_{\mathbb{R}^3} F(v) \, dx \right] \geq \frac{3}{2} c_{V_0} > 0.$$

On the other hand, using the Sobolev’s imbedding theorem and (4.6),

$$\|V\|_{L^\infty(\Lambda)} \int_{B_R(y_\varepsilon)} u_\varepsilon^2 \, dx + 2 \int_{B_R(y_\varepsilon)} f(u_\varepsilon)u_\varepsilon \, dx - 6 \int_{B_R(y_\varepsilon)} F(u_\varepsilon) \, dx$$
where $o(1) \to 0$ as $\varepsilon \to 0$, and we have used the fact that $|y_\varepsilon - \frac{x_0}{\varepsilon}| \geq \frac{\beta}{2\varepsilon}$. When we choose $d_0$ is small enough, we get a contradiction. So we prove the claim 1. Now, claim 1 can deduce that
\[ \lim_{\varepsilon \to 0} \int_{B_\varepsilon} |u_\varepsilon|^{p_1} \, dx = 0, \tag{4.10} \]
where $B_\varepsilon = B_{\frac{2\varepsilon}{\beta}}(\frac{2\varepsilon}{\beta}) \setminus B_{\frac{\varepsilon}{2\varepsilon}}(\frac{\varepsilon}{2\varepsilon})$ and $2 < p_1 < 6$. Indeed, since
\[ \sup_{y \in A_\varepsilon} \int_{B_1(y)} |u_\varepsilon|^2 \, dx \geq \sup_{y \in \mathbb{R}^3} \int_{B_1(y)} |u_\varepsilon \cdot \chi_{A_\varepsilon}^1|^2, \]
where $A_\varepsilon^1 = B_{\frac{\varepsilon}{2\varepsilon}}(\frac{\varepsilon}{2\varepsilon}) \setminus B_{\frac{\varepsilon}{2\varepsilon}}(\frac{\varepsilon}{2\varepsilon})$, by Lion’s concentration compactness principle, we obtain
\[ \int_{\mathbb{R}^3} |u_\varepsilon \cdot \chi_{A_\varepsilon}^1|^{p_1} \, dx \to 0 \text{ as } \varepsilon \to 0, \]
where $2 < p_1 < 6$. Since $B_\varepsilon \subset A_\varepsilon^1$ for $\varepsilon > 0$ small, (4.10) holds.

**Claim 2.** Let $u_{\varepsilon,1}(x) = \varphi(\varepsilon x - x_0)u_\varepsilon(x)$, $u_{\varepsilon,2} = (1 - \varphi(\varepsilon x - x_0))u_\varepsilon(x)$. Using (4.10) and direct computation, we can get that
\[ \int_{\mathbb{R}^3} F(u_\varepsilon) \, dx = \int_{\mathbb{R}^3} F(u_{\varepsilon,1}) \, dx + \int_{\mathbb{R}^3} F(u_{\varepsilon,2}) \, dx + o(1), \]
\[ \int_{\mathbb{R}^3} \nabla u_\varepsilon^2 \, dx \geq \int_{\mathbb{R}^3} \nabla u_{\varepsilon,1}^2 \, dx + \int_{\mathbb{R}^3} \nabla u_{\varepsilon,2}^2 \, dx + o(1), \]
\[ \int_{\mathbb{R}^3} V(\varepsilon x)|u_\varepsilon|^2 \, dx \geq \int_{\mathbb{R}^3} V(\varepsilon x)|u_{\varepsilon,1}|^2 \, dx + \int_{\mathbb{R}^3} V(\varepsilon x)|u_{\varepsilon,2}|^2 \, dx, \]
\[ \int_{\mathbb{R}^3} \psi_{u_\varepsilon}(u_\varepsilon)^2 \, dx \geq \int_{\mathbb{R}^3} \psi_{u_{\varepsilon,1}}(u_{\varepsilon,1})^2 \, dx + \int_{\mathbb{R}^3} \psi_{u_{\varepsilon,2}}(u_{\varepsilon,2})^2 \, dx, \]
\[ Q_\varepsilon(u_{\varepsilon,1}) = 0, \quad Q_\varepsilon(u_{\varepsilon,2}) = Q_\varepsilon(u_\varepsilon) \geq 0. \]
So we have
\[ J_\varepsilon(u_\varepsilon) \geq J_\varepsilon(u_{\varepsilon,1}) + J_\varepsilon(u_{\varepsilon,2}) + o(1). \tag{4.11} \]
Now, we claim that $\|u_{\varepsilon,2}\|_{H_\varepsilon} \to 0$ as $\varepsilon \to 0$. By (4.6), one has
\[ \|u_{\varepsilon,2}\|_{H_\varepsilon} \leq \|u_{\varepsilon,1} - \varphi(\varepsilon x - x_0)U_0(x - \frac{x_0}{\varepsilon})\|_{H_\varepsilon} + 2d_0 \]
\[ = \|u_{\varepsilon,1} - \varphi(\varepsilon x - x_0)U_0(x - \frac{x_0}{\varepsilon})\|_{H_\varepsilon(B_{\frac{2\varepsilon}{\beta}}(\frac{2\varepsilon}{\beta}))} + 2d_0. \]
We can check that 

\[ \left\| u_{\varepsilon,2} \right\|_{L^2(\mathbb{R}^3)} + 4d_0 \]

\[ = \left\| u_{\varepsilon,2} \right\|_{H^s(\mathbb{R}^3 \setminus B_{\varepsilon/2}(x_0))} + 4d_0 \]

\[ \leq C \left\| u_{\varepsilon} \right\|_{H^s(\mathbb{R}^3 \setminus B_{\varepsilon}(x_0))} + 4d_0 \]

\[ \leq C \| \varphi(x - x_0)U_0(x - \frac{x_0}{\varepsilon}) \|_{H^s(\mathbb{R}^3 \setminus B_{\varepsilon}(x_0))} + Cd_0 \]

(4.12)

\[ \leq C \| U_0(x - \frac{x_0}{\varepsilon}) \|_{H^s(\mathbb{R}^3 \setminus B_{\varepsilon}(x_0))} + Cd_0 \]

\[ \leq C \| U_0 \|_{H^s(B_{\varepsilon}(0))} + Cd_0 = C\varepsilon_0 + o(1), \]

as \( \varepsilon \to 0 \). So we obtain that \( \limsup_{\varepsilon \to 0} \| u_{\varepsilon,2} \|_{H^s} \leq C\varepsilon_0 \).

Using the facts that \( \langle J'_{\varepsilon}(u_\varepsilon), u_{\varepsilon,2} \rangle \to 0 \), as \( \varepsilon \to 0 \) and

\[ \langle Q'_{\varepsilon}(u_\varepsilon), u_{\varepsilon,2} \rangle = \langle Q'_{\varepsilon}(u_\varepsilon,2), u_{\varepsilon,2} \rangle \geq 0. \]

Combining with (f1) and (f2), we have

\[
\int_{\mathbb{R}^3} \nabla u_\varepsilon \cdot \nabla u_{\varepsilon,2} + V(\varepsilon x)u_\varepsilon u_{\varepsilon,2} dx \leq \int_{\mathbb{R}^3} f(u_\varepsilon)u_{\varepsilon,2} dx
\]

\[ \leq \frac{1}{2} \int_{\mathbb{R}^3} |u_\varepsilon|u_{\varepsilon,2} dx + C \int_{\mathbb{R}^3} |u_\varepsilon|^{p-1}u_{\varepsilon,2} dx + o(1). \]

Then by (4.10) and Sobolev’s embedding theorem that \( \| u_{\varepsilon,2} \|_{H^s} \leq C \| u_{\varepsilon,2} \|_{H^s} + o(1) \). Taking \( d_0 \) enough small, we deduce that \( \| u_{\varepsilon,2} \|_{H^s} = o(1) \). Then combine with (4.11), we get that

\[ J_{\varepsilon}(u_\varepsilon) \geq I_{\varepsilon}(u_{\varepsilon,1}) + o(1). \]

(4.13)

**Claim 3.** Set \( \tilde{w}_\varepsilon(x) = u_{\varepsilon,1}(x + \frac{x_0}{\varepsilon}) = \varphi(\varepsilon x)u_\varepsilon(x + \frac{x_0}{\varepsilon}) \), up to a subsequence, there exists \( \tilde{w} \in H^1(\mathbb{R}^3) \) such that

\[ \tilde{w}_\varepsilon \rightharpoonup \tilde{w} \text{ in } H^1(\mathbb{R}^3) \]

(4.14)

and

\[ \tilde{w}_\varepsilon(x) \to \tilde{w}(x) \text{ a.e. in } \mathbb{R}^3. \]

(4.15)

We can check that

\[ o(1) = \langle J'_{\varepsilon}(u_\varepsilon), \tilde{\varphi}(x - \frac{x_0}{\varepsilon}) \rangle \]

\[ = \int_{\mathbb{R}^3} \nabla u_\varepsilon(x + \frac{x_0}{\varepsilon}) \nabla \tilde{\varphi} + V(\varepsilon x + x_0)u_\varepsilon(x + \frac{x_0}{\varepsilon}) \tilde{\varphi} dx \]

\[ + \int_{\mathbb{R}^3} \psi_{\varepsilon}(x + \frac{x_0}{\varepsilon})u_\varepsilon(x + \frac{x_0}{\varepsilon}) \tilde{\varphi} dx - \int_{\mathbb{R}^3} f(u_\varepsilon(x + \frac{x_0}{\varepsilon})) \tilde{\varphi} dx \]

(4.16)

where \( \tilde{\varphi} \in C_0^\infty(\mathbb{R}^3) \). Here we have used the fact

\[ \int_{\mathbb{R}^3} \chi_\varepsilon(x)u_\varepsilon(x) \tilde{\varphi}(x - \frac{x_0}{\varepsilon}) dx = 0 \]

and \( \| u_{\varepsilon,2} \|_{H^s} \to 0 \) as \( \varepsilon \to 0 \). It implies that \( \tilde{w} \geq 0 \) satisfies that

- \( - \Delta \tilde{w} + V(x_0)\tilde{w} + \psi_\varepsilon \tilde{w} = f(\tilde{w}) \), \( x \in \mathbb{R}^3, \tilde{w} \geq 0. \)

(4.17)

**We claim that**

\[ \tilde{w}_\varepsilon \to \tilde{w} \text{ in } L^p(\mathbb{R}^3), \]

(4.18)
where $2 < p_1 < 6$.

Indeed, assuming the contrary, there exists $r > 0$ such that

$$
\lim \sup_{\varepsilon \to 0} \int_{B_{1}(z_{\varepsilon})} |\bar{w}_{\varepsilon} - \tilde{w}|^2 dx = 2r > 0.
$$

(4.19)

So when $\varepsilon$ enough small, there exists $z_{\varepsilon} \in \mathbb{R}^3$ such that

$$
\int_{B_{1}(z_{\varepsilon})} |\bar{w}_{\varepsilon} - \tilde{w}|^2 dx \geq r > 0.
$$

(4.20)

**Case 1.** \{\{z_{\varepsilon}\} is bounded, so we choose $\alpha > 0$ such that $|z_{\varepsilon}| \leq \alpha$, then for $\varepsilon > 0$ enough small,

$$
\int_{B_{\alpha+1}(0)} |\tilde{v}_{\varepsilon}|^2 \geq r > 0,
$$

(4.21)

where $\tilde{v}_{\varepsilon} = \bar{w}_{\varepsilon} - \tilde{w}$ and $\tilde{v}_{\varepsilon} \to 0$ in $H^1(\mathbb{R}^3)$. By Rellich imbedding theorem $[26, 41]$, one has

$$
\lim_{\varepsilon \to 0} \int_{B_{\alpha+1}(0)} |\tilde{v}_{\varepsilon}|^2 = 0,
$$

which contradicts to (4.21).

**Case 2.** \{\{z_{\varepsilon}\} is unbounded.** Without loss of generality, $\lim_{\varepsilon \to 0} |z_{\varepsilon}| = \infty$. Then, using (4.19),

$$
\liminf_{\varepsilon \to 0} \int_{B_{1}(z_{\varepsilon})} |\bar{w}_{\varepsilon}|^2 \geq r > 0,
$$

(4.22)

i.e.

$$
\liminf_{\varepsilon \to 0} \int_{B_{1}(z_{\varepsilon})} \left| \varphi(\varepsilon x)u_{\varepsilon}(x + \frac{x_0}{\varepsilon}) \right|^2 \geq r > 0.
$$

Since $\varphi(x) = 0$ for $|x| \geq \frac{3\beta}{\varepsilon}$, it implies that $|z_{\varepsilon}| \leq \frac{3\beta}{\varepsilon}$ for $\varepsilon > 0$ enough small. Moreover, we have $|z_{\varepsilon}| \leq \frac{\beta}{\varepsilon}$ for $\varepsilon > 0$ small. Indeed, If $|z_{\varepsilon}| > \frac{\beta}{\varepsilon}$, then $z_{\varepsilon} \in B_{\frac{3\beta}{\varepsilon}}(0) \setminus B_{\frac{\beta}{\varepsilon}}(0)$. By (4.7), one has

$$
\liminf_{\varepsilon \to 0} \int_{B_{1}(z_{\varepsilon})} |\tilde{v}_{\varepsilon}|^2 dx \leq \liminf_{\varepsilon \to 0} \sup_{z_{\varepsilon} \in B_{\frac{3\beta}{\varepsilon}}(0) \setminus B_{\frac{\beta}{\varepsilon}}(0)} \int_{B_1(z)} \left| u_{\varepsilon}(x + \frac{x_0}{\varepsilon}) \right|^2 dx
$$

$$
= \liminf_{\varepsilon \to 0} \sup_{z_{\varepsilon} \in A_{\varepsilon}} \int_{B_1(z)} \left| u_{\varepsilon}\right|^2 dx = 0,
$$

which is a contradiction. Therefore, $|z_{\varepsilon}| \leq \frac{\beta}{\varepsilon}$ for $\varepsilon > 0$ enough small. So we can assume that $\varepsilon z_{\varepsilon} \to z_0 \in B_{\frac{\beta}{2}}(0)$ and $\tilde{w}_\varepsilon(x) := \bar{w}_\varepsilon(x + z_{\varepsilon}) \to \tilde{w}(x)$ in $H^1(\mathbb{R}^3)$. By (4.22), it is easy to see that $\tilde{w} \neq 0$ and $\tilde{w}$ satisfies

$$
-\Delta \tilde{w} + V(x_0 + z_0)\tilde{w} + \psi \tilde{w} = f(\tilde{w}), \quad x \in \mathbb{R}^3, \quad \tilde{w} \geq 0.
$$

Similar to Claim 1, since $0 \in \mathcal{M}$, then

$$
\text{dist}(\{x_0 + z_0\}, \mathcal{M}) \leq \text{dist}(x_0, \mathcal{M}) + \text{dist}(z_0, \mathcal{M}) \leq \beta + \frac{\beta}{2}.
$$

So $x_0 + z_0 \in \Lambda$. For sufficiently large $R$, we have

$$
\lim \inf_{\varepsilon \to 0} \left[ \|V\|_{L^\infty(\bar{\Omega})} \int_{B_R(z_{\varepsilon})} \tilde{w}_{\varepsilon}^2 dx + 2 \int_{B_R(z_{\varepsilon})} f(\tilde{w}_\varepsilon)\tilde{w}_{\varepsilon} dx - 6 \int_{B_R(z_{\varepsilon})} F(\tilde{w}_\varepsilon) dx \right] \geq \frac{3}{2} \epsilon v_0 > 0.
$$

(4.23)
On the other hand, using the Sobolev’s imbedding theorem and (4.6),
\[ \| V \|_{L^\infty(\mathbb{A})} \int_{B_R(z_\varepsilon)} \tilde{w}_\varepsilon^2 dx + 2 \int_{B_R(z_\varepsilon)} f(\tilde{w}_\varepsilon) \tilde{w}_\varepsilon dx - 6 \int_{B_R(z_\varepsilon)} F(\tilde{w}_\varepsilon) dx \]
\[ = \| V \|_{L^\infty(\mathbb{A})} \int_{B_R(z_\varepsilon)} (u_\varepsilon(x + \frac{x_0}{\varepsilon}))^2 dx + 2 \int_{B_R(z_\varepsilon)} f(\tilde{w}_\varepsilon) \tilde{w}_\varepsilon dx - 6 \int_{B_R(z_\varepsilon)} F(\tilde{w}_\varepsilon) dx \]
\[ \leq C d_0 + C \int_{B_R(z_\varepsilon)} |\varphi(\varepsilon x) U_0(x)|^2 dx + C \int_{B_R(z_\varepsilon)} |\varphi(\varepsilon x) U_0(x)|^p dx \]
\[ \leq C d_0 + C \int_{B_R(z_\varepsilon)} |U_0(x)|^2 dx + C \int_{B_R(z_\varepsilon)} |U_0(x)|^p dx \]
\[ \leq C d_0 + o(1), \quad (4.24) \]
where \( o(1) \to 0 \) as \( \varepsilon \to 0 \), and we have used the fact that \( z_\varepsilon \to \infty \). We obtain a contradiction with (4.23) when choose \( d_0 \) is small enough. So we complete the proof of Claim 3.

In view of (4.13) and the assumption of Lemma 4.3, recall that \( \tilde{w}_\varepsilon(x) = u_{\varepsilon,1}(x + \frac{x_0}{\varepsilon}) \), then
\[ \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \tilde{w}_\varepsilon|^2 + V(\varepsilon x + x_0) \tilde{w}_\varepsilon^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} \psi_{\tilde{w}_\varepsilon} \tilde{w}_\varepsilon^2 dx - \int_{\mathbb{R}^3} F(\tilde{w}_\varepsilon) dx \leq c_{V_0} + o(1). \]

Using Lemma 2.1-(iii), (4.14), (4.15) and (4.18), we obtain
\[ \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \tilde{w}|^2 + V(x_0) \tilde{w}_\varepsilon^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} \psi_{\tilde{w}_\varepsilon} \tilde{w}_\varepsilon^2 dx - \int_{\mathbb{R}^3} F(\tilde{w}) dx \leq c_{V_0}, \]

i.e.
\[ I_{V(x_0)}(\tilde{w}) \leq c_{V_0}. \quad (4.25) \]
On the other hand, since \( \langle J'_\varepsilon(u_\varepsilon), u_{\varepsilon,1} \rangle \to 0 \), \( \| u_{\varepsilon,2} \|_{H^s} \to 0 \) as \( \varepsilon \to 0 \) and \( \langle Q'_\varepsilon(u_\varepsilon), u_{\varepsilon,1} \rangle = 0 \). Combining with \( \tilde{w}_\varepsilon(x) = u_{\varepsilon,1}(x + \frac{x_0}{\varepsilon}) \), we get
\[ \int_{\mathbb{R}^3} |\nabla \tilde{w}_\varepsilon|^2 + V(\varepsilon x + x_0) \tilde{w}_\varepsilon^2 dx + \int_{\mathbb{R}^3} \psi_{\tilde{w}_\varepsilon} \tilde{w}_\varepsilon^2 dx = \int_{\mathbb{R}^3} f(\tilde{w}_\varepsilon) \tilde{w}_\varepsilon dx + o(1). \]
At the same time, by (4.17), we get
\[ \int_{\mathbb{R}^3} |\nabla \tilde{w}|^2 + V(x_0) \tilde{w}_\varepsilon^2 dx + \int_{\mathbb{R}^3} \psi_{\tilde{w}_\varepsilon} \tilde{w}_\varepsilon^2 dx \]
\[ \leq \liminf_{\varepsilon \to 0} \left[ \int_{\mathbb{R}^3} |\nabla \tilde{w}_\varepsilon|^2 + V(\varepsilon x + x_0) \tilde{w}_\varepsilon^2 dx + \int_{\mathbb{R}^3} \psi_{\tilde{w}_\varepsilon} \tilde{w}_\varepsilon^2 dx \right] \]
\[ = \liminf_{\varepsilon \to 0} \int_{\mathbb{R}^3} f(\tilde{w}_\varepsilon) \tilde{w}_\varepsilon dx \]
\[ = \int_{\mathbb{R}^3} f(\tilde{w}) \tilde{w}_\varepsilon dx \]
\[ = \int_{\mathbb{R}^3} |\nabla \tilde{w}|^2 + V(x_0) \tilde{w}_\varepsilon^2 dx + \int_{\mathbb{R}^3} \psi_{\tilde{w}_\varepsilon} \tilde{w}_\varepsilon^2 dx, \]

hence, as \( \varepsilon \to 0 \),
\[ \int_{\mathbb{R}^3} V(\varepsilon x + x_0) \tilde{w}_\varepsilon^2 dx \to \int_{\mathbb{R}^3} V(x_0) \tilde{w}_\varepsilon^2 dx \text{ and } \int_{\mathbb{R}^3} |\nabla \tilde{w}_\varepsilon|^2 dx \to \int_{\mathbb{R}^3} |\nabla \tilde{w}|^2 dx. \quad (4.27) \]
Combining with (4.6) and the fact that $\|u_\varepsilon\|_{H_x} \to 0$ as $\varepsilon \to 0$, taking $d_0 > 0$ small, we can deduce that $\tilde{w} \neq 0$. Indeed, choose $2 < p_i < 6$, there exists a $r > 0$,
\[
\|\tilde{w}_\varepsilon(x)\|_{p_i} \geq \left\| u_{\varepsilon, 2}(x + \frac{x_0}{\varepsilon}) - \varphi(\varepsilon x)U_0(x) \right\|_p - \left\| u_{\varepsilon}(x) - \varphi(\varepsilon x - x_0)U_0(x - \frac{x_0}{\varepsilon}) \right\|_{p_i}
\geq \left\| \varphi(\varepsilon x)U_0(x) \right\|_{p_i} - \left\| u_{\varepsilon 2}(x + \frac{x_0}{\varepsilon}) \right\|_{p_i} - C d_0
> r > 0,
\]
when $d_0$ is enough small and $\varepsilon \to 0$. Using (4.18), it is easy to see that $\tilde{w}$ is nontrivial. By (4.17), one has
\[
I_{V(x_0)}(\tilde{w}) \geq c_{V(x_0)}. \tag{4.28}
\]
Since $x_0 \in \mathcal{M}^\beta \subset \Lambda$, (4.25) and (4.28) deduce that $V(x_0) = V_0$ and $x_0 \in \mathcal{M}$. It is clear that there exist $U \in S_{V_0}$ and $z_0 \in \mathbb{R}^3$ such that $\tilde{w}(x) = U(x - z_0)$. On the other hand, since
\[
\int_{\mathbb{R}^3} V(x_0) \tilde{w}_\varepsilon^2 dx \leq \int_{\mathbb{R}^3} V(\varepsilon x + x_0) \tilde{w}_\varepsilon^2 dx,
\]
by (4.27), we get that
\[
\tilde{w}_\varepsilon \to \tilde{w} \text{ in } H^1(\mathbb{R}^3),
\]
which implies that
\[
\left\| u_{\varepsilon} - \varphi(\varepsilon x - (x_0 + \varepsilon z_0))U(x - (\frac{x_0}{\varepsilon} + z_0)) \right\|_{H_x} \to 0 \text{ as } \varepsilon \to 0.
\]
Thus we complete the proof of Lemma. \qed

Lemma 4.4. Let $d_0$ be the number given in Lemma 4.3, then for any $d \in (0, d_0)$, there exists $\varepsilon_d > 0$, $\rho_d > 0$ and $w_d > 0$ such that
\[
\|J'_e(u)\|_{s, \varepsilon, R} \geq w_d > 0
\]
for all $u \in J^{\varepsilon_d + \rho_d}_{d} \cap (X^d \setminus X^d_\varepsilon) \cap H^1_0(B_R(0))$ with $\varepsilon \in (0, \varepsilon_d)$ and $R \geq R_0$. \[\varepsilon\]

Proof. Suppose by contradiction that the lemma does not hold, there exist $d \in (0, d_0)$, $\{\varepsilon_i\}$, $\{\rho_i\}$ with $\varepsilon_i, \rho_i \to 0$, $R_{\varepsilon_i} \geq R_0/\varepsilon_i$ and $u_i \in J^{\varepsilon_d + \rho_i}_{d} \cap (X^d \setminus X^d_\varepsilon) \cap H^1_0(B_{R_{\varepsilon_i}}(0))$ such that
\[
\|J'_{\varepsilon_i}(u_i)\|_{s, \varepsilon_i, R_{\varepsilon_i}} \to 0 \text{ as } i \to \infty.
\]
Using the Lemma 4.3-(i), then we can choose $\{y_i\}_{i=1}^{\infty} \subset \mathbb{R}^3$, $x_0 \in \mathcal{M}$, $U \in S_{V_0}$ such that
\[
\lim_{i \to \infty} |\varepsilon_i y_i - x_0| = 0 \quad \text{and} \quad \lim_{i \to \infty} \|u_i - \varphi(\varepsilon_i x - \varepsilon_i y_i)U(x - y_i)\|_{H_x} = 0,
\]
so we can deduce that $u_i \in X^d_{\varepsilon_i}$ when $i$ is large enough. This contradicts that $u_i \notin X^d_{\varepsilon_i}$. \[\varepsilon\]

Lemma 4.5. There exists $T_0 > 0$ with the following property: for any $\delta > 0$ small, there exist $\alpha_\delta > 0$ and $\varepsilon_\delta > 0$ such that if $J'_e(\gamma_\varepsilon(s)) \geq c_{V_0} - \alpha_\delta$ and $\varepsilon \in (0, \varepsilon_\delta)$, then $\gamma_\varepsilon(s) \in X^d_{\varepsilon_\delta}$, where $\gamma_\varepsilon(s) := W_{\varepsilon, st_0}$, $s \in [0, 1]$. \[\varepsilon\]

Proof. Firstly, we can find a $T_0 > 0$ such that for any $u \in H^1(\mathbb{R}^3)$,
\[
\|\varphi(\varepsilon x)u(x)\|_{H_x} \leq T_0 \|u(x)\|_{H^1(\mathbb{R}^3)}. \tag{4.29}
\]
Since $I_{V_0}((2x^2)U^*(tx))$ has unique maximum which arrive at $t = 1$, set
\[
\alpha_\delta = \frac{1}{4} \min\{c_{V_0} - I_{V_0}(s^2U^*(st_0x)) : s \in [0, 1], \|s^2U^*(st_0x) - U^*(x)\|_{H^1} \geq \delta\} > 0,
\]
we have
\[ I_{V_0}(s^2t_0^2U^\ast(st_0x)) \geq c_{V_0} - 2\alpha_\delta, \]
which implies that
\[ \|s^2t_0^2U^\ast(st_0x) - U^\ast(x)\|_{H^1} \leq \delta. \] (4.30)
Similar to the (4.1), we obtain that
\[ \max_{0 \leq \delta \leq 1} |J_{\varepsilon}((\gamma_\varepsilon(s)) - I_{V_0}(s^2t_0^2U^\ast(st_0x)))| \leq \alpha_\delta \] (4.31)
for all \( \varepsilon \in (0, \varepsilon_0) \). Thus if \( \varepsilon \in (0, \varepsilon_0) \) and \( J_{\varepsilon}((\gamma_\varepsilon(s)) \geq c_{V_0} - \alpha_\delta \), using (4.30) and (4.31), one has \( \|s^2t_0^2U^\ast(st_0x) - U^\ast(x)\| \leq \delta \), then by (4.29), we get that
\[ \|W_{\varepsilon, st_0}(x) - \varphi(\varepsilon x)U^\ast(x)\|_{H_s} = \|\varphi(\varepsilon x)s^2t_0^2U^\ast(st_0x) - \varphi(\varepsilon x)U^\ast(x)\|_{H_s} \leq T_0\|s^2t_0^2U^\ast(st_0x) - U^\ast(x)\|_{H^1} \leq T_0\delta. \]
Note that \( 0 \in M \), we get that \( \gamma_\varepsilon(s) := W_{\varepsilon, st_0} \in X^R_{\varepsilon, \delta} \).

For each \( R > R_0/\varepsilon \), set
\[ \gamma_\varepsilon(s) := W_{\varepsilon, st_0} \in H^1_R((B_\varepsilon(0))) \text{ for each } s \in [0, 1], \ X_\varepsilon \subset H^1_R((B_\varepsilon(0))). \]
Define
\[ c_{\varepsilon, R} := \inf_{\gamma \in T_{\varepsilon, R}} \max_{0 \leq \delta \leq 1} J_{\varepsilon}(\gamma(t)), \]
where
\[ T_{\varepsilon, R} := \{ \gamma \in C([0, 1], H^1_R((B_\varepsilon(0))) : \gamma(0) = 0, \gamma(1) = \gamma_\varepsilon(1) = W_{\varepsilon, st_0} \}. \]
Remark that \( \gamma_\varepsilon(s) := W_{\varepsilon, st_0} \in T_{\varepsilon, R}, c_{\varepsilon, R} \leq c_{\varepsilon, R} \leq \varepsilon \) and \( J_{\varepsilon}^{\varepsilon}_\varepsilon \bigcap X_\varepsilon \cap H^1_R((B_\varepsilon(0))) \neq \emptyset \) since \( \varphi(\varepsilon x)U^\ast(x) \in J_{\varepsilon}^{\varepsilon}_\varepsilon \bigcap X_\varepsilon \cap H^1_R((B_\varepsilon(0))). \)
Choosing \( d_1 > 0 \) such that \( T_0d_1 < \frac{d_0}{4} \) in Lemma 4.5 and fixing \( d = \frac{d_0}{4} := d_1 \) in Lemma 4.4. The following Lemma comes from [13].

**Lemma 4.6.** There exists \( \varepsilon > 0 \) such that for each \( \varepsilon \in (0, \varepsilon] \) and \( R > R_0/\varepsilon \), there exists a sequence
\[ \{v_{n, \varepsilon}^R\}_{n=1}^\infty \subset J_{\varepsilon}^{\varepsilon}_\varepsilon \bigcap X_\varepsilon \bigcap H^1_R((B_\varepsilon(0))) \]
such that \( J_{\varepsilon}(v_{n, \varepsilon}^R) \rightarrow 0 \) in \( (H^1_R((B_\varepsilon(0))))^{-1} \) as \( n \rightarrow \infty. \)

**Proof of Theorem 1.1. Step 1.** By Lemma 4.6, there exists \( \varepsilon > 0 \) such that for each \( \varepsilon \in (0, \varepsilon] \) and \( R > R_0/\varepsilon \), there exists a sequence
\[ \{v_{n, \varepsilon}^R\}_{n=1}^\infty \subset J_{\varepsilon}^{\varepsilon}_\varepsilon \bigcap X_\varepsilon \bigcap H^1_R((B_\varepsilon(0))) \]
such that \( J_{\varepsilon}(v_{n, \varepsilon}^R) \rightarrow 0 \) in \( (H^1_R((B_\varepsilon(0))))^{-1} \) as \( n \rightarrow \infty. \)
Since \( \{v_{n, \varepsilon}^R\} \) is bounded in \( H^1_R((B_\varepsilon(0))) \), up to a subsequence, we get
\[ v_{n, \varepsilon}^R \rightarrow v_\varepsilon^R \text{ weakly in } H^1_R((B_\varepsilon(0))), \]
\[ v_{n, \varepsilon}^R \rightarrow v_\varepsilon^R \text{ strongly in } L^s((B_\varepsilon(0))) \text{ for } 1 \leq s < 6, \]
(4.32)
\[ v_{n, \varepsilon}^R \rightarrow v_\varepsilon^R \text{ a.e. in } B_\varepsilon(0). \]
It is easy to check that \( v_\varepsilon^R \geq 0 \) and satisfies
\[
\begin{cases}
-\Delta v_\varepsilon^R + V(\varepsilon x)v_\varepsilon^R + \psi_{\varepsilon R}^R v_\varepsilon^R + 4(\int_{\mathbb{R}^3} \chi_\varepsilon(v_\varepsilon^R)^2 dx - 1) = f(v_\varepsilon^R), & x \in \mathbb{R}^3, \\
v_\varepsilon^R = 0 \text{ on } \partial B_{\varepsilon}(0) 
\end{cases}
\]
(4.33)
and we will assert that \( v_\varepsilon^R \in J_{\varepsilon}^{\varepsilon}_\varepsilon \bigcap X_\varepsilon \bigcap H^1_R((B_\varepsilon(0))) \) for \( d_0 > 0 \) small.
Indeed, we write that \( v^R_{n, \varepsilon} = u^R_{n, \varepsilon} + w^R_{n, \varepsilon} \) with \( u^R_{n, \varepsilon} \in X_\varepsilon \) and \( \| u^R_{n, \varepsilon} \|_{H_\varepsilon} \leq d_0 \). Since \( S_{V_0} \) is compact in \( H^1(\mathbb{R}^3) \), up to a subsequence, we can assume that \( u^R_{n, \varepsilon} \to u^R \) in \( H^1_0(B_R(0)) \) and \( w^R_{n, \varepsilon} \to w^R \) in \( H^1_0(B_R(0)) \) as \( n \to \infty \). Then we obtain \( v^R_{\varepsilon} = u^R_{\varepsilon} + w^R_{\varepsilon} \) with \( u^R_{\varepsilon} \in X_\varepsilon \) and \( \| w^R_{\varepsilon} \| \leq d_0 \), so \( v^R_{\varepsilon} \in X_{d_0} \). Using Fatou’s lemma and (4.32), we have

\[
\bar{c}_\varepsilon + \varepsilon \geq J_{\varepsilon}(v^R_{n, \varepsilon}) \geq J_{\varepsilon}(v^R_{\varepsilon}) + o(1)
\]
as \( n \to \infty \). Let \( n \to \infty \), we obtain \( v^R_{\varepsilon} \in J_{\varepsilon}^{\infty} \).

**Step 2.** Next, we claim that \( v^R_{\varepsilon} \to v_{\varepsilon} \in H_\varepsilon \cap X_{d_0} \cap J_{\varepsilon}^{6_{\varepsilon} + \varepsilon} \) as \( R \to \infty \) in \( H_\varepsilon \) for \( \varepsilon > 0 \) small but fixed. Since \( Q_{\varepsilon}(v^R_{\varepsilon}) \) is uniformly bounded for all \( \varepsilon > 0 \) small and \( R > \frac{R_0}{\varepsilon} \), we get

\[
\int_{\mathbb{R}^3 \setminus (A_{\varepsilon})} (v^R_{\varepsilon})^2 dx \leq C\varepsilon.
\]
(4.34)

We assert that \( \exists \bar{\varepsilon} > 0 \) such that for any \( \varepsilon \in (0, \bar{\varepsilon}] \) and \( R > \frac{R_0}{\varepsilon} \),

\[
\| v^R_{\varepsilon} \|_{L^\infty(\mathbb{R}^3)} \leq C.
\]

Otherwise, there exists \( \varepsilon \to 0, R_I > \frac{R_0}{\varepsilon} \) such that \( \| v^R_{\varepsilon} \|_{L^\infty(\mathbb{R}^3)} \to \infty \). By Lemma 4.3-(i), we can find \( \{y_i\} \subset \mathbb{R}^3 \), \( x_0 \in \mathcal{M}, U \subset S_{V_0} \) such that

\[
\lim_{i \to \infty} |\varepsilon_i y_i - x_0| = 0 \text{ and } \lim_{i \to \infty} \| v^R_{\varepsilon_i} - \varphi(\varepsilon_i x - \varepsilon_i y_i) U(x - y_i) \|_{H_{\varepsilon_i}} = 0.
\]

Combining with Brezis-Kato type argument and \( L^\infty \)-estimation, we can get that \( \| v^R_{\varepsilon_i} \|_{L^\infty(\mathbb{R}^3)} \leq C \), so we prove the assertion. Using the sub-solution estimate [14], we can deduce that

\[
\| v^R_{\varepsilon} \|_{L^\infty(B)} \leq C\varepsilon^\frac{1}{2}
\]
(4.35)

for all \( \varepsilon > 0 \) small and \( B := \{x \in \mathbb{R}^3 | |x| \geq \frac{R_0}{\varepsilon} + 2\} \). By the the comparison principle and elliptic estimates, we conclude that

\[
0 \leq v^R_{\varepsilon}(x) \leq C_1(\varepsilon) \exp(-C_2(\varepsilon)|x|) \text{ for all } |x| \geq \frac{R_0}{\varepsilon} + 2 \text{ and } R > \frac{R_0}{\varepsilon},
\]
(4.36)

where \( C_1(\varepsilon) \) and \( C_2(\varepsilon) \) are independent of \( R \). Since \( \{v^R_{\varepsilon}\} \) is bounded in \( H_\varepsilon \), we can assume \( R \to \infty \),

\[
v^R_{\varepsilon} \to v_{\varepsilon} \text{ weakly in } H_\varepsilon,
v^R_{\varepsilon} \to v_{\varepsilon} \text{ strongly in } L^s_{\text{loc}}(\mathbb{R}^3) \text{ for } 1 \leq s < 6,
v^R_{\varepsilon} \to v_{\varepsilon} \text{ a.e. in } \mathbb{R}^3.
\]
(4.37)

By (4.36), we can deduce that

\[
v^R_{\varepsilon} \to v_{\varepsilon} \text{ strongly in } L^s(\mathbb{R}^3) \text{ for } 1 \leq s < 6 \text{ as } R \to \infty.
\]

Using standard argument, we can prove the claim. Thus, \( v_{\varepsilon} \in H_\varepsilon \cap X_{d_0} \cap J_{\varepsilon}^{6_{\varepsilon} + \varepsilon} \) is nontrivial solution of

\[
-\Delta u + V(\varepsilon x) u^R + \psi_u u + 4(\int_{\mathbb{R}^3} \chi_\varepsilon(u)^2 dx - 1)_+ = f(u), \quad x \in \mathbb{R}^3.
\]

Since \( S_{V_0} \) is compact in \( H^1(\mathbb{R}^3) \), it is easy to check that \( 0 \notin X_{d_0} \) for \( d_0 > 0 \) small, thus \( v_{\varepsilon} \neq 0 \).

**Step 3.** For any sequence \( \{\varepsilon_j\} \) with \( \varepsilon_j \to 0 \), by Lemma 4.3-(ii), there exist, up to a subsequence, \( \{y_j\} \subset \mathbb{R}^3 \), \( x_0 \in \mathcal{M}, U \subset S_{V_0} \) such that

\[
\lim_{j \to \infty} |\varepsilon_j y_j - x_0| = 0 \text{ and } \lim_{i \to \infty} \| v_{\varepsilon_j} - \varphi(\varepsilon_j x - \varepsilon_j y_j) U(x - y_j) \|_{H_{\varepsilon_j}} = 0.
\]
(4.38)
which implies that when $j \to \infty$,
\[ w_{\varepsilon_j}(x) := v_{\varepsilon_j}(x + y_j) \to U(x) \text{ in } L^s(\mathbb{R}^3) \text{ for } s \in [2, 6]. \]

Using the standard arguments, we get that
\[ \lim_{|x| \to \infty} w_{\varepsilon_j}(x) = 0 \text{ uniformly for all } \varepsilon_j, \tag{4.39} \]
and
\[ w_{\varepsilon_j}(x) \leq C_4 \exp(-C_5|x|), \quad x \in \mathbb{R}^3, \]
where $C_4$ and $C_5$ are independent of $\varepsilon_j$.

Therefore,
\[ \varepsilon_j^{-1} \int_{\mathbb{R}^3 \setminus (\Lambda/\varepsilon_j)} v_{\varepsilon_j}^2(x) \, dx = \varepsilon_j^{-1} \int_{\mathbb{R}^3 \setminus (\Lambda/\varepsilon_j - y_j)} w_{\varepsilon_j}^2(x) \, dx \]
\[ \leq \varepsilon_j^{-1} \int_{\mathbb{R}^3 \setminus B_{\beta/\varepsilon_j}(0)} (C_4)^2 \exp(-2C_5|x|) \, dx \to 0, \]
as $j \to \infty$, i.e. $Q_{\varepsilon_j}(v_{\varepsilon_j}) = 0$ for $\varepsilon_j$ small. Thus $v_{\varepsilon_j}$ is a solution of (1.8). Set $w_\varepsilon(x) = v_{\varepsilon_j}(\frac{x}{\varepsilon_j})$, so $w_\varepsilon$ is a solution of (1.1).

Let $P_j$ be the a maximum point $w_{\varepsilon_j}$, similar to the proof in the Proposition 3.11, we can check that $\exists \, \beta \geq 0$ such that $w_{\varepsilon_j}(P_j) \geq \beta$, combining with (4.39), they implies that $\{P_j\}$ must be bounded.

Since $u_{\varepsilon_j}(x) = w_{\varepsilon_j}(x_j, y_j), x_j := \varepsilon_j P_j + \varepsilon_j y_j$ is a maximum point of $u_{\varepsilon_j}$. Using (4.38), $x_j \to x_0 \in \mathcal{M}$ as $j \to \infty$. Since the sequence $\{\varepsilon_j\}$ is arbitrary, we get the existence and concentration results in Theorem 1.1. The proof of the exponential decay of $u_\varepsilon$ is standard (see [15] and [22]), we omit it here. Thus we complete the proof of Theorem 1.1.

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REFERENCES

[1] A. Ambrosetti and D. Ruiz, Multiple bound states for the Schrödinger-Poisson problem, Commun. Contemp. Math., 10 (2008), 391–404.
[2] A. Azzollini and A. Pomponio, Ground state solutions for the nonlinear Schrödinger-Maxwell equations, J. Math. Anal. Appl., 345 (2008), 90–108.
[3] V. Benci and D. Fortunato, An eigenvalue problem for the Schrödinger-Maxwell equations, Topol. Methods Nonlinear Anal., 11 (1998), 283–293.
[4] V. Benci and D. Fortunato, Solitary waves of the nonlinear Klein-Gordon equation coupled with Maxwell equations, Rev. Math. Phys., 14 (2002), 409–420.
[5] J. Byeon and L. Jeanjean, Standing waves for nonlinear Schrödinger equations with a general nonlinearity, Arch. Ration. Mech. Anal., 185 (2007), 185–200.
[6] J. Byeon and L. Jeanjean, Standing waves with a critical frequency for nonlinear Schrödinger equations II, Calc. Var. Partial Diff. Equ., 18 (2003), 207–219.
[7] G. Cerami and G. Vaira, Positive solutions for some non-autonomous Schrödinger-Maxwell systems, J. Differ. Equ., 248 (2010), 521–543.
[8] S. T. Chen and X. H. Tang, Ground state sign-changing solutions for a class of Schrödinger-Poisson type problems in $\mathbb{R}^3$, Z. Angew. Math. Phys., 67 (2016), Art. 102, 18 pp.
[9] S. T. Chen and X. H. Tang, Improved results for Klein-Gordon-Maxwell systems with general nonlinearity, Discrete. Contin. Dyn. Syst., 38 (2018), 2333–2348.
[10] S. T. Chen and X. H. Tang, Geometrically distinct solutions for Klein-Gordon-Maxwell systems with superlinear nonlinearities, Appl. Math. Letters, 90 (2019), 188–193.
[11] S. T. Chen and X. H. Tang, Ground state solutions of Schrödinger-Poisson systems with variable potential and convolution nonlinearity, J. Math. Anal. Appl., 473 (2019), 87–111.
12. T. D'Aprile and J. C. Wei, On bound states concentrating on spheres for the Maxwell-Schrödinger equation, *SIAM J. Math. Anal.*, 37 (2005), 321–342.

13. G. M. Figueiredo, N. Ikoma and J. R. Santos Junior, Existence and concentration result for the Kirchhoff type equations with general nonlinearities, *Arch. Ration. Mech. Anal.*, 213 (2014), 931–979.

14. D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Grundlehren Math. Wiss., vol. 224, Springer, Berlin, 1983.

15. X. M. He, Multiplicity and concentration of positive solutions for the Schrödinger-Poisson equations, *Z. Angew. Math. Phys.*, 62 (2011), 869–889.

16. X. M. He and W. M. Zou, Existence and concentration of ground states for Schrödinger-Poisson equations with critical growth, *J. Math. Phys.*, 53 (2012), 023702, 19 pp.

17. Y. He and G. B. Li, Standing waves for a class of Schrödinger-Poisson equations in $\mathbb{R}^3$ involving critical Sobolev exponents, *Ann. Acad. Sci. Fenn. Math.*, 40 (2015), 729–766.

18. I. Ianni and G. Vaira, On concentration of positive bound states for the Schrödinger-Poisson problem with potentials, *Adv. Nonlinear Stud.*, 8 (2008), 573–595.

19. I. Ianni, Solutions of the Schrödinger-Poisson problem concentrating on spheres, part II: Existence, *Math. Models Methods Appl. Sci.*, 19 (2009), 877–910.

20. I. Ianni and G. Vaira, Solutions of the Schrödinger-Poisson problem concentrating on spheres, part I: necessary condition, *Math. Models Methods Appl. Sci.*, 19 (2009), 707–720.

21. L. Jeanjean, Existence of solutions with prescribed norm for semilinear elliptic equations, *Nonlinear Anal.*, 28 (1997), 1633–1659.

22. G. B. Li and S. S. Yan, Eigenvalue problems for quasilinear elliptic equations on $\mathbb{R}^N$, *Commun. Partial Differ. Equ.*, 14 (1989), 1291–1314.

23. P. L. Lions, Solutions of Hartree-Fock equations for Coulomb systems, *Comm. Math. Phys.*, 109 (1987), 33–97.

24. P. L. Lions, The concentration-compactness principle in the calculus of variations, The locally compact case, part 2, *Ann. Inst. H. Poincaré Anal. Non. Linéaire.*, 1 (1984), 223–283.

25. P. Markowich, C. Ringhofer and C. Schmeiser, *Semiconductor Equations*, Springer Monographs in Mathematics, Springer, BerlinCham, 2019.

26. N. S. Papageorgiou, V. D. Rădulescu and D. Repovš, *Nonlinear Analysis-Theory and Methods*, Springer, New York, 1990.

27. M. del Pino and P. Felmer, Local mountain pass for semilinear elliptic problems in unbounded domains, *Calc. Var. Partial Differ. Equ.*, 4 (1996), 121–137.

28. P. Pucci and J. Serrin, A general variational identity, *Indiana Univ. Math. J.*, 35 (1986), 681–703.

29. P. H. Rabinowitz, On a class of nonlinear Schrödinger equations, *Z. Angew. Math. Phys.*, 43 (1992), 270–291.

30. D. Ruiz, The Schrödinger-Poisson equation under the effect of a nonlinear local term, *J. Funct. Anal.*, 237 (2006), 655–674.

31. D. Ruiz, Semiclassical states for coupled Schrödinger-Maxwell concentration around a sphere, *Math. Models Methods Appl. Sci.*, 15 (2005), 141–164.

32. D. Ruiz and G. Vaira, Cluster solutions for the Schrödinger-Poisson-Slater problem around a local minimum of potential, *Rev. Mat. Iberoamericana.*, 27 (2011), 253–271.

33. J. Seok, On nonlinear Schrödinger-Poisson equations with general potentials, *J. Math. Anal. Appl.*, 401 (2013), 672–681.

34. J. J. Sun and S. W. Ma, Ground state solutions for some Schrödinger-Poisson systems with periodic potentials, *J. Differ. Equ.*, 260 (2016), 2119–2149.

35. X. H. Tang and S. T. Chen, Ground state solutions of Nehari-Pohozaev type for Schrödinger-Poisson problems with general potentials, *Discrete. Contin. Dyn. Syst.*, 37 (2017), 4973–5002.

36. X. H. Tang and S. T. Chen, Ground state solutions of Nehari-Pohozaev type for Kirchhoff-type problems with general potentials, *Calc. Var. PDE.*, 56 (2017), Art. 110, 25 pp.

37. X. H. Tang, X. Y. Lin and J. S. Yu, Nontrivial solutions for Schrödinger equation with local super-quadratic conditions, *J. Dyn. Differ. Equ.*, 31 (2018), 369–383.

38. X. F. Wang, On concentration of positive bound states of nonlinear Schrödinger equations, *Commun. Math. Phys.*, 153 (1993), 229–244.

39. J. Wang, L. X. Tian, J. X. Xu and F. B. Zhang, Existence and concentration of positive solutions for semilinear Schrödinger-Poisson systems in $\mathbb{R}^3$, *Calc. Var. PDE.*, 48 (2013), 243–273.
[40] J. Wang, L. X. Tian, J. X. Xu and F. B. Zhang, Existence of multiple positive solutions for Schrödinger-Poisson systems with critical growth, *Z. Angew. Math. Phys.*, 66 (2015), 2441–2471.

[41] M. Willem, *Minimax Theorems*, Birkhäuser, Berlin, 1996.

[42] J. Zhang, W. Zhang and X. H. Tang, Ground state solutions for Hamiltonian elliptic system with inverse square potential, *Discrete Contin. Dyn. Syst.*, 37 (2017), 4565–4583.

[43] X. Zhang and J. K. Xia, Semi-classical solutions for Schrödinger-Poisson equations with a critical frequency, *J. Differ. Equ.*, 265 (2018), 2121–2170.

[44] L. G. Zhao and F. K. Zhao, On the existence of solutions for the Schrödinger-Poisson equations, *J. Math. Anal. Appl.*, 346 (2008), 155–169.

[45] L. G. Zhao and F. K. Zhao, Positive solutions for Schrödinger-Poisson equations with a critical exponent, *Nonlinear Anal.*, 70 (2009), 2150–2164.

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E-mail address: math120701012cz@163.com (Z. Chen)
E-mail address: tangxh@mail.csu.edu.cn (X. Tang)
E-mail address: 18338766360@163.com (N. Zhang)
E-mail address: zhangjian433130@163.com (J. Zhang)