Real time statistical field theory

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We have written a \textit{Mathematica} program that calculates the integrand corresponding to any amplitude in the closed-time-path formulation of real time statistical field theory. The program is designed so that it can be used by someone with no previous experience with \textit{Mathematica}. It performs the contractions over the tensor indices that appear in real time statistical field theory and gives the result in the 1-2, Keldysh or RA basis. We have used the program to calculate the ward identity for the QED 3-point function, the QED 4-point function for two photons and two fermions, and the QED 5-point function for three photons and two fermions. In real time statistical field theory, there are seven 3-point functions, 15 4-point functions and 31 5-point functions. We produce a table that gives the results for all of these functions. In addition, we give a simple general expression for the KMS conditions between $n$-point green functions and vertex functions, in both the Keldysh and RA bases.

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I. INTRODUCTION

It is well known that calculations in real time statistical field theory are complicated by the proliferation of indices that results from the doubling of degrees of freedom (for a recent review see \cite{1}). This difficulty causes many people to avoid using real time finite temperature field theory, in spite of its significant advantages over the imaginary time formalism. Two of the major advantages of working in real time are that analytic continuations are not necessary, and that it is easy to generalize to non-equilibrium situations. In this paper we make a contribution towards reducing the technical difficulties associated with the real time formulation of statistical field theory.

We use the closed time path (CTP) formalism of real time statistical field theory \cite{2, 3}, which consists of a contour with two branches: one runs from minus infinity to infinity along the real axis, the other runs back from infinity to minus infinity just below the real axis. This contour results in a doubling of degrees of freedom. Physically, these extra contributions come from the additional processes that are present when the system interacts with a medium, instead of sitting in a vacuum. As a result of these extra degrees of freedom, \textit{n}-point functions have a tensor structure which results in calculational complexities that increase geometrically when one considers either calculations at higher loop order, or the calculation of higher \textit{n}-point functions. Statistical field theory can be formulated in different bases, which produce different representations of these tensors. There are three popular bases: the 1-2 basis, the Keldysh basis, and the RA basis. Most of the original work in this field was done in the 1-2 basis. The Keldysh basis has the advantage of being more easily adaptable to non-equilibrium situations. The RA basis produces particularly simple expressions in equilibrium.

This paper is organized as follows: In section \textbf{II} we review the 1-2 formalism. In section \textbf{III} we discuss the issue of basis transformations in general. Our discussion follows that of \cite{4}. In sections \textbf{V} and \textbf{VI} we give expressions for \textit{n}-point functions and vertex functions in the Keldysh and RA bases. In section \textbf{VII} we give the KMS relations in each of these bases. These equations give a set of relations between the various components of a given \textit{n}-point function or vertex function which hold in equilibrium, and are often useful for simplifying the expressions that result after contracting over indices. We note that some of these expressions have appeared previously in the literature, using a slightly different notation \cite{5, 6}. We give general expressions in the Keldysh and RA bases. We emphasize the simplicity of the expressions in the RA basis. In section \textbf{VIII} we discuss the \textit{Mathematica} computer program that we use to perform most of the calculations in this paper. This program is designed so that it can be used by someone with no previous experience with \textit{Mathematica}. It is available on the internet at www.brandonu.ca/physics/fugleberg/Research/Dick.html.

The program performs contractions over CTP indices and produces the integrand that corresponds to any \textit{n}-point

\footnotesize
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vertex diagram, in the 1-2, Keldysh or RA basis. Several options to simplify the results are built into the program and can be selected by the user as part of the input. For example, for calculations in equilibrium, the KMS conditions can be automatically implemented. We discuss the basic design of the program and explain in detail, using a simple example, how the program can be used. The program treats all fields as scalar bosons and from the beginning of the paper to the end of section VIII we discuss only scalar bosons. (where the subscripts \{i\} take values 1 or 2 to indicate which branch of the contour the corresponding time argument falls on, and the symbol \(P\) represents ordering along the closed time path. In what follows we will suppress the symbol \(P\).) In the 1-2 basis the 2-point function can be written as a 2

\[
G^{(n)}(x_1, \ldots x_n)_{b_1 \ldots b_n} := (-i)^{n-1} \langle P (\phi(x_1)_{b_1} \cdots \phi(x_n)_{b_n}) \rangle
\]

where the subscripts \(\{b_i\}\) take values 1 or 2 to indicate which branch of the contour the corresponding time argument falls on, and the symbol \(P\) represents ordering along the closed time path. In what follows we will suppress the symbol \(P\). In this paper we produce a complete set of expressions. Our results agree with the function were obtained in [7] and [8]. A partial set of expressions for bare 1-loop 3- and 4-point diagrams in the htl expresses hold when full corrected vertices and propagators are used. Results for some components of the 3-point function were denoted \(G\), except for the 2-point function, or the propagator, which will be called \(D\).

In the 1-2 basis the 2-point function can be written as a 2 \times 2 matrix:

\[
D_{(1-2)} = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}.
\]

The component \(D_{11}\) indicates a propagator for fields moving along the top branch of the contour, \(D_{12}\) is the propagator for fields moving from the top branch to the bottom branch, etc. In co-ordinate space these components are given by,

\[
\begin{align*}
D_{11}(x, y) &= -i \langle \mathcal{T}(\phi(x)\phi(y)) \rangle \\
D_{12}(x, y) &= -i \langle \phi(y)\phi(x) \rangle \\
D_{21}(x, y) &= -i \langle \phi(x)\phi(y) \rangle \\
D_{22}(x, y) &= -i \langle \mathcal{T}(\phi(x)\phi(y)) \rangle,
\end{align*}
\]

where \(\mathcal{T}\) is the usual time ordering operator and \(\mathcal{T}\) is the antichronological time ordering operator. These four components satisfy,

\[
\sum_{a=1}^{2} \sum_{b=1}^{2} (-1)^{a+b}D_{ab} = D_{11} - D_{12} - D_{21} + D_{22} = 0
\]

as a consequence of the identity \(\theta(x) + \theta(-x) = 1\), and thus only three components are independent.

The 3-point function in the 1-2 basis is a \(2 \times 2 \times 2\) tensor with components in co-ordinate space:

\[
\begin{align*}
G_{111}(x, y, z) &= -\langle \mathcal{T}(\phi(x)\phi(y)\phi(z)) \rangle \\
G_{112}(x, y, z) &= -\langle \phi(z)\mathcal{T}(\phi(x)\phi(y)) \rangle \\
G_{121}(x, y, z) &= -\langle \phi(y)\mathcal{T}(\phi(x)\phi(z)) \rangle \\
G_{211}(x, y, z) &= -\langle \phi(x)\mathcal{T}(\phi(y)\phi(z)) \rangle \\
G_{122}(x, y, z) &= -\langle \mathcal{T}(\phi(y)\phi(z))\phi(x) \rangle \\
G_{212}(x, y, z) &= -\langle \mathcal{T}(\phi(y)\phi(z)) \rangle.
\end{align*}
\]
The structure of higher \( n \)-point functions (like the propagator) and upper indices to vertices. Summations are carried out over pairs of repeated indices, where one of the indices is an upper index and the other is a lower index.

We will assign lower indices to which give relationships between different amplitudes.

These functions obey a relation similar to (4) and (6):

\[
G_{1111}(x, y, z, w) = i \langle T(\phi(x)\phi(y)\phi(z)\phi(w)) \rangle
\]

\[
G_{1112}(x, y, z, w) = i \langle T(\phi(x)\phi(y)\phi(z)) \rangle
\]

\[
G_{1121}(x, y, z, w) = i \langle T(\phi(x)\phi(y)\phi(z)) \rangle
\]

\[
G_{1211}(x, y, z, w) = i \langle \phi(y)T(\phi(x)\phi(z)) \rangle
\]

\[
G_{2111}(x, y, z, w) = i \langle \phi(x)T(\phi(y)\phi(z)) \rangle
\]

\[
G_{1122}(x, y, z, w) = i \langle T(\phi(z)\phi(w))T(\phi(x)\phi(y)) \rangle
\]

\[
\vdots
\]

These functions obey a relation similar to (4) and (6):

\[
\sum_{a=1}^{2} \sum_{b=1}^{2} \sum_{c=1}^{2} (-1)^{a+b+c+d} G_{abcd} = 0.
\]

The structure of higher \( n \)-point functions is similar. Truncated green functions are called vertices and will be denoted \( \Gamma \), except for the two point function which will be called \( \Pi \). They are defined by the equation:

\[
G_{b_1\cdots b_n} = G_{b_1b_1} \cdots G_{b_nb_n} \Gamma_{b_1\cdots b_n}.
\]

The vertex functions satisfy the constraint:

\[
\sum_{b_1=1}^{2} \sum_{b_2=1}^{2} \cdots \sum_{b_n=1}^{2} \Gamma_{b_1b_2\cdots b_n} = 0.
\]

III. BASIS TRANSFORMATIONS

A scattering amplitude is calculated by multiplying together a series of vertices and propagators. By convention we will assign lower indices to \( n \)-point functions (like the propagator) and upper indices to vertices. Summations are carried out over pairs of repeated indices, where one of the indices is an upper index and the other is a lower index.

The expression for a given scattering amplitude can be transformed to a different basis by performing a rotation. There are two commonly used bases: the Keldysh basis and the R/A basis. Both of these bases express results in terms of the components of the 1-2 propagator that have a direct physical interpretation: the retarded, advanced and symmetric propagators. These expressions are easy to obtain in coordinate space. Using (4) and (6) one can show:

\[
D_{\text{ret}}(x, y) = D_{11}(x, y) - D_{12}(x, y) = -i\theta(x_0 - y_0) \langle [\phi(x), \phi(y)] \rangle
\]

\[
D_{\text{adv}}(x, y) = D_{11}(x, y) - D_{21}(x, y) = -i\theta(y_0 - x_0) \langle [\phi(x), \phi(y)] \rangle
\]

\[
D_{\text{sym}}(x, y) = D_{11}(x, y) + D_{22}(x, y) = -i \langle [\phi(x), \phi(y)] \rangle.
\]

One advantage of the Keldysh basis is that it is easily generalizable to non-equilibrium situations. In equilibrium, the R/A basis produces particularly simple expressions for amplitudes, and for expressions like the KMS conditions, which give relationships between different amplitudes.
The rotation to a different basis is accomplished by matrix multiplication. We rotate lower indices by multiplying by a matrix $U$ and upper indices by multiplying by the matrix $V$. These matrices are related through the following equation.

$$V(k) = (U^T)^{-1}(-k).$$ \hfill (12)

We obtain expressions of the form:

$$U^a_b(k)U^b_c(-k)D_{ab}(k) \rightarrow D^\prime_{a,b}(k)$$  \hfill (13)

$$V^a_b(k_1)V^b_c(k_2)V^c_e(-k_1-k_2)\Gamma^{abc}(k_1,k_2,-k_1-k_2) \rightarrow (\Gamma^{\prime})^\bar{a},\bar{b}\bar{c}(k_1,k_2,-k_1-k_2)$$

where we have simplified the notation by writing a two point function of the form $D_{ab}(k,-k)$ as $D_{ab}(k)$. It is straightforward to see that amplitudes have the correct transformation properties. We look at the example shown in Fig. 1.

![Fig. 1: A typical amplitude](image)

The amplitude is represented by an expression of the form:

$$\Pi^{ab}(k_1) \sim \int dk_2 \Gamma^{ace}(k_1,k_2,-k_1-k_2)D_{cd}(-k_2)\Gamma^{bde}(-k_1,-k_2,k_1+k_2)D_{ef}(k_1+k_2)$$ \hfill (14)

Using (12) and (13) it is easy to see that this self energy transforms according to:

$$\Pi^{ab}(k_1) = V^a_d(k_1)V^b_c(-k_1)\Pi^{\bar{a}\bar{b}}(k_1).$$ \hfill (15)

In the sections below, indices in the 1-2 basis will be denoted $b_i$ and take the values 1 or 2. Keldysh indices will be written $\alpha_i$ and are assigned the values $\alpha = 1 := r$ and $\alpha = 2 := a$. R/A indices will be denoted $X_i$ and assigned the values $X = 1 := R$ and $X = 2 := A$. An $n$-point function in the 1-2 basis will be written:

$$G(k_1,k_2,k_3,\cdots k_n)_{b_1,b_2,b_3,\cdots b_n} := G_{b_1,b_2,b_3,\cdots b_n}$$ \hfill (16)

An $n$-point function in the Keldysh basis will be written:

$$G(k_1,k_2,k_3,\cdots k_n)_{\alpha_1,\alpha_2,\alpha_3,\cdots \alpha_n} := G_{\alpha_1,\alpha_2,\alpha_3,\cdots \alpha_n}$$ \hfill (17)

An $n$-point function in the R/A basis will be written:

$$G(k_1,k_2,k_3,\cdots k_n)_{X_1,X_2,X_3,\cdots X_n} := G_{X_1,X_2,X_3,\cdots X_n}$$ \hfill (18)

Note that the conservation of momentum gives $k_n = -(k_1+k_2+\cdots+k_{n-1})$ so that each $n$-point function depends on only $n-1$ independent momenta.

### IV. THERMAL FUNCTIONS

We write the general distribution function as $f(k)$ and define the symmetric and anti-symmetric combinations:

$$F_A(k) = f(k) - f(-k); \quad F_S(k) = f(k) + f(-k)$$ \hfill (19)
In the Keldysh representation it will be useful to define the functions:

$$\tilde{F}(k_i; \alpha_i) = \delta_{\alpha_i, a} F_A(k_i) - \delta_{\alpha_i, r} F_S(k_i); \quad \tilde{F}(k_i; \alpha_i) = \delta_{\alpha_i, r} F_A(k_i) - \delta_{\alpha_i, a} F_S(k_i).$$

To illustrate these definitions consider the expression:

$$\tilde{F}(k_1; \alpha_1) \tilde{F}(k_2; \alpha_2) \tilde{F}(k_3; \alpha_3) G(k_1, k_2, k_3) \alpha_1, \alpha_2, \alpha_3$$

If we take the case $\alpha_1 = \alpha_2 = r; \ \alpha_3 = a$ we obtain:

$$F_S(k_1) F_S(k_2) F_A(k_3) G(k_1, k_2, k_3)_{rra}.$$ 

For bosons in equilibrium these expressions become:

$$\tilde{F}(k_i) \to \tilde{N}(k_i); \quad \tilde{F}(k_i) \to \tilde{N}(k_i)$$

where we have defined:

$$\tilde{N}(k_i; \alpha_i) = \delta_{\alpha_i, a} N(k_i) + \delta_{\alpha_i, r}$$

$$\tilde{N}(k_i; \alpha_i) = \delta_{\alpha_i, r} N(k_i) + \delta_{\alpha_i, a}$$

$$N_i := N(k_i) = 1 + 2n(k_i)$$

Equilibrium functions satisfy:

$$N_1 + N_2 = 0 \text{ if } k_2 = -k_1$$

$$1 + N_1 N_2 + N_1 N_3 + N_2 N_3 = 0 \text{ if } k_3 = -k_1 - k_2$$

$$N_1 + N_2 + N_3 + N_4 + N_1 N_2 N_3 + N_2 N_3 N_4 + N_3 N_4 N_1 + N_4 N_3 N_2 = 0 \text{ if } k_4 = -k_1 - k_2 - k_3$$

$$\vdots$$

where the dots indicate that higher order expressions can be generated by iteration. (Expressions with the same properties can be defined for the fermion distribution function: $n_f(k) = 1/(e^{\beta k_0} - 1); \quad N_f(k) = 1 - 2n_f(k)$).

In order to write the KMS conditions in a concise way we define the function

$$C_n := C\{N_i\} = C(N_1, N_2, \cdots N_n) = \sum_{p=0}^{n} \frac{1}{2} \left[ 1 - (-1)^{n+p} \right] \cdot N(n; p)$$

where the symbol $N(n; p)$ means the following:

[a] start with $n$ indices $\{x_1, x_2, x_3 \cdots x_n\}$

[b] consider all possible subsets of these indices containing $p < n$ of the $x_i$’s (without considering order)

[c] for each of these subsets, take the product of the corresponding $N(x_i)$’s

[d] sum over all sets

In addition, we define $N(n; 0) = 1$. A few examples will illustrate this notation.

Example [1]: if $n = 3$ and $p = 2$ then the possible sets of $p$ are: $\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_1\}$ and the result is $N(3; 2) = N_1 N_2 + N_2 N_3 + N_3 N_1$.

Example [2]: if $n = 4$ and $p = 3$ then the possible sets of $p$ are: $\{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_3, x_4, x_1\}, \{x_4, x_1, x_2\}$ and the result is $N(4; 3) = N_1 N_2 N_3 + N_2 N_3 N_4 + N_3 N_4 N_1 + N_4 N_1 N_2$.

Example [3]: if $n = 3$ and $p = 1$ then the possible sets of $p$ are: $\{x_1\}, \{x_2\}, \{x_3\}$ and the result is $N(3; 1) = N_1 + N_2 + N_3$.

Below we write out the first few $C(N_1, N_2, \cdots N_n)$’s:

$$C(N_1) = 1$$

$$C(N_1, N_2) = N_1 + N_2$$

$$C(N_1, N_2, N_3) = 1 + N_1 N_2 + N_2 N_3 + N_3 N_1$$

$$C(N_1, N_2, N_3, N_4) = N_1 + N_2 + N_3 + N_4 + N_1 N_2 N_3 + N_2 N_3 N_4 + N_3 N_4 N_1 + N_4 N_3 N_2$$
Note that because of (25) each of these expressions is zero if the momenta satisfy $k_1 + k_2 + \cdots + k_n = 0$. Using (24) and (25) we have:

$$C(N_1, \cdots N_n) = 2^{n-1} \frac{n(k_1) \cdots n(k_n)}{n(k_1 + \cdots + k_n)}.$$  \hfill (28)

To write the KMS conditions for a given $n$-point function in a compact form, we will need to use $C$-functions of the form defined above, but with arguments that are not the full set $\{N_i\}$ with $i$ running from 1 to $n$. We will need the $C$ variables whose arguments are a subset of the $N_i$’s. When working in the Keldysh basis we want $C$-functions whose arguments are the set of $N_i$’s whose corresponding $\alpha_i$’s take the value $[i]$ $r$ or; $[ii]$ $a$. When working in the R/A basis we want $C$-functions whose arguments are the set of $N_i$’s whose the corresponding $X_i$’s take the value $[i]$ $R$ or; $[ii]$ $A$.

We define these modified $C$-functions as follows. In the Keldysh basis the set $\{\alpha_i\}$ contains $n$ variables. The number of $r$’s is defined to be $n_r$ and the number of $a$’s is defined to be $n_a$. Of course, we have $n = n_r + n_a$. We construct a set of $n_r$ variables: $\{N_i\delta_{\alpha_i,r}\}$ and a set of $n_a$ variables: $\{N_i\delta_{\alpha_i,a}\}$. Using these sets of $N$’s we define the corresponding $C$-functions:

$$C_{n_r} := C(\{N_i\delta_{\alpha_i,r}\})$$
$$C_{n_a} := C(\{N_i\delta_{\alpha_i,a}\}).$$ \hfill (29)

When working in the R/A basis we make the analogous definitions: The set $\{X_i\}$ contains $n$ variables. The number of $R$’s is $n_R$ and the number of $A$’s is $n_A$. We define:

$$C_{n_R} := C(\{N_i\delta_{X_i,R}\})$$
$$C_{n_A} := C(\{N_i\delta_{X_i,A}\}).$$ \hfill (30)

To clarify this notation, we look at the following example: Take $n = 7$ and consider the set $\{\alpha_i\} = \{r, r, a, r, a, a, r\}$. We have: $n_r = 4; C_{n_r} = C(N_1, N_2, N_4, N_7)$ and $n_a = 3; C_{n_a} = C(N_3, N_5, N_6)$.

V. THE KELDYSH BASIS

The rotation from the 1-2 representation to the Keldysh representation is accomplished by using the transformation matrix:

$$U_{Keldysh\rightarrow (1-2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \hfill (31)$$

Note that in this case \cite{12} gives $V = U$. In general the $n$-point function in the Keldysh representation is given by:

$$G_{\alpha_1, \cdots \alpha_n} = 2^{n/2 - 1} U_{\alpha_1 b_1} \cdots U_{\alpha_n b_n} G_{b_1 \cdots b_n}. \hfill (32)$$

The factor $2^{n/2 - 1}$ is introduced to produce simpler expressions for the vertices, following \cite{6} and \cite{10}. For example, using the notation defined in \cite{32}, the bare 3-point vertex in the Keldysh representation is written in tensor form:

$$\{\{0, -i\}, \{-i, 0\}, \{-i, 0\}, \{0, -i\}\}$$ \hfill (33)

This expression can be compared with the bare 3-point vertex in the 1-2 representation:

$$\{\{-i, 0\}, \{0, 0\}, \{0, 0\}, \{0, i\}\}$$ \hfill (34)

In addition, the appropriate components of the Keldysh vertex correspond directly to the retarded vertices (see for example Eqn. \hfill (11)).

A. 2-point Function

The 2-point function in the Keldysh basis is obtained from \cite{32}:

$$D_{\alpha_1, \alpha_2} = U_{\alpha_1 b_1} U_{\alpha_2 b_2} G_{b_1 b_2}.$$. \hfill (35)
We give three examples of the results that we obtain by summing over indices and using (6):

\[ \text{two components given in (40) correspond to similar expressions in coordinate space, but involve anti-commutators.} \]

The summation indicates that we sum over the combinations: \( (D_{11} + D_{22}, D_{11} - D_{22}) \). Using (5) it is easy to show in coordinate space that

\[ G_{rr} = G_{111} + G_{221} + G_{212} + G_{122} \]

\[ G_{rra} = G_{111} - G_{112} + G_{221} - G_{222} \]

\[ G_{raa} = G_{111} - G_{112} - G_{121} + G_{122} \]

Using (35) it is easy to show in coordinate space that

\[ G_{raa}(x_1, x_2, x_3) = G_{R_4}(x_1, x_2, x_3) \]

where \( G_{R_4} \) is the component of the 3-point function that is retarded with respect to the first leg and is given by the familiar expression:

\[ G_{R_4}(x_1, x_2, x_3) \]

\[ = (-i)^2 \left( \theta(t_1 - t_2) \theta(t_2 - t_3) (\theta(\phi(x_1), \phi(x_2)), \phi(x_3)) + \theta(t_1 - t_3) \theta(t_3 - t_2) (\theta(\phi(x_1), \phi(x_3)), \phi(x_2)) \right) \]

\[ = (-i)^2 \sum_{(t_a, t_b) \neq (t_2, t_3)} \theta(t_1, t_a) \theta(t_a, t_b) (\theta(\phi(x_1), \phi(t_a)), \phi(t_b)) \].

The summation indicates that we sum over the combinations: \( (t_a = t_2; t_b = t_3) \) and \( (t_a = t_3; t_b = t_2) \). The other two components given in (36) correspond to similar expressions in coordinate space, but involve anti-commutators.

### B. 3-point Function

The 3-point function in the Keldysh representation is obtained from (36):

\[ G_{a_1a_2a_3} = \sqrt{2} U_{a_1} b_1 U_{a_2} b_2 U_{a_3} b_3 G_{b_1b_2b_3}. \]

We give three examples of the results that we obtain by summing over indices and using (6):

\[ G_{rr} = G_{111} + G_{221} + G_{212} + G_{122} \]

\[ G_{rra} = G_{111} - G_{112} + G_{221} - G_{222} \]

\[ G_{raa} = G_{111} - G_{112} - G_{121} + G_{122} \]

Using (35) it is easy to show in coordinate space that

\[ G_{raa}(x_1, x_2, x_3) = G_{R_4}(x_1, x_2, x_3) \]

where \( G_{R_4} \) is the component of the 3-point function that is retarded with respect to the first leg and is given by the familiar expression:

\[ G_{R_4}(x_1, x_2, x_3) \]

\[ = (-i)^2 \left( \theta(t_1 - t_2) \theta(t_2 - t_3) (\theta(\phi(x_1), \phi(x_2)), \phi(x_3)) + \theta(t_1 - t_3) \theta(t_3 - t_2) (\theta(\phi(x_1), \phi(x_3)), \phi(x_2)) \right) \]

\[ = (-i)^2 \sum_{(t_a, t_b) \neq (t_2, t_3)} \theta(t_1, t_a) \theta(t_a, t_b) (\theta(\phi(x_1), \phi(t_a)), \phi(t_b)) \].

The summation indicates that we sum over the combinations: \( (t_a = t_2; t_b = t_3) \) and \( (t_a = t_3; t_b = t_2) \). The other two components given in (36) correspond to similar expressions in coordinate space, but involve anti-commutators.

### C. 4-point function

The 4-point function in the Keldysh representation is obtained from (36):

\[ G_{a_1a_2a_3a_4} = 2 U_{a_1} b_1 U_{a_2} b_2 U_{a_3} b_3 U_{a_4} b_4 G_{b_1b_2b_3b_4} \]

We give one example of the results that we obtain by summing over indices and using (6):

\[ G_{raaa} = G_{1111} - G_{1112} - G_{1121} + G_{1122} + G_{1211} + G_{1212} + G_{1221} - G_{1222} \]
In co-ordinate space this expression has the form:

\[ G_{\text{rada}}(x_1, x_2, x_3, x_4) = G_{\text{R1}}(x_1, x_2, x_3, x_4) \]

\[ = (-1)^3 \sum_{\{t_a, t_b, t_c\} = \{t_2, t_3, t_4\}} \theta(t_1, t_a) \theta(t_a, t_b) \theta(t_b, t_c) \left( [\phi(t_1), \phi(t_a), \phi(t_b), \phi(t_c)] \right) \]

where the sum is over all possible assignments of the variables \( t_2, t_3, t_4 \) to the variables \( t_a, t_b, t_c \) and the notation \( G_{\text{R1}}(x_1, x_2, x_3, x_4) \) indicates the 4-point function that is retarded with respect to the first leg. Expressions for higher \( n \)-point functions are obtained similarly.

VI. THE (R/A) BASIS

The matrix that performs rotations from the 1-2 basis to the R/A basis is:

\[ U_{(R/A)\leftarrow (1-2)} = \begin{pmatrix} -f(k) & f(k) \\ f(k)/f(-k) & f(k)/f(-k) \end{pmatrix} \]

We will find it more useful to obtain expressions in the RA basis by rotating from the Keldysh basis. The matrix that transforms from the Keldysh basis to the 1-2 basis is given by the inverse of \( U \). Combining with \( U \) we obtain the matrix that transforms from the Keldysh basis to the R/A basis:

\[ U_{(R/A)\leftarrow \text{Keldysh}} = U_{(R/A)\leftarrow (1-2)} (U_{\text{Keldysh} \leftarrow (1-2)})^{-1} \]

\[ = \begin{pmatrix} 0 & -\sqrt{2}f(k) \\ \sqrt{2}f(k) & \sqrt{2}f(k) \end{pmatrix} \]

The \( n \)-point function in R/A basis is given by:

\[ 2^n \Gamma^{-1} G_{X_1 \cdots X_n} = U_{(k_1) X_1} \cdots U_{(k_n) X_n} G_{\alpha_1 \cdots \alpha_n} \]

General expressions that relate R/A \( n \)-point functions and vertex functions to Keldysh functions are given below:

\[ G_{(k_1, \cdots k_n) X_1 \cdots X_n} \]

\[ = (-1)^n \prod_{l=1}^n \delta_{X_l A} \delta_{X_l R} f(k_l) \sum_{\alpha_l \in \{r, a\}} \left( \prod_{l=1}^n \left[ \hat{F}(k_l; \alpha_l) \delta_{X_l A} + \delta_{X_l R} \delta_{\alpha l} \right] \right) G_{(k_1, \cdots k_n) \alpha_1 \cdots \alpha_n} \]

\[ \Gamma_{(k_1, \cdots k_n) X_1 \cdots X_n} \]

\[ = (-1)^n \prod_{l=1}^n \delta_{X_l A} \delta_{X_l R} f(k_l) \sum_{\alpha_l \in \{r, a\}} \left( \prod_{l=1}^n \left[ \hat{F}(k_l; \alpha_l) \delta_{X_l A} + \delta_{X_l R} \delta_{\alpha l} \right] \right) \Gamma_{(k_1, \cdots k_n) \alpha_1 \cdots \alpha_n} \]

From \( 67, 68 \) and \( 69 \) we have

\[ G_{R \cdots R} = 0; \quad \Gamma^{A \cdots A} = 0 \]

These equations can be written in a simpler form in equilibrium. There are two kinds of simplifications that occur. We can replace general distribution functions with thermal distribution functions and use identities of the form \( 22, 24, 25, 28 \) for combining groups of distributions functions. We can also use KMS conditions. The KMS conditions are a set of equations which are valid only at equilibrium that relate various components of an \( n \)-point function or vertex function. These conditions will be derived in the next section. For completeness, we give below two general expressions: the first corresponds to \( 65 \) using thermal distribution functions, and the second comes from \( 69 \) using thermal distribution functions and the KMS conditions. The general equation obtained from \( 69 \) using thermal distribution functions is:

\[ 2^n - 1 (-1) n + 1 C_{n A} \left( \{N_i \delta_{X_i A} \} \right) G_{(k_1, \cdots k_n) X_1 \cdots X_n} \]

\[ = C_{n R} \left( \{N_i \delta_{X_i R} \} \right) \sum_{\alpha_l \in \{r, a\}} \left( \prod_{l=1}^n \left[ \hat{N}(k_l; \alpha_l) \delta_{X_l A} + \delta_{X_l R} \delta_{\alpha l} \right] \right) G_{(k_1, \cdots k_n) \alpha_1 \cdots \alpha_n} \]

\[ 2^n + 1 (-1) n + 1 C_{n R} \left( \{N_i \delta_{X_i R} \} \right) \Gamma_{(k_1, \cdots k_n) X_1 \cdots X_n} \]

\[ = C_{n A} \left( \{N_i \delta_{X_i A} \} \right) \sum_{\alpha_l \in \{r, a\}} \left( \prod_{l=1}^n \left[ \hat{N}(k_l; \alpha_l) \delta_{X_l A} + \delta_{X_l R} \delta_{\alpha l} \right] \right) \Gamma_{(k_1, \cdots k_n) \alpha_1 \cdots \alpha_n} \]
After applying the KMS conditions to the right hand side of this expression we obtain:

\[ 2^{nR-1}(-1)^{n+1}G(k_1, \ldots k_n)_{X_1 \cdots X_n} \]

\[ = \sum_{\alpha_i \in \{r,a\}} \left( \prod_{l=1}^{n} \left[ \vec{N}(k_l; \alpha_l) \delta_{X_l,R} + \delta_{X_l,A} \delta_{\alpha_1a} \right] \right) G^*(k_1, \ldots k_n)_{\alpha_1 \cdots \alpha_n}. \]

For clarity we give a few examples. We first give general expressions using (49). We next give the results obtained with thermal distribution functions using (51). Finally, we give the expressions obtained by applying the KMS conditions to the right hand side of the previous equations, or equivalently from using (52).

[i] The 2-point functions in the R/A formalism are given by:

\[ D_{RR} = 0 \]
\[ D_{RA} = -\mathcal{F}_S(k_2)D_{ar} \]
\[ D_{AR} = -\mathcal{F}_S(k_1)D_{ra} \]
\[ D_{AA} = (\mathcal{F}_S(k_1)\mathcal{F}_S(k_2)D_{rr} - \mathcal{F}_A(k_1)\mathcal{F}_S(k_2)D_{ar} - \mathcal{F}_S(k_1)\mathcal{F}_R(k_2)D_{ra}) / (2f(-k_1)f(-k_2)). \]

Using equilibrium distribution functions these results reduce to:

\[ D_{RR} = 0; \quad D_{RA} = D_{ar}^*; \quad D_{AR} = D_{ar}^*; \quad D_{AA} = 0, \]

which can be written in the familiar form:

\[ D = \begin{pmatrix} D_{RR} & D_{RA} \\ D_{AR} & D_{AA} \end{pmatrix} = \begin{pmatrix} 0 & D_{aro} \\ D_{ret} & 0 \end{pmatrix}. \]

[ii] The 3-point functions in the R/A formalism are given by:

\[ G_{RRA} = \frac{f(k_1)f(k_2)}{f(k_1 + k_2)} \mathcal{F}_S(k_3) G_{aar} \]
\[ G_{RAR} = \frac{f(k_1)f(k_3)}{f(k_1 + k_3)} \mathcal{F}_S(k_2) G_{ara} \]
\[ G_{ARR} = \frac{f(k_2)f(k_3)}{f(k_2 + k_3)} \mathcal{F}_S(k_1) G_{aaa} \]
\[ G_{AAR} = \frac{1}{2} \frac{f(-k_1 - k_2)}{f(-k_1)} \left( \mathcal{F}_S(k_1)\mathcal{F}_S(k_2)G_{rra} - \mathcal{F}_A(k_1)\mathcal{F}_S(k_2)G_{ara} - \mathcal{F}_S(k_1)\mathcal{F}_A(k_2)G_{rra} \right) \]
\[ G_{ARA} = \frac{1}{2} \frac{f(-k_1 - k_3)}{f(-k_3)} \left( \mathcal{F}_S(k_1)\mathcal{F}_S(k_3)G_{rra} - \mathcal{F}_A(k_1)\mathcal{F}_S(k_3)G_{ara} - \mathcal{F}_S(k_1)\mathcal{F}_A(k_3)G_{rra} \right) \]
\[ G_{RAA} = \frac{1}{2} \frac{f(-k_2 - k_3)}{f(-k_3)} \left( \mathcal{F}_S(k_2)\mathcal{F}_S(k_3)G_{rra} - \mathcal{F}_A(k_2)\mathcal{F}_S(k_3)G_{ara} - \mathcal{F}_S(k_2)\mathcal{F}_A(k_3)G_{rra} \right) \]
\[ G_{AAA} = \frac{1}{4} \frac{f(-k_1)f(-k_2)f(-k_3)}{f(-k_1 - k_2)f(-k_1 - k_3)f(-k_2 - k_3)} \left( \mathcal{F}_S(k_1)\mathcal{F}_S(k_2)\mathcal{F}_S(k_3)G_{rrr} \right. \]
\[ -\mathcal{F}_A(k_1)\mathcal{F}_A(k_2)\mathcal{F}_A(k_3)G_{arr} + \mathcal{F}_S(k_1)\mathcal{F}_A(k_2)\mathcal{F}_S(k_3)G_{rra} - \mathcal{F}_S(k_1)\mathcal{F}_A(k_3)\mathcal{F}_S(k_2)G_{rra} \]
\[ + \mathcal{F}_A(k_1)\mathcal{F}_A(k_2)\mathcal{F}_A(k_3)G_{aar} + \mathcal{F}_A(k_1)\mathcal{F}_S(k_2)\mathcal{F}_A(k_3)G_{ara} + \mathcal{F}_S(k_1)\mathcal{F}_A(k_2)\mathcal{F}_A(k_3)G_{rra}. \]

Using equilibrium distribution functions these results reduce to:

\[ G_{RRA} = -\frac{(N_1 + N_2)}{2} G_{aar}. \]
Applying KMS conditions we have:

\[ G_{RAR} = -\frac{(N_1 + N_3)}{2} G_{ara} \]
\[ G_{ARR} = -\frac{(N_2 + N_3)}{2} G_{raa} \]
\[ G_{AAR} = \frac{1}{(N_1 + N_2)} (G_{rra} + N_1 G_{ara} + N_2 G_{raa}) \]
\[ G_{ARA} = \frac{1}{(N_1 + N_3)} (G_{rar} + N_1 G_{aar} + N_3 G_{raa}) \]
\[ G_{RAA} = \frac{1}{(N_2 + N_3)} (G_{arr} + N_2 G_{aar} + N_3 G_{ara}) \]
\[ G_{AAA} = -\frac{1}{4 \, n(-k_1) n(-k_2) n(-k_3)} (G_{rrr} + N_1 G_{arr} + N_2 G_{rar} + N_3 G_{rra} + N_1 N_2 G_{aar} + N_1 N_3 G_{ara} + N_2 N_3 G_{raa}) \]

Applying KMS conditions we have:

\[ G_{RRA} = -\frac{1}{2} (G_{rra}^* + N_1 G_{ara}^* + N_2 G_{raa}^*) \]  \( (57) \)
\[ G_{RAR} = -\frac{1}{2} (G_{rar}^* + N_1 G_{aar}^* + N_3 G_{raa}^*) \]
\[ G_{ARR} = -\frac{1}{2} (G_{arr}^* + N_2 G_{aar}^* + N_3 G_{ara}^*) \]
\[ G_{AAR} = G_{aar}^* \]
\[ G_{ARA} = G_{ara}^* \]
\[ G_{RAA} = G_{raa}^* \]
\[ G_{AAA} = 0. \]

For the 4-point functions using equilibrium distribution functions we have:

\[ G_{RRRA} = \frac{C(N_1, N_2, N_3)}{4} G_{aaar} \]
\[ G_{RRAR} = \frac{C(N_1, N_2, N_4)}{4} G_{aara} \]
\[ G_{RARR} = \frac{C(N_1, N_3, N_4)}{4} G_{araa} \]
\[ G_{ARRR} = \frac{C(N_2, N_3, N_4)}{4} G_{raaa} \]
\[ G_{RRAA} = -\frac{1 \, C(N_1, N_2)}{2 \, C(N_3, N_4)} \left( G_{aaar} + N_3 G_{aaar} + N_4 G_{aaar} \right) \]
\[ G_{RARA} = -\frac{1 \, C(N_1, N_3)}{2 \, C(N_2, N_4)} \left( G_{arar} + N_2 G_{arar} + N_4 G_{arar} \right) \]
\[ G_{RRAA} = -\frac{1 \, C(N_1, N_4)}{2 \, C(N_2, N_3)} \left( G_{arr} + N_2 G_{arr} + N_3 G_{arr} \right) \]
\[ G_{ARRA} = \frac{1 \, C(N_2, N_3)}{2 \, C(N_1, N_4)} \left( G_{rra} + N_1 G_{rra} + N_3 G_{rra} \right) \]
\[ G_{ARAR} = \frac{1 \, C(N_2, N_4)}{2 \, C(N_1, N_3)} \left( G_{arr} + N_1 G_{arr} + N_3 G_{arr} \right) \]
\[ G_{AARR} = \frac{1 \, C(N_2, N_4)}{2 \, C(N_1, N_3)} \left( G_{arra} + N_1 G_{arra} + N_3 G_{arra} \right) \]
\[ G_{AAAR} = \frac{1 \, C(N_3, N_4)}{2 \, C(N_1, N_2)} \left( G_{rra} + N_1 G_{rra} + N_2 G_{rra} \right) \]
\[ G_{AAAR} = \frac{1 \, C(N_3, N_4)}{2 \, C(N_1, N_2)} \left( G_{rara} + N_1 G_{rara} + N_2 G_{rara} \right) \]
\[ G_{AAAR} = \frac{1 \, C(N_3, N_4)}{2 \, C(N_1, N_2)} \left( G_{rra} + N_1 G_{rra} + N_2 G_{rra} \right) \]
In this master equation, the set of variables \(\{n\}_{\alpha_i',a'}\) contains 2\(n\) when expressed in the R/A basis.

Applying KMS conditions we have:

\[
G_{RRRA} = \frac{1}{4} (G_{rrrr}^* + N_1 G_{arra}^* + N_2 G_{rraa}^* + N_3 G_{rrra}^* + N_1 N_2 G_{aaa}^* + N_1 N_3 G_{ara}^* + N_2 N_3 G_{aara}^* + N_3 N_4 G_{aarr}^*)
\]

\[
G_{RAAR} = \frac{1}{4} (G_{rar}^* + N_1 G_{ara}^* + N_2 G_{raa}^* + N_3 G_{ra}^* + N_1 N_2 G_{aar}^* + N_1 N_3 G_{ar}^* + N_2 N_3 G_{ara}^* + N_3 N_4 G_{a}^*)
\]

\[
G_{RAR} = \frac{1}{4} (G_{r}^* + N_1 G_{aar}^* + N_2 G_{ra}^* + N_3 G_{a}^* + N_1 N_2 G_{aar}^* + N_1 N_3 G_{ar}^* + N_2 N_3 G_{ara}^* + N_3 N_4 G_{a}^*)
\]

\[
G_{ARR} = \frac{1}{4} (G_{a}^* + N_1 G_{aar}^* + N_2 G_{ra}^* + N_3 G_{a}^* + N_1 N_2 G_{aar}^* + N_1 N_3 G_{ar}^* + N_2 N_3 G_{ara}^* + N_3 N_4 G_{a}^*)
\]

\[
G_{AARR} = \frac{1}{4} (G_{aa}^* + N_1 G_{aar}^* + N_2 G_{ra}^* + N_3 G_{a}^* + N_1 N_2 G_{aar}^* + N_1 N_3 G_{ar}^* + N_2 N_3 G_{ara}^* + N_3 N_4 G_{a}^*)
\]

\[
G_{AA} = G_{aa}^*
\]

\[
G_{A} = G_{a}^*
\]

\[
G_{A} = G_{a}^*
\]

\[
G_{AAA} = G_{aaa}^*
\]

\[
G_{AAAA} = 0
\]

**VII. KMS CONDITIONS**

The KMS conditions are a set of relations between the various components of a given \(n\)-point function that hold at equilibrium. The KMS conditions are often useful for simplifying the expressions that result from the contractions over indices, in any representation of real time statistical field theory. The KMS conditions have the simplest structure when expressed in the R/A basis.

**A. Keldysh Basis**

The KMS conditions in the Keldysh representation can be written:

\[
C_n (\{N_i \delta_{\alpha_i',a'}\}) \sum_{\alpha_i \in \{r,a\}} \left( \prod_{l=1}^{n} \left[ \tilde{N}(k_l; \alpha_l) \delta_{\alpha_l' r} + \delta_{\alpha_l' a} \delta_{\alpha_l a} \right] \right) G_{\alpha_1 \cdots \alpha_n}
\]

\[
= C_n (\{N_i \delta_{\alpha_i',r}\}) \sum_{\alpha_i \in \{r,a\}} \left( \prod_{l=1}^{n} \left[ \tilde{N}(k_l; \alpha_l) \delta_{\alpha_l' r} + \delta_{\alpha_l' a} \delta_{\alpha_l a} \right] \right) G_{\alpha_1 \cdots \alpha_n}^*
\]

In this master equation, the set of variables \(\{\alpha_i'\}\) are external variables that are not summed over. Thus, in principle, \(\text{[60]}\) contains \(2^n\) equations which come from the \(2^n\) choices of the sets \(\{\alpha_i'\}\). In fact, one half of these equations is
the complex conjugate of the other half and thus we have $2^{n-1}$ KMS conditions for each $n$-point function. We give several examples below:

$n = 2$

For the 2-point function we obtain $2^{n-1} |_{n=2} = 2$ equations:

$$D_{rr}(p) = D_{rr}^*(p) \quad \text{or} \quad D_{el}(p) = D_{adv}^*(p)$$
$$D_{rr}(p) = N(p)(D_{ra}(p) - D_{ar}(p)) \quad \text{or} \quad D_{sym}(p) = N(p)(D_{el}(p) - D_{adv}(p)).$$

$n = 3$

There are $2^{n-1} |_{n=3} = 4$ independent KMS conditions which are:

$$G_{rrr} + N_1 G_{arr} + N_2 G_{rar} + N_3 G_{rra} + N_1 N_2 G_{aar} + N_1 N_3 G_{ara} + N_2 N_3 G_{raa} = 0$$

$n = 4$

There are $2^{n-1} |_{n=4} = 8$ independent KMS conditions which are:

$$G_{rrrr} + N_1 G_{arr} + N_2 G_{rar} + N_3 G_{rra} + N_4 G_{rraa} + N_1 N_2 G_{aarr} + N_1 N_3 G_{arar} + N_1 N_4 G_{aara}$$

$$+ N_2 N_3 G_{aara} + N_2 N_4 G_{aarr} + N_3 N_4 G_{arar} + N_3 N_4 G_{arar} = 0$$

$n = 5$

There are $2^{n-1} |_{n=5} = 16$ independent KMS conditions. We write down three representative ones:

$$G_{rrrrr} + N_1 G_{arr} + N_2 G_{rar} + N_3 G_{rra} + N_4 G_{rraa} + N_5 G_{rraa}$$

$$+ N_1 N_2 G_{aarr} + N_2 N_3 G_{ara} + N_1 N_3 G_{ara} + N_1 N_4 G_{ara} + N_2 N_3 G_{ara}$$

$$+ N_1 N_4 G_{ara} + N_2 N_5 G_{ara} + N_3 N_4 G_{ara} + N_3 N_5 G_{ara} + N_4 N_5 G_{ara}$$

$$+ N_1 N_2 G_{aarr} + N_2 N_3 G_{ara} + N_1 N_3 G_{ara} + N_1 N_4 G_{ara} + N_2 N_3 G_{ara}$$

$$+ N_3 N_4 G_{ara} + N_3 N_5 G_{ara} + N_4 N_5 G_{ara} + N_1 N_2 G_{aarr} + N_2 N_3 G_{ara}$$

$$+ N_1 N_4 G_{ara} + N_2 N_5 G_{ara} + N_3 N_4 G_{ara} + N_3 N_5 G_{ara} + N_4 N_5 G_{ara} = 0$$
\[ \begin{align*}
+ N_1 N_3 G_{arara} + N_1 N_4 G_{araa} + N_2 N_3 G_{raara} + N_2 N_4 G_{raraa} \\
+ N_1 N_2 N_3 G_{aara} + N_1 N_2 N_4 G_{araa} + N_1 N_3 N_4 G_{raara} + N_2 N_3 N_4 G_{raraa} \\
= (N_1 + N_2 + N_3 + N_4 + N_1 N_2 N_3 + N_1 N_2 N_4 + N_1 N_3 N_4 + N_2 N_3 N_4) G_{aaar}^a \\
(N_4 + N_5) (G_{raraa} + N_1 G_{araa} + N_2 G_{rara} + N_3 G_{raaa} + N_1 N_2 G_{aaara} + N_1 N_3 G_{araaa} + N_2 N_3 G_{raaaa})
= (1 + N_1 N_2 + N_1 N_3 + N_2 N_3) (G_{aaar}^a + N_4 G_{aaara}^a + N_5 G_{aaara}^a)
\end{align*} \]

The KMS conditions for the vertex functions have almost exactly the same form. The master equation is obtained from (60) by interchanging the indices \( r \) and \( a \):

\[ C_{n_d}(\{N_i \delta_{\alpha_i' r}, \}) \sum_{\alpha_j \in \{r, a\}} \left( \prod_{j=1}^{\infty} \left[ \hat{N}(k_i; \alpha_l) \delta_{\alpha_i' r} + \delta_{\alpha_i' a} \delta_{\alpha_l r} \right] \right) G_{\alpha_1 \cdots \alpha_n}^{r a \cdots a} \]

\[ = C_{n_d}(\{N_i \delta_{\alpha_i' a}, \}) \sum_{\alpha_j \in \{r, a\}} \left( \prod_{j=1}^{\infty} \left[ \hat{N}(k_i; \alpha_l) \delta_{\alpha_i' r} + \delta_{\alpha_i' a} \delta_{\alpha_l r} \right] \right) (G^*)_{\alpha_1 \cdots \alpha_n}^{r a \cdots a} \tag{65} \]

**B. R/A Basis**

The KMS conditions have an even simpler form in the R/A representation. This can be anticipated by comparing the general structure of (61) and (60). The general expression for the KMS conditions in the R/A basis is:

\[ 2^{n_R - n_A} C_{n_d}(\{N_i \delta_{X_i A}, \}) G_{X_1 \cdots X_n} = (-1)^n C_{n_d}(\{N_i \delta_{X_i A}, \}) G_{X_1 \cdots X_n}^* \tag{66} \]

This master equation contains a series of equations where the variables \( X_i \) take all possible combinations of the values \( \{R, A\} \). We use the notation: \( A = R, \; \bar{R} = A \) so that \( n_{\bar{R}} = n_A \) and \( n_{\bar{A}} = n_R \). We give several examples below:

\[ n = 2 \] For the 2-point functions we have:

\[ D_{AR}(p) = D_{RA}(p); \; D_{AA} = 0. \tag{67} \]

\[ n = 3 \] For the 3-point functions we have:

\[ G_{AAA} = 0 \tag{68} \]

\[ 2C(N_3) G_{RRA} = - C(N_1, N_2) G_{AAR}^* \rightarrow 2G_{RRA} = -(N_1 + N_2) G_{AAR}^* \]

\[ 2C(N_2) G_{RAR} = - C(N_1, N_3) G_{AR}^* \rightarrow 2G_{RAR} = -(N_1 + N_3) G_{AR}^* \]

\[ 2C(N_1) G_{ARR} = - C(N_2, N_3) G_{AR}^* \rightarrow 2G_{ARR} = -(N_2 + N_3) G_{AR}^* \]

where the arrows indicate the results that are obtained by replacing the \( C \)-functions by their definitions in terms of thermal functions (see Eqn. (26)).

\[ n = 4 \] For the 4-point functions we have:

\[ G_{AAAA} = 0 \tag{69} \]

\[ C(N_3, N_4) G_{RRRA} = C(N_1, N_2) G_{AARR}^* \rightarrow (N_3 + N_4) G_{RRRA} = (N_1 + N_2) G_{AARR}^* \]

\[ C(N_2, N_4) G_{RRAA} = C(N_1, N_3) G_{ARAA}^* \rightarrow (N_2 + N_4) G_{RRAA} = (N_1 + N_3) G_{ARAA}^* \]

\[ C(N_2, N_3) G_{RAAA} = C(N_1, N_4) G_{ARAA}^* \rightarrow (N_2 + N_3) G_{RAAA} = (N_1 + N_4) G_{ARAA}^* \]

\[ 4G_{ARRR} = C(N_2, N_3, N_4) G_{ARAA} \rightarrow 4G_{ARRR} = (1 + N_2 N_3 + N_3 N_4 + N_4 N_2) G_{ARAA}^* \]

\[ 4G_{ARRR} = C(N_1, N_3, N_4) G_{ARAA} \rightarrow 4G_{ARRR} = (1 + N_1 N_3 + N_3 N_4 + N_4 N_1) G_{ARAA}^* \]

\[ 4G_{RARR} = C(N_1, N_2, N_4) G_{ARAA} \rightarrow 4G_{RARR} = (1 + N_1 N_2 + N_2 N_4 + N_4 N_1) G_{ARAA}^* \]

\[ 4G_{RRRA} = C(N_1, N_2, N_3) G_{ARAA} \rightarrow 4G_{RRRA} = (1 + N_1 N_2 + N_2 N_3 + N_3 N_1) G_{ARAA}^* \]

\[ n = 5 \] For the 5-point functions we give a few examples:

\[ G_{AAAAA} = 0 \tag{70} \]
Note that in every case one of the KMS equations requires the vanishing of the $G_{A_{-A}}$ component of the $n$-point function. The rest of the KMS equations are simple relations between pairs of off-diagonal components.

The KMS conditions for the vertex functions have almost exactly the same form. They are obtained from the same master equation, with the indices $R$ and $A$ interchanged:

$$2^n A^n C_n R^n C_n A^n R^n (\{N_i \delta X_{i,R}\}) \Gamma X_1 \cdots X_n = (-1)^n C_n R^n (\{N_i \delta X_{i,R}\}) (\Gamma)^{\bar{X}_1 \cdots \bar{X}_n} \quad (71)$$

VIII. A PROGRAM TO PERFORM THE CONTRACTION OF INDICES

A. Description of the Program

Our program calculates the integrand for any diagram by contracting indices. The calculation can be done in the 1-2, Keldysh or R/A basis. The basic strategy of the program is to treat a Feynman diagram as a tensor product of propagators, and bare and corrected vertices. In principle, this type of calculation can be done by hand, but when working with more than a few indices the process becomes extremely tedious. The symbolic manipulation program Mathematica is ideally suited to perform this kind of tensor calculation.

The program is divided into six main sections. Only the Input section needs to be edited by the user. The user enters some variables in this section, in order to specify the diagram that he wants to calculate. The rest of the program can be immediately executed, and the result is output in the last section. The user has the option to output the results to a file. The results are also defined functionally. Some basic functions are defined within the program that can be used interactively to manipulate the result.

When working in the RA basis the program always assumes thermal distribution functions, and thus the result is only valid in equilibrium. The user has the option to implement the KMS conditions, since both forms of the result can be useful. In the Keldysh basis, thermal distribution forms are only assumed if the KMS conditions are used. In the 1-2 basis the KMS conditions cannot be implemented in the current version of the program. Note that we have not used different notation to indicate equilibrium and non-equilibrium distributions (see Eqns. (23) and (24)): the output of the program will always contain distribution functions written in the form $n_p$ or $N_P$.

The program is designed to contract CTP indices and is written for scalar bosons. We set all coupling constants to one and use the notation:

- $\varepsilon = \text{i} \cdot \text{D}$
- $\triangleright = -\text{i}$
- $\times = -\text{i}$
- $\blacksquare = -\text{i} \Pi$
- $\bigtriangledown = -\text{i} \Gamma$
- $\bigcirc = -\text{i} \cdot M$
- $\blacklozenge = -\text{i} \cdot C$

![FIG. 2: Definitions of notation for propagator and vertices](image)

Note that additional numerical factors are introduced in the Keldysh representation, as defined in (32). The program can be used for other field theories with appropriate adjustments. The appropriate coupling constant(s) must be inserted by the user. In addition, any dirac, lorentz, or other group structure must be separately handled by the user.

We describe briefly the main sections of the program:

1. The Initialization section inputs the necessary Mathematica package.
2. The **Input** section is edited by the user to input specific parameters corresponding to the diagram he wants to calculate. This process is described in detail below.

3. The **Definitions** section establishes some basic definitions that will be used throughout the program.

4. The **Find Loops** section identifies all closed loops in the diagram. The variable \( \text{loopzerosubs} \) is printed out. The usefulness of this variable is explained in point 11 below.

5. The **Calculate Diagram** section performs the calculation.

6. The **Results** section outputs the results of the calculations and performs some basic manipulations to simplify them.

### B. Input section

We illustrate the **Input** section of the program with an example. Consider the diagram in Fig. 3.

![Feynman Diagram](image)

FIG. 3: Example Feynman diagram. Uppercase letters are momenta and lowercase letters label indices of the propagators and the vertices.

The **Input** section for this diagram is reproduced on the page that follows the itemized list below. Each entry is explained in the order it appears in the **Input** section.

1. **number of external legs, external momenta, external indices:** Specify the number of external legs, and the momentum and corresponding index for each leg.

2. **number of internal indices, internal indices:** Specify the number of internal indices and list them.

3. **list of momenta, list of indices:** For each bare vertex, list the incoming momentum for each leg and the corresponding index.

4. **list of momenta, list of indices:** For each corrected 3-point vertex, corrected 4-point vertex and corrected 5-point vertex, list the incoming momentum for each leg and the corresponding index.

5. **list of momenta, list of indices:** List the momentum and the corresponding pair of initial and final indices for each propagator.
   - In this example, there is one corrected 3-point vertex and one corrected 4-point vertex. The corresponding momentum arguments are listed in double set brackets, indicating that the one vertex is the first in a list of length one. The same notation is used in any case where multiple entries would create nested lists. This includes the momenta and indices of bare or corrected \( n \)-point vertices, with \( n \geq 3 \), and the indices of propagators. The momenta for any number of propagators (even if there is only one) are listed in single set brackets.

6. **choose basis (onetwo, Keldysh or RA):** Indicate the basis as either **onetwo**, **Keldysh** or **RA**.

7. **combination(s) to be evaluated:** ie. \( \text{ra, ... or AR, . or All} \): Indicate the set of external indices of the diagrams that are to be calculated or simply specify **All**.
8. **Simplify result? (option to use the Mathematica function ‘Simplify’)**: The user can choose to have the program apply the Mathematica function `Simplify` to the result (`simplifyIt = yes`). For smaller diagrams this produces neater results. For larger diagrams it can lead to significantly longer running times, without producing a result that is much more compact.

9. **use the KMS conditions**: In the Keldysh or RA bases, one can choose to enforce the KMS conditions (`useKMSconditions = yes`). In the 1-2 basis choosing (`useKMSconditions = yes`) has no effect on the output. If the user has chosen the Keldysh basis, he can enforce the KMS conditions for the propagator, but not for higher n-point vertices (`removeFs = yes`). Note that if the user chooses `useKMSconditions = yes` and `removeFs = no`, the first choice over-rides the second, and KMS conditions are used for all n-point functions including propagators.

10. **express answer in terms of RA expressions**: In the Keldysh basis, if the calculation was done using the KMS conditions, the user can request that the right-hand side of the final expression be written in the R/A basis (`InTermsOfRA = yes`), since this basis frequently produces more compact results. The default value of this parameter is “no” and produces results in which both sides of the equation are written in the same basis.

11. **remove terms which are zero after integration**: One can choose to eliminate terms which will vanish after integration (`removeZeros = yes`). This option only works in the Keldysh or RA bases (in the 1-2 basis choosing (`removeZeros = yes`) has no effect on the output). The program identifies closed loops in the diagram, and then looks for terms in which all of the propagators that form a closed loop have poles on the same side of the real axis. Choosing (`removeZeros = yes`) will cause the program to set all of these terms (except for tadpoles) to zero. The program automatically exempts tadpole loops, so that they are not incorrectly set to zero. The list of combinations of propagators which will be removed by this option are stored in (`loopzerosubs`) and printed in the Find Loops section of the program. The propagators that correspond to tadpoles are also listed. One can compare these lists to the original diagram as a check of the information given in the Input section. For our example we have:

\[
G_{ra}(L)G_{ra}(L + P + Q) \rightarrow 0; \quad G_{ar}(L)G_{ar}(L + P + Q) \rightarrow 0
\]  

The two expressions correspond to the one loop in our example diagram, with the momentum routed clockwise or counter-clockwise.

12. **replace C functions in terms of N functions**: If the KMS conditions are used, the program does the calculation in terms of the $C$-functions defined in Eqn. (26). One can choose to replace these $C$-functions with their definitions in terms of thermal functions (`replaceCs = yes`) - see Eqn. (26).

13. **specify name of output file using quotes**: The user can specify a filename to output the results to, or specify an empty string (""") for no output file.

14. **Directory for output file is directory of this notebook as specified below**: The directory of the output file is set to be the same as the directory of the original notebook. The user can specify another directory for the output file by rewriting this line.

### C. Error messages

Some checks have been implemented to detect possible input mistakes, and give error messages indicating the nature of the problem. A few examples are given below:

1. If the incoming momenta for any of the internal vertices, or for the external legs, do not sum to zero, the error message appears: “Momentum non-conserving vertex detected.”

2. If the combinations of external indices requested using “combination(s) to be evaluated” do not correspond to the basis specified using “choose basis (onetwo, Keldysh or RA),” an error message appears. For example, choosing the RA basis and requesting the combination $rra$ (corresponding to the vertex function $\Gamma^{rra}$) produces the error message: “Initialization Failed: Combination $rra$ is not specified in the RA basis.”

3. If the combinations of external indices requested do not have the same number of variables as the number of external legs, an error message will appear. For example, calculating a 3-point vertex function (as in our example) and requesting the combination $rraa$ (corresponding to the 4-point vertex $\Gamma^{rraa}$) produces the error message: “Initialization Failed: Combination $rraa$ does not have the correct number of indices.”
General::spell1
General::spell

Off[General::spell];
Off[General::spell1];

(* number of external legs: *) numExternalLegs = 3;
(* external momenta: *) externalMomenta = {P, Q, -P - Q};
(* number of internal indices: *) numInternalIndices = 4;
(* internal indices: *) internalIndices = {d, e, f, g};

(* list of momenta: *) bare3ptMomenta = {};
(* list of indices: *) bare3ptIndices = {};

(* list of momenta: *) bare4ptMomenta = {};
(* list of indices: *) bare4ptIndices = {};

(* list of momenta: *) Cor3ptMomenta = {{P + Q + L, -L, -P - Q}};
(* list of indices: *) Cor3ptIndices = {{f, g, c}};

(* list of momenta: *) Cor4ptMomenta = {{P, Q, -P - Q - L, L}};
(* list of indices: *) Cor4ptIndices = {{a, b, d, e}};

(* list of momenta: *) Cor5ptMomenta = {};
(* list of indices: *) Cor5ptIndices = {};

(* list of momenta: *) propagatorMomenta = {L, P + Q + L};
(* list of indices: *) propagatorIndices = {{g, e}, {d, f}};

(* choose basis (onetwo,
Keldysh or RA) *) Basis = Keldysh;

(* combination(s) to be evaluated: ie. {ra, ...} or {AR, .} or
All *) combinations = All;

(* Simplify result? (using Mathematica function:
Simplify) *) simplifyIt = no;

(* use the KMS conditions *)
useKMSconditions = no;
(* use KMS conditions for propagator only
(in the Keldysh basis) *) removeFs = yes;
(* express answer in terms of RA expressions
(in equilibrium and Keldysh only) *) removeZeros = yes;
(* replace C functions in terms of N functions *) replaceCs = yes;

(* specify name of file to output to using quotes *)
FileName = "leafA_example.txt";

(* Directory for output file is directory of this notebook as specified below *)
SetDirectory[DirectoryName[ToFileName[NotebookInformation[InputNotebook[]][1, 2]]]]
4. Each external index must appear one time only as a vertex index. Each internal index must appear one time as a vertex index and one time as a propagator index. If the indices are incorrectly entered, an error message will appear.

5. Because of the fact that vertices are defined with incoming momenta, for each propagator, the momentum corresponding to the second index must be the same as the momentum for the vertex leg with the matching index. Similarly, the momentum corresponding to the first index must be the negative of the momentum for the vertex leg with the matching index. If the momentum variables are incorrectly entered, an error message will appear.

### D. Interactive Manipulation of the Results

The result of the calculation is stored in the internal variable “diagram.” The user can extract and manipulate the results in several ways.

1. One can extract a specific result using the function DIAGRAM. For our example, evaluating DIAGRAM[a, r, r] gives the result for $\Gamma^{arr}$ in the Keldysh representation:

$$\Gamma^{arr} = \frac{1}{2} i G_{ar}(L) G_{ra}(L + P + Q) \Gamma^{arr}(P, Q, L, -L - P - Q)$$

2. In Eqn. (74) we define notation for the Keldysh basis so that each string of indices of the form $rrar \cdots$ corresponds to a single numerical index. A function called GAMMA is defined that will automatically supply the result of the calculation for a given numerical index. All vertices are translated into the notation defined in Eqn. (75). Propagators are written using the notation $D_{ra}(P) = D_{ret}(P) := r(P)$; $D_{ar}(P) = D_{adv}(P) := a(P)$. The function GAMMA is illustrated below. From Eqn. (77), the index ‘2’ corresponds to the string ‘arr’ for a 3-point vertex function. For our example, evaluating GAMMA[2] produces the translation of the result for $\Gamma^{arr}$ that is given in (78):

$$\Gamma^{(4, L + S, -L, -S)} - N_L \Gamma^{(2, L + S, -L, -S)} + N_{L+S} \Gamma^{(3, L + S, -L, -S)}$$

### IX. THE QED WARD IDENTITY FOR THE 4-POINT FUNCTION

In this section, we present a calculation of the ward identity for the QED 3-point vertex function, and the QED 4-point vertex function involving two photons and two fermions. Throughout this section we work in the Keldysh representation. As explained in the beginning of section VIII the goal of our program is provide compact results for the expressions that result from summing over CTP indices. The Dirac structure of fermions and any group structure associated with all fields, must be handled separately by the user. For the calculations done in this paper, there are no additional complications. The ward identities are derived by comparing groups of integrands, without evaluating the integrals themselves. Consequently, we can simply suppress all Dirac and Lorentz indices. For example, the result for the contribution to $\Sigma(2, p)$ shown in Fig. 6 is given by Eqn. (78). The first term is $-\frac{i}{2} \int \frac{d^4l}{(2\pi)^4} a(l) r(l+p) \Gamma^{(2, p, l+p)} \Gamma^{(4, l+p, p)}$. Including all indices, this factor gives a contribution:

$$-\frac{i}{2} \int \frac{d^4l}{(2\pi)^4} a_{\mu\nu}(l) \Gamma_{\beta\gamma}(2, l, l+p) r_r(\alpha' \beta') (l+p) \Gamma_{\alpha' \beta'}(4, l+p, p)$$

where $\{\mu, \nu\}$ are Lorentz indices and $\{\alpha, \alpha', \beta, \beta'\}$ are Dirac indices. Throughout the rest of this section we will leave all integrands in the form described above, with Dirac and Lorentz indices suppressed. In addition, we define $k + p + q = u$, $k + p = t$, $p + q = s$ and use the shorthand notation: $D_{ret}(p) = r(p)$, $D_{adv}(p) = a(p)$, $D_{sym}(p) = f(p)$.
In order to simplify the notation for the vertices, we replace each combination of the indices \( \{r, a\} \) by a single numerical index:

\[
\Gamma^{(n)} \alpha_1 \alpha_2 \cdots \alpha_n (p_1, p_2, \cdots, p_n) = \Gamma^{(n)}(i, p_1, p_2, \cdots, p_n)
\]  

(76)

We assign the choices of the variables \( \alpha_1 \alpha_2 \cdots \alpha_n \) to the variable \( i \) using the vector

\[
V_n = \left( \begin{array}{c} r_n \\ a_n \\ \vdots \\ r_2 \\ a_2 \\ \vdots \\ r_1 \\ a_1 \end{array} \right) \otimes \left( \begin{array}{c} r_1 \\ a_1 \end{array} \right)
\]

(77)

where the symbol \( \otimes \) indicates the outer product. For each \( n \), the \( i \)th component of the vector corresponds to a list of variables that is assigned the number \( i \). To simplify the notation we drop the subscripts and write a list like \( r_1 r_2 a_3 \) as \( r a r \). For clarity, the results are listed below.

[a] 2-point functions: \( rr \rightarrow 1, ar \rightarrow 2, ra \rightarrow 3, aa \rightarrow 4 \)

[b] 3-point functions: \( rrr \rightarrow 1, arr \rightarrow 2, rar \rightarrow 3, aar \rightarrow 4, rra \rightarrow 5, ara \rightarrow 6, raa \rightarrow 7, aaa \rightarrow 8 \)

[c] 4-point functions: \( rrrr \rightarrow 1, arrrr \rightarrow 2, rarrr \rightarrow 3, aarrr \rightarrow 4, raarr \rightarrow 5, ararr \rightarrow 6, aara \rightarrow 7, aaaaa \rightarrow 8 \)

[d] 5-point functions: \( rrrrr \rightarrow 1, arrrr \rightarrow 2, rarrr \rightarrow 3, aarrr \rightarrow 4, raarr \rightarrow 5, ararr \rightarrow 6, aara \rightarrow 7, aaarr \rightarrow 8, aaara \rightarrow 9, arraa \rightarrow 10, arara \rightarrow 11, aarar \rightarrow 12, arraa \rightarrow 13, raara \rightarrow 14, raara \rightarrow 15, raara \rightarrow 16 \)

We note that \( i = 1 \) corresponds to a vertex that is identically zero, for any number of external legs, as a consequence of the general relation \( \Gamma^{rr \cdots r} = 0 \) which can be obtained from (95).

To reduce the number of indices we will introduce separate names for the first five vertex functions and write:

\[
\Gamma^{(2)} = \Sigma; \quad \Gamma^{(3)} = \Gamma; \quad \Gamma^{(4)} = M; \quad \Gamma^{(5)} = C.
\]

(78)

The notation for the momentum arguments is as follows:

\[
\Sigma(p_{in}) ; \quad \Gamma^\mu(p_{in}, p_{out}) ; \quad M^{\mu\nu}(p_{in}, q_1^\mu, q_2^\nu, p_{out}) ; \quad C^{\mu\nu\tau}(p_{in}, q_1^\mu, q_2^\nu, q_3^\tau, p_{out})
\]

(79)

where \( p_{in} \) is the momentum of the incoming fermion, \( \{q_1, q_2, q_3\} \) are the momenta of the incoming photons, and \( p_{out} \) is the momentum of the outgoing fermion. Note that the momentum of the photon is not written for the 3-point vertex since it can be inferred from the momenta of the fermions. Similarly, the self energy is written \( \Sigma(p_{in}) \) instead of \( \Sigma(p_{in}, -p_{in}) \).

We begin by calculating the ward identity at the bare 1-loop level. In the next section, we verify that the same ward identities are satisfied by the complete set of graphs involving full corrected vertices.

### A. Bare 1-loop Diagrams

We start by looking at 1-loop diagrams with bare vertices, in order to determine the form of the ward identities. For the 2- and 3-point functions the graphs are shown in Figs. (5a) and (5b). For the 4-point function we have the box graph (Fig. (5c)), and the crossed version of the box graph where the two external photons are interchanged. For the five point function, the basic graph is show in Fig. (5d). There are six versions of this graph which correspond to the six possible permutations of the three external photons.
We give three specific examples below. The first set of numbers in the first line of (82) is \((2,2,2)\). Using (77) and (81) the corresponding ward identity is:

\[-ie^{\mu}G^{(1)}(i, p, p + q) = \Sigma^{(1)}(j_1, p) - \Sigma^{(1)}(j_2, p + q)\]

\[-ie^{\mu}M^{(1)}(i, p, q, k, p + q + k) = \Gamma^{(1)}_{\mu}(j_1, p, p + k) - \Gamma^{(1)}_{\mu}(j_2, p + q, k + p + q)\]

\[-ie^{\mu}C^{(1)}(i, p, q, k, s, p + q + k + s) = M^{(1)}_{\mu \nu}(j_1, p, k, s, k + p + s) - M^{(1)}_{\mu \nu}(j_2, p + q, k, s, k + p + q + s)\]

where the superscripts refer to the loop order of the graph. The ward identities are of the form

\[-i\Sigma^{(1)} = \text{self-energy graph}\]

\[-ie^{\mu}G^{(1)} = \sqrt{2} \left( \text{triangle graph} \right)\]

\[-ie^{\mu}M^{(1)} = 2 \left( \text{box graph + crossed box graph} \right)\]

\[-ie^{3}C^{(1)} = 2 \sqrt{2} \left( \text{six permutations of 5-point graph} \right)\]

where the indices \(\{i, j_1, j_2\}\) refer to the choices of the variables \(\alpha_1 \alpha_2 \cdots \alpha_n\) as defined in (17). We list below the sets of these indices for the 3-point, 4-point and 5-point functions.

\[\Gamma: \quad (2,2,2), (3,3,2), (4,4,1), (5,3,3), (6,4,4), (7,1,4), (8,2,3)\]  \hspace{1cm} (82)

\[M: \quad (2,2,2), (3,5,2), (4,6,1), (5,3,3), (6,4,4), (7,7,4), (8,8,3), (9,5,5), (10,6,6), (11,1,6), (12,2,5), (13,7,7), (14,8,8), (15,3,8), (16,4,7)\]

\[C: \quad (2,2,2), (3,9,2), (4,10,1), (5,3,3), (6,4,4), (7,11,4), (8,12,3), (9,5,5), (10,6,6), (11,13,6), (12,14,5), (13,7,7), (14,8,8), (15,15,8), (16,16,7), (17,9,9), (18,10,10), (19,1,10), (20,2,9), (21,11,11), (22,12,12), (23,3,12), (24,4,11), (25,13,13), (26,14,14), (27,5,14), (28,6,13), (29,15,15), (30,16,16), (31,7,16), (32,8,15)\]  \hspace{1cm} (83)

We give three specific examples below. The first set of numbers in the first line of (82) is \((2,2,2)\). Using (17) and (31) the corresponding ward identity is:

\[Q^{\mu}G^{(1)}_{\mu}(2, p, p + q) = \Sigma^{(1)}(2, p) - \Sigma^{(1)}(2, p + q)\]  \hspace{1cm} (83)

or

\[Q \cdot \Gamma^{(1)}_{arr}(p, p + q) = \Sigma^{(1)}_{ar}(p) - \Sigma^{(1)}_{ar}(p + q)\]

The first set of numbers in the second line of (82) is \((2,2,2)\). The corresponding ward identity is

\[Q^{\mu}M^{(1)}_{\mu \nu}(2, p, q, k, p + q + k) = \Gamma^{(1)}_{\mu \nu}(2, p, p + k) - \Gamma^{(1)}_{\mu \nu}(2, p + q, k + p + q)\]  \hspace{1cm} (84)

or

\[Q^{\mu} \left( M^{(1)}_{\mu \nu} \right)^{arr}_{arr}(p, p + k) = \left( \Gamma^{(1)}_{\mu \nu} \right)^{arr}_{arr}(p, k + p) - \left( \Gamma^{(1)}_{\mu \nu} \right)^{arr}_{arr}(p + q, k + p + q)\]

The last set of numbers in the last line of (82) is \((32,8,15)\). The corresponding ward identity is

\[Q^{\mu}C^{(1)}_{\mu \nu \rho \sigma}(32, p, q, k, s, p + q + k + s) = M^{(1)}_{\mu \nu}(8, p, k, s, k + p + s) - M^{(1)}_{\mu \nu}(15, p + q, k, s, k + p + q + s)\]

or

\[Q^{\mu} \left( C^{(1)}_{\mu \nu \rho \sigma} \right)^{aaaaa}_{aaaaa}(p, q, k, s, p + q + k + s) = \left( M^{(1)}_{\mu \nu} \right)^{aaaa}_{aaaa}(p, k, s, k + p + s) - \left( M^{(1)}_{\mu \nu} \right)^{aaaa}_{aaaa}(p + q, k, s, k + p + q + s)\]
B. Full Vertices

We verify that the ward identities derived above hold for the full 3-point and 4-point vertex functions.

1. 2-point vertex function

We give the results for the 2-point vertex function shown in Fig (6). These expressions will be needed to verify the ward identities for the 3-point vertex function.

\[
\Sigma(2, p) = -\frac{i}{2} \int \frac{d^4 l}{(2\pi)^4} \left[ a(l)r(l + p)\Gamma(2, p, l + p)\left(\Gamma(4, l + p, p) + \Gamma(3, l + p, p)N_F(l + p) - \Gamma(2, l + p, p)N_B(l)\right)\right]
\]

\[
\Sigma(3, p) = -\frac{i}{2} \int \frac{d^4 l}{(2\pi)^4} \left[ a(l + p)r(l)\Gamma(5, l + p, p)\left(\Gamma(7, p, l + p) - \Gamma(3, p, l + p)N_F(l + p) + \Gamma(5, p, l + p)N_B(l)\right)\right]
\]

\[
\Sigma(4, p) = -\frac{i}{2} \int \frac{d^4 l}{(2\pi)^4} \left[ a(l + p)r(l + p)\Gamma(5, l + p, p)\left(\Gamma(8, p, l + p) - \Gamma(4, p, l + p)N_F(l) + \Gamma(6, p, l + p)N_B(l)\right) + r(l + p)\Gamma(6, p, l + p)\left(\Gamma(8, l + p, p) + \Gamma(7, l + p, p)N_F(l + p) - \Gamma(6, l + p, p)N_B(l)\right)\right]
\]

C. 3-point vertex function

First we calculate all seven components of the three graphs shown in Fig (7) that contribute to the 3-point vertex function. We give the result for one example: \(\Gamma[2, p, p + q]\).

\[
\Gamma_{\text{triangle}}(2, p, p + q) = \frac{i}{2} \int \frac{d^4 l}{(2\pi)^4} \left[ a(l)r(l + p)\Gamma(2, p, l + p)\right.
\]

\[
(a(l + s)\Gamma(3, l + s, s)\left(\Gamma(6, l + p, l + s) + \Gamma(5, l + p, l + s)N_F(l + p) - \Gamma(2, l + p, l + s)N_B(l + s)\right) + r(l + s)\Gamma(2, l + p, l + s)\left(\Gamma(4, l + s, s) + \Gamma(3, l + s, s)N_F(l + s) - \Gamma(2, l + s, s)N_B(l)\right)\bigg] \]

FIG. 6: The 2-point vertex function

FIG. 7: The 3-point vertex function
Comparing with (85) we obtain

\[ \Gamma_{\text{leafA}}(2, p, p + q) = -\frac{i}{2} \int \frac{d^4l}{(2\pi)^4} \left[ a(l)r(l + p)M(2, p, q, l, l + s)(\Gamma(4, l + s, s) + \Gamma(3, l + s, s)N_F(l + s) - \Gamma(2, l + s, s)N_B(l)) \right] \]

\[ \Gamma_{\text{leafB}}(2, p, p + q) = -\frac{i}{2} \int \frac{d^4l}{(2\pi)^4} \left[ a(l)r(l + p)\Gamma(2, p, l + p)M(6, l + p, q, -l, s) + M(5, l + p, q, -l, s)N_F(l + p) - \Gamma(2, l + p, q, -l, s)N_B(l) \right] \]

Next, we verify the ward identities for the seven vertex functions. We give detailed results for one example: \( \Gamma[2, p, p + q] \). Starting from (86) and contracting with \( Q \) we obtain:

\[ Q \cdot \Gamma_{\text{triangle}}(2, p, p + q) = X[2] + Y[2] \]

\[ Q \cdot \Gamma_{\text{leafA}}(2, p, p + q) = -X[2] + x[2] \]

\[ Q \cdot \Gamma_{\text{leafB}}(2, p, p + q) = -Y[2] + y[2] \]

where

\[ X[2] = \frac{i}{2} \int \frac{d^4l}{(2\pi)^4} \left[ a(l)r(l + p + q)\Gamma(2, p, l + p) \right. \]

\[ \left. (\Gamma(4, l + p + q, q, p + q) + \Gamma(3, l + p + q, q, p + q)N_F(l + p + q) - \Gamma(2, l + p + q, q, p + q)N_B(l)) \right] \]

\[ Y[2] = -\frac{i}{2} \int \frac{d^4l}{(2\pi)^4} \left[ a(l)r(l + p)\Gamma(2, p, l + p) \right. \]

\[ \left. (\Gamma(4, l + p + q, q, p + q) + \Gamma(3, l + p + q, q, p + q)N_F(l + p) - \Gamma(2, l + p + q, q, p + q)N_B(l)) \right] \]

\[ x[2] = \frac{i}{2} \int \frac{d^4l}{(2\pi)^4} \left[ a(l)r(l + p + q)\Gamma(2, p + q, l + p + q) \right. \]

\[ \left. (\Gamma(4, l + p + q, q, p + q) + \Gamma(3, l + p + q, q, p + q)N_F(l + p + q) - \Gamma(2, l + p + q, q, p + q)N_B(l)) \right] \]

\[ y[2] = -\frac{i}{2} \int \frac{d^4l}{(2\pi)^4} \left[ a(l)r(l + p)\Gamma(2, p, l + p) \right. \]

\[ \left. (\Gamma(4, l + p, p) + \Gamma(3, l + p, p)N_F(l + p) - \Gamma(2, l + p, p)N_B(l)) \right] \]

Comparing with (85) we obtain

\[ Q \cdot \Gamma(2, p, p + q) = Q \cdot \left( \Gamma_{\text{triangle}}(2, p, p + q) + \Gamma_{\text{leafA}}(2, p, p + q) + \Gamma_{\text{leafB}}(2, p, p + q) \right) = \Sigma(2, p) - \Sigma(2, p + q) \]

which agrees with (83). The results for all components agree with the results listed in (82).

1. 4-point vertex function

We verify the ward identity for the 15 4-point vertex functions. We give detailed results for one example: \( M(2, p, q, k, p + q + k) \). There are five types of diagrams to consider. They are shown in Figs. (8), (9), (10), (11), (12).

![FIG. 8: The box and crossed-box diagrams](image-url)
Contracting with $Q$ and using Eqs. 86 and 87 we obtain:

$$\Gamma_{\text{seagull} - 1}(2, p, q, k, u) = \alpha[2] + A[2]; \quad \Gamma_{\text{seagull} - 2}(2, p, q, k, u) = \beta[2] + B[2]$$

(88)

$$\Gamma_{\text{leaftail} - 1}(2, p, q, k, u) = \gamma[2] + C[2]; \quad \Gamma_{\text{leaftail} - 3}(2, p, q, k, u) = \delta[2] + D[2]$$

$$\Gamma_{\text{jellyfish} - 1}(2, p, q, k, u) = \epsilon[2] + E[2]; \quad \Gamma_{\text{jellyfish} - 2}(2, p, q, k, u) = \phi[2] + F[2]$$

$$\Gamma_{\text{polywog}}(2, p, q, k, u) = G[2] + H[2]$$

$$\Gamma_{\text{leaftail} - 4}(2, p, q, k, u) = -A[2] - E[2]$$

$$\Gamma_{\text{box}}(2, p, q, k, u) = -G[2] - C[2]; \quad \Gamma_{\text{crossed-box}}(2, p, q, k, u) = -H[2] - D[2]$$

where

$$\alpha[2] = -\Gamma_{\text{leaftailB}}(2, p + q, u)$$

(89)
\[ \beta[2] = \Gamma_{\text{leaf}A}(2, p, t) \]
\[ \gamma[2] = -\Gamma_{\text{triangle}}(2, p + q, u) \]
\[ \delta[2] = \Gamma_{\text{triangle}}(2, p, t) \]
\[ \epsilon[2] = \Gamma_{\text{leaf}B}(2, p, t) \]
\[ \phi[2] = -\Gamma_{\text{leaf}A}(2, p + q, u) \]
\[ A[2] = -\frac{i}{2} \int \frac{d^4q}{(2\pi)^4} \left[ a(l)r(l + s)\Gamma(2, p, l + p)(M(6, l + s, k, -l, u) + M(3, l + s, k, -l, u)N_F(l + s) - M(1, l + s, k, -l, u)N_B(l)) \right] \]
\[ B[2] = \frac{i}{2} \int \frac{d^4l}{(2\pi)^4} \left[ a(l)r(l + t)\Gamma(2, p, k, l + t)\Gamma(4, l + u, u) + \Gamma(3, l + u, u)N_F(l + t) - \Gamma(2, l + u, u)N_B(l) \right] \]
\[ C[2] = -\frac{i}{2} \int \frac{d^4l}{(2\pi)^4} \left[ a(l)r(l + s)\Gamma(2, p, l + p) \right. \]
\[ \left. \frac{a(l + u)\Gamma(3, l + u, u)\Gamma(6, l + s, l + u) + \Gamma(5, l + s, l + u)N_F(l + s) - \Gamma(2, l + s, l + u)N_F(l + u) + r(l + u)\Gamma(2, l + s, l + u)\Gamma(4, l + u, u) + \Gamma(3, l + u, u)N_F(l + u) - \Gamma(2, l + u, u)N_B(l)) \right] \]
\[ D[2] = -\frac{i}{2} \int \frac{d^4l}{(2\pi)^4} \left[ a(l)r(l + p)\Gamma(2, p, l + p) \right. \]
\[ \left. \frac{a(l + t)\Gamma(3, l + u, u)\Gamma(6, l + p, l + t) + \Gamma(5, 2, p, l + t)N_F(l + p) - \Gamma(2, l + p, l + t)N_F(l + t)}{r(l + t)\Gamma(2, l + p, l + t)\Gamma(4, l + u, u) + \Gamma(3, l + u, u)N_F(l + t) - \Gamma(2, l + u, u)N_B(l)) \right] \]
\[ E[2] = \frac{i}{2} \int \frac{d^4l}{(2\pi)^4} \left[ a(l)r(l + p)\Gamma(2, p, l + p) \right. \]
\[ \left. \frac{a(l + u)\Gamma(3, l + u, u)\Gamma(6, l + s, l + u) + \Gamma(5, l + s, l + u)N_F(l + p) - \Gamma(2, l + s, l + u)N_F(l + u) + r(l + u)\Gamma(2, l + s, l + u)\Gamma(4, l + u, u) + \Gamma(3, l + u, u)N_F(l + u) - \Gamma(2, l + u, u)N_B(l)) \right] \]
\[ F[2] = -\frac{i}{2} \int \frac{d^4l}{(2\pi)^4} \left[ a(l)r(l + u)\Gamma(2, p, k, l + t)\Gamma(4, l + u, u) + \Gamma(3, l + u, u)N_F(l + u) - \Gamma(2, l + u, u)N_B(l)) \right] \]
\[ G[2] = \frac{i}{2} \int \frac{d^4l}{(2\pi)^4} \left[ a(l)r(l + p)\Gamma(2, p, l + p) \right. \]
\[ \left. \frac{a(l + u)\Gamma(3, l + u, u)\Gamma(6, l + p, l + t) + \Gamma(5, l + p, l + t)N_F(l + p) - \Gamma(2, l + p, l + t)N_F(l + u) + r(l + u)\Gamma(2, l + p, l + t)\Gamma(4, l + u, u) + \Gamma(3, l + u, u)N_F(l + u) - \Gamma(2, l + u, u)N_B(l)) \right] \]
\[ H[2] = -\frac{i}{2} \int \frac{d^4l}{(2\pi)^4} \left[ a(l)r(l + p)\Gamma(2, p, l + p) \right. \]
\[ \left. \frac{a(l + u)\Gamma(3, l + u, u)\Gamma(6, l + p, l + t) + \Gamma(5, l + p, l + t)N_F(l + p) - \Gamma(2, l + p, l + t)N_F(l + u) + r(l + u)\Gamma(2, l + p, l + t)\Gamma(4, l + u, u) + \Gamma(3, l + u, u)N_F(l + u) - \Gamma(2, l + u, u)N_B(l)) \right] \]

Using \( \Gamma = \Gamma_{\text{triangle}} + \Gamma_{\text{leaf}A} + \Gamma_{\text{leaf}B} \) and combining we have

\[ Q^\mu \cdot M_{\mu\nu}(2, p, q, k, u) = \Gamma_{\nu}(2, p, t) - \Gamma_{\nu}(2, p + q, u) \]

which agrees with [31]. The results for all components agree with the results listed in [29].

X. CONCLUSIONS

Calculations in real time statistical field theory are complicated by the extra indices that result from the doubling of degrees of freedom. Because of this technical problem, many people avoid the real time formalism of finite temperature field theory in spite of its significant advantages, as compared with the imaginary time formalism. Two of the major advantages of working in real time are the fact that analytic continuations are not necessary, and that it is easy to generalize to non-equilibrium situations. In this paper we have made a contribution towards reducing the technical difficulties associated with the real time formulation of statistical field theory. We have written a Mathematica program that performs the contractions over the tensor indices that appear in real time statistical field theory and determines the integrand corresponding to any amplitude. The program is designed so that it can be used by someone with no previous experience with Mathematica. It is available on the internet at www.brandonu.ca/physics/fugleberg/Research/Dick.html. It can be used in the 1-2, Keldysh or RA basis, and it can do calculations in or out of equilibrium.
We have used the program to calculate the QED ward identity for the 3-point function \((2^n - 1 = 7\) components), the 4-point function for two fermions and two photons \((2^n - 1 = 15\) components), and the 5-point function for two fermions and three photons \((2^n - 1 = 31\) components). Some of these identities have appeared previously in the literature, but the complete set of identities has not previously been published. The calculation therefore serves two purposes: it provides a check of the program, and it produces useful new information. We give a table that lists the results for the ward identities (Eqn. (82)). In addition, we give a simple general expression for the KMS conditions between \(n\)-point functions and vertex functions, in both the Keldysh and RA bases (Eqns. (60) and (66)).

[1] U. Kraemmer and A. Rebhan, Rept. Prog. Phys. 67 (2004) 351.
[2] P.C. Martin and J. Schwinger, Phys. Rev. 115 (1959) 1432.
[3] L.V. Keldysh, Sov. Phys. JETP 20 (1965) 1018.
[4] F. Gelis, Nucl. Phys. B508 (1997) 483.
[5] M.E. Carrington, Hou Defu and J.C. Sowiak, Phys. Rev. D 62 (2000) 065003.
[6] E. Wang and U. Heinz, Phys. Rev. D 66 (2002) 025008.
[7] M.E. Carrington, Hou Defu and M.H. Thoma, Phys. Rev. D 58 (1998) 085025.
[8] M.E. Carrington, Hou Defu, A. Hachkowski, D. Pickering and J.C. Sowiak, Phys. Rev. D 61 (2000) 25011.
[9] M.E. Carrington, Hou Defu, R. Kobes and U. Heinz, Phys. Rev. D 61 (2000) 085013; M.E. Carrington, Hou Defu, R. Kobes and U. Heinz, Phys. Rev. D 67 (2003) 049902.
[10] K.C. Chou, Z.B. Su, B.L. Hao and L. Yu, Phys. Rep. 118 (1985) 1.