Quantum Tomograms and Their Application in Quantum Information Science

Aleksey K Fedorov\(^1\),\(^2\),\(^3\) and Stanislav O Yurchenko\(^1\)

\(^1\) Center for Photonics and Infrared Engineering, Bauman Moscow State Technical University, 2nd Baumanskaya str., 5, 105005 Moscow, RUSSIA
\(^2\) Russian Quantum Center, Novaya str., 100, 143025, Skolkovo, Moscow region, RUSSIA
\(^3\) Visiting Research Associate at Institute for Quantum Information Science, University of Calgary, 2500 University Dr. NW, T2N 1N4, Calgary, Alberta, CANADA

E-mail: akfedorv@student.bmstu.ru, st.yurchenko@bmstu.ru

Abstract. This note is devoted to quantum tomograms application in quantum information science. Representation for quantum tomograms of continuous variables via Feynman path integrals is considered. Due to this construction quantum tomograms of harmonic oscillator are obtained. Application tomograms in causal analysis of quantum states is presented. Two qubit maximum entangled and "quantum-classical" states have been analyzed by tomographic causal analysis of quantum states.

1. Introduction

Tomographic representation of quantum states is one of the most developing areas of quantum physics [1]-[15]. The main idea of quantum tomography consists in using of nonnegative probability distribution functions (quantum tomograms). Quantum tomography is an equivalent to other approaches to quantum mechanics. In turn, quantum tomogram is directly related to Wigner function, Glauber-Sudarshan function and Husimi function [1]-[12]. This probability distribution functions are called "tomograms", because of their relation with Wigner function by Radon integral transformation, which is used in medicine computed tomography.

There are two main reasons for quantum tomography using. First, the tomograms of quantum states are measurable via method of balanced homodyne detection [13]-[15]. Current developments in continuous-variable quantum-state tomography presented in [15]. Experimental area of quantum tomography is closely connected with quantum state engineering. There are a lot of applications of this framework in quantum information technology [15]-[16].

Second, methods of quantum tomography are widely used for simulation [6], [7]. Most methods of quantum systems numerical simulation are based on solution equations obtained for not positive define functions (wave functions, Wigner function). This gives rise to problems associated with convergence of integrals ("sign problem") [6]. Quantum tomograms are positive defined functions. But numerical method based on quantum tomography requires solution of system of differential equations with local approximation (e.g., exponential). In this note, representation of quantum tomograms by Feynman path integral is used [1]. Feynman path integral representation of quantum tomogram is the basis for Path Integral Monte Carlo (PIMC) numerical methods with positive define quantum tomograms. PIMC method has significant computational advantages, e.g. in condensed matter physics [17].
2. Continuous variables quantum tomography

Quantum tomography is based on a linear canonical transformation of phase space. Transformation from one set of canonical variables (positions and momentums) \((q, p, t)\) to other \((Q, P, t)\) is the canonical one if and only if Jacobi matrix \(\mathcal{M}\) of this transformation is symplectic. It’s easy to show, that canonical transformation of phase space provides conservation of Poisson bracket in classical mechanics case and commutator (canonical commutation relation) in quantum mechanics case. Thus, Hamiltonian equations and principle of least action are valid for new canonical variables. Canonical transformation with symplectic matrix \(\mathcal{M}\) can be presented as follows

\[
\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} \mu & \eta \\ \eta & \mu \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}, \tag{1}
\]

where elements \(\mu, \eta, \tilde{\mu}, \tilde{\eta}\) of matrix \(\mathcal{M}\) are real numbers and determinant of \(\mathcal{M}\) is 1. Note, than canonical transformation is used in canonical quantization (see [20], [2]).

Symplectic tomogram \(\mathcal{T}(Q, \mu, \eta)\) of observable \(Q = \mu \hat{q} + \eta \hat{p}\) can be defined through complex wave function \(\psi(q)\) in position representation

\[
\mathcal{T}(Q, \mu, \eta) = |\hat{\mathcal{F}}_{\mu, \eta}[\psi(q)](Q)|^2, \quad m = h = \omega = 1
\]

where \(\hat{\mathcal{F}}_{\mu, \eta}\) is fractional Fourier transformation operator [12]

\[
\hat{\mathcal{F}}_{\mu, \eta}[\psi(q)](Q) = \frac{1}{\sqrt{2\pi|\eta|}} \exp \left[ \frac{i \mu q^2}{2|\eta|} \right] \int \psi(q) \exp \left[ i \left( \frac{\mu q^2}{2|\eta|} - \frac{Q q}{\eta} \right) \right] dq. \tag{2}
\]

A definition for symplectic tomogram is obtained from (1) and (2)

\[
\mathcal{T}(Q, \mu, \eta) = \frac{1}{2\pi|\eta|} \left| \int \psi(q) \exp \left[ i \left( \frac{\mu q^2}{2|\eta|} - \frac{Q q}{\eta} \right) \right] dq \right|^2. \tag{3}
\]

In accordance with definition (3), symplectic tomogram \(\mathcal{T}(Q, \mu, \eta)\) is nonnegative and normalize on \(Q\) function

\[
\mathcal{T}(Q, \mu, \eta) \geq 0, \quad \int \mathcal{T}(Q, \mu, \eta) dQ = 1, \tag{4}
\]

and, therefore, tomogram is a probability distribution function of variable \(Q\) with parameters \(\mu\) and \(\eta\). Position and momentum probability distribution functions could be calculated based on symplectic tomography

\[
\mathcal{T}(Q, 1, 0) = |\psi(q)|^2, \quad \mathcal{T}(Q, 0, 1) = |\psi(p)|^2. \tag{5}
\]

Symplectic tomograms are the most general case of probability distribution function in quantum tomography. Let consider two important examples. Starting form the orthogonal matrix \(\mathcal{M}\) case

\[
\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \tag{6}
\]

where \(\theta \in \mathbb{R}/2\pi\mathbb{Z}\) is a phase space rotation angle.

Using (3) expression for optical tomogram can be obtained

\[
\mathcal{T}(Q, \theta) = \frac{1}{2\pi|\sin \theta|} \left| \int \psi(q) \exp \left( i \cot \theta \frac{\mu q^2}{2} - \frac{Q q}{\sin \theta} \right) dq \right|^2. \tag{7}
\]
Optical tomography is extremely important for quantum states measurements in quantum optics with balanced homodyne detection technique and for quantum state reconstruction process [16]. Also, in quantum optics it is widely used a connection between Wigner function and quantum tomogram $\mathcal{T}(Q, \mu, \eta)$ by Radon transformation

$$\mathcal{W}(q, p) = \frac{1}{2\pi} \int \mathcal{T}(Q, \mu, \eta) \exp(i(Q - \mu q - \eta p))dQd\mu d\eta. \quad (8)$$

Tomogram is a first order homogeneity function. Lets consider inverse Radon transformation

$$\mathcal{T}(Q, \mu, \eta) = \int \mathcal{W}(q, p) \exp(-is(Q - \mu q - \eta p)) \frac{dsdqdqp}{4\pi^2}$$

and transformation variables $(Q, \mu, \eta) \mapsto (\lambda Q, \lambda \mu, \lambda \eta)$ with arbitrary real $\lambda$. Therefore

$$\mathcal{T}(\lambda Q, \lambda \mu, \lambda \eta) = \int \mathcal{W}(q, p) \exp(-is\lambda(Q - \mu q - \eta p)) \frac{dsdqdqp}{4\pi^2} = |\lambda|^{-1}\mathcal{T}(\varepsilon, \mu, \eta)$$

From Euler lemma for homogenates functions $\mathcal{T}(Q, \mu, \eta)$

$$\left(Q \frac{\partial}{\partial Q} + \mu \frac{\partial}{\partial \mu} + \eta \frac{\partial}{\partial \eta} + 1\right)\mathcal{T}(\varepsilon, \mu, \eta) = 0. \quad (9)$$

Thus, Fresnel tomogram [9] is a solution of equation (9)

$$\mathcal{T}_f(\dot{Q}, \dot{\eta}) = \frac{1}{|\mu|} f\left(\frac{Q}{\mu}, \frac{\eta}{\mu}\right),$$

where $f$ is an arbitrary function.

Quantum tomograms evolution equation is Fokker-Planck type equation. Fokker-Planck equation is a well-known stochastic differential equation of classical probability theory. For quantum system with Hamiltonian

$$\hat{H} = \frac{p^2}{2} + V(\hat{q})$$

Fokker-Planck equation is represented as

$$\frac{\partial \mathcal{T}}{\partial t} = \mu \frac{\partial \mathcal{T}}{\partial \eta} + i\left[-V\left(\hat{I}_Q \frac{\partial}{\partial \mu} + \frac{i\eta}{2} \frac{\partial}{\partial \mu}\right) - V\left(-\hat{I}_Q \frac{\partial}{\partial \mu} + \frac{i\eta}{2} \frac{\partial}{\partial \mu}\right)\right] \mathcal{T}, \quad (10)$$

where $\hat{I}_Q : f(Q) \rightarrow \int f(Q)dQ$ is an integrity operator on variable $Q$.

From (10) for free particle

$$\frac{\partial \mathcal{T}}{\partial t} - \mu \frac{\partial \mathcal{T}}{\partial \eta} = 0. \quad (11)$$

For harmonic oscillator

$$\frac{\partial \mathcal{T}}{\partial t} - \mu \frac{\partial \mathcal{T}}{\partial \eta} + \frac{\partial \mathcal{T}}{\partial \mu} = 0. \quad (12)$$

In case of parametric oscillator with time-dependent frequency $\omega(t)$

$$\frac{\partial \mathcal{T}}{\partial t} - \mu \frac{\partial \mathcal{T}}{\partial \eta} + \omega^2(t) \frac{\partial \mathcal{T}}{\partial \mu} = 0. \quad (13)$$
Figure 1. Optical tomograms for ground and first state of harmonic oscillator.

Let’s consider symplectic tomograms representation via Feynman path integral

\[
T(Q, \mu, \eta) = \frac{1}{2\pi |\eta|} \left| \int \psi(q_1) \exp \left[ i \left( S(q_1, q_2) + \frac{\mu^2}{2\eta} q_2^2 - \frac{Q}{\eta} q_2 \right) \right] D[q(t)] dq_1 dq_2 \right|^2, \tag{14}
\]

where \( S(q_1, q_2) \) is an action functional, \( \psi(q_1) \) is Cauchy problem initial condition for Schrodinger equation

\[
-\frac{1}{2} \Delta \psi(q, t) + V(q) \psi(q, t) = i \frac{\partial}{\partial t} \psi(q, t),
\]

and \( D[q(t)] \) is ”Lebesgue measure” [19]

\[
D[q(t)] = \lim_{N \to \infty} \prod_{j=1}^{N} \left( -\frac{2\pi i}{\Delta t_j} \right)^{3/2} d^3 q_j.
\]

Feynman path integral representation of optical tomograms and Fresnel tomograms can be obtained. Let consider harmonic oscillator as an example. Green function for Schrodinger equation in case of harmonic oscillator

\[
G_o(q_2, q_1, t) = \frac{1}{\sqrt{2\pi i \sin(t)}} \exp \left( i \frac{q_1^2 \cot(q_2^2 + q_1^2) - iq_1 q_2}{\sin(t)} \right).
\]

Substituting Green function in tomographic Feynman path integral solution for stationary states were obtained

\[
T_n(Q, \mu, \eta) = \frac{2^{-n}}{n! \sqrt{\pi (\mu^2 + \eta^2)}} \exp \left( -\frac{Q^2}{\mu^2 + \eta^2} \right) \mathcal{H}_n^2 \left( \frac{Q}{\sqrt{\mu^2 + \eta^2}} \right), \tag{15}
\]

where \( \mathcal{H}_n(x) \) is \( n \) order Hermite polynomials.

Therefore, quantum tomograms of ground state \( T_0(Q, \mu, \eta) \) has the following form

\[
T_0(Q, \mu, \eta) = \frac{1}{\sqrt{\pi (\mu^2 + \eta^2)}} \exp \left( -\frac{Q^2}{\mu^2 + \eta^2} \right). \tag{16}
\]
Tomogram $T_1(Q, \mu, \eta)$ of the first state

$$
T_1(Q, \mu, \eta) = \frac{2Q^2}{(\mu^2 + \eta^2)^{\frac{1}{2}} \pi \sqrt{\mu^2 + \eta^2}} \exp \left(-\frac{Q^2}{\mu^2 + \eta^2}\right).
$$

(17)

There are ground and the first state optical tomograms presented on fig. 1. Note, that for parameters $\mu = 1$ and $\eta = 0$ quantum tomogram is the ordinary probability distribution in position representation. These results are in agreement with results obtained in [8].

Feynman path integral is one of the main tool in theoretical physics [21]-[23]. Feynman path integral representation of quantum tomograms takes an important part in quantum tomography. First of all, series of perturbation theory can be constructed [3]. Second, this representation is closely connected with Feynman formula (representation of semigroups by limits of $n$-fold iterated integrals as $n \to \infty$) for Fokker-Planck equation [23] and path integral computation technique in filtering problem [24].

3. Quantum tomography for discrete variables.

Tomography for discrete variables has been proposed in [11]. Let consider density matrix $\rho^0$ for particle with 1/2 spin, which characterizing spin projection on $Oz$ axis. For arbitrary direction $Oz'$ which is characterized by identity vector $\vec{n} = \{ \sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta \}$, density matrix has the following form

$$
\rho(\vec{n}) = U(\vec{n}) \rho^0 U^\dagger(\vec{n}),
$$

(18)

where $U(\vec{n}) = U(\theta, \varphi)$ is an unitary rotation matrix

$$
U(\theta, \varphi) = \begin{pmatrix}
\cos(\theta/2) \exp(i\varphi/2) & \sin(\theta/2) \exp(i\varphi/2) \\
-\sin(\theta/2) \exp(-i\varphi/2) & \cos(\theta/2) \exp(-i\varphi/2)
\end{pmatrix},
$$

(19)

with Euler angels $\varphi, \theta$.

Tomograms of qubit state can be presented as follows

$$
T(m, \vec{n}) = \{ \rho(\vec{n}) \}_{mm},
$$

(20)

with $m = 1$ and $m = 2$ correspond to spin projections $| + 1/2, \vec{n} \rangle$ and $| - 1/2, \vec{n} \rangle$ respectively. In this way, only diagonal elements of density matrix are considered. They determine the probability to obtain a particular measurement result of spin projection on an arbitrary axis defined by the vector $\vec{n}$.

By definition, quantum tomogram $T(m, U)$ is nonnegative and normalized

$$
T(m, \vec{n}) \geq 0, \quad \sum_m T(m, \vec{n}) = 1.
$$

Therefore, the formula for von Neumann entropy in tomographic representation of quantum mechanics can be obtained in the following way

$$
S^t(\vec{n}) = -\sum_m T(m, \vec{n}) \log_2 T(m, \vec{n}).
$$

(21)

Index $t$ is fixed for tomographic functions. Tomographic entropy is generalization of von Neumann entropy

$$
S = - \text{Tr}[\rho \log(\rho)].
$$

Finally, for a two-part system with initial density matrix $\rho_{AB}^0$ tomogram can be presented in the form

$$
T(m_1, m_2, \vec{n}_1, \vec{n}_2) = \{ U(\vec{n}_1, \vec{n}_2) \rho_{AB}^0 U^\dagger(\vec{n}_1, \vec{n}_2) \}_{2m_1+m_2, 2m_1+m_2}.
$$

(22)
where matrix $U(\vec{n}_1, \vec{n}_2)$ can be defined by two unitary matrix tensor product

$$ U(\vec{n}_1, \vec{n}_2) = U(\vec{n}_1) \otimes U(\vec{n}_2). $$

Note, than Euler angles presents a rotation of space (19). Problem of representation for space rotation can be solved with quaternions. Construction of quantum tomography via quaternions is presented in [4].

4. Tomographic causal analysis of quantum states

Quantum causal analysis proposes an original approach, based on von Neuman entropy, which helps to understand causality of entangled asymmetric states [25], [26]. Mathematical formalization of causal analysis is bases on a pair of independence functions and on irreversible information flow velocity

$$ i_{A|B} = \frac{S_{A|B}}{S_A}, \quad i_{B|A} = \frac{S_{B|A}}{S_B}, \quad c_2 = k \frac{(1 - i_{A|B})(1 - i_{B|A})}{i_{A|B} - i_{B|A}}, \quad k = 1. $$

Tomographic causal analysis of quantum states is based on tomographic von Neumann entropy (21) for independence functions and for the linear velocity of irreversible information flow. In turn, the main idea quantum states tomographic causal analysis is to use tomographic relation (21) for von Neumann entropy [3].

First, let’s start from consideration of pure maximum entangled state

$$ |\psi\rangle_{AB} = 1/\sqrt{2}(|00\rangle_{AB} + |11\rangle_{AB}), $$

where $|0\rangle$ and $|1\rangle$ correspond to spin positive and negative projections on chosen axis. Thus, density matrix for this system has the following form

$$ \rho^0_{AB} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad \rho^0_A = \rho^0_B = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (23) $$

As the result, $S_{AB} = 0$, $S_A = S_B = 1$, and causality is upset $i_{A|B} = i_{B|A} = -1$, $|c_2| \to \infty$. This state is special, because density matrix of subsystems $A$ and $\rho_B$ are invariant to rotation matrix action

$$ U(\vec{n}_1)\rho^0_{AB}U^\dagger(\vec{n}_1) = U(\vec{n}_2)\rho^0_{AB}U^\dagger(\vec{n}_2) = \rho^0_A = \rho^0_B. $$

Finally, $S^0_A(\vec{n}_1) = S^0_B(\vec{n}_2) = 1$ and $i^0_{A|B} = i^0_{B|A}$. There are independence functions as the functions of angles $\theta_1$ and $\theta_2$ ($\varphi_1 = \varphi_2 = 0$ are fixed) on fig. 2.

System maximum correlation conditions, obtained after measurements, has been obtained in the following way

$$ i^t_{A|B} = i^t_{B|A} = 0. $$

This is a condition of using the same basis for measurements in both subsystem (with $\theta_1 = \theta_2$). These results of tomographic causal analysis are in agreement with the results of ordinary causal analysis of quantum states [27].

Second, let’s consider "quantum-classical" state with inequality for entropies

$$ S_B < S_{AB} < S_A, $$
with density matrix

$$\rho_{AB}^0 = q|\psi_1\rangle\langle\psi_1| + (1 - q)|\psi_2\rangle\langle\psi_2|,$$

where $|\psi_1\rangle = a|00\rangle_{AB} + \sqrt{1 - a^2}|11\rangle_{AB}$ and $|\psi_2\rangle = a|10\rangle_{AB} + \sqrt{1 - a^2}|01\rangle_{AB}$ — pure entangled states. In accordance with [4], parameters $q = 0.2$, $a^2 = 0.1$ are chosen.

Von Neumann entropy are $S_{AB} \simeq 0.722$, $S_A \simeq 0.827$, $S_B \simeq 0.469$. Causality is $c_2 \simeq 1.604$, in this way, subsystem $A$ is a "cause", subsystem $B$ is "effect". There are tomographic entropy and von Neumann entropy of subsystems $A$, subsystem $B$ and system for asymmetric "quantum-classical" state with fixed angles on fig. 3.

After tomographic transformation entropy becomes a function of angles $\theta_{1(2)}$: $S_{tA} = S_{tA}^r(\theta_1)$, $S_{tB} = S_{tB}^r(\theta_2)$. And entropy of system $S_{tAB} = S_{tAB}^r(n_1, n_2) = S_{tAB}^r(\theta_1, \varphi_1, \theta_2, \varphi_2)$. There are ordinary and tomographic velocities of irreversible information flow ($c_2$ and $c_2^t$) and information ($I$ and $I^t$) for asymmetric "quantum-classical" state is shown on fig. 4.

Therefore, tomographic quantum causal analysis is extremely useful for initial asymmetry.
Figure 4. Ordinary causal and tomographic velocities of irreversible information flow ($c_2$ and $\dot{c}_2$) and information ($I$ and $I^t$) for asymmetric “quantum-classical” state: $\varphi_1 = \varphi_2 = 0, \theta_1 = \theta_2 = \theta$ (a); $\varphi_1 = \varphi_2 = \theta_1 = 0$ (b)

investigation for system with unknown density matrix.

5. Conclusion
Application of quantum tomograms in quantum information science was discussed. Continuous variables quantum tomograms representation via Feynman path integrals was used. Tomograms construction of harmonic oscillator state was obtained. Application of tomograms in quantum states causal analysis was discussed. Two qubit maximum entangled pure and ”quantum-classical” states by tomographic causal analysis of quantum states was considered.

Acknowledgments
Authors thank A.I. Lvovsky, V.I. Man’ko and Yu.E. Lozovik for grateful discussion. Authors thank E.O. Kiktenko and S.M. Korotaev for grateful discussion about causal analysis of quantum states. This work was supported by Russian Foundation for Basic Research (projects 12-08-31104 and 12-08-33112).

References
[1] Fedorov A K, Yurchenko S O 2012 Symplectic Tomograms Presented through Feynman’s Path Integrals, J. BMSTU Nat. Science 45 29-37.
[2] Fedorov A K, Yurchenko S O 2011 Tomographic Methods in the Theory of Secondary Quantization, Sc. Educ.: Electr. Sci. Techn. Period. 13 62-67.
[3] Fedorov A K, Yurchenko S O 2012 Tomographic Series of Perturbation Theory, J. BMSTU Nat. Sc. Spec. Is. 6-12.
[4] Fedorov A K, Yurchenko S O 2012 Quaternion Representation of Spin Tomograms, J. BMSTU Nat. Sc. Spec. Is. 257-261.
[5] Kiktenko E O, Korotaev S M, Fedorov A K, Yurchenko S O 2012 Causal Analysis of Entangled States in Tomographic Representation of Quantum Mechanics, J. BMSTU Nat. Sc. Spec. Is. 75-85.
[6] Lozovik Yu E, Sharapov V A, Arkhipov A S 2005 Simulation of Tunneling in the Quantum Tomography, Phys. Rev. A 69 022116.
[7] Arkhipov A S Lozovik Yu E 2003 New Method of Quantum Dynamics Simulation Based on the Quantum Tomography, Physics Letters A 319 217–224.
[8] Man’ko O V, Man’ko V I 1999 Classical Propagator and Path Integral in the Probability Representation of Quantum Mechanics, J. Russ. Las. Res. 20 67-76.
De Nicola S, Fedele R, Man’ko M A, Man’ko V I 2005 Fresnel Tomography: A Novel Approach to Wave-Function Reconstruction Based on the Fresnel Representation of Tomograms, Theor. Math. Phys. 144 1206-1213.

Chernega V N, Man’ko V I 2007 Wave Function of the Harmonic Oscillator in Classical Statistical Mechanics, J. Russ. Las. Res. 28 535-547.

Andreev V A, Man’ko V I, Man’ko O V, Shchukin E V 2006 Tomography of Spin States, the Entanglement Criterion, and Bell’s Inequalities, Theor. Math. Phys. 146 140-151

Amosov G G, Dnestryan A I 2011 On the Spectrum of a Set of Integral Operators Determining the Symplectic Quantum Tomogram, Proc. MIPT 3 5-9

Beck M, Smizhey D T, Raymer M G 1993 Experimental Determination of Quantum-phase Distributions Using Optical Homodyne Tomography, Phys. Rev. A 48 890-893.

Smithey D T, Beck M, Raymer M G, Faridani A 1993 Measurement of the Wigner Distribution and the Density Matrix of a Light Mode Using Optical Homodyne Tomography: Application to Squeezed States and the Vacuum, Phys. Rev. A 70 1244-1247.

Lvovsky A I, Raymer M G 2009 Continuous-variable Optical Quantum-state Tomography, Rev. Mod. Phys. 81 299-322.

Anis A, Lvovsky A I 2012 Maximum-likelihood Coherent-state Quantum Process Tomography, New J. Phys. 14 105021.

Pollock E L, Ceperley D M 1987 Path-Integral Computation of Superfluid Densities, Phys. Rev. B 36 1244-1247.

Feynman R F, Hibbs A R 1965 Path Integral Computation of Superfluid Densities, New York, McGraw-Hill.

Garrod C 1966 Hamiltonian Path-Integral Methods, Rev. Mod. Phys. 38 483-494.

Bogoliubov N N., Shirkov D V 1959 The Theory of Quantized Fields, New York, Interscience.

Berezin F A 1972 Non-Wiener Functional Integrals, Theor. Math. Phys. 6 141-155.

Bottcher B, Butko Ya A, Schilling R L, Smolyanov O G 2011 Feynman Formulas and Path Integrals for Some Evolution Semigroups Related to τ-quantization, Russ. J. Math. Phys. 18 387-399.

Drozdov A N, Talkner P. 1998 Path Integrals for Fokker-Planck Dynamics With Singular Diffusion: Accurate Factorization for the Time Evolution Operator, J. Chem. Phys. 109 2080-2091.

Ovseevich A I 2008 Kalman Filter and Quantization, Prob. Inform. Transm. 44 53-71.

Kittenko E O, Korotaev S M 2012 Causal Analysis of Asymmetric Entangled States Under Decoherence, Phys. Lett. A. 376 820-823.

Korotaev S M, Kittenko E O 2012 Causality and Decoherence in the Asymmetric States, Phys. Scr. 85 055006.

Zyczkowski K, Horodecki P, Horodecki M, Horodecki R 2002 Dynamics of quantum entanglement, Phys. Rev. A 85 055006.