Characterizing the Integrality Gap of the Subtour LP for the Circulant Traveling Salesman Problem

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Abstract

We consider the integrality gap of the subtour LP relaxation of the Traveling Salesman Problem restricted to circulant instances. De Klerk and Dobre [11] conjectured that the value of the optimal solution to the subtour LP on these instances is equal to an entirely combinatorial lower bound from Van der Veen, Van Dal, and Sierksma [35]. We prove this conjecture by giving an explicit optimal solution to the subtour LP. We then use it to show that the integrality gap of the subtour LP is 2 on circulant instances, making such instances one of the few non-trivial classes of TSP instances for which the integrality gap of the subtour LP is exactly known. We also show that the degree constraints do not strengthen the subtour LP on circulant instances, mimicking the parsimonious property of metric, symmetric TSP instances shown in Goemans and Bertsimas [16] in a distinctly non-metric set of instances.

1 Introduction

The traveling salesman problem (TSP) is one of the most famous problems in combinatorial optimization. An input to the TSP consists of a set of n cities \( n := \{1, 2, \ldots, n\} \) and edge costs \( c_{ij} \) for each pair of distinct \( i, j \in [n] \) representing the cost of traveling from city \( i \) to city \( j \). Given this information, the TSP is to find a minimum-cost tour visiting every city exactly once. Throughout this paper, we implicitly assume that the edge costs are symmetric (so that \( c_{ij} = c_{ji} \) for all distinct \( i, j \in [n] \)) and interpret the \( n \) cities as vertices of the complete undirected graph \( K_n \) with edge costs \( c_e = c_{ij} \) for edge \( e = \{i, j\} \). In this setting, the TSP is to find a minimum-cost Hamiltonian cycle on \( K_n \).

With just this set-up, the TSP is well known to be NP-hard. An algorithm that could approximate TSP solutions in polynomial time to within any constant factor \( \alpha \) would imply P=NP (see, e.g., Theorem 2.9 in Williamson and Shmoys [36]). Hence more restricted assumptions are placed on the edge costs. If one assumes that edge costs are metric (i.e., \( c_{ij} \leq c_{ik} + c_{kj} \) for all distinct \( i, j, k \in [n] \)), it is known to be NP-hard to approximate TSP solutions in polynomial time to within any constant factor \( \alpha < \frac{123}{122} \) (see Karpinski, Lampis, and Schmied [20]). Conversely, the Christofides-Serdyukov algorithm [7, 33] outputs a Hamiltonian cycle that is at most a factor of \( \frac{3}{2} \) away from the optimal solution to any metric, symmetric instance.

For metric and symmetric edge costs, the Christofides-Serdyukov algorithm remains the state of the art. Significant work has gone into looking at more restricted sets of edge costs. For example, the \((1, 2)\)-TSP restricts \( c_{ij} \in \{1, 2\} \) for every edge \( \{i, j\} \) (see, e.g., Papadimitriou and Yannakakis [31], Berman and Karpinski [2], Karpinski and Schmied [21]). In graphic TSP, instead, the input corresponds to a connected, undirected graph \( G \) on vertex set \( [n] \), and for \( i, j \in [n] \), the cost \( c_{ij} \) is the length of the shortest \( i-j \) path in \( G \); approximation algorithms with stronger performance guarantees than the Christofides-Serdyukov algorithm are known in this case (see,
In circulant TSP, the matrix of edge costs $C = (c_{i,j})_{i,j=1}^n$ is circulant; the cost of edge $\{i, j\}$ only depends on $i - j \mod n$. Our assumption that the edge costs are symmetric and that $K_n$ is a simple graph implies that we can write our cost matrix in terms of $\left\lfloor \frac{n}{2} \right\rfloor$ parameters:

$$C = (c_{(i-j) \mod n})_{i,j=1}^n = \begin{pmatrix} 0 & c_1 & c_2 & c_3 & \cdots & c_{\left\lfloor \frac{n}{2} \right\rfloor} \\ c_1 & 0 & c_1 & c_2 & \cdots & c_2 \\ c_2 & c_1 & 0 & c_1 & \cdots & c_3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & \cdots & 0 \end{pmatrix},$$

with $c_0 = 0$ and $c_i = c_{n-i}$ for $i = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$. Importantly, in circulant TSP we do not implicitly assume that the edge costs are also metric. A circulant graph is a graph whose weighted adjacency matrix is circulant.

Circulant matrices have well-studied structure (see, e.g., Davis [10] and Gray [17]), and form an intriguing class of instances for combinatorial optimization problems. They seem to provide just enough structure to make a compelling, ambiguous set of instances: it is unclear whether or not a given combinatorial optimization problem should remain hard or become easy when restricted to circulant instances. Some classic combinatorial optimization problems become easy when restricted to circulant instances: in the late 70’s, Garfinkel [12] considered a restricted set of circulant TSP instances motivated by minimizing wallpaper waste and argued that, for these instances, the canonical greedy algorithm for TSP (the nearest neighbor heuristic) provides an optimal solution. In the late 80’s, Burkard and Sandholzer [1] showed that the decidability question for whether or not a symmetric circulant graph is Hamiltonian can be solved in polynomial time and showed that bottleneck TSP is polynomial-time solvable on symmetric circulant graphs. Bach, Luby, and Goldwasser (cited in Gilmore, Lawler, and Shmoys [15]) showed that one could find minimum-cost Hamiltonian paths in (not-necessarily-symmetric) circulant graphs in polynomial time. In contrast, Codenotti, Gerace, and Vigna [8] show that Max Clique and Graph Coloring remain NP-hard when restricted to circulant graphs and do not admit constant-factor approximation algorithms unless P=NP.

Because of this ambiguity, the complexity of circulant TSP has often been cited as an open problem (see, e.g., Burkhard [5], Burkhard, Deineko, Van Dal, Van der Veen, and Woeginger [6], and Lawler, Lenstra, Rinnooy Kan, and Shmoys [23]). It is not known if the circulant TSP is solvable in polynomial-time or is NP-hard, even when restricted to instances where only two of the edge costs $c_1, \ldots, c_{\left\lfloor \frac{n}{2} \right\rfloor}$ are finite: the two-stripe circulant TSP. (See Greco and Gerace [18]. Yang,
Burkard, Čela, and Woeginger [38] provide a polynomial-time algorithm for asymmetric TSP in circulant graphs with only two stripes having finite edge costs. The symmetric two-stripe circulant TSP is not, however, a special case of the asymmetric two-stripe version.) In addition to questions of minimizing wallpaper waste, circulant TSP has applications in reconfigurable network design (see Medova [24]).

Motivated by positive results on Hamiltonicity and minimum-cost Hamiltonian paths, Van der Veen, Van Dal, and Sierksma [35] developed two heuristic algorithms for circulant TSP. In the case where all costs $c_1, \ldots, c_{\lfloor n/2 \rfloor}$ are distinct, one heuristic provides tours within a factor of two of the optimal solution. In addition, Van der Veen, Van Dal, and Sierksma [35] give an explicit combinatorial formula as a lower bound for circulant TSP. Gerace and Greco [13] give a $2$-approximation algorithm for the general case of circulant TSP when costs may not be distinct. Gerace and Irving [14] give a $\frac{4}{3}$-approximation algorithm for circulant TSP when edge costs are also metric.

De Klerk and Dobre [11] consider several lower bounds for the circulant TSP, including the subtour elimination linear program (also referred to as the Dantzig-Fulkerson-Johnson relaxation [9] and the Held-Karp bound [19], and which we will refer to as the subtour LP). Let $V = [n]$ denote the set of vertices in $K_n$, and let $E$ denote the set of edges in $K_n$. For $S \subset V$, denote the set of edges with exactly one endpoint in $S$ by $\delta(S) := \{e = \{i, j\} : \{i, j\} \cap S = 1\}$ and let $\delta(v) := \delta(\{v\})$. The subtour LP is:

$$\min \sum_{e \in E} c_e x_e$$

subject to

$$\sum_{e \in \delta(v)} x_e = 2, \quad v = 1, \ldots, n$$

$$\sum_{e \in \delta(S)} x_e \geq 2, \quad S \subset V : S \neq \emptyset, S \neq V$$

$$0 \leq x_e \leq 1, \quad e \in E.$$

The constraints $\sum_{e \in \delta(v)} x_e = 2$ are known as the degree constraints, while the constraints $\sum_{e \in \delta(S)} x_e \geq 2$ are known as the subtour elimination constraints. When edge costs are metric (but not necessarily circulant), Wolsey [37] and Shmoys and Williamson [34] show that solutions to this linear program are within a factor of $\frac{3}{2}$ of the optimal, integer solution to the TSP.

De Klerk and Dobre [11] show that, in the context of circulant TSP, the subtour LP is at least as strong as the combinatorial lower bound of Van der Veen, Van Dal, and Sierksma [35]. They also conjecture that, on any instance of circulant TSP, the combinatorial lower bound of Van der Veen, Van Dal, and Sierksma [35] exactly equals the optimal solution to the subtour LP.

Our paper has two main results. First, we prove the conjecture of De Klerk and Dobre [11]. Second, we show that the integrality gap of the subtour LP is 2 for circulant TSP instances, making such instances one of the few non-trivial classes of TSP instances for which the integrality gap of the subtour LP is exactly known.

We begin, in Section 2, by reviewing major results and notation relevant to circulant TSP. In Section 3, we then state and prove our main theorem, showing that the combinatorial lower bound of Van der Veen, Van Dal, and Sierksma [35] exactly equals the optimal solution to the subtour LP. In proving this result, we provide an explicit optimal solution to the subtour LP on circulant instances. As a corollary, we show that the degree constraints do not strengthen the subtour LP on circulant instances, mimicking the parsimonious property of metric, symmetric TSP instances shown in Goemans and Bertsimas [16] in a distinctly non-metric set of instances. In Section 4, we complete our characterization of the integrality gap of the subtour LP and show that it is exactly 2 on circulant instances. The instances we use to show that the integrality gap is 2 are the same instances for which the crown inequalities (a certain class of facet-defining inequalities for the metric, symmetric TSP; see Naddef and Rinaldi [28]) were derived. We show that, unfortunately,
adding the crown inequalities to the subtour LP does not reduce the integrality gap when restricted to circulant TSP instances. This leads us to discuss and conjecture constraints whose addition to the subtour LP would lower its integrality gap on circulant instances.

Our results serve to motivate circulant TSP as a non-trivial class of TSP instances for which there is substantial number-theoretic and combinatorial structure. We hope our results reinvigorate broad interest in the circulant TSP, and thus we conclude by indicating several compelling open questions.

2 Circulant TSP: Notation and Background

Throughout this paper, we consider circulant TSP instances where \( V = [n] \) and let \( d := \left\lfloor \frac{n}{2} \right\rfloor \). We use \( \equiv_n \) to denote the mod-\( n \) equivalence relationship and assume all computations on the vertex set are done mod \( n \). In circulant TSP, all edges \( \{i, j\} \) such that \( i - j \equiv n k \) or \( i - j \equiv n (n - k) \) have the same cost \( c_k \). We refer to such edges as being in the \( k \)-th stripe, and we describe \( k \) as the length of the stripe. Classic algorithms and bounds for circulant TSP depend only on the ordering of the stripes with respect to their costs.

**Definition 2.1.** Let \( S \subset \{1, \ldots, d\} \). The circulant graph \( C(S) \) is the (simple, undirected, unweighted) graph including exactly the edges associated with the stripes \( S \). I.e., the graph with adjacency matrix

\[
A = (a_{ij})_{i,j=1}^n, \quad a_{ij} = \begin{cases} 1, & (i - j) \mod n \in S \text{ or } (j - i) \mod n \in S \\ 0, & \text{else.} \end{cases}
\]

For a set of stripes \( S \), the graph \( C(S) \) includes exactly the edges associated with those stripes.

Given such an input to circulant TSP, we associate a permutation \( \phi : [d] \to [d] \) that sorts the stripes in order of nondecreasing cost as well as a sequence that encodes the connectivity of \( C(\{\phi(1), \ldots, \phi(k)\}) \) for \( 1 \leq k \leq d \).

**Definition 2.2** (Van der Veen, Van Dal, and Sierksma \[35\]). Consider an instance of circulant TSP with edge costs \( c_1, \ldots, c_d \). A stripe permutation \( \phi : [d] \to [d] \) is a permutation such that \( c_{\phi(1)} \leq c_{\phi(2)} \leq \cdots \leq c_{\phi(d)} \). The \( g \)-sequence associated to \( \phi \) is \( g^\phi = (g^\phi_0, g^\phi_1, \ldots, g^\phi_d) \), recursively defined by

\[
g^\phi_i = \begin{cases} n, & i = 0 \\ \gcd (\phi(i), g^\phi_{i-1}), & \text{else.} \end{cases}
\]

Proposition 2.3 will allow us to interpret \( g^\phi_i \) is the number of components of \( C(\{\phi(1), \ldots, \phi(i)\}) \), the graph of all edges from the cheapest \( i \) stripes.

Note that, if edge costs are not distinct for a given instance of circulant TSP, there may be multiple associated stripe permutations. In this case, we will take \( \phi \) to be an arbitrary stripe permutation sorting the costs. In Van der Veen, Van Dal, and Sierksma \[35\], the \( g \)-sequence is denoted as \((\gcd(\phi(0)), \ldots, \gcd(\phi(d)))\) with \( \phi(0) := n \). In Greco and Gerace \[18\], \( \phi \) is referred to as a presentation.

An early result from Burkard and Sandholzer \[4\] characterizes when Hamiltonian cycles exist in circulant graphs:
Figure 1: Constructing a minimum-cost Hamiltonian Path via the nearest neighbor heuristic. In this case, \( n = 12 \), \( \phi(1) = 6 \) (thin edges), \( \phi(2) = 2 \) (thick edges), and \( \phi(3) = 3 \) (dotted edges). This process fully connects a component of \( C(\{\phi(1), \ldots, \phi(i)\}) \), uses an edge of length \( \phi(i + 1) \) to move to the new component of \( C(\{\phi(1), \ldots, \phi(i)\}) \), and recursively fully connects that component. When all possible edges of length \( \phi(1), \ldots, \phi(i), \phi(i + 1) \) have been added, the path connects a component of \( C(\{\phi(1), \ldots, \phi(i + 1)\}) \) and the process repeats using edges of length \( \phi(i + 2) \).

**Proposition 2.3** (Burkard and Sandholzer [4]). Let \( \{a_1, \ldots, a_t\} \subset [d] \) and let \( G = \gcd(n, a_1, \ldots, a_t) \). The circulant graph \( C(\{a_1, \ldots, a_t\}) \) has \( G \) components. The \( i \)th component, for \( 0 \leq i \leq G - 1 \), consists of \( n/G \) nodes \( \{i + \lambda G \mod n : 0 \leq \lambda \leq n/G - 1\} \).

\( C(\{a_1, \ldots, a_t\}) \) is Hamiltonian if and only if \( G = 1 \).

Set \( \ell := \min\{i : 1 \leq i \leq d, g_i^\phi = 1\} \).

By Proposition 2.3 the graph \( C(\{\phi(1), \ldots, \phi(\ell - 1)\}) \) is not Hamiltonian, while \( C(\{\phi(1), \ldots, \phi(\ell)\}) \) is. Hence any Hamiltonian tour uses an edge of cost at least \( c_{\phi(\ell)} \), and tours can be constructed where \( c_{\phi(\ell)} \) is the most expensive edge. Thus this proposition not only resolves Hamiltonicity in circulant graphs, but it also resolves bottleneck TSP in circulant graphs. In bottleneck TSP, the objective is to find a Hamiltonian tour for which the cost of the most expensive edge is minimized. Burkard and Sandholzer [4] use Proposition 2.3 to give a constructive algorithm for bottleneck TSP on circulant instances.

Moreover, Proposition 2.3 immediately gives rise to an easily solvable case of circulant TSP: if there exists a stripe permutation \( \phi \) such that \( g_i^\phi = 1 \), or equivalently, the length \( \phi(1) \) of a cheapest stripe is relatively prime to \( n \). For example, if \( n \) is prime, circulant TSP is easily solvable: you obtain a Hamiltonian tour by following edges of the cheapest stripe; after \( n \) edges you will have visited every node and returned to the start. These observations were first made in Garfinkel [12].

Proposition 2.3 can be used to solve the minimum-cost Hamiltonian path problem on circulant instances.

**Proposition 2.4** (Bach, Luby, and Goldwasser, cited in Gilmore, Lawler, and Shmoys [15]). Let \( c_1, \ldots, c_d \) be the edge costs of a circulant instance and let \( \phi \) be an associated stripe permutation. The minimum-cost Hamiltonian path has cost

\[
\sum_{i=1}^{\ell} (g_i^\phi - g_{i-1}^\phi)c_{\phi(i)}.
\]
Sketch. Van der Veen, Van Dal, and Sierksma [35] argue that the nearest neighbor heuristic constructs a Hamiltonian path using exactly $g_i - 1 - g_i$ edges from the $i$th cheapest stripe (see Figure 1). This path thus has cost

$$\sum_{i=1}^{\ell} (g_{i-1} - g_i) c_{\phi(i)}.$$  

The optimality of such a path can be seen by applying Kruskal’s algorithm [22] for minimum-cost spanning trees: For $1 \leq i \leq \ell$, Proposition 2.3 indicates that the graph $C\langle\{\phi(1), \phi(2), ..., \phi(i)\}\rangle$ has $g_i$ components. Hence, at most $n - g_i$ edges can be used from the cheapest $i$ stripes without creating a cycle. Kruskal’s algorithm will find a minimum-cost spanning tree using $n - g_1 = g_0 - g_1$ edges from the cheapest stripe, $g_1 - g_2$ edges from the second cheapest stripe, and in general $g_i - 1 - g_i$ edges from the $i$th cheapest stripe. This spanning tree thus also costs $\sum_{i=1}^{\ell} (g_{i-1} - g_i) c_{\phi(i)}$. Since any Hamiltonian path is itself a spanning tree, any Hamiltonian path must cost at least this much; the constructed Hamiltonian path achieves this lower bound and is therefore optimal.

Proposition 2.4 yields a natural lower bound on the optimal solution to circulant TSP instances: delete the most expensive edge of a Hamiltonian tour (of cost at least $c_{\phi(\ell)}$), and compare the resultant Hamiltonian path to a minimum-cost Hamiltonian path.

**Proposition 2.5 (Van der Veen, Van Dal, and Sierksma [35]).** Let $c_1, ..., c_d$ be the edge costs of a circulant instance and let $\phi$ be an associated stripe permutation. Any Hamiltonian tour costs at least

$$VDV := \left(\sum_{i=1}^{\ell} (g_{i-1} - g_i) c_{\phi(i)}\right) + c_{\phi(\ell)}.$$

**Proof.** Consider any Hamiltonian tour. Its most expensive edge costs at least $c_{\phi(\ell)}$. Removing this edge creates a Hamiltonian path which, by Proposition 2.3, has cost at least

$$\sum_{i=1}^{\ell} (g_{i-1} - g_i) c_{\phi(i)}.$$

If there are multiple stripe permutations associated with an instance (i.e., the $c_i$ are not all distinct), the lower bound is independent of the stripe permutation chosen. The lower bound is, moreover, tight as can be shown by considering any instance where the cheapest stripe has length relatively prime to $n$. For example the lower bound is tight for any instance where $\phi(1) = 1$.

De Klerk and Dobre [11] compare the VDV lower bound to several other well-known TSP bounds. In a series of numerical experiments, they provide evidence to conjecture that the VDV lower bound is exactly equal to the value of the optimal solution to the subtour LP:

**Conjecture 2.6 (De Klerk and Dobre [11]).** Let $c_1, ..., c_d$ be the edge costs of a circulant instance and let $\phi$ be an associated stripe permutation. Let $OPT_{\text{LP}}$ denote the optimal value of the subtour LP and VDV denote the value of the lower bound in Proposition 2.5. Then:

$$VDV = OPT_{\text{LP}}.$$  

1Start at some vertex and and follow a cheapest edge from that vertex. Then, recursively grow a Hamiltonian path by adding a cheapest edge from the most recently added vertex to a vertex that has not yet been visited.
De Klerk and Dobre [11] provide further evidence for this conjecture by showing the following.

**Theorem 2.7** (De Klerk and Dobre [11]). Let $c_1, \ldots, c_d$ be the edge costs of a circulant instance and let $\phi$ be an associated stripe permutation. Let $OPT_{LP}$ denote the optimal value of the subtour LP and $VDV$ denote the value of the lower bound in Proposition 2.7. Then:

$$VDV \leq OPT_{LP}.$$ 

**Sketch.** De Klerk and Dobre [11] relax the subtour LP by dropping the degree constraints. Denote by $OPT_{Relaxed}$ the value of an optimal solution to this LP, so that:

$$OPT_{Relaxed} = \min \sum_{e \in E} c_e x_e \quad \text{subject to} \quad \sum_{e \in \delta(S)} x_e \geq 2, \quad S \subset V : S \neq \emptyset, S \neq V$$

and

$$OPT_{Relaxed} \leq OPT_{LP}.$$ 

Any feasible solution to the dual of this relaxed LP thus also provides a lower bound on $OPT_{LP}$. The dual of this LP is:

$$\max \sum_{S \subset V : S \neq \emptyset, S \neq V} 2y_S$$

subject to

$$\sum_{S : e \in \delta(S)} y_S \leq c_e, \quad e \in E \quad \text{subject to} \quad S \subset V : S \neq \emptyset, S \neq V.$$ 

De Klerk and Dobre [11] provide a feasible solution to this dual of value equal to $VDV$ as follows. For $0 \leq k \leq \ell - 1$ and $1 \leq i \leq g_1^k$, let $C_k^i$ denote the $i$th connected component of the graph $C(\{\phi(1), \ldots, \phi(k)\})$ (with $C_1^0, \ldots, C_n^0$ corresponding to the $n$ isolated vertices). See Figure 2. Then:

$$y_S = \begin{cases} \frac{c_1(1)}{2}, & S \in \{C_1^0, \ldots, C_n^0\} \\ \frac{1}{\ell} (c_\phi(k+1) - c_\phi(k)), & S \in \{C_1^k, C_2^k, \ldots, C_n^k\}, k = 1, \ldots, \ell - 1 \\ 0, & \text{else} \end{cases}$$

To show that this solution has value $VDV$, add and group the terms by $c_\phi(i)$ to match the lower bound in Proposition 2.5

$$\sum_{S \subset V : S \neq \emptyset, S \neq V} 2y_S = g_0^\phi c_\phi(1) + \sum_{i=1}^{\ell-1} \sum_{j=1}^{i} (c_\phi(i+1) - c_\phi(i)) = \sum_{i=1}^{\ell} c_\phi(i) (g_i^{\phi} - g_{i-1}^{\phi}) + c_\phi(\ell) = VDV.$$ 

The feasibility of this solution follows by considering edges of each length. Consider an edge of length $\phi(i)$ with cost $c_\phi(i)$. The only subsets $S$ such that $e \in \delta(S)$ that we must consider are those $S$ such that $y_S$ has nonzero weight: those that are a $C_j^k$ for $0 \leq k \leq \ell - 1$ and $1 \leq j \leq g_1^k$. An edge of length $\phi(i)$ will either join exactly two distinct components of $C(\{\phi(1), \ldots, \phi(k)\})$ or be contained within a component of $C(\{\phi(1), \ldots, \phi(k)\})$. Moreover, if $k \geq i$ or $k \geq \ell$, then every edge of length $\phi(i)$ is contained within a component of $C(\{\phi(1), \ldots, \phi(k)\})$. The components of $C(\{\phi(1), \ldots, \phi(k)\})$ are then subsets of the components of $C(\{\phi(1), \ldots, \phi(k)\})$. Since $y_{C_1^k} = \cdots = y_{C_n^k}$, we can compute

$$\sum_{S : e \in \delta(S)} y_S \leq 2 \sum_{k=0}^{\min(i-1, \ell-1)} y_{C_1^k} = \begin{cases} c_\phi(i) \leq c_\phi(i), & i \leq \ell \\ c_\phi(\ell) \leq c_\phi(i), & i > \ell. \end{cases}$$

by simplifying the telescoping sum. □
Figure 2: The graph $C\langle \{6, 3\} \rangle$ for $n = 12$. If $\{\phi(1), \phi(2)\} = \{3, 6\}$, the three components are $C_1^2$, $C_2^2$, and $C_3^2$.

Theorem 2.7 can be interpreted as a statement bounding the integrality gap of the subtour LP on circulant instances. The integrality gap represents the worst-case ratio of the original problem’s optimal solution to the relaxation’s optimal solution.

**Definition 2.8.** Let $OPT_{TSP}(c_1, \ldots, c_d)$ denote the value of the optimal solution to the TSP for a circulant TSP instance with stripe costs $c_1, \ldots, c_d$, and let $OPT_{LP}(c_1, \ldots, c_d)$ denote the value of the optimal solution of the subtour LP for the same circulant instance. The **integrality gap** for the subtour LP on circulant instances is

$$\sup_{(c_1, \ldots, c_d) \in \mathbb{R}_\geq d} \frac{OPT_{TSP}(c_1, \ldots, c_d)}{OPT_{LP}(c_1, \ldots, c_d)}.$$  

This ratio is bounded below by 1, since the subtour LP is a relaxation of the TSP. For metric (but not necessarily circulant) instances, Wolsey [37] and Shmoys and Williamson [34] show that the integrality gap of the subtour LP is at most $\frac{3}{2}$. A corollary of Theorem 2.7 is that, on circulant (but not necessarily metric) instances, the subtour LP also has a bounded integrality gap.

**Corollary 2.9.** The integrality gap of the subtour LP restricted to circulant TSP instances is at most 2. That is,

$$\sup_{(c_1, \ldots, c_d) \in \mathbb{R}_\geq d} \frac{OPT_{TSP}(c_1, \ldots, c_d)}{OPT_{LP}(c_1, \ldots, c_d)} \leq 2,$$

**Proof.** Consider any circulant instance. Let $OPT_{TSP}$ denote the value of the optimal solution to the TSP on this instance, $OPT_{LP}$ denote the value of the optimal solution to the subtour LP on this instance, and let $VDV$ denote the value of the Van der Veen, Van Dal, and Sierksma [35] lower bound on this instance. By Theorem 2.7,

$$\frac{OPT_{TSP}}{OPT_{LP}} \leq \frac{OPT_{TSP}}{VDV}.$$  

Theorem 6.3 in Gerace and Greco [13] argues that $\frac{OPT_{TSP}}{VDV} \leq 2$, by constructing Hamiltonian tours of cost at most 2$VDV$. See Appendix A for details on this construction.

3 A Combinatorial Interpretation of the Subtour LP

In this section, we prove our first main result, answering Conjecture 2.6.
Theorem 3.1. Let $c_1, \ldots, c_d$ be the edge costs of a circulant instance and let $\phi$ be an associated stripe permutation. Let $\text{OPT}_{LP}$ denote the optimal value of the subtour LP and let $\text{VDV}$ denote the value of the lower bound in Proposition 2.5. Then:

$$\text{VDV} = \text{OPT}_{LP}.$$ 

Moreover, an optimal solution to the subtour LP is achieved by setting, for $1 \leq i \leq d$, the weight on every edge $e$ of length $\phi(i)$ to be

$$x_e = \begin{cases} 
\frac{g_{i-1}^\phi - g_i^\phi}{n}, & i \neq \ell, \phi(i) \neq \frac{n}{2} \\
\frac{g_{\ell-1}^\phi - g_\ell^\phi}{n}, & i \neq \ell, \phi(i) = \frac{n}{2} \\
\frac{g_i^\phi}{n}, & i = \ell, \phi(i) \neq \frac{n}{2} \\
\frac{g_{\ell-1}^\phi}{n}, & i = \ell, \phi(i) = \frac{n}{2}.
\end{cases}$$

The explicit $x_e$ values given in Theorem 3.1 spread out the weight placed by the Van der Veen, Van Dal, and Sierksma [35] bound,

$$\text{VDV} = \left( \sum_{i=1}^{\ell} (g_{i-1}^\phi - g_i^\phi)c_{\phi(i)} \right) + c_{\phi(\ell)}.$$

The coefficient of $c_{\phi(i)}$ is spread over all edges of length $\phi(i)$. For $n$ even and $\phi(i) = d$, there are only $\frac{n}{2}$ such edges; otherwise there are $n$ edges. As a result, we remark the following.

Remark 3.2. Let $x$ be defined as in Theorem 3.1. Then

$$\sum_{e \in E} c_e x_e = \text{VDV}.$$ 

Note also that the solution places zero weight on edges of length $\phi(\ell + 1), \ldots, \phi(d)$ as well as zero weight on edges of any length $\phi(i)$ such that $g_i^\phi = g_{i-1}^\phi$. The optimal solution $x$, therefore, only depends on the relative ordering of edge costs $\phi$, and specifically, those stripes $\phi(i)$ for which $C(\{\phi(1), \ldots, \phi(i)\})$ has fewer components than $C(\{\phi(1), \ldots, \phi(i-1)\})$.

To simplify our work that follows, we assume that the edges are ordered so that $g_0^\phi > g_1^\phi > \cdots > g_\ell^\phi = 1$. We can make this assumption without loss of generality: If $g_i^\phi = g_{i-1}^\phi$ for $i < \ell$, then zero weight is placed on any edge of length $\phi(i)$ by both the Van der Veen, Van Dal, and Sierksma [35] bound and in the edge weights in Theorem 3.1. Both the Van der Veen, Van Dal, and Sierksma [35] bound and the subtour LP solution we find in Theorem 3.1 thus remain the same on an instance where $c_{\phi(i)}$ is increased beyond $c_{\phi(\ell)}$. By applying this argument iteratively, we can obtain an instance of circulant TSP for which the $g$-sequence is strictly decreasing until it reaches 1, and which the Van der Veen, Van Dal, and Sierksma [35] bound and the subtour LP treat equivalently.

Our proof of Theorem 3.1 involves several steps. In Lemma 3.3 we show that the solution $x$ posited satisfies the degree constraints. We then use Lemmas 3.4 and 3.5 to characterize the components of $C(\{\phi(1), \ldots, \phi(i)\})$, for $1 \leq i \leq \ell - 1$, as maximally dense in Lemma 3.6. They satisfy the subtour elimination constraints with equality. To complete the proof, we look at arbitrary subsets $S \subset V$ in Proposition 3.6. Doing so involves casework wherein we consider ways to write $S$ as a disjoint union of circulant components.
Throughout the proof, we suppress the dependence of $g^\phi$ on $\phi$ to simplify notation. We fix $x \in \mathbb{R}^E$ to be the edge-weight vector where, for $1 \leq i \leq d$, the weight on every edge $e$ of length $\phi(i)$ is as stated in Theorem 3.1. For a set of edges $F \subseteq E$, $x(F)$ denotes the total weight of edges in $F$: $\sum_{e \in F} x_e$. Similarly, $E(S) := \{(i, j) : i, j \in S\}$

denotes the set of edges with both endpoints in $S$. For $0 \leq i \leq \ell - 1$ and $1 \leq k \leq g_i$, we use $C_k^i$ to denote the vertex set of the $k$th connected component of the graph $C(\{\phi(1), \ldots, \phi(i)\})$. See Figure 2 We let $C_i$ denote an arbitrary representative of $C_1^i, \ldots, C_{g_i}^i$. Finally, we use $\sqcup$ to denote a disjoint union or partition. That is $A = B \sqcup C$ means $A = B \cup C$ and $B \cap C = \emptyset$.

Lemma 3.3. For any vertex $v \in V$, $x(\delta(v)) = 2$.

Proof. Let $i < \ell$ and consider edges of length $\phi(i)$ incident to $v$. For $\phi(i) \neq n/2$, there are two edges of weight $\frac{2^{\ell} - g_i}{n}$: $(v, v + \phi(i))$ and $(v, v - \phi(i))$. For $\phi(i) = n/2$, there is one edge $(v, v + \phi(i)) = (v, v - \phi(i))$ of weight $2\frac{g_i - 1}{n}$. In either case, the total weight of edges of length $\phi(i)$ incident to $v$ is $2\frac{g_i - 1}{n}$. Analogously, the weight of edges of length $\phi(\ell)$ incident to $v$ is $2\frac{g_i - 1}{n}$. Thus

$$x(\delta(v)) = \sum_{i=1}^{\ell} \sum_{e \in \delta(v); \text{length}(e) = \phi(i)} x_e = 2 \frac{n}{n} \left( \sum_{i=1}^{\ell-1} (g_i - 1) + g_\ell - 1 \right) = 2 \frac{g_0}{n} = 2,$$

since $g_0 = n$. \hfill \Box

We next argue that, for a set of vertices $S = C_k^i$, the only edges within $E(S)$ that have nonzero weight are those of length $\phi(1), \ldots, \phi(i)$.

Lemma 3.4. Let $S = C_k^i$ where $0 \leq i \leq \ell - 1$ and $1 \leq k \leq g_i$. Let $e \in E(S)$. Then $x_e > 0$ implies $e$ is an edge in stripes $\phi(1), \ldots, \phi(i)$.

Proof. By Proposition 3.2, $S = \{v : v \equiv g_j, j\}$ for some $0 \leq j \leq g_i - 1$. Consider an edge of $e = \{v, v + \phi(t)\} \in E(S)$ of length $\phi(t)$ with $t > i$. Then $\phi(t) = c \cdot g_i$ for some $c \in \mathbb{N}$. Hence $g_t = \gcd(g_{t-1}, \phi(t)) = \gcd(g_{t-1}, c \cdot g_i) = g_{t-1}$, since $g_{t-1}$ divides $g_i$, and so $x_e = 0$. \hfill \Box

Lemma 3.4 lets us now show that the $C_k^i$ are maximally dense.

Lemma 3.5. Let $S = C_k^i$ for $0 \leq i \leq \ell - 1$ and $1 \leq k \leq g_i$. Then $x(\delta(S)) = 2$.

Proof. By Lemma 3.3, we can compute $x(E(S))$ by only summing up the weights of edges in the cheapest $i$ stripes. Consider any fixed $j$ with $j \leq i$ and $\phi(j) \neq n/2$. There are $n$ total edges of length $\phi(j)$ and, since $j \leq i$, none of these edges are in any $\delta(C^i)$. Thus each isomorphic component $C^i \in \{C_1^i, \ldots, C_{g_i}^i\}$ has $\frac{n}{g_i}$ edges of length $\phi(j)$ in $E(C^i)$, and each edge has weight $\frac{g_j - 1}{n} \frac{g_j}{g_i}$. If $\phi(j) = n/2$, then there are instead half as many edges, $\frac{n}{2g_i}$, each of double the weight, $2\frac{g_j - 1}{n} \frac{g_j}{g_i}$, in each $E(C^i)$. In either case,

$$\sum_{e \in E(S); \text{length}(e) = \phi(j)} x_e = \frac{n}{g_i} \frac{g_j - 1}{n} \frac{g_j}{g_i} = \frac{g_j - 1}{g_i}.$$


We can now compute:

\[ x(E(S)) = \sum_{j=1}^{i} \sum_{e \in E(S), \text{length}(e) = \phi(j)} x_e \]

\[ = \frac{1}{g_i} \sum_{j=1}^{i} (g_j - 1 - g_j) \]

\[ = \frac{g_0 - g_i}{g_i} \]

\[ = \frac{n}{g_i} - 1 \]

\[ = |C_i^j| - 1. \]

The lemma then follows because the degree constraints imply that \( x(\delta(S)) + 2x(E(S)) = 2|S| \), so that \( x(\delta(S)) = 2 \). \( \square \)

We now want to extend Lemma 3.5 to show that \( x(\delta(S)) \geq 2 \) for any \( S \subset V \), not just those corresponding to components connected by a set of cheapest stripes.

**Proposition 3.6.** Let \( S \subset V \) \( (2 \leq |S| \leq n - 2) \). Then \( x(\delta(S)) \geq 2 \).

Throughout the proof of Proposition 3.6, it will be helpful to instead treat our graph as a directed graph. Each edge from the \( i \)th stripe, \( i \neq n/2 \), is directed \( (v, v + i) \) (with the convention that \( v + i \) is taken mod \( n \)). If \( n \) is even, we treat each edge of length \( n/2 \) incident to \( v \) as two directed edges, \( (v, v + (n/2)) \) and \( (v + (n/2), v) \), each of which is assigned half the weight of an edge with length \( n/2 \). Thinking of our graph in this way means that every vertex \( v \) is incident to exactly two edges from each stripe \( i = 1, \ldots, d \), with one edge directed into \( v \) and one edge directed out of \( v \). Moreover, this simplifies the number of cases for \( x_e \); the weight on every edge \( e \) of length \( \phi(i) \) is now:

\[ x_e = \begin{cases} \frac{g_{\phi(i) - 1} - g_{\phi(i)}}{n}, & i \neq \ell \\ \frac{g_{\phi(i) - 1}}{n}, & i = \ell. \end{cases} \]

We still treat \( \delta(S) \) as the set of all edges with exactly one endpoint in \( S \), whether that edge is directed into or out of \( S \).

The proof of Proposition 3.6 will proceed by the following casework:

1. Edges of every length \( \phi(1), \ldots, \phi(\ell) \) appear in \( \delta(S) \). In this case, we argue that there must be two edges of each length in \( \delta(S) \), and the case follows from an argument akin to that of Lemma 3.3.

2. Edges of every length \( \phi(1), \ldots, \phi(k - 1) \) with \( 1 < k \leq \ell \) appear in \( \delta(S) \), but no edge of length \( \phi(k) \) appears in \( \delta(S) \). For this case, we partition \( S \) into maximally connected components of the graph \( C\{\phi(k)\} \). If a vertex in one such component is incident to an edge of length \( \phi(i) \) in \( \delta(S) \) with \( i < k \), we show that every vertex of that component must be incident an edge of length \( \phi(i) \) in \( \delta(S) \). Doing so allows us to then show that there are at least four distinct edges of length \( \phi(1) \) in \( \delta(S) \), each of which has a weight of at least \( \frac{1}{2} \), from which the case follows. (We actually show that there are four edges of each length \( \phi(i) \) in \( \delta(S) \) for \( 1 \leq i < k \), but all we need are the four length \( \phi(1) \) edges.)
3. No edge from any stripes of lengths \(\phi(1), \ldots, \phi(k-1)\) with \(1 < k \leq \ell\) appear in \(\delta(S)\), but an edge of length \(\phi(k)\) does. For this case we instead partition \(S = C_1^{k-1} \sqcup \cdots \sqcup C_m^{k-1}\) with \(1 \leq m < g_k-1\). This argument requires the most bookkeeping. For \(m = 1\), the result follows by Lemma 3.5. For \(m > 1\), we show that either there are a sufficient number of edges of length \(\phi(k)\) in \(\delta(S)\) for the result to follow directly, or there are a sufficient number of edges of each length \(\phi(i)\) for each \(i \geq k\) in \(\delta(S)\) for the result to follow directly.

Note that if no edges of any length \(\phi(1), \ldots, \phi(\ell)\) appear in \(\delta(S)\), then \(S = V\).

**Lemma 3.7** (Case 1). Let \(S \subset V\) be such that edges of every length \(\phi(1), \ldots, \phi(\ell)\) appear in \(\delta(S)\). Then \(x(\delta(S)) \geq 2\).

**Proof.** Consider an edge of length \(\phi(i)\) with \((v, v + \phi(i))\) in \(\delta(S)\). Following edges of length \(\phi(i)\) yields a cycle

\[
(v, v + \phi(i)), (v + \phi(i), v + 2\phi(i)), \ldots, \left(v + \frac{n}{\gcd(\phi(i), n)} - 1\right) \cdot \phi(i), v
\]

with at least two distinct edges of length \(\phi(i)\) in \(\delta(S)\). This is true for \(1 \leq i \leq \ell\), so the computations in Lemma 3.3 show that \(x(\delta(S)) \geq 2\).

**Lemma 3.8** (Case 2). Let \(S \subset V\) be such that edges of every length \(\phi(1), \ldots, \phi(k-1)\) with \(1 < k \leq \ell\) appear in \(\delta(S)\), but no edge of length \(\phi(k)\) appears in \(\delta(S)\). Then \(x(\delta(S)) \geq 2\).

**Proof.** By Proposition 2.3, the graph \(C\{\phi(k)\}\) consists of \(\gcd(n, \phi(k))\) equally sized components \(T_1, \ldots, T_{\gcd(n, \phi(k))}\); each component consists of a cycle on \(\frac{n}{\gcd(n, \phi(k))} \geq 2\) vertices using edges of length \(\phi(k)\). Since no edges of length \(\phi(k)\) are in \(\delta(S)\), \(S\) can be partitioned as

\[
S = T_1 \sqcup \cdots \sqcup T_m, \quad m < \gcd(n, \phi(k)).
\]

Consider a cycle \(T \in T_1, \ldots, T_m\) such that \(v \in T\), \((v, v + \phi(i))\) in \(\delta(S)\), and \(i < k\) (such an outgoing edge exists by an argument akin to that in Lemma 3.7).

**Claim 3.9.** Consider a cycle \(T \in T_1, \ldots, T_m\) such that \(v \in T\), \((v, v + \phi(i))\) in \(\delta(S)\), and \(i < k\). For every \(u \in T\), the edge \((u, u + \phi(i))\) in \(\delta(S)\).

To see this claim, note that \(u, v \in T\) implies that \(u \equiv_{\gcd(n, \phi(k))} v\) by Proposition 2.3. Hence, \(u + \phi(i) \equiv_{\gcd(n, \phi(k))} v + \phi(i)\) so that \(u + \phi(i), v + \phi(i)\) are in the same component \(C\{\phi(k)\}\). By assumption, this component is outside \(S\).

Hence, there are at least two distinct vertices \(u, v \in S\) with \(u + \phi(i), v + \phi(i) \notin S\). As in Lemma 3.7, each of these edges \((u, u + \phi(i))\) and \((v, v + \phi(i))\) also implies a distinct edge of length \(\phi(i)\) starting outside of \(S\) and returning to \(\delta(S)\) so that there are at least four edges of length \(\phi(i)\) in \(\delta(S)\). The total weight of these edges across stripes 1 up to \(k - 1\) is at least 2:

\[
x(\delta(S)) \geq 4 \sum_{i=1}^{k-1} \frac{g_i - 1}{n} = 4 \frac{n - g_k - 1}{n} \geq 2,
\]

since \(k > 1\) implies \(g_k - 1\) divides \(n\) and \(g_k - 1 < n\). As indicated in the outline of our proof, it is actually sufficient to only account for 4 edges of length \(\phi(1)\), corresponding to the first term in the sum; since each such edge has weight \(\frac{g_1 - 1}{n} = 1 - \frac{g_1}{n} \geq \frac{1}{2}\), their total weight is at least 2.
The last inequality happens since $g$.

Claim 3.11 (Case 3a) of Lemma 3.10: there are two distinct $C^k$ each intersecting $S$ nontrivially.

**Figure 3**: Claim 3.11 (Case 3a) of Lemma 3.10 there are two distinct $C^k$ each intersecting $S$ nontrivially.

**Lemma 3.10** (Case 3). Let $S \subset V$ be such that no edge from any stripes of lengths $\phi(1), \ldots, \phi(k-1)$ with $1 < k \leq \ell$ appear in $\delta(S)$, but an edge of length $\phi(k)$ does. Then $x(\delta(S)) \geq 2$.

**Proof.** For this case, we have a different decomposition of $S$. Because no edges of length $\phi(1), \ldots, \phi(k-1)$ are in $\delta(S)$,

$$S = C_1^{k-1} \sqcup \cdots \sqcup C_m^{k-1}, \quad 1 \leq m < g_{k-1}. $$

We may assume $m \geq 2$, as otherwise Lemma 3.10 is exactly Lemma 3.5. By assumption, at least one edge of length $\phi(k)$ is in $\delta(S)$. This edge must join a $C_i^{k-1}$, with $1 \leq i \leq m$ (i.e., one in $S$) to a $C_j^{k-1}$, $j > m$ (i.e., one not in $S$); this edge joins these $C_i^{k-1}$ and $C_j^{k-1}$ into a $C^k$ in the graph $C(\{\phi(1), \ldots, \phi(k)\})$.

We first consider a subcase where there are enough edges of length $\phi(k)$ in $\delta(S)$ for the degree constraints to hold, where the above joining essentially happens twice. That is, two distinct $C^{k-1}$ each get merged into a distinct $C^k$, and the two $C^k$’s both intersect nontrivially with $S$.

**Claim 3.11** (Case 3a). Suppose that $C_1^{k-1}, C_2^{k-1} \subset S$ with $C_1^{k-1} \neq C_2^{k-1}$. Suppose further that edges of length $\phi(k)$ join $C_1^{k-1}$ into $C^k_1$ and join $C_2^{k-1}$ into $C^k_2$, with $C_1^k \neq C_2^k$ and both $C_1^k, C_2^k$ intersecting $S$ nontrivially. See Figure 3. Then Lemma 3.10 holds for $S$.

This case follows as our previous arguments: each $C_i^k$, for $i = 1, 2$, has $2|C^{k-1}|$ edges of length $\phi(k)$ in $\delta(S)$: $|C^{k-1}|$ edges directed out of $S$, and $|C^{k-1}|$ edges directed into $S$. Then

$$x(\delta(S)) \geq \sum_{e \in \delta(S): \text{length}(e) = \phi(k)} x_e \geq 4|C_i^{k-1}|x_{\phi(k)} = 4 \frac{n}{g_{k-1}} \frac{g_{k-1} - g_k}{n} = 4 \left(1 - \frac{g_k}{g_{k-1}} \right) \geq 2. $$

The last inequality happens since $g_k \neq g_{k-1}$ and $g_k$ divides $g_{k-1}$.

The remaining subcase, Case 3b, is illustrated in Figure 4. $S$ contains at least two $C^{k-1}$ and has at least one edge of length $\phi(k)$ in $\delta(S)$. However, only one $C^k$ intersects nontrivially with $S$: either all $C^{k-1}$ in $S$ get joined into the same $C^k$, or there is one $C^k$ intersecting $S$ nontrivially and at least one $C^k$ completely contained in $S$. We make the following claim.

\[ \text{13} \]
Claim 3.12 (Case 3b). Suppose that $S$ contains at least two $C^{k-1}$ and has at least one edge of length $\phi(k)$ in $\delta(S)$. However, only one $C^k$ intersects nontrivially with $S$. Then for $k < t \leq \ell$, there are at least $2\frac{n}{g_{k-1}}$ edges of length $\phi(t)$ in $\delta(S)$ and Lemma 3.10 holds for $S$.

We first show that this claim is sufficient to imply Lemma 3.10. This follows by computing the total weight of the $2\frac{n}{g_{k-1}}$ edges of length $\phi(t)$ for $k < t \leq \ell$:

$$x(\delta(S)) \geq 2\frac{n}{g_{k-1}} \left( \frac{g_{k-1} - g_k}{n} + \cdots + \frac{g_{\ell-2} - g_{\ell-1}}{n} + \frac{g_{\ell-1}}{n} \right) = 2\frac{g_{k-1} - g_k}{g_{k-1}} = 2.$$ 

To complete the proof, we must argue that there are at least $2\frac{n}{g_{k-1}}$ edges of length $\phi(t)$ in $\delta(S)$. Let $C^k_a$ be the $C^k$ intersecting nontrivially with $S$ and let $C^{k-1}_a \subset S \cap C^k_a$. Note that all edges of length $\phi(t)$ go from $C^{k-1}_a$ to some other $C^{k-1}_b$ as in Claim 3.9 if $u \equiv g_{k-1} v$ then $u + \phi(t) \equiv g_{k-1} v + \phi(t)$. If $C^{k-1}_b \cap S = \emptyset$ as in Figure 5 then there are thus $|C^{k-1}_b| = \frac{n}{g_{k-1}}$ edges of length $\phi(t)$ starting in $S$ and ending outside of $S$, and thus $2\frac{n}{g_{k-1}}$ total edges of length $\phi(t)$ in $\delta(S)$.

Otherwise, $C^{k-1}_b \subset S$ as in Figure 6. This implies that $C^{k-1}_b$ must be joined into a $C^k_b \neq C^k_a$ by edges of length $\phi(k)$. By the assumptions of Case 3b, $C^k_b \subset S$. By our assumption that $C^k_a$ intersects $S$ nontrivially, there exists some $C^{k-1}_{a'} \subset C^k_a$ with $C^{k-1}_{a'} \cap S = \emptyset$. Let $u \in C^{k-1}_a$ and $v \in C^{k-1}_{a'}$ so that $u, v \subset C^k_b$ and $u \equiv g_{k-1} v$. Thus $u + \phi(t) \equiv g_{k-1} v + \phi(t)$ so that $u + \phi(t), v + \phi(t) \in C^k_b \subset S$. This implies that $(v, v + \phi(t)) \in \delta(S)$ and so that each vertex of $C^{k-1}_{a'}$ has an edge of length $\phi(t)$ crossing into $S$. This implies $\frac{n}{g_{k-1}}$ edges of length $\phi(t)$ from outside $S$ to inside $S$ and thus $2\frac{n}{g_{k-1}}$ total edges of length $\phi(t)$ in $\delta(S)$. This completes the proof of Claim 3.12 and thus of Lemma 3.10. 

Proof (Proposition 3.6). This proof follows immediately from Lemmas 3.7 and 3.8 and 3.10. 

Proof (Theorem 3.4). This proof follows immediately from Lemma 3.3 and Proposition 3.6.

We note that Theorem 3.1 together with the proof of Theorem 2.7 in De Klerk and Dobre [11], indicate the following result.
Corollary 3.13. The degree constraints do not strengthen the subtour elimination LP for circulant TSP. That is, letting $OPT_{\text{Relaxed}}$ denote the value of an optimal solution to the subtour LP relaxation obtained by dropping the degree constraints,

$$OPT_{\text{Relaxed}} = OPT_{LP} = VDV.$$

Proof. Our proof of Theorem 3.1 shows that $OPT_{\text{Relaxed}} \leq VDV$, while the proof of Theorem 2.7 shows that $VDV \leq OPT_{\text{Relaxed}}$. \qed

4 The Integrality Gap of the Subtour LP

Theorem 3.1 allows us to exactly characterize the integrality gap of the subtour LP on circulant instances by considering the Van der Veen, Van Dal, and Sierksma [35] bound. In this section we provide an example showing that this bound can be off by a factor of 2 asymptotically. This, together with Corollary 2.9, will imply our second main theorem.

Theorem 4.1. The integrality gap of the subtour LP restricted to circulant instances is exactly 2.

The example we use to prove this theorem is intimately related to the crown inequalities for the TSP, as we discuss in Section 5.

Proof. Corollary 2.9 implies that the integrality gap is at most 2. To prove the theorem it thus suffices to demonstrate an example where the Van der Veen, Van Dal, and Sierksma [35] bound is a factor of two away from the optimal TSP solution. For such an example, we take $n = 2^{k+1}$. Suppose that $c_1 = 1, c_d = 0$, and $c_i > 2^{k+1}$ otherwise. Then $\phi(1) = d$ and $\phi(2) = 1$, so that $g_1^\phi = d$, $g_i^\phi = 1$ for $i \geq 2$, and $\ell = 1$. By Theorem 3.1 the optimal solution to the subtour LP has cost

$$VDV = \left( \sum_{i=1}^\ell (g_{i-1}^\phi - g_i^\phi)c_{\phi(i)} \right) + c_{\phi(\ell)} = d \cdot 0 + d \cdot 1 = d = 2^k.$$
Now consider the optimal solution to the TSP. It cannot use any edges other than those of lengths 1 and $d$: we can find a tour of cost $2^{k+1}$ by just taking edges of length 1 (i.e. \{1, 2\}, \{2, 3\}, ..., \{n−1, n\}, \{n, 1\}), while edges of any length other than 1 or $d$ cost strictly greater than $2^{k+1}$. Now consider any Hamiltonian cycle using only edges from these cheapest two stripes, and consider it as a directed cycle as in Proposition 3.6. Suppose that it uses $s_1$ edges of length 1 (where we interpret a directed edge $(u, u+1)$ as having length $-1$), $s_{-1}$ edges of length $-1$ (where we interpret a directed edge $(u, u-1)$ as having length $-1$), and $n - s_1 - s_{-1}$ edges of length $d = 2^k$. Because $n$ is even, there is no difference between an edge of length $d$ and $-d$: $v + d \equiv_n v - d$.

**Claim 4.2.** Any Hamiltonian cycle satisfies

$$s_1 - s_{-1} \equiv_n 0.$$  

With the notation above, a Hamiltonian tour uses $n - s_1 - s_{-1}$ edges of length $d$. Since it starts and ends at the same vertex,

$$2^k (n - s_1 - s_{-1}) + s_1 - s_{-1} \equiv_n 0 \implies s_1 - s_{-1} \equiv_n 2^k (s_1 + s_{-1}).$$  

(1)

Since $n$ and $2^k$ are even, $(2^k (s_1 + s_{-1})) \mod n$ is even; for the left and right sides to have the same parity, $s_1 - s_{-1}$ must therefore also be even. Moreover

$$s_1 + s_{-1} = s_1 - s_{-1} + 2s_{-1}$$

so that $s_1 + s_{-1}$ is the sum of two even numbers and is therefore even. Consider again Equation (1). Since $s_1 + s_{-1}$ is even and $2^k = \frac{n}{2}$,

$$2^k (s_1 + s_{-1}) \equiv_n 0$$

(See Figure 7 for a picture of the corresponding subtour LP solution.)
Figure 7: An example of a class of instances showing that the integrality gap of the subtour LP restricted to circulant instances is at least 2. The dashed edges have weight 1/2 and cost 1, while the full edges have weight 1 and cost 0.

Thus Equation (1) implies

\[ s_1 - s_{-1} \equiv_n 0, \]

and Claim 4.2 follows.

Since \( s_1, s_{-1} \in [n] \), we have that

\[ s_1 - s_{-1} \in \{-n, 0, n\}. \]

The cases where \( |s_1 - s_{-1}| = n \) imply a tour only using edges of length 1; i.e., a tour of cost \( n = 2^{k+1} \). Thus we need only consider the case where \( s_1 = s_{-1} \). Here we analogize an argument from Theorem 5.2 in Greco and Gerace [13].

Claim 4.3. A tour using just edges of lengths 1, -1 and \( d \) visits \( \max\{s_1, s_{-1}\} + 1 \) components in of \( C\langle\{2^k\}\rangle \).

We note that the graph \( C\langle\{2^k\}\rangle \) using just edges of length \( 2^k \) has \( 2^k \) connected components \( C^1_1, C^1_2, ..., C^1_{2^k} \). We identify \( C^1_1 \) as consisting of the two vertices \( \{i, 2^k + i\} \) connected by a single edge of length \( 2^k \).

Let \( L = (e_1, ..., e_n) \) be a list of edges in any Hamiltonian tour using just edges of lengths 1, -1 and \( d \), so that \( e_i \in \{-1, 1, d\} \) for \( i = 1, ..., n \). From this list, we can bound the number of components of \( C\langle\{2^k\}\rangle \) visited: first, we can delete any edges of length \( d \): they do not cause us to change components of \( C\langle\{2^k\}\rangle \); any length 1 edge connects \( C^1_i \) to \( C^1_{i+1} \), while any length -1 edge connects \( C^1_i \) to \( C^1_{i-1} \) (regardless of whether or not any length \( d \) edges are used). Hence we need only consider the subsequence \( L' \) of \( L \) just consisting of edges of lengths 1 and -1 obtained by deleting the edges of length \( d \). Formally,

\[
L' = (e_{i_1}, ..., e_{i_k}) : i_1 < i_2 < \cdots < i_k, e_{i_j} \in \{\pm 1\}.
\]

We upper bound the number of components of \( C\langle\{2^k\}\rangle \) visited directly from \( L' \) as follows: Set \( U = 1 \), corresponding to starting at some component. Until \( L' \) is either all 1s or all -1s, find an occurrence of a 1 followed by a -1 in \( L' \) (or a -1 followed by a 1); delete these two elements and increment \( U \) by 1. Once this process terminates, increment \( U \) by \( |L'| \) (the number of 1s or -1s remaining when \( L' \) is either all 1s or all -1s). Note that, at the end, \( U = \max\{s_1, s_{-1}\} + 1 \).
provides an upper bound on the number of components of $C\langle\{2^k\}\rangle$ visited: Any time a 1 is followed by a −1 in $L$, the effect is to move from $C^1_i$ to $C^1_{i+1}$, then back to $C^1_i$. Hence we visit at most one new component, $C^1_{i+1}$. It is analogous anytime a −1 is followed by a 1. Thus Claim 4.3 holds.

Since any Hamiltonian cycle must visit every component of $C\langle\{2^k\}\rangle$, Claim 4.3 implies that we need

$$\max\{s_1, s_{-1}\} + 1 \geq \frac{n}{2} = 2^k.$$ 

That is, we need at least $2^k - 1$ length 1 edges, or $2^k - 1$ length −1 edges, to connect all components. Putting Claims 4.2 and 4.3 together, we find that we need

$$s_1, s_{-1} \geq 2^k - 1,$$

so that $OPT_{TSP} \geq 2^{k+1} - 2$. We can find such a tour to establish equality:

$$\{1,2\}, \{2,3\}, \ldots, \{2^{k-1}, 2^k\}, \{2^k, n\}, \{n, n-1\}, \ldots, \{2^{k+2}, 2^k+1\}, \{2^k+1, 1\}.$$

Thus

$$\frac{OPT_{TSP}}{VDV} = \frac{2^{k+1} - 2}{2^k} \to 2.$$ 

\[\square\]

5 Conclusions

Theorems 3.1 and 4.1 characterize the subtour LP when restricted to circulant instances: its optimal solution has an explicit combinatorial formulation given in Theorem 3.1 and based entirely on how connectivity changes from $C\langle\{\phi(1), \ldots, \phi(i-1)\}\rangle$ to $C\langle\{\phi(1), \ldots, \phi(i)\}\rangle$ for stripes $i = 1, \ldots, \ell$. Moreover, the integrality gap of the subtour LP on circulant instances is exactly two.

Our hope is also that this paper reinvigorates interest in several compelling open questions: What inequalities can be added to the subtour LP to strengthen its integrality gap on circulant instances? Are there stronger linear programs for circulant instances, and do they translate to metric, symmetric TSP?

With respect to adding inequalities to remove our bad instances (see Figure 7 for an example), we note that our instance achieving the worst-case integrality gap also appears in Naddef and Rinaldi [28] where they construct explicit facet-defining inequalities that remove it from the subtour LP in a non-circulant setting. These inequalities are the crown inequalities and take the form

$$\alpha^T x \geq \alpha_0 := 12s(s-1) - 2, \quad n = 4s,$$

where the weight $\alpha_e$ that $\alpha$ places on edge $e$ is based only on the length of edge $e$:

$$\alpha(v, v+j) = \begin{cases} 4s - 6 + j, & j < d \\ 2(s-1), & j = d. \end{cases}$$

Here, for example, the crown inequalities place a weight of $2(s-1)$ on each of the $d$ edges from the $d$th stripe, and a weight of $4s - 5$ on each edge in the first stripe. The subtour LP solution places a weight of 1 on each of the $d$ edges of length $d$, and 1/2 on each of the $n$ length 1 edges. Since $d = 2s$:

$$\alpha^T x = 2s(2s - 2) + \frac{1}{2}4s(4s - 5) = 2s(6s - 7) = 12s^2 - 14s < \alpha_0 = 12s^2 - 12s - 2.$$ 

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so that they are violated for any example where \( n = 4s \) and \( s > 1 \).

Unfortunately, adding these constraints does not reduce the integrality gap from 2. We can instead consider solutions to the subtour LP that place marginally less weight on the \( d \)-edges and marginally more weight on the 1-edges. If we let \( \lambda \) be the weight on the \( n \) edges of length 1 (on which \( \alpha \) places weight \( 4s - 5 = n - 5 \)), then \( 2 - 2\lambda \) is the weight on each of the \( d = \frac{n}{2} \) edges of length \( d \) (on which \( \alpha \) places a weight of \( 2s - 2 = \frac{n}{2} - 2 \)). The right hand side of the crown inequalities is \( 12\frac{n}{4} (\frac{n}{4} - 1) - 2 = \frac{3}{4}n^2 - 3n - 2 \), so we can solve for

\[
\lambda n(n-5) + (2-2\lambda) \frac{n}{2} \left( \frac{n}{2} - 2 \right) \geq \frac{3}{4}n^2 - 3n - 2 \rightarrow \lambda \geq \frac{n^2 - 4n - 8}{2n^2 - 12n} = \frac{1}{2} + \frac{2}{3n} + \frac{1}{3(n-6)}
\]

(assuming that \( n > 6 \)). Hence, setting

\[
\lambda = \frac{n^2 - 4n - 8}{2n^2 - 12n} = \frac{1}{2} + \frac{2}{3n} + \frac{1}{3(n-6)}
\]

suffices to find a solution that satisfies the subtour elimination constraints and the crown inequalities, but does not reduce the integrality gap.

**Proposition 5.1.** Adding the crown inequalities does not change the integrality gap of the subtour LP when restricted to circulant instances.

**Proof.** We take our solution above, setting

\[
\lambda = \frac{n^2 - 4n - 8}{2n^2 - 12n} = \frac{1}{2} + \frac{2}{3n} + \frac{1}{3(n-6)}
\]

and placing a weight \( \lambda \) on the 1-edges (the dashed edges in Figure 7) and \( 2 - 2\lambda \) on the edges of length \( d \) (the full edges in Figure 7). Note that this solution is still feasible for the subtour LP: we are taking a convex combination of the instance in Theorem 4 and the Hamiltonian cycle using just 1-edges. This thus lower bounds the integrality gap as:

\[
\frac{\text{OPT}_{\text{TSP}}}{\text{OPT}_{\text{LP}}} = \frac{n-2}{n\lambda} \rightarrow 2
\]
as \( n \rightarrow \infty \), where \( n = 2^{k+1} \).

We note that the ladder and chain inequalities (see Boyd and Cunningham [3], Padberg and Hong [30]) can similarly be added to remove the solutions constructed in Theorem 4 but do not reduce the integrality gap from 2.

We conjecture that the following inequalities are valid.

**Conjecture 5.2.** The following inequality, if valid, would strengthen the subtour LP in the symmetric circulant case. If \( 4|n \), then

\[
\sum_{i=1}^{n-1} \alpha_i \left( \sum_{e \in E: \text{length}(e) = i} x_e \right) \geq n - 2, \quad \alpha_i = \begin{cases} i, & \text{if } i \text{ odd} \\ d - i, & \text{if } i \text{ even.} \end{cases}
\]

Finally, as noted earlier, it is a major open question whether or not circulant TSP is polynomial-time solvable. The answer is not known even in the case where only two stripes have finite cost. It would be interesting to see if some of the tools developed recently for the metric TSP might be able to resolve this decades-long open question.
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A Appendix: Two Approximation for Circulant TSP

In this appendix, we sketch the 2-approximation algorithm for circulant TSP given in Gerace and Greco [13]. This 2-approximation algorithm is motivated by a heuristic Van der Veen, Van Dal, and Sierksma [35] developed for the case where every stripe has distinct cost. The algorithm only adds edges of length $g_i^\phi < g_{i-1}^\phi$. For simplicity of exposition, we'll suppress the dependence on $\phi$ and assume that

$$n = g_0 < g_1 < g_2 < \cdots < g_\ell = 1$$

as in Section 3.

A.1 Case 1: $g_{\ell-1}$ is Even

This algorithm is most straightforward when $g_{\ell-1}$ is even: First, it builds Hamiltonian paths on each component of $C\{\phi(1), \ldots, \phi(\ell - 1)\}$. It then deletes one edge from $g_{\ell-1} - 1$ of these paths. Finally, it adds $2(g_{\ell-1} - 1)$ of length $\phi(\ell)$. See Figure 8.

More specifically, construct a Hamiltonian path on the vertices in the component of $C\{\phi(1), \ldots, \phi(\ell - 1)\}$ containing vertex 1 using the nearest neighbor rule starting at vertex 1. Call this path $P_1$ and let $z$ be the other endpoint of $P_1$. Let $C_{\ell}^{\ell-1}$ be the component of $C\{\phi(1), \ldots, \phi(\ell - 1)\}$ containing vertex $1 + (i - 1)\phi(\ell)$ (as usual, here and throughout we implicitly consider all vertices mod $n$). Translate $P_1$ to a Hamiltonian path $P_i$ on the vertices in $C_{\ell}^{\ell-1}$: add $(i-1)\phi(\ell)$ to the label of every vertex in $P_1$. See Figure 8.

If $g_{\ell-1}$ is even, the algorithm deletes $g_{\ell-1} - 2$ edges: pick some edge $\{u, v\}$ in $P_1$. Delete the corresponding edge in each $P_2, P_3, \ldots, P_{g_{\ell-1}-1}$: delete the edge $\{u + (i - 1)\phi(\ell), v + (i - 1)\phi(\ell)\}$ from $P_i$. Form a Hamiltonian cycle on the entire vertex set by adding $2(g_{\ell-1} - 1)$ edges of length $\phi(\ell)$ as in Figure 8.

Specifically, add the following edges:

- Add the edges $\{1, 1 + \phi(\ell)\}, \{1 + 2\phi(\ell), 1 + 3\phi(\ell)\}, \ldots, \{1 + (g_{\ell-1} - 2)\phi(\ell), 1 + (g_{\ell-1} - 1)\phi(\ell)\}$. Also add the edges $\{z, z + \phi(\ell)\}, \{z + 2\phi(\ell), z + 3\phi(\ell)\}, \ldots, \{z + (g_{\ell-1} - 2)\phi(\ell), z + (g_{\ell-1} - 1)\phi(\ell)\}$. This adds $g_{\ell-1}$ edges of length $\phi(\ell)$.

Figure 8: Translations of a Hamiltonian path $P_1$ to other components of $C\{\{12, 6\}\}$ for a graph where $n = 24, \phi(1) = 12, \phi(2) = 6$, and $\phi(3) = 5$. In this example, $z = 7$. 
Figure 9: Constructing a Hamiltonian path when \( g_{\ell-1} \) is even. In this case, \( n = 24, \phi(1) = 12, \phi(2) = 6 \) and \( \phi(3) = 5 \). We pick \( \{u, v\} = \{1, 13\} \).

**Proposition A.1.** Consider any circulant instance where \( g_{\ell-1} \) is even. Let \( \text{OPT}_{\text{TSP}} \) denote the optimal cost of a Hamiltonian tour on the circulant instance. Then the above algorithm produces a Hamiltonian tour of cost at most \( 2\text{OPT}_{\text{TSP}} \).

**Sketch.** By construction, the above algorithm produces a Hamiltonian tour. We can analyze its cost in 3 steps:

1. When we start with \( g_{\ell-1} \) paths (each Hamiltonian on a component of \( C(\{\phi(1), ..., \phi(\ell-1)\}) \)), we have used all of the edges in a minimum-cost Hamiltonian path on \( [n] \) except those of length \( \phi(\ell) \). In total, these edges cost
   \[
   \sum_{i=1}^{\ell-1} (g_{i-1}^\phi - g_i^\phi)c_{\phi(i)}.
   \]

2. We then delete some edges (translates of \( \{u, v\} \)), which cannot increase the cost.

3. Finally, we add \( 2(g_{\ell-1} - 1) = 2(g_{\ell-1} - g_\ell) \) edges of cost \( \phi(\ell) \).

Hence, we end with a tour costing at most

\[
\sum_{i=1}^{\ell-1} (g_{i-1}^\phi - g_i^\phi)c_{\phi(i)} + 2(g_{\ell-1} - g_\ell)c_{\phi(\ell)} \leq 2\sum_{i=1}^{\ell} (g_{i-1}^\phi - g_i^\phi)c_{\phi(i)} \leq 2\text{OPT}_{\text{TSP}}.
\]

The second inequality follows because \( \sum_{i=1}^{\ell} (g_{i-1}^\phi - g_i^\phi)c_{\phi(i)} \) is the cost of a minimum-cost Hamiltonian path, which lower-bounds the cost of a Hamiltonian tour.

- Add the edges \( \{u + \phi(\ell), u + 2\phi(\ell)\}, \{u + 3\phi(\ell), u + 4\phi(\ell)\}, ..., \{u + (g_{\ell-1} - 3)\phi(\ell), u + (g_{\ell-1} - 2)\phi(\ell)\} \). Also add the edges \( \{v + \phi(\ell), v + 2\phi(\ell)\}, \{v + 3\phi(\ell), v + 4\phi(\ell)\}, ..., \{v + (g_{\ell-1} - 3)\phi(\ell), v + (g_{\ell-1} - 2)\phi(\ell)\} \). This adds \( g_{\ell-1} - 2 \) edges of length \( \phi(\ell) \).
A.2 Case 2: $g_{\ell-1}$ is Odd

If $g_{\ell-1}$ is odd, the algorithm of Gerace and Greco \[13\] proceeds similarly, but the analysis is more involved because the paths $P_1, ..., P_{g_{\ell-1}}$ cannot be connected into a Hamiltonian cycle as before. Instead, the algorithm recursively calls itself to produce a Hamiltonian cycle $H$ in component $C_{\ell}^{g_{\ell-1}}$, as explained below. As before, we take $P_1, ..., P_{g_{\ell-1}}$ to be Hamiltonian paths on the components of $C\langle\{\phi(1), ..., \phi(\ell-1)\}\rangle$, where the endpoints of $P_1$ are vertex 1 and vertex $z$, and each other $P_i$ is a translate of $P_1$. We take edge $\{u, v\}$ of length $\phi(\ell-1)$ in path $P_1$. Without loss of generality, we can assume $H$ contains edge $\{u, v\}$: $H$ contains some edge of length $\phi(\ell-1)$, and we can shift all the vertices in $H$ (adding some multiple of $g_{\ell-1}$ to each vertex) until that edge is $\{u, v\}$.

We then delete edge $\{u, v\}$ and its translates from $H, P_2, P_3, ..., P_{g_{\ell-1}-1}$ and add $2(g_{\ell-1} - 1)$ edges of length $\phi(\ell)$ as in Figure \[13\].

This recursive process will eventually reach one of two halting conditions:

1. It is called to find a Hamiltonian cycle on a component of $C\langle\{\phi(1), ..., \phi(t)\}\rangle$ where $\frac{g_{\ell-1}}{g_{\ell}}$ is even, in which case it proceeds as in Case 1. This cycle is then recursively used to create a Hamiltonian cycle on a component of $C\langle\{\phi(1), ..., \phi(t+1)\}\rangle$, and then on a component of $C\langle\{\phi(1), ..., \phi(t+2)\}\rangle$, and so on until it creates a Hamiltonian cycle on $C\langle\{\phi(1), ..., \phi(\ell-1)\}\rangle$ (following the process described above). Note that $\frac{g_{\ell-1}}{g_{\ell}}$ counts the number of components of $C\langle\{\phi(1), ..., \phi(t)\}\rangle$ that get merged into a component of $C\langle\{\phi(1), ..., \phi(t)\}\rangle$.

2. Otherwise, we recursively call the algorithm until it attempts to produce a Hamiltonian cycle on a component of $C\langle\{\phi(1)\}\rangle$, in which case the Hamiltonian cycle on $C\langle\{\phi(1)\}\rangle$ can be found by following edges of length $\phi(1)$ until a cycle is created. In the case where $\phi(1) = n/2$, we treat $\{1, 1+n/2\}$ as a cycle on $C\langle\{\phi(1)\}\rangle$ consisting of two length $d$ edges.

Proposition A.2. Consider any circulant instance where $g_{\ell-1}$ is odd. Let $OPT_{TSP}$ denote the optimal cost of a Hamiltonian tour on the circulant instance. Then the above algorithm produces a Hamiltonian tour of cost at most $2OPT_{TSP}$.

Sketch. By construction, the above algorithm produces a Hamiltonian tour. We analyze its cost inductively at each stage of the recursion.

Suppose the algorithm recurses until it finds a Hamiltonian cycle on a component of $C\langle\{\phi(1), ..., \phi(t)\}\rangle$ (where possibly $t = 1$). We claim that the cost of the Hamiltonian cycle produced on this component is at most

$$\frac{2}{g_{\ell}} \sum_{i=1}^{t} (g_{i+1} - g_i) c_{\phi(i)}.$$ 

Indeed, if the algorithm halts because $t = 1$, it produces a Hamiltonian cycle consisting of $\frac{n}{g_{\ell}}$ edges of cost $c_{\phi(1)}$ and

$$\frac{n}{g_{\ell}} c_{\phi(1)} \leq 2 \left( \frac{n}{g_{\ell}} - 1 \right) c_{\phi(1)} = \frac{2}{g_{\ell}} \sum_{i=1}^{t} (g_{i+1} - g_i) c_{\phi(i)}.$$ 

\[3\]Specifically:

- Add the edges $\{1 + \phi(1), 1 + 2\phi(1), 1 + 3\phi(1), 1 + 4\phi(1), ..., 1 + (g_{\ell-1} - 1)\phi(1)\}$. Also add the edges $\{z + \phi(1), z + 2\phi(1), z + 3\phi(1), z + 4\phi(1), ..., z + (g_{\ell-1} - 1)\phi(1)\}$. This adds $g_{\ell-1} - 1$ edges of length $\phi(1)$.
- Add the edges $\{u + \phi(1), u + 2\phi(1), u + 3\phi(1), u + 4\phi(1), ..., u + (g_{\ell-1} - 3)\phi(1), u + (g_{\ell-1} - 2)\phi(1)\}$. Also add the edges $\{v + \phi(1), v + 2\phi(1), v + 3\phi(1), v + 4\phi(1), ..., v + (g_{\ell-1} - 3)\phi(1), v + (g_{\ell-1} - 2)\phi(1)\}$. This adds $g_{\ell-1} - 1$ edges of length $\phi(1)$.
Figure 10: The 2-approximation algorithm for circulant TSP when \( g_{t-1} \) is odd. In this case, \( n = 30, \phi(1) = 15, \phi(2) = 5 \) and \( \phi(3) = 2 \). We find the Hamiltonian path \( P_1 = \{1,16\}, \{16,21\}, \{21,6\}, \{6,11\}, \{11,26\} \) so that, e.g., \( P_1 \{1,16\} = 1 \), \( P_1 \{16,21\} = 5 \) and \( P_1 \{21,6\} = 2 \). We pick \( \{u,v\} = \{18,23\} \), and edge of length \( \phi(2) = 5 \). Since \( g_{t-1} = g_2 = 5 \) is odd, we apply the recursive algorithm to find a Hamiltonian cycle on the vertices in \( P_1 \) (i.e., \( C_1^2 \)). This yields the cycle \( \{1,6\}, \{6,11\}, \{11,26\}, \{26,21\}, \{21,16\}, \{16,1\} \), including the edge \( \{u,v\} \), so we don’t need to shift it. We then delete the \( \{u,v\} \) and its translates from \( H, P_2, P_3, \) and \( P_4 \) and reconnect using the thick edges (of length \( \phi(3) = 2 \)). Bolded edges are of length \( \phi(i) \), while the dotted edges correspond to the edges from \( H \) (after \( \{u,v\} \) is removed).

If instead \( t > 1 \), we view the component of \( C(\{\phi(1), \ldots, \phi(t)\}) \) as the graph \( C'(\{\phi(1)/g_t, \phi(2)/g_t, \ldots, \phi(t)/g_t\}) \) with \( \frac{n}{g_t} \) vertices where edges of length \( \phi(i)/g_t \) have cost \( \phi(i)/g_t \). Since \( g_t = \gcd(n, \phi(1), \ldots, \phi(t)) \), this is a well-defined circulant graph\(^4\). Moreover, the algorithm reaching a base case of the recursion and \( t > 1 \) implies that \( \frac{g_{t-1}}{g_t} \) is even, so that the graph \( C'(\{\phi(1)/g_t, \phi(2)/g_t, \ldots, \phi(t-1)/g_t\}) \) with \( \frac{n}{g_t} \) vertices has an even number of components. Thus we can appeal to the analysis of the algorithm introduced in Appendix A.2 and, at the base case of recursion, the algorithm will produce a Hamiltonian tour on a component of \( C(\{\phi(1), \ldots, \phi(t)\}) \) of cost at most

\[
2 \sum_{i=1}^{t} \frac{g_{i-1} - g_i}{g_t} c_{\phi(i)} = \frac{2}{g_t} \sum_{i=1}^{t} (g_{i-1} - g_i) c_{\phi(i)}.
\]

\(^4\)Consider a component of \( C(\{\phi(1), \ldots, \phi(t)\}) \) whose smallest vertex is labeled \( i \). Any vertex in this component with label \( v \) can be relabeled with \( \frac{v-i}{g_t} \), which is an integer: \( v,i \) in the same component of \( C(\{\phi(1), \ldots, \phi(t)\}) \) implies \( v \equiv g_t \, i \). Any edge in this component is of length \( \phi(i) \) for \( 1 \leq i \leq t \), and

\[
u - v = \phi(t) \text{ if and only if } \frac{v-i}{g_t} - \frac{v}{g_t} = \frac{\phi(t)}{g_t}.
\]
We now analyze the algorithm inductively, claiming that at each subsequent iteration of the algorithm, it extends a Hamiltonian cycle on a component of \(C\langle\{\phi(1), \ldots, \phi(k)\}\rangle\) of cost at most \(\frac{2}{g_k} \sum_{i=1}^{k} (g_i - 1) c_{\phi(i)}\) to a Hamiltonian cycle on a component of \(C\langle\{\phi(1), \ldots, \phi(k+1)\}\rangle\) of cost at most \(\frac{2}{g_{k+1}} \sum_{i=1}^{k} (g_i - 1) c_{\phi(i)}\). We do so in the following steps:

1. By assumption, the Hamiltonian cycle on a component of \(C\langle\{\phi(1), \ldots, \phi(k)\}\rangle\) costs at most
   \[
   \frac{2}{g_k} \sum_{i=1}^{k} (g_i - 1) c_{\phi(i)}.
   \]

2. There are \(\frac{2}{g_{k+1}}\) components of \(C\langle\{\phi(1), \ldots, \phi(k)\}\rangle\) that get joined into a component of \(C\langle\{\phi(1), \ldots, \phi(k+1)\}\rangle\). The algorithm produces a minimum Hamiltonian path on the other \(\frac{2g_{k+1}}{g_k} - 1\) components of \(C\langle\{\phi(1), \ldots, \phi(k)\}\rangle\) that merge into \(C\langle\{\phi(1), \ldots, \phi(k+1)\}\rangle\). As in bounding the cost of base case of the recursion, each of these components is equivalent to the circulant graph \(C\langle\{\phi(1)/g_k, \phi(2)/g_k, \ldots, \phi(k)/g_k\}\rangle\) on \(\frac{n}{g_k}\) vertices so that the Hamiltonian path on each of these components will cost
   \[
   \sum_{i=1}^{k} \frac{g_i - 1}{g_k} c_{\phi(i)}.
   \]
   These paths, with our Hamiltonian cycle, together cost at most
   \[
   \frac{2}{g_k} \sum_{i=1}^{k} (g_i - 1) c_{\phi(i)} + \left(\frac{g_k}{g_{k+1}} - 1\right) \sum_{i=1}^{k} \frac{g_i - 1}{g_k} c_{\phi(i)} = \left(\frac{2}{g_k} + \frac{1}{g_{k+1}} - \frac{1}{g_k}\right) \sum_{i=1}^{k} (g_i - 1) c_{\phi(i)}
   \]
   \[
   = \frac{1}{g_{k+1}} \left(\frac{g_k}{g_{k+1}} + 1\right) \sum_{i=1}^{k} (g_i - 1) c_{\phi(i)}
   \]
   \[
   \leq \frac{2}{g_{k+1}} \sum_{i=1}^{k} (g_i - 1) c_{\phi(i)},
   \]
   since \(g_{k+1} \leq g_k\).

3. We then delete some edges, which cannot increase the cost.

4. Finally, we add \(2\left(\frac{g_k}{g_{k+1}} - 1\right)\) edges of length \(\phi(k + 1)\) to form the Hamiltonian cycle on a component of \(C\langle\{\phi(1), \ldots, \phi(k+1)\}\rangle\). In total, these edges cost
   \[
   2 \left(\frac{g_k}{g_{k+1}} - 1\right) c_{\phi(k+1)} = \frac{2}{g_{k+1}} (g_k - g_{k+1}) c_{\phi(k+1)}.
   \]
   Hence, we end with a Hamiltonian cycle on a component of \(C\langle\{\phi(1), \ldots, \phi(k+1)\}\rangle\) costing at most
   \[
   \frac{2}{g_{k+1}} \sum_{i=1}^{k} (g_i - 1) c_{\phi(i)} + \frac{2}{g_{k+1}} (g_k - g_{k+1}) c_{\phi(k+1)} = \frac{2}{g_{k+1}} \sum_{i=1}^{k+1} (g_i - 1) c_{\phi(i)},
   \]
   completing an inductive step.

Applying iteratively until we have a Hamiltonian cycle on the full instance, the total cost of this is at most
   \[
   2 \sum_{i=1}^{\ell} (g_i^\phi - g_i^\phi) c_{\phi(i)} \leq 2OPT_{TSP}.
   \]
The inequality again follows because \( \sum_{i=1}^t (g_{i-1}^\phi - g_i^\phi) c_{\phi(i)} \) is the cost of a minimum-cost Hamiltonian path, which lower-bounds the cost of a Hamiltonian tour. \( \Box \)