MACMAHON’S SUM-OF-DIVISORS FUNCTIONS, CHEBYSHEV
POLYNOMIALS, AND QUASI-MODULAR FORMS

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Abstract. We investigate a relationship between MacMahon’s generalized
sum-of-divisors functions and Chebyshev polynomials of the first kind. This
determines a recurrence relation to compute these functions, as well as proving
a conjecture of MacMahon about their general form by relating them to quasi-
modular forms. These functions arise as solutions to a curve-counting problem
on Abelian surfaces.

1. Introduction

The sum-of-divisors function $\sigma_k(n)$ is defined to be
$$\sigma_k(n) = \sum_{d|n} d^k.$$ For $k = 1$, this has as a generating function
$$A_1(q) = \sum_{k=1}^{\infty} \sigma_1(n) q^n = \sum_{k=1}^{\infty} \frac{q^k}{(1-q^k)^2}.$$ As a generalization of this notion, MacMahon introduces in the paper [5, pp. 303,
309] the generating functions
$$A_k = \sum_{0<m_1<\cdots<m_k} \frac{q^{m_1+\cdots+m_k}}{(1-q^{m_1})^2 \cdots (1-q^{m_k})^2},$$
$$C_k = \sum_{0<m_1<\cdots<m_k} \frac{q^{2m_1+\cdots+2m_{k-1}-k}}{(1-q^{2m_1-1})^2 \cdots (1-q^{2m_{k-1}-1})^2}.$$ These provide generalizations in the following sense.

Fix a positive integer $k$. We define $a_{n,k}$ to be the sum
$$a_{n,k} = \sum s_1 \cdots s_k$$
where the sum is taken over all possible ways of writing $n = s_1m_1 + \cdots + s_km_k$
with $0 < m_1 < \cdots < m_k$. Note that for $k = 1$ this is nothing but $\sigma_1(n)$, the usual
sum-of-divisors function. It can then be shown that we have
$$A_k(q) = \sum_{n=1}^{\infty} a_{n,k} q^n.$$ Similarly, we define $c_{n,k}$ to be
$$c_{n,k} = \sum_{s_1 \cdots s_k}$$
where the sum is over all partitions of $n$ into
\[ n = s_1(2m_1 - 1) + \cdots + s_k(2m_k - 1) \]
with, as before $0 < m_1 < \cdots < m_k$. For $k = 1$ this is the sum over all divisors whose conjugate is an odd number. As for the case of $a_{n,k}$, we have
\[ C_k(q) = \sum_{n=1}^{\infty} c_{n,k}q^n. \]
We recall also that Chebyshev polynomials \[1, p. 101\] are defined via the relation
\[ T_n(\cos \theta) = \cos(n \theta). \]
With these we form the following generating functions.
\[ F(x, q) := 2 \sum_{n=0}^{\infty} T_{2n+1}(x) q^{n^2+n} \]
\[ G(x, q) := 1 + 2 \sum_{n=1}^{\infty} T_{2n}(x) q^{n^2}. \]
The results of this paper are the following.

**Theorem 1.** We have the following equalities:
\[ F(x, q) = \left( q^2 + q^2 \right) \sum_{k=0}^{\infty} A_k(q) x^{2k+1} \]
\[ G(x, q) = \frac{(q; q)_{\infty}}{(-q; q)_{\infty}} \sum_{k=0}^{\infty} C_k(q) x^{2k} \]
where $(a; q)_{\infty} = \prod_{k=0}^{\infty} (1 - aq^k)$.

**Corollary 2.** The functions $A_k(q)$ and $C_k(q)$ can be written as
\[ A_k(q) = \frac{(-1)^k (2k+1)!(q; q)_{\infty}}{(2k)!(q; q)_{\infty}} \sum_{n=0}^{\infty} (-1)^n (2n+1) \frac{(n+k)!}{(n-k)!} q^{2n(n+1)} \]
\[ C_k(q) = \frac{(-1)^k (-q; q)_{\infty}}{(2k)!(q; q)_{\infty}} \sum_{n=1}^{\infty} (-1)^n 2n \frac{(n+k-1)!}{(n-k)!} q^{n^2}. \]

**Corollary 3.** The functions $A_k$ and $C_k$ satisfy the recurrence relations
\[ A_k(q) = \frac{1}{(2k+1)2k} \left( (6A_1(q) + k(k-1)) A_{k-1}(q) - 2q \frac{d}{dq} A_{k-1}(q) \right) \]
\[ C_k(q) = \frac{1}{2k(2k-1)} \left( (2C_1(q) + (k-1)^2) C_{k-1}(q) - q \frac{d}{dq} C_{k-1}(q) \right). \]
Our final result settles a long-standing conjecture of MacMahon. In MacMahon’s paper \[5, p. 328\] he makes the claim
The function $A_k = \sum a_{n,k} q^n$ has apparently the property that the coefficient $a_{n,k}$ is expressible as a linear function of the sum of the uneven powers of the divisors of $n$. I have not succeeded in reaching the general theory...

What we prove is the following.
Corollary 4. The functions $A_k(q)$ are in the ring of quasi-modular forms.

We will also discuss in section 3 some applications of this result to an enumerative problem involving counting curves on abelian surfaces.

2. Proofs

Proof of theorem 1. Beginning with the series $F(x, q)$, and letting $x = 2 \cos \theta$ we find

$$F(x, q) = 2 \sum_{n=0}^{\infty} T_{2n+1}(\cos \theta) q^{n^2+n}$$

$$= 2 \sum_{n=0}^{\infty} \cos ((2n+1)\theta) q^{n^2+n}$$

$$= \sum_{n=0}^{\infty} \left( e^{i(2n+1)\theta} + e^{-i(2n+1)\theta} \right) q^{n^2+n}$$

$$= \sum_{n=0}^{\infty} e^{i(2n+1)\theta} e^{n^2+n} + \sum_{n=0}^{\infty} e^{-i(2n+1)\theta} q^{n^2+n}$$

where in the latter sum, letting $n \mapsto -n - 1$ we obtain

$$F(x, q) = e^{i\theta} \sum_{n=-\infty}^{\infty} e^{2ni\theta} q^{n^2+n}.$$ 

Using the Jacobi triple product [1, p. 497, Thm 10.4.1] we see that this is equal to

$$e^{i\theta} \sum_{n=-\infty}^{\infty} e^{2ni\theta} q^{n^2+n} = e^{i\theta} (-e^{-2i\theta}; q^2)_\infty (-q^2 e^{2i\theta}; q^2)_\infty (q^2; q^2)_\infty$$

$$= \frac{(e^{i\theta} + e^{-i\theta}) (q^2; q^2)_\infty}{x} \prod_{m=1}^{\infty} \left( q + 2 \cos(2\theta) q^{2m} + q^{4m} \right)$$

$$= x(q^2; q^2)_\infty \prod_{m=1}^{\infty} \left( 1 - q^{2m} \right) \left( 1 + x^2 q^{2m} \right)$$

$$= x(q^2; q^2)_\infty \prod_{m=1}^{\infty} \left( 1 + x^2 \frac{q^{2m}}{(1-q^{2m})^2} \right)$$

$$= (q^2; q^2)_\infty \sum_{k=0}^{\infty} A_k(q^2) x^{2k+1}$$

and thus comparing coefficients of $x^{2k+1}$ yeilds the result.

We ply a similar trick for $G(x, k)$. In that case we have

$$G(x, q) = 1 + 2 \sum_{n>0} T_{2n}(\cos \theta) q^n$$

$$= 1 + 2 \sum_{n>0} \cos(2n\theta) q^n$$

$$= \sum_{n=-\infty}^{\infty} e^{2ni\theta} q^n.$$
which, again, by the Jacobi triple product yields
\[
\sum_{n=-\infty}^{\infty} e^{2n i \theta} q^n = (q^2; q^2)_\infty (-qe^{2i \theta}; q^2)_\infty (-qe^{-2i \theta}; q^2)_\infty \\
= (q^2; q^2)_\infty \prod_{m=1}^{\infty} \left(1 + \frac{2 \cos(2 \theta)}{q^{2m-1} + q^{4m-2}} \right) \\
= (q^2; q^2)_\infty \prod_{m=1}^{\infty} \left(1 - q^{2m-1} + x^2 q^{2n-1} \right) \\
= \frac{(q^2; q^2)_\infty (q; q^2)_\infty}{(-q; q)_\infty} \prod_{m=1}^{\infty} \left(1 + x^2 \frac{q^{2m-1}}{(1 - q^{2m-1})^2} \right) \\
= \frac{(q; q)_\infty}{(-q; q)_\infty} \sum_{k=0}^{\infty} C_k(q)x^{2k}
\]
which completes the theorem. \qed

To deduce Corollary 2, we begin by expanding the series \( F(x, q) \) (and similarly, \( G(x, q) \)) in powers of \( x \), i.e.
\[
F(x, q) = xf_0(q) + x^3 f_1(q) + x^5 f_2(q) + \cdots + x^{2k+1} f_k(q) + \cdots \\
G(x, q) = g_0(q) + x^2 g_1(q) + x^4 g_2(q) + \cdots + x^{2k} g_k(q) + \cdots.
\]

Now, it can be shown that the coefficients of \( x^{2k} \) in \( 2T_{2n}(\frac{1}{2}x) \) and of \( x^{2k+1} \) in \( 2T_{2n+1}(\frac{1}{2}x) \) are respectively given by
\[
2n(-1)^{n-k} \frac{(n + k - 1)!}{(n - k)!(2k)!} \quad (-1)^{n-k}(2n + 1) \frac{(n + k)!}{(n-k)!(2k+1)!}
\]
and thus we have
\[
f_k(q) = \frac{(-1)^k}{(2k+1)!} \sum_{n=0}^{\infty} (-1)^n (2n + 1) \frac{(n + k)!}{(n-k)!} q^{2n+k} \\
g_k(q) = \frac{(-1)^k}{(2k)!} 2 \sum_{n=1}^{\infty} (-1)^n n (n + k - 1)! \frac{(n + k)!}{(n-k)!} q^{2n^2}.
\]
As theorem 1 implies that \( f_k(q) = (q^2; q^2)_\infty A_k(q^2) \) and \( g_k(q) = \frac{(q^2; q^2)_\infty}{(-q^2; q^2)_\infty} C_k(q) \), we see that Corollary 2 follows.

Next, letting
\[
f_0(q) = \sum_{n=0}^{\infty} (-1)^n (2n + 1) q^{2n+k} = (q^2; q^2)_\infty \\
g_0(q) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} = \frac{(q; q)_\infty}{(-q; q)_\infty}
\]
and defining the operators \( D_\ell = q \frac{d}{dq} - \ell (\ell - 1) \) and \( D'_\ell = q \frac{d}{dq} - (\ell - 1)^2 \), we then have that

\[
\begin{align*}
    f_k(q) &= \frac{(-1)^k}{(2k+1)!} D_k \cdots D_1 f_0(q) \\
    g_k(q) &= \frac{(-1)^k}{(2k)!} D'_k \cdots D'_1 g_0(q).
\end{align*}
\]

From these formulae we note that the functions \( f_k, g_k \) satisfy the recursion relations

\[
\begin{align*}
    f_k(q) &= \frac{-1}{(2k+1)2k} \left( q \frac{d}{dq} - k(k - 1) \right) f_{k-1}(q) \\
    g_k(q) &= \frac{-1}{2k(2k-1)} \left( q \frac{d}{dq} - (k - 1)^2 \right) g_{k-1}(q).
\end{align*}
\]

Noting again that \( f_k(q) = (q^2; q^3)_k A_k(q^2) \) and \( g_k(q) = \frac{(q;q)_\infty C_k(q)}{(-q;q)_\infty} \), we now obtain the recurrence relation of Corollary 3 between the functions \( A_k(q) \) and \( C_k(q) \).

Our final result requires a bit of explanation. It is well-known that the ring of modular forms as a subring is the ring generated either by \( A_4, A_6 \), and \( A_2 \), or by \( q \frac{d}{dq} \) and \( E_2 \). Noting then that \( A_1(q) = \frac{1 - E_2(q)}{24} \), the recurrence relation from Corollary 3 implies that each \( A_k(q) \) lies in this ring, and hence the conclusion follows.

### 3. Applications

The functions \( A_k(q) \) and \( C_k(q) \) arise naturally in the following problem in enumerative algebraic geometry.

Let \( A \subset \mathbb{P}^N \) be a generic polarized abelian surface. There are then a finite number of hyperplane sections which are hyperelliptic curves of geometric genus \( g \) and having \( \delta = N - g + 2 \) nodes. The number of such curves, \( N(g, \delta) \) is independent of the choice of \( A \) and these numbers can be assembled into a generating function

\[
    F(x, u) = \sum_{g, \delta} N(g, \delta) x^g u^\delta.
\]

The coefficient of \( x^g \) in \( F \) is given by a certain homogeneous polynomial of degree \( g - 1 \) in the functions \( A_k(u^2) \) and \( C_k(u^2) \).

This formula is derived by relating hyperelliptic curves on \( A \) to genus zero curves on the Kummer surface \( A/\pm 1 \). The latter is computed using orbifold Gromov-Witten theory, the Crepant resolution conjecture \([2]\) and the Yau-Zaslow formula \([6]\) \([3]\). This will be described further in the second author's thesis.
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