Nonextensive quantum statistics and saturation of the PMD-SQS optimality limit in hadron-hadron scattering

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Abstract

In this paper, new results on the analysis in hadron-hadron scattering ($\pi N$, $K N$, $\bar{K} N$, etc) are obtained by using the nonextensive quantum entropy and principle of minimum distance in the space of quantum states (PMD-SQS). So, using $[S_{J}(p), S_{\theta}(q), S_{J\theta}(p, q)]$-Tsallis-like scattering entropies, the optimality as well as the nonextensive statistical behavior of the $[J$ and $\theta]$-quantum systems of states produced in hadronic scatterings are investigated in an unified manner. A connection between optimal states obtained from the principle of minimum distance in the space of quantum states (PMD-SQS) [17] and the most stringent (Max-Ent) entropic bounds on Tsallis-like entropies for quantum scattering, is established. The generalized entropic uncertainty relations as well as the correlation between the nonextensivities $p$ and $q$ of the $[J$ and $\theta]$-statistics are proved. New results on the experimental tests of the saturation of the PMD-SQS-optimality limit, as well as on the test of optimal entropic bands obtained by using the experimental pion-nucleon, kaon-nucleon, antikaon-nucleon phase shifts, are presented. The nonextensivity indices $p$ and $q$ are determined from the experimental entropies by a fit with the optimal entropies $[S_{J}^{o}(p), S_{\theta}^{o}(q), S_{J\theta}^{o}(p, q)]$ obtained from the principle of minimum distance in the space of states. In this way strong experimental evidences for the $p$-nonextensivities index in the range $p = 0.6$ with $q = p/(2p - 1) = 3$, is obtained from the experimental data of the $(\pi N, K N, \bar{K} N)$-scattering. The experimental evidence obtained here for the nonextensive statistical behavior of the $(J, \theta)$-quantum scatterings states in the above hadron-hadron scattering can be interpreted as an indirect manifestation of the presence of the quarks and gluons as fundamental constituents of the scattering system having the strong-coupling long-range regime required by the Quantum Chromodynamics.
1 Introduction

In the last time there is an increasing interest in the foundation of a new statistical theory [1,2] valid for the nonextensive statistical systems which exhibit some relevant long range interactions, the memory effects or multifractal structures. It is important to mention here that the Tsallis nonextensive statistical formalism [2] already has been successfully applied to a large variety of phenomena such as (see Ref. [3]): Levy-like and correlated anomalous diffusions, turbulence in electron plasma, self-graviting systems, cosmology, galaxy clusters, motion of Hydra viridissima, classical and quantum chaos, quantum entanglement, reassociation in folded proteins, superstatistics, economics, linguistic, etc. Here, is worth to mention the recent applications of nonextensive statistics to nuclear and high-energy particle physics, namely: electron-positron annihilations [4,5], quark-qluon plasma [6], hadronic collisions [7-12,13], nuclear collisions [14], and solar neutrinos [15,16].

In this paper, some new results on the optimal state analysis of hadron-hadron scattering ($\pi N, KN, \overline{KN}, \text{etc}$), obtained by using the nonextensive quantum entropy [7-9] and principle of minimum distance in the space of quantum states (PMD-SQS) [17], are presented. Then, using $[S_J(p), S_\theta(q), S_J\theta(p), S_J\theta(p,q)]$-Tsallis-like scattering entropies, the optimality as well as the nonextensive statistical behavior of the $[J$ and $\theta]$-quantum systems of states produced in hadronic scatterings are investigated in an unified manner. A connection between optimal states obtained from the principle of minimum distance in the space of quantum states (PMD-SQS) [17] and the most stringent (MaxEnt) entropic bounds on Tsallis-like entropies for quantum scattering, is established. The nonextensivity indices $p$ and $q$ are determined from the experimental entropies by a fit with the optimal entropies $[S^0_J(p), S^0_\theta(q), S^0_J\theta(p,q)]$. In this way strong experimental evidences for the $p$–nonextensivity index in the range $p = 0.6$ with $q = p/(2p - 1) = 3$, are confirmed from the experimental data of the principal hadron-hadron scatterings.

2 Optimality and nonextensive entropy for quantum scattering

2.1 Principle of minimum distance in the space of quantum states

Recently in [17] we described the essential features of the hadron-hadron scattering by using a new principle of optimum called principle of minimum distance in the space of quantum states (PMD-SQS). Then knowledge about the hadron-hadron scattering system (or more concretely, about partial amplitudes) are deduced by assuming that the scattering system behaves as to optimize some given measure of the system effectiveness, e.g., the distance in the Hilbert space of scattering states. Thus the behavior of the scattering system is completely specified by those variational variables (e.g., the partial scattering amplitudes) which are obtained by applying constrained optimization to its effectiveness. The PMD-SQS-optimum principle was formulated in a more general mathematical form by using reproducing kernel Hilbert spaces methods [17-19]. Then,
a new "analytic" quantum physics is developed in terms of the reproducing functions from the reproducing kernel Hilbert spaces (RKHS) of the transition amplitudes. In this new kind of analytic quantum physics the system variational variables are the partial transition amplitudes which are introduced by the development of S-matrix elements in terms of Fourier components, implied by the fundamental symmetry of the quantum interacting system. Here we discuss two very simple cases, namely, the application of PMD-SQS–optimal principle [17] to the (πN, KN, ¯KN)–scatterings.

Therefore, let \( f_{++}(x) \) and \( f_{+-}(x) \) be the scattering helicity amplitudes of the meson-nucleon scattering process:

\[
x = \cos(\theta), \theta \text{ being the c.m. scattering angle. The normalization of the helicity amplitudes } f_{++}(x) \text{ and } f_{+-}(x) \text{ is chosen such that the c.m. differential cross section } \frac{d\sigma}{d\Omega} (x) \text{ is given by}
\]

\[
\frac{d\sigma}{d\Omega} (x) = |f_{++}(x)|^2 + |f_{+-}(x)|^2
\]

Since we will work at fixed energy, the dependence of \( \sigma_{el} \) and \( \frac{d\sigma}{d\Omega} (x) \) on this variable was suppressed. Hence, the helicities of incoming and outgoing nucleons are denoted by \( \mu, \mu' \), and was written as (+),(-), corresponding to \( \left( \frac{1}{2} \right) \) and \( \left( -\frac{1}{2} \right) \), respectively. In terms of the partial waves amplitudes \( f_{J+} \) and \( f_{J-} \) we have

\[
f_{++}(x) = \sum_{J=\frac{1}{2}}^{J_{max}} (J + 1/2)(f_{J-} + f_{J+}) d^{\frac{1}{2}}_{J\frac{1}{2}} (x)
\]

\[
f_{+-}(x) = \sum_{J=\frac{1}{2}}^{J_{max}} (J + 1/2)(f_{J-} - f_{J+}) d^{\frac{1}{2}}_{\frac{1}{2}J} (x)
\]

where the \( d^J_{\mu\nu}(x)\)-rotation functions are given by

\[
d^{\frac{1}{2}}_{\frac{1}{2}J} (x) = \frac{1}{l+1} \cdot \left[ \frac{1+x}{2} \right]^\frac{1}{2} \left[ \frac{1}{2} P_{l+1} (x) - \frac{1}{2} P_{l} (x) \right]
\]

\[
d^{\frac{1}{2}}_{\frac{1}{2}J} (x) = \frac{1}{l+1} \cdot \left[ \frac{1-x}{2} \right]^\frac{1}{2} \left[ \frac{1}{2} P_{l+1} (x) + \frac{1}{2} P_{l} (x) \right]
\]

where \( \frac{1}{2} P_{l} \) are the derivatives of the Legendre polynomials.

Now, the elastic integrated cross section for the meson-nucleon scattering can be expressed in terms of partial amplitudes \( f_{J+} \) and \( f_{J-} \)

\[
\sigma_{el}/2\pi = \int_{-1}^{+1} dx \frac{d\sigma}{d\Omega} (x) = \sum_{J=\frac{1}{2}}^{J_{max}} (2J+1)(|f_{J+}|^2 + |f_{J-}|^2)
\]

Therefore, the variational variables for the \( (0^-1/2^+ \rightarrow 0^-1/2^+) \)–scatterings are the helicity amplitudes \( f_{J+} \) and \( f_{J-} \) while elastic integrated cross section expressed in terms of variational variables by Eq. (6) is taken as the measure of system effectiveness.

Moreover, the elastic integrated cross section is directly related to the concept of quantum distance in the space of states. If \( H \) be the Hilbert space of...
the scattering states, defined on the interval $S \equiv [-1, 1]$, with the inner product $< f, g >$ and the norm $||\cdot||$, given by

$$< f, g > = \int_{-1}^{+1} [f_{++}(x)g_{++}(x) + f_{+-}(x)g_{+-}(x)]dx$$

$$\sigma_{el}/2\pi = \int_{-1}^{+1} \frac{df}{dx}(x)dx = \int_{-1}^{+1} ||f_{++}(x)||^2 + ||f_{+-}(x)||^2|dx = ||f||^2$$

Then, the in general the distance $D(f, g)$ between any two scattering states $f, g \in H$ is given by:

$$D(f, g) = \min_{\phi} ||f - g \exp(-i\phi)|| = [||f||^2 + ||g||^2 - 2 |< f, g >|]^\frac{1}{2}$$

The value of the arbitrary phase $\Phi_{\text{min}}$ for which the distance function $||f - g \exp(-i\Phi)||$ is minimized is called minimum phase (see e.g. Refs. [37-40]) and is given by : $\exp(i\Phi_{\text{min}}) = < f, g > / |< f, g >|$.

If we take $g \equiv 0$ then

$$D(f, 0) = |f| = \left[\frac{\sigma_{el}}{2\pi}\right]^\frac{1}{2}$$

As is seen from the above definition, the quantum distances from the Hilbert space of the scattering amplitudes have just the dimensions of length (e.g. fm, cm, etc.). The detailed presentation of the PMD-SQS as well as some important results on application of reproducing kernel Hilbert space (RKHS) methods to the extremal problems of hadronic scattering can be found in Refs.[12,17,18]. Some definitions and PMD-SQS-predictions are presented without a proof in the Tables 1-2.

2.2 J-nonextensive statistics for the quantum scattering states

We define two kind of Tsallis-like scattering entropies. One of them, namely $S_J(p), p \in R$, is special dedicated to the investigation of the nonextensive statistical behavior of the angular momentum $J$-quantum states, and can be defined by [7]

$$S_J(p) = \left[1 - \sum_{J=\frac{1}{2}}^{J_{\text{max}}} (2J+1)p_J^p\right]/(p-1), \quad p \in R, \quad (7)$$

where the probability distributions $p_J$ are given by

$$p_J = \frac{|f_{J^+}|^2 + |f_{J^-}|^2}{\sum_{J=\frac{1}{2}}^{J_{\text{max}}} (2J+1)(|f_{J^+}|^2 + |f_{J^-}|^2)} \sum_{J=\frac{1}{2}}^{J_{\text{max}}} (2J+1) p_J = 1 \quad (8)$$

Here, it is important to present the following remark about geometric origin of the nonextensivity index $p\in R$.

**Remark 1:** Any Tsallis-like entropy of form (7) can be written in the equivalent form

$$S_J(p) = \left[1 - (||\varphi_J||_{2p})^{2p}\right]/(p-1),$$

where

$$\varphi_{J^\pm} = f_{J^\pm}/\sqrt{\sum_{J=\frac{1}{2}}^{J_{\text{max}}} (2J+1)(|f_{J^+}|^2 + |f_{J^-}|^2)}$$

with $||\varphi_J||^2_{2p} = p_J$, and $\{\varphi_J\} \in L_{2p}$, $||\varphi_J||_{2p} = [\sum_{J=\frac{1}{2}}^{J_{\text{max}}} (2J+1)p_J^p]^{\frac{1}{2p}} = [1 + (1-p)S_J(p)]^{\frac{1}{2p}}$.
and consequently the nonextensivity index is determined by the dimension $2p$ of the Hilbert space $L_{2p}$ of normalized partial helicity amplitudes $\{ J_{Jz}/\sqrt{\sigma_{el}/2\pi} \}$.

### 2.3 $\theta$-nonextensive statistics for the quantum scattering states

In similar way, for the $\theta$–scattering states considered as statistical canonical ensemble, we can investigate their nonextensive statistical behavior by using an angular Tsallis-like scattering entropy $S_{\theta}(q)$ defined as [7]

$$S_{\theta}(q) = \frac{1 - \int_{-1}^{1} dx |P(x)|^q}{(q - 1)}, \quad q \in R \quad (9)$$

where

$$P(x) = \frac{2\pi}{\sigma_{el}} \cdot \frac{d\sigma}{d\Omega}(x), \quad \int_{-1}^{1} P(x)dx = 1 \quad (10)$$

with $\frac{d\sigma}{d\Omega}(x)$ and $\sigma_{el}$ defined by Eqs.(2)-(3) and (6).

**Remark 2**: Any Tsallis-like entropy of form (9) can be written in the equivalent form

$$S_{\theta}(q) = \left[ 1 - \left( ||\phi||_{2q} \right)^{2q} \right] / (q - 1),$$

with $||\phi||_{2q}$ defined by Eq.(9).

$$\phi^{++}(x) = f_{++}(x)/\sqrt{\int_{-1}^{1} dx \left[ |f_{++}(x)|^2 + |f_{+-}(x)|^2 \right]}$$

$$\phi^{+-}(x) = f_{+-}(x)/\sqrt{\int_{-1}^{1} dx \left[ |f_{++}(x)|^2 + |f_{+-}(x)|^2 \right]}$$

with: $|\phi^{++}(x)|^2 + |\phi^{+-}(x)|^2 = P(x)$, and $\phi^{++}, \phi^{+-} \in L_{2q},$ $||\phi||_{2q} = \left[ \int_{-1}^{1} dx |P(x)|^q \right]^{1/2q} = [1 + (1 - q)S_{\theta}(q)]^{1/2q}$

and consequently the nonextensivity index $q$ is strictly determined by the dimension $2q$ of the Hilbert space $L_{2q}$ of the normalized helicity amplitudes $\{ \phi^{++}, \phi^{+-} \}$.

### 2.4 $[J\theta]$–Tsallis-like scattering entropies

Also we can define the following generalized Tsallis-like combined entropy [11,12]

$$S_{J\theta}(p,q) = \left[ 1 - \sum (2J + 1)p^p \int_{-1}^{1} dx |P(x)|^q \right] / (p - 1), \quad p \in R, \quad q \in R \quad (11)$$

The above Tsallis-like scattering entropies possesses two important properties. First, in the limit $k \to 1, k \equiv p, q$, the Boltzmann-Gibbs kind of entropies is recovered:

$$\lim_{p \to 1} S_J(p) = S_J(1) = - \sum (2J + 1)p_J \ln p_J \quad (12)$$

$$\lim_{q \to 1} S_{\theta}(q) = S_{\theta}(1) = - \int_{-1}^{1} dx P(x) \ln P(x) \quad (13)$$
Secondly, these entropies are called Tsallis-like scattering entropies, having the nonextensivity properties in the sense that

\[ S_{A+B}(k) = S_A(k) + S_B(k) + (1 - k)S_A(k)S_B(k), \quad k = p, q \in \mathbb{R} \]  

for any independent sub-systems \( A, B \) (\( p_{A+B} = p_A \cdot p_B \)). Hence, each of the indices \( p \neq 1 \) or \( q \neq 1 \) from the definitions (7) and (9) can be interpreted as measuring the degree nonextensivity.

**Remark 3:** Any Tsallis-like entropy of form (11) can be written in the following equivalent form

\[
S_{J\theta}(p, q) = \left[ 1 - (||\varphi_J||_{2p})^{2p} (||\phi||_{2q})^{2q} \right] / (p - 1)
\]

\[
\left( ||\varphi_J||_{2p} \right)^{2p} (||\phi||_{2q})^{2q} = [1 + (1 - p)S_{J\theta}(p, q)]
\]

### 2.5 The equilibrium distributions for the \([J]\)- and \([\theta]\)-systems of quantum scattering states

We next consider the maximum-entropy (MaxEnt) problem

\[
\max \{ S_J(p), S_\theta(q), S_{J\theta}(q), S_{J\theta}(p, q) \} \quad \text{when } \sigma_{cl} = \text{fixed and } \frac{d\sigma}{d\Omega} = \text{fixed} \]

as criterion for the determination of the "equilibrium" distributions \( p_{me}^J \) and \( p_{me}^\theta \) for the system of quantum states produced by the \((0^{-1/2} \to 0^{-1/2})\)-scattering. The equilibrium distributions, as well as the optimal scattering entropies for the quantum scattering of the spineless particles were obtained in Ref. [8-9]. For the \( J \)-quantum states, in the spin \((0^{-1/2} \to 0^{-1/2})\) scattering case, these distributions are given by:

\[
p_{me}^J = p_{me}^J = \frac{1}{2K_{1/2}^{1+1/2} + 1} = \frac{1}{(J_o + 1)^{1/2}}, \quad \text{for } \frac{1}{2} \leq J \leq J_o, \quad \text{and}
\]

\[
p_{me}^J = 0, \quad \text{for } J \geq J_o + 1 \]  

(16)

while, for the \( \theta \)-quantum states, these distributions are as follows

\[
P_{me}^\theta(x) = P_{me}^\theta(x) = \frac{[K_{1/2}^{1,0}(x, 1)]^2}{K_{1/2}^{1,0}(1, 1)}
\]  

(17)

where \( d_{1/2}^J(x) \) are the d-spin rotation functions (4)-(5) for the spin 1/2 particles, \( \dot{P}_{l}(x) \) are the derivatives of Legendre polynomials. The reproducing kernel [17-19] \( K_{1/2}^{1/2}(x, 1) \) is given by

\[
K_{1/2}^{1/2}(x, 1) = \frac{1}{2} \sum_{1/2}^{J_o} (2J + 1)d_{1/2}^J(x)
\]  

(18)

while the optimal angular momentum \( J_o \) is

\[
(J_o + 1)^2 - 1/4 = 2K_{1/2}^{1/2}(1, 1) = \frac{4\pi}{\sigma_{cl} d\Omega} \frac{d\sigma}{d\Omega}(1)
\]  

(19)
We note that results similar to (16)-(19) can be obtained with the constraint:
\[ \frac{d}{d\theta}(1) = \text{fixed} \] instead of \[ \frac{d}{d\theta}(p) = \text{fixed} \]

In the Table 2 we presented analytic formulas for both maximization problem of form (15).

**Proof:** In this case solving the problem (15) via Lagrange multipliers [20] we obtain that the singular solution \( \lambda_0 = 0 \) exists and is just given by the \([S_j^1(p), S_\theta^1(q), S_{\theta J}^1(p, q)]\) – optimal entropies corresponding to the PMD-SQS-optimal state (see Table 1). Indeed, the problem (15) is equivalent to the following unconstrained extremization problem [20]:

\[
\mathcal{L} \equiv \lambda_0 \{ S_J(p), S_\theta(q), S_{\theta J}(p, q) \} + \lambda_1 \{ \sigma_{cl}/4\pi - \sum (2J + 1) [ | f_J^- |^2 + | f_J^+ |^2 ] \} \\
+ \lambda_2 \{ \frac{d}{d\theta}(1) - \sum (2J + 1) \Re(f_J^+ + f_J^-)]^2 - \sum (2J + 1) \Im(f_J^+ + f_J^-)]^2 \} \rightarrow \text{extremum}
\]

Hence, the solution of the problem (20) in the singular case [20] \( \lambda_0 = 0 \) is reduced just to the solution of the minimum constrained distance in space of quantum states (PMD-SQS):

\[
\sum (2J + 1) \| f_J^- \|^2 + \| f_J^+ \|^2 \text{ when } \frac{d\sigma}{d\Omega} (+1) = \text{is fixed}
\]
with the optimal state solution

\[
f_{++}^{t+1}(x) = f_{++}(+1) \frac{K_{+}^{++}(x, +1)}{K_{++}(+1, +1)}, f_{++}^{t+1}(x) = 0
\]

Therefore, by a straightforward calculus we obtain that the solution of the problem (20) is given by

\[
S_J^1(p) = \left[ 1 - [2K_{+}^{++}(1, 1)]^{1-p} \right] /(p - 1), \quad (23)
\]

\[
S_\theta^1(q) = \left[ 1 - \int_{-1}^{+1} dx \left( \frac{K_{+}^{++}(x, 1)}{K_{++}(1, 1)} \right)^q \right] /(q - 1), \quad (24)
\]

\[
S_{\theta J}^1(p, q) = \left[ 1 - \left( \frac{1}{2K_{+}^{++}(1, 1)} \right)^{p-1} \int_{-1}^{+1} dx \left( \frac{K_{+}^{++}(x, 1)}{K_{++}(1, 1)} \right)^q \right] /(p - 1) \quad (25)
\]
for \( p > 0, q > 0 \), where the reproducing kernel \( K_{+}^{++}(x, 1) \) is given by Eq.(18).

### 2.6 Correlations between \([J]\) and \([\theta]\) - nonextensive statistics

A natural but fundamental question was addressed in Refs. [9-11], namely, what kind correlation (if it exists) is expected to be observed between the nonextensivity indices \( p \) and \( q \) corresponding to the \((p, J)\)-nonextensive statistics described by \( S_J(p) \) and \((q, \theta)\)-nonextensive statistics described by \( S_\theta(q) \)? So, in general, an answer at this question is difficult to give for all values of the nonextensivities
the parameter \( t \) to be correlated via the Riesz-Thorin relation

\[ \frac{1}{2p} + \frac{1}{2q} = 1, \quad \text{or} \quad q = p/(2p - 1) \]  

(26)

and the norm \( M \) of the Fourier transform [Eq. (3)-(4)] is expected to be bounded by

\[ M \equiv \frac{||Tf||_{L_{2q}}}{||f||_{L_{2p}}} \leq 2^{\frac{p-1}{2p}}, \quad \frac{1}{2} < p < 1 \text{ and } q = p/(2p - 1) \]  

(27)

Proof: In our case it was show that the result given by Eq. (26)-(27) is a direct consequence of the Riesz-Thorin interpolation theorem extended to the vector-valued functions. Indeed, let \( T \) be the Fourier transform defined by the helicity scattering amplitude (3) where the partial amplitudes are expressed as follow

\[ f_{Jz} = \frac{1}{2} \int_{-1}^{1+1} \left[ f_{++}(x)d_{\frac{J}{2},+}(x) \pm f_{+-}(x)d_{\frac{J}{2},-}(x) \right] dx \]  

(28)

Then, it was shown that:

\[ \sup\{|f_{++}|^2 + |f_{+-}|^2\}^{1/2} \leq \frac{1}{\sqrt{2}} \int_{-1}^{1+1} [||f_{++}(x)||^2 + ||f_{+-}(x)||^2]^{1/2} dx \]  

(29)

and it was used the Parseval’s formula

\[ \sum_{J}(2J + 1)(|f_{++}|^2 + |f_{+-}|^2) = \int_{-1}^{1+1} [||f_{++}(x)||^2 + ||f_{+-}(x)||^2] dx \]  

(30)

since \([d_{\frac{J}{2},+}(x)]^2 + [d_{\frac{J}{2},-}(x)]^2\)^{1/2} \leq 2^{1/2}. \) This means that we have \( T : L_1 \rightarrow L_{\infty} \) with the norm \( M_1 = 2^{-1/2} \) and \( T : L_2 \rightarrow L_2 \) with the norm \( M_2 = 1. \) Then, using the Riesz-Thorin interpolation theorem for the vector-valued functions (see J. Berth in Ref. [21]) \( T : L_p \rightarrow L_q \) with the norm \( M \) with \((1/p') = (1 - t)/1 + t/2, (1/q') = (1 - t)/\infty + t/2, \) and \( 0 < t < 1. \) Hence, eliminating the parameter \( t \) \( t = (1/2q) - (1/2p) + 1 \) and using the relations \( p' = 2p \) and \( q' = 2q, \) we get not only the condition (26) but also the norm - estimate (27) since according to Riesz-Thorin theorem [21] \( M \leq M_1^{1-t}M_2^t. \)

3 Numerical results

Now, for a systematic experimental investigation of the saturation of the optimality limits in hadron-hadron scattering is necessary to use the formulas from the Table 1 and the available experimental phase-shifts [22-24] to solve the following important problems:

\[ \text{8} \]
• To reconstruct the "experimental" pion-nucleon, kaon-nucleon and antikaon-nucleon scattering amplitudes;
• To obtain numerical values of the experimental scattering entropies \( S_J(q), S_\theta(q) \) from the reconstructed amplitudes;
• To obtain the numerical values of the optimal \( J_o \) from experimental scattering amplitudes and then, to calculate the numerical values for the PMD-SQS-optimal entropies \( S_{J}^{(q)}(q), S_{\theta}^{(q)}(q) \);
• To obtain numerical values for \( \chi^2_J(p) \) or/and \( \chi^2_{\theta}(q) \)-test functions given by

\[
\chi^2_X(k) = \sum_{i=1}^{n_{exp}} \left[ \frac{[S_X(k)]_i - [S_{X}^{(q)}(k)]_i}{|\Delta S_{X}^{(q)}|_i} \right]^2, \quad X \equiv J, \theta; \quad k \equiv p, q \tag{31}
\]

where

\[
\Delta S_{X}^{(q)}(k) = \left| [S_{X}^{(q)}(k)]_{J_o+1} - [S_{X}^{(q)}(k)]_{J_o-1} \right| \tag{32}
\]

are the values of the \( PMD-SQS-optimal \) entropies \( S_{X}^{(q)}(k)_{J_o \pm 1} \) calculated with the optimal angular momenta \( J_o \pm 1 \), respectively. Of course, this procedure is equivalent with assumption of an error of \( \Delta J_o = \pm 1 \) in estimation of the experimental values of the optimal angular momentum \( J_o \). The results obtained in this way are presented in the Fig. 1-3 and Table 3.

3.1 Nonextensivity index \( p \) for the statistics of \( J \)-quantum states

For the investigation of this important problem we use the experimental pion-nucleon [22] and kaon-nucleon [23] as well as antikaon-nucleon phase-shifts [24] for the calculation of \( [S_{J}(p)]_i, [S_{J}^{(q)}(p)]_i \) and \( |\Delta S_{J}^{(q)}|_i \) (see also Tables 1). The values of \( (\Delta S_{J}^{(q)})_i \) are calculated by assuming an error of \( \Delta J_o = \pm 1 \) in the estimation of the optimal angular momentum \( J_o \) from the experimental data [see Eq. (32)]. Then, by using Eq. (31) we can calculate the values of \( \chi^2_J(p) \). The numerical results obtained in this way for \( \chi^2_J(p)/n_D \) for different nonextensivities \( p \) in the interval \( 0.5 \leq k \leq 7.00 \) are presented in the Table 3, respectively. Hence, the results from Table 3 allow us to conclude that the statistics of the system of \( J \)-quantum states are superextensive (superadditive) with values of the nonextensivity index \( p \) in the interval \( 1/2 \leq p \leq 0.6 \). This experimental discovery can be compared with the recent results of Refs. [23-25] about the observed radial density profiles in pure-electron plasmas in Penning traps, which are also consistent with a value of the nonextensivity index around \( p = 1/2 \).

3.2 Nonextensivity index \( q \) for the statistics of \([\theta]\)-quantum states

In similar way, from the experimental pion-nucleon [20], kaon-nucleon [21] and antikaon-nucleon [22] phase-shifts, we obtain the experimental values of \( [S_{\theta}(q)]_i, [S_{\theta}^{(q)}(q)]_i \) and \( |\Delta S_{\theta}^{(q)}(q)|_i \); and, consequently, the experimental values of \( \chi^2_{\theta}(q)/n_D \) presented in Table 3 for the \([(\pi N)_{I=1/2,3/2}: (KN)_{I=0,1}; (KN)_{I=0,1}]-scatterings. \)
From the results of the Table 3 we conclude that the statistics of the system composed from $\theta$–quantum states are subextensive (subadditive) with an index $q \geq 3$.

3.3 Experimental evidence for $(1/2p+1/2q=1)$-nonextensivity correlation

Now, we can give an "experimental" answer to the fundamental question: what kind of correlation (if it exists) is expected to be observed between the nonextensivity indices $p$ and $q$ corresponding to the $(p,J)$-nonextensive statistics described by $S_J(p)$ and $(q,\theta)$–nonextensive statistics described by $S_\theta(q)$? [We remember that the "mathematical" answer is given by Eq. (26)]. Indeed, from Figs.1-3 as well as from the Table 3 we see that the experimental data on the scattering entropies $S_J(p)$ and $S_\theta(q)$ are simultaneously in excellent agreement ($CL > 99\%$) with the $[S_J^0(p), S_\theta^0(q)]$–optimal state predictions if the nonextensivities $p$ and $q$ of the $(J$ and $\theta)$–statistics are correlated via Riesz-Thorin relation: $1/p + 1/q = 2$ (or $q = p/(2p - 1)$). So, the best fit is obtained (see Tables 3) for the correlated pairs $p$ and $q = p/(2p - 1)$ with the values of $p$ in the range $p = 0.6$ and $q = p/(2p - 1) = 3$.

4 Conclusions

In this paper, by introducing $[S_J(p), S_\theta(q), S_J\theta(p,q)]$-Tsallis-like entropies, the saturation of the optimality limits as well as the nonextensive statistical behavior of the $(J$ and $\theta)$-quantum states produced in hadronic scatterings are investigated in an unified manner for the pure isospin $\pi N \to (\pi N)_{I=1/2,3/2};$ $KN \to (KN)_{I=0,1};$ $\overline{KN} \to (\overline{KN})_{I=0,1}$-scattering states. The main results and conclusions can be summarized as follows:

- Using the available experimental phase shifts analysis we calculated the numerical values for the $[S_J(p), S_\theta(q), S_J\theta(p,q)]$-Tsallis-like scattering entropies for the pure isospin $I$–scattering states: $[(\pi N)_{I=1/2,3/2}; (KN)_{I=0,1}; (\overline{KN})_{I=0,1}]$;
- We presented strong experimental evidence for the saturation of the $[S_J^0(p), S_\theta^0(q), S_J\theta^0(p,q)]$–PMD-SQS optimal limits for all nonextensive $(J, \theta, J\theta)$-statistical ensembles of quantum states produced in hadron-hadron scattering (see Figs.1-3 and Table 3). These results allow to conclude that the $[J]$-quantum system and $[\theta]$-quantum system are produced at "equilibrium" but with the $[1/2p + 1/2q]$-conjugated nonextensivities $p=0.6$ and $q = p/(2p - 1) = 3$ in all investigated isospin scattering states: $[(\pi N)_{I=1/2,3/2}; (KN)_{I=0,1}; (\overline{KN})_{I=0,1}]$. So the "geometric origin" of the nonextensivities $p$ and $q$ (as dimensions of the Hilbert spaces $L_{2p}$ and $L_{2q}$) as well as their correlations are experimentally confirmed with high accuracy ($CL > 99\%$);
- The strong experimental evidence obtained here for the nonextensive statistical behavior of the $(J, \theta)$–quantum scatterings states in the pion-nucleon, kaon-nucleon and antikaon-nucleon scatterings can be interpreted
as an indirect manifestation the presence of the quarks and gluons as fundamental constituents of the scattering system having the strong-coupling long-range regime required by the Quantum Chromodynamics.

Finally, we note that further investigations are needed since this saturation of optimality limits as well as nonextensive statistical behavior of the quantum scattering, emphasized here with high accuracy (CL > 99%), can be a signature of a new universal law of the quantum scattering.

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Table 1: The optimal distributions, reproducing kernels, optimal entropies, entropic bands, for the \((0^{-1/2^+} \rightarrow 0^{-1/2^+})\)–scattering

| Nr. | Name | \((0^{-1/2^+} \rightarrow 0^{-1/2^+})\)–scattering | See Ref. |
|-----|------|-------------------------------------------------|----------|
| 1   | Optimal inequalities | \(\frac{d\mathcal{M}}{dx}(x) \leq \frac{K_{\frac{1}{2} \pm \frac{1}{2}}(x, \pm 1)}{K_{\frac{1}{2} \pm \frac{1}{2}}(1, \pm 1)} \| f \|^2\) | [7] |
| 2   | Optimal states | \(f^{o+1}(x) = f(1)\frac{K_{\frac{1}{2} \pm \frac{1}{2}}(x, \pm 1)}{K_{\frac{1}{2} \pm \frac{1}{2}}(1, \pm 1)}\), \(f^{o-1}(x) = f(-1)\frac{K_{\frac{1}{2} \pm \frac{1}{2}}(x, \pm 1)}{K_{\frac{1}{2} \pm \frac{1}{2}}(-1, \pm 1)}\) | [11] |
| 3   | Reproducing kernels \(K_{\frac{1}{2} \pm \frac{1}{2}}(x, \pm 1)\) | \(K_{\frac{1}{2} \pm \frac{1}{2}}(x, +1) = \sum_{J_o} \left( J + \frac{1}{2} \right) d_{J}^{1} \left( x \right) \left( \begin{array}{c} J + \frac{1}{2} \end{array} \right) \left( \begin{array}{c} J - \frac{1}{2} \end{array} \right) \) \(K_{\frac{1}{2} \pm \frac{1}{2}}(x, -1) = \sum_{J_o} \left( J + \frac{1}{2} \right) d_{J}^{-1} \left( x \right) \left( \begin{array}{c} J + \frac{1}{2} \end{array} \right) \left( \begin{array}{c} J - \frac{1}{2} \end{array} \right) \) \(2K_{\frac{1}{2} \pm \frac{1}{2}}(\pm 1, \pm 1) = (J_o + 1)^2 - 1/4\) | [9-12, 17] |
| 4   | Optimal distributions \(P^{o \pm 1}(x)\) | \(P^{o+1}(x) = \frac{K_{\frac{1}{2} \pm \frac{1}{2}}(x, 1)}{K_{\frac{1}{2} \pm \frac{1}{2}}(1, 1)}\), \(P^{o-1}(x) = \frac{K_{\frac{1}{2} \pm \frac{1}{2}}(x, -1)}{K_{\frac{1}{2} \pm \frac{1}{2}}(-1, -1)}\) | [7-9] |
| 5   | Optimal distribution \(\{p_j^o\}\) | \(p_j^{o \pm 1} = \frac{2K_{\frac{1}{2} \pm 1}(\pm 1, \pm 1)}{\sum_{J_o} (J_o + 1)^2 - 1/4}\), for \(1/2 \leq J \leq J_o\) \(p_j^{o \pm 1} = 0\) for \(J \geq J_o + 1\) | [7-12] |
| 6   | Number of optimal states for \(y = 1\) | \(N_o = \sum (2J + 1) = (J_o + 1)^2 - 1/4 = 2K_{\frac{1}{2} \pm \frac{1}{2}}(\pm 1, \pm 1)\) | [7-8] |
| 7   | Optimal angular – momentum \(J_o\) | \(J_o = \left\{ \left[ \frac{4\pi}{\sqrt{\pi}} \sum_{\pm 1} \left( \begin{array}{c} J + \frac{1}{2} \end{array} \right) \left( \begin{array}{c} J - \frac{1}{2} \end{array} \right) \right]_{J_{\text{min}}}^{J_{\text{max}}} \right\}^{1/2} - 1\) | [7-8] |
| 8   | Optimal entropy \(S^{o \pm 1}_L(p)\) | \(S^{o \pm 1}_L(p) = \int_{-1}^{1} dx \left( \frac{K_{\frac{1}{2} \pm \frac{1}{2}}(x, \pm 1)}{K_{\frac{1}{2} \pm \frac{1}{2}}(1, \pm 1)} \right)^{1/2}\) | [9-12] |
| 9   | Optimal entropy \(S^{o \pm 1}_L(q)\) | \(S^{o \pm 1}_L(q) = \int_{-1}^{1} dx \left( \frac{K_{\frac{1}{2} \pm \frac{1}{2}}(x, \pm 1)}{K_{\frac{1}{2} \pm \frac{1}{2}}(1, \pm 1)} \right)^{1/2}\) | [9-12] |
| 10  | Optimal entropy \(S^{o \pm 1}_L(p)\) | \(S^{o \pm 1}_L(p) = \int_{-1}^{1} dx \left( \frac{K_{\frac{1}{2} \pm \frac{1}{2}}(x, \pm 1)}{K_{\frac{1}{2} \pm \frac{1}{2}}(1, \pm 1)} \right)^{1/2}\) | [9-12] |
| 11  | \(J\)–entropic band | \(0 \leq S_j(p) \leq S_j^{o \pm 1}(p)\) | [9-12] |
| 12  | \(\theta\)–entropic band | \(1 - K_{\frac{1}{2} \pm \frac{1}{2}}(1, 1)^{q - 1} \leq S_{\theta}(q) \leq S_{\theta}^{o \pm 1}(q)\) | [9-12] |
| 13  | \(J\theta\)–entropic band \((q = p)\) | \(1 - 2^{1-p} \leq S_{\theta}(q) \leq S_{\theta}^{o \pm 1}(q)\) | [9-12] |
| 14  | \(J\theta\)–entropic band \((q \neq p)\) | \(1 - 2^{1-q} \leq S_{\theta}^{o \pm 1}(q) \leq S_{\theta}(p) \leq S_{\theta}^{o \pm 1}(p)\) | [11, 12] |
Table 2: Examples of optimal angular distributions $P_{o\pm 1}(x)$, optimal logarithmic slope, optimal scaling variable and optimal scaling function

| $J_o$ | $P_{o\pm 1}(x) = \frac{\lambda^{1/2} + \frac{1}{2}x(1 \pm 1/2^2)x^{\mp 1/2}}{\lambda^{1/2} + \frac{1}{2}x(1 \pm 1/2^2)x^{\mp 1/2}}$ |
|-------|--------------------------------------------------------------------------------------------------|
| 1/2   | $\frac{1}{2}(1 \pm x)$                                                                            |
| 3/2   | $\frac{1}{3}(1 \pm x)x^2$                                                                        |
| 5/2   | $\frac{1}{5}(1 \pm x)(5x^2 - 1)^2$                                                                |
| 7/2   | $\frac{1}{7}(1 \pm x)(7x^2 - 3)^2x^2$                                                             |
| 9/2   | $\frac{1}{9}(1 \pm x)(21x^4 - 14x^2 + 1)^2$                                                        |
| 11/2  | $\frac{1}{11}(1 \pm x)(66x^4 - 60x^2 + 10)^2x^2$                                                   |
| 13/2  | $\frac{1}{13}(1 \pm x)(3003x^8 - 3465x^6 + 945x^4 - 35)^2$                                          |
| 15/2  | $\frac{1}{15}(1 \pm x)(51480x^{10} - 72072x^8 + 27720x^6 - 2520)^2x^2$                           |

**

1. Optimal logarithmic slope: $b_{o\pm 1} = \frac{\lambda^{1/2}}{\lambda^{1/2} + \frac{1}{2}x(1 \pm 1/2^2)x^{\mp 1/2}}$
2. Optimal scaling variable: $\tau_{o\pm 1} = 2 [\ln |t| b_{o\pm 1}]^{1/2} = \left(\frac{\lambda^{1/2}}{\lambda^{1/2} + \frac{1}{2}x(1 \pm 1/2^2)x^{\mp 1/2}}\right)^{1/2}$
3. Optimal scaling: $\frac{d\sigma_{o\pm 1}}{d\Omega} = \frac{P(x)}{P(1)} \approx \frac{2J_1(|\tau_{o\pm}|)}{J_1(|\tau_{o\pm}|)} = \text{Bessel function of first order}$
4. Optimal inequality: $b \geq \frac{d}{dt} \ln \left[\frac{d\sigma_{o\pm 1}}{d\Omega}(s,t)\right]_{t=0} \geq \frac{x}{4} \frac{d}{d\Omega}(\lambda^{1/2} + \frac{1}{2}x(1 \pm 1/2^2)x^{\mp 1/2}) - 1 = b_{o\pm 1}$

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Table 3: $\chi^2/n_D$ obtained from comparisons of the experimental scattering entropies: $S_J(p), S_\theta(q), S_{J\theta}(p,q)$, with the optimal entropies: $S_J^{o\pm 1}(p), S_\theta^{o\pm 1}(q), S_{J\theta}^{o\pm 1}(p,q)$, respectively, for the $(\pi N, KN)$-scattering (see the text)

| Hadron-hadron scattering | p | q | $\chi^2/n_D$ | $S_J(p)$ | $S_\theta(q)$ | $S_{J\theta}(p,q)$ | $\overline{S}_{J\theta}(p,q)$ |
|-------------------------|---|---|-------------|---------|-------------|----------------|--------------------------|
| $\pi N \rightarrow (\pi N)_{I=1/2}$ | 0.6 | 3.0 | 0.102 | 0.015 | 8.965 | 0.054 |
| 88 PSA | 1.0 | 1.0 | 0.649 | 0.648 | 124.2 | 0.721 |
| $P_{LAB}=0.02 \div 10$ GeV/c | 3.0 | 0.6 | 143.6 | 2.181 | 1.10 | 312.6 |
| $\pi N \rightarrow (\pi N)_{I=3/2}$ | 0.6 | 3.0 | 0.130 | 0.090 | 8.456 | 0.105 |
| 88 PSA | 1.0 | 1.0 | 0.691 | 1.059 | 89.44 | 0.147 |
| $P_{LAB}=0.02 \div 10$ GeV/c | 3.0 | 0.6 | 209.2 | 3.010 | 8.10 | 63.15 |
| $KN \rightarrow (KN)_{I=0}$ | 0.6 | 3.0 | 0.449 | 0.035 | 13.55 | 0.014 |
| 52 PSA | 1.0 | 1.0 | 0.146 | 0.494 | 33.12 | 0.068 |
| $P_{LAB}=0.1 \div 2.65$ GeV/c | 3.0 | 0.6 | 0.190 | 1.089 | 145.8 | 0.240 |
| $KN \rightarrow (KN)_{I=1}$ | 0.6 | 3.0 | 0.089 | 0.045 | 1.567 | 0.011 |
| 53 PSA | 1.0 | 1.0 | 0.259 | 0.586 | 0.485 | 0.050 |
| $P_{LAB}=0.05 \div 2.65$ GeV/c | 3.0 | 0.6 | 0.303 | 1.030 | 2160.3 | 3.113 |
| $KN \rightarrow (KN)_{I=0}$ | 0.6 | 3.0 | 0.168 | 0.009 | 5.267 | 0.026 |
| 50 PSA | 1.0 | 1.0 | 0.199 | 0.267 | 9.084 | 0.028 |
| $P_{LAB}=0.36 \div 1.34$ GeV/c | 3.0 | 0.6 | 11.38 | 0.551 | 31.56 | 0.150 |
| $KN \rightarrow (KN)_{I=1}$ | 0.6 | 3.0 | 0.062 | 0.010 | 4.254 | 0.001 |
| 50 PSA | 1.0 | 1.0 | 0.064 | 0.196 | 5.065 | 0.007 |
| $P_{LAB}=0.36 \div 1.34$ GeV/c | 3.0 | 0.6 | 16.00 | 0.391 | 31.40 | 0.787 |
Fig. 1: The experimental values of the Tsallis-like entropies $S_J(p)$ for $[(\pi N)_{I=1/2,3/2}; (KN)_{I=0,1,1}]; \bar{(KN)}_{I=0}$ scatterings, obtained from the available experimental phase-shifts [22-24], are compared with the PMD-SQS-optimal state predictions $S_{J}^{\text{opt}}(p)$ given in Table 1 (full curve). The saturation of the PMD-SQS (MaxEnt) optimal limits is evident.
Fig. 2: The experimental values of the Tsallis-like entropies $S_\theta(q)$ for $[(\pi N)_{l=1/2,3/2}; (KN)_{l=0,1} - (KN)_{l=0,1}]$ scatterings, obtained from the available experimental phase-shifts [22-24], are compared with the PMD-SQS-optimal state predictions $S_{\theta}^{o1}(q)$ given in Table 1 (full curve). The saturation of the PMD-SQS (MaxEnt) optimal limits is evident.
Fig 3: The experimental values of the Tsallis-like entropies $S_{J\theta}(p)$ for $[(\pi N)_{I=1/2,3/2}; (KN)_{I=0,1}; (KN)_{I=0,1}]$ scattering, obtained from the available experimental phase-shifts [22-24], are compared with the PMD-SQS-optimal state predictions $S^{opt}_{J\theta}(p)$ given in Table 1 (full curve). The saturation of the PMD-SQS (MaxEnt) optimal limits is evident.